Spin bit models from non-planar $\mathcal{N} = 4$ SYM

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Abstract

We study spin models underlying the non-planar dynamics of $\mathcal{N} = 4$ SYM gauge theory. In particular, we derive the non-local spin chain Hamiltonian corresponding to the generator of dilatations in the gauge theory at leading order in $g_{YM}^2 N$ but exact in $\frac{1}{N}$. States in our spin chain-like model are characterized by a spin-configuration as well as by a linking variable which describes how sites are connected in the chains. Joining and splitting of string/traces is implemented by a twist operator acting on the linking variable. The obtained model is used for the systematic study of non-planar anomalous dimensions and operator mixing in $\mathcal{N} = 4$ SYM. Beyond other, we identify a sequence of SYM operators for which corrections to the one-loop anomalous dimensions stop at the first $1/N$ non-planar order.

1 Introduction

The correspondence between Yang–Mills and string theories has by now a long history starting from [11] (see [2], for a recent review). The AdS/CFT proposal [3, 4] for a correspondence between $\mathcal{N} = 4$ super Yang–Mills theory (SYM) and superstring theory on $\text{AdS}_5 \times S^5$ is a remarkable realization of these ideas (for a review see [5]). Initially formulated in the $N \to \infty$ limit, the conjecture in its strong form extends to finite $N$. It relates the strongly coupled regime of $\mathcal{N} = 4$ SYM to the weakly coupled string theory and viceversa. This property, which makes out of this correspondence a very strong and efficient predictive tool, appeared to be an obstacle in proving the duality in itself.

In [6, 7], Berenstein, Maldacena and Nastase study the correspondence in the vicinity of some null geodesics of $\text{AdS}_5 \times S^5$, where the geometry looks like a gravitational plane wave [8]–[10]. On the CFT side this corresponds to focusing on SYM operators with a large $R$-charge. String theory appears to be solvable [11, 12] in such a background and it can be quantitatively compared with predictions coming from perturbative SYM computations [13]–[18] (see [19]–[22] for reviews on the BMN correspondence and references). Later, other limits based on spinning string solutions were proposed in [23]–[30]. In [31, 32] the

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correspondence was tested far from the BMN limit in the free SYM/tensionless string regime where holography relates the gauge theory to a higher spin gravity theory on $\text{AdS}_5 \times S^5$ (see [33]-[43] for related works).

At the same time, on the Yang–Mills side considerable progress was made in the study of the operator mixing and anomalous dimensions [16], [44]-[50]. As it was pointed out in [51], the dynamics in the sector of single-trace bosonic operators of SYM can be mapped into that of the Heisenberg SO(6) spin one model in such a way that the matrix of planar one-loop anomalous dimensions is identified with the Hamiltonian of the spin chain. The Bethe Ansatz techniques used for diagonalizing the Hamiltonian become then a powerful tool in determining anomalous dimensions in the gauge theory. In [49, 52] the result was generalized to the supersymmetric case.

The spectacular development of the understanding of SYM at large $N$ left somewhat behind the study of the nonplanar contributions. The latter is expected to correspond via AdS/CFT to taking into consideration the string production on the AdS side. String bits [53]-[55] were proposed as a model which mimics this feature out of (but not very far from) the BMN limit. Being a reasonably simple and good tool for computing some bosonic quantities, the string bit model suffers from definite consistency problems related to fermionic doubling [56]-[59].

On the SYM side the exact one-loop dilatation operator was derived in [49, 51, 60]. When non-planarity is taken into account, single and multi-trace operators get mixed. In the dual picture this should correspond to string-string interactions in AdS background which up the moment is not very well understood. Waiting for a better understanding of string physics on AdS space, one could hope to learn about string interactions there by exploiting the dual gauge theory picture. This is the main motivation for the present work. We build a map from the set of multi-trace operators to a model of spins and study the corresponding spin system which, as we will see, mixes the integrable spin approach and the string bits one. Such a theory can be called a spin bit model. Since it allows for dynamical splitting and joining of chains and its variable content is given by spins, the spin bit model differs from both the spin chain and the string bit models, though being a mixture of both. In particular, there is no fermion doubling and supersymmetry in the spin bit model is implemented in a consistent fashion. (In fact, it is inherited from the SYM theory.)

The spin bit Hamiltonian provides us with a powerful tool simplifying the study of SYM theory at the non-planar level. The spectrum of anomalous dimensions is obtained by straightforward diagonalization of the spin bit Hamiltonian. In the present work we apply this technique to a systematic study of anomalous dimensions in $\mathcal{N} = 4$ SYM. The results perfectly match those found by quantum field theory methods [62]-[65] and provide a compact and unifying description of these computations. This confirms the validity of the expression for the dilatation operator proposed in [60]. Beyond this, we identify a new sequence of eigenstates starting at $\Delta_0 = 8$ with anomalous dimensions given by a finite $1/N$ expansion (in fact having the order $(1/N)^1$).

In the limit $N \to \infty$ the spin Hamiltonian becomes the one of ordinary spin chain and is local and integrable. The Hamiltonian and the generator of the total spin are the first two charges, in the tower of commuting ones, predicted by integrability [60, 61]. Higher charges are given in terms of higher powers of next-to-nearest spin generators summed up over the chain. Corrections in $1/N$ spoil locality and integrability. The Hamiltonian and
its higher spin analogs can still be defined in terms of powers of spin generators but now the
next-nearest character is lost and corresponding charges are no longer commuting among
themselves. They can be thought of as broken symmetries of the would-be integrable
system. It would be nice to understand the role of these broken charges in the theory
near the “integrable” point $N \to \infty$.

Our paper is organized as follows. In section 2 we introduce the spin chain/gauge
theory dictionary. To the exact one-loop dilatation operator of $\mathcal{N} = 4$ SYM we associate
the Hamiltonian in the corresponding spin chain. In section 3 we apply the result to the
study of non-planar anomalous dimensions. We determine the exact one-loop characteris-
tic polynomials for the first few operators sitting in the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ closed subsectors
of $\mathcal{N} = 4$ SYM. Finally in section 4 we draw some conclusions. Appendices collect some
background material and tables that complement the discussion in the text.

2 Spin bits

In [51] Minahan-Zarembo have shown that planar one-loop corrections to anomalous di-
mensions of purely scalar operators in $\mathcal{N} = 4$ SYM can be effectively computed in terms
of an integrable system, the SO(6) spin one model. They proposed to use Bethe Ansatz
for its solution (see [66] for a review on Bethe Ansatz). In [49] the results were extended
to the supersymmetric case in terms of a $\mathfrak{psu}(2,2|4)$ integrable spin chain. Combining the
$\mathfrak{psu}(2,2|4)$ symmetry with the one-loop planar result, an expression for the non-planar
one-loop dilation operator was finally derived in [49].

In what follows we construct a map from multi-trace SYM operators to $\mathfrak{psu}(2,2|4)$ spin
states and rewrite the dilation operator as a Hamiltonian acting on ordered sets of these
spin states. This action has a non next-nearest character due to joining and splitting of
the chains.

2.1 Preliminaries

In this subsection we review the type of quantities we are dealing with as well as construct
the map to spin bit states.

We consider $\mathcal{N} = 4$ SYM theory with the following field content : $F_{\mu \nu}$, $\phi^i$, $\lambda^A_a$ and
$\bar{\lambda}_{A\dot{a}}$ which are, respectively, the gauge field, six scalars and gauginos$^1$. We are interested
in the description of gauge invariant (polynomial) multi-trace operators in this model. It is convenient to adopt a “philological” terminology. The above fields, as well as their
covariant derivatives, form gauge-covariant “letters” $W_A$ of the SYM “alphabet”

$$W_A = \{\nabla^{*} \phi, \nabla^{*} F, \nabla^{*} \lambda, \nabla^{*} \bar{\lambda}\}. \quad (2.1)$$

The components of $W_A$ transform in the so called “singleton” (infinite dimensional) rep-
resentation $V_F$ of the $\mathcal{N} = 4$ superconformal algebra (SCA) $\mathfrak{psu}(2,2|4)$. All elementary
fields and their derivatives can be obtained by acting with generators of the SCA on the
primary fields $\phi^i$ with $i = 1, \ldots 6$ running in the vector representation of $\mathfrak{so}(6)$. Out of the
letters $W_A$ one can build gauge invariant “words” (single-trace operators) which are

$^1$Here $i = 1, \ldots 6$, $A = 1, \ldots 4$, $\mu = 0, \ldots 3$, $a, \dot{a} = 1, 2$. The abbreviations $\nabla^{*} \phi, \nabla^{*} F, \ldots$ stand for
$\nabla_{\mu_1} \ldots \nabla_{\mu_s} \phi^{i\alpha}, \nabla_{\mu_1} \ldots \nabla_{\mu_s} F_{\mu_1 \mu_2}, \ldots$
traces of a sequence of \( W_A \), and out of “words” one can produce “sentences” which are sequences of “words” (multi-trace operators). For instance, out of \( \phi^i \) we can build the word \( \text{Tr} \phi^i_1 \ldots \phi^i_n \) and the sentences \( \text{Tr} \phi^i_1 \ldots \phi^i_{n_1} \text{Tr} \phi^j_1 \ldots \phi^j_{n_2} \ldots \).

To each SYM operator (sentence) of length \( L \) we can associate a state in a spin chain or set of chains of the same total length, with symmetry group \( \mathfrak{psu}(2,2|4) \) and spin states pointed to directions \( A_k \) in \( V_F \) (label \( k \) numbers the sites). A state/operator is specified by a spin sequence \( |A_1, \ldots, A_L\rangle \) and by an element \( \gamma \) of the \( S_L \) permutation group

\[
\gamma \equiv (\gamma_1 \gamma_2 \ldots \gamma_L) : \begin{pmatrix} a_1 & a_2 & \ldots & a_L \\ a_{\gamma_1} & a_{\gamma_2} & \ldots & a_{\gamma_L} \end{pmatrix}
\]

(2.2)
describing the way different sites in the chain are connected to each other. Precisely, using these data a generic multi-trace operator can be written as

\[
|A_1, \ldots, A_L; \gamma\rangle \leftrightarrow W_{A_{\gamma_1}}^{a_1} W_{A_{\gamma_2}}^{a_2} \ldots W_{A_L}^{a_L} = \text{Tr} \left( W_{A_{\gamma_1}} W_{A_{\gamma_2}} \ldots W_{A_L} \right).
\]

(2.3)

Here \( \gamma = (L_1)(L_2) \ldots (L_k) \) is a permutation made of smaller cyclic permutations of \( L_m \) elements. Generically, the permutation group splits in equivalence classes labelled by \( L_1, \ldots, L_k, \sum L_r = L \) of permutations consisting of cycles of respective lengths.

The correspondence between operators and spin states is one-to-one, up to covariant relabelling of the indices. (Cyclic symmetry of the traces is a particular case of this relabelling.) In fact, different choices of \( \gamma \in S_L \) give different operators (2.3), modulo the equivalence relation

\[
|A_{\sigma_1}, \ldots, A_{\sigma_L}; \sigma \cdot \gamma \cdot \sigma^{-1}\rangle \sim |A_1, \ldots, A_L; \gamma\rangle = |A_1, \ldots, A_L\rangle \otimes_{S_L} |\gamma\rangle
\]

(2.4)

where \( \otimes_{S_L} \) is the tensor product, modulo the action of \( \sigma \in S_L \). In particular, when \( \sigma = \gamma \) the equivalence (2.4) reflects the cyclicity of the trace

\[
|A_1, \ldots, A_L; \gamma\rangle \sim |A_{\gamma_1}, \ldots, A_{\gamma_L}; \gamma\rangle,
\]

(2.5)

In what follows, we do not restrict ourselves to the canonical form of the permutation, where \( \gamma = (L_1)(L_2) \ldots (L_k) \) sends each label to the immediate next one modulo cyclicity. However, the conjugation (2.4) can be used to always rearrange the labels in such a form.

The correspondence between SYM operators and spin bits allows one to map the dilatation operator into an operator acting on the spin space. This operator can be identified with the spin bit Hamiltonian.

In perturbation theory the anomalous dimensions of gauge invariant operators in \( \mathcal{N} = 4 \) SYM can be written as follows:

\[
\Delta(g_{\text{YM}}) = \sum_k H_k \lambda^k,
\]

(2.6)

\[\text{Here and below products in the permutation group are understood as } \gamma \cdot \sigma = \gamma \cdot (\sigma_1 \sigma_2 \ldots \sigma_L) = (\sigma_{\gamma_1} \sigma_{\gamma_2} \ldots \sigma_{\gamma_L}).\]
with $\lambda = \frac{g^2 \mathcal{N}}{16\pi^2}$ being the 't Hooft coupling. The coefficients in this expansion are given in terms of effective vertices, i.e. the operators $H_k$. They are determined by an explicit evaluation of the divergencies of two-point function Feynman amplitudes. In particular, when an operator $\mathcal{O}$ renormalizes multiplicatively, the operator \((2.6)\) becomes diagonal,

$$\Delta_{\mathcal{O}} = \Delta_0 + \frac{g^2 \mathcal{N}}{\pi^2} \gamma_{\mathcal{O}} + \ldots,$$

where $\gamma_{\mathcal{O}}$ is the one-loop anomalous dimension and dots stand for higher loop corrections.

The tree-level dimensions are the naive ones while one-loop vertices have been derived in \[49\]

$$
\begin{align*}
H_0 &= \Delta_{0A} \text{Tr} W_A \hat{W}^A, \\
H_2 &= -\frac{2}{N} \sum_{j=0}^{\infty} h(j) (P_{jC})_{CD} \text{Tr} [W_A, W^C] [W_B, W^D],
\end{align*}
$$

where

$$\hat{W}_A^{ab} = \frac{\partial}{\partial W_{ba}^A}$$

and the colons :: denote the fact that the derivatives $\hat{W}_A^{ab}$ never act on the letters from the same group inside the colons. $\Delta_{0A}$ are the classical dimensions of the elementary SYM fields, $\Delta_0 = 1$ for scalar fields $\phi^i$ and derivatives, $\Delta_0 = \frac{3}{2}$ for gauginos and $\Delta_0 = 2$ for $F_{\mu\nu}$. $(P_{j})_{CD}^{AB}$ is the $\text{psu}(2,2|4)$ projector to the irreducible module $V_j$ appearing in the expansion tensor product of two singletons $V_F$

$$V_F \times V_F = \sum_{j=0}^{\infty} V_j.$$  

The first modules $V_0, V_1, V_2$ contain the symmetric, antisymmetric and trace components in the tensor product of two SYM scalars and their superpartners. Higher modules $V_j$ contain spin $j - 2$ currents and their supersymmetric completions. Finally $h(j)$ is the harmonic number

$$h(j) = \sum_{s=1}^{j} \frac{1}{s}.$$  

The effective vertices \((2.8)\) are manifestly $\text{psu}(2,2|4)$ covariant. Higher loop contributions involve increasing number of derivatives and inserted letters.

### 2.2 The Hamiltonian

Here we derive the one-loop Hamiltonian $H_2$ in the spin chain variables. In order to do this, we apply the operator $H_2$ given in \((2.8)\) to the multi-trace operator corresponding to the spin bit state $|A_1, \ldots, A_L ; \gamma\rangle$. As $H_2$ is a second order differential operator one should apply multiply the Leibnitz rule. Thus, the result will be represented as a sum,

$$H_2 |A_1, \ldots, A_L ; \gamma\rangle = \sum_{k,l} H_{2,kl} |A_1, \ldots, A_k, \ldots, A_l, \ldots A_L ; \gamma\rangle,$$  

(2.11)
where $H_{2, kl}$ is the restriction of $H_2$ to the sites with numbers $k$ and $l$ only.

The two main types of terms emerging from application of (2.5)³:

$$\text{Tr}(W_A \hat{W}^C \hat{W}_B \hat{W}^D)(\ldots W_{A_k}^{a_{\gamma_k}} \ldots W_{A_i}^{a_{\gamma_i}} \ldots) = \delta^C_{A_k} \delta^D_{A_i} (\ldots W_{A_k}^{a_{\gamma_k}} \ldots W_{B}^{a_{\gamma_i}} \ldots) = \delta^C_{A_k} \delta^D_{A_i} |A_1 \ldots A..B..A_L \gamma \cdot \sigma_{kl} \rangle,$$

$$\text{Tr}(W_A \hat{W}^C \hat{W}_B \hat{W}^D)(\ldots W_{A_k}^{a_{\gamma_k}} \ldots W_{A_i}^{a_{\gamma_i}} \ldots) = \delta^C_{A_k} \delta^D_{A_i} \delta_{a_{\gamma_k}a_{\gamma_i}} (\ldots W_{A_k}^{b_{a_{\gamma_k}}} \ldots W_{B}^{a_{\gamma_i}} \ldots) = \delta^C_{A_k} \delta^D_{A_i} |A_1 \ldots A..B..A_L ; \gamma \cdot \sigma_{kl} \rangle,$$

(2.12)

Here $\sigma_{kl}$ denotes the pairwise permutation of the $k^{th}$ and $l^{th}$ elements, whereas $|\{A_1 .. A .. B .. A_L\}\rangle$ corresponds to the replacement of $W_{A_k}$ and $W_{A_i}$ by $W_A$ and $W_B$ respectively.

The other terms of the Hamiltonian can be obtained from (2.12) by the exchange $A \leftrightarrow B$ in the first equation and by the simultaneous exchanges $(A \leftrightarrow B, C \leftrightarrow D)$ in the second one. Using the equivalence (2.4) each of these two terms can be rewritten in the following form:

$$|\{A_1..B..A..A_L\} ; \gamma \cdot \sigma_{kl} \rangle = |\{A_1..A..B..A_L\} ; \gamma \cdot \sigma_{\gamma_{\gamma_i}} \rangle$$

(2.13)

In order to do this one has to use the property of permutation $\sigma_{kl} \gamma = \gamma \sigma_{\gamma_{\gamma_i}}$ to push the $\sigma$'s to the right of $\gamma$. For the second term one also needs to relabel the summation indices $k \leftrightarrow l$.

The four terms can be written in a compact form by introducing the “two-site Hamiltonian” $H_{kl}$ and the “twist” operator $\Sigma_{kl}$ acting on the spin and linking spaces respectively

$$H_{kl} |\{A_1 \ldots A_L\}\rangle = 4 \sum_j h(j) (P_j)^A_B |\{A_1 \ldots A \ldots B \ldots A_L\}\rangle,$$

$$\Sigma_{kl} |\gamma\rangle = \begin{cases} |\gamma \sigma_{kl}\rangle & \text{if } k \neq l \\ N |\gamma\rangle, & k = l. \end{cases}$$

(2.15)

Here $\Sigma_{kl}$ acts as a chain splitting and joining operator as illustrated in fig. 1 the factor $N$ in the case $k = l$ in eq. (2.15) appears because splitting a trace at the same place leads to a chain of length zero, whose corresponding trace is $\text{Tr} 1 = N$. It is important to note that the operators $\Sigma_{kl}$ act only on the linking variable, while $H_{kl}$ act on the spin space leaving the link variable unchanged. Therefore, the action of $H_{kl}$ commutes with those of $\Sigma_{mn}$.

Summing up all ingredients, the one-loop dilation operator acquires the following form:

$$H_2 = \frac{1}{2N} \sum_{k \neq l} H_{kl} (\Sigma_{\gamma_{\gamma_i}} + \Sigma_{\gamma_{\gamma_i}} - \Sigma_{kl} - \Sigma_{\gamma_{\gamma_i}}).$$

(2.16)

Alternatively, using the canonical form for $\gamma = (L_1)\ldots(L_m) : k_i \mapsto [k_i + 1] \equiv k_i + 1 \mod L_i$, with $i = 1, 2, \ldots, m$ and $k_i$ running inside the $i^{th}$ trace, eq. (2.11)³ can be rewritten as

$$H_2 = \frac{1}{2N} \sum_{k \neq l} H_{kl} (\Sigma_{[k+1],l} + \Sigma_{k,[l+1]} - \Sigma_{kl} - \Sigma_{i[k+1],[l+1]}).$$

(2.17)

³One can use the so called fusion and fission formulas from [60] : $\text{Tr} A \hat{W}_C B W_D = \delta_{CD} \text{Tr} A \text{Tr} B$ and $\text{Tr} A \hat{W}_C \text{Tr} W_D B = \delta_{CD} \text{Tr} AB$, where $A$ and $B$ are supposed not to depend on $W$'s.
Planar contributions come from terms involving $\Sigma_{kk}$ in (2.16, 2.17), i.e. $l = \gamma_k$ or $k = \gamma_l$.

$$H_{2, \text{planar}} = \sum_k H_{k\gamma_k} = \sum_k H_{k[k+1]}.$$  \hspace{1cm} (2.18)

Summarizing, the Hamiltonian (2.16) describes the exact one-loop anomalous dimensions matrix. As compared to ordinary spin chain description it contains a new dynamical variable described by a $L$-permutation group element $\gamma$. This “degree of freedom” describes the chain structure of the configuration and becomes trivial in the planar case.

There is a certain similarity between our model and string bits \cite{53, 54, 55} in what concerns splitting and joining of the chains/strings. Notice, however, that our Hamiltonian has a quite different look from the string bit model Hamiltonian. In contrast to the latter case, the values of the fields are taken in spin space, rather than a standard target space, which is the case for string bits. In particular, formulation of the fermionic sector is completely different. The spin bit model possesses an explicit supersymmetry, inherited directly from the super Yang-Mills model. Hence there is no doubling problem, since it would not be compatible with supersymmetry at the level of the spectrum. The reader may be puzzled by the absence of such phenomena, which make the supersymmetric string bit model inconsistent. However, as we showed in \cite{58}, relaxing the requirement that fermions form a worldsheet spinor structure allows one to formulate a self-consistent supersymmetric string-like discrete model with no doubling, which is probably the case in the present spin bit Hamiltonian.

3 Anomalous dimensions

In this section we apply the exact one-loop Hamiltonian (2.16) to the systematic study of non-planar corrections to anomalous dimensions and mixing for composite operators in $N = 4$ SYM.
Our results are in perfect agreement with previous computations performed via Feynman diagrams \cite{65, 64} and higher spin techniques \cite{62, 63}. Here, anomalous dimensions are derived by diagonalization of the dilatation operator represented by the spin bit Hamiltonian (2.16). This gives a compact and unified description of the previous results in the literature and extends easily to higher scaling dimension states. Once applied to SYM states, the Hamiltonian (2.16) is represented by a block-diagonal matrix which is easily diagonalizable e.g. by use of a computer.

We will focus on the closed \( \mathfrak{su}(2) \) and \( \mathfrak{sl}(2) \) subsectors of the full supersymmetric group \( \mathfrak{psu}(2, 2|4) \). The generalization to the \( \mathcal{N} = 4 \) supersymmetric spin chain is straight although technically more involved and will be briefly described in section 3.3.

We will display exact (one-loop) characteristic polynomials for the first few (in general multi-trace) operators belonging to the \( \mathfrak{su}(2) \) and \( \mathfrak{sl}(2) \) closed sectors of \( \mathcal{N} = 4 \) SYM. The anomalous dimensions are the zeroes of such polynomials. In some particular cases, where characteristic polynomials nicely factorize, an analytic form for the exact one-loop anomalous dimensions will be produced.

### 3.1 \( \mathfrak{su}(2) \) spin chain

The \( \mathfrak{su}(2) \) spin chain can be defined by first restricting to \( \mathfrak{so}(6) \) purely scalar operators and then choosing a \( \mathfrak{su}(2) \) subgroup inside \( \mathfrak{so}(6) \). In the case of \( \mathfrak{so}(6) \), there are three irreducible representations appearing in the expansion of the product of two spin one modules into irreducible components \cite{49}:

\[
\begin{align*}
(P_0)_{mn}^{pq} &= \frac{1}{2} (\delta^p_m \delta^q_n + \delta^q_m \delta^p_n) - \frac{1}{6} \delta_{mn} \delta^{pq}, & h(0) &= 0, \\
(P_1)_{mn}^{pq} &= \frac{1}{2} (\delta^p_m \delta^q_n - \delta^q_m \delta^p_n), & h(1) &= 1, \\
(P_2)_{mn}^{pq} &= \frac{1}{6} \delta_{mn} \delta^{pq}, & h(2) &= \frac{3}{2}.
\end{align*}
\]

Plugging this into the two-site Hamiltonian (2.14), one gets

\[
H_{\mathfrak{so}(6)}^{kl} = 2 - 2 P_{kl} + K_{kl},
\]

where \( K_{kl} \) and \( P_{kl} \) are respectively the trace and the permutation operators between the \( k \)th and \( l \)th sites. Restricting to the planar level (2.18), the Hamiltonian of the integrable \( \mathfrak{so}(6) \) spin chain is found \cite{51}.

Let us focus on the \( \mathfrak{su}(2) \) subsector, i.e. the \( \mathfrak{su}(2) \)\( j = \frac{1}{2} \) spin chain. This sector is spanned by holomorphic operators made out of only two complex scalars, let us say \( \phi_0 \) and \( \phi_1 \), transforming in the fundamental representation of \( \mathfrak{su}(2) \in \mathfrak{so}(6) \). This corresponds to restricting to SYM states with \( \mathfrak{su}(4) \sim \mathfrak{so}(6) \) Dynkin labels \( [n, \Delta_0 - 2n, n] \), with \( \Delta_0 \) denoting the naive conformal dimension (i.e. the number of letters) and \( n \) being a positive integer (representing the number of impurities, let us say \( \phi_1 \) in the \( \mathfrak{su}(2) \) highest weight states). Then, the \( \mathfrak{su}(2) \) spin is identified with the middle Dynkin label

\[
j = \frac{1}{2} \Delta_0 - n.
\]

The highest weight states \( [n, \Delta_0 - 2n, n] \) saturate at \( g_{YM} = 0 \) the BPS like unitarity bounds and sit in \( \frac{1}{2} \) - and \( \frac{1}{4} \) - BPS multiplets of the \( \mathcal{N} = 4 \) SCA for \( n = 0 \) and \( n \geq 1 \) respectively. When interactions are turned on \( g_{YM} \neq 0 \), anomalous dimensions are
generated and the bounds are no longer satisfied. Unprotected 1/4-BPS multiplets come together with semishort multiplets (sharing their dimension) to build long multiplets of the superconformal algebra. Protected 1/4-BPS multiplets were studied in [67] (see also [68]). Here we present a systematic description of anomalous dimensions and mixing for all (in general multitrace) 1/4-BPS, up to $\Delta_0 < 10$.

The two-site Hamiltonian follows from (3.2) by simply omitting the trace contribution

$$H_{kl} = 2 - 2 P_{kl},$$

The exact Hamiltonian is then obtained by plugging (3.3) into (2.16). Using (2.13) in terms of operators, i.e. $P_{kl} \Sigma_{kl} = \Sigma_{\gamma_k \gamma_l}$, one finds the $su(2)$ Hamiltonian:

$$H_2 = \frac{1}{N} \sum_{k \neq l} (2 - 2 P_{kl}) \Sigma_{k\gamma_l}$$

SYM states in the $su(2)$ sector are given by sequences of traces made out of $\phi_0$ and $\phi_1$. Since the Hamiltonian (2.16) act only by either permuting the fields $\phi_0$ and $\phi_1$, or by joining/splitting traces, then operators with different numbers of $\phi_0, \phi_1$ do not mix. The states can therefore be characterized by the conformal dimension $\Delta_0$, which is equal to the total number of fields, and by $n$, i.e. the number of impurities $\phi_1$. The anomalous dimension matrix can be found by first listing all inequivalent multi-trace operators for a fixed $\Delta_0, n$ and then acting upon them with the Hamiltonian.

The $su(2)$ symmetry can be used to reduce the entropy of the analysis. Indeed SYM states are organized in irreducible representations of $su(2)$, with all components in the same $su(2)$ multiplet sharing the same anomalous dimension. This is clearly the case since the Hamiltonian commutes with the operator of total spin. We can then focus on $su(2)$ highest weight states (h.w.s.). Irreducible representations of $su(2)$ are specified by Young tableaux with at most two rows. Therefore, we will use Young tableaux in order to represent our operators (see Appendix for details).

For example, the tableau $Y = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$ will stand for the following operator$^4$:

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} = \frac{1}{|Y|} \text{Tr} \phi_{(i_1 \phi_{i_2})\phi_{(i_3 \phi_{i_4})}}. $$

Here the tensor $\text{Tr} \phi_{i_1 \phi_{i_2}} \phi_{i_3 \phi_{i_4}}$ is projected according to the operator $\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$ by first symmetrizing along indices in the same row ($i_1 \leftrightarrow i_2, i_3 \leftrightarrow i_4$) and then antisymmetrizing indices along the columns ($i_1 \leftrightarrow i_3, i_2 \leftrightarrow i_4$). Notice that the two actions do not commute and therefore the resulting operator is no longer symmetric along the rows. $|Y|$ is a combinatorial factor (see formula (1.2) in Appendix B). Multi-trace operators will be represented in the Young tableaux by thick columns indicating where new traces start. Finally, we will not write the numbers in the boxes of the Young tableaux filled in the

$^4$ In our convention, the symmetrization ($\cdot$) and antisymmetrization symbols do not take into account the usual $\frac{1}{p!}$ factor: $T_{(kl)} = T_{kl} + T_{lk}$ and $T_{[kl]} = T_{kl} - T_{lk}$. 

9
As an illustration, let us consider an example of the sector of operators with \((\Delta, n) = (4, 2)\). The list of considered states is

\[
\begin{align*}
\text{Tr } \phi_0^2 \phi_1^2 &= |0011; (2341)\rangle \\
\text{Tr}(\phi_0 \phi_1)^2 &= |0101; (2341)\rangle \\
\text{Tr } \phi_0^2 \text{Tr } \phi_1^2 &= |0011; (2143)\rangle \\
(\text{Tr } \phi_0 \phi_1)^2 &= |0101; (2143)\rangle.
\end{align*}
\]

This sector is closed under the action of the Hamiltonian given by (3.4), with indices \(k\) and \(l\) running from 1 to 4. Let us consider first the double-trace state \(
\text{Tr } \phi_0^2 \text{Tr } \phi_1^2
\). When the Hamiltonian acts on this state one has to compute expressions as

\[
(1 - P_{32})\Sigma_{3\gamma_2}|0011; (2143)\rangle = (1 - P_{23})|0011\rangle \otimes \Sigma_{13}|(2143)\rangle
\]

\[
= (|0011\rangle - |0101\rangle) \otimes |(2143)\rangle
\]

\[
= (|0011\rangle - |0101\rangle) \otimes |(2341)\rangle
\]

\[
= |0011; (2341)\rangle - |0101; (2341)\rangle
\]

\[
= \text{Tr } \phi_0^2 \phi_1^2 - \text{Tr}(\phi_0 \phi_1)^2.
\]

All various terms in (2.16) give similar contributions. After summing them up, one ends with

\[
H_2 \text{Tr } \phi_0^2 \text{Tr } \phi_1^2 = \frac{16}{N} \text{Tr } \phi_0^2 \phi_1^2 - \frac{16}{N} \text{Tr}(\phi_0 \phi_1)^2.
\]

This kind of computation can easily be implemented with computer software as *Mathematica*.

In the same way, one finds

\[
\begin{align*}
H_2 \text{Tr } \phi_0^2 \phi_1^2 &= 4 \text{Tr } \phi_0^2 \phi_1^2 - 4 \text{Tr}(\phi_0 \phi_1)^2 \\
H_2 \text{Tr}(\phi_0 \phi_1)^2 &= -8 \text{Tr } \phi_0^2 \phi_1^2 + 8 \text{Tr}(\phi_0 \phi_1)^2 \\
H_2(\text{Tr } \phi_0 \phi_1)^2 &= -\frac{8}{N} \text{Tr } \phi_0^2 \phi_1^2 + \frac{8}{N} \text{Tr}(\phi_0 \phi_1)^2.
\end{align*}
\]

The anomalous dimensions \(\gamma\) are then given by \(\text{Det}(\gamma - \frac{H_2}{N})\). As a result one finds eigenvalue \(\gamma = 0\) (triple degenerate) and (non-degenerate) \(\gamma = 3/4\). Two out of the three \(\gamma = 0\) states sit in the completely symmetric \(j = 2\) \(\text{su}(2)\) multiplets with highest weight state\(^5\):

\[
\begin{align*}
\text{Tr } \phi_0^4 = |0\rangle_{\text{hws}} \\
\text{Tr } \phi_0^2 \phi_1^2 = |1\rangle_{\text{hws}}.
\end{align*}
\]

\(^5\)Here highest weight states correspond to Young tableaux \(Y\) with \(\Delta_0 - n\) boxes (filled with \(\phi_0\)'s) in the first row and \(n\) boxes (filled with \(\phi_1\) in the second row.
The remaining two eigenstates are \( \text{su}(2) \) singlets \((j = 0)\) can be written as

\[
j = 0 \quad \frac{4}{N} \text{hws} = \frac{2}{3} \text{Tr} \phi_0^2 \text{Tr} \phi_1^2 - \frac{2}{3} (\text{Tr} \phi_0 \phi_1)^2 - \frac{4}{3N} \text{Tr} \phi_0 [\phi_0, \phi_1] \phi_1
\]

\[
\text{hws} = \frac{1}{3} \text{Tr} \phi_0 [\phi_0, \phi_1] \phi_1.
\]

with \( \gamma = 0 \) and \( \gamma = 3/4 \) respectively. The latter corresponds to the \([2,0,2]\) scalar in the long Konishi multiplet.

In a similar way one proceeds for a different number of impurities \( n \). The one row tableau \[ \begin{array}{c} \hline \end{array} \] and \[ \begin{array}{c} \hline \end{array} \] collect all completely symmetric combinations with \( n = 0, 1, \ldots, 4 \) impurities with h.w.s. \( n = 0 \) given by (3.5). There are \( 2 \times 5 \) states of this type. Altogether they build two spin \( j = 2 \) \( \text{su}(2) \) multiplets. The anomalous dimension Hamiltonian acts trivially on these states since they are symmetric and therefore \( \Delta = \Delta_0 \).

Analogously, in the case of higher scaling dimensions, one first lists \( \text{su}(2) \) highest weight states for a given \( \Delta_0 \) and then diagonalizes the spin bit Hamiltonian (2.16,2.17) in the corresponding subspace. We collect in table 1 the characteristic polynomials for the first few \( \text{su}(2) \) states up to \( \Delta_0 = 7 \). The exact anomalous dimensions are the zeroes of these polynomials. In particular, the order of the polynomial gives the number of different (in general multi-trace) operators built out of \( n \phi_0 \)'s and \( \Delta_0 - n \phi_1 \)'s. Finally we display in the last column the corresponding eigenvalues at leading order in \( N \) (planar anomalous dimensions). In table 3, we collect the exact eigenstates and anomalous dimensions up to \( \Delta_0 = 6 \). Notice that although anomalous dimensions for states in this table do not receive \( \frac{1}{N} \) corrections, a non-trivial mixing between single and multi-trace operators is at work.

| \( \Delta_0 \) | \( n \) | \( \gamma_{\text{exact}} \) | \( \gamma_{\text{planar}} \) |
|---|---|---|---|
| 4 | 2 | \( \left( -\frac{3}{4} + x \right) x \) | \( 0, \frac{3}{4} \) |
| 5 | 2 | \( \left( -\frac{1}{2} + x \right) x \) | \( 0, \frac{1}{2} \) |
| 6 | 2 | \( x^3 \left( \frac{10}{N^2} - 15 - \frac{40}{N^2} x + 80 x - 128 x^2 + 64 x^3 \right) \) | \( 0^3, \frac{3}{4}, \frac{5+\sqrt{2}}{8} \) |
| 3 | \( -\frac{3}{4} + x \) | | \( \frac{3}{4} \) |
| 7 | 2 | \( x^4 \left( 9 + \frac{42}{N^2} x - 78 x - \frac{80}{N^2} x^2 + 232 x^2 - 288 x^3 + 128 x^4 \right) \) | \( 0^4, \frac{1}{4}, \frac{1}{2}, \left( \frac{3}{4} \right)^2 \) |
| 3 | \( -\frac{5}{8} + x \) | \( x^2 \left( \frac{-9+\sqrt{1+160}}{16} x + \frac{9-\sqrt{1+160}}{16} \right) \) | \( 0^2, \frac{1}{2}, \left( \frac{5}{8} \right)^2 \) |

Table 1: Characteristic polynomials for \( \text{su}(2) \) h.w.s. with \( \Delta \leq 7 \).

We display only \( \text{su}(2) \) highest weight states. More precisely a state in the table at

\[\text{Here and below we assume } N \text{ large enough } (N > \Delta_0), \text{ in order to avoid non trivial identifications between single and multi-trace operators. The generalization to small values of } N \text{ is straightforward. For instance, taking } N = 2 \text{ in (3.5), the first } j = 0 \text{ state vanishes identically, while the two } j = 0 \text{ states are related to each other and should be counted only once.} \]
$(\Delta_0, n)$ is the highest weight state of a spin $j = \frac{1}{2} \Delta_0 - n$ representation of $\mathfrak{su}(2)^7$. This implies that each characteristic polynomial for a state $(\Delta_0, n)$ in the table will be replicated at $\sum_{m=0}^{\Delta_0-2n} (\Delta_0, n + m)$. For example the polynomial at $(\Delta_0, n) = (6, 2)$ appears again at (6, 3) and (6, 4). Altogether they form a spin $j = 1 \mathfrak{su}(2)$ multiplet. These redundant states are not displayed.

In addition we omit multi-trace operators with $n = 0, 1$ number of impurities and their $\mathfrak{su}(2)$ descendants. They lead to states with exact conformal dimensions $\Delta_0$ protected from both loops and non-planar corrections. States with $n = 0$ sit in the vacuum multiplets with highest weight states $\text{Tr} \phi^3_0$. They correspond to all possible cuts of the chiral vacuum state $\text{Tr} \phi^{2\Delta_0}_0$. Similarly states with $n = 1$ arise from cuttings of $\text{Tr} \phi_1 \phi^{\Delta_0-1}_0$. They correspond to the highest weight states of $\mathfrak{su}(2)$ irreducible representations with spin $j = \frac{1}{2} \Delta_0$ and $j = \frac{1}{2} (\Delta_0 - 1)$ respectively. Generating functions for states with $n = 0, 1$ can be written as$^8$

$$
\begin{align*}
\text{For instance, at } (\Delta, n) = (4, 1), \text{ we find two “one impurity” states: } &\text{Tr} \phi^3_0 \phi_1 \text{ and } \text{Tr} \phi_0 \phi_1 \phi^2_0. \\
&\text{The full characteristic polynomial at } (\Delta, n) = (4, 2) \text{ is then given by the product of the polynomial in the table with } x^2 \text{ coming from the } \mathfrak{su}(2) \text{ descendants of the two } n = 1 \text{ states.}
\end{align*}
$$

The most interesting features start showing up at $n = 2$. Non-planar corrections first appear for $\Delta_0 = 4$ where a single trace can split into two double-trace operators, each made out of two letters. The $(\Delta_0, n) = (6, 2), (7, 2)$ cases were studied in $^{60, 50}$. The one-loop anomalous dimensions (the zeros of the characteristic polynomials) in these cases can be written as an infinite $\frac{1}{N}$-expansion.

In particular circumstances, the non-planar characteristic polynomials nicely factorize and an explicit form for the exact anomalous dimensions to all orders in $\frac{1}{N}$ can be written. Table 4 in the appendix collects some relevant examples where exact (to all order in $1/N$) expressions for one-loop anomalous dimensions can be written. The first case is the pair of states at $\Delta_0 = 7$ with $n = 3$ impurities where the infinite series of $\frac{1}{N}$ corrections reconstruct a square root $^{64}$. Even more interesting is a (presumably infinite) series of operators starting with $(\Delta_0, n) = (8, 3), (9, 4), (10, 3), \ldots$ whose one-loop anomalous dimensions get non-planar corrections only at the first order in $\frac{1}{N}$

$$
\Delta^{(8,3)}_\pm = 8 + \frac{g_\pi^2 M N}{\pi^2} \left( \frac{3}{4} \pm \frac{3}{4N} \right) \\
\Delta^{(9,4)}_\pm = 9 + \frac{g_\pi^2 M N}{\pi^2} \left( \frac{5}{8} \pm \frac{3}{4N} \right) \\
\Delta^{(10,3)}_\pm = 10 + \frac{g_\pi^2 M N}{\pi^2} \left( \frac{3}{4} \pm \frac{3}{2N} \right) \quad (3.7)
$$

$^7$su(2) descendants are given by $|i_1, \ldots, i_L\rangle \to \sum_k \delta_{i_k, 0} |i_1, \ldots, i_L\rangle |i_k\to 1$

$^8$The omission of the term $J = 1$ in the product corresponds to the fact that Tr $\phi_0 = 0$ in SU(N).
The presence of $1/N$ corrections rather than the more familiar $1/N^2$ may appear surprising. Notice however that such corrections come always in pairs with opposite signs and therefore the corresponding characteristic polynomials depend only on $1/N$. Non-planar corrections result in a splitting of the degenerate energy levels at planar order (see [44] for more details).

It is important to stress that non-planar corrections to the one-loop Hamiltonian are only of order $1/N$ corresponding to the joining-splitting string vertex $g_s \sim 1/N$. Anomalous dimensions (energy levels) follow from this Hamiltonian via diagonalization. On the string theory side the eigenstates (3.7) correspond to bound states which are a mixture of single and multi-string states.

For convenience of the reader, table 4 is also given in terms of traces, rather than with Young tableaux, in table ?? in the appendix.

### 3.2 $\text{sl}(2)$ spin chain

Now we consider states belonging to a $\text{sl}(2)$ subgroup. This subgroup is generated by a single scalar field component, say $\phi_0$, and all its covariant derivatives along e.g. the first direction: $\phi^n = \frac{1}{n!} D^n \phi_0$. A state in the $\text{sl}(2)$ sector is then specified by the sequence $|\{n_i\}\rangle \equiv \{\phi^{n_1}, \phi^{n_2}, \ldots, \phi^{n_L}\}$ and the linking variable $|\gamma\rangle$. Notice that unlike the $\text{su}(2)$ case, representations at each site are now infinite dimensional, i.e. $n_k$ runs from zero to infinity.

The two-site Hamiltonian is given by [49]:

$$H_{kl} \phi^m_k \phi^{n-m}_l = [h(m) + h(n - m)] \phi^m_k \phi^{n-m}_l - \sum_{m' = 0}^{n} \delta_{m \neq m'} \phi^m_k \phi^{m'-m}_l.$$

Plugging in (2.16) we can find the exact $\text{sl}(2)$ Hamiltonian. The spectrum of non-planar characteristic polynomials for the first few states in this sector is displayed in table 2 (see [62, 63] for previous results in this sector). Here again, states are labelled by the classical dimension $\Delta_0$ and the number of impurities, i.e. derivatives, $n$. Since each derivative contributes once to the dimension $\Delta_0$, the number of letters used in building the states in table 2 is $L = \Delta_0 - n$. Once more, we omit states with $n = 0, 1$ impurities and $SL(2)$ descendants. Now $\text{sl}(2)$ descendants of each line $(\Delta_0, n)$ span an infinite tower $\sum_{m=0}^{\infty} (\Delta_0 + m, n + m)$ of eigenstates found by acting with $m$ derivatives on a given eigenstate.

It is instructive to compare the $\text{su}(2)$ and $\text{sl}(2)$ tables 1 and 2. States in the two tables are often related by supersymmetry. If this is the case, their full non-planar characteristic polynomials of anomalous dimensions (not only the planar contributions) should coincide. Indeed, a simple inspection shows that the full $n = 2$ characteristic polynomials perfectly match. This is agreement with the results in [45] where $n = 2$ impurity states has been shown to belong to the so called "BMN supermultiplets" with highest weight state primary in the $[0, \Delta_0 - 2, 0]$ representation of $SU(4)$.

---

9A derivative corresponds to send $|n_1, \ldots, n_L\rangle \rightarrow \sum_k (n_k + 1) |n_1, \ldots, (n_k + 1), \ldots, n_L\rangle$. 

---

13
$\Delta_0$ | $\gamma_{\text{exact}}$ | $\gamma_{\text{planar}}$
--- | --- | ---
4 | $(-\frac{3}{4} + x)$ | $0, \frac{3}{4}$
5 | $(-\frac{1}{2} + x)$ | $0, \frac{1}{2}$
6 | $x^3 \left( \frac{10}{N^2} - 15 - \frac{40}{N^2} x + 80x - 128x^2 + 64x^3 \right)$ | $0^3, \frac{3}{4}, \frac{5 + \sqrt{5}}{8}$
3 | $(-\frac{15}{16} + x)^2$ | $(\frac{15}{16})^2$
4 | $(-\frac{25}{24} + x)$ | $\frac{25}{24}$
7 | $x^4 \left( 9 + \frac{42}{N^2} x - 78x - \frac{80}{N^2} x^2 + 232x^2 - 288x^3 + 128x^4 \right)$ | $0^4, \frac{1}{4}, \frac{1}{2}, (\frac{3}{4})^2$
3 | $(-\frac{3}{4} + x)^3$ | $(\frac{3}{4})^3$
4 | $(-\frac{3}{4} + x)$ | $\frac{3}{4}$

Table 2: Characteristic polynomials for $\mathfrak{sl}(2)$ highest weight states with $\Delta_0 \leq 7$.

### 3.3 Supersymmetric spin chain

For completeness we briefly describe next the generalization to the supersymmetric case. The two-site Hamiltonian for the $N = 4$ supersymmetric spin chain was derived in [45], in terms of the so called harmonic action. In this formalism SYM letters are represented by acting with any number of bosonic ($a_\alpha, b_\dot{\alpha}$) and fermionic oscillators ($c_r, d_\dot{r}$), $\alpha, \dot{\alpha}, r, \dot{r} = 1, 2$ on a Fock space vacuum $a^n a_\alpha b^n b_\dot{\alpha} c^l c_r d^l d_\dot{r} | 0 \rangle$ subjected to the condition

$$C = n_a - n_b + n_c - n_d = 0. \quad (3.8)$$

The two-site Hamiltonian reads [49]

$$H_{kl} | s_1, .. s_n \rangle = \sum_{s_i'} c_{n,n_{kl},n_{lk}} \delta_{C_k,0} \delta_{C_l,0} | s_1', .. s_n' \rangle. \quad (3.9)$$

Here $n$ is the total number of oscillators and $s_i, s_i' = k, l$ denote their position. Remarkably the Hamiltonian does not depend on the type of insertion but only on the positions of the insertions labelled by the sequences of $s_i$'s. Also, $\delta_{C_k,0}, \delta_{C_l,0}$ ensure that the $C = 0$ condition holds at each site. Finally the coefficients $c_{n,n_{kl},n_{lk}}$ are given by

$$c_{n,n_{kl},n_{lk}} = (-)^{1+n_{kl}n_{lk}} \frac{\Gamma(\frac{1}{2}(n_{kl} + n_{lk})) \Gamma(1 - \frac{1}{2}(n - n_{kl} - n_{lk}))}{\Gamma(1 + \frac{1}{2}n)},$$

$$c_{n,0,0} = h(\frac{1}{2}n),$$

with $n_{kl}, n_{lk}$ denoting the number of oscillators hopping from the $k$ to the $l$ site and viceversa.

The non-planar Hamiltonian is given in terms of (3.9) via (2.16, 2.17). The supersymmetry invariance of $H_2$ follows from the fact that $H_{kl}$ commutes with both $\mathfrak{psu}(2,2|4)$ [49] and $\Sigma_{k,l}$. The Hamiltonian $H_2$ determines the full non-planar corrections to one-loop anomalous dimensions in $\mathcal{N} = 4$ SYM theory.
4 Discussion

In this paper we apply spin chain techniques to the study of nonplanar corrections to the one-loop anomalous dimension matrix for composite operators in $\mathcal{N} = 4$ SYM. Eigenstates in the spin bit model describe, via AdS/CFT correspondence, bound states in String Field Theory, where string interactions are weighted by $1/N$.

While the planar approximation of this theory leads to the description in terms of integrable spin chains, taking into account the nonplanarity leads to the appearance of new degrees of freedom, i.e. the linking variable, in the spin chain model. Interactions result in dynamical fissions and fusions of the chains. This effect resembles the string splitting and joining in the string field theory. Also the discrete model we obtained has some similarity with spin network models introduced by Penrose \[69\] (for a review see \[70\]) in order to describe discrete gravity.

In order to demonstrate the power of the spin bit approach we use it to perform the computation of anomalous dimensions of composite operators in SYM theories. Thus, we reproduce a number of already known results and produce new ones, as well. Out of the total symmetry group $\mathfrak{psu}(2, 2|4)$ of the model we focus on closed subsectors corresponding to the following subgroups: $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$. We provide a detailed analysis of anomalous dimensions and mixing in these sectors. States in the two sectors are not completely independent but are related in many instances by supersymmetry. When this is the case, the exact anomalous dimensions in the two sectors match. Anomalous dimensions are encoded in characteristic polynomials, but an analytic form for the zeros is often hard to be extracted. Remarkably, in particular circumstances the characteristic polynomials nicely factorize and an analytic form for the exact anomalous dimensions can be written. This is the case for a pair of eigenstates at $\Delta_0 = 7$ with three impurities, where non-planar corrections in $\frac{1}{N^2}$ are summed up to reconstruct an exact square root \[64\]. Even more surprisingly, we identify a new sequence of paired operators where the conformal dimensions get corrected only at order $\frac{1}{N}$. The string interpretation of this result and whether it remains true also to higher loops still remains to be clarified.

Beyond the application to the study of non-planar corrections in $\mathcal{N} = 4$ SYM theory, the spin models under consideration here have their own interest as an example of a polymer model with dynamical splitting and joining and a nontrivial discrete model with supersymmetry. Eqs. (2.16,2.17) give a natural extension of any spin model (integrable or not) to account for decaying and fusions of the chain. This generalization can be thought as a sort of gauging of the global symmetry $k_i \rightarrow k_i + 1$ present in the planar (next-to-nearest) interaction (2.18). Indeed, this symmetry is enhanced in (2.16,2.17) to $k \rightarrow \sigma_k$, with $\sigma \in S_L$. The gauging is provided by the “connection” $\Sigma_{kl}$.

In our approach we did not use the whole power of Bethe Ansatz and integrability. Whether integrable techniques can be used efficiently at least in the framework of perturbation theory near the integrable point $N \rightarrow \infty$ remains to be seen. To this purpose one should compute the scalar products (formfactors) of Bethe states with arbitrary spin states. There is some approach in the literature to this issue\[10\] \[71\] (see \[72\] for a review of recent developments).

Another issue we left beyond our consideration is related to the fact that, at finite $N$, nonperturbative effects in SYM theory begin to take place. So far, we do not know what

\[10\] We thank N. Slavnov for pointing our attention to this research.
will be their effect on our analysis.

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A Young Tableaux

In this appendix, we collect some group theory material and explain our notations for Young tableaux used in section 3.1. A Young tableau is a row-decreasing diagram made out of boxes, representing the fundamental representation of a group G (here G = Sm or G = SU(m)). Tensor products $\Box^k$, i.e. tensors $T_{i_1...i_k}$ with $i_\sigma = 1, 2, \ldots, m$, decompose into a sum of irreducible representations of G characterized by Young tableaux specifying how indices are symmetrized or antisymmetrized.

We define the Young symmetrizer $S_{j_1...j_p; i_1...i_p}$ by:

$$S = S_{j_1...j_p; i_1...i_p} = \sum_{\sigma \in S_p} \delta_{j_1}^{i_\sigma_1} \delta_{j_2}^{i_\sigma_2} \cdots \delta_{j_p}^{i_\sigma_p}$$

and the Young antisymmetrizer $A_{j_1...j_p; i_1...i_p}$ by:

$$A = A_{j_1...j_p; i_1...i_p} = \sum_{\sigma \in S_p} \epsilon(\sigma) \delta_{j_1}^{i_\sigma_1} \delta_{j_2}^{i_\sigma_2} \cdots \delta_{j_p}^{i_\sigma_p} .$$

A Young tableau Y can be seen as the projection

$$T_Y^{T_{i_1...i_p}} = \frac{1}{|Y|} (AS)_{i_1...i_p; j_1...j_p} T_{j_1...j_p}$$

with $S(A)$ denoting the operator that (anti)symmetrizes indices in the same row (column): starting with a tensor $T_{i_1...i_p}$, we first apply symmetrizers according to the rows of Y, then antisymmetrizers according to the columns of Y. For example, applying $Y = \Box \Box \Box \Box$ on $T$ means that we first symmetrize the indices $(i_1, i_2, i_3)$ and $(i_4, i_5)$, then antisymmetrize the resulting tensor $T^S_{i_1i_2i_3i_4i_5} \equiv T_{i_1i_2i_3}(i_4i_5)$ on indices $[i_1, i_4]$ and $[i_2, i_5]$.

Notice that the two actions do not commute and therefore the resulting tensor is no longer symmetric on $(i_1, i_2, i_3)$ and $(i_4, i_5)$.

---

11. We do not put the usual renormalisation factor $\frac{1}{p!}$ for further simplicity.

12. In practice, this is equivalent to first antisymmetrize and then symmetrize, but acting on the positions of the indices appearing in the tensors rather than on their labels.
There are two combinatorial factors that characterize a tableau \( Y : |Y| \) and \( f_Y \). For example, we have

\[
|Y| = \begin{array}{c}
5 & 4 & 2 & 1 \\
2 & 1 \\
\end{array}
= 80
\]

\[
f_Y = \frac{m + 1}{m - 1} \frac{m + 2}{m - 2} \frac{m + 3}{m - 3} = (m - 1)m^2(m + 1)(m + 2)(m + 3).
\]

They depend only on the shape of the tableaux. Coefficient \( |Y| \) is given by the following hook formula: write the sum of the number of boxes to the bottom and to the right, plus one, inside each box of the Young tableau, then multiply all these numbers. The result gives the overall coefficient in \( \Box^k \) that ensures that the tableau is indeed a projector.

The two quantities determine the dimension \( d_Y \) of a tableau (the number of independent components in \( T^Y_{i_1, \ldots, i_k} \)), and the multiplicities \( n_Y \) of a given tableau shape in the tensor product \( \Box^k \).

\[
d_Y = \frac{f_Y}{|Y|} \quad n_Y = \frac{k!}{|Y|}
\]

\( \text{e.g.} \)

\[
\Box^3 = \Box + 2 \Box \Box + \Box
\]

\[
m^3 = \frac{1}{3!} m(m + 1)(m + 2) + \frac{1}{3} m(m - 1) + \frac{1}{3!} m(m - 1)(m - 2)
\]

Here \( n_Y \) are the coefficients in front of the tableaux, \( d_Y \) the dimensions. Using these conventions, we are now able to represent the \( \text{su}(2) \) eigenstates as Young tableaux by identifying these with their action on a tensor of the form \( Tr[\phi_{i_1}, \phi_{i_2}, \ldots, \phi_{i_k}] \), with \( i_p = 0, 1 \). As we are considering \( \text{su}(2) \), the only Young tableaux we should use have at maximum two rows. The following rules have been followed:

- In cases of multiple traces, thick columns in the Young tableaux give the positions where new traces start.

- After the projection, the indices \( i_p \) for which \( p \) is in the first \( (\Delta_0 - n) \) boxes take the value 0, while the \( i_p \) for which \( p \) is in the last \( n \) boxes are set to 1.

As an example we give, for the case \( (\Delta_0 = 6, n = 3) \), the procedure that constructs

\[
\begin{array}{c}
\boxed{0} \boxed{3} \boxed{1} \\
\boxed{0} \boxed{0} \\
\end{array}
\]

\[
\begin{align*}
T_{i_1i_2i_3i_4i_5i_6} &= \text{Tr} \phi_{i_1} \phi_{i_2} \text{Tr} \phi_{i_3} \phi_{i_4} \phi_{i_5} \phi_{i_6} \\
T^S_{i_1i_2i_3i_4i_5i_6} &= T_{(i_1i_2i_3i_4)(i_5i_6)} \\
T^Y_{i_1i_2i_3i_4i_5i_6} &= \frac{1}{|Y|} T^S_{i_1i_2i_3i_4i_5i_6} \\
&= T^Y_{000111}
\end{align*}
\]

The result is then

\[
\begin{align*}
\begin{array}{c}
\boxed{1} \boxed{2} \boxed{3} \boxed{4} \\
\boxed{0} \boxed{0} \\
\end{array}
&= \frac{1}{|Y|} \delta^{k_1k_2k_3k_4k_5k_6}_{000111} (\delta_{j_3j_4}^{j_1j_5} A_{j_1j_5}^{j_2j_6} A_{j_2j_6}^{k_1k_5} A_{k_1k_5}^{k_2k_6}) (S_{j_1j_2j_3j_4}^{i_1i_2i_3i_4} S_{j_5j_6}^{i_5i_6}) \text{Tr} \phi_{i_1} \phi_{i_2} \text{Tr} \phi_{i_3} \phi_{i_4} \phi_{i_5} \phi_{i_6} \\
&= \frac{1}{10} \text{Tr} \phi_1^2 \text{Tr} \phi_0^2 \phi_1 + \frac{1}{5} \text{Tr} \phi_0 \phi_1 \text{Tr} \phi_0^2 \phi_1^2 - \frac{2}{3} \text{Tr} \phi_0 \phi_1 \text{Tr}(\phi_0 \phi_1)^2 \\
&\quad + \frac{1}{10} \text{Tr} \phi_0^2 \phi_1^3.
\end{align*}
\]
For simplicity, we do not write the numbers in the boxes when these are filled in the natural order, e.g. $\begin{array}{c} n_1 \\ n_2 \end{array}$ $\equiv$ $\begin{array}{c} \to \\ \to \end{array}$.

For the $SU(2)$ case, a simple formula gives the $|Y|$ coefficient:

$$
| \begin{array}{c} n_1 \\ n_2 \end{array} | = \frac{(n_1 + 1)!}{n_1 - n_2 + 1}.
$$

Finally, the spin of such a tableau is given by $j = \frac{1}{2}(n_1 - n_2)$.

**B $su(2)$ anomalous dimension eigensystems**

Here we collect three $su(2)$ tables: table 3 lists all eigenstates for $\Delta \leq 6$, table 4 in the appendix collects the first few exact eigenstates for operators with rational $\gamma \neq 0$ anomalous dimensions, table 5 gives the translations of table 4 in the main text.
| $\Delta_0$ | n  | Eigenvectors                                                                 | $\gamma_{\text{exact}}$ |
|-----------|----|-----------------------------------------------------------------------------|--------------------------|
| 4         | 0  | $\begin{pmatrix} 1 \end{pmatrix}$                                         | 0                        |
|           | 2  | $\begin{pmatrix} 1 \end{pmatrix} - \frac{1}{N} \begin{pmatrix} 0 \end{pmatrix}$ | $\frac{3}{4}$            |
| 5         | 0  | $\begin{pmatrix} 1 \end{pmatrix}$                                         | 0                        |
|           | 1  | $\begin{pmatrix} 0 \end{pmatrix}$                                         |                          |
|           | 2  | $\begin{pmatrix} 1 \end{pmatrix} - \frac{2}{N} \begin{pmatrix} 0 \end{pmatrix}$ | $\frac{1}{2}$            |
| 6         | 0  | $\begin{pmatrix} 1 \end{pmatrix}$                                         | 0                        |
|           | 1  | $\begin{pmatrix} 0 \end{pmatrix}$                                         |                          |
|           | 2  | $\begin{pmatrix} 1 \end{pmatrix} + \frac{1}{N} \begin{pmatrix} 0 \end{pmatrix} - \frac{8}{N} \begin{pmatrix} 0 \end{pmatrix}$ |                          |
|           |    | $\begin{pmatrix} 0 \end{pmatrix}$                                         |                          |
|           |    | $\begin{pmatrix} 0 \end{pmatrix}$                                         |                          |
|           |    | $\begin{pmatrix} 0 \end{pmatrix}$                                         |                          |
|           |    | $\begin{pmatrix} 0 \end{pmatrix}$                                         |                          |
|           |    | $\begin{pmatrix} 0 \end{pmatrix}$                                         |                          |
|           |    | $\begin{pmatrix} 0 \end{pmatrix}$                                         |                          |
|           | 3  | $\begin{pmatrix} 0 \end{pmatrix}$                                         | $\frac{3}{4}$            |

Table 3: Eigenstates for $su(2)$ states with $\Delta \leq 6$. 
\[ \Delta_0 \quad n \quad \text{Eigenstates} \quad \gamma_{\text{exact}} \]

| \( \Delta_0 \) | \( n \) | \text{Eigenstates} | \( \gamma_{\text{exact}} \) |
|--------|------|-----------------|-----------------|
| 4      | 2    | \[ \begin{array}{c} 1 \end{array} \] | \( \frac{3}{4} \) |
| 5      | 2    | \[ \begin{array}{c} 1 \end{array} \] | \( \frac{1}{2} \) |
| 6      | 3    | \[ \begin{array}{c} 1 \end{array} \] | \( 0 \) |
| 7      | 3    | \[ \begin{array}{c} 1 \end{array} \] + \( \frac{2}{3} \) \[ \begin{array}{c} 1 \end{array} \] | \( \frac{5}{8} \) |
|        |      | \( \frac{1}{4} \left( 1 \pm \sqrt{1 + \frac{160}{N^2}} \right) \) | \( \frac{1}{16} \left( 9 \pm \sqrt{1 + \frac{160}{N^2}} \right) \) |
| 8      | 3    | \( (1 \pm \frac{2}{N}) \) \[ \begin{array}{c} 1 \end{array} \] + \( (1 \pm \frac{2}{N}) \) \[ \begin{array}{c} 1 \end{array} \] - \( (1 \pm \frac{4}{N}) \) \[ \begin{array}{c} 1 \end{array} \] | \( \frac{3}{4} \pm \frac{3}{4N} \) |
|        |      | \( \pm \left( 1 \pm \frac{3}{N} \right) \left( \begin{array}{c} 1 \end{array} + \begin{array}{c} 1 \end{array} + \begin{array}{c} 1 \end{array} \right) \) | |
| 9      | 4    | \[ \begin{array}{c} 1 \end{array} \] + \( \frac{1}{2} \) \[ \begin{array}{c} 1 \end{array} \] + \( \frac{1}{2} \) \[ \begin{array}{c} 1 \end{array} \] - \( \frac{1}{2} \) \[ \begin{array}{c} 1 \end{array} \] - \( \frac{1}{2} \) \[ \begin{array}{c} 1 \end{array} \] | \( \frac{5}{8} \pm \frac{3}{4N} \) |

Table 4: Analytic \( \mathfrak{su}(2) \) eigenstates with \( \gamma \neq 0 \) and \( \Delta < 10 \).
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