TOEPLITZ AND ASYMPTOTIC TOEPLITZ OPERATORS ON $H^2(\mathbb{D}^n)$

AMIT MAJI, JAYDEB SARKAR, AND SRIJAN SARKAR

Abstract. We initiate a study of asymptotic Toeplitz operators on the Hardy space $H^2(\mathbb{D}^n)$ (over the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$). We also study the Toeplitz operators in the polydisc setting. Our main results on Toeplitz and asymptotic Toeplitz operators can be stated as follows: Let $T_{z_i}$ denote the multiplication operator on $H^2(\mathbb{D}^n)$ by the $i$th coordinate function $z_i$, $i = 1, \ldots, n$, and let $T$ be a bounded linear operator on $H^2(\mathbb{D}^n)$. Then the following hold:

(i) $T$ is a Toeplitz operator (that is, $T = P_{H^2(\mathbb{D}^n)} M_\varphi |_{H^2(\mathbb{D}^n)}$, where $M_\varphi$ is the Laurent operator on $L^2(\mathbb{T}^n)$ for some $\varphi \in L^\infty(\mathbb{T}^n)$) if and only if $T_{z_i}^* T T_{z_i} = T$ for all $i = 1, \ldots, n$.

(ii) $T$ is an asymptotic Toeplitz operator if and only if $T = \text{Toeplitz} + \text{compact}$.

The case $n = 1$ is the well known results of Brown and Halmos, and Feintuch, respectively. We also present related results in the setting of vector-valued Hardy spaces over the unit disc.

1. Introduction

Although concrete bounded linear operators on Hilbert spaces exist in great variety and can exhibit interesting properties, one of the main concerns of function theory and operator theory has generally been the study of operators which are connected with the spaces of holomorphic and integrable functions. The class of Toeplitz and analytic Toeplitz operators have turned out to be one of the most important classes of concrete operators from this point of view.

Toeplitz operators on the Hardy space (or, on the $l^2$ space) were first studied by O. Toeplitz (and then by P. Hartman and A. Wintner in [16]). However, a systematic study of Toeplitz operators was triggered by the seminal paper of Brown and Halmos [4] on algebraic properties of Toeplitz operators on $H^2(\mathbb{D})$. Here $H^2(\mathbb{D})$ denote the Hardy space over the open unit disc $\mathbb{D}$ in $\mathbb{C}$. The study of Toeplitz operators on Hilbert spaces of holomorphic functions, like the Hardy space, the Bergman space and the weighted Bergman spaces, on domains in $\mathbb{C}^n$ is also one of the very active area of current research that brings together several areas of mathematics. For more information on this direction of research, we refer the reader to [3], [6], [8], [9], [10], [11], [17], [21] and the references therein.

Recall that the well-known Brown-Halmos theorem characterizes Toeplitz operators on $H^2(\mathbb{D})$ as follows (see the matricial characterization, Theorem 6 in [4]): Let $T$ be a bounded linear operator on $H^2(\mathbb{D})$. Then $T$ is a Toeplitz operator if and only if

$$T_{z_i}^* TT_{z_i} = T.$$
One of the main results of this paper is the following generalization of Brown-Halmos theorem (see Theorem 3.1): A bounded linear operator $T$ on $H^2(\mathbb{D}^n)$ is a Toeplitz operator if and only if
\[ T_{z_j}^* T T_{z_j} = T, \]
for all $j = 1, \ldots, n$ (see Section 2 for notation and background definitions).

The notion of Toeplitzness was extended to more general settings by Barría and Halmos [2] and Feintuch [12]. Also see Popescu [13] for Toeplitzness in the non-commutative setting. Accordingly, following Feintuch (and Barría and Halmos [2]) we shall say that a bounded linear operator $T$ on $H^2(\mathbb{D})$ is (uniformly) asymptotically Toeplitz if \( \{ T_{z_j}^m T T_{z_j}^m \}_{m \geq 1} \) converges in operator norm. The following theorem due to Feintuch [12] gives a remarkable characterization of asymptotically Toeplitz operators: A bounded linear operator $T$ on $H^2(\mathbb{D})$ is asymptotically Toeplitz if and only if $T = \text{Toeplitz} + \text{compact}$.

After a preliminary section (Section 2) on the Hardy space over unit polydisc, in Section 3, we introduce the asymptotic Toeplitz operators in polydisc setting (see Definition 3.3). In this general setting, to remedy the subtlety of the product domain $\mathbb{D}^n$, we modify the above condition by adding another natural condition. The main content of Section 4 is the following: Let $T, A \in \mathcal{B}(Q_\theta)$ for which
\[ C_{z_i}^* T C_{z_i} = T, \]
in norm, for some $A \in \mathcal{B}(Q_\theta)$ and for all $i = 1, \ldots, n$. In this general setting, to remedy the subtlety of the product domain $\mathbb{D}^n$, we modify the above condition by adding another natural condition. The main content of Section 4 is the following: Let $T, A \in \mathcal{B}(Q_\theta)$. Then $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \ldots, n$, if and only if $A = 0$. Moreover, the following are equivalent:
(i) $C_{z_i}^* T C_{z_i}^m \to A$ and $C_{z_i}^* (T - A) C_{z_i}^m \to 0$ in norm for all $i, j = 1, \ldots, n$;
(ii) $C_{z_i}^* T C_{z_i}^m \to 0$ in norm for all $i = 1, \ldots, n$;
(iii) $T$ is compact.

In Section 5, we study the above questions in the vector-valued Hardy space over the unit disc setting. To be precise, let $\mathcal{E}$ be a Hilbert space, and let $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ be an inner multiplier. Then the model space and the model operator are defined by $Q_\Theta = H^2_\Theta(\mathbb{D}) \ominus \Theta H^2_\Theta(\mathbb{D})$ and $S_\Theta = P_{Q_\Theta} T_{z_j}|_{Q_\Theta}$, respectively. We prove that for every $T \in \mathcal{B}(Q_\Theta)$, the
following holds: (i) $S_\Theta^{m}T S_\Theta = T$ if and only if $T = 0$, and (ii) $\{S_\Theta^{m}T S_\Theta^{n}\}_{m, n \geq 1}$ converges in norm if and only if $T$ is compact.

2. Preliminaries

Let $n \geq 1$ and $\mathbb{D}^n$ be the open unit polydisc in $\mathbb{C}^n$. In the sequel, $z$ will always denote a vector $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$.

The Hardy space $H^2(\mathbb{D}^n)$ over $\mathbb{D}^n$ is the Hilbert space of all holomorphic functions $f$ on $\mathbb{D}^n$ such that

$$\|f\|_{H^2(\mathbb{D}^n)} := \left( \sup_{0 \leq r < 1} \int_{\mathbb{T}^n} |f(re^{i\theta_1}, \ldots, re^{i\theta_n})|^2 \, d\theta \right)^{\frac{1}{2}} < \infty,$$

where $d\theta$ is the normalized Lebesgue measure on the torus $\mathbb{T}^n$, the distinguished boundary of $\mathbb{D}^n$. Let $(T_{z_1}, \ldots, T_{z_n})$ denote the $n$-tuple of multiplication operators by the coordinate functions $\{z_i\}_{i=1}^n$, that is,

$$(T_{z_i} f)(w) = w_i f(w),$$

for all $w \in \mathbb{D}^n$ and $i = 1, \ldots, n$. We will often identify $H^2(\mathbb{D}^n)$ with the $n$-fold Hilbert space tensor product of one variable Hardy space as $H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$. In this identification, $T_{z_i}$ can be represented as

$$I_{H^2(\mathbb{D})} \otimes \cdots \otimes T_{z_i} \otimes \cdots \otimes I_{H^2(\mathbb{D})},$$

for all $i = 1, \ldots, n$. Also one can identify the Hardy space (via the radial limits of functions in $H^2(\mathbb{D}^n)$) with the closed subspace of $L^2(\mathbb{T}^n)$ in the following sense: Let $\{e_k : k \in \mathbb{Z}^n\}$ be the orthonormal basis of $L^2(\mathbb{T}^n)$, where $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $e_k = e^{i\theta_1 k_1} \cdots e^{i\theta_n k_n}$. Then a function

$$\sum_{k \in \mathbb{Z}^n} a_k e_k \in L^2(\mathbb{T}^n),$$

is the radial limit function of some function in $H^2(\mathbb{D}^n)$ if and only if $a_k = 0$ whenever at least one of the $k_j$, $j = 1, \ldots, n$, is negative. In particular, the set of all monomials $\{z^k : k \in \mathbb{Z}^n\}$ form an orthonormal basis for $H^2(\mathbb{D}^n)$, where $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$ and $z^k = z_1^{k_1} \cdots z_n^{k_n}$ (cf. [1], [20]). We use $P_{H^2(\mathbb{D}^n)}$ to denote the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$, that is,

$$P_{H^2(\mathbb{D}^n)} \left( \sum_{k \in \mathbb{Z}^n} a_k e_k \right) = \sum_{k \in \mathbb{Z}^n_+} a_k e_k,$$

for all $\sum a_k e_k$ in $L^2(\mathbb{T}^n)$. For $\varphi \in L^\infty(\mathbb{T}^n)$, the Toeplitz operator with symbol $\varphi$ is the operator $T_\varphi \in \mathcal{B}(H^2(\mathbb{D}^n))$ defined by

$$T_\varphi f = P_{H^2(\mathbb{D}^n)}(M_\varphi f) \quad (f \in H^2(\mathbb{D}^n)),$$

where $M_\varphi$ is the Laurent operator on $L^2(\mathbb{T}^n)$ defined by $M_\varphi g = \varphi g$ for all $g \in L^2(\mathbb{T}^n)$. Therefore

$$T_\varphi = P_{H^2(\mathbb{D}^n)} M_\varphi|_{H^2(\mathbb{D}^n)}.$$
For the relevant results on Toeplitz operators on $H^2(\mathbb{D}^n)$ we refer the reader to [3, 6, 7, 19] and references therein.

The following lemma will prove useful in what follows.

**Lemma 2.1.** Let $H$ be a Hilbert space and $A \in \mathcal{B}(H)$ be a compact operator. If $R$ is a contraction on $H$, and if $R^m \to 0$ in strong operator topology, then $R^m A \to 0$ in norm.

**Proof.** This is a particular case of [3, 1.3 (d), page 3). □

In what follows, for each $k \in \mathbb{Z}^n_+$ and $l \in \mathbb{Z}^n$, we write $T^k_\nu = T^t_{z_1} \cdots T^t_{z_n}$, $M^l_\omega = M^t_{\nu_1} \cdots M^t_{\omega_n}$, $T^k_\nu = T^t_{z_1} \cdots T^t_{z_n}$ and $M^l_\omega = M^t_{\nu_1} \cdots M^t_{\omega_n}$.

### 3. Toeplitz operators in several variables

In the following we prove a generalization of Brown and Halmos characterization [3] of Toeplitz operators on $H^2(\mathbb{D})$. This result should be compared with the algebraic characterization of Guo and Wang [15] which states that $T$ in $\mathcal{B}(H^2(\mathbb{D}^n))$ is a Toeplitz operator if and only if $T_\varphi^* T_\varphi = T$ for all inner function $\varphi \in H\infty(\mathbb{D}^n)$.

**Theorem 3.1.** Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$. Then $T$ is a Toeplitz operator if and only if $T_\nu^* T_\nu = T$ for all $\nu = (1, \ldots, n)$.

**Proof.** For each $k \in \mathbb{Z}^n_+$, define $k_d \in \mathbb{Z}^n_{+,-}$ by $k_d = (k, \ldots, k)$. From $T^k_\nu T^k_\nu = T$, $\nu = (1, \ldots, n)$, we obtain that

$$T^k_\nu T^k_\nu = T,$$

which implies that

$$\langle T e_{i+k_d}, e_{j+k_d} \rangle = \langle TT^k_\nu e_i, T^k_\nu e_j \rangle = \langle T e_i, e_j \rangle,$$

for all $k \in \mathbb{Z}^n_+$ and $i, j \in \mathbb{Z}^n_{+,-}$. Now for each $l, m \in \mathbb{Z}^n$, there exists $t = (t_1, \ldots, t_n) \in \mathbb{Z}^n_{+,-}$ such that $l + k_d, m + k_d \in \mathbb{Z}^n_{+,-}$ for all $k_d \geq t$ (that is, $k \geq t$ for all $\nu = (1, \ldots, n)$). Hence setting

$$A_k = M^k_\nu T \mathcal{P}_{H^2(\mathbb{D}^n)} M^k_\nu,$$

for each $k \geq 1$, we have

$$\langle A_k e_l, e_m \rangle_{L^2(\mathbb{T}^n)} = \langle TP_{H^2(\mathbb{D}^n)} M^k_\nu e_l, M^k_\nu e_m \rangle_{L^2(\mathbb{T}^n)} = \langle TP_{H^2(\mathbb{D}^n)} e_{l+k_d}, e_{m+k_d} \rangle_{L^2(\mathbb{T}^n)},$$

and therefore, for all $k_d \geq t$, we have that

$$\langle A_k e_l, e_m \rangle_{L^2(\mathbb{T}^n)} = \langle T e_{l+k_d}, e_{m+k_d} \rangle_{H^2(\mathbb{D}^n)} = \langle T e_{l+t}, e_{m+t} \rangle_{H^2(\mathbb{D}^n)}.$$

This implies in particular that

$$\langle A_k e_l, e_m \rangle \to \langle T e_{l+t}, e_{m+t} \rangle$$

as $k \to \infty$.

Let the bilinear form $\eta$ on the linear span of $\{e_s : s \in \mathbb{Z}^n\}$ be defined by

$$\eta(e_l, e_m) = \lim_{k \to \infty} \langle A_k e_l, e_m \rangle,$$
for all \( l, m \in \mathbb{Z}^n \). Since \( \|A_k\| \leq \|T\| \), \( k \geq 1 \), it follows that \( \eta \) is a bounded bilinear form. Therefore, \( \eta \) can be extended to a bounded bilinear form (again denoted by \( \eta \)) on all of \( L^2(\mathbb{T}^n) \), and hence there exists a unique bounded linear operator \( A_\infty \) on \( L^2(\mathbb{T}^n) \) such that
\[
\eta(f, g) = \langle A_\infty f, g \rangle = \lim_{k \to \infty} \langle A_k f, g \rangle,
\]
for all \( f, g \in L^2(\mathbb{T}^n) \). Now let \( j \in \{1, \ldots, n\} \), \( l, m \in \mathbb{Z}^n \) and set
\[
\epsilon_j = (0, \ldots, 1, \ldots, 0)_{j^{th \ place}}.
\]
Then for all \( k \) sufficiently large (depending on \( l, m \) and \( j \) ), we have
\[
\langle (M^{l+\epsilon_j}_e T P H^2(\mathbb{D}^n), M^{m+\epsilon_j}_e) e_l^\epsilon, e_m^\epsilon \rangle_{L^2(\mathbb{T}^n)} = \langle TP H^2(\mathbb{D}^n) e_l^\epsilon + k_d + \epsilon, e_m^\epsilon + k_d + \epsilon \rangle_{L^2(\mathbb{T}^n)}
\]
\[
= \langle T e_l^k e_l^k + \epsilon, e_m^k + k_d + \epsilon \rangle_{H^2(\mathbb{D}^n)}
\]
\[
= \langle T_z \gamma TT_z, e_l^k e_l^k + \epsilon, e_m^k + k_d \rangle_{H^2(\mathbb{D}^n)}
\]
\[
= \langle T e_l^k e_l^k + \epsilon, e_m^k + k_d \rangle_{H^2(\mathbb{D}^n)}
\]
\[
= \langle A_k e_l^k, e_m^k \rangle_{L^2(\mathbb{T}^n)}.
\]
Therefore
\[
\langle A_\infty e_l^k, e_m^k \rangle_{L^2(\mathbb{T}^n)} = \lim_{k \to \infty} \langle (M^{l+\epsilon_j}_e T P H^2(\mathbb{D}^n), M^{m+\epsilon_j}_e) e_l^\epsilon, e_m^\epsilon \rangle_{L^2(\mathbb{T}^n)}
\]
\[
= \langle A_\infty e_l^k, e_m^k \rangle_{L^2(\mathbb{T}^n)},
\]
and consequently \( M^{l+\epsilon_j}_e A_\infty M^{m+\epsilon_j}_e = A_\infty \), that is, \( A_\infty M^{l+\epsilon_j}_e = M^{l+\epsilon_j}_e A_\infty \). Hence there exists \( \varphi \) in \( L^\infty(\mathbb{T}^n) \) such that \( A_\infty = M_\varphi \) [8]. Finally, we note that for \( f, g \in H^2(\mathbb{D}^n) \),
\[
\langle A_\infty f, g \rangle_{L^2(\mathbb{T}^n)} = \lim_{k \to \infty} \langle (M^{l+\epsilon_j}_e T P H^2(\mathbb{D}^n), M^{m+\epsilon_j}_e) e_l^\epsilon, e_m^\epsilon \rangle_{L^2(\mathbb{T}^n)}
\]
\[
= \lim_{k \to \infty} \langle T z \gamma TT_z, f, g \rangle_{H^2(\mathbb{D}^n)}
\]
that is,
\[
\langle A_\infty f, g \rangle_{L^2(\mathbb{T}^n)} = \langle T f, g \rangle_{H^2(\mathbb{D}^n)},
\]
and hence
\[
\langle P H^2(\mathbb{D}^n) A_\infty f, g \rangle_{H^2(\mathbb{D}^n)} = \langle A_\infty f, g \rangle_{L^2(\mathbb{T}^n)}
\]
\[
= \langle T f, g \rangle_{H^2(\mathbb{D}^n)}.
\]
Therefore, \( T = P H^2(\mathbb{D}^n) A_\infty |_{H^2(\mathbb{D}^n)} = P H^2(\mathbb{D}^n) M_\varphi |_{H^2(\mathbb{D}^n)} \), that is, \( T \) is a Toeplitz operator. Conversely, let \( \varphi \in L^\infty(\mathbb{T}^n) \) and \( T = P H^2(\mathbb{D}^n) M_\varphi |_{H^2(\mathbb{D}^n)} \). Then for \( f, g \in H^2(\mathbb{D}^n) \) and \( j = 1, \ldots, n \), we have
\[
\langle (T^* \varphi TT_z), f, g \rangle_{H^2(\mathbb{D}^n)} = \langle \varphi e^{i\theta} f, e^{i\theta} g \rangle_{L^2(\mathbb{T}^n)}
\]
\[
= \langle \varphi f, g \rangle_{L^2(\mathbb{T}^n)},
\]
that is,
\[
\langle (T^* \varphi TT_z), f, g \rangle_{H^2(\mathbb{D}^n)} = \langle P H^2(\mathbb{D}^n) M_\varphi |_{H^2(\mathbb{D}^n)} f, g \rangle_{H^2(\mathbb{D}^n)},
\]
and therefore \( T^* \varphi TT_z = T \) for all \( j = 1, \ldots, n \), as desired. \( \Box \)
We now characterize compact operators on $H^2(\mathbb{D}^n)$ in terms of the multiplication operators $\{T_{z_1}, \ldots, T_{z_n}\}$. This characterization was proved by Feintuch [12] in the case of $n = 1$.

**Theorem 3.2.** A bounded linear map $T$ on $H^2(\mathbb{D}^n)$ is compact if and only if $T_{z_i}^m T_{z_j}^m \to 0$ in norm for all $i, j \in \{1, \ldots, n\}$.

**Proof.** Let $T$ on $H^2(\mathbb{D}^n)$ be a bounded operator. First observe that for each $m \geq 1$, we have

$$T_{z_i}^m = I_{H^2(\mathbb{D})} - P_{F_m},$$

where

$$F_m = \mathbb{C} \oplus z\mathbb{C} \oplus \cdots \oplus z^{m-1}\mathbb{C},$$

is an $m$-dimensional subspace of $H^2(\mathbb{D})$. For each $m \geq 1$, set

$$F_m = \prod_{i=1}^n (I_{H^2(\mathbb{D}^n)} - T_{z_i}^m T_{z_i}^m).$$

Then

$$F_m = \prod_{i=1}^n (I_{H^2(\mathbb{D})} \otimes \cdots \otimes (I_{H^2(\mathbb{D})} - T_{z_i}^m T_{z_i}^m) \otimes \cdots \otimes I_{H^2(\mathbb{D})})
= \prod_{i=1}^n (I_{H^2(\mathbb{D})} \otimes \cdots \otimes P_{F_m} \otimes \cdots \otimes I_{H^2(\mathbb{D})})
= P_{F_m} \otimes \cdots \otimes P_{F_m},$$

which gives that $F_m$ is a finite rank operator and hence

$$\tilde{F}_m = TF_m + F_m T - F_m TF_m,$$

is a finite rank operator, $m \geq 1$. Moreover

$$T - \tilde{F}_m = T - (TF_m + F_m T - F_m TF_m)
= (I_{H^2(\mathbb{D}^n)} - F_m)T(I_{H^2(\mathbb{D}^n)} - F_m).$$

Finally, observe that

$$I_{H^2(\mathbb{D}^n)} - F_m = \sum_{1 \leq i_1 < \cdots < i_l \leq n} (-1)^{l+1} T_{z_{i_1}}^m \cdots T_{z_{i_l}}^m T_{z_{i_1}}^m \cdots T_{z_{i_l}}^m
= \sum_{1 \leq i_1 < \cdots < i_l \leq n} (-1)^{l+1} (T_{z_{i_1}}^m \cdots T_{z_{i_l}}^m)^m (T_{z_{i_1}} \cdots T_{z_{i_l}})^m,$$

for all $m \geq 1$. Hence, by hypothesis and the triangle inequality we have

$$\|T - \tilde{F}_m\| = \|(I_{H^2(\mathbb{D}^n)} - F_m)T(I_{H^2(\mathbb{D}^n)} - F_m)\| \to 0,$$

as $m \to \infty$, that is, $T$ is a compact operator.

The converse follows from Lemma 2.1. This completes the proof. □

In view of the preceding theorem, it seems reasonable to define asymptotic Toeplitz operators as follows (compare this with Feintuch [12] and Barriá and Halmos [2]):
Theorem 4.1. In the case for all $m \to \infty$ in norm, $1 \leq i, j \leq n$.

We close this section by characterizing asymptotic Toeplitz operators on $H^2(\mathbb{D})$ as analogous characterization of asymptotic Toeplitz operators on $H^2(\mathbb{D})$ (see [12] and also Theorem 5.4 in Section 5).

Theorem 3.4. Let $T$ be a bounded linear operator on $H^2(\mathbb{D})$. Then $T$ is an asymptotic Toeplitz operator if and only if $T$ is a compact perturbation of Toeplitz operator.

Proof. Let $A \in \mathcal{B}(H^2(\mathbb{D}))$, $T_{z_i}^*T^m_{z_i} \to A$ and $T_{z_i}^*(T - A)T^m_{z_j} \to 0$ in norm, as $m \to \infty$, and $1 \leq i, j \leq n$. Then for all $m \geq 1$,

$$\|A - T_{z_j}^* A T_{z_j}\| \leq \|A - T_{z_j}^* T_{z_j}^{(m+1)} T^m_{z_j}\| + \|T_{z_j}^* T_{z_j}^{(m+1)} T^m_{z_j} - T_{z_j}^* A T_{z_j}\|$$

yields $T_{z_j}^* A T_{z_j} = A$ for all $j = 1, \ldots, n$. Also by Theorem 3.2, $T - A$ is compact on $H^2(\mathbb{D})^n$. The converse follows from Lemma 2.1 and Theorem 3.1. This completes the proof. □

4. Quotient spaces of $H^2(\mathbb{D})$

The purpose of this section is to extend some of the results of Section 3 in the case when the ambient operator is the compression of $(T_{z_1}, \ldots, T_{z_n})$ to a joint $(T_{z_1}^\ast, \ldots, T_{z_n}^\ast)$-invariant subspace.

Let $Q$ be a joint $(T_{z_1}^\ast, \ldots, T_{z_n}^\ast)$-invariant subspace of $H^2(\mathbb{D})$. Set

$$C_{z_i} = P_Q T_{z_i} |_Q,$$

for all $i = 1, \ldots, n$. Note that $Q^\perp$ is a joint invariant subspace for $(T_{z_1}, \ldots, T_{z_n})$ and so

$$C_{z_i}^\ast = T_{z_i}^\ast|_Q \in \mathcal{B}(Q).$$

In the case $n = 1$, $C_{z}$ is called a Jordan block [15]. In the several variables quotient space setting, we have the following analogue of Theorem 3.4.

Theorem 4.1. Let $T, A \in \mathcal{B}(Q)$. Then $C_{z_i}^\ast T C_{z_i}^m \to A$ and $C_{z_i}^\ast (T - A) C_{z_j}^m \to 0$ in norm for all $i, j = 1, \ldots, n$ if and only if $T = A + K$, where $K \in \mathcal{B}(Q)$ is a compact operator and $C_{z_i}^\ast AC_{z_i} = A$ for all $i = 1, \ldots, n$.

Proof. We first note that, as in the proof of Theorem 3.4, the assumption $C_{z_i}^\ast T C_{z_i}^m \to A$ as $m \to \infty$ implies that

$$C_{z_i}^\ast AC_{z_i} = A,$$

for all $i = 1, \ldots, n$. Now it follows from the definition of $C_{z_i}$ that

$$C_{z_i}^m = T_{z_i}^m |_Q,$$

and hence

$$C_{z_i}^m (T - A) C_{z_j}^m = T_{z_i}^m (T - A) P_Q T_{z_j}^m |_Q,$$
for all \( i, j = 1, \ldots, n \) and \( m \geq 1 \). By once again using the fact that 
\[
P_Q T^m_{z_i} P_Q = P_Q T^m_{z_i},
\]
one sees that 
\[
T^m_{z_i} (T - A) P_Q T^m_{z_j} = T^m_{z_i} (T - A) P_Q T^m_{z_j} P_Q.
\]
Hence \( C^m_{z_i} (T - A) C^m_{z_j} \to 0 \) in \( \mathcal{B}(Q) \) if and only if \( T^m_{z_i} (T - A) P_Q T^m_{z_j} \to 0 \) in \( \mathcal{B}(H^2(\mathbb{D}^n)) \) as \( m \to \infty \).

Therefore, if \( C^m_{z_i} (T - A) C^m_{z_j} \to 0 \) as \( m \to \infty \) in norm for all \( i, j = 1, \ldots, n \), then \( T^m_{z_i} (T - A) P_Q T^m_{z_j} \to 0 \) in \( \mathcal{B}(H^2(\mathbb{D}^n)) \) as \( m \to \infty \), and consequently by Theorem 3.2 \( (T - A) P_Q \) is a compact operator on \( H^2(\mathbb{D}^n) \). Therefore
\[
(T - A) = (T - A) P_Q,
\]
is a compact operator on \( Q \), which proves the necessary part.

Conversely, let \( T - A \) be a compact operator on \( Q \) and \( C^*_i A C^*_j = A \) for all \( i = 1, \ldots, n \).

Since \( C^*_i \to 0 \) as \( m \to \infty \) in the strong operator topology, Lemma 2.1 implies that
\[
C^*_i (T - A) C^m_{z_j} \to 0,
\]
as \( m \to \infty \). In particular, for all \( i = 1, \ldots, n \)
\[
C^*_i T C^m_{z_j} \to C^*_i A C^m_{z_j}.
\]
But \( C^*_i A C^*_j = A \), \( i = 1, \ldots, n \), yields us
\[
C^*_i T C^m_{z_i} \to A.
\]

This completes the proof. \( \square \)

Considering the particular case \( Q_\theta = H^2(\mathbb{D}^n)/\theta H^2(\mathbb{D}^n) \), where \( \theta \in H^\infty(\mathbb{D}^n) \) is an inner function, we get the following result.

**Theorem 4.2.** Let \( \theta \in H^\infty(\mathbb{D}^n) \) be an inner function and \( Q_\theta = H^2(\mathbb{D}^n)/\theta H^2(\mathbb{D}^n) \) and \( A \in \mathcal{B}(Q_\theta) \). Then \( C^*_i A C^*_j = A \) for all \( i = 1, \ldots, n \), if and only if \( A = 0 \).

**Proof.** Let \( C^*_i A C^*_j = A \) for all \( i = 1, \ldots, n \). Since
\[
Q_\theta^\perp = \theta H^2(\mathbb{D}^n),
\]
is a joint invariant subspace for \( (T_{z_1}, \ldots, T_{z_n}) \), it follows that
\[
P_{Q_\theta} T^*_{z_i} |_{Q_\theta} = T^*_{z_i} P_{Q_\theta},
\]
and hence
\[
A P_{Q_\theta} = (C^*_i A C^*_j) P_{Q_\theta}
= (P_{Q_\theta} T^*_{z_i} |_{Q_\theta} A P_{Q_\theta} T_{z_i} |_{Q_\theta}) P_{Q_\theta}
= T^*_{z_i} A P_{Q_\theta} T_{z_i} P_{Q_\theta}
= T^*_{z_i} A P_{Q_\theta} T_{z_i}
= T^*_{z_i} (A P_{Q_\theta}) T_{z_i}.
\]
for all $i = 1, \ldots, n$. This and Theorem 3.1 implies that $AP_{Q_\theta}$ is a Toeplitz operator. Consequently, there exists $\psi \in L^\infty(\mathbb{T}^n)$ such that

$$AP_{Q_\theta} = T_\psi.$$ 

On the other hand, since $T_\theta$ is an analytic Toeplitz operator, it follows that

$$AP_{Q_\theta} T_\theta = 0.$$ 

Hence, using [Theorem 1, C. Gu [14]], we conclude that

$$T_{\psi \theta} = T_\psi T_\theta = AP_{Q_\theta} T_\theta = 0.$$ 

This completes the proof of the theorem. □

Summing up the above two results and Lemma 2.1, we have the following generalization of Theorem 1.2 in [5].

**Theorem 4.3.** For an inner function $\theta \in H^\infty(\mathbb{D}^n)$ and bounded linear operators $T$ and $A$ on $Q_\theta = H^2(\mathbb{D}^n)/\theta H^2(\mathbb{D}^n)$, the following are equivalent:

(i) $C^m z_i T C^m z_i \to A$ and $C^m (T - A) C^m z_i \to 0$ in norm for all $i, j = 1, \ldots, n$;

(ii) $C^m z_i T C^m z_i \to 0$ in norm for all $i = 1, \ldots, n$;

(iii) $T$ is compact.

For asymptotic Toeplitzness of composition operators on the Hardy space of the unit sphere in $\mathbb{C}^n$ we refer the reader to Nazarov and Shapiro [19], and Cuckovic and Le [7].

5. **Asymptotic Toeplitz operators on $H^2_\mathbb{D}$**

The main purpose of this section is to characterize the compact operators on the model space $H^2_\mathbb{D}/\Theta H^2_\mathbb{D}$, where $\Theta \in H^\infty(\mathbb{D})$ is an inner function. We note that this result for $p = 1$ case can be found in [5]. Moreover, our proof seems more shorter and conceptually different (for instance, compare Theorem 5.3 with Proposition 2.10 in [5]).

We begin with the definition of a Toeplitz operator with operator-valued symbol.

**Definition 5.1.** Let $\mathcal{E}$ be a Hilbert space. A bounded linear operator $T$ on $H^2_\mathcal{E}$ is said to be Toeplitz if there exists an operator-valued function $\Phi$ in $L^\infty_B(\mathbb{T})$ such that $T = P_{H^2_\mathcal{E}} M_\Phi |_{H^2_\mathcal{E}}$.

Here let us observe, before we proceed further, the following characterization of Toeplitz operators on a vector-valued Hardy space. The result is probably known to the experts but we were not able to find a reference in the literature. Since the result follows adapting similar concepts and techniques used in the proof of Theorem 3.1, we give a sketch of the proof.

**Theorem 5.2.** Let $\mathcal{E}$ be a Hilbert space and $T \in B(H^2_\mathcal{E})$. Then $T$ is a Toeplitz operator if and only if $T^*_z TT_z = T$. 

Proof. Note first that \( \{e_m \eta : m \in \mathbb{Z}, \eta \in \mathcal{E}\} \) is an orthogonal basis of \( L^2_\mathcal{E}(\mathbb{T}) \), where \( e_m = e^{im \theta} \), \( m \in \mathbb{Z} \). For each \( k \geq 1 \), set

\[
A_k = M^{*k}TPM^k,
\]

where \( M_{e^{i \theta}} \) is the bilateral shift on \( L^2_\mathcal{E}(\mathbb{T}) \) and \( P \) is the orthogonal projection from \( L^2_\mathcal{E}(\mathbb{T}) \) onto \( H^2_\mathcal{E}(\mathbb{D}) \). If \( T^*z^TT^z = T \) and \( k \in \mathbb{Z}_+ \), then

\[
\langle Te_{i+k} \eta, e_{j+k} \zeta \rangle = \langle Te_i \eta, e_j \zeta \rangle,
\]

for all \( i, j \geq 0 \). Then for each \( l, m \in \mathbb{Z} \), as in the proof of Theorem 3.1, there exists \( t \geq 0 \) such that \( l + k, m + k \geq 0 \) for all \( k \geq t \), and so

\[
\langle A_k e_l \eta, e_m \zeta \rangle \to \langle Te_{l+t} \eta, e_{m+t} \zeta \rangle,
\]

as \( k \to \infty \). Then

\[
(e_l \eta, e_m \zeta) \mapsto \lim_{k \to \infty} \langle A_k e_l \eta, e_m \zeta \rangle,
\]

defines a bounded bilinear form on the span of \( \{e_l \eta : l \in \mathbb{Z}, \eta \in \mathcal{E}\} \). Therefore, there exists \( A_\infty \in \mathcal{B}(L^2_\mathcal{E}(\mathbb{T})) \) such that

\[
\langle A_\infty f, g \rangle = \lim_{k \to \infty} \langle A_k f, g \rangle,
\]

for all \( f, g \in L^2_\mathcal{E}(\mathbb{T}) \). This yields

\[
A_\infty M_{e^{i \theta}} = M_{e^{i \theta}} A_\infty.
\]

Hence there exists a \( \Phi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T}) \) such that

\[
A_\infty = M_{\Phi},
\]

and hence

\[
T = P_{H^2_\mathcal{E}(\mathbb{D})} M_{\Phi} \mid_{H^2_\mathcal{E}(\mathbb{D})}.
\]

The proof of the converse part proceeds verbatim as that of Theorem 3.1. This completes the proof of the theorem. \( \square \)

Following Feintuch [12] we now define an asymptotic Toeplitz operator on a vector-valued Hardy space.

**Definition 5.3.** Let \( \mathcal{E} \) be a Hilbert space. A bounded linear operator \( T \) on \( H^2_\mathcal{E}(\mathbb{D}) \) is said to be an asymptotic Toeplitz operator if there exists \( A \in \mathcal{B}(H^2_\mathcal{E}(\mathbb{D})) \) such that \( T^*z^m TT^z \to A \) as \( m \to \infty \) in norm.

In the theorem below, we generalize the Feintuch’s characterization [12] (see also Theorem F, page 195, [19]) of asymptotic Toeplitz operators on Hardy space to asymptotic Toeplitz operators on Hardy space of finite multiplicity. However, the method of proof here is adapted from the original proof by Feintuch.

**Theorem 5.4.** Let \( T, A \in \mathcal{B}(H^2_\mathcal{E}(\mathbb{D})) \). Then \( T^*z^m TT^z \to A \) in norm if and only if \( A \) is a Toeplitz operator and \( (T - A) \) is compact.
Proof. It follows that
\[ \|T_z^{*(m+1)}TT_z^{m+1} - T_z^*AT_z\| \leq \|T_z^{*m}TT_z^m - A\| \to 0 \]
as \( m \to \infty \). This and the triangle inequality yields \( A = T_z^*AT_z \). Now let \( R_m = T_z^mT_z^m \) and \( Q_m = I - R_m \).

Further, let \( P_{C\ell} \) denote the orthogonal projection of \( H_2^p(D) \) onto the space of \( (C\ell\text{-valued}) \) constant functions. Since \( T_zT_z^* = I_{H_2^p(D)} - P_{C\ell} \), it follows that
\[ Q_m = \sum_{k=0}^{m-1} T_z^kP_{C\ell}T_z^{*k} \quad (m \geq 1). \]

Then \( Q_m, m \geq 1, \) is a finite rank operator, and therefore
\[ F_m = (T - A)Q_m + Q_m(T - A)Q_m = (m \geq 1), \]
is also a finite rank operator. Moreover
\[ (T - A) - F_m = R_m(T - A)R_m \quad (m \geq 1), \]
yields
\[ \|(T - A) - F_m\| = \|R_m(T - A)R_m\| \leq \|T_z^{*m}TT_z^m - A\| \to 0, \]
as \( m \to \infty \). So \( T - A \) is compact as desired.

The converse follows from Lemma 2.1. This completes the proof. \( \square \)

We have the following result in the model space setting.

Proposition 5.5. Let \( \Theta \in H_{B(\ell)}^\infty(D) \) be an inner multiplier and \( T \in B(Q_\Theta) \). Assume that \( \Theta(e^{i\theta}) \) is invertible a.e. Then \( S_\Theta^*TS_\Theta = T \) if and only if \( T = 0 \).

Proof. The proof goes exactly along the same lines as the proof of Theorem 4.2. Since
\[ TP_{Q_\Theta} = T_z^*(TP_{Q_\Theta})T_z, \]
it follows from Theorem 5.2 that \( TP_{Q_\Theta} \) is a Toeplitz operator. Consequently, there exists \( \Psi \in L_\infty(T) \) such that
\[ TP_{Q_\Theta} = T_\Psi. \]
Since \( T_\Theta \) is an analytic Toeplitz operator, again as in the proof of Theorem 4.2 it follows that
\[ T_\Psi T_\Theta = 0, \]
and hence
\[ \Psi T_\Theta = 0. \]
Since \( \Theta \) is invertible a.e., it follows that \( \Psi = 0 \) a.e. and hence \( T = 0 \). This completes the proof. \( \square \)

Not only is this proposition a considerable generalization of Proposition 2.10 of [5], but our proof is much simpler. The principal tool is the identity \( S_\Theta^* = T_z^*|Q_\Theta \).

We have the following characterization which generalizes the characterization of compact operators on \( Q_\Theta \) for \( p = 1 \) (see the implication (i) and (iii) in Theorem 1.2 in [5]).
Theorem 5.6. Let $\Theta \in H^\infty_0 (C_0)$ be an inner multiplier and $T \in B(Q_\Theta)$. Then $T$ is compact if and only if $\{S^{m \Theta}_m T S^{m \Theta}_m\}_{m \geq 1}$ converges in norm.

Proof. If $T$ is compact on $Q_\Theta$, then by Lemma 2.1 $\|S^{m \Theta}_m T S^{m \Theta}_m\| \to 0$ as $m \to \infty$. To prove the converse, let $A \in B(Q_\Theta)$ and $S^{m \Theta}_m T S^{m \Theta}_m \to A$, as $m \to \infty$, in norm. Then by the same argument used in the proof of Theorem 4.1, we have $S^{m \Theta}_m A S^{m \Theta}_m = A$. It now follows from Proposition 5.5 that $A = 0$ and therefore $S^{m \Theta}_m T P_{Q_\Theta} T S^{m \Theta}_m \to 0$ as $m \to \infty$. Now Theorem 5.4 implies that $T P_{Q_\Theta}$ is a compact operator on $H^2_0 (C_0)$. Therefore $T = T P_{Q_\Theta}$ is a compact operator on $Q_\Theta$. This completes the proof. □

Theorem 5.6 and Lemma 2.1 give us the following generalization of Theorem 1.2 in [5].

Theorem 5.7. Let $\Theta \in H^\infty_0 (C_0)$ be an inner multiplier and $T \in B(Q_\Theta)$. Then the following are equivalent:

(i) $\{S^{m \Theta}_m T S^{m \Theta}_m\}_{m \geq 1}$ converges in norm;
(ii) $S^{m \Theta}_m T S^{m \Theta}_m \to 0$ in norm;
(iii) $T$ is a compact operator.

Acknowledgement: We are grateful to Professor A. Bottcher for pointing out that our Lemma 2.1 is a special case of 1.3 (d) in the monograph by Bottcher and Silbermann [3]. The first author’s research work is supported by NBHM Post Doctoral Fellowship No. 2/40(50)/2015/R & D - II/11569. The second author is supported in part by NBHM (National Board of Higher Mathematics, India) grant NBHM/R.P.64/2014.

REFERENCES

[1] P. Ahern, E. Youssfi and K. Zhu, Compactness of Hankel operators on Hardy-Sobolev spaces of the polydisk. J. Operator Theory 61 (2009), 301-312.
[2] J. Barría and P. R. Halmos, Asymptotic Toeplitz operators. Trans. Amer. Math. Soc. 273 (1982), no. 2, 621-630.
[3] A. Bottcher and B. Silbermann, Analysis of Toeplitz operators. Springer-Verlag, Berlin, 1990.
[4] A. Brown and P. R. Halmos, Algebraic properties of Toeplitz operators. J. Reine Angew. Math. 213 (1963/1964), 89–102.
[5] I. Chalendar, W. T. Ross, Compact operators on model spaces, 2016 (arXiv:1603.01370).
[6] B. Choe, H. Koo and Y. Lee, Commuting Toeplitz operators on the polydisk. Trans. Amer. Math. Soc. 356 (2004), 1727-1749.
[7] Z. Cuckovic and T. Le, Toeplitzness of composition operators in several variables. Complex Var. Elliptic Equ. 59 (2014), 1351-1362.
[8] M. Dida and J. Eschmeier and D. Schillo, On Schatten-class perturbations of Toeplitz operators. J. Funct. Anal. 272 (2017), 2442-2462.
[9] X. Ding, Products of Toeplitz operators on the polydisk. Integral Equations Operator Theory. 45 (2003), 389-403.
[10] R. G. Douglas, Banach algebra techniques in operator theory. Second edition. Graduate Texts in Mathematics, 179. Springer-Verlag, New York, 1998.
[11] J. Eschmeier and K. Everard, Toeplitz projections and essential commutants. J. Funct. Anal. 269 (2015), 1115-1135.
[12] A. Feinstein, On asymptotic Toeplitz and Hankel operators. In The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988), volume 41 of Oper. Theory Adv. Appl., pages 241254. Birkhauser, Basel, 1989.
TOEPLITZ AND ASYMPTOTIC TOEPLITZ OPERATORS ON $H^2(\mathbb{D}^n)$

[13] G. Popescu, Free pluriharmonic functions on noncommutative polyballs. Anal. PDE 9 (2016), 1185-1234.
[14] C. Gu, Some algebraic properties of Toeplitz and Hankel operators on polydisk. Arch. Math. (Basel) 80 (2003), 393-405.
[15] K. Guo and K. Wang, On operators which commute with analytic Toeplitz operators modulo the finite rank operators. Proc. Amer. Math. Soc. 134 (2006), 2571-2576.
[16] P. Hartman and A. Wintner, The spectra of Toeplitz’s matrices. Amer. J. Math. 76 (1954), 867-882.
[17] S. Sun and D. Zheng, Toeplitz operators on the polydisk. Proc. Amer. Math. Soc. 124 (1996), 3351-3356.
[18] B. Sz.-Nagy and C. Foias, Harmonic Analysis of Operators on Hilbert Space. North-Holland, Amsterdam-London, 1970.
[19] F. Nazarov and H. Shapiro, On the Toeplitzness of composition operators. Complex Var. Elliptic Equ. 52 (2007), 193-210.
[20] W. Rudin, Function theory in polydiscs. W. A. Benjamin, Inc., New York-Amsterdam 1969.
[21] H. Upmeier, Toeplitz operators and index theory in several complex variables. Operator Theory: Advances and Applications, 81. Birkhauser Verlag, Basel, 1996.

Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India
E-mail address: amaji_pd@isibang.ac.in, amit.iitm07@gmail.com

Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India
E-mail address: jay@isibang.ac.in, jaydeb@gmail.com

Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India
E-mail address: srijan_rs@isibang.ac.in, srijansarkar@gmail.com