Several exact solutions for three dimensional Schrodinger equation involving inverse square and power law potentials

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Abstract
Several exact solutions for steady state Schrodinger equation in three dimensional space are derived in this paper. The potentials are taken to be sum of an inverse square potential and a power law potential. Different new exact solutions of Schrodinger equation are derived for this potential with zero energy. The solutions are derived in cartesian coordinates without separation of variables. Certain exact solutions for non-zero energy are also derived for Schrodinger equation with inverse square potential.

Keywords
Schrodinger equation, Exact solution, Zero and non-zero energy, Inverse Square Potential, Power Law Potential.

AMS Subject Classification
35A09, 35A24, 35Q40.

1 Introduction
Schrodinger equation is having many applications not only in modern physics but also in several other fields such as quantum information and econophysics. So exact solutions to Schrodinger equations are also having applications in these and several other fields. Availability of exact solutions for any partial differential equation which represent a physical system or phenomena are inevitable for a better understanding of behavior of the system or phenomena. They can also be utilized to check the correctness of approximate methods developed for obtaining specific solutions.

There are several exact solutions available in the literature for Schrodinger equation corresponding to the well known potentials such as Coulomb, harmonic oscillator inverse square, Mie-type, Eckart, Poschl-Teller, Morse, Woods-Saxen, Manning-Rosen, Scarf and Gendenstein potentials. In the case of such equations one can convert them into ordinary differential equations by suitable transformations. These equations can be solved using well known special functions such as Bessel, Hermite, Legendre, Heun, Whittaker and confluent hypergeometric functions. Some of the recent works involving exact solutions of Schrodinger equation are and involving approximate solutions are. Exact solutions of Schrodinger equation in three dimensional cases are obtained mainly by solving the corresponding radial part of the equation after assuming the solution in the separation of variables form. The motion of particles in inverse square potential has been discussed in.

There are several situations where Schrodinger equation
with zero energy can be applied [2, 4, 6, 14, 15, 19, 22]. Schrodinger equation can be simplified by assuming zero energy when discussing the case of scattering of ultracold particles. The zero energy eigenstates in the parabolic potential barrier play important role in statistical mechanics in Gelfand triplet[14]. Zero energy exact solutions are also motivated by studies in super symmetric quantum mechanics and they are also used in zero energy limit calculations in the study of loosely bound systems, scattering length and coupling parameters calculations[2].

Plan of the paper is as follows. In the next section certain exact solutions of Schrodinger equation with zero energy are derived. The potential is taken to be the sum of an inverse square potential and a power law potential. The required three dimensional solutions are obtained without separation of variables. In the third section certain exact solution of Schrodinger equation are derived with non-zero energy and in the case of inverse square potential. The exact solutions that we have derived in both cases are always bound state solutions.

### 2. Zero energy exact solutions

Most of the exact solutions available for Schrodinger equation in the literature are usually derived by the method of separation of variables in the spherical coordinates. This will lead to an ordinary differential equation corresponding to radial part. This equation is solved to find various exact solutions in the case of potentials which are functions of the radial vector \( r = \sqrt{x^2 + y^2 + z^2} \) only. But, in this paper we will find out certain exact solutions to the three dimensional Schrodinger partial differential equations at zero energy level in cartesian coordinates without applying the method of separation of variables. The potential that we consider here is the sum of inverse square potential and a power law potential. The corresponding equation is given by

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left( \frac{a}{x^2 + y^2 + z^2} - b \right) f(x,y,z) = 0 \tag{2.1}
\]

where \( a, b \) and \( k \) are arbitrary parameters. Assuming particular forms for the required exact solutions we will solve the Schrodinger equation (2.1). The exact solutions that we are interested to derive are the bound state solutions \( f(x,y,z) \) to Schrodinger equation which are zero at origin and also tends to zero as \( x \) or \( y \) or \( z \) tends to \( \pm \infty \).

#### 2.1 Case I

The equation (2.1) is converted into an ordinary differential equation by a suitable substitution. Putting \( u = x^2 + y^2 + z^2 \) in equation (2.1) we get an ordinary differential equation in terms of \( g(u) = f(x,y,z) \) given by

\[
4 \left( u \frac{d^2}{du^2} + 6 \frac{d}{du} \right) g(u) + \left( bu^k - \frac{a}{u} \right) g(u) = 0 \tag{2.2}
\]

Now we use the following change of variables to the independent and dependent variables

\[
z = \sqrt{bu^{k+1}} \frac{k + 1}{k + 1} \tag{2.3}
\]

and \( g(u) = u^{-\frac{1}{2}} w(z) \). On substitution and simplification the differential equation (2.2) becomes

\[
z^2 w''(z) + z w'(z) + w(z) \left( z^2 - \frac{4a + 1}{4(k + 1)^2} \right) = 0 \tag{2.4}
\]

This is a Bessel’s equation whose solutions in terms of Bessel functions of first and second kinds are given by

\[
c_1 J_{\frac{4a + 1}{4(k + 1)^2}} (z) + c_2 Y_{\frac{4a + 1}{4(k + 1)^2}} (z) \tag{2.5}
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants. Hence particular solutions to Schrodinger equation (2.1) are given by

\[
\left( x^2 + y^2 + z^2 \right)^{-\frac{1}{2}} \left( c_1 J_{\frac{4a + 1}{4(k + 1)^2}} \left( \sqrt{b} \left( x^2 + y^2 + z^2 \right)^{\frac{1}{2}} \right) \right) + \left( c_2 Y_{\frac{4a + 1}{4(k + 1)^2}} \left( \sqrt{b} \left( x^2 + y^2 + z^2 \right)^{\frac{1}{2}} \right) \right) \tag{2.6}
\]

For real numbers \( a \geq -1/4 \) and \( b > 0 \), the solution given by Bessel function of first kind satisfies the conditions for a bound state wave function, that is, they are going to zero as \( x, y \) or \( z \) tends to \( \pm \infty \). Also it becomes zero if all \( x, y \) and \( z \) are zeroes. Graphical representation of a particular wave function is given in figure 1 for fixed values for the variable \( z \).

#### 2.2 Case II

Now we seek a solution in the form of the function \( f(x,y,z) = xg(x^2 + y^2 + z^2) \). We convert the corresponding equation (2.1) into an ordinary differential equation by putting \( u = x^2 + y^2 + z^2 \). Then we get an ordinary differential equation in terms of \( g(u) \) given by

\[
4u g''(u) + 10 g'(u) + \left( bu^k - \frac{a}{u} \right) g(u) = 0 \tag{2.7}
\]

Here we use the change of variables to the independent and dependent variables given by equation (2.3) and \( g(u) = u^{-3/4} w(z) \). On substitution and simplification the differential equation (2.7) becomes

\[
z^2 w''(z) + z w'(z) + w(z) \left( z^2 - \frac{4a + 9}{4(1+k)^2} \right) = 0 \tag{2.8}
\]

This is again Bessel’s equation whose solutions in terms of Bessel functions of first and second kinds are given by

\[
c_1 J_{\frac{4a + 9}{4(1+k)^2}} (z) + c_2 Y_{\frac{4a + 9}{4(1+k)^2}} (z) \tag{2.9}
\]
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where $c_1$ and $c_2$ are arbitrary constants. Hence particular solutions to Schrodinger equation (2.1) are given by

$$x \left( x^2 + y^2 + z^2 \right)^{-\frac{3}{2}} \left( c_1 J_{\frac{\sqrt{b(x^2 + y^2 + z^2)}}{k + 1}} \left( \frac{\sqrt{b(x^2 + y^2 + z^2)}}{k + 1} \right)^{k+1} \right) + c_2 Y_{\frac{\sqrt{b(x^2 + y^2 + z^2)}}{k + 1}} \left( \frac{\sqrt{b(x^2 + y^2 + z^2)}}{k + 1} \right)^{k+1}$$

(2.10)

For any real numbers $a \geq -4/9$ and $b \geq 0$, Bessel function of first kind gives a bound state solution of wave equation. Its graphical representation is given in figure 2

2.3 Case III

Here we assume that a solution can be written in the form $f(x, y, z) = x y g(x^2 + y^2 + z^2)$. We convert the corresponding equation (2.1) into an ordinary differential equation by putting $u = x^2 + y^2 + z^2$. Then we get an ordinary differential equation in terms of $g(u)$ given by

$$4u g''(u) + 14g'(u) + \left( bu^k - \frac{a}{u} \right) g(u) = 0$$

(2.11)

Here we use the change of variables (2.3) and $g(u) = u^{-\frac{3}{2}} w(z)$. On substitution and simplification the differential equation (2.11) becomes

$$z^2 w''(z) + zw'(z) + w(z) \left( z^2 - \frac{25 + 4a}{4(k+1)^2} \right) = 0$$

(2.12)

This is a Bessel’s equation whose solutions in terms of Bessel functions of first and second kinds are given by

$$c_1 J_{\frac{\sqrt{b(z^2 - \frac{25 + 4a}{4(k+1)^2})}}{k + 1}} (z) + c_2 Y_{\frac{\sqrt{b(z^2 - \frac{25 + 4a}{4(k+1)^2})}}{k + 1}} (z)$$

(2.13)

where $c_1$ and $c_2$ are arbitrary constants. Hence particular solutions to Schrodinger equation (2.1) are given by

$$(xy) \left( x^2 + y^2 + z^2 \right)^{-\frac{3}{2}} \left( c_1 J_{\frac{\sqrt{b(x^2 + y^2)}}{k + 1}} \left( \frac{\sqrt{b(x^2 + y^2)}}{k + 1} \right)^{k+1} \right) + c_2 Y_{\frac{\sqrt{b(x^2 + y^2)}}{k + 1}} \left( \frac{\sqrt{b(x^2 + y^2)}}{k + 1} \right)^{k+1}$$

(2.14)

For any real numbers $a \geq -25/4$ and $b \geq 0$, Bessel function of first kind gives a bound state solution of wave equation. Its graphical representation is given in figure 3

2.4 Case IV

Here we assume that a solution can be written in the form $f(x, y) = (xyz) g(x^2 + y^2 + z^2)$. We convert the corresponding equation (2.1) into an ordinary differential equation by putting $u = x^2 + y^2 + z^2$. Then we get an ordinary differential equation in terms of $g(u)$ given by

$$4u g''(u) + 18g'(u) + \left( bu^k - \frac{a}{u} \right) g(u) = 0$$

(2.15)
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Here we use the change of variables given by (2.3) and $g(u) = u^{-\frac{3}{2}}w(z)$.

$$z^2w''(z) + z w'(z) + w(z) \left( z^2 - \frac{49 + 4a}{4(k + 1)^2} \right) = 0 \quad (2.16)$$

This is a Bessel’s equation whose solutions in terms of Bessel functions of first and second kinds are given by

$$c_1 J_{\frac{\sqrt{b(x^2 + y^2 + z^2)}}{k}}(z) + c_2 Y_{\frac{\sqrt{b(x^2 + y^2 + z^2)}}{k}}(z) \quad (2.17)$$

where $c_1$ and $c_2$ are arbitrary constants. Hence particular solutions to Schrodinger equation (2.1) are given by

$$(xyz) \left( x^2 + y^2 + z^2 \right)^{-\frac{3}{2}} \left( c_1 J_{\frac{\sqrt{b(x^2 + y^2 + z^2)}}{k}} + c_2 Y_{\frac{\sqrt{b(x^2 + y^2 + z^2)}}{k}} \right) \quad (2.18)$$

For any real numbers $a \geq -49/4$ and $b \geq 0$, Bessel function of first kind gives a bound state solution of wave equation. Its graphical representation is given in figure 4.

### 3. Non-zero energy solutions

Now we will derive certain exact solutions for three dimensional Schrodinger equation with non-zero energy. The potential that we consider here is the inverse square potential. The corresponding equation is given by

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left( \frac{a}{x^2 + y^2 + z^2} - E \right) f(x,y) \quad (3.1)$$

Assuming particular forms for the solution we will find out certain exact solution of the equation (3.1). We are interested to derive only the bound state solutions $f(x,y)$ to Schrodinger equation which are zero at origin and also tends to zero as $x$ or $y$ tends to $\pm \infty$. Since the procedure is same as in the previous section, the exact solutions are summarized below.

#### 3.1 Case I

We convert equation (3.1) into an ordinary differential equation by a suitable substitution. Putting $u = x^2 + y^2 + z^2$ in the corresponding equation we get an ordinary differential equation in terms of $g(u) = f(x,y,z)$ given by

$$4ug''(u) + 6g'(u) - \left( \frac{a}{u} - E \right) g(u) = 0 \quad (3.2)$$

Solving this, particular solutions to Schrodinger equation (3.1) are given by

$$(x^2 + y^2 + z^2)^{-\frac{1}{2}} \left( c_1 J_{\frac{\sqrt{E(x^2 + y^2 + z^2)}}{4\sqrt{a} + 1}} + c_2 Y_{\frac{\sqrt{E(x^2 + y^2 + z^2)}}{4\sqrt{a} + 1}} \right)$$

Figure 2. Representation of wave function given by equation (2.10) with $c_1 = 1, c_2 = 0, a = 1, b = 2, k = 3$ and and for $z = 0, z = 0.75$ and $z = 1.5$ respectively (Case II)
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Figure 3. Representation of wave function given by equation (2.14) with $c_1 = 1, c_2 = 0, a = 1, b = 2, k = 3$ and and for $z = 0, z = 0.75$ and $z = 1.5$ respectively (Case III)

Figure 4. Representation of wave function given by equation (2.18) with $c_1 = 1, c_2 = 0, a = 1, b = 2, k = 3$ and and for $z = 0.5, z = 1$ and $z = 1.5$ respectively (Case IV)
For real numbers \( a \geq -1/4 \), the solution given by Bessel function of first kind satisfies the conditions for a bounded state wave function, that is, they are going to zero as \( x, y \) or \( z \) tends to \( \pm \infty \). Also it becomes zero if all \( x, y \) and \( z \) become zero.—

### 3.2 Case II

Here we assume that a solution can be written in the form 
\[
f(x, y, z) = xg(x^2 + y^2 + z^2).
\]
Then we can derive particular solutions to Schrodinger equation (3.1) which are given by 
\[
x (x^2 + y^2 + z^2)^{-\frac{1}{2}} \left( c_1 J_{\frac{1}{2}} \sqrt{4\epsilon + \eta} \left( \sqrt{E (x^2 + y^2 + z^2)} \right) 
+ c_2 Y_{\frac{1}{2}} \sqrt{4\epsilon + \eta} \left( \sqrt{E (x^2 + y^2 + z^2)} \right) \right)
\]
\[
(3.4)
\]

For any real numbers \( a \geq -9/4 \), Bessel function of first kind gives a solution to wave equation which satisfies the conditions for a desirable wave function for all positive energy.

### 3.3 Case III

Next we assume that a solution can be written in the form 
\[
f(x, y, z) = xyg(x^2 + y^2 + z^2).
\]
Then we can derive particular solutions to Schrodinger equation (3.1) which are given by 
\[
xy (x^2 + y^2 + z^2)^{-\frac{1}{2}} \left( c_1 J_{\frac{1}{2}} \sqrt{4\epsilon + 25} \left( \sqrt{E (x^2 + y^2 + z^2)} \right) 
+ c_2 Y_{\frac{1}{2}} \sqrt{4\epsilon + 25} \left( \sqrt{E (x^2 + y^2 + z^2)} \right) \right)
\]
\[
(3.5)
\]

For any real numbers \( a \geq -25/4 \), here the Bessel function of first kind gives a solution to wave equation which satisfies the required conditions for a desirable wave function for all positive energy levels.

### 3.4 Case IV

Here we assume that a solution can be written in the form 
\[
f(x, y, z) = (xyz)g(x^2 + y^2 + z^2).
\]
Then we can derive particular solutions to Schrodinger equation (3.1) which are given by 
\[
(xy^2 + y^2 + z^2)^{-\frac{1}{2}} \left( c_1 J_{\frac{1}{2}} \sqrt{4\epsilon + 49} \left( \sqrt{E (x^2 + y^2 + z^2)} \right) 
+ c_2 Y_{\frac{1}{2}} \sqrt{4\epsilon + 49} \left( \sqrt{E (x^2 + y^2 + z^2)} \right) \right)
\]
\[
(3.6)
\]

Here also for any real numbers \( a \geq -49/4 \), the Bessel function of first kind gives a solution to wave equation which satisfies the required conditions for a desirable wave function for all positive energy levels.

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