Landau Level Mixing and Solenoidal Terms in
Lowest Landau Level Currents

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Abstract

We calculate the lowest Landau level (LLL) current by working in the full Hilbert space of a two dimensional electron system in a magnetic field and keeping all the non-vanishing terms in the high field limit. The answer a) is not represented by a simple LLL operator and b) differs from the current operator, recently derived by Martinez and Stone in a field theoretic LLL formalism, by solenoidal terms. Though that is consistent with the inevitable ambiguities of their Noether construction, we argue that the correct answer cannot arise naturally in the LLL formalism.
The microscopic theory of the quantum Hall effect (QHE) [1] is constructed around a large magnetic field (B) expansion which reflects the fact that the QHE is a strong field phenomenon. At integer QHE filling factors (ν) the strong field states are non-degenerate eigenstates of the kinetic energy—filled Landau levels—and further terms in the expansion, obtained by perturbing in the interaction, serve largely to renormalize excitation energies. At the fractional fillings the eigenstates of the kinetic energy are macroscopically degenerate and the lowest order solution requires degenerate perturbation theory in the interaction in a given Landau level. This is the step where the novel physics arises; inclusion of Landau level mixing is, as in the IQHE, qualitatively unimportant. It is worth emphasizing that the irrelevance of LL mixing is really an example of adiabatic continuity, for the ratio \((e^2/\epsilon l)/\bar{\hbar}\omega_c\) (\(l = \sqrt{\hbar c/eB}\) is the magnetic length, \(\hbar\omega_c = eB/mc\) is the cyclotron frequency calculated with the band effective mass and \(\epsilon\) is the background dielectric constant) that, naively, controls the mixing is \(O(1)\) in the experiments and is not negligible [2].

The solution of the degenerate problem for \(\nu < 1\), and by extension for other fractions, is greatly facilitated by restricting the Hilbert space to the lowest Landau level (LLL) and working with projected operators [3]. This procedure is not without its subtleties for restricting the Hilbert space restricts the intermediate states that occur in evaluating the product of a string of operators and hence the product of the projections does not, in general, equal the projection of the product; the simplest instance of this effect is that the projected particle co-ordinates obey the equal time commutator \([x, y] = il^2\). This letter is concerned with another, unphysical, instance that was first noted by Girvin, MacDonald and Platzman [4]. If \(H_L, \rho_L(r)\) and \(j_L(r)\) are the projected Hamiltonian, density and current operators then,

\[
\nabla \cdot j_L(r) = 0 \quad \text{while} \quad \partial_t \rho_L(r) = \frac{i}{\hbar} [H_L, \rho_L(r)] \neq 0
\]

thus violating continuity. These authors noted that for smoothly varying potentials \(V(r)\), \(\partial_t \rho_L(r)\) could be rearranged into the divergence of the drift current \(j_D(r) = (c/B^2)(B \times \nabla V(r))\rho_L(r)\). Subsequently, Sondhi and Kivelson [5] showed that the drift current could be obtained as a LL mixing contribution that survives at arbitrarily high fields. (Readers
unfamiliar with this issue may find it helpful to skim the concluding discussion at this point.)

Recently, Martinez and Stone [6] used a field theory incorporating the LLL constraint to construct a LLL Noether current for the potential scattering problem that (automatically) satisfies continuity. Subsequently, Rajaraman [7] derived the same operator as well as a form valid for interacting systems by working directly with the projected equations of motion as in the usual derivation of the Schrödinger current. In this communication we take a different tack: we calculate the LLL current by working in the full Hilbert space of the system and keeping all the non-vanishing terms in the high field limit. We show that our answer, which is the correct physical current, differs from the results in [6,7] by solenoidal terms that do not appear to arise naturally in the projected problem. Furthermore, the current in an arbitrary state, generically, has a (solenoidal) contribution that is not calculable in advance of diagonalizing the projected Hamiltonian and hence it is not possible to specify a useful form for a current operator in the LLL that reproduces the exact answer. We return to the implications of our results at the end.

**Projection:** We begin by reformulating, following [5], the high field dynamics as a purely LLL problem; this is an ancient technique, we provide details solely in the interests of clarity [8]. The Hamiltonian is,

\[ H = \frac{1}{\lambda} H_K + V \]  

(2)

where the kinetic term,

\[ H_K = \frac{1}{2m} \int d^2 r \ \psi^\dagger(r) \left[ \hbar i \nabla - eA_b(r) \right]^2 \psi(r) \]  

(3)

includes the vector potential \((A_b)\) of the uniform magnetic field while,

\[ V = \int d^2 r \ U(r)\psi^\dagger(r)\psi(r) \]

\[ + \frac{1}{2} \int d^2 r d^2 r' \ \psi^\dagger(r)\psi^\dagger(r')V(r-r')\psi(r')\psi(r) \]  

(4)

includes the interactions as well as any one-body potentials. The parameter \(\lambda\) serves, formally, to organize the large field expansion which is then a small \(\lambda\) expansion. The restriction to the LLL is achieved by constructing a hermitian generator \(T\) such that,
\[ \tilde{H} = e^{iT} H e^{-iT} \]  

has no matrix elements connecting purely LLL states with to those with some higher LL content. At small \( \lambda \), it is reasonable to assume that no level crossings occur among states separated at zeroth order by the cyclotron gap. Hence the eigenstates of \( \tilde{H}_L = P \tilde{H} P \), where \( P \) projects onto the LLL, are the lowest lying eigenstates of the full \( \tilde{H} \). (To avoid littering our equations with factors of \( P \), in the following we shall often denote the restriction of any operator to the LL, obtained by sandwiching an operator between two projectors, by the subscript \( L \).) Finally, the matrix elements of an operator \( O \) in the full Hilbert space are reproduced by those of \( \tilde{O}_L \) where,

\[ \tilde{O} = e^{iT} O e^{-iT}. \]  

To summarize: we trade the fluctuations to higher LLs for modified operators. The full solution is obtained by diagonalizing the restricted problem obtained in this fashion.

The generator \( T \) is constructed as a series in \( \lambda \),

\[ T = \sum_{k=1}^{\infty} \lambda^k T^k \]  

and yields in turn, the series

\[ \tilde{H} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \lambda^k H^k \]  

and

\[ \tilde{O} = \sum_{k=0}^{\infty} \lambda^k O^k. \]

Note that the leading order terms are of order \( \lambda \), \( \lambda^{-1} \), and \( \lambda^0 \) respectively. From Eq. \( \text{(5)} \) we see that the lowest order piece of \( T \) is specified by requiring that,

\[ \langle E | V + i [T^1, H_K] | L \rangle = 0 \]  

for all LLL states \( |L\rangle \) and states with higher LL content \( |E\rangle \); it is fully specified by the further choice \( T^1_L \equiv 0 \). It follows therefore that \( \tilde{H}_L = (\hbar \omega_c / 2\lambda) \tilde{N}_L + V_L + O(\lambda) \) where \( \tilde{N} \) is
the number operator and $\tilde{O}_L = O_L + O(\lambda)$; these are simply the naive projection onto the LLL.

**Current Operator:** We now consider the equation of continuity within this framework. First, a crucial point which was already made in \[5\]. Whereas the expansion in Eq. (9) holds for the density operator $\tilde{\rho}(r)$, the current operator derives from the kinetic energy,

$$\tilde{j}(r) = \frac{1}{\lambda} \delta H_K[A(r)] \bigg|_{A = A_b}$$

and hence its transform has a series,

$$\tilde{\tilde{j}}(r) = \frac{1}{\lambda} \sum_{k=0}^{\infty} \lambda^k \tilde{j}^k(r)$$

that begins with a term of order $1/\lambda$ \[3\]. Hence continuity,

$$-\partial_t \tilde{\rho}(r) \equiv \frac{i}{\hbar} [\tilde{\rho}(r), \tilde{H}] = \nabla \cdot \tilde{j}(r)$$

implies that the naively projected hamiltonian and density generate two equations:

$$\frac{i}{\hbar} [\tilde{\rho}^0_L(r), \tilde{H}_L^0] = \nabla \cdot \tilde{j}_L^0(r)$$

and

$$\frac{i}{\hbar} [\tilde{\rho}_L^1(r), \tilde{H}_L^1] = \nabla \cdot \tilde{j}_L^1(r).$$

As $\tilde{H}_L^0 \propto \hat{N}_L$, Eq. (14) shows that $\tilde{j}_L^0(r) \equiv \tilde{j}_L^1(r)$ is purely solenoidal. We also see that for continuity to hold, we must calculate the current to the lowest non-trivial order of the canonical transformation. Note that while the former is a consequence of the exact degeneracy of the LLL states, the “staggered” structure of the continuity equation is generic to all problems in which we project onto a subspace of the kinetic energy, perhaps in combination with another operator such as a periodic potential. In all such problems we need to compute the current beyond the lowest order.

Using Eq. (10) it is straightforward to write the following expression for the $O(\lambda^0)$ piece of the LLL current operator:

$$\tilde{j}_L^1(r) = P \tilde{j}(r) \frac{1}{\hbar \omega_c \hat{N}/2 - H_K} (1 - P) VP + PV(1 - P) \frac{1}{\hbar \omega_c \hat{N}/2 - H_K} \tilde{j}(r) P$$
Despite appearances, this is a purely LLL object and its magnetic field dependence only involves the magnetic length $l$; we have, however, failed to find a general form in which the sum over intermediate states has been carried out. We may verify directly that this expression obeys Eq. (15). We begin by noting that

\begin{equation}
\nabla \cdot \tilde{j}_L^1(r) = iP[\rho(r), H_K + V] \frac{1}{\hbar \omega_c N/2 - H_K} (1 - P) VP
+ iPV(1 - P) \frac{1}{\hbar \omega_c N/2 - H_K} [\rho(r), H_K + V] P.
\end{equation}

(17)

Using $[\rho(r), H_K + V] = [\rho(r), H_K - \hbar \omega_c N/2]$ we can rewrite the above as

\begin{equation}
\nabla \cdot \tilde{j}_L^1(r) = iP\rho(r)(1 - P) VP - iPV(1 - P)\rho(r) P
\end{equation}

(18)

which is equivalent to Eq. (15).

We have now constructed a current operator $(1/\lambda)\tilde{j}_L^0 + \tilde{j}_L^1$ that, along with the naively projected Hamiltonian and density, obeys the equation of continuity. Unfortunately, we still cannot use this operator in an unrestricted fashion. The problem arises when we try to consistently compute matrix elements to $O(\lambda^0)$ and are forced to keep track of the evolution of the states themselves in response to the $O(\lambda)$ term in $\tilde{H}_L$. For example, if $|\alpha, L\rangle$ are the eigenstates of $\hbar \omega_c N/2 + V_L$ with eigenvalues $\epsilon_\alpha$ and $|\alpha, L\rangle$ the corresponding eigenstates of $\tilde{H}_L$ then,

\begin{equation}
\langle \tilde{\alpha}|\tilde{j}_L^1(r)|\beta\rangle = \langle \alpha|\frac{1}{\lambda}\tilde{j}_L^0(r) + \tilde{j}_L^1(r)|\beta\rangle
+ \sum_{\gamma \neq \alpha} \frac{\langle \alpha|\tilde{H}_L^2|\gamma\rangle \langle \gamma|\tilde{j}_L^1(r)|\beta\rangle}{\epsilon_\alpha - \epsilon_\gamma}
+ \sum_{\gamma \neq \beta} \frac{\langle \alpha|\tilde{j}_L^0(r)|\gamma\rangle \langle \gamma|\tilde{H}_L^2|\beta\rangle}{\epsilon_\beta - \epsilon_\gamma}
\end{equation}

(19)

The extra terms are, as advertised, solenoidal and of a form that require that we diagonalize $V_L$ before we can compute them. We remark that in the full Hilbert space these terms arise from the computation of the first order pieces of the degenerate subspace wavefunctions—in contrast to the non-degenerate case these involve a contribution from within the subspace as well [10].

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Noether Current: We now show that the LLL Noether current differs from our exact high field answer deduced from Eq. (19). We do this by comparing explicit forms for the soluble problem of a linear potential

\[ V = -E \int d^2r \, y \, \psi^\dagger(r) \psi(r) \]  

(20)

The wavefunctions for this problem, in Landau gauge \( A = -By\hat{x} \) and after scaling lengths by \( l \), times by \( \omega_c^{-1} \) and energies by \( \hbar \omega_c \), are

\[ \psi_{ok}(r) = e^{ikx} e^{-(y+k-\lambda E)^2/2} \pi^{1/4} \]  

(21)

and they carry a current

\[ j_{ok}(r) = \hat{x} \left( y + k \right) e^{-(y+k-\lambda E)^2} \sqrt{\pi} \]  

\[ = \hat{x} \left( y + k \right) e^{-(y+k)^2} \sqrt{\pi} \left[ \frac{1}{\lambda} + 2E(y+k) + O(\lambda) \right] \]  

(22)

The expansion to \( O(\lambda) \) can also be derived directly from Eq. (19). In contrast the Noether current of Martinez and Stone [6,7] for a potential \( U(z,\bar{z}) \) is, in their complex co-ordinate notation:

\[ j^\bar{z}(z) = -i \sum_{n=1}^{\infty} \frac{2n}{n!} \partial_z^{n-1} \left[ \rho(z) \partial_z^n U(z,\bar{z}) \right] \]  

\[ j^z(z) = +i \sum_{n=1}^{\infty} \frac{2n}{n!} \partial_{\bar{z}}^{n-1} \left[ \rho(z) \partial_{\bar{z}}^n U(z,\bar{z}) \right] \]  

(23)

where we have ignored the \( O(\lambda^{-1}) \) purely projected piece. In our problem, \( U = E(\bar{z} - z)/2i \) and hence,

\[ j^\lambda(z) = \hat{x} \left( \frac{j^z + j^\bar{z}}{2} + \hat{y} \left( \frac{j^\bar{z} - j^z}{2i} \right) \right) \]  

\[ = \hat{x} E \rho(z). \]  

(24)

Now, within the LLL and in Landau gauge, there is precisely one state with a given x-momentum,

\[ \psi_{ok}(r) = e^{ikx} e^{-(y+k)^2/2} \pi^{1/4} \]  

(25)
which is therefore also an eigenstate of Eq. (20). Its Noether current is

\[ j^N_{\text{ok}}(r) = \hat{x} E e^{-\frac{(y+k)^2}{\pi}} \]  

which is evidently different (see Fig. 1) from the $O(E)$ piece of the exact result in Eq. (22). However, this difference is local; upon integration they both yield the drift current of a single particle.

It is also instructive to compare the various expressions for the currents when the states with $-\infty \leq k < 0$ are filled—a situation that approximates the edge of an ideal $\nu = 1$ state in a more realistic geometry. These have the forms (erfc is the complementary error function

\[ \tilde{j}^N_E(r) = \hat{x} \frac{E}{4\pi} \left[ \text{erfc}(-y) - \frac{2}{\sqrt{\pi}} ye^{-y^2} \right] \]  

for the $O(\lambda^0)$ currents, and

\[ j^E(r) = \hat{x} \frac{E}{4\pi} \text{erfc}(-y - E) + \frac{e^{-\frac{(y-E)^2}{4\pi^2}}}{4\pi^{3/2}} \]  

for the full current inclusive of the $O(\lambda^{-1})$ piece, and are shown in Fig. 2. These illustrate the general features that the drift of a uniform fluid in a slowly varying potential is represented correctly by both $O(\lambda^0)$ forms but that there is additional structure in the exact answer in regions of non-uniform density. An amusing feature of Fig. 2 is that the $O(\lambda^{-1})$ piece of the exact answer dominates the drift, and even changes the sign of the edge current; however, this contribution is non-dynamical and is not the subject of edge state theory \[12\].

Discussion: We begin by stating once more the problem that we have attempted to solve. In the high field limit it is clear that the eigenstates are constructed dominantly out of states in the LLL. However, as the states in the LLL are degenerate in kinetic energy the matrix elements of $\partial_\mu \rho(r) \equiv (i/\hbar)[H_K + V, \rho(r)]$ between them vanish, and hence cannot describe non-solenoidal current flows. The actual evaluation of the current shows that even among solenoidal flows \[13\] we get only the trivial piece arising from density gradients \[4\]. The issue
then is one of obtaining a formal description of the high field limit that allows for movement of charge, and in particular for the drift currents that flow in response to applied potentials, e.g. in a measurement of the Hall conductance!

The most straightforward procedure is to explicitly keep track of the pieces of the wave-functions in higher LLs that are evidently necessary for this task. A more elegant solution is to note that if we restrict even the intermediate states in the evaluation of $[H_K + V, \rho(r)]$ to the LLL, its matrix elements no longer vanish. Finally, we rewrite the commutator as $\nabla \cdot j^N(r)$ to obtain a purely LLL current operator. The current operator obtained in this fashion or equivalently by the Noether construction of Martinez and Stone is, however, ambiguous up to solenoidal terms.

Our canonical transformation analysis was intended to clarify two issues. First, by explicitly keeping track of the entire dynamics in a high-field expansion we hope we were able to make the mildly magical alteration of the commutator somewhat more palatable. Second, as there is a current operator in the full theory whose form is unambiguous on physical grounds, we wanted to check if the Noether current is its high field limit and we have concluded that it is not. Though the difference is solenoidal it produces structure on the scale of $l$ and hence cannot arise naturally in a purely LLL formalism. As we have already remarked, for nearly uniform flows in slowly varying, possibly internally generated, potentials both Eq. (22) and the Noether current reduce to the semi-classical drift—in other situations, however, the former is the correct answer.

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[9] Equivalently, this follows from \( \tilde{j}(\mathbf{r}) = \delta \tilde{H}/\delta \mathbf{A}(\mathbf{r}) \) and the expansion in Eq. (8).

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[12] Edge state theory deals with the coarse-grained drift at the edges or, more properly, with density fluctuations there. For a recent review see, X. G. Wen, Int. J. Mod. Phys., B 6, 1711 (1992).

[13] We note that any time independent flow is solenoidal.

[14] Of course, for calculations which are insensitive to the solenoidal ambiguity, such as the derivation of the edge algebra in [3], the Noether form can be useful.
FIGURES

FIG. 1. Current profiles in eigenstates of the linear potential for $E = 0.1$: high field limit (solid) and LLL Noether construction (dashed), both without the purely projected contribution. The corresponding probability density, in units of $1/l^2$, is numerically simply 10 times the Noether current.

FIG. 2. Current profiles at the non-interacting $\nu = 1$ edge for $E = 0.1$: high field limit (solid), LLL Noether construction (dashed), both without the purely projected contribution, and the exact answer (short-dashes), including the latter. The density is 10 times the Noether current.
