Quantum corrections to the entropy of Einstein-Maxwell dilaton-axion black holes

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ABSTRACT: We study the corrections to the entropy of Einstein-Maxwell dilaton-axion black holes beyond semiclassical approximations. We consider the entropy of the black hole as a state variable and derive these corrections using the exactness criteria of the first law of thermodynamics. We note that from this general framework the entropy corrections for “simpler” black holes like Schwarzschild, Reissner-Nordström and anti-de Sitter-Schwarzschild black holes follow easily. This procedure gives us the modified area law as well.
1. Introduction

In issues like black hole evaporation, Hawking radiation [1] and quantum tunneling we resort to semiclassical treatment to study changes in thermodynamical quantities. Thus quantum corrections to the Hawking temperature and the Bekenstein-Hawking area law for the Schwarzschild, anti-de Sitter Schwarzschild, Kerr [2] and Kerr-Newman [3, 4] black holes have been studied in the literature. Uncharged BTZ black holes have also been studied for these corrections [5, 6]. In our earlier work [7] we set up a general procedure for studying corrections to the entropy of charged and rotating black holes beyond semiclassical approximations. Corresponding modification in the Hawking temperature and the Bekenstein-Hawking area law were also presented.

Let us write the first law of thermodynamics for charged and rotating black holes. For the three parameters $M, J, Q$, the mass, angular momentum and charge of the black hole, respectively, this can be written as

$$dM = TdS + \Omega dJ + \Phi dQ,$$

(1.1)

where, $T$ is the temperature, $S$ entropy, $\Omega$ angular velocity and $\Phi$ electrostatic potential of the black hole. We can also write this as

$$dS(M, J, Q) = \frac{1}{T}dM - \frac{\Omega}{T}dJ - \frac{\Phi}{T}dQ.$$  

(1.2)

Compare this with the three dimensional differential of a function $f$,

$$df(x, y, z) = A(x, y, z)dx + B(x, y, z)dy + C(x, y, z)dz.$$

(1.3)

Now, this differential is exact if the following conditions hold

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}, \quad \frac{\partial A}{\partial z} = \frac{\partial C}{\partial x}, \quad \frac{\partial B}{\partial z} = \frac{\partial C}{\partial y}.$$  

(1.4)

Here we have

$$\frac{\partial f}{\partial x} = A, \quad \frac{\partial f}{\partial y} = B, \quad \frac{\partial f}{\partial z} = C,$$

(1.5)

and (1.3) can be integrated to yield $f$.

If we replace the $A, B, C$ of conditions (1.4) by $1/T, -\Omega/T, -\Phi/T$, in which case $M, J, Q$ will play the role of $x, y, z$, respectively, we note that in order for $dS$ to be an exact differential (1.2) the following conditions must be satisfied

$$\frac{\partial f}{\partial x} = A, \quad \frac{\partial f}{\partial y} = B, \quad \frac{\partial f}{\partial z} = C,$$

and (1.3) can be integrated to yield $f$. 

If we replace the $A, B, C$ of conditions (1.4) by $1/T, -\Omega/T, -\Phi/T$, in which case $M, J, Q$ will play the role of $x, y, z$, respectively, we note that in order for $dS$ to be an exact differential (1.2) the following conditions must be satisfied
\[ \frac{\partial}{\partial J} \left( \frac{1}{T} \right) = \frac{\partial}{\partial M} \left( -\frac{\Omega}{T} \right), \]  
(1.6)

\[ \frac{\partial}{\partial Q} \left( \frac{1}{T} \right) = \frac{\partial}{\partial M} \left( -\frac{\Phi}{T} \right), \]  
(1.7)

\[ \frac{\partial}{\partial Q} \left( -\frac{\Omega}{T} \right) = \frac{\partial}{\partial J} \left( -\frac{\Phi}{T} \right). \]  
(1.8)

This allows us to write entropy \( S(M, J, Q) \) in the integral form. We employed [7] this procedure to work out quantum corrections of entropy beyond the semiclassical limit, of the Kerr-Newman and the charged rotating BTZ black holes. Further, we showed that the (quantum) corrections for simpler black holes, found earlier using different techniques, can be easily recovered as special cases of that study. In this paper, after briefly describing our earlier results, we extend this analysis to the axially symmetric Einstein-Maxwell dilaton-axion black holes [8, 9]. The quantum corrections to the entropy of these black holes have been investigated using brick wall model [10]. The correction up to the second term only have been calculated in this paper. Our method is much simpler and more general at the same time. We have also presented the modified Bekenstein-Hawking area law. The leading order correction term is found to be logarithmic [11] and the higher order terms have ascending powers of inverse of the area.

2. The Kerr-Newman black hole

The Kerr-Newman spacetime in Boyer-Lindquist coordinates \((t, r, \theta, \phi)\) can be written as

\[
ds^2 = -\frac{\Delta^2}{\rho^2} (dt - asin^2\theta d\phi)^2 + \rho^2 \Delta^2 dr^2 + \rho^2 d\theta^2 + \frac{sin^2\theta}{\rho^2} (adt - (r^2 + a^2)d\phi)^2,
\]

where

\[
\Delta(r)^2 = (r^2 + a^2) - 2Mr + Q^2,
\]

\[
\rho^2(r, \theta) = r^2 + a^2 \cos^2\theta,
\]

\[
a = \frac{J}{M}.
\]
The inner and outer horizons for this metric are

\[ r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}. \tag{2.1} \]

The Hawking temperature is defined as

\[ T = \left( \frac{\hbar}{4\pi} \right) \frac{r_+ - r_-}{r_+^2 + a^2}, \] \tag{2.2}

which, in our case takes the form

\[ T = \left( \frac{\hbar}{2\pi} \right) \frac{\sqrt{M^4 - J^2 - Q^2 M^2}}{M \left( 2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2 M^2} \right)}. \] \tag{2.3}

The angular velocity [12], \( \Omega = a/(r_+^2 + a^2) \), takes the form

\[ \Omega = \frac{J}{M \left( 2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2 M^2} \right)}, \] \tag{2.4}

and the electrostatic potential, \( \Phi = r_+ Q/(r_+^2 + a^2) \), becomes

\[ \Phi = \frac{Q \left( M^2 + \sqrt{M^4 - J^2 - Q^2 M^2} \right)}{M \left( 2M^2 - Q^2 + 2\sqrt{M^4 - J^2 - Q^2 M^2} \right)}. \] \tag{2.5}

It is easy to see that these quantities for the Kerr-Newman black hole satisfy conditions (1.6)-(1.8), and therefore, \( dS \) is an exact differential. Thus we apply the procedure described in Section 1, and use the corrected form of the Hawking temperature [2]

\[ T_c = T \left( 1 + \sum \frac{\alpha_i \hbar^i}{(r_+^2 + a^2)^i} \right)^{-1}, \] \tag{2.6}

where \( \alpha_i \) correspond to higher order loop corrections to the surface gravity of black holes \( \mathcal{K} = 2\pi T \). The modified surface gravity [13] due to quantum effects becomes

\[ \mathcal{K} = \mathcal{K}_0 \left( 1 + \sum \frac{\alpha_i \hbar^i}{(r_+^2 + a^2)^i} \right)^{-1}. \] \tag{2.7}

Thus the entropy including the correction terms becomes
\[ S = \frac{\pi}{\hbar} (r^2 + a^2) + \pi \alpha_1 \ln (r^2 + a^2) + \sum_{k>2} \frac{\pi \alpha_{k-1} \hbar^{k-2}}{(2-k)(r^2 + a^2)^{k-2}} + \cdots. \quad (2.8) \]

Note that if we put charge \( Q = 0 \), we recover the corrections for the case of the Kerr black hole \([2]\). If the angular momentum is also zero we get results for the Schwarzschild black hole \((a = Q = 0)\). However, if only the angular momentum vanishes (i.e. \( a = 0 \)), we get corresponding corrections for Reissner-Nordström black hole, in which case the power series involve the charge \( Q \) also, in addition to \( M \).

Using the Bekenstein-Hawking area law relating entropy and horizon area, \( S = A/4\hbar \), where the area in our case is

\[ A = 4\pi(r^2 + a^2), \quad (2.9) \]

from (2.8) we obtain

\[ S = \frac{A}{4\hbar} + \pi \alpha_1 \ln A - \frac{4\pi^2 \alpha_2 \hbar}{A} - \frac{8\pi^3 \alpha_3 \hbar^2}{A^2} - \cdots, \quad (2.10) \]

which gives quantum corrections for the area law.

As regards the value of the prefactor \( \alpha_i \)'s there are different interpretations found in the literature. For example, some authors take \( \alpha_1 \) to be negative \([14]\), some positive integer \([15]\), while others find it to be zero even \([16]\).

3. Charged and rotating BTZ black hole

The Bañados-Teitelboim-Zanelli (BTZ) black hole \([17]\), which is \((1+2)\)-dimensional, when it is charged and rotating, can be written as \([18]\)

\[ ds^2 = -(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi Q^2 \ln r}{2} \ln r) dt^2 + (-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi Q^2 \ln r}{2} \ln r)^{-1} dr^2 + r^2 (d\phi - \frac{J}{2r^2} dt)^2, \quad (3.1) \]

where \( M \) is the mass, \( J \) the angular momentum, \( Q \) the charge of the black hole, and \( 1/l^2 = -\Lambda \) is the negative cosmological constant.

The event horizon and the inner horizon \( r_+ \) and \( r_- \) satisfy
\[-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi}{2} Q^2 \ln r = 0. \quad (3.2)\]

We write

\[f(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi}{2} Q^2 \ln r. \quad (3.3)\]

The angular velocity of the BTZ black hole is

\[\Omega = -\frac{\partial g_{\phi t}}{\partial \phi} \bigg|_{r=r_+} = \frac{J}{2r^2} \bigg|_{r=r_+}. \quad (3.4)\]

The event horizon is related with temperature \( T \) by [19, 20]

\[T = \frac{\hbar f'(r)}{4\pi} \bigg|_{r=r_+}, \quad (3.5)\]

where \( f'(r) \) denotes the derivative of \( f \) with respect to \( r \). The electric potential is given by [18]

\[\Phi = -\frac{\partial M}{\partial Q} \bigg|_{r=r_+} = -\pi Q \ln r_+. \quad (3.6)\]

With these thermodynamic quantities the BTZ black hole satisfies the first law of thermodynamics of the form (1.1) and the entropy with quantum corrections is given by the series

\[S = \frac{4\pi r_+}{\hbar} + 4\pi \alpha_1 \ln r_+ - \frac{4\alpha_2 \hbar \pi}{r_+} - \cdots. \quad (3.7)\]

Putting \( 8G_3 = 1 \), where \( G_3 \) is the three dimensional Newton’s gravitational constant, the area formula is

\[A = 2\pi r_+. \quad (3.8)\]

If we include \( G_3 \) this becomes

\[A = 16\pi G_3 r_+, \quad (3.9)\]

and the above result for entropy becomes
\[ S = \frac{A}{4\hbar G_3} + 4\pi\alpha_1 \ln A - \frac{64\alpha_2\hbar^2 G_3}{A} - \ldots. \] 

(3.10)

It may be pointed out here that these black holes are not physically significant, however, the above results provide useful mathematical insights for lower dimensional gravity theories.

4. Axially symmetric Einstein-Maxwell dilaton-axion black hole

The stationary axisymmetric Einstein-Maxwell black holes in the presence of dilaton-axion field are found in heterotic string theory [8]. In Boyer-Lindquist coordinates \((t, r, \theta, \phi)\), these are described by the line element [9]

\[
d s^2 = -\frac{\Sigma - a^2 \sin^2 \theta}{\Delta} dt^2 - \frac{2a \sin^2 \theta}{\Delta} \left[ (r^2 - 2Dr + a^2) - \Sigma \right] dt d\phi \\
+ \frac{\Delta}{\Sigma} dr^2 + \Delta d\theta^2 + \frac{\sin^2 \theta}{\Delta} \left[ (r^2 - 2Dr + a^2)^2 - \Sigma a^2 \sin^2 \theta \right] d\phi^2, \tag{4.1}
\]

where

\[
\Delta = r^2 - 2Dr + a^2 \cos^2 \theta, \tag{4.2}
\]

\[
\Sigma = r^2 - 2Mr + a^2. \tag{4.3}
\]

They have the electric charge

\[
Q = \sqrt{2\omega D(D - M)}, \text{ where } \omega = e^d. \tag{4.4}
\]

Here \(D\), \(M\), \(a\) and \(d\) denote the dilaton charge, mass, angular momentum per unit mass and the massless dilaton field, respectively, and \(m = M - D\) is the Arnowitt-Deser-Misner (ADM) mass of the black hole. The electrostatic potential is

\[
\Phi = \frac{-2DM}{Q(r^2 - 2Dr + a^2)}.
\]

The metric has singularities at \(r^2 - 2Dr + a^2 \cos^2 \theta = 0\). The outer and inner horizons are respectively
\[ r_\pm = \left( M - \frac{Q^2}{2\omega M} \right) \pm \sqrt{\left( M - \frac{Q^2}{2\omega M} \right) - a^2}. \] (4.5)

The outer horizon at \( r_+ \) is specified as a black hole horizon and is a null stationary 2-surface. The Killing vector normal to this surface is \( \chi^\alpha = t^\alpha + \Omega \phi^\alpha \) and it is null on the horizon. The angular velocity on the horizon is given by

\[ \Omega = \frac{J/M}{r_+^2 - 2Dr_+ + a^2} \]

or

\[ \Omega = \frac{J}{2M \left[ M(M + D) + \sqrt{M^2(M + D)^2 - J^2} \right]} \]

This horizon is generated by the Killing vector \( \chi^\alpha \), and the surface gravity \( \kappa \) associated with this Killing horizon is given [15] by

\[ \kappa^2 = \frac{-1}{2} \chi^\alpha;\beta \chi_{\alpha;\beta}. \] (4.6)

Using this definition of the surface gravity, it is easy to evaluate the temperature \( T = \kappa/2 \) associated with this horizon as,

\[ T = \frac{\hbar}{4\pi} \left[ \frac{(r_+ - M - D)}{(r_+^2 - 2Dr_+ + a^2)} \right], \] (4.7)

or

\[ T = \frac{\hbar}{4\pi} \left[ \frac{\sqrt{M^2(M + D)^2 - J^2}}{M[M(M + D) + \sqrt{M^2(M + D)^2 - J^2}]} \right]. \] (4.8)

One can easily check that the above thermodynamical quantities satisfy conditions (1.6)-(1.8). Thus the entropy differential \( dS \) is exact and we can evaluate the integral to work out the semiclassical entropy

\[ S(M, J, Q) = \int \frac{dM}{T} \]

\[ = \frac{4\pi}{\hbar} \int \frac{M \left[ M(M + D) + \sqrt{M^2(M + D)^2 - J^2} \right]}{\sqrt{M^2(M + D)^2 - J^2}} dM \]

\[ = \frac{2\pi}{\hbar} \left[ M(M + D) + \sqrt{M^2(M + D)^2 - J^2} \right]. \] (4.9)
Now, in order to work out the quantum corrections to this formula, we need to show that $T, \Omega, \Phi$ satisfy the following conditions involving corrections and which replace conditions (1.6)-(1.8) for exactness of entropy

\[
\frac{\partial}{\partial J} \frac{1}{T} \left( 1 + \sum \frac{\beta_i \hbar^i}{(r_i^2 - 2Dr_i + a^2)^i} \right) = \frac{\partial}{\partial M} \frac{-\Omega}{T} \left( 1 + \sum \frac{\beta_i \hbar^i}{(r_i^2 - 2Dr_i + a^2)^i} \right)
\]

\[
\frac{\partial}{\partial Q} \frac{1}{T} \left( 1 + \sum \frac{\beta_i \hbar^i}{(r_i^2 - 2Dr_i + a^2)^i} \right) = \frac{\partial}{\partial M} \frac{-\Phi}{T} \left( 1 + \sum \frac{\beta_i \hbar^i}{(r_i^2 + 2Dr_i + a^2)^i} \right)
\]

\[
\frac{\partial}{\partial Q} \frac{-\Omega}{T} \left( 1 + \sum \frac{\beta_i \hbar^i}{(r_i^2 - 2Dr_i + a^2)^i} \right) = \frac{\partial}{\partial J} \frac{-\Phi}{T} \left( 1 + \sum \frac{\beta_i \hbar^i}{(r_i^2 - 2Dr_i + a^2)^i} \right).
\]

We note that this is indeed the case and the entropy integral is simplified to

\[S(M, J, Q) = \int \frac{1}{T} \left( 1 + \sum \frac{\beta_i \hbar^i}{(r_i^2 - 2Dr_i + a^2)^i} \right) dM,\]

which can be written in the expanded form as

\[S(M, J, Q) = \int \frac{1}{T} dM + \int \frac{\beta_1 \hbar}{T(r_1^2 - 2Dr_1 + a^2)} dM + \int \frac{\beta_2 \hbar^2}{T(r_1^2 - 2Dr_1 + a^2)^2} dM + \int \frac{\beta_3 \hbar^3}{T(r_1^2 - 2Dr_1 + a^2)^3} dM + \cdots\]

\[= I_1 + I_2 + I_3 + I_4 + \cdots,\]

where the first integral $I_1$ has been evaluated in (4.9). We work out the other integrals one by one after substituting values from (4.5) and (4.8). Thus

\[I_2 = 2\pi \beta_1 \hbar \int \frac{M dM}{\sqrt{M^2(M + D)^2 - J^2}}\]

After making some appropriate substitution this can be evaluated as

\[I_2 = \pi \beta_1 \ln \left| M(M + D) + \sqrt{M^2(M + D)^2 - J^2} \right|,\]

which is nothing but
\[ I_2 = \pi \beta_1 \ln(r_+^2 - 2Dr_+ + a^2). \]  

(4.10)

The k-th integral \( I_k \) where \( k = 3, 4, \cdots \) can be evaluated as

\[ I_k = \int \frac{\beta_{k-1} \hbar^{k-1} dM}{T(r_+^2 - 2Dr_+ + a^2)^{k-1}} \]

\[ = \frac{2\pi}{\hbar} \int \frac{\beta_{k-1} \hbar^{k-1} M dM}{\sqrt{M^2 (M + D)^2 - J^2} \left[ M(M + D) + \sqrt{M^2 (M + D)^2 - J^2} \right]^{k-2}} \]

\[ = \frac{\pi \beta_{k-1} \hbar^{k-2}}{2 - k} \left[ M(M + D) + \sqrt{M^2 (M + D)^2 - J^2} \right]^{2-k}, k > 2, \]

or, in terms of \( r_+ \), this can be written as

\[ I_k = \frac{\pi \beta_{k-1} \hbar^{k-2}}{(2 - k)(r_+^2 - 2Dr_+ + a^2)^{k-2}}, k > 2. \]  

(4.11)

Thus the entropy including the correction terms becomes

\[ S = \frac{\pi}{\hbar} (r_+^2 - 2Dr_+ + a^2) + \pi \beta_1 \ln(r_+^2 - 2Dr_+ + a^2) \]

\[ + \sum_{k>2} \left( \frac{\pi \beta_{k-1} \hbar^{k-2}}{(2 - k)(r_+^2 - 2Dr_+ + a^2)^{k-2}} + \cdots \right). \]

The Bekenstein-Hawking entropy associated with this horizon is one quarter of the area of the horizon surface. It is important to note that unlike spherical geometry the horizon surface here is not simply a 2-sphere. The area of the horizon can be computed from the 2-metric on the horizon and it is given by

\[ A = 4\pi (r_+^2 - 2Dr_+ + a^2). \]  

(4.12)

Therefore the corresponding entropy associated with this horizon is

\[ S = \frac{A}{4\hbar}, \]

so that the modified area law takes the form

\[ S = \frac{A}{4\hbar} + \pi \beta_1 \ln A - \frac{4\pi^2 \beta_2 \hbar}{A} - \frac{8\pi^3 \beta_3 \hbar^2}{A^2} - \cdots. \]  

(4.13)
5. Conclusion

We write the first law of thermodynamics for black holes with three parameters, mass, charge and angular momentum. Taking this as a differential of entropy, we apply the criterion for exactness of differentials in three variables. This enables us to calculate quantum corrections in entropy for charged and rotating black holes beyond the semiclassical terms. We have briefly described results obtained earlier in the case of Kerr-Newman and charged BTZ black holes. Our main emphasis is to apply this analysis to axially symmetric Einstein-Maxwell black holes with dilaton-axion charge. Our procedure helps in evaluating the integrals involving higher order corrections of entropy. The first term in the power series is the semiclassical value and the leading order correction term is logarithmic. This is consistent with the results in the literature for different black holes, found by using quantum geometry techniques, field theoretic methods and the brick wall model. The higher order terms are in ascending powers of \((r_+^2 - 2D r_+ + a^2)^{-1}\). The modified Bekenstein-Hawking area law has also been derived. Here we have not required corrections in the angular momentum and the electrostatic potential.

An important feature of this procedure is that, in a sense, it unifies the earlier approaches and the results for ‘simpler’ situations can be recovered very easily. Corrections for the Kerr-Newman spacetime can be obtained by putting \(D = 0\). For non-rotating objects like Reissner-Nordstr"om black hole the series is in the powers of \((r_+^2)^{-1}\). If the charge and angular momentum both are put equal to zero, the corrections for the Schwarzschild black hole are obtained.

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