Linking numbers, quandles and groups

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Abstract

We introduce a quandle invariant of classical and virtual links, denoted $Q_{tc}(L)$. This quandle has the property that $Q_{tc}(L) \cong Q_{tc}(L')$ if and only if the components of $L$ and $L'$ can be indexed in such a way that $L = K_1 \cup \cdots \cup K_\mu$, $L' = K'_1 \cup \cdots \cup K'_\mu$ and for each index $i$, there is a multiplier $\epsilon_i \in \{-1, 1\}$ that connects virtual linking numbers over $K_i$ in $L$ to virtual linking numbers over $K'_i$ in $L'$: $\ell_{j/i}(K_i, K_j) = \epsilon_i \ell_{j/i}(K'_i, K'_j)$ for all $j \neq i$. We also extend to virtual links a classical theorem of Chen, which relates linking numbers to the nilpotent quotient $G(L)/G(L)^3$.

Keywords: group; linking number; quandle.

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1 Introduction

Linking numbers are among the oldest, and simplest, invariants of classical knot theory; as discussed by Ricca and Nipoti [15], Gauss mentioned linking numbers in his diary almost 200 years ago. Linking numbers were extended to virtual links by Goussarov, Polyak and Viro [5]. For our purposes, the most convenient way to describe linking numbers is to calculate them from link diagrams.

In this paper the term link diagram refers to an oriented virtual diagram; for a thorough discussion, we refer to the book of Manturov and Ilyutko [10]. Here is a summary. A link diagram is a subset $D$ of $\mathbb{R}^2$ obtained from the union of a finite number $\mu$ of oriented, piecewise smooth closed curves. The curves must be in general position, i.e., there are only finitely many (self-) intersections among them, and all of these (self-) intersections are transverse double points (crossings). Each crossing is either classical or virtual; if no crossing is virtual, then $D$ is a classical link diagram. The diagram is obtained from the union of the curves by: (a) at each classical crossing, removing a short piece of the underpassing segment on each side of the crossing, and (b) at each virtual crossing, drawing a small circle around the crossing. The result of removing the short pieces near classical crossings is to cut the original curves into arcs; the set of arcs in $D$ is denoted $A(D)$. These arcs are sometimes called the long arcs.
of $D$, to contrast with shorter arcs obtained by cutting also at overpasses, or at virtual crossings.

Two link diagrams are equivalent or of the same type if they are related through a finite sequence of four kinds of moves: detour moves affecting only virtual crossings, and classical Reidemeister moves affecting only classical crossings. Any of these moves provides a natural way to identify the closed curves underlying one diagram with the closed curves underlying the other diagram, so it makes sense to say that an equivalence class of link diagrams represents a $\mu$-component link type $L = K_1 \cup \cdots \cup K_\mu$. Reidemeister [14] showed that equivalence classes of classical diagrams represent link types in $S^3$, and Kuperberg [9] showed that equivalence classes of virtual link diagrams represent link types in thickened surfaces.

Linking numbers are defined using the notion of writhe of a classical crossing, illustrated in Fig. 1.

![Figure 1: Classical crossings of negative and positive writhe.](image)

\[ w(c) = -1 \quad w(c) = 1 \]

We refer to Ricca and Nipoti [15] for the history of the classical notion of linking numbers. The extension to virtual links given in Definition 1 was mentioned by Goussarov, Polyak and Viro [5]. It is a simple exercise to verify that these linking numbers are invariant under Reidemeister moves and detour moves, so they are link type invariants. In the classical case, it is always true that $\ell_{i/j}(K_i, K_j) = \ell_{j/i}(K_j, K_i)$, so we can use the simpler notation $\ell(K_i, K_j) = \ell(K_j, K_i)$. For virtuals, instead, the two linking numbers $\ell_{i/j}(K_i, K_j)$ and $\ell_{j/i}(K_j, K_i)$ are independent of each other.

When Kauffman [9] introduced virtual links, he extended many notions of classical knot theory to them. Among these notions were quandles, which had been introduced to classical knot theory by Joyce [8] and Matveev [11]. We recall the definition.

**Definition 2.** A quandle on a set $Q$ is specified by a binary operation $\triangleright$, with the following properties.
1. For each \( x \in Q \), \( x \circ x = x \).

2. For each \( y \in Q \), the formula \( \beta_y(x) = x \circ y \) defines a bijection \( \beta_y : Q \to Q \).

3. For all \( w, x, y \in Q \), \( (w \circ x) \circ y = (w \circ y) \circ (x \circ y) \).

If \( Q \) and \( Q' \) are quandles, then a function \( f : Q \to Q' \) is a quandle map if \( f(x \circ y) = f(x) \circ f(y) \) \( \forall x, y \in Q \). A bijective quandle map \( Q \to Q' \) is an isomorphism, a bijective quandle map \( Q \to Q \) is an automorphism, and the automorphisms of \( Q \) form a group \( \text{Aut}(Q) \) under function composition. The \( \beta_y \) maps mentioned in part 2 of Definition 2 are the translations of \( Q \). Notice that the third requirement of Definition 2 guarantees that the translations of \( Q \) are automorphisms of \( Q \). The subgroup of \( \text{Aut}(Q) \) generated by the translations is the translation group of \( Q \); we denote it \( \beta(Q) \).

**Figure 2**: The arcs incident at a classical crossing.

**Definition 3.** Let \( D \) be a link diagram representing a link \( L \). The quandle of \( L \) is the quandle generated by \( \{ q_a \mid a \in A(D) \} \), subject to the requirement that at each classical crossing as indicated in Fig. 2, \( q_{a_3} = q_{a_2} \circ q_{a_1} \). We denote this quandle \( Q(L) \).

It is not hard to see that \( Q(L) \) is invariant under Reidemeister moves and detour moves, up to isomorphism, so it provides a link type invariant.

Here is a central definition of the paper.

**Definition 4.** A quandle \( Q \) is translation-commutative if its translation group \( \beta(Q) \) is commutative.

As \( \beta(Q) \) is generated by the translations, \( Q \) is translation-commutative if and only if

\[
(x \circ y) \circ z = (x \circ z) \circ y \quad \forall x, y, z \in Q.
\]

(1)

It follows that translation-commutative quandles constitute a variety of quandles, in the sense discussed by Joyce [8, Sec. 10]. Therefore, every quandle has a canonical translation-commutative quotient, described below.

**Definition 5.** Let \( Q \) be a quandle. Then the translation-commutative quotient of \( Q \) is the quandle defined using generators and relations as follows. There is a generator \( x_q \) for each \( q \in Q \), there is a relation \( x_{pq} = x_p \circ x_q \) for each pair of elements \( p, q \in Q \), and there is a relation \( (x_p \circ x_q) \circ x_r = (x_p \circ x_r) \circ x_q \) for each triple of (not necessarily distinct) elements \( p, q, r \in Q \). We denote this quandle \( Q_{tc} \).
The mapping $q \mapsto x_q$ defines a canonical surjective quandle map $Q \to Q_{tc}$. It is easy to see that if $S \subseteq Q$ generates $Q$, then the image of $S$ generates $Q_{tc}$.

In Sec. 2, we show that translation-commutative quandles are much less complicated than general quandles. A translation-commutative quandle is completely determined by a family of subgroups of a free abelian group, and it is not hard to determine whether or not two families of subgroups determine isomorphic translation-commutative quandles.

If $L$ is a link, then we define $Q_{tc}(L)$ to be the translation-commutative quotient of $Q(L)$, i.e., $Q_{tc}(L) = Q(L)_{tc}$. Our main result is that $Q_{tc}(L)$ is strongly related to the linking numbers in $L$.

**Theorem 6.** Suppose $L$ and $L'$ are links. Then $Q_{tc}(L) \cong Q_{tc}(L')$ if and only if both of the following requirements are satisfied.

(a) $L$ and $L'$ have the same number of components.

(b) The components of $L$ and $L'$ can be indexed so that $L = K_1 \cup \cdots \cup K_\mu$, $L' = K'_1 \cup \cdots \cup K'_\mu$, and there are $\epsilon_1, \ldots, \epsilon_\mu \in \{-1, 1\}$ such that $\ell_{j/i}(K_i, K_j) = \epsilon_i \ell_{j/i}(K'_i, K'_j)$ for all $i \neq j \in \{1, \ldots, \mu\}$.

Notice in particular that if $L$ and $L'$ are any knots (i.e., both have $\mu = 1$), then the requirement regarding linking numbers in $L$ and $L'$ is satisfied vacuously, so Theorem 6 asserts that $Q_{tc}(L) \cong Q_{tc}(L')$. In fact, the translation-commutative quandle of every knot is a trivial one-element quandle.

As mentioned above, every quandle has a canonical translation-commutative quotient. Therefore, Theorem 6 directly implies the following.

**Corollary 7.** Let $\widetilde{Q}(L)$ be any invariant quandle of a link $L$, whose canonical translation-commutative quotient is $Q_{tc}(L)$. If $L$ and $L'$ are links with $\widetilde{Q}(L) \cong \widetilde{Q}(L')$, then $L$ and $L'$ satisfy the two requirements of Theorem 6.

There are at least two well-known invariant quandles that can play the role of $\widetilde{Q}(L)$ in Corollary 7: the full link quandle, $Q(L)$, and the medial quandle, $\text{AbQ}(L)$ in Joyce's notation [8].

In general, the sign changes $\epsilon_1, \ldots, \epsilon_\mu$ in Theorem 6 can vary independently. When $L$ and $L'$ are both classical links, however, the sign changes are tied together: if $\ell(K_i, K_j) \neq 0$, then $\epsilon_i$ must equal $\epsilon_j$. One way to make this interdependence explicit involves the following construction. See Fig. 3 for an example.

**Definition 8.** A link $L = K_1 \cup \cdots \cup K_\mu$ has a linking graph $\ell g(L)$, defined as follows.

- There is a vertex $v_i$ for each component $K_i$ of $L$ such that $\ell_{j/i}(K_i, K_j) \neq 0$ or $\ell_{i/j}(K_i, K_j) \neq 0$ for some $j \neq i \in \{1, \ldots, \mu\}$.

- Two vertices $v_i$ and $v_j$ are adjacent if $i \neq j$ and $\ell_{j/i}(K_i, K_j) \neq 0$ or $\ell_{i/j}(K_i, K_j) \neq 0$.

4
Note that in general, the number of vertices in \( \ell g(L) \) may be less than \( \mu \). (Indeed, \( \ell g(L) \) may be empty.) Also, \( \ell g(L) \) is a simple graph; there are neither loops nor parallel edges.

Recall that a \textit{connected component} of a graph is an equivalence class of vertices under the equivalence relation \( \sim \) generated by \( v_i \sim v_j \) whenever \( v_i \) and \( v_j \) are adjacent. We always use the phrase “connected component” for this notion, to avoid confusion with the components of a link. A graph with only one connected component is \textit{connected}. We consider the empty graph to be connected.

As classical links always have \( \ell_{j/i}(K_i, K_j) = \ell_{i/j}(K_i, K_j) \), Theorem 6 immediately implies the following.

\textbf{Theorem 9.} Let \( L \) and \( L' \) be classical links. Then \( Q_{tc}(L) \cong Q_{tc}(L') \) if and only if both of the following conditions hold.

(a) \( L \) and \( L' \) have the same number of components.

(b) The components of \( L \) and \( L' \) can be indexed in such a way that for each connected component \( C \) of \( \ell g(L) \), there is an \( \epsilon_C \in \{-1, 1\} \) such that \( \ell(K_i, K_j) = \epsilon_C \ell(K'_i, K'_j) \) whenever \( v_i \) is a vertex of \( C \).

In particular, if \( \ell g(L) \) has only one connected component then (b) requires an \( \epsilon \in \{-1, 1\} \) such that \( \ell(K_i, K_j) = \epsilon \ell(K'_i, K'_j) \forall i \neq j \in \{1, \ldots, \mu\} \).

In the classical case, Theorems 6 and 9 are the strongest results one could hope for. Even the full quandle \( Q(L) \) of a classical link – a very sensitive link invariant – does not detect linking numbers absolutely. The reason is that \( Q(L) \) is invariant under the combination of reversing the orientations of all the components of \( L \), and replacing \( L \) with its mirror image. Moreover, for a split link \( L \), \( Q(L) \) is not changed if the component orientations are reversed in just one split portion of \( L \), and that portion is replaced with its mirror image. The effect of replacing a split portion with the orientation-reversed mirror image is to multiply all linking numbers from the connected component(s) of \( \ell g(L) \) corresponding to that portion of the link by \(-1\).
Here is an outline of the paper. In Sec. 2 we provide a complete structure theory for translation-commutative quandles, and in Sec. 3 we use this structure theory to prove Theorem 6. In Sec. 4 we relate \( Q_{tc}(L) \) to the nilpotent quotient \( G(L)/G(L)_3 \) of the group of \( L \). In Sec. 5 we briefly discuss the connection between our results and those of Harrell and Nelson \[6\], who related linking numbers of two-component links to quandle counting invariants.

2 Translation-commutative quandles

In this section, we give a structure theory for translation-commutative quandles. If \( B \) is a nonempty set, we use \( \mathbb{Z}_B \) to denote the free abelian group on \( B \).

Definition 10. Let \( S = \{S_b \mid b \in B\} \) be a family of subgroups of \( \mathbb{Z}_B \), with \( b \in S_b \) \( \forall b \in B \). For each \( b \in B \), let \( A_b = \mathbb{Z}_B / S_b \). Then \( Q(S) \) is the disjoint union

\[
Q(S) = \bigcup_{b \in B} A_b,
\]

equipped with the operation \( \triangleright \) defined as follows: if \( x \in A_b \) and \( y \in A_c \), then

\[
x \triangleright y = x + (c + S_b) \in A_b.
\]

As a minor abuse of notation we often write \( x \triangleright y = x + c + S_b \), even though \( x + c \) is not well defined.

Proposition 11. \( Q(S) \) is a translation-commutative quandle.

Proof. If \( x \in A_b \) then as \( b \in S_b \),

\[
x \triangleright x = x + b + S_b = x + 0 + S_b = x.
\]

If \( y \in A_c \) then for each \( b \in B \), the map \( A_b \to A_b \) defined by \( x \mapsto x \triangleright y \) is a bijection, with inverse function given by \( x \mapsto x - (c + S_b) \) \( \in A_b \).

If \( x \in A_b \), \( y \in A_c \) and \( z \in A_d \), then \( y \triangleright z \in A_c \), so

\[
(x \triangleright y) \triangleright z = x + c + d + S_b = x + d + c + S_b
\]

\[
= (x + d + S_b) \triangleright (y \triangleright z) = (x \triangleright z) \triangleright (y \triangleright z).
\]

Also, formula (1) is satisfied because

\[
(x \triangleright y) \triangleright z = x + c + d + S_b = x + d + c + S_b = (x \triangleright z) \triangleright y.
\]

Recall that if \( x \) is an element of a quandle \( Q \), then the orbit of \( x \) in \( Q \) is the smallest subset \( Q_x \subseteq Q \) such that \( x \in Q_x \) and \( \beta^z_y = 1 \) \( \in Q_x \) \( \forall y \in Q \) \( \forall z \in Q_x \).

It is easy to see that for a quandle of type \( Q(S) \), the orbits are the subsets \( A_b \).

Also, Definition 10 implies that if \( b \in B \) and \( y, z \in A_b \), then \( \beta_y = \beta_z \). We use \( \beta_b \) to denote the map \( \beta_y \) for all \( y \in A_b \).
Lemma 12. Suppose \( n \in \mathbb{N}, b_1, \ldots, b_n \in B, \) and \( m_1, \ldots, m_n \in \mathbb{Z}. \) Let

\[
y = \sum_{i=1}^{n} m_i b_i \in \mathbb{Z}_B,
\]

and let

\[
\beta_y = \prod_{i=1}^{n} \beta_{b_i}^{m_i} \in \beta(Q(S)).
\]

Then for each \( b \in B, \) the following are equivalent to each other.

1. \( y \in S_b. \)
2. For some \( x \in A_b, \) \( \beta_y(x) = x. \)
3. For all \( x \in A_b, \) \( \beta_y(x) = x. \)

Proof. According to Definition 10 if \( x \in A_b \) then

\[
\beta_y(x) = \beta_{b_1}^{m_1} \cdots \beta_{b_n}^{m_n}(x) = x + y + S_b.
\]

Therefore, \( \beta_y(x) = x \) if and only if \( y + S_b = S_b. \)

If \( f : B \to B' \) is a function of sets, then we also use \( f \) to denote the homomorphism \( \mathbb{Z}_B \to \mathbb{Z}_{B'}, \) defined by \( f. \)

Proposition 13. Suppose \( B \) and \( B' \) are nonempty sets, \( \{S_b \mid b \in B\} \) is a family of subgroups of \( \mathbb{Z}_B \) with \( b \in S_b \) \( \forall b \in B, \) and \( \{S'_b \mid b' \in B'\} \) is a family of subgroups of \( \mathbb{Z}_{B'}, \) with \( b' \in S'_b \) \( \forall b' \in B'. \) Then \( Q(S) \equiv Q(S') \) if and only if there is a bijection \( f : B \to B' \) with \( f(S_b) = S'_f(b) \) \( \forall b \in B. \)

Proof. We modify notation used for \( Q(S) \) by using apostrophes when discussing \( Q(S'). \) For instance, if \( b' \in B' \) then \( A'_{b'} = \mathbb{Z}_{B'}/S'_{b'}. \) If there is a bijection \( f \) as described, then for each \( b \in B, \) \( f \) defines a group isomorphism \( A_b \to A'_{f(b)}. \)

These group isomorphisms define a quandle isomorphism \( Q(S) \to Q(S'). \)

For the converse, suppose \( g : Q(S) \to Q(S') \) is a quandle isomorphism. As an isomorphism, \( g \) must define a bijection between the orbits of \( Q(S) \) and the orbits of \( Q(S'), \) so \( g \) must define a bijection \( f : B \to B' \) with \( g(A_b) = A'_{f(b)} \) \( \forall b \in B. \)

As \( g \) is a quandle isomorphism, \( g(x \triangleright y) = g(x) \triangleright g(y) \) \( \forall x, y \in Q(S). \) It follows that \( g \circ \beta_{b_1}^{m_1} \cdots \beta_{b_n}^{m_n} = (\beta'_{f(b_1)})^{m_1} \cdots (\beta'_{f(b_n)})^{m_n} \circ g \)

whenever \( n \in \mathbb{N}, b_1, \ldots, b_n \in B \) and \( m_1, \ldots, m_n \in \mathbb{Z}. \) Now, suppose \( b \in B. \) According to 2, if \( x \in A_b, \) \( b_1, \ldots, b_n \in B \) and \( m_1, \ldots, m_n \in \mathbb{Z} \) then \( \beta_{b_1}^{m_1} \cdots \beta_{b_n}^{m_n}(x) = x \) if and only if \( (\beta'_{f(b_1)})^{m_1} \cdots (\beta'_{f(b_n)})^{m_n}(g(x)) = g(x). \) We deduce from Lemma 12 that \( \sum m_i b_i \in S_b \) if and only if \( \sum m_i f(b_i) \in S'_{f(b)}. \)

That is, \( f(S_b) = S'_{f(b)}. \)

\( \square \)
A special case of Proposition [13] will be particularly useful. Observe that if $C \subset B$ then there is a natural way to consider $\mathbb{Z}_C$ as a subgroup of $\mathbb{Z}_B$: simply identify each element $\sum n_ic_i \in \mathbb{Z}_C$ with the element $\sum n_ic_i \in \mathbb{Z}_B$ given by the same linear combination of generators.

**Corollary 14.** Suppose $B$ and $B'$ are nonempty sets, $s_b \in \mathbb{Z}_{B-b} \forall b \in B$, and $s'_b \in \mathbb{Z}_{B'-b'} \forall b' \in B'$. For each $b \in B$, let $S_b$ be the subgroup of $\mathbb{Z}_B$ generated by $\{b, s_b\}$, and for each $b' \in B'$, let $S'_{b'}$ be the subgroup of $\mathbb{Z}_{B'}$ generated by $\{b', s'_b\}$. Then $Q(S) \cong Q(S')$ if and only if there is a bijection $f : B \to B'$ with $f(s_b) = \pm s'_f(b) \forall b \in B$.

**Proof.** If $b \in B$ and $x, y \in \mathbb{Z}_{B-b}$, then $\{b, x\}$ and $\{b, y\}$ generate the same subgroup of $\mathbb{Z}_B$ if and only if $x = \pm y$. The corollary follows from this property and Proposition [13].

The next result will help us show that every translation-commutative quandle is isomorphic to a quandle of type $Q(S)$.

**Lemma 15.** Suppose $Q$ is a translation-commutative quandle, and $y \in Q$. Then $\beta_y = \beta_y \forall z \in Q_Q$.

**Proof.** If $y \in Q$ then according to Definition [2] for any $w, x \in Q$,

$$\beta_{\beta_y(x)}(w) = w \triangledown (y \triangledown x) = (\beta^{-1}_x(w) \triangledown y) \triangledown y = (\beta^{-1}_x(w) \triangledown y) \triangledown x.$$

According to formula [1], it follows that for every $w \in Q$,

$$\beta_{\beta_y(x)}(w) = (\beta^{-1}_x(y) \triangledown x) \triangledown y = w \triangledown y = \beta_y(w).$$

Hence $\beta_{\beta_y(x)} = \beta_y$.

**Theorem 16.** Let $Q$ be a translation-commutative quandle, and suppose there is a bijection $b \leftrightarrow Q_b$ between a set $B$ and the set of orbits of $Q$. Then there is a family $S = \{S_b \mid b \in B\}$ of subgroups of $\mathbb{Z}_B$ such that $Q \cong Q(S)$.

**Proof.** For each $b \in B$, choose a representative element $q_b \in Q_b$, and let $\beta_b = \beta_{q_b}$. According to Lemma [15], $\{\beta_b \mid b \in B\}$ includes all the $\beta$ maps of $Q$, so $\beta(Q)$ is an abelian group generated by $\{\beta_b \mid b \in B\}$. For each $b \in B$, then, there is a surjective function $f_b : \mathbb{Z}_B \to Q_b$ given by

$$f_b\left(\sum_{i=1}^{n} m_i b_i\right) = \left(\prod_{i=1}^{n} \beta_{b_i}^{m_i}\right)(q_b)$$

whenever $n \geq 0 \in \mathbb{Z}$, $b_1, \ldots, b_n \in B$ and $m_1, \ldots, m_n \in \mathbb{Z}$.

For each $b \in B$, let $S_b = \{\sum m_i b_i \in \mathbb{Z}_B \mid f_b(\sum m_i b_i) = q_b\}$. The fact that $Q$ is translation-commutative implies that $S_b$ is a subgroup of $\mathbb{Z}_B$, and $f_b$ induces a bijection between $\mathbb{Z}_B/S_b$ and $Q_b$. Also, the fact that $\beta_b(q_b) = \beta_{b_b}(q_b) = q_b$ implies that $b \in S_b$. It follows that taken together, the surjections $f_b$ induce an isomorphism $Q(S) \to Q$. 

\[ \square \]
3 Theorems 6 and 9

Let $D$ be a diagram of an oriented link $L = K_1 \sqcup \cdots \sqcup K_\mu$. The image in $D$ of each component $K_i$ consists of finitely many arcs, separated at classical crossings where $K_i$ is the underpassing component. For each $i$, choose a fixed arc $a_{i0}$ belonging to the image of $K_i$ in $D$.

If $a \in A(D)$ then for convenience, we use $x_a$ to denote the element $x_{qa} \in Q_{tc}(L)$.

Lemma 17. The quandle $Q_{tc}(L)$ has $\mu$ orbits. For each $i \in \{1, \ldots, \mu\}$, there is an orbit that contains all the elements $x_a$ such that $a \in A(D)$ belongs to the image of $K_i$ in $D$. This orbit does not contain any element $x_a$ such that $a \in A(D)$ belong to the image of $K_j$ in $D$, where $j \neq i$.

Proof. Consider a crossing of $D$ as pictured in Fig. 2. Definitions 3 and 5 tell us that $x_{a_3} = x_{a_2} \triangleright x_{a_1}$, so $x_{a_2}$ and $x_{a_3}$ belong to the same orbit of $Q_{tc}(L)$.

Now, choose any $i \in \{1, \ldots, \mu\}$. If we walk along the image of $K_i$ in $D$, starting at $a_{i0}$, then we ultimately encounter every arc of $D$ belonging to $K_i$. The observation of the previous paragraph applies each time we pass from one arc of $D$ to a different arc of $D$, so the $x_a$ elements corresponding to arcs of $K_i$ all belong to the same orbit of $Q_{tc}(L)$.

To show that these orbits are distinct, consider the trivial quotient quandle $Q_0$ of $Q_{tc}(L)$, i.e., the quandle obtained by adding $x_p \triangleright x_q = x_p \ \forall p, q \in Q$ to the requirements of Definition 5 with $Q = Q(L)$. It is clear that the definition of this quotient quandle is satisfied by the set $\{1, \ldots, \mu\}$, considered as a trivial quandle, with all $\beta$ maps equal to the identity map. The fact that there is a well-defined quotient map $Q_{tc}(L) \to Q_0$ guarantees that $Q_{tc}(L)$ has $\mu$ different orbits.

For each $i \in \{1, \ldots, \mu\}$, let $\beta_i = \beta_{x_{a_{i0}}}: Q_{tc}(L) \to Q_{tc}(L)$. According to Lemma 15, $\beta_1, \ldots, \beta_\mu$ are all the $\beta$ maps of $Q_{tc}(L)$. It follows that $Q_{tc}(L)$ can be described like this:

Proposition 18. $Q_{tc}(L)$ is the translation-commutative quandle generated by $\{x_a \mid a \in A(D)\}$, subject to these requirements.

- $Q_{tc}(L)$ has $\beta$ maps $\beta_1, \ldots, \beta_\mu$, corresponding to the components of $L$.

- At each classical crossing as indicated in Fig. 3 if the overpassing arc $a_1$ belongs to the image of $K_n$ in $D$, then $x_{a_3} = \beta_n(x_{a_3})$.

To describe $Q_{tc}(L)$ in greater detail, it is convenient to index the arcs and classical crossings of $D$. Suppose that for each $i \in \{1, \ldots, \mu\}$, $n_i$ is the number of arcs of $D$ belonging to $K_i$. We index these arcs $a_{i0}, a_{i1}, \ldots, a_{i(n_i-1)}, a_{in_i} = a_{i0}$ in order, as we walk along $K_i$ in the direction of its orientation. If $0 \leq m < n_i$, let $c_{im}$ be the classical crossing we traverse as we walk from $a_{im}$ to $a_{i(m+1)}$, let $K_{x(c_{im})}$ be the overpassing component of $L$ at $c_{im}$, and let $w(c_{im}) \in \{-1, 1\}$ be the writhe of $c_{im}$. See Fig. 4.
Proposition 18 implies that if \( w(c_{im}) = -1 \) then \( x_{a_{im}} = \beta_{\kappa(c_{im})}(x_{a_{(m+1)}}) \), and if \( w(c_{im}) = 1 \) then \( x_{a_{i(m+1)}} = \beta_{\kappa(c_{im})}(x_{a_{im}}) \). In either case, \( x_{a_{i(m+1)}} = \beta_{\kappa(c_{im})}(x_{a_{im}}) \). Therefore if \( 1 \leq i \leq \mu \) and \( 0 \leq m \leq n_i - 1 \),

\[
x_{a_{i(m+1)}} = \left( \prod_{k=0}^{m} \beta_{\kappa(c_{ik})}^{w(c_{ik})} \right) (x_{a_{i0}}).
\] (3)

An instance of (3) with \( 0 \leq m \leq n_i - 2 \) allows us to express the quandle element \( x_{a_{i(m+1)}} \) in terms of \( x_{a_{i0}}, \ldots, x_{a_{i0}} \). After we remove all of these redundant \( x_{a_{i(m+1)}} \) generators, only \( x_{a_{i0}}, \ldots, x_{a_{i0}} \) remain. We rename these elements \( x_1, \ldots, x_\mu \), for simplicity.

The instances of (3) that have not been used to remove redundant generators are the ones with \( m = n_i - 1 \). They tell us that for \( 1 \leq i \leq \mu \),

\[
x_i = \left( \prod_{j=1}^{\mu} \beta_{j/i}^{\ell_{j/i}(K_i, K_j)} \right) (x_{i0}).
\] (4)

We deduce yet another equivalent description of \( Q_{tc}(L) \):

**Proposition 19.** \( Q_{tc}(L) \) is the translation-commutative quandle generated by \( \{x_1, \ldots, x_\mu\} \), subject to the requirement that (4) holds for every \( i \in \{1, \ldots, \mu\} \).

Let \( B = \{b_1, \ldots, b_\mu\} \), and for each \( i \in \{1, \ldots, \mu\} \) let \( S_i(L) = S_{b_i}(L) \) be the subgroup of \( \mathbb{Z}_B \) generated by \( b_i \) and \( \ell_i \), where

\[
\ell_i = \sum_{j=1}^{\mu} \ell_{j/i}(K_i, K_j)b_j.
\] (5)

Let \( S(L) = \{S_{b_i}(L) \mid 1 \leq i \leq \mu\} \). Then Proposition 19 implies the following.

**Corollary 20.** The quandles \( Q_{tc}(L) \) and \( Q(S(L)) \) are isomorphic.

Theorem 6 follows from Corollaries 14 and 20. As discussed in the introduction, Theorem 9 follows directly from Theorem 6 and the fact that for a classical link, \( \ell_{j/i}(K_i, K_j) = \ell_{i/j}(K_i, K_j) \forall i \neq j \in \{1, \ldots, \mu\} \).
4 The quotient $G(L)/G(L)_3$

In this section we connect the ideas of the preceding sections to a theorem of Chen [3] involving the nilpotent quotient $G(L)/G(L)_3$, where $G(L) = \pi_1(S^3 - L)$ is the fundamental group of the complement of a classical link. We begin with a construction that dates back to the introduction of quandles to knot theory by Joyce [8] and Matveev [11].

**Definition 21.** If $Q$ is a quandle, then there is an associated group $\text{As}(Q)$, described using generators and relations as follows. There is a generator $g_q$ for each $q \in Q$, and for each pair $p, q \in Q$ there is a relation $g_{pq} = g_q g_p g_q^{-1}$.

We should mention that Joyce [8] used the notation $\text{Adconj}(Q)$ instead of $\text{As}(Q)$. Also, his quandle is the “opposite” of ours, i.e., it is defined by $g_{pq} = g_q^{-1} g_p g_q$ instead of $g_{pq} = g_q g_p g_q^{-1}$. We use the convention of Definition 21 so that if $L$ is a link with a diagram $D$, then the presentation of $\text{As}(G(L))$ provided by Definitions 3 and 21 is the same as the presentation of the group $G(L)$ given by Kauffman [9]. In the classical case, the same presentation was given by Fox [4].

Recall that if $G$ is a group and $H$ is a subgroup of $G$, then $[G, H]$ is the subgroup of $G$ generated by the set of commutators $[g, h] = ghg^{-1}h^{-1}$ with $g \in G$ and $h \in H$. The lower central series of $G$ is the sequence of normal subgroups $G = G_1 \supseteq G_2 \supseteq \ldots$ with $G_{n+1} = [G, G_n]$ for each $n \geq 2$, and $G$ is nilpotent of class $n$ if $G_n \neq \{1\} = G_{n+1}$. In particular, nontrivial abelian groups are nilpotent of class 1.

**Proposition 22.** Let $Q$ be a quandle. Then the quandle surjection $Q \rightarrow Q_{1c}$ induces a group surjection $\text{As}(Q) \rightarrow \text{As}(Q_{1c})$, whose kernel is the lower central series subgroup $\text{As}(Q)_3$.

**Proof.** The kernel of the induced surjection $\text{As}(Q) \rightarrow \text{As}(Q_{1c})$ is the normal subgroup of $\text{As}(Q)$ generated by the images of relators derived from the quandle relations $(x_p \triangleright x_q) \triangleright x_r = (x_p \triangleright x_r) \triangleright x_q$ for $p, q, r \in Q$.

The quandle relation $(x_p \triangleright x_q) \triangleright x_r = (x_p \triangleright x_r) \triangleright x_q$ provides the group relation $g_{x_p} g_{x_q} g_{x_p}^{-1} g_{x_q}^{-1} = g_{x_q} g_{x_r} g_{x_p}^{-1} g_{x_q}^{-1}$, which corresponds to the relator $g_{x_p} g_{x_q} g_{x_p}^{-1} g_{x_q}^{-1} g_{x_r} g_{x_p}^{-1} g_{x_q}^{-1} g_{x_r} g_{x_q}^{-1} g_{x_r}^{-1} g_{x_q}^{-1}$. Conjugating by $g_{x_p}^{-1} g_{x_q}^{-1}$, we see that this relator is equivalent to

$g_{x_p}^{-1} g_{x_q}^{-1} g_{x_p} g_{x_q} g_{x_p}^{-1} g_{x_q}^{-1} g_{x_r} g_{x_p}^{-1} g_{x_q}^{-1} g_{x_r} g_{x_q}^{-1} = [g_{x_p}^{-1}, [g_{x_r}, g_{x_q}^{-1}]]$.

As $\text{As}(Q)$ is generated by the inverses of the elements $g_x$ for $y \in Q$, the commutators $[g_{x_p}^{-1}, [g_{x_r}, g_{x_q}^{-1}]]$ generate $\text{As}(Q)_3$ as a normal subgroup of $\text{As}(Q)$.

**Corollary 23.** Let $L$ be a link with group $G(L)$. Then the nilpotent quotient group $G(L)/G(L)_3$ has a presentation with a generator $g_i$ for each $i \in \{1, \ldots, \mu\}$ and two kinds of relations.
1. If $i \in \{1, \ldots, \mu\}$, there is a relation

$$g_i = \left( \prod_{j=1}^{\mu} g_j^{\ell_{j/i}(K_i, K_j)} \right) g_i \left( \prod_{j=1}^{\mu} g_j^{\ell_{j/i}(K_i, K_j)} \right)^{-1}.$$ 

2. If $i, j, k \in \{1, \ldots, \mu\}$, there is a relation $[g_i, [g_j, g_k]] = 1$.

Proof. Proposition 22 implies that the nilpotent quotient $G(L)/G(L)_3$ is isomorphic to $\mathbb{A}_3(Q_{tc}(L))$. According to Proposition 19, the quandle $Q_{tc}(L)$ is determined by two properties: the quandle relations in (4) and the fact that $Q_{tc}(L)$ is translation-commutative. Using the notation $g_i = g_{x_i}$ for $1 \leq i \leq \mu$, the quandle relations in (4) provide the first kind of group relation mentioned in the statement of the corollary, and as noted in the proof of Proposition 22, the quandle relations in (1) provide group relations of the form $[g_i^{-1}, [g_j^{-1}, g_k^{-1}]] = 1$. The $-1$ exponents in these commutators are not important, because elements of the form $[g_i^{-1}, [g_j^{-1}, g_k^{-1}]]$ generate the same normal subgroup of the free group on $\{g_1, \ldots, g_\mu\}$ as elements of the form $[g_i, [g_j, g_k]]$.

For classical links with $\mu = 2$, the first kind of relation in Corollary 23 was mentioned by Reidemeister in his famous monograph [14, p. 45]. Reidemeister’s observation was extended to classical links of arbitrarily many components by Chen [1], who studied the lower central series of both $G(L)$ and the metabelian quotient $G(L)/G(L)^{\prime\prime}$; the lower central series quotients of $G(L)/G(L)^{\prime\prime}$ are called Chen groups in his honor. Chen provided presentations of the groups $G(L)/G(L)_k$ for all $k \geq 3$. Milnor [12] simplified Chen’s presentations, and used them to define the $\mu$-invariants. Only $G(L)/G(L)_3$ is important here, so this paper is not the right place to survey the extensive literature regarding these ideas. We refer to Hillman [7] and Papadima and Suciu [13] for more general discussions, and further references to the classical theory. Recently, Chrisman [2, Sec. 4] has extended the Chen-Milnor theory of nilpotent quotients and $\mu$-invariants to virtual links; although he does not single out $G(L)/G(L)_3$ for special attention, his results imply Corollary 23.

Sakuma and Traldi [16] used Chen’s presentation of $G(L)/G(L)_3$ to prove Theorem 25 below. Recall that an articulation point in a graph is a vertex whose removal results in a graph with strictly more connected components.

Definition 24. Let $L = K_1 \cup \cdots \cup K_\mu$ be a link, and let $\ell g(L)$ be the linking graph defined in Definition 8. Then $L$ has inseparable linking numbers if $\ell g(L)$ is connected, and has no articulation point.

For example, the link pictured in Fig. 3 of the introduction has inseparable linking numbers. No three-component sublink of this link has inseparable linking numbers, and every one- or two-component sublink has inseparable linking numbers. (In fact, every one- or two-component link has inseparable linking numbers.)
Theorem 25. ([16]) Let \( L = K_1 \cup \cdots \cup K_\mu \) and \( L = K'_1 \cup \cdots \cup K'_\mu \) be \( \mu \)-component classical links. Then these two properties are equivalent.

1. There is an isomorphism between \( G(L)/G(L)_3 \) and \( G(L')/G(L')_3 \), which is “meridian-preserving” in the sense that for each \( i \in \{1, \ldots, \mu\} \), the image of a Wirtinger generator of \( G(L)/G(L)_3 \) corresponding to \( K_i \) is a conjugate of a Wirtinger generator of \( G(L')/G(L')_3 \) corresponding to \( K'_i \).

2. Whenever \( K_{i_1} \cup \cdots \cup K_{i_\nu} \subseteq L \) is a sublink with inseparable linking numbers, either \( \ell(K_{i_j}, K_{i_k}) = \ell(K'_{i_j}, K'_{i_k}) \) \( \forall j \neq k \in \{1, \ldots, \nu\} \) or \( \ell(K_{i_j}, K_{i_k}) = -\ell(K'_{i_j}, K'_{i_k}) \) \( \forall j \neq k \in \{1, \ldots, \nu\} \).

The notion of “meridian-preserving isomorphism” used in Theorem 25 can be described using quandles. There are natural maps \( \eta : Q_{tc}(L) \to \text{As}(Q_{tc}(L)) \) and \( \eta' : Q_{tc}(L') \to \text{As}(Q_{tc}(L')) \), given by \( \eta(x) = g_x \) \( \forall x \in Q_{tc}(L) \) and \( \eta'(x') = g_{x'} \) \( \forall x' \in Q_{tc}(L') \). An isomorphism \( f : G(L)/G(L)_3 \to G(L')/G(L')_3 \) is meridian-preserving if and only if for each \( i \in \{1, \ldots, \mu\} \), \( f(\eta(x_i)) = \eta'(x'_i) \) for some element \( x'_i \) of the orbit of \( x_i \) in \( Q_{tc}(L') \).

Theorems 9 and 25 imply that \( Q_{tc}(L) \) (up to quandle isomorphisms) is a stronger invariant of classical links than \( G(L)/G(L)_3 \) (up to re-indexings of \( K_1, \ldots, K_\mu \) and meridian-preserving group isomorphisms). The reason is that Theorem 9 involves entire connected components of \( \ell g(L) \), while Theorem 25 involves only connected subgraphs of \( \ell g(L) \) that have no articulation points. These subgraphs can be much smaller than connected components.

To illustrate this point, choose \( \mu \geq 3 \), and consider the oriented versions of the \( \mu \)-component link pictured in Fig. 5. There are as many different oriented link types as there are lists of \( \mu - 1 \) linking numbers all equal to \( \pm 1 \), with a list \( \ell(K_1, K_2), \ldots, \ell(K_{\mu - 1}, K_\mu) \) considered equivalent to the reversed list \( \ell(K_{\mu - 1}, K_\mu), \ldots, \ell(K_1, K_2) \). It follows that there are always at least

\[
2 + (2^{\mu - 1} - 2)/2 = 2^{\mu - 2} + 1
\]

oriented link types, because there are always at least two lists that remain the same when reversed, namely \( 1, \ldots, 1 \) and \(-1, \ldots, -1\). For each of these links, the entire graph \( \ell g(L) \) is connected, but the only nontrivial connected subgraphs of \( \ell g(L) \) without articulation points are single edges. Theorem 9 tells us that the \( Q_{tc} \) quandles corresponding to two lists of linking numbers are isomorphic.

![Figure 5: A connected sum of \( \mu - 1 \) copies of the Hopf link.](image-url)
if and only if the components of the two links can be indexed so that the linking numbers differ only by a common factor of \( \pm 1 \). Therefore, there are always at least
\[
[(2^{n-2} + 1)/2] = 2^{n-3} + 1
\]
onisomorphic \( Q_{tc} \) quandles among these links. In contrast, Theorem 25 tells us that the nilpotent quotient groups \( G(L)/G(L)_3 \) are all isomorphic to each other, through meridian-preserving isomorphisms. In fact, even the \( G(L) \) groups are all isomorphic through meridian-preserving isomorphisms:

**Proposition 26.** The groups of all oriented versions of the link pictured in Fig. 9 are isomorphic to each other, through meridian-preserving isomorphisms.

**Proof.** Let \( L \) be one of these oriented links. Index the components of \( L \) as \( K_1, \ldots, K_\mu \) in order from left to right, and index the arcs of the diagram as in Fig. 5. For convenience, we use \( g_a \) to denote the element of \( G(L) = \text{As}(Q(L)) \) corresponding to an arc \( a \in A(D) \), rather than \( g_{a_1} \).

The two crossings involving \( K_1 \) and \( K_2 \) are on the far left. These two crossings provide two relations in \( G(L) \): one relation is \( g_{b_1}g_{a_1}g_{b_2}^{-1} = g_{a_1} \), and the other is either \( g_{a_1}g_{b_1}g_{a_1}^{-1} = g_{b_2} \) or \( g_{a_1}g_{b_2}g_{a_1}^{-1} = g_{b_2} \), depending on orientations. The first relation tells us that \( g_{a_1} \) and \( g_{b_1} \) commute with each other. Then either of the possible second relations tells us that \( g_{a_2} = g_{b_2}^{-1} \). With this equality in mind, one of the next two crossings tells us that \( g_{b_2}g_{a_2}g_{b_3} = g_{a_2} \), so \( g_{a_2} \) and \( g_{b_3} \) commute. The other crossing between \( K_2 \) and \( K_3 \) then provides a relation that guarantees \( g_{a_3} = g_{b_3} \). With this equality in mind, the relations provided by the next two crossings tell us that \( g_{a_3} \) and \( g_{b_3} \) commute, and \( g_{a_4} = g_{b_4} \).

Continuing in this vein, we ultimately conclude that no matter how the link components are oriented, \( G(L) \) is generated by \( g_{a_1}, \ldots, g_{a_\mu} \), subject to the relations
\[
g_{a_i}g_{a_{i+1}} = g_{a_{i+1}}g_{a_i}, \forall i \in \{1, \ldots, \mu - 1\}.
\]

## 5 A theorem of Harrell and Nelson

Generally speaking, the absence of a structure theory for arbitrary quandles makes it difficult to work with them. One way to deal with this difficulty is to study special varieties of quandles, which can be described more easily. For instance, knot theorists have considered Alexander, involutory, latin, medial and \( n \)-quandles; see [3] for a survey. The translation-commutative quandles we have discussed in this paper constitute a subvariety of the medial quandles.

Another way to deal with the intractability of quandles is to consider numerical invariants derived from them. One way to derive such invariants of a link \( L \) is to count the quandle maps from \( Q(L) \) into specified target quandles. In [6], Harrell and Nelson observed that if \( L = K_1 \cup K_2 \) is a two-component link, then there is a connection between the linking numbers \( \ell_{1/2}(K_1, K_2), \ell_{2/1}(K_1, K_2) \) and the number of quandle maps from \( Q(L) \) into quandles \( X_n \) of a particular type.

In the notation of Sec. 2, the quandle \( X_n \) may be defined as follows. Suppose \( n \geq 1 \in \mathbb{Z} \), let \( B = \{b_1, b_2\} \), and let \( S = \{S_{b_1}, S_{b_2}\} \), where \( S_{b_1} = \mathbb{Z}_B \) and \( S_{b_2} \).
is the subgroup of $\mathbb{Z}_B$ generated by $b_1$ and $nb_2$. Then $X_n = Q(S)$. Notice that $X_n$ is a translation-commutative quandle with a very simple structure: the first orbit has only one element, the second orbit has $n$ elements, $\beta_{b_1}$ cyclically permutes the elements of the second orbit, and $\beta_{b_2}$ is the identity map. Harrell and Nelson [6] proved the following.

**Proposition 27.** (6) Suppose $n \geq 2 \in \mathbb{Z}$. Let $L$ be a two-component link, and let $k$ be the number of quandle maps $Q(L) \rightarrow X_n$.

1. $k = n^2 + 1$ if $n$ divides neither $\ell_{1/2}(K_1, K_2)$ nor $\ell_{2/1}(K_1, K_2)$.
2. $k = n^2 + 1 + n$ if $n$ divides precisely one of $\ell_{1/2}(K_1, K_2)$, $\ell_{2/1}(K_1, K_2)$.
3. $k = n^2 + 1 + 2n$ if $n$ divides both $\ell_{1/2}(K_1, K_2)$ and $\ell_{2/1}(K_1, K_2)$.

As $X_n$ is translation-commutative, every quandle map $Q(L) \rightarrow X_n$ factors through the canonical surjection $Q(L) \rightarrow Q_{\text{tc}}(L)$. Using this fact, it is not difficult to deduce Proposition 27 from Corollary 20. For every two-component link $L$, there are $n^2 + 1$ non-surjective quandle maps $Q_{\text{tc}}(L) \rightarrow X_n$, which are constant on each orbit of $Q_{\text{tc}}(L)$. Moreover, if $i \neq j \in \{1, 2\}$ and $n$ divides $\ell_{j/i}(K_1, K_2)$, then there are $n$ surjective quandle maps $Q_{\text{tc}}(L) \rightarrow X_n$, which map $A_i = \mathbb{Z}_B/S_i(L)$ onto the $n$-element orbit of $X_n$ and map $A_j$ onto the one-element orbit of $X_n$.

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