Boundary Conditions for Singular Perturbations of Self-Adjoint Operators

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Abstract. Let $A : D(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be an injective self-adjoint operator and let $\tau : D(A) \rightarrow \mathcal{X}$, $\mathcal{X}$ a Banach space, be a surjective linear map such that $\|\tau \phi\|_{\mathcal{X}} \leq c \|A \phi\|_{\mathcal{H}}$. Supposing that Kernel $\tau$ is dense in $\mathcal{H}$, we define a family $A_{\tau}^\Theta$ of self-adjoint operators which are extensions of the symmetric operator $A_{\{\tau=0\}}$. Any $\phi$ in the operator domain $D(A_{\tau}^\Theta)$ is characterized by a sort of boundary conditions on its univocally defined regular component $\phi_{\text{reg}}$, which belongs to the completion of $D(A)$ w.r.t. the norm $\|A \phi\|_{\mathcal{H}}$. These boundary conditions are written in terms of the map $\tau$, playing the role of a trace (restriction) operator, as $\tau \phi_{\text{reg}} = \Theta Q \phi$, the extension parameter $\Theta$ being a self-adjoint operator from $\mathcal{X}'$ to $\mathcal{X}$. The self-adjoint extension is then simply defined by $A_{\tau}^\Theta \phi := A \phi_{\text{reg}}$. The case in which $A \phi = \Psi \ast \phi$ is a convolution operator on $L^2(\mathbb{R}^n)$, $\Psi$ a distribution with compact support, is studied in detail.

1. Introduction

Let

$A : D(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$

be a self-adjoint operator on the complex Hilbert space $\mathcal{H}$ (to prevent any misunderstanding we remark here that all over the paper we will avoid to identify a Hilbert space with its strong dual). As usual $D(A)$ inherits a Hilbert space structure by introducing the scalar product leading to the graph norm

$\|\phi\|^2_A := \langle \phi, \phi \rangle_{\mathcal{H}} + \langle A \phi, A \phi \rangle_{\mathcal{H}}$.

Considering then a linear bounded operator

$\tau : D(A) \rightarrow \mathcal{X}, \quad \tau \in \mathcal{B}(D(A), \mathcal{X}),$

$\mathcal{X}$ a complex Banach space, we are interested in describing the self-adjoint extensions of the symmetric operator $A_{\{\tau=0\}}$. In typical situations $A$ is a (pseudo-)differential operator on $L^2(\mathbb{R}^n)$ and $\tau$ is a trace (restriction) operator along some null subset $F \subset \mathbb{R}^n$ (see e.g. [1]-[4], [6]-[8], [16]-[19], [21], [22] and references therein).
Denoting the resolvent set of \( A \) by \( \rho(A) \), we define \( R(z) \in B(\mathcal{H}, D(A)) \), \( z \in \rho(A) \), by
\[
R(z) := (-A + z)^{-1}
\]
and we then introduce, for any \( z \in \rho(A) \), the operators \( \tilde{G}(z) \in B(\mathcal{H}, \mathcal{X}) \) and \( G(z) \in \tilde{B}(\mathcal{X}', \mathcal{H}) \) by
\[
\tilde{G}(z) := \tau \cdot R(z), \quad G(z) := C^{-1}_{\mathcal{H}} \cdot \tilde{G}(z^*)'.
\]
(1)

Here the prime \( ' \) denotes both the strong dual space and the (Banach) adjoint map, and \( C_{\mathcal{H}} \) indicates the canonical conjugate-linear isomorphism on \( \mathcal{H} \) to \( \mathcal{H}' \) (the reader is referred to section 2 below for a list of definitions and notations).

As an immediate consequence of the first resolvent identity for \( R(z) \) we have
\[
(z - w) R(w) \cdot G(z) = G(w) - G(z)
\]
and so
\[
\forall w, z \in \rho(A), \quad \text{Range}(G(w) - G(z)) \subseteq D(A).
\]
(3)

In [19, thm. 2.1], by means of a Krein-like formula, and under the hypotheses

\( \tau \) is surjective \hspace{1cm} (h1)
\[
\text{Range } \tau' \cap \mathcal{H}' = \{0\}, \hspace{1cm} (h2)
\]
we constructed a family \( A_{\tau}^{\ominus} \) of self-adjoint extension of \( A_{\{\tau=0\}} \) by giving its resolvent family. The hypothesis (h1) could be weakened, see [19], but here we prefer to use a simpler framework. In formulating (h2) we used the embedding of \( \mathcal{H}' \) into \( D(A)' \supseteq \text{Range } \tau' \) given by \( \varphi \mapsto \langle C^{-1}_{\mathcal{H}} \varphi, \cdot \rangle_{\mathcal{H}} \). Such an hypothesis is then equivalent to the denseness, in \( \mathcal{H} \), of the set \( \{\tau = 0\} \). Indeed there exists \( \ell \in \mathcal{X}' \) such that \( \tau' \ell \in \mathcal{H}' \) if and only if there exists \( \psi \in \mathcal{H} \) (necessarily orthogonal to Kernel \( \tau \)) such that for any \( \phi \in D(A) \) one has \( \langle \psi, \phi \rangle_{\mathcal{H}} = \ell(\tau \phi) \).

The advantage of the formula given in [19] over other approaches (see e.g. [20], [9], [10], [12] and references therein) is its relative simplicity, being expressed directly in terms of the map \( \tau \); moreover the domain of definition of \( A_{\tau}^{\ominus} \) can be described, interpreting the map \( \tau \) as a trace (restriction) operator, in terms of a sort of boundary conditions (see [19, remark 2.10]). In the case \( 0 \notin \sigma(A) \), \( \sigma(A) \) denoting the spectrum of \( A \), this description becomes particularly expressive since \( A_{\tau}^{\ominus} \phi \) can be simply defined by the original operator applied to the regular component of \( \phi \). Such a regular component \( \phi_0 \in D(A) \) is univocally determined by the natural decomposition which enter in the definition of \( D(A_{\tau}^{\ominus}) \) and it has to satisfy the boundary condition
\[
\tau \phi_0 = \Theta Q_{\phi}.
\]

More precisely, by (h1), (h2) and by [19, lemma 2.2, thm. 2.1, prop. 2.1, remarks 2.10, 2.12], we have the following
Theorem 1. Let $A : D(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be self-adjoint with $0 \notin \sigma(A)$, let $\tau : D(A) \rightarrow \mathcal{X}$ be continuous and satisfy (h1) and (h2). If $\Theta \in \mathcal{L}(\mathcal{X}', \mathcal{X})$ is self-adjoint, $G := G(0)$ and

$$D(A_G^\tau) := \{ \phi \in \mathcal{H} : \phi = \phi_0 + GQ_\phi, \phi_0 \in D(A), Q_\phi \in D(\Theta), \tau_0 \phi_0 = \Theta Q_\phi \},$$

then the linear operator

$$A_G^\tau : D(A_G^\tau) \subseteq \mathcal{H} \rightarrow \mathcal{H}, \quad A_G^\tau \phi := A\phi_0$$

is self-adjoint and coincides with $A$ on the kernel of $\tau$; the decomposition entering in the definition of its domain is unique. Its resolvent is given by

$$R_G^\tau(z) := R(z) + G(z) \cdot (\Theta + \Gamma(z))^{-1} \cdot \hat{G}(z), \quad z \in W_G^- \cup W_G^+ \cup \mathbb{C} \setminus \mathbb{R},$$

where

$$\Gamma(z) := \tau \cdot (G - G(z))$$

and

$$W_G^\pm := \{ \lambda \in \mathbb{R} \cap \rho(A) : \gamma(\pm \Gamma(\lambda)) > -\gamma(\pm \Theta) \}.$$

Remark 2. By (h1) one has $\mathcal{X} \simeq D(A) / \text{Kernel } \tau \simeq (\text{Kernel } \tau)^\perp$ and so

$$D(A) \simeq \text{Kernel } \tau \oplus \mathcal{X}.$$

This implies that $\mathcal{X}$ inherits a Hilbert space structure and we could then identify $\mathcal{X}'$ with $\mathcal{X}$. Even if this gives some advantage (see [19, remarks 2.13-2.16, lemma 2.4]) here we prefer to use only the Banach space structure of $\mathcal{X}$.

The purpose of the present paper is to extend the above theorem to the case in which $A$ is merely injective. Thus, denoting the pure point spectrum of $A$ by $\sigma_{pp}(A)$, we require $0 \notin \sigma_{pp}(A)$ but we do not exclude the case $0 \in \sigma(A) \setminus \sigma_{pp}(A)$; this is a typical situation when $A$ is a differential operator on $L^2(\mathbb{R}^n)$. In order to carry out this program we will suppose that the map $\tau$ has a continuous extension to $\hat{D}(A)$, the completion of $D(A)$ with respect to the norm $\|A\phi\|_\mathcal{H}$ (note that $\hat{D}(A) = D(A)$ when $0 \notin \sigma(A)$). This further hypothesis allows then to perform the limit $\lim_{\epsilon \rightarrow 0} G(i\epsilon) - G(z)$ (see lemma 3); thus an analogue on the above theorem 1 is obtained (see theorem 5). Such an abstract construction is successively specialized to the case in which $A\phi = \Psi \ast \phi$ is a convolution operator on $L^2(\mathbb{R}^n)$, where $\Psi$ is a distribution with compact support (so that this comprises the case of differential-difference operators). In this situation the results obtained in theorem 5 can be made more appealing (see theorem 11). The case in which $A = \Delta : H^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n), n > 4$, and $\tau$ is the trace (restriction) operator along a $d$-set with a compact closure of zero Lebesgue measure, $0 < n - d < 4$, is explicitly studied (see example 14). Of course, since $-\Delta$ is not negative, in this case one could apply theorem 1 to $-\Delta + \lambda, \lambda > 0$, and then define $-\Delta_G^\tau := (-\Delta + \lambda)_G^\tau - \lambda$. However this alternative definition
looks a bit artificial and has the drawback of giving rise to boundary conditions which depend on the arbitrary parameter $\lambda$. The starting motivation of this work was indeed the desire to get rid of such a dependence.

2. Definitions and notations

- Given a Banach space $X$ we denote by $X'$ its strong dual;
- $L(X, Y)$, resp. $\tilde{L}(X, Y)$, denotes the space of linear, resp. conjugate linear, operators from the Banach space $X$ to the Banach space $Y$; $L(X) := L(X, X)$, $\tilde{L}(X) := \tilde{L}(X, X)$.
- $B(X, Y)$, resp. $\tilde{B}(X, Y)$, denotes the (Banach) space of bounded, everywhere defined, linear, resp. conjugate linear, operators on the Banach space $X$ to the Banach space $Y$.
- Given $A \in L(X, Y)$ and $\tilde{A} \in \tilde{L}(X, Y)$ densely defined, the closed operators $A' \in L(Y', X')$ and $\tilde{A}' \in \tilde{L}(Y', X')$ the are the adjoints of $A$ and $\tilde{A}$ respectively, i.e.
  \[ \forall x \in D(A) \subseteq X, \quad \forall \ell \in D(A') \subseteq Y', \quad (A' \ell)(x) = \ell(Ax), \]
  \[ \forall x \in D(\tilde{A}) \subseteq X, \quad \forall \ell \in D(\tilde{A}') \subseteq Y', \quad (\tilde{A}' \ell)(x) = (\ell(\tilde{A}x))^* \]
where $\ast$ denotes complex conjugation.
- $J_X \in B(X, X'')$ indicates the injective map (an isomorphism when $X$ is reflexive) defined by $(J_X x)(\ell) := \ell(x)$.
- A closed, densely defined operator $A \in L(X', X) \cup \tilde{L}(X', X)$ is said to be self-adjoint if $J_X \cdot A = A'$.
- For any self-adjoint $A \in L(X', X) \cup \tilde{L}(X', X)$ we define
  \[ \gamma(A) := \inf \{ \ell(A\ell), \ \ell \in D(A), \ \|\ell\|_{X'} = 1 \} . \]
- If $H$ is a complex Hilbert space with scalar product (conjugate linear w.r.t. the first variable) $\langle \cdot, \cdot \rangle$, then $C_H \in B(H, H')$ denotes the isomorphism defined by $(C_Hy)(x) := \langle y, x \rangle$. The Hilbert adjoint of the densely defined linear operator $A$ is then given by $A^* = C_H^{-1} \cdot A' \cdot C_{H}$. 
- $\mathcal{F}$ and $\ast$ denote Fourier transform and convolution respectively.
- $\mathcal{D}'(\mathbb{R}^n)$ denotes the space of distributions and $\mathcal{E}'(\mathbb{R}^n)$ is the subspace of distributions with compact support.
- $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is the usual scale of Sobolev-Hilbert spaces, i.e. $H^s(\mathbb{R}^n)$ is the space of tempered distributions with a Fourier transform which is square integrable w.r.t. the measure with density $(1 + |x|^2)^s$. As usual the strong dual of $H^s(\mathbb{R}^n)$ will be represented by $H^{-s}(\mathbb{R}^n)$.
- $c$ denotes a generic strictly positive constant which can change from line to line.
3. Singular Perturbations and Boundary Conditions

Given the injective self-adjoint operator \( A : D(A) \subseteq \mathcal{H} \rightarrow \mathcal{H} \), we denote by \( \hat{D}(A) \) the Banach space given by the completion of \( D(A) \) with respect to the norm

\[
\| \phi \|_{(A)} := \| A\phi \|_{\mathcal{H}}.
\]

As usual \( D(A) \) will be treated as a (dense) subset of \( \hat{D}(A) \) by means of the canonical embedding \( I : D(A) \rightarrow \hat{D}(A) \) which associates to \( \phi \) the set of all Cauchy sequences converging to \( \phi \).

As in the introduction we consider then a continuous linear map

\[
\tau : D(A) \rightarrow X,
\]

\( X \) is a Banach space, and we will suppose that it satisfies, besides (h1) and (h2), the further hypothesis

\[
\| \tau \phi \|_X \leq c \| A\phi \|_{\mathcal{H}}.
\]

(h3)

By (h3) \( \tau \) admits an extension belonging to \( \mathcal{B}(\hat{D}(A), X) \); analogously \( A \) admits an extension belonging to \( \mathcal{B}(\hat{D}(A), \mathcal{H}) \). By abuse of notation we will use the same symbols \( \tau \) and \( A \) to denote these extensions.

Let us now take a sequence \( \{\epsilon_n\}_{1}^{\infty} \subset \mathbb{R} \) converging to zero. By functional calculus one has

\[
\| (-A \cdot R(i\epsilon_n) - I)\phi \|_{\mathcal{H}}^2 = \int_{\sigma(A)} d\mu_\phi(\lambda) \frac{\epsilon_n^2}{\lambda^2 + \epsilon_n^2}
\]

with \( \mu_\phi(\{0\}) = 0 \) since \( 0 \notin \sigma_{pp}(A) \). Thus

\[
1 \geq \frac{\epsilon_n^2}{\lambda^2 + \epsilon_n^2} \rightarrow 0, \quad \mu_\phi\text{-a.e.}
\]

and, by dominated convergence theorem,

\[
\mathcal{H} - \lim_{n \uparrow \infty} -A \cdot R(i\epsilon_n)\phi = \phi.
\]

So \( \{R(i\epsilon_n)\phi\}_{1}^{\infty} \) is a Cauchy sequence in \( D(A) \) with respect to the norm \( \| \cdot \|_{(A)} \).

We can therefore define \( R \in \mathcal{B}(\mathcal{H}, \hat{D}(A)) \) by

\[
R\phi := \hat{D} - \lim_{n \uparrow \infty} R(i\epsilon_n)\phi,
\]

and then \( K(z) \in \hat{B}(X', \hat{D}(A)) \) by

\[
K(z) := zR \cdot G(z).
\]

Alternatively, using (2), \( K(z) \) can be defined by

\[
K(z)\phi := \hat{D} - \lim_{n \uparrow \infty} (G(i\epsilon_n) - G(z)) \phi.
\]
This immediately implies, using (3),
\[ \forall w, z \in \rho(A), \quad \text{Range}(K'(w) - K(z)) \subseteq D(A) \]
and
\[ \forall w, z \in \rho(A), \quad K(w) - K(z) = G(z) - G(w). \]
(4)
Also note that
\[ -A \cdot K(z) = zG(z). \]
(5)

**Lemma 3.** The map
\[ \Gamma : \rho(A) \to \tilde{B}(X', X), \quad \Gamma(z) := \tau \cdot K(z) \]
satisfies the relations
\[ \Gamma(z) - \Gamma(w) = (z - w)\hat{G}(w) \cdot G(z) \]
(6)
and
\[ J_X \cdot \Gamma(z^*) = \Gamma(z)', \]
(7)

**Proof.** Since \( K(z) \) is the strong limit of \( G(\pm i\epsilon_n) - G(z) \), one has
\[ \forall \ell \in X', \quad \Gamma(z)\ell = \lim_{n \uparrow \infty} \hat{\Gamma}_n(z)\ell, \]
where
\[ \hat{\Gamma}_n(z) : X' \to X, \quad \hat{\Gamma}_n(z) := \tau \cdot \left( \frac{G(i\epsilon_n) + G(-i\epsilon_n)}{2} - G(z) \right). \]
Thus \( \Gamma(z) \) satisfies (6) and
\[ \forall \ell_1, \ell_2 \in X', \quad \ell_1(\Gamma(z^*)\ell_2) = (\ell_2(\Gamma(z)\ell_1))^* \]
(which is equivalent to (7)) since \( \hat{\Gamma}_n(z) \) does (see [19, lemma 2.2]).

Before stating the next lemma we introduce the following definition:
Given \( \phi \in \mathcal{H} \) and \( \psi \in \hat{D}(A) \), the writing \( \phi = \psi \) will mean that \( \phi \) is in \( D(A) \) and \( J\phi = \psi \).

**Lemma 4.** Given \( \phi \in \mathcal{H} \), \( z \in \rho(A) \), suppose there exist \( \psi \in \hat{D}(A) \) and \( Q \in X' \) such that
\[ \phi - G(z)Q = \psi + K(z)Q. \]
(8)
Then the couple \( (\psi, Q) \) is unique and \( z \)-independent.
**Proof.** Let $(\psi_1, Q_1), (\psi_2, Q_2)$ both satisfy (8). Then

$$G(z)(Q_2 - Q_1) = (\psi_1 - \psi_2) + K(z)(Q_1 - Q_2).$$

By (h2) and the definition of $G(z)$ one has $\text{Range} G(z) \cap D(A) = \{0\}$ and so $Q_1 - Q_2 \in \text{Kernel} G(z)$. But (h1) implies the injectivity of $G(z)$ (see [19, remark 2.1]). Therefore $(\psi_1, Q_1) = (\psi_2, Q_2)$. The proof is then concluded observing that $z$-independence follows by (4).

We now can extend theorem 1 to the case in which $A$ is injective:

**Theorem 5.** Let $A : D(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be self-adjoint and injective, let $\tau : D(A) \rightarrow \mathcal{X}$ satisfy (h1)-(h3). Given $\Theta \in \tilde{L}(\mathcal{X}', \mathcal{X})$ self-adjoint, let $D(A_{\Theta})$ be the set of $\phi \in \mathcal{H}$ for which there exist $\phi_{\text{reg}} \in \hat{D}(A), Q_{\phi} \in D(\Theta)$ such that

$$\phi - G(z)Q_{\phi} = \phi_{\text{reg}} + K(z)Q_{\phi}$$

and

$$\tau \phi_{\text{reg}} = \Theta Q_{\phi}.$$ 

Then

$$A_{\Theta} : D(A_{\Theta}) \subseteq \mathcal{H} \rightarrow \mathcal{H}, \quad A_{\Theta} \phi := A \phi_{\text{reg}},$$

is a self-adjoint operator which coincides with $A$ on the kernel of $\tau$ and its resolvent is given by

$$R_{\Theta}(z) := R(z) + G(z) \cdot (\Theta + \Gamma(z))^{-1} \cdot \tilde{G}(z), \quad z \in W_{\Theta}^{-} \cup W_{\Theta}^{+} \cup \mathbb{C} \backslash \mathbb{R}.$$ 

where

$$\Gamma(z) := \tau \cdot K(z).$$

**Proof.** For brevity we define

$$\phi_z := \phi - G(z)Q_{\phi}, \quad \phi \in D(A_{\Theta})$$

and

$$\Gamma_{\Theta}(z) := \Theta + \Gamma(z).$$

Then one has

$$\Gamma_{\Theta}(z)Q_{\phi} = \tau \phi_{\text{reg}} + \tau \cdot K(z)Q_{\phi} = \tau \phi_z.$$ 

Since $\Gamma(z)$ is a bounded operator satisfying (6) and (7), and $\Theta$ is self-adjoint, by (h1) and [19, prop. 2.1, remark 2.12], $\Gamma_{\Theta}(z)$ has a bounded inverse for any $z \in W_{\Theta}^{-} \cup W_{\Theta}^{+} \cup \mathbb{C} \backslash \mathbb{R}$. Therefore

$$Q_{\phi} = \Gamma_{\Theta}(z)^{-1} \cdot \tau \phi_z$$

and

$$D(A_{\Theta}) \subseteq \{ \phi \in \mathcal{H} : \phi = \phi_z + G(z) \cdot \Gamma_{\Theta}(z)^{-1} \cdot \tau \phi_z, \phi_z \in D(A) \}.$$ 

Let us now prove the reverse inclusion.
Given $\phi \in \mathcal{H}$,

$$\phi = \phi_z + G(z) \cdot \Gamma_{\Theta}(z)^{-1} \cdot \tau \phi_z, \quad \phi_z \in D(A),$$

we define $\phi_{\text{reg}} \in \hat{D}(A)$ by

$$\phi_{\text{reg}} := \phi_z - K(z) \cdot \Gamma_{\Theta}(z)^{-1} \cdot \tau \phi_z.$$

Thus one has

$$\tau \phi_{\text{reg}} = \tau \phi_z - \tau \cdot K(z) \cdot \Gamma_{\Theta}(z)^{-1} \cdot \tau \phi_z \quad = \tau \phi_z - \Theta \cdot \Gamma_{\Theta}(z)^{-1} \cdot \tau \phi_z = \Theta Q_{\phi},$$

with

$$Q_{\phi} := \Gamma_{\Theta}(z)^{-1} \cdot \tau \phi_z.$$

In conclusion

$$D(A_{\Theta}^{\tau}) = \{ \phi \in \mathcal{H} : \phi = \phi_z + G(z) \cdot \Gamma_{\Theta}(z)^{-1} \cdot \tau \phi_z, \phi_z \in D(A) \}.$$

Since, by (5),

$$A \phi_{\text{reg}} = A \phi_z - A \cdot K(z)Q_{\phi} = A \phi_z + zG(z)Q_{\phi}$$

we have

$$(-A_{\Theta}^{\tau} + z)\phi = (-A + z)\phi_z.$$

Thus $A_{\Theta}^{\tau}$ coincides with the operator constructed in [19, thm. 2.1]; therefore, by (h2), this operator is self-adjoint, has resolvent given by $R_{\Theta}(\Phi)$ and is equal to $A$ on the kernel of $\tau$. $\square$

4. Singular perturbations of convolution operators

Let $\Psi \in \mathcal{E}'(\mathbb{R}^n)$. By Paley-Wiener theorem we know that $\mathcal{F}\Psi$ is a smooth function which is, together with its derivatives of any order, polynomially bounded. Then we define the continuous convolution operator

$$\Psi \ast : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n), \quad \phi \mapsto \Psi \ast \phi,$$

and, supposing $\mathcal{F}\Psi$ real-valued, the restriction of this operator to the dense subspace

$$D(\widetilde{\Psi}) := \{ \phi \in L^2(\mathbb{R}^n) : \Psi \ast \phi \in L^2(\mathbb{R}^n) \}$$

provide us with the self-adjoint convolution operator

$$\widetilde{\Psi} : D(\widetilde{\Psi}) \subseteq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \widetilde{\Psi} \phi := \Psi \ast \phi.$$

Evidently $\widetilde{\Psi}$ is injective if and only if the set of real zeroes of $\mathcal{F}\Psi$ is a null set. From now on we will therefore suppose that $\mathcal{F}\Psi$ is real and has a null set of real zeroes. This implies, since $(\text{Range } \widetilde{\Psi})^\perp = \text{Kernel } \widetilde{\Psi}^*$, that $\widetilde{\Psi}$ has a dense range and, using (h3), the following lemma becomes then obvious:
Lemma 6. The map \( \tau : D(\tilde{\Psi}) \to X \) can be extended to
\[
D(\tilde{\Psi}) := \{ \varphi \in D'(\mathbb{R}^n) : \Psi \ast \varphi \in L^2(\mathbb{R}^n) \},
\]
by defining
\[
\tau \varphi := \lim_{n \to \infty} \tau \phi_n,
\]
where \( \{ \phi_n \}_1^\infty \subset D(\tilde{\Psi}) \) is any sequence such that
\[
L^2 - \lim_{n \to \infty} \Psi \ast \phi_n = \Psi \ast \varphi.
\]

Now we will moreover suppose that \( \mathcal{F}\Psi \) is slowly decreasing, i.e. (see [11]) we will suppose that there exist \( k > 0 \) such that for any \( \zeta \in \mathbb{R}^n \) we can find a point \( \xi \in \mathbb{R}^n \) such that
\[
|\zeta - \xi| \leq k \log(1 + |\zeta|),
\]
\[
|\mathcal{F}\Psi(\xi)| \geq (k + |\xi|)^{-k}.
\]
By [11, thm. 1] we know that \( \Psi : D'(\mathbb{R}^n) \to D'(\mathbb{R}^n) \) is surjective if and only if \( \mathcal{F}\Psi \) is slowly decreasing. This is certainly true when \( \tilde{\Psi} \) is a differential operator, i.e. when \( \mathcal{F}\Psi \) is a polynomial. The hypotheses we made on \( \mathcal{F}\Psi \) permit us to state the following

Lemma 7. Given \( \Psi \) as above, one has the identification
\[
\tilde{D}(\tilde{\Psi}) \simeq D(\tilde{\Psi}) / \sim,
\]
where
\[
\varphi_1 \sim \varphi_2 \iff \Psi \ast \varphi_1 = \Psi \ast \varphi_2.
\]
This identification is given by the isometric maps which to the equivalence class of Cauchy sequences \( \{ \phi_n \}_1^\infty \in \tilde{D}(\tilde{\Psi}) \) associates the equivalence class of distributions \( [\varphi] \in D(\tilde{\Psi}) / \sim \) such that
\[
L^2 - \lim_{n \to \infty} \Psi \ast \phi_n = \Psi \ast \varphi.
\]

Proof. Given \( \{ \phi_n \}_1^\infty \in \tilde{D}(\tilde{\Psi}) \), the sequence \( \{ \Psi \ast \phi_n \}_1^\infty \) is a Cauchy one in \( L^2(\mathbb{R}^n) \) and so it converges to some \( f \in L^2(\mathbb{R}^n) \). Then, by [11, thm. 1], there exists \( \varphi \in D(\tilde{\Psi}) \) such that \( \Psi \ast \varphi = f \). Conversely let \( \varphi \in D(\tilde{\Psi}) \); since \( \tilde{\Psi} \) has a dense range there exists a (unique in \( \tilde{D}(\tilde{\Psi}) \)) sequence \( \{ \phi_n \}_1^\infty \subset D(\tilde{\Psi}) \) such that \( \Psi \ast \phi_n \) converges in \( L^2(\mathbb{R}^n) \) to \( \Psi \ast \varphi \).

Defining
\[
\tilde{D}'(\mathbb{R}^n) := D'(\mathbb{R}^n) / \sim
\]
and then the sum of $\phi \in L^2(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ plus $\psi = [\varphi] \in \hat{\mathcal{D}}(\tilde{\Psi}) \simeq \mathcal{D}(\tilde{\Psi})/\sim \subset \hat{\mathcal{D}}'(\mathbb{R}^n)$ by

$$\phi + \psi := [\phi + \varphi] \in \hat{\mathcal{D}}'(\mathbb{R}^n),$$

we can introduce the linear operator

$$G : \mathcal{X}' \to \hat{\mathcal{D}}'(\mathbb{R}^n), \quad G := G(z) + K(z).$$

According to lemma 4, theorem 5 and the definition of $G$, for any $\phi \in D(\tilde{\Psi}_\Theta)$ we can give the unique decomposition

$$\phi = \phi_{\text{reg}} + GQ_\phi.$$ 

Thus we can define $D(\tilde{\Psi}_\Theta)$ as the set of $\phi \in L^2(\mathbb{R}^n)$ for which there exists $Q_\phi \in D(\Theta)$ such that

$$\phi - GQ_\phi =: \phi_{\text{reg}} \in \hat{D}(\tilde{\Psi})$$

and

$$\tau \phi_{\text{reg}} = \Theta Q_\phi.$$ 

**Lemma 8.** The definition of $G$ is $z$-independent and

$$\forall \ell \in \mathcal{X}', \quad G\ell = [G_\ast \ell],$$

where

$$G_\ast : \mathcal{X}' \to \mathcal{D}'(\mathbb{R}^n),$$

is any conjugate linear operator such that

$$-\Psi \ast G_\ast \ell = \tau^* \ell.$$  \hspace{1cm} (9)

Here $\tau^* : \mathcal{X}' \to \mathcal{D}'(\mathbb{R}^n)$ is defined by

$$\tau^* \ell(\varphi) := (\ell(\tau \varphi^*))^*, \quad \varphi \in C_0^\infty(\mathbb{R}^d).$$

**Proof.** $z$-independence is an immediate consequence of (4). By the definition of $G$ and by (5) there follows

$$-\Psi \ast G_\ast \ell = (-\Psi \ast + z)G(z)\ell$$

and the proof is concluded by the relation

$$(-\Psi \ast + z) \cdot G(z) = \tau^* \hspace{1cm} (10)$$

which can be obtained proceeding as in [19, remark 2.4].
Remark 9. By [11, thm. 1], as $\mathcal{F}\Psi$ is slowly decreasing, the equation (9) is always resoluble; in particular, denoting the fundamental solution of $-\Psi^*$ by $\mathcal{G}$, when the convolution $\mathcal{G} \ast \tau^* \ell$ is well defined (e.g. when $\tau^* \ell \in \mathcal{E}'(\mathbb{R}^n)$), one has

$$G_* : \mathcal{X}' \to \mathcal{D}'(\mathbb{R}^n), \quad G_* \ell = \mathcal{G} \ast \tau^* \ell.$$  

Analogously, denoting the fundamental solution of $-\Psi^* + z$ by $\mathcal{G}_z$, one has

$$G(z) : \mathcal{X}' \to L^2(\mathbb{R}^n), \quad G(z) \ell = \mathcal{G}_z \ast \tau^* \ell.$$  

Remark 10. Note that $\mathcal{P}_\phi \mathcal{G} \phi_\phi = \mathcal{G} \phi$ and $\tau \phi_\phi = \tau \phi$ for any $\phi \in \mathcal{D}(\mathcal{P})$ such that $\phi_\phi = [\phi]$. Here we implicitly used the extension given in lemma 6 and the identification given in lemma 7. This also implies that $\Gamma(z)$ in lemma 3 can be re-written as

$$\Gamma(z) = \tau \cdot (G_* - G(z)).$$  

By remark 9, when the convolution is well defined, one can also write

$$\Gamma(z) \ell = \tau ((\mathcal{G} - \mathcal{G}_z) \ast \tau^* \ell).$$  

In conclusion, by making use of the previous lemmata and remarks, we can restate theorem 5 in the following way:

**Theorem 11.** Let $\Psi \in \mathcal{E}'(\mathbb{R}^n)$ with $\mathcal{F}\Psi$ real-valued, slowly decreasing and having a null set of real zeroes, let $\mathcal{P} : D(\mathcal{P}) \subseteq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $\mathcal{P} \phi := \Psi \phi$, let $\tau : D(\mathcal{P}) \to \mathcal{X}$ satisfy (h1)-(h3). Given $\Theta \in \mathcal{L}(\mathcal{X}', \mathcal{X})$ self-adjoint, let $D(\mathcal{P}_\Theta)$ be the set of $\phi \in L^2(\mathbb{R}^n)$ for which there exists $Q_\phi \in D(\Theta)$ such that

$$\phi - G_* Q_\phi =: \phi_\phi \in \mathcal{D}(\mathcal{P})$$

and

$$\tau \phi_\phi = \Theta Q_\phi.$$  

Then

$$\mathcal{P}_\Theta : D(\mathcal{P}_\Theta) \subseteq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \mathcal{P}_\Theta \phi := \Psi \phi,$$

is a self-adjoint operator which coincides with $\mathcal{P}$ on the kernel of $\tau$ and its resolvent is given by

$$R_{\Theta}^\tau(z) := R(z) + G(z) \cdot (\Theta + \Gamma(z))^{-1} \cdot \mathcal{G}(z), \quad z \in W_{\Theta}^- \cup W_{\Theta}^+ \cup \mathbb{C} \setminus \mathbb{R},$$

where

$$\Gamma(z) := \tau \cdot (G_* - G(z)).$$
Remark 12. The boundary conditions and the operators \( \Gamma(z) \) and \( \tilde{\Psi} \) appearing in the previous theorem are independent of the choice of the representative (see lemma 8) \( G_* \) entering in the definition of \( \varphi_{\text{reg}} \). Indeed any different choice will not change the equivalence class to which \( \varphi_{\text{reg}} \) belongs, and both \( \tau \) and \( \tilde{\Psi} \) do not depend on the representative in such a class (see remark 10).

Remark 13. Proceeding as in [19, remark 2.4] one can give the following alternative definition of \( \tilde{\Psi} \) where only \( Q \phi \in D(\Theta) \) appears:

\[
\tilde{\Psi} \phi := \Psi \ast \phi + \tau \ast Q \phi .
\]

This is an immediate consequence of identity (10).

Example 14. Let us consider the case \( A = \Delta : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \). Obviously \( A \) is an injective convolution operator, thus we can apply to it the previous theorem.

A Borel set \( F \subset \mathbb{R}^n \) is called a \( d \)-set, \( d \in (0, n] \), if

\[
\exists c_1, c_2 > 0 : \forall x \in F, \forall r \in (0, 1), \quad c_1 r^d \leq \mu_d(B_r(x) \cap F) \leq c_2 r^d ,
\]

where \( \mu_d \) is the \( d \)-dimensional Hausdorff measure and \( B_r(x) \) is the closed \( n \)-dimensional ball of radius \( r \) centered at the point \( x \) (see [14, §1.1, chap. VIII]).

Examples of \( d \)-sets are \( d \)-dimensional Lipschitz submanifolds and (when \( d \) is not an integer) self-similar fractals of Hausdorff dimension \( d \) (see [14, chap. II, example 2]). Moreover a finite union of \( d \)-sets which intersect on a set of zero \( d \)-dimensional Hausdorff measure is a \( d \)-set.

In the case \( 0 < n - d < 4 \) we take as the linear operator \( \tau \) the unique continuous surjective (thus (h1) holds true) map

\[
\tau_F : H^2(\mathbb{R}^n) \to B^{2,2}_\alpha(F), \quad \alpha = 2 - \frac{n - d}{2}
\]

such that, for \( \mu_d \)-a.e. \( x \in F \),

\[
\tau_F \phi(x) = \left\{ \phi_F^{(j)}(x) \right\}_{|j| < \alpha} = \left\{ \lim_{r \downarrow 0} \frac{1}{\lambda_n(r)} \int_{B_r(x)} dy D^j \phi(y) \right\}_{|j| < \alpha} ,
\]

where \( j \in \mathbb{Z}^n_+ \), \( |j| := j_1 + \cdots + j_n \), \( D^j := \partial_{j_1} \cdots \partial_{j_n} \) and \( \lambda_n(r) \) denotes the \( n \)-dimensional Lebesgue measure of \( B_r(x) \). We refer to [14, thms. 1 and 3, chap. VII] for the existence of the map \( \tau_F \); obviously it coincides with the usual evaluation along \( F \) when restricted to smooth functions. The definition of the Besov-like (actually Hilbert, see remark 1) space \( B^{2,2}_\alpha(F) \) is quite involved and we will not reproduce it here (see [14, §2.1, chap. V]). In the case \( 0 < \alpha < 1 \)
(i.e. $2 < n - d < 4$) things simplify and $B^{2,2}_\alpha(F)$ can be defined (see [14, §1.1, chap. V]) as the Hilbert space of $f \in L^2(F; \mu_F)$ having finite norm

$$
\|f\|_{B^{2,2}_\alpha(F)}^2 := \|f\|_{L^2(F)}^2 + \int_{|x-y|<1} d\mu_F(x) d\mu_F(y) \frac{|f(x) - f(y)|^2}{|x-y|^{d+2\alpha}},
$$

where $\mu_F$ denotes the restriction of the $d$-dimensional Hausdorff measure $\mu_d$ to the set $F$. When $\alpha > 1$ and $F$ is a generic $d$-set the functions $\phi^{(j)}_F \in L^2(F; \mu_F)$ are not uniquely determined by $\phi^{(0)}_F$; contrarily we may then identify $\{\phi^{(j)}_F\} \cup \{\phi^{(0)}_F\}$ with the single function $\phi^{(0)}_F$. This is possible when $F$ preserves Markov’s inequality (see [14, §2, chap. II]). Sets with such a property are closed $d$-sets with $d > n - 1$ (see [14, thm. 3, §2.2, chap. II]), a concrete example being e.g. the boundary of von Koch’s snowflake domain in $\mathbb{R}^2$ (a $d$-set with $d = \log 4/\log 3$, see [24]). If $F$ has some additional differential structure then $B^{2,2}_\alpha(F) \simeq H^\alpha(F)$, where $H^\alpha(F)$ denotes the usual (fractional) Sobolev-Slobodeckii space. Some known cases where $B^{2,2}_\alpha(F) \simeq H^\alpha(F)$ (for any value of $\alpha > 0$) are the following:

- $F$ is the graph of a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^{n-d}$ (see [5, §20]);
- $F$ is a bounded manifold of class $C_\gamma$, $\gamma > \min(3, \max(1, \alpha))$, i.e. $F$ has an atlas where the transition maps are of class $C^k$, $k < \gamma \leq k + 1$, and have derivatives of order less or equal to $k$ which satisfy Lipschitz conditions of order $\gamma - k$ (see [13] for the case $\gamma > \max(1, \alpha)$ and see [5, §24] for the case $\gamma > 3$);
- $F$ is a connected complete Riemannian manifold with positive injectivity radius and bounded geometry, in particular a connected Lie group (see [23, §7.4.5, §7.6.1]).

Supposing now that $F$ has a compact closure, let $\chi$ be a smooth function with a compact support $B$ such that $\chi = 1$ on $F$. Then by Sobolev’s inequality one has (from now on $n > 4$),

$$
\|\tau_F \phi\|_{B^{2,2}_\alpha(F)}^2 = \|\tau_F \chi \phi\|_{B^{2,2}_\alpha(F)}^2 \leq c \|\chi \phi\|_{H^2(\mathbb{R}^n)}^2 \\
\leq c (\|\phi\|_{L^2(B)}^2 + \|\nabla \phi\|_{L^2(B)}^2 + \|\Delta \phi\|_{L^2(B)}^2) \\
\leq c (\|\phi\|_{L^{\frac{2n}{n-2}}(B)}^{2n} + \|\nabla \phi\|_{L^{\frac{2n}{n-4}}(B)}^{2n} + \|\Delta \phi\|_{L^2(\mathbb{R}^n)}^2) \\
\leq c \|\Delta \phi\|_{L^2(\mathbb{R}^n)},
$$

and so $\tau_F$ satisfies (h3). Moreover, since

$$
\mathcal{D}(\Delta) = \{\varphi \in \mathcal{D}'(\mathbb{R}^n) : \Delta \varphi \in L^2(\mathbb{R}^n)\} \subseteq H^2_{\text{loc}}(\mathbb{R}^n)
$$

and $\mathcal{F}$ is supposed to be compact, the extension of $\tau_F$ to $\mathcal{D}(\Delta)$ is again defined by (11). Denoting the dual of $B^{2,2}_\alpha(F)$ by $B^{2,2}_\alpha(F)$ (the space $B^{2,2}_\alpha(F)$ can be explicitly characterized in the case $0 < \alpha < 1$ or when $F$ preserves Markov’s
inequality, see [15]), hypothesis (h2) is equivalent to \( \tau'_F \ell \notin L^2(\mathbb{R}^n) \) for any \( \ell \in B^{2,2}_{-\alpha}(F) \backslash \{0\} \), where \( \tau'_F \ell \in H^{-2}(\mathbb{R}^n) \) is defined by

\[
\tau'_F \ell(\phi) := \ell(\tau_F \phi).
\]

Therefore, as the support of \( \tau'_F \ell \) is given by \( \bar{F} \), (h2) is certainly verified when \( \bar{F} \) has zero Lebesgue measure. Considering the fundamental solution of \( -\Delta \), given by

\[
G(x) = \frac{1}{(n-2)\sigma_n} \frac{1}{|x|^{n-2}},
\]

\( \sigma_n \) the measure of the unitary sphere in \( \mathbb{R}^n \), the convolution \( G \ast \tau'_F \ell \) is a well defined distribution as \( \tau'_F \) is in \( E'(\mathbb{R}^n) \). Therefore, by lemma 8 we can choose \( G \ast \tau'_F \ell \) to be the map

\[
G_* : B^{2,2}_{-\alpha}(F) \to D'(\mathbb{R}^n), \quad G_* \ell := G \ast \tau'_F \ell.
\]

Thus, by the previous theorem (and remark 13), supposing that the \( d \)-set \( F \) has a compact closure of zero Lebesgue measure, given any self-adjoint operator \( \Theta \in L(B^{2,2}_{-\alpha}(F), B^{2,2}_{\alpha}(F)) \), \( \Theta \in L(B^{2,2}_{\alpha}(F)) \) if one uses the identification \( B^{2,2}_{-\alpha}(F) \simeq B^{2,2}_{\alpha}(F) \), we have then the self-adjoint operator

\[
\Delta^F \phi := \Delta \varphi_{\text{reg}} \equiv \Delta \phi + \tau'_F Q \phi,
\]

where

\[
\phi = \varphi_{\text{reg}} + G \ast \tau'_F Q \varphi, \quad \varphi_{\text{reg}} \in D(\Delta), \quad Q \phi \in D(\Theta),
\]

and

\[
\left\{ \lim_{r \downarrow 0} \frac{1}{\lambda_n(r)} \int_{B_r(x)} dy \, D^j(\phi - G \ast \tau'_F Q \phi)(y) \right\}_{|j| < \alpha} = \Theta Q \phi(x),
\]

\(|j| = 0 \) if \( F \) preserves Markov’s inequality or \( B^{2,2}_{\alpha}(F) \simeq H^{\alpha}(F) \). When \( F = M \), \( M \) a compact Riemannian manifold, a natural choice for \( \Theta \) is given by \( \Theta = (-\Delta_{LB})^{-\alpha} \), where \( \Delta_{LB} \) denotes the Laplace-Beltrami operator. The case in which \( A = (\Delta - \lambda) : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \), \( \lambda > 0 \) (note that here \( 0 \notin \sigma(A) \), thus theorem 1 directly applies), and \( F \) is a plane circle, is treated, without giving boundary conditions, in [16] (also see [19, example 3.2] for connections with Birman-Krein-Vishik theory).

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