DERIVED CATEGORIES OF CYCLIC COVERS
AND THEIR BRANCH DIVISORS

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Abstract. Given a variety $Y$ with a rectangular Lefschetz decomposition of its derived category, we consider a degree $n$ cyclic cover $X \to Y$ ramified over a divisor $Z \subset Y$. We construct semiorthogonal decompositions of $D^b(X)$ and $D^b(Z)$ with distinguished components $A_X$ and $A_Z$, and prove the equivariant category of $A_X$ (with respect to an action of the $n$-th roots of unity) admits a semiorthogonal decomposition into $n - 1$ copies of $A_Z$.

1. Introduction

Let $Y$ be an algebraic variety with a line bundle $O_Y(1)$. Assume the bounded derived category of coherent sheaves $D^b(Y)$ is equipped with a rectangular Lefschetz decomposition with respect to $O_Y(1)$. In other words, assume an admissible subcategory $B \subset D^b(Y)$ is given such that

$$D^b(Y) = \langle B, B(1), \ldots, B(m - 1) \rangle$$

is a semiorthogonal decomposition, where $B(t)$ stands for the image of $B$ under the autoequivalence $\mathcal{F} \mapsto \mathcal{F} \otimes O_Y(t)$ of $D^b(Y)$.

Choose positive integers $n$ and $d$ such that $nd \leq m$. Let $f : X \to Y$ be a degree $n$ cyclic cover of $Y$, ramified over a Cartier divisor $Z$ in the linear system $|O_Y(nd)|$. Let $i : Z \hookrightarrow Y$ be the inclusion. Then the derived pullback functor $f^* : D^b(Y) \to D^b(X)$ is fully faithful on the subcategory $B(t)$ for any $t$, and the same is true of $i^* : D^b(Y) \to D^b(Z)$ provided $nd < m$. Moreover, denoting by $B_X(t) \subset D^b(X)$ and $B_Z(t) \subset D^b(Z)$ the images of $B(t)$, there are semiorthogonal decompositions

$$D^b(X) = \langle A_X, B_X, B_X(1), \ldots, B_X(m - (n - 1)d - 1) \rangle$$

and

$$D^b(Z) = \langle A_Z, B_Z, B_Z(1), \ldots, B_Z(m - nd - 1) \rangle.$$  

Here $A_X$ and $A_Z$ are defined as the right orthogonal categories to the copies of $B$ appearing in the semiorthogonal decompositions. The goal of this paper is to demonstrate a relation between $A_X$ and $A_Z$.

Explicitly, we consider the action of the group of $n$-th roots of unity $\mu_n$ on $X$ by automorphisms over $Y$. This action preserves $A_X$ since it preserves each category $B_X(t)$ in the

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above semiorthogonal decomposition. We denote by $A^{\mu_n}_X$ the category of $\mu_n$-equivariant objects of $A_X$. Our main result can be stated as follows (see Theorem 6.4 for a more precise formulation).

**Theorem 1.1.** There is a semiorthogonal decomposition of $A^{\mu_n}_X$ consisting of $n-1$ copies of the category $A_Z$.

**Remark 1.2.** Theorem 1.1 should hold more generally (with the same proof) if $Y$ is a Deligne–Mumford stack (e.g. a weighted projective space). We have chosen not to work in this generality since for some of the results we need, the references only treat the case of varieties.

If $n = 2$, the theorem gives an equivalence $A^{\mu_2}_X \simeq A_Z$. By a result of Elagin, we deduce in this case a “dual” equivalence (see Corollary 7.4 for a more precise formulation).

**Corollary 1.3.** If $n = 2$, there is an action of $\mathbb{Z}/2$ on $A_Z$ such that $A_X \simeq A_Z^{\mathbb{Z}/2}$.

In Proposition 7.9 we describe this $\mathbb{Z}/2$-action explicitly in terms of a natural “rotation functor” associated to $A_Z$.

If $n > 2$, there is still a description of $A_X$ in terms of $A_Z$, but it is more complicated. In this case, $A_X$ can be recovered as the $\mathbb{Z}/n$-equivariant category of a “gluing” of $n-1$ copies of $A_Z$. For $n = 3$, we speculate about a way to make this reconstruction result more explicit,

in terms of $A_Z$ and its associated “rotation functor.” We plan to return to this subject later.

We emphasize that in our results we do not assume the varieties involved to be smooth. Note that even if $Y$ is smooth (as will often be true in applications), the cover $X$ and its branch divisor $Z$ can easily be singular. We also never assume the varieties involved to be proper, as all the results are local with respect to $Y$.

We apply our main theorem to three cases — quartic double solids, Fano varieties of Picard number 1, degree 10, and coindex 3 (so-called Fano–Gushel–Mukai varieties), and cyclic cubic hypersurfaces.

**Organization of the paper.** In Sections 2 and 3 we discuss background material on semiorthogonal decompositions and group actions on categories. In Section 4 we give a semiorthogonal decomposition of the equivariant derived category of a cyclic cover. In Section 5 we establish the semiorthogonal decompositions of $D^b(X)$ and $D^b(Z)$ mentioned above. In Section 6 we prove Theorem 1.1. In Section 7 we introduce the rotation functors and prove the reconstruction results stated above. Finally, in Section 8 we apply our results to several examples.

**Conventions.** We work over an algebraically closed field $k$ of characteristic coprime to $n$. Varieties will be assumed integral and quasi-projective over $k$.

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2. Preliminaries on triangulated categories

In this paper, triangulated categories are $k$-linear and functors between them are $k$-linear and exact.
2.1. Derived categories of varieties. We use the following notation: For a variety $X$, we denote by $\mathcal{D}^b(X)$ the bounded derived category of coherent sheaves on $X$, regarded as a triangulated category. We refer to $\mathcal{D}^b(X)$ simply as the derived category of $X$. For a morphism of varieties $f : X \to Y$, we write $f_* : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ for the derived pushforward (provided $f$ is proper), and $f^* : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ for the derived pullback (provided $f$ has finite Tor-dimension). For $\mathcal{F}, \mathcal{G} \in \mathcal{D}^b(X)$, we write $\mathcal{F} \otimes \mathcal{G} \in \mathcal{D}^b(X)$ for the derived tensor product.

2.2. Semiorthogonal decompositions. We recall some well-known facts about semiorthogonal decompositions. We suggest the reader consult [2] and [3] for more details, or [16] for a short review of results.

Definition 2.1. A semiorthogonal decomposition of a triangulated category $\mathcal{I}$ is a sequence $A_1, \ldots, A_n$ of full triangulated subcategories such that:

1. $\text{Hom}(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{F} \in A_i$, $\mathcal{G} \in A_j$ if $i > j$.
2. For any $\mathcal{F} \in \mathcal{I}$, there is a sequence of maps $0 = \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 = \mathcal{F}$ such that $\text{Cone}(\mathcal{F}_i \to \mathcal{F}_{i-1}) \in A_i$.

If only condition (1) is required, we say the sequence $A_1, \ldots, A_n$ is semiorthogonal. We write $\mathcal{I} = \langle A_1, \ldots, A_n \rangle$ for a semiorthogonal decomposition with components $A_1, \ldots, A_n$.

Remark 2.2. Condition (1) of the definition implies the “filtration” in (2.1) and its “factors” are unique and functorial. The functors $\mathcal{I} \to A_i$, $\mathcal{F} \mapsto \text{Cone}(\mathcal{F}_i \to \mathcal{F}_{i-1})$ are called the projection functors of the semiorthogonal decomposition. We call the object $\text{Cone}(\mathcal{F}_i \to \mathcal{F}_{i-1})$ the component of $\mathcal{I}$ in $A_i$.

A full triangulated subcategory $A \subset \mathcal{I}$ is called right admissible if the embedding functor $\alpha : A \to \mathcal{I}$ has a right adjoint $\alpha^r : \mathcal{I} \to A$, left admissible if $\alpha$ has a left adjoint $\alpha^l : \mathcal{I} \to A$, and admissible if it is both right and left admissible.

If a semiorthogonal decomposition $\mathcal{I} = \langle A, B \rangle$ is given, then $A$ is left admissible and $B$ is right admissible. Vice versa, if $A \subset \mathcal{I}$ is right admissible then there is a semiorthogonal decomposition $\mathcal{I} = \langle A^\perp, A \rangle$ (2.1) and if $A$ is left admissible then there is a semiorthogonal decomposition $\mathcal{I} = \langle A, A^\perp \rangle$. (2.2)

Here $A^\perp$ is the right orthogonal to $A$, i.e. the full subcategory of $\mathcal{I}$ consisting of objects $\mathcal{F} \in \mathcal{I}$ such that $\text{Hom}(\mathcal{G}, \mathcal{F}) = 0$ for all $\mathcal{G} \in A$; the left orthogonal $A^\perp$ is defined similarly.

If $A \subset \mathcal{I}$ is right admissible, then for any object $\mathcal{F} \in \mathcal{I}$ there is a distinguished triangle $\alpha^l(\mathcal{F}) \to \mathcal{F} \to L_A(\mathcal{F})$, where $L_A(\mathcal{F})$ is defined as the cone of the counit morphism. The first and last vertices of this triangle give the components of $\mathcal{F}$ in $A$ and $A^\perp$ with respect to (2.1). Similarly, if $A \subset \mathcal{I}$ is left admissible, there is a distinguished triangle $\text{R}_A(\mathcal{F}) \to \mathcal{F} \to \alpha^r(\mathcal{F})$ with vertices the components of $\mathcal{F}$ in $A^\perp$ and $A$ with respect to (2.2).
The functors
\[ L_A : \mathcal{T} \to \mathcal{T} \quad \text{and} \quad R_A : \mathcal{T} \to \mathcal{T} \]
defined by the above formulas are called the left and right mutation functors of \( A \subset \mathcal{T} \). In what follows, when considering mutation functors we always assume \( A \) is admissible (even if this assumption is not explicitly stated). The functors \( L_A \) and \( R_A \) annihilate \( A \), and the restrictions
\[ L_A|_{-A} : ^{\perp}A \to A^{\perp} \quad \text{and} \quad R_A|_{A^{\perp}} : A^{\perp} \to ^{\perp}A \]
are mutually inverse equivalences. Furthermore, if \( A_1, \ldots, A_n \) is a semiorthogonal sequence of admissible subcategories of \( \mathcal{T} \), then \( \langle A_1, \ldots, A_n \rangle \subset \mathcal{T} \) is admissible and
\[
\begin{align*}
L_{\langle A_1, \ldots, A_n \rangle} & \cong L_{A_1} \circ L_{A_2} \circ \cdots \circ L_{A_n}, \quad (2.3) \\
R_{\langle A_1, \ldots, A_n \rangle} & \cong R_{A_n} \circ R_{A_{n-1}} \circ \cdots \circ R_{A_1}. \quad (2.4)
\end{align*}
\]

We are interested in mutation functors because they act on semiorthogonal decompositions:

**Proposition 2.3.** Let \( \mathcal{T} = \langle A_1, \ldots, A_n \rangle \) be a semiorthogonal decomposition. Then
\[
\mathcal{T} = \langle A_1, \ldots, A_{i-1}, L_{A_i}(A_{i+1}), A_i, A_{i+2}, \ldots, A_n \rangle,
\]
\[
\mathcal{T} = \langle A_1, \ldots, A_{i-2}, A_i, R_{A_i}(A_{i-1}), A_{i+1}, \ldots, A_n \rangle,
\]
are semiorthogonal decompositions with \( L_{A_i}(A_{i+1}) \cong A_{i+1} \) and \( R_{A_i}(A_{i-1}) \cong A_{i-1} \).

The following observation is useful for computing mutations: If \( A \subset \mathcal{T} \) is admissible and \( \mathcal{F} \in \mathcal{T} \), then to compute \( L_A(\mathcal{F}) \) it suffices to construct a distinguished triangle
\[ \mathcal{G} \to \mathcal{F} \to \mathcal{G}' \quad (2.5) \]
with \( \mathcal{G} \in A \) and \( \mathcal{G}' \in A^{\perp} \); indeed, then \( L_A(\mathcal{F}) \cong \mathcal{G}' \) by Remark 2.2. Similarly, if we construct a distinguished triangle
\[ \mathcal{H}' \to \mathcal{F} \to \mathcal{H} \quad (2.6) \]
with \( \mathcal{H} \in A \) and \( \mathcal{H}' \in ^{\perp}A \), then \( R_A(\mathcal{F}) \cong \mathcal{H}' \). We call a triangle as in (2.5) or (2.6) a mutation triangle. The following two lemmas can easily be proved using the description of mutation functors in terms of mutation triangles.

**Lemma 2.4.** Let \( \mathcal{T} = \langle A_1, \ldots, A_n \rangle \) be a semiorthogonal decomposition. Assume for some \( i \) the components \( A_i \) and \( A_{i+1} \) are completely orthogonal, i.e. \( \operatorname{Hom}(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}(\mathcal{G}, \mathcal{F}) = 0 \) for all \( \mathcal{F} \in A_i, \mathcal{G} \in A_{i+1} \). Then \( L_{A_i}(\mathcal{G}) = \mathcal{G} \) for any \( \mathcal{G} \in A_{i+1} \), and \( R_{A_{i+1}}(\mathcal{F}) = \mathcal{F} \) for any \( \mathcal{F} \in A_i \). In particular,
\[ \mathcal{T} = \langle A_1, \ldots, A_{i-1}, A_{i+1}, A_i, A_{i+2}, \ldots, A_n \rangle \]
is a semiorthogonal decomposition.

**Lemma 2.5.** Let \( F : \mathcal{T}_1 \to \mathcal{T}_2 \) be an equivalence of triangulated categories. Let \( A \subset \mathcal{T}_1 \) be an admissible subcategory. Then \( F \circ L_A \cong L_{F(A)} \circ F \) and \( F \circ R_A \cong R_{F(A)} \circ F \).
3. Group actions on triangulated categories

In this section we discuss (finite) group actions on categories. After recalling the definition of the equivariant category of a group action on an arbitrary category, we focus on the triangulated case. In Section 3.2 we describe several situations where the equivariant category is naturally triangulated. We also state a result of Elagin, which gives a semiorthogonal decomposition of the equivariant derived category of a variety induced by a decomposition of the non-equivariant derived category. In Section 3.3 we give a semiorthogonal decomposition of the equivariant category of a trivial action. Finally, in Section 3.4 we summarize some facts about the equivariant derived categories of varieties.

3.1. Equivariant categories. Suppose given a (right) action of a finite group $G$ on a category $\mathcal{C}$. In other words, suppose given:

- For every $g \in G$, an autoequivalence $g^*: \mathcal{C} \to \mathcal{C}$.
- For every $g, h \in G$, an isomorphism of functors $c_{g,h}: (gh)^* \sim h^* \circ g^*$, such that the diagram

\[
\begin{array}{ccc}
(fgh)^* & \xrightarrow{c_{fgh}} & h^* \circ (fg)^* \\
\downarrow{c_{f,gh}} & & \downarrow{h^*c_{f,g}} \\
(gh)^* \circ f^* & \xrightarrow{c_{g,h}f^*} & h^* \circ g^* \circ f^*
\end{array}
\]

commutes for all $f, g, h \in G$.

**Definition 3.1.** A $G$-equivariant object of $\mathcal{C}$ is a pair $(\mathcal{F}, \phi)$, where $\mathcal{F}$ is an object of $\mathcal{C}$ and $\phi$ is a collection of isomorphisms $\phi_g: \mathcal{F} \sim g^*(\mathcal{F})$ for $g \in G$, such that the diagram

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\phi_h} & h^*(\mathcal{F}) \\
& \phi_{gh} \downarrow & \downarrow{c_{g,h}(\mathcal{F})} \\
& & (gh)^*(\mathcal{F})
\end{array}
\]

commutes for all $g, h \in G$. The equivariant structure $\phi$ is often suppressed from the notation.

The equivariant category $\mathcal{C}^G$ of $\mathcal{C}$ is the category of $G$-equivariant objects of $\mathcal{C}$, with the obvious morphisms. If $\mathcal{C}$ is additive and $G$ acts by additive autoequivalences, then $\mathcal{C}^G$ is additive.

3.2. Triangulated equivariant categories. Now assume a finite group $G$ acts on a triangulated category $\mathcal{T}$ by exact autoequivalences. We will always assume in this situation that the order of $G$ is coprime to the characteristic of the base field $k$. The category $\mathcal{T}^G$ is additive and comes with a natural shift functor and a class of distinguished triangles. Namely, the shift functor on $\mathcal{T}^G$ is given by $(\mathcal{F}, \phi) \mapsto (\mathcal{F}[1], \phi[1])$, and a triangle in $\mathcal{T}^G$ is distinguished if and only if the underlying triangle in $\mathcal{T}$ is distinguished. In general, $\mathcal{T}$ being triangulated is not sufficient to guarantee this defines a triangulated structure on $\mathcal{T}^G$ (see [7] for a more detailed discussion). In case this does define a triangulated structure, we will simply say “$\mathcal{T}^G$ is triangulated.”

The category $\mathcal{T}^G$ is triangulated in certain situations, typically when it can be identified with a category which is a priori triangulated. First, this holds if $\mathcal{T} = \text{D}^b(X)$ for a variety $X$ and $G$ acts via automorphisms of $X$. Indeed, in this case $\mathcal{T}^G$ is equivalent to the bounded
derived category of $\text{Coh}(X)^G$ (see \[7, Theorem 9.6\]). Second, $\mathcal{T}^G$ is triangulated if $\mathcal{T}$ is a semiorthogonal component of $\text{D}^b(X)$ and $G$ acts via automorphisms of $X$ that preserve $\mathcal{T}$. In fact, in this case the following theorem shows $\mathcal{T}^G$ is a semiorthogonal component of $\text{D}^b(X)^G$, hence in particular triangulated.

**Theorem 3.2** \([8, 9]\). Let $X$ be a quasi-projective variety with an action of a finite group $G$. Assume $\text{D}^b(X) = \langle A_1, \ldots, A_n \rangle$ is a semiorthogonal decomposition which is preserved by $G$, i.e. each $A_i$ is preserved by the action of $G$. Then there is a semiorthogonal decomposition

$$\text{D}^b(X)^G = \langle A_1^G, \ldots, A_n^G \rangle. \quad (3.1)$$

**Proof.** The equivariant category $\text{D}^b(X)^G$ comes with an induced semiorthogonal decomposition by \([8, Theorem 6.3]\), and its components are equivalent to the equivariant categories $A_i^G$ by \([9, Proposition 3.10]\). □

3.3. **Trivial actions.** We say the action of $G$ on a category $\mathcal{C}$ is **trivial** if for each $g \in G$ an isomorphism of functors $\tau_g : \text{id} \sim g^*$ is given, such that

$$c_{g,h} \circ \tau_{gh} = h^* \tau_g \circ \tau_h$$

for all $g, h \in G$.

**Proposition 3.3.** Let $\mathcal{T}$ be a triangulated category with a trivial action of a finite group $G$. If $\mathcal{T}^G$ is also triangulated, then there is a completely orthogonal decomposition

$$\mathcal{T}^G = \langle \mathcal{T} \otimes V_0, \mathcal{T} \otimes V_1, \ldots, \mathcal{T} \otimes V_n \rangle, \quad (3.2)$$

where $V_0, \ldots, V_n$ is a list of the finite-dimensional irreducible representations of $G$.

**Proof.** Since the action of $G$ is trivial, a $G$-equivariant object of $\mathcal{T}$ is the same as an object $\mathcal{F} \in \mathcal{T}$ with a representation $G \to \text{Aut}(\mathcal{F})$. In particular, for any $\mathcal{F} \in \mathcal{T}$ and $V \in \text{Rep}(G)$ (the category of finite-dimensional representations of $G$), the tensor product $\mathcal{F} \otimes V$ is a $G$-equivariant object of $\mathcal{T}$. Moreover, given $\mathcal{F}, \mathcal{F}' \in \mathcal{T}$ and $V, V' \in \text{Rep}(G)$, then

$$\text{Hom}_{\mathcal{T}^G}(\mathcal{F} \otimes V, \mathcal{F}' \otimes V') \cong \text{Hom}_{\mathcal{T}}(\mathcal{F}, \mathcal{F}') \otimes \text{Hom}_{\text{Rep}(G)}(V, V').$$

Hence the functors $\mathcal{T} \to \mathcal{T}^G$ given by $\mathcal{F} \mapsto \mathcal{F} \otimes V_k$ are fully faithful with orthogonal essential images $\mathcal{T} \otimes V_k$. Finally, if $\mathcal{F} \in \mathcal{T}^G$ then

$$\mathcal{F} = \mathcal{F} \otimes_{k[G]} k[G] = \mathcal{F} \otimes_{k[G]} \left( \bigoplus_{k=0}^n V_k^\vee \otimes V_k \right) = \bigoplus_{k=0}^n (\mathcal{F} \otimes_{k[G]} V_k^\vee) \otimes V_k,$$

which proves the categories $\mathcal{T} \otimes V_k$ generate $\mathcal{T}^G$. □

We will apply Proposition 3.3 when the group $G$ acts on a variety $X$ and $\mathcal{T}$ is a semiorthogonal component of $\text{D}^b(X)$ preserved by $G$, so that $\mathcal{T}^G$ is triangulated by Theorem 3.2. In this situation, we define functors

$$i_k : \mathcal{T} \to \mathcal{T}^G, \quad \mathcal{F} \mapsto \mathcal{F} \otimes V_k, \quad (3.3)$$

$$\pi_k : \mathcal{T}^G \to \mathcal{T}, \quad \mathcal{F} \mapsto \mathcal{F} \otimes_{k[G]} V_k^\vee. \quad (3.4)$$
The above proof shows the $\pi_k$ are the projection functors for the semiorthogonal decomposition \([3.2]\). Since this decomposition is completely orthogonal, the functors $\pi_k$ are simultaneously left and right adjoint to $\iota_k$, and we have

$$
\pi_k \circ \iota_\ell = \begin{cases} 
\text{id}_{\Sigma} & \text{if } k = \ell, \\
0 & \text{if } k \neq \ell.
\end{cases}
$$

3.4. **Equivariant derived categories of varieties.** The usual functor formalism for categories of sheaves extends directly to the equivariant setting. We summarize the relevant facts here, referring to \([4]\) for a more detailed exposition. Let $G$ be a finite group and $f : X \to Y$ a $G$-equivariant morphism of varieties with $G$-actions. Then if $f$ is proper there is a derived pushforward functor

$$
f_* : \text{D}^b(X)^G \to \text{D}^b(Y)^G,
$$

and if $f$ is of finite Tor-dimension there is a derived pullback functor

$$
f^* : \text{D}^b(Y)^G \to \text{D}^b(X)^G.
$$

When both functors are defined, $f^*$ is left adjoint to $f_*$. Similarly, there is a derived tensor product for equivariant complexes $\mathcal{F}, \mathcal{G} \in \text{D}^b(X)^G$, which we denote by $\mathcal{F} \otimes \mathcal{G}$. These functors satisfy the same relations as in the non-equivariant setting, e.g. the projection formula

$$
f_*(\mathcal{F} \otimes f^* \mathcal{G}) \cong f_* (\mathcal{F}) \otimes \mathcal{G}
$$

(3.5)

holds for any $\mathcal{F} \in \text{D}^b(X)^G$ and $\mathcal{G} \in \text{D}^b(Y)^G$. Moreover, there is an equivariant Grothendieck duality, providing a right adjoint $f^! : \text{D}^b(Y)^G \to \text{D}^b(X)^G$ to $f_*$ when $f$ is proper. As in the non-equivariant setting, the functor $f^!$ admits an explicit description in certain cases, e.g. if $f : X \to Y$ is the inclusion of a divisor defined by an invariant section of a $G$-equivariant line bundle $\mathcal{L}$ on $Y$, then $f^!(\mathcal{F}) = f^* (\mathcal{F}) \otimes f^*(\mathcal{L})[-1]$.

4. **The equivariant derived category of a cyclic cover**

4.1. **Setup and notation.** Let $Y$ be an algebraic variety and $\mathcal{L}$ a line bundle on $Y$. Suppose $Z$ is a Cartier divisor in $Y$ defined by a section of $\mathcal{L}^n$. Let $f : X \to Y$ be the degree $n$ cyclic cover of $Y$ ramified over $Z$, i.e.

$$
X = \text{Spec}_Y (\mathcal{R}_Y), \quad \mathcal{R}_Y := \mathcal{O}_Y \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{-(n-1)},
$$

where the algebra structure on $\mathcal{R}_Y$ is given by the divisor $Z$.

Let $\mu_n$ denote the group of $n$-th roots of unity. Its dual group $\hat{\mu}_n$ (the group of characters) is a cyclic group. We identify $\hat{\mu}_n$ with $\mathbb{Z}/n$ by choosing a primitive character $\chi : \mu_n \to \mathbb{K}$ and associating $k \in \mathbb{Z}/n$ to $\chi^k \in \hat{\mu}_n$. Thus all irreducible representations of $\mu_n$ are given by

$$
V_0 = 1, \ V_1 = \chi, \ldots, \ V_{n-1} = \chi^{n-1},
$$

and are indexed by $\mathbb{Z}/n$.

We equip $Y$ with the trivial $\mu_n$-action. The group $\mu_n$ acts on the sheaf $\mathcal{R}_Y$ via the character $\chi^k$ on the summand $\mathcal{L}^{-k}$, so that as an equivariant sheaf

$$
\mathcal{R}_Y = (\mathcal{O}_Y \otimes 1) \oplus (\mathcal{L}^{-1} \otimes \chi) \oplus \cdots \oplus (\mathcal{L}^{-(n-1)} \otimes \chi^{n-1}).
$$

(4.1)

This induces an action of $\mu_n$ on $X$ such that $X/\mu_n = \text{Spec}_Y (\mathcal{R}_Y^{\mu_n}) = \text{Spec}_Y (\mathcal{O}_Y) = Y$. 

Theorem 4.1. The category of a blowup has a semiorthogonal decomposition which is similar to Orlov’s decomposition of the derived category of the projectivization of a vector bundle. On the other hand, the semiorthogonal decompositions.

Let $\mathcal{L}_X$ and $\mathcal{L}_Z$ be the pullbacks of $\mathcal{L}$ to $X$ and $Z$ respectively. By (4.1), the line bundle $\mathcal{L}_X \otimes \chi^{-1}$ has a $\mu_n$-invariant section on $X$. The zero locus of this section is the ramification divisor of the cover $X \to Y$, which can be identified with $Z$. Denoting by $i : Z \hookrightarrow Y$ and $j : Z \to X$ the embeddings of $Z$ as the branch divisor and ramification divisor, we have a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{j} & X \\
\downarrow{i} & & \mu_n \\
Y & \xrightarrow{f} & \\
\end{array}
\]

(4.2)

The embeddings $i$ and $j$ are $\mu_n$-equivariant if $Z$ is equipped with the trivial $\mu_n$-action. Moreover, they are proper and have Tor-dimension 1. For each $k \in \mathbb{Z}/n$, we define the functors $f_k^*, f_k$, $j_k$ as the compositions

\[
f_k^* : D^b(Y) \xrightarrow{i_k} D^b(Y)^{\mu_n} \xrightarrow{f^*} D^b(X)^{\mu_n},
\]

\[
f_k : D^b(X)^{\mu_n} \xrightarrow{f^*} D^b(Y)^{\mu_n} \xrightarrow{\pi_k} D^b(Y),
\]

\[
j_k : D^b(Y) \xrightarrow{i_k} D^b(Y)^{\mu_n} \xrightarrow{f} D^b(X)^{\mu_n},
\]

where $i_k$ and $\pi_k$ are given by (3.3) and (3.4). Then $f_k^*$ is the left adjoint of $f_k$, and $f_k^!$ is the right adjoint of $f_k^*$.

The morphism $f : X \to Y$ is $\mu_n$-equivariant with respect to the above actions. Moreover, it is proper and flat. For each $k \in \mathbb{Z}/n$, we define the functors $j_k^*, j_k$, $j_k^!$ as the compositions

\[
j_k^* : D^b(X)^{\mu_n} \xrightarrow{j^*} D^b(Z)^{\mu_n} \xrightarrow{\pi_k} D^b(Z),
\]

\[
j_k : D^b(Z) \xrightarrow{i_k} D^b(Z)^{\mu_n} \xrightarrow{j^*} D^b(X)^{\mu_n},
\]

\[
j_k^! : D^b(X)^{\mu_n} \xrightarrow{j^!} D^b(Z)^{\mu_n} \xrightarrow{\pi_k} D^b(Z).
\]

Again, $j_k^*$ is the left adjoint of $j_k$, and $j_k^!$ is the right adjoint of $j_k^*$. Note that since $Z$ is the zero locus of an invariant section of $\mathcal{L}_X \otimes \chi^{-1}$, we have an equivariant exact sequence

\[
0 \to \mathcal{L}_X^{-1} \otimes \chi \to \mathcal{O}_X \otimes 1 \to j_{*0} \mathcal{O}_Z \to 0.
\]

(4.3)

4.2. Semiorthogonal decompositions. The action of $\mu_n$ on $Z$ is trivial, hence by Proposition 3.3 we have a semiorthogonal decomposition

\[
D^b(Z)^{\mu_n} = (\iota_0(D^b(Z)), \iota_1(D^b(Z)), \ldots, \iota_{n-1}(D^b(Z))).
\]

(4.4)

It is similar to Orlov’s decomposition of the derived category of the projectivization of a vector bundle. On the other hand, the $\mu_n$-equivariant derived category of the cyclic cover $X$ has a semiorthogonal decomposition which is similar to Orlov’s decomposition of the derived category of a blowup.

Theorem 4.1. For each $k \in \mathbb{Z}/n$ the functors

\[
f_k^* : D^b(Y) \to D^b(X)^{\mu_n},
\]

\[
j_k : D^b(Z) \to D^b(X)^{\mu_n},
\]
are fully faithful. Moreover, for $k, \ell \in \mathbb{Z}/n$ we have:

\[(j_k, D^b(Z), j_\ell, D^b(Z)) \text{ is semiorthogonal if } k \neq \ell, \ell + 1, \quad (4.5)\]

\[(j_k, D^b(Z), f_\ell^* D^b(Y)) \text{ is semiorthogonal if } k \neq \ell, \quad (4.6)\]

\[(f_\ell^* D^b(Y), j_k, D^b(Z)) \text{ is semiorthogonal if } k \neq \ell - 1. \quad (4.7)\]

Finally, for each $k \in \mathbb{Z}/n$ there is a semiorthogonal decomposition

\[D^b(X)^{\mu_n} = \langle j_{k+1}, D^b(Z), j_{k+2}, D^b(Z), \ldots, j_{n-1}, D^b(Z), f_0^* D^b(Y), j_{0*} D^b(Z), j_{1*} D^b(Z), \ldots, j_{k-1*} D^b(Z) \rangle. \quad (4.8)\]

This decomposition is well-known to the experts. The case $n = 2$ was proved by Collins and Polishchuk in [5]. The general case follows from a result of Ishii and Ueda [13, Theorem 1.6], which gives a semiorthogonal decomposition of the derived category of a root stack. For completeness, we provide a proof here.

**Proof.** The functor $f_\ell^*$ is fully faithful since the composition with its right adjoint $f_{k*}$ satisfies $f_{k*} \circ f_\ell^* = \text{id}$. Indeed, by the projection formula (3.5) and (4.1), we have

\[f_{k*}(f_\ell^* F) = \pi_k(f_\ell(f_\ell^* F)) = \pi_k((F \otimes \chi^k) \otimes R_Y) = \pi_k((\mathcal{F} \otimes \chi^k) \otimes R_Y) = \mathcal{F}. \]

Similarly, to prove $j_{k*}$ is fully faithful we show $j_{k*} \circ j_{\ell*} = \text{id}$. For $\mathcal{F} \in D^b(Z)^{\mu_n}$, there is a distinguished triangle

\[\mathcal{F} \otimes \mathcal{L}_Z^{-1} \otimes \chi[1] \to \mathcal{F}, \]

which is the equivariant version of the standard triangle from [1] Lemma 3.3. If $\mathcal{F} = \iota_k(\mathcal{G})$ for $\mathcal{G} \in D^b(Z)$, the last vertex of this triangle is in the component $D^b(Z) \otimes \chi^k$ of $D^b(Z)^{\mu_n}$, while the first is in $D^b(Z) \otimes \chi^{k+1}$. Applying $\pi_k$ we deduce $j_{k*}(j_{k*}\mathcal{G}) = \mathcal{G}$, so that $j_{k*}$ is fully faithful.

To prove (4.5), by adjunction we must show $j_{k*} \circ j_{\ell*} = 0$ for $k \neq \ell, \ell + 1$. This follows by the same argument used to show $j_{k*} \circ j_{k*} = \text{id}$ above.

To prove (4.6), by adjunction we must show $j_{k*} \circ f_\ell^* = 0$ for $k \neq \ell$. Since $j_{k*} = \pi_k \circ j^*$, this is immediate from the fact that the image of $j^* \circ f_\ell^*$ lies in the component $D^b(Z) \otimes \chi^k$ of $D^b(Z)^{\mu_n}$.

To prove (4.7), by adjunction we must show $j_{k*} \circ f_\ell^* = 0$ for $k \neq \ell - 1$. If $\mathcal{F} \in D^b(X)^{\mu_n}$ then

\[j^*(\mathcal{F}) = j^*(\mathcal{F}) \otimes \mathcal{L}_X \otimes \chi^{-1}[1] = j^*(\mathcal{F}) \otimes \mathcal{L}_Z \otimes \chi^{-1}[1]. \]

So for $\mathcal{G} \in D^b(Y)$, we have

\[j^*(f_\ell^* \mathcal{G}) = j^*(\mathcal{G} \otimes \chi^k) \otimes \mathcal{L}_Z \otimes \chi^{-1}[1] = i^*(\mathcal{G} \otimes \mathcal{L}_Z \otimes \chi^k) \otimes \mathcal{L}_Z \otimes \chi^{-1}[1]. \]

Now applying $\pi_k$ proves the required vanishing.

Finally, we establish the semiorthomorph decomposition (4.8). By (4.5), (4.6), and (4.7), the components of the claimed decomposition are indeed semiorthogonal, so we must show the category $\mathcal{F}$ they generate is all of $D^b(X)^{\mu_n}$.

First we claim $j_{k*} D^b(Z) \subset \mathcal{F}$. For this we note that if $\mathcal{G} \in D^b(Z)$, then (4.1) implies $f_0^*(\mathcal{G})$ has a filtration with factors $j_{k*}(\mathcal{G} \otimes \mathcal{L}_Z^{-\ell})$ for $\ell = 0, 1, \ldots, n - 1$. Since $\mathcal{F}$ contains $f_0^*(\mathcal{G})$ and all of these factors for $\ell \neq k$, it follows that $\mathcal{F}$ also contains $j_{k*}(\mathcal{G} \otimes \mathcal{L}_Z^{-k})$. But the twist by a power of $\mathcal{L}_Z$ is an autoequivalence of $D^b(Z)$, so this proves the claim.

Now take any $\mathcal{F} \in D^b(X)^{\mu_n}$. To finish the proof we must show $\mathcal{F} \in \mathcal{F}$. The canonical morphism $f_0^* f_{0*} \mathcal{F} \to \mathcal{F}$ is an isomorphism on $X \setminus Z$ by [9] Corollary 5.3, since $X \setminus Z \to Y \setminus Z$
is a Galois cover. Hence the cone $\mathcal{F}'$ of this morphism is supported set-theoretically on $Z$. It follows that each cohomology of $\mathcal{F}'$ is supported set-theoretically on $Z$, and hence admits a filtration by sheaves supported on $Z$ scheme-theoretically. Moreover, this filtration can be chosen $\mu_n$-equivariantly. By (4.2) any $\mu_n$-equivariant sheaf scheme-theoretically supported on $Z$ can be written as a direct sum of sheaves contained in the categories $j_\ell\mathcal{D}^b(Z)$. Since $\mathcal{F}$ contains all these categories, it follows that $\mathcal{F}'$ and hence $\mathcal{F}$ are contained in $\mathcal{F}$.

5. Semiorthogonal decompositions induced by a Lefschetz decomposition

Let $Y$ be an algebraic variety with a line bundle $\mathcal{O}_Y(1)$, and assume a rectangular Lefschetz decomposition

$$D^b(Y) = (\mathcal{B}, \mathcal{B}(1), \ldots, \mathcal{B}(m-1))$$

is given (see Section 8 for examples of such $Y$). Here $\mathcal{B}(t)$ denotes the image of $\mathcal{B}$ under the “twist by $t$” autoequivalence

$$\mathcal{F} \mapsto \mathcal{F}(t) := \mathcal{F} \otimes \mathcal{O}_Y(t)$$

of $D^b(Y)$. We denote by $\beta : \mathcal{B} \to D^b(Y)$ the embedding functor. By (5.1) it has a left adjoint which we denote by $\beta^*$. Moreover, twisting (5.1) by $\mathcal{O}_Y(-(m-1))$, we see $\beta$ also has a right adjoint, which we denote by $\beta^!$.

For any variety $X$ mapping to $Y$, we define $\mathcal{O}_X(1)$ as the pullback of $\mathcal{O}_Y(1)$ and use the same notation as above for twists. We show (5.1) induces semiorthogonal decompositions of the derived categories of a cyclic cover of $Y$ and a divisor in $Y$.

**Lemma 5.1.** Let $n$ and $d$ be positive integers such that $(n-1)d < m$. Let $f : X \to Y$ be a degree $n$ cyclic cover of $Y$ ramified over a divisor in $|\mathcal{O}_Y(nd)|$.

1. The functor $f^* : D^b(Y) \to D^b(X)$ is fully faithful on the subcategory $\mathcal{B} \subset D^b(Y)$.
2. Denoting the essential image of $\mathcal{B}$ by $\mathcal{B}_X$, for each $0 \leq t \leq m - (n-1)d$ there is a semiorthogonal decomposition

$$D^b(X) = (\mathcal{B}_X, \ldots, \mathcal{B}_X(t-1), \mathcal{A}_X(t), \mathcal{B}_X(t), \ldots, \mathcal{B}_X(m - (n-1)d - 1)),$$

where $\mathcal{A}_X$ is the full subcategory of $D^b(X)$ consisting of all $\mathcal{G} \in D^b(X)$ such that $f_\ast \mathcal{G} \in (\mathcal{B}(-(n-1)d), \ldots, \mathcal{B}(-1))$.

**Remark 5.2.** The semiorthogonal decomposition (5.2) still holds for $(n-1)d = m$ — in this case there are no “trivial components” equivalent to $\mathcal{B}$ in $D^b(X)$, so $\mathcal{A}_X = D^b(X)$.

**Proof.** Let $\mathcal{F}, \mathcal{G} \in \mathcal{B}$. For any integers $r$ and $s$, adjunction gives

$$\text{Hom}(f^* \mathcal{F}(r), f^* \mathcal{G}(s)) = \text{Hom}(\mathcal{F}(r), f_\ast f^* \mathcal{G}(s)).$$

By the projection formula and (4.1) we have

$$f_\ast f^* \mathcal{G} = \mathcal{G} \otimes (\mathcal{O}_Y \oplus \mathcal{O}_Y(-d) \oplus \cdots \oplus \mathcal{O}_Y(-(n-1)d)).$$

Hence

$$\text{Hom}(f^* \mathcal{F}(r), f^* \mathcal{G}(s)) = \bigoplus_{a=0}^{n-1} \text{Hom}(\mathcal{F}(r), \mathcal{G}(s-ad)).$$

If $r = s$, the $a > 0$ summands vanish by semiorthogonality of (5.1) and the assumption $(n-1)d < m$. This proves $f^*$ is fully faithful on $\mathcal{B}(r)$. 
If $0 \leq s < r \leq m - (n - 1)d - 1$, all of the summands in (5.3) vanish, again by semiorthogonality of (5.1). This proves the sequence

$$
\mathcal{B}_X, \mathcal{B}_X(1), \ldots, \mathcal{B}_X(m - (n - 1)d - 1)
$$

is semiorthogonal. Note that $\mathcal{B}_X$ is admissible in $D^b(X)$. Indeed, since $\mathcal{B}$ is admissible in $D^b(Y)$, it suffices to observe $f^*$ has right and left adjoints; the existence of the right adjoint is obvious, while the left adjoint exists since the Grothendieck duality functor $f^!$ satisfies

$$
f^!(\mathcal{F}) = f^*(\mathcal{F}) \otimes \mathcal{O}_X((n - 1)d)
$$

and has a left adjoint. Hence for $0 \leq t \leq m - (n - 1)d$ there is a semiorthogonal decomposition

$$
D^b(X) = \langle \mathcal{B}_X, \ldots, \mathcal{B}_X(t - 1), A^{(t)}_X, \mathcal{B}_X(t), \ldots, \mathcal{B}_X(m - (n - 1)d - 1) \rangle,
$$

where $A^{(t)}_X \subset D^b(X)$ is the full subcategory of $\mathcal{G} \in D^b(X)$ such that for all $\mathcal{F} \in \mathcal{B}$ we have

$$
\text{Hom}(f^*\mathcal{F}(s), \mathcal{G}) = 0 \quad \text{for all } t \leq s \leq m - (n - 1)d - 1,
$$

$$
\text{Hom}(\mathcal{G}, f^*\mathcal{F}(s)) = 0 \quad \text{for all } 0 \leq s \leq t - 1.
$$

To finish the proof, we must show $A^{(t)}_X = A_X(t)$. By adjunction and (5.4), the above conditions can be rewritten as

$$
\text{Hom}(\mathcal{F}(s), f_*\mathcal{G}) = 0 \quad \text{for all } t \leq s \leq m - (n - 1)d - 1,
$$

$$
\text{Hom}(f_*\mathcal{G}, \mathcal{F}(s - (n - 1)d)) = 0 \quad \text{for all } 0 \leq s \leq t - 1,
$$

or equivalently

$$
f_*\mathcal{G} \in \langle B(-(n - 1)d), \ldots, B(t - 1 - (n - 1)d) \rangle \cap \langle B(t), \ldots, B(m - (n - 1)d - 1) \rangle^\perp.
$$

It follows from (5.1) twisted by $\mathcal{O}_Y(-(n - 1)d)$ that the above intersection of categories equals

$$
\langle B(t - (n - 1)d), \ldots, B(t - 1) \rangle.
$$

This is the twist by $\mathcal{O}_Y(t)$ of the category defining $A_X$, hence $A^{(t)}_X = A_X(t)$. □

Later we will also need the following strengthening of Lemma 5.1(1), which holds if $nd \leq m$ (as in the setup of Theorem 1.1).

**Lemma 5.3.** Let $n$ and $d$ be positive integers such that $nd \leq m$. Let $f : X \to Y$ be a degree $n$ cyclic cover of $Y$ ramified over a divisor in $|\mathcal{O}_Y(nd)|$. Then the restriction of the functor $f^* : D^b(Y) \to D^b(X)$ to the subcategory

$$
\langle \mathcal{B}, \mathcal{B}(1), \ldots, \mathcal{B}(d - 1) \rangle \subset D^b(Y)
$$

is fully faithful and induces an equivalence onto $\langle \mathcal{B}_X, \mathcal{B}_X(1), \ldots, \mathcal{B}_X(d - 1) \rangle \subset D^b(X)$.

**Proof.** The same argument as in the proof of part (1) of Lemma 5.1 works. □

Note that the action of $\mu_n$ on $D^b(X)$ preserves the decomposition (5.2) of Lemma 5.1. Moreover, since the twist by $t$ autoequivalence of $D^b(X)$ is $\mu_n$-equivariant, the category $\mathcal{B}_X(t)^{\mu_n}$ equals the category $\mathcal{B}_X^{\mu_n}(t)$ obtained from $\mathcal{B}_X^{\mu_n}$ by twisting by $\mathcal{O}_X(t)$; similarly, $A_X(t)^{\mu_n}$ equals $A_X^{\mu_n}(t)$. Hence applying Theorem 5.2 to (5.2) gives the following.
Lemma 5.4. Let $n$ and $d$ be positive integers such that $(n - 1)d < m$. Let $f : X \to Y$ be a degree $n$ cyclic cover of $Y$ ramified over a divisor in $|O_Y(nd)|$. For each $0 \leq t \leq m - (n - 1)d$ there is a semiorthogonal decomposition

$$D^b(X)^{\mu_n} = \langle B^\mu_X, \ldots, B^\mu_X(t - 1), A^\mu_X(t), B^\mu_X(t), \ldots, B^\mu_X(m - (n - 1)d - 1) \rangle.$$  

(5.5)

For a divisor in $Y$, we have the following analogue of Lemma 5.1.

Lemma 5.5. Let $e$ be an integer such that $1 \leq e < m$. Let $i : Z \hookrightarrow Y$ be the inclusion of a divisor in $|O_Y(e)|$.

1. The functor $i^* : D^b(Y) \to D^b(Z)$ is fully faithful on the subcategory $B \subset D^b(Y)$.

2. Denoting the essential image of $B$ by $B_Z$, for each $0 \leq t \leq m - e$ there is a semiorthogonal decomposition

$$D^b(Z) = \langle B_Z, \ldots, B_Z(t - 1), A_Z(t), B_Z(t), \ldots, B_Z(m - e - 1) \rangle,$$  

(5.6)

where $A_Z$ is the full subcategory of $D^b(Z)$ consisting of all $\mathcal{G} \in D^b(Z)$ such that $i_*\mathcal{G} \in \langle B(-e), \ldots, B(-1) \rangle$.

Remark 5.6. Again, the semiorthogonal decomposition (5.6) still holds for $e = m$ — in this case there are no “trivial components” equivalent to $B$ in $D^b(Z)$, so $A_Z = D^b(Z)$.

Proof. Let $\mathcal{F}, \mathcal{G} \in B$. For any integers $r$ and $s$, adjunction gives

$$\text{Hom}(i^*\mathcal{F}(r), i^*\mathcal{G}(s)) = \text{Hom}(\mathcal{F}(r), i_*i^*\mathcal{G}(s)).$$

On the other hand, we have a distinguished triangle

$$\mathcal{G}(s - e) \to \mathcal{G}(s) \to i_*i^*\mathcal{G}(s)$$

obtained by tensoring the resolution of $i_*O_Z$ on $Y$ with $\mathcal{G}(s)$. Applying $\text{Hom}(\mathcal{F}(r), -)$ gives a long exact sequence

$$\cdots \to \text{Hom}(\mathcal{F}(r), \mathcal{G}(s - e)) \to \text{Hom}(\mathcal{F}(r), \mathcal{G}(s)) \to \text{Hom}(\mathcal{F}(r), i_*i^*\mathcal{G}(s)) \to \cdots$$

Now the result follows by the same argument as in the proof of Lemma 5.1 using the above sequence in place of (5.3). \hfill $\square$

Remark 5.7. Lemmas 5.1 and 5.5 generalize directly to the situation where the Lefschetz decomposition (5.1) is not assumed to be rectangular. However, we will not need this generalization.

6. Proof of the main result

In this section, we prove Theorem 1.1. The functors embedding the $n - 1$ copies of the category $A_Z$ into $A^\mu_X$ are constructed explicitly in the course of the proof.

6.1. Setup. Recall the setup of the theorem: Let $Y$ be an algebraic variety with a line bundle $O_Y(1)$ and a length $m$ rectangular Lefschetz decomposition of its derived category as in (5.1). Choose positive integers $n$ and $d$ such that $nd \leq m$. We set $\mathcal{L} = O_Y(d)$ and consider a degree $n$ cyclic cover $f : X \to Y$ ramified over a Cartier divisor $Z$ in $|\mathcal{L}^n| = |O_Y(nd)|$, as in Section 1. This data fits into a commutative diagram (4.2).
6.2. Strategy of the proof. We start with one of the semiorthogonal decompositions of $\text{D}^b(X)^{\mu_n}$ provided by Theorem 4.1, namely
\[
\text{D}^b(X)^{\mu_n} = \langle f_0^*\text{D}^b(Y), j_0^*\text{D}^b(Z), j_1^*\text{D}^b(Z), \ldots, j_{n-2}^*\text{D}^b(Z) \rangle.
\] (6.1)
Taking into account the decomposition of $\text{D}^b(Y)$ given by (5.1) and of $\text{D}^b(Z)$ given by (5.6) with $t = 0$ and $e = nd$, we see $\text{D}^b(X)^{\mu_n}$ has a semiorthogonal decomposition with $m + (n-1)(m-nd) = nm - n(n-1)d$
copies of the category $\mathcal{B}$ and $n-1$ copies of the category $\mathcal{A}_Z$ as components. On the other hand, Lemma 5.4 gives
\[
\text{D}^b(X)^{\mu_n} = \langle \mathcal{A}_X^{\mu_n}, \mathcal{B}_X^{\mu_n}, \mathcal{B}_X^{\mu_n}(1), \ldots, \mathcal{B}_X^{\mu_n}(m - (n-1)d - 1) \rangle.
\] (6.2)
Note that the action of $\mu_n$ on $\mathcal{B}_X(t)$ is trivial for any $t$, so by Proposition 3.3 it follows
\[
\mathcal{B}_X^{\mu_n}(t) = \langle \mathcal{B}_X(t) \otimes 1, \ldots, \mathcal{B}_X(t) \otimes \chi^{n-1} \rangle.
\]
Hence the decomposition (6.2) has $n(m - (n-1)d)$ copies of $\mathcal{B}$ (the same number as above!) and one copy of $\mathcal{A}_X^{\mu_n}$ as components. To prove Theorem 1.1, we find a sequence of mutations transforming the $\mathcal{B}$-components of (6.1) into those of (6.2).

To concisely write the decompositions occurring in the proof, we introduce the following notation. Given subsets of “twists” $T \subset \mathbb{Z}$ and “weights” $W \subset \mathbb{Z}/n$, we define
\[
\mathcal{B}_X^W(T) = \langle \mathcal{B}_X(t) \otimes \chi^k \rangle_{t \in T, k \in W} \subset \text{D}^b(X)^{\mu_n}
\]
to be the triangulated subcategory generated by $\mathcal{B}_X(t) \otimes \chi^k$ for $t \in T, k \in W$, and we define
\[
\mathcal{B}_Z(T) = \langle \mathcal{B}_Z(t) \rangle_{t \in T} \subset \text{D}^b(Z)
\]
to be the triangulated subcategory generated by $\mathcal{B}_Z(t)$ for $t \in T$. If $a \leq b$ are integers, we write $[a, b]$ for the set of integers $t$ with $a \leq t \leq b$. We also set
\[
M = m - (n-1)d.
\]
With this notation, we can rewrite (6.1) as
\[
\text{D}^b(X)^{\mu_n} = \langle \mathcal{B}_X^0([0, m-1]), j_0^*\text{D}^b(Z), j_1^*\text{D}^b(Z), \ldots, j_{n-2}^*\text{D}^b(Z) \rangle
\] (6.3)
and (6.2) as
\[
\text{D}^b(X)^{\mu_n} = \langle \mathcal{A}_X^{\mu_n}, \mathcal{B}_X^{[0,n-1]}([0, M-1]) \rangle.
\] (6.4)

6.3. Mutations. Now we perform a sequence of mutations.

**Step 1.** Write the first component of the decomposition (6.3) as
\[
\mathcal{B}_X^0([0, m-1]) = \langle \mathcal{B}_X^0([0, M-1]), \mathcal{B}_X^0([M, M + d - 1]), \ldots, \mathcal{B}_X^0([m - d, m - 1]) \rangle,
\]
with $M$ copies of $\mathcal{B}$ in the first component and $d$ copies in each of the next $n - 1$ components. Note that the subcategory $j_{k*}\text{D}^b(Z) \subset \text{D}^b(X)^{\mu_n}$ is admissible since the functor $j_{k*}$ has both left and right adjoints $j_k^!$ and $j_k^*$; hence the mutation functors through this subcategory are well-defined. So for $a = 1, \ldots, n - 1$, we can successively right mutate the component $\mathcal{B}_X^0([m - ad, m - ad + d - 1])$ through
\[
\langle j_0^*\text{D}^b(Z), \ldots, j_{n-a-1}^*\text{D}^b(Z) \rangle
\]
in (6.3). To understand the result we need the following lemma.
Lemma 6.1. For any twist \( t \in \mathbb{Z} \) and weight \( k \in \mathbb{Z}/n \), we have
\[
R_{j_k^* D^b(Z)}(B_X^k(t)) = B_X^{k+1}(t-d).
\]

Proof. In fact, for any \( \mathcal{F} \in D^b(Y) \) we prove
\[
R_{j_k^* D^b(Z)}(f_k^* \mathcal{F}) \cong f_{k+1}^* \mathcal{F}(-d).
\]

Tensoring \((4.3)\) by \( f_k^* \mathcal{F} = f_0^* \mathcal{F} \otimes \chi^k \) and using the projection formula \( f_0^* \mathcal{F} \otimes j_0! O_Z \cong j_0! i^* \mathcal{F} \), we obtain a distinguished triangle
\[
f_{k+1}^* \mathcal{F}(-d) \rightarrow f_k^* \mathcal{F} \rightarrow j_{k*} i^* \mathcal{F}. \tag{6.5}\]

The last vertex is in \( j_{k*} D^b(Z) \) and the first is in \( f_{k+1}^* D^b(Y) \), so to show this is a mutation triangle \((2.0)\) it suffices to show the pair \((j_k^* D^b(Z), f_{k+1}^* D^b(Y))\) is semiorthogonal. But this holds by \((4.6)\). \(\square\)

By an iterated application of the lemma, the result of the above mutations is a semiorthogonal decomposition
\[
D^b(X)^{\mu_n} = \langle B_X^0([0, M-1]), j_0! A_Z(d), j_1! A_Z(d), \ldots, j_{n-2}! A_Z(d), j_n! B_Z([d, M-1]), B_X^2([M-d, M-1]), \ldots, B_X^{n-1}([M-d, M-1]) \rangle. \tag{6.6}\]

Step 2. Substitute for each copy of \( D^b(Z) \) in \((6.6)\) the \( t = 0 \) decomposition \((5.6)\) (with \( e = nd \)) twisted by \( O_Z(d) \) (if \( m = nd \) just take \( A_Z = D^b(Z) \):
\[
D^b(X)^{\mu_n} = \langle B_X^0([0, M-1]), j_0! A_Z(d), j_1! A_Z(d), \ldots, j_{n-2}! A_Z(d), j_n! B_Z([d, M-1]), B_X^2([M-d, M-1]), \ldots, B_X^{n-1}([M-d, M-1]) \rangle. \tag{6.7}\]

Step 3. Note that the subcategory \( B_X^k(t) \subset D^b(X)^{\mu_n} \) is admissible for all \( t \) and \( k \), since \( B(t) \) is admissible in \( D^b(Y) \) and the functor \( f_k^* \) has both right and left adjoints. So for \( k = 0, \ldots, n-2 \), we can successively left mutate the component \( j_{k*} A_Z(d) \) through the copies of \( B \) appearing to its left in \((6.7)\).

This gives
\[
D^b(X)^{\mu_n} = \langle \Phi_0(A_Z), \Phi_1(A_Z), \ldots, \Phi_{n-2}(A_Z), \mathcal{C}_{n-1} \rangle, \tag{6.8}\]
where \( \Phi_k(\mathcal{F}) = Lc_k(j_{k*} \mathcal{F}(d)) \) and
\[
\mathcal{C}_k = \langle B_X^0([0, M-1]), j_0! B_Z([d, M-1]), B_X^2([M-d, M-1]), \ldots, j_{k-1}! B_Z([d, M-1]), B_X^{k-1}([M-d, M-1]) \rangle. \tag{6.9}\]

Note that the functors \( \Phi_k : A_Z \rightarrow D^b(X)^{\mu_n} \) are fully faithful since \( j_{k*} A_Z(d) \) is left orthogonal to \( \mathcal{C}_k \). We shall see their images lie in \( A_Z^{\mu_n} \) and give the desired semiorthogonal decomposition.
6.4. The final argument. It remains to show the “$B$-part” $C_{n-1}$ of the decomposition (6.8) equals the “$B$-part” $B_X^{[0,n-1]}([0, M - 1])$ of the decomposition (6.4). We do this in Lemma 6.3, where we in fact establish a simple expression for each category $C_k$, which for $k = n - 1$ gives the desired equality of “$B$-parts.” We will need the following mutation lemma.

Lemma 6.2. Assume $nd < m$. For twists $s, t \in \mathbb{Z}$ and weights $k, \ell \in \mathbb{Z}/n$, we have

$$L_{B_X^{\ell}(s)}(j_{k,\ell}B_Z(t)) = \begin{cases} j_{k,\ell}B_Z(t) & \text{if } k \neq \ell \text{ or if } t < s < t + M - d, \\ B_X^{k+1}(t - d) & \text{if } k = \ell \text{ and } t = s. \end{cases}$$

(6.10)

Proof. The assumption $nd < m$ guarantees $B_Z$ is defined. For $k \neq \ell$ or $t < s < t + M - d$, it suffices to show the pair $(j_{k,\ell}B_Z(t), B_X^{\ell}(s))$ is semiorthogonal. If $k \neq \ell$, this holds by (4.16) since $B_X^{\ell}(s) = f_{s}^*(B(s))$. If $k = \ell$, note that by adjunction the desired semiorthogonality is equivalent to semiorthogonality of the pair $(B_Z(t), j_{k,\ell}B_X^{\ell}(s))$. Since $j_{k,\ell}B_X^{\ell}(s) = B_Z(t)$, this holds if $t < s < t + M - d$ by Lemma 5.5.

For $k = \ell$ and $t = s$, by definition of the category $B_Z(t)$ it is enough to check

$$L_{B_X^{\ell}}(j_{k,\ell}F(t)) \cong f_{k+1}^*(F(t - d)[1])$$

(6.11)

for any $F \in B$. Twisting the triangle (6.10) by $O_X(t)$ and then rotating, we obtain a triangle

$$f_{k}^*F(t) \to j_{k,\ell}^*F(t) \to f_{k+1}^*F(t - d)[1].$$

The first vertex is in $B_X^{k}(t)$ and the last is in $B_X^{k+1}(t - d)$, so to show this is a mutation triangle it suffices to show the pair $(B_X^{k+1}(t - d), B_X^{k}(t))$ is semiorthogonal. But this holds by the decomposition (6.2) since $m - (n - d) - 1 \geq d$. □

Now we can prove a simple formula for the categories $C_k$.

Lemma 6.3. For $0 \leq k \leq n - 1$, there is an equality

$$C_k = B_X^{[0,k]}([0, M - 1]).$$

(6.12)

Proof. If $m = nd$, the result is obvious. Indeed, in this case $M = d$ and there are no $B_Z$-components in (6.9). Thus from now on we assume $nd < m$.

The proof is by induction on $k$. For $k = 0$, there is nothing to prove. If the result holds for $k$, then

$$C_{k+1} = \langle B_X^{[0,k]}([0, M - 1]), j_{k}B_Z([d, M - 1]), B_X^{k+1}([M - d, M - 1]) \rangle.$$  

(6.13)

To show this equals $B_X^{[0,k+1]}([0, M - 1])$, we mutate each component of $j_{k}B_Z([d, M - 1])$ through a subset of the components of $B_X^{[0,k]}([0, M - 1])$. Namely, for $t = d, \ldots, M - 1$, we successively left mutate $j_{k}B_Z(t)$ through $B_X^{[0,k]}([t, M - 1])$. By Lemma 6.2 this transforms $j_{k}B_Z(t)$ into $B_X^{k+1}(t - d)$. The components of the resulting decomposition of $C_{k+1}$ thus coincide (up to a permutation) with the components of $B_X^{[0,k+1]}([0, M - 1])$. Hence these categories are equal. □

This finishes the proof of Theorem 1.1. To compactly write the formula for the embedding functors $\Phi_k$, we use the notation $T_E(F) = F \otimes E$

for the tensor product functor. Then the result we have shown is:
Theorem 6.4. The functors $\Phi_0, \Phi_1, \ldots, \Phi_{n-2} : A_Z \to A_X^{\mu_n}$ defined by

$$\Phi_k = L_B^{\mu_n} \circ j_k \circ T_{\emptyset}(d)$$

(6.14)

are fully faithful, and their essential images give a semiorthogonal decomposition

$$A_X^{\mu_n} = \langle \Phi_0(A_Z), \Phi_1(A_Z), \ldots, \Phi_{n-2}(A_Z) \rangle.$$

Below we give a simpler expression for the functors $\Phi_k$.

6.5. Simplifications of the functors $\Phi_k$. First we show the mutation functor in (6.14) can be simplified considerably.

Proposition 6.5. For $0 \leq k \leq n - 2$, there is an isomorphism of functors

$$\Phi_k \cong L_B^k([0,d-1]) \circ j_k \circ T_{\emptyset}(d).$$

(6.15)

Proof. The left mutation functor $L_B^{\mu_n}([0,M-1])$ factors into simpler pieces as follows. By (6.2) there is a decomposition

$$B_X^{\mu_n}([0,M-1]) = \langle B_X^k([0,d-1]), B_X^{0,k-1}([0,d-1]), B_X^{0,k}([d,M-1]) \rangle.$$

On the other hand, Lemma 5.3 implies the action of $\mu_n$ on $B_X([0,d-1])$ is trivial, so by the complete orthogonality in Proposition 3.3 there is a decomposition

$$B_X^{0,k}([0,M-1]) = \langle B_X^k([0,d-1]), B_X^{0,k-1}([0,d-1]), B_X^{0,k}([d,M-1]) \rangle.$$

Hence by (2.3) we get a factorization

$$L_B^{\mu_n}([0,M-1]) = L_B^k([0,d-1]) \circ L_B^{0,k-1}([0,d-1]) \circ L_B^{0,k}([d,M-1]).$$

Thus, to prove the proposition it suffices to show the mutation functors $L_B^{\mu_n}([d,M-1])$ and $L_B^{0,k-1}([0,d-1])$ act as the identity functor on the category $j_{k*}A_Z(d)$. This is a consequence of the following lemma.

Lemma 6.6. If $k \neq \ell$ or if $d \leq s \leq M - 1$, the pair $(j_{k*}A_Z(d), B_X^\ell(s))$ is semiorthogonal. In particular, in this case $L_B^\ell(s)$ is the identity functor on $j_{k*}A_Z(d)$.

Proof. By adjunction, this is equivalent to semiorthogonality of the pair $(A_Z(d), j_k^*f_\ell^*(B(s)))$. If $k \neq \ell$, then $j_k^*f_\ell^* = 0$, so this is clear. If $k = \ell$, then $j_k^*f_\ell^* = B_Z(s)$, and the required semiorthogonality follows from (5.6).

The proposition implies the functors $\Phi_k$ differ from each other by twists by characters:

Corollary 6.7. For $0 \leq k \leq n - 2$, there is an isomorphism of functors

$$\Phi_k \cong T^{\mu_n} \circ \Phi_0.$$

In particular, the semiorthogonal decomposition of Theorem 6.4 can be written as

$$A_X^{\mu_n} = \langle \Phi_0(A_Z), \Phi_0(A_Z) \otimes \chi, \ldots, \Phi_0(A_Z) \otimes \chi^{n-2} \rangle.$$
Proof. We have
\[
\Phi_k \cong L_{\mathbb{B}_X^k([0,d-1])} \circ j_k^* \circ T_{\mathcal{O}_Z(d)} \\
\cong L_{\mathbb{B}_X^k([0,d-1])} \otimes \chi_k \circ j_0^* \circ T_{\mathcal{O}_Z(d)} \\
\cong T_{\chi_k} \circ L_{\mathbb{B}_X^k([0,d-1])} \circ j_0^* \circ T_{\mathcal{O}_Z(d)} \\
\cong T_{\chi_k} \circ \Phi_0.
\]
The first and last isomorphisms hold by Proposition 6.5, the second by the definitions, and the third by Lemma 2.5.

7. Rotation functors and reconstruction results

7.1. Reconstruction. In [9] Elagin proved that, under certain conditions, an additive category equipped with a group action can be reconstructed from its equivariant category.

Theorem 7.1 ([9, Theorem 7.2]). Let $G$ be a finite abelian group. Let $\mathcal{C}$ be an additive idempotent complete category, linear over an algebraically closed field of characteristic coprime to $|G|$. Then there is an equivalence
\[ \mathcal{C} \cong (\mathcal{C}^G)^\hat{G}, \]
where characters $\chi \in \hat{G}$ act on $\mathcal{C}^G$ by the tensor product functors $T_\chi : \mathcal{C}^G \to \mathcal{C}^G$.

Remark 7.2. Suppose in the situation of Theorem 7.1 that $\mathcal{C}$ and $\mathcal{C}^G$ are triangulated. Then $(\mathcal{C}^G)^\hat{G}$ comes with a natural class of distinguished triangles, consisting of those triangles whose image under the forgetful functor $(\mathcal{C}^G)^\hat{G} \to \mathcal{C}^G$ are distinguished (see the discussion in Section 3.2). In fact, the equivalence $\mathcal{C} \cong (\mathcal{C}^G)^\hat{G}$ of the theorem respects the classes of distinguished triangles (in particular $(\mathcal{C}^G)^\hat{G}$ is triangulated). Indeed, unwinding Elagin’s construction of this equivalence, it follows that its composition with the forgetful functor $\mathcal{C} \cong (\mathcal{C}^G)^\hat{G} \to \mathcal{C}^G$ is triangulated. As this composition is triangulated, it follows that $\mathcal{C} \cong (\mathcal{C}^G)^\hat{G}$ respects distinguished triangles.

Theorem 7.1 applies to the category $\mathcal{A}_X$ with the action of the group $\mu_n$, where $X$ is as in Section 6. Note that in this case the dual group is $\hat{\mu}_n = \mathbb{Z}/n$.

Corollary 7.3. There is an equivalence $\mathcal{A}_X \cong (\mathcal{A}_{\mu_n}^X)^{\mathbb{Z}/n}$.

By Theorem 6.4 there is a semiorthogonal decomposition of $\mathcal{A}_{\mu_n}^X$ into $n - 1$ copies of $\mathcal{A}_Z$. In case $n = 2$, we have the following consequence of Theorem 7.1.

Corollary 7.4. Let $n = 2$ and let $\chi$ be the nontrivial character of $\mu_2$. The functor
\[ \tau = \Phi_0^{-1} \circ T_\chi \circ \Phi_0 : \mathcal{A}_Z \to \mathcal{A}_Z \]
induces a $\mathbb{Z}/2$-action on $\mathcal{A}_Z$, such that there is an equivalence
\[ \mathcal{A}_X \cong \mathcal{A}_Z^{\mathbb{Z}/2}. \]
The situation for \( n > 2 \) is more complicated. To recover \( A_X \) from \( A_Z \), we need the data of the gluing functors for the \( n-1 \) copies of \( A_Z \) in the decomposition of Theorem 6.4 together with the action of \( \hat{\mu}_n \) on the gluing of these categories (see Section 7.5 for more details).

In the rest of this section, we discuss some interesting autoequivalences of \( A_X, A^{\mu_n}_X, \) and \( A_Z \), which we call rotation functors. We use these rotation functors to identify more explicitly the functor \( \tau \) from Corollary 7.4 (see Proposition 7.9). Then, in case \( n = 3 \), we speculate about a way to reconstruct \( A_X \) in terms of \( A_Z \) and its associated rotation functor (see Section 7.5).

7.2. Rotation functors. We work in the following setup: \( Y \) is a variety with a rectangular Lefschetz decomposition as in (5.1); \( f : X \to Y \) is a degree \( n \) cyclic cover ramified over a divisor in \( |O_Y(nd)| \), where \((n-1)d \leq m\); and \( i : Z \to Y \) is the inclusion of a divisor in \( |O_Y(e)| \), where \( 1 \leq e \leq m \). This is the natural setup for defining the rotation functors. Later we specialize to the setup of Section 6.

The rotation functors are the following endofunctors:

\[
\begin{align*}
L_{B_X} \circ T_{O_X(1)} : D^b(X) & \to D^b(X), \\
L_{B^{\mu_n}_X} \circ T_{O_X(1)} : D^b(X)^{\mu_n} & \to D^b(X)^{\mu_n}, \\
L_{B_Z} \circ T_{O_Z(1)} : D^b(Z) & \to D^b(Z).
\end{align*}
\]

If \((n-1)d = m\) the category \( B_X \) is not defined, and if \( e = m \) the category \( B_Z \) is not defined. However, in these cases there are still natural definitions of the functors \( L_{B_X}, L_{B^{\mu_n}_X}, L_{B_Z}, \) under an additional technical assumption — the finiteness of the cohomological amplitude (see [18] Section 2.6) of the projection functor \( \beta \beta^! \) onto \( B \) (which holds automatically if \( Y \) is smooth). We discuss \( L_{B_Z} \), the other functors being similar. We take

\[
L_{B_Z} = \text{Cone}(i^* \beta \beta^! i_* \to \text{id}). \tag{7.2}
\]

To make sense of this as a functor, we note that under the above assumption of finiteness of cohomological amplitude, the projection functor \( \beta \beta^! \) can be represented as a Fourier–Mukai functor by [18] Theorem 7.1. It follows that \( i^* \beta \beta^! i_* \) is a Fourier–Mukai functor as well. Moreover, the morphism \( i^* \beta \beta^! i_* \to \text{id} \) is induced by a morphism of kernels of Fourier–Mukai functors. We define \( L_{B_Z} \) to be the Fourier–Mukai functor with kernel given by the cone of this morphism of kernels.

For the most part the reader can ignore the distinction between the \( e = m \) functor \( L_{B_Z} \) and the usual mutation functors, as they satisfy similar properties, e.g. for \( \mathcal{F} \in D^b(Z) \) there is a functorial distinguished triangle

\[
i^* \beta \beta^! i_* \mathcal{F} \to \mathcal{F} \to L_{B_Z}(\mathcal{F}),
\]

and the obvious analogue of Lemma 2.5 holds. The functor \( i^* \circ \beta : B \to D^b(Z) \) is in fact spherical and \( L_{B_Z} \) is the corresponding spherical twist, but we will not need this fact.

In what follows, when considering \( L_{B_X}, L_{B^{\mu_n}_X}, \) or \( L_{B_Z} \) in the boundary cases \((n-1)d = m\) or \( e = m \), we will tacitly assume the projection functor \( \beta \beta^! \) has finite cohomological amplitude. Again, this condition is automatic if \( Y \) is smooth.

Lemma 7.5. The rotation functors preserve the subcategories \( A_X, A^{\mu_n}_X, \) and \( A_Z \).

Proof. We give the proof for \( A_Z \), the other two cases being essentially the same. If \( e = m \), then \( A_Z = D^b(Z) \) and there is nothing to prove. Thus we assume \( e < m \), so that \( B_Z \) is defined
Proof. We start by rewriting both sides of (7.6). For the left side, we have functors

\[ D^b(Z) = \langle B_Z, A_Z(1), B_Z(1), \ldots, B_Z(m - e - 1) \rangle. \]

By Proposition 2.3, the functor \( L_{B_Z} \) is fully faithful on \( A_Z(1) \) and induces a semiorthogonal decomposition

\[ D^b(Z) = \langle L_{B_Z}(A_Z(1)), B_Z, B_Z(1), \ldots, B_Z(m - e - 1) \rangle. \]

Comparing this with (7.3) for \( t = 0 \), we deduce the claim. \( \square \)

The lemma shows the rotation functors restrict to endofunctors of \( A_X, A_X^{\mu_n} \), and \( A_Z \). In fact, the argument of the lemma shows these endofunctors are autoequivalences (with inverse the composition of a right mutation and a twist). We denote these autoequivalences by

\[ O_X : A_X \rightarrow A_X, \]

\[ O_X^{\mu_n} : A_X^{\mu_n} \rightarrow A_X^{\mu_n}, \]

\[ O_Z : A_Z \rightarrow A_Z. \]

The following theorem is an unpublished result of the first author.

**Theorem 7.6.** There is an isomorphism of functors

\[ O_Z^d \cong [2]. \] (7.3)

Moreover, if \( n = 2 \) (so \( d \leq m \)), there are isomorphisms of functors

\[ O_X^d \cong \sigma[1], \]

\[ (O_X^{\mu_n})^d \cong T_X[1], \] (7.4) (7.5)

where \( \sigma : A_X \rightarrow A_X \) is the involution induced by the involution of \( X \) over \( Y \).

**Remark 7.7.** If \( Y, X \), and \( Z \) are smooth and projective, \( n = 2 \), and \( \omega_Y = O_Y(-m) \), then Theorem 7.6 can be used to give an expression for a power of the Serre functors of \( A_X, A_X^{\mu_n} \), and \( A_Z \). In fact, this was the first author's original motivation for proving the theorem. For example, denoting \( S_{A_Z} \) and \( S_{D^b(Z)} \) the Serre functors of \( A_Z \) and \( D^b(Z) \), we have

\[ S_{A_Z}^{-1} \cong L_{B_Z, \ldots, B_Z(m - e - 1)} \circ S_{D^b(Z)}^{-1} \]

\[ \cong L_{B_Z, \ldots, B_Z(m - e - 1)} \circ T_{O_Z(m - e)}[-\dim(Z)] \]

\[ \cong O_Z^{m - e}[-\dim(Z)], \]

a power of which can be computed by the theorem (see [19] Theorem 5.3 and Remark 5.4).

### 7.3 Intertwining property

For the rest of this section, we assume the setup and notation of Section 6. In particular, from now on \( nd \leq m \) and \( Z \) denotes the branch locus of \( f : X \rightarrow Y \).

Here is a key observation about the rotation functors:

**Proposition 7.8.** The functor \( \Phi_0 : A_Z \rightarrow A_X^{\mu_n} \) defined by (6.14) intertwines the rotation functors \( O_Z \) and \( O_X^{\mu_n} \), i.e.

\[ O_X^{\mu_n} \circ \Phi_0 \cong \Phi_0 \circ O_Z. \] (7.6)

**Proof.** We start by rewriting both sides of (7.6). For the left side, we have

\[ O_X^{\mu_n} \circ \Phi_0 \cong L_{B_X^{\mu_n}} \circ T_{O_X(1)} \circ L_{B_X^{\mu_n}}(0, d - 1) \circ j_0^* \circ T_{O_Z(d)} \]

\[ \cong L_{B_X^{\mu_n}} \circ L_{B_X^{\mu_n}(1, d)} \circ j_0^* \circ T_{O_Z(d + 1)}. \]
The first isomorphism holds by the definition of \( \mathcal{O}^{\mu_n}_{B^X} \) and Proposition 6.5 and the second by Lemma 2.5 and the projection formula. Note that \( (\mathcal{B}^{\mu_n}_{X}, \mathcal{B}^0_X([1, d - 1])) \) is a semiorthogonal pair by (6.2) and our assumption \( nd \leq m \). (We caution the reader that the pair \( (\mathcal{B}^{\mu_n}_{X}, \mathcal{B}^0_X([1, d])) \) is not semiorthogonal if \( m = nd \).) The action of \( \mu_n \) on \( \mathcal{B}^0_X([0, d - 1]) \) is trivial by Lemma 5.3 so by the complete orthogonality in Proposition 8.3 there is a decomposition

\[
\langle \mathcal{B}^{\mu_n}_{X}, \mathcal{B}^0_X([1, d - 1]) \rangle = \langle \mathcal{B}^0_X([0, d - 1]), \mathcal{B}^1_X \rangle.
\]

Hence by (2.3) we have

\[
\mathcal{L}_{\mathcal{B}^{\mu_n}_{X}} \circ \mathcal{L}_{\mathcal{B}^0_X ([1,d])} \cong \mathcal{L}_{\mathcal{B}^{\mu_n}_{X}, \mathcal{B}^0_X([1,d-1])} \circ \mathcal{L}_{\mathcal{B}^0_X (d)} \cong \mathcal{L}_{\mathcal{B}^{\mu_n}_{X}, \mathcal{B}^0_X([0,d-1])} \circ \mathcal{L}_{\mathcal{B}^1_X} \circ \mathcal{L}_{\mathcal{B}^0_X (d)}.
\]

Combining this with the above, we have

\[
\mathcal{O}^{\mu_n}_{X} \circ \Phi_0 \cong \mathcal{L}_{\mathcal{B}^{\mu_n}_{X}, \mathcal{B}^0_X([0,d-1])} \circ \mathcal{L}_{\mathcal{B}^1_X} \circ \mathcal{L}_{\mathcal{B}^0_X (d)} \circ j_{0*} \circ \mathcal{T}_{\mathcal{O}_Z (d+1)}.
\]

(7.7)

Now we consider the right side of (7.6). First we note

\[
\Phi_0 \cong \mathcal{L}_{\mathcal{B}^0_X ([0,d-1])} \circ \mathcal{L}_{\mathcal{B}^1_X} \circ j_{0*} \circ \mathcal{T}_{\mathcal{O}_Z (d)}.
\]

Indeed, \( \mathcal{L}_{\mathcal{B}^1_X} \) is the identity on \( j_{0*} \mathcal{A}_Z (d) \) by Lemma 6.6 so this is equivalent to the isomorphism of Proposition 6.5. Using this and Lemma 2.5 we find

\[
\Phi_0 \circ \mathcal{O}_Z \cong \mathcal{L}_{\mathcal{B}^0_X ([0,d-1])} \circ \mathcal{L}_{\mathcal{B}^1_X} \circ j_{0*} \circ \mathcal{L}_{\mathcal{B}_Z} \circ \mathcal{T}_{\mathcal{O}_Z (d+1)}.
\]

(7.8)

To prove the proposition, by (7.7) and (7.8) it suffices to construct a morphism of functors

\[
\mathcal{L}_{\mathcal{B}^0_X (d)} \circ j_{0*} \rightarrow j_{0*} \circ \mathcal{L}_{\mathcal{B}_Z (d)}
\]

whose composition with \( \mathcal{L}_{\mathcal{B}^1_X} \) is an isomorphism. By Lemma 2.5 this is equivalent to constructing a morphism

\[
\mathcal{L}_{\mathcal{B}^0_X} \circ j_{0*} \rightarrow j_{0*} \circ \mathcal{L}_{\mathcal{B}_Z}
\]

whose composition with \( \mathcal{L}_{\mathcal{B}^1_X} \) is an isomorphism.

For this, consider the commutative diagram of functors

\[
\begin{array}{ccc}
\mathcal{L}_{\mathcal{B}^0_X} \circ j_{0*} & \rightarrow & j_{0*} \circ \mathcal{L}_{\mathcal{B}_Z} \\
\downarrow & & \downarrow \\
\mathcal{L}_{\mathcal{B}^0_X} & \rightarrow & \mathcal{L}_{\mathcal{B}^0_X}
\end{array}
\]

(7.9)

Here the two rows come from the definition of the mutation functors (or from (7.2) in case \( m = nd \)), and the left vertical arrow is induced by the isomorphism \( f_{0*} j_{0*} \cong i_* \) and the morphism \( f_{0*} \rightarrow j_{0*} j_{0*} f_{0*} \cong j_{0*} i_* \) obtained from the unit of the adjunction between \( j_{0*} \) and \( j_{0*} \). It is easy to check the left square commutes. All of the functors in the diagram are Fourier–Mukai functors and the arrows in the diagram come from morphisms of kernels. In the corresponding diagram of Fourier–Mukai kernels we can find a dashed arrow making the diagram commute, and this induces the dashed arrow in the above diagram.

Now we describe the cone of the morphism \( \mathcal{L}_{\mathcal{B}^0_X} \circ j_{0*} \rightarrow j_{0*} \circ \mathcal{L}_{\mathcal{B}_Z} \) applied to an object \( \mathcal{F} \in \mathcal{D}^b(Z) \). Set \( \mathcal{G} = \beta \beta^i \mathcal{F} \) so that \( \mathcal{G} \in \mathcal{B} \). Tensoring (1.3) with \( f_{0}^* \mathcal{G} \), we see the left column of diagram (7.9) applied to \( \mathcal{F} \) fits into a distinguished triangle

\[
f_{0}^* \mathcal{G} (-d) \otimes \chi \rightarrow f_{0}^* \mathcal{G} \rightarrow j_{0*}^* \mathcal{G}.
\]
By the octahedral axiom, diagram (7.9) applied to \( \mathcal{F} \) thus gives a distinguished triangle
\[
f^*_0 \mathcal{S}(-d) \otimes \chi[1] \to \mathbb{L}^n \mathcal{H}(j_0^* \mathcal{F}) \to j_0^* \mathbb{L} \mathcal{Z}(\mathcal{F}).
\]
The first vertex is contained in \( \mathcal{B}^1_X(-d) \), hence is killed by \( \mathbb{L} \mathcal{B}^{[1,n-1]}_X(-d) \). This proves the composition of \( \mathbb{L} \mathcal{B}^0_X \circ j_0^* \to j_0^* \circ \mathbb{L} \mathcal{Z} \) with \( \mathbb{L} \mathcal{B}^{[1,n-1]}_X(-d) \) is an isomorphism, as required. \( \square \)

7.4. The involution for \( n = 2 \). As we observed in Corollary 7.4 if \( n = 2 \) there is an involution \( \tau : \mathcal{A}_Z \to \mathcal{A}_Z \) such that \( \mathcal{A}_X \simeq \mathcal{A}_Z^{Z/2} \), where \( Z/2 \) acts by \( \tau \). Now we can give a simple formula for \( \tau \) in terms of the rotation functor for \( \mathcal{A}_Z \).

**Proposition 7.9.** If \( n = 2 \) then
\[
\tau \simeq \mathcal{O}_Z^d[-1]
\]
is an involution of \( \mathcal{A}_Z \) such that \( \mathcal{A}_X \simeq \mathcal{A}_Z^{Z/2} \), where \( Z/2 \) acts by \( \tau \).

**Remark 7.10.** The proposition is consistent with the isomorphism \( \mathcal{O}_Z^{2d} \cong [2] \) given by Theorem 7.6.

**Proof.** This follows by combining Proposition 7.8 with Theorem 7.6. Indeed, applying \( d \) times the intertwining property (7.6) we get an isomorphism
\[
(O_X^\mu)^d \circ \Phi_0 \cong \Phi_0 \circ \mathcal{O}_Z^d.
\]
Now \( (O_X^\mu)^d \cong T_\chi[1] \) by Theorem 7.6, so we have
\[
T_\chi \circ \Phi_0 \cong \Phi_0 \circ \mathcal{O}_Z^d[-1].
\]
Since \( \tau = \Phi_0^{-1} \circ T_\chi \circ \Phi_0 \) by (7.1), the result follows. \( \square \)

7.5. *Reconstruction for \( n > 2 \).* As we already mentioned, the reconstruction of \( \mathcal{A}_X \) from \( \mathcal{A}_Z \) is more involved when \( n > 2 \). First, Theorem 6.4 gives a semiorthogonal decomposition
\[
\mathcal{A}_X^\mu_n = \langle \Phi_0(\mathcal{A}_Z), \Phi_1(\mathcal{A}_Z), \ldots, \Phi_{n-2}(\mathcal{A}_Z) \rangle.
\]
In [20] Section 4 it is explained that given a semiorthogonal decomposition \( \mathcal{I} = \langle \mathcal{I}_1, \mathcal{I}_2 \rangle \), the category \( \mathcal{I} \) can be constructed as a “gluing” of the categories \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) along the “gluing functor” \( i_1^* i_2[1] : \mathcal{I}_2 \to \mathcal{I}_1 \), where \( i_p : \mathcal{I}_p \to \mathcal{I} \) are the embeddings. (Technically, we should assume \( \mathcal{I} \) has a DG enhancement and work with the DG version of the gluing functor, but we will suppress this point.) In our case, it follows that \( \mathcal{A}_X^\mu_n \) can be constructed as a gluing of \( n - 1 \) copies of \( \mathcal{A}_Z \). In fact, one can check that for each adjacent pair of components in the above decomposition, the glueing functor \( \mathcal{A}_Z \to \mathcal{A}_Z \) is given by the \( d \)-th power \( \mathcal{O}_Z^d \) of the rotation functor. The category glued from the \( n - 1 \) copies of \( \mathcal{A}_Z \) carries an action of \( \tilde{\mu}_n \) (induced by the action on \( \mathcal{A}_X^\mu_n \)), and it follows from Corollary 7.2 that \( \mathcal{A}_X \) is equivalent to the equivariant category for this action. Hence, in principle, \( \mathcal{A}_X \) can be entirely reconstructed from the category \( \mathcal{A}_Z \).

However, it is difficult to make this result explicit, because it is difficult to identify explicitly the action of \( \tilde{\mu}_n \) on the glued category. Ideally, we would have a direct description of \( \mathcal{A}_X \) in terms of \( \mathcal{A}_Z \) and the rotation functor \( \mathcal{O}_Z \) (as we have when \( n = 2 \)). In case \( n = 3 \), we propose to consider the category of distinguished triangles in \( \mathcal{A}_Z \) of the form
\[
A_0 \to \mathcal{O}_Z^d[1](A_1) \to \mathcal{O}_Z^{-2d}[2](A_2) \to \mathcal{O}_Z^{-3d}[3](A_0).
\]
Note that $\mathcal{O}_Z^{-3d} \cong [−2]$ by Theorem 7.6, so indeed $\mathcal{O}_Z^{-3d}[3](A_0) \cong A_0[1]$. Moreover, there is an action of $\mathbb{Z}/3$ on the above category of triangles, where the generator acts by sending $O_{A_1}$ to the triangle

$$A_1 \to \mathcal{O}_Z^{-d}[1](A_2) \to \mathcal{O}_Z^{-2d}[2](A_0) \to \mathcal{O}_Z^{-3d}[3](A_1)$$

obtained by applying $\mathcal{O}_Z^d[-1]$. We think a natural guess is that the category $A_X$ is equivalent to the $\mathbb{Z}/3$-equivariant category of the above category of triangles. Note that, a priori, it is not even clear the category of triangles is triangulated. In the future, we plan to return to this proposal for $n = 3$ and to consider the general case $n > 2$.

8. Applications

The main results of this paper can be applied to a cyclic cover of any variety having a rectangular Lefschetz decomposition of its derived category. Quite a number of such varieties are known — projective spaces (more generally projective bundles), Grassmannians $G(k,n)$ for $k$ and $n$ coprime [10] (and some of their linear sections), and some others (see [19] for a review). If we consider more generally Deligne-Mumford stacks (see Remark 1.2), there are other natural examples, e.g. weighted projective spaces (regarded as quotient stacks). Here we discuss only several examples of cyclic covers of varieties in the above list — quartic double solids, hyperelliptic Fano–Gushel–Mukai varieties, and cyclic cubic hypersurfaces.

8.1. Quartic double solids. Let $Y = \mathbb{P}^3$ and let $f : X \to Y$ be a quartic double solid, i.e. a double cover of $Y$ ramified over a quartic surface $Z \subset |\mathcal{O}_Y(4)|$. We have the standard semiorthogonal decomposition

$$\mathcal{D}^b(Y) = \langle \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2), \mathcal{O}_Y(3) \rangle.$$

Hence we are in the situation of Theorem 1.1 with $\mathcal{B} = \langle \mathcal{O}_Y \rangle$, $m = 4$, and $n = d = 2$. The semiorthogonal decompositions (5.2) and (5.6) for $t = 0$ are in this case

$$\mathcal{D}^b(X) = \langle A_X, \mathcal{O}_X, \mathcal{O}_X(1) \rangle \quad \text{and} \quad A_Z = \mathcal{D}^b(Z).$$

We note the Serre functor of $A_X$ satisfies $S_{A_X} \cong \sigma[2]$, where $\sigma$ is the involution generating the $\mathbb{Z}/2$-action on $A_X$ (this follows by an argument as in Remark 7.7). Applying Theorem 1.1 and Corollary 1.3, we conclude

$$A_X^{\mathbb{Z}/2} \simeq \mathcal{D}^b(Z) \quad \text{and} \quad A_X \simeq \mathcal{D}^b(Z)^{\mathbb{Z}/2}. \quad (8.1)$$

Here, by Proposition 7.9 the group $\mathbb{Z}/2$ acts on $\mathcal{D}^b(Z)$ by $\mathcal{O}_Z^2[−1]$.

This is exactly analogous to the relationship between the derived categories of an Enriques surface $S$ and its associated K3 surface $T$. Namely, there are equivalences

$$\mathcal{D}^b(S)^{\mu_2} \simeq \mathcal{D}^b(T) \quad \text{and} \quad \mathcal{D}^b(S) \simeq \mathcal{D}^b(T)^{\mathbb{Z}/2},$$

where $\mu_2$ acts on $\mathcal{D}^b(S)$ by $S_{\mathcal{D}^b(S)}[−2]$, i.e. by tensoring with $\omega_S$, and $\mathbb{Z}/2$ acts on $\mathcal{D}^b(T)$ by the covering involution. Thus (8.1) can be thought of as saying $A_X$ is a “noncommutative Enriques surface” obtained by taking the quotient of the K3 surface $Z$ by an involutive autoequivalence (which can be thought of as a “noncommutative automorphism”). This complements the results of [12], where it is shown that if $X^+$ is a small resolution of singularities of an Artin-Mumford quartic double solid, then the distinguished component $A_X^+ \subset \mathcal{D}^b(X^+)$ is equivalent to the distinguished component $A_S \subset \mathcal{D}^b(S)$ of an associated Enriques surface $S$. 

8.2. Fano–Gushel–Mukai varieties. Next we apply our results to the following class of varieties.

**Definition 8.1.** A Fano–Gushel–Mukai variety (FGM variety for short) is a smooth projective variety $X$, such that one of the following hold:

- $X$ is Fano with $\text{Pic}(X) \cong \mathbb{Z}$, $-K_X = (\dim(X) - 2)H$, and $H^{\dim X} = 10$,
  where $H$ is the ample generator of $\text{Pic}(X)$; or
- $X$ is a Brill–Noether general polarized K3 surface of degree 10.

We do not recall here the definition of a Brill–Noether general K3, as below we will focus on the Fano case. See [6] for a detailed discussion of the geometry of FGM varieties. The following theorem gives the classification of FGM varieties, originally due to Gushel [11] and Mukai [22].

Let $V$ be a 5-dimensional vector space and $G(2, V)$ the Grassmannian of 2-dimensional subspaces of $V$, embedded in $\mathbb{P}(\wedge^2 V) = \mathbb{P}^9$ via the Plücker embedding.

**Theorem 8.2** ([11, 22, 6]). Let $X$ be an FGM variety of dimension $N$. Then there is a morphism $f : X \rightarrow G(2, V)$ such that one of the following hold:

(a) We have $2 \leq N \leq 5$ and there is a linear subspace $P \cong \mathbb{P}^{N+4} \subset \mathbb{P}(\wedge^2 V)$ and a quadric hypersurface $Q \subset \mathbb{P}(\wedge^2 V)$ such that $f : X \rightarrow G(2, V)$ embeds $X$ as a smooth complete intersection $X = G(2, V) \cap P \cap Q$.

(b) We have $3 \leq N \leq 6$ and there is a linear subspace $P \cong \mathbb{P}^{N+3} \subset \mathbb{P}(\wedge^2 V)$ and a quadric hypersurface $Q \subset \mathbb{P}(\wedge^2 V)$ such that $Y = G(2, V) \cap P$ and $Z = Y \cap Q$ are smooth complete intersections, the image of $f$ is $Y$, and $f : X \rightarrow Y$ is the double cover of $Y$ ramified over $Z$.

Conversely, any variety as in (a) or (b) is an $N$-dimensional FGM variety.

Let $X$ be an FGM variety. We call the morphism $f : X \rightarrow G(2, V)$ of Theorem 8.2 the Gushel map of $X$. We say $X$ is ordinary if Theorem 8.2(a) holds, and hyperelliptic if Theorem 8.2(b) holds. If $X$ is hyperelliptic, we often regard the Gushel map as a morphism $f : X \rightarrow Y$, where $Y$ is as in Theorem 8.2(b).

From now on we assume $X$ is a hyperelliptic FGM variety of dimension $N \geq 3$. It follows from Theorem 8.2 that the target $Y$ of the Gushel map $f : X \rightarrow Y$ is an $N$-dimensional linear section of the Grassmannian $G(2, V)$, and the branch locus $Z \subset Y$ is an ordinary FGM $(N - 1)$-fold. By [15, Section 6.1], we have a semiorthogonal decomposition

$$D^b(Y) = \langle \mathcal{O}_Y, \mathcal{U}_Y^*, \ldots, \mathcal{O}_Y(N - 2), \mathcal{U}_Y^*(N - 2) \rangle,$$

where $\mathcal{U}_Y$ is the restriction to $Y$ of the tautological rank 2 bundle on $G(2, V)$. We set $\mathcal{B} = \langle \mathcal{O}_Y, \mathcal{U}_Y^* \rangle$, so that $D^b(Y)$ has a rectangular Lefschetz decomposition

$$D^b(Y) = \langle \mathcal{B}, \mathcal{B}(1), \ldots, \mathcal{B}(N - 2) \rangle,$$

of length $m = N - 1$. Since $X$ is a double cover of $Y$ ramified over a quadratic divisor, we are in the situation of Theorem [11] with $n = 2$ and $d = 1$. Thus, we have decompositions

$$D^b(X) = \langle A_X, \mathcal{B}_X, \ldots, \mathcal{B}_X(N - 3) \rangle \quad \text{and} \quad D^b(Z) = \langle A_Z, \mathcal{B}_Z, \ldots, \mathcal{B}_Z(N - 4) \rangle,$$
and equivalences

\[ A_X^{\mu_2} \simeq A_Z \quad \text{and} \quad A_X \simeq A_Z^{Z/2}. \tag{8.2} \]

Here, by Proposition 7.4, the group \( \mathbb{Z}/2 \) acts on \( A_Z \) by \( \mathcal{O}_Z[-1] \).

An interesting feature of this example is that the categories \( A_X \) and \( A_Z \) are the nontrivial components of FGM varieties of dimensions differing by one. As is discussed in [21], according to whether the dimension of a FGM variety is even or odd, its nontrivial component is a “noncommutative K3 surface” or a “noncommutative Enriques surface” (at the level of Serre functors this follows from Remark 7.7). Hence, the equivalences (8.2) can be interpreted in the same way as (8.1), except now the K3 is also “noncommutative.”

8.3. Cyclic hypersurfaces. We say a hypersurface \( X \subset \mathbb{P}^N = \mathbb{P}(V) \) of degree \( n \) is cyclic if it is invariant under the action of \( \mu_n \) induced by a representation of \( \mu_n \) on \( V \) such that

\[ V \cong (W \otimes 1) \oplus \chi, \]

where \( W \subset V \) is a subspace of codimension 1 and \( \chi \) is the generator of \( \hat{\mu}_n \). If we choose \( \mu_n \)-equivariant coordinates \( x_0, \ldots, x_N \) on \( V \) such that \( W \) is given by the equation \( x_0 = 0 \), then the equation of such a hypersurface has the form

\[ F(x_0, x_1, \ldots, x_N) = x^3_0 - G(x_1, \ldots, x_N). \]

Consequently, \( X \) is a cyclic covering of \( Y = \mathbb{P}(W) \) of degree \( n \) ramified over a hypersurface \( Z \subset \mathbb{P}(W) \) of degree \( n \) with equation \( G = 0 \). Since \( Y = \mathbb{P}(W) \) admits a rectangular Lefschetz decomposition of its derived category

\[ \mathcal{D}^b(Y) = (\mathcal{O}_Y, \mathcal{O}_Y(1), \ldots, \mathcal{O}_Y(N-1)), \]

we can apply our results with \( \mathcal{B} = (\mathcal{O}_Y), \ m = N, \ d = 1, \) and \( n \) the degree of the hypersurface. For \( n \leq N \), we obtain semiorthogonal decompositions

\[ \mathcal{D}^b(X) = \langle A_X, \mathcal{O}_X, \ldots, \mathcal{O}_X(N-n) \rangle, \]

\[ \mathcal{D}^b(Z) = \langle A_Z, \mathcal{O}_Z, \ldots, \mathcal{O}_Z(N-n-1) \rangle, \]

and a semiorthogonal decomposition of the equivariant category

\[ A_X^{\mu_n} = \langle A_Z, A_Z \otimes \chi, \ldots, A_Z \otimes \chi^{n-2} \rangle. \]

Here, we have suppressed the embedding functor (given by Theorem 6.3) of \( A_Z \) into \( A_X^{\mu_n} \).

Let us see what this gives for smooth cyclic cubic hypersurfaces of small dimension:

- If \( X \) is a cyclic cubic surface, then \( A_X = (\mathcal{O}_X)^{\perp} \subset \mathcal{D}^b(X) \). From the description of \( X \) as the blow-up of \( \mathbb{P}^2 \) in 6 points, it follows that \( A_X \) is generated by an exceptional collection of length 8. Further, \( Z \) is an elliptic curve and \( A_Z = \mathcal{D}^b(Z) \). We get a decomposition

\[ A_X^{\mu_3} = \langle \mathcal{D}^b(Z), \mathcal{D}^b(Z) \otimes \chi \rangle. \]

So, we have a category generated by an exceptional collection whose equivariant category decomposes into two copies of the derived category of an elliptic curve.

- If \( X \) is a cyclic cubic threefold, then \( A_X = (\mathcal{O}_X, \mathcal{O}_X(1))^{\perp} \subset \mathcal{D}^b(X) \). In particular, \( A_X \) is a fractional Calabi–Yau category of dimension 5/3 (see [14] Corollary 4.3) or argue as in Remark 7.7. Further, \( Z \) is a cubic surface and \( A_Z = (\mathcal{O}_Z)^{\perp} \subset \mathcal{D}^b(Z) \). In particular, \( A_Z \) is generated by an exceptional collection of length 8. We get a decomposition

\[ A_X^{\mu_3} = \langle A_Z, A_Z \otimes \chi \rangle. \]
So, we have a fractional Calabi–Yau category whose equivariant category decomposes into two copies of a fractional Calabi–Yau category generated by an exceptional collection of length 16. On the other hand, applying the reconstruction result of Corollary 7.3, we see the fractional Calabi–Yau category \( A_X \) can be obtained as the equivariant category of a category generated by an exceptional collection of length 16.

- If \( X \) is a cyclic cubic fourfold, then \( A_X = \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle \subset D^b(X) \). In particular, \( A_X \) is a K3 category (again by \([14]\) Corollary 4.3) or Remark 7.7. Further, \( Z \) is a cubic threefold and \( A_Z = \langle \mathcal{O}_Z, \mathcal{O}_Z(1) \rangle \subset D^b(Z) \) is a fractional Calabi–Yau category of dimension 5/3. We again get a decomposition

\[
A_X^{\mu_3} = \langle A_Z, A_Z \otimes \chi \rangle.
\]

So, we have a K3 category whose equivariant category decomposes into two copies of a fractional Calabi–Yau category of dimension 5/3. On the other hand, applying the reconstruction result of Corollary 7.3, we see the K3 category \( A_X \) can be obtained as the equivariant category of a category glued from two copies of the fractional Calabi–Yau category \( A_Z \).

We note that the above construction can be iterated. For instance, consider a double cyclic cubic fourfold \( X \), i.e. a cyclic cubic fourfold \( X \to \mathbb{P}^4 \) such that the branch locus \( Z \subset \mathbb{P}^4 \) is itself a cyclic cubic threefold. Concretely, in suitable coordinates \( X \) is cut out in \( \mathbb{P}^5 \) by an equation of the form

\[
F(x_0, \ldots, x_5) = x_0^2 + x_1^3 - G(x_2, x_3, x_4, x_5).
\]

The map \( X \to \mathbb{P}^4 \) is given by projection onto the \( x_1, \ldots, x_5 \) coordinates, and \( Z \subset \mathbb{P}^4 \) is defined by \( x_1^3 - G(x_2, x_3, x_4, x_5) \). The group \( \mu_3 \times \mu_3 \) acts on \( X \) by scaling the \( x_0 \) and \( x_1 \) coordinates, and the induced action on \( D^b(X) \) preserves the decomposition

\[
D^b(X) = \langle A_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle.
\]

It follows from the definitions that there is an equivalence

\[
A_X^{\mu_3 \times \mu_3} \simeq (A_X^{\mu_3})^{\mu_3}, \quad (8.3)
\]

where on the right side the inner \( \mu_3 \) acts by scaling on \( x_0 \) and the outer \( \mu_3 \) acts by scaling on \( x_1 \). By Theorem 6.4 we have a decomposition

\[
A_X^{\mu_3} = \langle \Phi_0(A_Z), \Phi_1(A_Z) \rangle. \quad (8.4)
\]

It is straightforward to see the functors \( \Phi_0, \Phi_1 : A_Z \to A_X^{\mu_3} \) are equivariant with respect to the \( \mu_3 \)-action on \( A_Z \) (induced by the cyclic cover structure of \( Z \)) and the \( \mu_3 \)-action on \( A_X^{\mu_3} \) described above. Hence, by a mild generalization of Elagin’s result Theorem 3.2 we obtain

\[
A_X^{\mu_3 \times \mu_3} \simeq (A_X^{\mu_3})^{\mu_3} = \langle \Phi_0(A_Z)^{\mu_3}, \Phi_1(A_Z)^{\mu_3} \rangle, \quad (8.5)
\]

where each component is equivalent to \( A_Z^{\mu_3} \). Combined with the description of \( A_Z^{\mu_3} \) above, we conclude \( A_X^{\mu_3 \times \mu_3} \) is generated by an exceptional collection of length 32.

Finally, we note that it is easy to see a double cyclic cubic fourfold \( X \) contains a pair of skew planes. The results of \([17]\) then apply to show \( A_X \simeq D^b(S) \) for a K3 surface \( S \). Thus the above gives a description of the equivariant derived category of a commutative K3 surface.
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