Discrepancies between decoherence and the Loschmidt echo

B. Casabone\textsuperscript{1}, I. García-Mata\textsuperscript{2}(a) and D. A. Wisniacki\textsuperscript{1}

\textsuperscript{1} Departamento de Física and IFIBA, FCEyN, UBA - Pabellón I Ciudad Universitaria, C1428EGA Buenos Aires, Argentina
\textsuperscript{2} Departamento de Física, Lab. TANDAR, Comisión Nacional de Energía Atómica - Av. del Libertador 8250, 1429 Buenos Aires, Argentina

received 6 January 2010; accepted in final form 24 February 2010
published online 26 March 2010

PACS 03.65.Yz – Decoherence; open systems; quantum statistical methods
PACS 03.67.-a – Quantum information
PACS 05.45.Mt – Quantum chaos; semiclassical methods

Abstract – The Loschmidt echo and the purity are two quantities that can provide invaluable information about the evolution of a quantum system. While the Loschmidt echo characterizes instability and sensitivity to perturbations, purity measures the loss of coherence produced by an environment coupled to the system. For classically chaotic systems both quantities display a number of —supposedly universal— regimes that can lead to think of them as equivalent quantities. We study the decay of the Loschmidt echo and the purity for systems with finite-dimensional Hilbert space and present numerical evidence of some fundamental differences between them.

Some of the latest breakthroughs in theoretical and experimental quantum physics have permitted among other things to explore and manipulate new states of matter – like Bose-Einstein condensates – and also manipulate small numbers of atoms or ions making it possible to test some of the assertions of relatively new areas of research like quantum information. Two of the main problems that affect the achievement of such advances are uncontrolled coupling to an environment and irreversibility caused by sensitivity to small perturbations of the quantum evolution.

The presence of a coupling to an external bath introduces decoherence [1]. By definition decoherence washes out interference terms due to quantum superposition. One way of characterizing the decrease of the interference terms caused by decoherence is by measuring the purity of the system as a function of time. For classically chaotic systems it was conjectured [2] and numerically shown [2–4] that for a certain range of values, the exponential decay of purity is independent of the coupling strength and is characterized by the Lyapunov exponent of the classical counterpart. Complementarily, to characterize irreversibility and instability arising from the chaotic nature of systems the Loschmidt echo (LE) —also known as fidelity— has been used [5–9]. The idea is to study the overlap as a function of time of two states evolving with slightly different evolution operators characterized by some perturbation parameter $\Sigma$. To avoid the singularity of a particular initial condition the LE is computed by averaging over many initial states. The averaging can be treated in analogy to the effects of decoherence and it was claimed [10] —or expected [11]— that at least for classically chaotic systems, since they exhibit the same decay rate regimes, they provide essentially the same information. Both the purity and the LE are of capital importance in experiments because they are readily measurable. In quantum information fidelity helps determine the accuracy of quantum gates and state preparation. Purity on the other hand characterizes how much the system is coupled to an environment or how much two parties of a system are entangled with each other.

In the present contribution the aforementioned quantities are explored for systems with finite-dimensional Hilbert space, and which have a classically chaotic counterpart. We present numerical evidence that, contrary to previous results [4,10], significant differences can exist between the —supposedly universal— behaviors of the LE and the purity. We will focus on the asymptotic regime as a function of perturbation (and decoherence) strength, both in the small perturbation regime as well as for larger perturbations. Differences in the short-time regime were already studied elsewhere (see, e.g., [12]). While for

\[(a)\text{E-mail: garciama@tandar.cnea.gov.ar}\]
small perturbations the LE shows the expected quadratic regime, we show that the decay rate of the purity, as a function of the coupling strength, depends —strongly— on the type of environment affecting the system. Furthermore, in the strong-perturbation regime, the LE presents an oscillatory behavior that can mask the Lyapunov decay [13–16]. A measurement of the fidelity decay in these regimes can thus give results that are far from the expected ones. Besides, the Lyapunov decay for the purity is only observed for special types of environment.

The systems we consider are quantum maps on the torus. Quantum maps provide a useful tool to understand universal properties of quantum chaotic systems —e.g. universal spectral statistics. In addition, there exist efficient quantum algorithms for some quantum maps that can be implemented with a small number of qubits [17,18]. This makes them ideal testbeds for current quantum computers. The quantized torus has associated an N-dimensional Hilbert space with Planck constant \( \hbar = 1/2\pi N \), and the position basis \( \{ q_i \}_{i=0}^{N-1} \) and momentum basis \( \{ p_i \}_{i=0}^{N-1} \) are related by the discrete Fourier transform (DFT). We consider quantum maps \( U \) whose classical counterpart are chaotic. For simplicity, we use maps whose evolution operator for one iteration can be written as follows:

\[
U = e^{i2\pi NT(p)} e^{-i2\pi NV(q)}. \tag{1}
\]

These kind of maps can be efficiently computed using the DFT (through the fast Fourier transform) and some have been implemented experimentally —e.g. [19]— and have efficient quantum algorithms —e.g. [17,18]. The corresponding classical map is

\[
p' = p - \frac{dV(q)}{dq}, \quad q' = q - \frac{dT(p')}{dp'} \pmod{1}. \tag{2}
\]

In particular for numerical calculations we use the cat map perturbed with a non-linear shear:

\[
p' = p + aq + 2\pi k(\cos[2\pi q] - \cos[4\pi q]) \pmod{1},
\]

\[
q' = q + b p',
\]

with \( a, b \) integers. This map is chaotic with largest Lyapunov exponent \( \lambda \approx \ln((2 + ab + \sqrt{ab(4 + ab)})/2)/2 \), for \( k \ll 1 \). The perturbation destroys the symmetries related to the arithmetic nature of cat maps —which are responsible for non-generic spectral statistics.

Let us first consider the study of the LE. For pure states the LE is defined as

\[
M(t) = |\langle \psi | U_t^* U_k^{| \psi } \rangle|^2. \tag{4}
\]

We define the perturbation strength as

\[
[50009-p2]
\]

In what follows \( t \) is an integer that represents the number of times that the map \( U \) is applied. Equation (4) is dubbed “echo” because it measures the overlap between a state evolved forwards up to time \( t \) with \( U \) and then backwards with the slightly perturbed operator \( U_{\Sigma} \). It can also be seen as a measure of the separation of two, initially identical states, evolved forwards with two slightly different evolution operators. If the classical dynamics is chaotic, there are three well-identified regimes for the LE as a function of time: parabolic or Gaussian for very short times; exponential for intermediate times followed by a saturation depending on the effective Hilbert space size. Here we focus on the decay rate \( \Gamma_{LE} \) as a function of \( \Sigma \) for the exponential decay regime. The decay rate can be extracted from the smooth curves (see fig. 1) obtained after performing an average over 1024 uniformly distributed initial coherent states, chosen randomly.

Figure 1 (left) shows the decay of the LE as a function of discrete time \( t \) for various values of \( \Sigma \). We see that after a few steps the decay is exponential. As expected for small perturbation strength (e.g. \( \Sigma/\hbar = 0.65884, 1.31768, 1.6471 \)) the decay rate is much smaller than \( \lambda \), but contrary to predictions [6,7] as we increase \( \Sigma \) (e.g. \( \Sigma/\hbar = 2.63536, 2.96478 \)) decay rates can reach values much larger than \( \lambda \). The Lyapunov decay appears for greater values of the perturbation (e.g. \( \Sigma/\hbar = 6.5884 \)). This complex behavior of the decay rate of the LE as a function of the perturbation is shown in detail in fig. 1 (right) where we plot the decay rate \( \Gamma_{LE} \) as a function of the rescaled strength of the perturbation \( \Sigma/\hbar \). We tested results for two different versions of the map of eq. (3) with different Lyapunov exponent (\( (\bigcirc) \ a = b = 2, \lambda = \ln(3 + 2\sqrt{2}) \); (red dots) \( a = b = 4, \lambda = \ln(9 + 4\sqrt{5}) \)). We can see that for small perturbation strength the behavior is, as expected, \( \Gamma_{LE} \propto \Sigma^2 \) —usually called Fermi-golden-rule (FGR) regime. For larger perturbation strengths, the decay rate is not as commonly predicted in the literature (see [8,9] and references therein) —with some exceptions, e.g. [13–16]— perturbation-independent behavior. We find oscillations behavior near the value \( \lambda \). These oscillations can be understood through the local density of states (LDOS). For finite-dimensional Hilbert space the LDOS grows quadratically with the perturbation up to a point where it starts to oscillate. If the mean value of the oscillatory part is comparable or smaller than the classical Lyapunov exponent, then the oscillatory behavior is reflected in the echo. If, on the contrary, the Lyapunov exponent is much smaller than the mean value of the oscillations of the LDOS, then no oscillations are appreciated in the LE [16]. The important thing to remark is that, after the FGR behavior, the decay of the LE is not perturbation independent. This can explain the difficulty to find the Lyapunov regime in echo experiments [15].

We now consider the evolution of our system in the presence of an environment. We explore the behavior of the purity for different types of environments. Interaction between system and environment produces global state
which is non-separable, i.e. entangled. Once we trace out the environment degrees of freedom the reduced density matrix obtained evolves non-unitaly with a consequence loss of coherence. One way to measure the effect of the decoherence produced by the environment is through the purity (see e.g. [20]) as a function of time

\[ P(t) = \text{tr}(\rho_t^2), \] (6)

were \( \rho_t \) is the reduced density matrix of the system. The purity is basis independent and measures the relative weight of the non-diagonal matrix elements. It can be used to measure how entangled are the two systems coupled together. If \( P(t) = 1 \), it means that the global system can be factorized into two separate systems and there is no entanglement. On the contrary for maximally entangled states the reduced density matrix has minimum purity and the state is maximally mixed. In the case of a \( N \)-dimensional system \( P(t) = 1/N \) for a maximally mixed state. As a function of time, after an initial short transient, the purity decays exponentially. Like the LE for long times it saturates to a minimum value given by \( \hbar \). We focus on the exponential decay and the dependence of the decay rate on the coupling parameter.

Instead of studying the evolution of system plus environment and then tracing the environment out, we model directly the effect of the environment as a map of density matrices, or superoperator which, for Markovian environment and weak coupling, can be written in Kraus operator sum form [21]. The decoherence models we use can be expressed as a weighed sum of unitary operations:

\[ \rho' \overset{\text{def}}{=} D_\varepsilon(\rho) = \sum_{p,q=0}^{N-1} c_{\varepsilon}(q,p) T_{qp} \rho T_{qp}^\dagger, \] (7)

where \( T_{qp} \) are the translation operators on the torus, \( c_{\varepsilon}(q,p) \) is a function of \( q \) and \( p \) and \( \varepsilon \) characterizes the strength. The Kraus form implies complete positivity and the trace is preserved if \( \sum_{q,p} c_{\varepsilon}(q,p) = 1 \). Furthermore, as \( T_{qp} \) are unitary, the identity is preserved, i.e. the map \( D_\varepsilon \) is unital. Although position and momentum operators are not well defined in finite-dimensional Hilbert space, translations can be defined as cyclic shifts [22]. In ref. [23] it is shown that a variety of noise superoperators can be implemented in the form of eq. (7). The interpretation is simple: with probability \( c_{\varepsilon}(q,p) \) every possible translation in phase space is applied to \( \rho \) (incoherently). The decoherent effect of \( D_\varepsilon \) is evident: suppose we have a Schrödinger-cat state that exhibits interference fringes in the Wigner function. Equation (7) written for the Wigner function of \( \rho \) results to be

\[ W'(Q,P) = \sum_{q,p} c_{\varepsilon}(q,p) W(Q - q, P - p). \] (8)

Then this incoherent sum of slightly displaced Wigner functions washes out fast oscillating terms leaving only the classical part.

The complete map with decoherence takes place then in two steps, the unitary followed by the non-unitary part \( \rho' = D_\varepsilon(U \rho U) \). This is an approximation that works
We want to take the square of the trace, so the first step is to calculate the trace of the density matrix $\rho = \rho_0 T_{t_1} \rho_0^\dagger T_{t_1}^\dagger$.

In continuous Hilbert space and in the presence of GDM type decoherence, the decay rate of the purity exhibits two different regimes as a function of the coupling parameter $\varepsilon$. For small values of $\varepsilon$ it is equivalent to that of the LE as a function of $\Sigma$ [4,10], i.e., the decay rate $\Gamma_\varepsilon$ depends quadratically for small $\varepsilon$. Then, after a critical value it becomes independent of the environment and results $\Gamma_\varepsilon = \lambda$ [2,4]. For large enough $\varepsilon$ values, the behavior of $\Gamma_\varepsilon$ in the case of quantum maps is the same. After a critical value the decay rate saturates to a constant value given by $\lambda$. However, as we show in fig. 2 (right), for $\varepsilon$ small the dependence of $\Gamma_\varepsilon$ is nowhere near quadratic. For the GDM (fig. 2, right) if $\varepsilon$ is very small, of order $1/N$, then the probability of applying any translation is negligibly small. Thus for $\varepsilon \lesssim 1/N$ there is no decoherence and the purity remains constant and equal to unity. For larger decoherence strengths, the purity decays exponentially but the dependence of $\Gamma_\varepsilon$ is not quadratic. We remark that all the calculations done for the purity do not need any kind of averaging. Figure 2 was obtained using a single Gaussian initial state.

We can derive an approximate analytic expression for the small $\varepsilon$ regime. If we assume $\partial_t \rho = \Delta_t \rho = \text{tr}(\rho_{n+1}) - \text{tr}(\rho_n) = -\Gamma_\varepsilon \text{tr}(\rho_n)$, then from eqs. (7) and (9), if $\varepsilon \ll 1$, we have

$$\rho' \approx c_\varepsilon(0,0)\rho + c_\varepsilon(0,1)T_{0,1}\rho T_{0,1}^\dagger + c_\varepsilon(1,0)T_{1,0}\rho T_{1,0}^\dagger$$

$$+ c_\varepsilon(-1,0)T_{-1,0}\rho T_{-1,0}^\dagger + c_\varepsilon(0,-1)T_{0,-1}\rho T_{0,-1}^\dagger. \quad (10)$$

We want to take the square of the trace, so the first approximation we take is $\text{tr}(\rho T_{i,j}\rho T_{i,j}^\dagger)_{i,j=0,1} = \text{tr}(\rho^2)$, and we neglect higher-order terms as well as higher-order translations (even $T_{1,-1}$). We also take into account the fact $c_\varepsilon(q,p)$ is symmetric around $(q = 0, p = 0)$. Thus we have $\text{tr}(\rho^2) = (c_\varepsilon(0,0)^2 + 4c_\varepsilon(0,1)^2 - 1)\text{tr}(\rho^2)$. Now, neglecting also higher-order terms in the normalization (remember that $\sum_{q,p} c_\varepsilon(q,p) = 1$), we get

$$\Delta \text{tr}(\rho^2) \approx -4 \frac{\exp(-2\pi^2/(\varepsilon N)^2) + 4 \exp(-2\pi^2/(\varepsilon N)^2)^2}{(1 + 4 \exp(-2\pi^2/(\varepsilon N)^2) + \ldots)^2} \text{tr}(\rho^2). \quad (11)$$

For small $\varepsilon$ we can of course neglect the terms coming from periodic boundary conditions. This expression reproduces very well the results obtained numerically (see fig. 2, right, solid line) in the $\varepsilon \approx O(1/N)$ region.
In order to attain the quadratic dependence of $\Gamma_\varepsilon$ for small coupling, observed in continuous Hilbert space, $c_\varepsilon(q,p)$ should have tails that decay slower than Gaussian, i.e. long-distance correlations in phase-space. We can for example take a well-known decoherence channel for quantum information processing, the depolarizing channel (DC) [26], which is also a convex sum of unitaries and can be simply written in terms of translations in phase space [23]:

$$D_{\varepsilon}^{DC} = (1 - \varepsilon)\rho + \frac{\varepsilon}{N^2} \sum_{q,p \neq 0} T_{qp}\rho T_{qp}^\dagger.$$  \hspace{1cm} (12)

In fig. 3 (left) we show the decay rate $\Gamma_\varepsilon$ for the DC. Following a similar reasoning as the one followed to obtain eq. (11) we get, for $\varepsilon \ll 1$, $\Gamma_\varepsilon = 2\varepsilon$ (see fig. 3, left, solid line). The DC is an extreme case to consider as phase space decoherence because it is highly non-local: with the same probability it implements every possible translation $T_{qp}$ ($q,p \neq 0$). Therefore, there is no reason to expect a Lyapunov regime in this case. In fact for $\varepsilon$ close to 1, the dynamics is dominated by the environment. The absence of Lyapunov regime (or any $\varepsilon$ independence) is clearly appreciated in fig. 3 (left). The non-locality of DC has also devastating effects on the entangling power of the algorithms that implement chaotic maps [27].

To reproduce the FGR quadratic regime we thus need a decoherence model which is peaked at $c_\varepsilon(0,0)$ and which has polynomially decaying tails. We propose to take a Lorentzian

$$c_\varepsilon(q,p) = \frac{1}{\pi A} \sum_{j,k=-x}^x \frac{\varepsilon N}{(\frac{q-j}{2N})^2 + (q-Nj)^2 + (p-Nk)^2}$$ \hspace{1cm} (13)

with $A$ the proper normalization for $\sum_{q,p} c_\varepsilon(q,p) = 1$. We will call this case Lorentz decoherence model (LDM). The sum is done to account for the periodicity of the torus (theoretically $x \to \infty$, practically $x$ is an integer much larger than 1). Equation (7) with $c_\varepsilon(q,p)$ given by eq. (13) defines a random process with Lorentzian weight. We can relate this to superdiffusion by Lévy flights. Long tail decoherence was also considered in ref. [28] where it was shown that the decoherence rates can be tuned to power law decay in cold atom experiments. In fig. 3 (right), we show $\Gamma_\varepsilon$ for the LDM. The quadratic dependence is clearly observed. As in the DC model the Lyapunov regime is not present. Larger $\varepsilon$ implies longer Lorentzian tails which, when periodized sum up to non-negligible non-local effects all over phase space. This is why for the LDM not only is the Lyapunov regime also not present but the decay rate of the purity continues to grow indefinitely. To obtain the so-called universal behavior — quadratic-FGR growth followed by constant-Lyapunov— a very specific model with large tails but sufficiently localized is needed. A combination of both GDM and LDM, so that the former dominates at larger $\varepsilon$ and the latter dominates for smaller $\varepsilon$ would yield both the FGR regime and the Lyapunov regime. Decoherence combining both Gaussian and Lorentzian processes was studied e.g. in [29].

To summarize, the LE and the purity for systems with finite-dimensional Hilbert space has been analyzed. We have shown that though they can exhibit qualitative similarities, they are fundamentally very different: the small-coupling regime for the purity is not quadratic but depends on the environment model. Moreover, while the large perturbation regime for the LE can present high amplitude oscillations around the classical Lyapunov exponent depending on the LDOS, for the purity it depends decidedly on the type of environment. Only
environments that act *locally* in phase space exhibit the—*independent*—Lyapunov regime. Thus, we remark that the LE and the purity provide intrinsically different information.

***

The authors acknowledge financial support from CONICET (PIP-6137), UBACyT (X237) and ANPCyT. DAW and IG-M are researchers of CONICET. Discussions with M. Saraceno are thankfully acknowledged.

REFERENCES

[1] Zurek W. H., *Rev. Mod. Phys.*, **75** (2003) 715.
[2] Zurek W. H. and Paz J. P., *Phys. Rev. Lett.*, **72** (1994) 2508.
[3] Monteoliva D. and Paz J. P., *Phys. Rev. Lett.*, **85** (2000) 3373.
[4] Petitjean C. and Jacquod P., *Phys. Rev. Lett.*, **97** (2006) 194103.
[5] Peres A., *Phys. Rev. A*, **30** (1984) 1610.
[6] Jalabert R. A. and Pastawski H. M., *Phys. Rev. Lett.*, **86** (2001) 2490.
[7] Jacquod Ph, Silvestrov P. G. and Beenakker C. W. J., *Phys. Rev. E*, **64** (2001) 055203.
[8] Gorin T., Prosen T., Seligman T. and Žnidarič M., *Phys. Rep.*, **435** (2006) 33.
[9] Jacquod P. and Petitjean C., *Adv. Phys.*, **58** (2009) 67.
[10] Cucchietti F., Dalvit D., Paz J. and Zurek W., *Phys. Rev. Lett.*, **91** (2003) 210403.
[11] Chaudhury S., Smith A., Anderson B. E., Ghose S. and Jessen P. S., *Nature*, **461** (2009) 768.
[12] Jacquod P., *Phys. Rev. Lett.*, **92** (2004) 150403.
[13] Wang W., Casati G. and Li B., *Phys. Rev. E*, **69** (2004) 025201.
[14] Wang W., Casati G., Li B. and Prosen T., *Phys. Rev. E*, **71** (2005) 037202.
[15] Andersen M., Kaplan A., Grünzweig T. and Davidson N., *Phys. Rev. Lett.*, **97** (2006) 104102.
[16] Ares N. and Wisniacki D. A., *Phys. Rev. E*, **80** (2009) 046216.
[17] Georgeot B. and Shepelyansky D. L., *Phys. Rev. Lett.*, **86** (2001) 5393.
[18] Lévi B. and Georgeot B., *Phys. Rev. E*, **70** (2004) 056218.
[19] Moore F. L., Robinson J. C., Bharucha C. F., Sundaram B. and Raizen M. G., *Phys. Rev. Lett.*, **75** (1995) 4598.
[20] Wisniacki D. A. and Toscano F., *Phys. Rev. E*, **79** (2009) 025203.
[21] Kraus K., *States Effects and Operations* (Springer-Verlag, Berlin) 1983.
[22] Schrödinger J., *Proc. Natl. Acad. Sci. U.S.A.*, **46** (1960) 570.
[23] AOLITA M. L., GARCIA-MATA I. and SARACENO M., *Phys Rev A*, **70** (2004) 062301.
[24] Strunz W. T. and Percival I. C., *J. Phys. A: Mathematical and General*, **31** (1998) 1801.
[25] Carvalho A. R. R., de Matos Filho R. L. and Davydovich L., *Phys. Rev. E*, **70** (2004) 026211.
[26] Nielsen M. A. and Chuang I. L., *Quantum Computation and Quantum Information* (Cambridge University Press) 2000.
[27] Garcia-Mata I., Carvalho A. R. R., Mintert F. and Buchleitner A., *Phys. Rev. Lett.*, **98** (2007) 120504.
[28] Schomerus H. and Lutz E., *Phys. Rev. Lett.*, **98** (2007) 260401.
[29] Vaccini B., *Phys. Rev. Lett.*, **95** (2005) 230402.