Length functions on groups and rigidity

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Abstract

Let $G$ be a group. A function $l : G \rightarrow [0, \infty)$ is called a length function if

1) $l(g^n) = |n|l(g)$ for any $g \in G$ and $n \in \mathbb{Z}$;

2) $l(hgh^{-1}) = l(g)$ for any $h, g \in G$; and

3) $l(ab) \leq l(a) + l(b)$ for commuting elements $a, b$.

Such length functions exist in many branches of mathematics, mainly as stable word lengths, stable norms, smooth measure-theoretic entropy, translation lengths on CAT(0) spaces and Gromov $\delta$-hyperbolic spaces, stable norms of quasi-cocycles, rotation numbers of circle homeomorphisms, dynamical degrees of birational maps, absolute values of Margulis invariants and so on. We study length functions on Lie groups, Gromov hyperbolic groups, arithmetic subgroups, matrix groups over rings and Cremona groups. As applications, we prove that every group homomorphism from an arithmetic subgroup of a simple algebraic $\mathbb{Q}$-group of $\mathbb{Q}$-rank at least 2, or a finite-index subgroup of the elementary group $E_n(R)$ ($n \geq 3$) over an associative ring, or the Cremona group $\text{Bir}(P^2_{\mathbb{C}})$ to any group $G$ having a purely positive length function must have its image finite. Here $G$ can be outer automorphism group Out($F_n$) of free groups, mapping class group $\text{MCG}(\Sigma_g)$, $\text{CAT}(0)$ groups or Gromov hyperbolic groups, or the group $\text{Diff}(\Sigma, \omega)$ of diffeomorphisms of a hyperbolic closed surface preserving an area form $\omega$.

0.1 Introduction

The rigidity phenomena have been studied for many years. The famous Margulis superrigidity implies any group homomorphism between irreducible lattices in semisimple Lie groups of real rank $\text{rk}_R(G) \geq 2$ are virtually induced by group homomorphisms between the Lie groups. Therefore, group homomorphisms from ‘higher’-rank irreducible lattices to ‘lower’-rank irreducible lattices normally have finite images. Farb, Kaimanovich and Masur [29] [10] prove that every homomorphism from an (irreducible) higher rank lattice into the mapping class group $\text{MCG}(\Sigma_g)$ has a finite image. Bridson and Wade [17] showed that the same superrigidity remains true if the target is replaced with the outer automorphism group Out($F_n$) of the free group. Mimura [50] proves that every homomorphism from Chevalley group over commutative rings to $\text{MCG}(\Sigma_g)$ or Out($F_n$) has a finite image. Many other rigidity results can be found, e.g. [53] [18] [19] [34] [58] and [57]. In this article, we study rigidity phenomena with the notion of length functions.

Let $G$ be a group. We call a function $l : G \rightarrow [0, \infty)$ a length function if

1) $l(g^n) = |n|l(g)$ for any $g \in G$ and $n \in \mathbb{Z}$;

2) $l(hgh^{-1}) = l(g)$ for any $h, g \in G$;

3) $l(ab) \leq l(a) + l(b)$ for elements $a, b$ satisfying $ab = ba$. 
Such length functions exist in geometric group theory, dynamical systems, algebra, algebraic geometry and many other branches of mathematics. For example, the following functions \( l \) are length functions (see Section 2 for more examples with details).

- (The stable word lengths) Let \( G \) be a group generated by a symmetric (not necessarily finite) set \( S \). For any \( g \in G \), the word length \( \phi_S(w) = \min \{ n \mid g = s_1s_2 \cdots s_n \} \) is the minimal number of elements of \( S \) whose product is \( g \). The stable length is defined as
  \[
l(g) = \lim_{n \to \infty} \frac{\phi_S(g^n)}{n}.
  \]

- (Stable norms) Let \( M \) be a compact Riemannian manifold and \( G = \text{Diff}(M) \) the diffeomorphism group consisting of all self-diffeomorphisms. For any diffeomorphism \( f : M \to M \), let
  \[
  \|f\| = \sup_{x \in M} \|D_x f\|,
  \]
  where \( D_x f \) is the induced linear map between tangent spaces \( T_x M \to T_{f(x)} M \). Define
  \[
l(f) = \max \left\{ \lim_{n \to +\infty} \frac{\log \|f^n\|}{n}, \lim_{n \to +\infty} \frac{\log \|f^{-n}\|}{n} \right\}.
  \]

- (Smooth measure-theoretic entropy) Let \( M \) be a \( C^\infty \) closed Riemannian manifold and \( G = \text{Diff}_2^\mu(M) \) consisting of diffeomorphisms of \( M \) preserving a Borel probability measure \( \mu \). Let \( l(f) = h_\mu(f) \) be the measure-theoretic entropy, for any \( f \in G = \text{Diff}_2^\mu(M) \).

- (Translation lengths) Let \((X, d)\) be a metric space and \( G = \text{Isom}(X) \) consisting of isometries \( \gamma : X \to X \). Fix \( x \in X \), define
  \[
l(\gamma) = \lim_{n \to \infty} \frac{d(x, \gamma^nx)}{n}.
  \]
  This contains the translation lengths on CAT(0) spaces and Gromov \( \delta \)-hyperbolic spaces as special cases.

- (Average norm for quasi-cocycles) Let \( G \) be a group and \( E \) be a Hilbert space with an \( G \)-action by linear isometries. A function \( f : G \to E \) is a quasi-cocycle if there exists \( C > 0 \) such that
  \[
  \|f(gh) - f(g) - gf(h)\| < C
  \]
  for any \( g, h \in G \). Let \( l : G \to [0, +\infty) \) be defined by
  \[
l(g) = \lim_{n \to \infty} \frac{\|f(g^n)\|}{n}.
  \]

- (Rotation numbers of circle homeomorphisms) Let \( \mathbb{R} \) be the real line and \( G = \text{Homeo}_\mathbb{Z}(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \text{ is a monotonically increasing homeomorphism such that } f(x+n) = f(x) + n \text{ for any } n \in \mathbb{Z} \} \). For any \( f \in \text{Homeo}_\mathbb{Z}(\mathbb{R}) \) and \( x \in [0, 1) \), the translation number is defined as
  \[
l(f) = \lim_{n \to \infty} \frac{f^n(x) - x}{n}.
  \]
(Asymptotic distortions) Let \( f \) be a \( C^{1+bv} \) diffeomorphism of the closed interval \([0, 1]\) or the circle \( S^1 \). ("bv" means derivative with finite total variation.) The asymptotic distortion of \( f \) is defined (by Navas [54]) as

\[
\ell(f) = \lim_{n \to \infty} \frac{1}{n} \text{var}(\log Df^n).
\]

This gives a length function \( l \) on the group \( \text{Diff}^{1+bv}(M) \) of \( C^{1+bv} \) diffeomorphisms for \( M = [0, 1] \) or \( S^1 \).

(Dynamical degree) Let \( \mathbb{C}P^n \) be the complex projective space and \( f : \mathbb{C}P^n \to \mathbb{C}P^n \) be a birational map given by

\[
(x_0 : x_1 : \cdots : x_n) \mapsto (f_0 : f_1 : \cdots : f_n),
\]

where the \( f_i \)'s are homogeneous polynomials of the same degree without common factors. The degree of \( f \) is \( \deg f = \deg f_i \). Define

\[
\ell(f) = \max\{ \lim_{n \to \infty} \log \deg(f^n)^{\frac{1}{n}}, \lim_{n \to \infty} \log \deg(f^{-n})^{\frac{1}{n}} \}.
\]

This gives a length function \( l : \text{Bir}(\mathbb{C}P^n) \to [0, +\infty) \). Here \( \text{Bir}(\mathbb{C}P^n) \) is the group of birational maps, also called Cremona group.

(Margulis invariant) Let \( \text{Aff}(\mathbb{R}^3) = \mathbb{R}^3 \rtimes \text{GL}_3(\mathbb{R}) \) be the affine automorphism group of the 3-dimensional space \( \mathbb{R}^3 \) and

\[
\phi : \Gamma \to \text{Aff}(\mathbb{R}^3),
\]

\[
\gamma \mapsto (u(\gamma), \Phi(\gamma)),
\]

a group homomorphism. Let \( B \) be the bilinear form on \( \mathbb{R}^3 \) defined by

\[
B(v, w) = v_1w_1 + v_2w_2 - v_3w_3,
\]

with the isometric group \( \text{O}(2, 1) \). Suppose that the linear part \( \Phi(\Gamma) \) is contained in \( \text{SO}(2, 1) \) and each non-trivial element \( \Phi(\gamma) \) has eigenvalues \( \lambda < 1 < 1/\lambda \) for some \( \lambda > 0 \) (i.e. \( \Phi(\gamma) \) is hyperbolic). Denote by \( x^-(\gamma), x^0(\gamma), x^+(\gamma) \) the corresponding unit eigenvectors such that \( B(x^0(\gamma), x^0(\gamma)) = 1 \) and \( (x^-(\gamma), x^0(\gamma), x^+(\gamma)) \) is positively oriented basis. The Margulis invariant (defined in [51, 52]) is

\[
\alpha_\phi : \Gamma \to \mathbb{R},
\]

\[
\gamma \mapsto B(x^0(\gamma), u(\gamma)).
\]

The absolute value function \( |\alpha_\phi| \) is a length function on \( \Gamma \).

The terminologies of length functions are used a lot in the literature (eg. [31], [23], [24]). However, they usually mean different things from ours (in particular, it seems that the subadditivity condition 3 has not been required only for commuting elements before).

Our first observation is the following result on the vanishing of length functions.

\textbf{Theorem 0.1} Let \( G_A = \mathbb{Z}^2 \rtimes_A \mathbb{Z} \) be an abelian-by-cyclic group, where \( A \in \text{SL}_2(\mathbb{Z}) \).

(i) When the absolute value of the trace \( |\text{tr}(A)| > 2 \), any length function \( l : \mathbb{Z}^2 \rtimes_A \mathbb{Z} \to \mathbb{R}_{\geq 0} \) vanishes on \( \mathbb{Z}^2 \).
When \(|\text{tr}(A)| = 2\) and \(A \neq I_2\), any length function \(l : \mathbb{Z}^2 \rtimes_A \mathbb{Z} \to \mathbb{R}_{\geq 0}\) vanishes on the direct summand of \(\mathbb{Z}^2\) spanned by eigenvectors of \(A\).

**Corollary 0.2** Suppose that the semi-direct product \(G_A = \mathbb{Z}^2 \rtimes_A \mathbb{Z}\) acts on a compact manifold by Lipschitz homeomorphisms (or \(C^2\)-diffeomorphisms, resp.). The topological entropy \(h_{\text{top}}(g) = 0\) (or Lyapunov exponents of \(g\) are zero, resp.) for any \(g \in \mathbb{Z}^2\) when \(|\text{tr}(A)| > 2\) or any eigenvector \(g \in \mathbb{Z}^2\) when \(|\text{tr}(A)| = 2\).

It is well-known that the central element in the integral Heisenberg group \(G_A\) (for \(A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\)) is distorted in the word metric. Lubotzky, Mozes, Ragunathan [45] obtain a general distortion result in the word metric for the semi-direct product \(\mathbb{Z}^2 \rtimes_A \mathbb{Z}\). When the Heisenberg group \(G_A\) acts on a \(C^\infty\) compact Riemannian manifold, Hu-Shi-Wang [37] proves that the topological entropy and all Lyapunov exponents of the central element are zero. These results are special cases of Theorem 0.1 and Corollary 0.2 by choosing special length functions.

We give characterizations of length functions on Lie groups. Our next result is that there is essentially only one length function on the special linear group \(\text{SL}_2(\mathbb{R})\):

**Theorem 0.3** Let \(G = \text{SL}_2(\mathbb{R})\). Any length function \(l : G \to [0, +\infty)\) continuous on the subgroup \(\text{SO}(2)\) and the diagonal subgroup is proportional to the translation function

\[
\tau(g) := \inf_{x \in X} d(x, gx),
\]

where \(X = \text{SL}_2(\mathbb{R}) / \text{SO}(2)\) is the upper-half plane.

More generally, we study length functions on Lie groups. Let \(G\) be a connected semisimple Lie group whose center is finite with an Iwasawa decomposition \(G = KAN\). Let \(W\) be the Weyl group, i.e. the quotient group of the normalizers \(N_K(A)\) modulo the centralizers \(C_K(A)\). Our second result shows that a length function \(l\) on \(G\) is uniquely determined by its image on \(A\).

**Theorem 0.4** Let \(G\) be a connected semisimple Lie group whose center is finite with an Iwasawa decomposition \(G = KAN\). Let \(W\) be the Weyl group.

(i) Any length function \(l\) on \(G\) that is continuous on the maximal compact subgroup \(K\) is determined by its image on \(A\).

(ii) Conversely, any length function \(l\) on \(A\) that is \(W\)-invariant (i.e. \(l(w \cdot a) = l(a)\)) can be extended to be a length function on \(G\) that vanishes on the maximal compact subgroup \(K\).

Using the notion of length functions, we treat the rigidity phenomena in a unified approach. A length function \(l : G \to [0, \infty)\) is called purely positive if \(l(g) > 0\) for any infinite-order element \(g\). A group \(G\) is called virtually poly-positive, if there is a finite-index subgroup \(H < G\) and a subnormal series

\[
1 = H_n \triangleleft H_{n-1} \triangleleft \cdots \triangleleft H_0 = H
\]

such that every finitely generated subgroup of each quotient \(H_i/H_{i+1}\) \((i = 0, \ldots, n - 1)\) has a purely positive length function. Our following results are on the rigidity of group homomorphisms.
Theorem 0.5 Let $\Gamma$ be an arithmetic subgroup of a simple algebraic $\mathbb{Q}$-group of $\mathbb{Q}$-rank at least 2. Suppose that $G$ is virtually poly-positive. Then any group homomorphism $f : \Gamma \to G$ has its image finite.

Theorem 0.6 Let $G$ be a group having a finite-index subgroup $H < G$ and a subnormal series

$$1 = H_n < H_{n-1} < \cdots < H_0 = H$$

satisfying that

(i) every finitely generated subgroup of each quotient $H_i/H_{i+1}$ ($i = 0, ..., n - 1$) has a purely positive length function; and

(ii) any torsion abelian subgroup in every finitely generated subgroup of each quotient $H_i/H_{i+1}$ ($i = 0, ..., n - 1$) is finitely generated.

Let $R$ be a finitely generated associative ring with identity and $E_n(R)$ the elementary subgroup. Suppose that $\Gamma < E_n(R)$ is finite-index subgroup. Then any group homomorphism $f : \Gamma \to G$ has its image finite when $n \geq 3$.

Corollary 0.7 Let $\Gamma$ be an arithmetic subgroup of a simple algebraic $\mathbb{Q}$-group of $\mathbb{Q}$-rank at least 2, or a finite-index subgroup of the elementary subgroup $E_n(R)$ ($n \geq 3$) for an associative ring $R$. Then any group homomorphism $f : E \to G$ has its image finite. Here $G$ is one of the following groups:

- a Gromov hyperbolic group,
- $\text{CAT}(0)$ group,
- automorphism group $\text{Aut}(F_k)$ of a free group,
- outer automorphism group $\text{Out}(F_k)$ of a free group,
- mapping class group $\text{MCG}(\Sigma_g)$ ($g \geq 2$), or
- the group $\text{Diff}(\Sigma, \omega)$ of diffeomorphisms of a closed surface preserving an area form $\omega$.

Theorem 0.8 Suppose that $G$ is virtually poly-positive. Let $R$ be a finitely generated associative ring of characteristic zero such that any nonzero ideal is of a finite index (e.g. the ring of algebraic integers in a number field). Suppose that $S < E_n(R)$ is a finite-index subgroup of the elementary group. Then any group homomorphism $f : S \to G$ has its image finite when $n \geq 3$.

Corollary 0.9 Let $R$ be an associative ring of characteristic zero such that any nonzero ideal is of a finite index. Any group homomorphism $f : E \to G$ has its image finite, where $E < E_n(R)$ is finite-index subgroup and $n \geq 3$. Here $G$ is one of the followings:

- a $\text{CAT}(0)$ group or more generally a semi-hyperbolic group,
- a group acting properly semi-simply on a $\text{CAT}(0)$ space, or
- a group acting properly semi-simply on a $\delta$-hyperbolic space.
Some relevant cases of Theorem 0.6 and Theorem 0.8 are already established in the literature, whose proofs are usually based on the distortions in the word length. Bridson and Wade [17] showed that any group homomorphism from an irreducible lattice in a semisimple Lie group of real rank $\geq 2$ to the mapping class group $\text{MCG}(\Sigma_g)$ must have its image finite. However, Theorem 0.5 can never hold when $\Gamma$ is a cocompact lattice, since a cocompact lattice has its stable word length purely positive. When the length functions involved in the virtually poly-positive group $G$ are required to be stable word lengths, Theorem 0.5 holds more generally for $\Gamma$ non-uniform irreducible lattices a semisimple Lie group of real rank $\geq 2$ (see Proposition 8.3). When the length functions involved in the virtually poly-positive group $G$ are given by a particular kind of quasi-cocycles, Theorem 0.5 holds more generally for $\Gamma$ with property TT (cf. Py [58], Prop. 2.2). Haettel [34] proves that any action of a high-rank lattice on a Gromov-hyperbolic space is elementary (i.e. either elliptic or parabolic). Guirardel and Horbez [34] prove that every group homomorphism from a high-rank lattice to the outer automorphism group of the torsion-free hyperbolic group has a finite image. Compared with these results, our target group $G$ and the source group $E_n(\mathbb{R})$ (can be defined over any non-commutative ring) in Theorem 0.6 are much more general. The inequalities of $n$ in Theorem 0.6, Theorem 0.8 and Corollary 0.7, Corollary 0.9 can not be improved, since $\text{SL}_2(\mathbb{Z})$ is hyperbolic. The group $\Gamma$ in Corollary 0.7 has Kazhdan’s property T (i.e. an arithmetic subgroup of a simple algebraic $\mathbb{Q}$-group of $\mathbb{Q}$-rank at least 2, or a finite-index subgroup of the elementary subgroup $E_n(R)$, $n \geq 3$, for an associative ring $R$ has Kazhdan’s property T by [26]). However, there exist hyperbolic groups with Kazhdan’s property T (cf. [33], Section 5.6). This implies that Corollary 0.7 does not hold generally for groups $\Gamma$ with Kazhdan’s property T. Franks and Handel [30] prove that any group homomorphism from a quasi-simple group containing a subgroup isomorphic to the three-dimensional integer Heisenberg group, to the group $\text{Diff}(\Sigma, \omega)$ of diffeomorphisms of a closed surface preserving an area form $\omega$, has its image finite (cf. Lemma 8.2).

We now study length functions on the Cremona groups.

**Theorem 0.10** Let $\text{Bir}(\mathbb{P}_k^n)$ ($n \geq 2$) be the group of birational maps on the projective space $\mathbb{P}_k^n$ over an algebraically closed field $k$. Any length function $l : \text{Bir}(\mathbb{P}_k^n) \to [0, +\infty)$ vanishes on the automorphism group $\text{Aut}(\mathbb{P}_k^n) = \text{PGL}_{n+1}(k)$.

When $n = 2$, a result of Blanc and Furter [11] (page 193 and Proposition 4.41) implies that there are three length functions $l_1, l_2, l_3$ on $\text{Bir}(\mathbb{P}_k^2)$ such that any element $g \in \text{Bir}(\mathbb{P}_k^2)$ satisfying $l_1(g) = l_2(g) = l_3(g) = 0$ is either finite or conjugate to an element in $\text{Aut}(\mathbb{P}_k^2)$. This implies that the automorphism group $\text{Aut}(\mathbb{P}_k^n)$ (when $k = 2$) is one of the ‘largest’ subgroups of $\text{Bir}(\mathbb{P}_k^2)$ on which every length function vanishes. The distortion of elements in the word metric is completely obtained by Cantat and Cornulier [22].

**Corollary 0.11** Let $G$ be a virtually poly-positive group. Any group homomorphism $f : \text{Bir}(\mathbb{P}_k^2) \to G$ is trivial, for an algebraically closed field $k$.

In particular, Corollary 0.11 implies that any quotient group of $\text{Bir}(\mathbb{P}_k^2)$ can act properly semisimply neither on a Gromov $\delta$-hyperbolic space nor a CAT(0) space. This is interesting, considering the following facts. There are (infinite-dimensional) hyperbolic
space and cubical complexes, on which Bir($\mathbb{P}_k^n$) acts isometrically (see [21], Section 3.1.2 and [47]). The Cremona group Bir($\mathbb{P}_k^n$) is sub-quotient universal: every countable group can be embedded in a quotient group of Bir($\mathbb{P}_k^n$) (see [21], Theorem 4.7). Moreover, Blanc-Lamy-Zimmermann [12] (Theorem E) proves that when $n \geq 3$, there is a surjection from Bir($\mathbb{P}_k^n$) onto a free product of two-element groups $\mathbb{Z}/2$. This means that Corollary 0.11 can never hold for higher dimensional Cremona groups.

The proofs of Theorems 0.3 and 0.4 are based the Jordan-Chevalley decompositions of algebraic groups and Lie groups. We will prove that any length function on a Heisenberg group vanishes on the central elements (see Lemma 5.2). This is a key step for many other proofs. Based on this fact, we prove Theorems 0.5, 0.8, 0.6 and 0.10 by looking for Heisenberg subgroups. In Section 1, we give some elementary facts on the length functions. In Section 2, we discuss typical examples of length functions. In later sections, we study length functions on Lie groups, algebraic groups, hyperbolic groups, matrix groups and the Cremona groups. Theorem 0.3 and Corollary 0.2 are proved in Section 4. Theorem 0.4 is proved in Section 6. Theorem 0.5 is proved in Section 7. Theorem 0.10 and Corollary 0.11 are proved in Section 10.

1 Basic properties of length functions

1.1 Length functions

Definition 1.1 Let $G$ be a group. A function $l : G \to [0, \infty)$ is called a length function if

1) $l(g^n) = |n|l(g)$ for any $g \in G$ and $n \in \mathbb{Z}$.
2) $l(hgh^{-1}) = l(g)$ for any $h, g \in G$.
3) $l(ab) \leq l(a) + l(b)$ for commuting elements $a, b \in G$, i.e. $ab = ba$.

Lemma 1.2 Any torsion element $g \in G$ has length $l(g) = 0$.

Proof. Note that $l(1) = 2l(1)$ and thus $l(1) = 0$. If $g^n = 1$, then $l(g) = l(1)/|n| = 0$. ■

Recall that a subset $V$ of a real vector space is a convex cone, if $av + bw \in V$ for any $v, w \in V$ and any non-negative real numbers $a, b \geq 0$.

Lemma 1.3 The set Func($G$) of all length functions on a group $G$ is a convex cone.

Proof. It is obvious that for two functions $l_1, l_2$ on $G$, a non-negative linear combination $al_1 + bl_2$ is a new length function. ■

Lemma 1.4 Let $f : G \to H$ be a group homomorphism between two groups $G$ and $H$. For any length function $l : H \to [0, \infty)$, the composite $l \circ f$ is a length function on $G$.

Proof. It is enough to note that a group homomorphism preserves the powers of elements, conjugacy classes and commutativity of elements. ■

Corollary 1.5 For a group $G$, let Out($G$) = Aut($G$)/Inn($G$) be the outer automorphism group. Then Out($G$) acts on the set Func($G$) of all length functions by pre-compositions

$$l \mapsto l \circ g,$$

where $l \in Func(G)$, $g \in Out(G)$. This action preserves scalar multiplications and linear combinations (with non-negative coefficients).
Proof. For an inner automorphism $I_g : G \to G$ given by $I_g(h) = ghg^{-1}$, the length function $l \circ I_g = l$ since $l$ is invariant under conjugation. Therefore, the outer automorphism group $\text{Out}(G)$ has an action on $\text{Func}(G)$. It is obvious that the pre-compositions preserve scalar multiplications and linear combinations with non-negative coefficients. ■

Definition 1.6 A length function $l : G \to [0, \infty)$ is primitive if it is not a composite $l' \circ f$ for a non-trivial surjective group homomorphism $f : G \to H$ and a length function $l' : H \to [0, \infty)$.

Lemma 1.7 Suppose that a length function $l : G \to [0, \infty)$ vanishes on a central subgroup $H < G$. Then $l$ factors through the quotient group $G/H$. In other words, there exists a homogeneous and conjugate-invariant function $l' : G/H \to [0, \infty)$ such that $l = l' \circ q$, where $q : G \to G/H$ is the quotient group homomorphism.

Proof. Write $G = \bigcup gH$, the union of left cosets. For any $h \in H$, we have $l(gh) \leq l(g) + l(h) = l(g)$ and $l(g) = l(ghh^{-1}) \leq l(gh)$. Therefore, $l(gh) = l(g)$ for any $h \in H$. Define $l'(gH) = l(g)$. Then $l'$ is a function on the quotient group $G/H$. The required property follows easily. ■

Corollary 1.8 Suppose that a group $G$ has a non-trivial finite central subgroup $Z(G)$. Any length function $l$ on $G$ factors through $G/Z(G)$.

Proof. This follows from Lemma 1.7 and Lemma 1.2 ■

Lemma 1.9 Let $G$ be a group. Suppose that any non-trivial normal subgroup $H \lhd G$ is of finite index. Then any non-vanishing length function $l : G \to [0, \infty)$ is primitive.

Proof. Suppose that $l$ is a composite $l' \circ f$ for a non-trivial surjective group homomorphism $f : G \to Q$ and a length function $l' : Q \to [0, \infty)$. By the assumption of $G$, the quotient group $Q$ is finite. This implies that $l'$ and thus $l$ vanishes, which is a contradiction. ■

Corollary 1.10 Let $\Gamma$ be an irreducible lattice in a connected irreducible semisimple Lie group of real rank $\geq 2$. Then any non-vanishing length function $l : \Gamma/Z(\Gamma) \to [0, \infty)$ is primitive.

Proof. By the Margulis-Kazhdan theorem (see [66], Theorem 8.1.2), any normal subgroup $N$ of $\Gamma$ either lies in the center of $\Gamma$ (and hence it is finite) or the quotient group $\Gamma/N$ is finite. The previous Lemma 1.9 implies that $l$ is primitive. ■

2 Examples of length functions

Let’s see a general example first. Let $G$ be a group and $f : G \to [0, +\infty)$ be a function satisfying $f(gh) \leq f(g) + f(h)$ and $f(g) = f(g^{-1})$ for any elements $g, h \in G$. Define $l : G \to [0, +\infty)$ by

$$l(g) = \lim_{n \to \infty} \frac{f(g^n)}{n}$$

for any $g \in G$. 
Lemma 2.1 The function $l$ is a length function in the sense of Definition 1.1.

Proof. For any $g \in G$, and natural numbers $n, m$, we have $f(g^{n+m}) \leq f(g^n) + f(g^m)$. This means that $\{f(g^n)\}_{n=1}^{\infty}$ is a subadditive sequence and thus the limit $\lim_{n \to \infty} \frac{f(g^n)}{n}$ exists. This shows that $l$ is well-defined.

From the definition of $l$, it is clear that $l(g^n) = |n| |l(g)|$ for any integer $n$. Let $h \in G$. We have

$$l(hgh^{-1}) = \lim_{n \to \infty} \frac{f(hgh^{-1})}{n} \leq \lim_{n \to \infty} \frac{f(h) + f(g^n) + f(h^{-1})}{n} = \lim_{n \to \infty} \frac{f(g^n)}{n} = l(g).$$

Similarly, we have $l(g) = l(h^{-1}(hgh^{-1})h) \leq l(hgh^{-1})$ and thus $l(g) = l(hgh^{-1})$. For commuting elements $a, b$, we have $(ab)^n = a^n b^n$. Therefore,

$$l(ab) = \lim_{n \to \infty} \frac{f((ab)^n)}{n} = \lim_{n \to \infty} \frac{f(a^n b^n)}{n} \leq \lim_{n \to \infty} \frac{f(a^n) + f(b^n)}{n} \leq l(a) + l(b).$$

\[ \square \]

Many (but not all) length functions $l$ come from these subadditive functions $f$.

2.1 Stable word lengths

Let $G$ be a group generated by a (not necessarily finite) set $S$ satisfying $s^{-1} \in S$ for each $s \in S$. For any $g \in G$, the word length $\phi_S(g) = \min\{n \mid g = s_1 s_2 \cdots s_n, \text{each } s_i \in S\}$ is the minimal number of elements of $S$ whose product is $g$. The stable length $l(g) = \lim_{n \to \infty} \frac{\phi_S(g^n)}{n}$. Since $\phi_S(g^n)$ is subadditive, the limit always exists.

Lemma 2.2 The stable length $l : G \to [0, +\infty)$ is a length function in the sense of Definition 1.1.

Proof. From the definition of the word length $\phi_S$, it is clear that $\phi_S(g h) \leq \phi_S(g) + \phi_S(h)$ and $\phi_S(g) = \phi_S(g^{-1})$ for any $g, h \in G$. The claim is proved by Lemma 2.1 \[ \square \]

When $S$ is the set of commutators, the $l(g)$ is called the stable commutator length, which is related to lots of topics in low-dimensional topology (see Calegari [20]).

2.2 Growth rate

Let $G$ be a group generated by a finite set $S$ satisfying $s^{-1} \in S$ for each $s \in S$. Suppose $\mid \cdot \mid_S$ is the word length of $(G, S)$. For any automorphism $\alpha : G \to G$, define $l'(\alpha) = \max\{|\alpha(s)|_S : s_i \in S\}$. Let $l(\alpha) = \lim_{n \to \infty} \frac{\log l'(\alpha^n)}{n}$. This number $l(\alpha)$ is called the algebraic entropy of $\alpha$ (cf. [11], Definition 3.1.9, page 114).

Lemma 2.3 Let $\text{Aut}(G)$ be the group of automorphisms of $G$. The function $l : \text{Aut}(G) \to [0, +\infty)$ is a length function in the sense of Definition 1.1.

Proof. Since $\alpha(s_i)^{-1} = \alpha^{-1}(s_i)$ for any $s_i \in S$, we know that $l'(\alpha) = l'(\alpha^{-1})$. For another automorphism $\beta : G \to G$, let $l'(\beta) = |\beta(s)|_S$ for some $s_i \in S$. Suppose that
\[
\beta(s_i) = s_i s_2 \cdots s_k \text{ with } k = l'(\beta). \quad \text{Then } |(\alpha \beta)(s_i)|_S = |\alpha(s_i)\alpha(s_2) \cdots \alpha(s_k)|_S \leq l'(\alpha)k.
\]
This proves that \( l'(\alpha \beta) \leq l'(\alpha)l'(\beta) \). The claim is proved by Lemma 2.1. \qed

Fix \( g \in G \). For any automorphism \( \alpha : G \to G \), define \( b_n = |\alpha^n(g)|_S \). Suppose that \( g = s_1 s_2 \cdots s_k \) with \( k = |g|_S \). Note that \( b_n = |\alpha^n(g)|_S = |\alpha^n(s_1)\alpha^n(s_2) \cdots \alpha^n(s_k)|_S \leq l'(\alpha^n)|g|_S \). Therefore, we have
\[
\limsup_{n \to \infty} \frac{\log b_n}{n} \leq l(\alpha).
\]
This implies that \( l(\alpha) \) is an upper bound for the growth rate of \( \{ |\alpha^n(g)|_S \} \). The growth rate is studied a lot in geometric group theory (for example, see [44] for growth of automorphisms of free groups).

### 2.3 Matrix norms and group acting on smooth manifolds

For a square matrix \( A \), the matrix norm \( \| A \| = \sup_{\| x \| = 1} \| Ax \| \). Define the stable norm \( s(A) = \lim_{n \to +\infty} \frac{\log \| A^n \|}{n} \). Since \( \| AB \| \leq \| A \| \| B \| \) for any two matrices \( A, B \), the sequence \( \{ \log \| A^n \| \}_{n=1}^{\infty} \) is subadditive and thus the limit exists.

**Lemma 2.4** Let \( G = \text{GL}_n(\mathbb{R}) \) be the general linear group. The function \( l : G \to [0, +\infty) \) defined by
\[
l(g) = \max \{ s(g), s(g^{-1}) \}
\]
is a length function in the sense of Definition 1.1.

**Proof.** From the definition of the matrix norm, it is clear that \( \log \| gh \| \leq \log \| g \| + \log \| h \| \) for any \( g, h \in G \). Then \( l(g) = \max \{ s(g), s(g^{-1}) \} \) is a length function by Lemma 2.1. \qed

Let \( M \) be a compact Riemannian manifold and \( \text{Diff}(M) \) the diffeomorphism group consisting of all self-diffeomorphisms. For any diffeomorphism \( f : M \to M \), let
\[
\| f \| = \sup_{x \in M} \| D_x f \|,
\]
where \( D_x f \) is the induced linear map between tangent spaces \( T_x M \to T_{f(x)} M \). Define
\[
l(f) = \max \left\{ \lim_{n \to +\infty} \frac{\log \| f^n \|}{n}, \lim_{n \to +\infty} \frac{\log \| f^{-n} \|}{n} \right\}.
\]
A similar argument as the proof of the previous lemma proves the following.

**Lemma 2.5** Let \( G \) be a group acting on a Riemannian manifold \( M \) by diffeomorphisms. The function \( l : G \to [0, +\infty) \) is a length function in the sense of Definition 1.1.

For an \( f \)-invariant Borel probability measure \( \mu \) on \( M \), it is well known (see [56]) that there exists a measurable subset \( \Gamma_f \subset M \) with \( \mu(\Gamma_f) = 1 \) such that for all \( x \in \Gamma_f \) and \( u \in T_x M \), the limit
\[
\chi(x, u, f) = \lim_{n \to \infty} \frac{1}{n} \log \| D_x f^n(u) \|
\]
exists and is called the Lyapunov exponent of \( u \) at \( x \). From the definitions, we know that \( \chi(x, u, f) \leq l(f) \) for any \( x \in \Gamma_f \) and \( u \in T_x M \).
2.4 Smooth measure-theoretic entropy

Let $T : X \to X$ be a measure-preserving map of the probability space $(X, \mathcal{B}, m)$. For a finite-sub-$\sigma$-algebra $A = \{A_1, A_2, \ldots, A_k\}$ of $\mathcal{B}$, denote by

\[ H(A) = -\sum m(A_i) \log m(A_i), \]
\[ h(T, A) = \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^{-i} A), \]

where $\bigvee_{i=0}^{n-1} T^{-i} A$ is a set consisting of sets of the form $\cap_{i=0}^{n-1} T^{-i} A_j$. The entropy of $T$ is defined as $h_m(T) = \sup h(T, A)$, where the supremum is taken over all finite sub-$\sigma$-algebra $A$ of $\mathcal{B}$. For more details, see Walters [62] (Section 4.4).

**Lemma 2.6** Let $M$ be a $C^\infty$ closed Riemannian manifold and $G = \text{Diff}^2_\mu(M)$ consisting of diffeomorphisms of $M$ preserving a Borel probability measure $\mu$. The entropy $h_\mu$ is a length function on $\text{Diff}^2_\mu(M)$ in the sense of Definition 1.1.

**Proof.** For any $f, g \in \text{Diff}^2_\mu(M)$ and integer $n$, it is well-known that $h_\mu(f^n) = |n|h_\mu(f)$ and $h_\mu(f) = h_\mu(gfg^{-1})$ (cf. [62], Theorem 4.11 and Theorem 4.13). Hu [30] proves that $h_\mu(fg) \leq h_\mu(f) + h_\mu(g)$ when $fg = gf$. □

2.5 Stable translation length on metric spaces

Let $(X, d)$ be a metric space and $\gamma : X \to X$ an isometry. Fix $x \in X$. Note that $d(x, \gamma_1 \gamma_2 x) \leq d(x, \gamma_1 x) + d(\gamma_1 x, \gamma_1 \gamma_2 x) = d(x, \gamma_1 x) + d(x, \gamma_2 x)$ and $d(x, \gamma_1 x) = d(x, \gamma_1^{-1} x)$ for any isometries $\gamma_1, \gamma_2$. Define

\[ l(\gamma) = \lim_{n \to \infty} \frac{d(x, \gamma^n x)}{n}. \]

For any $y \in X$, we have

\[ d(x, \gamma^n x) \leq d(x, y) + d(y, \gamma^n y) + d(\gamma^n y, \gamma^n x) \]
\[ = 2d(x, y) + d(y, \gamma^n y) \]

and thus $\lim_{n \to \infty} \frac{d(x, \gamma^n x)}{n} \leq \lim_{n \to \infty} \frac{d(y, \gamma^n y)}{n}$. Similarly, we have the other direction

\[ \lim_{n \to \infty} \frac{d(y, \gamma^n y)}{n} \leq \lim_{n \to \infty} \frac{d(x, \gamma^n x)}{n}. \]

This shows that the definition of $l(\gamma)$ does not depend on the choice of $x$.

**Lemma 2.7** Let $G$ be a group acting isometrically on a metric space $X$. Then the function $l : G \to [0, +\infty)$ defined by $g \mapsto l(g)$ as above is a length function in the sense of Definition 1.1.

**Proof.** This follows from Lemma 2.1. □
2.6 Translation lengths of isometries of CAT(0) spaces

In this subsection, we will prove that the translation length on a CAT(0) space defines a length function. First, let us introduce some notations. Let \((X, d_X)\) be a geodesic metric space, i.e. any two points \(x, y \in X\) can be connected by a path \([x, y]\) of length \(d_X(x, y)\). For three points \(x, y, z \in X\), the geodesic triangle \(\Delta(x, y, z)\) consists of the three vertices \(x, y, z\) and the three geodesics \([x, y], [y, z]\) and \([z, x]\). Let \(\mathbb{R}^2\) be the Euclidean plane with the standard distance \(d_{\mathbb{R}^2}\) and \(\Delta\) a triangle in \(\mathbb{R}^2\) with the same edge lengths as \(\Delta\). Denote by \(\varphi : \Delta \to \bar{\Delta}\) the map sending each edge of \(\Delta\) to the corresponding edge of \(\bar{\Delta}\). The space \(X\) is called a CAT(0) space if for any triangle \(\Delta\) and two elements \(a, b \in \Delta\), we have the inequality

\[d_X(a, b) \leq d_{\mathbb{R}^2}(\varphi(a), \varphi(b)).\]

The typical examples of CAT(0) spaces include simplicial trees, hyperbolic spaces \(\mathbb{H}^n\), products of CAT(0) spaces and so on. From now on, we assume that \(X\) is a complete CAT(0) space. Denote by Isom\((X)\) the isometry group of \(X\). For any \(g \in \text{Isom}(X)\), let

\[\text{Minset}(g) = \{ x \in X : d(x, gx) \leq d(y, gy) \text{ for any } y \in X \}\]

and let \(\tau(g) = \inf_{x \in X} d(x, gx)\) be the translation length of \(g\). When the fixed-point set \(\text{Fix}(g) \neq \emptyset\), we call \(g\) elliptic. When \(\text{Minset}(g) \neq \emptyset\) and \(d_X(x, gx) = \tau(g) > 0\) for any \(x \in \text{Minset}(g)\), we call \(g\) hyperbolic. The group element \(g\) is called semisimple if the minimal set \(\text{Minset}(g)\) is not empty, i.e. it is either elliptic or hyperbolic. A subset \(C\) of a CAT(0) space is convex, if any two points \(x, y \in C\) can be connected by the geodesic segment \([x, y] \subset C\). A group \(G\) is called \(\text{CAT}(0)\) if \(G\) acts properly discontinuously and cocompactly on a CAT(0) space \(X\). In such a case, any infinite-order element in \(G\) acts hyperbolically on \(X\). For more details on CAT(0) spaces, see the book of Bridson and Haefliger [16].

The following was proved by Ballmann-Gromov-Schroeder [4] (Lemma 6.6, page 83). The original proof was for Hardmard manifolds, which is also (well-known) holding for general cases. For completeness, we give details here.

**Lemma 2.8** Let \(\gamma : X \to X\) be an isometry of a complete \(\text{CAT}(0)\) space \(X\). For any \(x_0 \in X\), we have

\[\tau(\gamma) := \inf_{x \in X} d(\gamma x, x) = \lim_{k \to \infty} \frac{d(\gamma^k x_0, x_0)}{k}.\]

**Proof.** For any \(p = x_0 \in X\), let \(m\) be the middle point of \([p, \gamma p]\). We have that \(d(m, \gamma m) \leq \frac{1}{2}d(p, \gamma^2 p)\) by the convexity of length functions. Therefore, \(d(p, \gamma^2 p) \geq 2\tau(\gamma)\) and \(\tau(\gamma^2) \geq 2\tau(\gamma)\). Note that \(d(p, \gamma^2 p) \leq d(p, \gamma p) + d(\gamma p, \gamma^2 p) = 2d(p, \gamma p)\) and thus \(\tau(\gamma^2) \leq 2\tau(\gamma)\). Inductively, we have

\[2^n \tau(\gamma) \leq d(p, \gamma^{2^n} p) \leq 2^n d(p, \gamma p).\]

Note that the limit \(\lim_{k \to \infty} \frac{d(\gamma^k p, p)}{k}\) exists and is independent of \(p\) (see the previous subsection). Therefore, the limit \(\lim_{k \to \infty} \frac{d(\gamma^k p, p)}{k}\) equals to \(\tau(\gamma)\).

**Corollary 2.9** Let \(X\) be a complete \(\text{CAT}(0)\) space and \(G\) a group acting on \(X\) by isometries. For any \(g \in G\), define \(\tau(g) = \inf_{x \in X} d(x, gx)\) as the translation length. Then \(\tau : G \to [0, \infty)\) is a length function in the sense of Definition [1,1].

**Proof.** This follows from Lemma 2.8 and Lemma 2.7.
2.7 Translation lengths of Gromov $\delta$-hyperbolic spaces

Let $\delta > 0$. A geodesic metric space $X$ is called Gromov $\delta$-hyperbolic if for any geodesic triangle $\Delta xyz$ one side $[x, y]$ is contained a $\delta$-neighborhood of the other two edges $[x, z] \cup [y, z]$. Fix $x_0 \in X$. Any isometry $\gamma : X \to X$ is called elliptic if $\{\gamma^n x_0\}_{n \in \mathbb{Z}}$ is bounded. If the orbit map $\mathbb{Z} \to X$ given by $n \mapsto \gamma^n x_0$ is quasi-isometric (i.e. there exists $A \geq 1$ and $B \geq 0$ such that

$$\frac{1}{A} |n - m| - B \leq d_X(\gamma^n x_0, \gamma^m x_0) \leq A |n - m| + B$$

for any integers $n, m$), we call that $\gamma$ is hyperbolic. Otherwise, we call that $\gamma$ is parabolic.

Define $l(\gamma) = \lim_{n \to \infty} \frac{d(\gamma^n x_0, x_0)}{n}$. For any group $G$ acts isometrically on a $\delta$-hyperbolic space, the function $l : G \to [0, \infty)$ is a length function by Lemma 2.7. A finitely generated group $G$ is Gromov $\delta$-hyperbolic if for some finite generating set $S$, the Caley graph $\Gamma(G, S)$ is Gromov $\delta$-hyperbolic.

Any infinite-order element $g$ in a Gromov $\delta$-hyperbolic group is hyperbolic and thus has positive length $l(g) > 0$ (cf. [33], 8.1.D). For more details on hyperbolic spaces and hyperbolic groups, see the book [33] of Gromov.

2.8 Quasi-cocycles

Let $G$ be a group and $(E, \| \cdot \|)$ be a normed vector space with an $G$-action by linear isometries. A function $f : G \to E$ is a quasi-cocycle if there exists $C > 0$ such that

$$\|f(gh) - f(g) - gf(h)\| < C$$

for any $g, h \in G$. Let $l : G \to [0, +\infty)$ be defined by

$$l(g) = \lim_{n \to \infty} \frac{\|f(g^n)\|}{n}.$$ 

Note that $\|f(g^{n+m})\| \leq \|f(g^n)\| + \|f(g^m)\| + C$ for any integers $n, m \geq 0$. This general subadditive property implies that the limit $\lim_{n \to \infty} \frac{\|f(g^n)\|}{n}$ exists (see [60], Theorem 1.9.2, page 22). We call $l$ the average norm. Many applications of quasi-cocycles can be found in [60].

**Lemma 2.10** For any quasi-cocycle $f : G \to E$, the average norm $l$ is a length function.

**Proof.** For any natural number $n$, we have

$$\|f(1) - f(g^{-n}) - g^{-n}f(g^n)\| < C$$

and thus $\|\frac{l(1) - f(g^{-n}) - g^{-n}f(g^n)}{n}\| < \frac{C}{n}$. Taking the limit, we have $\lim_{n \to \infty} \|f(g^{-n})\|/n = \lim_{n \to \infty} \|f(g^n)\|/n$.

Therefore, for any $k \in \mathbb{Z}$, we have $l(g^k) = \lim_{n \to \infty} \|f(g^k n)\| = |k| l(g)$. For any $h \in G$, we have

$$\|f(h g^n h^{-1})\| \leq \|f(h)\| + \|f(h^{-1})\| + \|f(g^n)\| + 2C.$$ 

Therefore, we have

$$l(h g^{-1} h^{-1}) = \lim_{n \to \infty} \frac{\|f(h g^n h^{-1})\|}{n} \leq \lim_{n \to \infty} \frac{\|f(g^n)\|}{n} = l(g).$$

Similarly, we have $l(g) = l(h^{-1}(h g^{-1})h) \leq l(h g^{-1})$. When $g, h$ commutes, we have

$$l(gh) = \lim_{n \to \infty} \frac{\|f((gh)^n)\|}{n} = \lim_{n \to \infty} \frac{\|f(g^n h^n)\|}{n} \leq \lim_{n \to \infty} \frac{\|f(g^n)\| + \|f(h^n)\| + C}{n} = l(g) + l(h).$$
2.9 Rotation number

Let $\mathbb{R}$ be the real line and $\text{Homeo}_Z(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \text{ is a monotonically increasing homeomorphism such that } f(x + n) = f(x) + n \text{ for any } n \in \mathbb{Z} \}$. For any $f \in \text{Homeo}_Z(\mathbb{R})$ and $x \in [0, 1)$, the translation number is defined as

$$l(f) = \lim_{n \to \infty} \frac{f^n(x) - x}{n}.$$ 

It is well-known that $l(f)$ exists and is independent of $x$ (see [55], Prop. 2.22, p.31). Note that every $f \in \text{Homeo}_Z(\mathbb{R})$ induces an orientation-preserving homeomorphism of the circle $S^1$.

**Proposition 2.11** The absolute value of the translation number $|l| : \text{Homeo}_Z(\mathbb{R}) \to [0, \infty)$ is a length function in the sense of Definition 1.7.

**Proof.** For any $f \in \text{Homeo}_Z(\mathbb{R})$ and $k \in \mathbb{Z}\setminus\{0\}$, we have that

$$l(f^k) = \lim_{n \to \infty} \frac{f^{kn}(x) - x}{n} = k \lim_{n \to \infty} \frac{f^{kn}(x) - x}{nk} = kl(f).$$

For any $a \in \text{Homeo}_Z(\mathbb{R})$, we have that

$$|l(afa^{-1}) - l(f)| = \lim_{n \to \infty} \left| \frac{af^n(a^{-1}x) - f^n(x) + x}{n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{af^n(a^{-1}x) - f^n(a^{-1}x) + f^n(a^{-1}x) - f^n(x)}{n} \right|$$

$$= 0,$$

since $a - \text{id}_\mathbb{R}$ is bounded on $[0, 1]$ and $|f^n(a^{-1}x) - f^n(x)| \leq 2 + |a^{-1}x - x|$. For commuting elements $f, g \in \text{Homeo}_Z(\mathbb{R})$, we have that

$$l(fg) = \lim_{n \to \infty} \frac{f^n(g^n(x)) - x}{n} = \lim_{n \to \infty} \frac{f^n(g^n(x)) - g^n(x) + g^n(x) - x}{n}.$$ 

Suppose that $g^n(x) = k_n + x_n$ for $k_n \in \mathbb{Z}$ and $x_n \in [0, 1)$. Then

$$\lim_{n \to \infty} \frac{f^n(g^n(x)) - g^n(x) + g^n(x) - x}{n}$$

$$= \lim_{n \to \infty} \frac{f^n(x_n) - x_n + g^n(x) - x}{n}$$

$$= \lim_{n \to \infty} \frac{f^n(0) - 0 + g^n(x) - x}{n} = l(f) + l(g).$$

Therefore, we get $|l(fg)| \leq |l(f)| + |l(g)|$. ■

**Remark 2.12** It is actually true that the rotation number $l$ is multiplicative on any amenable group (see [55], Prop. 2.2.11 and the proof of Prop. 2.2.10, page 36). This implies that the absolute rotation number $|l|$ is subadditive on any amenable group. In other words, for any amenable group $G < \text{Homeo}_Z(\mathbb{R})$ and any $g, h \in G$ we have $|l(gh)| \leq |l(g)| + |l(h)|$.  

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2.10 Asymptotic distortions

Let $f$ be a $C^{1+bv}$ diffeomorphism of the closed interval $[0, 1]$ or the circle $S^1$. ("bv" means derivative with finite total variation.) The asymptotic distortion of $f$ is defined as

$$l(f) = \text{dist}_\infty(f) = \lim_{n \to \infty} \frac{1}{n} \text{var}(\log Df^n).$$

It’s proved by Eynard-Bontemps and Navas ([27], pages 7-8) that

1. $\text{dist}_\infty(f^n) = |n| \text{dist}_\infty(f)$ for all $n \in \mathbb{Z}$;
2. $\text{dist}_\infty(hf^{-1}h^{-1}) = \text{dist}_\infty(f)$ for every $C^{1+bv}$ diffeomorphism $h$;
3. $\text{dist}_\infty(f \circ g) \leq \text{dist}_\infty(f) + \text{dist}_\infty(g)$ for commuting $f, g$.

Therefore, the asymptotic distortion is a length function $l$ on the group $\text{Diff}^{1+bv}(M)$ of $C^{1+bv}$ diffeomorphisms for $M = [0, 1]$ or $S^1$.

2.11 Dynamical degrees of Cremona groups

Let $k$ be a field and $\mathbb{P}_k^n = k^{n+1}\backslash \{0\}/\{\lambda \sim \lambda x : \lambda \neq 0\}$ be the projective space. A rational map from $\mathbb{P}_k^n$ to itself is a map of the following type

$$(x_0 : x_1 : \cdots : x_n) \to (f_0 : f_1 : \cdots : f_n)$$

where the $f_i$’s are homogeneous polynomials of the same degree without common factor. The degree of $f$ is $\deg f = \deg f_1$. A birational map from $\mathbb{P}_k^n$ to itself is a rational map $f : \mathbb{P}_k^n \to \mathbb{P}_k^n$ such that there exists a rational map $g : \mathbb{P}_k^n \to \mathbb{P}_k^n$ such that $f \circ g = g \circ f = \text{id}$. The group $\text{Bir}(\mathbb{P}_k^n)$ of birational maps is called the Cremona group (also denoted as $\text{Cr}_n(k)$). It is well-known that $\text{Bir}(\mathbb{P}_k^n)$ is isomorphic to the group $\text{Aut}_k(k(x_1, x_2, \ldots, x_n))$ of self-isomorphisms of the field $k(x_1, x_2, \ldots, x_n)$ of the rational functions in $n$ indeterminates over $k$. The (first) dynamical degree $\lambda(f)$ of $f \in \text{Bir}(\mathbb{P}_k^n)$ is defined as

$$\lambda(f) = \max\{ \lim_{n \to \infty} \deg(f^n)^{1/n}, \lim_{n \to \infty} \deg(f^{-n})^{1/n} \}.$$
This checks the three conditions of the length function. ■
It is surprising that when $n = 2$ and $k$ is an algebraically closed field, the length function $l(f)$ is given by the translation length $\tau(f)$ on an (infinite-dimensional) Gromov $\delta$-hyperbolic space (see Blanc-Cantat [10], Theorem 4.4). Some other length functions are studied by Blanc and Furter [11] for groups of birational maps, e.g. the dynamical number $\delta$

2.12 Margulis invariants

Let $\text{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \text{GL}_n(\mathbb{R})$ be the affine group of the $n$-dimensional space $\mathbb{R}^n$. For a subgroup $G < \text{Aff}(\mathbb{R}^n)$, the linear part is $\Phi(G)$, where $\Phi : \mathbb{R}^n \rtimes \text{GL}_n(\mathbb{R}) \to \text{GL}_n(\mathbb{R})$ is the natural quotient map.

**Theorem 2.14** Let $G < \text{Aff}(\mathbb{R}^n)$. Suppose that

(i) the linear part $\Phi(G)$ lies in the isometric group $\text{Isom}(B)$ for a bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

(ii) any $1 \neq g \in \Phi(G)$ has 1 as its eigenvalue and the corresponding eigenspace is of dimension 1, spanned by a unit eigenvector $x_g \in \mathbb{R}^n$.

(iii) any commuting elements $a, b \in \mathfrak{l}(G)$ have a common eigenvector $0 \neq x_a = x_b \in \mathbb{R}^n$ corresponding to the eigenvalue 1.

Then $l : G \to \mathbb{R}_{\geq 0}$ given by $l((g, v)) = |B(x_g, v)|$ is a length function.

**Proof.** Note that the definition of $l$ does not depend on the choice the sign of $x_g$. For any integer $n > 0$, we have $(v, g)^n = (\sum_{i=0}^{n-1} g^i v, g^n)$ for any $(v, g) \in G$. Therefore,

$$B(x_g, \sum_{i=0}^{n-1} g^i v) = nB(x_g, v),$$

implying $l((v, g)^n) = nl(g, v)$. Note that $x_g = x_{g^{-1}}$ and

$$B(x_{g^{-1}}, -g^{-1}v) = B(x_g, -v),
\quad l((v, g)) = l((v, g)^{-1}).$$

For any $(u, h) \in G$, we have

$$(u, h)(v, g)(u, h)^{-1} = (u + hv - hgh^{-1}u, hgh^{-1}).$$

Note that $hgh^{-1}(hx_g) = hx_g$ and $x_{hgh^{-1}} = hx_g$. We have

$$l((u, h)(v, g)(u, h)^{-1}) = |B(hx_g, u + hv - hgh^{-1}u)|
= |B(x_g, h^{-1}u + v - gh^{-1}u)| = |B(x_g, v)|.

For any commuting pair $(u, a), (v, b) \in G$, we have

$$l((u, a)(v, b)) = l(u + av, ab) = |B(x_a, u + av)| = |B(x_a, u) + B(x_a, v)| \leq l((u, a)) + l((v, b)).$$

■
Corollary 2.15 Let $\alpha : \Gamma \to \mathbb{R}$ be the Margulis invariant defined in the section of Introduction. The absolute value function $|\alpha|$ is a length function.

Proof. By Theorem 2.14, it is enough to check the three conditions. If $\varphi(g)$ is hyperbolic (for an element $g \in \Gamma$), it has three real eigenvalues with one of them is 1. The first two conditions are obvious. Now we check the third condition. If $g, h$ are commuting elements, the linear parts $\varphi(g), \varphi(h)$ lie in a maximal split torus in $SO(2,1)$ and thus conjugate to elements in the standard torus simultaneously. Let $t$ be a nonzero real number. The adjoint representation of $SL(2, \mathbb{R})$ on the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ gives an embedding $f : PSL_2(\mathbb{R}) \to SO(2,1)$. Note that

$$\begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \begin{bmatrix} 1/t & 0 \\ 0 & t \end{bmatrix} = \begin{bmatrix} a & t^2b \\ c/t^2 & -a \end{bmatrix}.$$  

Therefore,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is a common eigenvector for elements in the torus

$$\{f(\begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix}) : t \in \mathbb{R}^*\}.$$  

This checked condition 3). The claim follows from Theorem 2.14. □

The Margulis invariant has essential applications in the study of properly discontinuously affine actions on the Euclidean space $\mathbb{R}^n$. For more information, see the survey article [1].

3 Groups with purely positive length functions

Definition 3.1 A length function $l$ on a group $G$ is said to be purely positive if $l(g) > 0$ for any infinite-order element $g \in G$.

In this section, we show that the (Gromov) hyperbolic group, mapping class group and outer automorphism groups of free groups have purely positive length functions. First, let us recall the relevant definitions.

A geodesic metric space $X$ is $\delta$-hyperbolic (for some real number $\delta > 0$) if for any geodesic triangle $\Delta xyz$ in $X$, one side is contained the $\delta$-neighborhood of the other two sides. A group $G$ is (Gromov) hyperbolic if $G$ acts properly discontinuously and cocompactly on a $\delta$-hyperbolic space $X$.

Definition 3.2 (i) An element $g$ in a group $G$ is called primitive if it cannot be written as a proper power $\alpha^n$, where $\alpha \in G$ and $|n| \geq 2$;

(ii) A group $G$ has unique-root property if every infinite-order element $g$ is a power of a unique (up to sign) primitive element, i.e. $g = \gamma^n = \gamma_1^m$ for primitive elements $\gamma, \gamma_1$ will imply $\gamma = \gamma_1^\pm$.

The following fact is well-known.

Lemma 3.3 A torsion-free hyperbolic group has unique-root property.
Proof. Let $G$ be a torsion-free hyperbolic group and $1 \neq g \in G$. Suppose that $g = \gamma^n = \gamma_1^m$ for primitive elements $\gamma$ and $\gamma_1$. The set $C_G(g)$ of centralizers is virtually cyclic (cf. [16], Corollary 3.10, page 462). By a result of Serre, a torsion-free virtually free group is free. Since $G$ is torsion-free, the group $C_G(g)$ is thus free and thus cyclic, say generated by $t$. Since $\gamma$ and $\gamma_1$ are primitive, they are $t^\pm$.

Remark 3.4 The previous lemma does not hold for general hyperbolic groups with torsions. For example, let $G = \mathbb{Z}/2 \times \mathbb{Z}$. We have $(0, 2) = (0, 1)^2 = (1, 1)^2$ and $(0, 1), (1, 1)$ are both primitive.

For a group $G$, let $P(G)$ be the set of all primitive elements. We call two primitive elements $\gamma, \gamma'$ are generally conjugate if there exists $g \in G$ such that $g\gamma g^{-1} = \gamma'$ or $g\gamma^{-1}g = \gamma'$. Let $CP(G)$ be the general conjugacy classes of primitive elements. For a set $S$, let $S_\mathbb{R}$ be the set of all real functions on $S$. The convex polyhedral cone spanned by $S$ is the subset

$$\{ \Sigma_{s \in S} a_s s \mid a_s \geq 0 \} \subset S_\mathbb{R}.$$

Lemma 3.5 Let $G$ be a torsion-free hyperbolic group. The set of all length functions on $G$ is the convex polyhedral cone spanned by the general conjugacy classes $CP(G)$.

Proof. Let $l$ be a length function on $G$. Then $l$ gives an element $\Sigma_{s \in S} a_s s$ in the convex polyhedral cone by $a_s = l(s)$. Conversely, for any general conjugacy classes $[s] \in CP(G)$ with $s$ a primitive element, let $l_s$ be the function defined by $l_s(s^+) = 1$ and $l_s(\gamma) = 0$ for element $\gamma$ in any other general conjugacy classes. For any $1 \neq g \in G$, there is a unique (up to sign) primitive element $\gamma$ such that $g = \gamma^n$. Define $l_s(g) = |n|l_s(\gamma)$. Then $l_s$ satisfies conditions (1) and (2) in Definition 1.1. Condition (3) is satisfied automatically, since any commuting pair of elements $a,b$ generates a cyclic group in a torsion-free hyperbolic group. Any element $\Sigma_{s \in S} a_s s$ gives a length function on $G$ as a combination of $a_s l_s$.

Lemma 3.6 Let $G$ be one of the following groups:

- automorphism group $\text{Aut}(F_k)$ of a free group;
- outer automorphism group $\text{Out}(F_k)$ of a free group or
- mapping class group $\text{MCG}(\Sigma_{g,m})$ (where $\Sigma_{g,m}$ is an oriented surface of genus $g$ and $m$ punctures);
- a hyperbolic group,
- a $\text{CAT}(0)$ group or more generally
- a semi-hyperbolic group,
- a group acting properly semi-simply on a $\text{CAT}(0)$ space,
- a group acting properly semi-simply on a $\delta$-hyperbolic space.

Then $G$ has a purely positive length function.
Proof. Note that hyperbolic groups and CAT(0) groups are semihyperbolic (see [16], Prop. 4.6 and Cor. 4.8, Chapter III.Γ). For a semihyperbolic group $G$ acting a metric space $X$ (actually $X = G$), the translation $\tau$ is a length function by Lemma 2.7. Moreover, for any infinite-order element $g \in G$, the length $\tau(g) > 0$ (cf. [16], Lemma 4.18, page 479). For group acting properly semisimply on a CAT(0) space (or a $\delta$-hyperbolic space), the translation $l(\gamma) = \lim_{n \to \infty} \frac{d(x, \gamma^n x)}{n}$ is a length function (cf. Lemma 2.7). For any hyperbolic $\gamma$, we get $l(\gamma) > 0$. For any elliptic $\gamma$, it is finite-order since the action is proper.

Alibegovic [2] proves that the stable word length of $\text{Aut}(F_n)$, $\text{Out}(F_n)$ are purely positive. Farb, Lubotzky and Minsky [28] prove that Dehn twists and more generally all elements of infinite order in $\text{MCG}(\Sigma_{g,m})$ have positive translation length.

Definition 3.7 A group $G$ is called poly-positive (or has a poly-positive length), if there is a subnormal series

$$1 = H_n \triangleleft H_{n-1} \triangleleft \cdots \triangleleft H_0 = G$$

such that every finitely generated subgroup of the quotient $H_i/H_{i+1}$ ($i = 0, \ldots, n-1$) has a purely positive length function.

Recall that a group $G$ is poly-free, if there is a subnormal series $1 = H_n \triangleleft H_{n-1} \triangleleft \cdots \triangleleft H_0 = G$ such that the successive quotient $H_i/H_{i+1}$ is free ($i = 0, \ldots, n-1$). Since a free group is hyperbolic, it has a purely positive length function. This implies that a poly-free group is poly-positive. A group is said to have a virtual property if a finite-index subgroup has the property.

Let $\Sigma$ be a closed oriented surface endowed with an area form $\omega$. Denote by $\text{Diff}(\Sigma, \omega)$ the group of diffeomorphisms preserving $\omega$ and $\text{Diff}_0(\Sigma, \omega)$ the subgroup consisting of diffeomorphisms isotopic to the identity.

Lemma 3.8 When the genus of $\Sigma$ is greater than 1, the group $\text{Diff}_0(\Sigma, \omega)$ and $\text{Diff}(\Sigma, \omega)$ are poly-positive.

Proof. This is essentially proved by Py [58] (Section 1). There is a group homomorphism

$$\alpha : \text{Diff}_0(\Sigma, \omega) \to H_1(\Sigma, \mathbb{R})$$

with $\ker \alpha = \text{Ham}(\Sigma, \omega)$ the group of Hamiltonian diffeomorphisms of $\Sigma$. Polterovich [57] (1.6.C.) proves that any finitely generated group of $\text{Ham}(\Sigma, \omega)$ has a purely positive stable word length. Since the quotient group $\text{Diff}(\Sigma, \omega)/\text{Diff}_0(\Sigma, \omega)$ is a subgroup of the mapping class group $\text{MCG}(\Sigma)$, which has a purely positive stable word length by Farb-Lubotzky-Minsky [28], the group $\text{Diff}(\Sigma, \omega)$ is poly-positive.

4 Vanishing of length functions on abelian-by-cyclic groups

We will need the following result proved in [31].
Lemma 4.1 Given a group $G$, let $l : G \to [0, +\infty)$ be a function such that
1) $l(e) = 0$;
2) $l(x^n) = |n| l(x)$ for any $x \in G$, any $n \in \mathbb{Z}$;
3) $l(xy) \leq l(x) + l(y)$ for any $x, y \in G$.

Then there exists a real Banach space $(\mathbb{B}, \|\|)$ and a group homomorphism $\varphi : G \to \mathbb{B}$ such that $l(x) = \|\varphi(x)\|$ for all $x \in G$. Furthermore, if $l(x) > 0$ for any $x \neq e$, one can take $\varphi$ to be injective, i.e., an isometric embedding.

Let $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ be an abelian-by-cyclic group, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(\mathbb{Z})$. We prove Theorem 0.1 by proving the following two theorems.

Theorem 4.2 When the absolute value of the trace $|\text{tr}(A)| > 2$, any length function $l : \mathbb{Z}^2 \rtimes_A \mathbb{Z} \to \mathbb{R}_{\geq 0}$ vanishes on $\mathbb{Z}^2$.

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(\mathbb{Z})$. Suppose that $t$ is a generator of $\mathbb{Z}$ and

$$t \begin{bmatrix} x \\ y \end{bmatrix} t^{-1} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

for any $x, y \in \mathbb{Z}$. Note that

$$(0, t^k)(v, 0)(0, t^k)^{-1} = (A^k v, 0)$$

for any $v \in \mathbb{Z}^2$ and $k \in \mathbb{Z}$. Therefore, an element $v \in \mathbb{Z}^2$ is conjugate to $A^k v$ for any integer $k$. Note that

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}, A^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a^2 + bc \\ ac + dc \end{bmatrix}$$

and

$$\begin{bmatrix} a^2 + bc \\ ac + dc \end{bmatrix} = (a + d) \begin{bmatrix} a \\ c \end{bmatrix} - (ad - bc) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$ 

Therefore, we have

$$|a + d| l(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = l((a + d) \begin{bmatrix} a \\ c \end{bmatrix}) = l(\begin{bmatrix} a^2 + bc \\ ac + dc \end{bmatrix} + (ad - bc) \begin{bmatrix} 1 \\ 0 \end{bmatrix}) \leq (1 + |ad - bc|)l(\begin{bmatrix} 1 \\ 0 \end{bmatrix}).$$

When $ad - bc = \pm 1$ and $|a + d| > 2$, we must have $l(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = 0$. Similarly, we can prove that $l(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = 0$. Since $l$ is subadditive on $\mathbb{Z}^2$, we get that $l$ vanishes on $\mathbb{Z}^2$. ■

Theorem 4.3 When the absolute value $|\text{tr}(A)| = |a + d| = 2, I_2 \neq A \in \text{SL}_2(\mathbb{Z})$, any length function $l : \mathbb{Z}^2 \rtimes_A \mathbb{Z} \to \mathbb{R}_{\geq 0}$ vanishes on the direct summand of $\mathbb{Z}^2$ spanned by eigenvectors of $A$. 

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Proof. We may assume that $A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \alpha \neq 0$. For any integer $k \geq 0$ and $v \in \mathbb{Z}^2$, we have

$$t^k vt^{-k} = A^k v.$$ 

Take $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to get that

$$t^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} kn \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$ 

Since the function $l|_{\mathbb{Z}^2}$ is given by the norm of a Banach space according to Lemma 4.1, we get that

$$k|n|l(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) \leq l(t^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{-k}) + l(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = 2l(\begin{bmatrix} 0 \\ 1 \end{bmatrix}).$$ 

Since $k$ is arbitrary, we get that $l(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = 0$. ■

Remark 4.4 When $A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$, the semidirect product $G = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is a Heisenberg group. A length function on $G/\mathbb{Z}(G) \cong \mathbb{Z}^2$ gives a length function on $G$. In particular, a length function of $G$ may not vanish on the second component $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{Z}^2 < G$.

Remark 4.5 When $A \in \text{SL}_2(\mathbb{Z})$ has $|\text{tr}(A)| < 2$, the matrix $A$ is of finite order and the semi-direct product $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ contains $\mathbb{Z}^3$ as a finite-index normal subgroup. Actually, in this case the group $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is the fundamental group of a flat 3-manifold $M$ (see [65], Theorem 3.5.5). Therefore, the group $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ acts freely properly discontinuously isometrically and cocompactly on the universal cover $\tilde{M} = \mathbb{R}^3$. This means the translation length gives a purely positive length function on $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$.

Lemma 4.6 Let $A \in \text{GL}_n(\mathbb{Z})$ be a matrix and $G = \mathbb{Z}^n \rtimes_A \mathbb{Z}$ the semi-direct product. Let $\sum_{i=0}^n a_i x^i$ be the characteristic polynomial of some power $A^k$. Suppose that for some $k$, there is a coefficient $a_i$ such that

$$|a_i| > \sum_{j \neq i} |a_j|.$$ 

Any length function $l$ of $G$ vanishes on $\mathbb{Z}^n$.

Proof. Let $t$ be a generator of $Z$ and $tat^{-1} = Aa$ for any $a \in \mathbb{Z}^n$. Note that for any integer $m$, we have $t^m at^{-m} = A^ma$ and $l(a) = l(A^ma)$. Note that

$$\sum_{i=0}^n a_i A^i = 0$$

and thus

$$\sum_{i=0}^n a_i A^i a = 0,$$

$$l(\sum_{i=0}^n a_i A^i a) = 0.$$
for any $a \in \mathbb{Z}^n$. Therefore,

$$|a_i|l(a) = |a_i|l(A^{kj}a) = l(\sum_{j \neq i} a_j A^{kj}a) \leq \sum_{j \neq i} |a_j|l(a).$$

This implies that $l(a) = 0$. ■

**Corollary 4.7** Let $G$ be a finitely generated nilpotent group, which is not virtually abelian. Any length function $l : G \to \mathbb{R}_{\geq 0}$ vanishes at an infinite-order element $g$.

**Proof.** The group $G$ contains a finite-index torsion-free nilpotent subgroup $H$. Suppose that $H$ is not abelian. Denote by $H_0 = H$ and $H_i = [H_{i-1}, H]$ for each positive integer $i$. Since $H$ is nilpotent, there is an integer $i = n \geq 1$ such that $H_n \neq 1$ is abelian. Choose $a, b \in H_{n-1}$ such that $[a, b] = c \neq 1 \in H_n$. Since $H_{n+1}$ is trivial, the elements $a, b$ commute with $c$. This implies that the subgroup generated by $a, b, c$ is a Heisenberg group. Theorem 0.1 implies that $l(c) = 0$.

**Remark 4.8** The idea of the proofs of Theorem 0.1 is already known for the word metric or the stable norms (for example, see [45][3][24]). Actually, Connor [24] obtained a general vanishing result for the stable norms on solvable groups. An analog will be of great interest for the vanishing of entropies and Lyapunov exponents (as in Corollary 0.2). Currently, we don’t know whether such an analog holds for general length functions.

**Proof of Corollary 0.2.** When the group action is $C^2$, define

$$l(f) = \max\left\{ \lim_{n \to +\infty} \frac{\log \sup_{x \in M} \|D_x f^n\|}{n}, \lim_{n \to +\infty} \frac{\log \sup_{x \in M} \|D_x f^{-n}\|}{n} \right\}$$

for any diffeomorphism $f : M \to M$. Lemma 2.5 shows that $l$ is a length function, which is an upper bound of the Lyapunov exponents. When the group action is Lipschitz, define

$$L(f) = \sup_{x \neq y} \frac{d(fx, fy)}{d(x, y)}$$

for a Lipschitz-homeomorphism $f : M \to M$. Since $L(fg) \leq L(f)L(g)$ for two Lipschitz-homeomorphisms $f, g : M \to M$, we have that $l(f) = \lim_{n \to +\infty} \frac{\max\{\log L(f^n)\}}{\log L(f^n)}$ gives a length function by Lemma 2.1. Note that $l(f) \geq h_{top}(f)$ (see [11], Theorem 3.2.9, page 124). The vanishings of the topological entropy $h_{top}$ and the Lyapunov exponents in Corollary 0.2 are proved by Theorem 0.1 considering these length functions. ■

## 5 Classification of length functions on nilpotent groups

The following lemma is a key step for our proof of the vanishing of length functions on Heisenberg groups.

**Lemma 5.1** Let 

$$G = \langle a, b, c \mid aba^{-1}b^{-1} = c, ac = ca, bc = cb \rangle$$

be the Heisenberg group. Suppose that $f : G \to \mathbb{R}$ is a conjugation-invariant function, i.e.

$$f(xgx^{-1}) = f(g)$$

for any $x, g \in G$. For any coprime integers (not-all-zero) $m, n$ and any integer $k$, we have

$$f(a^m b^n c^k) = f(a^m b^n).$$

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1.1) factors through the abelianization $g$ since $aba$ by Lemma 4.1, there is a Banach space $B$ such that $l(|ba^n| - |a^{-n}c^n|) = |b^n - c^n|$. Therefore, $a^nba^{-n}b^{-1} = e^n$ and $a^nba^{-n} = c^n$, $a^nba^{-n} = c^n$ for any integer $m$. This means $[a^n, b^n] = e^n$. For any coprime $m, n$, and any integer $k$, let $s, t \in \mathbb{Z}$ such that $ms + nt = k$. We have
\[
\begin{align*}
a^{-m}b^{-s}a^mb^s &= c^ms, \quad b^{-s}a^mb^s = a^mc^ms, \quad b^{-s}a^mb^s = a^nb^nc^ms \\
\end{align*}
\]
and
\[
\begin{align*}
a^lb^n - b^{-n} &= c^nt, \quad a^la^n - b^{-n} = b^n - c^nt, \quad a^la^mb^na^kt = a^mb^nc^mt.
\end{align*}
\]
Therefore,
\[
\begin{align*}
a^l(b^{-s}a^nb^n)c^{-nt} &= a^l(a^mb^nc^ms)c^{-nt} = a^mb^nc^nt + ms = a^mb^nc^k.
\end{align*}
\]

When $f$ is conjugation-invariant, we get $f(a^mb^n c^k) = f(a^mb^n)$ for any coprime $m, n$, and any integer $k$. □

**Lemma 5.2** Let $G = \langle a, b, c \mid aba^{-1}b^{-1} = b, ac = ca, bc = cb \rangle$ be the Heisenberg group. Any length function $l : G \to [0, \infty)$ (in the sense of Definition [1]) factors through the abelianization $G_{ab} := G/[G, G] \cong \mathbb{Z}^2$. In other words, there is a function $l' : G_{ab} \to [0, \infty)$ such that $l'(x^n) = |n|l'(x)$ for any $x \in G_{ab}$, any integer $n$ and $l = l' \circ q$, where $q : G \to G_{ab}$ is the natural quotient group homomorphism.

**Proof.** Let $H = \langle c \rangle \cong \mathbb{Z}$ and write $G = \bigcup gH$ the union of left cosets. We choose the representative $g_{ij} = a^ib^j$ with $(i, j) \in \mathbb{Z}^2$. Note that the subgroup $\langle g_{ij}, c \rangle$ generated by $g_{ij}, c$ is isomorphic to $\mathbb{Z}^2$ for coprime $i, j$. The length function $l$ is subadditive on $\langle g_{ij}, c \rangle$. By Lemma 4.1, there is a Banach space $B$ and a group homomorphism $\varphi : \langle g_{ij}, c \rangle \to B$ such that $l(g) = ||\varphi(g)||$ for any $g \in \langle g_{ij}, c \rangle$. Lemma 5.1 implies that
\[
||\varphi(g_{ij}) + \varphi(c^k)|| = ||\varphi(g_{ij})||
\]
for any integer $k$. Since
\[
||\varphi(g_{ij})|| = ||\varphi(g_{ij}) + \varphi(c^k)|| \geq |k||\varphi(c)|| - ||\varphi(g_{ij})||
\]
for any $k$, we have that $||\varphi(c)|| = l(c) = 0$. This implies that
\[
l(g_{ij}^nc) = ||\varphi(g_{ij}^nc)|| = l(g_{ij}^n)
\]
for any integers $m, n$. Moreover, for any integers $m, n$ and coprime $i, j$, we have $a^imb^nc^m = (a^ib^j)^nc^k$ for some integer $k$. This implies that
\[
l(a^imb^nc^m) = l((a^ib^j)^n) = |n|l(a^ib^j).
\]
Therefore, the function $l$ is constant on each coset $gH$. Define $l'(gH) = l(g)$. Since $l(g^k) = |k|l(g)$, we have that $l'(g^{kH}) = |k|l'(gH)$. The proof is finished. □

Denote by $S' = \{(m, n) \mid m, n$ are coprime integers, $(m, n) \neq (0, 0)\}$ be the set of coprime integer pairs and define an equivalence relation by $(m, n) \sim (m', n')$ if $(m, n) = \pm(m', n')$. Let $S = S'/\sim$ be the equivalence classes.
Theorem 5.3 Let $G = \langle a, b, c \mid aba^{-1}b^{-1} = c, ac = ca, bc = cb \rangle$ be the Heisenberg group. The set of all length functions $l : G \to [0, \infty)$ (in the sense of Definition 1.1) is the convex polyhedral cone

$$\mathbb{R}_{>0}[S] = \{ \sum_{s \in S} a_s s \mid a_s \in \mathbb{R}_{>0}, s \in S \}. $$

Proof. Similar to the proof of the previous lemma, we let $H = \langle c \rangle \cong \mathbb{Z}$ and write $G = \bigcup gH$ the union of left cosets. We choose the representative $g_{ij} = a^ib^j$ with $(i, j) \in \mathbb{Z}^2$. Let $T$ be the set of length functions $l : G \to [0, \infty)$. For any length function $l$, let $\varphi(l) = \sum_{s \in S} a_s s \in \mathbb{R}_{>0}[S]$, where $a_s = l(a^ib^j)$ with $(i, j)$ a representative of $s$. Note that

$$l(a^{-i}b^{-j}) = l(b^{-j}a^{-i}c^{ij}) = l(b^{-j}a^{-i}) = l(a^ib^j),$$

which implies that $a_s$ is well-defined. We have defined a function $\varphi : T \to \mathbb{R}_{>0}[S]$. If $\varphi(l_1) = \varphi(l_2)$ for two functions $l_1, l_2$, then $l_1(a^ib^j) = l_2(a^ib^j)$ for coprime integers $i, j$. Since both $l_1, l_2$ are conjugation-invariant, Lemma 5.1 implies that $l_1, l_2$ coincide on any coset $a^ib^jH$ and thus on the whole group $G$. This proves the injectivity of $\varphi$. For any $\sum_{s \in S} a_s s$, we define a function

$$l : G = \bigcup a^ib^jH \to [0, \infty).$$

For any coprime integers $i, j$, define $l(a^ib^jz) = a_s$ for any representative $(i, j)$ of $s$ and any $z \in H$. For any general integers $m, n$ and $z \in H$, define $l(a^mb^nz) = l(mn) = l(a^m b^n) = |\gcd(m, n)| l(a^m b^n)$ and $l(z) = 0$. From the definition, it is obvious that $l$ is homogeneous. Note that any element of $G$ is of the form $a^kb^sc^t$ for integers $k, s, t \in \mathbb{Z}$. For any two elements $a^kb^sc^t, a^kb^sc^t'$ we have the conjugation

$$a^kb^sc^t a^kb^sc^t' (a^kb^sc^t')^{-1} = a^kb^sc^{t''}$$

for some $t'' \in \mathbb{Z}$. Therefore, we see that $l$ is conjugation-invariant. The previous equality also shows that two elements $g = a^kb^sc^t, h = a^kb^sc^t'$ are commuting if and only if $ks' = k's$, which is also equivalent to that they lies simultaneously in $\langle a^ib^j, c \rangle$ for a pair of coprime integers $i, j$. By construction, we have $l(gh) = l(g) + l(h)$. This proves the surjectivity of $\varphi$. ■

6 Length functions on matrix groups

In this section, we study length functions on matrix groups $SL_n(\mathbb{R})$. As the proofs are elementary, we present them here in a separate section, without using profound results on Lie groups and algebraic groups. The following lemma is obvious.

Lemma 6.1 Let $G_{p,q} = \langle x, t : tx^pt^{-1} = x^q \rangle$ be a Baumslag-Solitar group. When $|p| \neq |q|$, any length function $l$ on $G$ has $l(x) = 0$.

Proof. Note that $|p|l(x) = l(x^p) = l(x^q) = |q|l(x)$, which implies $l(x) = 0$. ■

Let $V^n$ be a finite-dimensional vector space over a field $K$ and $A : V \to V$ a nilpotent linear transformation (i.e. $A^k = 0$ for some positive integer $k$). The following fact is from linear algebra (see the Lemma of page 313 in [3]. Since the reference is in Chinese, we repeat the proof here).
Lemma 6.2  I+A is conjugate to a direct sum of Jordan blocks with 1s along the diagonal.

Proof. We prove that V has a basis

\[ \{a_1, Aa_1, \ldots, A^{k_1-1}a_1, a_2, Aa_2, \ldots, A^{k_2-1}a_2, \ldots, a_s, \ldots, Aa_s, \ldots, A^{k_s-1}a_s\} \]

satisfying \( A^ka_i = 0 \) for each \( i \), which implies that the representation matrix of \( I + A \) is similar to a direct sum of Jordan blocks with 1 along the diagonal. The proof is based on the induction of \( \text{dim} \ V \). When \( \text{dim} \ V = 1 \), choose \( 0 \neq v \in V \). Suppose that \( Av = \lambda v \). Then \( A^kv = \lambda^kv = 0 \) and thus \( \lambda = 0 \). Suppose that the case is proved for vector spaces of dimension \( k < n \). Note that the invariant subspace \( AV \) is a proper subspace of \( V \) (otherwise, \( AV = V \) implies \( A^kV = A^{k-1}V = V = 0 \)). By induction, the subspace \( AV \) has a basis

\[ \{a_1, Aa_1, \ldots, A^{k_1-1}a_1, a_2, Aa_2, \ldots, A^{k_2-1}a_2, \ldots, a_s, \ldots, Aa_s, \ldots, A^{k_s-1}a_s\} \]

Choose \( b_i \in V \) satisfying \( A(b_i) = a_i \). Then \( A \) maps the set

\[ \{b_1, Ab_1, \ldots, A^{k_1}b_1, b_2, Ab_2, \ldots, A^{k_2}b_2, b_s, \ldots, A^{k_s}b_s\} \]

to the basis

\[ \{a_1, Aa_1, \ldots, A^{k_1-1}a_1, a_2, Aa_2, \ldots, A^{k_2-1}a_2, \ldots, a_s, \ldots, Aa_s, \ldots, A^{k_s-1}a_s\} \]

This implies that the former set is linearly independent (noting that \( A(A^{k_i}b_i) = 0 \)). Extend this set to be a \( V \)'s basis

\[ \{b_1, Ab_1, \ldots, A^{k_1}b_1, b_2, Ab_2, \ldots, A^{k_2}b_2, b_s, \ldots, A^{k_s}b_s, b_{s+1}, \ldots, b_{s'}\} \]

Note that \( Ab_i = 0 \) for \( i \geq s + 1 \) and \( A^{k_i+1}b_i = A^{k_i}a_i = 0 \) for each \( i \leq s \). This finishes the proof.

Corollary 6.3 Let \( A_{n \times n} \) be a strictly upper triangular matrix over a field \( K \) of characteristic \( \text{ch}(K) \neq 2 \). Then \( A^2 \) is conjugate to \( A \).

Proof. Suppose that \( A = I + u \) for a nilpotent matrix \( u \). Lemma 6.2 implies that \( A^2 \) is conjugate to a direct sum of Jordan blocks. Without loss of generality, we assume \( A \) is a Jordan block. Then \( A^2 = I + 2u + u^2 \). By Lemma 6.2 again, \( A^2 \) is conjugate to a direct sum of Jordan blocks with 1s along the diagonal. The minimal polynomial of \( A^2 \) is \( (x - 1)^n \), which shows that there is only one block in the direct sum and thus \( A^2 \) is conjugate to \( A \).

Recall that a matrix \( A \in \text{GL}_n(\mathbb{R}) \) is called semisimple if as a complex matrix \( A \) is conjugate to a diagonal matrix. A semisimple matrix \( A \) is elliptic (respectively, hyperbolic) if all its (complex) eigenvalues have modulus 1 (respectively, are \( > 0 \)). The following lemma is the complete multiplicative Jordan (or Jordan-Chevalley) decomposition (cf. [35], Lemma 7.1, page 430).

Lemma 6.4 Each \( A \in \text{GL}_n(\mathbb{R}) \) can be uniquely written as \( A = euh \), where \( e, h, u \in \text{GL}_n(\mathbb{R}) \) are elliptic, hyperbolic and unipotent, respectively, and all three commute.

The following result characterizes the continuous length functions on compact Lie groups.

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Lemma 6.5 Let $G$ be a compact connected Lie group and $l$ a continuous length function on $G$. Then $l = 0$.

Proof. For any element $g \in G$, there is a maximal torus $T \ni g$. For finite order element $h \in T$, we have $l(h) = 0$. Note that the set of finite-order elements is dense in $T$. Since $l$ is continuous, $l$ vanishes on $T$ and thus $l(g) = 0$ for any $g$. ■

Theorem 6.6 Let $G = \text{SL}_n(\mathbb{R})$ $(n \geq 2)$. Let $l : G \to [0, +\infty)$ be a length function, which is continuous on compact subgroups and the subgroup of diagonal matrices with positive diagonal entries. Then $l$ is uniquely determined by its images on the subgroup $D$ of diagonal matrices with positive diagonal entries.

Proof. For any $g \in \text{SL}_n(\mathbb{R})$, let $g = ehu$ be the Jordan decomposition for commuting elements $e, h, u$, where $e$ is elliptic, $h$ is hyperbolic and $u$ is unipotent (see Lemma 6.4) after multiplications by suitable powers of determinants. Then

$$l(g) \leq l(e) + l(h) + l(u).$$

For any unipotent matrix $u$, there is an invertible matrix $a$ such that $aua^{-1}$ is strictly upper triangular (see [35], Theorem 7.2, page 431). Lemma 6.3 implies that $u^2$ is conjugate to $u$. Therefore, $l(u) = 0$ by Lemma 6.1. Since $l$ vanishes on a compact Lie group (cf. Lemma 6.5), we have that $l(e) = 0$ for any elliptic matrix $e$. Therefore, $l(g) \leq l(h)$. Similarly,

$$l(h) = l(e^{-1}gu^{-1}) \leq l(g)$$

which implies $l(g) = l(h)$. Note that a hyperbolic matrix is conjugate to a real diagonal matrix with positive diagonal entries. ■

Let $h_{i1}(x)$ $(i = 2, \ldots, n)$ be an $n \times n$ diagonal matrix whose $(1, 1)$-entry is $x$, $(i, i)$-entry is $x^{-1}$, while other diagonal entries are 1s and non-diagonal entries are 0s.

Proof of Theorem 0.3. By Theorem 6.6, the length function $l$ is determined by its image on the subgroup $D$ generated by

$$h_{12}(x) = \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}, x \in \mathbb{R}_{>0}.$$ Take $x = e^t$, $t \in \mathbb{R}$. We have $l(h_{12}(e^\frac{k}{r})) = \frac{|k|}{r}l(h_{12}(e))$ for any rational number $\frac{k}{r}$. Since $l$ satisfies condition 2) of the definition and is continuous on $D$, we see that $l|_D$ is determined by the image $l(h_{12}(e))$ (actually, any real number $t$ is a limit of a rational sequence). Note that the translation function $\tau$ vanishes on compact subgroups and is continuous on the subgroup of diagonal subgroups with positive diagonal entries (cf. [16], Cor. 10.42 and Ex. 10.43, page 320). Therefore, $l$ is proportional to $\tau$. Actually, $\tau$ can be determined explicitly by the formula $\text{tr}(A) = \pm 2\cosh \frac{\tau(A)}{2}$ (for nonzero $\tau(A)$), where $\text{tr}$ is the trace and $\cosh$ is the hyperbolic cosine function (see [7], Section 7.34, page 173). This implies that $l(A)$ is determined by the spectral radius of $A$ (which could also be seen clearly by the matrix norm). ■

The subgroup $D = \text{SL}_n(\mathbb{R})$ of diagonal matrices with positive diagonal entries is isomorphic to $(\mathbb{R}_{>0})^{n-1}$ and $D$ is generated by the matrices $h_{i1}(x)$ $(i = 2, \ldots, n)$ whose $(1, 1)$-entry is $x$, $(i, i)$-entry is $x^{-1}$. Since $h_{i1}(x)$ $(i \neq 1)$ is conjugate to $h_{12}(x)$, a length function $l : \text{SL}_n(\mathbb{R}) \to [0, +\infty)$ is completely determined by its image on the convex hull spanned by $h_{12}(e), h_{13}(e), \ldots, h_{1n}(e)$ (see Theorem 7.10 for a more general result on Lie groups). Here $e$ is the Euler’s number in the natural exponential function.
Corollary 6.7 Let \( l : \text{SL}_2(\mathbb{R}) \to [0, +\infty) \) be a non-trivial length function that is continuous on the subgroup \( \text{SO}(2) \) and the diagonal subgroup. Then \( l(g) > 0 \) if and only if \( g \) is hyperbolic.

**Proof.** It is well-known that the elements in \( \text{SL}_2(\mathbb{R}) \) are classified as elliptic, hyperbolic and parabolic elements. Moreover, the translation length \( \tau \) vanishes on the compact subgroup \( \text{SO}(2) \) and the parabolic elements. The corollary follows from Theorem 6.3.

When the length function \( l \) is the asymptotic distortion function \( \text{dist}_{\infty} \), Corollary 6.7 is known to Navas [27] (Proposition 4).

## 7 Length functions on algebraic and Lie groups

For an algebraic group \( G \), let \( k[G] \) be the regular ring. For any \( g \in G \), let \( \rho_g : k[G] \to k[G] \) be the right translation by \( g \). The following is the famous Jordan (or Jordan-Chevalley) decomposition.

**Lemma 7.1** ([38], p.99) Let \( G \) be an algebraic group and \( g \in G \). There exists unique elements \( g_s, g_u \) such that \( g_s g_u = g_u g_s \), and \( \rho_{g_s} \) is semisimple, \( \rho_{g_u} \) is unipotent.

**Lemma 7.2** ([43]) Let \( G \) be a reductive connected algebraic group over an algebraically closed field \( k \). The conjugacy classes of unipotent elements in \( G \) are finite.

**Lemma 7.3** ([33], Theorem 3.4) If \( G \) is a reductive linear algebraic group defined over a field \( k \) and \( g \in G(k) \) then the set of conjugacy classes in \( G(k) \) which when base changed to the algebraically closed field \( \overline{k} \) are equal to the conjugacy class of \( g \) in \( G(\overline{k}) \) is in bijection with a subset of \( H^1(\overline{k}/k, Z(g)(k)) \), the Galois cohomology set.

**Definition 7.4** A field \( k \) is of type \( (F) \) if for any integer \( n \) there exist only finitely many extensions of \( k \) of degree \( n \) (in a fixed algebraic closure \( \overline{k} \) of \( k \)).

Examples of fields of type \( (F) \) include: the field \( \mathbb{R} \) of reals, a finite field, the field of formal power series over an algebraically closed field.

**Lemma 7.5** [Borel-Serre [17], Theorem 6.2] Let \( k \) be a field of type \( (F) \) and let \( H \) be a linear algebraic group defined over \( k \). The set \( H^1(\overline{k}/k, H(k)) \) is finite.

**Lemma 7.6** Let \( G(k) \) be a reductive linear algebraic group over a perfect field of type \( (F) \) and \( l \) a length function on \( G = G(k) \). Then \( l(g) = l(g_s) \), where \( g_s \) is the semisimple part of \( g \).

**Proof.** By the Jordan decomposition \( g = g_s g_u \), we have \( l(g) \leq l(g_s) + l(g_u) \) and \( l(g_s) \leq l(g) + l(g_u^{-1}) \). Note that for any integer \( n \), \( g_u^n \) is also unipotent. By the Lemma 7.2, Lemma 7.3 and Lemma 7.5, there are only finitely many conjugacy classes of unipotent elements. This implies that \( h g_u^{n_1} h^{-1} = g_u^{n_2} \) for distinct positive integers \( n_1, n_2 \), and some \( h \in G \). Therefore, we have \( n_1 l(g_u) = n_2 l(g_u) \), which implies that \( l(g_u) = 0 \) and thus \( l(g) = l(g_s) \).

A Lie group \( G \) is semisimple if its maximal connected solvable normal subgroup is trivial. Let \( \mathfrak{g} \) be its Lie algebra and let \( \exp : \mathfrak{g} \to G \) denote the exponential map. An element \( x \in \mathfrak{g} \) is real semi-simple if \( \text{Ad}(x) \) is diagonalizable over \( \mathbb{R} \). An element \( g \in G \) is
called hyperbolic (resp. unipotent) if \( g \) is of the form \( g = \exp(x) \) where \( x \) is real semisimple (resp. nilpotent). In either case, the element \( x \) is easily seen to be unique and we write \( x = \log g \). The following is the Jordan decomposition in Lie groups. An element \( e \in G \) is elliptic if \( \text{Ad}(e) \) is diagonalizable over \( \mathbb{C} \) with eigenvalues 1.

**Lemma 7.7** ([43], Prop. 2.1 and Remark 2.1)

1. Let \( g \in G \) be arbitrary. Then \( g \) may be uniquely written
   \[ g = e(g)h(g)u(g) \]
   where \( e(g) \) is elliptic, \( h(g) \) is hyperbolic and \( u(g) \) is unipotent and where the three elements \( e(g), h(g), u(g) \) commute.

2. An element \( f \in G \) commutes with \( g \) if and only if \( f \) commutes with the three components. Moreover, if \( f, g \) commutes, then
   \[ e(fg) = e(f)e(g), h(fg) = h(f)h(g), u(fg) = u(f)u(g). \]

**Lemma 7.8** ([25], Prop. 1.14.6, page 63) Let \( G \) be a connected semisimple Lie group whose center is trivial. Then there exists an integer \( n \geq 2 \) and an algebraic group \( G^* \in \mathfrak{gl}_n(\mathbb{C}) \) defined over \( \mathbb{Q} \) such that \( G \) is isomorphic to \( G^*_R \) (the connected component of \( G^*_R \) containing the identity) as a Lie group.

Let \( G = KAN \) be an Iwasawa decomposition. The Weyl group \( W \) is the finite group defined as the quotient of the normalizer of \( A \) in \( K \) modulo the centralizer of \( A \) in \( K \). For an element \( h \in A \), let \( W(h) \) be the set of all elements in \( A \) which are conjugate to \( h \) in \( G \).

**Lemma 7.9** ([43], Prop. 2.4) An element \( h \in G \) is hyperbolic if and only if it is conjugate to an element in \( A \). In such a case, \( W(h) \) is a single \( W \)-orbit in \( A \).

**Theorem 7.10** Let \( G \) be a connected semisimple Lie group whose center is finite with an Iwasawa decomposition \( G = KAN \). Let \( W \) be the Weyl group.

(i) Any length function \( l \) on \( G \) that is continuous on the maximal compact subgroup \( K \) is determined by its image on \( A \).

(ii) Conversely, any length function \( l \) on \( A \) that is \( W \)-invariant (i.e. \( l(w \cdot a) = l(a) \)) can be extended to be a length function on \( G \) that vanishes on the maximal compact subgroup \( K \).

**Proof.** (i) For any \( g \in G \), the Jordan decomposition gives \( g = ehu \), where \( e \) is elliptic, \( h \) is hyperbolic and \( u \) is unipotent and where the three elements \( e, h, u \) commute (cf. Lemma 7.7). We assume that \( G \) has the trivial center. By Lemma 7.8, the Lie group \( G \) is an algebraic group. Lemma 7.6 implies that \( l \) vanishes on unipotent elements and \( l(g) = l(eh) \). Since \( l \) vanishes on \( e \) (cf. Lemma 6.5), we have \( l(g) = l(h) \). Therefore, the function \( l \) is determined by its image on \( A \). Let \( Z \) be the center of \( G \). Then \( G/Z \) is connected with trivial center. For any \( z \in Z, g \in G \), we have \( l(z) = 0 \) and \( l(gz) = l(g) \). For any \([g] \in G/Z \), define \( l'(\[g]\)) = l(g) \). The length function \( l \) factors through a length function \( l' \) on \( G/Z \) (By Corollary 1.8 it is enough to show that \( l' \) is subadditive for
commuting pair \( a, b \in G/Z \). But \( l'(ab) = l(h_a h_b u_a u_b) \leq l(h_a u_a) + l(h_b u_b) = l'(a) + l'(b) \), where \( h_a, h_b, u_a, u_b \) are hyperbolic, unipotent parts of preimages of \( a', b' \), respectively.

(ii) Let \( l \) be a length function \( l \) on \( A \) that is \( W \)-invariant. We first extend \( l \) to the set \( H \) of all the conjugates of \( A \). For any \( g \in G, a \in A \), define \( l'(gag^{-1}) = l(a) \). If

\[
g_1 a_1 g_1^{-1} = g_2 a_2 g_2^{-1}
\]

for \( g_1, g_2 \in G, a_1, a_2 \in A \), then \( g_1^{-1} g_2 a_2 g_2^{-1} g_1 = a_1 \). By Lemma 7.9 there exists an element \( w \in W \) such that \( w \cdot a_2 = a_1 \). Therefore, we have \( l(a_1) = l(a_2) \) and thus \( l' \) is well-defined on the set \( H \) of conjugates of elements in \( A \). Such a set \( H \) is the set consisting of hyperbolic elements by Lemma 7.9. We then extend \( l' \) on the set of all conjugates of elements in \( K \). For any \( g \in G, k \in K \), define \( l'(gkg^{-1}) = 0 \). If

\[
g_1 k g_1^{-1} = g_2 a g_2^{-1}
\]

for \( g_1, g_2 \in G, k \in K, a \in A \), then \( g_1^{-1} g_2 a g_2^{-1} g_1 = k \). Then \( k \) is both hyperbolic and elliptic. The only element which is both elliptic and hyperbolic is the identity element. Therefore, we have \( k = a = 1 \) and

\[
l'(g_1 k g_1^{-1}) = l(g_2 a g_2^{-1}) = 1.
\]

This shows that \( l' \) is well-defined on the set of hyperbolic elements and elliptic elements. For any unipotent element \( u \in G \), define \( l'(u) = 0 \). For any element \( g \in G \), let \( g = ehu \) be the Jordan decomposition. Define \( l'(g) = l'(h) \).

We check the function \( l' \) is a length function on \( G \). The definition shows that \( l' \) is conjugate invariant. For any positive integer \( n \) and any \( g \in G \) with Jordan decomposition \( g = ehu \), we have \( g^n = e^n h^n u^n \) and thus \( l'(g^n) = l'(h^n) \). But \( h^n \) is hyperbolic and conjugate to an element in \( A \) (see Lemma 7.9). Therefore, we have \( l'(h^n) = |n|l'(h) \) and thus \( l'(g^n) = |n|l'(g) \). If \( g_1 = e_1 h_1 u_1 \) commutes with \( g_2 = e_2 h_2 u_2 \), then

\[
g_1 g_2 = e_1 e_2 h_1 h_2 u_1 u_2
\]

(cf. Lemma 7.7) and \( l'(g_1 g_2) = l'(h_1 h_2) \). Since \( h_1, h_2 \) are commuting hyperbolic elements, they are conjugate simultaneously to elements in \( A \). Therefore, we have

\[
l'(h_1 h_2) \leq l'(h_1) + l'(h_2)
\]

and

\[
l'(g_1 g_2) \leq l'(g_1) + l'(g_2).
\]

\[\blacksquare\]

**Remark 7.11** A length function \( l \) on \( A \) is determined by a group homomorphism \( f : A \to \mathbb{B} \), for a real Banach space \( (\mathbb{B}, \|\|) \), satisfying \( l(a) = \|f(a)\| \) (see Lemma 4.1). The previous theorem implies that a length function \( l \) on the Lie group \( G \) (that is continuous on compact subgroup) is uniquely determined by such a group homomorphism \( f : A \to \mathbb{B} \) such that \( \|f(a)\| = \|f(wa)\| \) for any \( a \in A \) and \( w \in W \), the Weyl group.

Let \( G \) be a connected semisimple Lie group whose center is finite with an Iwasawa decomposition \( G = KAN \). Let \( \exp : g \to G \) be the exponent map from the Lie algebra \( g \) with subalgebra \( \mathfrak{h} \) corresponding to \( A \).
Theorem 7.12 Suppose that \( l \) is a length function on \( G \) that is continuous on \( K \) and \( A \). Then \( l \) is determined by its image on \( \exp(v) \) (unit vector \( v \in h \)) in a fixed closed Weyl chamber of \( A \).

**Proof.** Let \( Z \) be the center of \( G \). Then \( G/Z \) is connected with trivial center. The length function \( l \) factors through a length function on \( G/Z \). For any \( g \in G \) we have \( g = ehu \), where \( e \) is elliptic, \( h \) is hyperbolic and \( u \) is unipotent and where the three elements \( e, h, u \) commute (cf. Lemma 7.7). By Lemma 7.8 and Lemma 7.6, we have \( l(g) = l(eh) \). Since \( l \) vanishes on \( e \) (cf. Lemma 6.5), we have \( l(g) = l(h) \).

Any element \( h \in A \) is conjugate to an element in a fixed Weyl chamber \( C \) (cf. [39], Theorem 8.20, page 254). For any element \( \exp(x) \in C \), with unit vector \( x \in h \), the one-parameter subgroup \( \exp(Rx) \) lies in \( A \). Since \( l \) is continuous on \( A \), the function \( l \) is determined by its image on \( \exp(Qx) \).

Note that \( l(e^m) = \frac{m}{n} l(e^n) = \frac{m}{n} l(e^x) \) for any rational number \( \frac{m}{n} \). The function \( l \) is determined by \( l(e^x) \), for all unit vectors \( x \) in the fixed closed Weyl chamber.

**Corollary 7.13** Let \( G \) be a connected semisimple Lie group whose center is finite of real rank 1. There is essentially only one length function on \( G \). In order words, any continuous length function is proportional to the translation function on the symmetric space \( G/K \).

**Proof.** When the real rank of \( G \) is 1, a closed Weyl chamber is of dimension 1. Therefore, the previous theorem implies that any continuous length function is determined by its image on a unit vector in a split torus.

8 Rigidity of group homomorphisms on arithmetic groups

Let \( V \) denote a finite-dimensional vector space over \( \mathbb{C} \), endowed with a \( \mathbb{Q} \)-structure. Recall that the arithmetic subgroup is defined as the following (cf. Borel [13], page 37).

**Definition 8.1** Let \( G \) be a \( \mathbb{Q} \)-subgroup of \( GL(V) \). A subgroup \( \Gamma \) of \( G_{\mathbb{Q}} \) is said to be arithmetic if there exists a lattice \( L \) of \( V_{\mathbb{Q}} \) such that \( \Gamma \) is commensurable with \( G_{\mathbb{Q}} \) \( L \) = \( \{ g \in G_{\mathbb{Q}} : gL = L \} \).

Let \( \Gamma \) be an arithmetic subgroup of a simple algebraic \( \mathbb{Q} \)-group of \( \mathbb{Q} \)-rank at least 2. Suppose that \( G \) is virtually poly-positive. We will prove Theorem 0.5, i.e. any group homomorphism \( f : \Gamma \rightarrow G \) has its image finite.

Recall that a group is quasi-simple, if any non-trivial normal subgroup is either finite or of finite index. The Margulis-Kazhdan theorem (see [66], Theorem 8.1.2) implies that an irreducible lattice (and hence) in a semisimple Lie group of real rank \( \geq 2 \) is quasi-simple.

**Lemma 8.2** Let \( \Gamma \) be a finitely generated quasi-simple group that contains a Heisenberg subgroup, i.e. there are elements torsion-free elements \( a, b, c \in \Gamma \) satisfying \( [a, b] = c, [a, c] = [c, b] = 1 \). Suppose that \( G \) is virtually poly-positive. Then any group homomorphism \( f : \Gamma \rightarrow G \) has its image finite.

**Proof.** Suppose that \( G \) has a finite-index subgroup \( H \) and a subnormal series

\[
1 = H_n \triangleleft H_{n-1} \triangleleft \cdots \triangleleft H_0 = H
\]
such that every finitely generated subgroup of $H_i/H_{i+1}$ has a purely positive length function. Without loss of generality, we assume that $H$ is normal. Let $f : \Gamma \to G$ be a homomorphism. The kernel of the composite

$$f_0 : \Gamma \xrightarrow{f} G \to G/H$$

is finitely generated. Suppose that the image of the composite

$$f_1 : \ker f_0 \xrightarrow{f} H \to H/H_1$$

has a purely positive length function $l$. After passing to finite-index subgroups, we may still suppose that $\ker f_0$ contains a Heisenberg subgroup $\langle a, b, c \rangle$. By Lemma 5.2, the length function $l$ vanishes on $f_1(c)$. Therefore $f_1(c^k) = 1 \in H/H_1$ for some integer $k > 0$. The normal subgroup $\ker f_1$ containing $c^k$ is of finite index. Now we have map $\ker f_1 \to H_1$ induced by $f$. An induction argument shows that $f$ maps some power $c^d$ of the central element of the Heisenberg subgroup into the identity $1 \in G$. Therefore, the image of $f$ is finite.

**Proof of Theorem 0.5.** It is well-known that $\Gamma$ contains a $\mathbb{Q}$-split simple subgroup whose root system is the reduced subsystem of the $\mathbb{Q}$-root system of $\Gamma$ (see [15], Theorem 7.2, page 117). Replacing $\Gamma$ with this $\mathbb{Q}$-subgroup, we may assume $\Gamma$ is $\mathbb{Q}$-split and the root system of $\Gamma$ is reduced. Because $\Gamma$ is simple and $\mathbb{Q}$-rank$(\Gamma) \geq 2$, we know that the $\mathbb{Q}$-root system of $\Gamma$ is irreducible and has rank at least two. Therefore, the $\mathbb{Q}$-root system of $\Gamma$ contains an irreducible subsystem of rank two, that is, a root subsystem of type $A_2, B_2, G_2$ (see [64], page 338). For $A_2$, choose $\{\alpha_1, \alpha_3\}$ as a set of simple roots (see Figure 1). Then the root element $x_{\alpha_1 + \alpha_3}(rs) = x_{\alpha_2}(rs)$ is a commutator $[x_{\alpha_1}(r), x_{\alpha_3}(s)]$, with $x_{\alpha_2}(rs)$ commutes with $x_{\alpha_1}(r), x_{\alpha_3}(s)$. For $G_2$, the long roots form a subsystem of $A_2$. For $B_2$, choose $\{\alpha_1, \alpha_4\}$ as a set of simple roots (see Figure 2). The long root element $x_{\alpha_4}(2rs)$ is a commutator $[x_{\alpha_2}(r), x_{\alpha_4}(s)]$ of the two short root elements, and $x_{\alpha_3}(2rs)$ commutes with $x_{\alpha_2}(r), x_{\alpha_4}(s)$ (cf. [38], Proposition of page 211). This shows that the arithmetic subgroup $\Gamma$ contains a Heisenberg subgroup. The theorem is then proved by Lemma 8.2.

$\blacksquare$

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If we consider special length functions, general results can be proved. When we consider the stable word lengths, the following is essentially already known (cf. Polterovich [57], Corollary 1.1.D and its proof).

**Proposition 8.3** Let $\Gamma$ be an irreducible non-uniform lattice in a semisimple connected, Lie group without compact factors and with finite center of real rank $\geq 2$. Assume that a group $G$ has a virtually poly-positive stable word length. In other words, the group $G$ has a finite-index subgroup $H$ and a subnormal series

$$1 = H_n \triangleleft H_{n-1} \triangleleft \cdots \triangleleft H_0 = H$$

such that every finitely generated subgroup of $H_i/H_{i+1}$ ($i = 0, 1, \ldots, n - 1$) has a purely positive stable word length. Then any group homomorphism $f : \Gamma \to G$ has its image finite.

**Proof.** Without loss of generality, we assume that $f$ takes its image in $H$. Since a lattice is finitely generated, $\Gamma$ has its image in $H_0/H_1$ finitely generated. When the image has a purely positive word length, any distorted element in $\Gamma$ must have a trivial image in $H_0/H_1$ (see Lemma 1.2). Lubotzky, Mozes and Raghunathan [46] prove that irreducible non-uniform lattices in higher rank Lie groups have non-trivial distortion elements (They prove the stronger result that there are elements in the group whose word length has logarithmic growth). Then a finite-index subgroup $\Gamma_0 < \Gamma$ will have image in $H_1$, since high-rank irreducible lattices are quasi-simple. An induction argument finishes the proof.

When we consider the length given by quasi-cocycles, the following is also essentially already known (cf. Py [58], Prop. 2.2, following Burger-Monod [18] [19]). Recall that a locally compact group has property TT if any continuous rough action on a Hilbert space has bounded orbits (see [53], page 172). Burger-Monod proves that an irreducible lattice $\Gamma$ in a high-rank semisimple Lie group has property TT.

**Proposition 8.4** Let $\Gamma$ be an irreducible lattice in a semisimple connected, Lie group without compact factors and with a finite center of real rank $\geq 2$. Assume that a group $G$ has a virtually poly-positive average norm for quasi-cocycles. In other words, the group $G$ has a finite-index subgroup $H$ and a subnormal series

$$1 = H_n \triangleleft H_{n-1} \triangleleft \cdots \triangleleft H_0 = H$$

such that every finitely generated subgroup of $H_i/H_{i+1}$ ($i = 0, 1, \ldots, n - 1$) has a purely positive length given by a quasi-cocycle with values in Hilbert spaces. Then any group homomorphism $f : \Gamma \to G$ has its image finite.

**Proof.** Note that a group $\Gamma$ has property TT if and only if $H^1(\Gamma; E) = 0$ and

$$\ker(H^2_b(\Gamma; E) \to H^2(\Gamma; E)) = 0$$

for any linear isometric action of $\Gamma$ on a Hilbert space $E$. Here $H^2_b(\Gamma; E)$ is the second bounded cohomology group. Suppose that $u : \Gamma \to E$ is a quasi-cocycle. There is a bounded map $v : \Gamma \to E$ and a 1-cocycle $w : \Gamma \to E$ such that

$$u = v + w,$$
by Proposition 2.1 of Py\cite{58}. Since $\Gamma$ has property T, there exists $x_0 \in E$ such that $w(\gamma) = \gamma x_0 - x_0$. Therefore, we have
\[
\|u(\gamma^n)\|_n = \|v(\gamma^n) + w(\gamma^n)\|_n = \|v(\gamma^n) + \gamma^n x_0 - x_0\|_n \\
\leq \|v(\gamma^n)\|_n + 2\|x_0\|_n \to 0.
\]

Without loss of generality, we assume that $G = H$. Suppose that any finitely generated subgroup of $H/H_1$ has a purely positive average norm $l$ given by a cocycle. The composite
\[
\Gamma \xrightarrow{f} H \xrightarrow{g} H/H_1
\]
has a finite-index kernel $\Gamma_0$, since $l$ vanishes on infinite-order elements of the image. This implies that $f(\Gamma_0)$ lies in $H_1$. A similar argument proves that $\ker f$ is of finite index in the general case. \[\square\]

9 Rigidty of group homomorphisms on matrix groups

9.1 Steinberg groups over finite rings

Recall that a ring $R$ is right Artinian if any non-empty family of right ideals contains minimal elements. A ring $R$ is semi-local if $R/\text{rad}(R)$ is right Artinian (see Bass’ K-theory book \cite{5} page 79 and page 86), where $\text{rad}(R)$ is the Jacobson radical. Let $n$ be a positive integer and $R^n$ the free $R$-module of rank $n$ with the standard basis. A vector $(a_1, \ldots, a_n)$ in $R^n$ is called right unimodular if there are elements $b_1, \ldots, b_n \in R$ such that $a_1 b_1 + \cdots + a_n b_n = 1$. The stable range condition $sr_m$ says that if $(a_1, \ldots, a_{m+1})$ is a right unimodular vector then there exist elements $b_1, \ldots, b_m \in R$ such that $(a_1 + a_{m+1} b_1, \ldots, a_m + a_{m+1} b_m)$ is right unimodular. It follows easily that $sr_m \Rightarrow sr_n$ for any $n \geq m$. A semi-local ring has the stable range $sr_2$ (\cite{5}, page 267, the proof of Theorem 9.1). A finite ring $R$ is right Artinian and thus has $sr_2$. The stable range
\[
sr(R) = \min\{m : R \text{ has } sr_{m+1}\}.
\]

Thus $sr(R) = 1$ for a finite ring $R$.

We briefly recall the definitions of the elementary subgroups $E_n(R)$ of the general linear group $GL_n(R)$, and the Steinberg groups $St_n(R)$. Let $R$ be an associative ring with identity and $n \geq 2$ be an integer. The general linear group $GL_n(R)$ is the group of all $n \times n$ invertible matrices with entries in $R$. For an element $r \in R$ and any integers $i, j$ such that $1 \leq i \neq j \leq n$, denote by $e_{ij}(r)$ the elementary $n \times n$ matrix with 1s in the diagonal positions and $r$ in the $(i, j)$-th position and zeros elsewhere. The group $E_n(R)$ is generated by all such $e_{ij}(r)$, i.e.
\[
E_n(R) = \langle e_{ij}(r) \mid 1 \leq i \neq j \leq n, r \in R \rangle.
\]

Denote by $I_n$ the identity matrix and by $[a, b]$ the commutator $aba^{-1}b^{-1}$.

The following lemma displays the commutator formulas for $E_n(R)$ (cf. Lemma 9.4 in \cite{19}).
Lemma 9.1 Let $R$ be a ring and $r, s \in R$. Then for distinct integers $i, j, k, l$ with $1 \leq i, j, k, l \leq n$, the following hold:

1) $e_{ij}(r + s) = e_{ij}(r)e_{ij}(s)$;
2) $[e_{ij}(r), e_{jk}(s)] = e_{ik}(rs)$;
3) $[e_{ij}(r), e_{kl}(s)] = I_n$.

By Lemma 9.1, the group $E_n(R)$ ($n \geq 3$) is finitely generated when the ring $R$ is finitely generated. Moreover, when $n \geq 3$, the group $E_n(R)$ is normally generated by any elementary matrix $e_{ij}(1)$.

The commutator formulas can be used to define Steinberg groups as follows. For $n \geq 3$, the Steinberg group $St_n(R)$ is the group generated by the symbols $\{x_{ij}(r) : 1 \leq i \neq j \leq n, r \in R\}$ subject to the following relations:

(St1) $x_{ij}(r + s) = x_{ij}(r)x_{ij}(s)$;
(St2) $[x_{ij}(r), x_{jk}(s)] = x_{ik}(rs)$ for $i \neq k$;
(St3) $[x_{ij}(r), x_{kl}(s)] = 1$ for $i \neq l, j \neq k$.

There is an obvious surjection $St_n(R) \to E_n(R)$ defined by $x_{ij}(r) \mapsto e_{ij}(r)$.

For any ideal $I \triangleleft R$, let $p : R \to R/I$ be the quotient map. Then the map $p$ induces a group homomorphism $p_* : St_n(R) \to St_n(R/I)$. Denote by $St_n(R, I)$ (resp., $E_n(R, I)$) the subgroup of $St_n(R)$ (resp., $E_n(R)$) normally generated by elements of the form $x_{ij}(r)$ (resp., $e_{ij}(r)$) for $r \in I$ and $1 \leq i \neq j \leq n$. In fact, $St_n(R, I)$ is the kernel of $p_*$ (cf. Lemma 13.18 in Magurn [49] and its proof). However, $E_n(R, I)$ may not be the kernel of $E_n(R) \to E_n(R/I)$ induced by $p$.

Lemma 9.2 When $n \geq \text{sr}(R) + 2$, the natural map $St_n(R) \to St_{n+1}(R)$ is injective. In particular, when $R$ is finite, the Steinberg group $St_n(R)$ is finite for any $n \geq 3$.

Proof. Let $W(n, R)$ be the kernel of the natural map $St_n(R) \to St_{n+1}(R)$. When $n \geq \text{sr}(R) + 2$, the kernel $W(n, R)$ is trivial (cf. Kolster [42], Theorem 3.1 and Cor. 2.10). When $n$ is sufficient large, the Steinberg group $St_n(R)$ is the universal central extension of $E_n(R)$ (cf. [63], Proposition 5.5.1. page 240). Therefore, the kernel $St_n(R) \to E_n(R)$ is the second homology group $H_2(E_n(R); \mathbb{Z})$. When $R$ is finite, both $E_n(R)$ and $H_2(E_n(R); \mathbb{Z})$ are finite. Therefore, the group $St_n(R)$ is finite for any $n \geq 3$. ■

9.2 Rigidity of group homomorphisms on matrix groups

Theorem 9.3 Suppose that $G$ is a group satisfying that

1) $G$ has a purely positive length function, i.e. there is a length function $l : G \to [0, \infty)$ such that $l(g) > 0$ for any infinite-order element $g$; and
2) any torsion abelian subgroup of $G$ is finitely generated.

Let $R$ be an associative ring with identity and $St_n(R)$ the Steinberg group. Suppose that $S < St_n(R)$ is a finite-index subgroup. Then any group homomorphism $f : St_n(R) \to G$ has its image finite when $n \geq 3$.
Proof. Since any ring $R$ is a quotient of a free (non-commutative) ring $\mathbb{Z}\langle X \rangle$ for some set $X$ and $\text{St}_n(R)$ is functorial with respect to the ring $R$, we assume without loss of generality that $R = \mathbb{Z}\langle X \rangle$. We prove the case $S = \text{St}_n(R)$ first. Let $x_{ij} = (x_{ij}(r) : r \in R)$, which is isomorphic to the abelian group $R$. Note that

$$[x_{12}(1), x_{23}(1)] = x_{13}(1)$$

and $x_{13}(1)$ commutes with $x_{12}(1)$ and $x_{23}(1)$. Lemma 5.2 implies any length function vanishes on $x_{13}(1)$. By Lemma 1.4, the length $l(f(x_{13}(1))) = 0$. Note that $x_{ij}(r)$ is conjugate to $x_{13}(r)$ for any $r \in R$ and $i, j$ satisfying $1 \leq i \neq j \leq n$. Since $l$ is purely positive, we get that $f(x_{12}(1))$ is of finite order. Let $I = \ker f|_{x_{12}}$. Then $I \neq \emptyset$, as $f(x_{12}(1))$ is of finite order. For any $x \in I$, and $y \in R$, we have

$$x_{12}(xy) = [x_{13}(x), x_{32}(y)].$$

Therefore,

$$f(x_{12}(xy)) = [f(x_{13}(x)), f(x_{32}(y))] = 1$$

and thus $xy \in I$. Similarly, we have $f(x_{12}(yx)) = f([x_{13}(y), x_{32}(x)]) = 1$. This proves that $I$ is a (two-sided) ideal. Note that $f(x_{12}) = R/I$ is a torsion abelian group. By the assumption 2), the quotient ring $R/I$ is finite. Let $\text{St}_n(R, I)$ be the normal subgroup of $\text{St}_n(R)$ generated by $x_{ij}(r), r \in I$. There is a short exact sequence

$$1 \to \text{St}_n(R, I) \to \text{St}_n(R) \to \text{St}_n(R/I) \to 1.$$ 

Since $R/I$ is finite, we know that $\text{St}_n(R/I)$ is finite by Lemma 0.2. This proves that $\text{Im} f$ is finite since $f$ factors through $\text{St}_n(R/I)$. For general finite-index subgroup $S$, we assume $S$ is normal in $\text{St}_n(R)$ after passing to a finite-index subgroup of $S$. A similar proof shows that $S$ contains $\text{St}_n(R, I)$ for some ideal $I$ with the quotient ring $R/I$ finite. Therefore, the image $\text{Im} f$ is finite. ■

Theorem 9.4 Suppose that $G$ is a group having a purely positive length function $l$. Let $R$ be an associative ring of characteristic zero such that any nonzero ideal is of a finite index (e.g. the ring of algebraic integers in a number field). Suppose that $S < \text{St}_n(R)$ is a finite-index subgroup of the Steinberg group. Then any group homomorphism $f : S \to G$ has its image finite when $n \geq 3$.

Proof. The proof is similar to that of Theorem 9.3 Let $I = \ker f|_{x_{12}}$, where $x_{12} = S \cap (x_{12}(r) : r \in R)$. Since $R$ is of characteristic zero and the length $l(f(x_{12}(k))) = 0$ for some integer $k$, we have $f(x_{12}(k))$ is of finite order. Therefore, $f(x_{12}(k')) = 1$ for some integer $k'$, which proves that the ideal $I$ is nonzero. Since $I$ is of finite index in $R$, we get that $\text{St}_n(R, I)$ is of finite index in $S$. This finishes the proof. ■

Since the natural map $\text{St}_n(R) \to E_n(R)$ is surjective, any group homomorphism $f : E_n(R) \to G$ can be lifted to be a group homomorphism $\text{St}_n(R) \to G$. Moreover, a finite-index subgroup $E$ of $E_n(R)$ is lifted to be a finite-index subgroup $S$ of $\text{St}_n(R)$. Theorem 0.6 and Theorem 0.8 follow from Theorem 9.3 and Theorem 9.4 by inductive arguments on the subnormal series as those of the proofs of Theorem 0.5.

Proof of Corollary 0.7 and Corollary 0.9 For Corollary 0.7 it is enough to check the two conditions for $G$ in Theorem 0.6. Lemma 3.6 proves that $G$ has a purely positive
length function. When \( G \) is a CAT(0) group, (i.e. \( G \) acts properly and cocompactly on a CAT(0) space), then any solvable subgroup of \( G \) is finitely generated (and actually virtually abelian, see the Solvable Subgroup Theorem of [16], Theorem 7.8, page 249).

When \( G \) is hyperbolic, it’s well-known that \( G \) contains finitely many conjugacy classes of finite subgroups and thus a torsion abelian subgroup is finite (see [16], Theorem 3.2, page 459). Birman-Lubotzky-McCarthy [9] proves that any abelian subgroup of the mapping class groups for orientable surfaces is finitely generated. Bestvina-Handel [8] proves that every solvable subgroup of Out\( (F_k) \) has a finite index subgroup that is finitely generated and free abelian. When \( G \) is the diffeomorphism group Diff\( (\Sigma, \omega) \), there is a subnormal series (see the proof of Lemma 3.5)

\[
1 < \text{Ham}(\Sigma, \omega) < \text{Diff}_0(\Sigma, \omega) < \text{Diff}(\Sigma, \omega),
\]

with subquotients in Ham\( (\Sigma, \omega) \), \( H_1(\Sigma, \mathbb{R}) \) and the mapping class group MCG\( (\Sigma) \). Any abelian subgroup of a finitely generated subgroup of these groups is finitely generated.

Corollary 0.9 follows from Theorem 0.4 and Lemma 3.6. ■

**Remark 9.5** An infinite torsion abelian group may act properly on a simplicial tree (see [16], Example 7.11, page 250). Therefore, condition 2) in Theorem 0.6 does not hold for every group \( G \) acting properly on a CAT(0) (or a Gromov hyperbolic) space. We don’t know whether condition 2) can be dropped.

## 10 Length functions on Cremona groups

Let \( k \) be a field and \( k(x_1, x_2, \ldots, x_n) \) be the field of rational functions in \( n \) indeterminates over \( k \). It is well-known that the Cremona group \( \text{Cr}_n(k) \) is isomorphic to the automorphism group \( \text{Aut}_k(k(x_1, x_2, \ldots, x_n)) \) of the field \( k(x_1, x_2, \ldots, x_n) \).

**Lemma 10.1** Let \( f : k(x_1, x_2, \ldots, x_n) \to k(x_1, x_2, \ldots, x_n) \) be given by \( f(x_1) = \alpha x_1, f(x_i) = x_i \) for some \( 0 \neq \alpha \in k \) and any \( i = 2, \ldots, n \). Then \( f \) lies in the center of a Heisenberg subgroup. In other words, there exists \( g, h \in \text{Cr}_n(k) \) such that \( [g, h] = ghg^{-1}h^{-1} = f \), \([g, f] = 1 \) and \([h, f] = 1 \).

**Proof.** Let \( g, h : k(x_1, x_2, \ldots, x_n) \to k(x_1, x_2, \ldots, x_n) \) be given by

\[
g(x_1) = x_1x_2, g(x_i) = x_i(i = 2, \ldots, n)
\]

and

\[
h(x_1) = x_1, h(x_2) = \alpha^{-1}x_2, h(x_j) = x_j(j = 3, \ldots, n).
\]

It can be directly checked that \([g, h] = f \), \([g, f] = 1 \) and \([h, f] = 1 \). ■

**Lemma 10.2** Let \( l : \text{Bir}(\mathbb{P}_k^n) \to [0, \infty) \) be a length function \( (n \geq 2) \). Then \( l \) vanishes on diagonal elements and unipotent elements of \( \text{Aut}(\mathbb{P}_k^n) = \text{PGL}_{n+1}(k) \).

**Proof.** Let \( g = \text{diag}(a_0, a_1, \ldots, a_n) \in \text{PGL}_{n+1}(k) \) be a diagonal element. Note that \( l \) is subadditive on the diagonal subgroups. In order to prove \( l(g) = 0 \), it is enough to prove that \( l(\text{diag}(1, \ldots, 1, a_i, 1, \ldots, 1)) = 0 \), where \( \text{diag}(1, \ldots, 1, a_i, 1, \ldots, 1) \) is the diagonal matrix with \( a_i \) in the \((i, i)\)-th position and all other diagonal entries are 1.
But diag(1, a, 1, ..., 1) is conjugate to diag(1, a, 1, ..., 1) for \( a = a_i \). Lemma 10.1 implies that diag(1, a, 1, ..., 1) lies in the center of a Heisenberg group. Therefore, \( l(\text{diag}(1, a, 1, ..., 1)) = 0 \) by Lemma 5.2. This proves \( l(g) = 0 \). The vanishing of \( l \) on unipotent elements follows from Corollary 6.3 when the characteristic of \( k \) is not 2. When the characteristic of \( k \) is 2, any unipotent element \( A = I + u \) (where \( u \) is nilpotent) is of finite order. This means \( l(A) = 0 \).

**Proof of Theorem 0.10.** When \( k \) is algebraically closed, the Jordan normal form implies that any element \( g \in \text{PGL}_n(k) \) is conjugate to the form \( s \) with \( s \) diagonal and \( n \) the strictly upper triangular matrix. Moreover, \( sn = ns \). Therefore, \( l(f) \leq l(s) + l(n) \). By Lemma 10.2, \( l(s) = l(n) = 0 \) and thus \( l(g) = 0 \).

**Proof of Corollary 0.11.** Let \( f : \text{Bir}(\mathbb{P}^2) \to G \) be a group homomorphism. Suppose that \( G \) has a purely positive length function \( l \). By Theorem 0.10, the purely positive length function \( l \) on \( G \) will vanish on \( f(\text{PGL}_3(k)) \). Since \( k \) is infinite and \( \text{PGL}_3(k) \) is a simple group, we get that \( \text{PGL}_3(k) \) lies in the \( \text{ker} f \). By the Noether-Castelnuovo Theorem, \( \text{Bir}(\mathbb{P}^2) \) is generated by \( \text{PGL}_3(k) \) and an involution. Moreover, the \( \text{Bir}(\mathbb{P}^2) \) is normally generated by \( \text{PGL}_3(k) \). Therefore, the group homomorphism \( f \) is trivial. The general case is proved by an inductive argument on the subnormal series of a finite-index subgroup of \( G \).

**Lemma 10.3** Let \( \text{Bir}(\mathbb{P}^n_R) \) \((n \geq 2)\) be the real Cremona group. Any length function \( l : \text{Bir}(\mathbb{P}^n_R) \to [0, \infty) \), which is continuous on \( \text{PSO}(n+1) < \text{Aut}(\mathbb{P}^n_R) \), vanishes on \( \text{PGL}_{n+1}(\mathbb{R}) \).

**Proof.** By Lemma 10.2, the length function \( l \) vanishes on diagonal matrices of \( \text{PGL}_{n+1}(\mathbb{R}) \). Theorem 0.4 implies that \( l \) vanishes on the whole group \( \text{PGL}_{n+1}(\mathbb{R}) \).

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