Optimal Controls of Stochastic Differential Equations with Jumps and Random Coefficients: Stochastic Hamilton–Jacobi–Bellman Equations with Jumps

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Abstract
We study the stochastic Hamilton–Jacobi–Bellman (HJB) equation with jump, which arises from a non-Markovian optimal control problem with a recursive utility cost functional. The solution to the equation is a predictable triplet of random fields. We show that the value function of the control problem, under some regularity assumptions, is the solution to the stochastic HJB equation; and a classical solution to this equation is the value function and characterizes the optimal control. With some additional assumptions on the coefficients, an existence and uniqueness result in the sense of Sobolev space is shown by recasting the stochastic HJB equation as a backward stochastic evolution equation in Hilbert spaces with the Brownian motion and Poisson jump.

Keywords Stochastic control · Dynamic programming · Stochastic HJB equation · Stochastic partial integral differential equation

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1 Introduction

Backward stochastic partial differential equations (BSPDEs) are natural generalizations of backward stochastic differential equations (BSDEs). The theory of BSDEs has been well developed, dating back to [5, 26] on the linear and nonlinear cases, respectively. A systematic account of the theory and application of BSDEs is available in [12, 37, 39] and the references therein. In the jump setting, BSDEs have also been studied by many authors (see, for example, [14, 30, 35]). In recent years, there has been growing interest on BSPDEs, partly driven by their extensive applications in stochastic control theory and mathematical finance. For instance, BSPDEs serve as adjoint equations in Pontryagin’s maximum principle when the controlled system is a stochastic partial differential equations. For this line of research, one can refer to [4, 16, 22, 24, 32, 33, 38]. Recently, [3] studied the option pricing problems under rough volatility models and proved that the value function satisfies a BSPDE.

The (backward) stochastic Hamilton-Jacobi-Bellman (HJB) equations, which is a class of fully nonlinear BSPDEs, was first introduced by Peng [27] in the study of stochastic control systems driven by Brownian motions, where the coefficients of the control systems are allowed to be random. The stochastic HJB equations have a stochastic control interpretation as the classical deterministic HJB equation of which the classical solution is the value function of the stochastic optimal control problem with deterministic coefficients. In [27], for the case where the diffusion term of the system does not contain the control variable, the existence and uniqueness of adapted solutions to the stochastic HJB equations driven by Brownian motions in the sense of Sobolev space were established by treating the equations as a class of backward stochastic evolution equations for Hilbert space valued processes. However, in general the existence and uniqueness results of the classical solutions to the stochastic HJB equations have been still an open problem.

This paper studies a stochastic HJB equation driven by both Brownian motions and Poisson jumps. More specifically, the stochastic HJB equation encountered in this paper is associated with a stochastic optimal control problem driven by Brownian motions and Poisson jumps simultaneously and with the recursive utility type cost functional, which is given by a BSDE. In economics and finance, recursive utilities have been used to disentangle the investor’s risk aversion and intertemporal substitution, see [11, 13]. Using the BSDE theory, the stochastic differential utility was extended to the case with multiple priors that distinguish risk and ambiguity in a unified framework, see [7]. Using dynamic programming principle, we first show that the value function, under some regularity assumptions, is the solution to the stochastic HJB equation. However, this result does not apply to a general case since in general the value function does not satisfy these regularity assumptions. On the other hand, we provide a verification theorem and show that if the stochastic HJB equation admits a classical solution which is a triplet of random fields and satisfies sufficient regularity conditions, then the first component of the triplet coincides with the value function of the optimal control problem. Prior to our work, Li and Peng [21] also studied the optimal control problem with jumps. Unlike our framework accommodating random coefficients, they focused on the case with deterministic coefficients, thereby proving the relation between the value function and the viscosity solution of (deterministic) HJB equation. Allowing
the coefficients to be random results in the main difficulty of our study. For example, in [21], the authors showed that the value function is continuous in $t$. Unfortunately, this is not the case in our framework. Moreover, the viscosity solution of BSPDEs is not well-studied. To our best knowledge, most existing works considered the control systems driven by Brownian motions only, see [29] and the references therein.

Instead of considering the viscosity solution for the most general case, we study the stochastic HJB equation with jumps in the sense of Sobolev spaces and prove the existence and uniqueness results in these spaces with some additional assumptions. To this end, the most important step is to first establish the existence and uniqueness results for a class of backward stochastic evolution equations with jumps in Hilbert spaces. Secondly, we recast the stochastic HJB equation with jumps as the backward stochastic evolution equation with jumps in Hilbert spaces, where the existence and uniqueness results are applicable. It is worth noting that Øksendal et al. [25] investigated a class of semi-linear backward stochastic partial differential equations with jumps which appear as adjoint equations in the maximum principle for optimal control of stochastic partial differential equations driven by Poisson jumps. Our formulation in the current paper is more general than theirs. Thus, the results in [25] can be covered by ours as special cases.

The rest of this paper is organized as follows. In Sect. 2, we introduce some notations and preliminary results, followed by the dynamic programming principle given in Sect. 3. In Sect. 4, we provide the optimal control interpretation of the stochastic HJB equations with jumps and a related verification theorem. Section 5 is devoted to studying the existence and uniqueness theory of adapted solutions to backward stochastic evolution equations with random jumps. With this result, we prove the existence and uniqueness results for a special class of the stochastic HJB equations with jumps.

2 Notations, Preliminary and Problem Formulation

In this section, we first introduce basic notations, standing assumptions, and a preliminary result on the essential infimum for a family of nonnegative random variables. Then, we formulate the stochastic control problem with jumps and random coefficients.

2.1 Notations and Preliminary

Let $T \triangleq [0, T]$ denote a fixed time interval of finite length, i.e., $T < \infty$. We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which all random objects are defined. The space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with a right-continuous, $\mathbb{P}$-complete filtration $\mathbb{F} \triangleq \{\mathcal{F}_t | t \in T\}$, to be specified later. Furthermore, we assume that $\mathcal{F}_T = \mathcal{F}$. Denote by $\mathbb{E}[-]$ the expectation with respect to $\mathbb{P}$, by $\mathcal{P}$ the predictable $\sigma$-algebra on $\Omega \times T$ associated with $\mathbb{F}$, and by $\mathcal{B}(\Lambda)$ the Borel $\sigma$-algebra of any topological space $\Lambda$. Let $\{W(t)|t \in T\} \triangleq \{(W_1(t), W_2(t), \cdots, W_d(t))^\top|t \in T\}$ be a $d$-dimensional standard Brownian motion under the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(E, \mathcal{B}(E), \nu)$ be a measure space with $\nu(E) < \infty$, and $\eta$ be a stationary Poisson point process with the
characteristic measure \( \nu \) (see [17] for details). Then, the counting measure induced by \( \eta \) is defined by

\[
\mu((0, t] \times A) \overset{\Delta}{=} \#\{s; s \leq t, \eta(s) \in A\}, \quad \text{for } t > 0, \ A \in \mathcal{B}(E),
\]

and \( \tilde{\mu}(de, dt) \overset{\Delta}{=} \mu(de, dt) - \nu(de)dt \) is a compensated Poisson random martingale measure which is assumed to be independent of the Brownian motion \( \{W(t)|t \in T\} \). Moreover, the filtration \( \mathbb{F} \) is the \( \mathbb{P} \)-augmentation of the natural filtration generated by the Brownian motion \( \{W(t)|t \in T\} \) and the Poisson random measure \( \{\mu((0, t] \times A)|t \in T, A \in \mathcal{B}(E)\} \). Let \( T_0 \) be the set of all stopping times bounded by \( T \). For any stopping time \( \tau \in T_0 \), denote by \( T_\tau \) the set of all stopping times in \( T_0 \) and greater than \( \tau \).

Let \( H \) be any Hilbert space. The inner product in \( H \) is denoted by \( \langle \cdot, \cdot \rangle \), and the norm in \( H \) is denoted by \( |\cdot|_H \) or \( |\cdot| \) whenever there is no risk of confusion. For a scalar-valued function \( \phi : \mathbb{R}^n \to \mathbb{R} \), we denote by \( \phi_x, D_x \phi \), or \( D\phi \) its gradient and \( \phi_{xx}, D_x^2 \phi \), or \( D^2 \phi \) its Hessian, which is a symmetric matrix. For a vector-valued function \( \phi : \mathbb{R}^n \to \mathbb{R}^k \) (with \( k \geq 2 \)), \( \phi_x \overset{\Delta}{=} (\frac{\partial \phi_i}{\partial x_j}) \) is the corresponding \((k \times n)\) Jacobian matrix. For any two stopping times \( \tau \) and \( \gamma \), the corresponding stochastic interval is defined by the following set

\[
[(\tau, \gamma)] \overset{\Delta}{=} \{(t, \omega) \in \mathbb{R} \times \Omega; \tau(\omega) \leq t \leq \gamma(\omega)\}.
\]

The following spaces will be frequently used throughout this paper: for any \( \tau \in T_0 \) and \( \gamma \in T_\tau \),

- \( M^{2,p}_\mathbb{F}(\tau, \gamma; H) \): the space of all \( H \)-valued and \( \mathbb{F} \)\( - \))\( \)predictable processes \( f = \{f(t, \omega)|t, \omega \in [\tau, \gamma]\} \) satisfying

\[
\|f\|_{M^{2,p}_\mathbb{F}(\tau, \gamma; H)} \overset{\Delta}{=} \mathbb{E}\left[\left(\int_\tau^\gamma |f(t)|^2 dt\right)^p\right] < \infty,
\]

whenever there is no risk of confusion, we write \( M^2_\mathbb{F}(\tau, \gamma; H) \overset{\Delta}{=} M^{2,1}_\mathbb{F}(\tau, \gamma; H) \);

- \( S^2_\mathbb{F}(\tau, \gamma; H) \): the space of all \( H \)-valued and \( \mathbb{F} \)\( - \))\( \)adapted càdlàg processes \( f = \{f(t, \omega)|t, \omega \in [\tau, \gamma]\} \) satisfying

\[
\|f\|_{S^2_\mathbb{F}(\tau, \gamma; H)} \overset{\Delta}{=} \sqrt{\mathbb{E}\left[\sup_{\tau \leq t \leq \gamma} |f(t)|^2\right]} < \infty;
\]

- \( M^{p,2}(E; H) \): the space of all \( H \)-valued measurable functions \( r = \{r(e)|e \in E\} \) defined on the measure space \( (E, \mathcal{B}(E); \nu) \) satisfying

\[
\|r\|^2_{M^{p,2}(E; H)} \overset{\Delta}{=} \int_E |r(e)|^2 \nu(de) < \infty;
\]
\[ M_{\mathcal{F}}^{\nu,2}(\tau, \gamma) \times E; H) \]: the space of all \( M_{\mathcal{F}}^{\nu,2}(E; H) \)-valued and \( \mathbb{F} \)-predictable processes \( r = \{r(t, \omega, e)| (t, \omega, e) \in [\tau, \gamma] \times E\} \) satisfying
\[
\int_{\tau}^{\gamma} \|r(t, \cdot)\|^{2}_{M_{\mathcal{F}}^{\nu,2}(E; H)} dt < \infty;
\]

\[ L^{2}(\Omega, \mathcal{G}, \mathbb{P}; H) \]: the space of all \( H \)-valued, \( \mathcal{G} \)-measurable, random variables \( \xi \) defined on \((\Omega, \mathcal{G}, \mathbb{P})\) satisfying
\[
\|\xi\|_{L^{2}(\Omega, \mathcal{G}, \mathbb{P}; H)} = \sqrt{\mathbb{E}[|\xi|^{2}]} ,
\]
where \( \mathcal{G} \) is a sub-algebra of \( \mathcal{F} \).

In what follows, we recall a classical theorem for the essential infimum of a family of nonnegative random variables in a probability space (see [18, Appendix A]).

**Lemma 2.1** Let \( X \) be a family of nonnegative integrable random variables defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Then there exists a random variable \( X^{*} \) such that
1. for all \( X \in X \), \( X \geq X^{*} \) a.s.;
2. if \( Y \) is another random variable satisfying \( X \geq Y \) a.s., for all \( X \in X \), then \( X^{*} \geq Y \) a.s..

The random variable \( X^{*} \), which is unique a.s., is called the essential infimum of \( X \), and is denoted by \( \text{ess inf}_{X \in X} X \).

Furthermore, if \( X \) is closed under pairwise minimum (i.e., \( X, Y \in X \) implies \( X \wedge Y \in X \)), then there exists a nonincreasing sequence \( \{Z_{n}\}_{n \in \mathbb{N}} \) of random variables in \( X \) such that \( X^{*} = \lim_{n \to \infty} Z_{n} \) a.s. Moreover, for any sub-algebra \( \mathcal{G} \) of \( \mathcal{F} \), the \( \mathcal{G} \)-conditional expectation is interchangeable with the essential infimum, that is,
\[
\mathbb{E}\left[\text{ess inf}_{X \in X} X \bigg| \mathcal{G}\right] = \text{ess inf}_{X \in X} \mathbb{E}[X | \mathcal{G}] \text{ a.s..} \quad (2.1)
\]

**Remark 2.1** The above result can be extended to the family \( X^{*} \) of random variables that are uniformly bounded from below by another random variable \( Y \), i.e., \( X \geq Y \), \( \forall X \in X \). For that purpose, we only need to apply Lemma 2.1 to the family \( \{X - Y | X \in X^{*}\} \) to get the desired result.

### 2.2 Statement of the Control Problem

Let \( U \) be a nonempty subset of \( \mathbb{R}^{m} \). In this paper, the admissible control is defined as follows:

**Definition 2.1** For any given \( \tau \in T_{0} \), a stochastic process \( u(\cdot) \) is said to be an admissible control on the interval \([\tau, T] \), if \( u(s) \in U \) for almost \((s, \omega) \in [0, T] \times \Omega \), \( u(\cdot) \) is an \( \{\mathcal{F}_{s}|s \in [0, T]\} \)-predictable process, and \( u(s) = 0 \) if \( s \leq \tau \).

The set of all admissible controls is denoted by \( \mathcal{U}[\tau, T] \).

In this paper, for any given initial state \( x \in \mathbb{R}^{n} \) and admissible control \( u(\cdot) \in \mathcal{U}[0, T] \), we consider the following controlled stochastic differential equation driven by the Brownian motion \( W \) and the Poisson random martingale measure \( \tilde{\mu} \):
Here our assumptions are stronger than the usual assumptions that

\[ \begin{aligned}
    \left\{ \begin{array}{l}
        dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s) + \int_E g(s, e, X(s-), u(s))\,\tilde{\mu}(ds, de), \quad 0 \leq s \leq T, \\
        X(0) = x_0 \in \mathbb{R}^n,
    \end{array} \right. \\
\end{aligned} \tag{2.2} \]

where the coefficients \(b, \sigma,\) and \(g\) are given random mappings satisfying the following assumption:

**Assumption 2.1**

1. The mappings \(b : T \times \Omega \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n\) and \(\sigma : T \times \Omega \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^{n \times d}\) are \(\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U)-\text{measurable};\) the mapping \(g : T \times \Omega \times E \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n\) is \(\mathcal{P} \otimes \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U)-\text{measurable}.\)

2. There exists a positive constant \(C\) and deterministic nonnegative function \(\rho(e)\) such that for all \((u, u', x, x') \in U \times U \times \mathbb{R}^n \times \mathbb{R}^n\) and a.e. \((t, \omega, e) \in T \times \Omega \times E,

\[ \begin{aligned}
    |b(t, x, u) - b(t, x', u')| &+ |\sigma(t, x, u) - \sigma(t, x', u')| \leq C(|x - x'| + |u - u'|), \\
    |g(t, e, x, u) - g(t, e, x', u')| &\leq \rho(e)(|x - x'| + |u - u'|), \\
    |b(t, x, u)| &+ |\sigma(t, x, u)| \leq C(1 + |x| + |u|), \\
    |g(t, e, x, u)| &\leq \rho(e)(1 + |x| + |u|), \\
\end{aligned} \]

and

\[ \int_E \exp \{\rho(e)\} v(de) < \infty. \]

**Remark 2.2** Here our assumptions are stronger than the usual assumptions that

\[ \int_E |g(t, e, x, u) - g(t, e, x', u')|^2 v(de) \leq C(|x - x'| + |u - u'|)^2, \]

and

\[ \int_E |g(t, e, x, u)|^2 v(de) \leq C(1 + |x| + |u|)^2. \]

The reason is that, to get the results in Lemmas 3.2 and 3.3, we need the following conditions

\[ \int_E |g(t, e, x, u) - g(t, e, x', u')|^p v(de) \leq C(|x - x'| + |u - u'|)^p, \]

and

\[ \int_E |g(t, e, x, u)|^p v(de) \leq C(1 + |x| + |u|)^p. \]

hold for \(p \geq 2,\) which can be derived from Assumption 2.1.
Furthermore, we impose the following assumption on the control region.

**Assumption 2.2** The control domain $U$ is a compact subset of $\mathbb{R}^m$.

Under Assumptions 2.1–2.2, for any initial value $X(0) = x_0 \in \mathbb{R}^n$ and admissible control $u(\cdot) \in \mathcal{Y}[0, T]$, the SDE (2.2) admits a unique strong solution satisfying $X^{0,x_0;u}(\cdot) \in S^p_{\mathcal{F}}(0, T; \mathbb{R}^n)$, for any $p > 1$. The solution $X(\cdot) \triangleq X^{0,x_0;u}(\cdot)$ to the SDE (2.2) is referred to as the state process corresponding to the initial state $x_0$ at time zero and the admissible control process $u(\cdot)$ and the pair of stochastic processes $(u(\cdot); X(\cdot))$ is referred to as an admissible pair.

For any admissible pair $(u(\cdot); X(\cdot))$, consider the following BSDE with jumps

$$
Y(t) = h(X(T)) + \int_t^T f(s, X(s), u(s), Y(s), Z(s)) \, ds - \int_t^T Z(s) \, dW(s) - \int_t^T \int_E K(s, e) \tilde{\mu}(ds, de), \quad 0 \leq t \leq T,
$$

where $f$, $h$, and $l$ are given random mappings satisfying the following assumption:

**Assumption 2.3**

(i) $f : T \times \Omega \times \mathbb{R}^n \times U \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$-measurable; $l : T \times \Omega \times E \to \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(E)$-measurable;

(ii) There exists a positive constant $C$ such that for almost all $(t, \omega) \in T \times \Omega$ and $(x, u, y, z, k), (x', u', y', z', k') \in \mathbb{R}^n \times U \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$,

$$
\begin{align*}
|f(t, x, u, y, z, k) - f(t, x', u', y', z', k')| + |h(x) - h(x')| & \leq C \left[ (1 + |x| + |x'| + |u| + |u'|)((x - x') + |u - u'|) + |y - y'| + |z - z'| + |k - k'| \right], \\
|f(t, x, u, y, z, k) + h(x)| & \leq C(1 + |x|^2 + |u|^2 + |y| + |z| + |k|).
\end{align*}
$$

(iii) $k \mapsto f(t, x, u, y, z, k)$ is non-decreasing for all $(t, x, u, y, z) \in T \times \mathbb{R}^n \times U \times \mathbb{R} \times \mathbb{R}^d$; there exists a positive constant $C$ such that $0 \leq l(t, e) \leq C(1 + |e|)$ for all $(t, e) \in T \times E$.

From Assumption 2.3, we see that (2.3) admits a unique solution $(Y, Z, K) \in S^2_{\mathcal{F}}(0, T; \mathbb{R}) \times \mathcal{M}^1_{\mathcal{F}}(0, T; \mathbb{R}^d) \times \mathcal{M}^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$. Moreover, Condition (iii) in Assumption 2.3 ensures that the comparison principle holds (see [21]). We may also denote the process as $(Y^{0,x_0;u}, Z^{0,x_0;u}, K^{0,x_0;u})$ to emphasize the dependence on the initial data and admissible control whenever necessary. The cost functional is defined by

$$
J(0, x_0; u(\cdot)) \triangleq Y(0).
$$

In this paper, we also need the following assumption for the coefficient on the jump part, which ensures related stochastic flows to be invertible.
Assumption 2.4 The map $\phi_{t,e,u}: x \mapsto x + g(t, e, x, u)$ is homeomorphic from $\mathbb{R}^n$ to $\mathbb{R}^n$ and the inverse map $\psi_{t,e,u}$ has uniformly linear growth and is uniformly Lipschitz continuous. Moreover, there exists a positive constant $\delta$ such that

$$\left| \det(I + D_x g(t, e, x, u)) \right| \geq \delta, \quad \forall (t, e, x, u) \in T \times E \times \mathbb{R}^n \times U. \quad (2.5)$$

Under Assumptions 2.1–2.3, it is easy to check that $|J(0, x_0; u(\cdot))| < \infty$. Thus, the cost functional (2.4) is well-defined. We are now ready to state our optimal control problem:

**Problem 2.2** Find an admissible control process $\bar{u}(\cdot) \in \mathcal{V}[0, T]$ such that

$$J(0, x_0; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{V}[0, T]} J(0, x_0; u(\cdot)) \quad (2.6)$$

subject to (2.2) and (2.3).

The admissible control $\bar{u}(\cdot) \in \mathcal{V}[0, T]$ satisfying (2.6) is called an optimal control process of Problem 2.2. Correspondingly, the state process $\bar{X}(\cdot)$ associated with $\bar{u}(\cdot)$ is called an optimal state process, and $(\bar{u}(\cdot); \bar{X}(\cdot))$ is called an optimal pair of Problem 2.2.

### 3 Bellman’s Dynamic Programming Principle with Jumps

One of the key features of Problem 2.2 is that all the coefficients in the state Eq. (2.2) and the cost functional (2.3) are stochastic processes or random variables. Therefore, Problem 2.2 is indeed a non-Markovian stochastic optimal control problem.

Pontryagin’s stochastic maximum principle and Bellman’s dynamic programming principle are two of the most important approaches to solving stochastic optimal control problems. In the former approach, a necessary condition of optimality can be obtained under certain regularity conditions of the system. On the other hand, the latter approach results in different versions of HJB equations, which can be used to characterize the optimal control. We refer readers to [35] for the general stochastic maximum principle for the control system driven by jump-diffusion processes. For a systematic account of the two approaches, one may refer to the monograph [39] and the references therein.

This paper is concerned with Bellman’s dynamic programming principle and the associated stochastic HJB equation with jumps. We first study the corresponding Bellman’s dynamic programming principle for Problem 2.2. Note that the initial time $t = 0$ and the initial state $X(0) = x_0$ are fixed in the formulation of Problem 2.2. The basic idea of the dynamic programming principle is, however, to consider a family of optimal control problems with different initial times and states, to establish the relation among these problems then, and finally to solve all these problems via a stochastic HJB equation.

To elaborate this idea precisely, we fix a set of initial data $(\tau, \xi) \in T_0 \times L^2(\Omega; \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)$. For any given admissible control $u(\cdot) \in \mathcal{V}[\tau, T]$, we consider the following state equation:
\[ \begin{aligned}
dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s) + \int_E g(s, e, X(s-), u(s))\tilde{\mu}(ds, de), \quad \tau \leq s \leq T, \\
X(\tau) = \xi.
\end{aligned} \]  

(3.1)

The cost functional is defined by

\[ J(\tau, \xi; u(\cdot)) \triangleq Y^{\tau, \xi; u} (\tau), \]  

(3.2)

where \((Y^{\tau, \xi; u}, Z^{\tau, \xi; u}, K^{\tau, \xi; u})\) is the solution of (2.3) on time interval \([\tau, T]\). Then, corresponding to the control system (3.1) and the cost functional (3.2), the optimal control problem parameterized by \((\tau, \xi)\in T_0\times L^{2}(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)\) is formulated as follows:

**Problem 3.1** \((D_{\tau, \xi})\). Find an admissible control process \(\bar{u}(\cdot)\in \mathcal{V}[\tau, T]\) such that

\[ J(\tau, \xi; \bar{u}(\cdot)) = \text{ess inf}_{u(\cdot)\in \mathcal{V}[\tau, T]} J(\tau, \xi; u(\cdot)). \]  

(3.3)

We denote the above optimal control problem by Problem \((D_{\tau, \xi})\) to stress the dependence on the initial state \((\tau, \xi)\). From Lemma 3.2 in the next subsection, for any initial data \((\tau, \xi)\in T_0\times L^{2}(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)\) and admissible control \(u(\cdot)\in \mathcal{V}[\tau, T]\), the state equation (3.1) has a unique strong solution \(X(\cdot)\equiv X^{\tau, \xi; u}(\cdot)\in S^2_{\mathcal{F}}(\tau, T; \mathbb{R}^n)\) and the cost functional (3.2) is well-defined. Furthermore, we can define the following conditional minimal value system

\[ \mathbb{V}(\tau, \xi) \triangleq \text{ess inf}_{u(\cdot)\in \mathcal{V}[\tau, T]} J(\tau, \xi; u(\cdot)). \]  

(3.4)

Clearly, for any \((\tau, \xi)\in T_0\times L^{2}(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n), \mathbb{V}(\tau, \xi)\) is an \(\mathcal{F}_\tau\)-measurable random variable.

### 3.1 Preliminary Results

In this subsection, we provide some preliminary results for the controlled SDE (3.1), which are needed in the following sections. The proofs of the first two lemmas can be found in [?].

**Lemma 3.2** Let Assumptions 2.1–2.3 hold. Given \(\tau \in T_0\) and \(p \geq 2\), SDE (3.1) admits a unique solution for any \(u(\cdot)\in \mathcal{V}[\tau, T]\) and \(\xi \in L^{p}(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)\). Moreover, there exists a positive constant \(C_p\) such that, for any \(\tau \in T_0\), \(u(\cdot), \bar{u}(\cdot)\in \mathcal{V}[\tau, T]\), and \(\xi, \bar{\xi} \in L^{p}(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)\), it holds that

\[ \mathbb{E} \left[ \sup_{\tau \leq s \leq T} |X^{\tau, \xi; u}(s)|^p \bigg| \mathcal{F}_\tau \right] \leq C_p(1 + |\xi|^p), \]  

(3.5)
and

\[ E \left[ \sup_{\tau \leq s \leq T} |X^{t,\xi;u}(s) - X^{t,\tilde{\xi};\tilde{u}}(s)|^p \right] \leq C_p \left( |\xi - \tilde{\xi}|^p + E \left[ \int_{\tau}^T |u(s) - \tilde{u}(s)|^p ds \right] \right). \tag{3.6} \]

Moreover, the solution \( X(\cdot) \) satisfies the flow property, i.e., for any \( t \leq \tau \leq \gamma \) and \( u \in \mathcal{V}[\tau, T] \),

\[ X^{t,x;u}(\gamma) = X^{\tau,X^{t,x;u}(\tau);u}(\gamma), \quad a.s. \]

**Lemma 3.3** Let Assumptions 2.1–2.4 hold. Then, the stochastic flow \( X^{t,x;u} \) is an onto homeomorphism for any \( t \) a.s.. Moreover, the gradient \( \partial X^{t,x;u} \) of the stochastic flow is the solution of the following SDE:

\[
d \partial X^{t,x;u}(s) = D_x b(s, X^{t,x;u}(s), u(s)) \partial X^{t,x;u}(s) ds \\
+ D_x \sigma(s, X^{t,x;u}(s), u(s)) \partial X^{t,x;u}(s) dW(s) \\
+ \int_E D_x g(s, e, X^{t,x;u}(s-), u(s)) \partial X^{t,x;u}(s-) \tilde{\mu}(ds, de), \quad \partial X^{t,x;u}(t) = I, \tag{3.7}
\]

where \( D_x \) denotes the gradient with respect to \( x \).

From the a priori estimates for linear SDEs, we immediately have that, for any \( p \geq 2 \), there exists a constant \( C_p \) such that

\[ E \left[ |\partial X^{t,x;u}(s)|^p \right] \leq C_p. \]

We also have the following lemma concerning the solution to BSDE (2.3).

**Lemma 3.4** Let Assumptions 2.1–2.3 hold. The solution \((Y^{\tau,\xi;u}, Z^{\tau,\xi;u}, K^{\tau,\xi;u})\) to BSDE (2.3) satisfies

\[ |Y(\tau)|^2 \leq C E \left[ |h(X(T))|^2 + \int_{\tau}^T |f(s, X(s), u(s), 0, 0, 0)|^2 ds \right] \mathcal{F}_\tau. \]

Hence, we have that

\[ |Y(\tau)| \leq C (1 + |\xi|^2). \]

**Proof** The first estimate is a standard result for the a priori estimates of BSDE with jumps, see [1, 35]. The second one can be obtained by using Lemma 3.2. \( \square \)

Then we immediately have the following result, whose proof is a straightforward application of the above lemmas and thus is omitted.
Lemma 3.5 Let Assumptions 2.1–2.3 be satisfied. Then, for any given \( \tau \in T_0, \xi, \tilde{\xi} \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n) \), and \( u(\cdot), \tilde{u}(\cdot) \in \mathcal{V}[\tau, T] \), we have
\[
\mathbb{J}(\tau, \xi; u) \leq C(1 + |\xi|^2), \tag{3.8}
\]
and
\[
|\mathbb{J}(\tau, \xi; u) - \mathbb{J}(\tau, \tilde{\xi}; \tilde{u})| \\
\leq C \left( 1 + |\xi| + |\tilde{\xi}|\right) |\xi - \tilde{\xi}| + \mathbb{E} \left[ \int^T_\tau \left( 1 + |u(s)| + |\tilde{u}(s)|\right) |u(s) - \tilde{u}(s)| ds \bigg| \mathcal{F}_\tau \right]. \tag{3.9}
\]
Therefore,
\[
|\mathbb{V}(\tau, \xi)| \leq C(1 + |\xi|^2), \tag{3.10}
\]
and
\[
|\mathbb{V}(\tau, \xi) - \mathbb{V}(\tau, \tilde{\xi})| \leq C(1 + |\xi| + |\tilde{\xi}|)|\xi - \tilde{\xi}|. \tag{3.11}
\]

3.2 Dynamic Programming Principle

In this subsection, we prove that the conditional minimal value system \( \{\mathbb{V}(\tau, \xi)|\tau \in T\} \) satisfies dynamic programming principle. For that purpose, we first introduce the concept of the so-called backward semigroup, which is first introduced by Peng [28]. Given the initial data \((\tau, \xi)\) with \( \tau \in T \) and \( \xi \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n) \), a stopping time \( \gamma \in T_\tau \), an admissible control process \( u(\cdot) \in \mathcal{V}[\tau, \gamma] \), and a real-valued random variable \( \eta \in L^2(\Omega, \mathcal{F}_\gamma, \mathbb{P}; \mathbb{R}) \), we define
\[
G_{s,\gamma}^{r,\xi;u(\cdot)}[\eta] \triangleq \tilde{Y}(s), \quad s \in [\tau, \gamma].
\]
where \((\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{K})\) is the solution to the following forward-backward system:
\[
\begin{cases}
    dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s) + \int_E g(s, e, X(s-), u(s))\tilde{\mu}(ds, de), \\
    dY(s) = -f(s, X(s), u(s), Y(s), Z(s)), \int_E K(s, e)\nu(de)ds + Z(s)dW(s) + \int_E K(s, e)\tilde{\mu}(ds, de), \quad \tau \leq s \leq \gamma, \\
    X(\tau) = \xi, \quad Y(\gamma) = \eta.
\end{cases}
\]
The main result is given in the following theorem.

Theorem 3.6 Under Assumptions 2.1–2.3, the conditional minimal value system \( \mathbb{V}(\tau, \xi) \) obeys the following dynamic programming principle: for any \( \tau \in T_0, \gamma \in T_\tau, \) and \( \xi \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n) \), it holds
\[
\mathbb{V}(\tau, \xi) = \text{ess inf}_{u(\cdot) \in \mathcal{V}[\tau, \gamma]} G_{r,\gamma}^{r,\xi;u(\cdot)}[\mathbb{V}(\gamma, X^{r,\xi;u(\cdot)}(\gamma))]. \tag{3.12}
\]
Before proving Theorem 3.6, we present the following lemmas.

**Lemma 3.7** Let Assumptions 2.1–2.3 be satisfied. Then for any initial data \((\tau, \xi) \in T_0 \times L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)\), the set \(\{J(\tau, \xi; u(\cdot)) | u(\cdot) \in \mathcal{V}[\tau, T]\}\) is closed under pairwise minimization. Consequently, there exists a sequence of admissible controls \(\{u_k(\cdot)\}_{k=1}^{\infty}\) such that \(\{J(\tau, \xi; u_k(\cdot))\}_{k=1}^{\infty}\) is non-increasing and

\[
\lim_{k \to \infty} J(\tau, \xi; u_k(\cdot))(\omega) = \mathbb{V}(\tau, \xi)(\omega), \quad \text{a.e.} \tag{3.13}
\]

Moreover, for any sub-algebra \(\mathcal{G}\) of \(\mathcal{F}_t\), the \(\mathcal{G}\)-conditional expectation is interchangeable with the essential infimum:

\[
\mathbb{E}[\mathbb{V}(t, \xi) | \mathcal{G}] = \text{ess inf}_{u(\cdot) \in \mathcal{V}[t, T]} \mathbb{E}[J(t, \xi; u(\cdot)) | \mathcal{G}], \quad \text{a.e.} \tag{3.14}
\]

**Proof** Given \(u_1(\cdot), u_2(\cdot) \in \mathcal{V}[\tau, T]\), letting \(A \equiv \{\omega | J(\tau, \xi; u_1(\cdot)) \leq J(\tau, \xi; u_2(\cdot))\}\), we have \(A \in \mathcal{F}_\tau\). Define \(v(\cdot) \equiv u_1(\cdot)1_A + u_2(\cdot)1_{A^c}\), which is an admissible control in \(\mathcal{V}[t, T]\). From the uniqueness for the solution to BSDE (2.3), it is easy to check that

\[
J(\tau, \xi; v(\cdot)) = J(\tau, \xi; u_1(\cdot)1_A + u_2(\cdot)1_{A^c}) = J(\tau, \xi; u_1(\cdot))1_A + J(\tau, \xi; u_2(\cdot))1_{A^c} \tag{3.15}
\]

Thus, the set \(\{J(\tau, \xi; u(\cdot)) | u(\cdot) \in \mathcal{V}[\tau, T]\}\) is closed under pairwise minimization. Moreover, one can also get that

\[
J(\tau, \xi; u(\cdot)) \geq -C(1 + |\xi|^2).
\]

Then, according to Remark 2.1, (3.13) and (3.14) follow directly from Lemma 2.1. The proof is completed. \(\square\)

Next, we prove that one can choose an admissible control that is of at most \(\varepsilon\) difference to the optimal value.

**Lemma 3.8** For any initial data \((\tau, \xi) \in T_0 \times L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)\) and \(\varepsilon > 0\), there exists an admissible control \(u^\varepsilon \in \mathcal{V}[\tau, T]\) such that

\[
J(\tau, \xi; u^\varepsilon(\cdot)) \leq \mathbb{V}(\tau, \xi) + \varepsilon, \quad \text{a.s.}
\]

**Proof** From Lemma 3.7, there exists a sequence of admissible controls \(\{u_k(\cdot)\}_{k=1}^{\infty}\) such that \(\{J(\tau, \xi; u_k(\cdot))\}_{k=1}^{\infty}\) is non-increasing and converges to \(\mathbb{V}(\tau, \xi)\) almost surely. Define the following sets as

\[
A_k \equiv \{J(\tau, \xi; u_k(\cdot)) \leq \mathbb{V}(\tau, \xi) + \varepsilon\}.
\]
Then, we have that $A_k \subset A_{k+1}$ and $\cup_{k=1}^{\infty} A_k = \Omega$. We construct the admissible control $u^\varepsilon$ as

$$u^\varepsilon(\cdot) = \sum_{k=1}^{\infty} u_k(\cdot) \mathbb{1}_{A_k \setminus A_{k-1}},$$

with $A_0 = \emptyset$. From the uniqueness for the solution to BSDE (2.3), we see that

$$\mathbb{J}(\tau, \xi; u^\varepsilon(\cdot)) \mathbb{1}_{A_k \setminus A_{k-1}} = \mathbb{J}(\tau, \xi; u_k(\cdot)) \mathbb{1}_{A_k \setminus A_{k-1}} \leq (\mathbb{V}(\tau, \xi) + \varepsilon) \mathbb{1}_{A_k \setminus A_{k-1}},$$

for any $k = 1, 2, \cdots$, which leads to the desired result.

\[\square\]

\textbf{Proof of Theorem 3.6} First, from the uniqueness for the solution of the forward-backward system, we have that for any initial data $(\tau, \xi)$, admissible control $u(\cdot) \in \mathcal{V}[\tau, T]$, and $\gamma \in T_\tau$, the following relation holds

$$G_{\tau, T}^{\tau, \xi; u}[h(X^{\tau, \xi; u}(T))] = G_{\tau, T}^{\tau, \xi; u}[Y^{\gamma, X^{\tau, \xi; u}(\gamma); u}(\gamma)], \quad \tau \leq s \leq \gamma.$$

Hence,

$$\mathbb{V}(\tau, \xi) = \text{ess inf} \ G_{\tau, T}^{\tau, \xi; u}[h(X^{\tau, \xi; u}(T))]$$

$$= \text{ess inf} \ G_{\tau, T}^{\tau, \xi; u}[Y^{\gamma, X^{\tau, \xi; u}(\gamma); u}(\gamma)] \geq \text{ess inf} \ G_{\tau, T}^{\tau, \xi; u}[\mathbb{V}(\gamma, X^{\tau, \xi; u}(\gamma))].$$

From Lemma 3.8, it holds that, for any $\varepsilon > 0$ and $u(\cdot) \in \mathcal{V}[\tau, T]$, there exists an admissible control $\tilde{u}(\cdot) \in [\gamma, T]$ such that

$$\mathbb{V}(\gamma, X^{\tau, \xi; u}(\gamma)) \geq Y^{\gamma, X^{\tau, \xi; u}(\gamma); \tilde{u}(\cdot) (\gamma)} - \varepsilon, \quad \text{a.s.}.$$ 

Combine the two controls $u(\cdot)$ and $\tilde{u}(\cdot)$ as

$$\tilde{u}(s) \triangleq \begin{cases} 
    u(s), & \tau \leq s \leq \gamma, \\
    \tilde{u}(s), & \gamma \leq s \leq T.
\end{cases}$$

Note that $\mathcal{V}[\tau, T]$ equipped with $L^2$ norm is a separable metric space. Lemma 2.1 in [31] states that $\tilde{u}(\cdot)$ is still admissible. We have

$$\mathbb{V}(\tau, \xi) \leq G_{\tau, T}^{\tau, \xi; \tilde{u}}[h(X^{\tau, \xi; \tilde{u}}(T))] = G_{\tau, T}^{\tau, \xi; \tilde{u}}[Y^{\gamma, X^{\tau, \xi; \tilde{u}}(\gamma); \tilde{u}}(\gamma)]$$

$$\leq G_{\tau, T}^{\tau, \xi; \tilde{u}}[\mathbb{V}(\gamma, X^{\tau, \xi; \tilde{u}}(\gamma))] + \varepsilon \leq G_{\tau, T}^{\tau, \xi; \tilde{u}}[\mathbb{V}(\gamma, X^{\tau, \xi; \tilde{u}}(\gamma))] + C\varepsilon,$$

where the last inequality is due to the estimate for BSDE with jumps. From the arbitrariness of $\varepsilon$, we get the desired result.

\[\square\]
We see that $V(\tau, \cdot)$ is a mapping from $L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R})$ to itself. One can get a random function from this mapping by restricting $V$ to the deterministic random variables as initial state values, i.e.,

$$V(t, x) \triangleq V(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^n.$$  \hfill (3.16)

This random function is called the value function for the optimal control problem. We shall see that $V(t, x)$ also satisfies dynamic programming principle. Similarly, for any control $u(\cdot)$, we define

$$J(t, x; u(\cdot)) \triangleq \mathcal{J}(t, x; u(\cdot)), \quad (t, x) \in [0, T] \times \mathbb{R}^n.$$  \hfill (3.17)

The following theorem is an analogous result of Theorem 3.6.

**Theorem 3.9** Let Assumptions 2.1–2.3 be satisfied. Then the value function $V(t, x)$ obeys the following dynamic programming principle: for any $0 \leq t \leq t + \delta \leq T$,

$$V(t, \xi) = \text{ess inf}_{u(\cdot) \in \mathcal{U}[t, T]} \mathcal{G}^{t, \xi; u}_{t, t+\delta}[V(t + \delta, X^{t, \xi; u}(t + \delta))].$$  \hfill (3.17)

Now we present some elementary properties of the cost functional and the value function. After that, Theorem 3.9 is an immediate result of these lemmas.

**Lemma 3.10** Let Assumptions 2.1–2.3 be satisfied. Then, for any given $(t, \xi) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$ and $u(\cdot) \in \mathcal{U}[t, T]$, we have

$$J(t, \xi; u(\cdot)) = \mathcal{J}(t, \xi; u(\cdot)).$$  \hfill (3.18)

**Proof** We first consider the case of a simple random variable:

$$\xi \triangleq \sum_{i=1}^N x_i 1_{A_i},$$  \hfill (3.19)

where $\{A_i\}_{i=1}^N$ is a finite partition of $(\Omega, \mathcal{F}_t)$ and $x_i \in \mathbb{R}^n$, for $1 \leq i \leq N$. In view of

$$\sum_{i=1}^N \Phi(x_i) 1_{A_i} = \Phi\left(\sum_{i=1}^N x_i 1_{A_i}\right),$$

we derive

$$\sum_{i=1}^N 1_{A_i} X^{t, x_i; u}(s) = \xi + \int_0^t b(s, \sum_{i=1}^N 1_{A_i} X^{t, x_i; u}(s), u(s))ds$$

$$+ \int_0^t \sigma(s, \sum_{i=1}^N 1_{A_i} X^{t, x_i; u}(s), u(s))dW(s)$$

\hfill Springer
\[
+ \int_0^t \int_E g(s, e, \sum_{i=1}^N 1_{A_i} X^{t,x_i;u}(s), u(s)) \tilde{\mu}(ds, de). 
\]

(3.20)

Thus, the uniqueness of the solution to the above SDE leads to

\[
X^{t,\xi;u}(s) = \sum_{i=1}^N 1_{A_i} X^{t,x_i;u}(s), \quad s \in [t, T].
\]

Therefore, we have

\[
J(t, \xi; u(\cdot)) = \mathbb{E} \left[ \int_t^T f(s, X^{t,\xi;u}(s), u(s)) ds + h(X^{t,\xi;u}(T)) \bigg| \mathcal{F}_t \right]
\]

\[
= \mathbb{E} \left[ \int_t^T f(s, \sum_{i=1}^N 1_{A_i} X^{t,x_i;u}(s), u(s)) ds + h(\sum_{i=1}^N 1_{A_i} X^{t,x_i;u}(T)) \bigg| \mathcal{F}_t \right]
\]

\[
= \sum_{i=1}^N 1_{A_i} \mathbb{E} \left[ \int_t^T f(s, X^{t,x_i;u}(s), u(s)) ds + h(X^{t,x_i;u}(T)) \bigg| \mathcal{F}_t \right]
\]

\[
= \sum_{i=1}^N 1_{A_i} J(t, x_i; u) = J(t, \sum_{i=1}^N 1_{A_i} x_i; u) = J(t, \xi; u).
\]

(3.21)

For a general \( \xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n) \), we can choose a sequence of simple random variables \( \{\xi_i\} \) such that

\[
\lim_{i \to \infty} \xi_i = \xi \quad \text{in} \quad L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n).
\]

The final desired result (3.18) follows from Lemma 3.5.

Remarking that Theorem 3.9 is a ‘weaker’ version of Theorem 3.6. The extension of Theorem 3.9 to any stopping times in place of \( t \) and \( t + \delta \) seems to be non-trivial. In many cases, it requires the continuity of the value function with respect to both \( t \) and \( x \). Li and Peng [21] considered similar optimal control problems but with deterministic coefficients. They proved that the value function, which is a deterministic function, is \( 1/2 \)-Hölder continuous with respect to \( t \). However, in our framework, it is not the case. In fact, from our later result, the value function is the solution to BSPDE and, thus, is only right continuous with left limits. Tang [34] proved that the result in Lemma 3.10 holds true also for random time \( \tau \). This will yield a stronger version of dynamic programming principle. However, their results rely on the linear-quadratic structure of the control problem and the aggregation of a \( \mathcal{T} \)-supermartingale family. See [34, 36] for details.
4 Stochastic HJB Equation with Jumps

In this section, we introduce a stochastic HJB equation driven by the Brownian motion and the Poisson random measure. This equation is associated with our optimal control problem (2.2) and is derived from the dynamic programming principle under sufficient smoothness assumptions on the value function. Let

\[ H(t, x, u, p, q, Q, A, k) \triangleq f(t, x, u, \sigma^\top p + q, k) + \left( p, b(t, x, u) \right) + \text{Tr}\left[ Q\sigma^\top(t, x, u) \right] + \frac{1}{2} \text{Tr}\left[ A\sigma\sigma^\top(t, x, u) \right] , \]

where \( \text{Tr}[\cdot] \) denotes the trace of a square matrix.

Now we introduce a fully nonlinear backward stochastic partial differential-integral equation driven by the Brownian motion \( B \) and the Poisson random measure \( \tilde{\mu} \). The differential form is given by

\[
\begin{aligned}
-dV(t, x) &= \text{ess inf}_{u \in U} \left\{ H(t, x, u, DV(t, x), \Phi(t, x), D\Phi(t, x), D^2 V(t, x), L(t, x, u, V, \Psi)) \ight. \\
&\quad+ \int_E [IV(t, e, x, u) - (g(t, e, x, u), DV(t, x))] v(de) \\
&\quad+ \int_E [I\Psi(t, e, x, u)] v(de) \bigg\} dt - \Phi(t, x)dW(t) - \int_E \Psi(t, e, x)\tilde{\mu}(dt, de),
\end{aligned}
\]

where the non-local operators \( L \) and \( I \) are, respectively, defined by

\[ L(t, x, u, \varphi, \psi) \triangleq \int_E \left(I\varphi(t, e, x, u) + \psi(t, e, x + g(t, e, x, u))\right) l(t, e) v(de), \]

and

\[ I\varphi(t, e, x, u) \triangleq \varphi(t, x + g(t, e, x, u)) - \varphi(t, x). \]

The above Eq. (4.2) is the stochastic HJB equation with jumps associated with Problem (2.2), whose solution consists of a triplet of random fields \((V, \Phi, \Psi)\).

Remark 4.1 When all the mappings involved in the state equation (2.2) and the cost functional (2.3) are deterministic, the value function \( V(t, x) \), i.e., the first component of the triplet \((V, \Phi, \Psi)\), becomes a deterministic function with respect to \((t, x)\), and the corresponding stochastic HJB equation degenerates to a deterministic nonlinear second-order partial differential equation, i.e., \( \Phi = 0 \) and \( \Psi = 0 \).

Now we give the definition of the predictable classical solution to the stochastic HJB Eq. (4.2).
Definition 4.1 A triplet of random fields \((V, \Phi, \Psi)\) is called a predictable classical solution to the stochastic HJB Eq. (4.2) if

(i) for each \(x \in \mathbb{R}^n\), \((t, \omega) \rightarrow V(t, \omega, x)\) is an adapted càdlàg process and for almost all \((t, \omega) \in T \times \Omega\), \(x \rightarrow V(t, x, \omega)\) is twice continuously differentiable;
(ii) for each \(x \in \mathbb{R}^n\), \((t, \omega) \rightarrow \Phi(t, \omega, x)\) is a predictable process and for almost all \((t, \omega) \in T \times \Omega\), \(x \rightarrow \Phi(t, x, \omega)\) is continuously differentiable;
(iii) for each \(x \in \mathbb{R}^n\), \((t, \omega, e) \rightarrow \Psi(t, \omega, e, x)\) is a \(\mathcal{P} \otimes \mathcal{B}(E)\)-measurable random field and for almost all \((t, \omega, e) \in T \times \Omega \times E\), \(x \rightarrow \Psi(t, x, \omega)\) is continuous;
(iv) the triplet of random fields \((V, \Phi, \Psi)\) satisfies (4.2), for all \((t, x) \in T \times \mathbb{R}^n\) a.s..

Proposition 4.1 Let Assumptions 2.1–2.4 be satisfied. Suppose that the value function \(V(t, x)\) of Problem 2.2 (see (3.16)) can be written as a semimartingale of the following form:

\[
V(t, x) = h(x) + \int_t^T \Gamma(s, x)ds - \int_t^T \Phi(s, x)dW(s) - \int_t^T \int_E \Psi(s, e, x)\tilde{\mu}(ds, de), \quad (t, x) \in T \times \mathbb{R}^n,
\]

where \((V, \Phi, \Psi)\) is a given triplet of random fields satisfying the regular conditions (i)-(iii) in Definition 4.1 and the random field in the drift term, i.e., \((t, \omega, x) \rightarrow \Gamma(t, \omega, x)\), is a \(\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n)\) measurable mapping. Assume that these four random fields satisfy the following regularity conditions:

(a) \(V, \Phi, \Psi, \Gamma\), and their involved partial derivatives with respect to \(x\) are continuous in \(x \in \mathbb{R}^n\);
(b) there exists a predictable process \(L \in M_{\mathcal{F}}^{2,2}(0, T; \mathbb{R})\) such that

\[
|V(t, x)| + |\Gamma(t, x)| + |\Phi(t, x)| + \int_E |\Psi(t, e, x)|\nu(de) \leq L_t(1 + |x|^2),
\]
\[
|DV(t, x)| + |D\Gamma(t, x)| + |D\Phi(t, x)| + \int_E |D\Psi(t, e, x)|\nu(de) \leq L_t(1 + |x|),
\]

and

\[
|D^2V(t, x)| \leq L_t.
\]

If, in addition, for each \((t, x)\), the optimal control \(u^{*,t,x}\) exists, then \((V, \Phi, \Psi)\) is a classical solution to the stochastic HJB Eq. (4.2).

Proof To prove \((V, \Phi, \Psi)\) is a classical solution to the stochastic HJB Eq. (4.2), by Definition 4.1, we only need to show the following equality holds for all \((t, x) \in [0, T] \times \mathbb{R}^n\) a.s.,
\[\Gamma(t, x) = \text{ess inf}_{u \in U} \left\{ H(t, x, u, DV(t, x), \Phi(t, x), D\Phi(t, x), D^2V(t, x), \mathcal{L}(t, x, u, V, \Psi)) + \int_E \mathcal{I}V(t, e, x, u) - (g(t, e, x, u), DV(t, x))v(de) + \int_E \mathcal{I}\Psi(t, e, x, u)v(de) \right\},\]

which implies that Condition (iv) in Definition 4.1 holds.

Let \(X_{0,x;u}^{0} \) be the state process corresponding to the control \(u(\cdot) \in \mathcal{V}[0, T]\) for Problem \((D_{0,x})\). Whenever there is no risk of confusion, we abbreviate \(X_{0,x;u}^{0}\) as \(X\). Applying the Itô-Ventzell formula to the value function (see [6] for Itô-Ventzell formula with jump processes), we obtain

\[
V(t + \delta, X(t + \delta)) - V(t, X(t)) = -\int_t^{t+\delta} \Gamma(s, X(s-))ds + \int_t^{t+\delta} \Phi(s, X(s-))dW(s) + \int_t^{t+\delta} \int_E \Psi(s, X(s-) + g(s, e, X(s-), u(s)))\tilde{\mu}(ds, de) + \int_t^{t+\delta} \mathcal{A}^uV(s, X(s-))ds + \int_t^{t+\delta} \left( DV(s, X(s-)), \sigma(s, X(s-), u(s))dW(s) \right) + \int_t^{t+\delta} \int_E \mathcal{I}V(s, e, X(s-), u(s))\tilde{\mu}(ds, de) + \int_t^{t+\delta} \int_E \mathcal{I}(s, e, X(s-), u(s)) - (DV(s, X(s-)), g(s, e, X(s-), u(s)))v(de)ds + \int_t^{t+\delta} \mathcal{I}(s, e, X(s-), u(s))v(de)ds,
\]

with

\[
\mathcal{A}^uV \triangleq (DV, b(t, x, u)) + \frac{1}{2} \text{Tr}[D^2V\sigma\sigma^\top(t, x, u)].
\]

On the other hand, consider the following BSDE:

\[
\begin{cases}
    dY(s) = -f(s, X(s), u(s), Y(s), Z(s)) + \int_E K(s, e)l(s, e)v(de)ds + Z(s)dW(s) + \int_E K(s, e)\tilde{\mu}(ds, de), \\
    Y(t + \delta) = V(t + \delta, X(t + \delta)).
\end{cases}
\]
From the dynamic programming principle (3.17), it holds that \( V(t, X(t)) \leq Y(t) \).

Define

\[
F(t, x, u) \triangleq - \Gamma(t, x) + H(t, x, u, DV(t, x), \Phi(t, x), D\Phi(t, x)) \\
+ \int_E \left[ TV(t, e, x, u) - (g(t, e, x, u), DV(t, x)) \right] v(de) \\
+ \int_E \left[ T\Psi(t, e, x, u) \right] v(de).
\]

Let

\[
Z'(s) \triangleq \Phi(s, X(s-)) + \sigma(s, X(s-), u(s))^\top DV(s, X(s-)),
\]

and

\[
K'(s, e) \triangleq TV(s, e, X(s-), u(s)) + \Psi(s, e, X(s-) + g(s, e, X(s-), u(s))).
\]

Then, we can see that \( V(s, X(s)) \) satisfies the following BSDE

\[
dV(s, X(s)) = - \left[ f(s, X(s), u(s), V(s, X(s)), Z'(s), \int_E K'(s, e)l(s, e)v(de)) \\
- F(s, X(s), u(s)) \right] ds + Z'(s)dW(s) + \int_E K'(s, e)\tilde{\mu}(ds, de).
\]

Following the comparison principle for jump BSDEs (see [1, Proposition 2.6]), we derive that

\[
0 \leq Y(t) - V(t, X(t)) = \mathbb{E} \left[ \int_t^{t+\delta} \xi_s F(s, X(s), u(s))ds \middle| \mathcal{F}_t \right],
\]

where \( \xi_s \) is the solution to the following linear SDE

\[
d\xi_s = \alpha_s \xi_s ds + \beta_s \xi_s dW(s) + \int_E \gamma(s, e)\xi_s - \tilde{\mu}(ds, de), \quad \xi_t = 1,
\]

with the coefficients \( \alpha, \beta, \) and \( \gamma \) being bounded processes. Their bounds are determined by the Lipschitz constants of \( f \) and the bounds on \( l \). Then, from the classical estimates for SDEs, we have that

\[
\mathbb{E} \left[ |\xi_s - 1|^2 \middle| \mathcal{F}_t \right] \leq C |t - s|. \tag{4.4}
\]
To highlight the dependence on $t$ and $\delta$, we denote $\xi$ as $\xi^{t,\delta}$. Then, we claim that for any $t$ and $\delta$,

$$\mathbb{E}\left[\int_t^{t+\delta} F(s, X(s), u(s))ds \bigg| \mathcal{F}_t\right] \geq 0, \text{ a.s..} \quad (4.5)$$

To see this, for fixed $t$ and $\delta$, similar to the above arguments we have that for any $n$ and $k \leq n$,

$$\mathbb{E}\left[\int_{t+\frac{k}{n}\delta}^{t+\frac{k+1}{n}\delta} \xi^{t+\frac{k}{n}\delta,\frac{\delta}{n}} F(s, X(s), u(s))ds \bigg| \mathcal{F}_t\right] \geq 0. \quad (4.6)$$

Then, from (4.6), we have

$$\mathbb{E}\left[\int_{t+\frac{k}{n}\delta}^{t+\frac{k+1}{n}\delta} F(s, X(s), u(s))ds \bigg| \mathcal{F}_t\right] = \mathbb{E}\left[\int_{t+\frac{k}{n}\delta}^{t+\frac{k+1}{n}\delta} \xi^{t+\frac{k}{n}\delta,\frac{\delta}{n}} F(s, X(s), u(s))ds \bigg| \mathcal{F}_t\right]
+ \mathbb{E}\left[\int_{t+\frac{k}{n}\delta}^{t+\frac{k+1}{n}\delta} (1 - \xi^{t+\frac{k}{n}\delta,\frac{\delta}{n}}) F(s, X(s), u(s))ds \bigg| \mathcal{F}_t\right]
\geq \mathbb{E}\left[\int_{t+\frac{k}{n}\delta}^{t+\frac{k+1}{n}\delta} (1 - \xi^{t+\frac{k}{n}\delta,\frac{\delta}{n}}) F(s, X(s), u(s))ds \bigg| \mathcal{F}_t\right].$$

Summing over $k$ in the above inequality, we have

$$\mathbb{E}\left[\int_t^{t+\delta} F(s, X(s), u(s))ds \bigg| \mathcal{F}_t\right] \geq \sum_{k=0}^{n-1} \mathbb{E}\left[\int_{t+\frac{k}{n}\delta}^{t+\frac{k+1}{n}\delta} (1 - \xi^{t+\frac{k}{n}\delta,\frac{\delta}{n}}) F(s, X(s), u(s))ds \bigg| \mathcal{F}_t\right]. \quad (4.7)$$

By Hölder’s inequality, we have

$$\left| \mathbb{E}\left[\int_{t+\frac{k}{n}\delta}^{t+\frac{k+1}{n}\delta} (1 - \xi^{t+\frac{k}{n}\delta,\frac{\delta}{n}}) F(s, X(s), u(s))ds \bigg| \mathcal{F}_t\right]\right| \leq \left( \mathbb{E}\left[\int_{t+\frac{k}{n}\delta}^{t+\frac{k+1}{n}\delta} (1 - \xi^{t+\frac{k}{n}\delta,\frac{\delta}{n}})^2ds \bigg| \mathcal{F}_t\right]\right)^{1/2}
\left( \mathbb{E}\left[\int_{t+\frac{k}{n}\delta}^{t+\frac{k+1}{n}\delta} |F(s, X(s), u(s))|^2ds \bigg| \mathcal{F}_t\right]\right)^{1/2}.$$
Thus,

\[
\sum_{k=0}^{n-1} E\left[ \int_{t+\frac{k+1}{n} \delta}^{t+\frac{k+1}{n} \delta} \left( 1 - \xi_s^{t+\frac{k+1}{n} \delta, \frac{s}{n}} \right) F(s, X(s), u(s)) ds \bigg| \mathcal{F}_t \right] \leq C \int_{t+\frac{k+1}{n} \delta}^{t+\frac{k+1}{n} \delta} (s - t + \frac{k}{n} \delta) ds = C \frac{\delta^2}{n^2}.
\]

This implies that

\[
\sum_{k=0}^{n-1} E\left[ \int_{t+\frac{k+1}{n} \delta}^{t+\frac{k+1}{n} \delta} \left( 1 - \xi_s^{t+\frac{k+1}{n} \delta, \frac{s}{n}} \right)^2 ds \bigg| \mathcal{F}_t \right] \leq \frac{\delta^2}{n}.
\]

Letting \( n \to \infty \), we can see that the right hand side of (4.7) converges to 0. This leads to (4.5). For fixed \( t \in [0, T] \) and any nonnegative random variable \( \eta \in \mathcal{F}_t \), it follows from (4.4) that

\[
E\left[ \eta E\left[ \int_{t}^{T} F(s, X(s), u(s)) \eta I_{(t, t+\delta)}(s) ds \bigg| \mathcal{F}_t \right] \right] \geq 0.
\]

Consequently, for any nonnegative simple process \( \phi \in M^2_{\mathcal{F}}(0, T; \mathbb{R}) \), we have that

\[
E\left[ \int_{0}^{T} F(s, X(s), u(s)) \phi_s ds \right] \geq 0.
\]

For any nonnegative process \( \psi \in M^2_{\mathcal{F}}(0, T; \mathbb{R}) \), there exists a sequence of nonnegative simple processes \( \phi^n \in M^2_{\mathcal{F}}(0, T; \mathbb{R}) \), \( n \in \mathbb{N} \), such that

\[
\lim_{n \to \infty} E\left[ \int_{0}^{T} |\phi^n_s - \psi_s|^2 ds \right] = 0.
\]
Hence,

\[
\lim_{n \to \infty} \left| \mathbb{E} \left[ \int_0^T F(s, X(s), u(s)) \phi^n_s ds \right] - \mathbb{E} \left[ \int_0^T F(s, X(s), u(s)) \psi_s ds \right] \right| \leq \lim_{n \to \infty} \left( \mathbb{E} \left[ \int_0^T |F(s, X(s), u(s))|^2 ds \right] \right)^{1/2} \left( \mathbb{E} \left[ \int_0^T |\phi^n_s - \psi_s|^2 ds \right] \right)^{1/2} = 0,
\]

which implies that

\[
\mathbb{E} \left[ \int_0^T F(s, X(s), u(s)) \psi_s ds \right] \geq 0.
\]

Noting the arbitrariness of the nonnegative process \(\psi\), we have that

\[
F(s, X(s), u(s)) \geq 0, \quad \text{for a.e. } s \in [0, T], \text{ a.s.}
\]

Given an admissible control \(u\), let \(X^x(s)\) be the stochastic flow generated by the SDE (2.2) with the initial condition \(X(0) = x\). From Lemma 3.2, with probability 1, for each \(s\), \(X(s)\) is a diffeomorphism of class \(C^1\). For each \(x_i\), we also have that

\[
F(s, X^x(s), u(s)) \geq 0, \quad \text{for a.e. } s \in \mathcal{T}, \text{ a.s.}
\]

Thus, it holds that for all \(x_i\),

\[
F(s, X^x(s), u(s)) \geq 0, \quad \text{for a.e. } s \in \mathcal{T}, \text{ a.s.}
\]

Since \(F(s, x, u)\) and \(X^x\) are continuous with respect to \(x\), we obtain that for all \(x\),

\[
F(s, X^x(s), u(s)) \geq 0, \quad \text{for a.e. } s \in \mathcal{T}, \text{ a.s.}
\]

From the growth condition of the coefficients and the value function, we see that

\[
|F(t, X^x(t), u(t))|^2 \leq C(1 + L_t^2)(1 + |X^x(t)|^4).
\]

Then,

\[
\mathbb{E} \left[ \int_0^T |F(t, X^x(t), u(t))|^2 dt \right] \\
\leq C \mathbb{E} \left[ \int_0^T (1 + L_t^2)(1 + |X^x(t)|^4) dt \right] \\
\leq C \mathbb{E} \left[ \sup_{0 \leq t \leq T} (1 + |X^x(t)|^4) \int_0^T (1 + L_t^2) dt \right]
\]

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\[ \leq C \left( \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} (1 + |X^x(t)|^4) \right)^{2} \right] \right)^{1/2} \left( \mathbb{E} \left[ (\int_0^T (1 + L^2_r)dt)^{2} \right] \right)^{1/2} \leq C (1 + |x|^4). \]

Now, let \( \phi \) be a smooth function such that
\[
\phi(x) = \begin{cases} 
1, & \text{for } |x| \leq 1; \\
0, & \text{for } |x| \geq 2; \\
\in [0, 1], & \text{otherwise.} 
\end{cases}
\]

For \( s \in [0, T] \), define \( \tilde{X} \) to be the inverse function of \( X \) and consider a random function
\[
g(s, x) \triangleq \xi(\tilde{X}^x(s))\varphi\left(\frac{x}{N}\right)\varphi\left(\frac{\tilde{X}^x(s)}{N}\right) |\det \partial_{\tilde{X}^x} \tilde{X}^x(s)|^{-1} p_s,
\]
where \( N \in \mathbb{N} \), \( p \) is an arbitrary non-negative, bounded, adapted process, and \( \xi \) is a smooth non-negative function with a compact support. We first show that \( \mathbb{E} [\int_0^T \int_{\mathbb{R}}^n F(s, \tilde{X}^x(s))g(s, x)dxds] \) is integrable. By Hölder’s inequality, it holds that
\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}}^n |F(s, \tilde{X}^x(s), u(s))g(s, x)|dxds \right] 
\leq \left( \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}}^n |F(s, \tilde{X}^x(s), u(s))|^2 \varphi\left(\frac{x}{N}\right)dxds \right] \right)^{1/2} 
\times \left( \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}}^n |\det \partial_{\tilde{X}^x} \tilde{X}^x(s)|^{-2} \xi^2(\tilde{X}^x(s))\varphi\left(\frac{x}{N}\right)\varphi^2\left(\frac{\tilde{X}^x(s)}{N}\right)p_s^2 dxds \right] \right)^{1/2}.
\]

For the first term on the right hand side, we have
\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}}^n |F(s, \tilde{X}^x(s), u(s))|^2 \varphi\left(\frac{x}{N}\right)dxds \right] 
\leq \int_{|x| \leq N+2} \mathbb{E} \left[ \int_0^T |F(s, \tilde{X}^x(s), u(s))|^2 ds \right] dx < \infty.
\]

Note that \( \tilde{X} \tilde{X}^x(s) = x \). Hence \( \partial_y \tilde{X}^y(s)|_{y=\tilde{X}^x(s)} \partial_x \tilde{X}^x(s) = I \), and thus \( |\det \partial_x \tilde{X}^x(s)|^{-1} = |\det \partial_y \tilde{X}^y(s)|_{y=\tilde{X}^x(s)} |. \) For the second term, it holds that

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denote by \( F \) the corresponding optimal control. From dynamic programming principle (3.17), we see that the equality holds in (4.5) when we replace the arbitrary control \( \xi \). This confirms that \( E \) this implies that Letting \( N \leq E \), the above inequality reduces to

\[
0 \leq E \left[ \int_0^T \int_{\mathbb{R}^n} F(s, X^x(s), u(s)) g(s, x) dx ds \right]
\]

\[
= E \left[ \int_0^T \int_{\mathbb{R}^n} F(s, X^x(s), u(s)) \xi(x) \varphi \left( \frac{x}{N} \right) \varphi \left( \frac{\hat{X}^x(s)}{N} \right) \partial_x \hat{X}^x(s) \right] ds dx ds
\]

\[
= E \left[ \int_0^T \int_{\mathbb{R}^n} F(s, x, u(s)) \xi(x) \varphi \left( \frac{\hat{X}^x(s)}{N} \right) \varphi \left( \frac{\hat{X}^x(s)}{N} \right) \partial_x \hat{X}^x(s) \right] ds dx ds
\]

Letting \( N \to +\infty \), the above inequality reduces to

\[
E \left[ \int_0^T \int_{\mathbb{R}^n} F(s, x, u(s)) \xi(x) p_x dx ds \right] \geq 0.
\]

From the arbitrariness of \( \xi \), \( p \), and \( u \), we have that

\[
\inf_u F(s, x, u) \geq 0, \quad \text{for all } x, ds \times \mathbb{P}-\text{a.s..} \quad (4.8)
\]

Next, we show that the equality holds. Given any \((t, x) \in T \times \mathbb{R}^n\), let \( u^{*_{t,x}} \) be the corresponding optimal control. From dynamic programming principle (3.17), we see that the equality holds in (4.5) when we replace the arbitrary control \( u \) with the optimal control \( u^{*_{t,x}} \). Following previous arguments, we see that

\[
F(s, X^{u^{*_{t,x}}}(s); u^{*_{t,x}}(s)) = 0, \quad \text{for a.e. } s \in [t, T], \text{a.s..}
\]

Denote by \( F(s, x) \triangleq \inf_u F(s, x, u) \). Then, we see that

\[
F(s, x, 0) \geq F(s, x) \geq 0.
\]

This implies that

\[
|F(s, x)| \leq |F(s, x, 0)| \leq CL_t(1 + |x|^2),
\]
which further yields that $F(\cdot, x) \in M_{\mathcal{F}}^{2,1}(0, T; \mathbb{R})$, for any $x$. Let $\zeta(t)$ be a mollifier defined on $[0, +\infty)$, i.e.,

$$
\zeta(t) = \begin{cases} 
C e^{-\frac{1}{1-t^2}}, & \text{if } t \leq 1, \\
0, & \text{otherwise}, 
\end{cases}
$$

with the constant $C$ chosen so that $\int_0^\infty \zeta(t) dt = 1$ and $\zeta_n(t) = n \zeta(nt)$. Define

$$
F_n(s, x) \triangleq \int_0^\infty \zeta_n(\tau) F(s + \tau, x) d\tau.
$$

We have that as $n \to +\infty$,

$$
\mathbb{E} \left[ \int_0^T F_n(s, x) ds \right] \to \mathbb{E} \left[ \int_0^T F(s, x) ds \right]. \quad (4.9)
$$

Note that

$$
F_n(s, x) = \int_0^\infty \zeta_n(\tau) F(s + \tau, x) d\tau \\
\leq \int_0^\infty \zeta_n(\tau) F(s + \tau, x, u^{*,s,x}(s + \tau)) d\tau \\
= \int_0^\infty \zeta_n(\tau) \left( F(s + \tau, x, u^{*,s,x}(s + \tau)) \\
- F(s + \tau, X^{u^{*,s,x},x}(s + \tau), u^{*,s,x}(s + \tau)) \right) d\tau.
$$

For simplicity, we abbreviate $(X^{u^{*,s,x},x}, u^{*,s,x})$ as $(X^*, u^*)$. From the assumptions of the current proposition, we have

$$
|F(s + \tau, x, u^{*,s,x}) - F(s + u, X^{u^{*,s,x},x}(s + \tau), u^{*,s,x}(s + \tau))| \\
\leq C L_{s+\tau} (1 + |x| + |X^*(s + \tau)| + |u^*(s + \tau)||X^*(s + \tau) - x|.
$$

Hence,

$$
\mathbb{E} \left| \int_0^\infty \zeta_n(\tau) (F(s + \tau, x, u^{*,s,x}) - F(s + \tau, X^{u^{*,s,x},x}(s + \tau), u^{*,s,x}(s + \tau))) d\tau \right| \\
\leq C \left( \mathbb{E} \left[ \int_0^\infty \zeta_n(\tau) L_{s+\tau} (1 + |x| + |X^*(s + \tau)| + |u^*(s + \tau)||X^*(s + \tau) - x|)^2 d\tau \right] \right)^{1/2} \\
\times \left( \mathbb{E} \left[ \int_0^\infty \zeta_n(\tau) L_{s+\tau} X^*(s + \tau) - x|^2 d\tau \right] \right)^{1/2} \\
\leq C \left( \mathbb{E} \left[ \int_0^\infty \zeta_n(\tau) L_{s+\tau}^2 d\tau \right] \right)^{1/2} \left( \mathbb{E} \left[ \int_0^\infty \zeta_n(\tau) (1 + |x| + |X^*(s + \tau)|)^2 d\tau \right] \right)^{1/2}.
$$
\[ +|u^*(s + \tau)|^4 d\tau \right)^{1/4} \]
\[ \times \left( \mathbb{E} \left[ \int_0^\infty \xi_n(\tau)|X^*(s + \tau) - x|^4 d\tau \right] \right)^{1/4}. \]

Then, we see that for all \( s \),
\[ \mathbb{E} \left[ \int_0^\infty \xi_n(\tau)|X^*(s + \tau) - x|^4 d\tau \right] \rightarrow 0, \]
and
\[ \mathbb{E} \left[ \int_0^\infty \xi_n(\tau)(1 + |x| + |X^*(s + \tau)| + |u^*(s + \tau)|)^4 d\tau \right] \]
is uniformly bounded with respect to \( n \). Moreover, it holds that, for almost \( s \) and as \( n \rightarrow \infty \),
\[ \mathbb{E} \left[ \int_0^\infty \xi_n(\tau)L_{s+\tau}^2 d\tau \right] \rightarrow \mathbb{E} \left[ L_s^2 \right]. \]

Hence,
\[ \lim \inf_{n \rightarrow \infty} \mathbb{E} [F_n(s, x)] \leq 0, \]
for almost \( s \). From (4.9), we have
\[ \mathbb{E} \left[ \int_0^T F(s, x) ds \right] \leq 0. \]
Combining with the fact that \( F(s, x) \geq 0 \), we obtain that
\[ F(s, x) = 0. \]
The proof is completed. \( \square \)

Next, we prove the verification theorem. That is, a classical solution to the stochastic HJB equation is the value function and characterizes the optimal control. The statement of this result is heavy, but its proof is standard and relies essentially on the Itô-Ventzell formula.

**Theorem 4.2** Let Assumptions 2.1–2.4 be satisfied. Suppose that a triplet of random fields \((\varphi, \phi, \psi)\) is a classical solution to the stochastic HJB equation (4.2), i.e.,
\[ \varphi(t, x) = h(x) + \int_t^T \inf_{u \in U} \left\{ H(s, x, u, D\varphi(s, x), \phi(t, x), D\phi(s, x), D^2\varphi(s, x), \right\}, \]
\[ (4.10) \]
Let \( z \) is an optimal control, i.e., \( V \)

\[
\int_E (I \varphi(s, e, x, u) + \psi(s, e, x + g(t, e, x, u))) l(s, e) v(de)
+ \int_E \left[ I \varphi(s, e, x, u) - (g(s, e, x, u), D \varphi(s, x)) \right] v(de)
+ \int_E \left[ I \psi(s, e, x, u) \right] v(de) ds
- \int_t^T \phi(s, x) dW(s) - \int_t^T \int_E \psi(s, e, x) \mu(ds, de),
\]

(4.11)

and satisfies the regularity condition (b) in Proposition 4.1. Moreover, for almost all \((t, \omega, x) \in [0, T] \times \Omega \times \mathbb{R}^n\), the infimum in (4.10) is achieved at a \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^k) \)-measurable random field \( \bar{u} : (t, \omega, x) \to \bar{u}(t, \omega, x) \) taking values in \( U \) such that for any given initial state \( X(t) = x \), the following feedback control system

\[
\begin{aligned}
dX(s) &= b(s, X(s), \bar{u}(s, X(s))) ds + \sigma(s, X(s), \bar{u}(s, X(s))) dW(s) \\
&\quad + \int_E g(s, e, X(s), \bar{u}(s, X(s)))) \mu(ds, de), \quad t \leq s \leq T, \\
X(t) &= x,
\end{aligned}
\]

(4.12)

has a unique strong solution \( \tilde{X}(\cdot) \) and \((\tilde{u}(\cdot), \tilde{X}(\cdot))\) is an admissible pair. Then \( \phi(t, x) = V(t, x) \) for all \((t, x) \in [0, T] \times \mathbb{R}^n \) a.s., and the feedback control \( \tilde{u}(\cdot, \tilde{X}(\cdot)) \) is an optimal control, i.e., \( V(t, x) = J(t, \tilde{u}(\cdot, \tilde{X}(\cdot))) \).

**Proof** Let \((v(\cdot), z(\cdot))\) be an arbitrary admissible control pair of Problem \((D_{t,x})\). That is, \( z(\cdot) \) solves the following stochastic differential equation:

\[
\begin{aligned}
dz(s) &= b(s, z(s), v(s)) ds + \sigma(s, z(s), v(s)) dW(s) + \int_E g(s, e, z(s), v(s))) \mu(ds, de), \quad t \leq s \leq T, \\
z(t) &= x.
\end{aligned}
\]

(4.13)

Define

\[
\Delta(s, x, u) \triangleq H(s, x, u, D \varphi(s, x), \phi(s, x), D \phi(s, x), D^2 \varphi(s, x),
\int_E (I \varphi(s, e, x, u) + \psi(s, e, x + g(s, e, x, u))) l(s, e) v(de)
+ \int_E \left[ I \varphi(s, e, x, u) - (g(s, e, x, u), D \varphi(s, x)) \right] v(de)
+ \int_E \left[ I \psi(s, e, x, u) \right] v(de).
\]

By applying the Itô-Ventzell formula to the random field \( \varphi(\cdot, x) \) and the state process \( z(\cdot) \) (see (4.10) and (4.13)), we get
From the comparison principle for BSDEs with jumps, the above inequality leads to

\[
\varphi(T, z(T)) = \varphi(t, x) - \int_t^T \inf_u \Delta(s, z(s), u) ds + \int_t^T (D\varphi(s, z(s)), b(s, z(s), v(s))) ds
\]

\[+ \int_t^T (D\phi(s, z(s)), \sigma(s, z(s), v(s))) ds + \frac{1}{2} \int_t^T \text{Tr}[D^2\varphi(s, z(s))\sigma\sigma^T(s, z(s), v(s))] ds
\]

\[+ \int_t^T \int_E \left[ \varphi(s, z(s) + g(s, e, z(s), v(s))) - \varphi(s, z(s)) - (g(s, e, z(s), v(s)), D\varphi(s, z(s))) \right] \nu(de)
\]

\[+ \int_t^T \int_E \left[ \varphi(s, z(s) + g(s, e, z(s), v(s))) - \varphi(s, z(s)) - (g(s, e, z(s), v(s)), D\varphi(s, z(s))) \right] \nu(de) ds
\]

\[+ \int_t^T \int_E \left[ \varphi(s, z(s) + g(s, e, z(s), v(s))) - \varphi(s, z(s)) - (g(s, e, z(s), v(s)), D\varphi(s, z(s))) \right] \nu(de) ds
\]

\[= \varphi(t, x) + \int_t^T \left[ \Delta(s, z(s), v(s)) - \inf_u \Delta(s, z(s), u) \right] ds
\]

\[+ \int_t^T f(s, x(s), z(s), \sigma^T(s, z(s), v(s))) D\varphi(s, z(s)) + \phi(s, z(s)) ds
\]

\[+ \int_t^T (I\varphi(s, e, z(s), v(s)) + \psi(s, e, z(s) - g(s, e, z(s), u(s))) I(s, e) v(ds)) ds
\]

\[+ \int_t^T \left[ \sigma^T(s, z(s), v(s)) D\varphi(s, z(s)) + \phi(s, z(s)) \right] dW(s)
\]

\[+ \int_t^T \int_E \left[ I\varphi(s, e, z(s), v(s)) + \psi(s, e, z(s) - g(s, e, z(s), u(s))) I(s, e) \right] \tilde{\nu}(ds, de)
\]

(4.14)

From the comparison principle for BSDEs with jumps, the above inequality leads to

\[
\varphi(t, x) \leq G_{t,T}^{x;u(\cdot)}[\varphi(T, z(T))] = G_{t,T}^{x;u(\cdot)}[h(z(T))] = J(t, x; u(\cdot)).
\]

(4.15)

Since \(v(\cdot)\) is arbitrary, taking the infimum in (4.15) gives

\[
\varphi(t, x) \leq V(t, x), \quad \text{a.s.}
\]

(4.16)

Finally, again applying the Itô-Ventzell formula to the random field \(\varphi(\cdot, x)\) (see (4.10)) and the state process \(\tilde{X}(\cdot)\) associated with the feedback control \(\tilde{u}(\cdot, \tilde{X}(\cdot))\) and taking conditional expectation with \(\mathcal{F}_t\), we obtain the equality in (4.14), thereby

\[
\varphi(t, x) = J(t, x; \tilde{u}(\cdot, \tilde{X}(\cdot))).
\]

(4.17)
Therefore, from (4.16) together with the definition of the value function \( V(t, x) \) (see (3.3)), we have

\[
V(t, x) \leq J(t, x; \bar{u}(\cdot, \bar{X}(\cdot))) = \varphi(t, x) \leq V(t, x).
\]

Consequently, we conclude that \( \varphi(t, x) \) coincides with the value function \( V(t, x) \) and \((\bar{u}(\cdot, \bar{X}(\cdot)), \bar{X}(\cdot))\) is an optimal pair. \( \square \)

5 Backward Stochastic Evolution Equations with Jumps

As in the deterministic case, the classical solution to the stochastic HJB equation does not exist in general cases. Thus, this section is devoted to the existence and uniqueness result for the stochastic HJB equation with jumps in the sense of Sobolev spaces. To this end, we need to recast the stochastic HJB equation with jumps as a class of backward stochastic evolution equations with jumps in Hilbert spaces. We refer readers to [8] for the general theory of stochastic evolution equations in Hilbert spaces.

5.1 Backward Stochastic Evolution Equations with Jumps

We first introduce the framework of a Gelfand triple under which the backward stochastic evolution equation is studied. The Brownian motion \( B \) and the Poisson random measure \( \tilde{\mu} \) are defined the same as in previous sections.

Let \( V \) and \( H \) be two separable (real) Hilbert spaces such that \( V \) is densely embedded in \( H \). The space \( H \) is identified with its dual space by the Riesz mapping. Then we can take \( H \) as a pivot space and get a Gelfand triple \( V \subset H = H^* \subset V^* \), where \( H^* \) and \( V^* \) denote the dual spaces of \( H \) and \( V \), respectively. Denote by \( \| \cdot \|_V, \| \cdot \|_H, \) and \( \| \cdot \|_{V^*} \) the norms of \( V, H, \) and \( V^* \), respectively, by \( \langle \cdot, \cdot \rangle_H \) the inner product in \( H \), and by \( \langle \cdot, \cdot \rangle \) the duality product between \( V \) and \( V^* \). Moreover, we write \( \mathcal{L}(V, V^*) \) the space of bounded linear transformations of \( V \) into \( V^* \).

Now we recall a version of Itô’s formula in Hilbert space which is frequently used in this section (see [15] for the proof).

**Lemma 5.1** Let \( \varphi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H) \). Let \( Y, Z, \) and \( \Gamma \) be three progressively measurable stochastic processes defined on \( T \times \Omega \) with values in \( V, H, \) and \( V^* \) such that \( Y \in S^2(0, T; V), Z \in M^2(0, T; H), \) and \( \Gamma \in M^2(0, T; V^*) \), respectively. Let \( R \) be a \( \mathcal{P} \otimes \mathcal{B}(E) \)-measurable stochastic process defined on \( T \times \Omega \times E \) with values in \( H \) such that \( R \in M_{\mathcal{P}}^{0, 2}(0, T; H) \). Suppose that for every \( \eta \in V \) and almost every \( (t, \omega) \in T \times \Omega \), it holds that

\[
(\eta, Y(t))_H = (\eta, \varphi)_H + \int_0^t (\eta, \Gamma(s))ds + \int_0^t (\eta, Z(s))_H dW(s)
\]

\[
+ \int_0^t \int_E (\eta, R(s,e))_H \tilde{\mu}(ds, de).
\]
Then, $Y$ is an $H$-valued strongly càdlàg $\mathcal{F}_t$-predictable process, satisfying
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} ||Y(t)||^2_H \right] \leq \infty,
\]
and the following Itô’s formula holds for the squared $H$-norm of $Y$:
\[
||Y(t)||^2_H
= ||\varphi||^2 + 2 \int_0^t \langle \Gamma(s), Y(s) \rangle ds + 2 \int_0^t \langle Z(s), Y(s) \rangle dW(s) + \int_0^t ||Z(s)||^2_H ds + \int_0^t \int_{\Omega} [||R(s, e)||^2_H + 2(Y(s), R(s, e))] \, \tilde{\mu}(ds, de) + \int_0^t \int_{\Omega} ||R(s, e)||^2_H v(de) ds.
\]
(5.1)

Here ’strongly’ means that the left and right limits of $Y(t)$ are taken in the strong topology of $H$, i.e., the norm topology.

Now we introduce a backward stochastic evolution equation with jumps (BSEEJ) in the Gelfand triple $(V, H, V^*)$ of the following form:
\[
dY(t) = \left[ A(t)Y(t) + B(t)Z(t) + F(t, Y(t), Z(t), R(t, \cdot)) \right] dt
+ Z(t)dW(t) + \int_{\Omega} R(t, e) \mu(dt, de), \quad t \in [0, T],
\]
(5.2)

where the coefficients $(A, B, F, \xi)$ are given mappings such that $A : [0, T] \times \Omega \to \mathcal{L}(V, V^*)$ is $\mathcal{P}/\mathcal{B}(\mathcal{L}(V, V^*))$-measurable; $B : [0, T] \times \Omega \to \mathcal{L}(H, V^*)$ is $\mathcal{P}/\mathcal{B}(\mathcal{L}(H, V^*))$-measurable; $F : [0, T] \times \Omega \times V \times H \times M^{\nu,2}(\nu; H) \to H$ is $\mathcal{P} \otimes \mathcal{B}(V) \otimes \mathcal{B}(H) \otimes \mathcal{B}(M^{\nu,2}(\nu; H))/\mathcal{B}(H)$-measurable; $\xi : \Omega \to H$ is $\mathcal{F}_T$-measurable. Furthermore, we assume that the coefficients $(A, B, F, \xi)$ satisfy the following conditions:

**Assumption 5.1**

(i) $F(\cdot, 0, 0, 0) \in M^2_{\mathbb{P}}(0, T; H)$ and $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$;

(ii) the operators $A$ and $B$ satisfy the super-parabolic condition, i.e., there exist constants $\alpha > 0$ and $\lambda$ such that
\[
2 \langle A(t)\phi, \phi \rangle + \lambda ||\phi||^2_H \geq \alpha ||\phi||^2_V + ||B^*(t)\phi||^2_H, \quad \forall t \in [0, T], \; \forall \phi \in V;
\]
(5.3)

(iii) the operators $A$ and $B$ are uniformly bounded, i.e., there exists a constant $C > 0$ such that
\[
\sup_{(t, \omega) \in [0, T] \times \Omega} ||A(t, \omega)||_{\mathcal{L}(V, V^*)} + \sup_{(t, \omega) \in [0, T] \times \Omega} ||B(t, \omega)||_{\mathcal{L}(H, V^*)} \leq C; \quad (5.4)
\]
(iv) $F$ is uniformly Lipschitz continuous in $(y, z, r)$, i.e., there exists a constant $C > 0$ such that for all $(y, z, r), (\tilde{y}, \tilde{z}, \tilde{r}) \in V \times H \times M^{v,2}(E; H)$ and a.e. $(t, \omega) \in \mathcal{T} \times \Omega$, 

$$
\|F(t, y, z, r) - F(t, \tilde{y}, \tilde{z}, \tilde{r})\|_H^2 \leq C (\|y - \tilde{y}\|_V^2 + \|z - \tilde{z}\|_H^2 + \|r - \tilde{r}\|_{M^{v,2}(E; H)}^2).
$$

(5.5)

For any set of $(A, B, F, \xi)$ satisfying Assumption 5.1, we call it a generator of the BSEEJ (5.2).

**Definition 5.1** A $V \times H \times M^{v,2}(E; H)$-valued, $\mathbb{P}$-predictable process $(Y(\cdot), Z(\cdot), R(\cdot, \cdot))$ is called a solution to the BSEEJ (5.2), if $(Y(\cdot), Z(\cdot), R(\cdot, \cdot)) \in M^2_{\mathcal{F}}(0, T; V) \times M^2_{\mathcal{F}}(0, T; H) \times M^{v,2}_{\mathcal{F}}(0, T; H)$ and, for every $\phi \in V$, it holds that

$$(Y(t), \phi)_H = (\xi, \phi)_H - \int_t^T \left\{ A(s)Y(s) + B(s)Z(s) + F(s, Y(s), Z(s), R(s, \cdot)), \phi \right\} dt$$

$$- \int_t^T (Z(s), \phi)_H dW(s) - \int_t^T \int_E (R(s, e), \phi)_H \tilde{\mu}(ds, de), \quad \text{for a.e. } t \in \mathcal{T}, \text{a.s.,}$$

(5.6)

or alternatively, $(Y(\cdot), Z(\cdot), R(\cdot, \cdot))$ satisfies the following Itô’s equation in $V^*$:

$$Y(t) = \xi - \int_t^T \left[ A(s)Y(s)ds + B(s)Z(s) + F(s, Y(s), Z(s), R(s, \cdot)) \right] ds - \int_t^T Z(s)dW(s) - \int_t^T \int_E R(s, e)\tilde{\mu}(ds, de), \quad t \in \mathcal{T}.$$  

(5.7)

**Theorem 5.2** (Continuous Dependence Theorem) If $(Y(\cdot), Z(\cdot), R(\cdot, \cdot))$ is the solution to the BSEEJ (5.2) corresponding to the generator $(A, B, F, \xi)$, then the following estimate holds:

$$E \left[ \sup_{0 \leq s \leq T} \|Y(t)\|_H^2 \right] + E \left[ \int_0^T \|Y(t)\|_V^2 dt \right]$$

$$+ E \left[ \int_0^T \|Z(t)\|_H^2 dt \right]$$

$$+ E \left[ \int_0^T \int_E \|R(t, e)\|_H^2 v(de) dt \right]$$

$$\leq K \left\{ E[\|\xi\|_H^2] \right\}$$

$$+ E \left[ \int_0^T \|F(t, 0, 0, 0)\|_H^2 dt \right],$$

(5.8)
where \( K \triangleq K(T, C, \alpha, \lambda) \) is a positive constant depending only on \( T, C, \alpha, \) and \( \lambda. \) Moreover, if \((\check{Y}(\cdot), \check{Z}(\cdot), \check{R}(\cdot, \cdot))\) is a solution to the BSEEJ (5.2) corresponding to another generator \((A, B, \bar{F}, \bar{\xi})\), then we have

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \|Y(t) - \check{Y}(t)\|_H^2 \right] + \mathbb{E}\left[ \int_0^T \|Y(t) - \check{Y}(t)\|_V^2 \, dt \right] \\
+ \mathbb{E}\left[ \int_0^T \|Z(t) - \check{Z}(t)\|_H^2 \, dt \right] + \mathbb{E}\left[ \int_0^T \int_E \|R(t, e) - \check{R}(t, e)\|_H^2 \, \nu(de) \, dt \right] \\
\leq K \left\{ \mathbb{E}[\|\xi - \bar{\xi}\|_H^2] \\
+ \mathbb{E}\left[ \int_0^T \|F(t, \check{Y}(t), \check{Z}(t), \check{R}(t, \cdot)) - \bar{F}(t, \check{Y}(t), \check{Z}(t), \check{R}(t, \cdot))\|_H^2 \, dt \right] \right\}.
\]

**Proof** If we take the generator \((A, B, \bar{F}, \bar{\xi}) = (A, B, 0, 0),\) then the corresponding solution to the BSEEJ (5.2) is \((\check{Y}(\cdot), \check{Z}(\cdot), \check{R}(\cdot, \cdot)) = (0, 0, 0).\) Hence, the estimate (5.8) follows from (5.9) immediately. Therefore, it suffices to prove (5.9). To simplify our notations, we denote by

\[
\hat{Y}(t) \triangleq Y(t) - \check{Y}(t), \quad \hat{Z}(t) \triangleq Z(t) - \check{Z}(t), \quad \hat{R}(t, e) \triangleq R(t, e) - \check{R}(t, e), \quad \hat{\xi} \triangleq \xi - \bar{\xi},
\]

\[
\hat{F}(t) \triangleq F(t, \check{Y}(t), \check{Z}(t), \check{R}(t, \cdot)) - \bar{F}(t, \check{Y}(t), \check{Z}(t), \check{R}(t, \cdot)), \quad \hat{\bar{F}}(t) \triangleq \bar{F}(t, Y(t), Z(t), R(t, \cdot)) - \bar{F}(t, \check{Y}(t), \check{Z}(t), \check{R}(t, \cdot)).
\]

From Lemma 5.1, we obtain

\[
\|\hat{Y}(t)\|_H^2 = \|\hat{\xi}\|^2 - 2 \int_t^T \langle A(s)\hat{Y}(s) + B(s)\hat{Z}(s) + \hat{F}(s), \hat{Y}(s) \rangle \, ds \\
- 2 \int_t^T \langle \hat{Z}(s), \hat{Y}(s) \rangle_H \, dW(s) - \int_t^T \|\hat{Z}(s)\|_H^2 \, ds \\
- \int_t^T \int_E \left[ \|\hat{R}(s, e)\|_H^2 + 2\langle \hat{Y}(s), \hat{R}(s, e) \rangle_H \right] \mu(ds, de) \\
- \int_t^T \int_E \|\hat{R}(s, e)\|_H^2 \, \nu(de) \, ds.
\]

Using the inequality \(2ab \leq \frac{1}{\epsilon} a^2 + \epsilon b^2, \forall a, b > 0, \epsilon > 0,\) we have

\[
- 2 \langle B(s)\hat{Z}(s), \hat{Y}(s) \rangle = -2\langle \hat{Z}(s), B^*(s)\hat{Y}(s) \rangle_H \\
\leq \frac{1}{1 + \epsilon_1} \|Z(s)\|_H^2 + (1 + \epsilon_1)\|B^*(s)\hat{Y}(s)\|_H^2, \\
- 2\langle \hat{F}(s), \hat{Y}(s) \rangle \leq \epsilon_2 \|\hat{F}(s)\|_H^2 + \frac{1}{\epsilon_2} \|Y(s)\|_H^2
\]
\[ \int_{\mathcal{H}} \varphi(t) \, d\mu(t) = C \varepsilon_1 \left( \| \hat{Y}(s) \|^2_V + \| \hat{Z}(s) \|^2_H + \| \hat{R}(s, \cdot) \|^2_{M^p(E; H)} + \frac{1}{\varepsilon_2} \| Y(s) \|^2_H \right), \]

and

\[-2 \langle \hat{F}(s), \hat{Y}(s) \rangle \leq \| \hat{Y}(s) \|^2_H + \| \hat{F}(s) \|^2_H. \]

Then, it implies that

\[
\frac{\| \hat{Y}(t) \|^2_H}{\| \hat{F}(s) \|^2_H} \leq \| \hat{Y}(s) \|^2_H - \frac{C \varepsilon_2}{1 + \varepsilon_1} \| \hat{Z}(s) \|^2_H - \frac{1}{\varepsilon_2} \| \hat{R}(s, \cdot) \|^2_{M^p(E; H)} ds \\
- 2 \int_t^T \langle \hat{Z}(s), \hat{Y}(s) \rangle dW(s) - \int_t^T \int_E \| \hat{R}(s, e) \|^2_H + 2 \langle \hat{Y}(s), \hat{R}(s, e) \rangle_H \tilde{\mu}(ds, de). 
\]

From Assumption 5.1, we have that

\[-2 \langle A(s) \hat{Y}(s), \hat{Y}(s) \rangle + (1 + \varepsilon_1) \| B^*(s) Y(s) \|^2_H \leq 2 \varepsilon_1 \langle A(s) \hat{Y}(s), \hat{Y}(s) \rangle + \lambda (1 + \varepsilon_1) \| \hat{Y}(s) \|^2_H - \alpha (1 + \varepsilon_1) \| \hat{Y}(s) \|^2_V \leq \lambda (1 + \varepsilon_1) \| \hat{Y}(s) \|^2_H - (\alpha + \alpha \varepsilon_1 - 2 C \varepsilon_1) \| \hat{Y}(s) \|^2_V. \]

Hence, we get

\[
\frac{\| \hat{Y}(t) \|^2_H}{\| \hat{F}(s) \|^2_H} + \int_t^T \left( 1 - \frac{C \varepsilon_2}{1 + \varepsilon_1} \| \hat{Z}(s) \|^2_H - \frac{1}{\varepsilon_2} \| \hat{R}(s, \cdot) \|^2_{M^p(E; H)} \right) ds \\
\leq \int_t^T \left( 1 + \frac{1}{\varepsilon_2} + \lambda (1 + \varepsilon_1) \right) \| \hat{Y}(s) \|^2_H - (\alpha + \alpha \varepsilon_1 - 2 C \varepsilon_1 - C \varepsilon_2) \| \hat{Y}(s) \|^2_V \\
+ \| \hat{F}(s) \|^2_H ds + 2 \int_t^T \langle \hat{Z}(s), \hat{Y}(s) \rangle dW(s) + \int_t^T \int_E \| \hat{R}(s, e) \|^2_H \\
+ 2 \langle \hat{Y}(s), \hat{R}(s, e) \rangle_H \tilde{\mu}(ds, de). \quad (5.10) 
\]

From the integrability condition of the solution in Definition 5.1, we know that \( \int_{\mathcal{H}} \varphi(t) \, d\mu(t) \) is a uniformly integrable martingale. Moreover, \( \int_t^T \| \hat{R}(s, e) \|^2_H + 2 \langle \hat{Y}(s), \hat{R}(s, e) \rangle_H \tilde{\mu}(ds, de) \) is also a uniformly integrable martingale. Taking expectations on both sides, we have
Using (5.10) and the Burkholder-Davis-Gundy inequality yields
\[
\mathbb{E} \left[ \| \hat{Y}(t) \|^2_H + \int_t^T \left(1 - \frac{1}{1 + \varepsilon_1} - C_{\varepsilon_2} \right) \| \hat{Z}(s) \|^2_H \right.
\]
\[
\left. + (1 - C_{\varepsilon_2}) \| \hat{R}(s, \cdot) \|^2_{M^{\frac{1}{2}}(E; H)} ds \right]
\]
\[
\leq \mathbb{E} \left[ \int_t^T \left(1 + \frac{1}{\varepsilon_2} + \lambda(1 + \varepsilon_1) \right) \| \hat{Y}(s) \|^2_H - (\alpha + \alpha \varepsilon_1 - 2C_{\varepsilon_1} - C_{\varepsilon_2}) \| \hat{Y}(s) \|^2_V
\]
\[
+ \| \hat{F}(s) \|^2_H ds \right].
\]

Choosing sufficiently small \( \varepsilon_1 \) and \( \varepsilon_2 \) such that
\[1 - \frac{1}{1 + \varepsilon_1} - C_{\varepsilon_2} > 0, \quad 1 - C_{\varepsilon_2} > 0, \quad \text{and} \quad \alpha + \alpha \varepsilon_1 - 2C_{\varepsilon_1} - C_{\varepsilon_2} > 0,\]
we finally get that
\[
\mathbb{E}[\| \hat{Y}(t) \|^2_H] + \mathbb{E} \left[ \int_t^T \| \hat{Y}(s) \|^2_V ds \right] + \mathbb{E} \left[ \int_t^T \| \hat{Z}(s) \|^2_H ds \right]
\]
\[
+ \mathbb{E} \left[ \int_t^T \int_E \| \hat{R}(s, e) \|^2_H V(de) ds \right]
\]
\[
\leq K(\lambda, \alpha) \left\{ \mathbb{E}[\| \hat{\xi} \|^2_H] + \mathbb{E} \left[ \int_0^T \| \hat{F}(s) \|^2_H ds \right] \right\}. \quad (5.11)
\]

Then using Grönwall’s inequality to (5.11) gives
\[
\sup_{0 \leq t \leq T} \mathbb{E}[\| \hat{Y}(t) \|^2_H] + \mathbb{E} \left[ \int_0^T \| \hat{Y}(t) \|^2_V dt \right] + \mathbb{E} \left[ \int_0^T \| \hat{Z}(t) \|^2_H dt \right]
\]
\[
+ \mathbb{E} \left[ \int_0^T \int_E \| \hat{R}(s, e) \|^2_H V(de) ds \right]
\]
\[
\leq K(\alpha, \lambda) \left\{ \mathbb{E}[\| \hat{\xi} \|^2_H] + \mathbb{E} \left[ \int_0^T \| \hat{F}(t) \|^2_H dt \right] \right\}. \quad (5.12)
\]

Using (5.10) and the Burkholder-Davis-Gundy inequality yields
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \| \hat{Y}(t) \|^2_H \right]
\]
\[
\leq K(\alpha, \lambda) \left\{ \mathbb{E}[\| \hat{\xi} \|^2_H] + \mathbb{E} \left[ \int_0^T \| \hat{Y}(t) \|^2_H dt \right] + \mathbb{E} \left[ \int_0^T \| \hat{F}(t) \|^2_H dt \right] \right\}
\]
\[
+ 2 \mathbb{E} \left[ \sup_{0 \leq t \leq T} \int_t^T (\| \hat{Y}(s), \hat{Z}(s) \|_H dW(s)) \right]
\]
\[
+ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \int_t^T \int_E \left( \| \hat{R}(s, e) \|^2_H + 2(\hat{Y}(s), \hat{R}(s, e))_H \right) \mu(ds, de) \right]
\]
\[
\leq K(\alpha, \lambda) \left\{ \mathbb{E}[\| \hat{\xi} \|^2_H] + \mathbb{E} \left[ \int_0^T \| \hat{Y}(t) \|^2_V dt \right] + \mathbb{E} \left[ \int_0^T \| \hat{F}(t) \|^2_H dt \right] \right\}.
\[ + \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| \hat{Y}(t) \|_H^2 \right] \\
+ K \mathbb{E} \left[ \int_0^T \| \hat{Z}(t) \|_H^2 dt \right] + K \mathbb{E} \left[ \int_0^T \int_E \| \hat{R}(s, e) \|_H^2 v(de) ds \right], \tag{5.13} \]

where the last inequality is obtained due to the fact that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_t^T (\hat{Y}(s), \hat{Z}(s))_H dW(s) \right| \right] \leq C \mathbb{E} \left[ \left( \int_0^T |(\hat{Y}(s), \hat{Z}(s))_H|^2 ds \right)^{1/2} \right] \\
\leq C \mathbb{E} \left[ \sup_{0 \leq s \leq T} \| \hat{Y}(s) \|_H \left( \int_0^T |Z(s)|_H^2 ds \right)^{1/2} \right] \\
\leq C \mathbb{E} \left[ \varepsilon \sup_{0 \leq s \leq T} \| \hat{Y}(s) \|_H^2 + \frac{1}{\varepsilon} \int_0^T |Z(s)|_H^2 ds \right]
\]

and

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_t^T \int_E \| \hat{R}(s, e) \|_H^2 + 2(\hat{Y}(s), \hat{R}(s, e))_H \right| \tilde{\mu}(ds, de) \right] \leq C \mathbb{E} \left[ \int_0^T \int_E \| \hat{R}(s, e) \|_H^2 + 2(\hat{Y}(s), \hat{R}(s, e))_H \right] v(de) ds \\
\leq C \mathbb{E} \left[ \varepsilon \sup_{0 \leq s \leq T} \| \hat{Y}(s) \|_H^2 + \left( 1 + \frac{1}{\varepsilon} \right) \int_0^T \int_E \| R(s, e) \|_H^2 v(de) ds \right].
\]

Combining (5.13) with (5.12) gives the desired result (5.9). \hfill \Box

**Theorem 5.3** (Existence and uniqueness theorem of BSEEJ) *Given a generator \((A, B, F, \xi)\) satisfying Assumption 5.1, the BSEEJ (5.2) has a unique solution \((Y(\cdot), Z(\cdot), R(\cdot, \cdot)) \in \mathcal{S}^2_{\mathbb{F}}(0, T; V) \times \mathcal{M}^2_{\mathbb{F}}(0, T; H) \times \mathcal{M}^{V,2}_{\mathbb{F}}(0, T; H)).*

To prove this result, we first consider a simple case when \(F\) is independent of \((Y, Z, R)\). To be more precise, we consider a BSEEJ with \(F\) replaced by a \(V^*\)-valued \(\mathbb{F}\)-predictable process \(F_0\) as follows:

\[
Y(t) = \xi - \int_t^T [A(s)Y(s) + B(s)Z(s) + F_0(s)] ds - \int_t^T Z(s) dW(s) \\\n- \int_t^T \int_E R(s, e) \tilde{\mu}(ds, de). \tag{5.14}
\]

Now we state the existence and uniqueness result of a solution to the BSEEJ (5.14).
Lemma 5.4 Suppose that the coefficients \((A, B, F_0, \xi)\) satisfy Assumption 5.1. Then the BSEEJ (5.14) has a unique solution in the sense of Definition 5.1.

Proof First of all, we fix a standard complete orthogonal basis \(\{e_i | i = 1, 2, 3, \ldots\}\) in the space \(H\) which is dense in the space \(V\). For any \(n\), consider the following finite-dimensional backward stochastic differential equation in \(\mathbb{R}^n\):

\[
y^n_i(t) = (\xi, e_i) - \int_t^T \left( \sum_{j=1}^n y^n_j(s) A(s) e_j, e_i \right) ds + \int_t^T \sum_{j=1}^n z^n_j(s) B(s) e_j, e_i - (F_0(s), e_i)_H \right) ds
- \int_t^T z^n_i(s) dW(s) + \int_t^T \int_E r^n_i(s, e) \tilde{\mu}(ds, de), \quad i = 1, 2, \ldots, n. \tag{5.15}
\]

Under Assumption 5.1, from the existence and uniqueness theory for the finite-dimensional BSDE with jumps, the above equation admits a unique strong solution \((y^n(\cdot), z^n(\cdot), r^n(\cdot, \cdot))\) such that

\[
(y^n(\cdot), z^n(\cdot), r^n(\cdot, \cdot)) \in S^2_{\mathbb{F}}(0, T; \mathbb{R}^n) \times M^2_{\mathbb{F}}(0, T; \mathbb{R}^n) \times M^{v, 2}_{\mathbb{F}}(0, T; \mathbb{R}^n),
\]

where \(y^n(\cdot) \triangleq (y^n_1(\cdot), \ldots, y^n_n(\cdot))\), \(z^n(\cdot) \triangleq (z^n_1(\cdot), \ldots, z^n_n(\cdot))\), and \(r^n(\cdot) \triangleq (r^n_1(\cdot), \ldots, r^n_n(\cdot))\).

Now we can define an approximation solution to (5.14) as follows: \(Y^n(\cdot) \triangleq \sum_{i=1}^n y^n_i e_i, Z^n(\cdot) \triangleq \sum_{i=1}^n z^n_i e_i, F^n(\cdot) \triangleq \sum_{i=1}^n (F(\cdot), e_i)_H e_i, R^n(\cdot) \triangleq \sum_{i=1}^n r^n_i e_i, \) and \(\xi^n \triangleq \sum_{i=1}^n (\xi, e_i)_H e_i\). Then, from Eq. (5.15), we see that

\[
(Y^n(t), e_i)_H = (\xi^n, e_i)_H
- \int_t^T \left( \langle A(s) Y^n(s), e_i \rangle + \langle B(s) Z^n(s), e_i \rangle + (F^n(s), e_i)_H \right) ds
- \int_t^T (Z^n(s), e_i)_H dW(s) - \int_t^T \int_E (R^n(s, e), e_i)_H \tilde{\mu}(ds, de), \quad i = 1, 2, \ldots, n. \tag{5.16}
\]

Now applying Itô formula to \(\|Y^n(t)\|_H^2\), we get

\[
\|Y^n(t)\|_H^2 = \|\xi^n\|^2 - 2 \int_t^T \langle A(s) Y^n(s) + B(s) Z^n(s) + F^n(s), Y^n(s) \rangle ds
- \int_t^T \|Z^n(s)\|_H^2 ds - \int_t^T \int_E \|R^n(s, e)\|_H^2 v(de) ds - 2 \int_t^T \langle Z^n(s), Y^n(s) \rangle dW(s)
- \int_t^T \int_E (\|R^n(s, e)\|_H^2 + 2\langle Y^n(s), R^n(s, e) \rangle_H) \tilde{\mu}(ds, de). \tag{5.17}
\]

Therefore, under Assumption 5.1, similar to the proof of the estimate (5.9), using Grönwall’s inequality and the Burkholder-Davis-Gundy inequality, we can easily get

\[\square\] Springer
the following estimate:

\[
\begin{align*}
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \|Y^n(t)\|_H^2 \right] &+ \mathbb{E}\left[ \int_0^T \|Y^n(t)\|_V^2 dt \right] \\
+ \mathbb{E}\left[ \int_0^T \|Z^n(t)\|_H^2 dt \right] &+ \mathbb{E}\left[ \int_0^T \int_E \|R^n(t, e)\|_H^2 v(de) dt \right] \\
\leq K \left\{ \mathbb{E}[\|\xi^n\|_H^2] + \mathbb{E}\left[ \int_0^T \|F^n(t)\|_H^2 dt \right] \right\} &\leq K \left\{ \mathbb{E}[\|\xi\|_H^2] + \mathbb{E}\left[ \int_0^T \|F(t)\|_H^2 dt \right] \right\}.
\end{align*}
\]

(5.18)

This inequality implies that there is a subsequence \( \{n'\} \) of \( \{n\} \) and a triplet \((Y(\cdot), Z(\cdot), R(\cdot, \cdot)) \in M^2_{\mathcal{F}}(0, T; V) \times M^2_{\mathcal{F}}(0, T; H) \times M^{v, 2}_{\mathcal{F}}(0, T; H) \) such that \( Y^n \to Y \) weakly in \( M^2_{\mathcal{F}}(0, T; V) \), \( Z^n \to Z \) weakly in \( M^2_{\mathcal{F}}(0, T; H) \), and \( R^n \to R \) weakly in \( M^{v, 2}_{\mathcal{F}}(0, T; H) \), respectively. Let \( \Pi \) be an arbitrary bounded random variable on \((\Omega, \mathcal{F})\) and \( \psi \) be an arbitrary bounded measurable function on \([0, T]\). From the equality (5.16), for any \( n \in \mathbb{N}^* \) and basis \( e_i \), where \( i \leq n \), we have

\[
\begin{align*}
\mathbb{E}\left[ \int_0^T \Pi \psi(t)(Y^n(t), e_i)_H dt \right] &= \mathbb{E}\left[ \int_0^T \Pi \psi(t) \left\{ \langle \xi^n, e_i \rangle_H - \int_t^T (A(s) Y^n(s), e_i) ds \\
&\quad + \langle B(s) Y^n(s), e_i \rangle + (F(s), e_i)_H ds \right\} dt \right] \\
&\quad - \int_t^T (Z^n(s), e_i)_H dW(s) - \int_t^T \int_E (R^n(s, e), e_i)_H \tilde{\mu}(ds, de) \right\} dt \right].
\end{align*}
\]

(5.19)

Now let \( n' \to \infty \) on the both sides of the above equation to get its limit. Firstly, from the weak convergence property of \( \{Y^n\}_{n=1}^{\infty} \) in \( M^2_{\mathcal{F}}(0, T; V) \), we have

\[
\begin{align*}
\lim_{n' \to \infty} \mathbb{E}\left[ \int_0^T \Pi \psi(t)(Y^{n'}(t), e_i)_H dt \right] &= \lim_{n' \to \infty} \mathbb{E}\left[ \int_0^T \mathbb{E}[\Pi|\mathcal{F}_t] \psi(t)(Y^{n'}(t), e_i)_H dt \right] \\
&= \lim_{n' \to \infty} \mathbb{E}\left[ \int_0^T (Y^{n'}(t), \mathbb{E}[\Pi|\mathcal{F}_t] \psi(t)e_i)_H dt \right] \\
&= \mathbb{E}\left[ \int_0^T (Y(t), \mathbb{E}[\Pi|\mathcal{F}_t] \psi(t)e_i)_H dt \right] \\
&= \mathbb{E}\left[ \int_0^T \Pi \psi(t)(Y(t), e_i) dt \right],
\end{align*}
\]

(5.20)
and
\[
\lim_{n' \to \infty} \mathbb{E} \left[ \int_t^T \Pi \langle A(s) Y_{n'}(s), e_i \rangle ds \right] = \lim_{n' \to \infty} \mathbb{E} \left[ \int_t^T \mathbb{E}[\Pi \mid \mathcal{F}_s] \langle A(s) Y_{n'}(s), e_i \rangle ds \right] \\
= \lim_{n' \to \infty} \mathbb{E} \left[ \int_t^T \langle A(s) Y_{n'}(s), \mathbb{E}[\Pi \mid \mathcal{F}_s] e_i \rangle ds \right] \\
= \lim_{n' \to \infty} \mathbb{E} \left[ \int_t^T \langle Y_n(s), A^*(s) \mathbb{E}[\Pi \mid \mathcal{F}_s] e_i \rangle ds \right] \\
= \mathbb{E} \left[ \int_0^T \langle Y(s), A^*(s) \mathbb{E}[\Pi \mid \mathcal{F}_s] e_i \rangle ds \right] \\
= \mathbb{E} \left[ \int_0^T \Pi \langle A(s) Y(s), e_i \rangle ds \right], \quad (5.21)
\]
where the orders of integration, expectation, and limit can be exchanged due to the integrability of related processes. More precisely, in view of (5.4) and (5.18), we conclude that
\[
\mathbb{E} \left[ \left| \int_t^T \Pi \langle A(s) Y_{n'}(s), e_i \rangle ds \right| \right] \leq C \left\{ \mathbb{E} \left[ \int_0^T ||Y_{n'}(s)||_V^2 ds \right] \right\}^{1/2} < C < \infty,
\]
where the constant \( C \) is independent of \( n' \). Hence from Fubini’s Theorem and Lebesgue’s Dominated Convergence Theorem, we have
\[
\mathbb{E} \left[ \int_0^T \Pi \psi(t) \int_t^T \langle A(s) Y_{n'}(s), e_i \rangle ds dt \right] \\
= \int_0^T \psi(t) \mathbb{E} \left[ \int_t^T \Pi \langle A(s) Y_{n'}(s), e_i \rangle ds \right] dt \\
\to \int_0^T \psi(t) \mathbb{E} \left[ \int_t^T \Pi \langle A(s) Y(s), e_i \rangle ds \right] dt. \quad (5.22)
\]
Similarly, we have
\[
\mathbb{E} \left[ \int_0^T \Pi \psi(t) \int_t^T \langle B(s) Y_{n'}(s), e_i \rangle ds dt \right] \\
\to \mathbb{E} \left[ \int_0^T \Pi \psi(t) \int_t^T \langle B(s) Y(s), e_i \rangle ds dt \right]. \quad (5.23)
\]

It is easy to see that \( M^2_{\mathcal{F}}(t, T; \mathbb{R}) \times M^{\nu,2}_{\mathcal{F}}(t, T; \mathbb{R}) \) and \( L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) \) are Hilbert spaces equipped with both strong and weak topologies. Since the stochastic integrals with respect to the Brownian motion \( W \) and the Poisson random martingale measure \( \tilde{\mu} \) are linear and continuous mappings in the strong topology from \( M^2_{\mathcal{F}}(t, T; \mathbb{R}) \times M^{\nu,2}_{\mathcal{F}}(t, T; \mathbb{R}) \) to \( L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) \), respectively, they are also continuous in the weak
topology. Therefore, it follows from the weak convergence property of $Z^n$ and $R^n$ that

$$
\lim_{n' \to \infty} E \left[ \prod_{t} \left( \int_{t}^{T} (Z^n(s), e_i)_H dW(s) + \int_{t}^{T} \int_{E} (R^n(s, e), e_i)_H \tilde{\mu}(ds, de) \right) \right] = E \left[ \prod_{t} \left( \int_{t}^{T} (Z(s), e_i)_H dW(s) + \int_{t}^{T} \int_{E} (R(s, e), e_i)_H \tilde{\mu}(ds, de) \right) \right].
$$

(5.24)

Moreover,

$$
\phi(t) E \left[ \prod_{t} \left( \int_{t}^{T} (Z^n(s), e_i)_H dW(s) + \int_{t}^{T} \int_{E} (R^n(s, e), e_i)_H \tilde{\mu}(ds, de) \right) \right] \\
\leq \frac{1}{2} \phi^2(t) E |\prod|^2 + C \left\{ E \left[ \int_{0}^{T} \|Z^n(s)\|_H^2 dt \right] \\
+ E \left[ \int_{0}^{T} \int_{E} \|R^n(s, e)\|_H^2 \nu(de) dt \right] \right\} \leq C. 
$$

(5.25)

Hence, by Fubini’s Theorem and Lebesgue’s Dominated Convergence Theorem, we have

$$
\lim_{n' \to \infty} E \left[ \int_{0}^{T} \phi(t) \prod_{t} \left( \int_{t}^{T} (Z^n(s), e_i)_H dW(s) + \int_{t}^{T} \int_{E} (R^n(s, e), e_i)_H \tilde{\mu}(ds, de) \right) dt \right] = \lim_{n' \to \infty} \int_{0}^{T} \phi(t) E \left[ \prod_{t} \left( \int_{t}^{T} (Z^n(s), e_i)_H dW(s) + \int_{t}^{T} \int_{E} (R^n(s, e), e_i)_H \tilde{\mu}(ds, de) \right) \right] dt \\
= \int_{0}^{T} \phi(t) E \left[ \prod_{t} \left( \int_{t}^{T} (Z(s), e_i)_H dW(s) + \int_{t}^{T} \int_{E} (R(s, e), e_i)_H \tilde{\mu}(ds, de) \right) \right] dt \\
= E \int_{0}^{T} \left[ \phi(t) \prod_{t} \left( \int_{t}^{T} (Z(s), e_i)_H dW(s) + \int_{t}^{T} \int_{E} (R(s, e), e_i)_H \tilde{\mu}(ds, de) \right) \right] dt. 
$$

(5.26)
Therefore, combining (5.20),(5.22), (5.23), and (5.26), and letting \( n' \to \infty \) in (5.19), we conclude that

\[
\begin{align*}
\mathbb{E} \left[ \int_0^T \Pi \psi(t) (Y(t), e_i)_H dt \right] &= \mathbb{E} \left[ \int_0^T \Pi \psi(t) \left\{ (\xi(t), e_i)_H - \int_t^T \left( \langle A(s)Y(s), e_i \rangle + \langle B(s)Y(s), e_i \rangle + (F(s), e_i)_H \right) ds \\
&\quad - \int_t^T (Z(s), e_i)_H dW(s) - \int_t^T \int_E (R(s, e), e_i)_H \tilde{\mu}(ds, de) \right\} dt \right].
\end{align*}
\]

(5.27)

This implies that for a.s. \((t, \omega) \in [0, T] \times \Omega, \)

\[
\begin{align*}
(Y(t), e_i)_H &= (\xi(t), e_i)_H \\
&\quad - \int_t^T \left( \langle A(s)Y(s), e_i \rangle + \langle B(s)Y(s), e_i \rangle + (F(s), e_i)_H \right) ds \\
&\quad - \int_t^T (Z(s), e_i)_H dW(s) - \int_t^T \int_E (R(s, e), e_i)_H \tilde{\mu}(ds, de).
\end{align*}
\]

(5.28)

thanks to the arbitrariness of \( \Pi \) and \( \psi(\cdot) \). Since the standard complete orthogonal basis \( \{e_i| i = 1, 2, 3, \ldots \} \) in \( H \) is dense in the space \( V \), it holds that for every \( \phi \in V \) and a.e. \((t, \omega) \in [0, T] \times \Omega, \)

\[
\begin{align*}
(Y(t), \phi)_H &= (\xi(t), \phi)_H \\
&\quad - \int_t^T \left( \langle A(s)Y(s), \phi \rangle + \langle B(s)Y(s), e_i \rangle + (F(s), \phi)_H \right) ds \\
&\quad - \int_t^T (Z(s), \phi)_H dW(s) - \int_t^T \int_E (R(s, e), \phi)_H \tilde{\mu}(ds, de).
\end{align*}
\]

(5.29)

Therefore, from Definition 5.1, we conclude that the triplet \((Y(\cdot), Z(\cdot), R(\cdot, \cdot))\) is the solution to the BSEEJ (5.37). Thus the existence is proved. The uniqueness is an immediate result of Theorem 5.2. \( \square \)

**Proof of Theorem 5.3** We first take an arbitrary process \( F_0(\cdot) \in M^2_{\mathbb{F}}(0, T; H). \) Given that \( \rho \in [0, 1], \) consider the following BSEEJ:

\[
\begin{align*}
Y(t) &= \xi - \int_t^T \left[ A(s)Y(s) + B(s)Z(s) + \rho F(s, Y(s), Z(s), R(s, e)) + F_0(s) \right] ds \\
&\quad - \int_t^T Z(s)dW(s) - \int_t^T \int_E R(s, e) \tilde{\mu}(ds, de).
\end{align*}
\]

(5.30)
Note that the coefficients \((A, B, \rho F + F_0, \xi)\) of the BSEEJ (5.30) satisfy Assumption 5.1 with the same constants \(\alpha, \lambda, C\). If we can prove that the BSEEJ (5.30) admits a unique solution for any \(\rho\) and \(F_0\), then setting \(\rho = 1\) and \(F_0(\cdot) \equiv 0\) yields that the BSEEJ (5.42) has a unique solution \((Y(\cdot), Z(\cdot), R(\cdot, \cdot)) \in S^2(0, T; V) \times M^2_\mathcal{F}(0, T; H) \times M^{v,2}_\mathcal{F}(0, T; H)\).

Suppose for some \(\rho = \rho_0\), the BSEEJ (5.30) has a unique solution \((Y(\cdot), Z(\cdot), R(\cdot, \cdot)) \in M^2(0, T; V) \times M^2(0, T; H) \times M^{v,2}(0, T; H)\), for any \(F_0(\cdot) \in M^2_\mathcal{F}(0, T; H)\). For another \(\rho\), the BSEEJ (5.30) can be rewritten as

\[
Y(t) = \xi - \int_t^T \left[ A(s)Y(s) + B(s)Z(s) + \rho_0 F(s, Y(s), Z(s), R(s, e)) + F_0(s) \right] \, ds - \int_t^T Z(s) \, dW(s) \\
+ (\rho - \rho_0) F(s, Y(s), Z(s), R(s, e)) \bigg] \, ds - \int_t^T \int_E R(s, e) \tilde{\mu}(ds, de).
\] (5.31)

For any \((y(\cdot), z(\cdot), r(\cdot, \cdot)) \in M^2(0, T; V) \times M^2(0, T; H) \times M^{v,2}(0, T; H)\), the following BSEEJ

\[
Y(t) = \xi - \int_t^T \left[ A(s)Y(s) + B(s)Z(s) + \rho_0 F(s, Y(s), Z(s), R(s, e)) + F_0(s) \right] \, ds \\
+ (\rho - \rho_0) F(s, y(s), z(s), r(s, e)) \bigg] \, ds \\
- \int_t^T Z(s) \, dW(s) - \int_t^T \int_E R(s, e) \tilde{\mu}(ds, de)
\] (5.32)

has a unique solution \((Y(\cdot), Z(\cdot), R(\cdot, \cdot)) \in M^2(0, T; V) \times M^2(0, T; H) \times M^{v,2}(0, T; H)\). Thus, we can define a mapping \(\mathcal{I}\) from \(M^2(0, T; V) \times M^2(0, T; H) \times M^{v,2}(0, T; H)\) onto itself such that \(\mathcal{I}(y(\cdot), z(\cdot), r(\cdot, \cdot)) = (Y(\cdot), Z(\cdot), R(\cdot, \cdot))\). Moreover, we see that \((Y(\cdot), Z(\cdot), R(\cdot, \cdot))\) is a solution to (5.31) if and only if it is a fixed point of \(\mathcal{I}\).

For any \((y_i(\cdot), z_i(\cdot), r_i(\cdot, \cdot)) \in M^2(0, T; V) \times M^2(0, T; H) \times M^{v,2}(0, T; H)\), we can find \((Y_i(\cdot), Z_i(\cdot), R_i(\cdot, \cdot)) \in M^2(0, T; V) \times M^2(0, T; H) \times M^{v,2}(0, T; H)\) through the mapping \(\mathcal{I}(y_i(\cdot), z_i(\cdot), r_i(\cdot)) = (Y_i(\cdot), Z_i(\cdot), R_i(\cdot))\), for \(i = 1, 2\). According to the a priori estimate (5.9) and the Lipschitz continuity of \(F\), we have

\[
\mathbb{E} \left[ \int_0^T \| Y_1(t) - Y_2(t) \|_V^2 \, dt \right] + \mathbb{E} \left[ \int_0^T \| Z_1(t) - Z_2(t) \|_H^2 \, dt \right] \\
+ \mathbb{E} \left[ \int_0^T \| R_1(t, e) - R_2(t, e) \|_H^2 \, v(de) \, dt \right] \\
\leq K |\rho - \rho_0|^2 \mathbb{E} \left[ \int_0^T \| F(t, y_1(t), z_1(t), r_1(t, \cdot)) \|_H \, dt \right]
\]
\[-F(t, y_2(t), z_2(t), r_2(t, \cdot)) \left\| H^2 dt \right]\]
\[\leq K|\rho - \rho_0|^2 \times \left\{ \mathbb{E} \left[ \int_0^T \| y_1(t) - y_2(t) \|^2_H dt \right] + \mathbb{E} \left[ \int_0^T \| z_1(t) - z_2(t) \|^2_H dt \right] + \mathbb{E} \left[ \int_0^T \| r_1(t, e) - r_2(t, e) \|^2_H v(de) dt \right] \right\}, \quad (5.33)\]

where $K \triangleq K(C, \lambda, \alpha, \rho_0)$ is a constant independent of $\rho$. If $|\rho - \rho_0| < \frac{1}{2\sqrt{K}}$, the mapping $I$ is strictly contractive in $M^2_\mathcal{F}(0, T; V) \times M^2_\mathcal{F}(0, T; H) \times M^{\nu,2}_\mathcal{F}(0, T; H)$, which admits a fixed point. Hence, it implies that the BSEEJ (5.30) with the coefficients $(A, B, \rho F + F_0, \xi)$ admits a unique solution $(Y(\cdot), Z(\cdot), R(\cdot, \cdot)) \in M^2_\mathcal{F}(0, T; V) \times M^2_\mathcal{F}(0, T; H) \times M^{\nu,2}_\mathcal{F}(0, T; H)$. From Lemma 5.4, the uniqueness and existence of a solution to the BSEEJ (5.30) is true for $\rho = 0$. Then starting from $\rho = 0$, we have that the BSEEJ (5.30) also admits a unique solution for $\rho \in \left[ \frac{i-1}{2\sqrt{K}}, \frac{i}{2\sqrt{K}} \right)$, $i = 1, 2, \cdots$. Therefore, setting $i = \lfloor 2\sqrt{K} \rfloor + 1$ and $F_0(\cdot) \equiv 0$ leads to that the BSEEJ (5.30) with the coefficients $(A, B, F, \xi)$, i.e., the BSEEJ (5.42), has a unique solution $(Y(\cdot), Z(\cdot), R(\cdot, \cdot)) \in M^2_\mathcal{F}(0, T; V) \times M^2_\mathcal{F}(0, T; H) \times M^{\nu,2}_\mathcal{F}(0, T; H)$. Moreover, from the a priori estimate (5.8), we obtain $Y(\cdot) \in S^2_\mathcal{F}(0, T; V)$. This completes the proof. \hfill \Box

### 5.2 Stochastic HJB Equation

In this subsection, we recast the stochastic HJB equation as a BSEEJ and then establish the existence and uniqueness result of a weak solution in the sense of the Sobolev space. We see that the super-parabolic condition (5.3) is crucial for BSEEJ theory. Due to the limitation of this approach, we can only deal with the special case, in which the coefficient $\sigma$ does not contain the control variable $u$ and a nondegeneracy assumption is imposed on $\sigma$.

Let

$$\sigma(\cdot) = (\sigma_1(\cdot), \sigma_2(\cdot), \cdots, \sigma_{d-1}(\cdot), \sigma_d(\cdot)) \quad (5.34)$$

and

$$\hat{\sigma}(\cdot) = (\sigma_1(\cdot), \sigma_2(\cdot), \cdots, \sigma_{d-1}(\cdot)).$$

Let us define a sub-filtration $\mathcal{G} \triangleq \{ \mathcal{F}(t) \mid t \in T \}$ of $\mathcal{F}$, which is a $\mathbb{P}$-augmentation of the natural filtration generated by the one-dimensional Brownian motion $W_d(\cdot)$ and the Poisson random measure $\tilde{\mu}(\cdot, \cdot)$. We assume that all the coefficients involved in Problem 2.2 are restricted to $\mathbb{G}$-predictable processes or $\mathcal{F}_T$-measurable. The cost
The corresponding stochastic HJB equation is formally given by the following form:

\[
\begin{align*}
\dot{Y}(s) &= - f(s, X^{t, \xi; u}(s), u(s), Y(s), Z(s), \int_E K(s, e) l(s, e) v(de)) ds \\
&+ \sum_{i=1}^d Z_i(s) dW_i(s) + \int_E K(s, e) \hat{\mu}(ds, de), \quad Y(T) = h(X^{t, \xi; u}(T)).
\end{align*}
\]

The corresponding stochastic HJB equation is formally given by the following form:

\[
\begin{cases}
-dV(t, x) = \left\{ \frac{1}{2} u[\sigma \sigma^T (t, x) D^2 V(t, x)] + \sum_{i=1}^d (\sigma_i(t, x), D\Phi_i(t, x)) + \inf_{u \in U} \left[ b(t, x, u), D V(t, x) \right] \\
&+ f(t, x, u, V, \sigma^T D V + \Phi, \int_E (\mathcal{I} V(t, e, x, u) + \Psi(t, e, x + g)) l(t, e) v(de)) \\
&+ \int_E [\mathcal{I} V(t, e, x, u) - (g(t, e, x, u), D V(t, x)) ] v(de) + \int_E \mathcal{I} \Psi(t, e, x, u) v(de)] \right\} dt \\
&- \sum_{i=1}^d \Phi_i(t, x) dW_i(t) - \int_E \Psi(t, e, x) \hat{\mu}(dt, de), \\
V(T, x) &= h(x).
\end{cases}
\]

Since the randomness of the coefficients only comes from \(W_d\) and \(\hat{\mu}\), it holds that \(\Phi_i = 0\) for \(i = 1, 2, \ldots, d - 1\). Thus, the above stochastic HJB equation reduces to

\[
\begin{cases}
-dV(t, x) = \left\{ \frac{1}{2} \text{tr}[\sigma \sigma^T (t, x) D^2 V(t, x)] + \langle \sigma_d(t, x), D\Phi_d(t, x) \rangle + \inf_{u \in U} \left[ b(t, x, u), D V(t, x) \right] \\
&+ f(t, x, u, V, \sigma^T D V + \Phi_d, \int_E (\mathcal{I} V(t, e, x, u) + \Psi(t, e, x + g)) l(t, e) v(de)) \\
&+ \int_E [\mathcal{I} V(t, e, x, u) - (g(t, e, x, u), D V(t, x)) ] v(de) + \int_E \mathcal{I} \Psi(t, e, x, u) v(de)] \right\} dt \\
&- \Phi_d(t, x) dW_d(t) - \int_E \Psi(t, e, x) \hat{\mu}(dt, de), \\
V(T, x) &= h(x).
\end{cases}
\]

Indeed, this is a Cauchy problem for semi-martingale backward stochastic partial differential equations in non-divergence form. Next we rewrite this equation in divergence form. Note

\[
\begin{align*}
\text{tr}[\sigma \sigma^T D^2 V(t, x)] &= \nabla \cdot [\sigma \sigma^T D V(t, x)] - (\nabla \cdot (\sigma \sigma^T))(t, x), \\
\langle \sigma_d, D\Phi_d(t, x) \rangle &= \nabla \cdot [\Phi_d \sigma_d] - \Phi_d \nabla \cdot \sigma_d,
\end{align*}
\]
where (with $\sigma = (\sigma_1, \ldots, \sigma_d)$, each $\sigma_i$ takes values in $\mathbb{R}^n$)
\[
\nabla \cdot \sigma = (\nabla \cdot \sigma_1, \ldots, \nabla \cdot \sigma_d)^\top.
\]

In view of the above reduction, we have the following divergence form of the BSPDE (i.e., stochastic HJB equation):

\[
\begin{aligned}
-dV(t, x) &= \left\{ \frac{1}{2} \nabla \cdot [\sigma \sigma^\top (t, x)DV(t, x)] + \nabla \cdot [\Phi_d(t, x)\sigma_d(t, x)] - (\nabla \cdot [\sigma \sigma^\top (t, x)], DV(t, x)) \\
&\quad - \nabla \cdot [\sigma_d(t, x)\Phi_d(t, x)] + \inf_{u \in U} \left[ \langle b(t, x, u), DV(t, x) \rangle \\
&\quad + f(t, x, u, V, \sigma^\top DV + \Phi_d, \int_E (\mathcal{I}V(t, e, x, u) + \Psi(t, e, x + g)) l(t, e) \nu(de)) \\
&\quad + \int_E [\mathcal{I}V(t, e, x, u) - (g(t, e, x, u), DV(t, x))] \nu(de) + \int_E \mathcal{I}\Psi(t, e, x, u) \nu(de) \right] \right\} dt \\
V(T, x) &= h(x).
\end{aligned}
\]

The following definition gives the generalized weak solution to Eqs (5.36) or (5.37).

**Definition 5.2** A triplet $(V, \Phi, \Psi) \in M_\mathcal{F}^2(0, T; V) \times M_\mathcal{F}^2(0, T; H) \times M_\mathcal{F}^2(0, T; H)$ is called an adapted weak solution to (5.36) or (5.37) if, for every $\phi \in H^1(\mathbb{R}^n)$ and a.e. $(t, \omega) \in [0, T] \times \Omega$, it holds that

\[
\begin{aligned}
\int_{\mathbb{R}^n} V(t, x)\phi(x) dx &= \int_{\mathbb{R}^n} h(x)\phi(x) dx + \int_t^T \int_{\mathbb{R}^n} \left\{ -\frac{1}{2} \sigma \sigma^\top (s, x)DV(s, x) \\
&\quad + \sigma_d(s, x)\Phi_d(s, x), D\phi(x) \right\} dx ds \\
&\quad + \left[ -\left( \nabla \cdot [\sigma \sigma^\top (s, x)], DV(s, x) \right) - \Phi_d(s, x)\nabla \cdot \sigma_d(s, x) \right] dx ds \\
&\quad + \inf_{u \in U} \left[ \langle b(s, x, u), DV(s, x) \rangle \\
&\quad + f(s, x, u, V, \sigma^\top DV + \Phi_d, \int_E (\mathcal{I}V(s, e, x, u) + \Psi(s, e, x + g)) l(s, e) \nu(de)) \\
&\quad + \int_E [\mathcal{I}V(s, e, x, u) - (g(s, e, x, u), DV(s, x))] \nu(de) \\
&\quad + \int_E \mathcal{I}\Psi(s, e, x, u) \nu(de) \right] \phi(s, x) \right\} dx ds \\
&\quad - \int_t^T \int_{\mathbb{R}^n} \Phi_d(s, x)\phi(x) dx dW_d(s) - \int_t^T \int_{\mathbb{R}^n} \Psi(s, e, x)\phi(x) dx \tilde{\mu}(ds, de).
\end{aligned}
\]

(5.38)
Assumption 5.2 The diffusion coefficient $\hat{\sigma}$ is uniformly positive:

$$\hat{\sigma} \hat{\sigma}^T (t, x) \geq 2\alpha I, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n,$$  \tag{5.39}$$

where $\alpha$ is a positive constant.

Next we recall some preliminaries of Sobolev spaces. For $m = 0, 1$, we define the space $H^m = \{ \phi : \partial_x^\alpha \phi \in L^2(\mathbb{R}^n) \}$, for any $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| = |\alpha_1| + \cdots + |\alpha_n| \leq m$ with the norm

$$\|\phi\|_m \triangleq \left\{ \sum_{|\alpha| \leq m} \int_{\mathbb{R}^n} |\partial_x^\alpha \phi(z)|^2 dz \right\}^{\frac{1}{2}}.$$

If we denote by $H^{-1}$ the dual space of $H^1$ and set $V = H^1, H = H^0, V^* = H^{-1}$, then $(V, H, V^*)$ is a Gelfand triple. We further need some assumptions on the coefficients.

Assumption 5.3 There exists a constant $C$ such that

- $|\sigma(t, x, u)| \leq C$, for any $(t, x, u) \in [0, T] \times \mathbb{R}^n \times U$;
- $\|b(t, \cdot, u)\|_H \leq C$ and $\|\int_{\mathbb{R}^d} g(t, e, \cdot, u) v(de)\|_H \leq C$, for any $(t, u) \in [0, T] \times U$;
- $f(\cdot, \cdot, 0, 0) \in M^2_\mathcal{F}(0, T; H)$.

Remark 5.1 Assumptions 5.1 and 5.2 are standard in the study of nonlinear BSPDE, see also [9]. However, we have to admit that it may exclude the Black-Scholes model. Since the Black-Scholes model has a specific structure, it can be studied by other methods. In some cases, it can be transformed to another model, which satisfies the assumptions imposed in the paper, via a change of variable, e.g., $x = \log s$.

With these assumptions, we can apply our previous result about the BSEEJ, which leads to the following theorem.

Theorem 5.5 Let Assumptions 2.1–2.4 and 5.2–5.3 be satisfied. Then the stochastic HJB equation (5.37) has a unique weak solution $(V, \Phi, \Psi) \in M^2_\mathcal{F}(0, T; V) \times M^2_\mathcal{F}(0, T; H) \times M^2_\mathcal{F}(0, T; H)$ in the sense of Definition 5.2.

Proof The proof is conducted by recasting the stochastic HJB equation (5.37) in the form of the backward stochastic evolution equation. Under Assumptions 2.1–2.3 and 5.2, we define the mappings $A : [0, T] \times \Omega \to \mathcal{L}(V, V^*), B : [0, T] \times \Omega \to \mathcal{L}(H, V^*), f : [0, T] \times \Omega \times V \times H \times V \times H \to H$, and $\xi : \Omega \to H$ by

$$\langle A(t)w, \varphi \rangle \triangleq \frac{1}{2} \int_{\mathbb{R}^n} \langle \sigma^T(t, x) D\varphi(t, x), \sigma^T(t, x) D\varphi(t, x) \rangle dx, \quad \forall \varphi, w \in V,$$

$$\langle B(t)\varphi, \varphi \rangle \triangleq \int_{\mathbb{R}^n} \langle \sigma_d(t, x) \varphi(t, x), D\varphi(t, x) \rangle dx, \quad \forall \varphi \in V, \phi \in H,$$  \tag{5.41}$$

where

- $\sigma(t, x, u) \in L^2(\mathbb{R}^d, \mathbb{R}^{d \times n})$ and $\sigma_d(t, x, u) \in L^2(\mathbb{R}^d, \mathbb{R}^{d \times d})$ for any $(t, x, u) \in [0, T] \times \mathbb{R}^n \times U$.
- $\xi(t, \omega, \cdot) \in L^2(\mathbb{R}^d, \mathbb{R}^{d \times d})$ and $\xi_d(t, \omega, \cdot) \in L^2(\mathbb{R}^d, \mathbb{R}^{d \times d})$ for any $(t, \omega) \in [0, T] \times \Omega$.
- $\xi := \xi(t, \omega, \cdot) \in L^2(\mathbb{R}^d, \mathbb{R}^{d \times d})$ and $\xi_d := \xi_d(t, \omega, \cdot) \in L^2(\mathbb{R}^d, \mathbb{R}^{d \times d})$.

The unique weak solution $(V, \Phi, \Psi)$ is defined as follows:

- $V(t, \omega, u) = \mathbb{E}[\int_0^T \xi(s, \omega, \Phi(s, \omega, u) - \Psi(s, \omega, u)) ds + \xi_d(\omega, \Phi(T, \omega, u))]$;
- $\Phi(t, \omega, u) = \mathbb{E}[\int_t^T \langle A(s)w, \varphi \rangle + \langle B(t)\varphi, \varphi \rangle + \xi(s, \omega, \Phi(s, \omega, u) - \Psi(s, \omega, u)) ds + \xi_d(\omega, \Phi(T, \omega, u)) |\mathcal{F}_t]$, for any $t \leq T$; $w \in V, \varphi \in V, \phi \in H$.
and

\[ F(t, w, \phi, \psi) \]

\[ \triangleq - \langle \nabla \cdot [\sigma \sigma^\top(t, x)], Dw(t, x) \rangle - \phi(t, x)\nabla \cdot [\sigma_d(t, x)] \]

\[ + \inf_{u \in U} \left\{ \langle b(t, x, u), Dw(t, x) \rangle \right\} \]

\[ + f(t, x, u, V, \sigma^\top Dw + \phi, \int_{E} (Iw(t, e, x, u) + \psi(t, e, x + g))l(t, e)v(de)) \]

\[ + \int_{E} [Iw(t, e, x, u) - (g(t, e, x, u), Dw(t, x))v(de) + \int_{E} I\psi(t, e, x, u)v(de)] \]

\[ \forall w \in V, \phi \in H, \psi \in M^1,2(E; H). \]

Using the above operators, we can rewrite the stochastic HJB equation (5.37) as the following BSEEJ:

\[
\begin{cases}
    dY(t) = [A(t)Y(t) + B(t)Z(t) + F(t, Y(t), Z(t), R(t, \cdot))]dt + Z(t)dW_{2}(t) + \int_{E} R(t, e)\tilde{\mu}(dt, de), \quad t \in [0, T], \\
    Y(T) = \xi.
\end{cases}
\]

(5.42)

Then, (5.38) can be written as the following abstract formula:

\[
(Y(t), \phi)_H
\]

\[
= (\xi, \phi)_H - \int_{t}^{T} \left\langle A(s)Y(s) + B(s)Z(s) + F(s, Y(s), Z(s), R(s, \cdot)), \phi \right\rangle ds \\
- \int_{t}^{T} (Z(s), \phi)_HDW(s) - \int_{t}^{T} \int_{E} (R(s, e), \phi)_H\tilde{\mu}(ds, de), \quad t \in [0, T].
\]

(5.43)

Therefore, we can apply the results in Subsection 6.1 to discuss the solvability of the stochastic HJB equation (5.36). In order to obtain the existence and uniqueness result for the backward stochastic evolution equation (5.44) by Theorem 5.3, we need to check that Assumption 5.1 is satisfied. Indeed, Assumptions 5.1(i) and 5.1 (iii) follow directly from Assumptions 2.1–2.3 and the definition of mappings \( A(t) \) and \( F(t, w, \varphi, \psi) \). Moreover, Assumption 5.1(ii) on the coercivity of \( A(t) \) is guaranteed by Assumption 5.2.

From Assumption 5.2, we have that for any \( \phi \in H^1 \),

\[
\langle A(t)\phi, \phi \rangle - \|B^s(t)\phi\|^2_H
\]

\[
= \frac{1}{2} \int_{R^n} \langle \sigma^\top(t, x)D\phi(t, x), \sigma^\top(t, x)D\phi(x) \rangle dx - \int_{R^n} |\sigma_d^\top(t, x)D\phi(t, x)|^2 dx
\]

\[
= \frac{1}{2} \int_{R^n} \langle \hat{\sigma}\sigma^\top(t, x)D\phi(t, x), D\phi(x) \rangle dx
\]

(5.44)
\[ \geq \alpha \int_{\mathbb{R}^n} \langle D\phi(t, x), D\phi(t, x) \rangle \, dx \]
\[ = \alpha \|\phi\|_V - \alpha \|\phi\|_H. \]

Now it remains to check Assumption 5.1(vi), i.e., the Lipschitz condition of the mapping \( F \), is satisfied. For notational simplicity, we denote by

\[
\begin{align*}
J_1(t, w, \phi) &\triangleq -\langle \nabla \cdot [\sigma \sigma^\top(t, \cdot)], Dw(\cdot) \rangle - \phi(\cdot) \nabla \cdot [\sigma_d(t, \cdot)], \\
J_2(t, u, w) &\triangleq (b(t, \cdot, u), Dw(\cdot)) - \int_E (g(t, e, \cdot, u), Dw(\cdot)) v(de), \\
J_3(t, u, w) &\triangleq \int_E [w(\cdot + g(t, e, \cdot, u)) - w(\cdot)] v(de), \\
J_4(t, u, \psi) &\triangleq \int_E [\psi(e, \cdot + g(t, e, \cdot, u)) - \psi(e, \cdot)] v(de), \\
f(t, u, w, \phi, \psi) &\triangleq f(t, \cdot, u, \sigma^\top Dw + \phi, \int_E (\mathcal{I}w(e, \cdot, u) + \psi(e, \cdot + g)) l(t, e) v(de)).
\end{align*}
\]

Using the above notations, we rewrite \( F \) as

\[
F(t, w, \phi, \psi) = J_1(t, w, \phi) + \inf_{u \in U} \left( J_2(t, u, w) + J_3(t, u, w) + J_4(t, u, \psi) + f(t, u, w, \phi, \psi) \right). (5.44)
\]

By the boundedness property of \( b, f, \pi, \sigma \) and their derivatives, we see that there exists a positive constant \( C \) such that for any \( w_1, w_2 \in V, \phi_1, \phi_2 \in H, u \in U \) and a.e. \((t, \omega) \in [0, T] \times \Omega, \)

\[
||J_1(t, w_1, \phi_1) - J_1(t, w_2, \phi_1)||_H \leq C ||w_1 - w_2||_V + ||\phi_1 - \phi_2||_H, \quad (5.45)
\]

and

\[
||J_2(t, u, w_1) - J_2(t, u, w_2)||_H \leq C ||w_1 - w_2||_V. \quad (5.46)
\]

Moreover, using the variable transformation and Assumption 2.4, for any \( w_1, w_2 \in V \), we have

\[
\begin{align*}
||J_3(t, u, w_1) - J_3(t, u, w_2)||_H^2 &\geq \int_{\mathbb{R}^n} \left| \int_E \left[ (w_1(x + g(t, e, x, u))) - w_1(x) \right.ight. \\
&\quad - (w_2(x + g(t, e, x, u))) - w_2(x) \left. \right] v(de) \right|^2 \, dx \\
&= \int_{\mathbb{R}^n} \left| \int_E \left[ (w_1(x + g(t, e, x, u))) - w_2(x + g(t, e, x, u))) \right. \right. \\
&\quad \left. \left. - (w_2(x + g(t, e, x, u))) - w_2(x + g(t, e, x, u))) \right] v(de) \right|^2 \, dx
\end{align*}
\]
\[ +(w_1(x) - w_2(x))v(de) \] dx
\[ \leq 2v(E) \int_{E} \int_{\mathbb{R}^n} \left| w_1(x + g(t, e, x, u)) - w_2(x + g(t, e, x, u)) \right|^2 dx v(de) \]
\[ + 2v(E) \int_{E} \int_{\mathbb{R}^n} \left| w_1(x) - w_2(y) \right|^2 dx v(de) \]
\[ = 2v(E) \int_{E} \int_{\mathbb{R}^n} \left| w_1(y) - w_2(y) \right|^2 |\det(I + \partial_x g(t, e, x, u)|^{-1} dy v(de) \]
\[ + 2v^2(E) ||w_1 - w_2||_H^2 \]
\[ \leq 2v^2(E)(1 + \delta^{-1}) ||w_1 - w_2||_H^2 \leq 2v^2(E)(1 + \delta^{-1}) ||w_1 - w_2||_V^2. \quad (5.47) \]

Similarly, using variable transformation and Assumption 2.4, we can easily obtain
\[ ||J_4(t, u, \psi_1) - J_4(t, u, \psi_2)||_H^2 \]
\[ \leq 2v(E)(1 + \delta^{-1}) ||\psi_1^1(x) - \psi_2^2(x)||^2_{M^{v,2}(E; H)}, \quad \forall (t, u) \in [0, T] \times U. \quad (5.48) \]

From previous result and Lipschitz continuity of \( f \), we also have that
\[ \| f(t, u, w_1, \phi_1, \psi_1) - f(t, u, w_2, \phi_2, \psi_2) \|_H^2 \]
\[ \leq C(\|w_1 - w_2\|_V^2 + \|\phi_1 - \phi_2\|_H^2 + \|\psi_1 - \psi_2\|_{M^{v,2}(E; H)}^2). \quad (5.49) \]

Finally, as in [27], the Lipschitz condition on the mapping \( F \) can be easily derived from the above inequalities (5.45)–(5.49). Consequently, an application of Theorem 5.3 shows that the stochastic HJB equation (5.36) has a unique solution. \( \square \)

**Remark 5.2** The above result can be extended to a more general case, where the randomness of the coefficients comes from part, but not all, of the Brownian motions. More precisely, let \( d_1 \) and \( d_2 \) be two integers such that \( d_1 \geq 1 \) and \( d_1 + d_2 = d \). Assume that all the coefficients are predictable with respect to the filtration generated by the Brownian motions \( (W_{d_1+1}, \ldots, W_d) \) and the Poisson random measure \( \tilde{\mu} \). We further assume that \( \tilde{\sigma}(\cdot) \overset{d}{=} (\sigma_1(\cdot), \sigma_2(\cdot), \ldots, \sigma_{d_1}(\cdot)) \) satisfies the non-degenerate assumption (5.39). Then, the corresponding stochastic HJB equation admits a Sobolev solution. The proof for such a result is almost the same as that of Theorem 5.5 with a minor modification. Thus, we do not repeat it here.

### 5.3 Connection with Value Function

In this subsection, we give a brief discussion on the connection between the Sobolev solution and the value function. First, let us assume that the coefficients \( \sigma, b, \) and the functional \( F \) are smooth. With the smoothness assumption, one can show that the Sobolev solution belongs to Sobolev space \( W_2^p \). The proof is similar to that in [10]. The key idea is that, the derivative \( D_x Y \) is also a solution to a BSPDE. Having proved that the solution belongs to \( W_2^p \), the embedding theorem yields that it also belongs to
a Hölder space and, hence, it is a classical solution. Using the Itô-Wentzell formula, one can show that this solution coincides with the value function with the additional condition that

$$u^*(t, x)$$

$$\triangleq \arg\min_{u \in U} \left[ \langle b(t, x, u), DV(t, x) \rangle + f(t, x, u, V, \sigma^\top DV + \Phi_d, \int_E (I \ V(t, e, x, u) + \Psi(t, e, x, u)) l(t, e) v(de)) \right. $$

$$+ \left. \int_E [I \ V(t, e, x, u) - (g(t, e, x, u), DV(t, x))] v(de) + \int_E I \Psi(t, e, x, u) v(de) \right]$$

is an admissible control. However, the smoothness assumption is very strict, especially for the nonlinear function $F$, since it is defined by taking infimum with respect to $u$. For the Markovian case, Krylov [19] proved the existence of the classical solution without smoothness condition on the coefficients. See also [23] for the case of integro-differential operators. Whether a similar result can be obtained for the non-Markovian system is still an open problem even for the Brownian case. In the other direction, if we know a priori that the optimal control $u^*(t, x)$ is regular enough, in the sense that it satisfies the Lipschitz continuity and linear growth condition, then we obtain a forward-backward system, whose connection to the Sobolev solution can be regarded as an extended Feynmann-Kac representation. Bally and Lesigne [2] considered the Markovian case, i.e., the connection to Sobolev solutions of PDEs. The extension to BSPDEs is also possible, but is left for our future research.

6 Conclusion

In this paper, we study the stochastic HJB equation with random coefficients and jumps. We prove that the value function is a solution to the stochastic HJB equation if some regularity assumptions are satisfied. The idea of the proof is motivated by the method used by Tang [34] who studied the Riccati equation for the stochastic LQ problem. The stochastic Riccati equation is a special example of the stochastic HJB equation. However, our results do not include the LQ problem as a special case since some assumptions, like compact control region and linear growth generator $f$, do not hold in that case. These technical assumptions ensure that our proof is rigorous. In Zhang et al. [36], the authors proved the solvability of backward stochastic Riccati equation with random jumps. Their basic idea is similar to ours, but the proof heavily relies on the linear-quadratic structure of their problem. Thus, the result of Zhang et al [36] is stronger than ours in the sense that the semi-martingale structure of the value function is proved in [36], while we make it as an assumption. To our best knowledge, the question that under which conditions the value function is a semi-martingale remains an open problem. We also consider the stochastic HJB equation in the Sobolev space under some non-degenerate assumption which is standard in the study of BSPDEs. For the most general case, a classical or Sobolev solution is hard to obtain. Instead, the
so-called viscosity solution is often considered. For the BSPDE case, there are only few papers on this topic. Interested readers are referred to [29] for more discussion.

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