SYMMETRY RESULTS FOR NONLINEAR ELLIPTIC OPERATORS
WITH UNBOUNDED DRIFT

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Abstract. We prove the one-dimensional symmetry of solutions to elliptic equations of the form 
\[ -\text{div}(e^{G(x)}a(|\nabla u|)\nabla u) = f(u)e^{G(x)} , \]
under suitable energy conditions. Our results holds without any restriction on the dimension of the ambient space.

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1. Introduction

In this paper we study the one-dimensional symmetry of solutions to nonlinear equations of the following type:

\[ \text{div}(a(|\nabla u|)\nabla u) + a(|\nabla u|)\langle \nabla G(x), \nabla u \rangle + f(u) = 0, \]

or in a more compact form

\[ -\text{div}(e^{G(x)}a(|\nabla u|)\nabla u) = f(u)e^{G(x)}, \]

where \( f \in C^1(\mathbb{R}) \), \( G \in C^2(\mathbb{R}^n) \) and \( a \in C^{1,1}_{\text{loc}}((0, +\infty)) \). We also require that the function \( a \) satisfies the following structural conditions:

\[ a(t) > 0 \quad \text{for any } t \in (0, +\infty), \]
\[ a(t) + a'(t)t > 0 \quad \text{for any } t \in (0, +\infty). \]

1 One could consider functions \( f \) which are only locally lipschitz continuous, as in [9]. To avoids inessential technicalities, we do not treat this case here.
Observe that the general form of (2) encompasses, as very special cases, many elliptic
singular and degenerate equations. Indeed, if \( G \equiv 0 \) and 
\[ a(t) = \begin{cases} \frac{1}{\sqrt{1 + t^2}} & \text{for any } t > 0 \\ \frac{1}{t^{p-2}} & 1 < p < +\infty \\ \end{cases} \]
then we obtain the \( p \)-Laplacian and the mean curvature equations respectively. Moreover, if 
\[ a(t) \equiv 1 \] and 
\[ G(x) = -|x|^2/2 \] equation (1) boils down to the classical Ornstein-Uhlenbeck operator for which we refer to
[1] and the references therein.

To prove the one-dimensional symmetry of solutions we follow the approach introduced
in [5] and further developed in [9].

Following [5, 9, 3], we define \( A : \mathbb{R}^n \to \text{Mat}(n \times n) \), \( \lambda_1 \in C^0((0, +\infty)) \), \( \lambda_G \in C^0(\mathbb{R}^2) \) as
follows:

\[
A_{hk}(\xi) := \frac{a'(|\xi|)}{|\xi|} \xi_h \xi_k + a(|\xi|) \delta_{hk} \quad \text{for any } 1 \leq h, k \leq n,
\]

(5)

\[
\lambda_1(t) := a(t) + a'(t)t \quad \text{for any } t > 0
\]

(6)

and

\[
\lambda_G(x) := \text{maximal eigenvalue of } \nabla^2 G(x).
\]

(7)

Definition 1.1. We say that \( u \) is a weak solution to (1) if \( u \in C^1(\mathbb{R}^n) \) and denoted by
\( \text{d} \mu = e^{G(x)} \text{d}x \)

(8)

\[
\int_{\mathbb{R}^n} \langle a(|\nabla u|) \nabla u, \nabla \varphi \rangle - f(u) \varphi \text{ d} \mu = 0 \quad \forall \varphi \in C^1_c(\mathbb{R}^n)
\]

and either (A1) or (A2) is satisfied, where :

(A1) \( \{ \nabla u = 0 \} = \emptyset \).

(A2) \( a \in C^0([0, +\infty)) \) and

the map \( t \to ta(t) \) belongs to \( C^1([0, +\infty)) \).

Notice that (8) is well-defined, thanks to (A1) or (A2).

Notice also that weak solutions to (1) are critical points of the functional

(9)

\[
I(u) := \int_{\mathbb{R}^n} \left( \Lambda(|\nabla u|) + F(u) \right) \text{d} \mu
\]

where \( F'(t) = -f(t) \), \( \text{d} \mu = e^{G(x)} \text{d}x \) and

\[
\Lambda(t) := \int_0^t a(|\tau|) \tau \text{d} \tau.
\]

The regularity assumption \( u \in C^1(\mathbb{R}^n) \) is always fulfilled in many important cases, like those involving the \( p \)-Laplacian operator or the mean curvature operator. For instance, when \( a(t) = t^{p-2}, 1 < p < +\infty \), any distribution solution \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty_{\text{loc}}(\mathbb{R}^n) \) is of class \( C^1 \), by the well-known results in [16, 22]). In light of this, and in view of the great
generality of the function \( a \), it is natural to work in the above setting.
Definition 1.2. Let \( h \in L^1_{\text{loc}}(\mathbb{R}^n) \) and let \( u \) be a weak solution to (1). We say that \( u \) is \( h \)-stable if

\[
\int_{\mathbb{R}^n} \langle A(\nabla u)\nabla \varphi, \nabla \varphi \rangle - f'(u)\varphi^2 \, d\mu \geq \int_{\mathbb{R}^n} a(|\nabla u|)h\varphi^2 \, d\mu \quad \forall \varphi \in C^1_c(\mathbb{R}^n).
\]

Remark 1.3. When \( a(t) \equiv 1 \), Definition 1.2 boils down to the \( h \)-stability condition introduced in [2, 3].

When \( h \equiv 0 \), then \( u \) satisfies the classical stability condition [5, 9, 11, 10], and we simply say that \( u \) is stable. In particular, every minimum point of the functional (9) is a stable solution to (1).

Let us also point out that, in view of (A1) or (A2), the integral

\[
\int_{\mathbb{R}^n} \langle A(\nabla u)\nabla \varphi, \nabla \varphi \rangle - f'(u)\varphi^2 - a(|\nabla u|)h\varphi^2 \, d\mu
\]

is well defined.\(^2\) In particular, under the condition (A2) the function \( A \) can be extended by continuity at the origin, by setting \( A_{hk}(0) := a(0)\delta_{hk} \).

We can now state our main symmetry results:

**Theorem 1.** Assume \( G \in C^2(\mathbb{R}^n) \) and \( h \in L^1_{\text{loc}}(\mathbb{R}^n) \) with \( h \geq \lambda G \). Let \( u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\}) \) with \( \nabla u \in H^1_{\text{loc}}(\mathbb{R}^n) \) be a \( h \)-stable weak solution to (1).

Assume that there exists \( C > 0 \) such that

\[
\lambda_1(t) \leq C a(t) \quad \forall t > 0,
\]

and one of the following conditions holds

(a) there exists \( C_0 \geq 1 \) such that \( \int_{B_R} a(|\nabla u|)|\nabla u|^2 \, d\mu \leq C_0 R^2 \) for any \( R \geq C_0 \),

(b) \( n = 2 \) and \( u \) satisfies \( a(|\nabla u|)|\nabla u|^2 e^G \in L^\infty(\mathbb{R}^2) \).

Then \( u \) is one-dimensional, i.e. there exists \( \omega \in S^{n-1} \) and \( u_0 : \mathbb{R} \to \mathbb{R} \) such that

\[
u(x) = u_0(\langle \omega, x \rangle) \quad \forall x \in \mathbb{R}^n.
\]

Moreover,

\[
\langle (h(x)I_n - \nabla^2 G(x))\nabla u, \nabla u \rangle = 0 \quad \forall x \in \mathbb{R}^n.
\]

In particular, if \( u_0 \) is not constant, there are \( C \) and \( g \) of class \( C^2 \) such that

\[
G(x) = C(\langle \omega, x \rangle) + g(x'),
\]

where \( x' := x - \langle \omega, x \rangle \omega \) and \( \lambda G(x) = h(x) = C''(\langle \omega, x \rangle) \) for all \( x \in \mathbb{R}^n \).

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\(^2\) cfr. also [9, footnote 1 at p. 742 and footnote 2 at page 743].
Remark 1.4. Paradigmatic examples satisfying the assumption (12) are the $p$-Laplacian operator, for any $p \in (1, +\infty)$, and the generalized mean curvature operator obtained by setting $a(t) := (1 + t^q)^{-\frac{1}{q}}$, with $q > 1$.

Theorem 2. Let $G(x) := -|x|^2/2$, $a(t) := t^{p-2}$ with $p > 1$ and let $u \in C^1(\mathbb{R}^n) \cap W^{1, \infty}(\mathbb{R}^n)$ be a monotone weak solution to (1), i.e., such that
\[
\partial_i u(x) > 0 \quad \forall x \in \mathbb{R}^n,
\]
for some $i \in \{1, \ldots, n\}$. Suppose that $u$ satisfies either (a) or (b) in Theorem 1. Then $u$ is one-dimensional. Moreover, if either $p = 2$ or $a(t) := (1 + t^q)^{-\frac{1}{q}}$ with $q > 1$, then the same conclusion holds for every monotone weak solution $u \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Theorem 3. Let $u$ be a bounded weak solution to
\[
\Delta u - \langle x, \nabla u \rangle + f(u) = 0
\]
with Morse index $k$. Then,
\begin{enumerate}
  \item if $k \leq 2$ then $u$ is one-dimensional;
  \item if $3 \leq k \leq n$ then $u$ is a function of at most $k - 1$ variables, i.e. there exists $C \in \text{Mat}(((k-1) \times n))$ and $u_0 : \mathbb{R}^{k-1} \to \mathbb{R}$ such that
\end{enumerate}
\[
u(x) = u_0(Cx) \quad \forall x \in \mathbb{R}^n.
\]

The result in Theorem 3 should be compared with the analysis in [14], where the author shows that a minimal surface in the Gauss space, with Morse index less than or equal to $n$, is necessarily a hyperplane through the origin. These minimal surfaces are important geometric objects as they correspond to self-shrinkers for the mean curvature flow, which are the model of generic singularities. Since the minimal surface equation in the Gauss space arises as singular limit, as $\epsilon \to 0$, of the equations
\[
\Delta u - \langle x, \nabla u \rangle - \frac{W'(u)}{\epsilon} = 0,
\]
where $W$ is a double-well potential (see for instance [23]), it is natural to ask if there exist bounded solutions to (17), with Morse index less than or equal to $n$, which are not one-dimensional.

2. A geometric Poincaré inequality

We recall the following result which has been proved in [9].

Lemma 2.1. For any $\xi \in \mathbb{R}^n \setminus \{0\}$, the matrix $A(\xi)$ is symmetric and positive definite and its eigenvalues are $\lambda_1(|\xi|), \ldots, \lambda_n(|\xi|)$, where $\lambda_1$ is as in (6) and $\lambda_i(t) = a(t)$ for every $i = 2, \ldots, n$. Moreover,
\[
\langle A(\xi) \xi, \xi \rangle = |\xi|^2 \lambda_1(|\xi|),
\]
and
\[
0 \leq \langle A(\xi)(V - W), (V - W) \rangle = \langle A(\xi)V, V \rangle + \langle A(\xi)W, W \rangle - 2 \langle A(\xi)V, W \rangle,
\]
for any $V, W \in \mathbb{R}^n$ and any $\xi \in \mathbb{R}^n \setminus \{0\}$. 

Lemma 2.2. Let $u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\})$ with $\nabla u \in H^1_{loc}(\mathbb{R}^n)$ be a weak solution to (1). Then for any $i = 1, \ldots, n$, and any $\varphi \in C_c(\mathbb{R}^n)$ we have

$$
\int_{\mathbb{R}^n} (A(\nabla u)\nabla u_i, \nabla \varphi) - a(|\nabla u|) (\nabla u, \nabla (G_i)) \varphi - f'(u)u_i \varphi \, d\mu = 0.
$$

Proof. By Lemma 2.2 in [9] we have

$$
\text{the map } x \to W(x) := a(|\nabla u(x)|)\nabla u(x) \text{ belongs to } W^{1,1}_{loc}(\mathbb{R}^n, \mathbb{R}^n),
$$

therefore, since $e^{G(x)} \in C^2(\mathbb{R}^n)$ we get

$$
W e^G \in W^{1,1}_{loc}(\mathbb{R}^n, \mathbb{R}^n).
$$

By Stampacchia’s Theorem (see, e.g. [18, Theorem 6.19]), we get $\partial_t(W e^G) = 0$ for almost any $x \in \{W e^G = 0\} = \{W = 0\}$, that is

$$
\partial_t(W e^G) = 0
$$

for almost any $x \in \{\nabla u = 0\}$. In the same way, by Stampacchia’s Theorem and (A2), it can be proven that $\nabla u_i(x) = 0$, and hence $A(\nabla u(x))\nabla u_i(x) = 0$, for almost any $x \in \{\nabla u = 0\}$. Moreover, the following relation holds (see [9] for the proof)

$$
\partial_t(W e^G) = (A(\nabla u)\nabla u_i + a(|\nabla u|)\nabla G_i) e^G \quad \text{on } \{\nabla u \neq 0\},
$$

and thanks to the previous observations

$$
\partial_t(W e^G) = (A(\nabla u)\nabla u_i + a(|\nabla u|)\nabla G_i) e^G \quad \text{a.e. in } \mathbb{R}^n.
$$

Applying (8) with $\varphi$ replaced by $\varphi_i$ and making use of (23) and (25), we obtain

$$
0 = \int_{\mathbb{R}^n} a(|\nabla u|) (\nabla u, \nabla \varphi_i) - f(u)\varphi_i \, d\mu
$$

$$
= \int_{\mathbb{R}^n} - (A(\nabla u)\nabla u_i, \nabla \varphi) - a(|\nabla u|) (\nabla u, \nabla \varphi) G_i \, d\mu
$$

$$
+ \int_{\mathbb{R}^n} f'(u)u_i \varphi + f(u)\varphi G_i \, d\mu
$$

$$
= \int_{\mathbb{R}^n} - (A(\nabla u)\nabla u_i, \nabla \varphi) - a(|\nabla u|) (\nabla u, \nabla (\varphi G_i)) \, d\mu
$$

$$
+ \int_{\mathbb{R}^n} a(|\nabla u|) (\nabla u, \nabla G_i) \varphi + f'(u)u_i \varphi + f(u)\varphi G_i \, d\mu.
$$

Recalling (8), applied with $\varphi$ replaced by $\varphi G_i$, we obtain the thesis. \qed

From now on, we use $A$ and $a$, as a short-hand notation for $A(\nabla u)$ and $a := a(|\nabla u|)$ respectively. In the following result we prove that every monotone solution to (1) is indeed $h$–stable.

Lemma 2.3. Assume that $u$ is a weak solution to (1) and that there exists $i \in \{1, \ldots, n\}$ such that

$$
u_i := \partial_i u(x) > 0 \quad \forall x \in \mathbb{R}^n$$

(26)
then $u$ is $h$–stable, with

$$h(x) := \langle \nabla u(x), \nabla G_i(x) \rangle_{u_i(x)}.$$  

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\psi := \varphi^2 / u_i$. We use (20) with $V := \varphi \nabla u_i / u_i$ and $W := \nabla \varphi$ to obtain that

$$\frac{2\varphi}{u_i} \langle A\nabla u_i, \nabla \varphi \rangle - \frac{\varphi^2}{u_i^2} \langle A\nabla u_i, \nabla u_i \rangle \leq \langle A\nabla \varphi, \nabla \varphi \rangle.$$  

From this and Lemma 2.2 we get

$$0 = \int \langle A\nabla u_i, \nabla \psi \rangle - a(\nabla u, \nabla G_i) \psi - f'(u)u_i \psi \, d\mu$$

$$= \int 2\frac{\varphi}{u_i} \langle A\nabla u_i, \nabla \varphi \rangle - \frac{\varphi^2}{u_i^2} \langle A\nabla u_i, \nabla u_i \rangle - a\frac{\varphi^2}{u_i^2} \langle \nabla u, \nabla G_i \rangle - f'(u)\varphi^2 \, d\mu$$

$$\leq \int \langle A\nabla \varphi, \nabla \varphi \rangle - a\frac{\varphi^2}{u_i^2} \langle \nabla u, \nabla G_i \rangle - f'(u)\varphi^2 \, d\mu.$$  

Notice that we can apply Lemma 2.2 since, in view of (26), $u$ has no critical points and thus it is of class $C^2$, by the classical regularity results.

The following Lemma can be proved using the same techniques implemented in [9, Lemma 2.4],

**Lemma 2.4.** Let $h \in L^1_{loc}(\mathbb{R}^n)$. Let $u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\})$ with $\nabla u \in H^1_{loc}(\mathbb{R}^n)$ be a $h$–stable weak solution to (1). Then, (10) holds for any $\varphi \in H^1_0(B)$ and for any ball $B \subset \mathbb{R}^n$. Moreover, under the assumptions of Lemma 2.2,

$$\int_{\mathbb{R}^n} (A(\nabla u)\nabla u_i, \nabla \varphi) - a(\nabla u, \nabla G_i) \varphi - f'(u)u_i \varphi \, d\mu = 0.$$  

for any $i = 1, \ldots, n$, any $\varphi \in H^1_0(B)$ and any ball $B \subset \mathbb{R}^n$.

**Proposition 2.5.** Let $h \in L^1_{loc}(\mathbb{R}^n)$ and $u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\})$ with $\nabla u \in H^1_{loc}(\mathbb{R}^n)$ be a $h$–stable weak solution to (1). Then, for every $\varphi \in C^1_c(\mathbb{R}^n)$ it holds

$$\int_{\mathbb{R}^n} a(|\nabla u|)h(x)|\nabla u|^2 \varphi^2 \, d\mu \leq \int_{\mathbb{R}^n} |\nabla u|^2 \langle A\nabla \varphi, \nabla \varphi \rangle + a(|\nabla u|) \langle \nabla^2 G \nabla u, \nabla u \rangle \varphi^2$$

$$+ \varphi^2 \left[ \langle A(\nabla u)|\nabla u|, \nabla |\nabla u| \rangle - \sum_{i=1}^n \langle A(\nabla u)\nabla u_i, \nabla u_i \rangle \right] \, d\mu.$$  

Proof. We start observing that by Stampacchia’s Theorem, since $\mu << \mathcal{L}^n$, we get

$$\nabla |\nabla u|(x) = 0 \quad \text{\mu - a.e. } x \in \{|\nabla u| = 0\},$$

$$\nabla u_j(x) = 0 \quad \text{\mu - a.e. } x \in \{|\nabla u| = 0\} \subseteq \{u_j = 0\},$$  

for any $j = 1, \ldots, n$. Let $\varphi \in C^1_c(\mathbb{R}^n)$ and $i = 1, \ldots, n$. Using (21) with test function $u_i \varphi^2$ and summing over $i = 1, \ldots, n$ we get

$$\int_{\mathbb{R}^n} \sum_{i=1}^n \langle A(\nabla u)\nabla u_i, \nabla (u_i \varphi^2) \rangle - f'(u)|\nabla u|^2 \varphi^2 \, d\mu = \int_{\mathbb{R}^n} a(|\nabla u|) \langle \nabla^2 G \nabla u, \nabla u \rangle \varphi^2 \, d\mu$$.
Using (10) with test function $|\nabla u| \varphi$ (note that this choice is possible thanks to Lemma 2.4) we then get

$$\int_{\mathbb{R}^n} a(|\nabla u|) h(x)|\nabla u|^2 \varphi^2 \, d\mu \leq \int_{\mathbb{R}^n} \left( \left( A(\nabla u(x))\nabla(|\nabla u| \varphi) \right) \cdot \nabla(|\nabla u| \varphi) \right) - f'(u)|\nabla u|^2 \varphi^2 \, d\mu$$

$$= \int_{\mathbb{R}^n} |\nabla u|^2 \left( A\nabla \varphi, \nabla \varphi \right) d\mu + \int_{\{\nabla u \neq 0\}} \varphi^2 \left( A|\nabla u|, \nabla|\nabla u| \right)$$

$$+ 2\varphi|\nabla u| \left( A\nabla u, \nabla|\nabla u| \right) - f'(u)|\nabla u|^2 \varphi^2 \, d\mu$$

and by (32) we conclude that

$$\int_{\mathbb{R}^n} a(|\nabla u|) h(x)|\nabla u|^2 \varphi^2 \, d\mu \leq \int_{\mathbb{R}^n} |\nabla u|^2 \left( A\nabla \varphi, \nabla \varphi \right) d\mu + \int_{\{\nabla u \neq 0\}} \varphi^2 \left( A|\nabla u|, \nabla|\nabla u| \right)$$

$$+ \int_{\{\nabla u \neq 0\}} \varphi^2 \left[ \left( A\nabla|\nabla u|, \nabla|\nabla u| \right) - \sum_{i=1}^n \left( A(\nabla u)\nabla u_i, \nabla u_i \right) \right] d\mu.$$ 

which is the thesis. \hfill \Box

**Remark 2.6.** Letting

$$L_{u,x} := \{ y \in \mathbb{R}^n \mid u(y) = u(x) \},$$

we denote by $\nabla T u$ the tangential gradient of $u$ along $L_{u,x} \cap \{ \nabla u \neq 0 \}$, and by $k_1, \ldots, k_{n-1}$ the principal curvatures of $L_{u,x} \cap \{ \nabla u \neq 0 \}$.

$$\langle A|\nabla u|, \nabla|\nabla u| \rangle - \sum_{i=1}^n \langle A(\nabla u)\nabla u_i, \nabla u_i \rangle = a \left[ |\nabla|\nabla u| |^2 - \sum_{i=1}^n |\nabla u_i|^2 \right] - a'|\nabla u| |\nabla T|\nabla u|^2$$

and using (6) we get

$$\langle A|\nabla u|, \nabla|\nabla u| \rangle - \sum_{i=1}^n \langle A(\nabla u)\nabla u_i, \nabla u_i \rangle$$

$$= -\lambda_1 |\nabla T|\nabla u|^2 - a(|\nabla u|) \left( \sum_{i=1}^n |\nabla u_i|^2 - |\nabla T|\nabla u|^2 - |\nabla u|^2 \right)$$

Notice that the quantity

$$\sum_{i=1}^n |\nabla u_i|^2 - |\nabla|\nabla u| |^2 - |\nabla T|\nabla u|^2$$

has a geometric interpretation, in the sense that it can be expressed in terms of the principal curvatures of level sets of $u$.

More precisely, the following formula holds (see \cite{9, 20, 21})
\[
\sum_{i=1}^{n} |\nabla u_i|^2 - |\nabla |\nabla u||^2 - |\nabla T|\nabla u||^2 = |\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \quad \text{on } L_{u,x} \cap \{\nabla u \neq 0\},
\]

so that (34) becomes
\[
\int_{\{\nabla u \neq 0\}} a(|\nabla u|) h(x)|\nabla u|^2 \varphi^2 + \left[\lambda_1 |\nabla T|\nabla u||^2 + a(|\nabla u|)\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \right] \varphi^2 \\
- a(|\nabla u|) \langle \nabla^2 G \nabla u, \nabla u \rangle \varphi^2 \, d\mu \\
\leq \int_{\mathbb{R}^n} \langle A \nabla \varphi, \nabla \varphi \rangle |\nabla u|^2 \, d\mu.
\]

Rearranging the terms, we obtain
\[
\int_{\{\nabla u \neq 0\}} a(|\nabla u|) \langle (h(x)I - \nabla^2 G) \nabla u, \nabla u \rangle \varphi^2 + \left[\lambda_1 |\nabla T|\nabla u||^2 + a(|\nabla u|)\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \right] \varphi^2 \, d\mu \\
\leq \int_{\mathbb{R}^n} \langle A \nabla \varphi, \nabla \varphi \rangle |\nabla u|^2 \, d\mu,
\]

where \( I \in \text{Mat}(n \times n) \) denotes the identity matrix.

Notice that from (38) we also obtain
\[
\int_{\{\nabla u \neq 0\}} a(|\nabla u|) \langle (h(x)I - \nabla^2 G) \nabla u, \nabla u \rangle \varphi^2 \, d\mu \leq \int_{\mathbb{R}^n} \langle A \nabla \varphi, \nabla \varphi \rangle |\nabla u|^2 \, d\mu.
\]

3. One-dimensional symmetry of solutions

In this section we will use (38) to prove several one-dimensional results for solutions to (1), following the approach introduced in [5] and then developed in [9]. Notice that, more recently, a similar approach has also been used to handle semilinear equations in Riemannian and subriemannian spaces (see [6, 7, 8, 12, 13, 19]) and also to study problems involving the Ornstein-Uhlenbeck operator [2], as well as semilinear equations with unbounded drift [3].

The following Lemma is proved in [9, 13].

**Lemma 3.1.** Let \( g \in L_{\text{loc}}^{\infty}(\mathbb{R}^n, [0, +\infty)) \) and let \( q > 0 \). Let also, for any \( \tau > 0 \),
\[
\eta(\tau) := \int_{B_\tau} g(x) \, dx.
\]

Then, for any \( 0 < r < R \),
\[
\int_{B_R \setminus B_r} \frac{g(x)}{|x|^q} \, dx \leq q \int_r^R \frac{\eta(\tau)}{|\tau|^{q+1}} \, d\tau + \frac{1}{R^q} \eta(R)
\]
Proof of Theorem 1.

Let us fix $R > 0$ (to be taken appropriately large in what follows) and $x \in \mathbb{R}^n$ and let us define

$$\varphi(x) := \begin{cases} 
1 & \text{if } x \in B_{\sqrt{R}} \\
\frac{2 \log(R/|x|)}{\log(R)} & \text{if } x \in B_R \setminus B_{\sqrt{R}} \\
0 & \text{if } x \in \mathbb{R}^n \setminus B_R,
\end{cases}$$

where $B_R := \{ y \in \mathbb{R}^n \mid |y| < R \}$. Obviously $\varphi \in \text{Lip}(\mathbb{R}^n)$ and

$$|
abla \varphi(x)| \leq C_2 \frac{\chi_{\sqrt{R},R}(x)}{\log(R)} |x|$$

for suitable $C_2 > 0$. Hence for every $R > e$, (38) together with $h \geq \lambda_G$ yields

$$\int_{\{ \nabla u \neq 0 \} \cap B_R} \left[ \lambda_1 |\nabla T| \nabla u|^2 + a(|\nabla u|) |\nabla u|^2 \sum_{j=1}^{n-1} k_j \right] \varphi^2 \ d\mu \leq \int_{\mathbb{R}^n} \langle A(\nabla u) \nabla \varphi, \nabla \varphi \rangle |\nabla u|^2 d\mu$$

therefore, by (12)

$$\int_{\{ \nabla u \neq 0 \} \cap B_R} \left[ \lambda_1 |\nabla T| \nabla u|^2 + a(|\nabla u|) |\nabla u|^2 \sum_{j=1}^{n-1} k_j \right] \varphi^2 \ d\mu \leq (1 + C) \int_{\mathbb{R}^n} a(|\nabla u|) |\nabla \varphi|^2 |\nabla u|^2 d\mu \leq \frac{(1 + C) C_2^2}{\log(R)^2} \int_{B_R \setminus B_{\sqrt{R}}} \frac{a(|\nabla u|)|\nabla u|^2}{|x|^2} d\mu.$$

Applying Lemma 3.1 with $g = a(|\nabla u|)|\nabla u|^2 e^G$ and $q = 2$, and recalling that

$$\int_{B_R} a(|\nabla u|)|\nabla u|^2 d\mu \leq C_0 R^2$$

for $R$ large, we obtain

$$\int_{\{ \nabla u \neq 0 \} \cap B_R} \left[ \lambda_1 |\nabla T| \nabla u|^2 + a(|\nabla u|) |\nabla u|^2 \sum_{j=1}^{n-1} k_j \right] \varphi^2 \ d\mu \leq \frac{(1 + C) C_0 C_2^2}{\log(R)^2} \left[ 2 \int_{R}^{1/R} \frac{1}{|\tau|} d\tau + 1 \right] \leq 2 \frac{(1 + C) C_0 C_2^2}{\log(R)}.$$

Therefore, sending $R \to +\infty$ in (45) we get

$$k_j(x) = 0 \quad \text{and} \quad |\nabla T| \nabla u|(x) = 0$$

for every $j = 1, \ldots, n - 1$ and every $x \in \{ \nabla u \neq 0 \}$. From this and Lemma 2.11 in [9] we get the one-dimensional symmetry of $u$.
Let us now suppose $n=2$ and $a(|\nabla u|)|\nabla u|^2 e^G \in L^\infty(\mathbb{R}^2)$. Taking in (38) the following test function
\begin{equation}
\varphi(x) = \max\left[0, \min\left(1, \frac{\ln R^2 - \ln |x|}{\ln R}\right)\right],
\end{equation}
recalling that $h \geq \lambda_G$ and following [9, Cor. 2.6], we then obtain
\[\int_{\{\nabla u \neq 0\} \cap B_R} \left[\lambda_1 |\nabla T|\nabla u|^2 + a(|\nabla u|)|\nabla u|^2 \sum_{j=1}^{n-1} k_j^2\right] \varphi^2 \, d\mu \leq C' \int_{B_{R^2} \setminus B_R} a(|\nabla u(x)|)|\nabla u|^2 e^G(x) \, dx\]
for some constant $C' > 0$. When $R \to +\infty$, since $a(|\nabla u|)|\nabla u|^2 e^G(x)$ is bounded, the r.h.s. term of the previous inequality goes to zero, and we conclude again that $u$ is one-dimensional.

Assume now that $u$ is not constant. If we take in (39) the same test functions as above, we get
\[\int_{\mathbb{R}^n} a(|\nabla u|) \left\langle (h(x)I_n - \nabla^2 G(x))\nabla u, \nabla u \right\rangle d\mu(x) = 0.\]
Using the fact that $u(x) = u_0(\langle \omega, x \rangle)$ and $a(t) > 0$ we obtain that $\left\langle (h(x)I_n - \nabla^2 G(x))\omega, \omega \right\rangle = 0$ for all $x$ such that $u_0'(\langle \omega, x \rangle) \neq 0$. Since $u$ is not constant and is a solution to the elliptic equation (1), the set of points such that $u_0'(\langle \omega, x \rangle) = 0$ has zero measure, so, by the regularity of $G$ we conclude that
\[\left\langle (h(x)I_n - \nabla^2 G(x))\omega, \omega \right\rangle = 0 \quad \forall \, x \in \mathbb{R}^n,\]
which gives (14) and (15).

As pointed out in [3], a Liouville type result follows from Theorem 1.

**Corollary 3.2.** Let $G, h, u$ satisfy the assumptions in Theorem 1. Assume further that $h \in C^0(\mathbb{R}^n)$ and $h(x) > \lambda_G(x)$ for some $x \in \mathbb{R}^n$. Then $u$ is constant. In particular, if $u$ is a stable solution, that is $h \equiv 0$, and $\lambda_G(x) < 0$ for some $x \in \mathbb{R}^n$, then $u$ is constant.

In the following lemma we give a sufficient condition for a solution $u$ to satisfy condition (a) in Theorem 1.

**Lemma 3.3.** Let $u$ be a weak solution to (1). Then, for each $\varphi \in C^1_c(\mathbb{R}^n)$,
\begin{equation}
\int_{\mathbb{R}^n} a(|\nabla u|)|\nabla u|^2 \varphi \, d\mu = -\int_{\mathbb{R}^n} a(|\nabla u|) \langle \nabla u, \nabla \varphi \rangle \, d\mu + \int_{\mathbb{R}^n} f(u)u \varphi \, d\mu.
\end{equation}
In particular, if $t \to ta(t) \in L^\infty((0, +\infty))$, $u \in L^\infty(\mathbb{R}^n)$ and $\mu(\mathbb{R}^n) < +\infty$ then there exists $C > 0$ such that
\begin{equation}
\int_{\mathbb{R}^n} a(|\nabla u|)|\nabla u|^2 \, d\mu \leq C.
\end{equation}

**Proof.** Clearly (48) follows by taking $u \varphi$ as test function in (8). Let us show (49). For every $R > 1$ let $\Phi_R \in C^\infty(\mathbb{R})$ be such that $\Phi_R(t) = 1$ if $t \leq R$, $\Phi_R(t) = 0$ if $t \geq R + 1$ and
Φ′_R(t) ≤ 3 for t ∈ [R, R + 1], and define φ(x) := Φ_R(|x|). Then |∇φ(x)| ≤ |Φ′_R(|x|)| ≤ 3, and (48) yields
\[ \int_{B_R} a(|\nabla u|)|\nabla u|^2 d\mu \leq 3 \int_{B_{R+1}\setminus B_R} a(|\nabla u|)|\nabla u||u| d\mu + \int_{B_{R+1}} |f(u)||u| d\mu ≤ C, \]
which gives (49) by letting R → +∞.

In the rest of the section we fix G(x) = −|x|^2/2. We start with a result which follows directly from Lemma 2.3.

**Lemma 3.4.** Let G(x) := −|x|^2/2 and assume that u is a monotone weak solution to (1), i.e. there exists i ∈ {1, ..., n} such that
\[ \partial_i u(x) > 0 \quad \forall x ∈ R^n, \]
then u ∈ C^2(R^n) and u is (−1)−stable.

**Proof of Theorem 2.** We start observing that u is (−1)−stable by Lemma 2.3. Since ∇^2G(x) = −Id we have
\[ −1 = h(x) = λ_G(x) = −1. \]
If a(t) = t^{p-2} for some p > 1 then
\[ λ_1(t) = (p-1)t^{p-2} = (p-1)a(t) \quad \forall t > 0 \]
and the conclusion follows by Theorem 1. If a(t) = (1 + t^q)^{-\frac{2}{q}} with q > 1 then
\[ λ_1(t) = (1 + t^q)^{-\frac{1}{q}} - (1 + t^q)^{-\frac{q+1}{q}} t^q ≤ a(t) \quad \forall t > 0, \]
\[ ta(t) ≤ 1 \quad \forall t > 0. \]
By Lemma 3.3 and (54) there exists C > 0 such that
\[ \int_{R^n} a(|\nabla u|)|\nabla u|^2 d\mu ≤ C. \]
Notice that, if a(t) = 1 for every t > 0, by Theorem [17, Theorem 4.1] we have u ∈ H^2(R^n, µ), so that (55) holds in this case, too. The conclusion follows by (53), (55) and Theorem 1.

4. Solutions with Morse index bounded by the euclidean dimension

In this section we will focus on the Ornstein-Uhlenbeck operator. More precisely, we will consider weak solutions u ∈ H^1(R^n, µ) ∩ L^\infty(R^n) to
\[ Δu − \langle x, \nabla u \rangle + f(u) = 0 \]
where f ∈ C^1(R), and we will prove some new symmetry results for solutions with Morse index k ≤ n. We recall that, by Theorem [17, Theorem 4.1], bounded weak solutions to (56) satisfy u ∈ H^2(R^n, µ) ∩ L^\infty(R^n).
A bounded weak solution $u$ to the Ornstein-Uhlenbeck operator has Morse index $k \in \mathbb{N}$ if $k$ is the maximal dimension of a subspace $X$ of $H^1(\mathbb{R}^n, \mu)$ such that

\begin{equation}
Q_u(\varphi) := \int_{\mathbb{R}^n} |\nabla \varphi|^2 - f'(u)\varphi^2 \, d\mu < 0 \quad \forall \varphi \in X \setminus \{0\}.
\end{equation}

**Remark 4.2.** Let $u$ be a bounded solution to (56) and let $L : H^2(\mathbb{R}^n, \mu) \to L^2(\mathbb{R}^n, \mu)$ be the linear operator defined as

\begin{equation}
L(v) := -\Delta v + \langle \nabla v, x \rangle - f'(u)v.
\end{equation}

Notice that $L$ is self-adjoint in $L^2(\mathbb{R}^n, \mu)$ with compact inverse, so that by the Spectral Theorem [15] there exists an orthonormal basis of $L^2(\mathbb{R}^n, \mu)$ consisting of eigenvectors of $L$, and each eigenvalue of $L$ is real. Then, $u$ has Morse index $k$ if and only if $L$ has exactly $k$ strictly negative eigenvalues, repeated according to their geometric multiplicity (see for instance [17, Theorem 4.1]).

The following Proposition is proved in [2, Lemma 3.2].

**Proposition 4.3.** Let $u$ be a bounded weak solution to (56). If for some $i = 1, \ldots, n$, $u_i$ is not identically zero then it is an eigenfunction of $L$ with eigenvalue $-1$, i.e.

\begin{equation}
\int_{\mathbb{R}^n} \langle \nabla u_i, \nabla \varphi \rangle + u_i \varphi - f'(u)u_i \varphi \, d\mu = 0, \quad \forall \varphi \in H^1(\mathbb{R}^n, \mu).
\end{equation}

We are now in a position to prove Theorem 3.

**Proof of Theorem 3.**

By [17, Theorem 4.1] every bounded weak solution to (56) belongs to $H^2(\mathbb{R}^n, \mu)$, hence $u_i \in H^1(\mathbb{R}^n, \mu)$ for all $i = 1, \ldots, n$. Therefore, using (59) with $u_i$ as test function we obtain

\begin{equation}
Q_u(u_i) = \int_{\mathbb{R}^n} |\nabla u_i|^2 - f'(u)u_i^2 \, d\mu = -\int_{\mathbb{R}^n} u_i^2 \leq 0, \quad \forall i = 1, \ldots, n.
\end{equation}

In particular

\begin{equation}
Q_u(u_i) < 0
\end{equation}

for every $i = 1, \ldots, n$ such that $u_i$ is not identically zero. Let $L$ be the operator defined in (58). If $k = 0$ then $u$ is stable, hence it is constant by Corollary 3.2. If $k = 1$ then, by Remark 4.2 and Proposition 4.3, it follows that $-1$ is the smallest eigenvalue of $L$, that is

\begin{equation}
\inf_{\varphi \in H^1(\mathbb{R}^n, \mu), ||\varphi||_{L^2(\mathbb{R}^n, \mu)} = 1} \left( \int_{\mathbb{R}^n} |\nabla \varphi|^2 - f'(u)\varphi^2 \, d\mu \right) = -1.
\end{equation}

Using (62) it follows that $u$ is $(-1)$--stable and therefore, by Theorem 1, $u$ is one-dimensional. Assume now $2 \leq k \leq n$ and define $S := \{i \in \{1, \ldots, n\} \mid u_i(x) \neq 0$, for some $x \in \mathbb{R}^n\}$ and $X := \text{span}_{i \in S} \{u_i\} \subset H^1(\mathbb{R}^n, \mu)$. Clearly,

\begin{equation}
Q_u(v) < 0 \quad \forall v \in X \setminus \{0\}
\end{equation}

therefore, by Definition 4.1, $X$ has dimension less or equal than $k$, i.e. there exists $I \subset S$ with $|I| \geq |S| - k$ such that $\{u_i\}_{i \in I}$ are linearly dependent [15]. This means that, up to an orthogonal change of variables, $u$ depends on at most $k$ variables. Let us assume by contradiction that $u$ is a function of exactly $k$ variables. We claim that $-1$ is the smallest
eigenvalue of $L$, as before. Indeed, if this is not the case, then there exist $\lambda < -1$ and $v \in H^1(\mathbb{R}^n, \mu)$, with $v \neq 0$, such that $L(v) = \lambda v$. Therefore, by the linear independence of eigenvectors associated to different eigenvalues, it follows that $Y := \text{span}\{u_i, v\}$ has dimension equal to $k + 1$ and $Q_u(w) < 0$ for every $w \in Y \setminus \{0\}$ which is in contradiction with the fact that $u$ has Morse index $k$. This proves that $u$ is a function of at most $(k-1)$ variables, as claimed.

□

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