Remarkable identities

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Abstract: In the paper a number of identities involving even powers of the values of functions tangent, cotangent, secans and cosecans are proved. Namely, the following relations are shown:

\[ \sum_{j=1}^{m-1} f^{2n} \left( \frac{\pi j}{2m} \right) = w_f(m), \]

\[ \sum_{j=0}^{m-1} f^{2n} \left( \frac{2j + 1}{4m - \pi} \right) = v_f(m), \]

\[ \sum_{j=1}^{m} f^{2n} \left( \frac{\pi j}{2m + 1} \right) = \tilde{w}_f(m), \]

where \( m, n \) are positive integers, \( f \) is one of the functions: tangent, cotangent, secans or cosecans and \( w_f(x), v_f(x), \tilde{w}_f(x) \) are some polynomials from \( \mathbb{Q}[x] \).

One of the remarkable identities is the following:

\[ \sum_{j=0}^{m-1} \sin^{-2} \left( \frac{2j + 1}{2m} \pi \right) = m^2, \quad \text{provided } m \geq 1. \]

Some of these identities are used to find, by elementary means, the sums of the series of the form \( \sum_{j=1}^{\infty} \frac{1}{j^n} \), where \( n \) is a fixed positive integer. One can also notice that Bernoulli numbers appear in the leading coefficients of the polynomials \( w_f(x), v_f(x) \) and \( \tilde{w}_f(x) \).

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In [7] the following formulas have been proved

\[ \sum_{j=1}^{m} \cot^2 \frac{\pi j}{2m + 1} = \frac{m(2m - 1)}{3}, \]

\[ \sum_{j=1}^{m} \sin^{-2} \frac{\pi j}{2m + 1} = \frac{2m(m + 1)}{3}, \quad (1) \]
where $m \in \mathbb{N}_1$. By $\mathbb{N}_k$ for a positive integer $k$ we mean $\mathbb{N} \setminus \{0, 1, 2, \ldots, k - 1\}$. The above identities were then used in an elementary proof of the formula $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

In this paper we develop the ideas from [7] to prove more generalized identities than (1). Next we use some of them to find the sum of $\sum_{k=1}^{\infty} \frac{1}{k^n}$, where $n \in \mathbb{N}_1$. The general identities given in this article yield, in particular, the following identity of uncommon beauty

$$\sum_{j=0}^{m-1} \sin^{-2} \frac{2j+1}{2m} \pi = m^2, \quad m \in \mathbb{N}_1.$$ 

Some elementary methods of finding the sums of the series of the form $\sum_{j=1}^{\infty} \frac{1}{j^n}$ may be found for example in [1], [3], [5], [6], [8].

We start by recalling some basic facts on symmetric polynomials in $m$ variables.

Put

$$\sigma_n = \sum_{j=1}^{m} x_j^n \quad \text{for } n \in \mathbb{N}_1,$$

$$\tau_k = \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq m} x_{j_1} x_{j_2} \ldots x_{j_k} \quad \text{for } k \in \{1, 2, \ldots, m\}.$$ 

Moreover, for the convenience set $\tau_k = 0$ for $k > m$.

The following lemma comes from [2].

**Lemma 1** (Newton). Let $n \in \mathbb{N}_1$, then

$$\sigma_n - \tau_1 \sigma_{n-1} + \tau_2 \sigma_{n-2} - \cdots + (-1)^{n-1} \tau_1 \sigma_1 + (-1)^n n \tau_n = 0. \quad (2)$$

In view of Lemma 1 we have

$$\sigma_n = \det \begin{pmatrix} (-1)^{n+1} n \tau_n & -\tau_1 & \tau_2 & \cdots & (-1)^{n-3} \tau_{n-3} & (-1)^{n-4} \tau_{n-4} & \cdots & (-1)^{n-2} \tau_{n-2} & (-1)^{n-2} \tau_{n-1} \\ (-1)^n (n-1) \tau_{n-1} & 1 & -\tau_1 & \cdots & (-1)^{n-4} \tau_{n-4} & \cdots & (-1)^{n-3} \tau_{n-3} & \cdots & (-1)^{n-2} \tau_{n-2} \\ (-1)^{n-1} (n-2) \tau_{n-2} & 0 & 1 & \cdots & (-1)^{n-5} \tau_{n-5} & \cdots & (-1)^{n-4} \tau_{n-4} & \cdots & (-1)^{n-3} \tau_{n-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -2 \tau_2 & 0 & 0 & \cdots & 1 & -\tau_1 & \cdots & \cdots & \cdots \\ \tau_1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad (3)$$

for every $n \in \mathbb{N}_1$. Indeed, putting in (2) instead of $n$ respectively $n-1, n-2, \ldots, 1$ we get, together with (2), the system of $n$ equations in $n$ variables: $\sigma_1, \ldots, \sigma_n$. Such a system is a Cramer’s system and by the Cramer’s rule we get (3).

From now on by $D_{\tan}$ and $D_{\cot}$ we denote the domains of the trigonometric functions tangent and cotangent, respectively.

**Lemma 2.** The following identities hold true:

(A) $\sin^{2m} x \over \cos^{m} x \cot x = \sum_{j=0}^{m} \binom{2m}{2j+1} (-1)^j \tan^{2j} x$, \quad $(m, x) \in \mathbb{N} \times (D_{\tan} \cap D_{\cot})$;

(B) $\cos^{2m} x \over \cos^{2m} x \cot x = \sum_{j=0}^{m} \binom{2m}{2j} (-1)^j \tan^{2j} x$, \quad $(m, x) \in \mathbb{N} \times D_{\tan}$;
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(C) \( \frac{\sin(2(m+1)x)}{\cos^{2(m+1)}x} \) cot \( x = \sum_{j=0}^{m} \left( \frac{2m+1}{2j+1} \right) (-1)^j \tan^{2j} x, \quad (m, x) \in \mathbb{N} \times (D_{\tan} \cap D_{\cot}); \)

(D) \( \frac{\sin(2m+1)x}{\sin^{2m+1}x} = \sum_{j=0}^{m} \left( \frac{2m+1}{2j+1} \right) (-1)^j \cot^{2m-2j} x, \quad (m, x) \in \mathbb{N} \times D_{\cot}; \)

(E) \( \cos(2m+1)x \) tan \( x = \sum_{j=0}^{m} \left( \frac{2m+1}{2j+1} \right) (-1)^j \tan^{2j} x, \quad (m, x) \in \mathbb{N} \times D_{\tan}; \)

(F) \( \cos(2m+1)x \) sin \( x = \sum_{j=0}^{m} \left( \frac{2m+1}{2j+1} \right) (-1)^j \cot^{2m-2j} x, \quad (m, x) \in \mathbb{N} \times (D_{\tan} \cap D_{\cot}). \)

Proof. It is a known fact that

\[
\sum_{j=0}^{k} \binom{k}{j} \cos^{k-j} x (i \sin x)^j = (\cos x + i \sin x)^k = \cos kx + i \sin kx
\]

for \( k \in \mathbb{N} \) and \( x \in \mathbb{R} \). Putting \( k = 2m \) in the above equation and comparing real and imaginary parts of the both sides we obtain (A) and (B). Similarly, with \( k = 2m+1 \) we get (C), (D), (E) and (F).

Now we prove the following result.

**Theorem 1.** For every \( m \in \mathbb{N}_2 \) and any \( n \in \mathbb{N}_1 \),

\[
\sigma_{n,m}(A) = \sum_{j=1}^{m} \tan^{2n} \frac{\pi j}{2m} = \sum_{j=1}^{m} \cot^{2n} \frac{\pi j}{2m},
\]

where \( \sigma_{n,m}(A) \) denotes the determinant given by (3) in which \( \tau_j = \left( \frac{2m}{2j} \right) \) for \( j \in \{1, 2, \ldots, n\} \).

Proof. Replace in the identity (A) of Lemma 2, \( \tan^2 x \) by \( t \) and set

\[
w_A(t) = \sum_{j=0}^{m} \left( \frac{2m}{2j+1} \right) (-1)^j t^j,
\]

then \( w_A(t) \) is a polynomial of order \( m-1 \) in the real variable \( t \).

On the other hand, substituting \( \frac{\pi l}{2m} \), where \( l \in \{1, 2, \ldots, m-1\} \), for \( x \) in (A) we get

\[
0 = \sum_{j=0}^{m} \left( \frac{2m}{2j+1} \right) (-1)^j \tan^{2j} \frac{\pi l}{2m}, \quad l \in \{1, 2, \ldots, m-1\}.
\]

Hence and by (4) we obtain

\[
w_A(t) = (-1)^{m-1} \left( \frac{2m}{2m-1} \right) \prod_{j=1}^{m-1} \left( t - \tan^2 \frac{\pi j}{2m} \right) = (-1)^{m-1} 2m \prod_{j=1}^{m-1} \left( t - \tan^2 \frac{\pi j}{2m} \right).
\]
This and the Vieta's formulas give

\[
\sum_{1 \leq k_1 < k_2 < \cdots < k_j \leq m-1} \tan^2 \frac{\pi k_1}{2m} \tan^2 \frac{\pi k_2}{2m} \cdots \tan^2 \frac{\pi k_j}{2m} = \left(\frac{2m}{a_j + 1}\right)
\]

and in view of (3) we have

\[
\sigma_{n,m}(A) = \sum_{j=1}^{m-1} \tan^{2n} \frac{\pi j}{2m}
\]

As \(\tan \frac{\pi j}{2m} = \cot \frac{\pi (m-j)}{2m}\) for \(j \in \{1, 2, \ldots, m-1\}\) we get

\[
\sum_{j=1}^{m-1} \tan^{2n} \frac{\pi j}{2m} = \sum_{j=1}^{m-1} \cot^{2n} \frac{\pi (m-j)}{2m} = \sum_{j=1}^{m-1} \cot^{2n} \frac{\pi j}{2m},
\]

which completes the proof. \(\square\)

**Theorem 2.** For every \(m, n \in \mathbb{N}_1\) the following identity holds true:

\[
\sigma_{n,m}(B) = \sum_{j=0}^{m-1} \tan^{2n} \frac{\pi j + 1}{4m} \pi = \sum_{j=0}^{m-1} \cot^{2n} \frac{\pi j + 1}{4m} \pi,
\]

where \(\sigma_{n,m}(B)\) denotes the determinant given by (3) in which \(\tau_j = \left(\frac{2m}{a_j}\right)\) for \(j \in \{1, 2, \ldots, n\}\).

**Proof.** Similarly as in the proof of Theorem 1, replace in the right hand side of the identity (B) of Lemma 2, \(\tan^2 x\) by \(t\) and set

\[
w_B(t) = \sum_{j=0}^{m} \left(\frac{2m}{2j}\right)(-1)^j t^j.
\]

Next, substitute \(\frac{2l+1}{4m} \pi\), where \(l \in \{0, 1, \ldots, m-1\}\), for \(x\) in (B). This yields

\[
0 = \sum_{j=0}^{m} \left(\frac{2m}{2j}\right)(-1)^j \tan^2 \frac{2l+1}{4m} \pi, \quad l \in \{0, 1, \ldots, m-1\}.
\]

Hence and by the definition of \(w_B(t)\) we get

\[
w_B(t) = (-1)^m \prod_{j=0}^{m-1} \left( t - \tan^2 \frac{2j+1}{4m} \pi \right),
\]

which in view of the Vieta’s formulas gives

\[
\sum_{1 \leq k_1 < k_2 < \cdots < k_j \leq m-1} \tan^2 \frac{2k_1 + 1}{4m} \tan^2 \frac{2k_2 + 1}{4m} \cdots \tan^2 \frac{2k_j + 1}{4m} = \left(\frac{2m}{2j}\right).
\]
By this and (3),

\[
\sigma_{n,m}(B) = \sum_{j=0}^{m-1} \tan^{2n} \frac{2j+1}{4m} \pi.
\]

Using the same argument as in the proof of Theorem 1 we get

\[
\sum_{j=0}^{m-1} \tan^{2n} \frac{2j+1}{2m} \pi = \sum_{j=0}^{m-1} \cot^{2n} \frac{2j+1}{4m} \pi
\]

and the proof is completed. \(\square\)

Using identities (C) and (D) of Lemma 2 and the same method as in proofs of Theorems 1 and 2 one may obtain

**Theorem 3.** For every \(m, n \in \mathbb{N}_1\) the following identity holds true:

\[
\sigma_{n,m}(C) = \sum_{j=1}^{m} \tan^{2n} \frac{\pi j}{2m+1},
\]

where \(\sigma_{n,m}(C)\) denotes the determinant given by (3) in which \(\tau_j = \binom{2m+1}{2j+1} \) for \(j \in \{1, 2, \ldots, n\}\).

**Theorem 4.** For every \(m, n \in \mathbb{N}_1\) the following identity holds true:

\[
\sigma_{n,m}(D) = \sum_{j=1}^{m} \cot^{2n} \frac{\pi j}{2m+1},
\]

where \(\sigma_{n,m}(D)\) denotes the determinant given by (3) in which \(\tau_j = \frac{1}{2m+1} \binom{2m+1}{2j+1} \) for \(j \in \{1, 2, \ldots, n\}\).

Finally, applying the same reasoning as in the proof of Theorem 1 from (E) and (F) of Lemma 2 we have

**Theorem 5.** For every \(m, n \in \mathbb{N}_1\) the following identity holds true:

\[
\sigma_{n,m}(E) = \sum_{j=0}^{m-1} \tan^{2n} \frac{2j+1}{2(2m+1)} \pi,
\]

where \(\sigma_{n,m}(E)\) denotes the determinant given by (3) in which \(\tau_j = \frac{1}{2m+1} \binom{2m+1}{2j+1} \) for \(j \in \{1, 2, \ldots, n\}\).

**Theorem 6.** For every \(m, n \in \mathbb{N}_1\) the following identity holds true:

\[
\sigma_{n,m}(F) = \sum_{j=0}^{m-1} \cot^{2n} \frac{2j+1}{2(2m+1)} \pi,
\]

where \(\sigma_{n,m}(F)\) denotes the determinant given by (3) in which \(\tau_j = \binom{2m+1}{2j} \) for \(j \in \{1, 2, \ldots, n\}\).
The following formulas
\[
\cot^{2n} x = \left(\frac{1 - \sin^2 x}{\sin^2 x}\right)^n, \quad \tan^{2n} x = \left(\frac{1 - \cos^2 x}{\cos^2 x}\right)^n
\]
yield

Lemma 3. The following identities hold true:

(G) \(\sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \sin^{2j-2n} x = (-1)^{n-1} + \cot^{2n} x, \quad (n, x) \in \mathbb{N}_1 \times D_{\cot}\);

(H) \(\sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \cos^{2j-2n} x = (-1)^{n-1} + \tan^{2n} x, \quad (m, x) \in \mathbb{N}_1 \times D_{\tan}\).

Lemma 4. Assume that \(n \in \mathbb{N}_1\) and \(x \in D_{\cot}\), then
\[
\frac{1}{\sin^{2n} x} = \det \begin{pmatrix} (-1)^{n-1} + \cot^{2n} x & -\binom{n}{1} & -\binom{n}{2} & \cdots & -(-1)^{n-1} \binom{n}{n-1} \\ (-1)^{n-2} + \cot^{2n-2} x & 1 & -\binom{n-1}{1} & \cdots & -(-1)^{n-2} \binom{n-1}{n-2} \\ (-1)^{n-3} + \cot^{2n-4} x & 0 & 1 & \cdots & -(-1)^{n-3} \binom{n-3}{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 + \cot^2 x & 0 & 0 & \cdots & 1 \end{pmatrix}.
\]

Proof. Replacing \(n\) in (G) (Lemma 3) by \(n-1, n-2, \ldots, 1\), respectively we get, together with (G), the system of \(n\) equations in \(n\) variables:

\[
\frac{1}{\sin^{2n} x}, \frac{1}{\sin^{2n-2} x}, \ldots, \frac{1}{\sin^2 x}.
\]

Such a system is a Cramer’s system and the assertion follows by the Cramer’s rule. □

Using (H) in the same manner as in Lemma 4 we obtain

Lemma 5. Let \(n \in \mathbb{N}_1\) and \(x \in D_{\tan}\), then
\[
\frac{1}{\cos^{2n} x} = \det \begin{pmatrix} (-1)^{n-1} + \tan^{2n} x & -\binom{n}{1} & -\binom{n}{2} & \cdots & -(-1)^{n-1} \binom{n}{n-1} \\ (-1)^{n-2} + \tan^{2n-2} x & 1 & -\binom{n-1}{1} & \cdots & -(-1)^{n-2} \binom{n-1}{n-2} \\ (-1)^{n-3} + \tan^{2n-4} x & 0 & 1 & \cdots & -(-1)^{n-3} \binom{n-3}{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 + \tan^2 x & 0 & 0 & \cdots & 1 \end{pmatrix}.
\]

To shorten notation from now on we set
\[
\mu(a_n, a_{n-1}, \ldots, a_1) = \det \begin{pmatrix} a_n & -\binom{n}{1} & -\binom{n}{2} & \cdots & -(-1)^{n-1} \binom{n}{n-1} \\ a_{n-1} & 1 & -\binom{n-1}{1} & \cdots & -(-1)^{n-2} \binom{n-1}{n-2} \\ a_{n-2} & 0 & 1 & \cdots & -(-1)^{n-3} \binom{n-3}{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & 0 & 0 & \cdots & 1 \end{pmatrix}.
\]
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thus the identities of Lemmas 4 and 5 can be written as
\[
\frac{1}{\sin^{2n}x} = \mu \left((-1)^{n-1} + \cot^{2n}x, (-1)^{n-2} + \cot^{2n-2}x, \ldots, 1 + \cot^2x\right) \quad (5)
\]
and
\[
\frac{1}{\cos^{2n}x} = \mu \left((-1)^{n-1} + \tan^{2n}x, (-1)^{n-2} + \tan^{2n-2}x, \ldots, 1 + \tan^2x\right), \quad (6)
\]
respectively.

**Theorem 7.** For every \(m \in \mathbb{N}_2\) and each \(n \in \mathbb{N}_1\) the following identity holds true:
\[
\begin{align*}
\sum_{j=1}^{m-1} \sin^{-2n} \frac{\pi j}{2m} &= \sum_{j=1}^{m-1} \cos^{-2n} \frac{\pi j}{2m} \\
&= \mu \left((-1)^{n-1}(m-1) + \sigma_{n,m}(A), (-1)^{n-2}(m-1) + \sigma_{n-1,m}(A), \ldots, (m-1) + \sigma_{1,m}(A)\right),
\end{align*}
\]
where the numbers \(\sigma_{k,m}(A)\) for \(k \in \{1, 2, \ldots, n\}\) are defined in Theorem 1.

**Proof.** In view of (5) we can write
\[
\sin^{-2n} \frac{\pi j}{2m} = \mu \left((-1)^{n-1} + \cot^{2n} \frac{\pi j}{2m}, (-1)^{n-2} + \cot^{2n-2} \frac{\pi j}{2m}, \ldots, 1 + \cot^2 \frac{\pi j}{2m}\right)
\]
for \(j \in \{1, 2, \ldots, m-1\}\). This by the definition of \(\mu\), properties od determinants and Theorem 1 gives
\[
\begin{align*}
\sum_{j=1}^{m-1} \sin^{-2n} \frac{\pi j}{2m} &= \mu \left((-1)^{n-1} + \cot^{2n} \frac{\pi j}{2m}, (-1)^{n-2} + \cot^{2n-2} \frac{\pi j}{2m}, \ldots, 1 + \cot^2 \frac{\pi j}{2m}\right) \\
&= \mu \left((-1)^{n-1}(m-1) + \sigma_{n,m}(A), (-1)^{n-2}(m-1) + \sigma_{n-1,m}(A), \ldots, (m-1) + \sigma_{1,m}(A)\right).
\end{align*}
\]
The same reasoning applies to the second identity.

Analysis similar to that in the proof of Theorem 7 and the use of Theorems 2 – 6 give
Theorem 8. For every $n, m \in \mathbb{N}_1$ the following identities holds true:

\[
\sum_{j=0}^{m-1} \sin^{-2n} \frac{2j+1}{4m} \pi = \sum_{j=0}^{m-1} \cos^{-2n} \frac{2j+1}{4m} \pi = \mu \left((-1)^{n-1}m + \sigma_{n,m}(B), (-1)^{n-2}m + \sigma_{n-1,m}(B), \ldots, m + \sigma_{1,m}(B)\right),
\]

\[
\sum_{j=1}^{m} \sin^{-2n} \frac{\pi j}{2m+1} = \mu \left((-1)^{n-1}m + \sigma_{n,m}(D), (-1)^{n-2}m + \sigma_{n-1,m}(D), \ldots, m + \sigma_{1,m}(D)\right),
\]

\[
\sum_{j=1}^{m} \cos^{-2n} \frac{\pi j}{2m+1} = \mu \left((-1)^{n-1}m + \sigma_{n,m}(C), (-1)^{n-2}m + \sigma_{n-1,m}(C), \ldots, m + \sigma_{1,m}(C)\right),
\]

\[
\sum_{j=0}^{m-1} \sin^{-2n} \frac{2j+1}{2(2m+1)} \pi = \mu \left((-1)^{n-1}m + \sigma_{n,m}(F), (-1)^{n-2}m + \sigma_{n-1,m}(F), \ldots, m + \sigma_{1,m}(F)\right),
\]

\[
\sum_{j=0}^{m-1} \cos^{-2n} \frac{2j+1}{2(2m+1)} \pi = \mu \left((-1)^{n-1}m + \sigma_{n,m}(E), (-1)^{n-2}m + \sigma_{n-1,m}(E), \ldots, m + \sigma_{1,m}(E)\right),
\]

where $\sigma_{k,m}(B), \sigma_{k,m}(C), \sigma_{k,m}(D), \sigma_{k,m}(E), \sigma_{k,m}(F)$ for $k \in \{1, 2, \ldots, n\}$ are defined in Theorems 2 – 6.

Now we show that the general identities from Theorems 1 – 8 yield some particular equalities, including the one considered by the authors as remarkable.

Theorem 9. If $m \in \mathbb{N}_1$, then

\[
\sum_{j=1}^{m-1} \sin^{-2} \frac{\pi j}{m} = \frac{m^2 - 1}{3}, \quad \text{provided } m \geq 2,
\]

\[
\sum_{j=1}^{m-1} \cot^{-2} \frac{\pi j}{m} = \frac{(m-1)(m-2)}{3}, \quad \text{provided } m \geq 2,
\]

\[
\sum_{j=0}^{m-1} \sin^{-2} \frac{2j+1}{2m} \pi = m^2, \quad \text{provided } m \geq 1,
\]
Proof. According to Theorem 1 we have

\[ \sum_{j=1}^{m-1} \tan^2 \frac{\pi j}{2m} = \sum_{j=1}^{m-1} \cot^2 \frac{\pi j}{2m} = \frac{1}{2m} \left( \binom{2m}{3} \right). \]  \hspace{1cm} (16)

On the other hand, in view of

\[ \tan^2 x + \cot^2 x = \frac{4}{\sin^2 2x} - 2 \]

we get

\[ \sum_{j=1}^{m-1} \tan^{2n} \frac{\pi j}{2m} + \sum_{j=1}^{m-1} \cot^{2n} \frac{\pi j}{2m} = 4 \sum_{j=1}^{m-1} \sin^{-2} \frac{\pi j}{m} - 2(m-1). \]

Combining this with (16) gives

\[ \sum_{j=1}^{m-1} \sin^{-2} \frac{\pi j}{m} = \frac{m^2 - 1}{3} \]

for \( m \geq 2 \). This proves (13).

To prove (14) notice that the identity

\[ \cot^2 x - \frac{1}{\sin^2 x} = -1 \]

yields

\[ \sum_{j=1}^{m-1} \cot^2 \frac{\pi j}{m} - \sum_{j=1}^{m-1} \sin^{-2} \frac{\pi j}{m} = -(m-1), \quad m \geq 2. \]

Thus by (13) we obtain (14).

Finally we show the remarkable (15). Theorem 8 leads to

\[ \sum_{j=0}^{m-1} \sin^{-2} \frac{(2j+1)\pi}{4m} = \sum_{j=0}^{m-1} \cos^{-2} \frac{(2j+1)\pi}{4m} = m + \binom{2m}{2} = 2m^2 \]  \hspace{1cm} (17)

for \( m \geq 1 \). Since

\[ \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{4}{\sin^2 2x} \]

we have

\[ \sum_{j=0}^{m-1} \sin^{-2} \frac{(2j+1)\pi}{4m} + \sum_{j=0}^{m-1} \cos^{-2} \frac{(2j+1)\pi}{4m} = 4 \sum_{j=0}^{m-1} \sin^{-2} \frac{(2j+1)\pi}{2m}, \quad m \geq 1, \]

which by (17) implies (15), and the theorem follows.

Next we use the the identities proved here to find the sums of the series of the form \( \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \), where \( n \in \mathbb{N}_1 \). We begin with the following lemma.
Lemma 6. Let \( n \in \mathbb{N}_1 \), then expression \( \sigma_{n,m}(A) \), defined in Theorem 1, is a value of some polynomial from \( \mathbb{Q}[x] \), where \( x = m \). The order of such a polynomial does not exceed \( 2n \).

Proof. The proof is by induction on \( n \). For \( n = 1 \) we have

\[
\sigma_{1,m}(A) = \frac{1}{2m} \binom{2m}{3} = \frac{2}{3} m^2 - m - \frac{1}{3},
\]

and the assertion follows. Fix \( n \geq 2 \) Assuming Lemma 6 to hold for any \( k \in \mathbb{N}_1, k \leq n - 1 \) we prove it for \( n \). By (2),

\[
\sigma_{n,m}(A) = \sum_{j=1}^{n-1} (-1)^{j-1} \tau_j \sigma_{n-j,m}(A) - (-1)^n n \tau_n,
\]

where \( \tau_j = \frac{1}{2m} \binom{2m}{2j+1} \) for \( j \in \{1, 2, \ldots, n\} \). Hence by the inductive assumption \( \sigma_{n,m}(A) \) is a value of some polynomial from \( \mathbb{Q}[x] \) of order not greater than \( 2n \) with \( x = m \), as claimed. \( \square \)

Theorem 10. For every \( n \in \mathbb{N}_1 \),

\[
\sum_{j=1}^{\infty} \frac{1}{j^{2n}} = \lim_{m \to \infty} \frac{\pi^{2n} \sigma_{n,m}(A)}{(2m)^{2n}},
\]

where \( \sigma_{n,m}(A) \) is defined in Theorem 1.

Proof. Observe that

\[
0 < \cot x < \frac{1}{x} < \frac{1}{\sin x} \quad x \in \left(0, \frac{\pi}{2}\right),
\]

thus

\[
\cot^{2n} \frac{\pi j}{2m} < \left(\frac{2m}{\pi j}\right)^{2n} < \frac{1}{\sin^{2n} \frac{\pi j}{2m}}
\]

and in consequence

\[
\sum_{j=1}^{m-1} \cot^{2n} \frac{\pi j}{2m} < \left(\frac{2m}{\pi}\right)^{2n} \sum_{j=1}^{m-1} \frac{1}{j^{2n}} < \sum_{j=1}^{m-1} \frac{1}{\sin^{2n} \frac{\pi j}{2m}}
\]

for \( n \in \mathbb{N}_1, m \in \mathbb{N}_2 \) and \( j \in \{1, 2, \ldots, n\} \). By the definitions of \( \sigma_{n,m}(A) \) and the function \( \mu \) we have

\[
\frac{\pi^{2n} \sigma_{n,m}(A)}{(2m)^{2n}} < \sum_{j=1}^{m-1} \frac{1}{j^{2n}} < \frac{\pi^{2n}}{(2m)^{2n}} m(-1)^{n-1}(m-1) + \sigma_{n,m}(A),
\]

\[
(-1)^{n-2}(m-1) + \sigma_{n-1,m}(A),
\]

\[
\ldots, m-1 + \sigma_{1,m}(A).
\]
The formula for \( \mu \) and the properties of determinants give

\[
\mu ( (-1)^{n-1}(m-1) + \sigma_{n,m}(A), \ldots, m-1 + \sigma_{1,m}(A) ) = (m-1)\mu((-1)^{n-1}, \ldots, (-1)^{-n}) + \mu(\sigma_{n,m}(A), \sigma_{n-1,m}(A), \ldots, \sigma_{1,m}(A))
\]

where \( C_1, \ldots, C_n \) are constants depending on \( n \). Hence by Lemma 6 and inequality (18) we obtain

\[
\lim_{m \to \infty} \pi^{2n} \sigma_{n,m}(A) \leq \sum_{j=1}^{\infty} \frac{1}{j^{2n}} \leq \lim_{m \to \infty} \frac{\pi^{2n} \sigma_{n,m}(A)}{(2m)^{2n}},
\]

which establishes the formula. \( \square \)

**Remark 1.** Note that in the proof Theorem 10 (the last step of the proof) we have actually proved more, namely that the order of the polynomial from Lemma 6 equals exactly \( 2n \). Indeed, if it was not true, we would have

\[
\lim_{m \to \infty} \pi^{2n} \sigma_{n,m}(A) = 0
\]

and consequently

\[
\sum_{j=1}^{\infty} \frac{1}{j^{2n}} \leq 0,
\]

which is impossible.

**Remark 2.** Treating \( \sigma_{n,m}(A) \) as a polynomial in \( m \) of order \( 2n \) we have

\[
\lim_{m \to \infty} \frac{\pi^{2n} \sigma_{n,m}(A)}{(2m)^{2n}} = a_{2n} \frac{\pi^{2n}}{4^n},
\]

where \( a_{2n} \) denotes the leading coefficient of \( \sigma_{n,m}(A) \). On the other hand,

\[
B_{2n} \frac{2^{2n-1} \pi^{2n}}{(2n)!} (-1)^{n-1} = \sum_{j=1}^{\infty} \frac{1}{j^{2n}}, \quad n \in \mathbb{N}_1,
\]

where \( B_{2n} \) stands for the \( 2n \)-th Bernoulli number (see [4], p.320). Thus we get the following relation between Bernoulli numbers and the coefficients of \( \sigma_{n,m}(A) \)

\[
a_{2n} = B_{2n} \frac{2^{4n-1} (-1)^{n-1}}{(2n)!}, \quad n \in \mathbb{N}_1.
\]

**Remark 3.** Similarly as Theorem 10 one can show that

\[
\sum_{j=1}^{\infty} \frac{1}{j^{2n}} = \lim_{m \to \infty} \frac{\pi^{2n} \sigma_{n,m}(B)}{(2m)^{2n}} = \lim_{m \to \infty} \frac{\pi^{2n} \sigma_{n,m}(D)}{(2m)^{2n}} = \lim_{m \to \infty} \frac{\pi^{2n} \sigma_{n,m}(F)}{(2m)^{2n}}, \quad n \in \mathbb{N}_1,
\]

where \( \sigma_{n,m}(B), \sigma_{n,m}(D) \) and \( \sigma_{n,m}(F) \) are defined in Theorems 2, 4 and 6, respectively.
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