Massive Kaluza-Klein Gravity

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Abstract

The non-Abelian Kaluza-Klein unification of gravitation with gauge fields theory is reformulated, with the inclusion of a massive spin-2 field defined by the extrinsic curvature. The internal space is non-compact, characterized by the group of rotations of vectors orthogonal to the space-time. The non-compactness of the internal space warrants the solution of the fermion chirality problem of the original Kaluza-Klein theory and makes it closer to the more recent Brane World paradigm, in special to the so called DGP model. However, the access of gravitation to the extra dimensions is defined by the mentioned massive spin-2 field obeying the Fierz-Pauli equation. The existence of a short range gravitational component makes possible to apply the modified Kaluza-Klein unification to the Tev scale of energies.

1 Introduction

The recent detection of the Higgs particle at the LHC gives a new support to the standard model of the fundamental interactions. It also hints that gravitation as the force acting on masses should be somehow included in that model. One strong proponent of such unification is the Kaluza-Klein

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theory, in which gauge fields are part of a higher dimensional metric space obeying the Einstein-Hilbert principle.

Gravitation has been traditionally neglected in the standard model because the so-called gravitational hierarchy, whereby the gravitational field is too weak to play a significant role in presence of gauge fields, except perhaps at the Planck regime. This hierarchy is a consequence of the presence of Newton’s gravitational constant $G$ in Einstein’s equations. However, in a higher-dimensional gravitational theory the coupling constant of gravitation with gauge fields and matter sources cannot be defined by Newton’s constant, simply because that constant depends on the dimensions of the space, namely 3 space dimensions. Therefore, in higher dimensions the gravitational hierarchy may be broken.

Independently of how such higher dimensional theory is formulated, it must be compatible with General Relativity which provides the link with Newton’s gravity. Therefore, we face the possibility that gravitation may be more complex than the current understanding of such force, as for example by providing two levels of interaction: one being given by the traditional massless Einstein-like gravitation responsible for the classical long range interactions, which we experience in our everyday life. The other is a massive strong gravity acting at short range, at the Tev scale of energies. Of the later we know very little, except perhaps a few hints from the new experimental high energy physics at the Tev scale. It is also possible that it has implications to extragalactic astrophysics.

Massive spin-2 fields were described by the Pauli-Fierz action in 1939 and its interaction with four-dimensional Einstein’s gravitation were considered by various authors in the 70’s, including the possibility of existence of a short range strong gravitational component and its effects on high energy hadron physics, but also as a possible explanation of the acceleration of the universe, avoiding the problems associated with the cosmological constant \[1, 2, 4\]. The purpose of this note is to revise Kaluza-Klein theory with the two new ingredients. Firstly, the internal space is not compact. Secondly the propagation of gravitation in the extra-dimensional space is a consequence of a massive component of gravitation. Let us start with a very brief review of the original non-Abelian Kaluza-Klein theory.

During the subsequent 20 year period after it was firstly proposed in 1963, the non-Abelian Kaluza-Klein theory was considered to be a serious candidate for a successful unification of the four fundamental interactions (gravitation and the standard gauge forces) at the Planck scale. The theory
was defined in a higher-dimensional space with product topology $V_4 \times B_N$, where $V_4$ is the space-time and $B_N$ is a compact internal space with typical diameter of the order of Planck’s length. The metric geometry of the total space is defined by the Einstein-Hilbert action, that could be decomposed in the four-dimensional Einstein-Hilbert action defined in the space-time $V_4$, plus the Yang-Mills action defined with respect to the group of symmetries of $B_N$.

Then, in 1984 a contradiction of the theory was noted, whereby its predictions at the electroweak limit of Tev scale of energies did not agree with the observed chiral motion of fermions. The incompatibility with the fermion chirality was soon understood to be a consequence of the mathematical structure of the higher dimensional space, which was postulated to be the topological product $V_4 \times B_n$ and the small size of $B_n$. This assumption was justified by the necessity that the extra dimensions could not be directly observed at any practical level of energy. It was understood that $B_n$ also introduced an additional mass to fermions in the theory, proportional to the inverse of Planck’s length.

In spite of the many efforts to save the theory presented at that time [5, 6, 7, 8], it was practically abandoned. The general feeling was that, even if the fermion chirality problem could be solved, we would still facing an even more difficult problem which was the quantization of a theory based on the Einstein-Hilbert principle.

However, the legacy of the Kaluza-Klein theory to theoretical physics is considerable, mainly because it opened a perspective for the existence of extra dimensions, beyond the four dimensions of the space-time. The literature produced on Kaluza-Klein theory is vast, and from these we have much to learn. For historical and technical reviews we suggest a look at [9, 10, 11].

A substantial contingent of researchers moved to the then infant theory of strings also based on the property of submanifolds embedded in a space with more than four dimensions.

The quantization of the geometry in Kaluza-Klein theory would be unavoidable because it incorporates a quantizable component, in the form of a gauge field. The presence of such field built in the metric geometry of the total space implies that the remaining components of the metric also would show some quantum fluctuations. Here we are using the same principle proposed by Ashtekar to use the group of holonomy of the triads to get an auxiliar $SU(2)$ field which would induce quantum fluctuations in the remaining geometry (This later developed into the presently very active loop
quantum gravity program) [12]. The difference is that here the gauge fields are already present as additional components of the Kaluza-Klein metric, so that we do not depend on the holonomy groups and its implications to the use of Wilson’s integral.

However, if a unification theory like Kaluza-Klein theory is to be effective, the hierarchy of the gravitational field must be resolved. Newton’s gravitational constant $G$ was derived from observations within the context of Newtonian mechanics, measured with absolute time separated from the three-dimensional distances. Therefore, $G$ is consistent with the topology of the Newtonian space-time given by the product $\mathbb{R}^3 \times \mathbb{R}$, so that the physical dimensions of $G$ are appropriate to convert the squared mass by the squared three-dimensional distance into the Newtonian gravitational force. When the same constant is imported into General Relativity its physical dimensionality does not change and it remains entirely compatible with the fact that Einstein’s equations are of hyperbolic nature, implying that the topology of the space-times is $\mathbb{R}^3 \times \mathbb{R}$ [13]. This is the same topology of the Newtonian space-time, the difference being that in General relativity the time is local and the space sections are not simultaneous sections. In spite of this, the value of $G$ determined by Newtonian mechanics, imposes an enormous energy difference between the energy levels of Einstein’s gravitation and those of the relativistic gauge fields.

On the other hand, as a consequence of the mathematical structure of Maxwell’s equations, or more generally of the Yang-Mills equations, together with the experimental evidences of the standard model of particles and fields, the topology of the (Minkowski) relativistic space-time is not separated as a product, but has an inseparable relation between time and space in a four dimensional integrated space-time.

Indeed, the gauge field strength $F = F_{\mu \nu} dx^\mu \wedge dx^\nu$ is a 2-form (or equivalently, a covariant rank 2 antisymmetric Maxwell-like tensor). Its components are $F_{\mu \nu} = [D_\mu, D_\nu]$, where $D_\mu = \partial_\mu + A_\mu$ is the gauge covariant derivative with respect to the components of the gauge potential $A_\mu$, written in the adjoint representation of the local gauge symmetry [14, 15, 16]. Using the notation of exterior product the Yang-Mills equations is written as

$$D \wedge F^* = 0 \quad \text{and} \quad D \wedge F = 4\pi j$$

where the star denotes the dual $F^* = F^{*\rho} dx^\rho \wedge dx^\mu$, $F^{*\rho} = \epsilon_{\mu\nu\rho\sigma} F^{\nu\sigma}$. Therefore, the 3-form $D \wedge F$ must equal to the current one-form. However,
three-forms and one-forms are isomorphic only in four dimensional spaces. On the other hand, the gravitational field defined by Einstein’s equations does not have the same type of gauge confinement to the four-dimensional space-time based on the dynamical equations. Therefore, differently from gauge fields, in principle the gravitational field may propagate also along extra dimensions if they exist.

Under the hypothesis of the space-time being a subspace of a larger host space defined by the Einstein-Hilbert principle, the host space have the same hyperbolic characteristics, but where with the associated topology is decomposed in the product $V_4 \times \mathcal{M}_N$, where $V_4$ is a four-dimensional physical space-time and $\mathcal{M}_N$ is the N-dimensional local orthogonal space generated by the extra dimensions.

2 Smoothing the Space-time

The characterization of Riemannian manifolds as topological spaces was established by O. Veblen and H. Whitehead, 77 years after Riemann’s paper and 25 years after Einstein’s use of the Riemannian geometry to describe gravitation. Such time gap reflects a degree of conceptual complexity in Riemann’s original paper [17, 18, 19]. Another topological characteristic in Riemann’s paper which remained obscure for some time was the notion of the shape of the manifold: In his presentation, Riemann commented that his geometry was not capable to distinguish between two different manifolds with zero curvature, as for example between a plane and a cylinder among an infinite choice of non-trivial flat Riemann manifolds. This topological deficiency cannot be ignored in Einstein’s gravitational theory because the flat space-time acts also as a ground state for gravitation.

A solution of such shape problem was conjectured in 1871 by L. Schläfli, suggesting that the notion of shape of an observed object cannot be decided intrinsically. He proposed that the Riemann curvature could provide an unambiguous measure of shape provided it could be compared with the curvature of another Riemannian manifold. In principle any other Riemannian manifold could act as a reference of shape as long as both manifolds could be somehow locally compared. This would require that, like the old Euclidean geometry, any n-dimensional Riemannian manifold could be locally embedded in another D-dimensional manifold. The embedding manifold itself could act as a background reference for shape. Although Schläfli’s solution is very
intuitive, it took a while until a complete formulation of the problem, which finally culminated in the derivation of the Gauss-Codazzi-Ricci equations for the embedding. These equations involve not only the metric but also the other two fundamental forms of differential geometry.

In General relativity the definition of shape is local, so that the mentioned embedding is also only local, characterized by a map which takes a neighborhood of a point of manifold $\tilde{V}_\alpha$ into the embedding space $V_D$. In the case of a space-time the local embedding is

$$\tilde{\mathbf{X}} : \tilde{V}_4 \rightarrow V_D$$

such that its components $\tilde{X}^A$, $A = 1 \cdots D$ are functions of the space-time coordinates. These functions must be differentiable and regular, so that $\tilde{\mathbf{X}}$ can be locally invertible, thus enabling the local recovery of the original space-time. Since the line element in space-time is the same, independently of the fact that the space-time is embedded or not, we must have

$$ds^2 = G_{AB} \tilde{X}^A_{,\mu} \tilde{X}^B_{,\nu} dx^\mu dx^\nu = \bar{g}_{\mu\nu} dx^\mu dx^\nu$$  \hspace{1cm} (1)

where $G_{AB}$ is the metric of the embedding space and $\bar{g}_{\mu\nu}$ is the metric of the space-time. The derivatives $\tilde{X}^A_{,\mu}$ define a basis of the tangent space to the four-dimensional embedded space-time. To complete the basis of the embedding space we need an additional $N = D - 4$ vectors $\eta^A_a$ which can be chosen to be orthogonal to $\tilde{V}_4$ and to themselves at each point. In this way we obtain a Gaussian reference frame in the embedding space $\{\tilde{X}^A_{,\mu}, \eta^A_a\}$ such that (Hereafter, Greek indices $\mu, \nu ...$ run from 1 to 4, capital case Latin indices run from 1 to D and small case Latin indices run from 5 to D):

$$\tilde{X}^A_{,\mu} \eta^B_b G_{AB} = \bar{g}_{\mu\nu}, \quad \tilde{X}^A_{,\mu} \eta^B_a G_{AB} = 0, \quad \eta^A_a \eta^B_b G_{AB} = \bar{g}_{ab}$$  \hspace{1cm} (2)

where $\bar{g}_{ab} = \epsilon \delta_{ab}$, where $\epsilon = \pm 1$ which defines the signature of the extra dimensions.

Writing the Riemann tensor of $V_D$ in this Gaussian frame and applying all index symmetries and curvature identities, the only remaining independent equations are the well known Gauss-Codazzi-Ricci equations. These form the integrability conditions for the embedding, but essentially they are the components of the Riemann tensor written in the Gaussian frame [20]:

$$\mathcal{R}_{ABCD} \tilde{X}^A_{,\alpha} \tilde{X}^B_{,\beta} \tilde{X}^C_{,\gamma} \tilde{X}^D_{,\delta} = \tilde{R}_{\alpha\beta\gamma\delta} - 2\bar{g}^{mn} \bar{k}_{\alpha[\gamma m} \bar{k}_{\delta]\beta n}$$  \hspace{1cm} (3)

$$\mathcal{R}_{ABCD} \tilde{X}^A_{,\alpha} \eta^B_b \tilde{X}^C_{,\gamma} \tilde{X}^D_{,\delta} = \bar{k}_{\alpha[\gamma b \delta]} - \bar{g}^{mn} \bar{A}_{[\gamma m b} \bar{k}_{\alpha]n \delta]}$$  \hspace{1cm} (4)

$$\mathcal{R}_{ABCD} \eta^A_b \tilde{X}^B_{,\beta} \tilde{X}^C_{,\gamma} \tilde{X}^D_{,\delta} = -2\bar{g}^{mn} A_{[\gamma m a} \bar{A}_{\delta]nb} - 2\bar{A}_{[\gamma a \beta} \bar{k}_{\delta]nb} - \bar{g}^{mn} \bar{k}_{[\gamma m a \delta]nb}$$  \hspace{1cm} (5)
where \( \bar{k}_{\mu \nu a} \) denote the components of the extrinsic curvature, one for each extra dimension with label “a”, and \( \bar{A}_{\mu ab} \) denote the components of the third fundamental form defined respectively by

\[
\bar{k}_{\mu \nu a} = -G_{AB} \eta^{A}_{a, \mu} \bar{X}^{B}_{\nu}, \quad \bar{A}_{\mu ab} = G_{AB} \eta^{A}_{a, \mu} \eta^{B}_{b} \tag{6}
\]

Notice that equations (3-5) represent only the conditions for the existence of the embedding of a space-time. No boundary conditions were provided to guarantee that the embedding is unique.

The local and isometric embedding of space-times can be determined in three different ways: First by just try and error, finding the functions \( \bar{X}^{A} \) so that (1), (2) and (3-5) are satisfied. In this approach the normal vectors need to be calculated in an ad-hoc manner so that the second and third differential forms may be determined. Several examples of such procedure are given in [21]. The so called Penrose embedding diagrams are simplified examples of such procedure represented by two dimensional graph while the remaining dimensions are taken to be zero.

The second procedure consists in analytically solving the integrability equations (3-5) based on the well known theorems of Janet and Cartan [22, 23]. Since those equations form a non-linear system of equations on its three variables \( \bar{g}_{\mu \nu}, \bar{k}_{\mu \nu a}, \bar{A}_{\mu ab} \), the assumption that the functions \( \bar{X}^{A}(x) \) are analytic, imply that all fundamental forms are also analytic functions. Some results are obtained for a local embedding. For example for a n-dimensional manifold it is found that at most \( D = n(n + 1)/2 \) are required for the embedding space. As far as the mathematical analysis on manifolds with positive defined metrics are concerned this is fine. However, for pseudo Euclidean manifolds not all theorems based on converging positive power series apply. In addition, we remind that in the real world the analytic assumption is difficult to attain.

\( ^{1} \)In the Randall-Sundrum brane-world model, the space-time is embedded in the five dimensional anti deSitter space \( AdS_{5} \). It is assumed that the space-time acts as a mirror boundary for the higher-dimensional gravitational field. This condition has the effect that the extrinsic curvature becomes an algebraic function of the energy-momentum tensor of matter confined to the four-dimensional embedded space-time. In more than five dimensions, which is our case in study, that boundary condition does not make sense because the extrinsic curvature acquire an internal index while the energy-momentum tensor does not have this degree of freedom. As we shall see later, the extrinsic curvature behaves as a dynamical field. Finally, our main objective here cannot be accomplished in the Randall-Sundrum model because \( A_{\mu ab} \) simply do not exists in the case of a single extra dimension.
The third procedure uses a non-trivial theorem by John Nash, stating
that the solution of the equations (3-5) was obtained from the supposition
that an initially given embedded manifold (or better, a space-time) $\bar{V}_4$ can be
smoothly deformed along the normals producing a new embedded manifold $V_4$, with new extrinsic curvature satisfying the condition

$$k_{\mu \nu a} = -\frac{1}{2} \frac{\partial g_{\mu \nu}}{\partial y^a}$$

where $g_{\mu \nu}$ and $y^a$ denote respectively the metric of the deformed space-time
and the extra-dimensional coordinates [24].

The smoothing condition is related to the notion of geometric deforma-
tion, which by turn is defined by a continuous displacement of points of the
original manifold, $\bar{V}_4$ following a flow of lines which cross orthogonally the
original manifold without losing its continuity and regularity, by means of
the Lie transport [25], so it may be called the Nash geometric flow. The
first report of the condition (7) appeared in 1920 in a book by J. Campbell,
postulated to prove that any space-time could be embedded in a Ricci-flat
five-dimensional space [26]. However, along his proof Campbell implicitly
used the analytic expansion [27], so that implicitly he was using Cartan’s
analytic embedding. The same expression was independently derived by J.
York in 1971 to implement the ADM foliation of space-time by 3-dimension
al surfaces [28]. To understand Nash’s smoothing deformation process, we
may use an analogy with another more recent smoothing process derived by
Richard Hamilton, and subsequently applied to the proof of the Poincaré
conjecture [29, 30].

In the derivation of the heat equation J. Fourier used two approaches
to measure the increase of temperature in a spherical body embedded in
a compact solid situated near source of heat: On the one hand, Fourier
considered the specific capacity of the body material to absorb heat per unit
of volume and time. On the other hand, he considered the flux of heat
flow lines per unit of area crossing orthogonally the body surface and the
embedded sphere. Assuming that there are no additional heat sources or
sinks inside the body, the comparison between the two measures of heat led
to the Fourier’s parabolic heat equation

$$\nabla^2 u = \frac{\partial u}{\partial t}$$

Between the initial surface and the end sphere we may draw an infinite se-
quence of surfaces always orthogonal to the flow lines. This procedure may
be photographed in a time sequence and the resulting film may be played back, producing a continuous deformation of the body’s surface, without singularities or cusps, converging at the end to the sphere.

To translate the above reasoning to Riemannian (intrinsic) geometry, consider the Ricci tensor written as

\[ R_{\mu\nu} = (\log \sqrt{g})_{,\mu\nu} - \Gamma^\rho_{\mu\nu,\rho} + \Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\nu\sigma} - \Gamma^\rho_{\mu\nu} (\log \sqrt{g})_{,\rho} \]

Therefore, in geodesic coordinates, the Ricci scalar becomes

\[ R = g^{\mu\nu} R_{\mu\nu} = \nabla^2 (\log \sqrt{g}) \quad (9) \]

In the intended analogy, instead of varying temperature with time, there is a surface change in an orthogonal direction \( y \). Thus, replacing the temperature \( u \) in (8) by \( u = \log \sqrt{g} \), we obtain

\[ g^{\mu\nu} R_{\mu\nu} = \nabla^2 (\log \sqrt{g}) = g^{\mu\nu} \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial t} \]

Comparing equations (9) and (2) and solving the resulting tensor equation in \( R_{\mu\nu} \), replacing \( t \) by an arbitrary coordinate \( y \), we obtain up to the addition of a traceless tensor, the Ricci flow condition (the minus sign corresponds to having the heat flow in the opposite direction of the shrinking surface.)

\[ R_{\mu\nu} = -\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial y} \quad (10) \]

It is a simple exercise to see that this condition is not compatible with Einstein’s equations.

The Nash deformation of embedded submanifolds follows a similar picture, but with the difference that flow lines are the orbits of the Lie transport of points in an initial manifold along its orthogonal directions as described by (12). However, the physical meaning of Nash’s flow is not related to Fourier’s heat equation as it is entirely geometrical. To derive the Nash’s smoothing condition, consider an infinitesimal displacement \( y^a \) of the points of an initial manifold. The Lie derivative of the coordinates \( \mathcal{X}^A \) along the orthogonal direction gives

\[ \mathcal{Z}^A(x^\mu, y^a) = \mathcal{X}^A(x^\mu) + y^a (\mathcal{L}_{y^a} \mathcal{X})^A \quad (11) \]
Notice that $\mathcal{L}_\eta \tilde{\eta} = 0$, so that these orthogonal vectors do not propagate. Consequently, the above coordinates to define a new set of embedding equations similar to those for (2):

$$Z^A,\mu Z^B,\nu G_{AB} = g_{\mu\nu}, \quad \tilde{Z}^A,\mu \tilde{\eta}^B, \nu G_{AB} = g_{\mu a}, \quad \eta^A, a \tilde{\eta}^B, b G_{AB} = g_{ab} \quad (12)$$

The deformed manifold $V_4$ defined by these equations has a new set of orthogonal vectors $\eta_a(x^\mu, y^a)$ which are not necessarily parallel to $\tilde{\eta}$. Consequently the new cross metric components $g_{\mu a}$ defined above are not necessarily zero.

By repeating this procedure starting from $Z^A,\mu$ and $\eta^A_a$ we obtain a continuous sequence of four-dimensional embedded manifolds $V_4$, each one being a small deformation of the preceding one. From (12) we obtain the new fundamental forms

$$g_{\mu\nu}(x, y) = Z^A,\mu Z^B,\nu G_{AB} = \bar{g}_{\mu\nu} - 2y^a k_{\mu\nu a} + y^a y^b \bar{g}^{\alpha\beta} k_{\mu\nu a} \bar{k}_{ab} + g^{cd} \bar{A}_{\mu ca} \bar{A}_{\nu db} \quad (13)$$

$$g_{\mu a}(x, y) = Z^A,\mu \eta^B, a G_{AB} = y^b A_{\mu ab} = A_{\mu a} \quad (14)$$

$$k_{\mu a}(x, y) = -\eta^A_a Z^B,\mu G_{AB} = -k_{\mu\nu a} - y^b \bar{g}^{\alpha\beta} k_{\mu\nu a} \bar{k}_{ab} - g^{cd} \bar{A}_{\mu ca} \bar{A}_{\nu db} \quad (15)$$

$$A_{\mu ab}(x, y) = \eta^A_a \eta^B, a G_{AB} = \bar{A}_{\mu ab}(x) \quad (16)$$

Taking the derivative of $g_{\mu\nu}(x, y)$ with respect to $y_a$ in (13) and comparing with (16), we obtain Nash’s smoothing condition (7).

The generalization of Nash’s theorem to manifolds with non-positive metrics (Lorentzian manifolds) is well known [31, 32]. In this generalization, it is shown that the maximum number of extra dimensions required for the local embedding is $n(n+3)/2$, as opposed to the result $n(n+1)/2$ predicted by the analytic embeddings. Furthermore, the metric signature of the embedding space is not free to be chosen, but it depends on the topological properties of the embedded space-time. A notorious example is given by the spherically symmetric space-times: for the Schwarzschild space-time, the embedding signature is $(4,2)$, but for its geodesically complete (the maximal analytical extension), known as the Kruskal space-time, the signature changes to $(5,1)$ [33]. Since the Schwarzschild space-time is a subset of its geodesically complete extension, this change of signature can only be explained by the inclusion of the topological difference between the two space-times: The Nash deformations break down at the Schwarzschild horizon because the extrinsic curvature is not defined there. On the other hand, the Nash embedding applies to the Kruskal metric from $r = 0$ to infinity.
As we have already mentioned, the metric signature of the four-dimensional space-times is a result of the electromagnetic theory, whose equations are invariant under the Poincaré group in the four-dimensional Minkowski space-time. Such perception of a single-time as represented by a negative metric component is carried over General Relativity without any further ado, but only to make it consistent with the Minkowski tangent space postulate of flat limit. Because our gauge field probes are all confined, there is no experimental support which may lead us to conclude that the minus signature of an extra dimensions correspond to a time coordinate. In face of such doubt, we may as in the theory of curves, reparametrize the extra dimensional orbits by its arclength $y^a \rightarrow s^a$, in which case the velocity vector has unit norm. If necessary, it is possible to multiply these arclengths by $\frac{1}{c}$ to make them truly time-like.

3 The Kaluza-Klein Geometry

In the 60’s D. W. Joseph Y. Ne’emann, proposed that the (internal) gauge groups are isomorphic to the group of rotations of the extra dimensional space of an embedding space of the space-time, generated by $N$ vector fields orthogonal to the space-time [34, 35]. This is interesting in the extent that all physics will be defined in the same geometric structure. In the application of this proposal to Kaluza-Klein theory, those vectors are not directly observable, so that the justification for a small compact internal space of the original Kaluza-Klein theory is no longer required. However, we will show below that the proposed geometrization of the gauge symmetry leaves an observable footprint on the space time.

The only stable ground state for the higher-dimensional gravitational field in the original Kaluza-Klein theory, is the plane-flat D-dimensional space [9]. Therefore, it is reasonable to use the plane-flat embedding space as the ground state of the present revision of Kaluza-Klein theory, as the background reference for curvature and also as the source of the internal symmetries as proposed by Joseph and Ne’emann.

Since the metric of the space-time is induced by the metric of the embedding space, it follows that the geometry of the latter space must be defined by the same Einstein-Hilbert principle:

$$\frac{\delta}{\delta G_{AB}} \int (R - k_g \mathcal{L}_m) \sqrt{G} d^Dv = 0$$

(18)
where $\mathcal{R}$ denotes the higher-dimensional scalar curvature; $\mathcal{L}_m$ is the Lagrangian of the sources and $\mathcal{G}$ denotes the positive determinant of the higher dimensional space-time. Since gravitation is not confined to the four-dimensional space-time, the gravitational coupling constant is not necessarily the same as that of General Relativity $k_g = 8\pi G$, replaced by the new coupling constant $k_g^*$ to be determined from high energy physics experiments, possibly at the Tev scale of energies. For consistency, the value $k_g$ must be restored in the limit of General Relativity.

The functional variation of (18) with respect to the metric $\mathcal{G}_{AB}$ gives the D-dimensional Einstein’s equations

$$R_{AB} - \frac{1}{2} RG_{AB} = k_g^* T^*_A B, \quad A, B = 1..D$$

(19)

where the source term in (19) represented by the energy-momentum tensor $T^*_A B$ is composed by known observable sources in the four-dimensional space-time.

Equations (19) are again of the hyperbolic type, but the associated topology is now $\mathbb{R}^4 \times \mathbb{R}^N$. Therefore, just like in the ADM 3+1 metric decomposition of space-times, the metric geometry defined by (19) decomposes into the four-dimensional space-time components and the $N$ extra-dimensional components. To find these components we simply write the metric solution of (19) $\mathcal{G}_{AB}$ in the Gaussian basis $\{Z^A_{\mu \nu}, \eta^A_a\}$ of the embedding space, obtained from equations (12-17), in the form of a $(4+N) \times (4+N)$ matrix as:

$$\mathcal{G}_{AB} = \begin{pmatrix} \tilde{g}_{\mu \nu} + g^{ab} A_{\mu a} A_{\nu b} & A_{\mu a} \\ A_{\nu b} & g_{ab} \end{pmatrix}$$

(20)

where we have denoted

$$\tilde{g}_{\mu \nu} = \bar{g}_{\mu \nu} - 2y^a k_{\mu \nu a} + y^a y^b \bar{g}^{\alpha \beta} k_{\mu a} k_{\nu b}$$

(21)

These are the components of (13), excluding the terms involving $A_{\mu ab}$, representing the metric of the space-time deformed by the extrinsic curvature alone. The components $A_{\mu a}$ represent just a different notation for cross metric components $g_{\mu a}$ given by (14).

As it is clear, (20) is similar to the metric ansatz used in the standard non-Abelian Kaluza-Klein theory. The difference is that here it is a direct consequence of the $4+N$ hyperbolicity of (19), which tells that at each
point of the host space we have a four-dimensional space-time and \(N\) orthogonal vector fields. Thus, it follows from the same arguments that the Einstein-Hilbert Lagrangian \([18]\) written for \([20]\) decomposes into the four dimensional Einstein-Hilbert term \([21]\), plus the Yang-Mills Lagrangian: corresponding to the third fundamental form \(A_{\mu ab}\):

\[
\mathcal{R} \sqrt{g} = \tilde{\mathcal{R}} \sqrt{-\tilde{g}} + \frac{1}{4} \text{tr} F_{\mu\nu} F^{\mu\nu} \sqrt{-\tilde{g}}
\]

(22)

where \(F_{\mu\nu} = [D_{\mu}, D_{\nu}]\), and \(D_{\mu} = I\partial_{\mu} + A_{\mu}\) are defined in the Lie algebra of the (pseudo) rotations group of the normal vectors \(\eta_a\).

Note that the standard basis of the Lie algebra of a rotation group can be expressed with two indices \(L^{ab}\) such that \([L^{ab}, L^{cd}] = f^{abcd} L_{mn}\), where the factors \(f\) are the structure constants. In such basis we write \(A_{\mu} = A_{\mu ab} L^{ab}\) and the components of the matrices

\[
D_{\mu a}^{\ b} = \delta_{\ a}^{\ b} \partial_{\mu} + A_{\mu a}^{\ b}
\]

(23)

On the other hand, sometimes it is more convenient to use the Killing basis of the Lie algebra with a single index \(K^a\), to write \(A_{\mu} = A_{\mu a} K^a\). The final result is independent of the choice of basis.

The manifold \(\tilde{V}_4\) is a deformation of \(\bar{V}_4\), produced by the extrinsic curvature alone, with metric is \([21]\), defined by the Lagrangian

\[
\tilde{\mathcal{R}} \sqrt{\tilde{g}} = [\mathcal{R} \sqrt{g} - (K^2 - h^2)] \sqrt{g}
\]

(24)

In general, the final deformed manifold \(V_4\) also contains the contribution of the third fundamental form. Its geometric components are all given by \([13\text{-}17]\).

Now we may prove Ne’emann conjecture: The gauge group for the field \(A_{\mu}\) defined by third fundamental form is the group of (pseudo)-rotations of the vectors \(\eta_a\) orthogonal to the space-time.

Indeed, the group of rotations of the orthogonal vectors is a subgroup of the group of isometries of the embedding space \(\mathcal{L}_\xi \mathcal{G}_{AB} = 0\). Therefore, consider an infinitesimal transformation of that subgroup, given by

\[
x'^\mu = x^{\mu}, \quad y'^a = y^{a} + \xi^a(x^{\mu}, y^{a})
\]

\(^2\)In those expressions the indices \(a, b, c \cdots\) above are raised and lowered by the orthogonal metric \(g_{ab}\) and the indices \(\mu, \nu, \rho, \cdots\) are raised and lowered with the metric of the \(V_4\) foliation and its inverse.
where the descriptor is defined by $\xi^a = \delta \theta^a_b (x^\mu) y^b$ for the infinitesimal parameters $\delta \theta^a_b (x^\mu)$, functions of the space-time coordinates. The transformation of each component $(A)$ of the orthogonal vector field $\eta_a$ is given by

$$\eta'_a(x) = \frac{\partial y^b}{\partial y'^a} \eta^A(x) = (\delta^b_a - \xi^b_a) \eta^A_b.$$

Killing’s equation for the entire embedding space $\xi_{(A,B)} = 0$ gives for the considered group $\xi_{(a,b)} = 0$, $\xi_{(\mu,\nu)} \equiv 0$, with solution $\xi^a = \delta^a_m y^m$.

Applying the above transformation to the definition of $A_{\mu ab}$ (given by (6)), we obtain after neglecting second order products of $\delta \theta$:

$$A'_{\mu ab} = G_{AB}(\delta^m_a - \delta^m_a) \eta^A_m [ (\delta^b_n - \delta^b_n) \eta^B_n ]_{,\mu} = A_{\mu ab} - 2 \delta_{[a} \delta^b_{A\mu]} n - \delta_{ab,\mu}$$

which is the typical transformation of a gauge potential for a local gauge group.

To complete the proof we need to show that the third fundamental form satisfy the Yang-Mills equations for the considered transformations. For that purpose we may use the Lagrangian (22), or more appropriately, the explicit Lagrangian, obtained by separating the contributions of $g_{\mu \nu}$, $k_{\mu \nu a}$ and $A_{\mu ab}$ in (22):

$$\mathcal{L} = [R + (K^2 - h^2)] \sqrt{g} - \frac{1}{4} tr F_{\mu \nu} F^{\mu \nu} \sqrt{g}$$

(25)

where the determinant of the extra dimensional metric is a common constant factor which was removed from this Lagrangian.

Next take the variation of the Lagrangian in (25) with respect to $A_{\mu ab}$. Since the (25) depends of that field only in the term $\frac{1}{4} tr F_{\mu \nu} F^{\mu \nu}$, the variation of it leads to the Yang-Mills equations

$$D_\mu F^{\mu \nu} = 4 \pi J^\nu$$

(26)

$$D_\mu F^{* \mu \nu} = 0$$

(27)

where $J^\mu$ is the Noether current. Noting that the components $A_{\mu ab}$ are written in the space generated by the rotations of the vectors $\eta_a$, these components are in the adjoint representation of the Lie algebra of the group of rotations of the vector orthogonal to $V_4$.

The second relevant result concerns the gravitational equations. These can also be derived from the variations of (25) with respect to the three components $g_{\mu \nu}$, $g_{\mu a}$ and $g_{ab}$. Instead, we find it more illustrative just to
write the higher-dimensional Einstein’s equations (19) in the Gaussian basis \( \{ Z^A, \eta^A \} \), obtaining

\[
(\mathcal{R}_{AB} - \frac{1}{2} \mathcal{G}_{AB}) Z^A_{\mu} Z^B_{\nu} = R_{\mu\nu} - \frac{1}{2} \mathcal{R} - Q_{\mu\nu} - T^Y_{\mu\nu} = \kappa^* T^*_{\mu\nu}
\]

(28)

\[
(\mathcal{R}_{AB} - \frac{1}{2} \mathcal{G}_{AB}) Z^A_{\mu} \eta^B_{\mu} = k^\rho_{\mu a, \rho} - h_{a, \mu} + A_{\rho ca} k^\rho_{\mu c} - A_{\mu ca} h^c = \kappa^* T^*_{\mu a}
\]

(29)

\[
(\mathcal{R}_{AB} - \frac{1}{2} \mathcal{G}_{AB}) \eta^A_{\alpha} \eta^B_{\beta} = \frac{1}{2} [R - (K^2 - h^2)] g_{ab} = \kappa^* T^*_{ab}
\]

(30)

where we have denoted

\[
Q_{\mu\nu} = g^{ab} (k^\rho_{\mu a} k_{\rho \nu b} - h_{a, \mu} k_{\rho \nu b}) - \frac{1}{2} (K^2 - h^2) g_{\mu\nu}
\]

(31)

Here \( h_{a} = g^{\mu \nu} k_{\mu a} \) represents the mean curvature of the four-dimensional space-time with respect to the \( \eta_a \) direction. If we consider all extra dimensions, we obtain \( h^2 = g^{ab} h_{a} h_{b} \). The Gaussian curvature of the space-time is \( K^2 = k^{\mu \nu a} k_{\mu \nu a} \). The last term in the left hand side term in (28) is due to the presence of \( A_{\mu ab} \) in (22), corresponding to the energy-momentum tensor of the gauge (Yang-Mills) field built from the components \( A_{\mu ab} \):

\[
T^Y_{\mu\nu} = (F^\alpha_{\mu} F^\beta_{\nu} g_{\alpha\beta} - \frac{1}{2} g_{\mu\nu} F^\alpha_{\rho} F_{\alpha\beta})
\]

Finally, \( T^*_{\mu\nu}, T^*_{\mu a}, \) and \( T^*_{ab} \) are the projections of \( T^*_{AB} \) on the tangent, cross and the normal directions of the space-time. Obviously, admitting that these are composed of ordinary matter they are confined and conserved in the four-dimensional space-time, so that it is natural to assume that \( T^*_{\mu a} = 0 \) and \( T^*_{ab} = 0 \). In this case, equations (29) and (30) become homogeneous and consequently the extrinsic curvature cannot be completely determined, requiring an additional equation.

4 The Spin-2 Extrinsic Curvature

On the physical side, the extrinsic curvature corresponds to a spin-2 field for each internal index, or more appropriately, to a multiplet of spin-2 fields, which are independent of the metric in each four-dimensional space-time of the foliation. Several massive spin 2 particles are known, composing a nonet as for example the f, \( A_2 \) and the \( K^* \) mesons. The gravitational field
is also a spin-2 field but since it has a long range, it must be a massless field. A Known theorem due to Soraj Gupta in 1960 and later reviewed by Stanley Deser an others in 1970, proved that any massless spin-2 field defined by a symmetric rank-2 tensor is necessarily a solution of an Einstein-like equations. The Gupta theorem essentially reverses the linear approximation of Einstein equations, applied to the massless Fierz-Pauli theory, in practice reconstructing the non-linear terms [2], [37].

In 1971, C. Isham and others, proposed that one such massive spin-2 field would act as an intermediate field between Einstein’s gravity and hadrons, as a solution of Gupta’s equation [4]. During that same period, the existence of a short range gravitational field with mass was considered as a possible modification of General Relativity, starting from the a non-linear equation derived from the Pauli-Fierz spin-2 action with mass, so that General relativity would be recovered in the zero mass limit [38], [39]. However, it was soon found that in this limit the theory gives a different theory containing ghosts and that does not agree with the observed gravitational light bending experiment [40], although it was argued that this could be corrected if the non-linear terms in the Fierz-Pauli equation would be taken into account in a special parametrization [41]. Such possibility led to a renewed interest in the construction of massive gravity in a four-dimensional theory with two independent metrics (a bi-metric theory), each one responding to a different set of field equations. However, there are still some issues as it has been argued that one of these equations would constrain the other [43].

In the following we use a different approach, where the extrinsic curvature of the space-time plays the role of such massive gravitational field. As we have seen, the existence of the extrinsic curvature is essential for the inclusion of gravitation in the standard model of unification, as well as to the explanation of the acceleration of the universe [36].

The Fierz-Pauli Lagrangian for a relativistic spin-2 field described by a symmetric rank two tensor field $H_{\mu \nu}$ in Minkowski’s space-time is

$$\mathcal{L} = \frac{1}{4} \left[ H_{\mu \nu} H^{\mu \nu} - H_{\nu \rho \mu \nu} H^{\nu \rho \mu \nu} - 2 H_{\mu \nu} H^{\mu \nu} + 2 H_{\nu \rho \mu \nu} H^{\nu \mu \rho \nu} - m^2 (H_{\mu \nu} H^{\mu \nu} - H^2) \right]$$

where $H = \eta^{\mu \nu} H_{\mu \nu}$ and $m$ is a constant. Assuming that $H_{\mu \nu}$ is trace free:

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3 The Lagrangian in a curved space-time with a non-minimal coupling with Einstein’s gravitation has been considered, but it is dependent on what type of additional terms should be included [44].
\( H = 0 \), and that it is divergent free: \( H_{\mu\nu,\mu} = 0 \), we obtain the Klein-Gordon equation \( (\Box^2 - m^2) H_{\mu\nu} = 0 \) so that \( m \) can be interpreted as the Klein-Gordon mass of the spin-2 field \( H_{\mu\nu} \). \[1\]

Applying the above Lagrangian to the extrinsic curvature \( k_{\mu\nu\rho} \) in the curved space-time \( V_4 \) with metric \( g_{\mu\nu} \), with minimal coupling with gravitation, the Fierz-Pauli Lagrangian becomes

\[
\mathcal{L} = \frac{1}{4} \left[ h_{a,\mu} h^{\alpha,\mu} - k_{\rho\alpha a,\mu} k^{\rho\alpha a,\mu} - 2 k_{\mu\nu a,\mu} h^{\alpha,\nu} + 2 k_{\nu\rho a,\mu} h^{\alpha,\rho} - m^2 (K^2 - k_{33}) \right] (33)
\]

Where the semicolon denotes the covariant derivative with respect to the space-time metric \( g_{\mu\nu} \). The Euler-Lagrange equations with respect to \( k_{\mu\nu a} \) are

\[
(\Box^2 - m^2) k_{\mu\nu a} - g_{\mu\nu} \Box^2 h_a + h_{a,\nu,\mu} + g_{\mu\nu} k_{\alpha\beta a} ;\alpha - k_{\sigma\mu a,\nu} ;\sigma - m^2 (k_{\mu\nu a} - h_a g_{\mu\nu}) (34)
\]

Here \( \Box^2 \) stands for the covariant D’alambertian operator.

The Klein-Gordon mass of the extrinsic curvature appear under two conditions \( h_a = g^{\mu\nu} k_{\mu\nu a} = 0 \) and \( k_{\mu\nu a} ;\nu = 0 \):

\[
(\Box^2 - m^2) k_{\mu\nu a} = 0 \quad (35)
\]

Since the space-time coordinates and the extra dimensional coordinates are independent, the second condition \( k_{\mu\nu a} ;\nu = 0 \) becomes trivial in view of (7). On the other hand, \( h_a = 0 \) is more specific, telling that the space-time is minimal in the sense of minimal area surfaces. In the case of an embedded Riemannian manifold, this occurs when the extrinsic curvature is totally intrinsic, proportional to the metric: \( k_{\mu\nu a} = \alpha_a g_{\mu\nu} \). However from (28) it follows that \( Q_{\mu\nu} = \Lambda g_{\mu\nu} \) where \( \Lambda \) is proportional to the cosmological constant and also proportional to the inverse of the curvature radius of the space-time. Therefore, we conclude that \( h_a = 0 \) corresponds to the infinite limit of the curvature radius which means that the Klein-Gordon mass term is characterized in the Minkowski space-time by the mass operator of the Poincaré group as one would expect.

**Summary:**

The current effort to define a short range component of the gravitational field has been motivated by the necessity to understand the acceleration of the universe as a correction to Einstein’s gravity, but also by high energy physics, notably to obtain an intermediator between Einstein’s massless gravitation
and the gauge and matter fields. However, as it was pointed out Einstein’s gravity in four dimensions seems to be a unique theory, that is a theory by its own, and not a limit of some massive spin-2 theory [42, 43].

On the other hand, considerations in higher dimensional gravity has shown that it is possible to break the gravitational hierarchy, at the same time that gravitation acquire greater degrees of freedom. In particular, Kaluza-Klein theory splits the higher dimensional metric in four-dimensional gravitation plus four-dimensional Yang-Mills gauge fields. A nice theory that did not work because of an ill justified geometrical construction based on the product topology $\mathbb{R}^4 \times B_n$, where $B_n$ is a small compact space.

Based on a proposition of Joseph and Ne’emann from the early 60’s and on the Nash theorem from mid 60’s and on the discussion of massive gravity from the early 70’s to the present day, we have completely revised Kaluza-Klein theory, by removing the compact space, introducing the rather successful theory of smooth manifold deformations, and using the extrinsic curvature as a massive spin-2 field that couples with Einstein’s gravitation and with gauge fields, and generalize the cosmological constant term $\Lambda g_{\mu\nu}$ to a cosmic tensor $Q_{\mu\nu}$ built with the extrinsic curvature. The cosmological constant case corresponds to the intrinsic limit of the theory, where the extrinsic curvature becomes intrinsic, proportional to the metric. It is also interesting to notice that since the extrinsic curvature refers to the propagation of the metric in the extra dimensions, the eventual ghost states will be outside the space-time.

The gauge group must be detailed by a phenomenological analysis of the symmetries required by the GUT scheme [46]. Since Nash’s embedding theorem for space-times can be implemented with $D = 14$ dimensions, this result may point to GUT based on a 45 parameter group like for example $SO(10)$ or equivalent. It is also possible to look for a larger gauge symmetry containing those 45 parameter groups.

It is interesting to note that the above proposed unification occurs only when we have more than one extra dimension (That is, six dimensions altogether). This is a limiting case where the gauge group is either $SO(2)$ or $SO(1, 1)$. The first case corresponds to the unification of gravitation with the electromagnetic field. This happens for example when we consider the Kruskal space-time and the second case corresponds to the Schwarzschild space-time (up to the horizon). In both cases the gravitational field are static so that the gauge fields $A_{\mu a b}$ and the strong gravitational field $f_{\mu \nu a}$ are also static. Therefore, in the $SO(2) \sim SU(2)$ case we obtain a very simple unification of the Schwarzschild gravitation with the electrostatic field, corre-
sponding to the existence of a static charge located at the point $r = o$ which is where the singularity of the metric is located. Even in the vacuum case, the two above cases do not follow from not solutions of the Einstein-Maxwell equations in four-dimensions due to the presence of the tensor $Q_{\mu\nu}$ representing the conserved energy of the massive gravitational field at Tev energy scale. A more interesting example of the unification may be constructed with the embedding of the Kerr space-time where magnetic field would also appear.

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