M-convexity of the minimum-cost packings of arborescences

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Abstract

The aim of this paper is to reveal the discrete convexity of the minimum-cost packings of arborescences and branching. We first prove that the minimum-cost packings of disjoint \( k \) branchings (minimum-cost \( k \)-branchings) induce an \( \text{M}^\natural \)-convex function defined on the integer vectors on the vertex set. The proof is based on a theorem on packing disjoint \( k \)-branchings, which extends Edmonds’ disjoint branchings theorem and is of independent interest. We then show the \( \text{M}^\natural \)-convexity of the minimum-cost \( k \)-arborescences, which provides a short proof for a theorem of Bernáth and Király (SODA 2016) stating that the root vectors of the minimum-cost \( k \)-arborescences form a base polyhedron of a submodular function. Finally, building upon the \( \text{M}^\natural \)-convexity of \( k \)-branchings, we present a new problem of minimum-cost root location of a \( k \)-branching, and show that it can be solved in polynomial time if the opening cost function is \( \text{M}^\natural \)-convex.

Keywords: Arborescence, Minimum-cost packing, Base polyhedron, M-convex function, Polynomial algorithm.

1 Introduction

Packing arborescences in digraphs, originating from the seminal work of Edmonds [12], is a classical topic in combinatorial optimization. Up to the present date, it has been actively studied and a number of generalizations have been introduced [3, 4, 5, 10, 14, 21, 28, 30, 31, 35, 43, 47, 50].

Hereafter, for a positive integer \( k \), we refer to the union of arc-disjoint \( k \) arborescences as a \( k \)-arborescence and that of arc-disjoint \( k \) branchings as a \( k \)-branching. Among the recent work on \( k \)-arborescences, Bernáth and Király [4] presented a theorem stating that the root vectors of the minimum-cost \( k \)-arborescences form a base polyhedron of a submodular function (Theorem 4), which extends a theorem of Frank [15] stating that the root vectors of \( k \)-arborescences form a base polyhedron. This theorem suggests a new connection of the minimum-cost \( k \)-arborescences (or \( k \)-branchings) to the theory of submodular functions [20] and discrete convex analysis [39]. However, this connection is not explored in [4]; the theorem is just obtained along the way to the main result of [4]. Indeed, while some discrete convexity of the minimum-cost branchings is analyzed in [42, 48, 49], to the best of our knowledge, discrete convexity of the minimum-cost packings of branchings has never been discussed in the literature.

In this paper, we reveal the discrete convexity of the minimum-cost \( k \)-branchings and minimum-cost \( k \)-arborescences. More precisely, we prove that the minimum-cost \( k \)-branchings induce an \( \text{M}^\natural \)-convex function defined on the integer vectors on the vertex set (Theorem 11). To prove this theorem, we derive a theorem on packing disjoint \( k \)-branchings (Theorem 10), which follows from an extension of Edmonds’ disjoint branchings theorem due to Bérczi and Frank [11] and is of independent interest.

We then show that the minimum-cost \( k \)-arborescences induce an \( \text{M} \)-convex function on the integer vectors on the vertex set (Theorem 13). This \( \text{M} \)-convexity provides a short proof for the theorem of

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Bernáth and Király [4]. The proof in [4] requires a primal-dual argument of the linear program for the minimum-cost $k$-arborescences, matroid-restricted $k$-arborescences [18], and near supermodular functions. Instead of these concepts, we exploit a fundamental property of $M$-convex functions: the set of the minimizers of an $M$-convex function is an $M$-convex set, namely a base polyhedron (Theorems 5 and 6).

Finally, as an application of the $M^k$-convexity of the minimum-cost $k$-branchings, we present a new problem of minimum-cost root location of a $k$-branching and show that it can be solved in polynomial time if the opening cost function is an $M^k$-convex function on the integer vectors on the vertex set.

The organization of the rest of this paper is as follows. In Section 2, we review fundamental results on arborescences, branchings, submodular functions, and $M(k)$-convex functions. In Section 3 we prove a theorem on packing disjoint $k$-branchings. Based on this theorem, in Section 4 we prove that the minimum-cost $k$-branchings induce an $M^k$-convex function. We then show that the minimum-cost $k$-arborescences induce an M-convex function, and present a short proof for the theorem of Bernáth and Király [4]. In Section 5 we describe the minimum-cost root location problem.

2 Preliminaries

In this section, we describe the definitions and fundamental results on branchings, arborescences, and submodular functions. For more details, the readers are referred to thorough survey papers [2] [29] and textbooks [19] [20] [39] [46].

2.1 Branchings, arborescences, and submodular functions

Let $D = (V, A)$ be a digraph. For an arc set $F \subseteq A$ and a vertex set $X \subseteq V$, let $\rho_F(X)$ denote the number of arcs in $F$ from $V \setminus X$ to $X$. If $X$ is a singleton $\{v\}$ for a vertex $v \in V$, then $\rho_F(\{v\})$ is abbreviated as $\rho_F(v)$.

An arc set $F \subseteq A$ is called a branching if $\rho_F(v) \leq 1$ for every $v \in V$ and $F$ contains no undirected cycle. For a branching $F \subseteq A$, a vertex $v$ with $\rho_F(v) = 0$ is called a root of $F$. If a branching $F$ has a unique root, then $F$ is called an arborescence.

Let $k$ be a positive integer. Recall that an arc set $F \subseteq A$ is called a $k$-arborescence if it is the union of $k$ arc-disjoint arborescences, and is called a $k$-branching if it is the union of $k$ arc-disjoint branchings. For a $k$-branching $F$, its root vector $r_F \in \mathbb{Z}_+^A$ is defined by

$$r_F(v) = k - \rho_F(v) \quad \text{for each } v \in V.$$ 

In other words, the integer $r_F(v)$ represents the number of branchings among the $k$ branchings in $F$ which has a root $v$. Note that, when a $k$-branching $F$ is given, the $k$ arc-disjoint branchings whose union forms $F$ are not uniquely determined, whereas the root vector $r_F$ of $F$ is well defined.

The following fundamental theorem [12] characterizes the existence of $k$-branchings with prescribed root vectors. For a vector $x \in \mathbb{R}^V$ and $X \subseteq V$, let $x(X) = \sum_{v \in X} x(v)$. For a positive integer $k$, let $[k]$ denote the set of positive integers less than or equal to $k$.

**Theorem 1** (Edmonds [12]). Let $D = (V, A)$ be a digraph and $k$ be a positive integer. For vectors $q_1, \ldots, q_k \in \{0, 1\}^V$, there exist arc-disjoint branchings $B_1, \ldots, B_k$ such that $r_{B_i} = q_i$ for each $i \in [k]$ if and only if $\rho_A(X) \geq |\{i \in [k] : q_i(X) = 0\}|$ for each nonempty set $X \subseteq V$.

Among many extensions of Theorem 1 here we describe that by Bérczi and Frank [1], which will be used in the next section.

**Theorem 2** (Bérczi and Frank [1]). Let $D = (V, A)$ be a digraph with $S = \{s_1, \ldots, s_p\} \subseteq V$ and $T = V \setminus S$ such that no arc in $A$ enters a vertex in $S$. For a vector $h \in \mathbb{Z}_+^V$, there exist $h(S)$ arc-disjoint
branchings in D such that h(s_i) of those are arborescences in the subgraph induced by \{s_i\} \cup T for each i ∈ [p] if and only if

\[ \rho_A(X) \geq h(S \setminus X) \quad \text{for each } X \subseteq V \text{ with } X \cap T \neq \emptyset. \]

Let \( c \in \mathbb{R}^A \) be a cost vector on the arc set A. For an arc set \( F \subseteq A \), its cost \( c(F) \) is defined by \( c(F) = \sum_{a \in F} c(a) \). A \( k \)-branching \( F \subseteq A \) is called a minimum-cost \( k \)-branching if \( c(F) \leq c(F') \) for every \( k \)-branching \( F' \subseteq A \). A minimum-cost \( k \)-arborescence is also defined in the same manner.

Let \( S \) be a finite set. A function \( b : 2^S \to \mathbb{R} \cup \{+\infty\} \) is submodular if it satisfies

\[ b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y) \text{ for each } X, Y \subseteq S. \]

A function \( p : 2^S \to \mathbb{R} \cup \{-\infty\} \) is called supermodular if \(-p\) is submodular.

For a submodular function \( b \) on \( 2^S \), the base polyhedron \( B(b) \) associated with \( b \) is described as

\[ B(b) = \{ x \in \mathbb{R}^S : x(X) \leq b(X) (X \subseteq S), x(S) = b(S) \}. \]

For a digraph \( D = (V, A) \) and an arc set \( F \subseteq A \), it is well known that \( \rho_F : 2^V \to \mathbb{Z}_+ \) is a submodular function. Connection between \( k \)-arborescences and submodular functions is observed in \([15, 19]\) the following theorem is a consequence of Theorem[1]

**Theorem 3** (Frank [19] Theorem 10.1.8]). Let \( D = (V, A) \) be a digraph. Then, the set of the root vectors of the \( k \)-arborescences is described as

\[ \{ r \in 2^V : r(X) \geq k - \rho_A(X) (\emptyset \neq X \subseteq V), r(V) = k \}. \]

Hereafter, in discussing \( k \)-arborescence, assume that \( D \) has a \( k \)-arborescence, i.e., the set \( \{ 1 \} \) is nonempty. Theorem 3 implies that the convex hull of the root vectors of the \( k \)-arborescences is a base polyhedron. A stronger theorem is proved in \([4]\):

**Theorem 4** (Bernáth and Király [43]). Let \( D = (V, A) \) be a digraph and \( c \in \mathbb{R}^A \) be a cost vector. Then, the convex hull of the root vectors of the minimum-cost \( k \)-arborescences is a base polyhedron.

In Section 4 we present a short proof for Theorem 4 building upon the M-convexity of the minimum-cost \( k \)-arborescences (Theorem 13).

### 2.2 M-convex functions

Let \( S \) be a finite set. For \( u \in S \), let \( \chi_u \in \mathbb{Z}^S \) denote a vector such that \( \chi_u(u) = 1 \) and \( \chi_u(v) = 0 \) for every \( v \in S \setminus \{u\} \). For a vector \( x \in \mathbb{R}^S \), define \( \text{supp}^+(x), \text{supp}^-(x) \subseteq S \) by \( \text{supp}^+(x) = \{ v \in S : x(v) > 0 \} \) and \( \text{supp}^-(x) = \{ v \in S : x(v) < 0 \} \).

A nonempty set \( B \subseteq \mathbb{Z}^S \) is called an M-convex set \([36, 39]\) if it satisfies the following exchange axiom:

For \( x, y \in B \) and \( u \in \text{supp}^+(x-y) \), there exists \( v \in \text{supp}^-(x-y) \) such that \( x - \chi_u + \chi_v \in B \) and \( y + \chi_u - \chi_v \in B \).

The following theorem shows a connection between M-convex sets and submodular functions.

**Theorem 5** (see Murota [36, 39]). A set \( B \subseteq \mathbb{Z}^S \) is an M-convex set if and only if it is the set of integer points in the base polyhedron \( B(b) \) associated with an integer-valued submodular function \( b : 2^S \to \mathbb{R} \cup \{+\infty\} \), where \( b(\emptyset) = 0 \) and \( b(S) < +\infty \)

For a function \( f : \mathbb{Z}^S \to \mathbb{R} \cup \{+\infty\} \), the effective domain \( \text{dom} f \) is defined as \( \text{dom} f = \{ x \in \mathbb{Z}^S : f(x) < +\infty \} \). Now a function \( f : \mathbb{Z}^S \to \mathbb{R} \cup \{+\infty\} \) is called an M-convex function \([36, 39]\) if it satisfies the following exchange axiom:

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For \( x, y \in \text{dom } f \) and \( u \in \operatorname{supp}^+(x - y) \), there exists \( v \in \operatorname{supp}^-(x - y) \) such that

\[
f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).
\]

The next theorem is a fundamental property of M-convex functions.

**Theorem 6** (Murota [36, 39]). For an M-convex function \( f : \mathbb{Z}^S \to \mathbb{R} \cup \{+\infty\} \), \( \arg\min f \) is an M-convex set if it is not empty.

M\(^\oplus\)-convex sets and M\(^\ominus\)-convex functions \([39, 41]\) generalize M-convex sets/functions. A nonempty set \( B \subseteq \mathbb{Z}^S \) is called an M\(^\oplus\)-convex set if it satisfies the following exchange axiom:

For \( x, y \in B \) and \( u \in \operatorname{supp}^+(x - y) \), it holds that \( x - \chi_u \in B \) and \( y + \chi_u \in B \), or there exists \( v \in \operatorname{supp}^-(x - y) \) such that \( x - \chi_u + \chi_v \in B \) and \( y + \chi_u - \chi_v \in B \).

It is known that an M\(^\oplus\)-convex set is precisely the set of the integer points in an integral generalized polymatroid \([17, 23]\), and the convex hull of an M\(^\oplus\)-convex set is an integral generalized polymatroid (see \([20, 39]\)).

A function \( f : \mathbb{Z}^S \to \mathbb{R} \cup \{+\infty\} \) is called an M\(^\ominus\)-convex function if it satisfies the following exchange axiom:

For \( x, y \in \text{dom } f \) and \( u \in \operatorname{supp}^+(x - y) \), it holds that

\[
f(x) + f(y) \geq f(x - \chi_u) + f(y + \chi_u),
\]

or there exists \( v \in \operatorname{supp}^-(x - y) \) such that

\[
f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).
\]

For M\(^\ominus\)-convex functions, a counterpart of Theorem 6 is established as follows.

**Theorem 7** (see Murota [39]). For an M\(^\ominus\)-convex function \( f : \mathbb{Z}^S \to \mathbb{R} \cup \{+\infty\} \), \( \arg\min f \) is an M\(^\ominus\)-convex set if it is not empty.

There is a strong relation between M-convex functions and M\(^\ominus\)-convex functions (see Murota [39] for details). We focus on the following relation, which plays a key role in our proof for Theorem 9. For an M\(^\ominus\)-convex function \( f : \mathbb{Z}^S \to \mathbb{R} \cup \{+\infty\} \) and an integer \( k \in \mathbb{Z} \), define a function \( f_k : \mathbb{Z}^S \to \mathbb{R} \cup \{+\infty\} \) by

\[
f_k(x) = \begin{cases} f(x) & (x(S) = k), \\ +\infty & (x(S) \neq k). \end{cases}
\]

Define \( \lambda, \mu \in \mathbb{Z} \) by \( \lambda = \min\{k : \text{dom } f_k \neq \emptyset\} \), \( \mu = \max\{k : \text{dom } f_k \neq \emptyset\} \).

**Theorem 8** (Murota and Shioura [41, Theorem 3.1]). For an integer \( k \) with \( \lambda \leq k \leq \mu \), it holds that \( \text{dom } f_k \neq \emptyset \) and \( f_k \) is an M-convex function.

A relation between M\(^\ominus\)-convex functions and minimum-cost branchings is presented by Takazawa [49]: the minimum-cost branchings induce an M\(^\ominus\)-convex function defined on the integer vectors on the vertex set \( V \).

**Theorem 9** (Takazawa [49]). Let \( D = (V, A) \) be a digraph and \( c \in \mathbb{R}^A \) be a cost vector. Then a function \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) defined by

\[
\text{dom } f = \{x \in \mathbb{Z}^V : x = r_F \text{ for some branching } F \subseteq A\},
\]

\[
f(x) = \begin{cases} \min\{c(F) : F \subseteq A \text{ is a branching with } r_F = x\} & (x \in \text{dom } f), \\ +\infty & (x \in \mathbb{Z}^V \setminus \text{dom } f) \end{cases}
\]

is an M\(^\ominus\)-convex function.
3 Packing disjoint $k$-branchings

In this section, we prove a theorem on packing disjoint $k$-branchings. This theorem is used in showing the $M^3$-convexity of $k$-branchings (Theorem 11), and is of independent interest as an extension of Theorem [1].

The following theorem characterizes the existence of $p$ arc-disjoint $k$-branchings with prescribed root vectors $q_1, \ldots, q_p \in \{0, 1, \ldots, k\}^V$. Here we prove this theorem by utilizing Theorem 2 while in Appendix A we present an alternative proof, which extends that for Theorem 1 by Lovász [34] and implies a strongly polynomial algorithm to find desired $k$-branchings.

**Theorem 10.** Let $D = (V, A)$ be a digraph, $k$ and $p$ be positive integers, and $q_1, \ldots, q_p \in \{0, 1, \ldots, k\}^V$ be vectors such that $q_i(V) \geq k$ for each $i \in [p]$. Then, there exist $p$ arc-disjoint $k$-branchings $F_1, \ldots, F_p$ such that $r_{F_i} = q_i$ for each $i \in [p]$ if and only if

$$\rho_A(X) \geq \sum_{i \in [p]} \max\{0, k - q_i(X)\} \quad (\emptyset \neq X \subseteq V). \quad (2)$$

*Proof.* From $D = (V, A)$, construct a new digraph $\tilde{D} = (\tilde{V}, \tilde{A})$ as follows: the vertex set $\tilde{V}$ is obtained by adding the set of new distinct vertices $S = \{s_1, \ldots, s_p\}$ to $V$, and the arc set $\tilde{A}$ is obtained by adding $q_i(v)$ parallel arcs from $s_i$ to $v$ for each $i \in [p]$ and $v \in V$. Then, it is straightforward to see that $D$ has the desired $p$ arc-disjoint $k$-branchings if and only if $D'$ has $kp$ arc-disjoint branchings such that $k$ of those are arc-subdivisions in the subgraph induced by $\{s_i\} \cup V$ for each $i \in [p]$. It follows from Theorem 2 that the latter is equivalent to the following condition:

$$\tilde{\rho}_A(\tilde{X}) \geq k \cdot |S \setminus \tilde{X}| \quad \text{for each } \tilde{X} \subsetneq \tilde{V} \text{ with } \tilde{X} \cap V \neq \emptyset, \quad (3)$$

where $\tilde{\rho}_A(\tilde{X})$ denotes the number of arcs in $\tilde{A}$ from $\tilde{V} \setminus \tilde{X}$ to $\tilde{X}$. By letting $X$ denote $\tilde{X} \cap V$, Condition (3) is equivalent to

$$\rho_A(X) \geq k \cdot |S \setminus X| - \sum_{i \in [p] : s_i \in S \setminus X} q_i(X)$$

$$= \sum_{i : s_i \in S \setminus X} (k - q_i(X)) \quad \text{for each } \tilde{X} \subsetneq V \text{ with } \tilde{X} \cap V \neq \emptyset. \quad (4)$$

Now fix $X \subseteq V$. Then, the right-hand side of (4) is maximized by $\tilde{X}^* = X \cup S^*$, where

$$S^* = \{s_i \in S \mid k - q_i(X) \leq 0\},$$

and it follows that

$$\sum_{i \in [p] : s_i \in S \setminus X^*} (k - q_i(X)) = \sum_{i \in [p]} \max\{0, k - q_i(X)\}.$$

This completes the proof for Theorem 10. \hfill \Box

**Remark 1.** Let us describe a couple of remarks on Theorem 10. First, the case $k = 1$ of Theorem 10 exactly coincides with Theorem 1. Second, Theorem 10 discusses $pk$ arc-disjoint branchings in total, but it essentially differs from Theorem 1 where $k$ is replaced with $pk$: we do not have $pk$ prescribed root vectors for all branchings. Moreover, Theorem 10 is properly stronger than the characterization of the existence of one $pk$-branching with root vector $\sum_{i \in [p]} q_i$: the arc set of the $pk$-branching should be partitioned into $k$-branchings $F_1, \ldots, F_p$ with root vectors $q_1, \ldots, q_p$. Finally, if $p = 2$, the condition (2) is equivalent to the following condition, which would also be intuitive:

$$q_1(X) + q_2(X) \geq 2k - \rho_A(X) \quad (\emptyset \neq X \subseteq V) \text{ and }$$

$$q_i(X) \geq k - \rho_A(X) \quad (\emptyset \neq X \subseteq V, i = 1, 2).$$
4 Discrete convexity of minimum-cost \(k\)-branchings and \(k\)-arborescences

In this section, by using Theorem 10 we prove that the minimum-cost \(k\)-branchings induce an \(M^2\)-convex function on \(\mathbb{Z}^V\), which extends Theorem 9. We then show that the minimum-cost \(k\)-arborescences induce an \(M\)-convex function on \(\mathbb{Z}^V\), and derive a short proof for Theorem 13.

Let \(D = (V, A)\) be a digraph and \(c \in \mathbb{R}^A\) be a cost vector. Define a function \(f_0 : \mathbb{Z}^V \to \mathbb{R} \cup \{\pm\infty\}\) associated with the minimum-cost \(k\)-branchings in the following manner. First, the effective domain \(\text{dom } f_0\) is defined by

\[
dom f_0 = \{x \in \mathbb{Z}^V : x = r_F \text{ for some } k\text{-branching } F \subseteq A\}. \tag{5}
\]

Then, the function value \(f_0(x)\) is defined by

\[
f_0(x) = \begin{cases} \min\{c(F) : F \subseteq A \text{ is a } k\text{-branching with } r_F = x\} & (x \in \dom f_0), \\ +\infty & (x \notin \mathbb{Z}^V \setminus \dom f_0). \end{cases} \tag{6}
\]

**Theorem 11.** The function \(f_0\) is an \(M^2\)-convex function.

**Proof.** Let \(x, y \in \dom f_0\) and \(u \in \text{supp}^+(x-y)\). Denote the \(k\)-branchings attaining \(f_0(x)\) and \(f_0(y)\) by \(F_x\) and \(F_y\), respectively. That is, \(x = r_{F_x}\), \(f_0(x) = c(F_x)\), \(y = r_{F_y}\), and \(f_0(y) = c(F_y)\). Let \(D' = (V, A')\) be a digraph on \(V\) containing each arc \(a \in A\) with multiplicity zero if \(a \in A \setminus (F_x \cup F_y)\), one if \(a \in (F_x \setminus F_y) \cup (F_y \setminus F_x)\), and two if \(a \in F_x \cap F_y\).

Let \(x' = x - \chi_u\) and \(y' = y + \chi_u\). Suppose that \(A'\) can be partitioned into two \(k\)-branchings \(F'_x\) and \(F'_y\) satisfying \(r_{F'_x} = x'\) and \(r_{F'_y} = y'\). It then follows that

\[f_0(x') + f_0(y') \leq c(F'_x) + c(F'_y) = c(F_x) + c(F_y) = f_0(x) + f_0(y).\]

Suppose that \(A'\) cannot be partitioned into two \(k\)-branchings \(F'_x\) and \(F'_y\) satisfying \(r_{F'_x} = x'\) and \(r_{F'_y} = y'\). Define two functions \(g, g' : \mathbb{Z}^V \to \mathbb{Z}_+\) by

\[
g(X) = \max\{0, k - x(X)\} + \max\{0, k - y(X)\} \quad (X \subseteq V),
\]

\[
g'(X) = \max\{0, k - x'(X)\} + \max\{0, k - y'(X)\} \quad (X \subseteq V).
\]

It is straightforward to see that

\[|g(X) - g'(X)| \leq 1 \quad (X \subseteq V). \tag{7}\]

It follows from Theorem 10 that \(\rho_{A'}(X) - g(X) \geq 0\) for each nonempty set \(X \subseteq V\) and there exists a nonempty set \(X' \subseteq V\) such that \(\rho_{A'}(X') - g'(X') \leq -1\). We then obtain from (7) that \(\rho_{A'}(X') - g'(X') = -1\), which is the minimum value of the submodular function \(\rho_{A'} - g'\). Let \(X^* \subseteq V\) be the inclusion-wise minimal set \(X'\) with \(\rho_{A'}(X') - g'(X') = -1\). Since \(\rho_{A'}(X^*) \geq g(X^*)\), it follows that \(u \in X^*\) and

\[
\max\{0, k - x'(X^*)\} = \max\{0, k - x(X)\} + 1, \quad \max\{0, k - y'(X^*)\} = \max\{0, k - y(X)\}. \tag{8}
\]

By (3), it holds that \(x(X^*) \leq k\), whereas \(y(X^*) \geq k\) follows from (9). Since \(u \in X^* \cap \text{supp}^+(x-y)\), these imply that there exists a vertex \(v \in X^* \cap \text{supp}^+(x-y)\). Then, for \(x'' = x - \chi_u + \chi_v\) and \(y'' = y + \chi_u - \chi_v\), it follows from the minimality of \(X^*\) that \(\rho_{A'}(X^*) \geq \max\{0, k - x''(X)\} + \max\{0, k - y''(X)\}\) for each nonempty set \(X \subseteq V\). Thus, it is derived from Theorem 10 that \(A'\) can be partitioned into two \(k\)-branchings \(F''_x\) and \(F''_y\) satisfying \(r_{F''_x} = x''\) and \(r_{F''_y} = y''\). It then holds that

\[f_0(x'') + f_0(y'') \leq c(F''_x) + c(F''_y) = c(F_x) + c(F_y) = f_0(x) + f_0(y).\]

Therefore, we have shown that \(f_0\) satisfies the exchange axiom of \(M^2\)-convex functions. \(\Box\)
Remark 2. The above proof of Theorem 11 is a nontrivial extension of the proof for Theorem 9 in \cite{49}. In \cite{49}, the $M^\mathbb{R}$-convexity of minimum-cost branchings is derived from the exchangeability of branchings \cite{45,46}, which is proved by using Edmonds’ disjoint branchings theorem \cite{12}. In our proof of Theorem 11, the $M^\mathbb{R}$-convexity of minimum-cost $k$-branchings is derived from the exchangeability of $k$-branchings, which is proved by using Theorem 11, an extension of Edmonds’ disjoint branchings theorem \cite{12}.

We also remark that Theorem 11 essentially requires the concept of $M^\mathbb{R}$-convexity. Indeed, the effective domain of the $M^\mathbb{R}$-convex function $f$ in Theorem 9 is contained in $\{0, 1\}^V$, and hence $f$ is essentially a \textit{valuated matroid} \cite{8,9}. In contrast, the effective domain of the $M^\mathbb{R}$-convex function $f_b$ induced by the minimum-cost $k$-branchings lies over $\{0, 1, \ldots, k\}^V$, and hence $f_b$ is an $M^\mathbb{R}$-convex function which is essentially beyond valuated matroids.

We now present two consequences of Theorem 11. Firstly, by Theorem 11, the following extension of Theorem 4 is immediately derived from Theorem 7.

\begin{corollary}
Let $D = (V, A)$ be a digraph and $c \in \mathbb{R}^A$ be a cost vector. Then, the set of the root vectors of the minimum-cost $k$-branchings is an $M^\mathbb{R}$-convex set. Equivalently, the convex hull of the root vectors of the minimum-cost $k$-branchings is an integral generalized polymatroid.
\end{corollary}

Secondly, we exhibit the discrete convexity of the minimum-cost $k$-arborescences. Similarly as the function $f_b$ defined by \cite{5} and \cite{6}, define a function $f_a: \mathbb{Z}^V \rightarrow \mathbb{R} \cup \{+\infty\}$ associated with the minimum-cost $k$-arborescences by

$$
\text{dom } f_a = \{x \in \mathbb{Z}^V: x = r_F \text{ for some } k\text{-arborescence } F \subseteq A\},
$$

$$
f_a(x) = \begin{cases} 
\min\{c(F): F \subseteq A \text{ is a } k\text{-arborescence with } r_F = x\} & (x \in \text{dom } f_a), \\
+\infty & (x \in \mathbb{Z}^V \setminus \text{dom } f_a). 
\end{cases}
$$

It is straightforward to see that the function $f_a$ is the restriction of $f_b$ to the hyperplane $x(V) = k$. We thus obtain the following theorem from Theorems 8 and 11.

\begin{theorem}
The function $f_a$ is an $M$-convex function.
\end{theorem}

We remark that an alternative proof for Theorem 13 can be obtained by adapting the above proof for Theorem 11 to $f_a$.

Now Theorem 13 leads to the following proof for Theorem 4.

\begin{proof}[Proof for Theorem 4]
By Theorems 6 and 13, the set $R \subseteq \mathbb{Z}^V$ of the root vectors of the minimum-cost $k$-arborescences, which is nothing other than argmin $f_b$, is an $M$-convex set. Then, by Theorem 8, $R$ is the set of integer points in the base polyhedron $B(b)$ associated with some integer-valued submodular function $b: \mathbb{Z}^V \rightarrow \mathbb{Z} \cup \{+\infty\}$. For an integer-valued submodular function $b$, it holds that $B(b)$ is exactly the convex hull of the integer points in $B(b)$. We thus conclude that the convex hull of $R$ is a base polyhedron $B(b)$. \hfill \square
\end{proof}

5 Application: Minimum-cost root location

As an application of Theorem 11 in this section we present a new problem of \textit{minimum-cost root location} of a $k$-branching, a variation of the facility location problem. We show that our problem admits a polynomial-time exact algorithm if the opening cost function is $M^\mathbb{R}$-convex.

The problem is formulated as follows. Let $D = (V, A)$ be a digraph, and suppose that a function $f_0: \mathbb{Z}^V \rightarrow \mathbb{Z} \cup \{+\infty\}$ representing the opening cost and an arc-cost vector $c \in \mathbb{Z}^A$ are given. In order to simplify the complexity issue, in this section we assume that $f_0$ and $c$ are integer-valued and an
evaluation oracle for $f_p$ is available. The objective of the problem is to find a $k$-branching $F \subseteq A$ with root vector $r_F \in \mathbb{Z}_+^V$ minimizing

$$f_p(r_F) + c(F). \quad (10)$$

Intuitively, we open $r_F(v)$ facilities at each vertex $v \in V$, and construct a $k$-branching $F$ so that each vertex $u \in V$ is served by $k$ facilities connected by a directed path to $u$. The total cost is the sum of the opening cost $f_p(r_F)$ and the connecting cost $c(F)$.

In a special case where $k = 1$ and the function $f_p$ is separable, i.e., $f_p(x) = \sum_{v \in V} f_v(x(v))$, where $f_v$ is a univariate function for each $v \in V$, the problem is easy: it is reduced to the minimum-cost branching problem by redefining the arc cost $c'(a) = c(a) + f_v(0) - f_v(1)$ for each arc $a \in A$ entering $v$ for each $v \in V$. Thus, it can be solved in strongly polynomial time [6, 7, 11, 22].

In more general cases, however, it is not trivial whether the problem is tractable or not. In the case where the function $f_p$ is $M^\#-$convex, building upon Theorem 11, we can solve the problem in polynomial time in the following manner.

Recall the function $f_0$ defined by (5) and (6). The objective function (10) is restated as $f_p(r_F) + f_0(r_F)$, which is the sum of two $M^\#-$convex functions by Theorem 11. Although the sum of two $M^\#-$convex function is no longer $M^\#-$convex in general, its minimization is a generalization of weighted matroid intersection [13, 16, 24, 32] or valued matroid intersection [37, 38], and can be done in polynomial time [26, 27, 39, 40]. After computing a minimizer $r^*$ of $f_p(r_F) + f_0(r_F)$, we can find an $k$-branching of minimum cost among those with root vector $r^*$ in polynomial time by a weighted matroid intersection algorithm (see, e.g., [2]).

Theorem 14. A $k$-branching $F \subseteq A$ minimizing (10) can be found in polynomial time if the function $f_p$ is $M^\#-$convex.

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References

[1] K. Bérczi and A. Frank: Variations for Lovász’ submodular ideas, Technical Report TR-2008-07, Egervár Research Group, Budapest, 2008, www.cs.elte.hu/egres.

[2] K. Bérczi and A. Frank: Packing arborescences, in S. Iwata, ed., RIMS Kôkyûroku Bessatsu, B23, Research Institute for Mathematical Sciences, Kyoto University, 2010, 1–31.

[3] K. Bérczi, T. Király and Y. Kobayashi: Covering intersecting bi-set families under matroid constraints, SIAM Journal Discrete Mathematics, 30 (2016), 1758–1774.

[4] A. Bernáth and T. Király: Blocking optimal $k$-arborescences, in Proceedings of the 27th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2016), SIAM, 2016, 1682–1694.

[5] A. Bernáth and G. Pap: Blocking optimal arborescences, Mathematical Programming, 161 (2017), 583–601.

[6] F. Bock: An algorithm to construct a minimum directed spanning tree in a directed network, in Developments in Operations Research, Gordon and Breach, 1971, 29–44.
[7] Y.J. Chu and T.H. Liu: On the shortest arborescence of a directed graph, *Scientia Sinica*, 14 (1965), 1396–1400.

[8] A.W.M. Dress and W. Wenzel: Valuated matroid: A new look at the greedy algorithm, *Applied Mathematics Letters*, 3 (1990), 33–35.

[9] A.W.M. Dress and W. Wenzel: Valuated matroids, *Advances in Mathematics*, 93 (1992), 214–250.

[10] O. Durand de Gevigney, V.H. Nguyen and Z. Szigeti: Matroid-based packing of arborescences, *SIAM Journal on Discrete Mathematics*, 27 (2013), 567–574.

[11] J. Edmonds: Optimum branchings, *Journal of Research National Bureau of Standards, Section B*, 71 (1967), 233–240.

[12] J. Edmonds: Edge-disjoint branchings, in R. Rustin, ed., *Combinatorial Algorithms*, Algorithmics Press, 1973, 91–96.

[13] J. Edmonds: Matroid intersection, *Annals of Discrete Mathematics*, 4 (1979), 39–49.

[14] Q. Fortier, C. Király, Z. Szigeti and S. Tanigawa: On packing spanning arborescences with matroid constraint, *Electronic Notes in Discrete Mathematics*, 61 (2017), 451–457.

[15] A. Frank: On disjoint trees and arborescences, in *Algebraic Methods in Graph Theory* (Colloquia Mathematica Societatis János Bolyai, 25), 1978, 159–169.

[16] A. Frank: A weighted matroid intersection algorithm, *Journal of Algorithms*, 2 (1981), 328–336.

[17] A. Frank: Generalized polymatroids, in A. Hajnal, L. Lovász and V.T. Sós, eds., *Finite and Infinite Sets Vol. I (Proceedings of the Sixth Hungarian Combinatorial Colloquium, Eger, 1981)*, 1984, 285–294.

[18] A. Frank: Rooted $k$-connections in digraphs, *Discrete Applied Mathematics*, 157 (2009), 1242–1254.

[19] A. Frank: *Connections in Combinatorial Optimization*, Oxford University Press, New York, 2011.

[20] S. Fujishige: *Submodular Functions and Optimization*, Annals of Discrete Mathematics, 58, Elsevier, Amsterdam, second edition, 2005.

[21] S. Fujishige: A note on disjoint arborescences, *Combinatorica*, 30 (2010), 247–252.

[22] D.R. Fulkerson: Packing rooted directed cuts in a weighted directed graph, *Mathematical Programming*, 6 (1974), 1–13.

[23] R. Hassin: Minimum cost flow with set-constraints, *Networks*, 12 (1982), 1–21.

[24] M. Iri and N. Tomizawa: An algorithm for finding an optimal “independent assignment”, *Journal of the Operations Research Society of Japan*, 19 (1976), 32–57.

[25] S. Iwata, L. Fleischer and S. Fujishige: A combinatorial strongly polynomial algorithm for minimizing submodular functions, *Journal of the ACM*, 48 (2001), 761–777.

[26] S. Iwata, S. Moriguchi and K. Murota: A capacity scaling algorithm for M-convex submodular flow, *Mathematical Programming*, 103 (2005), 181–202.

[27] S. Iwata and M. Shigeno: Conjugate scaling algorithm for Fenchel-type duality in discrete convex optimization, *SIAM Journal on Optimization*, 13 (2002), 204–211.
[28] N. Kakimura, N. Kamiyama and K. Takazawa: The $b$-branching problem in digraphs, in I. Potapov, P.G. Spirakis and J. Worrell, eds., 43rd International Symposium on Mathematical Foundations of Computer Science (MFCS 2018), LIPIcs 117, 2018, 12:1–12:15.

[29] N. Kamiyama: Arborescence problems in directed graphs: Theorems and algorithms, Interdisciplinary Information Sciences, 20 (2014), 51–70.

[30] N. Kamiyama, N. Katoh and A. Takizawa: Arc-disjoint in-trees in directed graphs, Combinatorica, 29 (2009), 197–214.

[31] C. Király: On maximal independent arborescence packing, SIAM Journal on Discrete Mathematics, 30 (2016), 2107–2114.

[32] E.L. Lawler: Matroid intersection algorithms, Mathematical Programming, 9 (1975), 31–56.

[33] Y.T. Lee, A. Sidford and S.C. Wong: A Faster Cutting Plane Method and its Implications for Combinatorial and Convex Optimization, in V. Guruswami, ed., IEEE 56th Annual Symposium on Foundations of Computer Science (FOCS 2015), IEEE Computer Society, 2015, 1049–1065.

[34] L. Lovász: On two minimax theorems in graph, Journal of Combinatorial Theory, Series B, 21 (1976), 96–103.

[35] T. Matsuoka and Z. Szigeti: Polymatroid-based capacitated packing of branchings, Technical report, METR 2017-15, University of Tokyo, 2017.

[36] K. Murota: Convexity and Steinitz’s exchange property, Advances in Mathematics, 125 (1996), 272–331.

[37] K. Murota: Valuated matroid intersection I: Optimality criteria, SIAM Journal on Discrete Mathematics, 9 (1996), 545–561.

[38] K. Murota: Valuated matroid intersection II: Algorithms, SIAM Journal on Discrete Mathematics, 9 (1996), 562–576.

[39] K. Murota: Discrete Convex Analysis, Society for Industrial and Applied Mathematics, Philadelphia, 2003.

[40] K. Murota: Matrices and Matroids for Systems Analysis, Springer, Berlin, softcover edition, 2010.

[41] K. Murota and A. Shioura: M-convex function on generalized polymatroid, Mathematics of Operations Research, 24 (1999), 95–105.

[42] K. Murota and K. Takazawa: Relationship of two formulations for shortest bibranchings, CoRR, abs/1706.02029, (2017).

[43] A. Schrijver: Min-max relations for directed graphs, Annals of Discrete Mathematics, 16 (1982), 261–280.

[44] A. Schrijver: A combinatorial algorithm minimizing submodular functions in strongly polynomial time, Journal of Combinatorial Theory, Series B, 80 (2000), 346–355.

[45] A. Schrijver: Total dual integrality of matching forest constraints, Combinatorica, 20 (2000), 575–588.

[46] A. Schrijver: Combinatorial Optimization—Polyhedra and Efficiency, Springer, Heidelberg, 2003.
[47]  Z. Szigeti and S. Tanigawa: An algorithm for the problem of minimum weight packing of arborescences with matroid constraints, Technical report, METR 2017-14, University of Tokyo, 2017.

[48]  K. Takazawa: Shortest bibranchings and valuated matroid intersection, *Japan Journal of Industrial and Applied Mathematics*, 29 (2012), 561–573.

[49]  K. Takazawa: Optimal matching forests and valuated delta-matroids, *SIAM Journal on Discrete Mathematics*, 28 (2014), 445–467.

[50]  K. Takazawa: The $b$-bibranching problem: TDI system, packing, and discrete convexity, *CoRR*, abs/1802.03235, (2018).

A Alternative proof for Theorem \[10\]

Here we show an alternative proof for Theorem \[10\] which extends that for Theorem \[1\] by Lovász \[34\]. We remark that this proof is different from that in \[28\], which also extends the proof of Lovász \[34\] to a different generalization of branchings.

Let us begin with a function which plays a key role in the proof. Let $D = (V, A)$ be a digraph, $k$ and $p$ be positive integers, and $q_1, \ldots, q_p \in \{0, 1, \ldots, k\}^V$ be vectors, which describe the root vectors of $k$-branchings $F_1, \ldots, F_p$. For each $i \in [p]$, define a function $g_i: 2^V \rightarrow \mathbb{Z}_+$ by

$$g_i(X) = \max\{0, k - q_i(X)\} \quad (X \subseteq V).$$

Then define a function $g: 2^V \rightarrow \mathbb{Z}_+$ by

$$g(X) = \sum_{i \in [p]} g_i(X) \quad (X \subseteq V).$$

It is clear from this definition that $g_i (i \in [p])$ and $g$ are nonincreasing functions. Moreover, we observe that these functions are supermodular.

**Lemma 15.** The function $g_i$ is supermodular for each $i \in [p]$, and thus $g$ is also supermodular.

**Proof.** It suffices to prove that $g_i$ is supermodular for each $i \in [p]$. Let $X, Y \subseteq V$ and denote $\alpha = q_i(X \cap Y), \beta = q_i(X \setminus Y), \gamma = q_i(Y \setminus X)$. Then, it holds that

$$g_i(X) = \max\{0, k - (\alpha + \beta)\}, \quad g_i(Y) = \max\{0, k - (\alpha + \gamma)\},$$

$$g_i(X \cup Y) = \max\{0, k - (\alpha + \beta + \gamma)\}, \quad g_i(X \cap Y) = \max\{0, k - \alpha\}.$$

If $\alpha + \beta \geq k$, we have that $g_i(X) = g_i(X \cup Y) = 0$ and hence supermodular inequality follows from $g_i(Y) \leq g_i(X \cap Y)$. The same argument holds in the case where $\alpha + \gamma \geq k$.

Suppose that $\alpha + \beta < k$ and $\alpha + \gamma < k$. Then

$$g_i(X) + g_i(Y) - g_i(X \cap Y) = (k - (\alpha + \beta)) + (k - (\alpha + \gamma)) - (k - \alpha)$$

$$= k - (\alpha + \beta + \gamma)$$

$$\leq g_i(X \cup Y),$$

and thus we obtain supermodular inequality. \hfill \Box

We now describe another proof for Theorem \[10\]
Alternative proof for Theorem. Necessity is obvious. We prove sufficiency by induction on
\[ \sum_{v \in V} g(\{v\}) = \sum_{v \in V} \sum_{i \in [p]} (k - q_i(v)). \]
Suppose that (2) holds. The case where \( q_i(v) = k \) for every \( v \in V \) and \( i \in [p] \) is trivial: \( F_i = \emptyset \) for each \( i \in [p] \). Without loss of generality, let \( q_i(v) \leq k - 1 \) for some vertex \( v \in V \).
Define a partition \( \{V^{(k)}, V^{(+)}, V^{(0)}\} \) of \( V \) by
\[ V^{(k)} = \{ v \in V : q_i(v) = k \}, \quad V^{(1)} = \{ v \in V : 1 \leq q_i(v) \leq k - 1 \}, \quad V^{(0)} = \{ v \in V : q_i(v) = 0 \}. \]

Let \( W \subseteq V \) be a inclusion-wise minimal set satisfying
\[ \rho_A(W) = g(W), \quad W \cap (V^{(k)} \cup V^{(1)}) \neq \emptyset, \quad W \setminus V^{(k)} \neq \emptyset. \]

Note that such \( W \) always exists, since \( W = V \) satisfies (11)-(13). Let
\[ W^{(k)} = W \cap V^{(k)}, \quad W^{(+)} = W \cap V^{(1)}, \quad W^{(0)} = W \cap V^{(0)}. \]

In order to apply induction, we prove that there exists an arc \( a = uv \in A \) such that \( v \in W^{(+)} \cup W^{(0)} \) and resetting \( A := A \setminus \{a\}, q_i(v) := q_i(v) + 1 \) retains (2).
Suppose \( W^{(0)} \neq \emptyset \). Then \( g_1(W^{(0)}) = k \), and hence it follows from (12) that
\[ g_1(W) < k = g_1(W^{(0)}). \]
Since \( g_1(W^{(0)}) \geq g_1(W) \) holds for each \( i \in [p] \), we obtain that \( \rho_A(W^{(0)}) \geq g(W^{(0)}) > g(W) = \rho_A(W) \).
This implies that there exists an arc \( a = uv \in A \) with \( u \in W^{(k)} \cup W^{(+)} \) and \( v \in W^{(0)} \).

The fact that resetting \( A := A \setminus \{a\} \) and \( q_i(v) := q_i(v) + 1 \) retains (2) can be verified as follows. Suppose to the contrary that \( X \subseteq V \) violates (2) after resetting. Then we have that \( \rho_A(X) = g(X) \) before resetting, and the resetting decreases \( \rho_A(X) \) by one while it does not change \( g(X) \). This implies that \( q_1(X) \geq k \) before resetting, \( u \in V \setminus X \), and \( v \in X \). Hence \( u \in W \setminus X \), and \( X \cap W \subseteq W \). We now prove that \( X \cap W \) satisfies (11)-(13) to derive a contradiction to the minimality of \( W \).

Before resetting, it holds that
\[ \rho_A(X \cap W) \leq \rho_A(X) + \rho_A(W) - \rho_A(X \cup W) \]
\[ \leq g(X) + g(W) - g(X \cup W) \leq g(X \cap W), \]
where the last inequality follows from Lemma. Since \( \rho_A(X \cap W) \geq g(X \cap W) \) by (2), the inequalities in (15) holds with equality, and in particular it holds that
\[ g(X \cap W) = \rho_A(X \cap W), \]
\[ g_i(X) + g_i(W) - g_i(X \cup W) = g_i(X \cap W) \quad (i \in [p]). \]

It follows from (17) that \( g_1(X \cap W) = g_1(W) \), since \( q_1(X) \geq k \) implies \( g_1(X) = 0 \) and \( g_1(X \cup W) = 0 \). By (14), we have \( g_1(X \cap W) < k \), and hence
\[ (X \cap W) \cap (V^{(k)} \cup V^{(1)}) \neq \emptyset \]
follows from the definition of \( g_1 \). Finally, we have
\[ (X \cap W) \setminus V^{(k)} \neq \emptyset \]
since $v \in (X \cap W) \setminus V^{(k)}$. Therefore, from (16), (18), and (19), we conclude that $X \cap W$ satisfies (11)–(13).

Suppose $W(0) = \emptyset$. By (13), we have $W^{(+)} \neq \emptyset$. If $|W^{(+)}| > 1$, then
\[
\sum_{v \in W^{(+)}} \rho_A(v) \geq \sum_{v \in W^{(+)}} g({v}) > g(W^{(+)}) \geq g(W) = \rho_A(W).
\]
This implies that there exists an arc $a = uv \in A$ with $u \in W$ and $v \in W^{(+)}$, and the above argument applies.

If $|W^{(+)}| = 1$, let $W^{(+)} = \{v\}$. Since $\rho_A$ is submodular and $g$ is supermodular, $\rho_A - g$ is submodular and hence there exists a unique minimal set $X^*$ which minimizes $\rho_A - g$ among the sets containing $v$. By (2) and (11), we have that $W$ minimizes $\rho_A - g(X^* = 0)$ and $X^* \subseteq W$. Suppose that $X^*$ violates (2) after resetting $A := A \setminus \{a\}$ for an arc $a$ entering $v$ and $q_1(v) := q_1(v) + 1$. This implies that the resetting does not change $g_1(X^*)$ and hence $g_1(X^*) = 0$ before resetting. Since $g_1({v}) = k - q_1(v) \geq 1$ by $v \in W^{(+)}$, we have that $g_1({v}) > g_1(X^*)$. Since $g_1({v}) \geq g_1(X^*)$ for each $i \in [p]$, it follows that $\rho_A({v}) > g({v}) \geq g(X^*) = \rho_A(X^*)$. We thus have an arc $a^* = uv \in A$ with $u \in X^*$. Then a new resetting of $A := A \setminus \{a^*\}$ and $q_1(v) := q_1(v) + 1$ retains (2).

Now we can apply induction and add $a$ to $F_1$. We now show that $F_1$ remains a $k$-branching when $a$ is added to complete the proof. It directly follows from the above argument that $\rho_{F_1}(X) \geq g_1(X)$ ($\emptyset \neq X \subseteq V$) is maintained throughout. Then, Theorem 1 implies that $F_1$ is a $k$-branching with root vector $q_1$.

Note that the above proof implies the following algorithm to find desired $k$-branchings $F_1, \ldots, F_p$. A $k$-branching $F_1$ can be constructed by repeatedly finding an arc $a \in A$ such that the resetting $A := A \setminus \{a\}$, $q_1(v) := q_1(v) + 1$ retains (2). This can be done by at most $|A|$ times of the minimization of a submodular function $\rho_A - g$ [25, 33, 44]. By iterating this procedure for each $i \in [p]$, we can find $F_1, \ldots, F_p$ in strongly polynomial time.