Reformulating String Theory with the $1/N$ Expansion

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ABSTRACT

We argue that string theory should have a formulation for which stability and causality are evident. Rather than regard strings as fundamental objects, we suggest they should be regarded as composite systems of more fundamental point-like objects. A tentative scheme for such a reinterpretation is described along the lines of 't Hooft’s $1/N$ expansion and the light-cone parametrization of the string.
The discovery of string theory represented a radical departure from the incremental development of quantum field theory, which holds that a physical theory should be formulated in terms of fields locally defined on space-time. Since the fundamental entities of string theory are one-dimensional extended objects, important physical features that are automatic in quantum field theory are not guaranteed. For example, in quantum field theory we can

1. Compute the energy to assess the stability of the theory, at least in the context of a weak-coupling (semi-classical) expansion.
2. Easily incorporate Poincaré invariance without disturbing (1).
3. Check that the theory is causal, because the fundamental entities are point particles (or partons), and interactions are manifestly local.

In string theory, we have Poincaré invariance and presumably general covariance. But stability and causality are certainly not manifest, and may be absent altogether. In perturbation theory, superstring theory is free of ghosts (one hallmark of acausality) and tachyons (a hallmark of instability), but the intrinsic nonlocality of its description obscures the global stability of the theory. (Expanding $\phi^3$ theory about a locally stable vacuum reveals no tachyons or ghosts in perturbation theory, but it is absolutely unstable.)

If stability and causality are present in string theory, I believe the most natural way to understand this would be to discover that string should not, at a fundamental level, be described in terms of one-dimensional objects at all, but rather as composite structures built from point-like entities.

There are many precedents for extended structures in quantum field theory:

1. Vortices in superconductors and superfluids.
2. Nielsen-Olesen vortices in spontaneously broken abelian gauge theories.
3. Solitons, for example Skyrmions and 't Hooft-Polyakov monopoles.

In all these cases, the extended structures exist alongside ordinary point-like excitations of the fields, whereas in string theory the extended structures are everything. Also these features can’t be fit into the framework of a string field loop expansion.
A more closely parallel local theory is quark confining QCD. Then all the particle states are those of composite structures (hadrons). Furthermore, as 't Hooft has shown,\(^{[1]}\) one can also establish an expansion of Yang-Mills theory completely parallel to the string field loop expansion: the expansion in powers of \(1/N_c\) with \(g^2N_c\) fixed. The leading order of this expansion would give scattering amplitudes displaying an infinite number of zero width resonances of arbitrarily high spin, just as the leading order of string theory. The analogy is so close, in fact, that many authors, myself included, have wondered whether string theory could be a \(1/N\) expansion of some QCD-like theory.

But it can’t be QCD itself because

1. It has gravitons. An old folk theorem, proved relatively recently (1980) by Weinberg and Witten,\(^{[2]}\) rules out the existence of massless spin two particles in theories with a conserved Lorentz covariant energy momentum tensor.

2. It has no trace of point-like (parton) structures.

3. Superstring theory has supersymmetry and is consistent only in space-time dimension \(D = 10 > 4\).

Therefore a candidate underlying theory for superstrings should

1. Be generally covariant (probably without gravitons, since the graviton should be a state of the string which is to be dynamically generated).

2. Suppress the finite energy-momentum components of the constituents.

3. Possess supersymmetry and be able to generate extra dimensions (assuming one starts in four dimensions).

Over a decade ago, Giles, McLerran, and I gave a fishnet\(^{[3]}\) description of string theory in which the suppression of finite momentum constituents was imposed by hand, and extra dimensions appeared through the promotion of an internal symmetry to an extra dimension.\(^{[4–8]}\)

Our description relied heavily on light-cone coordinates

\[
x^\pm = (x^0 \pm x^1)/\sqrt{2}
\]

\[
x^\pm = (x^2, \ldots, x^{d-1})
\]

where \(d\) is the spatial dimension. To describe dynamics, \(\tau = x^+\) is regarded as...
the evolution parameter, and it is also convenient to replace $x^-$ by its conjugate $P^+ \geq 0$. A single free particle is then described by a Schrödinger wave function $\psi_{P^+}(x, \tau)$ satisfying
\[
i \frac{\partial}{\partial \tau} \psi_{P^+}(x, \tau) = \frac{1}{2P^+}(-\nabla^2 + m^2)\psi_{P^+}(x, \tau).
\]

The light-cone string,\[9\] for which $x^+ = \tau$ and the density of the + component of momentum is the constant rest tension, $P^+ = T_0$, can be described in this language by discretizing the world-sheet spatial coordinate $\sigma$ to $M$ lattice sites so that $P^+ = M\epsilon T_0$, after which the states of a single string can be described by an $M$ particle wave function
\[
\psi_M(x_1, \cdots, x_M, \tau) = \psi_M(x_2, \cdots, x_M, x_1, \tau),
\]
governed by the Hamiltonian (note that $P^-$ is conjugate to $x^+$)
\[
P^- = \frac{1}{\epsilon} \sum_i \frac{1}{2T_0}(-\nabla_i^2 + T_0(x_{i+1} - x_i)^2).
\]
In string field theory one promotes $\psi_M$ to a dynamical variable through second quantization.\[10,11\]

But instead of second-quantizing $\psi_M$, one can just as well, and more easily, second-quantize the constituents. Following 't Hooft’s ideas on the $1/N$ expansion\[1\] we can achieve this by introducing an $N \times N$ matrix variable $a_{k\ell}(x)$ and its canonical conjugate $\bar{a}_{k\ell}(x) = a_{k\ell}(x)\dagger$ so that
\[
[a_{k\ell}(x), \bar{a}_{m\ell}(y)] = \delta^n_k \delta^\ell_m \delta(x - y).
\]
In the large $N$ limit, one can show that the singlet operators
\[
\bar{A}(x_1, \cdots, x_M) = \left(\frac{1}{N}\right)^{M/2} \text{tr}\{\bar{a}(x_1) \cdots \bar{a}(x_M)\}
\]
behave as creation operators for discretized strings.\[8\] For example, if one considers
a Hamiltonian
\[ P^- = \frac{1}{\epsilon} \int \frac{dx}{2T_0} \text{tr} \nabla \bar{a}(x) \cdot \nabla a(x) + \frac{1}{\epsilon N} \int dxdy V(x - y) \text{tr} \bar{a}(x)\bar{a}(y)a(y)a(x) \] (1)
and evaluates its action on a state
\[ |\psi\rangle = \left( \frac{1}{N} \right)^{M/2} \text{tr}\{\bar{a}(x_1) \cdots \bar{a}(x_M)\} \psi_M(x_1, \ldots, x_M) |0\rangle , \]
he finds
\[ P^- |\psi\rangle = \frac{1}{\epsilon} \sum_{i=1}^{M} (-\nabla_i^2/2T_0 + V(x_{i+1} - x_i)) |\psi\rangle + \frac{1}{N\epsilon} \sum_{i=1}^{M} \sum_{j \neq i, i+1} A(x_{i+1}, \ldots, x_{j-1})A(x_{j}, \ldots, x_i) |0\rangle V(x_i - x_j) \psi_M(x_1, \ldots, x_M) . \]
The $1/N$ term describes the production of a discretized string, so $1/N$ should be identified with the string coupling constant. In the limit that it vanishes ($N \to \infty$), the above equation describes a noninteracting discretized string provided that the potential $V$ is sufficiently attractive to bind. It need not be a long range harmonic force, because in the continuum limit $M \to \infty$ with $P^+ = M\epsilon$ fixed, the finite energy excitations will be those of a string. This is because the low energy excitations of an $M$ particle discretized string are $O(1/M) \times$ the two-body level spacing.

If we try to interpret the constituents of string as particles, we would say that they all carry a fixed infinitesimal unit of $P^+ = \epsilon T_0$. It is this feature that in the continuum limit forces the constituents to have all components of $P^\mu$ infinitesimal.

If we compare (1) to that of an ordinary quantum field theory, the main difference is that the annihilation operator would carry an additional label $P^+$, which could take any positive value. Then quartic terms involving one creation and three annihilation operators (and vice versa) would also be allowed. To obtain the effective Hamiltonian (1) from a field theory, the dynamics would have to suppress this $P^+$ degree of freedom. In asymptotically free QCD one can compute the constituent wave-function at short distances and see that there is no such suppression.
Assuming that one can find a justification for (1), it is not hard to understand how internal degrees of freedom can be promoted to extra “compactified” dimensions. This is illustrated by a $(\phi^\dagger \phi)^2$ example in which the constituents are endowed with an abelian charge.\textsuperscript{[7]} We showed that summing over all ways charge could flow through a fishnet diagram produced a 2-dimensional 6-vertex model which has a bosonized description in terms of a compactified extra dimension.

General covariance is absent in the above discussion. But conceivably, the suppression of the finite energy momentum components can be attributed to general covariance. For example, if a generally covariant action does not possess kinetic terms (such as the Einstein-Hilbert action) for the metric, it is stationary under variations of the metric only if the local energy momentum tensor vanishes,

\[ T_{\mu\nu}(x) = 0. \] (2)

Could this be the explanation for the suppression of finite momentum components of the string constituents? From the point of view of string theory, the reason one can work in light-cone gauge, and in particular can choose the $P^+$ density constant along the string is, of course, world-sheet reparametrization invariance. It is also well-known that world-sheet reparametrization invariance is intimately linked with general coordinate invariance, in the sense that the generators of world-sheet reparametrizations, the Virasoro operators $L_n$, provide the Ward-identities associated with target space general covariance. Perhaps these are hints that the fundamental theory we seek is a generally covariant theory in which dynamical gravitons only appear nonperturbatively.

This means the local theory underlying string theory should be either a theory with no curvature terms, as in induced gravity,\textsuperscript{[13–16]} or a topological field theory.\textsuperscript{[17]} However, in the former case, there are (perhaps insurmountable) technical obstacles to any kind of perturbative treatment of the theory. It is like the infinite Newton constant limit of ordinary gravity coupled to matter. Integrating over the metric (which is necessary for general covariance) in the absence of curvature
terms imposes, at least classically, the constraint (2). But it is far from clear how such a constraint can be consistently implemented in a (semi-classical) loop expansion, especially for space-time dimension greater than 2. The standard world-sheet BRST formalism of string theory shows how to handle this constraint in two space-time dimensions, at least for conformally invariant theories. This shows that the constraints are not necessarily inconsistent. Of course, in this case the issue of generating gravitons dynamically does not arise. Unfortunately the techniques that make the two dimensional case tractable do not carry over to higher dimensions.

Although we are envisioning a microscopic formulation of string theory of the above type, it may not be possible to formulate it properly at the quantum level without including the dynamical effects that generate gravitons (i.e. string formation). Once gravitons are generated the constraints should be interpretable as graviton field equations, which should resolve some of the conceptual problems associated with them. Because the graviton in string theory is just a state of the closed string, it is desirable that the dynamical mechanism which generates it also is responsible for string formation. Since the latter cannot occur in any finite order in perturbation theory, we should demand the same of the former. Divergent but renormalizable field theories on a curved background induce an Einstein-Hilbert term at finite loop order, so in such theories the graviton would be on a different footing than any strings that might form. The best hope for generating string theory along these lines would therefore seem to be a generally covariant version of an ultraviolet finite theory such as $N = 4$ supersymmetric Yang-Mills theory in four space-time dimensions.

This is all very speculative, but I think the issues of stability and causality are of such paramount importance that for string theory to make sense, it must have an alternative formulation in which these properties are not mysterious. I

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* The $CP(N - 1)$ model in two dimensions is somewhat reminiscent of this situation. There one starts with an abelian gauge field with no $F^2$ term which would imply the constraints $J^\mu(x) = 0$. But after solving the theory in the large $N$ limit, an $F^2$ term is generated and the constraint does not have to be directly dealt with.
have tried to suggest a way string theory could come from quantum field theory because with the latter one can easily assess these conditions. For example, the hamiltonian (1) requires $V$ to be attractive, *i.e.* negative, for string formation. With only the exhibited terms the energy would then be unbounded below and the theory unstable. One might try to avoid this by making the constituents fermions, or by adding other repulsive terms to the energy. Perhaps superstrings would require such modifications. Probably, string theory satisfies stability and causality in a much more subtle way; conceivably it doesn’t satisfy them at all (as argued by Woodard$^{[18]}$ in the context of string field theory).

Finally, let me mention that in low space-time dimension ($D \leq 1$), recent work$^{[19–22]}$ has shown how subcritical string theory can arise from a matrix “field theory.” This is a prototype, albeit a trivial one, of the reformulation of string theory I am suggesting. For $D = 1$ the relevant matrix model is given by a one-dimensional gauge field interacting with matter in the adjoint representation. This suggests looking for generally covariant versions of such theories in higher dimensions. Again supersymmetric gauge theories come to mind as they naturally incorporate extra fields in the adjoint representation.

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Since this preprint was circulated in late 1991, further insights and clarifications have developed, which we sketch below.

1. It should be emphasized that the system described by our model hamiltonian (1) is a collection of *nonrelativistic* point particles (“*String Bits*”) moving in one less spatial dimension than the strings do. The longitudinal dimension, corresponding to $x^-, P^+$, is nonexistent for string bits and only appears with string formation: varying $P^+$ is nothing other than varying the number of bits in a given string. This manufacture of an extra noncompact dimension is certainly an intriguing feature of our scheme.

2. Suppose our string bit model is taken seriously as the fundamental formulation of string theory. Thermal statistics of string bits includes a sum over
bit number. In the low temperature limit, when string states should dominate the thermal partition function, this bit number sum should become an integral over the $P^+$ value carried by each string. The measure of this integration is thus determined unambiguously. Remarkably, it turns out that this unambiguous measure is exactly the modular invariant one that leads to the usual expression for the finite temperature free energy.

3. If $\epsilon$ is fixed at a very tiny but finite number, our model of string is very like a polymer with a finite but very large $O(1/\epsilon)$ ionization energy. At extremely high temperatures, these polymers should be completely ionized, giving a high temperature estimate for the free energy $F \sim -N^2\pi\epsilon/12\beta^{1+(D-2)/2}$, where $D$ is the space-time dimension perceived by string. Interestingly, for $D = 4$, this matches the $(\text{temperature})^2$ behavior Atick and Witten\textsuperscript{[23]} have argued should characterize the high temperature phase of string theory. If we require this matching, any extra (compact) dimensions would have to arise from an internal degree of freedom carried by the string bits, along the lines of Ref.[7].

4. We have observed that the model hamiltonian written in (1) cannot be stable. A simple way to try to stabilize it in a way that would retain attractive nearest neighbor interactions would be to replace the second term with

$$\frac{1}{2\epsilon N} \int dxdy \mathcal{V}(x - y) \text{tr} : (\bar{a}(x)a(x) - a(x)\bar{a}(x))(\bar{a}(y)a(y) - a(y)\bar{a}(y)) :$$

where $\mathcal{V}$ is now taken to be positive (repulsive). It is easily shown that the nearest neighbor interactions on a given polymer are attractive for $N > 1$, whereas the interactions between non-nearest neighbors and between bits on different polymer chains are repulsive. Thus polymer formation is still favored as long as they are not too dense, and the hamiltonian is bounded from below.

5. As Gross and Mende\textsuperscript{[24]} and others have emphasized, string theory should be scale invariant at short distances. This suggests that the natural choice
for $\mathcal{V}$ would be $\mathcal{V}(x) = \lambda_0 \delta(x)$, which is scale invariant in 2 transverse dimensions. The attractive potential $-\lambda_0 \delta(x)$ binds with an infinite binding energy. However, with a short distance cutoff $1/\Lambda$ on the delta function the binding energy $B \sim \Lambda^2 e^{-4\pi/\lambda_0}$ has the required dependence on $\lambda_0 > 0$ for dimensional transmutation. As shown in Ref.[12] a random phase approximation relates this $B$ to the slope $\alpha'$ of the string Regge trajectories: $\alpha' \approx 1/\pi B \sqrt{12}$.

REFERENCES

1. G. ’t Hooft, *Nucl. Phys.* B72 (1974) 461.
2. S. Weinberg and E. Witten, *Phys. Lett.* 96B (1980) 59.
3. H. B. Nielsen and P. Olesen, *Phys. Lett.* 32B (1970) 203; B. Sakita and M. A. Virasoro, *Phys. Rev. Lett.* 24 (1970) 1146.
4. R. Giles and C. B. Thorn, *Phys. Rev.* D16 (1977) 366.
5. C.B. Thorn, *Phys. Rev.* D17 (1978) 1073.
6. K. Bardakci and S. Samuel, *Phys. Rev.* D16 (1977) 2500.
7. R. Giles, L. McLerran, C. B. Thorn, *Phys. Rev.* D17 (1978) 2058.
8. C.B. Thorn, *Phys. Rev.* D20 (1979) 1435.
9. P. Goddard, J. Goldstone, C. Rebbi, and C. B. Thorn, *Nucl. Phys.* B56 (1973) 109.
10. M. Kaku and K. Kikkawa, *Phys. Rev.* D10 (1974) 1823.
11. C. B. Thorn, in *Unified String Theories*, ed. M. Green and D. Gross, World Scientific Publishing Co. (1986).
12. C.B. Thorn, *Phys. Rev.* D19 (1979) 639.
13. Ya. B. Zel’dovich, *Zh. Eksp. Teor. Fiz. Pis’ma Red* 6 (1967) 883 [*JETP Lett.* 6 (1967) 316].
14. A. D. Sakharov, *Dok. Akad. Nauk. SSSR* **177** (1967) 70 [Sov. Phys. Dokl. **12** (1968) 1040].

15. S. L. Adler, Phys. Rev. **D14** (1976) 379; Phys. Rev. Lett. **44** (1980) 1567; Phys. Lett. **95B** (1980) 241; Rev. Mod. Phys. **54** (1982) 729.

16. A. Zee, Phys. Rev. Lett. **42** (1979) 417; Phys. Rev. **D23** (1981) 858; Phys. Rev. Lett. **48** (1982) 295; Phys. Lett. **109B** (1982) 183.

17. E. Witten, *Comm. Math. Phys.* **117** (1988) 353.

18. D. A. Eliezer and R. P. Woodard, *Nucl. Phys.* **B325** (1989) 389.

19. V. A. Kazakov, *Mod. Phys. Lett.* **A4** (1989) 2125.

20. E. Brezin and V. A. Kazakov, *Phys. Lett.* **236B** (1990) 144.

21. D. J. Gross and A. A. Migdal, Phys. Rev. Lett. **64** (1990) 127; Phys. Rev. Lett. **64** (1990) 717; Nucl. Phys. **B340** (1990) 333; Phys. Rev. Lett. **64** (1990) 717.

22. M. R. Douglas and S. H. Shenker, *Nucl. Phys.* **B335** (1990) 635.

23. J. J. Atick and E. Witten, *Nucl. Phys.* **B310** (1988) 291.

24. D. J. Gross and P. F. Mende, Phys. Lett. **197B** (1987) 129; Nucl. Phys. **B303** (1988) 407.