Generalizing Hierarchical Bayesian Bandits

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Abstract

A contextual bandit is a popular and practical framework for online learning to act under uncertainty. In many problems, the number of actions is huge and their mean rewards are correlated. In this work, we introduce a general framework for capturing such correlations through a two-level graphical model where actions are related through multiple shared latent parameters. We propose a Thompson sampling algorithm \( G\text{-HierTS} \) that uses this structure to explore efficiently and bound its Bayes regret. The regret has two terms, one for learning action parameters and the other for learning the shared latent parameters. The terms reflect the structure of our model as well as the quality of priors. Our theoretical findings are validated empirically using both synthetic and real-world problems. We also experiment with \( G\text{-HierTS} \) that maintains a factored posterior over latent parameters. While this approximation does not come with guarantees, it improves computational efficiency with a minimal impact on empirical regret.

1 Introduction

A contextual bandit [45, 32, 34, 12] is a popular sequential decision making framework where an agent interacts with an environment over \( n \) rounds. In round \( t \in [n] = \{1, \ldots, n\} \), the agent observes a context, takes an action accordingly, and receives a reward that depends on both the context and the taken action. The goal of the agent is to maximize the expected cumulative reward over \( n \) rounds. Expected rewards being unknown, the agent should find a trade-off between taking the action that maximizes the estimated reward using collected data (exploitation), and exploring other actions to improve their estimates (exploration). For instance, in online advertising, contexts can be features of users, actions can be products, and the expected reward can be the click-through rate (CTR).

There are two main approaches to addressing the exploration-exploitation dilemma in sequential decision making. The first are upper confidence bounds (UCBs) [2, 4], which rely on the optimism in the face of uncertainty principle. UCB algorithms construct confidence sets and take the action with the highest estimated reward within these sets. Thompson sampling (TS) [46, 42] follows the Bayesian paradigm and takes the action with the highest estimated reward under posterior-sampled model parameters. Thompson sampling is naturally randomized, highly flexible, and it has competitive empirical performance [40, 43, 11]. This work takes the second direction.

Efficient exploration in contextual bandits [30, 13, 34, 1, 3] is an important research direction, as their action space is often large and naive exploration may lead to suboptimal performance. In this work, we start from a basic observation that the expected rewards of actions are often correlated in real-world problems. To model this phenomenon, we study a structured bandit environment where each action parameter depends on one or multiple latent parameters. It follows that actions share information through these latent parameters. Taking an action shall teach the agent about its latent parameters, which consequently teaches it about other actions that share the same latent parameters.

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This structure arises naturally in various applications. For instance, consider the problem of finding the optimal drug design in early stage clinical trials [9, 17]. A drug is a combination of multiple components with specific dosage. Each component is associated to a latent parameter, and the drug parameter is a known combination of latent parameters weighted by their dosage. The drug parameter may be perturbed by noise to incorporate uncertainty due to model misspecification. After administering a drug to a subject (action), the agent receives a reward that quantifies its toxicity or efficacy, which depends on the drug parameter (action parameter) and the features of the subject, such as age and gender (context). The toxicity or efficacy of each component is unobserved (latent), and the actions are correlated since the candidate drugs in the same clinical trial often share components. Another example is page construction in a movie recommender system, where movies are organized into genres, and each movie can belong to multiple genres. The engagement with the movies is correlated as they can share the genres.

Our paper makes the following contributions. 1) We formalize a general framework for hierarchical Bayesian bandits represented by a two-level graphical model where each action is associated with a \( d \)-dimensional parameter that depends on one or multiple latent parameters. 2) We design a General Hierarchical Thompson Sampling algorithm called G-HierTS, which leverages the structure of our graphical model to be more computationally efficient. A closed-form solution is given for two Gaussian bandit instances. 3) We prove that the Bayes regret of G-HierTS is bounded by two terms: one is associated with learning the action parameters, while the other quantifies the cost of learning the latent parameters. Both terms reflect the structure of the environment and the quality of priors. 4) We show that variational inference on factored distributions can further improve computational efficiency of our approach with minimal impact on empirical regret. 5) We show empirically that G-HierTS performs well and is computationally efficient in both synthetic and real-world problems. When compared to prior works on hierarchical bandits, we generalize some [27, 7, 24] to contextual bandits, and all [27, 7, 48, 24, 49, 23] to multiple latent parameters. This extension is non-trivial, as the prior works assumed that action parameters are centered at a single latent parameter. In our setting, this does not hold even if we treated all latent parameters as a single one. Moreover, action parameters can depend on a small subset of latent parameters, inducing sparsity which is not captured by prior analyses.

### 2 Setting

For any positive integer \( n \), we define \([n] = \{1, 2, ..., n\}\). Unless specified, the \( i \)-th coordinate of a vector \( v \) is \( v_i \). When the vector is already indexed, such as \( v_j \), we write \( v_{j,i} \). Let \( a_1, \ldots, a_n \in \mathbb{R}^d \) be \( n \) vectors. We denote by \( a = (a_i)_{i\in[n]} \in \mathbb{R}^{nd} \) a vector of length \( nd \) obtained by concatenation of vectors \( a_1, \ldots, a_n \). We use \( \otimes \) to denote the Kronecker product.

We study a setting where a learning agent interacts with a contextual bandit over \( n \) rounds. In round \( t \in [n] \), the agent observes context \( X_t \in \mathcal{X} \), where \( \mathcal{X} \subseteq \mathbb{R}^d \) is a \( d \)-dimensional context space. After that, it takes an action \( A_t \) from an action set \( \mathcal{A} = [K] \), and then observes a stochastic reward \( Y_t \in \mathbb{R} \) that depends on both \( X_t \) and \( A_t \). We consider a structured problem where the expected rewards of actions are correlated. Specifically, each action \( i \in [K] \) is associated with an unknown \( d \)-dimensional action parameter \( \theta_{s,i} \in \mathbb{R}^d \). The correlations between the action parameters arise because they are derived from \( L \) shared unknown \( d \)-dimensional latent parameters, \( \psi_{s,\ell} \in \mathbb{R}^d \) for any \( \ell \in [L] \). Let \( \Psi_s = (\psi_{s,\ell})_{\ell\in[L]} \in \mathbb{R}^{Ld} \) be a concatenation of the latent parameters. The action parameter \( \theta_{s,i} \) depends on \( \Psi_s \) through the conditional prior distribution \( P_{0,i}: \theta_{s,i} | \Psi_s \sim P_{0,i}(\cdot | \Psi_s) \). The distribution \( P_{0,i} \) can capture sparsity, when action \( i \) depends only on a subset of latent parameters \( \Psi_s \); and also incorporate uncertainty due to model misspecification, when \( \theta_{s,i} \) is not a deterministic function of the latent parameters. Finally, the latent parameters \( \Psi_s \) are initially sampled from a joint hyper-prior \( Q_0 \) known by the learning agent and do not have to be independent. In summary, all variables in our environment are generated as

\[
\begin{align*}
\Psi_s &\sim Q_0, \\
\theta_{s,i} | \Psi_s &\sim P_{0,i}(\cdot | \Psi_s), \\
Y_t | X_t, \theta_{s,A_t} &\sim P(\cdot | X_t; \theta_{s,A_t}), \\
\end{align*}
\]

where \( P(\cdot | x; \theta_{s,a}) \) is the reward distribution of action \( a \) and context \( x \), which only depends on parameter \( \theta_{s,a} \). The terminology of prior and hyper-prior is borrowed from Bayesian statistics
We also define the concatenation of action parameters
\[ \sum_{i=1}^{K} \theta_{s,i}, \psi_{s,i} \] which is visualized in Figure 1. Sparsity in this example is manifested by missing arrows from parent (latent parameters) to child nodes (action parameters). A missing arrow from parent \( \psi_{s,i} \) to child \( \theta_{s,i} \) means that the parameter of action \( i \) is independent of the \( \ell \)-th latent parameter, which can be incorporated in \( P_{0,i} \). Note that the action parameters are also latent, because we only observe \( Y_t \). We do not refer to them as latent to avoid confusion between \( \psi_{s,i} \) and \( \theta_{s,i} \). Moreover, they are arguably easier to learn than \( \psi_{s,i} \), since they generate \( Y_t \). Therefore, they could be viewed as observed variables in previously studied models, such as the QMR-DT network [25].

Our setting can be represented by a two-level graphical model, where \( \psi_{s,1}, \ldots, \psi_{s,L} \) are parent nodes and \( \theta_{s,1}, \ldots, \theta_{s,K} \) are child nodes, which is visualized in Figure 1. Sparsity in this example is manifested by missing arrows from parent (latent parameters) to child nodes (action parameters). A missing arc from parent \( \psi_{s,i} \) to child \( \theta_{s,i} \) means that the parameter of action \( i \) is independent of the \( \ell \)-th latent parameter, which can be incorporated in \( P_{0,i} \). Note that the action parameters are also latent, because we only observe \( Y_t \). We do not refer to them as latent to avoid confusion between \( \psi_{s,i} \) and \( \theta_{s,i} \). Moreover, they are arguably easier to learn than \( \psi_{s,i} \), since they generate \( Y_t \). Therefore, they could be viewed as observed variables in previously studied models, such as the QMR-DT network [25].

Figure 1: Example of a graphical model induced by (1) in round \( t \in [n] \).

Now we provide an important special case of our setting. A simple yet powerful formulation to capture sparsity in conditional priors \( P_{0,i} \) is to assume that the parameter of action \( i \) depends on latent parameters through \( L \) known mixing weights \( b_{i} = (b_{i,1, \ldots, b_{i,L}}) \in \mathbb{R}^L \) as \( \theta_{s,i} | \Psi_s \sim P_{0,i} (\cdot | \sum_{\ell=1}^{L} b_{i,\ell} \psi_{s,\ell}, \nu) \), where \( P_{0,i} \) is now parametrized by \( \sum_{\ell=1}^{L} b_{i,\ell} \psi_{s,\ell} \). In this case, the sparsity is no longer implicit in \( P_{0,i} \) and is reflected by \( b_{i,\ell} = 0 \), when the parameter of action \( i \) is independent of latent parameter \( \psi_{s,\ell} \). We present two Gaussian instances as examples for this setting.

### 2.1 Multi-armed bandit with latent parameters

For scalar latent parameters, \( \psi_{s,\ell} \in \mathbb{R} \) holds for all \( \ell \in [L] \), a natural hyper-prior \( Q_0 \) is a multivariate Gaussian with mean \( \mu_{\Psi} \in \mathbb{R}^L \) and covariance \( \Sigma_{\Psi} \in \mathbb{R}^{L \times L} \). The conditional prior \( P_{0,i} \) is a univariate Gaussian with mean \( \sum_{\ell=1}^{L} b_{i,\ell} \psi_{s,\ell} = b_{i}^T \Psi_s \in \mathbb{R} \) and variance \( \sigma_{0,i}^2 > 0 \). This is a non-contextual setting (\( X_t = 1 \)) and thus the corresponding model reads

\[
\begin{align*}
\Psi_s & \sim \mathcal{N}(\mu_{\Psi}, \Sigma_{\Psi}), \\
\theta_{s,i} | \Psi_s & \sim \mathcal{N}(b_{i}^T \Psi_s, \sigma_{0,i}^2), \\
Y_t | A_t, \theta_{s,i} & \sim \mathcal{N}(\theta_{s,i} A_t, \sigma^2),
\end{align*}
\]

### 2.2 Contextual linear bandit with latent parameters

For \( d \)-dimensional latent parameters \( \psi_{s,\ell} \), a natural hyper-prior \( Q_0 \) is a multivariate Gaussian with mean \( \mu_{\Psi} \in \mathbb{R}^{Ld} \) and covariance \( \Sigma_{\Psi} \in \mathbb{R}^{Ld \times Ld} \). The conditional prior \( P_{0,i} \) is a multivariate Gaussian with mean \( \sum_{\ell=1}^{L} b_{i,\ell} \psi_{s,\ell} \in \mathbb{R}^d \) and covariance \( \Sigma_{0,i} \in \mathbb{R}^{d \times d} \). and the whole model is

\[
\begin{align*}
\Psi_s & \sim \mathcal{N}(\mu_{\Psi}, \Sigma_{\Psi}), \\
\theta_{s,i} | \Psi_s & \sim \mathcal{N}(\sum_{\ell=1}^{L} b_{i,\ell} \psi_{s,\ell}, \Sigma_{0,i}), \\
Y_t | X_t, \theta_{s,i} & \sim \mathcal{N}(X_t^T \theta_{s,i} A_t, \sigma^2),
\end{align*}
\]
Algorithm 1: G-HierTS: General Hierarchical Thompson Sampling

**Input:** Joint hyper-prior $Q_0$, conditional priors $P_0$.
Initialize $Q_1 ← Q_0$ and $P_1 ← P_0$.

for $t = 1, \ldots, n$ do
  Sample $\Psi_t ∼ Q_t$
  for $i = 1, \ldots, K$ do
    Sample $\theta_{t,i} ∼ P_{t,i}(· | \Psi_t)$
    $\Theta_t ← (\theta_{t,i})_{i∈[K]}$, take action $A_t ← \arg\max_{i∈[K]} r(X_t, i; \Theta_t)$, and receive reward $Y_t$
  Compute new posteriors $Q_{t+1}$ and $P_{t+1}$.

This model extends (2) to include contexts, and both induce a Gaussian graphical model [26].

The model in (3) can be used in a contextual combinatorial bandit with bandit feedback [39, 15], where the latent parameters $\psi_{s,ℓ}$ correspond to basic actions and action parameters $\theta_{s,ℓ}$ correspond to super actions, which are sets of basic actions. In round $t$, the agent observes a context $X_t$, takes a super action $A_t$, and receives a stochastic reward $\mathcal{N}(X_t^\top θ_{s,A_t}, σ^2)$. While the outcomes of basic actions in $A_t$ are unobserved, the expected reward of $A_t$ depends on the expected rewards of its basic actions, and thus super actions that share basic actions shall teach the agent about each other.

This setting can also model drug design in early clinical trials. Here the components $ℓ ∈ [L]$ and drugs $i ∈ [K]$ are associated to the latent parameters $\psi_{s,ℓ}$ and action parameters $\theta_{s,i}$, respectively. The weight $b_{i,ℓ}$ is the dosage of component $ℓ$ in drug $i$. The reward $\mathcal{N}(X_t^\top θ_{s,A_t}, σ^2)$ quantifies the toxicity or efficacy of a drug $A_t$ to a subject with features $X_t$.

3 Algorithm

In this work, we design a Thompson sampling algorithm [46, 40, 43, 11], which is a natural Bayesian solution to our problem. The sampling is hierarchical [35], to capture the structure, and potential sparsity, in our model. Before we present the algorithm, we need to introduce additional notation. Let $H_t = (X_t, A_t, Y_t)_{ℓ∈[t−1]}$ be the history of all interactions of the agent up to round $t$. $H_{t,i} = (X_t, A_t, Y_t)_{ℓ∈[t],A_{i,ℓ}=i}$ denotes the history of interactions of the agent with action $i$ up to round $t$.

Our algorithm is presented in Algorithm 1 and we call it G-HierTS. Since the latent parameters are shared by actions, their posteriors are not independent. For this reason, we maintain a single joint hyper-posterior $Q_1(Ψ) = \mathbb{P}(Ψ = Ψ | H_t)$ for all latent parameters $Ψ_t$, in round $t$. Moreover, we maintain a conditional posterior $P_{t,i}(θ | Ψ_t) = \mathbb{P}(θ_{s,i} = θ | H_{t,i}, Ψ_t)$ for each action $i ∈ [K]$ given $Ψ_t = Ψ$. G-HierTS samples hierarchically as follows. In round $t$, we first sample latent parameters $Ψ_t ∼ Q_t$. Then we sample each action parameter $θ_{t,i} ∼ P_{t,i}(θ | Ψ_t)$ individually. Note that this is equivalent to sampling from the exact posterior $\mathbb{P}(θ_{s,i} = θ | H_t)$, since

$$\mathbb{P}(θ_{s,i} = θ | H_t) = \int_Ψ \mathbb{P}(θ_{s,i} = θ, Ψ_t = Ψ | H_t) \, dΨ = \int_Ψ P_{t,i}(θ | Ψ_t) Q_t(Ψ) \, dΨ.$$  (4)

3.1 Posterior derivations

The posteriors are computed as follows. We first express the joint hyper-posterior $Q_t$ as

$$Q_t(Ψ) ∝ \mathbb{P}(H_t | Ψ_t = Ψ_t) Q_0(Ψ) = \prod_{i=1}^{K} \mathbb{P}(H_{t,i} | Ψ_t = Ψ_t) Q_0(Ψ),$$  (5)

$$= \prod_{i=1}^{K} \int_θ \mathbb{P}(H_{t,i}, θ_{s,i} = θ | Ψ_t = Ψ_t) \, dθ \, Q_0(Ψ) = \prod_{i=1}^{K} \int_θ L_{t,i}(θ) P_{0,i}(θ | Ψ_t) \, dθ \, Q_0(Ψ),$$

where $L_{t,i}(θ) = \mathbb{P}(H_{t,i}, θ_{s,i} = θ) = \prod_{(x,a,y)∈H_{t,i}} P(y | x; θ)$ is the likelihood of all observations of action $i$ up to round $t$ given that $θ_{s,i} = θ$.

Next, for any action $i ∈ [K]$, the conditional posterior $P_{t,i}$ is defined as

$$P_{t,i}(θ | Ψ_t) = \mathbb{P}(θ_{s,i} = θ | H_{t,i}, Ψ_t = Ψ_t) ∝ L_{t,i}(θ) P_{0,i}(θ | Ψ_t).$$  (6)
Note that $P_{t,i}$ is similarly sparse as $P_{0,i}$. Specifically, in any round $t$, $P_{t,i}$ and $P_{0,i}$ are parameterized by the same subset of latent parameters $\Psi$, since $L_{t,i}(\theta)$ does not depend on $\Psi$.

The joint hyper-posterior $Q_t$ and conditional posteriors $P_{t,i}$ have closed-form solutions in Gaussian models (Sections 2.1 and 2.2), allowing for efficient sampling and theoretical analysis. Beyond these, MCMC and variational inference [16] can be used to approximate $Q_t$ and $P_{t,i}$ at each sampling stage. Next we derive a closed-form solution for the joint hyper-posterior $Q_t$ and conditional posteriors $P_{t,i}$ for the contextual model in (3). After that, we show that a posterior approximation based on variational inference for factored Gaussians further improves computational efficiency. The exact and approximate posteriors for the multi-armed bandit model in (2) are given in Appendix B, and are a special case of the results in Sections 3.2 and 3.3. All the proofs are deferred to Appendix C.

## 3.2 Contextual linear bandit with latent parameters

Fix round $t \in [n]$. Let $S_{t,i} = \{\ell < t, A_\ell = i\}$ denote the rounds where action $i$ is taken up to round $t$. We also introduce $G_{t,i} = \sigma^{-2} \sum_{\ell \in S_{t,i}} X_\ell X_\ell^T \in \mathbb{R}^{d \times d}$ and $B_{t,i} = \sigma^{-2} \sum_{\ell \in S_{t,i}} Y_\ell X_\ell \in \mathbb{R}^d$, the outer product of the corresponding contexts and their sum weighted by rewards $Y_\ell$, respectively. Both $G_{t,i}$ and $B_{t,i}$ are scaled by the reward noise $\sigma$. Using these, the posterior is defined as follows.

**Proposition 1.** For any round $t \in [n]$, the joint hyper-posterior is a multivariate Gaussian $Q_t = \mathcal{N}(\bar{\mu}_t, \bar{\Sigma}_t)$, where

$$
\bar{\Sigma}_t^{-1} = \Sigma_0^{-1} + \sum_{i=1}^K b_i b_i^T \otimes (\Sigma_{0,i} + G_{t,i})^{-1},
$$

$$
\bar{\mu}_t = \bar{\Sigma}_t \left( \Sigma_0^{-1} \mu_0 + \sum_{i=1}^K b_i \otimes (\Sigma_{0,i} + G_{t,i})^{-1} G_{t,i}^{-1} B_{t,i} \right) .
$$

**Proposition 2.** For any round $t \in [n]$, action $i \in [K]$, and latent parameters $\Psi$, the conditional posterior is a multivariate Gaussian $P_{t,i}(\cdot \mid \Psi) = \mathcal{N}(\cdot; \bar{\mu}_{t,i}, \bar{\Sigma}_{t,i})$, where

$$
\bar{\Sigma}_{t,i}^{-1} = \Sigma_{0,i}^{-1} + G_{t,i}, \quad \bar{\mu}_{t,i} = \bar{\Sigma}_{t,i} \left( \Sigma_0^{-1} + \sum_{\ell=1}^L b_{t,i,\ell} \psi_{t,i,\ell} \right) + B_{t,i} .
$$

Note that while matrix $G_{t,i}$ may not be invertible, the above formulation is for ease of exposition only. In fact, $G_{t,i}^{-1}$ appears after using the Woodbury matrix identity to invert $\Sigma_{0,i}^{-1} + G_{t,i}$, which is well defined. We refer the reader to Appendix C.1.1 for more details.

The hyper-posterior is additive in actions and can be interpreted as follows. Each action is a single noisy observation in its estimate. The maximum likelihood estimate (MLE) of the parameter of action $i$, $G_{t,i}^{-1} B_{t,i}$, contributes to (7) proportionally to its precision, $(\Sigma_{0,i}^{-1} + G_{t,i}^{-1})^{-1}$. The contribution to the $\ell$-th latent parameter is weighted by $b_{t,i,\ell}$, which is the mixture weight used to generate $b_{\ast,i}$ in (3). The conditional posterior in (8) has a standard form. Note that its prior mean depends on latent parameters $\Psi$, which G-HierTS samples.

## 3.3 Posterior approximation

As discussed earlier, the number of actions $K$ is often much larger than the number of latent parameters $L$. However, $L$ can also be large. In this section, we show how to improve the computational efficiency of G-HierTS using factored distributions [8]. As an example, consider the model in (3) and assume that the joint hyper-posterior $Q_t$ factorizes. Specifically, $Q_t(\Psi) = \prod_{\ell=1}^L Q_{t,\ell}(\psi_{t,\ell})$, where $Q_{t,\ell}$ is the hyper-posterior of the $\ell$-th latent parameter $\psi_{t,\ell}$. It follows that for any round $t \in [n]$, the hyper-posterior $Q_{t,\ell}$ is also a multivariate Gaussian $Q_{t,\ell} = \mathcal{N}(\bar{\mu}_{t,\ell}, \bar{\Sigma}_{t,\ell})$, where

$$
\bar{\Sigma}_{t,\ell}^{-1} = \Sigma_{0,\ell}^{-1} + \sum_{i=1}^K b_{t,\ell,i}^2 (\Sigma_{0,i} + G_{t,i}^{-1})^{-1},
$$

$$
\bar{\mu}_{t,\ell} = \bar{\Sigma}_{t,\ell} \left( \Sigma_{0,\ell}^{-1} \mu_0 + \sum_{i=1}^K b_{t,\ell,i} (\Sigma_{0,i} + G_{t,i}^{-1})^{-1} G_{t,i}^{-1} B_{t,i} \right) .
$$

(9)
\( \Sigma_{\psi_i} \) is the \( \ell \)-th \( d \times d \) diagonal block of \( \Sigma_{\psi} \), and \( \mu_{\psi_i} \in \mathbb{R}^d \) are such that \( \mu_{\psi} = (\mu_{\psi_i})_{i \in [L]} \). The above formulation allows for individual sampling of the latent parameters, which improves the space and time complexity. The derivation of the above posterior is presented in Appendix C.2.

### 3.4 Alternative algorithm designs

It is unclear if modeling of latent parameters is needed since the Bayes regret in Section 2 does not explicitly depend on \( \Psi_s \). The most intuitive approach would be to discard the latent parameters and maintain a single joint posterior over all action parameters \( \Theta_s \in \mathbb{R}^{K \times d} \), as the actions are correlated. While this is possible, we argue that it is suboptimal when the number of latent parameters \( L \) is much smaller than the number of actions \( K \), which is common in practice.

The main advantage of G-HierTS is that sampling of latent parameters \( \Psi_t \sim Q_t \) allows us to use the conditional independence of actions given \( \Psi_s \), and model \( \theta_{s,i} | H_{t,i}, \Psi_s = \Psi_t \). This is more computationally efficient than modeling the joint posterior of all action parameters \( \Theta_s \mid H_t \) when \( K \gg L \). To see this, suppose that all posteriors are multivariate Gaussians. Then the space complexity of maintaining \( \Theta_s \mid H_t \) is \( O(K^2d^2) \), due to storing a \( Kd \times Kd \) covariance matrix, while that of G-HierTS is only \( O((L^2 + K)d^2) \), due to storing the covariances of the joint hyper-posterior \( Q_t \) and conditional posteriors \( P_{t,i} \). The time complexity also improves since posterior sampling requires covariance matrix inversions. For the joint posterior, it is \( O((K^3d^2) \), while sampling from \( Q_t \) and then from \( P_{t,i} \) takes only \( O((L^3 + K)d^2) \) time. When the factored hyper-posteriors are used (Section 3.3), both the space and time complexities improve, due to maintaining separate posteriors for each \( \psi_{s,t} \). Specifically, we get \( O((L + K)d^2) \) space and \( O((L + K)d^3) \) time complexity.

One can also discard the latent parameters and maintain \( K \) separate posteriors, one for each action parameter \( \theta_{s,i} \in \mathbb{R}^d \). While this greatly improves computational efficiency, it does not model that the actions are correlated, as it only considers \( \theta_{s,i} \mid H_{t,i} \) instead of \( \theta_{s,i} \mid H_t \). This leads to statistical inefficiency, due to the loss of information as the histories of other actions \( H_{t,j} \) are discarded. This observation is validated in our experiments in Section 5.

### 4 Analysis

As discussed earlier, we analyze G-HierTS in the setting of Section 2.2. Before we start, we need additional notation. For any matrix \( A \in \mathbb{R}^{d \times d} \), we use \( \lambda_1(A) \) and \( \lambda_d(A) \) to denote the maximum and minimum eigenvalues of \( A \), respectively. Let \( A_1, \ldots, A_n \in \mathbb{R}^{d \times d} \) be \( n \) matrices of dimension \( d \times d \). Then \( \text{diag}((A_i)_{i \in [n]}) \in \mathbb{R}^{nd \times nd} \) denotes the block diagonal matrix where \( A_1, \ldots, A_n \) are the main-diagonal blocks. Similarly, \( (A_i)_{i \in [n]} \in \mathbb{R}^{nd \times d} \) is the \( nd \times d \) matrix obtained by concatenation of \( A_1, \ldots, A_n \). Finally, we write \( \tilde{O} \) for the big-O notation up to polylogarithmic factors.

#### 4.1 Main result

We first formally state and discuss the main result. Then we sketch its proof in Section 4.2. The complete proof is deferred to Appendix D.

**Theorem 1.** For any \( \delta \in (0, 1) \), the Bayes regret of G-HierTS, for the model in (3), is bounded as

\[
BR(n) \leq 2n \log(1/\delta) \left( R_{\text{action}}(n) + R_{\text{latent}} \right) + \sqrt{\frac{2}{\pi} (\lambda_{1,0} + c_\Phi) \kappa_x K n \delta},
\]

where \( R_{\text{action}}(n) = K dc_1 \log \left( 1 + \frac{\kappa_{x,1} \lambda_{1,0}}{\sigma_x \kappa_x} \right), \quad R_{\text{latent}} = L dc_2 \log \left( 1 + \frac{\kappa_{x,1} \lambda_{1,0}}{\lambda_{d,0}} \right), \quad \text{and} \quad \lambda_{1,0} = \max_{i \in [K]} \lambda_1(\Sigma_{0,i}), \quad \lambda_{d,0} = \min_{i \in [K]} \lambda_d(\Sigma_{0,i}), \quad \lambda_{1,\Psi} = \lambda_1(\Sigma_{\Psi}), \quad \kappa_b = \max_{i \in [K]} \| h_i \|_2^2, \quad \kappa_x \geq \max_{i \in [n]} \| X_i \|_2^2,
\]

\[
c_1 = \frac{\kappa_x \lambda_{1,0}}{\log(1 + \sigma^{-2} \kappa_x \lambda_{1,0})}, \quad c_2 = \frac{c_\Phi (1 + \sigma^{-2} \kappa_x \lambda_{1,0})}{\log(1 + \sigma^{-2} \kappa_x \lambda_{1,0})}, \quad c_3 = \frac{K \kappa_x \lambda_{1,0}^2 \lambda_{1,\Psi} \kappa_b}{\lambda_{d,0}^2}.
\]

The second term in (10) is constant for \( \delta = 1/n \); in which case the above bound is \( \tilde{O}(\sqrt{n}) \) and thus optimal in the horizon \( n \). The sum \( R_{\text{action}}(n) + R_{\text{latent}} \) has a natural interpretation. The first
term, $R^{\text{action}}(n)$, corresponds to the action regression problem. This problem has $Kd$ parameters, the maximum prior width is $\sqrt{\lambda_{1,0}}$, the maximum length of feature vectors is $\sqrt{\kappa_0}$, and we have $n$ observations with noise $\sigma$. The dependence on these quantities in $R^{\text{action}}(n)$ is the same as in a corresponding linear bandit [36]. The second term, $R^{\text{latent}}$, corresponds to the latent regression problem. This problem has $Ld$ parameters, the maximum prior width is $\sqrt{\lambda_{1,0}}$, the maximum length of the action parameter weights is $\sqrt{\kappa_0}$, and each of $K$ actions can be viewed as an observation with noise $\lambda_{1,0}$. The dependence on these quantities in $R^{\text{latent}}$ mimics those in $R^{\text{action}}(n)$. Finally, $R^{\text{latent}}$ also captures sparsity in the model through $\kappa_0$, which gets smaller as $b_{i,t} \to 0$.

One shortcoming of our analysis is that we do not provide a matching lower bound. The only lower bound that we are aware of is $\Omega(\log^2 n)$ for a $K$-armed bandit (Theorem 3 of Lai [28]). Seminal works on Bayes regret minimization [40, 41] do not match it. Therefore, to argue that our bound is sensible, and reflects the problem structure, we compare G-HierTS to baselines that have more information or use less structure. We start with G-HierTS with a known hyper-parameter $\Psi$. This baseline has clearly more information than G-HierTS. In this case, $\lambda_1(\Sigma) = 0$ and therefore $R^{\text{latent}} = 0$. This means lower regret. Next we consider a lazy agent that does not know $\Psi$, and also does not model it. This means that only $\Theta$ is learned. The regret of this agent would be as in Theorem 1 with $R^{\text{latent}} = 0$. The difference is that conditional prior covariance $\Sigma_{0,i}$ would be replaced with the marginal prior covariance $\bar{\Sigma}_{0,i}$. As the marginal must account for the uncertainty of the not-modeled $\Psi$, we would have $\lambda_1(\Sigma_{0,i}) > \lambda_1(\bar{\Sigma}_{0,i})$. Since the terms appear in $R^{\text{action}}(n)$, and that grows with $K \gg L$, the regret of the lazy agent would ultimately be higher than that of G-HierTS.

4.2 Sketch of the proof

The first key idea in our proof is that the conditional posteriors $P_{t,i}$ in Proposition 2 can be written as a single joint conditional posterior $P_t(\Theta | \Psi) = \mathbb{P}(\Theta = \Theta | H_t, \Psi = \Psi) = N(\Theta; \tilde{\mu}_t, \tilde{\Sigma}_t)$, where $\tilde{\Sigma}_t = \text{diag}((\tilde{\Sigma}_{i,t}, i \in [K])) \in \mathbb{R}^{Kd \times Kd}$ and $\tilde{\mu}_t = (\tilde{\mu}_{i,t}, i \in [K]) \in \mathbb{R}^{Kd}$. Moreover, by (4), our sampling is equivalent to sampling from the exact posterior of all action parameters $P(\Theta = \Theta | H_t)$, which is also a multivariate Gaussian $P(\Theta = \Theta | H_t) = N(\Theta; \mu_t, \Sigma_t)$ with $\mu_t \in \mathbb{R}^{Kd}$ and $\Sigma_t \in \mathbb{R}^{Kd \times Kd}$. This follows from the fact that the joint conditional posterior $P_t$ and the hyper-posterior $Q_t$ are Gaussians, and the Gaussianity is preserved after marginalization [26].

Hong et al. [24] analyzed regret in the linear bandit setting with a single latent parameter. We extend their analysis to linear contextual bandits with multiple latent parameters. To include contexts, we introduce a joint vector representation of actions $A_t$ and contexts $X_t$ that we define as $A_t = (1_{A_{t,i} = 1}, X_t)_{i \in [K]} \in \mathbb{R}^{Kd}$. This allows us to rewrite the Bayes regret in terms of $A_t$ and $\Theta$, and consequently bound it following standard analyses [40, 24] as

$$BR(n) \leq \sqrt{2n \log(1/\delta)} \left( \mathbb{E} \left[ \sum_{t=1}^{n} \|A_t\|_2^2 \|\Sigma_t\|_2 \right] + \sqrt{\frac{2}{\pi} \lambda_{1,0} c_{\Psi} K n \delta} \right), \quad \forall \delta \in (0, 1).$$

Next, to account for multiple latent parameters, we represent them as a single $Ld$-dimensional vector $\Psi = (\psi_{s,t})_{s \in [L], t \in [K]} \in \mathbb{R}^{Ld}$, and observe that $\sum_{t=1}^{n} b_{i,t} \psi_{s,t} = \Gamma_i \Psi_s$, where $\Gamma_i = b_i^T \otimes I_d \in \mathbb{R}^{d \times Ld}$ for any $i \in [K]$. It follows that the model in (3) can be rewritten as

$$\Psi_s \sim N(\mu_{\Psi}, \Sigma_{\Psi}), \quad \Theta_s | \Psi_s \sim N(\Gamma \Psi_s, \Sigma_0), \quad Y_t | A_t, \Theta_s \sim N(A_t^T \Theta_s, \sigma^2), \quad \forall t \in [n],$$

where $\Gamma = (\Gamma_{i,t})_{i \in [K]} \in \mathbb{R}^{Kd \times Ld}$ and $\Sigma_0 = \text{diag}((\Sigma_{0,i})_{i \in [K]}) \in \mathbb{R}^{Kd \times Kd}$. This compact formulation can be seen as a generalization of prior works [6, 27, 7, 44, 48, 24, 38, 49] with a single latent parameter, in which case $L = 1$ and $\Gamma_i = I_d$ for all $i \in [K]$. The challenge in bounding (11) is that we need to bound a $\Sigma_{\Psi}$-norm, while we only know closed forms of $\Sigma_t$ and $\Sigma_0$ (Section 3.2). Using the total covariance decomposition (Lemma 3 in Appendix D.2), we express $\Sigma_t$ as

$$\Sigma_t = \tilde{\Sigma}_t + \tilde{\Sigma}_t \Sigma_0^{-1} \Gamma \Sigma_t \Gamma^T \Sigma_0^{-1} \Sigma_t.$$  

The first term in (13) captures uncertainty in $\Theta_s | \Psi_s$, while the second captures uncertainty in $\Psi_s$, weighted by $\tilde{\Sigma}_t$, $\Sigma_0$, and the mixing weights $\Gamma$. Note that the action parameters $\Theta_s$ in (12) are not
necessarily centered at latent parameters $\Psi_\ell$, and this shift also appears in the covariance matrix $\hat{\Sigma}$ in (13). To control it, we observe that $\Gamma^\top \Sigma \Gamma = D \otimes I_d$, where $D = (b_i^\top b_j)_{(i,j) \in [K] \times [K]} \in \mathbb{R}^{K \times K}$. Using the Gershgorin circle theorem [47], $\lambda_1(\Gamma^\top \Sigma \Gamma) \leq K \kappa_b$. Similarly, we get $\Gamma^\top \Sigma \Gamma = \sum_{i=1}^K \Gamma_i^\top \Gamma_i$, and in turn $\lambda_1(\Gamma^\top \Sigma \Gamma) \leq K \kappa_b$. This allows us to bound $\mathbb{E} \left[ \sum_{t=1}^n \| \mathbf{a}_t \|^2_{\hat{\Sigma}} \right]$ by $R_{\text{action}}(n) + R_{\text{latent}}$.

## 5 Experiments

We evaluate G-HierTS using synthetic and real-world problems, and compare it to baselines that either ignore or partially use latent parameters. Additional experiments are in Appendix F. In each plot, we report the averages and standard errors of the quantities.

### 5.1 Synthetic experiments

Our first experiments are on a synthetic Gaussian bandit, where we validate our theoretical findings from Section 4. We consider the setting in Section 2.2. The hyper-prior is set as $\mu_\psi = [0]_{Ld}$ and $\Sigma_\psi = 3I_{Ld}$, the conditional action covariance is $\Sigma_{0,i} = I_d$ for all $i \in [K]$, and the observation noise is $\sigma = 1$. We use this setting because learning of the latent parameters is most beneficial when they are more uncertain than the action parameters. The context $X_t$ is sampled uniformly from $[-1, 1]^d$. We run 20 simulations and sample the mixing weights $b_{i,\ell}$ uniformly from $[-1, 1]$ in each run.

In Figure 2, we report regret from three experiments with horizon $n = 2000$, where we vary $K$ and $d$. In each experiment, we compare G-HierTS, its factored posterior approximation G-HierTS-Fa (Section 3.3), and three baselines that we describe next. LinUCB [34] is a linear upper confidence bound algorithm and LinTS [3] is a linear Thompson sampling algorithm. Both ignore the latent structure. To have a fair comparison, we set the marginal prior covariance in LinTS, from which $\theta_{\psi, i}$ is sampled, as $\Sigma_{0,i} = \Sigma_{0,i} + \Gamma_{i} \Sigma_{\psi} \Gamma_{i}^\top \in \mathbb{R}^{d \times d}$ where $\Gamma_{i} = b_i^\top \otimes I_d$. Finally, HierTS [23] can incorporate latent information similarly to G-HierTS, but only has a single latent parameter, whose hyper-prior is set as the distribution of the average of the latent parameters. In all experiments, we observe that G-HierTS and G-HierTS-Fa outperform other baselines that ignore latent parameters or incorporate them partially. A higher $d$ or $K$ implies that the algorithm has to learn more parameters, and the regret increases as predicted by the theoretical analysis. We also highlight that the computational complexity of G-HierTS is comparable to the baselines. As an example, the run times of G-HierTS and LinTS in the first plot are 45 and 35 seconds, respectively.

### 5.2 Experiments on MovieLens dataset

For our second experiment, we study the problem of recommending movies using the MovieLens 1M dataset [29]. This dataset contains 1M ratings given by 6040 users to 3952 movies. We perform low-rank factorization of the rating matrix to obtain $d = 5$ dimensional representation for the users and movies. We use the movies as actions, and cluster them into $L = 5$ clusters using $k$-means. Each cluster center corresponds to a latent parameter $\psi_{\psi, \ell}$. The mixing weight $b_{i,\ell}$ for movie $i$ and cluster $\ell$ is the exponential of the negative squared distance of its vector from the cluster center. For each movie, the weights are normalized to add up to one. The weights capture the fact that a movie can be explained better by a closer cluster,

![Figure 2: Regret of G-HierTS on synthetic bandit problems with varying feature dimension $d$ and number of actions $K$.](image-url)
and we did not try to fine tune this mechanism. We compute the mean of the movie vectors and their variance along each dimension, that we denote by $\mu \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$, respectively. In LinTS, the prior mean of each action is $\mu$ and the covariance is $\text{diag}(v) \in \mathbb{R}^{d \times d}$. In G-HierTS, we set $\mu_\Psi = (\mu)_{e \in [|L|]} \in \mathbb{R}^{Ld}$, $\Sigma_\Psi = 0.75 \text{diag}((\text{diag}(v))_{e \in [|L|]} \in \mathbb{R}^{Ld \times Ld}$ and $\Sigma_{\psi,i} = 0.25 \text{diag}(v) \in \mathbb{R}^{d \times d}$. Roughly speaking, the marginal covariance is as in LinTS while we model that the latent parameters are more uncertain. The context $X_t$ is sampled uniformly from the user vectors. We run 20 simulations with $K = 100$ random movies in each run. The reward is generated according to (3).

We plot the regret of G-HierTS, LinTS, and HierTS up to $n = 5000$ rounds in Figure 3; as LinTS and HierTS are the most competitive baselines in Section 5.1. We observe that G-HierTS achieves lower regret by leveraging the latent structure. This is despite the fact that we did not fine tune the mixing weights. This attests to the robustness of Bayesian models to misspecification, in our case the conditional action priors.

6 Related work

Thompson sampling [46] is a popular exploration algorithm in practice [11, 42]. Russo and Van Roy [40] derived first Bayes regret bounds for TS. We apply TS to two-level graphical models with multiple latent variables. Many recent works [6, 27, 7, 44, 48, 24, 38, 49] applied TS to a two-level hierarchy with a single parent variable, in both meta- and multi-task learning. The main difference in our work is that we consider multiple latent variables, and develop both algorithmic and theory foundations for this setting. Our analysis extends the covariance decompositions proposed in Hong et al. [24] to contextual bandits with multiple latent parameters. Although information theory can be used to derive Bayes regret bounds [41, 36, 7], we are unaware of any for multiple latent variables.

Our setting assumes that there exists an underlying structure among the actions. Many such structures have been proposed and we review some below. In latent bandits [37, 22], a single latent variable indexes multiple candidate models. In structured finite-armed bandits [31, 21], each arm is associated with a known mean function. The mean functions are parameterized by a shared latent parameter, which is learned. TS was also applied to more complex models than in our work, such as graphical models [50] and a discretized parameter space [20]. While these frameworks are general, they are often not guaranteed to be computationally and provably statistically efficient at the same time. Meta- and multi-task learning with UCB algorithms have a long history in bandits [5, 19, 14, 10]. All of these works are frequentist, analyze a stronger notion of regret, and often lead to conservative algorithm designs. In contrast, our approach is Bayesian, we analyze the Bayes regret, and G-HierTS performs well as analyzed without any additional tuning.

7 Conclusion

We propose a hierarchical Bayesian bandit represented by a two-level graphical model where actions can depend on multiple latent parameters. This structure can be used to explore more computationally and sample efficiently, and we design a TS algorithm G-HierTS with these properties. G-HierTS performs well on both synthetic and real-world data, when implemented as analyzed.

Although G-HierTS can be applied to the general model in (1), we only focused on the case where the action parameter is a weighted combination of latent parameters. In Appendix E.1, we provide an extension to mixed linear models where the scalar weights $b_{i,k} \in \mathbb{R}$ are replaced by matrices $C_{i,k} \in \mathbb{R}^{d \times d}$ to capture finer dependencies. We also derive the posterior and Bayes regret bound for this setting. We leave non-linear dependencies and richer distributions for future works. This work also opens the door to designing and analyzing deeper hierarchies with $m$ levels, which we motivate and introduce in Appendix E.2. We expect the corresponding Bayes regret to decompose as $\sqrt{n \left( R_{\text{action}}(n) + \sum_{k=1}^{m} R_{\text{latent}}^k(n) \right)}$, where $R_{\text{action}}(n) = O(Kd)$, $R_{\text{latent}}^k(n) = O(L_k d)$, and $L_k$ is the number of latent parameters in the $k$-th level.
References

[1] Yasin Abbasi-Yadkori, David Pal, and Csaba Szepesvari. Improved algorithms for linear stochastic bandits. In Advances in Neural Information Processing Systems 24, pages 2312–2320, 2011.

[2] Rajeev Agrawal. Sample mean based index policies by o(log n) regret for the multi-armed bandit problem. Advances in Applied Probability, 27(4):1054–1078, 1995.

[3] Shipra Agrawal and Navin Goyal. Thompson sampling for contextual bandits with linear payoffs. In Proceedings of the 30th International Conference on Machine Learning, pages 127–135, 2013.

[4] Peter Auer, Nicolo Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multiarmed bandit problem. Machine Learning, 47:235–256, 2002.

[5] Mohammad Gheshlaghi Azar, Alessandro Lazaric, and Emma Brunskill. Sequential transfer in multi-armed bandit with finite set of models. In Advances in Neural Information Processing Systems 26, pages 2220–2228, 2013.

[6] Hamsa Bastani, David Simchi-Levi, and Ruihao Zhu. Meta dynamic pricing: Transfer learning across experiments. CoRR, abs/1902.10918, 2019. URL https://arxiv.org/abs/1902.10918.

[7] Soumya Basu, Branislav Kveton, Manzil Zaheer, and Csaba Szepesvari. No regrets for learning the prior in bandits. In Advances in Neural Information Processing Systems 34, 2021.

[8] Christopher M Bishop. Pattern Recognition and Machine Learning, volume 4 of Information science and statistics. Springer, 2006.

[9] Djallel Boureffouf and Irina Rish. A survey on practical applications of multi-armed and contextual bandits, 2019.

[10] Leonardo Cella, Alessandro Lazaric, and Massimiliano Pontil. Meta-learning with stochastic linear bandits. In Proceedings of the 37th International Conference on Machine Learning, 2020.

[11] Olivier Chapelle and Lihong Li. An empirical evaluation of Thompson sampling. In Advances in Neural Information Processing Systems 24, pages 2249–2257, 2012.

[12] Wei Chu, Lihong Li, Lev Reyzin, and Robert Schapire. Contextual bandits with linear payoff functions. In Proceedings of the 14th International Conference on Artificial Intelligence and Statistics, pages 208–214, 2011.

[13] Varsha Dani, Thomas Hayes, and Sham Kakade. Stochastic linear optimization under bandit feedback. In Proceedings of the 21st Annual Conference on Learning Theory, pages 355–366, 2008.

[14] Aniket Anand Deshmukh, Urun Dogan, and Clayton Scott. Multi-task learning for contextual bandits. In Advances in Neural Information Processing Systems 30, pages 4848–4856, 2017.

[15] Maria Dimakopoulou, Nikos Vlassis, and Tony Jebara. Marginal posterior sampling for slate bandits. In Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19, pages 2223–2229. International Joint Conferences on Artificial Intelligence Organization, 2019.

[16] Arnaud Doucet, Nando de Freitas, and Neil Gordon. Sequential Monte Carlo Methods in Practice. Springer, New York, NY, 2001.

[17] Audrey Durand, Charis Achilleos, Demetris Iakovides, Katerina Strati, Georgios D. Mitsis, and Joelle Pineau. Contextual bandits for adapting treatment in a mouse model of de novo carcinogenesis. In Proceedings of the 3rd Machine Learning for Healthcare Conference, volume 85, pages 67–82, 2018.

[18] Andrew Gelman, John Carlin, Hal Stern, David Dunson, Aki Vehtari, and Donald Rubin. Bayesian Data Analysis. Chapman & Hall, 2013.
[19] Claudio Gentile, Shuai Li, and Giovanni Zappella. Online clustering of bandits. In Proceedings of the 31st International Conference on Machine Learning, pages 757–765, 2014.

[20] Aditya Gopalan, Shie Mannor, and Yishay Mansour. Thompson sampling for complex online problems. In Proceedings of the 31st International Conference on Machine Learning, pages 100–108, 2014.

[21] Samarth Gupta, Shreyas Chaudhari, Subhojyoti Mukherjee, Gauri Joshi, and Osman Yagan. A unified approach to translate classical bandit algorithms to the structured bandit setting. CoRR, abs/1810.08164, 2018. URL https://arxiv.org/abs/1810.08164.

[22] Joey Hong, Branislav Kveton, Manzil Zaheer, Yinlam Chow, Amr Ahmed, and Craig Boutilier. Latent bandits revisited. In Advances in Neural Information Processing Systems 33, 2020.

[23] Joey Hong, Branislav Kveton, Sumeet Katariya, Manzil Zaheer, and Mohammad Ghavamzadeh. Deep hierarchy in bandits. arXiv preprint arXiv:2202.01454, 2022.

[24] Joey Hong, Branislav Kveton, Manzil Zaheer, and Mohammad Ghavamzadeh. Hierarchical Bayesian bandits. In Proceedings of the 25th International Conference on Artificial Intelligence and Statistics, 2022.

[25] Tommi S. Jaakkola and Michael I. Jordan. Variational probabilistic inference and the qmr-dt network. J. Artif. Int. Res., 10(1):291–322, 1999.

[26] Daphne Koller and Nir Friedman. Probabilistic Graphical Models: Principles and Techniques. MIT Press, Cambridge, MA, 2009.

[27] Branislav Kveton, Mikhail Konobeev, Manzil Zaheer, Chih-Wei Hsu, Martin Mladenov, Craig Boutilier, and Csaba Szepesvari. Meta-Thompson sampling. In Proceedings of the 38th International Conference on Machine Learning, 2021.

[28] Tze Leung Lai. Adaptive treatment allocation and the multi-armed bandit problem. The Annals of Statistics, 15(3):1091–1114, 1987.

[29] Shyong Lam and Jon Herlocker. MovieLens Dataset. http://grouplens.org/datasets/movielens/, 2016.

[30] John Langford and Tong Zhang. The epoch-greedy algorithm for contextual multi-armed bandits. In Advances in Neural Information Processing Systems 20, pages 817–824, 2008.

[31] Tor Lattimore and Remi Munos. Bounded regret for finite-armed structured bandits. In Advances in Neural Information Processing Systems 27, pages 550–558, 2014.

[32] Tor Lattimore and Csaba Szepesvari. Bandit Algorithms. Cambridge University Press, 2019.

[33] Chi-Kwong Li and Roy Mathias. The lidskii-mirsky-wielandt theorem – additive and multiplicative versions. Numerische Mathematik, 81, 2001.

[34] Lihong Li, Wei Chu, John Langford, and Robert Schapire. A contextual-bandit approach to personalized news article recommendation. In Proceedings of the 19th International Conference on World Wide Web, 2010.

[35] Dennis Lindley and Adrian Smith. Bayes estimates for the linear model. Journal of the Royal Statistical Society: Series B (Methodological), 34(1):1–18, 1972.

[36] Xiuyuan Lu and Benjamin Van Roy. Information-theoretic confidence bounds for reinforcement learning. In Advances in Neural Information Processing Systems 32, 2019.

[37] Odalric-Ambrym Maillard and Shie Mannor. Latent bandits. In Proceedings of the 31st International Conference on Machine Learning, pages 136–144, 2014.

[38] Amit Peleg, Naama Pearl, and Ron Meir. Metalearning linear bandits by prior update. In Proceedings of the 25th International Conference on Artificial Intelligence and Statistics, 2022.
[39] Idan Rejwan and Yishay Mansour. Top-k combinatorial bandits with full-bandit feedback. In ALT, pages 752–776, 2020.

[40] Daniel Russo and Benjamin Van Roy. Learning to optimize via posterior sampling. Mathematics of Operations Research, 39(4):1221–1243, 2014.

[41] Daniel Russo and Benjamin Van Roy. An information-theoretic analysis of Thompson sampling. Journal of Machine Learning Research, 17(68):1–30, 2016.

[42] Daniel Russo, Benjamin Van Roy, Abbas Kazerouni, Ian Osband, and Zheng Wen. A tutorial on Thompson sampling. Foundations and Trends in Machine Learning, 11(1):1–96, 2018.

[43] Steven Scott. A modern bayesian look at the multi-armed bandit. Applied Stochastic Models in Business and Industry, 26:639 – 658, 2010.

[44] Max Simchowitz, Christopher Tosh, Akshay Krishnamurthy, Daniel Hsu, Thodoris Lykouris, Miro Dudík, and Robert Schapire. Bayesian decision-making under misspecified priors with applications to meta-learning. In Advances in Neural Information Processing Systems 34, 2021.

[45] Aleksandrs Slivkins. Introduction to multi-armed bandits. Foundations and Trends® in Machine Learning, 12(1-2):1–286, 2019.

[46] William R. Thompson. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. Biometrika, 25(3-4):285–294, 1933.

[47] Richard S. Varga. Gerschgorin and His Circles. 2004.

[48] Runzhe Wan, Lin Ge, and Rui Song. Metadata-based multi-task bandits with Bayesian hierarchical models. In Advances in Neural Information Processing Systems 34, 2021.

[49] Runzhe Wan, Lin Ge, and Rui Song. Towards scalable and robust structured bandits: A meta-learning framework. CoRR, abs/2202.13227, 2022. URL https://arxiv.org/abs/2202.13227.

[50] Tong Yu, Branislav Kveton, Zheng Wen, Ruiyi Zhang, and Ole Mengshoel. Graphical models meet bandits: A variational Thompson sampling approach. In Proceedings of the 37th International Conference on Machine Learning, 2020.
Organization

The supplementary material is organized as follows. In Appendix A, we include a more visual notation and present some preliminary results that we use in our analysis. In Appendix B, we provide a closed-form solution for the hyper-posterior and conditional posteriors in multi-armed bandits with latent variables introduced in Section 2.1. We also include the corresponding approximate hyper-posterior with factored distributions. In Appendix C, we give the proofs for the hyper-posterior and conditional posteriors for all the settings that we considered in this paper. In Appendix D, we prove the Bayes regret upper bound that we provided in Theorem 1. In Appendix E, we discuss in details possible extensions of this work. In Appendix F, we present additional experiments.

A Preliminaries

In this section, we include additional notation and provide some basic properties of matrix operations.

A.1 Notation

For any positive integer $n$, we define $[n] = \{1, 2, \ldots, n\}$. We use $I_d$ to denote the identity matrix of dimension $d \times d$. Unless specified, the $i$-th coordinate of a vector $v$ is $v_i$. When the vector is already indexed, such as $v_i$, we write $v_{ij}$. Similarly, the $(i, j)$-th entry of a matrix $A$ is $A_{ij}$. Let $a_1, \ldots, a_n \in \mathbb{R}^d$ be $n$ vectors. We use $A = [a_1, a_2, \ldots, a_n] \in \mathbb{R}^{d \times n}$ to denote the $d \times n$ matrix obtained by horizontal concatenation of vectors $a_1, \ldots, a_n$ such that the $j$-th column of $A$ is $a_j$ and its $(i, j)$-th entry is $A_{ij} = a_{ij}$. We also denote by $a = (a_i)_{i \in [n]} \in \mathbb{R}^{nd}$ a vector of length $nd$ obtained by concatenation of vectors $a_1, \ldots, a_n$. $\text{Vec}(\cdot)$ denotes the vectorization operator. For instance, we have that $\text{Vec}([a_1, \ldots, a_n]) = (a_i)_{i \in [n]}$. For any matrix $A \in \mathbb{R}^{d \times d}$, we use $\lambda_1(A)$ and $\lambda_d(A)$ to denote the maximum and minimum eigenvalue of $A$, respectively. Let $A_1, \ldots, A_n$ be $n$ matrices of dimension $d \times d$. Then $\text{diag}((A_i)_{i \in [n]}) \in \mathbb{R}^{nd \times nd}$ denotes the block diagonal matrix where $A_1, \ldots, A_n$ are the main-diagonal blocks. Similarly, $(A_i)_{i \in [n]} \in \mathbb{R}^{nd \times d}$ is the $nd \times d$ matrix obtained by concatenation of $A_1, \ldots, A_n$. We use $\otimes$ to denote the Kronecker product.

Now we provide a more visual presentation of the notation above. Let $a_1 \in \mathbb{R}^d, \ldots, a_n \in \mathbb{R}^d$ be $n$ vectors of dimension $d$, and let $A_1 \in \mathbb{R}^{d \times d}, \ldots, A_n \in \mathbb{R}^{d \times d}$ be $n$ matrices of dimension $d \times d$. We have that

$$a_1, \ldots, a_n = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^{d \times n}, \quad (a_i)_{i \in [n]} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^{nd},$$

$$\text{diag}((A_i)_{i \in [n]}) = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_n \end{pmatrix} \in \mathbb{R}^{nd \times nd}, \quad (A_i)_{i \in [n]} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} \in \mathbb{R}^{nd \times d}.$$

Finally, let $A_{i,j} \in \mathbb{R}^{d \times d}$, for $i \in [n]$ and $j \in [m]$ be $nm$ matrices of dimensions $d \times d$. We use $(A_{i,j})_{(i,j) \in [n] \times [m]}$ to denote the $nd \times md$ block matrix where $A_{i,j}$ is the $(i, j)$-th block. We also provide a more visual presentation for this notation.

$$(A_{i,j})_{(i,j) \in [n] \times [m]} = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,m} \end{pmatrix} \in \mathbb{R}^{nd \times md}.$$
A.2 Preliminary results

In this section, we recall some basic properties of matrix operations.

1. The mixed-product property. We have that \((A \otimes B)(C \otimes D) = AC \otimes BD\) for any matrices \(A, B, C, D\) such that the products \(AC\) and \(BD\) exist.

2. Transpose. We have that \((A \otimes B)^\top = A^\top \otimes B^\top\) for any matrices \(A, B\).

3. Vectorization. Let \(A \in \mathbb{R}^{n \times m}, B \in \mathbb{R}^{m \times p}\), we have that the following equalities hold
\[
\text{Vec}(AB) = (I_p \otimes A) \text{Vec}(B) = (B^\top \otimes I_n) \text{Vec}(A).
\]

4. For any matrix \(A\), we have that \(I_1 \otimes A = A\).

5. For any matrices \(A\) and \(B\), we have that \(\lambda_1(A \otimes B) = \lambda_1(A)\lambda_1(B)\).

6. From [33], we have that for any matrix \(A\) and any positive semi-definite matrix \(B\) such that the product \(A^\top B A\) exists, the following inequality holds \(\lambda_1(A^\top B A) \leq \lambda_1(B)\lambda_1(A^\top A)\).

B Exact and approximate posteriors for multi-armed bandits

In this section, we provide the hyper-posterior and conditional posteriors for the multi-armed bandit setting introduced in Section 2.1 and summarized in (2). We also provide the approximation of the corresponding hyper-posterior with factored distributions. Note that the results below can be derived from Propositions 1 and 2 by setting \(d = 1\) and \(X_t = 1\) for all \(t \in [n]\).

B.1 Exact posterior for multi-armed bandits with latent variables

Fix round \(t \in [n]\), and recall that \(S_{t,i}\) are the rounds where action \(i\) is taken up to round \(t\). We introduce \(N_{t,i} = |S_{t,i}|\) as the number of times that action \(i\) is taken up to round \(t\) and \(B_{t,i} = \sum_{\ell \in S_{t,i}} Y_{t,\ell}\) as the total reward of action \(i\) up to round \(t\). Note that \(B_{t,i}\) and weight vectors \(b_i\) are unrelated. We derive in (14) and (15) the hyper-posterior \(Q_t\) and the conditional posteriors \(P_{t,i}\) for the model in (2).

Proposition 3. For any round \(t \in [n]\), the joint hyper-posterior is a multivariate Gaussian \(Q_t = \mathcal{N}(\mu_t, \Sigma_t)\), where
\[
\Sigma_t^{-1} = \Sigma_\Psi^{-1} + \sum_{i \in [K]} \frac{N_{t,i}}{N_{t,i} \sigma^2_{0,i} + \sigma^2} b_i b_i^\top, \quad \mu_t = \Sigma_t \left( \Sigma_\Psi^{-1} \mu_{\Psi} + \sum_{i \in [K]} \frac{B_{t,i}}{N_{t,i} \sigma^2_{0,i} + \sigma^2} b_i \right).
\]

Moreover, for any action \(i \in [K]\), and latent parameters \(\Psi_t\), the conditional posterior is a univariate Gaussian \(P_{t,i}(\cdot | \Psi_t) = \mathcal{N}(\cdot | \hat{\mu}_{t,i}, \hat{\sigma}^2_{t,i})\), where
\[
\hat{\sigma}^2_{t,i} = \frac{1}{\sigma^2} + \frac{N_{t,i}}{\sigma^2}, \quad \hat{\mu}_{t,i} = \frac{\Psi_t b_i}{\sigma^2} + \frac{B_{t,i}}{\sigma^2}.
\]

The hyper-posterior is additive in actions and can be interpreted as follows. Each action is a single noisy observation in its estimate. The maximum likelihood estimate (MLE) of the mean reward of action \(i\), \(B_{t,i}/N_{t,i}\), contributes to (14) proportionally to its precision, \(N_{t,i}/(N_{t,i} \sigma^2_{0,i} + \sigma^2)\). The contribution is weighted by \(b_i\), which are the weights used to generate \(\theta_{s,i}\). The conditional posterior in (15) has a standard form. Note that its prior mean depends on latent parameters \(\Psi_t\), which are sampled.

B.2 Approximate posterior for multi-armed bandits with latent variables

Consider the model in (2) and assume that the joint hyper-posterior \(Q_t\) in (14) factorizes. Specifically, \(Q_t(\Psi) = \prod_{\ell=1}^L Q_{t,\ell}(\psi_{t,\ell})\), where \(Q_{t,\ell}\) is the hyper-posterior of the \(\ell\)-th latent parameter \(\psi_{s,\ell}\). It follows that for any round \(t \in [n]\), the hyper-posterior \(Q_{t,\ell}\) is a univariate Gaussian
\[ Q_{t, \ell} = \mathcal{N}(\mu_{t, \ell}, \Sigma_{t, \ell}) \]

where

\[
\bar{\sigma}_{t, \ell}^{-2} = \sigma_{\psi}^{-2} + \sum_{i \in [K]} b_{i, \ell}^2 \frac{N_{i, i}}{\sigma_{0, i}^2 + \sigma^2},
\]

\[
\bar{\mu}_{t, \ell} = \sigma_{t, \ell}^{-2} \left( \sigma_{\psi}^{-2} \mu_{\psi} + \sum_{i \in [K]} b_{i, \ell} \frac{B_{i, i}}{N_{i, i} \sigma_{0, i}^2 + \sigma^2} \right).
\]

\( \sigma_{\psi}^2 > 0 \) is the \( \ell \)-th diagonal entry of \( \Sigma_{\psi} \), and \( \mu_{\psi} \) is the \( \ell \)-th entry of \( \mu_{\psi} \). The above formulation allows for individual sampling of the latent parameters, which improves the space and time complexity. After sampling \( \psi_{t, \ell} \sim Q_{t, \ell} \) for all \( \ell \in [L] \), we set \( \Psi_{t} = (\psi_{t, \ell})_{\ell \in [L]} \) and use the conditional posterior as expressed in (15).

C Posterior derivations

In this section, we provide the derivations of the hyper-posterior and conditional posteriors for the setting introduced in Section 2.2 and summarized in (3).

C.1 Exact posterior derivations

Here, we present the proof for Proposition 1 in Appendix C.1.1, and then the proof of Proposition 2 in Appendix C.1.2.

C.1.1 Exact hyper-posterior

Proof of Proposition 1 (derivation of \( Q_t \)). First, from basic properties of matrix operations in Appendix A.2, we have that \( \sum_{\ell \in [L]} b_{i, \ell} \psi_{\ell} = \Gamma_i \Psi \) where \( \Psi = (\psi_{\ell})_{\ell \in [L]} \in \mathbb{R}^{L \times d} \) and \( \Gamma_i = b_i^\top \otimes I_d \).

Thus our model can be written as follows:

\[
\Psi_s \sim \mathcal{N}(\mu_{\psi}, \Sigma_{\psi}) , \\
\theta_{s, i} \mid \Psi_s \sim \mathcal{N}(\Gamma_i \Psi, \Sigma_{0, i}) , \\
Y_{\ell} \mid X_{\ell}, \theta_{s, A_{\ell}} \sim \mathcal{N}(X_{\ell}^\top \theta_{s, A_{\ell}}, \sigma^2) , \\
\forall i \in [K] , \forall \ell \in [t] .
\]

(17)

It follows that the joint hyper-posterior in round \( t \) reads,

\[
Q_t(\Psi) \propto \prod_{i \in [K]} \int_{\theta_i} L_{t, i}(\theta_i) \mathcal{N}(\theta_i; \Gamma_i \Psi, \Sigma_{0, i}) \, d\theta_i \mathcal{N}(\Psi; \mu_{\psi}, \Sigma_{\psi}) , \\
= \prod_{i \in [K]} \int_{\theta_i} \left( \prod_{\ell \in S_t, i} \mathcal{N}(Y_{\ell}; X_{\ell}^\top \theta_{s, i}, \sigma^2) \right) \mathcal{N}(\theta_i; \Gamma_i \Psi, \Sigma_{0, i}) \, d\theta_i \mathcal{N}(\Psi; \mu_{\psi}, \Sigma_{\psi}) .
\]

(18)

Now we compute the quantity \( \int_{\theta_i} \left( \prod_{\ell \in S_t, i} \mathcal{N}(Y_{\ell}; X_{\ell}^\top \theta_{s, i}, \sigma^2) \right) \mathcal{N}(\theta_i; \Gamma_i \Psi, \Sigma_{0, i}) \, d\theta_i \) using Lemma 1. Precisely, we obtain that it is proportional to \( \mathcal{N}(\Psi; \bar{\mu}_{t, i}, \Sigma_{t, i}) \) where

\[
\Sigma_{t, i}^{-1} = \Gamma_i^\top (\Sigma_{0, i} + G_{t, i}^{-1})^{-1} \Gamma_i ,
\]

\[
\bar{\mu}_{t, i} = \Sigma_{t, i} (\Gamma_i^\top (\Sigma_{0, i} + G_{t, i}^{-1})^{-1} G_{t, i}^{-1} B_{t, i}) ,
\]

and

\[
G_{t, i} = \sigma^{-2} \sum_{\ell \in S_t, i} X_{\ell} X_{\ell}^\top ,
\]

\[
B_{t, i} = \sigma^{-2} \sum_{\ell \in S_t, i} Y_{\ell} X_{\ell} .
\]

This means that the hyper-posterior \( Q_t \) is the product of \( K + 1 \) multivariate Gaussian distributions \( \mathcal{N}(\mu_{\psi}, \Sigma_{\psi}), \mathcal{N}(\mu_{t, 1}, \Sigma_{t, 1}), \ldots, \mathcal{N}(\mu_{t, K}, \Sigma_{t, K}) \). Thus, the hyper-posterior \( Q_t \) is also a multivariate
Gaussian distribution $\mathcal{N}(\tilde{\mu}_t, \Sigma_t^{-1})$, where

$$\Sigma_t^{-1} = \Sigma_\Psi^{-1} + \sum_{i=1}^K \Sigma_{t,i}^{-1} = \Sigma_\Psi^{-1} + \sum_{i=1}^K \Gamma_i^T (\Sigma_{0,i} + G_{t,i}^{-1})^{-1} \Gamma_i,$$

$$\tilde{\mu}_t = \bar{\Sigma}_t \left( \Sigma_\Psi^{-1} \mu_\Psi + \sum_{i=1}^K \Gamma_i^T (\Sigma_{0,i} + G_{t,i}^{-1})^{-1} G_{t,i} B_{t,i} \right).$$

Moreover, we use the properties in Appendix A.2 and that $\Gamma_i = b_i^T \otimes I_d$ to rewrite the terms as

$$\Gamma_i^T (\Sigma_{0,i} + G_{t,i}^{-1})^{-1} \Gamma_i = b_i b_i^T \otimes (\Sigma_{0,i} + G_{t,i}^{-1})^{-1},$$

$$\Gamma_i^T (\Sigma_{0,i} + G_{t,i}^{-1})^{-1} G_{t,i} B_{t,i} = b_i \otimes ((\Sigma_{0,i} + G_{t,i}^{-1})^{-1} G_{t,i} B_{t,i}).$$

This concludes the proof. 

To reduce clutter, we fix an action $i \in [K]$ and a round $t \in [n]$ and drop sub-indexing by $i$ and $t$ in the following lemma. We stress that in this lemma, $\Gamma$ and $\Sigma_0$ refer to $\Gamma_i$ and $\Sigma_{0,i}$ for some $i \in [K]$, respectively, as opposed to the rest of the paper where $\Gamma = (\Gamma_i)_{i \in [K]}$ and $\Sigma_0 = \text{diag}((\Sigma_{0,i})_{i \in [K]}).$

In summary, there exist $i \in [K]$ and $t \in [n]$ such that we have the following correspondences:

$$\Gamma \leftarrow \Gamma_i, \quad \Sigma_0 \leftarrow \Sigma_{0,i}, \quad N \leftarrow N_{t,i}, \quad \theta \leftarrow \theta_i, \quad (X_\ell, Y_\ell)_{\ell \in [N]} \leftarrow (X_{\ell,i}, Y_{\ell,i})_{\ell \in S_{t,i}}.$$  

Lemma 1 (Gaussian posterior update). Let $\Gamma \in \mathbb{R}^{d \times 2d}, \Sigma_0 \in \mathbb{R}^{d \times d}$, and $\sigma^2 > 0$ then we have that

$$\int_\theta \left( \prod_{\ell=1}^N \mathcal{N}(Y_\ell; X_\ell^T \theta, \sigma^2) \right) \mathcal{N}(\theta; \Gamma, \Sigma_0) \, d\theta \propto \mathcal{N}(\Psi; \mu_N, \Sigma_N).$$

where

$$\Sigma_N^{-1} = \Gamma^T (\Sigma_0 + G_N^{-1})^{-1} \Gamma,$$

$$\mu_N = \Sigma_N \left( \Gamma^T (\Sigma_0 + G_N^{-1})^{-1} G_N^{-1} B_N \right),$$

and

$$G_N = \sigma^{-2} \sum_{k=1}^N X_k X_k^T, \quad B_N = \sigma^{-2} \sum_{k=1}^N Y_k X_k.$$

Proof. Let $v = \sigma^{-2}, \quad \Lambda_0 = \Sigma_0^{-1}.$ We denote the integral in the lemma by $f(\Psi)$. It follows that

$$f(\Psi) = \int_\theta \left( \prod_{\ell=1}^N \mathcal{N}(Y_\ell; X_\ell^T \theta, \sigma^2) \right) \mathcal{N}(\theta; \Gamma, \Sigma_0) \, d\theta,$$

$$\propto \int_\theta \exp \left[ -\frac{1}{2} v \sum_{\ell=1}^N (Y_\ell - X_\ell^T \theta)^2 - \frac{1}{2} (\theta - \Gamma \Psi) \Lambda_0 (\theta - \Gamma \Psi) \right] \, d\theta,$$

$$= \int_\theta \exp \left[ -\frac{1}{2} \left( v \sum_{\ell=1}^N (Y_\ell - 2Y_\ell^T X_\ell + (\theta^T X_\ell)^2) + \theta^T \Lambda_0 \theta - 2\theta^T \Lambda_0 \Gamma \Psi + (\Gamma \Psi)^T \Lambda_0 (\Gamma \Psi) \right) \right] \, d\theta,$$

$$\propto \int_\theta \exp \left[ -\frac{1}{2} \left( \theta^T \left( v \sum_{\ell=1}^N X_\ell X_\ell^T + \Lambda_0 \right) \theta - 2\theta^T \left( v \sum_{\ell=1}^N Y_\ell X_\ell + \Lambda_0 \Gamma \Psi \right) + (\Gamma \Psi)^T \Lambda_0 (\Gamma \Psi) \right) \right] \, d\theta.$$

To reduce clutter, let

$$G_\ell = v \sum_{\ell=1}^N X_\ell X_\ell^T, \quad V_N = (G_N + \Lambda_0)^{-1}, \quad U_N = V_N^{-1},$$

$$B_N = v \sum_{\ell=1}^N Y_\ell X_\ell \quad \text{and} \quad \beta_N = V_N (B_N + \Lambda_0 \Gamma \Psi).$$

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We have that $U_N V_N = V_N U_N = I_d$, and thus
\[
f(\Psi) \propto \int_{\theta} \exp \left[ -\frac{1}{2} \left( \theta^T U_N \theta - 2 \theta^T U_N V_N (B_N + \Lambda_0 \Gamma \Psi) + (\Gamma \Psi)^T \Lambda_0 (\Gamma \Psi) \right) \right] d\theta,
\]
\[
= \int_{\theta} \exp \left[ -\frac{1}{2} \left( \theta^T U_N \theta - 2 \theta^T U_N \beta_N + (\Gamma \Psi)^T \Lambda_0 (\Gamma \Psi) \right) \right] d\theta,
\]
\[
= \int_{\theta} \exp \left[ -\frac{1}{2} \left( (\theta - \beta_N)^T U_N (\theta - \beta_N) - \beta_N^T U_N \beta_N + (\Gamma \Psi)^T \Lambda_0 (\Gamma \Psi) \right) \right] d\theta,
\]
\[
\propto \exp \left[ -\frac{1}{2} \left( -\beta_N^T U_N \beta_N + (\Gamma \Psi)^T \Lambda_0 (\Gamma \Psi) \right) \right],
\]
\[
= \exp \left[ -\frac{1}{2} \left( -(B_N + \Lambda_0 \Gamma \Psi)^T V_N (B_N + \Lambda_0 \Gamma \Psi) + (\Gamma \Psi)^T \Lambda_0 (\Gamma \Psi) \right) \right],
\]
\[
\propto \exp \left[ -\frac{1}{2} \left( \Psi^T (\Lambda_0 - \Lambda_0 V_N \Lambda_0) \Gamma - 2 \Psi^T (\Gamma^T \Lambda_0 V_N B_N) \right) \right],
\]
\[
= \exp \left[ -\frac{1}{2} \Psi^T \Sigma_N^{-1} \Psi + \Psi^T \Sigma_N^{-1} \mu_N \right],
\]
where
\[
\Sigma_N^{-1} = \Gamma^T (\Lambda_0 - \Lambda_0 V_N \Lambda_0) \Gamma = \Gamma^T (\Lambda_0^{-1} + G^{-1}_N)^{-1} \Gamma,
\]
\[
\Sigma_N^{-1} \mu_N = (\Gamma^T \Lambda_0 V_N B_N) = \Gamma^T (\Sigma_0 + G^{-1}_N)^{-1} G^{-1}_N B_N.
\]

We use the Woodbury matrix identity to get the second equalities which concludes the proof. \hfill \Box

### C.1.2 Exact conditional posterior

**Proof of Proposition 2 (Derivation of $P_{t,i}$).** This proposition is a direct application Lemma 2; in which case we get that the posterior $P_{t,i}$ is a multivariate Gaussian distribution $N(\tilde{\mu}_{t,i}, \tilde{\Sigma}_{t,i})$, where
\[
\tilde{\Sigma}_{t,i}^{-1} = G_{t,i} + \Sigma_{0,i}^{-1},
\]
\[
\tilde{\mu}_{t,i} = \tilde{\Sigma}_{t,i} \left( B_{t,i} + \Sigma_{0,i}^{-1} \sum_{\ell=1}^{L} b_{i,\ell} \psi_{t,\ell} \right).
\]

To reduce clutter, we consider a fixed action $i \in [K]$ and round $t \in [n]$, and drop subindexing by $t$ and $i$ in Lemma 2. In summary, there exist $i \in [K]$ and $t \in [n]$ such that we have the following correspondences:
\[
b_\ell \leftrightarrow b_{i,\ell}, \quad \Sigma_0 \leftrightarrow \Sigma_{0,i}, \quad N \leftrightarrow N_{t,i}, \quad \theta \leftrightarrow \theta_i, \quad (X_\ell, Y_\ell)_{\ell \in [N]} \leftrightarrow (X_{t,\ell}, Y_{t,\ell})_{\ell \in s_{t,i}}.
\]

**Lemma 2.** Consider the following model:

- $\theta | \Psi \sim N \left( \sum_{\ell=1}^{L} b_{\ell} \psi_{t,\ell}, \Sigma_0 \right),$
- $Y_\ell | X_\ell, \theta \sim N \left( X_\ell^T \theta, \sigma^2 \right), \quad \forall \ell \in [N].$

Let $H = \{X_1, Y_1, \ldots, X_N, Y_N\}$ then we have that $P(\theta_* = \theta | \Psi_* = \Psi, H) = N \left( \theta; \tilde{\mu}_N, \tilde{\Sigma}_N \right)$, where
\[
\tilde{\Sigma}_N^{-1} = \sigma^{-2} \sum_{\ell=1}^{N} X_\ell X_\ell^T + \Sigma_0^{-1},
\]
\[
\tilde{\mu}_N = \tilde{\Sigma}_N \left( \sigma^{-2} \sum_{\ell=1}^{N} X_\ell Y_\ell + \Sigma_0^{-1} \sum_{\ell=1}^{L} b_{i,\ell} \psi_{t,\ell} \right).
\]
Proof. Let \( v = \sigma^{-2} \), \( \Lambda_0 = \Sigma_0^{-1} \). Then the conditional posterior decomposes as

\[
P(\theta_* = \theta | \Psi_* = \Psi, H) 
\propto \prod_{\ell=1}^N \mathcal{N}(Y_{\ell}; X_{\ell}^T \theta, \sigma^2) \mathcal{N}(\theta; \psi_{\ell} \psi_{\ell}; \Sigma_0),
\]

where

\[
\begin{align*}
&= \exp \left[ -\frac{1}{2} \left( v \sum_{\ell=1}^N (Y_{\ell}^2 - 2Y_{\ell} X_{\ell}^T \theta + (X_{\ell}^T \theta)^2) + \theta^T \Lambda_0 \theta - 2\theta^T \Lambda_0 \sum_{\ell=1}^L \psi_{\ell} \psi_{\ell} + \sum_{\ell=1}^L \psi_{\ell} \psi_{\ell} \right) \right], \\
&\propto \exp \left[ -\frac{1}{2} \left( \theta^T (v \sum_{\ell=1}^N X_{\ell} X_{\ell}^T + \Lambda_0) \theta - 2\theta^T \sum_{\ell=1}^L \psi_{\ell} \right) \right], \\
&\propto \mathcal{N} \left( \theta; \tilde{\mu}_N, (\tilde{\Lambda}_N)^{-1} \right),
\end{align*}
\]

where

\[
\tilde{\Lambda}_N = v \sum_{\ell=1}^N X_{\ell} X_{\ell}^T + \Lambda_0,
\]

\[
\tilde{\Lambda}_N \tilde{\mu}_N = v \sum_{\ell=1}^N X_{\ell} Y_{\ell} + \Lambda_0 \sum_{\ell=1}^L \psi_{\ell} \psi_{\ell}.
\]

\qed

C.2 Approximate hyper-posterior

To reduce clutter in the following proof, we consider a fixed round \( t \in [n] \), and drop subindexing by \( t \). It follows that \( Q = \mathcal{N}(\tilde{\mu}, \tilde{\Sigma}) \) corresponds to the hyper-posterior \( Q_t = \mathcal{N}(\tilde{\mu}_t, \tilde{\Sigma}_t) \) for some round \( t \).

**Proof of approximate posterior in Section 3.3.** Here we restrict the family of hyper-posteriors \( Q \) to factored distributions. Precisely, we first partition the elements of \( \Psi_* = (\psi_{*, \ell})_{\ell \in [L]} \) into \( L \) disjoint \( d \)-dimensional groups where each group corresponds to a latent parameter \( \psi_{*, \ell} \). We then suppose that \( Q \) factorizes across the \( L \) latent parameters, that is \( Q(\Psi) = \prod_{\ell \in [L]} Q(\psi_{\ell}) \), where \( Q(\psi_{\ell}) \) are obtained using variational inference techniques [8] as we show next. First, we know that \( Q(\Psi) = \mathcal{N}(\tilde{\mu}, \tilde{\Sigma}) \), where \( \tilde{\mu} \in \mathbb{R}^{Ld} \) and \( \tilde{\Sigma} \in \mathbb{R}^{Ld \times Ld} \). We write the mean and covariance by blocks as \( \tilde{\mu} = (\tilde{\mu}_t)_{t \in [L]} \) and \( \tilde{\Sigma} = (\Sigma_{t,i,j})_{(i,j) \in [L] \times [L]} \), such that \( \tilde{\mu}_t \in \mathbb{R}^d \) and \( \Sigma_{t,i,j} \in \mathbb{R}^{d \times d} \). Now fix \( t \in [L] \), from know results [8] the optimal factor \( Q_t(\psi_{\ell}) \) that optimizes the Kullback-Leibler divergence satisfies

\[
Q_t(\psi_{\ell}) \propto \exp \left( E_{j \neq \ell} \log Q(\Psi) \right),
\]

where \( E_{j \neq \ell} [:] \) denotes an expectation with respect to the distributions \( Q_j \) such that \( j \neq \ell \). Let \( \Lambda = \Sigma^{-1} = (\Lambda_{t,i,j})_{(i,j) \in [L] \times [L]} \), the expectation can be computed as

\[
Q_t(\psi_{\ell}) \propto \exp \left( E_{j \neq \ell} \left[ -\frac{1}{2} (\psi_{\ell} - \tilde{\mu}_t)^T \Lambda_{t,i,j} (\psi_{j} - \tilde{\mu}_j) \right] \right),
\]

\[
\propto \exp \left( E_{j \neq \ell} \left[ -\frac{1}{2} \psi_{\ell}^T \Lambda_{t,i,j} \psi_{j} + \psi_{\ell}^T \Lambda_{t,i,j} \tilde{\mu}_j - \psi_{\ell}^T \sum_{j \neq \ell} \Lambda_{t,i,j} (E[\psi_{j}] - \tilde{\mu}_j) \right] \right),
\]

\[
\propto \exp \left( E_{j \neq \ell} \left[ -\frac{1}{2} \psi_{\ell}^T \Lambda_{t,i,j} \psi_{j} + \psi_{\ell}^T \left( \Lambda_{t,i,j} \tilde{\mu}_j - \sum_{j \neq \ell} \Lambda_{t,i,j} (E[\psi_{j}] - \tilde{\mu}_j) \right) \right] \right). \quad (20)
\]

Thus, we have that

\[
Q_t(\psi_{\ell}) = \mathcal{N}(\psi_{\ell}; m_{\ell}, \Sigma_{t,i,j}),
\]

(21)
We stress that matrix $\Psi$ depends on the other optimal factors $Q_j$ for $j \neq \ell$. However, we can provide a closed-form solution in this Gaussian case if we set $m_\ell = \mathbb{E} [\psi_\ell] = \tilde{\mu}_\ell$ for all $\ell \in [L]$; in which case we get that $Q_\ell(\psi_\ell) = \mathcal{N}(\tilde{\mu}_\ell, \tilde{\Sigma}_\ell)$, where $\tilde{\mu}_\ell \in \mathbb{R}^d$ and $\tilde{\Sigma}_\ell \in \mathbb{R}^{d \times d}$ are such that $\tilde{\mu} = (\tilde{\mu}_\ell)_{\ell \in [L]}$ and $\tilde{\Sigma} = (\tilde{\Sigma}_{\ell,j})_{\ell,j \in [L]}$. Finally, to get the results in Section 3.3, we simply retrieve the respective $\tilde{\mu}_\ell \in \mathbb{R}^d$ and $\tilde{\Sigma}_{\ell,j} \in \mathbb{R}^{d \times d}$ from the mean and covariance of the exact posterior given in Proposition 1.

\[\Box\]

## D Regret proofs

In this section, we prove Theorem 1. To do so, we first provide a compact formulation of our problem in Appendix D.1. Next, we use the total covariance decomposition formula to derive the covariance in Appendix D.2. Finally, we provide some preliminary eigenvalues inequalities in Appendix D.3 to proceed with the regret proof in Appendix D.4.

### D.1 Problem reformulation for regret analysis

Here, we aim at rewriting our model in a compact form to simplify regret analysis. We first introduce $K$ random multivariate Gaussian variables $Z_i \sim \mathcal{N}(0, \Sigma_{0,i})$ for $i \in [K]$, and the following matrices

\[
\Psi_{\text{mat},*} = [\psi_{*,1}, \ldots, \psi_{*,K}] \in \mathbb{R}^{d \times L}, \quad B = [b_1, b_2, \ldots, b_K] \in \mathbb{R}^{L \times K},
\]

\[
\Theta_{\text{mat},*} = [\theta_{*,1}, \theta_{*,2}, \ldots, \theta_{*,K}] \in \mathbb{R}^{d \times K}, \quad Z_{\text{mat}} = [Z_1, Z_2, \ldots, Z_K] \in \mathbb{R}^{d \times K}.
\]

We stress that matrix $B$ and vector $B_{t,i}$ are unrelated. Now the results of vectorization of matrices $B$ and $Z_{\text{mat}}$ are denoted by $b$ and $Z$, respectively. Clearly, the results of vectorization of $\Theta_{\text{mat},*}$ and $\Psi_{\text{mat},*}$ are $\Theta_*$ and $\Psi_*$ as defined in Section 2

\[
\Psi_* = \text{Vec}(\Psi_{\text{mat},*}), \quad \Theta_* = \text{Vec}(\Theta_{\text{mat},*}), \quad Z = \text{Vec}(Z_{\text{mat}}), \quad b = \text{Vec}(B).
\]

Now we notice that $\sum_{\ell=1}^L b_i \psi_{*,\ell} = \Psi_{\text{mat},*} b_i$ and thus given matrix $\Psi_{\text{mat},*}$ we have that

\[
\theta_{*,i} = \Psi_{\text{mat},*} b_i + Z_i, \quad \forall i \in [K].
\]

Concatenating all the $K$ vectors in (22) horizontally leads to

\[
\Theta_{\text{mat},*} = \Psi_{\text{mat},*} B + Z_{\text{mat}}.
\]

We vectorize (23) to obtain

\[
\Theta_* = \Gamma \Psi_* + Z,
\]

It follows that

\[
\Theta_* \mid \Psi_* \sim \mathcal{N}(\Gamma \Psi_* , \Sigma_0),
\]

where $\Gamma = B^\top \otimes I_d \in \mathbb{R}^{Kd \times Ld}$ and $\Sigma_0 = \text{diag}((\Sigma_{0,i})_{i \in [K]}) \in \mathbb{R}^{Kd \times Kd}$. Also, note that $\Gamma = (\Gamma_i)_{i \in [K]}$ where $\Gamma_i = b_i^\top \otimes I_d$.

In round $t$, action $A_t$ and context $X_t$ are represented jointly by a vector $A_t = (\mathbb{1}_{\{A_t = i\}} X_t)_{i \in [K]} \in \mathbb{R}^{Kd}$. Similarly, the optimal action $A_{*,t}$ and context $X_t$ are also represented jointly by a vector $A_{*,t} = (\mathbb{1}_{\{A_{*,t} = i\}} X_t)_{i \in [K]} \in \mathbb{R}^{Kd}$.

\[
A_t = \begin{pmatrix}
\mathbb{1}_{\{A_t = 1\}} X_t \\
\mathbb{1}_{\{A_t = 2\}} X_t \\
\vdots \\
\mathbb{1}_{\{A_t = K\}} X_t
\end{pmatrix} \in \mathbb{R}^{Kd}, \quad A_{*,t} = \begin{pmatrix}
\mathbb{1}_{\{A_{*,t} = 1\}} X_t \\
\mathbb{1}_{\{A_{*,t} = 2\}} X_t \\
\vdots \\
\mathbb{1}_{\{A_{*,t} = K\}} X_t
\end{pmatrix} \in \mathbb{R}^{Kd}.
\]
We stress that the joint vector representations of actions and contexts are written in bold letters, as opposed to the index representations of actions which are written in regular letters.

Now we can rewrite the model in (3) as

\[
\begin{align*}
\Psi_\ast & \sim \mathcal{N}(\mu_\Psi, \Sigma_\Psi), \\
\Theta_\ast | \Psi_\ast & \sim \mathcal{N}(\Gamma \Psi_\ast, \Sigma_0), \\
Y_t | A_t, \Theta_\ast & \sim \mathcal{N}(A_t^T \Theta_\ast, \sigma^2).
\end{align*}
\]

(D.2) Derivation of \( P(\Theta_\ast = \Theta | H_t) \)

We introduce the following terms:

\[
G_t = \sigma^{-2} \sum_{i=1}^{t} A_i A_i^T, \quad B_t = \sigma^{-2} \sum_{i=1}^{t} Y_i A_t,
\]

\[
G_{t,i} = \sigma^{-2} \sum_{\ell \in S_{t,i}} X_{\ell}X_{\ell}^T, \quad B_{t,i} = \sigma^{-2} \sum_{\ell \in S_{t,i}} Y_{\ell}X_{\ell}.
\]

Recall that \( \Gamma_i = b_i^T \otimes I_d \in \mathbb{R}^{d \times Ld} \). It follows that \( G_t = \text{diag}((G_{t,i})_{i \in [K]}) \in \mathbb{R}^{Kd \times Kd} \), and

\[
B_t = (B_{t,i})_{i \in [K]} = \begin{pmatrix} B_{t,1} & B_{t,2} & \cdots & B_{t,K} \end{pmatrix} \in \mathbb{R}^{Kd}, \quad \Gamma_i = (\Gamma_i)_{i \in [K]} = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_K \end{pmatrix} \in \mathbb{R}^{Kd \times Ld}.
\]

Lemma 3 (Covariance of \( P(\Theta_\ast = \Theta | H_t) \)). Consider the model in (26) and let \( \Psi_\ast | H_t \sim \mathcal{N}(\bar{\mu}_t, \bar{\Sigma}_t) \), then we have

\[
\hat{\Sigma}_t = \text{cov} [\Theta_\ast | H_t] = \bar{\Sigma}_t + \bar{\Sigma}_t \Sigma_0^{-1} \Gamma \bar{\Sigma}_t^\top \Sigma_0^{-1} \bar{\Sigma}_t,
\]

where \( \hat{\Sigma}_t = \text{cov} [\Theta_\ast | \Psi_\ast, H_t] = (G_t + \Sigma_0^{-1})^{-1} \).

Proof. It follows from Proposition 2, the fact that actions are independent given latent parameters, and basic block matrices operations combined with (27) that

\[
\text{cov} [\Theta_\ast | \Psi_\ast, H_t] = \left( \sum_{i=1}^{t} A_i A_i^T + \Lambda_0 \right)^{-1} = (G_t + \Lambda_0)^{-1}
\]

\[
\mathbb{E} [\Theta_\ast | \Psi_\ast, H_t] = \text{cov} [\Theta_\ast | \Psi_\ast, H_t] \left( \sum_{i=1}^{t} Y_i A_t + \Lambda_0 \Gamma \Psi_\ast \right) = \text{cov} [\Theta_\ast | \Psi_\ast, H_t] (B_t + \Lambda_0 \Gamma \Psi_\ast)
\]

Note that \( \text{cov} [\Theta_\ast | \Psi_\ast, H_t] = (G_t + \Lambda_0)^{-1} \) does not depend on \( \Psi_\ast \), and by definition \( B_t \) is constant conditioned on \( H_t \). Thus we have that

\[
\mathbb{E} [\text{cov} [\Theta_\ast | \Psi_\ast, H_t] | H_t] = \text{cov} [\Theta_\ast | \Psi_\ast, H_t] = (G_t + \Lambda_0)^{-1}
\]

and

\[
\text{cov} [\mathbb{E} [\Theta_\ast | \Psi_\ast, H_t] | H_t] = \text{cov} [\text{cov} [\Theta_\ast | \Psi_\ast, H_t] \Lambda_0 \Gamma \Psi_\ast | H_t]
\]

\[
= (G_t + \Lambda_0)^{-1} \Lambda_0 \Gamma \text{cov} [\Psi_\ast | H_t] \Gamma^\top \Lambda_0 (G_t + \Lambda_0)^{-1}
\]

\[
= (G_t + \Lambda_0)^{-1} \Lambda_0 \Gamma \bar{\Sigma}_t^\top \Lambda_0 (G_t + \Lambda_0)^{-1}.
\]

Finally, total covariance decomposition concludes the proof. \( \square \)
We consider our model rewritten in (26) and the notation introduced in Appendices D.1 and D.2. As we explained in Section 4.2, the joint posterior distribution of all action parameters where \( \Theta \) is a standard normal variable. Thus, the second term in (33) can be bounded as
\[
\text{Fix history } A^*, t = \arg\max_{a \in A_t} \langle a^\top, \Theta^* - \hat{\Theta}_t \rangle \quad \text{and let the Cauchy–Schwarz inequality to obtain}
\]

\[
\lambda_1(\Gamma^\top) = \lambda_1(D) .
\]

Now let \( \kappa_b = \max_{i \in [K]} \|b_i\|^2 \). We use the Gershgorin circle theorem [47] to prove that there exists \( i \in [K] \) such that \( \lambda_1(D) \leq \|b_i\|^2 + \sum_{j \neq i} \|b_i\||b_j| \leq K \kappa_b \).

Similarly, we have \( \Gamma^\top \Gamma = \sum_{i=1}^K \Gamma_i^\top \Gamma_i \). Thus using Weyl’s inequality we get that
\[
\lambda_1(\Gamma^\top) \leq \sum_{i=1}^K \lambda_1(\Gamma_i^\top) \leq \sum_{i=1}^K \|b_i\|^2 \leq K \kappa_b .
\]

Moreover, we have that
\[
\lambda_{1, 0} = \lambda_1(\Sigma_0) = \max_{i \in [K]} \lambda_1(\Sigma_{0, i}) , \quad \lambda_{d, 0} = \lambda_d(\Sigma_0) = \min_{i \in [K]} \lambda_d(\Sigma_{0, i}) .
\]

It follows from (28) and (30) and properties in Appendix A.2 that
\[
\lambda_1(\Sigma_k) \leq \lambda_{1, 0} + K \frac{\lambda_2^2(\Sigma_\Psi)^k \kappa_b}{\lambda_{d, 0}^2} .
\]

### D.3 Preliminary eigenvalues results

Using basic block matrices operations, we know that \( \Gamma^\top = \Gamma_i^\top ((i, j) \in [K] \times [K]) \subseteq \mathbb{R}^{Kd \times Kd} \), where \( \Gamma_i^\top = (b_i^\top b_j) \otimes I_d = (b_i^\top b_j)I_d \). It follows that \( \Gamma^\top = D \otimes I_d \), where \( D = (b_i^\top b_j) ((i, j) \in [K] \times [K]) \subseteq \mathbb{R}^{K \times K} \). Thus,
\[
\lambda_1(\Gamma^\top) = \lambda_1(D) .
\]

It follows from (28) and (30) and properties in Appendix A.2 that
\[
\lambda_1(\Sigma_k) \leq \lambda_{1, 0} + K \frac{\lambda_2^2(\Sigma_\Psi)^k \kappa_b}{\lambda_{d, 0}^2} .
\]

### D.4 Regret proof

We consider our model rewritten in (26) and the notation introduced in Appendices D.1 and D.2. As we explained in Section 4.2, the joint posterior distribution of all action parameters \( \Theta^* \mid H_t \) is a multivariate Gaussian distribution of mean \( \hat{\Theta}_t \in \mathbb{R}^{Kd} \) and covariance \( \Gamma_t \in \mathbb{R}^{Kd \times Kd} \). Moreover, Lemma 3 allows us to express the covariance as \( \Sigma_t = \Sigma_t + \sum_{i=1}^K \Gamma_i^\top \Gamma_i^\top \Gamma_i \). Now let \( A_t \) and \( A_{*, t} \) be the joint vector representation (see Appendix D.1) of the context \( X_t \) with the taken action and the optimal action in round \( t \), respectively. First, \( \Theta_t \) is deterministic given \( H_t \). In addition, \( A_t \) and \( A_{*, t} \) are i.i.d. given \( H_t \). Thus, we have that
\[
\mathbb{E}[A_{*, t}^\top \Theta^* - A^\top \Theta] = \mathbb{E} \left[ \mathbb{E} \left[ A_{*, t}^\top (\Theta^* - \hat{\Theta}_t) \mid H_t \right] \right] + \mathbb{E} \left[ \mathbb{E} \left[ A_t^\top (\hat{\Theta}_t - \Theta^*) \mid H_t \right] \right] .
\]

Now, note that given \( H_t, \Theta^* \), \( \hat{\Theta}_t \) is a zero-mean multivariate random variable independent of \( A_t \), and thus \( \mathbb{E} \left[ A_t^\top (\Theta^* - \hat{\Theta}_t) \mid H_t \right] = 0 \). Thus we only need to bound the first term in (32). Let \( A_t \) be the set of all possible action-context vector representations \( A_t \subseteq \mathbb{R}^{Kd} \) in round \( t \) and let
\[
E_t(\delta) = \left\{ \forall a \in A_t : \langle a^\top, (\Theta^* - \hat{\Theta}_t) \rangle \leq \sqrt{2 \log(1/\delta)} \|a\|_{\Sigma_t} \right\} , \quad \forall \delta \in (0, 1) .
\]

Fix history \( H_t \), then we split the expectation over the two complementary events \( E_t(\delta) \) and \( \bar{E}_t(\delta) \), and use the Cauchy–Schwarz inequality to obtain
\[
\mathbb{E} \left[ A_{*, t}^\top (\Theta^* - \hat{\Theta}_t) \mid H_t \right] \leq \sqrt{2 \log(1/\delta)} \mathbb{E} \left[ \|A_{*, t}\|_{\Sigma_t} \mid H_t \right] + \mathbb{E} \left[ A_{*, t}^\top (\Theta^* - \hat{\Theta}_t) \mathbb{1} \{ \bar{E}_t(\delta) \} \mid H_t \right] .
\]

Now note that for any action-context vector representation \( a_t \), \( \langle a^\top, (\Theta^* - \hat{\Theta}_t) \rangle / \|a\|_{\Sigma_t} \) is a standard normal variable. Thus, the second term in (33) can be bounded as
\[
\mathbb{E} \left[ A_{*, t}^\top (\Theta^* - \hat{\Theta}_t) \mathbb{1} \{ \bar{E}_t(\delta) \} \mid H_t \right] \leq \frac{\lambda_{\max, t} K \delta}{\pi} .
\]
In (i) we use the Cauchy-Schwarz inequality and the fact that \( \bar{E}_t(\delta) \) implies \( \| \hat{\Theta}_t - \hat{\Theta}_t \|_{\infty} \geq \sqrt{2 \log(1/\delta)} \). In (ii), we set \( \lambda_{\max,t} = \max_{a \in A_t} \| a \|_{\Sigma_t} \). We combine (33) and (34) and use that \( A_t \) and \( A_{s,t} \) are i.i.d. given \( H_t \) to obtain

\[
E \left[ A_t^T (\Theta_s - \hat{\Theta}_t) \mid H_t \right] \leq \sqrt{2 \log(1/\delta)} E \left[ \| A_t \|_{\Sigma_t} \mid H_t \right] + \sqrt{\frac{2}{\pi} \lambda_{\max,t} K \delta}.
\]  

The bound in (35) holds for any history \( H_t \) and thus we take an additional expectation and obtain that

\[
E \left[ \sum_{t=1}^n A_t^T \Theta_s - A_t^T \Theta_s \right] \leq \sqrt{2 \log(1/\delta)} E \left[ \sum_{t=1}^n \| A_t \|_{\Sigma_t} \right] + \sqrt{\frac{2}{\pi} \lambda_{\max,t} K \delta},
\]

\[
\leq (i) \sqrt{2n \log(1/\delta)} E \left[ \sum_{t=1}^n \| A_t \|_{\Sigma_t} \right] + \sqrt{\frac{2}{\pi} \lambda_{\max,t} Kn \delta},
\]

\[
(ii) \leq \sqrt{2n \log(1/\delta)} E \left[ \sum_{t=1}^n \| A_t \|_{\Sigma_t} \right] + \sqrt{\frac{2}{\pi} \lambda_{\max,t} Kn \delta}.
\]

In (i), we use the Cauchy-Schwarz inequality and (ii) follows from the concavity of the square root. Finally, using (31) we get that

\[
\lambda_{\max,t} = \max_{a \in A_t} \| a \|_{\Sigma_t} \leq \sqrt{\lambda_1(\Sigma_t) \kappa_x} \leq \sqrt{\left( \lambda_{1,0} + \frac{\lambda_{1,0}^2 \lambda_1(\Sigma_\psi) \kappa_b}{\lambda_{d,0}} \right) \kappa_x}.
\]

Now we focus on the the term \( \sqrt{E \left[ \sum_{t=1}^n \| A_t \|_{\Sigma_t}^2 \right]} \) that we decompose and bound as

\[
\| A_t \|_{\Sigma_t}^2 = \sigma^2 A_t^T \hat{\Sigma}_t A_t, \]

\[
\sigma^2 \left( \sigma^{-2} A_t^T \hat{\Sigma}_t A_t + \sigma^{-2} A_t^T \hat{\Sigma}_t \Sigma_0^{-1} \Gamma \Sigma_t \Gamma^T \Sigma_0^{-1} \hat{\Sigma}_t A_t \right),
\]

\[
\leq c_1 \log (1 + \sigma^{-2} A_t^T \hat{\Sigma}_t A_t) + c \log (1 + \sigma^{-2} A_t^T \hat{\Sigma}_t \Sigma_0^{-1} \Gamma \Sigma_t \Gamma^T \Sigma_0^{-1} \hat{\Sigma}_t A_t),
\]

\[
= c_1 \log \det (I_{K_d} + \sigma^{-2} \hat{\Sigma}_t A_t A_t^T \hat{\Sigma}_t^T) + c \log \det (I_{L_d} + \sigma^{-2} \hat{\Sigma}_t A_t A_t^T \hat{\Sigma}_t^T \Sigma_0^{-1} \Gamma \Sigma_0^{-1} \hat{\Sigma}_t A_t A_t^T \hat{\Sigma}_t^T \Sigma_0^{-1} \Gamma).
\]

(i) follows from \( \hat{\Sigma}_t = \hat{\Sigma}_t + \Sigma_0^{-1} \Gamma \Sigma_t \Gamma^T \Sigma_0^{-1} \hat{\Sigma}_t \), and we use the following inequality in (ii)

\[
\hat{\Sigma}_t = \hat{\Sigma}_t + \Sigma_0^{-1} \Gamma \Sigma_t \Gamma^T \Sigma_0^{-1} \hat{\Sigma}_t,
\]

\[
x = \frac{x}{\log(1 + x)} \log(1 + x) \leq \left( \max_{x \in [0,u]} \frac{x}{\log(1 + x)} \right) \log(1 + x) = \frac{u}{\log(1 + u)} \log(1 + x),
\]

which holds for any \( x \in [0,u] \), where constants \( c_1 \) and \( c \) are derived as

\[
c_1 = \frac{\kappa_x \lambda_{1,0}}{\log(1 + \sigma^{-2} \kappa_x \lambda_{1,0})}, \quad c = \frac{c_\psi}{\log(1 + \sigma^{-2} c_\psi)}, \quad c_\psi = \frac{K \kappa_x \lambda_{1,0}^2 \lambda_1(\Sigma_\psi) \kappa_b}{\lambda_{d,0}^2},
\]

The derivation of \( c_1 \) uses that

\[
A_t^T \hat{\Sigma}_t A_t \leq \lambda_1(\Sigma_t) \| A_t \|_2^2 \leq \lambda_d^{-1}(\Sigma_0^{-1} + G_t) \kappa_x \leq \lambda_1(\Sigma_0^{-1}) \kappa_x = \lambda_1(\Sigma_0) \kappa_x = \lambda_{1,0} \kappa_x.
\]

The derivation of \( c \) follows from

\[
A_t^T \hat{\Sigma}_t \Sigma_0^{-1} \Gamma \Sigma_t \Gamma^T \Sigma_0^{-1} \hat{\Sigma}_t A_t \leq \lambda_t^2(\Sigma_t) \lambda_1^2(\Sigma_0^{-1} \lambda_1(\Gamma \Sigma_t \Gamma^T) \kappa_x \leq \frac{\lambda_t^2(\Sigma_t) \lambda_1(\Sigma_\psi) \lambda_1(\Gamma \Sigma_t \Gamma^T) \kappa_x}{\lambda_{d,0}^2},
\]

\[
\leq K \lambda_{1,0} \lambda_1(\Sigma_\psi) \kappa_b \kappa_x.
\]

Finally, in (iii) we use the Weinstein–Aronszajn identity. Now we focus on bounding the logarithmic terms in (36).
**First Term in (36)** We first rewrite this term as

$$
\log \det (I_{Kd} + \sigma^{-2} \tilde{\Sigma}^\frac{1}{2} A_i A_i^\top \tilde{\Sigma}^\frac{1}{2}) = \log \det (\hat{\Sigma}^{-1}_i + \sigma^{-2} A_i A_i^\top) - \log \det (\tilde{\Sigma}^{-1}_i),
$$

$$= \log \det (\hat{\Sigma}^{-1}_t) - \log \det (\tilde{\Sigma}^{-1}_i).$$

Then we sum over all rounds $t \in [n]$, and get a telescoping that leads to

$$
\sum_{t=1}^n \log \det (I_{Kd} + \sigma^{-2} \tilde{\Sigma}^\frac{1}{2} A_i A_i^\top \tilde{\Sigma}^\frac{1}{2}) = \sum_{t=1}^n \log \det (\hat{\Sigma}^{-1}_t) - \log \det (\tilde{\Sigma}^{-1}_t),
$$

$$= \log \det (\hat{\Sigma}^{-1}_n) - \log \det (\tilde{\Sigma}^{-1}_1) = \log \det (\Sigma^{-1}_0) - \log \det (\tilde{\Sigma}^{-1}_1),$$

$$\leq K d \log \left( \frac{1}{K d} \text{Tr} (\Sigma^{-1} \hat{\Sigma}^{-1} \tilde{\Sigma}^{-1}) \right) \leq K d \log \left( 1 + \frac{\kappa_x \lambda_1 (\Sigma_0) n}{\sigma^2} \right).$$

where we use the inequality of arithmetic and geometric means in (i).

**Second Term in (36)** First, we rewrite the covariance matrix of the hyper-posterior $\hat{\Sigma}_t$ using the compact notation introduced in Appendices D.1 and D.2. Precisely, we have that

$$
\hat{\Sigma}_t^{-1} = \Sigma^{-1}_0 + \Gamma^\top (\Sigma_0 + G_t^{-1})^{-1} \Gamma.
$$

(37)

Now let $u = \sigma^{-1} \tilde{\Sigma}^\frac{1}{2} A_i$. It follows from (37) that

$$
\hat{\Sigma}_{t+1}^{-1} - \hat{\Sigma}_t^{-1} = \Gamma^\top \left( (\Sigma_0 + (G_t + \sigma^{-2} A_i A_i^\top)^{-1})^{-1} - (\Sigma_0 + G_t^{-1})^{-1} \right) \Gamma
$$

$$\stackrel{(i)}{=} \Gamma^\top \left( \Sigma_0^{-1} - \Sigma_0^{-1} (\hat{\Sigma}_t^{-1} + \sigma^{-2} A_i A_i^\top)^{-1} \Sigma_0^{-1} - (\Sigma_0^{-1} - \Sigma_0^{-1} \hat{\Sigma}_t \Sigma_0^{-1}) \right) \Gamma
$$

$$= \Gamma^\top \left( \Sigma_0^{-1} (\hat{\Sigma}_t^{-1} - (\tilde{\Sigma}_t^{-1} + \sigma^{-2} A_i A_i^\top)^{-1} \Sigma_0^{-1}) \right) \Gamma
$$

$$= \Gamma^\top \left( \Sigma_0^{-1} \tilde{\Sigma}_t^{-1} (I_{Kd} - (I_{Kd} + \sigma^{-2} \tilde{\Sigma}_t^\frac{1}{2} A_i A_i^\top \tilde{\Sigma}_t^\frac{1}{2})^{-1} \tilde{\Sigma}_t^\frac{1}{2} \Sigma_0^{-1}) \right) \Gamma
$$

$$= \Gamma^\top \left( \Sigma_0^{-1} \tilde{\Sigma}_t^\frac{1}{2} (I_{Kd} - (I_{Kd} + u u^\top)^{-1} \tilde{\Sigma}_t^\frac{1}{2} \Sigma_0^{-1}) \right) \Gamma
$$

$$\stackrel{(ii)}{=} \Gamma^\top \left( \Sigma_0^{-1} \tilde{\Sigma}_t^\frac{1}{2} \frac{u u^\top}{1 + u^\top u} \tilde{\Sigma}_t^\frac{1}{2} \Sigma_0^{-1} \right) \Gamma
$$

$$= \sigma^{-2} \Gamma^\top \left( \Sigma_0^{-1} \tilde{\Sigma}_t^\frac{1}{2} \frac{A_i A_i^\top}{1 + u^\top u} \tilde{\Sigma}_t^\frac{1}{2} \Sigma_0^{-1} \right) \Gamma.$$

(38)

In (i) we use the Woodbury matrix identity in both terms and (ii) follows from the Sherman-Morrison formula. Now we have that $\|A_i\|^2 \leq \kappa_x$, thus

$$1 + u^\top u = 1 + \sigma^{-2} A_i^\top \tilde{\Sigma}_t A_i \leq 1 + \sigma^{-2} \kappa_x \lambda_1 (\Sigma_0) = c'.$$

This allows us to bound the second logarithmic term in (36) as

$$
\log \det (I_{Ld} + \sigma^{-2} \tilde{\Sigma}^\frac{1}{2} \Gamma^\top \Sigma_0^{-1} \tilde{\Sigma}_t A_i A_i^\top \tilde{\Sigma}_t \Sigma_0^{-1} \tilde{\Sigma}^\frac{1}{2} \Gamma),
$$

$$\leq c' \log \det (I_{Ld} + \sigma^{-2} \tilde{\Sigma}^\frac{1}{2} \Gamma^\top \Sigma_0^{-1} \tilde{\Sigma}_t A_i A_i^\top \tilde{\Sigma}_t \Sigma_0^{-1} \tilde{\Sigma}^\frac{1}{2} \Gamma/c'),
$$

$$\leq c' \left[ \log \det (\hat{\Sigma}_t^{-1} + \sigma^{-2} \Gamma^\top \Sigma_0^{-1} \hat{\Sigma}_t A_i A_i^\top \hat{\Sigma}_t \Sigma_0^{-1} \Gamma/c') - \log \det (\hat{\Sigma}_t^{-1}) \right],
$$

$$\leq c' \left[ \log \det (\hat{\Sigma}_t^{-1} + \sigma^{-2} \Gamma^\top \Sigma_0^{-1} \hat{\Sigma}_t A_i A_i^\top \hat{\Sigma}_t \Sigma_0^{-1} \Gamma - u^\top u \tilde{\Sigma}_t \Sigma_0^{-1} \Gamma) - \log \det (\hat{\Sigma}_t^{-1}) \right],
$$

$$\leq c' \left[ \log \det (\hat{\Sigma}_t^{-1}) - \log \det (\hat{\Sigma}_t^{-1}) \right].$$
In (i), we use the fact that \( \log(1 + x) \leq c' \log(1 + x/c') \) for any \( x \geq 0 \) and \( c' \geq 1 \). In (ii), we use the log product formula and that the \( \det \) is a multiplicative map. In (iii), we use that \( 1/(1 + u' u) \geq 1/c' \). Finally, (iv) follows from the fact that we have a rank-1 update of \( \Sigma_i^{-1} \) as we showed in (38).

Now we sum over all rounds and get telescoping:

\[
\sum_{t=1}^{n} \log \det \left( I_{Ld} + \sigma^2 \Sigma_t^{-1} \Gamma \Sigma_t^{-1} \Sigma_t L \Sigma_t^{-1} \Sigma_t^{-1} \Gamma \right) \leq c' \left[ \log \det(\Sigma_{n+1}^{-1}) - \log \det(\Sigma_1^{-1}) \right],
\]

\[
= c' \log \det(\Sigma_{n+1}^{-1} \Sigma_{n+1}^{-1} \Sigma_{n+1}^{-1} \Gamma),
\]

\[
\leq c' Ld \log \left( \frac{1}{Ld} \text{Tr}(\Sigma_{n+1}^{-1} \Sigma_{n+1}^{-1} \Sigma_{n+1}^{-1} \Gamma) \right),
\]

\[
\leq c' Ld \log(c) \left( 1 + \lambda_1(\Sigma_{n+1}^{-1} \Sigma_{n+1}^{-1} \Gamma) \right).
\]

In (i) we use the inequality of arithmetic and geometric means. In (ii) we bound all eigenvalues by the maximum eigenvalue. In (iii) we use the expression of \( \Sigma_i \) in (37), Weyl’s inequality, and basic properties in Appendix A.2.

We combine the upper bounds for both logarithmic terms and get

\[
E \left[ \sum_{t=1}^{n} \|A_t\|_2^2 \right] \leq K d c_1 \log \left( 1 + \frac{\kappa \lambda_1(\Theta_{0,1})}{\sigma^2 Kd} \right) + L dc c' \log \left( 1 + \frac{\lambda_1(\Sigma_{n+1}^{-1} \Gamma)}{\lambda_{d,0}} \right).
\]

Finally, we set \( c_2 = cc' \) and \( \lambda_1, \Psi = \lambda_1(\Sigma_1 \Psi) \), and use that \( \lambda_1(\Gamma^{-1} \Gamma) \leq K \kappa_0 \) as we showed in Appendix D.3, which concludes the proof.

**E Extensions**

While Algorithm 1 can be applied to the general two-level hierarchical setting introduced in Section 2, we only focused on cases where the dependencies of action parameters with latent parameters are captured through a linear combination in the theoretical analysis and experiments. In Appendix E.1, we provide an extension of our analysis to mixed linear models where weights \( b_{i,t} \) are replaced by matrices \( C_{i,t} \). In Appendix E.2, we introduce and motivate deeper hierarchies, and provide an intuition on the corresponding Bayes regret.

**E.1 Beyond linear combinations**

An effective way to capture fine-grained dependencies is to assume that the parameter of action \( i \) depends on latent parameters through \( L \) known matrices \( C_{i,t} \in \mathbb{R}^{d \times d} \) as \( \theta_{i,t} \mid \Psi \sim P_0, i \mid \sum_{t=1}^{L} C_{i,t} \psi_{i,t} \). This generalizes the setting considered in our analysis, which corresponds to the case where \( C_{i,t} = b_{i,t} I_d \). We first make the observation that \( \sum_{t=1}^{L} C_{i,t} \psi_{i,t} = C_i \Psi \), where \( C_i = [C_{i,1}, \ldots, C_{i,L}] \in \mathbb{R}^{d \times L} \). It follows that for any round \( t \in [n] \), the joint hyper-posterior is a multivariate Gaussian \( Q_t = \mathcal{N}(\tilde{\mu}_t, \tilde{\Sigma}_t) \), where

\[
\tilde{\Sigma}_t^{-1} = \Sigma_{0,t}^{-1} + \sum_{i=1}^{K} C_{i,t}^{-1} (\Sigma_{0,i}^{-1} + G_{i,t}^{-1})^{-1} C_{i,t},
\]

\[
\tilde{\mu}_t = \tilde{\Sigma}_t \left( \Sigma_{0,t}^{-1} \mu_0 + \sum_{i=1}^{K} C_{i,t}^{-1} (\Sigma_{0,i}^{-1} + G_{i,t}^{-1})^{-1} G_{i,t}^{-1} B_{i,t} \right).
\]

Moreover, for any round \( t \in [n] \) and action \( i \in [K] \), the conditional posterior is a multivariate Gaussian \( P_{t,i} = \mathcal{N}(\cdot | \Psi) = \mathcal{N}(\cdot | \tilde{\mu}_{t,i}, \tilde{\Sigma}_{t,i}) \), where

\[
\tilde{\Sigma}_{t,i}^{-1} = G_{t,i}^{-1} + \Sigma_{0,i}^{-1}, \quad \tilde{\mu}_{t,i} = \tilde{\Sigma}_{t,i} \left( B_{t,i} + \Sigma_{0,i}^{-1} \sum_{t=1}^{L} C_{i,t} \psi_{i,t} \right).
\]
Finally, our regret proof extends smoothly leading to a Bayes regret upper bound similar to the one we derived in Theorem 1. Let $C = (C_i)_{i \in [K]} \in \mathbb{R}^{K d \times L d}$, the corresponding Bayes regret is given in the following proposition.

**Proposition 4.** For any $\delta \in (0, 1)$, the Bayes regret of $G$-HierTS, for the mixed linear model in Appendix E.1, is bounded as

$$
BR(n) \leq 2 n \log (1/\delta) \left( \mathcal{R}_{\text{action}}(n) + \mathcal{R}_{\text{latent}}(n) \right) + \sqrt{\frac{2}{\pi} \left( \lambda_{1,0} + c_\psi \right) \kappa_x K n \delta},
$$

where $\mathcal{R}_{\text{action}}(n) = K d c_1 \log \left( 1 + \frac{n \kappa_x \lambda_{1,0}}{\sigma c_{\psi}^2 K d} \right)$, $\mathcal{R}_{\text{latent}} = L d c_2 \log \left( 1 + K \max_{\ell \in [K]} \lambda_{\psi,\ell,0} \right)$, and

$$
\begin{align*}
\lambda_{1,0} &= \max_{i \in [K]} \lambda_1 (\Sigma_{0,i}), & \lambda_{d,0} &= \min_{i \in [K]} \lambda_d (\Sigma_{0,i}), & \lambda_{1,\psi} &= \lambda_1 (\Sigma_{\psi}), \\
\kappa_{c_1} &= \max_{i \in [K]} \lambda_1 (C_i^\top C_i), & \kappa_{c_2} &= \lambda_1 (C^\top C), & \kappa_x &= \max_{i \in [n]} \|X_i\|_2^2,
\end{align*}
$$

$c_1 = \frac{\kappa_x \lambda_{1,0}}{\log (1 + \sigma^{-2} \kappa_x \lambda_{1,0})}$, $c_2 = \frac{e \psi (1 + \sigma^{-2} \kappa_x \lambda_{1,0})}{\log (1 + \sigma^{-2} c_\psi)}$, and $c_\psi = \frac{\kappa_x \kappa_{c_2} \lambda_{2,0}^2 \lambda_{1,\psi}}{\lambda_{d,0}}$.

The interpretation of this result is similar to Theorem 1. The only difference is that sparsity is now captured through constants $\kappa_{c_1}$ and $\kappa_{c_2}$.

### E.2 Beyond two-level hierarchy

To motivate deeper hierarchies, consider the problem of page construction in movie streaming services where $J$ movies are organized into $L$ genres. First, a genre $\ell \in [L]$ is associated to a parameter $\psi_{s,\ell} \in \mathbb{R}^d$. Moreover, each movie $j \in [J]$ is associated to a parameter $\phi_{s,j} \in \mathbb{R}^d$, which is a combination of genre parameters $\psi_{s,\ell}$ weighted by scalars that quantify how related is each genre to the $j$-th movie. Finally, page layouts are actions and they are seen as lists of movies. Each page layout $i \in [K]$ is associated to an action parameter $\theta_{s,i}$ which is also a combination of movies parameters $\phi_{s,j}$ weighted by a scalar that quantifies position bias. Precisely, this scalar is set to 0 if the corresponding movie is not present in the page, and it has high value if the movie is placed in a position with high visibility. This setting induces a three-level hierarchical model, for which we give a Gaussian example below.

$$
\begin{align*}
\Psi_s &\sim \mathcal{N}(\mu_{\psi}, \Sigma_{\psi}), \\
\phi_{s,j} \mid \Psi_s &\sim \mathcal{N}\left( \sum_{\ell=1}^L \theta_{s,\ell} \psi_{s,\ell}, \Sigma_{\phi,j} \right), & \forall j \in [J], \\
\theta_{s,i} \mid \Psi_s &\sim \mathcal{N}\left( \sum_{j=1}^J w_{i,j} \phi_{s,j}, \Sigma_{0,i} \right), & \forall i \in [K], \\
Y_t \mid X_t, \theta_{s,A_t} &\sim \mathcal{N}(X_t^\top \theta_{s,A_t}, \sigma^2), & \forall t \in [n],
\end{align*}
$$

(41)

where $\Psi_s = (\psi_{s,\ell})_{\ell \in [L]} \in \mathbb{R}^{Ld}$ and $\Phi_s = (\phi_{s,j})_{j \in [J]} \in \mathbb{R}^{Jd}$. $G$-HierTS samples hierarchically as follows. First, we sample $\Psi_s$ from the posterior of $\Psi_s \mid H_t$. We then sample $\phi_{s,j}$ individually from the posterior of $\phi_{s,j} \mid \Psi_s, H_t$. Finally, we sample $\theta_{s,i}$ individually from the posterior of $\theta_{s,i} \mid \Phi_s, H_t,i$. This three-level hierarchical sampling scheme improves computational and space complexity in the realistic case where $K \gg J \gg L$. Moreover, we expect the upper bound of the Bayes regret of (41) following our analysis to be decomposed in three terms $\sqrt{n(\mathcal{R}_{\text{action}}(n) + \mathcal{R}_{1} + \mathcal{R}_{2})}$, where $\mathcal{R}_{\text{action}}(n) = \tilde{O}(Kd)$, $\mathcal{R}_1 = \tilde{O}(Jd)$ and $\mathcal{R}_2 = \tilde{O}(Ld)$.

### F Additional experiments

We provide additional experiments where we evaluate $G$-HierTS using synthetic and real-world problems, and compare it to baselines that either ignore or partially use latent parameters. In each plot, we report the averages and standard errors of the quantities. Both settings are described in Section 5.
Figure 4: Regret of G-HierTS on synthetic bandit problems with varying feature dimension $d \in \{2, 5, 10\}$ and number of actions $K \in \{20, 50, 100\}$.

F.1 Synthetic experiments

In Figure 4, we report regret from nine experiments with horizon $n = 2000$, where we vary $K$ and $d$. In each experiment, we compare G-HierTS, its factored posterior approximation G-HierTS-Fa (Section 3.3), LinUCB [34], LinTS [3], and HierTS [23]. In all experiments, we observe that G-HierTS and G-HierTS-Fa outperform other baselines that ignore latent parameters or incorporate them partially. We also notice that the gain in performance becomes smaller when $K/L$ decreases.

F.2 Experiments on MovieLens dataset

We plot the regret of G-HierTS, LinTS, and HierTS up to $n = 5000$ rounds in Figure 5; as LinTS and HierTS are the most competitive baselines in Section 5.1. We observe that G-HierTS outperforms the other baselines. This is despite the fact that we did not fine tune the mixing weights, which attests to the robustness of G-HierTS to model misspecification. Similarly to the synthetic problems, we observe that the gap in performance between G-HierTS and other baselines is less significant when $K/L$ is small.
Figure 5: Regret of G-HierTS on the MovieLens dataset with varying number of latent parameters $L \in \{2, 5, 10\}$ and number of actions $K \in \{20, 50, 100\}$. 