Spectrum Degeneracy of General 
$(p = 2)$–Parasupersymmetric Quantum 
Mechanics and Parasupersymmetric 
Topological Invariants

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Abstract

A thorough analysis of the general features of $(p = 2)$ parasupersymmetric quantum mechanics is presented. It is shown that for both Rubakov–Spiridonov and Beckers–Debergh formulations of $(p = 2)$-parasupersymmetric quantum mechanics, the degeneracy structure of the energy spectrum can be derived using the defining parasuperalgebras. Thus the results of the present article is independent of the details of the Hamiltonian. In fact, they are valid for arbitrary systems based on arbitrary dimensional coordinate manifolds. In particular, the Rubakov–Spiridonov (R-S) and Beckers–Debergh (B-D) systems possess identical degeneracy structures. For a subclass of R-S (alternatively B-D) systems, a new topological invariant is introduced. This is a counterpart of the Witten index of the supersymmetric quantum mechanics.

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1 Introduction

There are two alternative definitions of \((p = 2)\)-parasupersymmetric quantum mechanics (PSQM). These are the original Rubakov–Spiridonov [1] and the Beckers–Debergh [2] formulations of \((p = 2)\)-PSQM. The Rubakov–Spiridonov (R-S) \((p = 2)\)–PSQM is defined by the parasuperalgebra:

\[
Q^3 = 0 \quad (1) \\
[H, Q] = 0 \quad (2) \\
Q^2 Q^\dagger + QQ^\dagger Q + Q^\dagger Q^2 = 4QH \quad (3)
\]

where \(Q\) is a parasupercharge and \(H\) is the Hamiltonian. The R-S \((p = 2)\)-PSQM has been studied [1] and its variations and generalizations to arbitrary orders \((p > 2)\) have been given by Khare [4]. The degeneracy structure of this type of PSQM has been worked out for specific examples and some classes of systems in ordinary one-dimensional quantum mechanics, [1, 4].

Similarly, the Beckers–Debergh (B-D) \((p = 2)\)-PSQM is defined by the parasuperalgebra:

\[
Q^3 = 0 \quad (4) \\
[H, Q] = 0 \quad (5) \\
[Q, [Q^\dagger, Q]] = 2QH \quad (6)
\]

Particular examples of this type of PSQM has been studied [2] in the context of one-dimensional quantum mechanics. The corresponding coherent states have also been constructed [3].

In this article the parasuperalgebras (1)-(3) and (4)-(6) are used to study the degeneracy structure of the most general R-S and B-D systems. Our strategy is analogous to the one used for the treatment of supersymmetric quantum mechanics (SQM), [1, 2].

One of the most intriguing aspects of SQM is its relation to the Atiyah–Singer index theorem [7] (For an up-dated account of this subject see [8] and the references therein.) As SQM can be viewed as the \((p = 1)\)-PSQM, one might be tempted to seek similar features of \((p = 2)\) or even \((p > 2)\)-PSQM. The first step in this direction is to explore the degeneracy structure of such systems. This is the main motivation behind our general treatment of \((p = 2)\)-PSQM.
Before pursuing the study of \((p = 2)\)-PSQM, we present a brief review of the relevant aspects of SQM in Sec. 2. The R-S and B-D \((p = 2)\)-PSQM are analyzed in Secs. 3 and 4, respectively. In Sec. 5, a subclass of R-S (alternatively B-D) \((p = 2)\)-PSQM is considered. For this class a topological invariant is introduced which resembles the Witten index of SQM. Sec. 6 includes the concluding remarks.

2 Supersymmetric Quantum Mechanics

The \((N = 1)\)-SQM is defined according to the superalgebra:

\[
\begin{align*}
\{Q, Q\} &= [H, Q] = 0, \\
\{Q, Q^\dagger\} &= 2\kappa H,
\end{align*}
\]

where \(Q\) and \(H\) stand for the supercharge and the Hamiltonian and \(\kappa\) is a positive real number. It is usually chosen to be 1. However, this convention does not agree with the conventions used in different approaches to \((p = 2)\)-PSQM.

Furthermore, there exist a self-adjoint involution \(\mathcal{T}\) on the Hilbert space \(\mathcal{H}\) which satisfies:

\[
\begin{align*}
\mathcal{T}^2 &= 1, \\
\{\mathcal{T}, Q\} &= [\mathcal{T}, H] = 0.
\end{align*}
\]

The involution \(\mathcal{T}\) introduces a double grading of the Hilbert space, i.e., it leads to a decomposition of the Hilbert space \(\mathcal{H}\) into its \(\pm 1\)-eigenspaces:

\[
\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+, \quad \text{with} \quad \mathcal{T}|\psi_\pm\rangle = \pm|\psi_\pm\rangle, \quad \forall|\psi_\pm\rangle \in \mathcal{H}_\pm.
\]

The involution \(\mathcal{T}\) is also called the chirality operator and denoted by \((-1)^F\). The \(\pm 1\)-eigenstates of \(\mathcal{T}\) are said to be even (for +) and odd (for –) elements of the Hilbert space. They are also said to have \(\pm\) chirality. For more details of the importance of the chirality operator see [10].

An important property of SQM is that its degeneracy structure can be easily obtained using the superalgebra \([\mathbb{I}]\)--\([\mathbb{S}]\) and the properties of
To start the analysis of the spectrum of SQM, one uses the eigenvalues of $H$ and (say) $Q_1$ to label the basic states $|E,q_1\rangle$, where

$$H|E,q_1\rangle = E|E,q_1\rangle$$

and

$$Q_1|E,q_1\rangle = q_1|E,q_1\rangle.$$  

Combining Eqs. (13), (14), and (13), one immediately finds:

$$E|E,q_1\rangle = H|E,q_1\rangle = \kappa^{-1}Q_1^2|E,q_1\rangle = \kappa^{-1}q_1^2|E,q_1\rangle,$$

which for $|E,q_1\rangle \neq 0$ implies $q_1 = \pm \sqrt{\kappa E}$. Thus

a) $E \geq 0$;

b) $E = 0$ is non-degenerate;

c) $E > 0$ are doubly degenerate.

Note that in practice there may exist other conserved quantities (observables) that would introduce further quantum numbers and thus correspond to additional degeneracies associated with each $|E,q_1\rangle$. We shall call these subdegeneracies. The existence of this conserved quantities overshadows the effectiveness of statement (b) above, but statement (c) is still worth investigating.

In addition, Eqs. (12), and (14) can be easily employed to show

$$Q_2|E,\pm \sqrt{\kappa E}\rangle = \sqrt{\kappa E} e^{\pm i\phi} |E,\mp \sqrt{\kappa E}\rangle. $$

In fact, the phase factors $e^{\pm i\phi}$ can be absorbed in the definition of $|E,\pm \sqrt{\kappa E}\rangle$, so that

$$Q_2|E,\pm \sqrt{\kappa E}\rangle = \sqrt{\kappa E} |E,\mp \sqrt{\kappa E}\rangle.$$  

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Since $Q_2$ anticommutes with $\tau$ (it is an odd operator), $|E, \pm \sqrt{\kappa E}\rangle$ are called superpartners. To motivate the use of the word superpartner, one can construct a basis of the $E > 0$ subspaces consisting of a bosonic (even) and a fermionic (odd) state vector. In the basis

$$\left\{ |E, \sqrt{\kappa E}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |E, -\sqrt{\kappa E}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

one has:

$$Q_1|_{\mathcal{H}_E} = \sqrt{\kappa E} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sqrt{\kappa E} \sigma_3, \quad Q_2|_{\mathcal{H}_E} = \sqrt{\kappa E} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sqrt{\kappa E} \sigma_1.$$  \hspace{1cm} (18)

Here $\mathcal{H}_E$ denotes the degeneracy subspace associated with the energy level $E$ (Note that $E > 0$). Using Eqs. (11) and (14) and the the fact that the chirality operator is self-adjoint, one can show that there are only two possibilities for $\tau$. These are

$$\tau|_{\mathcal{H}_E} = \eta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \eta \sigma_2,$$  \hspace{1cm} (19)

where in Eqs. (18) and (19), $\sigma_\alpha$ with $\alpha = 1, 2, 3$ are Pauli matrices, and $\eta = \pm 1$. The arbitrariness of the sign proves to be unimportant. An exchange of sign corresponds to an exchange of the eigenvalues ($\pm 1$) of $\tau$. In any case, $\det(\tau|_{\mathcal{H}_E}) = -1$.

Having found the matrix representation of $\tau$, one can diagonalize it and find a basis of the $E$-eigenspace consisting of the state vectors of definite chirality. These are

$$|E, +\rangle = \frac{1}{\sqrt{2}}(|E, \sqrt{\kappa E}\rangle + i|E, -\sqrt{\kappa E}\rangle)$$  \hspace{1cm} (20)

$$|E, -\rangle = \frac{1}{\sqrt{2}}(|E, \sqrt{\kappa E}\rangle - i|E, -\sqrt{\kappa E}\rangle),$$  \hspace{1cm} (21)

with $\tau|E, \pm\rangle = \pm|E, \mp\rangle$. Choosing the opposite sign for $\eta$ (respectively for $\tau$) leads to changing $|E, \pm\rangle \rightarrow |E, \mp\rangle$ in (20) and (21).

Therefore, in general each ($E > 0$)--level consists of two linearly independent vectors of opposite chirality. These are the true superpartners. Even if there are further subdegeneracies, (in view of the statement (c) above) the positive energy levels must include pairs of superpartners. Thus there must be equal number of bosonic and
fermionic states with positive energy. This is the very reason why the Witten index (=trace(τ)) is invariant under smooth deformations of the Hamiltonian [7] and how SQM is related to the topological invariants such as indices of elliptic operators [8]. A simple manifestation of the properties of the Witten index is demonstrated by (19) which clearly indicates trace(τ|_{\mathcal{H}_{E>0}})=0.

The main purpose of the present article is to follow a similar approach in the study of (p = 2)–PSQM.

3 Rubakov–Spiridonov (p = 2)–PSQM

Let us first express the R-S parasuperalgebra (1)–(3) in terms of the self-adjoint parasupercharges (11):

\[ Q_3^3 - \{Q_1, Q_2^3\} - Q_2 Q_1 Q_2 = 0 \]  
\[ Q_2^3 - \{Q_2, Q_1^3\} - Q_1 Q_2 Q_1 = 0 \]  
\[ [H, Q_1] = [H, Q_2] = 0 \]  
\[ Q_1^3 = 2Q_1 H \]  
\[ Q_2^3 = 2Q_2 H , \]

where (22) and (23) are equivalent to (1), (24) is equivalent to (2), and (25) and (26) are equivalent to (3). Clearly, Eq. (13) with \( \kappa = 2 \) is a special case of Eqs. (22)–(26). Thus, SQM is included in R-S (p = 2)–PSQM.

Combining Eqs. (22) and (25) and similarly Eqs. (23) and (26), one obtains the following useful relations:

\[ 2Q_1 H - \{Q_1, Q_2^3\} - Q_2 Q_1 Q_2 = 0 \]  
\[ 2Q_2 H - \{Q_2, Q_1^3\} - Q_1 Q_2 Q_1 = 0 . \]

Again one can appeal to Eq. (24) to choose \(|E, q_1\rangle\) of Eqs. (15) and (16) as basic state vectors. In view of Eqs. (25), (15) and (16), one has:

\[ (Q_1^3 - 2Q_1 H)|E, q_1\rangle = q_1(q_1^2 - 2H)|E, q_1\rangle . \]

Thus,

\[ \text{either } q_1 = 0 , \quad \text{or } q_1 = \pm \sqrt{2E} . \]  

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In particular if $E \leq 0$, one necessarily has $q_1 = 0$. Thus the non-positive energy levels – if they exist – are non-degenerate.\footnote{That is aside from the subdegeneracies.}

To study the positive energy levels, first we write Eqs. (27) and (28) in terms of their matrix elements:

\[
\langle E, q'_1 \mid 2q_1 E - (q_1 + q'_1)Q_2^2 - Q_2Q_1Q_2 \mid E, q_1 \rangle = 0 \quad \text{(30)}
\]

\[
\langle E, q'_1 \mid 2E - (q_1 + q'_1)^2 + q_1q'_1 \mid E, q'_1 \rangle \langle E, q'_1 \mid Q_2 \mid E, q_1 \rangle = 0. \quad \text{(31)}
\]

Next we analyze the following possibilities:

**Case (1):** $E > 0$ and $\langle E, q_1 = 0 \rangle \neq 0$;

**Case (2):** $E > 0$ and $\langle E, q_1 = 0 \rangle = 0$.

Consider Case (1). Then one can set $q_1 = q'_1 = 0$ in Eq. (31) and obtain:

\[
\langle E, 0 \mid Q_2 \mid E, 0 \rangle = 0, \quad \text{(32)}
\]

Therefore, either $E$ is non-degenerate and $Q_2 \mid E, 0 \rangle = 0$, or it is degenerate and there is some $q_1 \neq 0$ such that $\langle E, q_1 \rangle \neq 0$. We shall refer to these two cases as Case (1a) and Case (1b), respectively. Next we consider Case (1b).

In this case, Eq. (31) together with (29) lead to:

\[
\langle E, q_1 \mid Q_2 \mid E, q_1 \rangle = 0. \quad \text{(33)}
\]

Note that, \textit{a priori}, $q_1$ may take one or both possible values $\pm \sqrt{2E}$. In other words, there are again two possibilities:

I) $\mid E, \pm \sqrt{2E} \rangle \neq 0$

II) $\mid E, q_1 \rangle \neq 0$ and $\mid E, -q_1 \rangle = 0$, for $q_1 = \sqrt{2E}$, or $q_1 = -\sqrt{2E}$.

In general, according to Eq. (32):

\[
Q_2 \mid E, 0 \rangle = a_+ \mid E, \sqrt{2E} \rangle + a_- \mid E, -\sqrt{2E} \rangle. \quad \text{(34)}
\]

where $a_{\pm}$ are complex coefficients. Moreover, setting $q_1 = q'_1 = 0$ in Eq. (30), one finds:

\[
\langle E, 0 \mid Q_2Q_1Q_2 \mid E, 0 \rangle = 0. \quad \text{(35)}
\]
This relation and Eq. (34) can be used to show
\[ a^*_+ a_+ = a^*_a_-, \quad \text{if } |E, \pm \sqrt{2E}\rangle \neq 0; \quad (36) \]
\[ a_\pm = 0, \quad \text{if } |E, \sqrt{2E}\rangle = 0 \text{ or } |E, -\sqrt{2E}\rangle = 0. \quad (37) \]

Let us show that in fact Case (II) above does not occur, i.e., the vanishing of either of \(|E, \pm \sqrt{2E}\rangle\) implies the vanishing of the other. By contradiction assume \(|E, \sqrt{2E}\rangle \neq 0\) and \(|E, \sqrt{2E}\rangle = 0\) or \(|E, -\sqrt{2E}\rangle = 0\). Then according to (37), \(a_\pm = 0\) and \(Q^2|E, 0\rangle = 0\). This together with Eq. (33) imply, \(Q^2|E, \sqrt{2E}\rangle \propto |E, 0\rangle\). In fact, since \(Q^2|E, 0\rangle\) vanishes, \(Q^2|E, \sqrt{2E}\rangle = 0\). Hence \(Q^2|E, \sqrt{2E}\rangle = 0\) as well. Using the last equation and Eqs. (22) and (29), one is led to the result \(E = 0\). A similar argument shows that \(|E, \sqrt{2E}\rangle = 0\) and \(|E, -\sqrt{2E}\rangle \neq 0\) give rise to the same conclusion. This contradicts the hypothesis \((E > 0)\).

Thus this case is forbidden and the energy levels of type (1b) are triply degenerate.

To obtain the matrix representation of \(Q^2\) in Case (1b), one rewrites Eq. (33) in the form:
\[ Q^2|E, \sqrt{2E}\rangle = c_+|E, 0\rangle + f|E, -\sqrt{2E}\rangle \]
\[ Q^2|E, -\sqrt{2E}\rangle = c_-|E, 0\rangle + f^*|E, \sqrt{2E}\rangle. \quad (38) \]

The coefficients \(c_\pm\) and \(a_\pm\) can be related by computing \(|Q^2|E, q_1\rangle|^2\) and \(\langle E, q_1|Q^2|E, q_1\rangle\) and equating the results. This leads to
\[ c_+(c_+^* - a_+) = 0 \quad (39) \]
\[ c_- (c_-^* - a_-) = 0 \quad (40) \]
\[ |a_\pm|^2 = \frac{1}{2} (a_+ c_+ + c_- a_-). \quad (41) \]

Furthermore, setting \(q_1 = q_1' = \pm \sqrt{2E}\) in (30) and using (38), one finds:
\[ |f|^2 = 2(E - |c_\pm|^2). \quad (42) \]

Hence, \(|c_+| = |c_-|\). A final relation among \(a_\pm\), \(c_\pm\) and \(f\) is obtained by acting both sides of (24) on \(|E, 0\rangle\) and repeatedly using (14) and (38). This yields:
\[ c_+ c_- f^* + c_- c_+ f = 0. \quad (43) \]

In view of Eqs. (33), (41), and (44), there are two possible cases:

**Case (1.b.i):** \(a_\pm = c_\pm \neq 0\), in which case \(Q^2|E, 0\rangle \neq 0;\)
Case (1.b.ii): \( a_\pm = c_\pm = 0 \), in which case \( Q_2|E, 0\rangle = 0 \).

For convenience we introduce the real parameter \( \zeta \in [0, 1] \):

\[
\zeta := \frac{|a_\pm|}{\sqrt{E}} = \frac{|c_\pm|}{\sqrt{E}}.
\]

(44)

Then one has:

\[
a_\pm = \zeta \sqrt{E} e^{-i\gamma \pm} , c_\pm = \zeta \sqrt{E} e^{i\gamma \pm} , f = \sqrt{2E(1 - \zeta^2)} e^{i\varphi}.
\]

In view of these relations and Eq. (43), (34) and (38) take the form:

\[
Q_2|E, 0\rangle = \zeta \sqrt{E} \left( e^{-i\gamma_+}|E, \sqrt{2E}\rangle + e^{-i\gamma_-}|E, -\sqrt{2E}\rangle \right)
\]

(45)

\[
Q_2|E, \pm \sqrt{2E}\rangle = \sqrt{E} \left[ \zeta e^{i\gamma \pm}|E, 0\rangle + \sqrt{2(1 - \zeta^2)} e^{\pm i\varphi}|E, \mp \sqrt{2E}\rangle \right]
\]

(46)

\[
e^{i(\varphi - \gamma_+ + \gamma_-)} = \epsilon, \quad \text{with} \quad \epsilon = \pm.
\]

(47)

Redefining \( |E, \pm \sqrt{2E}\rangle \rightarrow e^{-i\gamma \pm}|E, \pm \sqrt{2E}\rangle \), one can eliminate all the phase factors. This yields

\[
Q_2|E, 0\rangle = \zeta \sqrt{E} \left( |E, \sqrt{2E}\rangle + |E, -\sqrt{2E}\rangle \right)
\]

(48)

\[
Q_2|E, \pm \sqrt{2E}\rangle = \sqrt{E} \left( \zeta |E, 0\rangle \pm i\epsilon \sqrt{2(1 - \zeta^2)} |E, \mp \sqrt{2E}\rangle \right)
\]

(49)

Remarkably, the defining relations (22)–(26) of R-S (\( p = 2 \))–PSQM , do not impose any restriction on \( \zeta \) and \( \epsilon \). Hence, unless further details of the system is known, they cannot be determined.

At this stage, we can try to find a representation of the parasupercarges. Using the basis

\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix},
\begin{pmatrix}
0 \\
\end{pmatrix}
\]

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one can easily express $Q_1$ and $Q_2$ as $3 \times 3$ matrices:

$$Q_1|_{\mathcal{H}_E} = \sqrt{2E} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \sqrt{2E}J_3^{(1)},$$

$$Q_2|_{\mathcal{H}_E} = \sqrt{2E} \begin{pmatrix} 0 & \frac{\zeta}{\sqrt{2}} & -i\epsilon\sqrt{1-\zeta^2} \\ \frac{\zeta}{\sqrt{2}} & 0 & \frac{\zeta}{\sqrt{2}} \\ i\epsilon\sqrt{1-\zeta^2} & \frac{\zeta}{\sqrt{2}} & 0 \end{pmatrix} \quad (50)$$

$$= \sqrt{E}\zeta J_1^{(1)} + i\epsilon\sqrt{2E(1-\zeta^2)} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

where $J_i^{(1)}$, with $i = 1, 2, 3$, are the three dimensional ($j = 1$) representation of the generators of $SU(2)$. One may also check that for all values of $\zeta$ and $\epsilon$, $\text{Spectrum}(Q_2|_{\mathcal{H}_E}) = \{0, \pm\sqrt{2E}\} = \text{Spectrum}(Q_1|_{\mathcal{H}_E}).$

Next step is to adopt a self-adjoint involution operator $\tau$ which satisfies (9) and (14) and use these equations to find its representation in the $E$-eigenspace. Using the self-adjointness of $\tau$ and imposing Eqs. (9) and (14), one finds

$$\tau|_{\mathcal{H}_E} = \begin{pmatrix} 0 & 0 & \tilde{\eta} \\ 0 & \eta & 0 \\ \tilde{\eta} & 0 & 0 \end{pmatrix},$$

where, for $\zeta = 0$ (Case (1.b.ii)), $\eta$ and $\tilde{\eta}$ are arbitrary signs, i.e., $\eta, \tilde{\eta} = \pm 1$, and for $\zeta \neq 0$ (Case (1b.i)), $\tilde{\eta} = -\eta = \pm 1$. The sign ambiguities in (51) is in a sense analogous to the case of SQM. However as we show below, there are some important differences.

Let us construct the states of definite chirality. These are

$$|E, \pm\rangle := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ \pm\tilde{\eta} \end{pmatrix}, \quad |E, \eta^\circ\rangle := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$ 

They satisfy:

$$\tau|E, \pm\rangle = \pm|E, \pm\rangle, \quad \tau|E, \eta^\circ\rangle = \eta|E, \eta^\circ\rangle.$$ 

(53)
In other words depending on the value of \( \eta = \pm 1 \), \( |E, \eta^0 = \pm^0\rangle \) is parabosonic or parafermionic. Note also that \( \eta = -\det(\tau |_{\mathcal{H}_E}) \). Thus choosing \( \det(\tau |_{\mathcal{H}_E}) = -1 \), as is the case in SQM, is equivalent to setting \( \eta = 1 \).

Next we display the action of \( Q_2 \) on the states of definite chirality:

\[
Q_2|E, \pm\rangle = \sqrt{2E} \left[ \mp i\bar{\eta}\sqrt{1 - \zeta^2} |E, \mp\rangle + \frac{1 \pm \bar{\eta}}{2}\zeta |E, \eta^0\rangle \right] \quad (54)
\]

\[
Q_2|E, \eta^0\rangle = \zeta \sqrt{E}|E, -\eta\rangle . \quad (55)
\]

There are two specially interesting cases. These are characterized by \( \zeta = 0 \) (Case (1.b.ii) ), for which

\[
Q_2|E, \pm\rangle = \mp i\bar{\eta}\sqrt{2E}|E, \mp\rangle , \quad Q_2|E, \eta^0\rangle = 0 , \quad (56)
\]

and \( \zeta = 1 \) for which

\[
Q_2|E, \pm\rangle = \sqrt{2E} \left( \frac{\eta + \bar{\eta}}{2} |E, \eta^0\rangle , \quad Q_2|E, \eta^0\rangle = \sqrt{E}|E, -\eta\rangle \right)
\]

\[
Q_2|\mathcal{H}_E = \sqrt{2E} \begin{pmatrix}
0 & 1 \\
\sqrt{2} & 0 \\
0 & \frac{1}{\sqrt{2}} \\
\end{pmatrix} = \sqrt{2E}J_1^{(1)} . \quad (57)
\]

For these two special cases \( Q_2 \) eliminates one of the states defined in (52). Moreover, as we shall see in Sec. 4, these two cases also appear in the B-D \((p = 2)-\)PSQM . The coincidence of \( Q_1 \) and \( Q_2 \) with the generators of \( SU(2) \) is analogous with SQM.

One must emphasize that unlike SQM, in \((p = 2)-\)PSQM different choices of \( \tau \) (different choices for \( \eta \) and \( \bar{\eta} \)) can lead to different systems. This is because here the energy levels do not consist of pairs of superpartners. In fact, in general the defining signs \( \eta \) and \( \bar{\eta} \) which appear in the expression (51) of \( \tau |_{\mathcal{H}_E} \) depend on \( E \). They may take different values for different energy levels. Thus as far as quantities such as the difference of the numbers of parabosonic and parafermionic states are concerned, the values of \( \eta = \eta(E) \) and \( \bar{\eta} = \bar{\eta}(E) \) are important. For \( \zeta \neq 0 \), requiring \( \det(\tau |_{\mathcal{H}_E}) = -1 \) for all \( E \), fixes \( \eta(E) = -\bar{\eta}(E) = 1 \).

This completes our analysis of Case (1). A summary of the results is in order:
Lemma 1 For $E > 0$ and $|E, 0\rangle \neq 0$, either $E$ is non-degenerate with $Q_2|E, 0\rangle = 0$ or it is triply degenerate with the basis $\{|E, 0\rangle, |E, \pm \sqrt{2E}\rangle\}$. In the latter case two of the three linearly independent state vectors are parasuperpartners.

Finally we analyze the case $E > 0$ and $|E, 0\rangle = 0$, i.e., Case (2). Again following a similar argument as given in the previous case, one can show that $|E, \sqrt{2E}\rangle \neq 0$ implies $|E, -\sqrt{2E}\rangle \neq 0$ and vice versa. Hence we can safely assume that in this case $E$ is doubly degenerate with the eigenbasis $\{|E, \pm \sqrt{2E}\rangle\}$.

Using Eq. (31), one can still show the validity of (33). However now Eq. (33) can be written in the form:

$$Q_2|E, \pm \sqrt{2E}\rangle = b_{\pm}|E, \mp \sqrt{2E}\rangle,$$

(58)

where $b_{\pm} \in \mathbb{C}$. Furthermore, in view of (58) and (20), one has

$$0 = (Q_2^3 - 2HQ_2)|E, \pm \sqrt{2E}\rangle = -b_{\pm}(2E - b_+ b_-)|E, \mp \sqrt{2E}\rangle.$$

This means that either $b_{\pm} = 0$ or $b_+ b_- = 2E$. In fact, using Eq. (30) one can easily show that the choice $b_{\pm} = 0$ leads to $E = 0$ and is consequently inadmissible. Therefore,

$$b_+ b_- = 2E.$$

(59)

On the other hand, Eqs. (58) and (59) can be employed to compute:

$$|b_{\pm}|^2 = \langle E, \pm \sqrt{2E} | Q_2^3 | E, \pm \sqrt{2E}\rangle = \langle E, \pm \sqrt{2E} | (Q_2^3 | E, \pm \sqrt{2E}\rangle = 2E,$$

so that $b_{\pm} = \sqrt{2E} \exp(\pm i\lambda)$. Here Eq. (27) is also used and $\lambda \in \mathbb{R}$. Again it is possible to absorb the phase factors in $|E, \pm \sqrt{2E}\rangle$, in which case Eq. (58) reads:

$$Q_2|E, \pm \sqrt{2E}\rangle = \sqrt{2E}|E, \mp \sqrt{2E}\rangle.$$

(60)

The matrix representation of $Q_1, Q_2$ and $\tau$ is identical with the case of SQM described in Sec. 2 (with $\kappa = 2$). Thus the energy levels of this type involve (up to subdegeneracies) a pair of parasuperpartners.

The following lemma summarizes the results of our analysis of Case (2).

Lemma 2 For $E > 0$ and $|E, q_1 = 0\rangle = 0$, $E$ is doubly degenerate with the basis $\{|E, \pm \sqrt{2E}\rangle\}$. It consists of a pair of parasuperpartners.
This concludes our treatment of the R-S \((p = 2)\)-PSQM. Note that even in the presence of other quantum numbers (subdegeneracies), the energy levels of type (1b) involve equal number of state vectors associated with the three basic (subdegenerate) states. Similarly, the energy levels of type (2) consist of an equal number of state vectors of opposite chirality. The latter is reminiscent of the inclusion of SQM in \((p = 2)\)-PSQM [1].

4 Beckers-Debergh \((p = 2)\)-PSQM

In terms of the self-adjoint parasupercharges (11), the B-D parasuperalgebra (4)-(6) is expressed by Eqs. (22), (23), (24) and

\[
3\{Q_1, Q_2^3\} - 2Q_1^3 &= 2Q_1 H \\ 3\{Q_2, Q_1^3\} - 2Q_2^3 &= 2Q_2 H .
\]

(61) (62)

Combining these relations with (22) and (23), one also has:

\[
Q_1^3 - 3Q_2Q_1Q_2 &= 2Q_1 H \\ Q_2^3 - 3Q_1Q_2Q_1 &= 2Q_2 H \\ Q_1Q_2^2 + Q_2^2Q_1 - 2Q_2Q_1Q_2 &= 2Q_1 H \\ Q_2Q_1^2 + Q_1^2Q_2 - 2Q_1Q_2Q_1 &= 2Q_2 H .
\]

(63) (64) (65) (66)

Again one can check that (13) with \(\kappa = 1/2\) is a special case of (61) and (62). Hence SQM is included in B-D \((p = 2)\)-PSQM as well.

In view of Eqs. (24), (15), (16), and (63), the following is a straightforward observation:

\[
Q_2Q_1Q_2|E, q_1\rangle = \frac{q_1(q_1^2 - 2E)}{3}|E, q_1\rangle .
\]

(67)

Next let us consider expressing Eq. (62) in terms of its matrix elements. A simple calculation yields:

\[
[(q_1 - q')^2 - 2E]|E, q'_1|Q_2|E, q_1\rangle = 0 .
\]

(68)

Similarly, using (22) and (57), one has

\[
(q_1 + q')|E, q'_1|Q_2^2|E, q_1\rangle = \left[2q_1(q_1^2 + E)/3\right] \delta_{q_1 q'_1} .
\]

(69)
Setting $q_1 = q'_1$ in the last equation, one finds:

$$q_1 \langle E, q_1 | Q_2^2 | E, q_1 \rangle = q_1 |Q_2|E, q_1\rangle|^2 = q_1(q_1^2 + E) / 3 . \tag{70}$$

Next, let us consider the negative energy levels. For $E < 0$, Eq. (68) requires $Q_2|E, q_1\rangle = 0$. This relation together with (22) imply $q_1 = 0$. Thus, the negative energy levels are non-degenerate.

For the $E = 0$ energy level, Eq. (68) implies

either $\langle 0, q'_1 | Q_2 | 0, q_1 \rangle = 0$, or $q_1 = q'_1$. \tag{71}

In any case, $Q_2$ must be diagonal in the $E = 0$ eigenspace:

$$Q_2|0, q_1\rangle = \kappa(q_1)|0, q_1\rangle . \tag{72}$$

Making use of this equation to simplify (69), one is led to:

$$\kappa^2(q_1 \neq 0) = q_1^2 / 3 . \tag{73}$$

Similarly, using (72) and (64), one obtains:

$$\kappa(q_1)|\kappa(q_1)^2 - 3q_1^2| = 0 \tag{74}$$

Eqs. (73) and (74) imply $q_1 = 0$, and $\kappa(q_1) = 0$. Hence, $E = 0$ also is non-degenerate.

The analysis of the positive energy levels is rather involved. We shall first state some general algebraic results and then explore the following cases:

**Case (1):** $|E, 0\rangle \neq 0$ and

\begin{align}
(1.a) & \quad Q_2|E, 0\rangle \neq 0 \\
(1.b) & \quad Q_2|E, 0\rangle = 0
\end{align}

**Case (2):** $|E, 0\rangle = 0$

Let us suppose that for some $q_1 \neq 0$, $|E, q_1\rangle \neq 0$. then according to Eq. (69):

$$Q_2^2|E, q_1 \neq 0\rangle = c_{q_1}|E, -q_1\rangle + \frac{q_1^2 + E}{3}|E, q_1\rangle , \tag{75}$$

where $c_{q_1} := \langle E, -q_1 | Q_2^2 | E, q_1 \rangle$ is a complex number. Note that $|E, -q_1\rangle$ may not exist, in which case one sets $|E, -q_1\rangle = 0$ and $c_{q_1} = 0$. It is possible that $|E, -q_1\rangle \neq 0$ but still $c_{q_1} = 0$. We shall next consider the case where $c_{q_1} = 0$. 

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Let \( q_1 \neq 0 \) and \( c_{q_1} = 0 \), then Eqs. (75) and (64) lead to
\[
Q_1 Q_2 |E, q_1\rangle = \frac{q_1^2 - 5E}{9q_1} [Q_2 |E, q_1\rangle] .
\] (76)

Therefore by definition (16),
\[
Q_2 |E, q_1\rangle = \xi(q_1) |E, \tilde{q}_1\rangle, \quad \text{with} \quad \tilde{q}_1 := \frac{q_1^2 - 5E}{9q_1} , \quad \xi(q_1) \in \mathbb{C} .
\] (77)

Next substitute (77) in (75). In view of \( c_{q_1} = 0 \), this yields:
\[
\frac{q_1^3 + E}{3} |E, q_1\rangle = Q_2^2 |E, q_1\rangle = \xi(q_1) \xi(\tilde{q}_1) |E, \frac{q_1^2 - 5E}{9q_1}\rangle ,
\]
and consequently:
\[
\frac{q_1^2 - 5E}{9q_1} = q_1 , \quad \text{so that} \quad q_1 = \pm \sqrt{E/2} .
\] (78)

Moreover, in view of (73), (77) and (78), \( \xi(q_1) = \sqrt{E/2} e^{i\gamma} \). Again redefining the phases of \( |E, \pm \sqrt{E/2}\rangle \), one is able to set \( \gamma = 0 \). The result is
\[
Q_2 |E, \pm \sqrt{E/2}\rangle = \sqrt{E/2} |E, \mp \sqrt{E/2}\rangle .
\] (79)
\[
Q_2^2 |E, \pm \sqrt{E/2}\rangle = (E/2) |E, \pm \sqrt{E/2}\rangle .
\] (80)

Note that the last two relations also imply
\[
|E, \sqrt{E/2}\rangle = 0 \quad \text{if and only if} \quad |E, -\sqrt{E/2}\rangle = 0 .
\] (81)

Next consider the case \( q_1 \neq 0 \) and \( c_{q_1} \neq 0 \). This means that necessarily \( |E, \pm q_1\rangle \neq 0 \). Let both sides of (64) act on \( |E, q_1\rangle \) from the left and use (75) to compute \( Q_2^2 |E, q_1\rangle \). This leads to
\[
Q_1 Q_2 |E, q_1 \neq 0\rangle = \frac{1}{3q_1} \left[ \left( \frac{q_1^2 + E}{3} - 2E \right) Q_2 |E, q_1\rangle + c_{q_1} Q_2 |E, -q_1\rangle \right] .
\]

Now multiplying both sides of this expression by \( Q_2 \) from the left and using Eqs. (77) and (73) to simplify the result, one arrive at:
\[
\left[ (8q_1^4 - 14q_1^2 E + 5E^2 - 9c_{q_1} c_{-q_1} )/3 \right] |E, q\rangle + 2c_{q_1} (2E - q_1^2) |E, -q_1\rangle = 0 .
\] (82)
Therefore,

\[ q_1 = \pm \sqrt{2E} \]  
\[ c_{q_1} c_{-q_1} = |c_{q_1}|^2 = E^2, \]

where we also have used the obvious identity \( c_{-q_1} = c_{q_1}^* \). At this stage we begin considering all possible cases.

First suppose that \( E > 0 \) and \( |E,0\rangle \neq 0 \), (Case (1)). Then acting both sides of (61) on \( |E,0\rangle \) from the left, one finds \( Q_1Q_2^2|E,0\rangle = 0 \). Thus, by definition (16), \( Q_2^2|E,0\rangle = \alpha|E,0\rangle \). Let us further assume that \( Q_2|E,0\rangle \neq 0 \), i.e., consider Case (1.a). Then the coefficient \( \alpha \) can be determined by Eq. (64). The result is \( \alpha = 2E \), so that

\[ Q_2^2|E,0\rangle = 2E|E,0\rangle. \]  

Moreover, making use of (62), one also finds:

\[ Q_1^2[Q_2|E,0\rangle] = 2E[Q_2|E,0\rangle]. \]  

Hence clearly

\[ \langle E,0|Q_2|E,0\rangle = 0. \]  

Since by the hypothesis \( Q_2|E,0\rangle \neq 0 \), there must be some \( q_1 \neq 0 \), such that \( \langle E,q_1|Q_2|E,0\rangle \neq 0 \). Multiplying \( \langle E,q_1 \rangle \) by both sides of (80), one immediately finds

\[ q_1 = \pm \sqrt{2E}. \]  

Hence in view of Eq. (87),

\[ Q_2|E,0\rangle = \tilde{a}_+|E,\sqrt{2E}\rangle + \tilde{a}_-|E,-\sqrt{2E}\rangle. \]  

Using the last equation and Eq. (77), i.e., \( Q_2Q_1|E,0\rangle = 0 \), one can further show

\[ \tilde{a}_+Q_2|E,\sqrt{2E}\rangle = \tilde{a}_-Q_2|E,-\sqrt{2E}\rangle. \]  

This in turn implies that the vanishing of either of \( |E,\pm \sqrt{2E}\rangle \) implies the vanishing of \( Q_2^2|E,0\rangle \) and consequently of \( Q_2|E,0\rangle \). This is inconsistent with the hypothesis of this case. Thus \( |E,\pm \sqrt{2E}\rangle \neq 0 \).

A remarkable consistency check on our analysis is to observe the coincidence of Eqs. (88) and (83). In particular, \( c_{\pm \sqrt{2E}} \neq 0 \).
To obtain the action of $Q_2$ on eigenstates of $E$, we proceed as follows. Since, $Q_2^2|E,0\rangle \neq 0$, Eqs. (89) and (90) imply $Q_2|E,\pm \sqrt{2E}\rangle \neq 0$. In fact, substituting (89) in (87) and using (90), one has:

\begin{align}
\hat{Q}_2|E,\pm \sqrt{2E}\rangle &= \sqrt{E} e^{i\tilde{\gamma}_\pm} |E,0\rangle \\
Q_2|E,0\rangle &= \sqrt{E} \left( e^{-i\tilde{\gamma}_+} |E,\sqrt{2E}\rangle + e^{-i\tilde{\gamma}_-} |E,-\sqrt{2E}\rangle \right) \tag{91}
\end{align}

where $e^{i\tilde{\gamma}_\pm} := \sqrt{E}/\tilde{a}_\pm$ are unimportant phase factors. A remarkable observation is that Eqs. (91) and (92) are identical with Eqs. (45) and (46) with $\zeta = 1$. Thus the analysis of Case (1.b) of the previous section – with $\zeta = 1$ – applies for the case considered here.

A summary of the analysis of Case (1.a) is given by

Lemma 3 For $E > 0$, $|E,0\rangle \neq 0$, and $Q_2|E,0\rangle \neq 0$, $E$ is triply degenerate with the eigenbasis $\{ |\tilde{E},0\rangle, |\tilde{E},\pm \sqrt{2E}\rangle \}$.

Next we consider Case (1.b), where $E > 0$, $|E,0\rangle \neq 0$ and $Q_2|E,0\rangle = 0$. In this case $E$ may either be non-degenerate with $|E,0\rangle$ representing the non-degenerate state vector, or it may be degenerate. In the latter case, there must be some $q_1 \neq 0$ with $|E,q_1\rangle \neq 0$. Suppose that $|E,-q_1\rangle = 0$. Then, $c_{q_1}$ of Eq. (73) must vanish. But in this case, according to Eqs. (78) and (81), $q_1 = \pm \sqrt{E/2}$ and $|E,-q_1\rangle = 0$ implies $|E,q_1\rangle = 0$. This is a contradiction. Hence, $|E,\pm q_1\rangle \neq 0$.

In fact, one can show that indeed the condition $Q_2|E,0\rangle = 0$ implies $c_{q_1} = 0$. To see this, we employ Eq. (68) which states that

$$Q_2|E,q_1\rangle = \mu |E,0\rangle + \nu |E,-q_1\rangle .$$

Then,

$$c_{q_1} := \langle E,-q_1 | \left( Q_2^2 |E,q_1\rangle \right) = \nu \langle E,-q_1 | Q_2|E,-q_1\rangle = 0 ,$$

where we have used $Q_2|E,0\rangle = 0$ and Eq. (68). In view of this observation we conclude:

Lemma 4 For $E > 0$, $|E,0\rangle \neq 0$ and $Q_2|E,0\rangle = 0$, $E$ is either non-degenerate or it is triply degenerate with the basis $\{ |E,0\rangle, |E,\pm \sqrt{2E}\rangle \}$.

Furthermore the basic eigenvectors are related via Eq. (79). This case is quite similar to the case (1.b.ii) of the previous section. In fact, our treatment of Case (1b) of the R-S ($p = 2$)–PSQM with $\zeta = 0$ applies
to Case (1.b) of B-D (p = 2)–PSQM. The only difference is that in the latter case we need to set ζ = 0 and multiply the right hand side of Eqs. (50) and (54) by 1/2.

This leaves us with Case (2), where E > 0 and |E, 0⟩ = 0. In this case, Q2|E, 0⟩ = 0 is trivially satisfied and the analysis of the previous case applies. In particular:

**Lemma 5** For E > 0 and |E, 0⟩ = 0, E is doubly degenerate with the basic state vectors, |E, ±√E/2⟩, being parasuperpartners.

This case is identical with the case of SQM (with κ = 1/2). This is a clear indication of the inclusion of SQM in B-D (p = 2)–PSQM.

This concludes our analysis of B-D (p = 2)–PSQM.

## 5 Parasupersymmetric Topological Invariants

In the previous sections it was shown that unlike SQM, the degeneracy structure of the (p = 2)–PSQM does not allow the Witten index (:=trace(τ)) or similar quantities to be invariant under the smooth deformations of the Hamiltonian. There are three obvious reasons justifying this remark. These are:

a) existence of negative energy levels;

b) existence of positive energy levels with ⟨Q1⟩E = ⟨Q2⟩E = 0 (these are referred to as non-degenerate levels in Secs. 3 and 4);

c) existence of both doubly and triply degenerate energy levels.

In order to find systems with invariants analogous to trace(τ) of SQM, one needs to find (p = 2)–PSQM systems for which none of the above obstacles occur. In this section we consider an example of a class of (p = 2)–PSQM systems which fulfill this requirement and therewith lead to “new” topological invariants.

Consider a R-S (p = 2)–PSQ Mechanical system whose Hamiltonian is given by

\[
H = \frac{\gamma}{2} \left[ (Q^1)^2 + (Q^1)^2 + \alpha (Q^1 Q^2 + Q^1 Q^2) \right]^+, \tag{93}
\]

\(^2\)The identification of such invariants with known or unknown topological invariants is the subject of further investigation.
where $\alpha$ and $\gamma > 0$ are real parameters. The system (93) with $\alpha = -1/2$, $\gamma = 1$, and $QQ^\dagger Q = Q^\dagger Q^2 Q^\dagger$ has been suggested by Khare [3] and investigated in the context of ordinary one-dimensional quantum mechanics for arbitrary $p$. Here we shall assume that $p = 2$ and that the system also satisfies the parasuperalgebra of R-S (1)–(3).

In terms of the self-adjoint parasupercargees (11), Eq. (93) is written in the form:

$$H = \frac{\gamma}{2} \left[ \left( \frac{1 + \alpha}{2} \right) (Q_1^2 + Q_2^2)^2 - \left( \frac{1 - \alpha}{2} \right) ([Q_1, Q_2])^2 \right]^{1/2}.$$  \hspace{1cm} (94)

Clearly, the case $\alpha = 1$ and $\gamma = 1/2$ corresponds to SQM with $\kappa = 2$.

In addition, the negative and “non-degenerate” positive energy levels are now forbidden. To see this it is sufficient to square both sides of (94), i.e., consider

$$4H^2 = \gamma^2 \left[ \left( \frac{1 + \alpha}{2} \right) (Q_1^2 + Q_2^2)^2 - \left( \frac{1 - \alpha}{2} \right) ([Q_1, Q_2])^2 \right],$$  \hspace{1cm} (95)

and note that according to the results of Sec. 3, all such states are eliminated by $Q_2$. Thus upon the action of both sides of (95) on such states, the right hand side vanishes whereas the left hand side does not.

Furthermore, it can be shown quite easily that the doubly degenerate levels of lemma 2 are also missing for all values of $\gamma$ except $1/2$. To see this, we use the representation of $Q_1$ and $Q_2$ to express the right hand side of (95). The left hand side equals $4E^2$ times the $2 \times 2$ unit matrix. A simple calculation proves our assertion, i.e., for $\gamma \neq 1/2$, Eq. (95) is not satisfied. Thus the energy levels cannot be doubly degenerate.

This leaves us only with the triply degenerate states. We must still check that for these states, Eq. (95) can indeed be satisfied. Otherwise Eq. (95) would be inconsistent with the R-S parasuperalgebra (1)–(3). Khare [3] has shown that for $\alpha = -1/2$, there are examples for which both (95) and (1)–(3) are fulfilled. In fact, as we shall demonstrate instantly, the condition $\alpha = -1/2$ also is necessary.

Substituting Eq. (50) in the right hand side of (95) and carrying out the calculations, one observes that Eq. (95) is satisfied only if $\alpha = -1/2$, $\gamma = 1$, and $\zeta = 1$. In this case, according to (50) and (57), the parasupercargees are represented by

$$Q_1|_{H_E} = \sqrt{2E}J_3^{(1)}, \quad Q_2|_{H_E} = \sqrt{2E}J_1^{(1)}, \quad \forall E > 0.$$  \hspace{1cm} (96)
Note the remarkable similarity between Eqs. (96) and (18).

Summarizing the results, one has

**Lemma 6** For the R-S \( (p = 2) \)-PSQM the condition that the Hamiltonian satisfies (92) implies that the energy eigenvalues are non-negative and that either \( \gamma = 1/2 \) and \( \alpha \) is arbitrary, or \( \gamma = 1 \) and \( \alpha = -1/2 \). In the former case, the positive energy levels consist of (para)superpartners. In the latter case the positive energy levels are triply degenerate with two of the eigenstates being parasuperpartners.

In fact, one can similarly show that a B-D parasupersymmetric system which satisfies (93) has precisely identical degeneracy structure. In this case, either \( \gamma = 2 \) and positive energy levels are doubly degenerate, or \( \gamma = 1 \) and \( \alpha = -1/2 \), in which case the positive energy levels are triply degenerate.

Now let us consider smooth deformations of the (parameters of the) Hamiltonian satisfying the R-S parasuperalgebra and Eq. (93) with \( \alpha = -\gamma/2 = -1/2 \). Then according to Lemma 6, the initial zero-energy states can only acquire positive energy in groups of three and vice versa a positive energy state can only collapse to the zero level if the other two states within the original (non-perturbed) level accompany it. Thus if the chirality operator \( \tau \) has the same signature, say \( \det(\tau|_{\mathcal{H}_E}) = -1 \), for all \( E > 0 \) then the quantity

\[
\Delta_{(p=2)} := n^{(\pi B)} - 2n^{(\pi F)} = n^{(\pi B)}_0 - 2n^{(\pi F)}_0 ,
\]

(97)

with

\[
\begin{align*}
n^{(\pi B)} & := \text{number of parabosonic states}; \\
n^{(\pi F)} & := \text{number of parafermionic states}; \\
n^{(\pi B)}_0 & := \text{number of parabosonic states of zero energy}; \\
n^{(\pi B)}_0 & := \text{number of parabosonic states of zero energy},
\end{align*}
\]

(98)

remains invariant under the deformation. \( \Delta_{(p=2)} \) is a generalization of the Witten index of SQM and in the above sense is a topological invariant. In fact, if the corresponding system is defined on a smooth manifold or a fiber bundle, i.e., the Hamiltonian and parasupercharges depend on the corresponding geometric structures (connection), then \( \Delta_{(p=2)} \) is a true topological invariant.
By a similar argument one can also show that if a R-S (B-D) system satisfies Eq. (93) with \( \gamma = 1/2 \) (resp. \( \gamma = 2 \)), then the Witten index := trace(\( \tau \)) = \( n^{(\pi B)} - n^{(\pi F)} = n_0^{(\pi B)} - n_0^{(\pi F)} \), is a topological invariant.

The topological invariants defined in this way are measures of the exactness of the parasupersymmetry. More precisely, their being non-zero implies the existence of zero-energy ground states and consequently the exactness of parasupersymmetry.

6 Conclusion

The Robakov–Spiridonov and Beckers–Debergh (\( p = 2 \)) parasupersymmetric quantum systems share identical general degeneracy structures. For the R-S (\( p = 2 \))–PSQM, we defined a continuous (\( \zeta \)) and a discrete parameter (\( \epsilon \)) for each triply degenerate energy level. These determine the action of \( Q_2 \) on the corresponding states. The triply degenerate energy levels of the B-D (\( p = 2 \))–PSQM have the same structure as those of the R-S systems with \( \zeta = 0 \) or \( \zeta = 1 \). Thus in this sense, the R-S (\( p = 2 \))–PSQM is more general than the B-D (\( p = 2 \))–PSQM. The results of the present article is consistent with the results obtained for specific examples considered in the literature [1, 2].

A possible direction of further investigation is to seek a physical interpretation for the parameters \( \zeta \) and \( \epsilon \) of the R-S (\( p = 2 \))–PSQM. Another direction is to try to employ similar methods to arbitrary (\( p > 2 \))–parasupersymmetry. The parasuperalgebra for general R-S (\( p > 2 \))–parasupersymmetry is considerably more complicated. But its variations [3] may be attacked by similar methods. As demonstrated for one-dimensional systems by Khare [4], the degeneracy structure of the systems with even \( p \) (respectively odd \( p \)) displays similar features. This makes the study of (\( p = odd \)) parasupersymmetry more interesting. In fact, one might hope that they involve similar or even more appealing phenomena than supersymmetric quantum mechanics.

The treatment of the (\( p = 2 \))–PSQM presented in this article uses only the defining parasuperalgebras. The only additional assumption
made here is that the state vectors belong to a Hilbert space. Therefore, the results obtained here are applicable to arbitrary quantum systems satisfying these parasuperalgebras. In particular, one might consider systems which are sensitive to the geometric structure of an arbitrary manifold or a fiber bundle.

We have also introduced a chirality (parasupersymmetric involution) operator \( \tau \), and studied its possible representations for different types of energy levels. The general degeneracy structure of the \((p = 2)-\text{PSQM}\) does not allow for the Witten index, i.e., \( \text{trace}(\tau) \), to be a topological invariant. The main obstacle is the possible existence of negative and positive “non-degenerate” \( (\langle Q_1 \rangle_E = \langle Q_2 \rangle_E = 0) \) energy levels and the fact that even the degenerate levels can be either two or three fold degenerate.

These obstacles do not survive if one considers R-S or B-D type parasupersymmetric systems whose Hamiltonian are given by Eq. (93). For such systems, all the positive energy levels are either two fold or three fold degenerate. For the systems with doubly degenerate positive energy levels the Witten index remains to be a topological invariant. For the systems with triply degenerate positive energy levels, the difference of the number of parabosonic states and twice of the number of parafermionic states is a topological invariant. In both cases the non-vanishing of the corresponding topological invariant is an indication of the exactness of the parasupersymmetry. The study of specific examples of these systems and the identification of the topological invariants introduced above is the subject of ongoing investigation.
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