REPRESENTATION THEORY OF 0-HECKE-CLIFFORD ALGEBRAS

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Abstract. The representation theory of 0-Hecke-Clifford algebras as a degenerate case is not semisimple and also with rich combinatorial meaning. Bergeron et al. have proved that the Grothendieck ring of the category of finitely generated supermodules of 0-Hecke-Clifford algebras is isomorphic to the algebra of peak quasisymmetric functions defined by Stembridge. In this paper we further study the category of finitely generated projective supermodules and clarify the correspondence between it and the peak algebra of symmetric groups. In particular, two kinds of restriction rules for induced projective supermodules are obtained. After that, we consider the corresponding Heisenberg double and its Fock representation to prove that the ring of peak quasisymmetric functions is free over the subring of symmetric functions spanned by Schur’s Q-functions.

Keywords: 0-Hecke-Clifford algebra, peak algebra, Heisenberg double

1. Introduction

As a $q$-deformation of the Sergeev algebra, the Hecke-Clifford algebra $HCl_n(q)$ is defined by G. Olshanski in [21] mixing the Hecke algebra $H_n(q)$ and the Clifford algebra $Cl_n$. When $q$ is generic, it satisfies the Schur-Sergeev-Olshanski super-duality with the quantum enveloping algebra of queer Lie superalgebras $q_n$. Moreover, the Grothendieck group of the tower of Hecke-Clifford algebras is isomorphic to the subalgebra of symmetric functions spanned by Schur’s Q-functions, parallel to the classical case of Sergeev algebras ($q = 1$) [10, §3.3]. For the root of unity case, Brundan and Kleshchev in [7] consider the (affine) Hecke-Clifford algebra and relate its representation category with the positive part of the universal enveloping algebra for the affine Kac-Moody algebra $\mathfrak{g} = A_2^{(2)}$. Recently, Mori in [20] extends the method of cellular algebras to the superalgebra setting to study the cellular representation theory of Hecke-Clifford algebras uniformly only requiring that $q$ is invertible. In a word, the representation theory of (affine) Hecke-Clifford algebras has been widely studied; see also [14],[29],[30], etc.

There still leaves a degenerate case when $q = 0$ and also with nice combinatorial aspect. The 0-Hecke-Clifford algebra $HCl_n(0)$ is first considered by Bergeron et al. in [3], where they mainly construct the simple supermodules of $HCl_n(0)$, and prove the Frobenius isomorphism between the Grothendieck group of the category of finitely generated supermodules of 0-Hecke-Clifford algebras and the Stembridge algebra of peak quasisymmetric functions. Since the cellular approach fails in this degenerate case, one needs different techniques to handle its representation theory. In this paper, we dually discuss the category of finitely generated projective supermodules. Note that the graded Hopf dual of the algebra...

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of peak quasisymmetric functions is the peak algebra of symmetric groups [25]. Here we confirm the Hopf dual pair between the Grothendieck groups of the above two supermodule categories and explicitly relate the projective one to the peak algebra. In fact, there already have many nice papers considering towers of algebras together with their corresponding Grothendieck groups, which provide a bunch of (combinatorial) Hopf algebras; see [7],[9],[12],[17],[24], etc.

Recently, Berg et al. in [1] construct a noncommutative lift of Schur functions, called the immaculate basis, and then in [2] find the correspondence of its dual basis to the category of finitely generated modules of 0-Hecke algebras under the Frobenius isomorphism defined in [17]. Inspired by their work, we construct a lift of Schur’s Q-functions to the peak algebra and thus extract a new basis from it in [15]. Dually we define a new basis, called the quasisymmetric Schur’s Q-functions, in the Stembridge algebra, whose expansion in the peak functions is expected to be positive based on concrete examples [15, Conjecture 4.15]. It implies the chance for a representation theoretical meaning of such basis on the supermodule category of 0-Hecke-Clifford algebras. Burying such target in mind, we want to further study the representation theory of 0-Hecke-Clifford algebras.

As an application, we also use the corresponding Heisenberg double and its Fock representation from the tower of 0-Hecke-Clifford algebras to prove that the ring of peak quasisymmetric functions is free over the subring of symmetric functions spanned by Schur’s Q-functions. Such method is purposed by Savage et al. in [24] to give a new proof of the freeness of the ring of quasisymmetric functions over the ring of symmetric functions. For further discussion of twisted version of Heisenberg doubles, one can refer to [22], [23].

The organization of the paper is as follows. In §2 we provide some notation, definitions for all combinatorial Hopf algebras that we involve. Some preliminaries on superalgebras and the terminology of towers of superalgebras are recalled. In §3 we focus on the representation theory of 0-Hecke-Clifford algebras. Their projective supermodules induced from those of 0-Hecke algebras are mainly considered. Finally we clarify a dual pair of graded Hopf algebra structures on two Grothendieck groups of finitely generated supermodules and projective supermodules of 0-Hecke-Clifford algebras, whose Frobenius superalgebra structures are also figured out. In §4 we further give the concrete relation between the Grothendieck groups defined in the previous section with the peak algebra of symmetric groups and its dual, extending the results in [3]. In particular for those induced projective supermodules, two restriction rules and the decomposition formula to indecomposable ones are given. As a final application, the freeness of the ring of peak quasisymmetric functions over the subring of symmetric functions spanned by Schur’s Q-functions is proved.

2. Preliminaries

2.1. Notation and definitions. Throughout this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 for simplicity. Denote by $\mathbb{N}$ (resp. $\mathbb{N}_0$) the set of positive (resp. nonnegative) integers. Given any $m,n \in \mathbb{N}$, let $[m,n] := \{m, m+1, \ldots, n\}$ if $m \leq n$ and $\emptyset$ otherwise. Also $[n] := [1,n]$ for short. Let $2^{[n]}$ be the set of subsets of $[n]$ and $\mathcal{C}(n)$ be the set of compositions of $n$, consisting of ordered tuples of positive integers summed up to $n$. 
We denote $\alpha \vdash n$ when $\alpha \in \mathfrak{C}(n)$. Let $\mathfrak{C} := \bigcup_{n \geq 1} \mathfrak{C}(n)$. Given $\alpha = (\alpha_1, \ldots, \alpha_r) \vdash n$, let $\ell(\alpha) = r$ be its length and define its associated descent set as

$$D(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{r-1}\} \subseteq [n-1].$$

Given a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$, we also define its associated descent set as

$$D(w) := \{i \in [n-1] : w_i > w_{i+1}\}$$

and descent composition $c(w) \vdash n$ such that $D(c(w)) = D(w)$. Note that $\mathfrak{C}(n) \to 2^{[n-1]}, \alpha \mapsto D(\alpha)$ is a bijection. The refining order $\leq$ on $\mathfrak{C}(n)$ is defined by

$$\alpha \leq \beta \text{ if and only if } D(\beta) \subseteq D(\alpha), \forall \alpha, \beta \vdash n.$$

In general, for $\alpha \in \mathbb{N}_0^n$, let $\ell(\alpha) = |\{i : \alpha_i > 0\}|$. Given $\alpha \vdash n$, the corresponding descent class of the symmetric group $\mathfrak{S}_n$ is defined by

$$\mathfrak{D}_\alpha := \{w \in \mathfrak{S}_n : D(w) = D(\alpha)\}.$$

For $\alpha = (\alpha_1, \ldots, \alpha_r) \vdash n$, define its three counterparts:

1. the reverse of $\alpha$, $\bar{\alpha} := (\alpha_r, \ldots, \alpha_1)$, such that $D(\bar{\alpha}) = \{i \in [n-1] : n - i \in D(\alpha)\}$.
2. the complement of $\alpha$, $\check{\alpha}^\circ \vdash n$ such that $D(\check{\alpha}^\circ) = [n-1]\setminus D(\alpha)$.
3. the conjugation of $\alpha$, $\check{\alpha} := \bar{\check{\alpha}^\circ}$.

We also recall the concept of peaks. A subset $P \subseteq [n]$ is called a peak set in $[n]$ if $P \subseteq [2, n-1]$ and $i \in P \Rightarrow i - 1 \notin P$. Denote by $\mathcal{P}_n$ the collection of peak sets in $[n]$, $\mathcal{P} := \bigcup_{n \geq 1} \mathcal{P}_n$, and $\emptyset_n$ the empty set $\emptyset$ in $\mathcal{P}_n$. Given $\alpha = (\alpha_1, \ldots, \alpha_r) \vdash n$, let

$$P(\alpha) := \{x \in [2, n-1] : x - 1 \notin D(\alpha), x \in D(\alpha)\}$$

be its associated peak set in $[n]$, while its associated valley set $V(\alpha) \subseteq [n]$ is defined by

$$V(\alpha) \cap [2, n] = \{x \in [2, n] : x - 1 \in D(\alpha), x \notin D(\alpha)\}$$

and $1 \in V(\alpha) \Leftrightarrow 1 \notin D(\alpha)$. Note that $|V(\alpha)| = |P(\alpha)| + 1$. Given a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$, its peak set

$$P(w) := \{i \in [2, n-1] : w_{i-1} < w_i > w_{i+1}\} = P(c(w)).$$

Define the algebra of noncommutative symmetric functions as the free associative $\mathbb{K}$-algebra generated by the symbols $H_n (n \in \mathbb{N})$ and denote it by $\text{NSym}$ [13]. Define another set of generators $E_n (n \in \mathbb{N})$ in $\text{NSym}$ by

$$\sum_{i=0}^{n} (-1)^i E_i H_{n-i} = \delta_{n,0}.$$

The algebra $\text{NSym} = \bigoplus_{n=0}^{\infty} \text{NSym}_n$ is a $\mathbb{Z}$-graded algebra under the gradation given by $\deg(H_n) = n$, where $\text{NSym}_n$ is the subspace of homogeneous elements of degree $n$. Let

$$H_\alpha := H_{\alpha_1} \cdots H_{\alpha_r}, E_\alpha := E_{\alpha_1} \cdots E_{\alpha_r}, \alpha = (\alpha_1, \ldots, \alpha_r) \vdash n.$$
If we change base from $\mathbb{K}$ to $\mathbb{Z}$, then both $\{H_n\}_{n\in\mathbb{N}}$ and $\{E_n\}_{n\in\mathbb{N}}$ are $\mathbb{Z}$-bases of NSym, called the noncommutative complete and elementary symmetric functions respectively. There exists another important $\mathbb{Z}$-basis $\{R_n\}_{n\in\mathbb{N}}$ of NSym, called the noncommutative ribbon Schur functions and are defined by

$$R_\alpha := \sum_{\beta \geq \alpha} (-1)^{|\beta|-|\alpha|} H_\beta.$$ 

With the coproduct defined by

$$(2.1) \quad \Delta(H_n) = \sum_{k=0}^{n} H_k \otimes H_{n-k},$$

NSym becomes a graded and connected Hopf algebra. Moreover, the graded Hopf dual of NSym is the algebra of quasisymmetric functions, denoted by QSym. It is a subring of the power series ring $\mathbb{K}[[x_1, x_2, \ldots]]$ in the commuting variables $x_1, x_2, \ldots$ and has a linear basis, the monomial quasisymmetric functions, defined by

$$M_\alpha := M_\alpha(x) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}, \quad \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathcal{C}.$$

In other words, $F_\alpha = \sum_{\beta \leq \alpha} M_\beta$. Meanwhile, the canonical pairing $\langle \cdot, \cdot \rangle$ between NSym and QSym is defined by

$$\langle H_\alpha, M_\beta \rangle = \langle R_\alpha, F_\beta \rangle = \delta_{\alpha, \beta}$$

for any $\alpha, \beta \in \mathcal{C}$.

Let $\Lambda$ be the graded ring of symmetric functions in the commuting variables $x_1, x_2, \ldots$, with integer coefficients, and $\Omega$ be the subring of $\Lambda$ generated by the symmetric functions $q_n (n \geq 1)$, which are defined by

$$\sum_{n \geq 0} q_n z^n = \prod_{i \geq 1} \frac{1 + x_i z}{1 - x_i z}.$$

Note that $\Omega_\mathbb{Q} := \Omega \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathbb{Q}[p_{2k-1} : k \in \mathbb{N}]$, where $p_n$’s are the power-sum symmetric functions. It has the following canonical inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle p_\lambda, p_\mu \rangle = z_1^\lambda z_{-1}^{\ell(\lambda)} \delta_{\lambda, \mu}$$

for any partitions $\lambda, \mu$ only with odd parts, where $z_1 := \prod_{i \geq 1} m_i!^{m_i}$ for $\lambda = (m_1, m_2, \ldots)$ (see [18, Ch. III, §8]). Now introduce two Hopf algebra epimorphisms. One is

$$\theta : \Lambda \rightarrow \Omega, \quad h_n \mapsto q_n, \quad n \geq 1,$$

such that $\theta(p_n) = (1 - (-1)^n)p_n$, $n \geq 1$, and the other is

$$\pi : \text{NSym} \rightarrow \Lambda, \quad H_n \mapsto h_n,$$
called the \textit{forgetful map} and satisfying
\[
\langle F, f \rangle = \langle \pi(F), f \rangle, \quad F \in \text{NSym}, \ f \in \Lambda.
\]

2.2. The peak subalgebra and its Hopf dual. Introduce
\[
Q_0 := 1, \quad Q_n := \sum_{k=0}^{n} E_k H_{n-k}, \quad n \geq 1
\]
and also \( Q_n := 0, \ n < 0 \) for convenience. Let \( Q_\alpha := Q_{\alpha_1} \cdots Q_{\alpha_r}, \ \alpha = (\alpha_1, \ldots, \alpha_r) \vdash n \).
Then \( Q_n(n \geq 1) \) satisfy the following \textit{Euler relations} (also called the \textit{generalized Dehn-Sommerville relation})
\[
\sum_{r+s=n} (-1)^r Q_r Q_s = 0, \ \forall n \geq 1,
\]
or equivalently,
\[
\sum_{i=1}^{n-1} (-1)^{i-1} Q_i Q_{n-i} = \begin{cases} 
2Q_n, & \text{if } n \text{ is even}, \\
0, & \text{if } n \text{ is odd}.
\end{cases}
\]

Let \( \text{Peak} \) be the Hopf subalgebra of \( \text{NSym} \) generated by \( Q_n(n \geq 1) \). Then \( \text{Peak} \cap \text{NSym} \) is isomorphic to the \textit{peak algebra} of the symmetric group \( S_n \) when endowed with the internal product \([3, 25]\). For any \( P \in \mathcal{P}_n \), define
\[
\Xi_P := \sum_{P(\alpha) = P} \sum_{D(\alpha) \subseteq P} \triangle (D(\alpha)+1) F_{\alpha}, \quad P \in \mathcal{P}_n
\]
where \( D \triangle (D+1) = D \setminus (D+1) \cup (D+1) \setminus D \) for any \( D = \{ D_1 < \cdots < D_r \} \subseteq [n-1] \) and \( D+1 := \{ x+1 : x \in D \} \).

There exists a surjective Hopf algebra homomorphism
\[
\Theta : \text{NSym} \rightarrow \text{Peak}, \quad H_n \mapsto Q_n, \ n \geq 1,
\]
called the \textit{descent-to-peak transform} \([16, \S 5]\), where \( \ker \Theta \) is the Hopf ideal of \( \text{NSym} \) generated by \( \sum_{r+s=n} (-1)^r H_r H_s, \ n \geq 1 \) \([6, \text{Theorem 5.4}]\).

Next we introduce the graded Hopf dual of Peak, the \textit{Stembridge algebra} \( \text{Peak}^* \) of peak quasisymmetric functions, defined in \([27]\). This is a Hopf subalgebra of \( \text{QSym} \), with a natural basis called the Stembridge’s \textit{peak functions}. They can be defined by \([27, \text{Prop. 3.5}]\)
\[
K_P := 2|P| + 1 \sum_{P \in \mathcal{P}_n} \sum_{P \in \mathcal{P}_n} F_{\alpha}, \quad P \in \mathcal{P}_n,
\]
where \( D \triangle (D+1) = D \setminus (D+1) \cup (D+1) \setminus D \) for any \( D = \{ D_1 < \cdots < D_r \} \subseteq [n-1] \) and \( D+1 := \{ x+1 : x \in D \} \).
By \([27, \text{Prop. 2.2}]\), we also have
\[
K_P = \sum_{P \in \mathcal{P}_n} 2^{f_{\alpha}} M_{\alpha}, \quad P \in \mathcal{P}_n
\]
and in particular, by \([27, (2.5)]\),

\[
K_{\theta_n} = q_n = 2 \sum_{\alpha \in M} F_{\alpha} = \sum_{\alpha \in M} 2^{f(\alpha)} M_{\alpha}.
\]

There also exists a surjective Hopf algebra homomorphism

\[
\theta : \text{QSym} \rightarrow \text{Peak}^*, \quad F_{\alpha} \mapsto K_{P(\alpha)},
\]
called the descent-to-peak map.

Note that Peak can be regarded as a noncommutative lift of \(\Omega\). The following commutative diagrams illustrate the situation.

\[
\begin{array}{ccc}
\text{NSym} & \xrightarrow{\Theta} & \text{Peak}^* \\
\downarrow & & \downarrow \\
\Lambda & \xrightarrow{\theta} & \Omega
\end{array}
\quad
\begin{array}{ccc}
\text{QSym} & \xrightarrow{\theta} & \text{Peak}^* \\
\downarrow & & \downarrow \\
\Lambda & \xrightarrow{\theta} & \Omega
\end{array}
\]

where the vertical maps in the second diagram are inclusions. Also, the graded Hopf dual pairing between Peak and \(\text{Peak}^*\) is defined by

\[
[\cdot, \cdot] : \text{Peak} \times \text{Peak}^* \rightarrow \mathbb{K}, \quad [\Xi_P, K_Q] = \delta_{P,Q}, \ P, Q \in \mathcal{P},
\]

which satisfies the following property \([25, \text{Cor. 5.6}.]\).

\[
(\Theta(F), f) = \langle F, \vartheta(f) \rangle = [\Theta(F), \vartheta(f)], \ F \in \text{NSym}, f \in \text{QSym}.
\]

### 2.3. Preliminaries on superalgebras.

We first recall some standard results about the representation theory of finite dimensional (associative) superalgebras, referring to \([7, 8]\). A superalgebra \(A\) over \(\mathbb{K}\) is a \(\mathbb{Z}_2\)-graded \(\mathbb{K}\)-vector space \(A = A_0 \oplus A_1\) which is also an algebra such that \(A_iA_j \subseteq A_{i+j}, i, j \in \mathbb{Z}_2\). Any superalgebra \(A\) considered here has the unit \(1 = 1_A \in A_0\). Given \(a \in A_i (i \in \mathbb{Z}_2)\), let the degree of \(a\) be \(|a| := i\). Homomorphisms between two superalgebras are usual algebra homomorphisms. For two \(\mathbb{Z}_2\)-graded \(\mathbb{K}\)-vector spaces \(V, W\), \(\text{Hom}_{\mathbb{K}}(V, W)\) is \(\mathbb{Z}_2\)-graded such that \(f \in \text{Hom}_{\mathbb{K}}(V, W)\) if \(f(V_i) \subseteq W_{i+j}, i, j \in \mathbb{Z}_2\). The base field \(\mathbb{K}\) serves as a one dimensional even space.

A left \(A\)-supermodule \(M\) is a \(\mathbb{Z}_2\)-graded \(\mathbb{K}\)-vector space \(M = M_0 \oplus M_1\) which is also an \(A\)-module such that \(A_iM_j \subseteq M_{i+j}, i, j \in \mathbb{Z}_2\). Given \(m \in M_i (i \in \mathbb{Z}_2)\), also let the degree of \(m\) be \(|m| := i\). A morphism \(f\) between two left \(A\)-supermodules \(M, N\) is a linear map such that \(f(am) = (-1)^{|a||m|}af(m), a \in A, m \in M\). We denote the \(\mathbb{Z}_2\)-graded \(\mathbb{K}\)-vector space of all such morphisms by \(\text{Hom}_{\mathbb{K}}(M, N)\). On the other hand, \(M^r := \text{Hom}_{\mathbb{K}}(M, \mathbb{K})\) has the following natural left \(A\)-supermodule structure:

\[
(af)(m) = (-1)^{|a||f|+|m|} f(am), a \in A, f \in M^r, m \in M,
\]

if \(M\) is a right \(A\)-supermodule, and the natural right \(A\)-supermodule structure: \((fa)(m) = f(am), f \in M^r\), if \(M\) is a left \(A\)-supermodule.

An \(A\)-supermodule is irreducible (or simple) if it is non-zero and has no non-zero proper super submodules. Then it is either irreducible as an ordinary \(A\)-module (called of type \(M\)) or else reducible as an \(A\)-module (called of type \(Q\)).
All the finitely generated $A$-supermodules together with their morphisms constitute a superadditive category, denoted by $A$-mod. Besides, all the finitely generated projective $A$-supermodules form a full subcategory of $A$-mod, denoted by $A$-pmod. There exists the parity change functor $\Pi : A$-mod $\to A$-mod such that $\Pi M$ is the same underlying vector space but with $(\Pi M)_i = M_{i+1}$ ($i \in \mathbb{Z}_2$) and the following new action:

$$a.m := (-1)^{|a|} am, \ a \in A, m \in M.$$  

Note that one can identify $\text{Hom}_A(M,N)_i$ with $\text{Hom}_A(M,\Pi N)_{i+1}$ for any $i \in \mathbb{Z}_2$, and obviously the map $M \to \Pi M$, $m \mapsto (-1)^{|m|} m$ is a homomorphism of usual $A$-modules.

Suppose that $\nu$ is an automorphism of a superalgebra $A$. For any $M \in A$-mod, we can twist the left action on $M$ with $\nu$ to define the following twisted left $A$-module, denoted by $^\nu M$:

$$a_\nu m := \nu(a)m, \ m \in M, a \in A.$$  

If $\nu$ is an unsign anti-automorphism of $A$, i.e. $\nu(ab) = \nu(b)\nu(a), a,b \in A$, then we can twist the right action of $A$ on $M^\nu$ with $\nu$ to define the following twisted left $A$-module, also denoted by $^\nu(M^\nu)$:

$$(a_\nu f)(m) := f(\nu(a)m), \ f \in M^\nu, m \in M, a \in A.$$  

For any right $A$-supermodule $M$, we use the notation $M^\nu$ instead for the right twisted module structure without confusion.

The category $A$-mod has the underlying even subcategory with the same objects but only even morphisms, which is an abelian category. Hence, we can define the corresponding Grothendieck group $K_0(A$-mod) to be the quotient of the free abelian group with all objects in $A$-mod as a basis by the subgroup generated by

1. $M_1 - M_2 + M_3$ for every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in the underlying even subcategory.
2. $M - \Pi M$ for every $M \in A$-mod.

Similarly we define the Grothendieck group $K_0(A$-pmod). For any $M \in A$-mod or $A$-pmod, its class in such Grothendieck group is denoted by $[M]$. Note that the natural embedding from $A$-pmod to $A$-mod induces the Cartan map

\begin{equation}
\chi : K_0(A$-pmod) $\to K_0(A$-mod),
\end{equation}

which describes the multiplicity of composition factors in projective modules. Also there is a canonical embedding

$$K_0(A$-mod) $\otimes_{\mathbb{Z}} K_0(B$-mod) $\to K_0((A \otimes B)$-mod), \ [M] \otimes [N] \to [M \otimes N],$$

which is similarly defined on pmod’s and an isomorphism when changing the base ring to $\mathbb{Q}$.

There is a natural bilinear form

\begin{equation}
\langle \cdot, \cdot \rangle : K_0(A$-pmod) $\times K_0(A$-mod) $\to \mathbb{Z}, \ \langle [P], [M] \rangle = \dim_{\mathbb{Q}} \text{Hom}_A(P, M).
\end{equation}

For the pair on $K_0((A \otimes B)$-pmod) $\times K_0((A \otimes B)$-mod) $\to \mathbb{Z}$, we have

\begin{equation}
\langle [P \otimes Q], [M \otimes N] \rangle = \langle [P], [M] \rangle \langle [Q], [N] \rangle.
\end{equation}
Remark 2.3.1. The pair defined in (2.12) only involves dimensions of homomorphism spaces but not their graded dimensions as in [23]. It makes the critical difference since the authors in [23] consider the structure of twisted dual Hopf algebras indeed.

If $A$ is a finite dimensional superalgebra, let $V_1, \ldots, V_r$ be a complete list of non-isomorphic simple $A$-supermodules. If $P_i$ is the projective cover of $V_i$ in $A$-mod for $i = 1, \ldots, r$, then $P_1, \ldots, P_r$ is a complete list of non-isomorphic indecomposable projective $A$-supermodules and we have

$$K_0(A\text{-mod}) = \bigoplus_{i=1}^{r} \mathbb{Z}[V_i], \ K_0(A\text{-pmod}) = \bigoplus_{i=1}^{r} \mathbb{Z}[P_i].$$

Note that

$$\langle [P_i], [M_j] \rangle = \begin{cases} 1, & \text{if } i = j \text{ and } M_i \text{ is of type } M, \\ 2, & \text{if } i = j \text{ and } M_i \text{ is of type } Q, \\ 0, & \text{otherwise}. \end{cases}$$ (2.14)

Given two superalgebras $A$ and $B$, the tensor product $A \otimes B$ has the superalgebra structure defined by the following twisted multiplication:

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}ac \otimes bd, \ a, c \in A, b, d \in B$$

and $|a \otimes b| := |a| + |b|$ for any homogeneous $a \in A, b \in B$. Given an $A$-supermodule $M$ and a $B$-supermodule $N$, the tensor product $M \otimes N$ has the $A \otimes B$-supermodule structure defined by:

$$(a \otimes b)(m \otimes n) = (-1)^{|b||m|}am \otimes bn, \ a, b \in A, m \in M, n \in N$$

and $|m \otimes n| := |m| + |n|$ for any homogeneous $m \in M, n \in N$.

Lemma 2.3.2 ([8, §2]). If $M$ is a finite dimensional irreducible $A$-supermodule of type $Q$, then there exist bases $\{v_1, \ldots, v_n\}$ for $M_0$ and $\{\overline{v}_1, \ldots, \overline{v}_n\}$ for $M_1$ such that $\text{span}_{\mathbb{K}}(v_1 + \overline{v}_1, \ldots, v_n + \overline{v}_n)$ and $\text{span}_{\mathbb{K}}(v_1 - \overline{v}_1, \ldots, v_n - \overline{v}_n)$ form two non-isomorphic irreducible $A$-modules. Moreover, the linear map $J_M \in \text{End}_{\mathbb{K}}(M)$ defined by $v_i \mapsto \overline{v}_i, \overline{v}_i \mapsto -v_i$ is an odd $A$-supermodule automorphism of $M$.

Lemma 2.3.3 (Schur’s lemma). If $M$ is a finite dimensional irreducible $A$-supermodule, then

$$\text{End}_A(M) = \begin{cases} \text{span}_{\mathbb{K}}(id_M), & \text{if } M \text{ is of type } M, \\ \text{span}_{\mathbb{K}}(id_M, J_M), & \text{if } M \text{ is of type } Q, \end{cases}$$

where $J_M$ is as in Lemma 2.3.2.

2.4. Towers of superalgebras. Now mainly for 0-Hecke-Clifford algebras, we introduce the following important definition (see [23, Def. 4.1]) which extends the original one in [5, §3]. Further discussion can be found in [4].

Definition 2.4.1. Let $A = \bigoplus_{i=0}^{n} A_i$ be a graded superalgebra over $\mathbb{K}$ with multiplication $\mu : A \otimes A \to A$. Then $A$ is called a tower of superalgebras if the following conditions hold:
(1) Each graded component $A_n$ is a finite dimensional superalgebra with unit $1_n$, and $A_0 \cong \mathbb{K}$.

(2) The restriction $\mu_{m,n} : A_m \otimes A_n \to A_{m+n}$ of multiplication $\mu$ is a homomorphism of superalgebras for all $m, n \geq 0$, sending $1_m \otimes 1_n$ to $1_{m+n}$.

(3) For all $m, n \geq 0$, $\mu_{m,n}$ induces a two-sided projective $A_m \otimes A_n$-module structure on $A_{m+n}$ defined by

$$(a \otimes b).c := \mu_{m,n}(a \otimes b)c, \quad c.(a \otimes b) := c\mu_{m,n}(a \otimes b)$$

for any $a \in A_m, b \in A_n, c \in A_{m+n}$.

For a tower $A = \bigoplus_{n \geq 0} A_n$ of superalgebras, we define the categories

$$A\text{-mod} := \bigoplus_{n \geq 0} A_n\text{-mod}, \quad A\text{-pmod} := \bigoplus_{n \geq 0} A_n\text{-pmod}.$$ 

and the corresponding Grothendieck groups

$$\mathcal{G}(A) := \bigoplus_{n \geq 0} K_0(A_n\text{-mod}), \quad \mathcal{K}(A) := \bigoplus_{n \geq 0} K_0(A_n\text{-pmod}).$$

Both the Cartan map (2.11) and the pair (2.12) can be linearly extended to one for towers of superalgebras. Define

$$\chi := \bigoplus_{n \geq 0} \chi_n : \mathcal{K}(A) \to \mathcal{G}(A),$$

where $\chi_n$ is the Cartan map (2.11) of $A_n (n \in \mathbb{N})$. And

$$\langle \cdot, \cdot \rangle : \mathcal{K}(A) \times \mathcal{G}(A) \to \mathbb{Z}$$

by

$$\langle [P], [M] \rangle := \begin{cases} \dim_\mathbb{K} \text{Hom}_{A_m}(P, M), & P \in A_n\text{-pmod}, M \in A_n\text{-mod} \text{ for some } n, \\ 0, & \text{otherwise.} \end{cases}$$

For any $r \in \mathbb{N}$, write $A_{n_1, \ldots, n_r} := A_{n_1} \otimes \cdots \otimes A_{n_r}$ and define

$$A\text{-mod}^{\otimes r} := \bigoplus_{n_1, \ldots, n_r \geq 0} A_{n_1, \ldots, n_r}\text{-mod}, \quad A\text{-pmod}^{\otimes r} := \bigoplus_{n_1, \ldots, n_r \geq 0} A_{n_1, \ldots, n_r}\text{-pmod}.$$ 

Note that the formula (2.13) can be further extended on $A\text{-pmod}^{\otimes r} \times A\text{-mod}^{\otimes r}$.

For any $\sigma \in \mathfrak{S}_r$, define $\tau_\sigma : A\text{-mod}^{\otimes r} \to A\text{-mod}^{\otimes r}$ to be the functor twisting module structure by the following superalgebra isomorphism from $A_{n_1, \ldots, n_r}$ to $A_{n_{\sigma^{-1}(1)}, \ldots, n_{\sigma^{-1}(r)}}$:

$$a_1 \otimes \cdots \otimes a_r \mapsto (-1)^{d_0} a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(r)},$$

where $d_0 := \sum_{i<j} a_{j}||a_{i}$. Write $\tau_{ij} := \tau_{\sigma_{ij}}$ for short. For any $M \in A_{n_{\sigma^{-1}(1)}, \ldots, n_{\sigma^{-1}(r)}}\text{-mod}$, we denote its twisted $A_{n_1, \ldots, n_r}$-supermodule structure by $\tau_{\sigma}M$.

Now we have the following result.
Lemma 2.4.2. Given $M_i \in A_n^{\text{mod}} (1 \leq i \leq r)$ and $\sigma \in \mathfrak{S}_n$, there exists the following (even) $A_n^{\text{super}}$-supermodule isomorphism

\[ \Psi : M_1 \otimes \cdots \otimes M_r \to \tau_\sigma(M_{\sigma^{-1}(1)} \otimes \cdots \otimes M_{\sigma^{-1}(r)}), \]

\[ m_1 \otimes \cdots \otimes m_r \mapsto (-1)^{|m_1||m_1|} \otimes \cdots \otimes m_{\sigma^{-1}(r)}, \]

where $d_0 := \sum_{1 \leq j < r, a \in A_0} |m_j||m_j|.$

Proof. On one hand,

\[ \Psi((a_1 \otimes \cdots \otimes a_r),(m_1 \otimes \cdots \otimes m_r)) = (-1)^{\sum_{1 \leq j < r} (|a_j||m_j|)} \Psi(a_1 m_1 \otimes \cdots \otimes a_r m_r) \]

\[ = (-1)^{\sum_{1 \leq j < r} (|a_j||m_j|)} \sum_{a \in A_0} \Psi(a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(r)} m_{\sigma^{-1}(1)} \otimes \cdots \otimes m_{\sigma^{-1}(r)}). \]

On the other hand,

\[ (a_1 \otimes \cdots \otimes a_r)_{\tau_\sigma} \Psi(m_1 \otimes \cdots \otimes m_r) \]

\[ = (-1)^{\sum_{1 \leq j < r, a \in A_0} (|a_j||m_j|)} (a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(r)}) (m_{\sigma^{-1}(1)} \otimes \cdots \otimes m_{\sigma^{-1}(r)}), \]

\[ = (-1)^{\sum_{1 \leq j < r, a \in A_0} (|a_j||m_j|)} \sum_{a \in A_0} a_{\sigma^{-1}(1)} m_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(r)} m_{\sigma^{-1}(r)}. \]

They are equal since

\[ \sum_{1 \leq j < r} |a_j||m_j| + \sum_{1 \leq j < r, a \in A_0} (|a_j||m_j| + |a_j||m_j|) \]

\[ = \sum_{1 \leq j < r} |a_j||m_j| + \sum_{1 \leq j < r, a \in A_0} |a_j||m_j| = \sum_{1 \leq j < r} |a_{\sigma^{-1}(j)}||m_{\sigma^{-1}(j)}| \]

in $\mathbb{Z}_2$. \(\square\)

Given an even homomorphism $f : B \to A$ of superalgebras, the usual induction and restriction functors are defined by

\[ \text{Ind}_B^A : B^{\text{mod}} \to A^{\text{mod}}, \quad \text{Ind}_B^A M := A^n \otimes_B N, \quad N \in B^{\text{mod}}, \]

\[ \text{Res}_B^A : A^{\text{mod}} \to B^{\text{mod}}, \quad \text{Res}_B^A M := \text{Hom}_A(A^n, M) \cong A \otimes_A M, \quad M \in A^{\text{mod}}, \]

where the left $B$-action on $\text{Hom}_A(A^n, M)$ is defined by $(bg)(a) = (-1)^{|b||g||a|} g(a)(b)$, $a \in A, b \in B, g \in \text{Hom}_A(A, M)$, and the above isomorphism is given by $g \mapsto 1_A \otimes g(1_A)$.

Now if a tower $A$ of superalgebras is fixed, we abbreviate $\text{Ind}_{A_{n+1}}^{A_m} A_{n}$ as $\text{Ind}_{m,n}^{m+n}$ and $\text{Res}_{A_{n+1}}^{A_m} A_{n}$ as $\text{Res}_{m,n}^{m+n}$, which base on the multiplication $\mu_{m,n} : A_m \otimes A_n \to A_{m+n}$. Define

\[ \text{Ind} := \bigoplus_{m,n \geq 0} \text{Ind}_{m,n}^{m+n}, \quad \text{Res} := \bigoplus_{m,n \geq 0} \text{Res}_{m,n}^{m+n}. \]

We are interested in those pair $(\mathcal{K}(A), \mathcal{G}(A))$, which forms a dual pair of graded Hopf algebras via such induction and restriction. For such duality, we also need the notion of Frobenius superalgebras (see [23, §6]).
Definition 2.4.3. A finite dimensional superalgebra $A$ is called a Frobenius superalgebra if one of the following three equivalent conditions holds:

(a) There is an even left $A$-supermodule isomorphism $\rho : A \to A^\dagger$.

(b) There exists a nondegenerate invariant even $\mathbb{K}$-bilinear form $(\cdot, \cdot) : A \times A \to \mathbb{K}$ such that $(a, b) = 0$ if $a \in A_0, b \in A_1$ or $a \in A_1, b \in A_0$, and also $(ab, c) = (a, bc)$, $a, b, c \in A$.

(c) There exists an even $\mathbb{K}$-linear map $tr : A \to \mathbb{K}$, called the trace map, such that $\ker tr$ contains no non-zero left ideals of $A$.

The relationship between these three conditions is as follows:

$$\rho(b)(a) = (-1)^{|a||b|}(a, b), \quad (a, b) = tr(ab), \quad a, b \in A.$$ 

There exists an even automorphism $\varphi$ of $A$ satisfying $(a, b) = (-1)^{|a||b|}(\varphi(b), a)$, $a, b \in A$. This automorphism is called the Nakayama automorphism of $A$.

Lemma 2.4.4. Let $A_i$ ($i = 1, 2$) be two Frobenius superalgebras with trace maps $tr_i$ ($i = 1, 2$) and Nakayama automorphisms $\varphi_i$ ($i = 1, 2$) respectively. Then $A_1 \otimes A_2$ is also a Frobenius superalgebra with trace map $tr_1 \otimes tr_2$ and Nakayama automorphism $\varphi_1 \otimes \varphi_2$.

For a tower $A$ of superalgebras such that each $A_i$ is a Frobenius superalgebra with Nakayama automorphism $\varphi_i$. We abuse the notation $\varphi_i$ to be the automorphism on $A_i$-mod and $A_i$-pmod twisting modules by $\varphi_i$ of $A_i$. Then $\varphi := \bigoplus_{n \geq 0} \varphi_n$ is an automorphism of $A$-mod and $A$-pmod, thus induces an automorphism on $\mathcal{K}(A)$ and $G(A)$.

Proposition 2.4.5. [23, Prop. 6.7] Under the above assumption, the induction is conjugate right adjoint to restriction with conjugation $\varphi$, i.e. for any $m, n \geq 0$, $M \in A_m$-mod, $N \in A_n$-mod, $L \in A_{m+n}$-mod, the following functorial isomorphism holds:

$$\text{Hom}_{A_{m+n}} \left( L, \text{Ind}^{m+n}_{m,n} (M \otimes N) \right) \cong \text{Hom}_{A_m \otimes A_n} \left( \varphi \text{Res}^{m+n}_{m,n} (L), M \otimes N \right),$$

where $\varphi \text{Res}^{m+n}_{m,n} := (\varphi_m \otimes \varphi_n) \circ \text{Res}^{m+n}_{m,n} \circ \varphi_{m+n}^{-1}$.

3. Representation theory of 0-Hecke-Clifford algebras

In this section we further study the representation theory of 0-Hecke-Clifford algebras in detail, referring to the discussion in [3, §5]. The 0-Hecke-Clifford algebra $HCl_n(0)$ of type A is an algebra generated by $T_i$, $1 \leq i \leq n - 1$; $c_j$, $1 \leq j \leq n$, where $T_i$’s generate the 0-Hecke algebra $H_n(0)$ with relations

$$T_i^2 = -T_i, \quad 1 \leq i \leq n - 1,$$

$$T_i T_j = T_j T_i, \quad |i - j| > 1,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad 1 \leq i \leq n - 2,$$

and $c_j$’s generate the Clifford algebra $Cl_n$ with relations

$$c_i^2 = -1, \quad 1 \leq i \leq n; \quad c_i c_j = -c_j c_i, \quad i \neq j,$$
while the two parts satisfy the following cross-relations

\[ T_i c_j = c_j T_i, \quad j \neq i, i + 1, \]
\[ T_i c_i = c_{i+1} T_i, \quad 1 \leq i \leq n - 1, \]
\[ (T_i + 1) c_{i+1} = c_i (T_i + 1), \quad 1 \leq i \leq n - 1. \]

Let \( \deg(T_i) = 0, \deg(c_j) = 1 \), then \( HCl_n(0) \) becomes a \( \mathbb{Z}_2 \)-graded superalgebra, with a linear basis \( \{ c_D T_w : D \subseteq [n], w \in \Sigma_n \} \), where \( c_D := c_{i_1} \cdots c_{i_r} \) for \( D = \{ i_1 < \cdots < i_r \} \) and \( T_w := T_{s_{j_1}} \cdots T_{s_{j_k}} \) for any reduced expression \( w = s_{j_1} \cdots s_{j_k} \). Throughout this paper, we only consider 0-Hecke(-Clifford) algebras, thus write \( \mathcal{H}_n := H_n(0), \mathcal{HCl}_n := HCl_n(0) \) for short.

First recall the main result about representation theory of 0-Hecke-Cartan diagram. Due to Gessel, we know that the Cartan map \( \chi \) satisfies the following commutative diagram [17, Prop. 5.9]:

\[ \mathcal{K}(\mathcal{H}) \xrightarrow{\chi} \mathcal{G}(\mathcal{H}) \]
\[ \text{Ch}' \downarrow \quad \text{Ch} \]
\[ \text{NSym} \xrightarrow{\pi} \text{QSym} \]

That is, \( \chi([P_\alpha]) = \sum_{\beta} \langle R_{\beta}, r_\alpha \rangle [S_\beta] \), where \( r_\alpha := \pi(R_\alpha) \in \Lambda \) is the \textit{ribbon Schur function} of shape \( \alpha \).
3.1. Irreducible supermodules of 0-Hecke-Clifford algebras. A complete set of non-isomorphic simple supermodules of $\mathcal{HC}_n$ are parameterized by peak sets and thus denoted by $\{HCIS_\rho\}$. They can be obtained from the induced modules

$$S_\alpha := \text{Ind}_{\mathcal{HC}_n}^{HClS_\alpha}, \alpha \vdash n.$$ 

Slightly modifying [3, Theorem 5.4], we have

**Theorem 3.1.1.** Let $\alpha \vdash n$, $V := V(\alpha) \subseteq [n]$ and $Cl_V$ be the subalgebra of $Cl_n$ generated by $\{c_{i}\}_{i \in V}$. For any homogeneous $c \in Cl_V$, define $f_c \in \text{End}_{\mathcal{HC}_n}(\tilde{S}_\alpha)$ by

$$f_c(c_D\eta_\alpha) = (-1)^{|D|}c_Dc_\alpha, D \subseteq [n].$$

If we linearly extend it to all $c \in Cl_V$, then the map $c \mapsto f_c$ is an even isomorphism from $Cl_V$ to $\text{End}_{\mathcal{HC}_n}(\tilde{S}_\alpha)$.

By [3, Th. 5.5, Cor. 5.6] we know that $\tilde{S}_\alpha \cong \tilde{S}_\beta$ (even isomorphism) if and only if $P(\alpha) = P(\beta)$ and

$$(3.4) \quad \tilde{S}_\alpha \cong HCIS_{P(\alpha)} \eta_{2^\lambda_\alpha}, I_\alpha := \left\lfloor \frac{|P(\alpha)|+1}{2} \right\rfloor.$$

In fact, we know that for $V(\alpha) = \{n_1, \ldots, n_s\}$, if define the mutually orthogonal even idempotents

$$(3.5) \quad e^\alpha_\epsilon := \frac{1}{2e}(1 + e_1 \sqrt{-1}c_{n_1}c_{n_2})\cdots(1 + e_{l_\alpha} \sqrt{-1}c_{n_{l_\alpha-1}}c_{n_{l_\alpha}}),$$

where $e = (e_1, \ldots, e_{l_\alpha}) \in \{\pm 1\}^{l_\alpha}$, then $Cl_\alpha e^\alpha_\epsilon \eta_\epsilon$ ($\epsilon \in \{\pm 1\}^{\lambda_\alpha}$) provide all the simple components of $\tilde{S}_\alpha$.

**Proposition 3.1.2.** For any peak set $P$, the irreducible supermodule $HCIS_\rho$ is of type $M$ when $|P|$ is odd, and of type $Q$ when $|P|$ is even.

**Proof.** By Theorem 3.1.1, we know that $\text{End}_{\mathcal{HC}_n}(\tilde{S}_\alpha) \cong Cl_V$ for any $\alpha \vdash n$ and $V := V(\alpha) \subseteq [n]$. Since $|V(\alpha)| = |P(\alpha)| + 1$, by Schur’s Lemma we only need to prove that there exists an odd automorphism of $\tilde{S}_\alpha$ stabilizing any simple component of it and only if if $|V|$ is odd.

First for any even idempotent $e^\alpha_\epsilon$ in $(3.5)$, it is easy to check that

$$e^\alpha_{\epsilon i}c_i = \begin{cases} c_i e^\epsilon_\alpha, & i \notin \{n_1, \ldots, n_{2l_\alpha}\}, \\ c_i e^\epsilon_{\gamma i}, & i \in \{n_1, \ldots, n_{2l_\alpha}\}, \end{cases}$$

where $V = \{n_1, \ldots, n_s\}$ and $e^{(i)}$ is obtained from $e$ by only changing the sign $e_j$ if $i \in \{n_{j-1}, n_j\} \subseteq V$. It means that the map $f_{e^\alpha} (E \subseteq V)$ defined in Theorem 3.1.1 stabilizes any simple component of $\tilde{S}_\alpha$ if and only if $|E \cap \{n_{j-1}, n_j\}|$ is even for all $j = 1, \ldots, l_\alpha$. Hence, if we need $f_{e^\alpha}$ to be odd and satisfies the stability condition, then $V$ must be odd too. \qed

Denote

$$\mathcal{G} := G(\mathcal{H}), \mathcal{K} := K(\mathcal{H}), \tilde{\mathcal{G}} := G(\mathcal{HC}), \tilde{\mathcal{K}} := K(\mathcal{HC}).$$

In [3] Bergeron et al. nicely define the following Frobenius isomorphism for the tower of 0-Hecke-Clifford algebras:

$$\tilde{\text{Ch}} : \tilde{\mathcal{G}} \rightarrow \text{Peak}^*, [\tilde{S}_\alpha] \mapsto K_{P(\alpha)},$$
which satisfies the following commutative diagrams (as a categorification of the descent-to-peak map):

\[ \begin{array}{ccc}
\mathcal{G} & \xrightarrow{\text{Ind}_{H}^{H_{n}}} & \tilde{\mathcal{G}} \\
\text{Ch} & \downarrow & \tilde{\text{Ch}} \\
\text{QSym} & \xrightarrow{\partial} & \text{Peak}^* \\
\text{QSym} & \xrightarrow{\theta} & \text{Peak}^* \\
\end{array} \]

\[ \begin{array}{ccc}
\tilde{G} & \xrightarrow{\text{Res}_{H}^{H_{n}}} & G \\
\text{Ch} & \downarrow & \text{Ch} \\
\text{Peak}^* & \xrightarrow{\partial} & \text{QSym} \\
\text{Peak}^* & \xrightarrow{\theta} & \text{QSym} \\
\end{array} \]

where \( \text{Ind}_{H}^{H_{n}} := \bigoplus_{n \geq 0} \text{Ind}_{H_{n}}^{H_{n}} \) and \( \text{Res}_{H}^{H_{n}} := \bigoplus_{n \geq 0} \text{Res}_{H_{n}}^{H_{n}} \).

### 3.2. Projective supermodules of 0-Hecke-Clifford algebras

In this subsection we consider the category of finitely generated projective supermodules of 0-Hecke-Clifford algebras, which is lack of discussion in [3]. So far it is not so easy to construct the indecomposable one directly, we similarly consider the induction from the projective modules of 0-Hecke algebras. Define

\[ \tilde{P}_{\alpha} := \text{Ind}_{H_{n}}^{H_{n}} P_{\alpha}, \alpha \vdash n, \]

which becomes a projective \( H_{n} \)-supermodule.

Since \( H_{n} \) is a free right \( H_{n} \)-module of rank \( 2^n \), we can write a basis for \( \tilde{P}_{\alpha} \) as \( \{c_{D}u_{\alpha} : D \subseteq [n], \alpha \in \mathcal{D}_{\alpha}\} \) for short. Via the defining relation of \( H_{n} \), we get the following commutation relations:

\[ T_{i}c_{D} = \begin{cases} 
\begin{array}{ll}
c_{D}T_{i}, & i, i + 1 \notin D, \\
c_{(D \setminus \{i\}) \cup \{i+1\}}T_{i}, & i \in D, i + 1 \notin D, \\
c_{(D \setminus \{i+1\}) \cup \{i\}}(T_{i} + 1) - c_{D}, & i \notin D, i + 1 \in D, \\
-c_{D}(T_{i} + 1) + c_{D \setminus \{i,i+1\}}, & i, i + 1 \in D, 
\end{array} \end{cases} \]

for any \( D \subseteq [n], i = 1, \ldots, n - 1 \). Combining these relations and the module action (3.2) of \( P_{\alpha} \), one can describe explicitly the module structure of \( \tilde{P}_{\alpha} \) for any \( \alpha \vdash n \).
For example, the module structure of $\tilde{P}_{12}$ can be depicted explicitly by the following graphs (separated into two $\mathbb{Z}_2$-graded components):

where we use $w \in \Xi_\alpha$ to represent $u_w$ and put dots on the heads of those numbers to represent basis elements $c_D u_w$, e.g. we abbreviate $c_{1,3}u_{213}$ as $213$.

By relation (3.8), one easily gets the following result.

**Lemma 3.2.1.** For any $w \in \Xi_\alpha$, $D \subseteq [n]$, 
\[
T_w c_D = (-1)^{l_w,D} c_D T_w + \sum_{i \in D} a_{E,i} c_E T_i
\]
for some integers $a_{E,i}$, where $l_{w,D} := \{i, j \in D : i < j, w(i) > w(j)\}$, $w(D) := \{w(i) : i \in D\}$ and $<$ stands for the Bruhat order of $\Xi_\alpha$.

### 3.3. Dual Hopf algebras arising from 0-Hecke-Clifford algebras

In this subsection, we check a series of axioms for the tower of 0-Hecke-Clifford algebras in order to show that $(\tilde{K}, \tilde{G})$ forms a dual pair of graded Hopf algebras. One of the key steps is to define a proper Hopf pairing.

First the Mackey property of 0-Hecke-Clifford algebras is easy to proved by mimicing the nice approach in [7, §2-h.] for affine Hecke-Clifford algebras. That guarantees both of the Grothendieck groups $\tilde{K}$ and $\tilde{G}$ to be Hopf algebras via the induction and the restriction. One can also refer to [11, Theorem 2.7], [24, Prop. 4.3] for the case of Hecke algebras.

For the sake of completeness, we sketch the proof steps as follows. For any $\alpha = (\alpha_1, \ldots, \alpha_r) \vdash n$, define $\mathcal{H}_\alpha := \mathcal{H}_{\alpha_1} \otimes \cdots \otimes \mathcal{H}_{\alpha_r}$, embedding as a parabolic subalgebra of $\mathcal{H}_n$. Similarly define the subalgebra $\mathcal{H}_\alpha$ of $\mathcal{H}_n$.

Given $\alpha, \beta \vdash n$, let $\Xi_\alpha := \{\varsigma_i : i \notin D(\alpha)\}$ be the Young subgroup of $\Xi_n$, $\mathcal{R}_\alpha$ denote the set of minimal length left $\Xi_\alpha$-coset representatives in $\Xi_n$, and $\mathcal{R}_\alpha^{-1}$ for the one corresponding to right $\Xi_\alpha$-coset. Then $\mathcal{R}_{\alpha,\beta} := \mathcal{R}_\alpha^{-1} \cap \mathcal{R}_\beta$ is the set of minimal length $(\Xi_\alpha, \Xi_\beta)$-double coset representatives in $\Xi_n$. For any $x \in \mathcal{R}_{\alpha,\beta}$, $\Xi_\alpha x \Xi_\beta x^{-1}$ and $x^{-1} \Xi_\alpha x \cap \Xi_\beta$ are Young subgroups of $\Xi_n$, thus we can let $\alpha \cap x \beta$ and $x^{-1} \alpha \cap \beta$ to be the two compositions of $n$ such that
\[
\Xi_\alpha \cap x \Xi_\beta x^{-1} = \Xi_{\alpha \cap x \beta}, \quad x^{-1} \Xi_\alpha x \cap \Xi_\beta = \Xi_{x^{-1} \alpha \cap \beta}.
\]

Now it needs several technical lemmas as follows.
Lemma 3.3.1. For any \( x \in \mathcal{R}_{\alpha, \beta} \), the subspace \( \mathcal{H}C_{x}T_{x}H_{\beta} \) of \( \mathcal{H}C_{\alpha} \) has basis \( \{ C_{D}T_{w} : D \subseteq [n], w \in \mathcal{Z}_{\alpha, \beta} \} \). Moreover,
\[
\mathcal{H}C_{\alpha} = \bigoplus_{x \in \mathcal{R}_{\alpha, \beta}} \mathcal{H}C_{x}T_{x}H_{\beta}.
\]

Fix some total order \(<\) refining the Bruhat order \(<\) on \( \mathcal{R}_{\alpha, \beta} \). For \( x \in \mathcal{R}_{\alpha, \beta} \), let
\[
\mathcal{B}_{\leq x} := \bigoplus_{y \in \mathcal{R}_{\alpha, \beta}, y \leq x} \mathcal{H}C_{x}T_{y}H_{\beta}, \quad \mathcal{B}_{< x} := \bigoplus_{y \in \mathcal{R}_{\alpha, \beta}, y < x} \mathcal{H}C_{x}T_{y}H_{\beta},
\]
and \( \mathcal{B}_{x} := \mathcal{B}_{\leq x} / \mathcal{B}_{< x} \). By Lemma 3.2.1, we know that \( \mathcal{B}_{\leq x} \) (resp. \( \mathcal{B}_{< x} \)) is invariant under right multiplication by \( C_{n} \). Hence, \( \{ \mathcal{B}_{\leq x} \}_{x \in \mathcal{R}_{\alpha, \beta}} \) is an \((\mathcal{H}C_{\alpha}, \mathcal{H}C_{\beta})\)-bimodule filtration of \( \mathcal{H}C_{\alpha} \).

Lemma 3.3.2. For any \( x \in \mathcal{R}_{\alpha, \beta} \), there exists an algebra isomorphism
\[
\phi = \phi_{x} : \mathcal{H}C_{x, \alpha} \otimes \mathcal{H}C_{x, \beta} \to \mathcal{H}C_{x},
\]
with \( \phi(T_{w}) = T_{x^{-1}w} \), \( \phi(c_{i}) = c_{x^{-1}(i)} \) for \( w \in \mathcal{Z}_{\alpha, \beta}, i \geq 1 \).

Lemma 3.3.3. View \( \mathcal{H}C_{\alpha} \) as an \((\mathcal{H}C_{\alpha}, \mathcal{H}C_{\alpha, \beta})\)-bimodule and \( \mathcal{H}C_{\beta} \) as an \((\mathcal{H}C_{x^{-1} \alpha, \beta}, \mathcal{H}C_{\beta})\)-bimodule. Then \( \mathcal{H}C_{\alpha}^{\beta} \) is an \((\mathcal{H}C_{\alpha, \beta}, \mathcal{H}C_{\beta})\)-bimodule and
\[
\Phi : \mathcal{B}_{x} \to \mathcal{H}C_{\alpha} \otimes \mathcal{H}C_{x, \beta} \otimes \mathcal{H}C_{\beta}, \quad uT_{x}v + \mathcal{B}_{< x} \mapsto u \otimes v, \ u \in \mathcal{H}C_{\alpha}, v \in \mathcal{H}C_{\beta}
\]
is an isomorphism of \((\mathcal{H}C_{\alpha}, \mathcal{H}C_{\beta})\)-bimodule.

Theorem 3.3.4 (Mackey Theorem). Let \( \alpha, \beta \vdash n \) and \( M \) be an \( \mathcal{H}C_{\beta} \)-module. Then the \( \mathcal{H}C_{x} \)-module \( \text{Res}^{\mathcal{H}C_{\alpha}}_{\mathcal{H}C_{\beta}} \text{Ind}^{\mathcal{H}C_{\alpha}}_{\mathcal{H}C_{\beta}} M \) admits an \( \mathcal{H}C_{x} \)-submodule filtration with subquotients isomorphic to \( \text{Ind}^{\mathcal{H}C_{\alpha}}_{\mathcal{H}C_{x}} \text{Res}^{\mathcal{H}C_{\alpha}}_{\mathcal{H}C_{x}^{-1} \alpha} \) one for each \( x \in \mathcal{R}_{\alpha, \beta} \).

Especially when \( \alpha, \beta \) both have two parts, Theorem 3.3.4 implies that \( \mathcal{G} \) has graded Hopf algebra structure under induction and restriction. The case for \( \tilde{K} \) is similar, since \( \mathcal{B}_{x} \) is projective as a left \( \mathcal{H}C_{\alpha} \)-module.

In order to prove the Hopf duality between \( \tilde{K} \) and \( \mathcal{G} \) by Prop. 2.4.5, we continue to show that \( \mathcal{H}C_{n} \) is a Frobenius superalgebra. It is straightforward to check that

Proposition 3.3.5. There exist two even algebra involutions \( \varphi, \varphi' \) for the 0-Hecke-Clifford algebra \( \mathcal{H}C_{n} \), defined by
\[
\varphi(T_{i}) = T_{n-i} + c_{n-i}c_{n+1-i}, \ i = 1, \ldots, n - 1,
\]
\[
\varphi(c_{j}) = -c_{n+1-j}, \ j = 1, \ldots, n.
\]
(3.9)

\[
\varphi'(T_{i}) = -(T_{n-i} + 1), \ i = 1, \ldots, n - 1,
\]
\[
\varphi'(c_{j}) = -c_{n+1-j}, \ j = 1, \ldots, n.
\]

There exist two unsigned even algebra anti-involutions \( \psi, \psi' \) of \( \mathcal{H}C_{n} \) defined by
\[
\psi(T_{i}) = T_{i} + c_{i}c_{i+1}, \ i = 1, \ldots, n - 1,
\]
\[
\psi(c_{j}) = -c_{j}, \ j = 1, \ldots, n.
\]
(3.10)
Proposition 3.3.6. The 0-Hecke-Clifford algebra \( \mathcal{HC}_n \) is a Frobenius superalgebra with even trace map

\[
\text{tr}_n : \mathcal{HC}_n \to \mathbb{K}, \quad \text{tr}_n(c_DT_w) = \delta_{D,0}\delta_{w,w_0}, \quad D \subseteq [n], w \in \Xi_n,
\]

where \( w_0 \) is the longest element of \( \Xi_n \). Moreover, \( \varphi \) is the corresponding Nakayama automorphism.

Proof. We only need to prove that \( \ker \text{tr}_n \) contains no non-zero left ideals. Suppose \( I \) is a non-zero left ideal of \( \mathcal{HC}_n \). We choose an element \( b = \sum_{w \in \Xi_n} b_{D,w}c_DT_w \in I \setminus \{0\} \) and let \( \sigma \) be a maximal length element in the set \( \{w \in \Xi_n : b_{D',w} \neq 0 \text{ for some } D' \subset [n]\} \). Then by Lemma 3.2.1 we have

\[
\text{tr}_n(c_{w_0^0}(D)T_w0^1b) = b_{D',0}\text{tr}_n(c_{w_0^0}(D)T_w0^1c_DT_0) = (-1)^{\frac{\ell(w)-1}{2}}(-1)^{\frac{\ell(w)-1}{2}}b_{D',0} \neq 0.
\]

Thus \( I \notin \ker \text{tr}_n \). To show that \( \varphi \) is the corresponding Nakayama automorphism, it suffices to show that

\[
(1) \quad \text{tr}_n(c_DT_wT_i) = \text{tr}_n((T_{w-i} + c_{n-i}c_{n+1-i})c_DT_w),
\]

\[
(2) \quad \text{tr}_n(c_DT_wc_j) = (-1)^{\frac{\ell(w)-1}{2}}\text{tr}_n(-c_{n+1-j}c_DT_w)
\]

for all \( i \in \{1, \ldots, n-1\}, j \in \{1, \ldots, n\}, D \subseteq [n] \) and \( w \in \Xi_n \).

For (1) we break the proof into four cases as in [24, Lemma 4.2]. When \( w = w_0s_i \),

\[
\text{tr}_n(c_DT_wT_i) = \text{tr}_n(c_DT_{w_0}T_i) = \delta_{D,0}. \quad \text{On the other hand,}
\]

\[
\text{tr}_n((T_{n-i} + c_{n-i}c_{n+1-i})c_DT_w) = \text{tr}_n(c_{w_0}(D)T_{n-i}T_w) + \text{tr}_n(c_{n-i}c_{n+1-i}c_DT_w)
\]

\[
= \text{tr}_n(c_{w_0}(D)T_{w_0}) = \delta_{D,0},
\]

where we use relation (3.8) for the first equality and the identity \( w_0s_i = s_{n-i}w_0 \) for the second last one.

When \( w = w_0 \), \( \text{tr}_n(c_DT_{w_0}T_i) = -\text{tr}_n(c_DT_{w_0}) = -\delta_{D,0} \). On the other hand, if \( D \neq \emptyset, \{n-i, n+1-i\} \), then \( \text{tr}_n((T_{n-i} + c_{n-i}c_{n+1-i})c_DT_w) = 0 \) by relation (3.8). Otherwise, for \( D = \emptyset \),

\[
\text{tr}_n((T_{n-i} + c_{n-i}c_{n+1-i})T_{w_0}) = -\text{tr}_n(T_{w_0}) = -1.
\]

For \( D = \{n-i, n+1-i\} \),

\[
\text{tr}_n((T_{n-i} + c_{n-i}c_{n+1-i})c_DT_{w_0}) = \text{tr}_n((-c_{n-i}c_{n+1-i}(T_{n-i} + 1))T_{w_0}) + \text{tr}_n(-T_{w_0})
\]

\[
= \text{tr}_n(-c_{n-i}c_{n+1-i}(T_{n-i} + 1))T_{w_0}) = 0.
\]

The rest two cases when \( \ell(w) \leq \ell(w_0) - 2 \) or \( \ell(w) = \ell(w_0) - 1 \) but \( w \neq w_0s_i \) are similar to check. For (2),

\[
\text{tr}_n(c_DT_wc_j) = \text{tr}_n(c_Dc_{w_j}T_w) = -\delta_{D,[w_{j+1}]}\delta_{w,w_0} = -\delta_{D,[n+1-j]}\delta_{w,w_0} = (-1)^{\frac{\ell(w)-1}{2}}\text{tr}_n(-c_{n+1-j}c_DT_w).
\]

□

Proposition 3.3.7. For any \( \alpha \vdash n \),

(1) we have a supermodule isomorphism between \( \bar{S}_\alpha \) and \( \varphi(\bar{S}_\alpha) \) of degree \( \bar{n} \), sending \( c_D\eta_\alpha^* \) to \( (-1)^{\bar{n}}\varphi(c_D)c_{[n]}\eta_\alpha \). In particular, \( \varphi(\mathcal{HC}_S P_{(\alpha)}) \cong \mathcal{HC}_S P_{(\alpha')} \).
(2) we have an even supermodule isomorphism between $\tilde{S}_{\alpha}^*$ and $\varphi'(\tilde{S}_{\alpha})$, sending $c_D\eta_{\alpha}$ to $\varphi'(c_D)\eta_{\alpha}$. In particular, $\varphi'(\text{HClS } P(\alpha)) \cong \text{HClS } P(\alpha^*)$.

(3) we have an even supermodule isomorphism between $\tilde{S}_{\alpha}$ and $\psi(\tilde{S}_{\alpha}^*)$, sending $c_D\eta_{\alpha}$ to $c_D\psi\xi_{\alpha}$, where $\xi_{\alpha}$ is the dual of $\eta_{\alpha}$ with respect to the standard basis $\{c_D\eta_{\alpha}\}$ of $\tilde{S}_{\alpha}$. In particular, $\psi(\text{HClS } P^*) \cong \text{HClS } P$ for any peak set $P$ in $[n]$.

(4) we have a supermodule isomorphism between $\tilde{S}_{\alpha}$ and $\psi(\tilde{S}_{\alpha}^*)$ of degree $\bar{n}$, sending $c_D\eta_{\alpha}$ to $(-1)^{\bar{n}D}c_D\psi\xi_{\alpha}$, where $\xi_{\alpha}$ is the dual of $c_D\eta_{\alpha}$ with respect to the standard basis $\{c_D\eta_{\alpha}\}$ of $\tilde{S}_{\alpha}$. In particular, $\psi'(\text{HClS } P^*) \cong \text{HClS } P$ for any peak set $P$ in $[n]$.

**Proof.** (1) From relations (3.1) and (3.8) we have

$$T_i-c[n]\eta_{\alpha} = (T_{n-i} + c_{n-i}c_{n+1-i})c[n]\eta_{\alpha} = -c[n](T_{n-i} + 1)\eta_{\alpha} = \begin{cases} -c[n]\eta_{\alpha}, & \text{if } i \in D(\alpha'), \\ 0, & \text{otherwise.} \end{cases}$$

Hence, there exists an $\mathcal{H}_{\alpha}$-module homomorphism from $S_{\alpha}$ to $\text{Res}^{\mathcal{H}_{\alpha}}_{\mathcal{H}_{\alpha}} \varphi(\tilde{S}_{\alpha})$, sending $\eta_{\alpha}$ to $c_D\eta_{\alpha}$. By the universal property of induction functors, we obtain an $\mathcal{H}_{\alpha}$-supermodule homomorphism from $\tilde{S}_{\alpha}$ to $\psi(\tilde{S}_{\alpha})$ sending $c_D\eta_{\alpha}$ to $(-1)^{\bar{n}D}\psi(c_D)\eta_{\alpha}$. It is obviously surjective thus an isomorphism by dimension argument.

Meanwhile, for any $i \in [2, n-1]$,

$$i \in P(\alpha) \Leftrightarrow i \in D(\alpha), i - 1 \notin D(\alpha) \Leftrightarrow n - i \notin D(\alpha'), n + 1 - i \in D(\alpha') \Leftrightarrow n + 1 - i \in P(\alpha'),$$

which means that $|P(\alpha)| = |P(\alpha')|$, thus $\text{HClS } P(\alpha') \cong \psi'(\text{HClS } P(\alpha))$ by (3.4).

(2) From relation (3.1) we have

$$T_i-\varphi\eta_{\alpha} = -(T_{n-i} + 1)\eta_{\alpha} = \begin{cases} -\eta_{\alpha}, & \text{if } i \in D(\alpha), \\ 0, & \text{otherwise.} \end{cases}$$

Hence, there exists an $\mathcal{H}_{\alpha}$-module homomorphism from $S_{\alpha}$ to $\text{Res}^{\mathcal{H}_{\alpha}}_{\mathcal{H}_{\alpha}} \varphi(\tilde{S}_{\alpha})$, sending $\eta_{\alpha}$ to $\eta_{\alpha}$. The rest of the proof is nearly the same as (1).

(3) By relation (3.8) we have

$$T_i-\varphi\xi_{\alpha}(c_D\eta_{\alpha}) = \xi_{\alpha}(T_i + c_i+c_{i+1})c_D\eta_{\alpha} = \delta_D\varphi\xi_{\alpha}(T_i-\eta_{\alpha})$$

$$= \begin{cases} -1, & \text{if } D = \emptyset, i \in D(\alpha), \\ 0, & \text{otherwise.} \end{cases}$$

That is, $T_i-\varphi\xi_{\alpha} = -\xi_{\alpha}$ if $i \in D(\alpha)$ and 0 otherwise. Hence, there exists an $\mathcal{H}_{\alpha}$-module homomorphism from $S_{\alpha}$ to $\text{Res}^{\mathcal{H}_{\alpha}}_{\mathcal{H}_{\alpha}} \varphi(\tilde{S}_{\alpha})$, sending $\eta_{\alpha}$ to $\xi_{\alpha}$. Again by the universal property of induction functors, we obtain an $\mathcal{H}_{\alpha}$-supermodule homomorphism from $\tilde{S}_{\alpha}$ to $\psi(\tilde{S}_{\alpha}^*)$, sending $c_D\eta_{\alpha}$ to $c_D\psi\xi_{\alpha}$. It is also obviously surjective thus an isomorphism by dimension argument.
That is, \( T_{i, \psi} \xi_\alpha = -\xi_\alpha \) if \( i \in D(\alpha) \) and 0 otherwise. Hence, there exists an \( \mathcal{H}_n \)-module homomorphism from \( S_\alpha \) to \( \text{Res}_\mathcal{H}_n \psi(\mathcal{S}_\alpha) \), sending \( \eta_\alpha \) to \( \xi_\alpha \). The rest of the proof is nearly the same as (3).

For any \( m, n \in \mathbb{N} \), denote \( \mathcal{H}_{m,n} := \mathcal{H}_m \otimes \mathcal{H}_n \), \( \mathcal{H}^{m,n} := \mathcal{H}_m \otimes \mathcal{H}_n \). Let \( \iota_n : \mathcal{H}_n \to \mathcal{H}^{0,n} \), \( \mu_{m,n} : \mathcal{H}_{m,n} \to \mathcal{H}_{m+n,n} \), \( \tilde{\mu}_{m,n} : \mathcal{H}_{m,n} \to \mathcal{H}^{m+n,n} \) be the natural embeddings.

**Proposition 3.3.8.** For the tower of 0-Hecke-Clifford superalgebras, we have an isomorphism of functors

\[
\varphi_{\text{Res}} \cong \tau_{12} \circ \text{Res}
\]

on \( \mathcal{H} \)-mod (hence also on \( \mathcal{H} \)-pmod).

**Proof.** For any \( m, n \in \mathbb{N} \), let \( \text{Res}_m^{m+n} := \text{Res}_{\mathcal{H}_m^{m+n}} \) and \( \varphi_{m,n} := \varphi_m \otimes \varphi_n \). It needs to prove that there exists an isomorphism of functors from \( \mathcal{H}_{m,n} \)-mod to \( \mathcal{H}^m \)-mod:

\[
\varphi_{m,n} \circ \text{Res}_m^{m+n} \circ \varphi_{m+n}^{-1} \cong \tau_{12} \circ \text{Res}_n^{m+n}.
\]

By the definition of the Nakayama automorphisms \( \varphi_n (n \in \mathbb{N}) \) in (3.9), we have

\[
\varphi_{m+n} \circ \tilde{\mu}_{m,n} = \tilde{\mu}_{n,m} \circ \varphi_{m,m} \circ f_{12},
\]

where \( f_{12} : \mathcal{H}_m \otimes \mathcal{H}_n \to \mathcal{H}_n \otimes \mathcal{H}_m \) is the flip isomorphism (2.17). Hence, the LHS of (3.11) is

\[
\tilde{\mu}_{m,n} \varphi_n (\mathcal{H}_m^{m+n}) \varphi_{m+n} \otimes \mathcal{H}_m^{m+n} = \varphi_{m+n} \varphi_{m,n} \varphi_{m+n} f_{12} (\mathcal{H}_m^{m+n}) \varphi_{m+n} \otimes \mathcal{H}_m^{m+n} = \tilde{\mu}_{n,m} \varphi_{m+n} \mathcal{H}_m^{m+n} \otimes \mathcal{H}_m^{m+n},
\]

which is exactly the RHS of (3.11).

**Proposition 3.3.9.** For the tower of 0-Hecke-Clifford superalgebras, we have an isomorphism of functors \( \text{Res} \cong \tau_{12} \circ \text{Res} \) on \( \mathcal{H} \)-pmod.

**Proof.** For any \( m, n \in \mathbb{N} \), let \( \iota_{m,n} := \iota_m \otimes \iota_n \). First note that the following isomorphism of functors from \( \mathcal{H}_{m+n} \)-mod to \( \mathcal{H}^{m,n} \)-mod holds:

\[
\tilde{\mu}_{m,n} (\mathcal{H}_m^{m+n}) \varphi_{m+n} \otimes \mathcal{H}_{m+n} = \varphi_{m+n} (\mathcal{H}_m^{m+n}) \varphi_{m+n} \otimes \mathcal{H}_{m+n} -.
\]

In particular, for \( \alpha = m + n \), we choose \( P_{\alpha} \in \mathcal{H}_{m+n} \)-pmod to get that

\[
\text{Res}_{\mathcal{H}_m^{m+n}} P_{\alpha} \cong \tilde{\mu}_{m,n} (\mathcal{H}_m^{m+n}) \varphi_{m+n} \otimes \mathcal{H}_{m+n} P_{\alpha} \cong (\mathcal{H}_m^{m+n}) \varphi_{m+n} \otimes \mathcal{H}_{m+n} \text{Res}_{\mathcal{H}_m^{m+n}} P_{\alpha}.
\]
Now by Lemma 2.4.2, we have
\[
\tau_{12} \mathrm{Res}_{\mathcal{H} \mathcal{C}_{m,n}}^\mathcal{H} \tilde{P}_a = \tau_{12} (\mathcal{H} (\mathcal{C}_{m,n})_{\mathcal{H}_{m,n}} \otimes \mathcal{H}_{m,n}) \mathcal{H} \mathcal{C}_{n,m}^\mathcal{H} \tilde{P}_a \\
\cong (\mathcal{H} (\mathcal{C}_{m,n})_{\mathcal{H}_{m,n}} \otimes \mathcal{H}_{m,n}) \mathcal{H} \mathcal{C}_{n,m}^\mathcal{H} \tilde{P}_a \\
\cong (\mathcal{H} (\mathcal{C}_{m,n})_{\mathcal{H}_{m,n}} \otimes \mathcal{H}_{m,n}) \mathcal{H} \mathcal{C}_{n,m}^\mathcal{H} \tilde{P}_a.
\]
As NSym is cocommutative, so is \( \mathcal{K}(\mathcal{H}) \) by the Frobenius map (3.3), i.e. \( \mathrm{Res} \cong \tau_{12} \circ \mathrm{Res} \) on \( \mathcal{H} \)-pmod, thus it also holds on \( \mathcal{H} \mathcal{C} \)-pmod by the above discussion. In fact, if write \( \left[ \mathrm{Res} \tilde{P}_a \right] = \sum C_{\alpha_1,\alpha_2}^\alpha [\tilde{P}_{\alpha_1}] \otimes [\tilde{P}_{\alpha_2}] \) (The explicit formula is described by the shuffle product, see \( [19],[28], (16)) \), then by (3.12) we simply have \( \left[ \mathrm{Res} \tilde{P}_a \right] = \sum C_{\alpha_1,\alpha_2}^\alpha [\tilde{P}_{\alpha_1}] \otimes [\tilde{P}_{\alpha_2}] \). □

By Prop. 2.4.5, 3.3.6, 3.3.8, 3.3.9, we finally conclude that \( (\tilde{\mathcal{K}}, \tilde{\mathcal{G}}) \) forms a dual pair of graded Hopf algebras with respect to the pair (2.16). That is, for \( P, Q \in \mathcal{H} \mathcal{C} \)-pmod, \( M, N \in \mathcal{H} \mathcal{C} \)-mod,
\[
(\mathrm{Ind}([P] \otimes [Q]), [M]) = ([P] \otimes [Q], \mathrm{Res}[M]), \\
([P], \mathrm{Ind}([M] \otimes [N])) = ([\mathrm{Res}[P], [M] \otimes [N]).
\]
It is easy to see that there exists an algebra involution \( \tilde{\varphi} \) of \( \mathcal{H}_n \) defined by
\[
\tilde{\varphi}(T_i) = T_{n-i}, \ i = 1, \ldots, n - 1.
\]
\( \tilde{\varphi} \) is also the Nakayama automorphism of \( \mathcal{H}_n \) [24, Lemma 4.2]. Now we have the following result

**Proposition 3.3.10.** For any \( \alpha \vdash n \),
\[
\tilde{\varphi} S_{\alpha} \cong S_{\tilde{\alpha}}, \ \tilde{\varphi} P_{\alpha} \cong P_{\tilde{\alpha}}.
\]

**Proof.** For the twisted module \( \tilde{\varphi} S_{\alpha} \), we have
\[
T_i \tilde{\varphi} \eta_{\alpha} = T_{n-i} \eta_{\alpha} = \begin{cases} -\eta_{\alpha}, & i \in D(\tilde{\alpha}), \\ 0, & \text{otherwise}, \end{cases}
\]
which implies the first isomorphism.

For the second one, we note that for any \( w = w_1 \cdots w_n \in \Xi_n \),
\[
w \in \Xi_n \Leftrightarrow w_i > w_{i+1}, \ i \in D(\alpha) \Leftrightarrow w_0 w_0 (i) > w_0 w_0 (i + 1), \ i \in D(\tilde{\alpha}) \Leftrightarrow w_0 w_0 w_0 \in \Xi_{n-1}.
\]
That is, \( w_0 \Xi_{n-1} = \Xi_{n-1} \) and also \( i \in D(w) \Leftrightarrow n - i \in D(w_0 w_0 w_0) \). Hence, by the module structure (3.2) of \( P_{\alpha} \), we know that for any \( w \in \Xi_{n-1} \),
\[
T_i \varphi u_w = T_{n-i} u_w = \begin{cases} -u_w, & n - i \in D(w^{-1}), \\ u_{s_{n-i} w}, & n - i \notin D(w^{-1}), s_{n-i} w \in \Xi_n, \\ 0, & \text{otherwise}, \end{cases}
\]
which is in \( \Xi_{n-1} \) and also \( \tilde{\varphi} u_w = u_{s_{n-i} w} \) by the above discussion.

\[
T_i \varphi u_w = T_{n-i} u_w = \begin{cases} -u_w, & i \in D((w_0 w_0 w_0)^{-1}), \\ u_{s_{n-i} w}, & i \notin D((w_0 w_0 w_0)^{-1}), w_0 s_{n-i} w w_0 = s_i w_0 w_0 \in \Xi_n, \\ 0, & \text{otherwise}. \end{cases}
\]
Hence, \( u_w \mapsto u_{w_0} w w_0 \), \( w \in \mathcal{D}_\alpha \) gives the second isomorphism, where \( \{ u_w : w \in \mathcal{D}_\alpha \} \) is the standard basis of \( \mathcal{P}_\alpha \).

4. From 0-Hecke-Clifford algebras to the peak algebra of symmetric groups

In this section we inherit the known result in [3] to clarify more explicitly the relation between \( \text{Peak, Peak}^* \) and the supermodule categories of 0-Hecke-Clifford algebras, especially the dual Hopf pair \( (\tilde{K}, \tilde{G}) \) discussed in the previous section. Then we consider the corresponding Heisenberg double in order to prove the freeness of \( \text{Peak}^* \) over \( \Omega \).

4.1. From \( \tilde{K} \) to Peak. First of all, we define the adjoint map of the Frobenius isomorphism \( \tilde{\text{Ch}} \) relating two non-degenerate pairs \( [\cdot, \cdot] \) in (2.9) and \( \langle \cdot, \cdot \rangle \) in (2.16). That is a Hopf isomorphism \( \tilde{\text{Ch}}^* : \text{Peak} \to \tilde{K} \) satisfying

\[
[ F, \tilde{\text{Ch}} ([M]) ] = \langle \tilde{\text{Ch}}^* ( F ), [M] \rangle, \quad F \in \text{Peak}, [M] \in \tilde{G}.
\]

The following result provides the explicit form of \( \tilde{\text{Ch}}^* \):

**Theorem 4.1.1.** For any composition \( \alpha \), we have

\[
(4.1) \quad \tilde{\text{Ch}}^* ( \Theta (R_\alpha) ) = [P_\alpha].
\]

Moreover, the following commutative diagram of Hopf algebras holds:

\[
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\text{Ind}^H_C} & \tilde{K} \\
\Big\vert & & \Big\vert \\
\text{Ch}^* & \xrightarrow{\tilde{\chi}^*} & \tilde{G} \\
\text{NSym} & \xrightarrow{\Theta} & \text{Peak} \\
\end{array}
\]

where \( \tilde{\chi} : \tilde{K} \to \tilde{G} \) is the Cartan map of 0-Hecke-Clifford algebras. In particular, \( \text{Im} \chi = \Omega \).

**Proof.** Given compositions \( \alpha, \beta \), we have

\[
\langle \tilde{\text{Ch}}^* ( \Theta (R_\alpha) ), [S_\beta] \rangle = \langle \Theta (R_\alpha), \tilde{\text{Ch}} ([S_\beta]) \rangle = \langle R_\alpha, \text{Ch} ( [\text{Res}^H_C S_\beta] ) \rangle
\]

\[
= \langle \text{Ch}^* (R_\alpha), [\text{Res}^H_C S_\beta] \rangle = \langle [P_\alpha], [\text{Res}^H_C S_\beta] \rangle
\]

\[
= \langle [\text{Ind}^H_C P_\alpha], [S_\beta] \rangle = \langle [P_\alpha], [S_\beta] \rangle,
\]

where the second equality is due to (2.10) and (3.7), and the second last one bases on the fact that induction functor is left adjoint to restriction. Since the pair \( \langle \cdot, \cdot \rangle \) is non-degenerate, we get the desired formula, equivalent to the left commutative square.

For the right commutative square, we only need to prove that

\[
\tilde{\text{Ch}} \circ \tilde{\chi} ( [P_\alpha] ) = \pi \circ \Theta (R_\alpha)
\]

by formula (4.1). The module structure (3.2) of \( P_\alpha ( \alpha \vdash n ) \) and Lemma 3.2.1 imply that if fix a total order \( \prec \) refining theBruhat order on \( \mathcal{D}_\alpha \), then \( \tilde{P}_\alpha \) has a super submodule filtration \( \{ \tilde{P}_w \}_{w \in \mathcal{D}_\alpha} \), where \( \tilde{P}_\alpha := \{ c_D u_z : D \subseteq [n], z \in \mathcal{D}_\alpha, w \leq z \} \). Also, the subquotient
\[ \hat{P}_\alpha / \sum_{w < \hat{S}_\alpha} \hat{P}_\alpha \cong \hat{S}_{\varepsilon(w^{-1})} \text{ for any } w \in \mathfrak{D}_\alpha. \] Hence,
\[
\overline{\text{Ch}} \circ \tilde{\chi}(\hat{P}_\alpha)] = \sum_{w \in \mathfrak{D}_\alpha} \overline{\text{Ch}}(\hat{S}_{\varepsilon(w^{-1})}) = \sum_{w \in \mathfrak{D}_\alpha} K_{P(w^{-1})}
\]
\[
= \sum_{\beta} |\{w \in \mathfrak{D}_n : w \in \mathfrak{D}_\alpha, w^{-1} \in \mathfrak{D}_\beta\}| K_{P(w^{-1})} = \sum_{\beta} \langle R_\beta, r_\alpha \rangle K_{P(\beta)}
\]
\[
= \sum_{P} \sum_{P(\beta) = P} R_\beta, \pi(\alpha) K_P = \sum_{P} \langle \Xi_P, r_\alpha \rangle K_P = \sum_{P} [\Xi_P, \theta(r_\alpha)] K_P
\]
\[
= \theta(r_\alpha)(2.8) \Theta(2.8) \Theta(R_\alpha)(2.8) \pi \Theta(R_\alpha),
\]
where the fourth equality is due to the following well-known formula of Gessel (see [17, Prop. 5.9]):
\[
\langle R_\beta, r_\alpha \rangle = \langle R_\alpha, r_\beta \rangle = |\{w \in \mathfrak{D}_n : w \in \mathfrak{D}_\alpha, w^{-1} \in \mathfrak{D}_\beta\}|.
\]

**Corollary 4.1.2.** For any composition \( \alpha \), we have the following decomposition formula:
\[
\hat{P}_\alpha \cong \bigoplus_{P(\beta) \subseteq D(\alpha) \Delta (D(\alpha) + 1)} HClP_{P(\beta)} \otimes 2^{P(\beta)} \cdot l_\beta := \left\{ \frac{|P(\beta)| + 1}{2} \right\},
\]
where we use \( HClP \) to denote the projective cover of \( HClP \) for any peak set \( P \).

**Proof.** From [3, (6)] we know that
\[
\Theta(R_\alpha) = \sum_{P \subseteq D(\alpha) \Delta (D(\alpha) + 1)} 2^{|P(\beta)| + 1} \Xi_P.
\]

Now by Theorem 4.1.1,
\[
\langle [\hat{P}_\alpha], [\hat{S}_\beta] \rangle = \langle \Theta(R_\alpha), \overline{\text{Ch}}([\hat{S}_\beta]) \rangle = \langle \Theta(R_\alpha), K_{P(\beta)} \rangle
\]
\[
= \begin{cases} 
2^{|P(\beta)| + 1}, & \text{if } P(\beta) \subseteq D(\alpha) \Delta (D(\alpha) + 1), \\
0, & \text{otherwise.}
\end{cases}
\]

Combining it with (2.14), (3.4) and Prop. 3.1.2, we get the desired decomposition of \( \hat{P}_\alpha \). □

In particular, the generator \( Q_n = \Theta(R_\alpha) \) corresponds to the projective simple supermodule \( [\hat{P}_\alpha] = [\hat{S}_\alpha] = [HClS_{\emptyset}] = [HClP_{\emptyset}] \) via \( \overline{\text{Ch}}^+ \) and
\[
\langle [\hat{P}_\alpha], [\hat{S}_\beta] \rangle = 2\delta_{P(\beta), \emptyset},
\]
also due to Schur’s Lemma as the simple supermodule \( HClS_{\emptyset} \) is of type Q. Note that in the classical case, \( q_n \in \Omega \) corresponds to the basic spin module \( C_n \) of the Sergeev algebra \( Cl_n \cong \mathbb{K} \mathfrak{S}_n \) under the Frobenius isomorphism [10, §3.3].

Now we abuse the notation to denote by \( \tilde{\varphi} \) the Hopf algebra anti-involution of \( N\text{Sym} \) such that \( \tilde{\varphi}(H_n) = H_n (n \in \mathbb{N}) \). Note that \( \tilde{\varphi}(R_\alpha) = R_\alpha \). Meanwhile, since \( \tilde{\varphi}(\text{Ker } \Theta) = \text{Ker } \Theta \), \( \tilde{\varphi} \) induces a Hopf algebra anti-involution of Peak, which we abuse to denote by \( \varphi \), then \( \varphi \circ \Theta = \Theta \circ \tilde{\varphi} \).

We are in the position to prove the following restriction rule.
Theorem 4.1.3. For any composition \( \alpha \), the following commutative diagram of Hopf algebras holds:

\[
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\text{Res}^H} & \mathcal{K} \\
\downarrow & & \downarrow \\
\text{Peak} & \xrightarrow{\varphi} & \text{NSym}
\end{array}
\quad \text{or} \quad
\begin{array}{ccc}
\tilde{\mathcal{K}} & \xrightarrow{\text{Res}^H} & \tilde{\mathcal{K}} \\
\downarrow & & \downarrow \\
\text{Peak} & \xrightarrow{\varphi} & \text{NSym}
\end{array}
\]

where \( \varphi : \text{Peak} \to \text{NSym} \) is the natural inclusion. Equivalently, for any composition \( \alpha \),

\[
[\text{Res}^H \bar{P}_\alpha] = \sum_{P(\beta) \subseteq D(\alpha)} 2^{n + 1} [P_\beta],
\]

where \( \bar{P}_\alpha \) is one of the two sums

\[
\bar{P}_\alpha = \sum_{D \subseteq \{\alpha\}} c_D P(n) \quad \text{or} \quad \sum_{D \subseteq \{\alpha\}} c_D P(n) \quad \text{or} \quad \sum_{D \subseteq \{\alpha\}} c_D P(n) ,
\]

For \( \bar{P}_\alpha = \tilde{S}(n) \), \( v_D := c_D P(n) \) form a basis. By (3.8)

\[
T_{i,V_D} = \begin{cases} 
-v_D + v_D(i,i+1), & \text{if } i, i+1 \in D, \\
-v_D + v_D(i+1,i+1), & \text{if } i \notin D, i+1 \in D, \\
0, & \text{otherwise.}
\end{cases}
\]

Define a partial order \( \triangleleft \) on \( 2^n \) such that D covers \( D' \) if \( i, i+1 \notin D' \) and \( D = D' \cup \{i, i+1\} \), or \( i \in D', i+1 \notin D' \) and \( D = (D' \setminus \{i\}) \cup \{i+1\} \) for some \( i \in [n-1] \). We denote such covering relation by \( D' \triangleleft D \). For any \( 0 \leq k \leq n-1 \), take \( D_{n,k} \) to be one of the two sets \( \{n-k+1, \ldots, n\} \setminus \{1\} \) and \( \{1\} \cup \{n-k+1, \ldots, n\} \) with odd cardinality. Define

\[
v_{n,k} := v_{D_{n,k}} + \sum_D \epsilon_D v_D \in \bar{P}_\alpha,
\]

where the sum is over those \( D \triangleleft D_{n,k} \) such that \( D \triangleleft_i \cdots \triangleleft_i D_{n,k} \) for some \( i_1, \ldots, i_r \in [n-k, n-1] \), and \( \epsilon_D \) is the sign of length of any chain in \( 2^n \) from \( D \) to \( D_{n,k} \).

Then

\[
T_{i,v_{n,k}} = \begin{cases} 
0, & \text{if } i \in [1, n-k-2], \\
-v_{n,k}, & \text{if } i \in [n-k, n-1].
\end{cases}
\]

Meanwhile, \( \mathcal{H}_{n-k+1} \) is generated by \( u_{1, \ldots, (n-k-1), n} \) and the projective \( \mathcal{H}_{n-k+1} \)-module \( P(n-k,1^r) \) is generated by \( u_{1, \ldots, (n-k-1), n} \). Now one can check that the homogeneous component \( \bar{P}_\alpha \) is generated by \( u_{1, \ldots, (n-k-1), n} \) and the isomorphism \( \mathcal{H}_{n-k+1} \rightarrow P(n-k,1^r) \) holds. Define \( D_{n,k} \) analogous to \( D_{n,k} \) but with even cardinality and then \( v_{n,k} \) analogous to \( v_{n,k} \). Then \( \bar{P}_0 = \bigoplus_{k=0}^{n-1} \mathcal{H}_{n-k+1} v_{n,k} \) and \( \mathcal{H}_{n-k+1} v_{n,k} \) as in the odd case. Finally, we prove formula (4.5) and thus the commutative diagrams.
4.1.4 Then by (4.1) and the commutative diagram, we get the desired restriction rule (4.4).

Example 4.1.4. For \( n = 5, k = 2 \), the diagram of \( P_{(3,1^2)} \) is as follows.

Now \( D_{5,2} = \{1, 4, 5\} \) and \( v_{5,2} = v_{(1,4,5)} - v_{(1,3,5)} - v_{(1)} + v_{(1,3,4)} \). It is straightforward to check the isomorphism \( H_5.v_{5,2} \cong P_{(3,1^2)}, v_{5,2} \mapsto u_{12543} \).

Next we give another kind of restriction rule for the induced projective modules. Identifying \( H_{n-1} \) (resp. \( H_{n-1}^\circ \)) with \( H_{n-1,1} \) (resp. \( H_{n-1,1}^\circ \)), we have the embedding \( \mu_n : H_{n-1} \to H_n \) (resp. \( \bar{\mu}_n : H_{n-1}^\circ \to H_n^\circ \)) defined from \( \mu_{n-1,1} \) (resp. \( \bar{\mu}_{n-1,1} \)).

Theorem 4.1.5. For any \( \alpha = (\alpha_1, \ldots, \alpha_r) \in n \), we have

\[
\bar{\mu}_n P_\alpha \cong \bigoplus_{\alpha \geq \beta \geq \alpha \vdash r} P_{(\alpha_0)} \cong \bigoplus_{\alpha \geq \beta \geq \alpha \vdash r} P_{(\alpha_0)} \oplus P_{(\alpha_2, \ldots, \alpha_r)} \oplus \beta \appa 1,1,
\]

where \( \alpha(i) := (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \ldots, \alpha_r) \), \( \alpha'(i) = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i} + 1, \alpha_{i+1} - 1, \alpha_{i+2}, \ldots, \alpha_r) \).

Proof. First note that we have the following isomorphism of functors on \( H_n \)-mod.

\[
\mu_n (H_{n-1}^{\circ} \otimes H_n) \cong (H_{n-1}^{\circ} \otimes H_n) \oplus (H_{n-1}^{\circ} \otimes H_n) \oplus (H_{n-1}^{\circ} \otimes H_n) \oplus (H_{n-1}^{\circ} \otimes H_n)
\]

\[
c_D T_w \otimes - \mapsto (c_D \otimes T_w \otimes -), \quad (c_D \otimes T_w \otimes -) \mapsto (0, c_D \otimes T_w \otimes -)
\]

for any \( D \subseteq [n-1], w \in \Xi_n \). In particular,

\[
\mu_n P_\alpha = \mu_n (H_{n-1}^{\circ} \otimes H_n) P_\alpha \cong (H_{n-1}^{\circ} \otimes H_n) P_\alpha \oplus (H_{n-1}^{\circ} \otimes H_n) P_\alpha \oplus (H_{n-1}^{\circ} \otimes H_n) P_\alpha
\]

which reduces formula (4.6) to

\[
\mu_n P_\alpha \cong \bigoplus_{\alpha \geq \beta \geq \alpha \vdash r} P_{(\alpha_0)} \oplus P_{(\alpha_2, \ldots, \alpha_r)} \oplus \beta \appa 1,1.
\]

For example, \( \mu_n P_{(1,2)^2} \cong P_{(2,2)} \oplus P_{(1,2)^2} \oplus P_{(1,3)} \oplus P_{(1,2,1)} \).

Recall that \( P_\alpha \) has basis \( \{u_w : w \in \Xi_n\} \) with module action (3.2). Let \( k_i := \alpha_i + \cdots + \alpha_n, i = 1, \ldots, r \). It is easy to see that the \( j \)'s such that \( w_j = n \) for some \( w = w_1 \cdots w_n \in \Xi_n \) are those satisfying \( j = k_i \) with \( i = 1 \) or \( 1 < i < r, \alpha_i > 1 \). Now for any such \( j = k_i \), denote
\[ \mathcal{D}_{\alpha}^{(i)} := \{ w \in \mathcal{D}_\alpha : w_j = n \} \text{ and } P_{\alpha}^{(i)} := \text{span}_K \{ u_w : w \in \mathcal{D}_{\alpha}^{(i)} \}. \]

We need to deal with three cases.

If \( 1 < j = k_i < n, \alpha_i > 1 \), then \( \{ u_w : w \in \mathcal{D}_{\alpha}^{(i)}, w_{j-1} > w_{j+1} \} \) spans the projective \( \mathcal{H}_{n-1} \)-module \( P_{\alpha_i} \). Modulo these vectors, the quotient space of \( P_{\alpha_i}^{(i)} \) spanned by \( \{ u_w : w \in \mathcal{D}_{\alpha}^{(i)}, w_j < w_{j+1} \} \) is isomorphic to another projective \( \mathcal{H}_{n-1} \)-module \( P_{\alpha_i} \). If \( j = k_1 = \alpha_1 = 1 \), then \( P_{\alpha_i}^{(i)} \) spans \( \mathcal{P}_{(\alpha_2, \ldots, \alpha_r)} \). If \( j = k_1 = n, \alpha_r > 1 \), then \( P_{\alpha_i}^{(r)} \equiv P_{\alpha_i} := P_{(\alpha_1, \ldots, \alpha_r - 1, \alpha_r - 1)} \). In summary, we get the desired formula. \( \square \)

4.2. Application: Peak* is free over \( \Omega \). Finally we consider the Heisenberg double arising from \( (\mathcal{K}, \mathcal{G}) \) in order to prove that Peak* is a free \( \Omega \)-module using the method of Savage et al. in [24]. In general, given a graded Hopf pair \( (\mathcal{K}(A), \mathcal{G}(A)) \), there exists a left action of \( \mathcal{K}(A) \) on \( \mathcal{G}(A) \) such that \( \mathcal{G}(A) \) is a \( \mathcal{K}(A) \)-module algebra. It is defined by

\[
\mathcal{K}(A) = \left\{ \begin{array}{ll}
\text{Hom}_A \left( P_{\gamma_{n-i-t}} \mathcal{R}_{A_{n-i} \otimes A_i}, M \right), & P \in \mathcal{P}_{A}, i \leq n, \text{mod } i = n, \\
0, & \text{otherwise},
\end{array} \right.
\]

where \( \gamma_{n-i-t} : A_i \to A_{n-i} \otimes A_i, a \mapsto 1_{n-i} \otimes a \) is the natural embedding. Note that for the Cartan map \( \chi, \mathcal{G}_{\text{proj}}(A) := \text{Im} \chi \) is stable under such action, thus a subalgebra of \( \mathcal{G}(A) \). Now one can define the following two kinds of Heisenberg doubles:

\[
\mathcal{H} := \mathcal{K}(A), \mathcal{H} := \mathcal{G}(A), \mathcal{H}^{\text{proj}} := \mathcal{G}_{\text{proj}}(A), \mathcal{H} := \mathcal{K}(A), \mathcal{H} := \mathcal{G}(A).
\]

The notation \# means smash product construction on \( H \otimes B \) from a Hopf algebra \( H \) and an \( H \)-module algebra \( B \).

Let

\[
\mathcal{H}^{-} := \mathcal{K}(A), \mathcal{H}^{+} := \mathcal{G}(A), \mathcal{H}^{\text{proj}} := \mathcal{G}_{\text{proj}}(A), \mathcal{H} := \mathcal{K}(A), \mathcal{H} := \mathcal{G}(A).
\]

Then \( \mathcal{H}^{+} \) becomes a left \( \mathcal{H} \)-module, called the lowest weight Fock representation, where \( \mathcal{H}^{+} \) acts by left multiplication and \( \mathcal{H}^{-} \) acts by formula (4.7). For any \( \mathcal{H} \)-module \( V, v \in V \) is called a lowest weight vacuum vector if \( H^{-}v = 0 \).

From now on, we focus on the case when the tower \( A = \mathcal{H} \mathcal{E} \).

**Lemma 4.2.1.** Suppose \( V \) is an \( \mathcal{H}^{\text{proj}} \)-module generated by a finite set of lowest weight vacuum vectors. Then \( V \) is a direct sum of lowest weight Fock representations.

**Proof.** By the Stone-von Neumann Theorem for Heisenberg doubles [24, Theorem 2.11] and Theorem 4.1.1, we have \( \mathcal{H}^{\text{proj}} \cdot v = \mathcal{H}^{+} \equiv \Omega \) as an irreducible \( \mathcal{H}^{\text{proj}} \)-module over \( K \) for any lowest weight vacuum vector \( v \in V \). Now the same argument in [24, Lemma 9.1] gives the complete reducibility of \( V \). \( \square \)

For any \( \beta = (\beta_1, \ldots, \beta_r), \gamma = (\gamma_1, \ldots, \gamma_s) \), let \( \beta \cdot \gamma := (\beta_1, \ldots, \beta_r, \gamma_1, \ldots, \gamma_s) \), then \( \Delta(M_\alpha) = \sum_{\beta, \gamma = \alpha} M_\beta \otimes M_\gamma \). Since \( \Theta \) is a Hopf algebra epimorphism, \( N_\alpha := \Theta(M_\alpha) (\alpha \in \mathcal{C}) \) span the Stembridge algebra Peak* and

\[
\Delta(N_\alpha) = \sum_{\beta, \gamma = \alpha} N_\beta \otimes N_\gamma.
\]
Define an increasing filtration of \( \text{h}_{\text{proj}} \)-submodules of Peak\(^*\) as follows. For \( n \in \mathbb{N}_0 \), let
\[
(\text{Peak}^*)^{(n)} := \sum_{\ell(\alpha) \leq n} \text{h}_{\text{proj}} \cdot N_\alpha.
\]
In particular, \( (\text{Peak}^*)^{(0)} := \Omega \) and by convention we also let \( (\text{Peak}^*)^{(-1)} := 0 \).

**Proposition 4.2.2.** The space Peak\(^*\) of peak quasisymmetric functions is free as an \( \Omega \)-module.

**Proof.** For any composition \( \alpha \) such that \( \ell(\alpha) = n \), by the grading argument of \([\cdot , \cdot]\),
\[
Q_m N_\alpha = \sum_{\beta \gamma = \alpha} [Q_m, N_\gamma] N_\beta \in (\text{Peak}^*)^{(n-1)}, \ m \in \mathbb{N}.
\]
Hence, in the quotient \( V_n := (\text{Peak}^*)^{(n)}/(\text{Peak}^*)^{(n-1)} \), such \( N_\alpha \)'s are lowest weight vacuum vectors. Clearly these vectors generate \( V_n \) and thus by Lemma 4.2.1,
\[
V_n = \bigoplus_{v \in L_n} \Omega \cdot v,
\]
where \( L_n \) is some collection of vacuum vectors in \( V_n \).

Consider the short exact sequence of \( \Omega \)-modules
\[
0 \to (\text{Peak}^*)^{(n-1)} \to (\text{Peak}^*)^{(n)} \to V_n \to 0.
\]
Since \( V_n \) is a free \( \Omega \)-module, the above sequence splits. Now \( (\text{Peak}^*)^{(0)} \cong \Omega \), so we know that all \( (\text{Peak}^*)^{(n)} \ (n \in \mathbb{N}_0) \) are free over \( \Omega \) by induction on \( n \). It means that we can choose nested sets of vectors in \( \text{Peak}^* \)
\[
L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots
\]
such that, for any \( n \in \mathbb{N}_0 \), \( (\text{Peak}^*)^{(n)} = \bigoplus_{v \in L_n} \Omega \cdot v \). Let \( \bar{L} = \bigcup_{n \in \mathbb{N}_0} \bar{L}_n \). Then \( \text{Peak}^* = \bigoplus_{v \in \bar{L}} \Omega \cdot v \).

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