An exactly solvable model for driven dissipative systems

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We introduce a solvable stochastic model inspired by granular gases for driven dissipative systems. We characterize far from equilibrium steady states of such systems through the non-Boltzmann energy distribution and compare different measures of effective temperatures. As an example we demonstrate that fluctuation-dissipation relations hold, however with an effective temperature differing from the effective temperature defined from the average energy.

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Dissipative many-particle systems are far from thermodynamic equilibrium, and a general theoretical description of their statistical mechanics is lacking, in contrast to systems in equilibrium, for which there is a well-established theory. In this Letter we propose a simple exactly solvable model, which provides a context in which certain questions concerning driven dissipative systems may be resolved unambiguously, such as whether different proposed “definitions” of temperature give the same value, as is the case for thermal equilibrium.

Granular materials have in recent years been considered a paradigm for dissipative open systems [1]. In such systems comprised of macroscopic particles, energy is dissipated via interactions, being transferred from macroscopic degrees of freedom (motion of grains) into microscopic degrees of freedom, and can not be transformed back. Continuous driving is needed in order to maintain such a system in a dynamic state. This driving may be realized, for instance, by gravity-driven flow down an incline [2], by continuous avalanches in a rotating drum [3], or in a vibrated container [4].

An oft-studied system is a homogeneously heated granular gas [5]: a collection of hard spheres (or disks in 2D) undergoing inelastic collisions and driven by a stochastic thermostat. This yields Langevin dynamics including a random uncorrelated force \( \mathbf{F}(t) \), satisfying \( \langle \mathbf{F} \rangle = 0 \) and \( \langle F_i(t)F_j(t') \rangle = 2\gamma T_B \delta(t-t')\delta_{ij} \), and a drag force \( -\gamma \mathbf{v} \).

This may be interpreted as coupling to a heat bath of temperature \( T_B \) with a coupling strength \( \gamma \). Were the system non-dissipative, it would reach equilibrium with temperature \( T_B \), irrespective of the details of the coupling, that is, independent of \( \gamma \).

A driven dissipative system reaches a non-equilibrium steady state, for which a granular temperature \( T_G \) may be defined as the average kinetic energy of the grains. Not only is \( T_G \) always smaller than \( T_B \), but its value depends on the details of the coupling with the bath [6, 7]. This steady state behaves statistically differently than an equilibrium state at the same effective temperature. The energy distribution deviates from the Boltzmann distribution, exhibiting overpopulated high energy tails [1]. Surprisingly, even though driven dissipative systems are not in equilibrium, fluctuation-dissipation (FD) relations often hold and may serve to define an effective temperature, \( T_{FD} \) [8], which has been found in numerical experiments to coincide with \( T_G \) [9].

Granular gases may be described theoretically by the Boltzmann equation of kinetic theory, however solutions exist only in terms of approximations valid for small deviations from the equilibrium Boltzmann distribution, that is for small inelasticity and low volume fraction [1]. A dissipative model for which some exact results have been found is the one-dimensional Maxwell-model [10], in which particles collide inelastically with a uniform collision rate, independent of velocity or location. Energy distributions have also been investigated numerically in a two-dimensional version of this model assuming a random impact parameter in every collision [11].

In this Letter we present a novel exactly solvable dissipative model, in which interactions occur randomly and redistribute energy randomly between the interacting particles. Our main results are:

- All moments of the model may be computed exactly.
- In the maximally dissipative limit, the generating function of the model may be solved exactly.
- FD relations hold with \( T_{FD} > T_G \).

Our model consists of a collection of \( N \) particles having energies \( E_i \), with a constant interaction rate between any two particles in the system. In every interaction two particles from the system are chosen at random and their energies are summed. In the case of conservative dynamics (analogous to elastic collisions) this total energy is repartitioned randomly with a uniform distribution between the two interacting particles [12], while for dissipative dynamics (as for inelastic collisions with a constant restitution coefficient) only a fraction \( \alpha \) of the total energy is repartitioned between the particles and the rest is dissipated out of the system. Additionally, we couple the system to a heat bath so that it reaches a steady state.

The simplicity of this model results from the fact that every particle in it is described only by its energy, as opposed, for example, to the 2d degrees of freedom per particle in a d-dimensional frictionless hard sphere gas.

By eliminating the momentum and spatial variables and using solely the energy we turn the vectorial collisions between particles into scalar interactions, and preclude...
spatial correlations.

For conservative dynamics ($\alpha = 1$) the system reaches an equilibrium state with the exponential Boltzmann distribution for the particles’ energies, $p(E) = T^{-1} \exp(-E/T)$, where the temperature (measured in units of energy) equals the average energy in the system $T = \langle E \rangle$. Dissipative dynamics ($\alpha < 1$) cause energy to decay, and in order to maintain the system in a steady state, we keep it in contact with a heat bath, which is constructed as an infinitely large system of particles obeying the conservative dynamics described above, kept in equilibrium at a temperature $\alpha$. The coupling of the dissipative system to the bath is through conservative interactions between a particle chosen at random from the system and a particle chosen at random from the bath, and is characterized by a coupling strength, $f$, which is defined as the fraction of the particle’s interaction that are with the bath out of all its interactions (with the bath and with other particles in the dissipative system). We will see that the steady state of the dissipative system depends on the bath through both $T_B$ and $f$.

The stochastic equation of motion for the evolution of the energy of particle $i$ during an infinitesimal time step $dt$ is hence given by

$$E_i(t + dt) = \begin{cases} \text{value}: & E_i(t) \\ \text{probability}: & \begin{cases} \frac{z(\alpha(E_i(t) + E_j(t)))}{1 - \Gamma dt} & (1 - f)\Gamma dt \\ \frac{z(E_i(t) + E_B)}{f\Gamma dt} & \end{cases} \end{cases},$$

(1)

where: $\Gamma$ is the interaction rate per particle per unit time (which does not affect the steady state but only the rate of approach to it); $j \in \{1, ..., N\}$ is the index of the particle with which particle $i$ may interact, chosen randomly at every interaction; $z \in [0, 1]$ is the fraction of repartitioned energy given to particle $i$ in the interaction, chosen randomly with a uniform distribution at every interaction; $E_B$ is the energy of the particle from the bath with which particle $i$ may interact, which at every interaction is chosen randomly from the equilibrium distribution in the bath, $p_B(E_B) = T_B^{-1} \exp(-E_B/T_B)$.

We first demonstrate how all moments of $p(E)$ may be evaluated exactly from the dynamical rule for general restitution coefficient $\alpha$ and coupling strength $f$. Later, we shall consider the generating function of $p(E)$, using a procedure which is formally possible for the general case, but which will be fully solved only for limiting cases.

The first moment of $p(E)$ is the average energy, which in analogy with granular materials is denoted as the granular temperature, $T_G \equiv \langle E \rangle$. This is evaluated by averaging Eq. (1) over the whole system:

$$T_G(t + dt) = (1 - \Gamma dt)T_G(t) + (1 - f)\Gamma dt \alpha T_G(t) + f\Gamma dt \left[ \frac{1}{2}(T_G(t) + T_B) \right].$$

(2)

In the steady state $T_G(t + dt) = T_G(t)$, and therefore

$$T_G = T_B / [2\alpha - 1 + 2(1 - \alpha)/f].$$

(3)

While the model described above is extremely simple, it is interesting to note that it captures, at least qualitatively, some aspects of an actual driven granular gas. In Eq. (3) we see that $T_G$ is always smaller than $T_B$ and depends not only on the dissipation through the restitution coefficient $\alpha$, but also on the details of the coupling to the bath through the coupling strength $f$ ($T_G$ coincides with $T_B$ only for the two non-dissipative limiting cases: conservative interactions ($\alpha = 1$) and interactions only with the bath ($f = 1$)). To see this for a real granular gas (in 2D, for convenience), we estimate the ratio $T_G/T_B$ by the following mean-field energy balance calculation (see also [14]). Consider a gas of grains of mass $m$, diameter $D$, and restitution coefficient $\epsilon$, at volume fraction $\Phi$. The mean time between collisions for a grain of energy $E$ is $\frac{8\Phi}{\sqrt{\pi m}}$, and the mean energy dissipated per collision is proportional to $(1 - \epsilon^2)/T_G$. Consequently, the average energy loss rate due to collisions is $\frac{K_T(1 - \epsilon^2)}{8\Phi \sqrt{\pi m} T_G^{3/2}}$ where $K$ is a dimensionless constant. This is balanced in the steady state by gain due to driving by, and loss due to friction with the bath, $2\gamma(T_B - T_G)$, yielding $T_B - T_G - AT_B^{-1/2}T_G^{3/2} = 0$, which may be solved for $T_G/T_B$ as a function of the single parameter $A(\epsilon, \Phi, C) = \frac{K_T}{8\Phi \sqrt{\pi m} \sqrt{\epsilon}}$, where we use the dimensionless coupling strength $C = \gamma D \sqrt{\frac{m}{8\Phi \sqrt{\pi}}}$. We note that results of molecular dynamics simulations agree with this estimate, as can be seen in Fig. 1. This granular gas energy balance is more complicated than the calculation leading to Eq. (3), since the rate of dissipative interactions in a granular gas depends not only on the coupling strength $C$ and volume fraction $\Phi$, but is dynamically determined by the typical energy $T_G$ through the grain velocities. Moreover, density and energy correlations in granular gases likely break the validity of such a mean field approximation, especially at high volume fraction and low restitution coefficient, or for higher moments of the energy.

For our model the frequency of dissipative interactions is determined by the controllable parameter $f$, and since our model is inherently correlation free, any moment of the energy may be exactly calculated by taking the average of any power of Eq. (1). This yields the following recursion relation for $\langle E^n \rangle$ in the steady state in terms of all lower moments and the known moments of $E_B$, $\langle E_B^n \rangle = m! T_B^n$:

$$\langle E^n \rangle = (n + 1 - f - 2\alpha^{n-1}(1 - f))^{-1} f \langle E_B^n \rangle + \sum_{m=1}^{n-1} \binom{n}{m} \langle E^m \rangle \langle f(E_B^{n-m} + (1 - f)\alpha^{n-m}) \rangle.$$

(4)

In order to obtain a full solution of the energy distribution $p(E)$, we introduce the generating function
g(\lambda) \equiv \langle e^{-\lambda E} \rangle = \int_0^\infty e^{-\lambda E} p(E) dE. Averaging the exponential of Eq. 11 over the whole system and considering the steady state yields

\langle e^{-\lambda E} \rangle = (1 - \Gamma dt) \langle e^{-\lambda E} \rangle + (1 - f) \Gamma dt \int_0^1 \langle e^{-\lambda \alpha E} \rangle^2 dz \\
+ f \Gamma dt \int_0^1 \langle e^{-\lambda E} \rangle \langle e^{-z\lambda E} \rangle dz, \quad (5)

or

\begin{align*}
g(\lambda) &= (1 - f) \int_0^1 g^2(z\lambda) dz + f \int_0^1 \frac{g(z\lambda)dz}{z\lambda T_B + 1}, \quad (6)
\end{align*}

where we have used the generating function of the exponential energy distribution in the bath, \( g_B(\lambda) \equiv \int_0^\infty e^{-\lambda E_B} T_B^{-1} e^{-E_B/T_B} dE_B = (\lambda T_B + 1)^{-1} \). We now transform the integral equation (5) into a differential equation by change of variables to \( z\lambda \) in both integrals, multiplication by \( \lambda \) and differentiation by \( \lambda \), yielding

\begin{equation}
\lambda \frac{dg(\lambda)}{d\lambda} = (1 - f) g(\alpha \lambda) + f \left( \frac{1}{\lambda T_B + 1} - 1 \right) g(\lambda). \quad (7)
\end{equation}

We first note that the generating function of the Boltzmann distribution, \( g_B(\lambda) \), solves Eq. 7 for the two non-dissipative limiting cases \( \alpha = 1 \) and \( f = 1 \). Furthermore, the solution for the limit of maximal dissipation \( \alpha = 0 \) is \( g(\lambda) = \text{exp}(1, 2, 2 - f, -\lambda T_B) \cdot (\lambda T_B + 1) \), where \( \text{exp} \) is the Gauss hypergeometric function. The theoretical results of these two limiting cases \( \alpha = 0 \) and \( \alpha = 1 \) are plotted in Fig. 2 together with results for \( 0 \leq \alpha \leq 1 \) obtained from a Monte-Carlo numerical simulation of the model with 1000 particles. The numerical results indicate that the theoretical results known for \( \alpha = 0 \) and \( \alpha = 1 \) bound the family of solutions for all intermediate values \( 0 < \alpha < 1 \). Note that the characteristic form of the energy distribution in this exactly solvable model qualitatively resembles that of granular gases (see Fig. 2b), for which only approximate solutions exist.

We will now use FD relations in order to demonstrate one observable deviation from equilibrium behaviour in dissipative systems due to their non-Boltzmann energy distribution. In order to measure FD relations in their simplest form we introduce an internal degree of freedom per particle, \( x_i \), which is coupled to a uniform external field, \( F \), so that the total energy of every particle is \( U_i = E_i - x_i F \), where \( E_i \) is now referred to as the kinetic energy. We consider the relation between the fluctuation in \( x \), \( \langle \Delta x^2 \rangle \equiv \langle x^2 \rangle - \langle x \rangle^2 \), and its susceptibility with respect to changes in \( F \), \( \chi \equiv \frac{\partial^2 G}{\partial F^2} \).

We assume driven dissipative dynamics for the kinetic energy together with non-dissipative exchange of kinetic energy and internal energy, as described by the following

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{a) The ratio \( T_G/T_B \) vs. the scaled parameter \( A(\epsilon, \Phi, C) \), defined in the text. Results from numerical simulations with different combinations of the restitution coefficient \( \epsilon \), the volume fraction \( \Phi \) and the dimensionless coupling strength \( C \) (symbols) agree with the mean field calculation (solid line). Results with 400 particles are shown, however similar results were obtained with up to 12800 particles. b) Normalised energy distributions from simulations with \( \Phi = 0.1 \) and \( C = 0.01 \) (dashed lines), compared to the Boltzmann distribution (thick gray line).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{The generating function (a) and normalized energy distribution (b) for coupling strength \( f = 0.5 \). Dashed lines are results of numerical simulations for various values of the restitution coefficient \( \alpha \), and thick gray lines are theoretical solutions for the two limiting cases.}
\end{figure}
equations of motion,

\[
E_i(t + dt) = \begin{cases} \text{value} : & E_i(t) \\ \text{probability} : & 1 - F D(t) \\ \end{cases}, \quad (8a)
\]

\[
z(D(t) + E_B) / (1 - F D(t)) + \alpha f(1 - h) \Gamma dt \\
z(E_i(t) + E_B) / \Gamma dt \\
z(E_i(t) - x_i(t)) / \Gamma dt
\]

\[
x_i(t + dt) = \begin{cases} \text{value} : & x_i(t) \\ \text{probability} : & 1 - f h \Gamma dt \\ \end{cases}, \quad (8b)
\]

where \( h \) is a parameter introduced to describe the ratio between interactions with the bath and interactions between the two types of degrees of freedom. By calculating the first two moments of Eq. (8b) one sees that

\[
\langle \Delta E \rangle = \chi T_{FD}, \quad \text{and} \quad \langle \Delta x^2 \rangle = \frac{\langle E^2 \rangle}{2\langle E \rangle^2}.
\]

That is, the FD relation, \( \langle \Delta E \rangle = \chi T_{FD} \), is satisfied with an effective temperature

\[
T_{FD} = \frac{\langle E^2 \rangle}{2\langle E \rangle^2},
\]

which probes the distribution of the kinetic energy \( E \), determined from Eq. (8a). Note that \( \chi \) and \( \langle \Delta x^2 \rangle \) diverge as \( F \to 0 \) (where \( T_{FD} \) is normally defined), however their ratio is finite and independent of \( F \).

\( T_{FD} \), which characterizes fluctuations, results from the second moment of the energy distribution, and is generally different from \( T_G \), which describes the first moment. \( T_{FD} \) and \( T_G \) coincide only if \( \langle E^2 \rangle = 2\langle E \rangle^2 \), which is the case for the Boltzmann distribution. For dissipative systems far from equilibrium \( T_{FD} \) is larger than \( T_G \), and their ratio is given for our model in the limit \( h \to 0 \) by

\[
\frac{T_{FD}}{T_G} = \frac{\langle E^2 \rangle}{2\langle E \rangle^2} = \frac{[2(2 - f) - 2\alpha(4 - f)(1 - f)]}{[f(3 - f - 2\alpha^2(1 - f))]}.
\]

In dissipative systems with strong coupling \( f \approx 1 \) and large restitution coefficient \( \alpha \approx 1 \) the energy distribution is close to exponential, hence the values of \( T_{FD} \) and \( T_G \) are similar (but not identical). Generally, the ratio \( T_{FD}/T_G \) may reach any value larger than one, and diverges for vanishing coupling strength \( f \to 0 \). As has recently been predicted by kinetic theory \( \text{[13]} \), we expect \( T_{FD} \) to be larger than \( T_G \) in granular gases as well, where the energy distribution is non-exponential. In the cases studied numerically \( \text{[8]} \) the energy distributions were only slightly non-exponential, resulting in small differences between \( T_{FD} \) and \( T_G \), which explains their seeming coincidence.

In conclusion, we have presented a simple dissipative model, solved it in terms of all energy moments in the general case, and obtained an exact expression for the generating function in the maximally dissipative limit. Although our model is inspired by granular gases, we believe it may have relevance to a broader class of driven dissipative systems. A dissipative system coupled to a heat bath is very different from a conservative system coupled to the same bath. Not only is the granular temperature lower than the bath temperature, but it also depends on the coupling details. In addition, FD relations hold with an effective temperature \( T_{FD} \) which characterizes the second moment of the energy distribution, and is larger than the granular temperature, \( T_G \). These exactly coincide in equilibrium, where the energy distribution is exponential, however generally differ in dissipative systems. We expect that careful numerical studies of granular gases will show this heretofore unobserved difference.

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