Abstract

In this paper we show that the existence of a primarily discrete space-time may be a fruitful assumption from which we may develop a new approach of statistical thermodynamics in pre-relativistic conditions. The discreetness of space-time structure is determined by a condition that mimics the Heisenberg uncertainty relations and the motion in this space-time model is chosen as simple as possible. From these two assumptions we define a path-entropy that measures the number of closed paths associated with a given energy of the system preparation. This entropy has a dynamical character and depends on the time interval on which we count the paths. We show that it exists an like-equilibrium condition for which the path-entropy corresponds exactly to the usual thermodynamic entropy and, more generally, the usual statistical thermodynamics is reobtained. This result derived without using the Gibbs ensemble method shows that the standard thermodynamics is consistent with a motion that is time-irreversible at a microscopic level. From this change of paradigm it becomes easy to derive a $H$–theorem. A comparison with the traditional Boltzmann approach is presented. We also show how our approach can be implemented in order to describe reversible processes. By considering a process defined simultaneously by initial and final conditions
a well defined stochastic process is introduced and we are able to derive a
Schrödinger equation, an example of time reversible equation.

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I. INTRODUCTION

Despite the large successes obtained in statistical physics during the last century we
have not been able to derive the second law of thermodynamics from statistical mechanics
although a large number of works have been devoted to this problem (for a review see for in-
stance (1)). In the Clausius version, the second law asserts the existence of a state function,
the entropy, that is a non-decreasing function of time for any closed system (2). The diffi-
culty in the derivation of this law is the inadequacy between the postulated time-reversible
behavior for the motion of particles at a microscopic level and the observable irreversible
behavior of macroscopic systems. In classical mechanics, we have to deal with fundamen-
tal questions as the existence of the Poincaré’s recurrence time (1). In quantum physics,
Landau and Lifshitz (3) suggested that the origin of irreversibility might be related to the
measurement process that introduces a difference between past and future. Another kind
of approach consists to associate the thermodynamic arrow of time with the cosmological
expansion of the universe (see for instance (4)).

Since the problem of irreversibility resists to any demonstration by more than one century
it is tempting to investigate this problem starting from a new point of view. In a discussion
about the foundations of statistical thermodynamics in terms of path integrals Feynman (5)
developed a given number of fundamental remarks leading to ask whether the existence of
the hamiltonian is needed to formulate classical statistical mechanics i.e. the classical limit
of quantum statistical physics. In parallel, a possible new foundation of statistical mechanics
directly in terms of path integral was conjectured. The elaborations of a statistical physics
without hamiltonian represents a new field of investigation (6). If we adopt this point of
view we have no more to fight with the Poincaré’s recurrence time (1) in classical physics but in quantum physics the Schrödinger equation can not be used as a starting point and the canonical form of the density matrix must be abandoned. Consequently new physical ingredients have to be introduced but where to find them?

We know that the description of the universe at the Planck’s scale requires a deep modification of our usual physical concepts. For instance, in this domain a promising attempt consists to replace point particles by strings or to use a discrete space-time structure instead of the differentiable manifold of the general relativity ... (7), (8). It is very tempting to see if such new postulates may also change our description of the world at a scale much larger than the Planck’s one and even in pre-relativistic conditions. It is interesting to note that the combination of gravitation and quantum physics has led to develop new thermodynamic concepts as the holographic principle (9) or the existence of a geometric entropy connected with the properties of the quantum vacuum. Such an entropy represents a very important discovery that leads to describe fundamental thermodynamic laws from vacuum fluctuations without any reference to a Gibbs-ensemble description (10).

This paper represents an attempt in which we want to show that the existence of a primarily discrete space-time may be a useful point of view from which we may develop a new approach of statistical thermodynamics.

Our approach is build up on several steps:

• 1. We assume the existence of a discrete space-time for which the structure is determined by relations that mimic the Heisenberg uncertainty relations. Fortunately, to describe usual thermodynamics the continuous limit of this space-time model is sufficient.

• 2. We assume that the motion of a particle in space-time is as simple as possible. This motion can be characterized by a real-valued function describing the transition from one space-time point to another.

• 3. Amongst all the quantities that we can introduce to characterize the space-time
structure and the associated dynamics we choose one of them that is similar to the standard thermal entropy. This path-entropy becomes identical to the thermal one if the energy needed in the system preparation equilibrates the mean energy calculated on the paths. This result obtained without any reference to the Gibbs ensemble method shows that standard thermodynamics is consistent with a motion at a microscopic level that is time-irreversible.

- 4. From the previous points it is easy to derive a H-theorem without any new assumption in the case of free particles. At first glance our derivation of a H-theorem is very different from the Boltzmann one nevertheless some comparisons are possible.

- 5. To be really convincing we must also be able to describe some situations that we may consider as reversible. Our approach is then implemented by the introduction of entry-exit conditions that create a past/future symmetry. Then we may describe the system by a complex valued function that verifies a Schrödinger equation, \( i.e. \) a time-reversible equation in the Wigner sense.

Note that only points 1 and 2 corresponds to real hypothesis.

The paper is organized as follows. In Section 2 we introduce a space-time model and in Section 3 we describe the motion in this space-time. In Section 4 we characterize the space-time by a path-entropy and a path-temperature that mimic the correspondent thermodynamic quantities. In Section 5 we introduce a like-equilibrium condition from which the previous entropy and temperature coincide exactly with the corresponding thermodynamic quantities. In Section 6 a H-theorem is demonstrated in the case of free particles and a comparison with Boltzmann derivation is presented. In the next Section the approach is implemented in order to derive a Schrödinger equation. In the last Section some comments and concluding remarks are presented.
II. MODEL OF A PRIMARILY DISCRETE SPACE-TIME

The choice between a discrete or a continuous version of the space-time structure has already been analyzed by Riemann in the classical world and more than thirty years ago Feynman presented some doubts concerning the continuum nature of space-time in the quantum domain (11). Today it is well accepted that the conventional notions of space and time break down at the Planck’s scale. Due to this various attempts to elaborate a quantum theory based on the existence of a discrete space-time have been proposed (see for instance (8), (12) and (13) for the references quoted therein).

A discrete space-time means that any length is built up from a finite number of the elementary length, Δx, and any time interval results from a series of individual ”ticks” of duration Δt. If a relation between Δx and Δt is expected, its precise form must depend on the accuracy with which we want to describe the world. To establish general aspects of the space-time structure it is possible to start from new uncertainty relations issued from string theory (see for instance (14)) or simple gedanken experiments (see (15) for a short review in this field). These relations show a minimum of uncertainty in the determination of positions that is interpreted as the existence of a minimum, Δx, in the distance between two points while Δt is the minimum time interval needed to characterize two separated points. In the pre-relativistic domain that we consider the velocity of light is assumed to be infinite and there is no gravitational effect accordingly the only one universal constant that we have to consider at the microscopic level is h. Thus, if a mass, m, is located in a region Δx the only one relation that we can introduce between Δx and Δt is (Δx)^2/Δt = h/m. Whatever the values of Δx and Δt we have immediately ΔxΔp = h and ΔtΔE = h/2 provided we use Δp = mΔx/Δt and ΔE = (1/2)m(Δx/Δt)^2. Thus our assumption (Δx)^2/Δt = h/m is equivalent to suppose that the fine structure of space-time is described by relations that mimic the Heisenberg uncertainty relations. This seems a natural choice, it is reasonable to expect that the discrete space-time structure contains some ingredients that are already familiar to us from our study of the world at a larger scale (16), this leads to a new kind of correspondence
principle between a discrete space-time and its continuum limit (17).

However the relation \((\Delta x)^2/\Delta t = \hbar/m\) or the uncertainty relations do not fix the precise values of \(\Delta x\) and \(\Delta t\). If we want to avoid particle creation by quantum fluctuations, \(\Delta x\) must be larger than the Compton wavelength \((\hbar/mc)\) and \(\Delta t \gg \hbar/mc^2\) where \(c\) is the velocity of light; thus, in a pre-relativistic world \((c \to \infty)\) there is no limitation from below for \(\Delta x\) and \(\Delta t\). Hence we are free to assume that both \(\Delta x\) and \(\Delta t\) tend to zero provided we keep the relation \((\Delta x)^2/\Delta t = \hbar/m\) in this limit. This is the starting point of our approach.

III. DYNAMICS IN SPACE-TIME

Although this is probably not needed, in this first attempt, we assume that the space-time points \((t_i, x_i)\) are located on the sites of a regular lattice, as in the chessboard problem investigated in (5). Here we assume that the motion is as simple as possible. A particle may jump, at random, from one site to one of its nearest neighbors. Thus, by definition, a path corresponds to a set of sites \((t_i, x_i)\); the values of \(t_i\) are such as \(t_{i+1} > t_i\) whatever \(i\) and the coordinate positions, \(x_{i+1}\) is necessarily one of the nearest neighbors of \(x_i\). The conditions over \(\Delta x\) and \(\Delta t\) defined above lead, in the limit \(\Delta x, \Delta t \to 0\), to a continuous diffusion process (18) for which the diffusion coefficient is \(D = \hbar/2m\). This diffusion process has a pure quantum origin, \(D \to 0\) if \(\hbar \to 0\). The random walk can be characterized by a real-valued continuous function, \(q_0(t_0, x_0; t, x)\) representing the density of transition probability to go from \((t_0, x_0)\) to \((t, x)\) when \(t \geq t_0\). From \(q_0(t_0, x_0; t, x)\) and a function \(\phi_0(x)\) defined for \(t = t_0\) we form the function \(\phi(t, x)\) according to

\[
\phi(t, x) = \int \phi_0(y)q_0(t_0, y; t, x)dy
\]  

which is the solution of the diffusion equation

\[-\partial\phi(t, x)/\partial t + D\Delta_x\phi(t, x) = 0\]  

verifying the initial-value problem \(\phi(t_0, x) = \phi_0(x)\). Note that \(q_0(t_0, x_0; t, x)\) is the fundamental solution of (2) in which \(\Delta_x\) is the laplacian operator taken at the point \(x\). In presence
of an external potential, \( u(t, x) \), we generalize (2) into

\[-\partial \phi(t, x)/\partial t + D \Delta \phi(t, x) - \frac{1}{\hbar} u(t, x) \phi(t, x) = 0 \] (3)

In contrast with (2), the fundamental solution of (3), \( q(t_0, x_0; t, x) \), cannot be normalized in general (19). Thus, \( q(t_0, x_0; t, x) \) is no more a transition probability density but it verifies the Chapman-Kolmogorov law of composition (19) and therefore it can be used to describe transitions in space-time. By using the Feynman-Kac formula, the fundamental solution of (3) can be written in terms of path integral. Then \( q(t_0, x_0; t, x) \) appears as a weighted sum of all the paths \( x(t) \) connecting the space-time points \((x_0, t_0)\) to \((x, t)\); the weight of a path is determined by

\[ A[x(t); t, t_0] = \int_{t_0}^{t} \left[ \frac{1}{2} m \left( \frac{dx(t')}{dt'} \right)^2 + u(t', x(t')) \right] dt'. \] (4)

and we have

\[ q(t_0, x_0; t, x) = \int D x(t) \exp \left[ -\frac{1}{\hbar} A[x(t); t, t_0] \right] \] (5)

Note that the integrand in (4) looks like the hamiltonian for a free particle in presence of an external potential and therefore \( A[x(t); t, t_0] \) will be called the hamiltonian action. However, \( A[x(t); t, t_0] \) is a formal writing (20), to calculate the path integral we have to discretize \( A[x(t); t, t_0] \) and the paths that contribute to the integral are those for which there is no derivative \( i.e. \) no velocity in the usual sense (21).

**IV. PATH-ENTROPY**

* A priori, to describe structure and dynamics in our space-time model we may introduce a lot of functions. However, since our main goal is to recover thermodynamics we focus on quantities that resemble as far as possible the thermodynamic ones. To describe equilibrium situations we restrict the external potential, \( u(t, x) \), to be a time-independent quantity, \( u(x) \).
Accordingly $A[x(t); t, t_0]$ becomes $A[x(t); \tau]$ with $\tau = t - t_0$. By analogy with the entropy that is the key quantity in thermodynamics we have to introduce a quantity that measures the order in space-time. Since the standard entropy is defined for given values of internal energy and volume we must consider that our space-time system is prepared with a given energy $U$ and occupies a volume $V$.

To characterize the space-time order, around each point $x_0$ we count the number of paths for which $x(\tau) = x_0$ and on which the hamiltonian action, $A[x(t); \tau]$, does not deviate too much from the action $\tau U$. This is realized by introducing the measure $\exp -\frac{1}{\hbar} [A[x(t); \tau] - \tau U]$. By analogy with the standard thermodynamics we define a path-entropy, $S_{path}$, according to

$$ S_{path} = k_B \ln \int dx_0 \int Dx(t) \exp -\frac{1}{\hbar} [A[x(t); \tau] - \tau U]. \quad (6) $$

Larger is $S_{path}$ larger is the number of paths, this means that larger are the acceptable fluctuations of the hamiltonian action around $\tau U$ and less strict is the space-time order. Thus $S_{path}$ can be used to characterize the order in space-time. However $S_{path}$ depends on $\tau$ that is, for the moment, a free parameter and due to this we can not conclude that $S_{path}$ corresponds to the thermodynamic entropy. $S_{path}$ can be also rewritten as

$$ S_{path} = \frac{k_B \tau}{\hbar} U + k_B \ln Z_{path} \quad (7) $$

with

$$ Z_{path} = \int dx_0 \int Dx(t) \exp -\frac{1}{\hbar} A[x(t); \tau] = \int dx_0 q(0, x_0; \tau, x_0) \quad (8) $$

in which we have taken $t_0 = 0$ and $q(0, x_0; \tau, x_0)$ is the fundamental solution of (3) for closed paths and a time interval $\tau$. $Z_{path}$ is the total number of closed paths that we may count during $\tau$ irrespective the value of $U$. Now we may choose $\tau$ as a function of $U$.

To $S_{path}$ we may associate a path-temperature $T_{path}$ by $\frac{dS_{path}}{dU} = \frac{1}{T_{path}}$ that relays a change of $U$ to a change of $S_{path}$. Similarly to $S_{path}$, $T_{path}$ is a well defined quantity but it can not be identified with the thermodynamic temperature, at this level. From (7) we have immediately
\[
\frac{1}{T_{\text{path}}} = \frac{k_B \tau}{\hbar} + \frac{k_B}{\hbar} [U + \hbar \frac{d}{d\tau} \ln Z_{\text{path}}] \frac{d\tau}{dU} \tag{9}
\]

To calculate \( \frac{d}{d\tau} \ln Z_{\text{path}} \) we use the expression of \( Z_{\text{path}} \) in terms of \( q(0, x_0; \tau, x_0) \) given in (8) and from the Chapman-Kolmogorov law of composition we may write

\[
q(0, x_0; \tau, x_0) = \int dx_0 q(0, x_0; \delta t, x_0) q(\delta t, x_0; \tau - \delta t, x_0) \tag{10}
\]

for any \( \delta t \) such as \( 0 < \delta t < \tau \). From (8), (10) and (3) we have

\[
\hbar \frac{d}{d\tau} \ln Z_{\text{path}} = \hbar \int dx_0 \left[ \frac{1}{Z_{\text{path}}} \int dx_0 q(0, x_0; \delta t, x_0) [D \Delta x_0 - \frac{1}{\hbar} u(x_0)] q(\delta t, x_0; \tau - \delta t, x_0) \right] \tag{11}
\]

The term associated with the external potential can be written

\[
\frac{1}{Z_{\text{path}}} \int dx_0 u(x_0) q(0, x_0; \tau, x_0) = \int dx_0 < u_P(x_0) >_{\text{path}} \tag{12}
\]

We may interpret \( < u_P(x_0) >_{\text{path}} \) as the potential energy at the point \( x_0 \) calculated as an average over the paths. To calculate the contribution of the laplacian in (11) we use the fact that (10) holds if \( \delta t \to 0 \), in that case \( \Delta x_0 \) operates only on the term which looks like the kinetic energy in \( A[x(t); \tau] \). A simple calculation gives

\[
\frac{1}{Z_{\text{path}}} \int dx_0 \int d(\delta x) q(0, x_0; \delta t, x_0 + \delta x) \left[ \frac{m}{2} \left( \frac{\delta x}{\delta t} \right)^2 - \frac{\hbar}{2 \delta t} \right] q(\delta t, x_0 + \delta x; \tau - \delta t, x_0) \tag{13}
\]

that we can rewrite as

\[
\int dx_0 \frac{m}{2} < \left( \frac{\delta x}{\delta t} \right)^2 >_{\text{path}} - \frac{\hbar}{2 \delta t} \tag{14}
\]

This expression represents the difference between the kinetic energy calculated as an average over the paths and the quantum fluctuation of energy corresponding to a time interval \( \delta t \) on which the kinetic energy is calculated. In the limits \( \delta x \to 0 \) and \( \delta t \to 0 \) we may replace \( q(0, x_0; \delta t, x_0 + \delta x) \) in (13) by the free particle approximation that only depends on \( \delta t \) and \( \delta x \). Moreover we may approximate \( q(\delta t, x_0 + \delta x; \tau - \delta t, x_0) \) by \( q(0, x_0; \tau, x_0) \) and its integral over \( x_0 \) gives \( Z_{\text{path}} \) as shown in (8). In the limits that we consider it is easy to see that

\[\frac{m}{2} < (\frac{\delta x}{\delta t})^2 >_{\text{path}} = \frac{\hbar}{2 \delta t}.\]

We may also check this result by performing an exact calculation but
using an explicit form of \( u(x) \). This result is also expected from the properties of the initial lattice on which we have \( \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 = \frac{\hbar}{2m} \). Thus the term in bracket in (14) is well defined in the limit \( \delta t \to 0 \). More generally, we write

\[
\frac{m}{2} < \left( \frac{\delta x}{\delta t} \right)^2 >_{\text{path}} = \frac{\hbar}{2\delta t} < u_K(x_0) >_{\text{path}}
\]  

in which \(< u_K(x_0) >_{\text{path}} \) is a well behaved function in the limit \( \delta t \to 0 \). In principle \(< u_K(x_0) >_{\text{path}} \) and \(< u_P(x_0) >_{\text{path}} \) can be calculated for a given potential \( u(x) \) and a given value of \( \tau \). Finally, we can rewrite (9) according to

\[
\frac{\hbar}{k_B T_{\text{path}}} = \tau + \left[ U - \int dx_0 (< u_K(x_0) >_{\text{path}} + < u_P(x_0) >_{\text{path}}) \right] \frac{d\tau}{dU}
\]  

This equation establishes a relation between \( \tau \) and \( T_{\text{path}} \) when \( u(x) \) and \( U \) are given.

V. THERMODYNAMICS

The square bracket in (16) contains \( U \), the specific properties of the system via the averages over the paths and the variable \( \tau \). We are free to choose a particular value, \( \tau^* \), of \( \tau \) in such a way that the quantity

\[
\int dx_a [< u_K(x_a) >_{\text{path}} + < u_P(x_a) >_{\text{path}}]
\]

that looks like the internal energy calculated over the paths is indeed equal to the energy of the system preparation i.e. \( U \), this is clearly an equilibrium condition since the energy that is brought to the system, \( U \), is transformed into the sum of the mean kinetic and potential energy associated with the paths. Thus, whatever the value of the derivative \( \frac{d\tau^*}{dU} \) from (16) we get \( \tau^* = \frac{\hbar}{k_B T^*} = \beta^* \hbar \) that fixes the value of \( T^* \) when \( \tau^* \) is given. The partition function (8) can be rewritten as

\[
Z_{\text{path}}^* = \int dx_a \int Dx(t) \exp \left( -\frac{1}{\hbar} \int_0^{\beta^* \hbar} \left[ \frac{1}{2} m \left( \frac{dx(t')}{dt'} \right)^2 + u(x(t')) \right] dt' \right).
\]
This expression is identical to the one of the standard partition function $Z$ obtained in \((5)\) starting from the canonical form of the density matrix provided $T^*$ is identified with the thermal temperature $T$. This is justified because our definition of $S_{\text{path}}$ and $T^*$ are in agreement with the zeroth law of thermodynamics. In order to prove this point we follow the Callen derivation of the zeroth law \((22)\). Let consider a system formed by two independent subsystems $S_1$ and $S_2$, they may differ both by the mass and the external potential and their preparation energy corresponds to $U_1$ and $U_2$. The equilibrium condition defined above can be written for the global system and for each subsystem taken separately. With our definition of $S_{\text{path}}$ it is easy to see that the entropy is the sum of entropies of each subsystem. Now for a fixed value of $U = U_1 + U_2$ we may imagine a virtual energy transfer $\delta U$ from one system to the other one. The requirement that, at the equilibrium, the total entropy is maximum relative to any $\delta U$ leads to the conclusion that the two subsystems must have the same value for $T^*$, in agreement with the zeroth law. The equilibrium condition also leads to the relation

$$\frac{d}{d\beta} \ln Z_{\text{path}}^* = -U$$

which is expected from standard statistical mechanics \((2)\). If we define $F$ according to $F = -k_B T \ln Z$ we see that \((7)\) is nothing else than the standard relation

$$F = U - T S_{\text{path}}^*$$

and consequently $S_{\text{path}}^*$ corresponds to the traditional thermal entropy.

Thus we have obtained an equivalent description of the thermodynamics directly in terms of paths without using Gibbs ensemble method. In this dynamic approach the motion associated with paths is not due to a hamiltonian but results from the primarily discreetness of the space-time. However our approach is more than a simple alternative description of thermodynamics as we shall see below. (To be short in what follows we drop the uperscript *)

In standard path integral formalism $\beta \hbar$ is considered as a \textit{formal time} \((5)\). Here, $\tau$ results from a combination of dynamics determined by \((3)\) and thermodynamics via a particular choice in the solution of \((16)\), therefore $\tau$ must have a strong physical meaning. From standard textbooks in statistical mechanics \((2)\) it is well known that there is no entropy on a short period of time. To have thermodynamics we must consider time intervals such that the quantum fluctuations do not exceed the order of magnitude of the typical thermal
energy. In the case of free particles the mean value of the kinetic energy is $1/2\beta$ and from the time-energy uncertainty relation $(\tau/2\beta = \hbar/2)$ we have $\tau = \beta\hbar$; we see that $\tau$ represents the relaxation time that we have to wait in order to relax the quantum fluctuations and to reach the thermal regime. Our derivation of $\tau$ is more general since independent of the external potential but it has the same physical meaning. In parallel, using the results given in (21), it is easy to see that the paths are located, in average, on a given volume in space; in absence of external potential this volume is a sphere of radius $\Lambda = \frac{\hbar}{(mk_BT)^{1/2}}$ corresponding to the thermal de Broglie wavelength, an expected result. However the most important result of our approach is that *equilibrium thermodynamics is consistent with a motion that is time-irreversible at the microscopic level since given by (3).* The interest of this result is to open a door from which we may establish a *$H$–theorem* without any new assumption.

**VI. DERIVATION OF A $H$–THEOREM**

In the Boltzmann approach of the *$H$–theorem* (1) a system of particles is first prepared in a non-equilibrium state by external constraints. At the time $t = t_0$ these constraints are removed and the system relaxes towards its equilibrium state. The Clausius version of the second law of thermodynamics asserts that during the system relaxation it exists a function that increases monotonically versus $(t - t_0)$ and tends to the thermal entropy when $(t - t_0)$ becomes infinite. In the Boltzmann work there is no external potential for $t \geq t_0$, the relaxation is driven by collisions between particles. Hereafter we mimic a similar description in our space-time model. Using an external potential we prepare the system in such a way that, at the initial time, $t = t_0$, the space points are weighted by a distribution function, $\phi_0(x)$, which is positive and normalized. Any quantity calculated in these conditions depends on $\phi_0(x)$, at least for $(t - t_0) \geq 0$ but finite. For $t \geq t_0$ the external potential is switched off, $u(x) = 0$, and the time evolution of $\phi(t, x)$ is given by (2). In this case $q(t_0, x_0; t, x)$ is a density of transition probability for $t > t_0$ and $\phi(t, x)$ defined according to (1) is the probability to be in $x$ at
Now, we introduce a function $H(t)$ that generalizes $S_{\text{path}}$ given by (6). After relaxation, the space is uniform since $u(x) = 0$ and the entropy must be given by (6) that we can rewrite as $S = k_B \ln(V \gamma(\tau))$ where $V$ is the volume of the system. During the relaxation process, at each time, $t$, and for each point, $x$, we count the number of closed paths that we can form between the instants $t$ and $t + \tau$, this number is still $\gamma(\tau)$. However, the total number of paths at this time and this position is $\gamma(\tau)/\phi(t, x)$, it reduces to $\gamma(\tau)V$ if $\phi(t, x) = 1/V$ i.e. in the case of a uniform system. We define the average, $S_{\text{total}}(t)$, of $\ln[\gamma(\tau)/\phi(t, x)]$ taken over the overall volume as

$$S_{\text{total}}(t) = k_B \int \phi(t, x) \ln[\frac{\gamma(\tau)}{\phi(t, x)}] dx$$

(19)

Now we consider the quantity $H(t) = (1/k_B)(S_{\text{total}}(t) - S)$ defined by

$$H(t) = - \int \phi(t, x) \ln(\phi(t, x)) dx - \ln[V]$$

(20)

where we have used the normalization of $\phi(t, x)$. The first term in (20) is positive since the integrand is negative due to the fact that $0 \leq \phi(t, x) \leq 1$. Thus, $H(t)$ represents the competition between two quantities of opposite sign. The solution of (3) in a volume $V$ has been given in (23); in the limit $(t - t_0) \to \infty$ it has been shown that $\phi(t, x) = 1/V$ and consequently $H(t) \to 0$. In addition, we can show that (23)

$$\frac{dH(t)}{dt} = (\frac{\hbar}{2m}) \int \frac{1}{\phi(t, x)} (\frac{d\phi(t, x)}{dx})^2 dx \geq 0$$

(21)

Thus, $H(t)$ is a monotonic increasing function of $t$ and $H(t)$ vanishes in the stationary regime obtained in the limit $(t - t_0)$ goes to infinite. When the time is running, $S_{\text{total}}(t)$ increases in a monotonic manner to reach a stationary value corresponding to the thermal entropy $S$. Thus a $H$–theorem is demonstrated.

The irreversible behavior results from the existence of a space-time, which is discrete at a fundamental level and induces a special dynamics not connected with a hamiltonian. Nowhere we consider collisions between particles as in the Boltzmann dynamics but we
investigate the paths performed by one mass. To characterize the dynamics on these paths we select a given point \((x_i, t_i)\) and for a time interval \(\delta t \leq \tau/2\) we consider the forward and backward velocities defined respectively as:

\[ V_+ = \frac{(x_{i+1} - x_i)}{\delta t} \quad \text{and} \quad V_- = \frac{(x_i - x_{i-1})}{\delta t} \]

To the product \(V_+V_-\) we associate \((21)\) an average over paths \(< V_+V_->\); it has been shown that \((21)\)

\[ < V_+V_- > = < \left( \frac{x_{i+1} - x_i}{\delta t} \right) \left( \frac{x_i - x_{i-1}}{\delta t} \right) > = -1/(m\beta) \quad (22) \]

This result, independent of \(\delta t\), leads to several remarks. First, for trajectories on which there is a velocity in the usual sense \(< V_+V_->\) is a positive quantity, here the negative value of \(< V_+V_->\) is the signature of the fractal character of the paths. Second, to the negative value of \(< V_+V_->\) we may associate a formal collision, which reverses the direction of the velocity, in average. From \((22)\) we see that the correlations are not destroyed after a formal collision in contrast with the molecular chaos hypothesis used by Boltzmann \((1)\). It is only in the limit \(T = 0\) that the velocities \(V_+\) and \(V_-\) become statistically independent. Third, a formal collision appears at any time and everywhere in space; in opposite to the Boltzmann dynamics we cannot consider two separated time scale, one for the duration of collisions and another one for time between collisions.

Of course to be very satisfying our approach must also describes some situations that we may consider as reversible.

**VII. TIME-REVERSIBLE PROCESSES AND THE SCHRÖDINGER EQUATION**

In the previous Sections we have considered the real-valued function

\[ \phi(t, x) = \int \phi_0(y)q(t_0, y; t, x)dy \quad (23) \]

that is the solution of the diffusion equation \((3)\) verifying the initial-value problem \(\phi(t_0, x) = \phi_0(x)\); in \((23)\), \(q(t_0, x_0; t, x)\) is the fundamental solution of \((3)\). This description is quite natural; we know the initial condition for \(t = t_0\) and describe the system for \(t \geq t_0\). However, it is possible to consider another function \(\hat{\phi}(t, x)\) defined according to
\[ \hat{\phi}(t, x) = \int q(t, x; t_1, y) \hat{\phi}_1(y) dy \] (24)

for \( t_0 \leq t \leq t_1 \). Using the properties of the fundamental solution it is easy to prove that \( \hat{\phi}(t, x) \) is the solution of the differential equation

\[ \frac{\partial \hat{\phi}(t, x)}{\partial t} + D \Delta \hat{\phi}(t, x) - \frac{1}{\hbar} u(t, x) \hat{\phi}(t, x) = 0 \] (25)

verifying the final condition \( \phi(t_1, x) = \hat{\phi}_1(x) \). Up to now the use of (3) or (25) is a question of convenience depending on the information we have on the system.

In what follows we consider a new physical situation in which the system is defined simultaneously by (3) and (25) i.e. by a dynamics defined at an initial and a final time, \( t_0 \) and \( t_1 \) respectively. Of course, for the entry-exit conditions \((\phi_0(x), \hat{\phi}_1(x))\) our problem is now to predict what happens for any \( t \) between \( t_0 \) and \( t_1 \). This is reminiscent of the lagrangian mechanics that is known to describe reversible processes. However in our case we must show that the entry-exit conditions introduce a well identified mathematical object as, for instance, a well defined stochastic process. The proof of that has been given by Nagasawa (19) and we will not reproduce the mathematical details here but we will focus mainly on the physical aspects.

The functions \( \phi(t, x) \) and \( \hat{\phi}(t, x) \) that describe the transition in space-time are not transition probability density and, up to now, there is no restriction on \( \phi_0(x) \) and \( \hat{\phi}_1(x) \). Hereafter we assume that \( \phi_0(x) \) and \( \hat{\phi}_1(x) \) are two non-negative real-valued functions that we choose such as

\[ \int \phi_0(y) dy q(t_0, y; t_1, x) dx \hat{\phi}_1(x) = 1 \] (26)

From the set \([\phi_0(x), \hat{\phi}_1(x), q(s, y; t, x)]\), the condition (26) and the Chapman-Kolmogorov law of composition, we can show that (19)

\[ \mu(t, x) = \phi(x) \hat{\phi}(x) \] (27)

is a non-negative and normalized quantity; we may consider \( \mu(t, x) \) as the probability distribution of a stochastic process. Moreover, from the same ingredients as above we can define
a probability measure noted $Q = [\phi_0 q \gg \gg q \hat{\phi}_1]$ (19). It is possible to put $Q$ in a more traditional form by introducing the new variables

$$p(s, y; t, x) = \phi(s, y)q(s, y; t, x)\frac{1}{\phi(t, x)} \text{ if } \phi(t, x) \neq 0 \text{ and } p(s, y; t, x) = 0 \text{ otherwise.}$$

(28)

and

$$\hat{p}(s, y; t, x) = \frac{1}{\phi(s, y)}q(s, y; t, x)\hat{\phi}(t, x) \text{ if } \hat{\phi}(s, y) \neq 0 \text{ and } \hat{p}(s, y; t, x) = 0 \text{ otherwise}$$

(29)

From their definitions, we can see that $p(s, y; t, x)$ and $\hat{p}(s, y; t, x)$ are positive and normalized according to

$$\int p(s, y; t, x) dy = 1 \text{ and } \int dx \hat{p}(s, y; t, x) = 1$$

(30)

Consequently we can consider $p(s, y; t, x)$ and $\hat{p}(s, y; t, x)$ as transition probability densities. The probability measure $Q$ can be rewritten as $Q = [\phi_0 \hat{\phi}_0 \hat{\phi}]$ that is the traditional Kolmogorov representation of a Markov process with the initial distribution $\mu(t_0, x) = \phi_0(x)\hat{\phi}_0(x)$ and the transition probability density $\hat{p}(s, y; t, x)$. $Q$ is also identical to a time reversed Markov process $Q = \ll p\phi_1 \hat{\phi}_1].$ Thus the probability measure noted $Q = [\phi_0 q \gg \gg q \hat{\phi}_1]$ is the Schrödinger representation of a Markov process. This representation has been introduced by Schrödinger (24) and later investigated by Kolmogorov and several other mathematicians (for a review in this field see (25)). In the Kolmogorov representation, the Markov process and its time reversed version define two positive semi-group that are in duality relative to the measure $\mu(t, x)dtdx$. The transition probability densities $p(s, y; t, x)$ and $\hat{p}(s, y; t, x)$ verify diffusion equations:

$$-\frac{\partial p(s, y; t, x)}{\partial t} + (\hbar/2m)\Delta_x p(s, y; t, x) + a(t, x)\nabla_x p(s, y; t, x) = 0$$

(31)

and

$$\partial \hat{p}(s, y; t, x)/\partial t + (\hbar/2m)\Delta_x \hat{p}(s, y; t, x) + \hat{a}(t, x)\nabla_x \hat{p}(s, y; t, x) = 0$$

(32)
while the density distribution of the Markov process obeys to the law
\[
\frac{\partial \mu(t, x)}{\partial t} + \nabla\left(\frac{\hat{a} - a}{2} \mu(t, x)\right) = 0 \tag{33}
\]
The drift functions \(a\) and \(\hat{a}\) are determined by the duality condition, this leads to the celebrated Kolmogorov result \(a(t, x) = (h/m) \nabla x \ln[\phi(t, x)]\) and \(\hat{a}(t, x) = (h/m) \nabla x \ln[\hat{\phi}(t, x)]\).
Thus the process defined by the entry-exit conditions and the equations (3) and (25) is the Schrödinger representation of a well defined Markov process. This representation is useful to show the existence of a superposition principle for Markov processes.

Instead of the two real-valued functions \(\phi(t, x)\) and \(\hat{\phi}(t, x)\), we can introduce two other real-valued functions
\[
R(t, x) = \frac{1}{2} \ln \phi(t, x) \hat{\phi}(t, x) \quad \text{and} \quad S(t, x) = \frac{1}{2} \ln \frac{\hat{\phi}(t, x)}{\phi(t, x)} \tag{34}
\]
that we can combined into one complex-valued function \(\Psi(t, x) = \exp[R(t, x) + iS(t, x)]\). It has been shown \(^{(19)}\) that \(\Psi(t, x)\) verifies a Schrödinger equation
\[
i\hbar \frac{\partial \Psi(t, x)}{\partial t} = -\frac{\hbar^2}{2m} \Delta_x \Psi(t, x) + V(t, x)\Psi(t, x) \tag{35}
\]
in which \(V(x, t)\) is related to \(u(t, x)\) according to
\[
V(t, x) - u(t, x) + 2\hbar \left[\frac{\partial S(t, x)}{\partial t} + D(\nabla S)^2(t, x)\right] = 0 \tag{36}
\]
Thus, when we go from the representation involving two real-valued functions to another one based on one complex-valued function \(u(t, x)\) has to be changed into \(V(t, x)\) as shown by (36). In addition to the previous results the equation for the complex conjugate \(\bar{\Psi}(t, x)\) of \(\Psi(t, x)\) has been established and we have the basic result
\[
\mu(t, x) = \phi(t, x) \hat{\phi}(t, x) = \Psi(t, x)\bar{\Psi}(t, x) \tag{37}
\]
that gives the physical interpretation of the product \(\Psi(t, x)\bar{\Psi}(t, x)\). Finally, the superposition principle of Markov processes shows that \(\Psi(t, x)\) verifies the usual superposition principle associated with a wave function. Since the Schrödinger equation is time reversible in the
Wigner sense, in this Section we have shown how to implement the description presented in previous Sections in order to be able to describe reversible processes.

The derivation of the Schrödinger equation that we have developed here is totally different from a recent approach in which the complex nature of the wave function is connected with the assumption that the space-time, by itself, has a fractal nature (26). Another route to derive the Schrödinger equation has been developed in a series of very interesting papers presented by G.N. Ord (see for instance (27) and (28)). In these papers the role of time-irreversibility is carefully analyzed and the reversibility of the Schrödinger equation is obtained by considering more informations on the paths but selecting a special projection of the processes that appears as time-reversible. Here the route that we have retained is the one that allows to treat on a similar footing statistical mechanics and quantum physics.

VIII. CONCLUDING REMARKS

The existence of a discrete space-time is one of the concepts used to describe the physics at the Planck’s scale. In this paper we show that the existence of a primarily discrete space-time can be also a powerful concept to describe the physics in a pre-relativistic world. Our work is based on two main assumptions: i) the discrete space-time structure is determined by relations that mimic the Heisenberg uncertainty relations and ii) the motion in this space-time model is as simple as possible. For the properties that we have in mind the continuous limit is sufficient. All quantities defined in this approach have a dynamical character, they depends on the dynamics on the paths and on the time interval, $\tau$, on which we observe the paths. We have shown that it exists a characteristic time interval, $\tau = \beta \hbar$, for which the mean value of the energy calculated over the closed paths corresponds exactly to the energy needed in the system preparation; for this like-equilibrium condition our results are identical to the standard ones expressed in terms of path integral. We have seen that $\tau$ represents the time that we have to wait in order that the quantum fluctuations do not exceed the thermal ones. Of course, it is possible to follow the system evolution for time intervals smaller than
τ but in this case we have not the usual thermodynamics.

Our approach relates thermodynamics to the existence of motions that are not time-reversible at the microscopic level. From this change of paradigm it becomes easy to derive a $H$–theorem. To the motion in space-time we associate formal collisions, they may appear everywhere and at any time as a consequence of the fractal character of the motion. Moreover, in contrast with the Boltzmann dynamics, a formal collision does not destroy the correlations in the velocities.

At least in the simple example investigated here we may say that we have demonstrated the Feynman’s conjecture i.e. the possibility to derive the statistical thermodynamics directly from the inspections of the paths and without using all the apparatus of quantum mechanics. In this respect there is a parallel with the black hole thermodynamics for which it has been shown that the main results can be derived from general features of the theory rather than the detailed form of the Einstein’s equation (29). In addition, it is interesting to underline that our $S_{\text{path}}$ is basically related to the space-time properties, it does not result from the counting of microstates as in standard thermodynamics. However, we can perform an exact mapping between $S_{\text{path}}$ and the thermal entropy, a similar mapping exists between the standard heat-bath formalism and the thermal behavior induced by vacuum polarization in presence of causal horizons (10). Of course, here we have no quantum vacuum and no horizon and therefore $S_{\text{path}}$ has not the same origin that the geometrical entropy introduced in the vicinity of a black hole, nevertheless from our approach we may emphasize some similarities between two different fields of investigations.

The results obtained in statistical thermodynamics are based on the properties of a real-valued function $\phi(t, x)$ that verifies a diffusion like equation (3) but this equation is not a simple imaginary-time version of the Schrödinger equation, no analytic continuation is invoked in our approach. To get the Schrödinger equation we force the system to have a time-reversible behavior for any time, from this we leave statistical thermodynamics and enter in quantum mechanics.
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