Update on Quarkonium Spectroscopy and $\alpha_{\text{strong}}$ from NRQCD

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NRQCD results for Upsilon and Charmonium using both dynamical and quenched configurations are presented. We investigate dependence on the light dynamical quark mass. Preliminary dynamical ($n_f = 2$) Charmonium data are combined with quenched results to extract the strong coupling constant $\alpha_{(n_f)}^{(P)}$ for the physical number of light dynamical quarks, $n_f = 3$. Good agreement is found with calculations based on the Upsilon system. We show that a discrepancy in $\alpha_{(n_f=0)}^{(P)}$, found between the Upsilon and Charmonium systems in the quenched theory, disappears upon extrapolating to the physical number of flavors. Results for the strong coupling constant $\alpha_{\overline{\text{MS}}}^{(P)}$ are presented and sources of systematic error investigated.

1. Introduction

The Nonrelativistic QCD (NRQCD) approach to heavy quarks on the lattice has been applied very successfully to the $\Upsilon$ and Charmonium systems in recent years. Good agreement is found between lattice simulations and observed quarkonium levels. These investigations have also enabled us to determine the $b$-quark pole- and $\overline{\text{MS}}$-masses and the strength of the strong coupling constant, $\alpha_s$.

During the past year we have continued studies of quarkonium systems with particular emphasis on refining our $\alpha_s$ determination. We have:

1. a new high statistics $\Upsilon$ study on $16^3 \times 32$ Kogut et al configurations with $n_f = 0$ at $\beta = 6.0$.
2. increased statistics for $\Upsilon$ studies on $16^3 \times 32$ HEMCGC configurations with $n_f = 2$, $am_q = 0.01$ staggered dynamical quarks at $\beta = 5.6$.
3. new $\Upsilon$ results using HEMCGC dynamical configurations with $am_q = 0.025$.
4. new (preliminary) Charmonium data using $16^3 \times 24$ dynamical MILC configurations ($n_f = 2$, $am_q = 0.0125$) at $\beta = 5.145$.

This has led to the following main results:

- an $\Upsilon$ spectroscopy update
- a look at the $m_q$-dependence of $\alpha_{(n_f=2)}^{(P)}$
- extrapolation of $\alpha_{(P)}^{(n_f=0)}$, $\alpha_{(P)}^{(2)} \rightarrow \alpha_{(P)}^{(n_f=3)}$ from both $\Upsilon$ and Charmonium.
- a $5 \sim 6$ sigma discrepancy at $n_f = 0$ is removed through unquenching and extrapolation to the physical $n_f = 3$.
- establishment of a reliable calculation for $\alpha_{\overline{\text{MS}}}^{(P)}$ at the $Z^0$ mass.

2. $\Upsilon$ in the Quenched Approximation

The NRQCD action, details of how heavy quark propagators are evaluated, the smearing functions and the fitting procedures used are described in detail in several publications. We will review them briefly.
With $v^2 \sim 0.1$, the b quarks in the $\Upsilon$ system are quite nonrelativistic. The splittings between spin-averaged levels are around 500GeV [$\mathcal{O}(M_b v^2)$], which is much smaller than the $\Upsilon$ mass [$\mathcal{O}(2M_b)$], indicating that a systemic expansion of the QCD Hamiltonian in powers of $v^2$ is appropriate. The continuum action density, correct through $\mathcal{O}(M_b v^4)$, is broken down according to

$$L_{\text{cont}} = \bar{\psi}(D_t + H_{0\text{cont}}^\dagger)\psi + \psi^\dagger \delta H_{\text{cont}}^\dagger \psi \quad (1)$$

$H_{0\text{cont}}^\dagger$ and $\delta H_{\text{cont}}^\dagger$ are given explicitly in ref.\[4\], we give here their lattice counterparts $H_0$ and $\delta H$:

$$H_0 = -\frac{\Delta_0^2}{2M_b^2} \quad \text{and}$$

$$\delta H = -c_1 \left(\frac{\Delta_0^2}{2M_b^2}\right)^2 + c_2 \frac{ig}{8(M_b^0)^2}(\Delta \cdot E - E \cdot \Delta) +$$

$$- c_3 \frac{g}{8(M_b^0)^2} \sigma \cdot (\Delta \times E - E \times \Delta) +$$

$$- c_4 \frac{2}{2M_b^0} \sigma \cdot B + c_5 \frac{a^2 \Delta_0^4}{24M_b^0} - c_6 \frac{a(\Delta_0^2)^2}{16n(M_b^0)^2}$$

The last two terms in $\delta H$ come from finite lattice spacing corrections to the lattice Laplacian and lattice time derivative, respectively.

The quark propagators are determined from evolution equations that specify the propagator value, for $t > 0$, in terms of the value on the previous timeslice:

$$G_1 = \left(1 - \frac{aH_0}{2n}\right)^n U_4^\dagger \left(1 - \frac{aH_0}{2n}\right)^n \delta_{x,0},$$

$$G_{i+1} = \left(1 - \frac{aH_0}{2n}\right)^n U_4^\dagger \left(1 - \frac{aH_0}{2n}\right)^n (1 - a\delta H)G_i$$

The quark propagators are combined with the smearing operator $\Gamma$ to produce a meson propagator

$$G_{\pi\pi}(p, t) = \sum_{y_1, y_2} \text{Tr} \left[ G_1^\dagger(y_2) \Gamma^{(sk)}(y_1 - y_2) G_1(y_1) \right] \times e^{-i\frac{p}{2} \cdot (y_1 + y_2)}$$

with

$$\tilde{G}_i(y) = \sum_x G_i(y - x) \Gamma^{(sc)}(x)e^{\frac{E_x}{M_b}t}$$

where the trace is over spin and color. $\Gamma^{(sc)}$, the smearing at the source is distinguished from that at the sink, $\Gamma^{(sk)}$.

Two main fitting procedures were used to extract energies and amplitudes. For the first procedure, a matrix of correlations with $n_{sc}, n_{sk} = 1, 2, 3$ for $S$ states and $n_{sc}, n_{sk} = 1, 2$ for spin singlet $P$ states was fit simultaneously, each correlation being fit to an ansatz of the form

$$G_{\pi\pi}(n_{sc}, n_{sk}; t) = \sum_{k=1}^{N_{\exp}} a(n_{sc}, k) a^*(n_{sk}, k) e^{-E_k t}$$

The second procedure involved fitting a row of smeared-local correlations, $n_{sc} = 1, 2, 3$ or $n_{sc} = 1, 2$ with $n_{sk} = \text{loc}$, simultaneously, each correlation being fit to an ansatz of the form

$$G_{\pi\pi}(n_{sc}, \text{loc}; t) = \sum_{k=1}^{N_{\exp}} b(n_{sc}, k) e^{-E_k t}$$

A comparison of the dimensionless $1S-1\overline{3}P$ and $1S-2S \Upsilon$ splittings, once $\mathcal{O}(a^2)$ gluonic corrections are made (using perturbation theory $a\delta E_g \sim 0.0036$ for the $1S$ and $\sim 0.0023$ for the $2S$ level), with experimental values for these splittings yields the following values for $a^{-1}$

$$1S - 1P: \quad a^{-1} = 2.59(6) \text{ GeV}$$

$$1S - 2S: \quad a^{-1} = 2.34(5) \text{ GeV}$$

These results represent an increased statistical accuracy of a factor of 2 over our previous results\[4\]. Taking an average of $a^{-1} = 2.4$ GeV, the updated NRQCD $\Upsilon$ spectrum is shown in figures 1 & 2 (see section 3). The agreement between figures 1 & 2 and their previous counterparts\[4\] is good - a significant shift $(2\sigma)$ is only observed for the $3S$ state. This may be an indication that previous statistical errors, which included an element of fitting uncertainty, were overestimates.
3. \( \Upsilon \) with Dynamical Quarks

We now have twice the statistics on \( n_f = 2 \) configurations as compared to a year ago [7] and have accumulated 6,400 meson propagators per channel and per choice of smearing at source and sink. As in the past we find it crucial to use multi-exponential fits to several correlations simultaneously, in order to extract ground state and one or two excited state energies for the S-states and the spin averaged P-states. We refer to [4,5] for details and mention here just that with higher statistics we are now much more sensitive to systematics. To give one example, the statistical errors of the dimensionless 1S-1P splitting are of order \( \sim 0.005 \) which is comparable to shifts expected from \( O(a^2) \) corrections to the gluonic action. The latter have been calculated perturbatively to be \( a\delta E_g \sim 0.0057 \) for the 1S and \( \sim 0.0034 \) for the 2S level. With several groups now working with \( a^2 \) improved gluonic actions one would eventually like to check \( a\delta E_g \) nonperturbatively. These corrections are crucial when we check for scaling by comparing \( \Upsilon \) simulation results at different \( \beta \) values.

By comparing the dimensionless 1S-1P and 1S-2S \( \Upsilon \) splittings with experiment, one determines \( a^{-1} \). We find

\[
\begin{align*}
1S - 1P: \quad & a^{-1} = 2.44(7) \text{ GeV} \\
1S - 2S: \quad & a^{-1} = 2.37(10) \text{ GeV}
\end{align*}
\]

Using an average \( a^{-1}=2.4 \text{GeV} \) we show our updated results for the NRQCD \( \Upsilon \) spectrum in figures 1 & 2. Upon focusing on the 1P and 2S levels in figure 1, one notices better agreement with experiment in the dynamical theory than in the quenched theory. This reflects the fact that \( a^{-1} \)'s from the 1S-1P and 1S-2S splittings disagree with each other (at the \( \sim 4 \)-sigma level) in the quenched theory. As discussed in reference [8] the effect of this discrepancy in \( \alpha_P \) disappears when one extrapolates to the \( n_f = 3 \) theory. This remains true also of our new \( n_f = 0 \) and \( n_f = 2 \) data. A similar phenomenon of different quantities giving different \( a^{-1} \)'s in the quenched theory, whose effects then disappear for the physical number of dynamical flavors, will be discussed and explained in section 5 where we compare \( \Upsilon \) and Charmonium results.

Figure 2 shows spin splittings. One sees that there are some differences between experiment and simulations in the \( \chi_b \) splittings. This is particularly noticeable in the quenched case and one is tempted to try to extrapolate in \( n_f \). Before carrying out that exercise, we need to reexamine our fine structure fits. The fitting procedure used for the P-state fine structure differs from what was done for the S-states and the \( 1P_1 \) level to get the 1S-1P and 1S-2S splittings. For the latter levels we used multi-exponential multi-correlation fits as mentioned above. In order to extract the small fine-structure splittings (these are of order 10 MeV = size of the statistical errors in the \( 1P_1 \) level) we looked at ratios between \( 3P_J \) and \( 1P_1 \) correlations. These were fit to a single exponential. In past work it was not possible to find a signal for excited state contributions to these ratios. With our current higher statistics data it
may be possible and necessary to take such contributions into account. Further work is required before one can sort out quenching versus relativistic corrections to the fine structure splittings.

The last topic to discuss in this section on dynamical Υ data, concerns dependence on the mass of the dynamical quarks. The data discussed above used HEMCGC dynamical configurations with \( n_f = 2 \) staggered fermions of dimensionless mass \( am_q = 0.01 \). With inverse lattice spacing \( a^{-1} \sim 2.4 \text{GeV} \), this corresponds to light quarks considerably heavier than the up- or down- quarks. This should not pose a problem as long as \( m_q << \) typical momenta in heavy-heavy systems of about \( 0.5 \sim 1 \text{GeV} \) (with the higher number applicable for Υ). Hence heavy quark physics with dynamical light quarks should be much less sensitive to extrapolations in the light quark mass than light quark physics. We have made the first attempt to test this hypothesis by repeating our Υ runs with \( am_q = 0.025 \) dynamical HEMCGC configurations. The inverse lattice spacings are found to be

\[
am_q = 0.025
\]

\begin{align*}
1S - 1P: & \quad a^{-1} = 2.28(15) \text{ GeV} \\
1S - 2S: & \quad a^{-1} = 2.17(12) \text{ GeV}
\end{align*}

Comparing with the \( a^{-1} \)'s from the \( am_q = 0.01 \) configurations, one finds a 1-sigma difference for the S-P and a 2-sigma change for the 1S-2S \( a^{-1} \). It is not clear with present statistics, whether we are observing a true systematic effect, although the shift downwards in the central values would be in the right direction for an \( am_q \) that has become too large. At given \( \beta \) increasing the dynamical \( am_q \) will reduce the difference in \( \beta \) between that and a quenched simulation of the same \( a^{-1} \).

An accurate estimate of \( a^{-1} \) is the crucial input for lattice determinations of the strong coupling constant \( [3,9] \). So what one is really interested in here is the \( m_q \) dependence of \( \alpha_s \) extracted from lattice quarkonium studies. We use an \( \alpha_P \) defined through the plaquette value,

\[
-lnW_{1,1} = \frac{4\pi}{3} \alpha_P \left( \frac{3.41}{a} \right) [1 - (1.185 + 0.070n_f)\alpha_P]
\]

with \( a^{-1} \) from quarkonium splittings setting the scale. Using the above \( a^{-1} \)'s and evolving the couplings perturbatively to a common reference scale of 8.2 GeV, one finds

**S-P**

\[
\alpha_P^{(n_f=2)}[8.2 \text{GeV}] = 0.1793(16) \quad am_q = 0.01 \\
\alpha_P^{(n_f=2)}[8.2 \text{GeV}] = 0.1760(35) \quad am_q = 0.025
\]

**1S-2S**

\[
\alpha_P^{(n_f=2)}[8.2 \text{GeV}] = 0.1777(23) \quad am_q = 0.01 \\
\alpha_P^{(n_f=2)}[8.2 \text{GeV}] = 0.1735(28) \quad am_q = 0.025
\]

One could try extrapolating to \( m_q \rightarrow 0 \). Perturbation theory tells us to extrapolate quadratically in \( m_q \). One then finds results very close to the \( am_q = 0.01 \) values (0.1799(20) and 0.1785(28) respectively from S-P and 1S-2S) More studies of the \( m_q \)-dependence in dynamical simulations of quarkonium systems are called for, with higher statistics and several \( m_q \) values. At the moment our calculations indicate that the effects of extrapolating from \( am_q = 0.01 \) down to realistic light quark masses are very small and less than our statistical errors. Hence, we will continue to use the \( a^{-1} \) and \( \alpha_P \) values from the \( am_q = 0.01 \) simulations for which we have the best statistics.

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**Figure 2. Υ SPIN SPLITTINGS**

Symbols have the same meanings as in Figure 1.
4. $\alpha_s$ Determination and Investigation of Systematic Errors

One of the simplest quantities to calculate in lattice QCD, and to yield a value for the strong coupling constant, is the expectation value of the $1 \times 1$ Wilson loop operator. The coupling constant, $\alpha_P$, is defined through the plaquette value by the perturbative relation in section 3. With this definition, $\alpha_P$ coincides through $\mathcal{O}((\alpha_P)^2)$ with the coupling $\alpha_V$, defined in refs. [10][11].

Using the $a^{-1}$ values calculated in sections 2 & 3 ($am_q = 0.01$ results from section 3) to set the scale and evolving the couplings perturbatively using the 2-loop formula to the common reference scale of 8.2 GeV, one obtains

$$\begin{align*}
1S - 1P & \\
\alpha_p(n_f=0) & |8.2\text{GeV}| = 0.1551(11) \\
\alpha_p(n_f=2) & |8.2\text{GeV}| = 0.1793(16)
\end{align*}$$

$$\begin{align*}
1S - 2S & \\
\alpha_p(n_f=0) & |8.2\text{GeV}| = 0.1505(9) \\
\alpha_p(n_f=2) & |8.2\text{GeV}| = 0.1777(23)
\end{align*}$$

To obtain physical results, one must extrapolate the couplings to $n_f = 3$. The inverse of the coupling $1/\alpha_p(n_f)$ is known to be almost linear for small changes in $n_f$, hence extrapolating the inverse couplings one obtains

$$\begin{align*}
\alpha_p(n_f=3) & |8.2\text{GeV}| = 0.1945(30) \quad 1S-1P \\
\alpha_p(n_f=3) & |8.2\text{GeV}| = 0.1953(43) \quad 1S-2S
\end{align*}$$

The concordance of these values is more readily appreciated from figure 5.

To enable comparison with other determinations we convert to the $\overline{\text{MS}}$ scheme through the relation

$$\begin{align*}
\alpha_{\overline{\text{MS}}}^{(n_f)}(Q) &= \alpha_p^{(n_f)}(e^{5/6}Q) \\
& \times \left[ 1 + \frac{2}{\pi} \alpha_p^{(n_f)} + \mathcal{O}((\alpha_p^{(n_f)})^2) \right] (2)
\end{align*}$$

The factor of $e^{5/6}$ absorbs the $n_f$ dependence in (at least) the $\alpha^2$ term. Substituting for the $\alpha_p^{(n_f=3)}$ values shown above, evolving perturbatively to a scale of 1.3 GeV, matching from 3 to 4 flavors of dynamical quarks, evolving once more to 4.3 GeV, matching from 4 to 5 flavors of dynamical quarks and finally evolving to the $M_{Z^0} = 91.2$ GeV scale, we obtain

$$\begin{align*}
\alpha_{\overline{\text{MS}}}^{(1S-2S)}(M_{Z^0}) &= 0.1152(24) \quad 1S-1P \\
\alpha_{\overline{\text{MS}}}^{(1S-2S)}(M_{Z^0}) &= 0.1154(26) \quad 1S-2S
\end{align*}$$

The error in these numbers is dominated by the $\alpha^3$ terms in equation 2. The central value of course has zero for this coefficient; one standard deviation allows this coefficient to have magnitude 1. Ref. [13] has shown that for an $n_f = 0$ theory, the coefficient of the $(\alpha_p^{(n_f)})^3$ term in equation 3 is 0.96, giving a central value for $\alpha_{\overline{\text{MS}}}^{(1S-2S)}$ of 0.117. The remaining uncertainty would then be the $n_f$ dependence of the $\alpha^3$ coefficient, and that may be considerably less than the previous uncertainty.

Although this coefficient is the main source of error, we have also tested the robustness of our result to other factors. In one such test, the $\mathcal{O}(a^2)$ gluonic corrections to the action were omitted, yielding the results

$$\begin{align*}
\alpha_{\overline{\text{MS}}}^{(1S-2S)}(M_{Z^0}) &= 0.1145(24) \quad 1S-1P \\
\alpha_{\overline{\text{MS}}}^{(1S-2S)}(M_{Z^0}) &= 0.1154(26) \quad 1S-2S
\end{align*}$$

The 1S-2S value is identical to that obtained previously with the corrections in place since the corrections to the 1S and 2S levels partially cancel each other. The 1S-1P value is slightly lower because there are no P-state $\mathcal{O}(a^2)$ gluonic corrections. So, as expected, although the spectrum results are accurate enough to distinguish the presence of this correction our final value for $\alpha$ is not.

The effect of tadpole-improvement was tested by using quenched results from [4] with $u_0 = 1$. We have no dynamical results without tadpole-improvement so we modified our existing dynamical results to simulate this situation. We adjusted them by the difference of the tadpole-improved and tadpole-unimproved quenched energies, since the effect of unquenching on this difference is a second order effect. The subsequent $a^{-1}$ values from the 1S-2S splitting gave rise to the result

$$\alpha_{\overline{\text{MS}}}^{(1S-2S)}(M_{Z^0}) = 0.1149(27) \quad 1S-2S$$
This result is very close to that obtained with tadpole-improvement since this has little effect on spin-independent splittings (it is crucial for spin splittings).

The effect of the correction terms, $\delta H$, in the action of equation was tested using quenched S-state energies from [4] with $\delta H = 0$ and the dynamical S-state energies used previously but with a modification equivalent to that described above for the omission of tadpole-improvement. This time we obtained the result

$$\alpha_{\overline{MS}}(M_{Z^0}) = 0.1165(27) \quad \text{1S-2S}$$

Although the value for $\alpha_{\overline{MS}}$ is larger than that for non-zero $\delta H$, the increase is masked by the uncertainty incurred when we converted from $\alpha_P$ to $\alpha_{\overline{MS}}$. It is clear that higher order relativistic corrections that we do not include would be completely invisible.

The final factors we investigated were the thresholds at which we matched from 3 to 4 and from 4 to 5 flavors of dynamical quarks [12]. Figure 3 shows values of $\alpha_{\overline{MS}}$ obtained from the 1S-1P splitting where the lower matching threshold (3 to 4 flavors) was varied around the 1.3GeV value while the upper matching threshold (4 to 5 flavors) was fixed at 4.3GeV. Figure 4 is similar; this time the upper matching threshold was varied around 4.3GeV with the lower fixed at 1.3GeV. For simplicity we used the values from ref. [12] of 1.3 GeV and 4.3 GeV for the quark masses, $m_q(m_q)$. As can be seen from the figures there are comfortable ranges of at least 1.5GeV where there is essentially no effect on our final result, in accord with the findings of ref. [12].

5. Charmonium with Dynamical Quarks and a New Estimate of $\alpha_s$

The $a^{-1}$ that feeds into lattice determinations of $\alpha_s$ can, in principle, be obtained from a variety of physical quantities. The reason level splittings in the $\Upsilon$ system are optimal, is that statistical errors can be reduced rather efficiently in these systems, and because systematic errors are also under good control. Another similar sys-
tem that can be used for $\alpha_s$ determinations is the Charmonium system \[9\]. Our collaboration has studied Charmonium in the quenched approximation using NRQCD heavy fermions on $\beta = 5.7$ UKQCD configurations. These results are covered by Christine Davies at this conference \[5\]. We now also have preliminary dynamical Charmonium data on $n_f = 2$ MILC \[8\] configurations at $\beta = 5.145$. Using the Charmonium spin averaged S-P splitting to determine $a^{-1}$, one ends up with a value for $\alpha^{(n_f=2)}_P$ of,

$$\alpha^{(n_f=2)}_P[8.2\text{GeV}] = 0.1758(36) \text{ Charm}$$

which should be compared with the $\Upsilon$ value of 0.1793(16) from the previous section.

$\alpha^{(n_f)}_P$ need not agree between Charmonium and $\Upsilon$ for $n_f = 0$ and $n_f = 2$. However there should be agreement for the physical number of dynamical flavors, $n_f = 3$. In fact based on the quenched results in section 4 and those discussed by C.Davies \[5\] one finds for $n_f = 0$

$$\alpha^{(n_f=0)}_P[8.2\text{GeV}] = 0.1480(13) \text{ Charm}$$
$$\alpha^{(n_f=0)}_P[8.2\text{GeV}] = 0.1551(11) \text{ } \Upsilon$$

We believe the 5 $\sim$ 6 sigma discrepancy between Charmonium and $\Upsilon$ is largely a quenching effect combined with the fact that the two quarkonium systems have different characteristic energy scales $q^*_\Upsilon$ and $q^*_C$ ($q^*_\Upsilon > q^*_C$). In a quenched theory $\alpha(q)$ will run incorrectly between the two $q^*$'s. As a consequence the $\Upsilon$ S-P splitting will be underestimated relative to the same splitting in Charmonium. Upon comparing with real experimental data the $\Upsilon$ simulation results will lead to a larger $a^{-1}$ than in Charmonium and hence also to a larger $\alpha_P[8.2\text{GeV}]$.

The $n_f = 2$ $\alpha^{(n_f=2)}_P$'s agree better between Charmonium and $\Upsilon$, although one does not expect complete agreement until one has extrapolated to the physical number of flavors $n_f = 3$. We extrapolate in $1/\alpha_P$ and obtain,

$$\alpha^{(n_f=3)}_P[8.2\text{GeV}] = 0.1940(67) \text{ Charm}$$
$$\alpha^{(n_f=3)}_P[8.2\text{GeV}] = 0.1945(30) \text{ } \Upsilon$$

So, extrapolation to the correct number of flavors has removed the discrepancy altogether. The $n_f$ dependence is summarized in figure 5. The errors on the Charmonium results are still large, but we are working towards improving them. The $\alpha^{(n_f=3)}_P$ from Charmonium and the updated value from $\Upsilon$ agree well with our previous published value \[6\].

In figure 5 we also show $\alpha^{(n_f)}_P$ from $\Upsilon$ 1S-1P splittings (the open circles). Again a discrepancy at $n_f = 0$ disappears upon going to the $n_f = 3$ theory. Arguing why, in the same quarkonium system, one expects a larger $\alpha^{(n_f=0)}_P$ from a 1S-1P determination than from the 1S-2S splitting, is somewhat subtle. It no longer suffices to talk about just one typical characteristic $q^*_\Upsilon$ as we did when comparing $\Upsilon$ with Charmonium. One must look more carefully into the details of individual binding energies for the S- and P-states and the
\( \alpha(q) \) values associated with them. As discussed in reference [7], in a theory with incorrect running of \( \alpha \) (a running that is too rapid) the 1S-1P splitting will again be underestimated relative to the 1S-2S splitting, leading to the phenomenon observed in figure 5. The important point here is that we have now observed subtle quenching effects in several ways and that in each case such effects are removed by including the appropriate number of dynamical light quarks.

6. Summary

We have carried out several tests/updates of lattice determinations of \( \alpha_s \) using quenched and dynamical Quarkonium simulations. These include investigations of the \( m_q \)-dependence, determination of \( \alpha_P^{(n_f=3)} \) from systems and/or splittings with different characteristic scales and investigation into the sources of systematic error of the \( \alpha_{\overline{MS}}(M_{Z^0}) \) calculation. These tests uphold previous determinations of \( \alpha^{(3)}_P \) and \( \alpha_{\overline{MS}} \) and give us further confidence in the reliability of those results. Our final value for \( \alpha_{\overline{MS}}(M_{Z^0}) \) remains 0.115(2) if the \( \alpha^3 \) coefficient in the matching to \( \overline{MS} \) is set to zero, and 0.117 if this coefficient takes its \( n_f = 0 \) value of 0.96.

This work is supported in part by the U.S. DOE, the NSF and by the UK PPARC. The numerical computations were carried out at NERSC, the Ohio Supercomputer Center and the Atlas Centre. We thank the HEMCGC and MILC collaborations for providing the dynamical configurations and UKQCD, Kogut et al and Kilcup et al for providing the quenched configurations for these studies.

REFERENCES

1. G. P. Lepage and B. A. Thacker, in Field Theory on the Lattice, Proceedings of the International Symposium, Seillac, France, 1987, edited by A. Billoire et al. [Nucl. Phys. B (Proc. Suppl.) 4 199, (1988)]; B. A. Thacker and G. P. Lepage, Phys. Rev. D 43, 196 (1991).
2. G. P. Lepage, L. Magnea, C. Nakhleh, U. Magnea and K. Hornbostel, Phys. Rev. D 46, 4052 (1992).
3. C.T.H.Davies and B.A.Thacker, Nucl.Phys. B405 (1993) 593.
4. C.T.H.Davies, K.Hornbostel, A.Langnau, G.P.Lepage, A.Lidsey, J.Shigemitsu and J.Sloan, Phys. Rev. D 50, 6963 (1994).
5. C.T.H.Davies, K.Hornbostel, G.P.Lepage, A.Lidsey, J.Shigemitsu and J.Sloan; “Precision Charmonium Spectroscopy from Lattice QCD”, submitted to Phys. Rev. D. [hep-lat 9506026]
6. C.T.H.Davies, K.Hornbostel, G.P.Lepage, A.Lidsey, J.Shigemitsu and J.Sloan; Phys. Rev. Lett. 73 (1994) 2654.
7. C.T.H.Davies, K.Hornbostel, G.P.Lepage, A.Lidsey, J.Shigemitsu and J.Sloan, Phys. Lett. B 345, 42 (1995).
8. The authors wish to thank HEMCG, MILC, UKQCD, Kogut et al and Kilcup et al collaborations for their generous provision of configurations.
9. A.El-Khadra, G.Hockney, A.Kronfeld and P. B.Mackenzie; Phys. Rev. Lett. 69 (1992) 729; M.Wingate, T. DeGrand, S.Collins and U. Heller, Phys. Rev. D52 (1995) 307; S. Aoki et al, Phys. Rev. Lett. 74 (1995) 22.
10. S.J.Brodsky, G.P.Lepage and P.B.Mackenzie; Phys. Rev. D28 (1983) 228.
11. G.P.Lepage and P.B.Mackenzie, Phys. Rev. D 48, 2250 (1993).
12. G.Rodrigo and A.Santamaria, Phys. Lett. B 313, 441 (1993).
13. M.Lüscher and P.Weisz; Phys. Lett. B349 (1995) 165.