HIGHER ORDER INVARIANTS OF IMMERSIONS
OF SURFACES INTO 3-SPACE

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ABSTRACT. We classify all finite order invariants of immersions of a closed orientable surface into \( \mathbb{R}^3 \), with values in any Abelian group. We show that they are all functions of order one invariants.

1. INTRODUCTION

Finite order invariants of stable immersions of a closed orientable surface into \( \mathbb{R}^3 \) have been defined in [N], where all order 1 invariants have been classified. In the present work we classify all finite order invariants of order \( n > 1 \), and show that they are all functions of the universal order 1 invariant constructed in [N].

The structure of the paper is as follows: In Section 2 we summarize the necessary background. We define finite order invariants of immersions of a closed orientable surface into \( \mathbb{R}^3 \). For given surface \( F \), regular homotopy class \( \mathcal{A} \) of immersions of \( F \) into \( \mathbb{R}^3 \), and Abelian group \( G \), we define \( V_n \) to be the group of all invariants on \( \mathcal{A} \) of order at most \( n \) with values in \( G \). We present a group \( \Delta_n = \Delta_n(G) \) and an injection \( u : V_n/V_{n-1} \to \Delta_n \). The question of classifying all finite order invariants then becomes the question of finding the image of \( u \). In Section 3 we state our classification. We specify a subgroup \( E_n \subseteq \Delta_n \) which we claim to be the image of \( u \). In Section 4 we show that \( u(V_n) \supseteq E_n \) by explicitly constructing a universal finite order invariant. In Section 5 we show that \( u(V_n) \subseteq E_n \). The proof relies on the result of Section 4.

2. BACKGROUND

In this section we summarize the background needed for this work, all of which may be found in [N]. Given a closed orientable surface \( F \), \( Imm(F, \mathbb{R}^3) \) denotes the space of all immersions of \( F \) into \( \mathbb{R}^3 \), with the \( C^1 \) topology. A CE point of an immersion \( i : F \to \mathbb{R}^3 \) is a point of self intersection of \( i \) for which the local stratum in \( Imm(F, \mathbb{R}^3) \) corresponding to the
self intersection, has codimension one. We distinguish twelve types of CEs which we name $E^0, E^1, E^2, H^1, H^2, T^0, T^1, T^2, T^3, Q^2, Q^3, Q^4$. This set of twelve symbols is denoted $\mathcal{C}$. A co-orientation for a CE is a choice of one of the two sides of the local stratum corresponding to the CE. All but two of the above CE types are nonsymmetric in the sense that the two sides of the local stratum may be distinguished via the local configuration of the CE, and for those ten CE types, permanent co-orientations for the corresponding strata are chosen once and for all. The two exceptions are $H^1$ and $Q^2$ which are completely symmetric.

We fix a closed orientable surface $F$ and a regular homotopy class $A$ of immersions of $F$ into $\mathbb{R}^3$, and denote by $I_n \subseteq A$ ($n \geq 0$) the space of all immersions in $A$ which have precisely $n$ CE points (the self intersection being elsewhere stable). In particular, $I_0$ is the space of all stable immersions in $A$.

For an immersion $i : F \to \mathbb{R}^3$ having a CE located at $p \in \mathbb{R}^3$, the degree $d_p(i) \in \mathbb{Z}$ of $i$ at $p$ is defined in [N]. Then $C_p(i)$ is the expression $R_m^a$ where $R^a \in \mathcal{C}$ is the symbol describing the configuration of the CE of $i$ at $p$ (one of the twelve symbols) and $m = d_p(i)$. $\mathcal{C}_n$ denotes the set of all un-ordered $n$-tuples of expressions $R_m^a$ with $R^a \in \mathcal{C}, m \in \mathbb{Z}$. (So $\mathcal{C}_n$ is the set of un-ordered $n$-tuples of elements of $C_1$.) A map $C : I_n \to \mathcal{C}_n$ is defined by $C(i) = [C_{p_1}(i), \ldots, C_{p_n}(i)] \in \mathcal{C}_n$ where $p_1, \ldots, p_n$ are the $n$ CE points of $i$. The map $C : I_n \to \mathcal{C}_n$ is surjective.

Let $\mathbb{G}$ be any Abelian group and let $f : I_0 \to \mathbb{G}$ be an invariant, i.e. a function which is constant on each connected component of $I_0$. Given an immersion $i \in I_n$, a temporary co-orientation for $i$ is a choice of co-orientation at each of the $n$ CE points $p_1, \ldots, p_n$ of $i$. Given a temporary co-orientation $\mathcal{F}$ for $i$, and a subset $A \subseteq \{p_1, \ldots, p_n\}$, $i_{\mathcal{F}, A} \in I_0$ is the immersion obtained from $i$ by resolving all CEs of $i$ as follows: The CEs at points of $A$ are resolved into the positive side with respect to $\mathcal{F}$, and those not in $A$ into the negative side. Given $i \in I_n$ and a temporary co-orientation $\mathcal{F}$ for $i$, $f^{\mathcal{F}}(i)$ is defined as follows:

$$f^{\mathcal{F}}(i) = \sum_{A \subseteq \{p_1, \ldots, p_n\}} (-1)^{n-|A|} f(i_{\mathcal{F}, A})$$

where $|A|$ is the number of elements in $A$. The statement $f^{\mathcal{F}}(i) = 0$ is independent of the temporary co-orientation $\mathcal{F}$ so we simply write $f(i) = 0$. An invariant $f : I_0 \to \mathbb{G}$ is called of finite order if there is an $n$ such that $f(i) = 0$ for all $i \in I_{n+1}$. The minimal such $n$ is called the order of $f$. The group of all invariants on $I_0$ of order at most $n$ is denoted $V_n$. 
Let \( f \in V_n \). If \( i \in I_n \) has at least one CE of type \( H^1 \) or \( Q^2 \) and \( \mathcal{F} \) is a temporary co-orientation for \( i \), then \( 2f^\mathcal{F}(i) = 0 \), and so \( f^\mathcal{F}(i) \) is independent of \( \mathcal{F} \). This fact is used to extend any \( f \in V_n \) to \( I_n \) as \( f^\mathcal{F}(i) \), where if \( i \) includes at least one CE of type \( H^1 \) or \( Q^2 \) then \( \mathcal{F} \) is arbitrary, and if all CEs of \( i \) are not of type \( H^1 \) or \( Q^2 \) then the permanent co-orientation is used for all CEs of \( i \). (If \( f \in V_n \) then we are not extending \( f \) to \( I_k \) for \( 0 < k < n \).) For \( f \in V_n \) and \( i, j \in I_n \), if \( C(i) = C(j) \) then \( f(i) = f(j) \), so any \( f \in V_n \) induces a well defined function \( u(f) : C_n \rightarrow \mathbb{G} \). The map \( f \mapsto u(f) \) induces an injection \( u : V_n/V_{n-1} \rightarrow C_n^* \) where \( C_n^* \) is the group of all functions from \( C_n \) to \( \mathbb{G} \). Finding the image of \( u \) for all \( n \) gives a classification of all finite order invariants, which is what we do in this work (Theorem 3.2). For order 1 invariants this has been done in [N].

A subgroup \( \Delta_n = \Delta_n(\mathbb{G}) \subseteq C_n^* \) which contains the image of \( u \) is defined as the set of functions in \( C_n^* \) satisfying relations which we write as relations on the symbols \( R_m^a \), e.g. \( T_m^0 = T_m^3 \) will stand for the set of all relations of the form \( g([T_m^0, R_{m}^{a_2}, \ldots, R_{n}^{a_n}]) = g([T_m^3, R_{m}^{a_2}, \ldots, R_{n}^{a_n}]) \) with arbitrary \( R_{m}^{a_2}, \ldots, R_{n}^{a_n} \in C_1 \). The relations defining \( \Delta_n \) are:

- \( E_m^2 = -E_m^0 = H_m^2 \), \( E_m^1 = H_m^1 \).
- \( T_m^0 = T_m^3 \), \( T_m^1 = T_m^2 \).
- \( 2H_m^1 = 0 \), \( H_m^1 = H_{m-1}^1 \).
- \( 2Q_m^2 = 0 \), \( Q_m^2 = Q_{m-1}^2 \).
- \( H_m^2 - H_{m-1}^2 = T_m^3 - T_m^2 \).
- \( Q_m^4 = Q_m^3 = T_m^3 - T_{m-1}^3 \), \( Q_m^3 - Q_{m}^2 = T_m^2 - T_{m-1}^2 \).

Let \( \mathbb{B} \subseteq \mathbb{G} \) be the subgroup defined by \( \mathbb{B} = \{ x \in \mathbb{G} : 2x = 0 \} \). To obtain a function \( g \in \Delta_1 \) one may assign arbitrary values in \( \mathbb{G} \) for the symbols \( \{ T_m^a \}_{a=2,3,m \in \mathbb{Z}} \), \( H_0^2 \) and arbitrary values in \( \mathbb{B} \) for the two symbols \( H_1^1, Q_0^2 \). Once this is done then the value of \( g \) on all other symbols is uniquely determined, namely:

1. \( H_m^1 = H_0^1 \) for all \( m \).
2. \( H_m^2 = H_0^2 + \sum_{k=1}^{m} (T_k^3 - T_k^2) \) for \( m \geq 0 \).
3. \( H_m^2 = H_0^2 - \sum_{k=m+1}^{0} (T_k^3 - T_k^2) \) for \( m < 0 \).
4. \( E_m^0 = -H_m^2 \), \( E_m^1 = H_m^1 \), \( E_m^2 = H_m^2 \) for all \( m \).
5. \( T_m^0 = T_m^3 \), \( T_m^1 = T_m^2 \) for all \( m \).
6. \( Q_m^2 = Q_0^2 \) for all \( m \).
7. \( Q_m^3 = Q_m^2 + T_m^2 - T_{m-1}^2 \) for all \( m \).
8. \( Q_m^4 = Q_m^3 + T_m^3 - T_{m-1}^3 \) for all \( m \).
A “universal” Abelian group $\mathbb{G}_U$ is defined by

$$\mathbb{G}_U = \langle \{ t_m^a \}_{a=2,3,m \in \mathbb{Z}}, h_0^2, h_1^1, q_0^2 \mid 2h_0^1 = 2q_0^2 = 0 \rangle.$$ 

Then the universal element $g_U^1 \in \Delta_1(\mathbb{G}_U)$ is defined by $g_U^1(T_m^a) = t_m^a$ $(a = 2, 3)$, $g_U^1(H_0^1) = h_0^1$, $g_U^1(Q_0^2) = q_0^2$ and the value of $g_U^1$ on all other symbols of $\mathcal{C}_1$ is determined by formulae 1-8 above. The main result of [N] is that for any closed orientable surface $F$, regular homotopy class $\mathcal{A}$ of immersions of $F$ into $\mathbb{R}^3$ and Abelian group $\mathbb{G}$, the injection $u : V_1/V_0 \to \Delta_1$ is surjective. This is shown by constructing an order one invariant $f_U^1 : I_0 \to \mathbb{G}_U$ with $u(f_U^1) = g_U^1$. Then for arbitrary $\mathbb{G}$, if $g \in \Delta_1(\mathbb{G})$ then $g = \phi \circ g_U^1$ for $\phi \in \text{Hom}(\mathbb{G}_U, \mathbb{G})$, and $f = \phi \circ f_U^1$ satisfies $u(f) = g$.

3. Statement of Classification

Let $X = \{ T_m^a \}_{a=2,3,m \in \mathbb{Z}} \cup \{ H_0^1 \}$, and $Y = X \cup \{ H_0^1, Q_0^2 \}$ so $X \subseteq Y \subseteq C_1$. We observe that as for $\Delta_1$, it is true for any $\Delta_n$ that a function $g \in \Delta_n$ may be assigned arbitrary values in $\mathbb{G}$ for any un-ordered $n$-tuple of elements of $X$ and arbitrary values in $\mathbb{B}$ for all $n$-tuples of elements of $Y$ which include $H_0^1$ or $Q_0^2$ at least once. Once this is done, the value of $g$ on all other $n$-tuples in $\mathcal{C}_n$ is uniquely determined by the procedure 1-8 of Section 2, applied $n$ times, independently to each of the $n$ entries of $g$. This is an evident fact that is somewhat obscured by the fact that we are dealing with un-ordered $n$-tuples. So, in this paragraph only, we revert to ordered notation. Let us index the elements of $Y$ with index $i$ and the elements of $\mathcal{C}_1 \supseteq Y$ with index $j$. Let us denote $g([i_1, \ldots, i_n])$ by $G_{i_1, \ldots, i_n}$ where now the $n$ indices are ordered, and so the expression $G_{i_1, \ldots, i_n}$ is symmetric i.e. unchanged when permuting the indices. The function of $G_{i_1, \ldots, i_n}$ produced by the procedure given by 1-8 of Section 2 applied to one of the indices of $G_{i_1, \ldots, i_n}$ is linear with integer coefficients say $y_j^i$ where for each $j$, $y_j^i \neq 0$ for only finitely many values of $i$. Application of the procedure on all $n$ indices of $G_{i_1, \ldots, i_n}$ is given by the sum:

$$F_{j_1, \ldots, j_n} = \sum_{i_1, \ldots, i_n} y_{j_1}^{i_1} y_{j_2}^{i_2} \cdots y_{j_n}^{i_n} G_{i_1, \ldots, i_n}.$$

With this notation it is clear that $F_{j_1, \ldots, j_n}$ thus obtained is a well defined symmetric function of $j_1, \ldots, j_n$ which is the unique extension of $G_{i_1, \ldots, i_n}$ to the indices $j_1, \ldots, j_n$, satisfying the relations defining $\Delta_n$.

We will now define $E_n \subseteq \Delta_n$ by two additional restrictions on the functions $g \in \Delta_n$. Thanks to the discussion of the previous paragraph, we may state the additional restrictions
in terms of the values of \( g \) on \( n \)-tuples of elements of \( Y \) only. Given an un-ordered \( n \)-tuple \( z \) of elements of \( Y \), we define \( m_{H^1_0}(z) \) and \( m_{Q^2_0}(z) \) as the number of times that \( H^1_0 \) and \( Q^2_0 \) appear in \( z \) respectively. We define \( r(z) \), (the repetition of \( H^1_0 \) and \( Q^2_0 \) in \( z \)), as

\[
r(z) = \max(0, m_{H^1_0}(z) - 1) + \max(0, m_{Q^2_0}(z) - 1).
\]

**Definition 3.1.** Given an Abelian group \( \mathbb{G}, E_n = E_n(\mathbb{G}) \subseteq \Delta_n(\mathbb{G}) \) is the subgroup consisting of all \( g \in \Delta_n(\mathbb{G}) \) satisfying the following two additional restrictions:

1. When \( n \geq 3 \), \( g \) must satisfy the relation \( H^1_0H^1_0Q^2_0 = H^1_0Q^2_0Q^2_0 \).
   By this we mean that \( g([H^1_0, H^1_0, Q^2_0, R^a_{d1}, \ldots, R^a_{dn}]) = g([H^1_0, Q^2_0, Q^2_0, R^a_{d1}, \ldots, R^a_{dn}]) \)
   for arbitrary \( R^a_{d1}, \ldots, R^a_{dn} \in Y \).
2. For any un-ordered \( n \)-tuple \( z \) of elements of \( Y \), \( g(z) \in 2^{r(z)}\mathbb{G} \), i.e. there exists an element \( a \in \mathbb{G} \) such that \( g(z) = 2^{r(z)}a \). (Note that whenever \( r(z) > 0 \) then in particular \( H^1_0 \) or \( Q^2_0 \) does appear in \( z \) so in fact we have \( g(z) \in \mathbb{B} \cap 2^{r(z)}\mathbb{G} \).)

In this work we prove:

**Theorem 3.2.** For any closed orientable surface \( F \), regular homotopy class \( \mathcal{A} \) of immersions of \( F \) into \( \mathbb{R}^3 \) and Abelian group \( \mathbb{G} \), the image of the injection \( u : V_n/V_{n-1} \rightarrow \Delta_n \) is \( E_n \).

Furthermore, for any \( g \in E_n \) there exists a function (not homomorphism) \( s : \mathbb{G}_U \rightarrow \mathbb{G} \) such that the invariant \( f = s \circ f^U_1 \) is of order \( n \) and satisfies \( u(f) = g \). (\( f^U_1 \) of Section 3.) It follows that all finite order invariants are functions of the order 1 invariant \( f^U_1 \).

4. Proof that \( u(V_n) \supseteq E_n \)

For convenience, we rename the generating elements of the group \( \mathbb{G}_U \).

We relabel \( \{t^m_n\}_{a=2,3,m\in \mathbb{Z}} \cup \{h^2_0\} \) as \( \{a_i\}_{i \in X} \) where \( X \) is a countable set of indices, and relabel \( h^1_0, q^2_0 \) as \( b, c \). We define algebraic structures \( K \subseteq L \subseteq M \), where \( L \) is a commutative ring, \( K \) is a subring of \( L \), and \( M \) is a module over \( K \). \( L \) is defined as the ring of formal power series with integer coefficients and variables \( \{a_i\}_{i \in X} \cup \{b, c\} \) and with relations

- \( b^2c = bc^2 \).
- \( 2b = 2c = 0 \).

We emphasize that though there is an infinite set of variables, any given power series may include only finitely many monomials of any given degree \( n \). Given a monomial \( p \), we define \( m_b(p) \) and \( m_c(p) \) as the multiplicity of \( b \) and \( c \) in \( p \) respectively. We define \( r(p) \), (the repetition
of $b$ and $c$ in $p$), as
\[ r(p) = \max(0, m_b(p) - 1) + \max(0, m_c(p) - 1). \]

$r(p)$ is preserved under the relations in $L$ and so is well defined on equivalence classes of monomials. The equivalence class of a monomial $p$ will be denoted $\overline{p}$. We note that an equivalence class $\overline{p}$ includes more than just the one monomial $p$ iff $m_b(p) \geq 1$, $m_c(p) \geq 1$ and $r(p) \geq 1$; it then includes precisely $r(p) + 1$ different monomials. When $\overline{p}$ includes only the one monomial $p$ then we will interchangeably write $p$ and $\overline{p}$. Now $K \subseteq L$ is defined to be the subring of power series including only the variables $\{a_i\}_{i \in X}$. On the other hand we extend $L$ to a larger structure $M$ which will be a module over the subring $K$, as follows: For each $\overline{p}$ for which $p$ is a monomial with coefficient 1, we adjoin a new element $\zeta_p$ satisfying the relation $2^{r(\overline{p})} \zeta_p = \overline{p}$. The new elements $\zeta_p$ will be considered monomials of the same degree as $\overline{p}$, and will appear as terms in our formal power series. (Indeed, one can think of $\zeta_p$ as $2^{-r(\overline{p})} p$.)

Note that if $r(p) = 0$ then $\zeta_p = p$, in particular, $\zeta_1 = 1$ and $\zeta_e = e$ for each generating variable $e$. Now $K$ acts on $M$ as follows: If $k \in K, p \in L$ are monomials then $k \cdot \zeta_p = \zeta_{kp}$. This is extended in the natural way to an action of power series in $K$ on power series in $M$.

We note that the whole of $L$ cannot act on $M$ in this way, since we would get contradictions such as $0 \neq b^2 = 2 \zeta_b^2 = 2b \cdot \zeta_b = 0$. In particular, we do not have a ring structure on $M$. For each $n \geq 0$ we denote by $K_n \subseteq L_n \subseteq M_n$ the additive subgroups of $K \subseteq L \subseteq M$ respectively generated by the monomials of degree $n$. (As mentioned, the degree of $\zeta_{\overline{p}} \in M$ is defined as the degree of $\overline{p}$). We note $L_1 = M_1$ is the Abelian group with generators $\{a_i\}_{i \in X} \cup \{b, c\}$ and relations $2b = 2c = 0$, i.e. the group $G_U$. We have $L_1 = K_1 \oplus S$ where $S \subseteq L_1$ is the four element subgroup generated by $b, c$. We now define a function $F : L_1 \to M$ as follows: We first define $F : K_1 \to K$ as the group homomorphism from the additive group $K_1$ to the multiplicative group of invertible elements in $K$, which is given on generators by
\[ F(a_i) = \sum_{n=0}^{\infty} a_i^n. \]

These are indeed invertible elements, giving
\[ F(-a_i) = \left( \sum_{n=0}^{\infty} a_i^n \right)^{-1} = 1 - a_i. \]

Note that for $x = \pm a_i$, $F(x) = 1 + x + T_2$ where $T_2$ stands for the “higher order terms” (or “tail”) of the given series, i.e. a power series including only monomials of degree at least 2. It follows that for any $x \in K_1$, $F(x) = 1 + x + T_2$. 



We then define $\mathcal{F} : S \to M$ explicitly on the four elements of $S$ as follows:

1. $\mathcal{F}(0) = 1$.
2. $\mathcal{F}(b) = \sum_{n=0}^{\infty} \zeta b^n$. 
3. $\mathcal{F}(c) = \sum_{n=0}^{\infty} \zeta c^n$. 
4. $\mathcal{F}(b + c) = 1 + b + c + \sum_{n=2}^{\infty} (\zeta b^n + \zeta c^n + \zeta b^n c^n)$.

Finally, $\mathcal{F} : L_1 \to M$ is defined as follows: Any element in $L_1$ is uniquely written as $k + s$ with $k \in K_1$, $s \in S$, and we define $\mathcal{F}(k + s) = \mathcal{F}(k) \mathcal{F}(s)$ where the product on the right is the action of $K$ on $M$. It follows that for any $k \in K_1, l \in L_1$: $\mathcal{F}(k + l) = \mathcal{F}(k) \mathcal{F}(l)$.

For any $(n + 1)$-tuple $(l; l_1, l_2, \ldots, l_n)$ of elements of $L_1$, we define

$$\mathcal{F}'(l; l_1, \ldots, l_n) = \sum_{A \subseteq \{1, \ldots, n\}} (-1)^{n-|A|} \mathcal{F}(l + \sum_{i \in A} l_i).$$

**Proposition 4.1.** For any $n + 1$-tuple $(l; l_1, \ldots, l_n)$ of elements of $L_1$:

$$\mathcal{F}'(l; l_1, \ldots, l_n) = l_1 l_2 \cdots l_n + T_{n+1}$$

where $T_{n+1}$ stands for higher order terms i.e. some power series in $M$ having only monomials of degree at least $n + 1$.

**Proof.** We first show that it is enough to prove the proposition for the case when all $l_i$ are generators or minus generators of $L_1$, i.e. of the form $\pm e$ where $e \in \{a_i\}_{i \in X} \cup \{b, c\}$. We prove this by induction on the sum of the lengths of $l_1, \ldots, l_n$ in terms of the generators. Say $l_1$ is not a generator and $l_1 = l_1' + l_1''$ where $l_1'$ and $l_1''$ have shorter length than $l_1$ in terms of the generators. Then

$$\mathcal{F}'(l; l_1, \ldots, l_n) = \sum_{A \subseteq \{1, \ldots, n\}} (-1)^{n-|A|} \mathcal{F}(l + \sum_{i \in A} l_i)$$

$$= \sum_{A \subseteq \{2, \ldots, n\}} (-1)^{n-|A|-1} \left( \mathcal{F}(l + l_1 + \sum_{i \in A} l_i) - \mathcal{F}(l + \sum_{i \in A} l_i) \right)$$

$$= \sum_{A \subseteq \{2, \ldots, n\}} (-1)^{n-|A|-1} \left( \mathcal{F}(l + l_1' + l_1'' + \sum_{i \in A} l_i) - \mathcal{F}(l + l_1' + \sum_{i \in A} l_i) \right) +$$

$$+ \sum_{A \subseteq \{2, \ldots, n\}} (-1)^{n-|A|-1} \left( \mathcal{F}(l + l_1' + \sum_{i \in A} l_i) - \mathcal{F}(l + \sum_{i \in A} l_i) \right)$$

$$= \mathcal{F}'(l; l_1', l_2, \ldots, l_n) + \mathcal{F}'(l; l_1', l_2', \ldots, l_n)$$

$$= l_1' l_2 \cdots l_n + T_{n+1} + l_1' l_2 \cdots l_n + T_{n+1} = l_1 l_2 \cdots l_n + T_{n+1}.$$
We are left with proving the proposition in case all \( l_i \) are generators or minus generators. We prove this by induction on \( n \). If one of the \( l_i \), say \( l_1 \), is in \( K_1 \), then

\[
\mathcal{F}'(l; l_1, \ldots, l_n) = \sum_{A \subseteq \{1, \ldots, n\}} (-1)^{n-|A|} \mathcal{F}(l + \sum_{i \in A} l_i)
\]

\[
= \sum_{A \subseteq \{2, \ldots, n\}} (-1)^{n-|A|-1} \left( \mathcal{F}(l + l_1 + \sum_{i \in A} l_i) - \mathcal{F}(l + \sum_{i \in A} l_i) \right)
\]

\[
= (\mathcal{F}(l_1) - 1) \sum_{A \subseteq \{2, \ldots, n\}} (-1)^{n-|A|} \mathcal{F}(l + \sum_{i \in A} l_i)
\]

\[
= (\mathcal{F}(l_1) - 1) \mathcal{F}'(l; l_2, \ldots, l_n)
\]

\[
= (l_1 + T_2)(l_2 \cdots l_n + T_n) = l_1 l_2 \cdots l_n + T_{n+1}.
\]

Note that the term multiplying on the left is indeed an element of \( K \).

So assume now that all \( l_1, \ldots, l_n \) are \( b \) and \( c \). Assume \( k \) of them are \( b \) and \( n - k \) of them are \( c \). We first deal with the case \( k = n \) i.e. \( l_1, \ldots, l_n \) are all \( b \) (the case \( k = 0 \) is identical).

Since \( 2b = 0 \) we get:

\[
\mathcal{F}'(l; b, b, \ldots, b) = \sum_{A \subseteq \{1, \ldots, n\}} (-1)^{n-|A|} \mathcal{F}(l + |A|b)
\]

\[
= \pm \left( \sum_{|A| \text{ odd}} \mathcal{F}(l + b) - \sum_{|A| \text{ even}} \mathcal{F}(l) \right) = \pm 2^{n-1} \left( \mathcal{F}(l + b) - \mathcal{F}(l) \right).
\]

since indeed there is an equal number of odd and even sized subsets of \( \{1, \ldots, n\} \) (e.g. since \( \sum_{i=0}^n (-1)^i \binom{n}{i} = 0 \)).

Now letting \( l = k + s \) with \( k \in K_1, s \in S \), we get:

\[
\pm 2^{n-1}(\mathcal{F}(k + s + b) - \mathcal{F}(k + s)) = \pm \mathcal{F}(k) 2^{n-1}(\mathcal{F}(s + b) - \mathcal{F}(s)) = \pm (1 + T_1) 2^{n-1}(\mathcal{F}(s + b) - \mathcal{F}(s)).
\]

Since multiplication by \( 1 + T_1 \in K \) leaves the lowest order term unchanged we may assume we have only \( \pm 2^{n-1}(\mathcal{F}(s + b) - \mathcal{F}(s)) \). If \( s = 0 \) or \( b \) then

\[
\pm 2^{n-1}(\mathcal{F}(s + b) - \mathcal{F}(s)) = \pm 2^{n-1} \sum_{m=1}^\infty \zeta_{b^m} = b^n + T_{n+1}
\]

since \( r(b^m) = m - 1 \) and so \( 2^{n-1} \zeta_{b^m} = 0 \) for \( m < n \) and \( 2^{n-1} \zeta_{b^m} = b^n \). (The \( \pm \) was dropped since \( 2b^n = 0 \).) If \( s = c \) or \( c + b \) then we get

\[
\pm 2^{n-1}(\mathcal{F}(s + b) - \mathcal{F}(s)) = \pm 2^{n-1} \left( b + \sum_{m=2}^\infty (\zeta_{b^m} + \zeta_{b^{m-1}-1}) \right) = b^n + T_{n+1}
\]
since again \( r(b^m) = m - 1 \), but also \( r(bc^{m-1}) = m - 2 \) and so \( 2^{n-1\zeta_{bc^{m-1}}} = 0 \) for \( m \leq n \).

We are left with the case of \( b \) appearing \( k \) time and \( c \) appearing \( n-k \) times, with \( 0 < k < n \). Since \( 2b = 2c = 0 \), we get:

\[
\mathcal{F}'(l; b, \ldots, b, c, \ldots, c) = \sum_{B \subseteq \{1, \ldots, k\}, C \subseteq \{k+1, \ldots, n\}} (-1)^{|B|-|C|} \mathcal{F}(l + |B|b + |C|c)
\]

\[
= \pm \left( \sum_{|B|\text{odd}, |C|\text{odd}} \mathcal{F}(l + b + c) - \sum_{|B|\text{odd}, |C|\text{even}} \mathcal{F}(l + b) \right. \\
- \sum_{|B|\text{even}, |C|\text{odd}} \mathcal{F}(l + c) + \sum_{|B|\text{even}, |C|\text{even}} \mathcal{F}(l) \left.
\right)
\]

\[
= \pm 2^{n-2} \left( \mathcal{F}(l + b + c) - \mathcal{F}(l + b) - \mathcal{F}(l + c) + \mathcal{F}(l) \right).
\]

As before we may assume \( l = s \in S \). For each of the four elements \( s \in S \) we get:

\[
\pm 2^{n-2} \left( \mathcal{F}(s + b + c) - \mathcal{F}(s + b) - \mathcal{F}(s + c) + \mathcal{F}(s) \right) = \pm 2^{n-2} \sum_{m=2}^{\infty} \zeta_{bc^{m-1}}
\]

\[
= b^{m-1} + T_{n+1} = b^{k}c^{n-k} + T_{n+1}
\]

since \( r(bc^{m-1}) = m - 2 \) and so \( 2^{n-2\zeta_{bc^{m-1}}} = 0 \) for \( m < n \), and \( 2^{n-2\zeta_{bc^{m-1}}} = b^{m-1} \).

\[\square\]

We now return to our original symbols \( \{t_m^a\}_{a=2,3, m \in \mathbb{Z}}, h_0^2, h_1^1, q_0^2 \) in place of \( \{a_i\}_{i \in X}, b, c \). Extending the definition of \( g_1^U \) of Section 3 \( (g_1^U : C_1 \to \mathbb{G}_U = L_1) \), let \( g_n^U \in E_n(M_n) \) be first defined on un-ordered \( n \)-tuples of elements of \( Y \) by \( g_n^U([R_1^{a_1}, \ldots, R_{n_{d_n}}^{a_{n_n}}]) = r_1^{a_1} \cdots r_n^{a_n} \in M_n \) where \( r_i \) is the lower case letter corresponding to the capital letter \( R_i \). (Note that the upper indices are not powers but part of the given symbols.) The value of \( g_n^U \) on all other \( n \)-tuples in \( C_n \) is then determined by the procedure discussed in the opening paragraph of Section 3. It follows that for any \( n \)-tuple \( [R_1^{a_1}, \ldots, R_{n_{d_n}}^{a_{n_n}}] \in C_n \), \( g_n^U([R_1^{a_1}, \ldots, R_{n_{d_n}}^{a_{n_n}}]) = g_1^U(r_1^{a_1})g_1^U(r_2^{a_2}) \cdots g_1^U(r_n^{a_n}) \) where the product is that in \( L \). By construction of \( M_n \) indeed \( g_n^U \in E_n(M_n) \).

Let \( \mathcal{F}_n : L_1 \to M_n \) be the projection onto \( M_n \) of \( \mathcal{F} : L_1 \to M \), and let \( f_n^U : I_0 \to M_n \) be the invariant given by \( f_n^U = \mathcal{F}_n \circ f_1^U \). (\( f_1^U : I_0 \to \mathbb{G}_U = L_1 \) appearing in the concluding paragraph of Section 2.) Let \( i \in I_m \), where \( m \) is either \( n \) or \( n + 1 \), with CEs at \( \{p_1, \ldots, p_m\} \) and let \( \mathfrak{S} \) be a temporary co-orientation for \( i \) which is the permanent co-orientation wherever
there is one. Then by Proposition 4.1 and since $u_{TAHL NOWIK}$

$$f_n'(i) = (f_n')^\mathcal{I}(i) = \sum_{A \subseteq \{p_1, \ldots, p_m\}} (-1)^{m-|A|} \mathcal{F}_n \circ f_1'(i_{\mathcal{I},A})$$

$$= \sum_{A \subseteq \{p_1, \ldots, p_m\}} (-1)^{m-|A|} \left( \mathcal{F}_n \left( f_1'(i_{\mathcal{I}}) + \sum_{p_j \in A} g_l(C_{p_j}(i)) \right) \right)$$

$$= \begin{cases} 0 & \text{when } m = n + 1 \\
g_1^U(C_{p_1}(i))g_1^U(C_{p_2}(i)) \cdots g_l^U(C_{p_n}(i)) = g_0^U(C(i)) & \text{when } m = n. \end{cases}$$

That is $f_n'$ is an invariant of order $n$ with $u(f_n') = g_n'$. Now for arbitrary Abelian group $\mathbb{G}$, if $g \in E_n(\mathbb{G})$ then there is $\phi \in \text{Hom}(M_n, \mathbb{G})$ such that $g = \phi \circ g_n'$. It is then clear that $f = \phi \circ f_n' (= \phi \circ \mathcal{F}_n \circ f_1')$ is an invariant of order $n$ with $u(f) = g$. This proves that $u(V_n) \supseteq E_n$ for any $\mathbb{G}$. To complete the proof of Theorem 3.2 it remains to show that $u(V_n) \subseteq E_n$. When this is established then $\mathcal{F} \circ f_1'$ may be named a “universal finite order invariant”.

5. PROOF THAT $u(V_n) \subseteq E_n$

For $x, y, n \geq 0$ we define $I_n^{x,y}$ to be the space of immersions in $I_{x+y+n}$ with $x$ designated CEs of type $H_0^1$ with choice of ordering on them, $y$ designated CEs of type $Q_0^2$ with choice of ordering on them, and a choice of co-orientation for these $x + y$ CEs. The remaining $n$ CEs may be of any type and they are not ordered nor co-oriented. So the same underlying immersion appears $2^{x+y}x!y!$ times in $I_n^{x,y}$ with different choices of ordering and co-orientations.

Also note that $I_n^{0,0} = I_n$.

An $(x, y)$-invariant is a function $f : I_0^{x,y} \to \mathbb{G}$ which is constant on the connected components of $I_0^{x,y}$. Given a temporary co-orientation $\mathcal{I}$ for the $n$ non-designated CEs of $i \in I_n^{x,y}$, we define $i_{\mathcal{I},A} \in I_0^{x,y}$ as before, resolving only non-designated CEs and keeping the order and co-orientation of the designated CEs. We may then define $f^\mathcal{I}(i)$ and invariants of order $n$ as before, and define $V_n^{x,y}$ to be the space of all $(x, y)$-invariants of order at most $n$. We define $C : I_n^{x,y} \to C_n$ as before, using the $n$ non-designated CEs. Again $C$ is surjective and induces an injection $u : V_n^{x,y} / V_{n-1}^{x,y} \to \Delta_n$. Indeed all arguments (appearing in [N]) showing that $u$ may be defined on $V_n$ and that $u(V_n) \subseteq \Delta_n$, are applicable in just the same way to show that the same is true for $V_n^{x,y}$. (As first step note that by [N], namely Proposition 3.4, proof of Proposition 3.5, and Remark 3.7, for any $i, j \in I_n^{x,y}$, $C(i) = C(j)$ iff there is an AB
Lemma 5.2. Given $n$, assume it is known that for any $k < n$, (and any $x, y$) $u(V_k^x)$ $\subseteq E_k$. Then for any $k < n$ and any $f \in V_k^x$ there exists $F \in V_k$ such that for any $i \in I_0^{x,y}$, $f(i) = F(i')$.

Proof. Proof by induction on $k$ ($< n$). By our assumption $u(f) \in E_k$. By Section 4, $u(V_k) \supseteq E_k$ and so there exists $G \in V_k$ with $u(G) = u(f)$. Let $h$ be the invariant on $I_0^{x,y}$ defined by $h(i) = f(i) - G(i')$. Then $h(h) = 0$ (u defined on $V_k^x$) so $h \in V_k^{x,y}$, so by the induction hypothesis there is $H \in V_{k-1}$ such that $h(i) = H(i')$ for all $i \in I_0^{x,y}$. $F = H + G$ is the required invariant on $I_0$.

Lemma 5.3. Given $n$, assume it is known that for any $k < n$, (and any $x, y$) $u(V_k^x)$ $\subseteq E_k$. Let $f \in V_k^x$ and let $i, j \in I_0^{x,y}$ be two immersions such that there is an AB equivalence between them (respecting ordering and co-orientations of the designated CEs) during which precisely two CEs occur, both of which are of type $H_0^1$. Then $f(i) = f(j)$.

The same is true for $Q_0^2$.

Proof. Given $f \in V_n^x$ we define $f^H \in V_{n-1}^{x+1,y}$ and $f^Q \in V_{n-1}^{x,y+1}$ as follows: For $i \in I_0^{x,y}$ let $f^H(i) = f(i^+) - f(i^-)$ where $i^+ \in I_0^{x,y}$ is the immersion obtained from $i$ by resolving the $(x+1)$th designated CE of type $H_0^1$ into the positive side determined by the chosen co-orientation, and the ordering and co-orientation on the remaining designated CEs remains as in $i$. Similarly $i^-$ is defined using the negative side of the co-orientation at the same CE.

In the same way $f^Q$ is defined on $i \in I_0^{y+1}$ using the $(y+1)$th designated CE of type $Q_0^2$. Indeed it is clear that $f^H \in V_{n-1}^{x+1,y}$ and $f^Q \in V_n^{x,y+1}$. We continue discussing $H_0^1$ but clearly all will be true for $Q_0^2$ as well. By our assumption and Lemma 5.2 there exists $G \in V_{n-1}$ such that $f^H(i) = G(i')$ for all $i \in I_0^{x+1,y}$. Let $J_t : F \to \mathbb{R}^3$ ($0 \leq t \leq 1$) be the AB equivalence
in the assumption of the lemma so \( J_0 = i, J_1 = j \) and assume the two CEs occur at times \( \frac{1}{3} \) and \( \frac{2}{3} \). We make \( J_\frac{1}{3} \) and \( J_\frac{2}{3} \) into elements of \( I^\circ_0 + 1 \) by announcing the additional CE that is occurring as the \((x + 1)\)th designated CE of type \( H^1_0 \). For \( J_\frac{1}{3} \) we choose the co-orientation of this \((x + 1)\)th CE to be represented by the motion of \( J_t \) through \( J_{\frac{1}{3}} \) with increasing time, whereas for \( J_\frac{2}{3} \) we use the motion of \( J_t \) with decreasing time. So \( J_\frac{1}{3} \) is on the positive side of both \( J_\frac{1}{3} \) and \( J_\frac{2}{3} \). The co-orientation and order on all other designated CEs of \( J_\frac{1}{3} \) and \( J_\frac{2}{3} \) are those of \( i \) which are continuously carried along the regular homotopy \( J_t \). Similarly \( J_\frac{2}{3} \) is made into an element of \( I^\circ_0 + 2 \) by continuously carrying the co-orientation and order of the CEs of \( i \) along \( J_t \). We get

\[
f(J_\frac{1}{3}) - f(i) = f^H(J_\frac{1}{3}) = G(J_\frac{1}{3}) = f^H(J_\frac{2}{3}) = f(J_\frac{2}{3}) - f(j)
\]

since \( G \) is defined on \( I_0 \) and \( J'_{\frac{1}{3}} \) and \( J'_{\frac{2}{3}} \) are in the same connected component of \( I_0 \). And so we get \( f(i) = f(j) \).

We now prove that \( u(V^n x y) \subseteq E_n \), by induction on \( n \) i.e. we assume it is true for any \( k < n \), and so the conclusion of Lemma 5.3 holds. Let \( i \in I^n x y \) having all its non-designated CEs from the set \( Y \) and located at \( p_1, \ldots, p_n \). If \( \Sigma \) is a temporary co-orientation for \( p_1, \ldots, p_n \) then by Lemma 5.3, \( f(i_{\Sigma,A}) = f(i_{\Sigma,B}) \) whenever \( A, B \subseteq \{p_1, \ldots, p_n\} \) have the same number mod 2 of points of type \( H^1_0 \) and the same number mod 2 of points of type \( Q^2_0 \), all other points being the same. It follows that in the expression for \( f^\Sigma(i) \), each value of \((-1)^{n-|A|} f(i_{\Sigma,A})\) appears \( 2^r \) times, where \( r = r(C(i)) \), and so we get the following:

**Lemma 5.4.** For \( i \) as above let \( R \subseteq \{p_1, \ldots, p_n\} \) be a subset which includes all the points which are not of type \( H_0^1 \) and \( Q_0^2 \), and includes only one of the points of type \( H_0^1 \) if such exists, and only one of the points of type \( Q_0^2 \) if such exists, then:

\[
f^\Sigma(i) = \sum_{A \subseteq \{p_1, \ldots, p_n\}} (-1)^{n-|A|} f(i_{\Sigma,A}) = 2^r \sum_{A \subseteq R} (-1)^{n-|A|} f(i_{\Sigma,A}).
\]

This proves that \( u(f) \) satisfies property 7 of Definition 5.1.

Now let \( i \in I^n x y \) having all its non-designated CEs from the set \( Y \) and located at \( \{p_1, \ldots, p_{n+1}\} \) and assume \( C_{p_1}(i) = C_{p_2}(i) = H^1_0 \) and \( C_{p_n}(i) = C_{p_{n+1}}(i) = Q^2_0 \). Given a temporary co-orientation \( \Sigma \) for \( i \), let \( i_0 \in I^n x y \) be the immersion obtained from \( i \) by resolving the CE at \( p_{n+1} \) into the negative side determined by \( \Sigma \). Let \( i_1 \in I^n x y \) be similarly defined using the point \( p_1 \). So the CEs of \( i_0 \) are \( \{p_1, \ldots, p_n\} \) and the CEs of \( i_1 \) are \( \{p_2, \ldots, p_{n+1}\} \). Let \( \Sigma_0, \Sigma_1 \) be the temporary co-orientations for \( i_0, i_1 \) respectively, which are the restrictions
of $\Sigma$, then for any $A \subseteq \{p_2, \ldots, p_n\}$, $(i_0)_{\Sigma,A} = (i_1)_{\Sigma,A}$. Let $R \subseteq \{p_2, \ldots, p_n\}$ be the set including $p_2, p_n$ and all points which are not of type $H_0^1$ and $Q_0^2$. Then this $R$ may be used as the $R$ appearing in Lemma 5.4, for both $i_0$ and $i_1$. Furthermore $r(C(i_0)) = r(C(i_1))$ which we denote $r$, so we get by Lemma 5.4:

$$f(i_0) = f^{\Sigma_0}(i_0) = 2r \sum_{A \subseteq R} (-1)^{n-|A|} f(i_{\Sigma,A}) = f^{\Sigma_1}(i_1) = f(i_1).$$

This proves that $u(f)$ satisfies property 1 of Definition 3.1 since $p_1, p_2, p_n$ are CEs of $i_0$ and $p_2, p_n, p_{n+1}$ are CEs of $i_1$, all other CEs being the same, and since any equality stated by 1 of Definition 3.1 may indeed be realized by an immersion $i \in \mathcal{I}_{n+1}$ with such set $p_1, \ldots, p_{n+1}$ of CEs. This completes the proof of Theorem 3.2.

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