Emergence of clustering, correlations, and communities in a social network model

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We propose a simple model of social network formation that parameterizes the tendency to establish acquaintances by the relative distance in a representative social space. By means of analytical calculations and numerical simulations, we show that the model reproduces the main characteristics of real social networks: large clustering coefficient, assortative degree correlations, and the emergence of a hierarchy of communities. Our results highlight the importance of communities in the understanding of the structure of social networks.

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A considerable effort has been devoted in recent years to the understanding of complex systems that can be described in terms of networks, in which vertices represent interacting units and edges stand for the presence of interactions between them. Examples of this new brand of complex networks have been found in systems as diverse as the Internet, the World-Wide-Web, foodwebs, and biological and social organizations (see and references therein).

While most of these so-called complex networks share many common traits that hint towards the possibility of common underlying structural principles, social networks seem to show some essential differences that place them apart from other technological or biological networks. The main differences between social and non-social networks can be summarized in the following three properties: (i) Clustering: The property of clustering can be measured by means of the clustering coefficient, defined as the probability that a pair of vertices with a common neighbor are also connected to each other. While most complex networks show a quite large level of clustering, it has been recently shown that in some cases the value of the clustering coefficient can be mostly accounted for by a simple random network model in which edges are placed at random, under the constraint of a fixed degree distribution \( \bar{P}(k) \) (defined as the probability that a vertex is connected to \( k \) neighbors, i.e., has degree \( k \)). For networks with a scale-free degree distribution of the form \( \bar{P}(k) \sim k^{-\gamma} \), this random construction can yield noticeable values of the clustering coefficient for finite networks, indicating that, in this case, the clustering could be a merely topological property. This construction, however, cannot explain the large clustering coefficient observed in social networks with a bounded, non-scale-free degree distribution. (ii) Degree correlations: It has been recently recognized that real networks show degree correlations, in the sense that the degrees at the end points of any given edge are not independent. In particular, this feature can be quantitatively measured by computing the average degree of the nearest neighbors of a vertex of degree \( k \), \( \bar{\kappa}_nn(k) \). In this sense, non-social networks exhibit disassortative mixing, implying that highly connected vertices tend to connect to vertices with small degree, and vice-versa. This property translates in a decreasing \( \bar{k}_nn(k) \) function. Social networks, on the other hand, display a strong assortative mixing, with high degree vertices connecting preferably to highly connected vertices, a fact that is reflected in an increasing \( \bar{k}_nn(k) \) function. It has been pointed out that, for finite networks, disassortative mixing can be obtained from a purely random model, by just imposing the condition of having no more than one edge between vertices. This observation implies that negative correlation can find a simple structural explanation; explanation that, on the other hand, does not apply to social networks, which must be driven by different organizational principles. (iii) Community structure: Social networks possess a complex community structure, in which individuals typically belong to groups or communities, with a high density of internal connections and loosely connected among them, that on their turn belong to groups of groups and so on, giving raise to a hierarchy of nested social communities of practice showing in some cases a self-similar structure.

Several authors have advocated this last property, the presence of a community structure, as the very distinguishing feature of social networks, responsible for the rest of the properties that differentiate those from non-social networks. In this spirit, in the present paper we propose a model of social networks in which each vertex (individual) has associated a position in a certain social space, whose coordinates account for the different characteristics that define their relative social position with respect to the rest of the individuals. Individuals establish social connections (acquaintances) with a probability decreasing with their relative social distance (properly defined in the social space). This
property yields as a result the presence of communities, defined as local clusters of individuals in a given social space neighborhood. For general forms of the connecting probability, the model yields networks of acquaintances with a non-vanishing clustering coefficient in the thermodynamic limit, plus general assortative correlations. For a certain range of connectivity probabilities, more-over, the model reproduces a community structure with self-similar properties. The model we propose resembles the hierarchical network model proposed in Ref. 13 (see also 14). Our approach differs, however, in the fact that hierarchies are not defined a priori, but they emerge as a result of the construction process.

Our model can be described as follows: Let us consider a set of $N$ disconnected individuals which are randomly placed within a social space, $\mathcal{H}$, according to the density $\rho(h)$, where vector $h_i \equiv (h_1^i, \ldots, h_{d_H}^i)$ defines the position of the $i$-th individual and $d_H$ is the dimension of $\mathcal{H}$. Each subspace of $\mathcal{H}$ (defined by the different coordinates of the vector $h$) represents a distinctive social feature, such as profession, religion, geographic location, etc. and, in general, it will be parametrized by means of a continuous variable with a domain growing with the size of the population. This choice is justified by the fact there are not two identical individuals and, thus, increasing the number of individuals also increases the diversity of the society. Even though it is not strictly necessary for our further development, we also assume that different subspaces are uncorrelated and, therefore, we can factorize the total density as $\rho(h) = \prod_{n=1}^{d_H} \rho_n(h^n)$. Assuming again the independence of social subspaces, we assign a connection probability between any two pairs of individuals, $h_i$ and $h_j$, given by

$$r(h_i, h_j) = \sum_{n=1}^{d_H} \omega_n r_n(h^n_i, h^n_j)$$

where $\omega_n$ is a normalized weight factor measuring the importance that each social attribute has in the process of formation of connections. The key point of our model is the concept of social distance across each subspace $\mathcal{H}$.

We assume that given two nodes $i$ and $j$ with respective social coordinates $h_i$ and $h_j$, it is possible to define a set of distances corresponding to each subspace, $d_n(h^n_i, h^n_j) \in [0, \infty)$, $n = 1, \ldots, d_H$. Moreover, we expect that the probability of acquaintance decreases with social distance. Therefore, we propose a connection probability

$$r_n(h^n_i, h^n_j) = \frac{1}{1 + \left[b_n^{-1} d_n(h^n_i, h^n_j)\right]^\alpha_n}$$

where $b_n$ is a characteristic length scale (that, eventually, will control the average degree) and $\alpha_n > 1$ is a measure of homophily 15, that is, the tendency of people to connect to similar people.

The degree distribution $P(k)$ of the network can be computed using the conditional probability $g(k|h)$ (propagator) that an individual with social coordinates $h$ has $k$ connections 17. We can thus write $P(k) = \int \rho(h) g(k|h) dh$, where $dh$ stands for the measure element of space $\mathcal{H}$. The propagator $g(k|h)$ can be easily computed using standard techniques of probability theory 17, leading to a binomial distribution

$$P(k) = \binom{N - 1}{k} \left(1 - \frac{k(h)}{N - 1}\right)^{N - 1 - k}$$

(3)

where $k(h)$ is the average degree of individuals with social coordinate $h$. For uncorrelated social subspaces, this average degree takes the form

$$\bar{k}(h) = (N - 1) \sum_{n=1}^{d_H} \omega_n \int \rho_n(h^n) r_n(h^n, h^n) dh^n.$$  

(4)

In the case of a sparse network—constant average degree—the propagator takes a Poisson form 17 and the degree distribution can simply be written as

$$P(k) = \frac{1}{k!} \int \rho(h) \bar{k}(h)^k e^{-\bar{k}(h)} dh$$

(5)

Therefore, if the population is homogeneously distributed in the social space, the degree distribution will be bounded, in agreement with the observations made in several real social systems 3 13 18 21.

The clustering coefficient is defined as the probability that two neighbors of a given individual are also neighbors themselves. Following 17, we first compute the probability that an individual with social vector $h$ is connected to an individual with vector $h'$, $p(h'|h)$. This probability reads $p(h'|h) = (N - 1) \rho(h') r(h', h') / \bar{k}(h)$. Given the independent assignment of edges among individuals, the clustering coefficient of an individual with vector $h$ is

$$c(h) = \int \int p(h'|h) r(h', h'') p(h''|h) dh' dh''$$

(6)

and the average clustering coefficient is simply given by

$$\langle c \rangle = \int \rho(h) c(h) dh$$

(7)

In order to test the behavior of our model, we consider the simplest case of a single social feature, i.e. $d_H = 1$. As we will see, even in this case our model presents several non-trivial properties, that are the signature of real social networks. Considering the space $\mathcal{H}$ to be the one-dimensional segment $[0, h_{max}]$, we assign individuals a random, uniformly distributed, position, i.e. $\rho(h) = 1/h_{max}$. In this way, the density of individuals in
FIG. 1: Left: Examples of typical networks generated for an average degree \( \langle k \rangle = 10 \), \( N = 250 \), \( \delta = 2 \), and different values of the parameter \( \alpha \). Right: Binary trees representing the community structure of the corresponding networks (see text).

The social space is given by \( \delta = N/h_{\text{max}} \). The distance between individuals is defined as \( d(h_i, h_j) = |h_i - h_j| \). Therefore, the controlling parameter in the model is the homophily parameter \( \alpha \). The left panel of Fig. 1 shows some typical examples of networks generated with our model, for different values of the parameter \( \alpha \).

The model, as defined above, is homogeneous in the limit \( h_{\text{max}} \gg 1 \), which means that all the vertex properties will eventually become independent of the social coordinate \( h \). Therefore, the average degree can be calculated as \( \langle k \rangle = \lim_{h_{\text{max}} \to \infty} \bar{k}(h = h_{\text{max}}/2) \) which leads to

\[
\langle k \rangle = \frac{2\delta \pi}{\alpha \sin \pi/\alpha}.
\] (8)

Thus, for fixed \( \delta \), we can construct networks with the same average degree and different homophily, \( \alpha \), by changing \( b \) according to the previous expression. For \( \alpha = 1 \) the average degree diverges because, in this case, there is a finite probability of connection to infinitely distant vertices. The clustering coefficient can be computed by means of Eq. (9), yielding

\[
\langle c \rangle = \frac{\alpha^2}{4\pi^2} f(\alpha) \frac{\sin^2 \pi}{\alpha}
\] (9)

where

\[
f(\alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx dy}{(1 + |x|^\alpha)(1 + |x - y|^\alpha)(1 + |y|^\alpha)}
\] (10)

Fig. 2 shows the perfect agreement between simulations of the model compared to the theoretical value Eq. (9), computed by numerical integration. We observe that the clustering coefficient vanishes when \( \alpha = 1 \), that is, for weakly homophilic societies, and converges to a constant value \( \langle c \rangle = 3/4 \) when \( \alpha \to \infty \) [22], which corresponds to a strongly homophilic society.

Regarding the degree correlations, at first sight one could conclude that, since the network is homogeneous in the social space \( H \), the resulting network is free of any correlations. However, numerical simulations of the average degree of the nearest neighbors as a function of the degree, \( k_{\text{nn}}(k) \), show a linear dependence on \( k \) and, consequently, assortative mixing by degree (see Fig. 2). This counterintuitive result is a consequence of the fluctuations of the density of individuals in the social space. Indeed, if individuals are placed in the space \( H \) with some type of randomness, they will end up forming clusters (communities) of close individuals, strongly connected among them. Therefore, an individual with large degree will most probably belong to a large cluster, and consequently its neighbors will have also a high degree.

Finally, we focus on the community structure displayed by our model. To this purpose, we use the algorithm proposed by Girvan and Newman (GN) [12] to identify communities in complex networks. The performance of
by a power law community size distribution, work becomes a perfectly hierarchical network characterized by a power law community size distribution, \( P(s) \sim s^{-2} \). In all the cases the size of the network is \( N = 1000 \). This algorithm relies on the fact that edges connecting different communities have high betweenness (a centrality measure of vertex and edges of the network \([19]\)), that is defined as the total number of shortest paths among pairs of vertices of the network that pass through a given vertex or edge \([20]\). The algorithm recursively identifies and cuts the edge with the highest betweenness, splitting the network until the single vertex level. The information of the entire process can be encoded into the binary tree generated by the splitting procedure. The advantage of using the binary tree representation is twofold, since the algorithm relies on the fact that edges connecting this hierarchical structure at low levels. These clusters are identified in the binary tree as the long branches with many leaves at the end of the tree.

To sum up, in this paper we have presented a model of social network with non-zero clustering coefficient in the thermodynamic limit, assortative degree mixing, and a hierarchical (self-similar) community structure. The origin of these properties can be traced back to the very presence of communities, due to the fluctuations in the position of individuals in social space. Our approach opens thus new views for a further understanding of the structure of complex social networks.

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21. This form of the degree distribution is due to the particular connection probability Eq. (2) considered. More
complex forms can yield different degree distributions, even with scale-free behavior.

[22] $f(\alpha \to \infty)$ can be exactly computed by noticing that the functions within the integrals approach, in this limit, to step functions.