A super-polynomial lower bound for learning nonparametric mixtures

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Abstract

We study the problem of learning nonparametric distributions in a finite mixture, and establish a super-polynomial lower bound on the sample complexity of learning the component distributions in such models. Namely, we are given i.i.d. samples from $f$ where

$$f = \sum_{i=1}^{k} w_i f_i, \quad \sum_{i=1}^{k} w_i = 1, \quad w_i > 0$$

and we are interested in learning each component $f_i$. Without any assumptions on $f_i$, this problem is ill-posed. In order to identify the components $f_i$, we assume that each $f_i$ can be written as a convolution of a Gaussian and a compactly supported density $\nu_i$ with $\text{supp}(\nu_i) \cap \text{supp}(\nu_j) = \emptyset$. Our main result shows that $\Omega((\frac{1}{\epsilon}C \log \log \frac{1}{\epsilon})$ samples are required for estimating each $f_i$. The proof relies on a fast rate for approximation with Gaussians, which may be of independent interest. This result has important implications for the hardness of learning more general nonparametric latent variable models that arise in machine learning applications.

1 Introduction

A hallmark of modern machine learning is the ability to train flexible, nonparametric latent variable models in high-dimensions. Examples of such models include variational autoencoders (VAEs, [38]), generative adversarial networks (GANs, [27]), normalizing flows [20, 54], and autoregressive models [62, 64]. At the opposite end of the spectrum are simple, parametric latent variable models such as the Gaussian mixture model (GMM, [51]). While the past two decades have born witness to an explosion in our understanding of the computational and statistical properties of GMMs (e.g. [15, 57, 5, 48, 35, 41, 25, 32, 18, 6]), the theoretical underpinnings of more general nonparametric latent variable models are less well understood. Given the widespread popularity and adoption of VAEs, GANs, etc., understanding the complexity of identifying and learning the latent structure of these models is a fundamental problem that has received surprisingly little attention. Although there have been substantial developments in our understanding of density estimation in these models [61, 9, 19, 7], our interest here is learning the underlying latent structure, which is a different and more difficult problem (see Section 1.2 for more details).

In this paper, we take a first step towards this understanding by establishing a fundamental hardness result for learning nonparametric latent variable models. Interpolating between simple parametric models on one extreme and complex nonparametric models at the other extreme, we consider arguably the simplest nonparametric latent variable model: a finite mixture of nonparametric distributions. In particular, we are interested in finite mixture models in which the mixture components themselves are allowed to be nonparametric:

$$f = \sum_{i=1}^{k} w_i f_i, \quad \sum_{i=1}^{k} w_i = 1, \quad w_i > 0. \quad (1)$$
Here the \( f_i \) are allowed to come from a flexible, nonparametric family of distributions. In particular, we will be interested in the problem of learning each \( f_i \) given i.i.d. samples from \( f \).

Our main result shows that super-polynomially many samples are required to distinguish the \( f_i \) (see Section 1.4).

Our motivation comes from latent variable models in machine learning. Consider a latent variable model with a discrete latent variable \( Z \) taking on \( k \) states. Then the observed data distribution \( P(X) \) can be written as

\[
P(X) = \sum_{i=1}^{k} P(Z = i)P(X | Z = i). \tag{2}
\]

Clearly, by identifying the prior on \( Z \) with the weights \( w_i := P(Z = i) \) and each class conditional distribution \( P(X | Z = i) \) with the density \( f_i \), the latent variable model (2) is just a special case of (1). Thus, the problem of learning general discrete latent variable models is a special case of the problem we consider. For example, in a typical discrete VAE [56, 63, 36, 44], we specify a fixed prior on \( P(Z) \)—i.e. the weights \( w_i \)—and learn a decoder \( P(X | Z) \)—i.e. the components \( f_i \). Similar reasoning applies to more complicated models with multivariate, discrete \( Z \).

Another important application is nonparametric clustering, where the goal is to partition a set of \( n \) data points into \( k \) clusters while making as few assumptions on the clusters as possible. In model-based clustering, we assume a mixture model as in (1), where each \( f_i \) represents a single “cluster”, and each sample is drawn from a randomly selected cluster with probability \( w_i \). The optimal clustering is then given by the Bayes optimal partition, which is defined by the resulting Bayes classifier. This problem has been well-studied in the literature [2, 37, 10, 40]. Other approaches to this problem include spectral clustering [19], hierarchical clustering [26, 10], mode-based clustering [11], and density-based clustering [33, 34, 55].

Given the apparent generality of the problem under consideration, it is worthwhile to understand the computational and sample complexity of recovering \((w_i, f_i)\) from \( f \). It is clear that without additional assumptions on the \( f_i \), this problem is ill-posed: There are infinitely many possible ways to write \( f \) as a mixture model of the form (1). There are various ways to overcome this degeneracy via restricting model capacity (e.g. by making parametric assumptions as in the GMM), assuming independence, or by assuming separation between components. Our approach is closest to the separation approach. Specifically, we will assume that each \( f_i \) can be written as a convolution of a Gaussian and a compactly supported density \( \nu \) with \( \text{supp}(\nu_i) \cap \text{supp}(\nu_j) = \emptyset \); see Section 1.4 for details. Clearly, learning more general nonparametric mixtures under separation conditions is at least as hard as this simplified problem.

### 1.1 Problem definition

Given \( \mu \in \mathbb{R} \), let \( g_\mu \) be the pdf of the standard Gaussian distribution centered at \( \mu \), i.e.

\[
g_\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}
\]

for any \( x \in \mathbb{R} \). Given \( S \subset \mathbb{R} \), let \( \mathcal{P}_S \) be the set of probability density functions (pdfs) of all probability distributions on \( S \) with respect to some fixed base measure. Given an interval \( I \subset \mathbb{R} \), let

\[
\mathcal{G}_I := \left\{ f \in \mathcal{P}_\mathbb{R} \mid f = \int_{\mu \in I} \nu(\mu) g_\mu d\mu \text{ where } \nu \in \mathcal{P}_I \right\}. \tag{3}
\]

Namely, \( \mathcal{G}_I \) is the collection of convolutions of a Gaussian \( g_0 \) with some distribution \( \nu \) whose support lies in the interval \( I \). We call such a distribution an interval Gaussian. Given \( k \) intervals \( I_1, \ldots, I_k \), we will frequently abuse notation by writing \( \mathcal{G}_i := \mathcal{G}_{I_i} \).
Now consider the pdf
\[ f = w_1 f_1 + w_2 f_2, \quad f_i \in G_i, \quad I_1 \cap I_2 = \emptyset, \quad w_i \geq 0, \quad w_1 + w_2 = 1 \quad (4) \]
where \( I_i = [a_i, b_i] \) are disjoint, closed, and bounded intervals. Since \( f_i \in G_i \), we can write \( f_i = \nu_i * g_0 \) for \( i = 1, 2 \), where \( \nu_i \in \mathcal{P}_{I_i} \). We let \( \nu = w_1 \nu_1 + w_2 \nu_2 \) denote the global mixing density, whence \( f = \nu * g_0 \).

Suppose we are given a set of samples drawn from \( f \). Then, what is the minimum number of samples required in order to estimate each component \( f_i \)? What are the fundamental limitations on estimating each component \( f_i \)? Formally, we have the following problem:

Let \( P \) be a set of \( n \) i.i.d. samples drawn from \( f = w_1 f_1 + w_2 f_2 \), where \( f \) is defined as in (4). For any small \( \varepsilon > 0 \), what is the threshold \( \tau_\varepsilon \) such that if \( n < \tau_\varepsilon \), then no algorithm taking \( P \) as the input returns two pdfs \( \tilde{f}_1, \tilde{f}_2 \) such that \( \| f_i - \tilde{f}_i \|_1 < \varepsilon \) with probability at least \( 1/2 \)?

Here, we are focusing on learning the components \( f_i \) and treating the weights \( w_i \) as nuisance parameters, however, as we show, this problem is still hard nonetheless: Even if the \( w_i \) are fixed and known, learning the \( f_i \) is hard (Theorem 2).

Naively, one might expect that standard approaches from the literature would suffice to establish a polynomial sample complexity upper bound for this problem, however, these approaches are insufficient as will be discussed in more detail in Section 1.3. In particular, without assuming strong separation conditions, it is not clear if there exist any efficient algorithms for this problem. Our main result will show that such an algorithm cannot exist.

Separation conditions In the sequel it will be important to distinguish two distinct regimes for this problem based on the amount of separation between the \( f_i \). Let \( R > 0 \) denote the minimum distance between the endpoints of any two intervals, and note that as \( R \) increases, the TV distance between the \( f_i \) also increases. Based on the behaviour of \( R \), we can distinguish two regimes:

- **Weak separation.** In this case, \( R = O(1) \), i.e. \( R \) is fixed as the accuracy improves (i.e. \( \varepsilon \to 0 \)).

- **Strong separation.** \( R \to \infty \) as \( \varepsilon \to 0 \), i.e. \( R \) increases as the accuracy improves.

Clearly, weak separation is the more difficult regime, and fewer results exist in the literature for this regime. To the best of our knowledge, no nontrivial lower bounds have been established under weak separation.

1.2 Parameter vs. density estimation

This problem cleanly encapsulates the simplest nonparametric setting in which modern latent variable models operate. To provide additional context for this problem, we recall that mixture modeling problems can be broadly classified into two general categories:

- **Parameter learning.** The most common example of parameter learning is for GMMs, in which case we seek to estimate the weights \( w_i \) and centers \( \mu_i \) (and possibly the variances \( \Sigma_i \)) for each component. In our nonparametric setting, recalling our definition of the mixing density \( \nu = w_1 \nu_1 + w_2 \nu_2 \), parameter learning would mean estimating \( \nu \), i.e. we find another pdf \( \nu' \) such that \( \nu \) and \( \nu' \) are close in say the Wasserstein distance. Under our assumptions, this is equivalent to deconvolution, which requires exponentially many samples (see Section 1.3).
• **Density estimation.** Here we estimate the mixture distribution \( f \) directly, i.e., we find another pdf \( f' \) such that \( f \) and \( f' \) are close in say the total variation or the Hellinger distance. This problem has been the subject of intense research in the statistics literature as well as in machine learning, most recently in the context of GANs [61, 9, 19, 7].

It is obvious that parameter learning implies density estimation; in particular, parameter learning is much harder than density estimation. These two general problems inspire an intriguing question:

• **Can we acquire any guarantee in between parameter learning and density estimation?** Instead of parameter learning or density estimation, we seek to learn the components \( f_i \) rather than the mixture distribution \( f \) or the mixing density \( \nu \). For this task, we do not need to estimate each \( \nu_i \). Of course, one could learn \( \nu \) in the traditional sense such as parameter learning; however, this is not necessary.

Since nonparametric density estimation can be done efficiently, what can be said about this “in-between” problem? This is the problem we seek to address here.

### 1.3 Limitations of existing approaches

In this section, we pause to describe some important limitations of existing approaches and explain why they are insufficient. To summarize the discussion below: One may easily construct an exponential-time algorithm based on deconvolution, or a polynomial-time algorithm under a strong separation condition. This leaves open the important question of whether or not there exists a polynomial-time algorithm under weak (i.e., \( R = O(1) \)) separation. The main contribution of this paper is a negative answer to this question (Theorem 2).

**Deconvolution** Given our problem definition, a simple approach to learning the \( f_i \) is via deconvolution: Since \( f = \nu \ast g_0 \), we can first directly learn the global mixing density \( \nu \), and then use this to approximate each \( \nu_i \) (for example by truncation). See Section 4 for more discussion. This approach requires only very weak separation conditions on the \( f_i \), however, learning \( \nu \) directly in the first place requires exponentially many samples [70, 23, 45]. Indeed, Gaussian deconvolution is a classical example of a statistical problem with an exponential lower bound on its sample complexity. This highlights the problems one faces when first seeking to estimate \( \nu \) instead of directly learning the components themselves.

**Tail clipping and strong separation** If the separation between the components grows with the sample size, then the problem becomes much easier. Concretely, suppose \( f = \sum_{i=1}^{k} w_i f_i \) with \( f_i \in \mathcal{G}_i \) and recall that \( R \) denotes the minimum distance between the endpoints of any two intervals. Then if \( R \) is sufficiently large (say \( \Omega(\sqrt{\log 1/\varepsilon}) \)), the components \( f_i \) can be estimated to within \( \varepsilon \)-accuracy with only polynomially many samples. This is because the tail probability mass beyond the bound of \( \Omega(\sqrt{\log 1/\varepsilon}) \) of \( f_i \) is less than \( \varepsilon \). Therefore, we only need to cut the real line into \( k \) intervals such that each interval contains one \( I_i \) and the endpoints of each interval is at least \( \Omega(\sqrt{\log 1/\varepsilon}) \) away from the endpoints of \( I_i \). Formally, we only need to find out \( k - 1 \) locations \( t_1, \ldots, t_{k-1} \) on \( \mathbb{R} \) such that, for each \( i \), \( t_{i-1} < \mu_i - r - \Omega(\sqrt{\log 1/\varepsilon}) \) and \( t_i > \mu_i + r + \Omega(\sqrt{\log 1/\varepsilon}) \) where \( I_i = [\mu_i - r, \mu_i + r] \). Then, we can simply apply any density estimation result and chop off the tails at \( t_i \). This approach will give us the desired result.

**Lemma 1** (folklore). For any \( \varepsilon > 0 \). Given samples from \( f = \sum_{i=1}^{k} w_i f_i \) with \( f_i \in \mathcal{G}_i \) and \( R = \Omega(\sqrt{\log 1/\varepsilon}) \), there is an efficient algorithm that outputs \( k \) weights \( w'_i \) and \( k \) pdfs \( f'_i \) such that \( |w_i - w'_i| \leq \varepsilon \) and \( \|f_i - f'_i\|_1 \leq \varepsilon \).

In fact, a similar result holds more generally (with appropriate modifications) when the components are simply subgaussian.
A significant drawback of this approach is that the resulting separation $R$ depends on the accuracy level $\varepsilon$—in particular, in order for $\varepsilon \to 0$, we must have $R \to \infty$. This implies that for any fixed $R > 0$, an $R$-separated mixture model cannot be consistently learned by such an algorithm. In this paper, we consider the case where $R = O(1)$ is fixed as $n$ increases.

1.4 Main result

Our main result shows that given a set of samples from a mixture of two interval Gaussians as in (4), estimating each interval Gaussian requires super-polynomially many samples. More formally, we will prove the following result:

**Theorem 2.** Let $P$ be a set of $n$ i.i.d. samples drawn from $f^*$, where $f^* = \frac{1}{2} f_1^* + \frac{1}{2} f_2^*$ such that $f_i^* \in \mathcal{G}_i$ and $I_1 \cap I_2 = \emptyset$. For any sufficiently small $\varepsilon > 0$, if $n < \left(\frac{1}{\varepsilon}\right)^C \log \log \frac{1}{\varepsilon}$ where $C$ is an absolute constant, then no algorithm taking $P$ as the input returns two pdfs $\hat{f}_1, \hat{f}_2$ such that $\|f_i^* - \hat{f}_i\|_1 < \varepsilon$ with probability at least $1/2$.

This theorem makes no assumptions on the separation $R$, and in particular holds in the weak separation regime with $R = O(1)$. The only implied assumption on $R$ is that $I_1 \cap I_2 = \emptyset$—i.e $R > 0$—which allows for fixed (i.e. weak) separation as $\varepsilon \to 0$. In particular, our result holds when $R = \Theta(1)$ stays bounded away from zero. When the separation between two intervals is $R$, the sample complexity lower bound becomes

$$\text{poly}\left(\frac{1}{\varepsilon}\right) \cdot 2^\Omega(m \log m), \quad \text{where} \quad m = \min\{R^2, \log \frac{1}{\varepsilon}\} / \log R.$$ 

This follows from a simple modification to the proof and hence we omit the details. When $R = \Omega(\sqrt{\log 1/\varepsilon})$ the above bound becomes a polynomial which matches the upper bound obtained by tail-clipping under the strong separation assumption (Lemma 1).

2 Related work

Although our main focus is on nonparametric mixtures as in (4) (or more generally, (1)), it is instructive to recall what is known for parametric mixtures such as GMMs. We begin by reviewing this material before discussing recent work on nonparametric mixtures. As mentioned before, there are two major formulations in learning GMMs: parameter learning and density estimation.

**Parameter learning**  The algorithmic results of learning the parameters of a mixture of Gaussians fall into two major categories: separation-based and moment-based. In separation-based methods, we devise algorithms whose running time is $\text{poly}(d, k, n)$ assuming that the means of the Gaussians are well separated. Of course, one would hope to minimize the separation. In moment-based methods, there are no explicit separation assumptions, however, the sample complexity and the time complexity depends exponentially on $k$. For separation-based methods, [10] showed how to learn a mixture of spherical Gaussians whose centers are $\Omega(\sqrt{d})$ apart from each other. This was then improved by [60] to separation is $\tilde{\Omega}(\min\{d, k\})^{1/4})$. Moreover, they also showed that, if the separation is $\Omega(1)$, we can learn a mixture of $k$ Gaussians in $\text{poly}(k^{O(k)}, n)$ time. Recently, [52] presented an efficient algorithm that, with $\Omega(\min\{\sqrt{d}, \sqrt{\log k}\})$ separation, learns the parameters of a mixture of $k$ spherical Gaussians. They also proved that when $d$ is $\Omega(\log k)$, this $\Omega(\log k)$ separation is tight for algorithms that use polynomially many samples. For moment-based methods, [38] showed that, assuming $k = O(1)$, there is an efficient algorithm that learns the parameters in $\text{poly}(d, \frac{1}{\varepsilon})$ time. [32] showed that, when $k = 2$, the sample complexity of learning the parameters is $\Theta(\frac{1}{\varepsilon^2})$. In the one-dimensional case, [69] devised an
algorithm with sample complexity $1/\varepsilon^{O(k)}$ that returns a set of $k$ centers where the Wasserstein distance is small which implies the returned centers are close to the true centers. Recently, [21] extended this result to high-dimensions.

**Density estimation** One can further categorize density estimation problems into two formulations: proper learning and improper learning, depending on whether the output is restricted to be a mixture of $k$ Gaussians.

**Proper learning.** For proper learning, [24] proposed an efficient algorithm that outputs a mixture of $k = O(1)$ axis-aligned Gaussians such that it is $\varepsilon$-close in KL divergence to the original distribution. However, the sample and time complexity depends on a parameter $L$, which is the bound on the means and the variances of the underlying Gaussians. Namely, their algorithm uses $\text{poly}(d, \frac{1}{\varepsilon}, L)$ samples and its running time is also $\text{poly}(d, \frac{1}{\varepsilon}, L)$.

[17] showed that, when the distribution is a mixture of two Gaussians in $\mathbb{R}$, by using $\tilde{O}(\frac{1}{\varepsilon})$ samples, their algorithm returns a mixture of two Gaussians that is $\varepsilon$-close in total variation to the underlying distribution in $\tilde{O}(\frac{1}{\varepsilon})$ time. Later, [68] proposed an algorithm that the running time is improved to $\tilde{O}(\frac{1}{\varepsilon^{30}})$. In the case of general $k$, [59] showed that, for a mixture of $k$ spherical Gaussians in $\mathbb{R}^d$, there is an algorithm that learns the distribution by using $n = O\left(\frac{d^3}{\varepsilon^5} \log^5 d\right)$ samples in $O(n^2d\log n + d^2(\frac{k}{\varepsilon^5} \log d)^k)$ time. In particular, if the distribution is one-dimensional, their algorithm uses $\tilde{O}(\frac{1}{\varepsilon})$ samples and its running time is $\tilde{O}\left(\frac{1}{\varepsilon^5}\right)$. [42] improved the result that, in one dimensional case, their algorithm has the running time $O((k \log \frac{1}{\varepsilon})^{O(k^4)} + \frac{k}{\varepsilon^5})$. Note that when $k$ is small, Li and Schmidt’s result beats Suresh et al.’s result.

**Improper learning.** For improper learning, [8] proved that given samples from a mixture of $k$ axis-aligned Gaussians in $\mathbb{R}^d$, there is an algorithm whose running time and sample complexity are $O\left(\frac{1}{\varepsilon^5}(kd)^d\right)$ and whose output is a mixture of $O\left(\frac{k}{\varepsilon^5}\right)$ axis-aligned Gaussians which is $\varepsilon$-close to the original mixture with high probability. [68] showed that, in one dimension, there is an efficient algorithm whose sample complexity is $\tilde{O}(\frac{1}{\varepsilon})$ and whose output is a mixture of $\tilde{O}\left(\min\left\{\frac{k^2}{\varepsilon^5}, \frac{k}{\varepsilon^5}\right\}\right)$ Gaussians which is $\varepsilon$-close to the original mixture with high probability. [12] showed that given any univariate distribution $p$ that is $\varepsilon$-close to a mixture $q$ of $k$ Gaussians, there is an efficient algorithm that uses $\tilde{O}(\frac{k}{\varepsilon})$ samples and outputs a pdf $h$ such that it is $O(\varepsilon)$-close to $p$ with high probability. Later, [1] improved the result of [12] by showing that there is an algorithm that uses $\tilde{O}(\frac{k}{\varepsilon})$ samples, runs in $\tilde{O}(\frac{k}{\varepsilon})$, and returns a pdf such that is $\varepsilon$-close to the original distribution.

**Nonparametric mixtures** Compared to learning GMMs, less is known about nonparametric mixtures. One strand of literature beginning with [60] assumes that each $f_i$ is a product distribution while allowing each marginal to be nonparametric. In this case, the parameters $(w_i, f_i)$ are identifiable, and consistent estimators can be constructed [31, 22, 30]. Recently there has been progress on learning algorithms for this model [14, 52, 43, 29, 28]. We note also related work on nonparametric mixtures in the statistics literature [58, 3, 50, 65].

Another approach is to sidestep learning each $f_i$ explicitly, and instead to focus on clustering. If the point set is drawn from a distribution whose pdf is $f = \sum_{i=1}^k w_i f_i$, we can view each cluster as the samples drawn from a certain component $f_i$. [2] and [30] showed that if a point set which is divided into $k$ clusters satisfies some separation conditions then there is an algorithm that classifies the point set with a low misclassification rate. [37] provide an algorithm to learn the weights, means, and covariances for general log-concave mixtures under strong separation. In [17], Mixon et al. used semidefinite programming to classify the samples drawn from a mixture of sub-Gaussians, also under strong separation. Unfortunately, if one wants to invoke these clustering results in our setting, the separation parameter $R$ will inevitably depend on $\varepsilon$. 6
3 Proof overview

As the construction of the problem instances used to establish the lower bound in Theorem 2 is nontrivial and somewhat involved, we outline the main ideas behind this construction and relegate the full proof to Appendix A.

3.1 Approximation by Gaussians

Our goal is to construct two mixtures \( f = \frac{1}{2}f_1 + \frac{1}{2}f_2 \) and \( f' = \frac{1}{2}f'_1 + \frac{1}{2}f'_2 \) such that the difference between the components \( f_i \) and \( f'_i \) is large compared to the difference between the mixtures \( f \) and \( f' \). More specifically, we want the following:

\[
\|f_1 - f'_1\|_1 > \varepsilon, \quad \|f_2 - f'_2\|_1 > \varepsilon, \quad \|f - f'\|_1 < o(\varepsilon^c) \quad \text{for any } c > 0. \tag{5}
\]

Suppose \( f_1, f'_1 \in G_1 \), \( f_2, f'_2 \in G_2 \), and \( I_1 := [a_i, b_i] \). One way for \( \|f - f'\|_1 \) to be bounded above is that \( f_1 - f'_1 \) should assign nontrivial mass outside of the interval \( I_1 \), and similarly \( f_2 - f'_2 \) should assign nontrivial mass outside of \( I_2 \). A simple way to accomplish this is to have

\[
f_1 - f'_1 \approx \lambda \cdot (g_{a_2} - g_{b_1}) \quad \text{and} \quad f_2 - f'_2 \approx \lambda \cdot (g_{b_1} - g_{a_2})
\]

for some small constant \( \lambda \). Consequently, we have

\[
f - f' \approx \lambda \cdot (g_{a_2} - g_{b_1}) + \lambda \cdot (g_{b_1} - g_{a_2}) = 0.
\]

But then

\[
L_1 := \frac{1}{\lambda} (f_1 - f'_1) + g_{b_1} \approx g_{a_2}, \quad L_2 := \frac{1}{\lambda} (f_2 - f'_2) + g_{a_2} \approx g_{b_1}
\]

where \( L_1 \) is a linear combination of Gaussians centred inside \( I_1 \). Note that \( L_1 \notin G_1 \) since \( L_1 \) is a linear combination not a convex combination; that is, it may have large and negative coefficients in its expansion. A similar argument applies to \( L_2 \), which is a linear combination of Gaussians centred inside \( I_2 \).

Thus, as long as we can construct \( L_1 \) and \( L_2 \) along with the appropriate rates in (5), we can achieve the desired goal. The key to this construction is the surprising fact that a single Gaussian centred anywhere can be approximated extremely well by a linear combination of Gaussians centred at points in an arbitrary interval. In fact, we have the following crucial lemma, which is proved in Appendix A.

Let \( \text{Grid}(\Delta) \) be a grid of cell width \( \Delta > 0 \) over the reals, i.e.

\[
\text{Grid}(\Delta) = \{ j \cdot \Delta \mid j \text{ is an integer} \}.
\]

**Lemma 3.** Given any \( g_a \) and any interval \( I \) (possibly not containing \( a \)), there exists a linear combination \( \sum_{\mu \in I \cap \text{Grid}(\Delta)} \alpha_\mu g_\mu \) such that

\[
\|g_a - \sum_{\mu \in I \cap \text{Grid}(\Delta)} \alpha_\mu g_\mu\|_1 < 2^{-\Omega((1/\Delta) \log(1/\Delta))}
\]

Two aspects of this result are noteworthy: The approximation occurs on an arbitrary, bounded interval, and the approximation rate is much faster than typical approximation results. If we let \( m = \frac{1}{\Delta} - 1 \), which is proportional to the number of grid points, then the approximation rate scales as \( O(m^{-\Omega(m)}) \), compared to the usual \( O(m^{-\Omega(1)}) \) or \( O(e^{-m}) \) found in the approximation literature.

The fact that translates of a Gaussian are dense in \( L^2(\mathbb{R}) \) dates back to classical results such as Wiener’s approximation theorem [67]. A more recent result can be found in [10]; it is
Figure 1: Graph of \( \Pi_V(g_{-1}) \) (red) and \( g_{-1} \) (green)

worth noting that these types of approximation results also encompass the subject of Tauberian theory \([39]\). So it is not surprising that such approximations are possible, however, these results stop short of proving explicit approximation rates as in Lemma 3. Determining this fast rate is a crucial step in our proof: For example, if this rate were merely exponential \( O(e^{-m}) \), the desired superpolynomial lower bound would not follow.

In order to exploit this lemma, we will show
\[
\|f - f'\|_1 \approx \|g - \sum_{\mu \in I \cap \text{Grid}(\Delta)} \alpha_\mu g_\mu\|_1 \leq 2^{-\Omega((1/\Delta) \log(1/\Delta))} \text{ while } \|f_i - f'_i\|_1 \geq 2^{-O(1/\Delta)}. \]
It implies that \( \|f_i - f'_i\|_1 \) decays no faster than an exponential while \( \|f - f'\|_1 \) decays much faster at a factorial rate. By choosing \( \Delta \) carefully (we preview that \( \frac{1}{\Delta} = \Theta(\log \frac{1}{\epsilon}) \)), this will show that \( \|f_i - f'_i\|_1 \) decays no faster than \( \epsilon \) while \( \|f - f'\|_1 \) decays much faster at a super-polynomial rate of \( \epsilon \), as in \([3]\). This is enough to establish the desired lower bound.

### 3.2 Example

More formally, consider the following construction as an example to guide intuition. Suppose we have an interval \( I \) and a grid \( \text{Grid}(\Delta) \) of cell width \( \Delta \). If we project any Gaussian onto the subspace spanned by the Gaussians centered at the grid points \( I \cap \text{Grid}(\Delta) \), the projection by definition is a linear combination of the Gaussians centered at the grid points. Moreover, the Gaussian being projected is close to the linear combination. In other words, any Gaussian can be approximated by a linear combination of Gaussians centered at the grid points. We have
\[
g_a \approx \Pi_V(g_a) = \text{the projection of } g_a \text{ onto } V
\]
for any \( a \in \mathbb{R} \) where \( V = \text{span}\{g_\mu \mid \mu \in I \cap \text{Grid}(\Delta)\} \). The quantity \( \Delta \) defines the approximation quality, i.e. the smaller \( \Delta \) is the better the approximation is. We emphasize it is a linear combination, which means the coefficients in the linear combination can be negative and large in magnitude. Namely, we have
\[
\Pi_V(g_a) = \sum_{\mu \in I \cap \text{Grid}(\Delta)} \alpha_\mu g_\mu
\]
for some coefficients \( \alpha_\mu \in \mathbb{R} \). For example, take \( a = -1, \Delta = 0.2 \) and \( I = [0,1] \). We have
\[
\Pi_V(g_{-1}) = 80.609g_{0.0} - 260.774g_{0.2} + 331.9g_{0.4} - 195.489g_{0.6} + 44.741g_{0.8} \approx g_{-1}.
\]
See Figure 1. It is worth noting that the sum of the coefficients \( \alpha_\mu \) is close to 1. The sum of the coefficients in the example is
\[
80.609 - 260.774 + 331.9 - 195.489 + 44.741 = 0.987 \approx 1.
\]
Now, we split this linear combination into two parts: the Gaussians with positive coefficients and the Gaussians with negative coefficients. These two parts are two unnormalized interval Gaussians (the sums of the coefficients are not 1 in magnitude). From the above observation, the difference of these two interval Gaussians is roughly the Gaussian \( g \).

Suppose we have two disjoint intervals of the same length, \( I_0 \) and \( I_1 \). We construct two mixtures of two interval Gaussians as follows. Pick any point in one of the intervals \( I_1 \), say \( a \in I_1 \). Then, project the Gaussian centered at \( a \), \( g_a \), onto the subspace spanned by the Gaussians centered at points in the intersection of another interval \( I_0 \) and a grid \( \text{Grid}(\Delta) \) of cell width \( \Delta \). The projection is a linear combination of the Gaussians at points in \( I_0 \cap \text{Grid}(\Delta) \), i.e. the projection is on \( \mathcal{V} = \text{span}\{g_\mu \mid \mu \in I_0 \cap \text{Grid}(\Delta)\} \). By symmetry, we repeat the above process except that this time we pick a point \( a' \) in \( I_0 \) and project \( g_{a'} \) onto the subspace spanned by the Gaussians centered at points in \( I_1 \cap \text{Grid}(\Delta) \), i.e. the projection is on \( \mathcal{V}' = \text{span}\{g_\mu \mid \mu \in I_1 \cap \text{Grid}(\Delta)\} \). Again, we split both linear combinations into two parts and group them accordingly. For example, take \( I_0 = [0, 1] \), \( I_1 = [-2, -1] \), \( a = -1 \), \( a' = 0 \) and \( \Delta = 0.2 \). We have

\[
\Pi_{\mathcal{V}}(g_{-1}) = 80.609g_0 - 260.774g_{0.2} + 331.9g_{0.4} - 195.489g_{0.6} + 44.741g_{0.8} \approx g_{-1}
\]

and, by symmetry,

\[
\Pi_{\mathcal{V}'}(g_0) = 80.609g_{-1} - 260.774g_{-1.2} + 331.9g_{-1.4} - 195.489g_{-1.6} + 44.741g_{-1.8} \approx g_0
\]

The two mixtures of two interval Gaussians are

\[
f = \frac{1}{2} \left( \frac{80.609}{457.25}g_{-1} + \frac{331.9}{457.25}g_{-1.4} + \frac{44.741}{457.25}g_{-1.8} \right) + \frac{1}{2} \left( \frac{80.609}{457.25}g_0 + \frac{331.9}{457.25}g_{0.4} + \frac{44.741}{457.25}g_{0.8} \right)
\]

and

\[
f' = \frac{1}{2} \left( \frac{0.987}{457.25}g_{-1} + \frac{260.774}{457.25}g_{-1.2} + \frac{195.489}{457.25}g_{-1.6} \right) + \frac{1}{2} \left( \frac{0.987}{457.25}g_0 + \frac{260.774}{457.25}g_{0.2} + \frac{195.489}{457.25}g_{0.6} \right).
\]

Note that the factor \( \frac{1}{457.25} \) is the normalizing factor and the factor 0.987 is the sum of all coefficients. The additional terms 0.987\( g_{-1} \) and 0.987\( g_0 \) in \( f' \) ensure that the difference between these two mixture is small as we will see below. We examine the difference between \( f_1 \) and \( f'_1 \) as follows:

\[
f_1 - f'_1 = \frac{80.609}{457.25}g_{-1} + \frac{331.9}{457.25}g_{-1.4} + \frac{44.741}{457.25}g_{-1.8} - \frac{0.987}{457.25} \left( g_{-1} - 260.774g_{-1.2} - 195.489g_{-1.6} \right)
\]

\[
\approx \frac{1}{457.25}g_0 - \frac{0.987}{457.25}g_{-1} \quad \text{from (7)}
\]

Similarly, from (6), we have

\[
f_2 - f'_2 \approx \frac{1}{457.25}g_{-1} - \frac{0.987}{457.25}g_0.
\]

Hence,

\[
f - f' = \frac{1}{2}(f_1 + f_2) - \frac{1}{2}(f'_1 + f'_2) \approx \frac{1}{2} \cdot \frac{1}{457.25} (1 - 0.987)(g_0 + g_{-1}).
\]
Now, we have
\[ \| f_1 - f'_1 \|_1 \approx C_1 \cdot \frac{1}{457.25}, \quad \| f - f' \|_1 \approx C_2 \cdot \frac{1}{457.25} (1 - 0.987), \]
where \( C_1 \) and \( C_2 \) are some constants.

In general, as \( \Delta \to 0 \) (recall that we took \( \Delta = 0.2 \) in the example), the term \( \| f_1 - f'_1 \|_1 \) is bounded from below by the inverse of the sum of the absolute value of the coefficients in the linear combination (\( \frac{1}{457.25} \) in the example) which decays slower than the rate of \( 1/(2^\Omega(1/\Delta)) \). On the other hand, the term \( \| f - f' \|_1 \) is bounded from above by the product of the inverse of the sum of the absolute value of the coefficients in the linear combination and the difference between 1 and the sum of the coefficients (\( \frac{1}{457.25} (1 - 0.987) \) in the example) which decays faster than the rate of \( 1/2^\Omega((1/\Delta) \log(1/\Delta)) \).

### 4 Discussion on the upper bound

In this section, we briefly discuss the sample complexity upper bound for this problem. Previously (see Section 1.3) we mentioned that deconvolution \([70, 23, 45]\) can be used to design an algorithm with exponential sample complexity.

This algorithm can be constructed as follows. Recall that \( f = \nu * g_0 \), where \( \nu \) is the global mixing density \( \nu = \frac{1}{2} \nu_1 + \frac{1}{2} \nu_2 \). This global mixing density \( \nu \) can be estimated directly via deconvolution; call this estimate \( \hat{\nu} \). Since we assume that \( \nu \) is supported on the union of disjoint intervals, we can split \( \hat{\nu} \) (e.g. by thresholding) into two parts, \( \hat{\nu}_1 \) and \( \hat{\nu}_2 \), such that \( \hat{\nu} = \frac{1}{2} \hat{\nu}_1 + \frac{1}{2} \hat{\nu}_2 \) such that \( \hat{\nu}_i * g_0 \approx f_i \). Although classical results on deconvolution assume the mixing density satisfies smoothness assumptions (and we do not require such assumptions), we can relax the mixing density by smoothing first. This may impose large errors (in \( \ell_1 \)) on the mixing density; however, the guarantee we are looking for is on the components \( f_i \) not on the component mixing densities (i.e. \( \nu_i \)) themselves. It turns out such relaxation does not impose large errors on the components. Hence, we can apply existing deconvolution techniques to solve the problem. Nonetheless, if implemented, this deconvolution technique would require exponentially many samples to estimate the mixing density, and this dependence is tight. For this reason, we omit the details of this particular construction. Indeed, this motivates the following interesting question:

**Is there a subexponential time algorithm for learning the component densities \( f_i \)?**

If such an algorithm exists, this would establish that while smooth, nonparametric deconvolution requires exponential time and samples, the problem of finite, nonparametric deconvolution is much easier. Nonetheless, our lower bound shows that “easier” is relative: Finite deconvolution still requires super-polynomially many samples.

### 5 Conclusion

In this paper, we studied the problem of learning mixture components from a nonparametric mixture model \( f = \sum_{i=1}^k w_i f_i \) where each component \( f_i \) is an interval Gaussian. Given samples drawn from a nonparametric mixture model of this form, we are interested in the sample complexity for estimating the components \( f_i \). It is worth noting that we do not need to estimate each mixing density of the component and we only estimate the density of each component. Roughly speaking, our guarantee lies in between parameter learning and density estimation. We showed that super-polynomially many samples are required to achieve this guarantee. To the best of our knowledge, no such nontrivial lower bounds have been established previously.

There are various ways to extend the result. As discussed in the previous section, it would be interesting to have a matching sample complexity upper bound. If such an upper bound
exists, we can conclude that the optimal sample complexity of this problem lies in between superpolynomial and subexponential, which is not common in learning theory. Other potential future directions include generalizing to the case of non-interval Gaussians, analyzing the bounds for the number of components in the mixture, and extensions to the case of multiple hidden variables.

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A Detailed proofs of the main theorem

In this section, we will prove a sample complexity lower bound for estimating components in a mixture of Gaussians. Recall that we have the following notations. $g_\mu$ is the standard Gaussian centered at $\mu$. Given any interval $I$, $G_I$ is the set $\{ f \in P_\mathbb{R} \mid f = \int_{\mu \in I} \nu(\mu) g_\mu d\mu \text{ where } \nu \in P_I \}$ which we call each element in this set an interval Gaussian. From now on, we will consider $\ell_2$ norm instead of $\ell_1$ norm for analytical convenience, and resolve this issue later. Given any $\Delta > 0$, recall that $\text{Grid}(\Delta)$ is the set $\{ j \cdot \Delta \mid j \text{ is an integer} \}$. Namely, it is an infinite grid of cell width $\Delta$ on the real line. Without loss of generality, we assume that $\frac{1}{\Delta}$ is an integer and denote the number $\frac{1}{\Delta} - 1$ by $m$. Here, we abuse the notations that we treat $g_\mu$ as a vector and define the inner product $\langle g_{\mu_1}, g_{\mu_2} \rangle = \int_{x \in \mathbb{R}} g_{\mu_1}(x) g_{\mu_2}(x) dx$ for any $\mu_1, \mu_2 \in \mathbb{R}$ in the usual way. By a straightforward calculation, we have $\langle g_{\mu_1}, g_{\mu_2} \rangle = \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}(\mu_1 - \mu_2)^2}$ for any $\mu_1, \mu_2 \in \mathbb{R}$.

As we mentioned before, the key observation in our construction is the fact that any Gaussian can be approximated by a linear combination of Gaussians centered at points inside an interval. The approximation is based on the projection of a Gaussian onto a specific subspace. For simplicity, we will illustrate how to express $g_{-1}$, the Gaussian centered at $-1$, as a linear combination of the Gaussians centered at points inside $[0, 1]$. It is easy to generalize our argument to the case of arbitrary centers and intervals.

We break the proof down into the following steps. In Section A.1 we will explicitly construct two mixtures of two interval Gaussians, $f$ and $f'$, along the lines of Section 3.2. In Section A.2 we will analyze the convergence rate of $\|f - f'\|_2$, $\|f_1 - f'_1\|_2$ and $\|f_2 - f'_2\|_2$. Finally, in Section A.3 we will state and prove our main theorem (Theorem 2).

A.1 Construction

Let $v$ be $g_{-1}$ and $u_i$ be $g_{i \cdot \Delta}$ for $i = 0, \ldots, m$. Let $\mathcal{V}$ be the subspace span$\{ g_\mu \mid \mu \in \text{Grid}(\Delta) \cap [0, 1] \}$. Let $\Pi_{\mathcal{V}}(v)$ be the projection of $v$ onto the subspace $\mathcal{V}$. We can express $\Pi_{\mathcal{V}}(v)$ as

$$\Pi_{\mathcal{V}}(v) = \sum_{i=0}^{m} \alpha_i u_i$$
for some coefficients $\alpha_i$. Recall that it is a linear combination and therefore each $\alpha_i$ can be negative and large in magnitude. We now split this linear combination into two parts: the Gaussians with positive coefficients and the Gaussians with negative coefficients. Let $J_+ = \{ i \mid \alpha_i \geq 0 \}$ and $J_- = \{ i \mid \alpha_i < 0 \}$. We rewrite $\Pi_V(v)$ as

$$\Pi_V(v) = \sum_{i \in J_+} \alpha_i u_i - \sum_{i \in J_-} (-\alpha_i) u_i.$$  

By symmetry, let $v' = g_0$ and $u'_i = g_{-1-i\Delta}$ for $i = 0, \ldots, m$.

Let $\mathcal{V}'$ be the subspace span $\{g_\mu \mid \mu \in \text{Grid}(\Delta) \cap [-2, -1]\}$. We rewrite $\Pi_V(v')$ as

$$\Pi_V(v') = \sum_{i \in J_+} \alpha_i u'_i - \sum_{i \in J_-} (-\alpha_i) u'_i.$$  

Let

$$C_{\Delta,+} = \sum_{i \in J_+} \alpha_i, \quad C_{\Delta,-} = - \sum_{i \in J_-} \alpha_i,$$

i.e. $C_{\Delta,+}$ is the sum of the positive coefficients and $-C_{\Delta,-}$ is the sum of the negative coefficients.

Now, we are ready to construct two mixtures of two interval Gaussians. They are

$$f = \frac{1}{2} \frac{1}{C_{\Delta,+}} \left( \sum_{i \in J_+} \alpha_i u_i \right) + \frac{1}{2} \frac{1}{C_{\Delta,+}} \left( \sum_{i \in J_+} \alpha_i u'_i \right)$$  

and

$$f' = \frac{1}{2} \frac{1}{C_{\Delta,+}} \left( \sum_{i \in J_-} (-\alpha_i) u_i + (C_{\Delta,+} - C_{\Delta,-}) u_0 \right) + \frac{1}{2} \frac{1}{C_{\Delta,+}} \left( \sum_{i \in J_-} (-\alpha_i) u'_i + (C_{\Delta,+} - C_{\Delta,-}) u'_0 \right)$$

To ease the notations, we define

$$f_1 = \frac{1}{C_{\Delta,+}} \left( \sum_{i \in J_+} \alpha_i u_i \right), \quad f'_1 = \frac{1}{C_{\Delta,+}} \left( \sum_{i \in J_-} (-\alpha_i) u_i + (C_{\Delta,+} - C_{\Delta,-}) u_0 \right)$$  

$$f_2 = \frac{1}{C_{\Delta,+}} \left( \sum_{i \in J_+} \alpha_i u'_i \right), \quad f'_2 = \frac{1}{C_{\Delta,+}} \left( \sum_{i \in J_-} (-\alpha_i) u'_i + (C_{\Delta,+} - C_{\Delta,-}) u'_0 \right)$$

and therefore we have

$$f = \frac{1}{2} f_1 + \frac{1}{2} f_2, \quad f' = \frac{1}{2} f'_1 + \frac{1}{2} f'_2$$  

(8)

### A.2 Analysis of the convergence rate

Recall that our objective is to show that $\|f - f'\|_2$ is small while $\|f_1 - f'_1\|_2$ and $\|f_2 - f'_2\|_2$ are large. We will now examine the terms $f_1 - f'_1$, $f_2 - f'_2$ and $f - f'$. For the term $f_1 - f'_1$,

$$f_1 - f'_1 = \frac{1}{C_{\Delta,+}} \sum_{i \in J_+} \alpha_i u_i - \frac{1}{C_{\Delta,+}} \left( \sum_{i \in J_-} (-\alpha_i) u_i + (C_{\Delta,+} - C_{\Delta,-}) u_0 \right)$$  

$$= \frac{1}{C_{\Delta,+}} \Pi_V(v) - \frac{C_{\Delta,+} - C_{\Delta,-}}{C_{\Delta,+}} u_0.$$
since the definition of \( \Pi_V(v) \) is \( \sum_{i \in \mathcal{I}_+} \alpha_i u_i - \sum_{i \in \mathcal{I}_-} (-\alpha_i) u_i \). Similarly,

\[
f_2 - f'_2 = \frac{1}{C_{\Delta,+}} \Pi_V(v') - C_{\Delta,+} - C_{\Delta,-} u'_0.
\]

Intuitively, we have \( \Pi_V(v) \approx v = u'_0 \) and \( \Pi_V(v') \approx v' = u_0 \). Also, the sum of the coefficients \( C_{\Delta,+} - C_{\Delta,-} \) is close to 1. By plugging them into the expressions, we have

\[
f_1 - f'_1 \approx \frac{1}{C_{\Delta,+}} u'_0 - \frac{1}{C_{\Delta,+}} u_0 = \frac{1}{C_{\Delta,+}} (u'_0 - u_0) = \frac{1}{C_{\Delta,+}} (g_0 - g_1)
\]

and it implies

\[
\|f_1 - f'_1\|_2 \approx \frac{1}{C_{\Delta,+}} \|g_0 - g_1\|_2 = \Omega\left(\frac{1}{C_{\Delta,+}}\right).
\]

For the term \( f - f' \), we have

\[
f - f' = \frac{1}{2C_{\Delta,+}} \Pi_V(v) - \frac{C_{\Delta,+} - C_{\Delta,-}}{2C_{\Delta,+}} u_0 + \frac{1}{2C_{\Delta,+}} \Pi_V(v') - \frac{C_{\Delta,+} - C_{\Delta,-}}{2C_{\Delta,+}} u'_0
\]

\[
= \frac{1}{2C_{\Delta,+}} \left( (\Pi_V(v) - u'_0) + (\Pi_V(v') - u_0) - (C_{\Delta,+} - C_{\Delta,-} - 1)(u_0 + u'_0) \right)
\]

Since \( \Pi_V(v) \) is the projection of \( v \) onto the subspace \( V \), we have \( \|\Pi_V(v)\|_2^2 + \|\Pi_V(v) - v\|_2^2 = \|v\|_2^2 \) by Pythagorean theorem. Let

\[
\beta_{\Delta} = \frac{\|v - \Pi_V(v)\|_2}{\|v\|_2}.
\]

The term \( \beta_{\Delta} \) defines how close is \( v \) to \( \Pi_V(v) \). By symmetry, \( \|v' - \Pi_V(v')\|_2 = \beta_{\Delta} \|v'\|_2 \). Hence,

\[
\|f - f'\|_2 \leq \frac{1}{2C_{\Delta,+}} \left( \|\Pi_V(v) - u'_0\|_2 + \|\Pi_V(v') - u_0\| + |C_{\Delta,+} - C_{\Delta,-} - 1| \|u_0 + u'_0\|_2 \right)
\]

\[
= \frac{1}{2C_{\Delta,+}} \left( \beta_{\Delta} \|v\|_2 + \beta_{\Delta} \|v'\|_2 + |C_{\Delta,+} - C_{\Delta,-} - 1| \|u_0 + u'_0\|_2 \right)
\]

\[
= O\left(\frac{1}{C_{\Delta,+}} \max\{\beta_{\Delta},|C_{\Delta,+} - C_{\Delta,-} - 1|\} \right).
\]

In other words, we reduce the problem of bounding the terms \( \|f - f'\|_2, \|f_0 - f'_0\|_2, \|f_1 - f'_1\|_2 \) to the problem of analyzing the terms \( \beta_{\Delta}, C_{\Delta,+}, C_{\Delta,-} \).

To analyze the terms \( C_{\Delta,+}, C_{\Delta,-} \) and \( \beta_{\Delta} \), we need to express these terms more explicitly. Recall that these terms are related to the coefficients of the linear combination for the projection of \( v \) onto the subspace \( V \). When we project a vector onto a subspace, it is useful to first find an orthogonal basis for the subspace. By Gram Schmidt process, we define the orthogonal basis \( \tilde{u}_0, \ldots, \tilde{u}_m \) as follows.

\[
\tilde{u}_0 = u_0 \quad \text{and} \quad \tilde{u}_i = u_i - \sum_{j=0}^{i-1} \frac{\langle u_i, \tilde{u}_j \rangle}{\langle \tilde{u}_j, \tilde{u}_j \rangle} \tilde{u}_j \quad \text{for} \quad i > 0
\]

Note that they are not normalized, i.e. \( \|\tilde{u}_i\|_2 \neq 1 \). Another way of expressing \( \Pi_V(v) \) is through the orthogonal basis \( \tilde{u}_0, \ldots, \tilde{u}_m \). Namely,

\[
\Pi_V(v) = \sum_{i=0}^{m} \langle v, \tilde{u}_i \rangle \frac{\tilde{u}_i}{\|\tilde{u}_i\|_2^2}
\]

The advantage of this expression is that we can compute the coefficients \( \langle v, \tilde{u}_i \rangle \) explicitly as we will show below. Lemma [4] gives an explicit formula for \( \langle g_a, \tilde{u}_i \rangle \) that depends on \( \Delta \) only and it further gives an explicit formula for the coefficients \( \langle v, \tilde{u}_i \rangle \|\tilde{u}_i\|_2^{-2} \).
Lemma 4. For any $a \in \mathbb{R}$, we have

$$\langle g_a, \tilde{u}_i \rangle = \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}a^2} e^{-\frac{i}{4}\Delta^2} \prod_{k=1}^{i} (e^{\frac{1}{2}a\Delta - \frac{(k-1)}{2}\Delta^2} - 1)$$

for $i = 0, 1, \ldots, m$. In particular, if we set $a = -1$ we have

$$\langle v, \tilde{u}_i \rangle = \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}} e^{-\frac{i}{4}\Delta^2} \prod_{k=1}^{i} (e^{-\frac{1}{2}\Delta - \frac{(k-1)}{2}\Delta^2} - 1)$$

and if we set $a = i \cdot \Delta$ we have

$$||\tilde{u}_i||^2 = \frac{1}{\sqrt{4\pi}} \prod_{k=1}^{i} (1 - e^{-\frac{k}{4}\Delta^2}).$$

Proof. We will prove the lemma by induction. Recall that

$$\tilde{u}_0 = u_0 \quad \text{and} \quad \tilde{u}_i = u_i - \sum_{j=0}^{i-1} \frac{\langle u_i, \tilde{u}_j \rangle}{\langle \tilde{u}_j, \tilde{u}_j \rangle} \tilde{u}_j \quad \text{for} \quad i > 0$$

Assuming that the statement holds for $j < i$, i.e.

$$\langle g_a, \tilde{u}_j \rangle = \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}a^2} e^{-\frac{j}{4}\Delta^2} \prod_{k=1}^{j} (e^{\frac{1}{2}a\Delta - \frac{(k-1)}{2}\Delta^2} - 1) \quad \text{for any} \quad a \in \mathbb{R}$$

Note that we also have $\langle g_a, u_i \rangle = \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}(a-i\Delta)^2} = \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}a^2} e^{\frac{1}{2}ia\Delta} e^{-\frac{1}{4}i^2\Delta^2}$. Now, we express $\langle g_a, \tilde{u}_i \rangle$.

$$\langle g_a, \tilde{u}_i \rangle = \langle g_a, u_i \rangle - \sum_{j=0}^{i-1} \frac{\langle u_i, \tilde{u}_j \rangle}{\langle \tilde{u}_j, \tilde{u}_j \rangle} \tilde{u}_j$$

$$= \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}a^2} e^{\frac{1}{2}ia\Delta} e^{-\frac{1}{4}i^2\Delta^2} - \sum_{j=0}^{i-1} \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}a^2} e^{-\frac{j}{4}\Delta^2} \prod_{k=1}^{j} (e^{\frac{1}{2}a\Delta - \frac{(k-1)}{2}\Delta^2} - 1) \frac{\langle u_i, \tilde{u}_j \rangle}{\langle \tilde{u}_j, \tilde{u}_j \rangle}$$

Consider the polynomial $P(y) = e^{-\frac{1}{4}i^2\Delta^2} y^i - \sum_{j=0}^{i-1} e^{-\frac{j}{4}\Delta^2} \prod_{k=1}^{j} (e^{-\frac{(k-1)}{2}\Delta^2} y - 1) \frac{\langle u_i, \tilde{u}_j \rangle}{\langle u_i, \tilde{u}_j \rangle}$. Namely, we can rewrite $\langle g_a, \tilde{u}_i \rangle$ as

$$\langle g_a, \tilde{u}_i \rangle = \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}a^2} P(e^{\frac{1}{2}a\Delta}) \quad \text{for any} \quad a \in \mathbb{R}$$

$P$ is a polynomial of degree $i$ and we will argue that $1, e^{\frac{1}{2}\Delta^2}, \ldots, e^{\frac{i-1}{2}\Delta^2}$ are its roots. By setting $a = j \cdot \Delta$, we have

$$\langle u_j, \tilde{u}_i \rangle = \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}j^2\Delta^2} P(e^{\frac{1}{2}j\Delta})$$

for $j = 0, \ldots, i - 1$. Since $u_j$ lies on span$\{\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_j\}$, we have $\langle u_j, \tilde{u}_i \rangle = 0$ by the definition of Gram Schmidt process. It implies that $P(e^{\frac{1}{2}i\Delta}) = 0$.

Since we have all roots of $P$, we can rewrite $P(y)$ as

$$P(y) = C \prod_{j=0}^{i-1} (y - e^{\frac{j}{2}\Delta})$$

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for some absolute constant $C$. By comparing the coefficient of $y^2$, we have $C = e^{-\frac{1}{2} \Delta^2}$. Therefore,

$$\langle g_a, \tilde{u}_i \rangle = \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}a^2} \mathcal{P}(e^{\frac{1}{2}a\Delta})$$

$$= \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}a^2} e^{-\frac{1}{2}i^2\Delta^2} \prod_{j=0}^{i-1} (e^{\frac{1}{2}a\Delta} - e^{\frac{1}{2}j\Delta})$$

$$= \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}a^2} e^{-\frac{1}{4}\Delta^2} \prod_{k=1}^{i} (e^{\frac{1}{2}a\Delta} - (\frac{k-1}{2})\Delta^2 - 1)$$

and we proved the statement.

Now, we are ready to analyze the terms $C_{\Delta,+}, C_{\Delta,-}$ and $\beta_{\Delta}$ explicitly through the orthogonal basis $\tilde{u}_i$ since Lemma 4 gives us an explicit formula in terms of $\Delta$ only. Lemma 5, Lemma 6 and Lemma 8 give us the bounds we need to bound the terms $\|f - f'\|_2$, $\|f_1 - f'_1\|_2$ and $\|f_2 - f'_2\|_2$.

**Lemma 5.** For any sufficiently small $\Delta > 0$, we have

$$C_{\Delta,+} + C_{\Delta,-} \leq 2^{O(1/\Delta)}$$

**Proof.** Recall that $\mathcal{V}$ is the subspace span\{u_0, \ldots, u_m\} and $\Pi_{\mathcal{V}}(g_a)$ is the projection of $g_a$ on the subspace $\mathcal{V}$. We have

$$\Pi_{\mathcal{V}}(g_a) = \sum_{i=0}^{m} \langle g_a, \tilde{u}_i \rangle \tilde{u}_i$$

From Lemma 4, we plug the expressions into each coefficient $\langle g_a, \tilde{u}_i \rangle$ in the above formula.

$$\langle g_a, \tilde{u}_i \rangle = \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}a^2} e^{-\frac{1}{4}\Delta^2} \prod_{k=1}^{i} (e^{\frac{1}{2}a\Delta} - (\frac{k-1}{2})\Delta^2 - 1)$$

Consider the polynomial $\mathcal{P}_i(y) = e^{-\frac{1}{4}\Delta^2} \prod_{k=1}^{i} (e^{\frac{1}{2}a\Delta} - (\frac{k-1}{2})\Delta^2 - 1)$. Namely, we can rewrite $\Pi_{\mathcal{V}}(g_a)$ as

$$\Pi_{\mathcal{V}}(g_a) = \sum_{i=0}^{m} e^{-\frac{1}{4}a^2} \mathcal{P}_i(e^{\frac{1}{2}a\Delta}) \tilde{u}_i$$

Recall that Gram Schmidt process gives us the following connection between $u_i$ and $\tilde{u}_i$.

$$\tilde{u}_0 = u_0 \quad \text{and} \quad \tilde{u}_i = u_i - \sum_{j=0}^{i-1} \frac{\langle u_i, \tilde{u}_j \rangle}{\langle \tilde{u}_i, \tilde{u}_j \rangle} \tilde{u}_j \quad \text{for } i > 0$$

Hence, we have

$$\Pi_{\mathcal{V}}(g_a) = \sum_{i=0}^{m} e^{-\frac{1}{4}a^2} \mathcal{P}_i(e^{\frac{1}{2}a\Delta}) \tilde{u}_i = \sum_{i=0}^{m} e^{-\frac{1}{4}a^2} \mathcal{P}_i(e^{\frac{1}{2}a\Delta}) \tilde{u}_i + e^{-\frac{1}{4}a^2} \mathcal{P}_m(e^{\frac{1}{2}a\Delta}) \tilde{u}_m$$

$$= \sum_{i=0}^{m-1} e^{-\frac{1}{4}a^2} \mathcal{P}_i(e^{\frac{1}{2}a\Delta}) \tilde{u}_i + e^{-\frac{1}{4}a^2} \mathcal{P}_m(e^{\frac{1}{2}a\Delta})(u_m - \sum_{i=0}^{m-1} \frac{\langle u_m, \tilde{u}_i \rangle}{\langle \tilde{u}_i, \tilde{u}_i \rangle} \tilde{u}_i)$$

$$= \sum_{i=0}^{m-1} e^{-\frac{1}{4}a^2} \mathcal{P}_i(e^{\frac{1}{2}a\Delta}) - \frac{\langle u_m, \tilde{u}_i \rangle}{\langle \tilde{u}_i, \tilde{u}_i \rangle} \mathcal{P}_m(e^{\frac{1}{2}a\Delta}) \tilde{u}_i + e^{-\frac{1}{4}a^2} \mathcal{P}_m(e^{\frac{1}{2}a\Delta}) u_m$$

a polynomial in $e^{\frac{1}{2}a\Delta}$ of degree $m$.
and if we recursively plug in $\tilde{u}_i$ then it is easy to see that we can further rewrite $\Pi_V(g_a)$ as

$$
\Pi_V(g_a) = \sum_{i=0}^{m} e^{-\frac{1}{4} i^2 \Delta^2} Q_i(e^{\frac{1}{2} i \Delta}) u_i
$$

for some polynomials $Q_i$ of degree $m$. Since $u_j \in V$, we have $\Pi_V(u_j) = u_j$ by definition. It implies that $e^{-\frac{1}{4} i^2 \Delta^2} Q_i(e^{\frac{1}{2} i \Delta}) = 0$ for all $j = 0, 1, \ldots, i - 1, i + 1, \ldots, m$ by the fact that $u_0, \ldots, u_m$ are linearly independent. It means that we have all roots of $Q_i$ and hence

$$
Q_i(y) = C_i \prod_{j=0,i\neq j}^{m} (y - e^{\frac{1}{2} j \Delta^2})
$$

for some absolute constant $C_i$. We also have $e^{-\frac{1}{4} i^2 \Delta^2} Q_i(e^{\frac{1}{2} i \Delta}) = 1$ and therefore

$$
Q_i(y) = e^{\frac{1}{4} i^2 \Delta^2} \frac{\prod_{j=0,i\neq j}^{m} (y - e^{\frac{1}{2} j \Delta^2})}{\prod_{j=0,i\neq j}^{m} (e^{\frac{1}{2} j \Delta^2} - e^{\frac{1}{2} j \Delta^2})}
$$

In particular,

$$
\alpha_i = e^{-\frac{1}{4} Q_i(e^{\frac{1}{2} \Delta})} = e^{-\frac{1}{4} e^{\frac{1}{4} i^2 \Delta^2} \prod_{j=0,i\neq j}^{m} (e^{\frac{1}{2} j \Delta^2} - e^{\frac{1}{2} j \Delta^2})}
$$

Furthermore, recall that $C_{\Delta,+}$ and $C_{\Delta,-}$ are defined as $C_{\Delta,+} = \sum_{i \in J_+} \alpha_i$ and $C_{\Delta,-} = \sum_{i \in J_-} -\alpha_i$ respectively. In other words, $C_{\Delta,+} + C_{\Delta,-} = \sum_{i=0}^{m} |\alpha_i|$. We need to give a bound for each $|\alpha_i|$. A useful inequality is $1 + x \leq e^x \leq 1 + x + x^2$ for any sufficiently small $x$. For $i > j$, the factors in the denominator of the fraction in $|\alpha_i|$ is

$$
|e^{\frac{1}{2} i \Delta^2} - e^{\frac{1}{2} j \Delta^2}| \geq 1 + \frac{1}{2} |i \Delta^2 - j \Delta^2 - \frac{1}{2} j \Delta^2 - \frac{1}{2} j \Delta| = \frac{1}{2} (i - j) \Delta^2 - \frac{1}{4} j^2 \Delta^4.
$$

Since $j \leq m = \frac{1}{\Delta} - 1 \leq \frac{1}{\Delta}$ which implies $j^2 \Delta^4 \leq \Delta^2 \leq (i - j) \Delta^2$, we have $|e^{\frac{1}{2} i \Delta^2} - e^{\frac{1}{2} j \Delta^2}| \geq \frac{1}{4} (i - j) \Delta^2$. Similarly, for $j > i$, we have $|e^{\frac{1}{2} j \Delta^2} - e^{\frac{1}{2} i \Delta^2}| \geq \frac{1}{4} (j - i) \Delta^2$. Hence,

$$
| \prod_{j=0,i\neq j}^{m} (e^{\frac{1}{2} i \Delta^2} - e^{\frac{1}{2} j \Delta^2}) | \geq \left( \frac{\Delta^2}{4} \right)^m \cdot i!(m - i)!
$$

On the other hand, the factors in the numerator of the fraction in $|\alpha_i|$ is

$$
|e^{-\frac{1}{4} \Delta^2} - e^{\frac{1}{2} j \Delta^2}| \leq 1 + \frac{1}{2} j \Delta^2 - \frac{1}{2} j \Delta = \frac{1}{2} (m + 1 + j) \Delta^2 + \frac{1}{4} j^2 \Delta^4
$$

Since $j \leq \frac{1}{\Delta}$ which implies $j^2 \Delta^4 \leq (m + 1 + j) \Delta^2$, we have $|e^{-\frac{1}{4} \Delta^2} - e^{\frac{1}{2} j \Delta^2}| \leq \frac{3}{4} (m + 1 + j) \Delta^2$. Hence,

$$
| \prod_{j=0,i\neq j}^{m} (e^{-\frac{1}{4} \Delta^2} - e^{\frac{1}{2} j \Delta^2}) | \leq \frac{1}{m + 1 + i} \left( \frac{3 \Delta^2}{4} \right)^m \cdot \frac{(2m + 1)!}{m!} \leq \left( \frac{3 \Delta^2}{4} \right)^m \cdot \frac{(2m + 1)!}{m!}
$$

Combining the above inequalities,

$$
|\alpha_i| \leq e^{-\frac{1}{4} \cdot \frac{1}{4} i^2 \Delta^2} \cdot 3^m \cdot (m + 1) \cdot \left( \frac{(2m + 1)!}{m!(m + 1)!} \right) \cdot \left( \frac{m!}{i!(m - i)!} \right) = 2^{O(1/\Delta)}
$$

and therefore

$$
C_{\Delta,+} + C_{\Delta,-} = 2^{O(1/\Delta)}.
$$
Lemma 6. For any sufficiently small $\Delta > 0$, we have

$$\beta_\Delta \leq \frac{1}{2^{\Omega((1/\Delta) \log(1/\Delta))}}$$

Proof. Recall that the definition of $\beta_\Delta$ is $\frac{\| \Pi_Y(v) - v \|^2_2}{\| v \|^2_2}$. Hence, $\beta_\Delta^2 = 1 - \frac{\| \Pi_Y(v) \|^2_2}{\| v \|^2_2}$. We first express $\frac{\| \Pi_Y(v) \|^2_2}{\| v \|^2_2}$ in an explicit formulation. Recall that

$$\Pi_Y(v) = \sum_{i=0}^m \langle v, \tilde{u}_i \rangle \tilde{u}_i.$$ 

By Pythagorean theorem, we have

$$\| \Pi_Y(v) \|^2_2 = \sum_{i=0}^m \langle v, \tilde{u}_i \rangle^2 \| \tilde{u}_i \|^2_2.$$ 

From Lemma 4 each term can be expressed as

$$\langle v, \| \tilde{u}_i \|^2_2 \rangle = e^{-\frac{1}{2} \cdot (1 - e^{-\frac{1}{2} \Delta^2})^2 \| \tilde{u}_i \|^2_2 \Delta^2}$$

To ease the notations, we consider the following sequences. For any $y < 1$,

$$S_0 = 1 - y^{(m+1)^2} \quad \text{and} \quad S_i = S_{i-1} - y^{(m+1)^2 + i}T_i \quad \text{for } i > 0$$

where $T_i = \prod_{j=1}^i \frac{(1-y^{m+j})^2}{1-y^j}$ and $m = \frac{1}{\Delta} - 1$. In other words, we replace $e^{-\frac{1}{2} \Delta^2}$ with $y$ and $1 - S_i$ is the sum of the first $i + 1$ terms of $\sum_{j=0}^m \langle v, \tilde{u}_i \rangle^2 \| \tilde{u}_i \|^2_2$ which means $S_m = 1 - \frac{\| \Pi_Y(v) \|^2_2}{\| v \|^2_2} = \beta_\Delta^2$.

Before we go into the detailed analysis, we first state a useful lemma which gives the explicit formulas for $S_i$ and $T_i$. We delay the proof to the end of this proof.

Lemma 7. Let $S_i$ and $T_i$ be the recurrence sequence defined in the proof of Lemma 6. We have

$$S_{i-1} = \prod_{j=1}^i (1 - y^{m+j}) \cdot \left( \sum_{(k_1, k_2, \ldots, k_{i}) \in \mathcal{I}_i} y^{\sum_{j=1}^i (m+j)k_j} \right)$$

$$T_i = \prod_{j=1}^i (1 - y^{m+j}) \cdot \left( \sum_{(k_1, k_2, \ldots, k_{i}) \in \mathcal{I}_i} y^{\sum_{j=1}^i jk_j} \right)$$

where $\mathcal{I}_i = \{(k_1, k_2, \ldots, k_{i}) \mid 0 \leq k_1 \leq k_2 \leq \cdots \leq k_1 + \cdots + k_{i} \leq m\}$.

In Lemma 4 we have the closed form of $S_i$ and $T_i$,

$$S_{i-1} = \prod_{j=1}^i (1 - y^{m+j}) \cdot \left( \sum_{(k_1, k_2, \ldots, k_{i}) \in \mathcal{I}_i} y^{\sum_{j=1}^i (m+j)k_j} \right)$$

$$T_i = \prod_{j=1}^i (1 - y^{m+j}) \cdot \left( \sum_{(k_1, k_2, \ldots, k_{i}) \in \mathcal{I}_i} y^{\sum_{j=1}^i jk_j} \right)$$

where $\mathcal{I}_i = \{(k_1, k_2, \ldots, k_{i}) \mid 0 \leq k_1 \leq k_2 \leq \cdots \leq k_1 + \cdots + k_{i} \leq m\}$. It is easy to see that $S_{i-1} \leq T_i$ since there is a one-to-one corresponding mapping between the terms $y^{\sum_{j=1}^i (m+j)k_j}$ and $y^{\sum_{j=1}^i jk_j}$ and $\sum_{j=1}^i (m+j)k_j \geq \sum_{j=1}^i jk_j$ with the fact $y < 1$. Now,

$$\beta_\Delta^2 = S_m \leq T_{m+1} = \prod_{j=1}^{m+1} \frac{(1 - y^{m+j})^2}{1 - y^j} = \prod_{j=1}^{m+1} \frac{(1 - e^{-\frac{1}{2} \Delta^2 - \frac{1}{2} j \Delta^2})^2}{1 - e^{-\frac{1}{2} j \Delta^2}} \quad \text{when } y = e^{-\frac{1}{2} \Delta^2}$$

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A useful inequality is $1 + x \leq e^x \leq 1 + 2x$ for any sufficiently small $x$. Then, we have

$$\prod_{j=1}^{m+1} (1 - e^{-\frac{j}{2} \Delta^2}) \geq \prod_{j=1}^{m+1} (1 - (1 - 2 \frac{j}{2} \Delta^2)) = \Delta^{2(m+1)} \cdot (m + 1)!$$

and, since $j - 1 \leq m = \frac{1}{\Delta} - 1 \leq \frac{1}{\Delta}$,

$$\prod_{j=1}^{m+1} (1 - e^{-\frac{j}{2} \Delta} - \frac{i}{2} \Delta^2)^2 \leq \prod_{j=1}^{m+1} (1 - (1 - 2 \frac{j}{2} \Delta - \frac{j-1}{2} \Delta^2))^2 \leq \Delta^{2(m+1)}$$

We conclude that

$$\beta \Delta \leq \sqrt{\frac{\Delta^{2(m+1)}}{\Delta^{2(m+1)} \cdot (m + 1)!}} = \sqrt{\frac{1}{(m + 1)!}} = \frac{1}{\sqrt{2\Omega(m \log m)}} = \frac{1}{\sqrt{2\Omega((1/\Delta) \log(1/\Delta))}}$$

\[
\text{of Lemma} \quad \square
\]

We first prove the expression for $T_i$. When $i = 1$, we have

$$T_1 = \frac{(1 - y^{m+1})^2}{1 - y} = (1 - y^{m+1})(\sum_{k_1=0}^{m} y^{k_1})$$

By induction, we have

$$T_{i+1} = T_i \cdot \frac{(1 - y^{m+1+i})^2}{1 - y^{i+1}}$$

$$= \prod_{j=1}^{i+1}(1 - y^{m+j}) \cdot \left( \sum_{(k_1,k_2,...,k_i) \in I_i} y^{\sum_{j=1}^{i} jk_j} \right) \cdot \frac{(1 - y^{m+1+i})^2}{1 - y^{i+1}}$$

$$= \prod_{j=1}^{i+1}(1 - y^{m+j}) \cdot \left( \sum_{(k_1,k_2,...,k_i) \in I_i} y^{\sum_{j=1}^{i} jk_j} \right) \cdot \frac{1 - y^{m+1+i}}{1 - y^{i+1}}$$

In other words, we want to prove

$$\left( \sum_{(k_1,k_2,...,k_i) \in I_i} y^{\sum_{j=1}^{i} jk_j} \right) \cdot \frac{1 - y^{m+1+i}}{1 - y^{i+1}} = \sum_{(k_1,k_2,...,k_{i+1}) \in I_{i+1}} y^{\sum_{j=1}^{i+1} jk_j}$$

We are now examining

$$\left( \sum_{(k_1,k_2,...,k_{i+1}) \in I_{i+1}} y^{\sum_{j=1}^{i+1} jk_j} \right) \cdot (1 - y^{i+1})$$

$$= \sum_{(k_1,k_2,...,k_{i+1}) \in I_{i+1}} y^{\sum_{j=1}^{i+1} jk_j} - \sum_{(k_1,k_2,...,k_{i+1}) \in I_{i+1}} y^{\sum_{j=1}^{i} jk_j + (i+1)(k_{i+1}+1)}$$

$$= \sum_{(k_1,k_2,...,k_{i}) \in I_{i}} \left( \sum_{k_{i+1}=0}^{\ell} y^{\sum_{j=1}^{i+1} jk_j} - \sum_{k_{i+1}=0}^{\ell} y^{\sum_{j=1}^{i} jk_j + (i+1)(k_{i+1}+1)} \right)$$

where $\ell = m - \sum_{j=1}^{i} k_j$. We fix the indices $k_1, \ldots, k_i$ and consider the summation with the index $k_{i+1}$. We have

$$\sum_{k_{i+1}=0}^{\ell} y^{\sum_{j=1}^{i+1} jk_j} - \sum_{k_{i+1}=0}^{\ell} y^{\sum_{j=1}^{i} jk_j + (i+1)(k_{i+1}+1)} = y^{\sum_{j=1}^{i} jk_j} - y^{\sum_{j=1}^{i} jk_j + (i+1)\ell}$$

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Note that
\[ \sum_{j=1}^{i} jk_j + (i + 1)(\ell + 1) = m + 1 + i + \sum_{i'=1}^{i} (m - \sum_{j=1}^{i'} k_j) \]
and, by change of variables, \( \mathcal{I}_i = \left\{ (k_2, \ldots, k_i, m - \sum_{j=1}^{i} k_j) \mid (k_1, k_2, \ldots, k_i) \in \mathcal{I}_i \right\} \). Therefore, we have
\[
\left( \sum_{(k_1, k_2, \ldots, k_i+1) \in \mathcal{I}_{i+1}} y^{i+1}_{\sum_{j=1}^{i+1} jk_j} \right) \cdot (1 - y^{i+1})
= \sum_{(k_1, k_2, \ldots, k_i) \in \mathcal{I}_i} \left( \sum_{k_{i+1}=0}^{\ell} y^{\sum_{j=1}^{i+1} jk_j} - \sum_{k_{i+1}=0}^{\ell} y^{\sum_{j=1}^{i} jk_j + (i+1)(k_{i+1}+1)} \right)
= \sum_{(k_1, k_2, \ldots, k_i) \in \mathcal{I}_i} \left( y^{\sum_{j=1}^{i} jk_j} - y^{\sum_{j=1}^{i} jk_j + (i+1)(\ell+1)} \right)
= \sum_{(k_1, k_2, \ldots, k_i) \in \mathcal{I}_i} \left( y^{\sum_{j=1}^{i} jk_j} - y^{m+1+i+\sum_{j=1}^{i'} (m-\sum_{j=1}^{i} k_j)} \right)
= \left( \sum_{(k_1, k_2, \ldots, k_i) \in \mathcal{I}_i} y^{\sum_{j=1}^{i} jk_j} \right) (1 - y^{m+1+i})
\]

Now, we will prove the expression for \( S_{i-1} \). When \( i = 0 \),
\[
S_0 = 1 - y^{(m+1)^2} = (1 - y^{m+1}) \left( \sum_{k_1=0}^{m} y^{k_1(m+1)} \right)
\]

By induction,
\[
S_i = S_{i-1} - y^{(m+1)^2+i} \mathcal{I}_i
= \prod_{j=1}^{i} (1 - y^{m+j}) \cdot \left( \sum_{(k_1, k_2, \ldots, k_i) \in \mathcal{I}_i} y^{\sum_{j=1}^{i} (m+j)k_j} \right)
- y^{(m+1)^2+i} \prod_{j=1}^{i} (1 - y^{m+j}) \cdot \left( \sum_{(k_1, k_2, \ldots, k_i) \in \mathcal{I}_i} y^{\sum_{j=1}^{i} jk_j} \right)
= \prod_{j=1}^{i} (1 - y^{m+j}) \cdot \left( \sum_{(k_1, k_2, \ldots, k_i) \in \mathcal{I}_i} y^{\sum_{j=1}^{i} (m+j)k_j} - \sum_{(k_1, k_2, \ldots, k_i) \in \mathcal{I}_i} y^{(m+1)^2+i+\sum_{j=1}^{i'} (m-\sum_{j=1}^{i} k_j)} \right)
\]

In other words, we need to prove
\[
\sum_{(k_1, k_2, \ldots, k_i) \in \mathcal{I}_i} y^{\sum_{j=1}^{i} (m+j)k_j} - \sum_{(k_1, k_2, \ldots, k_i) \in \mathcal{I}_i} y^{(m+1)^2+i+\sum_{j=1}^{i} jk_j}
= (1 - y^{m+1+i}) \left( \sum_{(k_1, k_2, \ldots, k_{i+1}) \in \mathcal{I}_{i+1}} y^{\sum_{j=1}^{i+1} (m+j)k_j} \right)
\]

Note that, by change of variables, \( \mathcal{I}_i = \left\{ (k_2, \ldots, k_i, m - \sum_{j=1}^{i} k_j) \mid (k_1, k_2, \ldots, k_i) \in \mathcal{I}_i \right\} \) and therefore
\[
\sum_{(k_1, k_2, \ldots, k_i) \in \mathcal{I}_i} y^{(m+1)^2+i+\sum_{j=1}^{i} jk_j} = \sum_{(k_1, k_2, \ldots, k_i) \in \mathcal{I}_i} y^{(m+1)^2+i+\sum_{j=1}^{i'} (m-\sum_{j=1}^{i} k_j)}.
\]
Also,
\[
(m + 1)^2 + i + \sum_{j=1}^{i'} (m - \sum_{j=1}^{i'} k_j) = \sum_{j=1}^{i} (m + j)k_j + (m + 1 + i)(m + 1 - \sum_{j=1}^{i} k_j)
\]
Hence, we have
\[
\sum_{(k_1, k_2, \ldots, k_i) \in I_i} y^{\sum_{j=1}^{i} (m+j)k_j} - \sum_{(k_1, k_2, \ldots, k_i) \in I_i} y^{(m+1)^2+i+\sum_{j=1}^{i} jk_j} \\
= \sum_{(k_1, k_2, \ldots, k_i) \in I_i} y^{\sum_{j=1}^{i} (m+j)k_j} - \sum_{(k_1, k_2, \ldots, k_i) \in I_i} y^{\sum_{j=1}^{i} (m+j)k_j + (m+1+i)(m+1-\sum_{j=1}^{i} k_j)} \\
= (1 - y^{m+1+i}) \left( \sum_{(k_1, k_2, \ldots, k_{i+1}) \in I_{i+1}} y^{\sum_{j=1}^{i+1} (m+j)k_j} \right)
\]
\]

Moreover, we want to analyze how close to 1 the term \(C_{\Delta,+} - C_{\Delta,-}\) is.

**Lemma 8.** For any sufficiently small \(\Delta > 0\), we have
\[
|C_{\Delta,+} - C_{\Delta,-} - 1| \leq \frac{1}{2^H((1/\Delta) \log(1/\Delta))}.
\]

**Proof.** By the definition of \(C_{\Delta,+}\) and \(C_{\Delta,-}\),
\[
C_{\Delta,+} = \sum_{i \in J_+} \alpha_i \quad \text{and} \quad C_{\Delta,-} = \sum_{i \in J_-} -\alpha_i
\]
where \(J_+ = \{i \mid \alpha_i \geq 0\}\) and \(J_- = \{i \mid \alpha_i < 0\}\). Also, the coefficients \(\alpha_i\) satisfy \(\Pi_Y(v) = \sum_{i=0}^{m} \alpha_i u_i\). If we take the integral,
\[
\int_{x \in \mathbb{R}} \Pi_Y(v) dx = \int_{x \in \mathbb{R}} \left( \sum_{i=0}^{m} \alpha_i u_i \right) dx = \sum_{i=0}^{m} \alpha_i = C_{\Delta,+} - C_{\Delta,-}
\]
by the fact that \(u_i = g_i \Delta\) are Gaussians and hence \(\int_{x \in \mathbb{R}} u_i dx = 1\). Since \(v = g_{-1}\) is also a Gaussian, it implies
\[
|C_{\Delta,+} - C_{\Delta,-} - 1| = |\int_{x \in \mathbb{R}} (\Pi_Y(v) - v) dx|.
\]
By triangle inequality, we have
\[
|C_{\Delta,+} - C_{\Delta,-} - 1| = |\int_{x \in \mathbb{R}} (\Pi_Y(v) - v) dx| \leq \int_{x \in \mathbb{R}} |\Pi_Y(v) - v| dx
\]
For any \(L > 0\), we split the integral into two parts.
\[
|C_{\Delta,+} - C_{\Delta,-} - 1| \\
\leq \int_{x \notin [-L,L]} |\Pi_Y(v) - v| dx + \int_{x \in [-L,L]} |\Pi_Y(v) - v| dx
\]
We first analyze the second term \(\int_{x \notin [-L,L]} |\Pi_Y(v) - v| dx\). By triangle inequality, we express the term \(|\Pi_Y(v) - v|\).
\[
|\Pi_Y(v) - v| = \left| \sum_{i=0}^{m} \alpha_i u_i - v \right| \leq \sum_{i=0}^{m} |\alpha_i| u_i + v
\]
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Since all Gaussians in \( \Pi_{\Omega}(v) - v \) centered in \([-1, 1]\), all integrals \( \int_{x \notin [-L, L]} u_i dx, \int_{x \notin [-L, L]} v dx \) are bounded by \( \int_{x \notin [-L-1, L-1]} \gamma_0 dx \). Namely, we have

\[
\int_{x \notin [-L, L]} |\Pi_{\Omega}(v) - v| dx \leq \int_{x \notin [-L, L]} \left( \sum_{i=0}^{m} |\alpha_i|u_i + v \right) dx \\
\leq (C_{\Delta, +} + C_{\Delta, -} + 1) \left( \int_{x \notin [-L-1, L-1]} g_0 dx \right)
\]

A straightforward calculation gives

\[
\int_{x \notin [-L-1, L-1]} g_0 dx = \int_{x \notin [-L-1, L-1]} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \leq \frac{4}{\sqrt{2\pi}(L-1)} e^{-\frac{1}{2}(L-1)^2}
\]

which means

\[
\int_{x \notin [-L, L]} |\Pi_{\Omega}(v) - v| dx \leq (C_{\Delta, +} + C_{\Delta, -} + 1) \frac{4}{\sqrt{2\pi}(L-1)} e^{-\frac{1}{2}(L-1)^2}
\]

Now, we analyze the first term \( \int_{x \in [-L, L]} |\Pi_{\Omega}(v) - v| dx \). By Cauchy inequality,

\[
\int_{x \in [-L, L]} |\Pi_{\Omega}(v) - v| dx \leq \sqrt{2L} \int_{x \in [-L, L]} (\Pi_{\Omega}(v) - v)^2 dx.
\]

Moreover,

\[
\int_{x \in [-L, L]} (\Pi_{\Omega}(v) - v)^2 dx \leq \int_{x \in \mathbb{R}} (\Pi_{\Omega}(v) - v)^2 dx = \|\Pi_{\Omega}(v) - v\|_2^2 = \beta_\Delta^2 \|v\|_2^2.
\]

It means

\[
\int_{x \in [-L, L]} |\Pi_{\Omega}(v) - v| dx \leq \sqrt{2L}\beta_\Delta \|v\|_2
\]

By taking \( L = \Theta\left( \sqrt{\log \frac{C_{\Delta, +} + C_{\Delta, -}}{\beta_\Delta}} \right) \), we have

\[
|C_{\Delta, +} - C_{\Delta, -} - 1| = O\left( \log \frac{C_{\Delta, +} + C_{\Delta, -}}{\beta_\Delta} \right)^{1/4} \beta_\Delta.
\]

By Lemma 5 and 6, we conclude that

\[
|C_{\Delta, +} - C_{\Delta, -} - 1| \leq \frac{1}{2\Theta(1/\Delta \log(1/\Delta))}
\]

\[
\square
\]

We want to show that \( \|f - f'\|_2 \) is small while \( \|f_1 - f'_1\|_2 \) and \( \|f_2 - f'_2\|_2 \) are large. As mentioned before, these terms \( \|f - f'\|_2, \|f_1 - f'_1\|_2 \) and \( \|f_2 - f'_2\|_2 \) can be expressed in terms of \( C_{\Delta, +}, C_{\Delta, -} \) and \( \beta_\Delta \). We have explicitly analyzed \( C_{\Delta, +}, C_{\Delta, -} \) and \( \beta_\Delta \). Hence, Lemma 9 gives the bounds for \( \|f - f'\|_2, \|f_1 - f'_1\|_2 \) and \( \|f_2 - f'_2\|_2 \).

**Lemma 9.** For any sufficiently small \( \Delta > 0 \), we have

\[
\|f - f'\|_2 \leq \frac{1}{2\Theta(1/\Delta \log(1/\Delta))} \quad \text{and} \quad \|f_1 - f'_1\|_2 \geq \frac{1}{2\Theta(1/\Delta)}
\]

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Hence, we have
\[
f_1 = \frac{1}{C_{\Delta,+}} \left( \sum_{i \in J_+} \alpha_i u_i \right), \quad f'_1 = \frac{1}{C_{\Delta,+}} \left( \sum_{i \in J_-} (-\alpha_i) u_i + (C_{\Delta,+} - C_{\Delta,-}) u_0 \right).
\]

Hence, we have
\[
f_1 - f'_1 = \frac{1}{C_{\Delta,+}} \sum_{i \in J_+} \alpha_i u_i - \frac{1}{C_{\Delta,+}} \left( \sum_{i \in J_-} (-\alpha_i) u_i + (C_{\Delta,+} - C_{\Delta,-}) u_0 \right)
= \frac{1}{C_{\Delta,+}} \Pi_v(v) - C_{\Delta,+} - C_{\Delta,-} u_0
\]
since the definition of \(\Pi_v(v)\) is \(\sum_{i=0}^m \alpha_i u_i = \sum_{i \in J_+} \alpha_i u_i - \sum_{i \in J_-} (-\alpha_i) u_i\). Now, we analyze \(\|f_1 - f'_1\|_2\).
\[
\|f_1 - f'_1\|_2 = \left\| \frac{1}{C_{\Delta,+}} \Pi_v(v) - C_{\Delta,+} - C_{\Delta,-} u_0 \right\|_2
= \left\| \frac{1}{C_{\Delta,+}} (\Pi_v(v) - v) - C_{\Delta,+} - C_{\Delta,-} - 1 u_0 + \frac{1}{C_{\Delta,+}} (v - u_0) \right\|_2
\geq \frac{1}{C_{\Delta,+}} \|v - u_0\|_2 - \frac{1}{C_{\Delta,+}} \|\Pi_v(v) - v\|_2 - \frac{|C_{\Delta,+} - C_{\Delta,-} - 1|}{C_{\Delta,+}} \|u_0\|_2 \quad (9)
\]
By Lemma 5 and Lemma 8, we have \(C_{\Delta,+} \leq 2^{O(1/\Delta)}\). For the first term \(\frac{1}{C_{\Delta,+}} \|v - u_0\|_2\), \(\frac{1}{C_{\Delta,+}} \geq 1/2^{O(1/\Delta)}\) and \(\|v - u_0\|_2 = \|g_1 - g_0\|_2 = \Omega(1)\). For the second term \(\frac{1}{C_{\Delta,+}} \|\Pi_v(v) - v\|_2\), \(\|\Pi_v(v) - v\|_2 = \beta \|v\|_2 \leq 1/2^{O((1/\Delta) \log(1/\Delta))}\) by Lemma 6 and hence \(\frac{1}{C_{\Delta,+}} \|\Pi_v(v) - v\|_2 \leq 1/2^{O((1/\Delta) \log(1/\Delta))}\). For the third term \(\frac{|C_{\Delta,+} - C_{\Delta,-} - 1|}{C_{\Delta,+}} \|u_0\|_2\), \(|C_{\Delta,+} - C_{\Delta,-} - 1| \leq 1/2^{O((1/\Delta) \log(1/\Delta))}\) by Lemma 8 and hence \(\frac{|C_{\Delta,+} - C_{\Delta,-} - 1|}{C_{\Delta,+}} \|u_0\|_2 \leq 1/2^{O((1/\Delta) \log(1/\Delta))}\). Plugging them into (9), we have
\[
\|f_1 - f'_1\|_2 \geq \frac{1}{2^{O(1/\Delta)}}
\]
Now, we will analyze \(\|f - f'\|_2\). Recall that
\[
f - f' = \frac{1}{2} (f_1 + f_2) - \frac{1}{2} (f'_1 + f'_2)
= \frac{1}{2 C_{\Delta,+}} \Pi_v(v) - C_{\Delta,+} - C_{\Delta,-} u_0 + \frac{1}{2 C_{\Delta,+}} \Pi_v(v') - C_{\Delta,+} - C_{\Delta,-} u_0'
= \frac{1}{2 C_{\Delta,+}} \left( (\Pi_v(v) - u'_0) + (\Pi_v(v') - u_0) - (C_{\Delta,+} - C_{\Delta,-} - 1)(u_0 + u'_0) \right).
\]
Hence, we have
\[
\|f - f'\|_2 \leq \frac{1}{2 C_{\Delta,+}} \left( \|\Pi_v(v) - u'_0\|_2 + \|\Pi_v(v') - u_0\| + (C_{\Delta,+} - C_{\Delta,-} - 1) \|u_0 + u'_0\|_2 \right)
= \frac{1}{2 C_{\Delta,+}} \left( \beta \|v\|_2 + \beta \|v'\|_2 + (C_{\Delta,+} - C_{\Delta,-} - 1) \|u_0 + u'_0\|_2 \right)
= O \left( \frac{1}{C_{\Delta,+}} \max \{ \beta, C_{\Delta,+} - C_{\Delta,-} - 1 \} \right).
\]
By Lemma 5, Lemma 6 and Lemma 8, we conclude that
\[
\|f - f'\|_2 \leq \frac{1}{2^{O((1/\Delta) \log(1/\Delta))}}.
\]
\[\Box\]
A.3 Main theorem

In our analysis, we have been using $\ell_2$ norm instead of $\ell_1$ norm for analytical convenience. We now resolve this issue in Lemma 10.

**Lemma 10.** We have

$$
\|f_1 - f'_1\|_2 \leq O(\sqrt{\|f_1 - f'_1\|_1}) \quad \text{and} \quad \|f - f'\|_1 = O(\|f - f'\|_2^{2/3})
$$

**Proof.** First, we have

$$
\|f_1 - f'_1\|_2^2 = \int_{x \in \mathbb{R}} (f_1(x) - f'_1(x))^2 \ dx \leq \int_{x \in \mathbb{R}} |f_1(x) - f'_1(x)| (f_1(x) + f'_1(x)) \ dx.
$$

Since $f_1$ and $f'_1$ are mixtures of Gaussians, $f_1(x), f'_1(x) \leq \frac{1}{\sqrt{2\pi}}$ for all $x \in \mathbb{R}$. Therefore, we have

$$
\|f_1 - f'_1\|_2 \leq \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \|f_1 - f'_1\|_1 = O(\|f_1 - f'_1\|_1).
$$

Also, in Lemma 6 of [50], they showed that

$$
\|f - f'\|_1 = O((E + E')^{1/3}\|f - f'\|_2^{2/3})
$$

where $E = \int_{x \in \mathbb{R}} |x|f(x) dx$ and $E' = \int_{x \in \mathbb{R}} |x|f'(x) dx$. We first bound the term $\int_{x \in \mathbb{R}} |x|g_\mu(x) dx$ for any $\mu \in \mathbb{R}$. We have

$$
\int_{x \in \mathbb{R}} |x|g_\mu(x) dx = \int_{x \geq 0} x g_\mu(x) dx - \int_{x \leq 0} x g_\mu(x) dx
$$

Note that $x g_\mu(x) = (x - \mu) g_\mu(x) + \mu g_\mu(x)$. Plugging it into the equation,

$$
\int_{x \in \mathbb{R}} |x|g_\mu(x) dx \leq \int_{x \in \mathbb{R}} |x|g_0(x) dx + \int_{x \in \mathbb{R}} |\mu| g_\mu(x) dx = O(|\mu|).
$$

Since all Gaussians in both $f$ and $f'$ are in $[-2, 1]$, we have both $E, E' = O(1)$. Therefore, we have $\|f - f'\|_1 = O(\|f - f'\|_2^{2/3})$.

Theorem 11 is the main theorem to show that estimating components from a mixture of Gaussians requires super-polynomially many samples. We reduce it to the problem of distinguishing two distributions given a finite number of samples. The two distributions are $f$ and $f'$ defined as in (8). From the previous lemmas, these two pdfs are very close in $\ell_1$ norm and the components in each corresponding pair are relatively far away in $\ell_1$ norm. Combining with the standard information theoretic lower bound, we will prove Theorem 11.

**Theorem 11** (Restated Theorem 2). Let $P$ be a set of $n$ i.i.d. samples drawn from $f^*$, where $f^* = \frac{1}{2}f_1^* + \frac{1}{2}f_2^*$ such that $f_i^* \in \mathcal{G}_i$ and $I_1 \cap I_2 = \emptyset$. For any sufficiently small $\varepsilon > 0$, if $n < (\frac{1}{2})^{\frac{C}{2} \log \log n}$ where $C$ is an absolute constant, then no algorithm taking $P$ as the input returns two pdfs $\hat{f}_1, \hat{f}_2$ such that $\|\hat{f}_1^* - \hat{f}_1\|_1 < \varepsilon$ with probability at least $1/2$.

**Proof.** Suppose there is an algorithm $\mathcal{A}$ that takes $P$ as the input and returns two pdfs $\hat{f}_1, \hat{f}_2$ such that $\|\hat{f}_1^* - \hat{f}_1\|_1 \leq \varepsilon$ with probability at least $\frac{1}{2}$. We reduce it to the problem of distinguishing $f$ and $f'$ defined as in (8). From Lemma 9 and Lemma 10 we have

$$
\|f - f'\|_1 \leq \frac{1}{2^{O((1/\Delta) \log (1/\Delta))}} \quad \text{and} \quad \|f_1 - f'_1\|_1 \geq \frac{1}{2^{O(1/\Delta)}}
$$

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for any sufficiently small $\Delta > 0$. By choosing $\frac{1}{\Delta} = \Theta(\log \frac{1}{\varepsilon})$, we have

$$
\|f - f'\|_1 \leq \varepsilon C_1 \log \frac{1}{\varepsilon}
$$

and

$$
\|f_1 - f'_1\|_1 \geq 3\varepsilon
$$

where $C_1$ is an absolute constant. If we are given a set of $n < (\frac{1}{\varepsilon})^{C \log \log \frac{1}{\varepsilon}}$ i.i.d. samples from one of $f$ and $f'$, we can apply the algorithm $A$ on these samples. From the assumption, $A$ returns two pdfs $\hat{f}_1, \hat{f}_2$ such that $\|f_i - \hat{f}_i\|_1 \leq \varepsilon$ or $\|f'_i - \hat{f}_i\|_1 \leq \varepsilon$. Since $\|f_1 - f'_1\|_1 \geq 3\varepsilon$, we can use $\hat{f}_1$ to determine which of $f$ and $f'$ the samples are drawn from. It implies that we can distinguish $f$ and $f'$ with $(\frac{1}{\varepsilon})^{C \log \log \frac{1}{\varepsilon}}$ samples while $\|f - f'\|_1 \leq \varepsilon C_1 \log \log \frac{1}{\varepsilon}$. It contradicts the standard information theoretic lower bounds. \qed