COMPACT GROUP ACTIONS WITH THE TRACIAL ROKHLIN PROPERTY

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Abstract. We define a “tracial” analog of the Rokhlin property for actions of second countable compact groups on infinite dimensional simple separable unital C*-algebras. We prove that fixed point algebras under such actions (and, in the appropriate cases, crossed products by such actions) preserve simplicity, Property (SP), tracial rank zero, the Popa property, and pure infiniteness. Our version of the tracial Rokhlin property is an exact generalization of the tracial Rokhlin property for actions of finite groups on classifiable C*-algebras (in the sense of the Elliott program), but for actions of finite groups on more general C*-algebras it may be stronger. We discuss several alternative versions of the tracial Rokhlin property. We give examples of actions of a totally disconnected infinite compact group on a UHF algebra, and of the circle group on a simple unital AT algebra, which have our version of the tracial Rokhlin property, but do not have the Rokhlin property, or even finite Rokhlin dimension with commuting towers.

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1. Introduction

Tracially AF C*-algebras, now known as C*-algebras with tracial rank zero (see [37]), were introduced in [36]. Roughly speaking, a C*-algebra has tracial rank zero if the local approximation characterization of AF algebras holds after cutting out a “small” approximately central projection. The term “tracial” comes from the fact that, in good cases, a projection $p$ is “small” if $\tau(p) < \varepsilon$ for every
tracial state $\tau$ on $A$. Simple C*-algebras with tracial rank zero are much more common than simple AF algebras, and the classification [39] of simple separable nuclear C*-algebras with tracial rank zero and satisfying the Universal Coefficient Theorem can be regarded as a vast generalization of the classification of AF algebras.

The use of the Rokhlin property for actions of finite groups on C*-algebras goes back at least to Herman and Jones in [23] and [24]. It was motivated by earlier work in von Neumann algebras, such as Jones [31]. Major advances on classification of Rokhlin actions of finite groups on C*-algebras which are classifiable in the sense of the Elliott program appear in [28] and [29]. The Rokhlin property can be viewed as a regularity condition for the group action, which can be used to show that various structural properties pass from a C*-algebra to its crossed product. This was first realized in Theorem 2.2 of [44]. The paper [40], and later work, followed up on this idea. See Theorem 2.6 of [43] for a summary of what was known at the time that survey was written. For an example of more recent work, see [50].

Actions of finite groups with the Rokhlin property are rare (unlike actions of $\mathbb{Z}$ with the Rokhlin property). See the discussion in [43], especially Section 3 there. The tracial Rokhlin property for actions of finite groups, introduced in [44], is related to the Rokhlin property in roughly the same way that tracial rank zero is related to the AF property. There are many more actions with the tracial Rokhlin property than there are Rokhlin actions. See Section 3 of [43], especially Example 3.12. Crossed products by actions of finite groups with the tracial Rokhlin property still preserve various structural properties of C*-algebras, although not as many as with the Rokhlin property. The original result was Theorem 2.6 of [44], for tracial rank zero. This result played a key role in the proof in [10] that the crossed products of irrational rotation algebras by the “standard” actions of $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$ are AF. Other preservation of structure theorems can be found in [26] (for compact groups), [1], [41] (for a more general setup than group actions), and [3] (using the weak tracial Rokhlin property).

In [26], Hirshberg and Winter introduced the Rokhlin property for actions of second countable compact groups on C*-algebras. Since then, crossed products by compact group actions with the Rokhlin property have been studied by several authors. In particular, permanence properties are proved in [26] and [13]. As with finite groups, Rokhlin actions of compact groups are rare, especially if the group is connected. For example, by Theorem 3.3(3) of [17], if $A$ is unital, $K_0(A)$ is finitely generated, and $A$ admits an action of the circle with the Rokhlin property, then $K_0(A) \cong K_1(A)$.

In this paper, we therefore extend the definition of the tracial Rokhlin property to actions of second countable compact groups on simple separable unital C*-algebras. Our property, which we call the tracial Rokhlin property with comparison, is formally stronger than the naive extension. We were unable to prove the desired permanence properties using the naive extension, and we doubt that it can be done. For actions of finite groups on simple separable unital C*-algebras with strict comparison, and in some other cases, the tracial Rokhlin property with comparison is equivalent to the tracial Rokhlin property, but this seems unlikely to be true in general. We then prove that fixed point algebras under such actions, and, in the appropriate cases, crossed products by such actions, preserve simplicity, Property (SP), tracial rank zero, the Popa property, and pure infiniteness. The fact that the crossed product is usually not unital causes some problems, and makes it
more convenient to work with the fixed point algebra. In addition, the Popa property is only defined for unital algebras. We further give examples of actions of both an infinite totally disconnected compact group and the circle group $S^1$ which have the tracial Rokhlin property with comparison but don’t have the Rokhlin property.

We briefly address a different generalization of the Rokhlin property, namely finite Rokhlin dimension with commuting towers. This concept was defined for actions of compact groups in [14], generalizing the version for finite groups in [27]. It is clear from [25] that even actions of finite groups which have finite Rokhlin dimension with commuting towers are relatively rare. On the other hand, for finite $G$, and under some restrictions on the algebra (infinite dimensional, simple, finite, unital, with strict comparison, and having at most countably many extreme quasitraces), finite Rokhlin dimension with commuting towers implies the tracial Rokhlin property [19]. We hope to address this question for compact groups in future work, but we point out here that we prove that our examples do not have finite Rokhlin dimension with commuting towers as in [14].

The paper is organized as follows. In the rest of this section, we present some notation, definitions, and basic lemmas that we will use throughout the paper. Section 2 contains the definition of the tracial Rokhlin property with comparison for compact groups, and its basic properties. In particular, we give a first version of a central sequence formulation of the definition, and give an averaging process which is the key technical tool for proofs of permanence properties.

In Section 3, we prove that the fixed point algebra and the crossed product of an infinite dimensional simple separable unital C*-algebra by an action of a compact group with the tracial Rokhlin property with comparison are again simple. We then give an improved version of a central sequence formulation of the tracial Rokhlin property with comparison. The rest of our permanence properties are in Section 4.

Section 5 discusses the relation between the tracial Rokhlin property with comparison and the tracial Rokhlin property. In particular, when the group is finite and under some reasonable conditions on the algebra, we prove that they are equivalent. Other variants of the tracial Rokhlin property are possible, and are suggested by the apparent failure of the naive generalization to do what is wanted. We discuss two of the most promising variants in Section 6, and describe what we can prove with them.

In Sections 7 and 8, we construct two examples of actions which have the tracial Rokhlin property with comparison. The first is an action of $(\mathbb{Z}_2)^N$ on the $3^\infty$ UHF algebra. The second is an action of the circle group $S^1$ on a simple unital AT algebra. These examples do not have the Rokhlin property, or even finite Rokhlin dimension with commuting towers. In both cases, the actions do have one of the alternate versions of the tracial Rokhlin property discussed in Section 6.

In Section 9 we give an easy nonexistence result for actions of $S^1$ with even the weakest form of the tracial Rokhlin property we consider.

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In the rest of this section, we collect some notation, definitions, and results that we need.

The C*-algebra of $n \times n$ matrices will be denoted by $M_n$. If $C$ is a C*-algebra, we write $C_+$ for the set of positive elements in $C$ and $C_{sa}$ for the set of selfadjoint elements in $C$. If $C$ is unital, we denote its tracial state space by $T(C)$, its set of normalized 2-quasitraces by $\text{QT}(C)$, and its unitary group by $U(C)$. Similarly, the set of unitary operators on a Hilbert space $\mathcal{H}$ is denoted $U(\mathcal{H})$. We denote the circle group by $S^1$, and identify it with the set of complex numbers of absolute value 1.

An action $\alpha : G \to \text{Aut}(A)$ of group $G$ on a C*-algebra $A$ is assumed continuous unless stated otherwise. Also, we denote by $A^\alpha$ the fixed point subalgebra of $A$ under $\alpha$.

**Notation 1.1.** Let $A$ be a C*-algebra, let $G$ be a topological group, and let $\alpha : G \to \text{Aut}(A)$ be an action of $G$ on $A$. We define

$$l^\infty(N, A) = \left\{ (a_n)_{n \in N} \in A^N : \sup_{n \in N} \|a_n\| < \infty \right\},$$

$$c_0(N, A) = \left\{ (a_n)_{n \in N} \in l^\infty(N, A) : \lim_{n \to \infty} \|a_n\| = 0 \right\},$$

and

$$A_\infty = l^\infty(N, A)/c_0(N, A).$$

We further let $\pi_A : l^\infty(N, A) \to A_\infty$ be the quotient map.

There are obvious actions $g \mapsto \alpha^\infty_g$ of $G$ on $l^\infty(N, A)$ and $g \mapsto \alpha_{\infty, g}$ of $G$ on $A_\infty$, which need not be continuous. We define

$$l_\alpha^\infty(N, A) = \left\{ a \in l^\infty(N, A) : g \mapsto \alpha^\infty_g(a) \text{ is continuous} \right\}$$

and

$$A_{\infty, \alpha} = \pi_A \left( l_\alpha^\infty(N, A) \right) \subseteq A_\infty.$$

The subalgebra $l_\alpha^\infty(N, A)$ is $\alpha^\infty$-invariant, so we still write $\alpha^\infty$ for the action of $G$ on $A_{\infty, \alpha}$. By construction, this action is continuous. We further still write $g \mapsto \alpha_{\infty, g}$ for the induced action on $A_{\infty, \alpha}$, which is also obviously continuous.

We identify $A$ in the obvious way with the subalgebra of $l_\alpha^\infty(N, A)$ consisting of constant sequences, and also with the image of this subalgebra in $A_{\infty, \alpha}$ under $\pi_A$. Then we can form the relative commutant algebra $A_{\infty, \alpha} \cap A' \subseteq A_{\infty, \alpha}$. It is clearly $\alpha^\infty$-invariant, and we also denote the restricted action on this subalgebra by $\alpha_{\infty}$.

We have

$$A_{\infty, \alpha} \cap A' = \left\{ \pi_A((a_n)_{n \in N}) \in A_\infty : (a_n)_{n \in N} \in l_\alpha^\infty(N, A) \text{ and } \lim_{n \to \infty} \|a_n a - a a_n\| = 0 \text{ for all } a \in A \right\}.$$

**Definition 1.2** ([25], Definition 1.3). Let $G$ be a compact group, and let $A$ and $D$ be unital C*-algebras. Let $\alpha : G \to \text{Aut}(A)$ and $\gamma : G \to \text{Aut}(D)$ be actions of $G$ on $A$ and $D$. Let $S \subseteq D$ and $D \subseteq A$ be subsets, and let $\varepsilon > 0$. A unital completely positive map $\varphi : D \to A$ is said to be an $(S, F, \varepsilon)$-approximately equivariant central multiplicative map if:

1. $\|\varphi(x y) - \varphi(x) \varphi(y)\| < \varepsilon$ for all $x, y \in S$.
2. $\|\varphi(x) a - a \varphi(x)\| < \varepsilon$ for all $x \in S$ and all $a \in F$.
(3) \( \sup_{g \in G} ||\varphi(\gamma_g(x)) - \alpha_g(\varphi(x))|| < \varepsilon \) for all \( x \in S \).

Ignoring the actions and omitting condition (3), we get the usual definition of an \((S, F, \varepsilon)\)-approximately central multiplicative map.

The following versions of approximate centrality and approximate multiplicativity are convenient.

**Definition 1.3.** Let \( A \) and \( D \) be unital C*-algebras, and let \( S \subseteq D \). A unital completely positive map \( \varphi : D \to A \) is said to be an \((n, S, \varepsilon)\)-approximately multiplicative map if whenever \( m \in \{1, 2, \ldots, n\} \) and \( x_1, x_2, \ldots, x_m \in S \), we have
\[
||\varphi(x_1x_2\cdots x_m) - \varphi(x_1)\varphi(x_2)\cdots \varphi(x_m)|| < \varepsilon.
\]

If also \( F \subseteq A \) is given, then \( \varphi \) is said to be an \((n, S, F, \varepsilon)\)-approximately central multiplicative map if, in addition, \( ||\varphi(x)a - a\varphi(x)|| < \varepsilon \) for all \( x \in S \) and all \( a \in F \).

**Lemma 1.4.** Let \( A \) and \( D \) be unital C*-algebras, let \( \varepsilon > 0 \), let \( S \subseteq D \) be compact, and let \( n \in \mathbb{N} \).

1. There exist \( \delta > 0 \) and a compact subset \( T \subseteq D \) such that whenever \( \varphi : D \to A \) is unital completely positive and is a \((T, \delta)\)-approximately multiplicative map, then \( \varphi \) is an \((n, S, \varepsilon)\)-approximately multiplicative map.
2. If in addition \( F \subseteq A \) is a compact subset, then there exist \( \delta > 0 \) and compact subsets \( T \subseteq D \) and \( E \subseteq A \) such that whenever \( \varphi : D \to A \) is unital completely positive and is a \((T, E, \delta)\)-approximately central multiplicative map, then \( \varphi \) is an \((n, S, F, \varepsilon)\)-approximately central multiplicative map.

**Proof.** The proof is routine, and is omitted. \( \square \)

**Notation 1.5.** If \( G \) is a locally compact group, we denote by \( \text{Lt} : G \to \text{Aut}(C_0(G)) \) the action of \( G \) on \( C_0(G) \) induced by the action of \( G \) on itself by left translation.

We also recall the following definitions related to Cuntz comparison. The first part is originally from [7].

**Definition 1.6.** Let \( A \) be a C*-algebra.

1. For \( a, b \in M_\infty(A)_+ \), we say that \( a \) is Cuntz subequivalent to \( b \) in \( A \), written \( a \sim_A b \), if there is a sequence \( (v_n)_{n=1}^\infty \) in \( M_\infty(A) \) such that \( \lim_{n \to \infty} v_n b v_n^* = a \).
2. We say that \( a \) and \( b \) are Cuntz equivalent in \( A \), written \( a \sim_A b \), if \( a \sim_A b \) and \( b \sim_A a \). This relation is an equivalence relation.
3. If \( \tau \in \text{QT}(A) \) and \( a \in M_\infty(A)_+ \), then \( d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n}) \).

**Lemma 1.7** ([47], Lemma 2.6). Let \( A \) be a simple C*-algebra, and let \( B \subseteq A \) be a nonzero hereditary subalgebra. Let \( n \in \mathbb{N} \), and let \( a_1, \ldots, a_n \in A_+ \setminus \{0\} \). Then there exists \( b \in B_+ \setminus \{0\} \) such that \( b \sim_A a_j \) for \( j = 1, \ldots, n \).

**Lemma 1.8** ([44], Lemma 1.10). Let \( A \) be an infinite dimensional simple unital C*-algebra with Property (SP). Let \( B \subseteq A \) be a nonzero hereditary subalgebra, and let \( n \in \mathbb{N} \). Then there exist nonzero Murray-von Neumann equivalent mutually orthogonal projections \( p_1, p_2, \ldots, p_n \in B \).
2. The tracial Rokhlin property with comparison

In this section we define the tracial Rokhlin property with comparison for actions of compact groups, and prove several equivalent versions and some basic facts. We begin with a brief reminder of the tracial Rokhlin property for actions of finite groups and the Rokhlin property for actions of compact groups. These are the properties we combine in this paper.

When restricted to finite groups, our version of the tracial Rokhlin property for actions of compact groups is stronger than the usual tracial Rokhlin property for actions of finite groups. We discuss this issue in more detail in Section 5, where we also discuss other possible definitions. We have been unable to prove the expected results just using the naive generalization of the tracial Rokhlin property for actions of finite groups.

In Lemma 2.15, we give a version of the definition (for separable C*-algebras) in terms of central sequence algebras. It will be improved later, but the proof of the improvement uses simplicity of the fixed point algebra, whose proof in turn uses the version in this section. We end this section with a theorem on existence of suitable maps from $A$ to $A^\alpha$, which in many proofs involving the tracial Rokhlin property with comparison is what we actually use.

**Definition 2.1** ([44], Definition 1.2). Let $A$ be an infinite dimensional simple unital C*-algebra, let $G$ be a finite group, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$. The action $\alpha$ has the **tracial Rokhlin property** if for every finite set $F \subseteq A$, every $\varepsilon > 0$, and every $x \in A_+$ with $\|x\| = 1$, there exist mutually orthogonal projections $p_g \in A$ for $g \in G$ such that, with $p = \sum_{g \in G} p_g$, we have:

1. $\|p_g a - ap_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$.
2. $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$.
3. $1 - p \not\prec_A x$.
4. $\|pxp\| > 1 - \varepsilon$.

Separability is assumed in Definition 1.2 of [44], but there is no reason to require separability.

We note in passing that, by Proposition 5.26 of [46], in (2), one can require $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.

Hirshberg and Winter defined the Rokhlin property for an action of a second countable compact group in Definition 3.2 of [26]. See the explanation before Definition 2.3 in [16].

**Definition 2.2** ([16], Definition 2.3). Let $A$ be a separable unital C*-algebra, let $G$ be a second countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$. Recalling Notation 1.5, we say that $\alpha$ has the **Rokhlin property** if there is an equivariant unital homomorphism

$$\varphi: (C(G), \text{Lt}) \to (A_{\infty, \alpha} \cap A', \alpha_\infty).$$

Separability is not assumed in [16], but the use of the central sequence algebra means that the definition is not appropriate for nonseparable C*-algebras. The right definition for the nonseparable case is the condition in the following lemma.

**Lemma 2.3.** Let $A$ be a separable unital C*-algebra, let $G$ be a second countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$. Then $\alpha$ has the Rokhlin property if and only if for every if for every finite set $F \subseteq A$, every
finite set $S \subseteq C(G)$, and every $\varepsilon > 0$, there exists a unital completely positive map
$
\varphi: C(G) \to A
$
which is $(F, S, \varepsilon)$-approximately equivariant central multiplicative (Definition 1.2).

Proof. The proof is standard, and is omitted. □

Definition 2.4. Let $A$ be an infinite dimensional simple unital C*-algebra, and let $\alpha: G \to \Aut(A)$ be an action of a second countable compact group $G$ on $A$.
The action $\alpha$ has the tracial Rokhlin property with comparison if for every finite
set $F \subseteq A$, every finite set $S \subseteq C(G)$, every $\varepsilon > 0$, every $x \in A_+$ with $\|x\| = 1$, and every $y \in (A^\alpha)_+ \setminus \{0\}$, there exist a projection $p \in A^\alpha$ and a unital completely
positive map $\varphi: C(G) \to pAp$ such that:
(1) $\varphi$ is an $(F, S, \varepsilon)$-approximately equivariant central multiplicative map (Def-
nition 1.2).
(2) $1 - p \precsim A x$.
(3) $1 - p \precsim A^\alpha y$.
(4) $1 - p \precsim A^\alpha p$.
(5) $\|pxp\| > 1 - \varepsilon$.

Definition 2.4 requires that $p$ be $\alpha$-invariant, a requirement not present in Def-
nition 2.1. But Definition 2.1 is unchanged if this requirement is added, by Lemma 1.17 of [44]. Definition 2.4 has two conditions which have no analogs in Def-
nition 2.1. Condition (3) is automatic for finite groups. We do not know whether it is automatic in general, but there is evidence to suggest that it isn’t. Condition (4) is automatic for finite groups under some additional conditions, for example, if $A$
has strict comparison. Even for finite groups, we do not know whether it is always automatic. For this reason, we don’t use the term “tracial Rokhlin property” in Definition 2.4. See Section 5 for these results, and more.

When $G$ is finite, and assuming suitable separability and nuclearity hypotheses, this definition can be reformulated in terms of the central sequence algebra. See Lemma 1.5 of [25].

For comparison, we give a reformulation of Definition 2.4 for finite groups which more closely resembles Definition 2.1.

Lemma 2.5. Let $A$ be an infinite dimensional simple unital C*-algebra, let $G$ be
a finite group, and let $\alpha: G \to \Aut(A)$ be an action of $G$ on $A$. Then $\alpha$ has the
tracial Rokhlin property with comparison if and only if for every finite set $F \subseteq A$, every $\varepsilon > 0$, every $x \in A_+$ with $\|x\| = 1$, and every $y \in A^\alpha_+ \setminus \{0\}$ there exist a projection $p \in A^\alpha$ and mutually orthogonal projections $(p_g)_{g \in G}$ such that:
(1) $p = \sum_{g \in G} p_g$.
(2) $\|p_g a - ap_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$.
(3) $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$.
(4) $1 - p \precsim A x$.
(5) $1 - p \precsim A^\alpha y$.
(6) $1 - p \precsim A^\alpha p$.
(7) $\|pxp\| > 1 - \varepsilon$.

Proof. Set $S = \{\chi_g: g \in G\} \subseteq C(G)$. It is easily seen that Definition 2.4 is equivalent (with a change in the value of $\varepsilon$) to the same statement but in which we always use this choice of $S$. 
Assume the conditions of the lemma. Let $F \in A$ be finite, let $\varepsilon > 0$, let $x \in A_+$ satisfy $\|x\| = 1$, and let $y \in A^\alpha \setminus \{0\}$. Let $p$ and $(p_g)_{g \in G}$ be as in the condition of the lemma for these choices. Then Conditions (2), (3), (4), and (5) in Definition 2.4 follow immediately. Define a unital homomorphism $\varphi: C(G) \to pAp$ by $\varphi(f) = \sum_{g \in G} f(g)p_g$ for $f \in C(G)$. The following calculations then show that $\varphi$ is $(F,S,\varepsilon)$-approximately equivariant central, and prove (1) in Definition 2.4. First, for $h \in G$, by (3) at the second step, we have
\[
\max_{g \in G} \|\varphi(\chi_{(gh)}) - (\alpha_g \circ \varphi)(\chi_{(h)})\| = \max_{g \in G} \|p_{gh} - \alpha_g(p_h)\| < \varepsilon.
\]
Second, for $g \in G$ and $a \in F$, by (2) at the second step, we have
\[
\|\varphi(\chi_{(g)})a - a\varphi(\chi_{(g)})\| = \|p_ga - ap_g\| < \varepsilon.
\]
For the other direction, assume that $\alpha$ has the tracial Rohlin property with comparison. Set $n = \text{card}(G)$. Let $F \subset A$ be finite, let $\varepsilon > 0$, let $x \in A_+$ satisfy $\|x\| = 1$, and let $y \in A^\alpha \setminus \{0\}$. Without loss of generality $\|a\| \leq 1$ for all $a \in A$. Set
\[
\varepsilon_0 = \min\left(\frac{1}{2n}, \frac{\varepsilon}{3}\right).
\]
Choose $\delta > 0$ so small that $\delta \leq \varepsilon_0$ and whenever $B$ is a C*-algebra and $b_1,b_2,\ldots,b_n \in B$ are selfadjoint and satisfy $\|b_j^2 - b_j\| < 3\delta$ for $j = 1,2,\ldots,n$ and $\|b_jb_k\| < \delta$ for distinct $j,k \in \{1,2,\ldots,n\}$, then there are mutually orthogonal projections $p_j \in B$ for $j = 1,2,\ldots,n$ such that $\|p_j - b_j\| < \varepsilon_0$ for $j = 1,2,\ldots,n$.

Apply Definition 2.4 with $\delta$ in place of $\varepsilon$, and with $F$, $x$, and $y$ as given, getting $p \in A^\alpha$ and $\varphi: C(G) \to pAp$ as there. In particular:
\[
\begin{align*}
(8) & \|\varphi(\chi_{(g)})^2 - \varphi(\chi_{(g)})\| < \delta \quad \text{for all } g \in G. \\
(9) & \|\varphi(\chi_{(g)})\varphi(\chi_{(h)})\| < \delta \quad \text{for all } g,h \in G \text{ with } g \neq h. \\
(10) & \|\varphi(\chi_{(g)})a - a\varphi(\chi_{(g)})\| < \delta \quad \text{for all } g \in G \text{ and all } a \in F. \\
(11) & \|\varphi(\chi_{(g)})\chi_{(h)} - \varphi(\chi_{(gh)})\| < \delta \quad \text{for all } g,h \in G.
\end{align*}
\]
By (8), (9), and the choice of $\delta$, there are mutually orthogonal projections $p_g$ for $g \in G$ such that $\|p_g - \varphi(\chi_{(g)})\| < \varepsilon_0$. Using (10), for $g \in G$ and $a \in F$ we get
\[
\|p_ga - ap_g\| \leq \|\varphi(\chi_{(g)})a - a\varphi(\chi_{(g)})\| + 2\|p_g - \varphi(\chi_{(g)})\| < \delta + 2\varepsilon_0 \leq \varepsilon.
\]
This is (2).

Using (11), for $g, h \in G$ we get
\[
\|\alpha_g(p_h) - p_{gh}\| \leq \|\varphi(\chi_{(gh)})\chi_{(h)} - \varphi(\chi_{(gh)})\| + \|p_h - \varphi(\chi_{(h)})\| + \|p_{gh} - \varphi(\chi_{(gh)})\|
\]
\[
< \delta + 2\varepsilon_0 \leq \varepsilon.
\]
This is (3). Also, since $\varphi(1) = 1$,
\[
\left\|\sum_{g \in G} p_g - \sum_{g \in G} \varphi(\chi_{(g)}) - p_g\right\| < n\varepsilon_0 < 1,
\]
so, since $\sum_{g \in G} p_g$ is a projection, we get $\sum_{g \in G} p_g = p$. This is (1). Conditions (4), (5), (6), and (7) are immediate. □

**Proposition 2.6.** Let $A$ be an infinite dimensional simple unital C*-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an action of a second countable compact group $G$ on $A$. If $\alpha$ has the Rohlin property, then $\alpha$ has the tracial Rohlin property with comparison.

**Proof.** By Lemma 2.3, one can take $p = 1$ in Definition 2.4. □
Definition 2.4 is unchanged if we merely assume that $F$ and $S$ are compact instead of finite, and require that the map \( \varphi : C(G) \to pAp \) be exactly equivariant and to be an \((n, F, S, \varepsilon)\)-approximately central multiplicative map.

**Lemma 2.7.** Let $A$ be an infinite dimensional simple unital C*-algebra, and let \( \alpha : G \to \text{Aut}(A) \) be an action of a second countable compact group $G$ on $A$. Then \( \alpha \) has the tracial Rokhlin property with comparison if and only if for every compact set $F \subseteq A$, every compact set $S \subseteq C(G)$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, every $x \in A_+$ with \( \|x\| = 1 \), and every $y \in (A^n)_+ \setminus \{0\}$, there exist a projection $p \in A^n$ and a unital completely positive map \( \varphi : C(G) \to pAp \) such that:

1. \( \varphi \) is an \((n, F, S, \varepsilon)\)-approximately central multiplicative map (Definition 1.3).
2. \( 1 - p \leq x \).
3. \( 1 - p \leq A^\circ \ y \).
4. \( 1 - p \leq A^\circ \ p \).
5. \( \|xp\| > 1 - \varepsilon \).

**Proof.** If in (1) we merely ask for an \((F, S, \varepsilon)\)-approximately central multiplicative map, the argument is the same as in Remark 1.4 of [25]. To get (1) as stated, use Lemma 1.4(2). \( \square \)

If $A$ is finite, then Condition (5) in Definition 2.4 and Condition (5) in Lemma 2.7 are redundant.

**Lemma 2.8.** Let $A$ be a finite infinite dimensional simple unital C*-algebra, and let \( \alpha : G \to \text{Aut}(A) \) be an action of a second countable compact group $G$ on $A$. Then \( \alpha \) has the tracial Rokhlin property with comparison if and only if for every finite set $F \subseteq A$, every finite set $S \subseteq C(G)$, every $\varepsilon > 0$, every $x \in A_+ \setminus \{0\}$, and every $y \in (A^n)_+ \setminus \{0\}$, there exist a projection $p \in A^n$ and a unital completely positive map \( \varphi : C(G) \to pAp \) such that:

1. \( \varphi \) is an \((F, S, \varepsilon)\)-approximately equivariant central multiplicative map.
2. \( 1 - p \leq x \).
3. \( 1 - p \leq A^\circ \ y \).
4. \( 1 - p \leq A^\circ \ p \).

Moreover, \( \alpha \) has the tracial Rokhlin property with comparison if and only if for every $\varepsilon$, $x$, and $y$ as above, every compact set $F \subseteq A$, every compact set $S \subseteq C(G)$, and every $n \in \mathbb{N}$, there exist a projection $p \in A^n$ and an equivariant unital completely positive map \( \varphi : C(G) \to pAp \) such that conditions (2), (3), and (4) hold, and \( \varphi \) is an \((F, S, \varepsilon)\)-approximately central multiplicative map.

**Proof.** The proof is the same as that of Lemma 1.16 of [44]. \( \square \)

**Lemma 2.9.** Let $A$ be an infinite dimensional separable simple unital C*-algebra. Let \( \alpha : G \to \text{Aut}(A) \) be an action of a second countable compact group $G$ on $A$ which has the tracial Rokhlin property with comparison. Then $A$ has Property (SP) or \( \alpha \) has the Rokhlin property.

**Proof.** Suppose $A$ does not have Property (SP). We verify the condition of Lemma 2.3. Let $F \subseteq A$ and $S \subseteq C(G)$ be finite, and let $\varepsilon > 0$. Choose $x \in A_+ \setminus \{0\}$ such that \( \|x\| = 1 \) and $xA_+$ contains no nonzero projection, and take $y = 1$. Apply Definition 2.4. Then $p = 1$, so the condition of Lemma 2.3 holds. \( \square \)
When $A$ is separable, we characterize the tracial Rokhlin property with comparison in terms of central sequences. (A better version will be given in Section 3.) We begin with a definition.

**Definition 2.10.** Let $A$ be a C*-algebra and let $\alpha: G \to \text{Aut}(A)$ be an action of a topological group $G$ on $A$. We say that a projection $p$ in $A_{\infty,\alpha}$ is $\alpha$-small if whenever $(q_n)_{n \in \mathbb{N}} \in l_\alpha^\infty(\mathbb{N}, A)$ is a sequence of projections in $A^\alpha$ which lifts $p$, then for every nonzero $x \in A_+$, there exist $N \in \mathbb{N}$ such that for every $n \geq N$ we have $q_n \lesssim_A x$. Without the action (that is, if $G = \{1\}$ and $\alpha$ is trivial), we say that $p$, now in $A_{\infty}$, is small.

It is enough to consider only one lift instead of all of them.

**Lemma 2.11.** Let $A$ be a C*-algebra and let $\alpha: G \to \text{Aut}(A)$ be an action of a topological group $G$ on $A$. Let $p \in A_{\infty}$ be a projection. Then $p$ is $\alpha$-small if and only if there is a sequence $(q_n)_{n \in \mathbb{N}} \in l_\alpha^\infty(\mathbb{N}, A)$ which lifts $p$ and such that for every nonzero $x \in A_+$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have $q_n \lesssim_A x$.

**Proof.** Only one direction needs proof. Assume the condition of the lemma, and let $(e_n)_{n \in \mathbb{N}} \in l_\alpha^\infty(\mathbb{N}, A)$ be another sequence of projections in $A^\alpha$ which lifts $p$. Then $\lim_{n \to \infty} \|e_n - q_n\| = 0$. So there is $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ the projections $q_n$ and $e_n$ are Murray-von Neumann equivalent. Then $q_n \lesssim_A x$ if and only if $e_n \lesssim_A x$. So, for every nonzero $x \in A_+$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have $e_n \lesssim_A x$.

**Lemma 2.12.** Let $A$ be a unital C*-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an action of a compact group $G$ on $A$. Let $p \in A_{\infty,\alpha}$ be an $\alpha$-invariant projection. Then there exists an $\alpha^\infty$-invariant projection $q \in l_\alpha^\infty(\mathbb{N}, A)$ such that $\pi_A(q) = p$.

**Proof.** Choose any positive element $a = (a_n)_{n \in \mathbb{N}} \in l_\alpha^\infty(\mathbb{N}, A)$ such that $\pi_A(a) = p$. By averaging over $G$, we may assume that $a$ is $\alpha^\infty$-invariant. Since $\pi_A(a)$ is a projection, we have $\lim_{n \to \infty} \|a_n^2 - a_n\| = 0$. Therefore there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\text{sp}(a_n) \cap \left(\frac{1}{3}, \frac{2}{3}\right) = \emptyset$. Define

$$b_n = \begin{cases} a_n & n \geq N \\ 0 & n < N. \end{cases}$$

One easily checks that $b = (b_n)_{n \in \mathbb{N}} \in l_\alpha^\infty(\mathbb{N}, A)$, that $\pi_A(b) = p$, and that $\text{sp}(b) \cap \left(\frac{1}{3}, \frac{2}{3}\right) = \emptyset$. Set $q = \chi_{[1/3, 2/3]}(b)$.

**Lemma 2.13.** Let $A$ be a C*-algebra, let $a \in A_+$, and let $p \in A$ be a projection. Suppose there is $v \in A$ such that $\|v^*aw - p\| < 1$. Then there is $w \in A$ such that $w^*aw = p$, and $p \lesssim_A a$.

**Proof.** The second statement follows from the first.

For the first, observe that $\|pv^*avp - p\| < 1$. Therefore $pv^*avp$ has an inverse in $pAp$. Call it $c$. Then $w = vpc^{1/2}$ satisfies $w^*aw = p$.

**Lemma 2.14.** Let $A$ be a C*-algebra, let $y_0, y_1 \in A_+$ satisfy $y_0y_1 = y_1$, and let $p \in A$ be a projection such that $p \lesssim_A y_1$. Then there exists $v \in A$ such that $\|v\| \leq 1$ and $v^*y_0v = p$.

**Proof.** Choose $w_0 \in A$ such that $\|w_0^*y_0w_0 - p\| < 1$. Lemma 2.13 provides $w \in A$ such that $w^*y_0w = p$. Set $v = y_1^{1/2}w$. Then $v^*v = p$ so $\|v\| \leq 1$. (If $p = 0$ then $\|v\| = 0$.) Moreover, $y_1^{1/2}y_0 = y_1^{1/2},$ so $v^*y_0v = p$. □
Lemma 2.15. Let $G$ be a second countable compact group, let $A$ be a simple separable infinite dimensional unital $C^*$-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$. Then $\alpha$ has the tracial Rokhlin property with comparison if and only if for every $x \in A_+$ with $\|x\| = 1$ and every $y \in (A^\alpha)_+ \setminus \{0\}$, there exist a projection $p \in (A_{\infty, \alpha} \cap A')^{\alpha_{\infty}}$ and a unital equivariant homomorphism $\psi: C(G) \to p(A_{\infty, \alpha} \cap A')p$,

such that the following hold:

1. $1 - p$ is $\alpha$-small in $A_{\infty, \alpha}$.
2. $1 - p \lesssim_{(A^\alpha)_+} y$.
3. $1 - p \lesssim_{(A^\alpha)_+} p$.
4. $\|xp\| = 1$.

We will later improve the statement to replace (2) with the statement that $1 - p$ is small in $(A^\alpha)_\infty$. (See Proposition 3.9.) For this, however, we need to know that $A^\alpha$ is simple, and the statement here is used in the proof of that fact. Once we have simplicity of $A^\alpha$, the proof will be essentially the same, and we will refer back to this proof for most of the steps.

Proof of Lemma 2.15. We first prove that the condition implies the tracial Rokhlin property with comparison.

Let $F \subseteq A$ and $S \subseteq C(G)$ be finite, let $\varepsilon > 0$, let $x \in A_+$ satisfy $\|x\| = 1$, and let $y \in A^\alpha \setminus \{0\}$. Choose $p$ and $\varphi$ as in the statement for $x$ and $y$ as given. Use Lemma 2.12 to choose an $\alpha^\infty$-invariant projection $q = (q_n)_{n \in \mathbb{N}} \in \ell^\infty_\alpha(N, A)$ such that $\pi_A(q) = p$. Then $\pi_A$ defines a surjective map from $q\ell^\infty_\alpha(N, A)q$ to $pA_{\infty, A, p}$. Use the Choi-Effros lifting theorem to choose a lift $\theta = (\theta_n)_{n \in \mathbb{N}}: C(G) \to q\ell^\infty_\alpha(N, A)q$ of $\psi$, consisting of unital completely positive maps $\theta_n: C(G) \to q_nAq_n$. We claim that the following properties hold:

5. $\lim_{n \to \infty} \|\theta_n(f_1f_2) - \theta_n(f_1)\theta_n(f_2)\| = 0$ for all $f_1, f_2 \in C(G)$.
6. $\lim_{n \to \infty} \|\theta_n(f) - a\theta_n(f)\| = 0$ for all $a \in A$ and all $f \in C(G)$.
7. $\lim_{n \to \infty} \sup_{g \in G} \|\theta_n \circ \text{Lt}_g(f) - (\alpha_g \circ \theta_n)(f)\| = 0$ for all $f \in C(G)$.

Properties (5) and (6) follow immediately from the fact that $\pi_A \circ \theta = \psi$ is a homomorphism and has range contained in $A_{\infty} \cap A'$. For (7), let $f \in C(G)$, and define $\rho_n: G \to [0, \infty)$ by

$$\rho_n(g) = \sup_{m \geq n} \|\theta_m \circ \text{Lt}_g(f) - (\alpha_g \circ \theta_m)(f)\|$$

for $g \in G$. We have $\lim_{n \to \infty} \rho_n(g) = 0$ for all $g \in G$ because $\pi_A \circ \theta = \psi$ is equivariant, and $\rho_n(g) \leq \rho(g)$ whenever $g \in G$ and $m \geq n$. Further let $\sigma: \ell^\infty_\alpha(N, A) \to \ell^\infty_\alpha(N, A)$ be the backwards shift, given by $\sigma((a_n)_{n \in \mathbb{N}}) = (a_2, a_3, \ldots)$. It is easy to see that the range of $\sigma$ is in fact contained in $\ell^\infty_\alpha(N, A)$. Now

$$\rho_n(g) = \|\sigma^{n-1}((\theta \circ \text{Lt}_g)(f) - (\alpha_g \circ \theta)(f))\|,$$

so $\rho_n$ is continuous. Therefore (7) follows from Dini’s Theorem. The claim is proved.

Let $\mu$ be normalized Haar measure on $G$. For $n \in \mathbb{N}$ and $f \in C(G)$, define

$$\gamma_n(f) = \int_G (\alpha_g^{-1} \circ \theta_n \circ \text{Lt}_g)(f) \, d\mu(g).$$
Then $\gamma_n$ is an equivariant unital completely positive map from $C(G)$ to $A$, and, using (7), one sees that

$$\lim_{n \to \infty} \|\gamma_n(f) - \theta_n(f)\| = 0 \text{ for all } f \in C(G).$$

Using (5), (6), and (8), choose $n_0 \in \mathbb{N}$ such that whenever $n \geq n_0$, we have $\|\gamma_n(f_1f_2) - \gamma_n(f_1)\gamma_n(f_2)\| < \varepsilon$ for all $f_1, f_2 \in C(G)$ and $\|\theta_n(f)a - a\theta_n(f)\| < \varepsilon$ for all $a \in A$ and $f \in C(G)$. Since $1 - p$ is a-small, there is $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$ we have $1 - q_n \ngeq_A x$. Since $1 - p \nleq_{(A^\alpha)^\infty} y$, by Lemma 2.13 there is $c \in (A^\alpha)^\infty$ such that $1 - p = c^*yc$. Choose $d = (d_n)_{n \in \mathbb{N}} \in l_\alpha^\infty(N, A)$ such that $\pi_{A}(d) = c$. Then $\lim_{n \to \infty} \|d_n^*yd_n - (1 - q_n)\| = 0$. Therefore there is $n_2 \in \mathbb{N}$ such that for every $n \geq n_2$ we have $\|d_n^*yd_n - (1 - q_n)\| < 1$. Since $1 - p \nleq_{(A^\alpha)^\infty} p$, there is $v \in (A^\alpha)^\infty$ such that $v^*v = 1 - p$ and $vv^* \leq p$. Then $1 - p = vv^*$. Choose $w = (w_n)_{n \in \mathbb{N}} \in l_\alpha^\infty(N, A)$ such that $\pi_{A}(w) = v$. Then $\lim_{n \to \infty} \|w_n^*qw_n - (1 - q_n)\| = 0$. Therefore there is $n_3 \in \mathbb{N}$ such that for every $n \geq n_3$ we have $\|w_n^*qw_n - (1 - q_n)\| < 1$. Also, there is $n_4 \in \mathbb{N}$ such that for every $n \geq n_4$ we have $\|q_nw_n\| > 1 - \varepsilon$. Set $n = \max(n_0, n_1, n_2, n_3, n_4)$, and in Definition 2.4 take $\varphi$ to be $\gamma_n$ and take $p$ to be $q_n$. All the conditions hold by construction, except that we use Lemma 2.13 to see that $1 - q_n \ngeq_{A^\alpha} y$ and $1 - q_n \nleq_{A^\alpha} q_n$.

We now show that the tracial Rokhlin property with comparison implies the existence of $\psi$. Since the Rokhlin property case is already known (see Theorem 1.7 of [18] and Theorem 2.11 of [13]), we may assume that $\alpha$ does not have the Rokhlin property. Therefore $A$ has Property (SP) by Lemma 2.9.

Let $x \in A_+$ satisfy $\|x\| = 1$ and let $y \in A^\alpha_+ \backslash \{0\}$. We may assume that $\|y\| = 1$. Choose dense sequences

$$f_1, f_2, \ldots \in C(G), \quad a_1, a_2, \ldots \in A,$$

and

$$x_1, x_2, \ldots \in \{a \in A_+: \|a\| = 1\}.$$

For $n \in \mathbb{N}$, use simplicity of $A$ and Lemma 1.7 to choose $z_n \in A_+ \backslash \{0\}$ such that

$$z_n \nearrow_A \left(x_k - \frac{1}{2}\right)_k,$$

for $k = 1, 2, \ldots, n$. Define $h_n, k_n: [0, 1] \to [0, 1]$ by

$$h_n(\lambda) = \begin{cases} \left(1 - \frac{\lambda}{n+1}\right)^{-1} & 0 \leq \lambda \leq 1 - \frac{1}{n+1} \\ 1 & 1 - \frac{1}{n+1} < \lambda \leq 1 \end{cases}$$

and

$$k_n(\lambda) = \begin{cases} 0 & 0 \leq \lambda \leq 1 - \frac{1}{n+1} \\ (n+1)\lambda - n & 1 - \frac{1}{n+1} < \lambda \leq 1. \end{cases}$$

Since $\|x\| = 1$, we have $\|k_n(x)\| = 1$, and there is a nonzero projection $r_n \in k_n(x)Ak_n(x)$. Then

$$h_n(x)r_n = r_n.$$

Since $A$ is simple, Lemma 1.7 provides $d_n \in (r_nAr_n) + \{0\}$ such that $d_n \ngeq_{A} z_n$. Use Property (SP) to choose a nonzero projection $e_n \in d_nA\overline{d_n}$. Then

$$e_n \leq r_n \quad \text{and} \quad e_n \nleq_{A} d_n \nleq_{A} z_n.$$
Also define \( y_0 = h_1(y) \) and \( y_1 = k_1(y) \), getting
\[
\|y_1\| = 1, \quad y_0y_1 = y_1, \quad \text{and} \quad y_0 \sim y.
\]
Applying Lemma 2.7 with
\[
S_n = \{f_1, f_2, \ldots, f_n\} \quad \text{and} \quad F_n = \{a_1, a_2, \ldots, a_n\}
\]
in place of \( S \) and \( F \), with \( \varepsilon = \frac{1}{n} \), with \( e_n \) in place of \( x \), and with \( y_1 \) in place of \( y \).

We get a projection \( q_n \in A^\alpha \) and an equivariant unital completely positive map \( \varphi_n : C(G) \to pAp \), such that:

(8) \( \varphi_n \) is an \( (F_n, S_n, \frac{1}{n}) \)-approximately central multiplicative map.

(9) \( 1 - q_n \lesssim_A e_n \).

(10) \( 1 - q_n \lesssim_A y_1 \).

(11) \( 1 - q_n \lesssim_A q_n \).

(12) \( \|q_ne_nq_n\| > 1 - \frac{1}{n} \).

Define an \( \alpha \)-invariant projection \( q \in l_\infty^\infty(N, A) \) by \( q = (q_n)_{n \in \mathbb{N}} \). Define \( \varphi : C(G) \to l_\infty^\infty(N, A) \) by \( \varphi(f) = (\varphi_n(f))_{n \in \mathbb{N}} \) for \( f \in C(G) \). Clearly \( \varphi \) is equivariant and unital completely positive. It follows from equivariance that the range of \( \varphi \) is contained in \( l_\infty^\infty(N, A) \). Set \( p = \pi_A(q) \) and \( \psi = \pi_A \circ \varphi \). Obviously \( \psi \) is equivariant. Using the fact that \( \|\varphi_n\| \leq 1 \) for all \( n \in \mathbb{N} \), density of the sequences in (2.1), and (2.7), it is easily seen that \( \psi \) is a homomorphism whose range commutes with the image of \( A \) in \( A_{\infty, \alpha} \).

We prove (1). By Lemma 2.11, it suffices to prove that for all nonzero \( t \in A_+ \) there exists \( N \in \mathbb{N} \) such that for every \( n \geq N \) we have \( 1 - q_n \lesssim A t \). We may assume that \( \|t\| = 1 \). Choose \( N \in \mathbb{N} \) such that \( \|x_N - t\| < \frac{1}{2} \). Then for \( n \geq N \) we have, using (9) at the first step, (2.5) at the second step, and (2.3) at the third step,
\[
1 - q_n \lesssim_A e_n \lesssim_A z_n \lesssim_A x_N - \left(\frac{1}{2}\right)_+ \lesssim_A t.
\]

For (2), use (2.6), (10), and Lemma 2.14 to choose \( w_n \in A^\alpha \) such that \( \|w_n\| \leq 1 \) and \( w_n^*y_0w_n = 1 - q_n \). Then \( v = \pi_A((w_1, w_2, \ldots)) \) satisfies \( v^*y_0v = 1 - p \). Since \( y_0 \sim_A y \) (by (2.6)), we conclude that \( 1 - p \lesssim (A^\alpha)_\infty y \).

To prove (3), by (11) there is \( t_n \in A^\alpha \) such that \( t_n^*t_n = 1 - q_n \) and \( t_n^*t_n \lesssim q_n \). Then \( s = \pi_A((t_1, t_2, \ldots)) \) satisfies \( s^*s = 1 - p \) and \( ss^* \leq p \).

Finally, we prove (4). It suffices to show that \( \lim_{n \to \infty} \|q_nxq_n\| = 1 \). For \( n \in \mathbb{N} \) we have \( \|x - h_n(x)\| < \frac{1}{n} \). Therefore \( \|q_nxq_n\| > \|q_nh_n(x)q_n\| - \frac{1}{n} \). Now \( e_n \leq h_n(x) \) by (2.5) and (2.4), so, by (12),
\[
\|q_nh_n(x)q_n\| \geq \|q_ne_nq_n\| > 1 - \frac{1}{n}.
\]
Therefore \( \|q_nxq_n\| > 1 - \frac{2}{n} \), and (4) follows. This completes the proof. \( \square \)

The following theorem is a combination of Theorem 1.7 of [18] and Theorem 2.11 of [13]. It is a key tool for transferring properties from the original algebra to the fixed point algebra. The proof follows that of Theorem 2.11 of [13], but there is a mistake there: the formula for \( \rho_n(f \otimes \alpha) \) isn’t additive in \( f \), so can’t be used to define a homomorphism as claimed there. The fix is taken from the proof of Theorem 1.7 of [18].

**Theorem 2.16.** Let \( A \) be an infinite dimensional simple separable unital C*-algebra, let \( G \) be a second countable compact group, and let \( \alpha : G \to \text{Aut}(A) \) be an action which has the tracial Rokhlin property with comparison. For every \( \varepsilon > 0 \),
every $n \in \mathbb{N}$, every compact subset $F_1 \subseteq A$, every compact subset $F_2 \subseteq A^\alpha$, every $x \in A_+$ with $\|x\| = 1$, and every $y \in (A^\alpha)_+ \setminus \{0\}$, there exist a projection $p \in A^\alpha$ and a unital completely positive contractive map $\psi: A \to pA^\alpha p$ such that:

1. $\psi$ is an $(n, F_1 \cup F_2, \epsilon)$-approximately multiplicative map (Definition 1.3).
2. $\|pa - ap\| < \epsilon$ for all $a \in F_1 \cup F_2$.
3. $\|\psi(a) - p\psi(a)p\| < \epsilon$ for all $a \in F_2$.
4. $1 - p \precsim_A x$.
5. $1 - p \precsim_{A^\alpha} y$.
6. $1 - p \precsim_{A^\alpha} p$.
7. $\|xp\| > 1 - \epsilon$.

Proof. Let $\epsilon, F_1, F_2, x,$ and $y$ be as in the statement.

We first claim that there are an $\alpha^\infty$-invariant projection $q \in l^\infty_\alpha(\mathbb{N}, A)$ and a unital completely positive map $\theta: C(G) \otimes A \to qS_{\alpha^\infty}(\mathbb{N}, A)q$ such that, with $r = \pi_A(q)$, the following hold:

1. $\theta$ is equivariant for the actions $g \mapsto \mathbb{L}t_g \otimes \alpha_g$ and $g \mapsto \alpha_g^\infty$.
2. $\pi_A \circ \theta$ is a homomorphism.
3. $(\pi_A \circ \theta)(1 \otimes a) = r\alpha(a)$ for all $a \in A$.
4. $1 - r$ is $\alpha$-small in $A_{\alpha^\infty}$.
5. $1 - r \precsim_{(A^\alpha)^\infty} y$.
6. $1 - r \precsim_{(A^\alpha)^\infty} r$.
7. $\|r x p\| = 1$.

To prove the claim, apply the “only if” part of Lemma 2.15 with $x$ and $y$ as given, getting $r \in (A_{\alpha^\infty} \cap A')_{\alpha^\infty}$ and a unital equivariant homomorphism $\beta: C(G) \to r(A_{\alpha^\infty} \cap A')r$, such that Conditions (1), (2), (3), and (4) of Lemma 2.15 hold. We write $\iota: A \to rA_{\alpha^\infty}r$ for the inclusion of $A$ in $A_{\alpha^\infty}$, using constant sequences, followed by the cutdown $b \mapsto rbr$. This map is a homomorphism, since $r \in A_{\alpha^\infty} \cap A'$. The ranges of $\beta$ and $\iota$ commute, so there is a unital homomorphism $\eta: C(G) \otimes A \to rA_{\alpha^\infty}r$ such that

$$\eta(f \otimes a) = \beta(f)\iota(a)$$

for all $f \in C(G)$ and $a \in A$. Use Lemma 2.12 to choose an $\alpha^\infty$-invariant projection $q = (q_m)_{m \in \mathbb{N}} \in l^\infty_\alpha(\mathbb{N}, A)$ such that $\pi_A(q) = r$. We now have Conditions (11), (12), (13), and (14) of the claim.

Using a partition of unity argument and second countability of $G$, choose positive integers $k(m)$, unital completely positive maps $\rho_m: C(G) \to \mathbb{C}^{k(m)}$, and unital completely positive maps $\sigma_m: C(G) \to \mathbb{C}^{k(m)}$, for $m \in \mathbb{N}$, such that for all $f \in C(G)$ we have $\lim_{n \to \infty} (\sigma_m \circ \rho_m)(f) = f$.

Fix $m \in \mathbb{N}$, and let $e_1, e_2, \ldots, e_{k(m)}$ be the minimal projections in $\mathbb{C}^{k(m)}$. For $j = 1, 2, \ldots, k(m) - 1$, choose $c_j \in q_{\alpha^\infty}(\mathbb{N}, A)q$ such that $\pi_A(c_j) = (\beta \circ \sigma_m)(e_j)$. Let $c_{k(m)}$ be the positive part of $q - \sum_{j=1}^{k(m)-1} c_j$. Then $\pi_A(c_{k(m)}) = (\beta \circ \sigma_m)(e_{k(m)})$ and, with $c = \sum_{j=1}^{k(m)} c_j \in q_{\alpha^\infty}(\mathbb{N}, A)q$, we have $c \geq q$. Taking functional calculus in $q_{\alpha^\infty}(\mathbb{N}, A)q$, for $j = 1, 2, \ldots, k(m)$ define $b_j = c^{-1/2}c_{j}^{-1/2}$. Then $b_j \geq 0$ and $\pi_A(b_j) = (\beta \circ \sigma_m)(e_j)$ for $j = 1, 2, \ldots, k(m)$, and $\sum_{j=1}^{k(m)} b_j = 1$. Identify $\mathbb{C}^{k(m)} \otimes A$ with $A^{k(m)}$, and define $\gamma_m: \mathbb{C}^{k(m)} \otimes A \to q_{\alpha^\infty}(\mathbb{N}, A)q$ by, identifying $a \in A$ with
the corresponding constant sequence in $l^\infty_\alpha(N, A)$,

$$\gamma_m(a_1, a_2, \ldots, a_{k(m)}) = \sum_{j=1}^{k(m)} b_j^{1/2} q a_j q b_j^{1/2}$$

for $a_1, a_2, \ldots, a_{k(m)} \in A$. Then one checks that $\gamma_m$ is a unital completely positive map such that $\pi_A \circ \gamma_m = (\beta \circ \sigma_m) \otimes \iota$. Therefore

$$\pi_A \circ \gamma_m \circ (\rho_m \otimes \text{id}_A) = \eta \circ [(\sigma_m \circ \rho_m) \otimes \text{id}_A].$$

This formula shows that the unital completely positive maps

$$\eta \circ [(\sigma_m \circ \rho_m) \otimes \text{id}_A] : C(G) \otimes A \to pA_{\infty, \alpha} \pi$$

have lifts to unital completely positive maps $C(G) \otimes A \to qA_{\infty, \alpha} q$. Since $C(G) \otimes A$ is separable and $\sigma_m \circ \rho_m \to \text{id}_{C(G)}$ pointwise, it follows that

$$\eta \circ [(\sigma_m \circ \rho_m) \otimes \text{id}_A] \to \eta$$

pointwise. Now Theorem 6 of [2] shows that $\eta$ lifts to a unital completely positive map $\theta_0 : C(G) \otimes A \to qA_{\infty, \alpha} q$. Since $\eta$ is equivariant, we can average $\theta_0$ over $G$ to get an equivariant unital completely positive map $\theta : C(G) \otimes A \to qA_{\infty, \alpha} q$ such that $\pi_A \circ \theta = \eta$. This gives (8) and (9) of the claim. Condition (10) follows from (2.8).

The claim is proved.

Let $\theta$ be as in the claim. For $b \in C(G, A)$, write $\theta(b) = (\theta_m(b))_{m=1}^\infty$. Then the maps $\theta_m : C(G, A) \to A$ are equivariant and unital completely positive. It follows from Proposition 2.3 of [13] that there is an isomorphism $\kappa : A \to C(G, A)^{\mathbb{N} \otimes \alpha}$ such that $\kappa(a) = 1 \otimes a$ for all $a \in A^{\infty}$. Then $\pi_A \circ \theta \circ \kappa$ is a homomorphism. Therefore there is $m_1 \in \mathbb{N}$ such that, for all $m \geq m_1$ and all $a_1, a_2, \ldots, a_m \in F_1 \cup F_2$, we have

$$\|(\theta_m \circ \kappa)(a_1 a_2 \cdots a_m) - (\theta_m \circ \kappa)(a_1 \kappa(a_2) \cdots (\theta_m \circ \kappa)(a_m))\| < \varepsilon.$$

Since $r \in A_{\infty, \alpha} \cap A'$, there is $m_2 \in \mathbb{N}$ such that for all $m \geq m_2$ and all $a \in F_1 \cup F_2$, we have $\|q_m a - a q_m\| < \varepsilon$. For $a \in A^{\infty}$ we have

$$(\pi_A \circ \theta \circ \kappa)(a) = \iota(a) = \text{rar},$$

so we can choose $m_3 \in \mathbb{N}$ such that for all $m \geq m_3$ and all $a \in F_2$, we have $\|q_m a - a q_m\| < \varepsilon$. Because $1 - r$ is $\alpha$-small (by (11)), there is $m_4 \in \mathbb{N}$ such that $1 - q_m \lhd_A x$ for all $m \geq m_4$. Using (12) and (13), arguments given after (8) in the proof of Lemma 2.15 show that there are $m_5, m_6 \in \mathbb{N}$ such that $1 - q_m \lhd_A y$ for all $m \geq m_5$ and $1 - q_m \lhd_A q m$ for all $m \geq m_6$. Use (14) to choose $m_7 \in \mathbb{N}$ such that for all $m \geq m_7$ we have $\|q_m x q_m\| > 1 - \varepsilon$. Take

$$m = \max(m_1, m_2, \ldots, m_7), \quad p = q_m, \quad \text{and} \quad \psi = \theta_m \circ \kappa.$$

These choices prove the theorem. \qed

3. Simplicity of the crossed product

In this section we prove simplicity of the fixed point algebra and crossed product by an action which has the tracial Rokhlin property with comparison.

For finite groups, the tracial Rokhlin property implies pointwise outerness by Lemma 1.5 in [44], so that the crossed product is simple by Theorem 3.1 in [35]. Theorem 3.1 in [35] is valid for general discrete groups. But, without discreteness of the group, even assuming compactness, pointwise outerness of the action does not imply simplicity of the crossed product. For example, consider the gauge action.
\(\gamma\) of the circle group \(S^1\) on \(O_\infty\), which in terms of the standard generators \(s_j\) is given by \(\gamma(\zeta s_j) = \zeta s_j\) for all \(\zeta\) in \(S^1\) and all \(j \in \mathbb{N}\). Then \(\gamma\) is pointwise outer by Theorem 4 of \([11]\). However, its strong Connes spectrum is \(\mathbb{N}\) (Remark 5.2 of \([34]\)), so, by Theorem 3.5 of \([34]\), its crossed product is not simple.

Instead, we use the tracial Rokhlin property with comparison, in the form of Theorem 2.16, to prove that the fixed point algebra is simple. (This part does not need Condition (4) in Definition 2.4.) We also use the tracial Rokhlin property with comparison to show that the action is saturated, so that results of \([20]\) give simplicity of the crossed product.

Simplicity of the fixed point algebra also allows us to give a nicer formulation of Lemma 2.15. This is in Proposition 3.9.

**Theorem 3.1.** Let \(A\) be a simple separable unital infinite dimensional C*-algebra, let \(G\) be a second countable compact group, and let \(\alpha: G \to \text{Aut}(A)\) be an action which has the tracial Rokhlin property with comparison. Then the fixed point algebra \(A^\alpha\) is simple.

**Proof.** Let \(I\) be a nonzero ideal in \(A^\alpha\). We claim that \(I\) contains an invertible element. This will prove the theorem.

The set \(J = \overline{AI A}\) is an ideal in \(A\). Since \(J \neq 0\) and \(A\) is simple, \(J = A\). Since \(A\) is unital, in fact \(AI A = A\). Therefore there are \(m \in \mathbb{N}, a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in A,\) and \(x_1, x_2, \ldots, x_m \in I\) such that \(\sum_{j=1}^{m} a_j x_j b_j = 1\).

Fix a nonzero positive element \(z \in I\). Set

\[
M = 1 + \max_{j=1,2,\ldots,m} \max\{\|a_j\|, \|b_j\|\} \quad \text{and} \quad \delta = \frac{1}{2(m M^2 + m)}.
\]

Also set

\[
F_1 = \{a_j, b_j, x_j: j = 1, 2, \ldots, m\} \cup \{x\}
\]

and

\[
F_2 = \{x_j: j = 1, 2, \ldots, m\} \cup \{x\}.
\]

Use Theorem 2.16 with \(\delta\) in place of \(\varepsilon\), with \(n = 3\), with 1 in place of \(x\), with \(z\) in place of \(y\), and with \(F_1\) and \(F_2\) as given. We get a projection \(p\) in \(A^\alpha\) and a unital completely positive map \(\psi: A \to p A^\alpha p\) such that:

1. \(\|\psi(abc) - \psi(a)\psi(b)\psi(c)\| < \delta\) for all \(a, b, c \in F_1\).
2. \(\|pa - ap\| < \delta\) for all \(a \in F_1 \cup F_2\).
3. \(\|\psi(a) - pap\| < \delta\) for all \(a \in F_2\).
4. \(1 - p \preceq_{A^\alpha} z\).

Define

\[
c = 1 - p + \sum_{j=1}^{m} \psi(a_j) px_j p \psi(b_j).
\]

We have \(1 - p \in I\) by (4), and \(x_1, x_2, \ldots, x_m \in I\) by construction, so \(c \in I\). It remains to show that \(c\) is invertible.
First, using (1) and (3) at the second step,
\[
\left\| \psi \left( \sum_{j=1}^{m} a_j x_j b_j \right) - \sum_{j=1}^{m} \psi(a_j) p x_j p \psi(b_j) \right\| 
\leq \sum_{j=1}^{m} \left\| \psi(a_j x_j b_j) - \psi(a_j) \psi(x_j) \psi(b_j) \right\| + \sum_{j=1}^{m} \left\| \psi(a_j) \right\| \left\| \psi(x_j) - p x_j p \right\| \left\| \psi(b_j) \right\| 
< m \delta + m M^2 \delta = \frac{1}{2}.
\]
Combining this estimate with
\[
\psi \left( \sum_{j=1}^{m} a_j x_j b_j \right) = \psi(1) = p,
\]
we get
\[
\left\| 1 - c \right\| = \left\| p - \sum_{j=1}^{m} \psi(a_j) p x_j p \psi(b_j) \right\| < \frac{1}{2}.
\]
So $c$ is invertible, as desired. \hfill $\Box$

We recall the definition of the strong Arveson spectrum of an action of a compact group on a $C^*$-algebra, and related concepts.

**Definition 3.2** ([20], Definition 1.1(b), the preceding discussion, and Definition 1.2(b)). Let $A$ be a $C^*$-algebra, let $\alpha: G \to \text{Aut}(A)$ be an action of a compact group $G$ on $A$, and let $\pi: G \to \text{U}(\mathcal{H}_\pi)$ be a unitary representation of $G$ on a Hilbert space $\mathcal{H}_\pi$. We define
\[
A_2(\pi) = \{ x \in B(\mathcal{H}_\pi) \otimes A: (\text{id}_A \otimes \alpha_g)(x) = x(\pi(g) \otimes 1_A) \text{ for all } g \in G \}. \]

Let $\hat{G}$ be a set consisting of exactly one representation in each unitary equivalence class of irreducible representations of $G$. For $E, F \subseteq A$ take
\[
EF = \text{span}\{ xy: x \in E \text{ and } y \in F \}.
\]
Then define the strong Arveson spectrum by
\[
\widetilde{Sp}(\alpha) = \{ \pi \in \hat{G}: A_2(\pi)^* A_2(\pi) = (B(\mathcal{H}_\pi) \otimes A)^{\text{Ad}(\pi) \otimes \alpha} \}.
\]
Finally, the strong Connes spectrum is the intersection over all nonzero $\alpha$-invariant hereditary subalgebras $B$ of the strong Arveson spectrum of the restriction of $\alpha$ to $B$.

**Lemma 3.3.** In the situation of Definition 3.2, for any finite dimensional unitary representation $\pi$ of $G$, the set $A_2(\pi)^* A_2(\pi)$ is a closed two sided ideal in $(B(\mathcal{H}_\pi) \otimes A)^{\text{Ad}(\pi) \otimes \alpha}$.

**Proof.** This is easily checked, and is in the discussion before Definition 1.1 of [20]. \hfill $\Box$

Recall (Definition 7.1.4 of [42]) that an action $\alpha: G \to \text{Aut}(A)$ of a compact group $G$ on a $C^*$-algebra $A$ is saturated if a suitable completion of $A$ is a Morita equivalence bimodule from $A^\alpha$ to $C^*(G, A, \alpha)$. We recall some essentially known results on saturation, the strong Arveson spectrum, and simplicity. We won’t actually use Condition (4); it is included primarily to give context.
Theorem 3.4. Let $A$ be a C*-algebra, and let $\alpha : G \to \text{Aut}(A)$ be an action of a compact group $G$ on $A$. Then following are equivalent:

1. $A^\alpha$ is simple and $\alpha$ is saturated.
2. $A^\alpha$ is simple and $\tilde{\text{Sp}}(\alpha) = \tilde{G}$.
3. $C^*(G, A, \alpha)$ is simple.
4. $A$ has no nontrivial $G$-invariant ideals and $\tilde{\Gamma}(\alpha) = \tilde{G}$.

When these conditions hold, $A^\alpha$ is isomorphic to a full hereditary subalgebra of $C^*(G, A, \alpha)$, and is strongly Morita equivalent to $C^*(G, A, \alpha)$.

Proof. The equivalence of (1) and (2) is Theorem 5.10 of [43], originally from the remarks after Lemma 3.1 of [20].

The equivalence of (3) and (4) is Theorem 5.10 of [20].

For the implication from (3) to (2), use the implication from (3) to (4) and $\tilde{\Gamma}(\alpha) \subseteq \tilde{\text{Sp}}(\alpha)$ to get $\tilde{\text{Sp}}(\alpha) = \tilde{G}$. Use the Corollary in [49] to get simplicity of $A^\alpha$. (Warning: the reference to Corollary 3.8 of the proof of the Proposition in [48] is to Corollary 3.3 of [48] in the published version, and there is also a Proposition 3.3 in [48].)

To finish, we prove that (2) implies (3), by comparing the proof of Corollary 7.1.5 of [42] with the proof of the Proposition in [49]. For $x \in A$ let $\tilde{x} \in C^*(G, A, \alpha)$ be as in Definition 7.1.2 of [42], and let $p \in M(C^*(G, A, \alpha))$ be the projection in the proof of the Proposition in [49]. A calculation shows that $p\tilde{x} = \tilde{x}$ for all $x \in A$, so also $\tilde{x}p = \tilde{x}$. Now, as in the proof of Corollary 7.1.5 of [42] at the first step,

\[ \{ (x, y)_{C^*(G, A, \alpha)} : x, y \in A \} = \{(\tilde{x})^*\tilde{y} : x, y \in A \} = \{ x^*py : x, y \in A \}. \]

So the first set spans a dense subspace of $C^*(G, A, \alpha)$ if and only if the last set does too. That the first spans a dense subspace of $C^*(G, A, \alpha)$ is the definition of saturation. Using the Corollary in [49] and simplicity of $A^\alpha$, having the last span a dense subspace of $C^*(G, A, \alpha)$ is equivalent to simplicity of $C^*(G, A, \alpha)$.

Assuming the conditions, $A^\alpha$ is isomorphic to a full hereditary subalgebra of $C^*(G, A, \alpha)$ by the Corollary in [49], and is strongly Morita equivalent to $C^*(G, A, \alpha)$ by (1).

Proof of Proposition 3.5. Let $A$ be an infinite dimensional simple separable unital C*-algebra. Let $\alpha : G \to \text{Aut}(A)$ be an action of a second countable compact group $G$ on $A$ which has the tracial Rokhlin property with comparison. Then $\alpha$ is saturated.

Proposition 3.5. Let $A$ be an infinite dimensional simple separable unital C*-algebra. Let $\alpha : G \to \text{Aut}(A)$ be an action of a second countable compact group $G$ on $A$ which has the tracial Rokhlin property with comparison. Then $\alpha$ is saturated.

Remark 3.6. In the proof of Proposition 3.5, we only use Conditions (1) and (4) of Definition 2.4.

To verify this formally, first observe that the proof does not use Conditions (4), (5), and (7) in Theorem 2.16. One must then modify the proof of the “only if” direction of Lemma 2.15 to show that Definition 2.4 without Conditions (2), (3), and (5) implies the condition of Lemma 2.15 without Conditions (1), (2), and (4). This is done simply by omitting the parts of the proof which use the omitted conditions. Then one must modify the proof of Theorem 2.16 to show that the condition of Lemma 2.15, without Conditions (1), (2), and (4), implies the conclusion of Theorem 2.16 without Conditions (4), (5), and (7). Again, this is done simply by omitting the parts of the proof which use the omitted conditions.

Proof of Proposition 3.5. Let $\tilde{G}$ be as in Definition 3.2, and let $\pi : G \to \text{U}(H_\pi)$ be in $\tilde{G}$. We need to prove that $(B(H_\pi) \otimes A)^{\text{Ad}(\pi) \otimes \alpha} \subseteq \mathcal{A} \tilde{\alpha}^\pi \text{Ad}(\tilde{\pi}) \mathcal{A} \tilde{\alpha}^\pi \tilde{\pi}$. By Lemma 3.3,
$A_2(\pi)^*A_2(\pi)$ is a closed two sided ideal in $(B(\mathcal{H}_\pi) \otimes A)^{\text{Ad}(\pi)}\otimes \alpha$. So it is enough to prove that $A_2(\pi)^*A_2(\pi)$ contains an invertible element.

Let $d_\pi = \dim(\mathcal{H}_\pi)$ and identify $\mathcal{H}_\pi$ with $\mathbb{C}^{d_\pi}$. Set $\delta = 1/(4d_\pi^2)$. An easy computation shows that $\pi \in C(G)_2(\pi)$. For $j,k = 1,2,\ldots,d_\pi$, let $\pi_{j,k} \in C(G)$ be the function $g \mapsto \pi(g)_{j,k}$ whose value is the $(j,k)$ matrix entry of $\pi(g)$. Apply Lemma 2.7 to $\alpha$ with $\delta$ in place of $\varepsilon$, with $n = 2$, with
\[
F = \{1_A\} \quad \text{and} \quad S = \{\pi_{j,k}, \pi_{j,k}^*: j,k = 1,2,\ldots,d_\pi\},
\]
and with $x = y = 1$. We get a projection $p \in A^\alpha$ such that $1 - p \not\precsim_{A^\alpha} p$ and an equivariant unital completely positive ($F,S,\delta$)-approximately central multiplicative map $\varphi: C(G) \to pAp$. Then
\[
\text{id}_{B(\mathcal{H}_\pi)} \otimes \varphi: B(\mathcal{H}_\pi) \otimes C(G) \to (1_{B(\mathcal{H}_\pi)} \otimes p)(B(\mathcal{H}_\pi) \otimes A)(1_{B(\mathcal{H}_\pi)} \otimes p)
\]
is an equivariant unital completely positive map.

Let $(e_{j,k})_{j,k=1,2,\ldots,n}$ be the standard system of matrix units for $B(\mathcal{H}_\pi)$. Set
\[
c = (\text{id}_{B(\mathcal{H}_\pi)} \otimes \varphi)(\pi) = \sum_{j,k=1}^{d_\pi} e_{j,k} \otimes \varphi(\pi_{j,k}) \in B(\mathcal{H}_\pi) \otimes pAp.
\]
Since $\text{id}_{B(\mathcal{H}_\pi)} \otimes \varphi$ is equivariant, an easy calculation shows that $c \in A_2(\pi)$, so that $c^*c \in A_2(\pi)^*A_2(\pi)$. Moreover, since $\pi^*\pi = 1$,
\[
(3.4) \quad \|c^*c - 1_{B(\mathcal{H}_\pi)} \otimes p\| \leq \sum_{j,k,l=1}^{\infty} \|\varphi(\pi_{k,j}^*)\varphi(\pi_{k,l}) - \varphi(\pi_{k,j}\pi_{k,l})\| < d_\pi^3\delta.
\]
Using $1_A - p \not\precsim_{A^\alpha} p$, choose $s \in A^\alpha$ such that $s^*s = 1_A - p$ and $ss^* \leq p$. Then $s^*ps = 1_A - p$. Clearly $1_{B(\mathcal{H}_\pi)} \otimes s \in (B(\mathcal{H}_\pi) \otimes A)^{\text{Ad}(\pi)}\otimes \alpha$. Using (3.4), we get
\[
\|(1_{B(\mathcal{H}_\pi)} \otimes s)^*c^*c(1_{B(\mathcal{H}_\pi)} \otimes s) - 1_{B(\mathcal{H}_\pi)} \otimes (1 - p)\| < d_\pi^3\delta.
\]
Therefore the element
\[
z = c^*c + (1_{B(\mathcal{H}_\pi)} \otimes s)^*c^*c(1_{B(\mathcal{H}_\pi)} \otimes s)
\]
satisfies
\[
(3.5) \quad \|z - 1_{B(\mathcal{H}_\pi)} \otimes 1\| < 2d_\pi^3\delta < 1.
\]
Also, since $A_2(\pi)^*A_2(\pi)$ is an ideal in $(B(\mathcal{H}_\pi) \otimes A)^{\text{Ad}(\pi)}\otimes \alpha$, it follows that
\[
(1_{B(\mathcal{H}_\pi)} \otimes s)^*c^*c(1_{B(\mathcal{H}_\pi)} \otimes s) \in A_2(\pi)^*A_2(\pi).
\]
Therefore $z \in A_2(\pi)^*A_2(\pi)$. So (3.5) implies that $A_2(\pi)^*A_2(\pi)$ contains an invertible element. This completes the proof. \qed

**Theorem 3.7.** Let $A$ be an infinite dimensional simple separable unital C*-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an action of a second countable compact group $G$ on $A$ which has the tracial Rokhlin property with comparison. Then the crossed product $C^*(G, A, \alpha)$ is simple.

**Proof.** The algebra $A^\alpha$ is simple by Theorem 3.1 and $\alpha$ is saturated by Proposition 3.5. So Condition (1) in Theorem 3.4 holds. \qed

**Corollary 3.8.** Under the hypotheses of Theorem 3.7, the algebras $C^*(G, A, \alpha)$ and $A^\alpha$ are stably isomorphic.
Proof. By Theorem 3.7, the condition in Theorem 3.4(3) holds. The conclusion follows from separability and the last statement in Theorem 3.4. □

We can also improve Lemma 2.15, replacing in Condition (2) there the requirement $1 - p \prec (A^o)_\infty y$ with the requirement that $1 - p$ be small in $(A^o)_\infty$.

**Proposition 3.9.** Let $G$ be a second countable compact group, let $A$ be a simple separable infinite dimensional unital C*-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$. Then $\alpha$ has the tracial Rokhlin property with comparison if and only if for every nonzero positive element $x$ in $A_\infty$ with $\|x\| = 1$, there exist a projection $p \in (A_\infty,\alpha \cap A^)'_{\alpha}$ and a unital equivariant homomorphism $\psi: C(G) \to p(A_\infty,\alpha \cap A^)'p$, such that the following hold:

1. $1 - p$ is $\alpha$-small in $A_\infty,\alpha$.
2. $1 - p$ is small in $(A^o)_\infty$.
3. $1 - p \prec (A^o)_\infty p$.
4. $\|\psi(x)\| = 1$.

**Proof.** We only describe the changes from the proof of Lemma 2.15.

To show that the tracial Rokhlin property with comparison implies the existence of $\psi$, in addition to the dense sequences (2.1) and (2.2), we choose a dense sequence $y_1, y_2, \ldots \in \{a \in A^+_\alpha: \|a\| = 1\}$.

For $n \in \mathbb{N}$, use simplicity of $A^o$ (Theorem 3.1) and Lemma 1.7 to choose $d_n \in A^+_\alpha \setminus \{0\}$ such that

$$d_n \prec_A \left(y_k - \frac{1}{2}\right)_+$$

for $k = 1, 2, \ldots, n$. In the application of Lemma 2.7 in the proof of Lemma 2.15, use $d_n$ instead of $y$. This replaces (10) in the proof of Lemma 2.15 with the requirement that $1 - q_n \prec_A d_n$. Let $p$ be as in that proof. To show that $1 - p$ is small in $(A^o)_\infty$, let $t \in A_+ \setminus \{0\}$; as there, it is enough to find $N \in \mathbb{N}$ such that for all $n \geq N$ we have $1 - q_n \prec_A t$. We may assume that $\|t\| = 1$. Choose $N \in \mathbb{N}$ such that $\|y_N - t\| < \frac{1}{2}$. Then $n \geq N$ implies

$$1 - q_n \prec_A d_n \prec_A \left(y_N - \frac{1}{2}\right)_+ \prec_A t,$$

as desired.

For the reverse direction, all parts of the proof of the corresponding part of Lemma 2.15 apply, except that we must use the hypothesis that $1 - p$ is small in $(A^o)_\infty$, instead of the hypothesis $1 - p \prec (A^o)_\infty y$, to show that for all sufficiently large $n$ we have $q_n \prec_A y$. Accordingly, let $n_0, n_1, n_2, n_3, n_4 \in \mathbb{N}$ be as in the proof of Lemma 2.15. Since $1 - p$ is small in $(A^o)_\infty$, there is $n_5 \in \mathbb{N}$ such that for every $n \geq n_5$ we have $1 - q_n \prec_A y$. Then take $n = \max(n_0, n_1, \ldots, n_5)$, and in Definition 2.4 take $\varphi$ to be $\gamma_n$ and take $p$ to be $q_n$. Lemma 2.13 is still used to prove that $1 - q_n \prec_A q_n$. □
4. Permanence properties

In this section, we prove that fixed point algebras and, where appropriate, crossed products by actions of compact groups on infinite dimensional simple separable unital C*-algebras which have the tracial Rokhlin property with comparison preserve Property (SP), tracial rank zero, Popa algebras, and pure infiniteness.

For finite group actions with the tracial Rokhlin property, the tracial rank zero case is Theorem 2.6 of [44]. For a second countable compact group and an action with the Rokhlin property, it is in Theorem 4.5 of [13]. Preservation of tracial rank zero (partly) answers a question posed after Theorem 4.5 in [13].

Lemma 4.1. Let $A$ be an infinite dimensional simple separable unital C*-algebra. Let $\alpha: G \to \text{Aut}(A)$ be an action of a second countable compact group $G$ on $A$ which has the tracial Rokhlin property with comparison, but does not have the Rokhlin property. Then $A^\alpha$ has Property (SP).

Proof. Suppose $A^\alpha$ does not have Property (SP). We verify the condition of Lemma 2.3. Let $F \subseteq A$ and $S \subseteq C(G)$ be finite, and let $\varepsilon > 0$. Choose $y \in A^\alpha_+ \setminus \{0\}$ such that $yA^\alpha y$ contains no nonzero projection, and take $x = 1$. Apply Definition 2.4. Then $p = 1$, so the condition of Lemma 2.3 holds. □

The following result could have been in [13]. However, our proof needs simplicity of $A$. Without that, we don’t know how to prove that the projection we construct is nonzero, and it remains open whether fixed point algebras and crossed products by actions of compact groups preserve Property (SP) in general.

Theorem 4.2. Let $A$ be a simple separable infinite dimensional unital C*-algebra, let $G$ be a second countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action which has the Rokhlin property. If $A$ has Property (SP), so do $A^\alpha$ and $C^*(G, A, \alpha)$.

Proof. We prove that $A^\alpha$ has Property (SP). Let $x \in A^\alpha_+ \setminus \{0\}$; we need to prove that $xA^\alpha x$ contains a nonzero projection. Without loss of generality $\|x\| = 1$. Define continuous functions $h,h_0: [0,1] \to [0,1]$ by

\[
h(\lambda) = \begin{cases} 0 & 0 \leq \lambda \leq \frac{3}{4} \\ 4\lambda - 3 & \frac{3}{4} \leq \lambda \leq 1 \end{cases}
\]

and

\[
h_0(\lambda) = \begin{cases} \frac{3}{4}\lambda & 0 \leq \lambda \leq \frac{3}{4} \\ 1 & \frac{3}{4} \leq \lambda \leq 1. \end{cases}
\]

Set $x_0 = h_0(x)$. Use Property (SP) to choose a nonzero projection $e \in h(x)Ah(x)$. Then

\[
\|x - x_0\| \leq \frac{1}{4} \quad \text{and} \quad x_0e = e.
\]

Since $A$ is simple, there are $m \in \mathbb{N}$ and $a_1,a_2,\ldots,a_m,b_1,b_2,\ldots,b_m \in A$ such that $\sum_{j=1}^m a_j b_j = 1$. Define

\[M = 1 + \max_{j=1,2,\ldots,m} \max(\|a_j\|,\|b_j\|)\quad\text{and}\quad \varepsilon_0 = \min\left(\frac{1}{24}, \frac{1}{2m(M^2 + M + 1)}\right).\]

Choose $\varepsilon > 0$ so small that $\varepsilon \leq \varepsilon_0$ and whenever $D$ is a C*-algebra and $a \in D_{sa}$ satisfies $\|a^2 - a\| < \varepsilon$, then there is a projection $q \in D$ such that $\|q - a\| < \varepsilon_0$. Set

\[F_1 = \{a_j, a_j e, b_j : j = 1,2,\ldots,m\} \cup \{e,x_0\} \quad\text{and}\quad F_2 = \{x_0\}.\]
Apply Theorem 2.11 of [13] with these choices of $F_1$, $F_2$, and $\epsilon$, getting a unital completely positive map $\psi: A \to A^\alpha$ such that:

1. $\|\psi(ab) - \psi(a)\psi(b)\| < \epsilon$ for all $a, b \in F_1$.
2. $\|\psi(x_0) - x_0\| < \epsilon$.

We have $\|\psi(e)^2 - \psi(e)\| < \epsilon$ by (1). Therefore the choice of $\epsilon$ provides a projection $q \in A^\alpha$ such that $\|q - \psi(e)\| < \varepsilon_0$.

We claim that $q \neq 0$. For $j = 1, 2, \ldots, m$ we get

\[
\|\psi(a_j e b_j) - \psi(a_j)q\psi(b_j)\| \\
\leq \|\psi(a_j e b_j) - \psi(a_j e)\psi(b_j)\| + \|\psi(a_j e) - \psi(a_j)\psi(e)\|\|\psi(b_j)\| \\
+ \|\psi(a_j)\|\|\psi(e) - q\|\|\psi(b_j)\| \\
< \varepsilon + M\varepsilon + M^2\varepsilon_0 \leq (M^2 + M + 1)\varepsilon_0.
\]

So

\[
\left\| 1 - \sum_{j=1}^{m} \psi(a_j)q\psi(b_j) \right\| \leq \sum_{j=1}^{m} \|\psi(a_j e b_j) - \psi(a_j)q\psi(b_j)\| < m(M^2 + M + 1)\varepsilon_0 = \frac{1}{2}.
\]

Therefore $q \neq 0$, as claimed.

We have, using (1) and $ex_0 = x_0 e = e$ at the second step,

\[
\|\psi(e)x_0\psi(e) - \psi(e)\| \\
\leq \|x_0 - \psi(x_0)\| + \|\psi(e)\psi(x_0) - \psi(e)\|\|\psi(e)\| + \|\psi(e)^2 - \psi(e)\| \\
< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.
\]

So

\[
\|qxq - q\| \leq \|x - x_0\| + 3\|q - \psi(e)\| + \|\psi(e)x_0\psi(e) - \psi(e)\| \\
< \frac{1}{4} + 3\varepsilon_0 + 3\varepsilon \leq \frac{1}{4} + 6\varepsilon_0 \leq \frac{1}{2}.
\]

Therefore $(qxq)^{-1/2}$ makes sense in $qA^\alpha q$, and $s = x^{1/2}(qxq)^{-1/2}$ is a partial isometry with $s^* s = q$. So $ss^*$ is a nonzero projection in $xA^\alpha x$. Thus $A^\alpha$ has Property (SP).

Corollary 3.8 implies that $C^*(G, A, \alpha)$ and $A^\alpha$ are stably isomorphic. Therefore $C^*(G, A, \alpha)$ also has Property (SP). \qed

**Corollary 4.3.** Let $G$ be a second countable compact group, let $A$ be a simple separable infinite dimensional unital C*-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$ which has the tracial Rokhlin property with comparison. If $A$ has Property (SP), so do $A^\alpha$ and $C^*(G, A, \alpha)$.

**Proof.** If $\alpha$ has the Rokhlin property, then $A^\alpha$ has Property (SP), by Theorem 4.2. Otherwise, apply Lemma 4.1 to get the same conclusion.

Corollary 3.8 implies that $C^*(G, A, \alpha)$ and $A^\alpha$ are stably isomorphic. Therefore $C^*(G, A, \alpha)$ also has Property (SP). \qed

We next consider tracial rank zero. We use the following version of its definition.

**Definition 4.4** (Definition 3.6.2 of [38]). Let $A$ be a simple unital C*-algebra. Then $A$ has tracial rank zero for every finite subset $F \subseteq A$, every $\varepsilon > 0$, and every $x \in A \setminus \{0\}$, there exist a nonzero projection $p \in A$, a finite dimensional C*-algebra $B$, and a unital homomorphism $\varphi: B \to pAp$, such that:
provides orthogonal\]  
\[\epsilon \]
\Choose \[\epsilon \]
\[B \]
units for \[\gamma \]
there is a homomorphism \[\beta \] satisfies \[\phi \] clearly assume that \[\alpha \] does not have the Rokhlin property. So \[\alpha \] \[\phi \]
\Set \[\gamma \] \[\beta \] \[\phi \]
Is written as \[A \]
Applying Definition \[\gamma \] \[\phi \]
\[A \]
action of \[G \]
\Theorem \[\gamma \] \[\beta \] \[\phi \]
This completes the proof. \qed  
\Theorem \[\gamma \] \[\beta \] \[\phi \]
Let \[G \] be a second countable compact group, let \[A \] be a simple separable infinite dimensional unital \[C^* \]-algebra, and let \[\alpha : G \to \text{Aut}(A) \] be an action of \[G \] on \[A \] which has the tracial Rokhlin property with comparison. If \[A \] has tracial rank zero, then \[A^\alpha \] and \[C^*(G, A, \alpha) \] have tracial rank zero.  
\Proof We first prove that \[A^\alpha \] has tracial rank zero. If \[\alpha \] has the Rokhlin property, then \[A^\alpha \] has tracial rank zero by Theorem 4.5 of [13]. We may therefore assume that \[\alpha \] does not have the Rokhlin property. So \[A^\alpha \] has Property (SP) by Lemma 4.1.  
The algebra \[A^\alpha \] is simple by Theorem 3.1. So let \[F \subseteq A^\alpha \] be a finite subset, let \[\epsilon > 0 \], and let \[x \in A^\alpha \setminus \{0\} \]. We may clearly assume \[\|x\| = 1 \] and \[\epsilon < 1 \]. Lemma 4.1 provides a nonzero projection \[q \in xA^\alpha x \], and Lemma 1.8 provides orthogonal nonzero projections \[q_1, q_2, q_3 \in A^\alpha \] such that \[q_1, q_2, q_3 \leq q \].  
Applying Definition 4.4, getting a nonzero projection \[p \in A \], a finite dimensional \[C^* \]-algebra \[B \], and a unital homomorphism \[\phi : B \to pAp \], such that:  
(1) \[\|ap - pa\| < \frac{\epsilon}{4} \] for all \[a \in F \].  
(2) \[\text{dist}(pap, \phi(B)) < \frac{\epsilon}{2} \] for all \[a \in F \].  
(3) \[1 - p \preceq q_1 \].  
We may clearly assume that \[\phi \] is injective. Choose \[s \in A \] such that \[s^*s = 1 - p \] and \[ss^* \leq q_1 \].  
Set \[\epsilon_0 = \frac{\epsilon}{40 \dim(B)} \].  
Let \[E \] be a system of matrix units for \[B \]. Use semiprojectivity of \[B \] to choose \[\epsilon_1 > 0 \] such that \[\epsilon_1 \leq \epsilon_0 \] and whenever \[D \] is a \[C^* \]-algebra and \[\beta : E \cup \{0\} \to D \] satisfies \[\beta(0) = 0 \] and \[\|\beta(v)\beta(w) - \beta(vw)\| < 5\epsilon_1 \] for all \[v, w \in E \cup \{0\} \], then there is a homomorphism \[\gamma : B \to D \] such that \[\|\gamma(w) - \beta(w)\| < \epsilon_0 \] for all \[w \in E \]. Choose \[\epsilon_2 > 0 \] such that \[\epsilon_2 \leq \epsilon_1 \]; whenever \[D \] is a \[C^* \]-algebra and \[b \in D_{sa} \] satisfies \[\|\gamma(w) - \beta(w)\| < \epsilon_2 \] for all \[w \in E \].
\[ \| b^2 - b \| < \varepsilon_2, \text{ then there is a projection } e \in D \text{ such that } \| e - b \| < \varepsilon_1; \text{ and also} \]
whenever \( D \) is a C*-algebra, \( e, f \in D \) are projections, and \( t \in D \) satisfies \( \| t^*t - e \| < \varepsilon_1 \) and \( \| ft^* - t^*f \| < \varepsilon_1 \), then \( e \preceq_D f \). Apply Theorem 2.16, getting a projection \( e_0 \in A^0 \) and a unital completely positive contractive map \( \psi : A \rightarrow e_0 A^0 e_0 \) such that:

1. \[ \| \psi(ab) - \psi(a)\psi(b) \| < \varepsilon_2 \text{ for all } a, b \in \varphi(E) \cup \{ p, s, s^*, ss^* \} \cup F \cup \{ q_1 \}. \]
2. \[ \| e_0 a - ae_0 \| < \varepsilon_2 \text{ for all } a \in \varphi(E) \cup \{ p, s, s^*, ss^* \} \cup F \cup \{ q_1 \}. \]
3. \[ \| \psi(a) - e_0ae_0 \| < \varepsilon_2 \text{ for all } a \in F \cup \{ q_1 \}. \]
4. \[ 1 - e_0 \preceq_{A^0} q_2. \]

By (4) with \( a = p \) and the choice of \( \varepsilon_2 \), there is a projection \( e \in e_0 A^0 e_0 \) such that

\[ \| e - \psi(p) \| < \varepsilon_1. \]

Define a completely positive contractive map \( \beta : B \rightarrow eA^0 e \) by

\[ \beta(x) = e(\psi \circ \varphi)(x)e \]
for \( x \in B \). For \( w \in E \cup \{ 0 \} \) we have, using (4.3), \( p\varphi(w) = \varphi(w) \), and (4) at the second step,

\[ \| e(\psi \circ \varphi)(w) - (\psi \circ \varphi)(w) \| \]
\[ \leq \| e - \psi(p) \| \| (\psi \circ \varphi)(w) \| + \| \psi(p)(\psi \circ \varphi)(w) - (\psi \circ \varphi)(w) \| \]
\[ < \varepsilon_1 + \varepsilon_2 \leq 2\varepsilon_1. \]

Similarly \( \| (\psi \circ \varphi)(w)e - (\psi \circ \varphi)(w) \| < 2\varepsilon_1 \). Combining these gives

\[ \| e(\psi \circ \varphi)(w) - (\psi \circ \varphi)(w)e \| < 4\varepsilon_1 \]
and

\[ \| \beta(w) - (\psi \circ \varphi)(w) \| < 4\varepsilon_1. \]

Now for \( v, w \in E \cup \{ 0 \} \) we have, also using \( \varphi(vw) = \varphi(v)\varphi(w) \) at the first step and (4) at the third step,

\[ \| \beta(v)\beta(w) - \beta(vw) \| \leq \| e(\psi \circ \varphi)(v)e(\psi \circ \varphi)(w)e - e(\psi \circ \varphi)(v)(\psi \circ \varphi)(w)e \|
\]
\[ + \| e(\psi \circ \varphi)(v)e(\psi \circ \varphi)(w)e - e(\psi \circ \varphi)(w)e \| \]
\[ \leq \| e(\psi \circ \varphi)(v)\| \| e(\psi \circ \varphi)(w) - (\psi \circ \varphi)(w)e \|
\]
\[ + \| e\| \| (\psi(\varphi(v))\psi(\varphi(w)) - (\psi(\varphi(v))\varphi(w)) \| \| e \|
\]
\[ < 4\varepsilon_1 + \varepsilon_2 \leq 5\varepsilon_1. \]

By the choice of \( \varepsilon_1 \), there is a homomorphism \( \gamma : B \rightarrow eA^0 e \) such that \( \| \gamma(w) - \beta(w) \| < \varepsilon_0 \) for all \( v, w \in E \). Lemma 4.5 implies that

\[ \| \gamma(x) - \beta(x) \| \leq \dim(B)\varepsilon_0 \| x \| \]
for all \( x \in B \).

We claim that \( \gamma \) is unital. Let \( E_0 \subseteq E \) be the set of diagonal matrix units. Then, using (4.8) and (4.6) at the second step,

\[ \| \gamma(1) - e \| \leq \| \gamma(1) - \beta(1) \| + \sum_{w \in E_0} \| \beta(w) - (\psi \circ \varphi)(w) \|
\]
\[ < \dim(B)\varepsilon_0 + 4\text{card}(E_0)\varepsilon_1 < \varepsilon < 1. \]

Therefore \( \gamma(1) \) is invertible in \( eA^0 e \), and the claim follows.
We now verify the conditions of Definition 4.4, with $e$ in place of $p$ and $\gamma$ in place of $\varphi$.

For Definition 4.4(1), let $a \in F$. Then, at the third step using (5), (6), (4.3), and (4),

\[ \|ea - \psi(pa)\| = \|ee_0 a - \psi(pa)\| \]
\[ \leq \|ee_0\|\|e a - ae_0\| + \|e\|\|e_0 ae_0 - \psi(a)\| \]
\[ + \|e - \psi(p)\|\|\psi(a)\| + \|\psi(p)\psi(a) - \psi(pa)\| \]
\[ < \varepsilon_2 + \varepsilon_1 + \varepsilon_2 \leq 4\varepsilon_1. \]

Similarly $\|ae - \psi(ap)\| < 4\varepsilon_1$. Therefore, using (1) at the second step,

\[ \|ea - ae\| \leq \|ea - \psi(pa)\| + \|ae - \psi(ap)\| + \|\psi\|\|pa - ap\| < 4\varepsilon_1 + 4\varepsilon_1 + \frac{\varepsilon}{2} \leq \varepsilon, \]

as desired.

For Definition 4.4(2), let $a \in F$. Using $ee_0 = e$ and (6), we first observe that

\[ \|e\psi(a)e - eae\| = \|e\psi(a) - e_0 ae_0 e\| < \varepsilon_2. \]

Use (2) to choose $b \in B$ such that $\|pap - \varphi(b)\| < \frac{\varepsilon}{2}$. Since $\frac{\varepsilon}{2} < 1$ and $\varphi$ is injective, we have $\|b\| \leq 2$. Therefore, using (4.4) at the first step, and (4.8), $\|pap - \varphi(b)\| < \frac{\varepsilon}{2}$, and (4.9) at the second step,

\[ \|\gamma(b) - eae\| \leq \|\gamma(b) - \beta(b)\| + \|e(\psi \circ \varphi)(b)e - \psi(a)e\| + \|\psi(a)e - eae\| \]
\[ \leq 2 \dim(B)\varepsilon_0 + \frac{\varepsilon}{2} + \varepsilon_2 < \varepsilon. \]

We prove Definition 4.4(3). Using $s^* s = 1 - p$ and $\psi(1) = e_0$ at the first step and (4) and (4.3) at the second step, we get

\[ \|\psi(s^* s) - (e_0 - e)\| \leq \|\psi(s^* s) - \psi(s^* s)\| + \|\psi(1 - p) - (e_0 - e)\| \]
\[ < \varepsilon_2 + \varepsilon_1 \leq 2\varepsilon_1. \]

Using $\psi(ss^*) \in e_0 A e_0$ and $ss^* = q_1 ss^*$ at the first step, and (5), (6), and (4) at the third step, we get

\[ \|q_1 \psi(ss^*) - \psi(ss^*)\| = \|q_1 e_0 \psi(ss^*) - \psi(q_1 ss^*)\| \]
\[ \leq \|q_1\|\|q_1 e_0 - e_0 q_1\|\|\psi(ss^*)\| + \|e_0 q_1 e_0 - \psi(q_1)\|\|\psi(ss^*)\| \]
\[ + \|\psi(q_1)\psi(ss^*) - \psi(q_1 ss^*)\| \]
\[ < \varepsilon_2 + \varepsilon_2 + \varepsilon_2 = 3\varepsilon_2. \]

Since $\|\psi(ss^*) - \psi(s)\psi(s^*)\| < \varepsilon_2$ by (4), this gives $\|q_1 \psi(s)\psi(s^*) - \psi(s)\psi(s^*)\| < 5\varepsilon_2 \leq 5\varepsilon_1$. The choice of $\varepsilon_1$ implies that $e_0 - e \not\leq A^* q_1$. Since $1 - e_0 \not\leq A^* q_2$ by (7), we get $1 - e \not\leq A^* q_1 + q_2 \leq q \not\leq A^* x$, as desired. Also, since $1 - e \not\leq A^* q_1 + q_2$, $A^*$ is finite (being a subalgebra of $A$), and $q_1 + q_2$ is orthogonal to the nonzero projection $q_3$, we have $e \neq 0$. We have proved that $A^*$ has tracial rank zero.

Theorem A.4 of [12], Theorem 3.7, and Theorem 3.4 now imply that $C^*(G, A, \alpha)$ has tracial rank zero.

The following definition is based on a condition in Theorem 1.2 of [6]. That theorem only considers unital algebras.

**Definition 4.7.** A simple separable unital $C^*$-algebra $A$ is called a *Popa algebra* if for every finite subset $F \subseteq A$ and every $\varepsilon > 0$ there are a nonzero projection $p \in A$,
a finite dimensional C*-algebra $B$, and a unital homomorphism $\varphi: B \to pAp$, such that the following hold:

1. $\|pa - ap\| < \varepsilon$ for all $a \in F$.
2. $\text{dist}(pap, \varphi(B)) < \varepsilon$ for all $a \in F$.

**Theorem 4.8.** Let $A$ be a simple, separable unital C*-algebra which is a Popa algebra, let $G$ be a second countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action which has the tracial Rokhlin property with comparison. Then $A^\alpha$ is Popa algebra.

**Proof.** The proof is the same as that of Theorem 4.6, except that the projection $e$ must be treated differently.

The algebra $A^\alpha$ is simple by Theorem 3.1. So let $F \subseteq A^\alpha$ be a finite subset, and let $\varepsilon > 0$. Without loss of generality $\varepsilon < \frac{1}{2}$. Apply Definition 4.7, getting a nonzero projection $p \in A$, a finite dimensional C*-algebra $B$, and a unital homomorphism $\varphi: B \to pAp$, such that:

1. $\|ap - pa\| < \frac{\varepsilon}{2}$ for all $a \in F$.
2. $\text{dist}(pap, \varphi(B)) < \frac{\varepsilon}{2}$ for all $a \in F$.

We may clearly assume that $\varphi$ is injective.

Since $A$ is simple, there are $m \in \mathbb{N}$ and $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in A$ such that $\sum_{j=1}^m a_jpb_j = 1$. Define

$$F_0 = \{a_j, b_j: j = 1, 2, \ldots, m\} \quad \text{and} \quad M = 1 + \max_{j=1,2,\ldots,m} \max(\|a_j\|, \|b_j\|).$$

Let $E$, $\varepsilon_0$, $\varepsilon_1$, and $\varepsilon_2$ be as in the proof of Theorem 4.6, except that we also require

$$\varepsilon_1 \leq \frac{1}{2m(M^2 + 1)}.$$

Apply Theorem 2.16, getting a projection $e_0 \in A^\alpha$ and a unital completely positive contractive map $\psi: A \to e_0A^\alpha e_0$ such that:

1. $\|\psi(ab) - \psi(a)\psi(b)\| < \varepsilon_2$ and $\|\psi(abc) - \psi(a)\psi(b)\psi(c)\| < \varepsilon_2$ for all $a, b, c \in \varphi(E) \cup \{p\} \cup F \cup F_0$.
2. $\|e_0a - ae_0\| < \varepsilon_2$ for all $a \in \varphi(E) \cup \{p\} \cup F \cup F_0$.
3. $\|e_0\| > 1 - \varepsilon_2$.

(These are not the same as in the proof of Theorem 4.6. The biggest change is in (6).) As in the proof of Theorem 4.6, there is a projection $e \in A^\alpha$ such that $\|e - \psi(p)\| < \varepsilon_1$.

We claim that $e \neq 0$. For $j = 1, 2, \ldots, m$ we have

$$\|\psi(a_jpb_j) - \psi(a_j)e\psi(b_j)\| \leq \|\psi(a_jpb_j) - \psi(a_j)e\psi(p)\psi(b_j)\| + \|\psi(a_j)\|\|\psi(p) - e\|\|\psi(b_j)\|$$

$$< \varepsilon_2 + M^2\varepsilon_1 \leq (M^2 + 1)\varepsilon_1.$$ 

Therefore

$$\left\|e_0 - \sum_{j=1}^m \psi(a_j)e\psi(b_j)\right\| = \left\|\sum_{j=1}^m \psi(a_jpb_j) - \sum_{j=1}^m \psi(a_j)e\psi(b_j)\right\|$$

$$< m(M^2 + 1)\varepsilon_1 < 1.$$ 

Since $e_0$ is a projection, and $e_0 \neq 0$ by (6) and $\varepsilon_2 < \frac{1}{2}$, the claim follows.
By the same reasoning as in the proof of Theorem 4.6, we now get a \( \|ae - ae\| < \varepsilon \) for all \( a \in F \), and a unital homomorphism \( \gamma: B \to eA^\alpha e \) such that \( \operatorname{dist}(eae, \varphi(B)) < \varepsilon \) for all \( a \in F \). This completes the proof. \( \square \)

We now turn to pure infiniteness.

**Theorem 4.9.** Let \( A \) be a purely infinite simple separable unital C*-algebra, let \( G \) be a second countable compact group, and let \( \alpha: G \to \text{Aut}(A) \) be an action of \( G \) on \( A \) which has the tracial Rokhlin property with comparison. Then \( A^\alpha \) and \( C^*(G, A, \alpha) \) are purely infinite.

**Proof.** We first prove that \( A^\alpha \) is purely infinite. If \( \alpha \) has the Rokhlin property, this follows from Proposition 4.10 of [13]. We may therefore assume that \( \alpha \) does not have the Rokhlin property. So \( A^\alpha \) has Property (SP) by Lemma 4.1.

The algebra \( A^\alpha \) is simple by Theorem 3.1. For pure infiniteness, it then suffices to show that for every \( z \in A^\alpha \setminus \{0\} \), the hereditary subalgebra \( zA^\alpha z \) contains a nonzero infinite projection. (This is the original definition of pure infiniteness for a simple C*-algebra, given before Proposition 1.6 of [8].) In fact, we find a nonzero properly infinite projection.

Set \( \varepsilon_0 = \frac{1}{100} \). Choose \( \varepsilon_1 > 0 \) so small that \( \varepsilon_1 \leq \varepsilon_0 \) and whenever \( D \) is a C*-algebra and \( a \in D_{sa} \) satisfies \( \|a^2 - a\| < 55\varepsilon_1 \), then there is a projection \( e \in D \) such that \( \|e - a\| < \varepsilon_0 \). Repeat: choose \( \varepsilon_2 > 0 \) so small that \( \varepsilon_2 \leq \varepsilon_1 \) and whenever \( D \) is a C*-algebra and \( a \in D_{sa} \) satisfies \( \|a^2 - a\| < 7\varepsilon_2 \), then there is a projection \( e \in D \) such that \( \|e - a\| < \varepsilon_1 \). Repeat again: choose \( \delta > 0 \) so small that \( \delta \leq \varepsilon_2 \) and whenever \( D \) is a C*-algebra and \( a \in D_{sa} \) satisfies \( \|a^2 - a\| < \delta \), then there is a projection \( e \in D \) such that \( \|e - a\| < \varepsilon_2 \).

Since \( A^\alpha \) has Property (SP), there is a nonzero projection \( q \in zA^\alpha z \). Since \( A \) is purely infinite, there are projections \( q_1, q_2 \in A \) such that

\[ q_1 \sim q_2 \sim q, \quad q_1 q_2 = 0, \quad \text{and} \quad q_1 + q_2 \leq q. \]

In particular, there are \( s_1, s_2 \in A \) such that

\[ s_2^* s_1 = q_1, \quad s_1 s_2^* = q, \quad s_2^* s_2 = q_2, \quad \text{and} \quad s_2 s_2^* = q. \]

Use Theorem 2.16 with \( \varepsilon = \delta \), \( x = q \), \( y = 1 \), \( n = 3 \),

\[ F_1 = \{ q, q_1, q_2, s_1, s_1^*, s_2, s_2^* \} \quad \text{and} \quad F_2 = \{ q \}, \]

obtaining a projection \( p \in A^\alpha \) and a unital completely positive map \( \psi: A \to pA^\alpha p \) which satisfy:

1. For all \( a, b, c \in F_1 \), we have
   \[ \|\psi(ab) - \psi(a)\psi(b)\| < \delta \quad \text{and} \quad \|\psi(abc) - \psi(a)\psi(b)\psi(c)\| < \delta. \]
2. \( \|pa - ap\| < \delta \) for all \( a \in F_1 \cup F_2 \).
3. For all \( a \in F_2 \), we have \( \|\psi(a) - pap\| < \delta \).
4. \( \|pqp\| > 1 - \delta. \)

Using condition (2) with \( a = q \), we have

\[ \|qpq\| < \delta \quad \text{and} \quad \|qpq - pqp\| < 2\delta. \]

By the first part of (4.10) and the choice of \( \delta \), there exists a projection \( r \in qA^\alpha q \subseteq zA^\alpha z \) such that

\[ \|r - qpq\| < \varepsilon_2. \]
Combining this, the second part of (4.10), and (4), we get
\[ \|r^*\| > \|q_{ppq}\| - \varepsilon_2 > 1 - 3\delta - \varepsilon_2 \geq 1 - 4\varepsilon_2 > 0. \]
Therefore
\[ (4.12) \quad r \neq 0. \]
Also, using (4.11), (4.10), and (3),
\[ (4.13) \quad \|r - \psi(q)\| \leq \|r - q_{ppq}\| + \|q_{ppq} - \psi(q)\| + \|\psi(q)\| < \varepsilon_2 + 2\delta + \delta \leq 4\varepsilon_2. \]

We claim that for all \( x \in F_1 \) we have
\[ (4.14) \quad \|\psi(x)r - r\psi(x)\| < 6\varepsilon_2. \]
To see this, from the definition of \( F_1 \) we see that \( qx = xq = x \) and \( \|x\| \leq 1. \) Now, using (3) and (1) at the third step,
\[ \|\psi(x)q_{ppq} - \psi(x)\psi(x)\| = \|\psi(x)q_{ppq} - \psi(xq)\| \leq \|\psi(x)\|\|q_{ppq} - \psi(q)\| + \|\psi(x)\|\|\psi(q) - \psi(x)\| < \delta + \delta = 2\delta. \]
Since \( x^* \) is also in \( F_1 \), taking adjoints gives \( \|pq\psi(x) - \psi(x)\| < 2\delta. \) So
\[ \|\psi(x)r - r\psi(x)\| < 2\|r - q_{ppq}\| + \|\psi(x)q_{ppq} - \psi(x)\| + \|\psi(x)\| < 2\varepsilon_2 + 2\delta + 2\delta \leq 6\varepsilon_2, \]
proving the claim.

The element \( b_1 = r\psi(q_1)r \) satisfies, using (4.14) and (1) at the third step,
\[ (4.15) \quad \|b_1^2 - b_1\| = \|r\psi(q_1)r\psi(q_1)r - r\psi(q_1)r\| \leq \|r\|\|\psi(q_1)r - r\psi(q_1)\|\|\psi(q_1)r\| + \|r\|\|\psi(q_1)^2 - \psi(q_1)\|\|r\| < 6\varepsilon_2 + 6\delta \leq 7\varepsilon_2. \]

By the choice of \( \varepsilon_2 \), there is a projection \( r_1 \in rA^\circ r \) such that
\[ (4.16) \quad \|r_1 - r\psi(q_1)r\| < \varepsilon_1. \]
Therefore also
\[ (4.17) \quad \|r_1 - r_1\psi(q_1)r_1\| = \|r_1(r_1 - r\psi(q_1)r)r_1\| < \varepsilon_1. \]

Set \( c_1 = r\psi(s_1)r_1. \) Then, using \( rr_1 = r_1 \) at the second step, and (4.14), (1), \( s_1^*s_1 = q_1 \), and (4.17) at the third step,
\[ (4.18) \quad \|c_1^*c_1 - r_1\| = \|r_1\psi(s_1)^*r\psi(s_1)r_1 - r_1\| \leq \|r_1\psi(s_1)^*\|\|r\psi(s_1) - \psi(s_1)\|\|r_1\| + \|r_1\|\|\psi(s_1)^*\|\|\psi(s_1) - \psi(q_1)\|\|r_1\| + \|r_1\psi(q_1)r_1 - r_1\| < 6\varepsilon_2 + \delta + \varepsilon_1 < 1. \]
Also, using \( s_1q_1s_2^* = q \) at the second step, and (4.16), (4.14) (twice), (1), and (4.13) at the third step,
\[
\|c_1^*c_1 - r\| = \|r\psi(s_1)r_1\psi(s_1)^*r - r\|
\leq \|r\psi(s_1)\||r_1 - r\psi(q_1)r\|\|\psi(s_1)^*r\|
+ \|r\|\|\psi(s_1)r - r\psi(s_1)\|\|\psi(q_1)r\psi(s_1)^*r\|
+ \|r\|\|\psi(s_1)\psi(q_1)\|\|\psi(s_2^*) - \psi(s_1^*)r\|\|r\|
+ \|r\|\|\psi(q) - r\|\|r\|
< \varepsilon_1 + 6\varepsilon_2 + \delta + 4\varepsilon_2 < 1.
\]

By (4.18) and (4.19), the element \( t_1 = c_1^*(c_1^*)^{-1/2} \) (functional calculus in \( rA^\alpha r \)) satisfies \( t_1^*t_1 = r_1 \) and \( t_1t_1^* = r \). Thus
\[
r_1 \sim r.
\]

Now define \( b_2 = (r - r_1)\psi(q_2)(r - r_1) \). We estimate, using \( q_1q_2 = 0 \) at the first step and (4.16), (4.14), and (1) at the second step,
\[
\|r_1\psi(q_2)\| \leq \|r_1 - r\psi(q_1)r\||\|\psi(q_2)\| + \|\psi(q_1)r - r\psi(q_1)\||\|\psi(q_2)\|
+ \|r\||\psi(q_1)\psi(q_2) - \psi(q_1q_2)\| \\
< \varepsilon_1 + 6\varepsilon_2 + \delta < 3\varepsilon_1.
\]

Taking adjoints gives \( \|\psi(q_2)r_1\| < 3\varepsilon_1 \). Therefore
\[
\|r\psi(q_2)r - b_2\| \leq \|r_1\psi(q_2)\|\|r - r_1\| + \|r\||\psi(q_2)r_1\| < 16\varepsilon_1.
\]

Now
\[
\|b_2^2 - b_2\| \leq 3\|b_2 - r\psi(q_2)r\| + \|r\psi(q_2)r\psi(q_2)r - r\psi(q_2)r\|
\]
We have
\[
\|r\psi(q_2)r\psi(q_2)r - r\psi(q_2)r\| < 7\varepsilon_2
\]
by the same reasoning as was used for (4.15), so
\[
\|b_2^2 - b_2\| < 3(16\varepsilon_1) + 7\varepsilon_2 < 55\varepsilon_1.
\]

By the choice of \( \varepsilon_1 \), there is a projection \( r_2 \in (r - r_1)A^\alpha(r - r_1) \) such that
\[
\|r_2 - r\psi(q_2)r\| < \varepsilon_0.
\]
Therefore also
\[
\|r_2 - r_2\psi(q_2)r_2\| = \|r_2(r_2 - r\psi(q_2)r_2)\| < \varepsilon_0.
\]

Set \( c_2 = r\psi(s_2)r_2 \). A calculation like (4.18), but using (4.22) to estimate \( \|r_2\psi(q_2)r_2 - r_2\| \), gives
\[
\|c_2^*c_2 - r_2\| < 6\varepsilon_2 + \delta + \varepsilon_0 < 1.
\]

A calculation like (4.19), using (4.21) to estimate \( \|r\psi(q_2)r - r_2\| \), gives
\[
\|c_2^*c_2 - r\| < \varepsilon_0 + 6\varepsilon_2 + 6\varepsilon_2 + \delta + 4\varepsilon_2 < 1.
\]

By (4.23) and (4.24), the element \( t_2 = c_2^*(c_2^*)^{-1/2} \) (functional calculus in \( rA^\alpha r \)) satisfies \( t_2^*t_2 = r_2 \) and \( t_2t_2^* = r \). Thus \( r_2 \sim r \). By (4.12) and (4.20), and since \( r_1r_2 = 0 \) and \( r_1 + r_2 \leq r \), this proves that \( r \) is a nonzero properly infinite projection. We have proved that \( A^\alpha \) is purely infinite.
Corollary 3.8 implies that $C^*(G, A, \alpha)$ and $A^\alpha$ are stably isomorphic. Therefore $C^*(G, A, \alpha)$ is also purely infinite. \hfill \Box

5. The naive tracial Rokhlin property

Our definition of the tracial Rokhlin property with comparison (Definition 2.4), applied to finite groups, is formally stronger than the tracial Rokhlin property for finite groups (Definition 2.1), as discussed after Definition 2.4. In this section we address the differences. Definition 5.1 below is given only to make discussion easier; it is not intended for general use. It is what one gets by just copying the definition of the tracial Rokhlin property for finite groups. We say at the outset that we know of no examples of actions, even of infinite compact groups, which have the naive tracial Rokhlin property but not the tracial Rokhlin property with comparison, although we believe they exist, and we offer some evidence below.

For actions of finite groups, we can show that the tracial Rokhlin property “almost” implies the tracial Rokhlin property with comparison.

Definition 5.1. Let $A$ be an infinite dimensional separable simple unital $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$. The action $\alpha$ has the naive tracial Rokhlin property if for every finite set $F \subseteq A$, every finite set $S \subseteq C(G)$, every $\varepsilon > 0$, and every $x \in A_+$ with $\|x\| = 1$, there exist a projection $p \in A^\alpha$ and a unital completely positive contractive map $\varphi: C(G) \to pA p$ such that:

1. $\varphi$ is an $(F, S, \varepsilon)$-equivariant central multiplicative map.
2. $1 - p \not\prec_A x$.
3. $\|p x p\| > 1 - \varepsilon$.

We prove that condition (3) of Definition 2.4 is automatic when the group is finite, regardless of what $A$ is.

Proposition 5.2. Let $A$ be an infinite dimensional separable simple unital $C^*$-algebra, let $G$ be a finite group, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$ which has the tracial Rokhlin property. Then for every finite set $F \subseteq A$, every $\varepsilon > 0$, every $x \in A_+$ with $\|x\| = 1$, and every $y \in A^\alpha \setminus \{0\}$, there exist a projection $p \in A^\alpha$ and mutually orthogonal projections $(p_g)_{g \in G}$ such that:

1. $p = \sum_{g \in G} p_g$.
2. $\|p_g a - a p_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$.
3. $\|\alpha_g(p_h) - p_{gh}\| < \varepsilon$ for all $g, h \in G$.
4. $1 - p \not\prec_A x$.
5. $1 - p \not\prec_{A^\alpha} y$.
6. $\|p x p\| > 1 - \varepsilon$.

Proof. Let $F \subseteq A$ be finite and let $\varepsilon > 0$. Let $x \in A_+$ with $\|x\| = 1$ and $y \in A^\alpha \setminus \{0\}$ be given. By Corollary 1.6 of [44], we know that $A^\alpha$ is simple and not of Type I. Therefore the same is true for $yA^\alpha y$. By Lemma 2.1 of [47] there is a positive element $z \in yA^\alpha y$ such that $0$ is a limit point of $\text{sp}(z)$. Use Lemma 1.7 to choose $x_0 \in A_+ \setminus \{0\}$ such that $x_0 \not\prec_A x$ and $x_0 \not\prec_A z$. Now apply Lemma 1.17 of [44] to $\alpha$ with $x_0$ in place of $x$, and with $F$ and $\varepsilon$ as given. We obtain mutually orthogonal projections $p_g$ for $g \in G$ such that, with $p = \sum_{g \in G} p_g$ (so that (1) holds), $p$ is $\alpha$-invariant, Conditions (2), (3), and (6) are satisfied, and $1 - p$ is Murray-von
Neumann equivalent to a projection in $x_0Ax_0$. Thus $1 - p \preceq_A x_0 \preceq_A x$, which is (4). Also,

$$1 - p \preceq_A x_0 \preceq_A z, \quad 1 - p, z \in A^\alpha, \quad \text{and} \quad 0 \in \text{sp}(z) \setminus \{0\}.$$ 

It follows from Lemma 3.7 of [3] that $1 - p \preceq_A \gamma$. Since $z \in yA^\alpha y$, it follows that $1 - p \preceq_A \gamma$, which is (5). This completes the proof. □

We now give some conditions on actions of finite groups under which Condition (4) of Definition 2.4 is automatic.

**Proposition 5.3.** Let $A$ be a stably finite infinite dimensional simple separable unital exact C*-algebra. Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$ which has the tracial Rokhlin property. If $A$ has strict comparison then $\alpha$ has the tracial Rokhlin property with comparison.

**Proof.** We verify the conditions of Lemma 2.5. The argument of the proof of Lemma 1.16 of [44] shows that it is enough to prove this without Condition (7). By Proposition 2.6, we may assume that $\alpha$ does not have the Rokhlin property. So $A^\alpha$ has Property (SP) by Lemma 4.1.

Let $F \subseteq A$ be finite, let $\varepsilon > 0$, let $x \in A_+$ satisfy $\|x\| = 1$, and let $y \in A^\alpha \setminus \{0\}$.

Without loss of generality $\|y\| \leq 1$.

Recall that $T(B)$ is the tracial state space of $B$. Set

$$\delta = \inf_{\tau \in T(A^\alpha)} \tau(y).$$

Then $\delta > 0$. Choose $n \in \mathbb{N}$ with $n > \max(3, 1/\delta)$. The algebra $C^*(G, A, \alpha)$ is simple by Corollary 1.6 of [44]. So $A^\alpha$ is simple by Theorem 3.4. Lemma 1.8 provides $n$ nonzero mutually orthogonal projections in $A$. Let $e$ be one of them.

Apply Lemma 1.7 to find $x_0 \in A_+$ such that

$$\|x_0\| = 1, \quad x_0 \preceq_A e, \quad \text{and} \quad x_0 \preceq_A x.$$

Apply Lemma 1.17 of [44] with $F$ and $\varepsilon$ as given, and $x_0$ in place of $x$, getting mutually orthogonal projections $p_0 \in A$ for $g \in G$ such that, in Lemma 2.5 and with $p = \sum_{g \in G} p_0$, Conditions (1), (2), (3), and (4) are satisfied with $x_0$ in place of $x$. Then Condition (4) as stated follows because $x_0 \preceq_A x$.

We claim that $\tau(1 - p) \leq \frac{1}{n}$ for all $\tau \in T(A^\alpha)$. For $\tau \in T(A)$, clearly $\tau(e) \leq \frac{1}{n}$, so $\tau(1 - p) \leq \frac{1}{n}$. Let $E : A \to A^\alpha$ be the conditional expectation given by averaging over $G$. If $\tau \in T(A^\alpha)$, then $\tau \circ E \in T(A)$, so $\tau(1 - p) = (\tau \circ E)(1 - p) \leq \frac{1}{n}$, proving the claim. It follows that for all $\tau \in T(A^\alpha)$, we have

$$d_\tau(1 - p) \leq \frac{1}{n} < \tau(y) \leq d_\tau(y) \quad \text{and} \quad d_\tau(1 - p) \leq \frac{1}{n} \leq \frac{1}{3} < \frac{2}{3} \leq d_\tau(p).$$

By Theorem 4.1 of [3], $A^\alpha$ has strict comparison. Since $A$ is exact, so is $A^\alpha$, by Proposition 7.1(i) of [33]. Therefore, by Theorem 5.11 of [22], all quasitraces on $A$ are traces. It follows that $1 - p \preceq_A \gamma$, which is Condition (5) of Lemma 2.5, and $1 - p \preceq_A \gamma$, which is Condition (6) of Lemma 2.5. □

We presume that exactness is not needed, but one needs to know that if $\tau \in QT(A^\alpha)$ then $\tau \circ E$ is a quasitrace.

To make the argument work in general, one needs to apply to $A^\alpha$ a positive answer to the following question.
Question 5.4. Let $B$ be a stably finite simple separable unital $\text{C}^*$-algebra. Does there exist $z \in B_+ \setminus \{0\}$ such that whenever a projection $q \in B$ satisfies $q \preceq_B z$, then $q \preceq_B 1 - q$?

This question seems hard, but the answer may well be negative.

We now prove that if $G$ is finite and $A$ is purely infinite simple, then condition (4) of Definition 2.4 is automatic.

Proposition 5.5. Let $A$ be an infinite dimensional simple separable unital $\text{C}^*$-algebra. Let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$ which has the tracial Rokhlin property. If $A$ is purely infinite then $\alpha$ has the tracial Rokhlin property with comparison.

Proof. We verify the condition of Lemma 2.5. So let $F \subseteq A$ be finite, let $\varepsilon > 0$, let $x \in A_+$ satisfy $\|x\| = 1$, and let $y \in A^\alpha \setminus \{0\}$. Without loss of generality $\varepsilon < 1$. Apply Lemma 1.17 of [44] with $F$, $\varepsilon$, and $x$ as given, getting mutually orthogonal projections $p_g \in A$ for $g \in G$ such that, in Lemma 2.5 and with $p = \sum_{g \in G} p_g$, Conditions (1), (2), (3), (4), and (7) are satisfied. The algebra $C^*(G, A, \alpha)$ is simple by Corollary 1.6 of [44]. So $A^\alpha$ is simple by Theorem 3.4. The action $\alpha$ is pointwise outer by Lemma 1.5 of [44], so Theorem 3 of [30] implies that $C^*(G, A, \alpha)$ is purely infinite. Separability and the last part of Theorem 3.4 now imply that $A^\alpha$ is purely infinite. Since $y \neq 0$, the relation $1 - p \preceq_{A^\alpha} y$ is automatic; this is Condition (5) of Lemma 2.5. Since $p \neq 0$ (from $\|p \varphi p\| > 1 - \varepsilon$), the relations $1 - p \preceq_{A^\alpha} p$ is automatic; this is Condition (6) of Lemma 2.5. \qed

6. THE MODIFIED TRACING ROKHLIN PROPERTY

The extra conditions in Definition 2.4 seem somewhat unsatisfactory, partly because there are two of them. We seem to need Condition (3) $(1 - p \preceq_{A^\alpha} y)$ in order to prove preservation of tracial rank zero (see the proof of Theorem 4.6), and we seem to need Condition (4) $(1 - p \preceq_{A^\alpha} p)$ in order to prove that the crossed product is simple (see the proof of Proposition 3.5). This has led us to consider other variants. In this section, we discuss the most promising of these, which we call the modified tracial Rokhlin property (again, a name intended for use only in this paper). There are actually two versions. We will explain what we can prove with them, and prove that, in a very special case, they are automatic for actions of finite groups with the tracial Rokhlin property. The example we construct in Section 7 has the strong modified tracial Rokhlin property, and the example in Section 8 has the modified tracial Rokhlin property but probably not the strong modified tracial Rokhlin property.

Some proofs in this section are sketchy.

Definition 6.1. Let $A$ be an infinite dimensional separable simple unital $\text{C}^*$-algebra, let $G$ be a second countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$. The action $\alpha$ has the modified tracial Rokhlin property if for every finite set $F_1 \subseteq A$, every finite set $F_2 \subseteq A^\alpha$, every finite set $S \subseteq C(G)$, every $\varepsilon > 0$, and every $x \in A_+$ with $\|x\| = 1$, there exist a projection $p \in A^\alpha$, a partial isometry $s \in A^\alpha$, and a unital completely positive contractive map $\varphi: C(G) \to pAp$, such that:

1. $\varphi$ is an $(F_1, S, \varepsilon)$-approximately equivariant central multiplicative map.
2. $1 - p \preceq_A x$.
for the modified tracial Rokhlin property is as follows.

Let $A$ be an infinite dimensional separable simple unital C*-algebra, let $G$ be a second countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$. The action $\alpha$ has the strong modified tracial Rokhlin property if for every finite set $F \subset A$, every finite set $S \subset C(G)$, every $\varepsilon > 0$, and every $x \in A_+$ with $\|x\| = 1$, there exist a partial isometry $s \in A^\alpha$, a projection $p \in A^\alpha$, and a unital completely positive contractive map $\varphi: C(G) \to pA_p$, such that:

1. $\varphi$ is an $(F, S, \varepsilon)$-approximately equivariant central multiplicative map.
2. $1 - p \preceq_A x$.
3. $s^* s = 1 - p$ and $ss^* \preceq p$.
4. $\|sa - as\| < \varepsilon$ for all $a \in F$.
5. $\|pxp\| > 1 - \varepsilon$.

The proof is essentially the same as that of Lemma 6.3. The analog of Lemma 6.3. is as follows.

Lemma 6.3. Let $G$ be a second countable compact group, let $A$ be a simple separable infinite dimensional unital C*-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$. Then $\alpha$ has the modified tracial Rokhlin property if and only if for every $x \in A_+$ with $\|x\| = 1$ there exist a projection $p \in (A_{\infty, \alpha} \cap A')^{\alpha\infty}$, a partial isometry $s \in (A_{\infty, \alpha} \cap (A^\alpha)'^{\alpha\infty})$, and a unital equivariant homomorphism $\psi: C(G) \to p(A_{\infty, \alpha} \cap A')p$ such that the following hold:

1. $1 - p$ is $\alpha$-small in $A_{\infty, \alpha}$.
2. $s^* s = 1 - p$ and $ss^* \preceq p$.
3. $\|pxp\| = 1$.

Proof. The proof is essentially the same as that of Lemma 2.15. The changes are similar to what is described in Remark 3.6, but one must also add arguments to deal with $s$. These are similar to arguments already in the proof of Lemma 2.15. $\Box$

The analog of Theorem 2.16 is as follows.

Proposition 6.4. Let $A$ be an infinite dimensional simple separable unital C*-algebra, let $G$ be a second countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action which has the modified tracial Rokhlin property with comparison. For every $\varepsilon > 0$, every $n \in \mathbb{N}$, every compact subset $F_1 \subseteq A$, every compact subset $F_2 \subseteq A^\alpha$, and every $x \in A_+$ with $\|x\| = 1$, there exist a projection $p \in A^\alpha$, a partial isometry $s \in A^\alpha$, and a unital completely positive contractive map $\psi: A \to pA^n p$ such that:

1. $\psi$ is an $(n, F_1 \cup F_2, \varepsilon)$-approximately multiplicative map (Definition 1.3).
2. $\|pa - ap\| < \varepsilon$ for all $a \in F_1 \cup F_2$.
3. $\|\psi(a) - p\psi(a)p\| < \varepsilon$ for all $a \in F_2$.
4. $1 - p \preceq_A x$. 
\( s^*s = 1 - p \) and \( ss^* \leq p \).

(6) \( \|sa - as\| < \varepsilon \) for all \( a \in F_1 \cup F_2 \).

(7) \( \|pxp\| > 1 - \varepsilon \).

**Proof.** Given Lemma 2.15, the proof is essentially the same as that of Lemma 2.15, in the same way that the proof of Lemma 2.15 is essentially the same as that of Lemma 2.15.

We can use this result to prove simplicity of the crossed product.

**Proposition 6.5.** Let \( A \) be an infinite dimensional simple separable unital C*-algebra. Let \( \alpha : G \to \text{Aut}(A) \) be an action of a compact group \( G \) on \( A \) which has the modified tracial Rokhlin property. Then the fixed point algebra \( A^\alpha \) is essentially the same as that of \( A \).

**Proof.** By Remark 3.6, the proof of Proposition 3.5 applies here.

**Theorem 6.6.** Let \( A \) be a simple separable infinite dimensional unital C*-algebra, let \( G \) be a second countable compact group, and let \( \alpha : G \to \text{Aut}(A) \) be an action which has the modified tracial Rokhlin property. Then the fixed point algebra \( A^\alpha \) is simple.

**Proof.** We proceed as in the proof of Theorem 3.1. As there, let \( I \) be a nonzero ideal in \( A^\alpha \). There are \( m \in \mathbb{N}, a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in A \), and \( x_1, x_2, \ldots, x_m \in I \) such that \( \sum_{j=1}^{m} a_j x_j b_j = 1 \). As there, define \( M, \delta > 0, F_1 \subseteq A, \) and \( F_2 \subseteq A^\alpha \) by (3.1), (3.2) and (3.3). Apply Proposition 6.4 with \( \delta \) in place of \( \varepsilon \), with \( n = 3 \), with \( 1 \) in place of \( x \), and with \( F_1 \) and \( F_2 \) as given. We get a projection \( p \) in \( A^\alpha \), a partial isometry \( s \) in \( A^\alpha \), and a unital completely positive map \( \psi : A \to pA^\alpha p \) such that:

1. \( \|\psi(abc) - \psi(a)\psi(b)\psi(c)\| < \delta \) for all \( a, b, c \in F_1 \).
2. \( \|pa - ap\| < \delta \) for all \( a \in F_1 \cup F_2 \).
3. \( \|\psi(a) - pap\| < \delta \) for all \( a \in F_2 \).
4. \( s^*s = 1 - p \) and \( ss^* \leq p \).

Define
\[
d = \sum_{j=1}^{m} \psi(a_j)px_j p\psi(b_j) \in pIp \subseteq I.
\]

The reasoning of the proof of Theorem 3.1 gives \( \|p - d\| < \frac{1}{2} \). Also \( s^*ds \in (1 - p)I(1 - p) \subseteq I \) and
\[
\|1 - p - s^*ds\| = \|s^*(p - d)s\| < \frac{1}{2}.
\]

Since \( p \) and \( 1 - p \) are orthogonal, we get \( \|1 - (d + s^*ds)\| < \frac{1}{2} \). So \( d + s^*ds \) is an invertible element of \( I \).

**Theorem 6.7.** Let \( A \) be an infinite dimensional simple separable unital C*-algebra, and let \( \alpha : G \to \text{Aut}(A) \) be an action of a second countable compact group \( G \) on \( A \) which has the modified tracial Rokhlin property. Then the crossed product \( C^*(G, A, \alpha) \) is simple.

**Proof.** The algebra \( A^\alpha \) is simple by Theorem 6.6 and \( \alpha \) is saturated by Proposition 6.5. So Condition (1) in Theorem 3.4 holds.

Crossed products by actions with the modified tracial Rokhlin property probably also preserve Property (SP) and pure infiniteness. We were not able to prove that
even crossed products by actions with the strong modified tracial Rokhlin property preserve tracial rank zero.

In the rest of this section, we prove that an action of a finite group $G$ on a UHF algebra $A$ which has the tracial Rokhlin property must in fact have the strong modified tracial Rokhlin property. We do not know to what extent this result can be generalized, even if we require only the modified tracial Rokhlin property.

**Notation 6.8.** Let $n \in \mathbb{N}$. Let $S_n$ denote the symmetric group on $n$ letters, and let $(e_{j,k})_{j,k=1,2,\ldots,n}$ be the standard system of matrix units for $M_n$. For $\sigma \in S_n$ and $\varepsilon \in \{-1,1\}^n$, let $v(\sigma, \varepsilon) = \sum_{j=1}^n \varepsilon_j e_{\sigma(j),j}$. We call the matrices $v(\sigma, \varepsilon)$ the signed permutation matrices in $M_n$.

**Lemma 6.9.** Let $A$ be a unital C*-algebra, let $n \in \mathbb{N}$, and let $\varphi: M_n \to A$ be a unital homomorphism. Following Notation 6.8, define a map $E: A \to A$ by

$$E(x) = \frac{1}{2^{\varepsilon n!}} \sum_{\sigma \in S_n} \sum_{\varepsilon \in \{-1,1\}^n} \varphi(v(\sigma, \varepsilon))x\varphi(v(\sigma, \varepsilon))^*.$$

Then $E$ is a conditional expectation from $A$ to the relative commutant $\varphi(M_n)' \cap A$. Moreover, for any $x \in A$, we have

$$\|E(x) - x\| \leq 2 \text{dist}(x, \varphi(M_n)' \cap A).$$

**Proof.** This is well known, and easy. One checks that the signed permutation matrices are a group, and that they span $M_n$. This is easily seen to imply all but the last sentence of the conclusion. For that, let $x \in A$. Set $\rho = \text{dist}(x, \varphi(M_n)' \cap A)$. Let $\varepsilon > 0$. Choose $y \in \varphi(M_n)' \cap A$ such that $\|y - x\| < \rho + \frac{\varepsilon}{2}$. Then $E(y) = y$, so

$$\|E(x) - x\| = \|E(x - y) - (x - y)\| \leq \|E(x - y)\| + \|x - y\| \leq 2\|x - y\| < 2\rho + \varepsilon.$$ 

This completes the proof. \hfill $\square$

By the reasoning in the proof of Lemma 2.5, the following result implies that the actions in the hypotheses have the strong modified tracial Rokhlin property. We emphasize that the action is not required to be a direct limit action.

**Proposition 6.10.** Let $A$ be a UHF algebra, let $G$ be a finite group, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$ which has the tracial Rokhlin property. Then for every finite set $F \subseteq A$, every $\varepsilon > 0$, every $x \in A_+$ with $\|x\| = 1$, there exist a partial isometry $s \in A^\alpha$, a projection $p \in A^\alpha$, and mutually orthogonal projections $p_g \in A$ for $g \in G$ such that:

1. $p = \sum_{g \in G} p_g$.
2. $\|p_g a - a p_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$.
3. $\|\alpha_g(p_h) - p_h\| < \varepsilon$ for all $g, h \in G$.
4. $1 - p \preceq_A x$.
5. $ss^* = 1 - p$ and $ss^* \leq p$.
6. $\|sa - as\| < \varepsilon$ for all $a \in F$.
7. $\|x_p\| > 1 - \varepsilon$.

**Proof.** Write $A = \varinjlim_m A_m$ with unital maps and with integers $d(m) \geq 2$ such that $A_m \cong M_{d(m)}$ for $m \in \mathbb{Z}_{\geq 0}$. Set $n = \text{card}(G)$. Let $\tau$ be the unique tracial state on $A$. Let $F \subseteq A$ be finite, let $\varepsilon > 0$, and let $x \in A_+$ satisfy $\|x\| = 1$. Without loss of generality $\|a\| \leq 1$ for all $a \in F$. Define

$$(6.1) \quad \delta_0 = \min \left(\frac{1}{3}, \frac{\varepsilon}{22 + 2\sqrt{n}}, \frac{\varepsilon}{6n}\right).$$

...
Choose \( \delta_1 > 0 \) so small that the following three conditions are satisfied. First, \( \delta_1 \leq \delta_0 \).

Second, whenever \( B \) is a \( \mathbb{C}^* \)-algebra, \( e \in B \) is a projection, \( y \in B \), \( ye = y \), and \( \| y^*y - e \| < 42\delta_1 \), then, with functional calculus evaluated in \( pBp \), the element \( z = y(y^*y)^{-1/2} \) exists and satisfies \( z^*z = e \) and \( \| z - y \| < \delta_0 \). Finally, whenever \( B \) is a unital \( \mathbb{C}^* \)-algebra, \( e, f \in B \) are projections, and \( \| e - f \| < 2\delta_1 \), then there is a unitary \( w \in B \) such that \( wew^* = f \) and \( \| w - 1 \| < \delta_1 \). Choose \( \delta_2 > 0 \) so small that

\[
\delta_2 \leq \min \left( \frac{1}{5\delta_1}, \frac{\delta_1}{5\delta_1}, \frac{\delta_1^2}{5\delta_1} \right),
\]

and also so small that whenever \( B \) is a \( \mathbb{C}^* \)-algebra, \( b \in B \) is selfadjoint, and \( \| b^2 - b \| < 60n\delta_2 \), then there is a projection \( e \in B \) such that \( \| e - b \| < \delta_1 \). Choose \( \delta > 0 \) so small that

\[
\delta \leq \delta_2,
\]

and also so small that whenever \( B \) is a \( \mathbb{C}^* \)-algebra and \( b_1, b_2, \ldots, b_n \in B \) are selfadjoint and satisfy \( \| b_j^2 - b_j \| < 3\delta \) for \( j = 1, 2, \ldots, n \) and \( \| b_jb_k \| < 3\delta \) for distinct \( j, k \in \{1, 2, \ldots, n\} \), then there are mutually orthogonal projections \( e_j \in B \) for \( j = 1, 2, \ldots, n \) such that \( \| e_j - b_j \| < \delta_2 \) for \( j = 1, 2, \ldots, n \). Set \( F_0 = \bigcup_{g \in G} \alpha_g(F) \). Choose \( m \in \mathbb{Z}_{\geq 0} \) such that for every \( a \in F_0 \) there is \( b \in A_m \) such that \( \| a - b \| < \delta_0 \). Let \( P \) be the image under some isomorphism \( M_{d(m)} \to A_m \) of the set of signed permutation matrices in \( M_{d(m)} \) (as in Notation 6.8). Define continuous functions \( h, h_0 : [0, 1] \to [0, 1] \) by

\[
h(\lambda) = \begin{cases} 0 & 0 \leq \lambda \leq 1 - \delta \\ \delta^{-1}(\lambda - 1 + \delta) & 1 - \delta \leq \lambda \leq 1 \end{cases}
\]

and

\[
h_0(\lambda) = \begin{cases} (1 - \delta)^{-1} \lambda & 0 \leq \lambda \leq 1 - \delta \\ 1 & 1 - \delta \leq \lambda \leq 1. \end{cases}
\]

Then \( h(x) \neq 0 \) since \( \| x \| = 1 \). Since \( A \) is a UHF algebra, there is a nonzero projection \( f \in h(x)A\overline{h(x)} \subseteq xAx \) such that

\[
\tau(f) < \frac{1}{n + 1}.
\]

Apply Definition 2.1 with \( P \) in place of \( F \), with \( \delta \) in place of \( \varepsilon \), and with \( f \) in place of \( x \). Call the resulting projections \( q_g \) for \( g \in G \), and set \( q = \sum_{g \in G} q_g \). Thus:

\[
\begin{align*}
8 & \| q_g v - v q_g \| < \delta \text{ for all } v \in P \text{ and all } g \in G. \\
9 & \| \alpha_g(q_h) - q_{gh} \| < \delta \text{ for all } g, h \in G. \\
10 & 1 - q \not<_{\text{A}} f. \\
11 & \| q f q \| > 1 - \delta.
\end{align*}
\]

It follows from (9) that

\[
\| \alpha_g(q) - q \| < n\delta
\]

for all \( g \in G \).

We claim that

\[
\| q x q \| > 1 - 3\delta.
\]
To prove the claim, first observe that
\[ h_0(x)^{1/2}h(x) = h(x) \quad \text{and} \quad \|h_0(x) - x\| < 2\delta. \]
It follows that \( h_0(x)^{1/2}f = f \). Therefore
\[ qfq = qh_0(x)^{1/2}fh_0(x)^{1/2}q \leq qh_0(x)q, \]
so, using (11) at the last step,
\[ \|q\| \geq \|qh_0(x)q\| - \|h_0(x) - x\| > \|qfq\| - 2\delta > 1 - 3\delta, \]
as claimed.

By Lemma 6.9, the formula
\[ E(x) = \frac{1}{2^{d(m)d(m)!}} \sum_{v \in P} v xv^* \]
defines a conditional expectation \( E : A \to A_m' \cap A \) such that
\[ \|E(x) - x\| \leq 2\text{dist}(x, A_m' \cap A) \]
for all \( x \in A \). Also clearly
\[ \|E(x) - x\| \leq \sup_{v \in P} \|vx - xv\|. \]
Combining (8) and (6.9), we get \( \|E_2(q_g) - q_g\| < \delta \) for all \( g \in G \). Therefore
\[ \|E_2(q_g)^2 - E_2(q_g)\| \]
\[ \leq \|E_2(q_g) - q_g\|\|E_2(q_g)\| + \|q_g\|\|E_2(q_g) - q_g\| + \|E(q_g) - q_g\| < 3\delta \]
and, if \( g \neq h \), then
\[ \|E_2(q_g)E_2(q_h)\| \leq \|E_2(q_g) - q_g\|\|E_2(q_h)\| + \|q_g\|\|E_2(q_h) - q_h\| < 2\delta. \]
By the choice of \( \delta \), there exist mutually orthogonal projections \( r_g \in A \) for \( g \in G \) such that \( \|r_g - E(q_g)\| < \delta_2 \) for \( j = 1, 2, \ldots, n \). Set \( r = \sum_{g \in G} r_g \). Using (6.4) at the last step,
\[ \|r_g - q_g\| \leq \|r_g - E(q_g)\| + \|E(q_g) - q_g\| < \delta_2 + \delta \leq 2\delta_2 \]
for all \( g \in G \), so
\[ \|r - q\| < 2n\delta_2. \]

Using (9) and (6.10) at the second step and (6.4) at the last step, for \( g, h \in G \) we get
\[ \|\alpha_g(r_h) - r_{gh}\| \leq \|\alpha_g(q_h) - q_{gh}\| + \|r_g - q_g\| + \|r_{gh} - q_{gh}\| \]
\[ < 2\delta_2 + 2\delta_2 + \delta \leq 5\delta_2. \]
Also using (6.6) at the second last step, for \( g \in G \),
\[ \|\alpha_g(r) - r\| \leq \|\alpha_g(r) - \alpha_g(q)\| + \|\alpha_g(q) - q\| + \|q - r\| \]
\[ < 2n\delta_2 + n\delta + 2n\delta_2 \leq 5n\delta_2. \]
Since \( 2n\delta_2 \leq 1 \) by (6.3), it follows from (6.11) that the projections \( 1 - r \) and \( 1 - q \) are Murray-von Neumann equivalent. Since \( 1 - q \preceq_A f \) by (10), using (6.5) we get
\[ \tau(1 - r) < \frac{1}{n + 1} \quad \text{and} \quad \tau(r) > \frac{n}{n + 1}. \]
For \( g \in G \), since \( 5\delta_2 \leq 1 \) by (6.3), it follows from (6.12) that \( \alpha_g(r_1) \sim r_g \). Uniqueness of \( \tau \) implies \( \tau \circ \alpha_g = \tau \). So \( \tau(r_g) = \tau(r_1) \). Hence

\[
\tau(r_1) = \frac{\tau(r)}{n} > \frac{1}{n+1} > \tau(1-r).
\]

Since \( A_m'^{'} \cap A \) is a UHF algebra whose unique tracial state is \( \tau|_{A_m'^{'} \cap A} \), there is \( s_0 \in A_m'^{'} \cap A \) such that \( s_0^*s_0 = 1-r \) and \( s_0s_0^* \leq r_1 \). Define

\[
(6.14) \quad c_0 = \frac{1}{\sqrt{n}} \sum_{g \in G} \alpha_g(s_0).
\]

For \( g \in G \), using (6.12) at the last step,

\[
\|r_g \alpha_g(s_0) - \alpha_g(s_0)\| = \|r_g \alpha_g(s_0) - \alpha_g(r_1s_0)\| \\
\leq \|r_g - \alpha_g(r_1)\| \|\alpha_g(s_0)\| < 5\delta_2.
\]

So, for \( h \neq g \), since \( r_g r_h = 0 \),

\[
(6.15) \quad \|r_g \alpha_h(s_0)\| < 5\delta_2
\]

and

\[
\|\alpha_g(s_0)^* \alpha_h(s_0)\| \leq \|\alpha_g(s_0)^* - \alpha_g(s_0)^*r_g\| + \|\alpha_h(s_0) - r_h \alpha_h(s_0)\| \\
< 5\delta_2 + 5\delta_2 = 10\delta_2.
\]

Now, using (6.13) at the last step,

\[
\left\|1 - r - \frac{1}{n} \sum_{g \in G} \alpha_g(s_0^* s_0)\right\| \leq \frac{1}{n} \sum_{g \in G} \|r - \alpha_g(r)\| < 5n\delta_2,
\]

so

\[
(6.16) \quad \left\|1 - r - c_0^* c_0\right\| \leq \left\|1 - r - \frac{1}{n} \sum_{g,h \in G} \alpha_g(s_0^* \alpha_h(s_0)\right\| \\
\leq \left\|1 - r - \frac{1}{n} \sum_{g \in G} \alpha_g(s_0^* s_0)\right\| + \frac{1}{n} \sum_{g \neq h} \|\alpha_g(s_0^* \alpha_h(s_0)\| \\
< 5n\delta_2 + 10n\delta_2 = 15n\delta_2.
\]

Since \( 15n\delta_2 \leq 1 \) by (6.3), we get

\[
(6.17) \quad \|c_0^* c_0\| \leq 1 + \|1 - r\| \leq 2 \quad \text{and} \quad \|c_0\| \leq \sqrt{2} < 2.
\]

It also follows that

\[
\|r c_0^* c_0 r\| = \|r [c_0^* c_0 - (1-r)] r\| \leq \|1 - r - c_0^* c_0\| < 15n\delta_2,
\]

so, using (6.3) at the last step,

\[
(6.18) \quad \|c_0(1-r) - c_0\| = \|c_0 r\| < \sqrt{15n\delta_2} \leq \delta_1.
\]

Also, using (6.14) at the first step (6.15) and (6.16) at the second last step, and

\[
(6.19) \quad \|r c_0 - c_0\| \leq \|r c_0 - c_0(1-r)\| + \|c_0(1-r) - c_0\| \\
< \frac{1}{\sqrt{n}} \sum_{g \in G} \|r \alpha_g(s_0) - \alpha_g(s_0)\| \\
\leq \frac{1}{\sqrt{n}} \sum_{g \in G} \left( \|r_g \alpha_g(s_0) - \alpha_g(s_0)\| + \sum_{h \in G \setminus \{g\}} \|r_h \alpha_g(s_0)\| \right) \\
< \sqrt{n}(5\delta_2 + (n-1) \cdot 5\delta_2) = 5n^{3/2} \delta_2 \leq \delta_1.
\]
Let \( a \in F \). Then for all \( g \in G \) there is \( b \in A_m \) such that \( \|a^{-1}(a) - b\| < \delta \), so \( s_0 \in A_m \cap A \) implies \( \|a^{-1}(a)s_0 - a_0a^{-1}(a)\| < 2\delta \). Thus, by (6.14),

\[
\|aa_0 - c_0a\| \leq \frac{1}{\sqrt{n}} \sum_{g \in G} \|aa_0(s_0) - a_0(s_0)a\|
\]

(6.21)

\[
= \frac{1}{\sqrt{n}} \sum_{g \in G} \|a^{-1}(a)s_0 - a_0\| < 2\sqrt{n}\delta.
\]

Setting \( r_0 = 1 - r \), giving \( r_0^2 = r_0 \), and using (6.17) and (6.18) at the second last step, we get

\[
\|(c_0^*c_0)^2 - c_0^2c_0\| < \|c_0^*c_0 - r_0\| \|c_0^*c_0 + r_0\| + \|c_0^*c_0 - r_0\| < 2(15n\delta_2) + 15n\delta_2 = 60n\delta_2.
\]

By the choice of \( \delta_2 \), there is a projection \( p_0 \in A^\alpha \) such that \( \|p_0 - c_0^*c_0\| < \delta_1 \). Set \( p = 1 - p_0 \). Using (6.17) at the second last step and (6.3) at the last step, we get

\[
\|p - r\| = \|p_0 - r_0\| < \|p_0 - r_0\| + \|c_0^*c_0 - r_0\| < \delta_1 + 15n\delta_2 < 2\delta_1 + \delta_1 = 5\delta_1.
\]

Define \( c = pc_0(1 - p) \). Then

\[
\|c\| \leq 2
\]

by (6.18). We have, using (6.22), (6.18), and (6.20) at the second step,

\[
\|pc_0 - c_0\| < \|pc_0 - c_0\| + \|r_0\| < 2\delta_1 \cdot 2 + \delta_1 = 5\delta_1,
\]

and, using (6.19) in place of (6.20),

\[
\|c_0(1 - p) - c_0\| < \|c_0\| \|p - r\| + \|c_0(1 - r) - c_0\| < 2 \cdot 2\delta_1 + \delta_1 = 5\delta_1.
\]

Therefore

\[
\|c - c_0\| < \|pc_0(1 - p) - c_0\| + \|pc_0 - c_0\| < 10\delta_1.
\]

Using this, (6.22), (6.23), and (6.18) at the second step,

\[
\|1 - p - c^*c\| \leq \|p - r\| + \|c^*c - c - c_0\| + \|c - c_0\| < 2\delta_1 + 2 \cdot 10\delta_1 = 10\delta_1.
\]

By the choice of \( \delta_1 \), it makes sense to evaluate \((c^*c)^{-1/2}\) in \((1 - p)A^\alpha(1 - p)\), and, moreover, if we define \( s = c(c^*c)^{-1/2} \), then

\[
s^*s = 1 - p \quad \text{and} \quad s - c < \delta_0.
\]

It follows from (6.22) and the choice of \( \delta_1 \) that there is a unitary \( u \in A \) such that \( uru^* = p \) and \( \|u - 1\| < \delta_0 \). Define \( p_g = ur_gu^* \) for \( g \in G \). We claim that \( (p_g)_{g \in \mathbb{G}} \) \( p \), and \( s \) satisfy the conditions in the conclusion of the proposition. Condition (1) is immediate. To prove (2) and (3), we start by observing that for \( g \in G \) we have

\[
\|p_g - r_g\| \leq 2\|u - 1\| < 2\delta_0.
\]

Now, for \( g \in G \) and \( a \in F \), choose \( b \in A_m \) such that \( \|b - a\| < \delta_0 \). Since \( r_g \in A_m \cap A \) and \( \|a\| \leq 1 \), and using (6.1) at the last step,

\[
\|p_ga - ap_g\| \leq 2\|p_g - r_g\| + 2\|a - b\| < 4\delta_0 + 2\delta_0 \leq \varepsilon.
\]
This is (2). Also, for \( g, h \in G \), at the second step using (6.26) on the first and last terms, (6.10) on the second and fourth terms, and (9) on the middle term, at the second last step using (6.4), (6.3), and (6.2), and at the last step using (6.1),
\[
\|\alpha_g(p_h) - p_{gh}\| \\
\leq \|p_h - r_h\| + \|r_h - q_h\| + \|\alpha_g(q_h) - q_{gh}\| + \|q_{gh} - r_{gh}\| + \|r_{gh} - p_{gh}\| \\
< 2\delta_0 + 2\delta_2 + \delta + 2\delta_2 + 2\delta_0 \leq 9\delta_0 \leq \varepsilon.
\]
Condition (3) is proved.

For Condition (4), use (6.22) and (6.11) at the third step, (6.3) at the fourth step, and (6.2) and (6.1) at the fifth step, to get
\[
\|(1 - p) - (1 - q)\| = \|p - q\| \leq \|p - r\| + \|r - q\| \\
< 2\delta_1 + 2n\delta_2 \leq 3\delta_1 \leq 1.
\]
Therefore, using (10) and \( f \in xAX \) at the second step, \( 1 - p \sim 1 - q \not\sim_A x \). To prove Condition (7), observe that, using (6.7) at the second step, (6.4) and (6.3) at the third step, and (6.2) and (6.1) at the fourth step, and using \( \|p - q\| < 3\delta_1 \) from the calculation above,
\[
\|pxp\| \geq \|qxq\| - 2\|p - q\| > 1 - 3\delta - 6\delta_1 \geq 1 - 7\delta_1 \geq 1 - \varepsilon.
\]
For Condition (5), we have \( s^*s = 1 - p \) by (6.25), and \( ss^* \leq p \) because \( pc = c \) implies \( ps = s \). Finally, we check Condition (6). For \( a \in F \), we have, using \( \|a\| \leq 1 \) at the first step, using (6.25), (6.24), and (6.21) at the second step, and using (6.4), (6.3), (6.2) and (6.1) at the third step,
\[
\|sa - as\| \leq 2\|s - c\| + 2\|c - c_0\| + \|c_0a - ac_0\| \\
< 2\delta_0 + 20\delta_1 + 2\sqrt{n}\delta \leq \varepsilon.
\]
This completes the proof. \( \square \)

7. AN ACTION OF A TOTALLY DISCONNECTED COMPACT GROUP

In this section we construct an action of a totally disconnected infinite compact group on a UHF algebra which has the tracial Rokhlin property with comparison and the strong modified tracial Rokhlin property, but does not have the Rokhlin property, or even finite Rokhlin dimension. In the next section, we construct an action of \( S^1 \) on a simple AT algebra which has the same properties, except that we only prove the modified tracial Rokhlin property.

We abbreviate \( \mathbb{Z}/n\mathbb{Z} \) to \( \mathbb{Z}_n \); the \( p \)-adic integers will not appear in this paper. The group is \( G = \prod_{n=1}^{\infty} \mathbb{Z}_2 \), and the action is the infinite tensor product of copies of the same action of \( \mathbb{Z}_3 \) on the \( 3^\infty \) UHF algebra. We give the example in Construction 7.1, and prove its properties in several results afterwards.

**Construction 7.1.** We start with a slight reformulation of Example 10.4.8 of [45].

For \( k \in \mathbb{N} \), set \( r(k) = \frac{1}{2}(3^k - 1) \). Define \( w_k \in U(M_{3^k}) \) to be the block unitary
\[
w_k = \left( \begin{array}{ccc} 0 & 1_{M_{r(k)}} & 0 \\
1_{M_{r(k)}} & 0 & 0 \\
0 & 0 & 1_c \end{array} \right) \in M_{3^k}.
\]

Set $B = \bigotimes_{k=1}^{\infty} M_{3^k}$, which is the $3^\infty$ UHF. Define
\[ \nu = \bigotimes_{k=1}^{\infty} \text{Ad}(w_k) \in \text{Aut}(B), \]
which is an automorphism of order 2. Let $\gamma: \mathbb{Z}_2 \to \text{Aut}(B)$ be the product type action generated by $\nu$.

Define $G = \prod_{n=1}^{\infty} \mathbb{Z}_2$ and $A = \bigotimes_{n=1}^{\infty} B$. Let $\alpha: G \to \text{Aut}(A)$ be the infinite tensor product action, determined by
\[ \alpha_{(h_1, h_2, \ldots)}(b_1 \otimes b_2 \cdots) = \gamma h_1(b_1) \otimes \gamma h_2(b_2) \otimes \cdots \]
for $h_1, h_2, \ldots \in \mathbb{Z}_2$ and $b_1, b_2, \ldots \in B$ with $b_n = 1$ for all but finitely many $n \in \mathbb{N}$.

To work effectively with this example, we set up some useful notation.

**Notation 7.2.** Given the notation in Construction 7.1, make the following further definitions. For $n \in \mathbb{N}$ set $B_n = B$, so that $A = \bigotimes_{m=1}^{\infty} B_m$, and set $A_n = \bigotimes_{m=1}^{n} B_m$, so that $A = \lim_{n \to \infty} A_n$. For $n, k \in \mathbb{N}$ set $C_{n,k} = M_{3^k}$, so that $B_n = \bigotimes_{k=1}^{n} C_{n,k}$, and set $B_{n,l} = \bigotimes_{k=1}^{l} C_{n,k}$, so that $B_n = \lim_{l \to \infty} B_{n,l}$. Further set $A_{n,l} = \bigotimes_{k=1}^{l} B_{n,k}$. We identify $A_n$ and $A_{n,l}$ with their images in $A$ and $B_{n,k}$ with its image in $B_n$.

Treat $G$ similarly: for $n \in \mathbb{N}$ set $H_n = \mathbb{Z}_2$, so that $G = \prod_{m=1}^{\infty} H_m$, and set $G_n = \prod_{m=1}^{n} H_m$, so that $G = \lim_{n \to \infty} G_n$. This gives
\[ C(G_n) = \bigotimes_{m=1}^{n} C(H_m), \quad \text{and} \quad C(G) = \lim_{n \to \infty} C(G_n) = \bigotimes_{m=1}^{\infty} C(H_m). \]

We identify $C(G_n)$ with its image in $C(G)$.

As an informal overview, write
\[ A = \bigotimes_{m=1}^{\infty} \left( \bigotimes_{k=1}^{\infty} M_{3^k} \right) = \bigotimes_{m=1}^{\infty} \left( \bigotimes_{k=1}^{\infty} C_{m,k} \right). \]

Then:

- $C_{n,l}$ uses the $(n, l)$ tensor factor.
- $B_n$ uses the $(n, k)$ tensor factors for $k \in \mathbb{N}$.
- $B_{n,l}$ uses the $(n, k)$ tensor factors for $k = 1, 2, \ldots, l$.
- $A_n$ uses the $(m, k)$ tensor factors for $m = 1, 2, \ldots, n$ and $k \in \mathbb{N}$.
- $A_{n,l}$ uses the $(m, k)$ tensor factors for $m = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, l$.

**Lemma 7.3.** Let $n \in \mathbb{N}$, let $A_1, A_2, \ldots, A_n$ be unital C*-algebras, and for $m = 1, 2, \ldots, n$ let $e_m \in A_m$ be a projection and let $\tau_m$ be a tracial state on $A_m$. Let $A = A_1 \otimes A_2 \otimes \cdots \otimes A_n$ (minimal tensor product), and set
\[ e = e_1 \otimes e_2 \otimes \cdots \otimes e_n \in A \quad \text{and} \quad \tau = \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n \in \text{T}(A). \]

Then
\[ \tau(1 - e) \leq \sum_{m=1}^{n} \tau_m(1 - e_m). \]
Proof. For $m = 1, 2, \ldots, n$ set $\lambda_m = \tau_m(e_m) \in [0, 1]$. We need to show that

\begin{equation}
1 - \prod_{m=1}^{n} \lambda_m \leq \sum_{m=1}^{n} (1 - \lambda_m).
\end{equation}

We do this by induction on $n$. The case $n = 1$ is immediate. For $n = 2$, the relation (7.2) becomes

\[1 - \lambda_1 \lambda_2 \leq 2 - \lambda_1 - \lambda_2.\]

This is equivalent to $(1 - \lambda_1)(1 - \lambda_2) \geq 0$, so the case $n = 2$ holds.

Assume now (7.2) holds for some $n \geq 2$, and $\lambda_1, \lambda_2, \ldots, \lambda_{n+1} \in [0, 1]$. Set $\mu = \prod_{m=1}^{n} \lambda_m$. Then $\mu \in [0, 1]$. Using the case $n = 2$ at the second step and the induction hypothesis at the third step, we get

\[1 - \prod_{m=1}^{n+1} \lambda_m = 1 - \mu \lambda_{n+1} \leq (1 - \lambda_{n+1}) + (1 - \mu) \leq \sum_{m=1}^{n+1} (1 - \lambda_m).
\]

This completes the proof. \qed

Lemma 7.4. Let the notation be as in Construction 7.1. Let $k \in \mathbb{N}$. Then there are isomorphisms

\[(M_{3^k})^{\text{Ad}(w_k)} \cong M_{r(k)} \oplus M_{r(k)+1}\]

and

\[(M_{3^k} \otimes M_{3^k+1})^{\text{Ad}(w_k) \otimes \text{Ad}(w_{k+1})} \cong M_{r(k^2+k)} \oplus M_{r(k^2+k)+1}.\]

The first isomorphism sends the projections

\[e_0 = \begin{pmatrix} 1_{M_{r(k)}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_{M_{r(k)}} & 0 \\ 0 & 0 & 0 \end{pmatrix},\]

(using the same block matrix decomposition as in (7.1)) to a projection of rank $r(k)$ in $M_{r(k)+1}$ and to the identity of $M_{r(k)}$ respectively. The map

\[\rho: (M_{3^k})^{\text{Ad}(w_k)} \to (M_{3^k} \otimes M_{3^k+1})^{\text{Ad}(w_k) \otimes \text{Ad}(w_{k+1})}\]

induced by $a \mapsto a \otimes 1$ induces maps $\rho_{i,j}: M_{r(k)+i} \to M_{r(k^2+k)+j}$ for $i, j \in \{0, 1\}$, and the corresponding partial embedding multiplicities $m_k(i,j)$ are given by

\[m_k(0,0) = m_k(1,1) = r(k+1) + 1 \quad \text{and} \quad m_k(0,1) = m_k(1,0) = r(k+1).\]

Proof. For any $k \in \mathbb{N}$, it is easy to check that $w_k$ is unitarily equivalent to

\[v_k = \text{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1) \in M_{3^k},\]

in which the diagonal entry 1 occurs $r(k)+1$ times and the diagonal entry $-1$ occurs $r(k)$ times. Therefore we can prove the lemma with $v_k$ and $v_{k+1}$ in place of $w_k$ and $w_{k+1}$. With this change, for example, the map $\nu: M_{r(k)} \oplus M_{r(k)+1} \to M_{3^k}$, given by $\nu(a_0, a_1) = \text{diag}(a_1, a_0)$, is easily seen to be an isomorphism from $M_{r(k)} \oplus M_{r(k)+1}$ to $(M_{3^k})^{\text{Ad}(w_k)}$. The rest of the proof is a computation with diagonal matrices and the dimensions of their eigenspaces, and is omitted. \qed

Theorem 7.5. The action $\alpha: G \to \text{Aut}(A)$ of Construction 7.1 has the tracial Rokhlin property with comparison and the strong modified tracial Rokhlin property, using the same choices of $p \in A$ and $\varphi: G \to pAp$. 

Proof. Let \( F \subseteq A \) and \( S \subseteq C(G) \) be finite sets, let \( \varepsilon > 0 \), let \( x \in A_+ \setminus \{0\} \) with \( \|x\| = 1 \), and let \( y \in A_+^\alpha \setminus \{0\} \). Without loss of generality we can assume \( \|\alpha\| \leq 1 \) for all \( \alpha \in F \), \( \|f\| \leq 1 \) for all \( f \in S \), and \( \varepsilon < 1 \). According to Definition 2.4, we need to find a projection \( p \in A^\alpha \) and a unital completely positive \( \varphi : C(G) \to pA_p \) such that:

1. \( \varphi \) is an \((F,S,\varepsilon)\)-approximately equivariant central multiplicative map.
2. \( 1 - p \precsim_A x \).
3. \( 1 - p \precsim_{A^\alpha} y \).
4. \( 1 - p \precsim_{A^\alpha} p \).
5. \( \|p_{xy}\| > 1 - \varepsilon \).

According to Definition 6.1, we also need to find a partial isometry \( s \in A^\alpha \) such that:

1. \( s^*s = 1 - p \) and \( ss^* \leq p \).
2. \( \|sa - as\| < \varepsilon \) for all \( a \in F \).

For the same reason as in the proof of Lemma 2.8 (and by Lemma 2.8 if we are only proving that \( \alpha \) has the tracial Rokhlin property with comparison), we can ignore (5). We can also ignore (4), since it follows from (6).

Since \( A \) is a UHF algebra, there is a nonzero projection \( q_1 \in \overline{xAx} \) and the unique tracial state \( \tau \) on \( A \) satisfies \( \tau(q_1) > 0 \). Set \( \delta_1 = \tau(q_1) \).

The action \( \gamma \) has the tracial Rokhlin property by Example 10.4.8 of [45]. Therefore \( B^\gamma \) is simple by Corollary 1.6 of [44]. Since \( \gamma \) is a direct limit action, \( B^\gamma \) is an AF algebra. It is easy to check that \( A^\alpha \) can be identified with \( \bigotimes_{m=1}^\infty B_m^{\varepsilon} \). It follows that \( A^\alpha \) is an AF algebra, which is simple because it is an infinite tensor product of simple C*-algebras. Therefore there is a nonzero projection \( q_2 \in A^\alpha \), and the number \( \delta_2 = \inf_{\tau \in T(A^\alpha)} \tau(q_2) \) satisfies \( \delta_2 > 0 \).

Following Notation 7.2, \( \bigcup_{n=1}^\infty A_n \) is dense in \( A \) and, for every \( n \in \mathbb{N} \), \( \bigcup_{l=1}^\infty B_{n,l} \) is dense in \( B_n \). Therefore there are \( N_1, L_0 \in \mathbb{N} \) and a finite subset \( F_0 \subseteq A_{N_1,L_0} \subseteq A \) such that for every \( a \in F \) there is \( b \in F_0 \) with \( \|a - b\| < \frac{1}{N_1} \), and also \( \|b\| \leq 1 \) for all \( b \in F_0 \). Similarly, there are \( N_2 \in \mathbb{N} \) and a finite subset \( S_0 \subseteq C(G_{N_2}) \subseteq C(G) \) such that for every \( f \in F \) there is \( c \in F_0 \) with \( \|f - c\| < \frac{1}{N_1} \) and also \( \|c\| \leq 1 \) for all \( c \in S_0 \). Set \( N = \max(N_1, N_2) \), and choose \( L \in \mathbb{N} \) so large that

\[
L \geq L_0 \quad \text{and} \quad \frac{2N}{3L} < \min \left( \delta_1, \delta_2, \frac{1}{2} \right).
\]

Let \( e_0, e_1 \in M_{3L+1} \) be as in Lemma 7.4, with \( k = L + 1 \). For \( m = 1, 2, \ldots, N \) define the projections

\[
e_0^{(m)} = 1_{B_{m,L}} \otimes e_0, \quad e_1^{(m)} = 1_{B_{m,L}} \otimes e_1 \in B_{m,L} \otimes M_{3L+1} = M_{3L+1} \subseteq B_m.
\]

Identify \( H_m = \mathbb{Z}_2 = \{0, 1\} \) with addition modulo 2, and for \( h = (h_1, h_2, \ldots, h_N) \in G_N = \prod_{m=1}^N H_m \), set

\[
e_h = e_h^{(1)} \otimes e_h^{(2)} \otimes \cdots \otimes e_h^{(N)}.
\]

These are mutually orthogonal projections. Define

\[
p = \sum_{h \in G_N} e_h \in A_{N,L+1} \subseteq A.
\]

There is a unital homomorphism \( \varphi_0 : C(G_N) \to pA_{n,L+1} \subseteq pA \) given by

\[
\varphi_0(f) = \sum_{h \in G_N} f(h) e_h \quad \text{for} \quad f \in C(G_N).
\]

Set \( K = \prod_{n=N+1}^\infty H_m \), so that \( G = G_N \times K \). Let \( \mu \) be normalized Haar measure on \( K \). Then there is a conditional expectation \( P : C(G) \to C(G_N) \), given by

\[
P(f)(h) = \int_K f(h,g) d\mu(g) \quad \text{for} \quad f \in C(G) \quad \text{and} \quad h \in G_N.
\]

The identification of
$C(G_N)$ as a subalgebra of $C(G)$ makes it invariant under the action $g \mapsto \mathbf{L} t_g$ of $G$ on $C(G)$, and, with this identification, $P$ is equivariant. Also, it is easily checked that $p \in A^ρ$ and that $ϕ_0$ is $G$-equivariant. Therefore $ϕ = ϕ_0 \circ P : C(G) \to pAp$ is an equivariant unital completely positive map.

Since $F_0 \subseteq A_{N,L}$, it follows that $ϕ(f)$ exactly commutes with all elements of $F_0$. Also, $ϕ(c_1c_2) = ϕ(c_1)ϕ(c_2)$ for all $c_1, c_2 \in S_0$, since $S_0 \subseteq C(G_N) \subseteq C(G)$. Let $a \in F$ and let $f_1, f_2 \in S$. Choose $b \in F_0$ and $c_1, c_2 \in S_0$ such that
\[
\|b - a\| < \frac{ε}{4}, \quad \|c_1 - f_1\| < \frac{ε}{4}, \quad \text{and} \quad \|c_2 - f_2\| < \frac{ε}{4}.
\]
Since all elements of $F, F_0, S$, and $S_0$ have norm at most 1, using $ϕ(c_1)b = bϕ(c_1)$ we get
\[
\|ϕ(f_1)α - αϕ(f_1)\| \leq 2\|b - a\| + 2\|f_1 - c_1\| < ε.
\]
Similarly,
\[
\|c_1c_2 - f_1f_2\| < \frac{ε}{2}
\]
so $\|ϕ(f_1f_2) - ϕ(f_1)ϕ(f_2)\| < ε$. We have proved (1).

For $m = 1, 2, \ldots, n$, set $p_m = e^{(m)}_0 + e^{(m)}_1 \in A_m$. Then $p_m$ is the image in $A_m$ of a projection $z_m \in C_{m,L+1} = M_{3L+1}$ such that $1 - z_m$ has rank 1. Therefore the unique tracial state $τ_m$ on $A_m$ satisfies $τ(1 - p_m) = 3^{-(L+1)}$. Since $p = p_1 \otimes p_2 \otimes \cdots \otimes p_N$, by Lemma 7.3 we have
\[
τ(1 - p) \leq \frac{N}{3L+1} < δ_1 = τ(q_1).
\]
Since UHF algebras have strict comparison, we get $1 - p \not\precsim q_1 \not\preceq x$, which is (2).

For the remaining conditions, for convenience set $T = (L + 1)(L + 2)$. Let $e_0, e_1$, and $ρ$ be as in Lemma 7.4, with $k = L + 1$, and let the components of $ρ$ in its codomain be
\[
ρ_j : M_{r(L+1)} \oplus M_{r(L+1)+1} \to M_{r(T)+j}
\]
for $j = 0, 1$. Using the ranks and partial embedding multiplicities given in Lemma 7.4, we see that $ρ_0(1 - e_0 - e_1) \in M_{r(T)}$ has rank $r(L + 2)$ and $ρ_1(1 - e_0 - e_1) \in M_{r(T)+1}$ has rank $r(L + 2) + 1$. The normalized traces of these are
\[
\frac{r(L + 2)}{r(T)} < \frac{2}{3L+1} \quad \text{and} \quad \frac{r(L + 2) + 1}{r(T) + 1} < \frac{2}{3L+1}.
\]
Set
\[
D = (M_{r(L+1)} \oplus M_{r(L+1)+1})^\otimes N \quad \text{and} \quad E = (M_{r(T)} \oplus M_{r(T)+1})^\otimes N,
\]
and consider $ρ^\otimes N : D \to E$. We can write
\[
E = \bigoplus_{j \in \{0, 1\}^N} M_{r(T)+j_1} \otimes M_{r(T)+j_2} \otimes \cdots \otimes M_{r(T)+j_N}.
\]
Call the $j$ tensor factor $E_j$. For $j = (j_1, j_2, \ldots, j_N) \in \{0, 1\}^N$, let $d_j$ be the image in $E_j$ of the corresponding summand of $ρ^\otimes N((e_0 + e_1)^\otimes N)$. Thus
\[
d_j = ρ_{j_1}(e_0 + e_1) \otimes ρ_{j_2}(e_0 + e_1) \otimes \cdots \otimes ρ_{j_N}(e_0 + e_1).
\]
By Lemma 7.3 and (7.3), $1 - d_j$ has normalized trace less than $2N \cdot 3^{-L-1}$. 

Use Lemma 7.4 to identify $D$ and $E$ with the subalgebras
\[
\left( \bigotimes_{m=1}^{N} C_{m, L+1} \right)^{\alpha} \quad \text{and} \quad \left( \bigotimes_{m=1}^{N} C_{m, L+1} \otimes C_{m, L+2} \right)^{\alpha},
\]
in such a way that $(e_0 + e_1)^{\otimes N}$ is identified with $p$. Under this identification, $E$ commutes exactly with all elements of $A_{N, L}$, hence with all elements of $F_0$.

Since $2N \cdot 3^{-L-1} < \frac{\theta}{2}$, we conclude that $1 - d_j \preceq_{E} d_j$. Therefore $1 - p \preceq_{E} p$, that is, there is $s \in E$ such that $s^* s = 1 - p$ and $ss^* \leq p$. We have $s \in A^o$ since $E \subseteq A^o$. Also, $s$ commutes exactly with all elements of $F_0$, so $\|as - sa\| < \frac{\theta}{2}$ for all $a \in F$. We have verified Conditions (6) and (7).

Since $2N \cdot 3^{-L-1} < \delta_2$, for every $\sigma \in T(E)$ we have $\sigma(1 - p) < \delta_2$. Every tracial state $\tau \in T(A^o)$ restricts to a tracial state on $E$, so $\tau(1 - p) < \delta_2$ for all $\tau \in T(A^o)$. Since simple AF algebras have strict comparison, we get $1 - p \preceq_{A^o} q_2$. Condition (3) follows.

**Proposition 7.6.** The action $\alpha : G \to \text{Aut}(A)$ of Construction 7.1 does not have finite Rokhlin dimension with commuting towers.

**Proof.** Suppose $\alpha$ has finite Rokhlin dimension with commuting towers. Then Proposition 3.10 of [14] implies that the action on $A$ of the first factor of $G$, called $H_1$ in Notation 7.2, also has finite Rokhlin dimension with commuting towers. However, $H_1 \cong \mathbb{Z}_2$, $A$ is the $3^\infty$ UHF algebra and, by Corollary 4.8(2) of [25], there is no action of $\mathbb{Z}_2$ on the $3^\infty$ UHF algebra which has finite Rokhlin dimension with commuting towers. \qed

### 8. An action of $S^1$

The purpose of this section is to construct a direct limit action of the group $S^1$ on a simple unital AT algebra which has the tracial Rokhlin property with comparison but does not have finite Rokhlin dimension with commuting towers.

The general construction, with unspecified partial embedding multiplicities (which, for properties we want, need to be chosen appropriately), is presented in Construction 8.3. For the purpose of readability, the properties asserted there, as well as others needed later, are proved in a series of lemmas.

The algebra $A$ in our construction will be a direct limit of algebras $A_n$ isomorphic to $C(S^1, M_{Nr_1(n)}) \oplus C(S^1, M_{r_1(n)})$. Up to equivariant isomorphism and exterior equivalence, the action of $\zeta \in S^1$ on $C(S^1, M_{Nr_1(n)})$ is rotation by $\zeta^N$ and its action on $C(S^1, M_{r_1(n)})$ is rotation by $\zeta$. It is technically convenient to present the first summand in a different way; the description above is explicit in Lemma 8.7. The action has the tracial Rokhlin property with comparison provided the image of the summand $C(S^1, M_{Nr_1(n)}) \subseteq A_n$ in $A$ can be made “arbitrarily small in trace” by choosing $n$ large enough.

The algebras $B_n$ and $B$ in parts (11), (12), (13), and (14) of Construction 8.3 are a convenient description of the fixed point algebras of $A_n$ and $A$; see Lemma 8.11.

We say here a little more about the motivation for the construction and possible extensions. If $G$ is finite, one can construct a direct limit action of $G$ on an AF algebra $\lim_n A_n$ by taking $A_n = M_{r_0(n)} \oplus C(G, M_{r_1(n)})$. The action on $C(G, M_{r_1(n)})$ is essentially translation by group elements. The partial map from $C(G, M_{r_1(n)})$ to $M_{r_0(n+1)}$ is the direct sum of the evaluations at the points of $G$. The action on $M_{r_0(n+1)}$ is inner, and must permute the images of the maps from $C(G, M_{r_1(n)})$.
appropriately; this leads to a slightly messy inductive construction of inner actions of \( G \) on the algebras \( M_{r_0(n)} \) and inner perturbations of the translation actions on \( C(G, M_{r_1(n)}) \).

When \( G \) is not finite, point evaluations can no longer be used, since equivariance forces one to use all of them or none of them. The algebra \( C(S^1, M_{N_{r_0(n)}}) \) with the action of rotation by \( \zeta^N \) is the codomain for a usable substitute for point evaluations. Something similar to the inductive construction of perturbations from above is needed, but the messiness can be mostly hidden by instead using the algebra \( R \) as in Construction 8.3.

Construction 8.3 can be generalized in several ways. One can replace \( C(S^1, M_{r_1(n)}) \) with \( C(X, M_{r_1(n)}) \) for a compact space \( X \) with a free action of \( S^1 \). To ensure simplicity, one will need to incorporate additional partial maps in the direct system, which can be roughly described as point evaluations at points of \( X/S^1 \). One can increase the complexity of the \( K \)-theory and the departure from the Rokhlin property by taking

\[
A_n = C(S^1, M_{N_2N_1r_0(n)}) \oplus C(S^1, M_{N_1r_1(n)}) \oplus C(S^1, M_{r_2(n)})
\]

with actions exterior equivalent to rotations by \( \zeta^{N_1}, \zeta^{N_2}, \zeta^N \), and \( \zeta \). One can use more summands, even letting the number of them approach infinity as \( n \to \infty \). One can also replace \( S^1 \) with \( (S^1)^m \). However, it is not clear how to construct an analogous action with \( S^1 \) replaced by a nonabelian connected compact Lie group.

We introduce some notation specifically for this section.

**Definition 8.1.** Let \( G \) be a group, let \( A \) and \( B \) be \( C^* \)-algebras, with \( B \) unital, let \( \alpha: G \to \text{Aut}(A) \) and \( \beta: G \to \text{Aut}(B) \) be actions of \( G \) on \( A \) and \( B \), and let \( \varphi, \psi: A \to B \) be equivariant homomorphisms. We say that \( \varphi \) and \( \psi \) are *equivariantly unitarily equivalent*, written \( \varphi \sim \psi \), if there is a \( \beta \)-invariant unitary \( u \in B \) such that \( u\varphi(a)u^* = \psi(a) \) for all \( a \in A \).

**Notation 8.2.** Let \( A \) and \( B \) be \( C^* \)-algebras, and let \( \psi: A \to B \) be a homomorphism. We let \( \psi^{(k)}: A \to M_k \otimes B \) be the map \( a \mapsto 1_{M_k} \otimes \psi(a) \), and we define

\[
\psi_n = \text{id}_{M_n} \otimes \psi: M_n \otimes A \to M_n \otimes B
\]

and

\[
\psi_n^{(k)} = \text{id}_{M_n} \otimes \psi^{(k)}: M_n \otimes A \to M_{kn} \otimes B.
\]

In particular, the “amplification map” from \( M_n(A) \) to \( M_{kn}(A) \), given by \( a \mapsto 1_{M_k} \otimes a \), is denoted by \( (\text{id}_A)^{(k)} \).

**Construction 8.3.** We choose and fix \( N \in \mathbb{N} \) with \( N \geq 2 \), \( \theta \in \mathbb{R} \setminus \mathbb{Q} \), \( r(0) = (r_0(0), r_1(0)) \in (\mathbb{N})^2 \), and, for \( n \in \mathbb{Z}_{\geq 0} \) and \( j, k \in \{0, 1\} \), numbers \( l_{j,k}(n) \in \mathbb{N} \). We suppress them in the notation for the objects we construct, and, in later results, we will impose additional restrictions on them.

We then define the following \( C^* \)-algebras, maps, and actions of \( S^1 \).

1. Define \( \beta: S^1 \to \text{Aut}(C(S^1)) \) by \( \beta(z)(f)(z) = f(\zeta z) \) for \( \zeta, z \in S^1 \). Further, for \( n \in \mathbb{N} \), identify \( C(S^1, M_n) \) and \( M_n(C(S^1)) \) with \( M_n \otimes C(S^1) \) in the obvious way, and let \( \beta_n: S^1 \to \text{Aut}(C(S^1), M_n) \) be given by \( \beta_n = \text{id}_{M_n} \otimes \beta \) for \( \zeta \in S^1 \). (The order of subscripts in \( \beta_{\lambda,n} \) is chosen to be consistent with Notation 8.2). To simplify notation, for \( \lambda \in \mathbb{R} \) we define

\[
\tilde{\beta}_\lambda = \beta_{\exp(2\pi i \lambda)} \quad \text{and} \quad \tilde{\beta}_{\lambda,n} = \beta_{\exp(2\pi i \lambda), n}.
\]
(2) Define
\[ \omega = \exp(2\pi i/N) \quad \text{and} \quad s = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \in M_N. \]

(3) Define
\[ R = \{ f \in C(S^1, M_N) : f(\omega z) = sf(z)s^* \text{ for all } z \in S^1 \}. \]

Then $R$ is invariant under the action $\beta_N$ of $S^1$ on $C(S^1, M_N)$ above. (See Lemma 8.4 below.) We define $\gamma : S^1 \to \text{Aut}(R)$ to be the restriction of this action. Further, for $n \in \mathbb{N}$, let $\gamma_n : S^1 \to \text{Aut}(M_n \otimes R)$ be the action $\gamma_{\zeta,n} = \text{id}_{M_n} \otimes \gamma_{\zeta}$ for $\zeta \in S^1$. Finally, for $\lambda \in \mathbb{R}$ we define $\gamma_{\lambda,n} = \gamma_{\exp(2\pi i\lambda), n}$.

(4) Let $\iota : R \to C(S^1, M_N)$ be the inclusion. Define $\xi : C(S^1) \to R$ by
\[ \xi(f)(z) = \text{diag}(f(z), f(\omega z), \ldots, f(\omega^{N-1}z)) \]
for $f \in C(S^1)$ and $z \in S^1$. For $n \in \mathbb{N}$ write
\[ l(n) = \begin{pmatrix} l_{0,0}(n) & l_{0,1}(n) \\ N l_{1,0}(n) & 2 N l_{1,1}(n) \end{pmatrix}. \]

For $n \in \mathbb{N}$ inductively define, starting with $r(0) = (r_0(0), r_1(0)) \in (\mathbb{N})^2$ as at the beginning of the construction,
\[ r(n+1) = l(n) r(n) \]
(usual matrix multiplication).

(6) For $n \in \mathbb{Z}_{\geq 0}$ set
\[ A_{n,0} = M_n^I (R), \quad A_{n,1} = M_{N_1}(C(S^1)), \quad \text{and} \quad A_n = A_{n,0} \oplus A_{n,1}. \]

Define an action $\alpha^{(n)} : S^1 \to \text{Aut}(A)$ (notation not in line with Notation 8.2) by, now following Notation 8.2, $\alpha^{(n)}_{\zeta} = \gamma_{\zeta, r_0(n)} \pm \gamma_{\zeta, r_1(n)}$.

(7) For $n \in \mathbb{Z}_{\geq 0}$ and $j, k \in \{0, 1\}$, define maps
\[ \nu_{n,0,0} : A_{n,0} \to M_{l_0,0}(r_0(n)) (R), \quad \nu_{n,0,1} : A_{n,1} \to M_{l_0,1}(r_1(n)) (R), \quad \nu_{n,1,0} : A_{n,0} \to M_{N l_1,0}(r_0(n)) (C(S^1)), \]
and
\[ \nu_{n,1,1} : A_{n,1} \to M_{N l_1,1}(r_1(n)) (C(S^1)) \]
as follows. Recalling Notation 8.2, set
\[ \nu_{n,0,0} = (\text{id}_R)^{l_0,0}_C, \quad \nu_{n,0,1} = c^{l_0,1}_C, \quad \nu_{n,1,0} = l_0^C, \]
and
\[ \nu_{n,1,1} = \text{diag} \left( \tilde{\beta}^{l_1,1}_{0,r_1(n)}, \tilde{\beta}^{l_1,1}_{1,r_1(n), r_1(n)}, \tilde{\beta}^{l_1,1}_{1,2/r_1(n), r_1(n)}, \ldots, \tilde{\beta}^{l_1,1}_{1,(N-1)/r_1(n), r_1(n)}, \right. \]
\[ \left. \tilde{\beta}^{l_1,1}_{1,r_1(n), r_1(n)}, \tilde{\beta}^{l_1,1}_{1,2/r_1(n), r_1(n)}, \tilde{\beta}^{l_1,1}_{1,3/r_1(n), r_1(n)}, \ldots, \tilde{\beta}^{l_1,1}_{1,(N-1)/r_1(n), r_1(n)} \right). \]
Lemma 8.5. The map \( \xi: C(S^1) \to R \) of Construction 8.3(4) is well defined and equivariant.

Proof. The first part is easy to check just by the definition of \( \xi \). For equivariance, it is enough and immediate to check on the usual generator of \( C(S^1) \). \qed
Lemma 8.6. The maps $\nu_{n,j,k} : A_{n,k} \to A_{n,j}$ of Construction 8.3(7) and $\nu_n : A_n \to A_{n+1}$ of Construction 8.3(8) are equivariant.

Proof. This is immediate from Lemma 8.4, Lemma 8.5, and the fact that $\beta_{z_1}$ commutes with $\beta_{z_2}$ for $z_1, z_2 \in S^1$. □

Lemma 8.7. Let $R \subseteq C(S^1, M_N)$ be as in Construction 8.3(3). Let $\tau : S^1 \to \text{Aut}(C(S^1))$ be the action $\tau(z)(f)(z) = f(\tau^{-1}z)$ for $z \in S^1$. Then there is an isomorphism $\psi : R \to C(S^1, M_N)$ satisfying the following conditions.

1. For every rank one projection $e \in R \subseteq C(S^1, M_N)$, the projection $\psi(e) \in C(S^1, M_N)$ has rank one.
2. The action $\zeta \mapsto \psi \circ \zeta \circ \psi^{-1}$ is exterior equivalent to the action $\zeta \mapsto \tau_{\zeta,N}$.
3. With $L_\phi$ as defined in 1.3 of [9], for $z \in S^1$ we have
   
   $L_{\psi \circ \phi}(z) = \{ y \in S^1 : y^N = z \}$ and $L_{\psi \circ \phi^{-1}}(z) = \{ z^N \}$.

The isomorphism in this lemma is not equivariant when $C(S^1, M_N)$ is equipped with the action $\zeta \mapsto \beta_{\zeta,N}$, or any action exterior equivalent to it.

Proof of Lemma 8.7. Choose a continuous unitary path $\lambda \mapsto s_\lambda$ in $M_N$, defined for $\lambda \in [0, 1]$, such that $s_0 = 1$ and $s_1 = s$. For any $\lambda \in \mathbb{R}$, choose $\lambda_0 \in [0, 1)$ such that $\lambda - \lambda_0 \in \mathbb{Z}$. Taking $n = \lambda - \lambda_0$, we then define $s_\lambda = s^n s_{\lambda_0}$. This function is still continuous. Moreover, for any $m \in \mathbb{Z}$,

\begin{equation}
   s_{\lambda + m} = s^{m+n}s_{\lambda_0} = s^m s^n s_{\lambda_0} = s^m s_\lambda.
\end{equation}

We claim that there is a well defined homomorphism $\psi : R \to C(S^1, M_N)$ such that, whenever $f \in R$ and $\lambda \in \mathbb{R}$, we have

$\psi(f)(e^{2\pi i \lambda}) = s_\lambda^* f(e^{2\pi i \lambda/N}) s_\lambda.$

The only issue is whether $\psi(f)(e^{2\pi i \lambda})$ is well defined. It is sufficient to prove that if $\lambda_0 \in [0, 1)$ and $n = \lambda - \lambda_0 \in \mathbb{Z}$, then the formulas for $\psi(f)(e^{2\pi i \lambda})$ and $\psi(f)(e^{2\pi i \lambda_0})$ agree. To see this, use the definition of $R$ at the second step to get

\begin{align*}
   s_\lambda^* f(e^{2\pi i \lambda/N}) s_\lambda &= s_0^* s_\lambda s^n f(e^{2\pi i \lambda_0/N}) s^n s_\lambda = s_0^* s_\lambda f(e^{2\pi i \lambda_0/N}) s_\lambda.
\end{align*}

as desired.

The construction of $\psi$ makes Part (1) obvious. Bijectivity is easy just by checking the definition. We now prove (2). For $\zeta \in S^1$, choose $\tau \in \mathbb{R}$ such that $e^{2\pi i \tau} = \zeta$, and define a function $\nu_\zeta \in C(S^1, M_N)$ by $\nu_\zeta(e^{2\pi i \lambda}) = s_\lambda^* s_{\lambda-N\tau}$ for $\lambda \in \mathbb{R}$. We claim that $\nu_\zeta$ is well defined. First, we must show that if $m \in \mathbb{Z}$ then

\begin{align*}
   s_m^* s_\lambda s_{\lambda+m-N\tau} = s_m^* s_{\lambda-N\tau}.
\end{align*}

This follows directly from (8.2). Second, we must show that if $e^{2\pi i \tau_1} = e^{2\pi i \tau_2}$, then

\begin{align*}
   s_{\lambda-N\tau_1} = s_{\lambda-N\tau_2}.
\end{align*}

For this, set $m = \tau_2 - \tau_1 \in \mathbb{Z}$, and use (8.2) and $s^N = 1$ to see that

\begin{align*}
   s_{\lambda-N\tau_2} = s_{\lambda-N\tau_1+N\tau_1} = s_{N \tau_1} s_{\lambda-N\tau_1} = s_{\lambda-N\tau_1}.
\end{align*}

The claim is proved.

It is now easy to check that $(\zeta, \lambda) \mapsto \nu_\zeta(e^{2\pi i \lambda})$ is continuous, so that $\zeta \mapsto \nu_\zeta$ is a continuous function from $S^1$ to the unitary group of $C(S^1, M_N)$.
We next claim that \( v_{\zeta_1, \zeta_2} = v_{\zeta_1} \beta_{\zeta_1 N, N} \psi(v_{\zeta_2}) \) for \( \zeta_1, \zeta_2 \in S^1 \). To do this, choose \( \tau_1, \tau_2 \in \mathbb{R} \) such that \( \zeta_1 = e^{2\pi i \tau_1} \) and \( \zeta_2 = e^{2\pi i \tau_2} \). Then \( \zeta_1 \zeta_2 = e^{2\pi i (\tau_1 + \tau_2)} \). So for \( \lambda \in \mathbb{R} \),
\[
 v_{\zeta_1} (e^{2\pi i \lambda}) \beta_{\zeta_1 N, N} (v_{\zeta_2}) (e^{2\pi i \lambda}) = v_{\zeta_1} (e^{2\pi i \lambda}) v_{\zeta_2} (e^{2\pi i (\lambda - N \tau_1)}) = s_{\lambda}^* \cdot s_{\lambda - N \tau_1} \cdot s_{\lambda - N \tau_2} \cdot s_{\lambda - N \tau_1 - N \tau_2} = v_{\zeta_1 \zeta_2} (e^{2\pi i \lambda}),
\]
proving the claim.

We have shown that \( \zeta \mapsto v_{\zeta} \) is a cocycle for the action \( \zeta \mapsto \beta_{\zeta N, N} \) of \( S^1 \) on \( C(S^1, M_N) \). Therefore the formula
\[
 \rho_\zeta(g) = v_{\zeta} \beta_{\zeta N, N}(g) v_{\zeta}^*
\]
defines an action of \( S^1 \) on \( C(S^1, M_N) \) which is exterior equivalent to \( \zeta \mapsto \beta_{\zeta N, N} = \tau_{\zeta, N} \).

To finish the proof of (2), we show that \( \psi \) is equivariant for the action \( \rho \) on \( C(S^1, M_N) \). Let \( f \in R \), let \( \lambda \in \mathbb{R} \), let \( \zeta \in S^1 \), and choose \( \tau \in \mathbb{R} \) such that \( \zeta = e^{2\pi i \tau} \). Then
\[
(\rho_\zeta \circ \psi)(f)(e^{2\pi i \lambda}) = s_{\lambda}^* s_{\lambda - N \tau} \psi(f)(\zeta^{-N} e^{2\pi i \lambda} \psi_{s_{\lambda - N \tau}} \psi_{s_{\lambda - N \tau}} s_{\lambda}) = s_{\lambda}^* \psi_{s_{\lambda - N \tau}} f(e^{2\pi i (\lambda - N \tau)/N}) s_{\lambda - N \tau} s_{\lambda} = s_{\lambda}^* f(\zeta^{-N} e^{2\pi i \lambda/N}) s_{\lambda} = (\psi \circ \gamma_\zeta)(f)(e^{2\pi i \lambda}).
\]
This completes the proof of (2).

For (3), we first observe that if \( f \in C(S^1) \) and \( \lambda \in \mathbb{R} \) then
\[
(\psi \circ \xi)(f)(e^{2\pi i \lambda}) = s_{\lambda}^* \text{diag}(f(e^{2\pi i \lambda/N}), f(\omega e^{2\pi i \lambda/N}), \ldots, f(\omega^{N-1} e^{2\pi i \lambda/N})) s_{\lambda}.
\]
Therefore
\[
 L_{\psi \circ \xi}(e^{2\pi i \lambda}) = \{e^{2\pi i \lambda/N}, \omega e^{2\pi i \lambda/N}, \ldots, \omega^{N-1} e^{2\pi i \lambda/N}\}.
\]
This is the same as the description in the statement.

For the second formula, one checks that for \( g \in C(S^1, M_N) \) and \( \lambda \in \mathbb{R} \), we have
\[
\psi^{-1}(g)(e^{2\pi i \lambda}) = s_{N \lambda} g(e^{2\pi i \lambda/N}) s_{N \lambda}^* = s_{N \lambda} g(e^{2\pi i \lambda/N}) s_{N \lambda}^*.
\]
Since \( \iota : R \to C(S^1, M_N) \) is just the inclusion, this gives \( L_{\psi \circ \iota}^{-1}(z) = \{z^N\} \) for \( z \in S^1 \), as desired.

**Lemma 8.8.** The algebra \( A \) of Construction 8.3(9) is a simple AT algebra.

**Proof.** Using Lemma 8.7, we can rewrite the direct system in Construction 8.3(8) as
\[
 A = \lim_{\rightarrow} \left[ C(S^1, M_{N \alpha(n)}) \oplus C(S^1, M_{r_1(n)}) \right],
\]
with maps \( \tilde{\nu}_n \), for \( n \in \mathbb{Z}_{\geq 0} \), obtained analogously to Construction 8.3(8) from
\[
\tilde{\nu}_{n,0,0} = (\text{id} C(S^1)_{N \alpha(n)}) \tilde{\nu}_{n,0,0} = \psi_{r_1(n)} \sigma_n(n) \sigma_n(n) \circ \nu_{n,0,1},
\]
\[
\nu_{n,1,0} = \nu_{n,1,0} \circ (\psi_{r_1(n)} \sigma_n(n))^{-1}, \quad \text{and} \quad \nu_{n,1,1} = \nu_{n,1,1},
\]
and with \( \tilde{\nu}_{n,m} = \nu_{n-1} \circ \tilde{\nu}_{n-2} \circ \cdots \circ \tilde{\nu}_m \):
\[
 C(S^1, M_{N \alpha(n)}) \oplus C(S^1, M_{r_1(n)}) \to C(S^1, M_{N \alpha(n)}) \oplus C(S^1, M_{r_1(n)}).
\]
It is now obvious that $A$ is an AT algebra. For simplicity, we use Proposition 2.1 of [9], with $L_k$ as defined in 1.3 of [9]. To make the notation easier, we take $X_n = S^1 \times \{0, 1\}$ (rather than $S^1 \mathbb{II} S^1$ as in [9]), and for $j \in \{0, 1\}$ identify $S^1 \times \{j\}$ with the primitive ideal space of $C(S^1, M_{r_j}(n))$.

For $z \in S^1$ it is immediate that

$$L_{\varphi_{n, 0, 0}}(z) = \{z\}$$

and

$$L_{\varphi_{n, 1, 1}}(z) = L_{\varphi_{n, 1, 1}}(z) = \{z, \omega z, \ldots, \omega^{N-1} z, e^{2\pi i \theta} z, e^{2\pi i \theta} \omega z, \ldots, e^{2\pi i \theta} \omega^{N-1} z\}.$$  

Also, using Lemma 8.7(3),

$$L_{\varphi_{n, 0, 1}}(z) = L_{\psi \circ \xi}(z) = \{y \in S^1 : y^N = z\}$$

and

$$L_{\varphi_{n, 1, 0}}(z) = L_{\tilde{\psi} \circ \tilde{\xi}}(z) = \{z^N\}.$$  

Putting these together, we get

$$L_{\varphi}(z, 0) = \{(z, 0)\} \cup \{(y, 1) : y \in S^1 \text{ and } y^N = z\}$$

and

$$L_{\varphi}(z, 1) = \{(z^N, 0)\} \cup \{(z, 1), (\omega z, 1), \ldots, (\omega^{N-1} z, 1), (e^{2\pi i \theta} z, 1), (e^{2\pi i \theta} \omega z, 1), \ldots, (e^{2\pi i \theta} \omega^{N-1} z, 1)\}.$$  

One checks that if $C$, $D$, and $E$ are finite direct sums of homogeneous unital C*-algebras, and $\Phi : C \to D$ and $\Psi : D \to E$ are unital homomorphisms, with primitive ideal spaces $X$, $Y$, and $Z$, then for $z \in Z$ we have

$$L_{\Psi \circ \Phi}(z) = \bigcup_{y \in L_{\Phi}(z)} L_{\Phi}(y).$$

The equations (8.3) and (8.4) show that $x \in L_{\tilde{\varphi}}(x)$ for any $x \in X_n$. Therefore, for any $m, n_1, n_2 \in \mathbb{Z}_{\geq 0}$ with $n_2 > n_1 > m$, and any $x \in X_m$,

$$L_{\tilde{\varphi}_{n_1, m}}(x) \subseteq L_{\tilde{\varphi}_{n_2, m}}(x).$$

It now suffices to prove that for every $m \in \mathbb{Z}_{\geq 0}$ and every $\varepsilon > 0$, there is $n > m$ such that for every $z \in S^1$ and $j \in \{0, 1\}$, the set $L_{\tilde{\varphi}_{n, m}}(z, j)$ is $\varepsilon$-dense in $S^1 \times \{0, 1\}$.

Choose $n > m + 2$ such that

$$\{e^{2\pi ik\theta} : k = 0, 1, \ldots, n - m - 2\}$$

and

$$\{e^{2\pi ikN\theta} : k = 0, 1, \ldots, n - m - 2\}$$

are both $\varepsilon$-dense in $S^1$.

We claim that if $z \in S^1$ is arbitrary, then $L_{\tilde{\varphi}_{n-1, m}}(z, 1)$ is $\varepsilon$-dense in $S^1 \times \{0, 1\}$. To prove the claim, first use (8.5) repeatedly and (8.4) to observe that for $z \in S^1$,

$$\{e^{2\pi ik\theta} z, 1) : k = 0, 1, \ldots, n - m - 2\} \subseteq L_{\tilde{\varphi}_{n-1, m+1}}(z, 1).$$

By (8.4) and (8.6), this set is contained in $L_{\tilde{\varphi}_{n-1, m}}(z, 1)$, and, since since multiplication by $z$ is isometric, it is $\varepsilon$-dense in $S^1 \times \{1\}$. Also, by (8.7), (8.5), and (8.4),

$$\{e^{2\pi ikN\theta} z^N, 0) : k = 0, 1, \ldots, n - m - 2\} \subseteq L_{\tilde{\varphi}_{n-1, m}}(z, 1),$$

and this set is $\varepsilon$-dense in $S^1 \times \{0\}$. The claim follows.

Now, for any $x \in S^1 \times \{0, 1\}$, the set $L_{\tilde{\varphi}_{n, m+1}}(x)$ contains at least one point in $S^1 \times \{1\}$ by (8.3) and (8.4). Using (8.5), it follows that $L_{\tilde{\varphi}_{n, m}}(x)$ is $\varepsilon$-dense in $S^1 \times \{0, 1\}$. Now simplicity of $A$ can be deduced from Proposition 2.1 of [9]. \qed
Lemma 8.9. The algebra $B$ of Construction 8.3(13) is a simple unital AF algebra.

Proof. That $B$ is a unital AF algebra is immediate from its definition. Simplicity follows from the corollary on page 212 of [5]. □

Lemma 8.10. Define

$$c = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)^2} \end{pmatrix}.$$ 

Then the formula

$$\varepsilon_0(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) = c^* \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) c$$

for $\lambda_0, \lambda_1, \ldots, \lambda_{N-1} \in \mathbb{C}$, defines an isomorphism from $\mathbb{C}^N$ to $R^\gamma$.

Proof. One checks that $c$ is unitary and

\begin{equation}
(8.8) \quad \text{csc}^* = \text{diag}(1, \omega^{-1}, \omega^{-2}, \ldots, \omega^{-(N-1)}).
\end{equation}

It is immediate that $R^\gamma$ is the set of constant functions in $C(S^1, M_N)$ whose constant value commutes with $s$. Let $D$ be the set of constant functions in $C(S^1, M_N)$ whose constant value commutes with $\text{csc}^*$. Then $a \mapsto c^* ac$ is an isomorphism from $D$ to $R^\gamma$. Also, by (8.8),

$$(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) \mapsto \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{N-1})$$

is an isomorphism from $\mathbb{C}^N$ to $D$. □

Lemma 8.11. There is a family $(\eta_n)_{n \in \mathbb{Z}_{\geq 0}}$ of isomorphisms $\eta_n: B_n \to (A_n)^{\alpha(n)}$ such that:

1. $\eta_n \circ \chi_{n,m} = \nu_{n,m} \circ \eta_m$ whenever $m, n, \in \mathbb{Z}_{\geq 0}$ satisfy $m \leq n$.
2. With $p_n$ as in Construction 8.3(10) and $q_n$ as in Construction 8.3(14), we have $\eta_n(q_n) = p_n$.
3. For all $n \in \mathbb{Z}_{\geq 0}$ and $j \in \{0, 1\}$, we have $\eta_n(B_{n,j}) = (A_{n,j})^{\alpha(n)}$.
4. The family $(\eta_n)_{n \in \mathbb{Z}_{\geq 0}}$ induces an isomorphism $\eta_\infty: B \to A^\alpha$.

We warn that the subscript in $\eta_n$ does not have the meaning taken from Notation 8.2.

Proof of Lemma 8.11. Since $\alpha$ is a direct limit action, the inclusions $(A_n)^{\alpha(n)} \to A_n$ induce an isomorphism $A^\alpha \to \varprojlim_n (A_n)^{\alpha(n)}$. Therefore (4) follows from the rest of the statement of the lemma.

Lemma 8.10 implies that $(\varepsilon_0)_{r_0(n)}: M_{r_0(n)}(\mathbb{C}^N) \to (A_{n,0})^{\alpha(n)}$ is an isomorphism. It is immediate that the embedding $\varepsilon_1: \mathbb{C} \to C(S^1)^{\beta}$ as constant functions is an isomorphism. Therefore

$$\eta_n^{(0)} = (\varepsilon_0)_{r_0(n)} \oplus (\varepsilon_1)_{r_1(n)}: B_{n,0} \oplus B_{n,1} \to (A_n)^{\alpha(n)}$$

is an isomorphism. Clearly (2) and (3) hold with $\eta_n^{(0)}$ in place of $\eta_n$.

Let $\iota^{S^1}$ and $\xi^{S^1}$ be the restriction and corestriction of $\iota$ and $\xi$ to the corresponding fixed point algebras, and similarly define $\nu_{n}^{S^1}$, $\nu_{n,j,k}^{S^1}$, etc. Then the inverses of the
maps $\eta_n^{(0)}$ implement an isomorphism from the direct system $((A_n)^{\alpha(n)})_{n \in \mathbb{Z}_{\geq 0}}$ to the direct system $(B_n)_{n \in \mathbb{Z}_{\geq 0}}$, with the maps $\sigma_n: B_n \to B_{n+1}$ taken to be

$$\sigma_n(a_0, a_1) = \left( \text{diag} \left( (\text{id}_{B_n})^{l_{0,0}(n)}(a_0),\right),
\left( \left( (\varepsilon_0)_{l_{0,1}(n)}(r_{1}(n)) \right)^{-1} \circ \left( \varepsilon_1^{10}(n) \right)^{l_{1,0}(n)} \circ (\varepsilon_1)_{r_{1}(n)} \right)(a_1)\right),
\text{diag} \left( \left( (\varepsilon_1)_{l_{1,0}(n)}(r_{1}(n)) \right)^{-1} \circ \left( \varepsilon_0^{11}(n) \right)^{l_{1,0}(n)} \circ (\varepsilon_0)_{r_{1}(n)} \right)(a_0),
\left( \text{id}_{B_n} \right)^{2Nl_{1,1}(n)}(a_1) \right).$$

The map $(\varepsilon_0)^{-1} \circ \varepsilon_1: \mathbb{C} \to \mathbb{C}^N$ is a unital homomorphism. There is only one such unital homomorphism, so this map is equal to the map $\mu$ in Construction 8.3(12). The map $(\varepsilon_1)^{-1} \circ \varepsilon_0: \mathbb{C}^N \to M_N$ is an injective unital homomorphism. Therefore it must be unitarily equivalent to the map $\delta$ in Construction 8.3(12). It now follows from the definitions (see Construction 8.3(13)) that for $n \in \mathbb{Z}_{\geq 0}$ there is a unitary $v_n \in B_{n+1}$ such that $\chi_n(b) = v_n \sigma_n(b) v_n^*$ for all $b \in B_n$. Inductively define unitaries $w_n \in (A_n)^{\alpha(n)}$ by $w_0 = 1$ and, given $w_n$, setting $w_{n+1} = \nu_n^{l_{1,1}(n)}(w_n)\eta_{n+1}^{l_{0,0}(n)}(v_n)^*$. Then define $\eta_n(b) = w_n \eta_n^{l_{1,1}(n)}(v_n)^* w_n^*$ for $b \in B_n$.

The conditions (2) and (3) hold as stated because they hold for the maps $\eta_n^{(0)}$. For $n \in \mathbb{Z}_{\geq 0}$, using $\nu_n^{l_{1,1}(n)} \circ \eta_n^{(0)} = \eta_{n+1}^{l_{1,1}(n)} \circ \sigma_n$, one gets $\nu_n^{l_{1,1}(n)} \circ \eta_n = \eta_{n+1} \circ \chi_n$. This implies (1).  \( \square \)

**Lemma 8.12.** In Construction 8.3, assume that $r_0(0) \leq r_1(0)$ and that for all $n \in \mathbb{Z}_{\geq 0}$ we have $l_{1,0}(n) \geq l_{0,0}(n)$ and $l_{1,1}(n) \geq l_{0,1}(n)$. Further assume that

$$\lim_{n \to \infty} \frac{l_{0,1}(n)}{l_{0,0}(n)} = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{l_{1,1}(n)}{l_{1,0}(n)} = \infty.$$  

Then the action $\alpha$ of Construction 8.3(8) has the tracial Rokhlin property with comparison.

The hypotheses are overkill. They are chosen to make the proof easy.

**Proof of Lemma 8.12.** Since $A$ is stably finite, by Lemma 1.15 in [44], we may disregard condition (5) in Definition 2.4.

We first claim that $0 < r_0(n) \leq r_1(n)$ for all $n \in \mathbb{N}$. This is true for $n = 0$ by hypothesis. For any other value of $n$, using $r_0(n-1) > 0$ and $r_1(n-1) > 0$, we have

$$r_0(n) = l_{0,0}(n-1) + l_{0,1}(n-1)r_1(n-1) \leq l_{0,0}(n-1) + l_{1,1}(n-1)r_1(n-1) = r_1(n).$$

Also, since $l_{0,0}(n-1) > 0$, the first step of this calculation implies that $r_0(n) > 0$.

Next, we claim that for every $n \in \mathbb{Z}_{\geq 0}$ and every tracial state $\tau$ on the algebra $B_{n+1}$ of Construction 8.3(11), with $\chi_n$ as in Construction 8.3(13) and $\eta_n$ as in Construction 8.3(14), we have

$$\tau(1 - \chi_n(q_n)) \leq \max \left( \frac{l_{0,0}(n)}{l_{0,1}(n)}, \frac{l_{1,0}(n)}{l_{1,1}(n)} \right).$$

To see this, we first look at the partial embedding multiplicities in Construction 8.3(12) to see that the rank of $1 - \chi_n(q_n)$ in each of the first $N$ summands (all
equal to $M_{r_0(n+1)}$ is $l_{0,0}(n)r_0(n)$, and the rank of $1 - \chi_n(q_n)$ in the last summand (equal to $M_{r_1(n+1)}$) is $l_{1,0}(n)r_0(n)$. Now

\[(8.10) \frac{l_{0,0}(n)r_0(n)}{r_0(n+1)} = \frac{l_{0,0}(n)r_0(n)}{l_{0,0}(n)r_0(n) + l_{0,1}(n)r_1(n)} \leq \frac{l_{0,0}(n)r_0(n)}{l_{0,1}(n)r_1(n)} \leq \frac{l_{0,0}(n)}{l_{0,1}(n)}\]

and

\[(8.11) \frac{l_{1,0}(n)r_0(n)}{r_1(n+1)} = \frac{l_{1,0}(n)r_0(n)}{Nl_{1,0}(n)r_0(n) + 2Nl_{1,1}(n)r_1(n)} \leq \frac{l_{1,0}(n)r_0(n)}{l_{1,1}(n)r_1(n)} \leq \frac{l_{1,0}(n)}{l_{1,1}(n)}\]

The number $\tau(1 - \chi_n(q_n))$ is a convex combination of the numbers in (8.10) and (8.11). The claim follows.

Now let $F \subseteq A$ and $S \subseteq C(S^1)$ be finite sets, let $\varepsilon > 0$, let $x \in A_+ \setminus \{0\}$, and let $y \in A^*_n \setminus \{0\}$. We may assume that $\|x\| \leq 1$ and $\|y\| \leq 1$, and that $\|f\| \leq 1$ for all $f \in S$. Set

\[(8.12) \quad \varepsilon_0 = \frac{1}{2} \min \left( \inf_{\tau \in F(A)} \tau(x), \inf_{\tau \in F(A^\tau)} \tau(y), \frac{1}{2} \right)\]

Choose $n \in \mathbb{Z}_{\geq 0}$ so large that there is a finite subset $F_0 \subseteq A_n$ with $\text{dist}(a, \nu_{\infty,n}(F_0)) < \frac{\varepsilon_0}{3}$ for all $a \in F$, and also so large that

\[(8.13) \quad \min \left( \frac{l_{0,1}(n)}{l_{0,0}(n)}, \frac{l_{1,1}(n)}{l_{1,0}(n)} \right) > \frac{1}{\varepsilon_0} \]

Let $p \in A$ be $p = \nu_{\infty,n}(p_n)$. Define

\[\varphi_0 : C(S^1) \to p_nA_np_n = M_{r_1(n)} \otimes C(S^1)\]

by $\varphi_0(g) = (0, 1 \otimes g)$ for $g \in C(S^1)$. Define $\varphi = \nu_{\infty,n} \circ \varphi_0 : C(S^1) \to pA_p$. Then $\varphi$ is an equivariant unital homomorphism. In particular, $\varphi$ is exactly multiplicative on $S$. Further, let $a \in F$ and $f \in S$. Choose $b \in F_0$ such that $\|a - \nu_{\infty,n}(b)\| < \frac{\varepsilon_0}{3}$. Then, using $\|f\| \leq 1$ and the fact that $\varphi_0(f)$ commutes with all elements of $A_n$, we have

\[\|\varphi(f)a - a\varphi(f)\| \leq 2\|a - \nu_{\infty,n}(b)\| < \varepsilon.\]

Part (1) of the definition is verified.

For the remaining three conditions, let $\tau$ be any tracial state on either $A$ or $A^\tau$. Let $(\eta_n)_{n \in \mathbb{Z}_{\geq 0}}$ be as in Lemma 8.11. Then $\tau \circ \nu_{\infty,n+1} \circ \eta_{n+1}$ is a tracial state on $B_{n+1}$. Combining this with (8.9), (8.13), (8.12), and $(\nu_{\infty,n+1} \circ \eta_{n+1} \circ \chi_n)(q_n) = p$, we get $\tau(1 - p) \leq \tau(x) \leq d_\tau(x)$ for all $\tau \in T(A)$ and $\tau(1 - p) \leq \tau(y) \leq d_\tau(y)$ for all $\tau \in T(A^\tau)$. Since simple unital AF algebras and simple unital AT algebras have strict comparison, it follows that $1 - p \precsim_A x$ and $1 - p \precsim_{A^\tau} y$. Since $\varepsilon_0 < \frac{\varepsilon}{2}$, similar reasoning gives $1 - p \precsim_{A^\tau} p$. \qed

We use equivariant K-theory to show that $\alpha$ does not have finite Rokhlin dimension with commuting towers. Recall equivariant K-theory from Definition 2.8.1 of [42]. For a unital C*-algebra $A$ with an action $\alpha : G \to \text{Aut}(A)$ of a compact group $G$, it is the Grothendieck group of the equivariant isomorphism classes of equivariant finitely generated projective right modules $E$ over $A$, with “equivariant” meaning that the module is equipped with an action of $G$ such that $g \cdot (\xi a) = (g \cdot \xi)\alpha_g(a)$ for $g \in G$, $\xi \in E$, and $a \in A$. Further recall the representation ring $R(G)$ of a compact group from page 113 of [51] and Definition 2.1.3 of [42] (it is $K_0^G(C)$, or the Grothendieck group of the isomorphism classes of finite dimensional representations of $G$), its augmentation ideal $I(G)$ from page 125 of [51]
Lemma 8.13. We have $K^{S^1, \beta}_1(C(S^1)) = 0$ and (with $R \subseteq C(S^1, M_N)$ as in Construction 8.3(3)) $K^{S^1}_*(R) = 0$.

Proof. By Theorem 2.8.3(7) of [42], we have

$$K^{S^1, \beta}_1(C(S^1)) \cong K_1(C^*(S^1, C(S^1), \beta)).$$

Since

$$C^*(S^1, C(S^1), \beta) \cong K(L^2(S^1),$$

we conclude $K^{S^1, \beta}_1(C(S^1)) = 0$.

For $K^{S^1}_*(R)$, since exterior equivalent actions of a compact group $G$ give isomorphic $R(G)$-modules $K^G(A)$ (Theorem 2.8.3(5) of [42]), by Lemma 8.7(2) it is sufficient to prove this for the action $\zeta \mapsto \gamma_\zeta$. By stability of equivariant K-theory (Theorem 2.8.3(4) of [42]), it suffices to prove this for action $\gamma$ of $S^1$ on $C(S^1)$. By [32] (or Theorem 2.8.3(7) of [42]), we have $K^{S^1}_1(C(S^1)) = K_1(C^*(S^1, C(S^1), \gamma))$. Corollary 2.10 of [21], with $G = S^1$ and $H = \mathbb{Z}_N$, tells us that

$$C^*(S^1, C(S^1), \gamma) \cong K(L^2(S^1)) \otimes C^*(\mathbb{Z}_N),$$

which has trivial $K_1$-group.

Lemma 8.14. There is an $R(S^1)$-module isomorphism $K^{S^1, \gamma}_0(C(S^1)) \cong \mathbb{Z}$, with the $R(S^1)$-module structure coming from the isomorphism $\mathbb{Z} \cong R(S^1)/I(S^1)$, and such that the class in $K^{S^1}_0(C(S^1))$ of the rank one free module is sent to 1 in $\mathbb{Z}$.

Proof. In Proposition 2.9.4 of [42], take $A = \mathbb{C}$, $G = S^1$, and $H = \{1\}$, and refer to the description of the map in the proof of that proposition.

Lemma 8.15. There is an $R(S^1)$-module isomorphism $K^{S^1, \gamma}_0(C(S^1)) \cong R(\mathbb{Z}_N)$, with the $R(S^1)$-module structure coming from the surjective restriction map $R(S^1) \rightarrow R(\mathbb{Z}_N)$. Moreover, the classes in $K^{S^1, \gamma}_0(C(S^1))$ of the equivariant finitely generated projective right $C(S^1)$-modules with underlying nonequivariant module $C(S^1)$ correspond exactly to the elements of $(\mathbb{Z}_N) \wedge \subseteq R(\mathbb{Z}_N)$.

Proof. In Proposition 2.9.4 of [42], take

$$A = \mathbb{C}, \quad G = S^1, \quad \text{and} \quad H = \{1, \omega, \ldots, \omega^{N-1}\} \cong \mathbb{Z}_N.$$
With these choices, $C(G \times_H \mathbb{C})$ is the set of $\omega$-periodic functions on $S^1$, with the action of $\zeta \in S^1$ being rotation by $\zeta$. With $\varphi: S^1 \to \text{Aut}(C(S^1))$ as in Lemma 8.7, this algebra is equivariantly isomorphic to $(C(S^1), \varphi)$ in an obvious way. From Proposition 2.9.4 of [42], we get $K_0^{S^1}(C(S^1)) \cong R(\mathbb{Z}_N)$. Using the description of the map in the proof of the proposition, the map sends the class of a $\varphi$-equivariant finitely generated projective right $C(S^1)$-module $E$ to the class, as a representation space of $H$, of its pushforward under the evaluation map $f \mapsto f(1)$. If $E$ is nonequivariantly isomorphic to $C(S^1)$, this pushforward is nonequivariantly isomorphic to $\mathbb{C}$. The only classes in $R(\mathbb{Z}_N)$ with underlying vector space $\mathbb{C}$ are those in $(\mathbb{Z}_N)^\wedge$. To check that an element $\tau \in (\mathbb{Z}_N)^\wedge$ actually arises this way, choose $l \in \mathbb{Z}$ such that $\tau(\omega) = \omega^l$. Then use the action of $S^1$ on $C(S^1)$ given by $(\zeta \cdot f)(z) = \zeta^l f(\zeta^{-l} z)$ for $\zeta, z \in S^1$ and $f \in C(S^1)$. One readily checks that this makes $C(S^1)$ a $\varphi$-equivariant right $C(S^1)$-module whose restriction to $\{1\}$ is $\mathbb{C}$ with the representation $\varphi$.

Lemma 8.16. There is an $R(S^1)$-module isomorphism $K_0^{S^1}(R) \cong R(\mathbb{Z}_N)$, with the $R(S^1)$-module structure coming from the surjective restriction map $R(S^1) \twoheadrightarrow R(\mathbb{Z}_N)$.

Proof. Using Lemma 8.7(1), it suffices to prove this for $C(S^1, M_N)$ and the action $\zeta \mapsto \rho_\zeta = \psi \circ \varphi_\zeta \circ \psi^{-1}$ in Lemma 8.7(2) in place of $R$ and $\gamma$.

So fix a rank one projection $e \in C(S^1, M_N)$ which is invariant under $\zeta \mapsto \psi \circ \varphi_\zeta \circ \psi^{-1}$. If the group action is ignored, $e$ is Murray-von Neumann equivalent a constant projection, so $E = eC(S^1, M_N)$ is nonequivariantly isomorphic to $C(S^1, \mathbb{C}^N)$ as a right $C(S^1, M_N)$-module. Let $\varphi: S^1 \to \text{Aut}(C(S^1))$ be as in Lemma 8.7, and write $\varphi_N$ for the action $\zeta \mapsto \varphi_{\zeta, N}$ on $(C(S^1, M_N)$. By Lemma 8.7(2), $\rho$ is exterior equivalent to $\varphi_N$. By Proposition 2.7.4 of [42], and the formula in the proof for the isomorphism, there is a (natural) isomorphism from $K_0^{S^1}(C(S^1))$ to $K_0^{S^1}(C(S^1, M_N))$ which sends the class of $E$ to the class of the same module with a different action of $G$. By stability in equivariant K-theory (Theorem 2.8.3(4) of [42]), there is an isomorphism

$$K_0^{S^1}(C(S^1, M_N)) \to K_0^{S^1}(C(S^1))$$

which maps the class of $E$ to the class of some equivariant module whose underlying nonequivariant module is $C(S^1)$. Combining this with Lemma 8.15, we have an isomorphism from $K_0^{S^1}(R)$ to $R(\mathbb{Z}_N)$ which sends $[e]$ to some element $\tau \in (\mathbb{Z}_N)^\wedge$. Multiplying by $\tau^{-1}$ gives an isomorphism from $K_0^{S^1}(R)$ to $R(\mathbb{Z}_N)$ which sends $[e]$ to $1 \in (\mathbb{Z}_N)^\wedge$.

Lemma 8.17. Identify $R(S^1) = \mathbb{Z}[\sigma, \sigma^{-1}]$ as before Lemma 8.13. Let $\iota$ and $\xi$ be as in Construction 8.3(4). There are isomorphisms of $R(S^1)$-modules $K_0^{S^1,\beta}(C(S^1)) \cong \mathbb{Z}$, via the surjective ring homomorphism which sends $\sigma \in R(S^1)$ to $1 \in \mathbb{Z}$, and $K_0^{S^1}(R) \cong \mathbb{Z}[\sigma]/(\sigma^N - 1)$, via the surjective ring homomorphism which sends $\sigma \in R(S^1)$ to $\sigma \in \mathbb{Z}$, with respect to which $\iota_*: K_0^{S^1}(C(S^1)) \to K_0^{S^1}(R)$ is determined by $\iota_*(1) = \iota_*(\sigma) = 1$ and $\xi_*: K_0^{S^1}(R) \to K_0^{S^1,\beta}(C(S^1))$ is determined by $\xi_*(1) = 1 + \sigma + \cdots + \sigma^{N-1}$.

Proof. Recall that $\sigma \in \widetilde{S^1}$ is the identity map $S^1 \to S^1$. The map $R(S^1) \to R(\mathbb{Z}_N)$ is well known to be surjective, and the image $\sigma$ of $\sigma$ in $R(\mathbb{Z}_N)$ satisfies $\sigma^N = 1$.
but no lower degree polynomial relations, so \( R(\mathbb{Z}_N) \cong \mathbb{Z}[\sigma]/(\sigma^N - 1) \). Now the isomorphism \( K_0^{S^1,\beta}(C(S^1)) \cong \mathbb{Z} \) is Lemma 8.14 and the isomorphism
\[
K_0^{S^1}(R) \cong \mathbb{Z}[\sigma]/(\sigma^N - 1)
\]
is Lemma 8.16. Fix a rank one invariant projection \( e \in R \subseteq C(S^1, M_N) \), gotten from Lemma 8.10.

By Lemma 8.16, the isomorphism (8.14) can be chosen to send the class \([eR]\) of the right module \( eR \) to 1. We have \( \iota_\ast([eR]) = [eC(S^1, M_N)] \), the class of some rank one free module, but, by Lemma 8.14, only one element of \( K_0^{S^1,\beta}(C(S^1)) \), namely \( 1 \in \mathbb{Z} \), comes from a rank one free module. So \( \iota_\ast(1) = 1 \). Since \( \sigma \) is the class of \( eR \) with a different action of \( S^1 \), we get \( \iota_\ast(\sigma) = 1 \) for the same reason.

By Lemma 8.10, \( \xi(1) \) is a sum of \( N \) rank one \( \gamma \)-invariant projections in \( R \). It follows from Lemma 8.16 that, under the isomorphism (8.14), each corresponds to some element of \( (\mathbb{Z}_N)^N \subseteq R(\mathbb{Z}_N) \), that is, to some power \( \tau^k \) with \( 0 \leq k \leq N - 1 \). So there are \( m_0, m_1, \ldots, m_{N-1} \in \{0, 1, \ldots, N - 1\} \) such that \( \xi_\ast(1) = \sum_{j=0}^{N-1} \tau^j m_j \).

Since \( \sigma \cdot 1 = 1 \) in \( K_0^{S^1,\beta}(C(S^1)) \cong \mathbb{Z} \), it follows that \( \sigma \cdot \xi_\ast(1) = \xi_\ast(1) \). The only possibility is then \( \xi_\ast(1) = 1 + \tau + \cdots + \tau^{N-1} \).

**Lemma 8.18.** In Construction 8.3, assume that for all \( n \in \mathbb{Z}_{\geq 0} \) the numbers \( l_{0,1}(n) \) and \( l_{1,0}(n) \) are both odd. Let \( A \) and \( \alpha : S^1 \to \text{Aut}(A) \) be as in Construction 8.3(9). Then there is an injective \( R(S^1) \)-module homomorphism from \( \mathbb{Z} [\sigma]/(\sigma^N - 1) \) to \( K_0^{S^1}(A) \).

The hypotheses are much stronger than necessary, but this statement is easy to prove.

**Proof of Lemma 8.18.** The inclusion of \( R = A_{0,0} \) in \( A_0 = A_{0,0} \oplus A_{0,1} \) is injective on equivariant K-theory. Since equivariant K-theory commutes with direct limits (Theorem 2.8.3(6) of [42]), it is now enough to prove that \( (\nu_n)_\ast : K_0^{S^1}(A_n) \to K_0^{S^1}(A_{n+1}) \) is injective for all \( n \in \mathbb{Z}_{\geq 0} \). Identify \( K_0^{S^1}(C(S^1)) \) and \( K_0^{S^1}(R) \), as well as \( \iota_\ast \) and \( \xi_\ast \), as in Lemma 8.17. Also, observe that, with these identifications, for all \( t \in \mathbb{R} \) the map \( (\beta_t)_\ast : K_0^{S^1}(C(S^1)) \to K_0^{S^1}(C(S^1)) \) becomes \( \text{id}_{\mathbb{Z}} \). Therefore injectivity of \( (\nu_n)_\ast \) is the same as injectivity of the map
\[
\Phi : \mathbb{Z}_N[\sigma]/(\sigma^N - 1) \oplus \mathbb{Z} \to \mathbb{Z}_N[\sigma]/(\sigma^N - 1) \oplus \mathbb{Z}
\]
which for \( m, m_0, m_1, \ldots, m_{N-1} \in \mathbb{Z} \) is given by
\[
\Phi \left( \sum_{k=0}^{N-1} m_k \sigma^k, m \right) = \left( \sum_{k=0}^{N-1} (l_{0,0}(n)m_k + l_{0,1}(n)m) \sigma^k, l_{1,0}(n) \sum_{k=0}^{N-1} m_k + 2nl_{1,1}(n)m \right).
\]

Suppose the right hand side of (8.15) is zero. For \( k = 0, 1, \ldots, N - 1 \) we then have \( l_{0,0}(n)m_k + l_{0,1}(n)m = 0 \). Therefore
\[
m_0 = m_1 = \cdots = m_{N-1} = -\frac{l_{0,1}(n)m}{l_{0,0}(n)}.
\]
Putting this in the second coordinate gives

$$0 = 2Nl_{1,1}(n) m - \frac{l_{1,0}(n)l_{0,1}(n)mN}{l_{0,0}(n)}.$$  

Since $N \neq 0$, this says $2l_{1,1}(n)l_{0,0}(n) = l_{1,0}(n)l_{0,1}(n)$ or $m = 0$. The hypotheses rule out the first, so $m = 0$, whence also $m_k = 0$ for $k = 0, 1, \ldots, N - 1$. Thus $(\nu_n)_*\sigma$ is injective. \hfill $\square$

**Corollary 8.19.** Under the hypotheses of Lemma 8.18, the action $\alpha$ does not have finite Rokhlin dimension with commuting towers.

**Proof.** Suppose $\alpha$ has finite Rokhlin dimension with commuting towers. By Corollary 4.5 of [14], there is $n \in \mathbb{N}$ such that $I(S^1)^nK_0^{S^1}(A) = 0$. Lemma 8.18 implies that $I(S^1)^nR(\mathbb{Z}_N) = 0$. Lemma 8.15 then says that

$$I(S^1)^nK_0^{S^1}\tau(C(S^1)) = 0.$$  

Since the underlying action of $S^1$ on $S^1$ is not free, this contradicts Theorem 1.1.1 of [42]. \hfill $\square$

It is also not hard to prove directly that $I(S^1)^nR(\mathbb{Z}_N) \neq 0$ for all $n \in \mathbb{N}$. With the notation of Lemma 8.17, there is a homomorphism $f: R(\mathbb{Z}_N) \to \mathbb{C}$ such that $f(\tau) = \omega$. Since $\sigma - 1 \in I(S^1)$ and the map $R(S^1) \to R(\mathbb{Z}_N)$ sends $\sigma$ to $\tau$, it is enough to show that $f((\tau-1)^n) \neq 0$ for all $n \in \mathbb{N}$. But $\mathbb{C}$ is a field and $f(\tau-1) \neq 0$.

**Theorem 8.20.** In Construction 8.3, choose $r_0(0) = r_1(0) = 1$ and for $n \in \mathbb{Z}_{\geq 0}$ choose

$$l_{0,0}(n) = l_{1,0}(n) = 1 \quad \text{and} \quad l_{0,1}(n) = l_{1,1}(n) = 2n + 1.$$  

Then $A$ is simple, the action $\alpha$ of Construction 8.3(8) has the tracial Rokhlin property with comparison, but $\alpha$ does not have finite Rokhlin dimension with commuting towers.

**Proof.** Simplicity is Lemma 8.8. With these choices, both Lemma 8.12 and Corollary 8.19 apply. \hfill $\square$

We now address the modified tracial Rokhlin property.

**Lemma 8.21.** Let the notation be as in Construction 8.3. Let $n \in \mathbb{Z}_{\geq 0}$. Define

$$D_1 = M_{l_{0,0}(n+1), l_{0,0}(n)+l_{0,1}(n+1)/l_{1,0}(n)}, \quad D_2 = M_{l_{0,1}(n+1), l_{1,0}(n)}, \quad D_3 = M_{l_{0,0}(n+1), l_{0,0}(n)+2Nl_{0,1}(n+1)/l_{1,1}(n)}, \quad D_4 = M_{l_{1,0}(n+1), l_{0,0}(n)+2Nl_{1,0}(n+1)/l_{1,1}(n)},$$  

and define

$$D_5 = M_{Nl_{1,0}(n+1), l_{0,1}(n)+4Nl_{1,1}(n+1)/l_{1,1}(n)}.$$  

and define

$$(8.16) \quad m_1 = N, \quad m_2 = N^2 - N, \quad m_3 = N, \quad m_4 = N, \quad \text{and} \quad m_5 = 1.$$  

Set

$$(8.17) \quad D = (D_1)^{m_1} \oplus (D_2)^{m_2} \oplus (D_3)^{m_3} \oplus (D_4)^{m_4} \oplus (D_5)^{m_5},$$  

and for $k \in \{1, 2, 3, 4, 5\}$ and $j = 1, 2, \ldots, n_j$, let $\pi_{k,j}: D \to D_k$ be the projection to the $j$ summand of $D_k$ in the definition of $D$. As usual, write the relative commutant of $\nu_{n+2,n}(A_n)^{(n)}$ in $\nu_{n+2,n}(A_{n+2})^{(n+2)}$ as $\nu_{n+2,n}(A_n)^{(n)} \cap (A_{n+2})^{(n+2)}$. Then

$$\nu_{n+1}(\mathcal{P}_{n+1}) \in \nu_{n+2,n}(A_n)^{(n)} \cap (A_{n+2})^{(n+2)}.$$
and there is an isomorphism
\[ \kappa: \nu_{n+2,n}( (A_n)^{\alpha(n)} ) \cap (A_{n+2})^{\alpha(n+2)} \to D \]
such that for \( k \in \{1, 2, 3, 4, 5\} \) and \( j = 1, 2, \ldots, m_j \), we have
\[ \text{rank}((\pi_{1,j} \circ \kappa \circ \nu_{n+1})(p_{n+1})) = \text{rank}((\pi_{2,j} \circ \kappa \circ \nu_{n+1})(p_{n+1})) = \text{rank}((\pi_{3,j} \circ \kappa \circ \nu_{n+1})(p_{n+1})) = \text{rank}((\pi_{4,j} \circ \nu_{n+1})(p_{n+1})) = 2Nl_0,1(n)l_{1,1}(n), \]
and
\[ \text{rank}((\pi_{5,j} \circ \kappa \circ \nu_{n+1})(p_{n+1})) = 4N^2l_{1,1}(n + 1)l_{1,1}(n). \]

\textbf{Proof.} Let the notation be as in parts (11), (12), (13), and (14) of Construction 8.3. By Lemma 8.11, it is enough to prove the lemma with the algebra \( B_t \) in place of \((A_t)^{\alpha(n)}\) for \( t = n, n + 1, n + 1 \), with \( \chi_t \) and \( \chi_{t,s} \) in place of \( \nu_t \) and \( \nu_{t,s} \), and with \( q_{n+1} \) in place of \( p_{n+1} \).

First, since \( q_{n+1} \) is in the center of \( B_{n+1} \), it commutes with the range of \( \chi_n \). So \( \chi_{n+1}(p_{n+1}) \in \chi_{n+2,n}(B_n)' \cap B_{n+2} \) is clear.

We change to more convenient notation for the structure of the algebras \( B_t \).

Write
\[ B_t = B_{t,0} \oplus B_{t,1} \oplus \cdots \oplus B_{t,N-1} \oplus B_{t,N}, \]
with
\[ B_{t,0} = B_{t,1} = \cdots = B_{t,N-1} = M_{r_0(n)} \quad \text{and} \quad B_{t,N} = M_{r_1(n)}. \]

Thus \( B_{t,0} \oplus B_{t,1} \oplus \cdots \oplus B_{t,N-1} \) is what was formerly called \( B_{t,0} \), and \( B_{t,N} \) is what was formerly called \( B_{t,1} \). The partial multiplicities in \( \chi_t \) of the maps \( B_{t,j} \to B_{t+1,k} \) are
\[ m_t(k, j) = \begin{cases} l_{0,0}(t) & j, k \in \{0, 1, \ldots, N - 1\} \\ l_{0,1}(t) & j = N \text{ and } k \in \{0, 1, \ldots, N - 1\} \\ l_{1,0}(t) & j \in \{0, 1, \ldots, N - 1\} \text{ and } k = N \\ 2Nl_{1,1}(t) & j - k = N. \end{cases} \]

An easy calculation now shows that the partial multiplicities in \( \chi_{n+2,n} \) of the maps \( B_{n,j} \to B_{n+2,k} \) are
\[ \tilde{m}(k, j) = \begin{cases} l_{0,0}(n + 1)l_{0,0}(n) + l_{0,1}(n + 1)l_{1,0}(n) & j, k \in \{0, 1, \ldots, N - 1\} \text{ and } j = k \\ l_{0,1}(n + 1)l_{1,0}(n) & j, k \in \{0, 1, \ldots, N - 1\} \text{ and } j \neq k \\ l_{0,0}(n + 1)l_{0,1}(n) + 2Nl_{0,1}(n + 1)l_{1,1}(n) & j = N \text{ and } k \in \{0, 1, \ldots, N - 1\} \\ l_{1,0}(n + 1)l_{0,0}(n) + 2Nl_{1,1}(n + 1)l_{1,0}(n) & j \in \{0, 1, \ldots, N - 1\} \text{ and } k = N \\ Nl_{1,0}(n + 1)l_{0,1}(n) + 4N^2l_{1,1}(n + 1)l_{1,1}(n) & j - k = N. \end{cases} \]

Recall that if
\[ \varphi: M_{r_1} \oplus M_{r_2} \oplus \cdots \oplus M_{r_s} \to M_{t_{r_1} + t_{r_2} + \cdots + t_{r_2}} \]
is unital with partial multiplicities \( l_1, l_2, \ldots, l_s \), then the relative commutant of the range of \( \varphi \) is isomorphic to \( M_{l_1} \oplus M_{l_2} \oplus \cdots \oplus M_{l_s} \), with the identity of \( M_{l_i} \) being, by abuse of notation, \( \varphi(1_{M_{l_i}}) \). Therefore the values of \( \tilde{m}(k, j) \) are the matrix sizes of the summands in \( \chi_{n+2, n}(B_n)' \cap B_{n+2} \). That the exponents \( m_j \) in (8.17) are given as in (8.16) follows simply by counting the number of times each case in (8.18) occurs.

The rank of the image of \( q_{n+1} \) in each summand is the contribution to \( \tilde{m}(k, j) \) from maps factoring through \( B_{n+1, N} \) (in the original notation, \( B_{n+1, 1} \)), as opposed to the other summands. That these numbers are in the statement of the lemma is again an easy calculation. \( \square \)

**Proposition 8.22.** In Construction 8.3, assume that \( r_0(0) \leq r_1(0) \) and that for all \( n \in \mathbb{Z}_{\geq 0} \) we have

\[
\begin{align*}
l_{1,0}(n) &\geq l_{0,0}(n), \\
l_{0,1}(n) &\geq l_{0,0}(n), \\
l_{1,1}(n) &\geq l_{0,1}(n), \quad \text{and} \quad l_{1,1}(n) \geq l_{1,0}(n).
\end{align*}
\]

Further assume that

\[
\lim_{n \to \infty} \frac{l_{0,1}(n)}{l_{0,0}(n)} = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{l_{1,1}(n)}{l_{1,0}(n)} = \infty.
\]

Then the action \( \alpha \) of Construction 8.3(8) has the tracial Rokhlin property with comparison and has the modified tracial Rokhlin property as in Definition 6.1. Moreover, given finite sets \( F \subseteq A, F_0 \subseteq A^\alpha, \) and \( S \subseteq C(G) \), as well as \( \varepsilon > 0, x \in A^+ \) with \( \|x\| = 1 \), and \( y \in A^\alpha \setminus \{0\} \), it is possible to choose a projection \( p \in A^\alpha \), a unital completely positive contractive map \( \varphi: C(G) \to pAp \), and a partial isometry \( s \in A^\alpha \), such that the conditions of both Definition 2.4 and Definition 6.1 are simultaneously satisfied.

As usual, the hypotheses are overkill.

**Proof of Proposition 8.22.** Since \( A \) is finite, as usual, the argument of Lemma 1.16 of [44] applies, and shows that it suffices to verify this without the condition \( \|pxp\| > 1 - \varepsilon \), simultaneously in both definitions.

The method used for the proof of Lemma 8.12 also applies here. The key new point is that in every summand of

\[
\nu_{n+2, n}(\left(A_n\right)^{\alpha(n)}') \cap (A_{n+2})^{\alpha(n+2)}
\]

as described in Lemma 8.21, the rank of the component of \( \nu_{n+1}(p_{n+1}) \) is greater than half the corresponding matrix size. Therefore the rank of the component of \( 1 - \nu_{n+1}(p_{n+1}) \) is less than the rank of the component of \( \nu_{n+1}(p_{n+1}) \). Rank comparison implies that there exists \( s_0 \in \nu_{n+2, n}(\left(A_n\right)^{\alpha(n)}') \cap (A_{n+2})^{\alpha(n+2)} \)

such that

\[
1 - \nu_{n+1}(p_{n+1}) = s_0^*s_0 \quad \text{and} \quad s_0s_0^* \leq \nu_{n+1}(p_{n+1}).
\]

Now set \( s = \nu_{\infty, n+1}(s_0) \). The rest of the proof is an easy computation. \( \square \)
9. Nonexistence

For actions of $S^1$, one can require in all variants of the definition the $(F, S, \varepsilon)$-equivariant central multiplicative map $\varphi$ be exactly a homomorphism and exactly equivariant. For direct limit actions of $S^1$, one can also require that it takes values in some algebra in the system. We state a slightly more general form (also covering finite abelian groups) in the following proposition. The statement about general actions is covered, using the trivial direct system in which all the algebras are $A$.

**Proposition 9.1.** Let $G$ be a (not necessarily connected) compact abelian Lie group such that $\dim(G) \leq 1$. Let $((A_n)_{n \in \mathbb{Z}_{\geq 0}}, (\nu_{n,m})_{m \leq n})$ be a direct system of unital C*-algebras with unital injective maps $\nu_{n,m} : A_m \to A_n$. Set $A = \lim_{n \to \infty} A_n$, with maps $\nu_{\infty,m} : A_m \to A$. Assume we are given actions $\alpha^{(n)} : G \to \text{Aut}(A_n)$ such that the maps $\nu_{n,m}$ are equivariant, and let $\alpha$ be the direct limit action $\alpha = \lim_{n \to \infty} \alpha^{(n)}$.

1. The action $\alpha$ has the Rokhlin property if and only if for every $m \in \mathbb{Z}_{\geq 0}$, every finite set $F \subseteq A_m$, every finite set $S \subseteq C(G)$, and every $\varepsilon > 0$, there exists $n \geq m$ and a unital equivariant homomorphism $\varphi : C(G) \to A_n$ such that $\|\varphi(f)\nu_{n,m}(a) - \nu_{n,m}(a)\varphi(f)\| < \varepsilon$ for all $f \in S$ and $a \in F$.

2. Suppose $A$ is simple. Then $\alpha$ has the tracial Rokhlin property with comparison if and only if for every $m \in \mathbb{Z}_{\geq 0}$, every finite set $F \subseteq A_m$, every finite set $S \subseteq C(G)$, every $\varepsilon > 0$, every $x \in A_+$ with $\|x\| = 1$, and every $y \in A_n^+ \setminus \{0\}$, there exist $n \geq m$, a projection $p \in A_n^+$, and a unital equivariant homomorphism $\varphi : C(G) \to pA_n p$ such that:
   
   (a) $\|\varphi(f)\nu_{n,m}(a) - \nu_{n,m}(a)\varphi(f)\| < \varepsilon$ for all $f \in S$ and $a \in F$.
   
   (b) $1 - \nu_{\infty,n}(p) \not\lesssim A x, 1 - \nu_{\infty,n}(p) \not\lesssim A y, \text{ and } 1 - p \not\lesssim A_n p$.
   
   (c) $\|\nu_{\infty,n}(p)x\nu_{\infty,n}(p)\| > 1 - \varepsilon$.

3. Suppose $A$ is simple. Then $\alpha$ has the naive tracial Rokhlin property if and only if for every $m \in \mathbb{Z}_{\geq 0}$, every finite set $F \subseteq A_m$, every finite set $S \subseteq C(G)$, every $\varepsilon > 0$, and every $x \in A_+$ with $\|x\| = 1$, there exist $n \geq m$, a projection $p \in A_n^+$, and a unital equivariant homomorphism $\varphi : C(G) \to pA_n p$ such that:
   
   (a) $\|\varphi(f)\nu_{n,m}(a) - \nu_{n,m}(a)\varphi(f)\| < \varepsilon$ for all $f \in S$ and $a \in F$.
   
   (b) $1 - \nu_{\infty,n}(p) \not\lesssim A x$.
   
   (c) $\|\nu_{\infty,n}(p)x\nu_{\infty,n}(p)\| > 1 - \varepsilon$.

**Proof.** In all three parts, the fact that the condition implies the appropriate property follows from the fact that any finite subset of $A$ can be approximated arbitrarily well by a finite subset of $\bigcup_{n=0}^{\infty} \nu_{\infty,n}(A_n)$. The other directions follow from equivariant semiprojectivity of $C(G)$, which is Theorem 4.4 of [16].

Let $A$ be a unital C*-algebra. It is known (in [15] see Theorem 2.17, Example 3.22, and Example 3.23) that the existence of an action of $S^1$ on $A$ with the Rokhlin property implies severe restrictions on $A$. One can in fact rule out at least direct limit actions on a simple unital AF algebra with even the naive tracial Rokhlin property.

**Proposition 9.2.** Let $A$ be an infinite dimensional simple unital AF algebra and let $G$ be a one dimensional compact abelian Lie group. There is no direct limit action of $G$ on $A$, with respect to any realization of $A$ as a direct limit of finite dimensional C*-algebras, which has the naive tracial Rokhlin property (Definition 5.1).
Proof. Let \( (A_n)_{n \in \mathbb{Z}_{\geq 0}}, (\nu_{n,m})_{m \leq n} \) be an equivariant direct system of finite dimensional C*-algebras with actions \( \alpha^{(n)} : G \to \text{Aut}(A) \) and such that the direct limit action \( \alpha = \lim_{\rightarrow} \alpha^{(n)} \) has the naive tracial Rokhlin property. Choose any projection \( x \in A \setminus \{0, 1\} \). Apply Proposition 9.1(3) with \( m = 0, F = \{1\}, S = \{1\}, \varepsilon = \frac{1}{2} \), and \( x \) as given. We get \( n \in \mathbb{Z}_{\geq 0}, \) a nonzero projection \( p \in A_{\alpha n} \), and a unital equivariant homomorphism \( \phi : C(G) \to pA_n p \). Since \( C(G) \) is \( G \)-simple and \( \phi \) is equivariant, \( \phi \) must be injective. This is a contradiction because \( G \) is infinite and \( A_n \) is finite dimensional. \( \square \)

Corollary 9.3. Let \( A \) be an infinite dimensional simple unital AF algebra and let \( G \) be a one dimensional compact abelian Lie group. There is no direct limit action \( G \) on \( A \), with respect to any realization of \( A \) as a direct limit of finite dimensional C*-algebras, which has the tracial Rokhlin property with comparison.

Proof. This is immediate from Proposition 9.2, because the tracial Rokhlin property with comparison implies the naive tracial Rokhlin property. \( \square \)

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