Classification of Subsets in Finite Projective Line Over Galois Field of Order Twenty-Seven

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Abstract. The principle objective in this paper is to computed the projectively inequivalent \( k \)-sets, \( k = 4, \ldots, 14 \) in the projective line \( PG(1,27) \) and then classified these \( k \)-set into \((k-1)\)-subsets and. Also, the group of projectivities of each \( k \)-set has been founded.

Keywords: Finite field, Finite projective line.

1. Introduction

Throughout the paper, let \( p \) be a prime and \( F_q \) be the Galois field with \( q = p^h \) elements and \( F_q^+ \) be \( F_q \) union infinity point, \( \infty \).

A \( k \)-set in the projective line over the Galois field of order \( q \), \( F_q \), is just a subset of the with \( k \) elements. A classification of a \( k \)-set or description means the types of the \( k \) of \((k-1)\)-subsets. For small \( q \), Hirschfeld in [1], the classification of \( PG(1,q) \) for \( 2 \leq q \leq 13 \) have been summarized, where a full classification of \( PG(1,11) \) has been done by Sadeh in [2] and of \( PG(1,13) \) has been done by Ali in [3].

In recent years, new researches on the classification of a projective line have done over different fields, \( F_q \) for example: Hirschfeld and Al-seraji in [4], gave a full classification of \( k \)-sets in \( PG(1,17) \). Al-seraji in [5,6] gave the inequivalent \( k \)-sets only in \( PG(1,16) \) and \( PG(1,23) \). Al-Zangana and Hirschfeld in [7], studied the geometry of line of order nineteen with its relation to the conic, where a full classification and its application to error correcting codes have been given. Al-Zangana in [8] using the relation between conic and projective line the spectrum sizes of \( k \)-sets on \( PG(1,23) \) are given as a direct results from this relation. Al-Zangana and Shehab in [9], investigated the geometry of \( PG(1,25) \) with its relation to the conic, where a full classification have been given.

The main objective of this paper is to classify the \( k \)-sets in the projective line over \( F_{27} \), where \( k = 4, \ldots, 14 \).
Notations:
\( G \times H \): the direct product of \( G \) and \( H \)
\( N \triangleright H \): a semi-direct product of \( H \) with \( N \) a normal subgroup of \( N \triangleright H \)

1: single group
\( S_n \): symmetric group of degree \( n \)
\( A_n \): alternating group of degree \( n \)
\( D_n \): dihedral group of order \( 2n \)
\( F_0 \): the cross-ratio is another value.

2. Projective Line Structure Over \( F_q \)
The \( q + 1 \) points \( P(t_0, t_1), t_i \in F_q \) of \( PG(1, q) \) are determined by the non-homogeneous coordinate \( t_0/t_1 \). The coordinate for \( P(1, 0) \) is infinity, so the points of \( PG(1, q) \) can be represented by the set \( F_q \cup \{ \infty \} = \{ \infty, \lambda_1, \lambda_2, \ldots, \lambda_q \mid \lambda_i \in F_q \} \). A projectivity \( T = M(A) \) of \( PG(1, q) \) which takes a point \( X \) to \( Y \) is given by a \( 2 \times 2 \) non-singular matrix \( A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \). If put \( s = y_0/y_1 \) and \( t = x_0/x_1 \), then its projective equation is \( s = (at + b)/(ct + d) \).

**Definition 2.1:** The cross-ratio \( \lambda = \{ P_1, P_2, P_3, P_4 \} \) of four ordered distinct points \( P_1, P_2, P_3, P_4 \in \) \( PG(1, q) \) with coordinates \( t_1, t_2, t_3, t_4 \) is

\[
\lambda = \{ P_1, P_2, P_3, P_4 \} = \{ t_1, t_2; t_3, t_4 \} = (t_1 - t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_1).
\]

The cross-ratio of the 24 permutations of \( \{ P_1, P_2, P_3, P_4 \} \) takes just six values \( \lambda, 1/\lambda, 1 - \lambda, 1/(1 - \lambda), (\lambda - 1)/\lambda, \lambda/(\lambda - 1) \) and no one of these values are \( \infty, 0 \) or 1.

**Definition 2.2:** An unordered set of four distinct points is called a tetrad. Let \( \lambda \) be the cross ratio of a given order, the tetrad is called

(i) harmonic, denoted by \( H \), if the cross-ratio are \(-1, 2, 1/2\).

(ii) equianharmonic, denoted by \( E \), if \( \lambda^3 + 1 = 0 \) has three solutions in \( F_q \) or \( \lambda = -1 \) is a unique solution of

\[
\lambda^2 - \lambda + 1 = 0 \text{ in } F_q
\]  

(iii) neither harmonic nor equianharmonic, denoted by \( N \), if the cross-ratio is another value.

The points of the standard frame, \( \Gamma_q(3) \), of \( PG(1, q) \) are \( P(1, 0), P(0, 1), P(1, 1) \) which can represented as \( \infty, 0 \) and 1. The cross-ratio \( \lambda \) of any 4-set through \( \Gamma_q(3) \) say \( \{ \infty, 0, 1, t \} \) is always \( \in F_q^+ \setminus \Gamma_q(3) \). Hence, there are three classes of 4-sets:

\[ \chi_1 = \{ \text{tetrads of type } H \}, \chi_2 = \{ \text{tetrads of type } E \}, \chi_3 = \{ \text{tetrads of type } N \}. \]

The two classes \( \chi_1 \) and \( \chi_2 \) are equal if \( p = 3 \) and the class \( \chi_3 \) can be partition into more classes depending on the value of \( q \).

**Theorem 2.3** [1]: (The Fundamental Theorem of Projective Geometry)

If \( \{ P_1, \ldots, P_{n+2} \} \) and \( \{ P'_1, \ldots, P'_{n+2} \} \) are sets of \( n + 2 \) points of \( PG(n, q) \) such that no \( n + 1 \) points chosen from the same set lie in a hyperplane, then there exists a unique projectivity \( \delta \) such that \( P_i \delta = P'_i \) for \( i = 1, \ldots, n + 2 \). For \( n = 1 \), then there is a unique projectivity transforming any three distinct points on a line to any other three.
The fundamental theorem of projective geometry in the case of line determine the matrix structure of the projectivity which can diagnosed by given two 4-sets having the same cross-ratio. Thus, projectivity is determined by the images of three points. If \( Q_i = P_i \) for \( i = 2,3,4 \) and \( P_i \) and \( Q_i \) have the respective coordinates \( t_i \) and \( s_i \), then \( T \) is given by

\[
\frac{(s - s_3)(s_2 - s_4)}{(s - s_4)(s_2 - s_3)} = \frac{(t - t_3)(t_2 - t_4)}{(t - t_4)(t_2 - t_3)}
\]

Therefore, it is enough to start with standard frame \( \Gamma_q(3) \), which has the symmetric group of order three, \( S_3 \) as a stabilizer, to construct a \( k \)-set, \( k \geq 4 \). The strategy that used to extension the \( k \)-set \( \mathcal{K} \) is by adding points from each orbit that comes from the action of stabilizer group of \( \mathcal{K} \) on \( \mathcal{K} \) itself. The classification of each \( k \)-set \( Q = \{a_1, ..., a_k\} \) means the type of the \( k \) of \( (k - 1) \)-subsets in the following order:

\[
\{a_1, ..., a_{k-1}\}, \{a_1, ..., a_{k-2}, a_k\}, ..., \{a_1, a_3, ..., a_{k-1}, a_k\}, \{a_2, ..., a_{k-1}, a_k\}.
\]

The main computing tool that was used in this thesis is the mathematical programming language GAP [10]. The main reference that used to know the stabilizer group is [11].

In the case of 4-set the number of distinct 4-set of each class \( \chi_i \) and the stabilizer group type are known as in the following theorem.

**Theorem 2.4** [1]: In \( PG(1, q) \), \( q = p^h \)

(i) the number of harmonic tetrads \( n_H \) and the stabilizer group type, SG-type, of each one are given below:

| \( p \)   | \( n_H \)          | SG-type |
|----------|--------------------|---------|
| \( p = 3 \) | \( q(q^2 - 1)/24 \) | \( S_4 \) |
| \( p > 3 \) | \( q(q^2 - 1)/8 \)  | \( D_4 \) |

(ii) the number of equianharmonic tetrads \( n_E \) and the stabilizer group type, SG-type, of each one are as in the following table.

| \( p \)   | \( n_E \)          | SG-type |
|----------|--------------------|---------|
| \( p = 3 \) | \( q(q^2 - 1)/24 \) | \( S_4 \) |
| \( p \equiv 1(\text{mod} 3) \) | \( q(q^2 - 1)/12 \) | \( A_4 \) |

3. **The projective line over \( F_{27} \)**

The projective line \( PG(1, 27) \) has 28 points and these points identified with \( F_{27}^+ = \{\infty, 0,1, \alpha, \alpha^2, ..., \alpha^{25}\} \) where \( \alpha \) is the primitive element of \( F_{27} \). A 4-set of type \( E \) and \( H \) are equal when \( q = 27 \) since, \( -1 \) is the unique solution to (1).

Let \( S \) be the set of all different 4-sets in \( PG(1, 27) \). Then the order of \( S \) is

\[
|S| = \binom{28}{4} = 20475.
\]

The order of each class is \( |z_1| = 819 \) and \( |z_2| = 19656/4 = 4914 = |z_3| = |z_4| = |z_5| \).
There are five classes, $\chi_i$, of 4-sets which form twenty-five 4-sets:

| Symbol | Class of $E = H$ tetrads | Elements of class |
|--------|--------------------------|------------------|
| $\chi_1$ | $I_{27}(3)U[a^{13}]$ | |
| $\chi_2$ | $I_{27}(3)U[a_1^1], a_1 = a, a^2, a^{10}, a^{16}, a^{24}, a^{25}$ |
| $\chi_3$ | $I_{27}(3)U[a_2^1], a_2 = a^2, a^{14}, a^6, a^{20}, a^{22}, a^{24}$ |
| $\chi_4$ | $I_{27}(3)U[a_3^1], a_3 = a^3, a^7, a^{11}, a^{15}, a^{19}, a^{21}$ |
| $\chi_5$ | $I_{27}(3)U[a_4^1], a_4 = a^4, a^9, a^{12}, a^{14}, a^{17}, a^{18}$ |

The number of projectively inequivalent 4-sets has been computed as summarized in the following theorem.

**Theorem 3.1:** On $PG(1,27)$, there are precisely five projectively distinct tetrads given with their stabilizer group in Table 1.

| Type | 4-sets | SG-type |
|------|--------|---------|
| $H$  | $I_{27}(3)U[a^{13}]$ | $S_n = (1/t + 1, a^{13}t)$ |
| $N_1$ | $I_{27}(3)U[a]$ | $V_4 = ((a^{13}/a^{13}t), (a^{13}t + a)/(a t + a))$ |
| $N_2$ | $I_{27}(3)U[a^3]$ | $V_4 = ((a^{16}/a^{13}t), (t + a^{13})/(t + a^{13}))$ |
| $N_3$ | $I_{27}(3)U[a^5]$ | $V_4 = ((a^{18}/a^{13}t), (t + a^{10})/(t + a^{13}))$ |
| $N_4$ | $I_{27}(3)U[a^8]$ | $V_4 = (a^{21}/a^{13}t, (t + a^{21})/(t + a^{13}))$ |

From the five 4-sets in Theorem 3.1, twenty-eight 5-sets are computed and then the inequivalence ones have been founded.

**Theorem 3.2:** On $PG(1,27)$, there are precisely eight distinct 5-sets, $P_i$, given with their stabilizer groups in Table 2.

| Type | 5-sets | SG-type | Types of 4-sets |
|------|--------|---------|-----------------|
| $P_1$ | $HN[\alpha]$ | $I$ | $HN_1N_1N_1N_1$ |
| $P_2$ | $N_1U[\alpha^2]$ | $Z_2 = (a^{15}/a t)$ | $N_1N_1N_1N_1N_1N_1$ |
| $P_3$ | $N_1U[\alpha^3]$ | $Z_2 = (a^{13}t + a)$ | $N_1N_1N_1N_1N_1N_2$ |
| $P_4$ | $N_1U[\alpha^6]$ | $Z_2 = ((a^{19}t + a^6)/(a^{13}t + a^6))$ | $N_1N_1N_1N_1N_1N_3$ |
| $P_5$ | $N_1U[\alpha^4]$ | $I$ | $N_1N_1N_1N_1N_1N_4$ |
| $P_6$ | $N_1U[\alpha^{12}]$ | $Z_2 = ((t + a^{14})/(t + a^{13}))$ | $N_1N_1N_1N_1N_1N_4$ |
| $P_7$ | $N_2U[\alpha^2]$ | $Z_2 = ((a^{10}t)/(a^{14}t + a^{23}))$ | $N_1N_1N_1N_1N_1N_3$ |
| $P_8$ | $N_2U[\alpha^3]$ | $Z_2 = ((a^{12}t + a^5))$ | $N_1N_1N_1N_1N_1N_4$ |

Adding the one point from the eight orbits of action the group $G_{P_1}$ on $P_1$, gave 102 distinct 6-sets. The projectively inequivalent 6-sets are computed and the results summarized below.

**Theorem 3.3** On $PG(1,27)$, there are 34 inequivalent 6-sets. Stabilizer group type and type of pentads are given in Table 3 and Table 4.
Table 3. Classification of 6-sets

| Type   | 6-sets                                                                 | Types of 5-sets                                      |
|--------|----------------------------------------------------------------------|-----------------------------------------------------|
| H₁     | \(P_1 \cup \{\alpha^2\}\)                                          | \(P_1 P_2 P_3 P_6 P_7 P_8\)                        |
| H₂     | \(P_1 \cup \{\alpha^3\}\)                                          | \(P_1 P_2 P_3 P_6 P_7 P_8\)                        |
| H₃     | \(P_1 \cup \{\alpha^4\}\)                                          | \(P_1 P_3 P_4 P_6 P_8\)                           |
| H₄     | \(P_1 \cup \{\alpha^5\}\)                                          | \(P_1 P_3 P_4 P_6 P_8\)                           |
| H₅     | \(P_1 \cup \{\alpha^6\}\)                                          | \(P_1 P_4 P_5 P_6 P_8\)                           |
| H₆     | \(P_1 \cup \{\alpha^7\}\)                                          | \(P_1 P_4 P_5 P_6 P_8\)                           |
| H₇     | \(P_1 \cup \{\alpha^8\}\)                                          | \(P_1 P_5 P_6 P_8\)                               |
| H₈     | \(P_1 \cup \{\alpha^9\}\)                                          | \(P_1 P_5 P_6 P_8\)                               |
| H₉     | \(P_1 \cup \{\alpha^{10}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₁₀    | \(P_1 \cup \{\alpha^{11}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₁₁    | \(P_1 \cup \{\alpha^{12}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₁₂    | \(P_1 \cup \{\alpha^{13}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₁₃    | \(P_1 \cup \{\alpha^{14}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₁₄    | \(P_1 \cup \{\alpha^{15}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₁₅    | \(P_1 \cup \{\alpha^{16}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₁₆    | \(P_1 \cup \{\alpha^{17}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₁₇    | \(P_1 \cup \{\alpha^{18}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₁₈    | \(P_1 \cup \{\alpha^{19}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₁₉    | \(P_1 \cup \{\alpha^{20}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₂₀    | \(P_1 \cup \{\alpha^{21}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₂₁    | \(P_1 \cup \{\alpha^{22}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₂₂    | \(P_1 \cup \{\alpha^{23}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₂₃    | \(P_1 \cup \{\alpha^{24}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |
| H₂₄    | \(P_1 \cup \{\alpha^{25}\}\)                                       | \(P_1 P_5 P_6 P_8\)                               |

Table 4. Stabilizer group types of 6-sets

| Type   | SG-type                |
|--------|------------------------|
| \(H_1\) | \(I\)                  |
| \(H_2\) | \(I\)                  |
| \(H_3\) | \(I\)                  |
| \(H_4\) | \(Z_2 = (\alpha^{13}t/t + 1, (\alpha^{14} + \alpha/\alpha^{16}t + \alpha))\) |
| \(H_5\) | \(Z_2 = (t + 1)/(t + \alpha^{15})\) |
| \(H_6\) | \(Z_2 = (t + \alpha^{15})/(\alpha^{13}t + \alpha^{15})\) |
| \(H_7\) | \(I\)                  |
| \(H_8\) | \(I\)                  |
| \(H_9\) | \(Z_2 = (1/\alpha^{15})\) |
| \(H_{10}\) | \(Z_2 = ((\alpha^{13}t/\alpha^{14} + \alpha)(\alpha^{14}t + \alpha))\) |
| \(H_{11}\) | \(I\)                  |
\[
\begin{array}{l}
H_{12} & z_2 = ((\alpha^{13} t + 1), (\alpha^{22} t + \alpha^{10} / \alpha^9 t + \alpha^9)) \\
H_{13} & I \\
H_{14} & Z_2 = ((\alpha^{13} t + \alpha^{13}), (t + \alpha^{14} / t + \alpha^{13})) \\
H_{15} & Z_2 = ((t / t + \alpha^{14}), (\alpha^{20} t + \alpha^{20} / \alpha^{19} t + \alpha^9)) \\
H_{16} & Z_2 = ((\alpha^{13}/\alpha^{13} t), (\alpha^{22} t + \alpha^{9} / \alpha^{21} t + \alpha^9)) \\
H_{17} & Z_2 = ((\alpha^{16}/\alpha^{13} t)) \\
H_{18} & I \\
H_{19} & Z_2 = ((\alpha^{24} t + \alpha^{13}) / (\alpha^{17} t + \alpha^{11})) \\
H_{20} & Z_2 = ((\alpha^{14} t + \alpha) / (\alpha^{13} t + \alpha)) \\
H_{21} & S_3 = ((\alpha^{13} / (t + \alpha^{13})), (\alpha^{24} t + \alpha^{17}) / (at + \alpha^{15})) \\
H_{22} & I \\
H_{23} & S_3 = ((\alpha^{14} t + 1) / (\alpha^{9} + 1), (\alpha^{17} / \alpha^{13} t)) \\
H_{24} & Z_2 = ((\alpha^{16} t + \alpha^3) / (\alpha^{15} t + \alpha^3)) \\
H_{25} & Z_2 = ((\alpha^{11} t + \alpha) / (\alpha^{10} t + \alpha^{24})) \\
H_{26} & I \\
H_{27} & Z_2 = ((\alpha^{20} / \alpha^{13} t)) \\
H_{28} & Z_2 = ((t + \alpha^{14}) / (t + \alpha^{13})) \\
H_{29} & Z_2 = ((\alpha^{15} t + \alpha^3) / (\alpha^{24} t + \alpha^3)) \\
H_{30} & S_3 = ((\alpha^{7} t + \alpha^{21}) / (t + \alpha^{13}), (\alpha^{21} / \alpha^{13} t)) \\
H_{31} & Z_2 = ((\alpha^{20} t + \alpha^7) / (\alpha^{13} t + \alpha^7)) \\
H_{32} & Z_2 = ((t + \alpha^2) / (t + \alpha^{13})) \\
H_{33} & Z_2 = ((t + \alpha^{14}) / (t + \alpha^{13})) \\
H_{34} & S_3 = ((\alpha^{4} t + 1) / (at + 1), (\alpha^{25} / \alpha^{13} t)) \\
\end{array}
\]

From the thirty-four projectively 6-sets, 326 7-sets formed. The projectively inequivalent 7-sets are computed and the results summarized in next theorem.

In the following theorem the parameters \([a_1, a_2, a_3, a_4, a_5, a_6, a_7]\) will write instead of the seven 6-subsets of each 7-set, where \(a_i\) refer to the index of the 6-set \(H_{a_i}\).

**Theorem 3.3** On \(PG(1,27)\), there are 73 inequivalent 7-sets. Stabilizer group type and type of pentads are given in Table 5.

**Table 5. Classification of 7-sets**

| Type | 7-sets | Type of 6-sets | SG-type |
|------|--------|----------------|---------|
| \(\Gamma_1\) | \(H_1 \cup \{a^1\}\) | [1, 2, 3, 17, 22, 34, 33] | I |
| \(\Gamma_2\) | \(H_1 \cup \{a^2\}\) | [1, 4, 12, 13, 26, 34, 13] | I |
| \(\Gamma_3\) | \(H_1 \cup \{a^3\}\) | [1, 5, 2, 8, 11, 22, 13] | I |
| \(\Gamma_5\) | \(H_1 \cup \{a^5\}\) | [1, 6, 3, 19, 28, 26, 18] | I |
| \(\Gamma_6\) | \(H_1 \cup \{a^6\}\) | [1, 7, 10, 11, 31, 3, 28] | I |
| \(\Gamma_7\) | \(H_1 \cup \{a^7\}\) | [1, 2, 9, 20, 32, 26, 31] | I |
| \(\Gamma_9\) | \(H_1 \cup \{a^{10}\}\) | [1, 1, 1, 21, 1, 1, 1] | \(S_3\) |
| \(\Gamma_9\) | \(H_1 \cup \{a^{11}\}\) | [1, 8, 6, 22, 20, 26, 31] | I |
| \(\Gamma_{10}\) | \(H_1 \cup \{a^{12}\}\) | [1, 9, 8, 22, 28, 26, 25] | I |
| \(\Gamma_{11}\) | \(H_1 \cup \{a^{14}\}\) | [1, 10, 11, 16, 9, 13, 6] | I |
| \(\Gamma_{12}\) | \(H_1 \cup \{a^{15}\}\) | [1, 11, 4, 1, 11, 33, 33] | \(Z_2\) |
| \(\Gamma_{13}\) | \(H_1 \cup \{a^{16}\}\) | [1, 12, 16, 22, 14, 2, 3] | I |
| \(\Gamma_{14}\) | \(H_1 \cup \{a^{17}\}\) | [1, 11, 7, 22, 27, 32, 26] | I |
| $\Gamma_15$ | $H_4 \cup \{\alpha^{18}\}$ | $[1, 8, 2, 21, 20, 17, 18]$ | $I$ |
| $\Gamma_16$ | $H_4 \cup \{\alpha^{19}\}$ | $[1, 13, 11, 20, 22, 22, 29]$ | $I$ |
| $\Gamma_17$ | $H_4 \cup \{\alpha^{20}\}$ | $[1, 13, 14, 11, 31, 32, 6]$ | $I$ |
| $\Gamma_18$ | $H_4 \cup \{\alpha^{21}\}$ | $[1, 3, 8, 19, 32, 17, 25]$ | $I$ |
| $\Gamma_19$ | $H_4 \cup \{\alpha^{22}\}$ | $[1, 14, 7, 8, 27, 13, 28]$ | $I$ |
| $\Gamma_20$ | $H_4 \cup \{\alpha^{23}\}$ | $[1, 7, 13, 2, 9, 3, 26]$ | $I$ |
| $\Gamma_21$ | $H_4 \cup \{\alpha^{24}\}$ | $[1, 15, 12, 18, 8, 2, 10]$ | $I$ |
| $\Gamma_22$ | $H_4 \cup \{\alpha^{25}\}$ | $[1, 16, 15, 17, 8, 4, 3]$ | $I$ |
| $\Gamma_23$ | $H_2 \cup \{\alpha^4\}$ | $[2, 3, 8, 23, 17, 26, 24]$ | $I$ |
| $\Gamma_24$ | $H_2 \cup \{\alpha^8\}$ | $[2, 4, 11, 7, 22, 32, 9]$ | $I$ |
| $\Gamma_25$ | $H_2 \cup \{\alpha^9\}$ | $[2, 5, 8, 24, 19, 22, 27]$ | $I$ |
| $\Gamma_26$ | $H_2 \cup \{\alpha^7\}$ | $[2, 6, 3, 25, 18, 33, 29]$ | $I$ |
| $\Gamma_27$ | $H_2 \cup \{\alpha^8\}$ | $[2, 7, 11, 24, 22, 17, 19]$ | $I$ |
| $\Gamma_28$ | $H_2 \cup \{\alpha^9\}$ | $[2, 2, 2, 2, 2, 2, 30]$ | $S_3$ |
| $\Gamma_29$ | $H_2 \cup \{\alpha^{15}\}$ | $[2, 11, 16, 8, 21, 22, 11]$ | $I$ |
| $\Gamma_30$ | $H_2 \cup \{\alpha^{17}\}$ | $[2, 11, 13, 17, 20, 26, 32]$ | $I$ |
| $\Gamma_31$ | $H_2 \cup \{\alpha^{18}\}$ | $[2, 8, 6, 18, 11, 3, 7]$ | $I$ |
| $\Gamma_32$ | $H_2 \cup \{\alpha^{19}\}$ | $[2, 13, 7, 18, 17, 33, 25]$ | $I$ |
| $\Gamma_33$ | $H_2 \cup \{\alpha^{22}\}$ | $[2, 14, 10, 3, 8, 4, 26]$ | $I$ |
| $\Gamma_34$ | $H_2 \cup \{\alpha^{23}\}$ | $[2, 7, 15, 23, 18, 3, 7]$ | $I$ |
| $\Gamma_35$ | $H_2 \cup \{\alpha^{24}\}$ | $[2, 15, 13, 25, 11, 26, 6]$ | $I$ |
| $\Gamma_36$ | $H_2 \cup \{\alpha^{25}\}$ | $[2, 16, 7, 18, 19, 13, 5]$ | $I$ |
| $\Gamma_37$ | $H_3 \cup \{\alpha^{6}\}$ | $[3, 5, 8, 18, 32, 28, 22]$ | $I$ |
| $\Gamma_38$ | $H_3 \cup \{\alpha^{8}\}$ | $[3, 7, 13, 26, 33, 26, 27]$ | $I$ |
| $\Gamma_39$ | $H_3 \cup \{\alpha^{11}\}$ | $[3, 8, 9, 24, 22, 18, 27]$ | $I$ |
| $\Gamma_40$ | $H_3 \cup \{\alpha^{16}\}$ | $[3, 12, 13, 7, 4, 6, 5]$ | $I$ |
| $\Gamma_41$ | $H_3 \cup \{\alpha^{17}\}$ | $[3, 11, 12, 8, 13, 29, 28]$ | $I$ |
| $\Gamma_42$ | $H_3 \cup \{\alpha^{19}\}$ | $[3, 13, 11, 25, 26, 31, 18]$ | $I$ |
| $\Gamma_43$ | $H_3 \cup \{\alpha^{20}\}$ | $[3, 13, 15, 24, 3, 13, 24]$ | $Z_2$ |
| $\Gamma_44$ | $H_3 \cup \{\alpha^{21}\}$ | $[3, 3, 3, 3, 34, 3, 3]$ | $S_3$ |
| $\Gamma_45$ | $H_3 \cup \{\alpha^{23}\}$ | $[3, 7, 10, 25, 13, 29, 5]$ | $I$ |
| $\Gamma_46$ | $H_5 \cup \{\alpha^7\}$ | $[5, 6, 9, 27, 29, 28, 31]$ | $I$ |
| $\Gamma_47$ | $H_5 \cup \{\alpha^8\}$ | $[5, 7, 12, 5, 24, 24, 7]$ | $Z_2$ |
| $\Gamma_48$ | $H_5 \cup \{\alpha^{15}\}$ | $[5, 11, 11, 27, 20, 19, 30]$ | $I$ |
| $\Gamma_49$ | $H_5 \cup \{\alpha^{22}\}$ | $[5, 14, 15, 8, 9, 7, 11]$ | $I$ |
| $\Gamma_50$ | $H_6 \cup \{\alpha^8\}$ | $[6, 7, 17, 30, 29, 25, 24]$ | $I$ |
| $\Gamma_51$ | $H_6 \cup \{\alpha^{14}\}$ | $[6, 10, 13, 13, 6, 33, 33]$ | $Z_2$ |
| $\Gamma_52$ | $H_7 \cup \{\alpha^1\}$ | $[7, 8, 13, 29, 19, 18, 22]$ | $I$ |
| $\Gamma_53$ | $H_7 \cup \{\alpha^5\}$ | $[7, 11, 8, 31, 18, 18, 24]$ | $I$ |
| $\Gamma_54$ | $H_7 \cup \{\alpha^{16}\}$ | $[7, 12, 9, 27, 11, 8, 19]$ | $I$ |
| $\Gamma_55$ | $H_7 \cup \{\alpha^{25}\}$ | $[7, 16, 8, 20, 20, 7, 8]$ | $Z_2$ |
| $\Gamma_56$ | $H_8 \cup \{\alpha^{18}\}$ | $[8, 8, 8, 8, 8, 23, 8]$ | $S_3$ |
| $\Gamma_57$ | $H_9 \cup \{\alpha^{17}\}$ | $[9, 11, 14, 20, 9, 20, 11]$ | $Z_2$ |
| $\Gamma_58$ | $H_9 \cup \{\alpha^{19}\}$ | $[9, 13, 13, 31, 32, 30, 33]$ | $I$ |
| $\Gamma_59$ | $H_9 \cup \{\alpha^4\}$ | $[17, 18, 23, 18, 17, 26, 26]$ | $Z_2$ |
| $\Gamma_60$ | $H_9 \cup \{\alpha^{11}\}$ | $[17, 22, 18, 18, 21, 17, 22]$ | $Z_2$ |
| $\Gamma_61$ | $H_9 \cup \{\alpha^{12}\}$ | $[17, 22, 26, 17, 22, 34, 26]$ | $Z_2$ |
| $\Gamma_62$ | $H_9 \cup \{\alpha^{16}\}$ | $[18, 19, 18, 29, 24, 29, 19]$ | $Z_2$ |
| $\Gamma_63$ | $H_9 \cup \{\alpha^{19}\}$ | $[18, 20, 18, 28, 24, 25, 27]$ | $I$ |
| $\Gamma_64$ | $H_9 \cup \{\alpha^{22}\}$ | $[18, 22, 26, 22, 18, 26, 30]$ | $Z_2$ |
Using the same technique, the projectively inequivalent $k$-sets, $k = 8, ..., 14$ have been founded and also the stabilizer groups of these $k$-sets are computed. The results are given in the following theorem.

Let $n_e$ denote the number of projectively inequivalent $k$-set.

**Theorem 3.4:** On $PG(1,27)$, the number of projectively inequivalent $k$-sets are as follows:

| $k$ | $n_e$ |
|-----|-------|
| 8   | 196   |
| 9   | 383   |
| 10  | 745   |
| 11  | 1142  |
| 12  | 1665  |
| 13  | 1976  |
| 14  | 2170  |

Table 6. 8-Set details

| SG-type | No. |
|---------|-----|
| $I$     | 128 |
| $Z_2$   | 54  |
| $V_4$   | 11  |
| $D_4$   | 3   |

Table 7. 9-Set details

| SG-type | No. |
|---------|-----|
| $I$     | 323 |
| $Z_2$   | 54  |
| $Z_3$   | 4   |
| $(Z_3 \times Z_3) \rtimes Z_2$ | 1 |

Table 8. 10-Set details

| SG-type | No. |
|---------|-----|
| $I$     | 600 |
| $Z_2$   | 125 |
| $Z_3$   | 4   |
| $V_4$   | 15  |
| $(Z_3 \times Z_3) \rtimes Z_2$ | 1 |

Table 9. 11-Set details

| SG-type | No. |
|---------|-----|
| $I$     | 1043|
| $Z_2$   | 99  |

Table 10. 12-Set details

| SG-type | No. |
|---------|-----|
| $I$     | 1449|
| $Z_2$   | 182 |
| $Z_3$   | 3   |
| $V_4$   | 21  |
| $S_3$   | 5   |
| $D_4$   | 3   |
| $A_4$   | 1   |
In the following examples, some $k$-sets have been chosen where $k = 9, 10, 12, 13, 14$ with unique largest size of stabilizer group.

Let $\zeta(n)$ be a subset of $PG(1, 27)$ of length $n$.

**Example 3.5:**

(i) There is unique 9-set

$$\zeta(9) = \{\infty, 0, 1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^{12}, \alpha^{18}\} = H_{17} \cup \{\alpha^4, \alpha^{12}, \alpha^{18}\}$$

with stabilizer group of type $(Z_3 \times Z_3) \rtimes Z_2 = \langle \alpha^7 / (\alpha^{16} t + \alpha^5), (\alpha^{13} t + \alpha^2) \rangle$.

(ii) There is a unique 10-set

$$\zeta(10) = \{\infty, 0, 1, \alpha^{13}, \alpha, \alpha^2, \alpha^{10}, \alpha^{16}, \alpha^{24}, \alpha^{25}\} = H_1 \cup \{\alpha^{10}, \alpha^{16}, \alpha^{24}, \alpha^{25}\}$$

with stabilizer group of type $(Z_3 \times Z_3) \rtimes Z_2 = \langle \alpha^{13} / (t + \alpha^{13}), (\alpha^{13} t + 1) \rangle$.

(iii) There is a unique 12-set

$$\zeta(12) = \{\infty, 0, 1, \alpha^{13}, \alpha, \alpha^2, \alpha^3, \alpha^6, \alpha^{21}, \alpha^{25}, \alpha^{11}, \alpha^{17}\} = H_1 \cup \{\alpha^3, \alpha^6, \alpha^{11}, \alpha^{17}, \alpha^{21}, \alpha^{25}\}$$

with stabilizer group of type $A_4 = \langle \alpha^{21} / (\alpha^{22} t + \alpha^{15}), (t + \alpha^8) / (t + \alpha^{13}) \rangle$.

(iv) There is a unique 13-set

$$\zeta(13) = \{\infty, 0, 1, \alpha^{13}, \alpha, \alpha^2, \alpha^3, \alpha^8, \alpha^9, \alpha^{10}, \alpha^{23}, \alpha^5, \alpha^7\} = H_1 \cup \{\alpha^3, \alpha^8, \alpha^9, \alpha^{10}, \alpha^{23}, \alpha^5, \alpha^2\}$$

with stabilizer group of type $D_{13} = \langle (\alpha^{21} / \alpha^{24} t + \alpha^{14}), (\alpha^{13} t + \alpha^3) \rangle$.

(vi) There is a unique 14-set

$$\zeta(14) = \{\infty, 0, 1, \alpha^{13}, \alpha, \alpha^2, \alpha^3, \alpha^6, \alpha^{14}, \alpha^{15}, \alpha^{10}, \alpha^{16}, \alpha^{19}, \alpha^{23}\} = H_1 \cup \{\alpha^3, \alpha^6, \alpha^{10}, \alpha^{14}, \alpha^{15}, \alpha^{16}, \alpha^{19}, \alpha^{23}\}$$

with stabilizer group of type $D_{14} = \langle (\alpha^{13} t, (\alpha^9 t + \alpha^{12}) / (\alpha^9 t + \alpha^9) \rangle$.

4. **Conclusion**

In this section, the results of the paper have been summarized in Table 12, which presents the number of all inequivalent $k$-sets in $PG(1, 27)$, $3 \leq k \leq 14$, and the type of their stabilizer groups. A cell $G : m$ means that $m$ of $k$-sets stabilized by the group of type $G$; $n_k$ means the number of inequivalent $k$-sets.

| SG-type | No. |
|---------|-----|
| $I$     | 1840|
| $Z_2$   | 125 |
| $Z_3$   | 4   |
| $S_3$   | 6   |
| $D_{13}$| 1   |

| SG-type | No. |
|---------|-----|
| $I$     | 1924|
| $Z_2$   | 224 |
| $V_4$   | 19  |
| $Z_3$   | 1   |
| $D_7$   | 1   |
| $D_{14}$| 1   |
Table 13. Results summary

| $k$ | $\tilde{n}_k$ | $G : m$ |
|-----|---------------|---------|
| 3   | 1             | $S_3 : 1$ |
| 4   | 5             | $S_4 : 1$ | $V_4 : 4$ |
| 5   | 8             | $I : 2$ | $Z_2 : 6$ |
| 6   | 34            | $I : 6$ | $Z_2 : 14$ | $V_4 : 6$ | $S_3 : 4$ |
| 7   | 73            | $I : 51$ | $Z_2 : 17$ | $S_3 : 4$ | $D_7 : 1$ |
| 8   | 196           | $I : 128$ | $Z_2 : 54$ | $V_4 : 11$ | $D_4 : 3$ |
| 9   | 382           | $I : 323$ | $Z_2 : 54$ | $Z_3 : 4$ | $(Z_2 \times Z_3) \rtimes Z_2 : 1$ |
| 10  | 745           | $I : 600$ | $Z_2 : 125$ | $Z_3 : 4$ | $V_4 : 15$ | $(Z_2 \times Z_3) \rtimes Z_2 : 1$ |
| 11  | 1142          | $I : 1043$ | $Z_2 : 99$ |
| 12  | 1665          | $I : 1449$ | $Z_2 : 182$ | $Z_3 : 3$ | $V_4 : 21$ | $S_3 : 6$ | $D_4 : 3$ | $A_4 : 1$ |
| 13  | 1976          | $I : 1840$ | $Z_2 : 125$ | $Z_3 : 4$ | $S_3 : 6$ | $D_4 : 1$ |
| 14  | 2170          | $I : 1924$ | $Z_2 : 224$ | $V_4 : 19$ | $Z_3 : 1$ | $D_7 : 1$ | $D_{14} : 1$ |

It is worth noting that the $k$-set, $\zeta(k)$, which stabilizer group type of large order appeared when classification of $\zeta(k)$ to its $(k - 1)$-sets, all of them are projectively equivalent to a unique type of $(k - 1)$-set. And a $k$-set, $\zeta(k)$, with single identity, $I$, most of the $(k - 1)$-subsets are projectively distinct. For example:

(i) The 6-sets $H_{21}, H_{30}$ and $H_{34}$ in Theorem 3.3 have six 5-subsets in one type with stabilizer group of type $S_3$.

The 6-sets $H_1, H_2, H_3, H_7, H_{19}, H_{11}, H_{13}, H_{19}, H_{22}$ and $H_{26}$ in Theorem 3.3 the most of the six 5-subsets are projectively distinct with stabilizer group of type $I$.

(ii) The unique 7-set, $\Gamma_7 = H_{28} \cup \{\alpha^{12}\}$, in Theorem 3.3 with stabilizer group of type $D_7$ has seven 6-subsets all of them are projectively equivalent to $H_{28}$. But, the 7-set, $\Gamma_7 = \{\infty, 0, 1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7\}$ has a single group as a stabilizer group and all the seven 6-subsets of it are projectively distinct.

(iii) The unique 9-set, $\zeta(9)$, in Example 3.5(i) with stabilizer group of type $(Z_3 \times Z_3) \rtimes Z_2$ has nine 8-subsets all of them are projectively equivalent to $H_{17} \cup \{\alpha^4, \alpha^{12}\}$.

The 9-set, $\zeta(9) = \{\infty, 0, 1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7\}$ has a single group as a stabilizer group and all the nine 8-subsets of it are projectively distinct.

(iv) The unique 10-set, $\zeta(10)$, in Example 3.5(ii) with stabilizer group of type $(Z_3 \times Z_3) \rtimes Z_2$ has ten 9-subsets of it projectively equivalent to $H_{17} \cup \{\alpha^{10}, \alpha^{16}, \alpha^{24}\}$, and one of them is projectively equivalent to $H_{17} \cup \{\alpha^4, \alpha^{12}, \alpha^{18}\}$.

The 10-set, $\zeta(10) = \{\infty, 0, 1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7\}$ has a single group as a stabilizer group and all the ten 9-subsets of it are projectively distinct.

Backed to the all previous works on the classification of projective lines the same idea will be deduced.

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