Synthesis of Boolean Functions with Clausal Abstraction

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Abstract. Dependency quantified Boolean formulas (DQBF) is a logic admitting existential quantification over Boolean functions, which allows us to elegantly state synthesis problems in planning and verification. In this paper, we lift the clausal abstraction algorithm for quantified Boolean formulas (QBF) to DQBF. Clausal abstraction for QBF is an abstraction refinement algorithm that operates on a sequence of abstractions that represent the different quantifiers. For DQBF we need to generalize this principle to partial orders of abstractions. The two challenges to overcome are: (1) Clauses may contain literals with incomparable dependencies, which we address by the recently proposed proof rule called fork extension, and (2) existential variables may have spurious dependencies, which we prevent by tracking (partial) Skolem functions during the execution. Our prototype implementation shows improved performance compared to previous algorithms.

1 Introduction

Planning and verification problems can be elegantly stated as the search for functions. For example, the termination of a program can be stated as the existence of a ranking function, and planning can be stated as the existence of a function mapping (discrete) observations to actions. The search for functions given such declarative specifications is often called the synthesis problem, and is considered to be an extremely hard algorithmic problem. Even modern SMT solvers are usually ineffective for these problems, if the function being searched for does not fall into some narrow pattern and has a domain that is too large to be enumerated. We focus on dependency quantified Boolean formulas (DQBF), which is probably the simplest logic admitting the existential quantification over Boolean functions, using so called Henkin quantifiers. While existing algorithms for DQBF suffer from poor performance [7,8,4], recent progress in proof systems for DQBF raises the hope that we may be able to exploit the same algorithmic insights that make SAT and QBF solvers effective algorithmic backends.

This paper is a first attempt at leveraging the fork resolution proof system [18] and at exploring the space of practical algorithms that it enables. We propose a new algorithm for DQBF that generalizes the idea of clausal abstraction [19], also known as clause selection [13]. Clausal abstraction has been very

\textsuperscript{\textdagger} Work primarily done while at UC Berkeley
successful for QBF, winning the recent editions of the annual QBF competition QBFEVAL [16]. The idea of clausal abstraction for QBF is to split the given quantified problem into a sequence of propositional problems, one for each quantifier in the quantifier prefix, and instantiate a SAT solver for each of them. The original quantified problem is then solved through the interaction between this sequence of SAT solvers. Thereby, clausal abstraction exploits the linear order of the quantifiers in QBF in prenex normal form. For DQBF, however, quantifiers can form an arbitrary partial order, which introduces two new challenges. (1) Clauses may now contain variables from incomparable quantifiers, and (2) variables may have spurious dependencies, i.e. they may only be able to satisfy all the constraints, if they depend on variables that are not allowed to by the Henkin quantifiers. Challenge (1) can be addressed by the recently introduced proof rule fork extension [18]. Our algorithm eagerly applies fork extension, which introduces new variables. (Through the implementation of the algorithm, we also identified a problem with the fork extension proof system for which we discuss two solutions in Section 3.) Challenge (2) can be addressed by exploiting the ease of certification in clausal abstraction algorithms and record the Skolem functions [19]. By recording partial Skolem functions during the execution, we make sure that the existential variables only depend on universals they are allowed to depend on.

The main contribution of this work is the generalization of clausal abstraction to DQBF. The experimental evaluation shows that clausal abstraction for DQBF outperforms existing algorithms on most benchmarks.

2 Preliminaries

Let \( \mathcal{V} \) be a finite set of propositional variables. We use the convention to denote universally quantified variables (short also universals) by \( x \) and existentially quantified variables (or existentials) by \( y \). The set of all universals is denoted \( \mathcal{X} \), and the set of all existentials is denoted \( \mathcal{Y} \). For sets of universals and existentials we use \( X \) and \( Y \), respectively. We consider DQBF of the form \( \forall x_1 \ldots \forall x_n . \exists y_1 (H_1) . \ldots \exists y_m (H_m) . \varphi \), that is, DQBF begin with universal quantifiers followed by Henkin quantifiers and the quantifier-free part \( \varphi \). A Henkin quantifier \( \exists y (H) \) introduces a new variable \( y \), like a normal quantifier, but also explicitly states the dependencies \( H \subseteq \mathcal{X} \) of the variable. (In QBF the dependencies of a variable are implicitly determined by the universal variables that occur before the quantifier in the quantifier prefix.) A literal \( l \) is either a variable \( v \in \mathcal{V} \) or its negation \( \lnot v \). We call the disjunction \( C = (l_1 \lor l_2 \ldots \lor l_n) \) over literals a clause, and assume w.l.o.g. that the propositional part of DQBFs are given as a conjunction of clauses, i.e., in conjunctive normal form (CNF). We call the propositional part of a DQBF in CNF the matrix and we use \( C_i \) to refer to clause \( i \) of the matrix where unambiguous. For convenience, we treat clauses also as a set of literals and we treat matrices as a set of clauses and use the usual set operations for their manipulation. We denote by \( \text{var}(l) \) the variable \( v \) corresponding to literal \( l \). For literals \( l \) of existential variables with dependency set \( H \) we define \( \text{dep}(l) = H \).
For literals of universal variables we define $\text{dep}(l) = \{\text{var}(l)\}$. We lift the operator $\text{dep}$ to clauses by defining $\text{dep}(C) = \bigcup_{l \in C} \text{dep}(l)$. We define $C_V$ for some clause $C$ and set of variables $V$ as the clause $\{ l \in C \mid \text{var}(l) \in V \}$. Given a subset of variables $V \subseteq V$, an assignment of $V$ is a function $\alpha : V \rightarrow \mathbb{B}$ that maps each variable $v \in V$ to either true ($\top$) or false ($\bot$). A partial assignment is a partial function from $V$ to $\mathbb{B}$, i.e., it may be undefined on some inputs. We use $\alpha \cup \alpha'$ as the update of partial assignment $\alpha$ with $\alpha'$: $(\alpha \cup \alpha')(v) = \begin{cases} \alpha'(v) & \text{if } v \in \text{dom}(\alpha'), \\ \alpha(v) & \text{otherwise.} \end{cases}$

Further we write $\alpha \subseteq \alpha'$ if $\alpha(v) = \alpha'(v)$ for every $v \in \text{dom}(\alpha)$. To restrict the domain of an assignment $\alpha$ to a set of variables $V$, we write $\alpha|_{V}$. We denote the set of assignments of a set of variables $V$ by $\mathcal{A}(V)$. A Skolem function $f_{y} : \mathcal{A}(\text{dep}(y)) \rightarrow \mathbb{B}$ maps an assignment of the dependencies of $y$ to an assignment of $y$. The truth of a DQBF $\Phi$ with matrix $\varphi$ is equivalent to the existence of a Skolem function $f_{y}$ for every variable $y$ of the existentially quantified variables $V$, such that substituting all existentials $y$ in $\varphi$ by their Skolem function $f_{y}$ results in a valid formula. We use $\Phi[\alpha]$ to denote the replacement of variables bound by $\alpha$ in $\Phi$ with the corresponding value.

### 3 Underlying Proof System

In this section we recall the Fork Resolution proof system, which underlies the algorithm proposed in this paper. We also discuss a problem with the completeness of Fork Resolution and suggest two ways to overcome the problem.

Fork Resolution consists of the well-known proof rules Resolution and Universal Reduction and introduces a new proof rule called Fork Extension [13].

Resolution allows us to merge two clauses as follows: Given two clauses $C_1 \lor v$ and $C_2 \lor \neg v$, we call $(C_1 \lor v) \otimes (C_2 \lor \neg v) = C_1 \lor C_2$ their resolvent with pivot $v$. The resolution rule states that $C_1 \lor v$ and $C_2 \lor \neg v$ imply their resolvent. Universal reduction allows us to drop universal variables from clauses when none of the existential variables in that clause may depend on them. Let $C$ be a clause, let $l \in C$ be a literal of a universal variable, and let $\neg l \notin C$. If for all existential variables $y$ in $C$ we have $\text{var}(l) \notin \text{dep}(y)$, universal reduction allows us to derive $C \setminus l$. Fork Extension allows us to split a clause $C_1 \lor C_2$ by introducing a fresh variable $y$. The dependency set of $y$ is defined as the intersection $\text{dep}(C_1) \cap \text{dep}(C_2)$ and represents the question whether $C_1$ or $C_2$ satisfies the original clause needs to be resolved based on the information that is available to both of them. Fork extension is usually only applied when $C_1$ and $C_2$ have incomparable dependencies ($\text{dep}(C_1) \nsubseteq \text{dep}(C_2)$ and $\text{dep}(C_1) \nsubseteq \text{dep}(C_2)$), as only then the dependency set of $y$ is smaller than those of $C_1$ and of $C_2$. We state the rule formally in Fig. [1].

**Fig. 1. Fork Extension**

\[
C_1 \cup C_2 \quad \text{y is fresh}
\]

\[
\exists y(\text{dep}(C_1) \cap \text{dep}(C_2)). \quad C_1 \cup \{y\} \land C_2 \cup \{\neg y\}
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For literals of universal variables we define $\text{dep}(l) = \{\text{var}(l)\}$. We lift the operator $\text{dep}$ to clauses by defining $\text{dep}(C) = \bigcup_{l \in C} \text{dep}(l)$. We define $C_V$ for some clause $C$ and set of variables $V$ as the clause $\{ l \in C \mid \text{var}(l) \in V \}$. Given a subset of variables $V \subseteq V$, an assignment of $V$ is a function $\alpha : V \rightarrow \mathbb{B}$ that maps each variable $v \in V$ to either true ($\top$) or false ($\bot$). A partial assignment is a partial function from $V$ to $\mathbb{B}$, i.e., it may be undefined on some inputs. We use $\alpha \cup \alpha'$ as the update of partial assignment $\alpha$ with $\alpha'$: $(\alpha \cup \alpha')(v) = \begin{cases} \alpha'(v) & \text{if } v \in \text{dom}(\alpha'), \\ \alpha(v) & \text{otherwise.} \end{cases}$

Further we write $\alpha \subseteq \alpha'$ if $\alpha(v) = \alpha'(v)$ for every $v \in \text{dom}(\alpha)$. To restrict the domain of an assignment $\alpha$ to a set of variables $V$, we write $\alpha|_{V}$. We denote the set of assignments of a set of variables $V$ by $\mathcal{A}(V)$. A Skolem function $f_{y} : \mathcal{A}(\text{dep}(y)) \rightarrow \mathbb{B}$ maps an assignment of the dependencies of $y$ to an assignment of $y$. The truth of a DQBF $\Phi$ with matrix $\varphi$ is equivalent to the existence of a Skolem function $f_{y}$ for every variable $y$ of the existentially quantified variables $V$, such that substituting all existentials $y$ in $\varphi$ by their Skolem function $f_{y}$ results in a valid formula. We use $\Phi[\alpha]$ to denote the replacement of variables bound by $\alpha$ in $\Phi$ with the corresponding value.

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For for winning regions of games can all be expressed as the existence of a function $f: \mathbb{B}^m \rightarrow \mathbb{B}^n$ such that for all tuples of inputs $x_1, \ldots, x_k \in \mathbb{B}^m$ some relation $\varphi(x_1, f(x_1), \ldots, x_k, f(x_k))$ over function applications of $f$ is satisfied. By the typical translation into DQBF this results in a formula with pairwise disjoint dependency sets plus the dependency set $\mathcal{X}$ for the Tseitin variables. In particular, we have never observed a dependency cycle in the available benchmarks.

Resolution is refutationally complete for propositional Boolean formulas. This means that for every propositional Boolean formula that is equivalent to false we can derive the empty clause using only Resolution. In the same way, Resolution and Universal Reduction (together they are called $Q$-resolution) are refutational complete for QBF. For DQBF, however, $Q$-resolution is not sufficient—it was proven to be sound but incomplete [1]. Fork Resolution addresses this problem by extending $Q$-resolution by the Fork Extension proof rule [18].

Unfortunately, the proof of the completeness of Fork Resolution relied on a hidden assumption that we uncovered by implementing and testing the algorithm proposed in this work. Consider the DQBF with prefix $\forall x_1, x_2, x_3, \exists y_1(x_1, x_2), \exists y_2(x_1, x_2), \exists y_3(x_1, x_3)$ and a clause $C = (y_1 \vee y_2 \vee y_3)$. Formally, $C$ is an information fork [18], i.e., it contains variables with incomparable dependencies. However, we cannot apply $\text{FEx}$ because any split of the clause into two parts $C = C_1 \lor C_2$ satisfies either $\text{dep}(C_1) \subseteq \text{dep}(C_2)$ or $\text{dep}(C_1) \supseteq \text{dep}(C_2)$. Fork Extension therefore fails its purpose in this case to eliminate all information forks as required by the proof of completeness in [18]. We say that information forks that Fork Extension cannot split with a literal with smaller dependency are a dependency cycle. It is easy to extend the example above to a formula for which Fork Resolution is incomplete (see Appendix [20]).

We see two ways to counter this problem. The first is to consider a normal form of DQBF that does not have dependency cycles. We can restrict to DQBFs where every incomparable pair of dependency sets must have an empty intersection. This fragment does not admit any dependency cycles and it is NExpTime-complete [17] and therefore could be used as a normal form of DQBF. It also guarantees that every dependency set that gets introduced through Fork Extension maintains this property (in fact, only variables with the empty dependency set can be created). In this way, the Fork Resolution proof system is indeed strong enough to serve as a proof system for DQBF. In fact, most applications already fall in this fragment. The (Boolean) synthesis of invariants, programs, or winning regions of games can all be expressed as the existence of a function $f: \mathbb{B}^m \rightarrow \mathbb{B}^n$ such that for all tuples of inputs $x_1, \ldots, x_k \in \mathbb{B}^m$ some relation $\varphi(x_1, f(x_1), \ldots, x_k, f(x_k))$ over function applications of $f$ is satisfied. By the typical translation into DQBF this results in a formula with pairwise disjoint dependency sets plus the dependency set $\mathcal{X}$ for the Tseitin variables [18]. In particular, we have never observed a dependency cycle in the available benchmarks.
The second approach is to avoid this normal form and strengthen Fork Extension in a way that allows us to break dependency cycles. The new rule Strong Fork Extension, depicted in Fig. 2, extends Fork Extension by the ability to introduce new universal literals $C_X$ to the two clauses that it produces. Intuitively, adding the literals $C_X$ restricts the Skolem function of $y$ to the case that all literals in $C_X$ are false. Hence $y$ does not need to explicitly depend on $dep(C_X)$. This allows us to remove $dep(C_X)$ from the dependency set of the freshly introduced variable $y$.

Lemma 1. The Strong Fork Extension rule is sound.

Proof. Given any Skolem function for the formula, we make a case split over the assignments to the universals: If a literal of $C_X$ is true, both produced clauses are true and the rule is trivially sound. If all literals of $C_X$ are false, the Strong Fork Extension is equivalent to Fork Extension [18].

Theorem 1. Strong Fork Resolution is sound and complete for DQBF.

Proof. The proof of completeness of Fork Resolution assumed that any information fork can be split with Fork Extension, by introducing new literals with smaller dependency sets [18]. Strong Fork Extension guarantees this property also for dependency cycles: we pick some universal variable $x$ of the dependency sets and split the original clause twice; once with $C_X = \{x\}$ and once with $C_X = \{\overline{x}\}$. This results in four clauses that together imply the original clause (such that the original clause can be dropped from the formula), and the two variables introduced have smaller dependency sets. The rest of the proof remains the same.

4 Clausal Abstraction for DQBF

In this section, we lift clausal abstraction to DQBF. For the remainder of this section, we assume w.l.o.g. that we are given a DQBF $\Phi$ with matrix $\varphi$ and (1) that $\varphi$ does not contain clauses with information forks and (2) that every clause is universally reduced.

The clausal abstraction approach [19] assigns existential and universal variables, where the order of assignments is determined by the quantifier prefix, until all clauses in the matrix are satisfied or there is a conflict, i.e., a set of clauses that cannot be satisfied simultaneously. Those variable assignments are generated by abstractions, one for every quantifier, that replaces for every clause $C$ in the matrix dependencies and non-dependencies by interface variables. In case of a conflict, the reason for this conflict is excluded by refining the abstraction at an outer quantifier. Before we go into algorithmic details, we state important invariants and how they are generalized for DQBF. Due to the assignment order based on the quantifier prefix, for QBF it holds that an existential variable is only assigned if, and only if, its dependencies are assigned. Due to Henkin quantifiers and the resulting incomparable dependencies, there is in general no linear
order of assignments. Thus, we weaken this invariant by requiring that for some existential variable \(y\), all of its dependencies have to be assigned before assigning \(y\). We ensure this by creating a graph-based data structure described in Sec. 4.1.

As an immediate consequence, and in contrast to QBF, an existential variable may change its value based on assignments of non-dependencies. To eliminate those spurious dependencies, we enhance the certification approach of clausal abstraction [19] to build, incrementally, Skolem functions for existential variables. Section 4.3 describes how Skolem functions are build, how they interfere with the algorithm, and when they are invalidated.

Lastly, we build an abstraction for every existential quantifier \(\exists Y\), that splits every clause \(C\) of the matrix into three parts, based on whether a literal \(l \in C\) is a dependency, non-dependency, or \(\text{var}(l) \in Y\), respectively. As all dependencies are guaranteed to be assigned when we ask the abstraction for a candidate assignment of variables \(Y\), the outer variables are abstracted into whether or not they satisfy the clause \(C\). Since the current quantifier may not be able to satisfy all remaining clauses, there is an assumption variable for every clause, indicating that the current quantifier assumes that an inner quantifier satisfies it. Conflicts are represented by a set of assumption variables that turned out to be not satisfiable by the inner quantifiers. Refinements are clauses over those assumption variables, requiring that at least one of those contained clauses is satisfied at the current quantifier. Since those refinements correspond to conflict clauses in search-based algorithms [21], we have to apply Fork Extension in case the conflict clause contains a information fork. Section 4.2 gives a formal description of the abstraction.

Example 2. We will use the following formula with the dependency sets \(\{x_1\}\), \(\{x_2\}\), and \(\{x_1, x_2\}\) as a running example throughout this section.

\[
\forall x_1, x_2. \exists y_1(x_1). \exists y_2(x_2). \exists y_3(x_1, x_2).
(\neg x_1 \lor x_2 \lor y_2 \lor y_3) \land (\neg y_1 \lor y_2 \lor y_3) \land (\neg y_1 \lor y_2 \lor y_3) \land (y_1 \lor y_3) \land (x_1 \lor y_1)
\]

4.1 Dependency Lattice and Quantifier Levels

To lift clausal abstraction to DQBF, we need to deal with partially ordered dependency sets. Furthermore, we have to consider more than just the dependency sets that are explicitly indicated in the formula due to possible applications of the Fork Extension rule. Given a DQBF \(\Phi\), the algorithm thus starts with closing the dependency sets under intersection, which can also be described as building the meet-semilattice \(\langle H, \subseteq \rangle\). That is, \(H\) contains all dependency sets in \(\Phi\) and we add \(H \cap H'\) to \(H\) for every \(H, H' \in H\) until a fixed point is reached. We call this meet-semilattice also the dependency lattice. For our running example, we have to add the empty dependency set, resulting in the dependency lattice depicted on the left of Fig. 3.
Quantifier Levels and Nodes. The algorithm maintains a data structure of quantifier levels, where each level contains nodes that bind the variables of the DQBF. We index levels by natural numbers $i$, starting with 0. There are two types of levels and nodes, universal and existential ones. Each universal level contains exactly one universal node $\langle \forall, X \rangle$ binding universal variables $X$. Existential levels may contain multiple existential nodes $\langle \exists, Y, H \rangle$ binding existential variables $Y$ with dependency set $H$. On the right of Fig. 3 is an example for the data structure obtained from the dependency lattice on its left. Before describing the construction of quantifier levels, we state its invariants after introducing additional notation. For some node $N$, let $\text{bound}(N)$ be the set of variables bound at $N$, i.e., the union of all $X$ where $\langle \forall, X \rangle$ is in an earlier level than node $N$. Let $\text{bound}_3(N)$ be the analogously defined set of bound existential variables.

The set of bound variables is $\text{bound}(N) := \text{bound}_3(N) \cup \text{bound}_\exists(N)$.

**Lemma 2.** The quantifier levels data structure has the following properties.

- Every variable is bound only once, i.e., for every variable $v$ in $\Phi$, there is only one node $\langle \forall, X \rangle$ or $\langle \exists, Y, H \rangle$ such that $v \in X$ or $v \in Y$.
- Every pair of nodes $\langle \exists, Y, H \rangle$ and $\langle \exists, Y', H' \rangle$ with $Y \neq Y'$ contained in an existential level have incomparable dependencies, i.e., $H \not\subseteq H'$ and $H \not\supseteq H'$.
- For every pair of nodes $\langle \exists, Y, H \rangle$ and $\langle \exists, Y', H' \rangle$ contained in existential levels $i$ and $j$ with $i < j$, it holds that either $H \subset H'$ or $H$ and $H'$ are incomparable.
- For every existential node $\langle \exists, Y, H \rangle$ it holds that $H \subseteq \text{bound}_3(\langle \exists, Y, H \rangle)$.
- There is a unique maximal $\langle \exists, Y, H \rangle$ with $H \supset H'$ for every other $\langle \exists, Y', H' \rangle$.

In the following, we describe the construction of quantifier levels from a dependency lattice. Every element of the dependency lattice $H \in \mathcal{H}$ makes one existential node, $\langle \exists, Y, H \rangle$, where $Y$ is the set of existential variables with dependency set $H$, i.e. $\text{dep}(y) = H$ for all $y \in Y$. Some existential nodes (like the root node in our example) may thus be empty initially. The existential levels are obtained by an antichain decomposition of the dependency lattice. The maximal element is added afterwards, if needed.

Universal nodes are placed such that the following existential nodes observe an over-approximation of their dependency set. Hence, a universal variable is introduced before the existential level it first appeared as dependency. This
is achieved by a top-down pass through the existential solver levels, adding a universal level with node $N = \langle \forall, X \rangle$ before existential level with nodes $\langle \exists, Y_1, H_1 \rangle, \ldots, \langle \exists, Y_k, H_k \rangle$ such that $X = \left( \bigcup_{1 \leq i \leq k} H_i \right) \setminus \text{bound}_\forall(N)$. Empty universal levels $\langle \forall, \emptyset \rangle$ can be omitted. Level numbers follow the inverse order of the dependency sets, such that the “outer” quantifiers have smaller level numbers than the “inner” quantifiers; see Fig. 3.

If the formula is a QBF, it holds that $\text{bound}_\forall(\langle \exists, Y, H \rangle) = H$. For QBF, this construction yields a strict alternation between universal and existential levels, but for DQBF existential levels can succeed each other, as shown in Fig. 3.

1: procedure $\text{Solve}(\text{DQBF } \Phi)$
2: levels ← build quantifier levels
3: init abstraction $\theta$ for every node in levels
4: $\alpha_V \leftarrow \{\}, \text{lvl} \leftarrow 0$
5: loop
6: match $\text{SolveLevel}(\text{lvl})$
7: CandidateFound $\Rightarrow$ lvl ← lvl + 1
8: Conflict(nextLvl) $\Rightarrow$ lvl ← nextLvl
9: Result(res) $\Rightarrow$ return res
10: procedure $\text{SolveLevel}(\text{lvl})$
11: if lvl is $\forall$ then return $\text{Solve}_\forall(\text{levels}[i])$
12: for each node $n$ in levels[i] do
13: if $\text{Solve}_\exists(n) = \text{Conflict}(\text{nextLvl})$ then
14: return Conflict(nextLvl)
15: return CandidateFound

Fig. 4. Main algorithm

**Algorithmic Overview.** The overall approach of the algorithm is to construct a formula $\theta$ for every node, represented by a SAT solver, that represents which clauses it can satisfy (for existential nodes) or falsify (for universal nodes). We describe their initialization in detail below. In every iteration of the main loop (Fig. 4) the algorithm either extends the variable assignment $\alpha_V$, which we assume to be globally accessible, or to refine one or multiple of the node formulas by adding an additional clause.

The nodes are responsible for determining candidate assignments to the variables bound at that node, or to give a reason why there is no such assignment. If a node is able to provide a candidate assignment, we extend $\alpha_V$ and proceed to the next level (Fig. 4 line 7). A conflict occurs when $\alpha_V$ definitely violated some clauses (existential conflict) or satisfies all clauses (universal conflict). When conflicts are inspected in $\text{Refine}$, they indicate a level that tells the main loop how far we have to jump back (Fig. 4 line 8). The last alternative in the main loop is that we have found a result, which allows us to terminate (line 9).

### 4.2 Initialization of the abstractions $\theta$.

The formula $\theta$ for each node represents how the node’s variables interact with the assignments on other levels. The algorithm guarantees that whenever we as a node generate a candidate assignment, all variables on outer (=smaller) levels have a fixed assignment, and thus some set of clauses is satisfied already. Existential nodes then try to satisfy more clauses with their assignment, while universal nodes try to find an assignment that make it harder to satisfy all clauses. An existential variable $y$ may not only depend on assignments of its dependencies, but also on assignments of existential variables with strict smaller dependency as they are in a strictly smaller level (see Section 4.1) and thus are
guaranteed to be assigned before $y$. We call this the extended dependency set, written $exdep(y)$, and it is defined as $dep(y) \cup \{y' \in Y \mid dep(y') \subseteq dep(y)\}$. For a set $Y \subseteq Y$, we define $exdep(Y) = \bigcup_{y \in Y} exdep(y)$.

The interaction of abstractions is established by a common set of clause satisfaction variables $S$, one variable $s_i \in S$ for every clause $C_i \in \varphi$. Given some existential node $\langle \exists, Y, H \rangle$ with extended dependency set $D = exdep(Y)$ and assignment $\alpha_V$ of outer variables $V$ (w.r.t. $\exists$), i.e., $V = bound(\langle \exists, Y, H \rangle)$, for every clause $C_i \in \varphi$ it holds that $\alpha_V \models C_i|_D$ if $s_i$ is assigned to true. This, however, is not enough for existential quantifiers as they have the choice to either satisfy the clause or assume that the clause will be satisfied by an inner quantifier. Thus, we add an additional type of variables $A$, called assumption variables, with the intended semantics that $a_i$ set to false at some existential quantifier $\exists X$ implies that the clause $C_i$ is satisfied at this quantifier (either by an assignment to $X$ or an outer assignment represented by an assignment $\alpha_S$ to the satisfaction variables $S$), formally, $\alpha_V \cup \alpha_S \models C_i|_{D \cup Y}$ if $a_i$ is false.

We are now going to define the abstraction that implements this intuition. Formally, for every node $\langle \exists, Y, H \rangle$ and every clause $C_i$, we define $C_{i,\varphi} := \{l \in C_i \mid \text{var}(l) \in exdep(Y)\}$, $C_{i,Y} := \{l \in C_i \mid \text{var}(l) \in Y\}$, and $C_{i,Y}^\varnothing := \{l \in C_i \mid \text{var}(l) \notin exdep(Y) \cup Y\}$. By definition, it holds that $C = C_{i,\varphi} \cup C_{i,Y} \cup C_{i,Y}^\varnothing$. $C_{i,\varphi}$ is the set of literals on which the current node may depend, $C_{i,Y}$ the set of literals bound at the current node, and $C_{i,Y}^\varnothing$ the set of literals on which the current node may not depend. The clausal abstraction $\theta$ for this node is defined as $\bigwedge_{C_i \in \varphi} (a_i \lor s_i \lor C_{i,Y}^\varnothing)$. Note, that $s_i$ and $a_i$ are omitted if $C_{i,\varphi} = \emptyset$ and $C_{i,Y}^\varnothing = \emptyset$, respectively.

Over time, the algorithm calls each node potentially many times for candidate assignments, and it adds new clauses learnt from refinements. The new clauses for existential nodes will only contain literals from assumption variables $A \subseteq A$, representing sets of clauses that together cannot be satisfied by the inner levels. The refinement $\bigvee_{a_i \in L} \overline{a_i}$ ensures that some clause $C_i$ with $a_i \in L$ is satisfied at this node.

Universal nodes $\langle \forall, X \rangle$ have the objective to falsify some clause. Thus, for every clause $C_i$, we add $s_i \lor \neg C_{i,\varphi} = \bigwedge_{l \in C_i} \overline{l} \lor s_i$ to $\theta$ for $\langle \forall, X \rangle$. Observe that universal nodes do not have separate sets of variables $A$ and $S$, but just one copy $S$. This is just a minor simplification, exploiting the formula structure of universal nodes. Note that $s_i$ set to false implies that $\alpha_X \not\models C_{i,\varphi}$. Refinements are represented as clauses $\bigvee_{s_i \in L} \overline{s_i}$ over literals in $S$.

In our running example, clauses 3-5 ($\overline{y} \lor x_2 \lor \overline{y_3})(y_1 \lor \overline{y_3})(x_1 \lor y_1)$ are represented at node $\langle \exists, \{y_1\}, \{x_1\} \rangle$ by clauses $(a_3 \lor \overline{y_3})(a_4 \lor y_1)(s_5 \lor y_1)$. Note especially, that $x_2 \notin dep(y_1) = \{x_1\}$, thus there is no $s$ variable in the first encoded clause, despite $x_2$ being assigned earlier in the algorithm (Fig. 3).

**Solving Levels and Nodes.** **SOLVELEVEL** in Figure 3 directly calls **SOLVE\_\forall** and **SOLVE\_\exists** on all the nodes in the level. For existential levels, if any node returns a conflict, the level returns that conflict (line 13).

We assume a SAT solver interface $\text{SAT}(\psi, \alpha)$ for matrices $\psi$ and assumptions (assignments) $\alpha$. It returns either $\text{Sat}(\alpha')$, which means the formula is satisfiable
with assignment $\alpha' \supseteq \alpha$, or Unsat($\alpha'$), which means the formula is unsatisfiable and $\alpha' \subseteq \alpha$ is an unsat core.

We process universal and existential nodes with the two procedures shown in Figure 5. The SAT solvers are used to generate a candidate assignment to the variables (lines 4 and 13) of that node, which is used to extend the (global) assignment $\alpha_V$ (lines 7 and 16). In case the SAT solver returns Unsat, the unsat core represents a set of clauses that cannot be satisfied (for existential nodes) or falsified (for universal nodes). The unsat core is then used to refine an outer node (lines 6 and 12) and we proceed with the level returned by Refine.

**Solving Existential Nodes.** There are some differences in the handling of existential and universal nodes that we look into now. The linear ordering of the levels in our data structure means that there may be variable assigned that that an existential node must not depend on. We therefore need to project the assignment $\alpha_V$ to those variables in the node’s dependency set. We define a function $prj_\exists : Y \times A(V) \to \alpha_S$ that maps variable assignments $\alpha_V$ to assignments of satisfaction variables $S$ such that $s_i$ is set to true if some literal $l \in C_i^<$ is assigned positively by $\alpha_V$. Thus, the projection function only considers actual dependencies of $\exists V Y H$:

$$prj_\exists(Y, \alpha_V)(s_i) = \begin{cases} \top & \text{if } \alpha_V \models C_i^< \\ \bot & \text{otherwise} \end{cases}$$

For our running example, at node $\exists \{y_2\}, \{x_2\}$, the projection for the first clause $(x_1 \lor \neg x_2 \lor y_2 \lor y_3)$ is $prj_\exists(\{y_2\},\{x_1 \mapsto \bot, x_2 \mapsto \top\})(s_1) = prj_\exists(\{y_2\},\{x_1 \mapsto \top, x_2 \mapsto \bot\})(s_1) = \bot$ whereas $prj_\exists(\{y_2\},\{x_1 \mapsto \bot, x_2 \mapsto \top\})(s_1) = prj_\exists(\{y_2\},\{x_1 \mapsto \top, x_2 \mapsto \bot\})(s_1) = \top$.

If the SAT solver returns a candidate assignment at the maximal existential node (i.e., a node on innermost level), we know that all clauses in this node have been satisfied, and we have therefore refuted the candidate assignment of some universal node. This is handled by calling Refine in line 9. For existential nodes we additionally have to check for consistency, which we come back to later (called in line 2).

**Solving Universal Nodes.** Similar to the projection for existential nodes, we need an (almost symmetric) projection for universal nodes (line 12). It has to
differ slightly from \( prj \), because we use just one set of variables \( S \) for universal nodes. A universal quantifier can not falsify the clause if it is already satisfied.

\[
prj^v(X, \alpha)(s_i) = \begin{cases} 
\top & \text{if } \alpha \models C_i|_{\text{bound}(\forall,X)} \\
\text{undef.} & \text{otherwise}
\end{cases}
\]

**Refinement.** Algorithm \( \text{refine} \) in Fig. 6 is called whenever there is a conflict, i.e. whenever it is clear that \( \alpha \) satisfies all clauses (\( sat \) conflict) or violates some clause (\( unsat \) conflict). In case there is a conflict at an existential node, we build the (universally reduced) conflict clause from \( \alpha_S \) in line 3. If this conflict clause contains a fork, we apply Fork Extension, and encode the newly created clauses and variables within their respective nodes. We update the abstractions with those fresh variables and clauses as discussed in Sec. 4.2. Additionally, we reset learned Skolem functions as they may be invalidated by the refinement.

In all other cases, we can apply the standard refinement for clausal abstraction [19] with the exception that we need to find the unique refinement node first (line 5). This backward search over the quantifier levels is shown in Figure 6. For an \( unsat \) conflict, we traverse the levels backwards until we find an existential node that binds a variable contained in the conflict clause. Because the conflict clause is fork-free, the target node of the traversal is unique. For a \( sat \) conflict, we do the same for universal nodes but the uniqueness comes from the fact that universal levels are singletons. We then add the refinement clause to the SAT solver at the corresponding node (lines 17 and 21) and proceed. For \( sat \) conflicts, we have to additionally learn Skolem functions (line 25). In case the conflict propagated beyond the root node, we can terminate the program.

### 4.3 Skolem Function Consistency

The algorithm described so far produces correct refutations in case the DQBF is false. For positive results, the consistency of Skolem functions of incomparable existential variables may be violated. Consider for example the formula \( \forall x_1 \forall x_2. \exists y_1(x_1). \exists y_2(x_2). \exists y_3(x_1, x_2). \varphi \) and assume that for the assignment \( \{ x_1 \rightarrow \bot, x_2 \rightarrow \bot \} \), there is a corresponding satisfying assignment \( \{ y_1 \rightarrow \bot, y_2 \rightarrow \bot, y_3 \rightarrow \bot \} \). If the next assignment is \( \{ x_1 \rightarrow \bot, x_2 \rightarrow \top \} \), then the assignment to \( y_1 \) has to be the same as before \( (y_1 \rightarrow \bot) \) as the value of its sole dependency \( x_1 \) has not changed.

We enhance the certification capabilities of [19] to build partial Skolem functions in our algorithm during solving. This function is then encoded (incrementally) in a SAT solver to be enforced for the next candidate generation. The function that the SAT solver encodes is an if-then-else chain, starting with the first learned entry [19]. Before generating a candidate assignment, we check if the Skolem function for the given assignment \( \alpha \) is already determined (line 2 in Fig. 5). If it is the case, we get an assignment \( \alpha_Y \) that is then assumed for the candidate generation. Note that in this case, the \texttt{sat} call in line 4 is guaranteed to return \texttt{Sat} (we already verified this assignment, otherwise it would
1: procedure Refine(res, α, node)
2: if node is existential then
3: \( C_{\text{conflict}} = \bigcup_{C_i | x} C_i \) \( \supseteq \) \( C_{\text{bound(node)}} \) \( \triangleright C_{\text{conflict}} \) is universally reduced
4: if \( C_{\text{conflict}} \) contains information fork then
5: fork elimination \( \Rightarrow \) add clauses and variables, update abstractions \( \theta \)
6: reset Skolem for all nodes
7: return Conflict(lvl = 0)
8: if next \( \leftarrow \) DetermineRefinementNode(res, α, node) then
9: return Conflict(next.level)
10: else \( \triangleright \) conflict at outermost \( \exists/\forall \) node
11: return Result(res)
12: procedure DetermineRefinementNode(res, α, lvl)
13: while lvl \( \geq \) 0 do
14: if res = unsat and lvl is existential then
15: for node \( \langle \exists, Y, H \rangle \) in levels[lvl] do
16: if \( C_i | X \neq \emptyset \) for some \( C_i \in \varphi \) with \( \alpha_S(s_i) = \top \) then
17: \( \theta \leftarrow \theta \wedge \bigvee_{C_i \in \varphi, \alpha_S(s_i) = \top} \overline{y} \) \( \triangleright \) refine abstraction
18: return \( \langle \exists, Y, H \rangle \)
19: else if res = sat and lvl is universal with node \( \langle \forall, X \rangle \) then
20: if \( C_i | X \neq \emptyset \) for some \( C_i \in \varphi \) with \( \alpha_S(s_i) = \top \) then
21: \( \theta \leftarrow \theta \wedge \bigvee_{C_i \in \varphi, \alpha_S(s_i) = \top} \overline{y} \) \( \triangleright \) refine abstraction
22: return \( \langle \forall, X \rangle \)
23: else if res = sat and lvl is existential then
24: for node \( \langle \exists, Y, H \rangle \) in levels[lvl] with \( H \subseteq \text{bound}_{\varphi}(\langle \exists, Y, H \rangle) \) do
25: LearnEntry(\( \alpha_S, \alpha_Y \)) \( \triangleright \) \( \alpha_Y \) from previous SAT call
26: lvl \( \leftarrow \) lvl - 1
27: return None

Fig. 6. Refinement algorithm that applies Fork Extension in case of information forks and the backward search algorithm to determine refinement node.

not have been learned). Further, Skolem function learning is only needed for existential nodes \( \langle \exists, Y, H \rangle \) with \( H \subseteq \text{bound}_{\varphi}(\langle \exists, Y, H \rangle) \), i.e., that observe an over-approximation of their dependency set. We reset Skolem functions in case we applied fork resolution (line 6 in Fig. 6) as the new clauses may affect already learned parts of the function. We, further, have to reset the clauses learned at universal nodes (line 7 in Fig. 7).

1: procedure checkSkolem(\( \alpha_Y \))
2: match sat(skolem, \( \alpha_Y \))
3: Sat(\( \lambda \)) \( \Rightarrow \) return empty assignment
4: if res = Sat then
5: return res
6: procedure resetSkolem
7: skolem \( \leftarrow \) new SAT solver instance
8: reset learned clauses at universal nodes
9: procedure learnEntry(\( \alpha_S, \alpha_Y \))
10: add \( \langle \bigwedge_{C_i | \alpha_S(s_i) = \top} C_i \rangle \) \( \rightarrow \alpha_Y \) to skolem

Fig. 7. Skolem Function Learning

We learn Skolem function entries on the backward search on sat conflicts, that is in line 26 in Fig. 6. When we determine the next universal node, we call LearnEntry(\( \alpha_S, \alpha_Y \)) in every existential node \( \langle \exists, Y, H \rangle \) with \( H \subseteq \text{bound}_{\varphi}(\langle \exists, Y, H \rangle) \) on the path to that node. In our example, when the base case of \( \langle \exists, \{y_3\}, \{x_1, x_2\} \rangle \), we learn Skolem entries at nodes \( \langle \exists, \{y_2\}, \{X_2\} \rangle \) and \( \langle \exists, \{y_1\}, \{x_1\} \rangle \) before refining at \( \langle \forall, \{x_1, x_2\} \rangle \).
As we build Skolem functions during execution for a subset of nodes, we can adapt the algorithm to certify positive results by constructing Skolem functions for all existential nodes. Further, as our solving algorithm is based on a resolution-style proof system, we can construct refutation proofs in case the instance is unsatisfiable (using a SAT solver that produces the propositional resolution proofs).

**Example.** We consider a possible execution of our algorithm on our running example. For the sake of readability, we combine unimportant steps and focus on the interesting cases. Further, we use the representation of assignments as cubes \((x\overline{y} \equiv \{x \mapsto \top, y \mapsto \bot\})\). Assume the following initial assignment \(\alpha_1 = x_1x_2y_1y_2\) before node \(\langle \exists, \{y_3\}, \{x_1, x_2\}\rangle\). The result of projecting function \(\text{prj}_3(\{y_3\}, \alpha_1)\) is \(s_1s_2s_3s_4s_5\) and the SAT solver (line 4 in Fig. 5) returns \(\text{Unsat}(\alpha'_1)\) with core \(\alpha'_1 = s_2\overline{s_4}\) as there is no way to satisfy both clauses \((s_2 \lor y_3)\) and \((s_4 \lor \overline{y_3})\).

The refinement algorithm (Figure 6) builds the conflict clause \(C_2 \otimes y_3, C_4 = (\overline{x_1} \lor y_2 \lor y_1)\) at line 8 which contains an information fork between \(y_1\) and \(y_2\). We have already seen in Example 1 that the fork can be eliminated resulting in fresh variable \(y_4\) with \(\text{dep}(y_4) = \emptyset\) and the clauses 6 and 7 \((\overline{x_1} \lor y_1)\lor (\overline{x_1} \lor y_4)\lor (\overline{x_1} \lor y_2)\).

Now, the root node contains variable \(y_4\), for which we assume assignment \(\{y_4 \mapsto \top\}\). For the same universal assignment as before \((x_1\overline{x_2})\), the assignment of \(y_2\) has to change to \(\{y_2 \mapsto \top\}\) due to the newly added clause, leading to \(\alpha_2 = x_1\overline{x_2}y_1y_2y_4\) before node \(\langle \exists, \{y_3\}, \{x_1, x_2\}\rangle\). The only unsatisfied clause is \(C_4\) which can be satisfied using \(\{y_3 \mapsto \bot\}\), leading to the base case (line 9 in Fig. 5). During refinement, we learn Skolem function entries \((x_1 \land y_4) \mapsto \overline{y_1}\) and \((\overline{x_2} \land y_4) \mapsto y_2\) as \(\text{prj}_3(\{x_1\}, \alpha_2)\) and \(\text{prj}_3(\{x_2\}, \alpha_2)\) assign \(s_1, s_5, s_6\) and \(s_3, s_6\) positively, respectively.

For the following universal assignment \(\{x_1 \mapsto \bot, x_2 \mapsto \bot\}\), the value of \(y_2\) is already determined by the partial Skolem function (line 2 in Fig. 5) to be positive. The algorithm continues without further existential conflict, determining that the instance is true.

**Correctness.** We sketch the correctness argument for the algorithm, which relies on formal arguments regarding correctness and certification of the clausal abstraction algorithm [19] and the subsequent analysis of the underlying proof system [21]. For soundness, the algorithm has to guarantee that existential variables are assigned consistently, that is for an existential variable \(y\) with dependency \(\text{dep}(y)\) it holds that \(f_y(\alpha) = f_y(\alpha')\) if \(\alpha |_{\text{dep}(y)} = \alpha' |_{\text{dep}(y)}\) for every \(\alpha\) and \(\alpha'\). For QBF, this property is maintained by the hierarchical quantifier prefix: every existential variable may depend on every prior bound variable. For DQBF, this is no longer the case, as we order the nodes such that the existential variables observe an over-approximation of their dependencies. The Skolem function learning on incomparable nodes makes sure that this property is guaranteed at every point during the execution of the algorithm. Completeness relies on the fact that the underlying proof system is refutationally complete for DQBF.

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3 More details can be found in the full version [20].
Table 1. Number of instances solved within 10 min.

| Benchmark            | Number of instances | dCAQE | iDQ   | HQS   | iProver |
|----------------------|---------------------|-------|-------|-------|---------|
| PEC1                 | 800                 | 662   | 50    | 365   | 179     |
| PEC2                 | 720                 | 371   | 218   | 243   | 225     |
| BoSy                 | 1216                | 922   | 927   | 878   | 955     |
| Safety Synthesis     | 461                 | 76    | 47    | 74    | 72      |
|                      | 3197                | **2031** | 1242 | 1560 | 1431    |

Progress is guaranteed as there are only finitely many different conflict clauses and, thus, only finitely many Skolem function resets.

5 Evaluation

We compare our prototype implementation, called dCAQE, against the publicly available DQBF solvers, iDQ [8], HQS [10], and iProver [15]. We ran the experiments on machines with a 3.6 GHz quad-core Xeon processor with timeout and memout set to 10 minutes and 32 GB, respectively. The instances were preprocessed with the DQBF preprocessor HQSPre [24]. We evaluate our solver on four DQBF case studies [6,10,4,3].

The first two benchmark sets consider the partial equivalence checking (PEC) problem [9], that is, the problem whether a circuit containing not-implemented (combinatorial) parts, so-called “black boxes”, can be completed such that it is equivalent to a reference circuit. The inputs to the circuit are modeled as universally quantified variables and the outputs of the black boxes as existentially quantified variables. Since the output of a black box should only depend on the inputs that are actually visible to the black box, we need to restrict the dependencies of the existentially quantified variables to subsets of the universally quantified variables. The benchmark sets PEC1 and PEC2 refer to [6] and [10], respectively. The second case study (BoSy) considers the problem of synthesizing sequential circuits from specifications given in linear-time temporal logic (LTL) [4]. The benchmarks were created using the tool BoSy [5] and the LTL benchmarks from the Reactive Synthesis Competition [12,11]. Each formula encodes the existence of a sequential circuit that satisfies the LTL specification. The last benchmark (Safety Synthesis) considers the problem of solving safety games [3] that encode circuit synthesis problems.

The results are presented in Table 1. Overall dCAQE solves most instances, with the result on the benchmark set PEC1 being most promising. We conjecture this is due to a dominance of UNSAT formulas in this benchmark set and the effectiveness of the resolution-based refutations that dCAQE is based on.

\footnote{The preprocessor did improve number of solved instances for every solver.}
6 Related Work

The satisfiability problem for DQBF was shown to be NExpTime-complete \cite{17}. Fröhlich et al. \cite{7} proposed a first detailed solving algorithm for DQBF based on DPLL. They already encountered many challenges of lifting QBF-based algorithms like Skolem function consistency, replay of Skolem functions, forks in conflict clauses, but solved them differently. Their algorithm, called DQDPLL, has some similarities to our algorithm (in the same way that clausal abstraction and QDPLL share the same underlying proof system \cite{21}) but also has some deficiencies that make the implementation “not perform very well” \cite{7}. We highlight a few differences which we believe to be crucial: (1) Our algorithm tries to maintain as much order as possible. Placing universal nodes at the latest possible allows us to apply the cheaper QBF refinement method more often. (2) We learn Skolem functions only if they have been verified to satisfy the clauses, while DQDPLL learns them on decisions. Consequently, in DQDPLL, learned Skolem functions become part of the clauses, thus, making conflict analysis more complicated and less effective as they may be undone during solving. We keep learned Skolem functions distinct from the clauses, all existential conflicts are thus valid during solving. (3) Skolem functions in DQDPLL are represented as clauses representing truth-table entries, thus, become quickly infeasible. In contrast, we use a separate certification mechanism as in QBF solvers \cite{19}. iDQ \cite{8} uses an instantiation-based algorithm which is based on the Inst-Gen calculus, a state-of-the art decision procedure for the effectively propositional fragment of first-order logic (EPR), which is also NExpTime-complete. HQS \cite{10} is an expansion based solver that expands universal variables until the resulting instance has a linear prefix and applies QBF solving afterwards. Wimmer et al. \cite{22} considered the problem of certifying Skolem functions produced by DQBF solvers. Bounded unsatisfiability \cite{6} asserts the existence of a partial (bounded) expansion tree that guarantees that no Skolem function exists. QBF preprocessing techniques have been lifted to DQBF \cite{23,24}.

Our solving technique is based on clausal abstraction \cite{19} (also called clause selection \cite{13}) for QBF, which can provide certificates \cite{19}. Later, it was shown that refutation in clausal abstraction can be simulated by Q-resolution \cite{21}. Lastly, this work also shows that the clausal abstraction algorithm for QBF can be used to solve QBFs incrementally while still being able to extract certificates.

7 Conclusions

We lifted the clausal abstraction algorithm to DQBF. This algorithm is the first to exploit the new Fork Resolution proof system and it significantly increases performance of DQBF solving on most benchmarks. In particular in the light of the past attempts to define search algorithms \cite{7} (which are closely related to clausal abstraction) for DQBF this is a surprising success. It appears that the fork extension proof rule was the missing piece in the puzzle to build search/abstraction algorithms for DQBF.
References

1. Balabanov, V., Chiang, H.K., Jiang, J.R.: Henkin quantifiers and boolean formulae: A certification perspective of DQBF. Theor. Comput. Sci. 523, 86–100 (2014). https://doi.org/10.1016/j.tcs.2013.12.020

2. Balabanov, V., Jiang, J.R.: Unified QBF certification and its applications. Formal Methods in System Design 41(1), 45–65 (2012). https://doi.org/10.1007/s10152-012-0152-6

3. Bloem, R., Könighofer, R., Seidl, M.: Sat-based synthesis methods for safety specs. In: Proceedings of VMCAI. LNCS, vol. 8318, pp. 1–20. Springer (2014). https://doi.org/10.1007/978-3-642-54013-4_1

4. Faymonville, P., Finkbeiner, B., Rabe, M.N., Tentrup, L.: Encodings of bounded synthesis. In: Proceedings of TACAS. LNCS, vol. 10205, pp. 354–370 (2017). https://doi.org/10.1007/978-3-662-54577-5_20

5. Faymonville, P., Finkbeiner, B., Tentrup, L.: Bosy: An experimentation framework for bounded synthesis. In: Proceedings of CAV. LNCS, vol. 10427, pp. 325–332. Springer (2017). https://doi.org/10.1007/978-3-319-63390-9_17

6. Finkbeiner, B., Tentrup, L.: Fast DQBF refutation. In: Proceedings of SAT. LNCS, vol. 8561, pp. 243–251. Springer (2014). https://doi.org/10.1007/978-3-319-09284-3_19

7. Fröhlich, A., Kovásznai, G., Biere, A.: A DPLL algorithm for solving DQBF. In: Proceedings of POS@SAT (2012)

8. Fröhlich, A., Kovásznai, G., Biere, A., Veith, H.: idq: Instantiation-based DQBF solving. In: Proceedings of SAT. EPiC Series in Computing, vol. 27, pp. 103–116. EasyChair (2014)

9. Gitina, K., Reimer, S., Sauer, M., Wimmer, R., Scholl, C., Becker, B.: Equivalence checking of partial designs using dependency quantified boolean formulae. In: Proceedings of ICCD. pp. 396–403. IEEE Computer Society (2013). https://doi.org/10.1109/ICCD.2013.6557071

10. Gitina, K., Wimmer, R., Reimer, S., Sauer, M., Scholl, C., Becker, B.: Solving DQBF through quantifier elimination. In: Proceedings of DATE. pp. 1617–1622. ACM (2015)

11. Jacobs, S., Basset, N., Bloem, R., Brenguier, R., Colange, M., Faymonville, P., Finkbeiner, B., Khalimov, A., Klein, F., Michaud, T., Pérez, G.A., Raskin, J., Sankur, O., Tentrup, L.: The 4th reactive synthesis competition (SYNT-COMP 2017): Benchmarks, participants & results. In: Proceedings of SYNT@CAV. EPTCS, vol. 260, pp. 116–143 (2017). https://doi.org/10.4204/EPTCS.260.10

12. Jacobs, S., Bloem, R., Brenguier, R., Khalimov, A., Klein, F., Könighofer, R., Kreber, J., Legg, A., Narodytska, N., Pérez, G.A., Raskin, J., Ryzhyk, L., Sankur, O., Seidl, M., Tentrup, L., Walker, A.: The 3rd reactive synthesis competition (SYNT-COMP 2016): Benchmarks, participants & results. In: Proceedings of SYNT@CAV. EPTCS, vol. 229, pp. 149–177 (2016). https://doi.org/10.4204/EPTCS.229.12

13. Janota, M., Marques-Silva, J.: Solving QBF by clause selection. In: Proceedings of IJCAI. pp. 325–331. AAAI Press (2015)

14. Kleine Büning, H., Karpinski, M., Flögel, A.: Resolution for quantified boolean formulas. Inf. Comput. 117(1), 12–18 (1995). https://doi.org/10.1006/inco.1995.1025

15. Korovin, K.: iprover - an instantiation-based theorem prover for first-order logic (system description). In: Proceedings of IJCAR. LNCS, vol. 5195, pp. 292–298. Springer (2008). https://doi.org/10.1007/978-3-540-71070-7_24
16. Narizzano, M., Pulina, L., Tacchella, A.: The qbfeval web portal. In: Logics in Artificial Intelligence. pp. 494–497. Springer Berlin Heidelberg (2006)
17. Peterson, G., Reif, J., Azhar, S.: Lower bounds for multiplayer non-cooperative games of incomplete information. Computers and Mathematics with Applications 41, 957–992 (2001)
18. Rabe, M.N.: A resolution-style proof system for DQBF. In: Proceedings of SAT. LNCS, vol. 10491, pp. 314–325. Springer (2017). https://doi.org/10.1007/978-3-319-66263-3_20
19. Rabe, M.N., Tentrup, L.: CAQE: A certifying QBF solver. In: Proceedings of FM-CAD. pp. 136–143. IEEE (2015)
20. Rabe, M.N., Tentrup, L.: Synthesis of boolean functions with clausal abstraction. CoRR abs/1808.08759 (2018), http://arxiv.org/abs/1808.08759
21. Tentrup, L.: On expansion and resolution in CEGAR based QBF solving. In: Proceedings of CAV. LNCS, vol. 10427, pp. 475–494. Springer (2017). https://doi.org/10.1007/978-3-319-63390-9_25
22. Wimmer, K., Wimmer, R., Scholl, C., Becker, B.: Skolem functions for DQBF. In: Proceedings of ATVA. LNCS, vol. 9938, pp. 395–411 (2016). https://doi.org/10.1007/978-3-319-46520-3_25
23. Wimmer, R., Gitina, K., Nist, J., Scholl, C., Becker, B.: Preprocessing for DQBF. In: Proceedings of SAT. LNCS, vol. 9340, pp. 173–190. Springer (2015). https://doi.org/10.1007/978-3-319-24318-4_13
24. Wimmer, R., Reimer, S., Marin, P., Becker, B.: HQSpre - an effective preprocessor for QBF and DQBF. In: Proceedings of TACAS. LNCS, vol. 10205, pp. 373–390 (2017). https://doi.org/10.1007/978-3-662-54577-5_21
A Incompleteness Example

We give an example to demonstrate that Fork Resolution is incomplete for general DQBF. The example formula is an extension of the incompleteness examples used in [2]. On the high level, the formula expresses the following:

\[ \forall x_1, x_2, x_3, \exists y_1(x_1, x_2), \exists y_2(x_2, x_3), \exists y_3(x_1, x_3). (x_1 \land x_2 \land x_3) \leftrightarrow (y_1 \oplus y_2 \oplus y_3) \]

In CNF, the formula looks as follows:

\[
\begin{align*}
(x_1 \lor x_1 \lor y_2 \lor \neg y_3) \land (x_1 \lor x_1 \lor y_3 \lor \neg y_2) \land \\
(x_1 \lor y_2 \lor y_1 \lor \neg x_1) \land (x_2 \lor x_1 \lor y_2 \lor \neg y_3) \land \\
(x_2 \lor x_1 \lor y_3 \lor \neg y_2) \land (x_2 \lor y_2 \lor y_3 \lor \neg x_1) \land \\
(x_3 \lor x_1 \lor y_2 \lor \neg y_3) \land (x_3 \lor x_1 \lor y_3 \lor \neg y_2) \land \\
(x_3 \lor y_2 \lor y_1 \lor \neg x_1) \land (x_1 \lor y_1 \lor y_2 \lor \neg y_3) \land \\
(x_1 \lor y_1 \lor \neg x_1 \lor \neg x_2 \lor \neg x_3) \land (x_1 \lor x_1 \lor x_2 \lor \neg x_3 \lor \neg y_2 \lor \neg y_3) \land \\
(y_2 \lor \neg x_1 \lor \neg x_2 \lor \neg x_3 \lor \neg y_1 \lor \neg y_3) \land (y_3 \lor \neg x_1 \lor \neg x_2 \lor \neg x_3 \lor \neg y_1 \lor \neg y_2) \\
\end{align*}
\]

Note that:

- The formula is false.
- Universal reduction cannot be applied to any clause.
- All resolvents are tautologies.
- Fork extension is not applicable.
- The formula does not contain the empty clause.

This means that Fork Resolution proof system is not strong enough to refute any DQBF. However, we want to emphasize that for the fragment of DQBF that admits only ordered or disjoint dependency sets, which is also \text{NExpTime}-complete, Fork Resolution is sound and complete, as we discussed in the proof system section in the paper.

The problem in the proof of completeness in [18] is that two conflicting definitions of information forks were given. In the introduction information forks are defined as clauses that contain two variables with incomparable dependencies (as it is used in this work). In Section 4 of [18] information forks were then defined again as clauses that consist of two parts \( C_1 \) and \( C_2 \) that have incomparable dependencies. The two definitions do not match for clauses that contain three or more variables with pairwise intersecting dependency sets. This led to the wrong assumption that all information forks (of the first kind) can be eliminated with the Fork Extension rule, which is not the case.
B Dependency Cycles

In the following, we formalize dependency cycles and show that they are the reason for incompleteness of fork extension. A clause \( C \) contains a dependency cycle of length \( k \geq 2 \), if there is a subset \( \{l_1, l_2, \ldots, l_k\} \subseteq C \) of existential literals such that \( \text{dep}(l_i) \cap \text{dep}(l_{i+1}) \neq \emptyset \) for all \( 1 \leq i \leq k \), with \( l_{k+1} = l_1 \).

Given a clause \( C \), the clause poset, written \( \text{poset}(C) \), is a partially ordered set \( \langle P, \subseteq \rangle \) where \( P \subseteq \mathcal{X} \) is the set of dependencies of existential literals in \( C \), i.e., \( \{ \text{dep}(y) \mid l \in C \wedge l \text{ is existential} \} \). If \( \text{poset}(C) \) contains more than one maximal element w.r.t. \( \subseteq \), \( C \) contains a information fork.

**Lemma 3.** Fork extension is applicable for clauses with information fork if, and only if, the clause does not contain a dependency cycle.

**Proof.** Assume \( C \) contains a dependency cycle, that is, there is a \( k > 2 \) and \( \{l_1, l_2, \ldots, l_k\} \subseteq C \) of existential literals such that \( \text{dep}(l_i) \cap \text{dep}(l_{i+1}) \neq \emptyset \) for all \( 1 \leq i \leq k \) \((l_{k+1} = l_0)\). W.l.o.g. we assume that \( \{l_1, l_2, \ldots, l_k\} \) are the only existential variables in \( C \). Let \( C_1 \cup C_2 \) be an arbitrary split containing at least one existential variable. Then, either \( \text{FEx} \) is not applicable \( (\text{dep}(C_1) \subseteq \text{dep}(C_2) \lor \text{dep}(C_1) \supseteq \text{dep}(C_2)) \) or applying it will lead to a clause with dependency cycle: let \( y \) be the fresh variable with \( \text{dep}(y) = \text{dep}(C_1) \cap \text{dep}(C_2) \). If \( C_1 \) contains a single existential literal \( l_1 \), then \( \text{dep}(y) \cap \text{dep}(l_{i+1}) \neq \emptyset \) and \( \text{dep}(y) \cap \text{dep}(l_{i-1}) \neq \emptyset \), i.e., the resulting clause has a dependency cycle of length \( k \). If \( C_1 \) contains \( j > 1 \) existential literals, then both resulting clauses contain a dependency cycle of length \( j + 1 \) and \( k - j + 1 \).

Assume \( C \) does not contain a dependency cycle, that is, there is a maximal element \( H \) in \( \text{poset}(C) \), such that there is a unique maximal element in the set of intersections with other maximal elements \( H^* = \max \subseteq \{ H \cap H' \mid H' \text{ is a maximal element of } \text{poset}(C) \} \). We use the fork extension rule \( \text{FEx} \) with \( C_1 = \{ l \in C \mid \text{dep}(l) \subseteq H, \text{dep}(l) \not\subseteq H^* \} \) and \( C_2 = C \setminus C_1 \).

C Correctness

We show the formal arguments for the base case and the refinements and show that existential assignments are always consistent.

The first lemma states the base case, i.e., that the abstraction for the maximal element is equisatisfiable to replacing the assignment of the bound variables \( \alpha_V \) in the matrix \( \varphi \).

**Lemma 4.** Let \( \langle \exists Y, H \rangle \) be the existential node corresponding to the unique maximal element and let \( \alpha_V \) be some assignment with \( V = \text{bound}(\langle \exists Y, H \rangle) \). Then, the SAT call (line 4) of \( \text{solve}_\exists(\langle \exists Y, H \rangle) \) returns \text{Sat} if, and only if, \( \varphi[\alpha_V] \) is satisfiable.

**Proof.** The abstraction \( \theta \) for the maximal element does not contain assumption literals, i.e., it has the form \( \theta = \bigwedge_{C_i \in \varphi} s_i \land \neg \neg C_i \). By definition of \( \alpha_S = \text{proj}_\exists(Y, \alpha_V) \), it holds that \( \theta[\alpha_S] = \bigwedge_{C_i \in \varphi_{\alpha_V \neq \alpha_S}} C_i \land \neg \neg C_i \).
Additionally, we state the following Lemma for non-maximal nodes.

**Lemma 5.** Let $\langle \exists, Y, H \rangle$ be an existential node and let $\alpha_S$ be some assignment of the satisfaction variables. It holds that $\theta[\alpha_S] = \bigwedge_{C_i \in \varphi, \alpha_S(s_i)=\bot} (C_i^c \lor a_i)$.

**Lemma 6.** Let $\langle \forall, X \rangle$ be an universal node and let $\alpha_S$ be some positive assignment of the satisfaction variables (i.e., a partial assignment containing only positive values). It holds that $\theta[\alpha_S] = \bigwedge_{C_i \in \varphi, \alpha_S(s_i)=\bot} (s_i \lor \neg C_i^c)$.

The following lemmata state that refinements are correct, i.e., that the clause contained in the refinement is satisfied, respectively, falsified.

**Lemma 7.** Let $\langle \exists, Y, H \rangle$ be some existential node and let $\alpha_V$ be some assignment with $V = \text{bound}(\langle \exists, Y, H \rangle)$. Let $\alpha$ be the assignment after a satisfiable call to the abstraction $\theta$ (line 12 of \textsc{solve}_\exists(\langle \exists, Y, H \rangle)). For every clause $C_i \in \varphi$ it holds that $a_i \rightarrow \bot$ implies that $\alpha_Y \uplus \alpha_V \models C_i$.

**Lemma 8.** Let $\langle \forall, X \rangle$ be some universal node and let $\alpha_V$ be some assignment with $V = \text{bound}(\langle \forall, X \rangle)$. Let $\alpha$ be the assignment after a satisfiable call to the abstraction $\theta$ (line 13 of \textsc{solve}_\forall(\langle \forall, X \rangle)). For every clause $C_i \in \varphi$ it holds that $s_i \rightarrow \bot$ implies that $\alpha_X \uplus \alpha_V \not\models C_i$.

**Proof.** Follows by the abstraction definitions and the projection functions.

As Skolem function consistency can be shown independently, we can assume that all assignments are consistent in the following proofs.

**Lemma 9.** Let $\Phi$ be a DQBF, let $N$ be a node, and let $\alpha_V$ be an assignment of prior bound variables, i.e., $V = \text{bound}(N)$. If $\Phi[\alpha_V]$ is satisfiable, \textsc{solve}_Q(N) eventually returns a sat conflict.

**Proof.** The proof is carried out by a structural induction over the quantifier levels. The base case is given in Lemma 4.

Let $N$ be an existential node $\langle \exists, Y, H \rangle$. As $\Phi[\alpha_V]$ is satisfiable, there is an assignment of the existential variables $\alpha_Y$ such that $\Phi[\alpha_V \cup \alpha_Y]$ is satisfiable. Assume this assignment is returned by the abstraction $\theta$. By IH, the next level returns a sat conflict. Assume, the assignment is different, then either the inner level returns a sat conflict as well, or an unsat conflict. Consider the latter case. As $\Phi[\alpha_V]$ is satisfiable, the conflict clause has to contain a variable in $Y$. The unsat conflict refinement eliminates at least one assignment to the assumption variables $\alpha_A$. Let $\psi$ be the conflict clause refined at $N$. $\alpha_Y$ is still a satisfying assignment of $\theta \land \psi$, thus, we are one step closer to finding $\alpha_Y$.

Let $N$ be a universal node $\langle \forall, X \rangle$. As $\Phi[\alpha_V]$ is satisfiable, for every assignment of the universal variables $\alpha_X$ it holds that $\Phi[\alpha_V \cup \alpha_Y]$ is satisfiable. Thus, by IH, the next level returns a sat conflict. Either, no universal variable is contained, thus, the algorithm returns the sat conflict as well, or a universal variable in $X$ is contained. The following refinement eliminates at least one assignment of $\alpha_S$, thus, bringing us one step closer to termination.
Lemma 10. Let $\Phi$ be a DQBF, let $N$ be a node, and let $\alpha_V$ be an assignment of prior bound variables, i.e., $V = \text{bound}(N)$. If $\Phi[\alpha_V]$ is unsatisfiable, $\text{solve}_Q(N)$ eventually returns an unsat conflict.

Proof. The proof is carried out by a structural induction over the quantifier levels. The base case is given in Lemma 4. The induction cases are analogous to Lemma 9.

Theorem 2. $\text{solve}(\Phi)$ returns sat if, and only if, $\Phi$ is satisfiable.

D Comparison to QBF solvers

Table 2. Results of the QBF competition QBFEval 2018

| solver     | total sat | unsat |
|------------|-----------|-------|
| CAQE       | 304       | 129   | 175   |
| DepQBF     | 280       | 119   | 161   |
| Portfolio QBF | 275     | 103   | 172   |
| dCAQE      | 263       | 99    | 164   |
| predyndep  | 258       | 116   | 142   |
| heritiq    | 252       | 90    | 162   |
| RAREQS     | 247       | 93    | 154   |
| iProver    | 210       | 79    | 131   |
| ijtihad    | 204       | 73    | 131   |
| GhostQ     | 181       | 92    | 89    |

Table 2 compares dCAQE to the results of the latest QBF competition (QBFEVAL 2018). We used a smaller timeout of 600 seconds compared to 900 seconds the competitions, but other apart from that our numbers are very close to the results of the competition. We observed that, despite the overhead due to handling a more expressive logic, and despite the lack of some optimizations, dCAQE performs still pretty close to CAQE, ranking fourth in the competition. It is also interesting to observe that dCAQE solves over 50 instances more than iProver on those QBF benchmarks.