Cesàro summation and multiplicative functions on a symmetric group

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November 7, 2008

Abstract

We investigate the summability in sense of Cesàro and its applications to investigation of the mean values of multiplicative functions on permutations.

Key words: Cesàro sums, Tauberian theorem, divergent series, multiplicative functions, symmetric group, random permutations.

1 Results

Let $S_n$ be the symmetric group. Each element $\sigma \in S_n$ can be decomposed into a product of independent cycles.

$$\sigma = \kappa_1\kappa_2...\kappa_\omega$$

this decomposition is unique up to the order of the multiplicands. We will call a function $f : S_n \to \mathbb{C}$ multiplicative if $f(\sigma) = f(\kappa_1)f(\kappa_2)...f(\kappa_\omega)$. In what follows we will assume that the value of $f$ on cycles depends only on the length of cycle, that is $f(\kappa) = \hat{f}(|\kappa|)$, where $|\kappa|$ - the order of cycle $\kappa$. Let $m_k(\sigma)$ be equal to the number of cycles in the decomposition $\sigma$ whose order is equal to $k$. Then obviously $m_1(\sigma) + 2m_2(\sigma) + ... + nm_n(\sigma) = n$. Thus $n$ complex number $\hat{f}(1), \hat{f}(2), ..., \hat{f}(n)$ completely determine the value of function $f$ on any permutation $\sigma \in S_n$

$$f(\sigma) = \hat{f}(1)^{m_1(\sigma)}\hat{f}(2)^{m_2(\sigma)}...\hat{f}(n)^{m_n(\sigma)}.$$  

On the group $S_n$ we will define the so called Ewens’s measure $\nu_{n,\theta}$ by means of formula

$$\nu_{n,\theta}(\sigma) = \frac{\theta^{k(\sigma)}}{\theta(n)},$$
where \( k(\sigma) = m_1(\sigma) + m_2(\sigma) + \ldots + m_n(\sigma) \), and \( \theta_n = \theta(\theta + 1)(\theta + n - 1) \).

We will investigate the mean values of multiplicative functions with respect to Ewens measure

\[
M_n(f) = \sum_{\sigma \in S_n} f(\sigma) \nu_{n,\theta}(\sigma).
\]

Since the number of \( \sigma \) such that \( m_k(\sigma) = s_k \) is equal to \( n! \prod_{j=1}^{n} \frac{1}{s_j j^{s_j}} \), therefore

\[
\nu_{n,\theta}(m_1(\sigma) = s_1, \ldots, m_n(\sigma) = s_n) = \frac{n!}{\theta(n)} \prod_{j=1}^{n} \left( \frac{\theta}{j} \right)^{s_j} \frac{1}{s_j!}.
\]

Hence

\[
M_n(f) = \binom{n + \theta - 1}{n}^{-1} \sum_{k_1+2k_2+\ldots+nk_n=n} \prod_{j=1}^{n} \left( \frac{\theta \hat{f}(j)}{j} \right)^{k_j} \frac{1}{k_j!}.
\]

It is easy to see that \( M_n(f) \) is equal to \( (n + \theta - 1)^{-1} N_n \), where \( N_n \) is defined by means of relation

\[
F(z) = \exp \left\{ \theta \sum_{j=1}^{\infty} \frac{\hat{f}(j)}{j} z^j \right\} = \sum_{m=0}^{\infty} N_m z^m.
\]

Since the numbers \( \hat{f}(j) \) with \( j > n \) do not influence the value of the coefficient of \( z^n \) therefore we will assume that \( \hat{f}(j) = 1 \) for \( j > n \). Therefore

\[
F(z) = \exp \left\{ \theta \sum_{j=1}^{\infty} \frac{\hat{f}(j)}{j} z^j \right\} = \sum_{j=0}^{\infty} N_j z^j = \frac{\exp\{\theta L_n(z)\}}{(1 - z)^{\theta}},
\]

here and in what follows \( L_n(z) = \sum_{j=1}^{n} \frac{\hat{f}(j)-1}{j} z^j \) and \( L_0(z) = 0 \).

The function \( F(z) \) is the product of two functions \( \exp\{\theta L_n(z)\} = \sum_{k=0}^{\infty} m_k z^k \) and \( \frac{1}{(1 - z)^{\theta}} = \sum_{k=0}^{\infty} \binom{n+\theta-1}{n} z^n \), therefore

\[
M_n(f) = \binom{n + \theta - 1}{n}^{-1} \sum_{j=0}^{n} m_j \binom{n - j + \theta - 1}{n - j}.
\] (1)

We will estimate the sum on the right hand side of the equation (1) by means of the following theorem.

**Theorem 1.1.** Let \( f(z) = \sum_{m=0}^{\infty} a_m z^m \) be analytic for \( |z| < 1 \). Let us denote

\[
S_\theta(f; n) = \sum_{k=1}^{n} k a_k \binom{n - k + \theta - 1}{n - k},
\]
then for fixed $\theta > 0$ we have
\[
\frac{1}{\binom{n+\theta-1}{n}} \sum_{k=0}^{n} a_k \left( n - k + \theta - 1 \right) - f(e^{-1/n}) - \frac{S_\theta(f; n)}{n^{(n+\theta-1)}}
\ll \frac{1}{n} \sum_{j=1}^{\infty} \left| S_\theta(f; j) \right| e^{-j/n} + \frac{1}{n^\theta} \sum_{j=n}^{\infty} \left| S_\theta(f; j) \right| e^{-j/n}.
\]

The constant in the symbol $\ll$ depends only on $\theta$.

The sum on the righthand side of (1), is called Cesàro mean with parameter $p = \theta - 1$. If for a given formal series $\sum_{k=0}^{\infty} a_k$ the Cesaro means with parameter $p$ converge to some number $A$, then we say that $\sum_{j=0}^{\infty} a_k$ is $(C, p)$ summable and its Cesàro sum is $A$ and write $(C, p) \sum_{j=0}^{\infty} a_j = A$.

From Theorem 1.1 we can deduce the following result, which is probably already known.

**Theorem 1.2.** Suppose $p > -1$. A series $\sum_{k=0}^{\infty} a_k$ is $(C, p)$ with summable and it’s $(C, p)$ sum is equal to $A$ if and only if

\[
\lim_{x \to 1^-} \frac{\sum_{k=0}^{\infty} a_k x^k}{x-1} = A, \quad (2)
\]

\[
\lim_{n \to \infty} \frac{S_{p+1}(f; n)}{n^{p+1}} = 0, \quad (3)
\]

where $f(x) = \sum_{j=0}^{\infty} a_j x^j$.

In the case when $\theta = 1$ Theorem 1.2 becomes the classical theorem of Tauber (see. e.g. [6],[7]). For this special case the proof of Theorem 1.1 can be obtained by modifying the proof of Tauber’s theorem. Let us define
\[
\mu_n(p) = \left( \frac{1}{n} \sum_{k=1}^{n} \left| \hat{f}(k) - 1 \right|^p \right)^{1/p}.
\]

Applying Theorem 1.1 we can easily prove the following result

**Theorem 1.3.** $p > \max \left\{ 1, \frac{1}{\theta} \right\}$ $|\hat{f}(j)| \leq 1,$

\[
M_n(f) = \exp \left\{ \theta \sum_{k=1}^{n} \frac{\hat{f}(k) - 1}{k} \right\} + O(\mu_n(p)),
\]

here the constant in symbol $O(.)$ depends only on $\theta$ and $p$.

The variants of Theorem 1.3 with less precise estimate of the remainder term have been proved in [2],[3],[4] and [5].
2 Proofs

Lemma 2.1. Let \( f(z) = \sum_{m=0}^{\infty} f_m z^m \) be analytic function in the region \( \Delta(\phi, \eta) = \{|z| < 1 + \eta, \ |\arg(z - 1)| > \phi\} \), where \( \eta > 0 \) and \( 0 < \phi < \pi/2 \). If
\[
|f(z)| \leq K_1|1-z|^{\alpha_1} + K_2|1-z|^{\alpha_2},
\]
for \( z \in \Delta(\phi, \eta) \) then there exists such a constant \( c = c(\alpha_1, \alpha_2, \eta, \phi) \) which is independent of \( K_1, K_2 \) and such that
\[
|f_n| \leq c(K_1n^{-\alpha_1-1} + K_2n^{-\alpha_2-1}).
\]

Proof. The same as of Theorem 1 of [1]. \( \square \)

Let us denote
\[
c_{m,j} = \sum_{s=0}^{m} \frac{(m-s+\theta-1)(s-\theta-1)}{s+j},
\]
for \( j \geq 1 \). Then the generating function of \( c_{m,j} \) will have the form
\[
F_j(z) = \sum_{m=0}^{\infty} c_{m,j} z^m = \frac{1}{(1-z)^\theta} \int_0^1 (1-xz)^\theta x^{j-1} dx.
\]

Lemma 2.2. We have the following estimates for \( c_{m,j} \):

(i) \( 0 \leq c_{m,j} \leq \frac{\theta}{j} e^{\theta m/j}, \ m \geq 1, \ c_{0,j} = \frac{1}{j} \);

(ii) \( c_{m,j} = \frac{(m+\theta-1)}{m} \int_0^1 (1-y)^\theta y^{j-1} dy + O\left(\frac{m^{\theta-2}}{j^\theta} + \frac{1}{m^2}\right) \).

Proof. Differentiating \( F_j(z) \) we obtain
\[
z F_j'(z) = \frac{\theta z F_j(z)}{1-z} + 1 - j F_j(z).
\]
Expanding both sides of the above equation into Taylor series and equating the coefficients of the same powers \( z^m \) we obtain
\[
c_{m,j} = \frac{\theta}{m+j} \sum_{s=0}^{m-1} c_{s,j}, \ m \geq 1
\]
and \( c_{0,j} = \frac{1}{j} \). This recurrent relation implies that
\[
0 < c_{m,j} \leq \frac{\theta}{j} \sum_{s=0}^{m-1} c_{s,j}, \ m \geq 1.
\]
Then

\[ 0 \leq c_{m,j} \leq b_{m,j}, \]

where \( b_{m,j} \) is solution of the recurrent equation

\[ b_{m,j} = \frac{\theta}{j} \sum_{s=0}^{m-1} b_{s,j}, \quad m \geq 1, \]

with initial condition \( b_{0,j} = \frac{1}{j} \). It is easy to check that

\[ b_{m,j} = \frac{\theta}{j^2} \left( 1 + \frac{\theta}{j} \right)^{m-1}, \quad m \geq 1. \]

Therefore applying inequality \( 1 + x \leq e^x \) we obtain the estimate (i)

\[ c_{m,j} \leq \frac{\theta}{j^2} \left( 1 + \frac{\theta}{j} \right)^{m-1} \leq \frac{\theta}{j^2} e^{\theta m/j}, \quad m \geq 1. \]

In order to prove estimate (ii) we will use Lemma [2.1] with \( \eta = 1/2 \) and \( \phi = \pi/4 \). We can represent \( F_j(z) \) as a sum of two functions

\[ F_j(z) = \frac{1}{(1-z)^\theta} \int_0^1 (1-x)^\theta x^{j-1} dx + G_j(z). \]

Let \( z \in \Delta(1/2, \pi/4), |z - 1| < 1/2 \). Then

\[ \int_0^1 (1 - z y)^\theta y^{j-1} dy - \int_0^1 (1 - y)^\theta y^{j-1} dy \]

\[ = \int_0^{1-|1-z|} (1 - y)^\theta \left( \left( 1 - y \frac{z - 1}{1 - y} \right)^\theta - 1 \right) y^{j-1} dy + \]

\[ + \int_{1-|1-z|}^1 (1 - y)^\theta (y^{j-1} - (1 - y)^\theta) y^{j-1} dy \]

\[ \ll \int_0^{1-|1-z|} (1 - y)^{\theta - 1} y^{j-1} dy + \int_{1-|1-z|}^1 y^{j-1} |1 - z|^\theta dy \]

\[ \ll |1 - z| \int_0^1 (1 - y)^{\theta - 1} y^{j-1} dy + |1 - z|^{\theta+1} \]

\[ \ll \frac{|1 - z|}{j^\theta} + |1 - z|^{\theta+1}. \]
It is easy to see that the obtained estimate holds in the whole region \( \Delta(\eta, \phi) \). Therefore for \( z \in \Delta(\eta, \phi) \)

\[
G_j(z) = \frac{1}{(1-z)^\theta} \int_0^1 \left( (1-yz)^\theta - (1-y)^\theta \right) y^{j-1} dy
\]

\[
\ll |1-z| + \frac{|1-z|^{1-\theta}}{j^\theta}.
\]

Applying Lemma 2.1 with \( f(z) = G_j(z) \) and taking into account (4) we obtain estimate (ii).

The lemma is proved.

\[
\Box
\]

Proof of Theorem 1.1 Since

\[
\sum_{k=1}^\infty S_\theta(f; k) z^k = \frac{zf'(z)}{(1-z)^\theta},
\]

then

\[
n \alpha_n = \sum_{k=1}^{n} S_\theta(f; k) \left( \frac{n-k-\theta-1}{n-k} \right), \quad n \geq 1.
\]

Therefore

\[
R_n := \sum_{k=0}^{n} a_k \left( \frac{n-k+\theta-1}{n-k} \right) - f(e^{-1/n}) \left( \frac{n+\theta-1}{n} \right)
\]

\[
= \sum_{k=1}^{n} \left( \frac{n-k+\theta-1}{n-k} \right) \frac{1}{k} \sum_{j=1}^{k} S_\theta(f; j) \left( \frac{k-j-\theta-1}{k-j} \right)
\]

\[
- \left( \frac{n+\theta-1}{n} \right) \sum_{k=1}^{n} e^{-k/n} \frac{1}{k} \sum_{j=1}^{k} S_\theta(f; j) \left( \frac{k-j-\theta-1}{k-j} \right)
\]

\[
= \sum_{j=1}^{n} S_\theta(f; j) c_{n-j,j} - \left( \frac{n+\theta-1}{n} \right) \sum_{j=1}^{\infty} S_\theta(f; j) \sum_{k=j}^{\infty} \frac{(k-j-\theta-1)e^{-k/n}}{k}.
\]

Suppose \( j > n/2 \), then

\[
\sum_{k=j}^{\infty} \frac{(k-j-\theta-1)e^{-k/n}}{k} = \sum_{s=0}^{\infty} \frac{(s-\theta-1)e^{-s/s}}{s+j} = \int_{0}^{e^{-1/n}} (1-x)^\theta x^{j-1} dx
\]

\[
= \int_{1/n}^{\infty} (1-e^{-y})^\theta e^{-y} dy \leq \int_{1/n}^{\infty} y^\theta e^{-y} dy
\]

\[
\ll \frac{e^{-j/n}}{jn^\theta}.
\]

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Applying the obtained estimate and Lemma 2.2 we obtain

\[ R_n - \frac{S_\theta(f; n)}{n} \]

\[ \ll \sum_{j \leq n/2} |S_\theta(f; j)| c_{n-j,j} - \left( \frac{n + \theta - 1}{n} \right) \sum_{k=j}^{\infty} \frac{(k-j-\theta-1)e^{-k/n}}{k} \]

\[ + \sum_{n/2 < j < n} |S_\theta(f; j)| c_{n-j,j} + \left( \frac{n + \theta - 1}{n} \right) \sum_{j>n/2} |S_\theta(f; j)| \sum_{k=j}^{\infty} \frac{(k-j-\theta-1)e^{-k/n}}{k} \]

\[ \ll \left( \frac{n + \theta - 1}{n} \right) \left( \frac{1}{n} \sum_{j \leq n/2} \frac{|S_\theta(f; j)|}{j^\theta} \right) + \frac{1}{n} \sum_{n/2 < j < n} \frac{|S_\theta(f; j)|}{n^\theta} \left( \frac{1}{n} \sum_{j>n} |S_\theta(f; j)| e^{-j/n} \right) \]

here we have used the fact that \( \left( \frac{n}{n} + \theta - 1 \right) = n^{\theta-1} \left( 1 + O \left( \frac{1}{n} \right) \right) \).

The theorem is proved.

**Proof of Theorem 1.2** The sufficiency of conditions (2) and (3) follows immediately from Theorem 1.1. The fact that Cesàro summability implies (2) and (3) is proved in [7].

**Proof of Theorem 1.3** Let us apply Theorem 1.1 with \( f(z) = \exp \{ L_n(z) \} = \sum_{k=0}^\infty m_k z^k \). Then

\[ \sum_{k=1}^\infty S_\theta(f; k) z^k = \frac{z f'(z)}{(1-z)^\theta} = F(z) \theta \sum_{k=1}^n (\hat{f}(k) - 1) z^k, \]

therefore

\[ S_\theta(f; m) = \sum_{k=1}^m (\hat{f}(k) - 1) N_{m-k}. \]

Since \( |N_k| \leq \binom{k+\theta-1}{k} \) then applying Cauchy inequality with parameters \( \frac{1}{p} + \frac{1}{q} = 1 \) we obtain

\[ |S_\theta(f; m)| \ll \left( \sum_{k=1}^m |\hat{f}(k) - 1|^p \right)^{1/p} \left( \sum_{k=1}^m k^{(\theta-1)q} \right)^{1/q} \]

\[ \ll m^\theta \left( \frac{n}{m} \right)^{1/p} \mu_n(p). \]
Applying Theorem 1.1 and using the estimate \( \exp\{\theta L_n(e^{-1/n})\} = \exp\{\theta L_n(1)\}(1 + O(\mu_n(p))) \) we get

\[
\frac{N_n}{n^{n+\theta-1}} - \exp \left\{ \theta \sum_{k=1}^{n} \hat{f}(k) - 1 \right\} \ll \mu_n(p) + \frac{\mu_n(p)}{n} \sum_{m=1}^{\infty} \left( \frac{n}{m} \right)^{1/p} e^{-m/n} \\
+ \mu_n(p) \frac{1}{n^\theta} \sum_{m=n}^{\infty} m^{\theta-1} \left( \frac{n}{m} \right)^{1/p} e^{-m/n} \\
\ll \mu_n(p).
\]

The theorem is proved. \( \square \)

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