The extended 1-perfect trades in small hypercubes

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Abstract

An extended 1-perfect trade is a pair \((T_0, T_1)\) of two disjoint binary distance-4 even-weight codes such that the set of words at distance 1 from \(T_0\) coincides with the set of words at distance 1 from \(T_1\). Such trade is called primary if any pair of proper subsets of \(T_0\) and \(T_1\) is not a trade. Using a computer-aided approach, we classify nonequivalent primary extended 1-perfect trades of length 10, constant-weight extended 1-perfect trades of length 12, and Steiner trades derived from them. In particular, all Steiner trades with parameters \((5, 6, 12)\) are classified.

Keywords: trades, bitrades, 1-perfect code, Steiner trades, small Witt design

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1. Introduction

Trades of different types are used to study, construct, and classify different kinds of combinatorial objects (codes, designs, matrices, tables, etc.). Trades are also studied independently, as some natural generalization of objects of the corresponding type. In the current paper, we classify small (extended) 1-perfect binary trades. The 1-perfect trades are objects related to 1-perfect codes (perfect codes with distance 3). A 1-perfect code is a set \(C\) of vertices of a graph such that \(|C \cap B| = 1\) for every ball \(B\) of radius 1. A 1-perfect trade is a pair \((T_0, T_1)\) of disjoint vertex sets of a graph such that \(|T_0 \cap B| = |T_1 \cap B| \in \{0, 1\}\) for every ball \(B\) of radius 1. Formally, the 1-perfect trades generalize the pairs of disjoint 1-perfect codes; in some cases, for every 1-perfect code \(C\) a disjoint mate \(C'\) can be explicitly constructed, e.g., \(C' = C + 100...0\) in a Hamming space.

In the theory of 1-perfect codes, trades play an important role for the construction of codes with different properties and the evaluation of their number. There are not so many works where the class of 1-perfect trades is studied independently [23], [30], [31]; however, the subsets of 1-perfect codes called i-components, or switching components, which are essentially a special kind of trade mates, are used in many constructions of such codes, see the surveys in [2], [9], [20], [25], [26].

In the binary case, when the graph is the \(n\)-cube, 1-perfect codes exist if and only if \(n + 1\) is a power of two (see, e.g., [13]), while 1-perfect trades exist for every odd \(n\) [31] (it can be easily established by the local analysis that the size \(n + 1\) of a ball must be even if a 1-perfect trade exists; see another explanation of this in Section 2.4). The last fact allows to consider recursive approaches in constructing and studying trades and to collect some experimental material for small \(n\). Note that one of the standard approaches to study 1-perfect binary codes is to consider extended 1-perfect codes, taking into account a natural bijection between these two classes; in this paper, we also favor the framework of extended 1-perfect codes and, respectively, extended 1-perfect trades.

As a part of the study of properties of the classified trades, we consider the connection between 1-perfect trades and Steiner trades. The Steiner trades are well known in the theory of combinatorial designs, and there is a lot of literature on this topic, see the surveys [8], [29]. All vertices of a 1-perfect trade at minimum
distance from some fixed “non-trade” vertex form a Steiner trade; we consider the question which Steiner trades can be derived from 1-perfect trades in such a way.

The paper is organized as follows. After the definitions (Section 2) and the preliminary results (Section 3), we classify primary extended 1-perfect trades of length 8. The next three sections are devoted to computational results. In Section 4 we classify the extended 1-perfect trades of length 10; in Section 5 we classify the constant-weight extended 1-perfect trades of length 12. The lists of trades are given in tables, together with some additional information (automorphism group, dual space, derived Steiner trades, connection with the Witt design). In Section 7 we describe a concatenation construction of extended 1-perfect trades, showing that small trades can be utilized to construct trades of larger lengths; the construction also demonstrates the connection of the 1-perfect trades with the latin trades, which are widely studied in the theory of latin squares [6] and latin hypercubes [24].

In our computer-aided classification, we used general principles described in [12]. The programs were written in c++ (early versions used sage [28]; nauty [19] was used to deal with automorphisms and isomorphisms.

2. Definitions

We consider simple graphs $G = (V(G), E(G))$. The distance $d(x, y)$ between two vertices $x$ and $y$ of a connected graph is defined as the minimum length of a path connecting $x$ and $y$. Two sets $C$ and $S$ of vertices of a graph are equivalent if there is an automorphism $\pi$ of the graph such that $\pi(C) = S$. Two pairs $(C_0, C_1)$ and $(S_0, S_1)$ of sets of vertices of a graph are equivalent if there is an automorphism $\pi$ of the graph such that either $\pi(S_0) = C_0$ and $\pi(S_1) = C_1$, or $\pi(S_0) = C_1$ and $\pi(S_1) = C_0$. The automorphism group, denoted $\text{Aut}(C)$, of a vertex set $C$ is defined as its stabilizer in the graph automorphism group.

2.1. Hamming graphs, halved $n$-cubes, and Johnson graphs

The Hamming graph $H(n, q)$ (if $q = 2$, the $n$-cube $H(n)$) is a graph whose vertices are the words of length $n$ over the alphabet $\{0, \ldots, q-1\}$, two words being adjacent if and only if they differ in exactly one position. The weight $\text{wt}(x)$ of a word $x$ is the number of nonzeros in $x$.

The halved $n$-cube $\frac{1}{2}H(n)$ is a graph whose vertices are the even-weight (or odd-weight) binary words of length $n$, two words being adjacent if and only if they differ in exactly two positions.

The Johnson graph $J(n, w)$ is a graph whose vertices are the weight-$w$ binary words of length $n$, two words being adjacent if and only if they differ in exactly two positions.

It is known (see, e.g., [4] Th. 9.2.1, [4] p. 265, [4] Th. 9.1.2) that any automorphism of $H(n)$, $\frac{1}{2}H(n \geq 5)$, or $J(n, w)$ is a composition of a coordinate permutation and a translation to some binary word $x$, which is arbitrary in the case of $H(n)$, even-weight for $\frac{1}{2}H(n)$, the all-zero $0^n$ or all-one $1^n$ for $J(2w, w)$, and only $0^n$ for $J(n, w)$, $n \neq 2w$. For a vertex set $C$, in addition to $\text{Aut}(C)$, we will use the notation $\text{Sym}(C)$, which denotes the set of all coordinate permutations that stabilize $C$.

The Hamming distance $d_H(x, y)$ between two words $x$ and $y$ of the same length is the number of coordinates in which $x$ and $y$ differ, i.e., the distance in the corresponding Hamming graph. Note that the graph distance in a Johnson graph or the halved $n$-cube is the half of the Hamming distance: $d(x, y) = d_H(x, y)/2$.

2.2. 1-perfect codes, extended 1-perfect codes, Steiner systems, latin hypercubes

A 1-perfect code is a set of vertices of $H(n)$ such that every radius-1 ball contains exactly one codeword.

An extended 1-perfect code is a set of vertices of $\frac{1}{2}H(n)$ such that every maximum clique contains exactly one codeword. Note that the maximum cliques in $\frac{1}{2}H(n)$, $n \geq 5$, are radius-1 spheres in $H(n)$. There is a one-to-one correspondence between the 1-perfect codes in $H(n - 1)$ and the extended 1-perfect codes in $\frac{1}{2}H(n)$: if, for some fixed $i \in \{1, \ldots, n\}$ (to be explicit, take, e.g., $i = n$), we delete the $i$th symbol from all codewords of an extended 1-perfect code, then the resulting set will be a 1-perfect code (inversely, the deleted symbol can be uniquely reconstructed as the modulo-2 sum of the other symbols). Extended 1-perfect codes in $\frac{1}{2}H(n)$ and 1-perfect codes in $H(n - 1)$ exist if and only if $n$ is a power of 2.
A Steiner $k$-tuple system $S(k−1, k, n)$ is a set of vertices of $J(n, k)$, $n \geq 2k$, such that every maximum clique contains exactly one word from the set. $S(2, 3, n)$ and $S(3, 4, n)$ are known as STS($n$) and SQS($n$), Steiner triple systems and Steiner quadruple systems, respectively. Note that every maximum clique in $J(n, k)$ consists of all weight-$k$ binary words of length $n$ adjacent in $H(n)$ with a given word of weight $k−1$ (in the case $n = 2k$, of weight $k−1$ or $k+1$).

There is a well-known connection between 1-perfect codes and Steiner systems. If a 1-perfect (extended 1-perfect) code of length $n$ contains the all-zero word $0^n$, then its weight-$3$ (weight-$4$, respectively) codewords form a STS($n$) (SQS($n$)), which is called derived from the code. However, in contrast to the 1-perfect codes, STS($n$)s exist for all $n \equiv 1, 3$ mod $6$, and SQS($n$)s exist for all $n \equiv 2, 4$ mod $6$. Clearly, such STS or SQS cannot be derived if $n+1$ (respectively, $n$) is not a power of $2$. The question if there exist non-derived STS($n$) (SQS($n$)) when $n+1$ (respectively, $n$) is a power of $2$ is a known open problem, which is solved only for $n \leq 16$ [21], [11, Satz 8.5].

The parameters $S(5, 6, 12)$ play a special role in our classification. A sextuple system $S(5, 6, 12)$ found in [6] and [32] is unique up to equivalence [32] and known as the small Witt design.

A latin hypercube (if $n = 3$, a latin square) is a set of vertices of $H(n, q)$ such that every maximum clique contains exactly one word from the set. A maximum clique of $H(n, q)$ consists of $q$ words differing in only one coordinate. Often, a latin hypercube is imagined an $(n−1)$-dimensional table of size $q \times \ldots \times q$ filled by the values of the last, $n$th, coordinate.

2.3. 1-perfect, extended 1-perfect, Steiner, and latin trades

A 1-perfect trade is a pair $(T_0, T_1)$ of disjoint nonempty sets of vertices of $H(n)$ such that for every radius-1 ball $B$ it holds

$$|T_0 \cap B| = |T_1 \cap B| \in \{0, 1\}. \quad (1)$$

An extended 1-perfect trade (an $S(k−1, k, n)$ trade, a latin trade) is a pair $(T_0, T_1)$ of disjoint nonempty sets of vertices of $H(n)$ (of $J(n, k)$ with $n \geq 2k$, of $H(n, q)$, respectively) such that (1) holds for every maximum clique $B$.

In what follows, trade always means one of the four considered types of trades. Each component $T_i$ of a trade $(T_0, T_1)$ is called a trade mate.

**Remark 1.** Often, trades are defined as unordered pairs $\{T_0, T_1\}$. In this paper, however, we find it convenient to use the ordered version of the definition. Also, it should be noted that there is a different terminology in the literature (especially, in the works on latin trades, see, e.g., [6]), where $(T_0, T_1)$ is called a bitrade, and each of $T_0, T_1$ is called a trade.

The volume of a trade $(T_0, T_1)$ is the cardinality of $T_0$ (equivalently, of $T_1$, as (1) implies $|T_0| = |T_1|$). The length of $(T_0, T_1)$ means the length of words $T_0$ and $T_1$ consist of. A trade $(T_0, T_1)$ is called primary if it cannot be partitioned into two trades $(T_0^0, T_1^0)$ and $(T_0^1, T_1^1)$, $T_0 = T_0^0 \cup T_0^1$, $T_1 = T_1^0 \cup T_1^1$. The role of trades in the study of (extended) 1-perfect codes, Steiner systems, latin squares and hypercubes is emphasized by the following fact: if $C_0$ and $C_1$ are different 1-perfect codes, extended 1-perfect codes, Steiner $k$-tuple systems, or latin hypercubes with the same parameters, then $(T_0, T_1)$, where $T_i = C_i \setminus C_{1−i}$, is a trade of the corresponding type. In particular, we have $C_1 = T_1 \cap C_0 \setminus T_0$, i.e., with a trade we can get one object from the other.

Extended 1-perfect trades and 1-perfect trades are in the same one-to-one correspondence as extended 1-perfect codes and 1-perfect codes. If $(T_0, T_1)$ is a 1-perfect or extended 1-perfect trade and $x \notin T_0, T_1$ is a word at distance $k$ from $T_0$, then the weight-$k$ words of $T_0 + x$ and $T_1 + x$ form an $S(k−1, k, n)$ trade, called derived from $(T_0, T_1)$. Latin trades can be used for the construction of trades of other types (see Section 7), the only place in this paper where the latin trades appear.

For trades consisting of words of weight $n/2$ (we call them constant-weight trades), there is the following simple but remarkable correspondence.

**Proposition 1.** A pair $(T_0, T_1)$ of weight-$k$ binary words of length $2k$ is a $S(k−1, k, 2k)$ trade if and only if it is an extended 1-perfect trade.
Proof. Let $G$ be the set of vertices of $J(2k,k)$, and also a subset of the vertex set of $\frac{1}{2}H(2k)$. For every maximum clique $B$ in $\frac{1}{2}H(2k)$, the set $B \cap G$ is either empty or a maximum clique in $J(2k,k)$. Moreover, every maximum clique in $J(2k,k)$ is represented in such a way. Trivially, $|T_i \cap B| = |T_i \cap (B \cap G)|$ holds for every subset $T_i$ of $G$. So, for such subsets, the definitions of $S(k-1,k,2k)$ trades and extended 1-perfect trades are equivalent. □

A tuple $(T_0, \ldots, T_{k-1})$ of $k \geq 2$ sets is called a $k$-way trade if every two different sets from it form a trade. The concepts defined above for the trades (length, volume, primary, derived) and Proposition 4 are naturally extended to $k$-way trades.

2.4. Characteristic functions

Via the characteristic functions, the trades of the considered types can be represented as eigenfunctions of the corresponding graphs with some special discrete restrictions. This allows the trades to be studied using approaches of algebraic combinatorics, see, e.g., [10]. An eigenfunction of a graph $G = (V,E)$ corresponding to an eigenvalue $\theta$ is a real-valued function $f$ over $V$ that satisfies the equation $\theta f(x) = \sum_{y \in \{y,x\} \in E} f(y)$ for every $x \in V$. The eigenvalues of the Hamming, Johnson, and halved $n$-cube graphs can be found, e.g., in [2], Th. 9.2.1], [3], Th. 9.1.2], [2], p. 264).

We define the characteristic function of a trade $(T_0,T_1)$ as the $\{0,1,-1\}$-function $\chi(T_0,T_1) \overset{\text{def}}{=} \chi_{T_0} - \chi_{T_1}$, where $\chi_T$ denotes the characteristic $\{0,1\}$-function of a vertex set $T$.

It is straightforward that the characteristic function of a 1-perfect, extended 1-perfect, $S(k-1,k,n)$, or latin trade is an eigenfunction of the corresponding graph with the eigenvalue $-1$, $-n/2$, $-n$, respectively. Moreover, the characteristic function of an extended 1-perfect trade, considered as a function over the vertex set of $H(n)$, is an eigenfunction of $H(n)$ with the eigenvalue 0.

The graph $H(n)$ has an eigenvalue $-1$ ($0$) if and only if $n$ is odd (even, respectively). This gives a necessary condition for the existence of 1-perfect trades (extended 1-perfect trades, respectively), which turns out to be sufficient, see, e.g., [30].

For the other two types of trades, there are no restrictions on the parameters: $-k$ is the smallest eigenvalue of all $J(n,k)$s, $n \geq 2k$; and $-n$ is the smallest eigenvalue of all $H(n,q)$s. For all parameters, $S(k-1,k,n)$ trades and latin trades exist, see, e.g., [3], [2].

2.5. The rank and the dual space

The rank is one of the characteristics of nonlinear codes that say how far a code is from being linear. In the theory of 1-perfect codes, the concept of rank plays an important role; the structure of arbitrary 1-perfect codes of small rank was studied in [1], [11]. In the current paper, we use the affine rank of binary codes, which is invariant under the automorphisms of the $n$-cube.

We consider the set of all binary words of length $n$ as a vector-space $F^n$ over the finite field of order 2, with coordinate-wise modulo-2 addition and multiplication by a constant.

Let $C$ be a set of binary words of length $n$ (a code). A binary word $x = (x_0, \ldots, x_{n-1})$ is said to be orthogonal (antiorthogonal) to $C$ if $x_0c_0 + \ldots + x_{n-1}c_{n-1} \equiv 0 \mod 2$ (respectively, $\equiv 1 \mod 2$) for all $(c_0, \ldots, c_{n-1})$ from $C$. The set of all binary words that are orthogonal or antiorthogonal to $C$ is called the dual space of $C$ (in affine sense) and denoted $C^\perp$. The (affine) rank, rank$(C)$, of a code $C \subseteq F^n$, is the minimum dimension of an affine subspace of $F^n$ including $C$. It is straightforward that rank$(C) + \text{rank}(C^\perp) = n$.

3. Preliminary results

The following two facts, Lemma 1 and its corollary, are well known.

Lemma 1. Let $\phi$ be an eigenfunction of $H(n)$ with the eigenvalue $n-2i$, $i \in \{0,1,\ldots,n\}$. For every vertex $x$ of $H(n)$, it holds $\phi(x + 1^n) = (-1)^i\phi(x)$, where $1^n$ is the all-one word.
Suppose Lemma 2 \cite{30}.

Let \((\sigma, k) > 0\), we conclude from Lemma 1 that such a trade is self-complementary, in the following sense.

Hence, they form a basis of the space of real-valued functions on the vertex set of \(H(n)\). Consequently, any eigenfunction \(\phi\) with the eigenvalue \(n - 2i\) is a linear combination of \(\phi_y\) with \(wt(y) = i\). So, it is sufficient to check that the statement of the lemma holds for every such \(\phi_y\):

\[
\phi_y(x + 1^n) = (-1)^{\sum_j y_j(x_j + 1) + \sum_j x_j + y_j(x_j + 1)} \neq (-1)^{\sum_j y_j + \sum_j y_j} \neq (-1)^i \phi_y(x).
\]

Since the characteristic function of any 1-perfect or extended 1-perfect trade (in particular, any \(S(k-1, k, 2k)\) trade, by Proposition \cite{10}) is an eigenfunction of the Hamming graph corresponding to the eigenvalue \(-1\) or 0 (see Section 2.2), we conclude from Lemma \cite{10} that such a trade is self-complementary, in the following sense.

**Corollary 1.** Let \((T_0, T_1)\) be a 1-perfect trade in \(H(n-1)\) or an extended 1-perfect trade in \(\frac{1}{2} H(n)\). Denote \(T_j \equiv T_j + 11\ldots = \{x + 11\ldots | x \in T_j\}, j = 0, 1\).

(i) If \(n \equiv 0 \mod 4\), then \(T_0 = T_0\) and \(T_1 = T_1\).

(ii) If \(n \equiv 2 \mod 4\), then \(T_0 = T_1\).

We see that there is an essential difference between the two cases, (i) and (ii). In case (ii), \(T_1\) is uniquely determined from \(T_0\). They are complementary to each other and thus equivalent. In case (i), each element of a trade \((T_0, T_1)\) is self-complementary, but \(T_0\) does not uniquely define \(T_1\) and vice versa; \(T_0\) and \(T_1\) can be nonequivalent in this case (see Section 5.1 for examples). In fact, for every \(n \equiv 1 \mod 4\), there exists a 3-way 1-perfect trade \cite{30}. For \(k > 3\), \(k\)-way trades can also exist.

The next lemma (a partial case of the construction in Section 7) guarantees that the number of different (and, in fact, the number of nonequivalent) 1-perfect trades is monotonic in \(n\). Here and everywhere in the paper, for a symbol \(\sigma\) and a set \(T\) of words, by \(T\sigma\) we denote the set of words obtained by appending \(\sigma\) to the words of \(T\) (for example, in the notation \(T_{00}\) below, this rule is applied twice).

**Lemma 2 \cite{30}**. Let \((T_0, T_1)\) be a (extended) 1-perfect trade in \(H(n)\). Then \((T_{00} \cup T_{11}, T_{01} \cup T_{10})\) is a (extended) 1-perfect trade in \(H(n+2)\).

The following easy-to-prove fact plays a crucial role in our classification algorithm.

**Proposition 2 \cite{16, Theorem 1}**. Suppose \(T_0, T_1\) are disjoint vertex sets of \(\frac{1}{2} H(x)\) (or \(J(n, k)\)). The pair \((T_0, T_1)\) is an extended 1-perfect trade (an \(S(k-1, k, n)\) trade, respectively) if and only if the subgraph induced by \(T_0 \cup T_1\) is bipartite with parts \(T_0, T_1\) and regular of degree \(n/2\) (of degree \(k\), respectively).

**Corollary 2.** An extended 1-perfect trade \((T_0, T_1)\) in \(\frac{1}{2} H(n)\) is primary if and only if the corresponding induced subgraph is connected. The same is true for the \(S(k-1, k, n)\) trades.

The next lemma is convenient for representing the dual space of a primary trade by a basis. Note that, for a primary extended 1-perfect or primary \(S(k-1, k, n)\) trade \((T_0, T_1)\), the union \(T_0 \cup T_1\) induces a connected subgraph of \(\frac{1}{2} H(n)\).

**Lemma 3.** Let \(C\) be a vertex set of \(\frac{1}{2} H(n)\) such that the induced subgraph is connected. Then the dual space of \(C\) is closed with respect to the coordinate-wise multiplication.

**Proof.** Let \(x, y \in C^\perp\), and let \(c\) and \(d\) be codewords of \(C\) differing in exactly two coordinates, \(i\) and \(j\). Denote by \(z\) the coordinate-wise multiplication of \(x\) and \(y\). Since \(x\) is orthogonal or antithogonal to \(\{c, d\}\), we have \(x_i = x_j\). Similarly, \(y_i = y_j\). It follows that \(z_i = z_j\), and we see that \(z\) is orthogonal or antithogonal to \(\{c, d\}\). From the connectivity, we get that \(z\) is orthogonal or antithogonal to \(C\). \(\square\)

Any space closed with respect to the coordinate-wise multiplication can be represented by the standard basis whose elements have mutually disjoint sets of non-zero coordinates.
4. Extended 1-perfect trades in \( \frac{1}{2}H(8) \)

As noted in Section 2, 4 extended 1-perfect trades in \( \frac{1}{2}H(n) \) exist only if \( n \) is even. An example of a trade can be constructed recursively, starting with the trivial trade \( \{\{00\}, \{11\}\} \) and applying the construction in Lemma 2. Before we start our classification for \( n = 8 \), we note that this is the first case when nonequivalent extended 1-perfect trades exist. Indeed, the case \( n = 2 \) is trivial. In the case \( n = 4 \), by Corollary 4 every trade mate includes a self-complementary pair of vertices; since every maximum clique intersects with such a pair, the volume is 2. If \( (T_0, T_1) \) is an extended 1-perfect trade in \( \frac{1}{2}H(6) \), then we can assume without loss of generality that 000000 \( \in T_0 \) and, in accordance with Proposition 2, 000011, 001100, 110000 \( \in T_1 \); then, by Corollary 4, \( T_0 = \{000000, 111100, 110011, 001111\} = T_1 + 1^6 \).

Now consider three extended 1-perfect codes of length 8, \( C_0 = (00001111, 00110011, 01010101, 11111111) \), \( C_1 = (10000111, 00110011, 01010101, 11111111) \), \( C_2 = (00001111, 00110011, 01010110, 11111111) \), where \( \langle \ldots \rangle \) denotes the linear span over the finite field of order 2. It is not difficult to check that \( (C_0 \setminus C_1, C_1 \setminus C_0) \), \( (C_0 \setminus C_2, C_2 \setminus C_0) \), and \( (C_1 \setminus C_2, C_2 \setminus C_1) \) are constant-weight extended 1-perfect trades of volume 8, 12, and 14, respectively. As we see from the following theorem, all nonequivalent primary extended 1-perfect trades are exhausted by these three constant-weight trades and two trades of volume 16 (each consisting of two extended 1-perfect codes).

Theorem 1. There are only 5 nonequivalent primary extended 1-perfect trades in \( \frac{1}{2}H(8) \), of volume 8, 12, 14, 16 and 16, respectively.

Proof. Step 1. Without loss of generality, we can assume that 00000000 \( \in T_0 \) and \( v_1 = 11000000 \), \( v_2 = 00110000 \), \( v_3 = 00001100 \), \( v_4 = 00000011 \in T_1 \). By Corollary 4, we also have \( 1^8 \in T_0 \) and \( v_1 + 1^8 \), \( v_2 + 1^8 \), \( v_3 + 1^8 \), \( v_4 + 1^8 \in T_1 \).

Step 2. Now consider the word \( v_1 = 11000000 \) from \( T_1 \). We know one of its neighbors in \( T_0 \), \( 0^8 \). There are 15 ways to choose the set \( \{x, y, z\} \) of three words adjacent to \( V_1 \) and not adjacent mutually and with \( 0^8 \) (indeed, \( v_1 + 0^8, v_1 + x, v_1 + y, v_1 + z \) must be weight-2 words with mutually disjoint sets of ones). Without loss of generality, it suffices to consider only three of them (each of the other cases can be obtained from these three by applying one of \( 4! \cdot 2^4 \) coordinate permutations stabilizing the collection of words chosen at Step 1):

- (a) In this case, no more words can be added to \( T_1 \) or \( T_2 \) as the subgraph generated by the set of chosen 16 vertices satisfies the condition of Proposition 2.
- (b) We already know all four \( T_0 \)-neighbors of \( v_2 = 00110000: 0^8, x, y + 1^8, z + 1^8 \). Consider \( v_3 = 00001100 \). We know two its neighbors from \( T_0 \): \( 0^8 \) and \( x = 00001111 \). The words 11001100 and 00111100 are adjacent to \( y \) and \( y + 1^8 \), respectively, and hence cannot belong to \( T_0 \). Consequently, either 10101100, 01101110 \( \in T_0 \), or 01011100, 01011100 \( \in T_0 \). Without loss of generality we consider the former case (all previously chosen words have coinciding values in the first two coordinates). Then, 01010011 and 10100011 are also in \( T_0 \) and in the neighborhood of \( v_4 \).
- (c) When \( x \) from \( T_0 \), we know its four neighbors from \( T_1 \): \( v_1, v_2, v_3 + 1^8, \) and \( v_4 + 1^8 \). Consider \( y = 11000101 \in T_0 \). We have 11000000, 11001111 \( \in T_1 \); the other two neighbors of \( y \) from \( T_1 \) are
  - (i) 10101010 and 01011010 or
  - (ii) 10011010 and 01101010

(note that the third subcase, 00001010 and 11111010, is not feasible).

In the first subcase, including also the complements, we get

\[
T_0 \supset \{00000000, 11111111, 11110000, 00001111, 11001010, 00110101,
11000101, 00011010, 10101000, 10100111, 10100011\},
\]
are two ways to choose the other two trade neighbors of 01011010; the complement is 10100101. At this stage, we have T_1 \supseteq \{11000000, 00110000, 00001100, 00000011, 10101010, 01010101, 01010101, 01010101, 11111110, 11111100\}.

We see that the words we have already form a trade; so, T_0 and T_1 do not contain more vertices. In the second subcase, including also the complements, we get

\[
T_0 \supseteq \{00000000, 11111111, 11110000, 00001111, 11001010, 00110101, 11000101, 00111010, 10101100, 01010011, 01011100, 10100011\},
\]
\[
T_1 \supseteq \{11000000, 00110000, 00001100, 00000011, 00111111, 11001111, 11110011, 11111110, 10011100, 01100101, 01101010, 10010101, 10010101, 10010101\}.
\]

Consider \(z = 10101100\). We know 00001100, 11111100 \(\subseteq T_1\). The other two neighbors of \(z\) in \(T_1\) can be 10100000, 10101111 (this way is not feasible as 11000000 \(\subseteq T_1\), 10101010, 10100101 (not feasible as 10011010 \(\subseteq T_1\)), or 10101001, 10100110, the only feasible way.

Similarly, considering 10011010 \(\subseteq T_1\), we find that 10011001, 10010110 \(\subseteq T_0\). Including also the complements, we have two extended 1-perfect codes:

\[
T_0 = \{00000000, 11111111, 11110000, 00001111, 11001010, 00110101, 11000101, 00111010, 10101100, 01010011, 01011100, 10100011, 10100101, 10100101, 10100101\},
\]
\[
T_1 = \{11000000, 00110000, 00001100, 00000011, 00111111, 11001111, 11110011, 11111100, 10011100, 01100101, 01101010, 10010101, 10010101, 10010101, 10010101\}.
\]

(c) 00000000, 11111111, 11101000, 11010101, 11000100, 00101101, 10011011, 00110110, 00111010 \(\subseteq T_0\).

Consider the word \(v_2 = 00110000 \subseteq T_1\) and its possible neighbors from \(T_0\). We know 00000000, 00111101 \(\subseteq T_0\). We see that 11110000 \(\not\subseteq T_0\) (because 11101000 \(\subseteq T_0\)). The remaining words are 10110100, 01110101, 10110001, 01110001. Without loss of generality, 10110100, 01110001 \(\subseteq T_0\). The complements 01001010 and 10000110 are also in \(T_0\).

Next, consider the neighborhood of \(v_3 = 00001100 \subseteq T_1\). We know 00000000, 00101101, 10001110 \(\subseteq T_0\); so, we deduce that 01011100 \(\subseteq T_0\). The complement 10100011 is the fourth neighbor of \(v_4\) in \(T_0\).

Now we know that

\[
T_0 \supseteq \{00000000, 11111111, 11101000, 11010010, 11000101, 00010111, 00101101, 00111010, 10110100, 01110001, 01010101, 10001110, 10100101, 10010101, 10010101, 10010101\}.
\]

Step 3 (case (c)). Consider the neighborhood of \(x = 11101000\). It contains \(v_1\) and \(v_4 + 1^8\) from \(T_1\). There are two ways to choose the other two trade neighbors of \(x\).

(i) 10101010 and 01101001 (and their complements 01010101 and 10010110) are in \(T_1\).

Then, \(y = 11010010\) is adjacent to \(v_1, v_3 + 1^8, 10010110\) from \(T_1\). So, the fourth trade neighbor is 01011010; the complement is 10110101. At this stage, we have

\[
T_1 \supseteq \{11000000, 00110000, 00001100, 00000011, 00111111, 11001111, 11111001, 11111100, 11111110, 10101010, 01101001, 01010101, 10010110, 10010101, 10100101\}
\]

and no more words can be added to \(T_0\) or \(T_1\). Then, \((T_0, T_1)\) is a trade of volume 14.

(ii) 01101010 and 10101001 (and their complements 10010101 and 10010110) are in \(T_1\).

Then, \(y = 11010010\) is adjacent to \(v_1, v_3 + 1^8, 10010110\) from \(T_1\). The fourth \(T_1\)-neighbor of \(y\) must be 10010110; the complement is 01101001. Next, 10110100 from \(T_0\) has neighbors \(v_2, v_4 + 1^8, 10010101\) from \(T_1\). The fourth neighbor from \(T_1\) must be 10100110; the complement is 01011001. Now, we know 16 words of \(T_1\). Since 16 is the cardinality of an extended 1-perfect code, no more words can be added. Two more words should be found in \(T_0\); it is not difficult to see that the only way is 01100110, 10011001. We
have two disjoint extended 1-perfect codes:

\[
T_0 = \{00000000, 11111111, 11101000, 11010010, 11000101, 00010111, 00101101, 00111010, \\
10110100, 01110001, 01001111, 10011110, 10100011, 01100110, 10011001, 01011001, 10101100, 01011001\},
\]

\[
T_1 = \{11000000, 00110000, 00001100, 00000011, 00111111, 11001111, 11110011, 11111100, \\
01101010, 10110100, 10010101, 01010110, 10011010, 01100101, 10100110, 01011001\}. \tag{3}
\]

It remains to show that the solutions (2) and (3) are nonequivalent. We count the number of words orthogonal to \(T_0 \cup T_1\) (it is easy to see that this number is invariant among equivalent codes). The unique, up to equivalence, extended 1-perfect code containing \(0^5\) is self-dual, that is, the set of all orthogonal words is the code itself. So, every orthogonal word belongs to \(T_0\); for each word of \(T_0\), it is easy to check if it is orthogonal to \(T_1\). As a result, for (2) and (3), the dual spaces are \(\{0^5, 1^5, 00011111, 11110000\}\) and \(\{0^5, 1^5\}\) respectively, which certifies that the corresponding trades are nonequivalent. \(\square\)

5. Extended 1-perfect trades in \(\frac{1}{2}H(10)\)

In this section, we describe a computer-aided classification of extended 1-perfect trades of length 10. The computation took a few seconds on a modern PC.

5.1. Algorithm

The algorithm described below is similar to the one used in the proof of Theorem 1. We omit the details concerning some natural improvement and show only the general approach. Essentially, it is the breadth-first search of a bipartite 5-regular induced subgraph of \(H(10)\) that takes into account the complementarity. Below, we consider \(T_0\) and \(T_1\) as lists of words, whose contents change during the run of the algorithm.

At step 1, we assume that \(T_0\) contains \(0^{10}\) and \(T_1\) contains \(v_1 = 1100000000, v_2 = 0011000000, v_3 = 0001100000, v_4 = 0000001100, v_5 = 0000000011;\) utilizing Corollary \(\Box\) we add \(1^{10}\) to \(T_1\) and \(v_i + 1^{10}\) to \(T_0\), \(i = 1, \ldots, 5\). Since any trade is equivalent to one with these words, these twelve words will not be changed during the search.

At step 2, for \(i\) from 1, \ldots, 5, we choose the lexicographically first collection of 5 mutually non-adjacent words in the neighborhood of \(v_i\) that are not adjacent to any word of \(T_0\). This implies that any word of \(T_0\) (say, \(0^{10}\)) adjacent to \(v_i\) is automatically chosen. The other chosen words are “new”, and we include them to \(T_0\), and their complements to \(T_1\). If, for some \(i\), there is no such collection of 5 words, we return to \(v_{i-1}\) and choose the next lexicographical alternative for it (if there is no such alternative, return to \(v_{i-2}\), and so on). When the 5 neighbors are chosen for all \(v_i, i = 1, \ldots, 5\), we come to the next step.

At step 3, for each word of \(T_0\) added at the previous step, we find 5 mutually non-adjacent neighbors that are not adjacent to any word of \(T_1\), add the chosen words that are new to \(T_1\), and add the complements to \(T_0\). Again, after considering all possibilities for a given vertex, we roll back to the previous vertex, which is at this or the previous step, and choose the next alternative for it.

Similarly, step 4, step 5, and so on.

If, at some step, we find that each included words in \(T_0\) (\(T_1\)) already has 5 neighbors in \(T_0\) (respectively, \(T_1\)), then we have found a trade. We add it to the list of found solutions and continue the search.

We finish this section by the pseudocode of the algorithm.

\[
\text{define RECURSION}(s): \# s is the step number} \\
\quad j := s \mod 2 \quad \# \text{the parity of the step} \\
\quad i := 1 - j \\
\quad \text{if } T_0^+ = T_1^+ = \emptyset: \\
\quad \quad \text{FOUND_SOLUTION() \# record the solution } (T_0, T_1), \text{ proceed isomorph rejection, }\ldots \\
\quad \text{else if } T_1^+ = \emptyset: \\
\quad \quad \text{RECURSION}(s + 1) \# \text{go to the next step} \\
\quad \text{else:}
\]
choose \( v \) from \( T_i^+ \)
\[
T_i^+ := T_i^+ \setminus \{ v \}
\]
for all 5-subsets \( N \) of the neighborhood of \( v \)
such that \( N \cup T_i \) is an independent set do:
\[
N^+ := N \setminus T_i \quad \# \text{ new vertices to add}
\]
\[
T_j^+ := T_j^+ \cup N^+
\]
\[
T_j := T_j \cup N^+
\]
\[
T_i := T_i \cup (N^+ + 1111111111)
\]
\[
\text{RECURSION}(s)
\]
\[
T_j^+ := T_j^+ \setminus N^+
\]
\[
T_j := T_j \setminus N^+
\]
\[
T_i^+ := T_i^+ \setminus (N^+ + 1111111111)
\]

\# now, the main part of the algorithm
\[
T_0 := \{ 0000000000, 1111111110, 1111101011, 1111100111, 1110111111, 1100111111, 0011111111 \}
\]
\[
T_1 := \{ 1111111111, 0000000111, 0000011100, 0001100000, 0011000000, 0110000000, 1100000000 \}
\]
\[
T_0^+ := \{ \} \quad \# T_0^+ \text{ and } T_1^+ \text{ keep the chosen vertices with the "unsolved" neighborhood}
\]
\[
T_1^+ := \{ 0000000011, 0000011100, 0001100000, 0111000000, 1110000000 \}
\]
\[
\text{RECURSION}(2)
\]

5.2. Results

There are 8 nonequivalent extended 1-perfect trades in \( \mathbb{H}(10) \). Below, for each of them, we list the component \( T_0 \), while \( T_1 \) is obtained by taking the complement for each word of \( T_0 \). For brevity, \( T_0 \) is represented in the form \( T_0 = K + R = \{ a + b : a \in K, b \in R \} \), where \( K \) (the kernel of \( T_0 \)) is the maximal linear subspace admitting such decomposition of \( T_0 \).

\( T_0 = \{ 0000000000, 1111111110, 1111101011, 1111100111, 1110111111, 1100111111, 0011111111 \} + 0101010101 \)

\( T_2 = \{ 1111111111, 0000000000, 0000111100, 0011011100, 0011100111, 0011111000, 1101001100, 1101011100, 1101101100, 1101111100, 1110000000, 1110001100, 1110010111, 1110011100, 1110100111, 1110110100, 1111000000, 1111001111, 1111010100, 1111011100, 1111100100, 1111110100, 1111111100 \} + 0101010101 \)

\( T_3 = \{ 1111111111, 0000000000, 0000111100, 0011011100, 0011100111, 0011111000, 1101001100, 1101011100, 1101101100, 1101111100, 1110000000, 1110001100, 1110010111, 1110011100, 1110100111, 1110110100, 1111000000, 1111001111, 1111010100, 1111011100, 1111100100, 1111110100, 1111111100 \} + 0101010101 \)

\( T_4 = \{ 1111111111, 0000000000, 0000111100, 0011011100, 0011100111, 0011111000, 1101001100, 1101011100, 1101101100, 1101111100, 1110000000, 1110001100, 1110010111, 1110011100, 1110100111, 1110110100, 1111000000, 1111001111, 1111010100, 1111011100, 1111100100, 1111110100, 1111111100 \} + 0101010101 \)

The trades \( T_1, T_2, T_3, \) and \( T_4 \) are constant-weight; the others, \( T_2a, T_2b, T_2c, \) and \( T_4c \), cannot be represented as constant-weight. Each mate of the trade \( T_4c \) is an optimal distance-4 code equivalent to the Best code \( [3] \).

Table \( [4] \) reflects some properties of the listed trades. For each trade \( (T_0, T_1) \), the second column contains...
its volume. The column “rank” contains the affine rank of $T_0 \cup T_1$, where the first summand is the rank of $T_0$ (as follows from Corollary 1 the second summand is either 0 or 1 for length 10 $\equiv 2$ mod 4).

The column $|\text{Aut}|$ contains the order of the automorphism group of $T_0 \cup T_1$. In this column, the last factor is the order of the stabilizer of a vertex of $T_0 \cup T_1$ under $\text{Aut}(T_0 \cup T_1)$; and the last two factors correspond to $|\text{Aut}(T_0)|$ (it can be seen from Corollary 1 that $|\text{Aut}(T_0 \cup T_1)| = 2 \cdot |\text{Aut}(T_0)|$). In seven (all but one) cases, the second factor coincides with $|T_0|$. For these seven trades, $T_0$ forms one orbit under the action $\text{Aut}(T_0)$; i.e., $\text{Aut}(T_0)$ acts transitively on $T_0$. For one trade, $T_{32c}$, $T_0$ is divided into two orbits of size 16.

The last column contains the coordinate orbits under the action of $\text{Aut}(T_0 \cup T_1)$. In particular, the number of orbits is the number of nonequivalent 1-perfect trades in $H(9)$ obtained by puncturing (deleting the same coordinate in all words) from a given extended 1-perfect trade in $\frac{1}{2}H(10)$. We see that the total number of nonequivalent primary 1-perfect trades in $H(9)$ is $15 = 1 + 2 + 2 + 2 + 3 + 3 + 1 + 1$.

In Table 2 we list all STS trades derived from extended 1-perfect trades of length 10. In Table 3.4], the authors list all nonequivalent STS trades of volume at most 9. As the result of the current search, we can say that the STS trades number 1 (of volume 4), 2, 4 (of volume 6), 5 (a pair of STS of volume 7), 7, 11–16 (of volume 8) are derived, 6 (of volume 7) and 10 (of volume 8) are not derived (numbers 3, 8, 9 in Table 3.4] are for 3- and 4-way trades), all STS trades of volume 9 are not derived from extended 1-perfect trades of length 10. The number in a cell of the table indicates how many times the STS trade $T_0 + x$, $T_1 + x$ form the corresponding STS trade). Note that derived STS trades are not necessarily primary.

It should be noted that if an STS trade is derived from extended 1-perfect trades of length $n + 1$, and has at least one constantly zero coordinate, then it is derived from 1-perfect trades of length $n$ (in our case, $n = 9$). This argument is applicable to the first seven STS trades in Table.

5.3. Validation of classification

To check the results, we recount the number of solutions that should be found by the algorithm in an alternative way. Double-counting is a standard way to validate computer-aided classifications of combinatorial objects, see [12].

Given a trade $(T_0, T_1)$, consider all graph automorphisms that send it to a solution. For every word $t$ from $T_0 \cup T_1$ and its five neighbors $t_1, t_2, t_3, t_4, t_5$ from $T_0 \cup T_1$, there is one translation $x \rightarrow x + t$ that sends $t$ to $0^{10}$ and $5! \cdot 2^5$ coordinate permutations that send $\{t_1, t_2, t_3, t_4, t_5\}$ to $\{v_1, v_2, v_3, v_4, v_5\}$. So, totally, there are $|T_0 \cup T_1| \cdot 5! \cdot 2^5$ graph automorphisms that make from $(T_0, T_1)$ one of the solutions of the algorithm above. Then, the number of different solutions equivalent to $(T_0, T_1)$ is

$$|T_0 \cup T_1| \cdot 5! \cdot 2^5 / |\text{Aut}(T_0 \cup T_1)|.$$ 

Summing this value over all found nonequivalent trades, we get 1817, the exact number of different solutions found by the computer.

| Name | Volume | Rank | $|\text{Aut}|$ | Coordinate orbits |
|------|--------|------|---------------|-------------------|
| T16  | 16     | $4+1$| $2 \cdot 16 \cdot 3840$| $\{0,1,2,3,4,5,6,7,8,9\}$ |
| T24  | 24     | $7+0$| $2 \cdot 24 \cdot 64$  | $\{0,1\}, \{2,3,4,5,6,7,8,9\}$ |
| T28  | 28     | $8+0$| $2 \cdot 28 \cdot 48$  | $\{0,1\}, \{2,3,4,5,6,7,8,9\}$ |
| T32a | 32     | $6+1$| $2 \cdot 32 \cdot 64$  | $\{0,1\}, \{2,3,4,5,6,7,8,9\}$ |
| T32b | 32     | $7+1$| $2 \cdot 32 \cdot 48$  | $\{0,1,6,9\}, \{2,3,4,5\}, \{7,8\}$ |
| T32c | 32     | $8+0$| $2 \cdot 16 \cdot 8$   | $\{0,2,4,5\}, \{1,3\}, \{6,7,8,9\}$ |
| T36  | 36     | $9+0$| $2 \cdot 36 \cdot 40$  | $\{0,1,2,3,4,5,6,7,8,9\}$ |
| T40  | 40     | $9+0$| $2 \cdot 40 \cdot 8$   | $\{0,1,2,3,4,5,6,7,8,9\}$ |

Table 1: Extended 1-perfect trades in the 10-cube
| No | blocks                          | T16 | T24 | T28 | T32a | T32b | T32c | T36 | T40 |
|----|--------------------------------|-----|-----|-----|------|------|------|-----|-----|
| 1  | 012, 034, 135, 245              | 320 | 48  | 0   | 0    | 0    | 16   | 0   | 0   |
|    | 013, 024, 125, 345              |     |     |     |      |      |      |     |     |
| 2  | 012, 034, 135, 146, 236, 245    | 0   | 64  | 0   | 0    | 0    | 0    | 0   | 0   |
|    | 013, 024, 126, 145, 235, 346    |     |     |     |      |      |      |     |     |
| 4  | 012, 034, 135, 246, 257, 367    | 0   | 128 | 112 | 0    | 0    | 0    | 0   | 0   |
|    | 013, 024, 125, 267, 346, 357    |     |     |     |      |      |      |     |     |
| 5  | 012, 034, 056, 135, 146, 236, 245 | 0   | 0   | 32  | 0    | 0    | 0    | 0   | 0   |
|    | 013, 025, 046, 126, 145, 234, 356 |     |     |     |      |      |      |     |     |
| 7  | 012, 034, 067, 135, 147, 236, 257, 456 | 0   | 0   | 0   | 0    | 0    | 0    | 0   | 0   |
|    | 013, 026, 047, 127, 145, 235, 346, 567 |     |     |     |      |      |      |     |     |
| 11 | 012, 034, 135, 146, 178, 236, 247, 258 | 0   | 0   | 0   | 0    | 0    | 64   | 0   | 0   |
|    | 013, 024, 126, 147, 158, 235, 278, 346 |     |     |     |      |      |      |     |     |
| 12 | 012, 034, 135, 147, 236, 258, 378, 468 | 0   | 0   | 0   | 0    | 32   | 0    | 0   | 0   |
|    | 014, 023, 125, 137, 268, 346, 358, 478 |     |     |     |      |      |      |     |     |
| 13 | 012, 034, 135, 146, 178, 236, 237, 379, 589 | 0   | 0   | 0   | 0    | 0    | 16   | 0   | 0   |
|    | 014, 023, 126, 137, 158, 346, 359, 789 |     |     |     |      |      |      |     |     |
| 14 | 012, 034, 067, 089, 135, 245, 246, 568, 579 | 0   | 24  | 0   | 0    | 0    | 0    | 0   | 0   |
|    | 013, 024, 046, 079, 125, 345, 567, 589 |     |     |     |      |      |      |     |     |
| 15 | 012, 034, 135, 146, 246, 257, 289, 368, 379 | 0   | 0   | 84  | 128  | 192  | 0    | 0   | 0   |
|    | 013, 024, 125, 268, 279, 346, 357, 389 |     |     |     |      |      |      |     |     |
| 16 | 012, 034, 135, 246, 257, 368, 589, 679 | 0   | 0   | 0   | 0    | 0    | 32   | 0   | 0   |
|    | 013, 024, 125, 267, 346, 358, 579, 689 |     |     |     |      |      |      |     |     |
|    | 023, 045, 124, 135, 258, 348, 068, 079, 169, 178 | 0   | 0   | 0   | 0    | 0    | 32   | 0   | 0   |
|    | 025, 034, 123, 145, 248, 358, 069, 078, 168, 179 |     |     |     |      |      |      |     |     |
| 17 | 035, 079, 048, 127, 145, 168, 269, 258, 349, 367, | 0   | 0   | 0   | 0    | 0    | 32   | 0   | 0   |
|    | 037, 058, 049, 125, 148, 167, 279, 268, 369, 345, |     |     |     |      |      |      |     |     |
| 18 | 017, 029, 038, 128, 145, 139, 235, 367, 468, 479, 578, 569 | 0   | 0   | 0   | 0    | 0    | 0    | 0   | 80  |
|    | 019, 037, 028, 125, 147, 138, 239, 356, 458, 579, 678, 469 |     |     |     |      |      |      |     |     |
|    | 017, 028, 039, 056, 129, 145, 168, 235, 247, 367, 469, 348 | 0   | 0   | 0   | 0    | 0    | 0    | 0   | 60  |
|    | 018, 029, 035, 067, 127, 149, 156, 238, 245, 347, 369, 468, |     |     |     |      |      |      |     |     |

Table 2: STS trades derived from extended 1-perfect trades of length 10; the numbers in the first column are given in accordance with [5, Table 3.4].
6. Extended 1-perfect trades in $\frac{1}{2}H(12)$

It is hardly possible to enumerate all primary extended 1-perfect trades in $\frac{1}{2}H(12)$ using the technique described above, even if we reject isomorphic partial solutions at some steps of the search. However, if we restrict the search by only the words of weight 6, the number of cases becomes essentially smaller and exhaustive enumeration becomes possible if we additionally apply isomorph rejection. The idea of this technique is standard: at some stage, we check the obtained partial solution and reject it if it is equivalent to a partial solution considered before. Similarly to the length-10 case, we fix one element of $T_0$, now it is 000000111111, and six its $T_1$-neighbors, $v_1$, . . . , $v_6$. After some experiments, it was decided to perform isomorph rejection after choosing the $T_0$-neighbors for $v_1$, $v_2$, $v_3$ and after choosing the $T_0$-neighbors for all $v_1$, . . . , $v_6$. The isomorph rejection reduced the total time of the algorithm run by the factor 1400, approximately. All calculation took an hour and a half using one core of a 3GHz personal computer. The classification was validated using the same approach as for the length 10 (Section 5.3); however, taking into account the isomorph rejection, each solution found by the computer was counted with the multiplicity equal to a partial solution considered before. Similarly to the length-10 case, we fix one element of $T_0$, now it is 000000111111, and six its $T_1$-neighbors, $v_1$, . . . , $v_6$. After some experiments, it was decided to perform isomorph rejection after choosing the $T_0$-neighbors for $v_1$, $v_2$, $v_3$ and after choosing the $T_0$-neighbors for all $v_1$, . . . , $v_6$. The isomorph rejection reduced the total time of the algorithm run by the factor 1400, approximately. All calculation took an hour and a half using one core of a 3GHz personal computer. The classification was validated using the same approach as for the length 10 (Section 5.3); however, taking into account the isomorph rejection, each solution found by the computer was counted with the multiplicity equal to a partial solution considered before.

### 6.1. Description of the trades

The results of the classification are the following. Up to equivalence, there are exactly 25 constant-weight extended 1-perfect trades in $\frac{1}{2}H(12)$ of the following volumes: 32, 48, 56, 68, 86, 72, 72, 72, 72, 80, 80, 92, 92, 92, 96, 96, 98, 102, 108, 108, 110, 110, 120, 120, 132. The data for generation of the trades can be found in the table below.

The first column contains the volume of the trade, sometimes followed by a letter, to form a unique “name”.

Representatives of the orbits of $T_0$ are in the column “$T_0^0$” of the table. The number in the index indicates the size of the orbit; sole number means that the orbit is self-complementary (i.e., each element is contained together with its complement); if the index ends with -2, then the complementary orbit should be additionally taken.

The column marked $\text{Sym}(T_0) \cap \text{Sym}(T_1)$ contains generators of this group, and, sometimes an information about its structure. The grayed generators are not necessary to induce $T_0$ (i.e., all orbits are induced by only the subgroup generated by non-grayed elements). In each case, the coordinates are ordered in such a way that the symmetry group can be represented in a convenient intuitive way (as much as possible, from the point of view of the author).

If $\text{Sym}(T_0 \cup T_1) = \text{Sym}(T_0) \cap \text{Sym}(T_1)$, then representatives of the orbits of $T_1$ are listed in the “$T_1$” column. Usually this means that $T_0$ and $T_1$ are nonequivalent; the only exception is the trade 80a, where $T_1 = T_0 + 000000111111$. The other case is $|\text{Sym}(T_0 \cup T_1)| = 2|\text{Sym}(T_0) \cap \text{Sym}(T_1)|$; then the column “$T_1$” contains an additional generator element. The same column contains information how to generate $T_2$ in the case when the trade $(T_0, T_1)$ can be expanded to a 3-way trade $(T_0, T_1, T_2)$.

The last column of the table contains: (1) the order of the automorphism group of the trade, represented in the form $|\{x \mid x + T_0 \cup T_1 = T_0 \cup T_1\}| \cdot |\text{Sym}(T_0 \cup T_1)|$, where $|\text{Sym}(T_0 \cup T_1)|$ is the stabilizer of 012 in $\text{Aut}(T_0 \cup T_1)$ (for all considered trades, the automorphism group happens to be the product of the translation group with Sym); (2) the standard basis of the dual space; (3) the mark “Witt” if the $T_0$ is a subset of $S(5,6,12)$ (see the next subsection); (4) the mark “no squares” for the unique trade whose graph has girth more than 4, i.e., 6.

| $T_0$ | $\text{Sym}(T_0) \cap \text{Sym}(T_1)$ | $T_1$ | properties |
|-------|----------------------------------|-------|-------------|
| 32    | (89)(ab) (8a)(9b) (02468)(13579) | (ab)  | $|\text{Aut}| = 64 \cdot 46080$ |
|       |                                  |       | dual: $2^{12}(10^{-2i}i = 0,1,2,3,4,5$ |
| \(T_0\) | \(\text{Sym}(T_0) \cap \text{Sym}(T_1)\) | \(T_1\) | \text{properties} |
|---|---|---|---|
| 48 | \(012\) (465) \(01\) (23) (45) (67) \(02\) (46) (89) \(89\) (ab) \(8a\) (9b) \(04\) (15) (26) (37) | \(ab\) | \(|\text{Aut}| = 16 \cdot 1536\) dual: \(1^4 0^8, 0^4 1^4 0^4, 0^8 1^2 0^2, 0^{10} 1^2\) |
| 56a | \(0123456\) \(01\) (23) (46) (89) \(89\) (ab) \(8a\) (9b) | \(ab\) | \(|\text{Aut}| = 8 \cdot 2688\) dual: \(1^8 0^4, 0^8 1^2 0^2, 0^{10} 1^2\) |
| 56b | \(0110011010110_{x8}\) \(00110101011_{x24}\) \(00110011011_{x24}\) \(000111110110\_{x4}\) \(001110001111_{x16}\) \(001110010110_{x24}\) \(00111001011_{x24}\) \(000111110101\_{x2}\) | \(01\) (23) (02) (13) \(048\) (159) (26a) (37b) \(04\) (15) (26) (37) \(01\) (45) (89) | \(|\text{Aut}| = 8 \cdot 768\) dual: \(1^4 0^8, 0^4 1^4 0^4, 0^8 1^4\) |
| 68 | \(0110011000111_{x2}\) \(00110101001_{x12}\) \(01010011101_{x12}\) \(000111010111_{x48}\) | \(01\) (23) (012) (465) \(048\) (159) (26a) (37b) \(04\) (15) (26) (37) \(01\) (45) (8a) \(T_2 : (01\) (45) (9a) | \(|\text{Aut}| = 8 \cdot 6912\) dual: \(1^4 0^8, 0^4 1^4 0^4, 0^8 1^4\) |
| 72a | \(01100101001_{x12}\) \(010101001_{x12}\) \(000111010111_{x48}\) | \(01\) (23) (45) (67) \(09\) (48) (57) (ab) \(123\) (567) (9ab) \(04\) (15) (26) (37) \(01\) (45) (89) | \(|\text{Aut}| = 4 \cdot 2880\) dual: \(1^{10} 0^2, 0^{10} 1^2\) Witt |
| 72b | \(011010101011_{x36}\) \(001101011001_{x36}\) | \(01\) (2) (87) (2365) \(09\) (48) (57) (ab) \(12\) (45) (78) (ab) \(03\) (14) (25) (69) (7a) (8b) \(06\) (17) (28) (39) (4a) (5b) \(12\) (78) | \(|\text{Aut}| = 4 \cdot 192\) dual: \(1^8 0^6, 0^6 1^6\) |
| 72c | \(011010101001_{x36}\) \(001101011000_{x36}\) | \(01\) (2) (82) (67) \(03\) (14) (25) (69) (7a) (8b) \(06\) (17) (28) (39) (4a) (5b) \(12\) (78) | \(|\text{Aut}| = 4 \cdot 144\) dual: \(1^6 0^6, 0^6 1^6\) |
| 80a | \(001111110100_{x20}\) \(000111101001_{x60}\) | \(01\) (2) (34) (6789a) \(05\) (23) (6b) (89) \(06\) (17) (28) (39) (4a) (5b) \(1342\) (79) (8a) | \(|\text{Aut}| = 4 \cdot 240\) dual: \(1^4 0^6, 0^6 1^6\) |
| 80b | \(00011111101_{x4}\) \(00110100011_{x12}\) \(00011100011_{x16}\) \(00110100101_{x24}\) \(0011010101_{x24}\) \(000111110110_{x4}\) | \(01\) (2) (34) (67) \(03\) (14) (25) (69) (7a) (8b) \(06\) (17) (28) (39) (4a) (5b) \(123\) (567) \(04\) (15) (27) (36) (ab) \(89\) (ab) \(8a\) (9b) \(01\) (45) (67) (ab) \(23\) (67) (a) | \(|\text{Aut}| = 4 \cdot 192\) dual: \(1^8 0^4, 0^8 1^4\) |
| 86 | \(00011111100_{x2}\) \(00011110110_{x12}\) \(00110011101_{x18}\) \(00101101101_{x18}\) \(00110111100_{x36}\) | \(01\) (2) (34) (6789a) \(05\) (14) (23) (6b) (7a) (8b) \(06\) (17) (28) (39) (4a) (5b) \(00\) (11) (000) (111) (100) \(001\) (011) (001) \(001\) | \(|\text{Aut}| = 2 \cdot 36\) dual: \(1^{12}\) |
| $T_0$          | $\text{Sym}(T_0) \cap \text{Sym}(T_1)$                  | $T_1$                     | properties                        |
|---------------|--------------------------------------------------------|---------------------------|-----------------------------------|
| 92a           | (01234)(6789a)                                          | 011111 100000×12          | $|\text{Aut}| = 2 \cdot 240$ dual: $1^{12}$ |
| 000000 111111×2 | (1342)(79a8)                                             | 000111 001111×20          |                                    |
| 001111 100000×30 | (05)(23)(6b)(89)                                        | 001111 101001×60          |                                    |
| 000111 100011×60 | (06)(17)(28)(39)(4a)(5b)                               |                           |                                    |
| 92b           | $PGL_2(5) \times C_2$                                   | 01 01 01 10 10 10×8       | $|\text{Aut}| = 2 \cdot 48$ dual: $1^{12}$ |
| 000000 111111×2 | (0213)(6879)                                             | 000111 111000×6           |                                    |
| 000011 111100×6 | (2435)(8a9b)                                             | 000111 001111×12          |                                    |
| 000111 001111×12 | (06)(17)(28)(39)(4a)(5b)                               |                           |                                    |
| 000111 110101×24 | $S_4 \times C_2$                                       | 000011 1111001×48         |                                    |
| 96a           | (0527)(1634)                                             | (02)(13)                  | $|\text{Aut}| = 2 \cdot 384$ dual: $1^{12}$ Witt |
| 000111 001111×11 | (048)(159)(26a)(37b)                                    |                           |                                    |
| 001111 100101×11 | (01)(23)(45)(67)(89)(ab)                               |                           |                                    |
| 96b           | (0213)(6879)                                             | 000111 001111×12          | $|\text{Aut}| = 2 \cdot 48$ dual: $1^{12}$ |
| 000111 001011×12 | (2435)(8a9b)                                             | 000011 111110×12          |                                    |
| 000011 111110×24 | (06)(17)(28)(39)(4a)(5b)                               |                           |                                    |
| 001111 101001×48 | $S_4 \times C_2$                                       | 000111 010101×10×48       |                                    |
| 98            | (012345)(6789a)                                          | 010101 101010×2           | $|\text{Aut}| = 2 \cdot 12$ dual: $1^{12}$ |
| 111111 000000×2 | (0213)(6879)                                             | 011100 110001×6           |                                    |
| 011011 001001×6 | (2435)(8a9b)                                             | 011100 011100×6           |                                    |
| 010101 001101×6 | (06)(17)(28)(39)(4a)(5b)                               |                           |                                    |
| 001111 101011×12 | $D_6$                                                   | 001101 011010×12          |                                    |
| 001111 101011×12 | $D_6$                                                   | 001110 101011×12          |                                    |
| 001101 100111×12 | $D_6$                                                   | 001101 101111×12          |                                    |
| 000101 100111×12 | $D_6$                                                   | 000101 100111×12          |                                    |
| 000101 100111×12 | $D_6$                                                   | 000101 1011001×12         |                                    |
| 0102          | (01234)(6789a)                                           | 010101 101010×12          | $|\text{Aut}| = 2 \cdot 60$ dual: $1^{12}$ |
| 10 01 01 01 10×12 | (01)(23)(45)(67)(89)(ab)                               | 001101 011110×30          |                                    |
| 10 01 11 00 01 10×30 | (01)(23)(45)(67)(89)(ab)                               |                           |                                    |
| 11 11 00 10 00 10×30 | (01)(23)(45)(67)(89)(ab)                               |                           |                                    |
| 108a          | (012)(345)(678)                                          | (06)(17)(28)(9b)          | $|\text{Aut}| = 2 \cdot 432$ dual: $1^{12}$ Witt |
| 000011 011111×542 | (01)(23)(45)(67)(89)(ab)                               |                           |                                    |
| 108b          | (01)(23)(45)(67)(89)(ab)                                | (06)(17)(28)(9b)          | $|\text{Aut}| = 2 \cdot 432$ dual: $1^{12}$ Witt |
| 000011 011111×36 | (01)(23)(45)(67)(89)(ab)                               |                           |                                    |
| 108a          | (01)(23)(45)(67)(89)(ab)                                | (06)(17)(28)(9b)          | $|\text{Aut}| = 2 \cdot 432$ dual: $1^{12}$ Witt |
| 000011010110×72 | (01)(23)(45)(67)(89)(ab)                               |                           |                                    |
| 0011 00110110×36 | (01)(23)(45)(67)(89)(ab)                               |                           |                                    |
| 108a          | (01)(23)(45)(67)(89)(ab)                                | (06)(17)(28)(9b)          | $|\text{Aut}| = 2 \cdot 144$ dual: $1^{12}$ Witt |
| 000011011111×110 | (0123456789a)                                           |                           |                                    |
| 110b          | (0123)(456789)                                          |                           | $|\text{Aut}| = 2 \cdot 1320$ dual: $1^{12}$ |
| 000101011111×110 | (0123456789a)                                           |                           |                                    |
| 110b          | (0123456789a)                                           |                           | $|\text{Aut}| = 2 \cdot 660$ dual: $1^{12}$ no squares |
| 000010111101×110 | (0123456789a)                                           |                           |                                    |
| 110b          | (0123456789a)                                           |                           |                                    |
### 6.2. Some additional results of the classification

#### 6.2.1. Small Witt design

The possible differences of two small Witt designs $S(5,6,12)$ were classified in [13]; the results in this paragraph show the place for these differences in our classification, but do not establish new facts. The trade of the maximum volume, 132, consists of two $S(5,6,12)$. Only 7 nonequivalent trades, with numbers 72b, 96a, 108a, 108b, 120a, 120b, 132, can be represented as the difference pair $(W_0 \setminus W_1, W_1 \setminus W_0)$ of two $S(5,6,12)$ $W_0$ and $W_1$. This was established by an additional run of the algorithm with $T_0$ being restricted by only the elements of a fixed $S(5,6,12)$ (the search was fast enough without the isomorph rejection for partial solutions).

**Corollary 3 ([13]).** Up to equivalence, there is only one pair of disjoint $S(5,6,12)$.

**Proof.** Two disjoint $S(5,6,12)$ systems $W_0$ and $W_1$ form a trade $(W_0, W_1)$. If it is not primary, then there is a trade $(V_0, V_1)$ such that $V_i \subset W_i$, $i = 0, 1$, and $0 < |V_0| \leq |W_0|/2 = 66$, which is not possible as the minimum trade included in an $S(5,6,12)$ has the volume 72. Since a primary trade of volume 132 is unique, the statement follows. □

#### 6.2.2. 3-way trades and no more

Four of the trades, 72a, 108a, 110a, and 110b, can be continued to 3-way trades $(T_0, T_1, T_2)$. It occurs that for given $T_0$ and $T_1$, the choice of $T_2$ is unique; it follows that for the considered parameters, no primary trades can be continued to $k$-way trades for $k > 3$. The 3-way trades from the trades 110a and 110b are the same; so, there are only three nonequivalent 3-way trades obtained by continuing primary trades with considered parameters.

#### 6.2.3. Derived STS trades

87 nonequivalent STS trades are derived from these extended 1-perfect trades. Among the STS trades of volume at most 9, only numbers 13 (of volume 8), 29, 30, 31, 33 (of volume 9), in the classification [7, Table 3.4] are not derived.

Only the trades 32 and 132 are STS-uniform, that is, each has only one derived STS trade, up to equivalence. Each of the three trades 72a, 110a, 110b has only two nonequivalent derived STS trades; the other 20 trades have 4 or more.

All STS 3-way trades of volume at most 9 (numbers 3, 8, 23, 25, and 28 in [7, Table 3.4]) are derived from the 3 found extended 1-perfect 3-way trades: number 3 is derived from the 3-way trade of volume 72; number 8, 23, 25 are derived from the 3-way trade of volume 108; number 28 is derived from the 3-way trades of volumes 72 and 108.

| $T_0$ | $\text{Sym}(T_0) \cap \text{Sym}(T_1)$ | $T_1$ | properties |
|-------|-------------------------------------|-------|------------|
| 120a  | $0011\ 0101\ 0101\times 24$ $0001\ 0011\ 1011\times 96$ | $(0527)(1634)$ $(048)(159)(26a)(37b)$ | $(02)(46)(9b)$ | $|\text{Aut}| = 2 \cdot 192$ dual: $1_{10}^2$ Witt |
| 120b  | $0111\ 1010\ 1000\times 60$ $0000\ 1001\ 1111\times 60$ | $(02468)(13579)$ $(01)(23)(48)(59)(6a)(7b)$ | $(01)(23)(45)(67)(89)(ab)$ | $|\text{Aut}| = 2 \cdot 120$ dual: $1_{12}^2$ Witt |
| 132   | $000001011111\times 132$ | $(0123456789a)$ $(13954)(267a8)$ $(0b)(1a)(25)(37)(48)(69)$ $\sim PSL_2(5)$ | $(0a)(19)(28)(37)(46)$ | $|\text{Aut}| = 2 \cdot 1320$ dual: $1_{12}^2$ Witt |

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[13]: Reference or citation not provided.
6.2.4. Other remarks

It should be noted that all 25 trades found are rather symmetric. The trade of volume 98 has the smallest automorphism group among all found trades. Its symmetry group is isomorphic to the dihedral group of order 12 and acts transitively on the coordinates. However, with the growth of length, it is naturally to conjecture the existence of primary trades with small automorphism groups, even consisting of the identity and the translation by the all-one vector only.

Trades 110a, 110b, and 132 consist of orbits of the same symmetry group (three orbits of cardinality 110, which form a 3-way trade, and two orbits of cardinality 132, Steiner sextuple systems S(5, 6, 12), the projective special group PSL₂(11), with the natural action on the 12 coordinates. The other orbits of this group are not connected with trades.

Trade 110b is the only trade whose graph (i.e., the subgraph of \(\frac{1}{2}H(12)\) induced by \(T_0 \cup T_1\)) has no cycles of length 4 (squares).

7. Construction of extended 1-perfect trades

In this section, we show how extended 1-perfect trades or \(k\)-way trades of small length can be used to construct trades of larger length. In particular, if the length of a trade is not a power of two, it obviously cannot be embedded into a pair of extended 1-perfect codes of the same length, but the question if it can be embedded after lengthening by the construction below is considerably more difficult.

The construction uses latin trades, whose construction is not discussed here; however, there is a simple example of a latin trade \((L_0, L_1)\) that can be used in the construction: \(L_j\) consists of all words of length \(m\) over the alphabet \(\{0, \ldots, q - 1\}\) with the sum of all coordinates being \(j\) modulo \(q\). The simplest case that works is \(q = 2\), and the simplest extended 1-perfect trade that can be used in the construction has length 2: \((\{00\}, \{11\})\); the corresponding partial case is described in Lemma 3. The construction below is a variant of the product construction of extended 1-perfect codes suggested in [22] and also inherit ideas of the generalized concatenation construction for error-correcting codes from [33]. The proof is straightforward and omitted here.

**Proposition 3.** Let \(M = (M_0, M_1)\) be a latin trade in \(H(m, q)\), and let for every \(i\) from 0 to \(m - 1\), only the symbols \(0, \ldots, q_i - 1, q_i \leq q, \) occur in the \(i\)th position of the words of \(M\). For each \(i\) from 0 to \(m - 1\), let \(T^{(i)} = (T_0^{(i)}, \ldots, T_{q_i - 1}^{(i)})\) be an extended 1-perfect \(q_i\)-way trade in \(\frac{1}{2}H(n_i)\). Then the pair \((T_0, T_1)\), where

\[
T_j = \{(c_0 \ldots c_m - 1) | c_i \in T_{b_{i}}^{(i)}, (b_0 \ldots b_{m - 1}) \in M_j\},
\]

is an extended 1-perfect trade in \(\frac{1}{2}H(n_0 + \ldots + n_{m - 1})\). Moreover,

1. if the trade \(M\) is primary and for every \(i\) from 0 to \(m - 1\) and every different \(j, j'\) from \(0, \ldots, q_i - 1\), the trade \((T_j^{(i)}, T_{j'}^{(i)})\) is primary too; \(T_j^{(i)}\) is primary too;
2. for every \(i\) from 0 to \(m - 1\), the word \(0^{n_i} \ldots 0^{n_i - 1}1^{n_i}, 0^{n_i + 1} \ldots 0^{n_i - 1}\) is dual to \((T_0, T_1)\).

Similarly, if we replace the latin trade by a latin \(k\)-way trade, then we obtain an extended 1-perfect \(k\)-way trade.

A special case of 1-perfect trade mates, subsets of 1-perfect binary codes called \(i\)-components, can be constructed using another approach, see e.g. [27]. In particular, for each length of form \(2^m - 1\), there are primary 1-perfect trades of volume \(2^m - m - 2\), i.e., half of the cardinality of a 1-perfect code (readily, the same is true for extended 1-perfect trades and codes of length \(2^m\)).

8. Concluding remarks

We presented some classification results on extended 1-perfect trades, obtained by computer search, for small parameters. In the conclusion, we would like to emphasize a connection of some of found trades with optimal codes.
The optimal distance-4 binary code of length 10 (it has 40 codewords, is known as the Best code \cite{3}, and it is unique up to equivalence \cite{17}) form an extended 1-perfect trade with its complement.

So, we found new illustrations to the fact that some good codes can be represented as algebraic-combinatorial objects like eigenfunctions (classical examples of such codes are the perfect codes; less obvious examples are, e.g., the binary \((n = 2^k - 3, 2^{n-k}, 3)\) and \((n = 2^k - 4, 2^{n-k}, 3)\) codes \cite{14, 15}. It would be quite interesting to continue classification of the extended 1-perfect trades in small dimensions and try to find more connections with good codes. However, the possibilities of the considered algorithm seem to be exhausted. The attempts to start it with larger parameters did not allow even to evaluate the time needed to complete the search. The number of the objects we search grows double-exponentially \(2^{2^O(n)}\), and the complexity of any algorithm finds a physical limit rather fast. Some hope to find interesting examples in larger lengths by computer search is related with the search of objects with some restrictions, for example, with a prescribed automorphism group.

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