Numerical schemes to reconstruct three-dimensional time-dependent point sources of acoustic waves

Bo Chen¹, Yukun Guo², Fuming Ma³ and Yao Sun¹, ⁴

¹ College of Science, Civil Aviation University of China, Tianjin, People’s Republic of China
² School of Mathematics, Harbin Institute of Technology, Harbin, People’s Republic of China
³ Institute of Mathematics, Jilin University, Changchun, People’s Republic of China

E-mail: syhf2008@gmail.com

Received 12 November 2019, revised 24 April 2020
Accepted for publication 1 May 2020
Published 19 June 2020

Abstract
This paper addresses the numerical simulation of three-dimensional time-dependent inverse source problems of acoustic waves. The reconstructions of both multiple stationary point sources and a moving point source are considered. The modified method of fundamental solutions (MMFS), which expands the solution utilizing the time convolution of Green’s function and the signal function, is proposed to solve the problem. Moreover, for the reconstruction of a moving point source, the MMFS is simplified as a simple sampling method at each time step. Numerical experiments are conducted to show the effectiveness of the proposed methods.

Keywords: inverse source problem, time-dependent, wave equation, modified method of fundamental solutions, sampling method

(Some figures may appear in colour only in the online journal)

1. Introduction

The inverse problems for partial differential equations appear in various fields of science and engineering and have been extensively studied in the past decades [17, 21, 23]. Among them, the inverse source problem, especially the identification of moving sources, has a wide range of applications, such as underwater sonar [26, 27], sound simulation and sound source localization [18, 28].

⁴Address for correspondence: College of Science, Civil Aviation University of China, 2898 Jinbei Road, Dongli District, Tianjin 300300, China.
For the reconstruction of stationary sources, the inverse source problems with sources $\delta(t)g(x)$ that are delta-like in time and of limited oscillation in space and sources $q(t)\delta(\partial G)$ that are oscillation in time and delta-like on the boundary of a region are considered in \cite{12,33}, respectively. A uniqueness analysis related to the Helmholtz equation with phaseless data is presented in \cite{37,38}. The stability analysis and identification of multiple point sources for the time-harmonic case are considered in \cite{2,3}. The conditional stability estimate of the wave equation on a line related to the inverse source problem is provided by \cite{10,11}. Multi-frequency inverse source problems are analysed in \cite{6,7,25,36}. An analysis of random sources can be seen in \cite{4,24}. Time-dependent inverse source problems in elastodynamics are analysed in \cite{5}.

For the reconstruction of a moving point source, direct identifications of the moving point source are studied in \cite{29,34}. An analysis of the moving point source when the velocity of the source is comparable to the speed of wave propagation can be seen in \cite{14}. The matched-filter imaging method and correlation-based imaging for small fast-moving debris with constant velocity are analysed in \cite{13}. A gesture-based input technique with an electromagnetic wave is analysed in \cite{20}.

The method of fundamental solutions (MFS) is a meshless method that expands the solution utilizing the fundamental solution \cite{1,9,31,35}. The property of the fundamental solution, or Green’s function, is the theoretical basis of the MFS. However, the Green’s function for the d’Alembert operator $c^{-2}\partial_{tt} - \Delta$ is

$$G(x,t,s) = \frac{\delta(t - c^{-1}|x - s|)}{4\pi|x - s|},$$

where $c > 0$ denotes the sound speed of the homogeneous background medium, $\partial_{tt}u = \frac{\partial^2u}{\partial t^2}$, $\Delta$ is the Laplacian in $\mathbb{R}^3$, and $\delta$ is the Dirac delta distribution. Since the Green’s function involves the Dirac delta distribution, the MFS is no longer feasible to solve the three-dimensional wave equation. Unable to be applied directly, the Green’s function for the d’Alembert operator usually appears in the time convolution

$$G(x,t,s) \ast \lambda(t) = \frac{\lambda(t - c^{-1}|x - s|)}{4\pi|x - s|},$$

where $\lambda(t)$ is a signal function. One of the most popular applications of $G(x,t,s) \ast \lambda(t)$ is the boundary integral equation method, which is a commonly used method \cite{8,16,30,32}. Therefore, instead of Green’s function, new bases $G(x,t,s) \ast \lambda(t)$ are employed in the modified method of fundamental solutions (MMFS) proposed in this paper. Moreover, the MMFS can be simplified to a simple sampling method at each time step to reconstruct a moving point source, in which the sampling method is a well-known method in the numerical computation of inverse problems \cite{15,19,21,22,39}.

The time convolution of Green’s function and the signal function is an invaluable tool for the analysis of time domain scattering problems. Therefore, the MMFS has important significance in the theory of time domain analysis. Moreover, the proposed methods are feasible to reconstruct both multiple stationary point sources and a moving point source. The numerical implementations of the proposed methods are simple, and extensive experiments are provided to show the effectiveness of the methods.

The outline of this paper is as follows. In section 2, the inverse source problem with multiple stationary point sources is considered. The uniqueness result is provided, and the MMFS is proposed. In section 3, the MMFS is applied to the inverse source problem with a moving point source. Moreover, the method is simplified as a simple sampling method at each time.
step. In section 4, numerical experiments are provided to show the effectiveness of the proposed methods. The concluding remarks are given in section 5.

2. Reconstruction of stationary point sources

Denote by \( \Omega \subset \mathbb{R}^3 \) a bounded convex open region. Consider the wave equation

\[
c^{-2} \partial_{tt} u(x, t) - \Delta u(x, t) = \lambda(t) \sum_{j=1}^{M} a_j \delta(x - s_j), \quad x \in \mathbb{R}^3, \ t \in \mathbb{R},
\]

where \( M \in \mathbb{N}^+ \) is a positive integer, \( s_j \in \Omega \) are stationary source points, and \( a_j \neq 0 \) are the intensities of the sources.

The source points \( s_j \) are assumed to be mutually distinct. The signal function \( \lambda(t) \) is assumed to be causal, which means \( \lambda(t) = 0 \) for \( t < 0 \). Thus the source term \( f(x, t) = 0 \) for \( t < 0 \), and the initial condition

\[
u(\cdot, 0) = \partial_t u(\cdot, 0) = 0 \quad \text{in} \ \mathbb{R}^3
\]

is a direct conclusion of the causality.

The inverse source problem (P1) under consideration is: determine the locations and intensities of the stationary points \( s_j \) from the measurement data \( u(x, t), \ x \in \partial \Omega, \ t \in \mathbb{R} \).

Below we prove the uniqueness of the solution to the inverse source problem (P1). We need some preparation before proving the uniqueness theorem.

**Remark 2.1.** Let \( S := \bigcup_{j=1}^{M} \{ s_j \} \) be a set of distinct points in a bounded convex open region \( \Omega \subset \mathbb{R}^3 \), where \( M \in \mathbb{N}^+ \) and \( M \geq 2 \). Assume that \( x_0 \in \partial \Omega \) and \( s_k \in S \) are a pair of points such that \( |x_0 - s_k| = \min_{s \in \partial \Omega, x \in S} |x - s| \). Then

\[
\min_{x \in S \setminus \{ s_k \}} |x_0 - s| > \min_{x \in \partial \Omega, x \in S} |x - s|.
\]

Note that remark 2.1 follows directly from the convexity. Then, to prove the uniqueness, we are going to prove the following statement: If there exist two source terms \( f_1 \) and \( f_2 \), for which the solutions of these two wave equations coincide on \( \partial \Omega \times \mathbb{R} \), then \( f_1 = f_2 \). Instead of proving the statement directly, first, we claim that there exists a special common source point of \( f_1 \) and \( f_2 \) in the following lemma.

**Lemma 2.2.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded convex open region, and let \( u(x, t) \) be a solution to

\[
c^{-2} \partial_{tt} u(x, t) - \Delta u(x, t) = \sum_{j=1}^{M_i} \sum_{j=1}^{M_j} (-1)^{j-1} a_j^{(0)} \lambda_j^{(0)}(t) \delta(x - s_j^{(0)}), \quad x \in \mathbb{R}^3, \ t \in \mathbb{R},
\]

where \( M_i \in \mathbb{N}^+ \), \( a_j^{(0)} \neq 0 \), \( \lambda_j^{(0)}(t) \in C(\mathbb{R}) \), and \( s_j^{(0)} \in \Omega \) are mutually distinct for \( i = 1, 2 \). Define

\[
S^{(0)} := \bigcup_{j=1}^{M_i} \{ s_j^{(0)} \}, \ i = 1, 2.
\]

Assume that

\[
u = 0 \quad \text{on} \ \partial \Omega \times \mathbb{R}
\]
and

\[
\begin{cases}
\lambda_j^{(i)}(t) = 0, & t \leq t_i, \\
\lambda_j^{(i)}(t) \neq 0, & t_i < t < t_i + \tau,
\end{cases} \quad j = 1, \ldots, M_i, \ i = 1, 2
\]  

(6)

for some \( t_i > 0 \) and \( \tau > 0 \). Then, \( t_1 = t_2 \), and there exists a point \( s_0 \in S^{(1)} \cap S^{(2)} \) such that

\[
\min_{x \in \partial \Omega, s \in S^{(1)}} |x - s| = \min_{x \in \partial \Omega, s \in S^{(2)}} |x - s|.
\]

**Proof.** Note that \( \lambda_j^{(i)}(t) \) are causal due to the assumption. Then, \( u(x, t) \) is the unique causal solution to (4) (refer to [30] for a discussion of the uniqueness), which is given by

\[
u(x, t) = \sum_{i=1}^{M} \sum_{j=1}^{M_i} (-1)^{i-1} a_j^{(i)} G(x, t; s_j^{(i)}) \ast \lambda_j^{(i)}(t), \quad x \in \mathbb{R}^3, \ t \in \mathbb{R}.
\]

As shown in figure 1, there exist \( x_0^{(i)} \in \partial \Omega \) and \( s_k^{(i)} \in S^{(i)} \) \((1 \leq k_i \leq M_i)\) that satisfy

\[
|x_0^{(i)} - s_k^{(i)}| = d_i := \min_{x \in \partial \Omega, s \in S^{(i)}} |x - s|.
\]

Define

\[
d_i' := \begin{cases} 
\min_{x \in S^{(i)}} |x_0^{(i)} - s|, & \text{if } M_i \geq 2, \\
d_i + 1, & \text{if } M_i = 1.
\end{cases} \quad i = 1, 2.
\]

Then, remark 2.1 implies \( d_i' > d_i \).

We claim that \( t_1 + c^{-1}d_1 = t_2 + c^{-1}d_2 \). Otherwise, without loss of generality, assume that \( t_1 + c^{-1}d_1 < t_2 + c^{-1}d_2 \). Then, we have

\[
u(x_0^{(1)}, t) = a_k^{(1)} \frac{\lambda_k^{(1)}(t - c^{-1}d_1)}{4\pi d_1}, \quad t \in (T_1, T_2),
\]

where \( T_1 = t_1 + c^{-1}d_1 \) and \( T_2 = \min\{t_1 + c^{-1}d_1', t_2 + c^{-1}d_2\} \). Note that \( \lambda_k^{(1)}(t - c^{-1}d_1) \) is non-trivial for \( t \in (T_1, T_2) \) according to (6). Then, (5) implies \( a_k^{(1)} = 0 \), which is a contradiction to \( a_j^{(i)} \neq 0 \).
Furthermore, define $d_3 := \min_{x \in \mathcal{S}^{(3)}} |x_0^{(1)} - s|$. Apparently $d_2 \leq d_3$. If $d_2 < d_3$, we have $t_1 + c^{-1}d_1 < t_2 + c^{-1}d_3$. Similar to the above discussion, a contradiction can be drawn, which implies $d_2 = d_3$ and

$$t_1 + c^{-1}d_1 = t_2 + c^{-1}d_3. \tag{7}$$

Then, according to the definitions of $d_2$ and $d_3$, there exists a point $s_{k_1}^{(2)} \in \mathcal{S}^{(2)}$ such that

$$\left| x_0^{(1)} - s_{k_1}^{(2)} \right| = d_3 = d_2 = \min_{x \in \partial \Omega, x \in \mathcal{S}^{(2)}} |x - s|.$$ 

Then, the convexity of the boundary $\partial \Omega$ implies that the points $x_0^{(1)}$, $s_{k_1}^{(1)}$ and $s_{k_2}^{(2)}$ are collinear.

Then, we assert that $t_1 = t_2$. If otherwise, without loss of generality, we assume that $t_1 > t_2$, which implies $d_1 < d_3$. Define

$$d'_1 := \begin{cases} \min_{x \in \mathcal{S}^{(2)}} \left\{ \left| x_0^{(1)} - s \right| \right\}, & \text{if } M_2 \geq 2, \\ d_3 + 1, & \text{if } M_2 = 1. \end{cases}$$

Again, remark 2.1 implies $d'_1 > d_3$. For a point

$$x^* \in \left\{ x \in \partial \Omega : 0 < \left| x - x_0^{(1)} \right| < \frac{1}{2} \min \{ d'_1 - d_1, d'_3 - d_3 \} \right\},$$

the collinearity of the points $x_0^{(1)}$, $s_{k_1}^{(1)}$ and $s_{k_2}^{(2)}$ implies

$$\left| x^* - s_{k_2}^{(2)} \right| - \left| x^* - s_{k_1}^{(1)} \right| < d_3 - d_1.$$ 

Then, (7) implies $t_2 + c^{-1} \left| x^* - s_{k_2}^{(2)} \right| < t_1 + c^{-1} \left| x^* - s_{k_1}^{(1)} \right|$ and

$$u(x^*, t) = -\frac{\lambda_{k_2}^{(2)}(t)}{4\pi} \left( t - c^{-1} \left| x^* - s_{k_2}^{(2)} \right| \right), \quad t \in (T_3, T_4),$$

where $T_3 = t_2 + c^{-1} \left| x^* - s_{k_2}^{(2)} \right|$ and $T_4 = \min \left\{ t_1 + c^{-1} \left| x^* - s_{k_1}^{(1)} \right|, t_2 + \frac{1}{2}c^{-1}(d_3 + d'_3) \right\}$. Again, $\lambda_{k_2}^{(2)} \left( t - c^{-1} \left| x^* - s_{k_2}^{(2)} \right| \right)$ is non-trivial for $t \in (T_3, T_4)$ according to (6). Then, (5) implies a contradiction to $d_1^{(0)} \neq 0$.

Furthermore, (7) implies $d_1 = d_3$, and the collinearity of the points $x_0^{(1)}$, $s_{k_1}^{(1)}$ and $s_{k_2}^{(2)}$ implies $s_{k_1}^{(1)} = s_{k_2}^{(2)}$. The proof is completed by denoting $s_0 = s_{k_1}^{(1)} = s_{k_2}^{(2)}$. \qed
Then, basing on lemma 2.2, we are going to prove that the terms in \( f_1 \) and \( f_2 \) corresponding to the common source point are exactly the same. Following the procedure, we can finally get \( f_1 = f_2 \), as shown in the following theorem.

**Theorem 2.3.** Assume that \( \Omega \subset \mathbb{R}^3 \) is a bounded convex open region. Let

\[
f_i(x, t) = \lambda_i^{(0)}(t) \sum_{j=1}^{M_i} a_j^{(i)} \delta(x - s_j^{(i)}) , \quad i = 1, 2
\]

be two source terms with \( M_i \in \mathbb{N}^+ \), \( s_j^{(i)} \in \Omega \), \( a_j^{(i)} \neq 0 \) and \( \lambda_i^{(0)}(t) \in C(\mathbb{R}) \), such that the corresponding solutions to (1) for \( f_1 \) and \( f_2 \) are \( u_1 \) and \( u_2 \), respectively. Assume that \( \lambda_i^{(0)}(t) \) are non-trivial causal functions, \( s_j^{(i)} \) are mutually distinct for \( i = 1, 2 \), and

\[
u_1 = u_2 \quad \text{on} \quad \partial \Omega \times \mathbb{R}.
\]

Then, \( f_1 = f_2 \), that is, \( M_1 = M_2 = M \), \( s_j^{(1)} = s_j^{(2)} \) and \( a_j^{(1)} \lambda_i^{(1)}(t) = a_j^{(2)} \lambda_i^{(2)}(t), j = 1, 2, \ldots, M \) for some permutation \( \pi(j) \) of \( 1, 2, \ldots, M \).

**Proof.** Define \( w := u_1 - u_2 \), which satisfies

\[
c^{-2} \partial_{tt} w - \Delta w = f_1 - f_2 \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}, \quad w = 0 \quad \text{on} \quad \partial \Omega \times \mathbb{R}.
\]

Note that the unique causal solution to the wave equation (8) is

\[
u(x, t) = \sum_{i=1}^{2} \sum_{j=1}^{M_i} (-1)^{i-1} a_j^{(i)} G(x, t; s_j^{(i)}) \ast \lambda_i^{(0)}(t), \quad x \in \mathbb{R}^3, \ t \in \mathbb{R}.
\]

Define \( S^{(i)} := \bigcup_{j=1}^{M_i} \{ s_j^{(i)} \} \) and \( d_i := \min_{x \in \partial \Omega, s \in S^{(i)}} |x - s|, i = 1, 2 \). Since \( \lambda_i^{(0)}(t) \in C(\mathbb{R}) \) are non-trivial causal functions, we have

\[
\begin{aligned}
\lambda_i^{(0)}(t) &= 0, \quad t \leq t_i, \\
\lambda_i^{(0)}(t) &\neq 0, \quad t_i < t < t_i + \tau, \quad i = 1, 2.
\end{aligned}
\]

for some \( t_i > 0 \) and \( \tau > 0 \). For simplicity of presentation, we define \( t_i \) as the ‘starting time’ of the signal \( \lambda_i^{(0)}(t) \). Then, lemma 2.2 implies \( t_1 = t_2 \) and there exist a pair of points \( s_i^{(3)} \in S^{(3)} := S^{(1)} \cap S^{(2)} \) and \( x_0 \in \partial \Omega \) such that

\[
|x_0 - s_i^{(3)}| = \min_{x \in \partial \Omega} |x - s_i^{(3)}| = \min_{x \in \partial \Omega, s \in S^{(1)}} |x - s| = \min_{x \in \partial \Omega, s \in S^{(2)}} |x - s|.
\]

Without loss of generality, assume that \( s_i^{(3)} = s_{k_1}^{(1)} = s_{k_2}^{(2)} \), where \( s_{k_1}^{(1)} \in S^{(1)} \) and \( s_{k_2}^{(2)} \in S^{(2)} \).

For any source point \( s_j^{(3)} \in S^{(3)}, j = 1, \ldots, M_3 (M_3 \geq 1) \), the corresponding signal function \( \zeta_j(t) \) takes the form

\[
\zeta_j(t) = a_j^{(3)} \lambda_i^{(1)}(t) + b_j^{(3)} \lambda_i^{(2)}(t), \quad j = 1, \ldots, M_3.
\]
where \( a_j^{(3)}, b_j^{(3)} \neq 0 \). If \( \zeta_j(t) \equiv 0 \), no waves are motivated by the source point \( s_j^{(3)} \). If \( \zeta_j(t) \) is non-trivial, (10) and (12) lead to

\[
\begin{cases}
\zeta_j(t) = 0, & t \leq t_j^{(3)}, \\
\zeta_j(t) \neq 0, & t_j^{(3)} < t < t_j^{(3)} + r_j^{(3)}
\end{cases}
\]

for some \( t_j^{(3)} \geq t_2 = t_1 \) and \( r_j^{(3)} > 0 \).

Then, we are going to prove that the terms in \( f_1 \) and \( f_2 \) corresponding to the common source point \( s_j^{(3)} = s_k^{(1)} = s_l^{(2)} \) are exactly the same, that is, \( \zeta_1(t) \equiv 0 \).

If \( \zeta_j(t) \equiv 0, \forall j = 1, \ldots, M_3 \), the assertion is obvious. Otherwise, if \( t_j^{(3)} = t_1 \) for all the non-trivial \( \zeta_j(t) \), assume that \( \zeta_1(t) \) is non-trivial. We consider the wave field \( u(x, t) \) for \( x = x_0 \). Combined with (11), a similar discussion as that in the proof of lemma 2.2 implies a contradiction.

Note that if \( t_j^{(3)} \neq t_1 \) for some \( 1 \leq j \leq M_3 \), a contradiction can be drawn directly. First, we claim that there exists a constant \( t_3 > t_2 \) such that \( t_j^{(3)} = t_3, \forall j^{(3)} \neq t_1 \). Otherwise, without loss of generality, assume that \( t_{l_2}^{(3)} > t_{l_1}^{(3)} > t_1 \) for some \( 1 \leq l_1, l_2 \leq M_3 \). Then, we acquire

\[
\begin{align*}
\zeta_{l_1}(t) &= \zeta_{l_2}(t) = 0, & t \in [t_1, t_{l_1}^{(3)}], \\
\zeta_{l_1}(t) &\neq 0, \zeta_{l_2}(t) = 0, & t \in \left[t_{l_1}^{(3)}, \min \left\{ t_{l_1}^{(3)} + r_{l_1}^{(3)}, t_{l_2}^{(3)} \right\}\right].
\end{align*}
\]

Note that \( a_{l_1}^{(3)}, b_{l_1}^{(3)} \neq 0 \) for \( i = 1, 2 \), and \( \lambda_i(t), i = 1, 2 \) are non-trivial continuous for \( t \in [t_1, t_{l_1}^{(3)}] \) according to (10). Then, (13) indicates \( \zeta_1(t) = \alpha \zeta_{l_2}(t) \) with some \( \alpha \neq 0 \), which is a contradiction to (14). Therefore, the ‘starting’ of any non-trivial signal function corresponding to a source point in \( S^{(1)} \cup S^{(2)} \) would be either \( t_1 \) or \( t_3 \). Then, lemma 2.2 implies \( t_3 = t_1 \), which is a contradiction to \( t_3 > t_1 \).

In conclusion, we have proved that \( s_j^{(3)} = s_k^{(1)} = s_l^{(2)} \) and

\[
\zeta_1(t) = a_{l_1}^{(1)} \lambda_1(t) - a_{l_2}^{(2)} \lambda_2(t) \equiv 0.
\]

Then, the wave field can be rewritten as

\[
u(x, t) = \sum_{j = 1}^{M_1-\pi} a_{j}^{(1)} G \left(x, t; s_j^{(1)} \right) \ast \lambda_1(t) - \sum_{j = 1}^{M_1-\pi} a_{j}^{(2)} G \left(x, t; s_j^{(2)} \right) \ast \lambda_2(t).\]

If \( M_1 \neq M_2 \), without loss of generality, assume that \( M_1 > M_2 \). Following the above procedure, we finally have

\[
u(x, t) = \sum_{j = 1}^{M_1-\pi} a_{j}^{(1)} G \left(x, t; s_j^{(1)} \right) \ast \lambda_1(t),\]

where \( a_j^{(1)} \neq 0 \). Again, a similar discussion as that in the proof of lemma 2.2 leads to a contradiction. Then, we have \( M_1 = M_2 = M \). Therefore, the above procedure finally implies \( s_j^{(3)} = s_{\pi,0}^{(2)} \) and \( a_j^{(1)} \lambda_1(t) = a_j^{(2)} \lambda_2(t), j = 1, 2, \ldots, M \) for some permutation \( \pi(j) \) of \( 1, 2, \ldots, M \). □
The classic MFS expands the solution utilizing Green’s function (refer to [31, 32, 35]). However, since the Green’s function for the d’Alembert operator involves the Dirac delta distribution, the MFS is no longer feasible to solve the three-dimensional wave equation. Hence, consider the expansion

$$u(x, t) = \sum_{l=1}^{N_c} c(z_l)G(x, t; z_l) \ast \lambda(t),$$  \hspace{1cm} (15)$$
where $N_c \in \mathbb{N}^*$, $z_l \in \Omega$ are the sampling points and $c(z_l)$ are unknown coefficients to be computed.

Expansion (15) leads to the first modified method of fundamental solutions (MMFS1). We introduce the following proposition concerning MMFS1.

**Proposition 2.4.** Assume that $\Omega \subset \mathbb{R}^3$ is a bounded convex open region. Let $u(x, t)$ be a causal wave field that solves

$$c^{-2} \partial_t \partial_t u(x, t) - \Delta u(x, t) = \lambda(t) \sum_{j=1}^{M} a_j \delta(x - s_j), \hspace{1cm} x \in \mathbb{R}^3, \hspace{0.2cm} t \in \mathbb{R},$$  \hspace{1cm} (16)$$
where $M \in \mathbb{N}^*$, $a_j \neq 0$, $s_j \in \Omega$ are mutually distinct source points and $\lambda(t) \in C(\mathbb{R})$ is a non-trivial causal signal function. Assume that the sampling points $z_l \in \Omega$, $l = 1, 2, \ldots, N_c$ and a group of corresponding constants $c(z_l)$ satisfy

$$\sum_{l=1}^{N_c} c(z_l)G(x, t; z_l) \ast \lambda(t) = u(x, t), \hspace{1cm} x \in \partial \Omega, \hspace{0.2cm} t \in \mathbb{R},$$  \hspace{1cm} (17)$$
where $N_c \in \mathbb{N}^*$. Define $S_d := \{s_j\}_{j=1}^{M}$ and $Z_d := \{z_l\}_{l=1}^{N_c}$. Then, $S_d \subset Z_d$. Moreover,

$$c(z_l) = \begin{cases} a_j, & z_l = s_j, \\ 0, & z_l \in Z_d \setminus S_d. \end{cases}$$

**Proof.** Notice that

$$u(x, t) = \sum_{j=1}^{M} a_j G(x, t; s_j) \ast \lambda(t)$$  \hspace{1cm} (18)$$
is the unique causal solution of the wave equation (16). In addition, the wave field

$$u'(x, t) = \sum_{l=1}^{N_c} c(z_l)G(x, t; z_l) \ast \lambda(t), \hspace{1cm} x \in \mathbb{R}^3, \hspace{0.2cm} t \in \mathbb{R}$$
is a causal solution of

$$c^{-2} \partial_t \partial_t u(x, t) - \Delta u(x, t) = \lambda(t) \sum_{l=1}^{N_c} c(z_l)\delta(x - z_l), \hspace{1cm} x \in \mathbb{R}^3, \hspace{0.2cm} t \in \mathbb{R}.$$

Moreover, (17) implies

$$u'(x, t) = u(x, t), \hspace{1cm} x \in \partial \Omega, \hspace{0.2cm} t \in \mathbb{R}.$$
Algorithm 1. MMFS1 to reconstruct stationary point sources.

Step 1. Choose a convex region $\Omega$, a signal function $\lambda(t)$, an integer $M$ and the locations $s_j (j = 1, \ldots, M)$ of the point sources. Collect the wave data $u(x_i, t_k)$ for the sensing points $x_i \in \partial \Omega (i = 1, \ldots, N_x)$ and the discrete time steps $t_k \in [0, T] (k = 1, \ldots, N_T)$, where $T$ is a chosen terminal time.

Step 2. Choose a sampling region $D \subset \Omega$ such that $s_j \in D$ and $D \cap \partial \Omega = \emptyset$. Select a grid of sampling points $z_l (l = 1, \ldots, N_z)$ in $D$. Compute $c(z_l)$ from

$$\sum_{l=1}^{N_z} c(z_l) (G \ast \lambda(t; x_i, z_l)) = u(x_i, t_k), \quad i = 1, \ldots, N_x, \quad k = 1, \ldots, N_T$$

Step 3. Mesh $c(z_l)$ on the sampling grid. The locations of the point sources are given by the locations of $z_l$ for which $c(z_l)$ are local maximum or minimum values.

Remark 2.5. It is a strong hypothesis that the chosen sampling points $z_l \in \Omega$ and the constants $c(z_l)$ solve (17) for $x \in \partial \Omega$, $t \in \mathbb{R}$. Nevertheless, the numerical experiments in section 4 show the effectiveness of MMFS1 even if the hypothesis is not satisfied.

The MMFS1 to solve the inverse source problem (P1) is shown in algorithm 1. The numerical application of algorithm 1 can be seen in section 4.

3. Reconstruction of a moving point source

In this section, the inverse source problem with a moving point source is considered. The wave equation is

$$c^{-2}\partial_t u(x, t) - \Delta u(x, t) = \lambda(t) \delta(x - s(t)), \quad x \in \mathbb{R}^3, \quad t \in [0, T],$$

where $T > 0$ and $s : [0, T] \to \Omega$ signifies the smooth trajectory of the moving point source. Denote by $v(t) = \frac{dx(t)}{dt}$, $t \in (0, T)$ the instantaneous velocity of the point source. Again, $\lambda(t)$ is causal and the initial condition follows from the causality.

The inverse source problem (P2) is: determine the trajectory $s(t)$ of the moving point source in (19) from the measurement data

$$u(x, t), \quad x \in \partial \Omega, \quad t \in [0, T].$$

MMFS1 can feasibly reconstruct stationary point sources. However, for a moving point source, the location of the point source changes over time. Thus, the coefficients $c(z_l)$ in...
MMFS1 should also depend on the time variable. Therefore, consider a new expansion
\[ u(x, t) = \sum_{i=1}^{N_t} c(t; z_i) G(x, t; z_i) * \lambda(t), \quad t \in [0, T], \]
where \( z_i \in \Omega \) are the sampling points and \( c(\cdot; z_i) \) are unknown functions depending on \( z_i \). The second modified method of fundamental solutions (MMFS2) is similar to algorithm 1 except that the new expansion (21) is employed and \( c(t_i; z_i), i = 1, \ldots, N_t \) should be computed for each time step \( t_k, k = 1, \ldots, N_T \).

Note that there is only one point source in this case. If \( |v| = 0 \), we have \( s(t_k) = 0 \) for some \( s_0 \in \Omega \). Then, proposition 2.4 implies
\[ c(t_k, z_i) = \begin{cases} 1, & z_i = s_0, \\ 0, & \text{otherwise.} \end{cases} \]

Then, we expect that \( G(x, t; s(t_k)) * \lambda(t) \) is the approximation of \( u(x, t_k) \) when \( |v| \) is small. On this basis, define the indicator function
\[ I(z, t) = \| u_0(x, t) - G(x, t; z) * \lambda(t) \|_{\partial \Omega}^{-1}, \quad z \in \Omega, \ t \in [0, T], \]
where \( \| \cdot \|_{\partial \Omega} \) is the \( L^2(\partial \Omega) \) norm with respect to \( x \). We have the following theorem concerning the indicator function (22).

**Theorem 3.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded convex open region. Assume that \( d_\Omega := \sup_{x, y \in \Omega} |x - y| \ll c, \ |v| \ll c \) and \( \lambda, s \in C^1[0, T] \). Let \( u_0(x, t) \) be the causal solution of the wave equation (19). For any fixed \( t \in [0, T] \), the indicator function (22) satisfies
\[ I(z, t) \gg 1 \quad \text{when} \ z \to s(t). \]

**Proof.** Note that when \( |v(t)| < c \) for \( t \in (0, T) \), the explicit solution to the wave equation (19) is given by (refer to [29])
\[ u_0(x, t) = \frac{\lambda(t)}{4\pi|x - s(t)| \left( 1 - \frac{(x - s(t))}{c|x - s(t)|} \right)}, \tag{23} \]
where the retarded time \( \tau \) satisfies \( t - \tau = c^{-1}|x - s(\tau)| \).

Under the assumptions \( d_\Omega \ll c, \ |v| \ll c \) and \( \lambda, s \in C^1[0, T] \), we assert that
\[ G(x, t; s(t)) * \lambda(t) = \frac{\lambda(t - c^{-1}|x - s(t)|)}{4\pi|x - s(t)|} \]
is an approximation of the solution (23). The proof of a similar conclusion can be seen in [34]. Though we use an arbitrary causal signal function \( \lambda(t) \) instead of the time-harmonic signal \( \lambda(t) = \sin(\omega_0 t) \) for some \( \omega_0 > 0 \), and though the function \( \lambda(t - c^{-1}|x - s(t)|) \) is occupied in \( G(x, t; s(t)) * \lambda(t) \) in this paper instead of \( \lambda(t) \), a similar discussion implies
\[ u_0(x, t) = G(x, t; s(t)) * \lambda(t) + O(\varepsilon(t)), \quad x \in \partial \Omega, \ t \in [0, T] \]
with some \( 0 < \varepsilon(t) \ll 1 \).

Then, the smoothness of the function \( G(x, t; z) * \lambda(t) \) with respect to \( z \) implies the conclusion. \( \square \)
Algorithm 2. The simplified scheme to reconstruct a moving point source.

Step 1  Choose a convex region $\Omega$, a signal function $\lambda(t)$ and the trajectory $s(t), t \in [0, T]$ of the moving point source. Collect the wave data $u(x_i, t_k)$ for the sensing points $x_i \in \partial \Omega, i = 1, \ldots, N_\Omega$ and the discrete time steps $t_k (k = 1, \ldots, N_T)$

Step 2  Choose a sampling region $D \subset \Omega$ such that $s(t) \subset D$ and $D \cap \partial \Omega = \emptyset$. Select a grid of sampling points $z_i (i = 1, \ldots, N_z)$ in $D$. For each time step $t_k$, compute

$$I(z_i, t_k) = \left( \sum_{i=1}^{N_z} ((G \ast \lambda)(x_i, t_k; z_i) - u(x_i, t_k))^2 \right)^{-1/2}
$$

Step 3  For each time step $t_k$, the location of $s(t_k)$ of the point source is approximated by the location of $z_i$ for which $I(z_i, t_k)$ is the global maximum value.

Then, MMFS2 is in fact equivalent to a simple sampling method. The simplified scheme is shown in algorithm 2. The numerical implementation of algorithm 2 is shown in section 4.

4. Numerical examples

In this section, we consider the numerical implementation of the proposed algorithms. The radiated field is collected for $t \in [0, T]$, where $T$ is the terminal time. The time discretization is

$$t_k = k \frac{T}{N_T}, \quad k = 1, 2, \ldots, N_T,$$

where $N_T \in \mathbb{N}$. Random noise is added to the data with

$$u_i = (1 + \epsilon r) u_i,$$

where $\epsilon > 0$ is the noise level and $r$ are uniformly distributed random numbers in $[-1, 1]$.

In all the experiments, the signal function $\lambda(t)$ is chosen as

$$\lambda(t) = \begin{cases} 0, & t < 0, \\ \sin(\omega t) e^{-0.3(t-3)^2}, & t \geq 0, \end{cases}$$

where $\omega \in \mathbb{R}_+$ denotes the centre frequency of the wave.

4.1. Reconstruction of multiple static point sources

Algorithm 1 is employed for the reconstruction of multiple static point sources. The synthetic wave field data is given by the analytic solution (18). We choose $c = 1$ and $T = 15$ in this subsection. The sampling points are chosen as $N_S \times N_S \times N_S$ uniform discrete points in $[0, 2] \times [0, 2] \times [0, 2]$ with $N_S \in \mathbb{N}$.

Example 1. We investigate the reconstruction of point sources with different intensities in this example. The source points are chosen as $(0.9, 0.6, 0), (1.4, -1.1, 0), (-1.1, -0.9, 0), (-1.1, -1.1, 0), (-0.1, 0.9, 0), (0.4, -0.4, 0)$ and $(-0.6, 0.6, 0)$ with relative intensities $4, 3, 3, 2, 2, -2$ and $-3$, respectively. The sensing points are chosen as

$$x(i, j) = (5 \sin \varphi_i \cos \theta_j, 5 \sin \varphi_i \sin \theta_j, 5 \cos \varphi_i)$$

with $\varphi_i = \frac{2i-1}{8} \pi, i = 1, 2, \ldots, 8$ and $\theta_j = \frac{4j-1}{4} \pi, j = 0, 1, \ldots, 7$. We choose $\omega = 10, \epsilon = 1\%$ and $N_T = 64$ in this example. The reconstructions with respectively $N_S = 23, 45, 89$ and 177 are of interest.
Note that when \( N_S = 177 \), there are 5545 233 sampling points and the same number of \( c(z_l) \) to be computed. Then, the size of the coefficient matrix while utilizing the conjugate gradient method is 5545 233 \( \times \) 5545 233, which is a big challenge for the RAM of the computer. In fact, when \( N_S \) is doubled, the size of the coefficient matrix is 64 times larger, and our computer with 8 GB of RAM has failed to store the coefficient matrix when \( N_S = 45 \). Therefore, the algorithm must be modified to get more accurate calculations.

We provide the following strategy to reduce the matrix size and accelerate the algorithm. The main idea of the strategy is multi-level calculation. Only the region where the calculation results of the previous level exceed a certain threshold is considered in the calculation of the next level.

(a) For \( N_S^{(n)} = 11 \times 2^n + 1 \), \( n = 1, \ldots, 4 \), denote

\[
\epsilon^{(n)}(i, j, k) = \begin{pmatrix}
-2 + \frac{4(i-1)}{N_S^{(n)}} - 1, & -2 + \frac{4(j-1)}{N_S^{(n)}} - 1, & -2 + \frac{4(k-1)}{N_S^{(n)}} - 1 \\
0
\end{pmatrix}, \quad i, j, k = 1, \ldots, N_S^{(n)}.
\]

(b) For \( n = 1 \), compute \( \epsilon^{(1)}(z_l(i, j, k)) \), \( i, j, k = 1, \ldots, N_S^{(1)} \) using algorithm 1.

(c) For \( n = 2, 3, 4 \), define \( N_S^{(0)} \times N_S^{(n)} \times N_S^{(n)} \) matrices \( \epsilon^{(n)}(i, j, k) \) and \( \epsilon^{(n)}(i, j, k) \). Denote

\[
\epsilon^{(n)}(i, j, k) = \begin{cases}
\epsilon^{(n-1)}(z_l(i', j', k')) & \text{if } 2i' - 1, 2j' - 1, 2k' - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Then, define

\[
\epsilon^{(n)}(i, j, k) = \max_{1 \leq i' \leq i+1, 1 \leq j' \leq j+1, 1 \leq k' \leq k+1} \epsilon^{(n)}(i', j', k'), \quad 2 \leq i, j, k \leq N_S^{(n)} - 1,
\]

otherwise.

Choose a threshold value \( \Theta^{(n)} \). Denote \( \epsilon^{(n)}(z_l(i, j, k)) = 0 \) if \( \epsilon^{(n)}(i, j, k) \leq \Theta^{(n)} \), and compute \( \{ \epsilon^{(n)}(z_l(i, j, k)) \mid \epsilon^{(n)}(i, j, k) > \Theta^{(n)} \} \) using algorithm 1.

In this experiment, we choose \( \Theta^{(n)} \equiv 0.05 \) for all \( n = 2, \ldots, 4 \). The reconstructions with \( N_S^{(n)} = 11 \times 2^n + 1 \) respectively for \( n = 1, 2, 3 \) and 4 can be seen in figure 2. In our computation, instead of the coefficient matrix of the size 5545 233 \( \times \) 5545 233, the matrices of the sizes 12167 \( \times \) 12167, 16585 \( \times \) 16585, 2888 \( \times \) 2888 and 4439 \( \times \) 4439 are involved to compute \( \epsilon^{(n)}(z_l(i, j, k)) \) for \( n = 1, 2, 3 \) and 4, respectively. As shown in figure 2, the proposed algorithm with the simplification strategy is feasible to reconstruct point sources with different intensities.

The specific data of the reconstructions with \( n = 1, 2, 3 \) and 4 is given by the following procedure.

(a) Denote \( p = 1 \). Choose a reference intensity \( T_r \in \mathbb{R}_+ \) and the total steps \( N_p \in \mathbb{N}^+ \).

(b) Find the global maximum point \( z_l(i_p, j_p, k_p) \) of \( \{ \epsilon^{(n)}(z_l(i, j, k)) \} \). Denote

\[
T^{(n)}(i_p, j_p, k_p) = \sum_{i=i_p-A^{(n)}, \ldots, i_p+A^{(n)}} \sum_{j=j_p-A^{(n)}, \ldots, j_p+A^{(n)}} \sum_{k=k_p-A^{(n)}, \ldots, k_p+A^{(n)}} \epsilon^{(n)}(z_l(i, j, k)),
\]

where \( A^{(n)} \in \mathbb{N}^+ \) is a chosen parameter depending on \( n \).

(c) If \( T^{(n)}(i_p, j_p, k_p) > T_r \), the corresponding sampling point \( z_l(i_p, j_p, k_p) \) is regarded as a source point with the intensity \( T^{(n)}(i_p, j_p, k_p) \). If \( T^{(n)}(i_p, j_p, k_p) < T_r \), ignore the point.
Figure 2. Reconstruction of 7 stationary point sources with different intensities, $\epsilon = 1\%$ (example 1). (a) Location of the sensors. (b) Sketch of the example. (c) Reconstruction with $N_S^{(1)} = 23$. (d) Reconstruction with $N_S^{(2)} = 45$. (e) Reconstruction with $N_S^{(3)} = 89$. (f) Reconstruction with $N_S^{(4)} = 177$.

(d) Denote $z^{(n)}(i_p, j_p, k_p) = 0$ for $i = i_p - A^{(n)}, \ldots, i_p + A^{(n)}$, $j = j_p - A^{(n)}, \ldots, j_p + A^{(n)}$ and $k = k_p - A^{(n)}, \ldots, k_p + A^{(n)}$.

(e) Redefine $p = p + 1$. If $p < N_p$, go back to step 2. If $p = N_p$, end the procedure.

**Remark 4.1.** Because of the noise and algorithmic error, the intensities of several sampling points around a source point would be non-trivial. Therefore, the reconstructed intensity $T^{(n)}(i_p, j_p, k_p)$ is given by the superposition of the intensities of several sampling points around $z^{(n)}(i_p, j_p, k_p)$.

In this example, we choose $T_r = 1, N_p = 10$ and $A^{(n)} = n + 1$ by trial and error. The specific data is shown in table 1. The reconstructions of locations and intensities are both reliable when $N_S$ is large enough. Since the point sources no. 3 and no. 4 are too close to each other, only one source point is identified when $N_S$ is 23 and 45.

**Example 2.** In this example, the reconstruction of the stationary point sources located at $(-1, 0, 0.9), (0, 0.9, -1), (-1, 0.9, 0)$ and $(1.1, 0, -1)$ with intensities $3, 2, -2$ and $-3$ is considered. Five cases are considered to illustrate the influence of the frequency, noise level and sensors. We choose $N_T = 64$ and $N_S = 177$ in all cases.

In cases 1 through 3, the sensors are chosen as all the sensors in (24). The parameters are chosen as $\{\omega = 15, \epsilon = 1\%\}, \{\omega = 5, \epsilon = 1\%\}$ and $\{\omega = 5, \epsilon = 3\%\}$ in cases 1, 2 and 3, respectively. Following example 1, we choose $T^{(n)} = 0.05$ in cases 1 and 2. However, if we also choose $T^{(n)} = 0.05$ in case 3, too many sampling points need to be computed. Therefore, in case 3, we choose $T^{(2)} = 0.2, T^{(3)} = 0.1$ and $T^{(4)} = 0.05$ in the computation.

In cases 4 and 5, we choose $\omega = 5, \epsilon = 1\%, T^{(2)} = 0.05, T^{(3)} = 0.05$ and $T^{(4)} = 0.025$. The sensors are chosen as the upper half of the sensors in (24) with $i = 1, 2, \ldots, 8, j = 0, 1, \ldots, 7$ and the left half with $i = 1, 2, \ldots, 4, j = 0, 1, \ldots, 3$ respectively in two cases.
Table 1. Reconstruction of the locations and intensities of the point sources in example 1.

| No. | Items          | The actual point sources | Reconstruction with $N_S^{(1)} = 23$ | Reconstruction with $N_S^{(2)} = 45$ | Reconstruction with $N_S^{(3)} = 89$ | Reconstruction with $N_S^{(4)} = 177$ |
|-----|----------------|--------------------------|---------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|
| 1   | Location       | (0.9, 0.6, 0)            | (0.91, 0.55, 0)                       | (0.91, 0.64, 0)                      | (0.91, 0.59, 0)                      | (0.89, 0.59, 0)                      |
|     | Intensity      | 4                        | 4.39                                  | 3.30                                 | 4.03                                 | 3.95                                 |
| 2   | Location       | (1.4, −1.1, 0)           | (1.45, −1.09, 0)                      | (1.36, −1.09, 0)                     | (1.41, −1.09, 0)                     | (1.39, −1.09, 0)                     |
|     | Intensity      | 3                        | 3.35                                  | 2.85                                 | 2.89                                 | 3.16                                 |
| 3   | Location       | (−1.1, −0.9, 0)          | (−1.09, −0.91, 0)                     | (−1.09, −0.91, 0)                    | (−1.09, −0.91, 0)                    | (−1.09, −0.91, 0)                    |
|     | Intensity      | 3                        | 5.65                                  | 4.94                                 | 4.28                                 | 3.32                                 |
| 4   | Location       | (−1.1, −1.1, 0)          | Null                                  | Null                                 | (−1.14, −1.14, 0)                    | (−1.11, −1.11, 0)                    |
|     | Intensity      | 2                        | Null                                  | Null                                 | 0.79                                 | 1.98                                 |
| 5   | Location       | (0.1, 0.9, 0)            | (0.18, 0.91, 0)                       | (0.09, 0.91, 0)                      | (0.09, 0.91, 0)                      | (0.11, 0.91, 0)                      |
|     | Intensity      | 2                        | 2.13                                  | 1.80                                 | 2.05                                 | 2.03                                 |
| 6   | Location       | (0.4, −0.4, 0)           | (0.36, −0.36, 0)                      | (0.45, −0.36, 0)                     | (0.41, −0.41, 0)                     | (0.41, −0.41, 0)                     |
|     | Intensity      | −2                       | −3.52                                 | −3.14                                | −1.80                                | −1.92                                |
| 7   | Location       | (−0.6, 0.6, 0)           | (−0.55, 0.55, 0)                      | (−0.55, 0.64, 0)                     | (−0.59, 0.59, 0)                     | (−0.59, 0.61, 0)                     |
|     | Intensity      | −3                       | −3.34                                 | −3.20                                | −3.02                                | −3.12                                |
Figure 3. Reconstruction of 4 stationary point sources with different intensities, $N_S = 177$ (example 2). (a) Sketch of the example. (b) Reconstruction with all the sensors, $\omega = 5, \epsilon = 1\%$. (c) Reconstruction with all the sensors, $\omega = 5, \epsilon = 1\%$. (d) Reconstruction with all the sensors, $\omega = 5, \epsilon = 1\%$. (e) Reconstruction with the upper half of the sensors, $\omega = 5, \epsilon = 1\%$. (f) Reconstruction with the left half of the sensors, $\omega = 5, \epsilon = 1\%$.

Table 2. Reconstructions of the intensities of the point sources in example 2.

| No. | Location       | Intensity | Case 1 | Case 2 | Case 3 | Case 4 | Case 5 |
|-----|----------------|-----------|--------|--------|--------|--------|--------|
| 1   | (-1, 0, 0.9)   | 3         | 2.92   | 3.03   | 3.34   | 3.10   | 2.99   |
| 2   | (0, 0.9, -1)   | 2         | 1.89   | 2.02   | 2.02   | 1.86   | 2.01   |
| 3   | (-1, 0.9, 0)   | -2        | -1.87  | -2.02  | -2.33  | -2.03  | -2.09  |
| 4   | (1.1, 0, -1)   | -3        | -2.88  | -3.01  | -3.23  | -3.05  | -3.11  |

The reconstructions can be seen in figure 3. The specific data of the reconstructions is given by the same procedure as that in example 1 with the same parameters. The reconstructions of locations are accurate in all cases, and there is no obvious difference between cases. Therefore, we provide only the reconstructions of the intensities in table 2.

4.2. Reconstruction of a moving point source

This subsection is concerned with the reconstruction of a moving point source. A numerical scheme based on algorithm 2 is employed. The synthetic wave field data is given by the analytic solution (23). We choose $c = 340$ and $T = 2\pi$ in this subsection.

Example 3. In this example, we consider the reconstruction of an arbitrary trajectory of a moving source in $\mathbb{R}^3$. The sensors are chosen to be the same as that in example 1. The sampling points are chosen as $N_T \times N_T \times N_T$ uniform discrete points in $[-3, 3] \times [-3, 3] \times [-3, 3]$. We choose $N_T = 64$ in this example. The reconstruction of the location given by
Reconstruction of \( s_1(t) \) in \( \mathbb{R}^3 \), \( \omega = 10 \), \( \epsilon = 5\% \) (example 3, case 1). (a) Trajectory of the source point. (b) The reconstruction. (c) The error of the reconstruction. (d) The modified reconstruction. (e) The error of the modified reconstruction. (f) The smooth reconstruction with the Fourier expansion of order 5.

\[
\begin{align*}
 s'(t_k) &:= \arg \max_{z \in B} I(t_k, z), \quad k = 1, 2, \ldots, N_T \\
 s_1(t) &:= (2 + 0.3 \cos 3t)(\cos t, \sin t, 0).
\end{align*}
\]

In Case 1, we choose \( N_S = 51 \), \( \omega = 10 \), \( \epsilon = 5\% \), and the trajectory of the moving source is chosen as \( s_1(t) = (2 + 0.3 \cos 3t)(\cos t, \sin t, 0) \). The reconstructions can be seen in figure 4. As shown in figure 4(b), the reconstructions \( s_1'(t_k) \) are close to the trajectory \( s_1(t) \) except for several discrete points. The error \( E(t_k) := |s_1(t) - s_1'(t_k)| \) at each time step \( t_k \) can be seen in figure 4(c). The error \( E(t_k) \) is large when \( \lambda(t_k) \) is near zero. Therefore, the following modification is provided after the reconstruction.

(a) If \( |\lambda(t_1)| < 10^{-4} \), redefine \( s'(t_1) = 2s'(t_2) - s'(t_3) \).
(b) If \( |\lambda(t_{N_T})| < 10^{-4} \), redefine \( s'(t_{N_T}) = 2s'(t_{N_T-1}) - s'(t_{N_T-2}) \).
(c) If \( |\lambda(t_k)| < 10^{-4} \) for any \( k = 2, \ldots, N_T - 1 \), redefine \( s'(t_k) = \frac{1}{2}(s'(t_{k-1}) + s'(t_{k+1})) \).

The modified reconstruction and the corresponding error are shown in figures 4(d) and (e), respectively. As we can see from figure 4(e), the error \( E(t_k) \) is small at each time step after the modification. Therefore, similar modifications are applied to all the experiments in the rest of this subsection.

The smooth reconstruction of the trajectory is given by the post-processing of the data \( s'(t_k) \) by a Fourier approximation. The truncated Fourier expansion of order \( N \in \mathbb{N}^* \) is employed such that

\[
s'(t) = a_0 + \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt),
\]

where

\[
a_0 = \frac{1}{N_T} \sum_{k=1}^{N_T} s'(t_k),
\]
Figure 5. The influence of $N_S$ and $\epsilon$ on the error $E(t)$ (example 3, Case 2). (a) The error $E(t_k)$ for $N_S = 26$, 51 and 301, $\epsilon = 5\%$. (b) The error $E(t_k)$ for $N_S = 26$, 51 and 301, $\epsilon = 3\%$. (c) The error $E(t_k)$ for $N_S = 26$, 51 and 301, $\epsilon = 0$. (d) The maximum and the mean value of $\{E(t_k), k = 1, 2, \ldots, N_T\}$ for $N_S \in [26, 301]$, $\epsilon = 0$, 3\% and 5\%.

$$a_n = \frac{2}{N_T} \sum_{k=1}^{N_T} s'(t_k) \cos n t_k, \quad n = 1, 2, \ldots, N,$$

$$b_n = \frac{2}{N_T} \sum_{k=1}^{N_T} s'(t_k) \sin n t_k, \quad n = 1, 2, \ldots, N.$$ 

A Fourier expansion of order 5 is employed to obtain the smooth reconstruction in this example.

In case 2, the influence of $N_S$ and $\epsilon$ on the error $E(t)$ is considered. We choose $N_S \in [26, 301]$ and $\epsilon = 0$, 3\% and 5\% to illustrate the error, and the other parameters are chosen to be the same as those in case 1. The error analysis is shown in figure 5, where $E_{\text{max}}$ and $E_{\text{mean}}$ are the maximum and the mean value of $\{E(t_k), k = 1, 2, \ldots, N_T\}$, respectively. As shown in the figure, when $N_S$ increases, the error decreases quickly to a certain value depending on $\epsilon$ and then stabilizes at the value. Therefore, when $\epsilon = 5\%$, choosing $N_S = 51$ is enough for the reconstruction since further increasing $N_S$ does not reduce the error significantly.

Remark 4.2. Note that the algorithm is time consuming if we need to compute $I(z_l, t_k)$ for each $z_l \in D$ to obtain the reconstruction $s'(t_k)$. Nevertheless, taking into consideration the
limited speed of the point source and the limited error of the reconstruction, we have
\[ |s(t_k) - s'(t_{k-1})| \leq \|v(t)\|_\infty \frac{T}{N_T} + \|E(t)\|_\infty. \]

That is, if $s'(t_{k-1})$ has been obtained, we have $s'(t_k) = \arg \max_{z_l \in B_k} I(t_k, z_l)$, where
\[ B_k := \{ x \in D : |x - s'(t_{k-1})| \leq \|v(t)\|_\infty T/N_T + \|E(t)\|_\infty \}. \]

In case 3, we choose $\omega = 5$, and the trajectory of the point source is chosen as $s_2(t) = 2(\sin 2t, \cos 2t, \frac{t}{\pi} - 1)$. The other parameters are chosen to be the same as those in case 1. The reconstructions can be seen in figure 6.

**Example 4.** As an addition to example 3, the reconstruction of a handwritten Chinese character ‘ai’ is considered. We choose $N_T = 128$ in this example, and the other parameters are chosen to be the same as those in example 3, case 1.

The smooth reconstruction in this example is also provided by the Fourier expansion. However, the Chinese character ‘ai’ has 5 strokes and cannot be reconstructed with a single smooth curve. Thus, smooth reconstruction is provided for each stroke. Since the point source moves faster in the gap between two strokes, we use the following strategy to provide the smooth reconstruction.

(a) Choose a reference distance $d_r$. If $\max \{|s'(t_{k-1}) - s'(t_k)|, |s'(t_{k+1}) - s'(t_k)|\} > d_r$ for any $k = 2, \ldots, N_T - 1$, classify $s'(t_k)$ as an end point of a stroke or a point between two strokes.

(b) Separate the strokes of the character, and provide the smooth reconstruction of each stroke using the Fourier expansion.

We choose $d_r = 0.3$ in this example. The reconstructions can be seen in figure 7. As shown in figure 7, the algorithm is feasible to reconstruct the character with noise level $\epsilon = 5\%$. The smooth reconstructions by the Fourier expansion with order 5 and order 3 are shown in figures 7(c) and (d), respectively.

**Remark 4.3.** The smooth reconstruction by the Fourier expansion of order 5 indeed shows more details of the reconstruction than that of order 3. However, some of the details are caused by noise. As shown in figures 7(c) and (d), the smooth reconstruction by the Fourier expansion of order 3 is better in this example than that of order 5.

**Example 5.** In this example, we are concerned about the reconstruction of the trajectory $s_3(t) = 2 \left( \frac{1}{\pi} \sin 2t, \frac{1}{\pi} \cos 2t, \frac{t}{\pi} - 1 \right)$ using 4 sensors. The sensing points are chosen as
Figure 7. Reconstruction of a handwritten Chinese character ‘ai’ (example 4). (a) Trajectory of the point source. (b) The reconstruction, $\epsilon = 5\%$. (c) The smooth reconstruction with the Fourier expansion of order 5. (d) The smooth reconstruction with the Fourier expansion of order 3.

Figure 8. Reconstruction of $s_3(t)$ with 4 sensors (example 5). (a) Sketch of the example. (b) The reconstruction, $\epsilon = 5\%$. (c) The smooth reconstruction with the Fourier expansion of order 3.

(3, 3, −3), (3, −3, −3), (−3, 3, −3) and (−3, −3, −3). The sampling points and the time discretization are chosen to be the same as those in example 3, case 1. The reconstructions can be seen in figure 8.

The error of the reconstruction with only 4 sensors is larger than that of example 3, case 1. Nevertheless, the smooth reconstruction ignores most of the error, and the algorithm still works well.
5. Conclusion

We have considered the numerical simulations of the time-dependent inverse source problems of acoustic waves. Modified method of fundamental solutions have been established to reconstruct both multiple stationary sources and a moving point source. Moreover, the second modified method of fundamental solutions to reconstruct a moving point source has been modified to a simple sampling method. Several numerical examples have been provided to show the effectiveness of the proposed methods.

Acknowledgments

The work of Bo Chen was supported by the NSFC (No. 11671170) and the Fundamental Research Funds for the Central Universities (Special Project for Civil Aviation University of China, No. 3122018L009). The work of Yukun Guo was supported by the NSFC (No. 11971133, 11601107 and 11671111). The work of Fuming Ma was supported by the NSFC (No. 11771180). The work of Yao Sun was supported by the NSFC (No. 11501566).

ORCID iDs

Yukun Guo https://orcid.org/0000-0003-4477-7666
Yao Sun https://orcid.org/0000-0002-9165-1735

References

[1] Ahmadabadi M N, Arab M and Ghaini F M M 2009 The method of fundamental solutions for the inverse space-dependent heat source problem Eng. Anal. Bound. Elem. 33 1231–5
[2] Alves C, Kress R and Serranho P 2009 Iterative and range test methods for an inverse source problem for acoustic waves Inverse Problems 25 055005
[3] Badia A E and Nara T 2011 An inverse source problem for Helmholtz’s equation from the Cauchy data with a single wave number Inverse Problems 27 105001
[4] Bao G, Chow S N, Li P and Zhou H 2014 An inverse random source problem for the Helmholtz equation Math. Comput. 83 215–33
[5] Bao G, Hu G, Kian Y and Yin T 2017 Inverse source problems in elastodynamics Inverse Problems 34 045009
[6] Bao G, Lin J and Triki F 2010 A multi-frequency inverse source problem J. Differ. Equ. 249 3443–65
[7] Bao G, Lu S, Rundell W and Xu B 2015 A recursive algorithm for multi-frequency acoustic inverse source problems SIAM J. Numer. Anal. 53 1608–28
[8] Chen B, Ma F and Guo Y 2017 Time domain scattering and inverse scattering problems in a locally perturbed half-plane Appl. Anal. 96 1303–25
[9] Chen B, Sun Y and Zhuang Z 2019 Method of fundamental solutions for a Cauchy problem of the Laplace equation in a half-plane Bound. Value Probl. 2019 1–14
[10] Cheng J, Ding G and Yamamoto M 2002 Uniqueness along a line for an inverse wave source problem Commun. PDE 27 2055–69
[11] Cheng J, Peng L and Yamamoto M 2005 The conditional stability in line unique continuation for a wave equation and an inverse wave source problem Inverse Problems 21 1993–2007
[12] De Hoop M V and Titielitz J 2015 An inverse source problem for a variable speed wave equation with discrete-in-time sources Inverse Problems 31 075007
[13] Fournier J, Garnier J, Papanicolaou G and Tsogka C 2017 Matched-filter and correlation-based imaging for fast moving objects using a sparse network of receivers SIAM J. Imag. Sci. 10 2165–216
[14] Garnier J and Fink M 2015 Super-resolution in time-reversal focusing on a moving source Wave Motion 53 80–93
[15] Guo Y, Hömberg D, Hu G, Li J and Liu H 2016 A time domain sampling method for inverse acoustic scattering problems J. Comput. Phys. 314 647–60
[16] Guo Y, Monk P and Colton D 2013 Toward a time domain approach to the linear sampling method Inverse Problems 29 095016
[17] Isakov V 1998 Inverse Problems for Partial Differential Equations (New York: Springer)
[18] Li J, Liu H, Shang Z and Sun H 2013 Two single-shot methods for locating multiple electromagnetic scatterers SIAM J. Appl. Math. 73 1721–46
[19] Li J, Liu H and Sun H 2018 On a gesture-computing technique using electromagnetic waves Inverse Problems Imaging 12 677–96
[20] Li J, Liu H and Zou J 2008 Multilevel linear sampling method for inverse scattering problems J. Comput. Phys. 30 1228–50
[21] Li J, Liu H and Zou J 2009 Strengthened linear sampling method with a reference ball SIAM J. Sci. Comput. 31 4013–40
[22] Li P 2011 An inverse random source scattering problem in inhomogeneous media Inverse Problems Imaging 27 035004
[23] Li P and Yuan G 2016 Increasing stability for the inverse source scattering problem with multi-frequencies Inverse Problems Imaging 11 745–59
[24] Lim P H and Ozard J M 1994 On the underwater acoustic field of a moving point source. i. range-independent environment J. Acoust. Soc. Am. 95 131–7
[25] Lim P H and Ozard J M 1994 On the underwater acoustic field of a moving point source. ii. range-dependent environment J. Acoust. Soc. Am. 95 138–51
[26] Matsumoto M, Tohyama M and Yanagawa H 2003 A method of interpolating binaural impulse responses for moving sound images Acoust Sci. Technol. 24 284–92
[27] Nakaguchi E, Inui H and Ohnaka K 2012 An algebraic reconstruction of a moving point source for a scalar wave equation Inverse Problems Imaging 28 065018
[28] Sun Y 2014 Modified method of fundamental solutions for the Cauchy problem connected with the Laplace equation Int. J. Comput. Math. 91 2185–98
[29] Sun Y 2017 Indirect boundary integral equation method for the Cauchy problem of the Laplace equation J. Sci. Comput. 71 469–98
[30] Ton B A 2003 An inverse source problem for the wave equation Nonlinear Anal. 55 269–84
[31] Wang X, Guo Y, Li J and Liu H 2017 Mathematical design of a novel input/instruction device using a moving emitter Inverse Problems 33 105009
[32] Wei T and Zhou D Y 2010 Convergence analysis for the Cauchy problem of Laplace’s equation by a regularized method of fundamental solutions Adv. Comput. Math. 33 491–510
[33] Zhang D and Guo Y 2015 Fourier method for solving the multi-frequency inverse source problem for the Helmholtz equation Inverse Problems 31 035007
[34] Zhang D and Guo Y 2018 Uniqueness results on phaseless inverse scattering with a reference ball Inverse Problems 34 085002
[35] Zhang D, Guo Y, Li J and Liu H 2018 Retrieval of acoustic sources from multi-frequency phaseless data Inverse Problems 34 094001
[36] Zhang D, Guo Y, Li J and Liu H 2019 Locating multiple multipolar acoustic sources using the direct sampling method Commun. Comput. Phys. 25 1328–56