A TWO-WEIGHT INEQUALITY FOR ESSENTIALLY WELL LOCALIZED OPERATORS WITH GENERAL MEASURES

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Abstract. We develop a new formulation of well localized operators as well as a new proof for the necessary and sufficient conditions to characterize their boundedness between $L^2(\mathbb{R}^n, u)$ and $L^2(\mathbb{R}^n, v)$ for general Radon measures $u$ and $v$.

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1. Introduction

We consider the boundedness of the integral operator

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy$$

acting from $L^2(\mathbb{R}^n, u)$ to $L^2(\mathbb{R}^n, v)$, that is, we want to characterize the following inequality

$$\|Tf\|_{L^2(\mathbb{R}^n, v)} \lesssim \|f\|_{L^2(\mathbb{R}^n, u)}$$

for all $f \in L^2(\mathbb{R}^n, u) \equiv \{ f : \int_{\mathbb{R}^n} |f|^2 u < \infty \}$. As is common in two-weight problems, we will consider the change of variables $d\sigma = \frac{1}{u} dx$, $F = \frac{F}{u}$ and $d\omega = v dx$, which allows us to instead characterize the boundedness of the operator $T(\sigma \cdot)$ from $L^2(\mathbb{R}^n, \sigma)$ to $L^2(\mathbb{R}^n, \omega)$, that is, we want to characterize the inequality

$$\|T(\sigma f)\|_{L^2(\mathbb{R}^n, \omega)} \lesssim \|f\|_{L^2(\mathbb{R}^n, \sigma)}.$$  

Nazarov, Treil and Volberg in [5] found necessary and sufficient conditions for this inequality in the case when $T$ is a so called well localized operator. The primary examples of such operators are band operators, the Haar shift, Haar multipliers and dyadic paraproducts as well as perfect dyadic operators.

In this paper we develop a new characterization of well localized operators and provide a new proof showing necessary and sufficient conditions for their boundedness in terms of

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Sawyer type testing on the operator. As in [5], we proceed with an axiomatic approach. Rather than assume that our operator $T$ can be represented as an integral operator (which is not always possible), we instead characterize the operator based on how it behaves on an orthonormal basis of Haar-type functions. However, our behavior of interest is a simple support condition.

For example, let $D$ be the standard dyadic grid in $\mathbb{R}$ and for each dyadic interval $I \in D$, let $I_R$ and $I_L$ denote the right and left halves of the interval, respectively. Define the Haar function $h^0_I \equiv \frac{1}{\sqrt{|I|}}(1_{I_R} - 1_{I_L})$ and the averaging function $h_I^1 \equiv \frac{1}{|I|}1_I$. Then an operator $T$ is said to be lower triangularly localized if there exists a constant $r > 0$ such that for all dyadic intervals $I, J \in D$ with $|I| \leq 2 |J|$, we have

$$\langle T(1_J), h^0_I \rangle = 0$$

if $I \not\subset J^{(r)}$ or if $|I| \leq 2^{-r} |J|$ and $I \not\subset J$. We say that $T$ is well localized if both $T$ and $T^*$ are lower triangularly localized.

Given a sequence $b = \{b_I\}_{I \in D}$ and a function $f \in L^2(\mathbb{R})$ we define the martingale transform

$$T_bf \equiv \sum_{I \in D} b_I \langle f, h^0_I \rangle h^0_I$$

and the paraproduct

$$P_bf \equiv \sum_{I \in D} b_I \langle f, h^1_I \rangle h^0_I.$$

A simple computation shows that these are both well localized with respect to the constant $r = 1$.

We also see that

$$T_b h^0_I = b_I h^0_I \text{ and } T^*_b h^0_I = b_I h^0_I$$

as well as

$$P_b h^0_I = \sum_{J \not\subset I} b_J \langle h^0_I, h_J^1 \rangle h^0_I \text{ and } P^* b h^0_I = b_I h^1_I.$$

Thus these operators satisfy a nice support condition when applied to any Haar function, namely, if $T$ is any of the operators above, we have supp$(T h^0_I) \subset I$.

For an additional example, we let $S$ be the Haar shift operator defined by

$$Sf \equiv \sum_{I \in D} b_I \langle f, h^0_I \rangle (h^0_{I_R} - h^0_{I_L}).$$

Then $S$ is well localized with associated constant $r = 2$, and we also have supp$(Sh^0_I) \subset I$ and supp$(S^* h^0_I) \subset I^{(1)}$, where $I^{(1)}$ denotes the dyadic parent of $I$. In the following section, we formally define this support condition, and in section 5 we show that this condition is in fact the same as the well localized condition up to a change in the constant $r$.

2. Definitions and Statement of Results

To define our orthonormal basis, let $D^n$ denote the dyadic grid in $\mathbb{R}^n$, and for any $F \in D^n$, define $D^*_k(F) \equiv \{ F' \in D^n : F' \subset F, \ell(F') = \frac{1}{2^k} \ell(F) \}$, where $\ell(F)$ denotes the side length of the cube $F$. We further define $D^n(F) = \bigcup_{k=0}^{\infty} D^*_k(F)$. From [6], we have the following lemma.

**Lemma 2.1.** Let $F \in D^n$. Then there are $2^n - 1$ pairs of sets $\{(E_{F,i}^1, E_{F,i}^2)\}_{i=1}^{2^n-1}$ such that
(1) for each $i$, $|E_{F,i}^1| = |E_{F,i}^2|$;
(2) for each $i$, $E_{F,i}^1$ and $E_{F,i}^2$ are non-empty unions of cubes from $D^n(F)$;
(3) for every $i \neq j$, exactly one of the following must hold:
   (a) $E_{F,i}^1 \cup E_{F,j}^2$ is entirely contained in either $E_{F,j}^1$ or $E_{F,j}^2$;
   (b) $E_{F,j}^1 \cup E_{F,i}^2$ is entirely contained in either $E_{F,i}^1$ or $E_{F,i}^2$;
   (c) $(E_{F,i}^1 \cup E_{F,i}^2) \cap (E_{F,j}^1 \cup E_{F,j}^2) = \emptyset$.

For simplicity, we let $E_{F,i} = E_{F,i}^1 \cup E_{F,i}^2$ and we will define
\[
\mathcal{H}^n \equiv \{ E_{F,i} : F \in D^n, 1 \leq i \leq 2^n - 1 \}
\]
to be the collection of all rectangles $E_{F,i}$. We note that for all $i$, $E_{F,i} \subseteq F$, however, $E_{F,i} \not\subseteq \bigcup_{k=1}^{\infty} D^n_k(F)$. We further note that for all $k = 0, 1, \ldots, n$, we have that
\[
F = \bigcup_{i=2^{k-1}}^{2^k - 1} E_{F,i}.
\]

For $r \geq 0$, we define $E_{F,i}^{(r)}$ to be the rectangle of volume $2^r |E_{F,i}|$ containing $E_{F,i}$.

We now define the Haar function $h_{F,i}^0$ and the averaging function $h_{F,i}^1$ associated with $E_{F,i}$ by
\[
h_{F,i}^0 = \frac{1}{\sqrt{|E_{F,i}|}} \left( 1_{E_{F,i}^1} - 1_{E_{F,i}^2} \right)
\]
and
\[
h_{F,i}^1 = \frac{1}{|E_{F,i}|} 1_{E_{F,i}}.
\]
The functions \( \{h_{F,i}^0\}_{F \in D^n, 1 \leq i \leq 2^n - 1} \) form an orthonormal basis for $L^2(\mathbb{R}^n)$.

Given a Radon measure $\sigma$, we let
\[
h_{F,i}^\sigma = \sqrt{\frac{\sigma(E_{F,i}^1)}{\sigma(E_{F,i})\sigma(E_{F,i}^2)}} 1_{E_{F,i}^1} - \sqrt{\frac{\sigma(E_{F,i}^2)}{\sigma(E_{F,i})\sigma(E_{F,i}^1)}} 1_{E_{F,i}^2}
\]
be the weight adapted Haar function if $\sigma(E_{F,i}^1), \sigma(E_{F,i}^2) > 0$ and we set $h_{F,i}^\sigma \equiv 0$ if either $\sigma(E_{F,i}^1) = 0$ or $\sigma(E_{F,i}^2) = 0$.

We will impose the following structure on this operator:

**Definition 2.2.** An operator is said to be essentially well localized if there exists an $r \geq 0$ such that for all $E_{F,i}$ the following properties hold:

\[
\text{supp}(T(\sigma h_{F,i}^\sigma)) \subseteq E_{F,i}^{(r)}; \quad \text{supp}(T^*(\omega h_{F,i}^\omega)) \subseteq E_{F,i}^{(r)}.
\]

We will establish the two-weight boundedness for any Radon measures $\sigma$ and $\omega$ by adapting the proof strategy found in characterizing the two-weight inequality for the Hilbert Transform (see [3] and [4]). We now state our main theorem.

**Theorem 2.3.** Let $T$ be essentially well localized for some $r \geq 0$. Let $\sigma$ and $\omega$ be two Radon measures on $\mathbb{R}^n$. Then

\[
\|T(\sigma f)\|_{L^2(\mathbb{R}^n, \sigma)} \lesssim C \|f\|_{L^2(\mathbb{R}^n, \sigma)}; \quad \|T^*(\omega f)\|_{L^2(\mathbb{R}^n, \sigma)} \lesssim C \|f\|_{L^2(\mathbb{R}^n, \omega)}
\]
if and only if for all $E_{F,i} \in \mathcal{H}^n$ and $E_{G,j} \in \mathcal{H}^n$ with $2^{-r} |E_{F,i}| \leq |E_{G,j}| \leq 2^r |E_{F,i}|$ and $E_{G,j} \cap E_{F,i}^{(r)} \neq \emptyset$, the following testing conditions hold:

\begin{align*}
(2.3) & \quad \|T(\sigma f)\|_{L^2(\mathbb{R}^n, \omega)} \lesssim C \|T\|_{L^2(\mathbb{R}^n, \sigma)}; \\
(2.4) & \quad \|T(\sigma f)\|_{L^2(\mathbb{R}^n, \omega)} \lesssim C \|T\|_{L^2(\mathbb{R}^n, \sigma)}; \\
(2.5) & \quad \|T(\sigma f)\|_{L^2(\mathbb{R}^n, \omega)} \lesssim C \|T\|_{L^2(\mathbb{R}^n, \sigma)}.
\end{align*}

Moreover, we have that $C \simeq C_1 + C_2 + C_3$.

Now because each $E_{F,i} \in \mathcal{H}^n$ is a union of cubes $Q \in \mathcal{D}^n$ and the boundedness of $T$ would imply a similar testing condition on cubes, we immediately have the following result.

**Theorem 2.4.** Let $T$ be essentially well localized for some $r \geq 0$. Let $\sigma$ and $\omega$ be two Radon measures on $\mathbb{R}^n$. Then

\begin{align*}
(2.6) & \quad \|T(\sigma f)\|_{L^2(\mathbb{R}^n, \omega)} \lesssim C \|f\|_{L^2(\mathbb{R}^n, \sigma)}; \\
(2.7) & \quad \|T(\sigma f)\|_{L^2(\mathbb{R}^n, \omega)} \lesssim C \|f\|_{L^2(\mathbb{R}^n, \sigma)}; \\
(2.8) & \quad \|T(\sigma f)\|_{L^2(\mathbb{R}^n, \omega)} \lesssim C \|f\|_{L^2(\mathbb{R}^n, \sigma)}.
\end{align*}

Moreover, we have that $C \simeq C_1 + C_2 + C_3$.

We are also able to easily extract global testing conditions by noting that the boundedness of $T$ immediately implies the global testing conditions (2.10) below, which then imply the local testing conditions (2.3) and (2.5). With this we state another corollary.

**Theorem 2.5.** Let $T$ be essentially well localized for some $r \geq 0$. Let $\sigma$ and $\omega$ be two Radon measures on $\mathbb{R}^n$. Then

\begin{align*}
(2.9) & \quad \|T(\sigma f)\|_{L^2(\mathbb{R}^n, \omega)} \lesssim C \|f\|_{L^2(\mathbb{R}^n, \sigma)}; \\
(2.10) & \quad \|T(\sigma f)\|_{L^2(\mathbb{R}^n, \omega)} \lesssim C \|f\|_{L^2(\mathbb{R}^n, \sigma)}; \\
(2.11) & \quad \|T(\sigma f)\|_{L^2(\mathbb{R}^n, \omega)} \lesssim C \|f\|_{L^2(\mathbb{R}^n, \sigma)}.
\end{align*}

Moreover, we have that $C \simeq C_1 + C_2$.

A similar result can be stated for cubes by (2.4).

**Theorem 2.6.** Let $T$ be essentially well localized for some $r \geq 0$. Let $\sigma$ and $\omega$ be two Radon measures on $\mathbb{R}^n$. Then

\begin{align*}
(2.12) & \quad \|T(\sigma f)\|_{L^2(\mathbb{R}^n, \omega)} \lesssim C \|f\|_{L^2(\mathbb{R}^n, \sigma)}; \\
(2.13) & \quad \|T(\sigma f)\|_{L^2(\mathbb{R}^n, \omega)} \lesssim C \|f\|_{L^2(\mathbb{R}^n, \sigma)}.
\end{align*}

Moreover, we have that $C \simeq C_1 + C_2$. 
3. Initial Reductions

Whenever there is no ambiguity, we will simply write $L^2(\sigma)$ instead of $L^2(\mathbb{R}^n, \sigma)$. We will also write $\sum_{E_F,i}$ rather than $\sum_{F \in D^n} \sum_{i: 1 \leq i \leq 2^n - 1}$. By duality, we will study the pairing $\langle Tf, g \rangle_\omega$, where

$$\langle f, g \rangle_\omega = \int_{\mathbb{R}^n} fg \omega.$$ 

We will also consider the martingale expansions of $f$ and $g$ with respect to $\sigma$ and $\omega$, respectively. Namely, $f = \sum_{E_F,i} \Delta^\sigma_{F,i} f$ and $g = \sum_{E_G,j} \Delta^\omega_{G,j} g$ where $\Delta^\sigma_{E_F,i} f = \hat{f}_\sigma(E_F,i) h^\sigma_{F,i}$.

We first make the assumption that $f$ and $g$ are finite linear combinations of indicator functions $1_{E_F,i}$ where $2^{-d} |Q_0| \leq |E_F,i| \leq |Q_0|$ for some cube $Q_0$ and some $d > 0$. We will obtain our estimates independent of $Q_0$ and $d$ and noting the density of simple functions in $L^2(\sigma)$ will give the result for general $f$ and $g$.

We now want to reduce to considering functions $f$ and $g$ compactly supported on a dyadic cube $Q_0 \in D^n$. To do this, for $1 \leq j \leq 2^n$, let $Q_j \in D^n$ be dyadic cubes in the $j$th orthant, respectively, so that $Q_0 \subseteq \bigcup_j Q_j$. Then we can write $f = \sum_j f 1_{Q_j}$ and similarly for $g$. So $\|f\|_{L^2(\sigma)}^2 = \sum_j \|f 1_{Q_j}\|_{L^2(\sigma)}^2$.

We now have

$$\langle T(\sigma f), g \rangle_\omega = \sum_{i,j} \langle T(\sigma f 1_{Q_i}), g 1_{Q_j} \rangle_\omega.$$ 

Analyzing the terms with $i \neq j$ gives

$$\langle T(\sigma f 1_{Q_i}), g 1_{Q_j} \rangle_\omega = \sum_{E_F,i \cap Q_i \neq \emptyset} \sum_{E_G,j \cap Q_j \neq \emptyset} \langle T(\sigma \Delta^\sigma_{E_F,i} f), \Delta^\omega_{E_G,j} g \rangle_\omega$$

where the cubes $E_{F,i}$ are in the $i$th orthant and the cubes $E_{G,j}$ are in the $j$th orthant. Now by property (2.1), this is zero by support considerations. Thus it suffices to show that we have $|\langle T(\sigma f 1_{Q_j}), g 1_{Q_j} \rangle_\omega| \lesssim C \|f 1_{Q_j}\|_{L^2(\sigma)} \|g 1_{Q_j}\|_{L^2(\omega)}$ because then we have

$$|\langle T(\sigma f), g \rangle_\omega| = \left| \sum_j \langle T(\sigma f 1_{Q_j}), g 1_{Q_j} \rangle_\omega \right| \lesssim C \sum_j \|f 1_{Q_j}\|_{L^2(\sigma)} \|g 1_{Q_j}\|_{L^2(\omega)} \lesssim C \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.$$ 

So with this, we assume $Q_0 \in D^n$. Now we can write

$$f = \sum_{E_F,i \subseteq Q_0} \Delta^\sigma_{E_F,i} f + \langle f \rangle^\sigma_{Q_0} 1_{Q_0},$$

as well as

$$\|f\|_{L^2(\sigma)}^2 = \sum_{E_F,i \subseteq Q_0} \|\Delta^\sigma_{E_F,i} f\|_{L^2(\sigma)}^2 + \|\langle f \rangle^\sigma_{Q_0}\|^2 \sigma(Q_0).$$
where $\langle f \rangle_{E,F,i}^\sigma = \frac{1}{\sigma(E,F,i)} \int_{E,F,i} f \, \sigma$ is the average of $f$ with respect to $\sigma$. With this we have

$$
\langle T(\sigma f), g \rangle_\omega = \sum_{E,F,i \subset Q_0} \left\langle T(\sigma \Delta_{E,F,i}^\sigma f), \Delta_{E,G,j}^\omega g \right\rangle_\omega + \sum_{E,F,i \subset Q_0} \left\langle T(\sigma \Delta_{E,F,i}^\sigma f), \langle g \rangle_{Q_0}^\omega 1_{Q_0} \right\rangle_\omega
$$

$$
+ \sum_{E,G,j \subset Q_0} \left\langle \langle f \rangle_{Q_0}^\sigma T(\sigma 1_{Q_0}), \Delta_{E,G,j}^\omega g \right\rangle_\omega + \left\langle \langle f \rangle_{Q_0}^\sigma T(\sigma 1_{Q_0}), \langle g \rangle_{Q_0}^\omega 1_{Q_0} \right\rangle_\omega.
$$

For the last three terms, we have the following lemma.

**Lemma 3.1.** The following estimates hold:

1. $\left| \sum_{E,F,i \subset Q_0} \left\langle T(\sigma \Delta_{E,F,i}^\sigma f), \langle g \rangle_{Q_0}^\omega 1_{Q_0} \right\rangle_\omega \right| \lesssim C_2 \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)}$;

2. $\left| \sum_{E,G,j \subset Q_0} \left\langle \langle f \rangle_{Q_0}^\sigma T(\sigma 1_{Q_0}), \Delta_{E,G,j}^\omega g \right\rangle_\omega \right| \lesssim C_1 \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)}$;

3. $\left| \left\langle \langle f \rangle_{Q_0}^\sigma T(\sigma 1_{Q_0}), \langle g \rangle_{Q_0}^\omega 1_{Q_0} \right\rangle_\omega \right| \lesssim C_1 \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)}$.

**Proof.** These are immediately controlled by Cauchy-Schwarz and applying the testing hypotheses. We will show the first estimate, and note that the remaining follow similarly.

$$
\left| \sum_{E,F,i \subset Q_0} \left\langle T(\sigma \Delta_{E,F,i}^\sigma f), \langle g \rangle_{Q_0}^\omega 1_{Q_0} \right\rangle_\omega \right| = \left| \langle g \rangle_{Q_0}^\omega \right| \left| \left\langle T \left( \sigma \sum_{E,F,i \subset Q_0} \Delta_{E,F,i}^\sigma f \right), 1_{Q_0} \right\rangle_\omega \right|
$$

$$
\leq \left| \langle g \rangle_{Q_0}^\omega \right| \left\| \sum_{E,F,i \subset Q_0} \Delta_{E,F,i}^\sigma f \right\|_{L^2(\sigma)} \left\| 1_{Q_0} T^\ast(\omega 1_{Q_0}) \right\|_{L^2(\sigma)}
$$

$$
\lesssim C_2 \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)}.
$$

Thus it is enough to control only the first term, so we consider functions $f$ and $g$ that have mean zero with respect to $\sigma$ and $\omega$, respectively.

4. **Proof of Main Theorem**

Throughout the proof, we will use the notation $\Pi(f,g) = \langle T(\sigma f), g \rangle_\omega$. We have

$$
\langle T(\sigma f), g \rangle_\omega = \sum_{E,F,i,E,G,j} \Pi(\Delta_{E,F,i}^\sigma f, \Delta_{E,G,j}^\omega g)
$$

$$
= \left( \sum_{2^{-r} |E,F,i| \leq |E,G,j|} + \sum_{|E,G,j| > 2^r |E,F,i|} + \sum_{|E,F,i| > 2^r |E,G,j|} \right) \Pi(\Delta_{E,F,i}^\sigma f, \Delta_{E,G,j}^\omega g)
$$

$$
= \sum_{k} \sum_{E,F,i} \Pi(\Delta_{E,F,i}^\sigma f, \Delta_{E,F,i,k}^\omega g) + \sum_{E,G,j \subset E_{F,i}^{(r)}} \Pi(\Delta_{E,F,i}^\sigma f, \Delta_{E,G,j}^\omega g)
$$

$$
+ \sum_{E,F,i \subset E_{G,j}^{(r)}} \Pi(\Delta_{E,F,i}^\sigma f, \Delta_{E,G,j}^\omega g)
$$

$$
= A(f,g) + B(f,g) + C(f,g)
$$
where we have used property (2.1) in the third equality to only consider rectangles with containment and where \( E_{F,i,k} \) is the \( k \)th rectangle \( E_{G,j} \) such that \( 2^{-r} |E_{F,i}| \leq |E_{G,j}| \leq 2^r |E_{F,i}| \) and \( \Pi(\Delta^\sigma_{E_{F,i}}, f, \Delta^\omega_{E_{G,j}}, g) \neq 0 \), for some ordering of this finite set. We note that all sets \( E_{F,i,k} \) will be contained in \( E^{(r)}_{F,i} \) and have length at least \( 2^{-r} |E_{F,i}| \), which gives that there are \( M = M(r, n) = \frac{2^n(2r+1) - 1}{2^n - 1} \) such sets. We will consider the first two sums only, as the third sum is symmetric to \( B(f, g) \).

\[
|A(f, g)| = \left| \sum_{k} \sum_{E_{F,i} \in D_n} \Pi(\Delta^\sigma_{E_{F,i}}, f, \Delta^\omega_{E_{F,i,k}}, g) \right| \\
= \left| \sum_{k} \sum_{E_{F,i} \in D_n} \hat{f}_{E_{F,i}}(E_{F,i}) \hat{g}_{E_{F,i,k}}(E_{F,i,k}) \Pi(h^\sigma_{E_{F,i}}, h^\omega_{E_{F,i,k}}, g) \right| \\
\leq \sum_{k} \sup_{E_{F,i}} \left| \Pi(h^\sigma_{E_{F,i}}, h^\omega_{E_{F,i,k}}, g) \right| \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} \\
\lesssim MC_3 \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}
\]

where we have used (2.5) in the final inequality. We now need only estimate

\[
B(f, g) = \sum_{E_{G,j} \supset E^{(r)}_{F,i}} \Pi(\Delta^\sigma_{E_{F,i}}, f, \Delta^\omega_{E_{G,j}}, g).
\]

We now define suitable stopping rectangles in \( H^n \). We initialize our construction with \( S_0 = Q_0 \), and we let \( S = \{S_0\} \). In the inductive step, for a minimal stopping rectangle \( S \), we let \( \text{ch}_S(S) \) be the set of all maximal \( H^n \) children \( S' \) of \( S \) such that the following holds:

(4.1) \[
\frac{1}{\omega(S')} \int_{S'} |g| d\omega > 2 \frac{1}{\omega(S)} \int_S |g| d\omega.
\]

We see immediately that

\[
\sum_{S' \in \text{ch}_S(S)} \omega(S') \leq \frac{1}{2} \omega(S),
\]

which gives us the Carleson condition

\[
\sum_{S \in S, S \subset Q} \omega(S) \leq 2 \omega(Q).
\]

We note that by the well-known dyadic Carleson embedding theorem (see [2]), this condition implies that for all \( g \in L^2(\omega) \)

(4.2) \[
\sum_{S \in S} \omega(S) |\langle g \rangle_S^\omega|^2 \lesssim \|g\|^2_{L^2(\omega)}.
\]

For every cube \( E_{F,i} \subset Q_0 \), we define the stopping parent

\[
\pi E_{F,i} = \min \{ S \in S : S \supset E_{F,i} \}.
\]

Now define the projections

\[
P^\omega_S g = \sum_{E_{G,j} : \pi E_{G,j} = S} \Delta^\omega_{E_{G,j}} g, \quad \hat{P}^\sigma_S f = \sum_{E_{F,i} : \pi E^{(r)}_{F,i} = S} \Delta^\sigma_{E_{F,i}} f.
\]
So \( f = \sum_{S \in \mathcal{S}} \tilde{P}_S^{\sigma} f \), and similarly \( g = \sum_{S \in \mathcal{S}} P_S^{\omega} g \). With this, we have

\[
B(f, g) = \sum_{s, s' \in \mathcal{S}} B(\tilde{P}_S^{\sigma} f, P_{s'}^{\omega} g) \\
= \sum_{s \in \mathcal{S}} B(\tilde{P}_S^{\sigma} f, \sum_{s' \in \mathcal{S}, s' \supset S} P_{s'}^{\omega} g) + \sum_{s, s' \in \mathcal{S}, s' \supset S} B(\tilde{P}_S^{\sigma} f, P_{s'}^{\omega} g) \\
= B_1(f, g) + B_2(f, g).
\]

We note that we do not get any contribution from the stopping cubes \( S \supset S' \) because we are reduced to the case \( E_{F, i}^{(r)} \subsetneq E_{G, j} \). We will now handle \( B_2(f, g) \) first:

\[
B_2(f, g) = \sum_{s, s' \in \mathcal{S}, s' \supset S} B(\tilde{P}_S^{\sigma} f, P_{s'}^{\omega} g) \\
= \sum_{s \in \mathcal{S}} B \left( \tilde{P}_S^{\sigma} f, \sum_{s' \in \mathcal{S}, s' \supset S} P_{s'}^{\omega} g \right) \\
= \sum_{s \in \mathcal{S}} \sum_{E_{G, j} \supset S} \Pi(\tilde{P}_S^{\sigma} f, 1_S \Delta_{E_{G, j}}^{\omega} g) \\
= \sum_{s \in \mathcal{S}} \langle g \rangle_S^{\omega} \Pi(\tilde{P}_S^{\sigma} f, 1_S).
\]

In the third and fourth equalities, we have used \( T(\sigma \tilde{P}_S^{\sigma} f) = 1_S T(\sigma \tilde{P}_S^{\sigma} f) \) by property (2.1) and further that

\[
\langle g \rangle_S^{\omega} 1_S = 1_S \sum_{E_{G, j} \supset S} \Delta_{E_{G, j}}^{\omega} \omega(S) g.
\]

With this we have

\[
|B_2(f, g)| \leq \sum_{S \in \mathcal{S}} \langle g \rangle_S^{\omega} \left\| \tilde{P}_S^{\sigma} f \right\|_{L^2(\omega)} \left\| 1_S T^*(\omega 1_S) \right\|_{L^2(\sigma)} \\
\leq C_2 \left\| f \right\|_{L^2(\sigma)} \left( \sum_{S \in \mathcal{S}} \langle g \rangle_S^{\omega} \omega(S) \right)^{1/2} \\
\lesssim C_2 \left\| f \right\|_{L^2(\sigma)} \left\| g \right\|_{L^2(\omega)}
\]

where the last inequality follows by the Carleson Embedding Theorem.
This leaves us only needing to estimate the term \( B_1(f, g) \). First, we will set \( B_S(f, g) = B(\tilde{P}_S f, P_S g) \). Then we have that \( B_1(f, g) = \sum_{S \in \mathcal{S}} B_S(f, g) \). We now have

\[
B_S(f, g) = \sum_{E_{F_i} \subseteq S, \pi E_{F_i} = \pi E_{G_j} = S} \Pi(\Delta^\sigma_{E_{F_i}}, f, \Delta^\omega_{E_{G_j}})
\]

\[
= \sum_{E_{F_i} \subseteq S, \pi E_{F_i} = S} \Pi \left( \Delta^\sigma_{E_{F_i}} f, \langle g \rangle_{\tau, E_{F_i}}^\omega, \mathbf{1}_{E_{F_i}} \right) - \sum_{E_{F_i} \subseteq S, \pi E_{F_i} = S} \Pi \left( \Delta^\sigma_{E_{F_i}} f, \langle g \rangle_{\tau, E_{F_i}}^\omega \mathbf{1}_{E_{F_i}} \right)
\]

\[
= I - II.
\]

Recalling by the stopping condition (4.1) we have that \( \| \langle g \rangle_{\tau, E_{F_i}} \| \leq \langle \| g \|_{\tau, E_{F_i}} \rangle \leq 2 \langle \| g \|_S \rangle \). With this, we have for the first term

\[
|I| = \left| \sum_{E_{F_i} \subseteq S, \pi E_{F_i} = S} \Pi \left( \Delta^\sigma_{E_{F_i}} f, \langle g \rangle_{\tau, E_{F_i}}^\omega, \mathbf{1}_{E_{F_i}} \right) \right|
\]

\[
\lesssim \sum_{E_{F_i} \subseteq S, \pi E_{F_i} = S} \langle \| g \|_S \rangle \left| \Pi \left( \Delta^\sigma_{E_{F_i}} f, \mathbf{1}_{E_{F_i}} \right) \right|
\]

\[
\lesssim \langle \| g \|_S \rangle \sum_{E_{F_i} \subseteq S, \pi E_{F_i} = S} \left\| \Delta^\sigma_{E_{F_i}} f \right\|_{L^2(\sigma)} \left\| \mathbf{1}_{E_{F_i}} \right\|_{L^2(\omega)}
\]

\[
\lesssim C_2 \langle \| g \|_S \rangle \left( \sum_{E_{F_i} \subseteq S, \pi E_{F_i} = S} \left\| \Delta^\sigma_{E_{F_i}} f \right\|_{L^2(\sigma)}^2 \right)^{1/2} \left( \sum_{E_{F_i} \subseteq S, \pi E_{F_i} = S} \omega(E_{F_i}, i) \right)^{1/2}
\]

\[
\lesssim C_2 \langle \| g \|_S \rangle \omega(S)^{1/2} \left\| \tilde{P}_S f \right\|_{L^2(\sigma)}.
\]
For the second term, we have

\[
|\Pi| \leq \langle |g| \rangle_S^\omega \left| \sum_{E_{F,i} \subseteq S, \pi E_{F,i} = S} \Pi(\Delta_{E_{F,i}}^\sigma, f, 1_S) \right|
\]

\[
= \langle |g| \rangle_S^\omega \Pi \left( \sum_{E_{F,i} \subseteq S, \pi E_{F,i} = S} \Delta_{E_{F,i}}^\sigma, f, 1_S \right)
\]

\[
\leq \langle |g| \rangle_S^\omega \|1_S T^*(\omega 1_S)\|_{L^2(\sigma)} \left\| \tilde{P}_S f \right\|_{L^2(\sigma)}
\]

\[
\lesssim C_2 \langle |g| \rangle_S^\omega \omega(S)^{1/2} \left\| \tilde{P}_S f \right\|_{L^2(\sigma)}.
\]

With this, we have finally that

\[
|B_1(f, g)| = \left| \sum_{S \in \mathcal{S}} B_S(f, g) \right|
\]

\[
\lesssim C_2 \sum_{S \in \mathcal{S}} \langle |g| \rangle_S^\omega \omega(S)^{1/2} \left\| \tilde{P}_S f \right\|_{L^2(\sigma)}
\]

\[
\lesssim C_2 \left\| f \right\|_{L^2(\sigma)} \left( \sum_{S \in \mathcal{S}} \langle |g| \rangle_S^\omega \omega(S) \right)^{1/2}
\]

\[
\lesssim C_2 \left\| f \right\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}
\]

where we again use the Carleson Embedding Theorem in the last inequality.

So we have now established \( C \lesssim C_1 + C_2 + C_3 \). To obtain the other inequality, we notice that we have

\[
C \geq \left\| T \right\|_{L^2(\sigma) \to L^2(\omega)} \sup_{E_{F,i}} \left\| 1_{E_{F,i}} T(\sigma 1_{E_{F,i}}) \right\|_{L^2(\omega)} = C_1.
\]

Similarly, we have \( C \geq \|T\|_{L^2(\sigma) \to L^2(\omega)} \gtrsim C_2 \) and \( C \gtrsim \|T\|_{L^2(\sigma) \to L^2(\omega)} \gtrsim C_3 \). This indeed gives

\[
C \simeq C_1 + C_2 + C_3.
\]

5. Well Localized Operators

We recall from [5] that well localized operators have the following definition:

**Definition 5.1.** \( T \) is said to be lower triangularly localized if there exists a constant \( r > 0 \) such that for all cubes \( R \) and \( Q \) with \( |R| \leq 2 \cdot |Q| \) and for all \( \omega \)-Haar functions on \( R h_R^\sigma \), we have

\[
\langle T(\sigma 1_Q), h_R^\omega \rangle = 0
\]

if \( R \not\subset Q^{(r)} \) or if \( |R| \leq 2^{-r} |Q| \) and \( R \not\subset Q \).

We say that \( T \) is well localized if both \( T \) and \( T^* \) are lower triangularly localized.
We will now show that well localized operators are essentially well localized. Let $T$ be a well localized operator associated with some $r > 0$. Fix a cube $E_{F,i}$ and let $E_{G,j}$ be any cube with $|E_{G,j}| = |E_{F,i}|$ and $E_{F,i}^{(r)} \cap E_{G,j} = \emptyset$. So

$$T(\sigma h_{F,i}^\sigma)_{1_{E_{G,j}}} = \sum_{E_{H,k} \subset E_{G,j}} \Delta^\omega_{H,k} T(\sigma h_{F,i}^\sigma) + \langle T(\sigma h_{F,i}^\sigma) \rangle_{E_{G,j}}^{\omega} 1_{E_{G,j}}.$$ 

Now since $T$ is well localized, it is immediate that $\langle T(\sigma h_{F,i}^\sigma) \rangle_{E_{G,j}}^{\omega} = 0$. So we have

$$T(\sigma h_{F,i}^\sigma)_{1_{E_{G,j}}} = \sum_{E_{H,k} \subset E_{G,j}} \Delta^\omega_{H,k} T(\sigma h_{F,i}^\sigma).$$ 

Now

$$\Delta^\omega_{H,k} T(\sigma h_{F,i}^\sigma) = \frac{\sqrt{\sigma(E_{F,i})}}{\sqrt{\sigma(E_{F,i})} \sigma(E_{F,i})} \Delta^\omega_{H,k} T(\sigma1_{E_{F,i}}) - \frac{\sqrt{\sigma(E_{F,i})}}{\sqrt{\sigma(E_{F,i})} \sigma(E_{F,i})} \Delta^\omega_{H,k} T(\sigma1_{E_{F,i}}).$$ 

For $E_{H,k} \subset E_{G,j}$, we clearly have $|E_{H,k}| \leq 2 |E_{F,i}|$ and $|E_{H,k}| \leq 2 |E_{F,i}|$. So applying the well localized property to each term gives $\Delta^\omega_{H,k} T(\sigma h_{F,i}^\sigma) = 0$.

Now for any rectangle $Q$ with $Q \cap E_{F,i}^{(r)} = \emptyset$, we can write $Q \subset \bigcup Q_k$ where $Q_k \cap E_{F,i}^{(r)} = \emptyset$ and $|Q_k| = |E_{F,i}|$. So we have $T(\sigma h_{F,i}^\sigma)_{1_Q} = 1_Q \sum_k T(\sigma h_{F,i}^\sigma)_{1_{Q_k}} = 0$. A similar calculation shows $T^*(\omega h_{F,i}^\omega)_{1_Q} = 0$. So we have that $T$ is essentially well localized.

This computation also gives the following characterization for essentially well localized operators.

**Theorem 5.2.** An operator $T$ is essentially well localized for some $r \geq 0$ if and only if for all $Q \in D^n$ and $E_{F,i} \in H^n$ with $|E_{F,i}| \leq 2 |Q|$ and $E_{F,i} \not\subset Q^{(r+1)}$, we have

$$\langle T(\sigma 1_Q), h_{F,i}^\omega \rangle = 0$$

and

$$\langle T^*(\omega 1_Q), h_{F,i}^\sigma \rangle = 0.$$

However, if $|E_{F,i}| \leq 2^{-(r+1)} |Q|$ and $E_{F,i} \not\subset Q$, then we have that $E_{F,i}^{(r)} \cap Q = \emptyset$. With this, we immediately have the following characterization of essentially well localized operators.

**Theorem 5.3.** An operator $T$ is essentially well localized for some $r \geq 0$ if and only if $T$ is well localized for $r + 1$.

Having an alternate characterization for well localized operators allows us to easily classify some operators as the following example shows.

**Definition 5.4.** An operator $T$ is said to be an essentially perfect dyadic operator if for some $r \geq 0$,

$$T(\sigma f)(x) = \int_R K(x, y)f(y)\sigma(y)dy$$

for $x \not\in \text{supp}(f)$, where

$$K(x, y) \leq \frac{1}{|x - y|}$$

and

$$|K(x, y) - K(x, y')| + |K(x, y) - K(x', y)| = 0.$$
whenever $x, x' \in I \in \mathcal{D}$, $y, y' \in J \in \mathcal{D}$ where $I^{(r)} \cap J = \emptyset$ and $I \cap J^{(r)} = \emptyset$.

If $r = 0$, we recover the perfect dyadic operators first introduced in [1].

Let $T : L^2(\sigma) \to L^2(\omega)$ be an essentially perfect dyadic operator and let $f = \sigma h_\ell^\sigma$ and $x \notin I^{(r)}$. Then for all $y, y' \in I$, we have $|K(x, y) - K(x, y')| = 0$. So $K(x, \cdot)$ is constant on $I$. With this we have

$$T(\sigma h_\ell^\sigma)(x) = \int_I K(x, y) h_\ell^\sigma(y) \sigma(y)dy = 0.$$

We can write

$$T^*(\omega g)(y) = \int_\mathbb{R} K(x, y) g(x) \omega(x) dx.$$

Now let $g = \omega h_j^\omega$ and let $y \notin J^{(r)}$. Then for all $x, x' \in J$, we have $|K(x, y) - K(x', y)| = |K(x, y) - K(x', y)| = 0$. So $K(\cdot, y)$ is constant on $J$. As above, we have

$$T^*(\omega h_j^\omega)(y) = \int_J K(x, y) h_j^\omega(x) \omega(x) dx = 0.$$

So we have that $T$ is essentially well localized.

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