ON THE RANK OF THE 2-CLASS
GROUP OF \(\mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{-1})\)

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Abstract. Let \(d\) be a square-free integer, \(k = \mathbb{Q}(\sqrt{d}, i)\) and \(i = \sqrt{-1}\). Let \(k_1^{(2)}\) be the Hilbert 2-class field of \(k\), \(k_2^{(2)}\) be the Hilbert 2-class field of \(k_1^{(2)}\) and \(G = \text{Gal}(k_2^{(2)}/k)\) be the Galois group of \(k_2^{(2)}/k\). Our goal is to give necessary and sufficient conditions to have \(G\) metacyclic in the case where \(d = pq\), with \(p\) and \(q\) are primes such that \(p \equiv 1 \pmod{8}\) and \(q \equiv 5 \pmod{8}\) or \(p \equiv 1 \pmod{8}\) and \(q \equiv 3 \pmod{4}\).

1. Introduction

Let \(k\) be an algebraic number field and let \(Cl_2(k)\) denote its 2-class group. Denote by \(k_2^{(1)}\) the Hilbert 2-class field of \(k\) and by \(k_2^{(2)}\) its second Hilbert 2-class field. Put \(G = \text{Gal}(k_2^{(2)}/k)\) and \(G'\) its derived group, then it's well known that \(C/G' \cong Cl_2(k)\). An important problem in Number Theory is to determine the structure of \(G\), since the knowledge of \(G\), its structure and its generators solve a lot of problems in number theory as capitulation problems, the finiteness or not of the towers of number fields and the structures of the 2-class groups of the unramified extensions of \(k\) within \(k_2^{(i)}\). In several times, the knowledge of the rank of \(G\) allows to know the structure of \(G\). In this paper, we give an example of this situation.

Let \(k = \mathbb{Q}(\sqrt{pq}, i)\), where \(p\) and \(q\) are two different primes, then the genus field of \(k\) is \(k^* = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{-1})\). According to [7] \(r_0\), the rank of the 2-class group of \(k\), is at most equal to 3. Moreover \(r_0 = 3\) if and only if \(p \equiv q \equiv 1 \pmod{8}\). Let \(G = \text{Gal}(k_2^{(2)}/k)\) be the Galois group of \(k_2^{(2)}/k\), where \(k_2^{(i+1)}\) is the Hilbert 2-class field of \(k_2^{(i)}\), with \(i = 0\) or 1 and \(k_2^{(0)} = k\). The Artin Reciprocity implies that \(r_0 = d(G)\), where \(d(G)\) is the rank of \(G\).

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In [7], the first and the second authors have shown that if \( q = 2 \), then \( G \) is metacyclic non abelian if and only if \( p = x^2 + 32y^2 \) and \( x \neq \pm 1 \) (mod 8). In this paper, we prove that if \( p \) and \( q \) are odd different primes, then \( r \), the rank of 2-class group of \( k^* \), helps to know in which case \( G \) is metacyclic.

2. The rank of the 2-class group of \( k^* \)

In what follows, we adopt the following notations: If \( p \equiv 1 \) (mod 8) is a prime, then \( \left( \frac{2}{p} \right)_4 \) will denote the rational biquadratic symbol which is equal to 1 or -1, according as \( 2^{\frac{p-1}{4}} \equiv \pm 1 \) (mod \( p \)). Moreover the symbol \( \left( \frac{p}{x} \right)_4 \) is equal to \((-1)^{\frac{p-1}{8}}\).

Let \( k \) be a number field and \( l \) be a prime; then \( \mathfrak{l}_k \) will denote a prime ideal of \( k \) above \( l \). We denote, also, by \( \left( \frac{x, y}{\mathfrak{l}_k} \right) \) (resp. \( \left( \frac{x}{\mathfrak{l}_k} \right) \)) the Hilbert symbol (resp. the quadratic residue symbol) for the prime \( \mathfrak{l}_k \) applied to \( (x, y) \) (resp. \( x \)). A 2-group \( H \) is said of type \( (2^{n_1}, 2^{n_2}, \ldots, 2^{n_s}) \) if it is isomorphic to \( \mathbb{Z}/2^{n_1} \times \mathbb{Z}/2^{n_2} \times \ldots \mathbb{Z}/2^{n_s} \), where \( n_i \in \mathbb{N} \). For all number field \( k \), \( h(k) \) will denote the 2-class number of \( k \). Finally, \( r_0 \) (resp. \( r \)) denotes the rank of the 2-class group of \( k \) (resp. \( k^* \)).

Lemma 1. If \( p \equiv 5 \) (mod 8) and \( q \equiv 3 \) (mod 4), then \( G \) is cyclic and \( r = 1 \).

Proof. If \( p \equiv 5 \) (mod 8) and \( q \equiv 3 \) (mod 4), then, according to [16], the 2-class group of \( k \) is cyclic, so \( G \) is an abelian group of rank 1. As \( k^* \) is an unramified extension of \( k \), then the 2-class group of \( k^* \) is also cyclic and \( r = 1 \). \( \square \)

Lemma 2. If \( p \equiv 1 \) (mod 8) and \( q \equiv 1 \) (mod 8), then \( G \) is a non-metacyclic group.

Proof. If \( p \equiv 1 \) (mod 8) and \( q \equiv 1 \) (mod 8), then, according to [16], the 2-class group of \( k \) is of rank 3, which is the rank of \( G \). This yields that \( G \) is not metacyclic, since the metacyclic groups are of ranks \( \leq 2 \). \( \square \)

According to the two Lemmas 1 and 2 it is interesting to assume, in what follows, that \( p \equiv 1 \) (mod 8) and \( q \equiv 5 \) (mod 8) or \( p \equiv 1 \) (mod 8) and \( q \equiv 3 \) (mod 4). So \( r_0 = 2 \) (see [16]). We continue with the following lemmas.

Lemma 3 ([11]). Let \( p \equiv 1 \) (mod 8) be a prime, then

\[
\left( \frac{i}{\mathbb{Q}((i))} \right) = 1 \quad \text{and} \quad \left( \frac{1+i}{\mathbb{Q}((i))} \right) = \left( \frac{2}{p} \right)_4 \left( \frac{p}{2} \right)_4.
\]
Lemma 4. Let $F = \mathbb{Q}(\sqrt{q}, i)$ where $q \equiv 5 \pmod{8}$ and $\varepsilon_q$ be the fundamental unite of $\mathbb{Q}(\sqrt{q})$, then

(i) \( \left( \frac{p, i}{l_F} \right) = 1 \) for all prime ideal $l_F$ of $F$.

(ii) \( \left( \frac{p, \varepsilon_q}{l_F} \right) = 1 \) for all odd prime ideal $l_F \neq p_F$ of $F$.

(iii) \( \left( \frac{p, \varepsilon_q}{p_F} \right) = \left( \frac{p, \varepsilon_q}{2_F} \right) = \begin{cases} 1, & \text{if } \left( \frac{p}{q} \right) = -1; \\ \left( \frac{p}{q} \right)_4 \left( \frac{q}{p} \right)_4, & \text{if } \left( \frac{p}{q} \right) = 1. \end{cases} \)

Proof. (i) Let $l_F$ be an odd prime ideal of $F$.

If $l_F \neq q_F$, then $l_F$ is a prime ideal of $F$ unramified in $F(\sqrt{q})$ (see the proof of the following theorem), hence

\[
\left( \frac{p, i}{l_F} \right) = \left( \frac{p}{l_F} \right)^{v(i)} = 1, \quad [12, \text{p. 205}].
\]

If $l_F = p_F$, the prime ideal of $F$ above $p$, then

\[
\left( \frac{p, i}{p_F} \right) = \left( \frac{i, p}{p_F} \right) = \left( \frac{i}{p_F} \right) \quad [12, \text{p. 205}]
\]
\[
= \left( \frac{i}{p_{Q(i)}} \right) \quad [12, \text{p. 205}]
\]
\[
= 1. \quad \text{(Lemma 3)}
\]

(ii) Same proof as in (i).

(iii) the inertia degree of $p_F$ is equal to 1 in $F/Q(\sqrt{p})$, which implies that

\[
\left( \frac{p, \varepsilon_q}{p_F} \right) = \left( \frac{\varepsilon_q, p}{p_F} \right) = \left( \frac{\varepsilon_q}{p_F} \right) \quad [12, \text{p. 205}]
\]
\[
= \left( \frac{\varepsilon_q}{p_{Q(\sqrt{q})}} \right) \quad [12, \text{p. 205}]
\]
Suppose that \((p, q) = 1\), then
\[
(p, q)_4 (p, q)_4. \quad \text{[8, p. 101]}
\]

Suppose that \((p, q) = -1\). With a same argument as above, we get:
\[
(p, ε_q l_F) = \left(\frac{N_{\mathbb{Q}(\sqrt{q}/\mathbb{Q})}(ε_q)}{p}\right) = \left(\frac{-1}{p}\right) = 1, \quad \text{[12, p. 205].}
\]

The product formula for the Hilbert symbol implies that \((\frac{p, ε_p}{p_F}) = (\frac{p, ε_p}{2_F})\). □

**Lemma 5.** Let \(F = \mathbb{Q}(\sqrt{q}, i)\) where \(q \equiv 3 \pmod{4}\) and \(ε_q\) be the fundamental unite of \(\mathbb{Q}(\sqrt{q})\). Then

(i) \(\left(\frac{p, i l_F}{l_F}\right) = 1\) for all prime ideal \(l_F\) of \(F\).

(ii) \(\left(\frac{p, \sqrt{iε_q}}{l_F}\right) = 1\) for all odd prime ideal \(l_F \neq p_F\) of \(F\).

(iii) \(\left(\frac{p, \sqrt{iε_q}}{p_F}\right) = \left(\frac{p, \sqrt{iε_q}}{2_F}\right) = \begin{cases} 1, & \text{if } \left(\frac{p}{q}\right) = -1; \\ \left(\frac{2}{p}\right)_4 \left(\frac{p}{2}\right)_4 \left(\frac{\sqrt{2ε_q}}{p_{\mathbb{Q}(\sqrt{q})}}\right), & \text{if } \left(\frac{p}{q}\right) = 1. \end{cases}\)

**Proof.** By a similar approach of previous lemma, we get (i) and (ii). For (iii), remark that \(2ε_q\) is a square in \(\mathbb{Q}(\sqrt{q})\) (see [4]), then \(iε_q\) is a square in \(F\), since \(2\sqrt{iε_q} = (1 + i)\sqrt{2ε_q}\). As the Hilbert symbol is a bilinear map with values in
$\{+1, -1\}$ and $2i = (1 + i)^2$, so

$$\left(\frac{p, \sqrt{\varepsilon q}}{\mathbb{Q}(\sqrt{q}, i)}\right) = \left(\frac{p, 2}{\mathbb{Q}(\sqrt{q}, i)}\right) \left(\frac{p, \sqrt{2\varepsilon q}}{\mathbb{Q}(\sqrt{q}, i)}\right) = \left(\frac{p, i}{\mathbb{Q}(\sqrt{q}, i)}\right) \left(\frac{p, \sqrt{2\varepsilon q}}{\mathbb{Q}(\sqrt{q}, i)}\right) = \left(\frac{1 + i}{\mathbb{Q}(\sqrt{q}, i)}\right) \left(\frac{\sqrt{2\varepsilon q}}{\mathbb{Q}(\sqrt{q}, i)}\right)$$

$$= \left(\frac{1 + i}{\mathbb{Q}(\sqrt{q}, i)}\right) \left(\frac{\sqrt{2\varepsilon q}}{\mathbb{Q}(\sqrt{q}, i)}\right) \left[12, p. 205\right]$$

Theorem 1. Let $p$ and $q$ be primes as above and $r$ be the rank of the 2-class group of $\mathbb{Q}(\sqrt{q}, \sqrt{p}, i)$.

(1) If $q \equiv 5 \pmod{8}$, then

$$r = \begin{cases} 
1, & \text{if } \left\langle \frac{p}{q} \right\rangle = -1; \\
2, & \text{if } \left\langle \frac{p}{q} \right\rangle = 1 \text{ and } \left(\frac{p}{q}\right)_4 = -\left(\frac{q}{p}\right)_4; \\
3, & \text{if } \left\langle \frac{p}{q} \right\rangle = 1 \text{ and } \left(\frac{p}{q}\right)_4 = \left(\frac{q}{p}\right)_4. 
\end{cases}$$

(2) If $q \equiv 3 \pmod{4}$, then, by putting $\eta = \left(\frac{\sqrt{2\varepsilon q}}{\mathbb{Q}(\sqrt{q}, i)}\right)$ if $\left(\frac{p}{q}\right) = 1$, we obtain

$$r = \begin{cases} 
1, & \text{if } \left\langle \frac{p}{q} \right\rangle = -1; \\
2, & \text{if } \left\langle \frac{p}{q} \right\rangle = 1 \text{ and } \left(\frac{2}{p}\right)_4 = -\left(\frac{p}{2}\right)_4 \eta; \\
3, & \text{if } \left\langle \frac{p}{q} \right\rangle = 1 \text{ and } \left(\frac{2}{p}\right)_4 = \left(\frac{p}{2}\right)_4 \eta. 
\end{cases}$$

Proof. Let $F$ denote the field $\mathbb{Q}(\sqrt{q}, i)$ defined above and $\varepsilon_q$ be the fundamental unit of the $\mathbb{Q}(\sqrt{q})$. According to [1], the unit group of $F$ is equal to

$$\left\langle i, \varepsilon_q \right\rangle, \quad \text{if } q \equiv 5 \pmod{8};$$

$$\left\langle i, \sqrt{i\varepsilon_q} \right\rangle, \quad \text{if } q \equiv 3 \pmod{4}.$$
As the class number of the $F$ is odd, then by the ambiguous class number formula (see [9]), we have:

$$r = t - e - 1,$$

where $t$ is the number of primes of $F$ that ramify in $k^*/F$ ($k^* = \mathbb{Q}(\sqrt{q}, \sqrt{p}, i)$ is the genus field of $k = \mathbb{Q}(\sqrt{pq}, i)$) and $e$ is determined by $2^e = [E_F : E_F \cap N_{k^*/F}(k^*)]$. The following diagram helps us to calculate the number $t$.

![Diagram](image)

**Figure 1.**

Let $l$ be a prime. Since the extension $k/k^*$ is unramified, then

$$e(l_F/l).e(l_{k^*}/l_F) = e(l_k/l).$$

As

$$e(l_F/l) = \begin{cases} 2 & \text{if } l = q \text{ or } 2, \\ 1 & \text{otherwise}, \end{cases}$$

and

$$e(l_k/l) = \begin{cases} 2 & \text{if } l = p, q \text{ or } 2, \\ 1 & \text{otherwise}, \end{cases}$$

so it is easy to see that

$$e(l_{k^*}/l_F) = \begin{cases} 2 & \text{if } l = p, \\ 1 & \text{otherwise}. \end{cases}$$

Similarly, we find that

$$f(l_{k^*}/l_F) = \begin{cases} 1 & \text{if } \left(\frac{p}{q}\right) = 1, \\ 2 & \text{if } \left(\frac{p}{q}\right) = -1. \end{cases}$$
Therefore
\[ t = \begin{cases} 
4 & \text{if } \left( \frac{p}{q} \right) = 1, \\
2 & \text{if } \left( \frac{p}{q} \right) = -1.
\end{cases} \]

Finally
\[ r = \begin{cases} 
3 - e & \text{if } \left( \frac{p}{q} \right) = 1, \\
1 - e & \text{if } \left( \frac{p}{q} \right) = -1.
\end{cases} \]

The Hasse norm theorem (see e.g. [12, theorem 6.2, p. 179]) implies that a unit \( \varepsilon \) of \( F \) is a norm of an element of \( F(\sqrt{p}) = k^* \) if and only if \( \left( \frac{p, \varepsilon}{p_F} \right) = 1 \), for all \( p_F \neq 2_F \) prime ideal of \( F \).

If \( q \equiv 3 \pmod{4} \), we conclude thanks to the previous lemma that
\[ e = \begin{cases} 
0, & \text{if } \left( \frac{p}{q} \right) = -1; \\
1, & \text{if } \left( \frac{p}{q} \right) = 1 \text{ and } \left( \frac{2}{p} \right)_4 = - \left( \frac{p}{2} \right)_4 \eta; \\
0, & \text{if } \left( \frac{p}{q} \right) = 1 \text{ and } \left( \frac{2}{p} \right)_4 = \left( \frac{p}{2} \right)_4 \eta.
\end{cases} \]

This completes the proof of our theorem. \( \square \)

3. Application

Let \( G = \text{Gal}(k_2^{(2)}/k) \) be the Galois group of the extension \( k_2^{(2)}/k \), where \( k = \mathbb{Q}(\sqrt{pq}, i) \) and \( p, q \) are distinct primes such that \( p \equiv 1 \pmod{8} \) and \( q \equiv 5 \pmod{8} \) or \( p \equiv 1 \pmod{8} \) and \( q \equiv 3 \pmod{4} \). In this section, we give an application of the previous theorem and we characterize the group \( G \). In particular, we will find results about \( G \) given by Azizi in [3] and [5].

**Theorem 2.** Let \( p \) and \( q \) be different primes defined as above, \( k = \mathbb{Q}(\sqrt{pq}, i) \), \( r \) be the rank of the 2-class group of \( \mathbb{Q}(\sqrt{q}, \sqrt{p}, i) \) and \( G = \text{Gal}(k_2^{(2)}/k) \). Then \( G \) is nonmetacyclic if and only if \( r = 3 \).

**Proof.** Since \( p \equiv 1 \pmod{8} \), then there exist two integers \( x \) and \( y \) such that \( p = x^2 + 16y^2 \). Put \( \pi_1 = x + 4yi, \pi_2 = x - 4yi, k_1 = k(\sqrt{\pi_1}) \) and \( k_2 = k(\sqrt{\pi_2}) \).

As \( \pi_1 \) and \( \pi_2 \) are ramified in \( k/Q(i) \), then the ideals generated by \( \pi_1 \) and \( \pi_2 \) are squares of ideals of \( k \). Note that \( x \) is odd, thus \( x \equiv \pm 1 \equiv i^2 \pmod{4} \),
then the two equations $\pi_i \equiv \xi^2$ are solvable in $k$. We conclude that the two extensions $k(\sqrt{\pi_1})/k$ and $k(\sqrt{\pi_2})/k$ are unramified. It is clear that $k_i \neq k^*$. Since $r_0 = d(G) = 2$, then $k_1$, $k_2$ and $k^*$ are precisely the three unramified quadratic extensions of $k$. Put $H_i = \text{Gal}(k_i(2)/k_i)$ where $i = 1, 2$ and $M = \text{Gal}(k^*_2/k^*)$. These three subgroups are the maximal subgroups of $G$. As $d(G) = 2$ and $k_1$ is isomorphic to $k_2$, then according to [17], $G$ is nonmetacyclic if and only if $d(M) = r = 3$.

**Lemma 6.** Let $k$ be an algebraic number field and $G = \text{Gal}(k(2)/k)$. Let $L$ be an extension of $k$ such that $M = \text{Gal}(k_2(2)/L)$ is a cyclic subgroup of $G$ of index 2. If four ideal classes of $k$ capitulate in $L$, then $G$ is abelian or dihedral group.

**Proof.** we say that an ideal class of $k$ capitulates in $L$ if it is in the kernel of the homomorphism $j : Cl_2(k) \rightarrow Cl_2(L)$ induced by extension of ideals from $k$ to $L$. Furthermore, the homomorphism $j$ corresponds, by the Artin reciprocity law to the group theoretical transfer $V : G/G' \rightarrow M$ (For further information on $V$, see for example [15]). Since $M$ is a cyclic subgroup of $G$ of index 2, then, according to [14], $G$ is isomorphic to one of the following groups:

1. Cyclic 2-group or 2-group of type $(2^n, 2^m)$.
2. The dihedral group.
3. The quaternion group.
4. The semidihedral group.
5. The modular 2-group.

If $G$ is one of the first four groups and four ideal classes of $k$ capitulate in $L$, then H. Kisilevsky has shown in [13] that $G$ is abelian or dihedral group. Next if $G$ is a modular 2-group, then $G = \langle x, y : x^{2^n-1} = y^2, y^{-1}xy = x^{1+2^{n-2}} \rangle$ and $M = \langle x \rangle$. An elementary calculation shows that $\ker(V) = \{G', yG'\}$, i.e. two ideal classes of $k$ capitulate in $L$. □

**Theorem 3.** Let $p$ and $q$ be different primes and $\eta$ be the number defined in Theorem [11] Put $k = \mathbb{Q}(\sqrt{pq}, i)$ and $G = \text{Gal}(k_2(2)/k)$.

1. If $q \equiv 5 \pmod{8}$ and $\left(\frac{p}{q}\right) = -1$, then $G$ is dihedral.
2. If $q \equiv 5 \pmod{8}$ and $\left(\frac{p}{q}\right) = 1$ and $\left(\frac{p}{\eta}\right)_4 = -\left(\frac{q}{p}\right)_4$, then $G$ is a non-abelian metacyclic group with $G/G'$ is of type $(2, 4)$. 
(3) If \( q \equiv 3 \pmod{8} \) and \( \left( \frac{p}{q} \right) = -1 \), then \( G \) is abelian or dihedral.

(4) If \( q \equiv 3 \pmod{4} \) and \( \left( \frac{p}{q} \right) = 1 \) and \( \left( \frac{2}{p} \right)_4 = -\left( \frac{p}{p} \right)_4 \eta \), then \( G \) is a metacyclic group.

Proof. Proceeding as in the proof of Theorem 7 of [2], we get that if \( q \equiv 5 \pmod{8} \), then \( h(\mathbf{k}^*) = \frac{h(-p)h(\mathbf{k})}{4} \), where \( h(-p) \) denotes the 2-class number of \( \mathbb{Q}(\sqrt{-p}) \). Since \( r_0 \), the rank of the 2-class group of \( k \), is equal to 2, then \( G \) is abelian if and only if \( h(\mathbf{k}^*) = \frac{h(\mathbf{k})}{2} \) (see [10]), this is equivalent to \( h(-p) = 2 \). Which is impossible since \( p \equiv 1 \pmod{8} \).

(1) If \( q \equiv 5 \pmod{8} \) and \( \left( \frac{p}{q} \right) = -1 \), then \( G \) is nonabelian. According to Theorem 1 the rank of the 2-class group of \( \mathbf{k}^* \) is \( r = 1 \), thus \( M = \text{Gal}(\mathbf{k}^{(2)}/\mathbf{k}^*) \) is cyclic. On the other hand, Azizi in [2] has shown, in this situation, that there are four ideal classes of \( \mathbf{k} \) capitulate in \( \mathbf{k}^* \), hence the Lemma 6 implies that \( G \) is a dihedral group.

(2) If \( q \equiv 5 \pmod{8} \), \( \left( \frac{p}{q} \right) = 1 \) and \( \left( \frac{2}{p} \right)_4 = -\left( \frac{p}{p} \right)_4 \), then \( G \) is nonabelian and the 2-class group of \( \mathbf{k} \) is of type \((2, 4)\) (see [4]). Moreover Theorem 1 yields that \( r = 2 \), then, according to the previous theorem, we have \( G \) is metacyclic.

(3) and (4) are proved similarly. \( \square \)

References

[1] A. Azizi, Unités de certains corps de nombres imaginaires et abéliens sur \( \mathbb{Q} \), Ann. Sci. Math. Québec 23 (1999), 87-93.

[2] A. Azizi, Capacitation of the 2-ideal Classes of \( \mathbb{Q}(\sqrt{p_1p_2}, i) \) Where \( p_1 \) and \( p_2 \) are primes such that \( p_1 \equiv 1 \pmod{8}, p_2 \equiv 5 \pmod{8} \) and \( \left( \frac{p_1}{p_2} \right) = -1 \), Lecture notes in pure and applied mathematics. vol. 208 (1999), 13-19.

[3] A. Azizi, Sur le 2-groupe de classe d'idéaux de \( \mathbb{Q}(\sqrt{d}, i) \), Rend. Circ. Mat. Palermo (2) 48 (1999), 71-92.

[4] A. Azizi, Sur la capitulation des 2-classes d'idéaux de \( k = \mathbb{Q}(\sqrt{2pq}, i) \), où \( p \equiv -q \equiv 1 \pmod{4} \), Acta. Arith. 94 (2000), 383-399.

[5] A. Azizi, Sur une question de Capacitation, Proc. Amer. Math. Soc. 130 (2002), 2197-2002.

[6] A. Azizi et M. Taous, Determination des corps \( \mathbf{k} = \mathbb{Q}(\sqrt{d}, i) \) dont le 2-groupes de classes est de type \((2, 4)\) ou \((2, 2, 2)\), Rend. Istit. Mat. Univ. Trieste. 40 (2008), 93-116.

[7] A. Azizi et M. Taous, Condition nécessaire et suffisante pour que certain groupe de Galois soit métacyclique, Ann. Math. Blaise Pascal. 16, No. 1 (2009), 83-92.
[8] A. Scholz, Über die Löbarkeit der Gleichung \( t^2 - Du^2 = -4 \), Math. Z. 39 (1934), 95-111.

[9] C. Chevalley, Sur la théorie du corps de classes dans les corps finis et les corps locaux, J. Fac. Sc. Tokyo, Sect. 1, t. 2, (1933), 365-476.

[10] E. Benjamin, F. Lemmermeyer and C. Snyder, Real quadratic fields with abelian 2-class field tower, J. Number Theory. 73 (1998), 182-194.

[11] F. Lemmermeyer, Reciprocity Laws, Springer Monographs in Mathematics, Springer-Verlag, Berlin 2000.

[12] G. Gras, Class field theory, from theory to practice, Springer Verlag 2003.

[13] H. Kisilevsky, Number fields with class number congruent to 4 mod 8 and Hilbert’s theorem 94, J. Number Theory 8 (1976), 271-279.

[14] H. Kurzweil, B. Stellmacher, The Theory of Finite Groups: An Introduction, Springer-Verlag, New York 2004.

[15] K. Miyake, Algebraic Investigations oh Hilbert’s Theorem 94, the Principal Ideal theorem and Capitulation Problem, Expos. Math. 7 (1989), 289-346.

[16] T. M. McCall, C. J. Parry, R. R. Ranalli, On imaginary bicyclic biquadratic fields with cyclic 2-class group, J. Number Theory. 53 (1995), 88-99.

[17] Y. Berkovich and Z. Janko, On subgroups of finite p-groups, Israel J. Math. 171 (2009), 29-49.

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