ON THE DISCRETENESS OF TRANSMISSION EIGENVALUES FOR THE MAXWELL EQUATIONS

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Abstract. In this paper, we establish the discreteness of transmission eigenvalues for Maxwell’s equations. More precisely, we show that the spectrum of the transmission eigenvalue problem is discrete, if the electromagnetic parameters \( \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \) in the equations characterizing the inhomogeneity and background, are smooth in some neighborhood of the boundary, isotropic on the boundary, and satisfy the conditions \( \varepsilon \neq \hat{\varepsilon}, \mu \neq \hat{\mu}, \) and \( \varepsilon/\mu \neq \hat{\varepsilon}/\hat{\mu} \) on the boundary. These are quite general assumptions on the coefficients which are easy to check. To our knowledge, our paper is the first to establish discreteness of transmission eigenvalues for Maxwell’s equations without assuming any restrictions on the sign combination of the contrasts \( \varepsilon - \hat{\varepsilon} \) and \( \mu - \hat{\mu} \) near the boundary, and allowing for all the electromagnetic parameters to be inhomogeneous and anisotropic, except for on the boundary where they are isotropic but not necessarily constant as it is often assumed in the literature.

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1. Introduction

The transmission eigenvalue problem is at the heart of inverse scattering theory for inhomogeneous media. This eigenvalue problem is a late arrival in scattering theory with its first appearance in [12], [20], in connection with injectivity of the relative scattering operator. Transmission eigenvalues are related to interrogating frequencies for which there is an incident field that doesn’t scatter by the medium. The transmission eigenvalue problem has a deceptively simple formulation, namely two elliptic PDEs in a bounded domain (one governs the wave propagation in the scattering medium and the other in the background that occupies the support of the medium) that share the same Cauchy data on the boundary, but presents a perplexing mathematical structure. In particular, it is a non-selfadjoint eigenvalue problem for a non-strongly elliptic operator, hence the investigation of its spectral properties becomes challenging. Roughly, the spectral properties depend on the assumptions on the contrasts in the media (i.e. the difference of the respective coefficients in each of the equations) near the boundary. Questions central to the inverse scattering theory include: discreteness of the spectrum that is closely related to the determination of the support of inhomogeneity from scattering data using linear sampling and factorization methods [5], location of transmission eigenvalues in the complex plane that is essential to the development of the time domain linear sampling method [10], and the existence of transmission eigenvalues as well as the accurate determination of real transmission eigenvalues from scattering data, which has became important since real transmission eigenvalues could be used to obtain information about the material properties of the scattering media. We refer the reader to [5] for a recent and self-contained introduction to the topic.
This paper concerns the discreteness and location of transmission eigenvalues in the scattering of time-harmonic electromagnetic waves by an inhomogeneous (possibly anisotropic) medium of bounded support. Let us introduce the mathematical formulation of the electromagnetic transmission eigenvalue problem. To this end, let \( \Omega \) be an open, bounded subset of \( \mathbb{R}^3 \) representing the support of the inhomogeneity, which we assume to be of class \( C^2 \).

Let \( \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \) be \((3 \times 3)\) symmetric, uniformly elliptic, matrix-valued functions defined in \( \Omega \) with \( L^\infty(\Omega) \) entries. A complex number \( \omega \) is called an eigenvalue of the transmission eigenvalue problem, or a transmission eigenvalue, associated with \( \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \) in \( \Omega \) if there exists a non-zero solution \((E, H, \hat{E}, \hat{H}) \in [L^2(\Omega)]^{12}\) of the following system

\[
\begin{align*}
\nabla \times E &= i\omega \mu H \quad \text{in } \Omega, \\
\nabla \times H &= -i\omega \varepsilon E \quad \text{in } \Omega,
\end{align*}
\]

\[
\begin{align*}
\nabla \times \hat{E} &= i\omega \hat{\mu} \hat{H} \quad \text{in } \Omega, \\
\nabla \times \hat{H} &= -i\omega \hat{\varepsilon} \hat{E} \quad \text{in } \Omega,
\end{align*}
\]

\[(1.2) \quad (\hat{E} - E) \times \nu = 0 \text{ on } \partial \Omega, \quad \text{and} \quad (\hat{H} - H) \times \nu = 0 \text{ on } \partial \Omega,
\]

where \( \nu \) denotes the outward unit normal vector to \( \partial \Omega \).

The main result that we prove in this paper is stated in Theorem \([1.1]\) below. For the reader convenience we first must clarify some terminology used in the formulation of this theorem. A \((3 \times 3)\) matrix-valued function \( M \) defined in a subset \( O \subset \mathbb{R}^3 \) is called isotropic at \( x \in O \) if it is proportional to the identity matrix at \( x \), i.e., \( M(x) = mI \) for some scalar \( m = m(x) \) where \( I \) denotes the \( 3 \times 3 \) identity matrix. In this case, for the notational ease, we also denote \( m(x) \) by \( M(x) \). If \( M \) is isotropic for \( x \in O \), then \( M \) is said to be isotropic in \( O \). Condition \((1.3)\) below is understood under the convention \( m(x) = M(x) \).

**Theorem 1.1.** Assume that

i) \( \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \) are of class \( C^1 \) in some neighborhood of \( \partial \Omega \),

ii) \( \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \) are isotropic on \( \partial \Omega \),

iii) \( (1.3) \quad \varepsilon \neq \hat{\varepsilon}, \quad \mu \neq \hat{\mu}, \quad \varepsilon/\mu \neq \hat{\varepsilon}/\hat{\mu} \) on \( \partial \Omega \).

The set of the transmission eigenvalues of \((1.1)\) and \((1.2)\) is discrete with \( \infty \) as the only possible accumulation point.

The analysis used in the proof of Theorem \([1.1]\) also allows us to obtain the following result on the transmission eigenvalue free region of the complex plane \( \mathbb{C} \).

**Proposition 1.1.** Assume that \( \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \) are of class \( C^1 \) in some neighborhood of \( \partial \Omega \), isotropic on \( \partial \Omega \), and

\[(1.4) \quad \varepsilon \neq \hat{\varepsilon}, \quad \mu \neq \hat{\mu}, \quad \varepsilon/\mu \neq \hat{\varepsilon}/\hat{\mu} \text{ on } \partial \Omega.
\]

For \( \gamma > 0 \), there exists \( \omega_0 > 0 \) such that if \( \omega \in \mathbb{C} \) with \( |\Im(\omega)| \geq \gamma |\omega|^2 \) and \( |\omega| \geq \omega_0 \), then \( \omega \) is not a transmission eigenvalue.

Here and and in what follows, for \( z \in \mathbb{C} \), let \( \Im(z) \) denote the imaginary part of \( z \).

**Remark 1.1.** Since \( \gamma > 0 \) can be chosen arbitrary small, the result of Proposition \([1.1]\) together the fact that \( \infty \) is the only accumulation point of the transmission eigenvalues proven in Theorem \([1.1]\), imply that all the transmission eigenvalues \( \omega \), but finitely many, lie in a wedge of arbitrary small angle.
The structure of spectrum of the transmission eigenvalue problem is better understood in the case of scalar inhomogeneous Helmholtz equations. In this case, the transmission eigenvalue problem can be stated as follows. Let $d \geq 2$ and $\Omega$ be an open, bounded Lipschitz subset of $\mathbb{R}^d$. Let $A_1, A_2$ be two $(d \times d)$ symmetric, uniformly elliptic, matrix-valued bounded function defined in $\Omega$ and $\Sigma_1$ and $\Sigma_2$ be two bounded positive functions defined in $\Omega$. A complex number $\omega$ is called an eigenvalue of the transmission eigenvalue problem, or a transmission eigenvalue, if there exists a non-zero solution $(u_1, u_2)$ of the system

\begin{equation}
\begin{aligned}
\text{div}(A_1 \nabla u_1) + \omega^2 \Sigma_1 u_1 &= 0 \quad \text{in } \Omega, \\
\text{div}(A_2 \nabla u_2) + \omega^2 \Sigma_2 u_2 &= 0 \quad \text{in } \Omega,
\end{aligned}
\end{equation}

\begin{equation}
u_1 = \nu_2, \quad A_1 \nabla \nu_1 \cdot \nu = A_2 \nabla \nu_2 \cdot \nu \quad \text{on } \partial \Omega.
\end{equation}

The discreteness of transmission eigenvalues for the Helmholtz equation has been investigated extensively in the literature. The first discreteness result appeared in [31], whereas [28] proves the state-of-the-art results on the discreteness of transmission eigenvalues for anisotropic background and inhomogeneity under most general assumptions on the coefficients using Fourier and multiplier approaches. More specifically, it is shown in [28] that the transmission eigenvalue problem for the Helmholtz equation has a discrete spectrum if the coefficients are smooth only near the boundary, and

i) $A_1(x)$, $A_2(x)$ satisfy the complementing boundary condition with respect to $\nu(x)$ for all $x \in \partial \Omega$, i.e., for all $x \in \partial \Omega$ and for all $\xi \in \mathbb{R}^d \setminus \{0\}$ with $\xi \cdot \nu = 0$, we have

$$\langle A_2 \nu, \nu \rangle \langle A_2 \xi, \xi \rangle - \langle A_2 \nu, \xi \rangle^2 \neq \langle A_1 \nu, \nu \rangle \langle A_1 \xi, \xi \rangle - \langle A_1 \nu, \xi \rangle^2,$$

ii) $\langle A_1(x) \nu(x), \nu(x) \rangle \Sigma_1(x) \neq \langle A_2(x) \nu(x), \nu(x) \rangle \Sigma_2(x)$ for all $x \in \partial \Omega$.

Additional results in [28], also include various combinations of the sign of contrasts $A_1 - A_2$ and $\Sigma_1 - \Sigma_2$ on the boundary. Previous results on discreteness can be found in [3, 21, 34] and references therein. We must emphasize that the conditions i) and ii) are more general than simply one sign contrasts $A_2 - A_1$ and/or $\Sigma_2 - \Sigma_1$ near the boundary. To complete the picture on the transmission eigenvalue problem in the scalar case, we remark that the first answer to the existence of transmission eigenvalues for one sign contrast in $\Omega$ was given in [30] where the authors showed the existence of a few real transmission eigenvalues for the index of refraction sufficiently large, followed by [3, 9], which prove the existence of infinite real transmission eigenvalues removing the size restriction on the index. Completeness of transmission eigenfunctions and first estimates on the counting function are shown in [32], [34] for $C^\infty$ boundary and coefficients since they use semiclassical analysis and pseudodifferential calculus. Again in $C^\infty$ setting, [35, 36] prove the sharpest known results in the scalar case on eigenvalue free zones and Weyl’s law for the scalar case improving an earlier result by [19].

The story of the transmission eigenvalue problem for Maxwell’s equations is not as complete as for the scalar case discussed above. Of the first results on discreteness is given by Haddar in [16] where it considers the case of $\mu = \hat{\mu} = \hat{\mu} = I$, and $\varepsilon - I$ invertible in $\Omega$. Chesnel in [11] employs the so-called $T$-coercivity to prove discreteness when $\varepsilon = \hat{\mu} = I$, and $\varepsilon - I$ and $\mu^{-1} - I$ are both greater than $c I$ or both less $- c I$ in $\Omega$ for some positive constant $c$ in a neighborhood of $\partial \Omega$. Cakoni et al. in [37] use an integral equation approach to study discreteness for the case when $\mu = \hat{\varepsilon} = \hat{\mu} = I$ and the matrix valued $\varepsilon$ becomes constant.
not equal to 1 near the boundary. Theorem 1.1 therefore adds to this list quite general conditions on the coefficients for which the discreteness holds. To our knowledge, our paper is the first to establish discreteness of transmission eigenvalues for Maxwell’s equations without assuming any restrictions on the sign combination of the contrasts $\varepsilon - \hat{\varepsilon}$ and $\mu - \hat{\mu}$ near the boundary (as long as they do not change sign up the boundary), and allowing for all the electromagnetic parameters to be inhomogeneous and anisotropic, except for on the boundary where they are isotropic but not necessarily constant as it is often assumed in the literature. For the case of electromagnetic transmission eigenvalue problems, other type of results are rather limited, and we refer the reader to [6] for the existence of real transmission eigenvalues and [18] for the completeness of eigenfunctions for the setting related to the one in [8] mentioned above.

The analysis in this paper is inspired by the concept of complementary conditions suggested by Agmon, Dougis, and Nirenberg in their celebrated papers [1, 2] for elliptic systems. For Maxwell’s equations, the complementary condition for the Cauchy problems has been recently investigated in [29] for general anisotropic coefficients in the context of negative index metamaterials. To be able to apply the theory of complementing conditions to the Maxwell equations, various forms of the Poincaré lemma and Helmholtz decomposition are used with a suitable implementation of local charts. The analysis in this paper is in the spirit of the one developed in [28]. The idea is to show that the following system

\begin{align*}
\nabla \times E &= i\omega \mu H + J_e \text{ in } \Omega, \\
\nabla \times H &= -i\omega \varepsilon E + J_m \text{ in } \Omega,
\end{align*}

is well-posed for some $\omega \in \mathbb{C}$ where $(J_e, J_m, \hat{J}_e, \hat{J}_m)$ is the input, which belongs to an appropriate functional space. Moreover, a key fact is to prove that the corresponding transformation which maps the input $(J_e, J_m, \hat{J}_e, \hat{J}_m)$ to the output $(E, H, \hat{E}, \hat{H})$ is compact. It is worth mentioning that the compactness is one of the crucial/critical difference between the study of the Maxwell equations and the Helmholtz equation. In our analysis, the functional space for the input is well-chosen so that the compactness property holds (see (4.20)) for $\omega$ in some domain. For example, these facts hold under the assumptions of Theorem 1.1 provided that $i\omega = |\omega|e^{i\pi/4}$, i.e., $\omega = |\omega|e^{-i\pi/4}$ and $|\omega|$ is large. To this end, we analyze the corresponding Cauchy problem with constant coefficients in a half-space (Proposition 3.1). Using the decay of Maxwell equations (Lemma 4.1), we can prove the uniqueness for (1.7) and (1.8). To establish the existence of a solution, the limiting absorption principle is used, and several processes involving the Fredholm theory of compact operator are applied. Again the choice of the functional space to ensure the compactness plays an important role here in order to use the Fredholm theory. Deriving (1.3) and handling the compactness are the key difference in the analysis of this paper and the one for the scalar case [28].

The Cauchy problem also naturally appears in the context of negative index materials after using reflections as initiated in [23]. The well-posedness and the limiting absorption principle for the Helmholtz equations with sign-changing coefficients was developed in [24] using the Fourier and multiplier approach. Recently, with Sil, the second author investigate these problems for the Maxwell equations [29]. Both papers [24, 29] deal with the
stability question of negative index materials and are the starting point for the analysis of the discreteness of transmission eigenvalues for the Helmholtz equation [28] and Maxwell’s equations in this work. Other aspects and applications of negative index materials involving the stability and instability the Cauchy problem (1.7) and (1.8) are discussed in [25, 26, 27] and the references therein.

The paper is organized as follows. In Section 2, we introduce several notations used frequently in this paper. Section 3 is devoted to the analysis in the half space. The main result in this section is Proposition 3.1. Condition (1.3) will appear very naturally there. Finally, we present the proof of Theorem 1.1 in Section 4. The choice of the right functional space plays an important role there so that the Fredholm theory can be applied. The proof of Proposition 1.1 is also given in this section.

2. Notations

The following notations are used frequently throughout the paper. Denote

\[ \mathbb{R}^3_+ = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3; x_3 > 0 \right\} \]

and

\[ \mathbb{R}^3_0 = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3; x_3 = 0 \right\}. \]

Let \( \Omega \) be a bounded, open subset of \( \mathbb{R}^3 \) and of class \( C^2 \), or \( \Omega = \mathbb{R}^3_+ \). We define the spaces

\[ H(\text{curl}, \Omega) = \left\{ u \in [L^2(\Omega)]^3; \nabla \times u \in [L^2(\Omega)]^3 \right\}, \]

\[ H_0(\text{curl}, \Omega) = \left\{ u \in H(\text{curl}, \Omega); u \times \nu = 0 \text{ on } \partial \Omega \right\}, \]

\[ H(\text{div}, \Omega) = \left\{ u \in [L^2(\Omega)]^3; \text{div} u \in L^2(\Omega) \right\}. \]

Set \( \Gamma = \partial \Omega \), and for \( s = -1/2, \text{ or } 1/2 \), define the trace space

\[ H^s_{\text{div}}(\Gamma) = \left\{ u \in [H^s(\Gamma)]^3; u \cdot \nu = 0 \text{ and } \text{div}_\Gamma u \in H^s(\partial \Omega) \right\}. \]

For a vector field \( u \) defined in a subset of \( \mathbb{R}^3 \), \( u_j \) denotes its \( j \)-th component for \( 1 \leq j \leq 3 \). We also denote, for \( s > 0 \),

\[ \Omega_s = \left\{ x \in \Omega; \text{dist}(x, \partial \Omega) < s \right\}. \]

3. Analysis on a half space

In order to simplify presentation, we let \( k \in \mathbb{C} \) be \( k := i\omega \). Let \( \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \) be four symmetric, uniformly elliptic matrix-valued functions defined in \( \mathbb{R}^3_+ \). In this section, we are interested in the following Cauchy problem for Maxwell’s equations in \( \mathbb{R}^3_+ \), with \( J_e, J_m, \hat{J}_e, \hat{J}_m \in L^2(\mathbb{R}^3_+) \) and \( f_e, f_m \in H_{\text{div}}^{-1/2}(\mathbb{R}^3_+) \),

\[ \begin{align*}
\nabla \times E &= k\mu H + J_e \quad \text{in } \mathbb{R}^3_+, \\
\nabla \times H &= -k\varepsilon E + J_m \quad \text{in } \mathbb{R}^3_+, \\
\nabla \times \hat{E} &= k\hat{\mu} \hat{H} + \hat{J}_e \quad \text{in } \mathbb{R}^3_+, \\
\nabla \times \hat{H} &= -k\hat{\varepsilon} \hat{E} + \hat{J}_m \quad \text{in } \mathbb{R}^3_+.
\end{align*} \]
and \[ (\hat{E} - E) \times e_3 = f_e \text{ on } \mathbb{R}^3_0, \quad \text{and} \quad (\hat{H} - H) \times e_3 = f_m \text{ on } \mathbb{R}^3_0. \]

We begin with proving the following lemma.

**Lemma 3.1.** Let \( \gamma > 0 \) and \( k \in \mathbb{C} \) with \( |\Im(k^2)| \geq \gamma |k|^2 \) and \( |k| \geq 1 \). Furthermore, let \( \Lambda \geq 1 \) and \( \varepsilon, \mu \) be two positive constants such that \( \Lambda^{-1} \leq \varepsilon, \mu \leq \Lambda \). For \( J_e, J_m \in [L^2(\mathbb{R}^3_+)]^3 \), there exists a unique solution \((E, H) \in \left[ L^2(\mathbb{R}^3) \right]^6 \) of the system

\[
\begin{aligned}
\nabla \times E &= k\mu H \text{ in } \mathbb{R}_+^3, \\
\nabla \times H &= -k\varepsilon E + J_m \text{ in } \mathbb{R}_+^3, \\
E \times e_3 &= 0 \text{ on } \mathbb{R}^3_0.
\end{aligned}
\]

Moreover, for some positive constant \( C \) depending only on \( \Lambda \) and \( \gamma \),

\[
\|(E, H)\|_{L^2(\mathbb{R}_+^3)} \leq C \frac{1}{|k|} \|J_m\|_{L^2(\mathbb{R}_+^3)},
\]

and if \( \text{div} \ J_m \in L^2(\Omega) \), then

\[
\|(E, H)\|_{H^1(\mathbb{R}_+^3)} \leq C \left( \|J_m\|_{L^2(\mathbb{R}_+^3)} + \frac{1}{|k|} \|\text{div} \ J_m\|_{L^2(\mathbb{R}_+^3)} \right).
\]

**Remark 3.1.** We emphasize here that the constant \( C \) appearing in (3.4) and (3.5) is independent of \( k \).

**Proof.** We have, from the system of \((E, H)\),

\[
\nabla \times (\nabla \times E) + k^2 \varepsilon \mu E = k\mu J_m \text{ in } \mathbb{R}_+^3.
\]

Multiplying (3.6) by \( \varphi \) (the conjugate of \( \varphi \)) with \( \varphi \in H_0(\text{curl}, \mathbb{R}_+^3) \), and integrating by parts yields

\[
\int_{\mathbb{R}_+^3} \langle \nabla \times E, \nabla \times \varphi \rangle + k^2 \varepsilon \mu \int_{\mathbb{R}^3} \langle E, \varphi \rangle = \int_{\mathbb{R}^3} k\mu \langle J_m, \varphi \rangle.
\]

Take \( \varphi = E \). Since \( |\Im(k^2)| \geq \gamma |k|^2 \) and \( |k| \geq 1 \), after considering the imaginary part and the real part of (3.7), we obtain

\[
\int_{\mathbb{R}_+^3} |\nabla \times E|^2 + |k|^2 |E|^2 \leq C \int_{\mathbb{R}_+^3} |J_m|^2,
\]

which implies (3.4) since \( \nabla \times E = k\mu H \) in \( \mathbb{R}_+^3 \). The uniqueness of \((E, H)\) follows.

To derive (3.5), we note that

\[
\|\nabla \times E\|_{L^2(\mathbb{R}_+^3)} \leq C \|J_m\|_{L^2(\mathbb{R}_+^3)},
\]

\[
\|\text{div} \ E\|_{L^2(\mathbb{R}_+^3)} \leq C \frac{1}{|k|} \|\text{div} \ J_m\|_{L^2(\mathbb{R}_+^3)},
\]

\[
\|E\|_{L^2(\mathbb{R}_+^3)} \leq C \|J_m\|_{L^2(\mathbb{R}_+^3)},
\]

\[
E \times e_3 = 0 \text{ on } \mathbb{R}^3_+.
\]

\(^1e_3 = (0, 0, 1) \in \mathbb{R}^3.\)
It follows from the Gaffney inequality, see e.g. [15 Theorems 3.7], [14 Theorem 1], that \( E \in H^1(\mathbb{R}^3_+) \) and

\[
\| E \|_{H^1(\mathbb{R}^3_+)} \leq C \| J_m \|_{L^2(\mathbb{R}^3_+)} + \frac{C}{|k|} \| \text{div} J_m \|_{L^2(\mathbb{R}^3_+)}.
\]

We also have

\[
\| \nabla \times H \|_{L^2(\mathbb{R}^3_+)} \leq C \| J_m \|_{L^2(\mathbb{R}^3_+)},
\]

\[
\| \text{div} H \|_{L^2(\mathbb{R}^3_+)} = 0,
\]

\[
\| H \|_{L^2(\mathbb{R}^3_+)} \leq C \| J_m \|_{L^2(\mathbb{R}^3_+)},
\]

and, since \( E \times e_3 = 0 \) on \( \mathbb{R}^3_+ \),

\[
H \cdot e_3 = 0 \text{ on } \mathbb{R}^3_+.
\]

It follows from Gaffney inequality again, see e.g. [15, Theorems 3.9], [14, Theorem 1], that

\[
\| \nabla \times \hat{E} \|_{L^2(\mathbb{R}^3_+)} \leq C \| J_{\hat{m}} \|_{L^2(\mathbb{R}^3_+)} + \frac{C}{|k|} \| \text{div} J_{\hat{m}} \|_{L^2(\mathbb{R}^3_+)},
\]

Combining (3.8) and (3.9) yields (3.5).

To prove the existence of \((E, H)\), we first apply the Lax-Milgram theory for variational formula given in (3.7) where the bilinear form is defined by the LHS and the linear functional is defined by the RHS in the Hilbert space \( H_0(\text{curl}, \mathbb{R}^3_+) \). We then derive that there exists a solution \( E \in H_0(\text{curl}, \mathbb{R}^3_+) \) of (3.6). Set

\[
H = \frac{1}{k\mu} \nabla \times E \text{ in } \mathbb{R}^3_+.
\]

Then

\[
\nabla \times E = k\mu H \text{ in } \mathbb{R}^3_+,
\]

and

\[
\nabla \times H = \frac{1}{k\mu} \nabla \times (\nabla \times E) = -k\varepsilon E + J_m \text{ in } \mathbb{R}^3_+.
\]

In other words, \((E, H) \in [L^2(\mathbb{R}^3_+)]^6\) is a solution of (3.6). The proof is complete. \( \square \)

We now state the main result of this section, which plays a key role in the proof of Theorem 1.1.

**Proposition 3.1.** Let \( \alpha \in \mathbb{C} \) with \( \alpha^2 \in \mathbb{R} \) and \( |\alpha| = 1, \gamma > 0, k \in \mathbb{C} \) with \( |\text{Im}(k^2)| \geq \gamma |k|^2 \), \( |\text{Im}(\alpha^2 k^2)| \geq \gamma |k|^2 \), and \( |k| \geq 1 \), and let \( \Lambda \geq 1 \) and \( \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \) be four positive constants such that

\[
\Lambda^{-1} \leq \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \leq \Lambda.
\]

Assume that, for some \( \Lambda_1 > 0 \)

\[
|\varepsilon - \hat{\varepsilon}| \geq \Lambda_1, \quad |\mu - \hat{\mu}| \geq \Lambda_1, \quad \text{and} \quad |\varepsilon/\mu - \hat{\varepsilon}/\hat{\mu}| \geq \Lambda_1.
\]

Let \( J_e, J_m, \hat{J}_e, \hat{J}_m \in H(\text{div}, \mathbb{R}^3_+) \) with \( (J_{e,3} - \hat{J}_{e,3}, J_{m,3} - \hat{J}_{m,3}) \in [H^{1/2}(\mathbb{R}^3_0)]^2 \), and let \( f_e, f_m \in H^{1/2}(\text{div}, \mathbb{R}^3_0) \). There exists a unique solution \((E, H, \hat{E}, \hat{H}) \in [L^2(\mathbb{R}^3)]^4\) of the system

\[
\begin{align*}
\nabla \times E &= k\mu H + J_e \text{ in } \mathbb{R}^3_+, \\
\nabla \times H &= -k\varepsilon E + J_m \text{ in } \mathbb{R}^3_+, \\
\nabla \times \hat{E} &= \alpha k\hat{\mu} \hat{H} + \hat{J}_e \text{ in } \mathbb{R}^3_+, \\
\nabla \times \hat{H} &= -\alpha k\hat{\varepsilon} \hat{E} + \hat{J}_m \text{ in } \mathbb{R}^3_+,
\end{align*}
\]
Moreover, we have

\[
\begin{align*}
(3.12) \quad C \left( \| (E, H, \hat{E}, \hat{H}) \|_{H^1(\mathbb{R}^3_+)} + |k| \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\mathbb{R}^3_+)} \right)
& \leq \| (J_e, J_m, \hat{J}_e, \hat{J}_m) \|_{L^2(\mathbb{R}^3_+)} + \frac{1}{|k|} \| (\text{div} \ J_e, \text{div} \ J_m, \text{div} \ \hat{J}_e, \text{div} \ \hat{J}_m) \|_{L^2(\mathbb{R}^3_+)} \\
& \quad + \frac{1}{|k|} \| (J_{e,3} - \hat{J}_{e,3}, J_{m,3} - \hat{J}_{m,3}) \|_{H^{1/2}(\mathbb{R}^3_+)} + |k|^{1/2} \| (f_e, f_m) \|_{L^2(\mathbb{R}^3_+)} \\
& \quad + \| (f_e, f_m) \|_{H^{1/2}(\mathbb{R}^3_+)} + \frac{1}{|k|} \| (\text{div}_\Gamma \ f_e, \text{div}_\Gamma \ f_m) \|_{H^{1/2}(\mathbb{R}^3_+)}.
\end{align*}
\]

for some positive constant C depending only on \( \gamma, \Lambda, \) and \( \Lambda_1. \)

Recall that, by our convention, \( J_{e,3}, J_{m,3}, \hat{J}_{e,3}, \hat{J}_{m,3} \) denote the third component of \( J_e, J_m, \hat{J}_e, \hat{J}_m. \)

It is worth noting that the constant \( C \) is independent of \( k. \)

**Proof.** Let \((E^1, H^1), (E^2, H^2), (\hat{E}^1, \hat{H}^1), (\hat{E}^2, \hat{H}^2) \in [L^2(\mathbb{R}^3_+)]^6\) be respectively the unique solutions of the following systems

\[
\begin{align*}
\nabla \times E^1 &= k\mu H^1 \quad \text{in} \ \mathbb{R}^3_+, \\
\nabla \times H^1 &= -k\varepsilon E^1 + J_m \quad \text{in} \ \mathbb{R}^3_+, \\
E^1 \times e_3 &= 0 \quad \text{on} \ \mathbb{R}^3_0, \\
\nabla \times E^2 &= k\mu H^2 + J_e \quad \text{in} \ \mathbb{R}^3_+, \\
\nabla \times H^2 &= -k\varepsilon E^2 \quad \text{in} \ \mathbb{R}^3_+, \\
H^2 \times e_3 &= 0 \quad \text{on} \ \mathbb{R}^3_0,
\end{align*}
\]

Applying Lemma 3.1, we obtain

\[
\begin{align*}
(3.13) \quad & \quad \| (E^1, H^1, E^2, H^2, \hat{E}^1, \hat{H}^1, \hat{E}^2, \hat{H}^2) \|_{H^1(\mathbb{R}^3_+)} \\
& \quad + |k| \| (E^1, H^1, E^2, H^2, \hat{E}^1, \hat{H}^1, \hat{E}^2, \hat{H}^2) \|_{L^2(\mathbb{R}^3_+)} \\
\quad \leq C \left( \| (J_e, J_m, \hat{J}_e, \hat{J}_m) \|_{L^2(\mathbb{R}^3_+)} + \frac{1}{|k|} \| (\text{div} \ J_e, \text{div} \ J_m, \text{div} \ \hat{J}_e, \text{div} \ \hat{J}_m) \|_{L^2(\mathbb{R}^3_+)} \right).
\end{align*}
\]

From the trace theory, we derive from (3.13) that

\[
\begin{align*}
(3.14) \quad & \quad \| (E^1, H^1, E^2, H^2, \hat{E}^1, \hat{H}^1, \hat{E}^2, \hat{H}^2) \|_{H^{1/2}(\mathbb{R}^3_+)} \\
\quad \leq C \| (E^1, H^1, E^2, H^2, \hat{E}^1, \hat{H}^1, \hat{E}^2, \hat{H}^2) \|_{H^1(\mathbb{R}^3_+)} \\
\quad \leq C \left( \| (J_e, J_m, \hat{J}_e, \hat{J}_m) \|_{L^2(\mathbb{R}^3_+)} + \frac{1}{|k|} \| (\text{div} \ J_e, \text{div} \ J_m, \text{div} \ \hat{J}_e, \text{div} \ \hat{J}_m) \|_{L^2(\mathbb{R}^3_+)} \right).
\end{align*}
\]
We have
\[
\| \text{div}_\Gamma (H^1 \times e_3) - \text{div}_\Gamma (\hat{H}^1 \times e_3) \|_{H^{1/2}(\mathbb{R}^3_+)} \\
= \| \nabla \times H^1 \cdot e_3 - (\nabla \times \hat{H}^1) \cdot e_3 \|_{H^{1/2}(\mathbb{R}^3_+)} \\
= \| (-k\varepsilon E^1 + J_m) \cdot e_3 - (-\alpha k\varepsilon \hat{E}^1 + \hat{J}_m) \cdot e_3 \|_{H^{1/2}(\mathbb{R}^3_+)} \\
\leq C|k| \| (E^1, \hat{E}^1) \|_{H^{1/2}(\mathbb{R}^3_+)} + C\| J_{m,3} - \hat{J}_{m,3} \|_{H^{1/2}(\mathbb{R}^3_+)}.
\]

It follows from (3.14) that
\[
(3.15) \quad \| \text{div}_\Gamma (H^1 \times e_3) - \text{div}_\Gamma (\hat{H}^1 \times e_3) \|_{H^{1/2}(\mathbb{R}^3_+)} \leq C|k| \left( \| (J_e, J_m, \hat{J}_e, \hat{J}_m) \|_{L^2(\mathbb{R}^3_+)} \right) \\
+ \frac{1}{|k|} \| (\text{div } J_e, \text{div } J_m, \text{div } \hat{J}_e, \text{div } \hat{J}_m) \|_{L^2(\mathbb{R}^3_+)} + \frac{1}{|k|} \| J_{m,3} - \hat{J}_{m,3} \|_{H^{1/2}(\mathbb{R}^3_+)}).
\]

Similarly, we obtain
\[
(3.16) \quad \| \text{div}_\Gamma (E^1 \times e_3) - \text{div}_\Gamma (\hat{E}^1 \times e_3) \|_{H^{1/2}(\mathbb{R}^3_+)} \leq C|k| \left( \| (J_e, J_m, \hat{J}_e, \hat{J}_m) \|_{L^2(\mathbb{R}^3_+)} \right) \\
+ \frac{1}{|k|} \| (\text{div } J_e, \text{div } J_m, \text{div } \hat{J}_e, \text{div } \hat{J}_m) \|_{L^2(\mathbb{R}^3_+)} + \frac{1}{|k|} \| J_{e,3} - \hat{J}_{e,3} \|_{H^{1/2}(\mathbb{R}^3_+)}).
\]

Using the fact, for \( u \in H^1(\mathbb{R}^3_+) \),
\[
(3.17) \quad \int_{\mathbb{R}^3_+} \| u \| \leq 2 \int_{\mathbb{R}^3_+} \| u \|_{L^2(\mathbb{R}^3_+)} \| \nabla u \|_{L^2(\mathbb{R}^3_+)},
\]
we have
\[
\| (E^1, H^1, E^2, H^2, \hat{E}^1, \hat{H}^1, \hat{E}^2, \hat{H}^2) \|_{L^2(\mathbb{R}^3_+)} \\
\leq C\| (E^1, H^1, E^2, H^2, \hat{E}^1, \hat{H}^1, \hat{E}^2, \hat{H}^2) \|_{L^2(\mathbb{R}^3_+)} \| (E^1, H^1, E^2, H^2, \hat{E}^1, \hat{H}^1, \hat{E}^2, \hat{H}^2) \|_{H^{1/2}(\mathbb{R}^3_+)}^{1/2}.
\]
This yields
\[
(3.18) \quad k^{1/2}\| (E^1, H^1, E^2, H^2, \hat{E}^1, \hat{H}^1, \hat{E}^2, \hat{H}^2) \|_{L^2(\mathbb{R}^3_+)} \\
\leq C|k| \| (E^1, H^1, \hat{E}^1, \hat{H}^1) \|_{L^2(\mathbb{R}^3_+)} + C\| (E^1, H^1, \hat{E}^1, \hat{H}^1) \|_{H^{1/2}(\mathbb{R}^3_+)}.\]

By considering \( (E - E^1 - E^2, H - H^1 - H^2, \hat{E} - \hat{E}^1 - \hat{E}^2, \hat{H} - \hat{H}^1 - \hat{H}^2) \), from (3.13), (3.14), (3.15), (3.16), and (3.15), w.l.o.g. one might assume that
\[
J_e = J_m = \hat{J}_e = \hat{J}_m = 0 \text{ in } \mathbb{R}^3_+.
\]
This will be assumed later on. Thus
\[
(\hat{E} - E) \times e_3 = f_e \text{ on } \mathbb{R}^3_+, \quad \text{ and } \quad (\hat{H} - H) \times e_3 = f_m \text{ on } \mathbb{R}^3_+.
\]
Using the identity for a vector field $A$
\[ \nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \Delta A, \]
we obtain the following equations for $E$ and $\hat{E}$
\[
\begin{align*}
\Delta E - k^2 \varepsilon \mu E &= 0 \text{ in } \mathbb{R}_+^3, \\
\Delta \hat{E} - \alpha^2 k^2 \hat{\varepsilon} \hat{\mu} \hat{E} &= 0 \text{ in } \mathbb{R}_+^3
\end{align*}
\]
(3.21)
(recall that here the coefficients are all constants). In the following, we denote the Fourier transform with respect to $(x_1, x_2) \in \mathbb{R}^2$ of an appropriate function $u : \mathbb{R}_+^3 \to \mathbb{C}$ by $u^F$, i.e.,
\[ u^F(\xi, x_3) = \frac{1}{2\pi} \int_{\mathbb{R}^2} u(x_1, x_2, x_3)e^{-i(x_1\xi_1 + x_2\xi_2)}
\]
dx_1 \, dx_2 \text{ for } (\xi, x_3) = (\xi_1, \xi_2, x_3) \in \mathbb{R}_+^3.

Similar notation is used for an appropriate function defined on $\mathbb{R}_0^3$.

Consider the first two equations of the system for $E$ and the first two equations of the system for $\hat{E}$ in (3.21). Solving these equations using the Fourier transform with respect to $(x_1, x_2)$ yields
\[ E_j^F(\xi, x_3) = a_j(\xi)e^{-x_3\sqrt{\xi^2 + k^2 \varepsilon \mu}} \text{ in } \mathbb{R}_+^3, \]
(3.22)
\[ \hat{E}_j^F(\xi, x_3) = \hat{a}_j(\xi)e^{-x_3\sqrt{\xi^2 + \alpha^2 k^2 \hat{\varepsilon} \hat{\mu}}} \text{ in } \mathbb{R}_+^3, \]
(3.23)
for $j = 1, 2$, where
\[ a_j(\xi) = E_j^F(\xi, 0) \text{ and } \hat{a}_j(\xi) = \hat{E}_j^F(\xi, 0) \text{ for } \xi \in \mathbb{R}^2. \]

We then have, with $a = (a_1, a_2)$ and $\hat{a} = (\hat{a}_1, \hat{a}_2)$,
\[ \hat{a}(\xi) - a(\xi) = h(\xi) \text{ where } h(\xi) = -f^F_e(\xi, 0) \times e_3. \]
(3.24)

Here and in what follows, we identify a vector $(y_1, y_2, 0) \in \mathbb{R}_0^3$ with $(y_1, y_2) \in \mathbb{R}^2$.

Since $\text{div } E = 0$ in $\mathbb{R}_+^3$, it follows that
\[ \partial_{x_3} E_3 = -(\partial_{x_1} E_1 + \partial_{x_2} E_2) \text{ in } \mathbb{R}_+^3. \]

This implies
\[ \partial_{x_3} E_3^F(\xi, x_3) = -i\xi_1 E_1^F(\xi, x_3) - i\xi_2 E_2^F(\xi, x_3) \text{ in } \mathbb{R}_+^3. \]

Using (3.22), we obtain
\[ \partial_{x_3} E_3^F(\xi, x_3) = -i\xi \cdot a(\xi)e^{-x_3\sqrt{\xi^2 + k^2 \varepsilon \mu}} \text{ in } \mathbb{R}_+^3. \]

We thus get
\[ E_3^F(\xi, x_3) = -\int_{x_3}^{\infty} i\xi \cdot a(\xi)e^{-s\sqrt{\xi^2 + k^2 \varepsilon \mu}} \, ds = \frac{i\xi \cdot a(\xi)e^{-x_3\sqrt{\xi^2 + k^2 \varepsilon \mu}}}{\sqrt{\xi^2 + k^2 \varepsilon \mu}} \text{ in } \mathbb{R}_+^3. \]
(3.25)

Similarly, we have
\[ \hat{E}_3^F(\xi, x_3) = \frac{i\xi \cdot \hat{a}(\xi)e^{-x_3\sqrt{\xi^2 + \alpha^2 k^2 \hat{\varepsilon} \hat{\mu}}}}{\sqrt{\xi^2 + \alpha^2 k^2 \hat{\varepsilon} \hat{\mu}}} \text{ in } \mathbb{R}_+^3. \]
(3.26)
Since $\hat{H} \times e_3 - H \times e_3 = f_m$ on $\mathbb{R}_0^3$, and $\nabla \times H = -k \varepsilon E$ and $\nabla \times \hat{H} = -\alpha k \hat{\varepsilon} \hat{E}$ in $\mathbb{R}_0^3$, it follows that

$$\alpha \hat{\varepsilon} \hat{E}_3 - \varepsilon E_3 = -\frac{1}{k} \text{div}_{\mathbb{R}_0^3} f_m \quad \text{on } \mathbb{R}_0^3.$$ 

Using (3.25) and (3.26), we derive that

$$\alpha \hat{\varepsilon} \hat{E}_3 - \varepsilon E_3 = -\frac{1}{k} \text{div}_{\mathbb{R}_0^3} f_m \quad \text{on } \mathbb{R}_0^3.$$ 

Combining (3.24) and (3.27), and noting $\varepsilon \hat{E}_3$, follows that

$$\alpha \hat{\varepsilon} \hat{E}_3 - \varepsilon E_3 = -\frac{1}{k} \text{div}_{\mathbb{R}_0^3} f_m \quad \text{on } \mathbb{R}_0^3.$$ 

We deduce that

$$|\xi \cdot \hat{a}(\xi)| \leq C \left( |\xi \cdot \hat{h}(\xi)| + \sqrt{|\xi|^2 + |k|^2} |g(\xi)| \right),$$

which yields, since $a = \hat{a} - h$,

$$|\xi \cdot a(\xi)| + |\xi \cdot \hat{a}(\xi)| \leq C \left( |\xi \cdot h(\xi)| + \sqrt{|\xi|^2 + |k|^2} |g(\xi)| \right).$$

We have, in $\mathbb{R}_+^3$,

$$k\mu H_1 = \partial_{x_3} E_3 - \partial_{x_2} E_2, \quad \alpha k \hat{\mu} \hat{H}_1 = \partial_{x_3} \hat{E}_3 - \partial_{x_2} \hat{E}_2.$$ 

Since $\hat{H}_1 - H_1 = f_{m,2} := f_m \cdot e_2$ with $e_2 = (0, 1, 0)$ on $\mathbb{R}_0^3$, it follows from (3.22), (3.23), (3.25), (3.26) that

$$\frac{1}{\alpha \hat{\mu}} \left( -\frac{\xi_2 \xi \cdot \hat{a}(\xi)}{\sqrt{|\xi|^2 + \alpha^2 k^2 \hat{\varepsilon} \hat{\mu}}} + \sqrt{|\xi|^2 + \alpha^2 k^2 \hat{\varepsilon} \hat{\mu} a_2(\xi)} \right)$$

$$= \frac{1}{\mu} \left( -\frac{\xi_2 \xi \cdot a(\xi)}{\sqrt{|\xi|^2 + k^2 \varepsilon \mu a_2(\xi)}} + \sqrt{|\xi|^2 + k^2 \varepsilon \mu a_2(\xi)} \right) + k f_{m,2}(\xi).$$
We derive from (3.24) that
\begin{equation}
\frac{1}{\alpha \hat{\mu}} \sqrt{\xi^2 + \alpha^2 \hat{k}^2 \varepsilon \hat{\mu} \hat{a}_2 (\xi)} - \frac{1}{\mu} \sqrt{\xi^2 + k^2 \varepsilon \mu \hat{a}_2 (\xi)}
\end{equation}
\begin{equation}
= \left( \frac{\xi \cdot \hat{a}(\xi)}{\alpha \hat{\mu} \sqrt{\xi^2 + \alpha^2 \hat{k}^2 \varepsilon \hat{\mu}}} - \frac{\xi \cdot a(\xi)}{\mu \sqrt{\xi^2 + k^2 \varepsilon \mu}} \right) \xi_2 - \frac{1}{\mu} \sqrt{\xi^2 + k^2 \varepsilon \mu h_2 (\xi) + k f^F_{m_2} (\xi)}.
\end{equation}
We thus obtain
\begin{equation}
\hat{a}_2 (\xi) = \frac{\mu \sqrt{\xi^2 + \alpha^2 \hat{k}^2 \varepsilon \hat{\mu}} + \alpha \hat{\mu} \sqrt{\xi^2 + k^2 \varepsilon \mu}}{(\mu^2 - \alpha^2 \hat{\mu}^2) \xi^2 + \alpha^2 \hat{k}^2 \varepsilon \hat{\mu} (\mu / \varepsilon - \hat{\mu} / \hat{\varepsilon})}
\end{equation}
\begin{equation}
\times \left\{ \left( \frac{\mu \xi}{\alpha \sqrt{\xi^2 + \alpha^2 \hat{k}^2 \varepsilon \hat{\mu}}} - \frac{\hat{\mu} \xi}{\sqrt{\xi^2 + k^2 \varepsilon \mu}} \right) \xi_2 - \hat{\mu} \sqrt{\xi^2 + k^2 \varepsilon \mu h_2 (\xi) + k f^F_{m_2} (\xi)} \right\}.
\end{equation}
Since $\mu \neq \hat{\mu}$, $\varepsilon / \mu \neq \hat{\varepsilon} / \hat{\mu}$, $\alpha^2 = \pm 1$, and $|\Im (k^2)| \geq \gamma |k|^2$, $|k| \geq 1$, we get
\begin{equation}
|\left( \mu^2 - \alpha^2 \hat{\mu}^2 \right) \xi^2 + \alpha^2 \hat{k}^2 \varepsilon \hat{\mu} (\mu / \varepsilon - \hat{\mu} / \hat{\varepsilon}) | \geq C (|\xi^2| + |k|^2).
\end{equation}
Using (3.32), we derive from (3.31) that
\begin{equation}
|\hat{a}_2 (\xi)| \leq \frac{C |\xi|}{|\xi^2 + |k|^2} \left( |\xi \cdot a(\xi)| + |k \cdot \hat{a}(\xi)| \right) + C \left( |h_2 (\xi)| + |f^F_{m_2} (\xi)| \right),
\end{equation}
which yields, since $\hat{a} = a = h$,
\begin{equation}
|a_2 (\xi)| + |\hat{a}_2 (\xi)| \leq \frac{C |\xi|}{|\xi^2 + |k|^2} \left( |\xi \cdot a(\xi)| + |k \cdot \hat{a}(\xi)| \right) + C \left( |h(\xi)| + |f^F_m (\xi)| \right).
\end{equation}
Combining (3.29) and (3.33) yields
\begin{equation}
(|a_2 (\xi)|^2 + |\hat{a}_2 (\xi)|^2) \sqrt{\xi^2 + |k|^2} \leq C (|h|^2 + |f^F_m|^2) \sqrt{\xi^2 + |k|^2} + C |g(\xi)|^2 |\xi|.
\end{equation}
From the definition of $g$ and $h$, we obtain
\begin{equation}
\int_{\mathbb{R}^2} (|a_2 (\xi)|^2 + |\hat{a}_2 (\xi)|^2) \sqrt{\xi^2 + |k|^2}
\end{equation}
\begin{equation}
\leq C \left( \|(f_e, f_m)\|_{H^{1/2}(\mathbb{R}^3)} + |k| \|(f_e, f_m)\|_{L^2(\mathbb{R}^3)} + \frac{1}{|k|^2} \|(\text{div}_G f_e, \text{div}_G f_m)\|_{H^{1/2}(\mathbb{R}^3)} \right).
\end{equation}
Similarly, we reach
\begin{equation}
\int_{\mathbb{R}^2} (|a_1 (\xi)|^2 + |\hat{a}_1 (\xi)|^2) \sqrt{\xi^2 + |k|^2}
\end{equation}
\begin{equation}
\leq C \left( \|(f_e, f_m)\|_{H^{1/2}(\mathbb{R}^3)} + |k| \|(f_e, f_m)\|_{L^2(\mathbb{R}^3)} + \frac{1}{|k|^2} \|(\text{div}_G f_e, \text{div}_G f_m)\|_{H^{1/2}(\mathbb{R}^3)} \right).
\end{equation}
On the other hand, from (3.22), (3.23), (3.26), and (3.20), we have
\begin{equation}
\int_{\mathbb{R}^2} |(\nabla E, \nabla \hat{E})|^2 + |k|^2 |(E, \hat{E})|^2 \leq C \int_{\mathbb{R}^2} |(a(\xi), \hat{a}(\xi)|^2 \sqrt{\xi^2 + |k|^2}.
\end{equation}
Combining (3.34), (3.35), and (3.36) yields

\[ (3.37) \quad \int_{\mathbb{R}^+} |(\nabla E, \nabla \hat{E})|^2 + |k|^2 |(E, \hat{E})|^2 \leq C \left( \| (f_e, f_m) \|^2_{H^{1/2}(\mathbb{R}^3)} + |k| \| (f_e, f_m) \|^2_{L^2(\mathbb{R}^3)} + \frac{1}{|k|^2} \| (\text{div}_r f_e, \text{div}_r f_m) \|^2_{H^{1/2}(\mathbb{R}^3)} \right). \]

Similarly, we obtain

\[ (3.38) \quad \int_{\mathbb{R}^+} |(\nabla H, \nabla \hat{H})|^2 + |k|^2 |(H, \hat{H})|^2 \leq C \left( \| (f_e, f_m) \|^2_{H^{1/2}(\mathbb{R}^3)} + |k| \| (f_e, f_m) \|^2_{L^2(\mathbb{R}^3)} + \frac{1}{|k|^2} \| (\text{div}_r f_e, \text{div}_r f_m) \|^2_{H^{1/2}(\mathbb{R}^3)} \right). \]

The conclusion now follows from (3.37) and (3.38). The proof is complete.

As a consequence of Proposition 3.1, we obtain

**Corollary 3.1.** Let \( \alpha \in \mathbb{C} \) with \( \alpha^2 \in \mathbb{R} \) and \( |\alpha| = 1 \), \( \gamma > 0 \), \( k \in \mathbb{C} \) with \( |\Im(k^2)| \geq \gamma |k|^2 \), \( |\Im(\alpha^2k^2)| \geq \gamma |k|^2 \), and \( |k| \geq 1 \), and let \( \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \in [L^\infty(\mathbb{R}^3_+)]^{3 \times 3} \) be symmetric, uniformly elliptic, and of class \( C^1 \). Let \( \Lambda \geq 1 \) be such that

\[ \Lambda^{-1} \leq \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \leq \Lambda \text{ in } B_1 \cap \mathbb{R}^3_+ \quad \text{and} \quad \| (\varepsilon, \mu, \hat{\varepsilon}, \hat{\mu}) \|_{C^1(\mathbb{R}^3_+ \cap B_1)} \leq \Lambda. \]

Assume that \( \varepsilon(0), \hat{\varepsilon}(0), \mu(0), \hat{\mu}(0) \) are isotropic, and for some \( \Lambda_1 \geq 0 \)

\[ |\varepsilon(0) - \hat{\varepsilon}(0)| \geq \Lambda_1, \quad |\mu(0) - \hat{\mu}(0)| \geq \Lambda_1, \quad \text{and} \quad |\varepsilon(0)/\mu(0) - \hat{\varepsilon}(0)/\hat{\mu}(0)| \geq \Lambda_1. \]

Let \( J_e, J_m, \hat{J}_e, \hat{J}_m \in H(\text{div}, \mathbb{R}^3_+) \) and assume that \((E, H, \hat{E}, \hat{H}) \in [L^2(\mathbb{R}^3)]^{12}\) is a solution of (3.10) and (3.11) with \( f_e = f_m = 0 \). There exist \( 0 < r_0 < 1 \) and \( k_0 > 1 \) depending only on \( \gamma, \Lambda, \) and \( \Lambda_1 \) such that if the supports of \( E, H, \hat{E}, \hat{H} \) are in \( B_{r_0} \cap \mathbb{R}^3_+ \) and \( |k| \geq k_0 \), then

\[ (3.39) \quad \| (E, H, \hat{E}, \hat{H}) \|_{H^1(\mathbb{R}^3_+)} + |k| \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\mathbb{R}^3_+)} \leq C \left( \| (J_e, J_m, \hat{J}_e, \hat{J}_m) \|_{L^2(\mathbb{R}^3_+)} \right) \]

\[ + \frac{1}{|k|} \| (\text{div}_r J_e, \text{div}_r J_m, \text{div}_r \hat{J}_e, \text{div}_r \hat{J}_m) \|_{L^2(\mathbb{R}^3_+)} + \left( \| (J_{e,3} - \hat{J}_{e,3}, J_{m,3} - \hat{J}_{m,3}) \|_{H^{1/2}(\mathbb{R}^3)} \right), \]

for some positive constant \( C \) depending only on \( \gamma, \Lambda, \) and \( \Lambda_1 \).

**Remark 3.2.** The constant \( C \) in (3.39) is independent of \( k \). Concerning the isotropic properties of \( \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \), we emphasize that \( \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \) are not required to be isotropic in \( B_1 \cap \mathbb{R}^3_+ \), we only assume that \( \varepsilon(0), \mu(0), \hat{\varepsilon}(0), \hat{\mu}(0) \) are.

Here and what follows \( B_r \), for \( r > 0 \) denotes the ball of radius \( r \) centered at the origin.

**Proof.** We rewrite (3.10) under the form

\[
\begin{aligned}
\nabla \times E &= k\mu(0)H + J_e^1 \text{ in } \mathbb{R}^3_+, \\
\nabla \times H &= -k\varepsilon(0)E + J_m^1 \text{ in } \mathbb{R}^3_+, \\
\nabla \times \hat{E} &= \alpha k\hat{\mu}(0)\hat{H} + \hat{J}_e^1 \text{ in } \mathbb{R}^3_+, \\
\nabla \times \hat{H} &= -\alpha k\hat{\varepsilon}(0)\hat{E} + \hat{J}_m^1 \text{ in } \mathbb{R}^3_+, 
\end{aligned}
\]

where, in \( \mathbb{R}^3_+ \),

\[
J_e^1(x) = J_e(x) + k(\mu(x) - \mu(0))H(x), \quad J_m^1(x) = J_m(x) - k(\varepsilon(x) - \varepsilon(0))E(x),
\]

\[
\hat{J}_e^1(x) = \hat{J}_e(x) + \alpha k\hat{\mu}(0)\hat{H}(x), \quad \hat{J}_m^1(x) = \hat{J}_m(x) - \alpha k\hat{\varepsilon}(0)\hat{E}(x).
\]
\[ \hat{J}^1_e(x) = \hat{J}_e(x) + \alpha k (\hat{\mu}(x) - \hat{\mu}(0)) \hat{H}(x), \quad \hat{J}^1_m(x) = \hat{J}_m(x) - \alpha k (\hat{\varepsilon}(x) - \hat{\varepsilon}(0)) \hat{\varepsilon}(x). \]

From Proposition 3.1, we obtain

\[ C \left( \| (E, H, \hat{E}, \hat{H}) \|_{H^1(\mathbb{R}_+^3)} + |k| \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\mathbb{R}_+^3)} \right) \]

\[ \leq \| (J^1_e, J^1_m, \hat{J}^1_e, \hat{J}^1_m) \|_{L^2(\mathbb{R}_+^3)} + \frac{1}{|k|} \| (\text{div } J^1_e, \text{div } J^1_m, \text{div } \hat{J}^1_e, \text{div } \hat{J}^1_m) \|_{L^2(\mathbb{R}_+^3)} \]

\[ + \frac{1}{|k|} \| (J^1_{e,3}, \hat{J}^1_{e,3}, J^1_{m,3}, \hat{J}^1_{m,3}) \|_{H^{1/2}(\mathbb{R}_0^3)}. \]

On the other hand, from the definition of \((J^1_e, J^1_m, \hat{J}^1_e, \hat{J}^1_m)\), one has

\[ C \| (J^1_e, J^1_m, \hat{J}^1_e, \hat{J}^1_m) \|_{L^2(\mathbb{R}_+^3)} \leq C \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\mathbb{R}_+^3)} + C r_0 \| k \|_{L^2(\mathbb{R}_+^3)} \]

\[ \frac{1}{|k|} \| (\text{div } J^1_e, \text{div } J^1_m, \text{div } \hat{J}^1_e, \text{div } \hat{J}^1_m) \|_{L^2(\mathbb{R}_+^3)} \leq \frac{C}{|k|} \| (\text{div } J_e, \text{div } J_m, \text{div } \hat{J}_e, \text{div } \hat{J}_m) \|_{L^2(\mathbb{R}_+^3)} \]

\[ + C \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\mathbb{R}_+^3)} + C r_0 \left\| \left( \nabla E, \nabla H, \nabla \hat{E}, \nabla \hat{H} \right) \right\|_{L^2(\mathbb{R}_+^3)}, \]

and

\[ \frac{1}{|k|} \| (J^1_{e,3}, \hat{J}^1_{e,3}, J^1_{m,3}, \hat{J}^1_{m,3}) \|_{H^{1/2}(\mathbb{R}_0^3)} \leq \frac{C}{|k|} \| (J^1_{e,3}, \hat{J}^1_{e,3}, J^1_{m,3}, \hat{J}^1_{m,3}) \|_{H^{1/2}(\mathbb{R}_0^3)} \]

\[ + C r_0 \| (E, H, \hat{E}, \hat{H}) \|_{H^1(\mathbb{R}_+^3)} + C \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\mathbb{R}_+^3)}. \]

Here in the last inequality, we involved the trace theory and used

\[ \left\| \left( (\mu(x) - \mu(0)) H, (\varepsilon(x) - \varepsilon(0)) E, (\hat{\mu}(x) - \hat{\mu}(0)) \hat{H}, (\hat{\varepsilon}(x) - \hat{\varepsilon}(0)) \hat{\varepsilon} \right) \right\|_{H^1(\mathbb{R}_+^3)} \]

\[ \leq C r_0 \| (E, H, \hat{E}, \hat{H}) \|_{H^1(\mathbb{R}_+^3)} + C \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\mathbb{R}_+^3)}. \]

Combining (3.40) – (3.43) yields

\[ C \left( \| (E, H, \hat{E}, \hat{H}) \|_{H^1(\mathbb{R}_+^3)} + |k| \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\mathbb{R}_+^3)} \right) \]

\[ \leq \| (J_e, J_m, \hat{J}_e, \hat{J}_m) \|_{L^2(\mathbb{R}_+^3)} + \frac{1}{|k|} \| (\text{div } J_e, \text{div } J_m, \text{div } \hat{J}_e, \text{div } \hat{J}_m) \|_{L^2(\mathbb{R}_+^3)} \]

\[ + \frac{1}{|k|} \| (J^1_{e,3}, \hat{J}^1_{e,3}, J^1_{m,3}, \hat{J}^1_{m,3}) \|_{H^{1/2}(\mathbb{R}_0^3)} \]

\[ (|k|r_0 + 1) \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\mathbb{R}_+^3)} + r_0 \| (E, H, \hat{E}, \hat{H}) \|_{H^1(\mathbb{R}_+^3)}. \]

Fix \( r_0 = \min\{C/4, 1/4\} \) where \( C \) is the constant in (3.44). Take \( k_0 \) such that \( k_0 r_0 \geq 1 \). One then can absorb the last two terms of the RHS of (3.44) by the LHS. The conclusion then follows.
4. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. We begin with a result which yields the uniqueness and the stability of (1.7) and (1.8).

**Proposition 4.1.** Let $\gamma > 0$, $\alpha \in \mathbb{C}$ with $\alpha^2 \in \mathbb{R}$ and $|\alpha| = 1$, and let $\varepsilon, \mu, \tilde{\varepsilon}, \tilde{\mu} \in [L^\infty(\Omega)]^{3 \times 3}$ be symmetric. Assume that there exist $\Lambda \geq 1$, $\Lambda_1 > 0$, and $s_0 > 0$ such that

$$\Lambda^{-1} \leq \varepsilon, \mu, \tilde{\varepsilon}, \tilde{\mu} \leq \Lambda \ a.e. \ in \ \Omega, \quad \| (\varepsilon, \mu, \tilde{\varepsilon}, \tilde{\mu}) \|_{C^1(\bar{\Omega}_{s_0})} \leq \Lambda,$$

and, for $x \in \partial \Omega$,

$$|\varepsilon(x) - \tilde{\varepsilon}(x)| \geq \Lambda_1, \quad |\mu(x) - \tilde{\mu}(x)| \geq \Lambda_1, \quad \text{and} \quad |\varepsilon(x)/\mu(x) - \tilde{\varepsilon}(x)/\tilde{\mu}(x)| \geq \Lambda_1.$$

There exist two positive constants $k_0 \geq 1$ and $C > 0$ depending only on $\Lambda$, $\Lambda_1$, $s_0$, $\gamma$, and $\Omega$ such that for $k \in \mathbb{C}$ with $|\Im(k^2)| \geq \gamma|k|^2$ and $|k| \geq k_0$, for every $(J_e, J_m, \hat{J}_e, \hat{J}_m) \in [H(\text{div}, \Omega)]^4$ with $(J_e \cdot \nabla \cdot \hat{J}_e, J_m \cdot \nabla \cdot \hat{J}_m) \in [H^{1/2}(\partial \Omega)]^2$, and for every solution $(E, H, \hat{E}, \hat{H}) \in [L^2(\Omega)]^{12}$ of

$$\begin{aligned}
\nabla \times E &= k\mu H + J_e \text{ in } \Omega, \\
\nabla \times H &= -k\varepsilon E + J_m \text{ in } \Omega,
\end{aligned}$$

$$\begin{aligned}
\nabla \times \hat{E} &= \alpha k\tilde{\mu} \hat{H} + \hat{J}_e \text{ in } \Omega, \\
\nabla \times \hat{H} &= -\alpha k\tilde{\varepsilon} \hat{E} + \hat{J}_m \text{ in } \Omega,
\end{aligned}$$

we have

$$\begin{aligned}
|k| \|(E, H, \hat{E}, \hat{H})\|_{L^2(\Omega)} + \|(E, H, \hat{E}, \hat{H})\|_{H^\gamma(\Omega_{s_0/2})} &\leq C \| (J_e, J_m, \hat{J}_e, \hat{J}_m) \|_{L^2(\Omega)} \\
+ \frac{C}{|k|} \|(\text{div } J_e, \text{div } J_m, \text{div } \hat{J}_e, \text{div } \hat{J}_m)\|_{L^2(\Omega)} + \frac{C}{|k|} \|(J_e \cdot \nabla \cdot \hat{J}_e, J_m \cdot \nabla \cdot \hat{J}_m)\|_{H^{1/2}(\partial \Omega)}.
\end{aligned}$$

Recall that $\Omega_s$ is given in (2.1).

**Proof.** We use local charts for $\Gamma = \partial \Omega$. In what follows, we denote $Q = (-1, 1)^3$, $Q_+ = Q \cap \mathbb{R}_+^3$, and $Q_0 = Q \cap \mathbb{R}_-^3$.

Let $m \geq 1$ and let $\varphi_\ell \in C^2_c(\mathbb{R}^3)$, $U_\ell \subset \mathbb{R}^3$ open ball, and $T_\ell : U_\ell \to Q$ with $1 \leq \ell \leq m$ be such that $T_\ell(U_\ell \cap \Omega) = Q_+$, and $T_\ell(U_\ell \cap \Gamma) = Q_0$, supp $\varphi_\ell \subset U_\ell$, and $\Phi = 1$ in a neighborhood of $\Gamma$, where

$$\Phi := \sum_{\ell=1}^m \varphi_\ell \text{ in } \mathbb{R}^3.$$

In what follows, we also assume that the diameter of the support of $\varphi_\ell$ is sufficiently small and $\nabla T_\ell(\varphi_\ell^{-1}(0))$ is a rotation, i.e., $(\nabla T_\ell \nabla T_\ell^T)(\varphi_\ell^{-1}(0)) = I$. Set, in $\Omega \cap U_\ell$,

$$(E^\ell, H^\ell, \hat{E}^\ell, \hat{H}^\ell) = (\varphi_\ell E, \varphi_\ell H, \varphi_\ell \hat{E}, \varphi_\ell \hat{H}),$$

and

$$(J^\ell_e, J^\ell_m, \hat{J}^\ell_e, \hat{J}^\ell_m) = (\varphi_\ell J_e + \nabla \varphi_\ell \times E, \varphi_\ell J_m + \nabla \varphi_\ell \times H, \varphi_\ell \hat{J}_e + \nabla \varphi_\ell \times \hat{E}, \varphi_\ell \hat{J}_m + \nabla \varphi_\ell \times \hat{H}).$$
We have
\begin{align}
(4.4) \quad \begin{cases}
\nabla \times E^\ell = k\mu H^\ell + J_e^\ell & \text{in } \Omega \cap U_\ell,
\nabla \times H^\ell = -k \varepsilon E^\ell + J_m^\ell & \text{in } \Omega \cap U_\ell,
\end{cases}
\begin{cases}
\nabla \times \hat{E}^\ell = \alpha k \mu \hat{H}^\ell + \hat{J}_e^\ell & \text{in } \Omega \cap U_\ell,
\nabla \times \hat{H}^\ell = -\alpha k \varepsilon \hat{E}^\ell + \hat{J}_m^\ell & \text{in } \Omega \cap U_\ell,
\end{cases}
\end{align}

\begin{align}
(4.5) \quad (\hat{E}^\ell - E^\ell) \times \nu = 0 & \text{ on } \partial \Omega \cap U_\ell, \text{ and } (\hat{H}^\ell - H^\ell) \times \nu = 0 \text{ on } \partial \Omega \cap U_\ell.
\end{align}

Given a diffeomorphism $T$ from an open $D$ onto an open $D'$, the following standard notations are used
\begin{align*}
\mathcal{T} \ast u(x') &= \nabla \mathcal{T}(x) u(x), \\
\mathcal{T}_a(x') &= \frac{\nabla \mathcal{T}(x) a(x) \nabla \mathcal{T}^T(x)}{\det \nabla \mathcal{T}(x)}, \quad \text{and} \quad \mathcal{T}_j(x') &= \frac{\nabla \mathcal{T}(x) j(x)}{\det \nabla \mathcal{T}(x)},
\end{align*}
with $x' = T(x)$, for a matrix-valued function $a$, and for vector fields $u$ and $j$ defined in $D$. Set, in $Q_+$,
\begin{align}
(E^\ell, H^\ell, \hat{E}^\ell, \hat{H}^\ell) &= \left( \mathcal{T}_\ast E^\ell, \mathcal{T}_\ast H^\ell, \mathcal{T}_\ast \hat{E}^\ell, \mathcal{T}_\ast \hat{H}^\ell \right), \\
(\varepsilon^\ell, \mu^\ell, \hat{\varepsilon}^\ell, \hat{\mu}^\ell) &= \left( \mathcal{T}_\ast \varepsilon^\ell, \mathcal{T}_\ast \mu^\ell, \mathcal{T}_\ast \hat{\varepsilon}^\ell, \mathcal{T}_\ast \hat{\mu}^\ell \right), \\
(J_e^\ell, J_m^\ell, \hat{J}_e^\ell, \hat{J}_m^\ell) &= \left( \mathcal{T}_\ast J_e^\ell, \mathcal{T}_\ast J_m^\ell, \mathcal{T}_\ast \hat{J}_e^\ell, \mathcal{T}_\ast \hat{J}_m^\ell \right).
\end{align}

By a change of variables, see e.g. [23, Lemma 7],
\begin{align}
(4.6) \quad \begin{cases}
\nabla \times E^\ell = k\mu H^\ell + J_e^\ell & \text{in } Q_+, \\
\nabla \times H^\ell = -k \varepsilon E^\ell + J_m^\ell & \text{in } Q_+,
\end{cases}
\begin{cases}
\nabla \times \hat{E}^\ell = \alpha k \mu \hat{H}^\ell + \hat{J}_e^\ell & \text{in } Q_+, \\
\nabla \times \hat{H}^\ell = -\alpha k \varepsilon \hat{E}^\ell + \hat{J}_m^\ell & \text{in } Q_+,
\end{cases}
\end{align}

\begin{align}
(4.7) \quad (\hat{E}^\ell - E^\ell) \times \nu &= 0 \text{ on } Q_0, \quad \text{and} \quad (\hat{H}^\ell - H^\ell) \times \nu = 0 \text{ on } Q_0.
\end{align}

Since $\nabla \mathcal{T}_\ell(\varphi^{-1}\varepsilon(0))$ is a rotation, and $\varepsilon^\ell, \mu^\ell, \hat{\varepsilon}^\ell, \hat{\mu}^\ell$ are isotropic on $\partial \Omega$, one has
\begin{align*}
\varepsilon^\ell(0), \mu^\ell(0), \hat{\varepsilon}^\ell(0), \hat{\mu}^\ell(0) \text{ are isotropic.}
\end{align*}

By considering the diameter of $\text{supp } \varphi^\ell$ sufficiently small, one can then apply Corollary 3.1 to $(E^\ell, H^\ell, \hat{E}^\ell, \hat{H}^\ell)$. We then obtain
\begin{align}
(4.8) \quad C \left( \left\| (E^\ell, H^\ell, \hat{E}^\ell, \hat{H}^\ell) \right\|_{H^1(Q_+)} + \left\| (E^\ell, H^\ell, \hat{E}^\ell, \hat{H}^\ell) \right\|_{L^2(Q_+)} \right)
\leq \left\| (J_e^\ell, J_m^\ell, \hat{J}_e^\ell, \hat{J}_m^\ell) \right\|_{L^2(Q_+)} + \frac{1}{|k|} \left( \text{div } J_e^\ell, \text{div } J_m^\ell, \text{div } \hat{J}_e^\ell, \text{div } \hat{J}_m^\ell \right)_{L^2(Q_+)}
\leq \frac{1}{|k|} \left\| (J_e^\ell \cdot e_3 - \hat{J}_e^\ell \cdot e_3, J_m^\ell \cdot e_3 - \hat{J}_m^\ell \cdot e_3) \right\|_{H^{1/2}(Q_0)}.
\end{align}

We have, by [22, Corollary 3.59],
\begin{align*}
\left\| (\text{div } J_e^\ell, \text{div } J_m^\ell, \text{div } \hat{J}_e^\ell, \text{div } \hat{J}_m^\ell) \right\|_{L^2(Q_+)} \leq C \left( \left\| (\text{div } J_e^\ell, \text{div } J_m^\ell, \text{div } \hat{J}_e^\ell, \text{div } \hat{J}_m^\ell) \right\|_{L^2(\Omega \setminus U_\ell)} \right)
\end{align*}
and we also obtain
\begin{align*}
\left\| (J_e^\ell \cdot e_3 - \hat{J}_e^\ell \cdot e_3, J_m^\ell \cdot e_3 - \hat{J}_m^\ell \cdot e_3) \right\|_{H^{1/2}(Q_0)} \leq C \left( \left\| (J_e^\ell \cdot \nu - \hat{J}_e^\ell \cdot \nu, J_m^\ell \cdot \nu - \hat{J}_m^\ell \cdot \nu) \right\|_{H^{1/2}(\partial \Omega \setminus U_\ell)} \right).
\end{align*}
We deduce from (4.8) that

\[
(4.9) \quad C \left( \| (E^\ell, H^\ell, \tilde{E}^\ell, \tilde{H}^\ell) \|_{H^1(\Omega \cap U_\ell)} + |k| \| (E^\ell, H^\ell, \tilde{E}^\ell, \tilde{H}^\ell) \|_{L^2(\Omega \cap U_\ell)} \right)
\]

\[
\leq \| (J_e^\ell, J_m^\ell, \tilde{J}_e^\ell, \tilde{J}_m^\ell) \|_{L^2(\Omega \cap U_\ell)} + \frac{1}{|k|} \| (\text{div} J_e^\ell, \text{div} J_m^\ell, \text{div} \tilde{J}_e^\ell, \text{div} \tilde{J}_m^\ell) \|_{L^2(\Omega \cap U_\ell)}
\]

\[
\quad + \frac{1}{|k|} \| (J_e^\ell \cdot \nu - \tilde{J}_e^\ell \cdot \nu, J_m^\ell \cdot \nu - \tilde{J}_m^\ell \cdot \nu) \|_{H^{1/2}(\partial \Omega \cap U_\ell)}.
\]

Take the sum with respect to \( \ell \). We then have, for some \( \tau_0 < s_0/4 \),

\[
(4.10) \quad C \left( \| (E, H, \hat{E}, \hat{H}) \|_{H^1(\Omega_{\tau_0})} + |k| \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\Omega_{\tau_0})} \right)
\]

\[
\leq \| (J_e, J_m, \tilde{J}_e, \tilde{J}_m) \|_{L^2(\Omega)} + \frac{1}{|k|} \| (\text{div} J_e, \text{div} J_m, \text{div} \tilde{J}_e, \text{div} \tilde{J}_m) \|_{L^2(\Omega)}
\]

\[
\quad + \frac{1}{|k|} \| (J_e \cdot \nu - \tilde{J}_e \cdot \nu, J_m \cdot \nu - \tilde{J}_m \cdot \nu) \|_{H^{1/2}(\partial \Omega)}
\]

\[
\quad + \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\Omega_{\tau_0}/2)} + \frac{1}{|k|} \| (E, H, \hat{E}, \hat{H}) \|_{H^1(\Omega_{\tau_0}/2)}.
\]

Applying Lemma 4.1 below, we have

\[
(4.11) \quad \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\Omega \setminus \Omega_{\tau_0}/2 \setminus \Omega_{\tau_0})} \leq c_1 e^{-c_2 |k|} \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\Omega)}
\]

for some positive constants \( c_1, c_2 \) depending only on \( \Lambda, \gamma, \tau_0, \) and \( \Omega \). Since \( (\varepsilon, \mu, \tilde{\varepsilon}, \tilde{\mu}) \in C^1(\Omega_3 \setminus \Omega_{\tau_0}/2) \), it follows from (4.11) that

\[
(4.12) \quad \| (E, H, \hat{E}, \hat{H}) \|_{H^1(\Omega_{\tau_0}/2 \setminus \Omega_{\tau_0})} \leq C |k| \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\Omega_{\tau_0} \setminus \Omega_{\tau_0}/2)}
\]

\[
\quad + C \| (J_e, J_m, \tilde{J}_e, \tilde{J}_m) \|_{L^2(\Omega)} + C \| (\text{div} J_e, \text{div} J_m, \text{div} \tilde{J}_e, \text{div} \tilde{J}_m) \|_{L^2(\Omega)}.
\]

Taking \( k_0 \) sufficiently large and \( |k| \geq k_0 \), from (4.11) and (4.12), one can absorb the last two terms of the RHS of (4.10) by the LHS of (4.10). We then have

\[
(4.13) \quad C \left( \| (E, H, \hat{E}, \hat{H}) \|_{H^1(\Omega_{\tau_0})} + |k| \| (E, H, \hat{E}, \hat{H}) \|_{L^2(\Omega_{\tau_0})} \right)
\]

\[
\leq \| (J_e, J_m, \tilde{J}_e, \tilde{J}_m) \|_{L^2(\Omega)} + \frac{1}{|k|} \| (\text{div} J_e, \text{div} J_m, \text{div} \tilde{J}_e, \text{div} \tilde{J}_m) \|_{L^2(\Omega)}
\]

\[
\quad + \frac{1}{|k|} \| (J_e \cdot \nu - \tilde{J}_e \cdot \nu, J_m \cdot \nu - \tilde{J}_m \cdot \nu) \|_{H^{1/2}(\partial \Omega)}.
\]

Using (4.11) and (4.13), we derive from (4.12) that

\[
(4.14) \quad C \| (E, H, \hat{E}, \hat{H}) \|_{H^1(\Omega_{\tau_0}/2 \setminus \Omega_{\tau_0})} \leq \| (J_e, J_m, \tilde{J}_e, \tilde{J}_m) \|_{L^2(\Omega)}
\]

\[
\quad + \frac{1}{|k|} \| (\text{div} J_e, \text{div} J_m, \text{div} \tilde{J}_e, \text{div} \tilde{J}_m) \|_{L^2(\Omega)} + \frac{1}{|k|} \| (J_e \cdot \nu - \tilde{J}_e \cdot \nu, J_m \cdot \nu - \tilde{J}_m \cdot \nu) \|_{H^{1/2}(\partial \Omega)}.
\]
Proof. Let \((s > 1, \Omega)\) as in the proof of Lemma 3.1, we have
\[
\Lambda^{-1} \leq \varepsilon, \mu, \bar{\varepsilon}, \bar{\mu} \leq \Lambda,
\]
for some \(\Lambda \geq 1\). Given \(J_e, J_m \in L^2(\Omega)\), let \((E, H) \in [L^2(\Omega)]^6\) be a solution of
\[
\begin{cases}
\nabla \times E = k\mu H + J_e \text{ in } \Omega, \\
\nabla \times H = -k\varepsilon E + J_m \text{ in } \Omega.
\end{cases}
\]

For all \(s > 0\), there exists two positive constants \(c_1\) and \(c_2\) depending only on \(\Lambda, \gamma, s, \) and \(\Omega\) such that
\[
\|(E, H)\|_{L^2(\Omega; \Omega_s)} \leq c_1 \exp(-c_2|k|)\|(E, H)\|_{L^2(\Omega_s)} + c_1\|(J_e, J_m)\|_{L^2(\Omega)}.
\]

Proof. Let \((E^1, H^1) \in [L^2(\Omega)]^6\) be the unique solution of
\[
\begin{cases}
\nabla \times E^1 = k\mu H^1 + J_e \text{ in } \Omega, \\
\nabla \times H^1 = -k\varepsilon E^1 + J_m \text{ in } \Omega,
\end{cases}
(4.15)
\]

As in the proof of Lemma 3.1, we have
\[
\|(E^1, H^1)\|_{L^2(\Omega)} \leq C\|(J_e, J_m)\|_{L^2(\Omega)}.
\]

Considering \((E - E^1, H - H^1)\), w.l.o.g., one might assume that \(J_e = J_m = 0\) in \(\Omega\). This is assumed from later on.

Fix \(\varphi \in C^2(\Omega)\) such that \(\varphi = cs\) in \(\Omega \setminus \Omega_s\) and \(\varphi = 0\) in \(\Omega_{s/2}\), and \(|\nabla \varphi| \leq c\) in \(\Omega\) where \(c\) is a small positive constant defined later (the smallness of \(c\) depends only on \(\Lambda\) and \(\Omega\), it is independent of \(s\)). Set \(\phi(x) = e^{k|\varphi(x)|}\) and \(E^1(x) = \phi(x)E(x)\) and \(H^1(x) = \phi(x)H(x)\) for \(x \in \Omega\). We have
\[
\begin{cases}
\nabla \times E^1 = k\mu H^1 + J^1_e \text{ in } \Omega, \\
\nabla \times H^1 = -k\varepsilon E^1 + J^1_m \text{ in } \Omega,
\end{cases}
\]
\[
J^1_e = \nabla \phi \times E \quad \text{and} \quad J^1_m = \nabla \phi \times H \text{ in } \Omega.
\]

Multiplying the first equation with \(H^1\), integrating by parts in \(\Omega \setminus \Omega_{s/2}\) for \(s/4 < \tau < s/2\), and using the second equation, we have
\[
\left| \int_{\Omega \setminus \Omega_{s/2}} k \langle \mu H^1, H^1 \rangle + \bar{k} \int_{\Omega \setminus \Omega_{s/2}} \langle \varepsilon E^1, E^1 \rangle \right|
\leq \int_{\Omega \setminus \Omega_{s/2}} |J^1_e| |H^1| + |J^1_m| |E^1| + C \int_{\partial(\Omega \setminus \Omega_{s/2})} (|E^1|^2 + |H^1|^2).
This yields

\begin{equation}
(4.16) \quad \left| \int_{\Omega \setminus \Omega_r} k^2 \langle \mu H^1, H^1 \rangle + |k|^2 \int_{\Omega \setminus \Omega_r} \langle \varepsilon E^1, E^1 \rangle \right| \leq |k| \int_{\Omega \setminus \Omega_r} |J^1_c||H^1| + |J^1_m||E^1| + C|k| \int_{\partial(\Omega \setminus \Omega_r)} (|E^1|^2 + |H^1|^2).
\end{equation}

By the definition of \( J^1_c, J^1_m \), and of \( E^1 \) and \( H^1 \),

\[ |J^1_c| \leq c|k||E^1|, \quad |J^1_m| \leq c|k||H^1| \text{ in } \Omega, \quad \text{and} \quad E^1 - E = H^1 - H = 0 \text{ in } \Omega_{s/2}, \]

we derive from (4.16) that, for \( c \) sufficiently small,

\[ \left| \int_{\Omega \setminus \Omega_r} k^2 \langle \mu H^1, H^1 \rangle + |k|^2 \int_{\Omega \setminus \Omega_r} \langle \varepsilon E^1, E^1 \rangle \right| \leq C|k| \int_{\partial(\Omega \setminus \Omega_r)} (|E|^2 + |H|^2). \]

The conclusion follows by taking \( \tau \) such that

\[ \int_{\partial(\Omega \setminus \Omega_r)} (|E|^2 + |H|^2) \leq C s^{-1} \int_{\Omega_s} (|E|^2 + |H|^2). \]

This yields

\[ \int_{\partial(\Omega \setminus \Omega_r)} (|E|^2 + |H|^2) \leq C s^{-1} \int_{\Omega_s} (|E|^2 + |H|^2), \]

and the conclusion follows by the definition of \( E^1 \) and \( H^1 \). The proof is complete. \( \square \)

**Remark 4.1.** The proof of Lemma 4.1 is quite standard, see e.g. [17, Theorem 2.2] for a variant dealing with the Helmholtz equation.

We now apply a limiting absorption principle. To this end, let us set

\[ \mathbb{H}_1(\Omega) = \left\{ (u, v) \in [H(\text{curl}, \Omega)]^2; (u - v) \times v = 0 \text{ on } \partial \Omega \right\}. \]

One can check that \( \mathbb{H}_1(\Omega) \) is a Hilbert space equipped with the natural scalar product induced from the one of \([H(\text{curl}, \Omega)]^2\).

Consider \((J_c, J_m, \hat{J}_c, \hat{J}_m) \in [L^2(\Omega)]^{12}\). Let \( \gamma > 0, k \in \mathbb{C} \) with \( |\Re(k^2)| \geq \gamma |k|^2 \) and \( |k| \geq 1 \). Set \( a = \text{sign } \Re(k^2) \). For \( \delta > 0 \) sufficiently small with respect to \( \gamma \), we claim that there exists a unique solution \((E^\delta, H^\delta, \hat{E}^\delta, \hat{H}^\delta) \in [L^2(\Omega)]^{12}\) of the system

\begin{align*}
\nabla \times E^\delta & = (1 - ia\delta)k\mu H^\delta + J_c \text{ in } \Omega, \\
\nabla \times H^\delta & = -(1 - ia\delta)k\varepsilon E^\delta + J_m \text{ in } \Omega, \\
\nabla \times \hat{E}^\delta & = -(1 - ia\delta)k\mu \hat{H}^\delta + \hat{J}_c \text{ in } \Omega, \\
\nabla \times \hat{H}^\delta & = (1 - ia\delta)k\varepsilon \hat{E}^\delta + \hat{J}_m \text{ in } \Omega,
\end{align*}

\begin{equation}
(18) \quad (\hat{E}^\delta - E^\delta) \times \nu = 0 \text{ on } \partial \Omega, \quad \text{and} \quad (\hat{H}^\delta - H^\delta) \times \nu = 0 \text{ on } \partial \Omega.
\end{equation}
Indeed, consider the following equation

\[(4.19) \quad \int_{\Omega} \langle (1 - i\alpha \delta)^{-1} \mu^{-1} \nabla \times E^\delta, \nabla \times \varphi \rangle + k^2 (1 - i\alpha \delta) \langle \varepsilon E^\delta, \varphi \rangle + \int_{\Omega} \langle (1 - i\alpha \delta)^{-1} \mu^{-1} \nabla \times \hat{E}^\delta, \nabla \times \varphi \rangle + k^2 (1 - i\alpha \delta) \langle \hat{\varepsilon} \hat{E}^\delta, \hat{\varphi} \rangle = \int_{\Omega} \langle (1 - i\alpha \delta)^{-1} \mu^{-1} J_e, \nabla \times \varphi \rangle + k \langle J_m, \varphi \rangle + \int_{\Omega} \langle (1 - i\alpha \delta)^{-1} \mu^{-1} \hat{J}_e, \hat{\varphi} \rangle - k \langle \hat{J}_m, \hat{\varphi} \rangle,\]

for all \((\varphi, \hat{\varphi}) \in H_1(\Omega)\). Note that, the absolute value of the imaginary part of the LHS of (4.19) is greater than \(C \delta \int_{\Omega} |(E^\delta, \nabla \times E^\delta, \hat{E}^\delta, \nabla \times \hat{E}^\delta)|^2\) for \(\delta\) sufficiently small. By Lax-Milgram’s theory, there exists a unique solution \((E^\delta, \hat{E}^\delta) \in H_1(\Omega)\) for (4.19). Set, in \(\Omega\),

\[H^\delta = k^{-1} (1 - i\alpha \delta)^{-1} \mu^{-1} \left( \nabla \times E^\delta - J_e \right) \quad \text{and} \quad \hat{H}^\delta = -k^{-1} (1 - i\alpha \delta) \mu^{-1} \left( \nabla \times \hat{E}^\delta - \hat{J}_e \right).\]

Considering \(\varphi, \hat{\varphi} \in C^1(\Omega)\) in (4.19), we obtain

\[\nabla \times H^\delta = -(1 + i\delta) k \varepsilon E^\delta + J_m \text{ in } \Omega \quad \text{and} \quad \nabla \times \hat{H}^\delta = -(1 - i\delta) k \hat{\varepsilon} \hat{E}^\delta + \hat{J}_m \text{ in } \Omega.\]

This in turn implies that

\[(\hat{H}^\delta - H^\delta) \times \nu = 0 \text{ on } \partial \Omega.\]

Therefore, the existence and the uniqueness of \((E^\delta, H^\delta)\) are established.

Next denote

\[(4.20) \quad H(\Omega) = \{(u, v, \hat{u}, \hat{v}) \in [L^2(\Omega)]^{12}; \text{div}(\varepsilon u) = \text{div}(\mu \nu) = \text{div}(\hat{\varepsilon} \hat{u}) = \text{div}(\hat{\mu} \hat{\nu}) = 0 \text{ in } \Omega, \quad \varepsilon u \cdot \nu - \hat{\varepsilon} \hat{u} \cdot \nu = \mu \nu \cdot \nu - \hat{\mu} \hat{\nu} \cdot \nu = 0 \text{ on } \partial \Omega\},\]

and let

\[\| (u, v, \hat{u}, \hat{v}) \|_H = \|(u, v, \hat{u}, \hat{v})\|_{L^2(\Omega)}.\]

One can check that \(H(\Omega)\) is a Hilbert space with the corresponding scalar product.

We finally have

**Proposition 4.2.** Let \(\gamma > 0, \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \in [L^\infty(\Omega)]^{3 \times 3}\) be symmetric. Assume that there exist \(\Lambda \geq 1, \Lambda_1 > 0, \text{ and } s_0 > 0\) such that

\[\Lambda^{-1} \leq \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \leq \Lambda \text{ in } \Omega, \quad \| (\varepsilon, \mu, \hat{\varepsilon}, \hat{\mu}) \|_{C^1(\Omega)} \leq \Lambda, \quad \varepsilon, \mu, \hat{\varepsilon}, \hat{\mu} \text{ are isotropic on } \partial \Omega,\]

and, for \(x \in \partial \Omega,\)

\[|\varepsilon(x) - \hat{\varepsilon}(x)| \geq \Lambda_1, \quad |\mu(x) - \hat{\mu}(x)| \geq \Lambda_1, \quad \text{and} \quad |\varepsilon(x)/\mu(x) - \hat{\varepsilon}(x)/\hat{\mu}(x)| \geq \Lambda_1.\]

There exist two positive constants \(k_0 \geq 1\) and \(C > 0\) depending only on \(\Lambda, \Lambda_1, \gamma, s_0, \text{ and } \Omega\) such that for \(k \in \mathbb{C}\) with \(|\Im(k^2)| \geq \gamma |k|^2\) and \(|k| \geq k_0, \text{ and for every } (J^1_e, J^1_m, \hat{J}^1_e, \hat{J}^1_m) \in H(\Omega)\),
there exists a unique solution \((E, H, \hat{E}, \hat{H}) \in \mathbf{H}(\Omega)\) of
\[
\begin{align*}
\nabla \times E &= k \mu H + J_e \text{ in } \Omega, \\
\nabla \times H &= -k \varepsilon E + J_m \text{ in } \Omega,
\end{align*}
\]
(4.21)
\[
\begin{align*}
\nabla \times \hat{E} &= k \mu \hat{H} + \hat{J}_e \text{ in } \Omega, \\
\nabla \times \hat{H} &= -k \varepsilon \hat{E} + \hat{J}_m \text{ in } \Omega,
\end{align*}
\]
(4.22)

where \((J_e, J_m, \hat{J}_e, \hat{J}_m) = (\mu J^1_m, \varepsilon J^1_e, \hat{\mu} \hat{J}^1_m, \hat{\varepsilon} \hat{J}^1_e)\). Moreover, \(\hat{E}, \hat{H}\) is an eigenfunction pair of \(T\) corresponding to the eigenvalue \(k = i\omega\). Hence, the spectrum
of the ITE problem is discrete, and the only possible accumulation point of the transmission eigenvalues is $\infty$ since they coincide with the eigenvalues of the inverse of $T$. □

Finally, we present

**Proof of Proposition 1.1.** Proposition 1.1 is just a consequence of Proposition 4.2 by noting that the solution given there is 0 if $(J_e, J_m, \tilde{J}_e, \tilde{J}_m) = 0$. □

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