HILBERT MODULAR FORMS AND $p$-ADIC HODGE THEORY

TAKESHI SAITO

Department of Mathematical Sciences
University of Tokyo

We consider the $p$-adic Galois representation associated to a Hilbert modular form. Carayol has shown that, under a certain assumption, its restriction to the local Galois group at a place not dividing $p$ is compatible with the local Langlands correspondence [C2]. In this paper, we show that the same is true for the places dividing $p$, in the sense of $p$-adic Hodge theory [Fo], as is shown for an elliptic modular form in [Sa]. We also prove that the monodromy-weight conjecture holds such representations.

We prove the compatibility by comparing the $p$-adic and $\ell$-adic representations for it is already established for $\ell$-adic representation [C2]. More precisely, we prove it by comparing the traces of Galois action and proving the monodromy-weight conjecture. The first task is to construct the Galois representation in purely geometric way in terms of etale cohomology of an analogue of Kuga-Sato variety and algebraic correspondences acting on it. Then we apply the comparison theorem of $p$-adic Hodge theory [Tj] and weight spectral sequence [RZ], [M] to compute the traces and monodromy operators in terms of the reduction modulo $p$. We obtain the required equality between traces by applying Lefschetz trace formula which has the same form for $\ell$-adic and for cristalline cohomology. We deduce the monodromy-weight conjecture from the Weil conjecture and a certain vanishing of global sections. The last vanishing result is an analogue of the vanishing of the fixed part $(\text{Sym}^{k-2}T_\ell E)^{S_{L_2}(\mathbb{Z}_\ell)}$ for $k > 2$ for the universal elliptic curve $E$ over a modular curve in positive characteristic.

We briefly recall the basic definitions on Hilbert modular forms in Section 1 and an $\ell$-adic representation associated to it in Section 2. The main compatibility result, Theorem 1, and the monodromy-weight conjecture, Theorem 2, are stated at the end of Section 2. We recall a cohomological construction of the $\ell$-adic representation in Section 3. After introducing Shimura curves in Section 4 and recalling its modular interpretation in Section 5, we give a geometric construction of the $\ell$-adic representation in Section 6. We extend the geometric construction to semi-stable...
models in Section 7 and prove Theorems 1 and 2 in Section 8 admitting Proposition 1. The last section 10 will be devoted to the proof of Proposition 1.

The strategy of the proof is the same as in the previous work in [Sa]. An essential part of the work consists of understanding the work of Carayol [C1], [C2]. The author thanks Prof. K. Fujiwara for the suggestion that the author’s earlier proof for totally real fields of odd degree should also work for those of even degree by using Carayol’s construction of ℓ-adic representation. He also thanks for Prof. F. Oort and Prof. A. de Jong for teaching him sufficient conditions for extension of abelian varieties.

Part of this work was done during a stay at JAMI in Johns Hopkins University in 1997, a stay at IHP in p-adic semester in 1997 and a stay at Paris-Nord in 1999. The author would like to thank their hospitality.

1. Hilbert modular form.

First, we briefly recall basic definitions on Hilbert modular forms slightly modifying those in [Sh]. Let $F$ be a totally real number field of degree $g > 1$ and $I = \{\sigma_1, \ldots, \sigma_g\}$ be the set of real embeddings $F \to \mathbb{R}$. We fix a multiweight $k = (k_1, \cdots, k_g, w) \in \mathbb{N}^{I+1}$ which is a $g + 1$-uple of integers satisfying the conditions $w \geq k_i \geq 2$ and $k_i \equiv w \mod 2$. The space $S^{(k)}_c$ of cusp forms of multiweight $k$ is defined as follows.

Let $X^I$ be the $g$-fold self product of the union $X = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$ of the upper and lower half planes. It has a natural left action of $GL_2(\mathbb{R})^I$ by

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d} = \left(\begin{array}{cc} a_i & b_i \\ c_i & d_i \end{array}\right) \in X^I$$

for $\gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = \left(\left(\begin{array}{cc} a_i & b_i \\ c_i & d_i \end{array}\right) \right) \in GL_2(\mathbb{R})^I$ and $\tau = (\tau_i) \in X^I$. It induces a left action of $GL_2(F)$ on $X^I \times GL_2(\mathbb{A}, f)$ defined by $\gamma(\tau, g) = (\gamma(\tau), \gamma g)$. Here $GL_2(F)$ is naturally embedded in $GL_2(\mathbb{A}, F) = GL_2(\mathbb{R})^I \times GL_2(\mathbb{A}, f)$. We also consider the right action of $GL_2(\mathbb{A}, f)$ on $X^I \times GL_2(\mathbb{A}, f)$ defined by $(\tau, g)g' = (\tau, gg')$.

A complex valued continuous function $f = f(\tau, g)$ on $X^I \times GL_2(\mathbb{A}, f)$ is said to be holomorphic if the function $\tau \mapsto f(\tau, g)$ on $X^I$ is holomorphic for each $g$ and the map $g \mapsto (\tau \mapsto f(\tau, g))$ is locally constant on $GL_2(\mathbb{A}, f)$. The actions of $GL_2(F)$ and of $GL_2(\mathbb{A}, f)$ on holomorphic functions on $X^I \times GL_2(\mathbb{A}, f)$ are defined as follows. For $\gamma \in GL_2(F)$ and a holomorphic function $f$ on $X^I \times GL_2(\mathbb{A}, f)$, we define $\gamma^{(k)} f = \gamma^* f$ to be

$$(\gamma^* f)(\tau, g) = \frac{\det(\gamma)^{w+k_i-2}}{(c\tau + d)^k} f(\gamma\tau, \gamma g) = \prod_i \frac{\det(\gamma_i)^{w+k_i-2}}{(c_i\tau_i + d_i)^{k_i}} f(\gamma\tau, \gamma g).$$
Here the image of $\gamma \in GL_2(F)$ in $GL_2(\mathbb{R})$ is denoted by $\left(\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\right)_i$. For $g' \in GL_2(\mathbb{A}_{F,f})$ and a holomorphic function $f$ on $X^I \times GL_2(\mathbb{A}_{F,f})$, we define $g^* f$ by $g'^* f(\tau, g) = f(\tau, gg')$.

For an open compact subgroup $K \subset GL_2(\mathbb{A}_{F,f})$, a modular form of multiweight $k$ and of level $K$ is defined to be a holomorphic function $f$ on $X^I \times GL_2(\mathbb{A}_{F,f})$ invariant under the action of $GL_2(F)$ and $K$. Let

$$
M_{C}^{(k), K} = \left\{ f \bigg| \begin{array}{l}
\text{holomorphic function on } X^I \times GL_2(\mathbb{A}_{F,f}) \text{ such that } \\
g^{(k)*} f = f \text{ for all } g \in GL_2(F) \text{ and } g^* f = f \text{ for all } g \in K
\end{array} \right\}
$$

be the space of modular forms of multiweight $k$ and of level $K$. We put $M_{C}^{(k)} = \bigcup K M_{C}^{(k), K}$ and call its element a modular form of multiweight $k$.

We recall the Fourier expansion of a modular form and the definition of the space of cusp forms. Let $\psi : \mathbb{A}_{F,f} \rightarrow \mathbb{C}^\times$ be the finite part of the additive character

$$
\mathbb{A}_F / F \xrightarrow{\text{Tr}_{F/Q}} \mathbb{A} / \mathbb{Q} \sim (\hat{\mathbb{Z}} \times \mathbb{R}) / \mathbb{Z} \rightarrow \mathbb{R} / \mathbb{Z} \xrightarrow{a \mapsto \exp(2\pi i a)}} \mathbb{C}^\times
$$

and $e_F$ be the function

$$
e_F(\tau, a) = \exp(2\pi i \sum_i \tau_i) \cdot \psi(a)
$$

on $X^I \times \mathbb{A}_{F,f}$. Let $f$ be a modular form of multiweight $k$. Then there exists a function $c_z(\sigma, g, f)$ on $(z, \sigma, g) \in F \times \{\pm 1\}^I \times GL_2(\mathbb{A}_{F,f})$ satisfying

$$
f \left( \tau, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g \right) = \sum_{z \in F} c_z(\text{sgn}(\text{Im } \tau), g, f) e_F(z\tau, zb)
$$

since $f \left( \tau, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g \right) = f \left( \tau + \beta, \begin{pmatrix} 1 & b + \beta \\ 0 & 1 \end{pmatrix} g \right)$ for $\beta \in F$. The Fourier coefficients $c_z(\sigma, g, f)$ are 0 unless $z = 0$ or $\text{sgn}(z)\sigma$ is totally positive. In fact, a modular form is necessarily holomorphic at cusps since we assume $[F : \mathbb{Q}] > 1$. A cusp form is defined to be a modular form $f$ satisfying $c_0(\sigma, g, f) = 0$ for all $(\sigma, g) \in \{\pm 1\}^I \times \mathbb{A}_{F,f}^\times$. Unless $k_i$ are constant, we have $c_0(\sigma, g, f) = 0$ and hence a modular form is necessarily a cusp form. In the following, for a cusp form $f$, we drop $\sigma$ in the notation and write $c_z(g, f) = c_z(\text{sgn}z, g, f)$. We say a cusp form $f$ is normalized if $c_1(1, f) = 1$. Let $S_{C}^{(k)}$ denote the space of cusp forms of multiweight $k$. For an open compact subgroup $K \subset GL_2(\mathbb{A}_{F,f})$, we put $S_{C}^{(k), K} = S_{C}^{(k)} \cap M_{C}^{(k), K}$.

We recall the definition of the Dirichlet series $L(f, s)$ associated to a cusp form $f$. Let $D^{-1} = \{b \in F|\text{Tr}_{F/Q}(O_F b) \subset \mathbb{Z}\}$ be the codifferent ideal and let $\hat{T} =$
\( \hat{O}_F \oplus D^{-1}\hat{O}_F \subset \hat{A}_{F,f}^2 \) be a lattice. For an integral ideal \( n \subset O_F \), we define an open compact subgroup \( K_1(n) \subset GL_{\hat{O}_F}(\hat{T}) \subset GL_2(\hat{A}_{F,f}) \) to be
\[
K_1(n) = \left\{ g \in GL_2(\hat{A}_{F,f}) \mid g\hat{T} = \hat{T}, g \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pmod{n\hat{T}} \right\}
= \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{A}_{F,f}) \mid a, d \in \hat{O}_F, b, c \in D^{-1}\hat{O}_F, c \in D\hat{O}_F, ad - bc \in \hat{O}_F^\times, a \equiv 1 \pmod{n\hat{O}_F}, c \equiv 0 \pmod{nD\hat{O}_F} \}.
\]

Let \( f \in S^{(k),K_1(n)}_C \) be a cusp form. For an idele \( d \in \hat{A}_{F,f}^\times \), the Fourier coefficients \( c_1 \left( \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, f \right) \) depends only on the fractional ideal \( m = (d^{-1}) \) and is 0 unless \( m \subset O_F \). Here \( (d^{-1}) \) denotes the fractional ideal \( d^{-1}\hat{O}_F \cap F \). We put \( c(m, f) = c_1 \left( \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, f \right) \) for an ideal \( m \subset O_F \) by taking an idele \( d \in \hat{A}_{F,f}^\times \) such that \( m = (d^{-1}) \).

Since \( c_z \left( \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, f \right) = \prod_i z_i^{-w_i k_i} c_1 \left( \begin{pmatrix} 1 & 0 \\ 0 & z^{-1}d \end{pmatrix}, f \right) \), we have
\[
f \left( \tau, \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \right) = \sum_{m \subset O_F} c(m, f) \sum_{z \in F^\times, zO_F = (d) m, \sgn z = 1} \prod_i z_i^{-w_i k_i - 2} e_F(z\tau, zd^{-1}b).
\]

We define the Dirichlet series by
\[
L(f, s) = \sum_{m \subset O_F} c(m, f)Nm^{1-s}.
\]

It follows from the strong approximation theorem that a cusp form \( f \in S^{(k),K_1(n)}_C \) is determined by the Fourier coefficients \( c(m, f) \) i.e. by the L-series \( L(f, s) \).

We consider the natural action \( g^* f(\tau, g') = f(\tau, g'g) \) of \( g \in GL_2(\hat{A}_{F,f}) \) on \( S^{(k)}_C \). For \( g \in GL_2(\hat{A}_{F,f}) \) and open compact subgroups \( K, K' \subset GL_2(\hat{A}_{F,f}) \) satisfying \( g^{-1}Kg \subset K' \), the map \( g^* \) sends \( S^{(k),K}_C \) into \( S^{(k),K'}_C \) : \( g^* : S^{(k),K}_C \to S^{(k),K'}_C \). For open compact subgroups \( K' \subset K \subset GL_2(\hat{A}_{F,f}) \), we have the trace map \( \text{Tr} : S^{(k),K'}_C \to S^{(k),K}_C \) defined by \( f \mapsto \sum_{g \in K/K'} g^* f \). Hence by taking the fixed part, we recover \( S^{(k),K}_C = S^{(k)}_C \) \( K \). For an open compact subset \( T \subset GL_2(\hat{A}_{F,f}) \) stable under the actions of an open compact subgroup \( K \subset GL_2(\hat{A}_{F,f}) \) on bothside, the action \( T^* : S^{(k),K}_C \to S^{(k),K}_C \) is defined by \( f \mapsto \sum_{g \in T} g^* f \). If \( T = K g K \) for \( g \in GL_2(\hat{A}_{F,f}) \), it is the same as the composite
\[
S^{(k),K}_C \xrightarrow{g^*} S^{(k),K \cap g K g^{-1}}_C \xrightarrow{\text{Tr}} S^{(k),K}_C.
\]
We define the Hecke operators. Let p be a maximal ideal of \( O_F \). First we consider the case where an open compact subgroup \( K \subset GL_2(\mathbb{A}_{F,f}) \) is the product \( K = K_p \times K^p \) of \( K_p = GL_{O_{F_p}}(T) \) for some \( O_{F_p} \)-lattice \( T \) and a prime-to-p part \( K^p \). We put

\[
T_p = \{ g \in GL_2(F_p) \mid gT \supset T, gT/T \simeq O_F/p \} \times K^p,
\]

\[
R_p = \{ g \in GL_2(F_p) \mid gT \supset T, gT/T \simeq (O_F/p)^{\otimes 2} \} \times K^p.
\]

They define endomorphisms also denoted by \( T_p, R_p \) on \( S^{(k),K}_C \) respectively. They are independent of the choice of \( K \)-stable lattice \( T \) and depends only on \( K \). If \( K \) is of the form \( K_q K^q \) as above for another prime \( q \), the operators \( T_p, R_p, T_q, R_q \) are commutative to each other. Note that \( T_p, R_p \) defined above are closely related to but slightly different from \( T(p), T(p,p) \) in [Sh]. It is analogous to those in [D1].

When \( K = K_1(n) \), for an arbitrary integral ideal \( m \subset O_F \), we put \( T_m = \{ g \in GL_2(\mathbb{A}_{F,f}) \mid g(\hat{O}_F \oplus D\hat{O}_F) \supset \hat{O}_F \oplus D\hat{O}_F, \det g^{-1} \hat{O}_F = m\hat{O}_F \} \)

\[
R_m = \left\{ \begin{array}{ll}
\left( \begin{array}{cc}
a^{-1} & 0 \\
0 & a^{-1}
\end{array} \right) & a \in \mathbb{A}_{F,f}^\times, \ a\hat{O}_F = m\hat{O}_F \\
\emptyset & \text{if } (n, m) = 1,
\end{array} \right.
\]

if otherwise.

and let \( T_m, R_m \) also denote the endomorphism on \( S^{(k),K_1(n)}_C \). When \( m = p \nmid n \), the two definitions are the same. The Hecke operators satisfy the formal equality

\[
\sum T_m N_m^{-s} = \prod_p (1 - T_p N_p^{-s} + R_p N_p^{1-2s})^{-1}.
\]

For a cusp form \( f \in S^{(k),K_1(n)}_C \), by an elementary computation, we see \( c(1, T_m(f)) = c(m, f) \cdot N m \). Suppose a cusp form \( f \in S^{(k),K_1(n)}_C \) is an eigenform for all the Hecke operators \( T_m, R_m \) and let \( \chi_f(m) \) be the eigenvalue of \( R_m \). Then since \( f = 0 \) if \( c(m, f) = 0 \) for all \( m \), we see \( c(1, f) \neq 0 \) and the eigenvalue of \( T_m \) is \( c(m, f) \cdot N m / c(1, f) \). Hence if \( f \) is further normalized, the \( L \)-series has the Euler product

\[
L(f, s) = \prod_p \left( 1 - c(p, f) N_p^{1-s} + \chi_f(p) N_p^{1-2s} \right)^{-1}.
\]

The local Euler factor is \( L_p(f, T) = 1 - c(p, f) N_p T + \chi_f(p) N_p T^2 \).

An irreducible subrepresentation \( \pi \) of the representation \( S^{(k)}_C \) of the adele group \( GL_2(\mathbb{A}_{F,f}) \) is called a cuspidal automorphic representation of multiweight \( k \). It is known that we have a direct sum decomposition \( S^{(k)}_C = \bigoplus \pi \pi \) where \( \pi \) runs cuspidal automorphic representations of multiweight \( k \). For a cuspidal automorphic
representation \( \pi \), there exists a largest ideal \( n \), called the level of \( \pi \), such that the \( K_1(n) \)-fixed part \( \pi^{K_1(n)} = \pi \cap S^{(k),K_1(n)} \) is non-zero. If \( n \) is the level of a cuspidal automorphic representation \( \pi \), the dimension of the space \( \pi \cap S^{(k),K_1(n)} \) is 1. A non-zero element in this space is an eigenform for all the Hecke operators \( T_m, R_m \). Hence, it is generated by a normalized cusp form \( f \). The Fourier coefficients \( c(m, f) \) is determined by the condition that \( c(m, f)Np = n \) is equal to the eigenvalue of the Hecke operator \( T_m \). We call such \( f \) a normalized eigen new form. Since the irreducible representation \( \pi \) is generated by \( f \), the correspondence \( \pi \leftrightarrow f \) between the cuspidal automorphic representations and the normalized new eigenforms is one-to-one. In particular, each irreducible factor \( \pi \) appears only once in the direct sum decomposition above. In the following, we let \( \pi_f \) denote the cuspidal automorphic representation generated by \( f \) for a normalized eigen new form \( f \).

Since a cuspidal automorphic representation \( \pi_f \) is irreducible, the center \( A_{\bar{F}, f} \subset GL_2(\mathbb{A}_{\bar{F}, f}) \) acts on \( \pi_f \) by the so-called central character \( \chi_{\pi_f} : A_{\bar{F}, f} \to \mathbb{C}^\times \). The conductor of the central character \( \chi_{\pi} \) divides the level of \( \pi_f \). For \( p \nmid n \), we have \( \chi_f(p) = \chi_{\pi_f}(p)^{-1} \). Since \( \chi_f|_{F^\times} = \chi_{\pi_f}^{-1}|_{F^\times} = N_{F/Q}^{-w-2} : F^\times \to \mathbb{C}^\times \), the character \( \chi_f \) is an algebraic Hecke character whose algebraic part is \( N_{F/Q}^{-w-2} \). Hence there is a character \( \epsilon_f : A_{\bar{F}, f}/F^\times \to \mathbb{C}^\times \) of finite order and of conductor dividing \( n \) such that \( \chi_f = N_{F/Q}^{-w-2} \cdot \epsilon_f \). Therefore, for \( p \nmid n \), the Euler factor is given by

\[
L_p(f, T) = 1 - c(p, f) \cdot Np \cdot T + \epsilon_f(p) \cdot Np^{-1} \cdot T^2.
\]

Note that it is slightly different from the Euler factor \( L_p(\pi_f, T) \). We take this definition in order to make the formula (2.1) simpler.

To define an \( L \)-structure \( S_L^{(k)} \) of \( S^{(k)} \) over an number field \( L \), we recall the description of the space \( S^{(k),K}_C \) of cusp forms in terms of automorphic bundles [Mi] Chap.III. Let \( S = (S_K)_K \) be the canonical model of the Hilbert modular variety. It is the canonical model \( Sh(G, X) \) over the reflex field \( \mathbb{Q} \) of the Shimura variety defined by \( G = GL_2,F \) regarded as an algebraic group over \( \mathbb{Q} \) and the \( G(\mathbb{R}) \)-conjugacy class \( X^f \) of the homomorphism

\[
h : \mathbb{C}^\times \to \mathbb{C},
\]

\[
a + b\sqrt{-1} \mapsto \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right), \ldots, \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right)
\]

(cf. loc. cit.). For an open compact subgroup \( K \subset GL_2(\mathbb{A}_{\bar{F}, f}) \), the set of complex points \( S_K(\mathbb{C}) \) is given as the double cosets

\[
S_K(\mathbb{C}) = GL_2(F) \backslash X^f \times GL_2(\mathbb{A}_{\bar{F}, f})/K.
\]

The projective system \( S = (S_K)_K \) has a natural action of the group \( GL_2(\mathbb{A}_{\bar{F}, f}) \).
We define an invertible sheaf $\omega^{(k)}$ on $S$ as an automorphic vector bundle $\mathcal{V}(\mathcal{J}^{(k)})$ as follows. Let $\tilde{X}^I = (\mathbb{P}^1)^I$ be the compact dual of $X^I$. It has a natural action of $G_{\mathbb{C}} = (GL_2,\mathbb{C})^I$. Let $\omega$ be the dual of the tautological quotient bundle on $\mathbb{P}^1$. The line bundle $\omega$ has a natural equivariant $GL_2$-action. We define an equivariant $G_{\mathbb{C}}$-action on the line bundle

$$\mathcal{J}^{(k)} = \bigotimes_{i \in I} pr_i^* \omega^{\otimes k_i - 2}$$

on $\tilde{X}^I$ as follows. For each $i \in I$, we define an equivariant action of $GL_2,\mathbb{C}$ on $\omega^{\otimes k_i - 2}$ to be the $\det -\frac{w-k_i}{2}$-times the tensor product of the natural action $\omega$. By taking the tensor product, we define a $G_{\mathbb{C}}$-equivariant action on $\mathcal{J} = \mathcal{J}^{(k)}$. Let $G^c$ denote the quotient of $G$ by $\text{Ker}(N_{\mathbb{K}/F} : F^\times \to \mathbb{Q}^\times)$. Here $F^\times$ denotes the center of $GL_2,F$ as an algebraic group over $\mathbb{Q}$. Since the center $(\mathbb{G}_m,\mathbb{C})^I \subset G_{\mathbb{C}}$ acts by the $-(w-2)$-nd power of the product character, it defines a $G_{\mathbb{C}}^c$-equivariant bundle. Hence as in [Mi] Chap.III, we obtain a $G(\mathbb{A}_F)$-equivariant vector bundle $\omega^{(k)} = \mathcal{V}(\mathcal{J}^{(k)})$ on $S(\mathbb{C}) = (S_K(\mathbb{C}))_K$.

We say an open compact subgroup $K \subset GL_2(\mathbb{A}_F,f)$ is sufficiently small if the following two conditions are satisfied.

1. The quotient $(gKg^{-1} \cap GL_2(F))/(gKg^{-1} \cap F^\times)$ does not have non-trivial element of finite order for all $g \in GL_2(\mathbb{A}_F,f)$.
2. $N_{\mathbb{F}/\mathbb{Q}}(K \cap F^\times)^{w-2} = 1$.

If $K$ satisfies the condition (1), the canonical map $S_{K,F} \to S_K$ is etale for an open subgroup $K' \subset K$ and $S_K$ is smooth over $\mathbb{Q}$. If $K$ satisfies the condition (2), the invertible sheaf $\omega^{(k)}$ is defined on $S_K(\mathbb{C})$. Following the definition, it is straightforward to check that the space $M_{K,\mathbb{C}}^{(k)}$ of modular forms is identified with the space of global sections $\Gamma(S_K(\mathbb{C}), \omega_{S_K} \otimes \omega^{(k)})$ for sufficiently small $K \subset GL_2(\mathbb{A}_F,f)$. Here $\omega_{S_K}$ denotes the canonical invertible sheaf $\Omega_{S_K}^{g}$.

Let $L \subset \mathbb{C}$ be a number field which contains all the conjugates of $F$. Then the $G^c$-equivariant bundle $\mathcal{J}^{(k)}$ is defined over $L$. The $G(\mathbb{A}_F)$-equivariant invertible sheaf $\omega^{(k)}$ has the canonical model $\omega^{(k)}_L$ defined over $L$ by [Mi] Chap.III Theorem 5.1.(a). Hence $M^{(k)}_K(\mathbb{C})$ has a natural $L$-structure $M^{(k)}_K(L) = \Gamma(S_K, \omega_{S_K} \otimes \omega^{(k)}_L)$. Further we put $S^{(k)}_K(L) = M^{(k)}_K(L) \cap S^{(k)}_K(\mathbb{C})$ and get $S^{(k)}_K(\mathbb{C}) = S^{(k)}_K(L) \otimes_L \mathbb{C}$. The last equality is deduced from the fact that the Fourier expansion is defined algebraically using the HBAV-analogue of Tate curve as in [Ka].

We describe the invertible sheaf $\omega^{(k)}$ in terms of the moduli problem. The Hilbert modular variety $S_K$ is the coarse moduli scheme for the following functor: To a scheme $S$ over $\mathbb{Q}$, associate the set of isomorphism classes of abelian schemes $A$ over $S$ upto isogeny of dimension $g$ endowed with a ring homomorphism $F \to \text{End}_S(A)$ such that $\text{Lie} A$ is an invertible $F \otimes_{\mathbb{Q}} \text{OS}$-module, together with a weak polarization.
respect to $F$ and with a level structure modulo $K$. Although it is not representable, for $K$ sufficiently small, there is an abelian scheme $a : A \to S_K$ up to isogeny with multiplication by $F$, which is almost the universal abelian scheme. The cotangent bundle $\omega = a_*\Omega^1_{A/S_K}$ is an invertible $O_{S_K} \otimes F$-module and its dual is Lie $A$. The higher direct image of the relative de Rham complex $H = R^1a_*\Omega^\bullet_{A/S_K}$ is a locally free $O_{S_K} \otimes \mathbb{Q}$ $F$-module of rank 2. By $F \otimes_\mathbb{Q} L = \prod_{i \in I} L$, we have direct sum decompositions $\omega \otimes_\mathbb{Q} L = \bigoplus_i \omega_i$ and $H \otimes_\mathbb{Q} L = \bigoplus_i H_i$. Here $\omega_i = \omega \otimes_{F, \sigma_i} L$, $H_i = H \otimes_{F, \sigma_i} L$ are locally free $O_{S_K} \otimes \mathbb{Q}$ $L$-modules of rank 1 and 2 respectively. Then we have

$$\omega^*_L = \bigotimes_i \left( (\lambda^2 H_i)^{\otimes \frac{w-k}{2}} \otimes \omega_i^{\otimes k_i-2} \right).$$

The representation $S_c^{(k)}$ of $GL_2(A_F, F)$ has an $L$-structure $S^{(k)}_L = \varprojlim_K S^{(k), K}_L$. For a normalized eigen new form $f$, we see that the field $L(f) = L(c(m, f), m \subset O_F)$ generated by the Fourier coefficients is of finite degree over $L$ and that the representation $\pi_f$ has an $L(f)$-structure $\pi_{f, L(f)}$. In fact, let $n$ be the level of $f$ and consider the Hecke algebra $T(n)_L = L[T_m, m \subset O_F] \subset \text{End}_L(S^{(k), K_1(n)}_L)$. Then $L(f)$ is the image of $T(n) \to \mathbb{C}$ defined by the action on the subspace of $S^{(k), K_1(n)}_L$ by $f$ and is of finite degree. The intersection $S^{(k), K_1(n)}_{L(f)} \cap \pi_f$ is identified with $\text{Hom}_{T(n)L(f)}(L(f), S^{(k), K_1(n)}_{L(f)})$ and is of dimension one over $L(f)$. The subrepresentation of $S^{(k)}_{L(f)}$ generated by this line gives an $L(f)$-structure $\pi_{f, L(f)}$. We have direct sum decomposition $S^{(k)}_L = \bigoplus_f \pi_{f, L(f)}$ where $f$ runs the conjugacy classes of eigen newforms $f$ over $L$. The Euler factors $L_p(f, T)$ have the coefficients in the number field $L(f)$.

2. $\ell$-adic representation associated to a Hilbert modular form: Main results.

We recall the definition of the $\ell$-adic representation associated to a Hilbert modular form. Let $F$ be a totally real number field and let $f$ be a normalized new eigen form of multweight $k$. Let $L \subset \mathbb{C}$ be a number field which contains all the conjugate of $F$ and take a finite place $\lambda$ of the number field $L(f)$ generated by the Fourier coefficients $c(m, f)$. Then an $\ell$-adic representation $\rho : G_F \to GL_2(L(f), \lambda)$ is said to be associated $f$, if the following condition is satisfied for almost all finite places $\mathfrak{p}$ of $F$.

The representation $\rho$ is unramified at $\mathfrak{p}$ and the eigen polynomial of the geometric Frobenius $F_{\mathfrak{p}}$ is equal to the Euler factor $L_p(f, T)$

$$\det(1 - \rho(F_{\mathfrak{p}})T) = L_p(f, T) = 1 - c(\mathfrak{p}, f) \cdot N\mathfrak{p} T + \epsilon_f(\mathfrak{p}) \cdot N\mathfrak{p}^{w-1} T^2.$$  

In practice, the finite subset of places to be omitted consists of those dividing the level $n$ of $f$ or the prime $\ell$ below $\lambda$. The existence is established by an accumulation of works of many people [O], [C2], [RT], [BR], [Ta1]. Since it is known to
be irreducible [Ta2] Proposition 3.1, Chebotarev density implies the uniqueness.

In the following, we recall a theorem of Carayol [C2] which asserts not only the existence but also gives a precise description of the restriction to the decomposition group $D_p = \text{Gal}(\overline{F}_p/F_p)$ at finite places $p \nmid \ell$ including those dividing the level $n$. The description is given in terms of local Langlands correspondence, recalled in the following.

Let $\pi_{f,L(f)}$ be the $L(f)$-structure of the cuspidal automorphic representation of $GL_2(\mathbb{A}_{F,f})$ associated to $f$. Let $\pi_{f,L(f)} = \bigotimes_p \pi_{f,L(f),p}$ be the factorization into the tensor product of irreducible admissible representations $\pi_{f,L(f),p} = \pi_{f,L(f)}^{K_1(n)}$ of $GL_2(F_p)$ over $L(f)$. Here $n$ is the level of $f$ and $K_1(n)$ is the prime-to-$p$ component of $K_1(n) = K_1(n)_p \cdot K_1(n)^p$. To attach an $L(f)$-rational representation of Weil-Deligne group to the $L(f)$-representation $\pi_{f,L(f),p}$ of $GL_2(F_p)$, we briefly recall the local Langlands correspondence.

To an irreducible admissible representation $\pi$ of $GL_2(F_p)$, the local Langlands correspondence associates an $F$-semi-simple representation $\pi F$ of the Weil-Deligne group $W(\overline{F}_p/F_p)$ of degree 2. An $F$-semi-simple representation of the Weil-Deligne group is a pair of a semi-simple representation $(\rho, V)$ of the Weil group $W(\overline{F}_p/F_p)$ with open kernel and a nilpotent endomorphism $N$ of $V$ satisfying $\rho(\sigma)N = N\rho(\sigma)^{-1} = N^{n(\sigma)}$. Here $Np$ is the norm of $p$ and $n : W(\overline{F}_p/F_p) \to \mathbb{Z}$ is the canonical surjection sending a geometric Frobenius in $W(\overline{F}_p/F_p)$ to 1. A representation $(\rho, N)$ of the Weil-Deligne group is called unramified if $\rho$ is unramified and $N = 0$.

Among several ways to normalize the local Langlands correspondence, here we consider the so-called Hecke correspondence: $\pi \mapsto \sigma_F(\pi)$ [De]. To describe the normalization, we give the definition for a spherical representation $\pi$ of $GL_2(F_p)$. Let $K = GL_2(O_p)$ and let $\pi_p$ be a prime element of $F_p$. We define $\tau_p$ and $\rho_p$ to be the eigenvalues of the action of the double cosets

$$T_p = \{ g \in GL_2(F_p) | gO^2_{F_p} \supset O^2_{F_p}, \det gO_{F_p} = p^{-1} \} = K \left( \begin{array}{cc} \pi_p^{-1} & 0 \\ 0 & 1 \end{array} \right) K,$$

$$R_p = \{ g \in F_p^\times | gO_{F_p} = p^{-1} \} \cdot GL_2(O_{F_p}) = K \left( \begin{array}{cc} \pi_p^{-1} & 0 \\ 0 & \pi_p^{-1} \end{array} \right) K$$

respectively on the 1-dimensional space $\pi^K$. Then the representation $\sigma_F(\pi)$ is the unramified semi-simple representation characterized by

$$\det(1 - Fr_p T : \sigma_F(\pi)) = 1 - Np^{-1}\rho_p^{-1} \tau_p T + Np^{-1}\rho_p^{-1} T^2.$$

It is the same as to require, for the dual representation $\sigma_F(\pi)$,

$$\det(1 - Fr_p T : \sigma_F(\pi)) = 1 - \tau_p T + \rho_p \cdot Np T^2.$$
If \( \pi \) is defined over a field \( L \) of characteristic 0, the representation \( \sigma_h(\pi) \) is defined over an algebraic closure \( \bar{L} \) and the isomorphism class is invariant under the Galois group \( \text{Gal}(\bar{L}/L) \). In other words, the representation \( \sigma_h(\pi) \) is \( L \)-rational, not necessarily realized over \( L \).

We apply the construction \( \pi \mapsto \sigma_h(\pi) \) to the local component \( \pi_{f,p} \) of a cuspidal automorphic representation. Thus, we obtain an \( F \)-semi-simple \( L(f) \)-rational representation \( \tilde{\sigma}_h(\pi_{f,p}) \) of the Weil-Deligne group \( W'\bar{F}_p/F_p \). For a prime \( p \nmid n(f) \), it is an unramified representation. The equality (1) above is then rephrased as

\[
\det(1 - \rho(F_pT)) = \det(1 - F_pT : \tilde{\sigma}_h(\pi_{f,p})).
\]

It is the same as to say that the semi-simplification of the unramified representation \( \rho_{f,\lambda}|_{W_p} \) is isomorphic to \( \tilde{\sigma}_h(\pi) \).

On the other hand, to an \( \ell \)-adic representation of the local Galois group \( G_p = \text{Gal}(\bar{F}_p/F_p) \), we attach a representation of the Weil-Deligne group \( W'\bar{F}_p/F_p \). First we consider the case where \( p \nmid \ell \). Let \( L_\lambda \) be a finite extension of \( \mathbb{Q}_\ell \). Let \( \rho : G_p \to GL_{L_\lambda}(V) \) be a continuous \( \ell \)-adic representation. Take a lifting \( F \in W(\bar{F}_p/F_p) \) of the geometric Frobenius and an isomorphism \( \mathbb{Z}_\ell(1) \to \mathbb{Z}_\ell \) and identify them. Let \( \ell \) : \( I_p \to \mathbb{Z}_\ell(1) \to \mathbb{Z}_\ell \) be the canonical surjection. Then, by the monodromy theorem of Grothendieck, there is a representation \( \rho' = (\rho', N) \) of the Weil-Deligne group \( W'(\bar{F}_p/F_p) \) characterized by the condition

\[
\rho(F^n\sigma) = \rho'(F^n\sigma) \exp(t_\ell(\sigma)N)
\]

for \( n \in \mathbb{Z} \) and \( \sigma \in I_p \). The isomorphism class of the representation \( (\rho', N) \) of the Weil-Deligne group is independent of the choice of the lifting \( F \) or the isomorphism \( \mathbb{Z}_\ell(1) \to \mathbb{Z}_\ell \) and is determined by \( \rho \).

For an \( \ell \)-adic representation \( \rho \) of \( \text{Gal}(\bar{F}/F) \), let \( \rho_p \) denote the restriction to \( \text{Gal}(\bar{F}_p/F_p) \). Let \( \rho'_p \) denote the representation of the Weil-Deligne group attached to \( \rho_p \) and let \( \rho_p^{F-ss} \) denote its \( F \)-semi-simplification.

**Theorem 0.** [C2] Let \( f \) be a normalized eigen newform of multiweight \( k \) and \( \lambda|\ell \) be a finite place of the number field \( L(f) \). We assume that, if the degree \( g = [F : \mathbb{Q}] \) is even, there exists a finite place \( \nu \) such that the \( \nu \)-factor \( \pi_{f,\nu} \) lies in the discrete series. Then there exists an \( \ell \)-adic representation

\[
\rho = \rho_{f,\lambda} : G_F \to GL_{L(f)_\lambda}(V_{f,\lambda})
\]

satisfying the following property:

For a finite place \( p \nmid \ell \), there is an isomorphism

\[
\rho_{f,\lambda,p}^{F-ss} \cong \tilde{\sigma}_h(\pi_{f,p})
\]
of representations of the Weil-Deligne group \( W(F_p/F_p) \).

**Remark.** Since the right hand side is \( L(f) \)-rational, Theorem implies that so is the left hand side. For \( p \mid n(f)\ell \), the isomorphism means that we have an equality

\[
\det(1 - Fr_p T : \rho_{f,\lambda}) = \det(1 - Fr_p T : \overline{\sigma_h}(\pi)) = L_p(f, T).
\]

Hence \( V_{f,\lambda} \) in Theorem 1 is the \( \ell \)-adic representation associated to \( f \).

In this paper, we study the case \( p \) divides \( \ell \). Let \( p \) be the characteristic of a finite place \( p \) of \( F \). Let \( F_{p,0} \) denoted the maximal unramified subfield in \( F_p \).

We describe the construction attaching a representation of Weil-Deligne group to a \( p \)-adic representation of the local Galois group due to Fontaine [Fo]. Let \( B_{st} \) be the ring defined by Fontaine. It is an \( \hat{F}_{p,0}^{nr} \)-algebra and admits a natural action of the absolute Galois group \( G_{F_p} \), a semi-linear action of the Frobenius \( \varphi \) and an action of the monodromy group \( \pi \). For an open subgroup \( J \subset I \) of the inertia, the fixed part \( B_{st}^J \) is the completion \( \hat{F}_{p,0}^{nr} \) of a maximal unramified extension of \( F_{p,0} \). In this paper, we neglect the filtration. Let \( L_\mu \) be a finite extension of \( \mathbb{Q}_p \) and consider a continuous \( p \)-adic representation \( \text{Gal}(F_p/F_p) \to GL_{L_\mu}(V) \) of finite degree. Let \( \hat{L}_{\mu}^{nr} \) denote the completion of the maximum unramified extension of \( L_\mu \). We choose an arbitrary factor of \( \hat{F}_{p,0}^{nr} \otimes \mathbb{Q}_p L_\mu \). It is the same thing as to fix an embedding \( \hat{F}_{p,0}^{nr} \to \hat{L}_{\mu}^{nr} \). For an \( L_\mu \)-representation \( G_{F_p} \to GL_{L_\mu}(V) \) of finite degree, we put

\[
D(V) = D_{pst}(V) = \bigcup_{J \subset I} (B_{st} \otimes V)^J \otimes (\hat{F}_{p,0}^{nr} \otimes \mathbb{Q}_p L_\mu) \hat{L}_{\mu}^{nr}.
\]

Here \( J \) runs the open subgroups of the inertia subgroup \( I = I_p \) and \( J \) denotes the \( J \)-fixed part. The union \( \bigcup_{J \subset I} (B_{st} \otimes V)^J \) is a \( \hat{F}_{p,0}^{nr} \otimes \mathbb{Q}_p L_\mu \)-module since \( B_{st}^J = \hat{F}_{p,0}^{nr} \). It is known that \( D(V) \) is an \( \hat{L}_{\mu}^{nr} \)-vector space of finite dimension and \( \dim_{\hat{L}_{\mu}^{nr}} D(V) \leq \dim_{L_\mu} V \). We say \( V \) is potentially semi-stable (pst for short) if we have an equality \( \dim_{\hat{L}_{\mu}^{nr}} D(V) = \dim_{L_\mu} V \).

For a pst-representation \( V \), Fontaine defines a natural representation [Fo] on \( D(V) \) of the Weil-Deligne group \( W(\hat{F}_p/F_p) \) as follows [Fo]. By the Galois actions on \( B_{st} \) and on \( V \), the quotient \( G_F/J \) acts on the \( J \)-fixed part \( (B_{st} \otimes V)^J \) for normal \( J \subset G_F \). Passing to the limit, we obtain an action of \( G_F \) acting on the \( \hat{F}_{p,0}^{nr} \otimes \mathbb{Q}_p L_\mu \)-module \( \bigcup_{J \subset I} (B_{st} \otimes V)^J \). The kernel is open in the inertia \( I_p \). This Galois action is semi-linear with respect to its natural action on \( \hat{F}_{p,0}^{nr} \) and the trivial action on \( L_\mu \). We modify it by using the Frobenius \( \varphi \) to get a \( \hat{F}_{p,0}^{nr} \otimes \mathbb{Q}_p L_\mu \)-linear action of the Weil group \( W(\hat{F}_p/F_p) \) as follows.
Let $\mathbb{F}_p$ denote the residue field of $p$. Recall that the Weil group $W(F_p/F_p)$ is the inverse image of the inclusion $\mathbb{Z} \rightarrow \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ sending 1 to the geometric Frobenius $F_{\mathbb{F}_p}$ by the canonical map $\text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \rightarrow \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$. Let $n : W(\overline{\mathbb{F}_p}/F_p) \rightarrow \mathbb{Z}$ be the canonical map and $q = p^n = Np$. Then by let $\sigma \in W(\overline{\mathbb{F}_p}/F_p)$ act on $D(V)$ by $(\varphi^\ell \cdot n(\sigma) \otimes 1) \circ \sigma \circ \sigma$, we get a $\mathbb{F}_p[\rho, \mathcal{O}_v]$-linear action. Taking the $L_{\mu}^{nr}$-component, we obtain an $\mathbb{F}_p$-linear representation $D(V)$ of the Weil group $W(\overline{\mathbb{F}_p}/F_p)$. The monodromy operator $\delta$ on $B_{st}$ induces an $\mathbb{F}_p$-linear nilpotent operator on $D(V)$ satisfying $\sigma N = Np^n(\sigma)N \sigma$ since $\varphi N = pN \varphi$. Thus an $L_{\mu}^{nr}$-linear action $\rho_{\mu, \pi, v}$ of the Weil-Deligne group on $D(V)$ is defined.

We apply the above construction $V \mapsto D(V)$ to the restriction $\rho_{f, \mu, p}$ of the $p$-adic representation associated to $\pi_f$ to the decomposition group $\text{Gal}(\overline{\mathbb{F}_p}/F_p)$ for a place $p|p$. Thus we obtain an $L(f)^{nr}$-representation $\rho_{f, \mu, p}$ of the Weil-Deligne group $W(\overline{\mathbb{F}_p}/F_p)$. Our main result is the following.

**Theorem 1.** Let the assumptions be the same as in Theorem 0 above and let $\mu$ be a place of $L(f)$ dividing the characteristic of a prime $\mathfrak{p}$ of $F$. Then, the representation $\rho_{f, \mu, p}$ of $\text{Gal}(\overline{\mathbb{F}_p}/F_p)$ is potentially semi-stable and there is an isomorphism

$$\rho_{f, \mu, p}^{F_{ss}} \simeq \hat{\delta}_h(\pi_{f, p})$$

of representations of the Weil-Deligne group $W(\overline{\mathbb{F}_p}/F_p)$.

**Remark.** By the semi-stability of $\rho_{f, \mu, p}$, the representation $\rho_{f, \mu, p}$ is of degree 2. Similarly as in the $\ell$-adic case, Theorem implies that the left hand side $\rho_{f, \mu, p}^{F_{ss}}$ is $L(f)$-rational.

By the argument using a quadratic base change as in [C2], we may assume there exists a finite place $v \neq p$ where $\pi_{f, v}$ lies in the discrete series in the case where $g = [F : \mathbb{Q}]$ is even.

We will prove Theorem 1 by comparing $p$-adic cohomology with $\ell$-adic cohomology. Let $\lambda$ be a place of $L(f)$ dividing a prime $\ell \neq p$. By Theorem 0 applied to $\rho_{f, \lambda, p}$, it is enough to compare $\rho_{f, \lambda, p}$ with $\rho_{f, \mu, p}$. More precisely, we prove the following.

**Claim 1.** Let the notation be as in Theorem. Let $p|p$ be a finite place of $F$ and let $\lambda$ and $\mu$ be places of $L(f)$ dividing $\ell \neq p$ and $p$ respectively. Then the following holds.

(0) The representation $\rho_{f, \mu, p}$ is potentially semi-stable.

(1) For $\sigma \in W^+ = \{ \sigma \in W(\overline{\mathbb{F}_p}/F_p) \mid n(\sigma) \geq 0 \}$, we have an equality in some finite extension of $L(f)$

$$\text{Tr} \rho_{f, \lambda, p}(\sigma) = \text{Tr} \rho_{f, \mu, p}(\sigma).$$
(2) Let $N_\lambda$ and $N_\mu$ be the nilpotent monodromy operators for $\rho_{\lambda, \pi, p}$ and $\rho_{\mu, \pi, p}$ respectively. Then $N_\lambda = 0$ if and only if $N_\mu = 0$.

By Lemma 1 [Sa], Theorem 1 follows from Claim 1. In (1), we may allow a finite extension since we already know that the left hand side is in $L(f)$.

The assertion (0) is a special case of (1) where $\sigma = 1$. We deduce the assertion (2) from (1) together with the monodromy-weight conjecture, Theorem 2 below, asserting that the monodromy filtration gives the weight filtration.

Let $V$ be a representation of the Weil-Deligne group $W_p$. We assume $N^2 = 0$. Then $0 \subset W_{-1} V = \text{Image } N \subset W_0 V = \text{Ker } N \subset W_1 V = V$ is a filtration by subrepresentations of $V$. It is called the monodromy filtration. For a lifting $F$ of the geometric Frobenius $\text{Fr}$, the eigenvalues are independent of a choice of lifting. We say an algebraic number is pure of weight $n$ if the complex absolute value of its conjugates are $Np^n$. Then, for an integer $n \in \mathbb{Z}$, we say that the monodromy filtration of $V$ is pure of weight $n$, if the eigenvalues of a lifting $F$ of $\text{Fr}$ acting on $\text{Gr}_i W(V)$ for each $i$ are algebraic numbers of weight $n + i$.

**Theorem 2.** Let the notation be as in Claim 1. Then the monodromy filtration of the representations $\rho_{f, \lambda, p}(F)$ and $\rho_{f, \mu, p}(F)$ of the Weil-Deligne group are pure of weight $w - 1$. In other words, the eigenvalue $\alpha$ of $\rho_{f, \lambda, p}(F)$ for an arbitrary lifting $F \in W(\bar{F}_p/F_p)$ of the geometric Frobenius is of weight $n$ where

$$n = \begin{cases} 
  w - 1 & \text{if } N = 0 \\
  w - 2 & \text{if } N \neq 0 \text{ and } \alpha \text{ is the eigenvalue on } \text{Ker } N \\
  w & \text{if } N \neq 0 \text{ and } \alpha \text{ is the eigenvalue on } \text{Coker } N.
\end{cases}$$

**Remark.** The assertion for the case $N \neq 0$ is easy since we know the determinant and $N : \text{Gr}_1 W(V)(1) \rightarrow \text{Gr}_{-1} W(V)$ is an isomorphism.

We show that Theorem 2 and assertion (1) in Claim 1 imply (2) in Claim 1. In fact, by (1), the eigenvalues of a lifting $F$ of Frobenius are the same for $\lambda$ and $\mu$. By Theorem 2, we distinguish the 2 cases $N = 0$ and $N \neq 0$ by their absolute values. Thus (2) follows from (1) and Theorem 2.

Thus Theorem 1 is reduced to the assertion (1) in Claim 1 and Theorem 2.

3. **Cohomological construction of the $\ell$-adic representation.**

Carayol constructs an $\ell$-adic representation associated to a Hilbert modular form by decomposing the etale cohomology $H^1(M_{K, \bar{F}}, \mathcal{F}_\lambda)$ of a Shimura curve with a coefficient sheaf $\mathcal{F}_\lambda$. Here, we briefly recall the construction with a slight modification. Using the construction, we give a statement, Claim 2, which implies the main results.
First we recall the definition of the Shimura curve. We choose and fix a real place \( \tau_1 \) of the totally real field \( F \) and regard \( F \) as a subfield of \( \mathbb{R} \subset \mathbb{C} \) by \( \tau_1 \). When the degree \( g = [F : \mathbb{Q}] \) is even, we also fix a finite place \( \nu_0 \). Let \( B \) be a quaternion algebra over \( F \) ramifying exactly at the other real places \( \{ \tau_2, \ldots, \tau_g \} \) if \( g = [F : \mathbb{Q}] \) is odd and at \( \{ \tau_2, \ldots, \tau_g, \nu \} \) if \( g \) is even.

Let \( G \) denote \( B^\times \) regarded as an algebraic group over \( \mathbb{Q} \). Let \( X \) be the \( G(\mathbb{R}) \)-conjugacy of the map

\[
\begin{align*}
h : \quad \mathbb{C}^\times & \to G(\mathbb{R}) = (B \otimes_{\mathbb{Q}} \mathbb{R})^\times \\
a + b\sqrt{-1} & \mapsto \left( \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right), 1, \ldots, 1 \right).
\end{align*}
\]

The conjugacy class \( X \) is naturally identified with the union \( \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}) \) of the upper and lower half planes. Let \( M = M(G, X) = (M_K)_K \) be the canonical model of Shimura variety defined for \( G \) and \( X \). It is defined over the reflex field \( F \). Here \( K \) runs the open compact subgroups of \( G(\mathbb{A}^f) = (B \otimes_F \mathbb{A}_{F, f})^\times \). Each \( M_K \) is a proper smooth but not necessarily geometrically connected curve over \( F \). Since the reciprocity map \( F^\times \to G^\times \) is the identity, the constant field \( K_0 \) of \( M_K \) is the abelian extension of \( F \) corresponding to the compact open subgroup \( \text{Nrd}_{B/F} K \subset \mathbb{A}_{F, f}^\times \). The projective system \( (M_K)_K \) has a natural right action of the finite adeles \( G(\mathbb{A}) \). For \( g \in G(\mathbb{A}^f) \) and open compact subgroup \( K, K' \subset G(\mathbb{A}^f) \) such that \( g^{-1}Kg \subset K' \), we have \( g : M_K \to M_{K'} \). The set of \( \mathbb{C} \)-valued points \( M_K(\mathbb{C}) \) are identified with the set of double cosets \( G(\mathbb{Q}) \backslash X \times G(\mathbb{A})^\times / K \). The action of \( G(\mathbb{Q}) = B^\times \) on \( X \) is induced by \( B^\times \to (B \otimes_{F, \tau_1} \mathbb{R})^\times \cong GL_2(\mathbb{R}) \). For \( g, K, K' \) as above, the map \( g : M_K(\mathbb{C}) \to M_{K'}(\mathbb{C}) \) is induced by \( (x, g_1) \mapsto (x, g_1 g) \).

We will define a smooth \( \mathcal{L} \)-sheaf \( \mathcal{F}_\lambda^{(k)} \) on the Shimura curve \( M \). It is the dual of the sheaf denoted \( \mathcal{F}_\lambda \) in [Ca]. We prefer the dual because it is related directly to a direct summand of a cohomology sheaf as we will see in later sections. Let \( k = ((k_1, \ldots, k_g), w) \) be a multiform and put \( n = n(k) = \prod_i (k_i - 1) \). The algebraic group denoted \( G^c \) in [Mi] Chap. III for our group \( G = B^\times \) is the quotient of \( G \) by \( \text{Ker}(N_{F/\mathbb{Q}} : F^\times \to \mathbb{Q}^\times) \). Here we identify algebraic groups over \( \mathbb{Q} \) and their \( \mathbb{Q} \)-valued points and \( F^\times \subset B^\times \) denotes the center of \( G \). In order to define the sheaf \( \mathcal{F}_\lambda^{(k)} \), We define a representation of algebraic group \( \rho = \rho^{(k)} : G \to GL_n \) factoring the quotient \( G^c \) as follows. We have \( B \otimes_{\mathbb{Q}} \mathbb{C} \cong M_2(\mathbb{C})^I \) where \( I = \{ \tau_1, \ldots, \tau_g \} \) is the set of embeddings \( F \to \mathbb{C} \). It induces an isomorphism \( G_{\mathbb{C}} \to GL^I_{2, \mathbb{C}} \). We define the morphism \( \rho = \rho^{(k)} : G \to GL_n \) to be the composite of this isomorphism with the tensor product \( \bigotimes_{i \in I} ((\text{Sym}^{k_i - 2} \otimes \det (w - k_i) / 2) \circ \tilde{p}_{\tau_i}) \). Here \( \tilde{p}_{\tau_i} \) denotes the contragradient representation of the \( i \)-th projection \( GL^I_{2, \mathbb{C}} \to GL_{2, \mathbb{C}} \). Since the restriction of the center \( F^\times \) is the multiplication by \( N_{F/\mathbb{Q}}^{-w - 2} \), it factors through the quotient \( \rho^{(k)} : G^c \to GL_n \).
We take a number field $L \subset \mathbb{C}$ where $\rho = \rho^{(k)} : G \to GL_n$ is defined. For example it is enough to take $L$ containing the conjugates of $F$ and splitting $B$. We identify $\{\tau_i : F \to L\} = \{\tau_i : F \to \mathbb{C}\}$ by the inclusion $L \to \mathbb{C}$. We define the smooth $L_\lambda$-sheaf $\mathcal{F}_\lambda^{(k)}$ on $M$ to be the $L_\lambda$-component of the smooth $L \otimes \mathbb{Q}_\ell$-sheaf $V_\ell(\rho^{(k)})$ attached to the representation $\rho^{(k)}$ (loc. cit. 6). We consider the inductive limit

$$H^1(M_F, \mathcal{F}_\lambda^{(k)}) = \lim_{\rightarrow} H^1(M_{K,F}, \mathcal{F}_{K,\lambda}^{(k)}).$$

By the natural action of $G(\mathbb{A}_F)$ on the projective system $(M_K, \mathcal{F}_{K,\lambda}^{(k)})_K$, it is a representation of $G(\mathbb{A}_F) \times \text{Gal}(\bar{F}/F)$. The structure as a birepresentation is described as follows.

**Lemma 1.** Let $k$ be a multiweight and $L \subset \mathbb{C}$ be a number field splitting $F$ and $B$ as above. Then, we have the following.

(1) Let $\pi_f$ be a cuspidal automorphic representation $\pi_f$ of $GL_2(\mathbb{A}_F)$ of multiweight $k$. If $g = [F : \mathbb{Q}]$ is even, we assume that the $v$-component $\pi_{f,v}$ is in the discrete series. Then the representation $\pi'_f$ of $G(\mathbb{A}_F)$ corresponding to $\pi_f$ by the Jacquet-Langlands correspondence has an $L(f)$-structure $\pi'_{f,L(f)}$.

(2) There exists an isomorphism

$$H^1(M_F, \mathcal{F}_\lambda^{(k)}) \simeq \bigoplus_f \left( \pi'_{f,L(f)} \otimes L(f) \bigoplus_{\lambda' | \lambda} V_{\lambda',f} \right),$$

of representations of $G(\mathbb{A}_F) \times \text{Gal}(\bar{F}/F)$. Here $f$ runs the conjugacy classes over $L$ of normalized eigen newforms of multiweight $k$, such that, when $g = [F : \mathbb{Q}]$ is even, the $v$-component $\pi_{f,v}$ lies in the discrete series.

Using Lemma 1, we reduce Theorems to a statement below, Claim 2, on the cohomology. Let $K \subset G(\mathbb{A}_F)$ be a sufficiently small open compact subgroup. We take a sufficiently divisible integral ideal $n$ of $O_F$, divisible by $p$ and by $v$ if $g$ is even.

We assume $K$ is of the form $K = K_nK^n$. Here $K_n \subset \prod_{\mathfrak{r}|n} B_{\mathfrak{r}}^\times$ is an open compact subgroup and $K^n = \prod_{\mathfrak{r}|n} GL_2(O_{F,\mathfrak{r}})$ for some isomorphism $\prod_{\mathfrak{r}|n} B_{\mathfrak{r}} \simeq \prod_{\mathfrak{r}|n} M_2(F_{\mathfrak{r}})$. Let $T^n = L[T_{\mathfrak{r}}; \mathfrak{r} \mid n]$ be the free $L$-algebra generated by the Hecke operators $T_{\mathfrak{r}}$ for $\mathfrak{r} \mid n$. We consider $H^1(M_{K,F}, \mathcal{F}_{\lambda}^{(k)})$ as a $T^n$-module.

**Claim 2.** Let $K \subset G(\mathbb{A}_F)$ be a sufficiently small open compact subgroup and let $n \subset O_F$ be a sufficiently divisible ideal. We assume $K = K_nK^n$ as above. Then,

(0) The representation $H^q(M_{K,F}, \mathcal{F}_\lambda^{(k)})$ of $G_{F_n}$ for $q = 0, 1, 2$ is potentially semistable.
(1) For $\sigma \in W^+$ and $T \in T^n$, we have equalities in a finite extension of $L$

$$\sum_{q=0}^{2} (-1)^q \text{Tr}(\sigma \circ T|H^q(M_{K,F},\mathcal{F}_\chi^{(k)})) = \sum_{q=0}^{2} (-1)^q \text{Tr}(\sigma \circ T|D(H^q(M_{K,F},\mathcal{F}_\mu^{(k)})))$$

(2) For the representations $H^1(M_{K,F},\mathcal{F}_\chi^{(k)})$ and $D(H^1(M_{K,F},\mathcal{F}_\mu^{(k)}))$ of the Weil-Deligne group $\mathcal{W}_{F_s}$, their monodromy filtrations are pure of weight $w-1$.

We prove that the assertions in Claim 2 imply the corresponding assertions (0) and (1) in Claim 1 and Theorem 2, admitting Lemma 1. Let $f$ be a normalized eigen new cuspform of multiweight $k$. By Lemma 1, we have a cuspidal automorphic representation $\pi_f'$ of $G'(\mathbb{A}_F)$ defined over $L(f)$. Replacing $L$ by $L(f)$ if necessary, we may assume $L = L(f)$. Let $K$ be a sufficiently small open compact subgroup satisfying $\pi_f^{rk} \neq 0$ and Claim 2. The representations $V_{f,\lambda}$ and $V_{f,\mu}$ are direct summands of $H^1(M_{K,F},\mathcal{F}_\chi^{(k)})$ and $H^1(M_{K,F},\mathcal{F}_\mu^{(k)})$ by Lemma 1 respectively. Hence the assertion (0) in Claim 2 implies the assertion (0) in Claim 1 and the assertion (2) in Claim 2 implies Theorem 2.

We show that the equality (1) of the traces in Claim 2 implies the equality (1) in Claim 1. First we show that the equality for the alternating sum implies the equality for each piece

$$\text{Tr}(\sigma \circ T|H^r(M_{K,E},\mathcal{F}_\chi^{(k)})) = \text{Tr}(\sigma \circ T|D(H^r(M_{K,E},\mathcal{F}_\mu^{(k)})))$$

for $r = 0, 1, 2$. In fact, it is sufficient to show the equality for $r = 0, 2$.

We show that $H^0 = H^2 = 0$ if $k \neq ((2, \cdots, 2), w)$. The fundamental group $\pi_1(M_{K,\mathcal{F}})$ of the geometric fiber is isomorphic to $\text{Ker}(\text{Nrd}_{B/F} : K \to \hat{O}_F^\times)$. Hence its Lie algebra generates $B^0 = \text{Ker}(\text{Trd}_{B/F} : B \to F)$ over $F$. Hence the Lie algebra is $B^0 \otimes_{\mathbb{Q}} L \simeq \mathfrak{s}l_2(L)g$ is also generated by the Lie algebra of $\pi_1(M_{K,\mathcal{F}})$ over $L$. It follows easily from this that the representation of $\pi_1(M_{K,\mathcal{F}})$ corresponding to the sheaf $\mathcal{F}_\chi$ hence the sheaf itself is irreducible. Hence its largest geometrically constant subsheaf and quotient sheaves are 0 unless $k = ((2, \cdots, 2), w)$.

We assume $k = ((2, \cdots, 2), w)$ and we show the equality for $r = 0, 2$. Then the sheaf $\mathcal{F}_\chi^{(k)}$ is defined by the character $N_{F/F}^{w-2} \circ \text{Nrd}_{B/F} : G \to \mathbb{G}_m$ and is isomorphic to the Tate twist $L_{\lambda}(-\frac{w-2}{2})$. It is sufficient to show the assertion for $H^0$ since $H^2 \simeq H^0(-1)$. Let $F_K = \Gamma(M_K, O)$ be the constant field of $M_K$. Then there is an isomorphism

$$H^0(M_K, \mathcal{F}_\chi^{(k)}) \simeq \lim_{\rightarrow} H^0(F_K, L_{\lambda}(-\frac{w-2}{2}))$$
of $G_F \times G(\mathbb{A}_f)$-module. On the right hand side, the Galois action is the natural one. The action of $G(\mathbb{A}_f)$ is defined by that induced by its action on $\varprojlim_K \text{Spec } F_K$ multiplied by the character

$$G(\mathbb{A}_f) \xrightarrow{\mu_{F/Q} \circ \text{Nrd}_{B/F}} \mathbb{A}_f^\times/\mathbb{Q}^{\times} \xrightarrow{\zeta} \hat{\mathbb{Z}}^\times \rightarrow \mathbb{Z}_\lambda^\times \subset L_\lambda^\times.$$  

From this, we easily deduce the equality for $r = 0$.

We deduce the equality (1) in Claim 1 from the equality above for $r = 1$. By the strong multiplicity one theorem, the image of the Hecke algebra $T^n$ in $\text{End}_L(S_L^K)$ is $\prod_f L(f')$ where $f'$ runs the conjugacy class of eigen newforms $f'$ as in Lemma 1 such that $\pi_{f', K} \neq 0$. Let $e \in T^n$ be an element whose image is the idempotent corresponding to the component $L(f) = L$. Then if we put $d = \dim \pi_{f, K}'$, we see that $e \cdot H^1(M_{K, \bar{F}}, \mathcal{F}_{\lambda}^{(k)})$ is isomorphic to the direct sum $\hat{\lambda}_h(\pi)^{\oplus d}$ by Lemma 1 (2). Hence we have

$$d \cdot \text{Tr}' \rho_{\lambda, f, p}(\sigma) = \text{Tr}(\sigma \circ e | H^1(M_{K, \bar{F}}, \mathcal{F}_{\lambda}^{(k)}))$$

$$d \cdot \text{Tr}' \rho_{\mu, f, p}(\sigma) = \text{Tr}(\sigma \circ e | D(H^1(M_{K, \bar{F}}, \mathcal{F}_{\mu}))).$$

Thus the equality (1) in Claim 1 follows from that in Claim 2. It is clear that the assertion (2) in Claim 2 implies Theorem 2.

Therefore Theorems 1 and 2 are reduced to Claim 2 and Lemma 1.

**Proof of Lemma 1.** We will define an admissible representation $S_L'$ of $G(\mathbb{A}_f)$ over $L$ satisfying the following properties

1') $S_L' \otimes_L \mathbb{C} \simeq \bigoplus_f \pi_f'$ as a representation of $G(\mathbb{A}_f)$ over $\mathbb{C}$.

2') $H^1(M_{\bar{F}}, \mathcal{F}_{\lambda}^{(k)}) \simeq S_L'^{\otimes 2} \otimes_L L_{\lambda}$ as a representation of $G(\mathbb{A}_f)$ over $L_{\lambda}$.

First we prove Lemma 1 assuming we have such a representation $S_L'$. We show $\pi_f'$ is defined over $L(f)$. If $g = [F : Q]$ is odd, we have $G(\mathbb{A}_f) \simeq \text{GL}_2(\mathbb{A}_{F, f})$ and $\pi_f = \pi_{f}'$ and there is nothing to prove. We show the case $g$ is even. It is enough to show that each factor $\pi_{f, r}$ of $\pi_f' = \bigotimes_v \pi_{f, v}'$ is defined over $L(f)$. Let $n$ be the level of $f$ and $K(n) = K_1(n) \cdot K_1(n)^r \subset \text{GL}_2(\mathbb{A}_{F, f})$. Then the representation $\pi_{f, r}$ is given as the fixed subspace $\pi_{f, r} = \pi_{f, r \otimes 1}^{K_1(n)^r}$ and is defined over $L(f)$. For $\tau \neq q$, we have $\pi_{f, r} = \pi_{f, q}'$, and it is defined over $L(f)$. Finally we consider the case $\tau = q$. Then by (1), we see that the intertwining space $\text{Hom}_{G(\mathbb{A}_f)}(\bigotimes_{\tau \neq q} \pi_{f, \tau, L(f)}', S_L \otimes_L L(f))$ is an $L(f)$-structure of $\pi_{f, q}'$.

Next we show the isomorphism (2). We put

$$V_{\lambda', f} = \text{Hom}_{G(\mathbb{A}_f)}(\pi_{f, L(f)}', L(f)_{\lambda'}, H^1(M_{\bar{F}}, \mathcal{F}_{\lambda}^{(k)})).$$
Then by (2'), each $V_{\lambda', f}$ is an $L(f)_{\lambda'}$-representation of $G_F$ of degree 2 and we have

$$H^1(M_{\bar{\varphi}}, \mathcal{F}^{(k)}_{\lambda}) \simeq \bigoplus_f \left( \pi'_f \otimes L(f) \bigoplus_{\lambda' | \lambda} V_{\lambda', f} \right).$$

Therefore it is sufficient to show that the $\ell$-adic representation $V_{\lambda', f}$ is associated to $f$: $V_{\lambda', f} \simeq \tilde{\sigma}_h(\pi_f)$. We may extend the scalar to $\tilde{L}_\lambda$. Hence it is enough to show that $H^1(M_{\bar{\varphi}}, \mathcal{F}^{(k)}_{\lambda}) \otimes \tilde{L}_\lambda$ admits a direct sum decomposition of the form

$$H^1(M_{\bar{\varphi}}, \mathcal{F}^{(k)}_{\lambda}) \otimes \tilde{L}_\lambda \simeq \bigoplus_{\pi} \pi \otimes \tilde{\sigma}_h(\pi).$$

In [C2], it is shown that for $\mathcal{F}_\lambda$, we have a direct sum decomposition of the form $H^1(M_{\bar{\varphi}}, \mathcal{F}_\lambda) \simeq \bigoplus_{\pi'} \pi' \otimes \tilde{\sigma}_h(\pi')$. Since $\mathcal{F}^{(k)}_{\lambda}$ here is the dual of $\mathcal{F}_\lambda$ there, we have $H^1(M_{\bar{\varphi}}, \mathcal{F}^{(k)}_{\lambda}) \simeq \bigoplus_{\pi'} \tilde{\pi'} \otimes \sigma_h(\pi')(-1)$ by Poincaré duality. Since $\tilde{\sigma}_h(\tilde{\pi'}) \simeq \sigma_h(\pi')(-1)$, the claim is proved.

We define the space $S'_L$. First we will define the automorphic vector bundle ([Mi] Chap.III) $\mathcal{V}(\mathcal{J})$ associated to a $G^c$-equivariant vector bundle $\mathcal{J} = \mathcal{J}^{(k)}$ on the compact dual $\hat{X}$ and its canonical model $\mathcal{V}(\mathcal{J})_L$. Then it will be defined as the limit of the spaces of global sections

$$S'_L = \Gamma(M \otimes_F L, \Omega^1_M \otimes \mathcal{V}(\mathcal{J})_L) = \lim_K \Gamma(M_K \otimes_F L, \Omega^1_M \otimes \mathcal{V}(\mathcal{J})_L).$$

We use the notation loc. cit. The compact dual $\hat{X}$ is $\mathbb{P}^1_C$ in our case. We define a $G^c$-equivariant vector bundle $\mathcal{J} = \mathcal{J}^{(k)}$ on $\hat{X}$ in the following way. Let $\omega$ be the dual of the tautological quotient bundle on $\hat{X} = \mathbb{P}^1_C$. We put $\mathcal{J}^{(k)} = \omega^\otimes k_1 - 2 \otimes \bigotimes_{i=2}^g \text{Sym}^{k_{i-2}}(C \otimes \mathbb{Z})$. We define the action of $G_C = GL_2, C^I$ on $\mathcal{J}^{(k)}$ by giving the action of each factor in the following way. The first factor $GL_2, C$ acts on $\hat{X}$ in the natural way. On $\omega^\otimes k_1 - 2$, we consider $\det^{-\frac{w-k_1}{2}}$-times the natural action. For $i \neq 1$, the $i$-th factor $GL_2, C^I$ acts on $\hat{X}$ trivially. On $\text{Sym}^{k_{i-2}}(C \otimes \mathbb{Z})$, we consider $\det^{-\frac{w-k_i}{2}}$-times the action induced by the contragradient action of $GL_2$. By taking the tensor product, we obtain a $G_C$-equivariant bundle $\mathcal{J} = \mathcal{J}^{(k)}$. Since the center $\mathbb{C}_m$ acts by $-(w-2)$-nd power of the product character, it defines a $G^c_\mathbb{C}$-equivariant bundle. It is clearly defined over the number field $L \supset F$. Hence by [Mi] Chap.III Theorem 5.1.(a), we obtain a $G(A_f)$-equivariant vector bundle $\mathcal{V}(\mathcal{J})_L$ on $M_L$. Thus the representation $S'_L = \Gamma(M \otimes L, \Omega^1_M \otimes \mathcal{V}(\mathcal{J})_L)$ is defined.

By the Jacquet-Langlands correspondence [JL], we have $S'_C = S'_L \otimes_L C \simeq \bigoplus_f \pi'_f$, where $f$ runs cuspidal automorphic representation of $GL_2(A_{F,f})$ of multiweight {$k$.
such that, if \([F : \mathbb{Q}]\) is even, the \(q\)-component \(\pi_{f,q}\) is in the discrete series. Hence the representation \(S'_L\) satisfies the property (1') above.

We show the isomorphism (2'). Attached to the representation \(\rho^{(k)} : G^c \to GL_n\) defined over \(L\), we have a local system \(\mathcal{F}^{(k)} = V(\rho^{(k)})\) of \(L\)-vector spaces on \(M(\mathbb{C})\). Since \(H^1(M_F, \mathcal{F}_\lambda^{(k)}) \simeq H^1(M(\mathbb{C}), \mathcal{F}^{(k)}) \otimes_L L_\lambda\), it is sufficient to show \(H^1(M(\mathbb{C}), \mathcal{F}^{(k)}) \otimes_L \mathbb{C} \simeq S'_L \oplus 2 \otimes \mathbb{C}\) as a representation of \(G(\mathbb{A}_f)\). We deduce it from Hodge decomposition, which is a generalization of the Eichler-Shimura isomorphism. Let \(\mathcal{F}^{(k)}_C = \mathcal{F}^{(k)} \otimes_L \mathbb{C}\). We regard it as a local system of \(\mathbb{R}\)-vector spaces endowed with a ring homomorphism \(\mathbb{C} \to \text{End}(\mathcal{F}^{(k)}_C)\). We consider the filtration on \(\mathcal{F}^{(k)}_C \otimes_{\mathbb{R}} O_{M(\mathbb{C})}\) defined by \(\rho \circ h_x\). It defines on \(\mathcal{F}^{(k)}_C\) a structure of variation of polarizable \(\mathbb{R}\)-Hodge structures of weight \(w - 2\).

We put \(\mathcal{F}^{(k)}_C \otimes_{\mathbb{C}} O_{M(\mathbb{C})} = \mathcal{V}^{(k)}(= V(\rho^{(k)}))\) and let \(\sigma : M(\mathbb{C}) \to M(\mathbb{C})\) denote the complex conjugate. We identify \(\mathcal{F}^{(k)}_C \otimes_{\mathbb{R}} O_{M(\mathbb{C})} = \mathcal{V}^{(k)} \oplus \sigma^* \mathcal{V}^{(k)}\). The Hodge filtration \(F^{w-2}(\mathcal{V}^{(k)} \oplus \sigma^* \mathcal{V}^{(k)})\) is given by \(\mathcal{V}(\mathcal{J}^{(k)}) \oplus \sigma^* \mathcal{V}(\mathcal{J}^{(k)})\). Hence the Hodge decomposition gives a \(G(\mathbb{A}_f)\)-equivariant isomorphism

\[
H^1(M(\mathbb{C}), \mathcal{F}^{(k)}) \simeq H^1(M(\mathbb{C}), \Omega^*_M \otimes \mathcal{V}^{(k)}) \\
\simeq H^0(M(\mathbb{C}), \Omega^1_M \otimes \mathcal{V}(\mathcal{J}^{(k)})) \oplus \sigma^* H^0(M(\mathbb{C}), \Omega^1_M \otimes \mathcal{V}(\mathcal{J}^{(k)})�).
\]

Since \(S'_C = H^0(M(\mathbb{C}), \Omega^1_M \otimes \mathcal{V}(\mathcal{J}^{(k)})) \) and its complex conjugate \(\sigma^* H^0(M(\mathbb{C}), \Omega^1_M \otimes \mathcal{V}(\mathcal{J}^{(k)})\) is identified with \(H^0(M(\mathbb{C}), \Omega^1_M \otimes \mathcal{V}(\sigma^* \mathcal{J}^{(k)}))\), it is enough to show that the \(G(\mathbb{A}_f)\)-equivariant bundle \(\mathcal{J}^{(k)}\) on \(X\) is isomorphic to its complex conjugate \(\sigma^* \mathcal{J}^{(k)}\). It follows immediately from that the \(GL_2\)-action on the tautological quotient bundle on \(P^1\) is defined over \(\mathbb{R}\) and the standard representation \(\mathbb{H}^x \to GL_2\) defined over \(\mathbb{C}\) is \(GL_2(\mathbb{C})\)-conjugate to its complex conjugate.

4. Shimura curves and sheaves on them.

We give a geometric construction of a certain pull-back of the sheaf \(\mathcal{F}^{(k)}\) using functoriality of Shimura varieties.

First, we recall the definition of several Shimura varieties introduced in [C1]. We take an imaginary quadratic field \(E_0 = \mathbb{Q}(\sqrt{-a})\). We fix an embedding \(E_0 \subset \mathbb{C}\). We assume that the prime \(p\) splits in \(E_0\). We put \(E = FE_0 = F \otimes_{\mathbb{Q}} E_0\) and \(D = B \otimes_F E = B \otimes_{\mathbb{Q}} E_0\). We consider the reductive group \(G'' = B^\times \times_{F^1} E^\times \simeq B^\times \cdot E^\times \subset D^\times\). Here and in the following, we use its \(\mathbb{Q}\)-valued points \(G(\mathbb{Q})\) to describe an algebraic group \(G\) over \(\mathbb{Q}\). As in [C1], the notation \(B^\times \times_{F^1} E^\times\) does not mean the fiber product but the amalgamate sum. Let \(G'\) be the inverse image of \(\mathbb{Q}^\times \subset F^\times\) by the map \(\nu = \text{Nrd}_{B/F} \times N_{E/F} : G'' \to F^\times\). We also consider tori \(T = E^\times\) and \(T_0 = E_0^\times\). We consider the \(G'(\mathbb{R})\)-conjugacy class \(X'\) (resp.
The conjugacy classes $X'$, $X''$ have natural structures of complex manifold and are isomorphic to the upper half plane $X^+$ and to the union of upper and lower half planes $X$ respectively. Let $M' = M(G', X')$, $M'' = M(G'', X'')$, $N = M(T, h_E)$ and $N_0 = M(T_0, h_0)$ be the canonical models of the Shimura varieties defined over the reflex fields $E, E, E$ and $E_0$ respectively. The reciprocity map $E^\times \to E^\times$ is the identity for $(T, h_E)$. For an open compact subgroup $K \subset \mathbb{A}_{E,f}^\times$, the canonical model $N_K$ is the spectrum of the abelian extension $E_K$ corresponding to $K$ by class field theory. The same thing holds for the canonical model of $N_0$.

We define morphisms between Shimura curves. We consider the morphism $\alpha : G \times T \to G''$ of algebraic groups inducing

$$B^X \times E^X \to (B \otimes E)^X : (b, e) \mapsto b \otimes N_{E/E_0}(e) \cdot e^{-1}$$

on $\mathbb{Q}$-valued points. Since $h'_\alpha = \alpha \circ (h \times h_E)$, it induces a homomorphism of Shimura varieties $M \times N \to M''$ defined over $E$. We let $\alpha$ also denote the morphism $M \times N \to M''$. The inclusion $G' \to G''$ induces a natural map $M' \to M''$ of Shimura varieties over $E$. Let $\beta : G \times T \to T_0$ be the morphism inducing $N_{E/E_0} \circ \text{pr}_2 : B^X \times E^X \to E_0^X$ on $\mathbb{Q}$-valued points. Since $h_0 = N_{E/E_0} \circ h$, a homomorphism of Shimura varieties $M \times N \to N_0$ defined over $E$ is induced. We also let the map $M \times N \to N_0$ be denoted by $\beta$. We consider the diagram

$$
\begin{array}{ccc}
M & \xleftarrow{\text{pr}_1} & M \times N \\
\beta \downarrow & & \alpha \downarrow \\
N_0
\end{array}
$$

of (weakly) canonical models of Shimura varieties over $E$.

We define an $L_\alpha$-sheaf $\mathcal{F}_\alpha'^{(k)}$ on $M''$ analogous to $\mathcal{F}_\alpha^{(k)}$. Let $k = ((k_1, \ldots, k_g), w)$ be the multiweight and put $n = n(k) = \prod_i (k_i - 1)$. The algebraic group denoted $G''_{ec}$ in [Mi] Chap. III for the group $G''$ is the quotient of $G''$ by $\text{Ker}(N_{F/\mathbb{Q}} : F^\times \to \mathbb{Q}^\times)$.
Here $F^\times$ is regarded as a subgroup of the center $Z(G'') = E^\times$. We define a representation of algebraic group $\rho = \rho''(k) : G'' \to GL_n$ factoring the quotient $G''_{uc}$ as follows. Recall that we have an isomorphism $B \otimes_\mathbb{Q} \mathbb{C} \simeq M_2(\mathbb{C})^I$. It induces an injection $G'_{\mathbb{C}} \to (GL_{2,\mathbb{C}} \times GL_{2,\mathbb{C}})^I$. For each $i \in I$, the first component corresponds to the inclusion $E_0 \to \mathbb{C}$ and the second one corresponds to its complex conjugate. We define the morphism $\chi$ corresponding to its complex conjugate. We define the morphism of the second projection. Their product $\chi \in \mathbb{C}$.

Here the first component corresponds to the injection with the tensor product $\otimes_{i \in I} ((\text{Sym}^{k_i-2} \otimes \det (w - k_i)/2) \circ \tilde{pr}_2,i)$. Here $\tilde{pr}_2,i$ denotes the contragradient representation of the $(2,i)$-th projection $(GL_{2,\mathbb{C}} \times GL_{2,\mathbb{C}})^I \to GL_{2,\mathbb{C}}$. Since the restriction to the subgroup $F^\times \subset G''$ is the scalar multiplication by $N_{F/\mathbb{Q}}(w-2)$, it factors through the quotient $\rho''(k) : G''_{uc} \to GL_n$. The morphism $\rho'' = \rho''(k) : G \to GL_n$ is defined over the composite field $LE_0$. Replacing $L$ by $LE_0$ if necessary, we assume it is defined over $L$.

We may also define it as follows. Let $p_2 : G'' \to G$ be the map defined over $E_0$ induced by the second projection on $(D \otimes_\mathbb{Q} E_0)^{\times} = D^\times \times D^\times$ corresponding to the conjugate $E_0 \to E_0$. Then we have $\rho''(k) = \rho(k) \circ p_2$.

We define the smooth $L\lambda$-sheaf $F_{(k)}^{\rho''}$ on $M''$ to be the $L\lambda$-component of the smooth $L \otimes \mathbb{Q}_\ell$-sheaf $\mathcal{V}_\ell(\rho''(k))$ attached to the representation $\rho''(k)$ (loc. cit. 6). By restriction, we obtain a smooth $L\lambda$-sheaf $F_{\lambda}^{(k)}$ on $M'$ attached to the representation $\rho(k) = \rho''(k)|_{G'}$.

We also define a sheaf $\mathcal{F}(\chi)_\lambda$ on $N_0$. The algebraic group $T_0$ in [Mi] Chap. III is $T_0$ itself. We define a character $\chi : T_0 \to \mathbb{G}_m$. Over $\mathbb{C}$, we have $T_{0,\mathbb{C}} \simeq \mathbb{G}_m \times \mathbb{G}_m$. Here the first component corresponds to the inclusion $E_0 \to \mathbb{C}$ and the second one corresponds to its complex conjugate. We define the morphism $\chi : T_0 \to \mathbb{G}_m$ to be the inverse of the first projection. We also define the morphism $\tilde{\chi}$ to be the inverse of the second projection. Their product $\chi_0 = \chi \tilde{\chi}$ is the inverse of the norm map $\chi_0 = N_{E/\mathbb{Q}}^{-1} : T_0 \to \mathbb{G}_m$. They are defined over $E_0 \subset L$. We define the smooth $L\lambda$-sheaf $\mathcal{F}(\chi)$ on $N_0$ to be the $L\lambda$-component of the smooth $L \otimes \mathbb{Q}_\ell$-sheaf $\mathcal{V}_\ell(\chi)$ attached to the representation $\chi$. The sheaf $\mathcal{F}(\chi_0)$ is defined similarly.

We have $\rho''(k) \circ \alpha = (\rho(k) \circ pr_1) \times (\tilde{\chi}^{(w-2)(g-1)} \circ N_{E/E_0} \circ pr_2)$. In other words, we have a commutative diagram

$$
\begin{array}{ccc}
G \times T & \xrightarrow{\alpha \times \beta} & G'' \times T_0 \\
\rho(k) \circ pr_1 \downarrow & & \downarrow \rho''(k) \times \chi^{(w-2)(g-1)} \chi_0^{-1} \\
GL_n & \xleftarrow{\text{product}} & GL_n \times \mathbb{G}_m
\end{array}
$$

of homomorphisms defined over $L$. By the commutativity of the diagram, we obtain an isomorphism of smooth $L\lambda$-sheaves

$$
pr^*_1 \mathcal{F}(k) \simeq \alpha^* \mathcal{F}(\chi)^{(w-2)(g-1)} \otimes \beta^* \mathcal{F}(\chi_0)^{-(g-1)(w-2)}
$$
on $M \times N$. The isomorphism is equivariant with respect to the action of $G(\mathbb{A}_f) \times T(\mathbb{A}_f)$.

The sheaf $\beta^*F(\chi_0)$ together with the action of $T(\mathbb{A}_f)$ on it is identified as follows. Let $\beta_1 : N \to N_0$ denote the map induced by $N_{E/E_0}$. It is sufficient to describe $\beta_1^*F(\chi_0)$. If we forget the action, it is just the Tate twist $L_\lambda(-1)$. The action of $T(\mathbb{A}_f)$ is that induced by the natural action of $T(\mathbb{A}_f)$ on $N$ multiplied by the character

$$T(\mathbb{A}_f) \xrightarrow{N_{E/E}} \mathbb{A}_f^\times /\mathbb{Q}^\times \xrightarrow{\sim} \hat{\mathbb{Z}}^\times \to \bar{\mathbb{Z}}_\ell^\times \subset L_\lambda^\times.$$

Thus the geometric construction of $pr_i^*F^{(k)}$ is reduced to that of $F^{(k)}$ and that of $\bar{\chi}$.

Before constructing $F^{(k)}$ geometrically, we will study its restriction $F^{(k)}$ to $M'$. We prepare some notations. We consider the representation $\rho' : G' \subset G'' \subset D^\times \xrightarrow{b\mapsto b^{-1}} D^\times \subset GL(D)$ defined over $\mathbb{Q}$. Since the algebraic group $G'^c$ for $G'$ is equal to $G'$ itself, the representation $\rho'$ gives rise to a smooth $\ell$-adic sheaf $F'_i$ on $M'$ for each prime $\ell$. It is a smooth sheaf of $D \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-modules of rank 1.

Recall that we have an isomorphism $D \otimes_{\mathbb{Q}} L \simeq (M_2(L) \times M_2(L))^I$. For each $i \in I$, the first component corresponds to the embedding $E_0 \subset L \subset \mathbb{C}$ and the second to its conjugate. For each $i \in I$, let $e_i \in D \otimes_{\mathbb{Q}} L$ denote the idempotent whose $(2,i)$-th component is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and the other components are 0 under the isomorphism above. For each finite place $\lambda \mid \ell$, we regard the $L_\lambda$-sheaf $F' \otimes_{\mathbb{Q}_\ell} L_\lambda$ as a $D \otimes_{\mathbb{Q}} L \simeq (M_2(L) \times M_2(L))^I$-module. For each $i \in I$, let $F'_i$ denote the $e_i$-part $e_i(F' \otimes_{\mathbb{Q}_\ell} L_\lambda)$. It is easy to see that

$$F^{(k)} = \bigotimes_{i \in I} (Sym^{k_i-2}F'_i \otimes (\det F'_i)^{\otimes \frac{w-k_i}{2}})$$

as a smooth $L_\lambda$-sheaf on $M'$ with an action of $G'/(\mathbb{A}_f)$.

In section 6.1, we will construct the sheaf $F'$ and the idempotents $e_i$ after recalling a modular interpretation of $M'$ in Section 5. We will also construct $F(\chi)$ on $N_0$ in a similar way. After that, we study the relation between $M'$ and $M''$ and extend $F'$ to $M''$ in section 6.2.

5. Modular interpretation of $M'$ and $N_0$.

We recall the modular interpretation of the Shimura curve $M'$ on the category of schemes over $E$ [C1] 2.3. In the notation of [D2] (4.9) and (4.13), we put $L = V = D$, let the involution $\ast$ on $D = B \otimes_F E$ to be the tensor product of the main involution
of $B$ and the conjugate of $E$ and let $\psi$ be the non-generate alternating form on $D$ defined by
$$\psi(x, y) = \text{Tr}_{E/Q}(\sqrt{-a} \text{Trd}_{D/E} xy^*).$$

Then the group $G$ in (4.9) loc. cit. is $G'$ here and $G_1$ in (4.13) is $G''$ here.

We prepare some terminology to formulate a moduli problem for $M'$. Let $O_D$ be a maximal order in $D$ stable under the involution $\ast$. An abelian scheme $A$ over a scheme $S$ is called an $O_D$-abelian scheme over $S$ when a ring homomorphism $m : O_D \to \text{End}(A)$ is given. When $S$ is a scheme over $\text{Spec}$ $E$, for an $O_D$-abelian scheme $A$ on $S$, we define direct summands $\text{Lie}^2 A \supset \text{Lie}^{1,2} A$ of the $O_D \otimes_{\mathbb{Z}} O_S = D \otimes_{\mathbb{Q}} O_S$-module $\text{Lie} A$ as follows. The submodule $\text{Lie}^2 A$ is defined to be the submodule on which the action of $E_0 \subset D$ and that of $E_0 \subset O_S$ are the conjugate to each other over $\mathbb{Q}$. Similarly $\text{Lie}^{1,2} A$ is the submodule where the action of $E \subset D$ and that of $E \subset O_S$ are the conjugate to each other over $F$. They are the same as the tensor products $\text{Lie}^2 A = \text{Lie} A \otimes_{E_0 \otimes E_0} E_0$, $\text{Lie}^{1,2} A = \text{Lie} A \otimes_{E \otimes E} E$ and hence are direct summands. If $A$ is an $O_D$-abelian scheme, the dual $A^\ast$ is considered as an $O_D$-abelian scheme by the composite map $m^\ast : O_D \to O_D^{\text{opp}} \to \text{End}(A^{\text{opp}}) \to \text{End}(A^\ast)$ where $\text{opp}$ denotes the opposite ring. A polarization $\theta \in \text{Hom}(A, A^\ast)^{\text{sym}}$ of an $O_D$-abelian scheme $A$ is called an $O_D$-polarization if it is $O_D$-linear.

Let $K \subset \hat{O}_D' \subset G'(\hat{A}_f)$ be a sufficiently small compact open subgroup. Take a maximal order $O_D$ of $D$ and let $\hat{O}_D = O_D \otimes \hat{\mathbb{Z}} \subset D \otimes A_f$ be the corresponding maximal order. We assume $K \subset \hat{O}_D'$. Let $\hat{T} \subset D \otimes A_f$ be a $\hat{O}_D$-lattice satisfying $\psi(\hat{T}, \hat{T}) \subset \hat{\mathbb{Z}}$. We define a functor $M'_{K', \hat{T}}$ on the category of schemes over $E$ as follows. For a scheme $S$ over $E$ let $M'_{K', \hat{T}}(S)$ be the set of isomorphism class of the triples $(A, \theta, \hat{k})$ where
1. $A$ is an $O_D$-abelian scheme on $S$ of dimension $4g$ such that $\text{Lie}^2 A = \text{Lie}^{1,2} A$ and that it is a locally free $O_S$-module of rank $2$.
2. $\theta \in \text{Hom}(A, A^\ast)^{\text{sym}}$ is an $O_D$-polarization of $A$.
3. $\hat{k}$ is a $K$-equivalent class of a $O_D \otimes \hat{\mathbb{Z}}$-linear isomorphism $k : \hat{T}(A) \to \hat{T}$ such that there exists a $\hat{\mathbb{Z}}$-linear isomorphism $k'$ making the diagram
$$\begin{array}{ccc}
\hat{T}(A) \times \hat{T}(A) & \overset{(1, \theta_\ast)}{\Rightarrow} & \hat{T}(A) \times \hat{T}(A^\ast) \\
\downarrow_{k \times k} & & \downarrow_{k'} \\
\hat{T} \times \hat{T} & \overset{\text{Tr}_\psi}{\Rightarrow} & \hat{\mathbb{Z}}
\end{array}$$

commutative.

It is shown in [C1] (2.3) (2.6.2) that the scheme $M'_{K'}$ represents the functor $M'_{K', \hat{T}}$.

It is easily checked that the functor is independent of a choice of $\hat{T}$ upto uniquely defined canonical isomorphism. Let $A_{K', \hat{T}}$ denote the universal abelian scheme over $M_{K'}$. They form a projective system $A = (A_{K', \hat{T}})_{K', \hat{T}}$. 


We give a modular interpretation of the action of $G'(\mathbb{A}_f)$ on $M'$ and on $A$. Let $g \in G'(\mathbb{A}_f)$ and $K, K' \subset G'(\mathbb{A}_f)$ be sufficiently small open subgroups satisfying $g^{-1}Kg \subset K'$. We take a maximal order $O_D$ and let $\hat{T}$ and $\hat{T}'$ be a $K$-stable $O_D \otimes \mathbb{Z}$-lattice and a $K'$-stable $O'_D \otimes \mathbb{Z}$-lattice of $V \otimes \mathbb{A}_f$ satisfying $g^{-1}\hat{T} \subset \hat{T}'$ and $\psi(\hat{T}, \hat{T}') \subset \hat{\mathbb{Z}}$. The functor

$$g_* : \mathcal{M}_K \to \mathcal{M}_{K'}, \quad [(A, \theta, k)] \mapsto [(A', \theta', k')]$$

is described as follows. (Ind-)Etale locally on $S$, we take an isomorphism $\hat{k} : \hat{T} \to \hat{T}(A)$ in the $K$-equivalent class $k$ and identify $\hat{T}(A)$ with $\hat{k}$. Let $g_* : A \to A'$ be the isogeny of $O_D$-abelian schemes such that $\hat{T}(A') = g\hat{T}' \supset \hat{T} = \hat{T}(A)$. The $K'$-equivalent class $k'$ is the class of the isomorphism $g : \hat{T}' \to g\hat{T}' = \hat{T}(A')$. The pair $(A', k')$ is independent of the choice of $\hat{k}$. The polarization $\theta'$ on $A'$ is the map making the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\nu^+(g)\theta} & A^* \\
g_* \downarrow & & \uparrow t_g \\
A' & \xrightarrow{\theta'} & A'^*
\end{array}$$

commutative. Here $\nu^+ : G'(\mathbb{A}_f) \to \mathbb{Q}^\times$ is the composite $G'(\mathbb{A}_f) \xrightarrow{\nu} \mathbb{A}_f^\times \to \mathbb{A}_f^\times / \hat{\mathbb{Z}}^\times \isom\mathbb{Q}^\times$. We have the universal $O_D$-isogeny $g : A_{K, \hat{T}} \to g^*A_{K', \hat{T}'}$ and a commutative diagram

$$\begin{array}{ccc}
A_{K, \hat{T}} & \xrightarrow{g_*} & A_{K', \hat{T}'} \\
\downarrow & & \downarrow \\
M_K' & \xrightarrow{g} & M_{K'}'
\end{array}$$

For later use, we will extend the action of $G'(\mathbb{A}_f)$ on $M'$ and on $A$ to a larger group $\hat{G}$. Let $G''(\mathbb{R})_+$ be the inverse image of $GL_2(\mathbb{R})^+ \subset GL_2(\mathbb{R}) \subset \mathbb{C}^\times$ by the first projection $G''(\mathbb{R}) \to GL_2(\mathbb{R}) \subset \mathbb{C}^\times$ and let $G''(\mathbb{Q})_+ = G''(\mathbb{Q}) \cap G''(\mathbb{R})_+ = \{ \gamma \in G''(\mathbb{Q}) | \nu(\gamma) \text{ is totally positive} \}$. We put $\hat{G} = G''(\mathbb{Q})_+ \cdot G'(\mathbb{A}_f) \subset G''(\mathbb{A}_f)$. We extend the action of $G'(\mathbb{A}_f)$ on $M'$ to an action of $\hat{G}$. For $g \in G''(\mathbb{A}_f)$ and open compact subgroups $K' \subset G'(\mathbb{A}_f)$ and $K'' \subset G''(\mathbb{A}_f)$ such that $g^{-1}K'g \subset K''$, let $g : M_{K'}' \to M_{K''}'$ denote the composite $M_{K'}' \to M_{K''}'$. For $g \in \hat{G}$ and open compact subgroups $K'_1, K'_2 \subset G'(\mathbb{A}_f)$ such that $g^{-1}K'_1g \subset K'_2$, the map $g : M_{K'_1}' \to M_{K'_2}'$ is defined as follows. We may take an open compact subgroup $K'' \supset K'_2$ of $G''(\mathbb{A}_f)$ such that the canonical map $M_{K'_2}' \to M_{K''}'$ is an open immersion (see Lemma 2 in section 6.2). Then since $M_{K'_1}(\mathbb{C}) = \hat{G}(\mathbb{Q}) \setminus \hat{G}(\mathbb{A}_f) \times X'/K' = G''(\mathbb{Q})_+ \setminus \hat{G} \times X'/K'$, the image of $g : M_{K'_1}' \to M_{K''}'$ is contained in $M_{K'_2}'$. Hence the
required map $M_{K_1'} \to M_{K_2'}$ is induced. The modular interpretation of the action of $\hat{G}$ on $M'$ is described in the same way as above. The only modification is that $\nu^+$ is extended to $\hat{G}$ as the composite $\hat{G} \xrightarrow{\nu^+} F^{\times+} \xrightarrow{\mathbb{A}_f^{\times}} F^{\times+} / \mathbb{Z}^{\times} \xrightarrow{\nu^+} F^{\times+}$.

Similarly, we have a modular interpretation for $N_0$ in terms of elliptic curves with complex multiplication by $O_{E_0}$. Let $H \subset \hat{O}_{E_0}^\times$ be a sufficiently small open subgroup. We take a fractional ideal $R \subset E_0$ satisfying $\text{Tr}_{E_0/\mathbb{Q}}(\sqrt{aR}) \subset \mathbb{Z}$. Let $\hat{R} = R \otimes \hat{O}_E$ be the corresponding ideal. We define a functor $\tilde{N}_{0,H,R}$ on the category of schemes over $E_0$ as follows. For a scheme $S$ over $E_0$, let $\tilde{N}_{0,H,R}(S)$ be the set of isomorphism of the pairs $(A, \tilde{k})$ where

1. $A$ is an elliptic curve endowed with a ring homomorphism $O_{E_0} \to \text{End}_S(A)$ such that the induced homomorphism $O_{E_0} \to \text{End}_{O_{S}}(\text{Lie}A) = O_S$ is the same as that defined by the structure morphism $S \to \text{Spec} E_0$.

2. $\tilde{k}$ is an $H$-equivalent class of an $\hat{O}_{E_0}$-isomorphism $k : \hat{T}(A) \to T$ such that there exists a $\hat{\mathbb{Z}}$-isomorphism $\tilde{k}'$ making the diagram

\[
\begin{array}{ccc}
\hat{T}(A) \times \hat{T}(A) & \xrightarrow{k \times k} & \hat{\mathbb{Z}}(1) \\
\downarrow & & \downarrow \tilde{k}' \\
\hat{R} \times \hat{R} & \xrightarrow{(x,y) \mapsto \text{Tr}_{E_0/\mathbb{Q}}(ax\bar{y})} & \hat{\mathbb{Z}}
\end{array}
\]

commutative.

It is easily checked that the functor $\tilde{N}_{0,H,R}$ is independent of a choice of $R$ up to uniquely determined canonical isomorphism.

By the theory of complex multiplication, for a sufficiently small $H$, the functor $\tilde{N}_{0,H,R}$ is represented by $N_H = \text{Spec} E_{0,H}$ where $E_{0,H}$ is the abelian extension corresponding to the open subgroup $H \subset \hat{\mathbb{A}}_{E_0}^{\times}$ by the isomorphism $\hat{\mathbb{A}}_{E_0,f}^{\times} / E^{\times} \simeq \text{Gal}(E_{0}^{ab}/E_0)$ of class field theory. Similarly as above, a natural action of $T_0(\hat{\mathbb{A}}_f) = \hat{\mathbb{A}}_{E_0,f}$ on the projective systems $N = (N_K)_K$ and on the universal CM elliptic curve $b : A_0 = (A_{0,T,K})_{T,K} \to N$ is defined.

### 6.1 Geometric construction on $M'$, $N_0$.

We show that the direct image $R^1a_*\mathbb{Q}_\ell$ of the universal abelian scheme $a : A \to M'$ gives the sheaf $\mathcal{F}'$. Using it, we construct the sheaf $\mathcal{F}(\chi)$ on $M'$ in a purely geometric way. We will also define geometrically $\mathcal{F}(\chi)$ on $N_0$.

Let $K' \subset \hat{O}_D$, $\hat{T}$ and the universal $O_D$-abelian scheme $a_{K'} : A_{K',\hat{T}} \to M'_{K'}$ be as in the modular interpretation in Section 5. By the ring homomorphism $O_D \to \text{End}_{M'_{K'}}(a_{K',\hat{T}})$, we regard the direct image $R^1a_{K',\hat{T}} \mathbb{Q}_\ell$ as a sheaf of $D \otimes \mathbb{Q}_\ell$-modules for every $\ell$. It is independent of the choice of lattice $\hat{T}$. A canonical action of $G'(\hat{\mathbb{A}}_f)$ is defined on the system of sheaves $R^1a_*\mathbb{Q}_\ell = (R^1a_{K',\hat{T}} \mathbb{Q}_\ell)_{K'}$. By the
modular interpretation, it is easy to see that the sheaf $R^1a_*\mathbb{Q}_\ell$ is isomorphic to the sheaf $\mathcal{F}'$ with the action of $G'(\mathbb{A}_f)$ defined at the end of Section 4. We will identify them in the following.

For each $i \in I$, let $e_i \in D \otimes \mathbb{Q} L$ be the idempotent defined at the end of Section 4. We regarded $R^1a_*L_\lambda$ as a sheaf of $D \otimes \mathbb{Q} L$-modules. Then $e_i \in D \otimes \mathbb{Q} L$ acts on it as a projector and the $e_i$-part $e_i \cdot R^1a_*L_\lambda$ is isomorphic to $\mathcal{F}'_i$. Since $D$ is generated by $1 + pO_D$, we may write each $e_i$ as an $L$-linear combination of elements in $1 + pO_D$. Therefore $e_i$ is an $L$-linear combination of endomorphisms of $A$ over $M'$ whose degrees are prime to $p$.

One finds easily an idempotent $e^{(k_i)} \in \mathbb{Q}[S_{w-2}]$ of the group algebra of a symmetric group such that the $e^{(k_i)}$-part $e^{(k_i)} \cdot \mathcal{F}'_i \otimes (\det \mathcal{F}'_i)^{\otimes \frac{w}{2}}$. The action of the symmetric group $S_{w-2}$ on $\mathcal{F}'_i \otimes (\det \mathcal{F}'_i)^{\otimes \frac{w}{2}}$ is induced by the action of it on the fiber product $a^{w-2} : A^{w-2} \to M'$ over $M'$ as permutations. One can also find easily a $\mathbb{Q}$-linear combination $e^{(k_i)}$ of the multiplications by prime-to-$p$ integers on $A$ such that $e^{(k_i)}R^1a_*\mathbb{Q}_\ell = R^1a_*\mathbb{Q}_\ell$ and $e^{(k_i)}R^q a_*\mathbb{Q}_\ell = 0$ for $q \neq 1$.

Taking their product, we obtain an algebraic correspondence $e'$ on the $(w-2)g$-fold self-fiber product $A^{(w-2)g}$ of $A \to M'$ with coefficients in $L$ satisfying the following conditions.

1. It is an $L$-linear combinations of permutations in $S_{g(w-2)}$ and endomorphisms of $A^{(w-2)g}$ as an abelian scheme over $M'$ whose degrees are prime to $p$.

2. It acts as an idempotent on the cohomology sheaf $R^q a_*^{(w-2)g}L_\lambda$ where $a^{(w-2)g}$ denotes the map $A^{(w-2)g} \to M'$. We have $e'R^q a_*^{(w-2)g}L_\lambda = \mathcal{F}'(k_i) q = (w-2)g$ and $e'R^q a_*^{(w-2)g}L_\lambda = 0$ otherwise.

Similarly, we construct $\mathcal{F}(\chi)$. Let $H \subset \hat{O}_{\mathbb{F}_0}^w \hat{\mathcal{R}}$ and the universal $O_{E_0}$-elliptic curve $b_H : A_{0,H,\hat{\mathcal{R}}} \to N_{0,H}$ be as in the modular interpretation in Section 5. By the ring homomorphism $O_{E_0} \to \text{End}_{N_{0,H}}(A_{0,H,\hat{\mathcal{R}}})$, we regard the direct image $R^1b_{H,\mathbb{Q}_\ell}$ as a sheaf of $E_0 \otimes \mathbb{Q}_\ell$-modules for every $\ell$. It is independent of the choice of lattice $\hat{\mathcal{R}}$. A canonical action of $\hat{\mathcal{A}}_{E_0,f}$ is defined on the system of sheaves $R^1b_{H,\mathbb{Q}_\ell} = (R^1b_{H,\mathbb{Q}_\ell})_H$. By the modular interpretation, it is easy to see that the sheaf $R^1b_{H,\mathbb{Q}_\ell}$ is isomorphic to the sheaf on $N_0$ associated to the inverse of the tautological representation $E^w \to GL_\mathbb{Q}(E) : t \mapsto t^{-1} \times$. We will identify them in the following.

Let $e_0 \in E_0 \otimes \mathbb{Q} L$ be the idempotent corresponding to the inclusion $E_0 \to L$. We regarded $R^1a_{0,*}L_\lambda$ as a sheaf of $E_0 \otimes \mathbb{Q} L$-modules. Then $e_0 \in E_0 \otimes \mathbb{Q} L$ acts on it as a projector and the $e_0$-part $e_0 \cdot R^1a_{0,*}L_\lambda$ is isomorphic to $\mathcal{F}(\chi)$. Similarly as above, we may write each $e_0$ as an $L$-linear combination of elements in $1 + pO_D$. Therefore $e_0$ is an $L$-linear combination of endomorphisms of $A_0$ over $N_0$ whose degrees are prime to $p$. Similarly as above, after modifying $e_0$ if necessary, we also have $e_0 \cdot R^q b_{,\lambda} = 0$ for $q \neq 1$. 

6.2 Geometric construction on $M''$.

We extend the geometric construction on $M'$ to $M''$. We first study the relation between them. Recall that $G'' = B^\times \times_{F^\times} E^\times$ and $G'$ is the inverse image of $\mathbb{Q}^\times \subset F^\times \times_{F^\times} E^\times$ by $\nu = \text{Nrd}_B/F \times N_{E/F} : G'' \to F^\times$. For an open compact subgroup $K'' \subset G''(\mathbb{A}_f)$ and for $g \in G''(\mathbb{A}_f)$, we put $K''g = G'(\mathbb{A}_f) \cap gK''g^{-1}$. Recall that $g : M'_{K''} \to M'_{K''}$ denotes the composition $M'_{K''} \to M''_{gK''g^{-1}} \xrightarrow{\bar{g}} M''_{K''}$. The double coset $G\backslash G''(\mathbb{A}_f)/K''_1 = F^\times + \mathbb{A}_f^\times /\mathbb{A}_f^\times \nu(K'')$ is finite. If $\Sigma \subset G''(\mathbb{A}_f)$ is a complete set of representatives, we have a finite etale surjection $\Pi g : \bigsqcup_{g \in \Sigma} M'_{K''_g} \to M''_{K''_1}$.

**Lemma 2.** Let $K'' \subset G''(\mathbb{A}_f)$ be a compact open subgroup and put $K' = K'' \cap G'(\mathbb{A}_f)$. Then for a sufficiently small open subgroup $K''_1 \subset K''$ containing $K'$ and for a complete set $\Sigma$ of representatives of the finite set $G\backslash G''(\mathbb{A}_f)/K''_1$, the map

$$\Pi g : \bigsqcup_{g \in \Sigma} M'_{K''_g} \to M''_{K''_1}$$

is an isomorphism.

**Proof.** Since it is an etale surjection, it is enough to show the map is injective on the $\mathbb{C}$-valued points. Since $\Sigma$ is a complete set of representatives, it is enough to consider each map $g$. Let $\bar{\nu} : G''(\mathbb{A}_f) \to \mathbb{A}_f^\times /\mathbb{A}_f^\times$ denote the map induced by $\nu$.

We claim that the equality $\bar{\nu}(K'') \cap (\mathbb{O}_F^\times /\mathbb{Z}^\times) = \bar{\nu}(K'' \cap \mathbb{O}_E^\times)$ implies the injectivity of the map $g : M'_{K''_g}(\mathbb{C}) \to M''_{K''_1}(\mathbb{C})$.

We prove Lemma admitting the claim. Namely, we prove that for a sufficiently small open subgroup $K''_1 \supset K'$ of $K''$, we have an equality $\bar{\nu}(K'') \cap (\mathbb{O}_F^\times /\mathbb{Z}^\times) = \bar{\nu}(K'' \cap \mathbb{O}_E^\times)$. Since $N_{E/F}(\mathbb{O}_E^\times)$ is of finite index in $\mathbb{O}_F^\times$, the right hand side $\bar{\nu}(K'' \cap \mathbb{O}_E^\times)$ is an open subgroup of the left hand side $\bar{\nu}(K'') \cap (\mathbb{O}_F^\times /\mathbb{Z}^\times)$. Hence, for a sufficiently small open subgroup $K''_1$ of $K''/K'$, we have $K''_1 \cap (\mathbb{O}_F^\times /\mathbb{Z}^\times) = K''_1 \cap \nu(K'' \cap \mathbb{O}_E^\times)$. For the corresponding open subgroup $K''_1 = K'' \cap \nu^{-1}(K'')$, this is nothing but the required equality $\bar{\nu}(K'') \cap (\mathbb{O}_F^\times /\mathbb{Z}^\times) = \bar{\nu}(K'' \cap \mathbb{O}_E^\times)$.

We prove the claim. Namely, we assume $\bar{\nu}(K'') \cap (\mathbb{O}_F^\times /\mathbb{Z}^\times) = \bar{\nu}(K'' \cap \mathbb{O}_E^\times)$ and prove the map $g : M'_{K''_g}(\mathbb{C}) \to M''_{K''_1}(\mathbb{C})$ is injective. Replacing $K''$ by $gK''g^{-1}$, it is enough to show that the map $M'_{K''}(\mathbb{C}) \to M''_{K''}(\mathbb{C})$ is injective for $K' = K'' \cap G'(\mathbb{A}_f) = \text{Ker}(\nu : K'' \to \hat{O}_F^\times /\hat{\mathbb{Z}}^\times)$. We consider the commutative diagram of exact sequences

$$\begin{array}{cccccc}
K'' \cap \mathbb{O}_E^\times & \longrightarrow & K'' / K' & \longrightarrow & K'' / (K'' \cap \mathbb{O}_E^\times)K' & \longrightarrow & 1 \\
\bar{\nu} \downarrow & & \bar{\nu} \downarrow \cap & & \downarrow & \\
1 & \longrightarrow & \mathbb{O}_F^\times /\mathbb{Z}^\times & \longrightarrow & \hat{O}_F^\times /\hat{\mathbb{Z}}^\times & \longrightarrow & \hat{O}_F^\times /\hat{\mathbb{Z}}^\times \mathbb{O}_F^\times.
\end{array}$$
The middle vertical arrow is injective by the definition of \(K'\). By the snake lemma, the equality \(\bar{\nu}(K'') \cap (\overline{O_F^\times} / \mathbb{Z}^\times) = \bar{\nu}(K'' \cap \overline{O_E^\times})\) is equivalent to the injectivity of the right vertical arrow. Since \(\overline{O_E^\times} / \hat{\mathbb{Z}}^\times \overline{O_F^\times}\) is a subgroup of \(\hat{\mathbb{A}}_{F,f}^\times / K_{Q,f}^\times \mathcal{F}^\times\), we get an exact sequence

\[
K'' / (K' \cap \overline{O_E^\times}) \rightarrow K'' / (K'' \cap \overline{O_E^\times}) \rightarrow \hat{\mathbb{A}}_{F,f}^\times / K_{Q,f}^\times \mathcal{F}^\times.
\]

We consider the commutative diagram

\[
\begin{array}{ccc}
M'(\mathbb{C}) &=& \lim_{\rightarrow} K', M'_{K'}(\mathbb{C}) \\
\downarrow & & \downarrow \\
\hat{\mathbb{Q}}^\times \backslash \hat{\mathbb{A}}_{Q,f}^\times &=& \hat{\mathcal{F}}^\times \backslash \hat{\mathbb{A}}_{F,f}^\times.
\end{array}
\]

The horizontal arrows are injective by Variante 1.15.1 and Lemma 1.15.3 [D2]. We have \(M'_{K'}(\mathbb{C}) = M'(\mathbb{C}) / K'\) and \(M''_{K''}(\mathbb{C}) = M''(\mathbb{C}) / K''\). From these facts, it is straightforward to show that the exactness above implies the injectivity of the canonical map \(M'_K(\mathbb{C}) \rightarrow M''_{K''}(\mathbb{C})\).

We extend the universal \(O_D\)-abelian scheme \(A\) on \(M'\) to an \(O_D\)-abelian scheme also denoted by \(A\) on \(M''\). Let \(K'' \subset G''(\mathbb{A}_f)\) be a sufficiently small open subgroup. We assume that the map \(\Pi \hat{g} : \coprod_{g \in \Sigma} M'_{K'_{\hat{g}}} \rightarrow M''_{K''_{\hat{g}}}\) in Lemma 2 is an isomorphism.

We take an \(\hat{O}_D\)-lattice \(\hat{T}\) in \(D \otimes \mathbb{A}_f\). For each \(g \in \Sigma\), we have a \(gO_Dg^{-1}\)-abelian scheme \(A_{K'_{\hat{g}}}g\hat{T}\) on \(M'_{K'_{\hat{g}}}\) since \(g\hat{T}\) is \(K'_{\hat{g}}\)-stable. We define an abelian scheme \(A_{K''_{\hat{g}}}g\hat{T}\) on \(M''_{K''_{\hat{g}}}\) to be \(A_{K'_{\hat{g}}}g\hat{T}\) on the image of \(M'_{K'_{\hat{g}}}\). We define an \(O_D\)-multiplication on \(A_{K'_{\hat{g}}}g\hat{T}\) as \(O_D \xrightarrow{\hat{g}^{-1}g\hat{T}} gO_Dg^{-1} \rightarrow \text{End}_{M'_{K'_{\hat{g}}}}(A_{K'_{\hat{g}}}g\hat{T})\) on \(M'_{K'_{\hat{g}}}\). By the action of \(\hat{G}\) described in the previous section, we see that the abelian scheme \(A_{K''_{\hat{g}}}g\hat{T}\) is independent of the choice of representatives \(\Sigma\). We also see by the action of \(\hat{G}\) that, for \(g \in G''(\mathbb{A}_f)\), compact open subgroups \(K'_{1}, K''_{1} \subset G''(\mathbb{A}_f)\) and \(K'_{1}''\)-stable \(\hat{O}_D\)-lattices \(\hat{T}_{1}\) satisfying \(g^{-1}K'_{1}g \subset K''_{1}\), we have an isogeny \(A_{K''_{\hat{g}}}g\hat{T}_{1} \rightarrow g^{*}A_{K'_{\hat{g}}}g\hat{T}_{2}\).

Thus we obtain an action of \(G''(\mathbb{A}_f)\) on the projective system \(\hat{A} = (A_{K''_{\hat{g}}}g\hat{T})\) over \(M'' = (M''_{K''_{\hat{g}}})_{K''}\).

On the \((w-2)g\)-fold self-fiber product \(A^{(w-2)g}\) of \(A \rightarrow M''\), we define an algebraic correspondence \(e'\) with coefficients in \(L\) exactly in the same way as in the case of \(M'\). Then, it is an \(L\)-linear combinations of permutations in \(\mathcal{S}_{g(w-2)}\) and endomorphisms of \(A^{(w-2)g}\) as an abelian scheme over \(M'\) whose degrees are prime to \(p\). Further, it acts as an idempotent on the cohomology sheaf \(R^ia_*^{(w-2)g}L\lambda\) where \(a^{(w-2)g}\) denotes the map \(A^{(w-2)g} \rightarrow M'\). We have

\[
e' R^ia_*^{(w-2)g}L\lambda = \bigotimes_i \text{Sym}^{k_i - 2}(e_i \cdot R^1a_*L\lambda) \otimes (\det e_i \cdot R^1a_*L\lambda) \otimes \frac{w-k}{2}.
\]
for \( q = (w - 2)g \) and \( e' R^q a^{(w-2)g} \lambda = 0 \) otherwise. By the modular interpretation of \( M' \), we see that the \( K''' \)-equivalent class of the isomorphism \( \hat{T} \to T(A_{K'''} \hat{T}) \) is well-defined. Passing to the limit, we obtain an isomorphism \( D \otimes A_f \to R^q a^\beta \mathbb{Q}_\ell \) on \( \lim_{K'''} M_{K'''} \). The isomorphism is compatible with the action of \( G''(A_f) \). On the left hand side \( D \otimes A_f \), the group \( G''(A_f) \subset (D \otimes A_f)^\times \) acts by the multiplication by the inverse of the main involution: \( t \mapsto t^{-1} \times \). Thus similarly as on \( M' \), we have

\[
e' R^{(w-2)q} a^{(w-2)q} L \lambda = \mathcal{F}^{m(k)}.
\]

6.3. Geometric construction on \( M \).

We will define an analogue \( c : X \to M \times N \) of Kuga-Sato variety and an algebraic correspondence \( e = e^{(k)} \) on \( X \) with coefficient in \( L \) satisfying the following property: It is an \( L \)-linear combination of endomorphisms of \( X \) as an abelian scheme over \( M \times N \) whose degree are prime to \( p \). It acts as an idempotent on the higher direct image \( R^q c^\ast \mathbb{Q}_\ell \otimes L = \prod_{\ell \in \mathbb{I}} R^q c^\ast L \lambda \). The image of the projector \( e \cdot R^q c^\ast L \lambda \) is a smooth \( L_\lambda \)-sheaf isomorphic to

\[
\alpha^\ast \mathcal{F}^{(k)} \otimes \beta^\ast F(\chi) \otimes ((w-2)(g-1)) = pr_1^\ast \mathcal{F}^{(k)} \otimes \beta^\ast F(\chi_0) \otimes ((w-2)(g-1))
\]

for \( q = q_0 = (2g - 1)(w - 2) \) and is 0 otherwise.

We define \( X \) to be the fiber product

\[
X = \alpha^\ast A^{g(w-2)} \times_{M \times N} \beta^\ast A^{(g-1)(w-2)}_0.
\]

Here \( \alpha^\ast A^{g(w-2)} \) denotes the base change by \( \alpha : M \times N \to M'' \) of the \( g(w-2) \)-fold self fiber product of \( A \to M'' \). Similarly \( \beta^\ast A^{g(w-2)}_0 \) denotes the base change by \( \beta : M \times N \to N_0 \) of the \( (g-1)(w-2) \)-fold self fiber product of \( A_0 \to N_0 \).

The symbol \( X \) denotes the projective system \( X = (X_{K,H,\hat{T},R})_{K,H,\hat{T},R} \) of abelian schemes over \( M \times N = (M_K \times N_H)_{K,H} \).

Next we define an algebraic correspondence \( e = e^{(k)} \) on \( X \). We have defined algebraic correspondences \( e' \) on \( A^{g(w-2)} \) on \( M'' \) and \( e_0 \) on \( A^{g(w-2)}_0 \) on \( N_0 \) and the end of subsections 6.2 and 6.1 respectively. Let \( e_0 \otimes (g-1)(w-2) = \prod_{i=1}^{(g-1)(w-2)} pr_i e_0 \) be the algebraic correspondence on the \( (g-1)(w-2) \)-nd self fiber product \( A^{(g-1)(w-2)}_0 \) defined as the product of the pull-back of the algebraic correspondence \( e_0 \) on \( A_0 \) by projections. We define an algebraic correspondence \( e \) on \( X \) as the product of the pull-back of \( e' \) by \( \alpha \) with the pull-back of \( e_0 \otimes (g-1)(w-2) \) by \( \beta \). Namely we put

\[
e = \alpha^\ast e' \times \beta e_0 \otimes (g-1)(w-2).
\]

Then it satisfies the required property stated in the beginning of this section.

Let \( H \subset A_{E,f}^\times \) be a sufficiently small open compact subgroup. Let \( m = nO_E \) be a sufficiently divisible integral ideal of \( O_E \). We assume \( H = H_m \cdot H_m \) is the product
of the prime-to-$m$ component $H^m = \prod_{s|\mathfrak{m}} O_{E,s}^\times$ with the $m$-primary component $H_m$. Let $T_0^m = L[P_s; \mathfrak{s} \nmid \mathfrak{m}]$ be the free $L$-algebra generated by the class $P_s$ of the inverse of prime element for $\mathfrak{s} \nmid \mathfrak{m}$. We consider $H^q(X_{K,H,T,R} \otimes_E \breve{E}, L_\lambda)$ as a $T^n \times T_0^m$-module and $H^0(N_{H,E}, \mathcal{F}(\chi_0))$ as a $T_0^m$-module.

Applying the Leray spectral sequence to $c : X_{K,H,T,R} \to M_K \times_F N_H$, we obtain the following.

**Lemma 3.** Let $K \subset G(\mathbb{A}_f)$ and $H \subset \mathbb{A}_{E,f}^\infty$ be sufficiently small open compact subgroups and let $\hat{T} \subset V \otimes \mathbb{A}_f$ and $\hat{R} \subset E_0 \otimes \mathbb{A}_f$ be an $\hat{O}_D$-lattice and an $\hat{O}_{E_0}$-lattice respectively. Let $X = X_{K,H,T,R}$ be the analogue of Kuga-Sato variety defined above. Then there is an algebraic correspondence $e$ on $X$ with coefficient in $L$ satisfying the following properties.

1. There exists elements $a_i \in L$, permutations $\tau_i \in S_{g(w-2)}$ of the first $g(w-2)$-factors in $X$ and endomorphisms $\varphi_i \in \text{End}_M X$ of degree prime to $p$ such that

$$e = \sum_i a_i \tau_i \varphi_i.$$

2. For each finite place $\lambda$ of $L$, the action of $e$ on $H^q(X_{K,H,T,R} \otimes E \breve{E}, L_\lambda)$ is a projector. Put $q_0 = (2g-1)(w-2)$. Then, there is an isomorphism

$$e \cdot H^q(X_{K,H,T,R} \otimes E \breve{E}, L_\lambda) \simeq H^{q-q_0}(M_K \otimes_F \breve{F}, \mathcal{F}_\lambda^{(k)} \otimes_{\mathbb{Q}_l} H^0(N_{H,E} \otimes \breve{E}, \mathcal{F}(\chi_0^{(g-1)(w-2)})).$$

The isomorphism is compatible with the actions of the absolute Galois group $G_E = \text{Gal}(\breve{E}/E)$ and of the Hecke algebra $T^n \otimes T_0^m$.

Using Lemma 3, we give a statement, Claim 3, in terms of $X$ and $e$, implying Claim 2 and hence Theorems. Let $q$ be a place of $E$ dividing $p$. By the assumption that $p$ splits in $E_0$, the local field $E_q$ is canonically isomorphic to $F_p$. We identify $F_p = E_q$ by the canonical isomorphism. Since we want to prove the assertions on the action of Galois group $\text{Gal}(\breve{E}_p/F_p)$, it is enough to consider the action of $\text{Gal}(E_q/E_q)$, induced by the isomorphism.

**Claim 3.** We keep the notation in Claim 2. Let $K \subset G(\mathbb{A}_f)$ and $H \subset \mathbb{A}_{E,f}^\infty$ be sufficiently small open compact subgroups. Let $X = X_{K,H,T,R}$ denote the analogue of Kuga-Sato variety. Then, the following holds.

1. The $p$-adic representation $H^q(X \otimes E \breve{E}_p, \mathbb{Q}_p)$ of $G_{E_p} = \text{Gal}(\breve{E}_p/E_p)$ is potentially semi-stable for all $q$.

2. Let $\sigma \in W^+ = \{\sigma \in W(\breve{E}_p/E_p)|\pi(\sigma) \geq 0\}, T \in T^n, P \in T_0^m$, $r \in \mathcal{S}_{g(w-2)}$ and let $\psi : X \to X$ be an endomorphism of degree prime to $p$. Then for the composite $\Gamma = T \circ R \circ \tau \circ \psi$ as an algebraic correspondence, we have an equality in $\mathbb{Q}$

$$\sum_q (-1)^q \text{Tr}(\sigma \circ \Gamma|H^q(X \otimes E \breve{E}_p, \mathbb{Q}_l)) = \sum_q (-1)^q \text{Tr}(\sigma \circ \Gamma|D(H^q(X \otimes E \breve{E}_p, \mathbb{Q}_p))).$$
(2) Let $e$ be the algebraic correspondence in Lemma 3 and let $\mu|p$ be a finite place of $L \supset E_0$. Then the monodromy filtration of the representations $e \cdot H^q(X \otimes_E \bar{E}_q, L_\lambda)$ and $D(e \cdot H^q(X \otimes_E \bar{E}_q, L_\mu))$ of the Weil-Deligne group $W(\bar{E}_q/E_q)$ are pure of weight $q$.

We deduce each assertion in Claim 2 from the corresponding assertion in Claim 3. Since we identify $F_q$ with the Weil-Deligne group, there is an isomorphism $\text{iso}_q$ compatible with the actions of the Galois group $G_E = \text{Gal}(\bar{E}/E)$ and of the Hecke algebra $T^n$. Hence the equality in (1) in Claim 3 implies the equality in (1) in Claim 2. Thus Theorems 1 and 2, are reduced to Claim 3.

We may deduce the assertion (0) using alteration $[dJ]$. We will give a proof without using alteration by constructing a semistable model of $\mathcal{H}$. For later use, we describe the Hecke operators $T_r \in T^n$ and $P_s \in T_0^n$ for primes $r \nmid n$ of $O_F$ and $s \mid m$ of $O_E$ respectively. Write $X = X_{K,H,T,T^*}$ and $M \times N = M_K \times N_H$ for short. For $r$, it is defined as $T_r = p_1 \circ q^* \circ P_{s}^*$ where $p_1, p_2, q$ are as in the diagram

\[
\begin{array}{cccccc}
X & \xleftarrow{p_1} & X_{K,g,H,T,R} & \xrightarrow{q} & X_{K,g,H,gT,R} & \xrightarrow{p_2} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M \times N & \xleftarrow{p_1} & M_K \times N_H & \xrightarrow{P_{s}^*} & M_K \times N_H & \xrightarrow{p_2} & M \times N.
\end{array}
\]

In the diagram, $g = g_r \in G(\mathbb{A}_f)$ is an element whose $r$-component is $\begin{pmatrix} \pi_r^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ and other components are $1$ and $K_g = K \cap gKg^{-1}$. The map $p_1$ is induced by the inclusion $K_g \rightarrow K$, the map $p_2 = g_*$ is induced by $g$ and the left and right squares are cartesian. The map $q$ is an isogeny corresponding to the inclusion $T \rightarrow gT$.

Similarly for $s$, the operator is defined as $P_s = q^* \circ P_{s}^*$ where $p_2, q$ are as in the diagram

\[
\begin{array}{cccccc}
X & \xrightarrow{q} & X_{K,H,gT,N_{E_0/E_0}gR} & \xrightarrow{P_{s}^*} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M \times N & \xrightarrow{P_{s}^*} & M \times N & \xrightarrow{p_2} & M \times N.
\end{array}
\]
In the diagram, $g \in A_{E,f}^\times$ denotes an element whose $s$-component is the inverse of a prime element $\pi_s^{-1}$ and the other components are 1. The map $p_2 = g_s$ is induced by $g$ and right square is cartesian. The map $q$ is an isogeny corresponding to the inclusions $\hat{T} \to \hat{gT}$ and $\hat{R} \to N_{E/E_0}g \cdot \hat{R}$.

7. Semi-stable model.

In the last section, we defined an analogue of Kuga-Sato variety as an abelian schemes on a Shimura curve. The goal, Lemma 4, in this section is to give their semi-stable models.

We prepare some terminology. Let $K,H,\hat{T},\hat{R}$ be as in the last section. We assume that each component of the generic fiber $M_K \otimes_F \hat{F}$ are of genus greater than 1. Then by the stable reduction theorem of curve, for a sufficiently large finite extension $V$ of the maximal unramified extension $\hat{F}^nr$, the base change $M_K \otimes_F V$ admits a semi-stable model (not necessarily connected) over the integer ring $O_V$. We do not need unramified extension but for later use, we will do it here. We take the minimal one among the semi-stable models over $O_V$ and denote it by $M_{K,O_V}$. Recall that we identified the local field $E_p$ with $F_p$. From now on, we consider $V$ as an extension of $E_p$ by this identification. Since $N_H$ is the disjoint union of the spectrum of finite extensions of $E$, the base change $(M_K \times_F N_H) \otimes_E V$ also admits a semi-stable model over the integer ring $O_V$. We also take the minimal one among them and write it by $(M_K \times_F N_H)_{O_V}$. We claim the following.

**Lemma 4.** Let $K,H$ and $V$ be as above.

1. Let $g \in G(\mathbb{A}_f), h \in A_{E,f}^\times$ and let $K_1 \subset gKg^{-1}, H_1 \subset H$ be open compact subgroups. Assume that the groups are of the form $K = K_pK^p, g^{-1}K_1g = K_p(g^{-1}K_1g)^p, H = H_qH^q$ and $H_1 = H_qH_1^q$. Then the pull-back of the map $(g,h)_* : M_{K_1} \times_F N_{H_1} \to M_{K} \times_F N_{H}$ to the base change over $\hat{V}$ is extended uniquely to a finite etale morphism $(g,h)_* : (M_{K_1} \times_F N_{H_1})_{O_V} \to (M_{K} \times_F N_{H})_{O_V}$ of the minimal semi-stable models.

2. Let $\hat{T}, \hat{R}$ be as above. Then the pull-back of the abelian scheme $X_{K,H,\hat{T},\hat{R}} \to M_K \times_F N_H$ to the base extension $(M_K \times_F N_H) \otimes_E V$ is extended uniquely to an abelian scheme over a semi-stable model.

3. Let $\hat{T}_1, \hat{R}_1$ be sublattices in $\hat{T}$ and $\hat{R}$ in (2) respectively. Assume that their $p$-components are the same. Then the pull-back of the isogeny $X_{K,H,\hat{T}_1,\hat{R}_1} \to X_{K,H,\hat{T},\hat{R}}$ on $M_K \times_F N_H$ to the base extension $(M_K \times_F N_H) \otimes_E V$ is extended uniquely to an etale isogeny over a semi-stable model.

**Proof.** (1). We may assume $g = 1$ and $h = 1$. Further we may assume $H = H_1$. In fact, the map $N_{H_1} \to N_H$ is unramified at $q$ by the assumption that there $q$-components are the same by class field theory. Further, it is sufficient to show that the map $M_{K_1} \to M_K$ is extended to a finite etale morphism of minimal semi-stable models $M_{K_1, O_V} \to M_{K, O_V}$. In fact, then the fiber product $M_{K_1, O_V} \times_{M_{K, O_V}} (M_K \times$
$N_H)_{O_V}$ is a semi-stable model of $(M_K \times N_H)_V$ and does not have a $(-1)$-curve. Hence it is the minimal semi-stable model $(M_K \times N_H)_{O_V}$ and $(M_K \times N_H)_{O_V} \to (M_K \times N_H)_{O_V}$ is finite etale.

In the case where the p-components of $K_p = K_{1,p}$ are $GL_2(O_{F,p})$, it is shown in [C1] Proposition 6.1 and in loc. cit. 6.2 that the canonical map $M_{K_1,F_p} \to M_{K,F_p}$ is extended to a finite etale morphism $M_{K_1,O_{F,p}} \to M_{K,O_{F,p}}$ of proper smooth models. We consider the general case. Let $K \supset K, K_1 \supset K_1$ be the groups obtained by replacing their p-components $K_p = K_{1,p}$ by $GL_2(O_{F,p})$ respectively. First, we show that the canonical map $M_{K,V} \to M_{K,V}$ is extended to the minimal semi-stable model $M_{K,O_V} \to M_{K,O_V}$. In fact, it is extended on a suitable blow-up. However, the exceptional divisors are contracted to points in the image and hence the map is defined on the semi-stable model. We consider the fiber product $M_{K_1,O_V} \times M_{K_1,O_V} M_{K,O_V}$. It is a semi-stable model of $M_{K_1,V}$ and does not have a $(-1)$-curve. Hence it is minimal and the assertion is proved.

(2). We assume there exists an open compact subgroup $K'' \subset G''(\mathbb{A}_f)$ containing $K'' \supset KH$ and satisfying the following conditions (a) and (b).

(a) $K''$ satisfies the conclusion of Lemma 2. Namely for a complete set $\Sigma$ of representatives $G\setminus G''(\mathbb{A}_f)/K''$, the map $\Pi g : \coprod_{g} M'_{K_2} \to M''_{K''}$ is an isomorphism.

To state the other condition, we identify the group $G'(\mathbb{Q}_p)$. By the assumption that $E_0$ splits at $p$, we have an isomorphism

\[ G'(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Q}_p^\times \times (B \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times \xrightarrow{\sim} \mathbb{Q}_p^\times \times GL_2(F_p) \times (B \otimes_{F} F_p^\times)^\times \]

\[ \cap \quad \cap \]

\[ G''(\mathbb{Q}_p) \xrightarrow{\sim} (F \otimes \mathbb{Q}_p)^\times \times (B \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times \]

[C1] (2.6.3). The second condition is the following.

(b) The intersection $K' = K'' \cap G'(\mathbb{A}_f)$ is of the form $K' = \mathbb{Z}_p^\times \times GL_2(O_{F,p}) \times K''_p \times K''_p$ for some choice of isomorphism as above.

It is shown in [C1] Proposition 5.4 using a modular interpretation that the condition (b) implies that $M'_{K_2}$ has good reduction over $O_{E,p}$ and the abelian scheme $A'_{K_2} \times \mathbb{T}$ on the generic fiber is extended to a (unique) proper smooth model $M'_{K_2,O_{E,p}}$.

We will recall the modular interpretation in Section 9. Hence by the condition (a), $M''_{K''}$ has also good reduction over $O_{E,p}$ and the abelian scheme $A''_K \times \mathbb{T}$ is extended to the proper smooth model $M''_{K'',O_{E,p}}$. By the same argument as in the proof of (1), the map $(M_K \times N_H)_V \to M''_{K'',V}$ is extended uniquely to a map $(M_K \times N_H)_O \to M''_{K'',O_{E,p}} \otimes O_V$. Hence we obtain the extension of an abelian scheme $X_{K,H,T,R}$ by taking the pull-back.

(3). Since $(M \times N)_O$ is normal, an endomorphism on the generic fiber is extended to the integral model by a theorem of Grothendieck [G].
In the proof of (3), we could also use the modular interpretation recalled in Section 9.

8. Proof of Theorems.

We prove Theorems 1 and 2 by showing the assertions in Claim 3. The argument is the same as in [Sa]. Let the notation be as in Claim 3. We fix sufficiently small open compact subgroups $K \subset G(\mathbb{A}_f)$, $H \subset \mathbb{A}^\times_{E_0,f}$, an $\hat{O}_D$-lattice $\hat{T}$ and $\hat{O}_{E_0}$-lattice $\hat{R}$. To simplify the notation, we will write $M \times N$ for $M_K \times N_H$ and $X$ for $X_{K,H,\hat{T},\hat{R}}$.

Recall that we identify $E_q \simeq F_q$.

We prove that the $p$-adic representation $H^q(X \otimes_{E_q} \hat{E}_q, \mathbb{Q}_p)$ of the Galois group $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is potentially semi-stable. Since we have a semi-stable model $X_{O_V}$ of the base change $X_V$ to an extension $V$ of $\mathbb{Q}_p$ by Lemma 4, we just apply the Cst-conjecture proved by Tsuji [Tj] to a semi-stable scheme $X_O$.

We compute $D_{pst}(H^q(X \otimes_{E_q} \hat{E}_q, \mathbb{Q}_p))$ in terms of the minimal semi-stable model $X_O$ of $X$ defined in Lemma 4. Let $Y$ denote the closed fiber of the minimal semi-stable model $X_O$ with the natural log structure. Then further by [Tj], we have a canonical isomorphism

$$D_{pst}(H^q(X \otimes_{E_q} \hat{E}_q, \mathbb{Q}_p)) \simeq H^q_{\log \text{crys}}(Y/W) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$ 

It follows from the functoriality of the comparison isomorphism for finite etale morphism and from the compatibility with the Poincaré duality that the isomorphism is compatible with the action of endomorphisms and permutations appeared in Claim 3. We define Hecke operators on the log crystalline cohomology and compare them with those on the left hand side induced by the Hecke operators on the etale cohomology. Let $\mathfrak{n} \subset O_F$ and $\mathfrak{m} \subset O_E$ be sufficiently divisible ideals as in Section 6.3. Let $\mathfrak{r} \nmid \mathfrak{n}$ be a prime ideal of $O_F$. Then the projections $p_1, p_2$ and the isogeny $q$ described at the end of section 7 is extended to a finite etale morphism of the minimal semi-stable model by Lemma 4. On log crystalline cohomology, we define the Hecke operator $T_\mathfrak{r}$ as the composite $p_1^* \circ q^* \circ p_2^*$. Similarly we define the Hecke operator $P_\mathfrak{s}$ for a prime ideal $\mathfrak{s} \nmid \mathfrak{m}$ of $O_E$ as the composite $q^* \circ p_2^*$. Then it follows from the functoriality that the isomorphism is compatible with the Hecke operators thus defined.

We define the Galois action on the log crystalline cohomology and compare it with that on the left hand side defined in Section 2. We may and do assume that the finite extension $V$ of $\hat{E}_q^{nr}$ is the completion of a Galois extension of $\mathbb{Q}_p$. We have a natural action of the Galois group $G_\mathbb{Q} = \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $V$ and hence on the base change $M_V$. Since the minimal semi-stable model is unique, the action of $G_\mathbb{Q}$ on the generic fiber $M_V$ is extended to the minimal semi-stable model $M_{O_V}$. Further it is uniquely extended to that on the abelian scheme $X_{O_V}$. It induces a semi-linear action of the Weil group $W_\mathbb{Q}$ on the log crystalline cohomology. By modifying the action of $\sigma \in W_{E_q}$ by $\varphi^n(\sigma) \circ \sigma$ as in Section 2 and together with $N$, we define a
linear action of the Weil-Deligne group $W_{E_q}$ on the log crystalline cohomology. We verify the compatibility of the isomorphism with the action of Weil-Deligne group defined above. By transport of the structure, it is compatible with the semi-linear action of the Weil group before modification. Since the comparison isomorphism is compatible with the action of $F$ and $N$, the compatibility is established.

Therefore, Claim 3 is reduced to the following.

Claim 4. Let the notation be as in Claim 3. Then, the following holds.

(1) Let $\sigma \in W^+ = \{ \sigma \in W(\bar{E}_q/E_q)|n(\sigma) \geq 0\}, T \in T^n, R \in T^m, \tau \in S_{w-2}$ and let $\psi: X \to X$ be an endomorphism of degree prime to $p$. Then for the composite $\Gamma = T \circ R \circ \tau \circ \psi$ as an algebraic correspondence, we have an equality in $\mathbb{Q}$

$$\sum_q (-1)^q Tr(\sigma \circ \Gamma|H^q(X \otimes E_\bar{q}, \mathbb{Q}_\ell)) = \sum_q (-1)^q Tr(\sigma \circ \Gamma|H^q_{\log \text{cris}}(Y/W)).$$

(2) Let $e$ be the algebraic correspondence in Lemma 3 and let $\lambda \nmid p, \mu | p$ be finite places of $L \supseteq E_0$. Then the monodromy filtration of the representations $e \cdot H^q(X \otimes E_{\bar{q}}, L_\lambda)$ and $e \cdot (H^q_{\log \text{cris}}(Y/W) \otimes \widehat{L}_{nr}^\mu)$ of the Weil-Deligne group $W(\bar{E}_q/E_q)$ are pure of weight $q$.

In (2), the tensor product is taken with respect to the map $W = O_{\bar{E}_q,0} \subset \widehat{E}_{nr,0}^{nr} \leftarrow \widehat{F}_{p,0}^{nr} \to \widehat{L}_{nr}^{nr}$ where the last map is that fixed in section 2. We prove Claim 4 by studying the weight spectral sequences for $\ell \neq p$ and for $p$.

We prove (1). First we compute the $\ell$-adic case. We consider the weight spectral sequence $[RZ], [I]$

$$E_1^{i,j} = \bigoplus_{k \geq \max(0, -i)} H^{j-2k}(Y^{(i+2)k}, \mathbb{Q}_\ell(-k)) \Rightarrow H^{i+j}(X, \mathbb{Q}_\ell).$$

for the semi-stable model $X_{O_V}$. Here $Y^{(i)}$ denotes the disjoint union of $i+1$ by $i+1$ intersections of the irreducible components of the closed fiber $Y = X \otimes_{O_{E/V}, q} \mathbb{F}_q$. The schemes $Y^{(i)}$ are projective and smooth over $\mathbb{F}_q$. Since the action of the Galois group $G_q$ is extended to the semi-stable model $X_{O_V}$, the spectral sequence is compatible with its action by transport of structure. It is also compatible with the action of Hecke operators, endomorphisms and permutations by the same argument as in the case of $p$-adic comparison isomorphism above. Hence the left hand side of the equality (1) is equal to

$$\sum_i (-1)^i \sum_{k=0}^i Nq^{n(\sigma)k} \sum_q (-1)^q Tr(\sigma \circ \Gamma|H^q(Y^{(i)}, \mathbb{Q}_\ell)).$$
For an element $\sigma \in W_{q}^{+}$ in the Weil group with $n(\sigma) \geq 0$, we define an endomorphism $\sigma_{\text{geom}}$ of $Y^{(i)}$ to be $\sigma_{\text{geom}} = \sigma \circ (\text{abs. Frob.})^{[F_{q}:F_{p}]} \cdot n(\sigma)$ for each $i$. It is a geometric endomorphism of a scheme $Y^{(i)}$ over the base field $\overline{F}_{q}$. Since the absolute Frobenius acts trivially on etale cohomology $H^{q}(Y^{(i)}, \mathbb{Q}_{\ell})$, we have $\sigma_{*} = \sigma_{\text{geom}*}$ as an operator acting it. Let $\Gamma_{\sigma}$ denote the composite of $\sigma_{\text{geom}}$ with $\Gamma$ as an algebraic correspondence and let $(\Gamma_{\sigma}, \Delta)$ be the intersection number. We apply the Lefschetz trace formula, to a proper smooth scheme $Y^{(i)}$ and an algebraic correspondence $\Gamma_{\sigma}$. Then we obtain

$$\sum_{q} (-1)^{q} \text{Tr}(\sigma \circ \Gamma|H^{q}(Y^{(i)}, \mathbb{Q}_{\ell})) = (\Gamma_{\sigma}, \Delta).$$

Next we compute the $p$-adic case. For log cristalline cohomology, we also have the weight spectral sequence [M]

$$E_{1}^{i,j} = \bigoplus_{k \geq \max(0, -i)} H^{j-2k}(Y^{(i+2k)}/W)(-k) \Rightarrow H_{\text{log cris}}^{i+j}(Y/W).$$

Here the Tate twist $(-k)$ means that we replace the Frobenius $\varphi$ by $p^{k} \varphi$. Since the maps involved in the definitions of the Hecke operators are finite etale, by the same argument as above, we see that the spectral sequence is compatible with the action of Hecke operators, endomorphisms and permutations. It is also compatible with the semi-linear action of the Galois group and the Frobenius operator. Hence by modifying it in the same way on the both side, it is also compatible with the linear action of the Weil group. For $\sigma \in W^{+}$ the modified action $\sigma_{*} \circ F^{n(\sigma)[F_{q}:F_{p}]}$ is the same as the action of the geometric endomorphism $\sigma_{\text{geom}} = \sigma \circ (\text{abs. Frob.})^{[F_{q}:F_{p}]} \cdot n(\sigma)$. Hence the right hand side of (1) is equal to

$$\sum_{i} (-1)^{i} \sum_{k=0}^{i} Nq^{n(\sigma)k} \sum_{q} (-1)^{q} \text{Tr}(\sigma_{\text{geom}*} \circ \Gamma|H_{\text{cris}}^{q}(Y^{(i)}/W)).$$

Again by the Lefschetz trace formula [GM][Gr], we have

$$\sum_{q} (-1)^{q} \text{Tr}(\sigma_{\text{geom}*} \circ \Gamma|H_{\text{cris}}^{q}(Y^{(i)}/W)) = (\Gamma_{\sigma}, \Delta).$$

Thus the both sides give the same answer and the equality in (1) is proved.

Finally we prove the assertion (2), the monodromy weight conjecture. The algebraic correspondence $e$ in Lemma 3 acts as an projector on the spectral sequences. We consider the $e$-part of them. We compute the $E_{1}$-terms of the $e$-part. We have $Y^{(k)} = \emptyset$ for $k > 1$ since the semi-stable model $X_{O}$ is proper smooth
over the semi-stable model $M_O$ of a curve. Let $C$ denote the closed fiber of the semi-stable model $M_O$. Then the disjoint union $C^{(i)}$ of the components is the same as the normalization of $C$ and the disjoint union $C^{(0)}$ of the intersections is the singular locus of $C$. To describe the $E_1$-terms we introduce some sheaves on $C^{(i)}$. For a place $\lambda|\ell$ of $L$, we define a smooth $L_\lambda$-sheaf $\mathcal{F}_\lambda^{(k)}$ to be

$$\bigotimes_i (\text{Sym}^{k_i-2} \otimes \det \frac{w}{2} (e_i R^1 a_* L_\lambda)) \otimes (e_0 R^1 b_* L_\lambda))^{\otimes (w-2)(g-1)}.$$ 

It is the restriction of the extension of $\mathcal{F}_\lambda^{(k)}$ on $M$ to $M_O$. Similarly for a place $\mu|p$ of $L$, we define an $\mathcal{F}$-isocrystal $\mathcal{E}_\lambda$. We consider $\mathcal{F}$-isocrystals $R^1 a_* O_{\text{crys}} \otimes W \lambda^{nr} \hat{L}^{nr}_\mu$ and $R^1 b_* O_{\text{crys}} \otimes W \lambda^{nr} \hat{L}^{nr}_\mu$ where the tensor product is taken as remarked after Claim 4. We regard them as an $O_D \otimes \mathbb{Z} L$-module and an $O_{E_0} \otimes \mathbb{Z} L$-module respectively. Then we define $\mathcal{E}_\lambda$ to be

$$\bigotimes_i (\text{Sym}^{k_i-2} \otimes \det \frac{w}{2} (e_i R^1 a_* O_{\text{crys}} \otimes W \lambda^{nr} \hat{L}^{nr}_\mu)) \otimes (e_0 R^1 b_* O_{\text{crys}} \otimes W \lambda^{nr} \hat{L}^{nr}_\mu))^{\otimes (w-2)(g-1)}.$$ 

Then similarly as in Lemma 3, we have $e R^q c_* L_\lambda = \mathcal{F}_\lambda^{(k)}$ if $q = q_0 = (w-2)(2g-1)$ and $= 0$ if otherwise. Also we have $e R^q c_* O_{\text{crys}} \otimes W \lambda^{nr} \hat{L}^{nr}_\mu = \mathcal{E}_\mu$ if $q = q_0$ and $= 0$ if otherwise. By the same argument as in Lemma 3 using the Leray spectral sequence, we see that there are only 5 non-vanishing $E_1$-terms

$$E_1^{-1,q_0+2} \quad E_1^{0,q_0+2} \quad E_1^{0,q_0+1} \quad E_1^{0,q_0} \quad E_1^{1,q_0}$$

where $q_0 = (2g-1)(w-2)$. Each term is described as follows. In the $\ell$-adic setting, we have

$$E_1^{0,q_0+q} = H^q(C^{(0)}, \mathcal{F}_\lambda^{(k)}), \quad E_1^{1,q_0} = E_1^{-1,q_0+2}(1) = H^q(C^{(1)}, \mathcal{F}_\lambda^{(k)}).$$

In the crystalline setting, we replace $\mathcal{F}_\lambda^{(k)}$ by $\mathcal{E}_\lambda^{(k)}$. The map $d_1^{-1,q_0+2}$ is the Gysin map and $d_1^{1,q_0}$ is the restriction map. By the Weil conjecture, the eigenvalues of a lifting of geometric Frobenius acting on each $E_1$-term $E_1^{i,j}$ are algebraic integer purely of weight $j$ and the spectral sequence degenerates at $E_2$-terms. The monodromy operator $N$ on the limit is induced by the canonical isomorphism $N : E_1^{-1,q_0+2}(1) \to E_1^{1,q_0}$ [RZ],[M]. Therefore the weight-monodromy conjecture is equivalent to that the isomorphism $N$ on the $E_1$-term induces an isomorphism on $E_2$-terms. Thus it is reduced to show

Claim 5. Let $q_0 = (2g-1)(w-2)$. The canonical map

$$N : \text{Ker}(E_1^{-1,q_0+2} \to E_1^{0,q_0+2})(1) \to \text{Coker}(E_1^{0,q_0} \to E_1^{1,q_0})$$
is an isomorphism.

First we prove it in the case where the multiweight \( k \) is of the form \( k = (2, 2, \cdots, 2, w) \). In this case, the sheaves \( \mathcal{F}_\lambda^{(k)} \) and \( \mathcal{E}_\mu^{(k)} \) are constant. Let \( I \) be the set of irreducible components and \( J \) be the set of singular points. Then it is enough to show that \( \text{Ker}(\mathbb{Q}^J \to \mathbb{Q}^I) \to \text{Coker}(\mathbb{Q}^I \to \mathbb{Q}^J) \) is an isomorphism. It is proved easily by extending the scalar to \( \mathbb{R} \).

We assume the multiweight \( k \) is not of the form \( k = (2, 2, \cdots, 2, w) \). To show Claim, we prove Proposition 1 below in the next section. To state it, we introduce a terminology. Take a sufficiently small open compact subgroup \( K'' \) such that \( M_{K''}'' \) has a proper smooth model \( M_{K''}'' \cdot \mathcal{O} \) and that \( KH \subset K'' \). We say an component \( C_i \) in \( C \) is ordinary, if it dominates a component of the closed fiber of \( M_{K''}'' \cdot \mathcal{O} \). Otherwise, we say it is supersingular.

**Proposition 1.** Let \( C_i \) be an ordinary irreducible component of \( C \). Then we have

\[
H^0(C_i, \mathcal{F}^{(k)}) = H^2(C_i, \mathcal{F}^{(k)}) = 0,
\]

\[
H^0(C_i, \mathcal{E}^{(k)}) = H^2(C_i, \mathcal{E}^{(k)}) = 0
\]

unless \( k = (2, 2, \cdots, 2, w) \).

Proof will be given in the next section.

We show Claim admitting Proposition 1. Let \( \Sigma \subset M_{K''}'' \cdot \mathcal{O} \) be the union of the image of supersingular components and of singular points. Then for each \( s \in \Sigma \) the sheaves \( \mathcal{F}_\lambda^{(k)} \) and \( \mathcal{E}_\mu^{(k)} \) are constant in the inverse image. Let \( I_x \) be the set of irreducible components and \( J_x \) be the set of singular points in the inverse image. Then it is reduced to that \( \text{Ker}(\mathbb{Q}^{J_x} \to \mathbb{Q}^{I_x}) \to \text{Coker}(\mathbb{Q}^{I_x} \to \mathbb{Q}^{J_x}) \) is an isomorphism which is proved in the same way as above.

**9. Vanishing of \( H^0 \).**

We prove Proposition 1. First we restate it in terms of the closed fiber of the proper smooth model \( M_{K''}'' \cdot \mathcal{O}_v \) of \( M_{K''}'' \) and Tate-modules. Let \( K' \subset G'((\mathbb{A}_f)) \) be a sufficiently small open subgroup satisfying the condition (c) in the proof of Lemma 4 (2) in Section 7: \( K' = \mathbb{Z}_p^\times \times GL_2(O_{F,p}) \times K'^p \times K'^p \). Then as is recalled there, Carayol has shown that \( M_{K'}' \) has good reduction and the abelian variety \( A_{K', \hat{T}}' \) is extended to the proper smooth model \( M_{K''}' \cdot \mathcal{O}_{E,q} \). Let \( C \) be an irreducible component of the geometric closed fiber \( M_{K''}' \cdot \mathcal{O}_{E,q} \otimes \overline{\mathbb{F}}_q \). We will define a smooth \( \ell \)-adic sheaf \( \mathcal{F}_\lambda^{\ast(k)} \) and an \( F \)-isocrystal \( \mathcal{E}_\lambda^{\ast(k)} \) on \( C \) in a similar way as \( \mathcal{F}_\lambda^{(k)} \). For a place \( \lambda | \ell \) of \( L \), we define a smooth \( L_\lambda \)-sheaf \( \mathcal{F}_\lambda^{\ast(k)} \) to be

\[
\bigotimes_i \left( \text{Sym}^{k_{i-2}}(e_i T_{\ell}(A) \otimes_{\mathbb{Z}_\ell} L_\lambda) \otimes (\text{det}(e_i T_{\ell}(A) \otimes_{\mathbb{Z}_\ell} L_\lambda))^{\otimes \frac{w \cdot k_i}{\ell}} \right).
\]
Here the idempotents \( e_i \in \text{End}_{M'}(A) \otimes L \) act on \( T_\ell(A) \otimes \mathbb{Z}_\ell L_\lambda \) by the covariant functoriality of Tate modules. We define an \( F \) crystal. Let \( T_\ell(A) \) denote the \( F \)-crystal associated to the \( p \)-divisible group \( A[p^\infty] \) on \( M' \). Let \( \mu \mid p \) be a place of \( L \). We regard the crystal \( T_\ell(A) \otimes \mathbb{W}\hat{L}_{\mu}^{nr} \) as an \( O_D \otimes \mathbb{Z}_\ell L \)-module by the covariant functoriality as above. For each \( i \), we define an \( F \)-isocrystal \( E_i \) as above. For each \( i \), we define an \( F \)-isocrystal \( E_i \) to be \( e_i(T_\ell(A) \otimes \mathbb{W}\hat{L}_{\mu}^{nr}) \) and put \( \mathcal{E}_\lambda^{(k)} = \bigotimes_i \left( \text{Sym}^{k_i-2} \mathcal{E}_i \otimes (\det \mathcal{E}_i)^{w_k-i} \right) \).

**Proposition 1'**. Let \( C' \to C \) be a finite covering of proper smooth curves and assume the multioweight \( k \) is not of the form \( (2, 2, \cdots, 2, w) \). For \( \lambda \mid \ell \neq p \), the pullback to \( C' \) of the smooth sheaf \( \mathcal{F}_\lambda^{(k)} \) has no non-trivial (geometrically) constant subsheaf or quotient smooth sheaf. For \( \mu \mid p \), the pullback to \( C' \) of the underlying isocrystal \( \mathcal{E}_\lambda^{(k)} \) has no non-trivial constant subsheaf or quotient isocrystal.

We show that Proposition 1' implies Proposition 1. Let \( C_i \) be as in Proposition 1. By the construction of \( \mathcal{F}_\lambda^{(k)} \) and \( \mathcal{E}_\lambda^{(k)} \) on \( C_i \), Proposition 1' implies a similar statement where we replace \( C', \mathcal{F}_\lambda^{(k)} \) and \( \mathcal{E}_\lambda^{(k)} \) by \( C_i, \mathcal{F}_\lambda^{(k)} \) and \( \mathcal{E}_\lambda^{(k)} \). By definition of \( H^0 \) and by Poincaré duality, it implies the assertion in Proposition 1.

**Proof of Proposition 1' for \( \lambda \nmid p \)**. First, we prove the \( \ell \)-adic case. The argument is similar to the proof of vanishing of \( H^0 \) and \( H^2 \) in the reduction of the equality (1) in Claim 1 to that in Claim 2 given in Section 3. It is enough to show that the image of the action of \( \pi_1(C) \) is sufficiently large. We show that the action on the Tate module defines a surjection \( \pi_1(C) \to SK'_{\ell} = \text{Ker}(\nu : K'_\ell \to \mathbb{Z}_\ell^\times \times O_{E_{p, \ell}}) \). Let \( V \) denote the maximal unramified extension of \( E_p \) and \( M'_{K', O_V} \) be the connected component of the proper smooth model whose closed fiber is \( C \). Since \( T_\ell(A) \) is locally constant on \( M'_{K', O_V} \), the map \( \pi_1(M'_{K', \tilde{V}}) \to (\hat{O}_D)^p \) factors through a surjection \( \pi_1(M'_{K', \tilde{V}}) \to \pi_1(M'_{K', O_V}) \simeq \pi_1(C) \). Since \( \pi_1(M'_{K', \tilde{V}}) \simeq \pi_1(M'_{K', \mathbb{C}}) \simeq SK'_{\ell} \), we obtain the surjection. The rest of the argument is identical to that in loc.cit and we will not repeat it here.

To proceed to the crystalline case, we recall the modular interpretation \([C1]\) due to Carayol of the integral model of \( M' \) over the integer ring \( O = O_{E_p} \). Let \( K' \subset G'(\mathbb{A}_f) \) be a sufficiently small open subgroup satisfying the condition (c) in the proof of Lemma 4 (2) in Section 7: \( K' = \mathbb{Z}_p^\times \times GL_2(O_{F,p}) \times K'_p \times K'^p \). We take an order \( O_D \subset D \) such that \( K' \subset \hat{O}_D^p \). We take an \( \hat{O}_D \)-lattice \( \hat{T} \subset D \otimes \mathbb{A}_f \). We assume they satisfy the following conditions:

- \( O_D \) is stable under the involution *;
- \( O_D \otimes \mathbb{Z}_p \) is maximal in \( D \otimes \mathbb{Q} \mathbb{Q}_p \);
- \( \text{Tr}_\psi(\hat{T}, \hat{T}) \subset \hat{\mathbb{Z}} \);
- \( \text{Tr}_\psi(\hat{T} \otimes_{\hat{O}_E} O_{E_p}, \hat{T} \otimes_{\hat{O}_E} O_{E_p}) \to \mathbb{Z}_p \) is perfect.
We put \( \hat{T}^p = \hat{T} \otimes_\mathbb{Z} \prod_{q \neq p} \mathbb{Z}_q \). It is a free \( \hat{O}_E^p \)-module of rank 4 and has a symmetric bilinear from \( \text{Tr}_B \psi : \hat{T}^p \times \hat{T}^p \to \mathbb{Z}_p \). We define a free \( \hat{O}_{F,p}^p = \prod_{p'|p \neq p} O_{F,p'} \)-module \( T^p \) of rank 4 as follows. By the isomorphism \( O_E \otimes \mathbb{Z}_p = \prod_{p'|p \neq p} (O_{F,p'} \times O_{F,p}) \), we have direct sum decomposition \( \hat{T} \otimes \mathbb{Z}_p = \prod_{p'|p \neq p} (\hat{T}^p_{p'} \times \hat{T}^p_{p}) \). Here the first factors correspond to the embedding \( O_{E_0} \to \mathbb{Z}_p \) fixed in Section 7. We put \( T^p = \prod_{p'|p \neq p} \hat{T}^p_{p'} \).

For an \( O_D \)-abelian scheme \( A \) on an \( O_{E,q} \)-scheme \( S \), we define direct summands \( \text{Lie}^2 A \) and \( \text{Lie}^{1,2} A \) of \( \text{Lie} A \) similarly as in Section 5. We define \( T^p (A) \) similarly as \( \hat{T}^p \) as above. On the category of schemes over \( O = O_{E_p} \), there is a proper smooth model \( M'_{K,O} \) over \( O \) of \( M_K' \), representing the functor \( S \mapsto \{ \text{isomorphism class of } (A, \theta, \tilde{k}) \} \) where

1. \( A \) is an \( O_D \)-abelian schemes of dimension 4g such that \( \text{Lie}^2 A = \text{Lie}^{1,2} A \) and it is a locally free \( O_S \)-module of rank 2.
2. \( \theta \in \text{Hom}(A, A^*) \) is an \( O_D \)-polarization of \( A \).
3. \( \tilde{k} = \{ k_p \}_{p \neq p} \) is a pair of a \( K_p \)-isomorphism of \( k^p : T^p (A) \to T^p \) and a \( K_p \)-isomorphism of \( k^p : T^p (A) \to T^p \) such that there exists a \( \hat{\mathbb{Z}}^p = \prod_{q \neq p} \mathbb{Z}_q \)-isomorphism \( k' \) making the diagram

\[
\begin{array}{ccc}
T^p (A) \times T^p (A) & \xrightarrow{(1, \theta, *)} & T^p (A) \times T^p (A) \\
\downarrow \scriptstyle{k \times k} & & \downarrow \scriptstyle{k'} \\
T^p \times T^p & \xrightarrow{T \psi} & \mathbb{Z}^p
\end{array}
\]

commutative.

In (3), the \( O_E \otimes \hat{\mathbb{Z}}^p \)-module \( T^p (A) \) is free of rank 4 and, by the condition (1), the \( \prod_{p'|p \neq p} O_{E,p'} \)-module \( T^p (A) \) is also free of rank 4. As is shown in [C1], the generic fiber \( M'_{K,O} \otimes O_{E,p} \) represents the restriction of the functor \( M'_K \) to the schemes over \( E_p \). Hence the smooth proper scheme \( M'_{K,O} \) is a model of the base change \( M'_K \otimes E_p \) and the universal abelian scheme \( A \) is a unique extension on \( M'_{K,O} \) of the pull-back.

We state Lemma 5 and 6 on the \( p \)-divisible group \( A[p^\infty] \) on \( C \). We will deduce Proposition 1’ in crystal case from the Lemmas. As in [C1] 2.6.3, we put

\[
T^p_p (A) = T^p_p (A) \otimes_{O_{E,p}} \prod_{q \neq p, q_0, p} O_{E,q'}.
\]

We identify \( \prod_{q \neq p, q_0, p} O_{E,q'} = \prod_{p'|p \neq p} O_{F,p} = O_{F,p}^p \) and regard \( T^p_p (A) \) as an \( O_{B,p}^p = \prod_{p'|p \neq p} O_{B,p} \)-module. By the modular interpretation recalled above, it
is a smooth etale sheaf on the proper scheme $M'_{K'}$, of $O_{E,p}^0 = \prod_{p' | p, \neq p} O_{B,p}$-modules of rank 1. Let $q_2 | p, \neq q$ be the other prime ideal of $O_E$ dividing $p$. We identify $O_{D,q_2} = O_{B,p}$ and take an isomorphism $O_{B,p} \simeq M_2(O_{F,p})$. Let $e \in O_{D,q_2}$ be the idempotent corresponding to \[
abla \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \simeq M_2(O_{F,p}).\] Similarly as in [C1] 5.4, let $E_\infty$ be the $p$-divisible group

$$E_\infty = e(A[p^\infty] \otimes_{O_E} \mathbb{Z}_p, O_{E,q_2}).$$

In the terminology of [C1] Appendix 1, it is an $O_{F,p}$-divisible group of height 2. As a $p$-divisible group, it is of height 2 if $[F_p : \mathbb{Q}_p]$ and of dimension 1. Let $\mathcal{E}_0$ and $T^p$ be the $F$-crystals associated to the $p$-divisible group $E_\infty$ and to the Tate module $T^p(A)$ respectively.

The $F$-isocrystal $\mathcal{E}_i^{(k)} = \bigotimes (\text{Sym}^{k_i} - 2 \mathcal{E}_i \otimes (\det \mathcal{E}_i)^{-k_i})$ is related to them in the following way. We regard $\mathcal{E}_0$ and $T^p$ as an $O_{F,p}$-module and an $O_{F,p}$-module respectively by the covariant functoriality. Let $I_1 \subset I = \{\tau_i : F \to \mathbb{L}_e\}$ be the subset $I_1 = \{\tau_i : F \to L\}$. Then for $i \in I_1$, the $F$-isocrystal $\mathcal{E}_i$ is isomorphic to $\mathcal{E}_0 \otimes_{O_{F,p}} \mathcal{L}_{\mu_r}^{n_r}$ with respect to $\tau_i : O_{F,p} \to \mathcal{L}_{\mu_r}^{n_r}$. Here we identify $O_{F,p}$ with $O_{E,q}$.

For $i \in I - I_1$, we take an isomorphism $B \otimes_{F} \mathbb{L}_e \simeq M_2(L)$ for the tensor product with respect to $\tau_i$ and let $e$ be the idempotent corresponding to \[
abla \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \simeq M_2(O_{F,p}).\] Then the $F$-isocrystal $\mathcal{E}_i$ is isomorphic to $e(T^p \otimes_{O_{F,p}} \mathcal{L}_{\mu_r}^{n_r})$ with respect to $\tau_i : O_{F,p} \to \mathcal{L}_{\mu_r}^{n_r}$.

Here we also identify $O_{E,q'} = O_{F,p'}$ for primes $p'|p, \neq p$ and $q' | q, \neq q_0$.

It is shown in [Ca] (6.7), (9.4.3) that there exists a finite nonempty set $\Sigma \subset C$ of closed points satisfying the following condition:

At each point in $\Sigma$, the $p$-divisible group $E_\infty$ is connected. On the complement $U = C - \Sigma$, the $p$-divisible group $E_\infty$ is an extension of an etale $p$-divisible group $E^\text{et}_\infty$ by an connected $p$-divisible group $E^\text{et}_\infty$. We call a point in $\Sigma$ a supersingular point and a point in $U$ an ordinary point. The $p$-divisible groups $E^\text{et}_\infty$ and $E^\text{et}_\infty$ have natural structures of $O_{F,p}$-modules. The Tate module $T(E^\text{et}_\infty)$ is a smooth sheaf of $O_{F,p}$-modules of rank 1.

**Lemma 5.** The morphism $\pi_1(U) \to O_{E,p}^\times \times O_{B,p}^\times$ defined by the smooth sheaf $T_p(E^\text{et}_\infty) \times T_p(A)$ of $O_{F,p} \times O_{B,p}$-modules of rank 1 defines a surjection

$$\pi_1(U) \to O_q^\times \times SK^\times.$$
Lemma 6. The extension of the underlying isocrystal

\[ 0 \to \mathcal{E}'_0 \otimes \mathbb{Q}_p \to \mathcal{E}_0 \otimes \mathbb{Q}_p \to \mathcal{E}''_0 \otimes \mathbb{Q}_p \to 0 \]

is non-trivial.

Proof of Proposition 1' for \( \lambda \nmid p \). The argument is similar to that in [Cr]. First we prove it admitting Lemmas 5 and 6. It is sufficient to show that, on the inverse image \( U' \subset C' \) of the ordinary locus \( U \), the restriction of \( \mathcal{E}^*(k) \) has no non-constant subisocrystal or quotient isocrystal. Before starting proof, note that an \( F \)-isocrystal is constant if and only if the underlying (iso)crystal is constant. In fact, if the underlying (iso)crystal \( \mathcal{E} \) is constant, the Frobenius pull-back \( F^* \mathcal{E} \) and the Frobenius map \( F : F^* \mathcal{E} \to \mathcal{E} \) defining the structure of \( F \)-isocrystal is constant. The only if part is trivial.

We put \( r = [F_q : \mathbb{Q}_p] \) and \( I_1 = \text{Hom} (F_q, L^\mu) = \{ \tau_1, \ldots, \tau_r \} \subset I = \text{Hom}(F, L) = \{ \tau_1, \ldots, \tau_g \} \). We define a decreasing filtration on the restriction of \( \mathcal{E}^*(k) \) on \( U \) with multiindex \( Z^{I_1} \) as follows. On \( \mathcal{E}_0 \), we define a filtration \( F^* \) on \( \mathcal{E}_0 \) by \( F^0 \mathcal{E}_0 = \mathcal{E}_0, F^1 \mathcal{E}_0 = \mathcal{E}'_0, F^2 \mathcal{E}_0 = 0 \). For each \( i \in I_1 \), it induces a filtration on \( \mathcal{E}_i \) and hence on \( \text{Sym}^{k_i-2} \mathcal{E}_i \) by the isomorphism \( \mathcal{E}_i \cong \mathcal{E}_0 \otimes_{O_{F^p}} L^\mu \). Taking symmetric powers and tensor product, we obtain a filtration on \( \mathcal{E}^*(k) = \mathcal{E}_0 \mathcal{E}^*(k) / \sum_{q' > q} F^{q'} \mathcal{E}^*(k) \) for each \( q = (q_1, \ldots, q_r) \in Z^{I_1} \).

We deduce from Lemma 5 that the isocrystal \( Gr^q_p \mathcal{E}^*(k) \) has no constant subisocrystal or quotient isocrystal except for at most one multi-index \( q = (q_1, \ldots, q_r) \) satisfying \( (k_1, \ldots, k_g) = (2q_1 + 2, \ldots, 2q_r + 2, 2, \ldots, 2) \). In the exceptional case, we will see that the graded piece is in fact constant. We compute the graded pieces. The graded pieces are computed as

\[
Gr^q_p \mathcal{E}^*(k) = \bigotimes_{i \in I_1} \left( (\det \mathcal{E}_i)^{\otimes \frac{w-k_i+q_i}{2}} \otimes (Gr^0_p \mathcal{E}_i)^{\otimes k_i-2-2q_i} \right) \otimes \bigotimes_{i \in I-I_1} \left( \text{Sym}^{k_i-2} \mathcal{E}_i \otimes (\det \mathcal{E}_i)^{\otimes \frac{w-k_i}{2}} \right)
\]

for \( 0 \leq q_i \leq k_i - 2 \) for \( i \in I_1 \) and is 0 otherwise. By the Weil-pairing of Drinfeld basis [C1] 9.2, the determinant isocrystal \( \mathcal{E}_i \) is geometrically constant for \( i \in I_1 \). Similarly but more easily, \( \det \mathcal{E}_i \) is also constant for \( i \in I-I_1 \). Therefore it is sufficient to show that the isocrystal \( \bigotimes_{i \in I_1} (Gr^0_p \mathcal{E}_i)^{\otimes k_i-2-2q_i} \otimes \bigotimes_{i \in I-I_1} \text{Sym}^{k_i-2} \mathcal{E}_i \) has no non-trivial constant subisocrystal or quotient isocrystal unless \( (k_1, \ldots, k_g) = (2q_1 + 2, \ldots, 2q_r + 2, 2, \ldots, 2) \).

The \( F \)-isocrystals \( Gr^0_p \mathcal{E}_i \) for \( i \in I_1 \) and \( \mathcal{E}_i \) for \( i \in I-I_1 \) are defined by smooth \( p \)-adic etale sheaves on \( U \). Let \( \mathcal{L}_i \) and \( \mathcal{F}_i \) be the corresponding smooth \( p \)-adic sheaves. Since \( F \)-isocrystal is constant if and only if the underlying crystal is
constant, it is reduced to show that the smooth $p$-adic sheaf $\bigotimes_{i \in I_1} \mathcal{L}_{i} \otimes_{\mathcal{F}_{i}} \mathcal{S}_{k \cdot i - 2 - 2q}$ is an irreducible smooth $p$-adic etale sheaf. It follows from the surjectivity, Lemma 5, of the map $\pi_{1}(U) \to SK'_{p}$ by the same argument as in the $\ell$-adic case.

We complete the proof by using Lemma 6. We assume that there exists a non-trivial constant subisocrystal of $E^{*}(k)$ for $(k_{1}, \ldots, k_{g}) \neq (2, \ldots, 2)$ and deduce a contradiction. The proof for the quotient is similar and is omitted. By the study of the graded pieces above, the proof is completed except for the case where $k_{i}$ are even for $i \in I_{1}$ and $k_{i} = 2$ for $i \in I - I_{1}$. We put $(k_{1}, \ldots, k_{g}) = (2q_{1} + 2, \ldots, 2q_{r} + 2, \ldots, 2)$ and assume $q = (q_{1}, \ldots, q_{r}) \neq 0$. By the computation of the graded pieces, if we had non-trivial constant subisocrystal, it should be contained in $F^{q}E^{*}(k)$ and mapped isomorphically to $Gr^{q}E^{*}(k)$. Namely, the extension $F^{q}E^{*}(k)$ of $Gr^{q}E^{*}(k)$ is split. Take an index $i \in I_{1}$ such that $q_{i} > 0$ and let $q', q''$ be the multi-index obtained from $q$ by replacing $q_{i}$ be $q_{i} + 1, q_{i} + 2$ respectively. Then the extension

$$0 \to Gr^{q'}E^{*}(k) \to F^{q}E^{*}(k)/F^{q''}E^{*}(k) \to Gr^{q}E^{*}(k) \to 0$$

is also split. Its extension class is $q_{i}$-times the class of extension in Lemma 6 and hence is non-zero. Thus we get a contradiction. We have proved that Lemmas 5 and 6 implies Proposition 1'.

We prove Lemmas 5 and 6 to complete the proof of Proposition 1' hence of Theorems 1 and 2. We prove Lemma 5 using a supersingular point which exists by [Ca] (9.4.3). Lemma 6 will be proved using an ordinary point.

**Proof of Lemma 5.** Since $T_{p}^{p}(A)$ is smooth on the proper smooth model $M'_{K', O_{E, q}}$, the same argument as in the $\ell$-adic case shows that we have a surjection $\pi_{1}(C) \to SK'_{p}$. Take a supersingular point $x \in \Sigma \neq \emptyset$ and let $I_{x}$ denote the inertia group. It is enough to show that the restriction $I_{x} \to O_{q}^\times$ is surjective. Let $U_{n}$ be the finite etale covering $U_{n} = \text{Isom}(O_{F, p}/p^{n}, \mathbb{E}_{n}^{et})$ of $U$ trivializing the $p^{n}$-torsion part $\mathbb{E}_{n}^{et}$. Here an isomorphism means an isomorphism of $O_{F, p}/p^{n}$-group schemes. The covering $U_{n}$ is an analogue of an Igusa curve. It is sufficient to show that $U_{n}$ is totally ramified at a supersingular point. Namely we show the following.

**Lemma 7.** Let $K_{x}$ denote the completion of the function field of $C$ at a supersingular point $x$. Then the base change $U_{n} \times_{C} \text{Spec } K_{x}$ is the spectrum of a totally ramified extension of $K_{x}$.

**Proof.** Let $E$ denote the formal group associated to the $p$-divisible group $\mathbb{E}_{\infty}$ over the completion $\hat{C} = \text{Spec } \hat{O}_{C, x}$. Let $\pi$ be a prime element of $O_{F, p}$. For an integer $n$, let $E^{(n)}$ denote the base change of $E$ by the $Np^{n}$-th power Frobenius and $F_{n}^{*} : E \to E^{(n)}$ be the $Np^{n}$-th power relative Frobenius over $\hat{C}$. Then the multiplication $[\pi^{n}] : E \to E$ is factorized as $[\pi^{n}] = V^{n} \circ F^{n}$ for a map $V^{n} : E^{(n)} \to E$. Outside the
closed point \(x\), the map \(V^n\) is etale and hence \(\text{Ker} V^n\) is a finite flat group scheme over \(\hat{C}\) extending the etale quotient \(E^n_{\text{et}}\) on the generic point. Let \(C_n = (\text{Ker} V^n)^x\) be the scheme of \(O_{F,p}/p^n\)-basis of \(\text{Ker} V^n\) in the sense of Drinfeld. Namely, it is a closed subscheme of \(\text{Ker} V^n\) representing the functor

\[
R \mapsto \{ s \in \text{Ker} V^n(R) \mid \sum_{a \in O_{F,p}/p^n} [as] = \text{Ker} V^n \text{ as a divisor in } E_R^{(n)} \}
\]

for a ring over \(\hat{O}_{C,x}\). Outside the closed point, the scheme \(C_n\) is the same as the base change of \(U_n\). Therefore, it is sufficient to show that \(C_n\) is regular and the inverse image of the closed point \(x\) by \(C_n \to \hat{C}\) contains only one point. The second assertion is clear since \(C_n\) is a closed subscheme of a local scheme \(\text{Ker} V^n\). We show that the intersection \(C_n \cap [0]\) of \(C_n\) with the zero-section \([0]\) of the formal group \(E^{(n)}\) is equal to \(\text{Spec} \kappa(x)\). This implies that \(C_n\) is regular since the zero section is a divisor in \(E^{(n)}\).

Let \(R = \Gamma(C_n \cap [0], O)\). It is an Artin \(\hat{O}_{C,x}\)-algebra. It is sufficient to show that the surjection \(\hat{O}_{C,x} \to R\) factors through the surjection \(\hat{O}_{C,x} \to \kappa(x)\). By the assumption, the zero-section is an \(O_{F,p}/p^n\)-basis of \(\text{Ker} V^n\). Hence, we have \(\text{Ker}[p^n] = \text{Ker} E^{2n}\) on \(R\) and an isomorphism \(E_R \simeq E_R^{(2n)} \simeq E_R^{(2mn)}\) for \(m \geq 1\).

Since \(R\) is Artinean, for sufficiently large \(m\), the map \(a \to a^{Nq^{2mn}}\) factors through \(R \to \kappa(x) \to R\) and we obtain \(E_R \simeq E_R^{(2mn)} \simeq E_x \otimes_{\kappa(x)} R\). This means that \(\hat{O}_{C,x} \to R\) factors through \(\kappa(x)\) since \(E_{\infty}\) over \(\hat{O}_{C,x}\) is the universal deformation of \(E_{\infty}|_x\), Proposition 5.4 [C1]. Thus we have proved Lemma 7 and hence Lemma 5.

To prove Lemma 6, we show

**Lemma 8.** Let \(\hat{C} = \text{Spec} \hat{O}_{C,x}\) be the completion at an ordinary closed point \(x \in U\). Let \([E] \in \text{Ext}^1(\mathbb{E}^\text{et}, \mathbb{E})\) be the class of \(E\) as an extension of \(O_{F,p}\)-divisible groups on \(\hat{C}\). Then the class \([E]\) is not torsion.

We derive it from the following statement proved in [C1] Proposition 5.4, App. Théorème 3.

**Lemma 9.** On the completion \(\hat{C}\) at an ordinary closed point, the connected part \(E^0\) is isomorphic to the pull-back of the Lubin-Tate formal group. The etale part \(E^\text{et}\) is isomorphic to the constant \(O_{F,p}\)-divisible group \(F_p/O_{F,p}\). The completion \(\hat{C}\) pro-represents the functor \(R \mapsto \text{Ext}_R(F_p/O_{F,p}, \mathbb{E}_0) = \mathbb{E}_0(R)\) on the category of Artin \(\mathbb{F}_p\)-algebras \(R\) together with a surjection \(R \to \kappa(x)\). It is isomorphic to \(E^0\) as a formal scheme. The extension \(E\) on \(\hat{C} = E^0\) is identified with the universal extension.

**Proof of Lemma 8.** We identify the formal schemes \(E^0 = \hat{C}\). By Lemma 9, the universal extension \(E\) corresponds to the identity \(C \to E^0\). Hence it is the universal section of the formal group \(E^0\) and is not torsion.
Proof of Lemma 6. It is enough to prove that the restriction to the completion at an ordinary closed point is not the trivial extension. Since the \( p \)-divisible groups \( E^o \) and \( E^{et} \) are constant on \( \hat{C} \), the \( F \)-isocrystals \( E' \otimes \mathbb{Q}_p, E'' \otimes \mathbb{Q}_p \) and hence their underlying crystals are constant there. If the extension of the underlying isocrystal was trivial, the underlying isocrystal and hence the \( F \)-isocrystal \( E \otimes \mathbb{Q}_p \) would be constant. It means that the extension class \([E] \in \text{Ext}^1(E^{et}, E^o)\) is torsion and contradicts with Lemma 8.

Thus the proof of Proposition 1’ and hence of Theorems 1 and 2 are now complete.

References

[BR] D. Blasius and J. Rogawski, Motives for Hilbert modular forms, Inventiones Math. 114 (1993), 55-87.
[C1] H. Carayol, Sur la mauvaise réduction des courbes de Shimura, Compositio Math. 59 (1986), 151-230.
[C2] _____, Sur les représentations \( \ell \)-adiques associées aux formes modulaires de Hilbert, Ann. Sci. ENS 19 (1986), 409-468.
[Cr] R. Crew, \( F \)-isocrystals and their monodromy groups, Ann. Sci. ENS 4 Ser. 25 (1992), 429-464.
[D1] P. Deligne, Formes modulaires et représentations \( \ell \)-adiques, Seminaire Bourbaki épisode 355, Lecture note in Math., vol. 179, Springer, 1969, pp. 139-172.
[D2] _____, Travaux de Shimura, Seminaire Bourbaki, Fév 1971, épisode 389, Lecture note in Math., vol. 244, Springer, 1971, pp. 123-165.
[D3] _____, Formes modulaires et représentations de \( \text{GL}(2) \), Modular forms of one variable II, Lecture note in Math., vol. 349, Springer, 1973, pp. 55-105.
[D3] _____, Les constantes des équations fonctionnelles des fonctions \( L \), Modular forms of one variable II, Lecture note in Math., vol. 349, Springer, 1973, pp. 501-595.
[D5] _____, Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques, Proc. Symp. pur Math., vol. 33-2, 1979, pp. 247-290.
[DM] P. Deligne-D. Mumford, The irreducibility of the space of curves of given genus, Publ. Math. IHÉS 36 (1969), 75-109.
[Fo] J.-M. Fontaine, Représentations \( \ell \)-adiques potentiellement semi-stables, Périodes \( p \)-adiques, Astérisque, vol. 223, 1994, pp. 321-348.
[GM] H. Gillet-W. Messing, Cycle classes and Riemann-Roch for crystalline cohomology, Duke Math. J. 55 (1987), 501-538.
[Gr] M. Gros, Classes de Chern et classes de cycles en cohomologie logarithmique, Bull. Soc. Math. France 113 (1985).
[G] A. Grothendieck, Un théorème sur les homomorphismes de schemas abéliens, Inventiones Math. 2 (1966), 59-78.
[HK] O. Hyodo-K. Kato, Semi-stable reduction and crystalline cohomology with logarithmic poles, Périodes \( p \)-adiques, Astérisque, vol. 223, 1994, pp. 221-268.
[I] L. Illusie, Autour du théorème de monodromie locale, Périodes \( p \)-adiques, Astérisque, vol. 223, 1994, pp. 9-58.
[JL] H. Jacquet and R. P. Langlands, Automorphic forms on \( GL_2 \), Springer LNM, vol. 114, Springer, 1970.
[Ka] N. Katz, \( p \)-adic \( L \)-functions for CM fields, Inventiones Math. 49 (1978), 199-297.
[Ku] Ph. Kutzko, *The local Langlands conjecture for GL(2)*, Ann. of Math. **112** (1980), 381-412.

[La] R. P. Langlands, *Modular forms and ℓ-adic representations*, Modular forms of one variable II, Lecture note in Math., vol. 349, 1973, pp. 361-500.

[Mi] J. S. Milne, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, Automorphic forms, Shimura varieties and L-functions, I, Academic Press, 1990, pp. 284-414.

[Mo] A. Mokrane, *La suite spectrale des poids en cohomologie de Hyodo-Kato*, Duke Math. J. **72** (1993), 301-377.

[O] M. Ohta, *On ℓ-adic representations attached to automorphic forms*, Japan J. of Math. **8-1** (1982), 1-47.

[R] M. Rapoport, *Compactifications de l’espace de modules de Hilbert-Blumenthal*, Compositio Math. **36** (1978), 255-335.

[RZ] M. Rapoport-T. Zink, *Ueber die lokale Zetafunktion von Shimuravarietäten, Monodromiefiltration und verschwindende Zyklen in ungleicher Characteristik*, Inventiones Math. **68** (1982), 21-201.

[RT] J. Rogawski and J. Tunnell, *On Artin L-functions associated to Hilbert modular forms of weight one*, Inventiones Math. **74** (1983), 1-42.

[S] T. Saito, *Modular forms and p-adic Hodge theory*, Inventiones Math. **129** (1997), 607-620.

[Sc] A. Scholl, *Motives for modular forms*, Inventiones Math. **100** (1990), 419-430.

[Sh] G. Shimura, *The special values of the zeta functions associated with Hilbert modular forms*, Duke Math. J. **45-3** (1978), 637-679.

[Ta1] R. Taylor, *On Galois representations associated to Hilbert modular forms*, Inventiones Math. **98** (1989), 265-280.

[Ta2] ———, *ibid II*, Conference on Elliptic curve and Modular forms (eds.) J. Coates and S. T. Yau (1995), 185-191.

[Tj] T. Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Inventiones Math. **137** (1999), 233-411.

Tokyo 153-8914 Japan

E-mail address: t-saito@ms.u-tokyo.ac.jp