Localization to delocalization crossover in a driven nonlinear cavity array

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Abstract
We study nonlinear cavity arrays where the particle relaxation rate in each cavity increases with the excitation number. We show that coherent parametric inputs can drive such arrays into states with commensurate filling that form non-equilibrium analogs of Mott insulating states. We explore the boundaries of the Mott insulating phase and the crossover to a delocalized phase with spontaneous first order coherence. While sharing many similarities with the Mott insulator to superfluid transition in equilibrium, the phase diagrams we find also show marked differences. Particularly the off diagonal order does not become long range since the influence of dephasing processes increases with increasing tunneling rates.

1. Introduction

Photons are not usually conserved in light–matter interactions. Consequently, there is no chemical potential for photons, and the rich vein of many-body quantum effects in equilibrium systems is seemingly lost to photonics. Some exceptions, where the concept of an effective chemical potential can be meaningfully applied to photons, include photon emission in semiconductors [1], photons confined in a cavity that couple to excitons and thermalize [2–4], and photons interacting with a nonlinear medium that form a Bose–Einstein condensate [5, 6]. Moreover, in recent years, settings where light–matter interactions can mediate strong photon–photon interactions have gained significant interest as these allow one to generate matter–like phases such as photonic fluids [4, 7] or even strongly correlated phases [8–10].

Since photons are bosons, a key question for many-body phenomena in strongly interacting photons or polaritons is whether a phase transition from a Mott insulator to a superfluid state [11], as in Bose–Einstein condensates [12–14] can be observed. Early theory investigations of the phase diagrams of interacting photons or polaritons in arrays of coupled cavities considered equilibrium scenarios by introducing a chemical potential, the physical realization of which remained an open question [9, 15–17]. Given the limited lifetime of photons trapped in a cavity, it is however more natural to explore many-body phases in a non-equilibrium setting taking into account input drives and dissipation. Following this route, auxiliary systems together with specific driving mechanisms have recently been considered to generate effective chemical potentials for photons [18–21] and resulting phase diagrams have been explored [22].

Here we show that a Mott phase can be generated in a nonlinear cavity array with dissipation by only employing a coherent parametric drive that is directly applied to the cavities and explore the crossover from this Mott insulator to a delocalized phase showing first order coherence between lattice sites.

A key feature of a Mott insulating phase is that the particle number is commensurate with the number of lattice sites, i.e. there is an integer number of particles on each lattice site and the number fluctuations are strongly suppressed. Such a situation cannot be achieved in a nonlinear resonator array with coherent driving at the excitation frequency. In the limit of very strong nonlinearity and vanishing hopping, where the Mott insulating regime is expected, each lattice site may be approximated as two level system where population inversion corresponding to unit filling cannot be generated via a coherent drive.
Difficulties in arranging for a commensurate filling at vanishing hopping are not the only challenge for exploring Mott insulating to superfluid transitions in driven dissipative systems. For coherent driving fields, the phase relation between the inputs at different lattice sites is necessarily fixed. Therefore any phase-coherence between light fields in distant cavities that is found can be attributed at least in part to the coherent input drives \(^{23}\) and it is not clear whether such coherence forms spontaneously as in equilibrium \(^{12-14}\).

To circumvent the obstacles impeding the formation of Mott insulating phases and the difficulties in studying the spontaneous formation of coherences in coherently driven cavity arrays, we here consider a coherent parametric drive, that resonantly drives the transition from the zero excitation to the two excitation state \(^{19,24}\), but is off resonance with all other transitions. Together with a cascade of decay processes, where the decay from a two excitation state to a single excitation state, \(\gamma_1\), is much faster than the decay of a single excitation state to a zero excitation state, \(\gamma_0\), this leads to a stationary state with a very high probability to find a single excitation in each lattice site. This state emerges because the pump and relaxation processes combine to form an effective incoherent drive from the zero excitation state to the single excitation state, see figure 1 for a sketch of a two-site model. More precisely, the probability to find a single excitation in each lattice site approaches unity in the limit of \(\gamma_0/\gamma_1 \to 0\).

For this arrangement, we investigate the crossover from this Mott phase with commensurate filling to a delocalized phase with incommensurate filling \(^{25-31}\). An important property of our model is that the excitations in level \(|1\rangle\) are generated via the fast decay at rate \(\gamma_1\). These excitations, which are responsible for the physics we find since level \(|2\rangle\) remains unoccupied to good accuracy, are therefore insensitive to the coherent nature of the drive because the fast decay at rate \(\gamma_1\) erases all coherence between lattice sites that is due to the coherent input. Any first order correlations between the lattice sites that we find in the stationary states can thus clearly be attributed to the formation of a superfluid component.

In the following, we first introduce the model we considered, then present our results and finish with conclusions and an outlook.

2. The model

We consider a system of \(N\) coupled nonlinear cavities in a one-dimensional array, governed by a Bose–Hubbard Hamiltonian, with an additional coherent parametric driving term. After moving into a suitable rotating frame, applying the rotating wave approximation and setting \(\hbar = 1\) we are left with the Hamiltonian

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_f + \mathcal{H}_\Omega, \tag{1}
\]
where,
\[ \mathcal{H}_0 = \sum_j \left[ \Delta \hat{a}_j^\dagger \hat{a}_j + \frac{U}{2} \hat{a}_j^\dagger \hat{a}_j \hat{a}_j \hat{a}_j^\dagger \right] \]
(2)
\[ \mathcal{H}_f = -J \sum_j \left[ \hat{a}_j \hat{a}_{j+1}^\dagger + \hat{a}_{j+1} \hat{a}_j^\dagger \right] \]
(3)
\[ \mathcal{H}_{\Omega} = \sum_j \left[ \Omega \frac{\sqrt{2}}{\sqrt{2}} \hat{a}_j^\dagger \hat{a}_j + \Omega^* \frac{\sqrt{2}}{\sqrt{2}} \hat{a}_j \hat{a}_j^\dagger \right] \]
(4)

Here \( \Delta = \omega - \omega_L / 2 \) is the detuning between the drive laser frequency \( \omega_L \), and the cavity frequency \( \omega \), \( U \) is the interaction strength, \( J \) is the hopping rate between sites, and \( \Omega \) is the drive strength. We tune the drive laser frequency to be in resonance with the two excitation frequency \( \omega_L = 2\omega + U \) which implies \( \Delta = -U/2 \).

We note that a similar model with two-photon driving has been considered in \([24]\), where a spontaneous symmetry breaking with a second order transition from a symmetric phase to a quasi-coherent state with a finite expectation value of the Bose field has been found using a mean-field approach. Here in contrast we are interested in the physics generated by the combination of a cascaded relaxation process and two-photon driving.

The dissipative environment we consider is characterized by a cascade of dissipation rates, so the dissipation rate \( \gamma_m \) from \( |m + 1 \rangle \rightarrow |m \rangle \) is greater than the dissipation rate \( \gamma_n \) from \( |n + 1 \rangle \rightarrow |n \rangle \) when \( m > n \). We describe this dissipation via a standard Lindblad-form master equation,
\[ \rho = -i[\mathcal{H}, \rho] + \sum_{m \geq 0} \mathcal{D}_m[\rho], \]
(5)
where,
\[ \mathcal{D}_m[\rho] = \gamma_m \sum_{j} \left[ 2\kappa_{m,j} \rho \kappa_{m,j}^\dagger - \kappa_{m,j} \kappa_{m,j}^\dagger \rho - \rho \kappa_{m,j} \kappa_{m,j}^\dagger \right] \]
(6)
with jump operators
\[ \kappa_{m,j} = |m\rangle \langle m+1| . \]
(7)

We note that our model assumes that dissipation is dominated by single particle losses and would reduce to the standard dissipator \( \sum_{j} \left[ 2\kappa_{j} \rho \kappa_{j}^\dagger - \kappa_{j} \kappa_{j}^\dagger \rho - \rho \kappa_{j} \kappa_{j}^\dagger \right] \) in the limit where all relaxation rates become equal, \( \gamma_m = \gamma \).

We further assume that \( U \gg \Omega \) so that the occupation of levels \( |m\rangle \) with \( m > 2 \) remains negligible and we may truncate our description to the subspace of at most two excitations on each site. It is thus sufficient to ensure that the dissipation rate from \( |2\rangle \rightarrow |1\rangle \) exceeds the dissipation rate from \( |1\rangle \rightarrow |0\rangle \), i.e. \( \gamma_1 \gg \gamma_0 \). Experimentally, such a ratio of dissipation rates can for example be achieved via Purcell enhancement of the relaxation on a specific transition via coupling to a lossy resonator, whose resonance frequency matches this particular transition \([32]\).

3. Results

For the model described by equation (5) with the Hamiltonian (1), we investigated the stationary states. In doing so, we scaled all parameters around the fast dissipation rate, which was fixed at \( \gamma_1 = 1 \). We first consider a small three site lattice to explore the Mott insulating phase and test the accuracy of our Hilbert space truncation. We then explore the degrading of the Mott insulator and crossover to a delocalized phase for larger lattices \( N = 11 \) and \( N = 15 \) using matrix product operator simulations.

3.1. Small anharmonic system

Before considering the large many-body system we looked at exact calculations for just three sites with periodic boundary conditions, where we extend our description to up to three excitations per site to test the validity of a truncation to three levels. This few site system also allows us to explore the Mott insulating phase via an exact numerical solution of the model. Since longer range correlations are absent in the Mott insulating phase, we expect these results to also apply for large systems in the limit of \( J \ll \gamma_1 \). Our findings in sections 3.2 and 3.3 confirm this expectation.

The equilibrium phase diagram for the Bose–Hubbard model is typically parameterized by the chemical potential and the hopping between sites. In our case, in contrast, the drive strength and dissipation rates balance out to create an effective chemical potential. For this reason we consider the drive strength and the hopping rate to be an appropriate parameterization to explore the crossover we are interested in.

Figure 2 shows the number density \( \langle n_2 \rangle \) and its variance \( \langle n_2^2 \rangle - \langle n_2 \rangle^2 \) for one site in this translation invariant system. Both quantities are plotted against the drive strength, \( \Omega \), and the coupling strength, \( J \). As can be seen from figure 2(a) there is a region, bounded by a black line at low hopping rate for which the density is unity with
good accuracy, and the variance $\langle n_2^2 \rangle - \langle n_2 \rangle^2 \ll 1$. This shows that there is a stationary state phase for our model with very similar properties as a Mott insulator in equilibrium systems. In contrast to the equilibrium case for one-dimensional lattices, the Mott lobe however does not have a cusp-like shape \([33]\).

To explore the boundaries of this phase and its crossover to a delocalized phase, with first order correlations between lattice sites, we also considered larger lattices with $N = 15$, which we discuss next.

### 3.2. Large anharmonic system

In order to explore the build up of correlations and analyze their length, we moved to considering a larger 15-site system, with up to two excitations per site. This was achieved using an implementation of the time evolving block decimation (TEBD) method \([34]\). Unlike the small system calculation, we here consider a system with open boundary conditions. Figure 3 shows the density of the middle site and its variance plotted against drive strength $\Omega$, and coupling strength $J$. It can be seen that the density remains close to one for sufficient drive strength $\Omega$, and again at low coupling strength $J$ we see a variance of much less than one, indicative of the Mott insulator state.

For the parameter region where the local excitation number fluctuations start to increase, it is an interesting question whether a transition to a superfluid state or BEC occurs. A signature of such a transition would be an increase in long range first order coherence as found in an equilibrium BEC. Since the excitations in level $|1\rangle$ in our model are generated via the fast decay at rate $\gamma_1$ they are insensitive to the coherent nature of the drive. In fact, the pump and relaxation processes combine to form an effective incoherent drive from the zero excitation state to the single excitation. This interpretation is evident from an approach that adiabatically eliminates the double excitation levels to derive an effective model for a chain of two level systems, see appendix. Any first order
correlations between the lattice sites that we find in the stationary states can thus clearly be attributed to the formation of a superfluid component. We therefore investigated these first order correlations as quantified by the \( g^{(1)}(i,j) \)-function,

\[
g^{(1)}(i,j) = \frac{\langle \hat{a}_i^\dagger \hat{a}_j \rangle}{\sqrt{\langle \hat{n}_i \rangle \langle \hat{n}_j \rangle}}.
\]  

(8)

It can be seen in figure 4(a) that first order correlations build up in the system as the coupling strength increases. The range of these correlations however does not increase monotonically with \( J \), but drops after reaching a peak. To obtain a correlation length we fitted an exponential \( \exp(-|j - j_\lambda/\lambda|) \) to the \( g^{(1)} \) data, where \( \lambda \) is the correlation length. The result is shown in figure 4(b). We attribute the non-monotonic behavior of the correlation length to the competition between the tunneling processes, which enhance long range coherence, and the increase of dephasing processes with increasing local density fluctuations, which reduce long range coherence. Specifically the local density fluctuations \( \langle n_j^2 \rangle - \langle n_j \rangle^2 \) increase with the tunneling rate \( J \) and cause more occupation of the double excitation levels, which in turn leads to a stronger contribution of the fast relaxation mechanism and thus enhanced dephasing.

A further characteristic of a Mott insulator is incompressibility. To explore whether the system becomes more compressible with increasing hopping rate, we thus calculated two-photon coincidences as quantified by the \( g^{(2)} \)-function,

\[
\text{Figure 4. The first order correlation } g^{(1)}(i,j) \text{ and the correlation length } \lambda. \text{ For this calculation, } \gamma_1 = 1, \text{ and } \gamma_0 = 0.1. \text{ The interaction strength, } U = 20, \text{ and so } \Delta = -10. \text{ The first order correlation is plotted for a range of coupling strengths at a fixed drive strength, } \Omega = 5. \text{ The correlation length was determined by an } \exp(-|j - j_\lambda/\lambda|) \text{ fit to the } g^{(1)} \text{ data.}
\]

\[
\text{Figure 5. Second order correlation between sites, } g^{(2)}(i,j) \text{ plotted against coupling strength. For this calculation, drive strength } \Omega = 5, \gamma_1 = 1, \text{ and } \gamma_0 = 0.1. \text{ The interaction strength, } U = 20, \text{ and so } \Delta = -10.
\]
Figure 5 shows that indeed the on-site density correlations as measured by $g(2)(i,j) = \frac{\langle \hat{a}_i^\dagger \hat{a}_j \hat{a}_i \hat{a}_j \rangle}{\langle \hat{n}_i \rangle \langle \hat{n}_j \rangle}$ increase monotonically across this region.

3.3. Large harmonic system

To show that our findings are not limited to the parameters considered so far and to further explore this class of systems, we considered a lattice with $\Delta = 0$ and hence $U = 0$. We again used a truncation to the subspace of at most two excitations and calculated the steady state using a TEBD code for an 11-site system with open boundary conditions. Figure 6 shows the middle site density, and density variance plotted against drive strength $\Omega$, and coupling strength $J$. Here we also find a region where the excitation density is approximately commensurate and its variance is much less than one, indicative of the Mott insulator state. This region is however shifted in the parameter space to a higher drive strength. In figures 7 and 8 we again show the correlation length against drive strength and hopping rate, and the first and second order correlations for a fixed drive strength.

The principle difference between the correlations in the harmonic and anharmonic case is the presence of troughs in the first order correlation on alternating sites in the harmonic case. This can be understood from the momentum basis representation of the master equation. In this representation, the system is modeled by the
Hamiltonian,
\[
\mathcal{H} = \sum_{k=0}^{N-1} \left[ \frac{\Omega}{\sqrt{2}} \hat{b}_k \hat{b}^\dagger_{N-k} + \frac{\Omega^*}{\sqrt{2}} \hat{b}^\dagger_k \hat{b}_{N-k} \right] + \sum_{k=0}^{N-1} \left[ \Delta - 2J \cos \left( \frac{2\pi k}{N} \right) \right] \hat{b}_k \hat{b}^\dagger_k + \sum_{(j,k,l,m)} \left[ \frac{U}{N} \hat{b}_j \hat{b}_k \hat{b}_l \hat{b}_m \right],
\]
where \( \gamma_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp(i \frac{2\pi k}{N}) \hat{b}_k \) and we have assumed periodic boundary conditions. The notation \((j,k,l,m)\) indicates that the indices range from 0 to \(N-1\) and follow the condition \(l + m - j - k = nN\), where \(n\) is some integer.

In the harmonic case where \(\Delta = -U/2 = 0\), it can be seen that in the momentum basis the detuning \(\Delta - 2J \cos \left( \frac{2\pi k}{N} \right)\) is zero for modes with \(k = nN/4\) (where \(n\) is some integer). As such the drive is resonant to these modes, and these momenta determine the correlation profile. In the anharmonic case in turn \(\Delta = -U/2\) and is large compared to \(2J\) and the mode with \(k = 0\) is closest to resonance with the drive.

4. Conclusion

In conclusion, we have considered nonlinear cavity arrays where the relaxation on the transition \(|m+1\rangle \rightarrow |m\rangle\) is greater than the relaxation on the transition \(|n+1\rangle \rightarrow |n\rangle\) for \(m > n\), and shown that these have a stationary state with similar properties as a Mott insulator if they are parametrically driven on the transition \(|0\rangle \rightarrow |2\rangle\) in each lattice site. We have also explored the crossover to a delocalized phase with increasing tunneling rate and find that first order coherence does build up spontaneously. In contrast to the equilibrium case, this first order coherence does not become long range and decreases for very large tunneling rates after reaching a peak. We attribute this non-monotonic behavior to dephasing processes that become more important as the number fluctuations increase as a consequence of enhanced tunneling. In future research it would be interesting to corroborate these trends by exploring regimes with even larger tunneling rates \(J\) (which difficult to access with our current numerics) and to investigate higher density Mott lobes for systems with a larger cascade of dissipation rates.

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Appendix. Adiabatic elimination

Following the prescription of [35] we performed an adiabatic elimination of the upper (two excitation) state \(|2\rangle\) in the limit where \(\gamma_1\) is dominant, \((\gamma_1 \gg \Omega, U, \Delta, \gamma_0)\). For simplicity we also chose to consider here the harmonic case, where \(\Delta = -U/2 = 0\). We separated the Liouvillian of the master equation (5) into two
components, \( \mathcal{L} = \mathcal{L}_0 + \nu \), where the subspace dynamics of interest takes place in the stationary states of \( \mathcal{L}_0 \), and \( \nu \) contains all the other terms of the master equation, which we treat as a perturbation to \( \mathcal{L}_0 \). In our case, \( \mathcal{L}_0 = \mathcal{D}_i(\rho) \) is the fast dissipator which does not act on levels \( |1\rangle \) and \( |0\rangle \). To second order in \( \Omega/\gamma_1 \) and \( J/\gamma_1 \) we find the two level master equation,

\[
\dot{\rho}_0 = -i[\mathcal{H}_f, \rho_0] + \frac{\gamma_0}{2} \sum_j \left[ 2\hat{a}_j^\dagger \rho_0 \hat{a}_j - (\hat{a}_j^\dagger \hat{a}_j) \rho_0 - \rho_0 (\hat{a}_j^\dagger \hat{a}_j) \right] + \sum_j \left[ 2\alpha_j^\dagger \rho_0 \alpha_j - (\alpha_j^\dagger \alpha_j) \rho_0 - \rho_0 (\alpha_j^\dagger \alpha_j) \right],
\]

where \( \rho_0 \) is a density matrix for the truncated subsystem of interest, \( \mathcal{H}_f \) is the hopping Hamiltonian as in section 2, and,

\[
\alpha_j = \frac{\gamma_0 \Omega}{\sqrt{\gamma_1}} \hat{a}_j - \frac{2J}{\sqrt{\gamma_1}} (\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_j \hat{a}_{j-1}).
\]

We note that in this two level model the drive now appears as an incoherent pump term with jump operator \( \frac{\gamma_0 \Omega}{\sqrt{\gamma_1}} \hat{a}_j \), and hopping transitions between upper levels on neighboring sites appear as a density activated dissipation with jump operator \( \frac{2J}{\sqrt{\gamma_1}} (\hat{a}_{j+1} \hat{a}_j + \hat{a}_j \hat{a}_{j-1}) \).

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