On Huisman’s conjectures about unramified real curves

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Abstract
Let $X \subset \mathbb{P}^n$ be an unramified real curve with $X(\mathbb{R}) \neq \emptyset$. If $n \geq 3$ is odd, Huisman \cite{9} conjectured that $X$ is an $M$-curve and that every branch of $X(\mathbb{R})$ is a pseudo-line. If $n \geq 4$ is even, he conjectures that $X$ is a rational normal curve or a twisted form of a such. Recently, a family of unramified $M$-curves in $\mathbb{P}^3$ that serve as a counterexample to the first conjecture was constructed in \cite{11}. In this note we construct another family of counterexamples that are not even $M$-curves. We remark that the second conjecture follows for generic curves of odd degree from the de Jonquières formula.

Keywords: real algebraic curve, $M$-curve, ramification.

1. Introduction

In this note, a real (algebraic) curve $X$ is assumed to be smooth, geometrically integral, embedded into the real projective space $\mathbb{P}^n$ for some $n \in \mathbb{N}^*$, and such that the set of real points $X(\mathbb{R})$ is nonempty. Since we assumed the set of real points $X(\mathbb{R})$ to be nonempty, it inherits the structure of an analytic manifold, which decomposes into a finite number of connected components, which are called (real) branches of $X$. We call such a branch an oval if its homology class in $H_1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2)$ is trivial, and a pseudo-line otherwise. By Harnack’s Inequality \cite{5}, we know $s \leq g + 1$, where $s$ is the number of branches and $g$ is the genus of $X$. We say that $X$
is an $M$-curve if it has the maximum number of branches, i.e., if $s = g + 1$. The curve $X$ is called nondegenerate if there is no real hyperplane $H \subset \mathbb{P}^n$ such that $X \subset H$. A nondegenerate curve $X$ is called unramified if, taken any real hyperplane $H$, we have

$$\text{wt}(H \cdot X) \leq n - 1,$$

whereby the weight of the intersection divisor $H \cdot X$ is defined to be

$$\text{wt}(H \cdot X) := \deg(H \cdot X - (H \cdot X)_{\text{red}}),$$

i.e., the degree of the difference between the latter and the reduced divisor (which contains each point of $H \cap X$ with multiplicity exactly one). Otherwise, it is called ramified. For example, if $X \subset \mathbb{P}^2$ is a plane curve, two different types of ramification can occur: Bitangents, lines that are tangent to the curve in two different points, and flex lines, lines that intersect the curve in some point with multiplicity (at least) three. A result due to Klein [10] implies that the numbers $f$ and $b$ of real flex lines and real bitangents respectively to a real plane curve of degree $d$ satisfy the inequality

$$f + 2b \geq d(d - 2).$$

In particular, every real plane curve of degree $d \geq 3$ is ramified. The analogous statement fails in odd-dimensional projective spaces: Huisman [9, Thm. 3.1] shows that in any $\mathbb{P}^n$ with $n \geq 3$ odd there are unramified $M$-curves of arbitrary large genus and degree. However, in the same article he makes the following conjecture:

**Conjecture 1 (Conjecture 3.4 in [9]).** Let $n \geq 3$ be an odd integer and $X \subset \mathbb{P}^n$ be an unramified real curve. Then $X$ is an $M$-curve and each branch of $X$ is a pseudo-line, i.e., it realizes the nontrivial homology class in $H_1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2)$. It was pointed out in [12, Rem. 2.7] that Conjecture 1 is false: The maximally writhed links constructed in [11] are instances of unramified $M$-curves in $\mathbb{P}^3$ but not all of them consist of pseudo-lines only.

We remark that Conjecture 1 has an interesting link to the following question on totally real divisor classes:

**Question 1.** Given a real curve $X$, determine the smallest natural number $N(X) \in \mathbb{N}^*$ such that any divisor of degree at least $N(X)$ is linearly equivalent to a totally real effective divisor, i.e., a divisor whose support consists of real points only.
It was shown by Scheiderer [15, Cor. 2.10] that such a number \(N(X)\) exists. Question \([1]\) was studied by Huisman [7] and Monnier [13], but it seems challenging to obtain results for curves with few branches. However, assuming Conjecture \([1]\) to be true, Monnier [13, Thm. 3.7] established a new bound for \(N(X)\) at least for \((M - 2)\)-curves (i.e., \(s = g - 1\)). We remark that Huisman [8] has shown Conjecture \([1]\) under more restrictive assumptions (namely nonspecial linearly normal curves having “many branches and few ovals”). The main objective of this note is to construct families of counterexamples to Conjecture \([1]\) living on the quadric hyperboloid in \(\mathbb{P}^3\) that are, in contrast to those from \([11]\), not even \(M\)-curves.

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2. **Unramified curves in three-space**

This section includes two explicit counterexamples and a method for constructing families of counterexamples. First, we present a simple idea for generating an unramified rational curve of degree 4 in the three-space. Since it has even degree, its single branch is an oval, and hence contradicts the second part of Conjecture \([1]\) though being an \(M\)-curve (as stated in the first part).

**Example 1.** The coefficients of the fourth power of a nonzero homogeneous real linear form in two variables define a rational normal curve

\[ \varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^4, [a : b] \mapsto [a^4 : 4a^3b : 6a^2b^2 : 4ab^3 : b^4]. \]

The real hyperplane \(H := \mathcal{V}((-2x_0 + x_1 - x_2 + x_3 - 2x_4))\) does not intersect the real image points of \(\varphi\); the intersection divisor

\[ H \cdot \text{im}(\varphi) = 2\rho^\sigma \]

with \(\rho = \varphi([e^{i\pi} : 1]^\sigma)\) consists of a complex conjugate pair with multiplicity two. The linear projection from the orthogonal complement of \(H\) defines a map

\[ \psi : \mathbb{P}^1 \rightarrow \mathbb{P}^3, [x : y] \mapsto [x^4 + 2x^3y : x^4 - 2x^2y^2 : x^4 + 2xy^3 : -x^4 + y^4]. \]
The image is a real curve in $\mathbb{P}^3$ of genus 0 and degree 4, hence the only branch is not a pseudo-line. The curve is indeed unramified, since intersecting with an arbitrary real hyperplane is equivalent to finding the homogeneous roots of a polynomial

$$
\lambda_0(x^4 + 2x^3y) + \lambda_1(x^4 - 2x^2y^2) + \lambda_2(x^4 + 2xy^3) + \lambda_3(-x^4 + y^4),
$$

which cannot be a single real point with multiplicity 4; thus the weight of the intersection with an arbitrary hyperplane is at most 2. The curve is cut out by a quadric of the Segre type and three cubics.

The following more general construction produces a series of unramified curves in $\mathbb{P}^3$ of any degree and any even genus. In contrast to [12, Rem. 2.7], we stress that these curves are constructed to be far away from being $M$-curves; they consist of exactly one branch.

**Construction 1.** Let $p, q \in \mathbb{R}[t]$ be strictly interlacing polynomials both of degree $d \in \mathbb{N}^*$, i.e., all complex zeros of $p$ and $q$ are real and between each two consecutive zeros of $p$ there is exactly one zero of $q$, see [14, §6.3]. The graph $Y$ of the fraction $p(t)/q(t)$ has the following shape in the real plane:

![Graph](image)

**Figure 1:** $p(t) = (t^2 - 1)(t - 3)$ and $q(t) = (t^2 - 4)t$

Let $X := \overline{Y}$ denote the Zariski closure of that graph in $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., $X$ is the image of the map

$$
\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, [x : y] \longmapsto ([x : y], [Q(x, y) : P(x, y)]),
$$
where \( P \) and \( Q \) denote the homogenizations of \( p \) and \( q \) (e.g., \( P(1,t) = p(t) \)). The curve \( X \) realizes the homology class \((1,d)\) in \( H_1(\Sigma,\mathbb{Z}) \) where \( \Sigma = (\mathbb{P}^1 \times \mathbb{P}^1)(\mathbb{R}) \). Indeed, since it is the graph of a function, any vertical real line intersects \( X \) in exactly one point and it follows for example from the Hermite–Kakeya Theorem [14, Thm. 6.3.8] that every real horizontal line intersects \( X \) in \( d \) simple real points. After we embed \( \mathbb{P}^1 \times \mathbb{P}^1 \) to \( \mathbb{P}^3 \) via the Segre embedding, the real intersection with a hyperplane has homology class \((1,1)\) (or \((1,-1)\)) and thus intersects \( X \) in

\[
\det \begin{pmatrix} 1 & 1 \\ 1 & d \end{pmatrix} = d - 1 \quad \text{or} \quad \det \begin{pmatrix} 1 & -1 \\ 1 & d \end{pmatrix} = d + 1
\]

different real points. In particular, the induced embedding of \( X \) to \( \mathbb{P}^3 \) is unramified. Thus up to now, we have constructed a family of examples of unramified, rational curves of degrees \( d + 1 \) in \( \mathbb{P}^3 \). Our curve from Example 1 also arises in that way for \( d = 3 \) (up to a change of coordinates). In order to obtain curves of positive genus, let \( G \) denote the Zariski closure of

\[
\bigcap_{j=1}^e (t^2 + t_j) \subset \mathbb{A}^2 \subset \mathbb{P}^1 \times \mathbb{P}^1
\]

for pairwise distinct positive \( t_j \) embedded as in the previous case via

\[
\mathbb{A}^2 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, (s,t) \longmapsto ([1:s],[1:t]).
\]

The union \( Z := X \cup G \) realizes the class \((2e+1,d)\) in the Picard group of \( \mathbb{P}^1 \times \mathbb{P}^1 \), i.e., it is defined by a bihomogeneous polynomial \( F \) of bidegree \((2e+1,d)\). We have \( Z(\mathbb{R}) = X(\mathbb{R}) \). We now observe that every real hyperplane \( H \) intersects \( Z \) in \( e \) different complex conjugate pairs, one for each pair of lines, as well as in at least \( d - 1 \) (for \( H \) of the type \((1,1)\)) or in exactly \( d + 1 \) (for \( H \) of the type \((1,-1)\)) real points. Thus \( H \) intersects \( Z \) in at least \( d + 2e - 1 \) different points. This remains true for any sufficiently small perturbation \( F_\epsilon \) of \( F \) with a smooth zero set \( Z_\epsilon \) and we can conclude that

\[
\deg(H \cdot Z_\epsilon)_{\text{red}} \geq d + 2e - 1,
\]

hence the weight of \( H \cdot Z_\epsilon \) is at most two for all real hyperplanes \( H \), i.e., \( Z_\epsilon \) is unramified. We further note that in \( \mathbb{P}^3 \) the curve \( Z_\epsilon \) has degree \( d + \)
2e + 1 and for ϵ sufficiently small it still realizes the homology class (1, d) in $H_1(\Sigma, \mathbb{Z})$. Finally, the constructed real curve has the one branch coming from $X$, which is an oval or pseudo-line depending on the parity of $d$, and no other branches. The genus is $2e \cdot (d - 1)$, see [6] Ch. V, Exp. 1.5.2.

**Remark 1.** By [11, Thm. 1], the rational curves of degrees $d + 1$ in $\mathbb{P}^3$ from above (i.e., before adding line-pairs) are maximally writhed knots. Thus for $e = 0$ our counterexamples agree with those from [12, Rem. 2.7]. Letting $d = 1$ and $e > 1$ we obtain a family of unramified rational curves of even degree that are not maximally writhed.

**Remark 2.** Each of our examples $Z_\epsilon$ from Construction 1 intersects any real horizontal line only in real points. This implies that we obtain a morphism $f : Z_\epsilon \rightarrow \mathbb{P}^1$ which is *totally real* in the sense that $f^{-1}(\mathbb{P}^1(\mathbb{R})) = Z_\epsilon(\mathbb{R})$. This implies that $Z_\epsilon$ is of *type I* in the sense that $Z_\epsilon(\mathbb{C}) \setminus Z_\epsilon(\mathbb{R})$ is not connected. We don’t know whether there are also counterexamples to Conjecture 1 that are not of type I.

**Remark 3.** We remark that the counterexamples of positive genus do not stay unramified after changing the base field to $\mathbb{C}$. In fact, the only nondegenerate curves that are unramified over $\mathbb{C}$ are the rational normal curves. This follows from the Plücker formulas, see [1, Exc. I.C-14].

**Example 2.** We consider the case $d = 2$ and $e = 1$. In this case our resulting curve $Z_\epsilon$ has genus two and degree five. In particular, the curve is embedded to $\mathbb{P}^3$ via a complete and nonspecial linear system. In the light of [8, Thm. 2], which says that Conjecture 1 holds for nonspecial rational and elliptic curves, this is a minimal counterexample with respect to the genus and being nonspecial.

Finally, we compute explicit equations for the case $d = 3$ and $e = 1$.

**Example 3.** For the following computations we used Macaulay2 [3]. We take $p(t) = (t^2 - 1)(t - 3)$ and $q(t) = (t^2 - 4)t$; the graph of the fraction is shown in Figure 1. The closure in $\mathbb{P}^1 \times \mathbb{P}^1$ consists of points coming from $\mathbb{A}^2$, i.e., from $([1 : t], [1 : p(t)/q(t)])$ for $t \in \mathbb{A}^1 \setminus \{-2, 0, 2\}$, together with four points at infinity connecting appropriate end points of the graph in $\mathbb{A}^2$. In the real projective three-space, the ideal of the resulting curve $X$ is defined by
the Segre quadric and three cubics. The closure of the complex conjugate pair of lines (setting $e = 1$ and $t_1 = 1$) in $\mathbb{P}^3$ is

$$G := \mathcal{V}_4 \left( \langle x_0 x_3 - x_1 x_2, x_0^2 + x_1^2, x_2^2 + x_3^2 \rangle \right).$$

The intersection of the ideal of $X$ with the ideal of $G$ is the ideal generated by $q := x_0 x_3 - x_1 x_2$ and the following polynomial (denoted by $h$ henceforth):

$$3x_0^3 + 3x_0 x_1^2 - x_0^2 x_2 - 3x_0 x_2^2 + x_2^3 + 4x_0^2 x_3 - x_0 x_1 x_3 + 4x_1^2 x_3 - x_2^2 x_3 - 3x_0 x_3^2 + x_2 x_3^2 - x_3^3.$$

We deform $h$ by $p := x_0^3 + x_1^3 + x_2^3 - x_3^3$. The result $Z_{\epsilon} := \mathcal{V}_4 (q, h + \epsilon p)$ for a small $\epsilon$ is an explicit example of an unramified curve of degree six, genus four, i.e., a canonical curve, with exactly one oval and of type I. Note that an unramified canonical curve of genus four must be of type I with one oval because otherwise it would have a real tritangent plane [4, Prop. 5.1].

![Figure 2: The curve $Z_{10-5}$ living on the hyperboloid](image)

3. Unramified curves in even-dimensional spaces

For the sake of completeness, we remark that Huisman [9] has also studied embeddings of $M$-curves into even-dimensional projective spaces. For certain such embeddings he computes the exact number of real inflection points which turns out to be nonzero for positive genus. Here by an inflection point of a curve $X \subset \mathbb{P}^n$ we mean a point $P \in X$ for which there is a hyperplane $H$ intersecting $X$ in $P$ with multiplicity at least $n + 1$. In particular, a curve having an inflection point is ramified. In his article, he conjectures the following general statement:
Conjecture 2 (Conjecture 4.6 in [9]). Let $n \geq 4$ be an even integer and $X \subset \mathbb{P}^n$ be an unramified real curve. Then $X$ is a rational normal curve or a twisted form of a rational normal curve (i.e., after changing the base field to $\mathbb{C}$, the curve $X_\mathbb{C}$ is a rational normal curve).

Again, we remark that Huisman [5] has shown Conjecture 2 under more restrictive assumptions (namely nonspecial linearly normal curves having “many branches and few ovals”).

Remark 4. We would like to point out that this conjecture is in fact true for generic curves of odd degree. Indeed, by the de Jonquières formula [11], p. 359] a generic nondegenerate curve in $\mathbb{P}^{2n}$ of degree $2d + 1$ and genus $g$ has

$$(2d + 1) \cdot (2n + 1) + 2n \cdot (2n + 1) \cdot (g - 1)$$

complex inflection points. Since this is clearly an odd number, there must be a real inflection point. Therefore, the curve is ramified.

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