Instability of small AdS black holes in sixth-order gravity

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Abstract

We investigate the stability analysis of AdS black holes in the higher dimensional sixth-order gravity. This gravity is composed of Ricci cubic gravity and Lee-Wick term. It indicates that the Ricci tensor perturbations exhibit unstable modes for small AdS black holes in Ricci cubic gravity by solving the Lichnerowicz-type linearized equation. We show that the correlated stability conjecture holds for the AdS black hole by computing all thermodynamic quantities in Ricci cubic gravity. Furthermore, we find a newly non-AdS black hole in Ricci cubic gravity by making use of a static eigenfunction of the Lichnerowicz operator.

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1 Introduction

Recently, Einsteinian cubic gravity of \( \mathcal{L}_{EC} = R - 2\Lambda_0 - \lambda \mathcal{P}/6 \) was shown to be neither topological nor trivial in four dimensions [1]. Here \( \mathcal{P} \) is sixth-order Riemann polynomials. Even though black hole solutions to this gravity have interesting properties [2, 3, 4], these belong to either numerical or approximate solutions. This means that the Einstein equation cannot be solved analytically and an obstacle to studying these black holes is a lack of an analytic solution.

On the other hand, the other sixth-order gravity is given by the Ricci cubic gravity [5]. It is composed of Einstein-Hilbert term and Ricci cubic polynomials \( \mathcal{R}_3 \), which has a similar property to the Ricci quadratic gravity including Ricci quadratic polynomials \( \mathcal{R}_2 \) in four dimensions [6]. We note that Ricci polynomials \( \mathcal{R}_3 \) are much more manageable than Riemann polynomials \( \mathcal{P} \). A crucial benefit of Ricci cubic gravity is that the \( n \)-dimensional AdS black hole to Einstein gravity is also a solution to this gravity. Furthermore, one may construct a covariant linearized gravity on the AdS black hole in Ricci cubic gravity, but the covariant linearized theory of Einsteinian cubic gravity is allowed only on a vacuum of AdS\(_n\) spacetimes. This is a main reason why we prefer the Ricci cubic gravity to the Einsteinian cubic gravity in the study of AdS black hole. A final candidate for constructing a general sixth-order gravity is the Lee-Wick term which contains a d’Alembertian (\( \Box \)). The Lee-Wick gravity of \( R + \alpha_0 G_{\mu\nu} \Box R^{\mu\nu} \) turned out to be a ghost-free and renormalizable theory around the Minkowski background [7]. However, the Lee-Wick term generates uncontrollable sixth-order terms in the linearized equation and thus, it might make the analysis of black hole stability untractable [8]. Anyway, we will begin with a sixth-order gravity including the Lee-Wick term for completeness.

If an AdS black hole is obtained from the sixth-order gravity, one asks naturally what this all mean for “physical black hole”? A physical black hole can be justified through analysis of the stability by solving \( \delta R_{\mu\nu}(h) = 0 \) in Einstein gravity [9, 10]. If one adopts the Regge-Wheeler gauge, two physical degrees of freedom for a massless spin-2 mode could be provided by splitting odd and even parities. If it is stable against the metric perturbation \( h_{\mu\nu} \), one may accept it as a physical black hole. Investigating the stability analysis of the SAdS black hole in Einstein gravity with a cosmological constant, one might use the linearized Einstein equation \( \delta G_{\mu\nu}(h) = 0 \) [11, 12]. However, the Regge-Wheeler prescription is limited to the second-order gravity and thus, it is no longer applicable to a higher-order
gravity. For a fourth-order gravity, the linearized Ricci tensor $\delta R_{\mu\nu}$ represents a massive spin-2 mode while the linearized Ricci scalar $\delta R$ denotes a massive spin-0 mode \cite{13, 14, 15}. The key to considering these variables is to reduce a fourth-order linearized gravity to a second-order linearized gravity without introducing auxiliary variables. It was shown that the small black hole in Ricci quadratic gravity (Einstein-Weyl gravity) is unstable against $s$-mode Ricci tensor perturbation, while the large black hole is stable against $s$-mode Ricci tensor perturbation \cite{16}. Actually, this was performed by comparing the linearized Ricci tensor equation with the linearized metric tensor equation around the five-dimensional black string where the Gregory-Laflamme (GL) instability appeared \cite{17}.

Furthermore, there was a close connection between thermodynamic instability and classical instability for the black strings/branes in the metric tensor formalism. This Gubser-Mitra proposal \cite{18} was known to be the correlated stability conjecture (CSC) \cite{19}, which states that the classical instability of a black string/brane with translational symmetry and infinite extent sets in precisely, when the corresponding system becomes thermodynamically unstable. It has shown that the CSC holds for the $n$-dimensional AdS black hole in Einstein-Weyl gravity by establishing a relation between the thermodynamic instability and the GL instability \cite{20}. In this case, the massiveness of massive spin-2 mode takes over a role of the black sting.

Recently, a newly non-Schwarzschild black hole solution has been found from Ricci quadratic gravity by using a static (negative-eigenvalue) eigenfunction of the Lichnerowicz operator appeared in the stability analysis of Schwarzschild black hole \cite{21}. This static eigenfunction plays two roles of a perturbation away from the Schwarzschild black hole along the new black hole and a threshold unstable mode lying at the edge of a domain of GL instability for small black holes. This implies that the stability analysis of Schwarzschild solution is closely related to finding a newly non-Schwarzschild solution in Ricci quadratic gravity.

In this work, we will investigate classical stability and thermodynamics for $n$-dimensional AdS black holes in a sixth-order gravity. Ignoring the Lee-Wick term, one might obtain a covariant linearized gravity on the $n$-dimensional AdS black hole background. We find that a small black hole with $r_+ < \sqrt{(n-3)/(n-1)}\ell$ is unstable against the Ricci tensor perturbation, whereas a large black hole with $r_+ > \sqrt{(n-3)/(n-1)}\ell$ is stable against the Ricci tensor perturbation. After computing the Wald entropy, we derive other thermody-
namic quantities which satisfy the first-law of thermodynamics in $n$-dimensional Ricci cubic gravity. Hence, we will establish a relation between the GL instability and the thermodynamic instability for AdS black holes in $n$-dimensional Ricci cubic gravity. Finally, we wish to find a newly non-AdS black hole in $n$-dimensional Ricci cubic gravity by making use of a static eigenfunction of the Lichnerowicz operator appeared in stability analysis of AdS black holes.

2 Sixth-order gravity

We start with the sixth-order gravity in $n(\geq 4)$-dimensional spacetimes \[5\]

\[
S_{SG} = \frac{1}{16\pi} \int d^n x \sqrt{-g} \left[ \mathcal{L}_{RC} + \mathcal{L}_{LW} \right],
\]

where $\mathcal{L}_{RC}$ is the Ricci cubic gravity composed of Einstein-Hilbert term and Ricci polynomials $\mathcal{R}_3$

\[
\mathcal{L}_{RC} = \kappa_n (R - 2\Lambda_0) + \mathcal{R}_3, \quad \mathcal{R}_3 = e_1 R^3 + e_2 R R_{\mu\nu} R^{\mu\nu} + e_3 R^3 R_{\mu\nu} R^{\mu\nu}
\]

and $\mathcal{L}_{LW}$ denotes the Lee-Wick term $\mathcal{L}_{LW}$ given by

\[
\mathcal{L}_{LW} = \gamma G_{\mu\nu} \Box R^{\mu\nu}.
\]

Here $\kappa_n = 1/G_n$ is the inverse of $n$-dimensional Newtonian constant, $\Lambda_0$ is the bare cosmological constant, and $\{e_1, e_2, e_3\}$ are three cubic parameters to specify Ricci polynomials $\mathcal{R}_3$. In addition, $\gamma$ is a parameter for the Lee-Wick term and $G_{\mu\nu}$ denotes the Einstein tensor. We regard the action (1) as a general sixth-order gravity constructed out of Ricci tensor and d’Alembert operator ($\Box = \nabla\mu \nabla^\mu$). We note that the Lagrangian of $\mathcal{L} = \kappa_n (R - 2\Lambda_0) + \mathcal{R}_2 + \mathcal{R}_3$ with $\mathcal{R}_2 = \tilde{e}_1 R^2 + \tilde{e}_2 R_{\mu\nu} R^{\mu\nu}$ has been used for studying the tricritical gravity in three dimensions \[22\]. However, we will find that a role of the Lee-Wick term $\mathcal{L}_{LW}$ is quite different from Ricci polynomials $\mathcal{R}_3$ in the study of the AdS black holes.
From the action (1), we derive the Einstein equation as

\[
P_{\mu\alpha\beta\gamma} R_{\nu}^{-\alpha\beta\gamma} - \frac{1}{2} g_{\mu\nu} L_{RC} - 2 \nabla^\alpha \nabla^\beta P_{\mu\alpha\beta\nu} \\
+ \gamma \left[ - \frac{1}{2} \left( R_{\mu\nu} \Box R + R \Box R_{\mu\nu} + \nabla_\mu R \nabla_\nu R - \frac{1}{2} g_{\mu\nu} (\nabla R)^2 \right) \right] \\
+ 2 \Box (R^\rho_{\alpha\nu\sigma} R^\sigma_{\rho}) + \Box^2 G_{\mu\nu} - 4 \nabla_\alpha R_{\nu\rho} \nabla^\sigma R^\rho_{\mu} \\
+ 2 (\nabla R_{\rho}(\nabla R_{\sigma}^\rho)) + R_{\nu\rho} \nabla_\nu R_{\sigma}^\rho + R_{\nu\rho} \nabla_\nu R_{\sigma}^\rho \\
+ \frac{1}{2} g_{\mu\nu} (\nabla R_{\rho})^2 - (\nabla R + 2 R_{\rho} R_{\sigma}^\rho) \nabla (\mu R_{\sigma}) \\
- \nabla_\mu R_{\rho\sigma} \nabla_\nu R_{\sigma}^\rho - 2 R_{\sigma}(\mu \Box R_{\sigma}) = 0.
\]

(4)

Here the \( P \)-tensor is given by

\[
P_{\mu\nu\rho\sigma} \equiv \frac{\partial L_{RC}}{\partial R_{\rho\sigma}} = \frac{\kappa_n}{2} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) + \frac{3e_1}{2} R^2 (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \\
+ \frac{e_2}{2} R_{\alpha\beta} R^{\alpha\beta} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) + \frac{e_2}{2} R(g_{\mu\rho} R_{\nu\sigma} - g_{\mu\sigma} R_{\nu\rho} - g_{\nu\rho} R_{\mu\sigma} + g_{\nu\sigma} R_{\mu\rho}) \\
+ \frac{3 e_3}{4} (g_{\mu\rho} R_{\nu\sigma} R_{\rho}^\gamma - g_{\nu\sigma} R_{\nu\gamma} R_{\rho}^\rho - g_{\nu\rho} R_{\nu\gamma} R_{\delta}^\nu + g_{\nu\sigma} R_{\nu\gamma} R_{\rho}^\rho).
\]

(5)

The first line of Eq. (1) comes from \( L_{RC} \) \[5\] and the other lines are generated from \( L_{LW} \) \[23\].

At this stage, we wish to note that equation of motion from the Lee-Wick term is different from (2.3c)+(2.3d) in Ref. \[24\], where a term of \( \Box (R^\rho_{\mu\sigma\nu} R_{\rho}^\sigma) \) containing Riemann tensor was absent.

For a static black hole with spherical symmetry, the metric has the form

\[
ds^2 = -h(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-2}^2.
\]

(6)

Notice that an AdS black hole solution obtained from the \( n \)-dimensional Einstein gravity is also a solution to Eq. (4), whose form is given by

\[
h(r) = f(r) = \bar{f}(r) = 1 - \left( \frac{r_0}{r} \right)^{n-3} - \frac{a}{n-1} r^2, \quad a = -\frac{n-1}{\ell^2}
\]

(7)

with \( \ell \) the curvature radius of AdS\(_n\) spacetimes. Hereafter we denote all background quantities (AdS solution) with the overbar as

\[
ds_{AdS}^2 = -\bar{f}(r) dt^2 + \frac{dr^2}{\bar{f}(r)} + r^2 d\Omega_{n-2}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu.
\]

(8)

A black hole mass parameter \( r_0 \) is determined as

\[
r_0 = \left[ r_+^{n-3} + \frac{r_-^{n-1}}{\ell^2} \right]^{-\frac{1}{n-3}}
\]

(9)
which is not exactly the horizon radius \( r_+ \). The background Ricci tensor and Ricci scalar are determined by

\[
\bar{R}_{\mu\nu} = \frac{2\Lambda}{n-2} \bar{g}_{\mu\nu} \equiv a \bar{g}_{\mu\nu}, \quad \bar{R} = na. \tag{10}
\]

Plugging (10) into (5), the background \( P \)-tensor takes a maximally symmetric form as

\[
\bar{P}_{\mu\nu\rho\sigma} = \frac{1}{2} \left[ \kappa_n + 3(n^2 e_1 + ne_2 + e_3)a^2 \right] (\bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma} \bar{g}_{\nu\rho}) \]

\[
= \frac{1}{2} \left[ \kappa_n + (\alpha + n\beta)a^2 \right] (\bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma} \bar{g}_{\nu\rho}) \tag{11}
\]

with introducing two new parameters

\[
\alpha = ne_2 + 3e_3, \quad \beta = 3ne_1 + 2e_2. \tag{12}
\]

The two parameters \( \alpha \) and \( \beta \) are enough to denote the Ricci polynomials \( R_3 \), instead of three parameters \( \{e_1, e_2, e_3\} \) on the AdS black hole background. The expression (11) will be used to derive the Wald entropy in section 5. Here, the effective cosmological constant \( \Lambda \) is related to the bare cosmological constant \( \Lambda_0 \) as

\[
\Lambda + \frac{4(n-6)}{3\kappa_n(n-2)}(\alpha + n\beta)\Lambda^3 = \Lambda_0. \tag{13}
\]

In the cases of \( n = 6 \) and \( \alpha = -n\beta \), one finds \( \Lambda = \Lambda_0 \) in the sixth-order gravity as \( n = 4 \) in the Ricci quadratic gravity. It is easy to show that the AdS black hole (8) to the \( n \)-dimensional Einstein equation of \( G_{\mu\nu} = \Lambda_0 \bar{g}_{\mu\nu} \) is also the solution to the sixth-order gravity when one substitutes (11) together with (10) into (4). However, the background Riemann tensor for the AdS black hole is not given by the AdS\( _n \)-curvature tensor

\[
\bar{R}_{\mu\nu\rho\sigma} \neq \bar{R}^{\text{AdS}}_{\mu\nu\rho\sigma} = \frac{a}{n-1}(\bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma} \bar{g}_{\nu\rho}), \tag{14}
\]

which states that the AdS black hole (8) is not a maximally symmetric vacuum.

Finally, we wish to comment that the Lee-Wick term does not generate any background values because of its covariant derivatives acting on \( \bar{R} \) and \( \bar{R}_{\mu\nu} \).

### 3 Linearized theory of sixth-order gravity

In order to perform the stability analysis, let us introduce the metric perturbation around the AdS black hole as

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{15}
\]
The linear stability of the black hole solution to Eq.(1) is usually sufficient for the stability at the any perturbative level. According to Refs. [13,15], it is possible to investigate the stability of AdS black hole by means of the second-order equation for the linearized Ricci tensor, instead of the fourth-order equation for the metric perturbation. Therefore, we may define the linearized Ricci tensor and scalar as

\[ \delta \tilde{R}_{\mu\nu} = \delta R_{\mu\nu} - a h_{\mu\nu}, \]

\[ \delta R = \delta (g^{\mu\nu} R_{\mu\nu}) = \bar{g}^{\mu\nu} \delta \tilde{R}_{\mu\nu}, \] (16)

where the conventionally linearized Ricci tensor and scalar are given by

\[ \delta R_{\mu\nu}(h) = \frac{1}{2} \left[ \bar{\nabla}^\rho \bar{\nabla}_\mu h_{\nu\rho} + \bar{\nabla}^\rho \bar{\nabla}_\nu h_{\mu\rho} - \bar{\nabla}^2 h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h \right], \] (17)

\[ \delta R(h) = \bar{\nabla}^\mu \bar{\nabla}_\mu h - \bar{\nabla}^2 h - a h \] (18)

with \( h = h^\rho_\rho \). Roughly speaking, the linearized Ricci tensor represents a healthy massive spin-2 mode, while the linearized Ricci scalar represents a healthy massive spin-0 mode. Conveniently, we may introduce the linearized Einstein tensor as

\[ \delta G_{\mu\nu} = \delta \tilde{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \delta R = \delta R_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \delta R - a h_{\mu\nu}, \] (19)

which is the linearization of the Einstein tensor

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} \bar{g}_{\mu\nu} + \frac{(n-2)a}{2} g_{\mu\nu}. \] (20)

By linearizing Eq.(1), one finds the linearized equation

\[ \kappa_n \delta G_{\mu\nu} + a^2(3\alpha + n\beta) \delta \tilde{R}_{\mu\nu} - \frac{1}{2} a^2(\alpha + (n-4)\beta) \bar{g}_{\mu\nu} \delta R \]

\[ - a \alpha \bar{\Delta}_L \delta G_{\mu\nu} - a(\alpha + 2\beta)(\bar{\nabla}_\mu \bar{\nabla}_\nu - \bar{g}_{\mu\nu} \bar{\Box}) \delta R \]

\[ + \gamma \left[ - \frac{na}{2} \bar{\Box} \left( \delta R_{\mu\nu} + \frac{1}{n} \bar{g}_{\mu\nu} \delta R \right) + 2 \bar{\Box} (\tilde{R}^\rho \mu_{\sigma\nu} \delta \tilde{R}^\sigma_\rho) \right] \]

\[ + \bar{\Box}^2 \left( \delta G_{\mu\nu} + a(1 - \frac{n}{2}) h_{\mu\nu} \right) - 4a^2 \left( \delta R_{\mu\nu} + \frac{1}{2} \bar{\nabla}_\mu \bar{\nabla}_\nu h \right) \] = 0, \] (21)

where the Lichnerowicz operator is defined by acting on linearized Ricci scalar and tensor, respectively,

\[ \bar{\Delta}_L \delta R = - \bar{\Box} \delta R, \] (22)

\[ \bar{\Delta}_L \delta \tilde{R}_{\mu\nu} = - \bar{\Box} \delta \tilde{R}_{\mu\nu} - 2 \tilde{R}^\rho_{\mu\sigma\nu} \delta \tilde{R}^\sigma_\rho + \tilde{R}^\rho_\mu \delta \tilde{R}_{\rho\nu} + \tilde{R}^\rho_\nu \delta \tilde{R}_{\rho\mu}. \] (23)
The linearized equation (21) is not only a fourth-order equation for \( \delta G_{\mu\nu} \), but also it becomes a sixth-order equation when expressing in terms of \( h_{\mu\nu} \). This contrasts to higher-order gravity without \( \Box \) which explains that any higher-order gravity could be mapped into the Ricci quadratic gravity in the linearized level. The linearized sixth-order theory described by (21) seems not to be handled appropriately because the Lee-Wick term generates uncontrollable sixth-order terms as well as explicit appearances of the metric perturbation \( h_{\mu\nu} \). In the case of \( a = 0 \), two six-order terms of \( \Box^2 \delta G_{\mu\nu} \) and \( 2 \Box (R^\rho_{\ \mu\sigma\nu} \delta \tilde{R}_\rho^\sigma) \) came from the Lee-Wick term [8].

Without the Lee-Wick term (with \( \gamma = 0 \)), however, we have a promising second-order equation for \( \delta G_{\mu\nu} \) because of the inclusion of the cosmological constant \( \Lambda_0 \) in \( L_{RC} \) (2). Putting \( \Lambda = 0 (\Lambda_0 = 0) \) yields Ricci-flat solution (Schwarzschild solution) on which Ricci polynomials give no contributions to the linearized equation, leading to \( \delta R_{\mu\nu} = 0 \). This is one reason why we included the cosmological constant in the beginning action (1), compared to the fourth-order gravity. This implies that for \( \gamma = 0 \), one has a covariant linearized gravity on the AdS black hole background. On the other hand, the case of \( \alpha = \beta = 0 \) leads to the Lee-Wick gravity, which is ghost-free and renormalizable around the Minkowski background. Even though the Lee-Wick gravity sounds good for the Minkowski background, it puts a serious difficulty on the stability analysis of AdS black holes, leading to an uncontrollable linearized equation. Hence, it seems legitimate to choose the \( \gamma = 0 \) case which corresponds to the Ricci cubic gravity in \( n \)-dimensional spacetimes.

Taking the trace of Eq.(21) with \( \gamma = 0 \) leads to the linearized Ricci scalar equation

\[
a \left[ n\alpha + 4(n - 1)\beta \right] \Box \delta R - \left[ (n - 2)\kappa_n + (n - 6)(\alpha + n\beta) a^2 \right] \delta R = 0.
\]  

(24)

One notes that Eq.(21) with \( \gamma = 0 \) is a coupled second-order equation for \( \delta \tilde{R}_{\mu\nu} \) and \( \delta R \), which seems difficult to be solved. One way to avoid this difficulty is to split Eq.(21) into the traceless and trace parts by choosing \( \alpha \) and \( \beta \) appropriately. For this purpose, we introduce a traceless Ricci tensor as

\[
\delta \hat{R}_{\mu\nu} = \delta \tilde{R}_{\mu\nu} - \frac{1}{n} g_{\mu\nu} \delta R, \quad \delta \hat{R} = 0.
\]  

(25)

Then, Eq.(21) with \( \gamma = 0 \) and Eq.(24) lead to

\[
a \alpha (\tilde{\Delta}_L - 2a + \mu_2^2) \delta \tilde{R}_{\mu\nu} = -a(\alpha + 2\beta) \left( \nabla_\mu \nabla_\nu - \frac{1}{n} g_{\mu\nu} \Box \right) \delta R, \quad \text{a}(26)
\]

\[
\frac{a}{2} \left[ n\alpha + 4(n - 1)\beta \right] (\Box - \mu_0^2) \delta R = 0, \quad \text{a}(27)
\]

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where the mass squared $\mu_2^2$ for massive spin-2 mode and the mass squared $\mu_0^2$ for massive spin-0 mode are given by
\[ \mu_2^2 = 2a - \frac{(3\alpha + n\beta)a^2 + \kappa_n}{a\alpha}, \quad \mu_0^2 = \frac{\kappa_n(n - 2) + (n - 6)(\alpha + n\beta)a^2}{a(n\alpha + 4(n - 1)\beta)}. \] (28)
We note that decoupling of all massive modes requires either $a = 0$ or $\alpha = \beta = 0$. The former case corresponds to the Ricci-flat spacetimes on which Ricci polynomials give no contribution to the linearized Ricci tensor equation. On the other hand, the latter case yields a quasi-topological gravity,
\[ \mathcal{L}_{\text{RC}}^{\alpha=\beta=0} = \mathcal{L}_{\text{QT}} = \kappa_n(R - 2\Lambda_0) + R^3 - \frac{3n}{2}RR_{\mu\nu}R^{\mu\nu} + \frac{n^2}{2}R_{\mu}^\rho R^\nu_{\rho} R_{\mu} \] (29)
whose linearized equation is exactly given by
\[ \delta G_{\mu\nu} = 0. \] (30)
Hence, it leads to a ghost-free gravity when perturbing around the Minkowski spacetimes. However, the ghost-free gravity is not all that we wish to consider in this work.

When choosing a condition of $\alpha = -2\beta$, a nontrivial decoupling between traceless and trace parts may occur. This case corresponds to the linearized fourth-order (Ricci quadratic) gravity because one parameter $e_3$ is redundant and it is represented by
\[ e_3 = -2ne_1 - \frac{(n + 4)e_2}{3}. \] (31)
In this case, Eqs. (26) and (27) lead to the massive spin-2 and massive spin-0 equations, separately,
\[ (\bar{\Delta}_L - 2a + M_n^2)\delta \bar{R}_{\mu\nu} = 0, \] (32)
\[ (\Box - M_0^2)\delta \bar{R} = 0. \] (33)
Here the mass squared $M_n^2$ and $M_0^2$ are given by
\[ M_n^2 = \frac{\kappa_n\ell^2}{(n - 1)\alpha} - \frac{(n - 1)(n - 2)}{2\ell^2}, \quad M_0^2 = \frac{\kappa_n\ell^2}{(n - 1)\alpha} - \frac{3(n - 6)}{2\ell^2}. \] (34)
However, the number of degrees of freedom (DOF) for $\delta \bar{R}_{\mu\nu}$ is not given by $(n + 1)(n - 2)/2$ of a massive spin-2 mode in $n$ dimensions because the contracted Bianchi identity of $\bar{\nabla}^\mu \delta G_{\mu\nu} = 0$ does not imply the transverse condition
\[ \bar{\nabla}^\mu \delta \bar{R}_{\mu\nu} = \left[ \frac{n - 2}{2n} \right] \bar{g}_{\mu\nu} \bar{\nabla}_\nu \delta R \rightarrow \bar{\nabla}^\mu \delta \bar{R}_{\mu\nu} = 0 \] (35)
because of $\delta R \neq 0$ in the $\alpha = -2\beta$ Ricci cubic gravity.

A desirable choice may be done by requiring the non-propagation of the Ricci scalar. Imposing the condition of $\alpha = -\left[4(n-1)/n\right]\beta$ on the linearized Ricci scalar equation \(\eqref{21}\), the non-propagating Ricci scalar is implemented as

$$\delta R = 0.$$ \(\eqref{36}\)

In other words, choosing

$$e_3 = -4(n-1)\left[e_1 + \frac{2e_2}{3n}\right],$$ \(\eqref{37}\)

one could achieve the non-propagation of the linearized Ricci scalar. Also, one finds from \(\eqref{28}\) that the mass squared $\mu_0^2$ of massive spin-0 mode blows up at $\alpha = -\left[4(n-1)/n\right]\beta$, which means that the massive spin-0 mode decouples from the theory. The absence of massive spin-0 mode is also required by an $a$-theorem \(\eqref{26}\). In addition, the ghost-free condition is imposed by the absence of massive spin-2 mode. The absence of both massive modes are required by the causality condition such that the resulting theory becomes a linearized quasi-topological gravity \(\eqref{30}\). On the other hand, one needs $\delta R \neq 0$ and $\delta R_{\mu \nu} \neq 0$ to achieve renormalizability around the Minkowski spacetimes.

Going back to the stability analysis of AdS black holes, it is necessary to have $\delta R = 0$. The ghost-problem is not present here because one adopts $\delta \hat{R}_{\mu \nu}$ to represent a massive spin-2 mode, instead of $h_{\mu \nu}$. Considering $\delta G_{\mu \nu}$ in \(\eqref{19}\) together with $\delta R = 0$, Eq.\(\eqref{21}\) leads to a massive spin-2 equation for $\delta \hat{R}_{\mu \nu}$ (so-called Lichnerowicz-type linearized equation) as

$$(\bar{\Delta}_L - 2a + M_n^2)\delta \hat{R}_{\mu \nu} = 0$$ \(\eqref{38}\)

with the mass squared

$$M_n^2 = \frac{\kappa_n \ell^4}{(n-1)\alpha} - \frac{(n-2)^2}{4\ell^2}.$$ \(\eqref{39}\)

Figure 1 depicts the mass $M_n$ for a massive spin-2 mode as a function of $\alpha$ and $n$. Its zero appears at $\alpha = \alpha_n = 4\kappa_n \ell^4/(n-1)(n-2)^2$. Zero values decrease as $n$ increases. We observe that there is no distinct features depending on dimensions $n$.

In the AdS black hole background, a simple criterion of the stability for massive spin-2 mode requires the positive mass squared of $M_n^2 > 0$, which implies that

$$0 < \alpha < \alpha_n.$$ \(\eqref{40}\)
Figure 1: Plots of mass $M_n$ for a massive spin-2 mode as a function of $\alpha$ with $l = 10$ and $\kappa = 1$ in the $\alpha = -[4(n-1)/n] \beta$ Ricci cubic gravity. $M_n\{n = 10, \cdots, 4\}$ graphs are located from left to right. For positive $M_n^2$, there is no tachyonic instability but the Gregory-Laflamme instability will appear for $0 < M_n < M_n^t$ with $M_n^t$ threshold mass.

The condition (40) ensures that there is no tachyonic instability for $\delta \hat{R}_{\mu\nu}$ propagating on the AdS black hole background [27]. However, $M_n^2 > 0$ is not a necessary and sufficient condition for getting a stable black hole. One needs a further analysis to complete the stability analysis of the black hole. In the next section, the Gregory-Laflamme (GL) instability will appear for $0 < M_n < M_n^t$ with $M_n^t$ threshold mass for GL instability. If $M_n^2 < 0 (\alpha > \alpha_n)$, one does not need a further analysis because it implies the tachyonic instability. In addition, the case of $M_n^2 = 0 (\alpha = \alpha_n)$ implies a critical gravity when one employs the transverse-traceless gauge for the metric perturbation $h_{\mu\nu}$, leading to another instability of log-gravity.

Considering Eq.(36), the contracted Bianchi identity yields a desired transverse condition

$$\nabla^\mu \delta \hat{R}_{\mu\nu} = 0.$$  \hspace{1cm} (41)

Hence, the DOF of $\delta \hat{R}_{\mu\nu}$ becomes $(n+1)(n-2)/2$ from counting of $(n+1)(n+2)/2 - n - 1$ in the $\alpha = -[4(n-1)/n] \beta$ Ricci cubic gravity.
4 Stability of AdS black hole

When one performs the stability analysis of the AdS black hole in $\alpha = \beta = 0$ Ricci cubic gravity (quasi-topological Ricci cubic gravity) [5], one uses the linearized equation (30). It turns out to be stable by making use of the Regge-Wheeler prescription [11, 12, 28]. In this case, the $s(l = 0)$-mode analysis is unnecessary to show the stability of AdS black holes because a massless spin-2 mode starts from $l = 2$.

Now, let us consider the $\alpha = -[4(n-1)/n] \beta$ Ricci cubic gravity. Its linearized equation (38) is second-order with respect to $\delta \hat{R}_{\mu \nu}$. Importantly, the traceless Ricci tensor $\delta \hat{R}_{\mu \nu}$ representing a massive spin-2 mode satisfies the transverse condition (41). This means that the transverse-traceless (TT) gauge condition of $\bar{\nabla}_\mu \delta \hat{R}_{\mu \nu} = 0$ and $\delta \hat{R} = 0$ is allowed for $\delta \hat{R}_{\mu \nu}$. Hence, its DOF is determined to be $(n+1)(n-2)/2$ as for $\delta G_{\mu \nu}$ in the $n$-dimensional Einstein-Weyl gravity. At this stage, the even-parity metric perturbation may be chosen for a single $s$-mode analysis of $\delta \hat{R}_{\mu \nu}$ in the $\alpha = -[4(n-1)/n] \beta$ Ricci cubic gravity and whose form is given by [29]

\[
\delta \hat{R}_{\mu \nu} = e^{\Omega t} \begin{pmatrix}
H_{ij}(r) & H_{it}(r) & H_{ir}(r) & 0 & 0 & \ldots \\
H_{ij}(r) & H_{it}(r) & H_{tr}(r) & 0 & 0 & \ldots \\
H_{rj}(r) & H_{tr}(r) & H_{rr}(r) & 0 & 0 & \ldots \\
0 & 0 & 0 & K(r) & 0 & \ldots \\
0 & 0 & 0 & 0 & \sin^2 \theta K(r) & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]  (42)

with $i = 5, \ldots, n$. It is suggested that $\delta \hat{R}_{\mu \nu}$ grows exponentially in time as $e^{\Omega t}$ with $\Omega > 0$, spatially vanishes at the AdS infinity, and it is regular at the future horizon.

Starting with 3 DOF of $H_{tt}, H_{tr},$ and $H_{rr}$, they are related to each other when using the TT gauge condition. In order to investigate the classical instability for $n$-dimensional AdS black hole, we define the new perturbed variables as

\[
H \equiv H_{tr}, \quad H_{\pm} \equiv \frac{H_{tt}}{f(r)} \pm f(r)H_{rr}.
\]  (43)
Then, one finds two coupled first-order equations

\[ H' = \left[ \frac{3 - n - (n - 1)r^2/\ell^2}{rf} - \frac{1}{r} \right] H + \frac{\Omega}{2f}(H_+ + H_-), \tag{44} \]
\[ H'_- = \frac{M_n^2}{\Omega} H + \frac{n - 2}{2r} H_+ + \left[ \frac{n - 3 + (n - 1)r^2/\ell^2}{2rf} - \frac{2n - 3}{2r} \right] H_. \tag{45} \]

In addition, a constraint equation is given by

\[ r^2\Omega \left[ 4r\Omega^2 - r(f')^2 + (n - 2)f f' + 2rfM_n^2 + 2r f f'' \right] H_- \]
\[ -\Omega r^2 f \left[ 2M_n^2 r + (n - 2)f' \right] H_+ - 2r^2 f \left[ 2(n - 2)\Omega^2 - 2M_n^2 f + rM_n^2 f' \right] H = 0. \tag{46} \]

We note that two first-order equations and constraint equation can be obtained by using the second-order linearized equation (38) and the TT gauge condition. At the AdS infinity of \( r \to \infty \), asymptotic solutions to Eqs. (44) and (45) are

\[ H^{(\infty)} = C_1(\infty) r^{-(n+1)/2 + \sqrt{M_n^2\ell^2 + (n-1)^2}/4} + C_2(\infty) r^{-(n+1)/2 - \sqrt{M_n^2\ell^2 + (n-1)^2}/4}, \]
\[ H_-^{(\infty)} = \tilde{C}_1(\infty) r^{-(n-1)/2 + \sqrt{M_n^2\ell^2 + (n-1)^2}/4} + \tilde{C}_2(\infty) r^{-(n-1)/2 - \sqrt{M_n^2\ell^2 + (n-1)^2}/4}, \tag{47} \]

where \( \tilde{C}_{1/2}^{(\infty)} \) are

\[ \tilde{C}_{1/2}^{(\infty)} = \frac{M_n^2}{(1 - n)/2 \pm \sqrt{M_n^2\ell^2 + (n-1)^2}/4} C_{1/2}^{(\infty)}. \tag{48} \]

At the horizon \( r = r_+ \), their solutions are given by

\[ H^{(r_+)} = C_1^{(r_+)} (r^{n-3} - r_+^{n-3})^{-1 + \tau(\rho, r_+)} + C_2^{(r_+)} (r^{n-3} - r_+^{n-3})^{-1 - \tau(\rho, r_+)}, \]
\[ H_-^{(r_+)} = \tilde{C}_1^{(r_+)} (r^{n-3} - r_+^{n-3})^{-\tau(\rho, r_+)} + \tilde{C}_2^{(r_+)} (r^{n-3} - r_+^{n-3})^{-\tau(\rho, r_+)}. \tag{49} \]

Here, \( \tilde{C}_{1/2}^{(r_+)} \) are given by

\[ \tilde{C}_{1/2}^{(r_+)} = \pm \frac{(n - 3)r_+^{n-3}\Omega \left( 2\Omega + f'(r_+) \right)}{2f'(r_+)(M_n^2r_+ \pm (n - 2)\Omega)} C_{1/2}^{(r_+)} . \tag{50} \]

It is noted that two boundary conditions for regular solutions correspond to \( C_1^{(\infty)} = 0 \) at infinity and \( C_2^{(r_+)} = 0 \) at the horizon. By eliminating \( H_+ \) in Eqs. (44) and (45) together
Figure 2: Ω graphs as function of $M_n$ for a small black hole with $r_+ = 1$, $\ell = 10$ and $n = 4, 5, \ldots, 10$ from left to right curve. Here one may read off the threshold mass $M_n^t$ from the points that curves of Ω intersect the positive $M_n$-axis.

with the constraint (46), one can find two coupled equations for $H$ and $H_-$. For given dimension $n \in [4, \ldots, 10]$, fixed $M_n$, and various values of Ω, we solve these coupled equations numerically. They yield possible values of Ω as functions of $M_n$. Fig. 2 shows that the curves of Ω intersect the positive $M_n$-axis at one place: $M_n = M_n^t$ where $M_n^t$ is a threshold mass for GL instability. By observing Fig. 2, we read off the threshold mass $M_n^t$ depending on the dimension $n$ as

\[
\begin{array}{c|cccccccc}
  n & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  \hline
  M_n^t/M_n & 0.86 & 1.26 & 1.57 & 1.83 & 2.07 & 2.29 & 2.49 \\
\end{array}
\]  

(51)

For all large black holes, Ω approaches the maximum value being less than $10^{-4}$. This implies that there is no unstable modes for large AdS black holes in $n$-dimensional Ricci cubic gravity. The other intersecting point at the origin $(M_n = 0, \alpha = \alpha_n)$ means that AdS black hole is unstable in the critical theory of Ricci cubic gravity. For given the curve, $M_n < M_n^t(M_n > M_n^t)$ implies that the AdS black hole is unstable (stable) against the
Ricci tensor perturbation. Observing Fig. 1, one easily finds that such unstable (stable) conditions are always met.

Now we are in a position to find the GL instability condition explicitly. For $r_+ = 1$ and $\ell = 10$, from (9), the mass parameter takes the form

$$r_0 = \left( \frac{101}{100} \right)^{\frac{1}{n-3}}$$

which implies that $r_0 = \{1.01, 1.005, 1.003, 1.002, 1.002, 1.002, 1.001\}$. Here, the corresponding $\tilde{k}_n = r_0 M_n^t$ is determined to be

$$\begin{bmatrix}
  n & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  \tilde{k}_n & 0.87 & 1.27 & 1.57 & 1.83 & 2.07 & 2.29 & 2.49
\end{bmatrix}.$$  

(53)

Hence, we propose the bound for unstable modes as

$$0 < M_n < \frac{\tilde{k}_n}{r_0} = M_n^t,$$

(54)

which indicates that the GL instability of small AdS black holes in the $\alpha = -[4(n-1)/n]\beta$ Ricci cubic gravity is due to the massiveness of $M_n \neq 0$, but not a feature of sixth-order gravity giving a ghost. The ghost may appear only when expressing the linearized equation in terms of the metric perturbation $h_{\mu \nu}$. It is worth noting that the ghost (unhealthy massive spin-2 mode) does not appear here because we adopt the linearized Ricci tensor $\delta \hat{R}_{\mu \nu}$ to represent a massive spin-2 mode.

Finally, there is no connection between thermodynamic instability and GL instability for AdS black holes in $n$-dimensional Ricci cubic gravity. However, the GL instability condition (massiveness) picks up a small AdS black hole with $r_+ < r_+^{(n)} = \sqrt{(n-3)/(n-1)}\ell$ which may be thermodynamically unstable in Ricci cubic gravity. Hence, we will explore a deep connection between the GL instability and thermodynamic instability of AdS black holes in Ricci cubic gravity. It is well known that the CSC proposed by Gubser-Mitra [18] does not hold for the SAdS black hole found in the second-order Einstein gravity, but it holds for the SAdS black hole found in the fourth-order Einstein-Weyl gravity [20]. In order to confirm the GL instability for small AdS black holes, we wish to explore thermodynamic property of the AdS black hole in Ricci cubic gravity.
5 Wald entropy and thermodynamic stability

First of all, all thermodynamic quantities (mass, heat capacity, Bekenstein-Hawking entropy, Hawking temperature, Helmholtz free energy) of AdS black hole in \( n \)-dimensional Einstein gravity with a cosmological constant were known to be \[^{[20]}\]

\[
m_n(r_+) = \frac{\Omega_{n-2}(n-2)}{16\pi G_n} r_+^{n-3} \left[ 1 + \frac{r_+^2}{\ell^2} \right],
\]

\[
C_n(r_+) = \frac{dm_n}{dT_H'} = \frac{\Omega_{n-2}(n-2)r_+^{n-2}}{4G_n} \left[ (n-1)r_+^2 + (n-3)\ell^2 \right],
\]

\[
S_{BH}(r_+) = \frac{\Omega_{n-2}}{2G_n} r_+^{n-3} \left[ 1 - \frac{r_+^2}{\ell^2} \right],
\]

\[
F_n(r_+) = m_n - T_H S_{BH} = \frac{\Omega_{n-2}}{4\pi G_n} r_+^{n-3} \left[ 1 - \frac{r_+^2}{\ell^2} \right],
\]

with the area of \( S^{n-2} \)

\[
\Omega_{n-2} = \frac{2\pi^{n-1}}{\Gamma\left(\frac{n-1}{2}\right)}.
\]

It is easy to check that the first-law of thermodynamics is satisfied as

\[
dm_n = T_H' dS_{BH},
\]

where ‘\( d \)’ denotes the differentiation with respect to the horizon radius \( r_+ \) only. Thermodynamic stability is determined by the heat capacity \( C_n \) which blows up at \( r_+ = r_+^{(n)} = \sqrt{(n-3)/(n-1)}\ell \). Fig. 3 indicates a typical shape of the heat capacity in \( n = 4 \) dimensions. It turns out that a small AdS black hole with \( r_+ < r_+^{(n)} \) is thermodynamically unstable because of \( C_n < 0 \), while a large AdS black hole with \( r_+ > r_+^{(n)} \) is thermodynamically stable because of \( C_n > 0 \). Classifying a black hole into either small or large black hole is determined solely by the \( n \)-dimensional Einstein gravity with a cosmological constant.

Now we wish to compute the Wald entropy of the AdS black hole in sixth-order gravity. The Wald entropy is defined by the following integral performed on \((n-2)\)-dimensional spacelike bifurcation surface \( \Sigma_{n-2} \) \([24, 30, 31, 32]\):

\[
S_W = -\frac{1}{8} \int_{x_+} d^{n-2}x_+ \sqrt{g(r_+)} g_{\mu\nu} g^{\gamma\delta} \left[ \frac{\partial L_{RC}}{\partial R_{\mu\nu\rho\sigma}} - \nabla_\gamma \frac{\partial L_{LV}}{\partial \nabla_\gamma R_{\mu\nu\rho\sigma}} \right]^{(0)}
\]

\[
= -\frac{1}{8} \int_{x_+} d^{n-2}x_+ \sqrt{g(r_+)} g_{\mu\nu} g^{\gamma\delta} \tilde{E}_{\mu\nu\rho\sigma},
\]

with
Figure 3: Plot of heat capacity $C_{n=4}$ with $l = 10$ in Einstein gravity. The heat capacity blows up at $r_+ = r_+^{(4)} = \ell/\sqrt{3} = 5.7$. The thermodynamic stability is based on this heat capacity. The small black hole with $r_+ < r_+^{(4)}$ has the negative heat capacity, whereas the large black hole with $r_+ > r_+^{(4)}$ has the positive heat capacity. This picture persists to $C^{\alpha=-[4(n-1)/n] \beta}$ with $\alpha < \alpha_n$ in the Ricci cubic gravity.

where $g^{\perp}_{\mu \rho}$ represents the metric projection onto the subspace orthogonal to the horizon. We note that the superscript (0) denotes that the functional derivative with respect to $R_{\mu \nu \rho \sigma}$ is evaluated on-shell. The background (on-shell) $P$-tensor $\bar{P}^{\mu \nu \rho \sigma}$ is given by (11). It is emphasized that there is no contribution to the entropy from the Lee-Wick term because of its covariant derivatives acting on $\bar{R} = n a$ and $\bar{R}_{\mu \nu} = a \bar{g}_{\mu \nu}$. Now, the Wald entropy takes the form

$$S_W = \frac{A_n}{4} \left[ 1 + (\alpha + n \beta) a^2 \right]$$

(59)

with the area of horizon $A_n = \Omega_n - 2 r_+^{n-2}$ and $\kappa_n = 1/G_n = 1$. Here it is worth noting that for $\alpha = -n \beta$, there is no contribution to entropy from the Ricci polynomials $\mathcal{R}_3$.

In the case of $\alpha = -[4(n-1)/n] \beta$ Ricci cubic gravity, the Wald entropy takes the form

$$S_{W^{\alpha=-[4(n-1)/n] \beta}} = \frac{A_n}{4} \left[ 1 - \frac{(n-1)(n-2)^2 \alpha}{4 \ell^4} \right].$$

(60)

Up to now, the Wald entropy $S_{W^{\alpha=-[4(n-1)/n] \beta}}$ and the Hawking temperature $T_H^n$ in (55) are known only. All thermodynamic quantities of $k = 0$ AdS plane black hole are computed in
Ref. [24]. In this case, it is known that there is a correction \( \sigma \) to thermodynamic quantities from Ricci polynomials, but there is no corrections from the Lee-Wick term. Here we suggest that the first-law should be satisfied as

\[
dM_\alpha = -\frac{4(n-1)/n}{\beta} T^n dS_W^\alpha = -\frac{4(n-1)/n}{\beta}.
\] (61)

in the \( \alpha = -\frac{4(n-1)/n}{\beta} \) Ricci cubic gravity. Making use of the first-law (61) together with the entropy (60) and the Hawking temperature in (55), one might derive the mass as

\[
M_\alpha = \frac{1}{\ell^2} \int_0^{r_+} T^n(r') dS_W^\alpha = -\frac{4(n-1)/n}{\beta} m_n(r_+),
\] (62)

where \( M_n^2 \) is the mass squared of massive spin-2 mode in (39). The other thermodynamic quantities of heat capacity and free energy are computed as

\[
C_\alpha = -\frac{4(n-1)/n}{\beta} = \frac{\alpha}{\beta} T^n dS_W^\alpha = -\frac{4(n-1)/n}{\beta} C_n,
\] (63)

\[
F_\alpha = -\frac{4(n-1)/n}{\beta} = M_\alpha - T^n dS_W^\alpha = -\frac{4(n-1)/n}{\beta} F_n.
\] (64)

Now we are in a position to mention the thermodynamic stability of the AdS black hole in \( \alpha = -\frac{4(n-1)/n}{\beta} \) Ricci cubic gravity. First, we consider the case of \( M_n^2 > 0 \) which is dominantly described by the Einstein-Hilbert term. Since the heat capacity \( C_\alpha = -\frac{4(n-1)/n}{\beta} \) blows up at \( r = r_+ = \sqrt{(n-3)/(n-1)} \ell \) [see Fig. 3], we might divide still the black holes into the small black hole with \( r_+ < r_+ \) and the large black hole with \( r_+ > r_+ \). Then, it is suggested that the small black hole is thermodynamically unstable because \( C_\alpha < 0 \), while the large black hole is thermodynamically stable because \( C_\alpha > 0 \). For the other case of \( M_n^2 < 0 \) which is dominated by Ricci polynomials, the situation reverses. On the thermodynamic side, the small black hole is thermodynamically stable because \( C_\alpha > 0 \), while the large black hole is thermodynamically unstable because \( C_\alpha < 0 \). However, this case is unacceptable because it corresponds to the unconventional thermodynamic stability.

One finds from (54) that for \( M_n^2 > 0 \), a small (large) black hole with \( r_+ < r_+ (r_+ > r_+) \) is unstable (stable) against the s-mode massive spin-2 perturbation \( \delta \hat{R}_{\mu\nu} \). The GL instability condition picks up the small AdS black hole which is thermodynamically unstable in \( \alpha = -\frac{4(n-1)/n}{\beta} \).
$-\left[4(n-1)/n\right]\beta$ Ricci cubic gravity. It shows clearly that the CSC \cite{18} holds for the AdS black hole found in the $\alpha = -\left[4(n-1)/n\right]\beta$ Ricci cubic gravity. However, it is legitimate to note that the CSC does not hold for $M_2^2 < 0$ because it corresponds to the tachyonic instability and its thermodynamic stability is unconventional.

6 Non-AdS black hole solution

It is very interesting to explore a non-AdS black hole solution in Ricci cubic gravity. For this purpose, we remind the reader that a static eigenfunction of Lichnerowicz operator has two roles in Ricci quadratic gravity \cite{21}: it plays a role of perturbation away from Schwarzschild black hole along a newly non-Schwarzschild black hole and it plays a role of threshold unstable mode lying at the edge of a domain of GR instability for a small Schwarzschild black hole. Inspired by this work, we will explore a newly non-AdS black hole solution in $\alpha = -\left[4(n-1)/n\right]\beta$ Ricci cubic gravity.

Considering Eq.\,(6), we choose a particular metric for obtaining a newly static solution

$$h(r) = \bar{f}(r) \left[1 + \epsilon \tilde{h}(r)\right], \quad f(r) = \bar{f}(r) \left[1 + \epsilon \tilde{f}(r)\right],$$

(65)

where $\epsilon$ is a parameter to control perturbed metric functions of $\tilde{h}(r)$ and $\tilde{f}(r)$. From Eqs.\,(36) and (38), it is known that the metric perturbation $h_{\mu\nu}$ is determined by two linearized equations around AdS black hole

$$\delta E^R \equiv \delta R(h_{\mu\nu}) = 0,$$

(66)

$$\delta E_{\mu\nu} \equiv (\bar{\Delta}_L - 2a + M_2^2)\delta \hat{R}_{\mu\nu}(h_{\mu\nu}) = 0.$$  

(67)

Here we wish to point out that Eqs.\,(66) and (67) are considered as static linearized equations in compared to those [Eqs.\,(36) and (38)] for stability analysis in section 4. Substituting (65) into Eq.\,(66), one obtains a second-order coupled equation for $\tilde{h}(r)$ with $\tilde{f}(r)$ as

$$\delta E^R = \left[-\frac{n(n-1)}{\ell^2}r^2 - (n-2)(n-3)\right] \tilde{f} - \frac{1}{2} \left[(n-3)r + \frac{n(n-1)}{\ell^2}r^3 + (n-1)rf\right] \tilde{f}'$$

$$-\frac{1}{2} \left[3(n-3)r + \frac{3(n-1)}{\ell^2}r^3 - (n-5)rf\right] \tilde{h}' - r^2 \tilde{f} \tilde{h}'' = 0.$$  

(68)

Now, plugging (65) into Eq.\,(67) leads to the fourth-order equation which is formidable to be solved directly. Explicitly, $tt$-component ($\delta E_{tt} = 0$) and $rr$-component ($\delta E_{rr} = 0$) of
Eq. (67) become fourth-order equations for $\tilde{h}(r)$ [third-order equation for $\tilde{f}(r)$], while $\theta\theta$-component ($\delta E_{\theta\theta} = 0$) is a third-order equation for $\tilde{f}(r)$ and $\tilde{h}(r)$. In order to eliminate a fourth-order term of $\tilde{h}^{(4)}(r)$, we combine two component equations like as $\delta E_{\alpha} + \delta E_{rr} = 0$, leading to a third-order equation for $\tilde{f}(r)$ and $\tilde{h}(r)$. Furthermore, making use of $\delta E_{\theta\theta} = 0$, $\delta E^r = 0$ and $(\delta E^R)' = 0$, the third-order equation is reduced to a second-order equation

\[-2r^2 ((n - 1)r^2/\ell^2 + (n - 3)) \tilde{f} - 2r^3 \tilde{f} \tilde{h}' + \frac{1}{M_n^2} \left[ ((n - 3)^2(n - 2 + 2(n - 1)r^2/\ell^2) \\
+ (n - 4)(n - 1)^2 r^4/\ell^4 - (n - 3)(n(n - 1)r^2/\ell^2 + (n^2 - 5n + 2)\tilde{f}) \right] \tilde{f}
+ \frac{r}{2} (5((n - 1)r^2/\ell^2 + n - 3)^2 - (6(n - 1)^2r^2/\ell^2 + 2(n - 3)(3n - 5))\tilde{f} + (n - 1)^2 \tilde{f}^2) \tilde{f}'
+ \frac{1}{2} (2(n - 3)(n - 1)r^3 \tilde{f}/\ell^2 - r((n - 1)r^2/\ell^2 + n - 3)^2 + (n - 3)(2(n - 1) - (n + 1)\tilde{f}) \tilde{f}' + ((n - 1)r^2/\ell^2 - (n - 1)\tilde{f} + n - 3) \tilde{f}'' \right] = 0, \tag{69}\]

which is solvable for finding a new static solution with appropriate boundary conditions. We note that the AdS black hole solution $\bar{f}(r)$ approaches asymptotically AdS like $r^2$ as $r \to \infty$. To accommodate this asymptotic behavior, we introduce a new coordinate of $z = \frac{r_+}{r}$ so that $f(r)$ and $h(r)$ become $f = f(z)$ and $h = h(z)$ whose region is covered by $0 < z \leq 1$. The location of $z = 1$ corresponds to the event horizon $r = r_+$, whereas $z \to 0$ implies $r \to \infty$. Hence, it seems to impose the AdS boundary conditions at $z = 0$. However, $f(z)$ and $h(z)$ being proportional to $z^{-2}$ are still divergent at $z = 0$. Instead, we pay attention to functions of $z^2 f(z)$ and $z^2 h(z)$ which are finite between $z = 0$ and $z = 1$ as

\[z^2 h(z) = z^2 f(z) \left(1 + \epsilon \tilde{h}(z)\right), \quad z^2 f(z) = z^2 f(z) \left(1 + \epsilon \tilde{f}(z)\right). \tag{70}\]

Now, we rewrite two relevant equations (68) and (69) in terms of $z$ as

\[\left[\frac{n(n - 1)}{\ell^2} \frac{r^2}{z^2} + (n - 2)(n - 3) \tilde{f} - \frac{z}{2} \left[ (n - 3) + \frac{n(n - 1)}{\ell^2} \frac{r^2}{z^2} + (n - 1)\tilde{f} \right] \tilde{f}'
- \frac{z}{2} \left[ 3(n - 3) + \frac{3(n - 1)}{\ell^2} \frac{r^2}{z^2} - (n - 9)\tilde{f} \right] \tilde{h}' + z^2 \tilde{f} \tilde{h}'' = 0 \tag{71}\]
and

\[-2\frac{r_+^2}{z^2} \left( (n-1) \frac{r_+^2}{z^2\ell^2} + (n-3) \right) \tilde{f} + \frac{2r_+^2}{z^2} \tilde{h}' + \frac{1}{M_n^2} \left[ \left( (n-3)^2 (n-2 + 2(n-1) \frac{r_+^2}{z^2\ell^2}) + (n-4)(n-1)^2 \frac{r_+^4}{z^2\ell^4} - (n-3)(n(n-1) \frac{r_+^2}{z^2\ell^2} + (n^2 - 5n + 2)) \right) \tilde{f} \right] \]

\[+ \frac{z}{2} \left( 5((n-1) \frac{r_+^2}{z^2\ell^2} + n-3)^2 - \left( \frac{6(n-1)^2 r_+^2}{z^2\ell^2} + 2(n-3)(3n-5) + \frac{4(n-1)z^2}{r_+^2} \right) \tilde{f} \right] \]

\[+ (n-1)^2 \tilde{f}^2 - \frac{4(n-1)}{\ell^2} \tilde{f} \]

\[+ \frac{z}{2} \left( 2(n-3)(n-1) \frac{r_+^2}{z^2\ell^2} \tilde{f} - \left( (n-1) \frac{r_+^2}{z^2\ell^2} + n-3 \right)^2 + (n-3) (2(n-1) \right)

\[- (n+1) \tilde{f} \tilde{h}' + \left( \frac{(n-1)z^2}{\ell^2} - \frac{(n-1)z^4}{r_+^2} \tilde{f} + \frac{(n-3)z^4}{r_+^2} \right) \tilde{f}'' \right] = 0. \quad (72)\]

Here the prime (') denotes a derivative with respect to \( z \).

In order to find a regular solution on the horizon \( z = 1 \), it would be better to write \( \tilde{f}(z) \) and \( \tilde{h}(z) \) as the Taylor expansion,

\[
\tilde{f}(z) = 1 + f_1(1-z) + f_2(1-z)^2 + \ldots, \quad (73)
\]

\[
\tilde{h}(z) = 1 + h_1(1-z) + h_2(1-z)^2 + \ldots \quad (74)
\]

Substituting (73) and (74) into (71) and (72), two coefficients of \( f_1 \) and \( h_1 \) are determined to be

\[
f_1 = \frac{3M_n^2 r_+^2}{4} - 2(n-3) \left[ (n-1) \frac{r_+^2}{\ell^2} + (n-2) \right],
\]

\[
h_1 = -\frac{M_n^2 r_+^2}{4} + \left( n^2 - 1 \right) \frac{r_+^2}{\ell^2} + (n-2)(n-3).
\]

We emphasize that for given \( r_+ \), a newly static solution is allowed only for a specific value of \( M_n \). For example, for \( r_+ = 1 \) and \( \ell = 10 \), we obtain \( M_n \) depending on \( n \) as

\[
\begin{array}{cccccccccc}
\hline
n & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
M_n & 0.86 & 1.26 & 1.57 & 1.83 & 2.07 & 2.29 & 2.49 \\
\hline
\end{array}
\]

which are exactly the threshold mass \( M_n^f \) found in [51] for GL instability. It shows a close connection between the stability of AdS black hole and a newly non-AdS black hole.
solution. This is so because we have solved the same linearized equations with different metric perturbations. In Fig. 4, we depict a newly non-AdS black hole solution in $n = 5$ dimensions by comparing with AdS black hole. We would like to mention that a non-Schwarzschild AdS black hole in $n = 4$ dimensions was found in Ricci quadratic gravity with a cosmological constant [33]. Other non-AdS black hole solutions for $5 < n \leq 10$ are obtained numerically.

7 Discussions

First of all, we discuss the two following issues.

i) Role of the Lee-Wick term in the stability analysis of AdS black hole

We have ignored perturbations of the Lee-Wick term because they induced some difficulty to make a progress on the stability analysis. A main reason is that the Lee-Wick term generates uncontrollable sixth-order terms as is shown by Eq.(21). On the other hand, there is no contribution to thermodynamic quantities from the Lee-Wick term, which might state that this term has nothing to do with thermodynamic stability. To this respect, it would be better to check thermodynamic quantities of the AdS plane black hole in the similar sixth-order gravity. There is also no contribution to thermodynamic quantities from the
Lee-Wick term [24]. Assuming the CSC, there may be a close connection between the classical instability and thermodynamic instability for a small AdS black hole found in the higher-order gravity. If the d’Alembertian (\(\Box\)) is present in the action explicitly, it generates uncontrollable higher-order terms in the linearized equation. However, these terms were absent in the thermodynamic quantities. Hence, one may explore this connection further in the sixth-order gravity.

ii) Ghost-free gravity and quantum gravity

On the Minkowski background, the ghost-free (unitary) condition is not compatible with the renormalizability in the higher-order gravity. Recently, it was known that the renormalizability is closely related to a regular Newtonian potential at the origin. A regular potential can be achieved if both massive spin-0 and spin-2 modes are present. This means that the renormalizability requires both massive spin-0 and spin-2 modes. Hence, although the quasi-topological gravity without two massive modes is ghost-free [5], it is not renormalizable. The \(\alpha = -\frac{[4(n - 1)]}{n}\beta\) Ricci cubic gravity without massive spin-0 mode is neither unitary nor renormalizable. A higher-order gravity of \(f(R)\) gravity including a massive spin-0 mode is unitary only, but it is not renormalizable. It is known that one theory of ghost-free and renormalizable gravities around Minkowski background is the infinite-derivative (non-local) gravity where a single d’Alembertian (\(\Box\)) in [3] is replaced by the exponential form of an entire function like \((e^{\Box} - 1)\) [34, 35]. The non-locality of infinitely many d’Alembert operators plays an important role in achieving a quantum gravity with the unitarity and renormalizability. However, increasing number of the d’Alembertian makes the stability analysis of black hole more obscured. It seems that ‘what makes the problem difficult in stability analysis of black hole’ is the simultaneous requirement of unitarity and renormalizability around the Minkowski spacetime [8].

In the AdS black hole background of quasi-topological gravity, the AdS black hole is stable against the metric perturbation \(h_{\mu\nu}\). We emphasize that the \(\alpha = -\frac{[4(n - 1)]}{n}\beta\) Ricci cubic gravity does not have a massive spin-0 mode \(\delta R\) and thus, a small (large) AdS black hole is unstable (stable) against the Ricci tensor perturbation \(\delta \hat{R}_{\mu\nu} = e^{\Omega t}\{\cdots\}_{\mu\nu}\). This was done by solving the Lichnerowicz-type linearized equation (38) with \(\delta R = 0\) (36) numerically. This implies that the presence of a massive spin-0 mode makes the analysis of stability difficult, in addition to a single d’Alembertian included in Lee-Wick term.

Lastly, we have obtained a newly non-AdS black hole in the \(\alpha = -\frac{[4(n - 1)]}{n}\beta\) Ricci
cubic gravity numerically by making use of a static eigenfunction of the Lichnerowicz operator appeared in stability analysis of AdS black holes. This means that a non-AdS black hole is found by solving the Lichnerowicz-type linearized equation (67) with $\delta R = 0$. Actually, this equation becomes a fourth-order equation when considering the static metric perturbation (65). Hence, it is almost unsolvable without $\delta R = 0$. This indicates a strong connection between the stability of AdS black hole and a newly non-AdS black hole solution in Ricci cubic gravity.

In conclusion, we have shown that one prefers the Ricci tensor ($R_{\mu\nu}$) to the d’Alembertian ($\Box$) in constructing a general sixth-order gravity by analyzing the AdS black holes. However, one should continue to explore the connection among unitarity, renormalizability, and stability of black hole in higher-order gravity theories. To this direction, a recent work [35] has shown that replacing $\Box$ by the Lichnerowicz operator $\Delta_L$ in the form factor of the non-local gravity which is renormalizable and unitary, produces a correct stability of Schwarzschild black hole.

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