Lower Bounds for the Distribution of Suprema of Brownian Increments and Brownian Motion Normalized by the Corresponding Modulus Functions

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Abstract  The Lévy-Ciesielski Construction of Brownian motion is used to determine non-asymptotic estimates for the maximal deviation of increments of a Brownian motion process \((W_t)_{t \in [0,T]}\) normalized by the global modulus function, for all positive \(\varepsilon\) and \(\delta\). Additionally, uniform results over \(\delta\) are obtained. Using the same method, non-asymptotic estimates for the distribution function for the standard Brownian motion normalized by its local modulus of continuity are obtained. Similar results for the truncated Brownian motion are provided and play a crucial role in establishing the results for the standard Brownian motion case.

Key words  Brownian motion; global and local moduli of continuity of Brownian motion; Lévy-Ciesielski construction of Brownian motion; law of the iterated logarithm.

1 Introduction

We present a unified method for establishing both local and global moduli of continuity for a Brownian motion process, \((W_t)_{t \geq 0}\). A most useful consequence of our process allows for explicit estimates which will be explained in more detail below. We briefly recall some basic properties of the Lévy-Ciesielski construction in section two. This construction compared to others is used most often to establish the continuity of sample paths; however, not until recently has it been exploited to show other properties. For example, J.P. Kahane in 1985 applied the orthonormal expansion to study slow and fast points of the Brownian motion process [5]. M. Pinsky offers
a simple proof of the existence of the modulus of continuity based on the Lévy-Ciesielski construction [9].

In section three, we exploit the Lévy-Ciesielski construction and the piecewise-linear truncated process \((W_t^n)_{t \in [0,1]}\) (the following section contains an explicit description of this process) over dyadic intervals to establish several results regarding the global modulus of continuity for Brownian motion.

Specifically for every \(\epsilon > 0\) and for every \(\delta > 0\), we determine an estimate for the maximal deviation of increments of a Brownian motion process, \(W_t\), normalized by the global modulus function. More explicitly, we have determined functions \(k\) and \(p_1\) so that for every \(\epsilon > 0\) and every \(\delta > 0\)

\[
P \left( \sup_{0 \leq s < t \leq 1} \frac{|W_t - W_s|}{g(\delta)} \leq 1 + k(\epsilon, \delta) \right) \geq 1 - p_1(\epsilon, \delta),
\]

where \(g(x)\) is the global modulus of continuity for Brownian motion, \(\sqrt{2xL^2}\). More significantly, we also establish uniform results over \(\delta\) for the global modulus of continuity. Specifically, we determine \(p_2\), so that for all \(\epsilon > 0\) and every \(\delta_0 > 0\)

\[
P \left( \sup_{\delta \leq \delta_0} \sup_{0 \leq s < t \leq 1} \frac{|W_t - W_s|}{g(\delta)} \leq 1 + k(\epsilon, \delta_0) \right) \geq 1 - p_2(\epsilon, \delta_0).
\]

Surprisingly, \(p_2\) is similar to \(p_1\) but with larger constants when \(\epsilon\) is bounded away from 0. It should not be a surprise that to establish the uniform result was computationally challenging to say the least.

Using a similar method, in section four, we establish the local modulus of continuity for the standard Brownian motion, with the same type of gains. That is, we construct functions \(l\) and \(q\) so that

\[
P \left( \sup_{\delta \leq \delta_0} \sup_{t \leq \delta} \frac{W_t}{h(t)} \leq 1 + l(\epsilon, \delta) \right) \geq 1 - q(\epsilon, \delta)
\]

where \(h(x) = \sqrt{2x \ln \ln \frac{1}{x}}\), the local modulus of continuity. In all cases the Lévy-Ciesielski representation reveals how and why the logarithmic terms appear in the corresponding \(g\) and \(h\) functions.

We stress the differences between our work and others. First, the results of this paper are not asymptotic results, for they hold for every \(\epsilon > 0\) AND for every \(\delta > 0\). Many papers have been written on this subject; to the best of our knowledge, all of which are of an asymptotic-type. See [3], [4], and [7], and more recently, [5], [2], and [9]. Their results take the form: for every \(\epsilon > 0\), there is \(\delta(\epsilon) > 0\). Transitioning from these asymptotics to the results presented here would require several steps of approximations, as
compared to our one. First $\delta(\varepsilon)$ must be estimated. Then, after rescaling, a second level of approximations would be required to express the LIL or modulus of continuity in terms of an arbitrary $\delta$. These multiple levels of estimation would certainly affect the constants involved in the asymptotic results. Our results are straightforward. We express Brownian motion by an appropriate infinite sum, then determine

$$
\sup_{0 \leq t < s \leq 1, |t-s| < \delta} \left| W^n_s - W^n_t \right| / g(s-t)
$$

and

$$
\sup_{t \leq s} \frac{W^n_t}{h(t)}
$$

exactly; our only estimate is of the tail. Second, our approach is unique in that the method works for both the global and local modulus and allow us to easily establish both Lévy’s modulus of continuity and the law of iterated logarithms. The only difference between the two proofs is where we split the process into a piecewise-linear truncated process and an infinite tail. Finally, in practice, the usefulness of our results is the ability to choose $\delta$ a priori and independently of $\varepsilon$, and to select $\varepsilon$ afterwards corresponding to a desired confidence level.

Remark 1 Maple™ was used in many of our calculations. The values obtained were rounded to at most three decimal places in a way which did not compromise the direction of any inequalities and was coarser than the precision level of the computer algebra system. Moreover, as we seek to determine estimates for probabilities we at times produce long strings of inequalities. Thus when we number an inequality we are referring to greatest quantity in the string.

Remark 2 Throughout the paper, we will use the convention: $L(x) := \ln x$ and $L_2(x) = \ln \ln x$; we also use the standard notation, $[x]$ to denote the greatest integer less than or equal to $x$.

2 The Lévy-Ciesielski Construction

Throughout this paper, we let $W = (W_t)_{0 \leq t \leq 1}$, a Brownian motion process over the unit interval and $t \to W_t$ be a realization of the process over the unit interval. The Lévy-Ciesielski construction of Brownian motion is based on the Haar expansion of the covariance function of a Brownian motion process $W$ in the Cameron-Martin space. Via an isomorphism, it leads to the following representation of the Brownian motion process:

$$
W_t = tX_0 + \sum_{j=0}^{\infty} 2^{-j} \sum_{k=0}^{2^j-1} A_{j,k}(t) X_{j,k}
$$
where $X_o$ and $X_{j,k}$, for all $j,k$, are independent, standard normal random variables and

$$A_{j,k}(t) = \min\{2^j t - k, 1 - 2^j t + k\} 1_{I_{j,k}}(t)$$

with $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$.

Let $W^n_t$ be the $n$th partial sum of $W_t$, which includes $tX_o$. The process $W^n = (W^n_t)$ possesses some interesting properties. First, $W^n_t$ and $W_t$ agree at the dyadics at the $(n+1)$th level; that is, $W^n_{k2^{-n-1}} = W_{k2^{-n-1}}$ for $k = 0, \ldots, 2^{-n-1}$. Moreover, for $t \in I_{n+1,k}$, the process $W^n_t$ is linear in $t$, i.e. $W^n_t = At + B$ where $A$ and $B$ are normal random variables. Therefore, the process, $W^n$ is equivalent to the piecewise-linear process, $W^n$, created by connecting the points $(k/2^n+1, W_{k2^n+1})$, $k = 0, \ldots, 2^n+1$ linearly. As mentioned above this was noticed by P. Lévy and we shall use this fact repeatedly throughout. A more thorough introduction to this expansion can be found in [10].

3 Global Maximal Deviations for Truncated Brownian Increments and Brownian Increments

In this section we develop several results regarding the global modulus of continuity for the truncated Brownian motion process and the process itself. First we obtain an estimate for the distribution function of the ratio between the truncated Brownian increment and the global modulus of continuity function $g(\delta)$. Using this result we establish an estimate for the distribution function of the maximal deviation for the ratio of the Brownian increment and $g(\delta)$. More specifically, for $\varepsilon, \delta > 0$, we determine the probability of the set

$$\left\{ \sup_{0 \leq t < s \leq 1} \frac{|W_s - W_t|}{g(\delta)r(\delta)} \leq \sqrt{1 + \varepsilon} \right\}$$

where $r(x) = 1 + 3.5\left(\frac{L_{1/2}}{|x|}\right)^{-\frac{1}{2}}$. As $\delta \to 0$, the function $\delta \to \sup_{|t-s| \leq \delta} \frac{|W_s - W_t|}{g(\delta)r(\delta)}$ is not necessarily monotonic. Therefore we establish an estimate for the probability of the set

$$\left\{ \sup_{\delta \leq \delta_o} \sup_{0 \leq t < s \leq 1} \frac{|W_s - W_t|}{g(\delta)o(\delta)} \leq \sqrt{1 + \varepsilon} \right\},$$

which is monotonic in $\delta_o$. 
3.1 Preliminaries

The first lemma is essential in estimating the probability of the set

\[
\left\{ \sup_{0 \leq t < s \leq 1 \atop |t-s| \leq \delta} \frac{|W^n_s - W^n_t|}{g(s-t)} > \sqrt{1 + \varepsilon} \right\}.
\]

The second is needed to uniformly estimate the tail of the truncated increment.

Notice that \(W^n_s\) and \(W^n_t\) are piecewise linear in \(s\) and \(t\), and therefore so is their difference; that is, \(W^n_s - W^n_t = At + Bs + C\) for some random variables \(A, B,\) and \(C\).

**Lemma 1** Let \(\delta_0 > \delta \geq 0\) and \(f : C \to R\) be defined by

\[
f(t, s) = \frac{|at + bs + c|}{g(s-t)}
\]

where \(a, b\) and \(c\) are constants and \(g\) is the global modulus of continuity function of a Brownian motion process and \(C\) is the convex set

\[
\{(t_1, t_2) \times [s_1, s_2] \cap \{(t, s) : \delta \leq |s - t| \leq \delta_0\}\}
\]

with \(t_1, s_1 > 0, t_2 < s_1,\) and \(s_2 - t_1 < 1\). Then \(f\) achieves its maximum at one of the extreme points of the convex set \(C\).

Moreover, the supremum of \(f\) over all \(s\) and \(t\) such that \(|s - t| < \delta\) is achieved at a value for \(s - t\) which is bounded away from zero. That is, for \(\delta > 0\),

\[
\sup_{(t,s) \in I_{n+1,k} \times I_{n+1,k+1} \atop |s-t| \leq 2^{-n} \wedge \delta} \frac{|at + bs + c|}{g(s-t)} = \sup_{(t,s) \in I_{n+1,k} \times I_{n+1,k+1} \atop 2^{-n} \leq |s-t| \leq 2^{-n} \wedge \delta} \frac{|at + bs + c|}{g(s-t)}.
\]

**Proof** Calculus.

The next lemma is a variation of a well-known fact from the theory of Gaussian processes: the maximum of a finite collection of identically distributed normal random variables essentially grows as the square root of the natural log of the cardinality of the collection \(\mathbb{I}^n\). Many variations of this lemma have been used in the past. For instance, in 1991, a version similar to the one presented here was used by Meyer [5].

**Lemma 2** For \(d > 0\)

\[
P\left( \max_{j \geq n} \frac{|X_{j,k}|}{\sqrt{L2^j}} > \sqrt{2(d + 1)} \right) \leq \frac{2^{-dn}}{(1 - 2^{-d}) \sqrt{\pi L2^n}}.
\]
Proof

\[
\mathbb{P}\left( \max_{j \geq n} \frac{|X_{j,k}|}{\sqrt{L2^j}} > \sqrt{2(d+1)} \right) \leq \sum_{j=n}^{\infty} 2^j \mathbb{P}\left( \frac{|N(0,1)|}{\sqrt{L2^j}} > \sqrt{2(d+1)} \right) \\
\leq \sum_{j=n}^{\infty} \frac{2^{-d_j}}{\sqrt{\pi(d+1)2^j}} \\
\leq \frac{2^{-dn}}{(1-2^{-d}) \sqrt{\pi L2^n}}.
\]

3.2 Global maximal deviations

In this subsection we establish three results. The first theorem estimates the distribution function of the maximal deviation between the ratio of the truncated increment and the global modulus function for a fixed \( \delta > 0 \). Based on that result, coupled together with the tail estimate, the second theorem estimates the distribution function for the maximal deviation of the ratio of the increment of the process \( W \) and the global modulus of continuity function for a fixed \( \delta \). Finally, we go beyond the results of the standard modulus of continuity by establishing a result which holds uniformly over \( \delta \). This last result allows us to establish rates of convergence; see [1].

**Theorem 1** For \( \varepsilon > 0 \) and \( n \geq 4 \), if \( 0 < \delta < 2^{-n-1} \) we have

\[
\mathbb{P}\left( \sup_{0 \leq t < \delta} \frac{|W_n^s - W_n^t|}{g(s-t)} > \sqrt{1+\varepsilon} \right) \leq \frac{3\delta^\varepsilon}{\sqrt{\pi L^{1+\delta}}},
\]

and if \( \delta \geq 2^{-n-1} \) then

\[
\mathbb{P}\left( \sup_{0 \leq t < \delta} \frac{|W_n^s - W_n^t|}{g(s-t)} > \sqrt{1+\varepsilon} \right) \leq \frac{2^{-(n+1)} K(\varepsilon, \delta, n)}{\sqrt{\pi L^{1+\delta}}},
\]

where

\[
K(\varepsilon, \delta, n) = 1 + 9(2)^{\varepsilon} + 4(2^{n+1}\delta)^{1+\varepsilon} + 2(2^{n+1}\delta)^{2+\varepsilon}.
\]

**Remark 3** The restriction on \( n \) is imposed to assure monotonicity of the functions involved. Without this restriction the constants involved would be greater.
Proof Let \( \varepsilon, \delta > 0 \) and \( n \geq 4 \). Define \( \delta_n = \min \{ \delta, 2^{-n-1} \} \). Let \( I_k \) denote the \( k \)-th dyadic interval at the level \( n+1 \); that is, \( I_k = I_{n+1,k} \). While \( W^n_s - W^n_t \) is piecewise linear over the set \( \{(t,s) \mid 0 \leq t < s \leq 1, |s-t| \leq \delta\} \), the increment is linear in \( s \) and \( t \) when \( (t,s) \in I_k \times I_{k+1} \). Therefore, we increase the size of the set

\[
\left\{ \sup_{0 \leq t < s \leq 1, |s-t| \leq \delta} \frac{|W^n_s - W^n_t|}{g(s-t)} > \sqrt{1 + \varepsilon} \right\}
\]

to gain linearity and obtain

\[
P\left( \sup_{0 \leq t < s \leq 1, |s-t| \leq \delta} \frac{|W^n_s - W^n_t|}{g(s-t)} > \sqrt{1 + \varepsilon} \right)
\leq \sum_{k=0}^{2^{n+1} - 1} \sum_{l=0}^{[2^{n+1}] + 1} P\left( \sup_{(t,s) \in I_k \times I_{k+l}} \frac{|W^n_s - W^n_t|}{g(s-t)} > \sqrt{1 + \varepsilon} \right).
\]

(1)

Fix \( k \) and consider different \( l \) for the set

\[
\left\{ \sup_{(t,s) \in I_k \times I_{k+l}} \frac{|W^n_s - W^n_t|}{g(s-t)} > \sqrt{1 + \varepsilon} \right\}.
\]

Note, if \( l = 0 \) or \( l = 1 \), \( |s-t| \) is not necessarily bounded away from zero and must be treated with care. If \( l > 1 \), we must be mindful of the shape of the underlying set.

If \( l = 0 \), \( s \) and \( t \) lie in the same dyadic interval \( I_j \), for \( j \leq n + 1 \). Thus, for each \( j \leq n + 1 \), there exists an integer \( k_j \) such that \( s, t \in [k_j2^{-j}, (k_j+1)2^{-j}) \). Moreover,

\[
W^n_s - W^n_t = (s-t) \left( X_0 + \sum_{j=0}^{n} 2^{j/2}(-1)^{\varepsilon_j(t)}X_{j,k_j} \right),
\]
where \( \varepsilon_j (t) \) is the \( j^{th} \) term in the binary expansion of \( t \) (and \( s \)). Hence,

\[
\mathbb{P} \left( \sup_{s, t \in I_k, |s-t| \leq \delta} \frac{|W^n_s - W^n_t|}{g(s-t)} > \sqrt{1 + \varepsilon} \right)
= \mathbb{P} \left( \sup_{s, t \in I_k, |s-t| \leq \delta} \frac{|s-t|}{\sqrt{2|s-t|L|x|}} \left| X_0 + \sum_{j=0}^{n} 2^{j/2} (-1)^{j(t)} X_{j,k} \right| > \sqrt{1 + \varepsilon} \right)
\leq \mathbb{P} \left( |N(0, 1)| > \sqrt{2(1 + \varepsilon)} L \frac{1}{\delta_n} \leq \frac{(\delta_n)^{1+\varepsilon}}{\sqrt{\pi L}} \right) \tag{2}
\]

Next if \( l = 1 \), the difference \( s - t \) is no more than \( 2^{-n} \) but is not necessarily bounded away from zero, so we consider two cases, \( \delta < 2^{-n-1} \) and \( \delta \geq 2^{-n-1} \).

If \( \delta < 2^{-n-1} \), again by Lemma \( \mathbb{I} \) maximum is achieved at one of the two points: \((k + 1)/2^{n+1}, (k + 1)/2^{n+1} - \delta \) or \((k + 1)/2^{n+1} - \delta, (k + 1)/2^{n+1} \) yielding

\[
\mathbb{P} \left( \sup_{(t, s) \in I_k \times I_{k+1}, |s-t| \leq 2^{-n} \wedge \delta} \frac{|W^n_s - W^n_t|}{g(s-t)} > \sqrt{1 + \varepsilon} \right) \leq 2 \frac{\delta^{1+\varepsilon}}{\sqrt{\pi L}} \tag{3}
\]

If \( \delta \geq 2^{-n-1} \), by Lemma \( \mathbb{I} \)

\[
\sup_{(t, s) \in I_k \times I_{k+1}, |s-t| < 2^{-n} \wedge \delta} \frac{|W^n_s - W^n_t|}{g(s-t)} = \sup_{(t, s) \in I_k \times I_{k+1}, 2^{-n} \leq |s-t| < 2^{-n} \wedge \delta} \frac{|W^n_s - W^n_t|}{g(s-t)}
\]

and maximum is achieved at one of the four points:

\((k/2^{n+1}, (k + 1)/2^{n+1}), (k/2^{n+1}, k/2^{n+1} + (2^{-n} \wedge \delta)), ((k + 2)/2^{n+1} - (2^{-n} \wedge \delta), (k + 2)/2^{n+1}), \) or \(((k + 1)/2^{n+1}, (k + 2)/2^{n+1}) \).

Thus

\[
\mathbb{P} \left( \sup_{(t, s) \in I_k \times I_{k+1}, |s-t| \leq \delta} \frac{|W^n_s - W^n_t|}{g(s-t)} > \sqrt{1 + \varepsilon} \right)
\leq 2 \left[ \frac{2^{-n(1+\varepsilon)/2}}{\sqrt{\pi L} 2^{n+1}} + \frac{2^{-n(1+\varepsilon)} \wedge \delta^{1+\varepsilon}}{\sqrt{\pi L} (2^n \wedge 1/\delta)} \right] \leq 4 \frac{2^{-n(1+\varepsilon)}}{\sqrt{\pi L} 2^n} \tag{4}
\]
Thus there are five extreme points yielding upper bound over the rectangle \( I_k \times I_{k+l} \) achieves its maximum at one of the four corner points, yielding the function

\[
W_n(s) = W_n(t)
\]

over the rectangle \( I_k \times I_{k+l} \). We employ Lemma 1 in each situation.

When \( l < 2n^+1 \), the function

\[
W_n(s) = W_n(t)
\]

achieves its maximum at one of the four corner points, yielding the function

\[
W_n(s) = W_n(t)
\]

Finally, if \( l > 1 \), we consider the three cases: \( 1 < l \leq 2n^+1 \), \( l = 2n^+1 \), and \( l = 2n^+1 + 1 \). Each implies a different underlying shape of the set over which supremum is considered. They form rectangles, a pentagon, and a triangle respectively. We employ Lemma 1 in each situation.

When \( 1 < l \leq 2n^+1 \), the function

\[
(t, s) \mapsto \frac{|W_n(s) - W_n(t)|}{g(s-t)}
\]

achieves its maximum at one of the four corner points, yielding the function

\[
W_n(s) = W_n(t)
\]

Finally, if \( l > 1 \), we consider the three cases: \( 1 < l \leq 2n^+1 \), \( l = 2n^+1 \), and \( l = 2n^+1 + 1 \). Each implies a different underlying shape of the set over which supremum is considered. They form rectangles, a pentagon, and a triangle respectively. We employ Lemma 1 in each situation.

When \( l = 2n^+1 \), the set \( I_k \times I_{k+l} \cap \{ (t, s) : |s-t| \leq \delta \} \) is a pentagon; thus there are five extreme points yielding

\[
P \left( \sup_{(t, s) \in I_k \times I_{k+l}} \frac{|W_n(s) - W_n(t)|}{g(s-t)} > \sqrt{1 + \varepsilon} \right)
\]

achieves its maximum at one of the four corner points, yielding the function

\[
W_n(s) = W_n(t)
\]

Finally, if \( l > 1 \), we consider the three cases: \( 1 < l \leq 2n^+1 \), \( l = 2n^+1 \), and \( l = 2n^+1 + 1 \). Each implies a different underlying shape of the set over which supremum is considered. They form rectangles, a pentagon, and a triangle respectively. We employ Lemma 1 in each situation.

When \( l = 2n^+1 + 1 \), the set \( I_k \times I_{k+l} \cap \{ (t, s) : |s-t| \leq \delta \} \) is a triangle; thus

\[
P \left( \sup_{(t, s) \in I_k \times I_{k+l}} \frac{|W_n(s) - W_n(t)|}{g(s-t)} > \sqrt{1 + \varepsilon} \right)
\]

achieves its maximum at one of the four corner points, yielding the function

\[
W_n(s) = W_n(t)
\]

Finally, if \( l > 1 \), we consider the three cases: \( 1 < l \leq 2n^+1 \), \( l = 2n^+1 \), and \( l = 2n^+1 + 1 \). Each implies a different underlying shape of the set over which supremum is considered. They form rectangles, a pentagon, and a triangle respectively. We employ Lemma 1 in each situation.
We incorporate the upper bounds obtained from inequalities 2 and 3 into (1) and see, for $0 < \delta < 2^{-n-1}$,

$$
\mathbb{P}
\left(
\sup_{0 \leq t < s \leq 1 \atop |s-t| \leq \delta} \frac{|W^n_s - W^n_t|}{g(s-t)} > \sqrt{1+\varepsilon}
\right) \leq \frac{3\delta^\varepsilon}{\sqrt{\pi L^\frac{1}{2}}}.
$$

From inequalities 2 and 4 for $\delta \geq 2^{-n-1}$, we have

$$
\mathbb{P}
\left(
\sup_{0 \leq t < s \leq 1 \atop |s-t| \leq \delta} \frac{|W^n_s - W^n_t|}{g(s-t)} > \sqrt{1+\varepsilon}
\right) \leq \frac{2^{-\varepsilon(n+1)}}{\sqrt{\pi L^{2n+1}}} + \frac{2^{-n+1}}{\sqrt{\pi L^{2n}}} + 2^{n+1} \sum_{l=2}^{[2n+1]} \mathbb{P}
\left(
\sup_{(t,s) \in I_k \times I_{k+l} \atop |s-t| \leq \delta} \frac{|W^n_s - W^n_t|}{g(s-t)} > \sqrt{1+\varepsilon}
\right).
$$

Using inequalities 5, 6 and 7, the sum in the previous inequality can be estimated.

$$
\sqrt{\pi L^\frac{1}{2}} \sum_{l=1}^{[2n+1]+1} \mathbb{P}
\left(
\sup_{(t,s) \in I_k \times I_{k+l} \atop |s-t| \leq \delta} \frac{|W^n_s - W^n_t|}{g(s-t)} > \sqrt{1+\varepsilon}
\right) \leq \sum_{l=1}^{[2n+1]} \left( \frac{l-1}{2n+1} \right)^{1+\varepsilon} + 2 \left( \frac{l}{2n+1} \right)^{1+\varepsilon} + \left( \frac{l+1}{2n+1} \right)^{1+\varepsilon} +
+ 2 \left( \frac{[2n+1] \delta}{2n+1} \right)^{1+\varepsilon} + 2 \left( \frac{[2n+1] \delta}{2n+1} - 1 \right)^{1+\varepsilon} + 4\delta^{1+\varepsilon}
$$

$$
\leq \left( \sum_{l=1}^{[2n+1]} \left( \frac{l}{2n+1} \right)^{1+\varepsilon} \right) + \sum_{l=2}^{[2n+1]} 2 \left( \frac{l}{2n+1} \right)^{1+\varepsilon} + \left( \sum_{l=3}^{[2n+1]} \left( \frac{l}{2n+1} \right)^{1+\varepsilon} \right) + 4\delta^{1+\varepsilon}.
$$

Hence, for $\delta \geq 2^{-n-1}$,

$$
\mathbb{P}
\left(
\sup_{0 \leq t < s \leq 1 \atop |s-t| \leq \delta} \frac{|W^n_s - W^n_t|}{g(s-t)} > \sqrt{1+\varepsilon}
\right) \leq \frac{2^{-\varepsilon(n+1)}}{\sqrt{\pi L^{2n+1}}} \left( 1 + 2^{1+\varepsilon} \sqrt{1 + \frac{1}{n}} + 2^{n+1} \left( 4 \sum_{l=1}^{[2n+1]} \left( \frac{l}{2n+1} \right)^{1+\varepsilon} + 4 \left( \delta \right)^{1+\varepsilon} \right) \right).
$$

$$
\leq \frac{2^{-\varepsilon(n+1)}}{\sqrt{\pi L^{2n+1}}} K(\varepsilon, \delta, n)
$$
where $K(\epsilon, \delta, n) = 1 + 9(2^\epsilon + 2 \left(2^{n+1}\delta\right)^{2+\epsilon} + 4 \left(2^{n+1}\delta\right)^{1+\epsilon}$.

Now we establish an estimate for the distribution function of the global maximal deviation of the ratio of the Brownian increment and the modulus of continuity function for a fixed $\delta$. For monotonicity as explained in Remark 3 we insist that $\delta \leq 2^{-2}$ and for future purposes, we will need $\delta \leq 2^{-5}$.

Without this restriction, the constant, $K_1(\epsilon)$ in the theorem below would be greater.

**Theorem 2** Let $0 < \delta \leq 2^{-5}$ and $\epsilon > 0$. Then

$$\Pr \left( \sup_{0 \leq t < s \leq 1, |s-t| \leq \delta} \left| \frac{W_s - W_t}{g(\delta)\sqrt{L^1(\delta)}} \right| > \sqrt{1 + \epsilon} \right) \leq K_1(\epsilon) \delta \epsilon \left( \frac{L^1(\delta)}{\delta} \right)^{\frac{3}{2}}$$

where

$$r(\delta) = \left( 1 + \frac{2.65}{\sqrt{L^1(\delta)}} \right)$$

and

$$K_1(\epsilon) = 27.95 + \frac{0.11}{\epsilon} 1_{(0,1)}(\epsilon)$$

**Proof** Let $\epsilon > 0$ and $0 < \delta \leq 2^{-5}$ and $n \geq 8$, so that $(n + 1)^{2^{-n}} < \delta \leq n2^{-n}$.

The proof is completed in two steps. First we estimate the size of the set

$$A_{\epsilon, \delta} = \left\{ \sup_{0 \leq t < s \leq 1, |s-t| \leq \delta} \left| \frac{W_s - W_t}{g(\delta)\sqrt{L^1(\delta)}} \right| \leq \sqrt{1 + \epsilon} \right\} \cap \left\{ \max_{j \geq n+1, 0 \leq k \leq 2^{j-1}} \left| X_{j,k} \right| \leq \sqrt{2(1 + \epsilon)} \right\}$$

using both Theorem 1 and Lemma 2. Then we show that on $A_{\epsilon, \delta}$,

$$\left| W_s - W_t \right| \leq g(\delta) r(\delta) \sqrt{1 + \epsilon}$$

for all $|s-t| \leq \delta$.

By Theorem 1 and Lemma 2 we have

$$\Pr \left( A_{\epsilon, \delta}^c \right) \leq \frac{2^{-\epsilon(n+1)}}{\sqrt{\pi L^1(\delta)}} \left( K(\epsilon, \delta, n) + \frac{1}{(1 - 2^{-\epsilon})} \right)$$

$$\leq \frac{2^{-\epsilon(n+1)}}{\sqrt{\pi L^1(\delta)}} \left( 1 + 9(2^\epsilon + 2 \left(2^{n+1}\delta\right)^{2+\epsilon} + 4 \left(2^{n+1}\delta\right)^{1+\epsilon} + \frac{1}{(1 - 2^{-\epsilon})} \right)$$

$$\leq \frac{\delta^\epsilon}{\sqrt{\pi L^1(\delta)}} \left( 4 \left(2^{n+1}\delta\right) + 2 \left(2^{n+1}\delta\right)^2 + \frac{1 + 9(2) + \frac{1}{(1 - 2^{-\epsilon})}}{(n+1)^{\epsilon}} \right)$$

$$\leq c\delta^\epsilon \left( \frac{L^1(\delta)}{\delta} \right)^{\frac{3}{2}} \left( 4 \left(2^4\right) + 2 \left(2^4\right)^2 + \frac{1 + 9(2) + \frac{1}{(1 - 2^{-\epsilon})}}{(n+1)^{\epsilon}} \right)$$
since $\frac{L^4}{\delta^2} \geq L^2 n$ and where $c = \left( \sqrt{\pi} \left( L 2^5 \right)^2 \right)^{-1}$.

To eliminate the dependency on $n$ in our above estimate of $\mathbb{P}(A_{c,\delta})$, we consider $0 < \varepsilon < 1$ and $\varepsilon \geq 1$. If $0 < \varepsilon < 1$, then
\[
\mathbb{P}(A_{c,\delta}) \leq c \delta^c \left( \frac{L^8}{\delta} \right)^{\frac{3}{2}} \left( 2^6 + 2^9 + 10 + \frac{2e^{-1}}{L^2 (2 - L^2)} \right)
\]
and if $\varepsilon \geq 1$, then
\[
\mathbb{P}(A_{c,\delta}) \leq c \delta^\varepsilon \left( \frac{L^8}{\delta} \right)^{\frac{3}{2}} \left( 2^6 + 2^9 + \frac{1}{3} + 9 \left( \frac{2}{9} \right)^\varepsilon \right).
\]
Thus for all $\varepsilon > 0$, we have
\[
\mathbb{P}(A_{c,\delta}) \leq \left( 27.95 + \frac{0.11}{\varepsilon \text{I}(0,1)}(\varepsilon) \right) \delta^\varepsilon \left( \frac{L^8}{\delta} \right)^{\frac{3}{2}}.
\]

Next we estimate $|W_s - W_t|$ on the set $A_{c,\delta}$. By Theorem 1 and Lemma 2 and recalling that $n \geq 8$, we note that $W$ restricted to $A_{c,\delta}$ yields
\[
|W_s - W_t| \leq |W^n_s - W^n_t| + \sum_{j=n+1}^{\infty} 2^{-j/2} \sum_{k=0}^{2j-1} |A_{j,k}(t) - A_{j,k}(s)| X_{j,k}
\]
\[
\leq g(s-t) \sqrt{1 + \varepsilon} + 2 \sum_{j=n+1}^{\infty} 2^{-j/2} \sqrt{j} \max_{j \geq n+1, 0 \leq k < 2j-1} |X_{j,k}| \sqrt{j}
\]
\[
\leq \sqrt{1 + \varepsilon} \left( g(s-t) + \sqrt{2L^2} \sum_{j=n+1}^{\infty} 2^{-j/2} \sqrt{j} \right)
\]
\[
\leq \sqrt{1 + \varepsilon} \left( g(s-t) + \sqrt{2L^2} \sqrt{\frac{n+1}{2n+1}} \sum_{j=0}^{\infty} 2^{-j/2} \sqrt{1 + \frac{j}{9}} \right)
\]
\[
\leq \sqrt{1 + \varepsilon} \left( g(s-t) + 2.65 \sqrt{\frac{n+1}{2n+1}} \right).\tag{8}
\]

since
\[
\sqrt{L^2} \sum_{j=0}^{\infty} 2^{-j/2} \sqrt{1 + \frac{j}{9}} \leq 2.65.
\]
Recall $\frac{n+1}{2n+1} < \delta$ and that the inequality (8) holds for all $|s - t| \leq \delta$, we have
\[
\sup_{|s-t| \leq \delta} \frac{|W_s - W_t|}{g(\delta)} \leq \sqrt{1 + \varepsilon} \left( 1 + \frac{2.65}{L^2} \right)
\]
on the set $A_{\varepsilon, \delta}$ whose probability is greater than

$$1 - K_1 (\varepsilon) \delta^\varepsilon \left( \frac{L_1^1}{\delta} \right)^{\frac{3}{2}}.$$

For practical purposes, results uniformly over $\delta$ are of interest, thus the results of the previous theorem are not as desirable. Moreover, the function

$$\delta \rightarrow \sup_{\{s-t\leq \delta\}} \frac{|W_s - W_t|}{g(\delta)}$$

is not necessarily monotonic which make establishing uniform results challenging. The theorem below addresses this need and challenge. We should note that its proof is similar to the proof of Theorem 2, and it yields the same rate in $\delta$.

Additionally, it may seem at first glance, the expressions $K_1$ and $K_2$ in these two theorems may look different. However, they really only differ near $\varepsilon$ zero. $K_1$, above behaves as $\varepsilon^{-1}$ near zero while $K_2$ behaves as $\varepsilon^{-3}$. Moreover their corresponding multipliers are extremely different as well with the coefficient of the $\varepsilon^{-1}$ being a hundredth of $\varepsilon^{-3}$. But as $\varepsilon$ moves away from zero, the two behave basically the same.

**Theorem 3** Let $0 < \delta_0 \leq 2^{-5}$ and $\varepsilon > 0$. Then

$$\mathbb{P} \left( \sup_{\delta \leq \delta_0} \sup_{0 \leq t < s \leq 1} \frac{|W_s - W_t|}{g(\delta)} > \sqrt{1 + \varepsilon} \right) \leq K_2 (\varepsilon) \delta_0^\varepsilon \left( \frac{L_1^1}{\delta_0} \right)^{\frac{3}{2}},$$

where

$$\frac{9.57}{\varepsilon^3} 1_{(0,2a]}(\varepsilon) + \left( \frac{14.59}{\varepsilon} + 9.9 \right) 1_{(2a, \infty]}(\varepsilon) + 24.05,$$

where $a = (8L_2 - 1)^{-1}$ and $r(\delta)$ is as in Theorem 2.

**Proof** Let $\delta_0 \leq 2^{-5}$ and $n$ be such that $(n+1)2^{-n-1} < \delta_0 \leq n2^{-n}$. Our choice of $\delta_0$ forces $n \geq 8$. Set

$$A_{\varepsilon, \delta_0} = \left\{ \begin{array}{l} \sup_{0 \leq t < s \leq 1} \sup_{(n+1)2^{-n-1} < |s-t| \leq \delta_0} \frac{|W_s - W_t|}{g(t-s)} \leq \sqrt{1 + \varepsilon} \\
\sup_{m \geq n+1} \sup_{0 \leq t < s \leq 1} \sup_{|s-t| \leq m2^{-m}} \frac{|W^m_s - W^m_t|}{g(t-s)} \leq \sqrt{1 + \varepsilon} \\
\max_{j > n} \frac{|X_{i,j,k}|}{\sqrt{2^j}} \leq \sqrt{2(1 + \varepsilon)} \end{array} \right\}$$
and define $\delta_m = m2^{-m}$. Define $S_1$, $S_2$, and $S_3$ so that $\|A_{\varepsilon,\delta_o}\| \leq S_1 + S_2 + S_3$ and the following holds. The first term, $S_1$, is derived from the first set used to create $A_{\varepsilon,\delta_o}$. The upper bound of its compliment is determined in the same fashion as the proof of Theorem 1 for $\delta > 2^{-n-1}$. Specifically,

$$S_1 \leq \frac{1}{\sqrt{\pi L_{\frac{1}{2}\delta_o}}} \left(2^{n+1} \left(4 \sum_{l=n+2}^{2n+1} \left(\frac{l}{2n+1}\right)^{1+\varepsilon} + 4 (\delta_o)^{1+\varepsilon}\right)\right)$$

since $(n+1)2^{-n-1} < \delta_o$. we could approximate this sum by an integral which would have resulted in a constant divided by $2 + \varepsilon$. This would be beneficial in practice when $\varepsilon$ is large. However the constants would more than double. Instead, we replace the sum with its greatest summand and see that

$$S_1 \leq \left(\frac{1}{\delta_o}\right)^2 \frac{4n(2^{n+1}\delta_o)^{1+\varepsilon}}{\sqrt{\pi L_{\frac{1}{2}\delta_o} \left(\frac{L}{\delta_o}\right)^2} 2^{(n+1)\varepsilon}}$$

$$\leq \left(\frac{1}{\delta_o}\right)^2 \frac{8n^2\delta_o^\varepsilon}{(nL2 - Ln)^2}$$

$$\leq \delta_o^\varepsilon \left(\frac{1}{\delta_o}\right)^2 \frac{8}{(L2 - \frac{Ln}{n})^2}$$

$$\leq \delta_o^\varepsilon \left(\frac{1}{\delta_o}\right)^4 \frac{8}{\sqrt{\pi} \left(\frac{L^2}{L2 - \frac{Ln}{n}}\right)^2} < 24.05\delta_o^\varepsilon \left(\frac{1}{\delta_o}\right)^4. \quad (9)$$
\[ S_2 \text{ and } S_3 \text{ are derived from the last two sets used to create } A_{\varepsilon, \delta_0}. \text{ Specifically, according to Theorem } 1 \text{ and Lemma } 2, \]

\[
P \left( A_{\varepsilon, \delta_0} \setminus \left\{ \sup_{0 \leq s < t \leq 1, |s-t| \leq \delta_0} \frac{|W^n_s - W^n_t|}{g(t-s)} \leq \sqrt{1 + \varepsilon} \right\} \right) \leq \sum_{m=n+1}^{\infty} \frac{2^{-m(m+1)} K(\varepsilon, \delta_m, m)}{\pi L \delta_m} + \frac{2^{-m(n+1)}}{(1-2^{-\varepsilon}) \sqrt{\pi L^2 n+1}} \]

\[
\leq \sum_{m=n+1}^{\infty} \frac{2^{-m(m+1)}}{\pi L \delta_m} (1 + 9 (2)^{2+\varepsilon} + 4 (2^{m+1} \delta_m)^{1+\varepsilon}) + \frac{2^{-m(n+1)}}{(1-2^{-\varepsilon}) \sqrt{\pi L^2 n+1}} \]

\[
= S_2 + S_3 \]

Consider \( S_2 \) by looking at the sum below for \( k = 1, 2 \).

\[
\sum_{m=n+1}^{\infty} 2^{-m} m^{k+\varepsilon} \leq 2^{-m(n+1)} (n+1)^{k+\varepsilon} \sum_{m=0}^{\infty} \frac{2^{-m}}{2^{-m x}} \left( 1 + \frac{m}{8} \right)^{k+\varepsilon}. \]

Define

\[
I_k(\varepsilon) = \sum_{m=0}^{\infty} 2^{-m} \left( 1 + \frac{m}{8} \right)^{k+\varepsilon} \quad (10) \]

for \( k = 1, 2 \). Although we could use \( I_1(\varepsilon) \leq I_2(\varepsilon) \), we estimate both, \( I_1(\varepsilon) \) and \( I_2(\varepsilon) \) to obtain better constants. The function \( f_k : [0, \infty) \to \mathbb{R} \) defined by

\[
f_k(x) = \left( 1 + \frac{x}{8} \right)^{\varepsilon+k} 2^{-\varepsilon x} \quad (11)\]

has only one maximum which is achieved at

\[ x_o = 8 \left( \frac{k+\varepsilon}{8\varepsilon L^2} - 1 \right) \]

provided \( \varepsilon \leq \frac{k}{8(L+1)} \); otherwise, the maximum appears at \( x = 0 \). Therefore when \( \varepsilon \leq \frac{k}{8(L+1)} \), we have

\[
I_k(\varepsilon) \leq \int_0^{x_o} \left( 1 + \frac{x}{8} \right)^{\varepsilon+k} 2^{-\varepsilon x} dx + \int_{x_o}^{\infty} \left( 1 + \frac{x-1}{8} \right)^{\varepsilon+k} 2^{-\varepsilon(x-1)} dx + f_k(x_o) \]

\[
\leq \int_0^{\infty} \left( 1 + \frac{x}{8} \right)^{\varepsilon+k} 2^{-\varepsilon x} dx + f_k(x_o). \]
Otherwise, when \( \varepsilon > ak \),

\[
I_k(\varepsilon) \leq \int_0^{\infty} \left( 1 + \frac{x}{8} \right)^{k+\varepsilon} 2^{-\varepsilon x} \, dx + 1.
\]

Thus we can combine both cases and create the upper bound

\[
I_k(\varepsilon) \leq \int_0^{\infty} \left( 1 + \frac{x}{8} \right)^{k+\varepsilon} 2^{-\varepsilon x} \, dx + f_k(x_0)1_{(0,a]}(\varepsilon) + 1_{(a,\infty)}(\varepsilon).
\]

One remark regarding \( f_k(x_0) \) which we will need later on for \( \varepsilon \leq ak \) is that

\[
f_k(x_0) = \left( \frac{k + \varepsilon}{8L^2} \right)^{k+\varepsilon} 2^{-8\left( \frac{k + \varepsilon}{8L^2} \right)}
\]

and thus,

\[
f_1(x_0) = \frac{0.16}{\varepsilon(1+\varepsilon)} \text{ and } f_2(x_0) = \frac{0.14}{\varepsilon(1+\varepsilon)}.
\]

In order to simplify the integral that appears in \( I_k(\varepsilon) \), we substitute \( z = 8L^2(1 + \frac{x}{8}) \). Then

\[
\int_0^{\infty} \left( 1 + \frac{x}{8} \right)^{k+\varepsilon} 2^{-\varepsilon x} \, dx = \frac{2^{8\varepsilon}}{e^{k+\varepsilon+1}(8L^2)^{k+\varepsilon}} \int_{8L^2}^{\infty} z^{k+\varepsilon} e^{-z} \, dz.
\]

Since integral 12 is not easily integrated, we employ a few tricks. Note

\[
z^{k+\varepsilon} e^{-z} = z^{k+\varepsilon} e^{-\frac{z}{8}} e^{-(1-\frac{1}{8})z}
\]

for all \( b \neq 0 \). The idea is that we will replace the first two factors of 13 with their maximums, and integrate the last factor of 13. The function

\[ z \to z^{k+\varepsilon} e^{-\frac{z}{8}} \]

attains maximum when \( z = b(k+\varepsilon) \) for any \( b > 0 \) and the last factor is integrable if \( b > 1 \). The best upper bound would result by choosing the minimal \( b \) however, our choice of \( b = 16L^2/ (8L^2 + 1) \) produces constants which are easy to manipulate and not far from the minimal constants..

Now we compute \( I_1(\varepsilon) \).

\[
I_1(\varepsilon) \leq \frac{1}{\varepsilon(e+2)(L^2)^{e+2}23^{-5\varepsilon}} \left( \int_{8L^2}^{\infty} z^{1+\varepsilon} e^{-z} \, dz \right) + f_k(x_0)1_{(0,a]}(\varepsilon) + 1_{(a,\infty)}(\varepsilon).
\]
Since we eventually combine results with \( I_2 \), we only consider the cases \( \varepsilon \leq 2a \) and \( \varepsilon > 2a \) instead of \( \varepsilon \leq a \) and \( \varepsilon > a \). When \( \varepsilon \leq 2a \),
\[
I_1 (\varepsilon) \leq \frac{1}{\varepsilon^{\varepsilon + 2}} (L_2)^{\varepsilon + 2} 2^{3 - 5 \varepsilon} \left( \int_{8L_2}^{\infty} z^{1 + \varepsilon} e^{-z} \, dz \right) + \max (f_1 (x_o), 1)
\]
\[
\leq \frac{1}{\varepsilon^{\varepsilon + 2}} \left( \frac{32}{(8L_2 + 1) e^1} \right)^{1 + \varepsilon} (1 + \varepsilon)^{1 + \varepsilon} \frac{2\pi \varepsilon}{8L_2 - 1} + \max (0.16 \varepsilon, \varepsilon^{\varepsilon + 2})
\]
\[
\leq \frac{1.15}{\varepsilon^{\varepsilon + 2}}
\]
It should be clear that the second factor in the second to the last line above is increasing in \( \varepsilon \); thus, the final result is obtained by substituting \( 2a \) for \( \varepsilon \).

For \( \varepsilon > 2a \), computation is much easier since the maximum of the function
\[
z \mapsto z^{1 + \varepsilon} e^{-z}
\]
occurs before the lower limit of integration. Therefore,
\[
I_1 (\varepsilon) \leq \frac{1}{\varepsilon^{\varepsilon + 2}} (L_2)^{\varepsilon + 2} 2^{3 - 5 \varepsilon} \left( \int_{8L_2}^{\infty} z^{1 + \varepsilon} e^{-z} \, dz \right) + 1
\]
\[
\leq \frac{(8L_2)^{1 + \varepsilon} e^{-\frac{8L_2}{1 + \varepsilon}}}{\varepsilon^{\varepsilon + 2}} (L_2)^{\varepsilon + 2} 2^{3 - 5 \varepsilon} \int_{8L_2}^{\infty} e^{-(1 - \varepsilon)} z \, dz + 1
\]
\[
\leq \frac{1}{\varepsilon L_2} + 1.
\]

For \( k = 2 \), we repeat the process. For \( \varepsilon \leq 2a \),
\[
I_2 (\varepsilon) \leq \frac{1}{\varepsilon^{\varepsilon + 3}} (L_2)^{\varepsilon + 3} 2^{6 - 5 \varepsilon} \left( \int_{8L_2}^{\infty} z^{2 + \varepsilon} e^{-z} \, dz \right) + f_2 (x_0)
\]
\[
\leq \frac{0.70}{\varepsilon^{\varepsilon + 3}}.
\]
And \( \varepsilon > 2a \),
\[
I_2 (\varepsilon) \leq \frac{1}{\varepsilon^{\varepsilon + 3}} (L_2)^{\varepsilon + 3} 2^{6 - 5 \varepsilon} \left( \int_{8L_2}^{\infty} z^{2 + \varepsilon} e^{-z} \, dz \right) + 1
\]
\[
\leq \frac{1}{\varepsilon L_2} + 1.
\]

We are now ready to bound \( S_2 \).
\[
S_2 = 4 \sum_{m=n+1}^{\infty} 2^{-cm} m^{1 + \varepsilon} + 2 \sum_{m=n+1}^{\infty} 2^{-cm} m^{2+\varepsilon}
\]
\[
\leq \frac{4 (2^c (n+1) (n + 1)^{1 + \varepsilon}) I_1 (\varepsilon) + 2 (2^c (n+1) (n + 1)^{2+\varepsilon}) I_2 (\varepsilon)}{\sqrt{\pi L_{\frac{1}{\delta_n+1}}}}
\]
\[
\leq \delta_o \left( L_{\frac{1}{\delta_o}} \right)^{\frac{3}{2}} \left( 4 (n + 1) I_1 (\varepsilon) + 2 (n + 1)^2 I_2 (\varepsilon) \right).
\]
We bound by substituting and get

$$S_2 \leq \delta_o \left( \frac{1}{\delta_o} \right)^{\frac{1}{2}} \left( \frac{4I_1(\varepsilon) + 2I_2(\varepsilon)}{\sqrt{\pi} \left( L2 - \frac{1+2\varepsilon}{9} \right) \left( \ln 2 - \ln \frac{9}{\pi} \right)^{\frac{1}{2}}} \right)$$

We bring in results for $I_1$ and $I_2$ and see for $\varepsilon \leq 2a$

$$S_2 \leq \delta_o \left( \frac{1}{\delta_o} \right)^{\frac{1}{2}} \left( \frac{4I_1(\varepsilon) + 2I_2(\varepsilon)}{\sqrt{\pi} \left( L2 - \frac{1+2\varepsilon}{9} \right) \left( L2 - \frac{9}{\pi} \right)^{\frac{1}{2}}} \right)$$

$$\leq 9.51 \delta_o \left( \frac{1}{\delta_o} \right)^{\frac{1}{2}}$$ \hspace{1cm} (14)

And if $\varepsilon > 2a$

$$S_2 \leq \delta_o \left( \frac{1}{\delta_o} \right)^{\frac{1}{2}} \left( \frac{4I_1(\varepsilon) + 2I_2(\varepsilon)}{\sqrt{\pi} \left( L2 - \frac{1+2\varepsilon}{9} \right) \left( L2 - \frac{9}{\pi} \right)^{\frac{1}{2}}} \right)$$

$$\leq 9.90 \delta_o \left( \frac{1}{\delta_o} \right)^{\frac{1}{2}} \left( \frac{1}{\varepsilon L2} + 1 \right).$$ \hspace{1cm} (15)

Next we move on to our upper bound for $S_3$.

$$S_3 = \frac{2^{-\varepsilon(n+2)} \left( 1 + 9 \left( 2^{\varepsilon} \varepsilon \right) \right)}{\sqrt{\frac{\pi L1}{\delta_{n+1}}} \left( 1 - 2^{-\varepsilon} \right) \sqrt{\pi L2^{n+1}} \left( 1 - 2^{-\varepsilon} \right)}$$

$$\leq \frac{\delta_o \left( \frac{1}{\delta_o} \right)^{\frac{1}{2}}}{(n+1)^{\varepsilon} \left( 1 - 2^{-\varepsilon} \right) \left( \frac{1}{\delta_{n+1}} \right)^{\frac{1}{2}}} \left( \frac{2^{-\varepsilon} + 9}{\sqrt{\frac{\pi L1}{\delta_{n+1}}} + \sqrt{\pi L2^{n+1}}} \right)$$

$$\leq \frac{\delta_o \left( \frac{1}{\delta_o} \right)^{\frac{1}{2}}}{9^{\varepsilon} \left( 1 - 2^{-\varepsilon} \right) \left( L2^{n+1} \right)^{\frac{1}{2}} \sqrt{\pi}} \left( \frac{2^{-\varepsilon} + 9}{\sqrt{L2^{n+1}} + \sqrt{L2^{2n}}} \right)$$

For consistency, we consider the cases $\varepsilon \leq 2a$ and $\varepsilon > 2a$. For $\varepsilon \leq 2a$,

$$S_3 \leq \frac{\delta_o \left( \frac{1}{\delta_o} \right)^{\frac{1}{2}}}{9^{\varepsilon} \left( \varepsilon L2 - \frac{(\varepsilon L2)^2}{2} \right) \left( L2^{n+1} \right)^{\frac{1}{2}} \sqrt{\pi}} \left( \frac{2^{-\varepsilon} + 9}{\sqrt{L2^{n+1}} + \sqrt{L2^{2n}}} \right)$$

$$\leq \frac{\delta_o \left( \frac{1}{\delta_o} \right)^{\frac{1}{2}}}{9^{\varepsilon} \left( L2 - \frac{(\varepsilon L2)^2}{2} \right) \left( L2^{n+1} \right)^{\frac{1}{2}} \sqrt{\pi}} \left( \frac{10}{\sqrt{L2^{n+1}}} + \frac{1}{\sqrt{L2^{2n}}} \right)$$
Note that the function
\[ \varepsilon \rightarrow \frac{\varepsilon^2}{9 \varepsilon \left( L^2 - \frac{\varepsilon^2}{2} \right)} \]
increases on the set \((0, 2a]\). Thus
\[ S_3 \leq 0.06 \frac{\delta^5 (L + \delta)^{3/2}}{\varepsilon^3} \tag{16} \]
on that set. And when \( \varepsilon > 2a \), we have
\[ S_3 \leq 0.30 \frac{\delta^5 (L + \delta)^{3/2}}{\varepsilon} \tag{17} \]

Using the bounds \([9, 15, 14, 17, 16]\) and considering the sets, determined by \( a \), in which \( \varepsilon \) may lie, we estimate
\[ \frac{1}{\delta^5 (L + \delta)^{3/2}} \mathbb{P}(A_{\varepsilon, \delta_0}^c) \]
above by
\[ \frac{9.57}{\varepsilon^3} \mathbb{1}_{(0, 2a]}(\varepsilon) + \left( \frac{14.59}{\varepsilon} + 9.9 \right) \mathbb{1}_{(2a, \infty)}(\varepsilon) + 24.05 \]

Finally, for \( W \) on the set \( A_{\varepsilon, \delta_0} \), we let \( 0 < \delta \leq \delta_0 \) and \( s \) and \( t \) be such that \( 0 < |s - t| < \delta \). Let \( m \) be the smallest integer so that \( \delta \leq \delta_m \). Clearly \( m \geq n \). By inequality \( \ref{inequality} \) and using the same approach as in Theorem \( \ref{theorem} \) we obtain
\[ |W_s - W_t| \leq |W_s^m - W_t^m| + \left| \sum_{j=m+1}^{\infty} 2^{-j/2} \sum_{k=0}^{2^j - 1} (A_{j,k}(t) - A_{j,k}(s)) X_{j,k} \right| \]
\[ \leq \sqrt{1 + \varepsilon g(\delta)} \left( 1 + \frac{2.65}{\sqrt{L + \delta}} \right). \]
The right hand side does not depend on our choice of \( s \) and \( t \), so
\[ \sup_{|s - t| \leq \delta} \frac{|W_s - W_t|}{g(\delta) r(\delta)} \leq \sqrt{1 + \varepsilon}, \]
which holds for every \( \delta \leq \delta_0 \) on the set \( A_{\varepsilon, \delta_0} \).
3.3 Consequences

In this subsection we easily extend the results of the previous subsection to Brownian motion on \([0, T]\) by using the scaling property of Brownian motion.

**Corollary 1** For \(T \geq 1\) and \(\delta \leq T2^{-5}\), we have

\[
P \left( \sup_{0 \leq t < s \leq T} \frac{|B_s - B_t|}{g(\delta) r(\delta, T)} \leq \sqrt{1 + \varepsilon} \right) \geq 1 - K_1(\varepsilon) \left( \frac{\delta}{T} \right)^{\varepsilon} \left( \frac{T}{\delta} \right)^{\frac{3}{2}}
\]

and

\[
P \left( \sup_{\delta \leq \delta_0} \sup_{0 \leq t < s \leq T} \frac{|B_s - B_t|}{g(\delta) r(\delta, T)} \leq \sqrt{1 + \varepsilon} \right) \geq 1 - K_2(\varepsilon) \left( \frac{\delta_0}{T} \right)^{\varepsilon} \left( \frac{T}{\delta_0} \right)^{\frac{3}{2}},
\]

where

\[
r(\delta, T) = r(\delta T) \sqrt{\frac{L_{\delta T}}{L_{\delta}}}
\]

and \(K_1, K_2\) and \(r\) are the same as in theorems 2 and 3.

**Proof** For \(T \geq 1\) and \(\delta \leq T2^{-5}\), the scaling property of Brownian motion yields

\[
P \left( \sup_{0 \leq t < s \leq T} \frac{|B_s - B_t|}{g(\delta) r(\delta, T)} \leq \sqrt{1 + \varepsilon} \right) = P \left( \sup_{0 \leq t < s \leq T} \frac{\sqrt{T} |B_s - B_t|}{g(\delta) r(\delta, T)} \leq \sqrt{1 + \varepsilon} \right)
\]

\[
= P \left( \sup_{\delta \leq \delta_0} \sup_{0 \leq t < s \leq T} \frac{|B_s - B_t|}{g(\frac{\delta}{T}) r(\frac{\delta}{T})} \leq \sqrt{1 + \varepsilon} \right). \tag{18}
\]

Similarly,

\[
P \left( \sup_{\delta \leq \delta_0} \sup_{0 \leq t < s \leq T} \frac{|B_s - B_t|}{g(\frac{\delta}{T}) r(\frac{\delta}{T})} \leq \sqrt{1 + \varepsilon} \right)
\]

\[
= P \left( \sup_{\delta \leq \delta_0} \sup_{0 \leq t < s \leq T} \frac{|B_s - B_t|}{g(\frac{T}{\delta}) r(\frac{T}{\delta})} \leq \sqrt{1 + \varepsilon} \right). \tag{19}
\]

The proof is complete by applying Theorem 2 to equation 18 and Theorem 3 to equation 19.
4 Local Maximal Deviations for Truncated Brownian Motion and Brownian Motion

In this section, we develop new results regarding the local modulus of continuity for Brownian motion. Our main contributions are finding estimates for the distribution function of the maximum of the ratio of a truncated Brownian motion process and the local modulus of continuity \( h \) and the maximum of the ratio of a Brownian motion process and the local modulus of continuity \( h \). It may seem more simple to estimate the distribution function of

\[
\sup_{t<\delta} \frac{W_t}{h(t)}
\]

then of

\[
\sup_{|s-t|\leq\delta \atop 0\leq t<s\leq 1} \frac{|W_s-W_t|}{g(\delta)}
\]

However, the subtle difference of the local modulus being evaluated at \( t \) and the global modulus being evaluated at \( \delta \) makes establishing the local case more challenging.

The first subsection contains one technical result designed to exploit the fact that \( W^n_t \) restricted to \( I_{n+1, k} \) is a linear function in \( t \). It is an analog of Lemma 1 adapted to treat the local maximal deviations. In the second subsection, we detail the main results.

4.1 Preliminaries

As in the case of the modulus of continuity, we examine \( W^n_t + (W_t - W^n_t) \). The following lemma is necessary in estimating the maximal deviation for the truncated process. The restriction of \( t_2 < 0.17 \) appearing in the next lemma assures monotonicity of the function \( f \). Without this condition, the results obtained would still hold true but with greater constants.

**Lemma 3** Let \( a \) and \( b \) be constants and \( f : [t_1, t_2] \to \mathbb{R} \), \( t_1 > 0 \), \( t_2 < 0.17 \) be defined by

\[
f(t) = \frac{at + b}{h(t)}
\]

Then the relative maxima of \( f \) must occur at the end points of \( [t_1, t_2] \).

**Proof** Calculus.
4.2 Local Maximal Deviations

In this subsection we develop two results. First we estimate the probability of the set
\[ \left\{ \sup_{t \leq \delta} \frac{W^n_t}{h(t)} \geq \sqrt{1 + \varepsilon} \right\}, \]
which, with a slight modification, gives an upper bound for the probability of the set
\[ \left\{ \sup_{2^{-n-1} t \leq 2^{-n} t} \frac{W^n_t}{h(t)} \geq \sqrt{1 + \varepsilon} \right\}. \tag{20} \]
Then, using the estimate of the size of the set 20 together with a modification of the uniform tail estimate established in Lemma 2, we derive the main result, an upper bound for the uniform maximal deviation from zero of the ratio of the Brownian motion process and its modulus function \( h(t) \).

Remark 4 The restriction \( \delta \leq 2^{-4} \) imposed in this subsection serves only one purpose: to make computations easier. Any other bound on \( \delta \) which is less than one will only change the constants.

**Theorem 4** Let \( \varepsilon > 0 \), \( n \in \mathbb{N} \) and \( 0 < \delta \leq 2^{-4} \). For \( n \) such that \( 0 < \delta < 2^{-n-1} \),
\[ \mathbb{P} \left( \sup_{t \leq \delta} \frac{W^n_t}{h(t)} \geq \sqrt{1 + \varepsilon} \right) \leq \frac{(L_{1/2})^{1-\varepsilon}}{2 \sqrt{\pi L_{2/1}}}, \]
and for \( n \) such that \( \delta \geq 2^{-n-1} \),
\[ \mathbb{P} \left( \sup_{t \leq \delta} \frac{W^n_t}{h(t)} \geq \sqrt{1 + \varepsilon} \right) \leq \left( \frac{L_{1/2}}{\pi L_{2/1}} \right)^{1-\varepsilon}. \]

**Proof** As in Theorem 1, we will use the fact that \( W^n_t \) is a linear function of \( t \) when restricted to the intervals \( I_k = I_{n+1,k} \), for \( k = 0, ..., \left\lfloor 2^n \right\rfloor \). Thus
\[ \mathbb{P} \left( \sup_{t \leq \delta} \frac{W^n_t}{h(t)} \geq \sqrt{1 + \varepsilon} \right) \leq \sum_{k=0}^{\left\lfloor 2^n \right\rfloor} \mathbb{P} \left( \sup_{t \in I_{n+1,k}} \frac{W^n_t}{h(t)} \geq \sqrt{1 + \varepsilon} \right) \tag{21} \]
Set \( \delta_n = \min \{ \delta, 2^{-n-1} \} \). We treat each \( k \) differently; \( k = 0, 0 < k < \left\lfloor 2^n \right\rfloor \), and \( k = \left\lfloor 2^n \right\rfloor \). Notice, when \( k = 0 \), \( W^n_t = tX_0 + \sum_{j=0}^{n} 2^{j/2} X_{j,0} \) and thus
\[ \mathbb{P} \left( \sup_{t \in I_0} \frac{\sqrt{t} \left( X_0 + \sum_{j=0}^{n} 2^{j/2} X_{j,0} \right)}{\sqrt{2 \pi L_{2/1}}} \geq \sqrt{1 + \varepsilon} \right) \leq \frac{(L_{1/2})^{1-\varepsilon}}{2 \sqrt{\pi L_{2/1}}}. \tag{22} \]
And for $0 < k < \lfloor 2^n + 1 \delta \rfloor$, $W^n_t$ can be written as $at + b$. By Lemma 3, we have the inequality

$$\mathbb{P} \left( \sup_{t \in I_k} W^n_t \geq \sqrt{1 + \varepsilon} \right) \leq \mathbb{P} \left( \frac{W^n_{(k+1)2^{-n-1}}}{h((k+1)2^{-n-1})} \geq \sqrt{1 + \varepsilon} \right) + \mathbb{P} \left( \frac{W^n_{k2^{-n-1}}}{h(k2^{-n-1})} \geq \sqrt{1 + \varepsilon} \right) \leq \frac{L_{n+1}}{\sqrt{\pi L_{2^{n+1}k+1}}}. \quad (23)$$

Lastly, for $k = \lfloor 2^n + 1 \delta \rfloor$, we apply Lemma 3 again to obtain

$$\mathbb{P} \left( \sup_{t \in I_{2^n + 1}} W^n_t \geq \sqrt{1 + \varepsilon} \right) \leq \frac{(L_{n+1})^{-1-\varepsilon}}{\pi L_{2^{n+1}k+1}}. \quad (24)$$

Next we look at small and large $\delta$ separately; that is, $0 < \delta < 2^{-n-1}$ and $\delta \geq 2^{-n-1}$. For $0 < \delta < 2^{-n-1}$, we incorporate inequalities (22), (23) and (24) into inequality (21) and see

$$\mathbb{P} \left( \sup_{t \leq \delta} W^n_t \geq \sqrt{1 + \varepsilon} \right) \leq \frac{(L_{n+1})^{-1-\varepsilon}}{2 \pi L_{2^{n+1}}}.$$ 

Similarly for $\delta \geq 2^{-n-1}$, we obtain

$$\mathbb{P} \left( \sup_{t \leq \delta} W^n_t \geq \sqrt{1 + \varepsilon} \right) \leq \sum_{k=0}^{[2^n+\delta]-1} \frac{(L_{n+1})^{-1-\varepsilon}}{\sqrt{\pi L_{2^{n+1}k+1}}} + \frac{(L_{n+1})^{-1-\varepsilon}}{\pi L_{2^{n+1}}} \leq \frac{([2^n+\delta]+1)(L_{n+1})^{-1-\varepsilon}}{\sqrt{\pi L_{2^{n+1}}}}.$$ 

We point out the nuances between the global and local cases briefly mentioned in the introduction of this section. First in Theorem 3, the bounds obtained on the probability for the truncated process are not summable over $n$. Thus we can not directly employ our methods used in Theorem 3. Also recall that according to Lemma 2, the tail behaves as $\sqrt{m2^{-m}}$. At the end of the proof of Theorem 3, the tail term in inequality 8 is divided by $g(m2^{-m})$ resulting in a term that tends to zero as $m \to \infty$. This doesn’t happen in the local case since we divide by $h(t)$, not $h(\delta)$. To resolve these issues we estimate the set

$$\left\{ \sup_{t \leq \delta} W^n_t \geq \sqrt{1 + \varepsilon} \right\}$$

by breaking up the interval $(0, \delta)$ into subintervals $J_n = [2^{-n-1}, 2^{-n})$ where each subinterval $J_n$ produces an estimate which is a general term of a
summable series. Also note that breaking up the process over the intervals $J_n$ and choosing $d$ from Lemma 2 to be $1 + \frac{2}{\varepsilon}$ yields terms of order of $n^{-1-\frac{2}{\varepsilon}} (Ln)^{-\frac{2}{\varepsilon}}$ for both the truncated process and the tail.

Denote $m(\varepsilon) = \left[\frac{\varepsilon}{2L2}L2^{m+1} + f(\varepsilon)\right] + m + 1$

where $f(\varepsilon) = \left(1 - (L2)^{-1}\right)1_{[0,1]}(\varepsilon)$.

**Corollary 2** For $\varepsilon > 0$, $m \in \mathbb{N}$,

$$\mathbb{P}\left(\sup_{t \in J_m} \frac{W^m_t}{h(t)} \geq \sqrt{1 + \varepsilon}\right) \leq \frac{2^{m(\varepsilon)-m-1} (L2^{m+2})^{-(1+\varepsilon)} \pi (1 + \varepsilon)^{m+1}}{\sqrt{1 + \varepsilon}} L2^{2m+1}.$$

**Proof**

$$\mathbb{P}\left(\sup_{t \in J_m} \frac{W^m_t}{h(t)} \geq \sqrt{1 + \varepsilon}\right) \leq \frac{2^{m(\varepsilon)-m-1} (L2^{m+2})^{-(1+\varepsilon)} \pi (1 + \varepsilon)^{m+1}}{\sqrt{1 + \varepsilon}} L2^{2m+1}.$$

We are ready to determine the main results of the section, the local maximal deviation of the ratio of the Brownian motion and the local modulus of continuity.

**Theorem 5** For $0 < \delta < 2^{-4}$ and $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{t \leq \delta} \frac{W_t}{h(t)} s(t, \varepsilon) \leq \sqrt{1 + \varepsilon}\right) > 1 - J(\varepsilon, \delta)$$

where

$$s(t, \varepsilon) = 1 + 3.61 \left(1_{(0,1]}(\varepsilon) \sqrt{\frac{1}{L2^\frac{1}{4}} \max\left\{\left(\frac{1}{L2^\frac{1}{4}}\right)^2, \left(\frac{1}{L2^\frac{1}{4}}\right)^2\right\}} + 1_{(1,\infty]}(\varepsilon) \right)$$

and

$$J(\varepsilon, \delta) = \min\left(\frac{1.302}{L2^\frac{1}{4}}, \frac{1.181L1_{(1,\infty]}(\varepsilon)}{L2^\frac{1}{4}}\right).$$

**Proof** As in Theorem 2 the proof is complete in two steps. First we estimate the size of the set

$$B_{\varepsilon, \delta} = \left\{\sup_{m \geq \varepsilon} \sup_{t \in J_m} \frac{W^m_t}{h(t)} \leq \sqrt{1 + \varepsilon}\right\} \cap \left\{\sup_{m \geq \varepsilon} \max_{0 \leq k < 2^{-m}} \frac{|X_{i,k}|}{L2^{-m}} \leq 2\sqrt{1 + \frac{1}{\varepsilon}}\right\},$$
using both Corollary 2 and Lemma 2. Then we show that on this set
\[
W_t \leq \sqrt{1 + \varepsilon}
\]
for all \( t \leq \delta \). Recall, we substitute \( d = 1 + 2\varepsilon \) in Lemma 2.

Let \( 0 < \delta < 2^{-4} \) and choose \( n \) so that \( 2^{-n-1} \leq \delta < 2^{-n} \). By Corollary 2 and Lemma 2, we establish that
\[
\Pr\left( B_{\varepsilon, \delta} \right) \leq \sum_{m=n}^{\infty} \left( \frac{2^{m+1}}{\sqrt{\pi} (1 + \varepsilon) L^2 2^{2m+1}} \right) + \frac{2^{-(1 + \frac{1}{2}) (m+1)}}{\sqrt{\pi} (2 + \frac{1}{2}) L^2 2^{2m+1}}
\]

To approximate this sum, we consider two cases: \( 0 < \varepsilon \leq 1 \) and \( \varepsilon > 1 \).

Consider \( 0 < \varepsilon \leq 1 \). By definition of \( m(\varepsilon) \), we see that the summand in expression 25 is no greater than
\[
\frac{2^{m+1}}{\sqrt{\pi} (1 + \varepsilon) L^2 2^{2m+1}} \leq \frac{2^{m+1}}{\sqrt{\pi} (1 + \varepsilon) L^2 2^{2m+1}} + \frac{2^{-(1 + \frac{1}{2}) (m+1)}}{\sqrt{\pi} (2 + \frac{1}{2}) L^2 2^{2m+1}}
\]

Thus the summand from expression 25 is bounded above by
\[
(m + 1)^{-1} \left( 2^{1 - \frac{1}{2}} + \frac{(m + 1)^{-1} \left( 2^{1 - \frac{1}{2}} + \frac{1}{2} \right)}{1 - 2^{1 - \frac{1}{2}} (m + 1)} \right)
\]

With some algebraic manipulation and by approximating a sum with an appropriate integral we estimate expression 25 from above by
\[
\frac{(L^2)^{-1/2}}{\varepsilon \sqrt{\pi} (1 + \varepsilon) L^2 2^{2m+1}} \left( 2^{1 - \frac{1}{2}} + \frac{2^{-1/2} (1 - \frac{1}{2})}{2^{1 - \frac{1}{2}} - 1} \left( 2 + \frac{\varepsilon}{5} \right) \right)
\]

Since \( 0 < \varepsilon \leq 1 \),
\[
\frac{1}{2^{1 - \frac{1}{2}} - 1} \leq \frac{1}{7 \cdot 2^{1 - \frac{1}{2}}}
\]

and the function
\[
\varepsilon \to \frac{2^{1 - \frac{1}{2}} + 2^{-1/2} (2 - \frac{1}{2}) \frac{\varepsilon}{5} (2 + \frac{1}{5})}{L (2) \sqrt{\pi} (1 + \varepsilon)}
\]
attains an absolute maximum of no more than 1.302 at $\varepsilon = 1$, producing the desired bound for expression 25. That is,

$$
\mathbb{P}(B^c_{\varepsilon, \delta}) \leq \frac{1.302}{\varepsilon \left( L^\frac{1}{3} \right)^2 \sqrt{L^\frac{1}{3}}} 
$$

when $0 < \varepsilon \leq 1$.

Consider $\varepsilon > 1$. As in the previous case, we find an upper bound for the summand of expression 25. Each summand is convex as a function of $m(\varepsilon)$. Recall that $m(\varepsilon)$ changes definition when $\varepsilon > 1$. Therefore, we may replace the greatest integer function in $m(\varepsilon)$ with either $\varepsilon^2 L^2 m + 1$ or $\varepsilon^2 L^2 m + 1 - 1$. We determine which of these produces a greater value for the summand of the infinite sum found in expression 25. For $\varepsilon^2 L^2 m + 1$ the summand is bounded above by

$$
\left( L^{2m+2} \right)^{-\left( \frac{5}{2} + 1 \right)} + \frac{\left( L^{2m+1} \right)^{-\left( \frac{5}{2} + 1 \right)}}{2 \left( 1 + \frac{\varepsilon}{2} \right) \left( 2 \left( 1 + \frac{\varepsilon}{2} \right) - 1 \right)}
$$

(26)

and for $\frac{5}{2} L^2 L^2 m + 1 - 1$, the summand is bounded above by

$$
\frac{\left( L^{2m+2} \right)^{-\left( \frac{5}{2} + 1 \right)}}{2} + \frac{\left( L^{2m+1} \right)^{-\left( \frac{5}{2} + 1 \right)}}{2 \left( 1 + \frac{\varepsilon}{2} \right) - 1}.
$$

(27)

Comparing expressions 26 and 27, we see the former is greater. Thus the sum of expression 25 is bounded above by

$$
\sum_{m=0}^{\infty} \left( \left( L^{2m+2} \right)^{-\left( \frac{5}{2} + 1 \right)} + \frac{\left( L^{2m+1} \right)^{-\left( \frac{5}{2} + 1 \right)}}{2 \left( 1 + \frac{\varepsilon}{2} \right) \left( 2 \left( 1 + \frac{\varepsilon}{2} \right) - 1 \right)} \right).
$$

(28)

Again we use algebraic manipulation and approximate a sum with an appropriate integral to see that we can bound expression 28 above by

$$
\frac{1}{\left( L^2 \right)^{\left( \frac{5}{2} + 1 \right)} (n + 1)^\frac{2}{5}} \left( \frac{5^{-1}}{2 \left( 1 + \frac{\varepsilon}{2} \right) \left( 2 \left( 1 + \frac{\varepsilon}{2} \right) - 1 \right)} + \left( \frac{1}{2^{\left( 1 + \frac{\varepsilon}{2} \right)} - 1} \right) \frac{2}{\varepsilon} \right).
$$

Substitute $x = \frac{2}{\varepsilon} + 1$ and note that the function

$$
f(x) = \frac{5^{-1}}{2^x (2^x - 1)} + \left( \frac{1}{2^x (2^x - 1)} \right) (x - 1)
$$

is increasing for $x \in (1, 3]$. Therefore $f(x) \leq f(3) \leq 2.04$ and hence

$$
\mathbb{P}(B^c_{\varepsilon, \delta}) \leq \frac{2.04}{\left( L^2 \right) \left( L^\frac{1}{3} \right)^2 \sqrt{\pi (1 + \varepsilon)} L^\frac{1}{3} \sqrt{\pi}} \leq \frac{1.18}{\left( L^\frac{1}{3} \right)^2 \sqrt{\pi}}
$$

when $\varepsilon \in [1, \infty)$. 

We are now ready to determine an upper bound for
\[
\frac{W_t}{h(t) \sqrt{1 + \varepsilon}}
on\]
on the set \(B_{\varepsilon, \delta}\) which will hold for all \(t \leq \delta\). In the statement of this theorem, the desired upper bound is referred to as \(s(\delta, \varepsilon)\).

Let \(t \leq \delta\) and \(m\) be such that \(2^{-(m+1)} < t \leq 2^m\). By Corollary 2 and Lemma 2 on the set \(B_{\varepsilon, \delta}\), we have
\[
W_t = W_t^{m(\varepsilon)} + \left(W_t - W_t^{m(\varepsilon)}\right) 
\leq h(t) \sqrt{1 + \varepsilon} + \frac{1}{2} \sum_{j=m(\varepsilon)+1}^{\infty} 2^{-j/2} \sqrt{L2^{j-m}} \sup_{j>m(\varepsilon)} \frac{|X_{j,k}|}{\sqrt{L2^{j-m}}} 
\leq h(t) \sqrt{1 + \varepsilon} + \sqrt{1 + \frac{1}{\varepsilon}} \sum_{j=m(\varepsilon)+1}^{\infty} 2^{-j/2} \sqrt{L2^{j-m}}. \quad (29)
\]

By reindexing, the expression 29 is equivalent to
\[
h(t) \sqrt{1 + \varepsilon} (1 + g(\varepsilon, t)),
\]
where
\[
g(\varepsilon, t) = \sqrt{1 + \varepsilon} \left(1 + \frac{1}{\varepsilon}\right) \sqrt{\frac{L2}{2^{m(\varepsilon)+1}}} \sum_{j=0}^{\infty} 2^{-j/2} \sqrt{m(\varepsilon) + 1 + j - m}.
\]

We focus our attentions on estimating the function \(g(\varepsilon, t)\). As in our estimation of \(B_{\varepsilon, \delta}\), we consider two cases, \(0 < \varepsilon \leq 1\) and \(\varepsilon > 1\).

**Remark 5** In our estimations below we must remove the greatest integer function appearing in \(m(\varepsilon)\). There are simple methods for doing so which result in worse constants. We provide a more detailed computation which results in smaller constants.

**Case 1** \(0 < \varepsilon \leq 1\).

Consider \(m(\varepsilon) = m\). We break down computations further by looking at the case where
\[
L_2^{-\frac{1}{2}} \geq \left(L_1^{-\frac{1}{2}}\right)^{\frac{1}{2}}
\]
and the case where this inequality does not hold.

When \(L_2^{-\frac{1}{2}} \geq \left(L_1^{-\frac{1}{2}}\right)^{\frac{1}{2}}\), \(g(\varepsilon, t)\) is bounded above by
\[
\frac{1}{\sqrt{2\varepsilon L_2^{-\frac{1}{2}}}} \sqrt{\frac{L2}{2^{m+1}}} \sum_{j=0}^{\infty} 2^{-j/2} \sqrt{1 + j} 
\leq \frac{1}{\sqrt{\varepsilon L_2^{-\frac{1}{2}}}} \sqrt{\frac{L2}{2}} \sum_{j=0}^{\infty} 2^{-j/2} \sqrt{1 + j} \leq 3.46 \frac{2^{-m}}{2^{m+1}}.
\]
When $L_2^t < (L_1^t)^{\frac{\varepsilon}{t}}$, we have $(L_2^{m+1})^{\frac{\varepsilon}{t}} \leq \frac{\varepsilon}{t}$ since $m(\varepsilon) = m$. Also

$$t \to \frac{(L_1^t)^{\frac{\varepsilon}{t}}}{t L_2^t} \quad (30)$$

is decreasing when $t < 2^{-4}$. (We use this fact repeatedly without mention throughout the rest of the paper.) Thus $g(\varepsilon, t)$ is bounded above by

$$\frac{\sqrt{(L_2^{m+1})^{\frac{\varepsilon}{t}} 2^{m+1}}}{\sqrt{2 \varepsilon (L_1^t)^{\varepsilon}}} L_2^{2m+1} \sqrt{\frac{L_2}{2^{m+1}} \sum_{j=0}^{\infty} 2^{-j/2} \sqrt{1 + j}} \leq \frac{3.61\sqrt{\varepsilon}}{L_2^t}$$

Next consider $m(\varepsilon) = m + k$, for $k \geq 1$ Again we consider two cases,

$$L_2^t \geq (L_1^t)^{\frac{\varepsilon}{t}} \quad \text{and} \quad L_2^t < (L_1^t)^{\frac{\varepsilon}{t}}.$$  

When $L_2^t \geq (L_1^t)^{\frac{\varepsilon}{t}}$, $g(\varepsilon, t)$ is bounded above by

$$\frac{1}{\sqrt{\varepsilon L_2^t}} \sqrt{\frac{L_2}{2^{k+1}}} \sum_{j=0}^{\infty} 2^{-j/2} \sqrt{k + 1 + j} \leq \frac{2.87}{\sqrt{\varepsilon L_2^t}} \quad (31)$$

since, for $k \geq 1$, $j \geq 0$,

$$k \to \frac{k + 1 + j}{2^k}$$

is decreasing.

When $L_2^t < (L_1^t)^{\frac{\varepsilon}{t}}$, we look at two subcases: $k = 1$ and $k \geq 2$.

$k = 1$ implies that the greatest integer function for $m(\varepsilon)$ disappears. Here $g(\varepsilon, t)$ is bounded above by

$$\frac{1}{\sqrt{\varepsilon (L_1^t)^{\varepsilon}}} \sqrt{\frac{(L_2^{m+1})^{\frac{\varepsilon}{t}}}{2 L_2^{2m+1}}} \sqrt{\frac{L_2}{2}} \sum_{j=0}^{\infty} 2^{-j/2} \sqrt{2 + j}. \quad (32)$$

With the restriction of $L_2^t < (L_1^t)^{\frac{\varepsilon}{t}}$, the function

$$m \to \frac{(L_2^{m+1})^{\frac{\varepsilon}{t}}}{L_2^{2m+1}}$$

on the set where $m(\varepsilon) = m + 1$ is increasing. So we substitute by the largest $m$ satisfying $m(\varepsilon) = 1$ and get

$$\frac{(L_2^{m+1})^{\frac{\varepsilon}{t}}}{(L_2^{2m+1})} \leq \frac{\varepsilon}{t}.$$  

Thus, expression $32$ is bounded by

$$\frac{3.44}{\sqrt{(L_1^t)^{\varepsilon}}}.$$
k \geq 2$ implies that the greatest integer function appearing in $m(\varepsilon)$ is greater than or equal to one. For $x = \frac{\varepsilon}{2} \log_2 L 2^{m+1} + 1 - \frac{1}{L^2}$, the function $x \rightarrow \frac{x+2+j}{2^x}$ is decreasing for $x \geq 0$, $j \geq 0$. Thus, by substituting $[x]$ with $x - 1$, $g(\varepsilon, t)$ is bounded above by

$$
\sqrt{\frac{L 2}{\varepsilon^2 L^2 \left(2 \pi \frac{L 2^{m+1}}{\varepsilon} + \frac{1}{m+2}\right)}} \sum_{j=0}^{\infty} 2^{-j/2} \sqrt{\frac{\varepsilon}{2L2} L 2^{m+1} - \frac{1}{L^2} + 2 + j}
$$

$$
\leq \frac{1}{\sqrt{\varepsilon (L^2)^2}} \sqrt{\frac{L 2}{2^2 - \frac{1}{L^2}}} \sum_{j=0}^{\infty} 2^{-j/2} \sqrt{\frac{\varepsilon}{2L2} + \frac{2 - \frac{1}{L^2} + j}{L 2^{m+1}}}
$$

$$
\leq \frac{1}{\sqrt{(L^2)^2}} \sqrt{\frac{L 2}{2^2 - \frac{1}{L^2}}} \sum_{j=0}^{\infty} 2^{-j/2} \sqrt{\frac{1}{2L2} + \frac{2 - \frac{1}{L^2} + j}{L 2^{m+1}}} \leq 3.34 \sqrt{(L^2)^2}.
$$

Case 2 $\varepsilon > 1$.

For $x = \left[\frac{\varepsilon}{2L2^{m+1}}\right]$, $x \rightarrow \frac{x+1}{2^x}$ decreases as long as $l \geq 2$ and $x \geq 0$. Therefore $g(\varepsilon, t)$ is estimated by

$$
\sqrt{\frac{1}{(L^2)^2}} \sqrt{\frac{L 2}{2}} \sum_{j=0}^{\infty} 2^{-j/2} \sqrt{\frac{1}{2L2} + \frac{1 + j}{L 2^{m+1}}} \leq 3.59 \sqrt{(L^2)^2}.
$$

(33)

Combining the various estimates from both cases, we have

$$
W \leq h(t) \sqrt{1 + \varepsilon} \left(1 + 3.61 \left(\frac{1}{\sqrt{\varepsilon \max \left\{\sqrt{\frac{L 2}{L^2}}, \sqrt{(L^2)^2}\right\}}} + \frac{1}{\sqrt{(L^2)^2}}\right)\right).
$$

Remark 6 The authors note that in their paper [1], the constants in Proposition 2.1 can be improved upon slightly by using the results of Theorem 5.

References

1. Dobric, V. and Marano, L.: Rates of convergence for Lévy’s modulus of continuity and Hlinčin’s law of the iterated logarithm. High Dimensional Probability III (J. Hoffmann-Jørgensen, M. Marcus, and J. Wellner, eds.), Progress in Probability. 55, Birkhäuser, Basel, 105-109 (2003)
2. Einmahl, U.: The Darling-Erdös theorem for sums of i.i.d. random variables. Probab. Theory Related Fields 82(2), 241-257 (1989)
3. Erdös, P.: On the law of the iterated logarithm. Ann. Math. 43, 419-436 (1942)
4. Gnedenko, B. V. and Kolmogorov, A.N.: Limit distributions for sums of independent random variables. Addison-Wesley Publishing Company, Inc., Cambridge, Mass (1954)
5. Kahane, J.P.: Some random series of functions. Cambridge University Press, Cambridge (1985)
6. Khoshnevisan, D., Levin, D. and Shi, Z.: Extreme-Value Analysis of the LIL for Brownian Motion. Electron Commun Probab 10 Paper 20, 196-206 (2005)
7. Lévy, P.: Théorie de l’Additiondes Variables Aléatoires. Gauthier-Villars, Paris (1937)
8. Meyer, Y.: Ondlettes et Opérateurs. Hermann, Paris (1990)
9. Pinsky, M.: A. Brownian continuity modulus via series expansions. J. Theoret. Probab. Vol. 14, No. 1, 261–266 (2001)
10. Steele, M.: Stochastic Calculus and Financial Applications (Stochastic Modelling and Applied Probability). Springer (2001)
11. Talagrand, M. and Ledoux, M.: Probability in Banach Spaces. Springer-Verlag (1980)