Multi-critical absorbing phase transition in a class of exactly solvable models

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Abstract – We study diffusion of hardcore particles on a one dimensional periodic lattice subjected to a constraint that the separation between any two consecutive particles does not increase beyond a fixed value \((n+1)\); initial separation larger than \((n+1)\) can however decrease. These models undergo an absorbing state phase transition when the conserved particle density of the system falls below a critical threshold \(\rho_c = 1/(n+1)\). We find that \(\phi_k\)s, the density of 0-clusters (0 representing vacancies) of size \(0 \leq k < n\), vanish at the transition point along with activity density \(\rho_a\). The steady state of these models can be written in matrix product form to obtain analytically the static exponents \(\beta_k = n - k\), \(\nu = 1 = \eta\) corresponding to each \(\phi_k\). We also show from numerical simulations that starting from a natural condition, \(\phi_k(t)\)s decay as \(t^{-\alpha_k}\) with \(\alpha_k = (n - k)/2\) even though other dynamic exponents \(\nu_t = 2 = z\) are independent of \(k\); this ensures the validity of scaling laws \(\beta = \alpha \nu\), \(\nu_t = z\nu\).

Introduction. – Absorbing state phase transition (APT)\(^1\) is the most studied non-equilibrium phase transition in last few decades. Unlike equilibrium counterparts, these systems do not obey the detailed balance condition, as the absorbing configurations of the system can be reached by the dynamics but can not be left. Thus by tuning a control parameter these systems can be driven from an active phase to an absorbing one where the dynamics ceases. On one hand the non-equilibrium dynamics generically makes analytical treatment of these systems highly nontrivial, giving rise to varied class of distributions as well as rich variety of novel correlations, and on the other hand the non-fluctuating disordered phase being unique to APT leads to a unconventional critical behaviour. The most robust universality class of APT is directed percolation (DP)\(^2\), which is observed in context of synchronization\(^3\), damage spreading\(^4\), depinning transition\(^5\), catalytic reactions\(^6\), forest fire\(^7\), extinction of species\(^8\) etc. Recently DP critical behaviour has been observed experimentally\(^9\) in liquid crystals. It has been conjectured by\(^10\) that in absence of any special symmetry, APT with a fluctuating scalar order-parameter belongs to DP.

Models involving more than one species of particles can have interesting features\(^11\)\(^12\). Some of these models also show multi-criticality in a sense that the density of different species may vanish at the critical point following power-laws with different exponents. In one dimensional coupled directed percolation process\(^11\) where the transmutation is hierarchical, the order-parameter exponents for different species are found to be \(\beta = 0.27, 0.11, \ldots\), with the first value being that of DP. A similar feature has been observed numerically in roughening transition occurring in growth models with adsorption, and desorption at boundaries\(^13\). In this article we show that simple diffusion of hardcore particles on a lattice can undergo a multi-critical absorbing phase transition when additional constraints or particle interactions are introduced.

The model we investigate here is a variant of the assisted hopping models where hardcore particles hop to one of the neighbours with rates that generally depend on the distance of the moving particle from its nearest occupied neighbour\(^14\)\(^15\); steady state weights of some of these models are known exactly\(^14\)\(^18\)\(^19\). We restrain only to a special case, where diffusion of particles are additionally constrained not to increase the inter-particle separation beyond a fixed positive integer \((n + 1)\). The steady state weights of the models in this class, parameterized by the

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integer \( n \), can be written in a matrix product form. This helps us obtaining the spatial correlation functions exactly. In particular, the density of 0-clusters of size \( 0 \leq k < n \) vanishes at the critical point following power-laws with \( k \)-dependent exponents. Thus, the cluster density \( \phi_k \) for each \( k \) can be considered as order-parameters of the system in addition to the natural order parameter \( \rho_a \), namely activity density. Our careful numerical study of the decay of \( \phi_k \) from a natural initial condition [20][22], which is hyperuniform [21], shows that the dynamical exponents \( \alpha, \nu, \zeta \) do satisfy scaling relations separately for each \( k \).

**The Model.** – The model is defined on a one dimensional periodic lattice of size \( L \) with sites labeled by \( i = 1, 2 \ldots L \). Each site can be occupied by at most one particle and correspondingly there is a site variable \( s_i = 1,0 \) that represents the presence or absence of the particle at site \( i \). The dynamics of the model is the given by,

\[
10^k 10^{m_1} \rightarrow 10^{k+1} 10^{m_1-1} \quad \text{if} \quad k < n, m \geq 1 \\
10^{k-1} 10^{m_1+1} \quad \text{if} \quad m < n, k \geq 1
\] (1)

where a particle moves to the right or left vacant neighbour, chosen independently, if the move does not increase inter-particle separation beyond \( n + 1 \) (\( n \) being a fixed integer parameter of the model). Clearly, the total number of particles \( N = \sum_{i=1}^{L} s_i \), or equivalently the density \( \rho = N/L \) is conserved. A schematic description of the dynamics is given in Fig. 1.

Alternatively, the dynamics of the model can be considered as constrained diffusion of hardcore particles. The constraint comes from the fact that the diffusing particle’s distance, measured from the nearest particle, does not exceed \( n+1 \). We further refer to this model as constrained diffusion model (CDM). In fact, recently a similar assisted hopping model has been introduced and solved exactly [14], where particle hopping depends on the inter-particle separation but unlike CDM particles there can hop by one or more steps across the empty regions.

In this constrained diffusion model, a particle which is surrounded from both sides by other particles, or by 0-clusters of size \( \geq n \) are inactive and can not move; all other particles are active. Thus, the system has many absorbing configurations where all particles are inactive. Important to note that the dynamics allows decrement of length of all 0-clusters but increment of only those having length less than \( n \). Thus it is evident that when \( \rho \approx 0 \), i.e. when average separation between neighbouring particles is large, all the small 0-clusters (size < \( n \)) of the system tend to grow in size until they reach a maximum \( n \). In this case, the number of particles are not enough to reorganize the distances between the neighbouring particles below \( (n+1) \) forcing the system to fall into an absorbing configuration.

On the other hand, for large density the system has a large number of clusters of size \( < n \) which would grow in expense of the larger ones, but all of them can not reach the maximum value \( n \). Thus, all large clusters (size > \( n \)), if present in the initial state, would eventually be destroyed and the system remains active forever; this is surely the case, when \( \rho > \frac{1}{n+1} \). Clearly one expects an absorbing phase transition to occur at some density \( \rho \leq \frac{1}{n+1} \). We see later (in Eq. (14)) that the critical density is in fact \( \rho_c = \frac{1}{n+1} \).

Let us consider the system with \( \rho > \frac{1}{n+1} \) where the steady state is certainly active. The initial configurations of the system in this case may consist of several 0-clusters of size \( > n \) but all these configurations are non-recurring as the system, once leaves these configurations by destroying the large clusters, never visit them again. The stationary state of the system only consists of configurations which are recurring, where all 0-clusters are of size \( n \) or less. Thereby in the steady state, if dynamics [14] allows a particle to move from left to right it also allows the reverse, i.e. a move from right to left. Since both hopping rates are unity, the steady state satisfies the detailed balance condition with a stationary weight \( w(C) = 1 \) for all recurring configurations. Thus, representing the configurations as \( C \equiv \{10^{m_1}10^{m_2} \ldots 10^{m_N}\} \), we have

\[
w(\{10^{m_1}10^{m_2} \ldots 10^{m_N}\}) = \begin{cases} 1 & \forall m_i \leq n \\ 0 & \text{otherwise} \end{cases}
\] (2)

where the second step ensures that the steady state weight of the non-recurring configurations are zero. The corresponding probability is then,

\[
P_N(\{s_i\}) = \frac{w(\{s_i\})}{\Omega_N} = \sum_{\{s_i\}} w(\{s_i\}) \delta(\sum_i s_i - N).
\] (3)

Here, \( \Omega_N \) is the number of recurring configurations of a system of size \( L \) having \( N \) particles. It is customary to work in the grand canonical ensemble (GCE) where density of the system can be tuned by a fugacity \( z \), the partition function in GCE is \( Z = \sum_{N=0}^{\infty} \Omega_N z^N \). To proceed further, we make an ansatz that the steady state weights of the configurations can be expressed as a matrix product form,

\[
w(\{10^{m_1}10^{m_2} \ldots 10^{m_N}\}) = \text{Tr}[D E^{m_1} \ldots D E^{m_N}],
\] (4)

where matrices \( D \) and \( E \) represents 1,0 respectively. All what we need for a matrix formulation to work is to find...
a representation of $D$ and $E$ that correctly generates the steady state weights given by Eq. (2). The matrix formulation is very useful here, as one can simply set

$$E^m = 0 \text{ for } m > n$$

(5)

to ensure that probability of all non-recurring configurations are 0. Further, let us assume that matrix $D = |\alpha\rangle\langle\beta|$, where $|\beta\rangle, |\alpha\rangle$ are yet to be determined. Now, the recurring configurations are equally likely if

$$\langle\beta|E^m|\alpha\rangle = 1 \text{ for } 0 \leq m \leq n.$$  

(6)

Together, Eqs. (5) and (6) are satisfied by the following $(n + 1)$ dimensional matrices

$$E = \sum_{k=1}^{n} |k\rangle\langle k+1|; \quad \alpha = \sum_{k=1}^{n+1} |k\rangle; \quad \beta = |1\rangle; \quad D = |\alpha\rangle\langle\beta|.$$  

(7)

Now, we can write a grand canonical partition function,

$$Z_L(z) = Tr[T(z)^L] \text{ where } T(z) = zD + E$$  

(8)

where fugacity $z$ controls the particle density $\rho$. The weight of the configuration having no particles is $Tr[E^L] = 0$ for $L > n$ (from Eq. (3)). Thus, $Z_L(z)$ is the sum of the weights of all other configurations which has at least one particle.

$$Z_L(z) = z \sum_{k=1}^{L} \text{Tr} \left[ E^{k-1}DT^{k-1} \right] = z \sum_{k=1}^{L} \langle\beta|T^{L-k}E^{k-1}|\alpha\rangle.$$  

For any specific $n$, $Z_L(z)$ can be calculated explicitly. We prefer to use a generating function (or, partition function of the system in variable length ensemble (VLE)),

$$Z(z, \gamma) = \sum_{L=1}^{\infty} \gamma^L Z_L(z) = \langle\beta|\frac{z\gamma}{z - \gamma T} \frac{1}{z - \gamma E}|\alpha\rangle,$$

$$\gamma^2 - zg(\gamma) \frac{g'(\gamma)}{1 - zg(\gamma)}; \quad g(x) = \sum_{k=0}^{x} x^{k+1} = x^{x+1} - 1$$  

(10)

where, together $z$ and $\gamma$, determine the macroscopic variables

$$\langle L \rangle = \frac{\gamma}{z} \frac{\partial Z}{\partial z} = 1 + \gamma^2 \frac{g'(\gamma)}{1 - zg(\gamma)} + \gamma \frac{g''(\gamma)}{g'(\gamma)}$$  

and

$$\langle N \rangle = z \frac{\partial Z}{\partial z} = \frac{1}{1 - zg(\gamma)}.$$  

(11)

The thermodynamic limit $\langle L \rangle \to \infty$, where VLE is expected to be equivalent to GCE, corresponds to $z \to 1/g(\gamma)$. And, in this limit, the particle density is

$$\rho(\gamma) = \frac{\langle N \rangle}{\langle L \rangle} = \frac{1}{\gamma} \frac{g(\gamma)}{g'(\gamma)}.$$  

(12)

Since both $g(\gamma)$, and $g'(\gamma)$ are polynomials of order $(n + 1)$ the density $\rho$ must be finite as $\gamma \to \infty$, which corresponds to the limit $z \to 0$, as $z = 1/g(\gamma)$.

$$\lim_{z \to 0} \rho(z) \equiv \lim_{\gamma \to \infty} \rho(\gamma) = \frac{1}{n + 1} + \frac{1}{(n + 1)^2} \frac{1}{\gamma} + O(\frac{1}{\gamma^2})$$  

(13)

This proves that the critical density is

$$\rho_c = \frac{1}{n + 1}.$$  

(14)

and the system goes to an absorbing state when $\rho < \rho_c$. Further, Eq. (13) indicates that, near the absorbing transition

$$\gamma^{-1} \simeq (n + 1)^2(\rho - \rho_c).$$  

(15)

In Fig. 2(a) we have plotted $\rho$ as a function of $\gamma^{-1}$ for $n = 2$, where the inset shows $z \equiv g(\gamma)^{-1}$ as a function of $\gamma^{-1}$. Figure 2(b) there shows the plot of $\rho(z)$. Clearly, both in the limit $z \to 0$ or equivalently when $\gamma \to \infty$, $\rho \to \frac{1}{\gamma}$ indicating that an absorbing phase transition occurs at $\rho_c = \frac{3}{2} \nu$.

**Multicriticality.** At the critical density $\rho_c$, all other $\phi$ exist as length $n$. Thus as $\rho \to \rho_c$, from above, i.e., in the active phase $\rho > \rho_c$, number of $\phi$-clusters having size $k < n$ must individually vanish. Defining density of such clusters as $\phi_k$, we have

$$\phi_k = \langle 10^k \rangle = \frac{\gamma^{k+2}z^2}{Z(z, \gamma)} \text{Tr}[DE^k D \frac{1}{L - \gamma T}]$$  

$$= \gamma^{k+2}z^2 \frac{\langle\beta|E^k|\alpha\rangle}{\langle\beta|\frac{1}{L - \gamma T}|\alpha\rangle} = \rho \gamma^{k+1}$$  

(16)

for $0 \leq k < n$. Here, in the last step we have used the fact that

$$\langle\beta|\frac{1}{L - \gamma T}|\alpha\rangle = \frac{g(\gamma)}{\gamma - zg(\gamma)} \quad \text{and} \quad \langle\beta|E^k|\alpha\rangle = 1.$$  

In the thermodynamic limit, $z \to g(\gamma)^{-1}$, we have

$$\phi_k = \rho \gamma^{k+1} \frac{g'(\gamma)}{g'(\gamma)}.$$  

(18)

and in the critical limit $\gamma \to \infty$, (where $g(\gamma) \simeq \gamma^{n+1}$),

$$\phi_k \simeq \gamma^{k-n} \simeq (n + 1)^{3-2k}(\rho - \rho_c)\beta_k; \quad \beta_k = n - k.$$  

(19)

In Fig. 3(a) we have plotted $\phi_k$s for $n = 2$, as a function of density $\rho$. Both $\phi_{0,1}$ vanishes as $\rho \to \rho_c = \frac{3}{2}$ and thus each of them can be considered as an order-parameter that describes the APT. However, $\phi_2$ does not vanish and at the critical point $\phi_2 = (1 - \rho_c)/2$ because there is an exact correspondence $1 - \rho = \sum_{k=0}^{n} \phi_k$ which holds for any $n, \gamma$. Also at $\gamma = 1$, which corresponds to density $\lambda = 2\nu$ (from Eq. (12)), all $\phi_k$ takes the same value $\frac{1}{n+1}$ (from Eq. (13)). Thus for $n = 2$, $\phi_k$s cross each other at $\rho = \frac{1}{\gamma}$. In Fig. 3(b) we have shown $\phi_k$s as a function of $\rho - \rho_c$ in log-scale; both $\phi_0$ and $\phi_1$ show power laws as a function of $\Delta = \rho - \rho_c$ in log-scale suggesting that $\phi_{0,1} \sim \Delta^{\delta_{0,1}}$ with $\beta_0 = 2$ and $\beta_1 = 1$. 

![Fig. 2: (a) For $n = 2$, the density $\rho$ and the fugacity $z \equiv g(\gamma)^{-1}$ (inset) are shown as a function of $\gamma^{-1}$ following Eq. (27). The parametric plot of $\rho$ as a function of $z$ is shown in (b). Multi-critical absorbing phase transition](image-url)
Fig. 3: (a) For $n = 2$, $\phi_k = (10^k 1)$ are shown as functions of $\rho$ for $k = 0, 1, 2$. Clearly, $\phi_{0,1}$ vanishes as $\rho \to \rho_c = 1/3$ whereas $\phi_2 \to (1 - \rho_c)/2$ (b) Log-scale plot of $\phi_{0,1,2}$ as a function of $\rho - \rho_c$ gives slope $\beta_k = 2 - k$. The dashed line corresponds to near critical approximation of $\phi_k x$, given by Eq. (19).

Coming back to the general $n$, all the $\phi_k$ with $k = 0, 1, \ldots, n - 1$ vanishes as $\rho \to \rho_c$ following $\phi_k \simeq (\rho - \rho_c)^{\beta_k}$ with exponents $\beta_k = n - k$. The natural question is then, whether other exponents associated with $\phi_k x$s there is a natural order-parameter $\rho_0$, the density of active particles, which conventionally characterizes the APT. Since in the steady state, inactive particles are surrounded from both sides by 0-clusters of size 0 or $\beta_k = 2 - k$. The dashed line corresponds to near critical approximation of $\phi_k x$, given by Eq. (19).

$$\lambda^{n+1} - 2 \sum_{k=0}^{n} \lambda^k = 0,$$

which is equivalent to $zg(\lambda) = \lambda^{n+2}$. Since $g(x)$ satisfies an identity $g(1/x) = \frac{g(x)}{x}$, using $z = g(\gamma)^{-1}$ one can check that $\lambda = \gamma^{-1}$ is one of the solution of the characteristic equation. Again, since the characteristic equation changes sign once, from Descartes’ sign rule we conclude that there is exactly one positive real eigenvalue; thus the largest eigenvalue of $T$ is $\lambda_1 = 1/\gamma$. Assuming that the eigenvalues $\{\lambda_k\}$ are ordered such that $\lambda_1 < |\lambda_2| \leq \ldots |\lambda_{n+1}|$ (mod $n$ is taken, as generically, the eigenvalues could be complex), we write,

$$\langle |\beta T^r| \alpha \rangle = A_1 \left( \lambda_1^{n+1} + \sum_{k=2}^{n+1} A_k \lambda_k^r \right)$$

where $A_k$ are constants, independent of $r$. Since, the correlation function $C(r)$ vanishes in $r \to \infty$ limit, we must have $A_1 = pg(\gamma)/\gamma$, which results in the asymptotic form of the correlation function as,

$$C(r) \simeq \rho^2 A_2 (\gamma \lambda_2)^r ; C_k(r) \simeq \phi_{k+2} A_2 (\gamma \lambda_2)^r.$$  

(24)

If $\lambda_2$ is complex, then $\lambda_1$ must be $\lambda_2^\ast$, because complex roots of real valued polynomials appear pairwise. Taking $\lambda_2, 3 = \lambda e^{\pm i\theta}$, the correlation functions can be written as,

$$C(r) \simeq \rho^2 A_2 (\gamma \lambda)^r \cos(\theta) ; C_k(r) \simeq \phi_{k+2} A_2 (\gamma \lambda)^r \cos(\theta).$$

(25)

Let us calculate the correlation functions explicitly for $n = 2$, where the eigenvalues of the transfer matrix $T = zD + E$, with $z^{-1} = g(\gamma) = \gamma + \gamma^2 + \gamma^3$ and $D, E \text{ given by Eq. (7)}$ are

$$\lambda = \left\{ \frac{1}{\gamma}, \lambda e^{\pm i\theta} ; \lambda = \frac{\gamma}{g(\gamma)} \right\}. \tan(\theta) = \frac{1}{\sqrt{1 + 3 + 3\gamma^2}}.$$  

(26)

This leads to,

$$\rho = \frac{1 + \gamma + \gamma^2}{2 + 3\gamma} \text{ and } \phi_k = \frac{\gamma^k}{1 + 2\gamma + 3\gamma^2}.$$  

(27)
Thus in this case the spatial correlation functions would show damped oscillations of period $2\pi/\theta$. We calculate the density correlation functions of CDM with $n = 2$ at density $\rho = 0.345$ which is close to the critical density $\rho_c = 1/3$ and plot $C(r)$ as a function of $r$ in Fig. 1. We compare this with the analytic results, using $\gamma = 10.8$ (corresponding to $\rho = 10^{-4}$ in Eq. (26)). The oscillations are consistent with $\theta = 1.03$ calculated from Eq. (26).

It is important to note that, for any $n$, all $C_b(r)$s have same $r$ dependence, suggesting an unique length scale $\xi = 1/\ln(\gamma \lambda)$. At the critical point ($\gamma \to \infty$), the eigenvalues $\lambda_{k, s}$ approach towards $\frac{1}{\sqrt{2n_k}}$, and thus $|\lambda_{k, s}| \to 1$, resulting in a diverging correlation length $\xi$. Near the critical point, we may write, to the leading order, $(\gamma \lambda - 1) \sim \frac{1}{\xi}$, thus, the correlation length $\xi \sim (\rho - \rho_c)^{-\nu}$, with $\nu = 1$. Also, since the correlation functions are expected to decay as $r^{-2+2/\nu}$, for this one dimensional model ($d = 1$) we get $\eta = 1$.

Now let us turn our attention to the dynamic exponents at the critical point. At the critical point, every particle has exactly $n$ vacant sites to their right. If we add an extra particle, it will break one of the $\mathcal{O}$-clusters into two, each having size $< n$, creating some active particles in the system. It is easy to see that these active particles would do unbiased random walk, exploring a typical region of size $\sqrt{t}$ in time $t$. Thus the dynamic exponent is $z = 2$. Now assuming that the scaling relations $\nu_z = \nu z$ we expect $\nu_1 = 2$.

Since $\phi_{k, s}$ vanish at the critical point, it is natural to expect that their decay from an active initial condition follow a power-law,

$$\phi_k(t) \sim t^{-\alpha_k}; \quad \alpha_k = \frac{\beta}{\nu_1} = \frac{n - k}{2}. \quad (28)$$

Of course, we have assumed scaling relations to hold here, when its validity is being doubted in similar models. Thus it is necessary that we verify from numerical simulations, whether the scaling relations are indeed valid here.

To measure the decay exponents at the critical density $\rho_c$, corresponding to any $\phi_k$, one must carefully choose initial configurations with some nonzero $\phi_k$, which possess natural correlations of the critical state. It has been argued in similar models that the critical absorbing state is hyperuniform, i.e., the variance of density in the critical state is sub-linear in volume (here length $L$). Usually densities in hyperuniform states are anti-correlated and thus it is useful to study decay from configurations which already posses the natural correlations of the critical state. Such natural initial conditions can be generated following the prescriptions given in Ref. [20]. In the restricted diffusion model, starting from the absorbing configuration, 1s separated by $n$ zeros, we allow particle to diffuse stochastically for a very short time (say, 0.1 MCS) to create an active state and then turn on the dynamics. The decay of $\phi_k(t)$ for $n = 2$ and 3 are plotted in Fig. 5(a) and (b) respectively in log-scale; they consistently show that $\alpha_k = (n - k)/2$.

We also calculate the dynamical exponent $z$ from the finite size corrections. At the transition point,

$$\phi_k(t, L) = t^{-\alpha_k}\mathcal{F}_k\left(\frac{t}{L^z}\right). \quad (29)$$

Starting from the natural initial condition, we measure $\phi_k(t, L)$ for different $L$ and plot $\phi_k(t, L) t^{\alpha_k}$ as a function of $t/L$ in Fig. 6(a) and (b) respectively for $k = 0$ and 1, taking $z = 2$. A good data collapse confirms that $z = 2$. Note that the fluctuations and a small deviation of $\alpha_0 = 1.02$ from expected value 1 can be blamed to the small numerical value of $\phi_0$.

### Mapping to misanthrope process.

We must mention that CDM can be mapped to misanthrope process in one dimension [23], where particles do not obey hardcore restriction and hop, one at a time, from a site (usually called box) to one of the neighbours with a rate that depends on the occupation number of both, the departure and the arrival site. In this mapping, 1s are considered as boxes carrying exactly one particle from a box to a neighbour with a restriction that the occupation number of both, the departure and the arrival site.

In this mapping, 1s are separated by $n$ zeros, we allow particle to diffuse stochastically for a very short time (say, 0.1 MCS) to create an active state and then turn on the dynamics. The decay of $\phi_k(t)$ for $n = 2$ and 3 are plotted in Fig. 5(a) and (b) respectively in log-scale; they consistently show that $\alpha_k = (n - k)/2$.

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Starting from the natural initial condition, we measure $\phi_k(t, L)$ for different $L$ and plot $\phi_k(t, L) t^{\alpha_k}$ as a function of $t/L$ in Fig. 6(a) and (b) respectively for $k = 0$ and 1, taking $z = 2$. A good data collapse confirms that $z = 2$. Note that the fluctuations and a small deviation of $\alpha_0 = 1.02$ from expected value 1 can be blamed to the small numerical value of $\phi_0$. 

![Fig. 5](image-url) 

**Fig. 5:** At the critical point the order-parameters $\phi_k(t)$ for $k = 0, 1, \ldots, n - 1$ decay as $t^{-\alpha_k}$ where $\alpha_k = \frac{n - k}{2}$. In (a) and (b) we show decay of $\phi_k(t)$s, from a natural initial condition (see text for details), for $n = 2$ and $n = 3$ respectively (respective system sizes are $3 \times 2^{14}$ and $2^{16}$).

![Fig. 6](image-url) 

**Fig. 6:** Scaling collapse of order-parameters $\phi_{0,1}$ for $n = 2$ following Eq. (29). At the critical point, $\phi_k(t) t^{\alpha_k}$ is an universal function of $tL^{-z}$. (a) and (b) shows data collapse respectively for $k = 0, 1$, for system size $L = 300 \times (1, 2, 4, 8)$. Here we take $z = 2$ and use $\alpha_0$ as a fitting parameter; data collapse is observed in (a) for $\alpha_0 = 1.02$ and (b) for $\alpha_1 = 0.5$. 

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picture, it is rather difficult to calculate the correlation functions in general, as the information of particle ordering is lost in the mapping. In such cases, it is useful to write the steady state in matrix product form [26], whenever possible.

Summary. – In summary we study diffusion of hardcore particles on a one dimensional periodic lattice, where particle movement is constrained such that the inter-particle separation is not increased beyond \((n+1)\). Thus particles which are surrounded from both sides either by other particles or by \(0\)-clusters of size \(\geq n\) are immobile or inactive, whereas all other particles are active. Thus initial distances between two neighbouring particles, if larger than \((n+1)\), can only decrease if one of the particle is active. This constrained diffusion model (CDM) undergoes an absorbing state phase transition when density is lowered below a critical value \(\rho_c = \frac{1}{\bar{\eta}}\). Interestingly, besides the activity density \(\rho_a\), the APT here can be characterized by the steady state densities of \(0\)-clusters of size \(0 \leq k < n\) (i.e. \(\phi_k = \langle T^k \rangle = \langle s_i^k \rangle\)) which vanish simultaneously at \(\rho_c\). We show that the steady state of CDM can be written as a matrix product, which helps us obtaining the static critical exponents exactly: \(\rho\) approaches \(\rho_c\) from the active side, \(\rho_a \sim (\rho - \rho_c)^{\beta / \nu}\) with \(\beta = 1\) whereas other order-parameters vanish as \(\phi_k \sim (\rho - \rho_c)^{\eta_k}\) with \(\eta_k = n - k\). This multicritical behaviour is characterized by correlation exponents \(\nu = 1 = \eta\), same for all \(\phi_k\)s as \(\langle s_i^k s_{i+r}^k \rangle \sim e^{-r/\xi}\) with \(\xi \sim (\rho - \rho_c)^{-1}\). The steady state dynamics of CDM in the active phase is only unbiased diffusion of particles, leading to a dynamical exponent \(z = 2\). Thus, assuming that the scaling relations \(v_i = z v\), \(\alpha = \beta / \nu\) hold, one expects that \(v_\ell = 2\) is independent of \(k\) whereas \(\alpha = \alpha_k = (n - k)/2\). We verified the scaling relations explicitly from careful Monte-Carlo simulations of the model by measuring \(z, \alpha\) for \(n = 2, 3\). In these simulations, the major difficulty is to choose initial conditions that retains natural correlations of the stationary state, which we overcome by using natural initial conditions [21].

Multicritical phase transitions are not specific to absorbing phase transitions. It has been observed in many other contexts. Some of the examples in equilibrium includes eight-vertex solid on solid models [27, 28], \(N\)-state Potts model [29], antiferromagnetic spin chains [30] etc. Also, this has been observed in multi-species directed percolation process [11] and in growth models with adsorption [13]. In all these models, the critical point could be characterized by many order-parameters, each corresponding to a particular kind of order - but they all vanish at the same critical point. Exactly solvable models are a step forward to understand the nature of transition. It would be interesting to look for perturbations which could produce different ordered phases of CDM at different densities.

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