On the Origin of Coulomb Pseudopotential: Two Wrongs Make a “Right”

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We explore the effect of Coulomb repulsion on the phonon-mediated BCS-type Cooper instability in the electron gas using a numeric approach that is controllably accurate at small to moderate values of the Coulomb parameter, $r_s \lesssim 2$. We demonstrate the necessity of breaking the net result into two distinctively different effects: the reduction of the Fermi-liquid renormalization factor and the change of the effective low-energy coupling. The latter quantity is shown to behave non-monotonically with an extremum at $r_s \approx 0.75$. Within the random-phase approximation, Coulomb interaction eventually starts to *enhance* the effective pairing coupling at $r_s > 2$ and the suppression of the critical temperature is entirely due to the renormalized Fermi-liquid properties. The onset of the enhancement is, however, sensitive to the approximation used for incorporating vertex corrections. Our results call for radical reconsideration of the widely accepted picture that the effect of Coulomb interactions reduces to a (weak) repulsive pseudopotential.

*Introduction.*—In the vast majority of low-temperature superconductors, the scenario of Cooper instability is of the emergent BCS type, implying quantitatively accurate low-energy effective description in terms of the two (partially related) parameters: the energy/frequency cutoff $\omega_0 \ll \epsilon_F$ ($\epsilon_F$ is the Fermi energy; $\hbar = 1$) and the dimensionless effective coupling constant $g \ll 1$. Within this effective BCS theory, the expression for the critical temperature reads

$$T_c = \omega_0 e^{-1/g}.$$  \hfill (1)

For the phonon-mediated Cooper instability, one has $\omega_0 \sim \omega_{ph}$, where $\omega_{ph}$ is a typical phonon frequency. (The exact choice of $\omega_0$ is a matter of convention, because it is always possible to change $\omega_0$ by an order-unity factor in combination with small change in $g$.)

The emergent BCS regime also implies that the constant $g$ can be decomposed into a product of two distinctively different factors—the pseudopotential $U$ and the Fermi liquid factor $f_{FL}$:

$$g \propto U f_{FL}. \quad \hfill (2)$$

The pseudopotential is understood as an amplitude of the dimensionless (weak) attractive coupling between bare electrons near the Fermi surface (FS) and $f_{FL}$ is defined by

$$f_{FL} = z^2 (m^*/m_0), \quad \hfill (3)$$

where $z$ is the quasiparticle residue, $m^*$ is the FS effective mass, and $m_0$ is the bare electron mass. It accounts for the fact that we are dealing with the correlated Fermi liquid rather than an ideal gas. Exponential sensitivity of the critical temperature to the small parameter $g$ implies that the positive-definite factor $f_{FL}$—if noticeably smaller than unity—can dramatically suppress the value of $T_c$.

The strength of Coulomb interaction is characterized by the dimensionless coupling parameter (the Wigner-Seitz radius)

$$r_s = \left( \frac{3}{4\pi a_0^2 n} \right)^{1/3},$$

where $n$ is the number density, and $a_0$ is the Bohr radius. Typical experimental values of $r_s \gtrsim 2$ correspond to a moderately strong interaction. Therefore *a priori* one might expect that repulsive Coulomb potential will simply eliminate the possibility of phonon-mediated pairing in materials. Meanwhile, the experiment tells us otherwise leading to the following chain of assumptions and arguments aimed at understanding this outcome. *Taking for granted* that effective (low-energy) Coulomb interaction is repulsive and *assuming* that $f_{FL}$ can be ignored, one appeals to the effect of static screening to explain why the suppression of $U$ is often moderate.

This reasoning lead to a widely accepted scenario, originally proposed by Bogoliubov, Tolmachev, and Shirkov [1], and repeated by Morel and Anderson [2], of logarithmic suppression of Coulomb repulsion near the FS. The net result is the so-called repulsive Coulomb pseudopotential

$$\mu^* = \frac{\mu}{1 + \mu \ln (\epsilon_F/\omega_0)}.$$  \hfill (4)

It is often used to estimate $T_c$ with the help of McMillan’s formula [3], which was developed as a semi-phenomenological fit reproducing results of large number of experiments [4, 5]. The main effect of the Coulomb pseudopotential is to reduce the magnitude of the phonon-mediated $U$ as $U \rightarrow U - \mu^*$, with most experiments suggesting that $\mu^* \in (0.1 \div 0.15)$. It should be noted that $\mu > 0$ was introduced in a rather uncontrolled fashion as some coupling constant characterizing
screened Coulomb repulsion at frequencies $\sim E_F$. If $\mu$ is computed from $\rho_0 4\pi e^2/k^2$ where $\rho_0$ is the ideal gas FS density of states per spin component, and $\kappa$ is the Thomas-Fermi screening momentum, then $\mu = 0.5$. The pseudopotential construction explains that the values of $\mu^*$ are small because the $E_F/\omega_0$ ratio is large, but neither the values of $\mu$ nor its sign were derived from first principles, not to mention that Coulomb interaction cannot be fully screened at finite frequencies.

Recent breakthrough in precise computation of Fermi liquid properties of the uniform electron gas [9] shows that the Fermi liquid factor $f_{FL}$ is significantly smaller than unity at $r_s > 2$; see Fig. 1. See also Ref. [10] for the RPA results in the same context.

In combination with the above-mentioned experimental fact that corrections to $g$ remain small this result forces one to reconsider the effect of Coulomb potential on $U$ at $r_s > 2$—it has to be far smaller than predicted by $\mu^*$ and even opposite in sign, i.e. Coulomb interactions in the s-wave channel might actually increase the amplitude of attractive $U$.

![FIG. 1. Fermi liquid factors $f_{FL}$ of the uniform electron gas computed using data reported in Ref. [9].]

In this Letter, we employ an implicit renormalization protocol and a generalized discrete Lehmann representation for extracting effective coupling constant and critical temperature from the gap function equation (5) to study the effect of Coulomb repulsion on $U$ and $T_c$.

The four-point vertex function in the Cooper channel is based on approximations that guarantee exact results in the $r_s \to 0$ limit and their quantitative accuracy up to $r_s \lesssim 2$. We reveal that the suppression of $U$ is maximal at $r_s \approx 0.75$, and the effect diminishes for larger values of $r_s$. Within the RPA, Coulomb interactions start to enhance attractive coupling at $r_s > 2$, but this result is sensitive to inclusion of vertex corrections. We discuss our findings in the context of earlier work suggesting/pointing to a possibility of pairing instability in the absence of electron-phonon coupling, i.e. exclusively on the basis of dynamically screened Coulomb repulsion [1, 12, 14].

**Model.**—The Hamiltonian of the uniform electron gas (UEG) on a neutralizing background is defined as:

$$H = \sum_{k\sigma} \epsilon_k a_{k\sigma}^\dagger a_{k\sigma} + \frac{1}{2} \sum_{q\neq 0} \sum_{k\sigma} \sum_{k'\sigma'} V_q a_{k+q\sigma}^\dagger a_{k'-q\sigma'}^\dagger a_{k'\sigma'} a_{k\sigma},$$

(4)

Here $a_{k\sigma}^\dagger$ is the creation operator of an electron with momentum $k$ and spin $\sigma = \uparrow, \downarrow$, $\epsilon_k = \frac{k^2}{2m_0} - \mu$, and $V_q = \frac{4\pi e^2}{\pi a^2}$ is the bare Coulomb interaction.

The gap function equation in the singlet channel reads:

$$\lambda \Delta_{\omega_n, k} = -T \sum_m \frac{d^d p}{(2\pi)^d} \Gamma_{\omega_n, k}^{\omega_m, p} G_{\omega_n, p} G_{-\omega_m, -p} \Delta_{\omega_m, p},$$

(5)
Here $\Gamma$ is the particle-particle irreducible four-point vertex, $G$ is the single particle Green’s function, $\Delta$ is the gap function, and $\lambda \equiv \lambda (T)$ is its eigenvalue. The critical temperature $T_c$ corresponds to the point where $\lambda_{\text{max}} (T) = 1$.

We consider two approximations for $\Gamma$ based on the the screened Coulomb interaction, both depending only on the momentum and energy transfer, $\Gamma_{\omega m, \mathbf{p}} = \Gamma (\omega_m - \omega_n, \mathbf{p} - \mathbf{k})$. The RPA form is standard: $\Gamma_{\text{RPA}} = (V^{-1} + \Pi_0 (\omega, q))^{-1}$, where $\Pi_0$ is the polarization function computed from the convolution of bare Green’s function. For simplicity we take the functional form of $\Pi_0$ encoded in the local field factors $\Gamma$ at critical temperature. To account for vertex corrections and estimate their role as a function of $r_s$ we employ the Kukkonen-Overhauser ansatz \cite{[15]} when

$$\Gamma_{\text{KO}} = V_q + V_+ (q)^2 Q_+ (\omega, q) - 3 V_- (q)^2 Q_- (\omega, q),$$

with $Q_{\pm} (\omega, q) = -(\Pi_0^0 (\omega, q) + V_\pm (\omega, q))^{-1}$ and $V_+ = (1 - G_+) V_0, V_- = - G_- V_0$. Here $\Gamma_{\text{KO}}$ is already projected on spin-singlet state as required by the fermionic parity. The higher-order vertex corrections neglected in RPA are encoded in the local field factors $G_{\pm} (q)$ for which we adopt the ansatz proposed by Takada \cite{[16]}

Finally, we introduce phonon-mediated interactions taken to have the same functional form as considered by Ashcroft and Richardson to study the very same problem of superconductivity in the UEG with electron-phonon coupling \cite{[14]}

$$\Gamma_{\text{ph}} (\omega, q) = - \frac{a \rho_0}{1 + (q/2k_F)^2} \frac{\omega_q^2}{\omega_q^2 + \omega_q^2},$$

with the phonon dispersion $\omega_q^2 = \omega_{ph}^2 (q/k_F)^2/(1 + (q/k_F)^2)$ and dimensionless coupling strength $a$. For every choice of the vertex function considered in this work the single particle self-energy was computed self-consistently from the convolution of $G$ and $\Gamma$.

**Implicit renormalization approach.**—For the simplest case when $\Gamma = \Gamma_{\text{ph}}$, the eigenvalue $\lambda (T)$ is a linear function of $\ln T$ at low temperature $T \ll \omega_{\text{ph}}$ that can be written as:

$$\lambda (T) = - g \ln (T/\omega_0).$$

As expected, the condition $\lambda_{\text{max}} (T_c) = 1$ leads to equation \cite{[14]}, and $T_c$ can be determined accurately by fitting the data even if calculations need to be stopped at $T \gg T_c$. When Coulomb interactions are included, screening and renormalization effects taking place in a broad frequency range above the phonon frequency ensure that $\lambda_{\text{max}} (T)$ is an unknown non-linear function of $\ln T$ that can be used neither for reliable extrapolation towards lower temperature nor evaluation of the effective low energy coupling $U$. The implicit renormalization (IR) approach of Ref. \cite{[11]} provides a solution to both problems by formulating an alternative eigenvalue problem. The gap function is decomposed into two complementary (low-frequency and high-frequency) parts, $\Delta = \Delta^{(1)} + \Delta^{(2)}$, with $\Delta_n^{(1)} = 0$ for $|\omega_n| > \Omega_c$, and $\Delta_n^{(2)} = 0$ for $|\omega_n| < \Omega_c$, and the eigenvalue problem is solved for $\Delta_n^{(1)}$ only. The condition $\lambda (T_c) = 1$ for the largest eigenvalue of the new problem remains exact.

As shown in Fig. 3 the IR formulation brings back nearly perfect linear dependence of $\lambda$ on $\ln T$ for a properly chosen frequency scale separation $\Omega_c$. The slope of the curve is the emergent low-frequency coupling strength $g$ while the vertical axis intercept determines the characteristic low-frequency scale $\omega_0$. Linear dependence is also crucial for accurate determination of $T_c$ from simulations performed at $T > T_c$ when $T_c$ is extremely low and the number of Matsubara frequency points required to solve the gap equation is large (this is done efficiently by the generalized Discrete Lehmann representation \cite{[17, 18]}, see Supplemental Material).

![FIG. 3. Temperature dependence of the largest eigenvalue $\lambda$ for $\Gamma = \Gamma_{\text{ph}} + \Gamma_{\text{RPA}}$ at $r_s = 2$. The emergent BCS linear flow with effective coupling constant $g$ and energy scale $\omega_0$ is represented by the dotted line.](image_url)
FIG. 4. Gap functions dependence on frequency at $k = k_F$ (left panel) and momentum at $\omega_n/\pi \tau$ (right panel) when for $\Gamma$ equals to either $\Gamma_{\text{ph}}$, or $\Gamma_{\text{RPA}}$ or $\Gamma_{\text{ph}} + \Gamma_{\text{RPA}}$ at $r_s = 3$ (with $\Delta(k_F, \pi \tau)$ normalized to unity).

getting progressively smaller with increasing $r_s$. The net effect on the critical temperature is also non-monotonic, but $T_c(r_s)$ behavior is not as dramatic because the increase in $U$ at $r_s > 1$ is partially compensated by the suppression of $f_{\text{FL}}$.

When vertex corrections are accounted for Coulomb interaction and $\Gamma = \Gamma_{\text{ph}} + \Gamma_{\text{KO}}$, the Fermi liquid factor $f_{\text{FL}}$ remains essentially the same for all values of $r_s$. However, changes in $U$ are relatively small (less than 20%) only for $r_s < 1$. The most significant difference is the shift of the point of the onset of the Coulomb enhancement of $U$: from $r_s \approx 2$ to $r_s \approx 3$. This result underlines the importance of approximation-free high-order diagrammatic calculations. Nevertheless, the non-monotonic behavior of $T_c$ and $U$ is a robust effect based on the dynamic screening mechanism that tends to make effective Coulomb interactions attractive at large $r_s$. It is completely overlooked in the Coulomb pseudopotential treatment. If we take our value of $T_c$ at $r_s = 2$ and try to reproduce it with the help of McMillan’s formula, the phenomenological parameter $\mu^*$ ends up to be close to 0.08.

There exists yet another fundamental reason for non-additive effects when two pairing mechanisms are combined (even if $f_{\text{FL}}$ factors are accounted for exactly). If $\lambda_{1,2}$ and $\Delta_i$ are the largest eigenvalue and its eigenvector for matrix $\Gamma_i$ and $\Delta_1 \neq \Delta_2$ then the largest eigenvalue of $\Gamma = \Gamma_1 + \Gamma_2$ is always smaller than $\lambda_1 + \lambda_2$. In Fig. 4, we show gap function solutions for $\Gamma_1 = \Gamma_{\text{ph}}$, $\Gamma_2 = \Gamma_{\text{RPA}}$ and $\Gamma = \Gamma_1 + \Gamma_2$ at $r_s = 3$. One can see that the eigenvector “mismatch” between these solutions is significant: while $\Delta_{\text{ph}}$ is sign-positive and monotonic, $\Delta_{\text{RPA}}$ changes sign both in the momentum and frequency domains and features a pronounced singularity at $k = k_F$.

Discussion and conclusion.—It is instructive to put our findings in the context of historic developments. That Coulomb interaction can induce Cooper instability through dynamic screening mechanism is known for decades. Early work [1] demonstrated that even if the Cooper channel coupling is repulsive at all frequencies, after its high-frequency part is renormalized to a smaller value the effective low-frequency potential might end up attractive. Later, Takada and others calculated critical temperatures of the UEG numerically using various approximate forms of the screened potential [13, 16, 20] featuring singular frequency/momentum dependence (ignored without justification by introducing parameter $\mu$). These results raise an obvious question: Why are phenomenological values of $\mu^*$ used in material science always repulsive if Coulomb interaction alone can be the pairing glue?

Several studies attempted to account for Coulomb effects on superconductivity beyond the Coulomb pseudopotential [14, 21, 22]. Most relevant to our study is the work by Richardson and Ashcroft [14] who calculated $T_c$ for several metals by treating the electron-phonon, Eq. (7), and Coulomb interactions on equal footing. They reported that in Lithium (with $r_s = 3.25$) the inclusion of Coulomb interaction leads to smaller $T_c$. Our results explain, that for large values of $r_s$ the suppression of $f_{\text{FL}}$ is significant and cannot be dismissed as prescribed by the McMillan’s formula. However, this fact was not well established at the time, and Ashcroft and Richardson tried to accommodate all effects into the framework of existing phenomenological treatment.

By separating Coulomb suppression of the Fermi liquid factor $f_{\text{FL}}$ from its contribution to the low-frequency pseudopotential $U$, we shed new light on the origin of small critical temperatures observed experimentally when compared to predictions of the Migdal-Eliashberg theory. We reveal that Coulomb contribution to $U$ changes from repulsive to attractive and conclude that the original interpretation of $\mu^*$ is incorrect and misleading in two ways: (i) The scenario of enhancement of attractive $U$ due to dynamic nature of screening is ignored, leading to the false impression that $\mu^*$ is always positive; (ii) strong renormalization of Fermi liquid properties is
ignored, while it can easily reduce effective coupling by a factor of two. These two mistakes partially compensate each other in the phenomenological treatment, yielding reasonable effective coupling constants $g$ within the freedom of choosing $\mu^*$. However, the actual microscopic picture behind the procedure is missed.

Failure to appreciate non-additivity of the phonon and Coulomb contributions to the effective coupling constant $g$—implied by the structure of Eq. (2) and also by the eigenvector mismatch, Fig. 4—can lead to qualitatively wrong conclusions. For example, Ref. [13] stated that RPA is a deficient approximation at $r_s > 2$ because it predicts an attractive pseudopotential in contradiction with the “experimentally established” $\mu^* > 0$. Proper account of all the aspects of the interplay between dynamically screened Coulomb repulsion and (alternative) pairing mechanisms may bring new insights in the search for new superconducting materials, especially in cases when McMillan’s equation fails qualitatively.

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Supplemental Material

SYMMETRIZED DISCRETE LEHMANN REPRESENTATION

Accurately reproducing singular properties of the imaginary-time Green’s function, \( G(\tau) \), at low temperature \( T = \beta^{-1} \) is a challenging numerical problem. The recently developed Discrete Lehmann representation (DLR) [1] offers a solution in the form of a compact ansatz for \( G(\tau) \). Here we briefly render DLR in its original form [1] along with a generalized version—symmetrized DLR (SDLR) proposed by some of us [2]. The SDLR approach is particularly convenient when working in the Matsubara-frequency representation (and demonstrates other notable advantages that are discussed elsewhere [2]).

The standard spectral (Lehmann) representation for \( G \) reads (here and below we do not show the dependence on momentum and spin variables):

\[
G(\tau) = \int_{-\infty}^{\infty} K(\tau, \omega) \rho(\omega) d\omega ,
\]

where \( \omega \) is the real frequency and \( \rho(\omega) \) is the spectral density—a real-valued non-negative function. The convolution kernel \( K(\tau, \omega) \) is a universal function that only depends on the particle statistics and temperature. This universality allows one to interpret [1] as a linear superposition of fixed functions \( K(\tau, \omega) \), with \( \omega \) labelling the functions and \( \rho(\omega) \) playing the role of the superposition coefficients.

The key observation behind DLR and similar approaches is that the “vectors” \( K(\tau, \omega) \) are massively correlated. As a result, the effective dimension of the functional space spanned by the set \( K(\tau, \omega) \) with realistic \( \rho(\omega) \)’s turns out to be as small as

\[
r \sim \log \frac{E_{uv}}{T} \log \frac{1}{\epsilon},
\]

where \( E_{uv} \) is the frequency cutoff beyond which the physical spectral density can be set to zero and \( \epsilon \) is the specified accuracy with which \( G \) is reproduced. The efficient representation of the Green’s function can then be achieved by identifying a discrete set of \( r \) optimal basis functions. According to [2], the number of basis functions increases very slowly with decreasing \( T \) and \( \epsilon \), and in practice only several dozens of basis functions are needed.

Discrete Lehmann Representation

Within the DLR approach [1], the basis functions \( K(\tau, \omega_k) \) correspond to \( r \) optimally chosen frequency points \( \omega_k \) \((k = 1, 2, ..., r)\) identified by a pivoted QR algorithm [1]. The Green’s function is then represented as

\[
G(\tau) \approx G_{DLR}(\tau) = \sum_{k=1}^{r} K(\tau, \omega_k) \tilde{\rho}_k ,
\]

i.e. the spectral density is replaced with \( r \) “quasiparticle poles.” For a given \( G(\tau) \) data set, the pole residues \( \tilde{\rho}_k \) are obtained from the least-squares fitting. (In practice, the fitting protocol is naturally implemented in the Matsubara-frequency representation; see below.)

The Green’s function \( \tau \)-dependence on the \([-\beta, \beta]\) interval is unambiguously fixed by its behavior on the \((0, \beta]\) interval. Within this interval, the kernel \( K(\tau, \omega) \) can be rendered the same for bosons and fermions [by absorbing statistics-dependent but \( \tau \)-independent factors into the “auxiliary” spectral density \( \rho(\omega) \)]:

\[
K(\tau, \omega) = e^{-\omega \tau} , \quad \tau \in (0, \beta) .
\]

While having simple exponential basis functions in the \( \tau \)-representation is convenient, the least-squares fitting procedure for obtaining \( \tilde{\rho}_k \) is most naturally implemented in the Matsubara-frequency representation because in diagrammatic calculations \( G \) is extracted from the Dyson equation solved at frequencies \( \xi_m = 2m\pi T \) for bosons and \( \xi_m = (2m + 1)\pi T \) for fermions \((m = 0, \pm 1, \pm 2, \ldots)\). The two representations are related by

\[
G(\xi_m) = \int_0^\beta e^{\xi_m \tau} G(\tau) d\tau .
\]

After \( G(\xi_m) \) is computed for a pre-selected finite set of frequencies the data is used for the least-squares fitting:

\[
\sum_{k=1}^{r} K(\xi_m, \omega_k) \tilde{\rho}_k \approx G(\xi_m) .
\]

Here \( K(\xi_m, \omega) \) is the Fourier transformed kernel [3]:

\[
K(\xi_m, \omega) = \frac{1}{i\xi_m - \omega} ,
\]

with \( \pm \) sign for fermions/bosons.

Symmetrized Discrete Lehmann Representation

We now discuss the Symmetrized Discrete Lehmann Representation based on the generic decomposition of \( G(\tau) \) into the sum of particle-hole symmetric, \( G^+(\tau) \), and particle-hole anti-symmetric, \( G^- (\tau) \), parts:

\[
G(\tau) = G^+(\tau) + G^-(\tau) ,
\]
where

\[ G^\pm (\tau) = \frac{G(\tau) \pm G(\beta - \tau)}{2}, \] (9)

are obeying \( G^\pm (\tau) = \pm G^\pm (\beta - \tau) \). Despite the fact that exact Lehmann representations for the two functions share the same spectral density \( \rho(\omega) \),

\[ G^\pm (\tau) = \int_{-\infty}^{\infty} K^\pm (\tau, \omega) \rho(\omega) d\omega, \] (10)

the discrete approximations are not supposed to be strictly related. Indeed, for discrete sums

\[ G^\pm_{\text{SDLR}} (\tau) \equiv \sum_{k=1}^{r} K^\pm (\tau, \omega_k^\pm) \hat{\rho}_k^\pm, \] (12)

the two sets of optimised frequencies, \( \omega_k^\pm \), obtained by the algorithm of Ref. 1, are not identical. Correspondingly, the basis functions and pole residues, \( \hat{\rho}_k^\pm \), also turn out to be different.

The utility of working with \( G^+ \) and \( G^- \) comes form the following properties:

\[ G^+(\xi_m) = \begin{cases} \text{Re} \ G(\xi_m), & \text{bosons}, \\ i \text{Im} \ G(\xi_m), & \text{fermions}, \end{cases} \] (13)

\[ G^-(\xi_m) = \begin{cases} i \text{Im} \ G(\xi_m), & \text{bosons}, \\ \text{Re} \ G(\xi_m), & \text{fermions}, \end{cases} \] (14)

\[ K^+(\xi_m, \omega) = \frac{2i\xi_m}{\omega^2 + \xi_m^2} (1 + e^{-\omega\beta}) \quad (\text{fermions}), \] (15)

\[ K^-(\xi_m, \omega) = \frac{2\omega}{\omega^2 + \xi_m^2} (1 + e^{-\omega\beta}) \quad (\text{fermions}), \] (16)

\[ K^+(\xi_m, \omega) = \frac{2\omega}{\omega^2 + \xi_m^2} (1 - e^{-\omega\beta}) \quad (\text{bosons}), \] (17)

\[ K^-(\xi_m, \omega) = \frac{2i\xi_m}{\omega^2 + \xi_m^2} (1 - e^{-\omega\beta}) \quad (\text{bosons}). \] (18)

It is easy to see that the problem of extracting the weights \( \hat{\rho}_k^\pm \) from \( G(\xi_m) \) splits into two independent real-valued problems, independently utilizing the real and imaginary parts of \( G(\xi_m) \).

We found that the original DLR scheme suffers from instabilities when the Dyson and self-energy equations are iterated, e.g. when implementing the so-called \( GW_0 \) approximation. The SDLR scheme is free of this problem. More details are provided in Ref. 2.

We implemented SDLR in a julia package open-sourced in the repository [https://github.com/numericalEFT/Lehmann.jl](https://github.com/numericalEFT/Lehmann.jl).

We also calculated the basis functions for various temperatures and accuracy bounds, and published them on [https://github.com/numericalEFT/Lehmann.jl/tree/main/basis](https://github.com/numericalEFT/Lehmann.jl/tree/main/basis).

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