THE TOPOLOGICAL TYPE OF SPACES CONSISTING OF CERTAIN METRICS ON LOCALLY COMPACT METRIZABLE SPACES WITH THE COMPACT-OPEN TOPOLOGY

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Abstract. For a separable locally compact but not compact metrizable space $X$, let $\alpha X = X \cup \{x_\infty\}$ be the one-point compactification with the point at infinity $x_\infty$. We denote by $EM(X)$ the space consisting of admissible metrics on $X$, which can be extended to an admissible metric on $\alpha X$, endowed with the compact-open topology. Let $c_0 \subset (0, 1)^\mathbb{N}$ be the space of sequences converging to 0. In this paper, we shall show that if $X$ is separable, locally connected and locally compact but not compact, and there exists a sequence $\{C_i\}$ of connected sets in $X$ such that for all positive integers $i, j \in \mathbb{N}$ with $|i - j| \leq 1$, $C_i \cap C_j \neq \emptyset$, and for each compact set $K \subset X$, there is a positive integer $i(K) \in \mathbb{N}$ such that for any $i \geq i(K), C_i \subset X \setminus K$, then $EM(X)$ is homeomorphic to $c_0$.

1. Introduction

Throughout this paper, spaces are separable metrizable, maps are continuous, but functions are not necessarily continuous. Let $\mathbb{R}$ be the space of real numbers with the usual metric, $\mathbb{N}$ be the set of positive integers, and $X$ be a locally compact space. We denote by $C(X^2)$ the space of continuous real-valued functions on $X^2$ equipped with the compact-open topology. In the case that $X$ is locally connected, $C(X^2)$ is a Fréchet space. Let $PM(X), M(X)$ and $AM(X)$ be the spaces of continuous pseudometrics, continuous metrics and admissible metrics on $X$ with the relative topology of $C(X^2)$, respectively. As is easily observed, $PM(X)$ is a convex non-negative cone, and $M(X)$ and $AM(X)$ are convex positive cones in $C(X^2)$. Recall that $M(X) = AM(X)$ when $X$ is compact. However, they are not necessarily coincident in general.

Example 1. On the half open interval $[0, 1)$ topologized by the usual metric, define a continuous metric $d$ as follows:

$$d(x, y) = \min\{|x - y|, 1 + x - y, 1 - x + y\}$$

for all $x, y \in [0, 1)$. Then $d$ is not admissible.

If $X$ is not compact, then it has the one-point compactification $\alpha X = X \cup \{x_\infty\}$ with the point at infinity $x_\infty$ topologized by the following collection

$$\{U, (X \setminus K) \cup \{x_\infty\} \mid U \text{ is open in } X, K \text{ is compact in } X\}.$$ 

In the paper, we shall investigate the topological type of the following subspace of $AM(X)$:

$$EM(X) = \{d \in AM(X) \mid d \text{ can be extended to an admissible metric on } \alpha X\}.$$
Topologies of function spaces have been studied in the theory of infinite-dimensional topology. We denote the Hilbert cube by $Q^2$.

Let
\[
c_0 = \left\{ (x(n))_{n \in \mathbb{N}} \in s \mid \lim_{n \to \infty} x(n) = 0 \right\},
\]
that is homeomorphic to several function spaces, refer to [13, 5, 15, 14, 8]. We will establish the following:

**Main Theorem.** Let $X$ be a locally connected but not compact space. If there is a sequence $\{C_i\}$ consisting of connected sets in $X$ such that for all $i, j \in \mathbb{N}$ with $|i - j| \leq 1$, $C_i \cap C_j \neq \emptyset$, and for each compact subset $K$ of $X$, there exists $i(K) \in \mathbb{N}$ such that for every $i \geq i(K)$, $C_i \subset X \setminus K$, then $EM(X)$ is homeomorphic to $c_0$.

2. Preliminaries

Given spaces $A \subset Y$, denote the interior of $A$ by $\text{int} A$ and the closure of $A$ by $\text{cl} A$. For functions $f : Z \to Y$ and $g : Z \to Y$, and for an open cover $U$ of $Y$, $f$ is said to be $U$-close to $g$ if for every $z \in Z$, there is $U \in U$ containing $f(z)$ and $g(z)$. A closed set $A \subset Y$ is a $Z$-set if for each open cover $U$ of $Y$, there exists a map $f : Y \to Y$ such that $f$ is $U$-close to the identity map and $f(Y) \cap A = \emptyset$. A $Z_{\sigma}$-set is a countable union of $Z$-sets. A $Z$-embedding is an embedding whose image is a $Z$-set. Given a class $\mathcal{C}$ of spaces, we call $Y$ to be strongly $\mathcal{C}$-universal if the following condition holds.

- Let $A$ be a space in $\mathcal{C}$ and $f : A \to Y$ be a map. Assume that $B$ is a closed set in $A$ and the restriction $f|_B$ is a $Z$-embedding. Then for each open cover $U$ of $Y$, there exists a $Z$-embedding $g : A \to Y$ such that $g$ is $U$-close to $f$ and $g|_B = f|_B$.

Let $\mathcal{C}_{\sigma}$ denote the class of spaces that are countable unions of closed subspaces belonging to $\mathcal{C}$. For spaces $Y \subset M$, $Y$ is homotopy dense in $M$ provided that $M$ has a homotopy $h : M \times [0, 1] \to M$ such that $h(M \times (0, 1]) \subset Y$ and $h(y, 0) = y$ for any $y \in M$. A space $Y$ is said to be a $\mathcal{C}$-absorbing set in $M$ if it satisfies the following conditions.

1. $Y \in \mathcal{C}_{\sigma}$ and is homotopy dense in $M$.
2. $Y$ is strongly $\mathcal{C}$-universal.
3. $Y$ is contained in some $Z_{\sigma}$-set in $M$.

The symbol $\mathcal{M}_2$ stands for the class of absolute $F_{\sigma\delta}$-spaces, that is, $Y \in \mathcal{M}_2$ if $Y$ is an $F_{\sigma\delta}$-set in any space $M$ which contains $Y$. It is known that $c_0$ is an $\mathcal{M}_2$-absorbing set in $s$. Theorem 3.1 of [4] shows the topological uniqueness of absorbing sets in $s$.

**Theorem 2.1.** For subspaces $Y, Z \subset s$, if the both $Y$ and $Z$ are $\mathcal{M}_2$-absorbing sets in $s$, then $Y$ and $Z$ are homeomorphic.

In the paper [10], the author investigates the topological types of $PM(X)$ and $M(X)$, which are endowed with the uniform convergence topology. By the same argument as it, we can establish the following theorem.

**Theorem 2.2.** Let $X$ be a locally connected space. If $X$ is not discrete, then both $PM(X)$ and $M(X)$ are homeomorphic to $s$.

A sequence $\{C_i\}$ of subsets in a space $Y$ is a simple chain if for any $i, j \in \mathbb{N}$ with $|i - j| \leq 1$, $C_i \cap C_j \neq \emptyset$. It is said that $Y$ is chain-connected to infinity if there exists a simple chain $\{C_i\}$ of connected subsets in $Y$ such that for every compact set $K \subset Y$, there is $i(K) \in \mathbb{N}$ such that for any $i \geq i(K)$, $C_i \subset Y \setminus K$. 
For a metric space $Y = (Y, d_Y)$, a subset $A \subset Y$ and a positive number $\delta > 0$, put
\[ B_{d_Y}(A, \delta) = \{ y \in Y \mid d_Y(y, A) < \delta \} \quad \text{and} \quad \overline{B}_{d_Y}(A, \delta) = \{ y \in Y \mid d_Y(y, A) \leq \delta \} . \]

When $A = \{ a \}$, we write $B_{d_Y}(a, \delta) = B_{d_Y}(\{ a \}, \delta)$ and $\overline{B}_{d_Y}(a, \delta) = \overline{B}_{d_Y}(\{ a \}, \delta)$ for simplicity. Moreover, diam$^Y_A$ denotes the diameter of $A$. For metric spaces $Y_i = (Y_i, d_{Y_i})$, $i = 1, \cdots, n$, we will use an admissible metric $d_{\prod_{i=1}^n Y_i}$ on $\prod_{i=1}^n Y_i$ defined by
\[ d_{\prod_{i=1}^n Y_i}(y_1, \cdots, y_n, z_1, \cdots, z_n) = \max_{1 \leq i \leq n} d_{Y_i}(y_i, z_i) \]
for any $(y_1, \cdots, y_n, z_1, \cdots, z_n) \in \prod_{i=1}^n Y_i$. We denote by Comp($Y$) the hyperspace consisting of non-empty compact sets in $Y$ endowed with the Vietoris topology. Note that the topology of Comp($Y$) is induced by the Hausdorff metric $\left( d_{Y} \right)_H$, that is defined as follows:
\[ \left( d_{Y} \right)_H(A, B) = \inf \{ r > 0 \mid A \subset B_{d_Y}(B, r), B \subset B_{d_Y}(A, r) \} \]
for all $A, B \in \text{Comp}(Y)$, see [13 Proposition 5.12.4]. Under our assumption, we have the following proposition, refer to [13 Corollary 1.11.4 and Proposition 1.11.13]:

**Proposition 2.3.** If $Y$ is a connected, locally connected and locally compact space, then so is Comp($Y$).

### 3. The Borel complexity of $EM(X)$ in $PM(X)$

In this section, it will be shown that $EM(X) \in \mathcal{M}_2$. Since $X$ is a separable locally compact metrizable space, we can write $X = \bigcup_{n \in \mathbb{N}} X_n$, where each $X_n$ is compact and $X_n \subset \text{int} X_{n+1}$. For positive integers $n, m \in \mathbb{N}$, set
\[ A(n, m) = \{ d \in PM(X) \mid d(X_n, X \setminus X_{n+1}) \geq 1/m \}, \]
which is closed in $PM(X)$. We shall prove the following:

**Proposition 3.1.** The subset $AM(X)$ is an $F_{\sigma\delta}$-set in $PM(X)$.

*Proof.* We can write
\[ AM(X) = M(X) \cap \left( \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A(n, m) \right) . \]
Remark that $M(X)$ is $G_\delta$ in $PM(X)$, refer to the proof of [7 Lemma 5.1], and therefore $AM(X)$ is $F_{\sigma\delta}$ in $PM(X)$. \( \square \)

Since $X$ is separable locally compact, $C(X^2)$ is completely metrizable. Hence the both closed subset $PM(X)$ and $G_\delta$ subset $M(X)$ of $C(X^2)$ are Baire spaces. Furthermore, the following holds:

**Proposition 3.2.** The following are equivalent:

1. $X$ is compact;
2. $AM(X)$ is a Baire space.

*Proof.* If $X$ is compact, $AM(X)$ coincides with the Baire space $M(X)$. We will prove the implication (2) \( \Rightarrow \) (1). Suppose that $X$ is not compact. Observe that $AM(X) \subset \bigcup_{m \in \mathbb{N}} A(1, m)$. To show that every $A(1, m) \cap AM(X)$ is nowhere dense, take any admissible metric $d \in A(1, m) \cap AM(X)$ and any neighborhood $U$ of $d$. We can choose $k \geq m$ so that if for any $(x, y) \in X^2_k$, $d(x, y) = \rho(x, y)$, then $\rho \in U$. Since $X$ is not compact, we may assume that $X_k \neq \emptyset$. Moreover, $X \setminus X_{k+1}$ is not empty.
Fixing points $w \in X_k$ and $z \in X \setminus X_{k+1}$, define an admissible metric $\rho : (X_k \cup \{z\})^2 \to [0, \infty)$ on $X_k \cup \{z\}$ as follows:

$$
\rho(x, y) = \begin{cases} 
    d(x, y) & \text{if } (x, y) \in X_k^2, \\
    d(x, w) + 1/(m + 1) & \text{if } x \in X_k \text{ and } y = z, \\
    d(y, w) + 1/(m + 1) & \text{if } x = z \text{ and } y \in X_k, \\
    0 & \text{if } x = y = z.
\end{cases}
$$

Due to Hausdorff’s metric extension theorem [6], $\rho$ can be extended to an admissible metric $\tilde{\rho} \in AM(X)$. Note that for each $(x, y) \in X_k^2$, $\tilde{\rho}(x, y) = \rho(x, y) = d(x, y)$, so $\tilde{\rho} \in U$. Moreover, we have

$$
\tilde{\rho}(X_k, X \setminus X_{k+1}) \leq \tilde{\rho}(w, z) = \rho(w, z) = 1/(m + 1) < 1/m,
$$

which implies that $\tilde{\rho} \notin A(1, m) \cap AM(X)$. Therefore $A(1, m) \cap AM(X)$ is nowhere dense, and hence $AM(X)$ is not a Baire space. This is a contradiction. Consequently, $(2) \Rightarrow (1)$ holds. □

When $X$ is not compact, the family $\{\alpha X \setminus X_n \mid n \in \mathbb{N}\}$ is an open neighborhood basis of the point $x_\infty$ in $\alpha X$ and $x_\infty$ is not isolated. The space $EM(X)$ can be represented as follows:

**Lemma 3.3.** If $X$ is not compact, then

$$
EM(X) = \left\{ d \in AM(X) \mid \lim_{n \to \infty} \text{diam}_d(X \setminus X_n) = 0 \right\}.
$$

*Proof.* It is easy to show that

$$
EM(X) \subset \left\{ d \in AM(X) \mid \lim_{n \to \infty} \text{diam}_d(X \setminus X_n) = 0 \right\}.
$$

Conversely, we will verify that the left hand side contains the right one. Fix any $d \in \{d \in AM(X) \mid \lim_{n \to \infty} \text{diam}_d(X \setminus X_n) = 0\}$. Taking $x_n \in X \setminus X_n$, we can obtain a Cauchy sequence $\{d(x, x_n)\} \subset \mathbb{R}$ for each $x \in X$ because $\lim_{n \to \infty} \text{diam}_d(X \setminus X_n) = 0$. Let $\overline{d} : (\alpha X)^2 \to [0, \infty)$ be a function defined by

$$
\overline{d}(x, y) = \begin{cases} 
    d(x, y) & \text{if } x, y \in X, \\
    \lim_{n \to \infty} d(x, x_n) & \text{if } x \in X \text{ and } y = x_\infty, \\
    \lim_{n \to \infty} d(y, x_n) & \text{if } x = x_\infty \text{ and } y \in X, \\
    0 & \text{if } x = y = x_\infty.
\end{cases}
$$

Observe that for any $x \in X$, $\overline{d}(x, x_\infty) > 0$. Indeed, $x \in X_m$ for some $m \in \mathbb{N}$. Since $d \in AM(X)$, for every $n \geq m + 1$,

$$
d(x, x_n) \geq d(X_m, X \setminus X_{m+1}) > 0,
$$

and hence

$$
\overline{d}(x, x_\infty) = \lim_{n \to \infty} d(x, x_n) \geq d(X_m, X \setminus X_{m+1}) > 0.
$$

As is easily observed, $\overline{d}$ is a metric on $\alpha X$. We show that $\overline{d} \in M(\alpha X)$, that is, $\overline{d}$ is continuous. Let any $(x, y) \in (\alpha X)^2$. When $(x, y) \in X^2$, the continuity of $\overline{d}$ at $(x, y)$ follows from the one of $d$. When $x \in X$ and $y = x_\infty$, for each $\epsilon > 0$, there exist a neighborhood $U \subset X$ of $x$ and a positive integer $m \in \mathbb{N}$ such that for any $z \in U$, $d(x, z) \leq \epsilon/2$, and $\text{diam}_d(X \setminus X_m) \leq \epsilon/4$. Verify that $\text{diam}_d(\alpha X \setminus X_m) = \text{diam}_d(X \setminus X_m)$. For every point $(z, w) \in U \times (\alpha X \setminus X_m)$, that is a neighborhood
of \((x, y)\) in \((\alpha X)^2\), we have that for each \(n \geq m\),
\[
|\overline{d}(z, w) - \overline{d}(x, y)| \leq |\overline{d}(z, w) - \overline{d}(z, x_n)| + |\overline{d}(z, x_n) - \overline{d}(x, x_n)| + |\overline{d}(x, x_n) - \overline{d}(x, y)|
\]
\[
\leq \overline{d}(w, x_n) + \overline{d}(z, x) + \overline{d}(y, x_n)
\]
\[
\leq \text{diam}_d(X \setminus X_m) + d(x, z) + \text{diam}_\overline{d}(\alpha X \setminus X_m)
\]
\[
= 2 \text{diam}_d(X \setminus X_m) + d(x, z) \leq \epsilon/2 + \epsilon/2 = \epsilon.
\]
Hence \(\overline{d}\) is continuous at \((x, y)\). Similarly, the continuity of \(\overline{d}\) at \((x, y)\), where \(x = x_\infty\) and \(y \in X\), is valid. When \(x = x_\infty\) and \(y = x_\infty\), for each \(\epsilon > 0\), there is \(m \in \mathbb{N}\) such that \(\text{diam}_d(X \setminus X_m) \leq \epsilon\). Then \((\alpha X \setminus X_m)^2\) is a neighborhood of \((x, y)\) and for each \((z, w)\) in \((\alpha X \setminus X_m)^2\),
\[
|\overline{d}(z, w) - \overline{d}(x, y)| = \overline{d}(z, w) \leq \text{diam}_\overline{d}(\alpha X \setminus X_m) = \text{diam}_d(X \setminus X_m) \leq \epsilon,
\]
which implies the continuity of \(\overline{d}\) at \((x, y)\). Thus \(\overline{d}\) is continuous. It follows from the compactness of \(\alpha X\) that \(\overline{d} \in AM(\alpha X)\). As a consequence, \(d\) is extended to the admissible metric \(\overline{d}\), so \(d \in EM(X)\).
The proof is completed. \(\square\)

It is known that a space \(Y \in \mathcal{M}_2\) if and only if there exists an embedding from \(Y\) into a completely metrizable space as an \(F_{\sigma\delta}\)-set, refer to [11] Theorem 9.6.

**Proposition 3.4.** Suppose that \(X\) is not a compact space. Then the subset \(EM(X)\) is \(F_{\sigma\delta}\) in \(PM(X)\), and hence it is in \(\mathcal{M}_2\).

**Proof.** As is easily observed,
\[
\left\{ d \in AM(X) \middle| \lim_{n \to \infty} \text{diam}_d(X \setminus X_n) = 0 \right\} = \bigcap_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \left\{ d \in AM(X) \mid \text{diam}_d(X \setminus X_n) \leq 1/m \right\},
\]
and for any \(m, n \in \mathbb{N}\), \(\{ d \in AM(X) \mid \text{diam}_d(X \setminus X_n) \leq 1/m \}\) is closed. According to Lemma 3.3 \(EM(X)\) is an \(F_{\sigma\delta}\)-set in \(AM(X)\). Combining it with Proposition 3.1 we have that \(EM(X)\) is \(F_{\sigma\delta}\) in \(PM(X)\). \(\square\)

4. **THE HOMOTOPIETY DENSITY OF \(EM(X)\) IN \(PM(X)\)**

In this section, we shall prove that \(EM(X)\) is homotopy dense in \(PM(X)\).

**Proposition 4.1.** Suppose that \(X\) is not a compact space. Then \(EM(X)\) is dense in \(PM(X)\).

**Proof.** According to the same argument as Proposition 1 of [10], \(AM(X)\) is dense in \(PM(X)\). It remains to show that \(EM(X)\) is dense in \(AM(X)\). For each \(d \in AM(X)\) and each neighborhood \(U\) of \(d\), we shall construct a metric \(\rho \in EM(X)\) such that \(\rho \in U\). There is a compact subset \(K \subset X\) such that if \(d|_{K^2} = \rho|_{K^2}\), then \(\rho \in U\). Define an admissible metric \(\rho|_{(K \cup \{x_\infty\})^2}\) on \(K \cup \{x_\infty\}\) as follows:
\[
\rho(x, y) = \begin{cases} 
  d(x, y) & \text{if } x, y \in K, \\
  d(x, x_\infty) + 1 & \text{if } x \in K \text{ and } y = x_\infty, \\
  d(y, x_\infty) + 1 & \text{if } x = x_\infty \text{ and } y \in K, \\
  0 & \text{if } x = y = x_\infty.
\end{cases}
\]
Due to Hausdorff’s metric extension theorem [6], the above metric can be extended to an admissible metric \(\rho\) on \(\alpha X\). The restriction \(\rho|_{X^2} \in EM(X)\) is the desired admissible metric such that \(\rho|_{X^2} \in U\). The proof is complete. \(\square\)

Applying Lemma 3.3 we will verify the convexity of \(EM(X)\).
Proposition 4.2. If $X$ is not a compact space, then $EM(X)$ is a convex subset of $C(X^2)$. Moreover, if $X$ is locally connected, then it is an AR.

Proof. To prove that $EM(X)$ is convex in $AM(X)$, take any $d, \rho \in EM(X)$ and any $t \in [0, 1]$. By the convexity of $AM(X)$, $(1-t)d + t\rho \in AM(X)$. Then for every $n \in \mathbb{N}$,

$$\text{diam}_{(1-t)d + t\rho}(X \setminus X_n) = \sup_{(x,y) \in (X \setminus X_n)^2} ((1-t)d(x, y) + t\rho(x, y))$$

$$\leq (1-t)\text{diam}_d(X \setminus X_n) + t \text{diam}_\rho(X \setminus X_n).$$

Due to Lemma 3.3, $\text{diam}_d(X \setminus X_n) \to 0$ and $\text{diam}_\rho(X \setminus X_n) \to 0$ as $n \to \infty$, and hence $\text{diam}_{(1-t)d + t\rho}(X \setminus X_n)$ is also converging to 0. Using Lemma 3.3 again, we get that $(1-t)d + t\rho \in EM(X)$. Therefore $EM(X)$ is convex in $AM(X)$, and hence so is in $C(X^2)$. When $X$ is locally connected, $C(X^2)$ is a Fréchet space. Then the latter part holds. □

We can show the following proposition.

Proposition 4.3. Let $X$ be a locally connected but not compact space. Then $EM(X)$ is homotopy dense in $PM(X)$.

Proof. The convex subset $PM(X) \subset C(X^2)$ contains $EM(X)$ as a dense convex subset by Propositions 4.1 and 4.2. Combining Theorem 6.8.9 with Corollary 6.8.5 of [11], we have that $EM(X)$ is homotopy dense in $PM(X)$. □

5. The $Z_\sigma$-set property of $EM(X)$ in $PM(X)$

This section is devoted to proving that $EM(X)$ is contained in some $Z_\sigma$-set in $PM(X)$. For a compact metric space $Y = (Y, d_Y)$, we shall consider the set of partial pseudometrics on compact sets in $Y$. Let

$$PPM(Y) = \bigcup \{PM(A) \mid A \in \text{Comp}(Y)\}$$

whose topology is defined as follows: Identifying each partial pseudometrics $d \in PPM(Y)$ with its graph

$$\{(x, y, d(x, y)) \mid x, y \in \text{dom } d \} \in \text{Comp}(Y \times Y \times \mathbb{R}),$$

where the symbol $\text{dom } d \in \text{Comp}(Y)$ stands for the domain of $d$, we can regard $PPM(Y)$ as a subspace of $\text{Comp}(Y \times Y \times \mathbb{R})$. Note that the Hausdorff metric $(d_{Y \times Y \times \mathbb{R}})_H$ is admissible on $PPM(Y)$. Here set

$$PAM(Y) = \bigcup \{AM(A) \mid A \in \text{Comp}(Y) \text{ and } A \text{ is non-degenerate} \} \subset PPM(Y).$$

Lemma 5.1. Let $Y = (Y, d_Y)$ be a compact metric space, $U$ be an open subset of $Y$, $K$ be a closed subset of $Y$, and $y_0 \in U$ and $y_\infty \in K$ be distinct points. Suppose that $Z$ is a space, and $f : Z \to \text{Comp}(Y)$, $g : Z \to \text{Comp}(Y)$ and $h : Z \to [0, \infty)$ are maps such that $y_0 \in f(z) \subset U$, $y_\infty \in g(z) \subset K$ and $f(z) \cap g(z) = \emptyset$ for every $z \in Z$. Moreover, let $d_0 : Z \to PM(U)$ and $d_\infty : Z \to PM(K)$ be maps. Then the function $\Phi : Z \to PPM(Y)$ is continuous, which is defined by

$$\text{dom } \Phi(z) = f(z) \cup g(z);$$

$$\Phi(z)(x, y) = \begin{cases} d_0(z)(x, y) & \text{if } x, y \in f(z), \\ d_\infty(z)(x, y) & \text{if } x, y \in g(z), \\ d_0(z)(x, y_0) + d_\infty(z)(y, y_\infty) + h(z) & \text{if } x \in f(z) \text{ and } y \in g(z), \\ d_0(z)(y, y_0) + d_\infty(z)(x, y_\infty) + h(z) & \text{if } x \in g(z) \text{ and } y \in f(z). \end{cases}$$
Proof. As is easily observed, \( \Phi(z) \in PPM(Y) \) for any \( z \in Z \). To verify the continuity of \( \Phi \), take any \( z \in Z \) and \( \epsilon > 0 \). Since \( f(z) \) is compact, there is \( \delta \in (0, \epsilon) \) such that \( \overline{B}_{d_Y}(f(z), \delta) \subset U \). Combining the continuity of \( d_0(z) \) and \( d_\infty(z) \) with the compactness of \( f(z) \) and \( g(z) \), we can assume that for each \((x, y) \in f(z)^2 \) and each \((x', y') \in \overline{B}_{d_Y}(x, \delta) \times \overline{B}_{d_Y}(y, \delta) \), \(|d_0(z)(x, y) - d_0(z)(x', y')| < \epsilon/6 \), and for each \((x, y) \in g(z)^2 \) and each \((x', y') \in \overline{B}_{d_Y}(x, \delta) \cap K \times \overline{B}_{d_Y}(y, \delta) \cap K \), \(|d_\infty(z)(x, y) - d_\infty(z)(x', y')| < \epsilon/6 \). Since \( f, g \) and \( h \) are continuous, we can choose a neighborhood \( V \subset Z \) of \( z \) so that if \( w \in V \), then \((d_Y)_H(f(z), f(w)) \leq \delta \), \((d_Y)_H(g(z), g(w)) \leq \delta \) and \(|h(z) - h(w)| < \epsilon/3 \). Moreover, take a neighborhood \( W \subset V \) of \( z \) such that for any \( w \in W \), if \((x, y) \in \overline{B}_{d_Y}(f(z), \delta)^2 \), then \(|d_0(z)(x, y) - d_0(w)(x, y)| < \epsilon/6 \), and if \((x, y) \in K^2 \), then \(|d_\infty(z)(x, y) - d_\infty(w)(x, y)| < \epsilon/6 \). Observe that for each \( w \in W \), \((d_Y \times Y \times R)(\Phi(z), \Phi(w)) < \epsilon \). Indeed, let any \((x, y, \Phi(z)(x, y)) \in \Phi(z) \). When \((x, y) \in f(z)^2 \), there exists a point \((x', y') \in f(w)^2 \) such that

\[
d_Y^2((x, y), (x', y')) = \max\{d_Y(x, x'), d_Y(y, y')\} \leq \delta < \epsilon
\]

because \((d_Y)_H(f(z), f(w)) \leq \delta \). Note that \(|d_0(z)(x, y) - d_0(z)(x', y')| < \epsilon/6 \). Moreover, \((x', y') \in \overline{B}_{d_Y}(f(z), \delta)^2 \), and hence \(|d_0(z)(x', y') - d_0(w)(x', y')| < \epsilon/6 \). Observe that

\[
|\Phi(z)(x, y) - \Phi(w)(x', y')| = |d_0(z)(x, y) - d_0(w)(x', y')| \\
\leq |d_0(z)(x, y) - d_0(z)(x', y')| + |d_0(z)(x', y') - d_0(w)(x', y')| < \epsilon/6 + \epsilon/6 < \epsilon.
\]

Thus we have that \((x', y', \Phi(w)(x', y')) \in \Phi(w) \) and

\[
d_Y^2((x, y, \Phi(z)(x, y)), (x', y', \Phi(w)(x', y'))) = \max\{d_Y^2((x, y), (x', y')), |\Phi(z)(x, y) - \Phi(w)(x', y')|\} < \epsilon.
\]

Similarly, when \((x, y) \in g(z)^2 \), there exists a point \((x', y') \in g(w)^2 \) such that \((x', y', \Phi(w)(x', y')) \in \Phi(w) \) and

\[
d_Y^2((x, y, \Phi(z)(x, y)), (x', y', \Phi(w)(x', y'))) < \epsilon.
\]

When \( x \in f(z) \) and \( y \in g(z) \), we can find points \( x' \in f(w) \) and \( y' \in g(w) \) so that \( d_Y(x, x') \leq \delta < \epsilon \) and \( d_Y(y, y') \leq \delta < \epsilon \) since \((d_Y)_H(f(z), f(w)) \leq \delta \) and \((d_Y)_H(g(z), g(w)) \leq \delta \). Recall that \(|d_0(z)(x, y_0) - d_0(z)(x', y_0)| < \epsilon/6 \) and \(|d_\infty(z)(y, y_\infty) - d_\infty(z)(y', y_\infty)| < \epsilon/6 \). Furthermore, \(|d_\infty(z)(x', y_0) - d_\infty(w)(x', y_0)| < \epsilon/6 \) and \(|d_\infty(z)(y', y_\infty) - d_\infty(w)(y', y_\infty)| < \epsilon/6 \) because \((x', y_0) \in \overline{B}_Y(f(z), \delta)^2 \) and \((y', y_\infty) \in K^2 \). Then

\[
|\Phi(z)(x, y) - \Phi(w)(x', y')| \\
= |\left(|d_0(z)(x, y_0) + d_\infty(z)(y, y_\infty) + h(z)\right) - \left(|d_0(w)(x', y_0) + d_\infty(w)(y', y_\infty) + h(w)\right)| \\
\leq |d_0(z)(x, y_0) - d_0(z)(x', y_0)| + |d_0(z)(x', y_0) - d_0(w)(x', y_0)| \\
+ |d_\infty(z)(y, y_\infty) - d_\infty(z)(y', y_\infty)| + |d_\infty(z)(y', y_\infty) - d_\infty(w)(y', y_\infty)| + |h(z) - h(w)| \\
< \epsilon/6 + \epsilon/6 + \epsilon/6 + \epsilon/6 + \epsilon/3 = \epsilon.
\]

Therefore we have

\[
d_Y^2((x, y, \Phi(z)(x, y)), (x', y', \Phi(w)(x', y'))) = \max\{d_Y^2((x, y), (x', y')), |\Phi(z)(x, y) - \Phi(w)(x', y')|\} < \epsilon.
\]

Similarly, when \( x \in g(z) \) and \( y \in f(z) \), there are points \( x' \in g(w) \) and \( y' \in f(w) \) such that \((x', y', \Phi(w)(x', y')) \in \Phi(w) \) and

\[
d_Y^2((x, y, \Phi(z)(x, y)), (x', y', \Phi(w)(x', y'))) < \epsilon.
\]
It follows that \( \Phi(z) \subset B_{d_{Y \times X \times \mathbb{R}}}(\Phi(w), \epsilon) \). By the same argument as the above, we can see that \( \Phi(w) \subset B_{d_{Y \times X \times \mathbb{R}}}(\Phi(z), \epsilon) \). Consequently, \((d_{Y \times X \times \mathbb{R}})_{H}(\Phi(z), \Phi(w)) < \epsilon\), which implies that \( \Phi \) is continuous.

**Remark 1.** In the above lemma, for each \( z \in Z \), if \( d_{0}(z) \) and \( d_{\infty}(z) \) are admissible on \( f(z) \) and \( g(z) \) respectively, and \( h(z) > 0 \), then \( \Phi(z) \in PAM(Y) \).

We will give a useful path on \( \text{Comp}(X) \) for the latter argument.

**Lemma 5.2.** Let \( X \) be connected and locally connected. Then there exists a map \( \xi : [0, \infty) \to \text{Comp}(X) \) satisfying the following conditions:

1. \( X = \bigcup \xi([0, \infty)) \);
2. For any \( 0 \leq s \leq t < \infty \), \( \xi(s) \subset \xi(t) \);
3. For each \( n \in \mathbb{N} \), \( \xi(n) \subset \text{int} \xi(n + 1) \).

**Proof.** We will prove this lemma in the case where \( X \neq \emptyset \). Write \( X = \bigcup_{n \in \mathbb{N}} X_{n} \), where each \( X_{n} \) is compact and \( X_{n} \subset \text{int} X_{n+1} \). We may assume that \( X_{1} \neq \emptyset \), so choose any point \( x_{0} \in X_{1} \). According to Proposition 2.3, the hyperspace \( \text{Comp}(X) \) is connected, locally connected and locally compact metric, and hence it is path-connected by Theorem 5.14.5 of [11]. Now we shall inductively construct a map \( \xi : [0, \infty) \to \text{Comp}(X) \) so that \( X_{n} \subset \xi(n) \) for each \( n \in \mathbb{N} \).

1. Firstly, since \( \text{Comp}(X) \) is path-connected, there is a path \( \xi_{1} : [0, 1] \to \text{Comp}(X) \) such that \( \xi_{1}(0) = \{x_{0}\} \) and \( \xi_{1}(1) = X_{1} \).
2. Then we can define a map \( \xi_{1}^{1} : [0, 1] \to \text{Comp}(X) \) by \( \xi_{1}^{1}(t) = \xi_{1}([0, t]) \) for all \( t \in [0, 1] \).
3. Moreover, by virtue of [13] Lemma 1.11.6 and Proposition 1.11.7, we can obtain a map \( \xi_{1}^{\prime} : [0, 1] \to \text{Comp}(X) \) defined by \( \xi_{1}^{\prime}(t) = \bigcup \xi^{1}(t) \) for each \( t \in [0, 1] \). As is easily observed, for any \( 0 \leq s \leq t \leq 1 \), \( \xi_{1}^{\prime}(s) \subset \xi_{1}^{1}(t) \) and \( X_{1} \subset \xi_{1}^{1}(1) \).
4. Then let \( \xi_{[0,1]} = \xi_{1}^{\prime} \).
5. Secondly, assume that \( \xi_{[0,n]} \) is obtained for some \( n \in \mathbb{N} \). Because \( \xi(n) \) is compact, there exists \( m \geq n + 1 \) such that \( \xi(n) \subset \text{int} X_{m} \). Due to the same argument as the above, extend \( \xi_{[0,n]} \) over \([0, n + 1] \) so that \( X_{m} \subset \xi(n + 1) \).

By the inductive construction, we can get the desired map \( \xi \).

From now on, the map \( \xi \) is as the above lemma and let \( X_{n} = \xi(n) \) for each \( n \in \mathbb{N} \). Moreover, if \( X \) is not empty, then fix a point \( x_{0} \in X_{1} \). We shall define a distance \( D \) between real-valued functions on \( X^{2} \) as follows:

\[
D(f, g) = \sup_{n \in \mathbb{N}} \min \left\{ \sup_{(x, y) \in X_{n}^{2}} |f(x, y) - g(x, y)|, 1/n \right\}
\]

for any \( f, g : X^{2} \to \mathbb{R} \). Note that \( D \) is an admissible metric on \( C(X^{2}) \), refer to [11] 1.1.3(7). We can see the following:

**Lemma 5.3.** Suppose that \( X \) is connected and locally connected, and that \( \epsilon > 0 \). For any \( f, g \in C(X^{2}) \), if \( f|_{\xi(1/\epsilon)^{2}} = g|_{\xi(1/\epsilon)^{2}} \), then \( D(f, g) \leq \epsilon \).

**Proof.** Let \( f, g \in C(X^{2}) \) with \( f|_{\xi(1/\epsilon)^{2}} = g|_{\xi(1/\epsilon)^{2}} \). In the case that \( n \leq 1/\epsilon \), \( X_{n} = \xi(n) \subset \xi(1/\epsilon) \), so

\[
\min \left\{ \sup_{(x, y) \in X_{n}^{2}} |f(x, y) - g(x, y)|, 1/n \right\} \leq \sup_{(x, y) \in X_{n}^{2}} |f(x, y) - g(x, y)| = \sup_{(x, y) \in X_{n}^{2}} |f(x, y) - f(x, y)| = 0.
\]

In the case that \( n > 1/\epsilon \),

\[
\min \left\{ \sup_{(x, y) \in X_{n}^{2}} |f(x, y) - g(x, y)|, 1/n \right\} \leq 1/n < \epsilon.
\]
Therefore we have
\[ D(f, g) = \sup_{n \in \mathbb{N}} \min \left\{ \sup_{(x,y) \in X_n^2} |f(x, y) - g(x, y)|, 1/n \right\} \leq \epsilon. \]

The proof is complete. □

We shall use the following extension theorem of partial metrics with various domains according to [3, Theorem 2.1].

**Theorem 5.4.** Let \( Y \) be compact. There exists a map \( e : PPM(Y) \to PM(Y) \) such that \( PAM(Y) \subseteq AM(Y) \).

For spaces \( Y \subseteq M \), the restriction \( r : PM(M) \to PM(Y) \) is continuous, which is defined by \( r(d) = d|_{Y^2} \) for all \( d \in PM(M) \). Note that \( r(AM(M)) \subseteq AM(Y) \). From now on, let \( e : e : PPM(\alpha X) \to PM(\alpha X) \) be the extension as in Theorem 5.4 and let \( r : PM(\alpha X) \to PM(X) \) be the restriction as the above.

**Proposition 5.5.** Let \( X \) be connected and locally connected, but not compact. The space \( AM(X) \) is contained in some \( Z_\sigma \)-set in \( PM(X) \), and hence so is \( EM(X) \).

**Proof.** Notice that \( AM(X) \subseteq \bigcup_{m \in \mathbb{N}} A(1, m) \). To show that \( A(1, m) \) is a \( Z_\sigma \)-set in \( PM(X) \) for every \( m \in \mathbb{N} \), fix any map \( \epsilon : PM(X) \to (0, 1) \). We will construct a map \( \Psi : PM(X) \to PM(X) \) so that \( \Psi(PM(X)) \cap A(1, m) = \emptyset \) and \( D(d, \Psi(d)) \leq \epsilon(d) \) for each \( d \in PM(X) \). Define a function \( \Phi : PM(X) \to PPM(\alpha X) \) satisfying the following conditions:

1. \( \text{dom} \Phi(d) = \xi(1/\epsilon(d)) \cup \{x_\infty\} \);
2. \( \Phi(d)(x, y) = \begin{cases} 
    d(x, y) & \text{if } x, y \in \xi(1/\epsilon(d)), \\
    d(x, x_0) + 1/(m + 1) & \text{if } x \in \xi(1/\epsilon(d)) \text{ and } y = x_\infty, \\
    d(y, x_0) + 1/(m + 1) & \text{if } x = x_\infty \text{ and } y \in \xi(1/\epsilon(d)), \\
    0 & \text{if } x = y = x_\infty.
\end{cases} \)

It follows from Lemma 5.1 that \( \Phi \) is continuous. Let \( \Psi = e\Phi \), that is a desired map. To verify it, take any \( d \in PM(X) \). Firstly, we shall show that \( \Psi(d) \not\in A(1, m) \). Since \( e\Phi(d) \) is a continuous pseudometric on \( \alpha X \) and \( x_\infty \) is not an isolated point, we can obtain \( x \in X \setminus X_2 \) so that \( e\Phi(d)(x, x_\infty) < 1/m - 1/(m + 1) \). Then
\[
\inf \{ \Psi(d)(y, z) \mid y \in X_1, z \in X \setminus X_2 \} \leq \Psi(d)(x_0, x) = e\Phi(d)(x_0, x) = e\Phi(d)(x_0, x) \\
\leq e\Phi(d)(x_0, x_\infty) + e\Phi(d)(x_\infty, x) \\
< 1/(m + 1) + 1/m - 1/(m + 1) = 1/m,
\]
which means that \( \Psi(d) \not\in A(1, m) \). Secondly, prove that \( D(d, \Psi(d)) \leq \epsilon(d) \). Remark that
\[
\Psi(d)(x, y) = e\Phi(d)(x, y) = e\Phi(d)(x, y) = \Phi(d)(x, y) = d(x, y)
\]
for any \( x, y \in \xi(1/\epsilon(d)) \). It follows from Lemma 5.3 that \( D(d, \Psi(d)) \leq \epsilon(d) \). We conclude that \( A(1, m) \) is a \( Z_\sigma \)-set in \( PM(X) \), so \( AM(X) \) is contained in the \( Z_\sigma \)-set \( \bigcup_{m \in \mathbb{N}} A(1, m) \). □

6. The strong \( \mathfrak{M}_2 \)-universality of \( EM(X) \)

In this section, we will verify the strong \( \mathfrak{M}_2 \)-universality of \( EM(X) \). For any \( c > 0 \), let
\[
Z(c) = \left\{ d \in PM(X) \mid \lim_{n \to \infty} \inf d(x_0, X \setminus X_n) \leq c \right\}.
\]

We have the following lemma.
Lemma 6.1. Let $X$ be connected and locally connected, but not compact. Suppose that $A$ is a closed set and $B$ is a Z-set in $PM(X)$, respectively. If $A \subset Z(c) \cup B$ for some $c > 0$, then $A$ is a Z-set.

Proof. For each map $\epsilon : PM(X) \to (0,1)$, we shall construct a map $\Psi : PM(X) \to PM(X)$ such that $\Psi(PM(X)) \cap (Z(c) \cup B) = \emptyset$ and $D(d, \Psi(d)) \leq \epsilon(d)$ for any $d \in PM(X)$. Since $B$ is a Z-set, there exists a map $\Phi_1 : PM(X) \to PM(X)$ such that $\Phi_1(PM(X)) \cap B = \emptyset$ and $D(d, \Phi_1(d)) \leq \epsilon(d)/2$. Letting

$$\delta(d) = \min\{\epsilon(d), D(\Phi_1(d), B)\}/2,$$

we define a function $\Phi_2 : PM(X) \to PPM(\alpha X)$ as follows:

1. $\text{dom } \Phi_2(d) = \xi(1/\delta(d)) \cup \{x_{\infty}\}$;
2. $\Phi_2(d)(x, y) = \begin{cases} d(x, y) & \text{if } x, y \in \xi(1/\delta(d)), \\ d(x, x_0) + c + 1 & \text{if } x \in \xi(1/\delta(d)) \text{ and } y = x_{\infty}, \\ d(y, x_0) + c + 1 & \text{if } x = x_{\infty} \text{ and } y \in \xi(1/\delta(d)), \\ 0 & \text{if } x = y = x_{\infty}. \end{cases}$

Due to Lemma 5.1, $\Phi_2$ is continuous. Then we can obtain the desired map $\Psi = r\epsilon\Phi_2\Phi_1$. By the same argument as Proposition 5.5, $D(\Phi_1(d), \Psi(d)) \leq \delta(d) = \min\{\epsilon(d), D(\Phi_1(d), B)\}/2$.

for each $d \in PM(X)$. Therefore

$$D(d, \Psi(d)) \leq D(d, \Phi_1(d)) + D(\Phi_1(d), \Psi(d)) \leq \epsilon(d)/2 + \delta(d) \leq \epsilon(d),$$

and $\Psi(PM(X)) \cap B = \emptyset$. It remains to prove that $\Psi(PM(X)) \cap Z(c) = \emptyset$. For each $d \in PM(X)$, there exists $m \in \mathbb{N}$ such that $x \in X \setminus X_m$, then $\epsilon\Phi_2\Phi_1(d)(x_{\infty}, x) < 1/2$. It follows that

$$\Psi(d)(x_0, x) = r\epsilon\Phi_2\Phi_1(d)(x_0, x) = \epsilon\Phi_2\Phi_1(d)(x_0, x) \geq \epsilon\Phi_2\Phi_1(d)(x_0, x_{\infty}) - \epsilon\Phi_2\Phi_1(d)(x_{\infty}, x) > c + 1 - 1/2 = c + 1/2$$

for any $k \geq m$ and any $x \in X \setminus X_k$. Hence $\liminf_{n \to \infty} \Psi(d)(x_0, X \setminus X_n) > c$, which means that $\Psi(PM(X)) \cap Z(c) = \emptyset$. As a result, $A$ is a Z-set. $\square$

We show the following:

Lemma 6.2. Let $X$ be a locally connected but not compact space. Then $X$ is chain-connected to infinity if and only if there is an arc $\sigma : [0,1] \to \alpha X$ such that $\sigma(0) = x_{\infty}$ and $\sigma((0,1]) \subset X$.

Proof. Firstly, we shall prove the “if” part. Taking an arc $\sigma : [0,1] \to \alpha X$ so that $\sigma(0) = x_{\infty}$ and $\sigma((0,1]) \subset X$, we can find $t_n \in (0,1]$ for each $n \in \mathbb{N}$ such that $\sigma([0,t_n]) \subset X \setminus X_n$ and $t_{n+1} < t_n$.

Let $C_i = \sigma([0,t_i])$, so $\{C_i\}$ is a simple chain consisting of connected subset in $X$. Moreover, for every compact set $K \subset X$, there is $i(K) \in \mathbb{N}$ such that $K \subset X_{i(K)}$, and hence if $i \geq i(K)$, then $C_i \subset X \setminus X_{i(K)} \subset X \setminus K$. It follows that $X$ is chain-connected to infinity.

Next, we show the “only if” part. Since $X$ is chain-connected to infinity, we can obtain a simple chain $\{C_i\}$ consisting of connected sets in $X$ so that for every $n \in \mathbb{N}$, there is $i(n) \in \mathbb{N}$ such that for any $i \geq i(n)$, $C_i \subset X \setminus X_n$. Put $D_n = \bigcup_{i \geq i(n)} C_i$ and observe that for any $n \in \mathbb{N}$, $D_n$ is connected and $D_{n+1} \subset D_n \subset X \setminus X_n$. Then we may assume that each $D_n$ is an open subset of $X$. Indeed, replace $D_1$ with the connected component in $X \setminus X_1$ containing $D_1$. Then the connected component $D_1$ is open because $X$ is locally connected. Suppose that $D_n$ is open for some $n \in \mathbb{N}$. Replacing $D_{n+1}$ with the connected component in $D_n \cap X \setminus X_n$ containing $D_{n+1}$, we have that $D_{n+1}$ is open due to the local connectedness of $X$. By induction, every $D_n$ is open. Fix any point
Proposition 6.4. Let \( X \) be connected and locally connected, but not compact. Suppose that \( X \) is chain-connected to infinity, then every \( \mathcal{M}_2 \)-set in \( \mathcal{M}_2 \)-spaces of \( X \) is a strong \( Z \)-set.

The space
\[
\mathcal{C}_1 = \left\{ (x(n))_{n \in \mathbb{N}} \in \mathbb{Q} \mid \lim_{n \to \infty} x(n) = 1 \right\}
\]
is an \( \mathcal{M}_2 \)-absorbing set in \( \mathbb{Q} \), see to [13], and hence it admits closed embeddings from spaces belonging to \( \mathcal{M}_2 \). Now we show the following:

Proposition 6.5. Let \( X \) be connected and locally connected, but not compact. Suppose that \( X \) is chain-connected to infinity. Then the space \( EM(X) \) is strongly \( \mathcal{M}_2 \)-universal.

Proof. Suppose that \( A \in \mathcal{M}_2 \), \( B \) is a closed set in \( A \), and \( f : A \to EM(X) \) is a map such that \( f|_B \) is a \( Z \)-embedding. For each open cover \( \mathcal{U} \) of \( EM(X) \), let us construct a \( Z \)-embedding \( h : A \to EM(X) \) such that \( h \) is \( \mathcal{U} \)-close to \( f \) and \( h|_B = f|_B \). By Proposition 4.3, \( EM(X) \) is an AR. Remark that \( X \) is not discrete. According to Proposition 4.4, \( f(B) \) is a strong \( Z \)-set in \( EM(X) \).

By virtue of [12], Proposition 2.8.12 and Lemma 2.8.10, we can assume that \( f(A \setminus B) \cap f(B) = \emptyset \) and \( f \) satisfies the following property:

(i) For each metric \( d \in f(B) \) and each sequence \( \{a_n\} \subset A \), if \( f(a_n) \) is converging to \( d \), then \( a_n \) is converging to \( f^{-1}(d) \).

Take a map \( \epsilon : EM(X) \to [0,1) \) such that
(ii) for any map $h : A \to EM(X)$, if $D(f(a), h(a)) \leq \epsilon f(a)$ for every $a \in A$, then $h$ is $U$-close to $f$.

(iii) for each $d \in EM(X)$, $\epsilon(d) \leq D(d, f(B))/2$, and $\epsilon(d) = 0$ if and only if $d \in f(B)$.

For each $k \in \mathbb{N}$, we put

$$A_k = \{ a \in A \mid 2^{-k} \leq \epsilon f(a) \leq 2^{-k+1} \}$$

and $\phi_k(a) = 2 - 2^k \epsilon f(a)$. Then $A \setminus B = \bigcup_{k \in \mathbb{N}} A_k$. Fixing a closed embedding $p : A \to c_1$, we can define a map $q^k : A_k \to [0, 1]$ by

$$q^k_i(a) = \begin{cases} 0 & \text{if } i = 1, \\ \epsilon f(a)(1 - \phi_k(a)) & \text{if } i = 2, \\ \epsilon f(a)(1 - \phi_k(a))p(a)(1) & \text{if } i = 3, \\ \epsilon f(a) & \text{if } i = 2j, j \geq 2, \\ \epsilon f(a)((1 - \phi_k(a))p(a)((i - 1)/2) + \phi_k(a)p(a)((i - 3)/2)) & \text{if } i = 2j + 1, j \geq 2. \end{cases}$$

For each $m \in \mathbb{N}$, let $\psi_m : \gamma([2^{-m}, 2^{-m+1}]) \to [0, 1]$ be a map defined by $\psi_m(x) = 2^m \gamma^{-1}(x) - 1$. Define a map $g_k : A_k \to PM(\{x_0\} \cup \gamma([0, 1]))$, $k \in \mathbb{N}$, as follows:

$$g_k(a)(x, x_0) = g_k(a)(x_0, x) = \begin{cases} \epsilon f(a) & \text{if } x = x_\infty, \\ \psi_{2k+i}(x)q^k(a) + (1 - \psi_{2k+i}(x))q^k_{i+1}(a) & \text{if } x \in \gamma([2^{-2k-i}, 2^{-2k-i+1}]), \\ 0 & \text{if } x = x_0 \text{ or } 2^{-2k} \leq \gamma^{-1}(x) \leq 1, \end{cases}$$

and for any $x, y \in \{x_0\} \cup \gamma([0, 1])$, $g_k(a)(x, y) = |g_k(a)(x, x_0) - g_k(a)(y, x_0)|$. Here we will observe that $g_k(a) = g_{k+1}(a)$ for all $a \in A_k \cap A_{k+1}$. Indeed, $g_k(a)(x_\infty, x_0) = \epsilon f(a) = g_{k+1}(a)(x_\infty, x_0)$, and $g_k(a)(x, x_0) = 0 = g_{k+1}(a)(x, x_0)$ for every $x \in \{x_0\} \cup \gamma([0, 1])$ with $x = x_0$ or $2^{-2k} \leq \gamma^{-1}(x) \leq 1$.

Because $\epsilon f(a) = 2^{-k}$, $q^k(a) = q^k_2(a) = q^k_3(a) = 0$. Thus for each $x \in \gamma([2^{-2k-1}, 2^{-2k-2}])$,

$$g_k(a)(x, x_0) = \psi_{2k+1}(x)q^k_2(a) + (1 - \psi_{2k+1}(x))q^k_3(a) = 0 = g_{k+1}(a)(x, x_0),$$

and for each $x \in \gamma([2^{-2k-2}, 2^{-2k-1}])$,

$$g_k(a)(x, x_0) = \psi_{2k+2}(x)q^k_2(a) + (1 - \psi_{2k+2}(x))q^k_3(a) = 0 = g_{k+1}(a)(x, x_0).$$

Furthermore, $q^k_3(a) = 0 = q^k_{2j+1}(a)$, $q^k_{2j+3}(a) = \epsilon f(a)p(a)(j) = q^k_{2j+1}(a)$ and $q^k_{2j+2}(a) = \epsilon f(a) = q^k_{2j+1}(a)$ for every $j \geq 1$, and hence for all $x \in \gamma([2^{-2k-i-2}, 2^{-2k-i-1}])$, $i \geq 1$,

$$g_k(a)(x, x_0) = \psi_{2k+i+2}(x)q^k_{i+2}(a) + (1 - \psi_{2k+i+2}(x))q^k_{i+3}(a) = \psi_{2(k+1)+i+1}(x)q^k_{i+1}(a) + (1 - \psi_{2(k+1)+i+1}(x))q^k_{i+1}(a) = g_{k+1}(a)(x, x_0).$$

It follows that $g_k(a) = g_{k+1}(a)$. Letting $g : A \setminus B \to AM(\{x_0\} \cup \gamma([0, 1]))$ be a map defined by

$$g(a)(x, y) = \begin{cases} g_k(a)(x, y) + |\gamma^{-1}(x) - \gamma^{-1}(y)| & \text{if } x, y \in \gamma([0, 1]), \\ g_k(a)(x_0, y) + \gamma^{-1}(y) + 1 & \text{if } x = x_0 \text{ and } y \in \gamma([0, 1]), \\ g_k(a)(x, x_0) + \gamma^{-1}(x) + 1 & \text{if } x \in \gamma([0, 1]) \text{ and } y = x_0, \\ g_k(a)(x, y) & \text{if } x = y = x_0, \end{cases}$$

where $a \in A_k$, we can obtain a function $g' : A \setminus B \to PAM(\alpha X)$ so that for any $a \in A \setminus B$,

(1) $\text{dom } g'(a) = \xi(1/\epsilon f(a)) \cup \eta(\epsilon f(a))$;

(2) $g'(a)(x, y) = \begin{cases} f(a)(x, y) & \text{if } x, y \in \xi(1/\epsilon f(a)), \\ g(a)(x, y) & \text{if } x, y \in \{x_0\} \cup \eta(\epsilon f(a)), \\ f(a)(x, x_0) + g(a)(y, x_0) & \text{if } x \in \xi(1/\epsilon f(a)) \text{ and } y \in \eta(\epsilon f(a)), \\ f(a)(y, x_0) + g(a)(x, x_0) & \text{if } x \in \eta(\epsilon f(a)) \text{ and } y \in \xi(1/\epsilon f(a)). \end{cases}$
The continuity of $g'$ follows from Lemma 5.1.

Let $h|_{A \setminus B} : A \setminus B \to EM(X)$ be a map defined by $h(a) = reg' a$. Note that $D(f(a), h(a)) \leq \epsilon f(a)$ by Lemma 5.3. According to (iii), the map $h$ can be extended to the desired map $h : A \to EM(X)$ by $h|_B = f|_B$, and verify that

$$h(A \setminus B) \subset EM(X) \setminus f(B) = EM(X) \setminus h(B).$$

By (ii), $h$ is $U$-close to $f$. It is remains to show that $h$ is a $Z$-embedding. To check the injectivity of $h$, fix any $a_1, a_2 \in A \setminus B$ with $h(a_1) = h(a_2)$, where we get some $k_1, k_2 \in \mathbb{N}$ such that $a_1 \in A_{k_1}$ and $a_2 \in A_{k_2}$ respectively. Let $k = \max\{k_i | i = 1, 2\}$. Remark that for each $i \in \mathbb{N}$ and for each $l = 1, 2,$

$$q_{2(k-k_l)+i+1}(a_l) = \psi_{2k+i}(x_{2k+i})q_{2(k-k_l)+i}(a_l) + (1 - \psi_{2k+i}(x_{2k+i}))q_{2(k-k_l)+i+1}(a_l)$$

$$= g_{k_l}(a_l)(x_{2k+i}, x_0) = g(a_l)(x_{2k+i}, x_0) - (2^{-2k-i} + 1)$$

so when $i = 3,$

$$\epsilon f(a_1) = q_{2(k-k_1)+4}(a_1) = q_{2(k-k_2)+4}(a_2) = \epsilon f(a_2).$$

Hence $a_1, a_2 \in A_k$ and $\phi_k(a_1) = \phi_k(a_2).$ In the case where $\phi_k(a_1) = 1,$

$$p(a_1)(j) = q_{2j+3}(a_1)/\epsilon f(a_1) = q_{2j+3}(a_2)/\epsilon f(a_2) = p(a_2)(j)$$

for any $j \in \mathbb{N}.$ In the case where $\phi_k(a_1) \neq 1,$

$$p(a_1)(1) = q_{2j+3}(a_1)/((1 - \phi_k(a_1)\epsilon f(a_1)) = q_{2j+3}(a_2)/((1 - \phi_k(a_2)\epsilon f(a_2))) = p(a_2)(1).$$

Assuming that $p(a_1)(j) = p(a_2)(j)$ for some $j \in \mathbb{N},$ we see that

$$p(a_1)(j+1) = (q_{2j+3}(a_1)/\epsilon f(a_1) - \phi_k(a_1)p(a_1)(j))/(1 - \phi_k(a_1))$$

$$= (q_{2j+3}(a_2)/\epsilon f(a_2) - \phi_k(a_2)p(a_2)(j))/(1 - \phi_k(a_2)) = p(a_2)(j+1).$$

By induction, $p(a_1)(j) = p(a_2)(j)$ for all $j \in \mathbb{N}.$ Consequently, $p(a_1) = p(a_2).$ The injectivity of $h$ follows from the one of $p.$

We prove that $h$ is a closed map. For any sequence $\{a_n\} \subset A$ and any metric $d \in EM(X)$ such that $h(a_n) \to d$ as $n \to \infty,$ we will choose a subsequence of $\{a_n\}$ converging to some point in $A.$ Notice that

$$D(f(a_n), d) \leq D(f(a_n), h(a_n)) + D(h(a_n), d) \leq \epsilon f(a_n) + D(h(a_n), d).$$

When $\epsilon f(a_n) \to 0,$ $D(f(a_n), d) \to 0.$ Due to the continuity of $\epsilon,$ $\epsilon(d) = 0,$ so $d \in f(B)$ by (iii). It follows from (i) that $a_n$ converges to $f^{-1}(d).$ When $\epsilon f(a_n) \to 0,$ we may replace $\{a_n\}$ with a subsequence in $A_k = (\epsilon f)^{-1}(\{2^{-k}, 2^{-k+1}\})$ for some $k \in \mathbb{N}.$ By the compactness of $[2^{-k}, 2^{-k+1}],$ we can also replace $\{a_n\}$ with a subsequence such that $\epsilon f(a_n)$ converges to some number $c \in [2^{-k}, 2^{-k+1}].$ For each $j \in \mathbb{N},$

$$c = \lim_{n \to \infty} \epsilon f(a_n) = \lim_{n \to \infty} (h(a_n)(x_{2(k+j)+1}, x_0) - (2^{-2(k+j)-1} + 1))$$

$$= d(x_{2(k+j)+1}, x_0) - (2^{-2(k+j)-1} + 1).$$

The metric $d \in EM(X)$ can be extended to $\overline{d} \in AM(\alpha X).$ Therefore

$$\overline{d}(x_\infty, x_0) = \lim_{j \to \infty} d(x_{2(k+j)+1}, x_0) = \lim_{j \to \infty} (c + (2^{-2(k+j)-1} + 1)) = c + 1.$$
In the case that $c = 2^{-k}$, $\lim_{n \to \infty} \phi_k(a_n) = 1$, so we can assume that $\phi_k(a_n) > 0$. Then
\[
\lim_{n \to \infty} p(a_n)(j) = \lim_{n \to \infty} (h(a_n)(x_{2(k+j+1)}, x_0) - (2^{-2(k+j+1)} + 1) - \epsilon f(a_n)(1 - \phi_k(a_n))p(a_n)(j + 1))/(\epsilon f(a_n)\phi_k(a_n))
\]
\[
= 2^k(d(x_{2(k+j+1)}, x_0) - (2^{-2(k+j+1)} + 1))
\]
for every $j \in \mathbb{N}$. Note that as $j \to \infty$,
\[
2^k(d(x_{2(k+j+1)}, x_0) - (2^{-2(k+j+1)} + 1)) \to 2^k(\overline{d}(x_\infty, x_0) - 1) = 2^k((2^{-k} + 1) - 1) = 1,
\]
which implies that $p(a_n)$ converges to $\{2^k(d(x_{2(k+j+1)}, x_0) - (2^{-2(k+j+1)} + 1))\} \in c_1$. Since $p$ is a closed embedding, $a_n$ is converging to some point of $A$. By the same argument, $a_n$ is converging to some point of $A$ in the case that $c = 2^{-k+1}$. When $c \in (2^{-k}, 2^{-k+1})$, it can be assumed that $\phi_k(a_n) \in (0, 1)$, and hence as $n \to \infty$,
\[
p(a_n)(1) = (h(a_n)(x_{2(k+1)}, x_0) - (2^{-2(k+1)} + 1))/(\epsilon f(a_n)(1 - \phi_k(a_n)))
\]
\[
\to (d(x_{2(k+1)}, x_0) - (2^{-2(k+1)} + 1))/(c(2^k c - 1)).
\]
Suppose that for some $j \in \mathbb{N}$, $p(a_n)(j)$ is converging and let $p_j = \lim_{n \to \infty} p(a_n)(j)$. Then
\[
\lim_{n \to \infty} p(a_n)(j + 1) = \lim_{n \to \infty} (h(a_n)(x_{2(k+j+1)}, x_0) - (2^{-2(k+j+1)} + 1) - \epsilon f(a_n)\phi_k(a_n)p(a_n)(j))/(\epsilon f(a_n)(1 - \phi_k(a_n)))
\]
\[
= (d(x_{2(k+j+1)}, x_0) - (2^{-2(k+j+1)} + 1) - c(2 - 2^k) p_j)/(c(2^k c - 1)).
\]
By induction, for all $j \in \mathbb{N}$, $p(a_n)(j)$ converges and denote $p_j = \lim_{n \to \infty} p(a_n)(j)$. Recall that $\{p_j\} \in \mathbb{Q}$. We will prove that $\{p_j\} \in c_1$. Supposing that $p_j \to 1$, we can choose a positive number $\lambda \leq 2 - 2^k c$ so that for any $j \in \mathbb{N}$, there is $i(j) \geq j$ such that $p_{i(j)} < 1 - \lambda$. For every $j \in \mathbb{N}$,
\[
(2^k c - 1)p_{j+1} + (2 - 2^k c)p_j = \lim_{n \to \infty} ((1 - \phi_k(a_n))p(a_n)(j + 1) + \phi_k(a_n)p(a_n)(j))
\]
\[
= \lim_{n \to \infty} 2^k d_{2(j+1)}(a_n)/\epsilon f(a_n)
\]
\[
= \lim_{n \to \infty} (h(a_n)(x_{2(k+j+1)}, x_0) - (2^{-2(k+j+1)} + 1))/c
\]
\[
= (d(x_{2(k+j+1)}, x_0) - (2^{-2(k+j+1)} + 1))/c.
\]
It follows that
\[
\lim_{j \to \infty} ((2^k c - 1)p_{j+1} + (2 - 2^k c)p_j) = \lim_{j \to \infty} (d(x_{2(k+j+1)}, x_0) - (2^{-2(k+j+1)} + 1))/c
\]
\[
= (\overline{d}(x_\infty, x_0) - 1)/c = ((c + 1) - 1)/c = 1.
\]
Thus there exist $j \in \mathbb{N}$ such that if $i \geq j$, then $1 - \lambda^2 \leq (2^k c - 1)p_{i+1} + (2 - 2^k c)p_i$. Hence
\[
1 - \lambda^2 \leq (2^k c - 1)p_{i(j)+1} + (2 - 2^k c)p_{i(j)} \leq (2 - 2^k c)p_{i(j)} + 2^k c - 1.
\]
Since $\lambda \leq 2 - 2^k c$, we get that
\[
1 - \lambda \leq 1 - \lambda^2/(2 - 2^k c) \leq p_{i(j)},
\]
which is a contradiction. Thus $\{p_j\} \in c_1$. Then $p(a_n)$ converges to $\{p_j\} \in c_1$ as $n \to \infty$, so $a_n$ is converging to some point of $A$ because $p$ is a closed embedding. It follows that $h$ is a closed map.
For every \( a \in A \setminus B, \) \( a \in A_k \) for some \( k \in \mathbb{N} \), and hence for any \( i \in \mathbb{N} \), since
\[
h(a)(x_{2(k+i)+1}, x_0) = g(a)(x_{2(k+i)+1}, x_0) = g_k(a)(x_{2(k+i)+1}, x_0) + (2^{-2(k+i)} - 1 + 1)
= q_{2(i+1)}(a) + (2^{-2(k+i)} - 1 + 1) = ef(a) + (2^{-2(k+i)} - 1 + 1) < 3,
\]
we have \( \liminf_{n \to \infty} h(a)(x_n, X \setminus X_n) \leq 3 \). According to Lemma 6.1, the image \( h(A) = h(A \setminus B) \) \( h(B) \), that is contained in \( Z(3) \) \( f(B) \), is a \( Z \)-set in \( EM(X) \). We conclude that \( h \) is a \( Z \)-embedding. □

7. Proof of Main Theorem

Now we shall prove Main Theorem.

Proof of Main Theorem. By Proposition 3.4, we have \( EM(X) \in \mathcal{M}_2 \subset (\mathcal{M}_2)_{\sigma}. \) Due to Propositions 4.3, \( EM(X) \) is homotopy dense in \( PM(X) \). We can decompose \( X = \bigoplus_{k \in K} Y_k \) into connected components for some \( K \in \mathbb{N} \). Since \( X \) is chain-connected to infinity, we can choose a positive integer \( k \in K \) and a path between \( x_{\infty} \) and some point of \( Y_k \) by Lemma 5.2. Hence \( Y_k \) is chain-connected to infinity and \( Y_k \cup \{ x_{\infty} \} \) is the one-point compactification of \( Y_k \). Remark that \( C(X^2) \) is homeomorphic to the product space \( \prod_{(i,j) \in K^2} C(Y_i \times Y_j) \) by virtue of the following homeomorphism:
\[
C(X^2) \ni f \mapsto (f|_{Y_i \times Y_j})_{(i,j) \in K^2} \in \prod_{(i,j) \in K^2} C(Y_i \times Y_j).
\]
Let \( PM_{(i,j)} = \{ d|_{Y_i \times Y_j} \mid d \in PM(X) \} \) and \( EM_{(i,j)} = \{ d|_{Y_i \times Y_j} \mid d \in EM(X) \} \), so \( PM(X) \) is homeomorphic to \( PM_{(k,k)} \times \prod_{(i,j) \in K^2 \setminus \{(k,k)\}} PM_{(i,j)} \) and \( EM(X) \) is homeomorphic to \( EM_{(k,k)} \times \prod_{(i,j) \in K^2 \setminus \{(k,k)\}} EM_{(i,j)} \), respectively. Note that for each \( (i,j) \in K^2 \), \( PM_{(i,j)} \) and \( EM_{(i,j)} \) are convex sets in a Fréchet space \( C(Y_i \times Y_j) \), so they are ARs, and moreover the product spaces \( \prod_{(i,j) \in K^2 \setminus \{(k,k)\}} PM_{(i,j)} \) and \( \prod_{(i,j) \in K^2 \setminus \{(k,k)\}} EM_{(i,j)} \) are also ARs. Observe that \( PM(Y_k) = PM_{(k,k)} \) and \( EM(Y_k) = EM_{(k,k)} \). Here we only verify the latter equality. Clearly, \( EM(Y_k) \supset EM_{(k,k)} \). To show that \( EM(Y_k) \subset EM_{(k,k)} \), let any \( d \in EM(Y_k) \). We use the same symbol for the extension of \( d \) on \( Y_k \cup \{ x_{\infty} \} \). Fix an admissible metric \( \rho \in AM(\alpha X) \), so we can define an extension \( \tilde{d} \in AM(\alpha X) \) of \( d \) as follows:
\[
\tilde{d}(x, y) = \begin{cases} 
  d(x, y) & \text{if } x, y \in Y_k \cup \{ x_{\infty} \}, \\
  \rho(x, y) & \text{if } x \in \alpha X \setminus Y_k,
  \\
  d(x, x_{\infty}) + \rho(y, x_{\infty}) & \text{if } x \in Y_k \text{ and } y \in X \setminus Y_k,
  \\
  d(y, x_{\infty}) + \rho(x, x_{\infty}) & \text{if } x \in X \setminus Y_k \text{ and } y \in Y_k.
\end{cases}
\]
Thus \( d \in EM_{(k,k)} \). By Proposition 5.5 there exists a \( Z_{\sigma} \)-set \( Z \) in \( PM(Y_k) \) that contains \( EM(Y_k) \). It follows from Proposition 3.5 that \( Z \times \prod_{(i,j) \in K^2 \setminus \{(k,k)\}} PM_{(i,j)} \) is a \( Z_{\sigma} \)-set in \( PM_{(k,k)} \times \prod_{(i,j) \in K^2 \setminus \{(k,k)\}} PM_{(i,j)} \), which means that \( EM(X) \) is contained in some \( Z_{\sigma} \)-set of \( PM(X) \). According to Proposition 6.5, \( EM(Y_k) \) is strongly \( \mathcal{M}_2 \)-universal. Combining Proposition 2.6 with Proposition 6.3 we can see that the product space \( EM_{(k,k)} \times \prod_{(i,j) \in K^2 \setminus \{(k,k)\}} EM_{(i,j)} \) is strongly \( \mathcal{M}_2 \)-universal, and therefore so is \( EM(X) \). Hence the space \( EM(X) \) is an \( \mathcal{M}_2 \)-absorbing set in \( PM(X) \). Combining this with Theorems 2.2 and 2.1 we conclude that \( EM(X) \) is homeomorphic to \( c_0 \). □

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