Annihilation of antiprotons on nuclear targets at low energies: a simple analysis.

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Abstract

We set up a plane wave impulse approximation (PWIA) formalism for the analysis of the annihilation cross sections of antinucleons on nuclear targets at very low momenta (below 100 MeV/c), where semiclassical approximations can’t be applied. Since, as we test here, PWIA fails in reproducing the unexpected “inversion” behavior of the $\bar{p}p$ and $\bar{p}$–nucleus annihilation cross sections found in a recent experiment\cite{1,2} we discuss some further possibilities, with a special attention to the optical potential model.
1 Introduction

Recents measurements\[1, 2\] of $\bar{p}$ annihilation on proton, deuteron and $^4$He targets show an interesting “inversion” pattern at projectile momenta below 60 MeV/c: total annihilation cross sections on Deuteron and $^4$He are smaller than the same cross sections on free proton targets. Such a behavior is not seen at larger momenta, where total annihilation cross sections increase with the target mass number $A$, accordingly to intuition. There are no low energy data on larger nuclei, to clarify whether the decrease of cross sections with $A$ is a systematic phenomenon or specifical of a few light nuclei. In the case of $^4$He target the phenomenon is particularly impressing if one takes into account that the Coulomb attraction introduces a further factor $\sim Z$ in any model for $\bar{p}$—nucleus annihilation, so a naive extimation of the ratio between low-energy $\bar{pp}$ and $\bar{p}^4$He annihilation cross section could be as small as $1/AZ = 1/8$. On the contrary, below 60 MeV/c the $\bar{p}$—D and $\bar{p}$—$^4$He annihilation cross sections are clearly smaller than $\bar{p} - p$ corresponding cross section. So, with $^4$He targets the most naive expectation largely overestimates the detected annihilation rate. A confirmation of these measurements comes from the recent data\[3\] on the widths and shifts of the electromagnetic levels of the antiprotonic deuterium. As demonstrated long ago\[4\] there is a relation between the width of a level in an antiprotonic atom and the $\bar{p}$ low energy annihilation rate on the corresponding nucleus. Although affected by large error bars, the comparison of the data on antiprotonic Hydrogen and Deuterium confirms the “inversion” behavior. For these last measurements a successful prediction existed\[5\] (see below for more details). In view of the possibility of a systematical analysis of $\bar{p}$ annihilation on complex nuclei at very low energies at the $\bar{p}$ accelerator AD at CERN\[6\], it is desirable to look for an understanding of what could be the properties of the interactions between $\bar{p}$ and nuclear matter, and to try to organize some model or framework aimed at this analysis. This will be the aim of this work, with special emphasis on the possible explainations of the inversion behavior.

A simple black sphere potential model\[7\] shows that the general properties of low energy scattering on strongly absorbing optical potentials don’t contraddict the idea of
“inversion”: the S-wave contribution to the reaction cross section on a spherical imaginary potential \( V(r) = iW \) for \( r < R \), and zero outside) is an increasing function of the strength parameter \( W \) only at small \( W \). When \( W \rightarrow \infty \) the S-wave reaction cross section tends to zero. However, we have played a little with this model, and we have seen that: (1) If the strength \( W \) is kept constant and \( R \) increased (which seems representative of a comparison between different nuclear targets) the cross section tends to a constant plateau (after an initial increase) instead of decreasing. (2) If Coulomb interactions are included, the constant plateau is substituted by a \( \sim Z \sim R^3 \) dependence. (3) The detected inversion behavior is present in a region where P-wave contributions are still large, perhaps dominant on nuclear targets, and where absorbing potentials with realistic shapes (we have tried Woods-Saxon reproducing known nuclear density properties) fail in reproducing the inversion. (4) Recently we have been informed\[8\] of a 3-body Faddeev-style analysis of the \( \bar{p} - D \) annihilation problem, where inversion results only when additional physics (of a kind which is not contained in the black sphere model) is included. So, although a simple optical potential model presents intrinsic features which depress nuclear cross sections, it does not give yet a satisfactory explanation of what we see.

A specific model for nuclear interactions, in this peculiar low energy domain, is still missing. From a fundamental point of view, it is not clear yet how much one can trust the concept of optical potential for describing hadronic annihilations (for a review about the theoretical developments in \( \bar{p} - p \) annihilation see e.g. ref.[9] and references therein). But even if one decides to trust it, it is desirable to have some physical connections between the nuclear potential and the underlying \( \bar{p} \)-nucleon interactions, not to reduce the physics of the problem just to a fit of radius, strength and a few more parameters. The traditional models used in nuclear physics for giving a meaning to the optical potential[10] can hardly be applied here: they are based on the idea of a soft global rearrangement reaction proceeding through a set of narrow excited states of the full compound “nucleus + projectile”. This point will be discussed in the last part of this work, but presently there are no signals of compound nucleus mechanisms. They should be accompanied by narrow resonances in the elastic channel and by complex fragmentation of the set of \( A - 1 \).
nucleons which don’t annihilate. Such processes should be present at any \( \bar{p} \) momentum of magnitude within the Fermi momentum (\( \approx 200 \text{ MeV} \)) or smaller. Narrow resonances have never been seen. Although we haven’t fragmentation data specific to the inversion region, at \( \bar{p} \) momenta \( \sim 100 \text{ MeV/c} \) there is no evidence\textsuperscript{[1]} of a reaction mechanism involving, at leading order, more than one target nucleon at a time: the number of annihilation processes leading to a fragmented residual is small enough to be explained in terms of final state interactions. The situation is in, some respect, reminiscent of high energy hadron-nucleus interactions, where the hard process is supposed to go mainly through single or multiple interaction between the projectile and single nucleons, and sudden residual fragmentation, when present, can be attributed to rescattering or to initial state (projectile independent) correlations between target nucleons. However, a simplifying feature of high energy processes is missing here: because of the long wavelength of the projectile, all the semiclassical approximations and intuition are lost. At large energies, short wavelengths allow for exploitation of the Glauber formalism\textsuperscript{[2]}. The lowest \( \bar{p} \) incident momentum at which data of \( \bar{p} \)–nucleus elastic and reaction cross sections were interpreted in a satisfactory way with a Glauber-style model was 600 MeV/c\textsuperscript{[3]}. At lower momenta (300 and 200 MeV/c) such a model is only able to reproduce gross features of the differential cross sections, although it still revealed effective in reproducing integrated cross sections. We are now interested in a kinematical region where the Glauber model is surely not reliable anymore, i.e. with projectile momentum starting from around 100 MeV/c and going down to 1 MeV/c. At the lower end of this range, nonresonant annihilation processes are S-wave dominated with any nuclear target. At momenta around 200 MeV/c the S-wave contribution is not the most relevant one (with nuclear targets). The “inversion” pattern, which appears below 60 MeV/c, seems to be associated with the transition to the S-wave dominance region, where surely the Glauber model can’t be applied.

The problem will be matched from different sides. We start from the easiest possible framework, i.e. the Plane Wave Impulse Approximation (PWIA). In the Impulse Approximation one sums coherently single interactions of the projectile with any one of the target nucleons (with the remaining \( A - 1 \) acting as spectators), multiple interaction effects are
disregarded and there is no reciprocal disturbance between scattering events on different nucleons. “Plane wave” means that no nuclear distortion of the waves participating to the elementary scattering event $\bar{p} + p$ (or $\bar{p} + n$) $\rightarrow$ “some final state” is considered. At some extent, we include Coulomb correction effects (a more systematical analysis of this point is still going on and will be presented elsewhere). These corrections are very relevant and lead to differences in magnitude between, e.g., $\bar{p} - D$ and $\bar{n} - D$ cross sections when the projectile momentum is much below 100 MeV/c. Coulomb effects even produce curious phenomena like a large difference between $\bar{p} - n$ cross sections in the two cases where the neutron target is free or bound to a deuteron. For simplicity, here we will not take care of spin and isospin effects. Presently our knowledge on the elementary annihilation processes does not make a discrimination among different spin and isospin channels very useful, although perhaps some information on the role of isospin is already available\cite{14} or will be soon\cite{15}. Waiting for systematic data on this point, in the following we will postulate approximative equality between $\bar{p} - p$ and $\bar{p} - n$ cross sections, apart for Coulomb effects. Despite all these approximations the IA formalism, as is evident in the following, is rather complicate.

Once Coulomb corrections are introduced, the most important limitation of the IA analysis is the lack of consideration for multiple interaction effects. So they will be discussed, at a qualitative level. We anticipate here that there are already some indications of the importance of these processes: (1) At larger energies, not to take rescattering into account leads to a wrong $A$ dependence of the annihilation rates, instead of the correct $A^{2/3}$ one. (2) At low energies, a Born leading order perturbative solution of the black sphere model does not produce inversion (at some extent the Born leading order, in a problem where the nucleus is approximated by a continuous potential, can be associated with discarding multiple scattering events in the realistic nuclear problem). (3) The strongest support to the role of multiple scattering in very low energy $\bar{p}$—nucleus physics comes from an analysis\cite{5} of the related problem of the transition widths in antiprotonic deuterium. There, terms over IA have been studied and found relevant. Double scattering terms are fundamental in decreasing the IA effects, and the results of that work are in
agreement with the recent data. (4) As evident in the following, IA does not lead to any inversion.

The fact that IA does not lead to inversion can perhaps be expected in advance: a sum of amplitudes will not be smaller than any of them, if they all combine with the same phase (taking into account that interference effects due to the different space positions of the scattering centers are usually lost in the total cross sections). Since we have not enough information on the relative sign of $\bar{p} - p$ and $\bar{p} - n$ interactions, assuming them as equal depresses our chances of producing any inversion from the very beginning.

Some interesting effects can be studied within the IA formalism. First, the IA allows one to calculate the Fermi motion smearing of any irregularity contained in the cross section for the interaction between any projectile and a single nucleon. This phenomenon is well known in many branches of nuclear physics and is simply due to the fact that bound nucleons move with random momentum $\sim 200$ MeV/c. So, a specific phenomenon (e.g. a resonance peak) which is present in the $\bar{p} - p$ cross section at a well defined value of the projectile momentum is spread through a kinematical region of range $\Delta p \sim p_{\text{Fermi}}$. We will show how this mechanism can produce an inversion behavior, and which conditions should be realized. According to the discussion accompanying fig.5 it will be evident that it is difficult, although not impossible, to support this explanation for the inversion.

Another interesting point examined here is the sensitivity of the IA results to the high momentum components of the nuclear momentum distribution. In the following it is evident that IA results are aligned with the intuitive expectations at the condition that the nuclear momentum distribution is dominated by those momenta which are typical of mean field theories ($\sim 100$ MeV/c). In the event of a certain consistence of the large momentum tail, the IA contribution to the annihilation rate is depressed by a small but observable factor $\sim 10\%$). This effect is evidently not sufficient to justify inversion, but it can be interesting in itself, since it suggests that the annihilation rates are sensitive to those elusive contributions to the nuclear momentum distribution which come from direct nucleon-nucleon interactions.

Finally, the last part of this work will be devoted to the study of the optical model in
its simplest form, i.e. the black sphere. We are pushed by the fact that, as anticipated, the optical potential model does predict nuclear cross sections which are much smaller that naive geometrical expectations, with inversion or saturation behaviors. Actually, although first intuition suggests absorption cross sections $\sim R^3$ or $R^2$ for any strong absorber of radius $R$, the general properties of low energy scattering suggest an $R^2$ law for the elastic total cross sections only, while esoenergetic reactions should roughly follow a behavior $\sigma \sim R/k$ \[16\]. The factor $1/k$ is a well known Bethe’s prediction (which can be motivated by the idea that the probability for a spontaneous reaction is proportional to the time that any particle spends inside the interaction region), and $R$ can be put there to give the right dimension to a cross section (as we see later, one can motivate this expectance in terms of the scattering length). The $R/k$ law roughly works for any “moderately reacting sphere”. The reaction properties of such a target can be expressed by an optical potential of the form $V = U - iW$. The imaginary part produces absorption of projectile flux according to the time dependence of the wavefunction $\exp(-iEt)$ (since inside the potential region $E = T + V$, where $T$ is the initial kinetic energy of the projectile). The surprise is that with large values of $W$ (uncommon in traditional nuclear physics, but perhaps suitable to describe the violent annihilation phenomenon) low energy reaction cross sections predicted by the optical model become much smaller than $R/k$, and show no $R$ dependence. We will analyze in detail the behavior of the wavefunction in the “strong optical model”, to understand how much of its predictions is related to the use of an optical potential and how much is general. The general conclusion will be that any model able to produce a “vacuum region” in the projectile flux will lead to a small reaction cross section, at the condition that the vacuum region surface diffuseness is small with respect to the vacuum region radius.

The work is organized as follows: In the next section the PWIA formalism will be described, leading to eq.(29) where a nuclear cross section for a specific transition is written in terms of the corresponding $\bar{p}$–nucleon cross sections. This equation is actually more suited for antineutron projectiles, while antiproton interactions at low energy are affected by strong Coulomb effects. These effects are discussed in the third section, where
eq. (29) is substituted by eq. (30). In the fourth section we discuss the main applications. First, we easily restore the high energy limit. Then, within certain approximations we introduce a low energy limit of total annihilation cross sections, in eq. (37). This equation expresses the ratio $R_A$ between the nuclear and single hadron annihilation cross sections in terms of the nuclear momentum distribution only (averaging on the details of the elementary $\bar{p}p$ and $\bar{p}n$ annihilations). Despite its simplicity, its calculation requires a many particle final state integration, so we limit ourselves to studying the simple case were two pions only are produced in the final state, presenting in fig. 4 the results of this calculation (which more or less confirm a priori expectations). For the nuclear momentum distribution, we adopt two simple mean-field examples. The last application of IA that we consider is the nuclear smearing of a resonance which could be found in the $\bar{p}p$ annihilation. In the fifth section we discuss qualitatively two effects which should be the subject of much more difficult works: the role of large momentum components of the nuclear distribution, and multiple scattering effects. In the further section we analyze the black sphere potential model, compare the two limiting cases of high and low energy projectiles, generalize some results and discuss their possible relation with the detected inversion data. The last section is just a short recollection of the main points.

2 General PWIA framework

We start from the amplitude $T_{A,n}(\vec{p}, \vec{\pi}_1, ... \vec{\pi}_N)$ for the reaction $A(\bar{p}, N\pi)(A - 1)_n$, where the annihilation of a $\bar{p}$ with momentum $\vec{p}$ takes place on a single nucleon inside a nucleus leaving the residual system of $A - 1$ nucleons in a state $n$, and producing a number $N$ of final state particles with momenta $\vec{\pi}_1, ..., \vec{\pi}_N$ (Pions and, in smaller number, Kaons, but in the following we will call them all “pions”, and the target nucleon will be called “proton”). We know from experimental evidence\cite{17} that the largest part of annihilation events imply production of up to seven pions). The elementary process $\bar{p} + p \rightarrow N\pi$ (or $\bar{p} + n \rightarrow N\pi$) is described by the $(N + 2)$-point amplitude $T(\vec{r}, \vec{r}_1, \vec{r}_3, ... \vec{r}_{N+2})$, where $\vec{r}$ refers to $\bar{p}$, $\vec{r}_1$ to the annihilated target nucleon, and $\vec{r}_3, ... \vec{r}_{N+2}$ to the produced pions.
The “missing” radius $\vec{r}_2$ will be used to describe the position of the center of mass of the residual $A-1$ nucleons, and $\vec{x}_2$ will represent the set of $3(A-1)$ coordinates of the residual $A-1$ nucleons with respect to their center of mass $\vec{r}_2$ (only $3(A-2)$ among these coordinates are independent).

$$T_{A,n}(\vec{p}, \pi_1, ... \pi_N) = \int e^{-i\sum_j \pi_j r_j} \Psi^*_A(r_1, x_2) T(r, r_1, r_j) e^{i\vec{p} r} \Psi_A(r_1, r_2, x_2)$$

$$d^3r d^3r_1 d^3r_2 d^3(A-2)x_2 d^3N r_j, \ (j = 3, .. N + 2). \quad (1)$$

In particular:

$$\Psi_A(r_1, r_2, x_2) \equiv \exp(iPX) \Phi_A(x_1, x_2), \ \Psi_{A-1}(r_2, x_2) \equiv \exp(iP_2r_2) \Phi_{A-1,n}(x_2), \quad (2)$$

where $\vec{P}$ and $\vec{P}_2$ are the momenta of the initial nucleus and of the final residual nuclear system, and $\vec{X}$ is the position of the center of mass of the initial nucleus.

Rigorously, in the previous matrix element Coulomb functions should appear, otherwise we are considering $\bar{n}$ scattering. Below we discuss the consequences implied when Coulomb interactions are properly taken into account.

In the calculations we will rely on the closure approximation on the internal states of the (unobserved) residual. The cross section $\sigma(\vec{p}, \pi_j)$, corresponding to a given choice of $\vec{p}, \pi_1, ..., \pi_N$, is proportional to $\sum_n |T_{A,n}|^2$:

$$\sum_n |T_{A,n}(p, \pi_1, ... \pi_N)|^2 = \sum_n \int dx_2 dx'_2 \Phi_{A-1,n}(x'_2) \Phi^*_{A-1,n}(x_2) \quad \quad \quad (3)$$

If the sum is on all the possible internal states of the residual, then:

$$\sum_n \Phi_{A-1,n}(x'_2) \Phi^*_{A-1,n}(x_2) = \delta^{3(A-2)}(\vec{x}_2 - \vec{x}'_2). \quad (4)$$

Rigorously the closure approximation is never exact, however the available energy in the game is high enough for a large continuum of residual states being accessible. From a practical point of view applying the closure means that, in the following integrations, all
variables \( r, R_1 \) etc. will have their primed counterpart \( r', R'_1 \) etc. with the exception of the \( 3(A-1) \) coordinates \( \vec{x}_2 \) (of which \( 3(A-2) \) only are integration variables), and the residual internal functions \( \Phi_{A-1,n} \) disappear. Then:

\[
\sigma(\vec{p}, \vec{\pi}_j) \propto \sum_n |T_{A,n}(p, \pi_1, \ldots \pi_N)|^2 = \int e^{i \sum_j \pi_j(r_j' - r_j) + ip(r - r') + i(P'_2r'_2 - P_2r_2)}
\]

\[
T^*(r', r'_1, r'_j)T(r, r_1, r_j)\Psi_A^*(r'_1, r'_2, x_2)\Psi_A(r_1, r_2, x_2) drdr'dr_1dr'_1dr_2dr'_2dx_2d^3N r_j d^3N r'_j
\]  

(5)

We clearly need to express everything in terms of the elementary scattering amplitude with all particles of well defined momentum. To get to this we need to decompose the bound nucleon state into Fourier components:

\[
\int d^3 r_2 d^3 r'_2 d^3(A-2) x_2 e^{-iP_2 r_2 +iP'_2 r'_2} \Psi_A^*(r'_1, r'_2, x_2)\Psi_A(r_1, r_2, x_2) = \int d^3 k d^3 k' F(k, k') e^{-ik'r'_1 +ikr_1}
\]

\[
\equiv \frac{1}{(2\pi)^6} \int d^3 k d^3 k' F(k, k')
\]

(6)

that implies:

\[
F(k, k') = \int d^3 r_1 d^3 r'_1 e^{ik'r'_1 -ikr_1} \int d^3 r_2 d^3 r'_2 d^3(A-2) x_2 e^{-iP_2 r_2 +iP'_2 r'_2}
\]

\[
\Psi_A^*(r'_1, r'_2, x_2)\Psi_A(r_1, r_2, x_2).
\]

(7)

Writing explicitly the nuclear wavefunction, according to eq.(6), and using the substitutions:

\[
\vec{r}_1 \equiv \vec{r}_2 + \vec{x}_1, \quad \vec{X} \equiv \vec{r}_1 + (A-1)\vec{r}_2 = \frac{\vec{r}_2 + \vec{x}_1}{A}.
\]

(8)

we obtain:

\[
F(k, k') = \int dr_1 dr'_1 dr^2 dr^2 dx^2
\]

\[
e^{-iP_2 r_2 + iP(r_2 + x_1/A) - ik(r_2 + x_1)} e^{iP'_2 r'_2 - iP(r'_2 + x'_1/A) + ik'(r'_2 + x'_1)} \Phi_A(x_1, x_2) \Phi_A^*(x'_1, x'_2) =
\]

10
\[
\begin{align*}
&= (2\pi)^6 \delta^3(P_2 + k - P) \delta^3(P'_2 + k' - P) S(k - P/A, k' - P/A), \\
\end{align*}
\]

where
\[
S(k, k') \equiv \int d^3x_1 d^3x'_1 e^{-ik_1 x_1 + ik'_1 x'_1} \int d^{3(A-2)} x_2 \Phi_A(x_1, x_2) \Phi^*_A(x'_1, x_2) \equiv \int dE S(k, k', E).
\]

The function \( S(k, k', E) \) is the nuclear spectral function for one nucleon removal\[^{18}\]. In a mean field approximation it has the known shell model form
\[
S(k, k', E)_{\text{sm}} = \sum_{\alpha} \delta(E - E_{\alpha}) \phi_{\alpha}(\vec{k}) \phi^*_{\alpha}(\vec{k}').
\]

However, due to few-body direct interactions, it is well known that the continuum deep energy components of \( S \) can play a strong role when large energy releases are involved in the elementary processes\[^{18}\].

In a treatment where initial and final state waves are affected by nuclear medium distortion (Distorted Wave Impulse Approximation DWIA) momentum exchanges affect initial and final states, and all the non-diagonal components of \( S(k, k') \) are involved. However, in a PWIA treatment the remaining part of the integral contains a function \( \delta^3(k - k') \), as we shall see later. So we can write from now on:

\[
\begin{align*}
F(k, k') & \to (2\pi)^6 \delta^3(P_2 + k - P) \delta^3(P'_2 - P_2) S(k_{P/A}, k - P/A) \to \\
& \to (2\pi)^6 \delta^3(P_2 + k) \delta^3(P'_2 - P_2) S(k, k).
\end{align*}
\]

The last equality follows from choosing the laboratory frame, where the target nucleus is at rest. Now we may rewrite:

\[
\sigma(\vec{p}, \vec{p}_j) \propto \sum_n |T_{A,n}(p, \pi_1, \ldots, \pi_N)|^2 = \int e^{i \sum_j \pi_j(r'_j - r_j) + i\vec{p}(r - r') + i(P'_2 r'_2 - P_2 r_2)}
\]

\[
T^*(r', r'_1, r'_j) T(r, r_1, r_j) \Psi^*_A(r'_1, r'_2, x_2) \Psi_A(r_1, r_2, x_2) dr dr' dr_1 dr'_1 dr_2 dr'_2 dx_2 dr_j dr'_j = 
\]

\[\text{11}\]
\[
\frac{1}{(2\pi)^6} \int d^3kd^3k'e^{i\sum_j \pi_j(r'_j-r_j)+ip(r-r')+i(kr_1-k'r'_1)}
\]

\[T^*(r',r'_1,r'_j)T(r,r_1,r_j)F(k,k')drdr'dr_1dr'_1. \quad (13)\]

In the previous equation it is easy to recognize the amplitude for the elementary process in momentum representation:

\[
G(p,k,\pi_j) = \int d^3rd^3r_1d^3N_rje^{-i\sum_j \pi_jr_j+ipr+k'r_1}T(r,r_1,r_j) \quad (14)
\]

However it is obvious that \(T\) will not depend on the overall position of the process in the space, so its Fourier transform \(G\) must contain a function \(\delta^3(\sum_j \pi_j - p - k)\). To explicitly show it, one must use relative coordinates. One possible choice is: \(\xi_\pi = \sum_j \vec{r}_j/N\) (center of mass of the outgoing pions), \(\xi_j = \vec{r}_j - \vec{\xi}_\pi\), \(\xi_p = (\vec{r}_1 + \vec{r})/2\) (center of mass of the colliding \(\bar{p}\) and \(p\)), \(\xi_1 = \vec{r}_1 - \vec{\xi}_2\), and finally \(\vec{\xi} = \vec{\xi}_\pi - \vec{\xi}_p\) and \(\vec{Y} = \alpha\vec{\xi}_\pi + \beta\vec{\xi}_p\) (\(\alpha = Nm_\pi/(Nm_\pi + 2m_p), \beta = 1 - \alpha\)). So \(\xi\) is the distance between the initial and final state centers of mass, and \(Y\) the “center of mass of the centers of mass”. Calculating \(G\) with this new set of coordinates, one patiently arrives to:

\[
G(p,k,\pi_j) = (2\pi)^3\delta^3(p+k-\sum \pi_j)T(p+k,p-k,\pi_j), \quad (15)
\]

\[
T(p+k,p-k,\pi_j) \equiv \int d^3\xi d^3\xi_1d^3N-1\xi_{j}e^{-i(p+k)\xi-i(p-k)\xi_1/2-i\sum_j \pi_j\xi_j/N}T(\xi,\xi_1,\xi_j) \quad (16)
\]

Although written in a more complicated form, the latter amplitude \(T\) corresponds to the function \(f(E,\theta)\) in the ordinary nonrelativistic scattering theory, where the problem is written with respect to relative coordinates, the target is given infinite mass, the projectile mass is changed to \(m_1m_2/(m_1+m_2)\), and the wavefunction of the relative motion is written in the form \(\exp(i\vec{p}\vec{r}) + f(E,\theta)\exp(ip'r)/r\). The amplitude \(f\) is the energy conserving scattering amplitude in momentum representation.

It is easier to see this if we start from eq.(13) and substitute the nuclear target with a free proton with momentum \(\vec{k}_o\). This is equivalent to performing the substitution:

\[
F(k,k') \rightarrow (2\pi)^6\delta^3(\vec{k} - \vec{k}_o)\delta^3(\vec{k}' - \vec{k}_o) \quad (17)
\]
in eq. (13), together with taking properly into account the change of the initial flux and of the final state phase space. To see this we rewrite eq. (13) in terms of the frequency of events $dW$ for a detected value of the recoil momentum of the residual $R$ (in the previous formalism closure has been applied on residual internal states only) and detected values of all the produced mesons:

$$dW(\bar{p} + A \rightarrow N\pi + R) = Cd\Phi_{N+1}\frac{1}{(2\pi)^6} \int d^3kd^3k'e^{i\sum_j \pi_j(r'_j - r_j) + ip(r - r') + i(kr_1 - k'r_1')}$$

$$T^*(r',r'_1,r'_j)T(r,r_1,r_j)F(k,k')drdr'dr_1dr'_1dr_jdr'_j$$  \hspace{1cm} (18)

where $C$ is a constant and $d\Phi_{N+1}$ is the phase space for emission of $N+1$ particles with energy conservation (momentum conservation is already present in the integral via the $\delta$ functions in $F$ and $G$). When the substitution (17) and the change of phase space are performed, eq. (18) becomes:

$$dW(\bar{p} + p \rightarrow N\pi) = Cd\Phi_N|G(p,k_o,\pi_j)|^2$$  \hspace{1cm} (19)

where $d\Phi_N$ is the phase space for emission of $N$ particles with energy conservation, and the constant $C$ is the same as in eq. (18). In the last equation (see eq. (13)) a squared delta function is present: $[\delta^3(p + k_o - \sum \pi_j)]^2$. The usual way [19] to treat this excess of singularity is by the substitution:

$$(2\pi)^6[\delta^3(p + k_o - \sum \pi_j)]^2 \rightarrow (2\pi)^3V\delta^3(p + k_o - \sum \pi_j)$$  \hspace{1cm} (20)

where $V$ is an overall normalization volume. This volume will disappear from the final expression for the cross section, so that one can put $V = 1$ from the very beginning. Taking into account eq. (15), we may rewrite eq. (19):

$$dW(\bar{p} + p \rightarrow N\pi) = Cd\Phi_N(2\pi)^3\delta^3(p + k_o - \sum \pi_j)|T(p + k_o, p - k_o, \pi_j)|^2.$$  \hspace{1cm} (21)
Dividing by the incoming flux one gets the corresponding cross section, which of course
is nonzero for conserved momentum and energy.

Exploiting the definition (14) of $G$, writing explicitly $F$ and $G$ according to eq.(12)
and eq.(15) and exploiting the two delta functions contained in $F$ for removing $k$ and $k’$
integrations, eq.(18) becomes:

\[
dW(\vec{p} + A \to N\pi + R) = Cd\Phi(N + 1)(2\pi)^6 \left\{ \delta^3(p + k - \sum \pi_j)^2 |T(p + k, p - k, \pi_j)|^2 S(k, k) \right\}_{k = -P_2} \quad (22)
\]

and once one of the two delta functions is removed (using the same trick as previously)
this last equation becomes:

\[
dW(\vec{p} + A \to N\pi + R) =
\]

\[
= Cd\Phi(N + 1)(2\pi)^3 \left\{ \delta^3(p + k - \sum \pi_j) |T(p + k, p - k, \pi_j)|^2 S(k, k) \right\}_{k = -P_2} . \quad (23)
\]

The same comments following eq.(21) can now be made about eq.(23). It is singular
on the momentum conservation shell, because the global recoil momentum of the resid-
ual is supposedly detected. The constant $C$ is the same in eq.(21) and eq.(23) if the
normalization of $S$ is chosen as:

\[
\int S(k, k)d^3k/(2\pi)^3 = A \quad (24)
\]

and this allows for comparison between nuclear and free proton cross sections.

However, it is unlikely that nuclear recoil information are present in the experiment,
so that we probably need to integrate the last equation on the phase space of the recoil
center of mass $d^3k/(2\pi)^3$. The correct phase space element would require a factor $1/2E$
$\approx 1/2M_r$. However we must take into account that, in a relativistic treatment, spinors
are normalized like $\bar{u}u = 2M$, so that the net effect of the $1/2E$ factor disappears from
the cross sections as far as we may neglect the kinetic energy of the residual center of mass (of course the phase space of the emitted mesons, considered in the following, will be relativistic). After this integration the cross section is not singular anymore on the 3-momentum conservation shell. It is still singular on the energy shell. The form of the energy conservation will be simplified by neglecting the energy of the nuclear recoil. We have:

\[
dW(\vec{p} + A \rightarrow N\pi + X) = C d\Phi_N(2\pi)^3 \left\{ |T(p + k, p - k, \pi_j)|^2 S(k, k) \right\}_{\vec{k} = \sum \pi_j - \vec{p}}
\]

To get to cross sections we still have to consider flux factors. This point is delicate: at the small momenta of our interest, the most evident behavior of the cross sections is dictated by the flux factor \(1/p\). For collisions between two particles with masses and 4-momenta \(m_1, m_2, p^\mu, P^\mu\), the relation between \(d\sigma\) and the above used \(dW\) is, neglecting spins, \(d\sigma = dWE_1E_2/I\) \((E_i\) in relativistic sense\) where \(I \equiv \sqrt{(p_\mu P^\mu)^2 - m^2_1m^2_2}\) (see e.g. [19], taking into account that notations for \(dW\) are not exactly the same). When the second particle is at rest (laboratory frame) the flux factor \(I/E_1E_2\) coincides exactly with the projectile velocity \(\beta_1\) in the laboratory frame. When the target is not at rest, but moves with nonrelativistic velocity, the flux factor coincides with the relative velocity up to relativistic corrections. So, neglecting spin factors, we can write the two comparable equations:

\[
d\sigma(\vec{p} + p \rightarrow N\pi) = \frac{C}{\beta_{rel}} d\Phi_N(2\pi)^3 \delta^3(p + k - \sum \pi_j) |T(p + k, p - k, \pi_j)|^2
\]

for the free proton target with momentum \(k\) \((\beta_{rel}\) becomes simply \(\beta\) when \(\vec{k}_o = 0\), and

\[
d\sigma(\vec{p} + A \rightarrow N\pi + X) = \frac{C}{\beta} d\Phi_N(2\pi)^3 \left\{ |T(p + k, p - k, \pi_j)|^2 S(k, k) \right\}_{\vec{k} = \sum \pi_j - \vec{p}}
\]

for a nuclear target at rest without detection of the residual. Writing:

\[
S(k, k)_{\vec{k} = \sum \pi_j - \vec{p}} = \int \frac{d^3k}{(2\pi)^3} S(k, k)(2\pi)^3 \delta^3(\vec{p} + \vec{k} - \sum \vec{\pi})
\]
and substituting this equation in eq. (27) we get the relation between the two cross sections:

$$d\sigma_{A,N}(\vec{p},0,\vec{\pi}_j) = \int \frac{d^3k}{(2\pi)^3} S(k,k) \frac{\beta(\vec{p},\vec{k})}{\beta(\vec{p},0)} d\sigma_{1,N}(\vec{p},\vec{k},\vec{\pi}_j)$$

(29)

where: $d\sigma_{A,N}(\vec{p},0,\vec{\pi}_j)$ is the cross section for annihilation of an antiproton with initial momentum $\vec{p}$ on a target nucleus at rest, with production of $N$ pions with detected momenta $\vec{\pi}_j$ in the phase space element $\delta(E_{fin} - E_{in}) \prod d^2\pi_j/[2E_j(2\pi)^3]$, and no information on the residual nucleus; $d\sigma_{1,N}(\vec{p},\vec{k},\vec{\pi}_j)$ is the cross section for annihilation between an antiproton with initial momentum $\vec{p}$ and a proton with momentum $\vec{k}$, with production of $N$ pions with detected momenta $\vec{\pi}_j$ in the same phase space element as before; $\beta(\vec{p},\vec{k})$ is the relative velocity between two incoming particles with momenta $\vec{p}$ and $\vec{k}$; $S(k,k)$ must be normalized so that $\int \frac{d^3k}{(2\pi)^3} S(k,k) = A$. The nuclear cross section $d\sigma_{A,N}$ is not momentum conserving.

The Coulomb corrected version of this equation (see below), together with its low energy application, is the main result of the calculations contained in this paper. The most evident observation is that it contains an uncoherent sum over the elementary scattering possibilities, i.e. the integral is over cross sections, not over amplitudes. It must be stressed that this lack of interference effects is due to the PWIA. In presence of rescattering we would find terms containing $S(k,k')$ with different $k$ and $k'$, and products $TT'$ of amplitudes corresponding to different processes.

### 3 Coulomb corrections

The above equation (29) is correct, within PWIA, for describing interactions of an $\bar{n}$ with a nucleus. For a $\bar{p}$ projectile it requires further relevant corrections because of Coulomb effects [21]. Coulomb attraction, at semiclassical level, implies focusing of the incoming $\bar{p}$ flux towards the scattering center. For pure S-wave low-energy annihilations on pointlike targets this implies a further $Z/\beta$ factor (times a $Z$-independent constant that we reabsorb in the target charge $Z$ not to overload formulas) in $d\sigma_{1,N}$ [16]. This correction has been
estimated since long from the ratio $|\psi_c/\psi_o|^2$ between the coulomb problem solution $\psi_c$ and the free motion wave $\psi_o$.

For the case of an extended target of nuclear size, with nonsingular charge distribution, we have numerically performed some test, representing the nucleus as a Wood-Saxon imaginary potential well, comparing the results of the two cases charged/neutral projectile. The charge distribution of the target is assumed homogeneous up to a radius $r_{\text{charge}} = 1.25 A^{1/3}$ fm. The detailed and systematic presentation of these Coulomb corrections will be the subject of a forthcoming paper. Here we report a simple example, in figs.1,2 and 3, and anticipate some general results.

The total cross section for $\bar{p}p$ annihilation can be well fitted in the range from 37 MeV/c to 70 MeV/c by an imaginary potential, of Woods-Saxon kind, with range 1.3 fm, diffuseness 0.6 fm and strength 33 MeV. The consequent cross section is showed in fig.1, together with the “uncharged” cross section. Center of mass corrections have been taken into account in the calculation. In the next two figures the behavior of the quantities $p\sigma$ and $p^2\sigma$ is showed for both “charged” and “uncharged” cross sections. In these nonrelativistic conditions momenta and velocities are proportional within a few percents. From the figures it is possible to see that both cross sections follow the $1/\beta$ standard law at the larger momenta of the considered range, but Coulomb deviations become more and more evident as the energy decreases, as already predicted[20]. The correcting factor becomes steadily proportional to $1/\beta$ for momenta below 10 MeV/c, as we can see by comparing figs. 2 and 3 at the left end of the scale. At larger momenta one can speak of a factor $F(\beta)$ some way between 1 (for larger $p$) and $Z/\beta$ (for smaller $p$). We have even tried electron screened potentials, but without seeing any relevant screening effect down to $p = 1$ MeV/c (probably screening will cause a cut off of the Coulomb effect at some small $p$, but this does not seem to happen in the considered range, even for heavy nuclei). When the reaction cross section is decomposed into partial wave contributions, the correcting factor seems to affect the same way all partial waves (for $p < 100$ MeV/c, we find a contribution of some percent in the $D$ wave). Despite the very small involved momenta and angular momenta, this behavior allows for a semiclassical interpretation in
Figure 1: Continuous line: the cross section (mb) for annihilation of $\bar{p}$ on a proton target, simulated by an imaginary potential of Woods-Saxon kind, with radius 1.3 fm, diffuseness 0.6 fm and strength 33 MeV, without inclusion of Coulomb interaction. Dotted line: the same, with Coulomb interaction included. The two curves are given as functions of the projectile momentum (MeV/c) in the laboratory frame. Center of mass corrections are included in the calculation. The parameters of the potential are chosen so that the dotted curve fits reasonably the available data in this region (see text). These data, explicitly reported in the figure, are taken from references [1], [21] and [22].
Figure 2: Continuous and dotted lines represent the same cross sections as in fig.1, but multiplied by $p$ (MeV/c), the projectile momentum in the laboratory.
Figure 3: Continuous and dotted line represent the same cross sections as in fig.1, but multiplied by $p^2$. 
terms of focusing of projectile trajectories. Similar conclusions remain valid even when we simulate much heavier nuclei.

The previous Coulomb corrections have been calculated assuming the nucleus as an absorbing potential. However, because the Coulomb focusing occurs on a scale which is sensibly larger than the nuclear radius and at $\vec{p}$ momenta which are not much larger than the Fermi momentum, its effect should not depend on the Fermi motion of the target constituents: indeed, during the time needed by Coulomb forces to deflect the incoming antiproton trajectories any of the target nucleons will change many times its momentum. In presence of Coulomb attraction, the previous eq. (29) is wrong, and must be rewritten as:

$$d\sigma_{A,N}(\vec{p},0,\vec{\pi}_j) = \int \frac{d^3 k}{(2\pi)^3} S(k,k) \frac{F_Z(\vec{p},\vec{k})\beta(\vec{p},\vec{k})}{F_1(\vec{p},0)\beta(\vec{p},0)} d\sigma_{1,N}(\vec{p},\vec{k},\vec{\pi}_j),$$

where the function $F(\vec{p},\vec{k})$ depends on $\vec{p}$ and $\vec{k}$ via the relative velocity $\beta(\vec{p},\vec{k})$, and satisfies the two limits $F(\beta) \to Z/\beta$ for small $\beta$ (practically for $\beta < 0.05$) and $F(\beta) \to 1$ at large $\beta$ (we know already that, for momenta of some hundreds MeV/c, $\sigma \sim 1/\beta$ as suggested by eq. (29) and by the general theory of inelastic processes in absence of Coulomb effects).

Some more general observations are necessary: First, our “uncharged antiproton” cross sections can’t be assumed to be exactly equal to antineutron cross sections on a proton target, because the $\bar{p}p$ and $\bar{n}p$ states have a different isospin composition: $\bar{n}p$ is a pure $I = 1$ state, while $\bar{p}p$ mixes $I = 1$ and $I = 0$ states. Working the other way round, by comparison with data one can extract the separate isospin contributions. On the other side, if the target is an isoscalar nucleus then our “uncharged antiproton” is equivalent to an antineutron.

Another relevant point is that in the case of a free neutron target there should be no Coulomb focusing effect. However, since experiments on $n\bar{p}$ scattering are normally performed on deuteron targets, the $n\bar{p}$ cross section is as much “Coulomb focused” as the $p\bar{p}$ one. So, in the following, if “free neutron target cross sections” are more or less explicitely used, they must be meant as extracted from $\bar{p}d$ scattering. Obviously, the
cross section for $\bar{n}p$ scattering (which is an actually performed experiment\cite{14, 15}, and where strong interaction are the same as in $\bar{p}n$ scattering) will be another thing, directly connected to the cross section for $\bar{p}$ scattering on a really free neutron. In practices, however, we will not distinguish between proton and neutron interactions in this work.

In principle, Coulomb interactions affect the final state also. But, since such effects are of practical importance when the velocity of the involved particles is $\beta << 1$, final state Coulomb effects would be relevant only when the total mass of the final state is $\approx 2M_p$, which seems to be a very rare event\cite{17}.

4 Some applications

First, some relevant limits have to be considered. For incoming $\bar{p}$ momenta sensibly larger than the Fermi momentum, i.e. sensibly larger than 200 MeV/c, $\beta(\vec{p}, \vec{k}) \approx \beta(\vec{p}, 0)$ and $d\sigma_{1,N}(\vec{p}, \vec{k}, \vec{\pi}_j) \approx d\sigma_{1,N}(\vec{p}, 0, \vec{\pi}_j)$, so that exploiting the normalization rule of $S(k,k)$ we simply get

$$\sigma_{A,N}(p) \approx A\sigma_{1,N}(p), \quad p >> 200 \text{ MeV/c}$$

for the total annihilation cross sections on nuclear and proton targets. The limit of the PWIA treatment is evident: no eclipse effect (related with multiple scattering and leading to a more realistic $A^{2/3}$ dependence\cite{13}) is present in this result.

Another interesting limit is at the opposite of the scale, for incoming $\bar{p}$ momenta clearly below the Fermi momentum, so for $p << 100$ MeV/c. Then eq.(30) can be approximated by

$$d\sigma_{A,N}(\vec{p}, 0, \vec{\pi}_j) = \frac{Z}{\beta^2(\vec{p}, 0)} \int \frac{d^3k}{(2\pi)^3} S(k,k)\beta^2(0, \vec{k}) d\sigma_{1,N}(0, \vec{k}, \vec{\pi}_j).$$

(32)

Now one can exploit the fact that $\beta^2(0, \vec{k}) d\sigma_{1,N}(0, \vec{k}, \vec{\pi}_j)$ is a regular function, dominated by the S-wave contribution for not too large $k$. Then we may write

$$\beta^2(0, \vec{k}) d\sigma_{1,N}(0, \vec{k}, \vec{\pi}_j) = C |T(0, k, \pi_j)|^2 (2\pi)^3 \delta^3(\vec{k} - \sum \vec{\pi}) \delta(E_{fin} - E_{in}) \prod \frac{d^3\pi_j}{(2\pi)^3 2E_j} \approx$$
\[ \approx C|T(0, 0, \pi_j)|^2 (2\pi)^3 \delta^3(\vec{k} - \sum \vec{\pi}) \delta(E_{\text{fin}} - E_{\text{in}}) \prod \frac{d^3\pi_j}{(2\pi)^3 2E_j} \]  

(33)

and

\[ d\sigma_{A,N}(\vec{p}, 0, \vec{\pi}_j) = \]

\[ = \frac{CZ}{\beta^2(\vec{p}, 0)} \int \frac{d^3k}{(2\pi)^3} S(k, k)|T(0, 0, \pi_j)|^2 (2\pi)^3 \delta^3(\vec{k} - \sum \vec{\pi}) \delta(E_{\text{fin}} - E_{\text{in}}) \prod \frac{d^3\pi_j}{(2\pi)^3 2E_j} \]  

(34)

Substituting \(|T(0, 0, \pi_j)|^2\) with some average on relative pion directions \(|T|^2\) (with the final aim of estimating total cross sections), and integrating on the pion phase space, the previous two equations become

\[ \sigma_{1,N} \approx \frac{1}{\beta^2(\vec{p}, 0)} C|T|^2 \int (2\pi)^3 \delta^3(\sum \vec{\pi}) \delta(\sum E_j - 2M_\rho) \prod \frac{d^3\pi_j}{(2\pi)^3 2E_j} \]  

(35)

\[ \sigma_{A,N} \approx \frac{Z}{\beta^2(\vec{p}, 0)} C|T|^2 \int S(k, k)_{\vec{k} = \sum \vec{\pi}} \delta(\sum E_j - 2M_\rho) \prod \frac{d^3\pi_j}{(2\pi)^3 2E_j} \]  

(36)

(approximately valid for \(p \ll 100\) MeV/c only).

The two reasons that justify extraction of an averaged \(|T|^2\) are: 1) now we are interested in total cross sections, so fine correlation effects between the momenta of the emitted pions don’t concern us too much, 2) we want to compare total cross sections with different targets and we expect that in the ratio the effects related with \(|T|^2\) are largely cancelled. Of course this procedure would hide the effect of a possible resonance peak of the \(\bar{p}p\) cross section decaying into a specific channel. So we roughly obtain

\[ \frac{\sigma_{A,N}}{\sigma_{1,N}} \approx Z \frac{\int[S_A(k, k)]_{\vec{k} = \sum \vec{\pi}} d\Phi_N}{\int (2\pi)^3 \delta^3(\vec{k})_{\vec{k} = \sum \vec{\pi}} d\Phi_N} \equiv Z R_A, \; p \ll p_{\text{Fermi}} \]  

(37)

The fact that this equation only contains nuclear quantities clearly shows its advantages and limitations. We can say that \(R_A\) gives some general information on the way the
nuclear structure may affect a largely esothermic reaction between a low energy hadronic projectile and a nucleon, with large (assumed) independence from the specific character of this reaction. The calculation of the integral in the numerator of $R_A$ is not easy in general, even numerically. In the peculiar case of two pions only in the final state, four of the six integrations can be performed analytically (there is full spherical symmetry in the emission of one of the two pions, at least in the limit $p \to 0$, and azimuthal symmetry for emission of the second pion; the integration over the energy of one of the two pions is removed by the energy shell condition). For the peculiar choice of a Gaussian distribution

$$S(k, k) \equiv A(2\pi)^3 \frac{r^3}{\pi^{3/2}} e^{-r^2k^2}$$

(38)

the integral in the numerator of $R_A$ can be fully calculated with the approximation of the steepest descent.

Direct numerical integration confirms the validity of this approximation for $r > 0.4$ fm: for such values of $r$ the integral is dominated by a narrow kinematical region around $|\vec{p}_1| = |\vec{p}_2|$, leading to a value of the integral which is $r$–independent. The result is that $R_A \simeq A$ for any reasonable value of the parameter $r$ (for any $r > 0.4$ fm). For unrealistically small $r$ the ratio tends to zero. For large enough nuclei ($^{12}$C onwards) $r$ simply means $1/p_{Fermi} \sim 1$ fm. For deuteron $r$ would be larger, although in this case a simple exponential shape for $S(k, k)$ is more reasonable. When the deuteron single particle momentum distribution $N(k)$ is represented in a semilog plot[22], it is almost a straight line with negative derivative $\sim -6$ between $k = 0$ and the $S – D$ interference minimum. If we choose the simple exponential law

$$S(k, k) \equiv A\frac{r^3}{8\pi} e^{-rk}$$

(39)

instead of the previous Gaussian one, a numerical calculation again gives $R_A \simeq A$ for $r > 1$ fm (3 percent of difference at $r = 1$ fm). At smaller $r$ the ratio $R_A$ is smaller than $A$, and the convergence to the limit $A$ is slower than in the Gaussian case: $R_A \simeq 0.7$ for
Figure 4: Continuous line: the ratio $R_A/A$ (see text) calculated in the two simple cases of a Gaussian distribution (dotted curve) and exponential distribution (continuous curve). $R_A/A$ is presented as function of the distribution parameter $r$ (fm).
$r = 0.4$ fm and 0.8 for $r = 0.5$ fm. But, for any reasonable value of $r$ we get the most obvious result. For deuteron $r \approx 6$ coincides approximately with twice the radius of the deuteron $S$–wave wavefunction (in deuteron $S(k,k) \equiv |\psi(k)|^2$). These results, showed in fig.4, discourage us from trying to implement a Montecarlo integral to estimate $R_A$ with a larger number of emitted pions. The feeling one gets from the above calculations is that the PWIA calculation does not reserve special surprises, at least for those values of $r$ which are reasonable in mean-field models. For a discussion of the role of small–$r$ components in $S(k,k)$ (coming from non-mean-field contributions) see the next section.

A “surprise” could arrive from the existence of a clear resonance peak in the $\bar{p}p$ cross section in the energy region which is just below the lower limit of the explored range. In the nuclear case Fermi motion would spread the effect of such a peak through a wide kinematical range, with relative advantage of the $\bar{p}p$ cross section in the energy region which is close to the resonance peak, and of the $\bar{p}$–nucleus cross section at larger energies. This effect can be estimated by eq.(29) or (30), it is well known in many different branches of nuclear physics and it is just a form of Doppler effect. In the following we show an example of such Fermi motion spread of a resonance peak. On the other side, we notice that it is possible to fit the presently available data on $\bar{p}p$ annihilation with curves that produce a progressive trend towards the above quoted non resonant $1/p^2$ law, without need of additional contributions. Written with respect to the momenta $\vec{p}$ and $\vec{k}$ of the colliding $\bar{p}$ and $p$ a resonance Breit-Wigner peak becomes (in nonrelativistic approximation and assuming onshellness of both particles)

$$
\beta^2 d\sigma = \frac{C}{(|\vec{p} - \vec{k}| - p_o)^2 + B^2} f(\vec{p}, \vec{k}, \vec{\pi}) d\Phi_N,
$$

where $B$ corresponds to $\Gamma/2$ in the momentum space, $C$ depends slowly on kinematics, $p_o$ is the momentum corresponding to the resonance energy for target at rest, and $f$ depends on the specific angular momentum of the channel where the resonance is present. The Breit-Wigner denominator contains the total center of mass energy of $\bar{p}$ and $p$. Since one of the two colliding particles is bound to a nucleus this concept is ill-defined, and
some prescription for off-shellness treatment is necessary. We have adopted the easiest
form, i.e. to treat the proton as a free particle with nonrelativistic energy \( m + k^2/2m \).
Another possible prescription, which expresses the exactly opposite point of view, is to fix
the proton energy to its mass plus a small negative constant, for any \( \vec{k} \). At a qualitative
level, results don’t depend too much on the choice of the offshellness prescription, since
the center of mass energy is anyway strongly dependent on the product \( \vec{p} \cdot \vec{k} \), which is
present in both cases.

Neglecting the differences coming from the phase space integration of the channel
function \( f \), from eq.(31) we see that the total annihilation cross sections for \( \bar{p}p \) and \( \bar{p}A \)
will be proportional to the two comparable factors:

\[
\beta^2 \sigma_1 = \frac{C}{(\vec{p} - p_o)^2 + B^2} \tag{41}
\]

\[
\beta^2 \sigma_A = Z \int \frac{d^3k}{(2\pi)^3} \frac{C}{(\vec{p} - \vec{k} - p_o)^2 + B^2} \tag{42}
\]

(we work assuming full low energy effect of the Coulomb interaction). As an example, in
fig.5 we show the two curves (41) and (42) calculated numerically with gaussian \( S(k,k) = A(r/\sqrt{\pi})^3exp(-r^2k^2) \), with a choice of resonance parameters \( B = 10 \text{ MeV/c}, p_o = 20 \text{ MeV/c} \), and nuclear parameters \( r = 2 \text{ fm}, A = 4 \text{ and } Z = 2 \).

It is easy, although long, to demonstrate approximately the obvious fact that at small
\( \vec{p} \) and for \( 1/B >> r \) (that means \( B << p_{Fermi} \)) the peak of the resonance is enlarged to
a size \( \sim p_{Fermi} \). One can approximate the Breit-Wigner shape \( 1/[(p - p_o)^2 - B^2] \) with a
Gaussian curve \( exp[-(p - p_o)^2/B^2]/B \). This approximation is good near the resonance
peak, and as wrong as a Breit-Wigner far from the peak (where, however, the fall of the
nuclear momentum distribution kills contributions in both cases). Then substituting this
approximation in eq.(42), together with neglecting terms of second order with respect to
the small parameters \( B/r \) and \( p/k \) allows for calculation of the integral, which is roughly
equal to a constant times \( exp(-r^2p^2) \).
Figure 5: Nuclear smearing of a resonant contribution in the cross section for $\bar{p}p$ annihilation into a given channel. The continuous and dotted curve represent eq. (41) and eq. (42) respectively (in mb), versus $p$ (MeV/c). We have chosen $B = 10$ MeV/c, $p_o = 20$ MeV/c, represented the nucleus via a Gaussian distribution with parameters corresponding to $^4$He (see text for details). The vertical scale is the same for both curves and corresponds to the choice $C = 4 \cdot 10^4$, for $\sigma$ expressed in mb and $p_o$ and $B$ in MeV/c (equivalent to $C = 1$ for $p_o$ and $B$ expressed in 1/fm).
Evidently in the figure we see something that can be exchanged for an inversion behavior if the data stop at 30 MeV/c. However, this such a peculiar situation is unlikely. In fact, we tested three further possibilities: (1) a larger resonance peak; (2) a set of many partially overlapping resonances. (3) a set of many clearly spaced resonance peaks. In case (1), for $B$ larger than a few tenth MeV/c the nuclear target cross section overcomes everywhere the single nucleon target one. So a phenomenon like the one shown in fig.5 is not seen but for really narrow resonances. In case (2) the behavior reproduces what one could expect from the analysis of the single resonance case: If the complete set of resonance peaks in $\bar{p}$–nucleon annihilation clearly rises from the background and is contained inside a narrow range (within a few tenth MeV/c) then the behavior is like in fig.5. If the set of resonance peaks of the $\bar{p}$–nucleon annihilation spreads its effects out of this range, then the situation is like in example (2), i.e. like the case of a single resonance with $B >> 10$ MeV/c. In the limiting case where the range occupied by the overlapping resonances is of the size of the Fermi momentum or larger we simply find the same ratio between $\bar{p}p$ and $\bar{p}A$ cross sections as in the nonresonant case. In case (3) the nuclear Fermi motion completely cancels any chance to distinguish different resonances. Even in the simple case of two narrow peaks only, if their distance is less than 200 MeV/c it is impossible, for one who looks at $\bar{p}$–nucleus data, to imagine that two resonances are present in the $\bar{p}$–nucleon cross section. A possible limiting case could be the one where in a wide region (of size larger than 100 MeV/c) many resonances are present in the $\bar{p}$–nucleon cross section, and although they are clearly spaced it is not experimentally possible to resolve them: then as in case (2) one will see two continuous curves whose ratio $R_A$ is the same as predicted for the nonresonant background. As a conclusion, an inversion behavior as the one experimentally seen at the lowest momenta of the explored range could be justified in terms of one or a few resonance peaks, but they all should be narrow ($B$ not much larger than 10 MeV/c) and their peaks should be all confined in the region $p < 30$ MeV/c and very clearly emerge from the background.

One more point should be stressed: in the averaging procedures leading to eq.(32) the effect of any irregularity in the $\bar{p}$–nucleon $|T|^2$ is lost. Such an equation could not be
used to show how a resonance peak is spread by Fermi motion. However, the example of the resonance peak suggests that any variation in $|T|^2$ can be the cause of an inversion behavior (originated by Fermi motion spreading) if the relative change of $|T|^2$ is large within a momentum range of a few tenths MeV/c.

5 Over Inpulse Approximation and Mean Field: a discussion

The fact that PWIA, within mean field approximations, does not explain inversion means that what we see is not a situation where annihilation takes place on one target nucleon only, with the remaining $A - 1$ just acting as spectators. Annihilation probably takes place on one nucleon mainly, however the other ones do play some more or less direct role in depressing its effectiveness. Here we limit ourselves to explanations involving nuclear physics in traditional sense, i.e. without considering phenomena related with the quark structure of a nucleon cluster. The common point is that both the consideration of components in $S(k,k)$ related with momenta $k \gg p_{Fermi}$, and the multiple scattering, are processes which directly involve more than one target nucleon, and have a large probability of leaving the residual nucleus in an excited or fragmented state. The data on the “inversion” region don’t classify final residual states. In addition, they involve Deuteron or $^4$He only, so a systematics of the behavior of the residual is not possible yet. A knowledge of the distribution of the residual states would be decisive in this respect. In the case of non IA processes (i.e. involving multiple interactions) a second scattering process accompanying the annihilation would transfer a large recoil momentum to a single spectator nucleon, increasing the probability of breaking the residual. In the case of an annihilation process within IA (i.e. directly involving a single target nucleon) but picking up a nucleon with initial momentum $k \gg p_{Fermi}$, this large momentum of a single nucleon is probably due to direct interaction with another nearby nucleon. This spectator nucleon would be left with large recoil momentum, just as in the rescattering case.
Let us examine first the role of $S(k,k)$: The Gaussian and exponential simple models we have used above are clearly approximations of distributions of mean-field character. However, the slower convergency of the results we get when a simple exponential law is used instead of a gaussian one suggests a certain dependence on the high momentum part of the distribution $S(k,k)$, i.e. for $k > P_{\text{Fermi}}$. At $k \sim 1/(0.5 \text{ fm})$, i.e. $k \sim 400 \text{ MeV/c}$, $S(k,k)$ receives large contributions by nuclear surface effects and short range correlations. The first effect can enter a mean field shell treatment, but not when gaussian functions are used, since no surface thickness parameter is present. The second one requires sophisticated nuclear models, beyond the mean field approximation. In practices the relevant points are that such effects depend on $A$ in a different way, and carry different dimensional scales. The mean field function introduces into any Fourier transform a large component at $k \sim R_{\text{nucleus}}^{-1} = 1/(r_0 A^{1/3})$, $r_0 \simeq 1.3 \text{ fm}$. So one may imagine a kind of $A e^{-b r_0 A^{1/3} k} (b \sim 1)$ dominating term. Surface effects and direct short range nucleon–nucleon interactions produce large components which are relevant at $k \sim 1/r_c$ or $1/r_s$, where the short range parameters $r_c$ and $r_s$ are smaller than $1 \text{ fm}$ (typically: $0.5 \text{ fm}$). The surface term could be $\sim A^{2/3} e^{-b r_s k} (b \sim 1)$, while the repulsion term is $\sim A^2 e^{-b r_c k}$ (i.e. proportional to the number of couples). Clearly, it is like having, in a nucleus, different momentum distributions living aside, and sharing a total normalization factor 1. In the case of deuteron the high momentum part of the momentum distribution coincides with the D-wave. In more complex nuclei, the calculation of the spectroscopic factors in $(e,e'p)$ reactions suggests that not much more than 60% of $\int S d^3k$ is filled by the mean field contributions. So a large part of $S(k,k)$ could be represented by terms damping on a scale $r \sim 0.5 \text{ fm}$, where the above ratio $R_A$ has been estimated to be smaller than $A$. However, fig.4 shows that $R_A \simeq 0.8$ for $r = 0.5 \text{ fm}$, so that even taking into account a 40% contribution by a term like $S \approx e^{r} (\text{exponent of } -r k)$ with $r \simeq 0.5 \text{ fm}$, the overall “large momentum decrease” would be contained within 10%. To estimate with more precision how much could be the real weight of such large $k$ terms in the phase space integral for the total annihilation cross section, a first interesting step could be a model calculation of the relative amounts of annihilations involving the deuteron S and D components. Experimentally it does not seem
easy to decide whether residual fragmentation (whenever found) originates in initial state
or in multiple scattering effects.

When multiple scattering effects are introduced the IA results could be largely modified. A long series of possible mechanisms should be studied, and we are doubtful about
the possibility to have all of them under control within short time since now. Within the
limits of our fantasy and knowledge, we list some of the possibilities:

1) Elastic shadowing, i.e. the eclipse effect between the nucleons. This effect is
surely present although its role at low energies is not so intuitive as in the corresponding
high-energy problem, where it may be identified with the decrease of the reaction
probability on a target nucleon due to flux removal in the previous reactions with another
one. At low energy the $\bar{p}$ position is not well defined inside the target nucleus, due to $\Delta x 
\sim 1/\Delta p \sim 1/p \to \infty$. So we can’t visualize easily the shape of the “shadow” that the
first target nucleon casts on the second one. We discuss below the possibility of treating
this mechanism within DWIA.

2) Inelastic shadowing, i.e. the intermediate (real or virtual) mesonic state created
by an initial annihilation is converted into $\bar{p} + p$ by the opposite process in a second step.
This transfers a part of the inelastic rate into the elastic one. The only sensible way to
treat this mechanism could be via a coupled channel model.

3) Inelastic antishadowing: in the first inelastic event a state is created which is
not allowed by the overall conservation rules, although it may temporarily exist
as a virtual state. A second scattering event converts this virtual state into a real and
allowed one, completing the annihilation process. One can speak of multistep annihilation
processes, peculiar of nuclear targets. It does not seem easy to make good quantitative
predictions on its effectiveness. Again, a coupled channel model could be the method.

4) Soft distortions of the projectile wave, produced by coherent interactions with the
nuclear matter. With “soft” we mean that they can involve excitation of the spectator
set of $A-1$ nucleons, but not drastic changes of the internal state of any of the individual
nucleons. These interactions could perhaps be treated via DWIA with traditional nuclear
optical potentials with a small imaginary part. In the momentum space, they transform
the $\bar{p}$ wavefunction $\psi(k)$ from a plane wave $\delta(k-p)$ into a distribution with peak at $k \approx p$ and fluctuation $\sim 200$ MeV/c. Intuition suggests that their effect should consist mainly in enhancing the kinematical spreading which is already intrinsic in PWIA.

5) Soft distortions of the projectile wave, produced by diffraction accompanying hard inelastic $\bar{p}-\text{nucleon}$ events. In practice, any annihilation creates a “vacuum region” in the projectile wavefunction $\psi$, and the borders of this region are characterized by large space derivatives of $\psi$, meaning high momentum components of the initial state wavefunction. About the possibility to treat this by DWIA, see below. We must remark that, as the analysis of the black sphere model contained in the next section shows, within that model this is the main cause of depression of the annihilation rate.

6) Soft distortions of the projectile wave $\psi$ produced by the above mechanism (2). Let us imagine that a $\bar{p} + p$ couple is converted by the first annihilation process into a set of 2 pions in a position $\vec{r}_1$, and by the opposite process into a couple $\bar{p} + p$ again, in the position $\vec{r}_2$. In nuclear matter the propagation properties of the two pion system from $\vec{r}_1$ to $\vec{r}_2$ are different from the corresponding ones of the $\bar{p} + p$ system. In the high energy regime it is relatively easy to calculate the phase difference associated with this mechanism once one knows the mass of the intermediate state. We don’t know a corresponding low energy theory.

7) Soft final state interactions (SFSI) of the meson system (the hard ones can be included into some of the previous cases). SFSI can be treated approximating the final meson plane waves with damped waves in Glauber style. These express the fact that any precisely identified meson state will be partly depleted by interactions with the residual nucleus. Although probably SFSI do not change much the total annihilation rate, it will enhance the number of annihilations accompanied by fragmentation of the residual. It will also introduce high momentum components into the meson wavefunctions. Due to the fact that the meson waves are already high momentum ones, this should be less relevant than the corresponding initial state mechanisms (4) and (5).

8) Multiple interactions between $\bar{p}$ and the same target nucleon. If these interactions are consecutive they are reabsorbed in the matrix element $T$ for the $\bar{p} - p$ interaction
used in the IA. If they are not consecutive they escape IA. At very low energy, because of 
delocalization of the wave functions, multiple interaction events on the same nucleon and 
backward scattering can be relevant. E.g. triple scattering on deuteron is not necessarily 
a rare event.

A common element which is present in some of the previous cases is the distortion of 
the incoming or outgoing waves, due to hard or soft interactions with the nuclear matter. 
In the target region this creates Fourier components of the wavefunction of $\bar{p}$ and of 
the final mesons which are not present asymptotically. So, even assuming a given vector 
$\vec{k}$ for the nuclear proton, the kinematical details of the annihilation are unknown. E.g. 
in the predictable case of strong absorption on the nuclear surface, the large gradients 
which accompany the fast damping of the $\bar{p}$ wavefunction will introduce high momentum 
components into it, partly shifting the reaction to a region where it can’t be considered 
as a “low energy” process.

In large nuclei some of the above described effects could be treated by DWIA. In this 
formalism the waves participating to the annihilation are not plane waves, but damped or 
somehow distorted waves. So, when interacting with one given nucleon, the projectile wave 
takes into account the flux removal, or the wave distortion, due to possibility that the $\bar{p}$ 
(or the final state mesons) have already interacted (or will interact) with another nucleon. 
Leaving aside formal problems, this treatment can be justified when it is sensible to write 
the interaction hamiltonian as a sum of two terms $H_1 + H_2$, where $H_1$ produces the matrix 
element $T$ used in the IA, and $H_2$ is responsible for the wave distortion. Supposedly, in 
our case $H_1$ would be the interaction between $\bar{p}$ and one nucleon, and $H_2$ the interaction 
between $\bar{p}$ (or the final mesons) and all the other ones. In practices and with all kinds of 
hadronic projectiles, DWIA normally means either the use of the Glauber formalism, or a 
distortion of the wavefunction by a suitably fitted optical potential. It can work as far as 
the average nuclear matter interactions with the projectile are known, which is absolutely 
not our case (they are exactly what we would like to understand here). In addition this 
kind of implementation of DWIA would mean an unpleasant mixing between two models, 
the IA and the optical potential model, in a regime where still both of them need separate
A proper study of inelastic shadowing or antishadowing is difficult, even at qualitative level, and would require a multi-channel model. In the simplest case one only takes into account two channels (elastic and “effective inelastic”). Although this model can produce strong inelastic shadowing and inversion the quantitative results depend critically on the coefficients (and phases) one chooses for the transition amplitudes between the elastic and the excited channels. Physical constraints over these coefficients are presently poor, making a quantitative analysis of low energy inelastic shadowing a difficult task. As far as only total annihilation rates are known, a many channel model is equivalent to the optical potential model. In the latter case one sums effectively over all the transitions (through both elastic and inelastic channels) leading to the elastic channel in the final state. All the information about the details of the inelastic channels is hidden in the optical potential (whose form can be formally derived starting from a set of coupled channel equations). In the many channel model one has to introduce specifical information about the inelastic channels. If this information is arbitrary the two models are in practices equivalent, although one can gain some physical interpretation of the phenomena hidden behind the optical potential.

Probably the easiest starting ground for a study of multiple interactions is a deuteron target. As we have anticipated, support to the role of non IA terms in the low energy interactions between $\bar{p}$ and nuclei comes from an analysis of the problem of the width and shift of the levels of the antiprotonic deuterium atom. In that work consistent cancellation has been found between effects which can be associated to single and double scattering. To complete the picture, recently we have been informed of a nonperturbative three-body analysis of $\bar{p}D$ scattering. In this work some reasonable forms for the $\bar{p}–$nucleon interaction have been postulated and 3-body equations solved for the system $p + n + \bar{p}$. It has been found that an annihilation $\bar{p}D$ cross section smaller than the $\bar{p}p$ one cannot be obtained with a purely absorbitive $\bar{p}–$nucleon interaction. It seems that more complex mechanisms are needed. This would decrease both the role of multiple interaction effects and of the optical model in its simplest form.
The obvious conclusion of this section is that, due to the difficulty in taking into account all the previously listed effects, these aspects of the problem will probably remain open for a long time.

6 The black sphere optical potential model

In ref.\cite{7} a black sphere potential \( V(r) = -iW \) for \( r < R \) was used to show that, increasing \( W \), one finds an initial increase followed by a definitive decrease of the annihilation rate, which tends to zero when \( W \to \infty \) (at very low energies). We have carried on some further calculations on the same easy model, which showed us that if one increases \( R \) at constant \( W \), the annihilation rate saturates fastly, leading to a constant S-wave contribution to the cross section.

Trust this potential model would imply to expect the same behavior when comparing \( \bar{n} \)-nucleus cross sections at different mass numbers. We could associate each nucleus with an imaginary potential well with radius \( R = A^{1/3}r_o \), \( r_o \approx 1.3 \) fm, and assume \( W = \) (unknown) constant for all nuclei. The last assumption means to consider the strength \( W \) as a local mean property of nuclear matter. Nuclear matter local properties are supposed to be similar in most of the nuclei (with the exception of the lightest ones). Then the consequence of the above reported saturation effect would be an \( A \)-independent \( \bar{n} \) annihilation cross section at low energies, and a \( \sim Z \sim A/2 \) dependence for \( \bar{p} \) annihilation (due to the Coulomb corrections).

To understand how much of this potential model can be general, we have analyzed the behavior of the wavefunction \textit{inside} the black sphere. As we explain a little more in detail below, we find that exponential damping of the \( \bar{p} \) density inside the black sphere happens on a scale \( R - r \sim 1/\sqrt{MW} \) (in natural units; \( M \) is the projectile reduced mass \( \sim 1 \) GeV, \( W \) is the strength of the imaginary part of the potential; we assumed zero real part). So, for any potential radius \( R \) sensibly larger than \( 1/\sqrt{MW} \) all of the incoming isotropical flux disappears before reaching the center of the target. This introduces a saturation mechanism, of the same kind that at higher energies causes the transition from
volume absorption to surface absorption.

For better understanding the peculiarities of the low energy problem, we would like to start by analyzing some mathematical properties that, in the high energy problem, lead to the “obvious” association between strong flux absorption by the target and a high reaction probability. Some of these key properties are not present in the corresponding problem at low energies.

First, the ideal geometry of the problem is spherical at low energies, and axial (with defined orientation) at large energies. From a certain point of view, the lowest possible border of the “high energy” region, (where a flux direction with back-front orientation can be clearly recognized inside the nuclear volume) may be associated with the presence of a consistent \(P\)–wave contribution. In the \(S\)–wave dominance regime, there is no preferred direction, and concerning orientation we can only speak of “inward” and “outward”. But with both S-wave and P-wave present with similar relevance, the direction of the nonzero average angular momentum identifies a preferential \(\hat{z}\)–axis in space, and the interference between the two waves \((S \sim 1, P \sim \cos(\theta))\) creates a difference between the \(\theta = 0\) and \(\theta = 180^\circ\) values of \(|\psi|^2\), which allows for identifying an orientation of the \(\hat{z}\)–axis. At really large energies, where many partial waves enter the scattering region \(l_{max} \sim Rp\) in natural units) the full projectile wavefunction \(\psi\) tends to the form \(e^{ipz}f(\vec{r})\), where \(\hat{z}\) is the incoming flux direction and \(f(\vec{r})\) depends on \(\vec{r}\) slowly enough to allow one to consider \(f\) as a constant over a space region of size \(1/p\). Assuming the nuclear center to be in the origin, with such a form we can say that \(\psi\) enters the nucleus on one side, and exit from the other side. Roughly, we can speak of an “entrance” surface portion \((z < 0)\) and of an “exit” surface portion \((z > 0)\). On both sides the values of the external wavefunction must match the values of the internal one. The internal function assumes rather different values in the two regions: large, at the “entrance”, small, at the “exit”. This can be seen exploiting the eikonal approximations:

\[
\nabla^2 + k^2 \approx \frac{\partial^2}{\partial z^2} + k^2 = \left(\frac{\partial}{\partial z} + ik\right)\left(\frac{\partial}{\partial z} - ik\right) \approx 2ik\left(\frac{\partial}{\partial z} - ik\right). \quad (43)
\]
These approximations reduce the Schroedinger equation to a first order simple equation with solution $exp(-mWz/k)$ in a region characterized by the uniform potential $V = iW$ (the other possible solution $exp(+mWz/k)$ is discarded on the basis of unitarity considerations). This creates the association “imaginary potential” $\rightarrow$ “desappearance of the flux”. For a black sphere potential, following the values of the high energy wavefunction along a line with impact parameter $\vec{b}$ we have the total damping $\psi_{exit}/\psi_{entrance} = exp(-mW\sqrt{R^2 - b^2}/k)$. So in quite a natural way the matching conditions oblige the external wavefunction to be large at the entrance side, and small on the exit side. Saturation of the bulk absorption appears for $R >> k/mW$: then the internal function is $\approx 0$ on the exit side whatever $R$, and we speak of surface absorption with $\sigma \sim R^2$.

In practical problems, we can sometimes speak of “high energy” down to a situation where only the S and P waves are important. In such a case the interference between the two waves must be effective enough to assure a relevant difference between $|\psi_{exit}|^2$ and $|\psi_{entrance}|^2$. In $\vec{p}$—nucleus the high energy approximations, as described in the introduction, can be pushed with some residual effectiveness down to momenta $\sim 100$ MeV/c.

In the case of the low energy S-wave dominated problem (with the same black sphere target) there are relevant differences both in the behavior of the internal wave and in the use of the matching conditions. First, there are no more possibilities of reducing the Schroedinger equation to a first order one. The solution becomes practically 1-dimensional with respect to the radius: $\psi(\vec{r}) \approx \psi(r)$. In the limit $\approx 0$ (or reasonably: whenever $k^2/2m << W$) the problem admits two solutions of the kind $exp(\pm \sqrt{-2imWR})/r \sim exp(\pm [(1 - i)\sqrt{mWR}]/r$ (from now on we will discuss of the behavior of $r\psi$ instead of $\psi$, so to avoid the accompanying factor $1/r$). As in any scattering problem, the overall normalization of the wavefunction is not relevant (the observables are determined by the ratios between incoming and outgoing fluxes). The asymptotic form of the external (S-wave) wavefunction must be of the form $r\psi = g\sin[k(r - a)]$, with constant $g$ and $a$ ($a$ is the scattering length). This function is the sum of an incoming and an outgoing wave $exp[\pm ik(r - a)]$. A nonzero imaginary part of $a$ means net flux absorption in the scattering, i.e. inelastic processes. The total elastic and inelastic parts of the cross section...
are proportional to $|a|^2$ and to $|Im(a)|/k$ (see e.g. [10]). On this ground one can say that at low energies the “geometric” naive expectations for the values of the cross sections are $\sigma_{el} \sim R^2$ and $\sigma_{react} \sim R/k$, reached when $Re(a) \sim Im(a) \sim R$.

The two boundary conditions to be satisfied are: (1) Finiteness in the center, which obliges one to consider both the above solutions with relative coefficient $-1$: $r\psi \sim exp\left(\pm(1-i)\sqrt{mWr}\right) - exp\left(-\mp(1-i)\sqrt{mWr}\right)$. (2) Matching with the total external solution in $r = R$ (the target surface). Because of the lack of meaning of the overall normalization of the solution, the matching condition is normally required for the logarithmic derivative of the solution: $[\psi'/\psi]_{\text{internal}} = [\psi'/\psi]_{\text{external}}$ (at $r = R$); $\psi_{\text{external}}$ is the sum of the inward and outward directed spherical waves.

We stress this point: The $S$--wave low energy problem only knows one matching place, “entrance” and “exit” now coincide. In their 1-dimensional forms (with respect to $z$ and $r$ variables respectively) the high energy problem is a “transition through a wall” problem, while the low energy one is a “reflection” problem.

As we have noticed above, now one can’t discard by assumption any of the two components $exp(\pm(1-i)\sqrt{mWr})/r$ of the internal solution. They are both necessary to satisfy the condition of finiteness in the center. Since their relative coefficient is $-1$, if $R >> 1/\sqrt{mW}$ one of the two terms only is relevant in $r = R$. This is the saturation of bulk absorption in the low energy problem. Intuition suggests that for any $R$ large enough to satisfy the saturation condition $R >> 1/\sqrt{mW}$ the full spherical incoming wave disappears inside the sphere, and it is not possible to remove more than the full flux. In semiclassical regime, this would imply a large flux absorption. This is not the case here. It may be verified, by direct calculation, that when the internal function at the matching point is composed by only one of the above two terms, matching can be fulfilled only if the inward and outward directed external waves have the same magnitude.

More insight in the black sphere subtleties can be reached by a comparison with a related problem: the scattering by a (very) hard core elastic repulsive potential $V(r) = +W$ for $r < R$, with $W >> k^2/2m$. In both problems the modulus of the internal solution behaves the same way, i.e. damps exponentially to zero on a scale $R - r \sim 1/\sqrt{mW}$.
and the logarithmic derivative (in the matching point) of the internal wavefunction is
very large for strong potentials, has positive real part, and imaginary part not larger in
magnitude than the real part (in the hard core case of course the imaginary part is zero).
This has the consequence that the scattering length is very close to $R$. We remind the
geometrical meaning of the scattering length: At very small $k$ and in the surroundings
of the matching point $r = R$, the external function $r \psi$ can be written in the form
$r \psi = (B/k) \sin[k(r-a)]$ and approximated by the straight line $B(r-a)$. The constant $a$
is the scattering length, and for $r = a$ the virtual prolongation of the linearized external solution
will reach zero (of course the internal real solution will not be zero at $r = a$, usually).
The practical key of nonrelativistic scattering problems is of course the possibility to
calculate somehow the internal solution. We suppose one has been able to do it, and has
normalized the internal wave in some arbitrary way. By applying the matching condition,
i.e. by requiring the equality of the external and internal logarithmic derivatives, one
determines all the parameters of the external wave, in particular $a$. If the logarithmic
derivative is large enough ($\psi'/\psi >> 1/R$) the virtual prolongation of the external solution
will reach zero for $r$ very close to $R$, so $a \approx R$. This is what happens with a very large
real repulsive potential. Incidentally we note that the S-wave boundary condition for the
free motion wavefunction (i.e. $r \psi_{\text{free}}(r) = \sin(kr) = 0$ for $r = 0$) defines the origin as
the “source” of the outgoing waves. In presence of a repulsive potential the position of
this source (which is now virtual) is “moved” to some point $r = a > 0$. For a repulsive
potential of infinite strength, $a = R$. In the case of a black sphere potential in saturation
condition, the reported form of the internal solution suggests that the real and imaginary
part of the logarithmic derivative are of the same scale at the matching radius. Then a
large logarithmic derivative implies $\Re(a) \approx R$, $\Im(a) \sim R - \Re(a)$ (the second equation
would not be necessarily valid if the imaginary part of the logarithmic derivative were
much larger than the real part; this corresponds to the case of a potential dominated by
a negative real part). In the limit of a very large $W$, $\Re(a) \to R$, $\Im(a) \to 0$, meaning
an elastic cross section close to the geometrical section, and reaction cross section $<<$
$R/k$, i.e. much smaller than what one could expect assuming a size $R$ of the absorbing
An approximate calculation, valid for $\sqrt{mW} \gg k$ and $\sqrt{mW} \gg 1/R$, gives $R - a \approx 0.5(1 + i)/\sqrt{mW}$. So, when the two conditions $mWR^2 \gg 1$ and $W \gg k^2/2m$ are realized, the black sphere and the hard core sphere lead to similar results, and we find that a potentially huge internal absorption just implies very good reflection of the incoming flux.

There is an interesting difference between the two problems: in the hard core problem, the internal fastly damped wave is a stationary wave, with no net flux. In the black sphere problem a flux $\propto -exp(2\sqrt{mW}r)$ (i.e. inward directed) is present inside the sphere. Despite this, real flux absorption is small for large $W$. This does not create contraddictions. In fact, crossing the matching point $d\psi/dr$ is continuous and regular, implying that the velocity is more or less the same on both sides, in the surroundings of $r = R$. However, due to the large and positive $Re(\psi'/\psi)$ in $r = R$, and to the smallness of $k$, the overall normalization constant $b$ of the external solution $bsin[k(r - a)]$ is much larger than any value of $\psi$ inside the potential region. For this reason, all the flux that is able to cross the potential surface is definitely lost, but this is much smaller than the incoming flux. Notice that, for large $k$, a large $\psi'/\psi$ in the matching point would not imply a large value of $b$.

We stress that in the above discussion the key points leading to a small flux removal are: (1) entering a well defined space region (the nucleus) the modulus of the projectile wavefunction becomes fastly much smaller than in the external space. More precisely we need $|\psi'/\psi| >> 1/R >> k$; (2) the imaginary part of the logarithmic derivative of the wavefunction at the target surface is not much larger in scale than the real part. As far as one believes that these two conditions are respected in general in the $\bar{p}$—nucleus annihilation problem, the potential can be completely forgotten and the above conclusions on inversion, saturation etc. remain valid. In other words, even though one does not believe to the reliability of an optical potential model for $\bar{p}$—nucleus interactions, the fact that a strong absorption of the projectile wave inside the target can be associated with an almost complete flux reflection outside seems to be completely general: it originates in the regularity properties of the projectile wavefunction at the surface of a strong absorber.
and not in the details of the absorption mechanism. This is in agreement with the final words of ref. [5], where the “repulsive properties of absorption” are explicitly cited to explain the opposite effect of single and double scattering terms.

Concerning magnitudes, for a target radius of 2 fm the saturation condition is reached for $W \sim 10$ MeV, and larger imaginary potentials or larger radii don’t imply more annihilations. The above discussed loss of semiclassical intuition in this problem prevents us from attributing a precise meaning to a given value of the imaginary part of an optical potential. However, intuition can content itself with the fact that the value of $W$ is associated with the spatial scale $1/\sqrt{mW}$ over which we assume the external flux to have been reasonably damped inside the nucleus. In this sense $1/\sqrt{mW}$ can be interpreted as the “free mean annihilation path in nuclear matter”, but this number has nothing to do with the rate of annihilation we really see. In the limit where this path tends to zero we see no annihilations at all.

The consequence of the previous discussion is that in systematic analysis of $\bar{n}$ annihilations on large-$A$ nuclei at low energies one should expect a roughly constant cross section. With $\bar{p}$ projectiles the cross section would be proportional to the nuclear charge $Z$. Behind this there is the assumption that small portions of hadronic matter (i.e. a single nucleon) have very large absorption properties.

With very light nuclei, structure details can be more important than nuclear matter bulk properties (e.g. one can’t speak of deep nucleons which are untouched by annihilation processes). Our analysis of the damping properties of the wavefunction relies on target continuity properties which could be absent in this case. Taking into account that Coulomb forces are not able, in very light nuclei, to accelerate $\bar{p}$ up to Fermi motion typical velocities, we could imagine two opposite limiting pictures. In the first one, before being subject to hard (elastic or not) nuclear interactions the antinucleon has no time to undergo relevant soft interactions, that would accelerate it to momenta $k \sim p_{\text{Fermi}}$. Then, due to the difference in velocity, we can speak of approximate adiabaticity, and say that $\bar{p}$ sees the nucleus more or less as a continuous target. In such a case the black sphere results could be applied to light nuclei too (very approximately, since the difference in
the velocity scale is not huge in the actual experimental kinematics\[1, 2\]). The opposite 
picture is the “compound nucleus” scenario: The antinucleon is able to mix with the 
target nucleons forming a “compound nucleus”, where all hadrons share similar veloci-
ties. In this case some nuclear-scale time would pass between the disappearance of the 
elastic channel and the formation of any final state, with presence of narrow resonances 
and large fragmentation probability for the residual nucleus. Presently we don’t see (or 
don’t resolve) resonances or many fragment final states (such phenomena could be present 
already at projectile momenta \(\sim p_{Fermi} \approx 200 \text{ MeV/c} \)). If the compound nucleus picture 
were realistic the above presented arguments should be largely modified.

Generalizing the last argument, if the \(\bar{p}–nucleon\) interaction is characterized also by 
a strong (elastic) attracting part, with the same range as the nucleon-nucleon interac-
tion, this could result in the formation of a global nuclear attracting well, with radius 
\(\sim\) the nuclear radius and diffuseness \(1/m_\pi\). Such an attracting well could produce nar-
row (\(\bar{p}–nucleus\)) resonances. As the potential model itself can show, by just taking a 
general potential with form \(V(r) = -(U + iW)\) and studying numerically some random 
solutions, the presence of resonances may largely affect the validity of all the previous 
conclusions. In particular these solutions may present “phantom” resonances. With this 
term we indicate cross section peaks which would be clear and narrow for \(W = 0\), and 
disappear with a nonzero \(W \sim U\) (which should be the case in our problem). Phantom 
resonances would not be easily detected, but enhance anyway the \(\bar{p}–nucleus\) annihilation 
cross section. Qualitatively the effect of a resonance can be understood by observing that 
any attractive interaction will act as a focusing device (as it happens with the Coulomb 
interaction), but in particular a narrow resonance may enlarge much the time spent by 
the projectile in the interaction region. In the previous language: it will enlarge the value 
of \(|\psi|^2\) inside the target, so decreasing the value of the logarithmic derivative at the sur-
face. This may increase largely the annihilation rate, and a systematical analysis of the 
phenomenon does not appear easy to us. For the case of a Deuteron target the problem 
is under examination in \[3\].

In the same work\[3\] it has been noted that the behavior of the black potential sphere
can be largely affected by the presence of a long exponential tail of the potential shape (we have assumed, in the previous discussion, a sharp potential surface). In the terms used in the previous discussion, where the potential is strong the $\bar{p}$ wavefunction $\psi$ will be fastly damped to zero. This creates a large relative variation of $\psi$ in the interaction surface region. But if this region diffuses over a region of size $\sim R$, $\psi$ will pass from its external value to zero in a region of size $R$, and the previous requirement $\psi'/\psi >> 1/R$ will not be realized. To see it in another way: the “black” sphere is surrounded by a large “grey” cloud, were annihilation may be much more effective (in agreement with the above established law: smaller potential $\rightarrow$ stronger annihilation rate). The condition of a diffuseness $\sim R$ will not be realized with heavy nuclei, but is possible with small compact nuclei like $^4$He, were $R$ is not much larger than $1/m_\pi$, the reasonable upper limit for any strong interaction diffuseness.

Concluding this section, on one side the study of the wavefunction suggests on a very general ground that a saturation behavior should be present and detectable in $\bar{p}$ and $\bar{n}$ annihilation on large-A nuclei, but on the other side it is still doubtful that this may be considered the satisfactory explanation of the inversion behavior that we actually see in very light nuclei. We must even remember that this saturation behavior means a systematic $\sim Z$ law for $\bar{p}$–nucleus annihilations. So the saturation could (roughly) be enough to explain the ratio between $\bar{p} - p$ and $\bar{p} - D$ data, but surely not the data on $^4$He.

7 Summary and conclusions

This work has been motivated by the need of understanding the apparently anomalous behavior of $\bar{p}$–nucleus annihilation cross sections at low energies.

Most of our efforts have been devoted to an analysis of the impulse approximation contribution to the cross section on nuclei. In general terms, and taking Coulomb effects into account, this has been expressed by eq. (30), with eq. (29) as reasonable approximation for momenta over $100\div200$ MeV/c.
For the limit of momenta below 100 MeV/c the ratio \( \equiv Z R_A \) between the nuclear and single nucleon annihilation cross sections has been estimated via eq. (37). This ratio, which has been calculated in an approximated way via an average on the squared modulus of the matrix element for the low energy \( \bar{p} \)–nucleon annihilation, has the peculiarity of depending on nuclear quantities only. Since, however, a complete calculation of eq. (37) requires a complicated numerical Montecarlo integration, and knowledge of the nuclear momentum distribution, we have just performed some estimation for the special case of two pion emission, within mean field nuclear distributions. These estimations simply reproduce the most expected PWIA behavior of cross sections, i.e. \( A \) times the elementary cross sections (\( AZ \) when Coulomb corrections are included). For momentum distributions which are large at large momenta (meaning the presence of strong non mean field contributions) we find a decrease with respect to the \( \sim A \) expectation, although the effect is not very marked (a 10\% correction, in magnitude).

PWIA allowed us to simulate the characteristic Fermi motion smearing of a resonance peak (fig. 5). The conclusion is that in principle this mechanism is able to produce inversion exactly the way we detect it (it is just a matter of tuning parameters), however some possible, but rather restrictive, conditions should be fulfilled. In particular, resonances in the \( \bar{p} \)–nucleon cross section should be very narrow and all regrouped within a small momentum range just a little below the lower limit of the experimentally analyzed region.

Accepting that IA (in practics: single scattering processes) is unable to explain both the experimental data and the potential model strangenesses, we feel like supporting the role of multiple scattering effects, as somehow indicated in [7] and [5]. We have discussed the possible physical origin of such effect, indicating several among the possible mechanisms (elastic shadowing, inelastic shadowing and antishadowing, and soft distortions of the \( \bar{p} \) wavefunction) which would probably require a separate style of analysis each. Even in the relatively simple case of a Deuteron target, an exhaustive calculation of the effect of inelastic multiple scattering looks presently prohibitive.

While discussing Coulomb interactions, we have perfectly reproduced the \( \bar{p}p \) annihi-
lation cross section with a very simple optical model (a pure imaginary potential well with Woods-Saxon shape, and standard parameters for radius and diffuseness). Taking into account that the optical model is also able to produce inversion, we have devoted a section of this work to an extensive analysis of the behavior of the solutions of the optical model in its simplest form, i.e. the black sphere. The conclusion is more general than this model would allow for: it appears that whenever in a spherical region of space the projectile wavefunction is heavily suppressed almost all of the incoming flux must be elastically reflected. This is an obvious result when the suppression mechanism is elastic repulsion, but we find that the same must happen even when the mechanism which creates a “vacuum hole” in the projectile flux is inelastic. This phenomenon looks more suitable for application to the case of heavy nuclear targets (where the general nuclear matter properties are more relevant than the details of the structure) than for the case of the light nuclei used in the performed experiment[1]. In addition it would anyway suggest a value of the $\bar{p} - ^4\text{He}$ annihilation rate larger than the $\bar{p} - p$ one (because of the stronger Coulomb attraction) instead of the observed inversion between the two.

So, although some explanations are available for the detected inversion mechanism, the problem is still open.

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