Besov regularity of non-linear parabolic PDEs on non-convex polyhedral domains

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Abstract

This paper is concerned with the regularity of solutions to parabolic evolution equations. We consider semilinear problems on non-convex domains. Special attention is paid to the smoothness in the specific scale \( B_{r,\tau}^{\frac{\tau}{d} + \frac{1}{p}} \) of Besov spaces. The regularity in these spaces determines the approximation order that can be achieved by adaptive and other nonlinear approximation schemes. We show that for all cases under consideration the Besov regularity is high enough to justify the use of adaptive algorithms. Our proofs are based on Schauder’s fixed point theorem.

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Contents

1 Introduction 2
2 Preliminaries 3
3 Sobolev, Kondratiev, and Besov spaces 4
4 Parabolic PDEs and operator pencils 7
  4.1 The fundamental parabolic problems 7
  4.2 Operator pencils 9
5 Regularity results in Sobolev and Kondratiev spaces 11
  5.1 Regularity results in Sobolev and Kondratiev spaces for Problem I 11
  5.2 Schauder’s fixed point theorem 13
  5.3 Regularity results in Sobolev and Kondratiev spaces for Problem II 14
6 Regularity results in Besov spaces 22
A Locally convex spaces 24
B An extension operator for Banach-valued Sobolev spaces 24

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1 Introduction

In this paper we derive regularity estimates in weighted Sobolev and Besov spaces of the solutions to evolution equations in non-smooth domains of polyhedral type $D \subset \mathbb{R}^3$, cf. Definition 3.2. Precisely, we investigate linear ($\varepsilon = 0$) and nonlinear ($\varepsilon > 0$) equations of the form

$$\frac{\partial}{\partial t} u + (-1)^m L(t, x, D_x) u + \varepsilon u^M = f \quad \text{in } [0, T] \times D,$$

(1.1)

with zero initial and Dirichlet boundary conditions, where $m, M \in \mathbb{N}$, and $L$ denotes a uniformly elliptic differential operator of order $2m$ with sufficiently smooth coefficients. Special attention is paid to the spatial regularity of the solutions to (1.1) in specific non-standard smoothness spaces, i.e., in the so-called adaptivity scale of Besov spaces

$$B^r_{r, \tau}(D), \quad \frac{1}{\tau} = \frac{r}{3} + \frac{1}{p}, \quad r > 0.$$

(1.2)

Our investigations are motivated by fundamental questions arising in the context of the numerical treatment of equation (1.1). It is our goal to mathematically justify that when solving parabolic PDEs on non-smooth domains adaptive schemes can outperform non-adaptive ones. In an adaptive strategy, the choice of the underlying degrees of freedom is not a priori fixed but depends on the shape of the unknown solution. In particular, additional degrees of freedom are only spent in regions where the numerical approximation is still 'far away' from the exact solution. Although the basic idea is convincing, adaptive algorithms are hard to implement, so that beforehand a rigorous mathematical analysis to justify their use is highly desirable. Since it has been shown that the achievable order of adaptive algorithms depends on the regularity of the target function in the specific scale of Besov spaces (1.2), whereas on the other hand it is the regularity of the solution in the scale of Sobolev spaces, which encodes information on the convergence order for nonadaptive (uniform) methods, we can draw the following conclusion: Adaptivity is justified, if the Besov regularity of the solution in the Besov scale (1.2) is higher than its Sobolev smoothness!

For the case of elliptic partial differential equations, a lot of positive results in this direction are already established [11–15, 17, 24, 25] inspired by the fundamental paper of Dahlke and DeVore [16]. It is well-known that if the domain under consideration, the right-hand side and the coefficients are sufficiently smooth, then there is no reason why the Besov smoothness should be higher than the Sobolev regularity [1]. However, on general Lipschitz domains and in particular in polyhedral domains, the situation changes completely: On these non-smooth domains, singularities at the boundary may occur that diminish the Sobolev regularity of the solution significantly [9, 10, 22, 23, 26] (but can be compensated by suitable weight functions).

To the best of our knowledge, not so many results in this direction are available for evolution equations so far. First results for the special case of the heat equation have been reported in [2, 4], but for a slightly different scale of Besov spaces. Inspired by these findings we studied parabolic equations of type (1.1) on polyhedral type domains in the paper [20] and its forerunner [19] (on polyhedral cones).

The results obtained in [19, 20] are very promising and indicate that (as in the elliptic case) the appearing boundary singularities do not influence the Besov regularity too much. However, as a drawback of the methods we used (Banach’s fixed point theorem), the nonlinear regularity results obtained in [20] Thms. 4.13, 5.6 only hold for convex domains so far, cf. the explanations given in [19] Rem. 3.8. The main reason for this were the restrictions we had to impose on the weight parameter $a$ appearing in the definition of the weighted Sobolev spaces $K^m_{p,a}(D)$ (so-called Kondratiev spaces), which we used for studying the regularity of the solution. In particular, the weight $\rho(x)^a$ involved measures only the distance to the singular set of the domain (i.e., the edges and vertices.
of the polyhedral type domains) and it turned out that there is no suitable $a$ satisfying all of our requirements in case of a non-convex domain.

However, since on these domains much more severe singularities might occur, we aim at removing this restriction on the parameter $a$. This will be done by imposing stronger assumptions on the right–hand side $f$, requiring that it is arbitrarily smooth with respect to time, i.e.,

$$f \in \bigcap_{l=0}^{\infty} H^l([0,T], L_2(D) \cap K_{\eta-2m}^{\eta-2m}(D)).$$

We already pursued this possibility in [20] when studying the linear equation (1.1) with $\varepsilon = 0$ and were able to weaken the restrictions on the parameter $a$ in order to allow a larger range. But unfortunately, the generalization of the results to nonlinear problems was not straightforward, since the right hand sides are not taken from a Banach or a quasi Banach space. Therefore, we now invoke Schauder’s fixed point theorem which allows us to work with the coarse topology of the locally convex topological vector spaces from the right hand side. The price to pay for this is that we only get existence but not uniqueness of the solutions.

Let us summarize our results: In the linear case $\varepsilon = 0$ it is known that if the right-hand side as well as its time derivatives are contained in specific Kondratiev spaces, then, for every $t \in [0,T]$ the spatial Besov smoothness of the solution to (1.1) is always larger than $2m$, provided that some technical conditions on the operator pencils are satisfied, see [20]. The reader should observe that the results are independent of the shape of the polyhedral domain, and that the classical Sobolev smoothness in the extreme case might be limited by $m$, see [29]. Therefore, for every $t$, the spatial Besov regularity can be more than twice as high as the Sobolev smoothness, which of course justifies the use of (spatial) adaptive algorithms. Moreover, for smooth domains and right-hand sides in $L_2$, the best one could expect would be smoothness order $2m$ in the classical Sobolev scale. So, the Besov smoothness on polyhedral type domains is at least as high as the Sobolev smoothness on smooth domains.

In this paper we generalize this result to nonlinear parabolic equations of the form (1.1). We prove that in an intersection of sufficiently small balls containing the solution of the corresponding linear equation, there exists a solution to (1.1) possessing the same Besov smoothness in the scale (1.2). The proof is performed by a technically quite involved application of Schauder’s fixed point theorem. The final result is stated in Theorem 6.1.

In conclusion, the results presented in this paper imply that for each $t \in (0,T)$ the spatial Besov regularity of the unknown solutions of the problems studied here is much higher than the Sobolev regularity, which justifies the use of spatial adaptive algorithms. This corresponds to the classical time-marching schemes such as the Rothe method. We refer e.g. to the monographs [28, 36] for a detailed discussion. Of course, it would be tempting to employ adaptive strategies in the whole space-time cylinder. First results in this direction have been reported in [35]. To justify also these schemes, Besov regularity in the whole space-time cylinder has to be established. This case will be studied in a forthcoming paper.

2 Preliminaries

We collect some notation used throughout the paper. As usual, we denote by $\mathbb{N}$ the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}^d$, $d \in \mathbb{N}$, the $d$-dimensional real Euclidean space with $|x|$, for $x \in \mathbb{R}^d$, denoting the Euclidean norm of $x$. By $\mathbb{Z}^d$ we denote the lattice of all points in $\mathbb{R}^d$ with integer components. For $a \in \mathbb{R}$, let $[a]$ denote its integer part and $a_+ := \max(a, 0)$.

Moreover, $c$ stands for a generic positive constant which is independent of the main parameters, but
its value may change from line to line. The expression $A \lesssim B$ means that $A \leq c B$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$.

Given two quasi-Banach spaces $X$ and $Y$, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding is bounded. By supp $f$ we denote the support of the function $f$. For a domain $\Omega \subset \mathbb{R}^d$ and $r \in \mathbb{N} \cup \{\infty\}$ we write $C^r(\Omega)$ for the space of all real-valued $r$-times continuously differentiable functions, whereas $C(\Omega)$ is the space of bounded uniformly continuous functions, and $\mathcal{D}(\Omega)$ for the set of test functions, i.e., the collection of all infinitely differentiable functions with compact support contained in $\Omega$. Moreover, $L^1_{\text{loc}}(\Omega)$ denotes the space of locally integrable functions on $\Omega$.

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d_0$ with $|\alpha| := \alpha_1 + \ldots + \alpha_d = r$, $r \in \mathbb{N}_0$, and an $r$-times differentiable function $u : \Omega \to \mathbb{R}$, we write

$$D^{(}\alpha\rangle u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} u$$

for the corresponding classical partial derivative as well as $u^{(k)} := D^{(k)}u$ in the one-dimensional case. Hence, the space $C^r(\Omega)$ is normed by

$$\|u|C^r(\Omega)\| := \max \sup_{|\alpha| \leq r \atop x \in \Omega} |D^{(\alpha)}u(x)| < \infty.$$  

Moreover, $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space of rapidly decreasing functions. The set of distributions on $\Omega$ will be denoted by $\mathcal{D}'(\Omega)$, whereas $\mathcal{S}'(\mathbb{R}^d)$ denotes the set of tempered distributions on $\mathbb{R}^d$.

The terms distribution and generalized function will be used synonymously. For the application of a distribution $u \in \mathcal{D}'(\Omega)$ to a test function $\varphi \in \mathcal{D}(\Omega)$ we write $(u, \varphi)$. The same notation will be used if $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$ (and also for the inner product in $L_2(\Omega)$). For $u \in \mathcal{D}'(\Omega)$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d_0$, we write $D^{\alpha}u$ for the $\alpha$-th generalized or distributional derivative of $u$ with respect to $x = (x_1, \ldots, x_d) \in \Omega$, i.e., $D^{\alpha}u$ is a distribution on $\Omega$, uniquely determined by the formula

$$(D^{\alpha}u, \varphi) := (-1)^{|\alpha|}(u, D^{(\alpha)}\varphi), \quad \varphi \in \mathcal{D}(\Omega).$$

In particular, if $u \in L^1_{\text{loc}}(\Omega)$ and there exists a function $v \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega v(x)\varphi(x)dx = (-1)^{|\alpha|} \int_\Omega u(x)D^{(\alpha)}\varphi(x)dx \quad \text{for all} \quad \varphi \in \mathcal{D}(\Omega),$$

we say that $v$ is the $\alpha$-th weak derivative of $u$ and write $D^{\alpha}u = v$. We also use the notation $\frac{\partial^k}{\partial x_j^k} u := D^\beta u$ as well as $D_{x_j}^k := D^\beta u$, for some multi-index $\beta = (0, \ldots, k, \ldots, 0)$ with $\beta_j = k$, $k \in \mathbb{N}$.

Furthermore, for $m \in \mathbb{N}_0$, we write $D^m u$ for any (generalized) $m$-th order derivative of $u$, where $D^0 u := u$ and $D^1 u := D^1 u$. Sometimes we shall use subscripts such as $D^m_x$ or $D^m_t$ to emphasize that we only take derivatives with respect to $x = (x_1, \ldots, x_d) \in \Omega$ or $t \in \mathbb{R}$.

### 3 Sobolev, Kondratiev, and Besov spaces

In this section we briefly collect the basics concerning weighted and unweighted Sobolev spaces as well as Besov spaces needed later on. In particular, we put $H^m = W^m_2$ and denote by $\tilde{H}^m$ the closure of test functions in $H^m$ and its dual space by $H^{-m}$. Moreover, $\mathcal{C}^{k,\alpha}, k \in \mathbb{N}_0$, stands for the usual Hölder spaces with exponent $\alpha \in (0, 1]$. The following generalized version of Sobolev’s embedding theorem for Banach-space valued functions will be useful, cf. [34, Thm. 1.2.5].

**Theorem 3.1 (Generalized Sobolev’s embedding theorem)** Let $1 < p < \infty$, $m \in \mathbb{N}$, $I \subset \mathbb{R}$ be some bounded interval, and $X$ a Banach space. Then

$$W^m_p(I, X) \hookrightarrow \mathcal{C}^{m-1,\frac{1}{p}}(I, X).$$  

(3.1)
Here the Banach-valued Sobolev spaces are endowed with the norm
\[ \|u|W_p^m(I, X)\| := \sum_{k=0}^{m} \|\partial_k u|L_p(I, X)\|^p \quad \text{with} \quad \|\partial_k u|L_p(I, X)\|^p := \int_I \|\partial_k u(t)|X\|^p dt, \]
whereas for the Hölder spaces we use
\[ \|u|C^{k,\alpha}(I, X)\| := \|u|C^k(I, X)\| + |u|^{(k)}|C^{\alpha}(I, X), \]
where \( \|u|C^k(I, X)\| = \sum_{j=0}^{k} \max_{t\in I} \|u|^{(j)}(t)|X\| \) and \( |u|^{(k)}|C^{\alpha}(I, X) = \sup_{s,t\in I, |t-s|^{\alpha}} \frac{|u|^{(k)}(t) - u^{(k)}(s)|X|}{|t-s|^{\alpha}}. \)

We collect some notation for specific Banach-space valued Lebesgue and Sobolev spaces, which will be used when studying the regularity of solutions of parabolic PDEs.

Let \( \Omega_T := [0, T] \times \Omega \). Then we abbreviate
\[ L_p(\Omega_T) := L_p([0, T], L_p(\Omega)). \]

Moreover, we put
\[ H^{m, l_*}(\Omega_T) := H^{l_*}(\Omega_T) \cap H^l([0, T], H^{-m}(\Omega)) \]
normed by
\[ \|u|H^{m, l_*}(\Omega_T)\| = \|u|H^{l_*}(\Omega_T)\| + \|u|H^l([0, T], H^{-m}(\Omega))\|. \]

**Kondratiev spaces** In the sequel we work to a great extent with weighted Sobolev spaces, the so-called Kondratiev spaces \( K^{m}_{p,a}(\Omega) \), defined as the collection of all \( u \in D'(\Omega) \), which have \( m \) generalized derivatives satisfying
\[ \|u|K^{m}_{p,a}(\Omega)\| := \left( \sum_{|\alpha| \leq m} \int_{\Omega} |\varrho(x)|^p(|\alpha| - a) |D^\alpha_x u(x)|^p dx \right)^{1/p} < \infty, \tag{3.2} \]
where \( a \in \mathbb{R}, 1 < p < \infty, m \in \mathbb{N}_0, \alpha \in \mathbb{N}^n_0, \) and the weight function \( \varrho : D \rightarrow [0, 1] \) is the smooth distance to the singular set of \( \Omega \), i.e., \( \varrho \) is a smooth function and in the vicinity of the singular set \( S \) it is equivalent to the distance to that set. Clearly, if \( \Omega \) is a polygon in \( \mathbb{R}^2 \) or a polyhedral domain in \( \mathbb{R}^3 \), then the singular set \( S \) consists of the vertices of the polygon or the vertices and edges of the polyhedra, respectively.

It follows directly from (3.2) that the scale of Kondratiev spaces is monotone in \( m \) and \( a \), i.e.,
\[ K^{m}_{p,a}(\Omega) \subseteq K^{m'}_{p,a}(\Omega) \quad \text{and} \quad K^{m}_{p,a}(\Omega) \subseteq K^{m}_{p,a'}(\Omega), \tag{3.3} \]
if \( m' < m \) and \( a' < a \).

Moreover, generalizing the above concept to functions depending on the time \( t \in [0, T] \), we define Kondratiev type spaces, denoted by \( L_q((0, T), K^{m}_{p,a}(\Omega)) \), which contain all functions \( u(x, t) \) such that
\[ \|u|L_q((0, T), K^{m}_{p,a}(\Omega))\| := \left( \int_{(0,T)} \left( \sum_{|\alpha| \leq m} \int_{\Omega} |\varrho(x)|^p(|\alpha| - a) |D^\alpha_x u(x, t)|^p dx \right)^{q/p} dt \right)^{1/q} < \infty, \tag{3.4} \]
with \( 0 < q \leq \infty \) and parameters \( a, p, m \) as above.
Kondratiev spaces on domains of polyhedral type  For our analysis we make use of several properties of Kondratiev spaces that have been proved in [18]. Therefore, in our later considerations, we will mainly be interested in the case that \( \mathcal{O} \) is a bounded domain of polyhedral type. The precise definition below is taken from Maz’ya and Rossmann [30, Def. 4.1.1].

**Definition 3.2** A bounded domain \( D \subset \mathbb{R}^3 \) is defined to be of polyhedral type if the following holds:

(a) The boundary \( \partial D \) consists of smooth (of class \( C^\infty \)) open two-dimensional manifolds \( \Gamma_j \) (the faces of \( D \)), \( j = 1, \ldots, n \), smooth curves \( M_k \) (the edges), \( k = 1, \ldots, l \), and vertices \( x^{(1)}, \ldots, x^{(l')} \).

(b) For every \( \xi \in M_k \) there exists a neighbourhood \( U_\xi \) and a \( C^\infty \)-diffeomorphism \( \kappa_\xi \) which maps \( D \cap U_\xi \) onto \( D_\xi \cap B_1(0) \), where \( D_\xi \subset \mathbb{R}^3 \) is a dihedron, which in polar coordinates can be described as

\[
D_\xi = K \times \mathbb{R}, \quad K = \{(x_1, x_2) : 0 < r < \infty, -\theta/2 < \varphi < \theta/2\},
\]

where the opening angle \( \theta \) of the 2-dimensional wedge \( K \) satisfies \( 0 < \theta \leq 2\pi \).

(c) For every vertex \( x^{(i)} \) there exists a neighbourhood \( U_i \) and a diffeomorphism \( \kappa_i \) mapping \( D \cap U_i \) onto \( K_i \cap B_1(0) \), where \( K_i \) is a polyhedral cone with edges and vertex at the origin.

**Remark 3.3** In the literature many different types of polyhedral domains are considered. A more general version which coincides with the above definition if \( d = 3 \) is discussed in [18]. Further variants of polyhedral domains can be found in Babuška, Guo [7], Bacuta, Mazzucato, Nistor, Zikatanov [8] and Mazzucato, Nistor [31].

Concerning pointwise multiplication the following result can be found in [18].

**Corollary 3.4** Let \( \frac{d}{2} < p < \infty \), \( m \in \mathbb{N}_0 \), and \( a \geq \frac{d}{p} - 2 \). Then there exists a constant \( c \) such that

\[
\|uv|K^m_{a,p}(D)\| \leq c\|u|K^{m+2}_{a+2,p}(D)\| \cdot \|v|K^m_{a,p}(D)\|
\]

holds for all \( u \in K^{m+2}_{a+2,p}(D) \) and \( v \in K^m_{a,p}(D) \).

**Besov spaces** Due to the different contexts Besov spaces arose from they can be defined/characterized in several ways, e.g. via higher order differences, via the Fourier-analytic approach or via decompositions with suitable building blocks, cf. [37, 38] and the references therein. Here we use the approach via higher order differences as can be found in [37, Sect. 2.5.12]. Let \( f \) be an arbitrary function on \( \mathbb{R}^d \), \( h \in \mathbb{R}^d \) and \( r \in \mathbb{N} \), then

\[
(\Delta^r_h f)(x) = f(x + h) - f(x) \quad \text{and} \quad (\Delta^{r+1}_h f)(x) = \Delta^1_h(\Delta^r_h f)(x)
\]

are the usual iterated differences. Given a function \( f \in L_p(\mathbb{R}^d) \) the \( r \)-th modulus of smoothness is defined by

\[
\omega_r(f, t)_p := \sup_{|h| \leq t} \|\Delta^r_h f | L_p(\mathbb{R}^d)\|, \quad t > 0.
\]
Then the Besov space $B^{s}_{p,q}(\mathbb{R}^d)$ contains all $f \in L_p(\mathbb{R}^d)$ such that

$$
\|f|_{B^{s}_{p,q}(\mathbb{R}^d)} := \|f|_{L_p(\mathbb{R}^d)} + \left( \int_0^1 t^{-sq} \omega_r(f,t)^q \frac{dt}{t} \right)^{1/q} < \infty,
$$

where $0 < p, q \leq \infty$, $s > 0$, and $r \in \mathbb{N}$ such that $r > s$. This definition is independent of $r$, meaning that different values of $r > s$ result in quasi-norms which are equivalent. Corresponding function spaces on domains $O \subset \mathbb{R}^d$ can be introduced via restriction, i.e.,

$$
B^{s}_{p,q}(O) := \left\{ f \in L_p(O) : \exists g \in B^{s}_{p,q}(\mathbb{R}^d), g|_O = f \right\},
$$

$$
\|f|_{B^{s}_{p,q}(O)} := \inf_{g|_O = f} \|f|_{B^{s}_{p,q}(\mathbb{R}^d)}.
$$

Our main tool when investigating the Besov regularity of solutions to the PDEs will be the following embedding result between Kondratiev and Besov spaces adapted to our needs, which is an extension of [24, Thm. 1]. For a proof we refer to [34, Thms. 1.4.12, 1.4.14].

**Theorem 3.5 (Embeddings between Kondratiev and Besov spaces)** Let $D \subset \mathbb{R}^3$ be some polyhedral type domain and assume $k, m, \gamma \in \mathbb{N}_0$. Furthermore, let $1 < p < \infty$, $s, a \in \mathbb{R}$, and suppose $0 \leq \gamma < \min(m, \frac{3}{2}s)$ and $a > \frac{3}{2}\gamma$, where $\delta$ denotes the dimension of the singular set (i.e., $\delta = 0$ if there are only vertex singularities and $\delta = 1$ if there are edge and vertex singularities). Then we have the continuous embedding

$$
H^k([0,T], \mathcal{C}^m_{p,a}(D)) \cap H^k([0,T], B^s_{p,p}(D)) \hookrightarrow H^k([0,T], B^{\gamma}_{\tau,\tau}(D)), \quad \text{where } \frac{1}{\tau} = \frac{\gamma}{3} + \frac{1}{p}. \ (3.5)
$$

**Remark 3.6** Note that for the adaptivity scale of Besov spaces $B^{\gamma}_{\tau,\tau}(D)$ appearing in Theorem 3.5, from the restriction on the parameters $\frac{1}{\tau} = \frac{\gamma}{3} + \frac{1}{p}$ together with $\tau \leq p$ and $p > 1$, we deduce

$$
\alpha = 3 \left( \frac{1}{\tau} - \frac{1}{p} \right) \geq 3 \left( \frac{1}{\tau} - \frac{1}{p} \right) + .
$$

In particular, for this range of parameters it is well known that the different approaches towards Besov spaces (such as the Fourier-analytic approach as well as decompositions via wavelets) actually coincide. Therefore, the Besov spaces on domains (as before the Kondratiev spaces) may be considered in the setting of distributions, i.e., as subsets of $\mathcal{D}'(O)$, and may contain 'functions' which take complex values. However, when considering the fundamental parabolic problems, we restrict ourselves to the real-valued setting: We assume the coefficients of the differential operator $L$ to be real-valued as well as the right-hand side $f$, therefore, the solutions are real-valued as well.

## 4 Parabolic PDEs and operator pencils

In this section we introduce the basic linear and nonlinear parabolic problems we will be concerned with in the sequel. Moreover, in order to state and proof our main results in Section 5 a short discussion of operator pencils is necessary.

### 4.1 The fundamental parabolic problems

Let $D$ denote some domain of polyhedral type in $\mathbb{R}^d$ according to Definition 3.2 with faces $\Gamma_j$, $j = 1, \ldots, n$. For $0 < T < \infty$ put $D_T = (0,T] \times D$ and $\Gamma_j \times [0,T] \times \Gamma_j = [0,T] \times \Gamma_j$.

We will use the regularity results obtained in [20] of the following linear parabolic problem.
Problem I (Linear parabolic problem in divergence form) Let $m \in \mathbb{N}$. We consider the following first initial-boundary value problem

\[
\begin{aligned}
&\left\{ \begin{array}{ll}
\frac{\partial}{\partial t} u + (-1)^m L(t, x, D_x) u = f & \text{in } D_T, \\
\frac{\partial^{k-1} u}{\partial x^{k-1}} \bigg|_{\Gamma_j, T} = 0, & k = 1, \ldots, m, \ j = 1, \ldots, n, \\
u \big|_{t=0} = 0 & \text{in } D.
\end{array} \right.
\end{aligned}
\]  

(4.1)

Here $f$ is a function given on $D_T$, $\nu$ denotes the exterior normal to $\Gamma_j, T$, and the partial differential operator $L$ is given by

\[
L(t, x, D_x) = \sum_{|\alpha|, |\beta| = 0}^{m} D^\alpha_x (a_{\alpha \beta}(t, x) D^\beta_x),
\]

where $a_{\alpha \beta}$ are bounded real-valued functions from $C^\infty(D_T)$ with $a_{\alpha \beta} = (-1)^{|\alpha|+|\beta|} a_{\beta \alpha}$. Furthermore, the operator $L$ is assumed to be uniformly elliptic with respect to $t \in [0, T]$, i.e.,

\[
\sum_{|\alpha|, |\beta| = m} a_{\alpha \beta} \xi^\alpha \xi^\beta \geq c |\xi|^{2m} \quad \text{for all } (t, x) \in D_T, \ \xi \in \mathbb{R}^d.
\]  

(4.2)

Let us denote by

\[
B(t, u, v) = \int_D \sum_{|\alpha|, |\beta| = 0}^{m} a_{\alpha \beta}(t, x)(D^\alpha_x u)(D^\beta_x v) dx
\]

the time-dependent bilinear form.

Moreover, for simplicity we set

\[
B_{\partial t_k}(t, u, v) = \sum_{|\alpha|, |\beta| \leq m} \int_D \frac{\partial a_{\alpha \beta}(t, x)}{\partial t^k} D^\beta_x u(t, x) D^\alpha_x v(t, x) dx.
\]  

(4.4)

Remark 4.1 (Assumptions on the time-dependent bilinear form) When dealing with parabolic problems we can assume w.l.o.g. that $B(t, \cdot, \cdot)$ satisfies

\[
B(t, u, u) \geq \mu \|u\|_{H^m(D)}^2
\]

for all $u \in \dot{H}^m(D)$ and a.e. $t \in [0, T]$. We refer to [32, Rem. 2.3.5] for a detailed discussion.

It is our intention to also study nonlinear versions of Problem I. Therefore, we modify (4.1) as follows.

Problem II (Nonlinear parabolic problem in divergence form) Let $m, M \in \mathbb{N}$ and $\varepsilon > 0$. We consider the following nonlinear parabolic problem

\[
\begin{aligned}
&\left\{ \begin{array}{ll}
\frac{\partial}{\partial t} u + (-1)^m L(t, x, D_x) u + \varepsilon u^M = f & \text{in } D_T, \\
\frac{\partial^{k-1} u}{\partial x^{k-1}} \bigg|_{\Gamma_j, T} = 0, & k = 1, \ldots, m, \ j = 1, \ldots, n, \\
u \big|_{t=0} = 0 & \text{in } D.
\end{array} \right.
\end{aligned}
\]  

(4.6)

The assumptions on $f$ and the operator $L$ are as in Problem I. When we establish Besov regularity results for Problem II, we interpret (4.6) as a fixed point problem and show that the regularity estimates for Problem I carry over to Problem II, provided that $\varepsilon$ is sufficiently small.
4.2 Operator pencils

In order to correctly state the global regularity results in Kondratiev spaces for Problems I and II, we need to work with operator pencils generated by the corresponding elliptic problems in the polyhedral type domain \( D \subset \mathbb{R}^3 \).

We briefly recall the basic facts needed in the sequel. For further information on this subject we refer to [27] and [30, Sect. 2.3, 3.2., 4.1]. On a domain \( D \subset \mathbb{R}^3 \) of polyhedral type according to Definition 3.2 we consider the problem

\[
\begin{align*}
Lu &= f \quad \text{in} \quad D, \\
\frac{\partial^{k-1} u}{\partial \nu^{k-1}} \bigg|_{\partial D} &= 0, \quad k = 1, \ldots, m.
\end{align*}
\]

(4.7)

The singular set \( S \) of \( D \) then is given by the boundary points \( M_1 \cup \ldots \cup M_l \cup \{ x^{(1)}, \ldots, x^{(l')} \} \). We do not exclude the cases \( l = 0 \) (corner domain) and \( l' = 0 \) (edge domain). In the last case, the set \( S \) consists only of smooth non-intersecting edges. Figure 2 gives examples of polyhedral domains without edges or corners, respectively.

![Figure 2: Corner domain \( D_c \) \((l = 0)\) and edge domain \( D_e \) \((l' = 0)\)](image)

The elliptic boundary value problem (4.7) on \( D \) generates two types of operator pencils for the edges \( M_k \) and for the vertices \( x^{(i)} \) of the domain, respectively.

1) **Operator pencil** \( A_\xi(\lambda) \) **for edge points:**

The pencils \( A_\xi(\lambda) \) for edge points \( \xi \in M_k \) are defined as follows: According to Definition 3.2 there exists a neighborhood \( U_\xi \) of \( \xi \) and a diffeomorphism \( \kappa_\xi \) mapping \( D \cap U_\xi \) onto \( D_\xi \cap B_1(0) \), where \( D_\xi \) is a dihedron.

Let \( \Gamma_{k_\pm} \) be the faces adjacent to \( M_k \). Then by \( D_\xi \) we denote the dihedron which is bounded by the half-planes \( \hat{\Gamma}_{k_\pm} \) tangent to \( \Gamma_{k_\pm} \) at \( \xi \) and the edge \( M_\xi = \hat{\Gamma}_{k_+} \cap \hat{\Gamma}_{k_-} \). Furthermore, let \( r, \varphi \) be polar coordinates in the plane perpendicular to \( M_\xi \) such that

\[
\hat{\Gamma}_{k_\pm} = \left\{ x \in \mathbb{R}^3 : r > 0, \; \varphi = \pm \frac{\theta_\xi}{2} \right\}.
\]

![Figure 3: Dihedron \( D_\xi \)](image)

We define the **operator pencil** \( A_\xi(\lambda) \) as follows:

\[
A_\xi(\lambda)U(\varphi) = r^{2m-\lambda}L_0(0, D_x)u,
\]

(4.8)
where \( u(x) = r^\lambda U(\varphi) \), \( \lambda \in \mathbb{C} \), \( U \) is a function on \( I_\xi := \left( -\frac{\theta_\xi}{2}, \frac{\theta_\xi}{2} \right) \), and

\[
L_0(\xi, D_x) = \sum_{|\alpha|=|\beta|=m} D^\alpha_x (a_{\alpha\beta}(\xi) D^\beta_x)\]

denotes the main part of the differential operator \( L(x, D_x) \) with coefficients frozen at \( \xi \). This way we obtain in \[4.8\] a boundary value problem for the function \( U \) on the 1-dimensional subdomain \( I_\xi \) with the complex parameter \( \lambda \). Obviously, \( A_\xi(\lambda) \) is a polynomial of degree \( 2m \) in \( \lambda \).

The operator \( A_\xi(\lambda) \) realizes a continuous mapping

\[
H^{2m}(I_\xi) \to L_2(I_\xi),
\]

for every \( \lambda \in \mathbb{C} \). Furthermore, \( A_\xi(\lambda) \) is an isomorphism for all \( \lambda \in \mathbb{C} \) with the possible exception of a denumerable set of isolated points, the spectrum of \( A_\xi(\lambda) \), which consists of its eigenvalues with finite algebraic multiplicities: Here a complex number \( \lambda_0 \) is called an eigenvalue of the pencil \( A_\xi(\lambda) \) if there exists a nonzero function \( U \in H^{2m}(I_\xi) \) such that \( A_\xi(\lambda_0) U = 0 \). It is known that the ‘energy line’ \( \text{Re}\lambda = m - 1 \) does not contain eigenvalues of the pencil \( A_\xi(\lambda) \). We denote by \( \delta_{\pm}(\xi) \) the largest positive real numbers such that the strip

\[
m - 1 - \delta_{\pm}(\xi) < \text{Re}\lambda < m - 1 + \delta_{\pm}(\xi)
\]

is free of eigenvalues of the pencil \( A_\xi(\lambda) \). Furthermore, we put

\[
\delta^{(k)}_{\pm} = \inf_{\xi \in M_k} \delta_{\pm}(\xi), \quad k = 1, \ldots, l.
\]

For example, concerning the Dirichlet problem for the Poisson equation on a domain \( D \subset \mathbb{R}^3 \) of polyhedral type, the eigenvalues of the pencil \( A_\xi(\lambda) \) are given by

\[
\lambda_k = k\pi/\theta_\xi, \quad k = \pm 1, \pm 2, \ldots,
\]

where \( \theta_\xi \) is the inner angle at the edge point \( \xi \), cf. \[34, \text{Ex. 2.5.2}\]. Therefore, the first positive eigenvalue is \( \lambda_1 = \frac{\pi}{\theta_\xi} \) and we obtain \( \delta_\pm = \frac{\pi}{\theta_\xi} \), cf. \[34, \text{Ex. 2.5.1}\].

2) Operator pencil \( \mathfrak{A}_i(\lambda) \) for corner points:

Let \( x^{(i)} \) be a vertex of \( D \). According to Definition \[3.2\] there exists a neighborhood \( U_i \) of \( x^{(i)} \) and a diffeomorphism \( \kappa_i \) mapping \( D \cap U_i \) onto \( K_i \cap B_1(0) \), where

\[
K_i = \{ x \in \mathbb{R}^3 : x/|x| \in \Omega_i \}
\]

is a polyhedral cone with edges and vertex at the origin. W.l.o.g. we may assume that the Jacobian matrix \( \kappa_i'(x) \) is equal to the identity matrix at the point \( x^{(i)} \). We introduce spherical coordinates \( \rho = |x|, \omega = \frac{x}{|x|} \) in \( K_i \) and define the operator pencil

\[
\mathfrak{A}_i(\lambda) U(\omega) = \rho^{2m-\lambda} L_0(x^{(i)}, D_x) U,
\]

where \( u(x) = \rho^\lambda U(\omega) \) and \( U \in \tilde{H}^m(\Omega_i) \) is a function on \( \Omega_i \). An eigenvalue of \( \mathfrak{A}_i(\lambda) \) is a complex number \( \lambda_0 \) such that \( \mathfrak{A}_i(\lambda_0) U = 0 \) for some nonzero function \( U \in \tilde{H}^m(\Omega_i) \). The operator \( \mathfrak{A}_i(\lambda) \) realizes a continuous mapping

\[
\tilde{H}^m(\Omega_i) \to H^{-m}(\Omega_i).
\]

Furthermore, it is known that \( \mathfrak{A}_i(\lambda) \) is an isomorphism for all \( \lambda \in \mathbb{C} \) with the possible exception of a denumerable set of isolated points. The mentioned enumerable set consists of eigenvalues with finite algebraic multiplicities.
Moreover, the eigenvalues of \( A_i(\lambda) \) are situated, except for finitely many, outside a double sector \(|\Re \lambda| < \varepsilon |\Im \lambda|\) containing the imaginary axis, cf. [27, Thm. 10.1.1]. In Figure 4 the situation is illustrated: Outside the yellow area there are only finitely many eigenvalues of the operator pencil \( A_i(\lambda) \).

Dealing with regularity properties of solutions, we look for the widest strip in the \( \lambda \)-plane, free of eigenvalues and containing the 'energy line' \( \Re \lambda = m - 3/2 \), cf. Assumption 5.3. From what was outlined above, information on the width of this strip is obtained from lower estimates for real parts of the eigenvalues situated over the energy line.

![Eigenvalues of operator pencil \( A_i(\lambda) \)](image)

**Figure 4: Eigenvalues of operator pencil \( A_i(\lambda) \)**

**Remark 4.2 (Operator pencils for parabolic problems)** Since we study parabolic PDEs, where the differential operator \( L(t, x, D_x) \) additionally depends on the time \( t \), we have to work with operator pencils \( A_\xi(\lambda, t) \) and \( A_i(\lambda, t) \) in this context. The philosophy is to fix \( t \in [0, T] \) and define the pencils as above: We replace (4.8) by

\[
A_\xi(\lambda, t)U(\varphi) = r^{2m-\lambda}L_0(t, 0, D_x)u,
\]

and work with \( \delta_+^{(\xi)}(t) \) and \( \delta_-^{(k)}(t) = \inf_{\xi \in M_k} \delta_+^{(\xi)}(t) \) in (4.9) and (4.10), respectively. Moreover, we put

\[
\delta_\pm^{(k)} = \inf_{t \in [0, T]} \delta_\pm^{(k)}(t), \quad k = 1, \ldots, l. \tag{4.12}
\]

Similar for \( A_i(\lambda, t) \), where now (4.11) is replaced by

\[
A_i(\lambda, t)U(\omega) = \rho^{2m-\lambda}L_0(t, x(i), D_x)u. \tag{4.13}
\]

5 Regularity results in Sobolev and Kondratiev spaces

This section presents regularity results for Problem II in Sobolev and Kondratiev spaces based on the corresponding findings for Problem I in [20]. They will form the basis for obtaining regularity results in Besov spaces later on via suitable embeddings.

5.1 Regularity results in Sobolev and Kondratiev spaces for Problem I

**Theorem 5.1 (Sobolev regularity with time derivatives)** Let \( l \in \mathbb{N}_0 \) and assume that the right hand side \( f \) of Problem I satisfies

\[
f \in H^l([0, T], H^{-m}(D)) \quad \text{and} \quad \partial_{t^k} f(0, x) = 0 \quad \text{for} \quad k = 0, \ldots, l - 1.
\]
Then the weak solution \( u \) in the space \( H^{m,1,\ast}(D_T) \) of Problem \( \text{[1]} \) in fact belongs to \( H^{m,l+1,\ast}(D_T) \), i.e., has derivatives with respect to \( t \) up to order \( l \) satisfying

\[
\partial_t^k u \in H^{m,1,\ast}(D_T) \quad \text{for} \quad k = 0, \ldots, l,
\]

and

\[
\sum_{k=0}^{l} \|\partial_t^k u|H^{m,1,\ast}(D_T)\| \leq C \sum_{k=0}^{l} \|\partial_t^k f|L_2([0,T],H^{-m}(D))\|,
\]

where \( C \) is a constant independent of \( u \) and \( f \).

**Remark 5.2** Note that the regularity results for the solution \( u \) in \( \text{[29, Thm. 2.1.}, \text{Lem. 3.1]} \) are slightly stronger than the ones obtained in Theorem 5.1 above (with the cost of also assuming more regularity on the right hand side \( f \)). By using similar arguments as in \( \text{[6, Lem. 4.3]} \) we are probably able to also show in our context that Theorem 5.1 can be strengthened in the sense that if \( f \in L_2([0,T],L_2(D)) \) then the weak solution \( u \) of Problem \( \text{[1]} \) belongs in fact to \( L_2([0,T],H^m) \cap H^1([0,T],L_2(D)) \). A corresponding generalization of Theorem 5.1 should also be possible in the spirit of \( \text{[6, Thm. 3.1]} \). However, for our purposes the above results on the Sobolev regularity are sufficient, so these investigations are postponed for the time being.

We need the following technical assumptions in order to state the regularity results in Kondratiev spaces.

**Assumption 5.3 (Assumptions on operator pencils)** Consider the operator pencils \( \mathfrak{A}_i(\lambda,t) \), \( i = 1, \ldots, l' \) for the vertices and \( A_{\xi}(\lambda,t) \) with \( \xi \in M_k \), \( k = 1, \ldots, l \) for the edges of the polyhedral type domain \( D \subset \mathbb{R}^3 \) introduced in Section 4.2. Moreover, we assume that \( t \in [0,T] \) is fixed. Let \( K_{p,b}(D) \) and \( K_{p,b}'(D) \) be two Kondratiev spaces, where the singularity set \( S \) of \( D \) is given by

\( S = M_1 \cup \ldots \cup M_l \cup \{x^{(1)}, \ldots, x^{(l')}\} \) and weight parameters \( b, b' \in \mathbb{R} \). Then we assume that the closed strip between the lines

\[
\text{Re}\lambda = b + 2m - \frac{3}{2} \quad \text{and} \quad \text{Re}\lambda = b' + 2m - \frac{3}{2}
\]

does not contain eigenvalues of \( \mathfrak{A}_i(\lambda,t) \). Moreover, \( b \) and \( b' \) satisfy

\[
-\delta_-(^k) < b + m < \delta_+^{(k)}, \quad -\delta_-(^k) < b' + m < \delta_+^{(k)}, \quad k = 1, \ldots, l,
\]

where \( \delta_+^{(k)} \) are defined \( \text{(4.12)} \).

**Remark 5.4** If \( l' = 0 \) we have an edge domain without vertices, cf. Figure 2. In this case condition \( \text{(5.1)} \) is empty. Moreover, if \( l = 0 \), we have a corner domain without edges, in which case condition \( \text{(5.2)} \) is empty. For further remarks and explanations concerning Assumption 5.3 we refer to \( \text{[19, Rem. 3.3]} \).

We recall the following regularity result from \( \text{[20, Thm. 4.11]} \), which will be discussed in Remark 5.6 below.

**Theorem 5.5 (Kondratiev regularity)** Let \( D \subset \mathbb{R}^3 \) be a domain of polyhedral type and \( \eta \in \mathbb{N} \) with \( \eta \geq 2m \). Moreover, let \( a \in \mathbb{R} \) with \( a \in [-m, m] \). Assume that the right hand side \( f \) of Problem \( \text{[1]} \) satisfies

\((i)\) \( f \in \bigcap_{\eta=0}^{\infty} H^1([0,T],L_2(D) \cap K_{p,b}'(D)). \)

12
Remark 5.6

where the constant is independent of $u$.

Furthermore, let Assumption 5.3 hold for weight parameters $b = a$ and $b' = -m$. Then for the weak solution $u \in \bigcap_{l=0}^{\infty} H^{m,l+1,s}(D_T)$ of Problem I we have

$$\partial_t u \in L_2([0,T], \mathcal{K}_{2,a+2m}^\eta(D)) \quad \text{for all} \quad l \in \mathbb{N}_0.$$ 

In particular, for the derivative $\partial_t u$ we have the a priori estimate

$$\sum_{k=0}^{l} \| \partial_t^k u \|_{L_2([0,T], \mathcal{K}_{2,a+2m}^\eta(D))} \lesssim \sum_{k=0}^{l+(\eta-2m)} \| \partial_t^k f \|_{L_2([0,T], \mathcal{K}_{2,a}^{\eta-2m}(D))} + \sum_{k=0}^{l+1+(\eta-2m)} \| \partial_t^k f \|_{L_2(D_T)}$$

where the constant is independent of $u$ and $f$.

Remark 5.6

- We remark that the results from Theorems 5.1 and 5.5 together yield that under the assumptions of Theorem 5.5 on the right hand side $f$ we have the following estimate for the solution $u = \tilde{L}^{-1} f$ of Problem II

$$\|u\|_{H^1([0,T], H^m(D) \cap \mathcal{K}_{2,a+2m}^\eta(D))} \lesssim \|f\|_{H^{1,1+n-2m}([0,T], L_2(D) \cap \mathcal{K}_{2,a}^{\eta-2m}(D))}.$$  

(5.3)

Moreover, a careful inspection of the proofs of Theorems 5.1 and 5.5 (which may be found in [34, Thms. 4.2.3, 4.2.12]) shows that for time independent coefficients the critical constants are independent of $\eta$.

- Note that in [20, Thm. 4.9, Thm. 4.11] we established two different kinds of regularity results in Kondratiev spaces of the linear Problem II. There we only used the results from [20, Thm. 4.9] when dealing with the nonlinear setting. However, in order to also deal with nonlinear problems on non-convex domains, we need to generalize the regularity results from Theorem 5.5 for Problem II. In Theorem 5.5 compared to [20, Thm. 4.9] we only require the parameter $a$ to satisfy $a \in [-m,m]$ and $-\delta_+^{(k)} < a + m < \delta_+^{(k)}$ independent of the regularity parameter $\eta$ which can be arbitrarily high. In particular, for the heat equation on a domain of polyhedral type $D$ (which for simplicity we assume to be a polyhedron with straight edges and faces where $\theta_k$ denotes the angle at the edge $M_k$), we have $\delta_+^{(k)} = \frac{\pi}{\theta_k}$, which leads to the restriction $-1 \leq a < \min\left(1, \frac{\pi}{\theta_k} - 1\right)$. Therefore, even in the extremal case when $\theta_k = 2\pi$ we can still take $-1 \leq a < -\frac{1}{2}$ (resulting in $u \in L_2([0,T], \mathcal{K}_{a+2}^\eta(D))$ being locally integrable since $a + 2 > 0$). Then choosing $\eta$ arbitrary high, we also cover non-convex polyhedral type domains with our results from Theorem 5.5.

5.2 Schauder’s fixed point theorem

We want to show that the regularity estimates in Kondratiev and Sobolev spaces of the linear Problem II stated in Theorems 5.1 and 5.5 carry over to Problem II provided that $\varepsilon$ is sufficiently small. In order to do this we interpret Problem II as a fixed point problem in the following way.

Let $\mathcal{D}$ and $\mathcal{S}$ be locally convex spaces ($\mathcal{D}$ and $\mathcal{S}$ will be specified in Theorem 5.8 below) and let $\tilde{L}^{-1} : \mathcal{D} \to \mathcal{S}$ be the linear operator defined via

$$\tilde{L}u := \partial_t u + (-1)^m Lu.$$  

(5.4)
Problem II is equivalent to
\[ \tilde{L}u = f - \varepsilon u^M =: Nu, \]
where \( N \) is a nonlinear operator. If we can show that \( N \) maps \( S \) into \( \mathcal{D} \), then a solution of Problem II is a fixed point of the problem
\[ (\tilde{L}^{-1} \circ N)u = u. \]

Our aim is to apply the following generalization of Schauder’s Fixed Point Theorem from [32, Thm. A].

**Theorem 5.7 (Schauder’s Fixed Point Theorem)** Let \( S_0 \) be a convex subset of a locally convex space \( S \) and \( T \) a continuous map of \( S_0 \) into a compact subset of \( S_0 \). Then \( T \) has a fixed point.

Therefore, Schauder’s fixed point theorem will guarantee the existence of a solution, if we can show that
\[ T := (\tilde{L}^{-1} \circ N) : S_0 \rightarrow S_0 \]
is continuous, \( T(S_0) \subset S_0 \) is compact. \( (5.5) \)

**5.3 Regularity results in Sobolev and Kondratiev spaces for Problem II**

Our main result is stated in the theorem below.

**Theorem 5.8 (Nonlinear Kondratiev regularity)** Let \( \tilde{L} \) and \( N \) be as described above. Assume the assumptions of Theorem 5.5 are satisfied and, additionally, we have \( m \geq 2 \) and \( a \geq -\frac{1}{2} \). Let
\[ \mathcal{D}_l := H^l([0,T], L_2(D) \cap K_{2,a}^{2m}(D)), \]
and consider the data space
\[ \mathcal{D} := \left\{ f \in \bigcap_{l=0}^{\infty} \mathcal{D}_l : \partial_t^l f(0,\cdot) = 0, \; l \in \mathbb{N}_0 \right\}. \]

Moreover, let
\[ S_k := H^k([0,T], \tilde{H}^m(D) \cap K_{2,a+2m}^0(D)), \]
and consider the solution space
\[ S := \bigcap_{k=0}^{\infty} S_k. \]

Suppose that \( f \in \mathcal{D} \) and \( \tilde{\eta} := \sup_{l \in \mathbb{N}_0} \| f \|_{\mathcal{D}_l} < \infty \). Moreover, we let \( r_0 > 1 \) and choose \( \varepsilon > 0 \) so small that
\[ \tilde{\eta}^{M-1}\| \tilde{L}^{-1} \|_M \leq \frac{r_0 - 1}{\varepsilon \cdot r_0^M}. \]

Then there exists a solution \( u \in S_0 \subset S \) of Problem II, where \( S_0 \) denotes the intersection of small balls around \( \tilde{L}^{-1} f \) (the solution of the corresponding linear problem) with radius \( R = (r_0 - 1)\tilde{\eta}\| \tilde{L}^{-1} \| > 0 \), i.e.,
\[ S_0 := \bigcap_{k=0}^{\infty} B_{k+1}(\tilde{L}^{-1} f, R), \] \( (5.7) \)
where \( B_{k+1}(\tilde{L}^{-1} f, R) \) denotes the closed ball in \( S_k \).
Now consider for some $k \in \mathbb{N}_0$. This holds in our setting, since for $I = \{r_0 - 1\} \tilde{n}_{k+1+\eta-2m} \|\tilde{L}^{-1}\| > 0$, we end up with the following condition for the sequence $(\tilde{n})$:}

$$
\tilde{n}_{k+1+\eta-2m} \max\left(\frac{2(M-1)}{\tilde{n}_{k+2+\eta-2m}}, 1\right) \leq 1 \quad \forall \, k \in \mathbb{N}_0.
$$

(5.8)

However, a closer inspection reveals that this condition is satisfied for $M = 1$ (corresponding to the linear case) if the sequence $(\tilde{n})$ is monotonically increasing with ‘controlled increments’ whereas for $M > 1$ (corresponding to the nonlinear case) the sequence $(\tilde{n})$ needs to be bounded. Moreover, since $\|f|\mathcal{D}\| := \sup_{t \in \mathbb{N}_0} \|f|\mathcal{D}_t\|$ is a norm on $\mathcal{D}$ one might wonder why we do not work with the Banach space structure of $(\mathcal{D}, \|\cdot|\mathcal{D}\|)$ and apply Banach’s fixed point theorem instead of Schauder’s in Theorem 5.8. But then the substitute of 5.12 that one needs to establish is

$$
\varepsilon \|(\tilde{L}^{-1} \circ N)(u) - (\tilde{L}^{-1} \circ N)(v)|H^I([0, T], X)\| \leq q \|u - v|H^{1+1+\eta-2m}([0, T], X)\|,
$$

(5.9)

for some $q < 1$ which has to be independent of the appearing derivatives $t \in \mathbb{N}_0$. However, the appearing constants in our estimates (e.g. see (5.17)) depend on the order of the derivatives taken. According to Lemma A.1 for the continuity in the ‘coarser’ topology of LC-spaces this independence is not necessary. This is precisely the reason why we work with Schauder’s fixed point theorem instead although our approach then comes with the prize that we do not get a unique fixed point.

**Proof:**  Step 1: (Construction of compact set $K \subset S$) We construct $S_0 \subset S$ and find a compact subset. The solution space

$$
S = \bigcap_{k=0}^{\infty} H^k([0, T], X), \quad X := K^\eta_{2, a+2m}(D) \cap \dot{H}^m(D),
$$

is a locally convex Hausdorff space $(S, \{p_k\}_{k \in \mathbb{N}})$ equipped with the following family of (semi-)norms

$$
p_k(u) := \|u|H^k([0, T], X)\|, \quad k \in \mathbb{N}_0.
$$

The mapping

$$
\text{id} : S = \bigcap_{k=0}^{\infty} H^{k+1}([0, T], X) \rightarrow \bigcap_{k=0}^{\infty} H^k([0, T], X),
$$

is continuous, which can be seen as follows: according to [21 Lem. 3.19(c)] a linear map $T : (S, \{p_k\}_{k \in \mathbb{N}}) \rightarrow (Y, \{q_k\}_{k \in \mathbb{N}})$ is continuous if, and only if, for every $k \in J$ there exists a finite index set $I_0 \subset I$ and some constant $C > 0$ such that

$$
q_k(Tu) \leq C \max_{l \in I_0} p_l(u) \quad \forall \, u \in S.
$$

This holds in our setting, since for $u \in S$ we have $\text{id}(u) \in \bigcap_{k=0}^{\infty} H^k([0, T], X)$, thus, $u \in H^{k_0}([0, T], X)$ for some $k_0 \in \mathbb{N}_0$ and

$$
q_{k_0}(u) = \|u|H^{k_0}([0, T], X)\| \leq \|u|H^{k_0+1}([0, T], X)\| = p_{k_0+1}(u).
$$

Now consider

$$
S_0 = \bigcap_{k=0}^{\infty} B_{k+1}(\tilde{L}^{-1} f, R),
$$

15
as defined in (5.7). We are going to show that
\[ K := \text{id}(S_0) = \text{id} \left( \bigcap_{k=0}^{\infty} B_{k+1}(\bar{L}^{-1} f, R) \right) \]
is a compact subset of S_0. First note that by the continuity of the embedding \( H^{k+1}([0, T], X) \hookrightarrow H^k([0, T], X) \) we have
\[ \bigcap_{k=0}^{\infty} B_{k+1}(\bar{L}^{-1} f, R) \subset B_{k_0+1}(\bar{L}^{-1} f, R) \hookrightarrow B_{k_0}(\bar{L}^{-1} f, R) \quad \text{for any } k_0 \in \mathbb{N}, \]
i.e., we deduce that \( \bigcap_{k=0}^{\infty} B_{k+1}(\bar{L}^{-1} f, R) \) is compact in \( B_{k_0}(\bar{L}^{-1} f, R) \) for all \( k_0 \in \mathbb{N} \). We proceed by constructing a diagonal sequence as follows: Let
\[ \{y_n\}_n \in \text{id} \left( \bigcap_{k=0}^{\infty} B_{k+1}(\bar{L}^{-1} f, R) \right) = \bigcap_{k=0}^{\infty} \text{id} \left( B_{k+1}(\bar{L}^{-1} f, R) \right), \quad (5.10) \]
where the equality of the sets follows from the injectivity of the identity. Hence, \( \{y_n\}_n \) belongs to the compact set \( \text{id} \left( B_{k+1}(\bar{L}^{-1} f, R) \right) \) in \( B_k(\bar{L}^{-1} f, R) \) for any \( k \in \mathbb{N} \) and therefore has a convergent subsequence \( \{y_{n_k}\}_l \) with limit in \( \text{id} \left( B_{k+1}(\bar{L}^{-1} f, R) \right) \) satisfying
\[ y_{n_k} \xrightarrow{l \to \infty} y^{(k)} \text{ in } H^k([0, T], X). \]
Since by (5.10) this subsequence also satisfies \( \{y_{n_k}\}_l \in \text{id} \left( B_{k+2}(\bar{L}^{-1} f, R) \right) \) which is compact in \( B_{k+1}(\bar{L}^{-1} f, R) \), again we find a convergent subsequence \( \{y_{n_{k,m}}\}_m \) of \( \{y_{n_k}\}_l \) with limit in \( \text{id} \left( B_{k+2}(\bar{L}^{-1} f, R) \right) \) satisfying
\[ y_{n_{k,m}} \xrightarrow{m \to \infty} y^{(k+1)} \text{ in } H^{k+1}([0, T], X). \]
Since \( k \in \mathbb{N} \) is a countable set, by continuing this procedure we can finally extract a subsequence denoted by \( \{y^*_n\}_n \) satisfying
\[ y^*_n \xrightarrow{n \to \infty} y^{(k)} \text{ in } H^k([0, T], X) \text{ for all } k \in \mathbb{N}. \]
The limits \( y^{(k)} \) and \( y^{(k+1)} \) coincide in \( H^k([0, T], X) \) for all \( k \in \mathbb{N} \), since
\[ \|y^{(k+1)} - y^{(k)}\|_{H^k([0, T], X)} \leq \|y^{(k+1)} - y^*_n\|_{H^k([0, T], X)} + \|y^{(k)} - y^*_n\|_{H^k([0, T], X)} \]
\[ \leq \|y^{(k+1)} - y^*_n\|_{H^{k+1}([0, T], X)} + \|y^{(k)} - y^*_n\|_{H^k([0, T], X)} \xrightarrow{n \to \infty} 0. \]
Therefore, the subsequence \( \{y^*_n\}_n \) has a unique limit \( y^* \) which by our construction belongs to \( \text{id} \left( B_{k+1}(\bar{L}^{-1} f, R) \right) \) for all \( k \in \mathbb{N} \), i.e.,
\[ y^*_n \xrightarrow{n \to \infty} y^* \in \bigcap_{k=0}^{\infty} \text{id} \left( B_{k+1}(\bar{L}^{-1} f, R) \right). \quad (5.11) \]
From (5.10) and (5.11) we deduce that \( K = \text{id}(S_0) \) is sequentially compact in \( S = \bigcap_{k=0}^{\infty} H^k([0, T], X) \), which is equivalent to \( K \) being compact if we can show that \( S \) is a complete metric space, cf. [21] Thm. 1.3.1, p. 13]. But this follows from standard arguments using on \( S \) the metric
\[ d(x, y) := \sum_{k \in \mathbb{N}_0} 2^{-k} \frac{p_k(x - y)}{1 + p_k(x - y)}. \]
Step 2: (Continuity of $\tilde{L}^{-1} \circ N$)

Let $u$ be the solution of the linear problem $L u = f$. From Theorems [5.1] and [5.5], see also [5.3], we know that $\tilde{L}^{-1} : D_{l+1+\eta-2m} \to S_l$ is a bounded operator. If $u^M \in D$ (this will immediately follow from our calculations in Step 2 as explained in Step 3 below), the nonlinear part $N$ satisfies the desired mapping properties, i.e., $N u = f - \varepsilon u^M \in D$. Hence, we can apply Theorem [5.5] now with right hand side $Nu$. Moreover, by Lemma A.1, the (nonlinear) map $\tilde{L}^{-1} \circ N : S_0 \to S$ is continuous if we can show for all $l \in \mathbb{N}_0$ and $u,v \in S_l$ that

$$\|(\tilde{L}^{-1} \circ N)(u) - (\tilde{L}^{-1} \circ N)(v)\| H^l([0,T],X) \leq c_l \|u - v\| H^{l+1+\eta-2m}([0,T],X).$$

(5.12)

Since $(\tilde{L}^{-1} \circ N)(u) = \tilde{L}^{-1}(f - \varepsilon u^M)$ we therefore have to show

$$\|\tilde{L}^{-1}(u^M - v^M)|H^l([0,T],X)\| \lesssim \|u - v\| H^{l+1+\eta-2m}([0,T],X).$$

We use Theorem [5.5] and the estimate (5.3), which yields

$$\|\tilde{L}^{-1}(u^M - v^M)|H^l([0,T],X)\| \leq \|\tilde{L}^{-1}\| \|u^M - v^M|D_{l+1+\eta-2m}\|
= \|\tilde{L}^{-1}\| \|u^M - v^M|H^{l+1+\eta-2m}([0,T],K_2\eta^{-2m}(D))\|
+ \|\tilde{L}^{-1}\| \|u^M - v^M|H^{l+1+\eta-2m}([0,T],L_2(D))\|
=: I + II$$

(5.13)

Moreover, we make use of the formula $u^M - v^M = (u - v) \sum_{j=0}^{M-1} u^j v^{M-1-j}$. Concerning the derivatives, applying Leibniz’s formula twice we see that

$$\partial^k (u^M - v^M) = \partial^k \left[(u - v) \sum_{j=0}^{M-1} u^j v^{M-1-j}\right]
= \sum_{w=0}^{k} \binom{k}{w} \partial^w (u - v) \cdot \partial^{k-w} \left(\sum_{j=0}^{M-1} u^j v^{M-1-j}\right)
= \sum_{w=0}^{k} \binom{k}{w} \partial^w (u - v) \cdot \left[\sum_{j=0}^{M-1} \sum_{r=0}^{k-w} \binom{k-w}{r} \partial^r u^j \cdot \partial^{k-w-r} v^{M-1-j}\right].$$

(5.14)

In order to estimate the terms $\partial^r u^j$ (and similar for $\partial^{k-w-r} v^{M-1-j}$) we apply Faà di Bruno’s formula

$$\partial^r (f \circ g) = \sum_{\kappa_1 \ldots \kappa_r} \frac{r!}{\kappa_1! \ldots \kappa_r!} (\partial_{\kappa_1 + \ldots + \kappa_r} f \circ g) \prod_{i=1}^{r} \left(\frac{\partial_{\kappa_i} g}{x^i!}\right)^{\kappa_i},$$

(5.15)

where the sum runs over all $r$-tuples of nonnegative integers $(\kappa_1, \ldots, \kappa_r)$ satisfying

$$1 \cdot \kappa_1 + 2 \cdot \kappa_2 + \ldots + r \cdot \kappa_r = r.$$  

(5.16)

In particular, from (5.16) we see that $\kappa_r \leq 1$, where $r = 1, \ldots, k$. Therefore, the highest derivative $\partial^r u$ appears at most once. We apply the formula to $g = u$ and $f(x) = x^j$ and make use of the embeddings (5.3) and the pointwise multiplier result from Corollary [3.4] (note that this leads to our restriction $a \geq \frac{d}{2} - 2 = -\frac{1}{2}$ for $d = 3$). This yields
$$\left\| \partial_t u^j |K_{2,a}^{\eta-2m}(D) \right\| \leq c_{r,j} \sum_{\kappa_1 + \ldots + \kappa_r \leq j, \ k_1 + 2^\cdot \kappa_2 + \ldots + \kappa_r = r} u^j - (\kappa_1 + \ldots + \kappa_r) \prod_{i=1}^r |\partial_t u|^{\kappa_i} |K_{2,a}^{\eta-2m}(D) |^{\kappa_i}$$

$$\leq \sum_{\kappa_1 + \ldots + \kappa_r \leq j, \ k_1 + 2^\cdot \kappa_2 + \ldots + \kappa_r = r} u |K_{2,a}^{\eta-2m+2}(D) |^{j-(\kappa_1 + \ldots + \kappa_r)} \left( \prod_{i=1}^r |\partial_t u|^{\kappa_i} |K_{2,a}^{\eta-2m+2}(D) |^{\kappa_i} \right)$$

$$\leq \sum_{\kappa_1 + \ldots + \kappa_r \leq j, \ k_1 + 2^\cdot \kappa_2 + \ldots + \kappa_r = r} u |K_{2,a}^{\eta-2m+2}(D) |^{j-(\kappa_1 + \ldots + \kappa_r)} \prod_{i=1}^r |\partial_t u|^{\kappa_i} |K_{2,a}^{\eta-2m+2}(D) |^{\kappa_i}.$$

(5.17)

For $\partial_{tk-w-r} v^{M-1-j}$ we obtain as in (5.17) with $K_{2,a}^{\eta-2m}(D)$ replaced by $K_{2,a+2}^{\eta-2m+2}(D)$ the estimate

$$\left\| \partial_{tk-w-r} v^{M-1-j} |K_{2,a+2}^{\eta-2m+2}(D) \right\|$$

$$\leq \sum_{\kappa_1 + \ldots + \kappa_r \leq M-1-j, \ k_1 + 2^\cdot \kappa_2 + \ldots + \kappa_r = k-w-r} \left( u \right)^{j-(\kappa_1 + \ldots + \kappa_r)} \prod_{i=1}^r |\partial_t v|^{\kappa_i} |K_{2,a+2}^{\eta-2m+2}(D) |^{\kappa_i},$$

(5.18)

(5.18)

since $m \geq 2$. Now (5.14), (5.17), and (5.18) together with Corollary 3.4 give for the first term in (5.13):

$$I = \left\| \hat{L}^{-1} \right\| \left\| u^M - v^M |H^{l+1+\eta-2m}([0,T], K_{2,a}^{\eta-2m}(D)) \right\|$$

$$\leq \left\| \hat{L}^{-1} \right\| \sum_{k=0}^{l+1+\eta-2m} \left( \int_0^T \left\| \partial_k \left( (u - v) \sum_{j=0}^{M-1} u^j v^{M-1-j} \right) |K_{2,a}^{\eta-2m}(D) \right\|^2 \, dt \right)^{1/2}$$

$$\leq \left\| \hat{L}^{-1} \right\| \sum_{k=0}^{l+1-\eta-2m} \sum_{k=0}^{M-1-k-w} \sum_{w=0}^{M-1-k-w} \sum_{j=0}^{r=0} \left( \int_0^T \left\| \partial_k (u - v) |K_{2,a}^{\eta-2m}(D) \right\|^2 \right. \left. \left( \prod_{i=1}^r |\partial_t v|^{\kappa_i} |K_{2,a+2}^{\eta-2m+2}(D) |^{\kappa_i} \right)^{1/2} \right)^{1/2}$$

$$\leq \left\| \hat{L}^{-1} \right\| \sum_{k=0}^{l+1+\eta-2m} \sum_{k=0}^{M-1-k-w} \sum_{w=0}^{M-1-k-w} \sum_{j=0}^{r=0} \left( \int_0^T \left\| \partial_k (u - v) |K_{2,a}^{\eta-2m}(D) \right\|^2 \right. \left. \left( \prod_{i=1}^r |\partial_t v|^{\kappa_i} |K_{2,a+2}^{\eta-2m+2}(D) |^{\kappa_i} \right)^{1/2} \right)^{1/2}$$

(5.19)
\[
\begin{align*}
&\lesssim \|\bar{L}^{-1}\| \sum_{k=0}^{l+1+\eta-2m} M \left( \int_0^T \|\partial_{t_k}(u-v)|K^{\eta}_{2,a}^m(D)\|^2 \right. \\
&\quad \left. \sum_{\kappa'_1 + \ldots \kappa'_k \leq \min(M-k)} \max_{w \in \{u,v\}} \left\| w|K^{\eta}_{2,a+2m}(D)\right\|^{2(M-1-(\kappa'_1+\ldots+\kappa'_k))} \right)^{1/2} \\
&\lesssim M\|\bar{L}^{-1}\| \left\| u-v|H^{l+1+\eta-2m}([0,T],K^{\eta}_{2,a+2m}(D))\right\|. 
\end{align*}
\]

We give some explanations concerning the estimate above. In (5.20) we used the fact that in the step before we have two sums with \(\kappa_1+\ldots+\kappa_j\leq j\) and \(\kappa_1+\ldots+\kappa_{k-w-r}\leq M-1-j\), i.e., we have \(k-w\) different \(\kappa_i\)’s which leads to at most \(k\) different \(\kappa_i\)'s if \(w=0\). We allow all combinations of \(\kappa_i\)'s and redefine the \(\kappa_i\)'s in the second sum leading to \(\kappa'_1,\ldots,\kappa'_k\) with \(\kappa'_1+\ldots+\kappa'_k\leq M-1\) and replace the old conditions \(\kappa_1+\ldots+r\kappa_r\leq r\) and \(\kappa_1+\ldots+(k-w-r)\kappa_{k-w-r}\leq k-w-r\) by the weaker ones \(\kappa'_1+\ldots+\kappa'_k\leq k\) and \(\kappa'_k\leq 1\). This causes no problems since the other terms appearing in this step do not depend on \(\kappa_i\) apart from the product term. There, the fact that some of the old \(\kappa_i\)’s from both sums might coincide is reflected in the new exponent \(4\kappa'_i\). From Theorem 3.1 we conclude that

\[
\begin{align*}
\quad u, v \in S &\implies H^{l+1+\eta-2m}([0,T],X) \implies C^{l+1+\eta-2m}([0,T],X) \implies C^{l+1+\eta-2m}([0,T],K^{\eta}_{2,a+2m}(D))
\end{align*}
\]

Thus, (5.21) yields

\[
\begin{align*}
I &= \|\bar{L}^{-1}\|\|u^M - v^M|H^{l+1+\eta-2m}([0,T],K^{\eta}_{2,a}^m(D))\| \\
&\lesssim \|\bar{L}^{-1}\| \max(R + \|\bar{L}^{-1}f|H^{l+2+\eta-2m}([0,T],X)\|, 1)^{2(M-1)}\|u-v|H^{l+1+\eta-2m}([0,T],X)\|.
\end{align*}
\]

We now turn our attention towards the second term in (5.13) and calculate

\[
\begin{align*}
II &= \|\bar{L}^{-1}\|\|(u^M - v^M)|H^{l+1+\eta-2m}([0,T],L_2(D))\| \\
&= \|\bar{L}^{-1}\|\|u-v\| \sum_{j=0}^{M-1} w^j v^{M-1-j} |H^{l+1+\eta-2m}([0,T],L_2(D))\| \\
&\lesssim \|\bar{L}^{-1}\| \sum_{k=0}^{l+1+\eta-2m} \|\partial_{t_k}(u-v)\| \sum_{j=0}^{M-1} w^j v^{M-1-j} |L_2(D_T)\| \\
&= \|\bar{L}^{-1}\| \sum_{k=0}^{l+1+\eta-2m} \left| \sum_{w=0}^{k} \frac{(k)}{w} \partial_{tw}(u-v) \cdot \sum_{j=0}^{M-1-k-w} \left| \sum_{r=0}^{k-w} \left( \sum_{j=0}^{M-1-k-w} \partial_{tr}w^j \cdot \partial_{t_{k-w-r}}v^{M-1-j} \right) \right| |L_2(D_T)\| \\
&\lesssim \|\bar{L}^{-1}\| \sum_{k=0}^{l+1+\eta-2m} \left| \sum_{w=0}^{k} \partial_{tw}(u-v) \cdot \sum_{j=0}^{M-1-k-w} \left| \sum_{r=0}^{k-w} \partial_{tr}w^j \cdot \partial_{t_{k-w-r}}v^{M-1-j} \right| |L_2(D_T)\|.
\end{align*}
\]
where we used Leibniz’s formula twice as in (5.14) in the second but last line. Again Faa di Bruno’s formula, cf. (5.15), is applied in order to estimate the derivatives in (5.23). We use a special case of the multiplier result from [33] Sect. 4.6.1, Thm. 1(i), which tells us that for \( m > \frac{3}{2} \) (by our assumptions \( m \geq 2 \)) we have

\[
\|uv\|_{L_2} \lesssim \|u^{H^m}\| \cdot \|v\|_{L_2}.
\]  

With this we obtain

\[
\left\| \partial_t^r u^j |L_2(D)\right\| \leq c_{r,j} \left\| \sum_{\kappa_1 + \ldots + \kappa_r \leq j} u^{j-(\kappa_1 + \ldots + \kappa_r)} \prod_{i=1}^r |\partial_t^i u|^\kappa_i |L_2(D)\right\| \\
\lesssim \sum_{\kappa_1 + \ldots + \kappa_r \leq j} \left\| u^{H^m(D)} \|^{j-(\kappa_1 + \ldots + \kappa_r)} \prod_{i=1}^{r-1} \|\partial_t^i u^{H^m(D)}\|^\kappa_i \|\partial_t^r u|L_2(D)\|^\kappa_r. 
\]

(5.25)

Similar for \( \partial_{k-w} v^{M-1-j} \). As before, from (5.16) we observe \( \kappa_r \leq 1 \), therefore the highest derivative \( u^{(r)} \) appears at most once. Note that since \( H^m(D) \) is a multiplication algebra for \( m > \frac{3}{2} \), we get (5.25) with \( L_2(D) \) replaced by \( H^m(D) \) as well. Now (5.24) and (5.25) inserted in (5.23) give

\[
II = \|\tilde{L}^{-1}\| \left\| u^{M} v^{\bar{M}} |H^{l+1+\eta-2m}([0, T], L_2(D))\right\| \\
\lesssim \|\tilde{L}^{-1}\| \sum_{k=0}^{l+1+\eta-2m} \left( \int_0^T \left\| \partial_k (u - v) \sum_{j=0}^{M-1} w^j v^{M-1-j} |L_2(D)\right\|^2 dt \right)^{1/2} \\
\lesssim \|\tilde{L}^{-1}\| \sum_{k=0}^{l+1+\eta-2m} k \sum_{w=0}^{k-w} \left( \int_0^T \left\| \partial_k (u - v) |H^m(D)\right\|^2 \sum_{j=0}^{M-1} \sum_{r=0}^{M-1-k-w} \left\| \partial_t^r u^j |H^m(D)\right\|^2 \|\partial_t^{k-w-r} v^{M-1-j} |H^m(D)\right\|^2 \right)^{1/2} dt \\
\lesssim \|\tilde{L}^{-1}\| \left\| u^{H^m(D)} \|^{2(\kappa_1 + \ldots + \kappa_r)} \prod_{i=1}^{r-1} \|\partial_t^i u^{H^m(D)}\|^\kappa_i \right\| \left\| v^{H^m(D)} \|^{2(M-1-j-(\kappa_1 + \ldots + \kappa_k-w-r))} \prod_{i=1}^{w-k-r} \|\partial_t^i v^{H^m(D)}\|^\kappa_i \right\|^{1/2} dt \\
\lesssim \|\tilde{L}^{-1}\| \sum_{k=0}^{l+1+\eta-2m} \left( \int_0^T \left\| \partial_k (u - v) |H^m(D)\right\|^2 dt \right)^{1/2} \\
M \sum_{\kappa_1 + \ldots + \kappa_k \leq \min\{M-1,k\}} \max_{w \in \{u,v\}} \left\| w^{H^m(D)} \|^{2(M-1-(\kappa_1 + \ldots + \kappa_k))} \max_{i=0,\ldots,l+1+\eta-2m} \left( \int_{[0,T]} \left\| \partial_t^i w^{H^m(D)}\right\|^{4\kappa_i} dt \right)^{1/2} \\
\lesssim \|\tilde{L}^{-1}\| \left\| u - v \right\|^{H^{l+1+\eta-2m}}([0, T], H^m(D)) \right\| \left( \max_{w \in \{u,v\}} \max_{i=0,\ldots,l+1+\eta-2m} \left\| \partial_t^i w \right\|^{L_\infty}([0, T], H^m(D))\right) \right\|^{2(M-1)}.
\]

(5.26)
For the redefinition of the $\kappa_i$'s in the second but last line in (5.26) we refer to the explanations given after (5.21). From Theorem 3.1 we see that
\[ u, v \in S \implies H^{l+2+\eta-2m}([0, T], X) \hookrightarrow C^{l+1+\eta-2m}([0, T], H^m(D)), \]
(5.27)
hence the term $\max_{i=0, \ldots, l+1+\eta-2m} \max \ldots ( \ldots)^2(M-1)$ in (5.26) is bounded by $\max(R + \|\tilde{L}^{-1}f|H^{l+2+\eta-2m}([0, T], X)|, 1)^2(M-1)$, since $u$ and $v$ are taken from $S_0$.
We obtain from (5.26) and (5.27),
\[ II = \|\tilde{L}^{-1}\|\|u^M - v^M|H^{l+1+\eta-2m}([0, T], L_2(D))\| \]
\[ \lesssim \|\tilde{L}^{-1}\| M \max(R + \|\tilde{L}^{-1}f|H^{l+2+\eta-2m}([0, T], X)|, 1)^2(M-1) \]
\[ \cdot \|u - v|H^{l+1+\eta-2m}([0, T], H^m(D))\|. \]
(5.28)
Now (5.13) together with (5.22) and (5.28) yields
\[ \|\tilde{L}^{-1}(u^M - v^M)|H^l([0, T], X)\| \]
\[ \lesssim \|\tilde{L}^{-1}\|\|u^M - v^M|H^{l+1+\eta-2m}([0, T], L_2(D) \cap K_2^{\eta-2m}(D))\| \]
\[ \lesssim M\|\tilde{L}^{-1}\| \cdot \|u - v|H^{l+1+\eta-2m}([0, T], X)| \cdot \max(R + \|\tilde{L}^{-1}f|H^{l+2+\eta-2m}([0, T], X)|, 1)^2(M-1) \]
\[ \lesssim M\|\tilde{L}^{-1}\| \cdot \|u - v|H^{l+1+\eta-2m}([0, T], X)| \cdot \max(R + \|\tilde{L}^{-1}|\tilde{\eta}|, 1)^2(M-1) \]
(5.29)
where we put $\tilde{\eta} := \sup_{t \in N_0} \|f|D_1\| = \sup_{t \in N_0} \|f|H^1([0, T], L_2(D) \cap K_2^{\eta-2m}(D))\|$. This proves (5.12) and shows the continuity of $\tilde{L}^{-1} \circ N : S_0 \to S$.

Step 3: The calculations in Step 2 show that $u^M \in D$: The fact that $u^M \in \bigcap_{k=0}^{\infty} D_k$ for $u \in S$ follows from the estimate (5.29). In particular, taking $v = 0$ in (5.29) we get an estimate from above for $\|u^M|D_k\|$ (or rather $\|u^M|D_{l+1+\eta-2m}\|$). The upper bound depends on $\|u|S_k\|$ and several constants which depend on $u$ but are finite whenever we have $u \in S$, see also (5.21) and (5.26). The dependence on $R$ in (5.29) comes from the fact that we choose $u \in S_0$ there. However, the same argument can also be applied to an arbitrary $u \in S$; this would result in a different constant $\tilde{c}$. In order to have $u^M \in D$, we still need to show that $\text{Tr}((\partial_x u^M)) = 0$ for all $k \in N_0$. This follows from similar arguments as in [20 Thm. 4.13]: Since $u \in S \hookrightarrow H^{k+1}([0, T], X) \hookrightarrow C^k([0, T], X)$ for any $k \in N_0$ we see that the trace operator $\text{Tr}((\partial_x u)) := (\partial_x u)(0, \cdot)$ is well defined for all $k \in N_0$. Using the initial assumption $u(0, \cdot) = 0$ in Problem II by density arguments ($C^\infty(D_T)$ is dense in $S$) and induction we deduce that $(\partial_x u)(0, \cdot) = 0$ for all $k \in N_0$. Moreover, since by Theorem 3.1
\[ u^M \in D_{k+1} \hookrightarrow H^{k+1}([0, T], L_2(D)) \hookrightarrow C^k([0, T], L_2(D)), \]
we see that the trace operator $\text{Tr}((\partial_x u^M)) := (\partial_x u^M)(0, \cdot)$ is well defined for $k \in N_0$. By (5.25) the term $\|((\partial_x u^M)(0, |L_2(D))\|$ is estimated from above by powers of $\|((\partial_x u)(0, |H^m(D))\|$, $l = 0, \ldots, k$. Since all these terms are equal to zero, this shows that $u^M \in D$.

Step 4: It remains to show that $(\tilde{L}^{-1} \circ N)(S_0) \subset S_0$, which holds if for some $l_0 \in N$ and all $l \geq l_0 \in N$ we have
\[ \|(\tilde{L}^{-1} \circ N)(u) - \tilde{L}^{-1}f|H^l([0, T], X)|\| = \varepsilon\|\tilde{L}^{-1}u^M|H^l([0, T], X)|\| \leq R. \]
We use the fact that $u \in S_0$ implies
\[ \|u|H^k([0, T], X)|\| \leq R + \|\tilde{L}^{-1}|\tilde{\eta}|, \quad \text{for all } k \in N_0, \]
21
and that for time independent coefficients $\tilde{L}^{-1} : \mathcal{D}_{l+1+\eta-2m} \to S_l$ is a bounded operator with norm independent of $l$, cf. (5.3). We use the fact that $H^k([0, \infty), X) = H^k([0, \infty), \tilde{H}^m(D) \cap K_{2,a+2m}^\eta)$ is a multiplication algebra for any $k \geq 2$ (this follows from [5, Cor. 6.2.4] together with [18, Cor. 5.11]). Then using the extension operator from Theorem B.1 gives the condition

$$
\varepsilon\|\tilde{L}^{-1}u^M|H^1([0, T], X)\| = \varepsilon\|\tilde{L}^{-1}u^M|S_l\|
\lesssim \varepsilon\|\tilde{L}^{-1}\|\|u^M|\mathcal{D}_{l+1+\eta-2m}\|
\lesssim \varepsilon\|\tilde{L}^{-1}\|\|u^M|S_{l+1+\eta-2m}\|
= \varepsilon\|\tilde{L}^{-1}\|\|u^M|H^{l+1+\eta-2m}([0, T], X)\|
\lesssim \varepsilon\|\tilde{L}^{-1}\|((Eu)^M|H^{l+1+\eta-2m}([0, \infty), X)\|)
\lesssim \varepsilon\|\tilde{L}^{-1}\|((Eu)|H^{l+1+\eta-2m}([0, \infty), X)\|)^M
\lesssim \varepsilon\|\tilde{L}^{-1}\|u^M|H^{l+1+\eta-2m}([0, T], X)\|^M \lesssim \varepsilon\|\tilde{L}^{-1}\|\|R + \tilde{L}^{-1}\|\tilde{\eta}\|^M \lesssim R.
$$

Choosing $R := (r_0 - 1)\|\tilde{L}^{-1}\|\tilde{\eta}$ leads to

$$
\tilde{\eta}^{M-1}\|\tilde{L}^{-1}\|^M \leq \frac{r_0 - 1}{\varepsilon \cdot r_0^M}.
$$

Thus, by applying Schauder’s fixed point theorem in a sufficiently small ball around the solution of the corresponding linear problem, we obtain a solution of Problem $\Pi$. \hfill \Box

**Remark 5.10** The restriction $m \geq 2$ in Theorem 5.8 comes from the fact that we require $s = m > \frac{q}{2} = \frac{3}{2}$ in (5.24). This assumption can probably be weakened, since we expect the solution to satisfy $u \in L_2([0, T], H^s(D))$ for all $s < \frac{3}{2}$, see also [20], Rem. 5.3] and the explanations given there. Moreover, the restriction $a \geq -\frac{1}{2}$ in Theorem 5.8 comes from Theorem 3.4(ii) that we applied. Together with the restriction $a \in [-m, m]$ we are looking for $a \in [-\frac{1}{2}, m]$ if the domain is a corner domain, e.g. a smooth cone $K \subset \mathbb{R}^3$ (subject to some truncation). For polyhedral cones with edges $M_k$, $k = 1, \ldots, l$, we furthermore require $-\delta_\bot^{(k)} < a + m < \delta_\bot^{(k)}$ from Theorem 5.5.

### 6 Regularity results in Besov spaces

Based on the work done in Section 5, in this section we finally come to the presentation of the regularity results in Besov spaces for Problem $\Pi$. It turns out that in all cases studied the Besov regularity is higher than the Sobolev regularity (by factor 3). This indicates that adaptivity pays off when solving these problems numerically.

Combining Theorem 5.8 (Nonlinear Kondratiev regularity) with the embeddings from Theorem 3.5, we derive the following result.

**Theorem 6.1 (Nonlinear Besov regularity)** Let the assumptions of Theorems 5.8 and 3.5 be satisfied. In particular, as in Theorem 5.8 for $\tilde{\eta} := \sup_{\ell \in \mathbb{N}_0} \|f|\mathcal{D}_\ell\| < \infty$ and $r_0 > 1$, we choose $\varepsilon > 0$ so small that

$$
\tilde{\eta}^{M-1}\|\tilde{L}^{-1}\|^M \leq \frac{r_0 - 1}{\varepsilon \cdot r_0^M}.
$$

\[6.1\]
Furthermore, for $\alpha > 0$ put
\[
B := \bigcap_{k=0}^{\infty} H^k([0, T], B_{r, \infty}^\alpha(D)), \quad \frac{1}{\tau} = \frac{\alpha}{3} + \frac{1}{2},
\]
and define $B_0$ as the intersection of small balls around $\tilde{L}^{-1} f$ (the solution of the corresponding linear problem) with radius $R = C(r_0 - 1)\tilde{\eta}\|\tilde{L}^{-1}\| > 0$, i.e.,
\[
B_0 := \bigcap_{k=0}^{\infty} \tilde{B}_{k+1}(\tilde{L}^{-1} f, R), \quad (6.2)
\]
where $\tilde{B}_{k+1}(\tilde{L}^{-1} f, R)$ denotes the closed ball in $H^k([0, T], B_{r, \infty}^\alpha(D))$. Then there exists a solution $u$ of Problem II, which satisfies
\[
u \in B_0 \subset B \quad \text{for all} \quad 0 < \alpha < \min \left( \frac{3}{\delta m}, \gamma \right),
\]
where $\delta$ denotes the dimension of the singular set of $D$.

Proof: This is a consequence of the regularity results in Kondratiev and Sobolev spaces from Theorem 5.8. To be more precise, Theorem 5.8 establishes the existence of a fixed point $u$ in
\[
S_0 \subset S := \bigcap_{k=0}^{\infty} H^k([0, T], \tilde{H}^m(D) \cap K_{2,a+2m}^\eta(D)).
\]
This together with the embedding results for Besov spaces from Theorem 3.5 completes the proof, since for all $k \in \mathbb{N},$
\[
\|u - \tilde{L}^{-1} f|H^k([0, T], B_{r, \infty}^\alpha(D))| \leq C\|u - \tilde{L}^{-1} f|H^k([0, T], K_{2,a+2m}^\eta(D) \cap H^m(D))| \leq C(r_0 - 1)\tilde{\eta}\|\tilde{L}^{-1}\| = R. \quad (6.3)
\]
Furthermore, it can be seen from (6.3) that the new constants $C$ appears when considering the radius $R$ around the linear solution where the problem can be solved compared to Theorem 5.8. \(\square\)

Remark 6.2 A few words concerning the parameters appearing in Theorem 6.1 (and also Theorem 5.8) seem to be in order. Usually, the operator norm $\|\tilde{L}^{-1}\|$ as well as $\varepsilon$ are fixed; but we can change $\tilde{\eta}$ and $r_0$ according to our needs. From this we deduce that by choosing $\tilde{\eta}$ small enough the condition (6.1) can always be satisfied. Moreover, it is easy to see that the smaller the nonlinear perturbation $\varepsilon > 0$ is, the larger we can choose the radius $R$ of the ball $B_0$.

Finally, combining Theorem 6.1 with Theorem 3.1 gives the following result.

Corollary 6.3 (Hölder-Besov regularity) Assume the assumptions of Theorem 6.1 are satisfied and let $B_0$ be defined as in (6.2). Then there exists a solution $u$ of Problem II, which satisfies
\[
u \in B_0 \subset C^\infty([0, T], B_{r, \infty}^\alpha(D)) \quad \text{for all} \quad 0 < \alpha < \min \left( \frac{3}{\delta m}, \gamma \right),
\]
where $\delta$ denotes the dimension of the singular set of $D$. 23
A Locally convex spaces

A locally convex topological vector space (denoted as LC-space in the sequel) is defined to be a vector space $X$ along with a family $\mathcal{P}_0$ of seminorms on $X$. This family of seminorms induces a canonical vector space topology on $X$, called the initial topology, making it into a topological vector space. By definition, it is the coarsest topology on $X$ for which all maps in $\mathcal{P}_0$ are continuous. In particular, the balls $B_p(x, r) := \{y \in X : p(x - y) < r\}$, where $x \in X$, $r > 0$, and $p \in \mathcal{P}_0$ constitute a subbasis of the topology.

Convergence in LC-spaces can be defined via the seminorms, i.e., a sequence $(x_i)_i$ converges in the underlying topology of a LC-space towards $x$ if, and only if, $p(x_i - x) \overset{i \to \infty}{\to} 0$ for all $p \in \mathcal{P}_0$.

In this paper we deal with LC-spaces for which the corresponding family of seminorms is countable. In particular, we want to investigate nonlinear mappings on these spaces and are interested in their continuity. For general topological spaces continuous maps are defined via open neighbourhoods as follows:

Let $X$ and $Y$ be topological spaces and let $f : X \to Y$. Then $f$ is continuous at $a \in X$ if, and only if, for every open neighbourhood $V$ of $f(a)$ there exists an open neighbourhood $U$ of $a$ such that $f(U) \subset V$.

The following lemma shows that for LC-spaces with a countable family of seminorms continuity of a map coincides with its sequential continuity.

Lemma A.1 (Continuity = Sequential continuity) A mapping $f : X \to Y$ of a countably seminormed space $X$ into a topological space $Y$ is continuous if, and only if, it is sequentially continuous, i.e., for each convergent sequence $x_i \to a$ also the image sequence $f(x_i) \to f(a)$ converges.

Proof: $(\Rightarrow)$ This follows immediately from the description of convergence in LC-spaces via seminorms.

$(\Leftarrow)$ Let us indirectly suppose that $f$ is not continuous. Then we find a point $a \in X$ and a neighbourhood $V$ of $f(a)$ such that $f^{-1}(V)$ does not contain any neighbourhood $U$ of $a$.

Now let $\{p_k\}_{k \in \mathbb{N}}$ be a countable family of seminorms and consider the sequence of neighbourhoods

$$U_n = \left\{ x \in X : p_k(a - x) < \frac{1}{n} \text{ for all } k < n \right\}.$$

Since $f$ does not map any of these neighbourhoods $U_n$ completely into $V$, for each $n$ there is an $x_n \in U_n$ with $f(x_n) \notin V$. But by construction we have $p_k(x_n - a) \to 0$ for $n \to \infty$, which shows the convergence $x_n \to a$ in $X$. The sequential continuity implies $f(x_n) \to f(a)$ so all except finitely many of the points $f(x_n)$ lie in a neighbourhood of $f(a)$. This is a contradiction since $f(x_n) \notin V$.

Remark A.2 One might also consult [21] Thm. 1.10(b)] in this context.

B An extension operator for Banach-valued Sobolev spaces

Let $X$ be a Banach space. We construct an extension operator for functions from $H^k([0, T], X)$ to $H^k(\mathbb{R}, X)$. The proof follows by modifying the standard case for real-valued functions appropriately.

For convenience to the reader we provide a short proof following [21] Thm. 6.10], since we could not find a reference in the literature.

Theorem B.1 (Extension operator) Let $k \in \mathbb{N}$. Then there is a continuous and linear extension operator

$$E : H^k([0, T], X) \to H^k(\mathbb{R}_+, X), \quad u(t) \mapsto (Eu)(t),$$

24
Moreover, we compute an operator from the interval \( \mathbb{R} \). The general assertion follows by density arguments since \( \|Eu\|_{H^k(\mathbb{R}_+, X)} \leq c \|u\|_{H^k([0, T], X)} \).

**Proof:**

**Step 1:** We start by extending an \( X \)-valued function from \( \mathbb{R}_+ = \{ x \in \mathbb{R} : x_n \geq 0 \} \) to \( \mathbb{R} \). Assume first that \( u \in C^k(\mathbb{R}_+, X) \) and let \( \lambda_1 > \lambda_2 > \ldots > \lambda_{k+1} > 1 \). We define the extension operator from the interval \( \mathbb{R}_+ \) to \( \mathbb{R} \) by some sort of reflection at the point 0 as follows:

\[
(Eu)(t) = \begin{cases} 
  u(t), & t \geq 0, \\
  \sum_{j=1}^{k+1} a_j (-\lambda_j t), & t < 0.
\end{cases}
\]

By construction \( E \) is linear with \((Eu)(t) = u(t)\) for \( t \in \mathbb{R}_+ \) and, in particular,

\[
\text{supp } EU \subset [-K, K] \times X \quad \text{if} \quad \text{supp } u \subset [0, K] \times X.
\]

(B.1)

The fact that \( EU \in C^k(\mathbb{R}, X) \) follows by considering the right and left derivatives, i.e.,

\[
\lim_{t \to 0^-} \partial_t Eu(t) = \partial_t u(0), \\
\lim_{t \to 0^+} \partial_t Eu(t) = \partial_t u(0) \sum_{j=1}^{k+1} a_j (-\lambda_j)^l,
\]

where \( l = 0, \ldots, k \). These derivatives coincide if \( \sum_{j=1}^{k+1} a_j (-\lambda_j)^l = 1 \) for all \( l = 0, \ldots, k \), which leads to a system of \( k+1 \) linear equations with coefficients from the Vandermonde determinant. Therefore, we can find a unique solution \( a_1, \ldots, a_{k+1} \) such that

\[
\partial_t Eu(t) \bigg|_{t \to 0^-} = \partial_t Eu(t) \bigg|_{t \to 0^+} \quad \text{for all} \quad l = 0, \ldots, k+1.
\]

Moreover, we compute

\[
\|Eu\|_{H^k(\mathbb{R}, X)}^2 = \sum_{l=0}^{k} \left\{ \int_0^\infty \|\partial_t^l u(t)\|X\|^2 dt + \int_{-\infty}^0 \left\| \sum_{j=1}^{k+1} a_j (-\lambda_j)^l (\partial_t^l u) (-\lambda_j t) \|X\|^2 dt \right\}
\]

\[
\leq \|u\|_{H^k(\mathbb{R}_+, X)}^2 + c \sum_{l=0}^{k} \|\partial_t^l u\|_{L^2(\mathbb{R}_+, X)}^2
\]

\[
\leq \|u\|_{H^k(\mathbb{R}_+, X)}^2.
\]

The general assertion follows by density arguments since \( C^k(\mathbb{R}_+, X) \) is dense in \( H^k(\mathbb{R}_+, X) \).

**Step 2:** Now we extend functions from \( H^k([0, T], X) \) to \( H^k(\mathbb{R}, X) \). For this choose a covering \((U_j, \Phi_j)_{j=0,1,2}\) of \([0, T]\) such that \( U_0 \subset \subset (0, T) \) and \( U_1 \) and \( U_2 \) are neighbourhoods of the boundary points 0 and \( T \), respectively. In particular, the functions \( \Phi_j \in C_0^\infty(U_j) \) constitute a subordinate resolution of unity: \( \sum_{j=0}^{2} \Phi_j(x) = 1 \) in \( I \supset \supset [0, T] \). Let \( \tau_2(t) = T - t \), then \( u_2 := (\Phi_2 u) \circ \tau_2^{-1} \) is a function in \( H^k(\mathbb{R}_+, X) \). Applying the extension operator from Step 1 we see that \( Eu_2 \in H^k(\mathbb{R}, X) \), where according to (B.1) \( \text{supp } Eu_2 \subset (-T/2, +T/2) \) if the support of \( u_2 \) (or \( \Phi_2 \), respectively) is small. The extension operator we are looking for is given by

\[
E_{[0, T]} u := (Eu_2) \circ \tau_2 + \Phi_1 u + \Phi_0 u,
\]

25
since by construction $E_{[0,T]} u = u$ in $[0,T]$ with supp $E_{[0,T]} u \subset \mathbb{R}_+$ (if the support of $\Phi_2$ is sufficiently small) and

$$\|E_{[0,T]} u|H^k(\mathbb{R}_+, X)\| \leq c\|u_j|H^k(\mathbb{R}_+, X)\| + \sum_{j=0}^1 \|\Phi_j u|H^k([0, T], X)\| \leq c\|u|H^k([0, T], X)\|.$$

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