Complexity of Hybrid Logics over Transitive Frames

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Abstract

This article examines the complexity of hybrid logics over transitive frames, transitive trees, and linear frames.

We show that satisfiability over transitive frames for the hybrid language extended with the downarrow operator \( \downarrow \) is NEXPTIME-complete. This is in contrast to undecidability of satisfiability over arbitrary frames for this language [2]. It is also shown that adding the @ operator or the past modality leads to undecidability over transitive frames.

This is again in contrast to the case of transitive trees and linear frames, where we show these languages to be nonelementarily decidable.

Moreover, we establish 2EXPTIME and EXPTIME upper bounds for satisfiability over transitive frames and transitive trees, respectively, for the hybrid Until/Since language. An EXPTIME lower bound is shown to hold for the modal Until language over both frame classes.

1 Introduction

Hybrid languages are extensions of modal logic that allow for naming and accessing states of a model explicitly. This renders hybrid logic an adequate representation formalism for many applications, where the basic modal and/or temporal languages do not suffice. Moreover, reasoning systems are easier to devise for hybrid than for modal logic.

Hybrid Logic, as well as the foundations of temporal logic, goes back to Arthur Prior [26]. Since then, many — more or less powerful — languages have been studied. Here we briefly introduce the extensions that shall concern us in this article.

Nominals are special atomic formulae that name states of models. They allow, for instance, for an axiom expressing irreflexivity, which cannot be captured by modal formulae: \( i \rightarrow \neg \Diamond i \).

The at operator @ can be used to directly jump to states named by nominals, independently of the accessibility relation. Hence, the above formula could also be written as \( @i \neg \Diamond i \).

With the help of the downarrow operator \( \downarrow \), it is possible to bind variables to states. Whenever \( \downarrow x \) is encountered during the evaluation of a formula, the variable \( x \) is bound to the current state \( s \). All occurrences of \( x \) in the scope of this \( \downarrow \) are treated like nominals naming \( s \). As an example, the formula \( \downarrow x . \neg \Diamond \Diamond x \) reads as: Name the current state \( x \) and make sure that it is not possible to go from \( x \) to \( x \) in exactly two steps. This is an axiom for asymmetry, another property not expressible in modal logic.

Combined with the @ operator, \( \downarrow \) leads to a very powerful language that can formulate many desirable properties and goes far beyond the scope of the simple nominal language. To give a more impressive example, we consider the Until operator. The formula \( \U(\varphi, \psi) \) reads as “there is a point in the future at which \( \varphi \) holds, and at all points between now and this point, \( \psi \) holds”. What the basic modal language is not able to express, can be achieved by the hybrid \( \downarrow @ \) language:

\[
\U(\varphi, \psi) \equiv \downarrow x . \Diamond \downarrow y . \varphi \land @x \Box (\Diamond y \rightarrow \psi).
\]

Besides more advanced temporal concepts such as “until” or “since”, hybrid temporal languages can express other desirable temporal notions such as “now”, “yesterday”, “today”, or “tomorrow”. Moreover, with hybrid logic one can capture many temporally relevant frame properties (besides the above...
Transitive Frames. Hybrid logic is interpreted over Kripke frames and models, as is modal logic. A frame consists of a set of states (points in time) and an accessibility relation $R$, where $x \mathrel{R} y$ says that $y$ is reachable from $x$ or, seen temporally, $y$ is in the future of $x$. We examine the computational complexity of satisfiability for several hybrid logics over transitive frames, transitive trees and linear frames.

Modal, hybrid, and first-order logics over transitive models have been studied recently in [3, 14, 32, 19, 20, 18, 11]. Although the complexity of hybrid (tense) logic has been extensively examined [7, 15, 2, 3, 13], there are highly expressive hybrid languages for whose satisfiability problems only results over arbitrary, but not over restricted, temporally relevant frame classes have been known.

We concentrate on transitive frames because transitivity is a property that the relations of many different temporal applications have in common, even if they differ in other properties such as tree-likeness, trichotomy, irreflexivity, or asymmetry. Transitivity can be seen as the minimal requirement in many applications, for example temporal verification.

But there are other reasons why this frame class is of interest, particularly in connection with computational complexity. In the special case of linear frames, nominals and $\circ$ can be simulated using the conventional modal operator and its converse. Hence, the basic hybrid language is as expressive over linear frames as the basic modal language. The $\downarrow$ operator is useless even on transitive trees, a representation of branching time. Over transitive frames, in contrast, these hybrid operators do make a difference. In this case, there are properties that can be expressed in the hybrid, but not in the modal language (see the irreflexivity example above). For this reason, the class of transitive frames can be regarded as a restricted frame class that is still general enough to separate hybrid from modal languages in terms of expressive power.

Yet another reason for considering precisely transitive frames will become clear in the next paragraph.

Complexity of Hybrid Logics. We use complexity classes NP, PSPACE, EXPTIME, NEXPTIME, $n$EXPTIME, $n \geq 2$, and coRE as known from [25]. A problem is nonelementarily decidable if it is decidable but not contained in any $n$EXPTIME.

It goes without saying that reasoning tasks for richer logics require more resources than those for simpler languages, such as the basic modal logic. We focus on one reasoning task, namely satisfiability. The modal and temporal satisfiability problems over arbitrary as well as over transitive frames are PSPACE-complete [23, 30]. If the “somewhere” modality $E$ is added, satisfiability becomes EXPTIME-complete over arbitrary frames [29]. For many, more restricted, frame classes, modal and temporal satisfiability is NP-complete [23, 21, 28]. In contrast, the known part of the complexity spectrum of hybrid satisfiability reaches up to undecidability.

Many complexity results for hybrid languages have been established in [2, 3]. It was proven in [2] that the hybrid language with nominals and $\circ$ has a PSPACE-complete satisfiability problem and that satisfiability for the hybrid tense language is EXPTIME-complete, even if $\circ$ or $E$ are added. The same authors show that these problems have the same complexity (or drop to PSPACE-complete or NP-complete, respectively) if the class of frames is restricted to transitive frames (or transitive trees, or linear frames, respectively) [2].

Moreover, they established EXPTIME-completeness of satisfiability for the hybrid Until/Since language. The complexity of this language over transitive frames and transitive trees, respectively, has been open. PSPACE-completeness over linear frames is known from [13]. We want to find out at which exact requirements to the frame classes the decrease from EXPTIME to PSPACE takes place.

Undecidability results for languages containing $\downarrow$ originate from [7, 15]. The strongest such result, namely for the pure nominal-free fragment of the $\downarrow$ language, is given in [2].

In recent work [34], it was demonstrated that decidability of the $\downarrow$ language can be regained by certain restrictions on the frame classes. Transitivity might be another property under which the $\downarrow$ language can be “tamed”, since it has already been observed that over transitive trees and linear orders, the $\downarrow$ operator on its own is useless.

New Road-Map Pages. This article establishes two groups of complexity results for hybrid languages over transitive frames, transitive trees, and linear frames.

First, we examine satisfiability of the hybrid $\downarrow$ language. Our most surprising result is the “taming” of this language over transitive frames: the satisfiability problem is NEXPTIME-complete. This high level of complexity is retained even over complete frames. We also show that enriching the language by the backward-looking modality $P$ or the $\circ$ operator leads to undecidability in the case of transitive frames.
The situation is different over transitive trees. Decidability, even for the richest ↓ language, is easy to see, but we will show it to be nonelementary if P or @ are added. For linear frames, this is already known in the temporal case. We prove that adding @ suffices to obtain nonelementary complexity.

As a second step, we consider satisfiability over transitive frames and transitive trees for the hybrid Until/Since-E language. We establish EXPTIME-hardness for the modal language extended with Until only. This is matched by an EXPTIME upper bound for the full language in the case of transitive trees. As for transitive frames, we give a 2EXPTIME upper bound.

Table 1 gives an overview of the satisfiability problems considered in this article (marked bold) and visualizes how our results arrange into a collection of previously known results. It makes use of the notation of hybrid languages introduced in Section 2. Complexity classes without addition stand for completeness results; “nonel.” stands for nonelementarily decidable. The work from which the results originate, is cited. Conclusions from surrounding results are abbreviated by “c.”.

| hybrid lang. | complexity over arbitrary frames | complexity over transitive frames | complexity over transitive trees | complexity over linear orders |
|--------------|----------------------------------|-----------------------------------|---------------------------------|-------------------------------|
| $\mathcal{H}L_{\alpha}$ | PSPACE [2] | PSPACE [3] | PSPACE [4] | NP [4] |
| $\mathcal{H}L_{F,P}$ | EXPTIME [2] | EXPTIME [3] | PSPACE [3] | NP [3] |
| $\mathcal{H}L_{E,P}$ | EXPTIME [3] | EXPTIME [3] | PSPACE [3] | NP [3] |
| $\mathcal{H}L_{\downarrow}$ | in 2EXPTIME [10], EXPTIME-hard [14] | EXPTIME [14,17] | PSPACE-hard [27] | |
| $\mathcal{H}L^{1}$ | coRE [2] | NEXPTIME [1] | PSPACE [3] | NP [13] |
| $\mathcal{H}L^{1,\alpha}$ | coRE [2] | coRE [11] | PSPACE [3] | NP [13] |
| $\mathcal{H}L_{F,P}^{1}$ | coRE [2] | coRE [11] | nonel. [12] | nonel. [13] |
| $\mathcal{H}L_{E,P}^{1}$ | coRE [2] | coRE (c.) | nonel. [12] | nonel. [13] |

Table 1. An overview of complexity results for hybrid logics. Numbers in round parentheses refer to the corresponding theorem.

Legend. This article is organized as follows. In Section 2 we give all necessary definitions and notations of modal and hybrid logic. We present the decidability and undecidability results for the hybrid ↓ languages in Sections 3 and 4. The hybrid Until/Since language is examined in Section 5. Section 6 contains some concluding remarks.

2 Modal and Hybrid Logic

We define the basic concepts and notations of modal and hybrid logic that are relevant for our work. The fundamentals of modal logic can be found in [6]; those of hybrid logic in [2, 5].

Modal Logic. Let PROP be a countable set of propositional atoms. The language $\mathcal{ML}$ of modal logic is the set of all formulae of the form

$$\varphi ::= p | \neg \varphi | \varphi \land \varphi' | \Box \varphi,$$

where $p \in$ PROP. We use the well-known abbreviations $\lor$, $\to$, $\leftrightarrow$, $\top$ (“true”), and $\bot$ (“false”), as well as $\Box \varphi ::= \neg \Diamond \neg \varphi$.

The semantics are defined via Kripke models. Such a model is a triple $\mathcal{M} = (\mathcal{M}, R, V)$, where $\mathcal{M}$ is a nonempty set of states, $R \subseteq \mathcal{M} \times \mathcal{M}$ is a binary relation—the accessibility relation—and $V : \text{PROP} \to \mathcal{P}(\mathcal{M})$ is a function—the valuation function. The structure $\mathcal{F} = (\mathcal{M}, R)$ is called a frame. Given a model $\mathcal{M} = (\mathcal{M}, R, V)$ and a state $m \in \mathcal{M}$, the satisfaction relation is defined by

$$\mathcal{M}, m \models p \quad \text{iff} \quad m \in V(p), \quad p \in \text{PROP},$$

$$\mathcal{M}, m \models \neg \varphi \quad \text{iff} \quad \mathcal{M}, m \not\models \varphi,$$

$$\mathcal{M}, m \models \varphi \land \psi \quad \text{iff} \quad \mathcal{M}, m \models \varphi \land \mathcal{M}, m \models \psi,$$

$$\mathcal{M}, m \models \Diamond \psi \quad \text{iff} \quad \exists n \in \mathcal{M}(mRn \& \mathcal{M}, n \models \psi).$$


A formula \( \varphi \) is satisfiable if there exist a model \( M = (M, R, V) \) and a state \( m \in M \), such that \( M, m \models \varphi \). If all states from \( M \) satisfy \( \varphi \), we write \( M \models \varphi \) and say that \( \varphi \) is globally satisfied by \( M \).

**Temporal Logic.** The language of temporal logic (tense logic) is the set of all formulae of the form

\[
\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi' \mid F \varphi \mid P \varphi,
\]

where \( p \in \text{PROP} \). It is common practice to use the abbreviations \( G \varphi := \neg F \neg \varphi \) and \( H \varphi := \neg P \neg \varphi \). Satisfaction for \( F \) and \( P \) formulae is defined by

\[
M, m \models F \psi \iff \exists n \in M(mRn & M, n \models \psi),
\]

\[
M, m \models P \psi \iff \exists n \in M(nRm & M, n \models \psi).
\]

Whenever one wants to speak not only of states accessible from the current state, but also of states “between” the current and some accessible state, one can make use of the binary operators \( U \) (“until”) and \( S \) (“since”), for which satisfaction is defined by

\[
M, m \models U(\varphi, \psi) \iff \exists n \in M(mRn & M, n \models \varphi & \forall s(mRsRn \Rightarrow M, s \models \psi)),
\]

\[
M, m \models S(\varphi, \psi) \iff \exists n \in M(nRm & M, n \models \varphi & \forall s(nRsRm \Rightarrow M, s \models \psi)).
\]

The \( U/S \) language is strictly stronger than the basic temporal language in the sense that \( F \) and \( P \) can be expressed by \( U \) and \( S \) (\( e.g. F \varphi = U(\varphi, \top) \)), but not vice versa.

In [3], a variant of the \( U/S \) operators, \( U^+ \) and \( S^+ \), is introduced. Satisfaction for \( U^+ \) (analogously for \( S^+ \)) is defined by

\[
M, m \models U^+(\varphi, \psi)
\]

\[
\text{iff } \exists n \in M(mRn & M, n \models \varphi & \forall s \in M(mR^+sR^+n \Rightarrow M, s \models \psi)),
\]

where \( R^+ \) is the transitive closure of \( R \). By means of these operators, they “simulated” transitive frames syntactically [3].

We go a step further and define another modification, \( U^{++} \) and \( S^{++} \), with the satisfaction relation

\[
M, m \models U^{++}(\varphi, \psi)
\]

\[
\text{iff } \exists n \in M(mR^+n & M, n \models \varphi & \forall s \in M(mR^+sR^+n \Rightarrow M, s \models \psi)),
\]

and analogously for \( S^{++} \). The resulting temporal language is an even closer simulation of transitivity, as we will see in Section 6.

**Hybrid Logic.** As indicated in the previous section, the hybrid language does not exist. Rather there are several extensions of the modal language allowing for explicit references to states and therefore being called hybrid. We introduce those hybrid languages that will interest us in this article. The definitions and notations are taken from [2, 3].

Let \( \text{NOM} \) be a countable set of *nominals*, \( \text{SVAR} \) be a countable set of *state variables*, and \( \text{ATOM} = \text{PROP} \cup \text{NOM} \cup \text{SVAR} \). It is common practice to write propositional atoms as \( p, q, \ldots \), nominals as \( i, j, \ldots \), and state variables as \( x, y, \ldots \). The full hybrid language \( \mathcal{HL}^{i,q} \) is the set of all formulae of the form

\[
\varphi ::= a \mid \neg \varphi \mid \varphi \land \varphi' \mid \Diamond \varphi \mid \Diamond t \varphi \mid \downarrow x \varphi,
\]

where \( a \in \text{ATOM}, t \in \text{NOM} \cup \text{SVAR} \), and \( x \in \text{SVAR} \).

A hybrid formula is called pure if it contains no propositional atoms; nominal-free if it contains no nominals; and a sentence if it contains no free state variables. (Free and bound are defined as usual; the only binding operator here is \( \downarrow \).)

A hybrid model is a Kripke model with the valuation function \( V \) extended to \( \text{PROP} \cup \text{NOM} \), where for all \( i \in \text{NOM}, |V(i)| = 1 \). Whenever it is clear from the context, we will omit the word “hybrid” when referring to models. In order to evaluate \( \downarrow \)-formulae, an assignment \( g: \text{SVAR} \rightarrow M \) for \( M \) is necessary. Given an assignment \( g \), a state variable \( x \) and a state \( m \), an \( x \)-variant \( g^x_m \) of \( g \) is defined by

\[
g^x_m(x') = \begin{cases} m & \text{if } x' = x, \\ g(x') & \text{otherwise.} \end{cases}
\]
For any atom $a$, let
\[ [V, g](a) = \begin{cases} \{g(a)\} & \text{if } a \in \text{SVAR}, \\ V(a) & \text{otherwise}. \end{cases} \]

The satisfaction relation for hybrid formulae is defined by
\[
\mathcal{M}, g, m \models a \iff m \in [V, g](a), \ a \in \text{ATOM},
\]
\[
\mathcal{M}, g, m \models \neg \varphi \iff \mathcal{M}, g, m \not\models \varphi,
\]
\[
\mathcal{M}, g, m \models \varphi \land \psi \iff \mathcal{M}, g, m \models \varphi \land \mathcal{M}, g, m \models \psi,
\]
\[
\mathcal{M}, g, m \models \Diamond \varphi \iff \exists n \in M(mRn \land \mathcal{M}, g, n \models \varphi),
\]
\[
\mathcal{M}, g, m \models \Diamond_{x} \varphi \iff \exists n \in M(\mathcal{M}, g, n \models \varphi \land [V, g](t) = \{n\}),
\]
\[
\mathcal{M}, g, m \models \downarrow x. \varphi \iff \mathcal{M}, g_{m}, m \models \varphi.
\]

A formula is satisfiable if there exist a model $\mathcal{M} = (M, R, V)$, an assignment $g$ for $\mathcal{M}$, and a state $m \in M$, such that $\mathcal{M}, g, m \models \varphi$.

We sometimes use the “somewhere” modality $E$ having the interpretation
\[ \mathcal{M}, g, m \models E \varphi \iff \exists n \in M(\mathcal{M}, g, n \models \varphi). \]

In this case, $\Diamond$ is needless, because $\Diamond_{t} \varphi$ can be expressed by $E(t \land \varphi)$. 

**First-Order Logic.** Modal and hybrid logic can be embedded into fragments of first-order logic. We will always use the standard notation of first-order logic.

We will make use of certain fragments of first-order logic and denote them in the style of \[10\]: $\text{ST}$ \([34]\) embeds hybrid logic into first-order logic and consists of two functions $\text{ST}_{x}$ and $\text{ST}_{y}$, defined recursively. Since $\text{ST}_{y}$ is obtained from $\text{ST}_{x}$ by exchanging $x$ and $y$, we only give $\text{ST}_{x}$ here.

\[
\text{ST}_{x}(p) = P(x), \quad \text{ST}_{x}(\Diamond \varphi) = \exists y(yRx \land \text{ST}_{y}(\varphi)),
\]
\[
\text{ST}_{x}(t) = t = x, \quad \text{ST}_{x}(\Diamond_{t} \varphi) = \exists y(y = t \land \text{ST}_{y}(\varphi)),
\]
\[
\text{ST}_{x}(\neg \varphi) = \neg \text{ST}_{x}(\varphi), \quad \text{ST}_{x}(\downarrow x. \varphi) = \exists v(x = v \land \text{ST}_{x}(\varphi)),
\]
\[
\text{ST}_{x}(\varphi \land \psi) = \text{ST}_{x}(\varphi) \land \text{ST}_{x}(\psi), \quad \text{ST}_{x}(E \varphi) = \exists y(\text{ST}_{y}(\varphi)),
\]

where $p \in \text{PROP}$, $t \in \text{NOM} \cup \text{SVAR}$, and $v \in \text{SVAR}$.

**Properties of Models and Frames.** Let $\mathcal{M} = (M, R, V)$ be a (Kripke or hybrid) model with the underlying frame $\mathcal{F} = (M, R)$. By $R^{+}$ we denote the transitive closure of $R$. For any subset $M' \subseteq M$, we write $R|M'$ and $V|M'$ for the restrictions of $R$ and $V$ to $M'$. We will refer to transitive frames or linear frames whenever we mean frames whose accessibility relation is transitive or a linear order, respectively. A linear order is an irreflexive, transitive, and trichotomous relation, where trichotomy is defined by

\[ (\forall xy(xRy \lor x = y \lor yRx)). \]

A frame $\mathcal{F}$ is a tree, if and only if it is acyclic and connected, and every point has at most one $R$-predecessor. A transitive tree is any $(M, R^{+})$, where $(M, R)$ is a tree.

**Satisfiability Problems.** Whenever we leave one or more operators out of the hybrid language, we omit the according superscript of $\mathcal{H}L$. If we proceed to a hybrid tense language, we add the suitable temporal operator(s) as subscript(s) to $\mathcal{H}L$. Analogously, when equipping the modal language with additional operators, we add them as sub- or superscripts to $\mathcal{M}L$.

For any hybrid language $\mathcal{H}L_{g}^{x}$, the satisfiability problem $\mathcal{H}L_{g}^{x}$-$\text{SAT}$ is defined as follows: Given a formula $\varphi \in \mathcal{H}L_{g}^{x}$, does there exist a hybrid model $\mathcal{M}$, an assignment $g$ for $\mathcal{M}$, and a state $m \in M$
3 Deciding $\mathcal{HL}^\downarrow$ over Transitive Frames

Areces, Blackburn, and Marx [2] proved that the downarrow operator $\downarrow$ turns the satisfiability problem for hybrid logics undecidable in general, even if no interaction with $@$ or $P$ is allowed.

We prove that undecidability vanishes if frames are required to be transitive.

**Theorem 1** The satisfiability problem for $\mathcal{HL}^\downarrow$ over transitive frames is complete for $\text{NEXPTIME}$.

Before we start with the proof, we have a first look at $\mathcal{HL}^\downarrow$ over transitive frames. Obviously, it has no finite model property. E.g., the following sentence requires a model containing an infinite chain of states labeled $p$.

$$p \land \lozenge p \land \Box \lozenge p \land \Box \downarrow x, \neg \Box x$$

Neither is it always possible to find a model that is a transitive tree. But, in some way, we can get close to this.

Although our models may contain cycles, transitivity ensures that all states in a cycle are pairwise connected. I.e., the subframe consisting of these states is complete. Therefore, we can view a model as consisting of maximal complete subframes and single states, that are connected in a transitive but acyclic fashion.

For every transitive model $\mathcal{M} = (M, R, V)$, we define its block tree $B(\mathcal{M}) = (M', R', V')$ as the structure obtained as follows. First, we replace each maximal complete subframe of $\mathcal{M}$, for short clique, with a single vertex. Second, we unravel the resulting structure into a (potentially infinite) transitive tree $T$. Then we replace each vertex of $T$ by (a copy of) the clique of $\mathcal{M}$ from which it is derived (Figure 1). Note that a block tree is not a tree, but we get a transitive tree if we view every clique as a node.

In the following, we are often interested in this underlying tree structure of a block tree and refer to the cliques of a block tree as *nodes*. For a state $s$, we denote its node by $u_s$. We say that a node $v$ is *below* a node $u$ if the states of $v$ are reachable from the states in $u$ (but not vice versa) and a node $v$ is a *child* of a node $u$, if $v$ is below $u$ but there is no node $w$ below $u$ and above $v$.

Likewise, we use the terms tree, subtree, and leaf for block tree, sub block tree, and leaf clique, respectively.

We have to be careful about how to treat nominals when unraveling a model. If $V(i) = \{s\}$ for a nominal $i$ and a state $s \in M$, we define $V'(i)$ to be the set of states from $M'$, that are copies via the unraveling of $s$. Therefore, $B(\mathcal{M})$ is not a model \[3\] as defined in Section \[2\] because nominals may hold at more than one state, but by viewing $i$ as a propositional atom it can be treated as a model. The satisfaction relation is not affected and it is therefore easy to see that this transformation preserves

\[2\] Note that we can always get a model for $\varphi$ from $B(\mathcal{M})$ by joining the states labeled with the same nominals, but this model might be different from $\mathcal{M}$. 

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**Figure 1:** A transitive model $\mathcal{M}$ and its block tree $B(\mathcal{M})$. The maximal complete subframes of $\mathcal{M}$ are marked by the dashed circles. Some edges caused by transitivity are left out for simplicity.
satisfaction of $\mathcal{HL}^1$-sentences: The relation associating each state of $\mathcal{M}$ with every copy in $B(\mathcal{M})$ is a quasi-injective bisimulation.$^3$

Lemma 2 For every transitive model $\mathcal{M}$, every state $s$ of $\mathcal{M}$, every copy $s'$ of $s$ in $B(\mathcal{M})$, and every $\mathcal{HL}^1$-sentence $\varphi$:

$$\mathcal{M}, s \models \varphi \iff B(\mathcal{M}), s' \models \varphi.$$ 

If, for some block tree $B$ and some state $s$ of $B$, $B, s \models \varphi$ holds, we refer to $B$ as a block tree model for $\varphi$.

Before we show how to use this tree-like structure to decide $\mathcal{HL}^1$ over transitive frames, we focus on complete subframes and show that their size can be bounded.

3.1 $\mathcal{HL}^1$ over Complete Frames

As complete subframes are a significant part of transitive models, we are now going to study the satisfiability problem of $\mathcal{HL}^1$ over complete frames.$^4$ The most important result for our purpose is an exponential-size model property of $\mathcal{HL}^1$ over complete frames.

We want to start by giving some insight why this property holds. In complete frames, the accessibility relation does not distinguish different states. Of course, states can be told apart if they are labeled differently by propositions. But the number of different labelings is exponentially bounded in the size of the formula. To use more states, we have to distinguish states labeled equally. This can only be done by assigning names to these states. But the number of states we can distinguish in this way is bounded by the number of different state variables and nominals used in the sentence.

While intuition is clear, we can prove this bound by observing that $\mathcal{HL}^1$ over complete frames is equivalent to the Monadic Class with equality ($\mathcal{MC}_=$), the fragment of first-order logic with only unary predicates, equality, and no function symbols.$^10$

Before we present this connection precisely, we have to make a note on models. Every hybrid model can be viewed as a relational structure for its first-order correspondence language. This first-order language usually contains a binary relation to reflect the accessibility relation. For complete frames, the accessibility relation is trivial and can be ignored, respectively added when going from $\mathcal{MC}_=$ to $\mathcal{HL}^1$.

Lemma 3 There are polynomial time functions mapping $\mathcal{HL}^1$ formulas $\varphi$ to $\mathcal{MC}_=$ formulas $\psi$ and vice versa such that $\varphi$ holds in a complete hybrid model $\mathcal{M}$ if and only if $\psi$ holds in the corresponding monadic structure.

Proof. The mapping from $\mathcal{HL}^1$ over complete frames to $\mathcal{MC}_=$ is based on the Standard Translation ST as defined in Section 2. The only rule of ST that uses the binary relation is the rule for the diamond operator: $ST_x(\Box \alpha) = \exists y (xRy \land ST_y(\alpha))$. But the right side can be reduced to $\exists y (ST_y(\alpha))$, since $xRy$ always holds on complete frames.

For the other direction, we give the rules of a reduction HT from $\mathcal{MC}_=$ to $\mathcal{HL}^1$.

$$\begin{align*}
\text{HT}(P(x)) &= \Diamond(x \land p) \\
\text{HT}(\neg \varphi) &= \neg \text{HT}(\varphi) \\
\text{HT}(\exists x. \varphi) &= \Diamond(\exists x. \text{HT}(\varphi))
\end{align*}$$

Note that both mappings can be computed in polynomial time and do not blow up formula size.$^\Box$

This result allows us to transfer complexity results and model properties for $\mathcal{MC}_=$ to $\mathcal{HL}^1$ over complete frames.

Theorem 4 $\mathcal{HL}^1$ over complete frames has the exponential-size model property and its satisfiability problem is complete for NEXPTIME.

The lower bound can be transferred directly to the case of transitive frames.

Corollary 5 The satisfiability problem for $\mathcal{HL}^1$ over transitive frames is hard for NEXPTIME.

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$^3$The notion of bisimulation has to be extended by requiring states carrying the same nominal to be related.

$^4$All results in this subsection hold for $\mathcal{HL}^1_{P}$, too.
Proof. We can force a transitive frame, more precisely, the subframe generated by the current state, to be complete by adding $\downarrow x, \Box \Diamond x$. In this way, we can give a reduction from the satisfiability problem of $MC=$ by mapping a formula $\varphi$ to $(\downarrow x, \Box \Diamond x) \land HT(\varphi)$. 

3.2 On Transitive Frames for $\mathcal{HL}^1$

Let us summarize what we have seen so far. For every $\mathcal{HL}^1$-formula satisfiable over transitive frames, instead of a transitive model we can consider its block tree. The size of the cliques in the block tree can be exponentially bounded in the size of the formula by Theorem 4.

The algorithm for testing $\mathcal{HL}^1$-satisfiability will essentially guess a model and verify that it is correct. As there is no finite model property, all models might be infinite. Nevertheless, we will show that, if the formula is satisfiable, there is always a model with a regular structure in which certain finite patterns are repeated infinitely often. This will allow us to find a finite representation of such a model.

To this end, Definition 6 captures the information about a state of a block tree that will be needed for the following. Intuitively, the $\varphi$-type of a state captures the information needed about its subtrees in order to evaluate any subformula of a given formula $\varphi$ at this state. Here, $\psi[\text{free}/\bot]$ is the sentence obtained from $\psi$ by replacing every free variable by $\bot$ and $\text{sub}(\varphi)$ is the set of all subformulæ of $\varphi$.

Definition 6 Let $\varphi$ be a $\mathcal{HL}^1$-sentence and $B = (M, R, V)$ a block tree model which is a model of $\varphi$. The $\varphi$-type of a state $s \in M$ is the set of all sentences from $\{\psi[\text{free}/\bot] \mid \Box \psi \in \text{sub}(\varphi)\}$ that hold at some state in the subtree rooted at $s$.

Note that states in the same clique have the same $\varphi$-type. Therefore, we can speak of the $\varphi$-type of a node. The type of a node is always a superset of the types of its children. More precisely, it is always the union of the types of the children together with the set of relevant formulæ which hold in the node itself.

When evaluating a subformula of a $\mathcal{HL}^1$-sentence $\varphi$ at some state $s$ of a block tree, all we need to know about states strictly below $u_s$ are the $\varphi$-types of the children of $u_s$. I.e., we can replace subtrees below $u_s$ by subtrees of the same $\varphi$-type. In the following lemma, for a block tree $B$ and two states $s_1, s_2$, $B[u_{s_1}/u_{s_2}]$ denotes the block tree resulting from $B$ by replacing the subtree rooted at $u_{s_1}$ by the subtree rooted at $u_{s_2}$. The result of this substitution is again a block tree.

Lemma 7 Let $\varphi$ be a $\mathcal{HL}^1$-sentence, $B = (M, R, V)$ a block tree model of $\varphi$ and $s_1$ and $s_2$ states of $M$ such that there is a path from $s_1$ to $s_2$ but not vice versa. For every formula $\psi \in \text{sub}(\varphi)$, every state $s_3$ of $M$ of the same $\varphi$-type as $s_2$, and every assignment $g$ that maps all free variables in $\psi$ to $s_1$ or states in $M$ preceding $s_1$:

$$B, g, s_1 \models \psi \iff B[u_{s_2}/u_{s_3}], g, s_1 \models \psi.$$

Proof. The proof is by induction on the structure of $\psi$. Most cases are trivial since $s_1$ is the only state that has to be considered, e.g., if $\psi = \psi_1 \land \psi_2$ we only need to know whether $\psi_1$ and $\psi_2$ hold at $s_1$.

The interesting case is $\psi = \Box \xi$. Here, we need to know whether $\xi$ holds at some state $s'$ reachable from $s_1$. This can only be affected by our substitution if $s'$ is in the replaced subtree, and therefore strictly below $s_1$. Thus, by our assumption on $g$, all free variables in $\psi$ are mapped to states different and not reachable from $s'$. We can conclude

$$B, g, s' \models \xi \iff B, g, s' \models \xi[\text{free}/\bot].$$

Hence $\xi$ holds at some state in the replaced subtree rooted at $u_{s_2}$, if and only if $\xi[\text{free}/\bot]$ is in the $\varphi$-type of $u_{s_2}$. The lemma follows because the new subtree has the same $\varphi$-type.

Note that we restricted the choice of $g$ only to those assignments that are relevant when evaluating the sentence $\varphi$.

We can use the previous lemma to get some nice restrictions on the block trees under consideration. E.g., we can assume that for every sentence in the type of a node, there is a witness in the node itself or in one of its children.

Lemma 8 Let $\varphi$ be a $\mathcal{HL}^1$-sentence satisfiable over transitive frames. Then there is a block tree model $B$ for $\varphi$, in which
3 DECIDING $\mathcal{HL}^↓$ OVER TRANSITIVE FRAMES

![Figure 2: An infinite block tree, its finite representation, and the infinite block tree obtained from this representation.](image)

- every node has at most $|\varphi|$ children,
- for every node $u$ with $\varphi$-type $t$ and every $\mathcal{HL}^↓$-sentence $\psi \in t$, $\psi$ holds at a state in $u$ or at a state in a child of $u$, and
- on every path from the root, infinite or ending at a leaf, every $\varphi$-type occurs only once or infinitely often.

Proof. Let $\varphi$ be a $\mathcal{HL}^↓$-sentence satisfiable over transitive frames and $B'$ a block tree model for $\varphi$.

A block tree satisfying the third condition can be obtained from $B'$ by applying Lemma 7. If there are two nodes $u$ and $v$ on a path, $v$ below $u$, that have the same $\varphi$-type, we can replace the subtree rooted at $u$ by the subtree rooted at $v$. This allows us to cut every finite repetition down to a single occurrence of a $\varphi$-type. The resulting structure is still a block tree model for $\varphi$.

Now, consider some node $u$ and its $\varphi$-type $t$. For every sentence $\psi \in t$, we select some state in the subtree rooted at $u$ such that $\psi$ holds at this state. Let $u_1, \ldots, u_k$ be the nodes of those states, that are not in $u$. It easy to see, following similar tracks as in Lemma 7, that the block tree obtained by removing the nodes below $u$ and inserting instead $u_1, \ldots, u_k$ as children of $u$ is again a block tree model of $\varphi$. Even more, the type of $u$ is not changed by this replacement.

By applying this argument top-down from the root, we get a block tree model for $\varphi$ satisfying the first two conditions. The third one is not affected by this transformation.

Note that some care is needed to make this approach work for infinite models. Basically, we must define a function that assigns to each node of the original model its set of witnesses. The resulting model is obtained by using this function in a straightforward fashion.

3.3 Deciding $\mathcal{HL}^↓$-SAT over Transitive Frames

We will now finish the proof of Theorem 1 by presenting a nondeterministic algorithm that decides $\mathcal{HL}^↓$-SAT over transitive frames in exponential time, basically by guessing and verifying the finite representation of a block tree model for a given $\mathcal{HL}^↓$-sentence $\varphi$.

Given a block tree $B$ with the properties of Lemma 8 we get a finite representation as follows. For each path of $B$ we consider the first node $v$ that has the same type as its parent node $u$. We replace the subtree below $v$ by a single state labeled with a reference to $u$, as shown in Figure 2. We need to keep $v$ because it might be the only witness for a formula in the $\varphi$-type of $u$ (cf. Lemma 8). Clearly, the resulting structure is finite.

By Lemma 7 and Lemma 8, we can get a block tree model from this representation by replacing each reference with the subtree rooted at the referenced node, i.e., essentially by an unraveling (Figure 2).

Due to Lemma 7 the size of the representation can be reduced even further. If there are two nodes $u$ and $v$ of the same $\varphi$-type which are both the first node of their type on their path from the root, we can replace the subtree rooted at $v$ with the subtree of $u$. I.e., whenever two nodes have the same $\varphi$-type, we can assume that their generated subtrees are equal. We have to check them only once.

This observation can be reflected in our representation by replacing every duplicate with a reference, as illustrated in Figure 8. This causes every type to appear at most twice in the representation, thus the number of nodes is at most exponential.
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Figure 3: Elimination of duplicates in the representation of a block tree.

$\mathcal{HL}^1\text{-SAT}(\varphi)$
1: Guess the finite representation $(M \cup C, R, V, f)$ of a block tree model for $\varphi$.
2: Guess a $\varphi$-type for every state in $C$.
3: Run MCFULL' on the states in $M$.
4: Compute the $\varphi$-type for every state in $M$ referenced by a state in $C$ via $f$.
5: Check for every state $s \in C$ that $f(s)$ has the same $\varphi$-type as $s$. If not, reject.
6: Accept iff $\varphi$ holds at some state in $M$.

Figure 4: Our algorithm for $\mathcal{HL}^1$-satisfiability.

Such a representation can be described by a structure $(M \cup C, R, V)$ such that the states in $C$ have no outgoing edges, and a function $f$ from $C$ to $M$. A state $s \in C$ stands for a repetition respectively duplication of the subtree rooted at $f(s)$, including states from $C$. This causes infinite repetition if $s$ is below $f(s)$. Note that a state in $C$ is a node of its own, in fact a leaf, and cannot be in the same complete subframe as a state in $M$.

Summing up, if $\varphi$ is satisfiable, there is a representation of a block tree model for $\varphi$ of size at most exponential in the length of $\varphi$. The first step of the algorithm presented in Figure 4 is to guess such a representation (step 1).

In order to obtain an algorithm which tests whether the representations indeed represents a model of $\varphi$, we describe how to modify the model checking algorithm MCFULL by Franceschet and de Rijke [12] to do so.

First, we deal with the states in $C$ (step 2), simply by guessing their $\varphi$-types. Next, the model checking algorithm MCFULL is used on the states in $M$ (step 3). We have to modify this algorithm in two respects. First, it has to use the information guessed for the states in $C$. Second, it should compute the $\varphi$-types of the states in $M$. To this end, it first evaluates the sentences resulting from subformulae of $\varphi$ by replacing free variables with $\bot$. We call this modified algorithm MCFULL'. The changes are straightforward.

After running MCFULL' the algorithm has computed for each state the set of formulae that hold at this state. These sets depend on the guesses in Step 2. Therefore, the algorithm has to verify their consistency. This can be done by comparing the $\varphi$-types of the states in $C$ with the $\varphi$-types of the referenced states (steps 4 and 5).

Finally, the algorithm checks if $\varphi$ holds at some state in $M$ (step 6).

Theorem 9 The algorithm $\mathcal{HL}^1\text{-SAT}$ presented in Figure 4 decides $\mathcal{HL}^1$-satisfiability over transitive frames nondeterministically in exponential time.

Proof. In Section 3.2 we have seen that for every satisfiable sentence $\varphi$ there is a block tree model as described in Lemma 8. We have also seen how to represent this block tree in a finite manner. The algorithm can guess this representation and the $\varphi$-types of the states in $C$. The computation of the $\varphi$-types of the states in $M$ works correctly, because we have a witness for every sentence in the type of some node in our representation. This is by Lemma 8 which ensures that witnesses are in the node of the state or in one of its children. Therefore, we cut below these witnesses when building the finite representation. Consequently, the algorithm will accept.

On the other hand, if the algorithm accepts, it is straightforward to construct a block tree model from the guessed representation. The only critical point for soundness is the verification of the $\varphi$-types guessed in Step 2, more precisely, the computation of the $\varphi$-types of the states in $M$. First, the $\varphi$-type
\[(xRy)^t = \exists abc (xRa \land bRa \land bRc \land yRc \land 0(x) \land 1(a) \land 2(b) \land 3(c) \land 0(y)),\]
\[(-\alpha)^t = -\langle \alpha \rangle^t,\]
\[(\alpha \land \beta)^t = \langle \alpha \rangle^t \land \langle \beta \rangle^t,\]
\[(\exists x \alpha)^t = \exists x (0(x) \land \alpha^t).\]

Figure 5: The translation function and a zig-zag transition.

of some state \(s \in M\) contains only sentences that hold at some state of the represented model below \(s\). This can be assured by looking only at states in \(M\) and not at states in \(C\). That the \(\varphi\)-type of \(s\) contains all sentences that hold below \(s\) can be assured by following the links represented by states in \(C\).

The first two steps of the algorithm can be performed in exponential time because the representation is of at most exponential size. That Step 3 runs in exponential time follows from Theorem 4.5 of [12], the truth of which is not affected by our modifications. The time bounds for the other steps follow again from the exponential size bound of the representation.

From Theorem [9] and Corollary [5] we can conclude Theorem [7].

4 Decidability of Richer Hybrid \(\downarrow\) Logics

This section is concerned with satisfiability over transitive frames, transitive trees, and linear frames for extensions of \(\mathcal{HL}^\downarrow\).

We investigate what happens if we extend the logic of the previous section with the \(\@\)-operator and/or the past modality \(P\). We will prove undecidability over transitive frames.

Therefore, we will consider these logics over more restricted frame classes, transitive trees and linear frames, where we will show them to be nonelementarily decidable.

4.1 Transitive Frames

Over transitive frames, we cannot sustain decidability if we enrich \(\mathcal{HL}^\downarrow\) with \(\@\) or the backward looking modality \(P\). We prove undecidability in both cases, making a detour via an undecidable fragment of first-order logic. The notation of such fragments is given in Section 2.

We proceed in two steps. First, we show that \([\text{all}, (4, 1)]\)-trans-SAT is undecidable. This is done by a reduction from \([\text{all}, (0, 1)]\)-SAT. The undecidability of the latter is a consequence of the undecidability of contained traditional standard classes [10]. The second step consists of reductions from \([\text{all}, (4, 1)]\)-trans-SAT to \(\mathcal{HL}^\downarrow, \alpha\)-trans-SAT and \(\mathcal{HL}^\uparrow, P\)-trans-SAT, respectively. To be more precise, the ranges of these reductions will be the fragments of the respective hybrid languages consisting of all nominal-free sentences.

Lemma 10 \([\text{all}, (4, 1)]\)-trans-SAT is undecidable.

Proof. In order to obtain the required reduction from \([\text{all}, (0, 1)]\)-SAT, we will transform a (not necessarily transitive) model satisfying \(\alpha\) into a transitive one. Simply taking the transitive closure in most cases adds new pairs to the interpretation of the relation and is not sufficient for keeping the information which pairs were in the “old” relation and which pairs were not. This problem does not arise if we instead use a variation of the zig-zag technique successfully applied in [3] for a reduction between a modal and a hybrid language. The core idea of this technique is to simulate an \(R\)-step \(t_1Rt_2\) in the original model \(M = (D, I)\) by a zig-zag transition in a model \(M' = (D', I')\), where \(I'(R)\) is transitive, as shown in Figure 5.

We define a translation function \((\cdot)^t\) using four extra predicate symbols \(0, 1, 2, 3\) according to Figure 5. The translation of the \(xRy\)-atoms exactly reflects the shown zig-zag transition. It is now straightforward to prove the following claim: For each formula \(\alpha\), \(\alpha\) is satisfiable iff \(f(\alpha)\) is satisfiable in some model that interprets \(R\) by a transitive relation.
Without loss of generality, we may assume that $\alpha$ has no free variables and that each variable is quantified exactly once. This can always be achieved by additional existential quantification and renaming, respectively.

"$\Rightarrow$". Suppose $\alpha$ is satisfied by some model $\mathcal{M} = (D, I)$. We construct a new model $\mathcal{M}^4 = (D^4, I^4)$, where $D^4 = D^0 \cup \cdots \cup D^3$ using $D^i = \{d^i \mid d \in D\}$, for $i = 0, 1, 2, 3$. The interpretation $I^4$ is defined by

$$I^4(R) = \{(x^0, x^1), (x^2, x^1), (x^2, x^3), (y^0, x^3) \mid (x, y) \in I(R)\} \quad \text{and} \quad I^4(P) = D^P, \quad P = 0, 1, 2, 3.$$ 

$I^4(R)$ codes an $I(R)$-transition from state $x$ to $y$ in $\mathcal{M}$ as a sequence of backward and forward transitions from $x_0$ to $y_0$ via $x_1, x_2, x_3$ as shown in Figure 5. It is easy to see that $I^4(R)$ is transitive, since there is no state with incoming and outgoing $I^4(R)$-edges.

We now show that for all subformulae $\beta(x_1, \ldots, x_m)$ of $\alpha$ and all $d_1, \ldots, d_m \in D$: $\mathcal{M} \models \beta[d_1, \ldots, d_m]$ iff $\mathcal{M}^4 \models \beta^t[d_0^1, \ldots, d_0^m]$. This immediately implies that $\mathcal{M}^4$ satisfies $\alpha^t$.

We proceed by induction on $\beta$. The base case, $\beta = xRy$, is clear from the construction of $I^4(R)$. The Boolean cases are obvious. For the case $\beta = \exists x \gamma$, we argue

$$\mathcal{M} \models \exists x \gamma[d_1, \ldots, d_m] \iff \exists d \in D(\mathcal{M} \models \gamma][d_1, \ldots, d_m, x \mapsto d])$$

The second and third line are equivalent due to the induction hypothesis. The equivalence of the third and fourth line follows from the definition of $R^4$; for the direction from below to above, one must take into account that because $x$ is interpreted by $d'$ and $0(x)$ is satisfied, $d'$ is indeed some $d^0$.

"$\Leftarrow$". Let $\mathcal{M} = (D, I)$ be a model satisfying $\alpha^t$, where $I(R)$ is transitive. We construct a new model $\mathcal{M}' = (D', I')$, where $D' = I(0)$ and

$$I'(R) = \{(d, e) \in (D')^2 \mid \exists abc \in D((d, a), (b, a), (b, c), (c, c) \in I(R))$$

& $a \in I(1) \& b \in I(2) \& c \in I(3))\}.$$

We now show that for all subformulae $\beta(x_1, \ldots, x_m)$ of $\alpha$ and all $d_1, \ldots, d_m \in D$: $\mathcal{M}' \models \beta[d_1, \ldots, d_m]$ iff $\mathcal{M} \models \beta^t[d_1, \ldots, d_m]$. This immediately implies that $\mathcal{M}'$ satisfies $\alpha$.

Again, the proof is via induction on $\beta$. The base case, $\beta = xRy$, is clear from the construction of $I'(R)$ and the fact that the translation of $xRy$ requires $0(x)$ and $0(y)$. The Boolean cases are obvious. For the case $\beta = \exists x \gamma$, we argue

$$\mathcal{M}' \models \exists x \gamma[d_1, \ldots, d_m] \iff \exists d \in D'(\mathcal{M}' \models \gamma][d_1, \ldots, d_m, x \mapsto d])$$

The induction hypothesis is applied between the second and third line. It is obvious that the third line implies the fourth; the backward direction is due to the fact that $x$ is interpreted by $d$ and $0(x)$ is satisfied, hence $d \in I(0)$.

This proves the above claim. Since $(\cdot)^t$ is an appropriate (even polynomial-time) reduction function, we have established undecidability for $\{all, (4, 1)\}$-trans-SAT.

\textbf{Theorem 11} $\mathcal{L}^{-, 0}_{(4, 1)}$-trans-SAT and $\mathcal{L}^\downarrow_{(4, 1)}$-trans-SAT are undecidable.
The composition of a sentence (see proof of Lemma 10). For the "spying" direction, suppose \( \alpha \) is satisfied by a model \( \mathcal{M} = (D, I) \). By adding the spy-point \( s \) to \( D \), we obtain the hybrid model \( \mathcal{M}^h = (M^h, R^h, V^h) \), where \( M^h = D \cup \{ s \} \), \( R^h = I(R) \cup \{ (s, d) | d \in D \} \), and \( V^h(p) = I(P) \). Clearly, \( \mathcal{M}^h \) satisfies \( \alpha \) at \( s \) — under any assignment, since \( f(\alpha) \) is a sentence. For the "eq" direction, suppose \( f(\alpha) \) is satisfied at state \( s \) of some hybrid model \( \mathcal{M} = (M, R, V) \). The composition of \( f(\alpha) \) enforces \( s \) to behave as the spy-point. It is easy to see that \( \mathcal{M}' = (M - \{ s \}, I) \), where \( I(R) = R|M_\lambda(s) \) and \( I(P) = V(p) \), satisfies \( \alpha \).

In the case of \( \mathcal{H}_{F,P}^\perp \), we must simulate the \( \otimes \) operator using \( P \), which is possible in the presence of a spy-point and transitivity. We simply re-define \( (\cdot)^t \) by

\[
(xRy)^t = P(i \land F(x \land Fy)), \quad (\neg \alpha)^t = \neg(\alpha^t), \\
(P(x))^t = P(i \land F(x \land p)), \quad (\alpha \land \beta)^t = \alpha^t \land \beta^t, \\
(\exists x \alpha)^t = \exists i \land \uparrow x \alpha^t.
\]

The rest of the proof is the same as for \( \mathcal{H}_{F,F}^\perp \).

4.2 Transitive Trees

Over transitive trees, where decidability of \( \mathcal{H}_{F}^\perp \) is trivial, even the extension \( \mathcal{H}_{F,F}^\perp \) is decidable. This is an immediate consequence of the decidability of the monadic second-order theory of the countably branching tree, S\( \omega \)S, \cite{19}. However, we have to face a nonelementary lower bound in both cases \( \mathcal{H}_{F,F}^\perp \) and \( \mathcal{H}_{F,P}^\perp \). This is obtained by a reduction from the nonelementarily decidable \( \mathcal{H}_{F,P}^\perp(\mathbb{N}, >)\)-SAT \cite{13}. In the latter notation, \( (\mathbb{N}, >) \) stands for the frame class consisting only of the frame \( (\mathbb{N}, >) \).

**Theorem 12** \( \mathcal{H}_{F,P}^\perp\text{-SAT}, \mathcal{H}_{F,F}^\perp\text{-SAT}, \text{and} \mathcal{H}_{F,F}^\perp\text{-tt-SAT} \) are nonelementarily decidable.

**Proof.** Decidability immediately follows from decidability of S\( \omega \)S, using the Standard Translation ST. For the nonelementary lower bound, we reduce \( \mathcal{H}_{F,P}^\perp(\mathbb{N}, >)\)-SAT to \( \mathcal{H}_{F,P}^\perp\text{-tt-SAT} \) and \( \mathcal{H}_{F,F}^\perp\text{-tt-SAT} \), respectively.

Let us first consider \( \mathcal{H}_{F,P}^\perp\text{-tt-SAT} \). The frame \( (\mathbb{N}, >) \) is a special case of a transitive tree. Our language is strong enough to enforce that a transitive tree model is based on \( (\mathbb{N}, >) \). All we have to do is to require two properties:

1. Every point has at most one direct successor.
2. The underlying frame is rooted.

Property (2) is expressed by PH\( \perp \). Property (1) can be formulated in the following way: For any point \( x \), whenever \( x \) has some successor, then we name one of the direct successors \( y \) and ensure that all direct successors of \( x \) satisfy \( y \). This translates as

\[
\lambda = F \leftarrow F^1 \downarrow y, P^1 G^1 y,
\]
where $F^1$, $P^1$, and $G^1$ can be expressed by means of $U$ and $S$, for example $F^1 \varphi \equiv U(\varphi, \bot)$. But $U(\varphi, \psi)$ can be simulated by $\downarrow x.F(\varphi \land H(P.x \to \psi))$; analogously for $S(\varphi, \psi)$.

Hence $\lambda$ is expressible in our language and of constant length. An appropriate reduction function $f$ is given by $f(\varphi) = \varphi \land \lambda \land H \land H \lambda \land \Phi \land \Phi \bot$. It is easy to observe that $\varphi$ is satisfiable in some linear model iff $f(\varphi)$ is satisfiable in some transitive tree.

In the case of $HL^{1,\alpha}$-tt-SAT, we first have to simulate the $P$ operator. This is done by a modified spy-point argument (for details of the spy-point technique see [12]). We simply label one point in the transitive tree by a fresh nominal $v$, and $i$. This is done in the following translation function $(\cdot)^t : HL_{F,P} \rightarrow HL^{1,\alpha}$.

$$a^t = a, \quad a \in ATOM, \quad (F \psi)^t = \diamond(\psi^t),$$
$$(-\psi)^t = \neg(\psi^t), \quad (P \psi)^t = \downarrow v.\diamond i.\diamond(\psi^t \land \diamond v),$$
$$(\psi_1 \land \psi_2)^t = \psi_1^t \land \psi_2^t, \quad (\downarrow x.\psi)^t = \downarrow x.(\psi^t).$$

It is easy to see that for each model $M$ based on $(\mathbb{N}, >)$, for each point $x \in \mathbb{N}$, and for each formula $\varphi \in HL_{F,P}$: whenever $i$ is true at the root 0, then $M, x \models \varphi \iff M, x \models \varphi^t$.

The point $s$ labelled $i$ represents the root of the frame $(\mathbb{N}, >)$. In the language $HL^{1,\alpha}$, it is not possible to express Property (2). This is in fact not necessary if we make sure that we never refer to the past of $s$ in our final translation of $\varphi$. Such a “wrong” reference can only appear when the $\mathbb{R}$ operator is used in connection with nominals occurring in $\varphi$. Let $NOM(\varphi) = NOM \cap Sub(\varphi)$, and let

$$\mu = \bigwedge_{j \in NOM(\varphi)} \mathbb{R} i . j.$$

Now the formula $\downarrow i.(\diamond \varphi^t \land \mu)$ does not contain any reference to any point before $s$.

It remains to ensure Property (1). This is done by replacing $\lambda$ by

$$\lambda' = \diamond \top \rightarrow \downarrow x.\diamond \top \rightarrow y.\mathbb{R}_x \square y$$

and again expressing $\diamond^1$ and $\square^1$ by means of $U$, which can be simulated as shown in Section [11]. Now, an appropriate reduction function is $f'$, where $f'(\varphi) = \downarrow i.(\diamond \varphi^t \land \mu \land \lambda' \land \square \lambda').$

4.3 Linear Frames

In the last part of this section we consider linear frames. A frame is called linear if it is irreflexive, transitive, and trichotomous. An important special case is the frame of the natural numbers with the usual ordering relation.

Hybrid $\downarrow$ languages over linear frames have already been addressed by Franceschet, de Rijke, and Schlingloff. They showed that satisfiability of $HL_{F,P}^{1,\alpha}$ is nonelementary, even over natural numbers. This result also holds for $HL_{F,P}^1$, because $\mathbb{R} i . \varphi$ can be simulated by

$$P(i \land \varphi) \lor (i \land \varphi) \lor F(i \land \varphi).$$

While complexity drops down to NP for $HL^1$, the last case, i.e. $HL_{F,P}^{1,\alpha}$, was left open. We will answer this question in the following.

Theorem 13 The satisfiability problem for $HL^{1,\alpha}$ over linear frames and over natural numbers is nonelementarily decidable.

Proof. Only the lower bound has to be shown. We do so by giving a reduction from the satisfiability problem of first order logic over strings, a problem long known to have nonelementary complexity [31].

Strings over a finite alphabet $\Sigma$ can be represented as

$$\{\{1, \ldots, n\}, \prec, (P_\sigma)_{\sigma \in \Sigma}\},$$

were $\prec$ is the usual ordering and $P_\sigma$ a unary relation for every $\sigma \in \Sigma$. As before, these structures can also be used for hybrid reasoning.
We give a translation $HT$ from first order logic into $\mathcal{HL}_{U,S}^{t}$, such that for every string $S$ and every first order sentence $\varphi$,

$$S \models \varphi \iff S' \models s.(HT(\varphi) \land \psi),$$

where $S'$ results from $S$ by adding a spy-point $s$ preceding all other states, what is ensured by $\psi$.

$$HT(P_\varphi(x)) = \circ_s(x \land p_\varphi) \quad HT(\neg \varphi) = \neg HT(\varphi)$$

$$HT(x = y) = \circ_s(x \land y) \quad HT(\varphi \land \psi) = HT(\varphi) \land HT(\psi)$$

$$HT(x < y) = \circ_s(x \land \Diamond y) \quad HT(\exists x.\varphi) = \circ_s(\Diamond x.HT(\varphi))$$

The only if direction is obvious. For the if direction, we have to state that $S'$ is a string and not just a linear frame. This is done by the formula

$$\psi = FL \land DISCRETE \land UNIQUE.$$

The precise meaning of $\psi$ is that the subframe generated by $s$, but without the state $s$ itself, is a string.

There has to be at least one state in the string and there has to be a start and an end of the string, i.e., a state only preceded by $s$ and a state without successor.

$$FL = (\Diamond \downarrow x.(\circ_s \Box \neg \Diamond x)) \land (\Diamond \Box \bot)$$

We also have to ensure that the frame is not dense.

$$DISCRETE = \Box(\Diamond \top \rightarrow (\downarrow x.\Diamond \downarrow y.\circ_x \Box \Diamond \neg y))$$

Finally, every state has to carry a unique label from the finite alphabet.

$$UNIQUE = \Box \bigwedge_{\sigma \in \Sigma} (\sigma \land \bigwedge_{\sigma' \neq \sigma} \neg \sigma')$$

All other properties of a string are already ensured by linearity. 

\[ \square \]

5 Hybrid Until/Since Logic over Transitive Frames and Transitive Trees

In this section, we will consider $\mathcal{HL}_{U,S}^{t,\Box}$-trans-SAT and $\mathcal{HL}_{U,S}^{t}$-tt-SAT. In [3] it was shown that $\mathcal{HL}_{U,S}^{\Box}$ is complete.

As for the lower bound, we establish a result as general as possible, namely EXPTIME-hardness of $\mathcal{ML}_{U}$-trans-SAT and $\mathcal{ML}_{U}$-tt-SAT.

**Theorem 14** $\mathcal{ML}_{U}$-trans-SAT and $\mathcal{ML}_{U}$-tt-SAT are EXPTIME-hard.

**Proof.** We will reduce the global satisfiability problem for $\mathcal{ML}$ to both our (local) problems $\mathcal{ML}_{U}$-trans-SAT and $\mathcal{ML}_{U}$-tt-SAT using the same reduction function. The global satisfiability problem is defined by

$$\mathcal{ML}$-GLOBSAT =

$$\{ \varphi \in \mathcal{ML} \mid \varphi \text{ is true in all states of some Kripke model } M \}.$$

Its EXPTIME-completeness is a direct consequence of the EXPTIME-completeness of $\mathcal{ML}^{E}$-SAT [29].

It may seem difficult to try reducing this problem over arbitrary frames to our satisfiability problem over transitive frames. The critical point lies in making a non-transitive model transitive: taking the transitive closure of its relation forces us to add new accessibilities that would disturb satisfaction of $\neg \Diamond$-formulae. Fortunately though, the $U$ operator can make us distinguish the accessibilities in the original model from those that have been added to make the relation transitive. Hence, a translation of $\Diamond \varphi$ should demand: “Make sure that the current state sees a state in which the translation of $\varphi$ holds, and that there is no state in between.” This translates as $U(\varphi^t, \bot)$ into the modal language.

To construct the required reduction, we define a translation function $(\cdot)^t : \mathcal{ML} \rightarrow \mathcal{ML}_{U}$ by

$$p^t = p, \quad p \in \text{PROP}, \quad (\varphi \land \psi)^t = \varphi^t \land \psi^t,$$

$$(\neg \varphi)^t = \neg (\varphi^t), \quad (\Diamond \varphi)^t = U(\varphi^t, \bot).$$

Using $(\cdot)^t$, we construct a reduction function $f : \mathcal{ML} \rightarrow \mathcal{ML}_{U}$ via $f(\varphi) = \varphi^t \land \Box \varphi^t$ (which is clearly computable in polynomial time). It is straightforward to prove the following two claims for each $\varphi \in \mathcal{ML}$.
(1) If $\varphi \in \mathcal{ML}$-GLOBSAT, then $f(\varphi) \in \mathcal{ML}_0$-tt-SAT.

(2) If $f(\varphi) \in \mathcal{ML}_0$-trans-SAT, then $\varphi \in \mathcal{ML}$-GLOBSAT.

Since each transitive tree is a transitive model, (1) and (2) imply the claim of this theorem.

(1). Suppose $\varphi$ is satisfied in all states of some Kripke model $\mathcal{M} = (M, R, V)$. By considering the submodel generated by some arbitrary state, we can assume w.l.o.g. that $\mathcal{M}$ has a root $w_0$.

Due to the tree model property [8] there exists a tree-like model (a model whose underlying frame is a tree) that satisfies $\varphi$ at all states, too. Hence we can suppose $\mathcal{M}$ itself to be tree-like. From this model, we construct $\mathcal{M}' = (M, R^+, V)$, which is clearly a transitive tree.

Because of the tree likeness of $\mathcal{M}$, we observe that for each pair $(w, v) \in R$, there exists no $u \in M$ between $w$ and $v$ in terms of $R^+$, i.e. no $u$ such that $wR^+u$ and $uR^+v$. By means of this observation, we show that for all states $m \in M$ and all formulae $\psi \in \mathcal{ML}$: $M, m \models \psi$ iff $M', m \models \psi^t$. This claim implies that $\mathcal{M}', w_0 \models \varphi^t \land \Box \varphi^t$. It is proven by induction on the structure of $\psi$. The only interesting case is $\psi = \Diamond \vartheta$, and the necessary argument can be summarized as follows.

$\mathcal{M}, m \models \Diamond \vartheta$

$\Leftrightarrow \exists n \in M(mRn \land \mathcal{M}, n \models \vartheta)$

$\Leftrightarrow \exists n \in M(mRn \land \mathcal{M}', n \models \vartheta^t)$

$\Leftrightarrow \exists n \in M(mR^+n \land \mathcal{M}', n \models \vartheta^t \land \neg \exists u \in M(mR^+u \land uR^+n))$

$\Leftrightarrow \mathcal{M}', m \models \top^t \land \bot$

In this argument, the equivalence of the first and the second line follows from the induction hypothesis. The second and third line are equivalent due to the above observation.

(2). Let $\mathcal{M} = (M, R, V)$ be a transitive model and $w_0 \in M$ such that $\mathcal{M}, w_0 \models f(\varphi)$. Again, we restrict ourselves to the submodel generated by $w_0$. Hence all states of $\mathcal{M}$ are accessible from $w_0$.

Define a new Kripke model $\mathcal{M}' = (M, R', V)$ from $\mathcal{M}$, where $R' = \{(w, v) \in R \mid \neg \exists u \in M(wRuRv)\}$. We show that for all states $m \in M$ and all formulae $\psi \in \mathcal{ML}$: $M', m \models \psi$ iff $\mathcal{M}, m \models \psi^t$. Again, we use induction on the structure of $\psi$ with the only interesting case $\psi = \Diamond \vartheta$ and the following argument.

$\mathcal{M}', m \models \Diamond \vartheta$

$\Leftrightarrow \exists n \in M(mR'n \land \mathcal{M}', n \models \vartheta)$

$\Leftrightarrow \exists n \in M((n = w_0 \text{ or } w_0Rn) \land mRn \land \neg \exists u(mRuRn) \land \mathcal{M}, n \models \vartheta^t)$

$\Leftrightarrow \mathcal{M}, m \models \top^t \land \bot$

The equivalence of the first and the second line is due to the fact that $\mathcal{M}$ is rooted, the definition of $R'$ as well as the induction hypothesis. Now, since $\mathcal{M}, w_0 \models \varphi^t \land \Box \varphi^t$, we conclude that for all states $x \in M$, $\mathcal{M}, x \models \varphi^t$. The previous claim implies that $\mathcal{M}'$ satisfies $\varphi$ at all states.

The upper bounds for $\mathcal{HL}_{U,5}^F$-trans-SAT and $\mathcal{HL}_{U}^{F,5}$-tt-SAT require separate treatment. As for $\mathcal{HL}_{U,5}$-trans-SAT, we use an embedding into an appropriate fragment of first-order logic. In order to eliminate transitivity, we “simulate” this semantic property by syntactic means, namely using the operators $U^+$ and $S^{++}$ defined in Section 2.

**Lemma 15** For any $X \subseteq \{\emptyset, E\}$, the problems $\mathcal{HL}_{U,5}^X$-trans-SAT and $\mathcal{HL}_{U,5}^{X,++}$-SAT are polynomially reducible to each other.

**Proof.** Either problem can be reduced to the other via a simple bijection $f: \mathcal{HL}_{U,5}^X \to \mathcal{HL}_{U,5}^{X,++}$ or its inverse, respectively. This function simply replaces every occurrence of $U$ (or $S$, respectively) in the input formula by $U^+$ (or $S^{++}$, respectively). Obviously, $f$ and $f^{-1}$ can be computed in polynomial time. It is straightforward to inductively verify the following two propositions.

(1) For every $\varphi \in \mathcal{HL}_{U,5}^X$: If $\varphi$ is satisfied in a state $m$ of some transitive model $\mathcal{M}$, then $\mathcal{M}, m \models f(\varphi)$.

(2) For all $\varphi \in \mathcal{HL}_{U,5}^{X,++}$: If $\varphi$ is satisfied in a state $m$ of some model $\mathcal{M} = (M, R, V)$, then the transitive model $\mathcal{M}' = (M, R^+, V)$ satisfies $f^{-1}(\varphi)$ at $m$. 


Now it is not difficult anymore to obtain a 2EXP\textsc{time} upper bound for $\mathcal{H}\mathcal{L}_{0,5}^{\oplus}$-\textsc{trans-sat} by an embedding into the loosely $\mu$-guarded fragment $\mu$\textsc{LGF} of first-order logic whose satisfiability problem is 2EXP\textsc{time}-complete \cite{17}. Only the $E$ operator requires a more careful analysis.

**Theorem 16** $\mathcal{H}\mathcal{L}_{0,5}^{E}$-\textsc{trans-sat} is in 2EXP\textsc{time}.

**Proof.** We first embed $\mathcal{H}\mathcal{L}_{0,5}^{\oplus}$ into the loosely $\mu$-guarded fragment $\mu$\textsc{LGF} of first-order logic \cite{17}. Since the satisfiability problem for $\mu$\textsc{LGF}-sentences is 2EXP\textsc{time}-complete \cite{17}, we obtain a 2EXP\textsc{time} upper bound for $\mathcal{H}\mathcal{L}_{0,5}^{\oplus}$-\textsc{trans-sat} by Lemma \cite{15}. As a second step, we will show a reduction from $\mathcal{H}\mathcal{L}_{0,5}^{\oplus}$-\textsc{trans-sat} to $\mathcal{H}\mathcal{L}_{0,5}^{E}$-\textsc{trans-sat}.

For the embedding into $\mu$\textsc{LGF}, we enhance the Standard Translation \textsc{st} (see Section \cite{2}) by the rule

$$\text{ST}_x \left( U^{++}(\varphi, \psi) \right) = \exists y [xR^+y \land \text{ST}_y(\varphi) \land \forall z ((xR^+z \land zR^+y) \rightarrow \text{ST}_z(\psi))]$$

and an analogous rule for $\text{ST}_x \left( S^{++}(\varphi, \psi) \right)$. $\text{ST}_y$ and $\text{ST}_z$ are defined by exchanging $x$, $y$, $z$ cyclically.

It remains to take care of the $R^+$ expressions. But $xR^+y$ can be expressed by

$$\left[ \text{LFP } W(x, y). (xRy \lor \exists z(zRy \land xWz)) \right] xy,$$

yielding a $\mu$\textsc{LGF}-sentence with three variables. (If $U^{++}$ operators are nested, variables can be "recycled".) The constants from the translations of nominals can be eliminated introducing new variables as shown in \cite{16}. The whole translation only requires time polynomial in the length of the input formula.

As for the more expressive language with $E$, we can embed the stronger language $\mathcal{H}\mathcal{L}_{0,5}^{E}$ into $\mathcal{H}\mathcal{L}_{0,5}^{\oplus}$ using a spy-point argument and exploiting the fact that we are restricted to transitive frames. A spy-point is a point $s$ that sees all other points and is named by the fresh nominal $i$. For details of this technique see \cite{21} [2].

By adding a spy-point to a transitive model, $E\varphi$ can be simulated by $\otimes_i \diamond \varphi$. Hence, if we take the translation $\cdot^t : \mathcal{H}\mathcal{L}_{0,5}^{E} \rightarrow \mathcal{H}\mathcal{L}_{0,5}^{\oplus}$ that simply replaces all occurrences of $E$ as shown, we obtain a reduction function $f : \mathcal{H}\mathcal{L}_{0,5}^{E}$-\textsc{trans-sat} $\rightarrow \mathcal{H}\mathcal{L}_{0,5}^{\oplus}$-\textsc{trans-sat} by setting $f(\varphi)i = i \land \neg \otimes i \land \diamond \varphi)^t$.

Clearly, $f$ is computable in polynomial time. It is straightforward to verify that $f$ is an appropriate reduction function: If $\varphi \in \mathcal{H}\mathcal{L}_{0,5}^{E}$ is satisfied at some point of some transitive model, add the spy-point $s$ and the according accessibilities. The new model satisfies $f(\varphi)s$. For the converse, if a transitive model satisfies $f(\varphi)$ at some point $s$, then $s$ must be a spy-point, and $\varphi^t$ is satisfied at another point $m$. Remove $s$ and the according accessibilities from the model and observe that it now satisfies $\varphi$ at $m$.

A note on the discrepancy between the upper and lower bound for $\mathcal{H}\mathcal{L}_{0,5}^{E}$-\textsc{trans-sat}. Since the 2EXP\textsc{time} result for $\mu$\textsc{LGF} in \cite{17} holds for sentences without constants only, constants — which arise from the translation of nominals — must be reformulated using new variables. This causes an unbounded number of variables in the first-order vocabulary, because we have no restriction on the number of nominals in our hybrid language.

Could we assume that the number of nominals were bounded, then the described reduction would yield guarded fixpoint sentences of bounded width. In this case, satisfiability is EXP\textsc{time}-complete \cite{17}. It is not known whether in the case of a bounded number of variables, but an arbitrary number of constants, satisfiability for $\mu$\textsc{LGF}-sentences also decreases from 2EXP\textsc{time} to EXP\textsc{time}, as it is the case for the fragment without the $\mu$ operator \cite{33}. If there were a positive answer to this question, an EXP\textsc{time} upper bound for our satisfiability problem would follow.

We now show that $\mathcal{H}\mathcal{L}_{0,5}^{E}$-\textsc{tt-sat} is in EXP\textsc{time}, using an embedding into $\mathcal{PDL}_{\text{tree}}$, the propositional dynamic logic for sibling-ordered trees introduced in \cite{21} [22]. Finite, node-labelled, sibling-ordered trees are the logical abstraction of XML (eXtensible Markup Language) documents. In \cite{1}, it was shown that satisfiability of $\mathcal{PDL}_{\text{tree}}$ formulae at the root of finite trees ($\mathcal{PDL}_{\text{tree-sat}}$) is decidable in EXP\textsc{time}.

Since we are going to give an embedding into $\mathcal{PDL}_{\text{tree}}$, we first introduce its syntax and semantics. $\mathcal{PDL}_{\text{tree}}$ is the language of propositional dynamic logic with four atomic programs $\text{left}$, $\text{right}$, $\text{up}$, and $\text{down}$ that are associated with the relations "left sister", "right sister", "parent", and "daughter" in trees. It consists of all formulae of the form

$$\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi' \mid (\pi) \varphi,$$
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where \( p \in \text{ATOM} \) and \( \pi \) is a program. Programs are defined by

\[
\pi ::= \text{left} | \text{right} | \text{up} | \text{down} | \pi'; \pi'' | \pi \cup \pi'' | \pi^* | \varphi?,
\]

where \( \varphi \) is a formula. We abbreviate \([\pi] \varphi := \neg(\pi)\neg \varphi \) and \( a^+ := a; a^* \) for atomic programs \( a \).

A \( \mathcal{PDL}_{\text{tree}} \) model is a multi-modal model \( M = (T, R_{\text{down}}, R_{\text{right}}, V) \), where \( T \) is a finite tree with an order relation on all immediate successors of any node, \( R_{\text{down}} \) is the successor relation and \( R_{\text{right}} \) is the “next-sister” relation. The set of relations is extended to arbitrary programs as follows:

\[
R_{\text{up}} = \overline{R_{\text{down}}}, \quad R_{\pi \cup \pi'} = R_{\pi} \cup R_{\pi'}, \quad R_{\pi'} = R_{\pi}^*, \quad R_{\varphi?} = \{(m, m) \mid M, m \models \varphi\}.
\]

The satisfaction relation for atomic formulae and Booleans is defined as for hybrid logic. The modal case is given by

\[
M, m \models (\pi) \varphi \iff \exists n \in T(m R_{\pi} n \& M, n \models \varphi).
\]

A formula \( \varphi \) is satisfiable if and only if there exists a model \( M = (T, R_{\text{down}}, R_{\text{right}}, V) \), such that \( M, m \models \varphi \), where \( m \) is the root of \( T \). For any \( \varphi \), let \( N(\varphi) \) be the set of all nominals occurring in \( \varphi \).

**Theorem 17** \( \mathcal{H}L_{0,5}^{\text{tt-SAT}} \) is in \textsc{Exptime}.

**Proof.** We reduce \( \mathcal{H}L_{0,5}^{\text{tt-trans-SAT}} \) to \( \mathcal{PDL}_{\text{tree}}^{\text{SAT}} \) and define a translation \((\cdot)^t : \mathcal{H}L_{0,5}^{\text{E}} \rightarrow \mathcal{PDL}_{\text{tree}} \) by

\[
\pi^t = \pi, \quad \text{up}^t = \text{up}, \quad \text{down}^t = \text{down}, \quad (E\varphi)^t = (\text{up}^*; \text{down}^* \) \varphi^t,
\]

\[
(\neg \varphi)^t = \neg(\varphi^t), \quad (U(\varphi, \psi))^t = (\text{down}^*; \varphi^t; \text{down}^* \) \varphi^t, \quad (S(\varphi, \psi))^t = (\text{up}^*; \varphi^t; \text{up}^* \) \varphi^t.
\]

(nominals are translated into atomic propositions).

Since \( \mathcal{PDL}_{\text{tree}} \) has no nominals, we must enforce that (the translation of) each nominal is true at exactly one point by requiring

\[
\nu(i) = (\text{down}^*)^t i \land [\text{down}^*]\left(i \rightarrow (\text{down}^*)^t i \land [\text{up}^*] \neg i \land [\text{up}^*; \text{left}^*; \text{down}^*] \neg i \land [\text{up}^*; \text{right}^*; \text{down}^*] \neg i\right)
\]

to hold for each nominal \( i \). As a reduction function, we have

\[
f(\varphi) = (\text{down}^*) \varphi^t \land \bigwedge_{i \in N(\varphi)} \nu(i).
\]

It is clear that \( f \) is computable in polynomial time and straightforward to show that \( f \) is an appropriate reduction function: Suppose, \( \varphi \) is satisfiable in some finite transitive tree model \( M = (M, R, V) \) based on the tree \( (M, R') \) with root \( w \). Then \( f(\varphi) \) is satisfiable in \( w \) of the \( \mathcal{PDL}_{\text{tree}} \) model based on the tree \( (M, R') \), equipped with the valuation \( V \). For the converse, if \( f(\varphi) \) is satisfied at the root of some \( \mathcal{PDL}_{\text{tree}} \) model \( M = (M, R_{\text{down}}, R_{\text{right}}, V) \), then \( \varphi^t \) is true at some point \( w \), and each nominal is true at exactly one point of \( M \). Hence \( (M, R_{\text{down}}^+, V) \) — where \( R_{\text{down}}^+ \) is the transitive closure of \( R_{\text{down}} \) — is a hybrid transitive tree model satisfying \( \varphi \) at \( w \).

Now there is one drawback in the reduction via \( f \). According to our definition of a tree, it is not necessary that a (transitive) tree is finite or has a root. A node can have infinitely many successors, or there may be an infinitely long forward or backward path from some point. For most practical applications these cases are certainly hardly of interest, but we strive for a more general result. If we do allow for infinite depth or width, the above translation into \( \mathcal{PDL}_{\text{tree}} \) — which is interpreted over finite, rooted trees — is not sufficient.

To overcome finiteness, it suffices to re-examine the proof for the \textsc{Exptime} upper bound of \( \mathcal{PDL}_{\text{tree}} \)-satisfiability in \cite{1}. This proof in fact covers a more general result, too, namely that satisfiability of \( \mathcal{PDL}_{\text{tree}} \) formulae over (not necessarily finite) trees is in \textsc{Exptime}.
To cater for the fact that “our” trees do not need to have roots, we first observe that satisfiability over rooted transitive trees is reducible to satisfiability over (arbitrary) transitive trees, because a root is expressible by \(PH\perp\) in our language. Since the lower bound from Theorem 14 holds with respect to rooted transitive trees, it also holds for arbitrary ones.

In order to obtain the upper bound with respect to arbitrary transitive trees, we propose a modification of the above reduction via \(f\). The basic idea is to turn the backward path from the node \(w\) (that is to satisfy \(\varphi\)) into a forward path, such that \(w\) becomes the root of the transformed model. Thus all predecessors of \(w\) (and their predecessors) become successors and must be marked by a fresh proposition \(\♭\). (See Figure 6.)

![Figure 6: Making predecessors successors.](image)

As a first step, we construct a new translation \((\cdot)^{♭}\) from \((\cdot)^{+}\) retaining all but the \(U/S\)-cases. For \(U/S\), we replace all occurrences of the programs \(\down\) and \(\up\) by programs that incorporate the new structure and the fact that for \(♭\)-nodes, their predecessors used to be their successors, and their \(♭\)-successors used to be their predecessors. We define

\[
(U(\varphi, \psi))^{♭} = ((\down'; \psi')^\ast; \down') \varphi^{♭}\quad \text{and} \quad (S(\varphi, \psi))^{♭} = ((\up'; \psi')^\ast; \up') \varphi^{♭},
\]

where

\[
dn' = (\down; \down; \down; \up) \cup (\down; \up; \up) \quad \text{and} \quad up' = (\up; \up; \down; \down; \down).
\]

Note that we do not change the translation of \(E\varphi\). The only thing that remains to do is to enforce that there is exactly one path at whose every node \(♭\) is true. This means that \(♭\) must be true at the root node and at exactly one successor of each node satisfying \(♭\). This can be expressed by

\[
\beta = \down^* \left( \down \rightarrow \left( \left[ \text{left}^* \right] \down \right) \right) \land \left( \down \rightarrow \left[ \text{down} \right] \down \right).
\]

It is now straightforward to show that \(f^{♭}\), given by \(f^{♭}(\varphi) = \varphi^{♭} \land \beta \land \down^* \nu(i)\), is indeed an appropriate reduction function.

(Note that \(\varphi^{♭}\) replaces \(\down^* \varphi^{♭}\), because we have turned \(w\) into the new root node.)

6 Conclusion

We have established two groups of complexity results for hybrid logics over three temporally relevant frame classes: transitive frames, transitive trees, and linear frames.

First, we have “tamed” \(\mathcal{H}L^{1}\) over transitive frames showing that \(\mathcal{H}L^{1-\text{trans-SAT}}\) is \(\text{NEXPTIME}\)-complete. The key step of our proof was to find a finite representation of transitive models for this logic. In contrast, we proved that \(\mathcal{H}L^{1,\alpha-\text{trans-SAT}}\) and \(\mathcal{H}L^{1,P-\text{trans-SAT}}\) are undecidable. In this context, the question arises whether the multi-modal variant of \(\mathcal{H}L^{1}\) over transitive frames is still decidable.

Over transitive trees, we showed three enrichments of \(\mathcal{H}L^{1}\) to be decidable, albeit nonelementarily, namely \(\mathcal{H}L^{1,\alpha-\text{tt-SAT}}\), \(\mathcal{H}L^{1,P,\alpha-\text{tt-SAT}}\), and \(\mathcal{H}L^{E,\alpha,P,\alpha-\text{tt-SAT}}\). Concerning linear frames, we obtained the same result for \(\mathcal{H}L^{1,\alpha-\text{lin-SAT}}\), an issue left open in [13].

In the third part of our work, we established an \(\text{EXPTIME}\) lower bound for \(\mathcal{M}L^{U-\text{trans-SAT}}\) and \(\mathcal{M}L^{U,\alpha-\text{tt-SAT}}\) and matched the latter with an \(\text{EXPTIME}\) upper bound for \(\mathcal{H}L_{U,S}^{E,\alpha-\text{tt-SAT}}\). This is the same
complexity as for satisfiability over arbitrary frames for the same language. As for $\mathcal{H}\mathcal{L}_{U,S}^{E}$-trans-SAT, we have given a 2EXPTIME upper bound. We conjecture EXPTIME-completeness.

Over linear frames, the complexity of hybrid U/S logic is still open. As a special case, satisfiability of $\mathcal{H}\mathcal{L}_{U,S}^{0}$ over $(\mathbb{N}, >)$ is known to be PSPACE-complete [13]. Moreover, in [27] it was shown that $\mathcal{ML}_{U}$ is PSPACE-complete with respect to general linear time. Over this frame class—which does not properly contain that of linear orders—the temporal hybrid languages from this paper could be re-examined in future work.

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