Propelinear 1-perfect codes from quadratic functions*

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Abstract. The 1-perfect code obtained by Vasil‘ev–Schönheim construction from a linear base code and a quadratic switching function is transitive and, moreover, propelinear. This gives at least \( \exp(cN^2) \) propelinear 1-perfect codes of length \( N \) over an arbitrary finite field, while an upper bound on the number of transitive codes is \( \exp(c(N \ln N)^2) \).

Keywords: perfect code, propelinear code, transitive code, automorphism group.

1. Introduction

Usually, a group code is defined as a subgroup of the additive group of a finite vector space. There are alternative approaches \([10, 13, 11, 6, 5, 7, 9]\) that allow to relate the codewords of a code with the elements of some group. Usually, the mapping from the group to the code is is required to satisfy some metric properties, because the distance is what is very important for error-correcting codes. One of the approaches considers so-called propelinear codes, introduced in \([13]\) for the binary space. The codewords of a propelinear code \( C \) are in one-to-one correspondence with a group \( G \) of isometries of the space that acts regularly on the code itself. In other words, given some fixed codeword \( v \in C \) (say, the all-zero word), every other codeword can be uniquely written as \( g(v) \), \( g \in G \). Every propelinear code is transitive; that is, it is an orbit of a group of isometries of the space (for a transitive code in general, this group is not required to act regularly).

In the current correspondence, we will prove that the number of nonequivalent propelinear codes with the same parameters, namely, the parameters of 1-perfect codes over an arbitrary finite field, grows at least exponentially with respect to the square of the code length (Corollary 1). By the order of the logarithm, this number is comparable with the total number of propelinear codes (Theorem 2). In contrast, there is only one (up to equivalence) linear 1-perfect code for each admissible length, but the number of non-linear 1-perfect codes grows doubly-exponentially \([16, 14]\).

For the case \( q = 2 \), an exponential lower bounds (with respect to the square root of the code length, to be more accurate) on the number of transitive and the number of propelinear 1-perfect codes were firstly established in \([12]\) and \([3]\), respectively. Here, we will show how to improve the lower bound and generalize it to an arbitrary prime power \( q \), using rather simple construction. Some other constructions of transitive and propelinear perfect codes can be found in \([1, 8, 15, 2]\).

Section 2 contains definitions and auxiliary lemmas. In Section 3 we formulate the main results of the correspondence. The main theorem is proven in Section 4. In Section 5

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we consider some remarks and examples concerning the structure of the group related to the code. In Section 6 we discuss a problem of studying functions that can result in transitive codes.

2. Preliminaries

Let $F$ be a finite field of order $q$; let $F^n$ be the vector space of all $n$-words over the alphabet $F$. An arbitrary subset of $F^n$ is referred to as a code. A code is linear if it is a vector subspace of $F^n$. A code $C \subset F^n$ is called 1-perfect if for every word $v$ from $F^n$ there is exactly one $c$ in $C$ agreeing with $v$ in at least $n - 1$ positions.

2.1. Vasil’ev–Schönheim construction

Let $H \subset F^n$, and let $f : H \rightarrow F$ be an arbitrary function. Define the set

$$C(H, f) = \{(v_0, p) : v_0 \in F^n, \sum_{a \in F} v_a = c \in H, p = \sum_{a \in F} |v_a| + f(c)\}$$

where $(v_0)_{a \in F}$ is treated as the concatenation of the words $v_a$ (which will be referred to as blocks) in some prefixed order, $|v_a|$ is the sum of all $n$ elements of $v_a$. If $H$ is a 1-perfect code, then $C(H, f)$ is a 1-perfect code in $F^{qn+1}$, known as a Schönheim code [14] (in the case $q = 2$, a Vasil’ev code [16]). Clearly, the set $C(H, f)$ essentially depends on the choice of the function $f$.

**Lemma 1.** For fixed $H$, different $f$ result in different $C(H, f)$.

**Proof.** The graph of the function $f$ can be reconstructed from the set $C(H, f)$:

$$\{(x, f(x)) : x \in H\} = \left\{\sum_{a \in F} v_a, p - \sum_{a \in F} |v_a| : ((v_0)_{a \in F}, p) \in C(H, f)\right\}.$$

Hence, $C(H, f) = C(H, f')$ implies $f = f'$. ▲

2.2. Automorphisms, equivalence, transitivity and propelinearity

The Hamming graph $G(F^n)$ is defined on the vertex set $F^n$; two words are connected by an edge if and only if they differ in exactly one position. It is known (see, e.g., [4, Theorem 9.2.1]) that every automorphism $\Pi$ of $G(F^n)$ is composed from a coordinate permutation $\pi$ and alphabet permutations $\psi_i$ in each coordinate: $\Pi(x) = (\psi_1(x_{\pi^{-1}(1)}), \ldots, \psi_n(x_{\pi^{-1}(n)})$. Two codes are said to be equivalent if there is an automorphism of $G(F^n)$ that maps one of the codes to the other. Note that the algebraic properties of the code, such as being a linear or affine subspace, are not invariant with respect to this combinatorial equivalence, in general. The automorphism group $\text{Aut}(C)$ of a code $C$ consists of all automorphisms of $G(F^n)$ that stabilize (fix set-wise) $C$. A code $C$ containing the all-zero word 0 is transitive if for every codeword $a$ there exists $\varphi_a \in \text{Aut}(C)$ that sends 0 to $a$. If, additionally, the set $\{\varphi_a : a \in C\}$ is closed under composition (that is, for all $a$ and $b$ from $C$ we have $\varphi_a \varphi_b = \varphi_c$, where $c = \varphi_a \varphi_b(0)$), then $C$ is a propelinear code, see e.g. [3].

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2.3. Quadratic functions

Assume $H$ is a subspace of $F^n$. A function $f : H \rightarrow F$ is called quadratic if it can be represented as a polynomial of degree at most 2.

We will use the following elementary property of the quadratic functions (actually, it is a characterizing property).

**Lemma 2.** Let $H$ be a subspace of $F^n$. If $f : H \rightarrow F$ is a quadratic function, then for every $c \in H$ there exist $\beta_0^c, \beta_1^c, \ldots, \beta_n^c \in F$ such that

$$f(x + c) = f(x) + \beta_0^c + \beta_1^c x_1 + \ldots + \beta_n^c x_n \quad \text{for all } x = (x_1, \ldots, x_n) \in H. \quad (1)$$

Moreover, $\beta_i^c, i \in \{1, \ldots, n\}$, depends linearly on $c$: $\beta_i^{c+d} = \beta_i^c + \beta_i^d$.

**Proof.** The difference of $(x_i + c_i)(x_j + c_j)$ and $x_i x_j$ has degree at most 1. Moreover, the coefficients at $x_i$ and $x_j$ in this difference depend linearly on $c$. Hence, the same is true for the difference of $p(x + c)$ and $p(x)$ for every polynomial $p$ of degree at most 2. \(\blacktriangleleft\)

**Lemma 3.** Let $H$ be an $m$-dimensional subspace of $F^n$. There are at least $q^{m^2/2}$ different quadratic functions from $H$ to $F$.

**Proof.** Obviously, a linear transformation of the space does not affect to the property of a function to be quadratic. Hence, we can assume without loss of generality that $H$ consists of the $n$-words that end with $n - m$ zeroes. Then, the number of different quadratic functions is the number of polynomials of degree at most 2 in $m$ first variables, i.e., $q^{m(m-1)/2 + m + m + 1}$ for $q > 2$ and $q^{m(m-1)/2 + m + 1}$ for $q = 2$ (when $x_i^2 \equiv x_i$). \(\blacktriangleleft\)

3. Main results

3.1. Lower bound

In the Section 4 we will prove the following theorem.

**Theorem 1.** If $H \subset F^n$ is a linear code and $f : H \rightarrow F$ is a quadratic function, $f(0) = 0$, then $C(H, f)$ is a propelinear code.

**Corollary 1.** The number of nonequivalent propelinear 1-perfect $q$-ary codes of length $N$ obtained by the Vasil’ev–Schönheim construction is at least $q^{N^2/2(1+o(1))}$.

**Proof.** As follows from Theorem 1, Lemma 1 and Lemma 3 the number of different propelinear 1-perfect codes of type $C(H, f)$ is at least $q^{m^2/2}$, where $m = n - \log_q (nq - n + 1)$ and $n$ is the length of $H$. Since $N = qn + 1$, we see that $q^{m^2/2} = q^{N^2/2q(1+o(1))}$. To evaluate the number of nonequivalent codes, we divide this number by the number $N!q^N = q^{N\log_q N(1+o(1))}$ of all automorphisms of $F^N$ and find that this does not affect on the essential part of the formula. \(\blacktriangleleft\)

3.2. Upper bound

To evaluate how far our lower bound on the number of transitive 1-perfect codes can be from the real value, we derive an upper bound:
Theorem 2. (a) The number of different transitive codes in \( F^N \) does not exceed \( 2^{(N \log_2 N)^2(1+o(1))} \). (b) The number of different propelinear codes in \( F^N \) does not exceed \( q^{N^2 \log_2 N(1+o(1))} \).

Proof. Since every subgroup of \( \text{Aut}(F^N) \) is generated by at most \( \log_2 |\text{Aut}(F^N)| \) elements, the number of subgroups is less than \( |\text{Aut}(F^N)|^{\log_2 |\text{Aut}(F^N)|} = 2^{(N \log_2 N)^2(1+o(1))} \) (recall that \( |\text{Aut}(F^N)| = (q!)^N N! = N^{N(1+o(1))} \)). Since every transitive code \( C \) containing 0 is uniquely determined by its automorphism group (indeed, \( C \) is the orbit of 0 under \( \text{Aut}(F^N) \)), statement (a) follows.

The automorphisms assigned to the codewords of a propelinear code \( C \) form a group of order \( |C| \leq q^N \). It is generated by at most \( \log_2 q^N = N \log_2 q \) elements; each of them can be chosen in less than \( |\text{Aut}(F^N)| = N^{N(1+o(1))} \) ways; (b) follows. ▲

4. Proof of Theorem 1

Let \( H \subset F^n \) be a linear code and let \( f : H \to F \) be a quadratic function. The key point in the proof is the following simple statement.

Lemma 4. Let \( f'(x) = f(x) + \beta x_j \) for some \( j \in \{1, \ldots, n\}, \beta \in F \). Then \( C(H, f') = \Pi^\beta \alpha C(H, f) \) where \( \Pi^\beta \alpha \) is the coordinate permutation that sends the \( j \)’th coordinate of the block \( v_{\alpha + \beta} \) to the \( j \)’th coordinate of the block \( v_{\alpha} \) for all \( \alpha \in F \) and fixes the other coordinates.

Proof. Let us consider the codeword \( x = ((v_{\alpha})_{\alpha \in F}, p) \) of \( C(H, f) \). It satisfies \( p = \sum_{\alpha \in F} \alpha |v_{\alpha}| + f(c) \). After the coordinate permutation \( \Pi^\beta \alpha \), we obtain the word \( y = \Pi^\beta \alpha x = ((u_{\alpha})_{\alpha \in F}, p) \) where for all \( \alpha \) the word \( u_{\alpha} \) coincides with \( v_{\alpha} \) in all positions except the \( j \)th, \( u_{\alpha, j} \) which is equal to \( v_{\alpha + \beta, j} \). Now we have

\[
p = \sum_{\alpha \in F} \alpha |v_{\alpha}| + f(c) = \sum_{\alpha \in F} \sum_{k \neq j} \alpha u_{\alpha, k} + \sum_{\alpha \in F} \alpha u_{\alpha, j} + f(c) \\
= \sum_{\alpha \in F} \sum_{k \neq j} \alpha u_{\alpha, k} + \sum_{\alpha \in F} \alpha u_{\alpha - \beta, j} + f(c) \\
= \sum_{\alpha \in F} \sum_{k \neq j} \alpha u_{\alpha, k} + \sum_{\alpha \in F} (\alpha + \beta) u_{\alpha, j} + f(c) \\
= \sum_{\alpha \in F} \sum_{k=1}^n \alpha u_{\alpha, k} + \beta \sum_{\alpha \in F} u_{\alpha, j} + f(c) = \sum_{\alpha \in F} \alpha |u_{\alpha}| + f(c) + \beta c_j,
\]

(we used that \( c = (c_1, \ldots, c_n) = \sum v_{\alpha} = \sum u_{\alpha} \) which proves that \( \Pi^\beta \alpha(x) \in C(H, f') \). ▲

Now denoting \( \Pi^c = \Pi_1^{c_1} \Pi_2^{c_2} \ldots \Pi_n^{c_n} \), where the coefficients \( \beta_j \) are from \( \Pi \), we get the following fact, which immediately proves the transitivity of the code:

Lemma 5. For every codeword \( w = ((w_{\alpha})_{\alpha \in F}, p) \) of \( C(H, f) \), the transform \( \Phi_w(v) = w + \Pi^c(v) \), where \( c = \sum_{\alpha \in F} w_{\alpha} \), is an automorphism of \( C(H, f) \), which sends the all-zero word to \( w \).

Proof. Consider \( v = ((v_{\alpha})_{\alpha \in F}, s) \) from \( C(H, f) \). It satisfies \( s = \sum_{\alpha \in F} \alpha |v_{\alpha}| + f(d) \), where \( d = \sum_{\alpha} v_{\alpha} \). Applying Lemma 4 with \( j = 1, \ldots, n \), we see that \( \Pi^c(v) = ((u_{\alpha})_{\alpha \in F}, s) \) satisfies \( s = \sum_{\alpha \in F} \alpha |u_{\alpha}| + f(d) + \beta_1 d_1 + \ldots + \beta_n d_n \), where \( d = (d_1, \ldots, d_n) = \sum \alpha u_{\alpha} \).
Adding \( w = ((w_\alpha)_{\alpha \in F}, p) \), we obtain \( w + \Pi^c(v) = ((w_\alpha + u_\alpha), r) \), where
\[
\begin{align*}
r &= \sum_{\alpha \in F} \alpha|u_\alpha| + f(d) + \beta_1^c d_1 + \ldots + \beta_n^c d_n + \sum_{\alpha \in F} \alpha|w_\alpha| + f(c) \\
&= \sum_{\alpha \in F} \alpha|u_\alpha + w_\alpha| + f(d + c) - \beta_0^c + f(c).
\end{align*}
\]
But \( f(c) = f(0) + \beta_0^c \), as we see from (1). Since \( f(0) = 0 \), we have proved that \( w + \Pi^c(v) \) belongs to \( C(H, f) \). ▲

So, we get the transitivity. It remains to prove that the set of \( \Phi_w, w \in C(H, f) \) is closed under composition.

**Lemma 6.** For every \( c, d \in H \) the composition \( \Pi^c \Pi^d \) equals \( \Pi^{c+d} \).

**Proof.** As follows directly from the definitions of \( \Pi^c \) and \( \Pi_i^\beta \),
\[
\Pi^c \Pi^d = \Pi_1^c \ldots \Pi_n^c \Pi_1^d \ldots \Pi_n^d = \Pi_1^{c+d} \Pi_2^{c+d} \ldots \Pi_n^{c+d}.
\]
By the definition of \( \Pi_i^\beta \), we have \( \Pi_i^{\beta c} \Pi_i^{\beta d} = \Pi_i^{\beta c + \beta d} \). But, by Lemma 2 \( \beta_i^c + \beta_i^d = \beta_i^{c+d} \). Finally, we have \( \Pi^c \Pi^d = \Pi_1^{\beta_{c+d}} \ldots \Pi_n^{\beta_{c+d}} = \Pi^{c+d} \). ▲

Now, consider \( w = ((w_\alpha)_{\alpha \in F}, p) \) and \( v = ((v_\alpha)_{\alpha \in F}, s) \) form \( C(H, f) \). Denote \( c = \sum_\alpha w_\alpha \) and \( d = \sum_\alpha v_\alpha \); observe that the permutation \( \Pi^c \) will not change the value of the last sum. Then,
\[
\Phi_w \Phi_v (\cdot) = w + \Pi^c(v + \Pi^d(\cdot)) = w + \Pi^c(v) + \Pi^c(\Pi^d(\cdot)) = u + \Pi^c(\cdot),
\]
where \( u = ((u_\alpha)_{\alpha \in F}, t) = w + \Pi^c(v), e = \sum_\alpha u_\alpha = c + d \). This completes the proof of the theorem.

## 5. Remarks and examples

As follows from the definition, to every codeword \( v \) of a propelinear code \( C \) there corresponds an automorphism \( \Phi_v \) of \( C \) and the set \( \{ \Phi_v : v \in C \} \) forms a subgroup of the automorphism group of \( C \). Although such subgroup, a **propelinear structure**, is not unique in general (see Remark 2 below), in the previous section we explicitly defined a variant of the choice of \( \Phi_v \) for every \( v \in C(H, f) \). Below, we provide two remarks with examples about the propelinear structure defined in the previous section.

**Remark 1.** For every \( v \in C(H, f) \), the element \( \Phi_v \) has the order 1, \( p \), or \( p^2 \), where \( p \) is the prime that divides \( q \). Indeed, every permutation \( \Pi^c \) is of order 1 or \( p \); hence, \( (\Phi_v)^p \) corresponds to the identity permutation and has the order 1 or \( p \).

As an example, we consider the (non-perfect) code \( C(H, f) \) constructed with the following parameters: \( q = 2, n = 2, H = F^2, f(x_1, x_2) = x_1 x_2 \). From (11) we find \( \beta_{01}^1 = 1, \beta_{00}^1 = 0, \beta_{10}^1 = 0, \beta_{11}^1 = 1, \beta_{21}^1 = 1 \). The group of automorphisms related with the propelinear code \( C(H, f) \) is generated by three elements \( \Phi_u, \Phi_v, \Phi_w \) with \( u = (110001), v = (100000), w = (10101) \) and the corresponding coordinate permutations \( \Pi^{11} = (13)(24), \Pi^{10} = (24), \Pi^{00} = \text{Id} \). The first element \( \Phi_u \) generates a cycle with the
corresponding codewords (00 00 0), (11 00 1), (11 11 0), (00 11 1). The second generating
element Φ_v adds four more codewords: (10 00 0), (00 01 1), (01 11 0), (11 10 1); the corre-
sponding automorphisms are of order 2. The group generated by Φ_u and Φ_v is described
by the orders of Φ_u, Φ_v and the identity Φ_uΦ_v = (Φ_u)^{-1}, and it is isomorphic to the
dihedral group D_4. The last generating element Φ_w commutes with all other elements
and has the order 2. It follows that the group of automorphisms related with C(H, f) is
isomorphic to the direct product D_4 × Z_2, where Z_2 is the cyclic group of order 2.

Remark 2. If H ≠ F^n, then there are more than one quadratic representations of every
quadratic function on H. The coefficients β_i^c and, as follows, the subgroup \{Φ_v : v ∈ C\}
of the automorphism group of the code depend on the representation; so, there are several
propelinear structures corresponding to the same code C(H, f). For example, the all-zero
function over H = \{000, 111\} (q = 2) can be represented as f(x_1, x_2, x_3) = 0 or, e.g., as
f(x_1, x_2, x_3) = x_1x_2 + x_1x_3. The resulting code is the same (a 1-perfect Hamming code
of length 7); but in the first case, the group is, of course, an additive subgroup of F^n
and isomorphic to Z_4^4, while the second representation leads to the group isomorphic to
Z_4 × Z_2^2.

6. Conclusion

For further development of the topic, it would be interesting to consider a wider class of
functions resulting in transitive (propelinear) codes. Such functions should have properties
similar to transitivity (propelinearity) of codes:

Problem. For a vector space V and a group A of linear permutations of V, find non-
quadratic functions f such that for every c from V there exists µ ∈ A meeting
f(µ(x) + c) = f(x) + l(x) for some affine l. For example, for constructing transitive 1-perfect codes as
above, we can take V = H and A ⊂ Aut(H).

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