Fractional strong matching preclusion for Cartesian product graphs

Bo Zhu\textsuperscript{1}, Tianlong Ma\textsuperscript{2}†

\textsuperscript{1}School of Mathematics and Statistics, Qinghai Normal University, Xining, Qinghai 810008, China
\textsuperscript{2}Department of Basic Research, Qinghai University, Xining, Qinghai, 810016, China

E-mails: zhuboqh@163.com; tianlongma@aliyun.com;

Abstract

The strong matching preclusion number of a graph, introduced by Park and Ihm in [13], is the minimum number of vertices and edges whose deletion results in a graph that has neither perfect matchings nor almost perfect matchings. As a generalization, the fractional strong matching preclusion number of a graph is the minimum number of edges and vertices whose deletion leaves the resulting graph without a fractional perfect matching. In this paper, we obtain the fractional strong matching preclusion number for Cartesian product graphs. As an application, the fractional strong matching preclusion number for torus networks is obtained.

Keywords: Matching; Fractional strong matching preclusion; Cartesian product; Torus networks

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1 Introduction

A matching $M$ in a graph is a set of pairwise non-adjacent edges. A perfect matching in a graph is a set of edges such that every vertex is incident with exactly one edge in this set. An almost-perfect matching in a graph is a set of edges such that every vertex except one is incident with exactly one edge in this set, and the exceptional vertex is incident to none. A set $F$ of edges in a graph $G = (V, E)$ is called a matching preclusion set if $G - F$ has neither a perfect matching nor an almost-perfect matching. The matching preclusion number of graph $G$, denoted by $mp(G)$, is the minimum number of edges whose deletion leaves the resulting graph without a perfect matching or an almost-perfect matching. The concept of matching preclusion was introduced by Birgham et al. [2]. Matching preclusion also has connections to a number of theoretical topics, including conditional connectivity and extremal graph theory, and further studied in [4–6, 9], with special attention given to interconnection networks.

In [13], the concept of strong matching preclusion was introduced. The strong matching preclusion number of a graph $G$, denoted by $smp(G)$, is the minimum number of vertices and edges whose deletion leaves the resulting graph without a perfect matching or an almost-perfect matching. A strong matching preclusion set is optimal if $|F| = smp(G)$. According to the definition of $mp(G)$ and $smp(G)$, we have that $smp(G) \leq mp(G) \leq \delta(G)$. If $G - F$ has an isolated vertex set and $F$ is an optimal strong matching preclusion set, then $F$ is a basic strong matching preclusion set. $F$ is a trivial strong matching preclusion set if, in addition, $G$ is even and $F$ has an even number of vertices, otherwise, $F$ is a nontrivial strong matching preclusion set. A strongly maximally matched even graph is strongly super matched if every optimal strong matching preclusion set is trivial.

Given a set of edges $M$ of $G$, we define $f^M$ to be the indicator function of $S$, that is, $f^M : E(G) \rightarrow \{0, 1\}$ such that $f^M(e) = 1$ if and only if $e \in M$. Let $X$ be a set of vertices of $G$. We define $\delta'(X)$ to be the set of edges with exactly one end in $X$. If $X = \{v\}$, we write $\delta'(v)$ instead of $\delta'(\{v\})$. (We note that although it is common to use $\delta(X)$ is the literature, we decided to use $\delta'(X)$ as it is also common to use $\delta(G)$ to denote the minimum degree of vertices in $G$.) With this notation, $f^M : E(G) \rightarrow \{0, 1\}$ is the indicator function of the perfect matching $M$ if $\sum_{e \in \delta'(v)} f^M(e) = 1$ for each vertex $v$ of $G$. If we replace “$=$” by “$\leq$” in the condition, then $M$ is a matching of $G$. Similarly, $f^M : E(G) \rightarrow \{0, 1\}$ is the indicator function of the almost perfect matching $M$ if $\sum_{e \in \delta'(v)} f^M(e) = 1$ for each vertex $v$ of $G$, except one vertex, say $v'$, and $\sum_{e \in \delta'(v')} f^M(e) = 0$. It is also common to use $f(X)$ to denote $\sum_{x \in X} f(x)$. It follows from the definition that $f^M(E(G)) = \sum_{e \in E(G)} f^M(e) = |M|$ for a matching $M$. In particular, $f^M(E(G)) = |V(G)|/2$ if $M$ is a perfect matching and $f^M(E(G)) = (|V(G)| - 1)/2$ if $M$ is an
almost perfect matching.

A standard relaxation from an integer setting to a continuous setting is to a continuous setting to replace the codomain of the indicator function from \{0,1\} to the interval \([0,1]\). Let \(f : E(G) \rightarrow [0,1]\). Then \(f\) is a fractional matching if \(\sum_{e \in \delta'(v)} f(e) \leq 1\) for each vertex \(v\) of \(G\); \(f\) is a fractional perfect matching if \(\sum_{e \in \delta'(v)} f(e) = 1\) for every vertex \(v\) of \(G\); and \(f\) is a fractional almost perfect matching if \(\sum_{e \in \delta'(v)} f(e) = 1\) for every vertex \(v\) of \(G\) except one vertex \(u\), and \(\sum_{e \in \delta'(u)} f(e) = 0\). Thus, if \(f\) is a fractional perfect matching, then:

\[
f(E(G)) = \sum_{e \in E(G)} f(e) = \frac{1}{2} \sum_{v \in V(G)} \sum_{e \in \delta'(v)} f(e) = \frac{|V(G)|}{2};
\]

and if \(f\) is a fractional almost perfect matching, then:

\[
f(E(G)) = \sum_{e \in E(G)} f(e) = \frac{1}{2} \sum_{v \in V(G)} \sum_{e \in \delta'(v)} f(e) = \frac{|V(G)| - 1}{2}.
\]

We know that although an even graph has no almost perfect matching, an even graph can have a fractional perfect matching. Recently, Liu and Liu in [10] introduced some natural and nice generalizations of the above concepts. An edge subset \(F\) of \(G\) is a fractional matching preclusion set (FMP set for short) if \(G - F\) has no fractional perfect matchings. The fractional matching preclusion number (FMP number for short) of \(G\), denoted by \(fmp(G)\), is the minimum size of FMP sets of \(G\), that is, \(fmp(G) = \min\{|F| : F\text{ is an FMP set}\}\). We refer the readers to [10][12] for more details and additional references.

**Proposition 1.1.** Let \(G\) be a graph. Then \(fmp(G) \leq \delta(G)\), where \(\delta(G)\) is the minimum degree of \(G\). Especially, if the number of vertices in \(G\) is even, then \(mp(G) \leq fmp(G)\).

If \(G\) is maximally matched, then \(fmp(G) = \delta(G)\). But, if the number of vertices in \(G\) is odd, \(mp(G)\) and \(fmp(G)\) do not have the same inequality relation. Some examples are given in [10].

A set \(F\) of edges and vertices of \(G\) is a fractional strong matching preclusion set (FSMP set for short) if \(G - F\) has no fractional perfect matchings. The fractional strong matching preclusion number (FSMP number for short) of \(G\), denoted by \(fsmp(G)\), is the minimum size of FSMP sets of \(G\), that is, \(fsmp(G) = \min\{|F| : F\text{ is an FSMP set}\}\).

**Proposition 1.2.** Let \(G\) be a graph. Then \(fsmp(G) \leq fmp(G) \leq \delta(G)\), where \(\delta(G)\) is the minimum degree of \(G\).

If \(fmp(G) = \delta(G)\), then \(G\) is fractional maximally matched; if, in addition, \(G - F\) has an isolated vertex for every optional fractional matching preclusion set \(F\), then \(G\) is fractional super...
matched. If $f\text{sm}(G) = \delta(G)$, then $G$ is fractional strongly maximally matched; if, in addition, $G - F$ has isolated vertices for every optimal fractional strong matching preclusion set $F$, then $G$ is fractional strongly super matched.

We refer to the book [1] for graph theoretical notations and terminology that are not defined here. Given a graph $G$, we let $V(G)$, $E(G)$, and $(u, v)$ denote the set of vertices, the set of edges, and the edge whose end vertices are $u$ and $v$, respectively. We use $G - F$ to denote the subgraph of $G$ obtained by removing all the vertices or the edges of $F$. We denote by $K_n$ the complete graph on $n$ vertices. If a cycle contains every vertex of $G$ exactly once, then the cycle is called a Hamiltonian cycle. If there is a Hamiltonian cycle in $G$, then $G$ is called Hamiltonian and if there is a Hamiltonian path between every two distinct vertices of $G$, then $G$ is called Hamiltonian connected.

2 Preliminaries

The following lemmas are useful in this paper to proof our main result.

Proposition 2.1. [14] A graph $G$ has a fractional perfect matching if and only if $i(G - S) \leq |S|$ for every set $S \subseteq V(G)$, where $i(G - S)$ is the number of isolated vertices of $G - S$.

Theorem 2.2. [15] Let $n \geq 3$ be an integer and $C_n$ be a cycle of length $n$. Then $s\text{mp}(C_n) = 2$.

Theorem 2.3. [3] Let $T(k_1, k_2, \ldots, k_n)$ be a torus with an even numbers of vertices. Then $mp(T(k_1, k_2, \ldots, k_n)) = 2n$. and each of its minimum MP sets is trivial.

Theorem 2.4. [10] Let $n \geq 3$. Then $f\text{sm}(K_n) = n - 2$.

Proposition 2.5. [14] The graph $G$ has a fractional perfect matching if and only if there is a partition $\{V_1, V_2, \ldots, V_n\}$ of the vertex set of $V(G)$ such that, for each $i$ the graph $G[V_i]$ is either $K_2$ or a Hamiltonian graph on odd number of vertices.

Proposition 2.6. [11] If a graph $G$ is Hamiltonian, then $G$ has a fractional perfect matching.

Lemma 2.7. [11] Let $G$ be fractional strongly super matching graph with $\delta(G) \geq 2$. If $F$ is a trivial FSMP set of $G$ and $G - F$ has an isolated vertex $v$, then $G - F - v$ has a fractional perfect matching.

Lemma 2.8. [16] Let $k \geq 4$ be an even integer. Then the following statements hold:

(a) $mp(\text{Torus}(n, n)) = 4$;

(b) every minimum matching preclusion set in Torus $(n, n)$ is trivial.
Lemma 2.9. Let $k_1 \geq 5$ and $k_2 \geq 5$ be two odd integers. Then $T(k_1, k_2)$ is super strongly matched.

3 Cartesian product

Let $G$ and $H$ be two simple graphs. Their Cartesian Product $G \square H$ is the graph with vertex set $V(G) \times V(H) = \{gh: g \in V(G), h \in V(H)\}$, in which two vertices $g_1h_1$ and $g_2h_2$ are adjacent if and only if $g_1 = g_2$ and $(h_1, h_2) \in E(H)$, or $(g_1, g_2) \in E(G)$ and $h_1 = h_2$. We view $G \square C_n$ as consisting of $n$ copies of $G$. Let these copies be $G_0, G_1, \ldots, G_{n-1}$ labeled along the cycle $C_n$. The edges between different copies of $G_i$ are called cross edges. It follows that each vertex $u$ of $G_i$ has two neighbors that not in $G_1$, called cross neighbors of $u$. Furthermore, the cross neighbors of $u$ are in different copies of $G$. If $u$ is in $G_0$, then its cross neighbors are in $G_1$ and $G_{n-1}$, respectively. In short, for each vertex in $G_i$, its cross neighbors are in adjacent copies of $G$.

Theorem 3.1. If graph $G$ is fractional strongly super matched and $C_n(n \geq 5)$ is a cycle with odd vertices, then $f_{smp}(G \square C_n) = \delta(G) + 2$, $(\delta(G) \geq 4)$. Moreover, every optimal fractional strongly matching preclusion set is trivial.

Proof. Let $F \subseteq V(G \square C_n) \cup E(G \square C_n)$ with $|F| = \delta(G) + 2$, and $F_i = F \cap (V(G_i) \cup E(G_i))$ for $0 \leq i \leq n - 1$. Suppose $|F_{i-1}| \geq |F_i|$, for $1 \leq i \leq n - 1$. We will prove that one of the following two cases holds: (i) $F$ is a trivial FSMP set of $G \square C_n$; (ii) $G \square C_n - F$ has a fractional perfect matching. We know that if $G_0 - \delta(G)$ has fractional perfect matching, then there is a partition $\{V_1, V_2, \ldots, V_{n}\}$ of the vertices set of $V(G_0 - \delta(G))$, such that the graph $(G_0 - \delta(G))|V_i|$ is either $K_2$ or a Hamiltonian graph with an odd number of vertices by Proposition 2.5. Four cases are considered in the following:

Case 1. $|F_0| = \delta(G) + 2$. Clearly $|F_i| = 0$ for $1 \leq i \leq n - 1$. Let $F'_0 = F_0 - \{\alpha, \beta\}$ for any $\alpha, \beta \in F_0$. Since $G_0$ is fractional strongly super matched, either $G_0 - F'_0$ has fractional perfect matching or $F'_0$ is a trivial FSMP set of $G_0$. Consider two subcases:

Subcase 1.1 $G_0 - F'_0$ has fractional perfect matching.

a. Suppose $\alpha, \beta$ are vertices of $K_2$ which is $\{(\alpha, \beta)\}$. Obviously, there is a fractional perfect matching $f_0$ in $G_0 - F'_0 - \{\alpha, \beta\}$. Thus $\bigcup_{i=0}^{n-1} f_i$ induce a fractional perfect matching of $G \square C_n - F$. Let $\alpha, \beta$ be vertices of $K_2$ which are $\{(\alpha, \alpha')\}$ and $\{(\beta, \beta')\}$. We assume that $u$ and $v$ are cross neighbors of $\alpha'$ and $\beta'$, respectively. By Proposition 2.5 $C_{k}^{k-1}(0) - F'_0 - \{\alpha, \alpha'\} - \{\beta, \beta'\}$ has fractional perfect matching $f_0$. From the definition of the $G \square C_n$, we know that $u$ and $v$ are either
in $G_1$ or in $G_{n-1}$. Let $F_i' = \{u, v\} \cap V(G_i)$ for $1 \leq i \leq n - 1$. It is clear that $|F_i'| \leq 2 < \delta(G)$, $(t = 1$ or $n - 1)$, and $|F_j'| = 0 < \delta(G)$, $(2 \leq j \leq n - 2)$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(\alpha', u), (\beta', v)\}$ induce a fractional perfect matching of $G \Box C_n - F$.

b. Suppose $\alpha, \beta$ are vertices of a subgraph of Hamiltonian graph of $(G_0 - F_0')[V_i]$ on an odd number of vertices. Then it is clear that there is at most one path on an odd number of vertices in subgraph by $V_i - \{\alpha, \beta\}$. We may assume that $v$ is the end vertex of the path on an odd number of vertices. Obviously, there is a fractional perfect matching in $G_0 - \{\alpha, \beta\} - v$. Let $v'$ be cross neighbor of $v$. Let $F_i' = \{v'\} \cap V(G_i)$ for $1 \leq i \leq n - 1$. It is clear that $|F_i'| \leq 1 < \delta(G)$, $(t = 1$ or $n - 1)$, and $|F_j'| = 0 < \delta(G)$, $(2 \leq j \leq n - 2)$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(v, v')\}$ induce a fractional perfect matching of $G \Box C_n - F$.

c. Suppose $\alpha$ is a vertex in $K_2$ which is $\{(\alpha, \alpha')\}$; $\beta$ is a vertex of a subgraph of Hamiltonian graph of $(G_0 - F_0')[V_i]$ on an odd number of vertices. There is a fractional perfect matching of the path on an even number of vertices of $(G_0 - F_0')[V_i] - \beta$. We assume that $\alpha''$ is a cross neighbor of $\alpha'$. Let $F_i' = \{\alpha''\} \cap V(G_i)$ for $1 \leq i \leq n - 1$. It is clear that $|F_i'| \leq 1 < \delta(G)$, $(t = 1$ or $n - 1)$ and $|F_j'| = 0 < \delta(G)$, $(2 \leq j \leq n - 2)$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(\alpha', \alpha'')\}$ induce a fractional perfect matching of $G \Box C_n - F$.

d. Suppose $\alpha, \beta$ are edges of $K_2$ which are $\{(v, w), (x, y)\}$. We assume that $v', w', x'$ and $y'$ are cross neighbors of $v, w, x$ and $y$, respectively. It is clear that there is a fractional perfect matching $f_0$ in $G_0 - \{v, w, x, y\}$. By the definition of networks $G \Box C_n$, we may let $v'$ and $w'$ be in $G_{n-1}$, $x'$ and $y'$ be in $G_1$. Let $F_i' = \{v', w', x', y'\} \cap V(G_i)$ for $1 \leq i \leq n - 1$. It is clear that $|F_i'| = 2 < \delta(G)$, $(t = 1$ or $n - 1)$ and $|F_j'| = 0 < \delta(G)$, $(2 \leq j \leq n - 2)$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(v, v'), (w, w'), (x, x'), (y, y')\}$ induce a fractional perfect matching of $G \Box C_n - F$.

e. Suppose $\alpha, \beta$ are edges of a subgraph of Hamiltonian graph of $(G_0 - F_0')[V_i]$ on an odd number of vertices. There are three cases in $(G_0 - F_0')[V_i] - \{\alpha, \beta\}$: (1) there is an isolate vertex and a path on an even number of vertices; (2) there is a path on odd vertices and a path on even vertices; (3) there are two paths on an odd number of vertices. We only need to prove the last case, the other two cases are similar. Let $v$ be an end vertex of Path$_1$ on an odd number of vertices and $w$ be an end vertex of Path$_2$ on an number odd of vertices. We may assume that $v'$ and $w'$ are cross neighbours of $v$ and $w$, respectively. Let $F_i' = \{v', w'\} \cap V(G_i)$ for $1 \leq i \leq n - 1$. It is clear that $|F_i'| \leq 2 < \delta(G)$, $(t = 1$ or $n - 1)$, and $|F_j'| = 0 < \delta(G)$, $(2 \leq j \leq n - 2)$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(v, v'), (w, w')\}$ induce a fractional perfect matching of $G \Box C_n - F$.

f. Suppose $\alpha$ is an edge $K_2$ of $(v, w)$; $\beta$ is an edge of a subgraph of Hamiltonian graph of $(G_0 - F_0')[V_i]$ on an odd number of vertices. There exists a path on an odd number of vertices in $(G_0 - F_0')[V_i] - \beta$. Let $x$ be an end vertex of the path on an odd vertices. We assume that
$v', w'$ and $x'$ are cross neighbours of $v, w$ and $x$, respectively. Let $F'_i = \{v', w', x'\} \cap V(C^{k-1}_n(i))$ for $1 \leq i \leq n-1$. It is clear that $|F'_i| \leq 3 < \delta(G)$, ($t = 1$ or $n-1$) and $|F'_j| = 0 < \delta(G)$, ($2 \leq j \leq n-2$). Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(v, v'), (w, w'), (x, x')\}$ induce a fractional perfect matching of $G \square C_n - F$.

$g.$ Suppose $\alpha$ is an vertex in $K_2$ of $\{(\alpha, \alpha')\}$; $\beta$ is an edge of a subgraph of Hamiltonian graph of $(G_0 - F'_0)[V_i]$ on an odd number of vertices. There exists a path on an odd number of vertices in $(G_0 - F'_0)[V_i] - \beta$. Let $x$ be an end vertex of the path on an odd vertices. We assume that $x'$ and $\alpha''$ are cross neighbours of $x$ and $\alpha'$, respectively. There is a fractional perfect matching $f_0$ in $G_0 - F'_0 - \{\alpha, \alpha', \beta, x\}$. Let $F'_i = \{\alpha'', x'\} \cap V(G_i)$ for $1 \leq i \leq n-1$. It is clear that $|F'_i| \leq 2 < \delta(G)$, ($t = 1$ or $n-1$) and $|F'_j| = 0 < \delta(G)$, ($2 \leq j \leq n-2$). Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(\alpha', \alpha''), (x, x')\}$ induce a fractional perfect matching of $G \square C_n - F$.

$h.$ Suppose $\alpha$ is an edge $K_2$ of $\{(v, w)\}$; $\beta$ is a vertex of a subgraph of Hamiltonian graph of $(G_0 - F'_0)[V_i]$ on an odd number of vertices. Obviously, there is fractional perfect matching of even vertices path of $(G_0 - F'_0)[V_i] - \beta$. Assume that $v'$ and $w'$ are cross neighbours of $v$ and $w$, respectively. Let $F'_i = \{v', w'\} \cap V(G_i)$ for $1 \leq i \leq n-1$. It is clear that $|F'_i| \leq 2 < \delta(G)$, ($t = 1$ or $n-1$), and $|F'_j| = 0 < \delta(G)$, ($2 \leq j \leq n-2$). Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(v, v'), (w, w')\}$ induce a fractional perfect matching of $G \square C_n - F$.

Subcase 1.2 $F'_0$ is a trivial FSMP set.

Let $G'_0 = G_0 - F'_0 - v$, where $v$ is an isolated vertex of $G_0 - F'_0$. By Lemma 2.7 we know that $G'_0$ has fractional perfect matching. Thus, the proof in Subcase 1.2. only needs to consider an isolated $\{v\}$ more than the proof in Subcase 1.1. From the proof in Subcase 1.1., it is clearly that $G \square C_n - F - v$ has a fractional perfect matching. In Subcase 1.1, there are at most four vertices that not matched in $G_0 - F$. Now, there are at most five vertices that not matched in $G_0 - F$, let us say $\{w_1, w_2, w_3, w_4, v\}$. So there is a fractional perfect matching $f_0$ in $G_0 - F - \{w_1, w_2, w_3, w_4, v\}$. Let $w'_1, w'_2, w'_3, w'_4$ and $v'$ be cross neighbors of $w_1, w_2, w_3, w_4$ and $v$, respectively. By the definition of $G \square C_n$, we may let $w'_1$ and $w'_2$ be in $G_{n-1}, w'_3, w'_4$ and $v'$ be in $G_1$. Let $F'_i = \{w'_1, w'_2, w'_3, w'_4, v'\} \cap V(G_i)$ for $1 \leq i \leq n-1$. It is obvious that $|F'_i| \leq 3 < \delta(G)$, for $1 \leq i \leq n-1$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(w_1, v'_1), (w_2, v'_2), (w_3, v'_3), (w_4, v'_4), (v, v')\}$ induce a fractional perfect matching of $G \square C_n - F$.

Case 2. $|F'_0| = \delta(G) + 1$. Clearly $|F'_i| = 1$, $|F'_j| = 0$ for $2 \leq i \leq n-1$. Let $F'_0 = F_0 - \{\alpha\}$ for every $\alpha \in F_0$. Since $G_0$ is fractional strongly super matched, either $G_0 - F'_0$ has fractional perfect matching or $F'_0$ is a trivial FSMP set of $G_0$. Let’s consider two subcases:

Subcase 2.1 $G_0 - F'_0$ has fractional perfect matching.
Suppose $\alpha$ is a vertex in $K_2$ of $\{(\alpha, \alpha')\}$. We assume that $\alpha''$ is a cross neighbor of $\alpha'$. It is clear that $G_0 - F_0' - \{\alpha, \alpha'\}$ has a fractional perfect matching $f_0$. Let $F'_i = (\{\alpha''\} \cap V(G_i)) \cup F_i$ for $1 \leq i \leq n - 1$. Obviously, $|F'_i| \leq 2 < \delta(G)$, $(1 \leq i \leq n - 1)$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $(\{\alpha', \alpha''\})$ induce a fractional perfect matching of $G \square C_n - F$. Let $\alpha$ be a vertex of a subgraph of Hamiltonian graph of $(G_0 - F_0')[V_i]$ on an odd number of vertices. It is obvious that $G_0 - F_0 - \{\alpha\}$ has a fractional perfect matching $f_0$. Then, $|F'_i| \leq 1 < \delta(G)$, $(1 \leq i \leq n - 1)$, thus $G_i - F'_i$ has fractional perfect matching $f_i$. Thus $\bigcup_{i=0}^{n-1} f_i$ induce a fractional perfect matching of $G \square C_n - F$.

Suppose $\alpha$ is an edge and $\alpha = (v, w)$. Let $\alpha$ be an edge of subgraph $K_2$ by partition $V_j$. There still exists a fractional perfect matching $f_0$ in $G_0 - F_0' - \{v, w\}$. Assume that $v'$ and $w'$ are cross neighbors of $v$ and $w$, respectively. Let $F'_i = (\{v', w'\} \cap V(G_i)) \cup F_i$ for $1 \leq i \leq n - 1$. Obviously, $|F'_i| \leq 3 < \delta(G)$, $(1 \leq i \leq n - 1)$, then $G_i - F'_i$ has a fractional perfect matching $f_i$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(v', w'), (w, w')\}$ induce a fractional perfect matching of $G \square C_n - F$. Suppose $\alpha$ is an edge of a subgraph of Hamiltonian graph of $(G_0 - F_0')[V_i]$ on an odd number of vertices. It is clear that there is a Hamiltonian path on an even number of vertices in subgraph by $V_i - \{v\}$ or $V_i - \{w\}$. We only need to prove the case of $V_i - \{v\}$. By Proposition 2.5, we can derive that $G_0 - F_0' - v$ has a fractional perfect matching $f_0$. We assume that $v'$ is a cross neighbor of $v$. Let $F'_i = (\{v'\} \cap V(G_i)) \cup F_i$ for $1 \leq i \leq n - 1$. Thus $|F'_i| \leq 2 < \delta(G)$, then $G_i - F'_i$ has a fractional perfect matching $f_i$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(v, v')\}$ induce a fractional perfect matching of $G \square C_n - F$.

Subcase 2.2 $F'_i$ is a trivial FSMP set.

This proof is similar to the proof of Subcase 1.2.

**Case 3.** $|F_0| = \delta(G)$. Clearly $|F'| = 2$ for $1 \leq i \leq n - 1$. Because $G_0$ is fractional strongly super matched, either $G_0 - F_0$ has a fractional perfect matching or $F_0$ is a trivial FSMP set. Suppose that $G_0 - F_0$ has a fractional perfect matching $f_0$. It is clear that $|F'| \leq 2 < \delta(G)$ for $1 \leq i \leq n - 1$, so $G_i - F_i$ has a fractional perfect matching $f_i$. Thus $\bigcup_{i=0}^{n-1} f_i$ induce a fractional perfect matching of $G \square C_n - F$. Suppose that $F_0$ is trivial FSMP set, let $v$ be an isolated vertex in $G_0 - F_0$. There is a fractional perfect matching in $G_0 - F_0 - v$ by Lemma 2.7. Assume that $v'$ is a cross neighbor of $v$. Let $F'_i = (\{v'\} \cap V(G_i)) \cup F_i$ for $1 \leq i \leq n - 1$. Thus $|F'_i| \leq 3 < \delta(G)$, $G_i - F'_i$ has fractional perfect matching $f_i$. Thus $\bigcup_{i=0}^{n-1} f_i$ and $\{(v, v')\}$ induce a fractional perfect matching of $G \square C_n - F$.

**Case 4.** $|F_0| \leq \delta(G) - 1$. Since $G_i$ is fractional strongly super matching, $G_i - F_i$ has fractional perfect matching. Thus $\bigcup_{i=0}^{n-1} f_i$ induce a fractional perfect matching of $G \square C_n - F$. 

\[\square\]
Now that we have completed the proof to show that \( G \square C_n \) is fractional strongly super matched. Let us look at some applications of this result to torus networks.

4 Application

The torus network is one of the most popular interconnection networks for massively parallel computing systems. The torus forms a basic class of interconnection networks. Let \( C_k \) be the cycle of length \( k \) with the vertex set \( \{0, 1, \ldots, k - 1\} \). Two vertices \( u, v \in V(C_k) \) are adjacent in \( C_k \) if and only if \( u = v \pm 1 (\text{mod} \ k) \). The torus \( T(k_1, k_2, \ldots, k_n) \) with \( n \geq 2 \) and \( k_i \geq 3 \) for all \( i \) is defined to be \( T(k_1, k_2, \ldots, k_n) = C_{k_1} \square C_{k_2} \square \ldots \square C_{k_n} \) with the vertex set \( \{u_1u_2\ldots u_n : u_i \in \{0, 1, \ldots, k_i - 1\}, 1 \leq i \leq n\} \). Two vertices \( u_1u_2\ldots u_n \) and \( v_1v_2\ldots v_k \) are adjacent in \( T(k_1, k_2, \ldots, k_n) \) if and only if there exists some \( j \in \{1, 2, \ldots, n\} \) such that \( u_j = v_j \pm 1 (\text{mod} \ k_j) \) and \( u_i = v_i \) for \( i \in \{1, 2, \ldots, k \} \setminus \{j\} \). Clearly, \( T(k_1, k_2, \ldots, k_n) \) is a connected 2\( k \)-regular graph consisting of \( k_1k_2\ldots k_n \) vertices. Let \( T(k_1, k_2) \) be a 2-dimensional torus, \( k_1 \geq 5 \) and \( k_2 \geq 5 \) are two odd integers. Then \( T(k_1, k_2) = C_{k_1} \square C_{k_2} \). We view \( C_{k_1} \square C_{k_2} \) as consisting of \( k_2 \) copies of \( C_{k_1} \). Let these copies be \( C_{k_1}^0, C_{k_1}^1, \ldots, C_{k_1}^{k_2-1} \) labeled along the cycle \( C_{k_2} \).

In this section, first we obtain \( mp(T(k_1, k_2, \ldots, k_n)) = 2n \) when \( T(k_1, k_2, \ldots, k_n) \) is a torus with an even number of vertices. Secondly we prove that \( f_{smp}(T(k_1, k_2)) = 4 \), \( (k_1 \geq 5 \) and \( k_2 \geq 5) \), and every optional fractional matching preclusion set is trivial. Then we can infer that \( f_{smp}(T(k_1, k_2, \ldots, k_n)) = 2n \) for \( n \geq 2 \), \( k_i \geq 5(\text{\( k_i \) is odd}) \).

**Theorem 4.1.** Let \( T(k_1, k_2, \ldots, k_n) \) be a torus with an even number of vertices. Then \( f_{mp}(T(k_1, k_2, \ldots, k_n)) = 2n \). Moreover, every optional fractional matching preclusion set is trivial.

**Proof.** From the definition of \( T(k_1, k_2, \ldots, k_n) \), it is a connected 2\( n \)-regular graph. By Theorem 2.3 we know that \( mp(T(n_1, n_2, \ldots, n_k)) = 2n \). It is clear that \( mp(T(k_1, k_2, \ldots, k_n)) \leq f_{mp}(T(k_1, k_2, \ldots, k_n)) \leq \delta(T(k_1, k_2, \ldots, k_n)) \) and \( mp(T(k_1, k_2, \ldots, k_n)) = \delta(T(k_1, k_2, \ldots, k_n)) = 2n \). By Theorem 2.3 each of \( T(k_1, k_2, \ldots, k_n) \) minimum MP sets is trivial, so every optimal fractional matching preclusion set is trivial. The proof is complete.

**Theorem 4.2.** Let \( k_1 \geq 5 \) and \( k_2 \geq 5 \) be two odd integers. Then \( f_{smp}(T(k_1, k_2)) = 4 \) and every optional fractional matching preclusion set is trivial.

**Proof.** In this part, first we introduce some notations. Let \( F \subseteq V(T(k_1, k_2)) \cup E(T(k_1, k_2)) \),
\( F_i = F \cap (V(C^i_{k_1}) \cup E(C^i_{k_1})) \) for \( 0 \leq i \leq k_2 - 1 \). Let \( F^V = F \cap V(T(k_1, k_2)) \), \( F^E = F \cap E(T(k_1, k_2)) \) and \( |F| = |F^V| + |F^E| = 4 \). It suffices to prove that \( T(k_1, k_2) - F \) has a fractional perfect matching or \( F \) is a trivial fractional matching preclusion set. We may assume that \( T(k_1, k_2) - F \) has a fractional perfect matching. If \( |F^V| \) is even, thus \( T(k_1, k_2) - F \) has a perfect matching by Lemma 2.9. So we only consider that \( |F^V| \) is odd. Without loss of generality, we assume \( |F_{i-1}| \geq |F_i| \) for \( 1 \leq i \leq k_2 - 1 \). If a graph contain an isolated vertex or a path with an odd number of vertices, then the graph has no a fractional perfect matching. So in this proof, we consider an isolated vertex as a path with an odd number of vertices.

**Case 1.** \( |F^V| = 3 \).

a. \( |F_0| = 4 \), clearly \( |F_1| = 0 \), for \( 1 \leq i \leq k_2 - 1 \). Since \( |F^V| = 3 \), it follows that \( |F^E| = 1 \). It is clear that \( C^0_{k_1} - F_0 \), \( (k_1 \text{ is odd and } k_1 \geq 5) \) exists at most four paths with an odd number of vertices. Suppose that \( v_1, v_2, v_3 \) and \( v_4 \) are end vertices of the four paths, respectively. Obviously, \( C^0_{k_1} - F_0 \setminus \{v_1, v_2, v_3, v_4\} \) has a fractional perfect matching \( f_0 \). By definition of \( T(k_1, k_2) \), each vertex of \( C^0_{k_1} \) has two cross neighbors, one is in the \( C^1_{k_1} \) and other one is in the \( C^{k_2-1}_{k_1} \). Assume that \( v'_1 \) and \( v'_2 \) in \( C^1_{k_1} \) are cross neighbors of \( v_1 \) and \( v_2 \), \( v'_3 \) and \( v'_4 \) in \( C^{k_2-1}_{k_1} \) are cross neighbors of \( v_3 \) and \( v_4 \). It is clear that \( C^1_{k_1} - \{v'_1, v'_2\} \) exists at most one path with an odd number of vertices; \( C^{k_2-1}_{k_1} - \{v'_3, v'_4\} \) exists at most one path with an odd number of vertices. Suppose that \( w_1 \) is end vertex of the odd path in \( C^1_{k_1} - \{v'_1, v'_2\} \); \( w_2 \) is an end vertex of the odd path in \( C^{k_2-1}_{k_1} - \{v'_3, v'_4\} \). Obviously, \( C^1_{k_1} - \{v'_1, v'_2\} - w_1 \) has a fractional perfect matching \( f_1 \); \( C^{k_2-1}_{k_1} - \{v'_3, v'_4\} - w_2 \) has a fractional perfect matching \( f_{k_2-1} \). Assume that \( w'_1 \) in \( C^2_{k_1} \) is a cross neighbor of \( w_1 \); \( w'_2 \) in \( C^{k_2-2}_{k_1} \) is a cross neighbor of \( w_2 \). Obviously, \( C^2_{k_1} - w'_1 \) has a fractional perfect matching \( f_2 \); \( C^{k_2-2}_{k_1} - w'_2 \) has a fractional perfect matching \( f_{k_2-2} \). At last, we can conclude that \( \bigcup_{i=0}^{k_2-1} f_i \) and \( \{(v_1, v'_1), (v_2, v'_2), (v_3, v'_3), (v_4, v'_4), (w_1, w'_1), (w_2, w'_2)\} \) induce a fractional perfect matching of \( T(k_1, k_2) - F \).

b. \( |F_0| = 3 \), \( |F_1| = 1 \), clearly \( |F_i| = 0 \), for \( 2 \leq i \leq k_2 - 1 \). If \( F_0 \) contains three vertices, then \( F_1 \) contains an edge. So \( C^0_{k_1} - F_0 \) exists at most two paths with an odd number of vertices; \( C^1_{k_1} - F_1 \) exists one path with an odd number of vertices. Suppose that \( v_1 \) and \( v_2 \) are end vertices of the two odd paths in \( C^0_{k_1} \); \( v_3 \) is end vertex of the odd path in \( C^1_{k_1} \). Obviously, \( C^0_{k_1} - F_0 \setminus \{v_1, v_2\} \) has a fractional perfect matching \( f_0 \); \( C^1_{k_1} - F_1 - v_3 \) has a fractional perfect matching \( f_1 \). Assume that \( v'_1 \) and \( v'_2 \) in \( C^{k_2-1}_{k_1} \) are cross neighbors of \( v_1 \) and \( v_2 \); \( v'_3 \) in \( C^2_{k_1} \) is cross neighbor of \( v_3 \). It is clear that \( C^2_{k_1} - v'_3 \) has a fractional perfect matching \( f_2 \). \( C^{k_2-1}_{k_1} - \{v'_1, v'_2\} \) exists at most one path with an odd number of vertices and let \( w \) be an end vertex of the odd path. Suppose \( w' \) in \( C^{k_2-2}_{k_1} \) is a cross neighbor of \( w \). It is easy to show that \( C^{k_2-2}_{k_1} - w' \) has a fractional perfect matching \( f_{k_2-2} \). At last, we can conclude that \( \bigcup_{i=0}^{k_2-1} f_i \) and \( \{(v_1, v'_1), (v_2, v'_2), (v_3, v'_3), (w, w')\} \)
induce a fractional perfect matching of $T(k_1, k_2) - F$. If $F_0$ contains two vertices and an edge, then $F_1$ contains a vertex. The proof of this case is similar to the one above. We can always find a fractional perfect matching in $T(k_1, k_2) - F$.

c. $|F_0| = 2$. Clearly, $|F_1| = 2$ or $|F_1| = 1$ and $|F_2| = 1$. It is easy to show that $C^0_{k_1} - F_0$ exists at most one path with an odd number of vertices. Let $w$ be an end vertex of the odd path and $w'$ be a cross neighbor of $w$ in $C^{k_2-1}_{k_1}$. When $|F_1| = 2$, let $F_1$ contain two vertices. Moreover, $C^1_{k_1} - F_1$ exists at most one path with an odd number of vertices. Assume that $v$ is an end vertex of the odd path. Suppose that $v$ has an fractional perfect matching $f_1$. Let $v'$ in $C^2_{k_1}$ be the cross neighbor of $v$. Obviously, $C^1_{k_1} - v'$ also has a fractional perfect matching $f_2$. At last, we can conclude that $\bigcup_{i=0}^{k_2-1} f_i$ and $\{(v, v'), (w, w')\}$ induce a fractional perfect matching of $T(k_1, k_2) - F$. When $|F_1| = 1$ and $|F_2| = 1$, let $F_1$ contain an edge. The proof is similar to the edge in $F_0$ or $F_2$, so we only need to proof that the edge is in $F_1$. $C^0_{k_1} - F_0$ exists a path with odd number of vertices and let $v_1$ be an end vertex of the odd path; $C^1_{k_1} - F_1$ also exists a path with an odd number of vertices and let $v_2$ be end vertex of the odd path. Suppose that $v'_1$ in $C^2_{k_1}$ is a cross neighbor of $v_1$; $v'_2$ in $C^3_{k_1}$ is a cross neighbor of $v_2$. Moreover, $C^2_{k_1} - F_2 - v'_2$ exists at most one path on an odd number vertices and let $v_3$ be the end vertex of the path. Assume that $v'_3$ in $C^3_{k_1}$ is a cross neighbor of $v_3$. At last, we can conclude that $\bigcup_{i=0}^{k_2-1} f_i$ and $\{(v_1, v'_1), (v_2, v'_2), (v_3, v'_3), (w, w')\}$ induce a fractional perfect matching of $T(k_1, k_2) - F$.

d. $|F_1| = 1$, for $0 \leq i \leq 3$ and $|F_j| = 0$, for $4 \leq j \leq k_2 - 1$. Let $F_2$ contain an edge, the proof is similar to the other case. Obviously, $C^0_{k_1} - F_0$ has a fractional perfect matching $f_0$ and $C^1_{k_1} - F_1$ has a fractional perfect matching $f_1$. $C^2_{k_1} - F_2$ exists a path with an odd number of vertices and let $v_1$ be an end vertex of the odd path. Suppose $v'_1$ in $C^3_{k_1}$ is a cross neighbor of $v_1$. So $C^3_{k_1} - F_3 - v'_1$ exists at most a path with an odd number of vertices and let $v_2$ be an end vertex of the odd path. Suppose that $v'_2$ in $C^4_{k_1}$ is a cross neighbor of $v_2$. At last, we can conclude that $\bigcup_{i=0}^{k_2-1} f_i$ and $\{(v_1, v'_1), (v_2, v'_2)\}$ induce a fractional perfect matching of $T(k_1, k_2) - F$.

Case 2. $|F^V| = 1$.

We can prove Case 2. the way we proved Case 1. It is easy to show that this case is identical to the corresponding Case 1. in the proof of Theorem 3.3, thus completing the proof.

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**Corollary 4.3.** Let $k_i \geq 5$ be an odd integer for $1 \leq i \leq n$. Then $fsmp(T(k_1, k_2, ..., k_n)) = 2n$. Moreover, $T(k_1, k_2, ..., k_n)$ is fractional super strong matched.

**Proof.** By applying Theorem 4.2 combined with Theorem 3.1, the proof of Corollary 4.3 is
5 Conclusion

The topic of fractional matching preclusion is getting more and more attention and we already know different research groups who are interested in this topic. The concept of fractional matching preclusion introduced in [10] is very useful in networks. In this paper, we obtained the result for Cartesian product graphs and apply this result to torus networks which is an important class of interconnection networks.

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