A Complete Surface Integral Method for Broadband Modeling of 3D Interconnects in Stratified Media

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Abstract—A surface integral equation solver is proposed for fast and accurate modeling of large-scale electromagnetic structures in stratified media. The single-source differential surface admittance operator, previously proposed for conductors in free space, is extended to conductors in stratified media, and accelerated with the adaptive integral method (AIM) for scalability. AIM is developed for a multilayer environment in a novel, generalized manner that poses no restrictions on layout of conductors, and requires no special grid refinement, unlike previous works. The multilayer Green’s function (MGF) is computed with a new series expansion-based method in the near-field, and with the discrete complex image method (DCIM) in the far-field. The proposed method is made robust over a wide frequency band by employing the augmented electric field integral equation. Realistic structures of different shapes and electrical sizes are successfully analyzed over a frequency range of 1 kHz–40 GHz. Results are validated against a commercial finite element tool, and demonstrate the accuracy and efficiency of the proposed method.

Index Terms—electromagnetic modeling, integrated circuit modeling, surface integral method, method of moments, multilayered media, acceleration.

I. INTRODUCTION

Electromagnetic (EM) simulation tools are essential in the design of modern integrated circuits (ICs), which are becoming increasingly intricate. Due to the high operating frequency and compact volume of most ICs, there is strong EM coupling and cross-talk between their constituent conductors. Circuit designers require quantitative predictions of this coupling, and therefore would benefit greatly from fast and accurate EM modeling of IC components over a broad frequency range, typically from DC to tens of gigahertz. Obtaining broadband EM models of on-chip interconnects is especially difficult because they exhibit strong field variations over such a wide frequency range, due to skin, proximity, and substrate effects. In the past, on-chip interconnects have primarily been analyzed through a 2D approach based on transmission line theory [1]. However, 2D techniques are only accurate when applied to long interconnects with a constant cross-section along the entire length. To model more sophisticated geometries, full-wave 3D EM solvers are necessary. Furthermore, given the complexity of modern on-chip interconnect networks, the speed and efficiency of the solver are of paramount importance.

Full-wave 3D characterization of interconnects requires solving for EM fields both inside (“interior problem”) and outside (“exterior problem”) the conductors. The interior problem is responsible for modeling skin and proximity effects. The exterior problem captures coupling between conductors, as well as the effect of the surrounding medium.

To model the interior problem, one class of formulations involves a volumetric discretization of fields within conductors, such as the finite element method (FEM) [2], [3], volumetric integral equation-based methods [4], [5], and the partial element equivalent circuit (PEEC) method [6], [7]. Despite their robustness over a wide frequency range, volumetric methods require a prohibitively large number of mesh elements when skin depth shrinks at high frequency.

In surface-based formulations, mesh elements exist only on the surfaces of conductors. This results in significantly smaller system matrices and thus faster computations without the need for adaptive meshing at high frequency. Instead, skin effect is taken into account through a surface-based operator, which can be derived analytically or numerically. Analytic surface operators based on the surface impedance boundary condition (SIBC) [8] are popular due to their simplicity. However, analytic operators are generally derived based on assumptions on geometry or frequency. For example, the SIBC is only accurate at high frequency, where skin depth is significantly smaller than the conductor cross-section. Numerical surface operators for general conductor geometries involve solving additional integral equations for the interior problem, and allow for accurate modeling of skin effect [9]–[15].

One general surface-based approach is the Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) formulation which was developed to model dielectrics [9]. However, this method is inaccurate when applied to conductors, due to the high contrast in material properties compared to surrounding media.

Alternative surface formulations that require only two integral equations have since been proposed, such as the generalized impedance boundary condition (GIBC) [14] and two-region surface integral equation methods [11], [12]. Still, these methods require solving at least two sets of unknowns on the surfaces of conductors: equivalent electric current density, and equivalent magnetic current density. This necessitates integral equation operators that involve gradients of the complicated layered media Green’s function. Instead, a formulation involving a single integral equation with simple operators would...
significantly improve the efficiency of surface-based methods.

The exterior problem for interconnect modeling poses two main challenges: modeling surrounding dielectric media that are typically stratified, and scaling the method to handle realistically large problems within reasonable computational resources.

A stratified dielectric medium would require no special treatment if its interfaces and boundaries were discretized along with the conductors within it, as in the PMCHWT formulation [16]. However, this leads to a dramatic increase in the number of mesh elements. Alternatively, the surrounding medium can be modeled using the multilayer Green’s function (MGF) [17], which does not require any discretization of layer interfaces. Despite its mathematical and numerical complexity, the MGF can be implemented quite efficiently using fitting techniques such as the discrete complex image method (DCIM) [18].

For realistic structures, the exterior problem typically requires acceleration techniques coupled with iterative methods to solve large problems with reasonable resources. Despite its popularity, the kernel-dependent nature of the multilevel fast multipole method (MLFMM) [19] makes it challenging for use with the MGF. The adaptive integral method (AIM) [20] is more amenable to inhomogeneous problems, but still requires careful treatment for stratified media. For example, certain techniques are limited to specific geometric configurations of source and test conductors or layers [21], [22]. General multilayer geometries have been successfully accelerated [23]. However, this comes at the expense of a significantly increased discretization of the AIM grid along the direction of stratification, and the need for an additional set of basis functions.

In this work, we present a complete method for accurate and efficient modeling of complex interconnect networks embedded in stratified dielectric media, while addressing the gaps and shortcomings of existing methods. Specifically, the contributions presented herein are:

- The single-source differential surface admittance operator formulation [13], [15] is extended for multiconductor systems in stratified media;
- The MGF computation in the near-field is accelerated with a Taylor series expansion of the Bessel function, which was shown to be significantly faster than DCIM [24];
- The proposed method is accelerated with AIM, which is developed for a multilayer environment in a novel, generalized manner: no restrictions are posed on the conductor layout, and no special refinement of the AIM grid is required, unlike other methods [23]. Therefore, the computational advantages of the traditional homogeneous AIM are demonstrably retained.

In addition to the above, we ensure robustness over a wide frequency band by employing the augmented electric field integral equation (aEFIE) [25], that avoids low-frequency issues of the EFIE and is well-conditioned from near-DC to tens of gigahertz.

This paper is organized as follows: Sec. II lays out the general problem statement to be addressed throughout this paper. The differential surface admittance operator is derived in Sec. III. The operator is incorporated into the exterior problem along with port definitions in Sec. IV. In Sec. V, we discuss the modeling of structures in the context of stratified media using DCIM and the series expansion of the Bessel function. The acceleration of the final system in a layered medium context is described in Sec. VI. Finally, the solver is validated and tested on realistic structures, and the results are presented and discussed in Sec. VII. The work is summarized and concluded in Sec. VIII.

II. PROBLEM STATEMENT

Throughout this paper, we consider a general structure consisting of $N_c$ conductors with an arbitrary geometry, embedded in $N_d$ perfect or lossy dielectric layers with stratification along the $z$ axis. The conductors are excited by any number of user-defined ports. The goal is to solve Maxwell’s equations in integral form with a surface-based triangle mesh on the conductors, and efficiently extract the scattering ($S$) matrix. Without loss of generality, we assume that each conductor can only reside in one dielectric layer, but conductors in the same or adjacent layers can be in contact with each other. If a conductor traverses multiple dielectric layers, it is split across layer interfaces, and conductors in contact with each other are handled explicitly, as shown in Sec. V-C.

A. Notation

Throughout this work, we consider time-harmonic fields. A time dependence of $e^{j\omega t}$ is assumed and suppressed. Field quantities are written with an overhead arrow, for example $\vec{a}(\vec{r})$. Primed coordinates represent source points, while unprimed coordinates represent field observation points. Matrices and column vectors are written in bold letters, such as $\mathbf{A}$, while dyadic quantities are written with a double bar overhead, as in $\overline{\mathbf{B}}$.

III. INTERIOR PROBLEM

Modeling lossy conductors necessitates the formulation of an interior problem to accurately capture the skin effect. We model the skin effect over a wide frequency range with the differential surface admittance operator [13], [15], which is computed for arbitrary shapes by applying the Stratton-Chu formulation and the equivalence principle, as described below.

A. Discretization of Fields and Currents

Consider a single conductor, with conductivity $\sigma_c$, permeability $\mu_c$, and permittivity $\varepsilon_c$. It is assumed here that the conductor is fully embedded in layer $l$ of the multilayer substrate. Treatment for conductors that traverse multiple layers is discussed in Sec. V-C. Fig. 1 shows the cross-section of a sample conductor in a stratified medium. The conductor’s surface is denoted by $S$. The tangential electric and magnetic fields on $S$ are expanded with RWG basis functions [26].
Equations (3) and (4) are also known as the electric field integral equation (EFIE) and magnetic field integral equation (MFIE), respectively.

Next, we substitute (11) and (12) into (13). In accordance with the standard method of moments procedure, the resulting equation is tested with \( \hat{n} \times \bar{f}_n(\bar{r}) \), to ensure that the \( L \) and \( K \) operators are well tested. This yields the matrix equation

\[
 j \omega \varepsilon_\eta \mathbf{L}_c \mathbf{E} + \eta_c \mathbf{K}_c \mathbf{H} = 0 ,
\]

where the \( (n, n') \) entries of \( \mathbf{L}_c \) and \( \mathbf{K}_c \) are given by

\[
 [\mathbf{K}_c]_{n, n'} = \left( \bar{f}_n(\bar{r}), \left[ K_c \left( \bar{f}_{n'}(\bar{r}') \right) \right](\bar{r}) \right) - \frac{1}{2} \left( \hat{n} \times \bar{f}_n(\bar{r}), \bar{f}_{n'}(\bar{r}) \right) ,
\]

\[
 [\mathbf{L}_c]_{n, n'} = \left( \bar{f}_n(\bar{r}), \left[ L_c \left( \bar{f}_{n'}(\bar{r}') \right) \right](\bar{r}) \right) ,
\]

and \( \langle \cdot \rangle \) denotes the inner product.

Matrix \( \mathbf{K}_c \) is full-rank and well-conditioned for highly conductive media, so we can express the tangential magnetic field in (8) as

\[
 \mathbf{H} = - [\mathbf{K}_c]^{-1} \mathbf{L}_c \mathbf{E} ,
\]

where \( \mathbf{Y}_{in} \) is the surface admittance matrix of the conductor. Even though (11) requires computing an LU factorization of a full matrix, the size of this matrix is typically small compared to the overall problem. Large conductors can be split up into smaller segments to significantly reduce the time required to compute (11). This is further discussed in Sec. V-C.

C. Equivalent Problem

Next, the equivalence principle (27) is applied to replace the conductor by its surrounding medium, and an equivalent electric current density \( \bar{f}_{eq}(\bar{r}) \), as shown in Fig. 1. The tangential electric field on \( S \) in the equivalent configuration is enforced to be the same as in the original problem, \( \hat{n} \times \bar{E}(\bar{r}) \).

The tangential magnetic field is modified to account for the change in material parameters in the equivalent problem, and is expanded with RWG basis functions as

\[
 \hat{n} \times \bar{H}_{eq}(\bar{r}) = \sum_{n=1}^{N_c} H_{eq,n} \bar{f}_n(\bar{r}) ,
\]

and the expansion coefficients \( H_{eq,n} \) are stored in vector \( \mathbf{H}_{eq} \).

Similar to (5)–(8), we again relate the tangential electric and magnetic fields on \( S \) in the equivalent problem by the Stratton-Chu formulation

\[
 j \omega \mu \hat{n} \times \left[ \mathbf{L}_l \left( \hat{n} \times \bar{H}_{eq}(\bar{r}) \right) \right] + \hat{n} \times \left[ K_l \left( \hat{n} \times \bar{E}(\bar{r}) \right) \right] = \frac{1}{2} \hat{n} \times \bar{E}(\bar{r}) ,
\]

\[
 j \omega \varepsilon_l \hat{n} \times \left[ \mathbf{L}_l \left( \hat{n} \times \bar{E}(\bar{r}) \right) \right] + \hat{n} \times \left[ K_l \left( \hat{n} \times \eta \bar{H}_{eq}(\bar{r}) \right) \right] = \frac{1}{2} \hat{n} \times \eta \bar{H}_{eq}(\bar{r})
\]

where \( k_l = \omega \sqrt{\mu_0 \varepsilon_l} \) and \( \eta = \sqrt{\mu_0 / \varepsilon_l} \) are the wave number and the intrinsic impedance of the \( l \)-th layer of the stratified
medium, respectively. Substituting (11) and (12) into (13) and (14), and testing with \( \hat{n} \times f_m(\vec{r}) \) yields the matrix equations,

\[
\begin{align*}
\vec{J} & = \nabla \times \vec{H}_{\text{eq}} - K_i \vec{E} = 0, \\
\eta L_i H_{\text{eq}} + j \omega \varepsilon L_i \vec{E} & = 0,
\end{align*}
\]

where \( L_i \) and \( K_i \) are the method of moment matrices defined by (9) and (10), but with the material properties of dielectric layer \( l \). The combined field integral equation (CFIE) is formed by adding equal parts of (15) and (16) to obtain

\[
\begin{align*}
\vec{J}_{\text{eq}} = -G_{ij}^{-1} G_{M} \vec{E},
\end{align*}
\]

We expand the tangential magnetic field via the surface admittance operator \( Y_{\text{out}} \) as

\[
\vec{H}_{\text{eq}} = -G_{M} \vec{E}. \tag{20}
\]

D. Equivalent Electric Current Density

According to the equivalence principle, the equivalent electric current density is

\[
\vec{J}_{\text{eq}}(\vec{r}) = \hat{n} \times \left[ \vec{H}_{\text{eq}}(\vec{r}) - \vec{H}(\vec{r}) \right]. \tag{21}
\]

We expand \( \vec{J}_{\text{eq}}(\vec{r}) \) with edge length normalized RWG basis functions,

\[
\vec{J}_{\text{eq}}(\vec{r}) = \sum_{n=1}^{N_e} J_n \vec{f}_n(\vec{r}), \tag{22}
\]

and store the coefficients \( J_n \) in vector \( \vec{J}_{\text{eq}} \). Since all three field quantities in (21) are expanded with RWG basis functions, we can express the discretized equivalent current as

\[
\vec{J}_{\text{eq}} = \vec{H}_{\text{eq}} - \vec{H}. \tag{23}
\]

Finally, by substituting (20) and (11) into (23) we obtain

\[
\begin{align*}
\vec{J} = \left[ Y_{\text{out}} - Y_{\text{in}} \right] \vec{E},
\end{align*}
\]

where \( Y_{\Delta} \) is the differential surface admittance operator that accurately models electromagnetic fields inside the conductor. This operator does not require a volumetric mesh and is applicable to conductors of arbitrary geometries.

E. Generalization to Multi-Conductor Systems

For a multi-conductor system with \( P \) conductors, the procedure described above is individually applied to each conductor. The differential surface admittance operator for each conductor is derived using the material properties of the layer in which it resides. The operators for each conductor are assembled into a global block diagonal matrix relating conductor-wise tangential electric fields and equivalent current densities,

\[
\begin{pmatrix}
J_1 \\
\vdots \\
J_P
\end{pmatrix}
= \begin{pmatrix}
Y_{\Delta}^{(1)} & \cdots & 0 \\
0 & \ddots & 0 \\
0 & 0 & Y_{\Delta}^{(P)}
\end{pmatrix}
\begin{pmatrix}
E_1 \\
\vdots \\
E_P
\end{pmatrix}. \tag{25}
\]

IV. Exterior Problem

We can now write down the EFIE for tangential fields in the exterior problem to capture the coupling between different conductors,

\[
\hat{n} \times \vec{E}(\vec{r}) + j \omega \mu \hat{n} \times \left( \vec{J}_{\text{eq}}(\vec{r}) \right) dS' = \vec{E}_{\text{inc}}, \tag{26}
\]

where \( \mu \) and \( k \) are the permeability and wave number of the material surrounding the conductor. \( \vec{E}_{\text{inc}} \) is the incident electric field, and \( \vec{E} \) is the total electric field. The electric field and equivalent current density are expanded with edge length normalized RWG basis functions as per (1) and (22). Equation (26) is then tested with \( \hat{n} \times f_n(\vec{r}) \) as before, leading to the following system of matrix equations:

\[
\vec{I}_{\text{orth,E}} + \vec{L} \vec{J}_{\text{eq}} = \vec{E}_{\text{inc}}, \tag{27}
\]

where \( \vec{L} \) represents the discretized operator \( \mathcal{L} \) in (5). Matrix \( \vec{I}_{\text{orth,E}} \) is the projection of \( \hat{n} \times \vec{f}_n(\vec{r}) \) on \( \vec{f}_n(\vec{r}) \).

The electric field \( \vec{E} \) can be eliminated from (27) using (25), obtaining

\[
(\vec{I}_{\text{orth}} Y_{\Delta}^{-1} + \vec{L}) \vec{J}_{\text{eq}} = \vec{E}_{\text{inc}}. \tag{28}
\]

Since \( Y_{\Delta} \) has a block-diagonal structure, its inverse can be computed quickly and stored at the time of generating \( Y_{\Delta}^{(p)} \) for each conductor \( p \).

A well-known pitfall of EFIE-based MOM formulations is low-frequency breakdown [28], which results from the poor conditioning and low rank of the system matrix. Since this issue occurs due to the imbalance between scalar and vector potential at low frequency, it can be mitigated by separating current and charge densities in the EFIE [23].

\[
\vec{E}(\vec{r}) + j \omega \mu \int_{S'} G(\vec{r}, \vec{r}') \vec{J}_{\text{eq}}(\vec{r}') dS' + \varepsilon^{-1} \nabla \int_{S'} G(\vec{r}, \vec{r}') \rho_s(\vec{r}') dS' = \vec{E}_{\text{inc}}, \tag{29}
\]

where \( \rho_s(\vec{r}') \) is the surface charge density on mesh triangles. Current and charge density can also be related via the continuity equation,

\[
\nabla \cdot \vec{J}_{\text{eq}}(\vec{r}') + j \omega \rho_s(\vec{r}') = 0. \tag{30}
\]

The charge density is discretized using area-normalized pulse basis functions, \( h_n(\vec{r}') \),

\[
\rho_s(\vec{r}') = \sum_{n=1}^{N_t} \rho_n h_n(\vec{r}'), \tag{31}
\]

and the coefficients \( \rho_n \) are stored in the vector \( \rho \). Term \( N_t \) is the total number of mesh triangles, and thus the total number of pulse basis functions. Equations (29) and (30) can now
be discretized the same way as before to yield an augmented EFIE system \[25\]

\[
\begin{bmatrix}
Z_{EM} - \mathbf{D}^T \mathbf{Z}_\Phi & 0 & 0 \\
\mathbf{D} & jk_0 \mathbf{I} & jk_0 \mathbf{Z}_{1K} \\
0 & Z_{K1} & 0 \\
0 & 0 & R
\end{bmatrix}
\begin{bmatrix}
\mathbf{J}_{eq} \\
\mathbf{c}_0 \rho \mathbf{J} \\
\mathbf{J}_I & \mathbf{J}_s
\end{bmatrix}
= \begin{bmatrix}
\mathbf{E}_{inc} \\
0 \\
0 \\
0
\end{bmatrix},
\]

(32)

where

\[Z_{EM} = jk_0 Z_A + \frac{1}{\eta_0} \mathbf{I}_{orth} \cdot \mathbf{Y}_\Delta^{-1}.\]

(33)

The entries of matrix blocks \(Z_A\) and \(Z_\Phi\) are defined as

\[
Z_{A,n}^{m,n} = \mu_r \int_{S_m} \mathbf{f}_m(r) \cdot \int_{S_n} \mathbf{G}(\mathbf{r}, \mathbf{r}') \mathbf{f}_n(r') \, dS_n \, dS_m,
\]

(34a)

\[
Z_{\Phi,n}^{m,n} = \varepsilon_r^{-1} \int_{T_m} h_m(\mathbf{r}) \int_{T_n} \mathbf{G}(\mathbf{r}, \mathbf{r}') h_n(r') \, dT_n \, dT_m,
\]

(34b)

where \(S_m\) is the support of \(\mathbf{f}_m(\mathbf{r})\), and \(T_m\) is the support of \(h_m(\mathbf{r})\). Constants \(\mu_r\) and \(\varepsilon_r\) are the relative permeability and permittivity of the surrounding medium, respectively. Matrix \(\mathbf{I}\) is the identity matrix, and \(\mathbf{D}\) is an incidence matrix that acts as a spatial derivative operator, as defined in \[29\].

Separating the scalar and vector potential parts of the EFIE in this way allows the system to be scaled so as to remove the frequency-dependent imbalance between matrices \(Z_A\) and \(Z_\Phi\). This leads to a system matrix that is well conditioned even at low frequency.

A. Port Handling

For interconnect problems, it is necessary to be able to extract scattering parameters of the network, given a set of terminals and ports. This requires coupling the above system to excitation and load circuit elements.

We define subsets of mesh triangles \(\{T_i\}_i\) as belonging to port terminals, where the index \(i = 1, 2 \ldots N_{term}\) and \(N_{term}\) is the number of terminals. The index \(t = 1, 2 \ldots N_{t,i}\) where \(N_{t,i}\) is the number of triangles in terminal \(i\). Each port is then naturally defined as a set of two terminals. Coupling to terminals is achieved by augmenting \(32\) with additional equations and unknowns \[30\].

\[
\begin{bmatrix}
Z_{EM} - \mathbf{D}^T \mathbf{Z}_\Phi & 0 & 0 \\
\mathbf{D} & jk_0 \mathbf{I} & jk_0 \mathbf{Z}_{1K} \\
0 & Z_{K1} & 0 \\
0 & 0 & R
\end{bmatrix}
\begin{bmatrix}
\mathbf{J}_{eq} \\
\mathbf{c}_0 \rho \mathbf{J} \\
\mathbf{J}_I & \mathbf{J}_s
\end{bmatrix}
= \begin{bmatrix}
\mathbf{E}_{inc} \\
0 \\
0 \\
0
\end{bmatrix},
\]

(35)

The additional equations in \[35\] relate \(J_i(r')\) to a Thévenin equivalent circuit consisting of a voltage source, \(V_s\), and a series resistance \(R\). This is achieved as follows:

- The continuity equation, which is the second row in \[35\], is modified for terminal triangles,

\[
\nabla \cdot \mathbf{J}_{eq}(r') + j\omega \rho_s(r') = J_i(r').
\]

(36)

Term \(J_i\) is the volume current density injected into the system from an external circuit, and is discretized on mesh triangles using area-normalized pulse basis functions. The additional vector of unknowns \(\mathbf{J}_I\) collects coefficients of the pulse basis expansion of \(J_i(r')\). Matrix \(Z_{1K}\) consists of ones in rows that correspond to terminal triangles, and zeros otherwise. Its purpose is to pick out and modify the continuity equations only for those triangles that are part of terminals.

- Terminal voltages are expressed in terms of scalar potentials,

\[
V_t = \frac{1}{\varepsilon} \int_S \mathbf{G}(\mathbf{r}, \mathbf{r}') \rho_s(r') \, dS',
\]

(37)

where \(V_t\) is the voltage at terminal \(t\). Assigning potentials to each terminal triangle, followed by expanding and testing with pulse basis functions leads to the third equation in \[35\], where terminal voltage coefficients are stored in \(V_t\). Matrix \(Z_{K1}\) consists of scalar potentials on terminal triangles, and is thus a subset of the matrix \(Z_\Phi\). Matrix \(\mathbf{C}\) consists of ones and zeros to enforce a constant scalar potential over all triangles that constitute a single terminal.

- The fourth row in \[35\] is obtained by applying Kirchoff’s voltage law to relate \(J_i\) and \(V_t\). A Thévenin-equivalent model for the external circuit is assumed between each pair of terminals that forms a port, with a source voltage \(V_s\) and series resistance \(R\). For a given port \(p\) with input terminal \(t_i\) and output terminal \(t_o\), this yields

\[
V_{t_i} - V_{t_o} = V_s + I_t R,
\]

(38)

where \(I_t\) is the current injected into the system by the external circuit. Since the volume current density \(J_i\) is discretized with area-normalized pulse basis functions defined on terminal triangles, its coefficients have units of amperes, and correspond directly to the injected circuit current \(I_t\). In \[35\], matrix \(\mathbf{P}\) contains the coefficients of \(V_{t_i}\) and \(V_{t_o}\), while \(\mathbf{R}\) contains resistances \(R\). Vector \(\mathbf{V}_s\) stores source voltages at each terminal.

B. Charge Neutrality

The system in \[35\] does not account for charge neutrality on conductors, which leads to loss of rank at low frequency. As suggested by Qian and Chew \[29\], we handle this by dropping one charge density unknown for each unconnected conductor in the structure. To account for conductors connected to each other via ports, or otherwise in contact with each other, an adjacency matrix is constructed to find each set of conductors that is isolated from the others. One charge density unknown is then dropped for each set of connected conductors, rather than each individual conductor. Mathematically, this is achieved by introducing mapping matrices \(\mathbf{F}\) and \(\mathbf{B}\), defined in \[29\]. These matrices map the full set of charge unknowns \(\rho\) to and from a reduced set of unknowns, \(\rho_r\). These matrices are incorporated into \[35\] to yield the final system,

\[
\begin{bmatrix}
Z_{EM} - \mathbf{D}^T \mathbf{Z}_0 \mathbf{B} & 0 & 0 \\
\mathbf{F} \mathbf{D} & jk_0 \mathbf{I}_r & jk_0 \mathbf{Z}_{1K} \mathbf{F} \mathbf{Z}_{1K} \\
0 & Z_{K1} \mathbf{B} & 0 \\
0 & 0 & R
\end{bmatrix}
\begin{bmatrix}
\mathbf{J} \\
\mathbf{c}_0 \rho_r \mathbf{J} \\
\mathbf{J}_I & \mathbf{J}_s
\end{bmatrix}
= \begin{bmatrix}
\mathbf{E}_{inc} \\
0 \\
0 \\
0
\end{bmatrix},
\]

(39)
V. CONDUCTORS IN STRATIFIED MEDIA

In order to model realistic structures embedded in layers of dielectric materials, the dyadic nature of the multilayer Green’s function (MGF) must be accounted for in (34a) and (34b).

A. MGF Formulation

In this work, we use the Michalski-Zheng formulation C [51], where the MGF is expressed as a dyadic term and a scalar term. Assuming that the dielectric layers are stacked along the z axis, the dyadic term is

$$
\mathbf{G} = \begin{bmatrix} G_{xx} & 0 & G_{xz} \\
0 & G_{yy} & G_{yz} \\
G_{zx} & G_{zy} & G_{zz} \end{bmatrix}.
$$

(40)

Expressions for each of the components in (40), as well as $G_\phi$, can be found in [41]. The reaction integrals in (34a) and (34b) now become

$$
Z_A^{n,n} = \mu_r \int_{S_m} f_m(\vec{r}) \cdot \int_{S_n} \mathbf{G}(\vec{r}, \vec{r}') f_n(\vec{r}') dS_m dS_n,
$$

(41a)

$$
Z_\phi^{n,n} = \varepsilon_r^{-1} \int_{S_m} h_m(\vec{r}) \int_{S_n} G_\phi(\vec{r}, \vec{r}') h_n(\vec{r}') dT_m dT_n.
$$

(41b)

It is well known that each component $G_{ij}$ in (40) can be expressed as a Sommerfeld integral over the complex $k_p$ plane. The scalar term and diagonal entries of (40) can be expressed as

$$
G_{ij}(\vec{r}, \vec{r}') = \frac{1}{2\pi} \int_0^\infty \tilde{G}_{ij}(k_p) J_0(k_p r) k_p dk_p,
$$

(42a)

$$
G_\phi(\vec{r}, \vec{r}') = \frac{1}{2\pi} \int_0^\infty \tilde{G}_\phi(k_p) J_0(k_p r) k_p dk_p,
$$

(42b)

while the off-diagonal terms in (40) can be written as

$$
G_{ij}(\vec{r}, \vec{r}') = \frac{1}{2\pi} \int_0^\infty \tilde{G}_{ij}(k_p) J_1(k_p r) k_p^2 dk_p.
$$

(43)

In each case, $\tilde{G}_{ij}$ is the spectral domain Green’s function component corresponding to $G_{ij}$. These spectral functions can be derived through a transmission line treatment of the layers [51], under the assumption that the layers extend infinitely along the lateral $(x$ and $y)$ directions. The functions $J_0$ and $J_1$ are Bessel functions of the first kind of order 0 and 1 respectively, and $k_p$ is the lateral wave vector in cylindrical coordinates. Additionally,

$$
\rho = \sqrt{(x-x')^2 + (y-y')^2},
$$

(44a)

$$
r = |\vec{r} - \vec{r}'| = \sqrt{\rho^2 + (z-z')^2}.
$$

(44b)

B. MGF Computation

Rather than evaluate the integrals in (42a), (42b) and (43) numerically, which can be quite time-consuming for large-scale structures, we use two different approaches to approximate the MGF: the discrete complex image method (DCIM) [18], [32] for far-field interactions, and a series expansion of the Bessel function [24] in the near-field. The distinction between near- and far-field interactions is clarified in the context of AIM-based acceleration in Sec. [VI]. It suffices to state here that the computational bottleneck is in near-field computations, for which we use the faster new method [24].

1) Far-field interactions with DCIM: We employ DCIM with two or three levels of sampling, depending on the number of layers and electrical size of the structure. Each component of the MGF can then be written analytically by employing the identities [34]

$$
e^{-jkr} = \int_0^\infty \frac{e^{-jk_{z'}(z-z')}}{jk_z} J_0(k_p \rho) k_p dk_p,
$$

(46)

and

$$
\rho (1 + jkr) \frac{e^{-jkr}}{r^3} = \int_0^\infty \frac{e^{-jk_{z'}(z-z')}}{jk_{z'}} J_1(k_p \rho) k_p^2 dk_p.
$$

(47)

This leads to

$$
G_{ij}(\vec{r}, \vec{r}') = \sum_{i=1}^{N_1} a_i e^{-jk_{R_{c,i}}/R_{c,i}},
$$

(48)

for diagonal components and the scalar potential, and

$$
G_{ij}(\vec{r}, \vec{r}') = \sum_{i=1}^{N_1} a_i \rho (1 + jk_{R_{c,i}}) e^{-jk_{R_{c,i}}/R_{c,i}}.
$$

(49)

for off-diagonal components, where $R_{c,i} = \sqrt{\rho^2 - b_i^2}$.

In the DCIM procedure, it is necessary to sample the spectral MGF sufficiently far along the real $k_p$ axis to accurately capture spatially near interactions. In order to achieve this without the need for a large number of sample points, we extract quasistatic terms from the spectral functions and add them back analytically in the spatial domain, using a process similar to [35]. This causes the spectral functions to decay to zero significantly faster along the real $k_p$ axis, thus precluding the need for more than a few hundred samples.

2) Near-field interactions with Bessel function expansion:

A more efficient MGF approximation is used to alleviate the near-field bottleneck [24]. Both types of Bessel functions are expanded with a Taylor series centered at $\rho k_p = 0$ [36],

$$
J_v(\rho k_p) = (0.5 \rho k_p)^v \sum_{i=0}^\infty \frac{(-0.25\rho^2 k_p^2)^i}{i! (v+i)!}.
$$

(50)

The Bessel function can then be written as a product of terms that depend only on $\rho$, and terms that depend only on the simulation frequency via $k_p$,

$$
J_v(\rho k_p) = \sum_{i=0}^\infty \rho^{v+2i} k_p^{v+2i} (0.5)^v \frac{(-0.25)^i}{i! (v+i)!}.
$$

(51)
The summation and \( \rho \)-dependent parts can be extracted out of the integral after substituting (51) into (42a), (42b) or (43),
\[
G_{ij}(k, r', r'') = \sum_{i=0}^{\infty} \pi^{v+2i} \frac{(0.5)^v (-0.25)^i}{i! (v+i)!} \int_0^{\infty} dk \rho(k)^{v+2i} \tilde{G}_{ij}(k, z, z') k^{v+i+1},
\]
(52)
where \( v = 0 \) or 1.

The integrand in (52) depends only on the simulation frequency and \((z, z')\). The Sommerfeld integral can thus be precomputed for a predetermined set of \(z-z'\) pairs. This is particularly advantageous for on-chip structures, which are relatively small in the direction of stratification. To precompute these semi-infinite integrals, we use the partition-extrapolation approach [37].

C. Current Continuity

Computing the differential surface admittance operator requires that a conductor be entirely situated in a single dielectric layer. To allow for this without sacrificing generality, we enforce continuity conditions that enable conductors to be in contact with each other within or across layers.

Continuity of currents is enforced by dropping one charge density unknown for each pair of contact triangles. Thus, the continuity equation for a pair of contact triangles is written as
\[
\nabla \cdot \tilde{J}_{eq,1}(r'') + \nabla \cdot \tilde{J}_{eq,2}(r'') + \omega \rho_{s,12}(r'') = 0,
\]
(53)
where \( \tilde{J}_{eq,1}(r'') \) is the equivalent current density on the contact triangle of one of the conductors, and \( \tilde{J}_{eq,2}(r'') \) is the current density on its counterpart. The term \( \rho_{s,12}(r'') \) represents the total charge density shared by the pair of contact triangles. This equation manifests itself only as a slight modification to the matrix \( D \) in (39).

VI. ACCELERATION AND SCALABILITY

Assembling the system in (39) for large problems with hundreds of thousands of unknowns would require prohibitively large amounts of memory and CPU time. Moreover, factorizing such a large dense matrix using direct methods such as LU decomposition would be impractical for the same reasons. This necessitates the use of acceleration techniques to speed up the computation of matrix elements, along with the use of an iterative solver to avoid the need to assemble the full system matrix. To accomplish this, we employ the adaptive integral method (AIM) [20] to project weakly-interacting matrix elements on to a regular 3D grid, and leverage fast Fourier transforms (FFTs) [38] to speed up the matrix-vector product in solving (39) iteratively.

In the case of homogeneous media, the translation-invariance of the Green’s function enables the use of 3D FFTs to accelerate interactions in all directions [38]. However, the MGF is only translation-invariant in the lateral directions, and thus amenable to 2D FFTs to accelerate interactions along the \( x \) and \( y \) directions. A 2D FFT-based method was initially proposed for the restrictive case of planar conductors lying in the \( xy \) plane [21]. A technique combining 2D and 3D FFTs has been explored, but is only applicable when all source and observation points lie in one layer [22]. A more general procedure has also been proposed, where 3D conducting objects of arbitrary shape are considered [23]. In this method, sources on the mesh are projected in two dimensions on the nearest \( xy \) grid, rather than three dimensions as in the homogeneous case. This requires an AIM grid that is very dense along the direction of stratification \((z)\) and makes the method more computationally expensive than the homogeneous version of AIM.

In this section, we propose a new AIM-based procedure where the grid has the same size and spacing as in the homogeneous case. Subsequently, we demonstrate that in the proposed approach, the presence of layered media does not detract significantly from the computational advantages of AIM, unlike previous works referenced above.

A. FFT-Accelerated Matrix Assembly

The principle behind AIM is to split the system matrix into “near-field” (NF) and “far-field” (FF) components, with the expectation that NF-related matrix entries contribute most strongly and thus must be computed accurately, while FF-related entries have a weaker contribution and thus can be approximated. In this formulation, the two dense matrices that involve reaction integrals and must be approximated are the vector potential matrix \( Z_A \) and the scalar potential matrix \( Z_\Phi \). Since the matrix \( Z_{k,1} \) is a subset of the rows of \( Z_\Phi \), its entries are also approximated via AIM by reusing the computations already performed for \( Z_\Phi \). These matrices can be written as
\[
Z_A = Z_{A,NF} + Z_{A,FF},
\]
(54a)
\[
Z_\Phi = Z_{\Phi,NF} + Z_{\Phi,FF},
\]
(54b)
where \( Z_{A,NF} \) and \( Z_{\Phi,NF} \) are the vector and scalar potential NF matrices whose entries are computed directly using (34a) and (34b). If the NF region is sufficiently small, (34a) and (34b) are sparse because they will only contain non-zero entries for a small fraction of source and test basis pairs. Memory requirements thus scale approximately linearly with problem size.

The AIM procedure exploiting FFT-based acceleration allows one to write the far-field matrices in the form [20]
\[
Z_{A,FF} = I_A HP_A,
\]
(55a)
\[
Z_{\Phi,FF} = I_\Phi HP_\Phi,
\]
(55b)
where polynomial-based projection matrices \( P_A \) and \( P_\Phi \) are responsible for projecting sources from the mesh to nodes of a regular grid superimposed on the structure, as shown in Fig. 2a. Matrices \( I_A \) and \( I_\Phi \) interpolate computed vector and scalar potentials from grid points back on to the original mesh basis functions. This is visualized in Fig. 2c. The process of generating and applying the interpolation and projection matrices has been thoroughly described in literature for homogeneous media [20, 38]. This process, as well as the choice of grid spacing, are identical in the proposed method and are not reiterated here.

The convolution matrix \( H \) encodes the Green’s function for computing interactions between source and test grid points.
For stratified media, the generation of \( \mathbf{H} \) requires special treatment. First, we define the following terms for convenience with reference to Fig. 2b:

- Let \( r_{i,j}^m \) represent each source grid point.
- Subscript \( i = 1, 2 \ldots N_x \) are indices through nodes along the \( x \) axis.
- Subscript \( j = 1, 2 \ldots N_y \) are indices through nodes along the \( y \) axis.
- Superscript \( m = 1, 2 \ldots N_z \) is an index through each \( xy \) source plane, along the \( z \) axis.
- \( N_x, N_y \) and \( N_z \) are the total number of grid points in the \( x, y \) and \( z \) directions, respectively.

The translation invariance of the MGF along \( x \) and \( y \) means that one only needs to consider one source grid point per \( xy \) plane. We can take \( \{ r_{i,j}^m \}, m = 1, 2 \ldots N_z \) as the set of source grid points for each \( xy \) grid plane. Each of these \( N_z \) source grid points interacts with every grid point. Let \( r_{n,j}^q \) represent each test grid point, where \( n = 1, 2 \ldots N_x \) is an index through each test layer. The total number of test grid points is \( N_x N_y N_z \). Thus the total number of interactions to be computed is \( N_x (N_y N_z)^2 \). This can more conveniently be understood as computing \( N_z \) stacks of \( N_x N_y \times N_z \) matrices, as shown in Fig. 2c. The entries of each matrix are the values of the Green’s function for that combination of source and grid points. Since there are \( 8 \) components in the MGF, including the scalar potential term, a total of \( 8 \) such matrix stacks are to be computed. Each entry of the convolution matrix is given by \( G_q(r_{i,j}^m, r_{n,j}^q) \) for each \( i, j, m \) and \( n \), where \( q = 1, 2 \ldots 8 \) is an index through each MGF component. Once computed and arranged in two-level Toeplitz format, we can leverage 2D FFTs on each of the \( N_x^2 \) 2D grids to compute all possible grid interactions in spatial frequency domain. The convolution matrix then has size \( 8 \times N_x \times N_y \times N_z \).

This form of the convolution matrix requires \( 8N_z \) times more memory than the homogeneous counterpart. However, typical structures of interest, particularly in interconnect modeling, are significantly larger along the \( x \) and \( y \) directions than along \( z \). In addition, the number of \( z \) grid points needed here does not need to be any larger than in the homogeneous case. Also, this phase of the procedure is generally not among the most expensive of a MoM-based code. Thus the added memory and time costs are negligible in the context of an entire simulation.

Note that the NF and FF matrices both have overlapping entries for NF terms, because in the generation of FF matrices, there was no provision made to exclude NF test grid points. Thus, additional pre-correction matrices \( Z_{A,C} \) and \( Z_{\Phi,C} \) are required to compensate for entries of the FF matrices that correspond to NF terms \(^{[38]}\). The pre-corrections are generated in the same way as the FF matrices, but for NF entries only, so that they can preemptively be subtracted from the NF matrices to cancel out the NF terms of FF matrices. To account for this, equations \(^{(54a)}\) and \(^{(54b)}\) become

\[
Z_A = Z_{A,NF} - Z_{A,C} + Z_{A,FF} \quad (56a)
\]

\[
Z_{\Phi} = Z_{\Phi,NF} - Z_{\Phi,C} + Z_{\Phi,FF} \quad (56b)
\]

It is clear that the proposed method does not require any special treatment of the AIM grid to accommodate stratified media. The projection and interpolation matrices stages are the same as in the free space case. Further, it is shown in Sec. VII that for practical interconnect applications, it is faster to compute several 2D FFTs rather than a single 3D FFT, as in the homogeneous case. Thus, including stratified media through the proposed approach does not reduce the performance gains of AIM.

B. Preconditioning

In order to speed up convergence of the iterative matrix solution of \(^{(57)}\), a good preconditioner is necessary. We employ a sparse right-preconditioner similar to the one in \(^{[25]}\).

\[
M = \begin{bmatrix}
\text{diag}(Z_{EM}) & \text{diag}(Z_{EM} B) & 0 & 0 \\
FD & jk_0 I_F & FZ_{IK} & 0 \\
0 & Z_{KJB} & 0 & \eta_0 I_C \\
0 & 0 & R & P
\end{bmatrix},
\]

where \( \text{diag}(\cdot) \) represents diagonal terms of the corresponding matrix. The preconditioner can be abbreviated as

\[
M = \begin{bmatrix}
[EM, d] & [21] \\
M_{21} & M_{22}
\end{bmatrix}
\]
TABLE I

| εr | µr | σ (S/m) | Height (µm) |
|----|----|---------|-------------|
| 2.1| 1.0| 0.0     | 50          |
| 12.5|1.0|0.0     |50           |

where \( Z_{EM,d} = \text{diag}(Z_{EM}) \), and

\[
M_{1,1} = \begin{bmatrix} \text{diag} \left( -D^T Z_{q} B \right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
M_{2,1} = \begin{bmatrix} F D & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T,
\]

\[
M_{2,2} = \begin{bmatrix} f k_0 I & F Z_{JK} & 0 \\ Z_{KC} & B & 0 \\ 0 & R & P \end{bmatrix}.
\]

Leveraging the Schur complement of \( Z_{EM,d} \), the exact inverse of this preconditioner can be written as [25]

\[
M^{-1} = \begin{bmatrix} Z_{EM,d}^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Delta^{-1} \begin{bmatrix} -M_{2,1} Z_{EM,d}^{-1} & 0 \end{bmatrix} \Delta^{-1} \begin{bmatrix} M_{2,1} & 0 \end{bmatrix} + \Delta^{-1} \begin{bmatrix} M_{2,1} Z_{EM,d}^{-1} & 0 \end{bmatrix} \Delta^{-1} \begin{bmatrix} M_{2,1} & 0 \end{bmatrix}
\]

where \( \Delta = M_{2,2} - M_{2,1} Z_{EM,d}^{-1} M_{1,2} \). Since \( Z_{EM,d} \) is diagonal, computing its inverse is trivial. The preconditioner is applied to the system matrix blocks during the matrix-vector product at each iteration. We employ the restarted-GMRES linear solver [39] provided through the PETSc software package [40].

VII. RESULTS

The robustness and performance of the proposed solver are demonstrated through three test cases which span a significant range of geometries, electrical sizes and computational complexity. All tests were conducted on a 3.6 GHz desktop computer with 24 GB of memory, on a single thread.

A. Inductor Coil

First, we consider a two-port inductor [41] embedded in a stack of two dielectric layers whose configuration and properties are given in Table I. The layers are bounded by free space above, and by an infinite ground plane below. The inductor geometry and computed surface current density distribution at 1 GHz are shown in Fig. 3. The maximum physical dimension of the inductor is 128.5 µm. The computed \( S \) parameters are shown in Fig. 4 and are validated against a commercial finite element solver (Ansys HFSS 18.2). Also shown are the \( S \) parameters obtained when using the Leontovich simple impedance boundary condition (SIBC) [8]. As expected, the SIBC performs well at high frequency, where skin effect is fully developed, but is unable to produce accurate results over a wide enough frequency band. The proposed method performs very well over an extremely wide frequency range. The profile data is shown in Table II, it is clear that the proposed method significantly outperforms HFSS in terms of time and memory. Over the entire frequency range, on average 42 iterations were required to reduce the relative residual norm below \( 10^{-6} \).

B. Rat-Race Divider

Next, we consider a four-port copper rat-race divider designed at 10 GHz. The purpose of this test case is to demonstrate the applicability of the proposed method to arbitrary geometries, thus the test is conducted in free space for simplicity. The geometry and computed surface current density distribution are shown in Fig. 5. The diameter of the structure is 24.7 mm, with trace thickness of 70.0 µm. The computed \( S \) parameters are shown in Fig. 6 for the proposed solver as well as HFSS and the SIBC. In this case, skin effect is fully developed around the design frequency, and so the proposed results match well against SIBC. Results are also in good agreement with HFSS. Profile data is shown in Table III. Once again, the proposed solver outperforms HFSS in terms of time and memory requirements.

C. On-chip Interconnect Network

To demonstrate applicability to on-chip applications, we consider a four-port network of 55 copper interconnects with a cross section of \( 1 \times 1 \) µm and conductor lengths of 150 µm on average. The geometry and current distribution are shown in Fig. 7. The interconnects are embedded in five lossy dielectric layers whose configuration and properties are given in Table IV. The layers are bounded by free space above and below. The \( S \) parameters are validated against HFSS, as shown in Fig. 8 and are in excellent agreement from 1 MHz to
TABLE II

Performance statistics for each test case averaged over all frequency points.

|                          | Inductor coil (Sec. VII-A) | Rat race divider (Sec. VII-B) | Interconnect network (Sec. VII-C) |
|--------------------------|----------------------------|------------------------------|----------------------------------|
| **Proposed**             | **HFSS**                   | **Proposed**                 | **HFSS**                         |
| Mesh elements            | 1,058                      | 2,236                        | 32,876                           |
| Memory used (GB)         | 0.28                       | 0.70                         | 2.6                              |
| CPU time (s)             | 45                         | 158                          | 1,996                            |

VIII. CONCLUSIONS

A complete surface integral solver is proposed to efficiently and accurately model lossy conductors in stratified dielectric media. A differential surface admittance operator is used to capture skin effect, and is generalized for the first time to multi-conductor systems embedded in layered media in any configuration. In order to efficiently model large structures, we have proposed an approach for applying the adaptive integral method (AIM) to layered media that does not require any special treatment of the mesh or the AIM grid. A novel series expansion-based technique is used to approximate the multilayer Green’s function in the near-field, while the discrete complex image method (DCIM) is harnessed for far-field interactions. Further, the augmented EFIE formulation is used to separate current and charge densities and obtain robust performance over a wide frequency range of 1 kHz–40 GHz. The accuracy and speed of the solver are demonstrated on three realistic structures including a large interconnect network. Scattering (S) parameters are accurately computed over the entire frequency range of interest, and are validated against a commercial finite element tool.

REFERENCES

[1] C. R. Paul, Analysis of Multiconductor Transmission Lines. Wiley, 2nd ed., 2007.
[2] J.-M. Jin, The Finite Element Method in Electromagnetics. IEEE Press, 2014.
[3] A. C. Cangellaris, J. L. Prince, and L. P. Vakanas, “Frequency-Dependent Inductance and Resistance Calculation for Three-Dimensional Structures in High-speed Interconnect Systems,” IEEE T. Compon. Hybr., vol. 13, pp. 154–159, March 1990.
[4] S. Omar and D. Jiao, “A New Volume Integral Formulation for Broadband 3-D Circuit Extraction in Inhomogeneous Materials with and without External Electromagnetic Fields,” IEEE Trans. Microw. Theory Tech., vol. 61, pp. 4302–4312, Dec 2013.
Fig. 8. Selected transmission ($S_{11}$) and reflection ($S_{12}$) parameters for the four-port on-chip interconnect in Sec. VII-C compared with HFSS.

[5] T. Moselhy and X. Hu and L. Daniel, “pFFT in FastMaxwell: A Fast Impedance Extraction Solver for 3D Conductor Structures Over Substrate,” in Conference on Design, Automation and Test, 2007.

[6] A. E. Ruehli, G. Antonini, L. Jiang, “Skin-Effect Loss Models for Time- and Frequency-Domain PEEC Solver,” Proc. IEEE, vol. 101, pp. 451–472, Feb 2013.

[7] A. E. Ruehli, G. Antonini, and L. Jiang, Circuit Oriented Electromagnetic Modeling Using the PEEC Techniques. IEEE Press, 2017.

[8] S. V. Yuferev and N. Ida, Circuit Oriented Electromagnetic-Circuit Simulation of Lossy Conductors,” Proc. IEEE, vol. 153, pp. 191–198, Apr 2006.

[9] Z. Zhu and B. Song and J. White, “Algorithms in FastImp: A Fast and Wide-Band Impedance Extraction Program for Complicated 3-D Geometries,” IEEE Trans. Comput.-Aided Design Integr. Circuits Syst., vol. 24, pp. 981–988, July 2005.

[10] M. S. Tong, G. Z. Yin, R. P. Chen, and Y. J. Zhang, “Electromagnetic Modeling of Packaging Structures With Lossy Interconnects Based on Two-Region Surface Integral Equations,” IEEE Trans. Compon., Packag., Manuf. Technol. A, vol. 4, pp. 1947–1955, Dec. 2014.

[11] W. Chai and D. Jiao, “Direct Matrix Solution of Linear Complexity for Surface Integral-Equation-Based Impedance Extraction of Complicated 3-D Structures,” Proc. IEEE, vol. 101, pp. 372–388, Feb. 2013.

[12] T. Moselhy and X. Hu and L. Daniel, “A Novel Single-Source Surface Integral Method to Compute Scattering from Dielectric Objects,” IEEE Antennas Wireless Propag. Lett., vol. 16, pp. 1715–1718, 2017.

[13] Z. G. Qian and W. C. Chew and R. Suaya, “Generalized Impedance Boundary Condition for Conductor Modeling in Surface Integral Equation,” IEEE Trans. Microw. Theory Tech., vol. 55, pp. 2354–2364, Nov. 2007.

[14] U. R. Patel, S. Sharma, S. Yang, S. V. Hum, P. Triverio, “Full-Wave Electromagnetic Characterization of 3D Interconnects Using a Surface Integral Formulation,” in 26th IEEE Conference on Electrical Performance of Electronic Packaging and Systems (EPEPS), (San Jose, CA), Oct. 2017.

[15] W. C. Gibson, The Method of Moments in Electromagnetics. CRC press, 2014.

[16] K. A. Michalski and J. R. Mosig, “Multilayered Media Green’s Functions in Integral Equation Formulations,” IEEE Trans. Antennas Propag., vol. 45, pp. 508–519, Mar 1997.

[17] M. I. Aksun, “A Robust Approach for the Derivation of Closed-Form Green’s Functions,” IEEE Trans. Microw. Theory Tech., vol. 44, pp. 651–658, May 1996.

[18] L. J. Jiang and W. C. Chew, “A Mixed-Form Fast Multipole Algorithm,” IEEE Trans. Antennas Propag., vol. 53, pp. 4145–4156, Dec 2005.

[19] E. Bleszynski, M. Bleszynski, and T. Jaroszewicz, “AIM: Adaptive Integral Method for Solving Large-Scale Electromagnetic Scattering and Radiation Problems,” Radio Science, vol. 31, no. 5, pp. 1225–1251, 1996.

[20] F. Ling, C.-F. Wang, and J.-M. Jin, “An Efficient Algorithm for Analyzing Large-Scale Microstrip Structures Using Adaptive Integral Method Combined with Discrete Complex-Image Method,” IEEE Trans. Microw. Theory Tech., vol. 48, pp. 832–839, May 2000.

[21] K. Yang and A. E. Yilmaz, “A Three-Dimensional Adaptive Integral Method for Scattering From Structures Embedded in Layered Media,” IEEE Trans. Geosci. Remote Sens., vol. 50, pp. 1130–1139, April 2012.

[22] V. Okhmatskovski, M. Yuan, I. Jeffrey, and R. Phelps, “A Three-Dimensional Pre-corrected FFT Algorithm for Fast Method of Moments Solutions of the Mixed-Potential Integral Equation in Layered Media,” IEEE Trans. Microw. Theory Tech., vol. 57, pp. 3505–3517, Dec 2009.

[23] S. Sharma, U. R. Patel, P. Triverio, “Accelerated Electromagnetic Analysis of Interconnects in Layered Media using a Near-Field Series Expansion of the Greens Function,” in 27th IEEE Conference on Electrical Performance of Electronic Packaging and Systems (EPEPS), (San Jose, CA), Oct. 2018.

[24] Z. G. Qian and W. C. Chew, “An Augmented Electric Field Integral Equation for High-Speed Interconnect Analysis,” Microwave and Optical Technology Letters, vol. 50, no. 10, pp. 2658–2662, 2008.

[25] S. M. Rao, D. R. Wilton, and A. W. Glisson, “Electromagnetic Scattering by Surfaces of Arbitrary Shape,” IEEE Trans. Antennas Propag., vol. 30, pp. 409–418, 1982.

[26] C. Balanis, Advanced Engineering Electromagnetics. John Wiley & Sons, 1989.

[27] Z. G. Qian and W. C. Chew, “A Quantitative Study on the Low Frequency Breakdown of EFIE,” Microwave and Optical Technology Letters, vol. 50, no. 5, pp. 1159–1162, May 1997.

[28] Z.-G. Qian and W. C. Chew, “Fast Full-Wave Surface Integral Equation Solver for Multiscale Structure Modeling,” IEEE Trans. Antennas Propag., vol. 57, pp. 3594–3601, November 2009.

[29] Y. Wang, D. Gope, V. Jandhyala, and C.-J. R. Shi, “Generalized Kirchhoff’s Current and Voltage Law Formulation for Coupled Circuit-Electromagnetic Simulation With Surface Integral Equations,” IEEE Trans. Microw. Theory Tech., vol. 52, pp. 1673–1682, July 2004.

[30] K. A. Michalski and D. Zheng, “Electromagnetic Scattering and Radiation by Surfaces of Arbitrary Shape in Layered Media. I. Theory,” IEEE Trans. Antennas Propag., vol. 38, pp. 335–344, Mar 1990.

[31] N. Kinayman and M. I. Aksun, “Efficient Use of Closed-Form Green’s Functions for the Analysis of Planar Geometries with Vertical Connections,” IEEE Trans. Microw. Theory Tech., vol. 45, pp. 593–603, May 1997.

[32] Y. Hua and T. K. Sarkar, “Generalized Pencil-of-Function Method for Extracting Poles of an EM System from its Transient Response,” IEEE Trans. Antennas Propag., vol. 37, pp. 229–234, Feb 1989.

[33] M. Yuan, T. K. Sarkar, and M. Salazar-Palma, “A Direct Discrete Complex Image Method from the Closed-Form Green’s Functions in Multilayered Media,” IEEE Trans. Microw. Theory Tech., vol. 54, pp. 1025–1032, March 2006.

[34] E. Simek, Q. H. Liu, and B. Wei, “Singularity Subtraction for Evaluation of Green’s Functions for Multilayer Media,” IEEE Trans. Microw. Theory Tech., vol. 54, pp. 216–225, Jan 2006.

[35] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. New York: Dover, tenth ed., 1964.

[36] K. A. Michalski and J. R. Mosig, “Efficient Computation of Sommerfeld Integral Tails - Methods and Algorithms,” Journal of Electromagnetic Waves and Applications, vol. 30, no. 3, pp. 281–317, 2016.

[37] Z. Zhu, B. Song, and J. K. White, “Algorithms in FastImp: A Fast and Wide-Band Impedance Extraction Program for Complicated 3-D Geometries,” IEEE Trans. Comput.-Aided Design Integr. Circuits Syst., vol. 24, pp. 981–988, July 2005.

[38] Y. Saad and M. Schultz, “GMRES: A Generalized Minimal Residual Algorithm for Solving Nonsymmetric Linear Systems,” SIAM Journal on Scientific and Statistical Computing, vol. 7, no. 3, pp. 856–869, 1986.

[39] S. Balay, S. Abhyankar, M. F. Adams, J. Brown, P. Brune, K. Buschelman, L. Dalcin, V. Eijkhout, W. D. Gropp, D. Kaushik, M. G. Knepley, D. A. May, L. C. McInnes, R. T. Mills, T. Munson, K. Rupp, P. Sanan, B. F. Smith, S. Zampini, H. Zhang, and H. Zhang, “PETSc Website,” 2018.

[40] T. Moselhy, X. Hu, and L. Daniel, “pFFT in FastMaxwell: A Fast Impedance Extraction Solver for 3D Conductor Structures over Substrate,” in Proceedings of the Conference on Design, Automation and Test, 2007.