UNIVERSAL CO-EXTENSIONS OF TORSION ABELIAN GROUPS

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Abstract. In [16], a theory of universal extensions in abelian categories is developed, moreover, the notion of Ext-universal object is presented. We show that an Ab3 abelian category which is Ext-small, satisfies the Ab4 condition if, and only if, each object of it is Ext-universal. In particular, this means that there are torsion abelian groups that are not co-Ext-universal in the category of torsion abelian groups. In this sense, we characterize all torsion abelian groups which are co-Ext-universal in such category. Namely, we show that such groups are the ones that admit a decomposition $Q \oplus R$, where $Q$ is injective and $R$ is a reduced group on which each $p$-component is bounded.

1. Introduction

In [13], Grothendieck axiomatizes the desirable properties that a category must have in order to be studied by means of homological algebra techniques. Specifically, axioms Ab1 and Ab2 define abelian categories, Ab3 consists of the existence of arbitrary coproducts, Ab4 the existence and exactness of arbitrary coproducts, and Ab5 the existence and exactness of direct limits. Additionally, the dual axioms are denoted as $\text{Ab}^n\ast$. Nowadays, an abelian category is said to be a Grothendieck category if it is Ab5 and admits a generator.

Given their importance, the $\text{Ab}^n\ast$ properties have been studied extensively. For example, it is known that Grothendieck categories are $\text{Ab}3\ast$ and have enough injectives. Moreover, it is known that these categories rarely are $\text{Ab}4\ast$ or have enough projectives. A classical result concerning the characterization of the $\text{Ab}4\ast$ condition in Grothendieck categories is a theorem of Roos (see [19, Theorem 1] or [20, Corollary 1.4]), which states that a Grothendieck category $\mathcal{G}$ is $\text{Ab}4\ast$ if and only if every object $X \in \mathcal{G}$ admits a projective effacement $P \to X$, where a projective effacement is an epimorphism $\alpha$ such that $\text{Ext}^1_{\mathcal{G}}(\alpha, G) = 0$ for all $G \in \mathcal{G}$ (see Section 4).

A similar (but dual) result has been proved recently by the second named author with Saorín and Virili in [16, Proposition 5.9]. In more detail, if the category is Ab4 and Ext-small, then every object is $\text{Ext}^1\ast$-universal. Recall that an object $B$ in an abelian category $\mathcal{A}$ is $\text{Ext}^1\ast$-universal if, for every $A \in \mathcal{A}$, there is a monomorphism $\alpha : A \to E$ such that $\text{Coker}(\alpha) \cong B^{(X)}$, for some set $X$, and $\text{Ext}^1_{\mathcal{A}}(B, \alpha) = 0$ (see Definition [5.1]). In this case, the short exact sequence $A \xrightarrow{\alpha} E \to B^{(X)}$ is called a universal extension of $B$ by $A$.

2000 Mathematics Subject Classification. 18g15, 18e10, 20k10, 20k25, 20k35, 20k40.

Key words and phrases. universal extension, abelian category, abelian groups, Ext-universal, abelian torsion group, cotorsion group.

The first named author was supported by a postdoctoral fellowship from DGAPA-UNAM.
The second named author was supported by CONICYT/FONDECYT/REGULAR/1200090.
In [3, Theorem 4.8], the first named author has shown that the condition Ab4 is equivalent to the natural morphism

\[ \Psi : \text{Ext}^1_A(\bigoplus_{i \in I} B_i, A) \to \prod_{i \in I} \text{Ext}^1_A(B_i, A), \]

being always bijective, for all object \( A \) and \( (B_i)_{i \in I} \) family of objects in such category (see Theorem 2.12). The aim of this paper is to show from this fact that, for Ext-small abelian categories, the condition Ab4 is equivalent to every object being Ext\(^1\)-universal. Moreover, we will revisit the aforementioned result of Roos to construct projective effacements from universal co-extensions, these latter being the dual notion of universal extensions. Having done so, we turn our attention to abelian categories that are Ab3 but not Ab4. Specifically, we will study the class of Ext\(^1\)-universal objects; our main goal is to find a characterization for this type of objects. In such a generality, our task seems to be unapproachable. However, we give a sufficient condition for an object to be Ext\(^1\)-universal in Lemma 5.3. In view of this result, we address our problem in a specific dual situation. Namely, we characterize the co-Ext\(^1\)-universal objects in the category of torsion abelian groups (see Theorem 6.9).

The paper is organized as follows. In Section 2 we introduce some notation and some facts about torsion pairs in the category of abelian groups, Grothendieck categories, and extensions in abelian categories. In particular, we will review the properties of the natural map \( \Psi \) mentioned above, and recall the basic properties of universal extensions. Among the results of this section, it is worth to highlight Proposition 2.5 in which we prove that, in a Grothendieck category \( G \), for every \( X \in G \) there is a regular cardinal \( \lambda \) such that \( \text{Ext}^1_G(X, -) : G \to \text{Ab} \) preserves \( \lambda \)-directed colimits. We will resume this result later when studying projective effacements.

In Section 3 we study universal extensions in Ab3 abelian categories. We will see that the existence of a universal extension of \( B \) by \( A \) is closely related to the bijectivity of the canonical morphism

\[ \Psi : \text{Ext}^1_A(B(I), A) \to \text{Ext}^1_A(B, A)^I, \]

for every set \( I \) (see Lemma 3.1). This will allow us to identify a special universal extension \( \eta \) with the property that the extension group \( \text{Ext}^1_A(B(X), A) \) turns out to be a cyclic right \( \text{End}_A(B(X)) \)-module generated by \( \eta \), where \( X := \text{Ext}^1_B(A, A) \) (see Theorem 3.5). Finally in Theorem 3.6 we will see that an Ext-small Ab3 abelian category is Ab4 if and only if every pair of objects in such category admits a universal extension.

In Section 4 we study projective effacements in a Grothendieck category. Specifically, in Proposition 4.3 we will use universal co-extensions to construct a projective effacement. As a corollary, we obtain a new proof of Roos’ theorem (see Corollary 4.4).

Section 5 is devoted to the study of the Ext\(^1\)-universal objects in an Ab3 abelian category. In particular, we prove that the class of Ext\(^1\)-universal objects is closed under coproducts and direct summands (see Corollary 5.5 and Corollary 5.7).

Finally, in Section 6 we will seek for a characterization of the co-Ext\(^1\)-universal objects in the category \( T_Z \) of abelian torsion groups. We achieve the desired characterization in two steps. In the first step, in Proposition 6.5 we will characterize the pairs of objects in \( T_Z \) that admit a universal co-extension. This will lead us to show that \( \bigoplus_{n \geq 1} \mathbb{Z}(p^n) \) is not co-Ext\(^1\)-universal in \( T_Z \) (see Corollary 6.7). In step...
two, exploiting the previous step and some facts from basic subgroups theory, we will show in Theorem 6.9 that a torsion group $V$ is a co-$\text{Ext}^1$-universal object in $\mathcal{T}_Z$ if and only if its reduced $p$-components are bounded.

2. Preliminaries

2.1. Abelian groups. We will denote by $\text{Ab}$ the category of abelian groups. Recall that $\text{Ab}$ is a hereditary abelian category, that is, $\text{Ext}^2_{\text{Ab}}(M, N) = 0$ for all $M, N \in \text{Ab}$ and $k > 1$. In particular, this means that every quotient of an injective abelian group is also injective. We also recall that every $G \in \text{Ab}$ is expressed as a direct sum $G = D \oplus R$, where $D$ is injective and $R$ is reduced (which means that $R$ has no nonzero injective subobjects). A group $G \in \text{Ab}$ is bounded if there is an integer $n \geq 1$ such that $0 = nG := \{ng \mid g \in G\}$. Finally, we point out that, for every positive integer $n$, $\mathbb{Z}(n)$ will denote the cyclic group of order $n$. For additional notation and results the reader is referred to [11][12].

2.2. Regular cardinals, directed sets and directed colimits. Recall that a cardinal $\lambda$ is regular if it is an infinite cardinal which is not a sum of less than $\lambda$ cardinals, all smaller than $\lambda$. In such case, a poset is called $\lambda$-directed if every subset of cardinality smaller than $\lambda$ has an upper bound. Moreover, a $\aleph_0$-directed poset is called directed set. On the other hand, recall that a category is small when the isomorphism classes of its objects form a set. In this sense, if $D$ is a $\lambda$-directed poset, for some regular cardinal $\lambda$, then $D$ can be viewed as a small category whose objects are the elements of $D$ and there is a unique morphism $\alpha \rightarrow \beta$ exactly when $\alpha \leq_D \beta$.

In what follows $D$ and $\mathcal{C}$ will denote a small category and a category, respectively. A functor $D \rightarrow \mathcal{C}$ will be called a $D$-diagram on $\mathcal{C}$. In this setting, the $D$-diagrams on $\mathcal{C}$ together with the respective natural transformations form a category, which will be denoted by $\text{Fun}(D, \mathcal{C})$. Moreover, the assignment $C \rightarrow \kappa^D(C)$ underlies a functor $\kappa^D: \mathcal{C} \rightarrow \text{Fun}(D, \mathcal{C})$, where $\kappa^D(C)$ is the $D$-diagram on $\mathcal{C}$ such that $d \mapsto C$ and $f \mapsto 1_C$, for each object $d$ and morphism $f$ of $D$, and for each morphism $h$ in $\mathcal{C}$, $\kappa^D_h := \kappa^D(h)$ denote the natural transformation given by the family of morphisms $(\kappa^D_{h,d})_{d \in D}$ with $\kappa^D_{h,d} = h$, for all $d \in D$. Such functor is called the constant diagram functor.

Let $F$ be a functor of $\text{Fun}(D, \mathcal{C})$, $C$ be an object of $\mathcal{C}$, and $\rho: F \rightarrow \kappa^D(C)$ be a natural transformation. Then, we will say that the pair $(C, \rho)$ is a $D$-colimit of $F$ when the following condition holds: for each natural transformation $\tau: F \rightarrow \kappa^D(C')$, there is a unique morphism $f: C \rightarrow C'$ in $\mathcal{C}$ such that $\kappa^D_f \circ \rho = \tau$. In such case, we use the following notation $C = \text{colim}_D(F)$ or $C = \text{colim}_D F(d)$. Now, if each $D$-diagram on $\mathcal{C}$ has a $D$-colimit, we say that $\mathcal{C}$ is $D$-co-complete. In this case, the functor $\text{colim}_D : \text{Fun}(D, \mathcal{C}) \rightarrow \mathcal{C}$ associating each $D$-diagram to its colimit is the left adjoint to the constant diagram functor $\kappa^D$. Moreover, the category $\mathcal{C}$ is called co-complete when it admits $D$-colimits for every small category $D$. Dually, we say that the category $\mathcal{C}$ is $D$-complete (resp. complete) if $\mathcal{C}^{op}$ is $D$-co-complete (resp. co-complete), for $D$ a small category. We say that a category is bicomplete if it is complete and co-complete. Lastly, let $\lambda$ be a regular cardinal and suppose that $D$ is a $\lambda$-directed poset. A $D$-diagram on $\mathcal{C}$ is called a $\lambda$-directed diagram, and its colimit (whenever it exists) is called a $\lambda$-directed colimit.
2.3. Grothendieck categories. In the sequel $\mathcal{A}$ is an abelian category. Given $A, B \in \mathcal{A}$, the symbol $\text{Ext}^{1}_{\mathcal{A}}(A, B)$ denotes the class of all equivalence classes of the short exact sequences in $\mathcal{A}$ of the form $\epsilon : B \rightarrow E \rightarrow A$, where $\hookrightarrow$ denotes a monomorphism in $\mathcal{A}$ and $\twoheadrightarrow$ an epimorphism in $\mathcal{A}$. In this sense, we will denote by $\tau$ the respective equivalence class associated to $\epsilon$. We recall that, in general, the class $\text{Ext}^{1}_{\mathcal{A}}(A, B)$ is not necessarily a set (see [10, Chapter 6, Exercise A]). When $\text{Ext}^{1}_{\mathcal{A}}(A, B)$ is a set for all $A, B \in \mathcal{A}$, the abelian category $\mathcal{A}$ is called Ext-small. Now, the morphisms in $\mathcal{A}$ induce additive assignments between the extensions in $\mathcal{A}$ via pullback and pushout. Namely, for each $f \in \text{Hom}_\mathcal{A}(X, A)$ the assignment $\eta \mapsto \eta \cdot f$ (resp. $\eta \mapsto f \cdot \eta$) induce an additive ‘map’ from $\text{Ext}^{1}_{\mathcal{A}}(A, B)$ (resp. $\text{Ext}^{1}_{\mathcal{A}}(B, X)$) to $\text{Ext}^{1}_{\mathcal{A}}(X, B)$ (resp. $\text{Ext}^{1}_{\mathcal{A}}(B, A)$).

On the other hand, recall that the category $\text{Fun}(D, \mathcal{A})$ is an abelian category, for each small category $D$. Let us recall the ‘hierarchy’ among abelian categories introduced by Grothendieck (see [13]). Concretely, we say that $\mathcal{A}$ is:

- **Ab3** if all set-indexed coproducts exist in $\mathcal{A}$ (equivalently, if it is cocomplete);
- **Ab4** if it is Ab3 and the functors $\text{colim}_D$ are exact, for each set $D$ viewed as the (small) category whose objects are the elements of $D$ and the only morphisms in it are the identity morphisms;
- **Ab5** if it is Ab3 and the functors $\text{colim}_D$ are exact, for each $D$ directed set viewed as a small category.
- **Grothendieck** if it is Ab5 and it has a **generator**, i.e. an object $G$ in $\mathcal{A}$ such that the functor $\text{Hom}_\mathcal{A}(G, -)$ is faithful.

We will denote by $\text{Abn}^*$ the property dual to Abn for each $n \in \{3, 4, 5\}$. Recall that a Grothendieck category is automatically bicomplete and it has enough injectives (see [17] Corollary 2.8.9, Corollary 3.7.10, Theorem 3.10.10).

**Definition 2.1.** Let $\mathcal{G}$ be a Grothendieck category and let $\lambda$ be a regular cardinal. Then, an object $X$ in $\mathcal{G}$ is said to be $\lambda$-**presentable** if $\text{Hom}_\mathcal{G}(X, -) : \mathcal{G} \rightarrow \text{Ab}$ preserves $\lambda$-directed colimits. In such case, we say that $X$ is $\lambda$-FP2 if $\text{Ext}^1_{\mathcal{G}}(X, -) : \mathcal{G} \rightarrow \text{Ab}$ preserves $\lambda$-directed colimits.

**Remark 2.2.** If $\lambda = \omega$, then $\lambda$-presentable objects are called finitely presented and $\lambda$-FP2 objects are called FP2 (see [6]). For additional information on $\lambda$-presentable objects, the reader is referred to [11].

Note that every $\lambda$-presentable (resp. $\lambda$-FP2) object in $\mathcal{G}$ is also $\mu$-presentable (resp. $\mu$-FP2), for each regular cardinal $\mu > \lambda$. Now, we recall the following fact.

**Remark 2.3.** It is well-known that, for every Grothendieck category $\mathcal{G}$, there exist a regular cardinal $\lambda_\mathcal{G}$ and a set $\mathcal{S}$ of $\lambda_\mathcal{G}$-presentable objects in $\mathcal{G}$ such that every object in $\mathcal{G}$ is a $\lambda_\mathcal{G}$-directed colimit of objects in $\mathcal{S}$ (i.e. the image of the respective $\lambda_\mathcal{G}$-directed diagram lives in $\mathcal{S}$) (see [13] Lemma 2.5.16)). In this case, we have that every object in $\mathcal{G}$ is a $\mu$-directed colimit of objects which are $\mu$-presentable, for each regular cardinal $\mu > \lambda_\mathcal{G}$ (see [13] Lemma 2.5.13)).

The following two results are inspired in [9] Lemma 2.2 and [9] Proposition 2.3 with the only difference that we replace colimits of functors $F : \mu \rightarrow \mathcal{G}$, that have an ordinal $\mu$ as their domain, with $\lambda$-directed colimits where $\lambda$ is a regular cardinal. While the proof is essentially the same, we consider worthwhile to state them as follows because, if $\lambda > \omega$, the notion of $\lambda$-presentable object cannot be defined using
Indeed, let $F: \mu \to A$ that have an ordinal $\mu$ as their domain (see [11 p.22]).

**Lemma 2.4.** Let $\mathcal{G}$ be a Grothendieck category. If we denote by $\text{Inj}(\mathcal{G})$ the class of all injective objects in $\mathcal{G}$, then there exists a regular cardinal $\lambda$ such that every $\lambda$-directed colimit of objects in $\text{Inj}(\mathcal{G})$ is also an injective object of $\mathcal{G}$.

**Proof.** It is a known fact that every object in $\mathcal{G}$ is $1$-presentable, for some regular cardinal $\lambda$ (e.g. see Lemma 2.5.13 and Proposition 2.5.16 in [14]). Let $G$ be a generator of $\mathcal{G}$. Since the lattice $\mathcal{L}(G) := \{\text{subobjects of } G\}$ is a set, there exist a regular cardinal $\lambda$ such that every $S \in \mathcal{L}(G)$ is $\lambda$-presentable. Now, let $F : D \to \mathcal{G}$ be a $\lambda$-directed diagram on $\mathcal{G}$ such that each $F(d)$ is an injective object of $\mathcal{G}$, for all $d \in D$. For each $S \in \mathcal{L}(G)$, we consider the following commutative diagram in $\mathcal{G}$, where $\iota_S : S \to G$ denote the respective inclusion:

$$
\begin{array}{ccc}
\text{colim}_D \text{Hom}_G(G, F(d)) & \xrightarrow{\text{colim}_D \text{Hom}_G(\iota_S, F(d))} & \text{colim}_D \text{Hom}_G(S, F(d)) \\
\downarrow & & \downarrow \\
\text{Hom}_G(G, \text{colim}_D F(d)) & \xrightarrow{\text{Hom}_G(\iota_S, \text{colim}_D F(d))} & \text{Hom}_G(S, \text{colim}_D F(d))
\end{array}
$$

Notice that $\text{colim}_D(\text{Hom}_G(\iota_S, F(d)))$ is an epimorphism since each $F(d)$ is an injective object of $\mathcal{G}$, for all $d \in D$. And hence, $\text{Hom}_G(\iota_S, \text{colim}_D F(d))$ is also an epimorphism (recall that $G$ and $S$ are $\lambda$-presentables). Therefore, it follows from [21 Proposition V.2.9] that $\text{colim}_D F(d)$ is an injective object of $\mathcal{G}$. □

Finally, we highlight the following fact.

**Proposition 2.5.** Let $\mathcal{G}$ be a Grothendieck category, $\lambda$ be a regular cardinal, and $X \in \mathcal{G}$. If $X$ is $\lambda$-presentable, then there is a regular cardinal $\mu \geq \lambda$ such that $X$ is $\mu$-FP2.

**Proof.** Let $X$ be a $\lambda$-presentable object of $\mathcal{G}$ and let $E$ be an injective cogenerator of $\mathcal{G}$. Consider the functor $\gamma : \mathcal{G} \to \mathcal{G}$, mapping an object $X$ to the product $E^{\text{Hom}_G(X,E)}$. In this case, such functor comes with a natural transformation $\nu : 1_G \to \gamma$, which is monomorphic. On the other hand, let $\kappa$ be a regular cardinal as in Lemma 2.4 and set $\mu := \max(\lambda, \kappa)$. We claim that $X$ is $\mu$-FP2. For this, it is enough to check that, the functor $\text{Ext}_G^1(X, -)$ preserves $\mu$-directed colimits. Indeed, let $F : D \to \mathcal{G}$ be a $\mu$-directed diagram on $\mathcal{G}$ and consider the following short exact sequence in $\text{Fun}(D, \mathcal{G})$

$$F \xrightarrow{\nu_F} \gamma \circ F \to C,$$

where $\nu_F$ is the natural transformation induced by $\nu$. Using now the Ab5 condition of $\mathcal{G}$, we get the following short exact sequence in $\mathcal{G}$:

$$\text{colim}_D F \xrightarrow{\nu_F} \text{colim}_D (\gamma \circ F) \to \text{colim}_D C,$$

Applying the functor $(X, -) := \text{Hom}_G(X, -)$ and considering the fact that $\text{colim}_D(\gamma \circ F) \in \text{Inj}(\mathcal{G})$ (by Lemma 2.4), we obtain the following commutative diagram with exact rows, where $^1(X, -) := \text{Ext}_G^1(X, -)$:
2.4. Torsion theories in \( \text{Ab} \). We are interested in highlighting some properties of the hereditary torsion classes in \( \text{Ab} \). Such properties are crucial for our purposes.

**Definition 2.6.** A **torsion pair** in \( \text{Ab} \) is a pair \( (\mathcal{T}, \mathcal{F}) \) of full subcategories such that \( \mathcal{T} = \{ X \in \text{Ab} : \text{Hom}_{\text{Ab}}(X, F) = 0 \text{ for all } F \in \mathcal{F} \} \) and \( \mathcal{F} = \{ X \in \text{Ab} : \text{Hom}_{\text{Ab}}(T, X) = 0 \text{ for all } T \in \mathcal{T} \} \). In such case, for each \( X \in \text{Ab} \) there is a (functorial) exact sequence in \( \text{Ab} \) of the form:

\[
\begin{array}{c}
\text{colim}_D(X, F) \rightarrow \text{colim}_D(X, \gamma \circ F) \rightarrow \text{colim}_D(X, C) \rightarrow 1(X, F) \\
\downarrow \quad \downarrow \quad \downarrow \\
(X, \text{colim}_D F) \rightarrow (X, \text{colim}_D(\gamma \circ F)) \rightarrow (X, \text{colim}_D C) \rightarrow 1(X, \text{colim}_D F)
\end{array}
\]

where the three vertical morphisms on the left are isomorphisms (since \( X \) is \( \mu \)-presentable) and therefore all vertical morphisms are isomorphisms by the Five Lemma, as desired. \( \square \)

Let us recall the following result that characterizes the hereditary torsion pairs in \( \text{Ab} \). We finish this subsection with the following remark and example.

**Proposition 2.7.** Let \( (\mathcal{T}, \mathcal{F}) \) be a torsion pair in \( \text{Ab} \). Then, the following assertions are equivalent:

1. \( \mathcal{T} \) is a hereditary torsion pair;
2. \( \mathcal{F} \) is of finite type;
3. \( \mathcal{T} = \mathcal{T}_Z := \{ M \in \text{Ab} : \text{Supp}(M) \subseteq Z \} \), for some sp-subset \( Z \) of \( \text{Spec}(\mathbb{Z}) \).

The following examples will be crucial in the paper.

**Example 2.8.** Let \( p \) be a prime number and, we put \( Z := \text{Spec}(\mathbb{Z}) \setminus \{0\} \) and \( Z_p := \{pZ\} \). Note that \( Z \) and \( Z_p \) are sp-subsets of \( \text{Spec}(\mathbb{Z}) \). In these cases, we have the following:

1. the class \( \mathcal{T}_Z = \{ M \in \text{Ab} : M \text{ is a torsion group} \} \) is the torsion class of a hereditary torsion pair in \( \text{Ab} \);
2. the class \( \mathcal{T}_p := \mathcal{T}_{Z_p} = \{ M \in \text{Ab} : \forall m \in M, \exists n \geq 0 \text{ such that } o(m) = p^n \} \) is the torsion class of a hereditary torsion pair in \( \text{Ab} \).

We finish this subsection with the following remark and example.
Remark 2.9. Every torsion class of a hereditary torsion pair in $\text{Ab}$ is clearly an abelian exact subcategory of $\text{Ab}$. Moreover, it is a Grothendieck category, where the coproducts in it are computed as in $\text{Ab}$, but the products in it are not computed as in $\text{Ab}$ (see Remark 6.2 and [10, Corollary 4.3(1,3)])

Example 2.10. The Grothendieck categories $\mathcal{T}_Z$ and $\mathcal{T}_{Z_p}$ are not an Ab4* abelian category. Indeed, note that the product in $\mathcal{T}_Z$ (resp. $\mathcal{T}_{Z_p}$) of the family of the canonical epimorphisms $\{ f_n : \mathbb{Z}_{p^n} \to \mathbb{Z}_p \}_{n \geq 1}$ can not be surjective.

2.5. Extensions, coproducts, and products. Let $\mathcal{A}$ be an abelian category and let $\{A_i\}_{i \in I}$ be a set of objects in $\mathcal{A}$ whose product (resp. coproduct) exists in $\mathcal{A}$. Now, we denote by $\pi^A_i : \prod_{i \in I} A_i \to A_i$ (resp. $\mu^A_i : A_i \to \prod_{i \in I} A_i$) the associated $i$-th projection (resp. inclusion). When there is an object $A$ in $\mathcal{A}$ such that $A_i = A$, for all $i \in I$, there exists a unique morphism $\Delta^A_i : A \to A^I$ (resp. $\nabla^A_i : A^I \to A$) such that $\pi^A_i \circ \Delta^A_i = 1_A$ (resp. $\nabla^A_i \circ \mu^A_i = 1_A$). We will refer to such morphism as the $I$-diagonal (resp. $I$-co-diagonal) morphism associated to $A$. On the other hand, for each object $B$ in $\mathcal{A}$ there is a natural map:

$$
\Psi : \text{Ext}^1_A\left(\bigoplus_{i \in I} A_i, B\right) \to \prod_{i \in I} \text{Ext}^1_A(A_i, B)
$$

declared as $\Psi(\mathfrak{c}) := \left( \varepsilon \cdot \mu^A_i \right)_{i \in I}$ (resp. $\Phi(\mathfrak{c}) := \left( \pi^A_i \cdot \varepsilon \right)_{i \in I}$). Now, we recall the main result in [3].

It is important to mention three aspects of the map $\Psi$. The first one: $\Psi$ is always injective, for all set of objects in $\mathcal{A}$ (see [4, Lemma 4.2]); the second one: when $\Psi$ is bijective, we can exhibit the correspondence rule of $\Psi^{-1}$, as shown in the following lemma; and the third one: when $\mathcal{A}$ is Ab3, $\mathcal{A}$ is Ab4 if and only if $\Psi$ is always bijective, as shown in the theorem below.

Lemma 2.11. Let $\mathcal{A}$ be an Ab3 (resp. Ab3*) abelian category, $A$ be an object in $\mathcal{A}$, $\{B_i\}_{i \in I}$ be a set of objects in $\mathcal{A}$, $\eta = \{ \eta_i : A \xrightarrow{f_i} E_i \xrightarrow{g_i} B_i \}_{i \in I}$ be a set of short exact sequences in $\mathcal{A}$, and $\nabla^A_i$ be the $I$-co-diagonal morphism. Consider the following pushout diagram.

$$
\begin{array}{ccc}
A^I & \xrightarrow{f_i} & E_i \\
\downarrow \nabla^A_i & & \downarrow \nabla' \\
A & \xrightarrow{f_n} & Z_{\eta} \\
\end{array}
\xrightarrow{g_n} \bigoplus_{i \in I} B_i
$$

Then, the following statements hold true:

(a) There is a short exact sequence $\eta' : A \xrightarrow{f'_i} X \xrightarrow{g'_i} \bigoplus_{i \in I} B_i$ such that $\Psi(\overline{\eta'}) = (\overline{\eta_i})_{i \in I}$ if and only if $f_n$ is a monomorphism.

(b) If $f_n$ is a monomorphism, then any short exact sequence $\eta'$ satisfying that $\Psi(\overline{\eta'}) = (\overline{\eta_i})_{i \in I}$ is equivalent to $\eta'' : A \xrightarrow{f_n} Z_{\eta} \xrightarrow{g_n} \bigoplus_{i \in I} B_i$.

(c) If $\mathcal{A}$ is Ab4, then $\Psi^{-1}(\overline{\eta_i})_{i \in I}) = \nabla^A_i \cdot (\bigoplus_{i \in I} \eta_i)$.

Proof.
(a) \( (\Rightarrow) \) Suppose that \( \Psi(\eta) = (\overline{\eta}_i)_{i \in I} \). Thus, we have the following commutative diagram, for all \( i \in I \):

\[
\begin{array}{c}
\eta_i : \ A & \xrightarrow{f_i} & E_i & \xrightarrow{g_i} & B_i \\
\downarrow \mu_i^A & & \downarrow \mu_i^E & & \downarrow \mu_i^B \\
\eta' : \ A & \xrightarrow{f'} & X & \xrightarrow{g'} & \bigoplus_{i \in I} B_i
\end{array}
\]

Now, by the universal property of coproducts there is a morphism \( \gamma \in \text{Hom}_A(\bigoplus_{i \in I} E_i, X) \) such that \( \mu_i^F = \gamma \circ \mu_i^E \), for all \( i \in I \). In particular, note that

\[
\gamma \circ \left( \bigoplus_{i \in I} f_i \right) = \gamma \circ \mu_i^A = \gamma \circ \mu_i^E \circ f_i = \mu_i^F \circ f_i = f' \circ \bigl( \bigoplus_{i \in I} f_i \bigr) \circ \mu_i^A \quad \forall i \in I.
\]

Once again from the universal property of coproducts, we get that \( \gamma \circ \left( \bigoplus_{i \in I} f_i \right) = f' \circ \bigl( \bigoplus_{i \in I} \bigl( \bigoplus_{i \in I} E_i \bigr) \bigr) \). Then, by the universal property of pushouts, there is a morphism \( \gamma' : Z_\eta \rightarrow X \) such that \( \gamma' \circ f_\eta = f' \) and \( \gamma' \circ \bigoplus_{i \in I} E_i = \bigoplus_{i \in I} B_i \). And hence, \( f_\eta \) is a monomorphism since \( f' \) is a monomorphism.

\( (\Leftarrow) \) Observe that we have the following commutative diagram with exact rows, for all \( i \in I \):

\[
\begin{array}{c}
\eta_i : \ A & \xrightarrow{f_i} & E_i & \xrightarrow{g_i} & B_i \\
\downarrow \mu_i^A & & \downarrow \mu_i^E & & \downarrow \mu_i^B \\
A^{(I)} & \xrightarrow{\bigoplus f_i} & \bigoplus E_i & \xrightarrow{\bigoplus g_i} & \bigoplus B_i \\
\downarrow & & \downarrow & & \downarrow \\
\eta'' : \ A & \xrightarrow{f_\eta} & Z_\eta & \xrightarrow{g_\eta} & \bigoplus_{i \in I} B_i
\end{array}
\]

Then, since that the top and the bottom rows are short exact sequences together with the fact that \( \nabla_i^A \circ \eta_i = 1_A \), it follows that \( \Psi(\eta'') = (\overline{\eta}_i)_{i \in I} \).

(b) It was proved above that \( \Psi(\eta') = (\overline{\eta}_i)_{i \in I} \) when \( f_\eta \) is monic. Therefore, (b) follows from [4, Lemma 4.2].

(c) When \( \mathcal{A} \) is Ab4, we get that the sequence \( \eta'' : A \xrightarrow{f_\eta} Z_\eta \xrightarrow{g_\eta} \bigoplus_{i \in I} B_i \) is exact and, by the proof of the item (a), \( \Psi(\eta'') = (\overline{\eta}_i)_{i \in I} \). Therefore, by Theorem 2.12 \( \Psi^{-1}(\overline{\eta}_i)_{i \in I} = \eta'' = \nabla_i^A \cdot \bigoplus_{i \in I} \eta_i \).

\[\square\]

**Theorem 2.12.** [3] Theorem 4.8 | Let \( \mathcal{A} \) be an Ab3 (resp. Ab3*) abelian category. Then, \( \mathcal{A} \) is Ab4 (resp. Ab4*) if and only if \( \Psi \) (resp. \( \Phi \)) is always bijective, for any set of objects in \( \mathcal{A} \).

### 2.6. Universal Extensions.

**Definition 2.13.** [16] Definition 5.6 | Let \( \mathcal{A} \) be an abelian category and \( A, B \in \mathcal{A} \). A universal extension of \( B \) by \( A \) is a short exact sequence

\[
A \xrightarrow{u} E \xrightarrow{p} B^{(X)}
\]

in \( \mathcal{A} \), for some non-empty set \( X \), such that one of the following equivalent statements hold true:
(a) $\text{Ext}^1_A(B, u) : \text{Ext}^1_A(B, A) \to \text{Ext}^1_A(B, E)$ is the zero morphism,
(b) $\text{Ext}^1_A(B, p) : \text{Ext}^1_A(B, E) \to \text{Ext}^1_A(B, B^{(X)})$ is injective,
(c) the connection morphism $\delta : \text{Hom}_A(B, B^{(X)}) \to \text{Ext}^1_A(B, A)$ is surjective.

For the keen reader, the term extension in the above definition may not seem precise. Perhaps a more appropriate definition would be that a universal extension is the equivalence class of an exact sequence that satisfies any of the above conditions. However, for the sake of simplicity, we will keep the above definition.

**Remark 2.14.** Let $A$ be an abelian category and $A, B \in A$.

(a) If $B$ is projective, then every exact sequence $A \xrightarrow{u} E \xrightarrow{p} B^{(X)}$ is a universal extension of $B$ by $A$.
(b) [16, Proposition 5.7] If $\text{Ext}^1_A(A, B)$ is a finitely generated $\text{End}_A(B)$-module, then there is a universal extension of $B$ by $A$.
(c) [16, Proposition 5.9] If $A$ is Ab4 and $\text{Ext}^1_A(A, B)$ is a set, then there is a universal extension of $B$ by $A$. Indeed, let $\{\eta_i : A \xrightarrow{d_i} E_i \xrightarrow{g_i} B\}_{i \in I}$ be a complete set of representatives of $\text{Ext}^1_A(B, A)$. It follows from Theorem 2.12 and Lemma 2.11 that an exact sequence representing $\Psi^{-1}(\{\eta_i\}_{i \in I})$ is a universal extension of $B$ by $A$.
(d) [16, Proposition 5.9] If there is a universal extension of $B$ by $A$, then $\text{Ext}^1_A(A, B)$ is a set. Indeed, this follows from condition (c) of Definition 2.13. Examples of abelian categories where one can find objects $A$ and $B$ such that $\text{Ext}^1_A(A, B)$ is not a set can be found in [10, Chapter 6, Exercise A] and [8, Lemma 1.1].

**Problem 2.15.** Let $A$ be an Ext-small Ab3 abelian category. Is the Ab4 condition equivalent to the existence of universal extensions?

In the next section we will show that the answer to this question is affirmative. Then, the next question arises.

**Problem 2.16.** Let $A$ be an Ext-small abelian category satisfying Ab3 but not Ab4. Is it possible to characterize the objects that admit universal extensions?

It is worth mentioning that throughout the article we will also be interested in studying the dual notion of universal extension. We will now present this notion for completeness.

**Definition 2.17.** Let $A$ be an abelian category and $A, B \in C$. A universal co-extension of $B$ by $A$ is a short exact sequence

$$B^X \xrightarrow{p} E \xrightarrow{u} A$$

in $A$, for some non-empty set $X$, such that one of the following equivalent statements hold true:

(a) $\text{Ext}^1_A(u, B) : \text{Ext}^1_A(A, B) \to \text{Ext}^1_A(E, B)$ is the zero morphism,
(b) $\text{Ext}^1_A(p, B) : \text{Ext}^1_A(E, B) \to \text{Ext}^1_A(B^X, B)$ is injective,
(c) the connection morphism $\delta : \text{Hom}_A(B^X, B) \to \text{Ext}^1_A(A, B)$ is surjective.

### 3. Ab4 vs. Universal Extensions

This section contains the main results of the article. Specifically, we will study the behavior of universal extensions in abelian Ab3 categories, and then characterize Ext-small Ab4 abelian categories through the existence of universal extensions.
3.1. Universal extensions in Ab3 abelian categories.

**Lemma 3.1.** Let $\mathcal{A}$ be an Ab3 abelian category and $A, B \in \mathcal{A}$. If there is a universal extension of $B$ by $A$, then $\Psi : \text{Ext}^1_\mathcal{A}(B^{(X)}, A) \to \text{Ext}^1_\mathcal{A}(B, A)^X$ is bijective for every set $X$. In particular, $\text{Ext}^1_\mathcal{A}(B^{(X)}, A)$ is a set for every set $X$.

**Proof.** Let $\eta : A \to E \to B^{(Y)}$ be a universal extension of $B$ by $A$. By [4, Lemma 4.2], it is enough to show that $\Psi$ is surjective. Let $(\overline{\eta})_{i \in X} \in \text{Ext}^1_\mathcal{A}(B, A)^X$. By Definition 2.13(c), we know that for every $i \in X$ there is a morphism $u_i : B \to B^{(Y)}$ such that $\eta \cdot u_i = \overline{\eta}$. Now, by the universal property of coproducts, gives a unique morphism $u : B^{(X)} \to B^{(Y)}$ such that $u \circ \mu^B_i = u_i$, for all $i \in X$. We claim that $\overline{\eta}' := \overline{\eta} \cdot u$ satisfies that $\Psi(\overline{\eta}') = (\overline{\eta})_{i \in X}$. Indeed, we have $\overline{\eta}' \cdot \mu^B_i = \overline{\eta} \cdot u_i = \overline{\eta}$. Therefore, $\Psi$ is bijective as desired. The final part follows from Remark 2.14(d). \hfill \Box

The following result is a direct consequence of the previous lemma together with the proof of the Lemma 2.11(a), for the set mentioned below.

**Corollary 3.2.** Let $\mathcal{A}$ be an Ab3 abelian category and $A, B \in \mathcal{A}$ be objects. Then, there exists a universal extension of $B$ by $A$ if, and only if, $X := \text{Ext}^1_\mathcal{A}(B, A)$ is a set and the natural map $\Psi : \text{Ext}^1_\mathcal{A}(B^{(X)}, A) \to \text{Ext}^1_\mathcal{A}(B, A)^X$ is bijective.

Before proceeding with the main theorem of this section, we prove the following useful and interesting results.

**Corollary 3.3.** Let $\mathcal{A}$ be an Ab3 abelian category, and $A, B \in \mathcal{A}$ such that $\text{Ext}^1_\mathcal{A}(A, B) = \{\overline{\eta}\}_{i \in X}$ is a set. If there is a universal extension of $B$ by $A$, then we can build $\eta : A \to E \to B^{(X)}$ a universal extension of $B$ by $A$ such that $\eta \cdot \mu^B_i = \overline{\eta}$, for all $i \in X$. Moreover, $\eta$ is the exact sequence described in Lemma 2.11.

**Proof.** By Lemma 3.1, $\Psi : \text{Ext}^1_\mathcal{A}(B^{(X)}, A) \to \text{Ext}^1_\mathcal{A}(B, A)^X$ is bijective. Therefore, there is $\overline{\eta} \in \text{Ext}^1_\mathcal{A}(B^{(X)}, A)$ such that $\Psi(\overline{\eta}) = (\overline{\eta})_{i \in X}$. This means that $\eta \cdot \mu^B_i = \overline{\eta}$, for all $i \in X$. \hfill \Box

**Definition 3.4.** Let $\mathcal{A}$ be an Ab3 abelian category, $A, B \in \mathcal{A}$ be objects that admit a universal extensions of $B$ by $A$, and $\eta$ the universal extension built in Corollary 3.3. We will say that $\eta$ is the **canonical universal extension** of $B$ by $A$.

**Theorem 3.5.** Let $\mathcal{A}$ be an Ab3 abelian category, $A, B \in \mathcal{A}$ be objects that admit a universal extension of $B$ by $A$, and $\eta : A \xrightarrow{\eta} E \xrightarrow{\gamma} B^{(X)}$ the canonical universal extension of $B$ by $A$. Then, $\text{Ext}^1_\mathcal{A}(B^{(X)}, A)$ is a cyclic right $\text{End}_\mathcal{A}(B^{(X)})$-module generated by $\overline{\eta}$.

**Proof.** Let $\{\eta_i : A \xrightarrow{\eta_i} E_i \xrightarrow{\gamma_i} B\}_{i \in X}$ be a complete set of representatives of $\text{Ext}^1_\mathcal{A}(B, A)$ (see Remark 2.14(d)). Given an extension $\overline{\eta}' \in \text{Ext}^1_\mathcal{A}(B^{(X)}, A)$ with $\eta' : A \xrightarrow{\eta'} E \xrightarrow{\gamma'} B^{(X)}$, we can define a function $\sigma : X \to X$ such that $\overline{\eta}' \cdot \mu^B_i = \overline{\eta}(\sigma(i))$, for all $i \in X$. From the aforementioned, we have the following commutative diagram with exact rows, for every $i \in X$: 
Hence, it follows from the universal property of pushouts that there is a morphism \( \eta : A \xrightarrow{f} E \xrightarrow{g} B^{(X)} \) such that \( \eta_{\sigma(i)} : A \xrightarrow{f_{\sigma(i)}} E_{\sigma(i)} \xrightarrow{g_{\sigma(i)}} B \). From the universal property of coproducts and of the following chain of compositions, we deduce that \( f \circ \nabla^A_X = \gamma \circ (\bigoplus_{i \in X} f_{\sigma(i)}) \).

Then, by the universal property of coproducts, there is \( \gamma \in \text{Hom}_A(\bigoplus_{i \in X} E_{\sigma(i)}, E) \) such that \( \gamma \circ \mu^E_{\sigma(i)} = \mu^A_{\sigma(i)} \), where \( \mu^E_{\sigma(i)} : E_{\sigma(i)} \rightarrow \bigoplus_{i \in X} E_{\sigma(i)} \) is the \( i \)-th canonical inclusion. From the universal property of coproducts and of the following chain of compositions, we deduce that \( f \circ \nabla^A_X = \gamma \circ (\bigoplus_{i \in X} f_{\sigma(i)}) \).

Now, by a similar argument to the proof of Lemma 2.11(c), we have the following pushout diagram.

\[
\begin{array}{c}
\bigoplus_{i \in X} \eta_{\sigma(i)} : & \bigoplus_{i \in X} f_{\sigma(i)} & \bigoplus_{i \in X} E_{\sigma(i)} & \bigoplus_{i \in X} g_{\sigma(i)} & B^{(X)} \\
\bigoplus_{i \in X} \eta' & \downarrow \nabla^A_X & \downarrow \nabla^E & \downarrow g' & \bigoplus_{i \in X} B \end{array}
\]

Hence, it follows from the universal property of pushouts that there is a morphism \( \gamma' : E' \rightarrow E \) such that \( \gamma' \circ f' = f \). And hence, by the universal property of cokernels, there is a morphism \( \gamma'' : B^{(X)} \rightarrow B^{(X)} \) such that \( \eta'' = \eta \cdot \gamma'' \). \qed

### 3.2. A characterization of Ab4 categories.

**Theorem 3.6.** Let \( A \) be an Ext-small abelian category which is Ab3. Then, \( A \) is Ab4 if, and only if, there is a universal extension of \( B \) by \( A \), for all \( A, B \in A \).

**Proof.** By [16, Proposition 5.9], we know that if \( A \) is Ab4 and Ext-small, then universal extensions exist for any pair of objects.

To prove the opposite implication, we consider an object \( A \in A \), a set \( \{B_i\}_{i \in X} \) of objects in \( A \) and its coproduct \( B := \bigoplus_{i \in X} B_i \). We will seek to show that, for every \( (\eta_i) \in \prod_{i \in X} \text{Ext}_A^1(B_i, A) \), the morphism \( f_H \) in the exact sequence \( A \xrightarrow{f_{\eta_i}} Z_H \xrightarrow{g_H} \bigoplus_{i \in X} B_i \) is a monomorphism, where \( H = \{\eta_i\}_{i \in X} \) (see Lemma 2.11). For this, we will use a universal extension of \( B \) by \( A \), say \( \eta : A \xrightarrow{a} D \xrightarrow{b} B^{(Y)} \). Consider the exact sequence \( \eta'_i := \eta_i \oplus \kappa_i \), for all \( i \in X \), where \( \kappa_i : 0 \rightarrow \bigoplus_{j \in X - \{i\}} B_j \xrightarrow{\oplus_{j \in X - \{i\}}} B_i \).

By Definition 2.13(c) there is a morphism \( u_i : B \rightarrow B^{(Y)} \) such that \( \eta'_{\eta_i} = \eta_{\eta'_i} \) for every \( i \in X \). This gives us the following commutative diagram.
Injective and projective effacements

Injective (resp. projective) effacements are a notion that arose from abstracting injective (resp. projective) objects [13, Section 1.10]. Among their applications we can mention that they are used to characterize Ab4 categories in a similar way to what we have done in the previous section (see [5, Theorem A], [13, Section 1.10, Remark 1], [15, Section 5, Example A], [20, Corollary 1.4], [19, Theorem 1]). Which leads us to ask whether there is a relationship between universal extensions and injective effacements. The aim of this section will be to explore this question.

**Definition 4.1.** [13, Definition 3.1] Let $\mathcal{A}$ be an abelian category, $A$ be an object of $\mathcal{A}$, and $S$ be a class of objects in $\mathcal{A}$. An $S$-injective effacement of $A$ is a monomorphism $\iota : A \rightarrow \overline{A}$ such that for any monomorphism $f : A \rightarrow B$ with $\text{Coker}(f) \in S$, there is a morphism $j : B \rightarrow \overline{A}$ such that $j \circ f = \iota$. An $S$-projective effacement is an epimorphism satisfying the dual condition. Moreover, we say that an $\mathcal{A}$-injective (resp. $\mathcal{A}$-projective) effacement is an injective (resp. projective) effacement.

**Remark 4.2.** Observe that a monomorphism $\iota : A \rightarrow \overline{A}$ is an $\mathcal{S}$-injective effacement if, and only if, $\text{Ext}_A^1(N, \iota) : \text{Ext}_A^1(N, A) \rightarrow \text{Ext}_A^1(N, \overline{A})$ is the zero morphism $\forall N \in S$ (see [20, Corollary 1.4]). And hence, the following statements are equivalent:

(a) we have an exact sequence $A \xrightarrow{\iota} \overline{A} \xrightarrow{\pi} C$, where $\iota$ is an $S$-injective effacement,
(b) $\text{Ext}_A^1(B, \iota) : \text{Ext}_A^1(B, A) \rightarrow \text{Ext}_A^1(B, \overline{A})$ is the zero morphism $\forall B \in S$,
(c) $\text{Ext}_A^1(B, \pi) : \text{Ext}_A^1(B, A) \rightarrow \text{Ext}_A^1(B, C)$ is injective $\forall B \in S$,
(d) the connection morphism $\delta : \text{Hom}_A(B, C) \rightarrow \text{Ext}_A^1(B, A)$ is surjective $\forall B \in S$. 

Here is the diagram:

$$
\begin{array}{ccc}
\eta_i : & A & \xrightarrow{f_i} & E_i & \xrightarrow{g_i} & B_i \\
\eta'_i : & A & \xrightarrow{1} & E_i & \oplus & B'_i & \xrightarrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} & B \\
\eta : & A & \xrightarrow{a} & D & \xrightarrow{b} & B^{(y)}
\end{array}
$$

where $B'_i := \bigoplus_{j \in X - \{i\}} B_i$. Now, by the universal property of coproducts, there is $\gamma \in \text{Hom}_A(\bigoplus_{i \in X} E_i, D)$ such that $\gamma \circ \mu_i^e = u'_i \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for all $i \in X$. Moreover, since

$$
\gamma \circ \left( \bigoplus_{i \in X} f_i \right) \circ \mu_i^e = \gamma \circ \mu_i^e \circ f_i = u'_i \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \circ f_i = a = a \circ \nabla_X^A \circ \mu_i^e, \forall i \in X
$$

we have that $\gamma \circ \left( \bigoplus_{i \in X} f_i \right) = a \circ \nabla_X^A$. And hence, by the universal property of pushouts between $\nabla_X^A$ and $\bigoplus_{i \in X} f_i$, there is a morphism $\gamma' : Z_H \rightarrow D$ such that $\gamma' \circ f_H = a$. Therefore, since $a$ is a monomorphism, $f_H$ is a monomorphism. This proves that $\mathcal{A}$ is Ab4 by Lemma 2.11 and Theorem 2.12. 

\[ \square \]
In particular, an exact sequence \( A \xrightarrow{\lambda} X \xrightarrow{\eta} B^{(X)} \) is a universal extension if, and only if, \( \lambda \) is an \( \{B\}\)-injective effacement. Therefore, we have that universal extensions are a special (and easy-to-use) kind of injective effacements.

In the rest of the subsection we study projective effacements in Grothendieck categories which are \( \text{Ab}^{4*} \). Concretely, we give a new proof of [20, Corollary 1.4]) using universal co-extensions.

**Proposition 4.3.** If \( \mathcal{G} \) be a Grothendieck category which is \( \text{Ab}^{4*} \), then every object of \( \mathcal{G} \) has a projective effacement.

**Proof.** Notice first that in \( \mathcal{G} \), there exists the universal co-extensions for any couple of objects in \( \mathcal{G} \) (see Theorem 3.10). On the other hand, let \( A \) be an object in \( \mathcal{G} \). From [14] Lemma 2.5.13, Proposition 2.5.16, we know that there is \( \kappa \)-regular cardinal such that \( A \) is \( \kappa \)-presentable. From Remark 2.3 and Proposition 2.5, we know that there is a regular cardinal \( \lambda \geq \kappa \) such that \( A \) is \( \lambda \)-FP2 and there is a set \( S \) of objects in \( \mathcal{G} \) such that every object in \( \mathcal{G} \) is a \( \lambda \)-directed colimit of objects in \( S \). Now, for each \( S \) in \( S \), we take a universal co-extension of \( S \) by \( A \), say \( \xi_S \in \text{Ext}_A^1(A, V_S) \), where \( V_S = S^{X_S} \) for some non-empty set \( X_S \). Using the \( \text{Ab}^{4*} \)-condition on \( \mathcal{G} \) together with the Theorem 2.12 we get that the canonical map \( \text{Ext}_A^1(A, \prod_{S \in S} V_S) \rightarrow \prod_{S \in S} \text{Ext}_A^1(A, V_S) \) is bijective. And hence, there is an extension \( \epsilon : \prod_{S \in S} V_S \xrightarrow{\iota} A \rightarrow A \) such that \( \pi_\lambda \cdot \epsilon = \xi_S \) for all \( B \in S \). We claim that \( \pi : A \rightarrow A \) is a projective effacement of \( A \). Indeed, let \( B \) be an object in \( \mathcal{G} \) and let \( F : D \rightarrow \mathcal{G} \) be a \( D \)-diagram on \( \mathcal{G} \) such that: (i) \( D \) is a \( \lambda \)-directed poset; (ii) \( F(d) \in S \), for all \( d \in D \); and (iii) \( \text{colim}_D F(d) = B \). Now, for each \( d \) in \( D \) we consider an extension in \( \mathcal{G} \) of the form:

\[
\eta_d : F(d) \xrightarrow{\iota_d} K_d \xrightarrow{\pi_d} A
\]

Thus, there is \( f_d \in \text{Hom}_\mathcal{G}(F(d), X_{F(d)}, F(d)) \) such that \( \pi_d \cdot \epsilon_{F(d)} = \xi_d \). In particular, we get \( \xi_d = (f_d \circ \pi_{F(d)}) \cdot \epsilon \). This fact shows that the canonical morphism \( \text{Ext}_A^1(A, F(d)) \xrightarrow{\xi_d} \text{Ext}_A^1((A, F(d)) \) is zero. On the other hand, observe that we have the following commutative diagram

\[
\begin{array}{ccc}
\text{colim}_D \text{Ext}_A^1(A, F(d)) & \xrightarrow{\xi_d} & \text{colim}_D \text{Ext}_A^1((A, F(d)) \\
\downarrow & & \downarrow \\
\text{Ext}_A^1(A, \text{colim}_D F(d)) & \xrightarrow{\xi_d} & \text{Ext}_A^1((A, \text{colim}_D F(d))
\end{array}
\]

where the morphism on the left is an isomorphism since \( A \) is \( \lambda \)-FP2. Hence, we deduce that the canonical morphism \( \text{Ext}_A^1(A, B) \rightarrow \text{Ext}_A^1((A, B)) \) is zero, since the morphism on the top of the diagram is null. □

**Corollary 4.4.** [20] Corollary 1.4] Let \( \mathcal{G} \) be a Grothendieck category. Then, \( \mathcal{G} \) is \( \text{Ab}^{4*} \) if, and only if, every object of \( \mathcal{G} \) has a projective effacement.

**Proof.** From Proposition 1.3 and the dual of Theorem 3.10 we only need to check that there exists the universal co-extensions for any couple of objects in \( \mathcal{G} \). Indeed, let \( A, B \) be objects in \( \mathcal{G} \) and let \( \pi : A \rightarrow A \) be a projective effacement of \( A \). Using the dual of Remark 4.2(d), we deduce that \( \text{Ext}_A^1(A, B) \) is a set. Therefore, we can
find a complete set
\[
\left\{ \eta_i : B \xrightarrow{a_i} E_i \xrightarrow{b_i} A \right\}_{i \in I}
\]
of representatives of \( \text{Ext}^1_G(A, B) \). Moreover, for each \( i \in I \) there is \( f_i \in \text{Hom}_G(\ker(\pi), B) \) such that \( f_i \cdot \epsilon = \eta_i \), where \( \epsilon := \ker(\pi) \xrightarrow{\iota} A \xrightarrow{\pi} B \). Now, it follows from the universal property of products that there is a morphism \( f : \ker(\pi) \to B \) such that \( \pi_B \circ f = f_i \), for all \( i \in I \). And hence, \( \epsilon_B := f \cdot \epsilon \) is a universal co-extension of \( B \) by \( A \) since \( \pi^B_B \cdot \epsilon_B = (\pi^B_B \circ f) \cdot \epsilon = f_i \cdot \epsilon = \eta_i \), for all \( i \in I \).

\[ \square \]

5. Ext-universal objects in Ab3 abelian categories

As we have seen in the subsection 3.2, if \( A \) is an Ab3 abelian category which is not Ab4, then there are objects \( B \) and \( A \) in \( A \) such that there is no universal extension of \( B \) by \( A \). The goal of this section is characterize those objects \( V \) such that there is always a universal extension of \( V \) by any other object in \( A \).

**Definition 5.1.** [16, Definition 5.6] Let \( A \) be an abelian category and \( A \in A \).

(a) An object \( V \) in \( A \) is said to be \( \text{Ext}^1 \)-universal by \( A \) when a universal extension of \( V \) by \( A \) exists in \( A \). The class of objects that satisfy being \( \text{Ext}^1 \)-universal by \( A \) will be denoted by \( \text{Ext}^1_{u, A}(A) \).

(b) An object \( V \) is said to be \( \text{Ext}^1 \)-universal if \( V \in \bigcap_{A \in A} \text{Ext}^1_{u, A}(A) \). The class of \( \text{Ext}^1 \)-universal objects in \( A \) will be denoted by \( \text{Ext}^1_{u, A} \).

**Remark 5.2.** Let \( A \) be an Ext-small Ab3 abelian category and let \( V \in A \) be an \( \text{Ext}^1 \)-universal object. It follows from Corollary 3.3 that, for every \( A \in A \), we can consider the canonical universal extension of \( V \) by \( A \).

**Lemma 5.3.** Let \( A \) be an Ab3 abelian category, and \( A, V \) be objects in \( A \) such that \( X := \text{Ext}^1_A(V, A) \) is a set. Consider the following statements:

(a) \( V \in \text{Ext}^1_{u, A}(A(X)) \);

(b) if \( \{ \eta_x : A \xrightarrow{f^X_x} E_x \xrightarrow{g_x} V \}_{x \in X} \) is a complete set of representatives of \( \text{Ext}^1_A(V, A) \), then \( \bigoplus_{x \in X} f^X_x : A(X) \to \bigoplus_{x \in X} E_x \) is monic, and hence \( \bigoplus_{x \in X} \eta_x \) is a short exact sequence;

(c) \( V \in \text{Ext}^1_{u, A}(A) \).

Then, \( (a) \Rightarrow (b) \) and \( (b) \Rightarrow (c) \).

**Proof.** \( (b) \Rightarrow (c) \) It follows straightforward from Theorem 2.12 [1] Lemma 4.2], Lemma 2.11 and Lemma 3.1.
(a) ⇒ (b) Consider the set of extensions \( \{\mu^x \cdot \eta_x\}_{x \in X} \). By Lemma 3.1 there is an extension \( \gamma \in \text{Ext}^1_\mathcal{A}(V^{(X)}, A^{(X)}) \) such that \( \gamma \cdot \eta_x = \mu^x \cdot \eta_x \), for all \( x \in X \).

As a consequence of this, we have the commutative diagram on the right, where \( \rho : A^{(X)} \twoheadrightarrow W \overset{\beta}{\rightarrow} V^{(X)} \). Now, by the universal property of coproducts, we get that there is \( \gamma \in \text{Hom}_\mathcal{A}(\bigoplus_{x \in X} E_x, W) \) such that \( \gamma \circ \mu^x = \mu^x \circ \mu^x \), for all \( x \in X \). Observe that \( \gamma \circ (\bigoplus_{x \in X} f_x) = f \). And hence, since \( f \) is monic, \( \bigoplus_{x \in X} f_x \) is a monomorphism.

\[ \square \]

Let us study how to find new \( \text{Ext}^1 \)-universal objects from old ones.

**Lemma 5.4.** Let \( \mathcal{A} \) be an Ab3 abelian category, \( \{V_i\}_{i \in I} \) be a set of objects in \( \mathcal{A} \) and \( A \in \mathcal{A} \). Then, the following statements hold true (here \( V := \bigoplus_{i \in I} V_i \) and \( X := \text{Ext}^1_{\mathcal{A}}(V, A) \)):

(a) If there is a universal extension of \( V_i \) by \( A \), for all \( i \in I \), then \( \text{Ext}^1_{\mathcal{A}}(V^{(S)}, A) \) is a set, for all set \( S \). In particular, \( X \) is a set.

(b) If \( V_i \in \text{Ext}_{\mathcal{A}}(A) \), for all \( i \in I \) and the natural maps

\[ \text{Ext}^1_{\mathcal{A}}(V^{(X)}, A) \to \prod_{i \in I} \text{Ext}^1_{\mathcal{A}}(V_i^{(X)}, A) \text{ and } \text{Ext}^1_{\mathcal{A}}(V, A) \to \prod_{i \in I} \text{Ext}^1_{\mathcal{A}}(V_i, A) \]

are bijective, then \( V \in \text{Ext}_{\mathcal{A}}(A) \).

**Proof.** Before proceeding with the proof, let us recall the following fact. Let \( S \) be a set. Consider the set \( \{V_s\}_{(i,s) \in I \times S} \subseteq \mathcal{A} \), where \( V_{i,s} := V_i \), for all \( s \in S \), and the coproduct \( V' := \bigoplus_{(i,s) \in I \times S} V_{i,s} \) with the canonical inclusions \( \alpha_{i,s} : V_{i,s} \to V' \). In one hand, we can see that, for \( i \) fixed, there is a unique morphism \( u_i : V_i^{(S)} \to V' \) such that \( u_i \circ v_{i,s} = \alpha_{i,s} \), for all \( s \in S \), where \( v_{i,s} : V_i \to V_i^{(S)} \) is the \( s \)-th canonical inclusion. Moreover, it can be shown that \( V' \) is the coproduct \( \bigoplus_{i \in I} V_i^{(S)} \) and that \( \{u_i : V_i^{(S)} \to V'\}_{i \in I} \) is the set of canonical inclusions.

On the other hand, we can see that, for \( s \) fixed, there is a unique morphism \( w_s : \bigoplus_{i \in I} V_{i,s} \to V' \) such that \( w_s \circ \mu_i^{(S)} = \alpha_{i,s} \), for all \( i \in I \), where \( \mu_i^{(S)} : V_i \to \bigoplus_{s \in S} V_i^{(S)} \) is the \( i \)-th canonical inclusion. Moreover, it can also be shown that \( V' \) is the coproduct \( \bigoplus_{i \in I} V_i^{(S)} \) and that \( \{w_s : V_i^{(X)} \to V'\}_{s \in S} \) is the set of canonical inclusions.

In other words, \( \bigoplus_{i \in I} \left( V_i^{(S)} \right) = \left( \bigoplus_{i \in I} V_i \right)^{(S)} \). We will use the notation presented in this paragraph throughout the proof for \( S = X \).

(a) Let \( S \) be a set. Observe that, for each \( i \in I \), \( \text{Ext}^1_{\mathcal{A}}(V_i, A) \) is a set by Definition 2.13(c). Thus, \( \text{Ext}^1_{\mathcal{A}}(V_i^{(S)}, A) \) also is a set by Lemma 3.1. Moreover, since there is an injective function

\[ \text{Ext}^1_{\mathcal{A}}(\bigoplus_{i \in I} V_i^{(S)}, A) \to \prod_{i \in I} \text{Ext}^1_{\mathcal{A}}(V_i^{(S)}, A) \]
Corollary 5.5. Let $\mathcal{A}$ be an Ab3 abelian category. Then, the following statements hold true:

(a) The class $\text{Ext}_{\mathcal{A}}(A)$ is closed under finite coproducts.

(b) If $\{V_i\}_{i \in I}$ is a set in $\text{Ext}_{\mathcal{A}}(A)$ and the natural map

$$\text{Ext}_{\mathcal{A}}^1 \left( \bigoplus_{i \in I} V_i^{(X)} \right) \to \prod_{i \in I} \text{Ext}_{\mathcal{A}}^1 (V_i^{(X)}, A)$$

is bijective for every set $X$ and every $a \in \mathcal{A}$, then $\bigoplus_{i \in I} V_i \in \text{Ext}_{\mathcal{A}}(A)$.

Lemma 5.6. Let $\mathcal{A}$ be an Ab3 abelian category, $\omega : A \xrightarrow{f} E \xrightarrow{g} V^{(X)}$ be a universal extension, $\eta : U \xrightarrow{a} V \xrightarrow{b} W$ be an exact sequence, and $(g' : E' \to U^{(X)}, e : E' \to E)$
be the pull-back of $g$ and $a^{(X)}$. If any of the following conditions hold true, then $\omega \cdot a^{(X)}$ is a universal extension of $U$ by $A$.

(a) $\text{Ext}^1_A(V, e)$ is injective and $\text{Ext}^1_A(a, A)$ is surjective.
(b) $\text{Ext}^1_A(U, e)$ is injective and $\text{Ext}^1_A(a, A)$ is surjective.
(c) If $\omega$ is the canonical universal extension and $a$ is a split monomorphism.

Proof. It is a known fact that, in this scenario, we get the following commutative diagram with exact rows.

Consider the diagram below, where $k(\cdot, -) := \text{Ext}^k_{\mathcal{A}}(\cdot, -)$, for all $k \in \{1, 2\}$. Observe that it is commutative since it is obtained through the left square in the diagram above. Moreover, the leftmost column is exact by the long exact sequence of homology induced by $\eta$. Now, since $\omega$ is a universal extension, we have that $\text{Ext}^1_{\mathcal{A}}(V, f) = 0$ by Definition 2.13(a). Let us prove that $\text{Ext}^1_{\mathcal{A}}(U, f') = 0$ in each case to conclude that $\omega \cdot a^{(X)}$ is a universal extension.

(a) If $\text{Ext}^1_{\mathcal{A}}(V, e)$ is injective, then $\text{Ext}^1_{\mathcal{A}}(V, f') = 0$ since $\text{Ext}^1_{\mathcal{A}}(V, e) \circ \text{Ext}^1_{\mathcal{A}}(V, f') = \text{Ext}^1_{\mathcal{A}}(V, f) = 0$. And thus,

$$\text{Ext}^1_{\mathcal{A}}(U, f') \circ \text{Ext}^1_{\mathcal{A}}(a, A) = \text{Ext}^1_{\mathcal{A}}(a, E') \circ \text{Ext}^1_{\mathcal{A}}(V, f') = 0.$$

Hence, if in addition $\text{Ext}^1_{\mathcal{A}}(a, A)$ is surjective, then $\text{Ext}^1_{\mathcal{A}}(U, f')$ is the zero map.

(b) It is proved in a similar way as (a).

(c) Since $a$ is a split monomorphism, there is $a' \in \text{Hom}_{\mathcal{A}}(V, U)$ such that $a'a = 1_U$. Let $\overline{\eta} \in \text{Ext}^1_{\mathcal{A}}(U, A)$. Observe that $\overline{\eta} \cdot a' \in \text{Ext}^1_{\mathcal{A}}(V, A)$. And hence, there is $i \in X$ such that $\overline{\eta} \cdot \mu^V_i = \overline{\eta} \cdot a'$. Therefore,

$$\overline{\eta} \cdot (a' \circ a) = \overline{\eta} \cdot (a^{(X)} \circ a) = \omega \cdot (a^{(X)} \circ \mu^V_i) = \omega \cdot (\mu^V_i \circ a) = \omega \cdot a^{(X)} \circ \mu^V_i.$$  
As a result, we have that $\text{Ext}^1_{\mathcal{A}}(U, f')$ is the zero map.

□

Corollary 5.7. Let $\mathcal{A}$ be an Ab3 abelian category and $V$ an $\text{Ext}^1$-universal object. Then, every direct summand of $V$ is a $\text{Ext}^1$-universal object.
6. Co-Ext\(^1\)-universal torsion groups

Throughout this section, \( p \) is a prime number and \( T \) is either \( T_Z \) or \( T_p \) (see Example 2.8). We recall that \( T \) is a Grothendieck category which is not Ab\(^4\) (see Remark 2.9 and Examples 2.8, 2.10). Thus, the abelian category \( T^{\text{op}} \) has objects which are not Ext\(^1\)-universal. In the sequel, we characterize those abelian groups in \( T \) which are Ext\(^1\)-universal in \( T^{\text{op}} \). For this task, we will study the dual definition of universal extension (see Definition 2.17) and Ext\(^1\)-universal object.

**Definition 6.1.** An abelian group \( G \) in \( T \) is said to be co-Ext\(^1\)-universal in \( T \) when a universal co-extension of \( G \) by any other abelian group in \( T \) exists in \( T \).

We recall that \( T_Z \) (resp. \( T_p \)) is the torsion class of a hereditary torsion pair in Ab (see Example 2.8(a)). We will denote by \( t : \text{Ab} \to \text{Ab} \) (resp. \( t_p : \text{Ab} \to \text{Ab} \)) the associated torsion radical. In this case, the functor \( t \) (resp. \( t_p \)) is left exact.

Moreover, let \( t \) denote either \( t \) or \( t_p \) in this section.

**Remark 6.2.** Note that for each extension \( \eta : N \xrightarrow{f} M \xrightarrow{g} K \) in Ab, we obtain an exact sequence \( t(\eta) : t(N) \xrightarrow{\overline{f}} t(M) \xrightarrow{\overline{g}} t(K) \) in Ab, where \( t(g) \) is not necessarily an epimorphism. On the other hand, we recall that the product in \( T \) of a family of abelian groups \( (G_i)_{i \in I} \subseteq T \) is given by \( t(\prod_{i \in I} G_i) \), where \( \prod_{i \in I} G_i \) is the product of such family in Ab, and the \( i \)-th canonical projections \( \pi_i^G := \pi_i \circ \iota_{\prod_{i \in I} G_i} : t(\prod_{i \in I} G_i) \to G_i \), where \( \iota_{\prod_{i \in I} G_i} : t(\prod_{i \in I} G_i) \to \prod_{i \in I} G_i \) is the canonical inclusion.

The following lemma establishes the relationship between the universal co-extensions in Ab and the universal co-extensions in \( T \).

**Lemma 6.3.** Let \( A, B \in T \) such that there is a universal co-extension of \( B \) by \( A \) in \( T \). If \( \eta : B^X \xrightarrow{f} E \xrightarrow{g} A \) is the canonical universal co-extension of \( B \) by \( A \) in Ab, then \( t(\eta) \) is a short exact sequence and it is the canonical universal co-extension of \( B \) by \( A \) in \( T \).

**Proof.** Let \( H = \left\{ \eta_i : B \xrightarrow{f_i} E_i \xrightarrow{g_i} A \right\}_{i \in \mathcal{I}} \) be a complete set of representatives of \( \text{Ext}_1^{\text{Ab}}(A, B) \). By the dual of Corollary 3.3 together with the Remark 6.2, we know that the canonical universal co-extension of \( B \) by \( A \) in \( T \) is the exact sequence \( \eta' : t(B^X) \xrightarrow{f'} E' \xrightarrow{g'} A \) built through the pullback of \( t(\Delta_X^A) \) and \( t(\prod_{i \in X} g_i) \) (the dual construction described in Lemma 2.11). Similarly, we know that \( \eta \) is built through the pullback of \( \Delta_X^A \) and \( \prod_{i \in X} g_i \). Therefore, we have the following commutative diagram.
Let us prove that \( t(g) \) is surjective and that \( \eta' \) is equivalent to \( t(\eta) \). For this, observe that \( \Delta_X \circ g' = (\prod_{i \in X} g_i) \circ \iota_{\prod_{i \in X} E, \circ \Delta'} \). And hence, by the universal property of pullbacks applied to \( \Delta_X \) and \( \prod_{i \in X} g_i \), there is \( \gamma \in \text{Hom}_{Ab}(E', E) \) such that \( \Delta' \circ \gamma = \iota_{\prod_{i \in X} E, \circ \Delta'} \) and \( g \circ \gamma = g' \). Moreover, since \( \Delta' \circ \gamma \circ f' = \Delta' \circ f \circ \iota_{B^X} \) and \( \Delta' \) is monic, we have that \( \gamma \circ f' = f \circ \iota_{B^X} \). Now, since \( E' \in \mathcal{T} \), there is \( \gamma' \in \text{Hom}_{Ab}(E', t(E)) \) such that \( \gamma = \iota_E \circ \gamma' \). Observe that \( \gamma' \circ f' = t(f) \) and \( t(g) \circ \gamma' = g' \) since \( \iota_E \circ \gamma' \circ f' = \iota_E \circ t(f), \Delta_X \circ t(g) \circ \gamma' = \Delta_X \circ g' \) and \( \Delta' \) are monic. Therefore, \( t(g) \) is surjective and that \( \eta' \) is equivalent to \( t(\eta) \). \( \square \)

**Proposition 6.4.** Let \( M \in \mathcal{T} \) be a cotorsion group (i.e. \( \text{Ext}^1_{Ab}(\mathbb{Q}/\mathbb{Z}, M) = 0 \)). Then, \( M \) is co-\( \text{Ext}^1 \)-universal in \( \mathcal{T} \).

**Proof.** Since \( M \) is a torsion cotorsion group, it follows from the Baer-Fomin Theorem that \( M = D \oplus B \), where \( D \) is an injective abelian group and \( B \) is a bounded abelian group (see [12] Chapter 15, Theorem 1.6]). By Remark 2.14(a), every injective torsion group is co-\( \text{Ext}^1 \)-universal in \( \mathcal{T} \). Hence, by Corollary 5.5(a), it is enough to show that \( B \) is co-\( \text{Ext}^1 \)-universal in \( \mathcal{T} \). Let \( A \in \mathcal{T} \). Consider a universal co-extension of \( B \) by \( A \) in \( \text{Ab} \), say \( \eta : B^X \rightarrow E \rightarrow A \). Observe that \( n \cdot B^X = 0 \), for some nonzero integer \( n \). Therefore, \( B^X, A \in \mathcal{T} \), and hence \( E \in \mathcal{T} \) because \( \mathcal{T} \) is closed under extensions in \( \text{Ab} \). Therefore, \( \eta \) is a universal co-extension of \( B \) by \( A \) in \( \mathcal{T} \). \( \square \)

**Proposition 6.5.** The following statements are equivalent for \( T, S \in \mathcal{T} \):

(a) there is a universal co-extension of \( T \) by \( S \) in \( \mathcal{T} \);
(b) \( \text{Ext}^1_{Ab}(S, t_{TX}) : \text{Ext}^1_{Ab}(S, t(T^X)) \rightarrow \text{Ext}^1_{Ab}(S, T^X) \) is an isomorphism, for every set \( X \);
(c) \( \text{Ext}^1_{Ab}(S, T^X/\overline{t(T^X)}) = 0 \), for every set \( X \).
Proof. Let \( X \) be a set and consider the exact sequence \( \mathcal{T}(T^X) \xrightarrow{t_{T^X}} T^X \xrightarrow{π_{T^X}} T^X/\mathcal{T}(T^X) \).
Applying \( \text{Hom}_{\mathcal{T}}(S, -) \), we get the exact sequence
\[
0 \to 1(S, t(T^X)) \xrightarrow{1(S, t_{T^X})} 1(S, T^X) \longrightarrow 1(S, T^X/\mathcal{T}(T^X)) \to 0
\]
where the last term is zero because \( \mathcal{A} \mathcal{B} \) is hereditary, and the first term is zero because \( S \in \mathcal{T} \) and \( T^X/\mathcal{T}(T^X) \) is torsion-free (with respect to \( \mathcal{T} \)). Now, observe \( \text{Ext}^1_{\mathcal{A} \mathcal{B}}(S, \mathcal{T}(T^X)) = \text{Ext}^1_{\mathcal{T}}(S, \mathcal{T}(T^X)) \), \( \text{Ext}^1_{\mathcal{A} \mathcal{B}}(S, T)^X = \text{Ext}^1_{\mathcal{T}}(S, T)^X \), and that the morphism
\[
\Phi_T : \text{Ext}^1_{\mathcal{T}}(S, \mathcal{T}(T^X)) \to \text{Ext}^1_{\mathcal{T}}(S, T)^X, \eta \mapsto \left((\pi_i \circ t_{T^X}) \cdot \eta\right)_{i \in X},
\]
is factored as \( \Phi_T = \Phi \circ \text{Ext}^1_{\mathcal{A} \mathcal{B}}(S, t_{T^X}) \), where \( \Phi : \text{Ext}^1_{\mathcal{A} \mathcal{B}}(S, T)^X \to \text{Ext}^1_{\mathcal{A} \mathcal{B}}(S, T)^X \) is the isomorphism defined as \( \Phi = \left((\pi_i \circ t_{T^X}) \cdot \eta\right)_{i \in X} \) (see Theorem 2.12). Hence, the following statements are equivalent: (i) \( \Phi_T \) is an isomorphism, (ii) \( \text{Ext}^1_{\mathcal{T}}(S, t_{T^X}) \) is an isomorphism, and (iii) \( \text{Ext}^1_{\mathcal{A} \mathcal{B}}(S, T^X/\mathcal{T}(T^X)) = 0 \). Now, it follows from the dual results of Lemma 5.1 and Corollary 5.2 that (i) holds for every set \( X \) if and only if (a) holds true. Therefore, statements (a), (b), and (c) are equivalent. \( \square \)

**Proposition 6.6.** The following statements are equivalent for \( T \in \mathcal{T} \):

(a) \( T \) is \( \text{co-Ext}^1 \)-universal in \( \mathcal{T} \);

(b) \( T^X/\mathcal{T}(T^X) \) is injective in \( \mathcal{A} \mathcal{B} \), for every set \( X \);

(c) every quotient of \( T \) is \( \text{co-Ext}^1 \)-universal in \( \mathcal{T} \).

**Proof.** Observe that (c) \( \Rightarrow \) (a) is trivial and that (b) \( \Rightarrow \) (a) follows straightforward from Lemma 6.3 a,c). For the proof of (a) \( \Rightarrow \) (b) recall that, for every abelian group \( G \) and every positive integer \( n \), we have that \( G/nG \cong \text{Ext}^1_{\mathcal{A} \mathcal{B}}(\mathbb{Z}(n), G) \) (see [12] p.267). Therefore, if \( T \) is \( \text{co-Ext}^1 \)-universal in \( \mathcal{T} \) and \( G = T^X/\mathcal{T}(T^X) \) for a set \( X \), then it follows from Lemma 6.3 a,c) that \( nG = G \) for all \( n > 0 \). And thus, \( G \) is injective (see [12] Chapter 4, Theorem 2.6). It remains to prove (a), (b) \( \Rightarrow \) (c). Let \( Q \) be a quotient of \( T \) and \( T \) be \( \text{co-Ext}^1 \)-universal in \( \mathcal{T}_Z \). Observe that, for every set \( X, Q^X/\mathcal{T}(Q^X) \) is a quotient of \( T^X/\mathcal{T}(T^X) \). Then, it follows from (b) that \( Q^X/\mathcal{T}(Q^X) \) is injective since quotients of injective groups are also injective in \( \mathcal{A} \mathcal{B} \). And hence, \( Q \) is \( \text{co-Ext}^1 \)-universal in \( \mathcal{T} \). \( \square \)

**Corollary 6.7.** Let \( p > 0 \) be a prime number. Then, \( \bigoplus_{n \in \mathbb{N}} \mathbb{Z}(p^n) \) is not \( \text{co-Ext}^1 \)-universal in \( \mathcal{T}_Z \) (resp. \( \mathcal{T}_p \)).

**Proof.** Consider \( R := \bigoplus_{n \in \mathbb{N}} \mathbb{Z}(p^n) \). By Proposition 6.3 a,b), it is enough to prove that \( R^N/t(R^N) \) is not injective. For this, observe that \( P := \prod_{n \in \mathbb{N}} \mathbb{Z}(p^n) \) is a quotient of \( R^N \). And thus, \( P/t(P) \) is a quotient of \( R^N/t(R^N) \). Therefore, we only need to prove that \( P/t(P) \) is not divisible. Let \( x := (x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \mathbb{Z}(p^n) \) with \( x_n = 1 + p^n \mathbb{Z} \), for all \( n \in \mathbb{N} \). If we assume that there is \( \alpha := (\alpha_n)_{n \geq 1} \in \prod_{n \in \mathbb{N}} \mathbb{Z}(p^n) \) such that \( x + t(P) = p\alpha + t(P) \), then there is \( m > 1 \) such that \( p^m(x - p\alpha) = 0 \), and thus, \( p^m \) divides \( p^n(1 - p\alpha) \), for all \( n \geq 1 \). This cannot be satisfied for \( n > m \). Therefore, \( P \) is not divisible. \( \square \)

**Fact 6.8.** Let \( p \) be a prime number. Recall that every \( A \in \mathcal{T}_p \) has a \( p \)-basic subgroup \( B \) (see [12] Chapter 5, Theorem 5.2]). That is, (i) \( B \) is a coproduct of cyclic groups, (ii) \( p^kB = B \cap p^kA \), for all \( k > 0 \), and (iii) \( p(A/B) = A/B \). We have the following properties for a \( p \)-basic subgroup \( B \) of a group \( A \in \mathcal{T}_p \).
(a) If $B$ is bounded and $A$ is reduced, then $A = B$. Indeed, if $B$ is bounded, then it follows from (ii) that $B$ is a direct summand of $A$ (see [12, Chapter 5, Theorem 2.5]). And hence, since $A$ is reduced, it follows from (iii) that $A = B$.

(b) In particular, if $A$ is reduced and not bounded, then $B$ is not bounded.

(c) Observe that, if $B$ is not bounded, then $B$ has a quotient isomorphic to $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}(p^n)$.

**Theorem 6.9.** The following statements are equivalent for $T \in T_{\mathbb{Z}}$:

(a) $T$ is co-$\text{Ext}^1$-universal in $T_{\mathbb{Z}}$;

(b) $t_p(T)$ is co-$\text{Ext}^1$-universal in $T_p$, for every prime number $p$;

(c) $t_p(T) = D_p \oplus R_p$, where $D_p$ is injective and $R_p$ is reduced and bounded, for every prime number $p$.

**Proof.** Let $\mathcal{P}$ be the set of prime numbers. We put $T_p := t_p(T)$, for every $p \in \mathcal{P}$; and note that $T = \bigoplus_{p \in \mathcal{P}} T_p$.

(a) $\Leftrightarrow$ (b) Let $X$ be a set. Observe that $T^X / t(t^X) \cong \bigoplus_{p \in \mathcal{P}} T_p^X / t(T_p^X)$ and hence, $T^X / t(t^X)$ is injective if and only if $T_p^X / t(T_p^X)$ is injective for all $p \in \mathcal{P}$ (see [12, Chapter 4, Theorem 3.1]). Therefore, the equivalence (a) $\Leftrightarrow$ (b) follows from Proposition 6.6(a,b).

(b) $\Rightarrow$ (c) Let $p \in \mathcal{P}$ and $T_p = D_p \oplus R_p$, where $D_p$ is injective and $R_p$ is reduced. If $R_p$ is not bounded, then it follows from [12, Chapter 5, Theorem 6.10] and Fact 6.8(c) that there is a quotient of $R_p$ isomorphic to $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}(p^n)$. And hence, by Proposition 6.6(a,c), we get that $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}(p^n)$ is co-$\text{Ext}^1$-universal in $T$, which contradicts Corollary 6.7.

(c) $\Rightarrow$ (b) Let $p \in \mathcal{P}$ and $T_p = D_p \oplus R_p$, where $D_p$ is injective and $R_p$ is reduced and bounded. By Corollary 5.3 and Proposition 6.6(a,b), it is enough to prove that $R_p$ is co-$\text{Ext}^1$-universal in $T_{\mathbb{Z}}$. And this follows from Proposition 6.6(a,b) since $R_p^X = t(R_p^X)$ for every set $X$. $\square$

**Acknowledgments**

The research presented in this article began in December 2021, when the first named author was a postdoctoral fellow at the Instituto de Matemática y Estadística Rafael Laguardia. The research project was continued and completed when the first author started a postdoctoral stay at Centro de Ciencias Matemáticas, UNAM Campus Morelia, in April 2022. The first author would like to thank all the academic and administrative staff of these institutions for their warm hospitality, and in particular Raymundo Bautista (CCM, UNAM), Marcelo Lanzilotta (IMERL) and Marco A. Pérez (IMERL) for all their support.

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