TOWARDS FIELD THEORY OF TURBULENCE

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Alexander Migdal
Department of Physics, New York University
726 Broadway, New York NY 10003

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ABSTRACT

We revisit the problem of stationary distribution of vorticity in three dimensional turbulence. Using Clebsch variables we construct an explicit invariant measure on stationary solutions of Euler equations with the extra condition of fixed energy flow/dissipation. The asymptotic solution for large circulation around large loops is studied as a WKB limit (instanton). The Clebsch fields are discontinuous across minimal surface bounded by the loop, with normal vorticity staying continuous. There is also a singular tangential vorticity component proportional to $\delta(z)$ where $z$ is the normal direction. Resulting flow has nontrivial topology. This singular tangent vorticity component drops from the flux but dominates the energy dissipation as well as the Biot-Savart integral for velocity field. This leads us to a modified equation for vorticity distribution along the minimal surface compared to that assumed in a loop equations, where the singular terms were not noticed. In addition to describing vorticity distribution over the minimal surface, this approach provides formula for the circulation PDF, which was elusive in the Loop Equations.

1 Introduction

Turbulence is well studied at a phenomenological level using numerical simulations of forced Navier-Stokes equations and fitting the data for distribution of various observables (such as moments of velocity and vorticity fields, as well as velocity circulation). The data suggest multi-fractal scaling laws implying some significant modifications of traditional Kolmogorov scaling by finite size vorticity structures with nontrivial distributions by shape, size and vorticity filling.

The microscopic theory, such as an effective Hamiltonian for the Gibbs distribution in ordinary critical phenomena, is missing. It is as though we already know the Newtonian dynamics but do not yet know the Boltzmann distribution. We can simulate the Navier-Stokes equations and average over time, but we lack basic definitions of stationary statistics for vorticity or velocity fields.

This statistics would be a fixed point of the evolution of the Hopf functional. If we knew such an analog of the Boltzmann law, we would be able to solve the theory analytically (at least in some extreme regime such as a large circulation limit for large loops). We would also have powerful Monte-Carlo methods with the Metropolis algorithm for fast simulation of this equilibrium statistics.

In this paper we are trying to fill this gap. We construct the distribution of vorticity and velocity in three dimensions which is manifestly conserved in Euler dynamics, while describing a steady energy cascade. It involves a two-component Clebsch field, as well as two auxiliary fields: one Bose field and one Majorana Grassmann field, both transforming as vectors in physical space $R_3$.

In the WKB limit the tails of the PDF for velocity circulation $\Gamma$ over large fixed loops $C$ are controlled by a classical field $\phi_{cl}(r)$ (instanton) concentrated around the minimal surface bounded by $C$.

The field is discontinuous across the minimal surface which leads to the delta function term for the tangent components of vorticity as a function of normal coordinate. The flux is still determined by the normal component of vorticity, which is smooth.
We study minimal surfaces in great detail in Appendix A and we derive explicit formulas for the Clebsch instanton in Appendix B. It has nontrivial topology, deserving further investigation.

As for the scaling area law \( \Gamma^2 \sim A_C \) that we derived in [1] from consistency of the loop equation, it now follows from simple power counting in the instanton equation.

The surprise here is an explicit form of the circulation PDF (involving two or three phenomenological parameters depending on the symmetry of the loop \( C \)). This PDF perfectly matches the DNS data at large circulations where this WKB solution applies.

## 2 Energy Flow as a Constraint to Vortex Statistics

As is well known, the energy is pumped into the turbulent flow from the largest scales (pipes, ships, etc.), and dissipated at the smallest scales due to viscosity effects. Let us see how that happens in some detail. Using Navier-Stokes equation

\[
v' = \nu \partial^2_v v - v \partial_v \nu v - \partial_v p; \partial_v v = 0
\]

we have

\[
\partial_t \int d^3 r \frac{1}{2} v^2 = \int d^3 r \nu v \partial^2_v v - v \partial_v (v \partial_v v + \partial_v p) = 0
\]

Compare the first term with \( \int d^3 r \omega^2 \alpha \):

\[
\int d^3 r \omega^2 \alpha =
\int d^3 r \left( \nu \partial_v \nu v - v \partial_v (v \partial_v v + \partial_v p) \right) =
\int d^3 r \nu \partial_v v \partial_v v + \int d^3 r \partial_v (v \partial_v v - \partial_v v)
\]

So, we have the balance of two terms cancelling each other in the time derivative of energy: dissipation and pumping.

The mean energy dissipation rate (used in similar context in [2] as a constraint to Navier-Stokes equations) is

\[
-\dot{E} = \nu \int d^3 r \omega^2 \alpha (r)^2
\]

This is to be compared with the same energy flow from large scales:

\[
-\dot{E} = \int d^3 r \nu (v \partial_v v + \partial_v p) + \nu \partial_v (v \partial_v v - \partial_v v)
\]

By the Stokes theorem, this reduces to the flow over the boundary \( \partial V \) of the integration box \( V \):

\[
-\dot{E} = \int_{\partial V} d\sigma \beta \left( v \beta \left( p + \nu v \partial_v v - \partial_v v \right) \right)
\]

Velocity is related to vorticity by the Biot-Savart law:

\[
v = -e_{\alpha \beta \gamma} \partial_v \int d^3 r' \frac{\omega_{\beta} (r')}{4\pi |r - r'|}
\]

In case there is no vorticity at the bounding sphere, the radial velocity would decrease as \( 1/|r|^3 \) at infinity:

\[
v = -\frac{r}{4\pi |r|^4} Q_{\alpha}
\]

\[
Q_{\alpha} = e_{\alpha \beta \gamma} \int d^3 r' r' \omega_{\beta} (r')
\]

In the case of an arbitrary bounding surface with the normal \( n_{\alpha} (r) \), there would be the leading global vortex term:

\[
v \propto \frac{n_{\alpha} (r)r_{\beta}}{|r|^3} e_{\alpha \beta \gamma} \int d^3 r' \omega_{\beta} (r')
\]
which would cancel out provided the net integral of vorticity vanishes:

\[ \int d^3 r' \omega_n(r') = 0 \]  
\[ (16) \]

The next term of multipole expansion of the Biot-Savart law will be the dipole:

\[ v_n \propto \frac{n_\alpha}{|r|^3} \epsilon_{\alpha\beta\gamma} \int d^3 r' r'_\beta \omega_\gamma(r') \]
\[ (17) \]

which is the term we keep. For a finite resulting flow after cancellation of powers of $|r|$, the pressure should behave as:

\[ p(r) \rightarrow |r| g(n) \]
\[ (18) \]

where $n_\alpha$ is the normal vector to the surface. This would correspond to finite force $\partial_\alpha p \sim 1$ which is distributed on a surface:

The resulting energy flux is:

\[ -\dot{E} \rightarrow Q_\alpha f_\alpha \]
\[ (19) \]

\[ f_\alpha = \lim_{|V| \rightarrow \infty} \int_{r \in \partial V} \frac{d\sigma(r)}{4\pi |r|^3} n_\alpha(r) p(r) \]
\[ (20) \]

This formula works for a generic bounding surface $S = \partial V$, in a limit when it is blown up to infinity. For a sphere, it becomes an average over unit vectors $n \in S_2$:

\[ f_\alpha = \int_{n \in S_2} \frac{d^2 n}{4\pi} n_\alpha g(n) = \langle n_\alpha g(n) \rangle_{n \in S_2} \]
\[ (21) \]

This random force $f_\alpha$ would have some unknown PDF depending upon the specific microscopic mechanism of energy pumping:

\[ dP(\vec{f}) = P(\vec{f}) d^3 f \]
\[ (22) \]

The natural assumption is that this PDF is Gaussian, in accordance with the Central Limit Theorem for an average of large number of uncorrelated forces on a surface of a remote sphere.

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1 This geometry, with finite cell confining vorticity and energy flow being pumped from a distant boundary surface, was recently realized in beautiful experiments by [3], where the vortex rings were initially shot from the eight corners of a glass cubic tank, and a stable vorticity cell (a confined vorticity "blob" in their terms) was created and observed and studied in the center of the tank. The energy was pumped in pulses from eight corners and the vorticity distribution inside the cell was consistent with K41 scaling. Reynolds numbers in that experiment were not large enough for our instanton, but at least the energy flow entering from the boundary and dissipating in a vortex cell inside was implemented and studied in real water.
The asymptotic behavior of the normal component of velocity is parameterized by the above vector $Q$ in (13). Note that this vector is not invariant with respect to translation of coordinates. By specifying this vector we effectively are specifying the center of mass of the vorticity distribution inside, relative to the center of the volume inside bounding surface.

Note that while the dissipation formula (8) has the space integral supported on high vorticity regions, the incoming energy flow is concentrated on the bounding surface. The energy balance (2) requires that these two values of energy flow are equal. All the energy pumped from the boundary dissipates by viscosity at small scales inside vorticity cells. This provides the relation between vorticity distribution and the random force:

$$-\int_V d^3r \nu \omega^2 \alpha(r) + f_{\alpha} \int_V d^3r e_{\alpha,\beta\gamma} r_{\beta} \omega_{\gamma}(r) = 0$$

(23)

### 3 Statistical Mechanics

It is significant that this relation involves distribution of vorticity in the cell, where all dissipation is taking place. The second term comes from the flow through the boundary at infinity, but it involves the vorticity inside the cell. All the boundary conditions at infinity are represented by a (Gaussian) random vector force $\vec{f}$.

Note that this is not the same as postulating the energy spectrum of the pumping forces. We have only one vector with Gaussian distribution with some unknown variance.

In the following section, we are going to add this constraint not as a delta function in microcanonical distribution, but rather as exponential factor, inserted in canonical distribution with corresponding Lagrange multiplier $\lambda$. The motivation is the same as in statistical mechanics. We assume there is a "thermostat" interacting with a subsystem, with subsystem exchanging energy flow with "thermostat".

This is not the Gibbs distribution, of course, and the term "thermostat" does not mean that this chemical potential $\lambda$ is related to the temperature.

Here is a physical picture we see as an origin of this thermodynamics. Consider a subsystem – single vorticity cell. The energy flowing through the infinite boundary is related to the net dipole moment in (14). This involves contributions from the other cells over the whole space, which act as a "thermostat".

The net energy flow constraint (23) tells us that net flow from the bounding surface is dissipated in all the cells, the subsystem as well as the thermostat. If we single out the dissipation inside the subsystem, then there is a missing piece, both the contribution of other cells to the net dipole moment (14) and the dissipation terms $\int_V d^3r \nu \omega^2 \alpha(r)$ inside these other cells.

Therefore, the equation (23) adds up from the subsystem and from the thermostat. If we single out the subsystem $\mathcal{E}$, there will be an extra fluctuating term $\mathcal{E}'$. The exponential distribution with Lagrange multiplier for the subsystem energy flow accounts for that extra term. The Lagrange multiplier comes about as logarithmic derivative of the phase space of the thermostat with respect to energy of the subsystem (in this case the energy flow).

Technically, we have (with $d\Gamma'$ representing the phase space element for the thermostat)

$$\int d\Gamma' \delta (\mathcal{E}' + \mathcal{E}) \propto \exp (S(-\mathcal{E}))$$

(24)

where $S(\mathcal{E})$ is an entropy (logarithm of total phase space volume of the hyper surface of energy flow constraints) of the thermostat. The statistical mechanics then proceeds with expanding this entropy in the (relatively small) contribution to the energy flow from the subsystem

$$S(-\mathcal{E}) \rightarrow S(0) - \lambda \mathcal{E}$$

$$\lambda = S'(0)$$

(25)

(26)

In case of microcanonical distribution, this entropy counted the volume of the energy hyper surface in phase space and we had $\lambda = \beta$.

In our case (see below) we are going to integrate over space of so called Generalized Beltrami Flows, so this entropy will count the volume of the hyper surface of energy flow constraint in the space of these flows. But the general philosophy of interaction between the thermostat and the subsystem via exchange of thermodynamic variables, fixed by certain fugacities (Lagrange multipliers for microscopic constraints) is the same here.
4 Generalized Beltrami Flow

Let us go deeper into the hydrodynamics.

We parameterize the vorticity by two-component Clebsch field \( \phi = (\phi_1, \phi_2) \in \mathbb{R}^2 \):

\[
\omega_\alpha = \frac{1}{2} \varepsilon_{\alpha\beta\gamma} e_{ij} \partial_\beta \phi_i \partial_\gamma \phi_j
\]  

(27)

The Euler equations are then equivalent to passive convection of the Clebsch field by the velocity field:

\[
\partial_t \phi_\alpha = -v_\alpha \partial_\alpha \phi_\alpha \tag{28}
\]

\[
v_\alpha(r) = \frac{1}{2} e_{ij} (\phi_i \partial_\alpha \phi_j)^\perp
\]  

(29)

Here \( V^\perp \) denotes projection to the transverse direction in Fourier space, or:

\[
V^\perp_\alpha(r) = V_\alpha(r) + \partial_\alpha \partial_\beta \int_{\mathbb{R}^3} d^3 r' \frac{V_\beta(r')}{4\pi |r - r'|}
\]  

(30)

One may check that projection (29) is equivalent to the Biot-Savart law (12).

The conventional Euler equations for vorticity:

\[
\partial_t \omega_\alpha = \omega_\beta \partial_\beta v_\alpha - v_\beta \partial_\beta \omega_\alpha \tag{31}
\]

follow from these equations.

The Clebsch field maps \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \) and the velocity circulation around the loop \( C \in \mathbb{R}^3 \):

\[
\Gamma(C) = \oint_C dr_\alpha v_\alpha = \oint_{\gamma_2} \phi_1 d\phi_2 = \text{Area}(\gamma_2)
\]  

(32)

becomes the oriented area inside the planar loop \( \gamma_2 = \phi(C) \). We discuss this relation later when we build the Clebsch instanton.

The most important property of the Clebsch fields is that they represent a \( p,q \) pair in this generalized Hamiltonian dynamics. The phase-space volume element \( D\phi = \prod_x d\phi_1(x) d\phi_2(x) \) is invariant with respect to time evolution, as required by the Liouville theorem. We will use it as a base of our distribution.

The generalized Beltrami flow (GBF) corresponding to stationary vorticity is described by \( G_\alpha(x) = 0 \) where:

\[
G_\alpha \overset{\text{def}}{=} \omega_\beta \partial_\beta v_\alpha - v_\beta \partial_\beta \omega_\alpha
\]  

(33)

These three conditions are in fact degenerate, as \( \partial_\alpha G_\alpha = 0 \). So, there are only two independent conditions, the same number as the number of local Clebsch degrees of freedom. However, as we see below, relation between vorticity and Clebsch field is not invertible.

There is some gauge invariance (canonical transformation in terms of Hamiltonian system, or area preserving diffeomorphisms geometrically):

\[
\phi_\alpha(r) \Rightarrow G_\alpha(\phi(r))
\]

(34)

\[
\det \frac{\partial G_\alpha}{\partial \phi_b} = \frac{\partial (G_1,G_2)}{\partial (\phi_1,\phi_2)} = 1.
\]  

(35)

These transformations manifestly preserve vorticity and therefore velocity.

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2I am grateful to Pavel Wiegmann for drawing my attention to this invariance.

3These variables and their ambiguity were known for centuries [4] but they were not utilized within hydrodynamics until pioneering work of Khalatnikov [5] and subsequent works of Kuznetzov and Mikhailov [6] and Levich [7] in early 80-ties. Modern mathematical formulation in terms of symplectomorphisms was initiated in [8]. Derivation of K41 spectrum in weak turbulence using kinetic equations in Clebsch variables was done by Yakhot and Zakharov [9].

In my work [10] the Clebsch variables were identified as major degrees of freedom in statistics of vortex cells and their potential relations to string theory was suggested. Finally, in recent work [11] I identified the surface degrees of freedom of the vortex cells as \( U(1) \) compactified critical \( c = 1 \) string in two dimension, which was exactly solved by means of matrix models.
In terms of field theory, this is an exact gauge invariance, rather than the symmetry of observables, much like color gauge symmetry in QCD. This is why back in the early 90-ties I referred to Clebsch fields as "quarks of turbulence". To be more precise, they are both quarks and gauge fields at the same time.

It may be confusing that there is another gauge invariance in fluid dynamics, namely the volume preserving diffeomorphisms of Lagrange dynamics. Due to incompressibility, the volume element of the fluid, while moved by the velocity field, preserved its volume. However, these diffeomorphisms are not the symmetry of the Euler dynamics, unlike the area preserving diffeomorphisms in Clebsch variables.

One could introduce gauge fixing, for example the one mapping some surface bounded by a loop $C$ inside a disk with the same area in Clebsch plane. We study the instanton in this gauge for the case of a planar loop in a later section of this paper. This gauge condition is linear and therefore it does not require any extra Faddeev-Popov ghosts.

The global description of the orbits of these symplectomorphisms is a hard mathematical problem which we do not address here. This subject deserves professional mathematical investigation.

Note also that our condition comes from the Poisson bracket with Hamiltonian $H = \int d^3r \frac{1}{2} v^2$.

\[ G_\alpha(r) = [\omega_\alpha, H] = \int d^3r \frac{\delta \omega_\alpha(r)}{\delta \phi_j(r')} \frac{\delta H}{\delta \phi_i(r')} = \int d^3r \frac{\delta \omega_\alpha(r)}{\delta \phi_j(r')} \frac{\delta H}{\delta \phi_i(r')} \]

We only demand that this integral vanish. The stationary solution for Clebsch would mean that the integrand vanishes with some unknown function $h$. The stationary solution for Clebsch field is being gauge transformed while convected by the flow. For the vorticity this means the circulation over large loop.

The GBF does not correspond to stationary Clebsch field: the more general equation

\[ \partial_t \omega_\alpha = \int d^3r \frac{\delta \omega_\alpha(r)}{\delta \phi_j(r')} \partial_t \phi_j(r) \]

\[ \partial_t \phi_i = -v_\alpha \partial_\alpha \phi_i + \epsilon_{ij} \frac{\partial h(\phi)}{\partial \phi_j} \]

with some unknown function $h(\phi)$ would still provide the GBF. The last term drops from here in virtue of infinitesimal gauge transformation $\delta \phi_a = \epsilon e_{ab} \frac{\partial h(\phi)}{\partial \phi_b}$ which leave vorticity invariant.

This means that Clebsch field is being gauge transformed while convected by the flow. For the vorticity this means the same GBF.

### 5 Our Main Conjecture

We propose the following grand canonical ensemble:

\[ dZ = dP(\tilde{f}) D\phi \delta_{FP} [F|\phi] \exp \left( -\lambda \left( \int_V d^3r v^2 - f_\alpha \int_V d^3r e_{\alpha\beta\gamma} r^\beta \omega^\gamma \right) \right) \]

where $\delta_{FP}$ is the Faddeev-Popov delta functional

\[ \delta_{FP}[F|\phi] = \det \frac{\delta G_\alpha}{\delta \phi_b} \int DU \exp \left( i \int d^3x U_a(x) G_\alpha(x) \right) \]

The functional determinant $\det \frac{\delta G_\alpha}{\delta \phi_a}$ compensates for transformation of our constraint $F$ with respect to evolution making our measure conserved as required by the Liouville theorem.

We need to be more specific here. What is the determinant of the operator where the left index is vector and the right one is Clebsch index? The left index transforms as vector under $O(3)$ rotations while the right index transforms covariantly under symplectomorphisms?

The only definition we found which satisfies desired symmetry properties is the following one. Consider Poisson bracket

\[ [G_\alpha(x), G_\beta(y)] = \int d^3z \frac{\delta G_\alpha(x)}{\delta \phi_a(x)} \epsilon_{ab} \frac{\delta G_\beta(y)}{\delta \phi_b(z)} \]

\[ \frac{\delta G_\alpha(x)}{\delta \phi_a(x)} = \epsilon_{ab} \frac{\delta G_\beta(y)}{\delta \phi_b(z)} \]
It is invariant with respect to symplectomorphisms as one can readily check.

\[ e_{ab} \frac{\partial G_a}{\partial \phi^a} \frac{\partial G_b}{\partial \phi^b} = e_{a'b'} \det \frac{\partial G_a}{\partial \phi^b} = e_{a'b'} \]  

From the point of view of matrix products in functional space this Poisson bracket is a product of three operators \( \delta F \delta \phi \times \hat{E} \times \delta F \delta \phi^T \), where \( \hat{E}_{a,b}(x, y) = e_{a,b} \delta(x - y) \). This makes determinant of Poisson bracket equal to the square of our determinant times \( \det \hat{E} = 1 \).

Henceforth our determinant can be defined as a pfaffian

\[ \det \frac{\delta G_{\alpha}}{\delta \phi^b} \equiv \sqrt{\det [G_{\alpha}(x), G_{\beta}(y)]} = \text{pf} ([G_{\alpha}(x), G_{\beta}(y)]) \]  

There are zero modes associated with conservation

\[ \frac{\partial}{\partial x_\alpha} [G_{\alpha}(x), G_{\beta}(y)] = 0, \quad \frac{\partial}{\partial y_\beta} [G_{\alpha}(x), G_{\beta}(y)] = 0 \]  

So this determinant formally would be zero, unless we project out these zero modes. Otherwise it is well defined invariant kernel with well defined eigensystem.

The Lagrange multiplier \( \lambda \) is conjugate to the energy flow constraint, so we have to use the thermodynamic relation

\[ \mathcal{E} = -\frac{\partial \log Z}{\partial \lambda} \]  

where \( \mathcal{E} \) is the energy flow from the "thermostat" to the subsystem under consideration.

Note that our distribution does not fix the scale of the Clebsch fields.

### 6 Finite Dimensional Examples

In a one dimensional rather than functional case, our distribution would be just the sum of delta functions of all the roots \( R \) of \( F \):

\[ \delta_{FP}[F|\phi] = \sum_{R:F[R]=0} \delta(\phi - R) \text{sign } F'(R) \]  

\[ \int D\phi \delta_{FP}[F|\phi]F[\phi] = \sum_{R:F[R]=0} F[R] \text{sign } F'(R) \]  

The finite dimensional analog of GBF would be the roots of \( F \) which are time-independent. Even if the function itself evolves with time, the Faddeev-Popov delta function projects on these conserved roots with unit coefficients.

Let us study our distribution for a simple example of \( N \) dimensional particle moving in the potential well with Hamiltonian:

\[ H(\vec{p}, \vec{q}) = \frac{\vec{p}^2}{2} + U(\vec{q}) \]  

the steady state equations would be simply with \( U_i = \partial_i U, U_{ij} = \partial_i \partial_j U \):

\[ \phi = (p_i, q_i) \]  

\[ G_{\alpha} = (-U_i, p_i) \]  

\[ [G_{\alpha}, G_{\beta}] = \begin{pmatrix} 0, U_{i,j} \\ -U_{j,i}, 0 \end{pmatrix} \]  

\[ J = |\det U_{ij}| \]

Following our prescription in this case would lead to the distribution:

\[ P(\vec{p}, \vec{q}) = \sum_{\vec{r} \in \nabla U(r) = 0} \delta(\vec{p}) \delta(\vec{q} - \vec{r}) \]

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4 We define pfaffian as a product of positive eigenvalues: \( \text{pf} M = \prod_{\lambda > 0} \lambda \) in every pair \( \pm \lambda \) in the spectrum of \( M \). The zero eigenvalues (zero modes) are excluded by appropriate gauge fixing.
which corresponds to a particle sitting at the local extrema \( r \) of the potential well with zero momentum. Even in this elementary example we see a complication. Consider axial symmetric potential of sombrero hat,

\[
U = \frac{1}{2} (q^2 - 1)^2
\]  

(56)

There is a maximum at the origin and degenerate minimum: a sphere \( q^2 = 1 \). We get determinant \( \det U_{ij} \propto (-1)^N \) at the maximum, positive \( \det U_{ij} \) at \( N = 1 \) and zero determinant at \( N > 1 \) at the minimum. The zero determinant at \( N > 1 \) comes because of the zero modes corresponding to rotations of this minimal sphere.

This is clearly not what we need: to reject the maximum and keep the minimum. To reject the maximum we need to demand that the whole matrix of second derivatives is positive definite, not just its determinant.

To remove the fictitious zero weight, let us add a linear force, which will act as gauge fixing

\[
U = \frac{1}{2} (q^2 - 1)^2 - \vec{f}.\vec{q}
\]

(57)

Now, at arbitrary \( f \) there will be only one stable minimum and we shall pick it, and we can tend \( \vec{f} \to 0 \).

7 Lyapunov Stability and Theta Factor

In general case, we have to fix the gauge\(^5\) and eliminate all the unstable GBF.

This Lyapunov stability of GBF is in fact determined by another kernel

\[
L_{\alpha\beta}(x, y) = \frac{\delta G_{\alpha}(x)}{\delta \omega_{\beta}(y)}
\]

(58)

which is not symmetric. For stability of our flow we need its eigenvalues (Lyapunov exponents) to all have negative or zero real part. There should not be any eigenvalues in the right semi-plane.

There is a simple identity which allows to count for a matrix \( \hat{L} \) the number of eigenvalues with positive real part (which we want to reject here)

\[
N_+ (\hat{L}) = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \frac{dz}{2\pi} \exp(i\epsilon z) \text{tr} \frac{1}{L + i\epsilon z}
\]

(59)

In our case this number must be zero, so that we introduce extra factor

\[
\Theta[\omega] = \theta \left( \frac{1}{2} - N_+ (\hat{L}) \right)
\]

(60)

Note that this formula does not rely on quantization of \( N_+ (\hat{L}) \) which may not be valid for operators in Hilbert space. Even if there is a continuous distribution of eigenvalues, this \( N_+ (\hat{L}) \) will remain positive in case there are some eigenvalues distributed in the right semi-plane. For any distribution in the left semi-plane including imaginary axis this \( N_+ (\hat{L}) \) would remain zero. For infinite number of eigenvalues in right semi-plane \( N_+ (\hat{L}) \to +\infty \) so that theta function still works.

If we introduce the extended operator \( \hat{L}(z) = \hat{L} + iz \)

\[
\text{tr} \frac{1}{\hat{L}(z)} = -i \partial_z \log \det \hat{L}(z)
\]

(61)

\[
N_+ (\hat{L}) = - \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \frac{dz}{2\pi i} \exp(i\epsilon z) \partial_z \log \det \hat{L}(z) =
\]

(62)

\[
\frac{1}{2\pi} \Delta_+ (z) \arg \det \hat{L}(z)
\]

(63)

Here \( \Delta_+ (z) \arg F(z) \) stands for the total phase acquired by \( F(z) \) when \( z \) goes around the anti-clockwise loop in upper semi-plane surrounding zeroes of \( F(z) \). In other words, it counts all eigenvalues of \( \hat{L} \) in the right semi-plane.

So, we have stability selection factor

\[
\Theta[\omega] = \theta \left( \pi - \Delta_+ (z) \arg \det \hat{L}(z) \right)
\]

(64)

\(^5\) Without gauge fixing our determinant will formally be zero due to the zero modes corresponding to gauge transformation.
Using Fourier Transform of theta function we finally find

\[
\Theta[\omega] = \int_{-\infty}^{\infty} \frac{dy}{2\pi i (iy + 1)} \exp \left( iy \left( \pi - \Delta_+^\omega \right) \arg \det \hat{L}(z) \right)
\]  

(65)

Coming back to our distribution with prescription (65) we see that the distribution is uniformly covering stable generalized Beltrami flows, and therefore is conserved in Euler dynamics. The gauge invariance remains unbroken at this stage. We do not know the general prescription of unambiguous gauge fixing, but in case of our instanton we can present a unique gauge condition (see below).

This is clearly not the Gibbs distribution (which would be undesirable). We are looking for an alternative fixed point of the PDF evolution which is capable of describing fixed energy flow instead of fixed energy.

As we shall see below, the GBF provides an adequately rich space of steady solutions that can incorporate energy flow.

The velocity circulation PDF is generated by the further constraint in (41):

\[
P(\Gamma|C) = \int D\phi \delta [G_\alpha] \text{pf} ([G_\alpha, G_\beta]) \Theta[\omega] \tag{66}
\]

\[
\delta \left( \Gamma - \oint_{\gamma_2} d\phi_1 d\phi_2 \right) \exp \left( -\lambda \left( \nu \int_V d^3 r \omega_\alpha(r)^2 - f_\alpha \int_V d^3 r e_{\alpha\beta\gamma}(r, \omega_\gamma(r)) \right) \right) \tag{67}
\]

By construction, this \( P(\Gamma|C) \) satisfies the Euler Loop equations, as they are equivalent to

\[
\left\langle \exp \left( i \gamma \oint d\vec{r} \vec{v} \right) \int d\vec{r} \times \vec{\omega} \right\rangle = 0
\]  

(68)

which reduces by the Stokes theorem to the flow of \( \nabla \times \vec{v} \times \vec{\omega} \) through the surface bounded by \( C \). This flow vanishes by virtue of steady equations of motion (31) for \( \omega \).

Moreover, the cancellation of the functional determinant between the delta function and Pfaffian means that our PDF reduces to the average over space of all stable GBF. To be more precise, we have constructed an invariant measure in this space.

8 Ghost Fields

With our modified Faddeev-Popov delta functional we can still use their ghost fields but to get Pfaffian we need one Grassmann field, not two:

\[
\int D\phi \text{pf} ([G_\alpha, G_\beta]) \delta [G_\alpha] = \int D\phi DU D\Psi \exp \left( i \left( \langle U_\alpha | G_\alpha \rangle + \langle \Psi_\alpha ||G_\alpha, G_\beta|| \Psi_\beta \rangle \right) \right) \tag{69}
\]

with \( \Psi_\alpha \) being Grassman field and \( \langle A|B \rangle, \langle A|X|B \rangle \) stands for vector and matrix products in functional space.

One may verify the simple re-scaling of Clebsch field leaves the measure invariant except for the random force PDF. The fields transform according to their dimensions:

\[
\phi_\alpha \Rightarrow \lambda \phi_\alpha \tag{70}
\]

\[
U_\alpha \Rightarrow \lambda^{-4} U_\alpha \tag{71}
\]

\[
\Psi_\alpha \Rightarrow \lambda^{-3} \Psi_\alpha \tag{72}
\]

The scale factors of \( \lambda \) emerging in the measure \( D\phi DU D\Psi \) will all cancel among themselves since our measure is dimensionless (the Grassmann variable measures transforms with inverse Jacobian). As for the phase counter \( \Theta[\omega] \) it is obviously invariant as the scaling of operator \( \hat{L} \) does not affect its phase.

The distribution for the random force will break this scale invariance. The same is true with respect to time reversal, which corresponds to the interchange of \( \phi_1, \phi_2 \). The distribution again does not change, but the random force will break this invariance, if its PDF is not even with respect to reflection \( f \Rightarrow -f \).

This representation of invariant measure with ghost fields is suitable for the perturbative expansion in a background of a classical solution (instanton), which, as we shall see, dominates the distribution in the case of large circulation around a large loop.
9 Clebsch Confinement

Let us look more closely at our functional integral. By naive counting of degrees of freedom it is just a number, as we have two degrees of freedom at each point in space and two independent local constraints, so that the whole integral reduces to a sum over solutions of these constraints, just as it did in the case of a particle in a potential well.

It is unfortunately not so simple: there is in fact a functional degeneracy of these constraints. First, one could shift vorticity by velocity times the arbitrary local scalar field $\vec{\omega} \Rightarrow \vec{\omega} + \phi(r)\vec{v}$ as long as $v_\alpha \partial_\alpha \phi = 0$ (meaning this field does not change along the flow). Also, from $\nabla \times \vec{v} \times \vec{\omega} = 0$ we can have $\vec{v} \times \vec{\omega} = \nabla F$ with arbitrary $F(x)$.

Naturally, we implied the ambiguity of the primary constraints as functionals of velocity and vorticity. As you start solving these constraints you will find that $F(x) = \nabla \left( p + \frac{1}{2} \vec{v}^2 \right)$. This does not change the fact that these constraints are degenerate, as they do not involve pressure $p(r)$ and are satisfied with arbitrary pressure.

As for the Clebsch field itself, it can be transformed by arbitrary local area-preserving diffeomorphism, as noted in the previous section.

There is, however, a limit where the functional integral reduces to a classical flow (instanton) up to the symplectomorphism. This is the limit of large circulation $\Gamma$ over a large loop $C$.

Let us first describe a qualitative physical picture of our instanton. It is similar in spirit to the magnetic monopole in 3-dimensional gauge theories. In these theories the ground state has condensate of monopoles there which leads to a dual Meissner effect of pushing electromagnetic field from the vacuum, leading to collapse of this field in thin flux tubes between charges.

This was the origin of confinement in 3D gauge theories, but of course, literally the same mechanism is absent here. There is no gauge invariance associated with velocity playing the role of vector potential. There is no $U(1)$ symmetry and no associated charges, and hence no monopoles either.

Our gauge symmetry involves the Clebsch fields and our analogues of monopoles are singular sheets in physical space where our gauge potential $\phi_\alpha$ become multi-valued. And our analog of confinement is confinement of Clebsches, and our analog of gluon field shrinking to minimal surfaces bounded by quark loops is the vorticity shrinking to minimal surface in case large circulation over large loop is present.

We expect confinement phenomenon here, except instead of magnetic monopoles we have found different singular solutions leading to condensation of vorticity (our analog of magnetic field).

Here is this picture of vorticity condensation.

Comparing our two constraints (energy dissipation and fixed circulation) we observe that to minimize dissipation in effective Hamiltonian $\lambda \nu \int d^3r \omega^2_\alpha$ at fixed circulation we need the vorticity to be concentrated in a thin layer (of viscous thickness $h \sim \nu$) around the minimal surface $S_{\text{min}}(C)$ with area $A_C$ surrounded by $C$ and directed along the normal $n_\alpha$ to this surface to maximize the flux.

There are, of course, other vorticity cells randomly distributed all over space, with their own energy dissipations. We are considering the energy dissipation per cell $E_{\text{cell}}$, assuming this cell covers the minimal surface bounded by the loop $C$.

10 Clebsch instanton

Let us study this instanton solution in more detail.

The basic clue is that the Clebsch field can be multi-valued without affecting uniqueness of the vorticity. An example was presented in [6]:

$$\omega_\alpha = A e_{ijk} e_{\alpha \beta \gamma} S_i \partial_\beta S_j \partial_\gamma S_k; \ S_1^2 = 1 \quad (73)$$

It can be rewritten in terms of our Clebsch fields in polar coordinates $\theta \in (0, \pi), \varphi \in (0, 2\pi)$ for the unit vector $S = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$:

$$\phi_1 = 2A \cos \theta; \quad \phi_2 = \varphi \ (\text{mod} \ 2\pi) \quad (74)$$

---

[6] Later we find out that in addition to smooth normal component of vorticity, providing the flux, there is also a singular tangent component, dropping from the flux but dominating the energy flow balance.
The second variable $\phi_2$ is multi-valued, but vorticity is finite and continuous everywhere. The helicity $\int d^3rv_\alpha\omega_\alpha$ was ultimately related to winding number of that second Clebsch field[7].

We found another case of multi-valued Clebsch fields with nontrivial topology which are relevant to large circulation asymptotic behavior.

Let us seek a solution for the Clebsch fields, with discontinuity across the minimal surface bounded by $C$. At each side $S_\pm$ of the surface the normal derivative of $\phi_i$ vanishes so that $\phi$ varies only in local tangent plane:

$$[n_i\partial_i\phi_a]_{S_\pm} = 0$$

however the values of $\phi_a^\pm$ differ, so that the discontinuity

$$\Delta \phi_a(r) = \phi_a^+ - \phi_a^- \neq 0$$

The tangent vorticity will vanish on both sides, so that vorticity would be directed at the oriented normal to the surface and will be continuous, as only values of Clebsch field are jumping, but not the tangent plane derivatives. This applies only to the limits of vorticity from above and below the minimal surface (see the next section).

Such surface cut out in $R_3$ is shown here for the loop shaped as soccer gates, Fig.5 with some symbolic thickness to stress that this is a hole cut in space. This is actual minimal surface for this bounding loop.

With $\phi_a(\xi)$ depending only on local coordinates $\xi = (\xi_1, \xi_2)$ on the minimal surface $r_\alpha = X_\alpha(\xi)$ we have:

$$\Gamma = \int_{S_{\min}(C)} d\sigma_\alpha(r)\omega_\alpha(r),$$

$$\omega_\alpha(r) = n_\alpha(r)\Omega(r)$$

$$\Omega(r) = \frac{1}{\sqrt{G}} \frac{\partial(\phi_1, \phi_2)}{\partial(\xi_1, \xi_2)}$$

where $G$ is determinant of the induced metric $G_{ij} = \partial_iX_\alpha\partial_jX_\alpha$; $i, j = 1, 2$. Geometrically, this $\Omega$ is the ratio of area element in Clebsch plane to that on a minimal surface.

It is important that this $\Omega(r)$ factor can be extended in linear vicinity of the surface. Namely, in the linear vicinity in the normal direction it does not depend upon the normal coordinate $z$ as it follows from our condition (76) on normal derivatives of Clebsch field (again, this excludes $z = 0$ where there are singular terms $\propto \delta(z)$)

$$n_\alpha\partial_\alpha\Omega(r) = 0$$

11 **Clebsch boundary conditions at the minimal surface**

Let us verify it. In linear vicinity of local tangent plane to the surface its equation reads (with $K_1, K_2$ being principal curvatures at this point)

$$z - \frac{K_1}{2}x^2 - \frac{K_2}{2}y^2 = 0$$

$$n_i = \frac{(-K_1x, -K_2y, 1)}{\sqrt{1 + K_1^2x^2 + K_2^2y^2}} = (0, 0, 1) + O(x, y)$$

$$\Omega = n_\alpha\omega_\alpha \rightarrow \frac{1}{2}e_{ij}e_{ab}\partial_i\phi_a\partial_j\phi_b + O(x, y)$$

$$n_\alpha\partial_\alpha\Omega(r) \rightarrow e_{ij}e_{ab}\partial_i\phi_a\partial_j\phi_b + O(x, y)$$

The mixed derivatives $\partial_i\partial_\alpha\phi_a$ vanish at $x = y = z = 0$ for our boundary conditions.

Self-consistency of this solution for Clebsch parameterization requires that this surface should be a minimal surface.

Indeed, let us assume that $\phi_a$ has a discontinuity along some surface, with normal derivatives vanishing on both sides of the cut in $R_3$. In this case we would have vorticity proportional as the normal $n_\alpha$ to that surface with coefficient $\Omega(r)$ depending only on the local tangent coordinates, no $z$ dependence in linear vicinity.

---

[7] To be more precise, it was Hopf invariant on a sphere $S_3$ instead of real space $R_3$ (see [6] for details).
The vorticity conservation $\partial_\alpha \omega_\alpha = 0$ would then lead to the equation

$$0 = \partial_\alpha \omega_\alpha = \partial_\alpha (n_\alpha \Omega) = \Omega \partial_\alpha n_\alpha + n_\alpha \partial_\alpha \Omega$$

(86)

The term $\partial_\alpha n_\alpha$ here involves the surface derivatives as in $n_\alpha \partial_\beta n_\alpha = \frac{1}{2} \partial_\beta n^2 = 0$. Therefore

$$\partial_\alpha n_\alpha = (\delta_{\alpha \beta} - n_\alpha n_\beta) \partial_\beta n_\alpha = -K_1 - K_2$$

(87)

which is the divergence in the tangent plane, or trace of external curvature tensor (see [1] for detailed discussion).

We see, that for our boundary condition, with vanishing normal derivatives of Clebsch field and therefore vorticity, we arrive at the Plateau equation for the minimal surface $K_1 + K_2 = 0$.

This is quite remarkable: Clebsch field is allowed to have jumps across minimal surface as long as its normal derivatives vanish at each side of this surface!

12 Master Equation

The loop equation in the minimal surface approximation was analyzed in detail and solved numerically by Mathematica® code in [1]. It was assumed in that paper that vorticity was smooth and dominated by a region close to the minimal surface, where it was directed towards normal.

As we see now, with Clebsch instanton, this assumption is modified in a quite convoluted way: the vorticity flux is still determined by smooth normal component of vorticity. However, there is a tangential vorticity in an infinitely thin layer around the minimal surface. Formally this tangential component comes as a delta function, related to the discontinuity of the Clebsch field.

In Appendix A we reproduce for readers convenience the equations of the minimal surface theory as presented in old paper [12]. We derive explicit representation for the solution of Plateau problem $\overrightarrow{r} = \overrightarrow{X}(\xi)$ in polar coordinates $\xi = (\rho, \alpha)$ for a unit disk in parameter space.

In Appendix B we use equations from that theory to analyze deeper the instanton solution and its discontinuity in the vicinity of the minimal surface. We derive these singular terms in vorticity in Appendix B.

As we learn from this solution, the vorticity has the structure

$$\omega \left( \overrightarrow{r} = \overrightarrow{X}(\xi) + z\overrightarrow{n}(\xi) \right) = \delta(z)2\pi n\overrightarrow{\nabla}\Phi(\xi) \times \overrightarrow{n}(\xi) + \overrightarrow{n}(\xi)\Omega(\xi) + O(z^2)$$

(88)

$$\Omega(\xi) = \frac{m}{\sqrt{\det G}} \frac{\partial \Phi(\xi)}{\partial \rho}$$

(89)

$$G_{ij} = \partial_i X_{\mu}(\xi) \partial_j X_{\mu}(\xi)$$

(90)

$$\overrightarrow{n} \propto \partial_\rho \overrightarrow{X} \times \partial_\alpha \overrightarrow{X}; \quad \overrightarrow{n}^2 = 1$$

(91)

The delta term comes from the normal discontinuity of $\phi_2$

$$\phi_2 \left( \overrightarrow{r} = \overrightarrow{X}(\xi) + z\overrightarrow{n}(\xi) \right) = m\alpha + 2\pi n\theta(z) + O(z^2); \quad m, n \in \mathbb{Z}$$

(92)

while the other component is continuous

$$\phi_1 \left( \overrightarrow{r} = \overrightarrow{X}(\xi) + z\overrightarrow{n}(\xi) \right) = \Phi(\xi) + O(z^2)$$

(93)

This delta term in vorticity is orthogonal to the normal vector and thus does not contribute to the flux through the minimal surface, so this flux is still determined by the second (regular) term and circulation is related to this $\Phi(\xi)$

$$\Gamma_C = \oint \phi_1 d\phi_2 = m \int_0^{2\pi} \Phi(1, \alpha) d\alpha$$

(94)

However, the Biot-Savart integral with this Clebsch instanton is dominated by the singular tangential component and is finite

$$v_\beta(\overrightarrow{r}) = 2\pi n \left( \delta_{\beta \gamma} \partial_\alpha - \delta_{\alpha \beta} \partial_\gamma \right) \int_{S_{\min}} d\sigma_\gamma(\xi) \partial_\alpha \Phi(\xi) \frac{1}{4\pi |\overrightarrow{X}(\xi) - \overrightarrow{r}|}$$

(95)

Now our GBF equations

$$\omega_\alpha \partial_\alpha v_\beta = v_\alpha \partial_\alpha \omega_\beta$$

(96)
will contain singular terms \( \propto \delta'(z) \) and \( \propto \delta(z) \). Cancellation of \( \delta'(z) \) terms yields:

\[
n_\alpha(\eta)v_\alpha \left( \vec{X}(\eta) \right) = 0 \tag{97}
\]

and the \( \delta(z) \) terms:

\[
F_\alpha \partial_\alpha v_\gamma = v_\beta \partial_\beta F_\gamma 
\]

\[
F_\alpha = -2\pi n e_\alpha \beta \gamma n_\beta \partial_\gamma \Phi \tag{99}
\]

Matching the regular terms would produce equations relating normal derivative of velocity at the surface with tangent derivatives of regular part of vorticity. These terms would depend upon unknown behavior of vorticity off the minimal surface, where we assume it going to zero outside a thin layer \( |z| \sim \nu \).

This vector \( \vec{F} \) lies to the local tangent plane and so does the velocity (no flow across the minimal surface). It is interesting that our flow is almost anti-Beltrami: vorticity and velocity are orthogonal at the surface, except for the \( \delta(z) \) term created by discontinuity of the Clebsch field, where they at least lie in the same tangent plane.

In real world, after viscous smoothing this would mean thin film of quasi-Beltrami flow in a smooth anti-Beltrami flow. But most of dissipation would be happening in this thin film, where both velocity and vorticity are distributed parallel to the minimal surface.

These equations (96) are restricting the dependence of the scalar function \( \Phi(\xi) \) along the minimal surface. Presumably, these vector equations are all dependent, so that a single scalar function can satisfy them. The key equation is of course the surface projection (95) of Biot-Savart integral. The degeneracy of these equations becomes obvious in the form of the Poisson brackets. As we noted above, the GBF equations will be satisfied provided the Clebsch master equation

\[
v_\alpha \partial_\alpha \phi_a = e_{ab} \frac{\partial h(\phi)}{\partial \phi_b} \tag{100}
\]

with arbitrary gauge function \( h(\phi) \).

The leading term in these equations near the minimal surface is still the normal flow restriction (97), which annihilates the \( \delta(z) \) term in (100). The next order terms will already involve the gauge function \( h(\phi) \).

These equations are quite different from those we deduced from the loop equations (1), because the singular terms in vorticity were missed there. The analytical and numerical investigations of these equations will be a subject of a separate publication.

### 13 Helicity

Let us now look at the helicity integral

\[
H = \int_{R_3 \setminus S_{\text{min}}} d^3r \vec{v} \vec{\omega} \tag{101}
\]

Note that in conventional form

\[
v_i = \phi_1 \partial_i \phi_2 + \partial_i \phi_3 \tag{102}
\]

there will be singular terms in velocity \( \propto \delta(z) \). However, the Biot-Savart integral (95) demonstrates that these singular terms cancel between \( \phi_2 \) and \( \phi_3 \) leaving finite resulting velocity field.

To avoid these fictitious singularity, let us rewrite velocity in an equivalent form

\[
v_i = -\phi_2 \partial_i \phi_1 + \partial_i \tilde{\phi}_3 \\
\tilde{\phi}_3 = \phi_1 \phi_2 + \phi_3 \tag{103}
\]

This \( \tilde{\phi}_3 \) is single-valued, unlike the \( \phi_3 \). The discontinuity of the first term is compensated by that of the second one. It can be written as an integral over the whole space

\[
\tilde{\phi}_3(r) = -\partial_3 \int d^3r' \phi_2(r') \partial_3 \phi_1(r') \frac{1}{4\pi |r - r'|} \tag{105}
\]
Now the singular component $\phi_2$ is not differentiated, so that there are no singularities. The helicity integral could now written as a map $R_3 \mapsto (\phi_1, \phi_2, \tilde{\phi}_3)$

\[
H = \int_{R_3 \setminus S_{\text{min}}} d^3r \left( -\phi_2 \partial_i \phi_1 + \partial_i \tilde{\phi}_3 \right) e_{ijk} \partial_j \phi_1 \partial_k \phi_2 = \tag{106}
\]

\[
\int_{R_3 \setminus S_{\text{min}}} d\phi_1 \wedge d\phi_2 \wedge d\tilde{\phi}_3 \tag{107}
\]

Here is the most important point. There is a surgery performed in three dimensional Clebsch space: an incision is made along the surface $\phi (S_{\text{min}})$ and $n$ more copies of the same space are glued to this incision like sheets of a Riemann surface (see Fig.1 for $n = 1$), except this time the three dimensional spaces are glued at the two dimensional boundary, rather than the two-dimensional Riemann sheets glued at one-dimensional cut in complex plane.

Best analogy: this minimal surface is a portal to other Clebsch universes like in science fiction movies. The winding number of $\phi_2$ when passing through that surface counts number of these parallel universes.

Integrating over $\phi_2$ in (107), using discontinuity $\Delta \phi_2 (S_{\text{min}}) = 2\pi n$ and then integrating $\int_{S_{\text{min}}} d\tilde{\phi}_3 \wedge d\phi_1$ we find a

\[
H = 2\pi n \oint_C \tilde{\phi}_3 d\phi_1 \tag{108}
\]

This is $2\pi$ times the number of parallel universes times the area of the portal.

One may wonder how can the pseudoscalar invariant like helicity be present in GBF: it is just the time reversal which is broken by energy flow, but not spacial parity.

The answer is that in virtue of the symmetry of the master equation there is always a GBF with an opposite helicity (negative $n$) and the same probability. We will take both solutions, instanton and anti-instanton into account when using the WKB methods to compute circulation PDF in the next sections.

One may also wonder how do we get the nontrivial helicity if the velocity is orthogonal to vorticity at the surface where all action is happening. There are two answers.

Formally, helicity is created just by the discontinuity of the Clebsch field by the tangent component of vorticity in the infinitely thin boundary layer. This delta function contributes to the helicity integral.

Another answer is that in the helicity integral over the remaining space $R_3 \setminus S_{\text{min}}$, the dot product $\vec{v} \cdot \vec{\omega}$ is not zero but but rather reduces to a total derivative of the phase field $\phi_2$. After cancellations of all internal terms this integral is proportional to the total phase change from one side of the surface to another, which is $2\pi n$.

Regardless how we compute helicity we observe that resulting loop integral (108) involves non-singular field $\tilde{\phi}_3$ which depends upon the behavior of the basic Clebsch field $\phi_1, \phi_2$ in the whole remaining space, not just in linear vicinity of the minimal surface.

Our main physical assumption was that vorticity was concentrated in a thin layer surrounding the minimal surface. There is a singular tangential component $\propto \delta(z)$ and smooth normal component. For the smooth component to rapidly decrease outside this thin layer, at least one of components of the base field $\phi_a (r)$ must go to zero outside this layer.

In the limit when the effective thickness of vorticity layer goes to zero the space integrals involving vorticity such as we have in Biot-Savart law and our dipole moment, will be dominated by the delta term and stay finite.

As for the field $\tilde{\phi}_3$ at the loop $C$ which is involved in helicity integral, it becomes arbitrarily small when effective thickness $h = \delta z$ goes to zero. Taking into account singularity of the Coulomb kernel we get an estimate $\tilde{\phi}_3 \sim h \log h \to 0$.

We observe that in the limit when the effective thickness of vorticity layer goes to zero, we have helicity integral going to zero. This happens in spite of existence of parallel Clebsch universes, just because the portal area shrinks to zero.

So, the helicity is not responsible for our vorticity distribution after all, nor it is relevant for distinguishing our instanton from some other Clebsch field.
14 Residue Of a Loop Singularity

We found another topological characteristics of our solution, which does not involve the behavior of Clebsch field outside minimal surface and stays finite in the limit when vorticity layer shrinks to the minimal surface.

Consider the circulation $\Gamma_{\delta C(\alpha)}$ around the infinitesimal loop $\delta C(\alpha)$ which encircles our loop at some point with angular variable $\alpha$ (Fig. 2). It is straightforward to compute

$$\Gamma_{\delta C(\alpha)} = \oint_{\delta C(\alpha)} \phi_1 d\phi_2 = 2\pi n \phi_1$$

(109)

Clearly, this circulation stays finite in a limit of shrinking loop $\delta C$ because of singular vorticity at the loop $C$.

Now, integrating this over $d\phi_2 = m d\alpha$ we get our original circulation

$$\oint \Gamma_{\delta C(\alpha)} d\phi_2(\alpha) = 2\pi n \oint \phi_1 d\phi_2 = 2\pi n \Gamma_C$$

(110)

This relation depends only on the properties of instanton in the linear vicinity of the minimal surface where we know it. Apparently it is valid for arbitrary contours $C$ as long as the Clebsch field corresponds to the minimal surface encircled by $C$. The circulation is gauge invariant but not topologically invariant, as it depends on the shape of $C$. The coefficient $2\pi n$ in front is, however, topologically invariant.

Geometrically the object made by rotating a loop $\delta C$ around a big loop $C$ represents a torus $T_C \in R_3$. This torus in $R_3$ is mapped into a torus in a direct product of a plane by a circle $T(T_C) \in R_2 \times S_1$, (Fig. 3).

We observe that our invariant is a limit of a volume inside this mapped torus $T(T_C)$ when the original torus $T_C$ in $R_3$ shrinks to a loop $C$. This volume is $2\pi n$ times the oriented area inside the loop $C = \phi(C)$ mapped from $R_3$ to $R_2$ by a Clebsch field.

Coming back to the picture of $n$ parallel Clebsch universes glued together at the portal of the minimal surface, we observe that this solid torus $D_C : dD_C = T(T_C)$ is covered $n$ times when the point $r$ covers original solid torus $D_C : dD_C = T_C$, and each volume of the solid torus in these parallel universes equals to $2\pi$ times oriented area inside the map of $C$.

This integral does to a loop in $R_3$ what a residue of a pole of holomorphic function does to a point in complex plane, so I am calling it a residue of a loop of singularities of two dimensional field $\phi_a(r) \in R_2$.

For any regular Clebsch field this solid torus volume would shrink to zero. We presented an explicit construction of a field where this volume stays finite and is quantized in the units of the oriented area inside the map $C = \phi(C)$ of the loop.

I am eager to hear professional mathematical opinion about this invariant.

15 Circulation PDF

Our master equations (96) were homogeneous: they did not normalize $\Phi(\xi)$. This normalization comes from the energy flow constraint, resulting in effective Hamiltonian in our grand canonical ensemble. As we show in Appendix B, our effective Hamiltonian becomes

$$H_{\text{eff}} = \lambda \int_{S_{\text{min}}} d\sigma(\xi) \left( \Lambda \overline{F}^2(\xi) - \overline{f} \left( (\vec{r} \vec{n}) \vec{F} - (\vec{r} \vec{F}) \vec{n} \right) \right)$$

(111)

$$\overline{F} = 2\pi n \vec{\nabla} \Phi \times \vec{n}$$

(112)

with some constant $\Lambda = \frac{\nu}{\sqrt{2\pi \hbar}}$ where $h \propto \nu$ is a thickness of the singular surface in full Navier-Stokes equation.

So, in a limit when this thickness goes to zero together with viscosity we have a finite result, just as in a point-split definition of the energy flow (Kolmogorov anomaly). In our case this anomaly reflects the fact that vorticity reaches large values $\omega \sim \frac{1}{h}$ in a narrow layer $|z| \sim h$ around the minimal surface, thus the contribution to the enstrophy grows as $\omega^2 h \sim \frac{1}{h}$ compensating the viscosity factor in front.

With the second term (vorticity dipole moment) in our effective Hamiltonian, as well as with the velocity field itself in Biot-Savart integral, each involving space integrals linear in viscosity field- the $\delta(z)$ term in viscosity leads to a finite integral over the minimal surface.
This justifies our assumptions about shrinking instanton solution— it follows from minimization of effective Hamiltonian. This Hamiltonian should be used to relate normalization of $\Phi(x)$ to the external random force $f$ which provides the energy flow. Let us now introduce an area-normalized solution $\Phi_0$, such that:

\[
\Phi(\xi) = Z \Phi_0(\xi) \quad (113)
\]

\[
\bar{F}(\xi) = Z \bar{F}_0(\xi) \quad (114)
\]

\[
\Lambda \int_{S_{\text{min}}} d\sigma(\xi) \bar{F}_0(\xi)^2 = A_C 
\]

\[
A_C = \int_{S_{\text{min}}} d\sigma(\xi) \quad (115)
\]

We find quadratic dependence of effective Hamiltonian of

\[
H_{\text{eff}} = \lambda \left( Z^2 A_C - Z \bar{f} \int_{S_{\text{min}}} d\sigma(\xi) \left( (\bar{r} \bar{n}) \bar{F}_0 - (\bar{r} \bar{F}_0) \bar{n} \right) \right) 
\]

Minimizing over $Z$ we find the following:

\[
Z = M_\alpha f_\alpha 
\]

\[
M_\alpha = \frac{1}{2 A_C} \int_{S_{\text{min}}} d\sigma(\xi) \left( (X_\beta n_\beta) F_0^\alpha - (X_\beta F_0^\beta) n_\alpha \right) 
\]

Here we recall that coordinate $X_\beta$ here is counted from the center of the bounding sphere used to define energy flow. This vector is part of our boundary conditions and it depends on the random force $f$ (see the next section).

Now, the vector $\bar{F}_0$ is normalized so that its mean square value over the surface scales as a constant. Estimating dimensions in (119) we find

\[
\bar{M} \sim \sqrt{A_C} 
\]

The circulation is related to $\Phi$ by (94), or in our new normalization

\[
\Gamma_C = M_\alpha f_\alpha m \int_0^{2\pi} \Phi_0(1, \alpha) d\alpha 
\]

This makes the circulation scale as $\sqrt{A_C}$ in agreement with the loop equation result [13].

The distribution of circulation will now follow from the distribution for $f$:

\[
P(\Gamma|C) = \int dP(\bar{f}) \delta (\Gamma - \Gamma_C(f)) 
\]

Note that the normalization factor $\int_0^{2\pi} \Phi_0(1, \alpha) d\alpha$ depends on external curvature of the minimal surface, leading to deviations from naive area rule for curved loops.

### 16 Gaussian Forces and Wilson Loop

Now, let us recall that the random force $f_\alpha$ adds up from the large number of uncorrelated small forces on the surface of remote sphere in our boundary conditions. As such, it would be a Gaussian random variable with symmetric isotropic distribution

\[
P(\bar{f}) \propto \exp \left( -\frac{1}{2} \frac{f_\alpha^2}{\sigma} \right) 
\]

As for the mean dipole moment $M_\alpha$ it would be zero in absence of the random forcing, due to translation invariance. Random forcing breaks this invariance and freezes the center of mass of the vorticity cell, which results in finite $M_\alpha$.

Our GBF distribution for the specific cell is translation invariant with respect to velocity and vorticity coordinates as well as the loop $C$ translation by any constant vector $\bar{r} \Rightarrow \bar{r} + \bar{a}$ but that translation would shift the vector $\bar{Q}^{\text{cell}} = \int_{\text{cell}} d^3 r \bar{r} \times \bar{\omega}$ by a vector $\bar{a} \times \int_{\text{cell}} d^3 r \bar{\omega}(r)$. 

True translation invariance would require shifting positions of ALL vortex cells by the same vector, not just the subsystem under consideration. Demanding that invariance relates our \( \vec{Q} \) to the net sum of all \( \vec{Q} \) for other cells of the thermostat.

That introduces some dependence of \( \vec{Q} \) and hence \( \vec{M} \) of the random force. You may write the relation like this

\[
Q_{\text{cell}}^{\text{cell}}(\vec{f}) + \sum_{\text{cell}^\prime \in \text{Thermostat}} Q_{\text{cell}^\prime}^{\text{cell}^\prime}(\vec{f}) = 0
\]

This relation holds at fixed force which places each center of vorticity cell depending on \( \vec{f} \) and that fixes each of dipole moments \( Q_{\text{cell}^\prime}^{\text{cell}^\prime}(\vec{f}) \). Now, this relation is translation invariant provided net vorticity vanishes

\[
\int_{\text{cell}} d^3r \vec{\omega} + \sum_{\text{cell}^\prime \in \text{Thermostat}} \int_{\text{cell}^\prime} d^3r \vec{\omega} = 0
\]

which is implied in our boundary conditions (we adjust pressure on the bounding surface so that we do not rotate fluid as whole). Coming back to the relation for a \( Q \) vector for subsystem

\[
Q_{\text{cell}}^{\text{cell}}(\vec{f}) = - \sum_{\text{cell}^\prime \in \text{Thermostat}} Q_{\text{cell}^\prime}^{\text{cell}^\prime}(\vec{f})
\]

With small force compared to the vorticity scales, one would expand (the zero order term vanishes by space symmetry):

\[
Q_{\text{cell}}^{\text{cell}}(\vec{f}) \to Q_{\alpha\beta} f_{\beta} + O(f^2)
\]

In case of the surface cell we are considering this implies linear law

\[
M_\alpha = \sqrt{A_C} \mu_{\alpha\beta} f_{\beta} + O(f^2)
\]

with some susceptibility tensor \( \mu_{\alpha\beta} \) depending on the shape of vortex cell (the minimal surface in our limit). The factor \( \sqrt{A_C} \) was introduced for correct normalization of the circulation.

Note that these leading linear terms preserve the time reversal symmetry (reflection of \( f \)), whereas the next, quadratic terms would already break this symmetry. The symmetry breaking effects as we shall see below, display themselves only for small circulations, and disappear in the tails of circulation PDF we are now studying.

In this case the PDF can be computed in explicit form. We have for the circulation

\[
\Gamma_C = \sqrt{A_C} \mu_{\alpha\beta} f_{\beta} m \int_0^{2\pi} \Phi_0(1, \alpha) d\alpha + O(f^3)
\]

Apparently, at fixed normalization of \( \Phi_0 \) there are two solutions of (96) for \( \Phi_0 \), with opposite signs, corresponding to positive and negative PDF tails:

\[
\Gamma_C = \pm \sqrt{A_C} \bar{\mu}_{\alpha\beta} f_{\beta} m \int_0^{2\pi} \Phi_0(1, \alpha) d\alpha + O(f^3)
\]

One may also say that the instanton (positive winding number \( n \)) has the same probability as anti-instanton (negative \( n \)), therefore there are two solutions \( \pm \vec{F}_0 \).

Note that only symmetric part of susceptibility tensor enters this relation for the circulation.

The Gaussian integral can be computed exactly for the Wilson loop

\[
W(\gamma, C) = \left\langle \exp \left( i \gamma \frac{\Gamma_C}{\sqrt{A_C}} \right) \right\rangle = \frac{1}{2} \left( \frac{1}{\sqrt{\prod_{i=1}^3 (1 - 2t \sigma_i \bar{\mu}_i)}} + \frac{1}{\sqrt{\prod_{i=1}^3 (1 + 2t \sigma_i \bar{\mu}_i)}} \right)
\]

where \( \bar{\mu}_i \) are three real positive eigenvalues of symmetric tensor \( \bar{\mu} \).

By definition the Wilson loop is the generating function for the moments \( \langle \Gamma^n \rangle \). Our solution leads to explicit finite sum of binomial terms for these moments. Asymptotically, they approach \( \Gamma(n + \frac{1}{2}) \) in proper normalization.
In general case these eigenvalues are all different. The Fourier integral
\[ P(\Gamma|C) = \int_{-\infty}^{+\infty} \frac{d\gamma}{2\pi \sqrt{A_C}} \exp \left( -i\gamma \frac{\Gamma}{\sqrt{A_C}} \right) W(\gamma, C) \] (133)
would be dominated by the nearest square root singularity in proper semi-plane depending of the sign of \( \Gamma \)
\[ P(\Gamma|C) \to \frac{\text{const}}{\sqrt{|\Gamma|}} \exp \left( -\frac{|\Gamma|}{2\bar{\mu}_1 \sqrt{A_C}} \right) \] (134)
corresponding to the largest eigenvalue \( \bar{\mu}_1 \).

17 Flat Loop

In case of flat loop there is extra symmetry which allows us to go further, with a slightly different result. Namely, in this case the minimal surface is a flat disk inside this loop, and so the tensor \( \mu_{\alpha\beta} \) is orthogonal to \( n_{\alpha} \)
\[ n_{\alpha} \bar{\mu}_{\alpha\beta} = 0 \] (135)
where \( n_{\alpha} \) is the normal to this plane. This makes one of the eigenvalues vanish.
\[ \mu_3 = 0 \] (136)
In that case the Fourier integral can be reduced to Bessel function of imaginary argument. In proper normalization \( (P(0|C) = 1) \)
\[ P(\Gamma|C) = \exp \left( -\frac{1}{2}(a_1 + a_2)|\Gamma| \right) I_0 \left( \frac{1}{2}(a_1 - a_2)\Gamma \right) \] (137)
\[ a_i = \frac{1}{2\bar{\mu}_1 \sigma \sqrt{A_C}} \] (138)
Asymptotically, at large \( |\Gamma|(a_2 - a_1) \) we have
\[ P(\Gamma|C) \to \frac{\exp (-a_1|\Gamma|)}{\sqrt{\pi(a_2 - a_1)|\Gamma|}} \] (139)
In symmetric case \( a_2 = a_1 \) we have \( I_0(0) = 1 \) and the spectrum is purely exponential. This is relevant to the square contour. In case when there is almost perfect symmetry in \( xy \) plane so that \( a_2 - a_1 \ll a_1 \) there is an intermediate regime when pre-exponential factor is far from constant (see Fig.4).

18 Comparison with Numerical Experiments

The first large-scale simulations of circulation PDF [14] showed nearly exponential decay on both sides, with different slopes. As for the dependence upon the shape and area of the loop, they observed the area law as well as the deviations from the K41 scaling \( \Gamma \sim A_C^{2/3} \). The moments \( M_p(C) = \langle |\Gamma_C|^p \rangle \) were measured in an inertial range of sizes of the loop \( C \). The results were interpreted as bi-fractal by these authors, in the sense that the K41 scaling law was transitioning to another with a lower scaling index \( \alpha < \frac{2}{3} \).

The area law (for the flat loop) implies that the dimensionless ratios of these moments (as well as the PDF tails in proper normalization of the slope) to, for instance, \( M_2^{\frac{2}{3}} \) \( p \gg 1, \) should not depend on the ratios of perimeter to the square root of area. As for the lower moments themselves, they show the clear dependence of this ratio. In particular, for the second moment this dependence is calculable\[ ^8 \] We wrote Mathematica\[ ^8 \] code computing the second moment as a function of this ratio.

Given the analysis for the circulation PDF in the previous sections we now see a source of dependence of the aspect ratio of the rectangle (two eigenvalues of the susceptibility tensor). We can no longer claim that higher moments ratios must be independent of the aspect ratio of rectangular loop.

The dependence of the dimensionless combinations of logarithms of moments \( \frac{1}{p} \log M_p - \frac{1}{p-1} \log M_{p-1} \) of the aspect ratios of rectangular loop is strong enough for small \( p \) but seems to fade out at larger \( p > 10 \) (Kartik Iyer, private

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\[ ^8 \] Victor Yakhot and Sasha Polyakov, private communication
communication). This supports the universality of area law for flat loops, though we have no more theoretical arguments for that universality.

As for the scaling index, the closer examination in collaboration with Kartik Iyer of the previous measurements [14] reveals the following picture. The effective scaling index $2\alpha_{eff}(p) = \frac{2}{p} \frac{\partial \log M_p(c)}{\partial \log A_C}$ starts at roughly $1.38$ at $p=0$, passes through the K41 value $\frac{4}{3} = 1.33$ at $p = 3$ and then drops to $1.2$ at $p = 10$. However, if one fits the whole curve after $p = 2$ it is evidently compatible with an asymptotic value of $2\alpha_{eff}(\infty) = 1$, corresponding to our prediction $\alpha = \frac{1}{2}$.

See Fig.6

The reason lower moments are closer to K41 than higher moments is that lower moments are dominated by the tip of the PDF, whereas higher ones are dominated by the tails. The minimal surface instanton applies only to the high moments $M_p(C), p \to \infty$.

So there is no contradiction between the scaling observed in DNS and the asymptotic area law with scaling index $\alpha = \frac{1}{2}$.

The exponential law we derived at the end of the previous section perfectly matches the DNS data for flat rectangular loop from [14], as we analyzed that data together with Kartik Iyer. Fig.7.

At low $\Gamma$ the breaking of time reversal is manifest because of the third moment, related to triple correlator of velocity field. However, at large $\Gamma$ with proper normalization of the left tail, the PDF become exponential and symmetric.

The apparent stretching of the exponential [14] is now understood as combination of a small $\Gamma$ effect and low statistics at the high end of the spectrum $|\Gamma| > 20000\nu$, PDF $< 10^{-10}$, which should be excluded from the fit. As we found out with Sreeni and Kartik, there are just about 100 events in these extreme data points, so we cannot expect them to be statistically meaningful.

As a general rule, the DNS based on single precision random number generators for Gaussian forces cannot have 10 digits statistical accuracy, regardless of number of the data points, because of a finite period of these pseudo-random sequences.

19 Discussion

I am stopping here at the most interesting place of our story. There are still some unsolved mysteries and some unproven conjectures.

First, it appears straightforward (though tedious) to verify the condition (31) in isotropic turbulence in numerical simulations of forced Navier-Stokes equations.

Second, the distribution of vorticity itself in the presence of large velocity circulation around a large loop can be measured in these simulations. Does it concentrate in our thin layer around the minimal surface?

Third, what is a topological meaning of our volume inside a map of shrinking torus [110], which characterizes our instanton?

Fourth, what is the value of a winding number $n$ of our instanton? The energy flux/dissipation is proportional to $n^2$ so that $n = \pm 1$ seems to follow from the minimization of effective Hamiltonian. Are there some deeper topological reasons to choose the winding number?

Fifth, what is the origin of the mean dipole moment $\bar{M}(f)$? We assumed that it linearly depends upon the Gaussian external force (presumed to be weak). This led to exponential decay of PDF, matching results of DNS, but we lack the microscopic computation of corresponding susceptibility tensor.

One can investigate relation of the present field theory to the String-Ising model [11]. This is quite intriguing at a conceptual level, but does not help in practical computations. The WKB expansion around the minimal surface instanton is the only tunnel with some light at its end.

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... simulations. I am grateful to Kartik Iyer for useful discussions of numerical data and providing the results of new DNS prior to publication.

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A Minimal surfaces

Let us present here the theory of the minimal surfaces from a point of view of a field theorist. The minimal surface can be described by parametric equation

\[ S : r_\alpha = X_\alpha (\xi_1, \xi_2) \]  

(140)

The function \( X_\alpha (\xi) \) should provide the minimum to the area functional

\[ A[X] = \int_S \sqrt{\sigma_{\mu\nu}} = \int d^2 \xi \sqrt{\text{Det} G} \]  

(141)

where

\[ G_{ab} = \partial_a X_\mu \partial_b X_\mu, \]  

(142)

is the induced metric. For the general studies it is sometimes convenient to introduce the unit tangent tensor as an independent field and minimize

\[ A[X, t, \lambda] = \int d^2 \xi \left( e_{ab} \partial_a X_\mu \partial_b X_\nu t_{\mu\nu} + \lambda (1 - t_{\mu\nu}^2) \right) \]  

(143)

From the classical equations we will find then

\[ t_{\mu\nu} = \frac{e_{ab}}{2\lambda} \partial_a X_\mu \partial_b X_\nu ; t_{\mu\nu}^2 = 1, \]  

(144)

which shows equivalence to the old definition.

For the actual computation of the minimal area it is convenient to introduce the auxiliary internal metric \( g_{ab} \)

\[ A[X, g] = \frac{1}{2} \int_S d^2 \xi \text{tr} g^{-1} G \sqrt{\text{Det} g}. \]  

(145)

The straightforward minimization with respect to \( g_{ab} \) yields

\[ g_{ab} \text{tr} g^{-1} G = 2G_{ab}, \]  

(146)

which has the family of solutions

\[ g_{ab} = \lambda G_{ab}. \]  

(147)

The local scale factor \( \lambda \) drops from the area functional, and we recover original definition. So, we could first minimize the quadratic functional \[ \] (145) with respect to \( X(\xi) \) (the linear problem), and then minimize with respect to \( g_{ab} \) (the nonlinear problem).

The crucial observation is the possibility to choose conformal coordinates, with the diagonal metric tensor

\[ g_{ab} = \delta_{ab} \rho, \quad g_{ab} = \delta_{ab}, \quad \sqrt{\text{Det} g} = \rho; \]  

(148)

after which the local scale factor \( \rho \) drops from the integral

\[ A[X, \rho] = \frac{1}{2} \int_S d^2 \xi \partial_a X_\mu \partial_a X_\mu. \]  

(149)

However, the \( \rho \) field is implicitly present in the problem, through the boundary conditions.

Namely, one has to allow an arbitrary parametrization of the boundary curve \( C \). We shall use the upper half plane of \( \xi \) for our surface, so the boundary curve corresponds to the real axis \( \xi_2 = 0 \). The boundary condition will be

\[ X_\mu (\xi_1, +0) = C (f(\xi_1)), \]  

(150)

where the unknown function \( f(t) \) is related to the boundary value of \( \rho \) by the boundary condition for the metric

\[ g_{11} = \rho = G_{11} = (\partial_1 X_\mu)^2 = C_{\mu} f'^2 \]  

(151)

As it follows from the initial formulation of the problem, one should now solve the linear problem for the \( X \) field, compute the area and minimize it as a functional of \( f(\cdot) \). As we shall see below, the minimization condition coincides with the diagonality of the metric at the boundary

\[ [\partial_1 X_\mu \partial_2 X_\mu]_{\xi_2 = +0} = 0 \]  

(152)

There is also an implied extra condition of uniqueness of parametrization, which needs \( f' > 0 \).
The linear problem is nothing but the Laplace equation $\partial^2 X = 0$ in the upper half plane with the Dirichlet boundary condition $[150]$. The solution is well known

$$X_\mu(\xi) = \int_{-\infty}^{+\infty} \frac{dt}{\pi} \frac{C_\mu(f(t)) \xi_2}{(\xi_1 - t)^2 + \xi_2^2}$$  \hspace{1cm} (153)

The area functional can be reduced to the boundary terms in virtue of the Laplace equation

$$A[f] = \frac{1}{2} \int d^2 \xi \partial_\alpha (X_\mu \partial_\alpha X_\mu) = -\frac{1}{2} \int_{-\infty}^{+\infty} d\xi_1 \left[ X_\mu \partial_2 X_\mu \right]_{\xi_2 = +0}$$  \hspace{1cm} (154)

Substituting here the solution for $X_\mu$ we find

$$A[f] = -\frac{1}{2\pi} \Re \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' \frac{C_\mu(f(t)) C_\mu(f(t'))}{(t - t' - i0)^2}$$  \hspace{1cm} (155)

This can be rewritten in a nonsingular form

$$A[f] = \frac{1}{4\pi} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' \frac{(C_\mu(f(t)) - C_\mu(f(t')))^2}{(t - t')^2}$$  \hspace{1cm} (156)

which is manifestly positive.

Another nice form can be obtained by integration by parts

$$A[f] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt f'(t) \int_{-\infty}^{+\infty} dt' f'(t') C_\mu'(f(t)) C_\mu'(f(t')) \log |t - t'|$$  \hspace{1cm} (157)

This form allows one to switch to the inverse function $\tau(f)$ which is more convenient for optimization

$$A[\tau] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} df \int_{-\infty}^{+\infty} df' C_\mu'(f) C_\mu'(f') \log |\tau(f) - \tau(f')|$$  \hspace{1cm} (158)

In the above formulas it was implied that $C(\infty) = 0$. One could switch to more traditional circular parametrization by mapping the upper half plane inside the unit circle

$$\xi_1 + i \xi_2 = \frac{1 - \omega}{1 + \omega}; \omega = re^{i\alpha}; r \leq 1.$$  \hspace{1cm} (159)

The real axis is mapped at the unit circle. Changing variables in above integral we find

$$X_\mu(\tau, \alpha) = \Re \int_{-\pi}^{\pi} \frac{d\theta}{\pi} C_\mu(\phi(\theta)) \left( \frac{1}{1 - r \exp(i\alpha - i\theta)} - \frac{1}{1 + \exp(-i\theta)} \right)$$  \hspace{1cm} (160)

Here

$$\phi(\theta) = f \left( \tan \frac{\theta}{2} \right).$$  \hspace{1cm} (161)

The last term represents an irrelevant translation of the surface, so it can be dropped. The resulting formula for the area reads

$$A[\phi] = \frac{1}{4\pi} \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\theta \left( C_\mu(\phi(\theta)) - C_\mu(\phi(\theta')) \right)^2$$  \hspace{1cm} (162)

or, after integration by parts and inverting parametrization

$$A[\theta] = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\phi' C_\mu'(\phi) C_\mu'(\phi') \log \left| \sin \frac{\theta(\phi) - \theta(\phi')}{2} \right|$$  \hspace{1cm} (163)

Let us now minimize the area as a functional of the boundary parametrization $f(t)$ (we shall stick to the upper half plane). The straightforward variation yields

$$0 = \Re \int_{-\infty}^{+\infty} dt \frac{C_\mu(f(t')) C_\mu'(f(t))}{(t - t' + i0)^2},$$  \hspace{1cm} (164)
which duplicates the above diagonality condition (152). Note that in virtue of this condition the normal vector \( n_\mu(x) \) is directed towards \( \partial Z_\mu \) at the boundary. Explicit formula reads

\[
n_\mu(C(f(t))) \propto \Re \int_{-\infty}^{\infty} dt' \frac{C_\mu(f(t'))}{(t - t' + i 0)^2} \tag{165}
\]

Let us have a closer look at the remaining nonlinear integral equation (164). In terms of inverse parametrization it reads

\[
0 = \Re \int_{-\infty}^{\infty} df \frac{C_\mu(f) C'_\mu(f')}{\tau(f) - \tau(f') + i 0} \tag{166}
\]

Introduce the vector set of analytic functions

\[
F_\mu(z) = \int_{-\infty}^{\infty} df \frac{C_\mu(f)}{\pi \tau(f) - z} \tag{167}
\]

which decrease as \( z^{-2} \) at infinity. The discontinuity at the real axis

\[
\Im F_\mu(\tau + i 0) = C'_\mu(f) f'(\tau) \tag{168}
\]

Which provides the implicit equation for the parametrization \( f(\tau) \)

\[
\int d\tau \Im F_\mu(\tau + i 0) = C_\mu(f) \tag{169}
\]

We see, that the imaginary part points in the tangent direction at the boundary. As for the boundary value of the real part of \( F_\mu(\tau) \) it points in the normal direction along the surface

\[
\Re F_\mu \propto n_\mu \tag{170}
\]

Inside the surface there is no direct relation between the derivatives of \( X_\mu(\xi) \) and \( F_\mu(\xi) \).

The integral equation (164) reduces to the trivial boundary condition

\[
F_\mu^2(t + i 0) = F_\mu^2(t - i 0) \tag{171}
\]

In other words, there should be no discontinuity of \( F_\mu^2 \) at the real axis. The solution compatible with analyticity in the upper half plane and \( z^{-2} \) decrease at infinity is

\[
F_\mu^2(z) = (1 + \omega)^4 P(\omega); \quad \omega = \frac{z}{t + z} \tag{172}
\]

where \( P(\omega) \) defined by a series, convergent at \( |\omega| \leq 1 \). In particular this could be a polynomial. The coefficients of this series should be found from an algebraic minimization problem, which cannot be pursued forward in general case. This minimization is also complicated by the extra requirement of uniqueness of parametrization of the boundary, i.e. \( f'(\tau) > 0 \), or, in terms of \( F_\mu(\tau) \)

\[
\Im F_\mu(\tau + i 0) C'_\mu(f) > 0 \tag{173}
\]

The flat loops are trivial though. In this case the problem reduces to the conformal transformation mapping the loop onto the unit circle. For the unit circle we have simply

\[
C_1 + i C_2 = \omega; \quad F_1 = i F_2 = -\frac{(1 + \omega)^2}{2}; \quad P = 0. \tag{174}
\]

Small perturbations around the circle or any other flat loop can be treated in a systematic way, by a perturbation theory.

### B Clebsch Instanton In Detail

As we discussed in the paper above, the Clebsch field can be discontinuous across the minimal surface as long as the normal derivatives vanish on each side.

Let us construct such an instanton solution for the minimal surface parametrized in (160). The explicit form of the loop parametrization \( \phi(\alpha) \) (not to be confused with our Clebsch field) is not relevant for our purpose. We consider this parametrization given. Obviously \( \phi'(\alpha) > 0 \) and \( \phi(\alpha) - \alpha \) is a periodic function

\[
\phi(\alpha) = \alpha + g(\alpha); \tag{175}
\]

\[
g(\alpha) = g(\alpha + 2\pi); \tag{176}
\]

\[
1 + g'(\alpha) > 0 \tag{177}
\]
The minimal surface \( \vec{r} = \vec{X}(\rho, \alpha) \) in our coordinates maps unit disk \( z = \rho e^{i\alpha}; \ 0 \leq \rho \leq 1 \) into \( R^3 \) with boundary condition

\[
\vec{X}(1, \alpha) = \vec{C}(\alpha + g(\alpha))
\]  

(178)

The solution for the Clebsch field in these polar coordinates in linear vicinity in normal coordinate \( z \) of the minimal surface reads

\[
\phi_1(\rho, \alpha, z) \to \Phi(\rho, \alpha)
\]  

(179)

\[
\phi_2(\rho, \alpha, z) \to m\alpha + 2\pi n\theta(z)
\]  

(180)

\[
m, n \in \mathbb{Z}
\]  

(181)

Discontinuity of \( \phi_2 \) at \( z = 0 \) leads to delta function in its gradient:

\[
\nabla \phi_2 \sim 2\pi n\delta(z)\vec{n}
\]  

(182)

which then leads to delta-function in vorticity

\[
\vec{\omega}(r) \sim \delta(z)\vec{F}(\rho, \alpha) \times \vec{n}(\rho, \alpha)
\]  

(183)

\[
\vec{F}(\rho, \alpha) = 2\pi n\vec{\nabla}\Phi(\rho, \alpha)
\]  

(184)

\[
\vec{n}(\rho, \alpha) \cdot \vec{F}(\rho, \alpha) = 0
\]  

(185)

Here \( \nabla \Phi(\rho, \alpha) \) is vector in a local tangent plane. Note that this term is orthogonal to \( \vec{n} \) and thus does not contribute to the flux (i.e. circulation \( \Gamma_C \)). It will however contribute to the dissipation \( \nu \int \omega^2 \), making it infinite and concentrated at the minimal surface, just as we assumed.

Viscose terms in Navier-Stokes equation will smear this delta singularity into a peak of thickness \( h \propto \nu \). After that the dissipation term will remain finite as for the smeared delta function \( \delta_h(z) = \frac{1}{\sqrt{\pi h}} \exp\left(-\frac{z^2}{h^2}\right) \) and become the surface integral

\[
\nu \int d^3r \left( \delta_h(z)\vec{F}(\xi) \times \vec{n}(\xi) \right)^2 \to \frac{\nu}{\sqrt{2\pi h}} \int_{S_{\text{min}}} d\sigma(\xi) \vec{F}^2(\xi)
\]  

(186)

The second term in our effective Hamiltonian in (66) also reduces to a surface integral

\[
f \int d^3r \vec{r} \times \omega(r) \to f \int_{S_{\text{min}}} d\sigma(\xi) \left( (\vec{r} \vec{n}) \vec{F} - (\vec{r} \vec{F}) \vec{n} \right)
\]  

(187)

with surface element related to induced metric

\[
d\sigma(\xi) = d^2\xi \sqrt{G}
\]  

(188)

\[
G = \det G_{ij};
\]  

(189)

\[
G_{ij} = \frac{\partial X_\mu}{\partial \xi_i} \frac{\partial X_\mu}{\partial \xi_j}
\]  

(190)

\[
\xi = (\rho, \alpha)
\]  

(191)

and circulation

\[
\Gamma_C = \oint \phi_1 d\phi_2 = m \int_0^{2\pi} \Phi(1, \alpha) d\alpha
\]  

(192)

Thus, with proper smearing of the dissipation integral our effective Hamiltonian remains finite for our singular instanton solution.

Let us summarize our instanton solution. We parametrize space coordinate in a surface reference frame

\[
\vec{r} = \vec{X}(\rho, \alpha) + z\vec{n}(\rho, \alpha)
\]  

(193)

\[
\vec{n}(\rho, \alpha) \propto \partial_\rho \vec{X}(\rho, \alpha) \times \partial_\alpha \vec{X}(\rho, \alpha); \ \vec{n}^2 = 1
\]  

(194)

This parametrization is valid only in the linear vicinity of the minimal surface. As such surfaces cannot self-intersect, this parametrization is unique.
The instanton solution in this vicinity

\[
\begin{align*}
\phi_1(r) &= \Phi(\rho, \alpha) + O(z^2) \\
\phi_2(r) &= m\alpha + 2\pi n\theta(z) + O(z^2) \\
\vec{\omega}(r) &= \delta(z)2\pi n\vec{\nabla}\Phi(\rho, \alpha) \times \vec{n}(\rho, \alpha) + \vec{n}(\rho, \alpha)\Omega(\rho, \alpha) + O(z^2) \\
\Omega(\rho, \alpha) &= \frac{m}{\sqrt{G}} \frac{\partial\Phi(\rho, \alpha)}{\partial \rho}
\end{align*}
\] (195) (196) (197) (198)

This solution is not defined in the whole \( \mathbb{R}^3 \) but rather only in a linear vicinity of a minimal surface. This is the region contributing to our effective Hamiltonian in approximation of zero viscosity.

We do not have any explicit solution outside that region within the Euler dynamics. This problem would require the analysis of the full Navier-Stokes equation.
Figure 1: The Riemann surface of $\sqrt{z}$
Figure 2: The infinitesimal loop $\delta C$ (red) encircling original loop $C$ (blue).
Figure 3: The solid torus mapped into Clebsch space
Figure 4: The pre-exponential factor $F(a_1 \Gamma) = P(\Gamma|C) \exp(a_1 \Gamma) \sqrt{a_1 \Gamma}$ for $a_2 = 1.01 a_1$
Figure 5: The singular surface in $\mathbb{R}^3$ bounded by soccer gate loop. The Clebsch field is discontinuous at this surface with normal derivatives vanishing on both sides. Vorticity is continuous and directed towards the local normal.
Figure 6: Here is the plot of effective index $2\alpha_{eff}(p)$ with the green line corresponding to our fit $2\alpha_{eff}(p) = 1 + 0.92\frac{\log p}{p}$. The data with error bars (red) was taken from [14].
8192$^3$ DNS, $R_\lambda = 1300$

Figure 7: The log plot of circulation PDF as measured in [14]. The tails are perfectly symmetric and approximately exponential down to PDF $\sim 10^{-10}$ within statistical errors. The linear fit of $\ln(PDF)$ is shown on a zoomed part.