LP Mixed Data Science : Outline of Theory

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0 Introduction

Statistics journals have great difficulty accepting papers unlike those previously published. For statisticians with new big ideas a practical strategy is to publish them in many small applied studies which enables one to provide references to work of others. This essay outlines the many concepts, new theory, and important algorithms of our new culture of statistical science called LP MIXED DATA SCIENCE. It provides comprehensive solutions to problems of data analysis and nonparametric modeling of many variables that are continuous or discrete, which does not yet have a large literature. It develops a new modeling approach to nonparametric estimation of the multivariate copula density. We discuss the theory which we believe is very elegant (and can provide a framework for United Statistical Algorithms, for traditional Small Data methods and Big Data methods). The utility of the theory will be demonstrated elsewhere by series of real applications.

1 LP Methods, Mixed Variables (X,Y), Mid-Distribution

By mixed variables we mean X, Y can be continuous or discrete. We will denote distribution functions by $F(x; X)$, $F(y; Y)$ and Quantile functions by $Q(u; X)$, $Q(v; Y)$, $0 < u < 1$, $0 < v < 1$.

One important mixed data problem is CLASSIFICATION: Y binary 0, 1; X continuous. The goal: Estimate nonparametrically $Pr[Y = 1 | X = x = Q(u; X)] = Pr[Y = 1 | F(X; X) = u]$ as a function of values $u$ of rank transform $F(X; X)$.

LP METHODS: L stands for extension of L rank statistics, L moments. P stands for Parzen, quantiles, mid-distribution, comparison densities, orthonormal score functions $T_j(x; X)$ custom built for each X as functions of mid-ranks. Mid-distribution is defined as $F_{\text{mid}}(x; X) = F(x; X) - .5p(x; X)$, probability mass function $p(x; X) = Pr[X = x]$, notation introduced by Parzen (1960) “Modern Probability Theory and its Applications”. Sample mid-distribution from sample of size $n$ computed in R by $F_{\text{mid}}(X; X) = (\text{Rank}(X) - .5)/n$.

LOOKING AT UNIVARIATE DATA: Quantile plot of sample with distinct values $x_j$ is scatter diagram ($u_j = F_{\text{mid}}(x_j; X), x_j = Q(u_j; X)$). We define Mid-quantile $Q_{\text{mid}}(u; X), 0 < u < 1$, linearly connect ($u_j, x_j$). Normal Q-Q plot is scatter diagram ($Q(u_j; Z), x_j$), where $Z \sim \text{Normal}(0, 1)$.

LOOKING AT BIVARIATE DATA (X,Y): We recommend to display three scatter plots $(X, Y)$, $(F_{\text{mid}}(X; X), Y)$, $(F_{\text{mid}}(X; X), F_{\text{mid}}(Y; Y))$.

COMPARISON DENSITY: To test if two continuous distributions $F(x)$ and $G(x)$ are equal we compare not their difference but their ratio by defining comparison distribution $D(u; G, F) = F(Q(u; G))$ with comparison density $d(u; G, F) = f(Q(u; G))/g(Q(u; G))$, likelihood ratio evaluated at $x = Q(u; G), u = G(x)$.
2 Score Function Construction, First Step of Modeling and Algorithm

For a distribution $F$ we construct orthonormal score functions $T_j(x; F)$ as functions of $x$; denoted $T_j(X; X)$ as transforms of $X$. Define $T_1(X; X) = T_1(X; F) = (F_{mid}(X; X) − .5)/\sigma(F_{mid}(X; X))$.

Theorem 1 (Parzen (2004)). $E[F_{mid}(X; X)] = .5$, and $\text{Var}[F_{mid}(X; X)] = (1/12)(1 − \sum_x p^3(x; X))$

For $Y \sim \text{Bernoulli}(p)$, $p = \text{Pr}[Y = 1]$, and $\text{Var}[F_{mid}(X; X)] = pq/4 = (1/12)(1 − p^3 − q^3)$.

Define Score functions $T_j(X; X) = T_j(X; F)$ constructed by Gram Schmidt orthonormalization of powers of $T_1(X; X)$. Define Score functions $S_j(u; X) = T_j(Q(u; X); X)$.

3 Legendre Polynomial Score Functions for $X$ Continuous

Orthonormal Legendre polynomials $\text{Leg}_j(u)$ on interval $0 < u < 1$ are constructed by Gram Schmidt orthonormalization of $1, u, u^2, \cdots$. First four Legendre polynomials are given by:

\begin{align*}
\text{Leg}_0(u) &= 1 \\
\text{Leg}_1(u) &= \sqrt{12}(u − .5) \\
\text{Leg}_2(u) &= \sqrt{5}(6u^2 − 6u + 1) \\
\text{Leg}_3(u) &= \sqrt{7}(20u^3 − 30u^2 + 12u − 1) \\
\text{Leg}_4(u) &= 3(70u^4 − 140u^3 + 90u^2 − 20u + 1)
\end{align*}

For $X$ continuous our score functions are $S_j(u; X) = \text{Leg}_j(u), T_j(x; X) = \text{Leg}_j(F(x; X))$.

We argue that we can model $\text{Pr}[Y = 1|X = x]$ or logodds $\text{Pr}[Y = 1|X = x]$ not as a linear function of powers of $x$ but as a linear function of orthonormal score functions of $F_{mid}(x; X)$ which yields model of logodds $\text{Pr}[Y = 1|X = Q(u; X)]$ as linear function of orthonormal score functions $S_j(u; X)$. We identify score functions before parameter estimation (which comes first, parameters or sufficient statistics?). Logistic regression algorithms can be applied to numerically compute parameters of identified score functions.

4 Mixed Statistical Data Science, Unify Small and Big Data

Data scientists dispute claim by statisticians that data science is just a sexier name for applied statistical science. They agree with Nate Silver (2013 JSM) that the ultimate goal is
quality applied research that gets practical answers. Our view is that data science has many cultures, including applied statistical science, machine learning, statistical learning. It is useful to distinguish between those who strive for utility, and those who aim for utility and elegance (applying RKHS (reproducing kernel Hilbert spaces), regularization, and sweep regression). Instruction in data science often presents not unified theory but algorithms (formulas for final answers) to be imitated rather than understood, including “looking at the dat”.

CHALLENGE! To understand differences between cultures of data science study their approach to two sample inference, classification, for data (Y binary 0−1, X continuous).

A VISION FOR FUTURE OF STATISTICS: We propose that applied statistical science can be more scientific (and less art) when it emphasizes:

(1) unified methods and graphical analysis that work for small data and big data (analogies between analogies), and

(2) awareness of the history and scope of statistical methods (confirmatory and exploratory), as outlined below.

A. Modern Probability: Think frequentist, compute axiomatically (Kolmogorov 1933)

Awareness of definition mixed conditional probability \( \Pr[Y = y | X = x] \) for \( Y \) discrete, \( X \) continuous. Parzen (1960) teaches modern probability without measure theory. NEW!

Mid-probability theory: mid-distribution inversion, convergence.

B. Parametric Inference (Objective): Think Bayesian (parameter probability), compute frequentist confidence quantiles to combine Fisher and Neyman.

C. Nonparametric Inference (Quantiles, Ranks): Think nonparametrically, compute parametrically by models selected nonparametrically using information criteria.

D. Statistical Data Science, Unify Cultures of Small and Big Data: COMPRESSION! goal of high dimensional data analysis, “statistics is like art, like dynamite, the goal is compression”; reduce number of influential variables, number of sufficient statistics that summarize massive Data.

5 Quantile Mechanics

The quantile function \( Q(u; X) \) of a random variable \( X \) is defined \( Q(u; X) = \inf\{ x : F(x; X) \geq u \} \), \( 0 < u < 1 \). Call \( u \) probable if there exists \( x \) such that \( F(x; X) = u \). Verify \( Q(u; X) \) equals \( x(u) \), smallest \( x \) such that \( F(x; X) = u \). Verify that the exact inverse property \( Q(F(x; X); X) = x \) holds for \( x = x(u) \). With probability one, an observed value of \( X \) equals \( x(u) \) for some probable \( u \).

Theorem 2. Any random variable \( X \) has the property that with probability 1, \( Q(F(X;X);X)=X \).
With probability 1, a function \( h(X) \) of \( X \) equals a function of rank transform \( F(X; X) \) since \( h(X) = hQ(F(X; X)) \), defining \( hQ(u) = h(Q(u; X)) \).

We now state a REMARKABLE fundamental theorem on conditional expectation by rank transform.

**Theorem 3.** With probability 1, \( E[Y|X] = E[Y|F(X; X)] = E[Y|F\text{mid}(X; X)] \).

This theorem has direct relevance to NONPARAMETRIC REGRESSION given by the following theorem.

**Theorem 4.** \( E[Y|X = x = Q(u; X)] \) can be approximated by a linear combination of orthonormal score functions \( T_j(X; X) \) with coefficients \( E[YT_j(X; X)] = LP(j, 0; X, Y) \).

**Theorem 5 (LP Representation).** \( E[g(Y)|X] = E[g(Y)|F(X; X)] = \sum_j T_j(X; X)E[g(Y)T_j(X; X)] \).

**Theorem 6.** \( Var(E[Y|X]) = \sum_{j>0} |LP(j, 0; X, Y)|^2 \).

We present below LP representation of \( Var(X) \) equivalent to letting \( Y = X \). We explore below consequences of LP representation of variance and concept of tail index of the distribution of a random variable \( X \).

**Theorem 7 (Parzen (1979)).** \( Q(u; g(X)) = g(Q(u; X)) \) for \( g \) quantile-like (non-decreasing and left continuous function).

**Theorem 8.** Let \( U \) denote Uniform(0,1) variable. In distribution \( X = Q(U; X) \). For \( X \) continuous, In distribution \( F(X; X) = U \) because \( F(Q(u; X); X) = u \), all \( u \). \( X \) discrete \( F(Q(u; X); X) = u \) for probable \( u \).

**Theorem 9 (Conditional Quantile).** Because \( Y = Q(F(Y; Y); Y) \) with probability 1 the conditional quantile function \( Q(u; Y|X) \) can be computed \( Q(u; Y|X) = Q(Q(u; F(Y; Y)|X); Y) \) by first computing the conditional given \( X \) quantile of the rank transform \( F(Y; Y) \).

We simulate this by estimating the comparison density \( d(v; Y, X|X) \), a very important concept defined below.

### 6 LP Moments of \( X \), Tail Index

Quantile formulas every statistician should know! Mean \( E[X] = E[Q(U; X)] \), Variance \( Var[X] = Var[Q(U; X)] \). Other measures of location and scale can be defined in terms of quartiles \( Q1, Q3 \), and median \( Q2 \). Mid quartile \( MQ = 0.5(Q1 + Q3) \), quartile deviation \( DQ = 2(Q3 - Q1) \) approximate slope at \( Q2 \). Next we introduce the concept of informative quantile function, a powerful exploratory data analysis tool. Define INFORMATIVE QUANTILE
QI(u; X), quantile of QI(X) = (X − MQ)/DQ. An observed value X is called Tukey outlier if |QI(X)| > 1.

GINI Method: Measure of scale of X continuous is \( \mathbb{E}[X(F(X; X) − .5)] = (1/4)\mathbb{E}[|X − X'|] \), where X and X' independent identically distributed. It has a long history of theory and application under name Gini coefficient. Definition for X discrete is not obvious; most accepted answer is equivalent to our general definition \( LP(1; X) = \mathbb{E}[X(F_{\text{mid}}(X; X) − .5)]/\sigma(F_{\text{mid}}(X; X)) \).

LP MOMENTS: \( LP(j; X) \) of X: X continuous, \( LP(j; X) = \mathbb{E}[X \text{Leg}_j(F(X; X))] \). All X define \( LP(j; X) = \mathbb{E}[XT_j(X; X)] \). For a distribution F with quantile Q, define LP moments \( LP(j; F) = \mathbb{E}[Q(U; F)T_j(Q(U; F); F)] \). Note for X continuous, \( T_j(Q(U; F); F) = \text{Leg}_j(U) \).

**Theorem 10** (LP Representation of Quantile Function). \( Q(u; X) = \sum_j S_j(u; X) LP(j; X) \).

LP Representation of Variance: \( \text{Var}[X] = \sum_{j > 0} |LP(j; X)|^2 \).

An empirical representation or estimator from data of \( Q(u; X) \) when X is continuous is \( \hat{Q}(u; X) = \sum_{\text{selected}_j} \text{Leg}_j(u)\hat{LP}(j; X) \).

**Theorem 11** (Orthonormal Representation of random variable X). With probability 1, \( X = \sum_j T_j(X; X) LP(j; X) \), and \( Z(X) = \sum_j T_j(X; X) LP(j; Z(X)) \).

TAIL BEHAVIOR OF DISTRIBUTIONS: Parzen (1979) classifies tails of distributions into short, long, medium (medium-short, medium-medium, medium-long). Normal is medium-short. A statistical joke: the tails (ends) justify the means (location estimator).

LP SHORT TAIL DISTRIBUTIONS. Recall \( Z(X) = (X − \mathbb{E}[X])/\sigma(X) \). LP Tail index of X is smallest m that \( \sum_{0 < j \leq m} |LP(j; Z(X))|^2 > .95 \).

Threshold .95 is chosen because Normal barely satisfies it. One could use threshold .99 to choose number of LP moments in an empirical representation of \( Q(u; X) \).

**Theorem 12.** For Z Normal(0,1), \( LP(1; Z) = \sqrt{12}\mathbb{E}[Q(U; Z)U] = \sqrt{12}\mathbb{E}[fQ(u; Z) = \sqrt{3/\pi} = .977, \text{Density quantile function } fQ(u; Z) = f(Q(u; Z); Z) \). Verify score \( J(u; Z) = −(fQ(u; Z))' = Q(u; Z) \).

**Theorem 13.** Normal X is short tailed. Uniform X has LP(1; Z(X)) = 1.

Goodness of fit test of Normality based on sample LP(1; Z(X)) is analogous to Shapiro Wilk test \( \mathbb{E}[Q(F_{\text{mid}}(X; X); Z)Z(X)] = \mathbb{E}[Z(X)\text{Hermite}_1(F_{\text{mid}}(X; X))] \).

L MOMENTS: Our concept of LP moments extends to discrete variables concept of L moment advocated by Hosking (1990)

LP CRITERION: Identify monotonic transform \( g(X) \) which is short tailed, i.e., satisfies \( \mathbb{E}[Z(g(X))Z(F_{\text{mid}}(X; X))] > .975 \).
7 LP Comoments of \((X, Y)\), Covariance Matrix of Score functions

Define \(LP(j, k; X, Y) = \mathbb{E}[T_j(X; X)T_k(Y, Y)]\).

Our concept LP comoment extends concept L comoment introduced by Serfling and Xiao (2007). For analogies with multivariate analysis use Covariance matrix \(KLP(X, Y)\) of vectors \(T_j(X; X)\) and \(T_k(Y, Y)\), selected \(j, k\). Note \(KLP(X; X)\) and \(KLP(Y; Y)\) are identity. Our criterion \(LPINFOR(X, Y)\) for independence of \(X\) and \(Y\), correspondence analysis, canonnical correlations squared are based on eigenvalues of \(LP\)-Coherence\((X,Y)\)=\(KLP(X, Y)KLP(Y, X)\).

\[LPINFOR(X, Y) = \sum_{j,k} |LP(j, k; X, Y)|^2 = \text{trace } LP\text{-Coherence}(X, Y).\]

**Theorem 14** (Correlation Representation of LP Comoments). From orthonormal representations for \(Z(X), Z(Y)\) obtain \(R(X, Y) = \sum_{j,k>0} LP(j; Z(X)) LP(j, k; X, Y) LP(k; Z(Y)), \) and \(R^2(X, Y) \leq \sum_{j,k} |LP(j, k; X, Y)|^2 = LPINFOR(X, Y).\)

\(LPINFOR(X, Y)\) defined above is an information measure of dependence.

**Theorem 15** (REMARKABLE approximate equality of Pearson and Spearman Correlation). For \(X, Y\) short tailed approximately \(R(X, Y) = LP(1; X) LP(1, 1; X, Y) LP(1; Y).\) This can be verified directly for \((X, Y)\) bivariate normal. Spearman Correlation can be defined for \(X, Y\) mixed \(RSPEARMAN(X, Y) = LP(1, 1; X, Y)\).

Spearman correlation equals Pearson correlation of mid-rank transform \(F_{\text{mid}}(X; X)\) and \(F_{\text{mid}}(Y; Y)\).

TIRES IN DATA: Estimation of Spearman correlation is difficult when data has many ties. Our definition of sample Spearman correlation avoids this problem.

**Theorem 16.** \(R(X, Y) = RSPEARMAN(X, Y)\) for \(X, Y\) both uniform or both binary \(0, 1\).

\(LPINFOR, CHISQUARED EMPIRICAL INFORMATION STATISTIC FOR INDEPENDENCE:\) We define data-driven nonlinear measure of dependence by \(LPINFOR(X, Y) = \sum_{\text{significant } j,k} |LP(j, k; X, Y)|^2.\)

**CHISQUARE TEST FOR INDEPENDENCE:** Also call \(LPINFOR\) Chi-square divergence to test independence of \(X\) and \(Y\), extension of usual chi-squared statistic which we express below as integral of square of discrete bivariate copula density function.

8 Skew-G Distribution, Goodness-of-fit

To estimate probability density \(f(x; X) = F'(x; X)\) of continuous \(X\) a popular method is kernel density estimation. A powerful alternative method starts with a parametric model \(G(x)\) with density \(g(x)\).
A model for unknown \( f(x; X) \), called Skew-G model, is \( f(x; X) = g(x)d(G(x)) \), where \( d(u) \) is the comparison density \( d(u) = d(u; G, F(; X)) = f(Q(u; G); X)/g(Q(u; G)) \). Probability density of G-transform \( G(X) \) with distribution function \( F(u; G(X)) = D(u) = D(u; G, F(; X)) = F(Q(u; G); X) \), called comparison distribution.

Estimator \( \hat{d} \) of \( d \), which provides estimator \( \hat{f} \) of \( f \), can be formed by orthonormal score function representation or by exponential (maximum entropy) model. Verify \( \sum d \hat{d} = \sum d \hat{d} \), and \( \hat{d} \) is inner product of density \( d \) and Leg\(_{j}(u) \) which is evaluated as the sample mean \( \mathbb{E}[\text{Leg}_j(G(X))] \). An empirical estimator of \( d(u) \) has representation \( \hat{d}(u) = \hat{d}(u; G, F) = \sum_{\text{selected}_j} \text{Leg}_j(u)\mathbb{E}[\text{Leg}_j(G(X))] \).

**COMPONENT TESTS GOODNESS OF FIT** of continuous \( G(x) \) to \( F(x; X) \) is tested by values of \( \mathbb{E}[\text{Leg}_j(G(X))] \) which we call component tests.

Define \( G \) COMPONENTS: \( \text{Comp}_j(X; G) = \mathbb{E}[T_j(X; G)] \), where expectation uses (sample) distribution of \( X \). Note \( \text{Comp}_j(X; G) = 0 \) when \( X \) has true distribution \( F \) equal to \( G \).

**Theorem 17** (Component Representation of Comparison Density). We have the following orthonormal expansion: \( d(u; G, F) = \sum_j T_j(Q(u; G); G) \text{Comp}_j(X; G) \).

**DISCRETE MODELING AND GOODNESS OF FIT:** When we observe a sample of discrete \( X \) with probable values \( x \), a null hypothesis is probability mass function \( g(x) \), usually denoted \( p_0(x) \). Sample probability mass function is \( \tilde{p}(x) \). Sample comparison density is \( \tilde{d}(u) = d(u; G, \tilde{F}) = \tilde{p}(Q(u; G))/g(Q(u; G)) \).

First component statistic is \( \mathbb{E}[Z(G^{\text{mid}}(X))] \). First estimator of true \( p(x) \) is \( \tilde{p}(x) = g(x) \left\{ 1 + Z(G^{\text{mid}}(x))\mathbb{E}[Z(G^{\text{mid}}(X))] \right\} \).

**EXAMPLE:** ED JAYNE’S DIE: Estimate probabilities of 6 sided die when observed sample mean of \( X \) equals 4.5; a fair die has population mean 3.5. Note we are not given value of \( n \), sample size.

**EXAMPLE SPARSE CHI-SQUARED** (large \( p \), small \( n \)). Let discrete \( X \) have \( p = 20 \) outcomes, Sample probabilities are \( .75, .25 \) for first two outcomes. Model for population model probabilities are \( .25 \) for first two outcomes and \( 1/36 \) other 18 outcomes. Sample size is \( n = 20 \). Test model and estimate from data observed true probabilities of outcomes using model as parametric start.

**9 Copula Density, Conditional Comparison Density, Comparison Probability Bayes Rule**

When \( X \) and \( Y \) are both continuous, or both discrete, their joint probability is described by joint probability density \( f(x, y; X, Y) \) or by joint probability mass function \( p(x, y; X, Y) \). When \( Y \) is discrete and \( X \) is continuous joint probability is described by either side of
identity \( \Pr[Y = y|X = x]f(x;X) = f(x;X|Y = y) \Pr[Y = y] \) which we call PRE-BAYES THEOREM.

BAYES RULE: \( \Pr[Y = y|X = x]/\Pr[Y = y] = f(x;X|Y = y)/f(x;X) \).

COMPARISON PROBABILITY: Define left side of Bayes rule to be \( \text{ComPr}[Y = y|X = x] \).

Define right side of Bayes rule to be \( \text{ComPr}[X = x|Y = y] \).

**Theorem 18** (Bayes Rule for MIXED X,Y). **Bayes Rule using Comparison Probability**

\[
\text{ComPr}[Y = y|X = x] = \text{ComPr}[X = x|Y = y].
\]

**CONDITIONAL COMPARISON DENSITY:** Let \( x = Q(u;X), y = Q(v;Y) \). Define conditional comparison density of \( X \) given \( Y \)

\[
d(u;X, X|Y = Q(v;Y)) = \text{ComPr}[X = Q(u;X)|Y = Q(v;Y)].
\]

Conditional comparison density of \( Y \) given \( X \)

\[
d[v;Y|X = Q(u;X)] = \text{ComPr}[Y = Q(v;Y)|X = Q(u;X)].
\]

**COPULA DENSITY:** \( \text{cop}(u, v; X, Y) \) to be common value of above conditional comparison densities. When \( X, Y \) jointly continuous copula density function equals

\[
\text{cop}(u, v; X, Y) = f(Q(u;X), Q(v;Y); X, Y)/f(Q(u;X); X)f(Q(v;Y); Y).
\]

Copula distribution \( \text{Cop}(u, v; X, Y) = F(Q(u;X), Q(v;Y); X, Y) \).

**MODELING (X,Y):** Estimate univariate marginal of \( X \), univariate marginal of \( Y \), joint copula density of \( (X, T) \)

**Theorem 19** (LP Representation of Copula Density).

\[
\text{cop}(u, v; X, Y) - 1 = \sum_{j,k>0} \text{LP}(j, k; X, Y) S_j(u; X) S_k(v; Y), \ 0 < u, v < 1.
\]

Equivalently,

\[
\int_{[0,1]^2} \text{du dv}(v;Y|X = Q(u;X))S_j(u; X)S_k(v; Y) = \text{LP}(j, k; X, Y).
\]

A proof of copula LP representation is provided by representations of conditional copula density and conditional expectations:

\[
d(v;Y|X = Q(u;X)) = \sum_k S_k(v; Y) \mathbb{E}[T_k(Y; Y)|X = Q(u;X)]
\]

\[
\mathbb{E}[T_k(Y; Y)|X = Q(u;X)] = \sum_j S_j(u; X)\mathbb{E}[T_j(X; X)T_k(Y; Y)]
\]

**MODELING DEPENDENCE IN PRACTICE:** The LP representation of the copula density provides data driven estimators of the copula density after constructing custom built score functions and LP comoments.
SLICE PLOTTING OF COPULA DENSITY: Plot for selected values of \( u \), as function of \( v \), cop\((u, v; X, Y) = d(v; Y, Y|X = Q(u; X)) \).

CONDITIONAL QUANTILE \( Q(v; Y, Y|X = Q(u; X)) \) can be simulated from conditional comparison density \( d(v; Y, Y|X = Q(u; X)) \).

10 Two Sample Inference, Unify Small and Big Data Modeling, Classification

Two sample inference is equivalent to \((X \text{ continuous}, Y \text{ binary } 0-1)\). A complete analysis estimates conditional comparison density \( d(u; X \text{ pooled sample}, X|Y = 1) \). Dependence of \( X \) and \( Y \) is measured by LPINFOR\((X, Y)\).

A quick measure of independence, equivalent to Wilcoxon linear rank statistic and Spearman correlation, is \( \text{LP}(1, 1; X, Y) = R(F^\text{mid}(X; X), I\{Y = 1\}) \), equal to \( E[Z(F^\text{mid}(X; X \text{ pooled})|Y = 1, X \text{ in sample 1})]\sqrt{\text{odds Pr}[Y = 1]} \). This statistic is asymptotically \( \mathcal{N}(0, 1/n) \) under null hypothesis of independence of \( X \) and \( Y \).

Traditional two sample Student \( t \) statistic to test equality of means \( \mathbb{E}[X|Y = 1] = \mathbb{E}[X|Y = 0] \) assuming equality of variances \( \text{Var}[X|Y = 1] = \text{Var}[X|Y = 0] \) is equivalent to \( T = R(X,Y)/\sqrt{1-R^2(X,Y)} \). \( R(X,Y) = \mathbb{E}[Z(X)Z(Y)] = \mathbb{E}[Z(X)|Y = 1]\sqrt{\text{odds Pr}[Y = 1]} \).

LPINFOR COMPRESSION: When there are many features \( X_m \) one wants to identify a small number of features to use to predict (classify) the value of \( Y \). For each feature \( X_m \) estimate LPINFOR\((X_m, Y)\). By plotting ranked values LPINFOR\((X_m, Y)\) one can start the process of identifying the features \( X_m \) which are most predictive of \( Y \).

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