Distributed Robust Control Synthesis for Safety and Fixed-Time Stability in Multi-Agent Systems

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Abstract—This paper presents a control synthesis approach for multi-agent systems modeled as a general class of nonlinear, control-affine dynamics under spatiotemporal constraints. We consider the problem of designing distributed controllers such that each agent reaches their respective goal set within a given fixed time, while always remaining in their respective safe set, which also encodes the maintenance of a safe distance from other agents. We assume that each agent can measure the states of any neighbor agent in their limited sensing radius with some bounded error. In the presence of this uncertainty, we use robust CBFs to encode inter-agent safety, while temporal specifications are encoded via Fixed-Time CLFs. We formulate a QP for each agent to compute its control input that fulfills the spatiotemporal specifications, and discuss the feasibility of the QP in the presence of control input constraints.

I. INTRODUCTION

Multi-agent systems arise in various applications, from distributed sensing and surveillance [1] to distributed load sharing for package transportation [2], [3]. Centralized algorithms, where the control inputs for all agents are computed centrally, typically become intractable as the number of agents increases. Decentralization of computation and control aims to mitigate the scalability issue. Decentralized motion planning approaches in particular, where the main objective is the generation of collision-free trajectories for multiple agents (e.g., unmanned aerial vehicles (UAVs)) to preassigned goal locations under limited sensing, communication, and interaction capabilities, have been studied by many researchers (see e.g., [4], [5]).

Inter-agent collision avoidance in distributed settings is a challenging problem, particularly in the case when the sensing capabilities of the agents are limited (i.e., agents have a finite sensing radius). In [6], the authors consider limited sensing radii for a pair of nonholonomic vehicles under cooperative and non-cooperative interactions. In their majority, earlier work in the literature [6]–[9] assumes perfect knowledge of the states of the agents and no sensing uncertainties. However, from a practical point of view, it is important to consider measurement errors to model for sensing uncertainty, and design robust control inputs that maintain safety. In our earlier work [10], we considered bounded disturbances in the measurement of the states of the neighboring agents, and designed a robust control using vector fields to maintain inter-agent safety. In our and most of the aforementioned work, the agents were assumed to be following linear dynamics [11], [12] or car-like nonlinear dynamics [6], [13]. In this paper, we extend these results for heterogeneous multi-agent systems modeled via a general class of nonlinear, control-affine dynamics.

Control barrier functions (CBFs) have been studied by many researchers to establish forward invariance of safe sets, thereby guaranteeing safety of the system trajectories [14], [15]. Similarly, control Lyapunov functions (CLFs) are used to guarantee convergence of the closed-loop trajectories to a desired state or set of states [14], [16], [17]. The development of fast optimization solvers has enabled online control synthesis using quadratic programs (QPs), where CLF and CBF conditions are encoded as linear constraints, with the objective of minimizing the norm of the control input [9], [14], [15] or its deviation from a nominal controller [18]. Most of the prior work on QP based control considers centralized settings, i.e., unlimited sensing capabilities, and no sensing uncertainty. The authors in [19], [20] use a network-based sensing model where the set of neighbors is fixed for each agent. We do not make such assumptions, and use a finite sensing-radius based model for inter-agent interactions.

This paper studies distributed control synthesis under spatiotemporal specifications, where each agent is required to reach a given goal set in a given fixed time, remain in a safe set for all times, and maintain a minimum safety distance with other agents. The agents are modeled via nonlinear, control-affine dynamics, that can be different among agents, i.e., we consider a heterogeneous set of agents in terms of their dynamical modeling. We assume that each agent has a limited sensing model that is erroneous, in the sense that each agent can measure the states of their neighboring agents within a bounded error. We define the notion of cooperative and non-cooperative neighbors based on whether the neighbor is contributing to maintaining inter-agent safety. We formulate a QP to compute the control input of each agent in a distributed manner, where the safety requirements are encoded as CBF constraints, and the fixed-time convergence requirement is encoded as a fixed-time CLF constraint (see [21] for details on fixed-time CLFs). In the proposed QP, each agent only needs to consider the safety constraints corresponding to the non-cooperative agents, thereby, reducing the problem complexity, and offering scalability with number of agents. In contrast to the work in [14], [18], [22], we discuss feasibility of the proposed QP in the presence of control input constraints, and show that the control input defined as the solution of the proposed QP solves the considered multi-objective problem.
II. Mathematical Preliminaries

A. Notation

In the rest of the paper, \( \mathbb{R} \) denotes the set of real numbers and \( \mathbb{R}_+ \) denotes the set of non-negative real numbers. We use \( ||x|| \) to denote the Euclidean norm of a vector \( x \in \mathbb{R}^n \). \( |x| \) denotes the absolute value when \( x \in \mathbb{R} \), and cardinality, or number of elements, when \( x \in 2^N \) is a set, for some positive integer \( N \). We use \( \partial S \) to denote the boundary of a closed set \( S \) and \( \text{int}(S) \) to denote its interior. The Lie derivative of a continuously differentiable function \( V : \mathbb{R}^n \to \mathbb{R} \) along a vector field \( f : \mathbb{R}^n \to \mathbb{R}^n \) at a point \( x \in \mathbb{R}^n \) is denoted as \( L_f V(x) = \frac{\partial V}{\partial x} f(x) \).

B. Problem formulation

Consider \( N \) agents whose dynamics are governed by

\[
\dot{x}_i(t) = f_i(x_i(t)) + g_i(x_i(t))u_i, \quad (1)
\]

where \( x_i \in \mathbb{R}^n \), \( u_i \in \mathbb{R}^m \) are the state and the control input vectors of agent \( i \), \( f_i : \mathbb{R}^n \to \mathbb{R}^n \) and \( g_i : \mathbb{R}^n \to \mathbb{R}^{n \times m} \) are continuous functions, \( i \in \{1, 2, \ldots, N\} \). Let \( h_{Si} : \mathbb{R}^n \to \mathbb{R} \) be a continuously differentiable function defining the safe set \( S_{Si} = \{ x | h_{Si}(x) \leq 0 \} \) for each agent \( i \), and \( h_{Gi} : \mathbb{R}^n \to \mathbb{R} \) a continuously differentiable function defining its goal set \( G_{Gi} = \{ x | h_{Gi}(x) \leq 0 \} \). We assume that the interior of the safe set of each agent \( i \), \( \text{int}(S_{Si}) = \{ x | h_{Si}(x) < 0 \} \), is non-empty. The problem formulation is stated below.

**Problem 1.** Find a control input \( u_i(t) \in U_i = \{ v \in \mathbb{R}^m; |u_{i,min_j} \leq v_j \leq u_{i,max_j}, j = 1, 2, \ldots, m \} \), \( t \geq 0 \), such that for all \( x_i(0) \in S_{Si} \):

- \( x_i(\tilde{T}) \in S_{Gi} \) for some user-defined \( \tilde{T} > 0 \), for all \( i = 1, 2, \ldots, N \);
- \( ||x_i(t) - x_j(t)|| \geq d_s \), for all \( t \geq 0 \), for all \( i \neq j \), where \( d_s > 0 \) is a user-defined safety distance;
- \( x_i(t) \in S_{Si} \), for all \( t \geq 0 \), for all \( i = 1, 2, \ldots, N \).

Note that the input constraints considered in Problem 1 can be written in compact form as \( U_i = \{ v | A_{ui}v \leq b_{ui} \} \). Each agent \( i \) has a limited sensing radius \( R_s > d_s \) and the set \( \mathcal{N}_i = \{ j ||x_j - x_i|| \leq R_s \} \) denotes the set of neighbors of agent \( i \).

C. Forward invariance of a set

We first review a necessary and sufficient condition for guaranteeing forward invariance of a set \( S_{Si} \), known as Nagumo’s Theorem.

**Lemma 1.** Let the solution of (1) exist and be unique in forward time. Then, for each agent \( i \), the set \( S_{Si} \) is forward invariant for the closed-loop trajectories of (1) for all \( x_i(0) \in S_{Si} \), if and only if the following condition holds:

\[
\inf_{u_i \in U_i} \{ L_{f_i} h_{Si}(x_i(t)) + L_{g_i} h_{Si}(x_i(t))u_i \} \leq 0, \quad \forall x_i \in \partial S_{Si}, \quad (2)
\]

where \( \partial S_{Si} \triangleq \{ x | h_{Si}(x) = 0 \} \) is the boundary of the safe set \( S_{Si} \).

In the related literature, a function that satisfies (2) is called a CBF [14]. Based on this, we make the following assumption so that Problem 1 is feasible.

**Assumption 1.** For each agent \( i \in \{1, 2, \ldots, N\} \), the condition (2) holds, for all \( x_i \in \partial S_{Si} \).

Similar assumptions have been used in literature either explicitly (see e.g. [16]) or implicitly (see e.g. [15]). We also need the following Lemma to prove our main results.

**Lemma 2.** If the initial conditions for an agent \( i \) are such that \( h_{Si}(x_i(0)) < 0 \) and the inequality

\[
\inf_{u_i \in U_i} \{ L_{f_i} h_{Si}(x_i(t)) + L_{g_i} h_{Si}(x_i(t))u_i \} \leq \alpha(-h_{Si}(x_i(t)))
\]

holds for some locally Lipschitz class-\( K^1 \) function \( \alpha \) for all \( t \geq 0 \), then for any \( T \geq 0 \), \( h_{Si}(x_i(t)) < 0 \) for all \( 0 \leq t \leq T \). The proof is provided in Appendix I. We note that the authors in [14] call a function \( h \) a zeroing-CBF for set \( S \) as defined in Lemma 2 if it satisfies (3) for all \( x \in S \).

III. Ideal case: Perfect measurements

We consider that each agent \( i \) has perfect knowledge of its neighbors’ states and their derivatives, i.e., the exact values of \( x_j, \dot{x}_j \) are known to agent \( i \), for all \( j \in \mathcal{N}_i \). Define \( h_{Si} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) as \( h_{Si}(x_i, x_j) = d_s^2 - ||x_i - x_j||^2 \) as the candidate CBF for the set \( S_{Si} \), so that safety of agent \( i \) with respect to any agent \( j \in \mathcal{N}_i \) can be encoded as \( h_{Si}(x_i, x_j) \leq 0 \). We drop the arguments of the functions \( f_i, g_i, h_{Si}, h_{S_{ij}} \) whenever clear from the context for the sake of brevity. We say that \( j \in \mathcal{N}_i \) is a cooperative neighbor to agent \( i \) if it is contributing to maintaining safe distance from \( i \), defined formally as follows.

**Definition 1.** For an agent \( i \in \{1, 2, \ldots, N\} \), an agent \( j \in \mathcal{N}_i \) is termed as a cooperative neighbor if the control input \( u_j \in U_j \) is such that the following holds:

\[
-2(x_j - x_i)^T \dot{x}_j \leq \alpha_c(-h_{S_{ij}}) + 2(x_i - x_j)^T \dot{x}_i,
\]

for all \( x_i, x_j \), for some \( \alpha_c \in \mathcal{K} \).

If there does not exist \( u_j \in U_j \) for an agent \( j \in \mathcal{N}_i \) such that (4) holds, then the agent \( j \) is termed as non-cooperative agent to agent \( i \). We denote the set of cooperative neighbors of agent \( i \) as \( \mathcal{N}^c_i \subseteq \mathcal{N}_i \) and non-cooperative neighbors as \( \mathcal{N}^n_i = \mathcal{N}_i \setminus \mathcal{N}^c_i \). Since the agent \( i \) has access to \( x_j, \dot{x}_j \) for \( j \in \mathcal{N}_i \), agent \( i \) can verify (4) to assess whether \( j \in \mathcal{N}_i \) is a cooperative neighbor or not. Later, when we formulate a QP to compute a control input to solve Problem 1, we make use of this information to exclude the cooperative neighbors from the safety constraints in the QP, which would help reduce the complexity of the problem and the computational cost of solving the QP.

We now present the first result that gives sufficient conditions using the candidate CBFs for inter-agent safety.

3Recall that a function \( \alpha \in \mathcal{K} \) if it is continuous, strictly increasing, and \( \alpha(0) = 0 \) (see [23], Definition 4.2 for more details).
Lemma 3. Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ be a class-\(\mathcal{K}\) function. If the holds

$$\inf_{u_i \in \mathcal{U}_i} \{ L_{f_i} h_{s_i} + L_{g_i} h_{s_i} u_i \} \leq \alpha(-h_{s_i}) - 2(x_i - x_j)^T \delta_j,$$

(5)

for all $j \in N_i^\alpha$, then there exists a control input $u_i \in \mathcal{U}_i$ such that the closed-loop trajectories satisfy $\|x_i(t) - x_j(t)\| \geq d_s$ for all $j \in N_i^\alpha$, for all $t \geq 0$.

Proof. For $j \in N_i^\alpha$, the time derivative of $h_{s_{ij}}$ reads

$$\dot{h}_{s_{ij}} = L_{f_i} h_{s_{ij}} + L_{g_i} h_{s_{ij}} u_i + 2(x_i - x_j)^T \delta_j.$$

Let $a_{ij} \triangleq L_{f_i} h_{s_{ij}} + L_{g_i} h_{s_{ij}} u_i$. Using (5), we obtain that there exists $u_i \in \mathcal{U}_i$ such that $\dot{h}_{s_{ij}} = a_{ij} + 2(x_i - x_j)^T \delta_j \leq \alpha(-h_{s_i})$. Thus, we have that $h_{s_{ij}}(x_i(t), x_j(t)) \leq 0$ for all $t \geq 0$, and so, $\|x_i - x_j\| \geq d_s$ for all $t \geq 0$ for all $j \in N_i^\alpha$.

Next, we discuss how to incorporate multiple safety constraints in a single QP formulation, and discuss its feasibility. Before presenting the main result, we need the following lemma.

Lemma 4. For each agent $i$, let $x_i(0) \in \text{int}(S_{S_i})$ be such that $\|x_i(0) - x_j(0)\| > d_s$ for all $j \in N_i^\alpha$. If there exists a continuous control input $u_i \in \mathcal{U}_i$ such that the following holds

$$L_{f_i} h_{s_i} + L_{g_i} h_{s_i} u_i \leq \alpha_1(-h_{s_i}),$$

(6)

$$L_{f_i} h_{s_i} + L_{g_i} h_{s_i} u_i \leq \alpha_2(-h_{s_i}),$$

(7)

for all $x_i \in S_{S_i}$, and for all $j \in N_i^\alpha$, where $\alpha_1$, $\alpha_2 \in \mathcal{K}$, then, under the effect of control input $u_i$, for any $T \geq 0$, the closed-loop trajectories of each agent $i$ satisfy $x_i(t) \in \text{int}(S_{S_i})$ and $\|x_i(t) - x_j(t)\| > d_s$ for all $t \leq T$, for all $j \in N_i^\alpha$.

The proof follows directly from Lemma 2. Lemma 4 guarantees that starting from the interior of the intersection of the safe sets, the closed-loop trajectories remain inside the intersection of the interior of the safe sets at all times. Now we are ready to present the QP to compute a control input $u_i$ for each agent $i$ that solves Problem 1. We can use the result of Lemma 3 to formulate constraints for inter-agent safety for the two classes of neighbors. Let $N_i^\alpha = |N_i^\alpha|$ denote the number of non-cooperative neighbors of agent $i$. Define $z_i = [v_i^T \; \delta_{i1} \; \delta_{i2} \; \{\delta_{ij}\}]^T \in \mathbb{R}^{|N_i^\alpha| + 2 + N_i^\alpha}$, and consider the following optimization problem

$$\min_{v_i, \delta_{i1}, \delta_{i2}, \{\delta_{ij}\}, z_i} \frac{1}{2} T H_i z_i + F_i^T z_i,$$

(8a)

subject to

$$A_{i1} v_i \leq b_{i1},$$

(8b)

$$L_{f_i} h_{G_i} + L_{g_i} h_{G_i} v_i \leq \delta_{i1} h_{G_i} - \alpha_1 \max(0, h_{G_i}) \gamma_{i1} - \alpha_2 \max(0, h_{G_i}) \gamma_{i2},$$

(8c)

$$L_{f_i} h_{S_i} + L_{g_i} h_{S_i} v_i \leq \delta_{i2} h_{S_i},$$

(8d)

$$L_{f_i} h_{s_{i,j}} + L_{g_i} h_{s_{i,j}} v_i \leq -\delta_{ij} h_{s_{i,j}} - 2(x_i - x_j)^T \delta_{j},$$

(8e)

where $H_i = \text{diag}\{w_{i,j}^T, w_{i,j}, w_{i,j}^T, \{w_{i,j}^T\}\}$ is a diagonal matrix consisting of positive weights $w_{i,j}, w_{i,j}, w_{i,j}^T, w_{i,j}^T > 0$ for each $p \in N_i \setminus N_i^\alpha$, $F_i = [\Omega_i q^T \; 0 \; 0_{N_i^\alpha}]$ with $q^T > 0$ and $\Omega_i \in \mathbb{R}^{q^T \times q}$ a column vector consisting of zeros. The parameters $\alpha_1$, $\alpha_2$, $\gamma_{i1}$, $\gamma_{i2}$ are fixed, and are chosen as $\alpha_1 = \frac{\mu_1}{2T}$, $\gamma_{i1} = 1 + \frac{1}{\mu_1}$ and $\gamma_{i2} = 1 - \frac{1}{\mu_1}$ with $\mu_1 > 1$. Below we explain what each constraint in the QP (8) encodes:

- (8b): control input constraint;
- (8c): fixed-time convergence of the closed-loop trajectories to the goal set $S_{G_i}$ [21, Theorem 6];
- (8d): forward-invariance of safe set $S_{S_i}$ (Lemma 1);
- (8e): safety with respect to non-cooperative agents (Lemma 3).

Remark 1. The QP (8) only considers inter-agent constraints corresponding to the non-cooperative neighbors. In practice, this number may be smaller compared to the total number of neighbors of an agent, and so, the number of the optimization variables in, and thereby the order of, the QP (8) may be lower than the one obtained from the approach in [9, 18]. Thus, the proposed QP offers scalability with the number of agents, as well as a structure that allows for solving for the control input faster as compared to the structure of the QPs in similar earlier work.

The constraints of the QP change every time that an agent $j$ either enters or leaves the sensing region of another agent $i$. For sake of brevity, and slight abuse of notation, we refer to these time instants, denoted as $t_{i,a}^j$, $l \geq 1$, as switching instants, given that the constraints of the QP change (“switch”) depending on the formed neighbor interactions. Thus, the solution of the QP (8) may not be continuous for all times, in general. Let $t_{i,a}^j < t_{i,a+1}^j$ be any two consecutive switching time instants such that the set of neighbors $N_i$ is constant for all $t_{i,a}^j \leq t \leq t_{i,a+1}^j$. Continuity of the solution of the QP (8) in $[t_{i,a}^j, t_{i,a+1}^j]$ for an agent $i$ can still be guaranteed under some conditions (see Remark 3). We are now ready to present our first main result. Let the solution of (8) be denoted as $z_i^*(\cdot) = [v_i^*(\cdot)^T \; \delta_{i1}^*(\cdot) \; \delta_{i2}^*(\cdot) \; \{\delta_{ij}^*(\cdot)\}]^T$.

Theorem 1. The following holds for each agent $i$:

(i) The QP (8) is feasible for all $x_i \in S_{S_i} \setminus S_{G_i}$, such that $\|x_i - x_j\| > d_s$ for all $j \in N_i^\alpha$.

(ii) If the solution $z_i^*$ is piecewise continuous in its arguments, then for all $x_i(0) \in S_{S_i}$ such that $\|x_i(0) - x_j(0)\| > d_s$ for all $j \in N_i^\alpha$, the closed-loop trajectories of (1) under the effect of control input $u_i = v_i^*$ satisfy $x_i(t) \in S_{S_i}$ and $h_{s_{i,j}}(x_i(t), x_j(t)) < 0$, i.e., $\|x_i(t) - x_j(t)\| > d_s$ for all $j = 1, 2, \ldots, N$, for all $t \geq 0$ such that $x(t) \notin S_{G_i}$.

(iii) If $\max_{0 \leq \tau \leq T} \delta_{ij}(x_i(\tau)) \leq 0$, then the control input defined as $u_i = v_i^*$ guarantees convergence of the closed-loop trajectories to the goal set $S_{G_i}$, within time $T$, i.e., the control input $u_i = v_i^*$ solves Problem 1.

Proof. Part (i): Since $x_i \notin S_{G_i}$ and $\|x_i - x_j\| > d_s$ for all $j \in N_i^\alpha$, we have that $h_{s_{i,j}}(x_i), h_{G_i}(x_i) \neq 0$. We consider the cases when $x_i \in \partial S_{S_i}$ and $x_i \in \text{int}(S_{S_i})$ separately.

For $x_i \in \text{int}(S_{S_i})$, we have that $h_{G_i}(x_i) \neq 0$. Since the control input constraint set $U_i$ is non-empty, we can choose $v_i = \hat{v}_i \in U_i$. Then, we can choose $\delta_{ij} =$
Before proceeding to the case with with measurement errors, we provide some discussion on Theorem 1.

**Remark 1.** Theorem 1 guarantees that starting from the intersection of the interiors of the inter-agent safety sets, all the agents remain in the interior of these sets, which, with the help of slack variables in (8d)-(8e), guarantees recursive feasibility of the QP. The case when the trajectories of agents start on the intersection of the boundaries of more than one safe sets, i.e., \( \exists j, k \in N_i, j \neq k \) such that \( \|x_i - x_j\| = \|x_i - x_k\| = d_s \), is much more involved, requires stronger viability assumptions, and is left for future work.

**Remark 3.** We impose continuity requirements on the solution of the QP (8) because the traditional Nagumo’s viability theorem require uniqueness of the solution of the dynamical system in order to guarantee forward invariance of a set. Some of the prior work e.g., [9], [14], [21] discusses conditions under which the solution of QPs are continuous, or even Lipschitz continuous. More recently, utilizing the concept of strong invariance and tools from non-smooth analysis (see [24]), the authors in [25] discuss forward invariance of a set requiring that the control input is only measurable and locally bounded.

**Remark 4.** Note that the result in part (iii) of Theorem 1 requires \( \sigma_{i1} \leq \sigma_2 \) so that the control input \( u_i \) solves the convergence requirement of Problem 1. When this condition does not hold, the closed-loop trajectories, while still satisfying safety requirements, may not converge to goal set within the required time \( T \), or from any arbitrary initial condition \( x_i(0) \notin S_G \), (see [21] for a detailed discussion on the matter). This inherently implies that the agents might encounter a deadlock situation. The authors in [26] characterize various types of deadlocks depending upon the value of the slack variables in the optimization problem, and discuss methods of resolving some of the deadlock situations. While important from the practical point of view, this analysis is out of the scope of this paper, and is left open for future investigation.

**IV. Realistic case: imperfect measurements**

In this section, we relax the assumption that each agent has perfect knowledge of its neighbors’ states and their time derivatives. Rather, we consider that each agent obtains an estimate of the states of its neighbors \( j \in N_i \), and their derivatives, denoted as \( \hat{x}_i \) and \( \hat{x}_j \), respectively. We make the following assumption.

**Assumption 2.** There exists \( 0 \leq \epsilon_1 \ll d_s \), such that \( \|x_j - \hat{x}_j\| \leq \epsilon_1 \) holds for all \( j \in N_i \) and all \( i \in \{1, 2, \ldots, N\} \). We also need the following assumption on the dynamics of each agent \( i \), as well as on the boundedness on the rate of change of their states.

**Assumption 3.** The functions \( f, g \) are continuously differentiable and Lipschitz continuous with Lipschitz constants \( L_F, L_G > 0 \), i.e.,

\[
\left| \frac{\partial f_i}{\partial x}(x) \right| \leq L_F, \quad \left| \frac{\partial g_i}{\partial x}(x) \right| \leq L_G,
\]

for all \( i = 1, 2, \ldots, N \). Also, there exists \( \epsilon_2 > 0 \) such that the \( \|x_i(t)\| \leq \epsilon_2 \) for all \( t \geq 0 \), for all \( i = 1, 2, \ldots, N \).
Next, we present a robust CBF formulation to guarantee inter-agent safety in the presence of measurement errors.

A. Robust Inter-agent Safety

Let \( d_{ij} = \|x_i - x_j\| \) denote the actual distance between agents \( i \) and \( j \), and \( \bar{d}_{ij} = \|x_i - \hat{x}_j\| \) denote the distance perceived by agent \( i \) with respect to its neighbor agent \( j \) in \( \mathcal{N}_i \). Note that while \( d_{ij} = \bar{d}_{ij} \), one in general has \( d_{ij} \neq \bar{d}_{ij} \).

Using the triangle inequality along with Assumption 2, one obtains \( d_{ij} - \epsilon_1 \leq \bar{d}_{ij} \leq d_{ij} + \epsilon_1 \). Define \( \hat{h}_{S_{ij}}(x_i(t), \hat{x}_j(t)) = (d_{ij} + \epsilon_1)^2 - \|x_i(t) - \hat{x}_j(t)\|^2 \), for all \( i, j = 1, 2, \ldots, N \), \( i \neq j \) as the candidate for a robust CBF for inter-agent safety. We present a sufficient condition for inter-agent safety using the functions \( \hat{h}_{S_{ij}} \).

**Lemma 5.** Let \( i \in \{1, 2, \ldots, N\} \) and \( j \in \mathcal{N}_i \). If \( \hat{h}_{S_{ij}}(x_i, \hat{x}_j) \leq 0 \) (respectively, < 0), then \( h_{S_{ij}}(x_i, x_j) \leq 0 \) (respectively, < 0).

**Proof.** Using \( \hat{h}_{S_{ij}}(x_i, \hat{x}_j) \leq 0 \), we obtain that \((d_{ij} + \epsilon_1)^2 - \|x_i - \hat{x}_j\|^2 \leq 0 \), i.e., \( \|x_i - \hat{x}_j\| \geq d_{ij} + \epsilon_1 \). Using the triangle inequality and Assumption 2, we have that \( \|x_i - \hat{x}_j\| \leq \|x_i - x_j\| + \|x_j - \hat{x}_j\| \leq \|x_i - x_j\| + \epsilon_1 \), thus, we obtain that \( d_{ij} + \epsilon_1 \leq \|x_i - \hat{x}_j\| \leq \|x_i - x_j\| + \epsilon_1 \), which implies that \( \|x_i - x_j\| \geq d_{ij} \), i.e., \( h_{S_{ij}}(x_i, x_j) \leq 0 \). The proof for the strict inequality follows immediately.

Lemma 5 states that \( \hat{h}_{ij} \) being negative (respectively, non-positive) implies that \( h_{ij} \) is negative (respectively, non-positive), or equivalently, agent \( i \) is at a safe distance from agent \( j \). In order to develop results similar to Lemma 3, we need the notion of cooperative agents. In the presence of measurement errors, we define the notion of *robustly cooperative agent* as follows.

**Definition 2.** For an agent \( i \in \{1, 2, \ldots, N\} \), an agent \( j \in \mathcal{N}_i \) is termed as a *robustly cooperative* the following holds

\[
2(x_i - \hat{x}_j)^T \hat{x}_j \leq \alpha(-\hat{h}_{S_{ij}}) - 2(x_i - \hat{x}_j)^T \hat{x}_i
\]

for all \( x_i, \hat{x}_j \), for some \( \alpha \in \mathbb{R} \).

**Remark 5.** Note that in the absence of sensing errors, i.e., \( \hat{x}_j = x_j \), (9) results into (4), and so, Definition 1 is a special case of Definition 2 with \( \epsilon_1 = 0 \).

With slight abuse of notation, we denote the set of robustly cooperative, and non-cooperative agents as \( \mathcal{N}_C \) and \( \mathcal{N}_P \), respectively, in the rest of the paper. Next, we present robust version of Lemma 3 using the robust CBF candidates \( \hat{h}_{ij} \).

Define \( \hat{a}_{ij}(u_i) = L_{ij} \hat{h}_{S_{ij}} + L_{ij} \hat{h}_{S_{ij}} u_i \).

**Lemma 6.** Consider an agent \( i \in \{1, 2, \ldots, N\} \) and \( j \in \mathcal{N}_i \). If the following holds

\[
\inf_{u_i \in \mathcal{U}_i} \bar{a}_{ij}(u_i) \leq \alpha(-\hat{h}_{S_{ij}}) - 2(x_i - \hat{x}_j)^T \hat{x}_j,
\]

for all \( i \in \mathcal{N}_P \), then there exists a control input \( u_i \in \mathcal{U}_i \) such that the closed-loop trajectories satisfy \( \|x_i(t) - x_j(t)\| \geq d_{ij} \) for all \( j \in \mathcal{N}_C \), for all \( t \geq 0 \).

**Proof.** The time derivative of \( \hat{h}_{S_{ij}} \) reads

\[
\dot{\hat{h}}_{S_{ij}} = L_{ij} \hat{h}_{S_{ij}} + L_{ij} \hat{h}_{S_{ij}} u_i + 2(x_i - \hat{x}_j)^T \hat{x}_j.
\]

For \( j \in \mathcal{N}_P \) from (10), we have that there exists \( u_i \in \mathcal{U}_i \) such that \( \hat{h}_{S_{ij}} = a_{ij} + 2(x_i - \hat{x}_j)^T \hat{x}_j \leq \alpha(-\hat{h}_{S_{ij}}) \). Thus, we have that

\[
\dot{\hat{h}}_{S_{ij}}(x_i(t), \hat{x}_j(t)) \leq \alpha \left( -\hat{h}_{S_{ij}}(x_i(t), \hat{x}_j(t)) \right),
\]

which implies that \( \hat{h}_{S_{ij}}(x_i(t), \hat{x}_j(t)) \leq 0 \) for all \( t \geq 0 \), for all \( j \in \mathcal{N}_P \). Hence, using Lemma 5, we have that \( \|x_i(t) - x_j(t)\| \geq d_{ij} \) for all \( j \in \mathcal{N}_P \) and for all \( t \geq 0 \).

B. QP based control design

We present a QP formulation to compute the control input \( u_i \) for each agent \( i \) in order to solve Problem 1. Consider the following optimization problem:

\[
\min_{v_i, s_i, \delta_{ij}((s_{ij}), (s_{ij}))} \frac{1}{2} x_i^T H_i x_i + F_i^T x_i
\]

s.t.

\[
L_i h_{C_i} + L_i h_{G_i} v_i \leq \delta_{ij} h_{G_i} - \alpha_1 \max(0, h_{G_i}) \gamma_1 - \alpha_2 \max(0, h_{G_i}) \gamma_2,
\]

\[
L_i \hat{h}_{s_i} + L_i \hat{h}_{s_i} \hat{v}_i \leq -\bar{d}_{ij} \hat{s}_{ij},
\]

\[
L_i \hat{h}_{s_{ij}} + L_i \hat{h}_{s_{ij}} v_i \leq -\delta_{ij} \hat{h}_{s_{ij}} - (x_i - \hat{x}_j)^T \hat{x}_j,
\]

\[j \in \mathcal{N}_i,\]

where all the parameters are defined as in (8). Let the solution of the QP (11) be denoted as \( \hat{z}_i \). The main result of this section can now be stated.

**Theorem 2.** The following holds for each agent \( i \):

(i) The QP (11) is feasible for all \( x_i \in S_{S_i} \setminus S_{G_i} \) such that \( \|x_i - x_j\| > d_{ij} + 2\epsilon_1 \) for all \( j \in \mathcal{N}_i \);

(ii) If the solution \( \hat{z}_i \) is piecewise continuous in its arguments, then for all \( x_i(0) \in S_{S_i} \) such that \( \|x_i(0) - x_j(0)\| > d_{ij} + 2\epsilon_1 \) for all \( j \in \mathcal{N}_i \), the closed-loop trajectories of (1) under the effect of control input \( u_i = \hat{v}_i \) satisfy \( x_i(t) \in S_{S_i} \) and \( h_{S_{ij}}(x_i(t), x_j(t)) < 0 \) for all \( i = 1, 2, \ldots, N \), for all \( t \geq 0 \) such that \( x(t) \in S_{G_i} \);

(iii) If \( \max_{0 \leq t \leq T} \delta_{ij}(x_i(t)) \leq 0 \), then the control input defined as \( u_i = \hat{v}_i \) solves Problem 1.

Note that \( d_{ij} + 2\epsilon_1 < \|x_i - x_j\| \leq \|x_i - \hat{x}_j\| + \epsilon_1 \), which implies that \( \|x_i - x_j\| > \|x_i - \hat{x}_j\| + \epsilon_1 \), i.e., \( h_{S_{ij}} < 0 \). Using this and the similar set of arguments as in the proof of Theorem 1, the proof can be provided, and hence, it is omitted here.

V. SIMULATION RESULTS

Similar to [25], we consider a set of \( N \) agents modelled via single integrator dynamics

\[
\dot{x}_i = u_i,
\]

where \( x_i, u_i \in \mathbb{R}^2 \), with the parameters for Problem 1 chosen as: \( N = 10, R_c = 1.5, d_s = 0.5, T = 5 \).2 The

2As argued in [25], the linear single-integrator dynamics can be locally mapped into the nonlinear unicycle model. For the sake of simplicity, we consider single-integrator dynamics in the simulations. However the consideration of more complicated nonlinear dynamics is part of our ongoing work.
initial locations of the agents are chosen randomly in a neighborhood of a circle of radius 10, and the goal locations are chosen as \( x_{gi} = x_{10-i}(0) \), i.e., antipodal agents are tasked to exchange their positions. The choice of initial conditions and goal locations also ensures maximum inter-agent interaction. The input constraints are considered as \( \mathcal{U} = \{ u = [u_1, u_2]^T | |u_1| \leq 10, |u_2| \leq 10 \} \), for all \( i = 1, 2, \ldots, 10 \). We choose \( \epsilon = 0.2 \) as the measurement error bound for each agent, and \( T = 5 \) as the required time of convergence. We also considered individual safety constraints for each agent \( i \), given as \( h_{si} \leq 0 \), where \( h_{si} \) is an equation of ellipse, centered at the origin, such that the initial and the goal location of agent \( i \) are inside this ellipse. Figure 1 shows the boundary of the safe set for agent 1, 4, 8 and 10.

Figure 1 plots the paths traced by the agents for one set of initial conditions. The circular markers represent the initial conditions while star markers denote the goal locations of the respective agents. It can be seen that the agents reach their goal locations.

Figure 2 shows the variation of the control input for each agent with time for the paths in Figure 1. It can be seen that the control inputs of all the agents satisfy the input constraint at all times.

Figure 3 plots the inter-agent distance at every time instant, where solid lines plot the actual minimum inter-agent distance, and red dotted line depicts the safe distance \( d_s \). It is evident that the agents are able to maintain the safe distance from each other, even in the presence of the measurement errors.

We studied the effect of the number of non-cooperative agents in the environment. Figure 4 plots the sum total of the path lengths traced by agents for various number of non-cooperative agents \( N_{nc} \) among themselves. It can be seen that the total path length increase with increase in the number of non-cooperative agents, before saturating to a maximum value.

Figure 5 plots minimum inter-agent distance at each time instance for various values of \( \epsilon \in [0.05, 0.5] \) (\( \epsilon \) increases from blue to red). It can be seen that the inter-agent safety is maintained for all considered values of \( \epsilon \). In conclusion, the simulation results demonstrate that the proposed QP solves the considered multi-agent motion planning involving safety and convergence requirements, under the effect of both input constraints and measurement uncertainties.
We considered a multi-objective motion planning problem for distributed multiagent systems where the objectives for each agent include maintaining safe distance from other agents, remaining in a safe set, and reach a goal set within a fixed time. We also considered input constraints for each agent, as well measurement errors in the agent’s limited sensing. We defined the notion of cooperative and non-cooperative neighbors, and formulated a QP, incorporating safety and convergence constraints using slack variables so that its feasibility is guaranteed. The notion of cooperative neighbors is utilized to reduce the number of safety constraints so that the proposed method is scalable with number of agents. Finally, we showed that under certain conditions, control input defined as the solution of the proposed QP solves the multi-objective problem, even in the presence of measurement errors.

As mentioned in Remark 3, the requirement of continuity of the solution of the QP can be relaxed by utilizing tools from nonsmooth analysis. We would like to use these tools in the context of the problem considered in this paper, and study the minimal set of requirements that can guarantee forward-invariance of safe sets in a distributed multiagent setting.

VI. Conclusion

We considered a multi-objective motion planning problem for distributed multiagent systems where the objectives for each agent include maintaining safe distance from other agents, remaining in a safe set, and reach a goal set within a fixed time. We also considered input constraints for each agent, as well measurement errors in the agent’s limited sensing. We defined the notion of cooperative and non-cooperative neighbors, and formulated a QP, incorporating safety and convergence constraints using slack variables so that its feasibility is guaranteed. The notion of cooperative neighbors is utilized to reduce the number of safety constraints so that the proposed method is scalable with number of agents. Finally, we showed that under certain conditions, control input defined as the solution of the proposed QP solves the multi-objective problem, even in the presence of measurement errors.

As mentioned in Remark 3, the requirement of continuity of the solution of the QP can be relaxed by utilizing tools from nonsmooth analysis. We would like to use these tools in the context of the problem considered in this paper, and study the minimal set of requirements that can guarantee forward-invariance of safe sets in a distributed multiagent setting.

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APPENDIX I
PROOF OF LEMMA 2

Proof. Consider the differential equation
\[ \dot{y} = -\alpha(y), \quad y(0) = y_0 = -h_{S_i}(x_i(0)) > 0. \] (12)

Using [23, Lemma 4.4], we have that the solution of (12) are unique. Since \( \alpha \) is locally Lipschitz, we have that
\[ |y(t)| \geq |y(0)|e^{-Lt}, \]
where \( L \) is Lipschitz constant for \( \alpha \). Thus, for any \( 0 < \epsilon < y_0 \), the following inequality holds
\[ |y(t)| \geq |y(0)|e^{-LT} \geq y(0)e^{-LT} = \epsilon \] (13)
for all \( t \leq T \) where \( T = T(\epsilon) = \frac{1}{L} \log \frac{y_0}{\epsilon} > 0 \). Thus, for all \( t \leq T \), we have that \( |y(t)| \geq \epsilon > 0 \). Since \( y_0 > 0 \), we have that \( |y(t)| = y(t) \geq \epsilon \). Thus, for \( z(t) = -y(t) \), we have that \( z(t) \leq -\epsilon \) for all \( t \leq T \). We also have that
\[ \dot{z} = -\dot{y} = \alpha(y) = \alpha(-z). \] (14)

Comparing (14) with \( h_{S_i} \leq \alpha(-h_{S_i}) \) with \( z(0) = -y(0) = h_{S_i}(x_i(0)) \) and using Comparison Lemma, we have that
\[ h_{S_i}(x_i(t)) \leq z(t) \leq -\epsilon < 0, \]
for all \( 0 \leq t \leq T \), which completes the proof. \( \blacksquare \)