The thermodynamical limit of general $gl(N)$ spin chains: II. Excited states and energies

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Received 7 November 2007
Accepted 19 December 2007
Published 18 January 2008

Abstract. We consider the thermodynamical limit of a $gl(N)$ spin chain with arbitrary representation at each site of the chain. We study the first excitations (with holes and new strings) above the vacuum and compute the corresponding densities of the Bethe roots as well as their energy.

Keywords: algebraic structures of integrable models, thermodynamic Bethe ansatz

ArXiv ePrint: 0710.5904
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1. Introduction

Integrable quantum spin chains are quantum mechanics models which can be solved exactly. Certainly, among them, the most studied is the spin $\frac{1}{2}$ Heisenberg spin chain (also called the XXX chain) [1] solved by Bethe [2] using his celebrated ansatz. However, in recent applications as condensed matter experiments [3]–[5] or string theory (for recent reviews, see [6, 7] and references therein), more involved models show up. These models deal with higher rank Lie algebras such as $gl(N)$ (the XXX chain being associated to $gl(2)$) and/or higher spin (i.e. higher dimensional representation) on each site, see e.g. [8]–[18]. They also include spin chains with impurities [19]–[21] and alternating spin chains [22]–[25]. In most papers, each model is investigated on its own. Obviously, in opposition to these case by case studies, it would be of very much interest to develop a general framework which allows one to solve all these models simultaneously. It is provided by the study of an ‘algebraic version’ of spin chains which encompasses a generic spin chain where each sites may carry different representations of $gl(N)$. The resolution of this general model started with the computation of the Bethe equations in [12, 26, 27]. Let us note that a very similar approach has been developed in [28, 29] and [30] where an ‘algebraic version’ of Bethe vectors is computed, in term of Lax operator (first case) or
Drinfeld currents (second case). Then, the vacuum state of this model was constructed in [31] in the thermodynamical limit (i.e. when the length $L$ of the chain tends to infinity) when the representations on the chain are characterized by rectangular Young tableaux (of arbitrary height and width). The vacuum was proved to be of spin 0 and formed by filled Fermi seas.

In this paper, we carry on the study started in [31] with the calculation of the first excited states above the vacuum state. This construction follows and generalizes the results obtained for example for the fundamental $gl(N)$ spin chain [26,32,33], for homogeneous $gl(N)$ spin chain with any symmetric representation [34], and also for alternating spin chains, as they were dealt in [35,36] for XXX Heisenberg chain or in [22] for $XXZ$-type chain.

The plan of the paper is the following. We first remind in section 2 the results obtained in [31]. Let us stress that the rest of the paper heavily relies on the results given there. In section 3, we consider hole excitations above the vacuum state, and the modifications these excitations induce on the densities of Bethe roots in the thermodynamical limit and within the string hypothesis. We also compute the form of some hole excitations valid for any type of spin chains, as well as hole configurations corresponding to ‘small’ excitations, and valid for a large class of spin chains (including all $gl(2)$ spin chains and $gl(N)$ alternating ones). More general excitations where new strings are added to the vacuum configuration are dealt in section 4, and we compute again the corresponding densities (theorem 4.1). We give a form for the excitation leading to a state with ‘small’ (but non-trivial) spin. The Bethe equations relating the Bethe roots for holes and the ones for new strings are also computed (theorem 4.2). In section 5, we give a general form for a $L_0$-local Hamiltonian and compute the energies for the excitations presented in sections 3 and 4. Finally, appendix is devoted to the proof of theorem 4.2.

2. Notation and summary of previous results

We give, in this section, a review of known results such that this paper be self-contained.

2.1. Bethe ansatz equations and string hypothesis

In the papers [12,26,27], the Bethe ansatz equations have been established for spin chains where each site may carry a different representation of $gl(N)$. To solve these equations, it is usual to use the string hypothesis which states that, when $L$, the length of the chain, tends to $\infty$, the solutions gather into $\nu_m^{(j)}$ strings of length $2m + 1$ ($m \in \frac{1}{2}\mathbb{Z}_+$) of the following form

$$\lambda_{m,k}^{(j)} + i\alpha, \quad \alpha = -m, -m + 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, N - 1$$

(2.1)

where $k = 1, \ldots, \nu_m^{(j)}$ and $\lambda_{m,k}^{(j)}$, the center of the string, is real. It is well known that the string hypothesis is valid only in the thermodynamical limit, i.e. when $L \to \infty$, and in particular describes correctly the ground state and the lowest-lying excited states [37,38]. The order of correction (due to large but finite $L$) to the physical values computed within this hypothesis has been estimated (by numerical [39]–[41] and/or analytical methods [42,43]). In particular, it has been proved that for the vacuum state or for spin chains with spin in fundamental representation, the corrections behave as $e^{-aL}$ (with
The valence, \( P \), of a state is completely characterized by the data of the numbers of strings, \( \Phi \), where the parameters \( Q \) correspond to rectangular Young tableau. They are thus characterized by two integers (energy of these states within the string hypothesis.

We will follow the lines given in [31] and consider representations which have a 

\[
\sum_{\ell=1}^{L} \delta_{j,\ell} \Phi^{(m)}_{a_{\ell}} \left( \lambda_{m,k}^{(j)} - 2\pi Q_{m,k}^{(j)} \right) = \sum_{p \in \frac{1}{2} \mathbb{Z}^+} \sum_{\ell=1}^{p} \Phi^{(p,m)}_{-1}(\lambda_{m,k}^{(j)} - \lambda_{p,\ell}^{(j-1)}) + \sum_{\ell=1}^{p} \Phi^{(p,m)}_{2}(\lambda_{m,k}^{(j)} - \lambda_{p,\ell}^{(j)}) + \sum_{\ell=1}^{p} \Phi^{(p,m)}_{-1}(\lambda_{m,k}^{(j)} - \lambda_{p,\ell}^{(j+1)})
\]

where the parameters \( Q_{m,k}^{(j)} \) are quantum numbers. We have introduced the functions:

\[
\Phi^{(m)}_{p}(\lambda) = \sum_{\alpha=-m}^{m} \varphi_{p+2\alpha}(\lambda), \quad p \in \mathbb{Z}^+, \quad m \in \frac{1}{2} \mathbb{Z}^+
\]

\[
\Phi^{(m,n)}_{2}(\lambda) = \varphi_{2m+2\alpha+2}(\lambda) + \varphi_{2|m-n|}(\lambda) + 2 \sum_{\alpha=|m-n|+1}^{m+n} \varphi_{2\alpha}(\lambda), \quad m, n \in \frac{1}{2} \mathbb{Z}^+
\]

\[
\Phi^{(m,n)}_{-1}(\lambda) = - \sum_{\alpha=|m-n|}^{m+n} \varphi_{2\alpha+1}(\lambda)
\]

\[
\varphi_{p}(\lambda) = 2 \arctan \left( \frac{2\lambda}{p} \right), \quad p \in \mathbb{Z}, \quad p \neq 0 \quad \text{and} \quad \varphi_{0}(\lambda) = 0.
\]

A state is completely characterized by the data of the numbers of strings, \( \nu_{a}^{(j)} \), and the quantum numbers, \( Q_{m,k}^{(j)} \). We can show that these quantum numbers are constrained by [31]:

\[
Q^{(j)}_{m,k} \in [-Q_{m,max}^{(j)}, Q_{m,max}^{(j)}]
\]

\[
Q_{m,max}^{(j)} = \frac{1}{2} \left( \nu_{m}^{(j)} - 1 + \sum_{\ell=1}^{L} \delta_{j,\ell} \min(2m+1, a_{\ell}) - \sum_{n \in (1/2) \mathbb{Z}^+} \min(2m+1, 2n+1) (2\nu_{n}^{(j)} - \nu_{n}^{(j-1)} - \nu_{n}^{(j+1)}) \right).
\]

The valence, \( P_{m}^{(j)} \), is the number of allowed quantum numbers i.e. \( P_{m}^{(j)} = 2Q_{m,max}^{(j)} + 1. \)

doi:10.1088/1742-5468/2008/01/P01015
2.2. Regular spin chains: definition and notation

We want to consider the thermodynamical limit (i.e. $L \to +\infty$) but it seems impossible to deal with an infinite number of different representations. Thus, following the computations done in [31], we will work with $L_0$-regular closed spin chains, that is to say periodic spin chain (of length $L$) such that the representation (the $gl(N)$ spin) at site $\ell$ is the same as the one at site $\ell + L_0$, keeping $L_0$ finite when $L \to +\infty$. Of course, for a $L_0$-regular spin chain, there is at most $L_0$ different Young tableaux. However, among a subchain of length $L_0$, the same representation may appear several times. In this subchain, inequivalent representations may differ by distinct values of $j_\ell$ and/or $a_\ell$. We define $L \leq L_0$, the number of representations which have different values of $a_\ell$, and called $\bar{a}_\alpha$, $\alpha = 1, \ldots, L$ these values. Accordingly, we define the ordered set:

$$\mathcal{N} = \left\{ n_\alpha = \frac{\bar{a}_\alpha - 1}{2} \text{ s.t. } 1 \leq \alpha \leq L \right\} \subset \frac{1}{2}\mathbb{Z}_{\geq 0} \quad \text{with } n_1 < n_2 < \cdots < n_L. \quad (2.8)$$

As a convention, we will also introduce $n_0 = -\frac{1}{2}$ and $n_{L+1} = +\infty$. The complementary set of $\mathcal{N}$ is

$$\bar{\mathcal{N}} = \frac{1}{2}\mathbb{Z}_{\geq 0} \setminus \mathcal{N} = \{ m \in \frac{1}{2}\mathbb{Z}_{\geq 0} \text{ s.t. } m \notin \mathcal{N} \}. \quad (2.9)$$

We also introduce the sets of indices defined by:

$$I_\alpha = \{ \ell \in [1, L_0] \text{ s.t. } a_\ell = \bar{a}_\alpha \}, \quad \forall \alpha \in [1, L] \quad (2.10)$$

such that

$$L_0 = \sum_{\ell=1}^{L_0} (\cdots)_\ell = \sum_{\alpha=1}^{L} \sum_{\ell' \in I_\alpha} (\cdots)_{\ell'}. \quad (2.11)$$

The cardinal $|I_\alpha|$ corresponds to the multiplicity of $\bar{a}_\alpha$ within a subset of $L_0$ sites. We define also

$$J_{\alpha,j} = \sum_{\ell \in I_\alpha} \delta_{j,j_\ell} \quad (2.12)$$

which corresponds to the multiplicity of the representation $(\bar{a}_\alpha, j)$ within a subset of $L_0$ sites. We thus have the property

$$L_0 = \sum_{\alpha=1}^{L} \sum_{j=1}^{N-1} J_{\alpha,j}. \quad (2.13)$$

We will also call $J$ the greatest common divisor (gcd) of the $J_{\alpha,j}$’s. In most of the cases (i.e. as soon as a representation appears only once in the subchain of length $L_0$), $J$ is in fact equal to 1.
2.3. Bethe equations for the vacuum state

Now, we can write down the Bethe equations in the thermodynamical limit for different states and, in particular, the vacuum state which is, by definition, the state such that $P_m^{(j)} - \nu_m^{(j)}$ vanish for any $m$ and $j$. In [31], we prove that this defines the state uniquely and one can show that it has spin zero (i.e. it is a trivial $gl(N)$ representation). This corresponds to the following choice of parameters

$$\nu_n^{(j)} = \frac{L}{NL_0} \sum_{\ell \in I_n} \min(j, j_\ell) \left( N - \max(j, j_\ell) \right) \quad \text{for } 1 \leq \alpha \leq \mathcal{L} \quad \text{and} \quad \nu_n^{(j)} = 0$$

for $n \in \mathcal{N}$

$$Q_m^{(j)} = k - \frac{1}{2} (\nu_m^{(j)} + 1) \quad \text{for } k = 1, \ldots, \nu_m^{(j)} \quad \text{and} \quad n \in \mathcal{N}.$$  \hspace{1cm} (2.14)

We will consider the thermodynamical limit $L \to \infty$, keeping $L_0$ finite, of the Bethe equations (2.2) for a $L_0$-regular spin chain and for the vacuum state. Then, we obtain the following set of equations for the densities, $\sigma_n^{(j)}(\lambda) \ (m \in \mathcal{N})$ of the center of the strings $\{\lambda_{m,k}^{(j)} | k = 1, \ldots, \infty\}$:

$$\sum_{n \in \mathcal{N}} \left\{ \int_{-\infty}^{\infty} d\lambda \, \sigma_n^{(j-1)}(\lambda) \Psi_{-1}^{(m,n)}(\lambda - \lambda) + \int_{-\infty}^{\infty} d\lambda \, \sigma_n^{(j)}(\lambda) \Psi_{2}^{(m,n)}(\lambda - \lambda) \right. $$

$$+ \left. \int_{-\infty}^{\infty} d\lambda \, \sigma_n^{(j+1)}(\lambda) \Psi_{-1}^{(m,n)}(\lambda - \lambda) \right\}$$

$$= -2\pi \sigma_n^{(j)}(\lambda_0) + \frac{1}{L_0} \sum_{\alpha = 1}^{\mathcal{L}} \left( \sum_{\ell \in I_n} \delta_{j,j_\ell} \right) \Psi_{-1}^{(m,n)}(\lambda_0)$$

$$\forall \lambda_0 \in ]-\infty, \infty[, \ \forall j = 1, \ldots, N - 1, \ \forall m \in \mathcal{N}$$  \hspace{1cm} (2.16)

where $\Psi(\lambda)$ are derivative of $\Phi(\lambda)$.

2.4. Densities for the vacuum state

To solve this set of equations, we perform a Fourier transform, with the following choice for the normalization

$$\hat{f}(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip\lambda} f(\lambda) \, d\lambda$$

and encompass them in a matrix. Finally, we obtain the following form of the BAE:

$$2\pi \hat{\Psi}(p) \hat{\Sigma}(p) = \Lambda(p)$$  \hspace{1cm} (2.18)

where the $(N - 1)\mathcal{L} \times (N - 1)\mathcal{L}$ matrix $\hat{\Psi}(p) = -A(p) \otimes \hat{\Psi}_{-1}(p)$ and $A(p)$ is a $(N - 1) \times (N - 1)$ tridiagonal matrix with the non-vanishing entries

$$[A(p)]_{jj} = 2 \cosh \left| \frac{p}{2} \right| \quad \text{and} \quad [A(p)]_{jj+1} = -1 = A(p)_{j+1j}$$  \hspace{1cm} (2.19)
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and \( \hat{\Psi}_{-1}(p) \) is a \( \mathcal{L} \times \mathcal{L} \) matrix such that \( [\hat{\Psi}_{-1}(p)]_{\alpha,\beta} = \hat{\Psi}^{(n_{\alpha},n_{\beta})}_{-1}(p) \). We have also introduced the \((N - 1)\mathcal{L}\) vectors

\[
\hat{\Sigma}(p) = \sum_{j=1}^{N-1} \sum_{\alpha=1}^{\mathcal{L}} \hat{\sigma}^{(j)}_{n_{\alpha}}(p) \ e^{(N-1)}_{j} \otimes e^{(\mathcal{L})}_{\alpha} \quad \text{and}
\]

\[
\Lambda(p) = \frac{1}{L_0} \sum_{j=1}^{N-1} \sum_{\alpha,\beta=1}^{\mathcal{L}} \sum_{\ell \in I_{\beta}} \delta_{j,\ell} \hat{\Psi}^{(n_{\alpha})}_{\beta}(p) e^{(N-1)}_{j} \otimes e^{(\mathcal{L})}_{\alpha}
\]

where \( e^{(p)}_{i} \) is the canonical basis of \( C^p \). Then, inverting (2.18) and performing an inverse Fourier transform, we get the densities of the centers of strings for the vacuum state:

\[
\sigma^{(k)}_{n_{\alpha}}(\lambda) = \frac{1}{NL_0} \sum_{\ell \in I_{\alpha}} \sum_{q=|(k-j)\ell+1|/2}^{(k+j-1)/2} \sin(2q\pi/N) \cosh((2\pi/N)\lambda) - \cos(2q\pi/N) \quad 1 \leq \alpha \leq \mathcal{L}.
\]

These densities allow us to compute physical quantities such as the vacuum energy.

We will deal with sums that runs on integers or half-integers, with an increment which can be either 1 or \( \frac{1}{2} \). To avoid confusion between these two types of sums, we will denote with a prime, \( \sum' \), the sums that increment by \( \frac{1}{2} \) steps, keeping the usual sum, \( \sum \) for the ones with step 1. Hence, we have for instance the relations (for \( m \) integer):

\[
\sum_{n=0}^{m} = \sum_{n=0}^{m} + \sum_{n=1/2}^{m-1/2} \quad \text{and} \quad \sum_{n=1/2}^{m} = \sum_{n=1}^{m} + \sum_{n=1}^{m+1/2}.
\]

The aim of the present paper is to consider the first excitations above the vacuum state (as defined above). There will be of two different types: holes, which correspond to a configuration where some of the strings defining the vacuum state have been removed, and new strings which may be added once holes have been done. We will treat the BAEs for these excitations, that is to say computes (in the thermodynamical limit) the new densities of the Bethe roots. We will also compute the energies for these excited states and give ‘local’ Hamiltonians.

3. Excited states: holes

In this section, we consider the first excitations above the vacuum state. We keep the string hypothesis, being interested in the \( 1/L \) contributions to the energy. They are constructed from the vacuum by modifying the root distribution. For such a purpose, we will create holes (i.e. remove some strings) in the filled seas of the vacuum, and possibly add new strings. We first deal with holes created in the \( j \)th sea, suppressing some strings of length \( a_{p}, \ p = 1, 2, \ldots \). The parameters \( a_{p} \) have to be some of the \( a_{\ell} \) parameters defining the representations of the spin chain (i.e. \( (a_{p} - 1)/2 \in N \)).

3.1. Valences and spin

Let \( \{\nu^{(j)}_{n}\} \) be the configuration of the vacuum. The configuration corresponding to an excited state reads, for \( n \in N \) and \( 1 \leq j \leq N - 1 \),

\[
\tilde{\nu}^{(j)}_{n} = \nu^{(j)}_{n} - \mu^{(j)}_{n}
\]

\[\text{doi:10.1088/1742-5468/2008/01/P01015} \]
where the sets \( \{ \mu_n^{(j)} | n \in \mathcal{N} \} \), \( 1 \leq j \leq N - 1 \), label the seas where perturbations to the vacuum configuration are introduced. Let us stress that \( v_n^{(j)} \) as well as \( \mu_n^{(j)} \) are a priori rational numbers (depending on \( L \), the length of the chain) while \( \nu_n^{(j)} \) must be integers (they correspond to the number of \( n \)-string in the sea \( j \)). In the following, we will use

\[
v_m^{(j)} = 2\mu_m^{(j)} - \mu_m^{(j-1)} - \mu_m^{(j+1)} \quad m \in \mathcal{N}, \quad j = 1, \ldots, N - 1
\]

with the convention \( \mu_n^{(0)} = \mu_n^{(N)} = 0 \).

The corresponding valences are given by

\[
\tilde{P}_n^{(j)} = \nu_n^{(j)} + \sum_{m \in \mathcal{N}} \min(2n + 1, 2m + 1) v_m^{(j)}.
\]  

They correspond to the number of allowed quantum numbers, while \( \nu_n^{(j)} \) denotes the number of parameters which are indeed used. Thus, for \( n \in \mathcal{N} \),

\[
\mathcal{D}_n^{(j)} = \tilde{P}_n^{(j)} - \nu_n^{(j)} = \sum_{m \in \mathcal{N}} \min(2n + 1, 2m + 1) v_m^{(j)}
\]

is the number of admissible quantum numbers \( Q_{n,k}^{(j)} \) which are not used. They provide the number (for each sea and each length of string) of free parameters attached to the excited state under consideration, i.e. the number of holes (formed by strings of length \( 2n + 1 \)) in the \( j \)th sea. Thus, they must be positive. Of course, some of the numbers \( \mathcal{D}_n^{(j)} \) may vanish. The spin of such a state is easily computed:

\[
S_j = \sum_{m \in \mathcal{N}} (2m + 1) v_m^{(j)} = \mathcal{D}_n^{(j)}. \]

It is equal to the number of holes in the \( j \)th sea of the longest string.

In order to determine the first hole excitations of the model, one should look for the values of \( v_m^{(j)} \) such that \( \tilde{P}_n^{(j)}, \mathcal{D}_n^{(j)} \) and \( S_j \) are positive integers and \( \mathcal{D}_n^{(j)} \) and/or \( S_j \) have the smallest value. Such task seems impossible to solve on full general grounds, so that we will treat two slightly different problems: (i) find excitations that exist whatever the type of chain; (ii) find ‘small’ excitations for a subclass of spin chain. The two following lemmas correspond to these two problems.

**Lemma 3.1.** The hole excitations given by, for \( 1 \leq \alpha \leq \mathcal{L} \),

\[
v_n^{(j)} = \frac{J_{\alpha,j}}{J} = \frac{1}{J} \sum_{\ell \in I_{\alpha}} \delta_{j,j_{\ell}},
\]

where \( J_{\alpha,j} \) and \( J \) are given in (2.12) and following, exists for any \( L_0 \)-regular spin chain. They have spin and hole numbers given by (for \( m \in \mathcal{N} \) and \( 1 \leq j \leq N - 1 \)):

\[
\mathcal{D}_n^{(j)} = \frac{1}{J} \sum_{\ell=1}^{L_0} \delta_{j,j_{\ell}} \min(2m + 1, a_{\ell}) \quad \text{and} \quad S_j = \frac{1}{J} \sum_{\ell=1}^{L_0} \delta_{j,j_{\ell}} a_{\ell}.
\]
These values generalize the ones obtained in [15], for the spin s XXX model, and in [33], for the $gl(N)$ spin chain built on fundamental representations (see also examples below). Moreover, since $\nu_{\alpha}^{(j)}$ are integers, equations (3.4) and (3.5) clearly show that the number of holes $D_{\alpha}^{(j)}$ and the spin $S_j$ for this state are also integers. It leads, from the equation (3.2), to the following values (for $1 \leq \alpha \leq \mathcal{L}$ and $1 \leq j \leq N - 1$):

\[
\mu_{\alpha}^{(j)} = \frac{1}{N\mathcal{J}} \sum_{\ell \in I_{\alpha}} \min(j, j_\ell)(N - \max(j, j_\ell))
\]

\[
\tilde{\nu}_{\alpha}^{(j)} = \frac{L - (L_0/\mathcal{J})}{NL_0} \sum_{\ell \in I_{\alpha}} \min(j, j_\ell)(N - \max(j, j_\ell)).
\]

We deduce that the corresponding excitation exists when the length of the chain, $L$, is such that $(L - (L_0/\mathcal{J}))$ is a multiple of $NL_0$. Such a length exists because $\mathcal{J}$ is also a divisor of $L_0$, see equation (2.13). The factor $1/\mathcal{J}$ may look weird at first sight. However, it may be explained by remarking that the Bethe equations do not depend on the choice of the order of the representations\(^4\) on the chain. Therefore, by reordering the representations, we may transform a $L_0$-regular spin chain to a $L_0/\mathcal{J}$-regular spin chain with the same excitations. This may explain the presence of the denominator $\mathcal{J}$ in the above formulae. These excitations correspond to the highest representation in the decomposition of the tensor product of the representations entering in the $L_0/\mathcal{J}$-regular chain. The Young tableau corresponding to this spin is the juxtaposition of the Young tableaux (in decreasing order of $\mathcal{J}$) of all representations of the $L_0/\mathcal{J}$-regular chain (see third example below and figure 1).

We now turn to the case (ii). The aim is to encompass a wide class of spin chain (i.e. a wide class of possible representations on the chain), and in particular for $sl(2)$ spin chains, it will encompass all types of representation, provided\(^5\) $L_0 > 1$. For $gl(N)$ spin chains, the representations entering the $L_0$-regular chain will have rectangular Young tableaux of same height $j_0$ and arbitrary width $a_\ell$.

**Lemma 3.2.** For $L_0$-regular spin chains with representations such that $j_\ell = j_0$, $\forall \ell$, and such that $(NL_0)$ is even, the following values define ‘small’ hole excitations:

\[
v_{\alpha}^{(j)} = \frac{N}{2} \delta_{j,j_0} \epsilon_{\alpha} \quad \text{with } \epsilon_{\alpha} = \begin{cases} +1 & \text{when } \alpha = \mathcal{L}, \mathcal{L} - 2, \ldots \vspace{1mm} \\
-1 & \text{when } \alpha = \mathcal{L} - 1, \mathcal{L} - 3, \ldots \end{cases}
\]

The hole numbers and spin are given by equations (3.4) and (3.5).

It leads, from equations (3.2), to the following values (for $1 \leq \alpha \leq \mathcal{L}$ and $1 \leq j \leq N - 1$):

\[
\mu_{\alpha}^{(j)} = \frac{\epsilon_{\alpha}}{2} \min(j, j_0)(N - \max(j, j_0))
\]

\[
\tilde{\nu}_{\alpha}^{(j)} = \frac{2L - \epsilon_{\alpha} NL_0}{2NL_0} \min(j, j_0)(N - \max(j, j_0)).
\]

\(^4\) On the other hand, the order is crucial in the computation of conserved quantities deduced from the transfer matrix: their explicit form may be completely different even though they have the same spectrum. In the same way, the order does matter for the calculation of Bethe vectors and correlation functions.

\(^5\) When $L_0 = 1$, one deals with spin $sl(2)$ spin chain, whose excitations are treated in lemma 3.1.
We deduce that the corresponding excitation exists when the length of the chain, $L$, is such that $(L \pm NL_0/2)$ is a multiple of $NL_0$. It is obviously possible only when $NL_0$ is even. This is in particular ensured for any $sl(2)$ spin chains and also for alternating $gl(N)$ spin chains.

**Remark 3.1.** Apart from the excitations introduced in lemmas 3.1 and 3.2, there are many other possible excitations. In particular, any multiple of these excitations is allowed:

\[ \tilde{\nu}_m^{(j)} = k \mu_m^{(j)}, \quad \forall m \in \mathcal{N}, \quad j = 1, \ldots, N - 1, \quad k \in \mathbb{Z}_+ \]  

(3.13)

where $\mu_m^{(j)}$ correspond to excitations of lemmas 3.1 or 3.2. For instance, the spin 1 excitation of [36] corresponds to $k = 2$ for an excitation of lemma 3.2.

**Remark 3.2.** For spin chains with a given length such that the vacuum exists (as determined in [31]), $\nu_n^{(j)}$ are integers, so that excitations must have integer $\mu_n^{(j)}$. We will say that the corresponding excitations lie in the vacuum sector. It is the point of view developed for example in [10, 15].

**Examples**

(1) We consider a $gl(N)$ spin chain where all sites carry the fundamental representation $\mathcal{N}$. Since the vacuum state is built on real strings only, it is the only type of strings one can suppress. Then,

\[ \tilde{\nu}_0^{(j)} = \frac{L(N - j)}{N} - \mu^{(j)}; \quad \tilde{\nu}_n^{(j)} = 0, \quad n > 0 \]  

(3.14)

and we get

\[ S_j = \mathcal{D}_0^{(j)} = 2\mu^{(j)} - \mu^{(j-1)} - \mu^{(j+1)}, \]  

(3.15)

in accordance with the results given above.

The excitation carrying the $sl(N)$ fundamental representation $\tilde{S} = (1, 0, 0, \ldots, 0)$ corresponds to the values given in lemma 3.1:

\[ \mu^{(j)} = \frac{N - j}{N}. \]  

(3.16)

The number of quasi-particles is $\mathcal{D}_0^{(j)} = \delta_{n,0} \delta_{j,1}$, as expected. This ‘first’ excitation lies in a spin chain with length such that $L - 1$ is a multiple of $N$. We remind that the vacuum is lying in a spin chain with $L$ multiple of $N$.

(2) For a $sl(2)$ spin chain with a spin $s$ representation on each site, the vacuum is built on $s \frac{1}{2}$ strings only. Thus, we get

\[ \tilde{\nu}_{s-1/2} = \frac{L}{2} - \mu \quad \text{and} \quad \tilde{\nu}_n = 0, \quad \forall n \in \frac{1}{2}\mathbb{Z}, \quad n \neq s - \frac{1}{2}. \]  

(3.17)

The resulting excited state has spin

\[ S/2 = 2s \mu \]  

and a number $\mathcal{D}_n = 4s \mu \delta_{n,s-1/2}$ of spin $\frac{1}{2}$ ‘elementary particles’ [10]. The hole excitation of lemma 3.1 corresponds to $\mu = \frac{1}{2}$.

\[ \text{doi:10.1088/1742-5468/2008/01/P01015} \]
They lead to non-vanishing hole numbers. They lead to a unique non-vanishing hole number $S$ given by

$$S = a_1 (2\mu_{n_1}^{(j)} - \mu_{n_1}^{(j-1)} - \mu_{n_1}^{(j+1)}) + a_2 (2\mu_{n_2}^{(j)} - \mu_{n_2}^{(j-1)} - \mu_{n_2}^{(j+1)}).$$

(3) If one considers an alternating spin chain, where the sites $2\ell + 1$ carry an sl($N$) representation defined by $(a_1, j_1)$ and the sites $2\ell$ carry a representation $(a_2, j_2)$. We suppose that $a_1 < a_2$, so that $a_\ell = 2n_\ell + 1$. Hole excitations have configurations:

$$\tilde{\nu}^{(j)}_n = \nu^{(j)}_n \quad \text{for } n \neq n_1, n_2$$

$$\tilde{\nu}^{(j)}_{n_k} = \nu^{(j)}_{n_k} - \mu^{(j)}_{n_k} \quad \text{for } k = 1, 2.$$  

The spin is

$$S_j = a_1 (2\mu_{n_1}^{(j)} - \mu_{n_1}^{(j-1)} - \mu_{n_1}^{(j+1)}) + a_2 (2\mu_{n_2}^{(j)} - \mu_{n_2}^{(j-1)} - \mu_{n_2}^{(j+1)}).$$

The excitations of lemma 3.1 are given by

$$\mu^{(j)}_{n_k} = \begin{cases} \frac{j(N - j_k)}{N} & j \leq j_k \quad k = 1, 2. \\ \frac{j_k(N - j)}{N} & j_k \leq j \end{cases}$$

They lead to non-vanishing hole numbers

$$\mathcal{D}^{(j)}_{n_1} = (2n_1 + 1)(\delta_{j,j_1} - \delta_{j,j_2})$$

$$\mathcal{D}^{(j)}_{n_2} = (2n_2 + 1)(\delta_{j,j_1} + (2n_2 + 1)\delta_{j,j_2}).$$

The spin $S = (S_1, \ldots, S_{N-1})$ is given by

$$S_j = 0 \quad \text{for } j \neq j_1, j_2 \quad \text{and} \quad S_{j_k} = a_k \quad \text{for } k = 1, 2$$

corresponding to a representation with Young tableau given in figure 1 (drawn for $j_1 > j_2$).

If we suppose furthermore that $j_1 = j_2 = j_0$, one can consider excitations as in lemma 3.2:

$$\mu^{(j)}_{n_k} = \begin{cases} (-1)^k \frac{j(N - j_0)}{2} & j \leq j_0 \quad k = 1, 2. \\ (-1)^k \frac{j_0(N - j)}{2} & j_0 \leq j \end{cases}$$

They lead to a unique non-vanishing hole number $\mathcal{D}^{(j)}_{n_2} = N(n_2 - n_1)\delta_{j,j_0}$ and have a spin $S = (S_1, \ldots, S_{N-1})$ with

$$S_j = 0 \quad \text{for } j \neq j_0 \quad \text{and} \quad S_{j_0} = N(n_2 - n_1).$$

3.2. Thermodynamical limit

For $m \in \mathcal{N}$, the quantum numbers for an excited state are

$$\tilde{Q}_{m,k}^{(j)} = k - \frac{1}{2}(\tilde{\nu}^{(j)}_m + 1) + \frac{1}{2} \sum_{d=1}^{\mathcal{D}_{m}^{(j)}} \text{sign}(k - k_{m,d}^{(j)}) \quad k = 1, \ldots, \tilde{\nu}_m^{(j)},$$

where we have introduced the sign function

$$\text{sign}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases}$$

DOI:10.1088/1742-5468/2008/01/P01015
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Figure 1. Young tableau of the hole excitation for an alternating spin chain \((a_1, j_1)\)-(\(a_2, j_2\)).

The numbers \( k^{(j)}_{m,d} \) are integers characterizing the holes.

They must fulfill the constraint

\[
1 \leq k^{(j)}_{m,d} \leq \tilde{\nu}^{(j)}_m.
\]

As before, we take the thermodynamical limit of the Bethe ansatz equation and we obtain (see e.g. [15]):

\[
\sum_{n \in \mathbb{N}} \left\{ \int_{-\infty}^{\infty} d\lambda \left( s^{(j)-1}_n(\lambda) + s^{(j)+1}_n(\lambda) \right) \Psi^{(m,n)}_-(\lambda_0 - \lambda) + \int_{-\infty}^{\infty} d\lambda \, s^{(j)}_n(\lambda) \Psi^{(m,n)}_2(\lambda_0 - \lambda) \right\}
\]

\[
= -2\pi \left( s^{(j)}_0(\lambda_0) + \frac{1}{L} \sum_{d=1}^{D^{(j)}_m} \delta(\lambda_0 - \tilde{\lambda}^{(j)}_{m,d}) \right) + \frac{1}{L} \sum_{\alpha=1}^{L} \left( \sum_{\ell \in I_{\alpha}} \delta_{j,\ell} \right) \Psi^{(m)}_{a_\alpha}(\lambda_0)
\]

\[
\forall \lambda_0 \in ]-\infty, \infty[, \; \forall j = 1, \ldots, N - 1, \; \forall m \in \mathcal{N}
\]

(3.30)

where \( s^{(j)}_m(\lambda) \) is the density of \((2m + 1)\)-strings in the sea \( j \) for the considered excited state. The parameter \( \tilde{\lambda}^{(j)}_{m,d} \) is the image of \((\widetilde{Q}_{m,k^{(j)}_{m,d}} - 1)/L \) in the limit \( L \to \infty \).

Performing a Fourier transform, the Bethe ansatz equations become

\[
2\pi \hat{\Psi}(p) \hat{S}(p) = \Lambda(p) - \frac{1}{L} \Delta(p),
\]

where \( \hat{\Psi}(p), \Lambda(p) \) are given in section 2 and

\[
\Delta(p) = \sum_{j=1}^{N-1} \sum_{\alpha=1}^{L} \sum_{d=1}^{D^{(j)}_{n_\alpha}} \exp(ip\tilde{\lambda}_{n_\alpha,d}^{(j)}) e_j^{(N-1)} \otimes e_\alpha^{(L)}
\]

(3.32)

\[
\hat{S}(p) = \sum_{j=1}^{N-1} \sum_{\alpha=1}^{L} \hat{s}_{n_\alpha}^{(j)}(p) e_j^{(N-1)} \otimes e_\alpha^{(L)}.
\]

(3.33)

By linearity, the solutions take the following form

\[
\hat{S}(p) = \hat{\Sigma}(p) + \frac{1}{L} \hat{R}(p)
\]

(3.34)

where \( \hat{\Sigma}(p) \) gathers the densities for the vacuum (2.20) and \( \hat{R}(p) = \sum_{j=1}^{N-1} \sum_{\alpha=1}^{L} \hat{\rho}_n^{(j)}(p) e_j^{(N-1)} \otimes e_\alpha^{(L)} \) corresponds to the densities of holes. It is solution of the following linear system

\[
2\pi \hat{\Psi}(p) \hat{R}(p) = -\Delta(p).
\]

(3.35)
Inverting the matrix $\hat{\Psi}(p)$, we get the solution

$$\hat{R}(p) = \frac{1}{2\pi} \left( A(p)^{-1} \otimes (\hat{\Psi}_1(p))^{-1} \right) \Delta(p)$$

(3.36)

where the inverse of the matrix $A(p)$ is given by (see e.g. [32])

$$(A(p)^{-1})_{ij} = \frac{\sinh((N - \max(i,j))(|p|/2)) \sinh(\min(i,j)(|p|/2))}{\sinh(N(|p|/2)) \sinh(|p|/2)}.$$  

(3.37)

The inverse of the $L \times L$ matrix $\hat{\Psi}_1(p)$ is provided by the following lemma (we remind that we ordered $n_1 < n_2 < \cdots < n_L$).

**Lemma 3.3.** The inverse of $\hat{\Psi}_1(p)$ is a symmetric tridiagonal matrix. Its diagonal elements are given by, for $1 \leq \alpha \leq L$

$$(\hat{\Psi}_1(p))^{-1}_{\alpha\alpha} = -\frac{\sinh(|p|/2) \sinh(|p|(n_{\alpha+1} - n_{\alpha-1}))}{\sinh(|p|(n_{\alpha} - n_{\alpha-1})) \sinh(|p|(n_{\alpha+1} - n_{\alpha}))}$$

(3.38)

where, by convention, we set $n_0 = -\frac{1}{2}$ and $n_{L+1} = +\infty$. We also used the limit $\sinh(\infty - a)/\sinh(\infty - b) = \exp(b - a)$. The non-vanishing non-diagonal elements read, for $1 \leq \alpha < L - 1$,

$$(\hat{\Psi}_1(p))^{-1}_{\alpha,\alpha+1} = \frac{\sinh(|p|/2)}{\sinh(|p|(n_{\alpha+1} - n_{\alpha}))} = (\hat{\Psi}_1(p))^{-1}_{\alpha+1,\alpha}.$$  

(3.39)

In the particular case $L = 1$, this matrix reduces to a number:

$$\left(\hat{\Psi}_1(p)\right)^{-1} = -\sinh\left(\frac{|p|}{2}\right) \frac{e^{|p|(n_1+1/2)}}{\sinh(|p|(n_1 + 1/2))}.$$  

(3.40)

**Proof.** Direct computation on the product $(\hat{\Psi}_1(p))^{-1} \hat{\Psi}_1(p)$.

Let us introduce useful functions, for $1 \leq \alpha \leq L$,  

$$\hat{h}_{j,k}^{(\alpha)}(p) = -\frac{1}{2\pi} \frac{\sinh((N - \max(j,k))(|p|/2)) \sinh(\min(j,k)(|p|/2))}{\sinh(N(|p|/2)) \sinh((n_{\alpha+1} - n_{\alpha})|p|)}$$

(3.41)

and

$$\hat{g}_{j,k}^{(\alpha)}(p) = \hat{h}_{j,k}^{(\alpha)}(p) \frac{\sinh((n_{\alpha+1} - n_{\alpha-1})|p|)}{\sinh((n_{\alpha} - n_{\alpha-1})|p|)}$$

(3.42)

where the convention $n_0 = -1/2$ and $n_{L+1} = \infty$ has been applied.

**Theorem 3.4.** Let us consider a $L_0$-regular spin chain based on $gl(N)$. For the excited states corresponding to holes in the filled seas, the densities are, for $1 \leq j \leq N - 1$ and $1 \leq \alpha \leq L$,

$$s_{n\alpha}^{(j)}(\lambda) = \sigma_{n\alpha}^{(j)}(\lambda) + \frac{1}{L} \rho_{n\alpha}^{(j)}(\lambda)$$

(3.43)
where $\sigma^{(j)}_{nm}(\lambda)$ are the densities (2.21) for the vacuum and the corrections at order $1/L$ are

$$
\rho^{(j)}_{nm}(\lambda) = \sum_{k=1}^{N-1} \left( - \sum_{d=1}^{D_{nm}^{(k-1)}} h^{(\alpha-1)}_{jk}(\lambda - \tilde{\lambda}^{(k)}_{nm,d}) + \sum_{d=1}^{D_{nm}^{(k)}} g^{(\alpha)}_{jk}(\lambda - \tilde{\lambda}^{(k)}_{nm,d}) - \sum_{d=1}^{D_{nm}^{(k+1)}} h^{(\alpha)}_{jk}(\lambda - \tilde{\lambda}^{(k)}_{nm+1,d}) \right). 
$$

(3.44)

The functions $h^{(\alpha)}_{jk}(\lambda)$ and $g^{(\alpha)}_{jk}(\lambda)$ are the inverse Fourier transform of (3.41) and (3.42) respectively. By convention, we used $D_{nm}^{(0)} = D_{-1/2}^{(k)} = 0$ and $D_{nm}^{(k)} = D_{\infty}^{(k)} = 0$.

The corrections to the densities, for the particular case $L = 1$, reduce to

$$
\rho^{(j)}_{n1}(\lambda) = \sum_{k=1}^{N-1} \sum_{d=1}^{D_{n1}^{(k)}} g_{jk}(\lambda - \tilde{\lambda}^{(k)}_{n1,d})
$$

(3.45)

where $g_{jk}(\lambda) = g^{(1)}_{jk}(\lambda)$ (for $L = 1$, $n_0 = -1/2$, $n_2 = \infty$).

**Proof.** We give the proof only for $\alpha = 1$. The other density corrections are obtained similarly. Projecting the linear system (3.36) on its first component and using the lemma 3.3, we get

$$
\hat{\rho}^{(j)}_{n1}(p) = -\frac{1}{2\pi} \sum_{k=1}^{N-1} \sum_{d=1}^{D_{n1}^{(k)}} \left( \sum_{d=1}^{D_{n1}^{(k)}} \hat{g}^{(1)}_{jk}(p) \exp(\text{i}p\tilde{\lambda}^{(k)}_{n1,d}) - \sum_{d=1}^{D_{n1}^{(k)}} \hat{h}^{(1)}_{jk}(p) \exp(\text{i}p\tilde{\lambda}^{(k)}_{n2,d}) \right) 
$$

(3.46)

Performing an inverse Fourier transform and using the conventions $n_0 = -\frac{1}{2}$ and $D_{n0}^{(k)} = 0$, we get the result (3.44) for $\alpha = 1$.

Let us remark that the explicit expressions of $h^{(1)}_{jk}(\lambda)$ and $g^{(1)}_{jk}(\lambda)$ are not needed, since physical quantities are computed via their Fourier transform (see for instance section 5.2).

### 3.3. Examples

1. For a $gl(N)$ fundamental spin chain, the corrections to the real root densities are

$$
\hat{\rho}^{(j)}_{0}(p) = -\frac{1}{2\pi} \sum_{k=1}^{N-1} \sum_{d=1}^{D_{0}^{(k)}} \frac{\sinh((N - \max(j,k))(|p|/2)) \sinh(\min(j,k)(|p|/2)) e^{(|p|/2) + \text{i}p\tilde{\lambda}^{(k)}_d}}{\sinh(N(|p|/2)) \sinh(|p|/2)}
$$

(3.47)

where we noted $\tilde{\lambda}^{(k)}_d$ for $\tilde{\lambda}^{(k)}_{0,d}$.

2. For a $gl(2)$ spin $s$ chain, the corrections to the root densities of length $2s$ are

$$
\hat{\rho}_{s-1/2}(p) = -\frac{1}{2\pi} \sum_{d=1}^{D_{s-1/2}} \frac{\sinh^2(|p|/2)}{\sinh(|p|) \sinh(s|p|)} e^{s|p| + \text{i}p\tilde{\lambda}_d}
$$

(3.48)

where $\tilde{\lambda}_d \equiv \tilde{\lambda}^{(1)}_{s-(1/2),d'}$.

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(3) For a $gl(N)$ alternating spin chain with $n_1 < n_2$, we get

$$\tilde{\rho}_{n_1}^{(j)}(p) = -\frac{1}{2\pi} \sum_{k=1}^{N-1} \frac{\sinh((N - \max(j, k))(|p|/2)) \sinh(\min(j, k)(|p|/2))}{\sinh(N(|p|/2)) \sinh((n_2 - n_1)|p|)}$$

$$\times \left\{ \sinh((n_2 + 1/2)|p|) \sum_{d=1}^{D_{n_1}^{(k)}} e^{ip\tilde{\lambda}_{n_1,d}^{(k)}} - \sum_{d=1}^{D_{n_2}^{(k)}} e^{ip\tilde{\lambda}_{n_2,d}^{(k)}} \right\}$$

and

$$\tilde{\rho}_{n_2}^{(j)}(p) = -\frac{1}{2\pi} \sum_{k=1}^{N-1} \frac{\sinh((N - \max(j, k))(|p|/2)) \sinh(\min(j, k)(|p|/2))}{\sinh(N(|p|/2)) \sinh((n_2 - n_1)|p|)}$$

$$\times \left\{ -\sum_{d=1}^\nu e^{ip\tilde{\lambda}_{n_1,d}^{(k)}} + \exp((n_2 - n_1)|p|) \sum_{d=1}^\nu e^{ip\tilde{\lambda}_{n_2,d}^{(k)}} \right\}.$$ (3.49)

4. General excited states

We now turn to a more general excited state. It corresponds to the case where holes are created in the filled seas and new strings added in other seas. As in the rest of the paper, we stick to the string hypothesis, being interested only in the first contributions of the excitations to the energies.

4.1. Valences and spins

The configuration of an excited state is characterized by

$$\tilde{\nu}_n^{(j)} = \nu_n^{(j)} - \mu_n^{(j)} \quad \text{for } n \in \mathcal{N} \quad \text{and} \quad \tilde{\nu}_n^{(j)} \geq 0 \quad \text{for } n \in \overline{\mathcal{N}},$$ (4.1)

where we kept the notation $\nu_n^{(j)}$, $n \in \mathcal{N}$, for the vacuum configuration. The set $\overline{\mathcal{N}}$ (complementary to $\mathcal{N}$) has been introduced in (2.9). Obviously, if $\tilde{\nu}_n^{(j)} = 0, \forall n \in \overline{\mathcal{N}}$, we recover the previous case with holes only.

The corresponding valences and spins are given by, for $n \in \frac{1}{2}\mathbb{Z}_{\geq 0}$,

$$\tilde{\nu}_n^{(j)} = \nu_n^{(j)} + \sum_{m \in \mathcal{N}}' \min(2n + 1, 2m + 1)(2\nu_m^{(j)} - \mu_m^{(j-1)} - \mu_m^{(j+1)})$$

$$- \sum_{m \in \overline{\mathcal{N}}} \min(2n + 1, 2m + 1)(2\tilde{\nu}_m^{(j)} - \tilde{\nu}_m^{(j-1)} - \tilde{\nu}_m^{(j+1)}).$$ (4.2)

$$S_j = \sum_{m \in \mathcal{N}}' \sum_{m \in \overline{\mathcal{N}}} \min(2m + 1)(2\nu_m^{(j)} - \mu_m^{(j-1)} - \mu_m^{(j+1)})$$

$$- \sum_{m \in \overline{\mathcal{N}}} \sum_{m \in \mathcal{N}}' (2\tilde{\nu}_m^{(j)} - \tilde{\nu}_m^{(j-1)} - \tilde{\nu}_m^{(j+1)}).$$ (4.3)

Obviously, demanding the valences and the spin to be positive imposes constraints between the number of holes, $D_m^{(j)} = \tilde{\nu}_m^{(j)} - \nu_m^{(j)} (m \in \mathcal{N})$, and the number of new strings, $\tilde{\nu}_n^{(j)} (n \in \overline{\mathcal{N}})$. We study in more details particular excitations in section 4.4.

doi:10.1088/1742-5468/2008/01/P01015
The new strings of length $2n + 1$ with $n \in \mathcal{N}$ are characterized by quantum numbers $\{Q^{(j)}_{n,k} | 1 \leq k \leq \tilde{\nu}^{(j)}_{n}\}$ and by the Bethe roots $\{\lambda^{(j)}_{n,k} | 1 \leq k \leq \tilde{\nu}^{(j)}_{n}\}$ (these sets may be empty). These quantum numbers are bounded, namely, for $n \in \mathcal{N}$,

$$\frac{1 - \tilde{P}^{(j)}_{n}}{2} \leq Q^{(j)}_{n,k} \leq \frac{\tilde{P}^{(j)}_{n} - 1}{2}. \quad (4.4)$$

Let us emphasize the difference between $\lambda^{(j)}_{n,k} (n \in \mathcal{N})$, Bethe roots for the new strings (the ones added to the vacuum configuration), and $\tilde{\lambda}^{(j)}_{n,k} (n \in \mathcal{N})$, missing Bethe roots in the filled seas (holes in the vacuum configuration).

### 4.2. Densities for a general excited state

In this section, we compute the densities for the general excited states in the thermodynamical limit. This is done in two steps: first, the Bethe equations for $n \in \mathcal{N}$ provide the correction to the densities in terms of the new Bethe roots $\lambda^{(j)}_{n,k}, n \in \mathcal{N}$ (see theorem 4.1). Second, the Bethe equations for $n \in \mathcal{N}$ couple these new Bethe roots with the holes in the filled seas (see theorem 4.2). These relations depend on the choice of the quantum numbers $\{Q^{(j)}_{n,k} | n \in \mathcal{N}, 1 \leq k \leq \tilde{\nu}^{(j)}_{n}\}$.

Using again linearity of the Bethe equations, the densities for these excited states rewrite:

$$t^{(j)}_{n}(\lambda) = s^{(j)}_{n}(\lambda) + \frac{1}{L} c^{(j)}_{n}(\lambda), \quad n \in \mathcal{N}, \quad (4.5)$$

$$t^{(j)}_{m}(\lambda) = \frac{1}{L} \sum_{\ell=1}^{\tilde{\nu}^{(j)}_{m}} \delta(\lambda - \lambda^{(j)}_{\ell,m}), \quad m \in \mathcal{N}, \quad (4.6)$$

where $s^{(j)}_{n}(\lambda) = \sigma^{(j)}_{n}(\lambda) + (1/L)p^{(j)}_{n}(\lambda)$ are the densities when only holes are created in the vacuum configuration, as given in theorem 3.4, while $c^{(j)}_{n}(\lambda)$ are the corrections due to the new added strings. We have assumed that the number of these new strings remains finite even in the limit $L \to \infty$, hence the form (4.6).

From the Bethe equations for $m \in \mathcal{N}$ and performing a calculation similar to the one of section 3, one shows that $\lambda^{(j)}_{n}(\lambda)$ must satisfy the following equations

$$\sum_{n \in \mathcal{N}} \left\{ \int_{-\infty}^{\infty} d\lambda \left( c^{(j-1)}_{n}(\lambda) + c^{(j+1)}_{n}(\lambda) \right) \Psi_{-1}^{(m,n)}(\lambda_{0} - \lambda) + \int_{-\infty}^{\infty} d\lambda \, c^{(j)}_{n}(\lambda) \Psi_{2}^{(m,n)}(\lambda_{0} - \lambda) \right\}$$

$$+ 2\pi c^{(j)}_{m}(\lambda_{0}) + \sum_{n \in \mathcal{N}} \left\{ \sum_{\ell=1}^{\tilde{\nu}^{(j-1)}_{n}} \Psi_{-1}^{(m,n)}(\lambda_{0} - \lambda^{(j-1)}_{n,\ell}) + \sum_{\ell=1}^{\tilde{\nu}^{(j)}_{n}} \Psi_{2}^{(m,n)}(\lambda_{0} - \lambda^{(j)}_{n,\ell}) \right\} \Psi^{(m,n)}_{-1}(\lambda_{0} - \lambda^{(j+1)}_{n,\ell}) \right\} = 0$$

$$\forall \lambda_{0} \in ]-\infty, \infty[, \quad \forall j = 1, \ldots, N - 1, \quad \forall m \in \mathcal{N}. \quad (4.7)$$
The densities for a general excited state are given by (4.6) and

\[
m \in \mathcal{N} \mapsto \begin{cases} 
\gamma^-(m) = n_\alpha \in \mathcal{N} \cup \{n_0 = -\frac{1}{2}, n_{\mathcal{L}+1} = \infty\} \\
\gamma^+(m) = n_{\alpha+1} \in \mathcal{N} \cup \{n_0 = -\frac{1}{2}, n_{\mathcal{L}+1} = \infty\}
\end{cases}
\]

with \(n_\alpha \leq m < n_{\alpha+1}\). \hspace{1cm} (4.8)

In words, the functions \(\gamma^\pm\) associate to any element in \(\mathcal{N}\), the two closest numbers belonging to \(\mathcal{N} \cup \{-\frac{1}{2}, \infty\}\).

We also need the following functions, for \(m \in \mathcal{N}\),

\[
\tilde{\omega}_m^\pm(p) = \frac{\sinh(p|m - \gamma^\pm(m)|)}{\sinh(p(\gamma^+(m) - \gamma^-(m)))}.
\]

Their inverse Fourier transform is

\[
\omega_m^\pm(\lambda) = \frac{\pi}{\gamma^+(m) - \gamma^-(m)} \frac{\sin\left(\frac{\pi|m - \gamma^\pm(m)|}{\gamma^+(m) - \gamma^-(m)}\right)}{\cosh\left(\frac{\pi\lambda}{\gamma^+(m) - \gamma^-(m)}\right) + \cos\left(\frac{\pi|m - \gamma^\pm(m)|}{\gamma^+(m) - \gamma^-(m)}\right)}.
\]

Let us remark that, for \(m \in \mathcal{N}\) finite and such that \(\gamma^+(m) = \infty\), these functions reduce to

\[
\omega_m^+(\lambda) = \frac{2(m - \gamma^-(m))}{\lambda^2 + (m - \gamma^-(m))^2} \quad \text{and} \quad \omega_m^-(\lambda) = 0.
\]

Now, we can give the corrections to the densities for a general excited state.

**Theorem 4.1.** The densities for a general excited state are given by (4.6) and by

\[
t_n^{(j)}(\lambda) = s_n^{(j)}(\lambda) + (1/L) c_n^{(j)}(\lambda), \quad n \in \mathcal{N}
\]

where \(s_n^{(j)}(\lambda)\) are given in theorem 3.4. The corrections \(c_n^{(j)}(\lambda)\) are given by, for \(1 \leq \alpha \leq \mathcal{L}\) and \(1 \leq j \leq N - 1\),

\[
c_n^{(j)}(\lambda) = -\frac{1}{2\pi} \left( \sum'_{n_\alpha-1 < m < n_\alpha} \omega_m^- (\lambda - \lambda_{m,\ell}) + \sum'_{n_\alpha < m < n_{\alpha+1}} \omega_m^+ (\lambda - \lambda_{m,\ell}) \right).
\]

We remind that \(n_0 = -1/2\) and \(n_{\mathcal{L}+1} = \infty\). By convention, we also set \(\omega_0^- (\lambda) = 0\).

**Proof.** Performing a Fourier transform of equation (4.7) and inverting the linear system, we get, for \(1 \leq \alpha \leq \mathcal{L}\),

\[
\tilde{c}_{n_\alpha}^{(j)}(p) = -\frac{1}{2\pi} \sum'_{m \in \mathcal{N}} \tilde{\nu}_m^{(j)}(\omega_m^- (\lambda_{m,\ell})) \sum'_{\ell=1}^\mathcal{L} (\hat{\Psi}_{-1}(p))^{-1}_{\alpha\beta} \tilde{\Psi}_{-1}^{(n_\alpha, m)}(p).
\]

\[\text{doi:10.1088/1742-5468/2008/01/P01015}\]
The final result is obtained through inverse Fourier transform of this last equality.

In theorem 4.1, the densities have been expressed in terms of the parameters $\lambda_{m,l}^{(j)}$ ($m \in \mathcal{N}$). These parameters are implicit functions of $\lambda_{n,l}^{(j)}$, as stated in the following theorem.

**Theorem 4.2.** The relations between the Bethe roots corresponding to holes in the filled seas and the ones corresponding to new strings are given by, for $m \in \mathcal{N}$,

$$2\pi Q_{m,k}^{(j)} = \sum_{d=1}^{D_{\gamma^{m}(m)}} \Omega_{m}^{+}(\lambda_{m,k}^{(j)} - \lambda_{\gamma^{+}(m),d}^{(j)}) + \sum_{d=1}^{D_{\gamma^{-}(m)}} \Omega_{m}^{-}(\lambda_{m,k}^{(j)} - \lambda_{\gamma^{-}(m),d}^{(j)})$$

$$- \sum_{r=\gamma^{-}(m)+1/2}^{\gamma^{+}(m)-1/2} \left\{ \sum_{\ell=1}^{j} \tilde{\nu}_{m,k}^{(j-1)} F_{1}^{(r,m)}(\lambda_{m,k}^{(j)} - \lambda_{r,\ell}^{(j-1)}) + \sum_{\ell=1}^{j} \tilde{\nu}_{m,k}^{(j)} F_{2}^{(r,m)}(\lambda_{m,k}^{(j)} - \lambda_{r,\ell}^{(j)}) \right\}$$

$$+ \sum_{\ell=1}^{j+1} \tilde{\nu}_{m,k}^{(j+1)} F_{1}^{(r,m)}(\lambda_{m,k}^{(j)} - \lambda_{r,\ell}^{(j+1)})$$

where the functions $\Omega_{m}^{\pm}$ are a primitive of $\omega_{m}^{\pm}$

$$\Omega_{m}^{\pm}(\lambda) = 2 \arctan \left[ \tan \left( \frac{\pi |m - \gamma^{\pm}(m)|}{2 \gamma^{+}(m) - \gamma^{-}(m)} \right) \right]$$

and the functions $F_{a}^{(r,m)}(\lambda)$ are

$$F_{-1}^{(r,m)}(\lambda) = - \sum_{q=|r-m|+1/2}^{r+m-2\gamma^{-}(m)-1/2} \Gamma_{q}^{(r)}(\lambda)$$

$$F_{2}^{(r,m)}(\lambda) = \begin{cases} 
\Gamma_{2m-2\gamma^{-}(m)}^{(r)}(\lambda) + 2 \sum_{q=1}^{2m-2\gamma^{-}(m)} \Gamma_{q}^{(r)}(\lambda) & \text{if } m = r \\
\Gamma_{r+m-2\gamma^{-}(m)}^{(r)}(\lambda) + \Gamma_{|r-m|}^{(r)}(\lambda) + 2 \sum_{q=|r-m|+1}^{r+m-2\gamma^{-}(m)} \Gamma_{q}^{(r)}(\lambda) & \text{if } m \neq r 
\end{cases}$$

$$\Gamma_{q}^{(r)}(\lambda) = -2 \arctan \left[ \tan \left( \frac{\pi \frac{q - \gamma^{+}(r) + \gamma^{-}(r)}{2 \gamma^{+}(r) - \gamma^{-}(r)}}{\lambda} \right) \right].$$
we get

\[
\Phi_{\alpha}(r-n_{\ell}) - \Phi_{\beta}(r-n_{m}) = \varphi_{2q}(\lambda) \tag{4.21}
\]

and (see equations (2.3)–(2.6))

\[
\Phi_{\alpha}(r-m_{\ell}) = \varphi_{2q}(\lambda) \tag{4.22}
\]

Notice also that \(\Phi_{\alpha}(r,m) = \Phi_{\beta}(m,r)\) when \(\gamma^{-}(m) = \gamma^{-}(r)\), which is always the case in the relations where these functions are used.

4.3. Examples

(1) For \(gl(N)\) fundamental spin chain, we get \(\mathcal{N} = \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}\) and \(\gamma^{-}(m) = 0\), \(\gamma^{+}(m) = \infty\) for any \(m \in \mathcal{N}\). Thus, the densities for the real Bethe roots reduce to

\[
\phi_{0}^{(j)}(\lambda) = \frac{2}{\pi} \sum_{m=1/2}^{\infty} \sum_{\ell=1}^{r} \frac{m}{(\lambda-\lambda_{m,\ell}^{(j)})^{2} + m^{2}}. \tag{4.23}
\]

The constraints (4.16) become, for \(m \in \mathcal{N}\),

\[
2\pi Q_{m,k}^{(j)} = \sum_{d=1}^{\rho_{m}^{(j)}} \varphi_{2m}(\lambda_{m,k}^{(j)} - \tilde{\lambda}_{d}^{(j)}) - \sum_{r=1}^{\infty} \left\{ \sum_{\ell=1}^{\rho_{r,m}^{(j-1)}} \Phi_{-1}^{(r-1/2, m-1/2)}(\lambda_{m,k}^{(j)} - \lambda_{r,\ell}^{(j-1)}) + \sum_{\ell=1}^{\rho_{r,m}^{(j+1)}} \Phi_{-1}^{(r-1/2, m-1/2)}(\lambda_{m,k}^{(j)} - \lambda_{r,\ell}^{(j+1)}) \right\}. \tag{4.24}
\]

We remind that \(\tilde{\lambda}_{d}^{(j)}\) stand for \(\tilde{\lambda}_{0,d}^{(j)}\).

For the particular case \(\nu_{m}^{(j)} = 0\) for any \(m \geq 1\) (i.e. only strings of length 2 are added), we get

\[
2\pi Q_{1/2,k}^{(j)} = \sum_{\ell=1}^{\rho_{1/2}^{(j)}} \varphi_{1}(\lambda_{1/2,k}^{(j)} - \lambda_{1/2,\ell}^{(j-1)}) - \sum_{\ell=1}^{\rho_{1/2}^{(j)} + 1} \varphi_{1}(\lambda_{1/2,k}^{(j)} - \lambda_{1/2,\ell}^{(j)}), \tag{4.25}
\]

We recover a previous result computed in \([33]\).

(2) For a \(gl(2)\) spin \(s\) spin chain, we get \(\mathcal{N} = \{0, \frac{1}{2}, \ldots, s-\frac{3}{2}, s-1, s, s+\frac{1}{2}, \ldots\}\). Thus \(\gamma^{+}(m) = s - \frac{1}{2}, \gamma^{-}(m) = -\frac{1}{2}\) for \(0 \leq m \leq s - 1\) and \(\gamma^{+}(m) = \infty, \gamma^{-}(m) = s - \frac{1}{2}\) for

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\[ s \leq m. \] The densities are

\[
c_s(\lambda) = -\sum_{m=0}^{s-1} \sum_{t=1}^{\nu_m} \frac{1}{2s} \frac{\sin(\pi(2m+1)/2s)}{\cosh(\pi(\lambda - \lambda_{m,t})/s) + \cos(\pi(2m+1)/2s)} - \sum_{m=s}^{\infty} \sum_{t=1}^{\nu_m} \frac{1}{\pi (\lambda - \lambda_{m,t})^2 + (m - s + 1/2)^2},
\]

(4.26)

The constraints become, for \( m = 0, \frac{1}{2}, \ldots, s - 1, \)

\[
2\pi Q_{m,k} = 2 \sum_{d=1}^{D_{s-1/2}} \arctan \left[ \tan \left( \frac{\pi m}{2s - 1} \right) \tanh \left( \frac{\pi (\lambda_{m,k} - \tilde{\lambda}_d)}{2s - 1} \right) \right] - \sum_{r=1/2}^{s-1} \sum_{t=1}^{r} \tilde{\nu}_r (\lambda_{m,k} - \lambda_{r,t})
\]

(4.27)

and, for \( m = s, s + 1/2, \ldots, \)

\[
2\pi Q_{m,k} = 2 \sum_{d=1}^{D_{s-1/2}} \arctan \left( \frac{\lambda_{m,k} - \tilde{\lambda}_d}{m - s + 1/2} \right) - \sum_{r=s}^{\infty} \sum_{t=1}^{\nu_r} \Phi_2^{(r,s,m-s)} (\lambda_{m,k} - \lambda_{r,t})
\]

(4.28)

keeping the notation \( \tilde{\lambda}_d \equiv \tilde{\lambda}_{s-1/2,d} \) for the Bethe roots of the holes.

(3) For an alternating spin chain with \( n_1 < n_2, \) we get \( \gamma^-(m) = -\frac{1}{2}, \ \gamma^+(m) = n_1 \) for \( 0 \leq m \leq n_1 - \frac{1}{2}, \ \gamma^-(m) = n_1, \ \gamma^+(m) = n_2 \) for \( n_1 + \frac{1}{2} \leq m \leq n_2 - \frac{1}{2} \) and \( \gamma^-(m) = n_2, \ \gamma^+(m) = \infty \) for \( n_2 + \frac{1}{2} \leq m. \) Thus, the densities are

\[
c_{n_1}^{(j)}(\lambda) = -\frac{1}{2} \sum_{m=0}^{n_1-1/2} \sum_{t=1}^{\nu_m} \frac{1}{n_1 + 1/2} \frac{\sin \left( \frac{\pi m + n_1 + 1/2}{n_1 + 1/2} \right)}{\cosh \left( \frac{\pi (\lambda - \lambda_{m,t})}{n_1 + 1/2} \right) + \cos \left( \frac{\pi m + n_1 + 1/2}{n_1 + 1/2} \right)} - \frac{1}{2} \sum_{m=n_1+1/2}^{n_2-1/2} \sum_{t=1}^{\nu_m} \frac{1}{n_2 - n_1} \frac{\sin \left( \frac{\pi m + n_2 - n_1}{n_2 - n_1} \right)}{\cosh \left( \frac{\pi (\lambda - \lambda_{m,t})}{n_2 - n_1} \right) + \cos \left( \frac{\pi m + n_2 - n_1}{n_2 - n_1} \right)}
\]

(4.29)

and

\[
c_{n_2}^{(j)}(\lambda) = -\frac{1}{2(n_2 - n_1)} \sum_{m=n_1+1/2}^{n_2-1/2} \sum_{t=1}^{\nu_m} \frac{\sin \left( \frac{\pi m + n_2 - n_1}{n_2 - n_1} \right)}{\cosh \left( \frac{\pi (\lambda - \lambda_{m,t})}{n_2 - n_1} \right) + \cos \left( \frac{\pi m + n_2 - n_1}{n_2 - n_1} \right)} - \frac{1}{\pi} \sum_{m=n_2+1/2}^{\infty} \sum_{t=1}^{\nu_m} \frac{m - n_2}{(\lambda - \lambda_{m,t})^2 + (m - n_2)^2}.
\]

(4.30)

The constraints between the roots are still given by theorem 4.2. Since it is a rather cumbersome equation, we refrain from giving its explicit form.
4.4. Excitations within the vacuum sector

We consider an excitation in the vacuum sector, as defined in remark 3.2: its hole configuration is given by \( \{ \tilde{\mu}_m^{(j)}, m \in \mathcal{N}, j = 1, \ldots, N - 1 \} \), where \( \tilde{\mu}_m^{(j)} = N \mu_m^{(j)} \) and \( \{ \mu_m^{(j)} \} \) describes the hole configuration as given in lemma 3.1. To this configuration, if we suppose that \( n_1 > 0 \), one can add new strings of the following type.

We first introduce (for \( m \in \mathcal{N} \)):
\[
\tilde{w}_m^{(j)} = \sum_{\alpha=1}^{L} \delta_{m, \tilde{n}_{\alpha}} v_n^{(j)} \text{ where } \tilde{n}_{\alpha} = \max(m \in \mathcal{N}, m < n_{\alpha}).
\] (4.31)

We remind that \( v_m^{(j)} = 2 \tilde{\nu}_m^{(j)} - \tilde{\nu}_m^{(j-1)} - \tilde{\nu}_m^{(j+1)} \) (\( m \in \mathcal{N} \)) and \( \tilde{w}_n^{(j)} = 2 \tilde{\nu}_n^{(j)} - \tilde{\nu}_n^{(j-1)} - \tilde{\nu}_n^{(j+1)} \).

The set \( \{ \mu_m^{(j)}, \tilde{\nu}_n^{(j)}, m \in \mathcal{N}, n \in \mathcal{N}, j = 1, \ldots, N - 1 \} \) define a configuration with a spin
\[
S_j = \sum_{\alpha=1}^{L} 2(n_{\alpha} - \tilde{n}_{\alpha}) v_{n_{\alpha}}
\] (4.32)
and hole numbers
\[
\mathcal{D}_m^{(j)} = 0 \quad \text{if } m < n_1
\]
\[
\mathcal{D}_m^{(j)} = \sum_{\alpha=\beta_1+1}^{\beta_2} 2(m - \tilde{n}_{\alpha}) v_n^{(j)} + \sum_{\alpha=\beta_2+1}^{L} 2(n_{\alpha} - \tilde{n}_{\alpha}) v_{n_{\alpha}} \quad \text{if } n_{\beta_1} \leq m \leq \tilde{n}_{\beta_2}
\]
\[
\mathcal{D}_m^{(j)} = \sum_{\alpha=1}^{L} 2(n_{\alpha} - \tilde{n}_{\alpha}) v_{n_{\alpha}} \quad \text{if } n_L \leq m.
\] (4.33)

Above, \( \beta_1 \) and \( \beta_2 \) depend on \( m \) and are such that
\[
n_{\beta_1} \leq m < n_{\beta_1+1} \quad \text{and} \quad \tilde{n}_{\beta_2-1} < m \leq \tilde{n}_{\beta_2}.
\] (4.34)

This construction generalizes the construction given in [10] for the spin s XXX spin chain.

Example

For the \( gl(2) \) spin s spin chain, the lemma 3.1 hole excitation is given by \( \tilde{\nu}_{s-1/2} = (L - 2)/2 \), leading to \( \tilde{w}_n = 2\delta_{s-1,n} \), i.e. \( \tilde{\nu}_{s-1} = 1 \). Then we get an excited state with spin \( S/2 = 1 \), corresponding to the adjoint representation of \( sl(2) \). As already mentioned, we recover the results given in [10].

5. Hamiltonian and energies

5.1. \( L_0 \)-local Hamiltonians

Definition 5.1. An Hamiltonian is called \( L_0 \)-local when it possesses only interactions between neighbours separated by at most \( L_0 - 1 \) sites.

This definition is quite natural. It is used in the following:

Conjecture 5.2. For any \( L_0 \)-regular spin chain there always exists an Hamiltonian which is \( L_0 \)-local.
This conjecture is supported by theorem 5.3 given below. To formulate it, we first introduce local representations of \( gl(N) \). On each site \( \ell \) we denote the representation of the \( gl(N) \) algebra by

\[
\mathcal{E}_{1,\ell} = \sum_{i,j=1}^{N} E_{ij} \otimes e^{(\ell)}_{ij}
\]

(5.1)

where the indices label the auxiliary and/or quantum spaces, the generator acting as identity operator in the other spaces. Accordingly, the Yangian representation at site \( \ell \) will be \( T_{1,\ell}(\lambda) = \lambda \mathbb{I}_{N} \otimes \mathbb{I} + i \mathcal{E}_{1,\ell} \).

**Theorem 5.3.** For a \( gl(N) \) spin chain, \( L_0 \)-regular, with a fundamental representation at sites \( 1 + q \, L_0, \, q \in \mathbb{Z}_+ \), the Hamiltonian \( H = -i \, t(0)^{-1} \, t'(0) \) is \( L_0 \)-local and takes the form:

\[
H_{(1,2,\ldots,L)} = \sum_{p=1}^{p} \left\{ \prod_{\ell=2}^{L_0} \mathcal{E}_{1+qL_0,\ell+(q-1)L_0}^{-1} \left( \prod_{\ell=2}^{L_0} \mathcal{E}_{1+(q-1)L_0,\ell+(q-1)L_0} \right) \right\} \left( \prod_{\ell=2}^{L_0} \mathcal{E}_{1+qL_0,\ell} \right) \left( \prod_{\ell=2}^{L_0} \mathcal{E}_{1+(q-1)L_0,\ell} \right) \right\}

(5.2)

where the total number of sites is \( L = pL_0 \).

**Proof.** We consider a \( gl(N) \) spin chain, \( L_0 \)-regular, with a fundamental representation at sites \( 1 + q \, L_0, \, q \in \mathbb{Z}_+ \). We define the Hamiltonian as

\[
H = -i \, t(0)^{-1} \, t'(0) \quad \text{with} \quad t'(\lambda) = \frac{d}{d\lambda} t(\lambda).
\]

(5.3)

Starting with \( t(\lambda) = tr_0(T_{01}(\lambda) \ldots T_{0L}(\lambda)) \), we decompose \( t'(0) \) into three terms, depending whether the derivation applies on \( T_{01}(\lambda) \), on \( T_{0,\ell}(\lambda) \), \( 1 < \ell < L_0 + 1 \) or on \( T_{0,\ell}(\lambda) \), \( L_0 < \ell \). After multiplication by \( t(0)^{-1} \), the first two terms can be easily computed, while in the last one, we recognize the Hamiltonian of the same type of spin chain, but where the first \( L_0 \) sites have been suppressed. Then, we get the recursion formula for the Hamiltonian:

\[
H_{(1,2,\ldots,L)} = H_{(L_0+1,L_0+2,\ldots,L)} + \left( \prod_{\ell=2}^{L_0} \mathcal{E}_{1+L_0,\ell} \right) \left( \prod_{\ell=2}^{L_0} \mathcal{E}_{1,\ell} \right) P_{1,1+L_0}

- \sum_{\ell_1=2}^{L_0} \left( \prod_{\ell=2}^{L_0} \mathcal{E}_{1+L_0,\ell} \right) \left( \prod_{\ell=2}^{L_0} \mathcal{E}_{1,\ell} \right) \left( \prod_{\ell=2}^{L_0} \mathcal{E}_{1+L_0,\ell} \right).

(5.4)

This recursion is easily solved, and we get (5.2). \( \square \)

Of course, depending on the representations sitting at sites \( \ell \neq 1 + qL_0 \), further simplifications may arise in the expression (5.2), see examples below.

doi:10.1088/1742-5468/2008/01/P01015
Note that once the representations of the spin chain are fixed, \( \mathcal{E}_{1,\ell} \) becomes a matrix, which can be easily inverted. In a more algebraic way, one may compute \( (\mathcal{E}_{1,\ell})^{-1} \) using the quantum comatrix in the Yangian \( Y(N) \).

When no fundamental representation occurs on the chain, one has to deal with a fusion procedure, in order to get auxiliary space with higher dimension (see e.g. [28, 46]). It is reasonable to think that to get a \( L_0 \)-local Hamiltonian, the auxiliary space has to be isomorphic to one of the representation on the chain. Then, the transfer matrix will be of different type. However, the Bethe equations (and thus the results presented above) will remain unchanged, since all fused transfer matrices commute among themselves.

**Examples**

(1) For \( gl(N) \) fundamental spin chain, one recovers the usual spin chain Hamiltonian

\[
H = \sum_{\ell=1}^{L} P_{\ell,\ell+1}.
\]

(5.5)

(2) The spin \( s \) \( gl(2) \) chain does not enter the hypothesis. However, the existence of a local (1-local) Hamiltonian is known [15, 47].

(3) For an alternating spin chain (with \( L = 2p \)), with fundamental representations on odd sites, one gets an Hamiltonian with next-to-nearest neighbour interactions (i.e. 2-local in our terminology):

\[
H = \sum_{\ell=1}^{L/2} \{(\mathcal{E}_{2\ell+1,2\ell})^{-1} \mathcal{E}_{2\ell-1,2\ell} P_{2\ell-1,2\ell+1} - \mathcal{E}_{2\ell+1,2\ell}\}.
\]

(5.6)

If furthermore, one takes a \( gl(2) \) spin chain, the matrices \( \mathcal{E} \) and \( \mathcal{E}^{-1} \) are easy to compute. Indeed, in a \( sl(2) \) representation of spin \( s \) (square matrices of size \( 2s+1 \)), one has

\[
\begin{align*}
\pi_s(e_3) &= \sum_{n=1}^{2s+1} (s + 1 - n) E_{nn}^{(s)}; & \pi_s(e_+) &= \sum_{n=1}^{2s} \sqrt{n(2s + 1 - n)} E_{n,n+1}^{(s)} \\
\pi_s(e_-) &= \sum_{n=1}^{2s} \sqrt{n(2s + 1 - n)} E_{n+1,n}^{(s)}; & \pi_s(1) &= \sum_{n=1}^{2s+1} E_{nn}^{(s)} = 1_{2s+1} 
\end{align*}
\]

where \( E_{ij}^{(s)} \) are the \( (2s + 1) \times (2s + 1) \) elementary matrices with 1 at position \((i, j)\) and 0 elsewhere. In this representation, one has

\[
[e_3, e_\pm] = \pm e_\pm; \quad [e_+, e_-] = 2e_3 \quad \text{and} \quad \text{tr}(e_+ e_-) = \text{tr}(e_3^2) = \frac{s(s + 1)(2s + 1)}{3}.
\]

(5.8)

Hence, we get

\[
2 \mathcal{E}_{2\ell+1,2\ell} = 1_{2} \otimes 1_{2s+1} + 4 \left( \sigma_3 \otimes \pi_s(e_3) + \frac{1}{2} (\sigma_+ \otimes \pi_s(e_-) + \sigma_- \otimes \pi_s(e_+)) \right) \equiv I + 4 X_s
\]

(5.9)
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where we have introduced the Pauli matrices

$$
\sigma_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}; \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
$$ (5.10)

Due to the properties of Pauli matrices, we have (whatever the value of $s$)

$$
X^2_s = \frac{1}{4} \mathbb{I}_2 \otimes C_s - \frac{1}{2} X_s
$$

where $C_s$ is the value of the Casimir operator in the spin $s$ representation:

$$
C_s = \pi_s \left( e^2_3 + \frac{1}{2} (e_- e_+ + e_+ e_-) \right) = s(s+1) \mathbb{I}_{2s+1}.
$$ (5.12)

This implies that

$$
(E_{2\ell+1,2\ell})^2 = \left( s + \frac{1}{2} \right)^2 \mathbb{I}_2 \otimes \mathbb{I}_{2s+1} \Rightarrow (E_{2\ell+1,2\ell})^{-1} = \frac{1}{(s+1/2)^2} E_{2\ell+1,2\ell}.
$$ (5.13)

Using the notation

$$
\vec{\sigma}_{2\ell-1} \cdot \vec{S}_{2\ell} = \sigma^{(2\ell-1)}_3 \otimes \pi^{(2\ell)}_s (e_3) + \sigma^{(2\ell-1)}_+ \otimes \pi^{(2\ell)}_s (e_+ - e_-) + \sigma^{(2\ell-1)}_- \otimes \pi^{(2\ell)}_s (e_+)
$$

where $\sigma^{(2\ell-1)}_{3,\pm}$ and $\pi^{(2\ell)}_s (e_{3,\pm})$ are representation of $e_{3,\pm}$ at site $2\ell-1$ (spin $\frac{1}{2}$) and $2\ell$ (spin $s$) respectively, and obvious generalizations of this notation to other cases, we can rewrite the Hamiltonian as

$$
H = \frac{-1}{2(s+(1/2))^2} \sum_{\ell=1}^{L/2} \left\{ s(s+1) \vec{S}_{2\ell} \cdot \vec{\sigma}_{2\ell+1} - \vec{\sigma}_{2\ell-1} \cdot \vec{S}_{2\ell} - \vec{\sigma}_{2\ell-1} \cdot \vec{\sigma}_{2\ell+1}
- 4(\vec{S}_{2\ell} \cdot \vec{\sigma}_{2\ell+1})(\vec{\sigma}_{2\ell-1} \cdot \vec{S}_{2\ell}) - 4(\vec{S}_{2\ell} \cdot \vec{\sigma}_{2\ell+1})(\vec{\sigma}_{2\ell-1} \cdot \vec{\sigma}_{2\ell+1})
- 16(\vec{\sigma}_{2\ell} \cdot \vec{\sigma}_{2\ell+1})(\vec{\sigma}_{2\ell-1} \cdot \vec{S}_{2\ell})(\vec{\sigma}_{2\ell-1} \cdot \vec{\sigma}_{2\ell+1}) \right\}
$$

which describes an alternating $gl(2)$ spin chains with spins $\frac{1}{2}$ and $s$ and next-to-nearest neighbour interaction.

5.2. Energies

Motivated by the result given above, we focus here on the case of $L_0$-regular spin chain containing at least one fundamental representation. This implies that $n_1 = 0$ (we remind that the $n_i$’s are ordered). In this case, the energy (eigenvalue of the Hamiltonian (5.2)) per site for a state characterized by the densities $t^{(j)}_m(\lambda)$, $j = 1, \ldots N - 1$, $n \in \frac{1}{2} \mathbb{Z}_+$ (as given in (4.5) and (4.6)) reads

$$
E - E_+ = \sum_{m \in \mathbb{Z}_{+}/2}^{'} \int_{-\infty}^{+\infty} d\lambda t^{(1)}_m(\lambda) \frac{2m + 1}{\lambda^2 + ((2m + 1)/2)^2}
$$

where $E_+$ is the energy of the pseudo-vacuum

$$
E_+ = -\frac{2}{L_0} \sum_{\alpha=1}^{L} \sum_{\ell \in I_\alpha} \frac{1}{a_\alpha + j_\ell}.
$$ (5.17)

doi:10.1088/1742-5468/2008/01/P01015
In particular, for the vacuum state (i.e. when densities are given by (2.21)), one gets [31]
\[ E_{\text{vac}} - E_+ = \frac{2}{NL_0} \sum_{\alpha=1}^{L} \sum_{\ell \in l_\alpha} \left\{ \psi \left( \frac{a_\alpha - j_\ell + 1}{2N} + 1 \right) - \psi \left( \frac{a_\alpha + j_\ell}{2N} \right) \right\} \] (5.18)

where \( \psi(x) = (d/dx) \ln \Gamma(x) \) is the Euler digamma function. The energy of the excited states studied in this paper, in the thermodynamical limit, are given by the following theorem.

**Theorem 5.4.** For a \( L_0 \)-regular spin chain with, at least, one fundamental representation, the energy per site of the excited states introduced in section 4, in the thermodynamical limit, is
\[ E - E_+ = E_{\text{vac}} + \frac{2\pi}{NL} \sum_{k=1}^{N-1} \sum_{d=1}^{D^0(k)} \frac{\sin (k\pi/N)}{\cos(k\pi/N) - \cosh(2\pi/N)^{(k)} \tilde{\lambda}_0,d).} \] (5.19)

**Proof.** The new terms in the energy, due to the modifications of the densities in the seas \( m \in \mathcal{N} \), are, respectively,
\[ \Delta^h E = \sum_{m \in \mathcal{N}}' \int_{-\infty}^{+\infty} d\lambda \rho_{m}^{(1)}(\lambda) \frac{2m + 1}{\lambda^2 + ((2m + 1)/2)^2} \] and
\[ \Delta^s E = \sum_{m \in \mathcal{N}}' \int_{-\infty}^{+\infty} d\lambda \epsilon_{m}^{(1)}(\lambda) \frac{2m + 1}{\lambda^2 + ((2m + 1)/2)^2} \] (5.20)

which can be computed via the Plancherel’s theorem
\[ \Delta^h E = 2\pi \sum_{m \in \mathcal{N}}' \int_{-\infty}^{+\infty} dp \rho_{m}^{(1)}(p) e^{-(2m+1)/2|p|} \] and
\[ \Delta^s E = 2\pi \sum_{m \in \mathcal{N}}' \int_{-\infty}^{+\infty} dp \epsilon_{m}^{(1)}(p) e^{-(2m+1)/2|p|}. \] (5.21)

The energy provided by the densities (4.6) of new strings read
\[ \Delta^n E = \sum_{m \in \mathcal{N}}' \sum_{\ell=1}^{D_{(k)}^0} \frac{2m + 1}{(\lambda_{m,\ell}^{(1)})^2 + ((2m + 1)/2)^2}. \] (5.22)

Using the explicit forms (3.44) of the functions \( \tilde{\rho}_{m}^{(1)}(p) \), we can show that, in the expression of \( \Delta^h E \), the coefficient of \( \exp(ip\tilde{\lambda}^{(k)}_{n_\alpha,d}) \) identically vanishes for all \( n_\alpha \) but \( n_1 \). Thus, one obtains (reminding that \( n_1 = 0 \)):
\[ \Delta^h E = - \sum_{k=1}^{N-1} \sum_{d=1}^{D^0(k)} \int_{-\infty}^{+\infty} dp \frac{\sinh((N - k)(|p|/2))}{\sinh((N|p|)/2)} e^{ip\tilde{\lambda}^{(k)}_0,d}. \] (5.23)
The contribution to the energy of complex holes was proven to be null (at order 1 in accordance with the calculations done in [42] for the alternating holes created in filled seas of non-real strings have a zero energy. This result is in scope of this article since we must obtain the eigenvalues of the transfer matrices with different representation in the auxiliary space.

Therefore, the holes in the filled seas of real strings can be interpreted as particle-like excitations with energy $\delta^{(k)}(\tilde{\lambda}_{0,d}^{(k)})$ and rapidity $\tilde{\lambda}_{0,d}^{(k)}$. These relations may allow one to compute dispersion relations of the considered excited states. However, we must compute their momentum, associated to the $L_0$-step shift operator. This computation is not in the scope of this article since we must obtain the eigenvalues of the transfer matrices with different representation in the auxiliary space.

Moreover, since only real holes appear in the expression of $\Delta E$, one can say that the holes created in filled seas of non-real strings have a zero energy. This result is in accordance with the calculations done in [42] for the alternating $sl(2)$ spin chain, where the contribution to the energy of complex holes was proven to be null (at order $1/L$).

Appendix. Proof of theorem 4.2

The parameters $\lambda_{n,\ell}^{(j)}$ satisfy the following constraints which are provided by the Bethe equations:

$$
\sum_{n \in \mathbb{N}}' \left\{ \int_{-\infty}^{\infty} d\lambda \left( t_n^{(j-1)}(\lambda) + t_n^{(j+1)}(\lambda) \right) \Phi_{-1}^{(m,n)}(\lambda_{m,k}^{(j)} - \lambda) + \int_{-\infty}^{\infty} d\lambda t_n^{(j)}(\lambda) \Phi_2^{(m,n)}(\lambda_{m,k}^{(j)} - \lambda) \right\}
= -\frac{1}{L} \sum_{q \in \mathbb{N}}' \left\{ \int_{-\infty}^{\infty} d\lambda \left( \Phi_1^{(q,m)}(\lambda_{m,k}^{(j)} - \lambda_{q,\ell}^{(j-1)}) + \Phi_2^{(q,m)}(\lambda_{m,k}^{(j)} - \lambda_{q,\ell}^{(j)}) \right) \right. \\
+ \sum_{\ell=1}^{\tilde{\rho}_q^{(j+1)}} \Phi_1^{(q,m)}(\lambda_{m,k}^{(j)} - \lambda_{q,\ell}^{(j+1)}) \right\}
+ \frac{1}{L_0} \sum_{\alpha=1}^{L} \left\{ \sum_{\ell \in I_\alpha} \delta_{j,j_\ell} \Phi_3^{(m,n)}(\lambda_{m,k}^{(j)}) - \frac{2\pi}{L} Q_{m,k}^{(j)} \right\} 
\tag{A.1}
$$

Similarly, using the explicit forms (4.13) of $\tilde{c}_m^{(1)}(p)$, we can simplify $\Delta^s E$:

$$
\Delta^s E = -\sum_{m \in \mathbb{N}}' \sum_{k=1}^{N} \int_{-\infty}^{\infty} dp e^{-((2m+1)/2)p} e^{ip\lambda_{m,k}^{(1)}}.
\tag{5.24}
$$

Then, performing an inverse Fourier transform of (5.23) and (5.24), we get the results (5.19) by remarking that $\Delta^s E = -\Delta^n E$. □

We would like to emphasize that although the studied spin chains are quite general, the final expression (5.19) for the energy are very simple. Indeed, this energy have the same shape for any $\tilde{\lambda}_{0,d}^{(k)}$, and can rewritten as:

$$
E - E_+ = E_{vac} + \frac{1}{L} \sum_{k=1}^{N} \sum_{d=1}^{\tilde{\rho}_k^{(j)}} \delta^{(k)}(\tilde{\lambda}_{0,d}^{(k)})
$$

where

$$
\delta^{(k)}(\lambda) = \frac{2\pi}{N} \frac{\sin(k\pi/N)}{\cos(k\pi/N) - \cosh((2\pi/N)\lambda)}.
\tag{5.25}
$$

Appendix. Proof of theorem 4.2

The parameters $\lambda_{n,\ell}^{(j)}$ satisfy the following constraints which are provided by the Bethe equations:

$$
\sum_{n \in \mathbb{N}}' \left\{ \int_{-\infty}^{\infty} d\lambda \left( t_n^{(j-1)}(\lambda) + t_n^{(j+1)}(\lambda) \right) \Phi_{-1}^{(m,n)}(\lambda_{m,k}^{(j)} - \lambda) + \int_{-\infty}^{\infty} d\lambda t_n^{(j)}(\lambda) \Phi_2^{(m,n)}(\lambda_{m,k}^{(j)} - \lambda) \right\}
= -\frac{1}{L} \sum_{q \in \mathbb{N}}' \left\{ \int_{-\infty}^{\infty} d\lambda \left( \Phi_1^{(q,m)}(\lambda_{m,k}^{(j)} - \lambda_{q,\ell}^{(j-1)}) + \Phi_2^{(q,m)}(\lambda_{m,k}^{(j)} - \lambda_{q,\ell}^{(j)}) \right) \right. \\
+ \sum_{\ell=1}^{\tilde{\rho}_q^{(j+1)}} \Phi_1^{(q,m)}(\lambda_{m,k}^{(j)} - \lambda_{q,\ell}^{(j+1)}) \right\}
+ \frac{1}{L_0} \sum_{\alpha=1}^{L} \left\{ \sum_{\ell \in I_\alpha} \delta_{j,j_\ell} \Phi_3^{(m,n)}(\lambda_{m,k}^{(j)}) - \frac{2\pi}{L} Q_{m,k}^{(j)} \right\} 
\tag{A.1}
$$

DOI:10.1088/1742-5468/2008/01/P01015
where \( m \in \mathbb{N} \), \( 1 \leq k \leq \tilde{\nu}^{(j)}(N) \) and \( 1 \leq j \leq N - 1 \). To simplify this relation, we replace the densities \( t_n^{(l)}(\lambda) \) in the lhs of (A.1) by their values obtained from (3.43) and (4.12):

\[
t_n^{(l)}(\lambda) = \sigma_n^{(l)}(\lambda) + \frac{1}{L} \rho_n^{(l)}(\lambda) + \frac{1}{L} \zeta_n^{(l)}(\lambda).
\] (A.2)

The three terms of the above sum are reduced in the three following lemmas:

**Lemma A.1.** For \( m \in \mathbb{N} \) and \( \lambda_0 \in \mathbb{R} \), we have the identity

\[
\sum_{n \in \mathbb{N}}' \left\{ \int_{-\infty}^{\infty} d\lambda \left( \sigma_n^{(j-1)}(\lambda) + \sigma_n^{(j+1)}(\lambda) \right) \Phi_n^{(m,n)}(\lambda_0 - \lambda) + \int_{-\infty}^{\infty} d\lambda \sigma_n^{(j)}(\lambda) \Phi_n^{(m,n)}(\lambda_0 - \lambda) \right\}
\]

\[
= \frac{1}{L_0} \sum_{a=1}^{\mathcal{L}} \left( \sum_{\ell \in I_a} \delta_{j,\ell} \right) \Phi_n^{(m,n)}(\lambda_0).
\] (A.3)

**Proof.** We differentiate w.r.t. \( \lambda_0 \) the lhs of (A.3), perform a Fourier transform and use the explicit form of the Fourier transform of the vacuum densities (2.21). Then, we remark that it is equal to the Fourier transform of the derivative of the rhs of (A.3) using, in particular,

\[
h_{j-1,k}(x) + h_{j+1,k}(x) - 2 \cosh(x) h_{j,k}(x) = -\delta_{j,k} \sinh(Nx) \sinh(x)
\]

where \( h_{j,k}(x) = \sinh(x(N - \max[j,k])) \sinh(x \min[j,k]) \). (A.4)

This proves the equality up to a constant, which is fixed by considering the value \( \lambda_0 = 0 \) in the equation, and remarking that the densities are even functions while \( \Phi_j^{(m,n)}(\lambda) \) is odd.

**Lemma A.2.** For \( m \in \mathbb{N} \) and for \( \lambda_0 \in \mathbb{R} \), we have the equality

\[
\sum_{n \in \mathbb{N}}' \left\{ \int_{-\infty}^{\infty} d\lambda \left( \rho_n^{(j-1)}(\lambda) + \rho_n^{(j+1)}(\lambda) \right) \Phi_n^{(m,n)}(\lambda_0 - \lambda) + \int_{-\infty}^{\infty} d\lambda \rho_n^{(j)}(\lambda) \Phi_n^{(m,n)}(\lambda_0 - \lambda) \right\}
\]

\[
= -\left( \sum_{d=1}^{\mathcal{D}(m)} \Omega_{n}^{+}(\lambda_0 - \tilde{\lambda}_{n}^{(j)}(m,d)) + \sum_{d=1}^{\mathcal{D}(m)} \Omega_{n}^{-}(\lambda_0 - \tilde{\lambda}_{n}^{(j)}(m,d)) \right).
\] (A.5)

**Proof.** Performing a derivation w.r.t. \( \lambda_0 \) and a Fourier transform of the lhs of (A.5), we get, using expression (3.44), the equality (A.5) up to a constant. Considering the limit \( \lambda_0 \to \infty \), one shows that the constant vanishes. □
Lemma A.3. For $m \in \mathbb{N}$, and for $\lambda_0 \in \mathbb{R}$, we have

$$
\sum_{n \in \mathbb{N}}' \left\{ \int_{-\infty}^{\infty} d\lambda \left( e_n^{(j-1)}(\lambda) + e_n^{(j+1)}(\lambda) \right) \Phi_{-1}^{(m,n)}(\lambda_0 - \lambda) 
+ \int_{-\infty}^{\infty} d\lambda c_n^{(j)}(\lambda) \Phi_{-1}^{(m,n)}(\lambda_0 - \lambda) \right\}
$$

$$
= - \sum_{q \in \mathbb{N}}' \left\{ \sum_{\ell=1}^{\gamma^+(m)-1/2} \Phi_{-1}^{(q,m)}(\lambda_0 - \lambda_{q,\ell}^{(j-1)}) + \sum_{\ell=1}^{\gamma^+(m)+1/2} F_{-1}^{(r,m)}(\lambda_0 - \lambda_{r,\ell}^{(j-1)}) \right\}
$$

$$
+ \sum_{\ell=1}^{\gamma^+(m)-1/2} \sum_{\ell=1}^{\gamma^+(m)+1/2} \Phi_{-1}^{(q,m)}(\lambda_0 - \lambda_{q,\ell}^{(j+1)}) + \sum_{\ell=1}^{\gamma^+(m)+1/2} F_{-1}^{(r,m)}(\lambda_0 - \lambda_{r,\ell}^{(j+1)})
$$

where $F_{-1}^{(r,m)}(\lambda)$ and $F_{-1}^{(r,m)}(\lambda)$ are given by (4.18) and (4.19).

Proof. Performing a derivation w.r.t. $\lambda_0$ and a Fourier transform of the lhs of (A.6), we get, using expression (4.13)

$$
- \sum_{q \in \mathbb{N}}' \left\{ \sum_{\ell=1}^{\gamma^+(m)-1/2} \tilde{\Phi}_{-1}^{(q,m)}(p) \exp(ip\lambda_{q,\ell}^{(j-1)}) + \sum_{\ell=1}^{\gamma^+(m)+1/2} \tilde{\Phi}_{-2}^{(q,m)}(p) \exp(ip\lambda_{q,\ell}^{(j+1)}) \right\}
$$

$$
+ \sum_{\ell=1}^{\gamma^+(m)-1/2} \sum_{\ell=1}^{\gamma^+(m)+1/2} \tilde{\Phi}_{-1}^{(q,m)}(p) \exp(ip\lambda_{q,\ell}^{(j+1)})
$$

$$
+ \sum_{\ell=1}^{\gamma^+(m)-1/2} \sum_{\ell=1}^{\gamma^+(m)+1/2} \tilde{f}_{-1}^{(r,m)}(p) \exp(ip\lambda_{r,\ell}^{(j-1)}) + \sum_{\ell=1}^{\gamma^+(m)+1/2} \tilde{f}_{-2}^{(r,m)}(p) \exp(ip\lambda_{r,\ell}^{(j)})
$$

$$
+ \sum_{\ell=1}^{\gamma^+(m)-1/2} \sum_{\ell=1}^{\gamma^+(m)+1/2} \tilde{f}_{-1}^{(r,m)}(p) \exp(ip\lambda_{r,\ell}^{(j+1)})
$$

where

$$
\tilde{f}_{-1}^{(r,m)}(p) = \frac{\sinh(|p|(\gamma^+(m) - \max(r, m))) \sinh(|p|(\gamma^+(m) - \min(r, m)))}{\sinh(|p|/2) \sinh(|p|(\gamma^+(m) - \gamma^-(m)))}
$$

$$
\tilde{f}_{-2}^{(r,m)}(p) = -2 \frac{\sinh(|p|(\gamma^+(m) - \max(r, m))) \sinh(|p|(\gamma^+(m) - \min(r, m)))}{\tanh(|p|/2) \sinh(|p|(\gamma^+(m) - \gamma^-(m)))} - \delta_{r,m}.
$$

\text{doi:10.1088/1742-5468/2008/01/P01015}
To obtain the explicit forms (4.18) and (4.19), one can perform the inverse Fourier transform by the theorem of residue and the integration. Remark that it is simpler to check these forms through a derivation and a direct Fourier transform. This fixes the equality up to a constant. Considering the limit $\lambda_0 \to \infty$, one shows that the constant vanishes. □

Finally, using relations (4.5), (A.3), (A.5) and (A.6) to simplify Bethe equations (A.1), we obtain the relation (4.16), which ends the proof of theorem 4.2.

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doi:10.1088/1742-5468/2008/01/P01015