An error estimate for the Gauss-Jacobi-Lobatto quadrature rule

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Abstract
An error estimate for the Gauss-Lobatto quadrature formula for integration over the interval \([-1,1]\), relative to the Jacobi weight function \(w^{\alpha,\beta}(t) = (1-t)^\alpha (1+t)^\beta\), \(\alpha, \beta > -1\), is obtained. This estimate holds true for functions belonging to some Sobolev-type subspaces of the weighted space \(L^1_{w^{\alpha,\beta}}([-1,1])\).

Keywords: Gauss-Lobatto formula; Jacobi weight function

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1. Introduction
For a Jacobi weight function \(w^{\alpha,\beta}(t) = (1-t)^\alpha (1+t)^\beta\), \(\alpha, \beta > -1\), on the interval \([-1,1]\) we consider the \((n+2)\)-point Gauss-Lobatto rule

\[
\int_{-1}^{1} f(t)w^{\alpha,\beta}(t)dt = w_0 f(-1) + \sum_{k=1}^{n} w_k f(t_k) + w_{n+1} f(1) + e_n(f)
\]

which is exact for polynomials of degree at most \(2n+1\), i.e.

\[e_n(f) = 0, \quad \forall f \in \mathbb{P}_{2n+1}\]

being \(\mathbb{P}_n\) the set of all algebraic polynomials on \([-1,1]\) of degree at most \(n\).

It is well known that the interior quadrature nodes \(t_k, k = 1, \ldots, n\), are the zeros of the Jacobi polynomial of degree \(n\) orthonormal with respect to the Jacobi weight \(w^{\alpha+1,\beta+1}(t) = (1-t)^2w^{\alpha,\beta}(t)\). The weights of the formula (1) are given by

\[
w_k = \int_{-1}^{1} l_k(t)w^{\alpha,\beta}(t)dt, \quad k = 0,1,\ldots,n+1
\]

where, setting from now on \(t_0 = -1\) and \(t_{n+1} = 1\), \(l_k(t)\) denotes the \((k+1)\)-th Lagrange fundamental polynomial associated to the system of nodes \(\{t_0, t_1, \ldots, t_n, t_{n+1}\}\).

By standard arguments, it can be easily proved that the quadrature error \(e_n(f)\) satisfies the following estimate

\[|e_n(f)| \leq 2 \left( \int_{-1}^{1} w^{\alpha,\beta}(t) dt \right) E_{2n+1}(f)_\infty, \quad \forall f \in C([-1,1]), \]

where

\[E_n(f)_\infty = \inf_{P \in \mathbb{P}_n} \|f - P\|_\infty\]
denotes the error of best approximation of a function \( f \in C([-1, 1]) \) by means of polynomials of degree at most \( n \) with respect to the uniform norm.

The aim of the present paper is to provide a new error estimate for less regular functions belonging to some Sobolev-type subspaces of the weighted space \( L_w^{1, \alpha, \beta}([-1, 1]) \). A similar estimate is proved in [3] for the classical Gauss-Jacobi quadrature formula, in [1] for the Gauss-Lobatto rule with respect to the Legendre weight \( w^{0,0} \) and, more recently, in [2] for the Gauss-Radau formula with respect to a general Jacobi weight \( w^{\alpha, \beta} \).

2. Notation and preliminary results

2.1. Notation

For a general weight function \( w(t) \) on \([-1, 1]\) and \( 1 \leq p < +\infty \), let \( L_w^p \) denote the weighted space of all real-valued measurable functions \( f \) on \([-1, 1]\) such that
\[
\|f\|_{L_w^p} = \|fw\|_p = \left( \int_{-1}^1 |f(t)w(t)|^p dt \right)^{\frac{1}{p}} < +\infty,
\]
and let \( W_w^p(w) \) be the following weighted Sobolev-type subspaces of \( L_w^p \)
\[
W_w^p(w) = \left\{ f \in L_w^p : f^{(r-1)} \in AC(-1, 1), \|f^{(r)}\varphi^r w\|_p < +\infty \right\},
\]
where \( r \in \mathbb{N}, r \geq 1, \varphi(t) = \sqrt{1-t^2} \) and \( AC(-1,1) \) is the collection of all functions which are absolutely continuous on every closed subset of \((-1,1)\), equipped with the norm
\[
\|f\|_{W_w^p(w)} = \|fw\|_p + \|f^{(r)}\varphi^r w\|_p.
\]

For a function \( f \in L_w^p \), the error of the best approximation of \( f \) in \( L_w^p \) by polynomials of degree at most \( n \) is defined as
\[
E_n(f)_{w,p} = \inf_{P \in P_n} \|f - P\|_{L_w^p}.
\]

Fixed a Jacobi weight \( w^{\gamma,\delta}(t) = (1-t)^\gamma(1+t)^\delta \), \( \gamma, \delta > -1 \), we will denote by \( x_n^{\gamma,\delta} \) and \( \lambda_n^{\gamma,\delta}, k = 1, \ldots, n \), the nodes and coefficients of the corresponding \( n \)-point Gauss-Jacobi quadrature rule on \([-1, 1]\) and by
\[
p_n^{\gamma,\delta}(t) = \gamma_n^{\gamma,\delta}t^n + \text{lower degree terms}, \quad \gamma_n^{\gamma,\delta} > 0,
\]
the Jacobi polynomial of degree \( n \) orthonormal w.r.t. \( w^{\gamma,\delta}(t) \) having positive leading coefficient.

In the sequel \( \mathcal{C} \) will denote a positive constant which may assume different values in different formulas. We write \( \mathcal{C} = \mathcal{C}(a,b,...) \) to say that \( \mathcal{C} \) is dependent on the parameters \( a, b, ... \) and \( \mathcal{C} \neq \mathcal{C}(a,b,...) \) to say that \( \mathcal{C} \) is independent of them. Moreover, we will write \( A \sim B \), if there exists a positive constant \( \mathcal{C} \) independent of the parameters of \( A \) and \( B \) such that \( 1/\mathcal{C} \leq A/B \leq \mathcal{C} \).

2.2. Preliminary results

It is well known that for functions \( f \) belonging to \( W_w^p(w) \), the following Favard inequality
\[
E_n(f)_{w,p} \leq \frac{\mathcal{C}}{n} E_{n-1}(f)_{w,p}, \quad (3)
\]
is fulfilled for a positive constant \( \mathcal{C} \) independent of \( n \) and \( f \) (see, for example, [3] (2.5.22), p. 172). By iteration of inequality (3), it follows that, for \( f \in W_w^p(w), r \geq 1 \), the estimate
\[
E_n(f)_{w,p} \leq \frac{\mathcal{C}}{n^r} E_{n-r}(f^{(r)})_{w,p}, \quad \mathcal{C} \neq \mathcal{C}(n, f) \quad (4)
\]
holds true.

Let us recall that the knots $x_{n,k}^{\gamma,\delta}$ and Christoffel numbers $\lambda_{n,k}^{\gamma,\delta}$ of the Gaussian quadrature formula corresponding to the Jacobi weight $w_{\gamma,\delta}$ satisfy the following properties (see, for instance, [3, (4.2.4), p. 249])

$$x_{n,k+1}^{\gamma,\delta} - x_{n,k}^{\gamma,\delta} \sim \sqrt{1 - \frac{t^2}{n}}, \quad x_{n,k}^{\gamma,\delta} \leq t \leq x_{n,k+1}^{\gamma,\delta},$$  

(5)

and (see [3, (14), p.673])

$$\lambda_{n,k}^{\gamma,\delta} \sim \sqrt{1 - \left(x_{n,k}^{\gamma,\delta}\right)^2 w_{\gamma,\delta}(x_{n,k}^{\gamma,\delta})}$$  

(6)

uniformly for $1 \leq k \leq n$, $n \in \mathbb{N}$. Moreover, for the orthonormal polynomials $\{p_n^{\gamma,\delta}(t)\}_n$ one has that (see [7, (12.7.2), p. 309])

$$\frac{\gamma_{n}^{\gamma,\delta}}{\gamma_{n-1}^{\gamma,\delta}} \sim 1, \quad \text{as} \quad n \to \infty,$$  

(7)

and (see, for instance, [5, p. 170]) the equality

$$\frac{1}{p_{n-1}^{\gamma,\delta}(x_{n,k}^{\gamma,\delta})} = \frac{\gamma_{n}^{\gamma,\delta}}{\gamma_{n-1}^{\gamma,\delta}} \frac{\lambda_{n,k}^{\gamma,\delta}}{\lambda_{n,k-1}^{\gamma,\delta}} \frac{\gamma_{n}^{\gamma,\delta}}{\gamma_{n}^{\gamma,\delta}} \left(\frac{p_{n}^{\gamma,\delta}}{p_{n}^{\gamma,\delta}(x_{n,k}^{\gamma,\delta})}\right)',$$  

(8)

holds true. Furthermore (see [5, Corollary 9.34, p. 171])

$$p_{n}^{\gamma,\delta}(1) \sim n^{\alpha+\frac{1}{2}}, \quad |p_{n}^{\gamma,\delta}(-1)| \sim n^{\beta+\frac{1}{2}},$$  

(9)

uniformly for $n \in \mathbb{N}$ and, more generally, [5, (4.4.49)-(4.2.30), p. 255])

$$|p_{n}^{\gamma,\delta}(t)| \sim n^{\alpha+\frac{1}{2}}, \quad 1 - \frac{C}{n^2} \leq t \leq 1,$$  

(10)

$$|p_{n}^{\gamma,\delta}(t)| \sim n^{\beta+\frac{1}{2}}, \quad -1 \leq t \leq -1 + \frac{C}{n^2}.$$  

(11)

3. Main results

Lemma 3.1. The nodes and the weights of the Gauss-Lobatto quadrature formula (1) satisfy the following relations

$$w_0 \leq C \Delta t_0 w_{\alpha,\beta}(t_1),$$  

(12a)

$$w_k \sim \left\{ \begin{array}{ll} \Delta t_k w_{\alpha,\beta}(t_k), & k = 1, \ldots, n-1, \\
\Delta t_{k-1} w_{\alpha,\beta}(t_k), & k = n \end{array} \right.,$$  

(12b)

$$w_{n+1} \leq C \Delta t_n w_{\alpha,\beta}(t_n),$$  

(12c)

where $\Delta t_k = t_{k+1} - t_k$, $k = 0,1,\ldots,n$ and $C \neq C(n)$.

Proof. First, let us observe that the weights of the formula (1), given in (2), have the following alternative representation

$$w_0 = \frac{1}{1 - t_0} \int_{-1}^{1} \frac{p_{n+1,\alpha+1,\beta+1}^{\alpha,\beta+1}(t)}{p_{n}^{\alpha+1,\beta+1}(t_0)} w_{\alpha,\beta+1}(t) dt,$$  

(13a)

$$w_k = \frac{1}{1 - t_k^2} \int_{-1}^{1} \frac{p_{n+1,\alpha+1,\beta+1}^{\alpha,\beta+1}(t)}{(t - t_k) \left(p_{n}^{\alpha+1,\beta+1}(t_k)\right)'} w_{\alpha,\beta+1}(t) dt, \quad k = 1, \ldots, n,$$  

(13b)

$$w_{n+1} = \frac{1}{1 + t_{n+1}} \int_{-1}^{1} \frac{p_{n+1,\alpha+1,\beta+1}^{\alpha,\beta+1}(t)}{p_{n}^{\alpha+1,\beta+1}(t_{n+1})} w_{\alpha,\beta+1}(t) dt.$$  

(13c)
For $k = 1, \ldots, n$, since
\[ w_k = \frac{\lambda^{n+1,\beta+1}}{1 - \left(x_{n,k}^{\alpha+1,\beta+1}\right)^2}, \]
using (6) and (5) for $\gamma = \alpha + 1$ and $\delta = \beta + 1$, we can deduce that
\[ w_k \sim \begin{cases} w^{\alpha,\beta}(t_k) \Delta t_k & k = 1, \ldots, n-1 \\ w^{\alpha,\beta}(t_k) \Delta t_{k-1} & k = n \end{cases} \]
i.e. (12a). Now, in order to prove (12c), proceeding in an analogous way, we start writing $w_{n+1}$ as follows
\[ w_{n+1} = \frac{1}{2} \left( \frac{p_n^{n+1,\beta+1}}{p_n^{n+1,\beta+1}(t_n)} \right)^2 \int_{t-n}^{t-n+1} \left( \frac{p_n^{n+1,\beta+1}(t)}{p_n^{n+1,\beta+1}(t_n)} \right)^2 \frac{(t-t_n)^2}{1-t} w^{n+1,\beta+1}(t) dt. \]

Taking into account (13b) and (7), (9) and (11) (all applied with $\gamma = \alpha + 1$ and $\delta = \beta + 1$), we can rewrite the first coefficient $w_0$ in (13a) as follows
\[ w_0 = \frac{1}{2} \left( \frac{p_n^{n+1,\beta+1}}{p_n^{n+1,\beta+1}(t_n)} \right)^2 \int_{t-n}^{t-n+1} \left( \frac{p_n^{n+1,\beta+1}(t)}{p_n^{n+1,\beta+1}(t_n)} \right)^2 \frac{(t-t_n)^2}{1-t} w^{n+1,\beta+1}(t) dt. \]

Being $(t-t_n)^2/(1+t) \leq C$ and using (8) (with $\gamma = \alpha + 1$ and $\delta = \beta + 1$) for $k = 1$ (we recall that in our notation $x_{n+1,\beta+1} \equiv t_1$), we get
\[ w_0 \leq C \left[ \frac{p_n^{n+1,\beta+1}}{p_n^{n+1,\beta+1}(t_n)} \right]^2 \lambda^{n+1,\beta+1}_{\alpha_{n+1}} \]
\[ = C \left[ \frac{1}{p_n^{n+1,\beta+1}(t_n)} \right]^2 \left( \frac{\lambda^{n+1,\beta+1}_{\alpha_{n+1}}}{p_n^{n+1,\beta+1}(t_n)} \right)^2 \frac{1}{\lambda^{n+1,\beta+1}_{\alpha_{n+1}}}. \]

Now, in virtue of (11), it is
\[ \lambda^{n+1,\beta+1}_{\alpha_{n+1}} \sim \frac{1 - x_{n,1}^{\alpha+1,\beta+1}}{n} \frac{\left(1 + x_{n,1}^{\alpha+1,\beta+1}\right)^{\beta+\frac{2}{2}}} \sim \frac{1}{n^{2\beta+4}}. \]

Taking into account (14), and (7), (9) and (11) (all applied with $\gamma = \alpha + 1$ and $\delta = \beta + 1$), we deduce the following estimate of $w_0$
\[ w_0 \leq C \frac{n^{2\beta+4}}{n^{2\beta+6}} = \frac{C}{n^{2\beta+2}}. \]

On the other hand,
\[ \Delta t_0 w^{\alpha,\beta}(t_1) = \frac{1}{n} \left(1 - x_{n,1}^{\alpha+1,\beta+1}\right)^\alpha \left(1 + x_{n,1}^{\alpha+1,\beta+1}\right)^{\beta+1} \sim \frac{1}{n^{2\beta+2}} \]

which, combined with (15), gives (12b). In order to prove (12c), proceeding in an analogous way, we start writing $w_{n+1}$ as follows
\[ w_{n+1} = \frac{1}{2} \left( \frac{p_n^{n+1,\beta+1}}{p_n^{n+1,\beta+1}(t_{n+1})} \right)^2 \int_{t-n+1}^{t-n} \left( \frac{p_n^{n+1,\beta+1}(t)}{p_n^{n+1,\beta+1}(t_n)} \right)^2 \frac{(t-t_n)^2}{1-t} w^{n+1,\beta+1}(t) dt. \]
Then, using \((t - t_n)^2 / (1 - t) \leq C\), \(\mathcal{O}\) and \(\mathcal{O}\) for \(\gamma = \alpha + 1\), \(\delta = \beta + 1\) and \(k = n\), \(\mathcal{O}\), \(\mathcal{O}\) and \(\mathcal{O}\) also with \(\gamma = \alpha + 1\) and \(\delta = \beta + 1\), we deduce the estimate

\[
w_{n+1} \leq C \left[ \frac{(p_{n+1}^{\alpha+\beta+1})'(t_n)}{p_{n+1}^{\alpha+\beta+1}(t_{n+1})} \right]^2 \lambda_{n+1,\beta+1} \sim \frac{1}{n^{2\alpha+2}}
\]

with \(C \neq C(n)\). Since

\[
\Delta t_n w^{\alpha,\beta}(t_n) \sim (1 - x_{n,n}^{\alpha+1,\beta+1})^{\alpha+1}(1 + x_{n,n}^{\alpha+1,\beta+1})^\beta \sim \frac{1}{n^{2\alpha+2}}
\]

we can, finally, conclude that also \(\mathcal{O}\) holds true.

Using the previous lemma we are able to prove our main result.

**Theorem 3.1.** For \(f \in W^1_r(w^{\alpha,\beta})\), \(r \geq 1\), the error of the Gauss-Lobatto quadrature formula \(\mathcal{O}\) satisfies the following estimate

\[
|e_n(f)| \leq \frac{C}{(2n)!} E_{2n+1-r}(f^{(r)}) \varphi^{w^{\alpha,\beta}}
\]

where \(\varphi(t) = \sqrt{1 - t^2}\) and \(C \neq C(f, n)\) is a positive constant.

**Proof.** We start by proving the estimate \(\mathcal{O}\) in the case \(r = 1\). First, we are going to show that

\[
\sum_{k=0}^{n+1} w_k |f(t_k)| \leq C ||f w^{\alpha,\beta}||_1 + \frac{C}{n} \int_{t_{n-1}}^{t_n} |f'(t)| \varphi(t) w^{\alpha,\beta}(t) dt.
\]

Taking into account \(\mathcal{O}\), \(\mathcal{O}\) and \(\mathcal{O}\) we have

\[
\sum_{k=0}^{n+1} w_k |f(t_k)| \leq C \left[ \Delta t_0 w^{\alpha,\beta}(t_1) |f(t_0)| + \sum_{k=1}^{n-1} \Delta t_k w^{\alpha,\beta}(t_k) |f(t_k)| + \Delta t_{n-1} w^{\alpha,\beta}(t_n) |f(t_{n-1})| \right].
\]

In virtue of the following inequality

\[
\frac{(b - a) |f(a)|}{(b - a) |f(b)|} \leq \int_a^b |f(t)| dt + (b - a) \int_a^b |f'(t)| dt,
\]

it follows that

\[
\sum_{k=0}^{n+1} w_k |f(t_k)| \leq C \left[ w^{\alpha,\beta}(t_1) \left( \int_{t_0}^{t_1} |f(t)| dt + \Delta t_0 \int_{t_0}^{t_1} |f'(t)| dt \right) + \sum_{k=1}^{n-1} w^{\alpha,\beta}(t_k) \left( \int_{t_k}^{t_{k+1}} |f(t)| dt + \Delta t_k \int_{t_k}^{t_{k+1}} |f'(t)| dt \right) + w^{\alpha,\beta}(t_n) \left( \int_{t_n}^{t_{n+1}} |f(t)| dt + \Delta t_n \int_{t_n}^{t_{n+1}} |f'(t)| dt \right) + w^{\alpha,\beta}(t_n) \left( \int_{t_n}^{t_{n+1}} |f(t)| dt + \Delta t_n \int_{t_n}^{t_{n+1}} |f'(t)| dt \right),
\]

from which, being for \(k = 0, 1, \ldots, n\),

\[
1 \pm t_k \sim 1 \pm t \sim 1 \pm t_{k+1}, \quad t_k \leq t \leq t_{k+1},
\]

(18)
and, taking into account (5) and also that

$$\Delta t_0 \sim \frac{\sqrt{1 - t^2}}{n}, \quad -1 < t \leq t_1,$$

$$\Delta t_n \sim \frac{\sqrt{1 - t^2}}{n}, \quad t_n \leq t < 1,$$

we deduce

$$\sum_{k=0}^{n+1} w_k |f(t_k)| \leq C \left[ \int_{t_0}^{t_1} |f(t)| w^{\alpha,\beta}(t) dt + \frac{1}{n} \int_{t_0}^{t_1} |f'(t)| w^{\alpha,\beta}(t) dt \right]$$

$$+ \sum_{k=1}^{n-1} \left( \int_{t_k}^{t_{k+1}} |f(t)| w^{\alpha,\beta}(t) dt + \frac{1}{n} \int_{t_k}^{t_{k+1}} |f'(t)| w^{\alpha,\beta}(t) dt \right)$$

$$+ \int_{t_0}^{t_n} |f(t)| w^{\alpha,\beta}(t) dt + \frac{1}{n} \int_{t_0}^{t_1} |f'(t)| w^{\alpha,\beta}(t) dt$$

$$= C \left[ \int_{-1}^{1} |f(t)| w^{\alpha,\beta}(t) dt + \frac{1}{n} \int_{-1}^{1} |f'(t)| w^{\alpha,\beta}(t) dt \right]$$

i.e. (17). Since the $(n+1)$ quadrature formula (11) is exact for any polynomial of degree at most $2n + 1$, for $P \in \mathbb{P}_{2n+1}$ we have

$$|e_n(f)| = |e_n(f - P)|$$

$$\leq \left| \int_{-1}^{1} (f - P)(t) w^{\alpha,\beta}(t) dt \right| + \sum_{k=0}^{n+1} w_k |(f - P)(t_k)|$$

$$\leq \| (f - P) w^{\alpha,\beta} \|_1 + \sum_{k=0}^{n+1} w_k |(f - P)(t_k)|.$$

Now, combining the previous inequality with (17) and the following relation (see, for instance, [3, p. 339], [4, p. 286])

$$\|(f - P)' \varphi w^{\alpha,\beta}\|_1 \leq C(2n + 2)\|(f - P) w^{\alpha,\beta}\|_1 + C_1 E_{2n}(f') \varphi w^{\alpha,\beta},$$

we can deduce

$$|e_n(f)| \leq C \|(f - P) w^{\alpha,\beta}\|_1 + \frac{C}{n} \|(f - P)' \varphi w^{\alpha,\beta}\|_1$$

$$\leq C \|(f - P) w^{\alpha,\beta}\|_1 + \frac{C}{n} E_{2n}(f') \varphi w^{\alpha,\beta}.1.$$  

Taking the infimum over $P \in \mathbb{P}_{2n+1}$ and using the Favard inequality (4) we get

$$|e_n(f)| \leq C E_{2n+1}(f) w^{\alpha,\beta},1 + \frac{C}{n} E_{2n}(f') \varphi w^{\alpha,\beta},1 \leq \frac{C}{2n} E_{2n}(f') \varphi w^{\alpha,\beta},1. \quad (19)$$

The estimate (16) for $r > 1$ can be deduced from (19) by iterating the application of the Favard inequality.

\[ \square \]

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