Existence results, smoothness properties and numerical methods for equations involving the generalized fractional derivatives of Caputo type

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Abstract In this paper, we obtain a full characterization of the solutions of equations involving the generalized Caputo fractional derivative, such as existence, uniqueness, smoothness, series expansion, etc. These results are proved by establishing an equivalence theorem allowing the comparison of systems involving the generalized Caputo fractional derivative and those involving the standard Caputo fractional derivative. The case of singularities at the origin is also tackled. Furthermore, we derive various numerical methods of high orders that approximate the generalized fractional operator and we show that the optimal convergence rate can be reached if an appropriate graded mesh is considered. Several examples and numerical tests are performed to validate our theoretical study.

Keywords: Generalized fractional derivatives, Caputo derivatives, Hadamard derivatives, Volterra integral equations, Fractional calculus, Numerical methods.

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1 Introduction

Fractional integrals and derivatives are somehow extensions to real orders of the integrals and derivatives of integer orders. Subsequently, they provide an alternative framework to model various physical phenomena which involve nonlocal effects or memory effects [29, 30, 33], such as viscoelasticity, fluid mechanics, quantum mechanics, dynamical systems, control theory, signal processing, etc. [13, 23, 24, 25, 32, 36]. There exists in the literature several fractional derivatives operators such as the derivative of Riemann-Liouville, Caputo, Hadamard, Erdélyi-Kober, Grünwald-Letnikov, Marchaud, Riesz, . . . (see e.g. [16, 17, 21, 35] and references therein).
In 2014, U.N. Katugampola introduced in [20] a modification of the Erdélyi-Kober derivative [22, 28] yielding to a new fractional derivative which generalizes into a single form the Riemann-Liouville and the Hadamard derivatives. More recently, Jarad et al. introduced and studied the generalized fractional derivative in the sense of Caputo [18]. Particularly, the authors showed that this new generalized Caputo derivative converges toward the Caputo (resp. the Caputo-Hadamard) derivative when a parameter denoted \( \rho \) tends to one (resp. tends to zero) (see [18, Theorem 3.11] or Theorem 1 hereafter). In physical framework, these new generalized fractional derivatives could be more flexible than the classical Riemann-Liouville and Hadamard derivatives since they allow more freedom degrees. Indeed, it is shown in [4] that the generalized fractional derivatives applied to fractional chaotic equations can improve security of image encryption results. Another potential application of the generalized derivative has also been suggested in quantum mechanics [3].

Since finding an explicit solution of a fractional differential system is generally not an easy task, the use of numerical methods can be helpful to approximate such a solution. There exists numerous methods to solve the classical fractional equations (see e.g. [1, 6, 7, 8, 9, 11, 26, 27] among others). However, only few methods that allow the approximation of the solutions of fractional systems involving the generalized derivative can be found in the literature [2, 4]. Moreover, the convergence rate of these methods is not given. Recently, the authors in [37] have derived a numerical method for linear generalized fractional differential equations. They have shown that the proposed scheme is of order \( 1 - \alpha \) (see [37, Theorem 2.2]). We propose in this paper to develop numerical methods to solve the fractional generalized Caputo systems with optimal orders, i.e. that are independent of the choice of the parameter \( \rho \).

The paper is organized as follows. After some preliminaries in section 2, we establish an equivalence theorem in section 3 that allows us to derive several properties in concern with the existence, the uniqueness and the regularity of the solutions of the generalized Caputo fractional systems. In section 4 we perform several numerical tests where we constantly obtain the optimal convergence rate which is in total agreement with our main results.
Let $a \geq 0$, $n \in \mathbb{N}$ and $\rho > 0$. In [19] it is shown that the following Cauchy integral formula holds

$$I_{n,\rho}^a + u(t) := \int_a^t s_1^{\rho - 1} \int_a^{s_1} s_2^{\rho - 1} \cdots \int_a^{s_{n-1}} s_n^{\rho - 1} u(s_n) ds_n ds_{n-1} \cdots ds_1 = \frac{\rho^{1-n}}{\Gamma(n)} \int_a^t \frac{s^{\rho - 1} u(s)}{(t^{\rho} - s^{\rho})^{1-n}} ds.$$  

This suggests to extend the formula to real values of $n$. The left-sided generalized fractional integral $I_{\alpha,\rho}^a + u(t)$ of order $\alpha > 0$ is defined for any $t > a$ by [19]:

$$I_{\alpha,\rho}^a + u(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho - 1} u(s)}{(t^{\rho} - s^{\rho})^{1-\alpha}} ds. \quad (1)$$

If we define the operator $\gamma := t^{1-\rho} \frac{d}{dt}$ then it is clear that $\gamma^n = \gamma \circ \cdots \circ \gamma$ (n times) is a left inverse of $I_{\alpha,\rho}^a$. Similarly, the generalized fractional derivative is defined as the left inverse of the generalized integral.

**Definition 1** Let $\alpha > 0$ and $\rho > 0$ be a given real numbers and let $n = [\alpha] + 1$. Then the generalized fractional derivative relative to the generalized integral (1) is given for $t > a$ by

$$D_{\alpha,\rho}^a + u(t) := \gamma^n \left( I_{\alpha-n,\rho}^a + u(t) \right) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left( t^{1-\rho} \frac{d}{dt} \right)^n \int_a^t \frac{s^{\rho - 1} u(s)}{(t^{\rho} - s^{\rho})^{\alpha-n+1}} ds.$$  

Let us emphasize here that although the definition of the generalized fractional derivatives seems very close to the well known Erdélyi-Kober derivatives, it is in fact not possible to obtain the former derivatives as a direct consequence of the latter ones (see e.g. Proposition 2 hereafter or [18, 19, 20, 28] for more details). The generalized fractional integrals and derivatives satisfy the semigroup, the composition and the inverse property. More precisely, if we denote $X^p_{c}(a,b)$ ($c \in \mathbb{R}$, $1 \leq p \leq \infty$) the set of real valued Lebesgue measurable functions $u$ on $[a, b]$ for which $\|u\|_{X^p_{c}(a,b)} < \infty$ where

$$\|u\|_{X^p_{c}(a,b)} := \left( \int_a^b |t^c u(t)|^p \frac{dt}{t} \right)^{1/p} \quad \text{for} \quad 1 \leq p < \infty$$

and

$$\|u\|_{X^\infty_{c}(a,b)} := \text{ess sup}_{x \in [a,b]} [x^c |u(x)|].$$

then we have the following proposition (see [19, 20]).
Proposition 1 Let $a \geq 0$, $\alpha > 0$, $\beta > 0$ and $\rho > 0$. Then for $u \in X_{c}^{p}(a, b)$ ($1 \leq p \leq \infty$ and $c \leq \rho - 1$) we have

1. $T_{a+}^{\alpha} T_{a+}^{\beta} u = T_{a+}^{\alpha + \beta} u$ and $D_{a+}^{\alpha} D_{a+}^{\beta} u = D_{a+}^{\alpha + \beta} u$.

2. $D_{a+}^{\alpha} T_{a+}^{\beta} u = T_{a+}^{\beta - \alpha} u$ if $\alpha < \beta$.

3. $D_{a+}^{\alpha} T_{a+}^{\alpha} u = u$.

Lemma 1 Let $\alpha > 0$, $n = \lfloor \alpha \rfloor + 1$ and $\beta > -1$, then

$$D_{a+}^{\alpha} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\beta} = \frac{\Gamma(1 + \beta)}{\Gamma(1 + \beta - \alpha)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\beta - \alpha}$$

and

$$D_{a+}^{\alpha} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\alpha - i} = 0, \quad i = 1, \ldots, n.$$

Proof. See [18] Lemma 2.8.

Recently, the authors in [18] introduced and investigated the generalized fractional derivative in Caputo sense. First, they defined the space

$$AC_{\gamma}^{n}[a, b] = \{ f : [a, b] \to C, \gamma^{n-1} f \in AC[a, b] \}$$

with $\gamma := t^{1-\rho} \frac{d}{dt}$, $\gamma^{n} = \gamma \circ \cdots \circ \gamma$ ($n$ times), $\gamma^{0} = id$ and $AC[a, b]$ is the set of absolutely continuous functions on $[a, b]$.

Definition 2 Let $b > a \geq 0$ and let $\alpha > 0$ and $n = \lfloor \alpha \rfloor + 1$. The left generalized Caputo fractional derivative of order $\alpha$ of a function $u \in AC_{\gamma}^{n}[a, b]$ is defined for $t > a$ by [18]:

$$cD_{a+}^{\alpha} u(t) := D_{a+}^{\alpha} \left( u(t) - \sum_{k=0}^{n-1} \frac{(\gamma^k u)(a)}{k!} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{k} \right).$$

The following result gives an explicit formula of the generalized Caputo fractional derivative.

Proposition 2 Let $Re(\alpha) > 0$ and $n = \lfloor Re(\alpha) \rfloor + 1$. Let $u \in AC_{\gamma}^{n}[a, b]$, where $0 \leq a < b < +\infty$. 
1. If $\alpha \not\in \mathbb{N}_0$, then

\[ cD^{\alpha,\rho}_{a+} u(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^t \frac{s^{\rho-1} (\gamma^n u)(s)}{(t^\rho - s^\rho)^{\alpha-n+1}} ds = \mathcal{I}^{n-\alpha,\rho}_{a+} (\gamma^n u)(t) \]

2. If $\alpha = n \in \mathbb{N}_0$, then

\[ cD^n_{a+} u(t) = D^n_{a+} u(t) = (\gamma^n u)(t). \]

Proof. See [18, Theorem 3.2].

The generalized Caputo fractional derivative satisfies the following composition property.

**Proposition 3** Let $\alpha > 0$ and $n = \lfloor \alpha \rfloor + 1$. For $u \in AC^n_{\gamma}[a,b]$ we have

\[ \mathcal{I}^{\alpha,\rho}_{a+} (cD^{\alpha,\rho}_{a+} u)(t) = u(t) - \sum_{k=0}^{n-1} \left( \frac{\gamma^k u}{k!} \right) \left( \frac{t^\rho - a^\rho}{\rho} \right)^k. \]

Proof. See Theorem 3.6 in [18].

**Lemma 2** Let $\alpha > 0$ and $n = \lfloor \alpha \rfloor + 1$. Then for any $\rho > 0$

\[ cD^{\alpha,\rho}_{a+} \left( \frac{x^\rho - a^\rho}{\rho} \right)^k = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} \left( \frac{x^\rho - a^\rho}{\rho} \right)^{\beta-\alpha}, \quad \beta > n-1. \]

\[ cD^{\alpha,\rho}_{a+} \left( \frac{x^\rho - a^\rho}{\rho} \right)^k = 0, \quad k = 0, 1, \ldots, n-1. \]

Proof. See Lemma 3.8 and 3.9 in [18].

The limit cases when $\rho$ goes to 0 or 1 have also been investigated in [18]. The authors obtained the following result.

**Theorem 1** Let $\alpha > 0$ and $n = \lfloor \alpha \rfloor + 1$. Then for $t > a$

1. \[ \lim_{\rho \to 1} cD^{\alpha,\rho}_{a+} u(t) = cD^\alpha_{a+} u(t). \]

2. \[ \lim_{\rho \to 0^+} cD^{\alpha,\rho}_{a+} u(t) = cH^\alpha_{a+} u(t). \]
where \( cD_{a+}^\alpha \) is the Caputo modified Riemann-Liouville fractional derivative \([21]\) and \( cH_{a+}^\alpha \) is the Caputo modified Hadamard fractional derivative \([14]\) defined respectively by

\[
cD_{a+}^\alpha u(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds
\]

and

\[
cH_{a+}^\alpha u(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t \left( \log \frac{t}{s} \right)^{n-\alpha-1} \left[ \left( \frac{d}{ds} \right)^n u \right] (s) \frac{ds}{s}.
\]

(2)

\[\text{Proof. See [18, Theorem 3.11].}\]

\[\square\]

3 Main results

Let \( T > a \geq 0 \), \( \alpha \) and \( \rho \) two positive reals, and let \( n = \lfloor \alpha \rfloor + 1 \). Consider the following generalized Caputo fractional differential system

\[
\begin{aligned}
&cD_{a+}^{\alpha,\rho} u(t) = f(t, u(t)), \quad t \in (a, T) \\
(\gamma^k u)(a) = a_k, \quad k = 0, 1, \ldots, n-1
\end{aligned}
\]

(3)

where \( f \) is a given function and \( a_k \in \mathbb{R} \) for \( k = 0, 1, \ldots, n-1 \). The following proposition gives an implicit integral representation of the solution of (3).

**Proposition 4** Assume \( f \) is continuous over \([a, T] \times \mathbb{R} \). Then a function \( u \in AC^n_{\gamma}[a, T] \) is a solution of (3) if and only if \( u \) is a solution of the nonlinear Volterra integral equation of the second kind

\[
\begin{aligned}
u(t) &= \sum_{k=0}^{n-1} \frac{a_k}{k!} \left( \frac{t^\rho - a^\rho}{\rho} \right)^k + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} f(s, u(s)) \frac{ds}{(t^\rho - s^\rho)^{1-\alpha}} \\
\end{aligned}
\]

(4)

with \( a_k = (\gamma^k u)(a) \) for \( k = 0, 1, \ldots, n-1 \).

\[\text{Proof. } \Rightarrow \text{ Applying the generalized fractional integral (1) on both sides of equation (3) and using Proposition 3 we get the result.}\]

\[\Rightarrow \text{ Let } u \text{ given by (4), then we have by (1)}\]

\[
u(t) = \sum_{k=0}^{n-1} \frac{a_k}{k!} \left( \frac{t^\rho - a^\rho}{\rho} \right)^k + T_{a+}^{\alpha,\rho} f(t, u(t)).
\]
Applying the generalized Riemann-Liouville derivative operator and using Proposition 1 we obtain
\[ D_{a+}^{\alpha,\rho} u(t) = D_{a+}^{\alpha,\rho} \left( \sum_{k=0}^{n-1} \frac{\gamma^k u}{k!} \left( \frac{t^\rho - a^\rho}{\rho} \right)^k \right) + f(t, u(t)). \]
The result follows immediately from the definition of the generalized fractional derivative in Caputo sense.

Lemma 3 Let \( n \in \mathbb{N} \) and suppose that a given function \( u \) is \( n \) times derivable in \( t \in \mathbb{R}_+ \). Let \( 0 < \rho \leq \frac{1}{n} \) and define \( v(t) = u(t^{1/\rho}) \). Then
\[ v^{(n)}(t^\rho) = \rho^{-n} \sum_{i=1}^{n} \lambda_{i,n} t^{i-n^\rho} u^{(i)}(t) = \rho^{-n} (\gamma^u(t)) \]
where \( (\lambda_{i,j})_{i,j} \) is the sequence given recursively by:
\[ \lambda_{i,j} = \begin{cases} 
0 & \text{if } i = 0 \text{ or } i > j \\
1 & \text{if } i = j \\
\lambda_{i-1,j-1} + (i - (j-1)\rho) \lambda_{i,j-1} & \text{if } 1 \leq i < j.
\end{cases} \]

Proof. See A.

Remark 1 Lemma 3 holds also true for any value of \( \rho > 0 \) if \( t > 0 \). Therefore, we shall assume in the sequel that
\[(\rho > 0 \text{ if } a > 0) \quad \text{or} \quad \left(0 < \rho \leq \frac{1}{n} \text{ if } a = 0\right). \quad (\mathbb{H}) \]

Corollary 1 Let \( b > a \geq 0, n \in \mathbb{N} \), and \( \rho \) satisfying \( (\mathbb{H}) \). Let \( v(t) = u(t^{1/\rho}) \) then
\[ u \in AC^n_{\gamma}[a, b] \iff v \in AC^n[\bar{a}, \bar{b}] \]
with \( \bar{a} = a^\rho \) and \( \bar{b} = b^\rho \).

Proof. Using Lemma 3 we obtain
\[ v \in AC^n[\bar{a}, \bar{b}] \iff \exists c \in \mathbb{R} \text{ and } g \in L^1(\bar{a}, \bar{b}) \text{ s.t. } v^{(n-1)}(t) = c + \int_{\bar{a}}^{t} g(s) \, ds \quad \forall \, t \in [\bar{a}, \bar{b}] \]
\[ \iff v^{(n-1)}(t^\rho) = c + \int_{a^{\rho}}^{t^\rho} g(s) \, ds \quad \forall \, t \in [a, b] \]
\[ \iff \rho^{1-n} \gamma^{n-1} u(t) = c + \int_{a^{\rho}}^{t^\rho} g(s) \, ds \quad \forall \, t \in [a, b]. \]
Use the change of variable $s = \xi^\rho$ and denote $\varphi(\xi) = \rho^n \xi^{\rho-1} g(\xi^\rho) \in L^1(a, b)$, one deduce

$$v \in AC^n[\bar{a}, \bar{b}] \iff \exists \bar{c} \in \mathbb{R} \text{ and } \varphi \in L^1(a, b) \text{ s.t. } \gamma^{n-1} u(t) = \bar{c} + \int_a^t \varphi(\xi) \, d\xi \quad \forall t \in [a, b]$$

$$\iff u \in AC^n_\gamma[a, b].$$

We are now able to state the main results of this work.

**Theorem 2 (Equivalence theorem)** Let $\rho$ satisfying $(\mathcal{H})$. A function $u \in AC^n_\gamma[a, T]$ is a solution of (3) if and only if $u(t) = \bar{u}(t^\rho)$ with $\bar{u} \in AC^n[\bar{a}, \bar{T}]$ is a solution of the Caputo fractional differential system

$$\begin{cases}
  cD^\alpha_{\bar{a}}, \bar{u}(t) = \bar{f}(t, \bar{u}(t)), & t \in (\bar{a}, \bar{T}) \\
  \bar{u}^{(k)}(\bar{a}) = \bar{a}_k, & k = 0, 1, \ldots, n-1
\end{cases}$$

with $\bar{a} = a^\rho$, $\bar{T} = T^\rho$, $\bar{a}_k = \rho^{-k} a_k$ and $\bar{f}(t, x) = \rho^{-\alpha} f(t^1/\rho, x)$.

**Proof.** Let $u \in AC^n_\gamma[a, T]$ be the solution of (3), then from (4) we have $\forall t \geq \bar{a} = a^\rho$

$$u(t^{1/\rho}) = \sum_{k=0}^{n-1} \frac{a_k}{k!} \left( \frac{t - \bar{a}}{\rho} \right)^k + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\bar{a}}^t \frac{s^{\rho-1} f(s, u(s))}{(t - s)^{1-\alpha}} \, ds.$$

Using the change of variable $\xi = s^\rho$, one deduce

$$u(t^{1/\rho}) = \sum_{k=0}^{n-1} \frac{a_k}{k!} \left( \frac{t - \bar{a}}{\rho} \right)^k + \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \int_{\bar{a}}^t \frac{f(\xi^{1/\rho}, u(\xi^{1/\rho}))}{(t - \xi)^{1-\alpha}} \, d\xi.$$

Define $\bar{u}(t) = u(t^{1/\rho})$ for $t \in [\bar{a}, \bar{T}]$, then

$$\bar{u}(t) = \sum_{k=0}^{n-1} \frac{\bar{a}_k}{k!} (t - \bar{a})^k + \frac{1}{\Gamma(\alpha)} \int_{\bar{a}}^t \frac{f(\xi^{1/\rho}, \bar{u}(\xi^{1/\rho}))}{(t - \xi)^{1-\alpha}} \, d\xi,$$

which means that $\bar{u}$ solves (5) (see [10] Lemma 6.2 or [21] Theorem 3.24). Moreover, Corollary 1 states that $\bar{u} \in AC^n[\bar{a}, \bar{T}]$. All what remains is to check that the initial conditions hold true. Applying Lemma 3, we obtain

$$\bar{u}^{(k)}(\bar{a}) = \bar{u}^{(k)}(a^\rho) = \rho^{-k} \left( \gamma^k u \right)(a) = \bar{a}_k$$

which ends the proof.
Theorem 2 allows us to derive several information about the existence, uniqueness and regularity of the solutions of system (3) since these properties are known for systems of the form (5) (see e.g. [10, 21, 31]). For instance, one can prove the following:

**Theorem 3** Let $\alpha > 0$, $n = \lfloor \alpha \rfloor + 1$ and $0 < \rho \leq 1/n$. Let $a_0, \ldots, a_{n-1}$ in $\mathbb{R}$ and $T > 0$. Let the function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be continuous and fulfill a Lipschitz condition with respect to its second variable. Then there exists a uniquely defined function $u \in AC^n[0, T]$ solving the initial value problem (3).

**Proof.** The function $\bar{f} : (t, x) \mapsto \rho^{-\alpha} f(t^{1/\rho}, x)$ is continuous over $[0, T^\rho] \times \mathbb{R}$ and fulfill a Lipschitz condition with respect to its second variable. It follows from (the proof of) Theorem 6.8 in [10] that there exists a unique function $\bar{u} \in AC^n[0, T^\rho]$ solution of the system (5). According to Theorem 2, the function $u : t \mapsto \bar{u}(t^\rho) \in AC^n[0, T]$ is a solution of the system (3). Finally, the uniqueness of $u$ is a direct consequence of the uniqueness of $\bar{u}$.

**Theorem 4** Let $\alpha \in (0, 1)$ and $0 < \rho \leq 1$. Assume that

(A1) The function $(t, x) \mapsto f(t^{1/\rho}, x)$ is of class $C^1$ over $[0, T^\rho] \times \mathbb{R}$.

(A2) The function $(t, x) \mapsto \frac{\partial f}{\partial x}(t^{1/\rho}, x)$ is locally Lipschitz continuous in $x$.

(A3) The unique continuous solution of (3) exists on $[0, T]$.

Then the solution $u$ of (3) is of class $C[0, T] \cap C^1(0, T]$.

**Proof.** From Theorem 2 here above and Theorem 1 in [31] one deduce that $\bar{u} \in C[0, T^\rho] \cap C^1(0, T^\rho]$. The result follows from the identity $\bar{u}(t) = u(t^{1/\rho})$.

**Theorem 5** Suppose there exists two relatively prime integers $p \geq 1$ and $q \geq 2$ such that $\alpha = p/q$. Consider the problem (3) and suppose that $f$ can be written in the form $f(t, u) = \bar{f}(t^{p/q}, u)$ with $0 < \rho \leq 1/n$ and $\bar{f}$ is analytic in a neighborhood of $(0, a_0)$. Then the solution $u$ can be written in the form $u(t) = \bar{u}(t^{p/q})$ with $\bar{u}$ is analytic in a neighborhood of 0.
Corollary 2  Under the assumptions of Theorem 5, the solution \( u \) of problem (3) can be written in a neighborhood of the origin in the form

\[ u(t) = \sum_{i=0}^{+\infty} \bar{u}_i t^{i/q} \]

with \( \bar{u}_i \in \mathbb{R} \). Moreover, \( \bar{u}_i = 0 \) if \( i < p \) and \( i/q \notin \mathbb{N}_0 \).

The proofs of Theorem 5 and Corollary 2 follow directly from Theorem 2 here above and Theorem 6.32 and Corollary 6.34 in [10]. For completeness, we give a direct but more technical proof of Theorem 5 in B.

Example 1: Let \( a \geq 0, \alpha \in (0, 1), \rho \) satisfying (H), \( \lambda \) and \( u_a \in \mathbb{R} \), and consider the system

\[
\begin{cases}
\mathcal{C}D^{\alpha, \rho}_a u(t) = \lambda u(t), & t > a, \\
u(a) = u_a.
\end{cases}
\] (6)

According to Theorem 2 we have \( u(t) = \bar{u}(t^\rho) \) where \( \bar{u} \) is the solution of the system

\[
\begin{cases}
\mathcal{C}D^{\alpha}_a \bar{u}(t) = \lambda \rho^{-\alpha} \bar{u}(t), & t > \bar{a}, \\
\bar{u}(\bar{a}) = u_a.
\end{cases}
\] (7)

with \( \bar{a} = a^\rho \). Since the solution of (7) is the well known Mittag-Leffler function (see e.g. [21, Eq. (2.1.57)]), i.e. \( \bar{u}(t) = u_a E_{\alpha} (\lambda \rho^{-\alpha} (t - \bar{a})^\alpha) \) then we deduce that the solution of (6) is

\[ u(t) = u_a E_{\alpha} (\lambda \rho^{-\alpha} (t^\rho - a^\rho)^\alpha) . \]

Let us remark that such a result has also been obtained in [5] by using a different approach (see [5, Example 1]).

Example 2: Let \( a \geq 0, \alpha > 0, n = \lfloor \alpha \rfloor + 1, \rho \) satisfying (H), \( m \in \mathbb{N} \) and \( \lambda, a_0, \ldots, a_{n-1} \in \mathbb{R} \). Consider the system

\[
\begin{cases}
\mathcal{C}D^\alpha_a u(t) = \lambda t^m, & t > a, \\
(\gamma^k u)(a) = a_k, & k = 0, \ldots, n - 1
\end{cases}
\] (8)

Theorem 2 asserts that the solution of (8) writes \( u(t) = \bar{u}(t^\rho) \) where \( \bar{u} \) is the solution of the system

\[
\begin{cases}
\mathcal{C}D^\alpha_a \bar{u}(t) = \lambda \rho^{-\alpha} t^m, & t > \bar{a}, \\
\bar{u}^{(k)}(\bar{a}) = \bar{a}_k, & k = 0, \ldots, n - 1
\end{cases}
\]
with \( \bar{a} = a^\rho \) and \( \bar{a}_k = \rho^{-k} a_k, \ k = 0, \ldots, n - 1 \). It can be shown using Property 2.16 in [21] (or Lemma 2 with \( \rho = 1 \)) that

\[
\bar{u}(t) = \sum_{k=0}^{n-1} \frac{\bar{a}_k}{k!} (t - \bar{a})^k + \lambda \rho^{-\alpha} m! \sum_{k=0}^{m} \frac{\bar{a}^{m-k}}{(m-k)! \Gamma(\alpha+k+1)} (t - \bar{a})^{k+\alpha}
\]

yielding

\[
u(t) = \sum_{k=0}^{n-1} \frac{a_k}{k!} \left( \frac{t^\rho - a^\rho}{\rho} \right)^k + \lambda m! \sum_{k=0}^{m} \frac{\rho^k a^{\rho(m-k)}}{(m-k)! \Gamma(\alpha+k+1)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{k+\alpha}.
\]

(9)

Using Lemma 2 and the binomial formula one can check that the function \( u \) given by (9) is indeed the solution of system (8). Notice also that this result is in total agreement with Theorem 5 and Corollary 2 in case \( a = 0 \).

### 4 Numerical methods for the generalized Caputo derivative

Providing a numerical scheme that approximate the solution of a system involving the generalized Caputo derivative operator with an optimal convergence order is a challenging task. Indeed, the parameter \( \rho \) appearing in the definition of the generalized derivative operator could impair the convergence rate of the scheme since the solution may not be smooth at the origin (see Corollary 2). Looking in the literature, we found only few numerical methods dealing with the generalized Caputo derivative operator [2, 4]. Moreover, the convergence rates of these methods are not established. Recently, a low order numerical scheme has been derived in [37] to approximate the solution of linear generalized fractional differential equations. On another hand, several schemes can be found for the approximation of the Caputo derivative operator (see e.g. [1, 2, 7, 9, 15, 26, 27] among others). We claim that these methods can be adapted to numerically solve the generalized fractional systems if the equivalence Theorem 2 is applied. The main advantage of discretizing the equivalent system rather than the original one is that the convergence orders of these methods remain unchanged. Of course, such an approach will lead to approximated values \( u_j \) of \( u \) at nodes of the form \( t_j^{1/\rho}, \ j \in \mathbb{N} \) (i.e graded mesh) if the solution of the system involving the Caputo derivative \( \bar{u} \) is approximated at nodes \( t_j, \ j \in \mathbb{N} \). In what follows, we apply some methods relative to the standard Caputo derivative which we took from the literature in order to evaluate numerically the generalized derivative in Caputo sense.
4.1 Example 1

Let \( \alpha \in (0,1) \) and \( \nu \geq 0 \). We consider the generalized fractional system

\[
\begin{cases}
  cD_{a}^{\alpha,\rho}u(t) = t^\nu, & t \in (0,1), \\
  u(0) = 0.
\end{cases}
\] (10)

The exact solution of (10) is given by \( u(t) = \rho^{-\alpha} \frac{\Gamma(1 + \nu/\rho)}{\Gamma(1 + \nu/\rho + \alpha)} t^{\rho\alpha + \nu} \) (see Lemma 2). On another hand, according to Theorem 2, the solution of (10) can also be written as

\[
u(t) = \bar{u}(t^\rho)
\]

where \( \bar{u} \) is the solution of the system

\[
\begin{cases}
  cD_{0}^{\alpha}\bar{u}(t) = \rho^{-\alpha} t^{\nu/\rho}, & t \in (0,1), \\
  \bar{u}(0) = 0.
\end{cases}
\] (11)

We use a finite difference method introduced in [26] to approach the solution \( \bar{u} \) of (11). In this method, the Caputo-derivative operator is approximated at nodes \( t_n = n \Delta t \), \( 1 \leq n \leq N \) as follows

\[
cD_{0}^{\alpha}\bar{u}(t_n) \simeq \frac{1}{\Gamma(1 - \alpha)\Delta t^\alpha} \sum_{j=0}^{n-1} (\bar{u}_{j+1} - \bar{u}_j) b_{n-j}
\] (12)

with \( b_{n-j} = (n-j)^{1-\alpha} - (n-j-1)^{1-\alpha} \) (see [26] Eq. (3.1)). It is proven that this method is of order \( 2 - \alpha \) for \( \alpha \in (0,1) \). Table 1 shows the difference between the analytic solution \( u \) and the numerical solution \( \bar{u}_n \) in \( \ell^\infty \)-norm defined by

\[
\|e\|_{\ell^\infty} = \sup_{1 \leq n \leq N} |e_n| \quad \text{with} \quad e_n := u(t_n^{1/\rho}) - \bar{u}_n
\]

for various values of \( \alpha \) and \( \rho \). We notice that the convergence order is approximately equal to \( 2 - \alpha \) for any value of \( \rho \) as expected by theory.

4.2 Example 2

Let \( \alpha \in (0,1) \) and \( m \in \mathbb{N} \). We consider the function \( u(t) = t^{\rho m} \) solution of the system

\[
\begin{cases}
  cD_{a}^{\alpha,\rho}u(t) = \rho^\alpha \frac{\Gamma(1 + m)}{\Gamma(1 + m - \alpha)} t^{\rho(m-\alpha)}, & t \in (0,1), \\
  u(0) = 0
\end{cases}
\] (13)
which is equivalent according to Theorem \[2\] to the system
\[
\begin{aligned}
\frac{cD_0^\alpha \bar{u}(t)}{\Gamma(1 + m)} &= \frac{\Gamma(1 + m)}{\Gamma(1 + m - \alpha)} t^{m-\alpha}, \quad t \in (0, 1), \\
\bar{u}(0) &= 0
\end{aligned}
\]
(14)

We use a finite difference method of order $3 - \alpha$ introduced in [1] to approach the solution $\bar{u}$ of (14). In this method, the Caputo-derivative operator is approximated at a node $t_{n+\sigma} = (n + \sigma) \Delta t$, with $\sigma = 1 - \frac{\alpha}{2}$ and $0 \leq n \leq N - 1$ as follows (see [1, Eq. (27)])

\[
\begin{aligned}
\frac{cD_0^\alpha \bar{u}(t_{n+\sigma})}{\Gamma(2 - \alpha)\Delta t^{\alpha}} &\approx \sum_{j=0}^{n} (\bar{u}_{j+1} - \bar{u}_j) c_{n-j}^{(\alpha, \sigma)}
\end{aligned}
\]

with $c_{0}^{(\alpha, \sigma)} = a_0^{(\alpha, \sigma)}$ for $n = 0$ and for $n \geq 1$

\[
\begin{aligned}
c_{j}^{(\alpha, \sigma)} &= \begin{cases}
a_0^{(\alpha, \sigma)} + b_1^{(\alpha, \sigma)}, & j = 0 \\
a_j^{(\alpha, \sigma)} + b_{j+1}^{(\alpha, \sigma)} - b_j^{(\alpha, \sigma)}, & 1 \leq j \leq n - 1 \\
a_n^{(\alpha, \sigma)} - b_n^{(\alpha, \sigma)}, & j = n
\end{cases}
\end{aligned}
\]

where

\[
\begin{aligned}
a_0^{(\alpha, \sigma)} &= \sigma^{1-\alpha}, \quad a_j^{(\alpha, \sigma)} = (j + \sigma)^{1-\alpha} - (j - 1 + \sigma)^{1-\alpha}, \quad j \geq 1
\end{aligned}
\]

and

\[
\begin{aligned}
b_j^{(\alpha, \sigma)} &= \frac{1}{2 - \alpha} \left[ (j + \sigma)^{2-\alpha} - (j - 1 + \sigma)^{2-\alpha} \right] - \frac{1}{2} \left[ (j + \sigma)^{1-\alpha} + (j - 1 + \sigma)^{1-\alpha} \right].
\end{aligned}
\]

Table 2 shows the $\ell^\infty$-norm of the difference between the exact and the numerical solutions for various values of $\alpha$ and $\rho$. We notice that these errors - and consequently the convergence orders - are totally independent of $\rho$ and are close to theoretical value $3 - \alpha$. This result is actually expected since the solution of the equivalent system (14) is given by $\bar{u}(t) = t^m$ and hence does not depend on $\rho$, even though the solution $u$ of system (13) do depend on $\rho$. Figure 1 shows the behavior of the solutions of system (13) for two different values of $\rho$. Let us emphasize that the convergence order of the method is optimal despite that the solution of system (13) is not smooth at the origin (case $\rho = 1/6$ for instance). This is of great advantage since the numerical methods are in general less accurate in case of singular solutions and a particular care have to be carried out to overcome this lack of precision, for example by using the concept of graded meshes [34] or by adding some non-polynomial functions to the numerical solution which reflect the singularity of the exact solution [7, 12]. Our approach is able to directly handle such difficulties without using any mesh transformations or having additional costs if the solution of the equivalent system is smooth enough.
4.3 Example 3

Let $\alpha \in (0, 1)$, $a \geq 0$ and $a_0 \in \mathbb{R}$. We consider the system

$$\begin{cases}
\frac{cD_a^\alpha u(t)}{\rho} = \rho^\alpha (u(t) + t^\rho - a^\rho), & t > a, \\
u(a) = a_0,
\end{cases} \quad (15)$$

which is equivalent to the system

$$\begin{cases}
\frac{cD_a^\alpha \bar{u}(t)}{\bar{a}} = \bar{u}(t) + t - \bar{a}, & t > \bar{a}, \\
\bar{u}(\bar{a}) = a_0
\end{cases} \quad (16)$$

with $\bar{a} = a^\rho$. One can check that the solution of (16) is given by

$$\bar{u}(t) = a_0 E_\alpha ((t - \bar{a})^\alpha) + (t - \bar{a})^{1+\alpha} E_{\alpha,\alpha+2} ((t - \bar{a})^\alpha)$$

(see also [21, Chapter 4]). It follows by Theorem 2 that the function

$$u(t) = a_0 E_\alpha ((t^\rho - a^\rho)^\alpha) + (t^\rho - a^\rho)^{1+\alpha} E_{\alpha,\alpha+2} ((t^\rho - a^\rho)^\alpha)$$

is the solution of (15). Now we aim at finding a numerical approximation of $u$ or similarly of $\bar{u}$ since $u(t) = \bar{u}(t^\rho)$. In view of Proposition 4 one can seek for $\bar{u}$ as a solution of the Volterra integral equation of a second kind

$$\bar{u}(t) = a_0 + \frac{1}{\Gamma(\alpha)} \int_\bar{a}^t \frac{\bar{u}(s) + s - \bar{a}}{(t - s)^{1-\alpha}} ds. \quad (17)$$

We use an explicit Euler method to approach the solution of (17). This method writes

$$\bar{u}_{n+1} = a_0 + \bar{F}_n(t_{n+1}) + \frac{\Delta t^\alpha}{\alpha} f(t_n, u_n), \quad n \geq 0$$

with $t_n = \bar{a} + n\Delta t$, $\bar{u}_0 = a_0$ and $f(t, u) = u + t - \bar{a}$. The term $\bar{F}_n(t_{n+1})$ being an approximation at node $t_{n+1}$ of the integral

$$F_n(t) := \frac{1}{\Gamma(\alpha)} \int_\bar{a}^t \frac{f(s, \bar{u}(s))}{(t - s)^{1-\alpha}} ds.$$

(see e.g [27]). Notice that $F_n(t_{n+1})$ does not involve a singular kernel and it can thus be approximated by any standard quadrature formula. For sake of simplicity, we chose a trapezoidal rule and we obtain after some simplifications

$$\bar{u}_{n+1} = a_0 + \frac{\Delta t^\alpha}{2\Gamma(\alpha)} \left( \sum_{j=0}^{n-1} d_{j+1} f(t_{j+1}, \bar{u}_{j+1}) + d_j f(t_j, \bar{u}_j) \right) + \frac{\Delta t^\alpha}{\alpha} f(t_n, \bar{u}_n)$$

with $d_j = (n - j + 1)^{\alpha-1}$. Table 3 shows the $\ell^\infty$ error between the analytic and the numerical solutions for different values of $\alpha$ and $\rho$. Here again we found that the convergence rate of this one stage Runge-Kutta method does not depend on $\rho$ and is approximately equal to $\alpha$, which is expected since $\bar{u}$ is only $\alpha$-Hölder continuous.
4.4 Example 4

Let us consider the system

\[
\begin{align*}
\begin{cases} 
^{cD}_{a} \alpha^{\rho} u(t) &= t \sin u(t), \quad t \in (a, T), \\
 u(a) &= u_a, 
\end{cases}
\end{align*}
\tag{18}
\]

which is equivalent to the system

\[
\begin{align*}
\begin{cases} 
^{cD}_{a} \bar{u}(t) &= \rho^{-\alpha} t^{1/\rho} \sin \bar{u}(t), \quad t \in (\bar{a}, \bar{T}), \\
 \bar{u}(a) &= u_a. 
\end{cases}
\end{align*}
\tag{19}
\]

Since there is no known solution for the system (18) nor the system (19), we shall compare our approach to another numerical method which directly solves the system (18). We chose a method proposed by Almeida et al. in [2] where the authors proved that the solution \( u(t) \) can be estimated by \( u_N(t) \) for a sufficiently large \( N \in \mathbb{N} \), with

\[
u_N(t) = u_a + A_N \left( t^\rho - a^\rho \right)^\alpha f(t, u_N(t)) - \sum_{k=1}^{N} B_{N,k} \left( t^\rho - a^\rho \right)^{\alpha-k} V_{k,N}(t) \tag{20}
\]

where \( f(t, u) = t \sin u \),

\[
A_N = \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)\Gamma(\alpha-1)} \sum_{k=0}^{N} \frac{\Gamma(k - 1 + \alpha)}{k!}.
\]

\[
B_{N,k} = \frac{\rho^\alpha \Gamma(k - 1 + \alpha)}{\Gamma(2-\alpha)\Gamma(\alpha-1)(k-1)!}
\]

and

\[
V_{k,N}(t) = \int_{a}^{t} s^{\rho-1} \left( s^\rho - a^\rho \right)^{k-1} f(s, u_N(s)) \, ds.
\]

For the approximation of \( \bar{u} \), we use the finite difference scheme (12) introduced in example 1. Figure 2 shows a comparison between the two obtained numerical solutions. One can notice the good superposition of these solutions which confirm once more our claimed results.

4.5 Example 5

We end up this section with an example that highlights the performance of our proposed method even in case \( \rho \to 0 \), that is the approximation of the Caputo-Hadamard
derivative. Let $\alpha \in (0,1)$, $T > 0$ and consider the system

$$
\begin{cases}
  c^\mathcal{H}_1^\alpha u(t) = \frac{(\log t)^{1-\alpha}}{\Gamma(2-\alpha)}, & t \in (1,T) \\
  u(1) = 0,
\end{cases}
$$

(21)

where $c^\mathcal{H}_1^\alpha$ is the Caputo modified Hadamard fractional derivative defined by (2). The solution of (21) is given by $u(t) = \log t$. In virtue of Theorem 1, problem (21) can be approximated as follows

$$
\begin{cases}
  c^D_1^{\alpha,\rho} u(t) = \frac{(\log t)^{1-\alpha}}{\rho \Gamma(2-\alpha)}, & t \in (1,T) \\
  u(1) = 0,
\end{cases}
$$

(22)

for sufficiently small value $\rho$. Now using the equivalence Theorem 2, we rewrite problem (22) as

$$
\begin{cases}
  c^D_1^\alpha \bar{u}(t) = \frac{(\log t)^{1-\alpha}}{\rho \Gamma(2-\alpha)}, & t \in (1,T^\rho) \\
  \bar{u}(1) = 0,
\end{cases}
$$

(23)

We perform several tests by setting $\alpha = 1/2$ and $\rho = 10^{-i}$, $i = 1, \ldots, 7$ and we use the finite difference scheme (12) to approximate $\bar{u}$. Figure 3 displays the error in $L^\infty$-norm between the analytic solution of equation (21) and the numerical solution relative to system (23) function of $\rho$. As theoretically expected in Theorem 1, the error converges toward zero as $\rho$ gets smaller. Figure 4 displays the analytic and the numerical solutions for $\rho = 10^{-7}$ and $T = 100$. One can clearly notice the good agreement between the two solutions over the graded mesh $(t_i)_{0 \leq i \leq N}$ with $t_i := (1 + (T^\rho - 1) i/N)^{1/\rho}$ and $N \in \mathbb{N}^*$ is a given number.

**Conclusion**

In this paper, we derived several properties in concern with the generalized fractional derivatives in Caputo sense [18]. This can be achieved by establishing a key theorem that allows the comparison between the solutions of systems involving the generalized Caputo derivatives and the standard Caputo derivatives. In addition, we performed several numerical tests that highlight our study. The numerical methods we derive are based on the approximation of the standard Caputo derivatives applied to an equivalent system which can be obtained easily using the equivalence Theorem 2. Subsequently,
we obtain a numerical solution over a graded mesh with an optimal convergence rate (i.e. that is independent on the parameter $\rho$). The technique we propose is easy to implement and can be applied to any fractional differential system involving the generalized fractional derivative without additional costs in comparison with standard Caputo fractional’s schemes. An example in case $\rho$ tends to zero, and which corresponds to the approximation of the Caputo-Hadamard derivative, is also given at the end and illustrates the good efficiency of our approach.

A Proof of Lemma 3

The second identity has already been established in [5, Lemma 2]. We shall focus on the first equality, which we prove by induction. The result is trivial for $n = 1$. Moreover, we have

$$u^{(n)}(t^\rho) = (u^{(n-1)})'(x)|_{x=t^\rho}$$

$$= \rho^{1-n} \sum_{i=1}^{n-1} \lambda_{i,n-1} x^{i/\rho-n+1} u^{(i)}(x^{1/\rho}) |_{x=t^\rho}$$

$$= \rho^{1-n} \left( \sum_{i=1}^{n-1} (i/\rho - n + 1) \lambda_{i,n-1} x^{i/\rho-n} u^{(i)}(x^{1/\rho}) + \rho^{-1} \lambda_{i,n-1} x^{(i+1)/\rho-n} u^{(i+1)}(x^{1/\rho}) \right) |_{x=t^\rho}$$

$$= \rho^{-n} \left( \sum_{i=1}^{n-1} (i - (n-1) \rho) \lambda_{i,n-1} t^{i-n\rho} u^{(i)}(t) + \sum_{i=2}^{n} \lambda_{i-1,n-1} t^{i-n\rho} u^{(i)}(t) \right).$$

Notice that $\lambda_{n,n-1} = \lambda_{0,n-1} = 0$ by definition. It follows

$$u^{(n)}(t^\rho) = \rho^{-n} \sum_{i=1}^{n} (\lambda_{i,n-1} (i - (n-1) \rho) + \lambda_{i-1,n-1} t^{i-n\rho} u^{(i)}(t))$$

$$= \rho^{-n} \sum_{i=1}^{n} \lambda_{i,n} t^{i-n\rho} u^{(i)}(t)$$

and the proof is completed.
B Proof of Theorem 5

We follow the methods used by Lubich \[27\] and Diethelm \[10\]. Let us consider the function
\[
u(t) = \sum_{i=0}^{+\infty} \bar{u}_i t^{\rho i/q} \tag{24}
\]
with \(u(0) = a_0\). Our aim is to find the coefficients \(\bar{u}_i, i \in \mathbb{N}_0\) such that the series is convergent and solves problem (3) (or equivalently problem (4)). Since \(\bar{f}\) is analytic in a neighborhood of \((0, a_0)\) then
\[
f(t, u(t)) = \bar{f}(t^{\rho/q}, u(t)) = \sum_{j,k=0}^{+\infty} f_{jk} t^{\rho j/q} (u(t) - a_0)^k.
\]
Substituting \(u\) by expression (24), we get by (4)
\[
\sum_{i=0}^{+\infty} \bar{u}_i t^{\rho i/q} = \sum_{k=0}^{+\infty} \frac{a_k}{k! \rho^k} t^{\rho k} + \rho^{1-\alpha} \int_0^t s^{\rho-1} (t^\rho - s^\rho)^{-\alpha-1} \sum_{j,k=0}^{+\infty} f_{jk} s^{\rho j/q} \left( \sum_{i=1}^{+\infty} \bar{u}_i s^{\rho i/q} \right)^k ds.
\]
If we rearrange the sum in the parentheses as follows
\[
\left( \sum_{i=1}^{+\infty} \bar{u}_i s^{\rho i/q} \right)^k = \sum_{\ell=0}^{+\infty} \left( \sum_{i_1 + \cdots + i_k = \ell} \bar{u}_{i_1} \cdots \bar{u}_{i_k} \right) s^{\rho \ell/q}
\]
then, assuming uniform convergence of the series, we get
\[
\sum_{i=0}^{+\infty} \bar{u}_i t^{\rho i/q} = \sum_{k=0}^{+\infty} \frac{a_k}{k! \rho^k} t^{\rho k} + \rho^{1-\alpha} \sum_{j,k=0}^{+\infty} f_{jk} \left( \sum_{i_1 + \cdots + i_k = \ell} \bar{u}_{i_1} \cdots \bar{u}_{i_k} \right) \int_0^t s^{\rho(1+\frac{j+k}{q})-1} ds.
\]
Using that \(\alpha = p/q\), it is easy to show that
\[
\int_0^t s^{\rho(1+\frac{j+k}{q})-1} (t^\rho - s^\rho)^{-\alpha-1} ds = \frac{1}{\rho} B\left(\alpha, \frac{j+\ell}{q} + \frac{1}{\rho}\right) t^{\rho \frac{j+\ell+p}{q}}
\]
where \(B(\cdot)\) denotes the beta function. It follows that
\[
\sum_{i=0}^{+\infty} \bar{u}_i t^{\rho i/q} = \sum_{k=0}^{+\infty} \frac{a_k}{k! \rho^k} t^{\rho k} + \frac{1}{\rho^{\alpha}} \sum_{j,k=0}^{+\infty} f_{jk} \left( \frac{\Gamma\left(\frac{j+\ell}{q} + \frac{1}{\rho}\right)}{\Gamma\left(\frac{j+\ell+p}{q} + \frac{1}{\rho}\right)} \sum_{i_1 + \cdots + i_k = \ell} \bar{u}_{i_1} \cdots \bar{u}_{i_k} \right) t^{\rho \frac{j+\ell+p}{q}}.
\]
Comparing both sides of the last equality, one can deduce that
\[
\begin{align*}
&\text{• if } i < p \text{ then } \bar{u}_i \text{ is given by} \\
&\quad \bar{u}_i = \begin{cases} 
\frac{a_{i/q}}{(i/q)! \rho^{i/q}} & \text{if } i = 0, q, 2q, \ldots, (n-1)q \\
0 & \text{otherwise},
\end{cases} \tag{25}
\end{align*}
\]
• if \( i \geq p \) then \( \bar{u}_i \) is given by

\[
\bar{u}_i = \frac{1}{\rho^\alpha} \sum_{j+\ell=i-p}^{+\infty} \sum_{k=0}^{+\infty} f_{jk} \left( \frac{\Gamma \left( \frac{j+\ell+1}{q} + \frac{1}{\rho} \right)}{\Gamma \left( \frac{j+\ell+p+1}{q} + \frac{1}{\rho} \right)} \right) \prod_{i_1+\ldots+i_k=\ell} \bar{u}_{i_1} \ldots \bar{u}_{i_k}.
\] (26)

Notice that the last sum in (26) depends on \( \ell \) which is smaller than \( i \) due to the first sum. Hence equation (26) gives a recurrence formula to compute \( \bar{u}_i \) for \( i \geq p \). We conclude that equations (25) and (26) uniquely determine \( \bar{u}_i \) for all \( i \in \mathbb{N}_0 \).

The remaining step is to prove that the series defined in (24) with \( \bar{u}_i \) given by (25) and (26) is uniformly convergent, and hence is analytic, in a neighborhood of the origin. In fact, we will prove that this series is absolutely convergent in the neighborhood of zero. Let us consider the function

\[
F(t^{\rho/q}, u(t)) = \sum_{j,k=0}^{+\infty} |f_{jk}| t^{\rho j/q} \left( u(t) - |a_0| \right)^k.
\]

Since \( \tilde{f} \) is analytic, then this series is convergent. Consider now the Volterra equation

\[
U(t) = \sum_{k=0}^{n-1} \frac{|a_k|}{k! \rho^k} t^{\rho k} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1} F(s, U(s))}{(t^\rho - s^\rho)^{1-\alpha}} ds.
\] (27)

The solution \( U \) of (27) can be computed in series form using exactly the same techniques as above, and hence \( U \) is a majorant of \( u \), with non negative coefficients in its series expansion. Let us now define the partial sum

\[
\tilde{U}_\ell(t) = \sum_{i=0}^{\ell} U_i t^{\rho i/q}.
\]

Then we have

\[
\tilde{U}_{\ell+1}(t) \leq \sum_{k=0}^{n-1} \frac{|a_k|}{k! \rho^k} t^{\rho k} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1} F(s, \tilde{U}_\ell(s))}{(t^\rho - s^\rho)^{1-\alpha}} ds.
\] (28)

Indeed, expanding the right hand side in series form will generate exactly the same low order coefficients as those of the left hand side up to the term \( t^{\rho (\ell+1)/q} \), and thus all these terms will cancel. Inequality (28) follows then from the fact that the coefficients of the series expansion of the quantity in the right hand side of (28) are all non negative.

Now we shall prove that \( |\tilde{U}_\ell| \) is bounded in a neighborhood of 0 for all \( \ell \in \mathbb{N}_0 \). Let \( \varepsilon > 0 \)}
and define $C_1 = \sum_{k=0}^{n-1} |a_k| \varepsilon^{\rho^k}/(k! \rho^k)$. Define $C_2 = \max_{(t,u)\in[0,\varepsilon] \times [0,2C_1]} |F(t,u)|/\Gamma(\alpha)$ and set $r = \min\{\varepsilon, (\rho^\alpha C_1/C_2)^{\frac{1}{\alpha}}\}$. We will show by induction on $\ell$ that

\[ |\tilde{U}_\ell(t)| \leq 2C_1 \quad \forall \ell \in \mathbb{N}_0 \text{ and } t \in [0,r]. \quad (29) \]

The result is obvious for $\ell = 0$. Assume inequality (29) is true up to order $\ell$, we have by (28)

\[
|\tilde{U}_{\ell+1}(t)| = \tilde{U}_{\ell+1}(t) \leq \sum_{k=0}^{n-1} \frac{|a_k|}{k! \rho^k} t^{\rho^k} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t s^{\rho-1} F(s, \tilde{U}_\ell(s)) \frac{ds}{(t^\rho - s^\rho)^{1-\alpha}} \\
\leq \sum_{k=0}^{n-1} \frac{|a_k|}{k! \rho^k} r^{\rho^k} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \max_{s\in[0,t]} |F(s, \tilde{U}_\ell(s))| \int_0^t s^{\rho-1} \frac{ds}{(t^\rho - s^\rho)^{1-\alpha}} \\
\leq C_1 + \frac{(t^\rho/p)^\alpha}{\Gamma(\alpha)} \max_{(t,u)\in[0,\varepsilon] \times [0,2C_1]} |F(t,u)| \\
\leq C_1 + (r^\rho/p)^\alpha C_2 \\
\leq 2C_1
\]

which ends the proof.

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Figure 1: Exact solution (red line) and numerical solution (blue circles) with $\rho = 5 \ln(2)/4$ (left) and $\rho = 1/6$ (right).

Figure 2: Comparison between the numerical solution of the system (18) obtained with Almeida et al. algorithm (20) (red solid line) and the numerical solution of the system (19) obtained using the finite difference approximation (12) (blue circles). The parameters used are $\alpha = 0.5$, $\rho = 0.75$, $a = 0.25$ and $T = 4$. 
Figure 3: Error in $L^\infty$ norm function of $\rho$ in log-scale for example 5 with exact solution $u(t) = \log t$, $t \in [1, 100]$ and $\alpha = 1/2$.

Figure 4: Exact solution (red solid line) and numerical solution (blue circles) for example 5. The numerical solution of the system (23) is obtained using the finite difference approximation (12). The parameters used are $\alpha = 0.5$, $\rho = 10^{-7}$ and $T = 100$. 
\[ \alpha = 0.9 \]

\[ \alpha = 0.5 \]

\[ \alpha = 0.2 \]

| \( N \) | \( \| e \|_{\ell^\infty} \) | Order | \( \| e \|_{\ell^\infty} \) | Order | \( \| e \|_{\ell^\infty} \) | Order |
|-----|------------------|------|------------------|------|------------------|------|
| \( 2^4 \) | 2.7511 (-02) | – | 7.6009 (-03) | – | 1.5803 (-03) | – |
| \( 2^5 \) | 1.2854 (-02) | 1.0978 | 2.7918 (-03) | 1.4024 | 4.8924 (-04) | 1.6916 |
| \( 2^6 \) | 6.0082 (-03) | 1.0972 | 1.0129 (-03) | 1.4627 | 1.4926 (-04) | 1.7127 |
| \( 2^7 \) | 2.8073 (-03) | 1.0977 | 3.6449 (-04) | 1.4745 | 4.5042 (-05) | 1.7285 |
| \( 2^8 \) | 1.3111 (-03) | 1.0984 | 1.3044 (-04) | 1.4825 | 1.3478 (-05) | 1.7407 |

| \( \rho = 0.9 \) | \( \rho = \pi/6 \approx 0.52 \) | \( \rho = 1/3 \) |
|-----|------------------|------|------------------|------|------------------|------|
| \( 2^4 \) | 4.9318 (-02) | – | 1.6160 (-02) | – | 4.1353 (-03) | – |
| \( 2^5 \) | 2.2986 (-02) | 1.1014 | 6.0450 (-03) | 1.4186 | 1.3189 (-03) | 1.6487 |
| \( 2^6 \) | 1.0738 (-02) | 1.0980 | 2.2211 (-03) | 1.4445 | 4.1094 (-04) | 1.6824 |
| \( 2^7 \) | 5.0183 (-03) | 1.0975 | 8.0617 (-04) | 1.4621 | 1.2594 (-04) | 1.7061 |
| \( 2^8 \) | 2.3444 (-03) | 1.0980 | 2.9019 (-04) | 1.4741 | 3.8134 (-05) | 1.7236 |

| \( \rho = 1/3 \) | \( \rho = \pi/6 \approx 0.52 \) | \( \rho = 1/3 \) |
|-----|------------------|------|------------------|------|------------------|------|
| \( 2^4 \) | 8.0872 (-02) | – | 2.9986 (-02) | – | 8.7349 (-03) | – |
| \( 2^5 \) | 3.7523 (-02) | 1.1079 | 1.1421 (-02) | 1.3926 | 2.8685 (-03) | 1.6065 |
| \( 2^6 \) | 1.7505 (-02) | 1.1000 | 4.2509 (-03) | 1.4258 | 9.1173 (-04) | 1.6536 |
| \( 2^7 \) | 8.1788 (-03) | 1.0978 | 1.5567 (-03) | 1.4493 | 2.8340 (-04) | 1.6858 |
| \( 2^8 \) | 3.8216 (-03) | 1.0977 | 5.6377 (-04) | 1.4653 | 8.6708 (-05) | 1.7086 |

Table 1: Error and convergence order for example 1 with exact solution \( u(t) = \rho^{-\alpha} \frac{\Gamma((1+2/\rho)/\rho)}{\Gamma(1+2/\rho+\alpha)} t^{\alpha+2}, t \in [0, 1] \) for various values of \( \alpha \) and \( \rho \).
\[ \alpha = 0.9 \]

\[ \alpha = 0.5 \]

\[ \alpha = 0.2 \]

| \( N \)     | \( \|e\|_{\ell^{\infty}} \) | Order | \( \|e\|_{\ell^{\infty}} \) | Order | \( \|e\|_{\ell^{\infty}} \) | Order |
|------------|-----------------|-------|-----------------|-------|-----------------|-------|
| 2^4        | 7.5367 (−04)    | −     | 2.2310 (−04)    | −     | 5.5932 (−05)    | −     |
| 2^5        | 1.7804 (−04)    | 2.0817| 4.2271 (−05)    | 2.3999| 9.1469 (−06)    | 2.6123|
| 2^6        | 4.1848 (−05)    | 2.0890| 7.8387 (−06)    | 2.4310| 1.4539 (−06)    | 2.6534|
| 2^7        | 9.8052 (−06)    | 2.0935| 1.4325 (−06)    | 2.4521| 2.2639 (−07)    | 2.6830|
| 2^8        | 2.2931 (−06)    | 2.0962| 2.5919 (−07)    | 2.4665| 3.4715 (−08)    | 2.7051|

| \( \rho = 5 \ln 2/4 \approx 0.86 \) |
| \( \rho = 1/6 \) |

| \( N \)     | \( \|e\|_{\ell^{\infty}} \) | Order | \( \|e\|_{\ell^{\infty}} \) | Order | \( \|e\|_{\ell^{\infty}} \) | Order |
|------------|-----------------|-------|-----------------|-------|-----------------|-------|
| 2^4        | 7.5367 (−04)    | −     | 2.2310 (−04)    | −     | 5.5932 (−05)    | −     |
| 2^5        | 1.7804 (−04)    | 2.0817| 4.2271 (−05)    | 2.3999| 9.1469 (−06)    | 2.6123|
| 2^6        | 4.1848 (−05)    | 2.0890| 7.8387 (−06)    | 2.4310| 1.4539 (−06)    | 2.6534|
| 2^7        | 9.8052 (−06)    | 2.0935| 1.4325 (−06)    | 2.4521| 2.2639 (−07)    | 2.6830|
| 2^8        | 2.2931 (−06)    | 2.0962| 2.5919 (−07)    | 2.4665| 3.4715 (−08)    | 2.7051|

Table 2: Error and convergence order for example 2 with exact solution \( u(t) = t^{3\rho} \), \( t \in [0, 1] \) for various values of \( \alpha \) and \( \rho \).
\[ \alpha = 0.9, \quad \alpha = 0.75, \quad \alpha = 0.5 \]

| \( \rho \) | \( N \) | \( \| e \|_{\ell^\infty} \) | Order | \( \| e \|_{\ell^\infty} \) | Order | \( \| e \|_{\ell^\infty} \) | Order |
|---|---|---|---|---|---|---|---|
| 0.9 | \( 2^4 \) | 5.5781 \((-03)\) | – | 4.1575 \((-02)\) | – | 8.3167 \((-01)\) | – |
| | \( 2^5 \) | 3.0903 \((-03)\) | 0.8520 | 2.5318 \((-02)\) | 0.7156 | 5.6434 \((-01)\) | 0.5595 |
| | \( 2^6 \) | 1.6838 \((-03)\) | 0.8761 | 1.5203 \((-02)\) | 0.7358 | 3.8009 \((-01)\) | 0.5702 |
| | \( 2^7 \) | 9.0993 \((-04)\) | 0.8879 | 9.0757 \((-03)\) | 0.7442 | 2.5805 \((-01)\) | 0.5587 |
| | \( 2^8 \) | 4.8971 \((-04)\) | 0.8938 | 5.4047 \((-03)\) | 0.7478 | 1.7692 \((-01)\) | 0.5446 |
| 0.5 | \( 2^4 \) | 3.0849 \((-03)\) | – | 2.3453 \((-02)\) | – | 4.5055 \((-01)\) | – |
| | \( 2^5 \) | 1.6930 \((-03)\) | 0.8656 | 1.4196 \((-02)\) | 0.7243 | 3.0941 \((-01)\) | 0.5422 |
| | \( 2^6 \) | 9.1820 \((-04)\) | 0.8827 | 8.5066 \((-03)\) | 0.7388 | 2.1189 \((-01)\) | 0.5462 |
| | \( 2^7 \) | 4.9505 \((-04)\) | 0.8912 | 5.0750 \((-03)\) | 0.7452 | 1.4590 \((-01)\) | 0.5384 |
| | \( 2^8 \) | 2.6611 \((-04)\) | 0.8955 | 3.0218 \((-03)\) | 0.7480 | 1.0109 \((-01)\) | 0.5294 |
| 0.05 | \( 2^4 \) | 3.2737 \((-04)\) | – | 2.9506 \((-03)\) | – | 6.3763 \((-02)\) | – |
| | \( 2^5 \) | 1.7664 \((-04)\) | 0.8901 | 1.7644 \((-03)\) | 0.7418 | 4.4834 \((-02)\) | 0.5081 |
| | \( 2^6 \) | 9.4977 \((-05)\) | 0.8951 | 1.0519 \((-03)\) | 0.7462 | 3.1566 \((-02)\) | 0.5089 |
| | \( 2^7 \) | 5.0982 \((-05)\) | 0.8976 | 6.2624 \((-04)\) | 0.7482 | 2.2159 \((-02)\) | 0.5077 |
| | \( 2^8 \) | 2.7343 \((-05)\) | 0.8988 | 3.7257 \((-04)\) | 0.7492 | 1.5602 \((-02)\) | 0.5061 |

Table 3: Error and convergence order for example 3 with exact solution \( u(t) = a_0 E_\alpha ((t^\rho - a^\rho)^\alpha) + (t^\rho - a^\rho)^{1+\alpha} E_{\alpha,\alpha+2} ((t^\rho - a^\rho)^\alpha) \), \( t \in [a,T] \) for various values of \( \alpha \) and \( \rho \). The parameters used are: \( a_0 = -1, a = 0.5 \) and \( T = 1 \).