Five-Loop Critical Temperature Shift in Weakly Interacting Homogeneous Bose-Einstein Condensate

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Using variational perturbation theory, we calculate the shift in the critical temperature \( T_c \) up to five loops to lowest order in the scattering length \( a \) and find \( \Delta T_c / T_c^{(0)} \approx (1.14 \pm 0.11) \, a n^{1/3} \), where \( n \) is the particle density. Our result is lower than the latest Monte Carlo result \((1.32 \pm 0.02) \, a n^{1/3}\).

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The effect of a small repulsive interaction upon the critical temperature \( T_c \) of a Bose-Einstein condensate (BEC) has been a matter of controversy for many years, and the various results are converging only very slowly towards a common answer. Here we want to contribute to the ongoing discussion with a further result obtained from a resummation of Feynman diagrams up to five loops.

The interacting Bose-Einstein condensate is described by the euclidean action

\[
A_E = \int_0^\beta d\tau \int d^3x \left\{ \psi^\dagger(x, \tau) \left( \frac{1}{2M} \nabla^2 - \mu \right) \psi(x, \tau) - \frac{2\pi \alpha}{M} [\psi^\dagger(x, \tau) \psi(x, \tau)]^2 \right\},
\]

where \( M \) is the mass of the bosons, \( \beta \) the inverse temperature in natural units with \( \hbar = k_B = 1 \), \( a \) is the \( s \)-wave scattering length, and \( \mu \) the chemical potential. The free system has a transition temperature

\[
T_c^{(0)} = \frac{2\pi}{M} \left( \frac{n}{\zeta(3/2)} \right)^{1/3},
\]

where \( n \) is the particle density. A small relative shift of \( T_c \) with respect to \( T_c^{(0)} \) can be calculated from the general formula

\[
\frac{\Delta T_c}{T_c^{(0)}} = -\frac{2}{3} \frac{\Delta n}{n^{(0)}},
\]

where \( n^{(0)} \) is the particle density in the free condensate and \( \Delta n \) its change at \( T_c \) caused by the small interaction. For small \( a \), this behaves like \( 1/2 \)

\[
\frac{\Delta T_c}{T_c^{(0)}} = c_1 a n^{1/3} + [c_2 \ln(a n^{1/3}) + c_2] a^2 n^{2/3} + O(a^3 n).
\]

where \( c_2 = -64 \pi \zeta(1/2)/3 \zeta(3/2)^{5/3} \approx 19.7518 \) can be calculated perturbatively, whereas \( c_1 \) and \( c_2 \) require nonperturbative techniques since infrared divergences at \( T_c \) make them basically strong-coupling results. The standard technique to reach this regime is based on a resummation of perturbation expansions using the renormalization group \[3, 4\], first applied in this context by Ref. \[6\]. Recently, however, it has been shown by calculating the best known critical exponent \( \alpha \) of superfluid helium from Satellite experiments \[7\] that the accuracy of strong-coupling results can be surpassed by much simpler variational perturbation theory \[4, 5, 6\].

Up to now, \( c_2 \) has been inferred only from Monte Carlo data to be \( c_2 \approx 75.7 \pm 0.4 \). In order to find the leading coefficient \( c_1 \), one may take advantage of an important simplification due to the fact that \( \Delta n \) can be calculated from the classical limit of the field theory, which is governed by the three-dimensional action

\[
A_{3d} = \beta \int d^3x \left\{ \psi^\dagger(x) \left( -\frac{1}{2M} \nabla^2 - \mu \right) \psi(x) + \frac{2\pi \alpha}{M} [\psi^\dagger(x) \psi(x)]^2 \right\},
\]

This is a special case \( N = 2 \) of the more general \( O(N) \)-invariant \( \phi^4 \) field theory

\[
A_\phi = \int d^3x \left[ \frac{1}{2} \nabla \phi \nabla \phi + \frac{1}{2} m^2 \phi^2 + \frac{u}{4!} (\phi^2)^2 \right],
\]
where the $N$-component field $\phi = (\phi_1, \phi_2, \ldots, \phi_N)$ is related to the original field $\psi$ for $N = 2$ by $\psi(x) = \sqrt{MT}[\phi_1(x) + i\phi_2(x)]$. The square mass is $m^2 = -2M\mu$, and the quartic coupling is $u = 48\pi aMT$. Using this relation, the shift of the critical temperature \([3]\) can be found from the formula

$$\frac{\Delta T_c}{T_c^{(0)}} \approx -\frac{2}{3} \frac{MT_c^{(0)}}{m} \langle \Delta \phi^2 \rangle = -\frac{4\pi}{3} \frac{(MT_c^{(0)})^2}{n} \frac{4!}{u} \left( \frac{\Delta \phi^2}{u} \right) a = -\frac{4\pi}{3} \frac{(2\pi)^2}{\zeta(3/2)} \frac{1}{4!} \frac{4!}{4!} \left( \frac{\Delta \phi^2}{u} \right) a n^{1/3},$$

(7)

corresponding in Eq. \([3]\) to

$$c_1 \approx -1103.09 \left( \frac{\Delta \phi^2}{u} \right).$$

(8)

The three-dimensional theory is superrenormalizable and requires only mass counterterms which shift the original bare mass $m$ to the renormalized mass $m_r$. A calculation of the Feynman diagrams in Fig. 1 yields the following five-loop perturbation expansion for the expectation value $\langle \phi^2/u \rangle$ \([10, 11]\)

$$\left\langle \frac{\phi^2}{u} \right\rangle = F(u) \equiv -\frac{N m_r}{4\pi u} - a_2 \frac{N}{2} \frac{(2 + N)}{18 (4\pi)^3} \frac{u}{m_r} + a_3 \frac{N}{108 (4\pi)^5} \left( \frac{u}{m_r} \right)^2$$

$$- a_{41} \frac{N}{324 (4\pi)^7} \frac{(2 + N)^2}{40 + 32 N + 8 N^2 + N^3} + a_{42} \left( \frac{N}{648 (4\pi)^7} \right) \frac{u^4}{m_r} + a_{43} \left( \frac{N}{324 (4\pi)^7} \right) \frac{(2 + N)^2}{44 + 32 N + 5 N^2} + a_{44} \left( \frac{N}{324 (4\pi)^7} \right) \frac{u^4}{m_r} + a_{45} \left( \frac{N}{324 (4\pi)^7} \right) \frac{(2 + N)^2}{44 + 32 N + 5 N^2} + \ldots,$$

(9)

where $a_2 \equiv \log(4/3)/2 \approx 0.143841$ and the other constants are only known numerically \([12]\):

$$a_3 = 0.642144, \quad a_{41} = -0.115069, \quad a_{42} = 3.128107, \quad a_{43} = 1.63, \quad a_{44} = -0.624638, \quad a_{45} = 2.39.$$

(10)

Writing the above expansion up to the $L$th term as $F_L(u) = \sum_{i=1}^{L} f_i(u/4\pi m_r)^i$, the expansion coefficients for the relevant number of components $N = 2$ are \([12]\):

$$f_{-1} = -126.651 \times 10^{-4}, \quad f_0 = 0, \quad f_1 = -4.04837 \times 10^{-4}, \quad f_2 = 2.39701 \times 10^{-4}, \quad f_3 = -1.80 \times 10^{-4}.$$

(11)

We need the value of the series $F_L(u)$ in the critical limit $m_r \to 0$, which is obviously equivalent to the strong-coupling limit of $F_L(u)$. As mentioned above, this limit should be most accurately found with the help of variational perturbation theory \([4, 8, 9]\). We form the sequence of truncated expansions $F_L(u)$ for $1, 2, 3$ and replace each term

$$(u/m_r)^\ell \to K^L[1 - 1]_{L-\ell}^{-\ell/2}$$

(12)

where the symbol $[1 - 1]^r_k$ is defined as the binomial expansion of $(1 - 1)^r$ truncated after the $k$th term

$$[1 - 1]^r_k \equiv \sum_{i=0}^{k} \binom{r}{i} (-1)^i = (-1)^k \binom{r - 1}{k}.$$

(13)

FIG. 1: Diagrams contributing to the expectation value $\langle \phi^2 \rangle$. 

The approximants $W_{1,2,3}^{QM}$ have extrema $W_{1,2,3}^{QM} \approx -0.00277, +0.00405, -0.0029$, corresponding, via (5), to $c_1 \approx 3.059, -4.46, 3.01$. These values have previously been obtained in Ref. [10] in a much more complicated way via a so-called $\delta$-expansion. Note the negative sign of the second approximation arising from the fact that an extremum exists only at negative $K$. According to our rules of variational perturbation theory one should, in this case, use the saddle point at positive $K$ which would yield $W_{2}^{QM} = 0.00153$ corresponding to $c_1 \approx 1.69$ rather than $-4.46$, leading to the more reasonable approximation sequence $c_1 \approx 3.059, 1.69, 3.01$, which shows no sign of convergence. In $W_{3}^{QM}$, there is also a pair of complex extrema from which the authors of Ref. [10] extract the real part $\text{Re} W_{3}^{QM,\text{complex}} \approx -0.00134$ corresponding to $c_1 \approx 1.48$, which they state as their final result. There is, however, no acceptable theoretical justification for such a choice [14].

This lack of convergence is not astonishing since naive quantum mechanical variational perturbation theory is inapplicable to field theory, contrary to ubiquitous statements in the literature [15]. A simple but essential modification is necessary to allow for the well-known fact that there are anomalous dimensions in the critical regime of fluctuating fields. This modification was discovered in Ref. [8] and tested by the fact it reproduces in $D = 4 - \epsilon$ dimensions exactly the known $\epsilon$ expansions of renormalization group theory [16]. In $D = 3$ dimensions, it leads to the most accurate critical exponents so far (see in particular Chapters 19 and 20 in the textbook [3]).

The correct procedure goes as follows: We form the logarithmic derivative of the expansion (11):

$$
\beta (u) \equiv \frac{\partial \log F(u)}{\partial \log u} = -1 + 2 \frac{f_1}{u \nu} + \left( \frac{u}{m_{\nu}} \right)^2 + 3 \frac{f_2}{u \nu} + \left( \frac{u}{m_{\nu}} \right)^3 + \left( \frac{u}{m_{\nu}} \right)^4 + \ldots .
$$

(14)

In order for $F(u)$ to go to a constant in the critical limit $m_{\nu} \to 0$, this function must go to zero in the strong-coupling limit $u \to \infty$. Writing the expansion as $\beta_L (u) = -1 + \sum_{l=2}^{L} b_l (u/4 \pi m_{\nu})^l$, the coefficients are

$$b_2 = 0.0639293, \quad b_3 = -0.056778, \quad b_4 = 0.0548799.
$$

(15)

The sums $\beta_L (u)$ have to be evaluated for $u \to \infty$ allowing for the universal anomalous dimension $\omega$ by which the physical observables of $\phi^4$-theories approach the scaling limit [3, 4]. The approach to the critical point $A + B (m_{\nu}/u)^{\omega'}$ where $\omega' = \omega/(1 - \eta/2)$ [14]. The exponent $\eta$ is the small anomalous dimension of the field while $\omega$ is related to the famous Wegner exponent [4] of renormalization group theory $\Delta \equiv \omega \nu$. Here it appears in the variational expression for the strong-coupling limit which is found [8, 4] by replacing $(u/m_{\nu})^l$ by $K^{-1/(l-1)} q^{l-2}/L_{-l}^{q-2}$, where $q \equiv 2/\omega'$. Thus we obtain the variational expressions

$$
W_{3}^{\beta} = -1 + \left( \frac{2 f_1}{f_{-1}} + \frac{f_1 q}{f_{-1}} + \frac{3 f_2}{f_{-1}} \right) K^3
$$

(16)

$$
W_{4}^{\beta} = -1 + \left( \frac{2 f_1}{f_{-1}} + \frac{3 f_2}{f_{-1}} + \frac{3 f_2}{f_{-1}} + \frac{f_1 q}{f_{-1}} + \frac{9 f_2 q}{2 f_{-1}} \right) K^3 + \frac{1}{2} \left( \frac{3 f_2}{f_{-1}} + \frac{9 f_2 q}{2 f_{-1}} \right) K^3 + \frac{1}{2} \left( \frac{3 f_2}{f_{-1}} + \frac{9 f_2 q}{2 f_{-1}} \right) K^3 + \frac{1}{2} \left( \frac{3 f_2}{f_{-1}} + \frac{9 f_2 q}{2 f_{-1}} + \frac{4 f_3}{f_{-1}} \right) K^4
$$

(17)

The first has a vanishing extremum at $\omega'_3 = 0.592$, the second has neither an extremum nor a saddle point. However, a complex pair of extrema lies reasonably close to the real axis at $\omega'_4 = 0.635 \pm 0.116$, whose real part is not far from the true exponent of approach $\omega'_\infty = 0.81$ [15, 4], to which $L_{-\infty}$ will converge for order $L \to \infty$ [8]. Given these $\omega'$-values, we now form the variational expressions $W_L$ from $F_L$ by the replacement $(u/m_{\nu})^l \to K^{-l} q_{-l}^{q-2}/L_{-l}^{q-2}$, which are

$$W_2 = f_{-1} \left( 1 - \frac{3}{4} q + \frac{1}{8} q^2 \right) K^{-1} + f_1 K,
$$

(18)

$$W_3 = f_{-1} \left( 1 - \frac{11}{13} q + \frac{1}{4} q^2 - \frac{1}{48} q^3 \right) K^{-1} + f_1 \left( 1 + \frac{q}{2} \right) K + f_2 K^2,
$$

(19)

$$W_4 = f_{-1} \left( 1 - \frac{25}{24} q + \frac{35}{96} q^2 - \frac{5}{96} q^3 + \frac{1}{384} q^4 \right) K^{-1} + f_1 \left( 1 + \frac{3}{4} q + \frac{1}{8} q^2 \right) K + f_2 (1 + q) K^2 + f_3 K^3.
$$

(20)
The lowest function \( W_2 \) is optimized with the naive growth parameter \( q = 1 \) since to this order no anomalous value can be determined from the zero of the beta function \([13]\). The optimal result is \( W_{2,\text{opt}} = -\sqrt{\log(4/3)/6}/8\pi^2 \approx -0.00277 \) corresponding to \( c_1 \approx 3.06 \). The next function \( W_3 \) is optimized with the above determined \( q_3 = 2/\omega_3 \) and yields \( W_{3,\text{opt}} \approx -0.000976 \) corresponding to \( c_1 \approx 1.078 \). Although \( \omega_3 \) is not real we shall insert its real part into \( W_4 \) and find \( W_{4,\text{opt}} \approx -0.000957 \) corresponding to \( c_1 \approx 1.057 \). The three values of \( c_1 \) for \( \bar{L} \equiv L - 1 = 1, 2, 3 \) can well be fitted by a function \( c_1 \approx 1.053 + 2/\bar{L}^6 \) (see Fig. 2). Such a fit is suggested by the general large-\( L \) behavior \( a + be^{-cL^{1-\omega'}} \) which was derived in Refs. \([13]\). Due to the smallness of \( 1 - \omega' \approx 0.2 \), this can be replaced by \( \approx a' + b'/\bar{L}^* \).

Alternatively, we may optimize the functions \( W_{1,2,3} \) using the known precise value of \( q_{\infty} = 2/\omega'_{\infty} \approx 2/0.81 \). Then \( W_2 \) turns out to have no optimum, whereas the others yield \( W_{3,\text{opt}} \approx -0.000554, -0.000735 \), corresponding via Eq. \([8]\) to \( c_1 \approx 0.580, 0.773 \). If these two values are fitted by the same inverse power of \( \bar{L} \), we find \( c_1 \approx 0.83 - 14/\bar{L}^6 \). From the extrapolations to infinite order we estimate \( c_{1,\infty} \approx 0.92 \pm 0.13 \).

![Figure 2](image.png)

**FIG. 2**: The three approximants for \( c_1 \) plotted against the order of variational approximation \( \bar{L} \equiv L - 1 = 1, 2, 3 \), and extrapolation to the infinite-order limit.

This result is to be compared with latest Monte Carlo data which estimate \( c_1 \approx 1.32 \pm 0.02 \)[17] [18]. Previous theoretical estimates are \( c_1 \approx 2.90 \)[15], 2.33 from a \( 1/N \)-expansion \([21]\), 1.71 from a next-to-leading order in a \( 1/N \)-expansion \([21]\), 3.059 from an inapplicable \( \delta \)-expansion \([22]\) to three loops, and 1.48 from the same \( \delta \)-expansion to five loops, with a questionable evaluation at a complex extremum \([10]\) and some wrong expansion coefficients (see \([12]\)). Remarkably, our result lies close to the average between the latest and the first Monte Carlo result \( c_1 \approx 0.34 \pm 0.03 \) in Ref. \([23]\).

As a cross check of the reliability of our theory consider the result in the limit \( N \to \infty \). Here we must drop the first term in the expansion \([21]\) which vanishes at the critical point (but would diverge for \( N \to \infty \) at finite \( m_r \)). The remaining expansion coefficients of \( \langle \phi^2/u \rangle / N \) in powers of \( Nu/4\pi m_r \) are

\[
f_1 = -6.35917 \times 10^{-4}, \quad f_2 = 4.7315 \times 10^{-4}, \quad f_3 = -3.84146 \times 10^{-4}.
\]

Using the \( N \to \infty \) limit of \( \omega' \) which is equal to 1 implying \( q = 2 \) in Eqs. \([19]\) and \([20]\), we obtain the two variational approximations

\[
W_{2,\infty} = -0.00127183K + 0.00047315K^2, \quad W_{3,\infty} = -0.00190775K + 0.00141945K^2 - 0.000384146K^3,
\]

whose optima yield the approximations \( c_1 \approx 1.886 \) and 2.017, converging rapidly towards the exact large-\( N \) result 2.33 of Ref. \([20]\), with a 10% error.

Numerically, the first two \( 1/N \)-corrections found from a fit to large-\( N \) results obtained by using the known large-\( N \) expression for \( \omega' = 1 - 8(8/3\pi^2N) + 2(104/3 - 9\pi^2/2)(8/3\pi^2N)^2 \)[24] produce a finite-\( N \) correction factor \( (1 - 3.1/N + 30.3/N^2 + \ldots) \), to be compared with \( (1 - 0.527/N + \ldots) \) obtained in Ref. \([21]\).

Since the large-\( N \) results can only be obtained so well without the use of the first term we repeat the evaluations of the series at the physical value \( N = 2 \) without the first term, where the variational expressions for \( f \) are

\[
W_2 = f_1 \left( 1 + \frac{q}{2} \right) K + f_2 K^2,
\]

\[
W_3 = f_1 \left( 1 + \frac{3}{4} q + \frac{1}{8} q^2 \right) K + f_2 (1 + q) K^2 + f_3 K^3.
\]

The lowest order optimum lies now at \( W_{2,\text{opt}} = -f_1^2(2+q^2)/16 f_2^2 \), yielding \( c_1 \equiv 0.942 \) for the exact \( q = 2/0.81 \). To next order, an optimal turning point of \( W_3 \) yields \( c_1 \approx 1.038 \).
At this order, we can derive a variational expression for the determination of $\omega'$ using the analog of Eq. (14), which reads

$$\beta(u) \equiv \frac{\partial \log F(u)}{\partial \log u} = 1 + \frac{f_2}{f_1} \frac{u}{m_r} + \left( \frac{2f_3}{f_1} - \frac{f_2^2}{f_1^2} \right) \left( \frac{u}{m_r} \right)^2 + \ldots.$$  \hfill (24)

After the replacement (12) we find

$$W_3^\beta = 1 + \frac{f_2(1 + q/2)}{f_1} K + \left( \frac{2f_3}{f_1} - \frac{f_2^2}{f_1^2} \right) K^2 + \ldots$$  \hfill (25)

whose vanishing extremum determines $\omega' = 2/q$ as being

$$\omega'_3 = \left( \frac{2\sqrt{2f_1f_3/f_2^2 - 1} - 1}{2} \right)^{-1} \approx 0.675,$$  \hfill (26)

leading to $c_1 \approx 1.238$ from an optimal turning point of $W_3$. There are now too few points to perform an extrapolation to infinite order. From the average of the two highest-order results we obtain our final estimate: $c_1 \approx 1.14 \pm 0.11$, such that the critical temperature shift is

$$\frac{\Delta T_c}{T_c^{(0)}} \approx (1.14 \pm 0.11) a n^{1/3}.$$  \hfill (27)

This lies reasonably close to the Monte Carlo number $c_1 \approx 1.32 \pm 0.02$.

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Note added in proof:
Since this paper appeared on the Los Alamos server last October, the Feynman diagrams where recalculated more accurately and extended to six loops in Ref. [25]. The more accurate $f_3$ has increased by about 6% to

$-1.92 \times 10^{-4}$, and the new expansion coefficient is $f_4 = 1.873 \times 10^{-4}$. Using the new numbers to perform a variational evaluation fits of the first type including the anomalous first term $f_{-1}$ we find the curves shown in Fig. 2 yielding $c_1 \approx 0.98 \pm 0.06$, only slightly larger than than the previous five-loop number $0.92 \pm 0.13$. The upper curve is now associated with the self-consistent $\omega'$-values $\omega'_3 = 0.626$ and $\omega'_4 = 0.745$, the last being almost equal to the exact value 0.81.

The evaluation of the second type where the anomalous term is omitted yields now $c_1 = 0.942, 1.056, 1.120$, for $\omega' = 0.81$, and $c_1 = 1.409, 1.384$ for the variationally determined $\omega'_3 = 0.621$ and 0.638, as shown in in Ref. [25]. Combining the last two results one obtains the larger value $c_1 \approx 1.23 \pm 0.12$, lying closer to the Monte Carlo number $c_1 \approx 1.32 \pm 0.02$ than our five-loop result [24].

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