Enhanced Power Graphs of Finite Groups
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Abstract
The enhanced power graph $G_e(G)$ of a group $G$ is the graph with vertex set $G$ such that two vertices $x$ and $y$ are adjacent if they are contained in a same cyclic subgroup. We prove that finite groups with isomorphic enhanced power graphs have isomorphic directed power graphs. We show that any isomorphism between power graphs of finite groups is an isomorphism between enhanced power graphs of these groups, and we find all finite groups $G$ for which Aut$(G_e(G))$ is abelian, all finite groups $G$ with $|\text{Aut}(G_e(G))|$ being prime power, and all finite groups $G$ with $|\text{Aut}(G_e(G))|$ being square free. Also we describe enhanced power graphs of finite abelian groups. Finally, we give a characterization of finite nilpotent groups whose enhanced power graphs are perfect, and we present a sufficient condition for a finite group to have weakly perfect enhanced power graph.

1 Introduction
The directed power graph of a semigroup was introduced by Kelarev and Quinn [10] and [11] as the simple digraph whose vertex set is the universe of the semigroup, and with $x \rightarrow y$ if $y$ is a power of $x$. The (undirected) power graph of a semigroup was defined by Chakrabarty in [7] as the simple graph whose vertex set is the universe of the semigroup, and with $x$ and $y$ being adjacent if one of them is a power of the other. In [5] Cameron and Ghosh proved that finite groups with isomorphic directed power graphs have the same numbers of elements of each order, and that finite abelian groups with isomorphic power graphs are isomorphic. Later in [6] Cameron proved that finite groups with isomorphic power graphs have isomorphic directed power graphs. The authors of [1] and [9] independently proved that the power graph of any group of bounded exponent is perfect. In [12] Shitov proved that chromatic number of the power graph of any semigroup is at most countable. Commuting graph of a group appears to be defined in [3]. It is the simple graph with the universe of the group being its vertex set, in which $x$ and $y$ are joined if $x$ and $y$ commute. Note that usually the vertex set of the commuting graphs contains only non-central elements of the group. In [1] the authors introduced the enhanced power graph of a group whose vertex set is the universe of the group, and with $x$ and $y$ being adjacent if $x$ and $y$ are contained in a same cyclic group. Note that the enhanced power graph of a group lies as a subgraph, between the power graph and the commuting graph of the group, if the vertex set of the commuting graph is entire universe of the group. In [1] the authors proved that maximal cliques of the enhanced power graph of a group is a cyclic subgroup or a locally cyclic subgroup of the group.

In this paper we study properties of the enhanced power graph of a finite group, and its relations to the power graph and the directed power graph of the group.

In Section 2 we provide some basic definitions and notations which will be used in this paper. Section 3 is motivated by the results of [5] and [6], and in this section we prove that finite groups with isomorphic enhanced power graphs
have isomorphic directed power graphs. This implies that finite groups have isomorphic enhanced power graphs if and only if they have isomorphic directed power graphs if and only if they have isomorphic power graphs. We also give an efficient way to determine the number of elements of the group of any order from the enhanced power graph. Finally, we show that it is possible to reconstruct any finite abelian group from its enhanced power graph.

Section 4 is motivated by [5]. Here we extend Cameron’s proof from [6] in order to show that an isomorphism of the power graph of a finite group is an isomorphism of the enhanced power graph of that group too. We also find all finite groups $G$ with abelian $\text{Aut}(G_e(G))$, all finite groups $G$ such that $|\text{Aut}(G_e(G))|$ is square free, and all finite groups $G$ such that $|\text{Aut}(G_e(G))|$ is prime power.

In Section 5 we give a description of enhanced power graphs of finite abelian $p$-groups, which enables us to describe enhanced power graphs of all finite abelian groups. Given a finite graph, we provide some necessary conditions for it to be the enhanced power graph of a group. Also, we give an algorithm which for a finite graph satisfying these conditions, were it the enhanced power graph of a finite group with a unique $p$-Sylow subgroup, returns the enhanced power graph of its $p$-Sylow subgroup.

Finally in Section 6, which is motivated by [1], shows that there are groups which do not have perfect enhanced power graph. Furthermore, we give a characterization of finite nilpotent groups whose enhanced power graphs are perfect. Also for a class of groups, which contains all finite nilpotent groups, we prove that all of them have weakly perfect enhanced power graphs.

2 Basic notions and notations

**Graph** $\Gamma$ is a structure $(V(\Gamma), E(\Gamma))$, or simply $(V, E)$, where $V$ is a set, and $E \subseteq V^{[2]}$ is a set of two-element subsets of $V$. Here $V$ is the set of vertices, and $E$ is the set of edges. For $x, y \in V$ we say that $x$ and $y$ are adjacent in $\Gamma$ if $(x, y) \in E$, and we denote it with $x \sim \gamma y$, or simply $x \sim y$. We say that graph $\Delta = (V_1, E_1)$ is a **subgraph** of graph $\Gamma = (V_2, E_2)$ if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. $\Delta$ is an **induced subgraph** of $\Gamma$ if $V_1 \subseteq V_2$, and $E_1 = E_2 \cap V_1^2$. In this case we also say that $\Delta$ is an induced subgraph of $\Gamma$ by $E_1$, and we denote that with $\Delta = \Gamma[E_1]$. **Strong product** of graphs $\Gamma$ and $\Delta$ is the graph $\Gamma \boxtimes \Delta$ such that

$$(x_1, y_1) \sim_{\Gamma \boxtimes \Delta} (x_2, y_2) \text{ if } (x_1 = x_2 \land y_1 \sim_\Delta y_2)$$

$$\lor (x_1 \sim_\Gamma x_2 \land y_1 = y_2)$$

$$\lor (x_1 \sim_\Gamma x_2 \land y_1 \sim_\Delta y_2).$$

**Closed neighborhood** of a vertex $x$ in a graph $\Gamma$, denoted with $\overline{N}_\Gamma(x)$, or simply $\overline{N}(x)$, is the set $\overline{N}_\Gamma(x) = \{y \mid y \sim_\Gamma x \lor y = x\}$.

**Directed graph** (or **digraph**) $\vec{\Gamma}$ is a structure $(V(\vec{\Gamma}), \vec{E}(\vec{\Gamma}))$, or simply $(V, E)$, where $V$ is a set, and $E$ is an irreflexive relation on $V$. Here two $V$ and $E$ are the set of vertices and the set of edges, respectively. If $(x, y) \in E$, then we say that $y$ is a direct successor of $x$, or that $x$ is a direct predecessor of $y$, and we denote that with $x \rightarrow_{\vec{\Gamma}} y$, or simply $x \rightarrow y$.

Throughout this paper, we shall denote algebraic structures, such as groups, with bold capitals, while we will denote their universes with respective regular
capital letters. For \( x, y \in G \), where \( G \) is a group, we shall write \( x \approx_G y \), or simply \( x \approx y \), if \( \langle x \rangle = \langle y \rangle \). Also, \( o(x) \) shall denote the order of an element \( x \) of a group.

**Definition 1** Directed power graph of a group \( G \) is the digraph \( \vec{G}(G) \) whose vertex set is \( G \), and there is a directed edge from \( x \) to \( y \), \( y \neq x \), if \( y \in \langle x \rangle \). We shall denote that there is a directed edge from \( x \) to \( y \) in \( \vec{G}(G) \) with \( x \sim_G y \), or simply \( x \sim y \).

Power graph of a group \( G \) is the graph \( G(G) \) whose vertex set is \( G \), and \( x \) and \( y \), \( x \neq y \), are adjacent if either \( y \in \langle x \rangle \) or \( x \in \langle y \rangle \). We shall denote that \( x \) and \( y \) are adjacent in \( G(G) \) with \( x \sim_G y \), or simply \( x \sim y \).

Enhanced power graph of a group \( G \) is the graph \( \vec{G}_e(G) \) whose vertex set is \( G \), and such that \( x \) and \( y \), \( x \neq y \), are adjacent if there exists \( z \in G \) such that \( x, y \in \langle z \rangle \). We shall denote that \( x \) and \( y \) are adjacent in \( \vec{G}_e(G) \) with \( x \sim_G y \), or simply \( x \sim y \).

We shall write \( x \equiv_G y \), or simply \( x \equiv y \), if \( \overline{\vec{G}}_e(G)(x) = \overline{\vec{G}}_e(G)(y) \).

**Lemma 2** Let \( G \) and \( H \) be groups. Then:

1. \( \vec{G}_e(G \times H) \leq \vec{G}_e(G) \otimes \vec{G}_e(H) \);

2. Let \( G \) and \( H \) be torsion groups. Then \( \vec{G}_e(G \times H) = \vec{G}_e(G) \otimes \vec{G}_e(H) \) if and only if \( \gcd(o(g), o(h)) = 1 \) for all \( g \in G \) and \( h \in H \).

**Proof.** 1. If \( (x_1, y_1) \sim_{G \times H} (x_2, y_2) \), then \( (x_1, y_1), (x_2, y_2) \in \langle (x_3, y_3) \rangle \), which implies \( x_1, x_2 \in \langle x_3 \rangle \) and \( y_1, y_2 \in \langle y_3 \rangle \). Now \( (x_1, y_1) \sim_{\vec{G}_e(G) \otimes \vec{G}_e(H)} (x_2, y_2) \) trivially follows, since we have \( x_1 \neq x_2 \) or \( y_1 \neq y_2 \). This proves the inclusion.

2. Suppose that \( \gcd(o(g), o(h)) = 1 \) for all \( g \in G \) and \( h \in H \). One inclusion has already been proved, so we only need to prove the other one. Let us assume that \( (x_1, y_1) \sim_{\vec{G}_e(G) \otimes \vec{G}_e(H)} (x_2, y_2) \). This implies \( x_1, x_2 \in \langle x_3 \rangle \) and \( y_1, y_2 \in \langle y_3 \rangle \) for some \( x_3 \in G \) and \( y_3 \in H \). Since \( x_3 \) and \( y_3 \) have relatively prime orders, it follows \( (x_1, 1), (1, y_3) = (x_3, y_3) \), which implies \( (x_1, y_1), (x_2, y_2) \in \langle (x_3, y_3) \rangle \). Because either \( x_1 \neq x_2 \) or \( y_1 \neq y_2 \), we have \( (x_1, y_1) \approx_{G \times H} (x_2, y_2) \), which proves the other inclusion. This proves one implication. Let us prove the other.

Assume that there are elements \( g_0 \in G \) and \( h_0 \in H \) with \( \gcd(o(g_0), o(h_0)) = k > 1 \). Since \( \vec{G}_e(K) \sqcup [L] = \vec{G}_e(L) \) for any groups \( K \) and \( L \) with \( L \leq K \), we have:

\[
(\vec{G}_e(G \times H))([g_0] \times [h_0]) = \vec{G}_e([g_0] \times [h_0]) = K_{o(g_0) \cdot o(h_0)} = K_{o(g_0)} \otimes K_{o(h_0)} = \vec{G}_e([g_0]) \otimes \vec{G}_e([h_0]) = (\vec{G}_e(G) \otimes \vec{G}_e(H))([g_0] \times [h_0]).
\]

Therefore, \( \vec{G}_e(G \times H) = \vec{G}_e(G) \otimes \vec{G}_e(H) \), which proves the other implication as well.

\[ \square \]

### 3 Enhanced power graph and the power graph

It is obvious that, if two groups have isomorphic directed power graphs, then they have isomorphic power graphs. Peter Cameron proved in [1] that the opposite holds too.
Theorem 3 (Theorem 2 in [2]). If $G$ and $H$ are finite groups whose power graphs are isomorphic, then their directed power graphs are also isomorphic.

We will prove that, if two finite groups have isomorphic enhanced power graphs, then they have isomorphic directed power graphs too. One can easily see that the opposite implication holds as well.

Note that there are non-isomorphic groups which have isomorphic power graphs. Peter Cameron and Shamik Ghosh gave several examples of such groups in [3], with some of them being finite. By our results from this section these finite groups have isomorphic enhanced power graphs. Before proceeding we will introduce the following lemma which was proved in [1].

Lemma 4 (Lemma 36 in [1]). A maximal clique in the enhanced power graph is either a cyclic subgroup or a locally cyclic subgroup.

Lemma 5 Let $G$ be a finite group, and let $Cl$ be the set of all maximal cliques of $G_e(G)$. Then for any $x, y \in G$:

1. $N(x) \subseteq N(y)$ in $G_e(G)$ if and only if $x \in C \Rightarrow y \in C$ for all $C \in Cl$;
2. $x \equiv y$ if and only if $x \in C \Leftrightarrow y \in C$ for all $C \in Cl$;
3. If $(x) = \langle y \rangle$, then $x \equiv y$;
4. Relation $\equiv$ defined with: $[x]_\equiv \leq [y]_\equiv$ if and only if $y \in C \Rightarrow x \in C$ for all $C \in Cl$, is an order on $G/\equiv$;
5. $\langle x \rangle_\equiv = \cap \{C \in Cl \mid [x]_\equiv \subseteq C\}$;
6. Any $\equiv$-class contains either $\varphi(m)$ or no elements of order $m$.\(^1\)

\(^1\) $\varphi$ represents the Euler function.
Let \( \mathcal{D} \) be the set of \( \equiv \)-classes \( [y]_{\equiv_{G}} \) with the property \( m \mid |([y]_{\equiv})| \). Suppose that \( [x]_{\equiv} \) is minimal in \( \mathcal{D} \). By 5, there is an element \( z \in ([x]_{\equiv}) \) with \( o(z) = m \), and by \( \mathcal{H} \) and 5, it follows \( [z]_{\equiv} \leq [x]_{\equiv} \). Were it \( [z]_{\equiv} < [x]_{\equiv} \), then \( [x]_{\equiv} \) would not be minimal in \( \mathcal{D} \). Therefore, \( z \in [x]_{\equiv} \). Let us prove now the other implication as well. If \( y \in [x]_{\equiv} \) with \( o(y) = m \), then \( (y) \leq ([x]_{\equiv}) \), and \( [x]_{\equiv} \in \mathcal{D} \). Suppose that \( [x]_{\equiv} \) is not minimal in \( \mathcal{D} \). Then there is \( x_{0} \in G \) with \( [x_{0}]_{\equiv} < [x]_{\equiv} \) and \( [x_{0}]_{\equiv} \) being minimal in \( \mathcal{D} \). Therefore, there is \( y_{0} \in [x_{0}]_{\equiv} \) with \( o(y_{0}) = m \), and we have \( (y_{0}) \leq ([x_{0}]_{\equiv}) \leq ([x]_{\equiv}) \). Also, since \( y_{0} \neq y \), by 3 we get \( (y_{0}) \neq (y) \), and \( (y) \) and \( (y_{0}) \) are two different cyclic subgroups of order \( m \) of the cyclic subgroup \( ([x]_{\equiv}) \). This is a contradiction, so \( [x]_{\equiv} \) is minimal in \( \mathcal{D} \), and the equivalence has been proved.

**Theorem 6** Let \( G \) and \( H \) be finite groups. If \( G_{e}(G) \cong G_{e}(H) \), then \( \tilde{G}(G) \cong \tilde{G}(H) \).

**Proof.** Let \( G \) and \( H \) be finite groups with \( G_{e}(G) \cong G_{e}(H) \), and let \( \psi \) be the isomorphism. Let \( C_{i}^{G}, C_{i}^{H} \), ..., \( C_{n}^{G} \) be all maximal cliques of \( G_{e}(G) \), and let \( C_{i}^{G}, C_{i}^{H}, \ldots, C_{n}^{H} \) be all maximal cliques of \( G_{e}(H) \) with \( \psi(C_{i}^{G}) = C_{i}^{H} \) for all \( i \leq n \). For \( X \subseteq G \) and \( Y \subseteq H \) of the same sizes we will say \( X \) and \( Y \) are corresponding if \( X \subseteq C_{i}^{G} \iff Y \subseteq C_{i}^{H} \) for all \( i \leq n \).

Let \( [x]_{\equiv_{G}} \) be a \( \equiv_{G} \)-class, \( y = \psi(x) \), and let \( m \in \mathbb{N} \). Then \( [y]_{\equiv_{H}} \) is corresponding \( \equiv_{H} \)-class to \( [x]_{\equiv_{G}} \). By Lemma 5, for any \( m \in \mathbb{N} \), \( [x]_{\equiv_{G}} \) contains an element of order \( m \) if and only if \( [y]_{\equiv_{H}} \) does, while both \( [x]_{\equiv_{G}} \) and \( [y]_{\equiv_{H}} \) contain at most one \( \approx \)-class made up of elements of order \( m \). Therefore, for any \( \approx_{G} \)-class \( [x_{0}]_{\equiv_{G}} \) there is a corresponding \( \approx_{H} \)-class \( [y_{0}]_{\equiv_{H}} \subseteq [y]_{\equiv_{H}} \) with \( o(x_{0}) = o(y_{0}) \). Let \( \vartheta : G \to H \) be a bijection which maps any \( \approx_{G} \)-class \( [x]_{\equiv_{G}} \) onto its corresponding \( \approx_{H} \)-class \( [y]_{\equiv_{H}} \) with \( o(x) = o(y) \). Let us prove that \( \vartheta \) is a digraph isomorphism from \( \tilde{G}(G) \) to \( \tilde{G}(H) \). Suppose that \( x_{0} \to_{G} x_{1} \), and \( \vartheta(x_{0}) = y_{0} \) and \( \vartheta(x_{1}) = y_{1} \). Then \( o(x_{1})|o(x_{0}) \), and \( x_{0}, x_{1} \in C_{i}^{G} \), for some \( i \in n \). Since \( y_{0} \) and \( y_{1} \) are in corresponding \( \approx_{H} \)-classes to \( \approx_{G} \)-classes of \( x_{0} \) and \( x_{1} \), respectively, we get \( y_{0}, y_{1} \in C_{i}^{H} \). Because \( y_{0} \) and \( y_{1} \) are contained in a cyclic subgroup, and \( o(y_{1})|o(y_{0}) \), it follows that \( y_{0} \to_{H} y_{1} \). This proves the theorem.

**Corollary 7** For all finite groups \( G \) and \( H \) the following conditions are equivalent:

1. Undirected power graphs of \( G \) and \( H \) are isomorphic;
2. Directed power graphs of \( G \) and \( H \) are isomorphic;
3. Enhanced power graphs of \( G \) and \( H \) are isomorphic.

The proof of the above corollary follows from Theorems 5 and 6.

In [5] Peter Cameron and Shamik Ghosh proved that finite abelian groups with isomorphic power graphs are isomorphic. Therefore, by Theorem 1 in [5] and Corollary 7 we get the following corollary.

**Corollary 8** Let \( G \) and \( H \) be finite abelian groups such that \( G_{e}(G) \cong G_{e}(H) \). Then \( G \cong H \).
In [5] they also proved that, for any \( k \in \mathbb{N} \), finite groups with isomorphic directed power graphs have the same numbers of elements of order \( k \). Therefore, Proposition 4 in [5] and Corollary[7] give us the next corollary.

**Corollary 9** Let \( G \) and \( H \) be finite groups with \( G_e(G) \cong G_e(H) \). Then for every \( m \in \mathbb{N} \) \( G \) and \( H \) have the same numbers of elements of order \( m \).

Now, for any \( m \in \mathbb{N} \) we will determine from the enhanced power graph how many elements of order \( m \) does the group have.

Let \( X \) be a finite set. For \( A, B \in \mathcal{P}(X) \), we say \( A \approx_k B \) if \( k \mid |A \cap B| \). We also define \( \Phi_m : \mathcal{P}(X) \to \mathbb{N} \) in the following way:

\[
\Phi_m(A) = \begin{cases} 
\varphi(m), & m \mid |A| \smallskip \\
0, & m \nmid |A|, 
\end{cases}
\]

where \( \varphi \) is the Euler function.

**Lemma 10** Let \( G \) be a finite group, and let \( \mathcal{C}_1 \) be the set of all maximal cliques of \( G_e(G) \) of sizes divisible by \( m \). Then \( \approx_m \) is an equivalence relation on \( \mathcal{C}_1 \).

**Proof.** Obviously \( \approx_m \) is reflexive and symmetric relation. It remains to prove transitivity. Suppose that \( C_1 \approx_m C_2 \) and \( C_2 \approx_m C_3 \). By Lemma[3] \( C_1, C_2, \) and \( C_3 \) are cyclic subgroups of \( G \). Since \( C_1 \approx_m C_2 \), there is a cyclic subgroup \( D_1 \leq C_1 \cap C_2 \) of order \( m \). Similarly there is a cyclic subgroup \( D_2 \leq C_2 \cap C_3 \).

Since the cyclic group \( C_2 \) has a unique subgroup of order \( m \), it follows that \( D_1 = D_2 \leq C_1 \cap C_3 \). Therefore, \( C_1 \approx_m C_3 \). \( \square \)

For any group \( G \), and \( X \subseteq G \), let us denote with \( X^{(m)} \) the set of elements of order \( m \) contained in \( X \), and let us denote with \( X^{(m)} \) the set of elements in \( X \) whose orders are divisors of \( m \). Also, throughout this paper \( \mathbb{N}_n \) shall denote the set \{1, 2, ..., \( n \)\}.

**Proposition 11** Let \( G \) be a finite group. Let \( \mathcal{C}_1 = \{C_1, C_2, ..., C_n\} \) be the set of all maximal cliques of \( G_e(G) \), and let \( \mathcal{D} = \{C \in \mathcal{C} \mid \|C\| \} \). Then:

1. \( |G^{(m)}| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{I \subseteq \mathbb{N}_n, \|I\| = k} \Phi_m\left( \bigcap_{i \in I} C_i \right) \);

2. \( |G^{(m)}| = \varphi(m) \cdot |\mathcal{D}| \approx_m | \).

**Proof.** 1. By Lemma[3] a maximal clique of \( G_e(G) \) is a maximal cyclic subgroup of \( G \). Therefore, \( C_1, C_2, ..., C_n \) are all maximal cyclic subgroups of \( G \). Since \( G^{(m)} = \bigcup_{i=1}^{n} C_i^{(m)} \), we have the following:

\[
|G^{(m)}| = \left| \bigcup_{i=1}^{n} C_i^{(m)} \right| = \sum_{k=1}^{n} \left( (-1)^{k+1} \sum_{I \subseteq \mathbb{N}_n, |I| = k} \left| \bigcap_{i \in I} C_i^{(m)} \right| \right)
= \sum_{k=1}^{n} \left( (-1)^{k+1} \sum_{I \subseteq \mathbb{N}_n, |I| = k} \left| \bigcap_{i \in I} C_i^{(m)} \right| \right).
\]
Now, for every $k \in \mathbb{N}$, a cyclic group of order $k$ has $\Phi(k)$ elements of order $m$, so we have:

$$|G^{(m)}| = \sum_{k=1}^{m} \left(\prod_{i=1}^{k} (\Phi_{p_i}(C_i)) \right),$$

which proves $1$.

2. By Lemma 11 the relation $\approx_m$ is an equivalence relation on $\mathcal{D}$. Let $\mathcal{D}_0$ be a $\approx_m$-class, and let $\mathcal{D}_0 \in \mathcal{D}_0$. Then $\mathcal{D}_0$ is a maximal cyclic subgroup of $\mathcal{G}$ with $m|\mathcal{D}_0|$, so $\mathcal{D}_0$ has a unique cyclic subgroup $\mathcal{M}$ of order $m$. What is more, $\mathcal{M}$ is the unique cyclic subgroup of $\mathcal{D}$ for all $\mathcal{D} \approx_m \mathcal{D}_0$. This implies that all maximal cliques from $\mathcal{D}_0$ contain the same $\varphi(m)$ elements of order $m$, the generators of $\mathcal{M}$. Beside that, each element of order $m$ is contained in at least one maximal cyclic subgroup whose order is divisible by $m$, and consequently in at least one maximal clique from $\mathcal{D}$. Also, if there is an $x$ of order $m$ such that $x \in E, F$ for $E, F \in \mathcal{D}$, then $(x) \leq E, F$, which implies $E \approx_m F$. This proves the equality. □

In the next proposition, given the enhanced power graph of an abelian group, we will give a description of this abelian group.

**Proposition 12** Let $\mathcal{G}$ be a finite abelian group with $|\mathcal{G}| = \prod_{i=1}^{n} p_i^{k_i}$, where $p_i$ are all different prime numbers. For every $m|\mathcal{G}|$ let $\mathcal{C}_m$ be the set of all maximal cliques of $\mathcal{G}_c(\mathcal{G})$ with sizes divisible by $m$. Then:

$$\mathcal{G} \cong \prod_{i=1}^{n} \prod_{j=1}^{k_i} (\mathcal{C}_{p_i}) \cdot \mathcal{G}^{2L(p_i,j)-L(p_i,j-1)-L(p_i,j+1)},$$

where $L(p,j) = \log_{p} \left( \sum_{k=0}^{j} \varphi(p^k) : |\mathcal{C}_{p^k} / \approx_{p^k} | \right)$.

**Proof.** Since $\mathcal{G}$ is an abelian group, $\mathcal{G} \cong \prod_{i=1}^{n} \mathcal{G}_{p_i}$, where, for each $i \in \mathbb{N}$, $\mathcal{G}_{p_i}$ is the unique $p_i$-Sylow subgroup of $\mathcal{G}$. Therefore, it is sufficient to prove the statement for any finite abelian $p$-group.

Let $\mathcal{G}$ be a finite abelian $p$-group. Then $\mathcal{G} \cong \prod_{i=1}^{n} \mathcal{C}_{p_i}$, where $l_i > 0$ for all $i \in n$. We shall call the cyclic groups $\mathcal{C}_{p_i}$ factors of $\mathcal{G}$. It is easy to see that

$$|\mathcal{G}^{(p^j)}| = \prod_{i=1}^{n} p^j \cdot \mathcal{G} \cdot \sum_{l_i \geq j} \min\{j,l_i\} = p^{\sum_{l_i \geq j} \min\{j,l_i\}},$$

for any $j \geq 0$. Therefore, for any $j \geq 1$, there are $\sum_{l_i \geq j} \min\{j,l_i\} - \sum_{l_i \geq j} \min\{j-1,l_i\} = \log_{p} |\mathcal{G}^{(p^j)}| - \log_{p} |\mathcal{G}^{(p^j)}| = \log_{p} |\mathcal{G}^{(p^j-1)}| \mathcal{G}^{(p^j-1)}|$ factors of $\mathcal{G}$ whose orders are at least $p^j$, which is by Proposition 12 equal to $L(p,j) - L(p,j - 1)$. Now, the number of factors of $\mathcal{G}$ whose orders are exactly $p^j$ is $L(p,j) - L(p,j - 1) - L(p,j + 1) - L(p,j)) = 2L(p,j) - L(p,j - 1) - L(p,j + 1)$. This proves the proposition. □

Note that there are non-isomorphic groups with same numbers of elements of each order. Therefore, given $\mathcal{G}_c(\mathcal{G})$, if we do not know whether $\mathcal{G}$ is abelian, we may not be able to recognize the group $\mathcal{G}$. Pair of groups of order 27 given by Cameron and Ghosh are one such example: $\mathcal{C}_3 \times \mathcal{C}_3 \times \mathcal{C}_3$ and the group with presentation $\mathcal{G} = \langle x, y | x^3 = y^3 = [x, y]^3 = 1 \rangle$. 
4 Automorphism groups of enhanced power graphs

As we have seen, power graph of a finite group determines the directed power graph of the group. It is not hard to see that an automorphism of power graph of a group may not be an automorphism of directed power graph of a group. However, as we shall prove, an automorphism of power graph of a finite group is an automorphism of enhanced power graph of the group.

Theorem 13 Let $G$ and $H$ be finite groups. Then any isomorphism from $G(G)$ to $G(H)$ is an isomorphism from $G_e(G)$ to $G_e(H)$.

Proof. Here we shall refer to some results of Cameron from [6].

If $G$ is a generalized quaternion group, then its enhanced power graph and power graph are equal since it is a $p$-group, implying $\text{Aut}(G(G)) = \text{Aut}(G_e(G))$. Also, if $G$ is a cyclic group, then its enhanced power graph is a complete graph, and then we obviously have $\text{Aut}(G(G)) \subseteq \text{Aut}(G_e(G))$.

So for the rest of the proof we shall assume that $G$ is neither generalized quaternion group, nor cyclic group. By Proposition 4 of [6] the identity element of $G$ is the only one being adjacent to all other elements of $G(G)$.

Both relations $\equiv$ and $\approx$ are easily seen to be equivalence relations, with $\equiv$-classes being recognizable in the graph. Moreover, in Proposition 5 of [6] Cameron showed what is the relation between the two equivalences. $\approx$ is a refinement of $\equiv$, and each non-identity $\equiv$-class $C$ is of one of the following forms:

1. $C$ is a $\approx$-class;
2. $C = \{x \in \langle y \rangle \mid o(x) \geq p^s\}$, where $o(y) = p^r$, $1 \leq s \leq r - 1$, and $p$ is a prime number.

Further in his proof he also shows that we can recognize the $\equiv$-classes of type (2) in the power graph, and recognize the prime number related to that $\equiv$-class.

For any $x \in G$ the size of $[x]_\equiv$ is $\varphi(o(x))$, where $\varphi$ is the Euler function. It is also known that $\varphi(\prod_{j=1}^l p_j^{i_j}) = \prod_{j=1}^l (p_j^{i_j-1}(p_j - 1))$ for any prime numbers $p_1, p_2, \ldots, p_l$ and any $i_1, i_2, \ldots, i_l > 0$. Therefore, $n|m$ implies $\varphi(n)|\varphi(m)$, while equality only holds when $n = m$, or $n$ is odd and $m = 2n$. This implies that, for any $x$ and $y$, $x \to y$ if and only if:

$$x \sim y, \text{ and } [y]_\equiv < [x]_\equiv, \text{ or}$$

$$x \sim y, \ |[y]_\equiv| =|[x]_\equiv|, \text{ and } x \sim z \text{ for some } z \text{ such that}$$

$$[z]_\equiv = \{z\}, \text{ and } N(z) \neq G, \text{ or}$$

$$x \approx y. \quad (3)$$

Note that $z$ in the condition (3) is an involution, so this one covers the case when $|[x]_\equiv| = 2|[y]_\equiv$ and $|[y]_\equiv$ is an odd number. In this case $[z]_\equiv = [z]_\approx$.

It is easily seen that if $x_0 \to y_0$, then $x \to y$ for any $x \approx x_0$ and $y \approx y_0$. This is true even for $\equiv$-classes, unless $x_0$ and $y_0$ are in the same class of type (2). Suppose that $[y_0]_\equiv$ is of type (2), and $x_0 \to y_0$ and $x_0 \not\equiv y_0$. Then $x_0 \sim y$, i.e. $x_0 \to y$ or $y \to x_0$. But $y \to x_0$ and $x_0 \to y_0$ would imply that $x_0 \in [y_0]_\equiv$. We have shown that $x_0 \to y_0$ and $x_0 \not\equiv y_0 \equiv y$ implies $x_0 \to y$. In similar way can be proved that $x_0 \to y_0$ and $x \equiv x_0 \not\equiv y_0$ implies $x \to y_0$. From this we can
easily conclude $x_0 \to y_0$ implies $x \to y$ for all $x \equiv x_0$ and $y \equiv y_0$, whenever $x_0$ and $y_0$ are not in a same $\equiv$-class of type $(2)$.

Thanks to the above said for any $x, y \in G$, whenever $x \not\sim y$, we can decide from $\hat{G}(G)$ whether $x \to y$. Now we prove that for any $x, y \in G$ we can decide whether $x \sim y$. Indeed, if $x \sim y$, then $x \not\sim y$. On the other hand, if $x \not\sim y$, then $x \not\sim y$ if and only if there exists $z \in G$ for which $z \rightarrow x$, $z \rightarrow y$, and $x \not\equiv y \not\equiv z \not\equiv x$. Given that, for any finite group $G$, it is decidable from $\hat{G}(G)$ whether $x \sim y$.

Now, let $G$ and $H$ be finite groups, and let $\varphi : G \rightarrow H$ be a graph isomorphism from $\hat{G}(G)$ to $\hat{G}(H)$. Since whether $x \not\sim y$ is decidable from $\hat{G}(G)$, and whether $\varphi(x) \not\sim_H \varphi(y)$ is decidable from $\hat{G}(H)$, it follows that $\varphi$ is also a graph isomorphism from $\hat{G}_e(G)$ to $\hat{G}_e(H)$. \hfill $\square$

**Corollary 14** Let $G$ be a finite group. Then

$$\text{Aut}(G) \subseteq \text{Aut}(\hat{G}(G)) \subseteq \text{Aut}(\hat{G}(G)) \subseteq \text{Aut}(\hat{G}_e(G)).$$

**Proof.** One can easily see that $\text{Aut}(G) \subseteq \text{Aut}(\hat{G}(G)) \subseteq \text{Aut}(\hat{G}(G))$, and the inclusion $\text{Aut}(\hat{G}(G)) \subseteq \text{Aut}(\hat{G}_e(G))$ is a direct consequence of Theorem 14. \hfill $\square$

Note that in the above theorem those inclusions for some groups can also be strict. For example $\text{Aut}(C_5) \subset \text{Aut}(\hat{G}(C_5)) \subset \text{Aut}(\hat{G}(C_5))$, and $\text{Aut}(\hat{G}(C_5)) \subset \text{Aut}(\hat{G}_e(C_5))$, so no equality holds in general.

**Corollary 15** The Klein group is the only non-trivial finite group $G$ such that $\text{Aut}(G) = \text{Aut}(\hat{G}_e(G))$.

**Proof.** Suppose that $G$ is a finite non-trivial group with $\text{Aut}(G) = \text{Aut}(\hat{G}_e(G))$. By Theorem 14 it follows $\text{Aut}(G) = \text{Aut}(\hat{G}(G))$, which implies $G \cong C_2 \times C_2$. On the other hand, it is easily checked that $\text{Aut}(C_2 \times C_2) = \text{Aut}(\hat{G}(C_2 \times C_2))$. \hfill $\square$

**Lemma 16** If $\Gamma$ and $\Delta$ are any two graphs, then $\text{Aut}(\Gamma)$ can be embedded into $\text{Aut}(\Gamma \boxtimes \Delta)$.

**Proof.** Let us define the mapping $\Phi : \varphi \mapsto \tilde{\varphi}$ with $\tilde{\varphi}((x, y)) = (\varphi(x), y)$ for all $\varphi \in \text{Aut}(\Gamma)$. Let $\varphi_0 \in \text{Aut}(\Gamma)$, and let $\tilde{\varphi}_0 = \Phi(\varphi_0)$. Obviously $\tilde{\varphi}_0$ is a bijection. Also, for all $x_1, x_2 \in V(\Gamma)$ and $y_1, y_2 \in V(\Delta)$ we have $(x_1, y_1) \sim (x_2, y_2)$ if and only if $(\varphi_0(x_1), y_1) \sim (\varphi_0(x_2), y_2)$ if and only if $\tilde{\varphi}_0((x_1, y_1)) \sim \tilde{\varphi}_0((x_2, y_2))$, so $\tilde{\varphi}_0 \in \text{Aut}(\Gamma \boxtimes \Delta)$. Besides, the implication $\varphi \neq \psi \Rightarrow \Phi(\varphi) \neq \Phi(\psi)$ obviously holds, and it is easily seen that $\Phi : \text{Aut}(\Gamma) \rightarrow \text{Aut}(\Gamma \boxtimes \Delta)$ is a homomorphism. Therefore, $\Phi$ is an injective homomorphism, which proves the statement of the lemma. \hfill $\square$

**Theorem 17** $C_2$ is the only non-trivial finite group whose enhanced power graph has abelian automorphism group.

**Proof.** Let $G$ be a non-trivial finite group, such that $\text{Aut}(\hat{G}_e(G))$ is abelian. If $G$ is not nilpotent, then $G/\text{Z}(G)$ is not abelian, and neither is $\text{Aut}(\hat{G}_e(G))$ since $G/\text{Z}(G) \cong \text{Inn}(G) \leq \text{Aut}(\hat{G}_e(G))$.

Since $G$ is nilpotent, by Theorem 5.39 in [14] its Sylow subgroups are unique for all prime divisors of $|G|$, and $G$ is their direct product. By Lemma 3 and
Lemma 10 it is sufficient to prove that the only non-trivial $p$-group with abelian $G_e(G)$ is $C_2$.

If a $p$-group $G$ has an element of order at least 5, then there are $x, y, z \in G$ so that $(x) = (y) = (z)$. Then permutations $(x, y)$ and $(y, z)$ on $G$ are automorphisms of $G_e(G)$ which do not commute. Let $G$ be a $p$-group of order 3. If $G \cong C_3$, then $Aut(G_e(G)) \cong S_3$. If it is non-cyclic, then $|G| \geq 9$ and $G$ has exponent 3, and there are elements $x, y, z$ that generate three different cyclic subgroups. Then $(x, y)(x^2, y^2), (y, z)(y^2, z^2) \in Aut(G_e(G))$, and they do not commute. Now let $G$ be a 2-group of order 4. If there is an element $G$ of order 2 that belongs to exactly one cyclic subgroup $\langle x \rangle$ of order 4, then $(x, x^2), (x^2, x^3) \in Aut(G_e(G))$, and $(x, x^2)$ and $(x^2, x^3)$ do not commute. If there is no such element of order 2, then there are $x, y \in G$ of order 4 so that $|\langle x \rangle \cap \langle y \rangle| = 2$. Then $(x, x^3)$ and $(x, y)(x^3, y^3)$ are automorphisms of $G_e(G)$ that do not commute. Finally, it is easy to notice that $C_2$ is the only group of exponent 2 having abelian automorphism group of its enhanced power graph. This proves the theorem.

Theorem 19 $C_2$, $C_3$, and the Klein group are the only non-trivial finite groups whose automorphism groups of their enhanced power graphs have square free orders.

Proof. Let $G$ be a group with $|Aut(G_e(G))|$ being square free. If $G$ is cyclic, then $G \cong C_2$ or $G \cong C_3$, so assume that $G$ is non-cyclic. Then for all $x \in G$ we have $o(x) < 5$, because $o(x) \geq 5$ implies $C_2 \times C_2 \cong ((x, x^{-1}), (x^2, x^{-2})) \leq Aut(G_e(G))$. Also, $G$ has at most two elements of order larger than 2. Otherwise there would be $x$ and $y$ with $o(x), o(y) > 2$ and $\langle x \rangle \neq \langle y \rangle$, and then $C_2 \times C_2 \cong ((x, x^{-1}), (y, y^{-1})) \leq Aut(G_e(G))$. If the non-cyclic group $G$ had exactly one cyclic subgroup of order 3 or 4, then it would have at least three more involutions, and $C_2 \times C_2$ could again be embedded into $Aut(G_e(G))$. So the exponent of $G$ is 2. Because $G$ does not have more than 3 involutions, it follows $G \cong C_2 \times C_2$.

Theorem 19 $C_2$ and $C_4 \times C_2$ are the only non-trivial finite groups whose automorphism groups of their enhanced power graphs have prime power orders.

Proof. Let $G$ be a non-trivial finite group for which $|Aut(G_e(G))|$ is a prime power. Since $Aut(G_e(G))$ contains a transposition on $G$, $Aut(G_e(G))$ is a 2-group.

If $G$ is not nilpotent, then $G/Z(G)$ is not a $p$-group, and because $G/Z(G) \cong Inn(G) \leq Aut(G_e(G))$, $Aut(G_e(G))$ is not a $p$-group either. It follows that $G$ is a nilpotent group. But no Sylow subgroup of $G$ contains an element $x$ with $o(x) \geq 5$. Otherwise there would be $x, y, z \in G$ so that $\langle x \rangle = \langle y \rangle = \langle z \rangle$ and $x \neq y \neq z \neq x$, and then the permutation $(x, y, z)$ would be an automorphism of $G_e(G)$. It follows by Theorem 5.39 in [14] that $G = P_2 \times P_3$, where $P_2$ and $P_3$ are the 2-Sylow subgroup of $G$ whose exponent is less than 8, and the 3-Sylow subgroup of $G$ whose exponent is less than 9, respectively. Next, by Lemma 2 we get $G_e(G) = G_e(P_2) \times G_e(P_3)$.

To show that $P_3$ is trivial, it is sufficient to show that otherwise $Aut(G_e(P_3))$ would contain an element of order of 3, because by Lemma 10 it implies that $G_e(G)$ also contains an element of order 3. If $P_3 \cong C_3$, then $Aut(G_e(P_3)) \cong S_3$. If the $P_3$ of exponent 3 is non-cyclic, then it has three elements $x, y,$ and
positive integers. Let \( l \) be a positive integer. Let \( G \) be a cyclic group of order \( 3 \). It follows that \( |G| \leq 8 \). Now it is easy to see that \( G \) is isomorphic to \( C_2 \times C_4 \times C_2 \), and we have \( \text{Aut}(G) \cong C_2 \) and \( \text{Aut}(C_4 \times C_2) \cong C_2 \times C_2 \times C_2 \times C_2 \).

5 The structure of enhanced power graphs of finite abelian groups

In this section we will describe enhanced power graphs of finite abelian groups. For start we will describe enhanced power graphs of finite abelian groups. To that end we introduce notions of rooted \( p \)-trees and \( p \)-semitrees. We will prove that \( p \)-semitrees are exactly enhanced power graphs of finite abelian groups.

A rooted tree is a tree whose one vertex is designated as the root. For two vertices \( x \) and \( y \) of a rooted tree we write \( x < y \) if the unique path from \( y \) to the root passes through \( x \). If \( x \) and \( y \) are adjacent and \( x < y \), then \( y \) is a child of \( x \), and \( x \) is the parent of \( y \). The height of vertex \( x \) is the distance between \( x \) and the root.

For tuples \( \overline{x} = (x_1, x_2, \ldots, x_n) \) and \( \overline{y} = (y_1, y_2, \ldots, y_n) \) of non-negative integers we say \( \overline{x} \leq \overline{y} \) if \( x_i \leq y_i \) for all \( i \leq n \). Let \( \overline{a} = (a_1, a_2, \ldots, a_n) \) be a tuple of positive integers. Let \( \overline{b} = (b_1, b_2, \ldots, b_n) \) and \( \overline{c} = (c_1, c_2, \ldots, c_n) \) be tuples of integers with \( \overline{b}, \overline{c} \in \overline{a} \). Then \( \overline{a} \)-width of the tuple \( \overline{b} \), or simply width of \( \overline{b} \), is the number \( |\{ i \in \mathbb{N}_n \mid b_i \neq a_i \}| \). We shall denote it with \( w(\overline{b}) \), or simply \( w(\overline{b}) \). \( \overline{a} \)-height of tuple \( \overline{b} \), or simply height of \( \overline{b} \), is the number \( \max\{a_i - b_i \mid i \in \mathbb{N}_n \} \). We shall denote it with \( h(\overline{b}) \), or simply \( h(\overline{b}) \). We say that \( \overline{c} \) is an \( \overline{a} \)-successor of \( \overline{b} \), or simply a successor of \( \overline{b} \), if \( \overline{c} \neq \overline{a} \), and:

- \( c_i = b_i - 1 \) if \( b_i \neq a_i \), and
- \( c_i \in \{a_i, a_i - 1\} \) if \( b_i = a_i \).

If \( \overline{c} \) is a successor of \( \overline{b} \), then \( \overline{b} \) is a predecessor of \( \overline{c} \).

The rooted \( p \)-tree associated to a tuple \( \overline{a} = (a_1, a_2, \ldots, a_n) \) of positive integers, denoted with \( T_p(\overline{a}) \), is the rooted tree \( T \) whose vertices can be labeled with tuples from \( \mathbb{N}^i \) in the following way (here we allow more vertices to be labeled with a same tuple):

- The root of \( T \) is labeled with \( \overline{a} \);
- The root has exactly \( \frac{p^n-1}{p-1} \) children which are all labeled with successors of \( \overline{a} \) in the following way: For each successor \( \overline{b} \) of \( \overline{a} \) there are exactly \( (p-1)^w(\overline{b})-1 \) children of the root labeled with \( \overline{c} \);
- If a vertex \( x \) of \( T \) is labeled with \( \overline{c} \neq \overline{a} \), with \( c_i \neq 0 \) for all \( i \in n \), then it has exactly \( p^{n-1} \) children which are all labeled with successors
of \( \mathfrak{T} \) in the following way: For each successor \( \overline{d} \) of \( \mathfrak{T} \) there are exactly
\[ p^w(\sigma) - 1 (p - 1)^{w(\tau) - w(\sigma)} \]
children of \( x \) labeled with \( \overline{d} \).

- If a vertex \( x \) of \( T \) is labeled with \( \mathfrak{T} \neq \mathfrak{T} \), with \( c_i = 0 \) for some \( i \leq n \), then \( x \) has no children.

It is easily seen that any two rooted trees satisfying the above conditions are isomorphic, so \( T_p(\mathfrak{T}) \) is well defined.

The \( p \)-semitree associated to the tuple \( \mathfrak{T} \), denoted by \( S_p(\mathfrak{T}) \), is the graph constructed from \( T_p(\mathfrak{T}) \) by adding an edge between every two \( x \) and \( y \) for which \( x < y \), and by replacing each vertex of height \( k > 0 \) by \( K_{p^{k-1}(p-1)} = K_{p} \).

**Proposition 20** A graph is the enhanced power graph of a finite abelian \( p \)-group if and only if it is a \( p \)-semitree.

**Proof.** The theorem obviously holds if the dimension of the tuple \( \mathfrak{T} \) is 1, because, for any \( m \in \mathbb{N} \), \( \mathfrak{T} = (m) \) if and only if \( S_p(\mathfrak{T}) \cong K_{p^m} \), which is the enhanced power graph of \( C_{p^m} \). So we assume its dimension is at least 2.

Let \( \mathfrak{T} = (a_1, a_2, ..., a_n) \) be a tuple of positive integers, and let \( G = \prod_{i=1}^{n} C_{p^{a_i}} \).

Since every finite abelian abelian group is isomorphic to a direct product of cyclic groups, it is sufficient to prove that \( S_p(\mathfrak{T}) \cong G_0(G) \). It is easily seen that \( S_p(\mathfrak{T}) \cong G_p(G) \) if and only if \( T_p(\mathfrak{T}) \) is, as a poset, isomorphic to the poset \( P \) of cyclic subgroups of \( G \). To show that there is an isomorphism \( \psi : T_p(\mathfrak{T}) \to P \) which maps each vertex labeled with \( \mathfrak{T} = (c_1, c_2, ..., c_n) \) into a cyclic subgroup whose \( i \)-th projection has order \( p^{c_i} \) for each \( i \leq n \), it is sufficient to prove the following:

1. \( G \) has exactly \( \frac{p^n - 1}{p-1} \) cyclic subgroups of order \( p \);
2. For \( M \subseteq \mathbb{N}_n \), there are exactly \( (p-1)^{|M| - 1} \) cyclic subgroups of \( G \) of order \( p \) whose \( i \)-th projection is non-trivial if and only if \( i \in M \);
3. If \( H \) is a cyclic subgroup of \( G \), and \( H_i \) its \( i \)-th projection for all \( i \leq n \), then \( H \) is a maximal cyclic subgroup of \( G \) if and only if \( |H_i| = p^{k_i} \) for some \( i \leq n \);
4. If \( H \) is a non-maximal cyclic subgroup of \( G \) of order \( p^k > 1 \) whose \( i \)-th projection is \( H_i \) of order \( p^{k_i} \) for all \( i \leq n \), then \( H \) is subgroup of exactly \( p^{n-1} \) cyclic subgroups of \( G \) of order \( p^{k+1} \). What is more, if \( L = \{ i \in \mathbb{N}_n \mid h_i \neq 0 \} \) and \( L \subseteq M \subseteq \mathbb{N}_n \), then \( H \) is subgroup of exactly \( p^{L|-1}(p-1)^{|M|-|L|} \) cyclic subgroups of order \( p^{k+1} \) whose \( i \)-th projection is non-trivial if and only if \( i \in M \).

\[ G = \prod_{i=1}^{n} C_{p^{a_i}} \] has \( p^n - 1 \) elements of order \( p \), so it has \( \frac{p^n - 1}{p-1} \) cyclic subgroups of order \( p \). This proves (1). To prove (2) notice that for \( M \subseteq \mathbb{N}_n \) there are \( (p-1)^{|M|} \) of elements of order \( p \) whose \( i \)-th projection is non-trivial if and only if \( i \in M \). This implies that there are \( (p-1)^{|M| - 1} \) cyclic subgroups of \( G \) of order \( p \) whose \( i \)-th projection is non-trivial if and only if \( i \in M \).

By a root of degree \( p \) of \( x \) we mean any element \( y \) with \( y^p = x \); we also say that \( y \) is a \( p \)-th root of \( x \). Notice that in a cyclic \( p \)-group any element has \( p \) roots of degree \( p \) if it is not a generator of the cyclic group, and no \( p \)-th roots otherwise. Now let \( H = (\mathfrak{T}) = \langle (x_1, x_2, ..., x_n) \rangle \), and let \( H_i \) be \( i \)-th projection.
of $H$. Then $H_i = \langle x_i \rangle$ for all $i \leq n$, and we have: $H$ is a non-maximal cyclic subgroup of $G$ if and only if $\mathfrak{p}$ has a $p$-th root in $G$ if and only if $x_i$ has a $p$-th root in $C_{p^r_i}$, for all $i \leq n$ if and only if $|H_i| < p^{b_i}$ for all $i \leq n$. This proves (3).

Let $H = (\mathfrak{T}) = (\langle x_1, x_2, ..., x_n \rangle)$ be a non-maximal cyclic subgroup of $G$ of order $p^h > 1$, and let $H_i$ be the $i$-th projection of $H$ of order $p^{h_i}$ for all $i \leq n$. Then $H_i = \langle x_i \rangle$ for all $i \leq n$. Now for each $i \leq n$ the group $\langle x_i \rangle$ is a non-maximal cyclic subgroup of $C_{p^{r_i}}$, and $x_i$ has $p$ roots of degree $p_i$, so $\mathfrak{T}$ has $p^{h_i}$ roots of degree $p$. As in each cyclic subgroup of $G$ of order $p^{h+1}$ containing $\langle \mathfrak{T} \rangle$ as a subgroup has $p$ roots of degree $p$, it follows that $\langle \mathfrak{T} \rangle$ is subgroup of $p^{h+1}$ cyclic subgroups of order $p^{h+1}$. This proves the first part of (4). Let us prove the second part as well.

Let $L = \{ i \in \mathbb{N}_n \mid h_i > 0 \}$, and let $L \subseteq M \subseteq \mathbb{N}_n$. Then the number or $p$-th roots of $\mathfrak{T}$, whose $i$-th projection is non-trivial if and only if $i \in M$, is $p^{|L|} |(p - 1)^{|M|} - |L|$. Again, since each cyclic subgroup of order $p^{h+1}$ containing $H$ as a subgroup contains $p$ roots of degree $p$ of $\mathfrak{T}$, $H$ is a subgroup of $p^{|L|} |(p - 1)^{|M|} - |L|^{p}$ cyclic subgroups of order $p^{h+1}$ whose $i$-th projection is non-trivial if and only if $i \in M$. This proves the second part of (4), and the theorem has been proved.

Note that there are non-abelian finite groups whose enhanced power graphs are isomorphic to a $p$-semitree. The group with representation $G = \langle x, y \mid x^3 = y^3 = [x, y]^3 = 1 \rangle$, mentioned in [5], of order 27 and exponent 3 has enhanced power graph isomorphic to the 3-semitree $S_3(1, 1, 1)$. Another example would be $M_{16}$ with representation $M_{16} = \langle a, x \mid a^8 = x^2 = 1, xax^{-1} = a^5 \rangle$ whose enhanced power graph is isomorphic to $S_2(3, 1)$.

**Theorem 21** A graph $\Gamma$ is the enhanced power graph of a finite abelian group if and only if $\Gamma \cong \bigotimes_{i=1}^n S_{p_i}(\mathfrak{T}_i)$ for some tuples $\mathfrak{T}_i$ of positive integers and for some pairwise different primes $p_1, p_2, ..., p_n$.

**Proof.** The theorem follows from Theorem 20 and Lemma 2. 

In the continuation of this section we will give a graph property which, for any finite group $G$, is satisfied by $G_\Gamma(G)$ if and only if $G$ is nilpotent.

The following theorem is a well known fact about nilpotent groups.

**Theorem 22** Let $G$ be a finite group. Then the following conditions are equivalent:

1. $G$ is nilpotent;
2. $G$ is the direct product of its Sylow subgroups;
3. $G$ has a unique $p$-Sylow subgroup, for all prime divisors $p$ of $|G|$.

In the next proposition we will prove that from a finite group’s enhanced power graph we can conclude whether the group is nilpotent.

**Proposition 23** Let $G$ be a finite group of order $p_1^{k_1} p_2^{k_2} ... p_n^{k_n}$, where $p_1, p_2, ..., p_n$ are pairwise different primes. For each $i \leq n$ and $j \in \{0, 1, ..., k_i\}$, let $D_{j_i}$ be the set of all maximal cliques of $G_\Gamma(G)$ of sizes divisible by $p_i$. Then:
1. **G** has a unique $p_i$-Sylow subgroup if and only if $\sum_{j=0}^{k_i} (\varphi(p_i^j) \cdot |\mathcal{D}_i^j|) = p_i^{k_i}$.

2. **G** is nilpotent if and only if $\sum_{j=0}^{k_i} (\varphi(p_i^j) \cdot |\mathcal{D}_i^j|) = p_i^{k_i}$ for all $i \leq n$.

Proof. Let $i \leq n$, and suppose that $\sum_{j=0}^{k_i} (\varphi(p_i^j) \cdot |\mathcal{D}_i^j|) = p_i^{k_i}$. By Proposition 11, for any $j$, $\varphi(p_i^j) \cdot |\mathcal{D}_i^j| = p_i^{k_i}$. This proves that $\varphi(p_i^j) \cdot |\mathcal{D}_i^j| = p_i^{k_i}$, which implies that $\varphi(p_i^j) \cdot |\mathcal{D}_i^j| = p_i^{k_i}$ for all $i \leq n$. Therefore, $\sum_{j=0}^{k_i} (\varphi(p_i^j) \cdot |\mathcal{D}_i^j|) = p_i^{k_i}$, which is the number of elements of order $p_i$ in $G$. Since the number of such elements is $p_i^{k_i}$, the proof is complete.

The following lemma provides some necessary conditions for a finite graph to be the enhanced power graph of a group.

**Lemma 24** Let $\Gamma$ be a finite graph of order $p_1^{k_1}p_2^{k_2}...p_n^{k_n}$. Let $\mathcal{C} = \{C_1, C_2, ..., C_n\}$ be the set of all maximal cliques of $\Gamma$, and let $\mathcal{B} = \{\bigcap_{C_i \in M} C_i \mid M \in \mathcal{P}(\mathcal{C})\}$ be the set of all intersections of maximal cliques. If $\Gamma$ is the enhanced power graph of a finite group, then:

(E1) $\bigcap_{\mathcal{C} \in \mathcal{C}} \neq \emptyset$;

(E2) $|\mathcal{C}| \mid p_1^{k_1}p_2^{k_2}...p_n^{k_n}$ for any $C \in \mathcal{C}$;

(E3) For any $B_1, B_2 \in \mathcal{B}$: If $B_1, B_2 \subseteq C$ for some $C \in \mathcal{C}$, then $|B_1 \cap B_2| = \gcd(|B_1|, |B_2|)$.

Proof. Suppose that $\Gamma = \mathcal{G}_e(G)$ for a finite group $G$. By Lemma 3, all elements of $\mathcal{B}$ and $\mathcal{C}$ are cyclic subgroups of $G$. (E1) follows from the fact that all subgroups of $G$ contain the identity element of $G$. $|G|$ is divisible by order of any subgroup of $G$, which implies (E2).

Suppose that $B_1, B_2 \in \mathcal{B}$ and $B_1, B_2 \subseteq C$ for some $C \in \mathcal{C}$. Then $B_1, B_2$ and $C$ are cyclic groups with $B_1, B_2 \leq C$, and the order of $B_1 \cap B_2$ is $\gcd(|B_1|, |B_2|)$. This proves (E3).

Notice that for any finite graph, with the set of maximal cliques $\mathcal{C}$, and the set of all intersections of maximal cliques $\mathcal{B}$, (E3) implies the following two conditions:

(E4) For any $B_1, B_2 \in \mathcal{B}$: $B_1 \subseteq B_2$ implies $|B_1||B_2|$;

(E5) For any $B_1, B_2 \in \mathcal{B}$: If $|B_1||B_2|$ and $B_1, B_2 \subseteq C$ for some $C \in \mathcal{C}$, then $B_1 \subseteq B_2$.

Therefore, (E4) and (E5) are also necessary conditions for a finite graph to be the enhanced power graph of a group.

However, conditions (E1) and (E3) are not sufficient for a graph to be the enhanced power graph. For example, the graph in Figure 1 satisfies (E1) and (E3) although it is not enhanced power graph of any finite group. Namely, $G_8$,
Figure 1:

$C_4 \times C_2$, $C_2 \times C_2 \times C_2$, $D_8$, and $Q_8$ are the only groups of order 8, and the graph from Figure 1 is isomorphic to none of their enhanced power graphs.

For a group $G$ and a prime $p$ we shall denote with $G_p$ the set \{ $x \in G \mid (\exists i \in N_0)(o(x) = p^i)$ \}. Note that $G_p$ may not be a subgroup of $G$. However, if $G$ has the unique $p$-Sylow subgroup $P$, then $P = G_p$.

We will prove that the enhanced power graph of a finite group with a unique $p$-Sylow subgroup determines the enhanced power graph of its $p$-Sylow subgroup.

For this reason we will introduce the notion of $p$-component of a graph.

**Definition 25** Let $\Gamma$ be a finite graph, and let $p$ be a prime number. Let $C_{\text{Gl}} = \{ C_1, C_2, \ldots, C_n \}$ be the set of all maximal cliques of $\Gamma$, and let $B = \{ \bigcap_{i \in M} C_i \mid M \in \mathcal{P}^+(N_n) \}$ be the set of all intersections of maximal cliques of $\Gamma$. A $p$-component of $\Gamma$, denoted with $\Gamma_p$, is any induced subgraph of $\Gamma$ with property:

For each $B \in B$, of size $p^k l$ with $p \not| l$,

\[ B \text{ contains exactly } p^k \text{ vertices of } \Gamma_p. \]  

(4)

Now we give an algorithm which for the enhanced power graph of a finite group having a unique $p$-Sylow subgroup returns the enhanced power graph of the $p$-Sylow subgroup.

**Proposition 26** Let $\Gamma$ be a finite graph satisfying conditions $[E1]$ and $[E3]$, and let $p$ be a prime divisor of $|\Gamma|$. Then there is an algorithm for constructing a $p$-component of $\Gamma$.

**Proof.** Let $C_1, C_2, \ldots, C_n$ be all maximal cliques of $\Gamma$, and let us denote $\bigcap_{i \in M} C_i$ with $B_M$ for every $M \in \mathcal{P}^+(N_n)$. Let $n_{[p]} = \max \{ p^k \mid k \in N_0 \text{ and } p^k | n \}$ for any $n \in N$. We claim that the following algorithm constructs a $p$-component of $\Gamma$.

1. Mark $|B_{N_n}_{[p]}$ vertices from $B_{N_n}$, and cross out $N_n$.

2. Choose a set $M_1 \in \mathcal{P}^+(N_n)$ whose all supersets but itself have been crossed out. Then mark more vertices from $B_{M_1} \setminus \bigcup_{M \supset M_1} B_M$, if any is needed, in order for $B_{M_1}$ to contain exactly $|B_{M_1}_{[p]}$ marked vertices, and cross out $M_1$.

3. Repeat the step 2. until all non-empty subsets of $N_n$ have been crossed out.

4. Construct $\Gamma'$ as subgraph of $\Gamma$ induced by the set of all marked vertices.
It is sufficient to prove that, for any \( M_1 \in \mathcal{P}^+ \) whose all supersets but itself have been crossed out, \( |B_{M_1}|_p \) is at least the sum of \( |B_{M_1} \setminus \bigcup_{M \supsetneq M_1} B_i| \) and the number of vertices from \( \bigcup_{M \supsetneq M_1} B_i \) that have been marked. Namely, were it true, then the algorithm finishes, and it outputs \( \Gamma' \) which satisfies (1).

Let \( M_1 \in \mathcal{P}^+(N_n) \), and let us denote all subsets \( B_M \) for which \( M_1 \subseteq M \subseteq N_n \) with \( B_1, B_2, \ldots, B_m \). Suppose that all strict supersets of \( M_1 \) have been crossed out, but that \( M_1 \) has not. Then each \( B_i \) contains exactly \( |B_i|_p \) marked vertices. Let \( p^{k_1} = |B_{M_1}|_p \), and let \( p^{k_2} = \max\{|B_i|_p \mid i \leq m\} \). Notice that \( p^{k_2} \leq p^{k_1} \) by (4). Without loss of generality let \( |B_1|_p = p^{k_2} \). We claim that all marked vertices from each \( B_i \) belong to \( B_1 \). Namely, for all \( i \leq m \) we have \( B_i \subseteq B_{M_1} \subseteq C_j \) for some \( j \leq n \), so by (3) it follows \( |B_i \cap B_{M_1}| = |B_i|_p \). Therefore, all marked vertices from \( B_i \) belong to \( B_1 \), which proves the claim. For this reason, \( \bigcup_{i=1}^{m} B_i \) contains exactly \( p^{k_2} \) marked vertices, and it remains for us to show that \( p^{k_1} \leq |B_{M_1} \setminus \bigcup_{i=1}^{m} B_i| + p^{k_2} \).

Let \( G \) be a cyclic group of order \(|B_{M_1}|_p \), and let \( G_i \) be the subgroup of \( G \) of order \(|B_i|_p \) for all \( i \leq m \). Notice that \(|G_i \cap C_j| = \gcd(|G_i|, |G_j|) \). Then by (3) and by the inclusion-exclusion principle we get

\[
|B_1 \cup B_2 \cup \ldots \cup B_m| = \sum_{i=1}^{m} (-1)^{i+1} \left| \bigcap_{j \in J} B_i \right| = \sum_{i=1}^{m} (-1)^{i+1} \sum_{J \subseteq \{1, \ldots, n\}} \left| \bigcap_{j \in J} G_i \right|
\]

because \( \bigcup_{i=1}^{m} G_i \) contains an element of \( G \) of order \( p^i \) for no \( i > k_2 \). Therefore, \( p^{k_1} \leq |B_{M_1} \setminus \bigcup_{i=1}^{m} B_i| + p^{k_2} \), and it follows that one can mark \( p^{k_1} - p^{k_2} \) vertices from \( B_{M_1} \setminus \bigcup_{M \supsetneq M_1} B_M \), and then for any \( M \supsetneq M_1 \) there would be exactly \(|B_{M_1}|_p \) marked vertices in \( B_M \).

Therefore, the algorithm eventually crosses over all non-empty subsets of \( N_n \), and it marks vertices of \( \Gamma \) so that each intersection of maximal cliques \( B \) contains exactly \(|B|_p \) marked vertices. Then the subgraph \( \Gamma' \) of \( \Gamma \) induced by the set of marked elements is a \( p \)-component of \( \Gamma \).

\[\square\]

**Lemma 27** Let \( G \) be a finite group, and let \( p \) be a prime divisor of \(|G|\). Then the subgraph of \( G_e(G) \) induced by \( G_p \) is a \( p \)-component of \( G_e(G) \).

**Proof.** Let \( \mathcal{C}l \) be the set of all maximal cliques of \( G_e(G) \), and let \( \mathcal{B} \) be the set of all intersections of maximal cliques of \( G_e(G) \). By Lemma 3 \( \mathcal{B} \) is a cyclic subgroup of \( G \) for each \( B \in \mathcal{B} \). Therefore, if \(|B|_p = p^k \) for some \( k \) and \( l, p \nmid l \), then \( B \) has a unique subgroup of order \( p^k \), so \( B \) contains exactly \( p^k \) elements whose orders are powers of \( p \).

\[\square\]

**Lemma 28** Let \( G \) be a finite group, and let \( p \) be a prime divisor of \(|G|\), and let \( \Gamma = G_e(G) \). Then all \( p \)-components of \( \Gamma \) are isomorphic to \( \Gamma(G_p) \).

**Proof.** Let \( \Delta_1 \) and \( \Delta_2 \) be two \( p \)-components of \( \Gamma \). For any \( x, y \in G \) we say \( x \equiv y \) if and only if \( \overline{N}_\Gamma(x) = \overline{N}_\Gamma(y) \). By Lemma 3 (2), \( x \equiv y \) if and only if \( x \in \mathcal{C}l \Rightarrow y \in \mathcal{C}l \) for all \( \mathcal{C}l \in \mathcal{C}l \). By Lemma 3 (4) \( G/ \equiv \) can be ordered in the following way: \( [x]_{\equiv} \leq [y]_{\equiv} \) if \( y \in \mathcal{C}l \Rightarrow x \in \mathcal{C}l \) for all \( \mathcal{C}l \in \mathcal{C}l \). By Lemma 3 (1, and 5) \( [x]_{\equiv} \leq [y]_{\equiv} \) if and only if \( \overline{N}_\Gamma(y) \subseteq \overline{N}_\Gamma(x) \) if and only if \( ([y]_{\equiv}) \leq ([x]_{\equiv}) \).
Height of an element $x$ of a poset is the maximal length of a chain whose greatest element is $x$. By induction by height of $\equiv$-class in the poset $(G/\equiv, \leq)$, we prove that each $\equiv$-class contains the same number of vertices of $\Delta_1$ and $\Delta_2$.

The least element of $G/\equiv$ is $\bigcap_{i=1}^n C_i$, and by (4) it has the same numbers of vertices of $\Delta_1$ and $\Delta_2$. Now let $D \in G/\equiv$. Then by (4) $\langle D \rangle = \bigcap \{ C \in Cl \mid D \subseteq C \}$ has the same numbers of vertices of $\Delta_1$ and $\Delta_2$. By the induction hypothesis all $\equiv$-classes $E < D$ have the same numbers of vertices of $\Delta_1$ and $\Delta_2$. Note that for all such $E$ we have $E \subseteq \langle D \rangle$. This implies that $D$ contains the same numbers of vertices of $\Delta_1$ and $\Delta_2$. This proves the claim that each $\equiv$-class contains the same number of vertices of $\Delta_1$ and $\Delta_2$.

Now, because the transposition of $x$ and $y$ is an automorphism of $\Gamma$ whenever $x \equiv y$, it follows that $\Delta_1$ and $\Delta_2$ are isomorphic by composition of transpositions of vertices belonging to same $\equiv$-classes. Since $\Gamma(G_p)$ is a $p$-component of $\Gamma$, the lemma has been proven.

**Theorem 29** Let $G$ be a finite group with a unique $p$-Sylow subgroup $G_p$ for a prime $p$. Then $G_c(G_p)$ is isomorphic to the $p$-component of $G_c(G)$.

**Proof.** The theorem follows directly from Lemma 28. □

If a finite graph $\Gamma$ of size $p_1^{n_1}p_2^{n_2}\ldots p_k^{n_k}$ satisfies (E1)-(E3), then a $p_i$-component $\Gamma_i$ can be constructed as described in Proposition 26 for any $i \leq k$. In this case $\Gamma$ is the enhanced power graph of a finite abelian group if and only if $\Gamma \cong \bigboxtimes_{i=1}^k \Gamma_{p_i}$, where $\Gamma_{p_i}$ is a $p_i$-semitrees for each $i$. If there are some $\Gamma_{p_i}$’s which are not $p_i$-semitrees, then we can not tell whether $\Gamma$ is an enhanced power graph of a finite group.

To check whether $\Gamma_p$ is a $p$-semitree, i.e. the enhanced power graph of a finite abelian $p$-group, we can, as in Lemma 11, figure how many elements of any order would the group have. From this information we can, as in Proposition 12, conclude to which tuple $\overline{a}$ would $\Gamma_p$ be related as a $p$-semitree. Then remains to check whether $\Gamma_p \cong S_p(\overline{a})$.

If $\Gamma$ does not satisfy any of (E1)-(E3), then it is not the enhanced power graph of any group.

### 6 Perfectness of enhanced power graph

A graph $\Gamma$ is called perfect if for every finite induced subgraph $\Delta$ of $\Gamma$ the chromatic number of $\Delta$ is equal to the maximum size of a clique of $\Delta$. A graph $\Gamma$ is Berge graph if neither $\Gamma$ nor its complement contain an odd-length cycle of size at least 5 as an induced subgraph. By The Strong Perfect Graph Theorem, which was proved in [8], a graph $\Gamma$ is perfect if and only if $\Gamma$ is a Berge graph. For that reason, the perfect graphs and the Berge graphs make up the same class of graphs. We shall denote the clique number of $\Gamma$ with $\omega(\Gamma)$, and the chromatic number of $\Gamma$ with $\chi(\Gamma)$. A graph is called weakly perfect if $\chi(\Gamma) = \omega(\Gamma)$.

**Example 30** There is an abelian finite group whose enhanced power graph is not perfect.
Proof. Let $G = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle b_1 \rangle \times \langle b_2 \rangle \times \langle c_1 \rangle \times \langle c_2 \rangle$, where orders of $a_1$, $a_2$, $b_1$, and $b_2$, and $c_1$ and $c_2$ are 2, 3, and 5, respectively. Then its elements $a_1 b_1$, $b_1 c_2$, $a_2 b_2$, and $a_1 c_1$ form the pentagon $\Pi$ as an induced subgraph of $G_e(G)$. Indeed, we have $a_1 b_1, b_1 c_2, b_1 c_2, a_2 b_2, b_2, a_1 c_1 \in \langle a_1 b_1, b_1 c_1 \rangle$, and $a_2 b_2$, $a_2 b_1$, $b_1 c_2$, and $b_1 c_2$ are not adjacent in $G_e(G)$.

Let us assume that $a_1 b_1, b_1 c_2, a_2 b_2$, and $a_1 c_1$ form the pentagon $\Pi$ as an induced subgraph of $G_e(G)$. This proves one implication.

For a group $G$, let us define relation $\sim_G$ on $G$ so that $x \sim_G y$ if $x = y$ or $x \sim y$.

**Theorem 31** A finite nilpotent group has perfect enhanced power graph if and only if it has at most two non-cyclic Sylow subgroups.

Proof. Let us assume that $H_1$, $H_2$, and $H_3$ are three different non-cyclic Sylow subgroups of a nilpotent group $G$. We shall prove that $G_e(H_1)$ contains a pentagon as induced subgraph, where $H = H_1 \times H_2 \times H_3$. By Theorem [22] orders of $H_1$, $H_2$, and $H_3$ are pairwise relatively prime. Let $a_1$ be an element of $H_1$ of maximal order, and let $a_2 \in H_1 \setminus \langle a_1 \rangle$. Then $a_1$ and $a_2$ are not adjacent in $G_e(H_1)$. In the same way can find elements $b_1, b_2 \in H_2$, and $c_1, c_2 \in H_3$ which are not adjacent in $G_e(H_2)$ and $G_e(H_3)$, respectively. By this, and by Lemma [2] it follows that the elements $a_1 b_1$, $b_1 c_2, a_2, b_2$, and $a_1 c_1$ in $H$ form a pentagon as induced subgraph of $G_e(H)$. This obviously implies that $G_e(G)$ also contains a pentagon as an induced subgraph. This proves one implication.

Suppose now that $G = \prod_{j=1}^n K_j$, where $K_1, K_2, \ldots, K_n$ have pairwise co-prime orders. Assume that $K_1, K_4, \ldots, K_n$ are cyclic subgroups of $G$, and let us denote $K_1 \times K_2$ with $K$. We shall prove that $G_e(K)$ does not contain any odd-length cycle of size at least 5, nor a complement of an odd-length cycle of size at least 5, as an induced subgraph. In the proof of this theorem we shall refer to these graphs as forbidden graphs.

First we shall note that $a \sim_P b \sim_P c \sim_P d$, where $P$ is finite a $p$-group $P$, implies $a \sim_P c$ or $b \sim_P d$. Indeed, since $b$ and $c$ are contained in a same cyclic $p$-group, one is a power of the other. If $c$ is a power of $b$, then $a \sim_P c$, while if $b$ is a power of $c$, then we have $b \sim_P d$.

Suppose now that $G_e(K)$ contains an odd-length cycle of size at least 5 as an induced subgraph. Let the cycle be $a_1 b_1, a_2 b_2, \ldots, a_{2k+1} b_{2k+1}$, where $a_i \in H_1$ and $b_i \in H_2$ for all $i$. Then we have

$$a_1 b_1 \sim_K a_2 b_2 \sim_K a_3 b_3 \sim_K \ldots \sim_K a_{2k+1} b_{2k+1} \sim_K a_1 b_1,$$

which by Lemma [2] implies

$$a_1 \sim_K a_2 \sim_K a_3 \sim_K \ldots \sim_K a_{2k+1} \sim_K a_1,$$

and

$$b_1 \sim_K b_2 \sim_K b_3 \sim_K \ldots \sim_K b_{2k+1} \sim_K b_1.$$
Also, we have $a_1b_1 \neq K a_3b_3$, which implies $a_1 \neq K a_3$ or $b_1 \neq K b_3$. This way we get:

\[
\begin{align*}
  a_1 & \neq K a_3, \\
  a_2 & \neq K a_4, \\
  & \quad \vdots \\
  a_{n-1} & \neq K a_1, \\
  a_{2k+1} & \neq K a_2
\end{align*}
\]

which can be observed as an odd-sized cycle colored into two colors: the cycle is $(1, 3), (2, 4), (3, 5), \ldots, (2k+1, 2k+2)$, while a pair $(i, i+2)$ is colored with a only if $a_i \neq K a_{i+2k+1}$, and it is colored with $b$ only if $b_i \neq K b_{i+2k+2}$. As such cycle has two subsequent elements, without loss of generality we can assume that we have $a_1 \neq K a_3$ and $a_2 \neq K a_4$. On the other hand, as seen above, $a_1 \neq K a_3$, $a_2 \neq K a_4$ implies $a_1 \neq K a_3$ or $a_2 \neq K a_4$. We’ve got a contradiction, so $\mathcal{G}_e(K)$ does not contain an odd-length cycle of size at least five.

Next we show that $\mathcal{G}_e(K)$ does not contain the complement of an odd-sized cycle of size at least 7. Note that we already have the result for the size 5. Therefore, we can assume that we have $a_1 \neq K a_3$ or $a_2 \neq K a_4$. On the other hand, as seen above, $a_1 \neq K a_3$, $a_2 \neq K a_4$ implies $a_1 \neq K a_3$ or $a_2 \neq K a_4$. We’ve got a contradiction. Therefore, $\mathcal{G}_e(K)$ does not contain any of the forbidden subgraphs as an induced subgraph.

Notice that $C = \prod_{i=3}^{n} K_i$ is a cyclic group. Suppose that $G = K \times C$ contains a forbidden subgraph. Now by Lemma 2, because $x \neq C y$ for any $x, y \in C$, it follows that $K$ contains a forbidden graph, which is a contradiction. Therefore, $G_e(K)$ does not contain any of the forbidden subgraphs as an induced subgraph. So by The Strong Perfect Graph Theorem we get that any finite nilpotent group with at most two non-cyclic Sylow subgroups contains a forbidden graph as an induced subgraph. So by The Strong Perfect Graph Theorem we get that any finite nilpotent group with at most two non-cyclic subgroups has a perfect enhanced power graph.

On the other hand, the following theorem implies that there are many groups which do not have perfect enhanced power graph, but whose clique number and chromatic number are equal.

**Theorem 32** If a finite group $G$ has an element whose order is equal to the exponent of $G$, then $G_e(G)$ is weakly perfect.
Proof. Let $G$ be a finite group which has an element whose order is equal to the exponent of $G$. Let $x$ be such an element, and let $o(x) = n$. Then $\omega(G_e(G)) = n$. We can color the elements of $\langle x \rangle$ with $n$ colors. Now, for any $\approx$-class $D$ of $G$ there is a $\approx$-class $C \subseteq \langle x \rangle$, such that $C$ and $D$ contain elements of the same order. Therefore, we can color the elements of $D$ with the same colors the elements of $C$ were colored with. Suppose now that all edges of $G_e(G)$ have been colored in this manner, and suppose that $y$ and $z$ are colored with a same color. Then $o(y) = o(z)$ and $y \not\approx z$, which implies $y \not\sim z$. For that reason, in this coloring of the graph $G_e(G)$ there is no pair of adjacent vertices colored with a same color, and we have $\chi(G_e(G)) = \omega(G_e(G))$. 

Corollary 33 If $G$ is a finite nilpotent group, then $G_e(G)$ is weakly perfect.

Proof. Let $G$ be a finite nilpotent group. Then it is a direct product of its Sylow subgroups, implying $G$ has an element whose order is equal to the exponent of $G$. Therefore, by Theorem 32, it follows that $G_e(G)$ is weakly perfect. \hfill \Box

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