Statistics of work done in degenerate parametric amplification process

Hari Kumar Yadalam and Upendra Harbola
Department of Inorganic and Physical Chemistry,
Indian Institute of Science, Bangalore, 560012, India.

We study statistics of work done by two classical electric field pumps (two-photon and one-photon resonant pumps) on a quantum optical oscillator. We compute moment generating function for the energy change of the oscillator, interpreted as work done by the classical drives on the quantum oscillator starting out in a thermalized Boltzmann state. The moment generating function is inverted, analytically when only one of the pumps is turned on and numerically when both the pumps are turned on, to get the probability function for the work. The resulting probability function for the work done by the classical drive is shown to satisfy transient detailed and integral work fluctuation theorems. Interestingly, we find that, in order for the work distribution function to satisfy the fluctuation theorem in presence of both the drivings, relative phase of drivings need to be shifted by $\pi$, this is related to the broken time reversal symmetry of the Hamiltonian.

INTRODUCTION

Work done by external forces on isolated mesoscopic systems, unlike their macroscopic counterparts, is subject to fluctuations [2–4] due to the smallness of the system size. These fluctuations could be due to the uncertainty of the initial state and due to the quantum nature of evolution and measurement process. Despite the noisy nature of the work done by the external force on the nanoscale system, these fluctuations exhibit marvelous symmetry property which links the frequency of certain amount of work done by the external force on the system to the frequency that the same amount of work is extracted by the external drive. Further, these symmetry properties of probability function for work done by the external force on the system, termed as work-fluctuation-theorems, are one of the first class of fluctuation theorems discovered for out of equilibrium systems [3–6]. This fluctuation relation states that, for a driven system, the probability that an external force extracts certain amount of work from the system is finite but exponentially suppressed compared to the probability that exactly the same amount of work is performed on the system. In this sense fluctuation theorems promote the inequality of second law of thermodynamics (for the dissipated work) to an equality [7]. These results are universal in the sense that only ingredients that are sufficient to establish these relationships are the equilibrium canonical nature of the initial state and the microscopic reversibility of the underlying dynamics, they are insensitive to the nature of microscopic details of the system [2–4]. Nevertheless, the probability function for work done by external force on the system is not universal and depends on the microscopic details of the system. Work distribution function has been computed for a variety of situations for both classical [1–8] and quantum systems [9–24]. Experimental measurement of work distribution and subsequent demonstration of Jarzynski-Crooks fluctuation theorems for classical systems is well established [1–8,25–30]. While for quantum systems, there is no work operator, it was initially confusing to define work in quantum case [3,31–33]. Subsequently, a two-point measurement protocol was proposed [3,32–35] to define work in a single realization. This was crucial for proving quantum versions of fluctuation theorems [2–8,32,33]. It has been challenging to implement two-point measurement protocol experimentally. However there have been some recent theoretical proposals of experiments [36–38] and implementations [39–41] of work measurements in quantum systems either directly through two-point measurement protocol or indirectly.

In this work we consider, two-point measurement protocol to study work statistics in generating displaced squeezed thermal states of quantum optical oscillator by driving the optical oscillator starting in Gibbs state by two classical pumps resonant with two-photon and one-photon transition. The scenario where only two photon resonant pump is on corresponds to the standard degenerate parametric amplification process. This work is motivated by recent interesting proposals [42–55] of using squeezed thermal reservoirs in quantum heat engines to surpass standard Carnot efficiency. Squeezed thermal states of light [56–58] can be realized using well established parametric amplification process [59–71]. We assume that the oscillator is isolated from the environment during the driving process. We thus interpret the change in the energy of the oscillator as work performed by the classical drives. We compute the work-distribution function for this process. The work distribution function is shown to satisfy quantum version of Jarzynski-Crooks fluctuation theorem [5,32,35] [72,73]. We note that work statistics has been studied in generating squeezed thermal state of a harmonic oscillator in Refs. [10,11], where classical driving is modeled through temporal modulation of harmonic oscillator frequency and analytic results for work statistics were obtained approximately under limiting conditions. In Ref. [9], work statistics has been studied in generating a displaced thermal state of a harmonic oscillator. In this work we consider a general process where a quantum optical oscillator is driven by two classical pumps, one resonant with the two-photon transition and other resonant with the one-photon transition. We obtain exact result for the moment generating func-
tion for work and show that interference between the two drivings affects the work statistics in a non-trivial way and plays important role in satisfying the fluctuation theorem. In the limit when only one of the pumps is on, we analytically invert the moment generating function to get work distribution function. Our results can be tested experimentally using classical optical simulation of quantum dynamics as proposed in Ref. [40] and experimentally implemented in Ref. [43].

In the next section, we describe the model system and obtain generating function for work done within the two-point measurement scheme and verify fluctuation theorems for work. The generating function is inverted to compute the probability distribution function in Sec. [III] and cumulants of work are analyzed. We conclude in Sec. [III].

I. GENERATING FUNCTION FOR THE WORK

We consider a quantum optical process of driving an isolated optical oscillator by two classical pumps [75]. We assume that one of the classical pumps has the same frequency as that of the optical oscillator ($\frac{\omega}{2\pi}$) and the second pump has twice the frequency of the optical oscillator. We treat the coupling between classical pumps and quantum optical oscillator within rotating wave approximation. We assume that the initial state of the optical oscillator is thermal state (i.e., oscillator is kept in contact with thermal reservoir till time zero). Note that the driving is non-adiabatic with respect to the natural frequency of the oscillator. The Hamiltonian describing the evolution of quantum optical oscillator driven by classical pumps (written in the interaction picture) [75] is,

$$\hat{H} = \hbar \left( z_1 b - z_1^* b^\dagger \right) + \frac{i}{2} \hbar \left( z_2 b^\dagger b - z_2^* b b^\dagger \right),$$

where $z_1/z_2$ is the product of coupling constant and electric field amplitude of one-photon/two-photon resonant classical pump and $b/b^\dagger$ are the annihilation/creation operator for the quantum optical oscillator. We interpret the energy change of the quantum oscillator as the work done by the classical pumps on it [32][33][78][79]. With these assumptions, the work done by the classical drivens is proportional to the number of quanta ($n \in \mathbb{Z}$) of energy exchanged with the quantum oscillator. The probability distribution for the number of quanta exchanged by the classical pumps with the quantum optical oscillator during time period '$t'$ is given as,

$$P[n; t] = \int_0^{2\pi} \frac{\partial}{\partial \chi} \mathcal{Z}[\chi; t] e^{-i\chi n},$$

with $\mathcal{Z}[\chi; t]$ being the moment generating function for the work done, obtained within the two-point measurement protocol [2][3][34][35][74]: the optical oscillator is kept in contact with a thermal reservoir till time 0, it is then disconnected from the reservoir and the first energy measurement is made. It is then subjected to driving by two classical pumps till time $t$. At time $t$, classical drives are turned off and the second energy measurement is made. The differences in the energies measured initially and finally is the work done in a single realization. With this two-point measurement protocol, $\mathcal{Z}[\chi; t]$ is given as,

$$\mathcal{Z}[\chi; t] = \text{Tr} \left[ \mathcal{U}_\chi(t, 0) \rho(0) \mathcal{U}_{-\chi}(0, t) \right],$$

with $\rho(0) = \frac{e^{-\beta H_0}}{\text{Tr}[e^{-\beta H_0}]}$ is the initial density matrix of the optical oscillator which is assumed to be at the thermal state, and $\mathcal{U}_\chi(t_1, t_2) = e^{-i\chi \hat{H}_\chi(t_1 - t_2)}$ is the twisted evolution operator in the interaction picture with

$$\hat{H}_\chi = i\hbar \left( z_1 e^{i\chi} b^\dagger b - z_1^* e^{-i\chi} b b^\dagger \right) + \frac{i}{2} \hbar \left( z_2 e^{i\chi} b^\dagger b - z_2^* e^{-i\chi} b b^\dagger \right).$$

In order to compute $\mathcal{Z}[\chi; t]$, it is convenient to work with the Weyl generating function, $G_\chi[\zeta, \zeta^*; t]$, defined (in the interaction picture) as [75][80][81],

$$G_\chi[\zeta, \zeta^*; t] = \text{Tr} \left[ e^{i(\zeta^* b^\dagger b + \zeta b^\dagger b^\dagger)} \mathcal{U}_\chi(t, 0) \rho(0) \mathcal{U}_{-\chi}(0, t) \right],$$

and then, $\mathcal{Z}[\chi; t] = G_\chi[0, 0; t]$. Using standard techniques from quantum optics literature [75][80][81], it can be shown that (using Eqs. (3) & (4)) $G_\chi[\zeta, \zeta^*; t]$ satisfies the following evolution equation,

$$\frac{\partial}{\partial t} G_\chi[\zeta, \zeta^*; t] = -\frac{1}{2} \left( \frac{\partial}{\partial \zeta} \right)^T \left( \begin{array}{cc} \mathcal{A} & \mathcal{B} \\
\mathcal{C} & \mathcal{D} \end{array} \right) \left( \begin{array}{c} \zeta \\
\zeta^* \end{array} \right) + \left( \begin{array}{c} d_1 \\
d_2 \end{array} \right)^T \left( \begin{array}{c} \zeta \\
\zeta^* \end{array} \right),$$

where $V^T$ represents transpose of a vector $V$ and,

$$\mathcal{A} = \left( \begin{array}{cc} -i \frac{\omega}{2} \sin(\chi) & 0 \\
0 & -i \frac{\omega}{2} \sin(\chi) \end{array} \right),$$

$$\mathcal{B} = \left( \begin{array}{cc} 0 & z_2^* \cos(\chi) \\
z_2 \cos(\chi) & 0 \end{array} \right),$$

$$\mathcal{C} = \left( \begin{array}{cc} -2iz_2^* \sin(\chi) & 0 \\
0 & -2iz_2 \sin(\chi) \end{array} \right),$$

$$d_1 = \left( iz_1 \cos\left( \frac{\omega}{2} \right) iz_1^* \cos\left( \frac{\omega}{2} \right) \right)^T,$$

$$d_2 = \left( 2z_2^* \sin\left( \frac{\omega}{2} \right) 2z_2 \sin\left( \frac{\omega}{2} \right) \right)^T.$$
\( \mathbf{G}_\chi[\zeta, \zeta^*; t | \bar{\zeta}, \bar{\zeta}^*; 0] = \)
\( e^{\int_0^t dt_1 [\bar{U}_{22}(t_1)^T D_1 - U_{11}(t_1)^T D_2] + \int_0^t dt_2 [\bar{U}_{22}(t_2)^T D_1 - U_{11}(t_2)^T D_2] - \frac{i}{2} \left( \frac{\zeta^*}{\zeta} \right)^T U_{12}(t) U_{12}^+(t) \left( \frac{\zeta}{\zeta^*} \right)^T \frac{\zeta^*}{\zeta} \right) + \int_0^t dt_1 [\bar{U}_{22}(t_1)^T D_1 - U_{11}(t_1)^T D_2] } \)
\( \pi \sqrt{-\det[\bar{U}_{22}(t)]} \)
\( e^{-\frac{i}{2} \left[ \left( \frac{\zeta^*}{\zeta} \right)^T U_{22}(t) \left( \frac{\zeta}{\zeta^*} \right)^T U_{22}(t)^T \right] \left( \frac{\zeta}{\zeta^*} \right)^T U_{22}(t) \left( \frac{\zeta}{\zeta^*} \right)^T + \int_0^t dt_1 \left[ \bar{U}_{22}(t_1)^T D_1 - U_{11}(t_1)^T D_2 \right] } \),

where \( \left( \frac{U_{11}(t)}{U_{21}(t)} \right) = e^{\left( \begin{array}{cc} B & -A \\ C & -B^T \end{array} \right) \cdot t} \).

Using Eq. [9] along with Eq. [10] in \( Z[\chi; t] = \mathbf{G}_\chi[0, 0; t] \) and performing \( \bar{\zeta} \) integrals, we get,

\[
Z[\chi; t] = \frac{1}{\sqrt{\det[\bar{U}_{22}(t) + U_{22}(t)D]}} \nonumber
\]
\( e^{\int_0^t dt_1 \int_0^t dt_2 [\bar{U}_{22}(t_1)^T D_1 - U_{11}(t_1)^T D_2]^T D [\bar{U}_{22}(t_1)^T D + U_{22}(t_1)] U_{11}(t_1)^T D_2 - U_{11}(t_1)^T D_2]} \)
\( e^{-\int_0^t dt_1 \int_0^t dt_2 [\bar{U}_{22}(t_2)^T D_1 - U_{11}(t_2)^T D_2]^T U_{12}(t_2) U_{12}^+(t_2) \left( \frac{\zeta}{\zeta^*} \right)^T U_{22}(t_2)^T D_2 - U_{11}(t_2)^T D_2]} \)
\( e^{\frac{i}{2} \int_0^t dt_1 \int_0^t dt_2 [\bar{U}_{22}(t_1)^T D_1 - U_{11}(t_1)^T D_2]^T U_{12}(t_2) U_{12}^+(t_2) \left( \frac{\zeta}{\zeta^*} \right)^T U_{22}(t_2)^T D_2 - U_{11}(t_2)^T D_2]} \)
\( e^{-\frac{i}{2} \int_0^t dt_1 \int_0^t dt_2 [\bar{U}_{22}(t_1)^T D_1 - U_{11}(t_1)^T D_2]^T U_{12}(t_2) U_{12}^+(t_2) \left( \frac{\zeta}{\zeta^*} \right)^T U_{22}(t_2)^T D_2 - U_{11}(t_2)^T D_2]} \),

The final expression for the moment generating function for work can obtained by using explicit expressions for \( U_{xy}(t) \). This gives,

\[
Z[\chi, \phi; t] = e^{\int_0^t dt \left( \frac{\sinh(\frac{t}{2} \zeta^*)}{\frac{t}{2}} \right)^2 \left[ (1 + f)(e^{i\chi} - 1) + f(e^{-i\chi} - 1) \right] \left[ \cosh(\frac{t}{2} \zeta^*) + \cosh(\phi) \sinh(\frac{t}{2} \zeta^*) \right] \left[ (1 + f)(e^{i\chi} - 1) + f(e^{-i\chi} - 1) \right]} \nonumber
\]
\[ \sqrt{1 - \sinh^2(\frac{t}{2} \zeta^*) \left[ (1 + f)^2 (e^{i2\chi} - 1) + f^2 (e^{-i2\chi} - 1) \right]} \),

where \( \phi = 2 \text{Arg}(z_1) - \text{Arg}(z_2) \).

It is clear that \( Z[\chi, \phi; t] \neq Z[-\chi + i\beta \epsilon, \phi; t] \), which is related to the broken time-reversal symmetry of the Hamiltonian \( \mathbf{H} \neq \mathbf{T} \mathbf{H} \mathbf{T}^{-1} \) (\( \mathbf{T} \) is the time-reversal operator). To recover Jarzynski-Crooks-Buchkov-Kuzolev fluctuation theorem, work distribution for time forward and backward trajectories need to be compared. The backward evolution \( \mathbf{H} \neq \mathbf{T} \mathbf{H} \mathbf{T}^{-1} \) is governed by \( \mathbf{T} \mathbf{H} \mathbf{T}^{-1} = -i \hbar \left[ z_1^* b^+ - z_1 b \right] - \frac{i}{2} \left[ z_1^* b^+ b - z_2 b^+ b^+ \right] \neq \mathbf{H} \).

In order to recover the time-reversal symmetry, we also need to change \( \phi \) to \( \pi - \phi \). This leads to the Gallavotti-Cohen symmetry for the work moment generating function : \( Z[\chi, \phi; t] = Z[-\chi + i\beta \epsilon, \pi - \phi; t] \). This, in turn, leads to the transient work fluctuation theorem : \( P^{[\pi - \phi, \pi - \phi]}_{[\pi - \phi, \pi - \phi]} = e^{\beta \epsilon n} \Rightarrow (e^{-\beta \epsilon n}) = 1 \). But for cases where only one of the pumps is present (i.e., \( z_1 = 0 \) or \( z_2 = 0 \), \( Z[\chi, \phi; t] \) becomes independent of \( \phi \). This is because, for the case when only one of the pumps is present, the phase of the electric field can be gauged out (for the initial thermal state) and doesn’t appear in the expression for \( Z[\chi, \phi; t] \). When both the pumps are present, the phases of both the pumps cannot be gauged out. Hence work statistics is unaffected by the phase of the classical fields for the cases \( z_1 = 0 \) or \( z_2 = 0 \) which is similar to the result noted in Ref. [9] where the phases of the classical field is shown to have no influence on the work statistics during the process of coherent displacement of harmonic oscillator from a thermal state. For the cases when only
one of the pumps is present, the moment generating function and hence the probability distribution function for the work for time forward and backward processes is the same.

In the next section, we discuss the cases \( z_1 = 0 \) and \( z_2 = 0 \) separately and present work distribution for the

\[
P[n; t] = \oint_{|\xi|=1} \frac{d\xi}{2\pi i \xi^{n+1}} \left[ \frac{1 - \frac{1}{\xi} (0)^2}{(1 - \xi (0)^2)} \right] \left[ \frac{1 - \frac{1}{\xi} (0)^2}{(1 - \xi (0)^2)} \right]^{\alpha(\phi, t)(\xi-1)} e^{\left(\frac{f}{1+2f}\right)[1+\frac{|z_2|^2}{2\phi}] \left\{ \left(1+\frac{f}{1+2f}\right)[1+\frac{|z_2|^2}{2\phi}] \right\}^n} e^{-\frac{|z_1|^2 t^2}{2(1+2f)}}.
\]

where

\[
\alpha(\phi, t) = |z_1|^2 \left( \frac{\sinh(|z_2| t)}{|z_2|^2} \right)^2 \left[ \cosh(|z_2| t) + \cos(\phi) \sinh(|z_2| t) \right],
\]

and

\[
\xi_{\pm}(\phi) = \frac{1}{2} \pm \sqrt{1 + 4f (1 + f) \cos^2(\phi) \tanh^2(|z_2| t)} \cos(\phi) \tanh(|z_2| t)
\]

For the case \( z_1 \neq 0 \neq z_2 \), we were not able to invert the moment generating function analytically to get probability function for work. Below we present analytical results for \( z_1 = 0 \) and \( z_2 = 0 \) cases and then discuss numerical results for the general case.

A. \( z_2 = 0 \) case

Taking \( z_2 \to 0 \) limit of Eq. (12), the moment generating function of work is,

\[
Z[\chi; t] = e^{|z_1|^2 t^2 \left\{ \left(1 + \frac{f}{1+2f}\right)[1+\frac{|z_2|^2}{2\phi}] \right\}^n} e^{-\frac{|z_1|^2 t^2}{2(1+2f)}}.
\]

This moment generating function corresponds to a bi-poissonian stochastic process. The above expression for \( Z[\chi; t] \) is a special case (resonant drive) of more general expression for work generating function derived in Ref. 3 for the general displacement drive. Cumulant generating function for work, \( \ln Z[\chi; t] \), clearly scales quadratically with time \( t \) and hence all cumulants (obtained using \((-i)^n \left( \frac{d}{d\chi} \right)^n \ln Z[\chi; t] \mid_{\chi=0} \) scale as \( t^2 \). First two cumulants of work are: \( \langle n \rangle = |z_1|^2 t^2 \) and \( \langle (n - \langle n \rangle)^2 \rangle = |z_1|^2 t^2 (1 + 2f) \).

\[ |z_1|^2 t^2 \] is modified Bessel function of first kind of order 0 of variable \( z \). In the zero temperature limit, \( \beta \to \infty \Rightarrow f \to 0 \), \( P[0; t] = e^{-|z_1|^2 t^2} \), which means, if the

II. STATISTICS OF THE WORK DONE

It is convenient to represent \( P[n; t] \), defined in Eq. (3) along with Eq. (12) as a contour integral around unit circle in complex plane \( \mathbb{C} \) as,

\[
P[n; t] = \oint_{|\xi|=1} \frac{d\xi}{2\pi i \xi^{n+1}} |z_1|^2 t^2 \frac{1}{1 + \frac{|z_2|^2}{2\phi}} \left\{ \left(1 + \frac{f}{1+2f}\right)[1+\frac{|z_2|^2}{2\phi}] \right\}^n e^{-\frac{|z_1|^2 t^2}{2(1+2f)}}.
\]
oscillator’s initial state is the ground state, number of microscopic realizations where no net work is performed by the displacement/linear drive on the oscillator decays with time as a Gaussian function. Further, in the zero temperature limit
\[ P[n; t] = \frac{|z_1|^{2n} t^{2n}}{\Gamma[1 + n]} e^{-|z_1|^2 t^2} \Theta[n], \]  
(20)
where \( \Theta[n] = 1 \) iff \( n \geq 0 \) else \( \Theta[n] = 0 \). This means, if the oscillator’s initial state is ground state, there are no microscopic realizations where work is extracted by the classical drive from the oscillator. Further the term \( t^{2n} \) competes with the exponential decay \( (e^{-|z_1|^2 t^2}) \), shifting the value of most probable work \( (n) \) to higher values for larger times.

**B. \( z_1 = 0 \) case**

For \( z_1 = 0 \) case, the moment generating function for work can be obtained by taking \( z_1 \to 0 \) limit of Eq. (12). This gives,
\[ Z[\chi; t] = \frac{1}{\sqrt{1 - \sinh^2(|z_2| t)}} \left[ (1 + f)^2 (e^{i2\chi} - 1) + f^2 (e^{-i2\chi} - 1) \right]. \]  
(21)
This leads to an average work given by, \( \langle n \rangle = (1 + 2f) \sinh^2(|z_2| t) \geq 0 \). Thus, on average, work is done on the quantum oscillator and grows exponentially in time. This is to be contrasted with the quadratic time-dependence for the \( z_2 = 0 \) case discussed above. Second cumulant of work distribution is \( \langle (n - \langle n \rangle)^2 \rangle = \left( 1 + (1 + 2f)^2 \cosh(2|z_2| t) \right) \sinh^2(|g_2| t) \geq 0 \). This indicates that the distribution function becomes broader exponentially with time. Similar to the previous section, the probability function for the work is expressed as the complex contour integral (Eq. (13)) as,
\[ P[n; t] = \oint_{|\xi| = 1} \frac{d\xi}{2\pi i} \frac{1}{\xi^{n+1}} \sqrt{\frac{1 - \frac{1}{\xi_-(0)^2}}{1 - \frac{1}{\xi_+(0)^2}}} \sqrt{\frac{1 - \frac{1}{\xi_-^2(0)^2}}{1 - \frac{1}{\xi_+^2(0)^2}}} \]  
(22)
Noticing \( |\xi_-^2(0)| < 1 < |\xi_+^2(0)| \) (for \( 0 < |z_2| t < \infty \) and \( 0 < f < \infty \)), we observe that the function with square-root in the above integrand is a multivalued complex function with four branch points at \( \pm \xi_-^2(0) \) and \( \pm \xi_+^2(0) \). A single valued branch can be chosen for this function by defining the branch cut as union of two straight lines joining \( -\xi_-^2(0) \) with \( +\xi_-^2(0) \) and \( -\xi_+^2(0) \) with \( +\xi_+^2(0) \) (through \( \infty \)). With this choice, we get a single valued function which is analytic in the strip \( |\xi_-^2(0)| < |\xi| < |\xi_+^2(0)| \). Hence the above complex integral just gives the Laurent expansion coefficients of \( \sqrt{\frac{1 - \xi_-^2(0)^2}{1 - \xi_+^2(0)^2}} \) expanded around \( \xi = 0 \). The final expression for the distribution is obtained as,
\[ P[2n; t] = \frac{1 + f}{f} n \left[ \left( 1 - \frac{1}{\xi_+^2(0)^2} \right) \left( 1 - \frac{1}{\xi_-^2(0)^2} \right) - \xi_-(0) \right] \]  
\[ \frac{\Gamma[\frac{1}{2} + |n|]}{\Gamma[1 + |n|]} \frac{1}{P^2} \left[ \frac{1}{2} \xi_+^2(0) + |n|, 1 + |n|; \left( \frac{\xi_-^2(0)^2}{\xi_+^2(0)} \right)^2 \right], \]  
(23)
and
\[ P[2n + 1; t] = 0. \]  
(24)
It is important to observe here that there are no microscopic realizations where classical drive does odd quanta of work on the optical oscillator. Further, similar to \( z_2 = 0 \) case, the probability function for work takes same form for both the forward and backward processes. The above explicit expression for \( P[n; t] \) satisfies the detailed work fluctuation theorem and hence the integral fluctuation theorem.

We find that the work distribution is always maximum at \( n = 0 \). The probability that no work is performed on the optical oscillator is given by,
\[ P[0; t] = \frac{2}{\pi} \sqrt{\left( 1 - \frac{1}{\xi_+^2(0)^2} \right) \left( 1 - \xi_-^2(0)^2 \right)} K \left[ \left( \frac{\xi_-^2(0)^2}{\xi_+^2(0)^2} \right)^2 \right], \]  
(25)
where \( K[\cdot] \) is the complete elliptic integral of first kind \( S2 [S3] \). For the zero temperature case \( (\beta \to \infty \Rightarrow f \to 0) \), \( \xi_-(0) = 0 \) and \( \xi_+^2(0) = \cosh^2(|z_2| t) \), which makes, \( P[0; t] = \text{sech}(|z_2| t) \) indicating that, for long time, number of microscopic realizations where no work is done on the oscillator are exponentially suppressed. Further, in the limit of zero temperature,
\[ P[2n; t] = \frac{1}{\sqrt{\pi}} \frac{\Gamma[\frac{1}{2} + n]}{\Gamma[1 + n]} \text{sech}(|z_2| t) \text{tanh}^{2n}(|z_2| t) \Theta[n], \]  
(26)
indicating that there are no microscopic realizations where work is extracted from the system, this is intuitive, since for zero temperature case, oscillators initial state is ground state and so it is not possible to extract any work from it. For large time \( |z_2| t \to \infty \), \( P[2n \geq 0; t] \) decays as an exponential in time. This behavior is different from \( z_2 = 0 \) case discussed previously, where \( P[n \geq 0; t] \) decays as a Gaussian with time.
For large fluctuations (large $n$) probability can be approximated by

$$P[2n; t] = \left(\frac{1 - \xi_+(0)^2}{1 - |\xi_+(0)|^2}\right)^\frac{1}{2} \left(\frac{1 - \xi_-(0)^2}{1 - |\xi_-(0)|^2}\right)^{1/2} e^{-2n\xi_+(0) t} \frac{1}{\sqrt{|n|}}.$$  

(27)

Thus the distribution function falls off exponentially in tails with different rates determined by $|\xi_+(0)|$ and $|\xi_-(0)|$. It shows that the large fluctuations in $n$ (or work) are suppressed exponentially.

The probability weight for smaller values of $n$ falls quickly as time $(|z_2|t)$ increases, however, since $\xi_+(0)$ approaches to unity, the weight for larger values of $n$ increases. Further for large times $(|z_2|t \to \infty)$, $\xi_+(0) \to 1$ and $\xi_-(0) \to 1$ $(\xi_+(0)^2 \xi_-(0)^2 = \left(\frac{f}{1 + f}\right)^2$ for any time), making the distribution function more flatter for positive $n$ with time, but for negative 'n', tails decay with finite rate even for long time.

**C. $z_1 \neq 0 \neq z_2$ case**

For the general case, the moment generating function of work is given in Eq. (12), the first two cumulants of work are given as:

$$\langle n \rangle = (1 + 2f) \sinh^2(|z_2|t) + \alpha(\phi, t)$$

(28)

and

$$\langle (n - \langle n \rangle)^2 \rangle = \left(1 + (1 + 2f)^2 \cosh(2|z_2|t)\right) \sinh^2(|z_2|t) + (1 + 2f) \alpha(\phi, t) \cosh(3|z_2|t) + \cos(\phi) \sinh(3|z_2|t).$$

(29)

It is interesting to note that the $\phi$ independent terms of both the cumulants are the same as that of $z_1 = 0$ case. This is because cumulant generating function is sum of the cumulant generating functions for $z_1 = 0$ case and another $\phi$ dependent term. Note that the two contributions to both the cumulants are positive, but for large times $|z_2|t \to \infty$, the second $\phi$ dependent terms grow exponentially (the first term always grows exponentially) with time except for the case $\phi = \pi$ where they saturate to a finite value or decay to zero respectively.

We numerically invert the generating function for the general case to obtain probability distribution. The probability distribution function for work done $P[n, \phi; t]$ obtained analytically for cases $z_1 = 0$ and $z_2 = 0$ and numerically for the general case for different values of $\phi$ at different times for fixed values of $|z_1|$, $|z_2|$ and $f$ is shown in Fig. [1].

For $z_2 = 0$ (case A), the distribution is more or less symmetric and the average work roughly corresponds to the peak position which increases quadratically with time. The distribution function behaves very differently for $z_1 = 0$ (case B), where the peak of the distribution is always fixed at zero work $(n = 0)$ while the average work increases exponentially with time leading to the asymmetric distribution.

**III. CONCLUSION**

We have computed the statistics of work done by two classical drives, one-photon and two-photon resonant pumps, on the quantum optical oscillator (a variant of degenerate parametric amplification process). This simple model allows us to obtain exact analytic expression for the moment generating function for work. When only one of the drives is present, the probability function for work is analytically obtained. Our results show very different behavior of the work distribution when only individual drivings are present. We found that for the case when only one of the pumps are present, work statistics is not influenced by the phase of the drive. When both drives are present, the relative phase between the drives influences the work statistics. Further for recovering the Jarzynski-Crooks fluctuation theorem, phase has to be reflected around $\pi$ (i.e., $\phi \to \pi - \phi$) which is related to the broken time reversal symmetry of the Hamiltonian.

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**APPENDIX**

**A. Solution of Eq. (6)**

Here we present sketch of the method used for solving the parabolic partial differential equation (Eq. (6)) with the initial condition $G_\chi[\zeta, \zeta^*; t]|_{t=0} = \tilde{G}[\zeta, \zeta^*; 0]$ is given. If $\mathcal{A} = 0$, the above equation is equivalent to the standard Ornstein-Uhlenbeck equation, whose solution can be obtained by the method of characteristics. For general $\mathcal{A}$, the transformation $\tilde{G}_\chi[\zeta, \zeta^*; t] = e^{-\frac{1}{2} \left(\frac{\zeta^*}{\zeta}\right)^T \mathcal{U}_{xy}(t) \left(\frac{\zeta^*}{\zeta}\right)} \tilde{G}_\chi[\zeta, \zeta^*; t]$ can be used to eliminate the quadratic term. Here $\mathcal{U}_{xy}(t)$ are square
With this, \( \tilde{G}_\lambda[\zeta, \zeta^*; t] \) satisfies the following PDE,

\[
\left( \begin{array}{cc}
U_{11}(t) & U_{12}(t) \\
U_{21}(t) & U_{22}(t)
\end{array} \right) = e^{\left( \begin{array}{cc}
B & -A \\
C & -B^T
\end{array} \right) t}.
\]  

(30)
\[
\frac{\partial}{\partial t} \tilde{G}_\chi[\zeta, \zeta^*; t] = \left[ \frac{1}{2} \left( \frac{\partial}{\partial \zeta} \zeta^* \right)^T \begin{bmatrix} B - U_{12}(t)U_{22}(t)^{-1}C \end{bmatrix} \right] \left( \frac{\partial}{\partial \zeta^*} \zeta \right) + \left( d_1 - U_{12}(t)U_{22}(t)^{-1}d_2 \right) \left( \frac{\partial}{\partial \zeta^*} \zeta \right) \tilde{G}_\chi[\zeta, \zeta^*; t],
\]

This PDE can be solved by the method of characteristics to get \( \tilde{G}_\chi[\zeta, \zeta^*; t] \). Using this, \( G_\chi[\zeta, \zeta^*; t] \) is given as,

\[
G_\chi[\zeta, \zeta^*; t] = \int_{\zeta \in C} d^2 \tilde{G}_\chi[\zeta, \zeta^*; t|\tilde{\zeta}, \tilde{\zeta}^*; 0] G[\tilde{\zeta}, \tilde{\zeta}^*; 0],
\]

with \( G_\chi[\zeta, \zeta^*; t|\tilde{\zeta}, \tilde{\zeta}^*; 0] \) given in Eq. [10].

REFERENCES

[1] Herbert B. Callen. *Thermodynamics and an introduction to Thermostatistics*. Wiley, 2013.
[2] Massimiliano Esposito, Upen德拉 Harbola, and Shaul Mukamel. Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems. *Reviews of modern physics*, 81(4):1665, 2009.
[3] Michele Campisi, Peter Hänggi, and Peter Talkner. Colloquium: Quantum fluctuation relations: Foundations and applications. *Reviews of modern physics*, 83(3):771, 2011.
[4] Udo Seifert. Stochastic thermodynamics, fluctuation theorems and molecular machines. *Reports on Progress in Physics*, 75(12):126001, 1999.
[5] C Jarzynski. Nonequilibrium Equality for Free Energy Differences. *Physical Review Letters*, 78(14):2690, 1997.
[6] C Jarzynski. Nonequilibrium work theorem for a system strongly coupled to a thermal environment. *J. Stat. Mech.: Theory Exp.*, 2004:P00005, 2004.
[7] Eliran Boksenbojm, Bram Wynants, and Christopher Jarzynski. Nonequilibrium thermodynamics at the microscope: Work relations and the second law. *Physica A*, 389(20):4406, 2010.
[8] Rainer Klages, Wolfram Just, and Christopher Jarzynski. Nonequilibrium Statistical Physics of Small Systems: Fluctuation Relations and Beyond. Wiley, 2013.
[9] Peter Talkner, P. Sekhar Burada, and Peter Hänggi. Statistics of work performed on a forced quantum oscillator. *Phys. Rev. E*, 78:011115, 2008.
[10] Sebastian Deffner and Eric Lutz. Nonequilibrium work distribution of a quantum harmonic oscillator. *Physical Review E*, 77(2):021128, 2008.
[11] Sebastian Deffner, Obinna Abah, and Eric Lutz. Quantum work statistics of linear and nonlinear parametric oscillators. *Chemical Physics*, 375(2-3):200, 2010.
[12] Juyeon Yi, Peter Talkner, and Michele Campisi. Nonequilibrium work statistics of an aharonov-bohm flux. *Phys. Rev. E*, 84:011138, 2011.
[13] Ian J Ford, David S Minor, and Simon J Binnie. Symmetries of cyclic work distributions for an isolated harmonic oscillator. *European Journal of Physics*, 33(6):1789, 2012.
[14] H. T. Quan and Christopher Jarzynski. Validity of nonequilibrium work relations for the rapidly expanding quantum piston. *Phys. Rev. E*, 85:031102, 2012.
[15] Van A. Ngo and Stephan Haas. Demonstration of jarzynski’s equality in open quantum systems using a stepwise pulling protocol. *Phys. Rev. E*, 86:031127, 2012.
[16] Spyros Sotiriadis, Andrea Gambassi, and Alessandro Silva. Statistics of the work done by splitting a one-dimensional quasicondensate. *Phys. Rev. E*, 87:052129, 2013.
[17] Liu Fei and Ouyang Zhong-Can. Nonequilibrium work inequalities in isolated quantum systems. *Chinese Physics B*, 23(7):070512, 2014.
[18] Alison Leonard and Sebastian Deffner. Quantum work distribution for a driven diatomic molecule. *Chemical Physics*, 446:18, 2015.
[19] Matteo Brunelli, Andre Xuereb, Alessandro Ferrari, Gabriele De Chiara, Nicolai Kiesel, and Mauro Paternostro. Out-of-equilibrium thermodynamics of quantum optomechanical systems. *New Journal of Physics*, 17(3):035016, 2015.
[20] Christopher Jarzynski, H. T. Quan, and Saar Rahav. Quantum-classical correspondence principle for work distributions. *Phys. Rev. X*, 5:031038, 2015.
[21] Marcin Lobejko, Jerzy Luczka, and Peter Talkner. Work distributions for random sudden quantum quenches. *Phys. Rev. E*, 95:052137, 2017.
[22] Ralf Blattmann and Klaus Mølmer. Macroscopic realism
of quantum work fluctuations. *Phys. Rev. A*, 96:012115, 2017.

[23] Ignacio García-Mata, Augusto J. Roncaglia, and Diego A. Wisniacki. Quantum-to-classical transition in the work distribution for chaotic systems. *Phys. Rev. E*, 95:050102, 2017.

[24] Bin Wang, Jingning Zhang, and H. T. Quan. Work distributions of one-dimensional fermions and bosons with dual contact interactions. *Phys. Rev. E*, 97:052136, 2018.

[25] Jan Liphardt, Sophie Dumont, Steven B Smith, Ignacio Tinoco, and Carlos Bustamante. Equilibrium information from nonequilibrium measurements in an experimental test of jarzynski’s equality. *Science*, 296(5574):1832, 2002.

[26] Delphine Collin, Felix Ritort, Christopher Jarzynski, Steven B Smith, Ignacio Tinoco Jr, and Carlos Bustamante. Verification of the crooks fluctuation theorem and recovery of rna folding free energies. *Nature*, 437(7056):231, 2005.

[27] Frédéric Douarche, Sergio Ciliberto, Artyom Petrosyan, and Ivan Rabi. An experimental test of the jarzynski equality in a mechanical experiment. *EPL (Europhysics Letters)*, 70(5):593, 2005.

[28] Valentin Blickle, Thomas Speck, Laurent Helden, Udo Seifert, and Clemens Bechinger. Thermodynamics of a colloidal particle in a time-dependent nonharmonic potential. *Physical review letters*, 96(7):070603, 2006.

[29] Sergio Ciliberto, Sylvain Joubaud, and Artem Petrosyan. Fluctuations in out-of-equilibrium systems: from theory to experiment. *Journal of Statistical Mechanics: Theory and Experiment*, 2010(12):P12003, 2010.

[30] O-P Saira, Y Yoon, T Tanttu, Mikko Möttönen, DV Averin, and Jukka P Pekola. Test of the jarzynski and crooks fluctuation relations in an electronic system. *Physical review letters*, 109(18):180601, 2012.

[31] A. E. Allahverdyan and Th. M. Nieuwenhuizen. Fluctuations of work from quantum subensembles: The case against quantum work-fluctuation theorems. *Phys. Rev. E*, 71:066102, 2005.

[32] Peter Talkner, Eric Lutz, and Peter Hänggi. Fluctuation theorems: Work is not an observable. *Physical Review E (R)*, 75(5):050102, 2007.

[33] Michele Campisi, Peter Talkner, and Peter Hänggi. Quantum bochkov-kuzovlev work fluctuation theorems. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 369(1935):291, 2011.

[34] Jorge Kurchan. A Quantum Fluctuation Theorem. *arXiv preprint cond-mat/0007360*, 2000.

[35] Hal Tasaki. Jarzynski Relations for Quantum Systems and Some Applications. *arXiv preprint cond-mat/0009244*, 2000.

[36] Gerhard Huber, Ferdinand Schmidt-Kaler, Sebastian Deffner, and Eric Lutz. Employing trapped cold ions to verify the quantum jarzynski equality. *Phys. Rev. Lett.*, 101:070403, 2008.

[37] M. Heyl and S. Kehrein. Crooks relation in optical spectra: Universality in work distributions for weak local quenches. *Phys. Rev. Lett.*, 108:190601, 2012.

[38] R. Dorner, S. R. Clark, L. Heaney, R. Fazio, J. Goold, and V. Vedral. Extracting quantum work statistics and fluctuation theorems by single-qubit interferometry. *Phys. Rev. Lett.*, 110:230601, 2013.

[39] L. Mazzola, G. De Chiara, and M. Paternostro. Measuring the characteristic function of the work distribution. *Phys. Rev. Lett.*, 110:230602, 2013.

[40] MAA Talarico, PB Monteiro, EC Mattei, EI Duzzioni, PH Souto Ribeiro, and LC Céleri. Work distribution in a photonic system. *Physical Review A*, 94(4):042305, 2016.

[41] Tiago B. Batalhão, Alexandre M. Souza, Laura Mazzola, Ruben Auccaise, Roberto S. Sarthour, Ivan S. Oliveira, John Goold, Gabriele De Chiara, Mauro Paternostro, and Roberto M. Serra. Experimental reconstruction of work distribution and study of fluctuation relations in a closed quantum system. *Phys. Rev. Lett.*, 113:140601, 2014.

[42] Shuoming An, Jing-Ning Zhang, Mark Um, Dingshun Lv, Yao Lu, Junhua Zhang, Zhang-Qi Yin, HT Quan, and Kihwan Kim. Experimental test of the quantum jarzynski equality with a trapped-ion system. *Nature Physics*, 11(2):193, 2015.

[43] R. Medeiros de Araújo, T Höffner, R Bernardi, DS Tasca, MPJ Lavery, MJ Padgett, A Kanaan, LC Céleri, and PH Souto Ribeiro. Experimental study of quantum thermodynamics using optical vortices. *Journal of Physics Communications*, 2(3):035012, 2018.

[44] Andrew Smith, Yao Lu, Shuoming An, Xiang Zhang, Jing-Ning Zhang, Zongping Gong, HT Quan, Christopher Jarzynski, and Kihwan Kim. Verification of the quantum nonequilibrium work relation in the presence of decoherence. *New Journal of Physics*, 20(1):013008, 2018.

[45] XL Huang, Tao Wang, XX Yi, et al. Effects of reservoir squeezing on quantum systems and work extraction. *Physical Review E*, 86(5):051105, 2012.

[46] Johannes Roßnagel, Obinna Abah, Ferdinand Schmidt-Kaler, Kilian Singer, and Eric Lutz. Nanoscale heat engine beyond the carnot limit. *Physical review letters*, 112(3):030602, 2014.

[47] Obinna Abah and Eric Lutz. Efficiency of heat engines coupled to nonequilibrium reservoirs. *EPL (Europhysics Letters)*, 106(2):20001, 2014.

[48] Robert Alicki and David Gelbwaser-Klimovsky. Nonequilibrium quantum heat machines. *New Journal of Physics*, 17(11):115012, 2015.

[49] Wolfgang Niedenzu, David Gelbwaser-Klimovsky, Abraham G Kofman, and Gershon Kurizki. On the operation of machines powered by quantum non-thermal baths. *New Journal of Physics*, 18(8):083012, 2016.

[50] Gonzalo Manzano, Fernando Galve, Roberta Zambrini, and Juan MR Parrondo. Entropy production and thermodynamic power of the squeezed thermal reservoir. *Physical Review E*, 93(5):052120, 2016.

[51] Jan Klaers, Stefan Faeßl, Atac Imamoglu, and Emre Togan. Squeezed thermal reservoirs as a resource for a nanomechanical engine beyond the carnot limit. *Physical Review X*, 7(3):031044, 2017.

[52] Bijay Kumar Agarwalla, Jian-Hua Jiang, and Dvira Seifert, and Clemens Bechinger. Thermodynamics of a nanomechanical engine beyond the carnot limit. *Physical Review Letters*, 101:070403, 2008.

[53] M. Heyl and S. Kehrein. Crooks relation in optical spectra: Universality in work distributions for weak local quenches. *Phys. Rev. Lett.*, 108:190601, 2012.

[54] R. Dorner, S. R. Clark, L. Heaney, R. Fazio, J. Goold, and V. Vedral. Extracting quantum work statistics and fluctuation theorems by single-qubit interferometry. *Phys. Rev. Lett.*, 110:230601, 2013.

[55] L. Mazzola, G. De Chiara, and M. Paternostro. Measuring the characteristic function of the work distribution. *Phys. Rev. Lett.*, 110:230602, 2013.

[56] MAA Talarico, PB Monteiro, EC Mattei, EI Duzzioni, PH Souto Ribeiro, and LC Céleri. Work distribution in a photonic system. *Physical Review A*, 94(4):042305, 2016.

[57] Tiago B. Batalhão, Alexandre M. Souza, Laura Mazzola, Ruben Auccaise, Roberto S. Sarthour, Ivan S. Oliveira, John Goold, Gabriele De Chiara, Mauro Paternostro, and Roberto M. Serra. Experimental reconstruction of work distribution and study of fluctuation relations in a closed quantum system. *Phys. Rev. Lett.*, 113:140601, 2014.
engine efficiency bound beyond the second law of thermodynamics. Nature communications, 9(1):165, 2018.

[55] Gonzalo Manzano. Squeezed thermal reservoir as a generalized equilibrium reservoir. Phys. Rev. E, 98:042123, 2018.

[56] AI Lvovsky. Squeezed light. Photonics Vol. 1 : Fundamentals of Photonics and Physics, page 121, 2015.

[57] Roman Schnabel. Squeezed states of light and their applications in laser interferometers. Phys. Rep., 684:1, 2017.

[58] Lisa Barsotti, Jan Harms, and Roman Schnabel. Squeezed vacuum states of light for gravitational wave detectors. Rep. Prog. Phys., 82(1):016905, 2018.

[59] W. H. Louisell, A. Yariv, and A. E. Siegman. Quantum fluctuations and noise in parametric processes. i. Phys. Rev., 124:1646, 1961.

[60] J. P. Gordon, W. H. Louisell, and L. R. Walker. Quantum fluctuations and noise in parametric processes. ii. Phys. Rev., 129:481, 1963.

[61] B. R. Mollow and R. J. Glauber. Quantum theory of parametric amplification. i. Phys. Rev., 160:1076, 1967.

[62] B. R. Mollow and R. J. Glauber. Quantum theory of parametric amplification. ii. Phys. Rev., 160:1097, 1967.

[63] M. T. Raiford. Degenerate parametric amplification with time-dependent pump amplitude and phase. Phys. Rev. A, 9:2060, 1974.

[64] L Mišta and J Peřina. Quantum statistics of parametric amplification. Czechoslovak Journal of Physics B, 28(4):392–404, 1978.

[65] K. Wódkiewicz and M. S. Zubairy. Effect of laser fluctuations on squeezed states in a degenerate parametric amplifier. Phys. Rev. A, 27:2003, 1983.

[66] Mark Hillery and M. S. Zubairy. Path-integral approach to the quantum theory of the degenerate parametric amplifier. Phys. Rev. A, 29:1275, 1984.

[67] David D. Crouch and Samuel L. Braunstein. Limitations to squeezing in a parametric amplifier due to pump quantum fluctuations. Phys. Rev. A, 38:4696, 1988.

[68] CJ Mertens, TAB Kennedy, and S Swain. Many-body quantum theory of the optical parametric oscillator. Physical Review A, 48(3):2374, 1993.

[69] Fernando Galve and Eric Lutz. Nonequilibrium thermodynamic analysis of squeezing. Phys. Rev. A, 79:055804, 2009.

[70] PB Acosta-Humánez, SI Kryuchkov, E Suazo, and SK Suslov. Degenerate parametric amplification of squeezed photons: Explicit solutions, statistics, means and variances. Journal of Nonlinear Optical Physics & Materials, 24(02):1550021, 2015.

[71] Ulrik L Andersen, Tobias Gehring, Christoph Marquardt, and Gerd Leuchs. 30 years of squeezed light generation. Physica Scripta, 91(5):053001, 2016.

[72] Gavin E. Crooks. Nonequilibrium Measurements of Free Energy Differences for Microscopically Reversible Markovian Systems. Journal of Statistical Physics, 90(5):1481, 1998.

[73] Gavin E. Crooks. Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences. Physical Review E, 60(3):2721, 1999.

[74] T Monnai. Unified treatment of the quantum fluctuation theorem and the Jarzynski equality in terms of microscopic reversibility. Physical Review E, 72(2):027102, 2005.

[75] Marlan O. Scully and M. Suhail Zubairy. Quantum Optics. Cambridge University Press, 1997.

[76] Girish S. Agarwal. Quantum Optics. Cambridge, 2013.

[77] J. C. Garrison and R. Y. Chiao. Quantum Optics. Oxford, 2013.

[78] Christopher Jarzynski. Comparison of far from equilibrium work relations. Comptes Rendus Physique, 8(5):495, 2007.

[79] Jordan Horowitz and Christopher Jarzynski. Comparison of work fluctuation relations. Journal of Statistical Mechanics, 2007(11):P11002, 2007.

[80] Howard J Carmichael. An open systems approach to quantum optics: lectures presented at the Université Libre de Bruxelles. Springer, 2009.

[81] Howard J Carmichael. Statistical Methods in Quantum Optics 1: Master Equations and Fokker-Planck Equations. Springer, 2003.

[82] George B. Arfken, Hans J. Weber, and Frank E. Harris. Mathematical Methods for Physicists. Elsevier, 2013.

[83] L Melville Milne-Thomson, M Abramowitz, and I A Stegun. Handbook of mathematical functions. Dover, 1972.

[84] I. S. Gradshteyn and I. M. Ryzhik. Tables of Integrals, Series and Products. Academic Press, 2013.

[85] Hari K Yadalam and Upendra Harbola. "statistics of heat transport across squeezed thermal baths", 2019. To be communicated.