HIGHER-ORDER CORRECTIONS TO
THE LARGE-\(N_c\) BOUND ON \(M_\eta/M_{\eta'}\)

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ABSTRACT

Next-to-leading \(1/N_c\) corrections to the upper bound on \(M_\eta/M_{\eta'}\) recently obtained by Georgi are considered. These corrections are just what is needed to reconcile the bound with the observed \(\eta\) and \(\eta'\) masses.

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Recently Georgi\textsuperscript{[1]} has observed an amusing fact concerning the large-$N_c$ approximation to QCD\textsuperscript{[2]} when applied to the $\eta$–$\eta'$ system\textsuperscript{[3]}. To lowest nontrivial order in $1/N_c$ and in the quark masses he has found that \( \frac{M^2_{\eta'}}{M^2_{\eta}} \leq \frac{3-\sqrt{3}}{3+\sqrt{3}} + O\left(\frac{m_u d}{m_s}\right) \). The experimental number is higher than this upper bound. Consequently it is mathematically impossible (and not just inaccurate) to fit the experimentally observed masses within this approximation.

The purpose of the present short note is to show that higher orders in $1/N_c$ eliminate this bound. This is of course no surprise since higher orders means new operators with unknown coefficients, so that the freedom in parameter space is larger, making it possible to avoid the constraints that lead to the bound of ref. [1]. It should be noticed, however, that it is because the bound in [1] is very close to the experimental number that this is possible. Corrections in $1/N_c$, being corrections, should be “small”, and it is hard to believe that they could fix this problem if Georgi’s bound had turned out to be very different from the experimental number.

Large-$N_c$ arguments provide a beautiful explanation of what used to be known as the $U(1)$ puzzle\textsuperscript{[3]}; they offer us an interesting way (if not the only one) to get a handle on the physics of the $\eta'$ from first principles, i.e. from QCD. It would have been very disturbing if the bound had remained after higher $1/N_c$ corrections were included, so it was necessary to check that this indeed does not happen. On the other hand it was already found in ref. [4] that some of the physics of the $\eta'$, such as the decay $\eta' \rightarrow \eta \pi \pi$, cannot be described, even qualitatively, without going to higher orders in $1/N_c$.

The limit $N_c \rightarrow \infty$ offers a consistent way to turn off the anomaly. In this limit, and in a world of massless quarks, the $\eta'$ truly becomes the Goldstone boson of the $U(1)_A$ symmetry that is seen at the level of the QCD Lagrangian. It is then expected that the real world will be reached from this limit by means of a combined perturbative expansion in the quark mass, $m_q$, and $1/N_c$.

We shall now show that higher-order corrections in the quark mass alone cannot reconcile the lowest-order bound obtained by Georgi with the empirical masses
because this bound persists, in fact, to all orders in $m_q$ in the limit $N_c \to \infty$, and not only to first order as originally derived in ref. [1]. Therefore, higher-order $1/N_c$ corrections are absolutely essential for this reconciliation\(^\dag\).

Let us start with the Lagrangian of QCD for massive quarks in the limit $N_c \to \infty$. Because there are no OZI violating $q\bar{q}$ annihilation diagrams in this limit, there exists a symmetry $U(1)_q \times U(1)_{\bar{q}}$ for each quark flavor that transforms independently quarks and antiquarks. Therefore the mass matrix, $\mathcal{M}$, for the $\pi^0$ ($= \frac{u\bar{u} - d\bar{d}}{\sqrt{2}}$), $\eta_8$ ($= \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}}$) and $\eta_0$ ($= \frac{u\bar{u} + d\bar{d} + s\bar{s}}{\sqrt{3}}$) mesons must read, in the basis $(u\bar{u}, d\bar{d}, s\bar{s})$,

$$\mathcal{M}^2 = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}.$$  \hspace{1cm} (1)

Non-diagonal entries vanish because they originate from symmetry-violating transitions in which a quark with a given flavor comes in but does not get out.

If we further take the reasonable limit $m_u = m_d = 0, m_s \neq 0$, then it turns out that there exists a further $SU(2)_A \subset SU(2)_L \times SU(2)_R$ symmetry rotating the up and down quarks. Under this symmetry the $\pi^0$ meson gets shifted by a constant amount, proportional to its decay constant, and therefore any mass term (i.e. $\pi^0 - \pi^0$, $\pi^0 - \eta_8$ and $\pi^0 - \eta_0$) must vanish. Hence $A$ and $B$ must be zero and the mass matrix $\mathcal{M}$ reads

$$\mathcal{M}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C \end{pmatrix}.$$ \hspace{1cm} (2)

If one now wishes to include the anomaly as the lowest-order correction in $1/N_c$,

\(^\dag\) The following discussion can be considered as a generalization of the results obtained in ref. [1] and originated from an illuminating comment by G. Veneziano that we gratefully acknowledge.
one obtains
\[ M^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C \end{pmatrix} + \frac{a}{N_c} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (3) \]
where \( a \) is the parameter that measures the strength of the anomaly. This mass matrix is exactly of the same form as that of ref. [1]* and leads to the same mass ratio:
\[ \frac{M^2_{\eta}}{M^2_{\eta'}} = \frac{3 + R - \sqrt{9 - 2R + R^2}}{3 + R + \sqrt{9 - 2R + R^2}}, \quad (4) \]
with \( R \equiv CN_c/a \). This mass ratio is maximized for \( R = 3 \) and one obtains
\[ \frac{M^2_{\eta}}{M^2_{\eta'}} \leq \frac{3 - \sqrt{3}}{3 + \sqrt{3}} + \mathcal{O}\left( \frac{m_{u,d}}{m_s} \right) \quad (5) \]
as in ref. [1]. However, from this derivation we see that this result is valid to all orders in the quark mass in the limit \( N_c \to \infty \).

The above discussion tells us that consideration of \( 1/N_c \) corrections will be crucial when discussing modifications to the bound (5). As a matter of fact what one has is a combined series expansion in the quark mass and \( 1/N_c \). To lowest nontrivial order, contributions to the Goldstone boson mass matrix are due to operators of order \( m_q \) and \( a/N_c \) (i.e. the anomaly). To next-to-leading order, one must certainly take into account corrections of order \( 1/N_c \) to the previous operators but also corrections of order \( m_q^2 \), at least in principle. Because of the above discussion, however, contributions that are quadratic in the quark mass will not affect the bound and may consequently be disregarded. This makes the following analysis considerably simpler.

Since we will not deal with any strong CP violation effect, we shall set \( \theta_{QCD} = 0 \). Therefore we shall next consider the quadratic part of the Lagrangian describing the \( \eta-\eta' \) system to next-to-leading order in \( 1/N_c \). It can be obtained from [4]

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* In ref. [1] \( C \) was approximated by its lowest-order value, i.e. \( C = m_s \times \text{const.} \).
\[ \mathcal{L} = \mathcal{L}_0 + \delta \mathcal{L} \]

\[ \mathcal{L}_0 = \frac{f_\pi^2}{4} \left[ \text{Tr} \partial_\mu U \partial^\mu U^\dagger + \text{Tr} (\chi U^\dagger + U \chi^\dagger) + \frac{a}{4 N_c} \left( \text{Tr} \log U - \text{Tr} \log U^\dagger \right)^2 \right] , \]

\[ \delta \mathcal{L} = \frac{f_\pi^2}{4} \left[ \frac{2a}{3 N_c} \text{Tr} U^\dagger \partial_\mu U \text{Tr} U^\dagger \partial^\mu U + \right. \\
\left. + \frac{\epsilon}{2 \sqrt{2 N_c}} \left( \text{Tr} \log U - \text{Tr} \log U^\dagger \right) \text{Tr} (\chi U^\dagger - \chi^\dagger U) \right] + ... , \]  

where \( ^\dagger \ U = \exp \left( \frac{-i \sqrt{2} \Phi(x)}{f_\pi} \right) \) with \( \Phi(x) = \frac{\phi^0}{\sqrt{3}} + \frac{\bar{\lambda} \phi}{\sqrt{2}} \) and \( \bar{\lambda} \) are the eight Gell-Mann matrices, \( f_\pi \simeq 93 \text{MeV} \) is the pion decay constant; \( \chi = 2 B_0 M \) where \( B_0 \) is a parameter related to the quark condensate in QCD and \( M \) is the quark mass matrix. In this Lagrangian \( a, \alpha \) and \( \epsilon \) are parameters of \( O(N_c^0) \). Then the first two terms in \( \mathcal{L}_0 \) are of \( O(p^2 N_c^0) \) and the term proportional to \( a \) is of \( O(a N_c^{-1}) \). The Lagrangian \( \mathcal{L}_0 \) is to be considered the leading-order Lagrangian, and \( \delta \mathcal{L} \) is the \( 1/N_c \) correction to it. Contributions of order \( a/N_c^2 \) to the mass matrix can be absorbed in a redefinition of \( a \).

Using the Lagrangian (6), it is straightforward to compute the mass matrix in the \( (\eta_8, \eta_0) \) basis. Neglecting terms proportional to the up and down quark masses but not to the strange quark mass, one finds \( ^\ddagger \)

\[ M^2 = \frac{4}{3} M_K^2 \begin{pmatrix} 1 & -\frac{y}{\sqrt{2}} \\ -\frac{y}{\sqrt{2}} & \frac{y^2}{2} + x \end{pmatrix} , \]  

where

\( ^\dagger \) We follow the notation of ref. [6].

\( ^\ddagger \) Notice that the term in eq. (6) proportional to \( \alpha \) affects the mass matrix through the normalization of the kinetic term.
\[ y \equiv 1 + \delta y = \frac{1 - \frac{3\epsilon}{\sqrt{2}N_c}}{\sqrt{1 - \frac{2\alpha}{N_c}}} \approx 1 - \frac{1}{N_c} \left( \frac{3\epsilon}{\sqrt{2}} - \alpha \right) + \mathcal{O}\left(\frac{1}{N_c^2}\right) , \quad (8) \]

\[ x \equiv \frac{9a}{4M_K^2N_c(1 - \frac{2\alpha}{N_c})} \approx \frac{9a}{4M_K^2N_c} \left( 1 + \frac{2\alpha}{N_c} + \mathcal{O}\left(\frac{1}{N_c^2}\right) \right) . \quad (9) \]

Amusingly, although our Lagrangian (6) has three unknown parameters \((a, \epsilon\) and \(\alpha)\) to start with (\(B_0m\) is fixed through the kaon mass), the mass matrix (7) depends only on two combinations of them, i.e. \(x\) and \(y\).

In the spirit of the \(1/N_c\) expansion one should take \(\delta y\) as a small parameter and expand in it. The \(\eta\) and \(\eta'\) masses are then determined by the conditions

\[ \frac{3(M_{\eta}^2 + M_{\eta'}^2)}{4M_K^2} = \frac{3}{2} + x + \delta y , \quad (10) \]

\[ \frac{9M_{\eta}^2M_{\eta'}^2}{16M_K^4} = x . \quad (11) \]

These equations yield \(\delta y \approx -0.35\) and \(x \approx 2.57\) when the masses \(M_K \approx 495\text{MeV}, M_{\eta} \approx 547\text{MeV}, M_{\eta'} \approx 958\text{MeV}\) are used. It is clear that the system (10) + (11) has always one solution for \(x\) and \(\delta y\), once the masses for the pseudoscalars are given; the mass matrix (7) can thus fit the \(\eta\) and \(\eta'\) masses, and the bound of ref. [1] is overcome. As a matter of fact, taking the mass matrix (7) one easily obtains

\[ \frac{M_{\eta}^2}{M_{\eta'}^2} = \frac{1 + x + y^2/2 - \sqrt{(1 + x + y^2/2)^2 - 4x}}{1 + x + y^2/2 + \sqrt{(1 + x + y^2/2)^2 - 4x}} . \quad (12) \]

This expression has a maximum when varied with respect to \(x\) (\(\sim\) the anomaly to quark mass ratio), keeping \(y\) fixed. One can understand this on physical grounds:
for $x \rightarrow \infty$ the mass ratio (12) goes to zero because the $\eta'$ mass becomes infinite. Furthermore, when $x \rightarrow 0$ the mass ratio also goes to zero because the $\eta$ mass vanishes since the situation of the $U(1)_A$ problem is reproduced. So there must be an intermediate value of $x$ at which eq. (12) has a maximum. Moreover, in this eq. (12) $y$ equals unity only when next-to-leading terms in $1/N_c$ are neglected. In this case one can see that eq. (12) reaches its maximum at $x_0 = 3/2$ and one then obtains the bound \cite{5} (5). However, if next-to-leading terms are included one has instead that $y \approx 1 + \delta y$ with $|\delta y| \sim 1/N_c << 1$. Expanding eq. (12) in $\delta y$ about $x_0$ one obtains
\begin{equation}
\frac{M^2_\eta}{M^2_{\eta'}} \leq \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \left(1 - \frac{2}{\sqrt{3}} \delta y\right) .
\end{equation}
A negative value for $\delta y$ of order $1/N_c \sim 0.3$ is more than enough for eq. (13) to be satisfied experimentally.

One must also assess the consistency of the $1/N_c$ expansion. First of all, the size of the $1/N_c$ corrections, $|\delta y| \simeq 0.35$, is indeed of order $1/N_c$. Secondly, within our approximation one finds that $f^{2}_{\eta_0} = f^{2}_{\pi}(1 - \frac{2\alpha}{N_c})$ for the decay constants. The analysis of ref. \cite{8} obtains that $f_{\eta_0} \simeq f_{\pi}$, which would suggest that $\alpha$ is small. Then the result for $x$ leads essentially to the same value of $a$ and the same estimate for the topological susceptibility as the original work of Veneziano \cite{7}. Notice that this is a consequence of the fact that eq. (11) is $y$-independent, which in turn stems from the particular $y$-dependence of the mass matrix (7). However, were we to compute the $\eta-\eta'$ mixing angle, we would obtain around $10^\circ$, i.e. half the experimental number. This is by now a well-known fact that results from a fortuitous approximate cancelation in the expression for this angle, which occurs in lowest non-trivial order in $1/N_c$ and $m_q$. This makes the $\eta-\eta'$ mixing angle a very sensitive parameter whose calculation can be reconciled with experiment only after corrections of $O(m^2_q)$ are included, as suggested in ref. \cite{8}. These corrections will also modify the relation $f_{\pi} = f_{\eta_0}$. However, a full calculation including next-to-leading terms in $1/N_c$ and terms of $O(m^2_q)$, although interesting, is beyond the scope of the present short note.
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