Neutrino mass characteristics in a phenomenological $3 + 2 + 1$ model

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The basic characteristics used for the description of both Dirac and Majorana massive neutrinos are studied. Currently available experimental data for these characteristics are presented with an evidence of possible anomalies beyond the Standard Model with three light neutrinos. Special attention has been paid to ways to determine the neutrino mass absolute scale. In accordance with the available data, the permissible values of the neutrino mass matrix elements are found numerically against the minimal mass of neutrino. Some phenomenological relations for the masses, the angles and the $CP$-violating phases of the neutrino mixing matrix are discussed, and the values of the neutrino mass characteristics and the Dirac $CP$-violating phase are evaluated for a model of bimodal neutrinos. The estimations made for masses of sterile neutrinos, the values of the neutrino mass characteristics and the Dirac $CP$-violating phase can be used for interpretation and prediction of the results of various neutrino experiments.

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I. INTRODUCTION

During the past years, a number of new interesting experimental results were obtained in the neutrino physics. To such results, one can refer the deviation from zero (more than 5$\sigma$) of the reactor mixing angle $\theta_{13}$ [1] and the experimental indications on possible existing of new anomalies for the number of neutrinos and antineutrinos in different processes [2]. The first result is important for determination of both the neutrino mixing matrix and the neutrino mass matrix [3], and also for justification of the experimental search of $CP$ violation in the lepton sector [4]. As is known, the experimental data confirming lepton $CP$ violation are absent to the present. The second result can be connected with existence of light sterile neutrinos, to which one can refer, for example, SU(2)$_L$ singlet right neutrinos. The existence of right neutrinos is beyond the Standard Model (SM) of weak and electromagnetic interactions. However, the discovery of oscillations of atmospheric, solar, reactor and accelerator neutrinos, as well as extremely smallness of the neutrino mass definitely points to violation of the conservation laws of the leptonic numbers $L_e$, $L_\mu$, and $L_\tau$ and existence of right neutrinos.

One of the unsettled outstanding problem in the neutrino physics is the question about the neutrino nature, that is the question, does neutrino belong to the Dirac or the Majorana particle type? These two types of elementary particles were introduced in the particle physics in 1928 by Dirac [5] and in 1937 by Majorana [6], respectively. In the present, considerable efforts by both experimentalists and theorists are concentrated to solve this problem. In this, it can be unexpected the lack of unambiguous answer. For example, neutrinos can simultaneously possess by both the Dirac and the Majorana properties [7–10]. One of the models, in which such situation is realized, is the model of bimodal (schizophrenic) neutrinos [11–13].

The other question requiring its speedy solution is the question about the absolute scale of neutrino masses. As is known, from the oscillation experiments with neutrinos it is possible to determine the absolute values of the differences of the neutrino mass in squares, rather than the mass values themselves. This circumstance gives rise to the problem of the neutrino mass hierarchy, as well as to the problem of the absolute mass scale. In spite of these problems can be distinctly solved only with using the results of the future special experiments, the different variants of their solutions in the framework of the phenomenological models are also proposed [14–17].

In the current paper, a phenomenological $3 + 2 + 1$ neutrino model is proposed for solution of the problems in the neutrino physics noted above. This model involves three active light neutrinos and also three sterile neutrinos, from which one sterile neutrino is comparatively heavy, but two other are light sterile neutrinos [18]. It is important that light sterile neutrinos may be practically degenerate by mass with light active neutrinos, so that they can jointly form quasi Dirac neutrinos, thus realizing the bimodal neutrino states with two Majorana and two quasi Dirac neutrinos. On the basis of available experimental data, the acceptable ranges for neutrino mass characteristics in this model are found numerically. Allowing for the cosmological restrictions on total number of neutrinos [19], it is easy to reduce the number of sterile neutrinos in the framework of the model under consideration, that is to pass from the $3 + 2 + 1$ model to the $3 + 1 + 1$, or even the $3 + 1$ model. However, such a reduction should be caused by forcible reasons. In this connection, note that the experimentally obtained cosmolog-
The determination of the neutrino mass absolute values and the mixing parameters, as well as ascertainment of the neutrino type (Dirac vs Majorana) are at present the basic problems of the neutrino physics. Solutions of these problems require active experimental and theoretical studies of both the neutrino mass observables, which define the absolute mass scale, and the neutrino oscillation characteristics, which characterize mixing of the neutrino states with different masses. It is expected that the comprehensive solution of the problem of the neutrino nature, as well as of the neutrino masses and mixing parameters will be given in the future Grand Unification Theory (GUT), which does not exist at present in its conventional form. A required direction of the further development of the SM can also be found by study of different correlations between experimental values of the neutrino masses and mixing parameters, to which we refer the angles of mixing and the CP-violating phases.
The discovery of such interrelationships can play an important role in finding out the ways to expand the SM and to successfully develop at last the following consistent GUT. For this, the data received in the current and the future experiments on determination the neutrino characteristics (PLANK, KATRIN, GERDA, CUORE, BOREXINO, Double CHOOZ, SuperNEMO, KamLand-Zen, EXO, etc.) will undoubtedly play a decisive role.

With using the only oscillation experiments with atmospheric, solar, reactor and accelerator neutrinos it is impossible to determine the neutrino mass absolute values, as well as the type, either Majorana or Dirac, of the neutrino. However, the experimental data obtained in the neutrino oscillation experiments indicate the violation of the conservation laws of the leptonic numbers $L_e$, $L_\mu$, $L_\tau$, and, besides, by virtue of deviation from zero of two oscillation parameters $\Delta m_{12}^2$ and $\Delta m_{13}^2$ (with $\Delta m_{ij}^2 = m_i^2 - m_j^2$) they indicate the existence at least two nonzero and different neutrino masses. Below we present the experimental values of the mixing angles and the oscillation parameters, which determine three-flavor oscillations of the light neutrinos. Together with the standard uncertainties on the level of $1\sigma$, these data obtained as a result of a global analysis of the latest high-accuracy measurements of the oscillation parameters [1] are as follows

\begin{align}
\sin^2 \theta_{12} &= 0.307^{+0.018}_{-0.016}, \\
\sin^2 \theta_{23} &= \begin{cases} 
NH : & 0.386^{+0.024}_{-0.025} \\
IH : & 0.392^{+0.021}_{-0.022}
\end{cases}, \\
\sin^2 \theta_{13} &= \begin{cases} 
NH : & 0.0241^{+0.0025}_{-0.0023} \\
IH : & 0.0244^{+0.0025}_{-0.0023}
\end{cases}, \\
\Delta m_{21}^2 / 10^{-5} \text{eV}^2 &= 7.54^{+0.26}_{-0.22}, \\
\Delta m_{31}^2 / 10^{-3} \text{eV}^2 &= \begin{cases} 
NH : & 2.47^{+0.06}_{-0.10} \\
IH : & -2.49^{+0.07}_{-0.11}
\end{cases}.
\end{align}

Since only the absolute value of $\Delta m_{31}^2$ is known, it is possible to arrange the absolute values of the neutrino masses by two ways, namely, as

a) $m_1 < m_2 < m_3$ and b) $m_3 < m_1 < m_2$.

These two cases correspond to so called the normal hierarchy (NH) and the inverse hierarchy (IH) of the neutrino mass spectrum, respectively. Unfortunately, the $CP$-violating phases $\alpha$, $\beta$ and $\delta$, as well as the neutrino mass absolute scale are unknown at present.

Three groups of the experimental data associated with the neutrino are sensitive to the neutrino absolute mass scale, namely, they are the data on $\beta$ decay, the data on neutrinoless double-$\beta$ decay, and the data obtained as a result of the cosmological observations. So, to determine the neutrino absolute mass scale it is necessarily to determine experimentally at least one from three mass observables of the neutrino, namely, either the mean cosmological mass $m_\alpha$ of the active neutrinos, or the $\beta$ decay neutrino mass $m_\beta$, or the effective double-$\beta$ decay neutrino mass $m_{\beta\beta}$, which are defined as follows

\begin{align}
m_\alpha &= \frac{1}{3} \sum_{i=1,2,3} |m_i|, \\
m_\beta &= \left( \sum_{i=1,2,3} |U_{ei}|^2 m_i^2 \right)^{1/2}, \\
m_{\beta\beta} &= \left| \sum_{i=1,2,3} U_{ei}^2 m_i \right|,
\end{align}

where $U_{ei}$ are the elements of the Pontecorvo–Maki–Nakagawa–Sakata matrix. Generally, we will call both the neutrino mass observables and the neutrino masses as the mass characteristics of the neutrino. With help of the corresponding data from these three groups of the experiments indicated above, three mass observables $m_\alpha$, $m_\beta$, and $m_{\beta\beta}$ one by one can be determined, respectively. Currently, the upper limits are only obtained for the mass observables, namely, $m_\alpha < 0.2$ eV [19], $m_\beta < 2.2$ eV [22], and $m_{\beta\beta} < 0.35$ eV [23, 24], where in the last case the limit should be increased approximately one and a half or even twice to take into account the uncertainties in the nuclear matrix elements governing this limitation. Note that the limit for $m_\beta$ given above, which was obtained in the experiments carried out in Troitsk and Mainz on the electron spectrum measurements in the tritium $\beta$ decay, is planned to be improved up to 0.2 eV in the scheduled KATRIN experiment [22].

As we know, the right neutrinos are sterile particles and are the candidates, along with the other possibilities, to the specific particles of the dark matter. There can be several types of the dark matter particles. We consider only the right neutrinos $\nu_R$. If $\nu_R$ together with the $\nu_L$ form the Dirac mass term in the Lagrangian, then $\nu_R$ will be degenerate in mass with $\nu_L$ thus being the light particles with the mass less than, at least, 0.3 eV. The other possibility is that the masses of $\nu_R$ can be larger than the masses of $\nu_L$. For example, they can belong to the range from 0.3 eV up to 3 eV. The right neutrinos of such a kind can be called as heavy sterile neutrinos. Besides, the neutrinos with masses from 3 eV up to 3 GeV and with masses more than 3 GeV can be called as extra-heavy and super-heavy neutrinos, respectively. The existence of super-heavy right neutrinos can be used both for explanation of high value of the invisible mass of the Universe, and for the explanation of small masses of the left neutrinos. Moreover, with the super-heavy right neutrinos, the observed baryon asymmetry of the Universe can be explained [25].

Recently [26], the corrected calculations of the spectrum of reactor antineutrino were provided, which result in higher calculated values of the fluxes of these particles. Thus, the experimental data indicate on the antineutrino deficiency in the measurements of the antineutrino fluxes on distances lower than 100 m from the particle source. These distances from the source should be considered as small. The currently available indications on the antineutrino deficiency on small distances can result in reactor anomaly. Rather like anomalies were observed in calibration measurements for the experiments GALLEX and
III. EVALUATION OF THE NEUTRINO MASS ABSOLUTE SCALE FROM THE POSSIBLE VALUES OF THE NEUTRINO MASS OBSERVABLES

Let us consider the neutrino mass matrix given by Eq. (3). The upper diagonal matrix element $M_{11}$ of this matrix enters the relation for the probability of the nuclear neutrinoless double-beta decay $(A,Z) \to (A,Z + 2) + 2e$, if the decay occurs with the assistance of the Majorana light neutrinos. In this case the absolute value of $M_{11}$ coincides with $m_{\beta\beta}$ from Eq. (6c), i.e., with the neutrino effective mass. The nuclear half-life period $T_{1/2}^{\beta\beta}$ is inversely proportional to $m_{\beta\beta}^2$ [32]. Note that the discovery of the neutrinoless double-beta decay is practically the only way to determine whether neutrinos are of the Dirac-type or the Majorana-type particles. The discovery of this decay could make it possible to determine the neutrino absolute mass scale with the aid of the $m_{\beta\beta}$ experimental value.

The explicit expression for $m_{\beta\beta}$ through the neutrino masses and the matrix elements of the neutrino mixing matrix $U_{\text{PMNS}}$ is given in Eq. (6c). In the present, on the basis of the experimental data given in Eqs. (4) on the mixing parameters of neutrino oscillations, and with the adjusting values of the $m_{\beta\beta}$, it can be possible to estimate the absolute scale of the neutrino mass spectra with the normal and inverse hierarchy. Indeed, $m_{\beta\beta}$ is expressed through the neutrino masses and the neutrino mixing parameters as follows

$$m_{\beta\beta} = |c_{12}^2 c_{13}^2 m_1 + s_{12}^2 c_{13}^2 e^{2i\alpha} m_2 + s_{13}^2 e^{2i(\beta - \delta)} m_3|.$$  \hspace{1cm} (7)

With using Eq. (7) and the experimental data from Eqs. (4), the explicit dependences of $m_{\beta\beta}$ versus the minimal neutrino mass $m_0$, that is either versus $m_1$ in the NH case or versus $m_3$ in the IH case can be determined. Besides, the similar dependences can be found for absolute values of other matrix elements $M_{ij}$. Figure 1(a) exhibits the dependences of $m_{\beta\beta}$ versus $m_0$ jointly for both NH and IH cases at zero value of the neutrino $CP$-violating phases. The allowance for the nonzero $CP$-violating phases alters the valid (from the point of view of available experimental data) range of variation of $m_{\beta\beta}$ versus $m_0$. For example, the characteristic limitation of the $m_{\beta\beta}$ values from the bottom arises at some values of the $CP$-violating phase $\delta$. Such behavior of $m_{\beta\beta}$ for both NH and IH cases and for $\delta = 45^\circ$ is presented in Fig. 1(b).

It is interesting that increase of accuracy of the numerical calculations together with the new experimental data results in division of the range of the minimal neutrino mass in the NH case on two sub-ranges, where zeroth values of $m_{\beta\beta}$ can be reached. At values of $m_0$ between these two sub-ranges, $m_{\beta\beta}$ is restricted from the bottom. These new features of behavior of $m_{\beta\beta}$ as a function of $m_0$ were not noted previously.

The possible structure of the neutrino mass matrix, as well as applicability of some additional assumptions about values of its matrix elements, which are used in the calibration measurements [27, 28]. In these models, the typical values of $\Delta m^2$ are of the order of 1 eV$^2$. 

SAGE [27, 28]. These anomalies can be called as calibration ones. The most recent experimental data indicating that the angle $\theta_{13}$ is noticeably deviated from zero [cf. Eq. (4c)] enhance the probability of the reactor and calibration anomalies. In this, the estimation $\sin^2 2\theta_{14} \lesssim 0.04$ for the mixing angle $\theta_{14}$ of the active and sterile neutrinos is obtained in the $3s + 1s$ model with single sterile neutrino [18, 29]. Note that, besides the $3s + 1s$ model, the $3s + 2s$ model with three active and two sterile neutrinos is used for explanation of the observed anomalies of the spectra of neutrinos and antineutrinos, including the neutrino spectra anomalies in the LSND experiment [30], and then in the MiniBooNE experiment [31] and in the calibration measurements [27, 28]. In these models, the typical values of $\Delta m^2$ are of the order of 1 eV$^2$. 

FIG. 1. (Color online) The variation ranges of the effective neutrino mass $m_{\beta\beta}$ as a function of the minimal neutrino mass $m_0$ at $\delta = 0$ (a) and at $\delta = 45^\circ$ (b), at $\alpha = 0, \beta = 0$ and at the values of the oscillation parameters from Eqs. (4). The shaded areas restricted by the solid and the dashed curves correspond to the NH and IH cases, respectively.
FIG. 2. (Color online) The ranges of absolute values of the matrix elements $M_{11}$ (a), $M_{22}$ (b) and $M_{33}$ (c) versus the minimal neutrino mass $m_0$ at $\sin^2\theta_{13} = 0$, $\delta = 0$, $\alpha = 0$, $\beta = 0$. The values of the other oscillation parameters coincide with the data from Eqs. (4). The shaded areas restricted by the solid and the dashed curves correspond to the NH and IH cases, respectively.

a number of phenomenological models [33, 34], can be obtained numerically. For calculations of the absolute values of the matrix elements $M_{ij}$, the approximation of $s_{13} = 0$ can be safely used. Figures 2 and 3 exhibit the absolute values of diagonal and off-diagonal elements $M_{ij}$, respectively, versus the minimal neutrino mass $m_0$ for both NH and IH cases. Note that in the applied approximation of $s_{13} = 0$, the main features of behavior of $|M_{ij}|$ remain invariable, but the fine peculiarities such a division of the $m_0$-range, where $m_{\beta\beta}$ vanishes, on the two sub-ranges disappear (cf. Fig. 1(a) and Fig. 2(a) for $|M_{11}|$). These dependences permit us to conceive of the structure of mass matrix $M_{ij}$ at different values of $m_0$. For example, at $m_0 \approx 3$ meV (the mean values of $M_{ij}$...
are also presented in meV)

\[ M^{(NH)} = \begin{pmatrix} 2 & 3 & 2 \\ 3 & 15 & 14 \\ 2 & 14 & 12 \end{pmatrix}, \quad M^{(IH)} = \begin{pmatrix} 30 & 3 & 2 \\ 3 & 20 & 10 \\ 2 & 10 & 20 \end{pmatrix}. \quad (8) \]

The different structures of matrix \( M_{ij} \) and different assumptions about the values of its matrix elements were considered in different models \([33-36]\). For instance, the conditions of vanishing of the individual matrix elements, as well as of its spur and determinant were analyzed. Figures 2 and 3 for the absolute values of the matrix elements \( M_{ij} \) can be used for inspection of different models of the neutrino mass matrix structure.

As can be seen from Fig. 1, while specifying the permissible values of \( m_{\beta\beta} \) it can be possible to determine the eventual intervals of the values of the minimal neutrino mass \( m_0 \), and then the absolute values of the other neutrino masses. Note that such calculations of the neutrino mass absolute values are sufficiently precise at small values of \( m_{\beta\beta} \), in spite of uncertainties of the experimental data. They are consistent with the results of the papers \([37, 38]\) in the limits of the precision of the presented graphs and the data used, in which the conditions resulting in the values of \( m_{\beta\beta} \) greater than \( 10^{-3} \) eV were considered. As is seen from Fig. 1(a), the IH spectrum can not be realized for \( \delta = 0 \) at \( m_{\beta\beta} < 0.01 \) eV that was noted before in Ref. \([37]\). Besides, as a result of the latter work, there is the restriction of the NH spectrum at small values of \( m_{\beta\beta} \) at some values of \( \delta \) [cf. with Fig. 1(b)], as well as at \( \delta = 0 \) in a small intermediate range [cf. with Fig. 1(a)].

In Figures 4(a) and (b) we present the mean cosmological mass \( m_a \) of active neutrinos and the \( \beta \) decay neutrino mass \( m_\beta \), respectively, versus the minimal neutrino mass \( m_0 \), which were obtained with the latest experimental data from Eqs. (4) on the oscillation characteristics of the neutrino.

The obtained intervals of the possible values of \( m_{\beta\beta} \) can be used for planning the experiments on search of the neutrinoless nuclear double-beta decay and for interpretation of the results obtained with allowance for the CP violation. The neutrino mass observables \( m_a \) and \( m_\beta \) can be used for the experiments with the results dependent on the neutrino mass absolute values.

**IV. PHENOMENOLOGICAL RELATIONS FOR THE NEUTRINO MASSES**

As is well-known, the problem of origin of the mass of the fundamental fermions is still unsolved. In the SM, these masses are originated due to Yukawa couplings between the fundamental fermion fields and the Higgs fields. Since the actual properties of the Higgs boson are still unknown in detail, the generation mass mechanism accepted in the SM should be considered as a hypothesis.

Moreover, the values of the neutrino masses are so small that, probably, the mechanism of the neutrino mass generation is primarily connected with the possible Majorana nature of the neutrino, rather than with the Higgs boson properties. In this case, the immediate basic problem is the simulation of the mass generation mechanism of the Majorana neutrinos. In absence of the appropriate theory of this phenomenon, the problem can be considered on the phenomenological level \([17]\). Let us suppose that there are several different contributions to the mass of the neutrino, and two of them are most significant. It can be assumed that the first contribution results in the mass of the left light neutrino, which is specified by the Majorana mass term in the Lagrangian as

\[ L_m = -\bar{\nu}_L M_{\nu} \nu_L / 2 + \text{h.c.} \quad (9) \]

In this case, with the Higgs mechanism of the mass generation, the Higgs sector of the SM should be changed and expanded. In what follows, the contribution of the type, which can be associated with Eq. (9), will be taken...
into account with the help of a new phenomenological parameter \( \xi \). The second contribution can be connected with the so-called seesaw mechanism, which is realized under inclusion into the theory the heavy right neutrinos \( N_i \), with \( i = 1, 2, 3 \). This mass term is of the form

\[
M'_\nu = -M_D^T M_R^{-1} M_D,
\]

where \( M_D \) is the matrix of the Dirac terms in the neutrino masses, \( M_R \) is the typical value of the right neutrino masses, which establishes a new scale associated with the right neutrino masses. Let us suppose that \( M_D \) is proportional to the mass matrix of the charged leptons, that is \( M_D = \sigma M_l \), where \( \sigma \) is of the order of unity, \( M_l = \text{diag}\{m_e, m_\mu, m_\tau\} \), and \( M_R = \sigma^2 M \). Then, for estimations of the neutrino masses \( m_i \), the following phenomenological formula can be used [17]:

\[
m_i = \pm \xi - m_R^2 / M,
\]

with \( m_R \) the masses of three charged leptons. With the help of Eq. (11) and the data from Eqs. (4), it is easily to obtain the absolute values of the neutrino masses \( m_i \) and the typical scales of \( \xi \) and \( M \) in eV for both the NH and the IH cases. They are as follows, respectively,

\[
\begin{align*}
  m_1 &\approx 0.0693, m_2 \approx 0.0698, m_3 \approx 0.0851, \\
  \xi &\approx 0.0693, M \approx 2.0454 \times 10^{19}, (NH), \\
  m_1 &\approx 0.0775, m_2 \approx 0.078, m_3 \approx 0.0606, \\
  \xi &\approx 0.0775, M \approx 2.2872 \times 10^{19}, (IH).
\end{align*}
\]

V. PHENOMENOLOGICAL RELATIONS FOR THE ANGLES AND THE CP-VIOLATING PHASES OF THE NEUTRINO MIXING MATRIX AND THE MODEL OF BIMODAL NEUTRINO

The upper diagonal matrix element \( m_{33} \) of the neutrino mass matrix is connected with the probability of the neutrinoless double-beta decay. At the same time, two other diagonal matrix elements can be equal in absolute values if the \( \mu - \tau \) symmetry is taken into account. It does not contradict to a number of models of the neutrino mass matrix [33, 34] and to the approximate \( \mu - \tau \) symmetry [39], as well as to estimations obtained in Sec. III for the matrix elements \( M_{ij} \) [see. e.g., Eq. (8)]. However, even in this approximation it is impossible to obtain without additional assumptions the estimations of \( m_{33} \) in the presence of the \( CP \) violation\(^1\). To obtain such estimations, one can consider the case of bimodal neutrino [11–13, 17], when the neutrino is neither the Dirac one nor the Majorana one but has simultaneously both quasi Dirac and Majorana properties. Indeed, a pair of quasi degenerate Majorana neutrinos can form the states of quasi, pseudo, and imaginary Dirac particles. Quasi Dirac particles consist of a pair of one active and one sterile neutrino, pseudo Dirac particles consist of a pair of active neutrinos, and imaginary Dirac particles consist of a pair of sterile neutrinos. Since the investigations of such particles are only starting now, the terminology given above is not completely established. Such particles have the properties of the Dirac particles and become entirely the Dirac particles under full degeneration.

In the model of bimodal neutrino it is usually assumed that the states of neutrino with a certain mass involve entirely the Dirac particles under full degeneration.

\(^1\) Without the \( CP \) violation, the estimations for \( m_{33} \) can be easily obtained as 0.07 (NH) and 0.08 eV (IH) at the values of neutrino masses from Eqs. (12), while it will be equal to 0.005 (NH) and 0.05 eV (IH) at the mass values from Eqs. (13).
the NH case, the second and the third neutrino mass states are the quasi Dirac ones, while in the IH case they are correspondingly the first and the second mass neutrino states.

Then, the condition of the $\mu - \tau$ symmetry, i.e., the equality of $m_{\mu\mu}$ and $m_{ee}$ results in the relation, which in the NH case reads as

$$|s_{12}^2 c_{23}^2 + 2s_{12} c_{12} s_{23} c_{23} s_{13} e^{i\delta} + c_{12}^2 s_{23}^2 c_{13} e^{2i\delta}|$$

$$= |s_{12}^2 s_{23}^2 - 2s_{12} c_{12} s_{23} c_{23} s_{13} e^{i\delta} + c_{12}^2 s_{23}^2 c_{13} e^{2i\delta}|,$$ (15)

while in the IH case it is as follows

$$s_{23}^2 = c_{23}^2.$$ (16)

Equation (15) permits one to determine the CP-violating phase $\delta$, while Eq. (16) determines the angle $\theta_{23}$. With using the data from Eqs. (4) we can obtain that $\delta \approx 100^\circ$ in the NH case, in contrast to the frequently used value of $\delta = 0^\circ$. However, the obtained value of $\delta$ is different also from the value $\delta = 180^\circ$ [1]. For this reason we consider that Eq. (15) is not fulfilled, that is the condition of the $\mu - \tau$ symmetry results in preferability of the IH case of the neutrino mass spectrum. In this case, it is possible to estimate the values of all the neutrino mass observables $m_\alpha$, $m_\beta$ and $m_{\beta\beta}$, if the minimal neutrino mass is of the order of $\lambda$. In eV, they are as follows

$$m_\alpha \approx 0.034, \quad m_\beta \approx 0.049, \quad m_{\beta\beta} \approx 0.00005.$$ (17)

The dependences of $m_{\beta\beta}$ versus the minimal neutrino mass $m_0$ in the model of bimodal neutrinos are shown in Fig. 5, in the range of $m_0$ between 0 and 1 eV. The characteristic feature of these results is the absence of the bottom limitation for $m_{\beta\beta}$ in both NH and IH cases, because in the bimodal neutrinos case $m_{\beta\beta}$ depends only on just one term involving $m_0$, that is $m_1$ or $m_3$ in the NH or IH case, respectively.

VI. CONCLUSION

The properties of the neutrino are rather mysterious, and both intensive theoretical and experimental studies are necessary for determination of the nature and the characteristics of these elementary particles. The construction and development of adequate phenomenological models of neutrino, which generalize the SM in the neutrino sector is one of the ways to interpret and predict the experimental results, as well as to develop the future GUT. At present, one of the first-priority problem in both the theoretical and the experimental investigations is the ascertainment of the neutrino type, that is the Dirac type or the Majorana type. It may occur rather unexpected that, possibly, this problem has no unambiguous solution and the neutrino can be bimodal, as was considered in the present paper. The other important problems, which should be solved in the process of the theoretical and experimental investigations are the determination of the absolute mass scale, the characteristics of the CP violation of the neutrino, and the different correlations between the numerous neutrino parameters.

In the present paper, the possible values of the neutrino mass observables $m_\alpha$, $m_\beta$ and $m_{\beta\beta}$ were calculated on the basis of the most recent experimental data. It was found that the minimal mass neutrino range, where $m_{\beta\beta}$ vanishes is divided into two sub-ranges with limitation from the bottom between them for the $m_{\beta\beta}$ values in the case of the normal hierarchy. It takes place both for the neutrino mass spectrum in the process of neutrinoless double-beta decay with CP violation, and even in the intermediate range of $m_0$ for the decays with CP conservation. For the investigation of the neutrino properties, the $3 + 2 + 1$ phenomenological neutrino model with three active neutrinos and three sterile neutrinos was proposed. The model permits reducing the number of sterile neutrinos, if the model-independent experimental restrictions on their number will be established. In the framework of this model and with allowance for the recent experimental data, the values of the neutrino mass observables $m_\alpha$, $m_\beta$ and $m_{\beta\beta}$ were obtained, and also the estimations of masses $m_{\alpha i} \equiv M_i$ of the light sterile neutrinos [see Eqs. (14)] were made. As is known, for experimental determination of the observables $m_{\alpha i}$, $m_{\beta i}$ and $m_{\beta\beta}$, numerous experiments are currently carried out and planned, namely, the experiments on search the neutrinoless double-beta decay, on determination of the form of the tritium $\beta$ decay spectrum, as well as the cosmological observations. The theoretical estimations of the values of $m_{\alpha i}$, $m_{\beta i}$, $m_{\beta\beta}$ and $m_{\alpha i}$ given above in this paper can be used for the interpretation and prediction of the results of these experiments.
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