On a Closed Binding Curve of One-holed Torus

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Abstract

Given a closed binding curve $\gamma$ of a surface $\Sigma$, any equivalence class of marked complete hyperbolic structure can be decomposed into polygons (possibly with a puncture) with sides being hyperbolic geodesic segments. When $\Sigma$ is a one-holed torus and $\gamma = A^3B^2$, we show that any equivalence class of marked complete hyperbolic structure gives rise to an equilateral bigon with a puncture and a hexagon with equal opposite sides. In particular, we give a new coordinates of the Fricke Space of the one-holed torus.
1 Introduction

The length function of a simple closed curve on the Teichmüller Space $\mathcal{T}_\Sigma$ (Fricke Space $\mathcal{F}_\Sigma$) of a closed surface $\Sigma$ has been studied by S.Kerckhoff [Ke80] and S.Wolpert [Wo87] since 1980’s, and it plays an important role in understanding the geometry of $\mathcal{T}_\Sigma$. One of the most important features of these length functions is that they are convex along Earthquake Paths (or Weil-Petersson geodesics) [Ke80, Wo87].

**Definition 1.1** (Fricke Space). $\mathcal{F}_\Sigma := \{(f, X) \mid \Sigma \xrightarrow{f} X \text{ is a diffeomorphism}\} / \sim$, where $X$ is a complete hyperbolic surface and $(f, X) \sim (g, Y)$ if there exists a hyperbolic isometry $i : X \rightarrow Y$ such that the following diagram commutes up to isotopy.

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{f} & X \\
\| & \downarrow i & \\
\Sigma & \xrightarrow{g} & Y
\end{array}
$$

**Definition 1.2** (Closed Binding Curves). Let $\Sigma$ be a surface (possibly with punctures), a closed curve $\gamma$ is binding if $\gamma$ intersects itself in a minimal position and $\Sigma - \gamma$ is a union of disjoint disks (possibly with a puncture).

Let $\gamma$ be a closed binding curve of $\Sigma$, the length function $l_\gamma : \mathcal{F}_\Sigma \rightarrow \mathbb{R}$ is not only strictly convex along Earthquake Paths (or Weil-Petersson geodesics) but also proper. Consequently, it has a unique minimum at some marked hyperbolic structure $[f_\gamma, X_\gamma] \in \mathcal{F}_\Sigma$.

Also, for any $[f, X] \in \mathcal{F}_\Sigma$, the unique closed hyperbolic geodesic isotopic to $f(\gamma)$ will cut $X$ into polygons(possibly with a puncture) with sides being hyperbolic geodesics, and conversely given these polygons $[f, X]$ can be reconstructed. Therefore these polygons may provide invariants and combinatoric ways to study $\mathcal{F}_\Sigma$.

![Figure 1: The closed binding curve $A^3B^2$ in the one-holed torus.](image)

When $\Sigma_{1,1}$ is the one-holed torus and $\gamma = A^3B^2$ as shown in Figure 1, any $[f, X] \in \mathcal{F}_{\Sigma_{1,1}}$ can be decomposed into

- A bigon with side lengths $a$ and $b$ and angles $\alpha$ and $\beta$, which forms a cusp region.
• A Hexagon with side lengths $a, c, d, b, c'$ and $d'$ and angles $\pi - \beta, \alpha, \pi - \beta$, $\pi - \alpha, \beta$ and $\pi - \alpha$ such that $c = c'$ and $d = d'$.

as in Figure 2

Figure 2: The punctured bigon and hexagon in a decomposition.

Let $\mathcal{P} := \{ \text{compatible pairs of a punctured bigon and a hexagon} \}$, then there is a 1-1 correspondence

$$\mathcal{P} \xrightarrow{g} \mathcal{F}_{\Sigma_1,1}$$

which comes from gluing and cutting. Let $\mathcal{P}_0 \subset \mathcal{P}$ be the subset consisting of those with $\alpha = \beta$ and $a = b$.

**Definition 1.3 (Length Function on $\mathcal{P}$).** The length function $l : \mathcal{P} \rightarrow \mathbb{R}$ is given by $a + b + c + d$.

Therefore the diagram

$$\mathcal{P} \xrightarrow{g} \mathcal{F}_{\Sigma_1,1}$$

$$\downarrow l \quad \downarrow l$$

$$\mathbb{R} \quad \mathbb{R}$$

commutes by construction of $l$.

## 2 Main Theorems

**Theorem 2.1.** $\mathcal{P}_0 = \mathcal{P}$. In particular, for any $[f, X] \in \mathcal{F}_{\Sigma_1,1}$,

$$\alpha = \beta \in \left(0, \frac{2\pi}{3}\right),$$

and

$$a = b = \log \left( \frac{1 + \cos \left( \frac{\alpha}{2} \right)}{1 - \cos \left( \frac{\alpha}{2} \right)} \right).$$
Figure 3: The region of $\mathcal{V}$.

First, we need the following lemmas.

**Lemma 2.1** (Existence). There exists an injective map $j : \mathcal{V} \to \mathcal{P}_0$, where

$$\mathcal{V} := \left\{ (t, s) \in \mathbb{R}^2 \left| -\log \left( \frac{t}{1-t} \right) < s < \log \left( \frac{t}{1-t} \right), \frac{1}{2} < t < 1 \right. \right\}.$$

This lemma will be proved in Section 3.

**Remark**

- $s$ will be the displacement of the mid point of $a$ from the common perpendicular of $a$ and $b$.
- $t$ will be $\cos(\frac{\alpha}{2})$.

**Lemma 2.2** (Properness). The pullback of the length function $j^*(l) : \mathcal{V} \to \mathbb{R}$ is proper.

This lemma will be proved in Section 4.

**Theorem 2.2.** $g \circ j : \mathcal{V} \to \mathcal{F}_{\Sigma_1,1}$ is a diffeomorphism.

**Proof of Theorem 2.2.** Note that $\mathcal{F}_{\Sigma_1,1}$ is diffeomorphic to $\mathbb{R}^2$, hence $g \circ j$ is an injective local diffeomorphism.

In addition $g \circ j$ is proper, since $j^*(l)$ and $l_\gamma$ are proper by Lemma 2.2 and the following diagram commutes.

$$\begin{array}{c}
\mathcal{V} \xrightarrow{g \circ j} \mathcal{F}_{\Sigma_1,1} \\
\downarrow j^*(l) \quad \downarrow l_\gamma \\
\mathbb{R} \quad \mathbb{R}
\end{array}$$

Therefore $g \circ j$ is a diffeomorphism by *Invariance of Domain* (See [Ha]).
Proof of Theorem 2.1. \( g \circ j \) is onto by Theorem 2.2, hence \( P_0 = P \). Since 
\[
\cos \left( \frac{\alpha}{2} \right) = t \in \left( \frac{1}{2}, 1 \right) \text{ (see remark following lemma 2.1)},
\]
\[
\alpha = \beta \in \left( 0, \frac{2\pi}{3} \right)
\]
and
\[
a = b = \log \left( \frac{1 + \cos \left( \frac{\alpha}{2} \right)}{1 - \cos \left( \frac{\alpha}{2} \right)} \right)
\]
follows from the following lemma.

Lemma 2.3. If \( a = b \) and \( \alpha = \beta \in (0, \pi) \), then
\[
a = \log \left( \frac{1 + \cos \left( \frac{\alpha}{2} \right)}{1 - \cos \left( \frac{\alpha}{2} \right)} \right).
\]

Proof. Use the upper half plane model with the hyperbolic metric
\[
ds^2 = \frac{dx^2 + dy^2}{y^2}.
\]
The length of the geodesic segment \( a \) is given by
\[
a = \int_{\frac{\alpha}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sin(\theta)} = \log \left( \frac{1 + \cos \left( \frac{\alpha}{2} \right)}{1 - \cos \left( \frac{\alpha}{2} \right)} \right).
\]

Figure 4: Relation between \( a \) and \( \alpha \) in the cusp.
3 Existence of Compatible Pairs from \( \mathcal{V} \)

In this section, we will prove Lemma 2.1 i.e.

There exists an injective map \( j: \mathcal{V} \rightarrow \mathcal{P}_0 \), where

\[
\mathcal{V} := \left\{ (t, s) \in \mathbb{R}^2 \mid -\log \left( \frac{t}{1-t} \right) < s < \log \left( \frac{t}{1-t} \right), \frac{1}{2} < t < 1 \right\}. \tag{1}
\]

Given \( t \in \left( \frac{1}{2}, 1 \right) \), Lemma 2.3 also guarantees the existence of the punctured bigon with

\[
\alpha = \beta = 2 \cos^{-1}(t),
\]

\[
a = b = \log \left( \frac{1+t}{1-t} \right). \tag{2}
\]

We divide \( \mathcal{V} \) into three parts as also shown in Figure 3.

- \( I = \{ (t, s) \in \mathcal{V} \mid -\frac{1}{2} \log \left( \frac{1+t}{1-t} \right) < s < \frac{1}{2} \log \left( \frac{1+t}{1-t} \right) \} \)
- \( II = \{ (t, s) \in \mathcal{V} \mid s < -\frac{1}{2} \log \left( \frac{1+t}{1-t} \right) \text{ or } s > \frac{1}{2} \log \left( \frac{1+t}{1-t} \right) \} \)
- \( III = \{ (t, s) \in \mathcal{V} \mid s = \pm \frac{1}{2} \log \left( \frac{1+t}{1-t} \right) \} \)

Injectivity of \( j \) will follow from our construction.

3.1 Existence of Type I Hexagons

For any \((t, s)\) in

\[
I = \left\{ (t, s) \in \mathcal{V} \mid -\frac{1}{2} \log \left( \frac{1+t}{1-t} \right) < s < \frac{1}{2} \log \left( \frac{1+t}{1-t} \right) \right\} \tag{3}
\]

we are going to construct the hexagon as shown in Figure 4.

- First of all, \( \alpha \) and \( a \) is determined by (2) and let

\[
a_1 := \frac{a}{2} + s = \frac{1}{2} \log \left( \frac{1+t}{1-t} \right) + s
\]

\[
a_2 := \frac{a}{2} - s = \frac{1}{2} \log \left( \frac{1+t}{1-t} \right) - s. \tag{4}
\]

Therefore by (1) and (3),

\[
a_1, a_2 \in \begin{cases} 
\left( \frac{1}{2} \log \left( \frac{1+t}{1-t} \right) - \log \left( \frac{1}{1-t} \right), \frac{1}{2} \log \left( \frac{1+t}{1-t} \right) + \log \left( \frac{1}{1-t} \right) \right), & t \in \left( \frac{1}{2}, \frac{\sqrt{2}}{2} \right) \\
(0, \log \left( \frac{1+t}{1-t} \right)), & t \in \left[ \frac{1}{2}, \frac{1}{1} \right)
\end{cases} \tag{5}
\]
Second, we need to show Proposition 3.1. The geodesic ray along $c$ does not intersect the common perpendicular on the right; and the geodesic ray along $d$ does not intersect the common perpendicular on the left either. (see Figure 5)

Recall the following fact from hyperbolic geometry (See [Ra]).

**Lemma 3.1.** The area of a hyperbolic triangle (possibly with ideal vertices) is given by

$$\pi - \alpha - \beta - \gamma$$

where $\alpha, \beta, \gamma$ are the inner angles. In particular, $\alpha + \beta + \gamma < \pi$.

**Proof of Proposition 3.1.** When $t \in \left[\frac{\sqrt{2}}{2}, 1\right]$ (i.e. $\alpha \in \left(0, \frac{\pi}{2}\right)$), the proposition is obviously true from Lemma 3.1. Therefore we may assume $t \in \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$ (i.e. $\alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$).

From Figure 6, the Euclidean length of $r$, which is the least distance to keep away from intersecting, is given by

$$\frac{\cos(\pi - \alpha)}{1 + \sin(\pi - \alpha)} = \frac{-\cos(\alpha)}{1 + \sin(\alpha)}.$$
Therefore using the Poincaré disk model with the hyperbolic metric

\[ ds^2 = \frac{4(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2}, \]

and hence \( a_1 > r \) and \( a_2 > r \) from (5). Therefore the proposition is true as well. \( \square \)

- And last, we are ready to show the existence of Type I hexagons.

Figure 7: Existence of Type I Hexagons.

Proof. Without loss of generality, we may assume \( s \leq 0 \), then \( a_1 \leq a_2 \).

See Figure 7. Since the blue line does not intersect the horizontal line by Proposition 3.1, and \( a_1 \leq a_2 \), there is a magenta line such that its vertex is on the blue line. They form a quadrilateral with angles \( \pi - \alpha, \frac{\pi}{2}, \frac{\pi}{2} \) and some nonzero angle, therefore its area is less than \( \alpha \). (When \( s = 0 \), it is a degenerate quadrilateral with area 0 < \( \alpha \).

Continue parallel translating the magenta line to the right, then there is a brown line such that it intersects the blue line at infinity. They form a pentagon with angles \( \pi - \alpha, \frac{\pi}{2}, \frac{\pi}{2}, \pi - \alpha \) and 0, therefore its area is equal to 2\( \alpha \).
Therefore there is a unique red line between the magenta line and the brown line, such that the area bounded by it together with the blue line is exactly $\alpha$, hence the angle between the blue line and the red line is $\alpha$.

By doubling the pentagon formed by the blue line and the red line, we find the desired Type I hexagon.

3.2 Existence of Type II Hexagons

For any $(t, s)$ in

$$II = \left\{ (t, s) \in \mathcal{V} \mid s < -\frac{1}{2} \log \left(\frac{1+t}{1-t}\right) \text{ or } s > \frac{1}{2} \log \left(\frac{1+t}{1-t}\right) \right\}. \quad (7)$$

From (1) and (7), $t \in \left(\frac{\sqrt{2}}{2}, 1\right)$ (i.e. $\alpha \in (0, \frac{\pi}{2})$). And without loss of generality, we assume $s < 0$ in the following.

We are going to construct the hexagon as shown in Figure 8.

![Figure 8: Shape of a Type II Hexagon.](image)

• First of all, we need to construct the right triangle at the right corner of Figure 8.

**Proposition 3.2.** There exists a right triangle with a side of length $-s - \frac{\alpha}{2}$ and the other adjacent angle being $\alpha$.

**Proof.** Since

$$-\log \left(\frac{t}{1-t}\right) < s < -\frac{1}{2} \log \left(\frac{1+t}{1-t}\right)$$

from (1) and (7) and

$$a = \log \left(\frac{1+t}{1-t}\right),$$
from (2),

\[-s - \frac{a}{2} \in \left(0, \log \left(\frac{t}{1-t}\right) - \frac{1}{2} \log \left(\frac{1+t}{1-t}\right)\right). \quad (8)\]

Use the same picture as in Figure 6 but replacing the angle \(\pi - \alpha\) with \(\alpha\), the least distance from being intersecting is given by

\[r' = \log \left(\frac{1 + \sin(\alpha) + \cos(\alpha)}{1 + \sin(\alpha) - \cos(\alpha)}\right)\]

\[= \log \left(\frac{1 + 2t\sqrt{1-t^2} + (2t^2 - 1)}{1 + 2t\sqrt{1-t^2} - (2t^2 - 1)}\right)\]

\[= \log(t) - \frac{1}{2} \log(1-t) - \frac{1}{2} \log(1+t). \quad (9)\]

Then \(-s - \frac{a}{2} < r'\) by (8), hence there exists a unique such right triangle. 

Note in Figure 8 that \(c_1\) is parallel to \(c_2\) along the common perpendicular and the geodesic along \(d\) does not intersect the common perpendicular on the left since \(\pi - \alpha > \frac{\pi}{2}\). Then we are ready to show the existence of Type II hexagons.

\[\text{Indicated angles are } \pi - \alpha\]

\[\text{area} = \alpha\]

\[\text{area} < \alpha\]

\[\text{area} = 2\alpha\]

\[\theta\]

\(-s - \frac{a}{2}\]

Figure 9: Existence of Type II Hexagons.

**Proof.** See Figure 9. Parallel translating \(c_2\) along the horizontal line to the left, there is a *magenta line* such that it meets the red line at the vertex, together with \(c_2\) and the horizontal line they form a quadrilateral with angles \(\theta, \pi - \theta, \pi - \alpha\) and some nonzero angle, therefore its area is less than \(\alpha\).

Continue parallel translating the *magenta line* to the left, then there is a *brown line* such that it intersects the red line at infinity. They form a
pentagon with angles $\theta, \pi - \theta, \pi - \alpha, \pi - \alpha$ and 0, therefore its area is equal to $2\alpha$.

Therefore there is a unique blue line between the magenta line and the brown line, such that the area inscribed by it together with the red line and $c_2$ is exactly $\alpha$, hence the angle between the blue line and the red line is $\alpha$.

By doubling the pentagon formed by the blue line, the red line and $c_2$, we find the desired Type II hexagon.

\[ \square \]

### 3.3 Existence of Type III Hexagons

For any $(t, s)$ in

\[ III = \left\{ (t, s) \in V \mid s = \pm \frac{1}{2} \log \left( \frac{1 + t}{1 - t} \right) \right\} \]

(10)

From (1) and (7), $t \in \left( \sqrt{2}, 1 \right)$ (i.e. $\alpha \in \left( 0, \frac{\pi}{2} \right)$). And without loss of generality, we assume $s = -\frac{1}{2} \log \left( \frac{1 + t}{1 - t} \right)$ in the following.

We are going to construct the hexagon as shown in Figure 10.

**Figure 10: Shape of a Type III Hexagon.**

- First of all, like Proposition 3.1 the geodesic ray along $d$ does not intersect the common perpendicular on the left either in this case, since $\pi - \alpha > \frac{\pi}{2}$.
- Then we are ready to show the existence of Type III hexagons.

**Proof.** See Figure 11. There is a magenta line such that it meets the red line at the vertex, they form a triangle with angles $\frac{\pi}{2} - \alpha, \frac{\pi}{2}$ and some nonzero angle, therefore its area is less than $\alpha$. 

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Indicated angles are $\frac{\pi}{2} - \alpha$

area $= \frac{\alpha}{2}$

$\pi - \alpha$

area < $\alpha$

Figure 11: Existence of Type III Hexagons.

Continue parallel translating the magenta line to the left, then there is a brown line such that it intersects the red line at infinity. They form a quadrilateral with angles $\frac{\pi}{2} - \alpha, \frac{\pi}{2}, \pi - \alpha$ and 0, therefore its area is equal to $2\alpha$.

Therefore there is a unique blue line between the magenta line and the brown line, such that the area inscribed by it together with the red line is exactly $\alpha$, hence the angle between the blue line and the red line is $\alpha$.

By doubling the quadrilateral formed by the blue line and the red line, we find the desired Type III hexagon.

This concludes the proof of Lemma 2.1.

4 Properness of the Length Function

In this section, we will prove Lemma 2.2 i.e.

The pullback of the length function $j^*(l): \mathcal{V} \rightarrow \mathbb{R}$ is proper, where $l$ is given by $a + b + c + d$.

If suffices to show that if any sequence $\{(t_n, s_n)\} \subset \mathcal{V}$ leaves any compact set of $\mathcal{V}$, then $j^*(l)(\{(t_n, s_n)\}) \rightarrow \infty$.

We may assume $\{t_n\}$ converges to $\hat{t} \in \left[\frac{1}{2}, 1\right]$. Then there are four cases to consider

- $\hat{t} = 1$
- $\hat{t} \in \left(\frac{\sqrt{2}}{2}, 1\right)$
- $\hat{t} \in \left[\frac{1}{2}, \frac{\sqrt{2}}{2}\right]$  
- $\hat{t} = \frac{\sqrt{2}}{2}$

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4.1 Case: $\hat{t} = 1$

Proof. Since $t_n \to \hat{t} = 1$, from (2)

$$a_n = \log \left( \frac{1 + t_n}{1 - t_n} \right) \to \infty,$$

and note that $j^*(l)(t_n, s_n) > a_n$ hence

$$j^*(l)(t_n, s_n) \to \infty.$$

\[ \square \]

4.2 Case: $\hat{t} \in \left( \frac{\sqrt{2}}{2}, 1 \right)$

Proof. In this case, without loss of generality we may assume $s_n < 0$, \{(t_n, s_n)\} \subset II and \{s_n\} converges to $\hat{s} = -\log \left( \frac{1}{\hat{t}} \right)$.

We consider again the right triangle as in Figure 12. We lift all the geodesic segments $-s_n - \frac{\hat{\alpha}}{2}$'s on the vertical line at the origin as in Figure 12. Since

$$-\hat{s} - \frac{\hat{\alpha}}{2} = \log \left( \frac{\hat{t}}{1 - \hat{t}} \right) - \frac{1}{2} \log \left( \frac{1 + \hat{t}}{1 - \hat{t}} \right) = r' > 0$$

from (9). The geodesic along $c_2$ is intersecting the horizontal line at infinity, hence $c_2 = \infty$.

Since $(t_n, s_n) \to (\hat{t}, \hat{s})$,

$$\alpha_n \to \hat{\alpha}$$

$$-s_n - \frac{\alpha_n}{2} \to -\hat{s} - \frac{\hat{\alpha}}{2}$$

then

$$c_{2,n} \to \hat{c}_2 = \infty.$$

Since $j^*(l)(t_n, s_n) > c_n > c_{2,n}$,

$$j^*(l)(t_n, s_n) \to \infty.$$

\[ \square \]
4.3 Case: $\tilde{t} \in \left[\frac{1}{2}, \frac{\sqrt{2}}{2}\right]

Proof. In this case, without loss of generality we may assume $s_n \leq 0$, $\{(t_n, s_n)\} \subset I$ and $\{s_n\}$ converges to $\hat{s} = -\log \left(\frac{\tilde{t}}{1-\tilde{t}}\right)$.

![Figure 13: $t_n \to \tilde{t} \in \left[\frac{1}{2}, \frac{\sqrt{2}}{2}\right]$.](image)

We lift all the geodesic segments $a_{1,n}$'s on the vertical line at the origin as in Figure 13. Since

$$\hat{a}_1 = \frac{\hat{a}}{2} + \hat{s} = \frac{1}{2} \log \left(\frac{1 + \tilde{t}}{1 - \tilde{t}}\right) - \log \left(\frac{\tilde{t}}{1 - \tilde{t}}\right) = r > 0$$

from (5) and (6), the geodesic along $\hat{c}$ intersects the horizontal line at infinity, hence $\hat{c} = \infty$.

For any $N > 0$, we can choose a brown line geodesic, which is perpendicular to the horizontal line, such that the distance between the vertical line and the brown line is greater than $N$.

Since $(t_n, s_n) \to (\tilde{t}, \hat{s})$,

$$\alpha_n \to \hat{\alpha}$$

$$a_{1,n} \to \hat{a}_1.$$ 

Therefore the geodesic along $c_n$ has to intersect the brown line, when $n$ is large enough.

Note that for any Type II hexagon, $c$ and $a_2$ are parallel (See Figure 5). Therefore the geodesic along $a_{2,n}$ has to be on the right hand side of the brown line. Therefore $a_{1,n} + c_n + d_n + a_{2,n} > N$ by triangular inequality. Let $N \to \infty$,

$$a_{1,n} + c_n + d_n + a_{2,n} \to \infty.$$

Since $j^*(l)((t_n, s_n)) > a_n + c_n + d_n = a_{1,n} + c_n + d_n + a_{2,n}$,

$$j^*(l)((t_n, s_n)) \to \infty.$$ 

\qed
4.4 Case: \( \hat{t} = \frac{\sqrt{2}}{2} \)

Proof. In this case, without loss of generality we may assume \( s_n < 0, \) \( \{(t_n, s_n)\} \) is either completely contained in \( I, \) \( II \) or \( III \) and \( \{s_n\} \) converges to

\[
\hat{s} = -\log \left( \frac{\hat{t}}{1 - \hat{t}} \right) = -\log \left( \sqrt{2} + 1 \right).
\]

From (2)

\[
\hat{\alpha} = 2 \cos^{-1}(\hat{t}) = \frac{\pi}{2}
\]

\[
\hat{a} = \log \left( \frac{1 + \hat{t}}{1 - \hat{t}} \right) = 2 \log \left( \sqrt{2} + 1 \right).
\]

• If \( \{(t_n, s_n)\} \subset I, \) from (5)

\[
\hat{a}_1 = \frac{\hat{a}}{2} + \hat{s} = 0.
\]

Therefore

\[
\alpha_n \to \frac{\pi}{2}
\]

\[
a_{1,n} \to 0.
\]

Figure 14: \( \alpha_n \to \frac{\pi}{2} \) and \( a_{1,n} \to 0. \)

We lift all the geodesic segments \( a_{1,n} \)'s on the vertical line at the origin as in Figure 14. For any \( N > 0, \) we can choose a brown line geodesic, which is perpendicular to the horizontal line, such that the distance between the vertical line and the brown line is greater than \( N. \)

Then when \( n \) is large enough, the geodesic along \( c_n \) has to intersect the brown line. Since the area bounded by \( c_n \) and the brown line approaches to 0, the geodesic along \( a_{2,n} \) has to be on the right hand side of the brown line when \( n \) is even larger enough. Therefore \( a_{1,n} + c_n + d_n + a_{2,n} > N \) by triangular inequality. Let \( N \to \infty, \)

\[
a_{1,n} + c_n + d_n + a_{2,n} \to \infty.
\]
Since $j^*(l)((t_n,s_n)) > a_n + c_n + d_n = a_{1,n} + c_n + d_n + a_{2,n}$,

$$j^*(l)((t_n,s_n)) \rightarrow \infty.$$ 

- If $\{(t_n,s_n)\} \subset II$,

\[ \hat{s} - \frac{\hat{a}}{2} = 0. \]

Therefore

\[ \alpha_n \rightarrow \frac{\pi}{2}, \]

\[ a_n + (-s_n, \frac{a_n}{2}) \rightarrow 2 \log \left( \sqrt{2} + 1 \right). \]

Indicated angles are $\pi - \alpha$

Figure 15: $\phi_n \rightarrow 0$ and $a_n + (-s_n, \frac{a_n}{2}) \rightarrow 2 \log \left( \sqrt{2} + 1 \right)$.

We lift all the geodesic segments $-s_n - \frac{a_n}{2}$'s on the vertical line at the origin as in Figure 15. Since $\phi_n < \theta_n < \pi - \alpha_n - \frac{\pi}{2}$,

$\phi_n \rightarrow 0$.

Consider the right triangle bounded by the magenta line, the red line and the horizontal line. The length of the magenta line approaches to infinity by the Law of Sine, hence

$$c_{1,n} + d_n \rightarrow \infty$$

by triangular inequality. Since $j^*(l)((t_n,s_n)) > c_n + d_n > c_{1,n} + d_n$,

$$j^*(l)((t_n,s_n)) \rightarrow \infty.$$ 

- If $\{(t_n,s_n)\} \subset III$, See Figure 16. The same argument above can be applied to this case as well.

This concludes the properness of $j^*(l)$. 

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Let $A := \{(t,0) \mid t \in \left(\frac{1}{2}, 1\right)\} \subset V$.

Since $j^*(l)$ has a unique minimum and it is even on $s$, the minimum is obtained at some $(t_0, 0) \in A$.

In fact, $j^*(l) \mid_A$ is quite explicit.

**Theorem 5.1.** $j^*(l) \mid_A : \left(\frac{1}{2}, 1\right) \to \mathbb{R}$ is given by

$$t \mapsto 2 \log \left(\frac{\sqrt{5}t^2 - 1 + 2t^2}{(2t - 1)(1-t)}\right).$$

**Corollary 5.1.** $j^*(l)$ has a unique minimal at $t_0 = \frac{3\sqrt{5}}{10}$. Hence the hyperbolic structure with the minimal $\gamma$ length is given by the hexagon with

$$\alpha = \beta = 2 \cos^{-1}\left(\frac{3\sqrt{5}}{10}\right),$$

$$a = b = \log\left(\frac{29 + 12\sqrt{5}}{11}\right),$$

$$c = d = \log\left(\frac{21 + 8\sqrt{5}}{11}\right).$$

We need the following Lemmas from [Ra].

**Lemma 5.1.** Let $Q$ be a hyperbolic convex quadrilateral with two adjacent right angles, opposite angles $\alpha$, $\beta$, and sides of length $c$, $d$ between $\alpha$, $\beta$ and the right angles, respectively. Then

$$\cosh(c) = \frac{\cos(\alpha) \cos(\beta) + \cosh(d)}{\sin(\alpha) \sin(\beta)}.$$
Lemma 5.2. Let $Q$ be a hyperbolic convex quadrilateral with three right angles and fourth angle $\gamma$, and let $a$, $b$ the lengths of sides opposite the angle $\gamma$. Then
\[ \cos(\gamma) = \sinh(a)\sinh(b). \]

![Diagram](image)

Figure 17: A quarter of a hexagon for $t \in \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$.

Proof of Theorem 5.1. We first want to find the formula for $t \in \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$. From Figure 17 and use the above lemmas,
\[
\begin{align*}
cosh(v) & = \cosh \left( \frac{a}{2} \right) \sin(\pi - \alpha) \\
cosh(h_1) & = \cosh(c_1) \sin(\pi - \alpha) \\
\cos(\pi - \alpha) & = \sin(h_1) \sinh(v) \\
cosh(h_2) & = \cosh(c_2) \sin \left( \frac{a}{2} \right) \\
\cos \left( \frac{a}{2} \right) & = \sin(h_2) \sinh(v).
\end{align*}
\]

Then
\[
\begin{align*}
\cos^2(\alpha) & = (\cosh^2(c_1) \sin^2(\alpha) - 1) \left( \cosh^2 \left( \frac{a}{2} \right) \sin^2(\alpha) - 1 \right) \\
\cos^2 \left( \frac{a}{2} \right) & = \left( \cosh^2(c_2) \sin^2 \left( \frac{a}{2} \right) - 1 \right) \left( \cosh^2 \left( \frac{a}{2} \right) \sin^2(\alpha) - 1 \right).
\end{align*}
\]

Use (2)
\[
\begin{align*}
cosh^2(c_1) & = \frac{t^2}{(4t^2 - 1)(1 - t^2)} \\
cosh^2(c_2) & = \frac{5t^2 - 1}{(4t^2 - 1)(1 - t^2)}.
\end{align*}
\]

Hence
\[
\begin{align*}
c_1 & = \log \left( \frac{(2t + 1)(1 - t)}{\sqrt{(4t^2 - 1)(1 - t^2)}} \right) \\
c_2 & = \log \left( \frac{\sqrt{5t^2 - 1} + 2t^2}{\sqrt{(4t^2 - 1)(1 - t^2)}} \right).
\end{align*}
\]
Then
\[ c = c_1 + c_2 = \log \left( \frac{\sqrt{5t^2 - 1} + 2t^2}{(2t - 1)(1 + t)} \right). \] (14)

It can be shown that (14) works for all \( t \in \left( \frac{1}{2}, 1 \right) \). Therefore
\[ j^*(l) = a + b + c + d = 2a + 2c = 2 \log \left( \frac{\sqrt{5t^2 - 1} + 2t^2}{(2t - 1)(1 - t)} \right). \] (15)

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\[ \log(t) - \log(1-t) \]

\[ \frac{\log(1+t) - \log(1-t)}{2} \]

\[ -\frac{\log(1+t) - \log(1-t)}{2} \]

\[ -(\log(t) - \log(1-t)) \]