On Rangasamy’s outsourcing algorithm for solving quadratic congruence equations

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Abstract

Outsourcing computation is a desired approach for IoT (Internet of Things) devices to transfer their burdens of heavy computations to those nearby, resource-abundant cloud servers. Recently, Rangasamy presented a passive attack against two outsourcing algorithms proposed by Zhang et al. for solving quadratic congruence equations, which is widely used in IoT applications. Furthermore, he also proposed a modified algorithm to fix these schemes and claimed that his algorithm was correct and enabled secure and verifiable delegation of solving quadratic congruence equations in IoTs. However, we show that Rangasamy’s modified algorithm has a flaw which makes it incorrect and also propose some further attacks to break the security claim, even when the flaw has been corrected.

Keywords: Cloud computing, Secure outsourcing, Quadratic congruence equations, Internet of Things

1. Introduction

The internet of things (IoT), which is one of emerging technologies in the Fourth Industrial Revolution, is likely to play a dominant role in what emerges...
post-pandemic. Information is collected, stored and shared across the internet by networked ‘smart’ physical objects, such as RFID tags, sensors, mobile phones etc. By 2025, it is predicted that 41.6 billion devices will be capturing data on how we live, work, move through our cities, operate and maintain the machines on which we depend according to the World Economic Forum’s State of the Connected World report. However, almost all of these deployed devices have limited computing and storage capacity. As a result, outsourcing computation is a desired approach for these devices to transfer their burdens of heavy computations to those nearby, resource-abundant cloud servers by pay-as-you-go model.

With the rapid development of cloud computing, a large number of cloud servers offering computing and storage make it possible to securely outsource computation tasks. However, for the sake of business interests, equipment breakdown and etc., there are security risks in outsourcing computation. So it must meet three requirements, which are high efficiency, input/output privacy and verifiability.

Solving quadratic congruence equations, which is to find root of \( x^2 \equiv a \mod n \) given a quadratic residue \( a \) modulo \( n \), is widely used in cryptographic constructions, such as Rabin Cryptosystem. Meanwhile, this public-key encryption scheme is suitable in IoT applications. However, it needs time \( O(\log^3 p) \) for the quadratic congruence problem with prime modulus \( p \), which makes the computation overloaded for those IoT devices.

Therefore, many scholars have been studying how to securely outsource the problem of solving quadratic congruence equations to cloud servers. Recently, Zhang et al. proposed two outsourcing algorithms, SoSQC1 and SoSQC2, for this problem, and claimed that all the original inputs and output cannot be exposed by the cloud servers in their algorithms. However, Li et al. presented some attacks to show that the SoSQC1 and SoSQC2 schemes are insecure since all the inputs and outputs can be efficiently recovered by just a curious server. They also presented another algorithm to fix these two algorithms. In another independent work, Rangasamy also questioned the security of SoSQC1 and
SoSQC2 schemes and presented a passive attack to recover the inputs and output when the protocol is executed more than once. Furthermore, Rangasamy also proposed a modified algorithm and claimed that his new algorithm was correct and enabled secure and verifiable delegation of solving quadratic congruence equations in IoTs.

However, in this note, we show that the modified outsourcing algorithm proposed by Rangasamy is incorrect. The legal client usually cannot get the correct answer for his outsourcing task when completing the modified algorithm with an honest server. Furthermore, we show that Rangasamy’s modified algorithm is insecure even when the flaw has been corrected. By proposing some attacks inspired by [6], we show that a curious server can successfully recover the original inputs and real output of the outsourcing task, which should be kept secret.

2. Description of Rangasamy’s modified algorithm

In this section, we describe Rangasamy’s modified algorithm [7], which fixes Zhang et al.’s algorithm SoSQC2 [5]. Rangasamy’s algorithm is also based on Cipolla’s algorithm, which is usually used to find a solution of the quadratic congruence equation $x^2 \equiv n \mod p$, and shown as Algorithm 1.

**Algorithm 1 Cipolla’s Algorithm**

**Input:** an odd prime $p$, and a quadratic residue $n \in \mathbb{F}_p$.

**Output:** $x$, such that $x^2 \equiv n \mod p$.

1. Find an $a \in \mathbb{F}_p$ until $(a^2 - n)^\frac{p-1}{2} \equiv -1 \mod p$, which means that $a^2 - n$ is a quadratic nonresidue modulo $p$.
2. Compute the root $x \equiv (a + \sqrt{w})^{\frac{p+1}{2}} \mod p$ in the field $\mathbb{F}_{p^2} = \mathbb{F}_p(\sqrt{w})$, where $w = a^2 - n$.
3. **return** $x$.

Next we describe Rangasamy’s algorithm as follows, in which the client outsources the task of solving quadratic congruence equation $x^2 \equiv n \mod p$ to the server.
1. The client randomly picks a large prime $q$, whose bit-length is the same as that of $p$ and random integers $r_1, r_2, k$ ($k$ must be small enough to ensure the efficiency) in $\mathbb{F}_p$, and then computes:

$$n' = n - r_1 p,$$

$$d' = (p - 1)/2 - k,$$

$$d'_2 = \frac{p + 1}{2} + r_2(p - 1),$$

$$p' = pq.$$  

Then the client sends $(n', d', d'_2, p')$ to the server.

2. The server selects integer $a \in \mathbb{F}_{p'}$ and calculates $R'_1 \equiv (a^2 - n')^{d'} \mod p'$. Then $(a, R'_1)$ is sent back to the client.

3. The client computes $R_1 \equiv R'_1 \mod p$ and checks whether or not $(a^2 - n')^k \mod p$. If so, a message “Y” is returned to the server; otherwise, “N” is returned.

4. Upon receiving “N”, the server repeats Step 2 by selecting another $a$ until receiving “Y”. Then the server calculates

$$R'_2 \equiv (a + \sqrt{a^2 - n'})^{d'_2} \mod p',$$

and sends $R'_2$ to the client.

5. The client computes

$$x \equiv R'_2 \mod p,$$

and checks whether or not $x^2 \equiv n \mod p$.

3. **Rangasamy’s algorithm is incorrect**

Unfortunately, we have to say that Rangasamy’s modified algorithm is incorrect, that is, even when the algorithm is executed honestly, the result $x$ computed by the client in Equation (6) cannot pass validation for correctness.
Denote \( w = a^2 - n \). Note that \( R'_2 \mod p \equiv (a + \sqrt{w})^{\frac{p+1}{2} + r_2(p-1)} \mod p \) computed in Equation (6) may not be equal to \( (a + \sqrt{w})^{\frac{p+1}{2}} \mod p \) that is desired by Cippolla’s algorithm (Step 2 in Algorithm 1), since \( R'_2 \mod p \in \mathbb{F}_p(\sqrt{w}) \) may not fall into \( \mathbb{F}_p \) due to the fact that the order of the multiplicative group in \( \mathbb{F}_p(\sqrt{w}) \) is \( p^2 - 1 \) instead of \( p - 1 \). Hence the correctness of Rangasamy’s algorithm does not hold.

In the following, we take outsourcing \( x^2 \equiv 9 \mod 83 \) as a counterexample. The client first randomly chooses \( q = 97, r_1 = 21, r_2 = 73, k = 13 \), computes

\[
\begin{align*}
n' &= n - r_1p = -1734, \\
d' &= (p - 1)/2 - k = 28, \\
d'_2 &= \frac{p+1}{2} + r_2(p-1) = 6028, \\
p' &= pq = 8051,
\end{align*}
\]

and sends them to the cloud server.

On receiving parameters \( (n', d', d'_2, p') \), the cloud server selects integer \( a = 3345 \) and sends to the client the value

\[
R'_1 \equiv (a^2 - n')^{d'} \mod p' \equiv 11190759^{28} \mod 8051 \equiv 3927.
\]

The client checks

\[
(a^2 - n')^6 \cdot R'_1 \equiv 11190759^{13} \cdot 3927 \mod 83 \equiv -1,
\]

and return “Y” to the server.

Receiving “Y”, the cloud server computes

\[
R'_2 \equiv (a + \sqrt{a^2 - n'})^{d'_2} \mod p' \equiv (3345 + \sqrt{11190759})^{6028} \mod 8051 \equiv 3935 \sqrt{7920} + 5592
\]

and sends \( 3935 \sqrt{7920} + 5592 \) to the client.

Finally, the client computes

\[
x \equiv R'_2 \mod p \equiv 34 \sqrt{35} + 31
\]

and finds that \( x^2 \equiv n \mod p \) doesn’t hold since \( x^2 \equiv 33 \sqrt{35} + 4 \), which is not even an integer.
To ensure correctness, it seems one should at least set
\[ d'_2 = \frac{(p+1)}{2} + r_2(p^2 - 1) \]
by following Rangasamy’s idea. However, we next show that the algorithm is still insecure.

4. Rangasamy’s algorithm is insecure

The input/output privacy requires that the outsourcing algorithm should keep the original input \( n, p \) and the correct output \( x \) secret to anyone except the client. However, we next show that for Rangasamy’s modified algorithm, these can be recovered efficiently by a curious server or an eavesdropper.

Note that if we could recover \( p \), then we can easily recover \( n \equiv n' \mod p \) by Equation (1) and recover \( x \) by solving the quadratic congruence equation. Therefore we just show how to recover \( p \) in the following.

4.1. Recovering \( p \) from \( d'_2 \)

By Equation (3), we can get \( 2d'_2 - 2 = (p - 1)(1 + 2r_2) \) which is exactly a multiple of \( p - 1 \). Moreover, this still holds even we set \( d'_2 = \frac{(p+1)}{2} + r_2(p^2 - 1) \), since now we have \( 2d'_2 - 2 = (p - 1)(2r_2(p + 1) + 1) \).

Note that the order of the cyclic multiplicative group \( \mathbb{F}_p^* \) is \( p - 1 \) since \( p \) is a prime. Hence, for any \( b \in \mathbb{F}_p^* \), we have \( b^{2d'_2 - 2} \equiv 1 \mod p \), from which we can get \( p | (b^{2d'_2 - 2} - 1) \). With the fact that \( p | p' \), we immediately gets

\[ p | g = \gcd((b^{2d'_2 - 2} - 1), p') = \gcd((b^{2d'_2 - 2} - 1) \mod p', p'). \]

If \( b^{2d'_2 - 2} - 1 \mod p' \neq 0 \), then \( p = g \) since the positive factors of \( p' \) are in the set \( \{1, p, q, p'\} \).

From this observation above, the curious adversary can choose random integer \( b \) until \( b^{2d'_2 - 2} - 1 \mod p' \neq 0 \). Then he can immediately obtain the secret modulus \( p \) by calculating \( g \). Due to the randomness of \( r_2 \) and \( b \), the probability of \( b^{2d'_2 - 2} - 1 \mod p' \neq 0 \) is very high.
We randomly generated 100 instances on personal laptop to verify the effect of our attack, in which the bit lengths of randomly chosen \( p, q \) are 512 bits and the bit length of \( k \) is 80. In our experiments, we successfully recovered \( p \) with all instances, that is, the success probability is 100%.

Remark 1. In fact, if the client executes the modified algorithm to solve \( x_1^2 \equiv n_1 \mod p \) and \( x_2^2 \equiv n_2 \mod p \) respectively as assumed in [7], then the adversary knows queries \((n_1', d', d_2', p')\) and \((n_2', d', d_2', p')\), and Rangasamy’s idea [7] can be directly employed to attack his modified algorithm, since \( 2d_2' - 2 \) and \( 2\bar{d}_2' - 2 \) are the multiples of \( p - 1 \) in the above two executions and \( \gcd(2d_2' - 2, 2\bar{d}_2' - 2) \) will leak \( p - 1 \) with high probability. An analysis similar to that in [7] shows that asymptotically the probability should be at least greater than 81.1%, the probability that two "random" odd integer are coprime.

We also generated 100 random instances when the client executes the modified algorithm twice to verify the effect of this attack. The parameters are set as in our attack. Finally, we successfully recovered \( p \) with probability 93%.

4.2. A simple attempt to change \( d_2' \) again

Based on the attacks above, we should force \( 2d_2' - 2 \) to be not a multiple of \( p - 1 \). A simple idea to fix it is to substitute previous \( d_2' = (p + 1)/2 + r_2(p^2 - 1) - k_1 \) with \( d_2' = (p + 1)/2 + r_2(p^2 - 1) - k_1 \) with small \( k_1 \).

However, we have to point out that we still should be careful with the choice of \( r_2 \) in such a case. Again assume the adversary knows queries \((n_1', d', d_2', p')\) and \((n_2', \bar{d}', \bar{d}_2', p')\) as in [7], where \( d_2' = (p + 1)/2 + r_2(p^2 - 1) - k_1 \) and \( \bar{d}_2' = (p + 1)/2 + \bar{r}_2(p^2 - 1) - \bar{k}_1 \). If \( r_2 \) and \( \bar{r}_2 \) are small enough, saying less than \( p \), then we have

\[
\left| \frac{d_2'}{d_2'} - \frac{r_2}{\bar{r}_2} \right| < \frac{1}{O(p^2)}
\]

and continued fractions method [8] may be an effective way to obtain \( r_2 \) and \( \bar{r}_2 \) since \( \frac{r_2}{\bar{r}_2} \) may be a best rational approximation of \( \frac{d_2'}{d_2'} \).

For example, suppose \( p = 691 \) and we generate \( d_2' = 325641678 \) where \( r_2 = 682, k_1 = 28 \) and \( \bar{d}_2' = 313704683 \) where \( \bar{r}_2 = 657, \bar{k}_1 = 23 \) in the first and second
executions, respectively. We computed the best rational approximations of \( \frac{d_2'}{d_2''} \) by the continued fractions method with Sagemath and get the sequence \[ 1, 27/26, 82/79, 109/105, 191/184, 682/657, 28153/27121, 28835/27778, 114658/110455, 4271181/4114613, 34284106/33027359, 72839393/70169331, 325641678/313704683 \] which contains the real \( \frac{r_2}{r_2'} = \frac{682}{657} \).

After recovering \( r_2 \), we can recover \( p \) from \( d_2' = (p + 1)/2 + r_2(p^2 - 1) - k_1 \) by the method similar to that in the following section when \( k_1 \) is small enough.

4.3. Recovering \( p \) from \( d' \) with small \( k \)

Even \( d_2' \) can be fixed in the modified algorithm, we have to show that it is still insecure with small \( k \) in Equation 2.

Note that in Step 3 of Rangasamy’s algorithm, the client must check whether \( (a^2 - n')k \cdot R_1 \equiv -1 \mod p \) or not, which means that \( k \) can not be too large since a large \( k \) will cost the client too much resource. Usually the bit length of \( k \) is set to be 80 to ensure 80-bit security as in 5. However, small \( k \) will lead some risk to leak \( p \).

From Equation (2), we can get \( d' = (p - 1)/2 - k \) for some small \( k \) in \( \mathbb{F}_p \). Thus, \( k \) is the root of \( g(x) = 2x + 2d' + 1 - p' \mod p \) where \( p' = pq \). Since 2 is coprime to odd \( p' \), we can define another polynomial \( f(x) = x + d' + \frac{1-p'}{2} \). It is apparent that \( k \) is also the root of \( f(x) \mod p \). Then we can recover \( k \) by Coppersmith’s algorithm in polynomial time if \( k \leq \sqrt{p} \). More precisely, we have

**Theorem 1 (Coppersmith algorithm 10).** Let \( f(x) \) be a univariate monic polynomial of degree \( \delta \), and \( N \) be an integer with unknown factorization. Assume that \( N \) has a divisor \( b \geq N^\beta \), where \( 0 < \beta \leq 1 \). Then all solutions \( x_0 \) for the equation \( f(x) \equiv 0 \mod b \) with \( |x_0| \leq cN^{\frac{\delta^2}{2\delta}} \) can be found in time \( O(c\delta^5 \log^9 N) \).

Taking \( p' \) as \( N \), \( p \) as \( b \) in the theorem, we can get \( \beta \approx \frac{1}{2} \) and \( \delta = 1 \) for \( f(x) \) and then the bound \( \sqrt{p} \) holds. Once \( k \) is gotten, \( p \) can be efficiently recovered from \( d' \).

To validate the effectiveness of our attack, we randomly generated 100 instances, in which \( p \) and \( q \) are 512 bits and \( k \) is 80 bits. We succeeded in
recovering \( k \) for all the experiments. Moreover, we also tested for the case when \( p, q \) are 1024 bits and \( k \) is 256 bits. 100 random instances were generated and we succeeded in all the experiments again.

5. Conclusion

In this note, we show that Rangasamy’s modified outsourcing algorithm for solving quadratic congruence equations has a flaw. Moreover, we present some attacks against it to show that all the inputs and output can be recovered efficiently, which breaks the security claim. We suggest the fixed algorithm in [6] as a candidate secure outsourcing algorithm for solving quadratic congruence equations.

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