HIGH TEMPERATURE TAP UPPER BOUND FOR THE FREE ENERGY OF MEAN FIELD SPIN GLASSES

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Abstract. This work proves an upper bound for the free energy of the Sherrington-Kirkpatrick model and its generalizations in terms of the Thouless-Anderson-Palmer (TAP) energy. The result applies to models with spherical or Ising spins and any mixed p-spin Hamiltonian with external field or with a non-linear spike term. The bound is expected to be tight to leading order at high temperature, and is non-trivial in the presence of an external field. For the proof a geometric microcanonical method is employed, in which one covers the spin space with sets, each of which is centered at a magnetization vector $m$ and whose contribution to the partition function is bounded in terms of the TAP energy at $m$.

1. Introduction

This article proves an upper bound for the free energy of spherical or Ising mixed p-spin spin glass models in terms of the maximum of their Thouless-Anderson-Palmer (TAP) energy. This is a step towards the computation of the free energy of spin glass models completely within a geometric, microcanonical TAP framework. Though its goal is similar, as explained below this approach is fundamentally different from other recent TAP approaches in the mathematical literature.

To formally state the results consider for $N \geq 1$ an inner product on $\mathbb{R}^N$ given by $\langle a, b \rangle = \frac{1}{N} \sum a_i b_i$, so that the corresponding norm $\| \cdot \|$ satisfies $\|a\|^2 = \frac{1}{N} \sum_{i=1}^N a_i^2$. Let $B_N (r) = \{ \sigma \in \mathbb{R}^N : \|\sigma\| \leq r \}$ and $B_N^o (r) = \{ \sigma \in \mathbb{R}^N : \|\sigma\| < r \}$ denote the closed resp. open ball of radius $r$ and let $B_N = B_N (1)$ and $B_N^o = B_N^o (1)$. Due to our convention $\{-1,1\}^N \subset B_{N-1}$.

Let $H_N (\sigma)$ be any mixed p-spin Hamiltonian on $B_N$, that is a centered Gaussian process indexed by the spin vectors $\sigma \in B_N$ with covariance

$$\mathbb{E} [H_N (\sigma) H_N (\sigma')] = N \xi (\langle \sigma, \sigma' \rangle) \text{ for } \sigma, \sigma' \in B_N,$$

for a power series $\xi (x) = \sum_{p \geq 0} a_p x^p$ with $a_p \geq 0$ and $\xi (1) < \infty$. The inverse temperature is denoted by $\beta \geq 0$.

Let $K \in \{1, \ldots, N\}$ and $U$ be a linear subspace of $\mathbb{R}^N$ of dimension $K$, and let $P_U$ denote projection onto $U$. Let $f_N : B_N \to \mathbb{R}$ be a function representing a generalized

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external field such that \( f_N(\sigma) = f_N(P^U \sigma) \) for all \( \sigma \in B_N \), and write

\[
H_N^{f}(\sigma) = H_N(\sigma) + f_N(\sigma) \quad \text{for} \quad \sigma \in B_N.
\]

With \( f_N(\sigma) = h \sum_{i=1}^{N} \sigma_i \) (and \( K = 1 \)) we obtain a standard linear external field and with \( f_N(x) = \frac{h}{N} (\sum_{i=1}^{N} \sigma_i)^2 \) a quadratic spike as studied for instance in [Ben+18].

Let the Onsager term of the TAP energy be given by

\[
\text{On}(q) = \xi(1) - (1 - q) \xi'(q) - \xi(q), q \in [0, 1].
\]

The TAP energy of the mixed \( p \)-spin model with Ising spins is

\[
H_{\text{TAP}}^{\text{Ising}}(m) = \beta H_N^{f}(m) + I_{\text{Ising}}(m) + \frac{\beta^2}{2} \text{On}(\|m\|^2),
\]

for the entropy term

\[
I_{\text{Ising}}(m) = -\sum_{i=1}^{N} J(m_i),
\]

where \( J \) is the binary entropy function given by

\[
J(m) = \frac{1 + m}{2} \log (1 + m) + \frac{1 - m}{2} \log (1 - m).
\]

Our main result for the Ising spin model is the following.

**Theorem 1.1 (Ising TAP Upper Bound).** For all \( \delta, L \in (0, \infty) \) there is a constant \( c_1 = c_1(\delta, L) \) such that the following holds. Let \( N \geq 1 \) and \( E \) be the uniform distribution on \( \{-1, 1\}^N \). Assume that \( 0 \leq \beta, \sqrt{\xi''(1)} \leq L \). Let \( U \) have dimension \( 1 \leq K \leq c_1 N/\log N \) and \( f_N : B_N \to \mathbb{R} \) be Lipschitz on \( B_N \) with respect to \( \| \cdot \| \) with Lipschitz constant at most \( LN \). Then

\[
\mathbb{P}\left( \log E \left[ \exp \left( \beta H_N^{f}(\sigma) \right) \right] \leq \sup_{m \in (-1, 1)^N} H_{\text{TAP}}^{\text{Ising}}(m) + \delta N \right) \geq 1 - c_1^{-1} e^{-c_1 N}.
\]

For the spherical mixed \( p \)-spin model the TAP energy is given by

\[
H_{\text{TAP}}^{\text{sph}}(m) = \beta H_N^{f}(m) + I_{\text{sph}}(m) + \frac{\beta^2}{2} \text{On}(\|m\|^2),
\]

for the entropy term

\[
I_{\text{sph}}(m) = \frac{N}{2} \log (1 - \|m\|^2),
\]
and our main result is the following. Let \( S_{N-1} (r) = \{ \sigma \in \mathbb{R}^N : \| \sigma \| = r \} \) denote the sphere of radius \( r \), and write \( S_{N-1} = S_{N-1} (1) \).

**Theorem 1.2 (Spherical TAP Upper Bound).** For all \( \delta, L \in (0, \infty) \), there is a constant \( c_2 = c_2 (\delta, L) \) such that the following holds. Let \( N \geq 1 \) and \( E \) be the the uniform distribution on \( S_{N-1} \). Assume that \( 0 \leq \beta, \sqrt{\xi''(1)} \leq L \). Let \( U \) have dimension \( 1 \leq K \leq c_2 N / \log N \) and \( f_N : B_N \to \mathbb{R} \) be Lipschitz on \( B_N \) with respect to \( \| \cdot \| \) with Lipschitz constant at most \( L N \).

Then

\[
\mathbb{P} \left( \log E \left[ \exp \left( \beta H_N^f (\sigma) \right) \right] \leq \sup_{m \in B_N} H_{\text{TAP}}^{\text{ph}} (m) + \delta N \right) \geq 1 - c_2^{-1} e^{-c_2 N}.
\]

The bounds (1.7) and (1.10) for the free energy are expected to be tight to leading order at high temperature. In the absence of an external field the bounds are no stronger than the annealed upper bound and are thus trivial, but in the presence of an external field obtaining such upper bounds is a difficult problem. In future work the author plans to rigorously compute the maximal TAP energy for some of these models (building on results such as \([\text{ABC}13; \text{Fyo15}; \text{Sub17a}; \text{ZSA21}; \text{Bel+22}] \) ), which combined with the this article will yield concrete upper bounds for the free energy.

An important question is whether a similar upper bound that is tight at all temperatures can be proven, in addition to a matching lower bound. Such bounds must involve conditions which rule out some \( m \)'s, including at least the Plefka condition \([\text{TAP}77; \text{Ple82}] \) of the physics literature. For the special case of the spherical 2-spin model with linear external field (i.e. \( E \) is uniform on \( S_{N-1} \), \( \xi (x) = x^2 \) and \( f_N (\sigma) = h \sum_{i=1}^N \sigma_i \) ) the work \([\text{BK19}] \) shows that in fact for all \( \beta, h \geq 0 \)

\[
\log E \left[ \exp \left( \beta H_N^f (\sigma) \right) \right] = \sup_{m; \beta (1 - \| m \|^2) \leq c_2} H_{\text{TAP}}^{\text{ph}} (m) + o (N),
\]

where the condition on \( m \) is precisely Plefka’s condition for this model. Thus it shows that matching upper and lower bounds hold at any temperature once the Plefka condition is added to the sup. An extension of this result to the present setting would pave the way for the computation of the free energy of a larger class of spin glass models, including at low temperature, using a geometric TAP approach.

Theorems 1.1-1.2 follow from a more general bound for a general spin reference measure \( E \) on the sphere \( S_{N-1} \), which need not be a product measure. To state it define the general entropy function

\[
I_{E,\delta} (m) = \inf_{\lambda \in \mathbb{R}^N, \| \lambda \| = 1} \log E \left[ \{ \sigma \in S_{N-1} : \langle \lambda, \sigma - m \rangle \geq -\delta \} \right] \in [-\infty, 0],
\]

for any probability measure \( E \) on \( S_{N-1} \), and a general TAP energy by

\[
H_{\text{TAP}}^{E,\delta} (m) = \beta H_N^f (m) + I_{E,\delta} (m) + \frac{\beta^2}{2} \text{On} \left( \| m \|^2 \right).
\]

Our general bound is the following.
Theorem 1.3 (General TAP Upper Bound). For any $\delta \in (0,1), \beta \geq 0, 1 \leq K \leq N, L \in (0, \infty)$ there is a constant $\kappa = \kappa (\delta, \beta, K, L)$ such that the following holds. Let $N \geq 1$, and $E$ be any probability measure on $S_{N-1}$. Assume $\sqrt{\xi' '(1)} \leq L$. Let $U$ have dimension $K$ and $f_N : B_N \to \mathbb{R}$ be Lipschitz on $B_N$ with respect to $\| \cdot \|$ with Lipschitz constant at most $LN$. Then it holds that

$$(1.13) \quad \mathbb{P} \left( \log E \left[ \exp \left( \beta H_N^f (\sigma) \right) \right] \leq \frac{1}{N} \sup_{m \in B_N} H_{E, \delta}^{E, \delta} (m) + \delta \right) \geq 1 - \kappa e^{-\frac{1}{2}N}.$$  

More explicitly, the constant $\kappa$ satisfies $\kappa \leq \bar{\kappa}$ for $\bar{\kappa} = c \max (K, (\beta L/\delta)^8)$ and a universal constant $c$.

Note that the only difference between (1.4), (1.8) and (1.12) is the corresponding entropy term $I_{\text{Ising}}, I_{\text{sph}}, I_{E, \delta}$. If $E$ is the uniform measure on $\{-1,1\}^N$, as in Theorem 1.1, it turns out that $I_{E, \delta}(m)$ is $-\infty$ if $m$ is sufficiently far from $(-1,1)^N$, and otherwise it can be bounded above in terms of $I_{\text{Ising}}(m)$. Similarly if $E$ is the uniform measure on the sphere $S_{N-1}$, as in Theorem 1.2, it turns out that $I_{E, \delta}(m)$ can be bounded above in terms of $I_{\text{sph}}(m)$. These bound will be proved and used to derive Theorems 1.2-1.3 from the general Theorem 1.3 in Sections 7-8.

The proof of Theorem 1.3 is based on covering the sphere with sets, each centered on a “magnetization” vector $m \in \mathbb{R}^N$, on which one can expand the Hamiltonian $H_N (\sigma)$ around $m$, giving rise to a recentered “effective” Hamiltonian $H_{E, \delta}^N (\sigma)$ (see (2.18) and (3.1)-(3.2)) for which the external field effectively vanishes. The integral of the Gibbs factor $\exp(\beta H_N^f (\sigma))$ over the set of the cover centered at $m$ can be bounded above by $\exp(H_{E, \delta}^N (m) + o(N))$ using an annealed upper bound (Markov inequality). The number of sets in the cover will be seen to grow slowly with $N$, and therefore a union bound over all the sets will be enough to obtain the upper bound (1.13). The approach can be seen as a TAP-informed sophisticated moment method. A more detailed sketch is provided in Section 2.

1.1. Related work and historical remarks. Mean field-spin glasses were introduced in [SK75] as toy models of the properties of exotic magnetic alloys, and they [KTJ76; CS92; Tal00; CL04; Tal06b] and related models have since become paradigmatic examples of complex systems [MPV87b; KR98; DSS15; DS19; MM09]. Their investigation can alternatively be thought of as the study of the extrema of highly correlated high dimensional random fields.

The partition function is the integral

$$Z_N = E \left[ \exp \left( \beta H_N^f (\sigma) \right) \right],$$

over the spins $\sigma$ against a reference measure $E$ (the uniform distribution on $\{-1,1\}^N$ for the Ising spin model, and the uniform measure on the sphere $S_{N-1}$ for the spherical model). The free energy is the exponent

$$F_N = \frac{1}{N} \log Z_N.$$
The spin vector $\sigma$ under the Gibbs measure $\mathcal{G}(A) = E[1_A \exp(\beta H_N^f(\sigma))] / Z_N$ models for instance the aforementioned exotic materials. Computing the free energy for large $N$ is a first step towards determining the behavior of $\sigma$ under the Gibbs measure, which is the ultimate goal of studying these models.

The TAP energy was introduced in [TAP77] for the purpose of solving the Sherrington-Kirkpatrick model. Presumably the original motivation was to devise a framework to - among other things - compute the free energy. In physics, the computation of the free energy was however achieved by the replica symmetry breaking ansatz and the replica method of Parisi [Par80; Par79], and in mathematics by the interpolation method of Guerra [Gue03] together with the methods of Talagrand, Aizenman-Sim-Starr and Panchenko [ASS03; Tal06c; Tal06a; Pan14; Che13; Pan13], which are very different approaches. The study of the TAP energy has played a complementary role in the analysis of the model in physics [MPV87a; BM80; DY83; GM84; KPV93; CS95; Cav+03] and mathematics [Cha10; AJ19b; AJ19a; CP18; CPS21; Tal11, Section 1.7], rather than being fully developed as a stand-alone solution of it (see however recent work mentioned below).

Initial steps towards the development of the TAP approach as a stand-alone solution were taken in [Kis16; BK19]. As mentioned above, the article [BK19] computed the free energy of the 2-spin spherical model at all temperatures and linear external fields in terms of the TAP energy. This article takes a further step, by giving an upper bound for the free energy in terms of the TAP energy in a much more general setting.

An alternative approach to computing the free energy using the TAP energy was initiated by Subag [Sub18; CPS21; CPS22; Sub21]. This involves properties of the limiting Gibbs measure such as the concept of “multisample overlap” and is therefore very different from the approach of the current article which is microcanonical and works on the level of spin configurations for finite $N$. Indeed, note that Theorems 1.1-1.3 are all quantitative finite $N$ statements.

An earlier stream of work also initiated by Subag computes the free energy of certain spherical spin glasses at low enough temperature [Sub17b; BSZ20]. These, and also [AJ21], use a moment method and are more similar in spirit to the present approach. The eventual goal of the present research project is to compute the free energy for all models and all temperatures using a TAP-informed moment method.

Another TAP approach involving an iterative solution of the TAP equations (critical point equations of the TAP energy) was initiated by Bolthausen [Bol14; Bol19; BY21]. The iterative construction of a cover of the sphere in the present article bears some similarity to this iterative solution of the TAP equations. Those works also use a moment method. The way iteration and moment method are used is however quite different; the aforementioned articles construct one sequence of iterates that converge to the conjecturally unique TAP solution at high temperature, while we here construct a hierarchy of iterates whose associated sets cover the whole sphere. Morally speaking, one of the iterates in our hierarchy should be the iterate that Bolthausen’s algorithm produces.

A further difference compared to the aforementioned works is that here bounding the free energy in terms of the TAP energy is neatly decoupled from the study of the
behavior of the TAP energy itself (indeed, the present work deals only with the former and leaves the latter for later research).

1.2. Overview of article. In Section 2 we give a detailed sketch of the proofs of Theorems 1.1-1.3, motivating the construction of the cover of $S_{N-1}$. In Section 3 we formally introduce the recentered Hamiltonian and study its law. In Section 4 we give the iterative construction of magnetizations, which we then use in Section 5 to construct the cover of $S_{N-1}$. Then in Section 6 the construction is used to prove the general TAP upper bound Theorem 1.3. In Section 7 the Ising upper bound Theorem 1.1 is derived from the general result, and in Section 8 the spherical upper bound Theorem 1.2 is similarly derived. The appendix contains some basic results about the Hamiltonian that follow from the classical theory of Gaussian processes.

We use $c$ to denote unspecified positive constants, whose numerical value may be different each time the notation $c$ is used, even within the same formula. The standard inner product is denoted by $a \cdot b$ and the standard norm by $|\cdot|$, so that $a \cdot b = \sum_{i=1}^{N} a_i b_i = N \langle a, b \rangle$ and $|a| = \sqrt{\sum_{i=1}^{N} a_i^2} = \sqrt{N}||a||$ for $a, b \in \mathbb{R}^N$.

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2. Sketch of proof

As the construction of the magnetizations and cover in Sections 4-5 is quite involved, this section gives a detailed sketch of the proofs which motivates it.

The goal is to use annealed upper bounds (i.e. the Markov inequality) to obtain a bound for the free energy that is tight to leading order, even in the presence of an external field.

2.1. Sphere with linear external field. First let us sketch a direct proof of the bound with a linear external field, that is with

\begin{equation}
H^f_N (\sigma) = H_N (\sigma) + Nh \langle \sigma, u_1 \rangle,
\end{equation}

for a $|| \cdot ||$-unit vector $u_1$, and for the spherical model where $E$ denotes the uniform measure on the sphere $S_{N-1}$ (cf. Theorem 1.2).

The standard annealed upper bound for the free energy $F_N$ is obtained from

\begin{equation}
Z_N \leq E \left[ E \left[ \exp \left( \beta H^f_N (\sigma) \right) \right] \right] e^{o(N)},
\end{equation}

and is a simple consequence of the Markov inequality. If $h = 0$ the integral on the RHS equals

\begin{equation}
\exp \left( \frac{1}{2} \text{Var} \left( \beta H^f_N (\sigma) \right) \right) = \exp \left( N \frac{\beta^2}{2} \xi (1) \right).
\end{equation}

If the covariance is $\xi (x) = \sum_{p \geq 0} a_p x^p$ with $a_0 = a_1 = 0$ and $h = 0$ then (2.2) gives a bound for the free energy that is tight to leading order at high temperature (for small enough $\beta$ this can be verified by proving a matching lower bound using
a simple second moment method). By contrast, if at least one of \( a_0, a_1, h \) are non-zero then the law of the Hamiltonian \( H_N^f (\sigma) \) is that of \( A_0 + A_1 \cdot \sigma + \tilde{H}_N (\sigma) \) where \( A_0 \sim \mathcal{N} (0, Na_0) \), \( A_1 \sim \mathcal{N} (hu_1, Na_1I) \) and \( \tilde{H}_N (\sigma) \) are independent, and \( \tilde{H}_N (\sigma) \) is a Hamiltonian with covariance function \( \tilde{\xi} (x) = \sum_{p \geq 2} a_p x^p \). A non-vanishing global shift \( A_0 \) or a non-vanishing (random) external field \( A_1 \cdot \sigma \) will both individually cause the annealed upper bound (2.2) to overestimate the free energy to leading order.

In this sketch we are interested in rectifying this to obtain a bound that is tight - at least for some \( \beta \) - when \( \xi \) only has terms of order 2 and higher, but \( h > 0 \). Roughly speaking we do this by covering the sphere with a finite number of regions where the effective external field vanishes.

Define the partition function restricted to a region \( A \) by
\[
Z_N (A) = E [1_A \exp (\beta H_N (\sigma) + N\beta h \langle \sigma, u_1 \rangle)].
\]

For the “equator”
\[
E = \{ \sigma : |\langle \sigma, u_1 \rangle| \leq \eta \},
\]
we have the bound
\[
Z_N (E) \leq E [\exp (\beta H_N (\sigma))] e^{N\beta h \eta}.
\]

Applying the annealed bound as in (2.2)-(2.3) to the integral on the RHS yields
\[
E [\exp (\beta H_N (\sigma))] \leq \exp \left( N \frac{\beta^2}{2} \xi (1) + o (N) \right) \overset{(1.3)}{=} \exp \left( N \frac{\beta^2}{2} \text{On} (0) + o (N) \right) \overset{(1.8)}{=} \exp \left( H_{\text{TAP}}^{\text{sph}} (0) + o (N) \right),
\]
so we obtain from (2.6)
\[
Z_N (E) \leq \exp \left( H_{\text{TAP}}^{\text{sph}} (0) + N\beta h \eta + o (N) \right).
\]

This will give a bound for \( \frac{1}{N} \log Z_N (E) \) that is at high temperature tight to leading order in the limits \( N \to \infty \) and then \( \eta \downarrow 0 \), since the covariance \( \xi \) of \( H_N (\sigma) \) only has terms of order 2 and higher.

Eq. (2.8) bounds the contribution of the equator \( E \) to the partition function in terms of the TAP energy at \( m = 0 \). To get a bound for the actual partition function \( Z_N = Z_N (S_{N-1}) \) we need to also bound the contribution of \( E^c \). It is natural to decompose \( E^c \) according to the value of \( \langle \sigma, u_1 \rangle \) using the sets
\[
D_{(\alpha_1)} = \{ \sigma \in S_{N-1} : \langle \sigma, u_1 \rangle \in (|\alpha_1|, |\alpha_1| + \varepsilon) \times \text{sign} (\alpha_1) \} \text{ for } \alpha_1 \in (-1, 1),
\]
(the somewhat unusual expression on the RHS is used because it’s convenient to have \( |\langle \sigma, u_1 \rangle| > |\alpha_1| \) for \( \sigma \in D_{(\alpha_1)} \)). Defining the \( \varepsilon \)-spaced grid
\[
I_{\varepsilon, \eta} = (\varepsilon \mathbb{Z}) \cap (-1, 1) \setminus \left[ -\frac{\eta}{2}, \frac{\eta}{2} \right],
\]
\[
D_{(\alpha_1)} = \{ \sigma \in S_{N-1} : \langle \sigma, u_1 \rangle \in (|\alpha_1|, |\alpha_1| + \varepsilon) \times \text{sign} (\alpha_1) \} \text{ for } \alpha_1 \in (-1, 1),
\]
(the somewhat unusual expression on the RHS is used because it’s convenient to have \( |\langle \sigma, u_1 \rangle| > |\alpha_1| \) for \( \sigma \in D_{(\alpha_1)} \)).
we have for \( \varepsilon \leq \eta/2 \)
\[
S_{N-1} = E \cup \left( \bigcup_{\alpha_1 \in I_{\varepsilon, \eta}} D_{(\alpha_1)} \right).
\]

On each \( D_{(\alpha_1)} \) the external field term is essentially constant (for small \( \varepsilon \)), so from (2.4) we can bound
\[
Z_N \left( D_{(\alpha_1)} \right) \leq E \left[ 1_{D_{(\alpha_1)}} \exp \left( \beta H_N (\sigma) \right) \right] e^{N\beta h_{\alpha_1} + O(\varepsilon N)}.
\]

One may approximate the set \( D_{(\alpha_1)} \) by the set
\[
\tilde{D} = \{ \sigma : \langle \sigma, u_1 \rangle = \alpha_1 \} \subset D_{(\alpha_1)}.
\]

Let \( E^{\tilde{D}} \) denote the uniform measure on \( \tilde{D} \). Using that the Hamiltonian has Lipschitz constant with respect to \( \| \cdot \| \) of order \( N \) with high probability (see (A.23)) the right-hand side of (2.12) can be shown to equal
\[
E \left[ D_{(\alpha_1)} \right] E^{\tilde{D}} [\exp (\beta H_N (\sigma))] e^{N\beta h_{\alpha_1} + O(\varepsilon N)}.
\]

It is natural to now apply the annealed upper bound as in (2.2)-(2.3) to \( E^{\tilde{D}} [\exp (\beta H_N (\sigma))] \). Unfortunately, this will not give a tight bound, essentially because of the presence of an effective external field, as we describe below. To see this at the level of the covariance of the Hamiltonian, let
\[
m_{(\alpha_1)} = \alpha_1 u_1,
\]
be the “center” of the sets \( \tilde{D}, D_{(\alpha_1)} \) and consider for \( \sigma \in \tilde{D} \) the change of variables
\[
\sigma = m_{(\alpha_1)} + \hat{\sigma} \text{ for } \hat{\sigma} \in \text{span}(m_{(\alpha_1)})^\perp \cap S_{N-1} \left( \sqrt{1 - \alpha_1^2} \right).
\]

An easy computation shows that the process \( \hat{\sigma} \rightarrow H_N (m_{(\alpha_1)} + \hat{\sigma}) \) has covariance function
\[
z \rightarrow \xi (\alpha_1^2 + z) = \xi (\alpha_1^2) + \xi' (\alpha_1^2) z + \frac{1}{2} \xi'' (\alpha_1^2) z^2 + \ldots,
\]
which is a power series that for \( \alpha_1 \neq 0 \) has terms of order 0 and 1 in \( z \). Therefore by the discussion after (2.3) an annealed upper bound for \( E^{\tilde{D}} [\exp (\beta H_N (\sigma))] \) can not be tight. The origin of these problematic terms of (2.17) can be understood by defining for any \( m \) with \( \| m \| < 1 \) a recentered Hamiltonian \( H_N^m (\hat{\sigma}) \) by the expansion
\[
H_N (m + \hat{\sigma}) = H_N (m) + \nabla H_N (m) \cdot \hat{\sigma} + H_N^m (\hat{\sigma}).
\]

It turns out that for fixed \( m \)
\[
H_N (m), \left( \nabla H_N (m) \cdot \hat{\sigma} \right)_{\hat{\sigma}, \hat{\sigma}, m = 0}, \left( H_N^m (\hat{\sigma}) \right)_{\hat{\sigma}, \hat{\sigma}, m = 0}
\]
are independent, and \( (H_N^m (\hat{\sigma}))_{\hat{\sigma}, \hat{\sigma}, m = 0} \) has covariance function
\[
z \rightarrow \xi_{\| m \|^2} (z), \text{ where } \xi_{\| m \|^2} (z) = \xi (q + z) - z \xi' (q) - \xi (q),
\]
(see Lemma 3.2). In the covariance function \( \xi_{\| m \|^2} (z) \) the first and second order terms that appear in (2.17) are removed by construction, so \( H_N^{m_{(\alpha_1)}} (\hat{\sigma}) \) is a Hamiltonian for which
an annealed upper bound can be tight in the absence of an external field. However $H_N (\sigma)$ on the set $\tilde{D}$ consists as seen in (2.18) of not only the well-behaved $H_N^{m(\alpha_1)} (\hat{\sigma})$, but also of a random mean $H_N (m_{(\alpha_1)})$ and a random external field $\nabla H_N (m_{(\alpha_1)})$. With

$$h_{\text{eff}} = \nabla H_N (m_{(\alpha_1)}),$$

we get from (2.18)

$$E^{\tilde{D}} \left[ \exp (\beta H_N (\sigma)) \right] = \exp (\beta H_N (m_{(\alpha_1)})) E^{\tilde{D}} \left[ \exp \left( \beta H_N^{m(\alpha_1)} (\hat{\sigma}) + N \beta \langle h_{\text{eff}}, \hat{\sigma} \rangle \right) \right].$$

The presence of an external field in the integral on the RHS again confirms that an annealed upper bound can not be tight. But if we limit ourselves to the equator inside $D_{(\alpha_1)}$, namely

$$\mathcal{E}' = \{ \sigma \in D_{(\alpha_1)} : |\langle \hat{\sigma}, h_{\text{eff}} \rangle| \leq \eta \},$$

we can similarly to in (2.6) eliminate the external field term via

$$E^{\tilde{D}} \left[ 1_{\mathcal{E}'} \exp \left( \beta H_N^{m(\alpha_1)} (\hat{\sigma}) + N \beta \langle h_{\text{eff}}, \hat{\sigma} \rangle \right) \right] \leq E^{\tilde{D}} \left[ \exp \left( \beta H_N^{m(\alpha_1)} (\hat{\sigma}) \right) \right] e^{N \beta \eta}.$$

After this the annealed upper bound

$$E^{\tilde{D}} \left[ \exp \left( \beta H_N^{m(\alpha_1)} (\hat{\sigma}) \right) \right] \leq \exp \left( \frac{\beta^2}{2} \text{Var} \left( H_N^{m(\alpha_1)} (\hat{\sigma}) \right) + o (N) \right) = \exp \left( N \frac{\beta^2}{2} \xi_{\alpha_1} (1 - q_{(\alpha_1)}) + o (N) \right),$$

will be tight to leading order (for small $\beta$), where the $\hat{\sigma}$ on the RHS of the first line is an arbitrary $\hat{\sigma} \in \text{span} \left( m_{(\alpha_1)} \right)^\bot \cap S_{N-1} (\sqrt{1 - \alpha_1^2})$ and

$$q_{(\alpha_1)} = \alpha_1^2 = \| m_{(\alpha_1)} \|^2.$$

For the continuation of the construction is however more convenient to define the equator not with respect to $h_{\text{eff}}$ as in (2.20) but with respect to a $\| \cdot \|$-unit vector $u_{(\alpha_1),2}$ (different for each $\alpha_1$) that is perpendicular to $u_1$ such that

$$\text{span} \left( u_1, u_{(\alpha_1),2} \right) = \text{span} \left( u_1, h_{\text{eff}} \right) = \text{span} \left( u_1, \nabla H_N (m_{(\alpha_1)}) \right),$$

i.e.

$$\mathcal{E}_{(\alpha_1)} = \{ \sigma \in D_{(\alpha_1)} : |\langle \hat{\sigma}, u_{(\alpha_1),2} \rangle| \leq \eta \}.$$

For $\mathcal{E}_{(\alpha_1)}$ one also has as in (2.21)

$$E^{\tilde{D}} \left[ 1_{\mathcal{E}_{(\alpha_1)}} \exp \left( \beta H_N^{m(\alpha_1)} (\hat{\sigma}) + N \langle h_{\text{eff}}, \hat{\sigma} \rangle \right) \right] \leq E^{\tilde{D}} \left[ \exp \left( \beta H_N^{m(\alpha_1)} (\hat{\sigma}) \right) \right] e^{c N},$$

for a constant $c$ depending on $\beta$, which with approximations like (2.13), (2.14) and (2.22) with $\mathcal{E}_{(\alpha_1)}$ in place of $D_{(\alpha_1)}$ and a set $\tilde{\mathcal{E}}_{(\alpha_1)} = \{ \sigma : \langle \sigma, u_1 \rangle = \alpha_1, \langle \sigma, u_{(\alpha_1),2} \rangle = 0 \}$ in place of $\tilde{D}$ will be seen to give

$$Z_N (\mathcal{E}_{(\alpha_1)}) \leq \exp \left( \beta H_N (m_{(\alpha_1)}) + N \beta h_{\alpha_1} \right) \times E \left[ \mathcal{E}_{(\alpha_1)} \right] \exp \left( N \frac{\beta^2}{2} \xi_{\alpha_1} (1 - q_{(\alpha_1)}) + c (\varepsilon + \eta) N \right).$$
By (2.15) and (2.1) the first term equals \( \exp(\beta H_N^f(m_{(a_1)}) \). Turning to the “entropy” term \( E[\mathcal{E}_{(a_1)}] \), note that \( \mathcal{E}_{(a_1)} \) is approximately a sphere of dimension \( N \) of radius \( \sqrt{1 - \alpha_1^2} \) (the aforementioned \( \tilde{E}_{(a_1)} \) is exactly this). Essentially since the surface area of spheres in dimension \( M \) scale like \( r^M \), where \( r \) is the radius, it turns out that

\[
E[\mathcal{E}_{(a_1)}] \leq (1 - \alpha_1^2)^{N+o(N)} \overset{\text{(2.23)}}{=} \exp\left(\frac{N}{2} \log (1 - \|m_{(a_1)}\|^2) + o(N)\right) = \exp\left(I_{\text{sph}}(m_{(a_1)}) + o(N)\right).
\]

Note finally that by (1.3) and (2.19)

\[
(2.29) \quad \text{On } (q) = \xi_q (1 - q) \text{ for all } q \in [0, 1],
\]

so the last factor on the right-hand side of (2.27) equals \( \exp\left(\frac{\beta^2}{2} \text{On}(\|m_{(a_1)}\|^2)\right) \). In this fashion one can obtain from (2.27) that

\[
Z_N(\mathcal{E}_{(a_1)}) \leq \exp\left(H_N^f(m_{(a_1)}) + I_{\text{sph}}(m_{(a_1)}) + N\frac{\beta^2}{2} \text{On}(\|m_{(a_1)}\|^2)\right)e^{(q+\varepsilon)N+o(N)} \overset{\text{(1.8)}}{=} \exp\left(H_{\text{TAP}}^{\text{sph}}(m_{(a_1)}) + c(\eta + \varepsilon)N\right),
\]

which is a bound on the contribution of the equator \( \mathcal{E}_{(a_1)} \) inside \( D_{(a_1)} \) to the partition function in terms of the TAP energy at \( m = m_{(a_1)} \). Note that the entropy term \( \frac{N}{2} \log (1 - \|m\|^2) \) of \( H_{\text{TAP}}^{\text{sph}} \) arose from the measure of the equator \( \mathcal{E}_{(a_1)} \) under the reference measure in (2.28), and the Onsager term \( \frac{\beta^2}{2} \text{On}(\|m\|^2) \) from the annealed upper bound for the partition function of the recentered Hamiltonian \( H_N^{m_{(a_1)}} \) on \( \mathcal{E}_{(a_1)} \) as in (2.22).

Combining (2.8) and (2.30) one arrives at the bound

\[
Z_N\left(\mathcal{E} \cup \left( \bigcup_{a_1 \in I_{\varepsilon, \eta}} \mathcal{E}_{(a_1)} \right) \right) \leq \left\{ \exp\left(H_{\text{TAP}}^{\text{sph}}(0)\right) + \sum_{a_1 \in I_{\varepsilon, \eta}} \exp\left(H_{\text{TAP}}^{\text{sph}}(m_{(a_1)})\right) \right\} e^{(q+\varepsilon)N}.
\]

This is an improvement on (2.8), but to obtain the desired bound on \( Z_N = Z_N(S_{N-1}) \) we still need to bound the contribution of the complement of the region \( \mathcal{E} \cup (\bigcup_{a_1 \in I_{\varepsilon, \eta}} \mathcal{E}_{(a_1)}) \).

Similarly to how we decomposed \( \mathcal{E}^c \) using sets \( D_{(a_1)} \) in (2.9)-(2.11), this complement can be decomposed into sets

\[
D_{(a_1,a_2)} = \{ \sigma \in D_{(a_1)} : \langle \sigma, u_{(a_1)},2 \rangle \in (|\alpha_2|, |\alpha_2| + \varepsilon) \times \text{sign}(\alpha_2) \},
\]

so that

\[
(2.33) \quad S_{N-1} = \mathcal{E} \cup \left( \bigcup_{a_1 \in I_{\varepsilon, \eta}} \mathcal{E}_{(a_1)} \right) \cup \left( \bigcup_{a_1,a_2 \in I_{\varepsilon, \eta}} D_{(a_1,a_2)} \right).
\]

Letting

\[
m_{(a_1,a_2)} = \alpha_1 u_1 + \alpha_2 u_{(a_1),2} = m_{(a_1)} + \alpha_2 u_{(a_1),2},
\]

(2.34)
(2.15)) and using the change of variables $\sigma = m_{(\alpha_1, \alpha_2)} + \dot{\sigma}$ one can decompose the Hamiltonian on $D_{(\alpha_1, \alpha_2)}$ as
\begin{equation}
H_N (\sigma) = H_N (m_{(\alpha_1, \alpha_2)}) + \nabla H_N (m_{(\alpha_1, \alpha_2)}) \cdot \dot{\sigma} + H_N^{m_{(\alpha_1, \alpha_2)}} (\dot{\sigma}),
\end{equation}
(cf. (2.16) and (2.18)). Here $H_N^{m_{(\alpha_1, \alpha_2)}} (\dot{\sigma})$, $\dot{\sigma} \in \text{span} \left( u_1, u_{(\alpha_1, 2)} \right) \cap B_N$, is a Hamiltonian with covariance function without terms of order 0 or 1 but the presence of the effective external field $\nabla H_N (m_{(\alpha_1, \alpha_2)})$ again means that an annealed upper bound on $Z_N (D_{(\alpha_1, \alpha_2)})$ will not be tight for all $\alpha_2$. To improve the situation we may define $u_{(\alpha_1, \alpha_2), 3}$ as a unit vector perpendicular to $u_1, u_{(\alpha_1, 2)}$ such that
\begin{equation}
\text{span} \left( u_1, u_{(\alpha_1, 2)}, u_{(\alpha_1, \alpha_2), 3} \right) = \text{span} \left( u_1, u_{(\alpha_1, 2)}, \nabla H_N (m_{(\alpha_1, \alpha_2)}) \right),
\end{equation}
(cf. (2.24)) and define the equator
\begin{equation}
E_{(\alpha_1, \alpha_2)} = \{ \sigma \in D_{(\alpha_1, \alpha_2)} : \| \sigma - m_{(\alpha_1, \alpha_2)} \| \leq \eta \}
\end{equation}
inside $D_{(\alpha_1, \alpha_2)}$, similarly to (2.25). On $E_{(\alpha_1, \alpha_2)}$ one can bound the effective external field term in (2.35) by $N^2 \eta$, and apply the same annealed bounds as above in (2.22) on each region $E_{(\alpha_1, \alpha_2)}$. Since also $E_{(\alpha_1, \alpha_2)}$ is essentially a lower dimensional sphere as in (2.28) (2.38)
\begin{equation}
E \left[ E_{(\alpha_1, \alpha_2)} \right] \leq \exp \left( \frac{N}{2} \log \left( 1 - \| m_{(\alpha_1, \alpha_2)} \| \right)^2 \right) e^{o(N)} = \exp \left( I_{\text{sph}} (m_{(\alpha_1, \alpha_2)}) \right) e^{o(N)}.
\end{equation}
In this way one can obtain
\begin{equation}
Z_N (E_{(\alpha_1, \alpha_2)}) \leq \exp \left( H_N^f (m_{(\alpha_1, \alpha_2)}) + I_{\text{sph}} (m_{(\alpha_1, \alpha_2)}) + N \frac{\alpha^2}{2} \text{On} \left( \| m_{(\alpha_1, \alpha_2)} \| \right)^2 \right) e^{(\eta + \varepsilon)N} \overset{(1.8)}{=} \exp \left( H_{\text{TAP}}^{\text{sph}} (m_{(\alpha_1, \alpha_2)}) + c (\eta + \varepsilon) N \right),
\end{equation}
(cf. (2.30)), and from this one can improve on the bound (2.31) to get
\begin{equation}
Z_N \left( E \cup \bigcup_{\alpha_1 \in I_{\varepsilon, \eta}} E_{(\alpha_1)} \cup \bigcup_{\alpha_1, \alpha_2 \in I_{\varepsilon, \eta}} E_{(\alpha_1, \alpha_2)} \right) \leq e^{(\eta + \varepsilon)N} \times \left\{ \exp \left( H_{\text{TAP}}^{\text{sph}} (0) \right) + \sum_{\alpha_1 \in I_{\varepsilon, \eta}} \exp \left( H_{\text{TAP}}^{\text{sph}} (m_{(\alpha_1)}) \right) + \sum_{\alpha_1, \alpha_2 \in I_{\varepsilon, \eta}} \exp \left( H_{\text{TAP}}^{\text{sph}} (m_{(\alpha_1, \alpha_2)}) \right) \right\}.
\end{equation}
We can naturally continue this construction for any number $M$ of iterations, giving rise for each $k = 1, \ldots, M$ and $\alpha \in I_{\varepsilon, \eta}$ to

- a direction $u_{\alpha, k+1}$, such that with the notation $u_{\alpha, l} = u_{(\alpha_1, \ldots, \alpha_{l-1}, \alpha_l)}$, $m_{\alpha, l} = m_{(\alpha_1, \ldots, \alpha_l)}$ we have that
\begin{equation}
m_{\alpha, k+1} = \alpha_1 u_1 + \alpha_2 u_{\alpha, 2} + \ldots + \alpha_k u_{\alpha, k} + \alpha_{k+1} u_{\alpha, k+1},
\end{equation}
(cf. (2.15) and (2.34)) and
\begin{equation}
u_{\alpha_1, 1}, \ldots, u_{\alpha_1, k+1} \text{ is an orthonormal basis of } \text{span} \left( u_1, \nabla H_N (m_{\alpha, 1}), \ldots, \nabla H_N (m_{\alpha, k}) \right),
\end{equation}
HIGH TEMPERATURE T AP UPPER BOUND

(2.24) and (2.36)),

• a set

\[ D_\alpha = \{ \sigma \in S_{N-1} : \langle \sigma, u_{\alpha,l} \rangle \in (|\alpha_l|, |\alpha_l| + \varepsilon) \times \text{sign}(\alpha_l), l = 1, \ldots, k \} , \]

(cf. (2.9) and (2.32))

• an equator (cf. (2.5) (2.25) and (2.37))

(2.44)

\[ E_\alpha = \{ \sigma \in D_\alpha : |\langle \sigma - m_\alpha, u_{\alpha,k+1} \rangle| \leq \eta \} , \]

such that similarly to (2.28) and (2.38)

(2.45)

\[ E[\mathcal{E}_\alpha] \leq \exp \left( \frac{N}{2} \log \left( 1 - \|m_\alpha\|^2 \right) + o(N) \right) = \exp \left( I_{\text{sph}}(m_\alpha) + o(N) \right) , \]

(for \( M \) growing slowly with \( N \)) and similarly to (2.30) and (2.39)

(2.46)

\[ Z_N(\mathcal{E}_\alpha) \leq \exp \left( H_{\text{TAP}}^\text{sph}(m_\alpha) + c(M\varepsilon + \eta)N \right) , \]

(the formal definitions appear in Definitions 4.1, 4.4, 5.1). In (2.46) it is crucial that while the error involving \( \varepsilon \) compounds at most \( M \) times since it comes from “continuity” errors as in (2.14) in \( k \leq M \) dimensions (see (2.43)), the error involving \( \eta \) comes only from one dimension as in (2.26) (see (2.44)) and is therefore not multiplied by \( M \) (see Lemma 6.6).

Furthermore using the notations \( m() = 0 \) and \( \mathcal{E()} = \mathcal{E} \) one has

(2.47)

\[ S_{N-1} = \left( \bigcup_{k=0}^{M} \bigcup_{\alpha \in T_{\varepsilon,\eta}^k} \mathcal{E}_\alpha \right) \cup \left( \bigcup_{\alpha \in T_{\varepsilon,\eta}^{M+1}} D_\alpha \right) , \]

(cf. (2.11), (2.33)) and as a simple consequence of (2.46) the bound

(2.48)

\[ Z_N \left( \bigcup_{k=0}^{M} \bigcup_{\alpha \in T_{\varepsilon,\eta}^k} \mathcal{E}_\alpha \right) \leq e^{c(M\varepsilon + \eta)N} \sum_{k=0}^{M} \sum_{\alpha \in T_{\varepsilon,\eta}^k} \exp \left( H_{\text{TAP}}^\text{sph}(m_\alpha) \right) , \]

for the contribution of the first set on the RHS of (2.47) to the partition function. Seemingly, the problem remains that we have no bound for the contribution \( Z_N(\bigcup_{\alpha \in T_{\varepsilon,\eta}^{M+1}} D_\alpha) \) of the second set on the RHS of (2.47). But we now argue that for large enough \( M \), the sets \( D_\alpha, \alpha \in T_{\varepsilon,\eta}^{M+1} \) are in fact empty. This is because if \( \alpha \in T_{\varepsilon,\eta}^{M+1} \) then in the construction we have the recursively defined basis from (2.42) such that if \( \sigma \in D_\alpha \) then \[ |\langle \sigma, u_{\alpha,k} \rangle| \geq \alpha_k \geq \frac{2}{M} \] for \( k = 1, \ldots, M \) (recall (2.43) and (2.10)). This implies that \( \|\sigma\|^2 \geq M^{\frac{2}{4}} \). Recall that \( D_\alpha \subset S_{N-1} \), so any \( \sigma \in D_\alpha \) satisfies \( \|\sigma\|^2 = 1 \). Thus if \( M = c\eta^{-2} \) for \( c \) large enough so that \( M^{\frac{2}{4}} > 1 \) then we have \( D_\alpha = \emptyset \), and in fact

(2.49)

\[ S_{N-1} = \bigcup_{0 \leq k \leq c\eta^{-2}} \bigcup_{\alpha \in T_{\varepsilon,\eta}^k} \mathcal{E}_\alpha . \]
It will then follow from (2.48) that

\[
Z_N(S_{N-1}) \leq e^{c(\eta^{-2} \varepsilon + \eta)N} \sum_{0 \leq k \leq cn^{-2}} \sum_{\alpha \in I_{k,\eta}} \exp \left( H_{\text{TAP}}^{\text{sph}}(m) \right).
\]

As the number of summands is bounded in \(N\) this will imply that

\[
Z_N(S_{N-1}) \leq \exp \left( \sup_m H_{\text{TAP}}^{\text{sph}}(m) + c(\eta^{-2} \varepsilon + \eta)N + o(N) \right).
\]

Since first \(\eta\) and then \(\varepsilon\) can be made arbitrarily small this is the desired bound, cf. (1.10). This sketch can be formalized to yield a proof of Theorem 1.2 in the case of linear external field.

2.2. General spike term. It is straight-forward to adapt the argument to a general Lipschitz spike term \(f_N(\sigma) = f_N(P^U\sigma)\) for a linear subspace \(U\) of dimension \(K\). One starts the iteration with a set of \(\| \cdot \|\)-orthonormal initial directions \(u_1, \ldots, u_K\) whose span is \(U\), rather than just one initial direction \(u_1\), and in the first step decomposes \(S_{N-1}\) into sets

\[
D_{(\alpha_1)} = \{ \sigma \in S_{N-1} : \langle \sigma, u_l \rangle \in (|\alpha_{1,l}|, |\alpha_{1,l}| + \varepsilon) \times \text{sign}(\alpha_l), l = 1, \ldots, K \},
\]

for \(\alpha_1 = (\alpha_{1,1}, \ldots, \alpha_{1,K}) \in (-1,1)^K\) instead of (2.9), and defines

\[
m_{(\alpha_1)} = \alpha_{1,1}u_1 + \ldots + \alpha_{1,K}u_K,
\]

instead of (2.15). Instead of (2.12) one now has

\[
Z_N(D_{(\alpha_1)}) \leq E \left[ 1_{D_{(\alpha_1)}} \exp \left( \beta H_N(\sigma) \right) \right] e^{\beta f_N(m_{(\alpha_1)})+\varepsilon KN},
\]

by the Lipschitz assumption on \(f_N\). The subsequent directions \(u_{(\alpha_1),2}, u_{(\alpha_1),3}, \ldots\) are constructed one at a time just as in Section 2.1 except that they are chosen orthogonal to all of \(u_1, \ldots, u_K\) and not just \(u_1\), and the equators \(E_{\alpha}\) are constructed by the same formula (2.44) for \(k \geq 1\). In this way one obtains

\[
S_{N-1} = \bigcup_{1 \leq k \leq cn^{-2}} \bigcup_{\alpha \in I_{k,\eta} \times \mathbb{I}^{k-1}_{\varepsilon,\eta}} E_{\alpha},
\]

rather than (2.49) and

\[
Z_N(S_{N-1}) \leq e^{c(\eta^{-2} \varepsilon + \eta)N} \sum_{1 \leq k \leq cn^{-2}} \sum_{\alpha \in I_{k,\eta} \times \mathbb{I}^{k-1}_{\varepsilon,\eta}} \exp \left( H_{\text{TAP}}^{\text{sph}}(m) \right),
\]

rather than (2.50), which implies (2.51) also for a general spike term. This sketch can be formalized to yield a proof of Theorem 1.2 for such a general spike.
2.3. Ising (or general) reference measure $E$. In the sketch above the TAP energy for the spherical model arose essentially because the entropy estimate (2.28) holds for the spherical reference measure $E$. If $E$ is e.g. the Ising reference measure (uniform measure on $\{-1, 1\}^\mathbb{N}$) then to obtain an upper bound like (2.30) with $H_{\text{TAP}}^{\text{Ising}}$ instead of $H_{\text{TAP}}^{\text{sph}}$ one should instead have that $E[\mathcal{E}_\alpha]$ is at most
\[
\exp (I_{\text{Ising}}(m)) = \exp \left(- \sum_{i=1}^{N} J(m_i) + o(N) \right),
\]
where $J$ is the binary entropy from (1.6). Such an upper bound however does not hold for sets $\mathcal{E}_\alpha$ as defined in e.g. (2.25) of the sketch above: a simple demonstration of this phenomenon is the fact that
\[
E[\{\sigma : \langle \sigma - m, \lambda \rangle \approx 0 \}] \gg \exp \left(- \sum_{i=1}^{N} J(m_i) \right),
\]
for $E$ the uniform measure on $\{-1, 1\}^\mathbb{N}$, except when $m_1 = \ldots = m_N$. However, it can be shown (under appropriate technical conditions) that there exists a unit vector $\lambda = \lambda_m$ such that
\[
E[\{\sigma : \langle \sigma - m, \lambda \rangle \approx 0 \}] \leq \exp \left(- \sum_{i=1}^{N} J(m_i) + o(N) \right),
\]
(see Lemma 7.2). This suggests the way to extend the sketch in Sections 2.1-2.2 above to Ising reference measures: in each step of the iteration one must decompose not only as in (2.34) and (2.41) in one new direction $u_{\alpha,k+1} = u_{\alpha,k+1,1}$ whose span with the previous vectors includes $\nabla H_N(m_{\alpha,k})$; rather one must include a second direction $u_{\alpha,k+1,2}$ whose span with the others includes also $\lambda_m$. Thus we let each $\alpha_k, k \geq 2$, be a pair of numbers in $I_{\varepsilon,\eta}^2$ that dictate an increment in span$(u_{\alpha,k,1}, u_{\alpha,k,2})$ so that
\[
m_{\alpha,k+1} = m_{\alpha,k} + \alpha_{k,1} u_{\alpha,k+1,1} + \alpha_{k,2} u_{\alpha,k+1,2},
\]
and define the sets $D_\alpha$ for $\alpha \in I_{\varepsilon,\eta}^K \times (I_{\varepsilon,\eta}^2)^{k-1}$ with respect to both the directions as
\[
D_\alpha = \{\sigma \in D_{(\alpha_1)} : \langle \sigma, u_{\alpha,l,j} \rangle \in ([\alpha_{l,j}], [\alpha_{l,j}] + \varepsilon) \times \text{sign}(\alpha_{l,j}), l = 2, \ldots, k, j = 1, 2\},
\]
(for $D_{(\alpha_1)}$ as in (2.52)) and similarly for the the equators
\[
\mathcal{E}_\alpha = \{\sigma \in D_\alpha : |\langle \sigma - m_{\alpha}, u_{\alpha,k+1,j} \rangle| \leq \eta \text{ for } j = 1, 2\}
\]
(cf. (2.44)). The equator $\mathcal{E}_\alpha$ is then contained in $\{\sigma : \langle \sigma - m_{\alpha}, \lambda_{m_{\alpha}} \rangle \approx 0 \}$, which means that by (2.54)
\[
E[\mathcal{E}_\alpha] \leq \exp \left(- \sum_{i=1}^{N} J((m_{\alpha})_i) + o(N) \right) = \exp (I_{\text{Ising}}(m_{\alpha}) + o(N))
\]
instead of (2.45), and then
\[ Z_N (E_\alpha) \leq \exp \left( \beta H^f_N (m_\alpha) + I_{\text{Ising}} (m_\alpha) + \frac{\beta^2}{2} \text{On} \left( \| m_\alpha \|^2 \right) + c (M \varepsilon + \eta) N \right), \]

(1.4) instead of (2.46). In this way the sketch in Sections 2.1-2.2 above can be modified to yield a proof of the inequality
\[ Z_N (S_{N-1}) \leq e^{(\eta^{-2} \varepsilon + \eta) N} \sum_{1 \leq k \leq \eta^{-2}} \sum_{\alpha \in I^{k}_{\varepsilon} \times I^{(k-1)}_{\varepsilon}} \exp \left( H^\text{Ising}_{\text{TAP}} (m_\alpha) \right), \]

instead of (2.53). This then implies (2.51) with \( H^\text{Ising}_{\text{TAP}} \) instead of \( H^\text{sph}_{\text{TAP}} \). This sketch can be formalized into a direct proof of Theorem 1.1.

Alternatively - and this is the approach taken below - the sketch can be turned into a formal proof of the TAP upper bound Theorem 1.3 for a general reference measure \( E \) by defining as in (1.11) the entropy term \( I_{E, \delta} (m) \) of the general TAP energy essentially as the LHS of (2.54) minimized over \( \lambda \). From this general result the spherical and Ising bounds can be derived by bounding \( I_{E, \delta} \) above by \( I_{\text{sph}} \) and \( I_{\text{Ising}} \) respectively.

3. The recentered Hamiltonian and its law

In this section we formally define the recentered Hamiltonian and study its law.

Recall that \( H_N (\sigma), \sigma \in B_N \) is a Gaussian process with covariance given by (1.1) for a power series \( \xi \) with non-negative coefficients and \( \xi (1) < \infty \). By Lemma A.10 in the appendix the process \( H_N (\sigma), \sigma \in B_N, \) exists and is almost surely differentiable on \( B_N^e \).

For any \( m \in B_N \) and \( \hat{\sigma} \in \mathbb{R}^N \) with \( \| \hat{\sigma} \|^2 \leq 1 - \| m \|^2 \) we define the recentered Hamiltonian by
\[ H^m_N (\hat{\sigma}) = H_N (m + \hat{\sigma}) - \nabla H_N (m) \cdot \hat{\sigma} - H_N (m), \]
so that
\[ H_N (m + \hat{\sigma}) = H_N (m) + \nabla H_N (m) \cdot \hat{\sigma} + H^m_N (\hat{\sigma}). \]

We furthermore define the effective external field
\[ h_{\text{eff}} (m) = \nabla H_N (m). \]

Note that we then have
\[ H^f_N (m + \hat{\sigma}) = H^f_N (m) + h_{\text{eff}} (m) \cdot \hat{\sigma} + H^m_N (\hat{\sigma}) + (f_N (m + \hat{\sigma}) - f_N (m)), \]

and since \( f_N (\sigma) = f_N (P^U \sigma) \) we have for \( \hat{\sigma} \in U^\perp \)
\[ H^f_N (m + \hat{\sigma}) = H^f_N (m) + h_{\text{eff}} (m) \cdot \hat{\sigma} + H^m_N (\hat{\sigma}). \]

Remark 3.1. If \( f_N \) is differentiable its recentering \( f^m_N \) and the recentering \( (H^f_N)^m \) of \( H^f_N \) are well-defined. Then an elegant alternative way to write this decomposition is to define
\[ h_{\text{eff}} (m) = \nabla H^f_N (m) = \nabla H_N (m) + \nabla f_N (m), \]
so that (3.4) can be replaced by

\[(3.5) \quad H^I_N (m + \hat{\sigma}) = H^I_N (m) + h_{\text{eff}} (m) \cdot \hat{\sigma} + \left( H^I_N \right)^m (\hat{\sigma}).\]

Ultimately in the proof we will apply (3.4) only when \(P^U \hat{\sigma}\) is small enough to make \(f_N (m + \hat{\sigma}) - f_N (m), f_N^m (\hat{\sigma})\) and \(\nabla f_N (m) \cdot \hat{\sigma}\) all negligible. As the benefit of writing (3.5) is purely aesthetic, and it has the drawback of requiring the assumption that \(f_N\) is differentiable rather than Lipschitz we use (3.4) instead.

If \(r > 0\) and \(\Sigma \subset B_N (r)\) we say that a centered Gaussian process \((g (\sigma))_{\sigma \in \Sigma}\) has covariance function \(\xi : [-r^2, r^2] \rightarrow [0, \infty)\) if

\[(3.6) \quad \mathbb{E} [g (\sigma) g (\sigma')] = N \xi (\langle \sigma, \sigma' \rangle) \text{ for all } \sigma, \sigma' \in \Sigma.\]

If \(V \subset \mathbb{R}^N\) is a linear space, \(\Sigma = V \cap B_N (r)\) and \(g\) is differentiable on \(V \cap B^\circ_N (r)\) we define the recentering of \(g^m\) (generalizing (3.1)) by

\[(3.7) \quad g^m (\hat{\sigma}) = g (m + \hat{\sigma}) - \nabla g (m) \cdot \hat{\sigma} - g (m) \text{ for } \hat{\sigma} \in \Sigma - m,\]

where the gradient \(\nabla g (m)\) is understood to be a vector in \(V\). If \(\xi : [-r^2, r^2] \rightarrow [0, \infty)\) is a covariance function and \(q \in [0, r^2]\) we define

\[(3.8) \quad \xi_q : [- (r^2 - q), r^2 - q] \rightarrow [0, \infty), \quad \xi_q (z) = \xi (q + z) - \xi (q) z - \xi (q).\]

The next lemma shows that the components of the decomposition in (3.7) are independent on the “slice” perpendicular to \(m\), and gives the law of the recentered process \(g^m\) on the slice. It follows by computing covariances of \(g\) and its partial derivatives, and the definition (3.7). Let \(\partial_r\) denote the radial derivative.

**Lemma 3.2.** Let \(N \geq 1, r > 0, V \subset \mathbb{R}^N\) be a linear space and let \((g (\sigma))_{\sigma \in \Sigma}\) for \(\Sigma = V \cap B_N (r)\) be a centered Gaussian field with covariance function \(\xi\), differentiable on \(V \cap B^\circ_N (r)\). For any \(m \in \Sigma\) let

\[(3.9) \quad \tilde{\Sigma} = \{ \hat{\sigma} \in \Sigma - m : \hat{\sigma} \cdot m = 0 \} = \tilde{V} \cap B_N \left( \sqrt{r^2 - \|m\|^2} \right),\]

for the linear space \(\tilde{V} = V \cap \{ \hat{\sigma} : \hat{\sigma} \cdot m = 0 \}\). Then the three objects

\[(3.10) \quad \left( g (m), \partial_r g (m) \right), \quad P^{\text{span}(m)^\perp} \nabla g (m), \quad (g^m (\hat{\sigma}))_{\hat{\sigma} \in \tilde{\Sigma}} \text{ are independent},\]

and \((g^m (\hat{\sigma}))_{\hat{\sigma} \in \tilde{\Sigma}}\) is a centered Gaussian process with covariance function \(\xi_{\|m\|^2}\).

**Proof.** The second equality in (3.9) is elementary. Since the covariance in (3.6) depends only on the inner product we can by rotating \(\mathbb{R}^N\) assume w.l.o.g. that \(V\) is the span of the first \(\dim (V)\) basis vectors and \(m\) is a multiple of \((1, \ldots, 0)\). We then use the covariance formulas for the partial derivatives of \(g\) from Lemma A.10. Since \(m_i = 0\) for \(i \geq 2\) one obtains from (A.19) that \(\partial_i g, i = 2, \ldots, \dim (V)\) are IID with variance \(\xi' (\|m\|^2)\), and that \(\partial_i g (m) = \partial_i g (m)\) is independent of \(\partial_i g, i = 2, \ldots, \dim (V)\). Also for \(\hat{\sigma}\) with \(\hat{\sigma} \cdot m = 0\) we obtain from (A.20)

\[(3.11) \quad \mathbb{E} [g (m + \hat{\sigma}) \partial_i g (m)] = (|m| \delta_{i1} + \hat{\sigma}_i) \xi' (\|m\|^2) \text{ for } i = 1, \ldots, \dim (V).\]
This implies that \( g(m) \) is independent of \( \mathcal{P}^{\text{span}(m)^\perp} \nabla g(m) \) by considering the case \( \hat{\sigma} = 0 \).
Thus we have shown that \((g(m), \partial_r g(m))\) and \( \mathcal{P}^{\text{span}(m)^\perp} \nabla g(m) \) are independent. Also any \( \hat{\sigma} \) perpendicular to \( m \)
\[
\mathbb{E}[g^m(\hat{\sigma})g(m)] = \mathbb{E}[(g(m + \hat{\sigma}) - \nabla g(m) \cdot \hat{\sigma} - g(m))g(m)]
\]
and similarly for \( i = 1, \ldots, \dim(V) \)
\[
\mathbb{E}[g^m(\hat{\sigma})\partial_i g(m)] = (|m| \delta_{i1} + \hat{\sigma}_i)\xi'(\|m\|^2) - \hat{\sigma}_i \xi'(\|m\|^2) - |m| \delta_{i1} \xi'(\|m\|^2) = 0.
\]
Thus the three objects in (3.10) are indeed independent.
Lastly one gets exploiting the independencies that for all \( \hat{\sigma}, \hat{\sigma}' \) perpendicular to \( m \),
\[
\mathbb{E}[g^m(\hat{\sigma})g^m(\hat{\sigma}')] = \mathbb{E}[g(m + \hat{\sigma})g^m(\hat{\sigma}')] = N\xi(\|m\|^2 + (\hat{\sigma}, \hat{\sigma}')) - N\xi'(\|m\|^2)(\hat{\sigma}, \hat{\sigma}') - N\xi(\|m\|^2)
\]
which shows that \((g^m(\hat{\sigma}))_{\hat{\sigma} \in \Sigma}\) is a centered Gaussian process with covariance function \( \xi_{\|m\|^2} \) as claimed.

Later we will use the following natural relations for recentered processes which follow directly from the definition (3.7) and hold for any \( N \geq 1, r > 0 \) and any process \( g : V \cap B_N^r \to \mathbb{R} \), differentiable on \( V \cap B_N^r \), and any \( a, b \in V \cap B_N^r \), \( \hat{\sigma} \in V \cap B_N \) such that \( a + b \in V \cap B_N \), \( a + b + \hat{\sigma} \in V \cap B_N \):
\[
(3.12) \quad \nabla g^a(b) = \nabla g(a + b) - \nabla g(a),
\]
(where the gradient is understood to be a vector in \( V \)) and
\[
(3.13) \quad (g^a)^b = g^{a+b}.
\]
Similarly it follows from (3.8) that
\[
(3.14) \quad (\xi_q)_{q'} = \xi_{q+q'} \text{ for all } \xi, q, q'.
\]

4. Iterative construction of magnetizations

In this section we iteratively construct magnetizations in \( B_N^0 \) which will be used in the next section be used to construct a cover of \( S_{N-1} \).

Let
\[
(4.1) \quad I_\varepsilon = \varepsilon \mathbb{Z} \cap (-1, 1),
\]
be a grid of evenly spaced points at distance \( \varepsilon \) covering \((-1, 1)\). For \( \varepsilon > 0 \) we “round down” numbers in \( \mathbb{R} \) to numbers in \( \varepsilon \mathbb{Z} \) with the operation
\[
(4.2) \quad [x]_\varepsilon = \begin{cases} 
\text{the } y \in \varepsilon \mathbb{Z} \text{ s.t. } x \in (y, y + \varepsilon) & \text{if } x > 0, \\
0 & \text{if } x = 0, \\
\text{the } y \in \varepsilon \mathbb{Z} \text{ s.t. } x \in [y - \varepsilon, y] & \text{if } x < 0.
\end{cases}
\]
Note that then
\[(4.3) \quad |x - [x]|_\epsilon \leq \varepsilon, |x| \geq ||x||_\epsilon \quad \text{for all} \quad x \in \mathbb{R} \quad \text{and} \quad |x| > ||x||_\epsilon \quad \text{for all} \quad x \in \mathbb{R} \setminus \{0\}.
\]

Next we define the space of increments that will be used in the construction of the cover. Recall that $K$ is the dimension of the linear space $U \subset \mathbb{R}^N$.

**Definition 4.1** (Space of increments). Define for each $1 \leq k \leq \frac{N-K}{2} + 1$ and $\varepsilon > 0$
\[(4.4) \quad A_k = \left\{ \alpha \in I^K_{\varepsilon} \times (I^2_{\varepsilon})^{k-1} : |\alpha| < 1 \right\},
\]
(\text{where} \quad |\alpha| = \sum_{i=1}^{k} |\alpha_k|^2 = \sum_{i=1}^{k} \sum_{j=1}^{A_i} \alpha_{k,j}^2 \quad \text{for} \quad A_1 = K \quad \text{and} \quad A_l = 2 \quad \text{for} \quad l \geq 2),
\]
with the convention that $(I^2_{\varepsilon})^0 = \{()\}$, where () is the sequence of length 0. Also let
\[A = \bigcup_{1 \leq k \leq \frac{N-K}{2} + 1} A_k.
\]

In the decomposition of the sphere we will decompose in a direction that is roughly speaking the normal of the hyperplane passing though $m$ that has minimal measure under $E$ (cf. (1.11), Section 2.3). We now define this normal $\lambda^\delta_m$ for any $m$.

**Definition 4.2** (Minimal entropy hyperplane around $m$). For any $m \in B_N$, $\lambda \in \mathbb{R}^N$ and $\delta > 0$ let
\[r_\delta (m, \lambda) = \log E [\langle \lambda, \sigma - m \rangle \geq -\delta],
\]
and
\[(4.5) \quad \lambda^\delta_m \in \left\{ \lambda \in S_{N-1} : r_\delta (m, \lambda) \leq \inf_{\bar{\lambda} \in \mathbb{R}^N : ||\bar{\lambda}|| = 1} r_\delta (m, \bar{\lambda}) + \delta \right\},
\]
be picked according to some arbitrary $m$-measurable rule. \hspace{1cm} \Box

Note that
\[(4.6) \quad \log E [\langle \lambda^\delta_m, \sigma - m \rangle \geq -\delta] \leq \inf_{\lambda : ||\lambda|| = 1} r_\delta (m, \lambda) + \delta \overset{\text{(1.11)}}{\leq} I_{E,\delta} (m) + \delta.
\]

**Remark 4.3.** For natural measures $E$ (like the uniform measure on $\{-1,1\}^N$ or $S_{N-1}$) the infimum in $\inf_{\lambda : ||\lambda|| = 1} r_\delta (m, \lambda)$ is achieved, and we could simply define $\lambda^\delta_m$ as a minimizer. Furthermore for $E$ uniform on $S_{N-1}$ and $m \neq 0$ the infimum is uniquely attained and in this case the parameter $\delta$ is unnecessary and one could define
\[(4.7) \quad \lambda^\delta_m = \lambda_m = \frac{m}{||m||}.
\]
In general however the infimum may not be achieved, and hence we define $\lambda^\delta_m$ in (4.5) as a vector that almost achieves it. \hspace{1cm} \Box
The next definition gives an iterative construction of a magnetization vector \( m_\alpha \) from a vector \( \alpha \in \mathcal{A}_k \) of increment magnitudes lying in \( I_\varepsilon \). It also constructs an associated subspace \( \overrightarrow{V}_\alpha \) of \( \sigma \) for which \( \sigma \cdot m_\alpha = 0 \). The crucial aspect of the construction is that the next set of increments for the magnetization covers the whole span of the gradient of the Hamiltonian and the normal of the hyperplane of smallest entropy, both evaluated at the current point of the iteration. Furthermore the increments are orthogonal to the previous increments, and the subspace \( V_\alpha \) consists of vectors orthogonal to all previous increments and to the gradient and minimal entropy hyperplane normal of \( m_\alpha \).

Let
\[
(4.8) \quad u_1, \ldots, u_K \in \mathbb{R}^N \text{ be an } \langle \cdot, \cdot \rangle \text{-orthonormal basis of } U,
\]
(chosen according to some arbitrary rule).

**Definition 4.4** (Magnetization \( m_\alpha \) and basis \( u_{\alpha,l,j} \) associated to \( \alpha \in \mathcal{A}_k \)).

Let \( \varepsilon, \delta > 0 \), \( 1 \leq k \leq \frac{N-K}{2} + 1 \) and \( \alpha \in \mathcal{A}_k \). We define \((m_{\alpha,l})_{1 \leq l \leq k}\) and \((u_{\alpha,l})_{1 \leq l \leq k+1}\)
where \( u_{\alpha,1} = (u_{\alpha,1,1}, \ldots, u_{\alpha,1,K}) \) and \( u_{\alpha,l} = (u_{\alpha,l,1}, u_{\alpha,l,2}) \) for \( l \geq 2 \) and \( u_{\alpha,l,j} \in \mathbb{R}^N \), via a recursion. It starts with
\[
(4.9) \quad u_{\alpha,1,j} = u_j \text{ for } j = 1, \ldots, K,
\]
and
\[
(4.10) \quad U_{\alpha,1} = \text{span} \left( u_{\alpha,1,1}, \ldots, u_{\alpha,1,K} \right) \text{ and } V_{\alpha,1} = U_{\alpha,1}^\perp,
\]
and
\[
(4.11) \quad m_{\alpha,1} = \sum_{j=1}^{K} \alpha_{1,j} u_{\alpha,1,j}.
\]

For the first step, consider the space
\[
(4.12) \quad \text{span} \left( \nabla H_N \left( m_{\alpha,1} \right), \lambda_\delta^{\varepsilon} m_{\alpha,1} \right) \cap V_{\alpha,1}.
\]
Let \( u_{\alpha,2,1}, u_{\alpha,2,2} \) be an \( \langle \cdot, \cdot \rangle \)-orthonormal basis of this space, or (if its dimension is less than two) of an arbitrary two-dimensional subspace of \( V_{\alpha,1} \) containing it, and let
\[
(4.13) \quad U_{\alpha,2} = \text{span} \left( U_{\alpha,1}, u_{\alpha,2,1}, u_{\alpha,2,2} \right) \text{ and } V_{\alpha,1} = U_{\alpha,1}^\perp.
\]
Then let
\[
(4.14) \quad m_{\alpha,2} = m_{\alpha,1} + \alpha_{2,1} u_{\alpha,2,1} + \alpha_{2,2} u_{\alpha,2,2}.
\]
For \( 3 \leq l \leq k \) consider the space
\[
(4.15) \quad \text{span} \left( \nabla H_N \left( m_{\alpha,l-1} \right), \lambda_\delta^{\varepsilon} m_{\alpha,l-1} \right) \cap V_{\alpha,l-1}.
\]
Let \( u_{\alpha,l,1}, u_{\alpha,l,2} \) be an \( \langle \cdot, \cdot \rangle \)-orthonormal basis of a two-dimensional subspace of \( V_{\alpha,l-1} \) containing \( (4.15) \), and
\[
(4.16) \quad U_{\alpha,l} = \text{span} \left( U_{\alpha,1}, u_{\alpha,r,1}, u_{\alpha,r,2} : 2 \leq r \leq l \right) \text{ and } V_{\alpha,l} = U_{\alpha,l}^\perp.
\]
Next, let
\[
(4.17) \quad m_{\alpha,l} = m_{\alpha,l-1} + \alpha_{l,1} u_{\alpha,l,1} + \alpha_{l,2} u_{\alpha,l,2},
\]
This recursively defines \((m_{\alpha,k})_{1 \leq l \leq k}\) and \((u_{\alpha,l})_{1 \leq l \leq k}\).

Define
\begin{equation}
(4.18) \quad m_{\alpha} = m_{\alpha,k},
\end{equation}
and
\begin{equation}
(4.19) \quad U_{\alpha} = U_{\alpha,k} \text{ and } V_{\alpha} = V_{\alpha,k} = U_{\alpha}^\perp.
\end{equation}
Finally define \(u_{\alpha,k+1,1}, u_{\alpha,k+1,2}\) as an \(\langle \cdot, \cdot \rangle\)-orthonormal basis of a two-dimensional subspace of \(V_{\alpha}\) containing
\begin{equation}
(4.20) \quad \text{span } (\nabla H_N (m_{\alpha}), \lambda_{m_{\alpha}}^i) \cap V_{\alpha},
\end{equation}
and let \(u_{\alpha,k+1} = (u_{\alpha,k+1,1}, u_{\alpha,k+1,2})\) and
\begin{equation}
(4.21) \quad \overline{U}_{\alpha} = U_{\alpha,k+1} = \text{span } (U_{\alpha}, u_{\alpha,k+1,1}, u_{\alpha,k+1,2}) \quad \text{and } \quad \overline{V}_{\alpha} = V_{\alpha,k+1} = U_{\alpha}^\perp.
\end{equation}

Remark 4.5.

(1) For pedagogical reasons we carried out the first step of the iteration in \((4.12)-(4.14)\) explicitly, but note that it’s precisely the general step from \((4.15)-(4.17)\) for \(l = 2\). Thus \((4.15)-(4.17)\) in fact hold for \(2 \leq l \leq k\), and not just \(3 \leq l \leq k\).

(2) Whenever a basis of a space \(\text{span } (a, b)\) for random vectors \(a, b\) is chosen above, it is understood to be chosen by a rule which is measurable with respect to \(a, b\). □

We collect here some simple consequences of this construction which will be used in the proofs that follow. Recalling \((4.8), (4.15)\) and the sentence below it, note that
\begin{equation}
(4.22) \quad u_{\alpha,l,j} \text{ for } 1 \leq l \leq k + 1, 1 \leq j \leq \mathcal{A}_l, \text{ are } \langle \cdot, \cdot \rangle\text{-orthonormal for each } \alpha \in \mathcal{A}_k,
\end{equation}
and from also \((4.19)\)
\begin{equation}
(4.23) \quad U_{\alpha} = \text{span } (u_{\alpha,l,j} : 1 \leq l \leq k, 1 \leq j \leq \mathcal{A}_l), \quad V_{\alpha} = U_{\alpha}^\perp.
\end{equation}
We have for all \(\alpha \in \mathcal{A}_k\) and \(1 \leq l \leq k\) that
\begin{equation}
(4.24) \quad m_{\alpha,l} \overset{(4.11),(4.17)}{=} \sum_{r=1}^{l} \sum_{j=1}^{\mathcal{A}_j} \alpha_{r,j} u_{\alpha,r,j} \overset{(4.4),(4.22)}{\in} B_N^\circ,
\end{equation}
so that
\begin{equation}
(4.25) \quad m_{\alpha,l} \in U_{\alpha,l} = V_{\alpha,l}^\perp \text{ for } l = 1, \ldots, k,
\end{equation}
and in particular (recall \((4.18)\))
\begin{equation}
(4.26) \quad m_{\alpha} = \sum_{l=1}^{k} \sum_{j=1}^{\mathcal{A}_l} \alpha_{l,j} u_{\alpha,l,j} \in B_N^\circ,
\end{equation}
and (recall \((4.19)\))
\begin{equation}
(4.27) \quad m_{\alpha} \in U_{\alpha,k} = U_{\alpha}.
\end{equation}
Also
\begin{equation}
\nabla H_N (m_{\alpha,l}), \lambda^\delta_{m_{\alpha,l}} \in U_{\alpha,l+1} = V^\perp_{\alpha,l+1}
\end{equation}
(see (4.15) and below it, (4.20) and above it and (4.23)) and in particular
\begin{equation}
h_{\text{eff}} (m_{\alpha,l}) = \nabla H_N (m_{\alpha,l}), \lambda^\delta_{m_{\alpha,l}} \in \overline{U}_{\alpha,l} = \overline{V}^\perp_{\alpha,l},
\end{equation}
where (see (4.21))
\begin{equation}
\overline{U}_{\alpha,l} = U_{\alpha,k+1} = \text{span} (u_{\alpha,l,j} : 1 \leq l \leq k, 1 \leq j \leq A_l) \supset U_{\alpha,l}
\end{equation}
and the inclusions are strict. Also
\begin{equation}
\mathbb{R}^N = \overline{U}_{\alpha} \oplus \overline{V}_{\alpha}.
\end{equation}
Furthermore, note from (4.17) that for each \( l \leq k \) the vector \( m_{\alpha,l} \) depends on \( \alpha \in \mathcal{A}_k \) only through \( (\alpha_l)_{1 \leq r \leq l} \) and \( u_{\alpha,l} \) depends on \( \alpha \) only through \( (\alpha_l)_{1 \leq r \leq l-1} \). Thus the bases are “nested” in the sense that for all \( 1 \leq l \leq (k+1) \land (k'+1) \)
\begin{equation}
\text{if } \alpha \in \mathcal{A}_k, \alpha' \in \mathcal{A}_{k'} \text{ with } \alpha_l = \alpha'_l \text{ for } 1 \leq r \leq l-1,
\end{equation}
\[ u_{\alpha,l,j} = u_{\alpha',l,j} \text{ for } 1 \leq j \leq A_l. \]
Lastly (from below (4.15) and above (4.20)) for \( l = 2, \ldots, k+1 \)
\begin{equation}
u_{\alpha,l} \text{ is measurable wrt. to } m_{\alpha,l-1} \text{ and } P^{V_{\alpha,l-1}} \nabla H_N (m_{\alpha,l-1}).
\end{equation}

**Remark 4.6.**
(1) Note that if we were treating only the case when \( E \) is the uniform measure on the sphere then we could replace \( \lambda^\delta_m \) with \( \lambda_m = \frac{m}{|m|} \) in the construction, cf. (4.7). Since \( m_{\alpha,l} \in U_{\alpha,l} \) the spaces (4.15), (4.20) then always have dimension at most 1. Thus in the spherical case the above construction could be simplified by omitting \( \alpha_l, \hat{\alpha}_l \) for all \( l \geq 2 \) and defining \( \mathcal{A}_k \) as a subset of \( I^K_{\varepsilon} \times I^{k-1}_{\varepsilon} \) rather than as in (4.4) (cf. the sketch in Sections 2.1, 2.2).

(2) Though we find it more convenient not to use this fact, it is easy to see that \( \nabla H_N (m_{\alpha,l}) \in U_{\alpha,l} \) occurs with probability 0 (except when \( m_{\alpha,l} = 0 \)), so that with probability one the spaces (4.15), (4.20) have dimension at least 1. \( \square \)

Define
\begin{equation}
q_{\alpha} = \|m_{\alpha}\|^2 (4.22),(4.26) |\alpha|^2.
\end{equation}
Another important feature of the construction is that conditionally on \( u_{\alpha,l}, l = 1, \ldots, k+1 \), the law of \( H^{m_{\alpha}}_N (\hat{\alpha}) \) restricted to \( \overline{V}_{\alpha} \) is that of a centered Gaussian process with covariance function \( \xi_{q_{\alpha}} \). For \( \alpha \in \mathcal{A}_1 \) this follows from Lemma 3.2 with \( g = H_N \) and \( m = m_{\alpha} = m_{\alpha,1} \). For \( \alpha \in \mathcal{A}_k, k \geq 2 \), it essentially follows by repeatedly applying Lemma 3.2, as we now show. Let
\begin{equation}
\mathcal{R}_\alpha = \sigma (P^{V_{\alpha,l}} \nabla H_N (m_{\alpha,l}), l = 1, \ldots, k; u_l, l = 2, \ldots, k+1).
\end{equation}
Note that under \( \mathbb{P} (\cdot | \mathcal{R}_\alpha) \) the objects \( \nabla_{\alpha}, m_{\alpha} \) are deterministic (recall (4.26), (4.30)).
Lemma 4.7. For $1 \leq k \leq \frac{N - K}{2} + 1$ and $\alpha \in A_k$ almost surely the $\mathbb{P}(\cdot | \mathcal{R}_\alpha)$-law of $(H_N^m(\hat{\sigma}))_{\sigma \in B_N(\sqrt{1 - q_{a,l}})}$ is that of a centered Gaussian process with covariance function $\xi_{q_{a,l}}$.

Proof. Let for $l = 0, \ldots, k$,

$$R_{\alpha,l} = \sigma \left( P V_{a,r} \nabla H_N(m_{a,r}) , r = 1, \ldots, l; u_r, r = 2, \ldots, l + 1 \right),$$

and

$$\Sigma_l = V_{a,l+1} \cap B_N(\sqrt{1 - q_{a,l}})$$

where $q_{a,l} = \|m_{a,l}\|^2$, with the convention $m_{a,0} = 0$ and $q_{a,0} = 0$. We prove by induction that for $l = 0, \ldots, k$

$$(4.36) \mathbb{P}(\cdot | \mathcal{R}_{\alpha,l})$$

is a CGPD($\Sigma_l, \xi_{q_{a,l}}, q_{a,l}$)

where “CGPD($\Sigma, \xi, q$)” is a shorthand for “centered Gaussian process with index set $\Sigma \subset B_N(\sqrt{1 - q})$ and covariance function $\xi$, differentiable on $\Sigma \cap B_N^c(\sqrt{1 - q})$.” The case $l = k$ implies the claim of the lemma.

The case $l = 0$ of (4.36) holds trivially (note that $H_N^0(\sigma) = H_N(\sigma)$ for all $\sigma$, that $R_{\alpha,0}$ is the trivial $\sigma$-algebra and recall that $V_{a,1}$ is deterministic).

To prove the induction step we will use that for $s = 0, \ldots, k - 1$

$$(4.37) \nabla H_N(m_{a,s+1}) - \nabla H_N(m_{a,s}) = \nabla H_N^{m_{a,s}}(m_{a,s+1} - m_{a,s}),$$

which follows from (3.12)

$$(4.38) H_N^{m_{a,s}}(\sigma) = (H_N^m(\hat{\sigma}))^{m_{a,s}}(m_{a,s})$$

which follows from (3.13), both with $V = \mathbb{R}^N, g = H_N, a = m_{a,s}, b = m_{a,s+1} - m_{a,s}$.

Now assume that (4.36) holds for $l = s$. Note that $m_{a,s}, u_{a,s+1}, m_{a,s+1} + V_{a,s+1}, \Sigma_s$ are deterministic under $\mathbb{P}(\cdot | \mathcal{R}_{\alpha,s})$. Consider the process $g(\hat{\sigma}) = H_N^{m_{a,s}}(\hat{\sigma})$ for $\hat{\sigma} \in \Sigma$ under $\mathbb{P}(\cdot | \mathcal{R}_{\alpha,s})$. By applying Lemma 3.2 to it with $r = \sqrt{1 - q_{a,s}}, \tilde{V} = V_{a,s+1}$ and

$$m = m_{a,s+1} - m_{a,s} \stackrel{(4.17)}{=} \text{span}(u_{a,s+1,1}, u_{a,s+1,2}),$$

under $\mathbb{P}(\cdot | \mathcal{R}_{\alpha,s})$ and using that

$$q_{a,s} + \|m\|^2 \stackrel{(4.22),(4.24)}{=} q_{a,s+1}$$

we obtain that the process $(g^m(\hat{\sigma}), \hat{\sigma} \in \Sigma)$ for

$$\Sigma = V_{a,s+1} \cap \{ \hat{\sigma} : \hat{\sigma} \cdot m = 0 \} \cap B_N(\sqrt{1 - q_{a,s+1}}),$$

is independent of $\nabla g(m) = P V_{a,s+1} \nabla H_N^{m_{a,s}}(m)$ and is a CGPD($\xi_{q_{a,s+1}}, \Sigma, q_{a,s+1}$). By (4.37) we have that $\nabla g(m)$ and $P V_{a,s+1} \nabla H_N(m_{a,s+1})$ are deterministically related under $\mathbb{P}(\cdot | \mathcal{R}_{\alpha,s})$, so under this measure $(g^m(\hat{\sigma}), \hat{\sigma} \in \Sigma)$ and $P V_{a,s+1} \nabla H_N(m_{a,s+1})$ are independent. By (4.33) the process $(g^m(\hat{\sigma}), \hat{\sigma} \in \Sigma)$ is then independent of $(P V_{a,s+1} \nabla H_N(m_{a,s+1}), u_{a,s+2})$ under $\mathbb{P}(\cdot | \mathcal{R}_{\alpha,s})$. Thus $(g^m(\hat{\sigma}, \hat{\sigma} \in \Sigma)$ is a CGPD($\xi_{q_{a,s+1}}, \Sigma, q_{a,s+1}$) also under $\mathbb{P}(\cdot | \mathcal{R}_{\alpha,s})$.

Next under this measure $u_{a,s+2}, V_{a,s+2}, \Sigma_{s+1}$ are deterministic, and $\Sigma \subset V_{a,s+2} \cap \{ \hat{\sigma} : \hat{\sigma} \cdot u_{a,s+1,j} = 0, j = 1, 2 \} \cap B_N(\sqrt{1 - q_{a,s+1}}) = \Sigma_{s+1}$ (recall (4.16) and thus $(g^m(\hat{\sigma}), \hat{\sigma} \in \Sigma_{s+1}$) is a CGPD($\xi_{q_{a,s+1}}, \Sigma_{s+1}, q_{a,s+1}$) under $\mathbb{P}(\cdot | \mathcal{R}_{\alpha,s+1})$. Finally by (4.38) this in fact
means that \( H_{N}^{m_{a,s+1}}(\hat{\sigma}), \hat{\sigma} \in \Sigma_{s+1} \) is CGPD \((\xi_{q_{a,s+1}}, \Sigma_{s+1}, q_{a,s+1})\) under \( \mathbb{P}(\cdot | R_{s+1}) \). Thus (4.36) holds for \( l = s + 1 \). This completes the proof of (4.36) by induction. \( \square \)

5. Construction of a cover of \( S_{N-1} \)

In this section we construct “equator” sets \( E_{\alpha} \) that form a cover of \( S_{N-1} \), each associated to a magnetization vector \( m_{\alpha} \) from the previous section. A crucial part of the definition is the condition \( |\langle \sigma, u_{\alpha,k+1,j} \rangle| \leq \eta \), which makes the effective external field on \( E_{\alpha} \) almost vanish, as will be proven in Proposition 6.3 below, and also makes the entropy of the sets \( E_{\alpha} \) bounded by the entropy function \( I_{E,\delta} \), as proven in Proposition 6.2 below.

**Definition 5.1.** (Regions \( E_{\alpha} \subset D_{\alpha} \) of \( S_{N-1} \) associated to \( m_{\alpha} \)) Fix \( \varepsilon > 0 \) and \( \eta > 0 \). For each \( 1 \leq k \leq \frac{N-K}{2} + 1 \) and \( \alpha \in A_{k} \) define the (random) sets

\[
D_{\alpha} = \{ \sigma \in S_{N-1} : |\langle \sigma, u_{\alpha,l,j} \rangle| \varepsilon = \alpha_{l,j} \text{ for } 1 \leq l \leq k, 1 \leq j \leq A_{l} \},
\]

and

\[
E_{\alpha} = \{ \sigma \in D_{\alpha} : |\langle \sigma, u_{\alpha,k+1,j} \rangle| \leq \eta \text{ for } j = 1, 2 \}.
\]

\( \square \)

The next definition essentially constructs the cover, by specifying the vectors \( \alpha \) of the sets \( E_{\alpha} \) that should be included.

**Definition 5.2.** (Index set \( A_{\varepsilon,\eta} \) of cover of \( S_{N-1} \)) For any \( 0 < \varepsilon, \eta < 1 \) let

\[
A_{\varepsilon,\eta} = \bigcup_{1 \leq k \leq 5\eta^{-2}} A_{k}.
\]

\( \square \)

Note from (4.4) and the fact that \( |I_{\varepsilon}| \leq 2\varepsilon^{-1} \) (recall (4.1)) that \( A_{k} \) are finite sets of cardinality bounded in \( N \), and for \( \varepsilon, \eta \in (0, 1) \),

\[
|A_{k}| \leq (2\varepsilon^{-1})^{K+2(k-1)} \text{ and } |A_{\varepsilon,\eta}| \leq (2\varepsilon^{-1})^{K+10\eta^{-2}}.
\]

We now show that \( (E_{\alpha})_{\alpha \in A_{\varepsilon,\eta}} \) is indeed a cover for \( S_{N-1} \).

**Proposition 5.3.** (\( (E_{\alpha})_{\alpha \in A_{\varepsilon,\eta}} \) is cover of \( S_{N-1} \)) It holds for \( 0 < \eta < 1, 0 < \varepsilon \leq \eta/2 \) and all \( \delta > 0 \) that

\[
S_{N-1} = \bigcup_{\alpha \in A_{\varepsilon,\eta}} E_{\alpha} \text{ almost surely.}
\]

**Proof.** Let \( \sigma \in S_{N-1} \). Define \( \alpha_{1} \in I_{\varepsilon}^{K} \) by

\[
\alpha_{1,1} = |\langle \sigma, u_{1} \rangle| \varepsilon, \ldots, \alpha_{1,K} = |\langle \sigma, u_{K} \rangle| \varepsilon,
\]

and define the random sequence \( \alpha_{l}, 2 \leq l \leq \frac{N-K}{2} + 1 \), recursively via

\[
\alpha_{l,1} = |\langle \sigma, u_{(\alpha_{1}, \ldots, \alpha_{l-1}),l,1} \rangle| \varepsilon \text{ and } \alpha_{l,2} = |\langle \sigma, u_{(\alpha_{1}, \ldots, \alpha_{l-1}),l,2} \rangle| \varepsilon,
\]

\( \square \)
where the $u_{(\alpha_1, \ldots, \alpha_{l-1}), l}$ are constructed in Definition 4.4. Let

\[(5.8) \quad k \text{ be the smallest positive integer such that } |\alpha_{k+1,1}|, |\alpha_{k+1,2}| \leq \frac{\eta}{2}, \]

and let $\alpha = (\alpha_1, \ldots, \alpha_k)$. By the “nesting” property (4.32) of the basis $u_{\alpha,l,j}$ we have

\[u_{(\alpha_1, \ldots, \alpha_{l-1}), l,j} = u_{\alpha,l,j} \text{ for all } 2 \leq l \leq k+1, 1 \leq j \leq 2,\]

which together with (4.9) means that (5.6)-(5.7) imply

\[(5.9) \quad |\langle \sigma, u_{\alpha,k,j} \rangle| \leq \alpha_{l,j} \text{ for all } 1 \leq l \leq k+1, 1 \leq j \leq A_t.\]

Thus $\alpha_{l,j} \in I_\varepsilon$ for all $l, j$ and since the $u_{\alpha,l,j}$ are orthonormal (recall (4.22))

\[(5.10) \quad \sum_{l=1}^{k} \sum_{j=1}^{A_t} \alpha_{l,j}^2 \leq \sum_{l=1}^{k} \sum_{j=1}^{A_t} |\sigma, u_{\alpha,l,j} |^2 \leq \| \sigma \|^2 = 1.\]

This implies that $\alpha \in A_k$ (see its definition (4.4)), and then (5.9) implies that $\sigma \in D_\alpha$ (see its definition (5.1)).

Furthermore, since $|\alpha_{k+1,1}|, |\alpha_{k+1,2}| \leq \eta$ the equality (5.9) implies that $|\langle \sigma, u_{\alpha,k+1} \rangle| \leq \eta/2 + \varepsilon \leq \eta$ so that $\sigma \in E_\alpha$ by (5.2).

We now show that also $\alpha \in A_{\varepsilon, \eta}$, thus completing the proof of (5.5). The construction of $\alpha$ (recall (5.8)) implies that

\[(5.11) \quad |\alpha_{l,1}| > \frac{\eta}{2} \text{ or } |\alpha_{l,2}| > \frac{\eta}{2} \text{ for } l = 2, \ldots, k.\]

Thus we have

\[(k - 1) \frac{\eta^2}{4} \leq \sum_{l=2}^{k} \sum_{j=1}^{A_l} \alpha_{l,1}^2 \leq 1,\]

showing that

\[(5.12) \quad k \leq 1 + 4\eta^{-2} \leq 5\eta^{-2},\]

and thus that $\alpha \in A_{\varepsilon, \eta}$ (see (5.3)).

**Remark 5.4.** The proposition could be somewhat strengthened, since one can prove that it also holds with

\[\left\{ \alpha \in A_{\varepsilon, \eta} : \alpha_l \notin \left[ -\frac{\eta}{2}, \frac{\eta}{2} \right] \text{ for } 2 \leq l \leq 5\eta^{-2} \right\},\]

in place of $A_{\varepsilon, \eta}$. Indeed, for instance for any $\alpha \in A_1$ the set $E_\alpha$ is itself contained in $\cup_{\alpha \in A_k} E_{\alpha}$ for any $k \geq 2$, so the spin configurations in $E_\alpha$ are contained in the RHS of (5.5) many times over. In the sketch of Section 2.1 this double-counting is avoided (see (2.10), (2.49)), but in the proof we do not bother with this as it would slightly complicate the definition of $A_{\varepsilon, \eta}$ and does not otherwise simplify the proof or strengthen the result. \qed
6. Proof of General TAP Upper Bound

In this section we complete the proof of the general TAP upper bound Theorem 1.3 using the construction from the previous two sections.

Recall that $E$ is the measure in the statement of Theorem 1.3. Proposition 5.3 implies

$$E \left[ \exp \left( \beta H_N^J (\sigma) \right) \right] \leq \sum_{\alpha \in \mathcal{A}_{\varepsilon, \eta}} E \left[ 1_{\mathcal{E}_\alpha} \exp \left( \beta H_N^J (\sigma) \right) \right],$$

which we will use in the proof of Theorem 1.3. Define for all $\alpha \in \mathcal{A}$ such that $E \left[ \mathcal{E}_\alpha \right] > 0$ the (random) measures

$$E^m_\alpha [A] = \frac{E \left[ A \cap \mathcal{E}_\alpha \right]}{E \left[ \mathcal{E}_\alpha \right]}, A \subset S_{N-1} \text{ measurable},$$

so that (6.1) can be written as

$$E \left[ \exp \left( \beta H_N^J (\sigma) \right) \right] \leq \sum_{\alpha \in \mathcal{A}_{\varepsilon, \eta} : E \left[ \mathcal{E}_\alpha \right] > 0} E \left[ \mathcal{E}_\alpha \right] E^m_\alpha \left[ \exp \left( \beta H_N^J (\sigma) \right) \right].$$

In what follows we will give bounds for $E \left[ \mathcal{E}_\alpha \right]$ and $E^m_\alpha \left[ \exp (\beta H_N^J (\sigma)) \right]$. We first consider $E \left[ \mathcal{E}_\alpha \right]$. To bound it we use the next lemma, which shows that if $\sigma \in \mathcal{E}_\alpha$ then the increment $\hat{\sigma} = \sigma - m_\alpha$ is almost orthogonal to $\overline{U}_\alpha$, i.e. almost lies in $\overline{V}_\alpha$. It will also be used to show that the effective external field on $\mathcal{E}_\alpha$ almost vanishes in Proposition 6.3.

Lemma 6.1. For any $0 < \eta \leq K^{-1/2}$ and $0 < \varepsilon \leq \eta^2$ it holds almost surely for all $\alpha \in \mathcal{A}_{\varepsilon, \eta}$ that

$$\sigma \in \mathcal{E}_\alpha \implies \| P_{\overline{U}_\alpha} \hat{\sigma} \| = \| P_{\overline{V}_\alpha} \sigma - m_\alpha \| = \| P_{\overline{V}_\alpha} \hat{\sigma} - \hat{\sigma} \| = \| P_{\overline{V}_\alpha} \sigma - \hat{\sigma} \| \leq 4 \eta,$$

where $\hat{\sigma} = \sigma - m_\alpha$.

Proof. The equality of the first four expressions is elementary and holds for any $\sigma \in \mathbb{R}^N$ since $P_{\overline{V}_\alpha} \hat{\sigma} + P_{\overline{U}_\alpha} \hat{\sigma} = \hat{\sigma}$ for all $\hat{\sigma}$ by (4.31), and $m_\alpha \in U_\alpha \subset \overline{U}_\alpha$ (see (4.27), (4.30)).

To show the inequality fix $1 \leq k \leq 5 \eta^{-2}$ and $\alpha \in \mathcal{A}_k$. By (4.22), (4.30) we have for any $\sigma \in \mathbb{R}^N$

$$\| P_{\overline{U}_\alpha} \hat{\sigma} \|^2 = \sum_{l=1}^{k+1} \sum_{j=1}^{A_l} \langle \hat{\sigma}, u_{\alpha,l,j} \rangle^2 = \sum_{l=1}^{k+1} \sum_{j=1}^{A_l} \langle \sigma - m_\alpha, u_{\alpha,l,j} \rangle^2.$$

By (4.22), (4.26) we have $\langle m_\alpha, u_{\alpha,l,j} \rangle = \alpha_{l,j}$ for $1 \leq l \leq k, 1 \leq j \leq A_l$, and $\langle m_\alpha, u_{\alpha,k+1,j} \rangle = 0$ for $j = 1, 2$. Thus the right-hand side can be written as

$$\sum_{l=1}^{k} \sum_{j=1}^{A_l} (\langle \sigma, u_{\alpha,l,j} \rangle - \alpha_{l,j})^2 + \{ \langle \sigma, u_{\alpha,k+1,1} \rangle^2 + \langle \sigma, u_{\alpha,k+1,2} \rangle^2 \}.$$

By the definition (5.1) of $D_\alpha$ and (4.3), and the definition (5.2) of $\mathcal{E}_\alpha$, this is at most

$$(K + 2 (k - 1)) \varepsilon^2 + 2 \eta^2.$$

Using $k \leq 5 \eta^{-2}, K \leq \eta^{-2}$ and $\varepsilon \leq \eta^2$ this can be bounded by $16 \eta^2$, giving the claim. □
For \( \alpha \in \mathcal{A}_k \) we refer to the set
\[
(m_\alpha + \nabla_\alpha) \cap B_N = m_\alpha + B_N \left( \sqrt{1 - \|m_\alpha\|^2} \right) \cap \nabla_\alpha,
\]
as a “slice” of the ball centered at \( m_\alpha \) of dim \( \nabla_\alpha = N - K - 2k \). Since \( \sigma \in \mathcal{E}_\alpha \) satisfies \( \|P_{\nabla_\alpha} \sigma - m_\alpha\| \leq 4\eta \) by the previous lemma we have
\[
\mathcal{E}_\alpha \subset \bigcup_{m \in \overline{\nabla}_\alpha, \|m - m_\alpha\| \leq 4\eta} (m + \nabla_\alpha) \cap B_N \text{ for } \alpha \in \mathcal{A}_{\varepsilon, \eta},
\]
where the right-hand side can be thought of as a “thick slice” centered at \( m_\alpha \).

The next result uses the previous lemma and the fact that \( \overline{\mathcal{U}}_\alpha \) contains the normal of an (almost) minimal entropy hyperplane around \( m_\alpha \) to bound \( E[\mathcal{E}_\alpha] \) in terms of \( I_{E,\delta}(m_\alpha) \).

**Proposition 6.2.** For \( I_{E,\delta} \) as in (1.11) and any \( \delta > 0, 0 < \eta \leq \min\left(K^{-1/2}, \delta/4\right) \) and \( 0 < \varepsilon \leq \eta^2 \) it holds that
\[
E[\mathcal{E}_\alpha] \leq e^{I_{E,\delta}(m_\alpha) + \delta} \text{ for all } \alpha \in \mathcal{A}_{\varepsilon, \eta} \text{ almost surely.}
\]

**Proof.** Fix \( \alpha \in \mathcal{A}_{\varepsilon, \eta} \). Since \( \lambda_{m_\alpha}^\delta \in \overline{\mathcal{U}}_\alpha = \nabla_\alpha^\perp \) (recall (4.29)) and \( \|\lambda_{m_\alpha}^\delta\| = 1 \) (recall Definition 4.2) we have for any \( \sigma \)
\[
|\langle \lambda_{m_\alpha}^\delta, \sigma - m_\alpha \rangle| = |\langle \lambda_{m_\alpha}^\delta, \hat{\sigma} \rangle| = |\langle \lambda_{m_\alpha}^\delta, P_{\nabla_\alpha} \hat{\sigma} - \hat{\sigma} \rangle| \leq \|P_{\nabla_\alpha} \hat{\sigma} - \hat{\sigma}\|,
\]
and by Lemma 6.1
\[
\|P_{\nabla_\alpha} \hat{\sigma} - \hat{\sigma}\| \leq 4\eta \leq \delta,
\]
for any \( \sigma \in \mathcal{E}_\alpha \) if \( \eta \leq \delta/4 \), so that
\[
\langle \lambda_{m_\alpha}^\delta, \sigma - m_\alpha \rangle \geq -\delta \text{ for all } \sigma \in \mathcal{E}_\alpha.
\]
Therefore
\[
E[\mathcal{E}_\alpha] \leq E\left[\langle \lambda_{m_\alpha}^\delta, \sigma - m_\alpha \rangle \geq -\delta \right] \stackrel{(4.6)}{=} \exp \left(I_{E,\delta}(m_\alpha) + \delta\right). \tag{6.45}
\]

The previous proposition will allow us to deal with the term \( E[\mathcal{E}_\alpha] \) in (6.3). We now turn to the other term on the RHS of (6.3), namely the normalized partition function \( E_{m_\alpha}^N[\exp(\beta H_N^\varepsilon(\sigma))] \). We first show that the effective external field on \( \mathcal{E}_\alpha \) can be neglected. This is one of the most important properties of the cover \( (\mathcal{E}_\alpha)_{\alpha \in \mathcal{A}_{\varepsilon, \eta}} \), together with the fact that it consists of a small number of sets.

**Proposition 6.3** (Effective external field on \( \mathcal{E}_\alpha \) vanishes for \( \alpha \in \mathcal{A}_{\varepsilon, \eta} \)).

For \( 0 < \eta \leq K^{-1/2} \) and \( 0 < \varepsilon \leq \eta^2 \) it holds a.s. for all \( \alpha \in \mathcal{A}_{\varepsilon, \eta} \) that
\[
\sigma \in \mathcal{E}_\alpha \implies |\langle \hat{\sigma}, h_{eff}(m_\alpha) \rangle| \leq 4\eta\|h_{eff}(m_\alpha)\| \text{ where } \hat{\sigma} = \sigma - m_\alpha.
\]

\( \Box \)
Proof. Since $\overrightarrow{U}_\alpha \oplus \overrightarrow{V}_\alpha$ is an orthogonal decomposition of $\mathbb{R}^N$ (see (4.31))
\[
|\langle \hat{\sigma}, h_{\text{eff}}(m_\alpha) \rangle| \leq \left| \left\langle P^{\overrightarrow{V}_\alpha} \hat{\sigma}, h_{\text{eff}}(m_\alpha) \right\rangle \right| + \| P^{\overrightarrow{U}_\alpha} \hat{\sigma} \| \| h_{\text{eff}}(m_\alpha) \|.
\]
The first term on the RHS vanishes since $h_{\text{eff}}(m_\alpha) \in \mathcal{U}_\alpha$ (recall (4.29)), and by Lemma 6.1 we have $\| P^{\overrightarrow{U}_\alpha} \hat{\sigma} \| \leq 4\eta$ so (6.5) follows.

In the remainder of the proof we will assume that $\xi'''(1) < \infty$ and work on the event (6.6)
\[
\mathcal{L}_N = \left\{ \sup_{m \in B^\circ_N} \| \nabla H_N(m) \| \leq c_\xi, \frac{|H_N(m) - H_N(\tilde{m})|}{N} \leq c_\xi \| m - m' \| \forall m, m' \in B^\circ_N \right\},
\]
for $c_\xi = c\sqrt{\xi'''(1)}$, which by Lemma A.11 and $\xi'(1) \leq \xi''(1) \leq \xi'''(1)$ satisfies
(6.7) \[
\mathbb{P}(\mathcal{L}_N) \leq e^{-N} \text{ for all } N \geq 1,
\]
for a large enough universal $c$. We now use the decomposition (3.4) to show that the normalized partition function $E^{m_\alpha}[\exp(\beta H_N^f(\sigma))]$ can be bounded in terms of the partition function of the recentered Hamiltonian $H^{m_\alpha}_{N,\xi}$ on $\mathcal{E}_\alpha$.

**Proposition 6.4** (Decomposing Hamiltonian on thick slice). Let $\beta \geq 0, L > 0$. If $\sqrt{\xi'''(1)} \leq L$ and if $f_N : B_N \to \mathbb{R}$ satisfies $f_N(\sigma) = f_N(P^{U}\sigma)$ for all $\sigma$ and is Lipschitz with respect to $\| \cdot \|$ with Lipschitz constant at most $LN$, it holds for $0 < \eta \leq K^{-1/2}$ and $0 < \varepsilon \leq \eta^2$ that
\[
E^{m_\alpha}[\exp(\beta H_N^f(\sigma))] \leq e^{\beta H_N^f(m_\alpha)}E^{m_\alpha}[\exp(\beta H_N^{m_\alpha}(\hat{\sigma}))]e^{\eta\beta LN} \text{ for all } \alpha \in \mathcal{A}_{\varepsilon,\eta},
\]
on the event (6.6).

Proof. Fix $\alpha \in \mathcal{A}_{\varepsilon,\eta}$. We use the decomposition (3.4) with $\sigma = m_\alpha + \hat{\sigma}$, i.e.
\[
H_N^f(m_\alpha + \hat{\sigma}) = H_N^f(m_\alpha) + h_{\text{eff}}(m_\alpha) \cdot \hat{\sigma} + (f_N(m_\alpha + \hat{\sigma}) - f_N(m_\alpha)) + H^{m_\alpha}_{N,\xi}(\hat{\sigma}).
\]
Since $f_N(x)$ depends only on $P^{U}x$, the Lipschitz assumption on $f_N$ implies that
\[
|f_N(m_\alpha + \hat{\sigma}) - f_N(m_\alpha)| \leq LN \| P^{U} \hat{\sigma} \| \text{ for all } \hat{\sigma}. \]
We have for $\sigma \in \mathcal{E}_\alpha \subset \mathcal{D}_\alpha$
\[
\| P^{U}_1 \hat{\sigma} \|^2 = \sum_{i=1}^{K} (\langle \sigma, u_i \rangle - \langle m_{\alpha,1}, u_i \rangle)^2 \leq \sum_{i=1}^{K} (\langle \sigma, u_i \rangle - \langle 1, u_i \rangle)^2 \leq \varepsilon^2 K
\]
so since $\varepsilon \leq \eta^2$ and $\eta \leq K^{-1/2}$
\[
|f_N(m_\alpha + \hat{\sigma}) - f_N(m_\alpha)| \leq NL\eta \text{ for all } \alpha \in \mathcal{A} \text{ and } \hat{\sigma} \in \mathcal{E}_\alpha - m_\alpha.
\]
Also $\| h_{\text{eff}}(m_\alpha) \| = \| \nabla H_N(m_\alpha) \| \leq c_\xi \leq cL$ on the event (6.6) so that by Proposition 6.3 we obtain that on that event
\[
|h_{\text{eff}}(m_\alpha) \cdot \hat{\sigma}| = N |\langle h_{\text{eff}}(m_\alpha), \hat{\sigma} \rangle| \leq NcL\eta \text{ for all } \alpha \in \mathcal{A}_{\varepsilon,\eta} \text{ and } \hat{\sigma} \in \mathcal{E}_\alpha - m_\alpha.
\]
The claim (6.8) follows from (6.9), (6.10) and (6.11). \qed
We now aim to bound the normalized partition function $E^{m_\alpha}[\exp(\beta H^{m_\alpha}_N(\hat{\sigma}))]$ of the recentered Hamiltonian from the RHS of (6.8). To do so we will first move from the “thick slices” $E_\alpha$ to the “thin slices” $m_\alpha + \Sigma_\alpha$, where

$$\Sigma_\alpha = B_N \left( \sqrt{1 - \|m_\alpha\|^2} \right) \cap \overline{V}_\alpha \text{ for } \alpha \in \mathcal{A}. \hspace{1cm} (6.12)$$

Note that for $\sigma \in m_\alpha + \Sigma_\alpha$ we have $\sigma = m_\alpha + \hat{\sigma}$ with $\hat{\sigma} \in \overline{V}_\alpha$, the latter being only approximately true for a $\sigma \in E_\alpha$.

For $\sigma \in E_\alpha$ we define

$$\hat{\tau}_\alpha(\sigma) = \sqrt{1 - \|m_\alpha\|^2} \frac{P^{\overline{V}_\alpha}\sigma}{\|P^{\overline{V}_\alpha}\sigma\|} \in \Sigma_\alpha, \hspace{1cm} (6.13)$$

(with the convention $\frac{P^{\overline{V}_\alpha}\sigma}{\|P^{\overline{V}_\alpha}\sigma\|} = 0$ if $P^{\overline{V}_\alpha}\sigma = 0$) to be a point in $\Sigma_\alpha$ that approximates $\hat{\sigma}$ well, as shown by the next lemma.

**Lemma 6.5** (Projecting to “thin slice”). For $0 < \eta \leq K^{-1/2}$ and $0 < \varepsilon \leq \eta^2$ we have almost surely for all $\alpha \in \mathcal{A}_{\varepsilon, \eta}$ and $\sigma \in E_\alpha$ that

a) $||P^{\overline{V}_\alpha}\sigma|| - \sqrt{1 - \|m_\alpha\|^2} \leq 8\eta^{1/4}. \hspace{1cm} (6.14)$

b) $\|\sigma - (m_\alpha + \hat{\tau}_\alpha(\sigma))\| = \|\hat{\sigma} - \hat{\tau}_\alpha(\sigma)\| \leq 12\eta^{1/4}$.\n
**Proof.**

a) Since $U_\alpha \oplus \overline{V}_\alpha$ is an orthogonal decomposition of $\mathbb{R}^N$ (see (4.31))

$$||P^{\overline{V}_\alpha}\sigma|| = \sqrt{1 - ||P^{\overline{U}_\alpha}\sigma||^2}. \hspace{1cm} (6.15)$$

Since $\|m_\alpha\|, \|\sigma\| \leq 1$ it holds that

$$\left|\|m_\alpha\|^2 - ||P^{\overline{V}_\alpha}\sigma||^2\right| = \left|\|m_\alpha\| + ||P^{\overline{V}_\alpha}\sigma||\right| \left|\|m_\alpha\| - ||P^{\overline{V}_\alpha}\sigma||\right| \leq 2 \left|\|m_\alpha\| - ||P^{\overline{V}_\alpha}\sigma||\right|, \hspace{1cm} (6.16)$$

so that by Lemma 6.1

$$\left|\|m_\alpha\|^2 - ||P^{\overline{U}_\alpha}\sigma||^2\right| \leq 8\eta, \hspace{1cm} \text{ for } \sigma \in E_\alpha. \hspace{1cm} (6.17)$$

for $\sigma \in E_\alpha$. Now if $\|m_\alpha\|^2 \geq 1 - \eta^{1/2}$, then

$$\left|\|P^{\overline{V}_\alpha}\sigma\| - \sqrt{1 - \|m_\alpha\|^2}\right| \leq \|P^{\overline{V}_\alpha}\sigma\| + \sqrt{1 - \|m_\alpha\|^2} \hspace{1cm} (6.17)$$

$$\leq \sqrt{1 - \|m_\alpha\|^2} + 8\eta + \sqrt{1 - \|m_\alpha\|^2} \hspace{1cm} \leq \sqrt{\eta^{1/2} + 8\eta + \eta^{1/2}} \hspace{1cm} \leq \sqrt{4\eta^{1/4}}. \hspace{1cm} (6.17)$$

If on the other hand $\|m_\alpha\|^2 \leq 1 - \eta^{1/2}$ then

$$\|P^{\overline{V}_\alpha}\sigma\| - \sqrt{1 - \|m_\alpha\|^2} \leq \sqrt{1 - \|P^{\overline{V}_\alpha}\sigma\|^2} - \sqrt{1 - \|m_\alpha\|^2} \hspace{1cm} (6.15)$$

$$= \sqrt{1 - \|m_\alpha\|^2} \left( \sqrt{1 + \frac{\|m_\alpha\|^2 - \|P^{\overline{V}_\alpha}\sigma\|^2}{1 - \|m_\alpha\|^2}} - 1 \right). \hspace{1cm} (6.18)$$
and
\[
\frac{\|m_\alpha\|^2 - \|P_{\mu_\alpha}\|^2}{1 - \|m_\alpha\|^2} \leq \frac{8\eta}{\eta^{1/2}} = 8\eta^{1/2}.
\]

Since \(|\sqrt{1 + x - 1}| \leq |x|\) for all \(x \geq -1\) we thus have that
\[
\left| \frac{\sqrt{1 + \|m\|^2 - \|P_{\mu_\alpha}\|^2}}{1 - \|m\|^2} - 1 \right| \leq 8\eta^{1/2},
\]
and so from (6.18)
\[
\left\| \|P_{\mu_\alpha}\| - \sqrt{1 - \|m_\alpha\|^2} \right\| \leq \sqrt{1 - \|m_\alpha\|^2} 8\eta^{1/2} \leq 8\eta^{1/2}.
\]

Combing this with (6.17) gives (6.14).

b) It holds that
\[
\|\sigma - (m_\alpha + \hat{\tau}_\alpha (\sigma))\| = \|\hat{\sigma} - \sqrt{1 - \|m_\alpha\|^2} \frac{P_{\mu_\alpha}}{\|P_{\mu_\alpha}\|}\|
\leq \|\hat{\sigma} - P_{\mu_\alpha}\| + \|P_{\mu_\alpha}\| - \sqrt{1 - \|m_\alpha\|^2} \leq 12\eta^{1/4}.
\]

Let now
\[
E^{\Sigma_\alpha} \text{ denote the law of } \hat{\tau}_\alpha (\sigma) \text{ under } E^{\mu_\alpha}.
\]

The next lemma bounds the partition function on the “thick slice” \(\mathcal{E}_\alpha\) by that on the “thin slice” \(m_\alpha + \Sigma_\alpha\), using the previous lemma to bound the error made when approximating \(\hat{\sigma}\) by \(\hat{\tau}_\alpha (\hat{\sigma})\).

**Lemma 6.6 (From “thick slice” to “thin slice”).** Let \(\beta \geq 0, L > 0\). If \(\sqrt{\xi^m (1)} \leq L\), \(0 < \eta \leq K^{-1/2}\) and \(0 < \varepsilon \leq \eta^2\) then on the event (6.6)

\[
E^{\mu_\alpha} \left[ \exp (\beta H^{\mu_\alpha}_N (\hat{\sigma})) \right] \leq E^{\Sigma_\alpha} \left[ \exp (\beta H^{\mu_\alpha}_N (\hat{\sigma})) \right] e^{\varepsilon^{1/4}B LN} \text{ for all } \alpha \in \mathcal{A}_{\varepsilon, \eta}.
\]

**Proof.** From (3.1) it holds for any \(\hat{\sigma}, \hat{\tau} \in B_\alpha := B_N \left( \sqrt{1 - \|m_\alpha\|^2} \right)\) and any \(\alpha\) that

\[
H^{\mu_\alpha}_N (\hat{\sigma}) - H^{\mu_\alpha}_N (\hat{\tau}) = H_N (m_\alpha + \hat{\sigma}) - H_N (m_\alpha + \hat{\tau}) - \nabla H_N (m_\alpha) \cdot (\hat{\sigma} - \hat{\tau}).
\]

Thus on the event (6.6) we have
\[
|H^{\mu_\alpha}_N (\hat{\sigma}) - H^{\mu_\alpha}_N (\hat{\tau})| \leq cLN \|\hat{\sigma} - \hat{\tau}\| \text{ for all } \hat{\sigma}, \hat{\tau} \in B_\alpha.
\]

Recall from (6.15) that \(\hat{\tau}_\alpha (\sigma) \in B_\alpha\) for all \(\sigma \in \mathcal{E}_\alpha\). Thus (6.21) together with Lemma 6.5 b) imply that on the event (6.6)

\[
E^{\mu_\alpha} \left[ \exp (\beta H^{\mu_\alpha}_N (\hat{\sigma})) \right] \leq E^{\mu_\alpha} \left[ \exp (\beta H^{\mu_\alpha}_N (\hat{\tau}_\alpha (\sigma))) \right] e^{\varepsilon^{1/4}B LN}.
\]
But by the definition (6.19) of $E^{\Sigma_\alpha}$ we have

\begin{equation}
E^{m_\alpha} [\exp (\beta H^m_N (\hat{\tau}_\alpha (\sigma)))] = E^{\Sigma_\alpha} [\exp (\beta H^m_N (\sigma))],
\end{equation}

which gives the claim (6.20). \hfill \Box

We have thus reduced the proof of Theorem 1.3 to bounding the partition function $E^{\Sigma_\alpha} [\exp (\beta H^m_N (\sigma))]$ on the “thin slice” $\Sigma_\alpha$. Next we do so uniformly over all $m_\alpha, \alpha \in \mathcal{A}_{\varepsilon, \eta}$. These bounds give rise to the Onsager correction in the TAP free energy. We obtain the bound by a simple Markov inequality for the partition function over each slice $\Sigma_\alpha$, and since the number of $m_\alpha$’s is small a union bound shows that the upper bound holds for all $m_\alpha$ simultaneously with high probability. Since the recentered Hamiltonian has no external field the Markov inequality will give a tight upper for small enough $\beta$. Recall the notation $q_\alpha = \|m_\alpha\|^2$ from (4.34).

**Proposition 6.7** (Onsager correction). For any $\varepsilon, \eta > 0$ it holds for $N, \beta, E$ as in Theorem 1.3 and any $\delta > 0$ and

\begin{equation}
O_{N, \delta, \varepsilon, \eta} = \left\{ \begin{array}{l}
E^{\Sigma_\alpha} [\exp (\beta H^m_N (\hat{\sigma}))] \leq e^{N\frac{\beta^2}{2} \xi_{\eta}(1-q_\alpha)+\delta N} \\
\text{for all } \alpha \in \mathcal{A}_{\varepsilon, \eta} \text{ s.t. } E[\mathcal{E}_\alpha] > 0
\end{array} \right\},
\end{equation}

that

\[ P(O_{N, \delta, \varepsilon, \eta}) \leq |\mathcal{A}_{\varepsilon, \eta}| e^{-\delta N}. \]

**Proof.** Fix $\alpha \in \mathcal{A}$ and consider

\begin{equation}
P \left( E^{\Sigma_\alpha} [\exp (\beta H^m_N (\hat{\sigma}))] \geq e^{N\frac{\beta^2}{2} \xi_{\eta}(1-q_\alpha)+\delta N} | \mathcal{R}_\alpha \right),
\end{equation}

for $\mathcal{R}_\alpha$ from (4.35). To lighten notation drop the $\alpha$ subscript and write $\Sigma = \Sigma_\alpha$, $m = m_\alpha$, $q = q_\alpha = \|m_\alpha\|^2$, $\mathcal{R} = \mathcal{R}_\alpha$ and $\mathcal{V} = \mathcal{V}_\alpha$. Note that $m$ and $\mathcal{V}$, and therefore $\Sigma$, are deterministic functions of $u_{\alpha,l,j}$, $1 \leq l \leq k+1, 1 \leq j \leq \Lambda_l$, are thus deterministic under the measure $P(\cdot | \mathcal{R})$ (see (4.26) for $m$ and (4.30) for $\mathcal{V}$, and (6.12) for $\Sigma = \Sigma_\alpha$).

By Markov’s inequality

\[ P \left( E^{\Sigma_\alpha} [\exp (\beta H^m_N (\hat{\sigma}))] \geq e^{N\frac{\beta^2}{2} \xi_{\eta}(1-q)+\delta N} | \mathcal{R}\right) \leq E \left[ E^{\Sigma_\alpha} [\exp (\beta H^m_N (\hat{\sigma}))] | \mathcal{R}\right] e^{-N\frac{\beta^2}{2} \xi_{\eta}(1-q)-\delta N} = E^{\Sigma_\alpha} \left[ E \left[ \exp (\beta H^m_N (\hat{\sigma})) | \mathcal{R}\right] \right] e^{-N\frac{\beta^2}{2} \xi_{\eta}(1-q)-\delta N}. \]

Lemma 4.7 implies that for fixed $\hat{\sigma} \in \Sigma$ the $P(\cdot | \mathcal{R})$-law of $H^m_N (\hat{\sigma})$ is that of a centered normal of variance $\xi_{\eta}(1-q)$, so that

\[ E \left[ \exp (\beta H^m_N (\hat{\sigma})) | \mathcal{R}\right] = e^{N\frac{\beta^2}{2} \xi_{\eta}(1-q)} \text{ for any } \hat{\sigma} \in \Sigma. \]

Thus in fact for all $\alpha \in \mathcal{A}$ it holds that

\[ P \left( E^{\Sigma_\alpha} [\exp (\beta H^m_N (\hat{\sigma}_\alpha))] \geq e^{N\frac{\beta^2}{2} \xi_{\eta}(1-q)+\delta N} \right) \leq e^{-\delta N}. \]

A union bound over $\alpha \in \mathcal{A}_{\varepsilon, \eta}$ and (2.29) completes the proof. \hfill \Box
We are now ready to prove the general TAP upper bound. As already mentioned we do so by splitting the partition function into integrals over each set $E_\alpha$ in the cover, normalizing these integrals and recentering the Hamiltonian in them, and using the previous results to bound the partition function on the slices by the Onsager correction.

**Proof of Theorem 1.3.** Assume

\[(6.25)\quad 0 < \eta \leq \min \left( K^{-1/2}, \frac{1}{2} \right) \quad \text{and} \quad 0 < \varepsilon \leq \eta^2 \leq \frac{\eta}{2}.\]

We work on the event $\mathcal{L}_N \cap \mathcal{O}_{N,\delta/2,\varepsilon,\eta}$, where $\mathcal{L}_N$ is the event from (6.6) and $\mathcal{O}_{N,\delta/2,\varepsilon,\eta}$ is the event from Proposition 6.7. By (6.7) and Proposition 6.7 we have

\[(6.26)\quad \Pr\left( \mathcal{L}_N \cap \mathcal{O}_{N,\delta/2,\varepsilon,\eta} \right) \geq 1 - 2 |A_{\varepsilon,\eta}| e^{-\frac{\varepsilon}{2}N},\]

(note that $\delta \in (0, 1)$ and $|A_{\varepsilon,\eta}| \geq 1$).

By Proposition 5.3 and the definition of $E^{m_\sigma}$ we have

\[(6.27)\quad E \left[ \exp \left( \beta H_N^f (\sigma) \right) \right] \leq \sum_{\alpha \in A_{\varepsilon,\eta}} E \left[ 1_{E_\alpha} \exp \left( \beta H_N^f (\sigma) \right) \right].\]

By Proposition 6.2 this is, provided $\eta \leq \frac{\delta}{6}$ and $N \geq 6$ (so that $\delta \leq \frac{\delta}{6}N$) bounded by

\[(6.28)\quad \sum_{\alpha \in A_{\varepsilon,\eta}} \exp \left( I_{E,\delta} (m_\alpha) + \frac{\delta}{6} N \right) E^{m_\sigma} \left[ \exp \left( \beta H_N^f (\sigma) \right) \right].\]

By Proposition 6.4 this is at most

\[(6.29)\quad \sum_{\alpha \in A_{\varepsilon,\eta}} \exp \left( \beta H_N^f (m_\alpha) + I_{E,\delta} (m_\alpha) + \left( \frac{\delta}{6} + c n \beta L \right) N \right) E^{m_\sigma} \left[ \exp \left( \beta H_N^m (\sigma) \right) \right],\]

By Lemma 6.6 this is at most

\[(6.30)\quad \sum_{\alpha \in A_{\varepsilon,\eta}} \exp \left( \beta H_N^f (m_\alpha) + I_{E,\delta} (m_\alpha) + \left( \frac{\delta}{6} + c n \beta L \right) N \right) E^{m_\sigma} \left[ \exp \left( H_N^{m_\sigma} (\sigma) \right) \right],\]

on the event $\mathcal{L}_N$. On the event $\mathcal{O}_{N,\delta/2,\varepsilon,\eta}$ from Proposition 6.7 this is bounded by

\[(6.31)\quad \sum_{\alpha \in A_{\varepsilon,\eta}} \exp \left( I_{E,\delta}^T (m_\alpha) + \left( \frac{4 \delta}{6} + c n \beta L \right) N \right) \leq \sup_{m \in B_N^{m_\sigma}} H_{TAP} (m) + \left( \frac{4 \delta}{6} + c n \beta L \right) N.\]

Set $\hat{\kappa} = c \max(K, (\beta L/\delta)^8) \geq 1$ for some universal large $c$, and $\eta = (\hat{\kappa})^{-1/2}$ and $\varepsilon = \eta^2$. We then get that (6.25) and all inequalities above hold if $N \geq 6$, and in addition and
in addition \( c\eta^{1/4}\beta L \leq \bar{\kappa}^{-8} \beta L \leq \frac{\delta}{6} \) so that the previous line is at most

\[
|A_{\varepsilon,\eta}| \exp\left(\sup_{m \in B_N^c} H_{\text{TAP}}^{E,\delta}(m) + \frac{5\delta}{6} N\right).
\]

Furthermore \( K \leq \eta^{-2} \) so by (5.4)

\[
|A_{\varepsilon,\eta}| \leq (2\eta^{-2})^{11\eta^{-2}} = (2\bar{\kappa})^{11\bar{\kappa}} \leq \frac{\delta}{6} N.
\]

for \( N \geq \delta^{-1} 66\bar{\kappa} \log (2\bar{\kappa}) \geq 6 \). Then (6.31) is at most

\[
\exp\left(\sup_{m \in B_N^c} H_{\text{TAP}}^{E,\delta}(m) + \delta N\right).
\]

with probability at least \( 1 - 2 (2\bar{\kappa})^{11\bar{\kappa}} e^{-\frac{\delta}{2} N} \) (recall (6.26) and (6.32)), provided \( N \geq \delta^{-1} 66\bar{\kappa} \log (2\bar{\kappa}) \). If \( N \leq \delta^{-1} 66\bar{\kappa} \log (2\bar{\kappa}) \) then \( (2\bar{\kappa})^{11\bar{\kappa}} e^{-\frac{\delta}{2} N} \geq 1 \) so the claim (1.13) follows for all \( N \geq 1 \) with \( \bar{\kappa} = \frac{3\bar{\kappa}}{2} \).

7. Bounding the Ising entropy and proof of Ising TAP upper bound

In this section we derive the TAP upper bound for the Ising mixed SK model from the general TAP upper bound, by explicitly bounding the entropy function \( I_{E,\delta}(m) \) of (1.11) by the entropy function \( I_{\text{Ising}}(m) \) from (1.5).

For this it is convenient to extend the definition of \( J \) from (1.6) so that

\[
J(m) = \begin{cases} 
\frac{1+m}{2} \log (1+m) + \frac{1-m}{2} \log (1-m) & \text{if } |m| \leq 1, \\
\log 2 & \text{if } |m| \geq 1.
\end{cases}
\]

We will need the following simple bound.

**Lemma 7.1.** For all \( m, \tilde{m} \in \mathbb{R} \)

\[
|J(m) - J(\tilde{m})| \leq |m - \tilde{m}| \log \frac{2e}{|m - \tilde{m}|} \wedge 1.
\]

**Proof.** Since \( 0 \leq J(m) \leq \log 2 \) for all \( m \) we trivially have

\[
|J(m) - J(\tilde{m})| \leq |m - \tilde{m}| \log 2 \text{ for } |m - \tilde{m}| \geq 1.
\]

From the shape of \( J'(x) = \frac{1}{2} \log \frac{1+x}{1-x} \) (for \( x \in (-1, 1) \), otherwise \( J'(x) = 0 \)) we see that if \( |m - \tilde{m}| \leq 1 \) then

\[
|J(\tilde{m}) - J(m)| \leq \frac{1}{2} \int_{1-|m-\tilde{m}|}^1 \log \frac{1+x}{1-x} dx \leq \frac{1}{2} \int_0^{|m-\tilde{m}|} \log \frac{2}{z} dz = \frac{1}{2} |m-\tilde{m}| \log \frac{2e}{|m-\tilde{m}|}.
\]

Combining (7.3) and (7.4) yields (7.2). \( \square \)

The required bound on the entropy function is the following. Let \( d(x, A) = \inf_{y \in A} \|x - y\| \) for \( x \in \mathbb{R}^N, A \subset \mathbb{R}^N \).

**Lemma 7.2** (Entropy for Ising reference measure). Let \( E \) be the uniform measure on \( \{-1, 1\}^N \) and let \( I_{E,\delta} \) be as in (1.11). For any \( N \geq 1, \delta \in (0, 1) \) the following holds.
a) For all $m \in B_N$

$$I_{E, \delta} (m) \leq I_{\text{Ising}} (m) + cN \delta \log \delta^{-1}. \tag{7.5}$$

b) For $m \in B_N$ such that $d (m, [-1, 1]^N) > \delta$ it holds that

$$I_{E, \delta} (m) = -\infty. \tag{7.6}$$

Proof. a) Note that for all $\lambda \in \mathbb{R}, m \in \mathbb{R}$ and $i = 1, \ldots, N$

$$\log E [\exp (\lambda (\sigma_i - m))] = \log (\cosh (\lambda)) \tag{7.7}$$

Also

$$\inf_{\lambda \in \mathbb{R}} \{\log \cosh (\lambda) - \lambda m\} = \begin{cases} -J (m) & \text{if } |m| \leq 1, \text{ achieved for } \lambda = \text{atanh} (m), \\ -\infty & \text{if } |m| > 1, \text{ achieved for } \lambda \to \pm \infty. \end{cases} \tag{7.8}$$

Now for $m \in B_N$ we have using the exponential Chebyshev inequality

$$I_{E, \delta} (m) \overset{(\ref{1.11})}{=} \inf_{\lambda \in \mathbb{R}^N, |\lambda|=1} \log E \left[ \lambda \cdot (\sigma - m) \geq -\delta \sqrt{N} \right] \leq \inf_{r>0} \inf_{\lambda \in \mathbb{R}^N, |\lambda|=1} \log \left( E \left[ \exp \left( r \lambda \cdot (\sigma - m) \right) \right] e^{r\delta \sqrt{N}} \right) \tag{7.9}$$

$$= \inf_{\lambda \in \mathbb{R}^N} \left\{ \log E \left[ \exp \left( \lambda \cdot (\sigma - m) \right) \right] + |\lambda| \delta \sqrt{N} \right\} \overset{(\ref{7.7})}{=} \inf_{\lambda \in \mathbb{R}^N} \left\{ \sum_{i=1}^{N} \log \cosh (\lambda_i) - \lambda_i m_i + |\lambda| \delta \sqrt{N} \right\}.$$ 

By (7.8) the choice $\lambda_i = \text{atanh} (m_i)$ or $\lambda_i \to \pm \infty$ would be optimal for sum in the inf, but if some coordinates of $m$ are above or close to $\pm 1$ it makes the term $|\lambda| \delta \sqrt{N}$ explode. Therefore we choose

$$\lambda_i = \text{atanh} (\tilde{m}_i) \text{ where } \tilde{m}_i = \begin{cases} 1 - \delta & \text{if } m_i \geq 1 - \delta, \\ m_i & \text{if } m_i \in [- (1 - \delta), 1 - \delta], \\ - (1 - \delta) & \text{if } m_i \leq - (1 - \delta). \end{cases} \tag{7.10}$$

We have

$$|\lambda| \delta \sqrt{N} \leq \text{atanh} (1 - \delta) \delta N \leq c\delta \log \delta^{-1} N \text{ for all } \delta \in (0, 1), N \geq 1,$$

and

$$\log \cosh (\lambda_i) - \lambda_i m_i \overset{\text{(7.10)}}{=} \log \cosh (\lambda_i) - \lambda_i \tilde{m}_i \overset{\text{(7.8)}, \text{(7.10)}}{=} - J (\tilde{m}_i).$$

so we obtain from (7.9)

$$I_{E, \delta} (m) \leq - \sum_{i=1}^{N} J (\tilde{m}_i) + c\delta \log \delta^{-1} N.$$ 

Recalling that $I_{\text{Ising}} (m) = - \sum_{i=1}^{N} J (m_i)$ and noting that $J (\tilde{m}_i) = J (m_i)$ if $|m_i| \leq 1 - \delta$, and otherwise

$$|J (m_i) - J (\tilde{m}_i)| \overset{\text{(7.1), \text{(7.10)}}}{\leq} |J (1) - J (1 - \delta)| \overset{\text{(7.2)}}{\leq} c\delta \log \delta^{-1},$$

the claim (7.5) then follows.
b) Since $[-1, 1]^N$ is convex, if $d(m, [-1, 1]^N) > \delta$ then there is a hyperplane separating \( m \) from $[-1, 1]^N$ at \( \| \cdot \| \)-distance greater than $\delta$ from $m$. This means that there exists a $\lambda \in S_{N-1}$ such that
\[
\langle \lambda, \tilde{m} - m \rangle < -\delta \text{ for all } \tilde{m} \in [-1, 1]^N.
\]
This in particular holds for any $\sigma \in \{-1, 1\}^N$ in place of $\tilde{m}$, so
\[
E \left[ \langle \lambda, \sigma - m \rangle \geq -\delta \right] = 0,
\]
so that (7.6) follows from the definition (1.11) of $I_{E, \delta}$. $\square$

We also need the following continuity estimate for $I_{\text{Ising}}$.

**Lemma 7.3.** It holds for all $m, \tilde{m} \in B_N$ that
\[
|I_{\text{Ising}}(m) - I_{\text{Ising}}(\tilde{m})| \leq 2N\|m - \tilde{m}\| \log \frac{4e}{\|m - \tilde{m}\|}.
\]

**Proof.** By (7.2) we have for all $m, \tilde{m} \in \mathbb{R}^N$,
\[
(7.11) \quad |I_{\text{Ising}}(m) - I_{\text{Ising}}(\tilde{m})| \leq \sum_{i=1}^{N} |m_i - \tilde{m}_i| \log \frac{2e}{|m_i - \tilde{m}_i| \wedge 1}.
\]
Note that
\[
(7.12) \quad \sum_{i=1}^{N} |m_i - \tilde{m}_i| \leq \sqrt{N} |m - \tilde{m}| = N\|m - \tilde{m}\|.
\]
Letting $f(x) = x \log \frac{2e}{x}$ the RHS of (7.11) is thus bounded by
\[
\sum_{i=1}^{N} f (|m_i - \tilde{m}_i| \wedge 1) + N\|m - \tilde{m}\| \log (2e).
\]
Note that $f(x)$ is concave on $[0, \infty)$ so
\[
\sum_{i=1}^{N} f (|m_i - \tilde{m}_i| \wedge 1) \leq N f \left( \frac{1}{N} \sum_{i=1}^{N} |m_i - \tilde{m}_i| \wedge 1 \right).
\]
Also $f$ is increasing on $[0, 2]$ and
\[
\frac{1}{N} \sum_{i=1}^{N} |m_i - \tilde{m}_i| \wedge 1 \leq \frac{1}{N} \sum_{i=1}^{N} |m_i - \tilde{m}_i| \overset{(7.12)}{\leq} \|m - \tilde{m}\| \leq 2,
\]
so that
\[
\sum_{i=1}^{N} f (|m_i - \tilde{m}_i| \wedge 1) \leq N f (\|m - \tilde{m}\|) = N\|m - \tilde{m}\| \log \left( \frac{2e}{\|m - \tilde{m}\|} \right).
\]
Combining these gives the claim. $\square$

From this and the continuity estimates of $H_N$ and $f_N$ we obtain the following continuity estimate for $H_{\text{TAP}}^{\text{Ising}}(m)$. 

Lemma 7.4. Let $L, \beta, \xi$ and $f_N$ be like in Theorem 1.1. There is a constant $c$ such that on the event $\mathcal{L}_N$ from (6.6) it holds for all $m, \tilde{m} \in B_N^2$ that
\begin{equation}
|H_{TAP}^{Ising}(m) - H_{TAP}^{Ising}(\tilde{m})| \leq c (1 + L^3) N \|m - \tilde{m}\| \log \frac{c}{\|m - \tilde{m}\|}
\end{equation}

Proof. We treat each term of $H_{TAP}^{Ising}(m)$ separately.

Lemma 7.3 gives the sufficient bound for the entropy term $I_{Ising}$.

On the event (6.6) we have
\begin{equation}
|\beta H_N(m) - \beta H_N(\tilde{m})| \leq c L^2 N \|m - \tilde{m}\| \text{ for all } m, \tilde{m} \in B_N^2.
\end{equation}

By the Lipschitz assumption
\begin{equation}
|\beta f_N(m) - \beta f_N(\tilde{m})| \leq L^2 N \|m - \tilde{m}\| \text{ for all } m, \tilde{m} \in B_N.
\end{equation}

For the Onsager term, note that from (1.3) it follows that On $(q)$ is Lipschitz with constant bounded by $c \xi''(1) \leq c L$ on $[0,1]$, so that since
\begin{equation}
\|m\|^2 - \|\tilde{m}\|^2 \leq \|m\| + \|\tilde{m}\| \|m\| - \|\tilde{m}\| \leq 2 \|m - \tilde{m}\|
\end{equation}

we have
\begin{equation}
\left|N \frac{\beta^2}{2} \text{On}(\|m\|^2) - N \frac{\beta^2}{2} \text{On}(\|\tilde{m}\|^2)\right| \leq c L^3 N \|m - \tilde{m}\| \text{ for all } m, \tilde{m} \in B_N.
\end{equation}

Combing all of the above imply (7.13), since for a large enough $c$ we have that $\log \frac{c}{\|m - \tilde{m}\|} \geq 1$ for $m, \tilde{m} \in B_N$.

Finally we derive the TAP upper bound for the Ising SK model from the general TAP upper bound.

Proof of Theorem 1.1. Let $\tilde{\delta} > 0$. By the general bound Theorem 1.3 with $\tilde{\delta}$ in place of $\delta$ we obtain that
\begin{equation}
\mathbb{P} \left( \log E \left[ \exp \left( \beta H_N^f(\sigma) \right) \right] \leq \sup_{m \in B_N} H_{TAP}^{E,\tilde{\delta}}(m) + \tilde{\delta} N \right) \geq 1 - \bar{\kappa} e^{-\frac{\tilde{\delta}}{2} N},
\end{equation}

where $\bar{\kappa} = c \max(K, (\beta L / \tilde{\delta})^8) \leq c \max(K, L^6 / \tilde{\delta}^8)$.

By the definitions (1.4), (1.12) of $H_{TAP}^{Ising}(m)$ and $H_{TAP}^{E,\delta}$ and Lemma 7.2 with $\tilde{\delta}$ in place of $\delta$
\begin{equation}
\sup_{m \in B_N^2} H_{TAP}^{E,\delta}(m) \leq \sup_{m \in B_N^2 : d(m, (-1,1)^N) \leq \tilde{\delta}} H_{TAP}^{Ising}(m) + c N \tilde{\delta} \log \tilde{\delta}^{-1}.
\end{equation}

Using Lemma 7.4 we obtain that on the event $\mathcal{L}_N$
\begin{equation}
\sup_{m \in B_N^2 \setminus (-1,1)^N : d(m, (-1,1)^N) \leq \delta} H_{TAP}^{Ising}(m) \leq \sup_{m \in (-1,1)^N} H_{TAP}^{Ising}(m) + c (1 + L^3) N \tilde{\delta} \log \frac{c}{\delta}.
\end{equation}

By picking $\delta > 0$ small enough depending on $\delta, L$, combing (7.14)-(7.16) and using (6.7) we obtain that
\begin{equation}
\mathbb{P} \left( \log E \left[ \exp \left( \beta H_N^f(\sigma) \right) \right] \leq \sup_{m \in B_N^2} H_{TAP}^{Ising}(m) + \delta N \right) \geq 1 - \bar{\kappa} e^{-\frac{\tilde{\delta}}{2} N} - e^{-N}.
\end{equation}
Now by picking $c_1 \leq \frac{4}{3}$ and small enough so that $(cK)^r K \leq e^{\frac{4}{3}N}$ for all $K \leq c_1 N/\log N$ and $N \geq 1$, and $1 + (cL^1/\delta^8)^{cL^1/\delta^8} \leq c_1^{-1}$, we get that the right-hand side is at least $1 - c_1^{-1}e^{-c_1 N}$, giving the claim (1.7).

8. Bounding the spherical entropy and proof of spherical TAP upper bound

In this section we derive the TAP upper bound for the spherical mixed SK model from the general TAP upper bound, by explicitly bounding the entropy function $I_{E,\delta}$ from (1.11) in terms of by $I_{sph}$ from (1.9).

Lemma 8.1 (Entropy for spherical reference measure). Let $E$ be the uniform measure on $S_{N-1}$ and let $I_{E,\delta}$ be as in (1.11). For all $\delta \in (0, 1)$ and all large enough $N$ depending on $\delta$ we have for all $m \in B_N^c$

\[(8.1) \quad I_{E,\delta}(m) \leq \frac{N}{2} \log \left(1 - \|m\|^2 + 2\delta \|m\|\right) + \delta N \leq I_{sph}(m) + N\delta \left(1 + \frac{\|m\|}{1 - \|m\|^2}\right).\]

Proof. Firstly simply by choosing $\lambda = \frac{m}{\|m\|}$ in (1.11) we have that

\[(8.2) \quad I_{E,\delta}(m) \leq E \left[\langle \sigma, \frac{m}{\|m\|} \rangle \geq \|m\| - \delta\right].\]

Next by [BK19], (2.8) (whose $E$ is the uniform distribution on $\{\sigma \in \mathbb{R}^N : |\sigma| = 1\}$) we have for any $u$ with $\|u\| = 1$ that

\[E \{\sigma : \langle \sigma, u \rangle \geq \alpha\} = \int_0^1 \frac{1}{\sqrt{\pi}} \frac{\Gamma(N/2)}{\Gamma(N/2 - 1/2)} \left(1 - x^2\right)^{N/2 - 3} dx \leq \frac{N}{2\pi} \left(1 - \alpha^2\right)^{N/2},\]

where we used that $\frac{\Gamma(N/2)}{\Gamma(N/2 - 1/2)} \leq \sqrt{\frac{N}{2}}$. Taking the log of both sides we get that for all $\alpha \leq 1 - \delta$

\[\log E \{\sigma : \langle \sigma, u \rangle \geq \alpha\} \leq \frac{N}{2} \log \left(1 - \alpha^2\right) + \delta N,
\]

since $\log \sqrt{\frac{N}{2\pi} (1 - \alpha^2)^{-N/2}} \leq \delta N$ for all $N$ large enough for such $\alpha$. Applying this to the right-hand side of (8.2) with $\alpha = \|m\| - \delta$ gives the first inequality of (8.1). Recalling (1.9), the second inequality is elementary. \[\square\]

We now we give the proof of Theorem 1.2 from the general result Theorem 1.3. We will use that by (A.8) and $\xi(1) \leq \xi'(1)$

\[(8.3) \quad \mathbb{P} \left(\sup_{m \in B_N} |H_N(m)| \geq c \sqrt{\xi'(1)}\right) \leq e^{-N} \text{ for all } N \geq 1,
\]

for a large enough universal $c$. 
Proof of Theorem 1.2. By the general bound Theorem 1.3 with \( \tilde{\delta} \) in place of \( \delta \) we obtain that for any \( \delta > 0 \)

\[
\mathbb{P}\left( \log E \left[ \exp \left( \beta H_N^f (\sigma) \right) \right] \leq \sup_{m \in B_N} H_{\text{TAP}}^{E,\delta} (m) + \tilde{\delta} N \right) \geq 1 - \tilde{\kappa}^\delta e^{-\frac{\tilde{\delta}^2}{4} N},
\]

where \( \tilde{\kappa} = c \max(K, (\beta L/\tilde{\delta})^8) \leq c \max(K, L^6/\tilde{\delta}^8) \). Let \( \gamma \in (0, 1) \) to be fixed later. Lemma 8.1 implies that for any \( \tilde{\delta} \in (0, 1) \) it holds for \( N \geq c(\tilde{\delta}) \) that

\[
I_{E,\tilde{\delta}} (m) \leq I_{\text{sph}} (m) + cN \frac{\tilde{\delta}}{\gamma}, \quad \text{for } m \in B_N^c \text{ such that } 1 - \|m\|^2 \geq \gamma.
\]

Thus

\[
\sup_{m: \|m\|^2 \leq 1 - \gamma} H_{\text{TAP}}^{E,\delta} (m) \leq \sup_{m: \|m\|^2 \leq 1 - \gamma} H_{\text{TAP}}^{\text{sph}} (m) + \frac{c \tilde{\delta}}{\gamma} N.
\]

To deal with the supremum over \( m \) such that \( 1 - \|m\|^2 \leq \gamma \) we note that the crude bound

\[
\mathbb{P}\left( \sup_{m \in B_N^c} \left\{ \beta H_N^f (m) + \frac{N}{2} \beta^2 f_N (0) + \frac{N}{2} \log \left( \gamma + 2 \tilde{\delta} \right) \right\} \leq \beta f_N (0) + cL^2 N \right) \geq 1 - e^{-N},
\]

holds by the bound (8.3) (recall \( \xi' (1) \leq \xi'' (1) \)), the Lipschitz assumption on \( f_N \) and that fact that \( \text{On} \) is bounded by \( \bar{\xi} (1) \) (see its definition (1.3)). Also for \( m \) such that \( 1 - \|m\|^2 \leq \gamma \) it holds by Lemma 8.1 for \( N \geq c(\tilde{\delta}) \) that

\[
I_{E,\tilde{\delta}} (m) \leq N \left( \frac{1}{2} \log \left( \gamma + 2 \tilde{\delta} \right) + \tilde{\delta} \right).
\]

Thus first choosing \( \gamma, \tilde{\delta} \) small enough depending on the \( cL^2 \) in (8.6) we have for such \( m \)

\[
H_{\text{TAP}}^{E,\tilde{\delta}} (m) \leq \beta f_N (0) + cL^2 N + \frac{N}{2} \log \left( \gamma + 2 \tilde{\delta} \right) \leq \beta f_N (0) \leq H_{\text{TAP}}^{\text{sph}} (0),
\]

on the event in (8.6), which implies

\[
\sup_{m \in B_N^c : \|m\|^2 > 1 - \gamma} H_{\text{TAP}}^{E,\delta} (m) \leq H_{\text{TAP}}^{\text{sph}} (0).
\]

Then picking \( \tilde{\delta} \) possibly even smaller depending on \( \delta \) so that \( c \frac{\tilde{\delta}}{\gamma} \leq \delta \), we obtain from (8.4), (8.5) and (8.6) that

\[
\mathbb{P}\left( \log E \left[ \exp \left( \beta H_N^f (\sigma) \right) \right] \leq \sup_{m \in B_N} H_{\text{TAP}}^{\text{sph}} (m) + \delta N \right) \geq 1 - \bar{\kappa}^\delta e^{-\frac{\delta^2}{4} N} - e^{-N}.
\]

Now by picking \( \tilde{c}_2 \leq \frac{\tilde{\delta}}{4} \) and small enough so that \( \bar{\kappa}^\delta \leq (cK)^{cK} \leq e^{\frac{c}{2} N} \) for all \( K \leq \bar{c}_2 N / \log N \) and \( N \geq 1 \), and also \( 1 + \left( cL^6/\bar{\delta}^8 \right) cL^6/\bar{\delta}^8 \leq \bar{c}_2^{-1} \) we get that the right-hand side is at least \( 1 - \bar{c}_2^{-1} e^{-\bar{c}_2 N} \), proving the claim (1.10) with \( c_2 = \bar{c}_2 \) for \( N \geq c(\delta) \). By possibly making \( c_2 \) even smaller depending on \( \delta \) the claim (1.10) can be made to hold for all \( N \geq 1 \). \( \square \)
Appendix A.

Here we collect some basic properties about the random field $H_N$ that follow from the classical theory of Gaussian processes.

We furthermore use the inner product $a \cdot b = a \cdot b/N$ for $a, b \in \mathbb{R}^N$ and the norm $\| \cdot \| = |\cdot|/\sqrt{N}$. Recall that $B_N(r) \subset \mathbb{R}^N$ is the closed and $B_N^2(r) \subset \mathbb{R}^N$ the open ball of radius $r$ in the $\| \cdot \|$-norm, and $S_N-1(r)$ is the the sphere of $\| \cdot \|$-radius $r$. The first lemma gives the existence of $H_N$.

**Lemma A.1.** Let $r > 0$. If $\xi(x) = \sum_{p \geq 0} a_p x^p$ is a power series with non-negative coefficients $a_p \geq 0$ such that $\xi(r^2) < \infty$ and $N \geq 1$ then there exists a centered Gaussian process $(H_N(\sigma))_{\sigma \in B_N(r)}$ with covariance

\begin{equation}
E[H_N(\sigma)H_N(\sigma')] = N\xi(\langle \sigma, \sigma' \rangle) \quad \text{for all } \sigma, \sigma' \in B_N(r).
\end{equation}

**Proof.** If $\xi(r^2) < \infty$ then $\xi(q) < \infty$ for all $q \in [-r^2, r^2]$, so $\xi(\langle \sigma, \sigma' \rangle)$ is well-defined for all $\sigma, \sigma' \in B_N(r)$. By Schoenberg’s theorem the function $\langle \sigma, \sigma' \rangle \rightarrow N\xi(\langle \sigma, \sigma' \rangle)$ is positive semi-definite [Sch42], so by standard existence results (e.g. [RY99, Chapter 1, Proposition 3.7]) one can construct the Gaussian process $H_N$. □

In the rest of the appendix we will show that $H_N$ is also a smooth function on $B_N^2(r)$, and provide useful regularity estimates. The first one is the following. Let $\| \cdot \|_{L^2}$ denote the $L^2$ norm on the linear space of random variables.

**Lemma A.2.** For any $r, \xi, N, H_N$ as in Lemma A.1 and $0 < s \leq r$ we have

\begin{equation}
\xi(\| \sigma \|^2) + \xi(\| \sigma' \|^2) - 2\xi(\langle \sigma, \sigma' \rangle) \leq 8s\xi'(s^2)\| \sigma - \sigma' \|,
\end{equation}

and

\begin{equation}
\| H_N(\sigma) - H_N(\sigma') \|_{L^2}^2 \leq 8s\xi'(s^2)N\| \sigma - \sigma' \|,
\end{equation}

for $\sigma, \sigma' \in B_N(s)$.

**Remark A.3.** When $s = r$ and $\xi'(r^2) = \infty$ we interpret the RHS of (A.2) and (A.3) as $\infty$, so that the statements are vacuous. Below we use the same interpretation in (A.7), (A.8) and (A.13).

**Proof.** Let $\Delta = \sigma' - \sigma$ and

\[ f(\lambda) = \xi(\| \sigma \|^2) + \xi(\| \sigma + \lambda \Delta \|^2) - 2\xi(\langle \sigma, \sigma + \lambda \Delta \rangle), \lambda \in [0, 1]. \]

We have $f(0) = 0$ and

\[ \xi(\| \sigma \|^2) + \xi(\| \sigma' \|^2) - 2\xi(\langle \sigma, \sigma' \rangle) = f(1) = \int_0^1 f'(\lambda) d\lambda. \]

Furthermore

\[ f'(\lambda) = \xi'(\| \sigma + \lambda \Delta \|^2) (2 \langle \sigma, \Delta \rangle + 2\lambda \| \Delta \|^2) - 2\xi'(\langle \sigma, \sigma + \lambda \Delta \rangle) \langle \sigma, \Delta \rangle, \]

so that

\[ \sup_{\lambda \in [0,1]} |f'(\lambda)| \leq \xi'(s^2) (4|\langle \sigma, \Delta \rangle| + 2\| \Delta \|^2). \]
Since $4 \| \langle \sigma, \Delta \rangle \| + 2 \| \Delta \|^2 \leq 4 \| \sigma \| \| \Delta \| + 4 s \| \Delta \| \leq 8 s \| \Delta \|$ the claim (A.2) follows.

The estimate (A.3) is an immediate consequence of (A.2) and (A.1). \hfill \Box

We encapsulate some classical results on the regularity of Gaussian processes in the following lemma.

**Lemma A.4.** There is a universal constant $c$ such that the following holds. Let $N \geq 1$, $s > 0$ and let $T \subset B_N(s)$ be a set. For $a \in (0, \infty)$ assume that $X_\sigma, \sigma \in T$, is a centered Gaussian process such that

\[(A.4) \quad \| X_\sigma - X_{\sigma'} \|_{L^2}^2 \leq a N \| \sigma - \sigma' \| \quad \text{for all } \sigma, \sigma' \in T.\]

Then $(X_\sigma)_{\sigma \in T}$ is almost surely continuous, and

\[(A.5) \quad \mathbb{E} \left( \sup_{\sigma \in T} |X_\sigma| \right) \leq c N \sqrt{sa},\]

and

\[(A.6) \quad \mathbb{P} \left( \sup_{\sigma \in T} |X_\sigma| \geq u \right) \leq e^{-\frac{u^2}{8 \sup_{\sigma \in T} \mathbb{E}[X_\sigma^2]}} \quad \text{for all } u \geq 2c\sqrt{sa}.\]

**Proof.** We use Dudley’s entropy bound [AW09, Theorem 1.18, Theroem 2.10]. Consider the distance $d(\sigma, \sigma') = \| X_\sigma - X_{\sigma'} \|_{L^2}$ on $T$. From (A.4) we have $d(\sigma, \sigma') \leq \sqrt{a} N^{1/4} | \sigma - \sigma'|^{1/2}$. Thus if one covers $B_N(s)$ with Euclidean balls of $|\cdot|$-radius $\varepsilon/(c_1 N^{1/4})^2$ then balls of $d$-radius $\varepsilon$ centered at the same points also cover $B_N(s) \supseteq T$. We have

\[
\frac{\text{Vol} \left( \left\{ \sigma : |\sigma| \leq s \sqrt{N} \right\} \right)}{\text{Vol} \left( \left\{ \sigma : |\sigma| \leq \frac{1}{2} \left( \varepsilon / (\sqrt{a} N^{1/4}) \right)^2 \right\} \right)} = \left( \frac{s \sqrt{N}}{\frac{1}{2} \left( \varepsilon / (\sqrt{a} N^{1/4}) \right)^2} \right)^N = (c_1 N / \varepsilon^2)^N,
\]

with $c_1 = 2sa$. Thus $B_N(s)$ can be covered with at most at most $N(\varepsilon) = \max((c_1 N / \varepsilon^2)^N, 1)$ Euclidean balls of $|\cdot|$-radius $(\varepsilon/(c_1 N^{1/4}))^2$. We have

\[
\int_0^\infty \sqrt{\log N(\varepsilon)} = \sqrt{2N} \int_0^{\sqrt{c_1 N}} \sqrt{\log \frac{\sqrt{c_1 N}}{\varepsilon}} d\varepsilon = N \sqrt{2c_1} \int_0^1 \sqrt{\log \frac{1}{\varepsilon}} d\varepsilon.
\]

This is finite, thus [AW09, Theorem 1.18] implies that $X_\sigma, \sigma \in T$, is continuous. Also [AW09, Theroem 2.10] implies (A.5). Lastly the Borell-TIS inequality [AW09, Theorem 2.8] implies (A.6). \hfill \Box

Applying this to $H_N$ yields the following.

**Lemma A.5.** There is a universal constant $c$, such that for any $r, \xi, N, H_N$ as in Lemma A.1 we have that $H_N(\sigma)$ is continuous almost surely for $\sigma \in B_N^\circ(r)$ (and if $\xi'(r^2) < \infty$ also for $\sigma \in B_N(r)$) and for $0 \leq s \leq r$ it holds that

\[(A.7) \quad \mathbb{E} \left[ \sup_{\sigma \in B_N(s)} |H_N(\sigma)| \right] \leq c s \sqrt{\xi'(s^2)} N,\]
and

\[
\mathbb{P} \left( \sup_{\sigma \in B_N(s)} |H_N(\sigma)| \geq u \right) \leq e^{-\frac{u^2}{8\xi'(s^2)N}} \text{ for all } u \geq 2cs\sqrt{\xi'(s^2)N}. \tag{A.8}
\]

**Proof.** Provided \( \xi'(s^2) < 1 \) the continuity of \( H_N(\sigma), \sigma \in B_N(s), (A.7) \) and (A.8) follow by applying Lemma A.4 with \( T = B_N(s), X_\sigma = H_N(\sigma) \) and \( a = 8s\xi'(s^2) \), since we have the bound (A.3). If \( \xi'(r^2) < \infty \) then with \( s = r \) continuity on \( B_N(r) \) follows. If \( \xi'(r^2) = \infty \) then continuity on \( B_N(s) \) for all \( s < r \) (note \( \xi'(s^2) < \infty \) for \( s < r \)) implies continuity on \( B_N^0(r) \).

We now turn to the derivatives of \( H_N \). For a multi-index \( \alpha \in \{1, \ldots, N\}^l \) let \( \partial_\alpha \) denote the corresponding partial derivative, and write \( \partial_\alpha^x \) for the partial derivative with respect to a variable \( x \). Let \( e_1, \ldots, e_N \) denote the standard basis vectors of \( \mathbb{R}^N \). The next lemma will be applied with \( C \) the covariance of a Gaussian process to prove differentiability.

**Lemma A.6.** If \( 0 < s < r \) and \( C : B_N^0(r) \times B_N^0(s) \rightarrow \mathbb{R} \) has derivatives up to fourth order that are Lipschitz functions and \( i, j \in \{1, \ldots, N\} \), then there exists a Lipschitz function \( R : B_N^0(s) \times B_N^0(s) \times (0, r-s)^2 \rightarrow \mathbb{R} \) that satisfies \( R(\sigma, \sigma', \eta_1, \eta_2) = O(\text{snorm}) \) such that for all \( \sigma, \sigma' \in B_N^0(s) \) and \( 0 < \eta, \eta' \leq r-s \)

\[
(1) \quad \frac{1}{\eta_1\eta_2} \sum_{s_1, s_2 \in \{0, 1\}} (-1)^{s_1+s_2} C(\sigma + s_1\eta_1 e_i, \sigma + s_2\eta_2 e_j) = \partial_\sigma^i \partial_\sigma'^j C(\sigma, \sigma') + R(\sigma, \sigma', \eta, \eta'),
\]

where the constant in the \( O \)-term, the Lipschitz constant of \( R \) and \( R \) itself depend on \( C, i, j \).

**Proof.** The LHS can be written as

\[
\frac{1}{\eta_2} \sum_{s_2 \in \{0, 1\}} (-1)^{s_2} \frac{C(\sigma + \eta_1 e_i, \sigma + s_2\eta_2 e_j) - C(\sigma, \sigma + s_2\eta_2 e_j)}{\eta_1}.
\]

By Taylor’s theorem with integral remainder in the \( \sigma \) variable this equals

\[
\frac{1}{\eta_2} \sum_{s_2 \in \{0, 1\}} (-1)^{s_2} \left( \partial_\sigma^i C(\sigma, x + s_2\eta_2 e_j) + \eta_1 \int_0^1 (1-t) (\partial_\sigma^i)^2 C(\sigma + t\eta_1 e_i, \sigma' + s_2\eta_2 e_j) dt \right)
\]

\[
= \frac{1}{\eta_2} \sum_{s_2 \in \{0, 1\}} (-1)^{s_2} \sum_{g_1 \in \{0, 1\}} \eta_1^{g_1} \int_0^1 (\partial_\sigma^i)^{1+g_1} C(\sigma + t\eta_1 e_i, \sigma' + s_2\eta_2 e_j) dt.
\]

Next applying the same theorem in \( \sigma' \) this equals

\[
\sum_{g_1, g_2 \in \{0, 1\}} \eta_1^{g_1} \eta_2^{g_2} \int_0^1 \int_0^1 (\partial_\sigma'^j)^{1+g_2} (\partial_\sigma'^i)^{1+g_1} (1-t_1)(1-t_2) C(\sigma + t_1\eta_1 e_i, \sigma' + t_2\eta_2 e_j) dt dt_2.
\]

The summand when \( g_1 = g_2 = 0 \) is \( \partial_\sigma^i \partial_\sigma'^j C(\sigma, \sigma') \), and the other summands are \( O(\text{snorm}) \) and Lipschitz in \( \eta_1, \eta_2, \sigma, \sigma' \) by the assumption on \( C \). With this lemma we can prove the existence of the derivatives of \( H_N \).
Lemma A.7. Let \( r, \xi, N, H_N \) be as in Lemma A.1. Then \( H_N \) is almost surely smooth on \( B_N^o (r) \) and \((\partial_\alpha H_N (\sigma))_{\sigma \in B_N^o (r), \alpha \in \cup_{i=1}^N (1, \ldots, N)^i} \) is a centered Gaussian process with covariance
\[
\mathbb{E} [\partial_\alpha H_N (\sigma) \partial_{\alpha'} H_N (\sigma')] = N \partial_\alpha^\sigma \partial_{\alpha'}^\sigma \xi (\langle \sigma, \sigma' \rangle) .
\]

Proof. Since \( \xi \) is a convergent power series in \( B_N^o (r) \), the function \((\sigma, \sigma') \rightarrow \xi (\langle \sigma, \sigma' \rangle) \) is infinitely differentiable. Consider the statement
\[
\partial_\alpha H_N (\sigma) \text{ exists and is continuous for all } \sigma \in B_N^o (r) , |\alpha| \leq k, \\
and is a centered Gaussian process with covariance (A.10).
\]

If this holds for all \( k \) then it implies the claim of the lemma, since existence and continuity of partial derivatives implies differentiability.

To prove (A.11) we use induction on \( k \). The case \( k = 0 \) follows since \( H_N \) is continuous on \( B_N^o (r) \) by Lemma A.5. Assume (A.11) holds for \( k \leq l \). Fix an \( s \in (0, r) \). Consider for any \( \alpha \) with \(|\alpha| = l \) and any \( i = 1, \ldots, N \), \( 0 < \eta < r - s, \sigma \in B_N^o (s) \)
\[
\Delta_{\alpha, i} (\sigma, \eta) = \frac{\partial_\alpha H_N (\sigma + \eta e_i) - \partial_\alpha H_N (\sigma)}{\eta}.
\]

Note that for \( 0 < \eta, \eta' \leq r - s, \sigma, \sigma' \in B_N^o (s) \) and \( \alpha, \alpha' \) with \(|\alpha|, |\alpha'| \leq l \) the covariance
\[
N^{-1} \mathbb{E} [\Delta_{\alpha, i} (\sigma, \eta_1) \Delta_{\alpha', j} (\sigma', \eta_2)] = N \partial_\alpha^\sigma \partial_{\alpha'}^\sigma C (\sigma, \sigma') + R (\sigma, \sigma', \eta, \eta'),
\]
where \( R (\sigma, \sigma', \eta, \eta') \) is \( O (|\eta| + |\eta'|) \) and Lipschitz. This implies that \( \|\Delta_{\alpha, i} (\sigma, \eta) - \Delta_{\alpha, i} (\sigma, \eta')\|_2 \leq O (|\eta| + |\eta'|) \). Thus \( \Delta_{\alpha, i} (\sigma, \eta) \) for \( \eta \downarrow 0 \) is a 2\(^\text{nd} \) Cauchy sequence, and there exists a random variable \( D^i \partial_\alpha H_N (\sigma) \) such that \( \Delta_{\alpha, i} (\sigma, \eta) \rightarrow D^i \partial_\alpha H_N (\sigma) \) in \( L^2 \), as \( \eta \rightarrow 0 \). Also \( D^i \partial_\alpha H_N (\sigma) \) is a centered Gaussian since it is the limit of centered Gaussians, and jointly Gaussian with \( \partial_\alpha H_N (\sigma), \sigma \in B_N^o (s), 0 < \eta < r - s \). Next define
\[
g_i (\sigma, \eta) = \begin{cases} 
\Delta_{\alpha, i} (\sigma, \eta) & \text{if } \eta \neq 0, \\
D^i \partial_\alpha H_N (\sigma) & \text{if } \eta = 0. 
\end{cases}
\]

Then \( g_i (\sigma, \eta) \) is a Gaussian process on \( T = B_N^o (s) \times (- (r - s), r - s) \subset B_{N+1} \left( \sqrt{s^2 + r^2} \right) \).

By (A.12) and the Lipschitz property of \( R \)
\[
\|g_i (\sigma, \eta) - g_i (\sigma', \eta')\|_2 \leq c \sqrt{|\sigma - \sigma'|^2 + |\eta - \eta'|^2},
\]
first for \( \eta, \eta' \neq 0 \) and by the \( L^2 \) convergence to \( D^i \partial_\alpha H_N (\sigma) \) resp. \( D^i \partial_\alpha H_N (\sigma') \) for all \( \eta, \eta' \). Therefore by Lemma A.4 the process \( g_i (\sigma, \eta) \) is almost surely continuous. Thus the limit \( \lim_{\eta \downarrow 0} \Delta_{\alpha, i} (\sigma, \eta) \) exists and equals \( D^i \partial_\alpha H_N (\sigma) \) for all \( \sigma \), almost surely. Then for all \( i \) and \( \sigma \in B_N^o (s) \) the derivative \( \partial_\alpha \partial_\beta H_N (\sigma) = g_i (\sigma, 0) \) exists and is continuous. Since this holds for all \( s < r \) it also holds for \( s = r \), proving all of (A.11) for \( l = k + 1 \) except for the formula for the covariance. The latter then follows since for any \( \sigma, \sigma' \in B_N^o (r), i, j = 1, \ldots, N, \alpha, \alpha' \) with \(|\alpha| = |\alpha'| \leq l \) and \( \eta, \eta' \) small enough we have
\[
\mathbb{E} [\Delta_{\alpha, i} (\sigma, \eta) \Delta_{\alpha', j} (\sigma', \eta')] \xrightarrow{(A.12)} N \partial_\alpha^\sigma \partial_{\alpha'}^\sigma C (\langle \sigma, \sigma' \rangle) = \partial_\alpha^\sigma \partial_{\alpha'}^\sigma \partial_{\alpha'}^\sigma \xi (\langle \sigma, \sigma' \rangle) \text{ as } \eta \rightarrow 0.
\]
Lastly we give a basic regularity estimate for the derivatives of $H_N$. The spectral norm of $\nabla^k H_N (\sigma) = (\partial_{i_1 \ldots i_k} H_N (\sigma))_{i_1, \ldots, i_k = 1, \ldots, N}$ viewed as a tensor is $\sup_{v: \|v\| = 1} |\nabla^k H_N (\sigma) v^{\otimes k}|$ where

\[
\nabla^k H_N (\sigma) v^{\otimes k} = \sum_{i_1, \ldots, i_p = 1}^N \partial_{i_1 \ldots i_p} H_N (\sigma) v_{i_1} \ldots v_{i_p}.
\]

We have the following.

**Lemma A.8.** Let $r, \xi, N, H_N$ be as in Lemma A.1, and $s < r$ and $k \geq 0$. We have for all $a, b \in B_N (s)$ and $v, w \in S_N (s)$

\[
\begin{align*}
\|\nabla^k H_N (a) v^{\otimes k} - \nabla^k H_N (b) w^{\otimes k}\|_2^2 &\leq c_k N s^{4k+1} \xi^{(2k+1)} (s^2) \sqrt{\|v - w\|^2 + \|a - b\|^2}, \\
\end{align*}
\]

where $c_k = c (k + 1)! 2^k$, and

\[
\begin{align*}
\sup_{v \in S_N (s)} \mathbb{E} \left[ (\nabla^k H_N (a) v^{\otimes k})^2 \right] &= N s^{2k} \sum_{l=0}^k \binom{k}{l} (k - l)! s^{2l} \xi^{(k+1)} (\|a\|^2).
\end{align*}
\]

**Proof.** By Lemma A.7

\[
\begin{align*}
\mathbb{E} \left[ (\nabla^k H_N (a) v^{\otimes k}) (\nabla^k H_N (b) w^{\otimes k}) \right] &= N \sum_{i_1 \ldots i_k} \partial_{i_1} \ldots \partial_{i_k} \nabla^k \xi (\langle a, b \rangle) v_{i_1} \ldots v_{i_k} w_{j_1} \ldots w_{j_k}.
\end{align*}
\]

We have

\[
\partial_{j_1} \ldots \partial_{j_k} \xi (\langle a, b \rangle) = N^{-k} a_{j_1} \ldots a_{j_k} \xi^{(k)} (\langle a, b \rangle),
\]

and by the product rule

\[
\partial_{i_1} a_{j_1} \ldots a_{j_k} \xi^{(k)} (\langle a, b \rangle) = \sum_{r=1}^k \delta_{i_k j_r} \prod_{l \neq r} a_{j_l} \xi^{(k)} (\langle a, b \rangle) + N^{-1} a_{j_1} \ldots a_{j_k} b_{i_k} \xi^{(k+1)} (\langle a, b \rangle).
\]

Applying also the other derivatives $\partial_{i_1} \ldots \partial_{i_{k-1}}$ we get a large sum, which can be expressed as follows. Letting $I$ denote the set of indices $r$ of $i_r$ where the derivative $\partial_{i_r}^a$ is applied to one of $a_{j_1}, \ldots, a_{j_k}$, and letting $\pi (r)$ denote the index of the factor it is applied to, we get that

\[
\begin{align*}
N \partial_{i_1}^a \ldots \partial_{i_k}^a \partial_{j_1}^b \ldots \partial_{j_k}^b \xi (\langle a, b \rangle) &= N^{1-k} \partial_{i_1}^a \ldots \partial_{i_k}^a a_{j_1} \ldots a_{j_k} \xi (\langle a, b \rangle) \\
&= \sum_{I} \sum_{\pi: I \to \{1, \ldots, k\}} \left( \prod_{r \in I} \delta_{i_r j_{\pi (r)}} \right) \left( \prod_{r \notin I} b_{i_r} \right) \left( \prod_{r \notin \pi (I)} a_{j_r} \right) N^{1-(2k-|I|)} \xi^{(2k-|I|)} (\langle a, b \rangle),
\end{align*}
\]

where $I$ is summed over all subsets of $\{1, \ldots, k\}$ and $\pi$ over all injective maps from $I$ to $\{1, \ldots, k\}$. Applying the sums over $i_1, \ldots, i_k, j_1, \ldots, j_k$ from (A.15) we get

\[
\begin{align*}
\mathbb{E} \left[ (\nabla^k H_N (a) v^{\otimes k}) (\nabla^k H_N (b) w^{\otimes k}) \right] &= N \sum_{I} \sum_{\pi: I \to \{1, \ldots, k\}} G_{|I|} (a, v, b, w),
\end{align*}
\]

where

\[
G_{n} (a, v, b, w) = \langle v, w \rangle^n \langle a, w \rangle^{k-n} \langle b, v \rangle^{k-n} \xi^{(2k-n)} (\langle a, b \rangle).
\]
From this (A.14) follows. It also implies that
\begin{equation}
(A.16) \quad \| \nabla^k H_N (a) w^{\otimes k} - \nabla^k H_N (b) w^{\otimes k} \|^2_{L^2} = N \sum I \sum_{x \in I \rightarrow \{1, \ldots, k\}} \{ G_{\xi | I} (a, v, a, v) + G_{\xi | I} (b, w, b, w) - 2 G_{\xi | I} (a, b, w) \}.
\end{equation}
Using that \( \|a\|, \|b\|, \|v\|, \|w\| \leq s \sqrt{N} \) and \( |x^l - y^l| \leq |x - y| \max (|x|, |y|)^{l-1} \) we get
\[
\begin{align*}
|\langle v, v \rangle^n - \langle v, w \rangle^n| & \leq |\langle v, v \rangle - \langle v, w \rangle| k s^{2(n-1)} \leq k \|v - w\| s^{2n-1}, \\
|\langle a, w \rangle^k - \langle a, v \rangle^k| & \leq |\langle a, a \rangle - \langle a, v \rangle| k s^{2(k-n-1)} \leq k \|a - v\| s^{2k-2n-1}, \\
|\langle a, v \rangle^k - \langle b, v \rangle^k| & \leq |\langle a, a \rangle - \langle b, a \rangle| k s^{2(k-n-1)} \leq k \|a - b\| s^{2k-2n-1}, \\
|\langle a, v \rangle^k - \langle b, w \rangle^k| & \leq |\langle a, a \rangle - \langle b, w \rangle| k s^{2(k-n-1)} \leq k (\|a - b\| + \|v - w\|) s^{2k-2n-1}.
\end{align*}
\]
From these we obtain
\[
G_n (a, v, a, v) + G_n (b, w, b, w) - 2 G_n (a, b, w) \\
\leq c k (\|a - b\| + \|v - w\|) s^{4k-2n-1} \xi (2k-n) (s^2) \\
+ s^{4k-2n} \left\{ \xi (2k-n) (\langle a, a \rangle) + \xi (2k-n) (\langle b, b \rangle) - 2 \xi (2k-n) (\langle a, b \rangle) \right\}.
\]
Next by (A.2) and the inequality \( \xi (l) (s^2) \leq s^{2m} \xi (l+m) (s^2) \) this is at most
\[
k c k (\|a - b\| + \|v - w\|) s^{4k+1} \xi (2k+1) (s^2).
\]
Thus from (A.16) and \( \sum I \sum_{x \in I \rightarrow \{1, \ldots, k\}} 1 \leq 2^k k! \) the claim (A.13) follows. \( \square \)

From this we derive the following estimates for the spectral norm of \( \nabla^k H_N (\sigma) \).

**Lemma A.9.** There is a universal constant \( c \) such that the following holds. Let \( r, \xi, N, H_N \) be as in Lemma A.1, and \( 0 < s < r \). Then with \( c_k = c (k+1)2^k \)
\begin{equation}
(A.17) \quad \mathbb{E} \left[ \sup_{\sigma \in B_N (s)} \sup_{v : \|v\| = 1} | \nabla^k H_N (\sigma) v^{\otimes k} | \right] \leq \sqrt{c_k N s^{k+1}} \sqrt{\xi (2k+1) (s^2)},
\end{equation}
and letting \( w^2 = s^{2k} \sum_{l=0}^{k} \binom{k}{l} (k-l)! s^{2l} \xi (k+l) (s^2) \) also
\begin{equation}
(A.18) \quad \mathbb{P} \left[ \sup_{\sigma \in B_N (s)} \sup_{v : \|v\| = 1} | \nabla^k H_N (\sigma) v^{\otimes k} | \geq u \right] \leq e^{-u^2 / s^{2k+1}} \text{ for } u \geq 2 \sqrt{c_k N s^{k+1}} \sqrt{\xi (2k+1) (s^2)}.
\end{equation}

**Proof.** Let \( T = B_N (s) \times S_{N-1} (s) \). It follows from Lemma A.7 that \( \nabla^k H_N (\sigma) v^{\otimes k}, (\sigma, v) \in T \), is a centered Gaussian process. The claims then follow from (A.13) and Lemma A.4 with \( 2N \) in place of \( N \), \( 2s \) in place of \( s \), \( T \subset B_{2N} (\sqrt{2s}) \) and \( a = c_k N s^{4k+1} \xi (2k+1) (s^2) \), after dividing by \( s^k \) to normalize \( v \). \( \square \)

In this article we use the following special cases. Recall that \( B_N = B_N (1), B_N^o = B_N^\circ (1) \).

**Lemma A.10.** If \( \xi (x) = \sum_{p \geq 0} a_p x^p \) is a power series with non-negative coefficients \( a_p \geq 0 \) such that \( \xi (1) < \infty \) and \( N \geq 1 \), then there exists a centered Gaussian process \( (H_N (\sigma))_{\sigma \in B_N} \) with covariance
\[
\mathbb{E} [H_N (\sigma) H_N (\sigma')] = N \xi (\langle \sigma, \sigma' \rangle) \text{ for all } \sigma, \sigma' \in B_N,
\]
which is almost surely differentiable in $B_N^\circ$. Also $(H_N(\sigma), \nabla H_N(\sigma))$ is a centered Gaussian process satisfying for all $i, j = 1, \ldots, N$ and $\sigma, \sigma' \in B_N^\circ$

\begin{equation}
\mathbb{E}[\partial_i H_N(\sigma) \partial_j H_N(\sigma')] = \delta_{ij} \xi'(\langle \sigma, \sigma' \rangle) + \frac{\sigma_j \sigma'_i}{N} \xi''(\langle \sigma, \sigma' \rangle)
\end{equation}

and

\begin{equation}
\mathbb{E}[H_N(\sigma) \partial_i H_N(\sigma')] = \sigma_i \xi'(\langle \sigma, \sigma' \rangle).
\end{equation}

Proof. These are a special case of Lemmas A.1, A.7. □

Lemma A.11. For $\xi, N, H_N$ as in Lemma A.10 with $\xi'(1) < \infty$ it holds that

\begin{equation}
\mathbb{P}\left( \sup_{\sigma \in B_N} |H_N(\sigma)| \geq u \right) \leq e^{-\frac{u^2}{8N \xi'(1)}} \text{ for all } u \geq c\sqrt{\xi'(1)N},
\end{equation}

and if also $\xi''(1) < \infty$ it holds for all $u \geq c\xi''(1)$ that

\begin{equation}
\mathbb{P}\left( \sup_{\sigma \in B_N^\circ} \|\nabla H_N(\sigma)\| \geq u \right) \leq e^{-\frac{u^2}{8(\xi''(1)+\xi'(1))}N},
\end{equation}

and

\begin{equation}
\mathbb{P}(\exists \sigma, \sigma' \in B_N^\circ \text{ s.t. } |H_N(\sigma) - H_N(\sigma')| \geq uN\|\sigma - \sigma'\|) \leq e^{-\frac{u^2}{8(\xi''(1)+\xi'(1))}N}.
\end{equation}

Proof. The bound (A.21) is a special case of (A.8), and (A.22) is the case $k = 1$ of Lemma A.9 and implies (A.23) via the mean value theorem. □

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