Quantum Corrections to the Thermodynamics of Charged 2-D Black Holes

by

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ABSTRACT

We consider one-loop quantum corrections to the thermodynamics of a black hole in generic 2-dimensional dilaton gravity. The classical action is the most general diffeomorphism invariant action in 1+1 space-time dimensions that contains a metric, dilaton, and Abelian gauge field, and having at most second derivatives of the fields. Quantum corrections are introduced by considering the effect of matter fields conformally coupled to the metric and non-minimally coupled to the dilaton. Back reaction of the matter fields (via non-vanishing trace conformal anomaly) leads to quantum corrections to the black hole geometry. Quantum corrections also lead to modifications in the gravitational action and hence in expressions for thermodynamic quantities. One-loop corrections to both geometry and thermodynamics (energy, entropy) are calculated for the generic theory. The formalism is then applied to a charged black hole in spherically symmetric gravity and to a rotating BTZ black hole.
1 Introduction

There exists a strong analogy between the properties of black holes and conventional thermodynamical systems [1]. In this analogy the entropy of the black hole is directly proportional to the surface area of its event horizon and literature refers to this quantity as the Bekenstein-Hawking entropy [2]. Meanwhile, the temperature of the black hole is proportional to the surface gravity of its horizon. In spite of this established correspondence, there is still a lack of understanding of precisely what accounts for black hole entropy which is a pure geometrical quantity. In a usual statistical mechanical system the entropy is explained in terms of the degrees of freedom of its microscopic constituents. However a black hole has a limited number of such degrees of freedom, as demonstrated by the so called ”No-Hair” theorems [3].

In recent literature, varied attempts have been made to derive black hole entropy on statistical mechanical principles, with varying degrees of success [4][5][6][7]. For instance, Strominger and Vafa [4] counted the degeneracy of soliton bound states for extremal black holes in string theory. In a very different and more geometrical approach, Carlip [5] counted horizon edge states in a gauge theory formulation of 2+1 anti-deSitter gravity. Another approach that has been investigated involves Sakharov’s theory of induced gravity [8] following a proposal by Jacobson [9]. Recent work along these lines to generate black hole entropy has been done by Frolov, Fursaev, and Zelnikov [6].

The success of a number of very diverse approaches seems to suggest that the correct explanation for black hole entropy may in some sense be universal. That is, it should depend explicitly on neither the macroscopic gravitational form nor on a hidden microscopic quantum theory [10]. Consequently, it may prove beneficial to study as wide a range of theories as feasible and in doing so look for model independent features. Such observations could potentially provide valuable insight as to the geometrical origins of black hole entropy. To this end, we examine the thermodynamic properties of black holes in generic dilaton gravity coupled to an Abelian gauge field in 1+1 dimensions. This provides a very extensive class of models which allow for black hole solutions. Even with the 2-dimensional limitation, many such models are seen to have direct physical significance. For instance, in the spherically symmetric reduction of 4-dimensional Einstein-Maxwell gravity the dilaton
scalar field corresponds to radial distance. Also it has been shown that
black hole solutions of constant curvature gravity in 2-dimensions (Jackiw-
Teitelboim[31]) are in fact projections of BTZ black holes described by 2+1
gravity with axial symmetry [11].

In recent work we have studied the classical thermodynamic properties of
generic dilaton gravity via a Hamiltonian partition function method [12][13].
In the present paper we calculate the thermodynamics so as to include one-
loop corrections. The approach we use here is based on York’s Euclidean-
action method [14][15] which in turn follows from the Gibbons-Hawking
path integral formalism [16]. This entails taking the black hole to be in
a state of thermal equilibrium with evaporated radiation and then relating
the periodicity of Euclidean time with the inverse thermodynamic temper-
ature. First-order quantum corrections are introduced into the procedure
via a technique applied by Frolov, Israel, and Soldukin [17] in the study of
spherically symmetric charged black holes. The basic idea is to add to the
classical action a correction corresponding to the one-loop effective action
obtained by integrating out matter fields coupled to the metric and non-
minimally coupled to the dilaton. This one loop effective action is a suitable
generalization [18, 19, 21, 22] of the Polyakov action obtained from the 2-D
conformal (or trace) anomaly [22]. The effect of these quantum corrections on
the black hole thermodynamics is two-fold. First, it modifies the black hole
geometry due to the non-vanishing one-loop effective stress-energy tensor.
Secondly, the surface terms which give rise to the black hole free energy also
acquire quantum corrections. As a result the formulae relevant to calculating
thermodynamic quantities (energy and entropy) are modified as well. Our
results will hopefully provide insight into the nature of such corrections for
generic dilaton gravity as well as provide the template for closer examination
of a myriad of specific theories.

This paper is arranged as follows. In Section 2 we introduce the ac-
tion for generic 2-dimensional dilaton gravity coupled to an Abelian gauge
field. Here we present the most general solution to the field equations and
by way of the Euclidean action approach [14] we are able to describe black
hole thermodynamic properties at a classical level. In Section 3 we intro-
duce one-loop quantum corrections due to matter fields propagating on a
curved background. The resulting modifications to the black hole geometry
are deduced by applying the formalism of Frolov et al. [17]. In Section 4
we calculate the quantum corrections to black hole energy and entropy. In
Sections 5 and 6 we apply our results to the specific examples of charged black holes in spherically symmetric gravity and rotating BTZ black holes respectively. For simplicity cases consider minimal coupling of matter fields with the dilaton however the formalism to be presented is readily extendable to more general coupling scenarios. Section 7 summarizes the paper and considers future prospects for related work.

2 Classical Theory

In two spacetime dimensions the Einstein tensor vanishes identically. Consequently, the construction of a dynamical theory of gravity with no more than two derivatives of the metric in the action requires the introduction of a scalar field, namely the dilaton. Recent works have demonstrated that the dilaton is more than a lagrange multiplier but significant in determining both the symmetries and topologies of the solution [25].

Here we consider the most general Lorentzian action functional depending on the metric tensor $g_{\mu\nu}$, dilaton scalar field $\phi$ and Abelian gauge field $A_{\mu\nu}$ in two spacetime dimensions [26][27]:

$$W[g, \phi, A] = \frac{1}{2G} \int d^2 x \sqrt{-g} \left[ D(\phi) R(g) + \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{l^2} V(\phi) - \frac{2G}{4} Y(\phi) F_{\alpha\beta} F_{\alpha\beta} \right]$$ (1)

where $G$ is the dimensionless 2-d Newton constant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $l$ is a fundamental constant of dimension length. Also $D(\phi), V(\phi)$, and $Y(\phi)$ are arbitrary functions of the dilaton field.

Variation of the action with respect to the metric, dilaton field and gauge field respectively leads to the following set of field equations.

$$- 2 \nabla_\alpha \nabla_\beta D(\phi) + \nabla_\alpha \phi \nabla_\beta \phi + g_{\alpha\beta} \left( 2 \Box D(\phi) - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{l^2} V(\phi) \right)$$

$$+ \frac{G}{2} Y(\phi) F_{\mu\nu} F_{\mu\nu} - 2GY(\phi) F_{\alpha\gamma} F_{\beta\gamma} = 0$$ (2)

1Quantum corrections to the thermodynamics of the BTZ black hole and some classes of 2d charged black holes have been previously considered using different methods in [23] while [24] has examined quantum gravitational corrections to the entropy of the BTZ black hole.
\[
- \Box \phi + \left[ R \frac{\delta D}{\delta \phi} + \frac{1}{l^2} \frac{\delta V}{\delta \phi} - \frac{G}{2} \frac{\delta Y}{\delta \phi} F_{\alpha \beta}^a F_{\alpha \beta}^a \right] = 0 \tag{3}
\]

\[
\nabla_\beta (Y(\phi) F^{\alpha \beta}) = 0 \tag{4}
\]

Directly solving Maxwell’s equation Eq. (4) yields

\[
F = \frac{\sqrt{-g} q}{Y(\phi)} \tag{5}
\]

where \( F \) is defined implicitly by \( F_{\mu \nu} = F_{\epsilon_{\mu \nu}} \) and \( q \) is a constant that corresponds to Abelian charge. Next we define an “effective” potential \( \tilde{V}(\phi, q) \) such that

\[
\tilde{V}(\phi, q) = V(\phi) - Gl^2 \frac{q^2}{Y(\phi)} \tag{6}
\]

The action and remaining field equations Eqs. (2, 3) can be rewritten as follows:

\[
W[g, \phi, q] = \frac{1}{2G} \int d^2 x \sqrt{-g} \left[ D(\phi) R(g) + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{l^2} \tilde{V}(\phi, q) \right] \tag{7}
\]

\[
- 2 \nabla_\alpha \nabla_\beta D(\phi) + \nabla_\alpha \phi \nabla_\beta \phi + g_{\alpha \beta} \left( 2 \Box D(\phi) - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{l^2} \tilde{V}(\phi, q) \right) = 0 \tag{8}
\]

\[
- \Box \phi + R \frac{\delta D}{\delta \phi} + \frac{1}{l^2} \frac{\delta \tilde{V}}{\delta \phi} = 0 \tag{9}
\]

Obtaining the solutions for an action of this form has been well documented in prior works \[12\][27] so only a brief account will be presented here. First the action is reparameterized thereby eliminating the kinetic term (requires \( D(\phi) \neq 0 \) and \( \frac{dD}{d\phi} \neq 0 \) for any admissable value of \( \phi \)).

\[
\tilde{\phi} = D(\phi) \tag{10}
\]

\[
\Omega^2 = \exp \frac{1}{2} \int \frac{d\phi}{dD/d\phi} \tag{11}
\]

\[
\tilde{g}_{\mu \nu} = \Omega^2(\phi) g_{\mu \nu} \tag{12}
\]

\[
\nabla(\tilde{\phi}, q) = \tilde{V}(\phi, q)/\Omega^2(\phi) \tag{13}
\]
The reparameterized action is then as follows:

\[ W[\bar{g}, \phi, q] = \frac{1}{2G} \int d^2x \sqrt{-\bar{g}} \left[ \phi R(\bar{g}) + \frac{1}{l^2} \nabla(\phi, q) \right] \]  

(14)

The timelike killing vector for the resultant field equations is easily identifiable. It is found to be:

\[ \bar{K}^\mu = l\epsilon^{\mu\nu} \partial_\nu \phi \]  

(15)

With norm given by

\[ |\bar{K}|^2 = \bar{j}(\phi) - 2GlM \]  

(16)

where

\[ \bar{j}(\phi) = \int^\phi d\phi \nabla(\phi, q) \]  

(17)

and \( M \) is a constant of integration identified as the mass observable.

Next we choose a local coordinate system in which \( \phi \) and hence \( \bar{\phi} \) have spatial dependence only. The final solutions in these static coordinates are then obtained by exploiting the form of the killing vector. These are found to be:

\[ \bar{\phi} = \frac{x}{l} \]  

(18)

\[ ds^2 = -\bar{g}(x)dt^2 + \bar{g}^{-1}(x)dx^2 \]  

(19)

where

\[ \bar{g}(x) = \bar{j}(\phi) - 2GlM \]  

(20)

We can then re-express this solution in terms of the original parameterization as follows:

\[ \phi = D^{-1}(\frac{x}{l}) \]  

(21)

\[ ds^2 = -g(x)dt^2 + g^{-1}(x)\Omega^{-4}(\phi(x))dx^2 \]  

(22)

where

\[ g(x) = \frac{1}{\Omega^2(\phi(x))} \left[ \bar{j}(\frac{x}{l}) - 2GlM \right] \]  

(23)

The necessary condition for a given theory to admit black hole configurations is the existence of apparent horizons. That is, spacetime curves of the form \( \phi(x, t) = \phi_o \) (constant) where \( \phi_o \) satisfies \( g(\phi_o; M, q) = 0 \). The nature of a given black hole solution can be revealed by considering \( dg/d\phi \).
evaluated at these event horizons $\phi = \phi_0$. For a fixed value of mass $M$ there may exist critical values of charge $q(M)$ so that this derivative vanishes. For such critical values the function $g(\phi; M, q)$ may have either a local extremum or point of inflection at the horizon. If it is an extremum the norm of the killing vector does not change sign when passing through the event horizon. As $q$ is varied away from its critical value either the horizon will disappear or two event horizons (inner and outer) will appear. The latter case signifies the presence of an extremal black hole when $q$ is at its critical value. For a point of inflection the norm of the killing vector does change sign but as $q$ is varied away from its critical value one expects the formation of either one or three horizons [27].

For the subsequent (thermodynamic) analysis we consider the Euclidean sector such that $t \to it$. Hence we re-write the action Eq.(7) with respect to the Euclidean metric tensor:

$$W_E = -\frac{1}{2G} \int d^2 x \sqrt{g} \left[ D(\phi) R(g) + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{l^2} \tilde{V}(\phi, q) \right] - \frac{1}{G} \oint_{\text{outer boundary}} dt \gamma D(\phi) \nabla_\alpha n^\alpha$$

(Henceforth the subscript E on the Euclidean action will be implied). The second integral in Eq.(24) is the extrinsic curvature boundary term. It is included so that when second derivatives of the metric are cancelled off (via appropriate integration by parts) then the resulting total divergences on the outer boundary will be cancelled off as well [46]. Here we define $n_\mu$ as the outward unit vector normal to the outer boundary enclosing the black hole and $\gamma$ as the induced metric appropriate for evaluating the line integral.

We re-write the Euclidean static metric in the following form

$$ds^2 = g(x)dt^2 + e^{-2\lambda(x)} g^{-1}(x) dx^2$$

where $e^{\lambda(x)} = \Omega^2(\phi(x))$. For this metric it follows that

$$\sqrt{g} = e^{-\lambda(x)}$$

and

$$R = -e^{\lambda(x)} \left( e^{\lambda(x)} g'(x) \right)'$$

where $'$ indicates differentiation with respect to $x$. The coordinates $t, x$ describe a disc and will be taken to range between the limits $x_+ \leq x \leq L$. 

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and $0 \leq t \leq 2\pi \beta$. Here $x = x_+$ represents the black hole horizon (ie. - $g(x_+) = 0$), $x = L$ is the outer boundary of the black hole (ie. - box size), and $\beta$ is the asymptotic inverse temperature. It can be shown that regularity of the solution requires the absence of a conical singularity which leads to the following condition:

$$\beta = \frac{2e^{-\lambda(x_+)} g'(x_+)}{2}$$  \hspace{1cm} (28)

Note that application of standard thermodynamics requires using the inverse temperature of the box $\beta$ which is ”red-shifted” from the previously defined quantity such that:

$$\beta = \frac{g^{1/2}(L)}{L^{2/3}} \beta$$  \hspace{1cm} (29)

For this metric it also follows that the extrinsic curvature (defined by the boundary term of the action) can be expressed [17]:

$$\gamma \nabla_\alpha n^\alpha = \frac{1}{2} e^{\lambda(L)} g'(L)$$  \hspace{1cm} (30)

Using these results Eqs. [25-30] we can express the Euclidean action functional (Eq.(24)), with respect to the generic static metric giving:

$$W = -\frac{\pi \beta}{G} \int_{x_+}^{L} dx \left( D'(x) e^{\lambda(x)} g'(x) + \frac{e^{\lambda(x)}}{2} g(x)(\phi'(x))^2 + \frac{e^{-\lambda(x)}}{l^2} \tilde{V}(x) \right) - \frac{D(x_+)}{G \beta}$$  \hspace{1cm} (31)

We can use this form of the action to derive thermodynamic properties of interest. These include the free energy $F = (2\pi \beta)^{-1} W$, energy $E = (2\pi)^{-1} \partial_\beta W$, and entropy $S = (\beta \partial_\beta - 1) W$.

$$F = -\frac{1}{2G g^{1/2}(L)} \int_{x_+}^{L} dx \left( D'(x) e^{\lambda(x)} g'(x) + \frac{e^{\lambda(x)}}{2} g(x)(\phi'(x))^2 + \frac{e^{-\lambda(x)}}{l^2} \tilde{V}(x) \right) - \frac{D(x_+)}{G \beta}$$  \hspace{1cm} (32)

$$E = -\frac{1}{2G g^{1/2}(L)} \int_{x_+}^{L} dx \left( D'(x) e^{\lambda(x)} g'(x) + \frac{e^{\lambda(x)}}{2} g(x)(\phi'(x))^2 + \frac{e^{-\lambda(x)}}{l^2} \tilde{V}(x) \right)$$  \hspace{1cm} (33)
\[ S = \frac{2\pi}{G} D(x_+) \]  

(34)

Since the box temperature is taken to be \( T = 2\pi \beta \) from Eqs. [32-34] we obtain the result \( F = E - TS \). At the extremum of free energy (or equivalently action) \( \delta F = 0 \) and hence the second law of thermodynamics immediately follows.

It is possible and convenient to re-express the action (Eq.(31)) in a form which, except for surface terms, vanishes on shell. Defining \( G_{\alpha\beta} \) to be the left hand side of Eq.(8) and re-writing with respect to the coordinate system defined by metric Eq.(25) yields:

\[
G_{\alpha\beta} = -2\delta_\alpha^x \delta_\beta^x D''(x) + 2\Gamma_\alpha^x D'(x) + g_{\alpha\beta} \left[ 2e^{\lambda(x)} (e^{\lambda(x)} g(x) D'(x))' - \frac{1}{2} g(x) e^{2\lambda(x)} (\phi'(x))^2 - \frac{1}{l^2} \tilde{V}(x) \right] = 0
\]

(35)

In the case where both tensor indices represent time coordinate denoted by 0 (and note that \( \Gamma_0^0 = -\frac{1}{2} e^{2\lambda} g g' \)) we get:

\[
G_0^0 = -e^{\lambda(x)} \left[ -e^{\lambda(x)} g'(x) D'(x) - 2g(x) (e^{\lambda(x)} D'(x))' + \frac{1}{2} g(x) e^{\lambda(x)} (\phi'(x))^2 + \frac{e^{-\lambda(x)}}{l^2} \tilde{V}(x) \right]
\]

(36)

Using this result to substitute for the second and third terms in the integrand of Eq.(31):

\[
W = -\frac{\pi \beta}{G} \int_{x_+}^{L} dx \left[ -e^{-\lambda(x)} G_0^0 + 2 \left( g(x) e^{\lambda(x)} D'(x) \right)' \right] - \frac{2\pi}{G} D(x_+) \]

(37)

Since the second term in the integrand is a total derivative and \( g(x_+) = 0 \) it follows that:

\[
W = \frac{\pi \beta}{G} \int_{x_+}^{L} e^{-\lambda(x)} G_0^0 dx - \frac{2\pi \beta}{G} e^{\lambda(L)} g(L) D'(L) - \frac{2\pi}{G} D(x_+)
\]

(38)

Reconsider the energy \( E = (2\pi)^{-1} \partial_\beta W \). Since thermodynamic quantities are presumed to be calculated for equilibrium configurations (i.e. on shell) here
we can set $G_0^0 = 0$ giving an energy which reduces to an outer boundary surface term:

$$ E = -\frac{1}{G} e^{\lambda(L)} g^{\frac{1}{2}}(L) D'(L) $$ (39)

This expression is typically divergent as $L \to \infty$. (This follows from the divergence of the Euclidean action as the outer boundary goes to infinity). To resolve this dilemma we compare the energy of Eq.(39) with that of a carefully selected background geometry [28]. The background metric will be taken here to represent the asymptotic geometry of the black hole. Hence we define $g_0 = \lim_{L \to \infty} g(L)$ and the "subtracted energy" is given by:

$$ E_{\text{sub}} = \frac{1}{G} e^{\lambda(L)} D'(L) \left[ g_0^{\frac{1}{2}} - g^{\frac{1}{2}}(L) \right] $$ (40)

We can justify this choice of background by noting the agreement between this result with that attained for a Hamiltonian partition function approach in a prior study [12].

3 Quantum Corrected Black Hole Geometry

In the path integral approach to black hole thermodynamics the matter fields can be integrated out yielding an effective action which depends only on fields in the classical action. Hence one-loop quantum effects can be taken into account by adding a quantum counterpart $\hbar \Gamma$ to the classical gravitational action $W_{CL}$ (Eq.(24)) such that (assuming no matter coupling to the Abelian gauge field):

$$ W [g, \phi, q] = W_{CL} [g, \phi, q] + \hbar \Gamma [g, \phi] $$ (41)

Variation of this complete action yields the quantum corrected field equations which may be solved perturbatively. Variation of the action with respect to the metric gives us:

$$ G_{\alpha\beta}(g, \phi, q) + \hbar T_{\alpha\beta}(g, \phi) + O(\hbar^2) = 0 $$ (42)

where $G_{\alpha\beta} = \frac{\delta W_{CL}}{\delta g^{\alpha\beta}}$ is again given by the left hand side of Eq.(8) and $T_{\alpha\beta} = \frac{\delta \Gamma}{\delta g^{\alpha\beta}}$. 

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The general form of the one loop effective action is \[18, 20\]:

\[
\Gamma = \frac{1}{12} \int d^2x \sqrt{g} \left[ a R + b(\phi)(\nabla \phi)^2 R + c(\phi) R - \ln(\mu^2) b(\phi)(\nabla \phi)^2 \right]
\]

(43)

The first term is the usual trace anomaly that arises for minimally coupled scalars, while the next two terms are contributions to the anomaly from the non-minimal coupling to the dilaton. The last term contains an arbitrary scale factor, \(\mu\), and comes from the conformally invariant part of the effective action\[18\]. It can be obtained using a Schwinger-DeWitt type expansion\[23\].

We will show that the scale factor \(\mu\) does not affect the final thermodynamic quantities. \(a\) is a constant (we will set \(a = 1\) for simplicity), while \(b(\phi)\) and \(c(\phi)\) are determined by the specific form of the coupling between the matter fields and the dilaton. For example, in dimensionally reduced spherically symmetric gravity, \(b\) and \(c\) are constants\[19, 21, 18, 20\]. Here we will treat them as arbitrary local functions of the dilaton.

It is important for the following thermodynamic analysis to put the non-local expression Eq.(43) for \(\Gamma\) in local form. We do this by introducing a pair of scalar fields \(\psi\) and \(N\), and writing:

\[
\Gamma = \frac{1}{12} \int d^2x \sqrt{g} \left[ (\psi + N) R + (\nabla N) \cdot (\nabla \psi) + b(\nabla \phi)^2 (\psi - \ln(\mu^2)) + c\phi R \right]
\]

\[
+ \frac{1}{6} \oint_{\text{outer boundary}} dt \gamma (\psi + N + c(\phi)) \nabla_\alpha n^\alpha
\]

(44)

where an extrinsic curvature surface term has been added in analogy to the classical case. It is straightforward to show that variation of Eq.(44) yields the following field equations for the scalars:

\[
\psi = \frac{1}{\Box} R
\]

(45)

and

\[
N = \frac{1}{\Box} (R + b(\nabla \phi)^2)
\]

(46)

Substituting these equations back into Eq.(44) yields precisely Eq.(43). Note that in the 2-dimensional minimally coupled case \((b = 0)\), \(N\) reduces to \(\psi\) and only a single scalar field need be introduced.

\[2\] We are grateful to S. Odintsov for pointing out the necessity for including this term.
Before proceeding we show it is possible to solve explicitly for $\psi(x)$ at the classical level (as is appropriate for this analysis). This is achieved by conformally mapping the coordinate space described by the static Euclidean metric Eq.(25) to a flat disc of radius $z_0$ and curvature $R = \Box \psi$. This disc may be expressed

$$ds^2 = e^{-\psi(z)}(z^2 d\theta^2 + dz^2)$$  \hspace{1cm} (47)

where the disc coordinates $\theta$ and $z$ are taken to range between $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq z_0$. Substituting the Euclidean static metric Eq.(25) into the left hand side gives

$$g(x)dt^2 + g^{-1}(x)e^{-\lambda(x)}dx^2 = e^{\psi(z)}(z^2 d\theta^2 + dz^2)$$  \hspace{1cm} (48)

where $t = \beta \theta$ and $x_+ \leq x \leq L$. The following relations follow directly from Eq.(48):

$$g(x)dt^2 = e^{-\psi}z^2 d\theta^2$$
$$= \frac{e^{-\psi}z^2}{\beta^2}dt^2$$  \hspace{1cm} (49)

$$g^{-1}(x)e^{-2\lambda}dx^2 = e^{-\psi}dz^2$$  \hspace{1cm} (50)

Using Eq.(49) to solve for $z$ and Eq.(50) to solve for $dz$ gives us:

$$z = \beta \sqrt{g}e^{\psi/2}$$  \hspace{1cm} (51)

$$dz = \frac{e^{-\lambda}e^{\psi/2}}{\sqrt{g}}dx$$
$$= \frac{e^{-\lambda}z}{\beta g}dx$$  \hspace{1cm} (52)

Dividing Eq.(52) by Eq.(51) and integrating for given boundary conditions yields:

$$\ln \left( \frac{z_0}{z} \right) = \frac{1}{\beta} \int_x^L \frac{dx}{g(x)} e^{-\lambda(x)}$$  \hspace{1cm} (53)

Using equation Eq.(51) to re-write the left-hand side of Eq.(53) as a function of $x$ and solving for $\psi(x)$:

$$\psi(x) = \psi(L) - \frac{2}{\beta} \int_x^L dx \frac{e^{-\lambda(x)}}{g(x)} + \ln g(L) - \ln g(x)$$  \hspace{1cm} (54)

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To find an explicit expression for $\psi(L)$ consider the calculation of the proper time evaluated for a closed path on the boundary of the disc at $x = L(z = z_o)$:

$$\oint_{\theta=0}^{2\pi} \sqrt{g(L)} \, dt = \oint_{\theta=0}^{2\pi} z_o e^{-\psi(L)/2} \, d\theta$$

(55)

Integrating and solving for $\psi(L)$:

$$\psi(L) = -2 \ln \left( \frac{\beta z_o}{z_o} \right) - \ln g(L)$$

(56)

Substituting Eq.(56) into Eq.(54):

$$\psi(x) = -\ln g(x) - \frac{2}{\beta} \int_x^L dx \frac{e^{-\lambda(x)}}{g(x)} - 2 \ln \left( \frac{\beta}{z_o} \right)$$

(57)

To solve for $N$ we repeat the prior analysis except here we map to a flat disc described in the form:

$$ds^2 = e^{-N(z)} + \frac{1}{\beta^2} [b(\nabla \phi)^2] \left( z^2 d\theta^2 + dz^2 \right)$$

(58)

This results in the following:

$$N(x) = -\ln g(x) - \frac{2}{\beta} \int_x^L dx \frac{e^{-\lambda(x)}}{g(x)} - 2 \ln \left( \frac{\beta}{z_o} \right) + \frac{1}{\beta^2} \left[ b(\phi)(\nabla \phi(x))^2 \right]$$

(59)

Because of the non-local form of the last term this is not a satisfactory result so we integrate $\Box N = R - b(\nabla \phi)^2$ giving

$$N(x) = N(L) - \ln g(x) + \ln g(L)$$

$$\quad - \int_x^L dx \frac{e^{-\lambda(x)}}{g(x)} \left[ C - \int_x^L d\tau \frac{b e^{\lambda(x)}(\phi'(\tau))^2}{g(\tau)} \right]$$

(60)

where $N(L)$ and $C$ are arbitrary constants of integration. The constant $N(L)$ does not affect the thermodynamic quantities in the subsequent analysis, so without loss of generality we set $N(L) = \psi(L)$. The remaining constant must in principle be determined by experiment. However, we adopt the ansatz that $N(x)$ should reduce to $\psi(x)$ when $b = 0$, and that the geometry
should uniquely determine both $\psi$ and $N$. With these conditions $N$ reduces to:

$$N(x) = \psi(x) + \int_x^L d\tilde{x} \frac{e^{-\lambda(\tilde{x})}}{g(\tilde{x})} \int_x^L d\tilde{\tau} b e^{\lambda(\tilde{\tau})}(\phi'(\tilde{\tau}))^2 g(\tilde{\tau})$$  \hspace{1cm} (61)

If we signify $g_{CL}$ as the classical metric and $g = g_{CL} + \delta g$ as the one-loop quantum corrected metric it can be shown (by perturbative expansion) that the following form of the field equation Eq.(42) is valid to first order:

$$G_{\alpha\beta}(g) + \hbar T_{\alpha\beta}(g_{CL}) = 0$$  \hspace{1cm} (62)

Where $G_{\alpha\beta}$ is given by the left hand side of Eq.(8) and $T_{\alpha\beta}$ can be obtained from the variation of Eq.(44). We find:

$$T_{\alpha\beta} = \frac{G}{3} \left[ \nabla_\alpha \nabla_\beta (\psi + N) - \frac{1}{2} (\nabla_\alpha N \nabla_\beta \psi + \nabla_\alpha \psi \nabla_\beta N) 
\right. 
- g_{\alpha\beta}(R + \Box N - \frac{1}{2}(\nabla N) \cdot (\nabla \psi)) 
- b(\psi - \ln(\mu^2))(\nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta}(\nabla \phi)^2) 
- (g_{\alpha\beta} \Box c(\phi) - \nabla_\alpha \nabla_\beta c(\phi)) \right]$$  \hspace{1cm} (63)

An explicit expression for $T_{\alpha\beta}$ in terms of the metric is then obtained by substituting for $\psi$ and $N$ via Eq.(57) and Eq.(61) respectively. It should be noted that the resulting equation can be equivalently obtained by direct functional differentiation of the action in its non-local form (Eq.(43)) \[29].

We again take the dilaton as representing the spatial coordinate so that the geometric corrections are manifested in the metric. Solving the field Eq.(62) yields an explicit form of the quantum corrected metric. In analogy to the formalism presented by Frolov et al. [17] we adapt the classical static metric (Eqs.[22-23]) to the quantum corrected case as follows:

$$ds^2 = g(x)e^{2w(x)}dt^2 + g^{-1}(x)\Omega^{-4}(\phi(x))dx^2$$  \hspace{1cm} (64)

$$g(x) = \frac{1}{\Omega^2(\phi(x))} \left[ \frac{f(x)}{l} - 2Glm - 2Glm(x) \right]$$  \hspace{1cm} (65)

Here $m(x)$ is the first order quantum correction to the classical mass $M$ and we have introduced a metric function $w(x)$ which vanishes in the classical limit (and where functions $f$ and $\Omega^2$ are as defined by Eqs.[11,13,17]).
We now solve $G_{\alpha\beta} = -hT_{\alpha\beta}$ by first finding expressions for quantum quantities $m(x)$ and $w(x)$ in terms of components of the tensor $T_{\alpha\beta}$. Using Eq.(8) for $G_{\alpha\beta}$ gives us:

$$-2\nabla_{\alpha}\nabla_{\beta}D + \nabla_{\alpha}\phi\nabla_{\beta}\phi + g_{\alpha\beta}\left[2\Box D - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{l^2}\tilde{V}\right] = -hT_{\alpha\beta} \quad (66)$$

Using the fact that the solution only depends on $x$ and the definition of covariant derivative:

$$-2\delta_{\alpha}^{\alpha}\delta_{\beta}^{\beta}D'' + 2\Gamma_{\alpha\beta}^{\gamma}D' + \delta_{\alpha}^{\alpha}\delta_{\beta}^{\beta}(\phi')^2 + g_{\alpha\beta}\left[2\Box D - \frac{g^{xx}(\phi')^2}{2} - \frac{1}{l^2}\tilde{V}\right] = -hT_{\alpha\beta} \quad (67)$$

The off diagonal components (i.e. $\alpha = x$, $\beta = t$) of the above equation vanish identically. For the case in which both indices $\alpha, \beta$ represent the time coordinate:

$$-g^{xx}g'_{tt}D' + g_{tt}\left[2\Box D - \frac{g^{xx}(\phi')^2}{2} - \frac{1}{l^2}\tilde{V}\right] = -hT_{tt} \quad (68)$$

Now we re-express the left hand side with respect to the metric defined by Eqs.[64-65]. First note that by using $D = \frac{\nabla\phi}{l}$ (Eq.(21)) we can evaluate $\Box D$ to give

$$\Box D = \frac{\Omega^2}{l}(\Omega^2 g w' + \tilde{j} - 2Glm') \quad (69)$$

and so:

$$e^{2w}\left[-\frac{2\Omega^4 g^2 w'}{l} - \frac{\Omega^2 g}{l}(\tilde{j} - 2Glm)' + \frac{g^2\Omega^2}{l}(\Omega')^2 + \frac{2g\Omega^2}{l}(g\Omega^2 w' + \tilde{j} - 2Glm') - \frac{\Omega^4 g^2}{2}(\phi')^2 - \frac{g}{l^2}\tilde{V}\right] = -hT_{tt} \quad (70)$$

From Eqs.[13][17]

$$\tilde{j} = \frac{1}{l}\frac{\tilde{V}}{\Omega^2} \quad (71)$$

and from Eqs.[11][21] :

$$(\Omega')^2 = \frac{l\Omega^2}{2}(\phi')^2 \quad (72)$$
Using these 2 results in Eq.(70) and solving for $m'$ (and using $T_{tt} = g_{tt}T_t^t$) :

$$m' = \frac{\hbar}{2G\Omega^2 T_t^t}$$  \hspace{1cm} (73)

Now for the case in which both tensor indices in Eq.(67) represent the spatial coordinate:

$$-2D'' + g^{xx}g'_{xx}D' + (\phi')^2 + g_{xx} \left[ 2\Box D - \frac{g^{xx}}{2}(\phi')^2 - \frac{1}{l^2} \tilde{V} \right] = -\hbar T_{xx}$$  \hspace{1cm} (74)

Using the metric (Eqs.[64,65]) along with Eq.(69) :

$$-\frac{1}{lg\Omega^2}(\tilde{J} - 2GlM)' - \frac{(\Omega^2)'}{l\Omega^2} + \frac{(\phi')^2}{2}$$
$$+ \frac{2}{lg\Omega^2}(g\Omega^2w' + \tilde{J} - 2Gl'm') - \frac{\tilde{V}}{l^2g\Omega^t} = -\hbar T_{xx}$$  \hspace{1cm} (75)

Using Eqs.[71,72] and solving for $w'$ :

$$w' = \frac{l}{2} \left( \frac{2G}{g\Omega^2}m' - \hbar T_{xx} \right)$$  \hspace{1cm} (76)

Substitute for $m'$ via Eq.(73) and use $T_{xx} = g_{xx}T_x^x$ :

$$w' = \frac{lh}{2g\Omega^4}(T_t^t - T_x^x)$$  \hspace{1cm} (77)

Eq.(73) and Eq.(74) provide the first order quantum corrections to the geometry. Note that consistency of the perturbative expansion requires $T_{\alpha\beta}$ in the above expressions to be evaluated on the classical solution.

Next we explicitly evaluate the tensor components $T_t^t$ and $T_x^x$. In terms of the classical static metric as expressed by Eq.(25) the non-vanishing terms are given by (after some simplification):

$$T_t^t = \frac{G}{3} \left[ \frac{1}{2} g'e^{2\lambda}(\psi' + N') + \frac{1}{2} ge^{2\lambda}N'\psi' + 2e^\lambda(e^\lambda g')' ight.$$
$$- bge^{2\lambda}(\phi')^2(1 - \frac{1}{2}(\psi - \ln(\mu^2))) - e^{2\lambda}(ge'' + ge'\lambda' + \frac{1}{2}g'c') \left. \right]$$  \hspace{1cm} (78)
\[ T_x^x = \frac{G}{3} \left[ g e^{2\lambda}(\psi + N)'' + \left( \frac{1}{2} g' + g\lambda' \right) e^{2\lambda}(\psi + N)' - \frac{1}{2} g e^{2\lambda}N'\psi' + 2e^\lambda(e^\lambda g)' \
- b e^{2\lambda}(\phi')^2 (1 + \frac{1}{2}(\psi - \ln(\mu^2))) - \frac{1}{2}e^{2\lambda}g'c' \right] \] (79)

Substituting for \( \psi \) (Eq.(57)) and \( N \) (Eq.(61)) and further simplification gives:

\[ T_t^t = \frac{G e^{2\lambda}}{6g} \left[ 4g e^{-\lambda}(e^\lambda g)' - (g')^2 + \frac{4}{\beta^2}e^{-2\lambda} \right. \]
\[ -2bg^2(\phi')^2 \left( 1 + \frac{1}{2} \ln g + \frac{1}{\beta} \int_x^L dx \frac{e^{-\lambda}}{g} + \ln\left( \frac{\beta}{\mu} \right) \right) \]
\[ -2\frac{e^{-2\lambda}}{\beta} \int_x^L dx \ln g(\phi')^2 - 2g \left( gc'' + gc'\lambda' + \frac{1}{2}g'c' \right) \] (80)

\[ T_x^x = \frac{G e^{2\lambda}}{6g} \left[ (g')^2 - 4 \frac{e^{-2\lambda}}{\beta^2} \right. \]
\[ -2bg^2(\phi')^2 \left( 1 + \frac{1}{2} \ln g - \frac{1}{\beta} \int_x^L dx \frac{e^{-\lambda}}{g} - \ln\left( \frac{\beta}{\mu} \right) \right) \]
\[ +2\frac{e^{-2\lambda}}{\beta} \int_x^L dx \ln g(\phi')^2 - gg'c' \] (81)

Substituting these results into Eq.(73) and Eq.(77) gives us the desired explicit expressions for the first order quantum corrected mass \( M(x) = M_{CL} + m(x) \) and metric function \( w(x) \). Integrating and using \( \Omega^2 = e^\lambda \) leads to the results:

\[ M(x) = M_{CL} + \frac{\hbar}{6} \int_x^L dx e^\lambda \left[ 2e^{-\lambda}(e^\lambda g)' - \left( \frac{g'}{2g} + \frac{2e^{-2\lambda}}{\beta^2g} \right) \right. \]
\[ -bg(\phi')^2 \left( 1 + \frac{1}{2} \ln g + \frac{1}{\beta} \int_x^L dy \frac{e^{-\lambda}(y)}{g(y)} + \ln\left( \frac{\beta}{\mu} \right) \right) \]
\[ -\frac{e^{-2\lambda}}{\beta g} \int_x^L dy \ln g(\phi'(y))^2 - \left( gc'' + gc'\lambda' + \frac{1}{2}g'c' \right) \] (82)
Here we have imposed the condition \( w(L) = 0 \) and have absorbed the lower limit of Eq. (82) into the constant \( M_{CL} \).

Also of importance (particularly for the evaluation of the quantum corrected entropy) is evaluation of the first order quantum shift in the horizon and hence in the horizon value of the dilaton field. To this purpose we define \( \Delta \phi^+ = \phi^+ - \phi_{+CL} \) where \( \phi^+ \) and \( \phi_{+CL} \) are the quantum corrected and classical horizon values of the dilaton field respectively. Because the norm of the killing vector (Eq. (16)) must vanish at the horizon it follows that the above fields must satisfy:

\[
\begin{align*}
  j(\phi^+) - 2M_{CL}Gl - 2m(\phi^+)Gl &= 0 \\
  j(\phi_{+CL}) - 2M_{CL}Gl &= 0
\end{align*}
\]

Expanding \( j(\phi^+) \) about \( \phi_{+CL} \) (to first order) and using Eq. (71) to evaluate the derivative of \( j \) gives:

\[
\begin{align*}
  j(\phi^+) &= j(\phi_{+CL}) + \left. \frac{1}{l} \frac{\tilde{V}(\phi^+)}{\Omega^2(\phi^+)} \right|_{\phi_{+CL}} \Delta \phi^+ \\
  \Delta \phi^+ &= \frac{2G\lambda^2 m(\phi)\Omega^2(\phi)\phi'}{\tilde{V}(\phi)} \bigg|_{\phi_{+CL}}
\end{align*}
\]
4 Quantum Corrections to Black Hole Thermodynamics

Here we calculate the thermodynamical quantities $E = (2\pi)^{-1}\partial_\beta W$ and $S = (\overline{\beta}\partial_\overline{\beta} - 1)W$ for the action functional Eq.(41) which describes the one loop quantum corrected black hole configuration:

$$W = W_{CL}[g] + \hbar \delta W_{CL}|_{g_{CL}} \delta g + \hbar \Gamma_{CL} + O[\hbar^2]$$

(88)

Recall Eq.(38) for the classical action $W_{CL}[g_{CL}]$. This included a term with an integrand proportional to $G_{00}$ and an inner and outer surface term. It is possible and convenient to derive an analogous expression for the quantum effective action $\Gamma$. Rewriting Eq.(44) for $\Gamma$ in terms of the static classical metric Eq.(25), using Eq.(30) to evaluate the extrinsic curvature boundary term and integrating by parts leads to:

$$\Gamma = \pi \beta \frac{1}{6} \int_{x_+}^{L} dx \left[ e^{\lambda_0} \left( \psi + N + c \right) + e^{\lambda_0} \psi' + 4(e^{\lambda_0} g')' + 2 \left( e^{\lambda_0} g (\psi - N) \right)' - 2(\psi c)' \right]$$

(89)

Now recall Eq.(78) for $T^0_0 = T^t_t$. Re-writing this result (making use of definitions of $\Box \psi$ and $\Box N$ and rearranging) yields:

$$T^0_0 = \frac{G e^\lambda}{6} \left[ e^{\lambda_0} \left( \psi + N + c \right) + e^{\lambda_0} \psi' + 4(e^{\lambda_0} g')' - 2(\psi c)' \right]$$

(90)

Using this result to substitute for the integrand in Eq.(89) yields:

$$\Gamma = \frac{\pi \beta}{G} \int_{x_+}^{L} dx \left[ e^{-\lambda_0} T^0_0 - \frac{2G}{3}(e^{\lambda_0} g')' - \frac{G}{3} (e^{\lambda_0} g (\psi - N))' + \frac{G}{3} (\psi c) \right]$$

(91)

Since three of the four terms in the integrand are total derivatives we get:
\[ \Gamma = \frac{\pi \beta}{G} \int_{x_+}^{L} dx e^{-\lambda T_0} \left( -\frac{2\pi \beta}{3} e^{\lambda(L)} g'(L) - \frac{\pi \beta}{3} e^{\lambda(L)} g(L) (\psi'(L) - N'(L) - c'(L)) + \frac{\pi}{3} (\psi(x_+) + N(x_+) + c(x_+)) \right) \]  

where we have used \( g'(x_+) e^{\lambda(x_+)} = 2/\beta \) and discarded the irrelevant constant term which results. When we combine this result for \( \Gamma \) with the first order quantum corrected form for \( W_{CL} \) into Eq.(88) we obtain an integral with integrand proportional to \( G_0^0(g) + \bar{h} T_0^0(g_{CL}) \) along with boundary terms. Since the integrand vanishes on shell according to the field equation (Eq.(62)) we are left with only surface contributions to \( W_{on \ shell} \). These are found to be

\[ W_{on \ shell} = -2\pi \left[ \frac{\beta}{G} e^{\lambda(L)} g(L) D'(\phi(L)) + \frac{D}{G} (\phi(x_+)) + \frac{\hbar}{3} e^{\lambda(L)} g_{CL}'(L) \right. \]

\[ \left. + \frac{\hbar}{6} e^{\lambda(L)} g_{CL}(L) (\psi'(L) - N'(L) - c'(L)) - \frac{g}{6} (\psi(x_+) + N(x_+) + c_{CL}(x_+)) \right] \]  

(93)

where the surface contributions from \( W_{CL} \) are obtained by generalizing Eq.(38). Note that \( g(x) \) and \( \phi(x_+) \) in the first two terms refer to the quantum corrected solutions whereas the remaining terms are defined with respect to classical geometry. Evaluation of thermodynamic quantities is then straightforward giving:

\[ E = -\frac{e^{\lambda(L)}}{G} g^{\frac{3}{2}}(L) D'(\phi(L)) - \frac{\hbar}{3} e^{\lambda(L)} g_{CL}' \frac{4}{3}(L) g_{CL}(L) \]

\[ + \frac{\hbar}{6} e^{\lambda(L)} g_{CL}^\frac{1}{3}(L) c'(L) \]  

(94)

\[ E_{sub} = E(g; g_{CL}) - E(g_0; g_{0CL}) \]  

(95)

\[ S = \frac{2\pi}{G} D(\phi(x_+)) - \frac{\hbar}{6} 2\pi [2\psi(x_+) + c_{CL}(x_+)] + \int_{x_+}^{L} dx \frac{e^{-\lambda(x)}}{g_{CL}(x)} \int_{x}^{L} d\bar{x} e^{\lambda(\bar{x})} \phi'(\bar{x}) g(\bar{x}) \]  

(96)
Where \( g_0 \) and \( g_{0\text{CL}} \) represent the background geometry and are the metric fields evaluated at \( x = L \to \infty \). Note that the left-most terms in the expressions for energy and entropy have classical forms but have implied quantum corrections due to geometry. On the other hand, the remaining terms all vanish in the classical (\( \hbar \to 0 \)) limit.

5 Quantum Corrections in Spherically Symmetric Reduced Gravity Theory

Next we want to use the preceding formalism to examine a specific theory. Here we consider the form of action obtained from the spherically symmetric reduction of 4-dimensional Einstein-Maxwell gravity to a 2- dimensional dilaton model \[30\]. We will specifically examine the minimal case \( b = c = 0 \) so that \( N = \psi \). This case in particular was studied by Frolov et al. \[17\] and we find our results to be in agreement.

We proceed by considering an effective action of the following form (Note that we neglect writing the extrinsic curvature term for sake of brevity but its inclusion is implied.):

\[
W_{\text{CL}} = -\frac{1}{2G l^2} \int d^2 x \sqrt{g} \left[ \frac{r^2}{2} R(g) + (\nabla r)^2 + \left( 1 - \frac{Q^2}{r^2} \right) \right]
\]

Where \( G l^2 = G^{(4)} \) is the square of the 3+1 dimensional Planck length and where the “effective” charge \( Q \) has dimensions of length. Comparison with the form of the classical action (Eq.(14)) leads to the following identifications:

\[
\phi = \frac{\sqrt{2} r}{l} \tag{98}
\]

\[
D(\phi) = D(r) = \frac{r^2}{2l^2} \tag{99}
\]

\[
\tilde{V}(\phi, q) = \tilde{V}(r, Q) = 1 - \frac{Q^2}{r^2} \tag{100}
\]

The classical solution Eqs.\[21-23\] can then be expressed:

\[
x = \frac{r^2}{2l} \tag{101}
\]
\[ \Omega^2(r) = e^{\lambda(r)} = \frac{r}{L} \]  
(102)

\[ \overline{j} \left( \frac{x}{L} \right) = \overline{j}(r) = \frac{r}{L} \left( 1 + \frac{Q^2}{r^2} \right) \]  
(103)

\[ g(r) = 1 - \frac{2Gl^2M}{r} + \frac{Q^2}{r^2} \]  
(104)

Note that the constant of integration in the evaluation of the integral defined in Eq.(11) for \( \Omega^2 \) is selected to be \(-2\ln \sqrt{2}\). This yields a metric function \( g(r) \) which goes to 1 as \( r \to \infty \), which is the correct asymptotic behaviour of the metric in spherically symmetric gravity. For subsequent calculations it will often be convenient to express the metric function \( g(r) \) in the following form (Here we consider solutions only for which two real, distinct horizons exist, i.e. \( (l^2GM)^2 > Q^2 \cdot \))

\[ g(r) = \frac{1}{r^2}(r - r_+)(r - r_-) \]  
(105)

where \( r_\pm \) represents the outer(+) and inner (-) horizons given by:

\[ r_\pm = l^2GM \pm \sqrt{(l^2GM)^2 - Q^2} \]  
(106)

Before proceeding to evaluate the quantum corrected quantities, we consider the classical energy and entropy. The classical energy (Eq.(39)) in terms of \( r \) for this theory becomes:

\[ E = -\frac{1}{G}e^{\lambda(r=L)}g^{\frac{1}{2}}(r = L) \frac{dD(r)}{dr} \bigg|_{r=L} \frac{dr}{dx} \]

\[ = -\frac{L}{Gl^2} \sqrt{1 - \frac{2Gl^2M}{L} + \frac{Q^2}{L^2}} \]  
(107)

Above and for the remainder of this section \( L \) is taken to be the value of \( r \) at the outer boundary. Clearly this energy is divergent as \( L \to \infty \). Hence we apply the standard subtraction procedure as defined by Eq.(40) to give:

\[ E_{sub} = \frac{L}{Gl^2} \left( 1 - \sqrt{1 - \frac{2Gl^2M}{L} + \frac{Q^2}{L^2}} \right) \]  
(108)
Taking the asymptotic ($L \to \infty$) limit yields the expected result $\lim_{L \to \infty} (E_{\text{sub}}) = M$. The classical entropy (Eq.(34)) is:

$$S = \frac{\pi r^2}{Gl^2} \quad (109)$$

Next we calculate the quantum corrected quantities in spherically symmetric theory. Recall we consider the minimal case $b = c = 0$ so that $N = \psi$. To avoid confusion classical-specific quantities will be labelled with the subscript "CL". Re-writing Eq.(82) for quantum corrected mass $M(x)$ in terms of $r$ and substituting Eq.(105) for the classical metric gives us:

$$M(r) = M_{CL} + \frac{\hbar}{6} \int^r dr \left( \frac{d^2 g_{CL}}{dr^2} - \frac{(\frac{dg_{CL}}{dr})^2}{2g_{CL}} + \frac{2}{\beta_{CL}^2 g_{CL}} \right)$$

$$= M_{CL} + \frac{\hbar}{6} \int^r dr \left[ \frac{10}{r^4} (r-r_{+CL})(r-r_{-CL}) - \frac{6}{r^3} (2r-r_{+CL} - r_{-CL}) \right]$$

$$+ \frac{3}{r^2} - \frac{(r-r_{-CL})}{2r^2(r-r_{+CL})} - \frac{(r-r_{+CL})}{2r^2(r-r_{-CL})}$$

$$+ \frac{2}{(r-r_{+CL})(r-r_{-CL})} \beta_{CL}^2 \right] \quad (110)$$

Integrating and using Eq.(28)

$$\beta_{CL} = \left. \frac{2e^{-\lambda(r)}}{(\frac{dg_{CL}}{dr})(\frac{dr}{dx})} \right|_{r=r_{+CL}} = \frac{2r_{+CL}^2}{(r_{+CL} - r_{-CL})} \quad (111)$$

gives us the following

$$M(r) = M_{CL} + \frac{\hbar}{6} [A \ln(r - r_{-CL}) + B \ln(r) + C(r)] \quad (112)$$

where:

$$A = \frac{(r_{+CL} - r_{-CL})^2 (r_{+CL} + r_{-CL})(r_{+CL}^2 + r_{-CL}^2)}{2r_{+CL}^4 r_{-CL}^2} \quad (113)$$

$$B = -\frac{(r_{+CL} - r_{-CL})^2 (r_{+CL} + r_{-CL})}{2r_{+CL}^2 r_{-CL}^2} \quad (114)$$
C(r) = \frac{2r}{\beta^2_{CL}} + \frac{(r_{+CL} - r_{-CL})^2}{2rr_{+CL}r_{-CL}} + 2\left(\frac{r_{+CL} + r_{-CL}}{r^2} - \frac{10r_{+CL}r_{-CL}}{3r^3}\right) \tag{115}

Note that \(A + B = 4M_{CL}Gl^2/\beta^2_{CL}\). Consider the quantum corrected mass for some special cases. For \(M(r = L)\) for large \(L\) then \(\ln(L - r_{-CL}) \sim \ln(L)\) and using the prior property for \(A\) and \(B\) gives:

\[M(L) \sim M_{CL} + \frac{\hbar}{3\beta^2_{CL}}[L + 2M_{CL}Gl^2 \ln(L)] \tag{116}\]

Also consider the case of an uncharged black hole. The classical metric function becomes \(g_{CL} = (1 - r_{+CL}/r)\) where \(r_{+CL} = 2M_{CL}Gl^2\) and \(\beta_{CL} = 2r_{+CL}\). An analogous calculation to that presented above then gives for the uncharged case:

\[M(r) = M_{CL} + \frac{\hbar}{12} \left[ \frac{r}{r^2_{+CL}} + \frac{7r_{+CL}}{2r^2} - \frac{1}{r} + \frac{\ln(r)}{r_{+CL}} \right] \tag{117}\]

For the case of an extremal black hole \(r_{-CL} \to r_{+CL}\) and \(\beta_{CL} \to \infty\). Consequently \(A\) and \(B\) vanish and \(C(r)\) reduces to:

\[C(r) = 4r_{+CL}/r^2 - 10r^2_{+CL}/3r^3 \tag{118}\]

Next consider the metric function \(w(x)\). Re-writing Eq.(113) in terms of \(r\) and then substituting Eq.(103) for the classical metric, using Eq.(111) for the inverse asymptotic temperature and finally integrating gives the result

\[w(r) = \frac{\hbar Gl^2}{6} (F(L) - F(r)) \tag{119}\]

where:

\[F(r) = -\frac{\left[3r^2_{+CL} + 2r_{+CL}r_{-CL} + 3r^2_{-CL}\right]}{r^2_{+CL}r^2_{-CL}} \ln(r) \]

\[+ \frac{\left[3r^4_{+CL} + 2r^3_{+CL}r_{-CL} + 2r^2_{+CL}r^2_{-CL} + 2r_{+CL}r^3_{-CL} - r^4_{-CL}\right]}{r^4_{+CL}r^2_{-CL}} \ln(r - r_{-CL}) \]

\[+ \frac{4}{r^2} + \frac{4(r_{+CL} + r_{-CL})}{rr_{+CL}r_{-CL}} - \frac{(r^4_{+CL} - r^4_{-CL})}{r^4_{+CL}r_{-CL}(r - r_{-CL})} \tag{120}\]
As for the quantum corrected mass we consider some special cases. For an uncharged black hole the function $F(r)$ takes the simpler form:

$$F(r) = \frac{3}{2r^2} + \frac{2}{rr_{+CL}} - \frac{1}{r^2_{+CL}} \ln(r)$$  \hspace{1cm} (121)$$

If $L$ is large, we can write:

$$e^{2w(r)} \sim \left( \frac{r}{L} \right)^{\frac{3hG^2}{r^2_{+CL}}} \exp \left( -\frac{hGl^2}{3} \left( \frac{3}{2r^2} + \frac{2}{rr_{+CL}} \right) \right)$$  \hspace{1cm} (122)$$

In the extremal black hole limit the function $F(r)$ reduces to:

$$F(r) = \frac{8}{r^2_{+CL}} \ln\left( \frac{r-r_{+CL}}{r} \right) + \frac{4}{r^2} + \frac{8}{rr_{+CL}}$$  \hspace{1cm} (123)$$

Consequently at the extremal black hole horizon $F(r_+) \to -\infty$ so that $e^{2w(r_+)} \to \infty$.

Next we examine the quantum corrected energy. Revising Eq.(94) for reduced spherically symmetric gravity:

$$E = -\frac{L}{Gl^2} \sqrt{1 - \frac{2Gl^2M(L)}{L} + \frac{Q^2}{L^2}}$$

$$-\frac{h}{3} L^2 \left( 2Gl^2 M_{CL} - \frac{2Q^2}{L} \right) \sqrt{1 - \frac{2Gl^2 M_{CL}}{L} + \frac{Q^2}{L^2}}$$  \hspace{1cm} (124)$$

Where $M(r = L)$ is given by Eq.(117). Consider the case of large box size $L$. Clearly the second part of the expression is small relative to the first. So we consider the first term only and substitute for the quantum corrected mass (for large $r = L$) by way of Eq.(116):

$$E \sim -\frac{L}{Gl^2} \sqrt{1 - \frac{2Gl^2 M_{CL}}{L} - \frac{2hGl^2}{3\beta^2_{CL}} - \frac{4h(Gl^2)^2 M_{CL}}{3\beta^2_{CL}L} \ln(L) + \frac{Q^2}{L^2}}$$  \hspace{1cm} (125)$$

As in the calculation of classical energy we again apply the standard subtraction procedure of comparing the divergent quantity with that of a background
defined by the metric $g_0 = \lim_{L \to \infty} g(r = L)$. Since $g(L)$ for large $L$ is the quantity inside the square root sign in Eq.(125) it follows that $g_0 = 1 - \frac{2hG_l^2}{3\beta_{CL}}$ and since $E[g_0] = -\frac{L \dot{g}}{G_l^2}$ the subtracted energy is given by:

$$E_{sub} \sim \frac{L}{G_l^2} \left[ \sqrt{1 - \frac{2hG_l^2}{3\beta_{CL}^2}} - \sqrt{1 - \frac{2G_l^2M_{CL}}{L}} - \frac{2hG_l^2}{3\beta_{CL}^2} - \frac{4h(G_l^2)^2M_{CL} \ln(L)}{3\beta_{CL}^2L} + \frac{Q^2}{L^2} \right]$$

(126)

The approach we use here is to first fix $L$ and expand the square roots with respect to the perturbative factor $\hbar$. Then we take $L$ to be large and expand with respect to $\frac{1}{L}$. Then eliminating all $O\left(\frac{1}{L^2}\right)$ terms inside the square brackets leaves:

$$E_{sub} \sim M_{CL} + \frac{hG_l^2M_{CL}}{3\beta_{CL}^2} (2 \ln(L) + 1)$$

(127)

Note that the first order quantum correction to the energy can be attributed to temperature effects since $\beta_{CL}$ represents the asymptotic inverse temperature.

Next consider the quantum corrected value of the horizon radius $r_+$. For this purpose we define $\Delta r_+ = r_+ - r_{+CL}$ and as previously defined $m(r) = M(r) - M_{CL}$. The quantum corrected metric must vanish at $r = r_+$ and this relation can be expressed as follows:

$$0 = g(r_+) = g_{CL}(r_+) - \frac{2G_l^2m(r_+)}{r_+}$$

(128)

Expanding to first order about $r_+ = r_{+CL}$ and using $g_{CL}(r_{+CL}) = 0$ and expressing $(dg_{CL}/dr)_{r=r_{+CL}}$ in terms of $\beta_{CL}$ (Eq.(111)) gives:

$$\Delta r_+ = \frac{\beta_{CL} G_l^2 m(r_{+CL})}{r_{+CL}}$$

(129)

Note that $m(r_{+CL})$ contains a factor of $\hbar$ (see Eq.(82)). From this result we can calculate the quantum correction to the horizon area which is proportional to $r_+^2 = (r_{+CL} + \Delta r_+)^2$ and hence to first order:
Finally in this section we evaluate the quantum correction to the entropy. For this theory the entropy (Eq.(96)) is:

$$S = \frac{\pi r_+^2}{Gl^2} - \frac{2\pi}{3} \psi(r_{+CL})$$

(131)

Making use of the preceding result for $r_+^2$ (Eq.(130)) we can write:

$$S = S_{CL} + 2\pi \beta_{CL} m(r_{+CL}) - \frac{\hbar}{3} \psi(r_{+CL})$$

(132)

Revising Eq.(17) for this theory gives us:

$$\psi(r_{+CL}) = -\ln g_{CL}(r_{+CL}) - \frac{2}{\beta_{CL}} \int_{r_+}^L \frac{dr}{g_{CL}(r)} - 2 \ln \left( \frac{\beta_{CL}}{z_0} \right)$$

(133)

Using Eq.(103) for $g_{CL}$, Eq.(111) for $\beta_{CL}$, integrating the middle term, and simplification yields:

$$\psi(r_{+CL}) = \frac{r_{-CL}^2}{r_{+CL}^2} \ln \left( \frac{L - r_{-CL}}{r_{+CL} - r_{-CL}} \right) - \ln \left( \frac{L - r_{+CL}}{r_{+CL} - r_{-CL}} \right)$$

$$- \frac{(r_{+CL} - r_{-CL})}{r_{+CL}^2} (L - r_{+CL}) - 2 \ln \left( \frac{r_{+CL}}{z_0} \right)$$

(134)

The third term in $\psi(r_{+CL})$ can be interpreted [17] as the contribution to the entropy of a two-dimensional hot gas in a box size $L - r_{+CL}$ and temperature $(2\pi \beta_{CL})^{-1} = (r_{+CL} - r_{-CL})/4\pi r_{+CL}^2$. Hence we subtract off this contribution to obtain the quantum corrected black hole entropy:

$$S = S_{CL} + 2\pi \beta_{CL} m(r_{+CL}) - \frac{\hbar}{3} \ln \left( \frac{L - r_{-CL}}{r_{+CL} - r_{-CL}} \right)$$

$$+ \frac{\hbar}{3} \ln \left( \frac{L - r_{+CL}}{r_{+CL} - r_{-CL}} \right) + \frac{4\pi}{3} \ln \left( \frac{r_{+CL}}{z_0} \right)$$

(135)

We next consider some special cases. In the case of large box size $L$ the entropy reduces to:

$$S \sim S_{CL} + 2\pi \beta_{CL} m(r_{+CL}) + \frac{2\pi}{3} \left( 1 - \frac{r_{-CL}^2}{r_{+CL}^2} \right) \ln \left( \frac{L}{r_{+CL} - r_{-CL}} \right) + \frac{4\pi}{3} \ln \left( \frac{r_{+CL}}{z_0} \right)$$

(136)
For an uncharged black hole then

\[ S = S_{CL} + 2\pi \beta_{CL} m(r_{+CL}) + \frac{2\pi}{3} \ln \left( \frac{L r_{+CL}}{z_0^2} \right) \]  

(137)

where \( m(r_{+CL}) \) can be evaluated using Eq.(117). Finally, in the extremal black hole limit

\[ S = S_{CL} + 2\pi \beta_{CL} m(r_{+CL}) + \frac{2\pi}{3} \ln \left( \frac{r_{+CL}^2}{z_0^2} \right) \]

(138)

where \( m(r_{+CL}) = \hbar/9r_{+CL} \) in the extremal case however \( \beta_{CL} \to \infty \) so the entropy is divergent in this limit.

6 Quantum Corrections in Jackiw-Teitelboim Theory

In this section we examine the Achucarro-Ortiz black hole [11], which is a solution to the field equations for Jackiw-Teitelboim gravity[31]. This theory can be obtained by imposing axial symmetry in 2+1 dimensional gravity, so that the Achucarro- Ortiz black hole corresponds to the projection of the BTZ axially symmetric black hole [32] into 1+1-dimensional spacetime. The Jackiw-Teitelboim field equations can be derived from an effective action of the form [11]

\[ W_{CL} = - \int d^2x \sqrt{g} \Lambda^\frac{1}{2} \left[ rR(g) + \Lambda r - \frac{J^2}{2r^3} \right] \]

(139)

where \( \Lambda \) is the cosmological constant (dimension length\(^{-2} \)) and \( J \) is an “effective charge” (dimension length) which describes the angular momentum of the BTZ black hole. Note that there is no kinetic term in this action so it is of the form (Eq.(14)) without the need for a field reparametrization. This leads to the following identification (provided we set \( 2G = 1 \) and \( l = \Lambda^{-\frac{1}{2}} \)):

\[ \overline{\phi} = \Lambda^\frac{1}{2} r \]

(140)

\[ D(\overline{\phi}) = D(r) = \Lambda^\frac{1}{2} r \]

(141)
\[ \tilde{V}(\phi, q) = \tilde{V}(r, J) = \Lambda_{\Lambda}^{\frac{1}{2}} (r - \frac{J^2}{2\Lambda r^3}) \]  

(142)

The classical solution Eqs.\[18–20\] can then be expressed:

\[ x = r \]  

(143)

\[ \tilde{j}(\phi) = \tilde{j}(r) = \frac{\Lambda r^2}{2} + \frac{J^2}{4r^2} \]  

(144)

\[ g(r) = \frac{\Lambda r^2}{2} - \frac{M}{\Lambda^2} + \frac{J^2}{4r^2} \]  

(145)

Here we consider only solutions for which two real, distinct horizons exist (i.e. \( M^2 > \Lambda^2 J^2 / 2 \)) so it will prove convenient to express the metric Eq.(145) in the following form

\[ g(r) = \frac{\Lambda}{2} \left( r^2 - r_+^2 \right) \left( r^2 - r_-^2 \right) \]  

(146)

where \( r_+ \) and \( r_- \) represent the inner and outer horizons, respectively, given by:

\[ r_+^2 = \frac{1}{\Lambda^{3/2}} \left[ M \pm \sqrt{M^2 - \Lambda^2 J^2 / 2} \right] \]  

(147)

Note that because the action is already in reparameterized form we set \( \Omega^2 = 1 \) (or equivalently \( \lambda = 0 \)) in the previously derived results.

The classical, subtracted energy for this theory, as described by Eq.(40), is given by:

\[ E_{\text{sub}} = 2 \frac{\Lambda}{2} L \left( 1 - \sqrt{1 - \frac{2M}{\Lambda^2 L^2} + \frac{J^2}{2\Lambda L^4}} \right) \]  

(148)

while the classical entropy (Eq.(34)) is:

\[ S = 4\pi \Lambda^{\frac{1}{2}} r_+ \]  

(149)

Next we calculate the quantum corrected quantities in Jackiw-Teitelboim theory with minimal coupling (\( b = c = 0 \)) as for SSG in the prior
section. Henceforth, purely classical quantities will be labelled with the subscript “CL.” Eq.(82) for the quantum corrected mass $M(x = r)$ gives us (substituting for classical metric Eq.(146)):

$$M(r) = M_{CL} + \frac{\Lambda h}{6} \int dr \left[ \frac{4}{r^2} (r_{+CL}^2 + r_{-CL}^2) ight.$$

$$+ \frac{5}{r^4} (r^2 - r_{+CL}^2)(r^2 - r_{-CL}^2) - \frac{(2r^2 - r_{+CL}^2 - r_{-CL}^2)}{(r^2 - r_{+CL}^2)}$$

$$- \frac{(2r^2 - r_{+CL}^2 - r_{-CL}^2)}{(r^2 - r_{-CL}^2)} + \frac{4\Lambda^{-2}r^2}{\beta_{CL}^2 (r^2 - r_{+CL}^2)(r^2 - r_{-CL}^2)} \right]$$

Integrating and using via Eq.(28)

$$\beta_{CL} = \left. \frac{2}{(\frac{dg_{CL}}{dr})} \right|_{r=r_{+CL}} = \frac{2}{\Lambda r_{+CL} \left( 1 - \frac{r_{-CL}^2}{r_{+CL}^2} \right)}$$



(150)

(151)

gives us the following result

$$M(r) = M_{CL} + \frac{h\Lambda}{G} \left[ A (\ln(r - r_{-CL}) - \ln(r + r_{-CL})) + B(r) \right]$$

where:

$$A = \frac{r_{+CL}^6 - 3r_{+CL}^4 r_{-CL}^2 + 3r_{-CL}^2 r_{+CL}^4 - r_{-CL}^6}{2r_{+CL}^2 r_{-CL}(r_{+CL}^2 - r_{-CL}^2)}$$

$$B(r) = r + \frac{r_{+CL}^2}{r} + \frac{r_{-CL}^2}{r} - \frac{5r_{+CL}^2 r_{-CL}^2}{3r^3}$$

(153)

(154)

Next consider the quantum corrected mass for some special cases. For $M(r = L)$ and large $L$ then coefficient of $A \sim 0$ and we are left with:

$$M(L) \sim M_{CL} + \frac{h\Lambda}{6L} \left[ 1 + \frac{r_{+CL}^2 + r_{-CL}^2}{L^2} \right]$$

(155)

Also we consider the “chargeless” case. The classical metric function becomes $g_{CL} = \frac{A}{2}(r^2 - r_{+CL}^2)$ where $r_{+CL}^2 = 2M\Lambda^{-3/2}$ and $\beta_{CL} = 2/\Lambda r_{+CL}$. Repeating the above calculation yields:

$$M(r) = M_{CL} + \frac{h\Lambda}{6r}$$

(156)
For the extremal black hole case \( r_{-CL} \to r_{+CL} \) and \( \beta_{CL} \to \infty \). The coefficient of \( A \) vanishes and \( B(r) \) reduces to:

\[
B(r) = r + 2 \frac{r^2_{+CL}}{r} - \frac{5}{3} \frac{r^4_{+CL}}{r^3} \tag{157}
\]

Next consider the metric function \( w(x = r) \). Applying Eq.(83) for this theory and using Eq.(146) for the classical metric and Eq.(151) for the inverse asymptotic temperature gives:

\[
w(r) = \frac{\hbar}{6\Lambda^2} \left( F(L) - F(r) \right) \tag{158}
\]

where:

\[
F(r) = \frac{4}{r} - \frac{r(r^2_{+CL} - r^2_{-CL})}{r^2_{+CL}(r^2 - r^2_{-CL})} + \ln\left( \frac{r - r_{-CL}}{r + r_{-CL}} \right) \left[ \frac{3r^4_{+CL} - 2r^2_{+CL}r^2_{-CL} - r^4_{-CL}}{2r_{-CL}r^2_{+CL}(r^2_{+CL} - r^2_{-CL})} \right] \tag{159}
\]

For the uncharged case we find using the revised metric function discussed above that \( w(r) = 0 \) for all allowable \( r \). Meanwhile for the extremal limit \( F(r) \) reduces to \( 4/r \) except at the horizon where the right most term in Eq.(159) becomes a divergent quantity. Hence in this limit \( F(r_+) \to -\infty \) and as in the preceding section the factor \( e^{2w(r_+)} \) is divergent.

Next we consider the quantum corrected energy. Revising Eq.(94) for Jackiw-Teitelboim theory:

\[
E = -2^{\frac{1}{2}}\Lambda L \left[ \sqrt{1 - \frac{2M(L)}{\Lambda^2 L^2}} + \frac{J^2}{2\Lambda L^2} + \frac{\hbar^2}{3\Lambda^2 L} \left( \frac{1 - \frac{J^2}{2\Lambda L^2}}{1 - \frac{2M_{CL}}{\Lambda^2 L} + \frac{J^2}{2\Lambda L^4}} \right) \right] \tag{160}
\]

Applying the usual background subtraction procedure (Eq.(93)) then gives:

\[
E_{sub} = 2^{\frac{1}{2}}\Lambda L \left[ 1 - \sqrt{1 - \frac{2M(L)}{\Lambda^2 L^2}} + \frac{J^2}{2\Lambda L^2} + \frac{\hbar^2}{3\Lambda^2 L} \left( \frac{1 - \frac{J^2}{2\Lambda L^2}}{1 - \frac{2M_{CL}}{\Lambda^2 L} + \frac{J^2}{2\Lambda L^4}} \right) \right] \tag{161}
\]
In the case of large box size \( L \) the second part vanishes relative to the first. Consequently, for large \( L \) the primary contribution to the quantum shift in energy is a result of the shift in mass as described by Eq.(153). So to first order in \( \hbar \) and to zero’th order in \( \frac{1}{L} \) we find:

\[
E_{\text{sub}} \sim (E_{\text{sub}})_{\text{CL}} + \frac{2\frac{7}{2} h \Lambda^{\frac{7}{2}}}{6}
\]  

(162)

Following the procedure for calculating the quantum correction to the horizon radius \( \Delta r_+ = r_+ - r_{+\text{CL}} \) which was introduced in the previous section we find

\[
\Delta r_+ = \frac{\beta_{\text{CL}} m(r_{+\text{CL}})}{2\Lambda^{\frac{7}{2}}}
\]  

(163)

where \( m(r) = M(r) - M_{\text{CL}} \) is given by Eq.(152). Furthermore the first order quantum correction to the horizon area can be obtained from:

\[
r_+^2 = r_{+\text{CL}}^2 + \frac{\beta_{\text{CL}} r_{+\text{CL}} M(r_{+\text{CL}})}{\Lambda^{\frac{7}{2}}}
\]  

(164)

Finally, in this section we determine the quantum correction to entropy. For Jackiw-Teitelboim theory the dilaton generic entropy (Eq.(96)) becomes:

\[
S = 4\pi \Lambda^{\frac{7}{2}} r_+ - \frac{\hbar 2\pi}{3} \psi(r_{+\text{CL}})
\]  

(165)

Making use of the preceding result \( \Delta r_+ \) (Eq.(163)):

\[
S = S_{\text{CL}} + 2\pi \beta_{\text{CL}} m(r_{+\text{CL}}) - \hbar \frac{2\pi}{3} \psi(r_{+\text{CL}})
\]  

(166)

From (Eq.(57)) we get:

\[
\psi(r_{+\text{CL}}) = -\ln g_{\text{CL}}(r_{+\text{CL}}) - \frac{2}{\beta_{\text{CL}}} \int_{r_{+\text{CL}}}^L \frac{dr}{g_{\text{CL}}(r)} - 2\ln \left( \frac{\beta_{\text{CL}}}{z_0} \right)
\]  

(167)

Using Eq.(146) for \( g_{\text{CL}} \), Eq.(151) for \( \beta_{\text{CL}} \), integrating the middle term and simplifying yields:

\[
\psi(r_{+\text{CL}}) = -\frac{r_{-\text{CL}}}{r_{+\text{CL}}} \ln \left[ \frac{(r_{+\text{CL}} - r_{-\text{CL}})(L + r_{-\text{CL}})}{(r_{+\text{CL}} + r_{-\text{CL}})(L - r_{-\text{CL}})} \right] + \ln \left( \frac{r_{+\text{CL}}^2 - r_{-\text{CL}}^2}{r_{+\text{CL}}^2} \right) - \ln \left( \frac{L - r_{+\text{CL}}}{L + r_{+\text{CL}}} \right) + \ln \left( \frac{\Lambda z_0^2}{8} \right)
\]  

(168)
So the complete quantum corrected entropy is obtained by substituting Eq. (168) for \( \psi(r_{+CL}) \) and \( m(r_{+CL}) \) via Eq. (152) back into Eq. (166). For large \( L \) this result reduces to (subtracting off the constant term):

\[
S = S_{CL} + 2\pi \beta_{CL}m(r_{+CL}) + \frac{\hbar 2\pi r_{-CL}}{3} \ln \left( \frac{r_{+CL} - r_{-CL}}{r_{+CL} + r_{-CL}} \right) \\
- \frac{\hbar 2\pi}{3} \ln \left( \frac{r_{+CL}^2 - r_{-CL}^2}{r_{+CL}^2} \right)
\]

(169)

For an uncharged black hole then the entropy is given by:

\[
S = S_{CL} + \hbar \frac{2\pi}{3} \ln \left( \frac{L - r_{+CL}}{L + r_{+CL}} \right)
\]

(170)

Note that using Eq. (156) the \( m(r_{+CL}) \) term reduces to a constant which we subtract off. Finally, in the extremal black hole limit

\[
S = S_{CL} + 2\pi \beta_{CL}m(r_{+CL})
\]

(171)

where \( m(r_{+CL}) = \frac{2\hbar}{\beta_{CL}}r_{+CL} \) in the extremal case however \( \beta_{CL} \to \infty \) so the entropy is divergent in this limit.

7 Conclusions

We have calculated the one-loop quantum corrections for generic dilaton gravity coupled to an Abelian gauge field. Both corrections to the black hole geometry and black hole thermodynamics were studied in detail. We then applied our generic results to the special cases of charged black holes in spherically symmetric gravity and rotating BTZ black holes. The former case enabled us to verify our results by comparison with the tree-level calculations of Braden et al. [15] and the one-loop corrections of Frolov et al. [17]. Study of BTZ black holes is of particular interest due to recent revelations of a possible connection between string inspired black holes and BTZ geometry [33].

Although our quantum corrected results can in principle be integrated exactly, numerical analysis will be required for rigorous study of particular theories. Such an analysis is in progress. Our hope is that ultimately such
studies will lead to a better understanding of quantum thermodynamical pro-
cesses associated with black holes and hence insight into the deep mysteries
surrounding quantum gravity.

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