Spectral Theory for Systems of Ordinary Differential Equations with Distributional Coefficients

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I am reporting on joint work with

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Introduction
Spectral theory and the Fourier transform

• To describe heat conduction Fourier (1822) considered the problem

\[ \phi_t = \phi_{xx}, \quad \phi'(0, t) = \phi'(L, t) = 0, \quad \phi(x, 0) = \phi_0(x) \]

• Separating variables and introducing the separation constant \( \lambda \) leads the boundary value problem

\[ -y'' = \lambda y, \quad y(0) = y'(L) = 0 \]

with eigenfunctions \( y_n = \cos(k_n x) \) and eigenvalues \( \lambda_n = k_n^2 = (n\pi/L)^2 \).

• Then

\[ \phi_0(x) = \sum_{n=0}^{\infty} c_n \cos(k_n x) \]

for appropriate Fourier coefficients whenever \( \phi_0 \in L^2((0, L), dx) \).
Generalizations

• Sturm and Liouville (1830s)

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- Krein (1952) treated $p = 1$, $\nu = 0$ but $r$ a positive measure.

- Savchuk and Shkalikov (1999) studied a Schrödinger equation with distributional potential $\nu$.

- Eckhardt, Gesztesy, Nichols, and Teschl (2013) generalized further and developed a spectral theory for the equation
  \[-(p(y' - sy))' - sp(y' - sy) + vy = \lambda ry\]
  on an interval $(a, b)$ when $1/p$, $\nu$, $s$, and $r$ are real-valued and locally integrable and $r > 0$. 
• It is useful to note that any of these equations can be realized as a system:

\[ Ju' + qu = \lambda wu \]

where \( u_1 = y, \ u_2 = p(y' - sy) \) and

\[
J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} v & -s \\ -s & -1/p \end{pmatrix}, \quad \text{and} \quad w = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}.
\]
Hurdles to overcome

• The definiteness condition

\[ Ju' + qu = 0 \text{ and } wu = 0 \text{ (or } \|u\| = 0) \text{ implies } u \equiv 0 \]

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  - Consider graphs: \((u, f) \in T_{\text{max}} \text{ if and only if } u \in \text{BV}_{\text{loc}} \text{ and } Ju' + qu = wf\)
Hurdles to overcome

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- The DE gives, in general, only relations not operators.
  - Consider graphs: \((u, f) \in T_{\text{max}}\) if and only if \(u \in \text{BV}_{\text{loc}}\) and
    \[ Ju' + qu = wf \]
  - Fortunately, there is an abstract spectral theory for linear relations
    \(\text{(Arens 1961, Orcutt 1969)}.\)
Our goal: allowing for rougher coefficients

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- If $u$ were even rougher one can not define $qu$ anymore.
- In the presence of discrete components of $q$ and $w$ existence and uniqueness of solutions become an issue.
Hypotheses for this work

We consider the equation $Ju' + qu = wf$ posed on $(a, b)$ and require the following:

- System size is $n \times n$.
- $J$ is constant, invertible, and skew-hermitian.
- $q$ and $w$ are hermitian distributions of order 0 (measures).
- $w$ non-negative (giving rise to the Hilbert space $L^2(w)$ with scalar product $\langle f, g \rangle = \int f^* wg$).
- Additional conditions to be discussed later (probably only technical).
Differential equations
Interpreting the differential equation

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- $u \in BV_{\text{loc}}$ implies $qu$ and $wu$ are distributions of order 0.

- Thus each term in

$$Ju' + qu = \lambda wu + wf$$

is a distribution of order 0.
Why balanced solutions?

We will look for solutions among the balanced solutions of locally bounded variation.

• If $F = tF^+ + (1 - t)F^-$ and $G = tG^+ + (1 - t)G^-$ for some fixed $t$

$$\int_{[x_1, x_2]} (F dG + G dF) = (FG) + (x_2) - (FG) - (x_1) + (2t - 1) \int_{[x_1, x_2]} (G^- - G^+) dF.$$
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- We call $(F^+ + (1 - t)F^-)/2$ balanced.
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- We call $(F^+ + F^-)/2$ balanced.
Existence and uniqueness of solutions

• If $Q$ or $W$ have a jump at $x$ the differential equation requires

$$J(u^+(x) - u^-(x)) + \left(\Delta_q(x) - \lambda \Delta_w(x)\right) \frac{u^+(x) + u^-(x)}{2} = \Delta_w(x) f(x)$$

where $\Delta_q(x) = q(\{x\}) = Q^+(x) - Q^-(x)$ (similar for $w$).
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- Equivalently, $B_+(\lambda, x)u^+(x) - B_-(\lambda, x)u^-(x) = \Delta w(x)f(x)$ where

$$B_\pm(x, \lambda) = J \pm \frac{1}{2}(\Delta q(x) - \lambda \Delta w(x)).$$
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• Without an existence and uniqueness theorem there is no variation of constants formula.
Existence of solutions

- Consider $\lambda = 0$. The points where $B_{\pm}(x)$ are not invertible are discrete.
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• Consider $\lambda = 0$. The points where $B_{\pm}(x)$ are not invertible are discrete.

• If there are only finitely many such points, a solution of $Ju' + qu = wf$ exists when

$$B\tilde{u} = F(f)$$

where

$$B = \begin{pmatrix}
-B_-(x_1)U_0(x_1) & B_+(x_1) & 0 & \cdots & 0 \\
0 & -B_-(x_2)U_1(x_2) & B_+(x_2) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -B_-(x_N)U_{N-1}(x_N) & B_+(x_N)
\end{pmatrix}$$

and the $U_j$ is a fundamental system in $(x_j, x_{j+1})$.  

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- One has to check whether $F(f) \in \text{ran } B$. 

\[ T_{\text{max}} = T^*_{\text{min}} \]
Maximal and minimal relation

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- \( \mathcal{T}_{\text{min}} \) and \( \mathcal{T}_{\text{min}} \).
- \( \mathcal{T}^* = \{(v, g) : \forall (u, f) \in T : \langle v, f \rangle = \langle g, u \rangle\} \).
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- \( T^* = \{(v, g) : \forall (u, f) \in T : \langle v, f \rangle = \langle g, u \rangle\} \).
- \( T^*_{\text{min}} = T_{\text{max}} \)
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Approach to a proof

• To show $T_{\text{max}} \subset T_{\text{min}}^*$ is simply an integration by parts.
Approach to a proof

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- For the converse two additional facts are required:

\[\int_{\xi_2}^{\xi_1} f^* \hat{w} v = \langle f, v \rangle = \langle u, g \rangle = \int_a^b u^* w g = \int_{\xi_2}^{\xi_1} f^* w v_1 = \int_{\xi_1}^{\xi_2} f^* w v_1\]
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  • Given $g \in L^2(w)$ the DE $Jv' + qv = wg$ has a solution $v_1$. 

  $\langle f, v \rangle = \langle u, g \rangle$ and partial integration give
  \[ \int_{\xi_2}^{\xi_1} f \bar{\cdot} w v = \langle f, v \rangle = \langle u, g \rangle = \int_a^b u \bar{\cdot} w g = \int_{\xi_2}^{\xi_1} f \bar{\cdot} w v_1 \]

  $v - v_1 \in K_0$ and hence $Jv' + qv = wg$ on $(\xi_1, \xi_2)$. 

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  • Given $g \in L^2(w)$ the DE $Jv' + qv = wg$ has a solution $v_1$.
  • Restrict to $[\xi_1, \xi_2]$ and define $K_0 = \{k : Jk' + qk = 0\}$ and $T_0 = \{([u], [f]) : Ju' + qu = wf, u(\xi_1) = u(\xi_2) = 0\}$. Then $\text{ran}(T_0) = L^2(w|_{[\xi_1, \xi_2]}) \ominus K_0$. 

$\langle f, v \rangle = \langle u, g \rangle$ and partial integration give

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• Suppose $([v], [g]) \in T_{\text{min}}^*$ and $([u], [f]) \in T_0$, extend the latter to $([u], [f]) \in T_{\text{min}}$. 

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  • Restrict to $[\xi_1, \xi_2]$ and define $K_0 = \{k : Jk' + qk = 0\}$ and $T_0 = \{([u], [f]) : Ju' + qu = wf, u(\xi_1) = u(\xi_2) = 0\}$.
    Then $\text{ran}(T_0) = L^2(w_{|[\xi_1,\xi_2]}) \ominus K_0$.

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- $[v - v_1] \in K_0$ and hence $Jv' + qv = wg$ on $(\xi_1, \xi_2)$. 

Additional details on the needed facts

• Existence of solutions for $Ju' + qu = wg$ may be shown if $g$ is in the range of $T_{\text{min}}^*$. 

• On to Fact 2:
  • To show $\text{ran} T_0 \subset L^2(w|\xi_1,\xi_2) \ominus K_0$ is simply an integration by parts and the fact that elements of $\text{dom} T_0$ vanish at the endpoints.

• For the converse we need to construct a solution $u$ of $Ju' + qu = wf$ if $f \in L^2(w|\xi_1,\xi_2) \ominus K_0$.

• This time $f \perp K_0$ allows to show existence of the sought solution.
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Spectral theory (expansion in eigenfunctions)
Extra conditions

- Set

\[ \Lambda = \{ \lambda \in \mathbb{C} : \text{det}(J \pm \frac{1}{2}(\Delta_q(x) - \lambda \Delta_w(x))) = 0\} , \]

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- \( \Lambda \cap \mathbb{R} = \emptyset \).

- \( \Lambda \) discrete.
Boundary conditions

- Deficiency indices: \( n_\pm = \dim \{(u, \pm iu) \in T_{\text{max}}\} \).
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- \( T \) is a self-adjoint restriction of \( T_{\max} \) if and only if \( T = \ker A \) and
Boundary conditions

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(v^*Ju)^-(b) - (v^*Ju)^+(a) = \langle v, f \rangle - \langle g, u \rangle.
\]

• \((u, f) \in \ker A\) if and only if \( 0 = (g_j^*Ju)^-(b) - (g_j^*Ju)^+(a) = 0 \) for \( j = 1, \ldots, n_{\pm} \).
The resolvent and Green’s function

- If \([u, f] \in T_{\text{max}}\) and if the definiteness condition is violated, the class \([u]\) may have many balanced representatives in \(BV_{\text{loc}}\).

\[
\text{Define } E : T_{\text{max}} \to BV_{\text{loc}} : ([u], [f]) \mapsto u.
\]

\[
\text{Define } E_\lambda : L^2(w) \to BV_{\text{loc}} : f \mapsto E(u, \lambda u + f) \text{ where } u = R_\lambda f \text{ whenever } \lambda \in \rho(T) \text{ (will not distinguish below)}.
\]

Each component of \(f \mapsto (R_\lambda f)(x)\) is a bounded linear functional.

Green’s function: 
\[
(R_\lambda f)(x) = \langle G(x, \cdot, \lambda) \ast f, w \rangle = \int G(x, \cdot, \lambda)wf.
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The resolvent and Green’s function

• If \([u, f] \in T_{\text{max}}\) and if the definiteness condition is violated, the class \([u]\) may have many balanced representatives in \(BV_{\text{loc}}\).

• However, there is a unique balanced representative \(u\) such that \(u(x_0)\) is perpendicular to \(N_0 = \{v(x_0) : Jv' + qv = 0 \& \ wv = 0\}\).
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• Each component of \(f \mapsto (R_{\lambda}f)(x)\) is a bounded linear functional.

• Green’s function: \((R_{\lambda}f)(x) = \langle G(x, \cdot, \lambda)^*, f \rangle = \int G(x, \cdot, \lambda)wf\).
Properties of Green’s function I

- The variation of constants formula: if $\lambda \not\in \Lambda$ and $x > x_0$

$$ (R_\lambda f)^-(x) = U^-(x, \lambda)(u_0 + J^{-1} \int_{(x_0, x)} U(\cdot, \lambda^*)wf) $$

where $u_0 = (R_\lambda f)(x_0)$ and $U(\cdot, \lambda)$ is a fundamental matrix with $U(x_0, \lambda) = I$. 

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Properties of Green’s function I

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• Assume that $f$ is compactly supported so that $u$ satisfies the homogeneous equation near $a$ and $b$. Then $u_0$ has to be chosen so that
  • $R_\lambda f$ is in $L^2(w)$ near both $a$ and $b$,
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$$(R_\lambda f)^-(x) = U^-(x, \lambda)(u_0 + J^{-1}\int_{(x_0,x)} U(\cdot, \lambda)*wf)$$

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  • $R_\lambda f$ satisfies the boundary conditions (if any), and
  • $(\mathbb{I} - P)u_0 = 0$ where $P$ is the orthogonal projection onto $N_0^\perp$.

• This gives rise to a (rectangular) linear system

\[
F(\lambda)u_0 = \int \left[ (b_-(\lambda)\chi_{(a, x_0)} + b_+(\lambda)\chi_{(x_0, b)})U(\cdot, \lambda)^*wf \right]
\]

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Properties of Green’s function II

- $F$ has a left inverse $F^\dagger$.

$$u_0 = \int (PF^\dagger b_-(\lambda)\chi_{(a,x_0)} + PF^\dagger b_+(\lambda)\chi_{(x_0,b)})U(\cdot, \overline{\lambda})^*wf.$$
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- $M = M_{\pm}$ on $\text{span}(B_+ \cup B_-)$
Properties of Green’s function II

• $F$ has a left inverse $F^\dagger$.

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• $M_+ = M_- \text{ on } B_+ \cap B_-.$

• $M = M_\pm \text{ on } \text{span}(B_+ \cup B_-)$

• On $\text{span}(B_+ \cup B_-)^\perp = N_0$ we set $M = 0.$
Properties of Green’s function III

Then

\[(R_\lambda f)(x) = U(x, \lambda) M(\lambda) \int_{(a,b)} U(\cdot, \bar{\lambda})^* w f\]

\[- \frac{1}{2} U(x, \lambda) J^{-1} \int_{(a,b)} \text{sgn}(\cdot - x) U(\cdot, \bar{\lambda})^* w f\]

\[+ \frac{1}{4} (U^+(x, \lambda) - U^-(x, \lambda)) J^{-1} U(x, \bar{\lambda})^* \Delta w(x) f(x)\]
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• $\Lambda \cap \mathbb{R}$ is empty.
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\]

- \(\Lambda \cap \mathbb{R}\) is empty.

- The Fourier transform \((\mathcal{F}f)(\lambda) = \int_{(a,b)} U(\cdot, \bar{\lambda})^*wf\) is analytic on \(\mathbb{R}\).
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• Last two terms of \(R_\lambda f\) are also analytic on \(\mathbb{R}\).

• All singularities and hence all spectral information is contained in \(M\).
The $M$-function

- $M$ is analytic away from $\mathbb{R}$ and $\Lambda$
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The $M$-function

- $M$ is analytic away from $\mathbb{R}$ and $\Lambda$
- $\text{Im } M / \text{Im } \lambda \geq 0$
- $\Lambda$ is a discrete set

\[ M(\lambda) = A\lambda + B + \int \left( \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) \nu(t) \]

where $\nu = N'$ and $N$ a non-decreasing matrix.
The $M$-function

- $M$ is analytic away from $\mathbb{R}$ and $\Lambda$
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The $M$-function

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- $M$ is a Herglotz-Nevanlinna function

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The Fourier Transform

- \((\mathcal{F}f)(\lambda) = \int U(\cdot, \lambda)^*wf\) if \(f \in L^2(w)\) is compactly supported and \(\lambda \notin \Lambda\).
The Fourier Transform

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• \(\mathcal{H}_\infty = \{ f : (0, f) \in T \}\) is the kernel of \(\mathcal{F}\). \(\mathcal{H}_0 = L^2(w) \ominus \mathcal{H}_\infty\).
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- $\mathcal{F} \circ \mathcal{G} = 1$ and $\mathcal{G} \circ \mathcal{F}$ is the projection onto $\mathcal{H}_0$. 
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- \((u, f) \in T\) if and only if \((\mathcal{F}f)(t) = t(\mathcal{F}u)(t)\).
Thank you