FINITE-SIZE SCALING OF THE DOMAIN WALL ENTROPY FOR THE 2D ±J
ISING SPIN GLASS

Ronald Fisch
382 Willowbrook Dr.
North Brunswick, NJ 08902
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The statistics of domain walls for ground states of the 2D Ising spin glass with +1 and -1 bonds are studied for \( L \times L \) square lattices with \( L \leq 20 \), and \( x = 0.25 \) and 0.5, where \( x \) is the fraction of negative bonds, using periodic and/or antiperiodic boundary conditions. Under these conditions, almost all domain walls have an energy \( E_{dw} \) equal to 0 or 4. The probability distribution of the entropy, \( S_{dw} \), is found to depend strongly on \( E_{dw} \). The results for \( S_{dw} \) when \( E_{dw} = 4 \) agree with the prediction of the droplet model. Our results for \( S_{dw} \) when \( E_{dw} = 0 \) agree with those of Saul and Kardar. In addition, we find that the distributions do not appear to be Gaussian in that case.

The special role of \( E_{dw} = 0 \) domain walls is discussed, and the discrepancy between the prediction of Amoruso, Hartmann, Hastings and Moore and the result of Saul and Kardar is explained.

I. INTRODUCTION

There continues to be a controversy about the nature of the Ising spin glass. The Sherrington-Kirkpatrick model, with its infinite-range interactions between the spins, is described by the Parisi replica-symmetry breaking mean-field theory. To understand models with short-range interactions on finite-dimensional lattices, however, it is necessary to include the effects of interfaces, which do not exist in a well-defined way in an infinite-range model. The droplet model of Fisher and Huse which starts from the domain-wall renormalization group ideas of McMillan and Bray and Moore, and studies the properties of interfaces, provides a very different viewpoint on the spin-glass phase. More recently, numerical results indicate that the actual situation for three-dimensional (3D) models combines elements of both pictures in a nontrivial way.

In two dimensions (2D), the spin-glass phase is not stable at finite temperature. Because of this, it is necessary to treat cases with continuous distributions of energies (CDE) and cases with quantized distributions of energies (QDE) separately.

In three or more space dimensions, where a spin-glass phase is believed to occur at finite temperature, the general framework of thermodynamics requires that the CDE and the QDE should be treated on the same footing. The way this comes about is that in these cases the typical domain wall energy increases as a positive power of the size of the lattice. Thus the quantization energy becomes a negligible fraction of the domain wall energy for large lattices. All bond distributions behave in a qualitatively similar way, except that the QDE have finite ground state entropies.

Amoruso, Hartmann, Hastings and Moore have recently proposed that in 2D there is a relation

\[ d_f = 1 + \frac{3}{4(3 + \theta_E)}, \]  

where \( d_f \) is the fractal dimension of domain walls, and \( \theta_E \) is the exponent which characterizes the scaling of the domain wall energy with size. For continuous energy distributions the existing numerical estimates of \( d_f \) and \( \theta_E \) satisfy Eqn. (1).

It is known from the droplet theory that for the QDE, which have a positive entropy at zero temperature,

\[ d_f = 2\theta_S. \]  

\( \theta_S \) is the exponent for the scaling of domain wall entropy with size. Thus, for the QDE, Eqn. (1) provides a relation between the scaling of the energy and the entropy of domain walls. It is not known how to calculate \( d_f \) directly for the QDE case, so we need to use Eqn. (2) to verify Eqn. (1) in that case.

For the QDE, it is known that \( \theta_E = 0.15\). Then using Eqn. (1) gives \( d_f = 5/4 \), or using Eqn. (2), \( \theta_S = 5/8 \). The calculation of \( \theta_S \) by Saul and Kardar found \( \theta_S = 0.49\pm0.02 \). Since \( d_f \) cannot be less than 1, this result was interpreted as a strong indication that \( \theta_S = 1/2 \).

In this work we will demonstrate that Eqn. (1) actually does work for both continuous and quantized distributions in 2D. The actual behavior of the QDE probability distributions under finite-size scaling turns out to be more subtle than what has been assumed until recently. As pointed out by Wang, Harrington and Preskill, domain walls of zero energy which cross the entire sample play a special role when the energy is quantized.

We will analyze data for the domain wall energy, \( E_{dw} \), and the domain wall entropy, \( S_{dw} \), for the ground states (GS) of 2D Ising spin glasses obtained using methods from earlier work. We will demonstrate that for \( L \times L \) square lattices the Edwards-Anderson (EA) model with a ±J bond distribution has a strong correlation between \( E_{dw} \) and \( S_{dw} \) for the GS domain walls. Because of this correlation, we will need to treat domain walls of different energies as distinct classes, whose entropies scale in different ways as \( L \) is increased. We will find that the scaling parameter identified by Saul and Kardar is the one associated with domain walls having \( E_{dw} = 0 \). It is not, however, the one which controls the dominant behavior for large \( L \).
II. THE MODEL

The Hamiltonian of the EA model for Ising spins is

$$H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j ,$$  

where each spin $S_i$ is a dynamical variable which has two allowed states, +1 and -1. The $\langle ij \rangle$ indicates a sum over nearest neighbors on a simple square lattice of size $L \times L$. We choose each bond $J_{ij}$ to be an independent identically distributed quenched random variable, with the probability distribution

$$P(J_{ij}) = x\delta(J_{ij} + 1) + (1-x)\delta(J_{ij} - 1) ,$$

so that we actually set $J = 1$, as usual. Thus $x$ is the concentration of antiferromagnetic bonds, and $(1-x)$ is the concentration of ferromagnetic bonds.

The data analyzed here used an ensemble in which, for a given value of $x$, every $L \times L$ random lattice sample had exactly $(1-x)L^2$ positive bonds and $xL^2$ negative bonds. Details of the methods used to calculate the GS energies and the numbers of GS have been described earlier and the data analyzed here were used before to obtain other properties of the model.

III. GROUND STATE DOMAIN WALLS

The GS entropy is defined as the natural logarithm of the number of ground states. For each sample the GS energy and GS entropy were calculated for the four combinations of periodic (P) and antiperiodic (A) toroidal boundary conditions along each of the two axes of the square lattice. We will refer to these as PP, PA, AP and AA. In the spin-glass region of the phase diagram, the variation of the sample properties for changes of the boundary conditions is small compared to the variation between different samples of the same size, except when $x$ is close to the ferromagnetic phase boundary and the ferromagnetic correlation length becomes comparable to $L$.

We define domain walls for the spin glass as it was done in the seminal work of McMillan. We look at differences between two samples with the same set of bonds, and the same boundary conditions in one direction, but different boundary conditions in the other direction. Thus, for each set of bonds we obtain domain wall data from the four pairs (PP,PA), (PP,AP), (AA,PA) and (AA,AP). The reader should remember that the term “domain wall”, as used in this work, refers only to this procedure. Saul and Kardar follow the same procedure used in this work, but use the term “defect” instead of “domain wall”.

It is important to realize that the meaning of a domain wall is very different when the GS entropy is positive, as in the model we study here, as compared to the standard case of a doubly degenerate ground state. In the standard case one can identify a line of bonds which forms a boundary between regions of spins belonging to the two different ground states. It is not possible to do that when there are many ground states. Despite this, we continue to use the term “domain wall”.

Due to the $P(J_{ij})$ we have chosen and the fact that we have only used even values of $L$, the energy is always a multiple of 4. Thus, the energy difference, $E_{dw}$, for any pair must also be a multiple of 4. The sign of $E_{dw}$ for a pair is essentially arbitrary for the values of $x$ we will study here, which are deep in the spin-glass region of the phase diagram. Thus we can, without loss of generality, choose all of the domain-wall energies to be non-negative. For $x = 0.25$ and $x = 0.5$, it turns out, crudely speaking, that about 75% of the time we find $E_{dw} = 0$, and 25% of the time $E_{dw} = 4$. For a given $L$, the fraction of $E_{dw} = 0$ domain walls, $f_0$, is about 3% higher at $x = 0.5$ than at $x = 0.25$. It is interesting to note that Wang, Harrington and Preskill use an analytical argument to predict that $f_0$ is approximately 0.75, independent of $x$, in the spin-glass regime. The value of $f_0$ tends to increase slightly as $L$ increases, but our statistical errors are too large for a good estimate of the finite-size scaling behavior to be made. Our results for $x = 0.5$ are consistent with the results of Amoruso et al.

Our calculated statistics for these values of $x$, as a function of $L$, are given in Table I and Table II. Estimating the statistical uncertainties in these numbers is not trivial, due to the fact that the values of $E_{dw}$ obtained from the same set of bonds with the four different pairs of boundary conditions are not statistically independent. An upper bound on the statistical uncertainties is obtained by counting the number of samples, rather than the number of pairs.

No domain walls with energies greater than 8 were observed at any $L$ for these values of $x$. This, however, does not have much fundamental significance. The probability distribution for $E_{dw}$ is also a function of the aspect ratio of the lattice.

The domain wall entropy, $S_{dw}$, is defined, by analogy to $E_{dw}$, to be the difference in the GS entropy when the

| $L$  | $N_{AF}$ | $N_{sam}$ | $n_0$ | $n_4$ | $n_8$ | $f_0$ | $f_4$ |
|-----|---------|-----------|------|------|------|------|------|
| 6   | 18      | 200       | 556  | 242  | 2    | 0.695| 0.305 |
| 8   | 32      | 200       | 584  | 214  | 2    | 0.730| 0.268 |
| 10  | 50      | 200       | 587  | 212  | 1    | 0.734| 0.265 |
| 12  | 72      | 200       | 577  | 220  | 3    | 0.721| 0.275 |
| 14  | 98      | 200       | 579  | 218  | 3    | 0.724| 0.272 |
| 16  | 128     | 200       | 573  | 226  | 1    | 0.716| 0.282 |
| 18  | 162     | 133       | 385  | 146  | 1    | 0.724| 0.274 |
| 20  | 200     | 200       | 598  | 200  | 2    | 0.748| 0.250 |
TABLE II: Domain wall energy statistics for $x = 0.5$. Column labels as in Table I.

| $L$ | $N_{AF}$ | $N_{sam}$ | $n_0$ | $n_4$ | $n_8$ | $f_0$ | $f_4$ |
|-----|----------|-----------|-------|-------|-------|-------|-------|
| 6   | 36       | 400       | 1158  | 442   | 0     | 0.724 | 0.276 |
| 8   | 64       | 400       | 1159  | 440   | 1     | 0.724 | 0.275 |
| 10  | 100      | 400       | 1210  | 388   | 2     | 0.756 | 0.242 |
| 12  | 144      | 400       | 1226  | 374   | 0     | 0.766 | 0.234 |
| 14  | 196      | 400       | 1189  | 410   | 1     | 0.743 | 0.256 |
| 16  | 256      | 400       | 1214  | 384   | 2     | 0.759 | 0.240 |
| 18  | 324      | 400       | 1234  | 366   | 0     | 0.771 | 0.229 |
| 20  | 400      | 238       | 737   | 214   | 1     | 0.774 | 0.225 |

The calculated skewness of these essentially symmetric distributions is, naturally, consistent with zero, but their kurtosis is not. For $x = 0.25$ the calculated kurtosis for different values of $L$ varies between 1.0 and 2.5, with no apparent trend in the $L$ dependence. For $x = 0.5$, where the number of samples is about twice as large, the kurtosis varies between 1.35 and 2.05, again without any apparent trend with $L$. If we average over $L$, we obtain $1.58 \pm 0.18$ for $x = 0.25$ and $1.70 \pm 0.10$ for $x = 0.5$. From this we conclude that the $E_{dw} = 0$ distributions are not Gaussian, and that they probably do not become Gaussian even as $L$ becomes large. The results are consistent with a probability distribution which has a kurtosis of about 1.7, independent of $L$ and $x$. Since the model is critical at $T = 0$ for both values of $x$, this result is not surprising.

When $E_{dw}$ is not zero, the relative signs of $E_{dw}$ and $S_{dw}$ are not arbitrary. Having chosen $E_{dw}$ to be nonnegative, we then find that, when $E_{dw}$ is positive, it turns out that $S_{dw}$ is almost always positive. In Fig. 2 we show the behavior of the average value of $S_{dw}$ for the cases where $E_{dw} = 4$, as a function of $L$. We see that for $E_{dw} = 4$, the average value of $S_{dw}(L)$ grows approximately as $L^{0.63}$, for both values of $x$. More precisely, a least-squares fit to the form

$$S_{dw}(L) = A L^{\theta_s}$$

(5)

gives

$$\theta_s = 0.639 \pm 0.036$$

(6)

for $x = 0.25$, and

$$\theta_s = 0.628 \pm 0.027$$

(7)

for $x = 0.5$. These numbers should be compared to the prediction of droplet theory Eqn. (2). Because of the large GS degeneracy in the $\pm J$ Ising spin glass, one does not know how to compute $d_f$ directly for this model.

For the Ising spin glass with a Gaussian bond distribution, however, this calculation in 2D is straightforward, and the result is $d_f = 1.27 \pm 0.01$. It would be desirable to improve the accuracy of our estimate of $\theta_s$ sufficiently to demonstrate that $\theta_s$ is really different for the CDE and the QDE, as predicted by Eqn. (1). The predicted numerical difference is small, however, so this will be difficult.

The skewness and kurtosis of the $S_{dw}$ distributions for
$E_{dw} = 4$ Domain Walls

![Graph showing average entropy divided by its standard deviation vs. L for $E_{dw} = 4$ domain walls, for the cases $x = 0.25$ and $x = 0.5$.](image)

FIG. 3: Average entropy divided by its standard deviation vs. $L$ for the $E_{dw} = 4$ domain walls, for the cases $x = 0.25$ and $x = 0.5$.

$E_{dw} = 4$ are both small, in contrast to the $E_{dw} = 0$ case. It is likely that these distributions become Gaussian in the large $L$ limit.

The result of Saul and Kardar\cite{18,19} obtained by looking at the distribution of $S_{dw}$ for all values of $E_{dw}$ combined, was $\beta_S = 0.49 \pm 0.02$. To obtain this exponent, Saul and Kardar fit the data at small values of $S_{dw}$. For practical purposes, this part of the data belongs to the $E_{dw} = 0$ component. It is clear that if our extrapolations from small $L$ are correct, however, eventually, at large enough $L$, the width of the total $S_{dw}$ distribution will be dominated by the $E_{dw} = 4$ component. Although they did not identify it, the peak in the probability distribution of $S_{dw}$ due to the $E_{dw} = 4$ component is clearly visible in Fig. 19 of the 1994 paper of Saul and Kardar\cite{18,19}.

In Fig. 3 we show the values of the ratio of average value of the $S(L)$ distribution for $E_{dw} = 4$ divided by the rms width of the distribution. Except for $L = 6$, where the ratios are smaller, we see that this ratio is about 2.9 for $x = 0.25$, and about 3.2 for $x = 0.5$. Thus this characteristic of the distributions seems to depend on $x$. However, it seems likely that the ratio is actually $L$-dependent, and that the large $L$ limit is universal, independent of $x$.

IV. DISCUSSION

What we have learned is that in this model there are two distinct classes of domain walls, the $E_{dw} = 0$ domain walls and the $E_{dw} > 0$ domain walls. For $x = 0.25$ or $x = 0.5$, the only type of $E_{dw} > 0$ domain walls which we need to worry about are the $E_{dw} = 4$ ones. If we wanted to study values of $x$ of 0.15 or less\cite{20} or values of the aspect ratio less than one\cite{21}, then we would need to consider larger values of $E_{dw}$. It should be expected that these are qualitatively similar to the $E_{dw} = 4$ case, as explained below.

As we have seen, the $E_{dw} = 4$ domain walls behave in a way which is essentially consistent with the predictions of the droplet model, but the $E_{dw} = 0$ domain walls do not. This difference in behavior cannot be understood within the droplet model. It must be related to the special feature of the model caused by the quantization of the energy spectrum, which is the positive GS entropy. Another fact which we need to use is the recently discovered result\cite{22} that the average GS entropy is an increasing function of the GS energy.

For an $E_{dw} > 0$ domain wall, a contribution to $S_{dw}$ comes from this shift in the average GS entropy with the shift in the GS energy. What remains to be understood is why $S_{dw}$ should scale with increasing $L$ in the way predicted by the droplet model. The conventional derivation of the droplet model\cite{23} appears to use the assumption that the GS is unique, up to a reversal of the entire state, in an essential way. The extension of the droplet model to the more general case was given by Fisher and Huse\cite{24}.

As long as $E_{dw} > 0$, the two boundary conditions which we are comparing are not on an equal footing. As Wang, Harrington and Preskill\cite{25} express the situation, the $E_{dw} > 0$ domain wall does not destroy the topological long-range order. However, in the $E_{dw} = 0$ case the two boundary conditions are on an equal footing, and the topological order is destroyed. Therefore the $E_{dw} = 0$ class of domain walls can be expected to behave in a special way, which differs from the prediction of the droplet model.

It is well established that this model displays large corrections to finite-size scaling\cite{26}. The reader may object that nothing reliable about the large $L$ behavior can be learned from data which is restricted to $L \leq 20$. However, our results indicate that the reason for the large corrections to scaling is that the differences in the scaling exponents for different classes of domain walls is small. When each class is analyzed separately, this effect is eliminated. Thus, going out to only $L = 20$ is not as bad as it might appear at first. A more significant limitation in our work is that the number of samples we have data for is much smaller than we would like.

We also wish to point out that there is a natural similarity between the $E_{dw} = 0$ domain walls in the 2D Ising spin glass with $\pm J$ bonds and the large-scale low energy excitations which have been found\cite{27,28} in the three-dimensional version of the same model. A similar, and possibly related large-scale low energy excitation has also been seen in the 3D random-anisotropy $XY$ model\cite{29}. Similar ideas have also been discussed by Hatano and Gubernatis\cite{30}.

The domain-wall renormalization group of McMillan\cite{31} is based on the idea that we are studying an effective coupling constant which is changing with $L$. For the case of a continuous distribution of bonds, we can use the energy as the coupling constant. For the quantized energy case,
what we should really do is a slight generalization of this idea. We should think of the coupling constant as the free energy at some infinitesimal temperature. When we do this, the entropy contributes to the coupling constant. And as we have seen, the distribution of $E_{dw}$ rapidly becomes essentially independent of $L$ as $L$ becomes large. Under these conditions, it becomes natural to treat each value of $E_{dw}$ as a separate class, representing a different coupling constant.

Therefore, since $S_{dw}$ increases as a positive power of $L$ for $E_{dw} = 4$, and presumably for all other (finite) positive values, this coupling constant must eventually be controlled by $S_{dw}$. And this coupling constant is also the one described by the droplet model.

The $E_{dw} = 0$ class behaves differently, and therefore represents a different coupling constant. It is natural to wonder if topological long-range order can be related to replica-symmetry breaking, and if the $E_{dw} = 0$ domain walls can be described by the replica-symmetry breaking theory. We will not attempt to do this here.

The argument that all classes other than the $E_{dw} = 0$ class behave in the same way should certainly be checked. The way to do this would be to do calculations with odd values of $L$. In this way we would obtain results for the $E_{dw} = 2$ class, and possibly also for the $E_{dw} = 6$ class.

The computing effort used to obtain the results described here is rather modest by current standards. The algorithm is capable of generating data for greater numbers of samples out to somewhat larger values of $L$ (perhaps to $L = 32$) with a reasonable effort. The method of Lukic et al. appears to be more powerful, and has been used by them to generate energy-entropy statistics for ground states of this model out to $L = 50$.

V. SUMMARY

We have studied the statistics of domain walls for ground states of the 2D Ising spin glass with +1 and -1 bonds for $L \times L$ square lattices with $L \leq 20$, and $x = 0.25$ and 0.5, where $x$ is the fraction of negative bonds, using periodic and/or antiperiodic boundary conditions. Under these conditions, almost all domain walls have an energy $E_{dw}$ equal to 0 or 4. The probability distribution of the entropy $S_{dw}$ is found to depend strongly on $E_{dw}$. The results for $S_{dw}$ when $E_{dw} = 4$ agree with the prediction of Amoroso, Hartmann, Hastings and Moore. Our results for $S_{dw}$ when $E_{dw} = 0$ agree with those of Saul and Kardar, but in addition we find that the distributions are not Gaussian in that case, even in the limit of large $L$. Due to the special role of the $E_{dw} = 0$ domain walls, we can understand the difference between the scaling exponent found by Saul and Kardar and the prediction of Eqn. (1).

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\footnotetext[1]{ron@princeton.edu}

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