Dynamical black holes with symmetry in Einstein–Gauss–Bonnet gravity

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Abstract

We explore various aspects of dynamical black holes defined by a future outer trapping horizon in \(n(\geq 5)\)-dimensional Einstein–Gauss–Bonnet gravity. In the present paper, we assume that the spacetime has symmetries corresponding to the isometries of an \((n - 2)\)-dimensional maximally symmetric space and the Gauss–Bonnet coupling constant is non-negative. Depending on the existence or absence of the general relativistic limit, solutions are classified into GR and non-GR branches, respectively. Assuming the null energy condition on matter fields, we show that a future outer trapping horizon in the GR branch possesses the same properties as that in general relativity. In contrast, that in the non-GR branch is shown to be non-spacelike with its area non-increasing into the future. We can recognize that this peculiar behavior arises from the fact that the null energy condition necessarily leads to the null convergence condition for radial null vectors in the GR branch, but not in the non-GR branch. The energy balance law yields the first law of a trapping horizon, from which we can read off the entropy of a trapping horizon reproducing Iyer–Wald's expression. The entropy of a future outer trapping horizon is shown to be non-decreasing in both branches along its generator.

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1. Introduction

Black holes in our universe are commonly considered to be formed by the collapse of massive stars in the final stage of their lives. Since one would expect the gravitational collapse to settle down to equilibrium states at late times, stationary black holes are thus of great physical interest and intensively studied in the literature. One major achievement of the study of
stationary black holes is the uniqueness theorem: the only stationary vacuum black hole in an asymptotically flat spacetime is the Kerr black hole, and it is completely specified by its mass and angular momentum (see, e.g. [1]). Toward the proof of the uniqueness theorem of black holes, the ‘rigidity theorem’ [2–5] plays an essential role. This theorem insists that a rotating black hole must be axi-symmetric: the stationary black-hole event horizon is the Killing horizon. Killing horizons exhibit thermodynamical properties [6–11], which strongly suggests the intimate association between classical general relativity, quantum theory and statistical mechanics.

However, black holes in our universe rarely reach equilibrium. They evolve by absorbing stars and galactic remnants, or by coalescing. In these fully dynamic processes, one cannot identify the location of the event horizon at each time because it is determined by the global structure of spacetime. Over the past few decades, some local definitions of horizons have provided useful and powerful implements for the analysis of dynamical aspects of non-stationary black holes. Among other things, trapping horizons [12–15] defined by Hayward, isolated horizons [16, 17] and dynamical horizons [18–21] defined and developed by Ashtekar and his coworkers, provide a quasi-local characterization of black holes. (See [22] for a review of the quasi-local horizons.) In contrast to the event horizon, neither of the above requires knowledge of the entire future. Related works have revealed that these horizons also exhibit laws of black-hole dynamics analogous to those of the Killing horizon, irrespective of the highly dynamical settings. These new concepts of horizons involve applications to numerical relativity, quantum gravity and so on.

Gravitation physics in higher dimensions is a prevalent subject of current research motivated by string theory. Higher-dimensional general relativity is obtained by the lowest order of the Regge slope expansion of strings. Even in general relativity, black holes in higher dimensions expose a sharp difference from those in four dimensions [23]. The next stringy compensation yields the quadratic Riemann curvature terms in the heterotic string case [24]. In order for the graviton amplitude to be ghost-free, a special combination of the remaining curvature-squared terms is required to be the renormalizable Gauss–Bonnet term [25]. These higher-curvature terms come into play in extremely curved regions. Black holes and singularities are one of the best testbeds for demonstrating the effects of higher curvature terms. To elucidate the nature of black holes and singularities in Einstein–Gauss–Bonnet gravity will aid in understanding the higher-dimensional, stringy corrected theory of gravity. This is the main subject of the present paper.

We explore the dynamics of black holes in $n(\geq 5)$-dimensional Einstein–Gauss–Bonnet gravity by taking particular note of the trapping horizon. (As another approach, black-hole dynamics in the framework of the isolated horizon were addressed in [26].) The spacetime is supposed to have symmetries corresponding to the isometries of an $(n - 2)$-dimensional maximally symmetric space, which is also assumed to be compact to make physical quantities finite. The energy–momentum tensor of matter fields is left arbitrary except for suitable energy conditions. Since the trapping property is inherently a local notion, it is suitable to manipulate basic equations by means of (quasi-) local quantities. As shown in [27], our quasi-local mass defined geometrically [28] makes the field equations rather tractable. This is the generalization of the Misner–Sharp quasi-local mass [29] and shares similar properties with the four-dimensional counterpart [27, 30]. The mass of a trapping horizon is shown to obey an isoperimetric inequality similar to that of Penrose and gives an upper or lower bound in some cases. Solutions in Einstein–Gauss–Bonnet gravity are classified into two classes in general: the GR branch (having a general relativistic limit) and the non-GR branch (having no general relativistic limit). Our main arguments show that, under the null energy condition, a future outer trapping horizon in the GR branch possesses the same properties as that in general
relativity. On the other hand, the non-GR-branch solutions behave rather pathologically under the null energy condition. We also unveil the laws of black-hole dynamics and discuss the area and entropy laws.

The rest of the present paper is constituted as follows. In the following section, a concise overview of Einstein–Gauss–Bonnet gravity, the definition of our quasi-local mass and basic equations are given. Section 3 focuses on the clarification of the dynamical properties of trapping horizons. Various types of trapping horizons are scrutinized for each branch, and subsequently the black-hole dynamics are discussed. Concluding remarks and discussions including future prospects are summarized in section 4.

Our basic notations follow [31]. The conventions of curvature tensors are \[ \nabla_\mu, \nabla_\nu \] V^\nu = R^\mu_{\nu\rho\sigma} V^\nu and \( R_{\mu\nu} := R^\sigma_{\mu\nu\sigma} \). The Minkowski metric is taken to be the mostly in plus sign, and Roman indices run over all spacetime indices. We adopt the units in which only the \( n \)-dimensional gravitational constant \( G_n \) is retained.

2. Preliminaries

We begin with a brief description of Einstein–Gauss–Bonnet gravity in the presence of a cosmological constant. The action in \( n(\geq 5) \)-dimensional spacetime is given by

\[
S = \int d^n x \sqrt{-g} \left( \frac{1}{2\kappa_n^2} (R - 2\Lambda + \alpha L_{GB}) \right) + S_{\text{matter}},
\]

where \( R \) and \( \Lambda \) are the \( n \)-dimensional Ricci scalar and the cosmological constant, respectively. \( S_{\text{matter}} \) in equation (2.1) is the action for matter fields and \( \kappa_n := \sqrt{8\pi G_n} \), where \( G_n \) is the \( n \)-dimensional gravitational constant. The Gauss–Bonnet term \( L_{GB} \) comprises the combination of the Ricci scalar, Ricci tensor \( R_{\mu\nu} \) and Riemann tensor \( R_{\mu\nu\rho\sigma} \) as

\[
L_{GB} := R^2 - 4R_{\mu\nu} R_{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}.
\]

In four-dimensional spacetime, the Gauss–Bonnet term does not contribute to the field equations since it becomes a total derivative. \( \alpha \) with the dimension of length-squared is the coupling constant of the Gauss–Bonnet term. We assume \( \alpha \geq 0 \) throughout this paper, as motivated by string theory. The gravitational equation derived from the action (2.1) is

\[
G^\mu_{\nu} + \alpha H^\mu_{\nu} + \Lambda \delta^\mu_{\nu} = \kappa_n^2 T^\mu_{\nu},
\]

where

\[
G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,
\]

\[
H_{\mu\nu} := 2 \left[ R R_{\mu\nu} - 2 R_{\mu\alpha} R^\alpha_{\nu} - 2 R_{\mu\alpha\beta\gamma} R^\alpha_{\beta\gamma} R^\nu_{\mu\alpha\beta\gamma} + R^{\alpha\beta\gamma\delta} R_{\mu\alpha\beta\gamma} R^\nu_{\delta\mu\alpha\beta\gamma} \right] - \frac{1}{2} g_{\mu\nu} L_{GB}
\]

and \( T_{\mu\nu} \) is the energy–momentum tensor of matter fields. The field equations (2.3) contain up to the second derivatives of the metric and linear in that term.

Suppose the \( n \)-dimensional spacetime \( (\mathcal{M}^n, g_{\mu\nu}) \) to be a warped product of an \( (n - 2) \)-dimensional constant curvature space \( (K^{n-2}, \gamma_{ij}) \) and a two-dimensional orbit spacetime \( (M^2, g_{ab}) \) under the isometry of \( (K^{n-2}, \gamma_{ij}) \). Namely, the line element is

\[
g_{\mu\nu} \ dx^\mu \ dx^\nu = g_{ab}(y) \ dy^a \ dy^b + r^2(y) \ dy_{ij}(z) \ dz^i \ dz^j,
\]

where \( a, b = 0, 1; i, j = 2, \ldots, n - 1 \). Here \( r \) is a scalar on \( (M^2, g_{ab}) \) with \( r = 0 \) defining its boundary, and \( \gamma_{ij} \) is the unit metric on \( (K^{n-2}, \gamma_{ij}) \) with its sectional curvature \( \kappa = \pm 1, 0 \). We assume that \( (\mathcal{M}^n, g_{\mu\nu}) \) is strongly causal and \( (K^{n-2}, \gamma_{ij}) \) is compact. Since the rank-two symmetric tensors on the maximally symmetric space are proportional to the metric tensor,
the symmetry of the background spacetime determines the structure of the energy–momentum tensor as

\[ T_{\mu \nu} \, dx^\mu \, dx^\nu = T_{ab}(y) \, dy^a \, dy^b + p(y) r^2(y) \gamma_{ij} \, dz^i \, dz^j, \]  

(2.7)

where \( p(y) \) is a scalar function on \((M^2, g_{ab})\).

The generalized Misner–Sharp mass [28] is a scalar function on \((M^2, g_{ab})\) with the dimension of mass such that

\[ m := \frac{(n-2)V_{n-2}^k}{2\kappa_n^2} \left\{ -\tilde{\Lambda} r^{n-1} + r^{n-3} \left[ k - (Dr)^2 \right] + \tilde{\alpha} r^{n-5} \left[ k - (Dr)^2 \right]^2 \right\}, \]  

(2.8)

where \( \tilde{\alpha} := (n-3)(n-4)a, \tilde{\Lambda} := 2\Lambda/[(n-1)(n-2)], D_a \) is a metric compatible linear connection on \((M^2, g_{ab})\) and \((Dr)^2 := g^{ab}(D_a r)(D_b r)\). \( V_{n-2}^k \) is the area of the unit \((n-2)\)-dimensional space of constant curvature. The quasi-local mass is defined by the quasi-local geometrical quantity on the boundary of a spatial surface and dependent only on the metric and first derivatives. It can also be derived by the locally conserved energy flux, from which the quasi-local mass is recognized as a total amount of energy enclosing the spatial surface [27]. The equations in the following analysis can be transcribed in a comprehensible form by using the quasi-local mass. Physical properties of the quasi-local mass were elucidated in [27] and partial results thereof will be used in the succeeding arguments.

In our analysis, it is suitable to write the line element in the double-null coordinates as

\[ ds^2 = -2e^{-f(u,v)} \, du \, dv + r^2(u,v) \gamma_{ij} \, dz^i \, dz^j. \]  

(2.9)

Null vectors \( \partial/\partial u \) and \( \partial/\partial v \) are taken to be future-pointing. The expansions of two independent future-directed radial null geodesics are defined as

\[ \theta_+ := (n-2)r^{-1}r_u, \]

(2.10)

\[ \theta_- := (n-2)r^{-1}r_v, \]  

(2.11)

where a comma denotes the partial derivative. Note that the values of \( \theta_+ \) and \( \theta_- \) are not the geometrical invariants since the null coordinates \( u \) and \( v \) have a residual rescaling freedom such as \( u \to U = U(u), v \to V = V(v) \). An invariant combination is \( e^f \theta_+ \theta_- \), which characterizes the trapping horizon as will be mentioned in the next section. The function \( r \) on the other hand, has a geometrical meaning as an areal radius: the area of symmetric subspace is given by \( V_{n-2}^k r^{n-2} \). Then, the quasi-local mass \( m \) is expressed in a double-null form as

\[ m = \frac{(n-2)V_{n-2}^k}{2\kappa_n^2} \left\{ -\tilde{\Lambda} r^2 + \left( k + \frac{2}{(n-2)^2} r^2 e^f \theta_+ \theta_- \right) \right\} + \tilde{\alpha} r^{-2} \left( k + \frac{2}{(n-2)^2} r^2 e^f \theta_+ \theta_- \right)^2 \]  

(2.12)

and the stress–energy tensor \( T_{\mu \nu} \) as

\[ T_{\mu \nu} \, dx^\mu \, dx^\nu = T_{uu}(u, v) \, du \, du + 2T_{uv}(u, v) \, du \, dv + T_{vv}(u, v) \, dv \, dv + p(u, v) \gamma_{ij} \, dz^i \, dz^j. \]  

(2.13)

The governing field equations (2.3) are

\[ (r_{,uu} + f_a r_{,u}) \left[ 1 + \frac{2\tilde{\alpha}}{r^2} \left( k + 2e^f r_{,a} r_{,u} \right) \right] = -\frac{\kappa_n^2}{n-2} r T_{uu}, \]  

(2.14)

\[ (r_{,vv} + f_v r_{,v}) \left[ 1 + \frac{2\tilde{\alpha}}{r^2} \left( k + 2e^f r_{,a} r_{,v} \right) \right] = -\frac{\kappa_n^2}{n-2} r T_{vv}, \]  

(2.15)
\[ r_{ru} + (n-3)r_{uu} + \frac{n-3}{2}k e^{-f} + \frac{\bar{a}}{2r^2}[\ln(n-5)k^2 e^{-f} + 4r_{ru}(k + 2e^f r_{rr})] \]
\[ + 4(n-5)r_{uv}(k + e^f r_{ru}) - \frac{n-1}{2} \bar{\Lambda}r^2 e^{-f} = \frac{\kappa_n^2}{n-2} r^2 T_{uv}. \] (2.16)

\[ r^2 f_{uu} + 2(n-3)r_{uu} + k(n-3)e^{-f} - (n-4)r_{ru} + \frac{2\bar{a}e^{-f}}{r^2} [e^f (k + 2e^f r_{rr}) r^2 f_{uv} \]
\[ - (n-8)r_{ru} \bar{\Lambda}] + 2r^2 e^{2f} \{(f_{ru} r_{u} + r_{uu})(f_{uv} r_{v} + r_{vv}) - (r_{uv})^2 \}
\[ + (n-5)(k + 2e^f r_{rr})^2] = \kappa_n^2 r^2 (T_{uv} + e^{-f} p). \] (2.17)

The variation of \( m \) is determined by these equations as

\[ m_{v} = \frac{1}{n-2} V_n^{k-2} e^{f/2} r^{n-1} (T_{uv} \theta_v - T_{vv} \theta_u), \] (2.18)

\[ m_{u} = \frac{1}{n-2} V_n^{k-2} e^{f/2} r^{n-1} (T_{uv} \theta_v - T_{vv} \theta_u). \] (2.19)

These variation formulae are the same as those in general relativity and therefore have several practical advantages.

In this paper, we do not specify the particular stress–energy tensor of matter fields. Alternatively, we impose energy conditions. The null energy condition for the matter field implies

\[ T_{uu} \geq 0, \quad T_{vv} \geq 0, \] (2.20)

while the dominant energy condition implies

\[ T_{uu} \geq 0, \quad T_{vv} \geq 0, \quad T_{uv} \geq 0, \] (2.21)

which assures that a causal observer measures the non-negative energy density and the energy flux is a future-directed causal vector. The dominant energy condition implies the null energy condition, but the converse is not true.

Unlike the general relativistic case, the quasi-local mass (2.12) is quadratic in \( e^f \theta_v \ldots \), so they do not have one-to-one correspondence. Solving equation (2.12) inversely, we obtain

\[ \frac{2}{(n-2)^2} r^2 e^f \theta_v \theta_u = -k - \frac{r^2}{2\bar{a}} \left( 1 + \frac{8k^2 \bar{\alpha} m}{(n-2) V_n^{k-2} r^{n-1}} + 4\bar{a} \bar{\Lambda} \right). \] (2.22)

There are two families of solutions corresponding to the sign in front of the square root in equation (2.22), stemming from the quadratic curvature terms in the action. We call the family having the minus (plus) sign the GR-branch (non-GR-branch) solution. Note that the GR-branch solution has a general relativistic limit as \( \alpha \to 0 \) but the non-GR branch does not. Throughout this paper, the upper sign is used for the GR branch.

We also assume, in addition to \( \alpha \geq 0 \), the range of \( \alpha \) as

\[ 1 + 4\bar{a} \bar{\Lambda} \geq 0 \] (2.24)

in order to avoid the zero-mass solution becoming unphysical. Equation (2.24) gives a restriction when \( \bar{\Lambda} \) is negative. If condition (2.24) is satisfied, the (anti-) de Sitter space
with effective cosmological constant \( \Lambda_{\text{eff}} = (1 \pm \sqrt{1 + 4\tilde{\alpha} \tilde{\Lambda}})/(2\tilde{\alpha}) \) solves the vacuum field equations.

Under above conditions, it follows from equation (2.22) that the quasi-local mass has a non-positive lower bound
\[
m \geq -\frac{(n-2)(1 + 4\tilde{\alpha} \tilde{\Lambda}) V_{n-2}^{k} r^{n-1}}{8\kappa_{n}^{2} \tilde{\alpha}} =: m_{b}.
\]
(2.25)

When the equality holds in equation (2.25), the two branches coincide. We call the points where \( m = m_{b} \) holds branch points.

An immediate consequence of the variation formulae is the constancy of the quasi-local mass in the absence of matter fields. If \( 1 + 4\tilde{\alpha} \tilde{\Lambda} \neq 0 \) and \((Dr)^2 = 0\), the general solution [27, 32] is given by the generalized Boulware–Deser–Wheeler solution [33]
\[
ds^2 = -F(r) \ dt^2 + F^{-1}(r) \ dr^2 + r^2 \gamma_{ij} \ dz^{i} \ dz^{j},
\]
(2.26)

where
\[
F(r) := k + \frac{r^2}{2\tilde{\alpha}} \left[ 1 \mp \sqrt{1 + \frac{8\kappa_{n}^{2} \tilde{\alpha}m}{(n-2)V_{n-2}^{k} r^{n-1}} + 4\tilde{\alpha} \tilde{\Lambda}} \right].
\]
(2.27)

Analysis in [34] provides a complete classification of the global structure of the generalized Boulware–Deser–Wheeler solution. (See [35] for the charged case.) The event horizon in this vacuum spacetime is the simplest example of the trapping horizon discussed below.

3. Trapping horizon and dynamical black hole

The event horizon \( H^{+} \) of a black hole is determined by the global structure of a spacetime as \( H^{+} = J^{-} (\mathcal{I}^{+}) \), where \( J^{-} \) and \( \mathcal{I}^{+} \) denote a causal past and a future null infinity, respectively. Namely, one has to know the entire future of a spacetime to identify black-hole regions. However, it is rare to solve exactly the field equations due to its nonlinearity, and therefore event horizons are of little use in identifying a black hole from a practical viewpoint. To overcome this difficulty, one may use a quasi-local notion of horizons, which is more easily handled than the event horizon\(^3\). The notion of trapping horizons was originally introduced by Hayward [12, 30]. To begin with, we recapitulate the definitions. We defer to [19, 20] for the comparison with dynamical horizons.

**Definition 1.** A trapped (untrapped) surface is a compact spatial \((n-2)\)-surface with \( \theta_{+} \theta_{-} > (<) 0 \).

**Definition 2.** A trapped (untrapped) region is the union of all trapped (untrapped) surfaces.

**Definition 3.** A marginal surface is an \((n-2)\)-surface with \( \theta_{+} \theta_{-} = 0 \).

Without loss of generality, we set \( \theta_{+} \) to be zero on a marginal surface. We also fix the orientation of the untrapped region such that \( \theta_{+} > 0 \) and \( \theta_{-} < 0 \) from hereafter. This means that \( (\partial/\partial v) \) and \( (\partial/\partial u) \) are pointing outward and inward, respectively.

\(^3\) Even if the stationarity and the dominant energy condition are assumed, the whole picture of the event horizon remains unclear in the Einstein–Gauss–Bonnet gravity. In general relativity, a powerful and useful theorem, called the rigidity theorem, is established [2–5]: the event horizon in a stationary spacetime is a Killing horizon. The Killing horizon is totally geodesic, and the surface gravity is constant over the horizon. Since the proof of rigidity heavily made use of Einstein equations, it has not been certain that the proof would proceed in parallel for the Einstein–Gauss–Bonnet gravity.
**Definition 4.** A marginal surface is future if \( \theta^- < 0 \), past if \( \theta^- > 0 \), bifurcating if \( \theta^- = 0 \), outer if \( \theta^{+,u} < 0 \), inner if \( \theta^{+,u} > 0 \), and degenerate if \( \theta^{+,u} = 0 \).

**Definition 5.** A trapping horizon is the closure of a hypersurface foliated by future or past, outer or inner marginal surfaces.

By definition, the notion of trapping horizons does not make any reference to the infinite future, nor the asymptotic structure. In contrast to event horizons, trapping horizons are meaningful even in the spatially compact spacetime.

Among all classes, the future outer trapping horizon is the most relevant in the context of black holes [12, 30]. In this case, the definition expresses the idea that the ingoing null rays should be converging, \( \theta^- < 0 \), and the outgoing null rays should be instantaneously parallel on the horizon, \( \theta^+ = 0 \), diverging just outside the horizon and converging just inside, \( \theta^{+,u} < 0 \).

Since trapping horizons and event horizons are conceptually different, one may suspect that there is no immediate relationship between them. However, as is well known, a trapped region certainly arises in the process of a black-hole formation from the gravitational collapse of a massive body. Now by the same arguments of proposition 9.2.1 of [3], under the null convergence condition together with the weak cosmic censorship, we conclude \( \theta^+ \geq 0 \) on \( H^+ \), i.e., trapped regions cannot be seen from the future null infinity. It then follows in general relativity that the trapping horizon coincides with or resides inside the event horizon under the null energy condition. Thus, they are mutually associated in physically reasonable circumstances.

The nature of trapping horizons in general relativity has been well appreciated together with energy conditions, which directly implies the null convergence condition. But in other theories of gravity, the relation between the convergence and energy conditions is not immediate via the field equations. For this reason, it is not apparent \textit{a priori} that trapping horizons in Einstein–Gauss–Bonnet gravity have the same properties as those in general relativity. To elucidate this is the main goal of this section.

### 3.1. Mass of the trapping horizon

Because the concept of a trapping horizon is quasi-local, a quasi-local mass is adopted to evaluate the mass of a black hole. The dynamical nature of a black hole was studied in [30] in the four-dimensional spherically symmetric case in general relativity without \( \Lambda \), in which case a trapping horizon is succinctly described by the Misner–Sharp mass. Before moving on to the details, we review some of the basic properties of our quasi-local mass (2.8). See [27] for the proof.

**Proposition 1** (asymptotic behavior). In the asymptotically flat spacetime, \( m \) reduces to the higher-dimensional Arnowitt–Deser–Misner (ADM) mass [36] at spatial infinity.

**Proposition 2** (monotonicity). If the dominant energy condition holds, \( m \) is non-decreasing (non-increasing) in any outgoing (ingoing) spacelike or null direction on an untrapped surface.

**Proposition 3** (positivity). If the dominant energy condition holds on an untrapped spacelike hypersurface with a regular center, then \( m \geq 0 \) holds there, where the regular center denotes a central point \( r = 0 \) with \( k - (Dr)^2 = O(r^2) \) in that neighborhood.

These properties support the well-posedness of the quasi-local mass. The asymptotic value correctly denotes the total energy, while monotonicity means that the mass contained within a spatial surface is non-decreasing outwardly. Positivity is not immediately manifest.
because of the negative contribution of gravitational potential. It should be stressed that since
a regular center is always trapped for \( k = -1 \), we cannot conclude the positivity of \( m \) in this
case, while the case where \( k = 1 \) guarantees proposition 3. In the case where \( k = 0 \), the
assumption in the proposition constraints on the metric form around the regular center.

Let us now look at the relation between the areal radius and the quasi-local mass of a
trapping horizon. From equation (2.12), the mass of the trapping horizon with a radius \( r = r_h \)
is given by

\[
m_h(r) := \frac{(n - 2)V_{n-2}^k}{2\kappa_n^2} \left( k + \frac{\tilde{\alpha}k^2}{\chi^2} - \tilde{\Lambda} \chi^2 \right).
\]

In the special case with \( 1 + 4\tilde{\alpha} \tilde{\Lambda} = 0 \), we have

\[
m_h(r) = \frac{(n - 2)V_{n-2}^k}{8\tilde{\alpha}\kappa_n^2} \chi^{-5}(2\tilde{\alpha}k + \chi^2)^2 \geq 0.
\]

The succeeding two propositions are shown by direct calculations from the definition
(2.22). These statements are not shared in the general relativistic case.

**Proposition 4** (absence of trapping horizons). An \((n - 2)\)-surface is necessarily untrapped,
and trapping horizons are absent in the non-GR-branch solution for \( k = 0 \) and 1. In the
GR-branch (non-GR-branch) solution for \( k = -1 \) with \( r^2 < (>)2\tilde{\alpha} \), an \((n - 2)\)-surface is
always trapped (untrapped), and trapping horizons are absent.

**Proposition 5** (trapping). In the GR-branch solution for \( k = 1 \), 0 and for \( k = -1 \) with
\( r^2 \geq 2\tilde{\alpha} \) (In the non-GR-branch solution for \( k = -1 \) with \( r^2 \leq 2\tilde{\alpha} \), an \((n - 2)\)-surface is
trapped if and only if \( m > (<) m_h(r) \), marginal if and only if \( m = m_h(r) \) and untrapped if and
only if \( m < (>) m_h(r) \).

Here we have implicitly assumed that the branch points are regular, so that trapping
horizons can appear at the minimal (maximal) areal radius \( r_h = \sqrt{2\tilde{\alpha}} \) in the GR-branch
(non-GR-branch) solutions for \( k = -1 \). However, the branch points become singular in most
cases, as we shall see in proposition 12. In the analysis below, we do not further consider
trapping horizons with \( r_h = \sqrt{2\tilde{\alpha}} \) for \( k = -1 \). Inasmuch as some equations become trivial
at these points, propositions in the next subsections cannot be established. The exclusion
of this special case is rather technical than physically unrealizable since our approach fails in the
above special situation.

Now we turn to the task of inspecting the relation between \( m_h \) and \( r_h \), which can be
completely understood from the result in [34] for the generalized Boulware–Deser–Wheeler
solution. (The variable \( M \) in [34] is related to \( m_h \) as \( M \equiv 2\kappa_n^2 m_h/[(n - 2)V_{n-2}^k] \). The \( m_h=r_h \)
diagram is of great advantage in identifying the number of horizons and their types.

**Proposition 6** (horizon mass in general relativity). In general relativity, the mass of the
trapping horizon \( m_h \) satisfies the inequalities in table 1, where \( r_{ex(GR)} := [k(n - 3)/[(n - 1)\tilde{\Lambda}]]^{1/2} \) and \( m_{ex(GR)} := k(n - 2)V_{n-2}^k \chi^{-3}/[(n - 1)\kappa_n^2] \).

**Proof.** See section 3 in [34].

The equality \( m_h = m_{ex(GR)} \) attains when the two trapping horizons are coincident, which
produces the degenerate trapping horizon.

As discussed in [34], both the \( n = 5 \) case and the \( 1 + 4\tilde{\alpha} \tilde{\Lambda} = 0 \) case require special
treatment in Einstein–Gauss–Bonnet gravity. This may be attributed to the fact that \( n = 5 \) is
the lowest dimension in which the Gauss–Bonnet term becomes nontrivial, and \( 1 + 4\tilde{\alpha} \tilde{\Lambda} = 0 \)
Table 1. Mass of the trapping horizon in general relativity.

| $k$  | $k = 1$ | $k = 0$ | $k = -1$ |
|------|---------|---------|----------|
| $\Lambda = 0$ | $m_h > 0$ | $m_h = 0$ | $m_h < 0$ |
| $\Lambda > 0$ | $m_h \leq m_{ex(GR)}$ | $m_h < m_B$ | $m_h < m_B$ |
| $\Lambda < 0$ | $m_h > 0$ | $m_h > 0$ | $m_h > m_{ex(GR)}$ |

Table 2. Mass of the trapping horizon in Einstein–Gauss–Bonnet gravity for $n \geq 5$ and $1 + 4\alpha \Lambda > 0$. Note that the inequality (2.25) may give a more severe constraint in the case with $k = -1$ and $\Lambda < 0$ in the GR branch.

| $k$  | $k = 1$ | $k = 0$ | $k = -1$ |
|------|---------|---------|----------|
| $\Lambda = 0$ | $m_h > 0$ | $m_h = 0$ | $m_h < m_B$ |
| $\Lambda > 0$ | $m_h \leq m_{ex(r_{ext}(-1))}$ | $m_h < m_B$ | $m_h < m_B$ |
| $\Lambda < 0$ | $m_h > 0$ | $m_h \geq m_{ex(r_{ext}(-1))}$ | $n/a$ |

Table 3. Mass of the trapping horizon in Einstein–Gauss–Bonnet gravity for $n = 5$ and $1 + 4\alpha \Lambda > 0$. Note that the inequality (2.25) may give a more severe constraint in the case with $k = -1$ and $\Lambda < 0$ in the GR branch.

| $k$  | $k = 1$ | $k = 0$ | $k = -1$ |
|------|---------|---------|----------|
| $\Lambda = 0$ | $m_h > m_{crit}$ | $m_h = 0$ | $m_h < m_B$ |
| $\Lambda > 0$ | $m_h \leq m_{ex(r_{ext}(-1))}$ | $m_h < m_B$ | $m_h < m_B$ |
| $\Lambda < 0$ | $m_h > m_{crit}$ | $m_h \geq m_{ex(r_{ext}(-1))}$ | $n/a$ |

Table 4. Mass of the trapping horizon in Einstein–Gauss–Bonnet gravity for $n \geq 5$ and $1 + 4\alpha \Lambda > 0$.

| $k$  | $k = 1$ | $k = 0$ | $k = -1$ |
|------|---------|---------|----------|
| $n = 5$ | $m_h > m_{crit}$ | $m_h > 0$ | $m_h \geq 0$ |
| $n \geq 6$ | $m_h > 0$ | $m_h \geq 0$ | $n/a$ |

is the special combination of Lovelock coefficients, which yields the Chern–Simons gravity for $n = 5$ [37].

Proposition 7 (horizon mass in Einstein–Gauss–Bonnet gravity). The mass of the trapping horizon $m_h$ satisfies the inequalities in table 2, 3 and 4 for $n \geq 6$ with $1 + 4\alpha \Lambda > 0$, $n = 5$ with $1 + 4\alpha \Lambda > 0$ and $n \geq 5$ with $1 + 4\alpha \Lambda = 0$, respectively, where

\[
m_{ex}(x) := \frac{k(n - 2)\sqrt{n - 2}}{(n - 1)\alpha^2}(x^2 + 2\alpha k)x^{n-5},
\]

\[
m_{crit} := \frac{3\alpha V_3^{1/3}}{\kappa_5^{2/3}},
\]
\[ m_B := -\frac{(2\tilde{\alpha})(n-3)/2(n-2)V_{n-2}^{-1}(1 + 4\tilde{\alpha} \tilde{A})}{4\kappa_n^2}, \quad (3.5) \]

\[ r_{ex} := \left( -\frac{k(n-5)\tilde{\alpha}}{n-3} \right)^{1/2}, \quad (3.6) \]

\[ r_{ex(k)} := \left[ -\frac{n-3}{2(n-1)\Lambda} \left\{ -k \pm |k| \sqrt{1 + \frac{4\tilde{\alpha}(n-1)(n-5)}{(n-3)^2}} \right\} \right]^{1/2}. \quad (3.7) \]

**Proof.** See section 4 in \[34\]. □

The above propositions imply an upper or a lower bound for the mass of the trapping horizon in some cases. Although the class of trapping horizons is not specified here, it surely gives a constraint for the mass of a black hole defined by a trapping horizon.

Next, we show the following mass inequality in Einstein–Gauss–Bonnet gravity.

**Proposition 8** (mass inequality). *If the dominant energy condition holds, then \( m \geq (\leq) m_h(r_h) \) holds in the GR branch (non-GR branch) on an untrapped spacelike hypersurface of which the inner boundary is a marginally trapped surface with radius \( r_h \).*

**Proof.** By proposition 2, we have \( m \geq m|_{r=r_h} \equiv m_h(r_h) \) on the untrapped spacelike hypersurface. □

The positivity of \( m \) in the untrapped region with a regular center was shown in proposition 3. On the other hand, proposition 8 claims that in the GR branch there may be a more severe lower bound on \( m \) on the untrapped hypersurface of which the inner boundary is a marginally trapped surface. For \( k = 1 \) and \( \Lambda \leq 0 \), for example, there is a positive lower bound on \( m \). If there is a black or white hole with area \( A_{n-2} r_h^n \), where \( A_{n-2}(:= V_{n-2}) \) is the area of a unit \((n-2)\)-sphere, the mass–energy measured outside the hole satisfies an isoperimetric inequality

\[ m \geq \frac{(n-2)A_{n-2}}{2\kappa_n^2} r_h^{n-3} \left( 1 + \frac{\tilde{\alpha}}{r_h^2} - \frac{\Lambda r_h^2}{2} \right) =: m_{irr}. \quad (3.8) \]

\( m_{irr}(>0) \) represents the minimal mass of a black hole or white hole, corresponding to the irreducible mass. For \( n = 4, k = 1, \Lambda = 0 \), the above inequality becomes \( \sqrt{4\pi r_h^2}/16\pi \leq G_4 m \), which is comparable to the Penrose inequality \[38–40\]. If the untrapped surface extends to spacelike infinity in the asymptotically flat case, propositions 1 and 8 prove the special case of the positive mass theorem for black holes \[41\]. On the other hand, when \( \Lambda \) is positive, \( m_h(r_h) \) may be negative for some \( r_h > 0 \), and then this result does not give a stronger lower bound on \( m \).

**3.2. Properties of the trapping horizon**

In this subsection, we investigate properties of the trapping horizon. Among all classes, a future outer trapping horizon defines a dynamical black hole and is particularly important.

In four-dimensional general relativity, the topology of an outer trapping horizon is restricted to either a two-sphere or a two-torus if we assume \( \Lambda \geq 0 \) and the dominant energy condition \[12\], which is the correspondent of Hawking’s topology theorem\(^4\) \[2, 3\].

\(^4\) Event horizons with toroidal topology cannot be realized in four dimensions if the null convergence condition, \( \mathcal{J} \simeq \mathbb{R} \times S^2 \) (this condition automatically holds if the spacetime is asymptotically flat or anti-de Sitter) and the weak cosmic censorship are assumed, as a consequence of the topological censorship \[42\].
A major difficulty encountered in Einstein–Gauss–Bonnet gravity is whether we can draw appropriate information on spacetime curvatures just from the energy condition. Under the present spacetime ansatz (2.6), however, it is rather straightforward.

**Proposition 9** (topology). *An outer trapping horizon must have a topology of non-negative curvature in the GR branch if \( \Lambda \geq 0 \) and the dominant energy condition is satisfied.*

**Proof.** Evaluation of equation (2.16) on a trapping horizon gives

\[
\frac{(n-2)k}{2r_h^2} \left[ n - 3 + (n - 5)\tilde{\alpha}kr_h^{-2} \right] = \kappa_n^2 e^{\ell} T_{uu} + \Lambda - e^{\ell} \dot{\theta}_{s,u} \left( 1 + \frac{2\tilde{\alpha}}{r_h^2} \right). \tag{3.9}
\]

Now let \( \Lambda \geq 0 \) and the dominant energy condition be assumed; then it follows from proposition 4 that the right-hand side of equation (3.9) is nonnegative for an outer trapping horizon in the GR branch. For \( k = -1 \), since we have \( r_h^2 > 2\tilde{\alpha} \) by proposition 4, the left-hand side of equation (3.9) becomes negative, which yields inconsistency. Thus, the only case where \( k = 1 \) or 0 is possible. \( \square \)

Note that an outer trapping horizon with the \( k = 0 \) topology can appear if and only if \( \Lambda = 0 \) and \( e^{\ell} = 0 \) on the trapping horizon. These black holes are considered to be non-generic and therefore rare to develop because they occur under highly restrictive conditions.

Unfortunately, there is no sign control of \( k \) when \( \Lambda < 0 \) in the GR branch, in which case various topologies are allowed. In the non-GR branch, any class of trapping horizons must have a topology of negative curvature, as shown in proposition 4, irrespective of the energy conditions and the sign of \( \Lambda \).

The following lemma is used in the proof for later propositions. Observe that trapping horizons coinciding with the branch points are excluded from our consideration.

**Lemma 1.** *If the null energy condition holds, \( \theta_{s,v} \leq (\geq) 0 \) is satisfied on the trapping horizon in the GR (non-GR) branch.*

**Proof.** From equation (2.15), we obtain

\[
2\theta_{s,v} \left( 1 + \frac{2\tilde{\alpha}}{r_h^2} \right) = -\kappa_n^2 T_{vv} \tag{3.10}
\]

on the trapping horizon, which gives \( \theta_{s,v} (r_h^2 + 2\tilde{\alpha}) \leq 0 \) by the null energy condition. Consequently, the above lemma follows from proposition 4. \( \square \)

Now let \( \xi^u (\partial / \partial x^u) = \xi^u (\partial / \partial u) + \xi^v (\partial / \partial v) \) be the generator of the trapping horizon. Since the trapping horizon is foliated by the marginal surfaces,

\[
\mathcal{L}_{\xi} \theta_s = \theta_{s,v} \xi^v + \theta_{s,u} \xi^u = 0 \tag{3.11}
\]

holds on the trapping horizon. If the trapping horizon is null (\( \xi^u = 0 \), \( \theta_{s,v} = 0 \)) is concluded. Then equation (3.10) signifies that there is no energy inflow \( T_{vv} = 0 \) across the null trapping horizon, irrespective of energy conditions.

**Proposition 10** (signature law). *Under the null energy condition, an outer (inner) trapping horizon in the GR branch is non-timelike (non-spacelike), while it is non-spacelike (non-timelike) in the non-GR branch.*

5 By ‘trapping horizon is non-timelike’ we mean that the generator of the trapping horizon \( \xi^u \), everywhere orthogonal to a foliation of marginal surfaces and preserving the foliation, is non-timelike.
Proof. From equation (3.11), we have
\[ \xi^u = -\frac{\theta_{\nu,v}}{\theta_{\nu,u}} \xi^v \] (3.12)
on the trapping horizon. Thus, by lemma 1, \( \xi^v \xi^u \leq (\geq) 0 \) is satisfied on the outer (inner) trapping horizon in the GR branch, while \( \xi^v \xi^u \geq (\leq) 0 \) is satisfied on the outer (inner) trapping horizon in the non-GR branch.

**Proposition 11** (trapped side). Let the null energy condition be assumed. Then, the outside (inside) region of a future inner trapping horizon is trapped (untrapped) in the GR branch, and the outside (inside) region of a future outer trapping horizon is untrapped (trapped) in the non-GR branch. To the contrary, the future (past) domain of a future outer trapping horizon is trapped (untrapped) in the GR branch, and the future (past) domain of future inner trapping horizon is untrapped (trapped) in the non-GR branch. \( \Box \)

**Propositions 10 and 11** mean that a future outer trapping horizon in the GR branch is a one-way membrane being matched to the concept of a black hole as a region of no escape. On the other hand, the non-GR branch is diametrically opposed. One might hope that a future outer trapping horizon in the non-GR branch also deserves to be called a black-hole horizon since proposition 11 shows that it is an inner boundary of untrapped surfaces. However, proposition 10 claims that it does not capture the idea that a black hole is a one-way membrane. A light ray emanating from a point on a future outer trapping horizon can propagate into both sides of it since it can be timelike. Then there naturally arises a question: what causes such an antithetical and pathological behavior in the non-GR branch?

We digress here to discuss this issue further. The two branches stem from the quadratic terms in curvature and are confluent at the branch points. The plus–minus sign in equation (2.22) makes the respective branches quite different. The following lemma answers the above question.

**Lemma 2.** If \( T_{\mu\nu}k^\mu k^\nu \geq 0 \) is satisfied for a radial null vector \( k^\mu \), \( R_{\mu\nu}k^\mu k^\nu \geq (\leq) 0 \) holds in the GR (non-GR) branch.

**Proof.** Using equations (2.14)–(2.17) together with the expressions of the Ricci tensors, we obtain
\[ T_{\mu\nu}k^\mu k^\nu = \frac{1}{k^2 R_{\mu\nu}} k^\mu k^\nu \left[ 1 + \frac{2\bar{g}}{r} (k + 2e^f r_{,uv}) \right] 
+ \frac{8\bar{g} e^{-f}}{k^2 r^4} k^\mu k^\nu \left[ \frac{r^4 e^{2f}}{(n-2)^2} R_{uv} - (k + 2e^f r_{,uv} - e^f r_{,uv})^2 \right] \] (3.13)
for a general null vector $k^\mu$. For a radial null vector, where $k^\mu(\partial/\partial x^\mu) = k^\mu(\partial/\partial u)$ or $k^\mu(\partial/\partial v)$, equations (3.13) and (2.22) combine to give

$$R_{\mu\nu}k^\mu k^\nu \sqrt{1 + \frac{8\kappa^2}{(n-2)v_{n-2}^4} + 4\alpha} = \kappa^2 T_{\mu\nu}k^\mu k^\nu. \quad (3.14)$$

This lemma shows that the null convergence condition $R_{\mu\nu}k^\mu k^\nu \geq 0$ fails in the non-GR branch if the null energy condition is strictly satisfied $T_{\mu\nu}k^\mu k^\nu > 0$. The Vaidya-type radiating solution gives such an example [43, 44]. It signals that solutions in the non-GR branch behave badly under the null energy condition, since properties of the geometry are determined not by energy conditions but by the convergence condition, as seen in the Raychaudhuri equation.

In the non-GR-branch solution, gravity effectively acts repulsively for the positive energy particles. Lemma 2 is most convincing to account for the peculiarity of the non-GR-branch solution.

In this paper, trapping horizons coinciding with the branch points are excluded from our considerations. This decision is strongly supported by the following proposition.

**Proposition 12** (branch singularity). If the null energy condition is strictly satisfied at least for a radial null vector, the branch points are curvature singularities.

**Proof.** Let $k^\mu = k^\alpha(\partial/\partial x^\alpha)^\mu$ be a radial null vector. It then follows from equation (3.14) that we have $R_{\mu\nu}k^\mu k^\nu \to \pm \infty$ at the branch points, where the inside of the square root in equation (3.14) vanishes. □

In proposition 12, the null energy condition must be strictly satisfied for a radial null vector; however, the appearance of a singularity is not necessarily due to the presence of matter fields. Even in the vacuum case, the generalized Boulware–Deser–Wheeler solution with $1 + 4\alpha > 0$ has a branch singularity where the Kretschmann scalar $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ diverges. As it now stands, we have no definite answer for how generic the curvature singularity is when $T_{\mu\nu} = 0$ holds at the branch points. We leave this to future investigation.

Here we also show the following proposition as another consequence of lemma 2, claiming that, as in general relativity, caustics develop in a congruence of a radial null geodesic in the GR branch if the convergence occurs anywhere.

**Proposition 13** (caustics). Let $k^\mu$ be the tangent to an affinely parametrized radial null geodesic, and let the null energy condition hold. If the expansion $\theta := \nabla_\lambda k^\mu$ takes the negative value $\theta_0$ at any point on a geodesic in the congruence, then $\theta \to -\infty$ along that geodesic within the affine length $\lambda \leq (n-2)/|\theta_0|$ in the GR branch, provided that the geodesic is extended to this parameter value.

**Proof.** The Raychaudhuri equation for an affinely parametrized radial null geodesic with tangent $k^\mu = k^\alpha(\partial/\partial x^\alpha)^\mu$ is written as

$$\frac{d\theta}{d\lambda} = -\frac{1}{n-2}\theta^2 - R_{\mu\nu}k^\mu k^\nu, \quad (3.15)$$

where $\lambda$ is an affine parameter. By lemma 2, the above equation gives

$$\frac{d\theta}{d\lambda} + \frac{1}{n-2}\theta^2 \leq 0 \quad (3.16)$$

in the GR branch, implying

$$\frac{d}{d\lambda}(\theta)^{-1} \geq \frac{1}{n-2} \quad (3.17)$$

and hence
\[ \theta(\lambda) \leq \frac{(n-2)\theta_0}{(n-2) + \lambda \theta_0}, \]  
(3.18)
where \( \theta_0 \) is the initial value of \( \theta \). If \( \theta_0 < 0 \), then equation (3.18) gives \( \theta \to -\infty \) within an affine parameter \( \lambda \leq (n-2)/|\theta_0| \), provided that the geodesic can be extended that far. □

Let us now turn to a discussion of the area law of a trapping horizon. The area of a black-hole event horizon is non-decreasing into the future under the null convergence condition [2], which is transcribed into the null energy condition, via Einstein equations, in general relativity. Then, how about the trapping horizon? It should be emphasized that the proof of Hawking’s area theorem relies on the Raychaudhuri equation along the null geodesic generator of the event horizon. We extrapolate that the area theorem fails for the future outer trapping horizon in the non-GR branch. The next proposition shows that this is indeed the case.

**Proposition 14** (area law). Under the null energy condition, the area of a future outer (inner) trapping horizon is non-decreasing (non-increasing) along the generator of the trapping horizon in the GR branch, while it is non-increasing (non-decreasing) in the non-GR branch.

**Proof.** The derivation of the area
\[ A(r) := V^{k}_{n-2} r^{n-2} \]  
(3.19)
along the generator of a trapping horizon \( \xi^\mu \) is given by
\[ \mathcal{L}_\xi A = (n-2) r^{n-3} V^{k}_{n-2} (r, u \xi^u + r, v \xi^v), \]
\[ = r^{n-2} V^{k}_{n-2} \partial_r \xi^u, \]  
(3.20)
where the second equality is evaluated on the trapping horizon. Here we fix the orientation such that \( \xi^u > 0 \), which guarantees the non-spacelike (spacelike) trapping horizon to be future-directed (outgoing). Then, we obtain \( \mathcal{L}_\xi A \geq (\leq)0 \) on the future outer (inner) trapping horizon in the GR branch and on the future inner (outer) trapping horizon in the non-GR branch by proposition 10. □

Here we note that the above area law takes the meaning of ‘time evolution’ only when the trapping horizon is non-spacelike. By proposition 10, such a trapping horizon is inner and outer in the GR and non-GR branches, respectively. On the other hand, if the trapping horizon is null, an outer trapping horizon in the GR branch and an inner trapping horizon in the non-GR branch have this property. When the generator of the trapping horizon is spacelike, the theorem simply says that it is outward pointing.

In closing this subsection, we give an example of dynamical spacetimes containing a future outer trapping horizon in figure 1. It represents the transition from a generalized Boulware–Deser–Wheeler black hole with mass \( m_1 \) to another generalized Boulware–Deser–Wheeler black hole with mass \( m_2 (> m_1) \) by an incident null dust fluid with positive energy density.

### 3.3. Black-hole dynamics

Black-hole thermodynamics is now established for a stationary spacetime in general relativity (see e.g., [8, 31]). Even in non-stationary and highly dynamical situations, analogous laws hold for a trapping horizon in the four-dimensional general relativistic case [12–15, 30]. It may be tempting to hope that similar results go through in other theories of gravity. The aim
of this subsection is to address the issue of black-hole dynamics in Einstein–Gauss–Bonnet gravity.

A well-defined mass should satisfy the first law, representing the energy conservation. This is validated also for the quasi-local mass \(2.8\) [27]. We define a scalar

\[
P := - \frac{1}{2} T^{a}{}_{a},
\]

and a vector

\[
\psi^{a} := T^{a}{}_{b} D^{b} r + P D^{a} r
\]
on \((M^{2}, g_{ab})\), where the contraction is taken over on the two-dimensional orbit space. It is also convenient to use the areal volume

\[
V := \frac{V_{n-2}}{n-1} r^{n-1}
\]
satisfying \(D_{a} V = A D_{a} r\), where \(A\) is defined by equation \(3.19\). By using the field equations (see appendix of [27]), we obtain

\[
dm = A \psi_{a} dx^{a} + P dV.
\]

This is the unified first law [45] corresponding to the energy balance law, which reduces to the variation formulae \((2.18)\) and \((2.19)\) in the double null coordinates. In the form \((3.24)\), the physical meaning of each term is more readily recognizable. The first term represents an energy flux, while the second an external work [45–47]. Assuming the dominant energy condition, we have \(P \geq 0\). \(\psi^{a}\) corresponds to the quasi-localization of the Bondi–Sachs energy loss [48], but its interpretation in odd spacetime dimensions remains unclear [49, 50]. We will not discuss this issue because it is beyond the scope of the present paper. We anticipate that the evaluation of the unified first law \((3.24)\) on the trapping horizon gives the first law of black-hole mechanics. To this end, we must read off various ‘thermodynamical quantities’.

We here follow in the footsteps of Killing horizons, which have been fully studied in the literature and are now well established. Let \(\xi^{\mu}\) be a horizon-generating Killing field of a Killing horizon. The surface gravity, \(\kappa\), of a Killing horizon is defined by [8, 31]

\[
\xi^{\nu} \nabla_{\nu} \xi^{\mu} = \kappa \xi^{\mu},
\]
where the equality is evaluated on the Killing horizon. $\kappa$ measures the non-affinity of the Killing field and remains constant over the horizon if it has a regular bifurcation surface [51]. What plays the role of a horizon-generating Killing field for a trapping horizon? We shall embrace the generalized Kodama vector (simply the Kodama vector, hereafter) as a substitute [52, 53], defined by

$$K^a = -\epsilon^{ab} D_br,$$

(3.26)

where $\epsilon_{ab}$ is a volume element of $(M^2, g_{ab})$. As shown in [27], the Kodama vector is intimately related to our definition of quasi-local mass. It follows immediately by definition that

$$K^a K_a = -(Dr)^2,$$

(3.27)

so that the Kodama vector generates a preferred time-evolution vector field in the untrapped region. On the trapping horizon, $K^a$ becomes null and is given by $K^a = D^a r$. A naive prescription for defining the surface gravity of a trapping horizon is to replace $\kappa$ by $\kappa_{TH}$ and $\zeta^\mu$ by $K^a$ in equation (3.25) and to evaluate the equality at the trapping horizon. Simple calculations show

$$K^b D_b K_a = (D^2r) D_a r - (D^b r) D_b D_a r,$$

$$K^b D_b K_a = \frac{1}{2} (D^2 r) D_a r - (D^b r) D_b D_a r,$$

$$= \frac{r \kappa^2}{n-2} \left( 1 + \frac{2\tilde{\alpha}}{r^2} (k - (Dr)^2) \right)^{-1} \tilde{\psi}_a,$$

(3.29)

where we have used two-dimensional identities $\epsilon_{ab} \epsilon^{cd} \equiv -2\delta^c_{[a} \delta^d_{b]}$ and

$$(D_a D_b r - \frac{1}{2} g_{ab} D^2 r) D^2 r \equiv (D_a D^r r)(D_b D_r r) - \frac{1}{2} g_{ab} (D_c D_d r)(D^c D^d r)$$

(3.30)

to derive these equations. Note that at this stage equalities in these equations are not restricted on the trapping horizon. Equation (3.29) reveals that $\psi^a$ vanishes if $K^a$ is a Killing vector on $(M^2, g_{ab})$, implying that $K^\mu = K^a (\partial/\partial x^a)^\mu$ is a hypersurface-orthogonal Killing vector on $(M^n, g_{\mu\nu})$. This fact also lends support to the physical interpretation of $\psi^a$. Since $\psi_a K^a = T_{ab} K^b$ on the trapping horizon where $D^r r = K^a$ holds, $\psi^a$ is not in general proportional to $K^a$ in a dynamical setting. Then the surface gravity of a trapping horizon should be defined by $K^b D_b K_a = \kappa_{TH} K_a$. Thus we have

$$\kappa_{TH} = \frac{1}{2} D^2 r = -\frac{1}{2} e^{ab} D_a K_b,$$

(3.31)

where the evaluation is performed on the trapping horizon. Note that equation (3.31) is expressed in a purely geometrical way and confirms that the surface gravity vanishes for a degenerate trapping horizon. Note also that even along the Kodama vector, the surface gravity is not constant in general, which reflects the non-equilibrium situation.

After some manipulations, we can rewrite the unified first law (3.24) as

$$A \psi_a = \frac{D^2 r}{2 \kappa^2_n} \left\{ D_a A + 2(n-2)\tilde{\alpha} V^k \frac{r^{n-5}}{n-2} [k - (Dr)^2] D_a r \right\}$$

$$+ \frac{(n-2)\tilde{\alpha} V^k}{r^{n-6}} \frac{k^2}{2 \kappa^2_n} [k - (Dr)^2]^2 - k^2] D_a r$$

$$+ r^{n-3} D_a \left[ \frac{m}{r^{n-2}} + \frac{(n-2) V^k}{2 \kappa^2_n} \left( \hat{A} r^2 - \frac{\tilde{\alpha} k^2}{r^2} \right) \right].$$

(3.32)
The second and third terms on the right-hand side of equation (3.32) vanish along the trapping horizon (see equation (3.1)), where \((Dr)^2 = 0\). Thus we obtain the desired first law for a trapping horizon

\[
A \xi^a \psi_a = \frac{\kappa_{TH}}{\kappa_n^2} \xi^a D_a \left[ A \left( 1 + \frac{2(n-2)\hat{a}k}{(n-4)r^2} \right) \right],
\]

(3.33)

where \(\xi^a\) denotes the generator of a trapping horizon. Since the unified first law (3.24) gives

\[
\mathcal{L}_\xi m = A \psi_a \xi^a + P \mathcal{L}_\xi V,
\]

(3.34)

the first term on the right-hand side is regarded as \(A \psi_a \xi^a \equiv T_{TH} L_{S_{TH}}\), where \(T_{TH}\) and \(S_{TH}\) are the temperature and entropy of a trapping horizon, respectively. Then, by identifying the temperature with \(T_{TH} := \frac{\kappa_{TH}}{2\pi}\) as for a Killing horizon, the entropy of a trapping horizon \(S_{TH}\) is obtained from equation (3.33) as

\[
S_{TH} = \frac{2\pi A(r_h)}{\kappa_n^2} \left[ 1 + \frac{2(n-2)\hat{a}k}{(n-4)r_h^2} \right],
\]

\[
S_{TH} = \frac{V_n^{n-2} r_h^{n-2}}{4G_n} \left[ 1 + \frac{2(n-2)(n-3)\hat{a}k}{r_h^2} \right].
\]

(3.35)

This coincides with Iyer and Wald’s definition of dynamical black-hole entropy [10, 26], which has several plausible properties among other things. Their entropy is independent of the potential ambiguity of the Lagrangian and associated with a Noether charge. Moreover, it agrees with a non-stationary perturbation of the entropy of a stationary black hole and reduces to the entropy of a stationary black hole in the stationary case. For the generalized Boulware–Deser–Wheeler solution (2.26), the black-hole entropy is given by the replacement of \(r_h\) by \(r_+\) in equation (3.35) [54–57], where \(r_+\) is the root of \(F(r_+) = 0\) in equation (2.27) denoting the location of the black-hole event horizon. (See also [58].) The first term on the right-hand side of equation (3.35) is one quarter of the surface area (in \(G_n = 1\) units) thus the second term represents a deviation from the one in general relativity. This discrepancy gives rise to the negative entropy for the \(k = -1\) case, which is a characteristic property of any higher-curvature gravitational theories because such a deviation traces back its origin to the terms in the Lagrangian other than the Einstein–Hilbert term [10]. Observe that equation (3.35) does not reproduce the general relativistic result in four dimensions, in which the Gauss–Bonnet term does not alter the dynamics. The correct expression in four dimensions is obtained by setting \(n = 4\) in equation (3.32) and integrating it.

In proposition 14, we have shown the area law under the null energy condition, implying the entropy-increasing law in the general relativistic case. It deserves to be noted that, since the entropy of a trapping horizon \(S_{TH}\) is not simply proportional to the area in Einstein–Gauss–Bonnet gravity, the entropy law—corresponding to the second law of black-hole mechanics—is quite nontrivial.

**Proposition 15 (entropy law).** Under the null energy condition, the entropy of a future outer (inner) trapping horizon is non-decreasing (non-increasing) along the generator of the trapping horizon in both branches.

**Proof.** From equation (3.33), we obtain

\[
\mathcal{L}_\xi S_{TH} = \frac{V_n^{n-2} r_h^{n-4} (r_h^2 + 2\hat{a}k)}{4G_n} \xi^\mu
\]

(3.36)

along the generator of the trapping horizon. Repeating the identical procedure given in proposition 14, the result follows from proposition 4. \(\square\)
Table 5. Properties of the future trapping horizon under the null energy condition. Each quoted term denotes that it has the meaning of time evolution only if the trapping horizon is null, since the area and entropy laws are formulated along the generator of the trapping horizon.

|                      | GR branch | Non-GR branch |
|----------------------|-----------|---------------|
|                      | Future outer | Future inner | Future outer | Future inner |
| Signature            | Non-timelike | Non-spacelike | Non-spacelike | Non-timelike |
| Trapped side         | Future      | Exterior      | Interior      | Past         |
| Area law             | ‘Non-decreasing’ | Non-increasing | Non-increasing | ‘Non-decreasing’ |
| Entropy law          | ‘Non-decreasing’ | Non-increasing | Non-decreasing | ‘Non-decreasing’ |

It turns out that the dynamical entropy $S_{TH}$ increases while the area of a future outer trapping horizon decreases in the non-GR branch (trapping horizons occur for only $k = -1$ and $r_4^2 < 2\tilde{\alpha}$). This is the exceptional case, appearing only in Einstein–Gauss–Bonnet gravity. When $K^a$ is a Killing vector on $(M^4, g_{ab})$, we have $\psi^a = 0$, as has been mentioned before, and equation (3.33) yields the constancy of the entropy along the trapping horizon.

In closing this section, we summarize the results obtained here in table 5.

4. Summary and discussion

In this paper, we investigated several aspects of dynamical black holes in Einstein–Gauss–Bonnet gravity. Properties of trapping horizons were elucidated for their types and branches. Let us summarize the upshots briefly. We supposed that the spacetime has symmetries corresponding to the isometries of an $(n-2)$-dimensional constant curvature space. We also assumed the Gauss–Bonnet coupling $\alpha$ to be in the range $\alpha \geq 0$ and $1 + 4\tilde{\alpha}\Lambda \geq 0$, where the first is motivated by string theory and the second is to avoid ghosts.

The quasi-local mass of a trapping horizon was shown to obey an inequality summarized in tables 1–4. In the GR branch with $k = 1$ and $\Lambda \leq 0$, in particular, the quasi-local mass on an untrapped hypersurface with a marginal surface as an inner boundary has a positive lower bound under the dominant energy condition, corresponding to the value of the mass on the marginal surface. This isoperimetric inequality is similar to the Penrose inequality and establishes the positive mass theorem of a black hole in the case where $k = 1$ and $\Lambda \leq 0$.

Trapping horizons in the GR branch were shown to inherit characteristic properties of those in general relativity. If the dominant energy condition and $\Lambda \geq 0$ hold, the topology of outer trapping horizons must have a non-negative curvature. A future outer trapping horizon is non-timelike under the null energy condition, embodying the idea that a black-hole horizon is a one-way membrane. Then, the area and entropy laws indicate that a black hole grows and mimics the second law of thermodynamics, respectively.

In contrast, in the non-GR branch, trapping horizons have some features which may run counter to our intuition. The non-spacelike character of a future outer trapping horizon does not display characteristics of a black hole as a region of no escape. Besides that, the area of a future outer trapping horizon is non-increasing into the future under the null energy condition, which does not follow our intuition, either.

To see this more concretely, let us consider the Hawking evaporation of a black hole, in which the null energy condition is violated. A black hole in the GR branch continues to lose its mass and reduce its area. In other words, the signature of a trapping horizon becomes non-
spacelike and shrinks. Whereas in the non-GR branch, a black hole defined by a future outer trapping horizon increases its size as it ‘evaporates’. A fundamental cause of this arises from the sign-flip in equation (3.14) for a radial null vector $k^\mu$, which makes the non-GR-branch solutions quite eccentric. But we have not explicitly shown whether this sign change is special to radial null vectors or an artifact of our spacetime ansatz (2.6).

We also investigated black-hole dynamics. In four-dimensional general relativity, trapping horizons exhibit laws analogous to black-hole thermodynamics. Since their derivation made full use of the Einstein equations, it is nontrivial in other theories. But as shown in [27], the unified first law strongly suggests the first law of a trapping horizon in Einstein–Gauss–Bonnet gravity. Taking the Kodama vector as an analog of the null generator of a Killing horizon, we defined the surface gravity of a trapping horizon by following in the footsteps of a Killing horizon. The first law of a trapping horizon states that the energy inflow across the trapping horizon is compensated for by the entropy gain. The resultant dynamical black-hole entropy does not coincide with one quarter of its area (in $G_a = 1$ units), again reproducing Iyer–Wald’s result. The disagreement with the general relativistic case causes some annoying but interesting issues. For the $k = -1$ case, for example, the entropy can be negative. More interestingly, the entropy of a future outer trapping horizon in the non-GR branch is non-decreasing while its area is non-increasing. This is preferable in view of the second law of black-hole thermodynamics. But the question remains open whether the generalized second law holds in this system.

It should also be observed that the null trapping horizon represents equilibrium configurations. Under the null energy condition, its area and entropy are invariant in time, and moreover energy inflow through the horizon is absent as we have shown below equation (3.11).

We excluded from our consideration the trapping horizons coincident with the branch points. These trapping horizons occur for $k = -1$, and the areal radius takes its minimal (maximal) value $r_h = \sqrt{2\alpha}$ in the GR (non-GR) branch. Since the left-hand side of equation (3.10) becomes identically zero, the following propositions could not be established. We may truncate these exceptional trapping horizons from our analysis if the null energy conditions are strictly satisfied, because equation (3.14) shows that these trapping horizons are singular. But under other circumstances, we have no definite answer. The only fact we have at present is that, from equation (3.36), the entropy remains constant along the generator of these trapping horizons, owing to the vanishing of $\psi_a$ at that point (see equation (3.29)). But we have not understood so far how much generality and physical significance these trapping horizons have.

We conclude this paper by commenting on the generalization of the present work. When we do not assume the present spacetime symmetries (2.6), a more general definition of quasi-local mass is required. A naive prescription is to generalize the Hawking mass [59], which should satisfy monotonicity and positivity, and represent the higher-dimensional ADM mass at spatial infinity in the asymptotically flat case. And moreover, this generalization should satisfy the energy balance law as equation (3.24). We envisage that the extra work terms due to the gravitational radiation should be accompanied in equation (3.33), as in the general relativistic case [14, 15]. These speculations are challenging but interesting issues for future investigations. Our analysis should be helpful to those pursuing general properties of trapping horizons in Einstein–Gauss–Bonnet gravity. Extensions into Lovelock gravity [60] are also an intriguing subject. We anticipate that the analysis in the present paper can be extended in Lovelock gravity and also our quasi-local mass formalism will be of use in analyzing the characteristic singularity structure [28, 43, 61]. These prospects are left for possible future investigations.
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