Justification Logics in a Fuzzy Setting

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Abstract. Justification Logics provide a framework for reasoning about justifications and evidences. Most of the accounts of justification logics are crisp in the sense that agent’s justifications for a statement is convincing or is not. In this paper, we study fuzzy variants of justification logics, in which an agent can have a justification for a statement with a certainty degree between 0 and 1. We replaced the classical base of the justification logics with some known fuzzy logics: Hajek’s basic logic, Łukasiewicz logic, Gödel logic, product logic, and rational Pavelka logic. In all of the resulting systems we introduced fuzzy models (fuzzy possible world semantics with crisp accessibility relation) for our systems, and established the soundness theorems. In the extension of rational Pavelka logic we also proved a graded-style completeness theorem.

Keywords: Justification logic, Fuzzy logic, Rational Pavelka logic.

1 Introduction

Justification logics is a family of logics used to reason about justifications and reasons for our belief or knowledge. Justification logics began with Artemov’s logic of proofs [1, 2], however various systems for justification logics have been introduced in the literature (see e.g. [3, 13]). Justification logics are extensions of classical propositional logic by expressions of the form $t : A$, where $A$ is a formula and $t$ is a justification term. Regarding epistemic models of justification logics (cf. [7, 16]), the formula $t : A$ is interpreted as “$t$ is a justification (or evidence) for $A$”, and some of the justification logics also enjoy arithmetical interpretation of the operator “$:$”, in which $t : A$ can be read as “$t$ is a proof of $A$ in Peano arithmetic.” Note that mathematical proofs are certain justifications for theorems. But in real life reasoning we often deal with uncertain justifications.

It seems the only work that addresses uncertain justifications for belief is Milnikel’s logic of uncertain justifications [14]. He replaced the justification assertions $t : A$ with $t : r A$, where $r \in \mathbb{Q} \cap (0, 1]$, with the intended meaning “I have (at least) degreee $r$ of confidence in the reliability of $t$ as evidence for belief in $A$.” Nevertheless, Milnikel’s logic of uncertain justifications has (two-valued)
classical propositional logic as its logical part. We think that the true logic of uncertain justifications should be based on fuzzy logics. Fuzzy logics are well-known frameworks with the aim of inference under vagueness.\footnote{In this paper, we use the term fuzzy logic in the narrow sense distinguished by Zadeh in [20] (from the broad sense), namely fuzzy logic is a logical system which aims at a formalization of approximate reasoning.}

This paper tries to give a synthesis of justification logic and fuzzy logic (both systems are considered in the propositional level in this paper). We extended justification logic and fuzzy logic to fuzzy justification logic, a framework for reasoning on justifications under vagueness. We replaced the classical base of the justification logic with some known fuzzy logics: Hajek’s basic logic, Lukasiewicz logic, G"odel logic, product logic, and rational Pavelka logic (cf. [10]). In all of the resulting systems we introduced fuzzy models (fuzzy possible world semantics with crisp accessibility relation) for our systems, and established the soundness theorems. In the extension of rational Pavelka logic we also proved a graded-style completeness theorem.

Our systems are t-norm based, that means given an arbitrary t-norm all connectives have associated truth degree functions which are defined from that t-norm (cf. [9]). The language of fuzzy justification logics includes expressions of the form $t : A$, but with a different interpretation. The formula $t : A$ can be interpreted as “$t$ is an evidence for believing $A$ with a certainty degree taken from the interval $[0, 1]$.” In other words, $t$ may be an evidence for $A$ that is not fully convincing and is only plausible to some extent. This allows us to reason about those facts that we do not have complete information or knowledge about them. Moreover, in fuzzy justification logics which are extensions of rational Pavelka logic (see Section 5) we are able to define justification assertions of the form $t : r A$, $t : r A$, and $t : r A$ that can be read respectively as “$t$ is a justification for believing $A$ with certainty degree at least $r$”, “$t$ is a justification for believing $A$ with certainty degree at most $r$”, and “$t$ is a justification for believing $A$ with certainty degree $r$.”

The paper is organized as follows. In Section 2, we review the axioms and rules of the basic justification logic, and its semantics. In Section 3, we review various t-norm based fuzzy logics. In Section 4, we introduce basic fuzzy justification logic and its fuzzy models. We also introduce Lukasiewicz, G"odel, and product justification logics and their models. In Section 5 we will develop a fuzzy justification logic whose language contains truth constants, and prove soundness and graded-style completeness theorems.

## 2 Basic justification logic

In this section, we recall the axiomatic formulation of basic justification logic $J$ and its semantics. The language of $J$ is an extension of the language of propositional logic by expressions of the form $t : A$, where $A$ is a formula and $t$ is a justification term. Justification terms are built up from justification variables
$x_1, x_2, x_3, \ldots$ and justification constants $c, c_1, c_2, \ldots$ using binary operations ‘,’ and ‘+’. More formally, justification terms and formulas of $J$ are formed by the following grammar:

$$t ::= x_i \mid c_i \mid t \cdot t \mid t + t,$$

$$A ::= p \mid \bot \mid A \rightarrow A \mid t : A,$$

where $p$ is a propositional variable and $\bot$ is the truth constant falsity. The set of all propositional variables, all terms and all formulas of $J$ are denoted respectively by $P$, $Tm$, and $Fm_J$.

Next we describe the axioms and rules of $J$.

**Definition 1.** Axiom schemes of $J$ are:

- **PC.** Finite set of axiom schemes for classical propositional calculus,

- **Appl.** $s : (A \rightarrow B) \rightarrow (t : A \rightarrow (s \cdot t) : B)$,

- **Sum.** $s : A \rightarrow (s + t) : A$, $s : A \rightarrow (t + s) : A$.

**Rules of inference:**

- **MP.** Modus Ponens, from $\vdash A$ and $\vdash A \rightarrow B$, infer $\vdash B$.

- **IAN.** Iterated Axiom Necessitation, $\vdash c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A$, where $A$ is an axiom instance of $J$, $c_{i_j}$’s are justification constants and $n \geq 1$.

We now proceed to the definition of Constant Specifications.

**Definition 2.** A constant specification $CS$ for $J$ is a set of formulas of the form $c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A$ ($n \geq 1$), where $c_{i_j}$’s are justification constants and $A$ is an axiom instance of $J$, such that it is downward closed: if $c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A \in CS$, then $c_{i_{n-1}} : \ldots : c_{i_1} : A \in CS$.

Let $J_{CS}$ be the fragment of $J$ where the Iterated Axiom Necessitation rule only produces formulas from the given $CS$.

In the rest of this section we introduce Fitting models for $J$.

**Definition 3.** A Fitting model $\mathcal{M} = (\mathcal{W}, R, \mathcal{E}, \mathcal{V})$ for justification logic $J_{CS}$ (or an $J_{CS}$-model for short) consists of a non-empty set of possible worlds $\mathcal{W}$, an accessibility relation $R$ which is a binary relation on $\mathcal{W}$, i.e. $R \subseteq \mathcal{W} \times \mathcal{W}$, a truth valuation $\mathcal{V} : \mathcal{W} \times P \rightarrow \{0, 1\}$ and an admissible evidence function $\mathcal{E} : \mathcal{W} \times Tm \times Fm_J \rightarrow \{0, 1\}$ meeting the following conditions (we use the notation $\mathcal{E}_w(t, A)$ instead of $\mathcal{E}(w, t, A)$):

$\mathcal{E}1.$ If $\mathcal{E}_w(s, A \rightarrow B) = 1$ and $\mathcal{E}_w(t, A) = 1$, then $\mathcal{E}_w(s \cdot t, B) = 1$.

$\mathcal{E}2.$ If $\mathcal{E}_w(s, A) = 1$, then $\mathcal{E}_w(s + t, A) = \mathcal{E}_w(t + s, A) = 1$.

$\mathcal{E}3.$ If $c : F \in CS$, then $\mathcal{E}_w(c, F) = 1$.

The truth valuation $\mathcal{V}$ extends uniquely to all formulas, i.e. $\mathcal{V} : \mathcal{W} \times Fm_J \rightarrow \{0, 1\}$, as follows (we use the notation $\mathcal{V}_w(A)$ instead of $\mathcal{V}(w, A)$):

1. $\mathcal{V}_w(\bot) = 0$,
2. $\mathcal{V}_w(A \rightarrow B) = 1$ iff $\mathcal{V}_w(A) = 0$ or $\mathcal{V}_w(B) = 1$,
3. \( V_w(t : A) = 1 \) iff \( E_w(t, A) = 1 \) and for every \( v \in \mathcal{W} \) with \( wRv \), \( V_v(A) = 1 \).

We say that a formula \( A \) is \( J_{CS} \)-valid (denoted by \( J_{CS} \models A \)) if for every \( J_{CS} \)-model \( M = (\mathcal{W}, R, E, V) \) and every \( w \in \mathcal{W} \) we have \( V_w(A) = 1 \).

The intended meaning of the admissible evidence function \( E \) is as follows: \( E_w(t, A) = 1 \) means that \( t \) is an admissible justification for \( A \) in \( w \). Thus condition \( V3 \) means: the formula \( t : A \) is true in a world \( w \) of a model \( M \) if and only if \( t \) is an admissible justification for \( A \) in \( w \), and \( A \) is true in all \( w \)-accessible worlds.

The proof of the following theorem can be found in [3].

**Theorem 1.** For a given constant specification \( CS \), the basic justification logic \( J_{CS} \) is sound and complete with respect to their \( J_{CS} \)-models, i.e., \( J_{CS} \models A \) iff \( J_{CS} \vdash A \).

Among various papers on the justification logics, it seems the only work that addresses uncertain justifications for belief is Milnikel’s logic of uncertain justifications \( J^U \) (cf. [14]). He replaced the justification assertions \( t : \tau A \), where \( r \in \mathbb{Q} \cap (0, 1] \), with the intended meaning “I have (at least) degree \( r \) of confidence in the reliability of \( t \) as evidence for belief in \( A \).” Milnikel considered the following principles:

1. \( \tau : r (A \rightarrow B) \rightarrow (t : \tau r, A \rightarrow s \cdot t : \tau r, B) \).
2. \( s : r A \rightarrow s + t : \tau r, A, s : r A \rightarrow t + s : \tau r A \).
3. \( t : \tau r A \rightarrow t : \tau r A, \text{ where } r \leq r' \).

along with the rule IAN with the conclusion: \( \vdash c_{i_n} : 1 c_{i_{n-1}} : 1 \ldots : 1 c_{i_1} : 1 A \).

Note that \( J^U \) is based on the classical propositional logic. Although axioms and rules of Milnikel’s logic is plausible, we think that the true logic of uncertain justifications is based on fuzzy logics. This is the approach we take in the following sections. Particularly, in Section 5 we will develop a fuzzy justification logic whose language contains truth constants, and show that all the principles of \( J^U \) is valid in it.

### 3 T-norm based fuzzy logics

In this section we recall the Hajek’s fuzzy basic logic \( BL \) from [10]. Formulas of \( BL \) are built by the following grammar:

\[
A ::= p | 0 | A \& A | A \rightarrow A,
\]

where \( p \in \mathcal{P} \), \( \& \) is the strong conjunction, and 0 is the truth constant for falsity. Further connectives are defined as follows:

\[
\neg A ::= A \rightarrow 0 \\
A \& B ::= A \& (A \rightarrow B) \\
A \lor B ::= ((A \rightarrow B) \rightarrow B) \lor ((B \rightarrow A) \rightarrow A) \\
A \equiv B ::= (A \rightarrow B) \& (B \rightarrow A) \\
A \leftrightarrow B ::= (A \rightarrow B) \& (B \rightarrow A)
\]

\[\text{for a more detailed exposition of other fuzzy logics consult [8].}\]
Definition 4. Axiom schemes of BL are:

BL1. \((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))\)
BL2. \((A \& B) \rightarrow A\)
BL3. \((A \& B) \rightarrow (B \& A)\)
BL4. \((A \& (A \rightarrow B)) \rightarrow (B \& (B \rightarrow A))\)
BL5a. \((A \rightarrow (B \rightarrow C)) \rightarrow ((A \& B) \rightarrow C)\)
BL5b. \(((A \& B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))\)
BL6. \(((A \rightarrow B) \rightarrow C) \rightarrow (((B \rightarrow A) \rightarrow C) \rightarrow C)\)
BL7. \(0 \rightarrow A\)

Modus Ponens is the only inference rule.

We list here some theorems of BL that are useful in later sections (for proofs cf. [10]).

Lemma 1. The following are theorems of BL:

1. \(1\).
2. \(A \rightarrow (B \rightarrow A)\).
3. \((A \& B) \rightarrow (A \& B)\).
4. \(A \& B \rightarrow A\).
5. \((A \rightarrow B) \rightarrow (A \rightarrow (A \& B))\).
6. \((A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))\).
7. \(((A_1 \rightarrow B_1) \& (A_2 \rightarrow B_2)) \rightarrow ((A_1 \& A_2) \rightarrow (B_1 \& B_2))\).
8. \((A \rightarrow B) \lor (B \rightarrow A)\).

The definition of the classical valuations can easily be extended to a many-valued realm. The standard set of truth degrees is the real interval \([0, 1]\). Now a truth valuation is a mapping \(V: \mathcal{P} \rightarrow [0, 1]\). In order to define the truth functions of the strong conjunction \(\&\) and implication \(\rightarrow\) we need the notion of triangular norm (or t-norm). A binary operation \(*\) on the interval \([0, 1]\) is a t-norm if it is commutative, associative, non-decreasing and 1 is its unit element.

Definition 5. A t-norm is a binary operation \(*: [0, 1]^2 \rightarrow [0, 1]\) satisfying the following conditions. For all \(x, y, z \in [0, 1]\):

\(- \ x \ast y = y \ast x,\)
\(- \ (x \ast y) \ast z = x \ast (y \ast z),\)
\(- \ x \leq y \implies x \ast z \leq y \ast z\)
\(- \ 1 \ast x = x.\)

\(*\) is a continuous t-norm if it is a t-norm and is a continuous mapping of \([0, 1]^2\) into \([0, 1]\).

Example 1. Well-known examples of continuous t-norms are the followings (where \(x, y \in [0, 1]\)):

1. Lukasiewicz t-norm: \(x \ast_L y = \max(0, x + y - 1)\)
2. Gödel t-norm: \(x \ast_G y = \min(x, y)\)
3. **Product t-norm**: \( x \ast y = x \cdot y \)

It is well-known that (see eg. [10]) each continuous t-norm is a combination of Lukasiewicz, Gödel and product t-norms (in a sense).

**Definition 6.** A residuum \( \Rightarrow \) of a continuous t-norm \( \ast \) is defined as follows:\(^3\)

\[
x \Rightarrow y = \max\{z | x \ast z \leq y\},
\]

where \( x, y \in [0, 1] \).

From the definition of residuum, for all \( x, y, z \in [0, 1] \), we have

\[
x \ast z \leq y \text{ iff } z \leq x \Rightarrow y \text{ iff } x \leq z \Rightarrow y.
\]

As it is shown in [10], for each continuous t-norm \( \ast \) and its residuum \( \Rightarrow \), the following holds:

\[
x \Rightarrow y = 1 \text{ iff } x \leq y,
\]

for all \( x, y \in [0, 1] \).

**Example 2.** The residuum of continuous t-norms of Example 1 are the followings (where \( x, y \in [0, 1] \)):

1. **Lukasiewicz implication**:

\[
x \Rightarrow_L y = \begin{cases} 
1, & x \leq y \\
1 - x + y, & x > y
\end{cases}
\]

or, \( x \Rightarrow_L y = \min(1, 1 - x + y) \).

2. **Gödel implication**:

\[
x \Rightarrow_G y = \begin{cases} 
1, & x \leq y \\
y, & x > y
\end{cases}
\]

3. **Product (or Goguen) implication**:

\[
x \Rightarrow_P y = \begin{cases} 
1, & x \leq y \\
\frac{y}{x}, & x > y
\end{cases}
\]

Note that the only continuous implication of Example 2 is the Lukasiewicz implication.

The following lemma is useful in later sections (we leave it to the reader to verify the details of the proof).

**Lemma 2.** For all \( x, x', y, y' \in [0, 1] \), we have:

1. If \( x' \leq x \), then \( x \Rightarrow_L y \leq x' \Rightarrow_L y \).
2. If \( y \leq y' \), then \( x \Rightarrow_L y \leq x \Rightarrow_L y' \).
3. \( (x \Rightarrow_L x') \ast_L (y \Rightarrow_L y') \leq (x \ast_L y) \Rightarrow_L (x' \ast_L y') \).

\(^3\) In fact this is well-defined only if the t-norm \( \ast \) is left-continuous.
Now suppose * is an arbitrary t-norm. One can construct a t-norm based propositional calculus $\text{PC}(\ast)$ (in the same language of $\text{BL}$) and interpret the strong conjunction $\&$ and implication $\rightarrow$ as t-norm $\ast$ and its residuum $\Rightarrow$. Now a truth valuation $\mathcal{V}$ (relative to $\ast$) can be extended to all formulas as follows:

\[
\mathcal{V}(\bar{0}) = 0.
\]

\[
\mathcal{V}(A \rightarrow B) = \mathcal{V}(A) \Rightarrow \mathcal{V}(B).
\]

\[
\mathcal{V}(A \& B) = \mathcal{V}(A) \ast \mathcal{V}(B).
\]

It can be shown that

\[
\mathcal{V}(A \land B) = \min(\mathcal{V}(A), \mathcal{V}(B)).
\]

\[
\mathcal{V}(A \lor B) = \max(\mathcal{V}(A), \mathcal{V}(B)).
\]

**Definition 7.** A formula $A$ is a 1-tautology of $\text{PC}(\ast)$ iff $\mathcal{V}(A) = 1$ for each truth valuation $\mathcal{V}$ of $\text{PC}(\ast)$. A formula $A$ is a t-tautology iff $\mathcal{V}(A) = 1$ for each truth valuation $\mathcal{V}$ and each continuous t-norm $\ast$.

It is shown in [10] that every theorem of $\text{BL}$ is a t-tautology, the converse (completeness of $\text{BL}$) is shown in [5]. Thus $\text{BL}$ is a common base of all the logics $\text{PC}(\ast)$.

**Theorem 2.** A formula $F$ is provable in $\text{BL}$ iff it is a t-tautology.

In the next definition, axiom systems of $\text{PC}(\ast)$ for t-norms of Example 1 are given (cf. [11]).

**Definition 8.** From the basic logic $\text{BL}$ (in the same language of $\text{BL}$) one gets a complete axiomatization of:

1. the /suppress Lukasiewicz logic $\text{L}$ if one adds the axiom scheme:

\[
\neg\neg A \rightarrow A
\]

(L)

2. the G"{o}del logic $\text{G}$ if one adds the axiom scheme

\[
A \rightarrow (A \& A)
\]

(G)

3. the product logic $\Pi$ if one adds the axiom scheme

\[
\neg\neg A \rightarrow ((A \rightarrow (A \& B)) \rightarrow (B \& \neg\neg B))
\]

(P)

**Definition 9.** Let $L$ denote either $\text{L}$, $\text{G}$, or $\Pi$. A formula $A$ is a 1-tautology of $L$ iff $\mathcal{V}(A) = 1$ for each truth valuation $\mathcal{V}$ of $L$.

**Theorem 3.** Let $L$ denote either $\text{L}$, $\text{G}$, or $\Pi$. A formula $F$ is provable in $L$ iff it is a 1-tautology of $L$.

Note that G"{o}del logic $\text{G}$ proves $A \& B \equiv A \land B$. Further, $\text{G}$ is an extension of intuitionistic logic by axiom BL6, or by the linearity axiom $(A \rightarrow B) \lor (B \rightarrow A)$. Moreover, classical propositional logic is equivalent to either $\text{L} \cup \text{G}$, or $\Pi \cup \text{G}$, or $\text{L} \cup \Pi$ (cf. [10]).

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4 A many valued logic is called t-norm based if the truth function of all its connectives are defined from a particular t-norm, using possibly some truth constants (see e.g. [9]).
4 T-norm based justification logics

In this section we introduce t-norm based justification logics. First we consider the basic justification logic $J$ on the bases of fuzzy basic logic $BL$ (instead of classical propositional logic). We call the resulting system $BLJ$. Then by adding axioms $L$, $G$, and $P$ from Definition 8 we obtain extensions of Łukasiewicz, Gödel and product logic.

The language of $BLJ$ is an extension of the language of $BL$ by justification assertions $t : A$, where justification term $t$ is defined by the same grammar as in $J$. Thus, formulas of $BLJ$ are built by the following grammar:

$$A ::= p \mid \bar{0} \mid A \& A \mid A \rightarrow A \mid t : A,$$

where $p \in P$, and $t \in Tm$. Other connectives $\neg, \land, \lor, \equiv, \leftrightarrow$ are defined as in Section 3. $Fm_{BLJ}$ denotes the set of all formulas of $BLJ$.

$BLJ$ are axiomatized by adding axiom schemes Appl and Sum (from Definition 1) to axiom schemes of $BL$ (axioms BL1-BL7 from Definition 4). The rules of inference of $BLJ$ are the same as those of $J$, i.e. MP and IAN (from Definition 1), with the difference that the axiom instance $A$ in the Iterated Axiom Necessitation rule should be now an axiom instance of $BLJ$.

A constant specification $CS$ for $BLJ$ is a set of formulas of the form $c_{i_n} : c_{i_{n-1}} : \ldots : c_{i_1} : A$ ($n \geq 1$), where $c_{i_j}$’s are justification constants and $A$ is an axiom instance of $BLJ$, such that it is downward closed. Let $BLJ_{CS}$ be the fragment of $BLJ$ where the Iterated Axiom Necessitation rule only produces formulas from the given $CS$.

The definition of F-models for $J$ can be easily extended to give models for $BLJ$, by replacing the set of truth values $\{0, 1\}$ to $[0, 1]$. In fact, we define F-models for $BLJ$ in such a way that the truth valuation $V$ and admissible evidence function $E$ in many-valued logic $BLJ$ be a generalization of those in classical two-valued logic $J$.

**Definition 10.** Let $*$ be a t-norm and $\Rightarrow$ be its residuum. A structure $\mathcal{M}_* = (W, R, E, V)$ (relative to $*$) is a fuzzy Fitting model for $BLJ_{CS}$ (or simply is a $BLJ_{CS}$-model), if $W$ is a non-empty set of possible worlds, $R$ is a (two-valued) accessibility relation on $W$, $V$ is a truth valuation of propositional variables in each world $V : W \times P \rightarrow [0, 1]$, and $E$ is a fuzzy admissible evidence function $E : W \times Tm \times Fm_{BLJ} \rightarrow [0, 1]$ satisfying the following conditions:

**FE1.** If $E_w(s, A \rightarrow B) \Rightarrow E_w(t, A) \leq E_w(s \cdot t, B)$.

**FE2.** $E_w(s, A) \leq E_w(t + s, A)$, $E_w(s, A) \leq E_w(s + t, A)$.

**FE3.** If $c : F \in CS$, then $E_w(c, F) = 1$, for every $w \in W$.

The truth valuation $V$ extends uniquely to all formulas of $BLJ$ as follows:

$V1. \quad V_w(\bar{0}) = 0$,

$V2. \quad V_w(A \rightarrow B) = V_w(A) \Rightarrow V_w(B)$,

$V3. \quad V_w(A \& A) = V_w(A)$.

$\quad$ We continue to write $E_w(t, A)$ for $E(w, t, A)$, and $V_w(A)$ for $V(w, A)$. 

8
3. \( V_w(A \& B) = V_w(A) \ast V_w(B) \),
4. \( V_w(t : A) = E_w(t, A) \ast V_w^c(A) \),

where
\[
V_w^c(A) = \inf \{ V_v(A) \mid wRv \}.
\]

A formula \( A \) is **BLJC**-valid if \( V_w(A) = 1 \), for every continuous t-norm \( \ast \) and every **BLJC**-model \( M_* = (W, R, E, V) \) and every \( w \in W \).

Note that conditions **FE1** - **FE3** in the above definition are generalizations of conditions **E1** - **E3** of Definition 3. The intended meaning of fuzzy admissible evidence function \( E \) is as follows: \( t \) is an admissible evidence for \( A \) in \( w \) with the certainty degree \( E_w(t, A) \). Moreover, \( E_w(t, A) = 1 \) means that \( t \) is an admissible evidence for \( A \) in \( w \) (as in the classical case). It seems that conditions **FE1**-**FE3** are meaningful. For example, condition **FE2** means that the certainty degree of \( t + s \) (or \( s + t \)) for \( A \) in \( w \) is greater or at least equal to that of \( t \) for \( A \) in \( w \). In other words, if we strengthen a justification (or reason) of a fact by adding another justification, then the certainty degree of our justification will be increased.

The definition of \( V_w^c \) indeed comes from the evaluation of modal formulas in many-valued modal logics with crisp accessibility relation (see e.g. [6, 19])
\[
V_w(\Box A) = \inf \{ V_v(A) \mid wRv \}.
\]

Note that in many-valued modal logics it is not difficult to show that (see e.g. [19])
\[
V_w(\Box(A \rightarrow B)) \ast V_w(\Box A) \leq V_w(\Box B)
\]
for every world \( w \). This shows that the modal axiom K, \( \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \), is valid in many-valued modal logics with crisp accessibility relation. It is shown in [4] that in general the axiom K is not valid in many-valued modal logics.

Next, we establish the soundness theorem for **BLJ**.

**Theorem 4.** For any constant specification CS for **BLJ**, if \( F \) is provable in **BLJC**\(_\text{CS} \), then \( F \) is **BLJC**\(_\text{CS} \)-valid.

**Proof.** Assume that \( F \) is provable in **BLJC**\(_\text{CS} \). By induction on the proof of \( F \) we show that \( F \) is **BLJC**\(_\text{CS} \)-valid. Let \( \ast \) be an arbitrary t-norm and \( M_* = (W, R, E, V) \) be an arbitrary **BLJC**\(_\text{CS} \)-model relative to \( \ast \) and \( w \) be an arbitrary world in \( W \). We have to show that \( V_w(F) = 1 \). First suppose that \( F \) is an axiom.

- By Theorem 2, all axioms of **BL** are t-tautologies. Thus, the result is known for axioms of **BL**.
- Suppose \( F \) is \( s : (A \rightarrow B) \rightarrow (t : A \rightarrow (s \cdot t) : B) \). By **FE1**, we have
\[
E_w(s, A \rightarrow B) \ast E_w(t, A) \leq E_w(s \cdot t, B).
\]

On the other hand, from (1) we have
\[
V_w^c(A \rightarrow B) \ast V_w^c(A) \leq V_w^c(B).
\]
Hence, by the properties of t-norms
\[ E_w(s, A \to B) * V^\square_w(A \to B) * E_w(t, A) * V^\square_w(A) \leq E_w(s \cdot t, B) * V^\square_w(B). \]
Thus, by \( V4 \),
\[ V_w(s : (A \to B)) * V_w(t : A) \leq V_w(s \cdot t : B), \]
or
\[ V_w(s : (A \to B) \leq V_w(t : A) \Rightarrow V_w(s \cdot t : B), \]
that is
\[ V_w(s : (A \to B) \leq V_w(t : A \to s \cdot t : B). \]
Therefore, \( V(F) = 1. \)

• Suppose \( F \) is \( s : A \to (s + t) : A. \) By \( F \mathcal{E}2, \)
\[ E_w(s, A) \leq E_w(s + t, A). \]
Hence, by the properties of t-norms
\[ E_w(s, A) * V^\square_w(A) \leq E_w(s + t, A) * V^\square_w(A). \]
Thus, by \( V4, \)
\[ V_w(s : A) \leq V_w(s + t : A). \]
Therefore, \( V_w(F) = 1. \) The case for \( s : A \to (t + s) : A \) is similar.

Now suppose \( F \) is obtained from \( G \) and \( G \to F \) by MP. It is easy to verify that if \( V_w(G) = V_w(G \to F) = 1 \) then \( V_w(F) = 1. \)

Finally, suppose \( F = c_i_n : c_i_{n-1} : \ldots : c_i_1 : A \) is obtained by the rule IAN. Thus \( c_i_n : c_i_{n-1} : \ldots : c_i_1 : A \in CS. \) Since \( CS \) is downward closed we have
\[ c_i_j : c_i_{j-1} : \ldots : c_i_1 : A \in CS, \]
for every \( 1 \leq j \leq n. \) Since \( \mathcal{M} \) is a BLJ_{CS}-model, we have
\[ E_w(c_i_j, c_i_{j-1} : \ldots : c_i_1 : A) = 1, \]
for every \( 1 \leq j \leq n. \) It is easy to prove, by induction on \( j, \) that
\[ V_v(c_i_j : c_i_{j-1} : \ldots : c_i_1 : A) = 1, \]
for every \( 1 \leq j \leq n \) and for every \( v \in W. \) Therefore
\[ V_w(c_i_n : c_i_{n-1} : \ldots : c_i_1 : A) = 1. \]

\( \square \)

It remains open whether the converse of the above theorem, that is the completeness theorem, does hold. Let us now introduce Mkrtchyan models (or M-models for short) for BLJ which are singleton fuzzy Fitting models.
Definition 11. Let * be a t-norm and ⇒ be its residuum. A structure \( \mathcal{M}_* = (\mathcal{E}, \mathcal{V}) \) (relative to *) is a fuzzy M-model for BLJCS (or BLJCS-M-model), if \( \mathcal{V} \) is a truth valuation of propositional variables \( \mathcal{V} : \mathcal{P} \rightarrow [0,1] \), and \( \mathcal{E} \) is a fuzzy admissible evidence function \( \mathcal{E} : \mathcal{T}m \times \mathcal{F}m_{\text{BLJ}} \rightarrow [0,1] \) satisfying the following conditions:

1. If \( \mathcal{E}(s, A \rightarrow B) \cdot \mathcal{E}(t, A) \leq \mathcal{E}(s \cdot t, B) \).
2. \( \mathcal{E}(s, A) \leq \mathcal{E}(t + s, A), \mathcal{E}(s, A) \leq \mathcal{E}(s + t, A) \).
3. If \( c : F \in \mathcal{CS} \), then \( \mathcal{E}(c, F) = 1 \).

The truth valuation \( \mathcal{V} \) extends uniquely to all formulas of BLJ as follows:

1. \( \mathcal{V}(\bar{0}) = 0 \),
2. \( \mathcal{V}(A \rightarrow B) = \mathcal{V}(A) \Rightarrow \mathcal{V}(B) \),
3. \( \mathcal{V}(A \& B) = \mathcal{V}(A) \cdot \mathcal{V}(B) \),
4. \( \mathcal{V}(t : A) = \mathcal{E}(t, A) \).

A formula \( A \) is BLJCS-M-valid if \( \mathcal{V}(A) = 1 \) for every continuous t-norm * and every BLJCS-M-model \( \mathcal{M}_* = (\mathcal{E}, \mathcal{V}) \).

The soundness theorem of BLJ with respect to M-models is a consequence of Theorem 4.

Theorem 5. For any constant specification \( \mathcal{CS} \) for BLJ, if \( F \) is provable in BLJCS, then \( F \) is BLJCS-M-valid.

Now let us define extensions of BLJ.

Definition 12. The following logics have the same language as BLJ.

1. LJ is obtained from BLJ by adding the axiom scheme L.
2. GJ is obtained from BLJ by adding the axiom scheme G.
3. ΠJ is obtained from BLJ by adding the axiom scheme P.

In the rest of this section let \( \mathcal{L} \) denote either LJ, GJ, or ΠJ. Note that the axiom instance \( A \) in the Iterated Axiom Necessitation rule in \( \mathcal{L} \) should be now an axiom instance of \( \mathcal{L} \). The definition of constant specifications \( \mathcal{CS} \) for \( \mathcal{L} \) and \( \mathcal{LCS} \) are similar to those of J.

Definition 13. An BLJCS-model \( \mathcal{M}_* = (\mathcal{W}, \mathcal{R}, \mathcal{E}, \mathcal{V}) \) is an

1. LJ-model if the t-norm * is \( *_L \).
2. GJ-model if the t-norm * is \( *_G \).
3. ΠJ-model if the t-norm * is \( *_P \).

A formula \( A \) is LCS valid if \( \mathcal{V}_w(A) = 1 \) for each LCS-model \( \mathcal{M} = (\mathcal{W}, \mathcal{R}, \mathcal{E}, \mathcal{V}) \) and every \( w \in \mathcal{W} \).

Now soundness of LJ, GJ, and ΠJ with respect to their models can be obtained easily from Theorems 3 and 4.

Theorem 6. For a given constant specification \( \mathcal{CS} \) for \( \mathcal{L} \), if \( F \) is provable in LCS, then \( F \) is LCS-valid.

M-models for LJ, GJ, and ΠJ can be defined similar to M-models of BLJ, and the soundness theorem also holds for M-validity.
5 Fuzzy justification logics with truth constants

In classical propositional logic the truth constants ⊤ (truth) and ⊥ (falsity) can be defined respectively by \( p \lor \neg p \) and \( p \land \neg p \), for some propositional variable \( p \). But in many-valued logics (introduced in Section 4) we cannot define all truth values \([0, 1]\) in our object language, and thus they may be added explicitly to the language. A well-known fuzzy logic with truth constants was introduced by Pavelka [18] (for a recent account of this logic see [17]). Hájek [10] gives a simple formulation of this logic, called Rational Pavelka Logic \( \text{RPL} \), in which rational numbers in \([0, 1]\), as truth constants, are added to the language of Lukasiewicz logic \( \mathbf{L} \). In this section, we first recall \( \text{RPL} \), and then extend it with justification assertions, and prove a soundness and a graded-style completeness theorem.

The language of \( \text{RPL} \) is an extension of the language of Lukasiewicz logic \( \mathbf{L} \) by truth constants \( \bar{r} \), for each rational number \( r \in \mathbb{Q} \cap [0, 1] \). More precisely, formulas of \( \text{RPL} \) is constructed by the following grammar:

\[
A ::= p \mid \bar{r} \mid A \land A \mid A \to A,
\]

where \( p \in \mathcal{P} \) and \( r \in \mathbb{Q} \cap [0, 1] \). Formulas of the form \( \bar{r} \to A \) are called graded formulas. Axioms and rules of \( \text{RPL} \) is given by adding the following bookkeeping axiom schemes for truth constants to axioms and rules of \( \mathbf{L} \):

\[
\begin{align*}
\text{TC1.} & \quad (\bar{r} \to \bar{r'}) \equiv r \Rightarrow_L r', \\
\text{TC2.} & \quad (\bar{r} \land r') \equiv r \ast_L r',
\end{align*}
\]

for all \( r, r' \in \mathbb{Q} \cap [0, 1] \).

Note that if \( r \leq r' \), then \( r \Rightarrow_L r' = 1 \). Therefore, by TC1, \((\bar{r} \to \bar{r'}) \equiv \overline{r \Rightarrow_L r'} = 1 \). Since \( \text{RPL} \vdash 1 \), we can conclude that \( \text{RPL} \vdash \bar{r} \to \bar{r'} \).

A truth valuation \( \mathcal{V} \) of formulas of \( \text{RPL} \) is defined similar to that of \( \mathbf{L} \), with the following addition:

\[
\mathcal{V}(\bar{r}) = r
\]

for any \( r \in \mathbb{Q} \cap [0, 1] \). By this definition, it is obvious that a graded formula \( \bar{r} \to A \) is true iff the truth value of \( A \) is at least \( r \), i.e.

\[
\mathcal{V}(\bar{r} \to A) = 1 \text{ iff } r \leq \mathcal{V}(A).
\]

A truth valuation \( \mathcal{V} \) of \( \text{RPL} \) is a model of a finite set of \( \text{RPL} \)-formulas \( T \), if \( \mathcal{V}(A) = 1 \) for all \( A \in T \). The following strong completeness theorem for \( \text{RPL} \) is proved in [10].

**Theorem 7.** Let \( T \) be a finite set of \( \text{RPL} \)-formulas, and \( F \) be an \( \text{RPL} \)-formula. \( \text{RPL} \vdash F \) iff \( F \) is true in every model of \( T \).

Now by adding justification assertions to \( \text{RPL} \), we give a Pavelka based justification logic. We denote this system by \( \text{RPLJ} \). Formulas of \( \text{RPLJ} \) are constructed by the following grammar:

\[
A ::= p \mid \bar{r} \mid A \land A \mid A \to A \mid t : A,
\]

\[ \tag{12} \]
where \( p \in \mathcal{P} \), \( r \in \mathbb{Q} \cap [0, 1] \), and \( t \in Tm \) (here \( Tm \) is the same set of justification terms of basic justification logic \( J \) in Section 2). The connectives \( \neg, \land, \lor, \equiv, \leftrightarrow \) are defined as in BL.

Thanks to truth constants, the following useful operators can be defined in the language of RPLJ:

\[
\begin{align*}
t : r & \quad A := \bar{r} \rightarrow t : A \\
t : r & \quad A := t : A \rightarrow \bar{r} \\
t : r & \quad A := \bar{r} \leftrightarrow t : A = (t : r) \land (t : \bar{r})
\end{align*}
\]

Informally, \( t : r A \), \( t : r A \), and \( t : r A \) can be read respectively as (see Lemma 8)

- “\( t \) is a justification for believing \( A \) with certainty degree at least \( r \)”.
- “\( t \) is a justification for believing \( A \) with certainty degree at most \( r \)”.
- “\( t \) is a justification for believing \( A \) with certainty degree \( r \)”.

Formulas of the form \( t : r A \), \( t : r A \), or \( t : r A \) are called \textit{graded justification assertions}.

**Definition 14.** Pavelka based justification logic RPLJ is given by adding the axiom schemes Sum and Appl (from Definition 1) and the following rule to axioms and rules of RPL:

\[ \text{GIAN. Graded Iterated Axiom Necessitation Rule:} \]

\[ \vdash c_{i_{1}} : c_{i_{n-1}} : \cdots : c_{i_{1}} : A, \]

where \( A \) is an axiom instance of RPLJ, \( c_{ij} \)'s are arbitrary justification constants and \( n \geq 1 \).

Note that in contrast to the language of BLJ, in RPLJ we are able to express explicitly the certainty degree of a justification for a statement. Particularly, in BLJ for an axiom instance \( A \), by rule IAN, we could deduce \( c : A \), for some constant \( c \), which means that \( c \) is an evidence for \( A \) with a certainty degree taken form the interval \([0, 1]\). But in RPLJ, we could deduce \( c : A \) which means that \( c \) is a completely convincing evidence for \( A \), and this is what we expect.

**Definition 15 (Constant specification).** A constant specification \( CS \) is a set of formulas of the form

\[ c_{i_{n}} : c_{i_{n-1}} : \cdots : c_{i_{1}} : A, \]

where \( n \geq 1 \), \( c_{ij} \)'s are arbitrary justification constants and \( A \) is an axiom instance of RPLJ, such that it is downward closed.

\(^6\) A similar approach but in the framework of modal logics was given by Mironov [15]. Mironov presented a fuzzy modal logic with modalities \( \Box_{a} \), where \( a \) is a member of a complete lattice, so that \( \Box_{a} A \) is read “the plausibility measure of \( A \) is equal to \( a \).”

13
Definition 16. A constant specification $CS$ is axiomatically appropriate if for each axiom instance $A$ of RPLJ there is a constant $c$ such that $c : A \in CS$, and in addition if $F \in CS$, then $c : F \in CS$ for some constant $c$.

For a constant specification $CS$, $RPLJ_{CS}$ is a fragment of RPLJ in which the rule GIAN only produces formulas from the given $CS$. For a set of formulas $T$, formula $A$ and constant specification $CS$, we write $T \vdash_{CS} A$ if $A$ is derived from $T$ in $RPLJ_{CS}$.

Lemma 3. Given an arbitrary constant specification $CS$, the following statements are theorems of $RPLJ_{CS}$. For all RPLJ-formulas $A$, all $t \in Tm$, and all $r, r' \in \mathbb{Q} \cap [0, 1]$:

1. $t : 1 \ A$.
2. $t : 0 \ A$.
3. $\neg t : r' \ A \rightarrow t : r A$.
4. $\neg t : r \ A \rightarrow t : r' A$.
5. $t : r A \rightarrow t : r A$, where $r \leq r'$.
6. $t : 1 A \equiv t : 1 A$.
7. $t : 1 A \rightarrow t : 0 A$.
8. $t : r A \lor t : r' A$.

Proof. 1. By Lemma 1 clause 2,

$$\bar{1} \rightarrow (t : A \rightarrow \bar{1}).$$

Since $\bar{1}$ is provable in $RPLJ_{CS}$ (Lemma 1 clause 1), by MP,

$$t : A \rightarrow \bar{1}.$$ 

That is, $t : 1 A$.

2. $t : 0 A$ is an instance of axiom BL7, i.e. $\bar{0} \rightarrow t : A$.

3. Note that $\neg t : r A$ is an abbreviation of $(t : A \rightarrow \bar{r}) \rightarrow \bar{0}$. Now by axiom BL6,

$$(t : A \rightarrow \bar{r}) \rightarrow ((\bar{r} \rightarrow t : A) \rightarrow \bar{0}),$$

or, in our notations, $\neg t : r A \rightarrow \neg t : r A$. Then, by axiom L of Lukasiewicz logic and axiom BL1 using MP, we obtain $\neg t : r A \rightarrow t : r A$.

4. Similar to clause 3.

5. As mentioned above, if $r \leq r'$, then $+_{CS} \bar{r} \rightarrow \bar{r}'$, and therefore by axiom BL1 and MP, we obtain

$$(\bar{r}' \rightarrow t : A) \rightarrow (\bar{r} \rightarrow t : A).$$

That is, $t : r A \rightarrow t : r A$.

6. By Lemma 1 clause 4, $t : 1 A \rightarrow t : 1 A$. For the converse, by Lemma 1 clause 2, we have

$$t : 1 A \rightarrow (t : 1 A \rightarrow t : 1 A).$$
From this and clause 1 using MP, we obtain \( t : 1 \ A \rightarrow t \downarrow A \). Hence, by Lemma 1 clause 5 using MP,

\[ t : 1 \ A \rightarrow (t : 1 \ A \land t : 1 A). \]

That is, \( t : 1 A \rightarrow t \downarrow A \).

7. By clause 6, \( t : 1 A \rightarrow t : 1 A \), or \( t : 1 A \rightarrow (t : 1 A \land t : 1 A) \) is provable in RPLJCS.

By Lemma 1 clause 6,

\[ (t : 1 A \rightarrow (t : 1 A)) \rightarrow (t : 1 A \rightarrow t : 1 A) \]

is provable in RPLJCS, from this by MP we infer \( \overline{t} : 1 A \rightarrow t : 1 A \). And thus, \( t : 1 A \rightarrow t : 1 A \) is provable in RPLJCS.

8. By Lemma 1 clause 8, \((\overline{t} \rightarrow t : A) \lor (t : A \rightarrow \overline{t})\). That is \( t : 1 A \land t : 1 A \).

Lemma 4. Given an arbitrary constant specification CS, the following rules are admissible in RPLJCS. For all RPLJ-formulas A and B, all \( t \in Tm \), and all \( r, r' \in Q \cap [0,1] \):

1. Graded Modus Ponens:

\[
\frac{\overline{r} \rightarrow (A \rightarrow B) \quad \overline{r'} \rightarrow A}{r * L r' \rightarrow B} \quad \text{GMP}
\]

2. Justified Graded Modus Ponens:

\[
\frac{r \rightarrow s : (A \rightarrow B) \quad \overline{r'} \rightarrow t : A}{r * L r' \rightarrow s \cdot t : B} \quad \text{JGMP}
\]

or

\[
\frac{s : r \rightarrow (A \rightarrow B) \quad t : r' \rightarrow A}{s \cdot t : r * L r' \rightarrow B} \quad \text{JGMP}
\]

3. Monotonicity:

\[
\frac{s : r \rightarrow A}{s + t : r \rightarrow A} \quad Mon_1 \quad \frac{s : r \rightarrow A}{t + s : r \rightarrow A} \quad Mon_2
\]

Proof. The proof of clause 1 can be found in [10]. For clause 2, suppose \( \overline{r} \rightarrow s : (A \rightarrow B) \) and \( \overline{r'} \rightarrow t : A \) are provable in RPLJCS. Thus,

\[ (\overline{r} \rightarrow s : (A \rightarrow B)) \land (\overline{r'} \rightarrow t : A) \]

is provable in RPLJCS. Hence, from Lemma 1 clause 7 using MP,

\[ (\overline{r} \land \overline{r'}) \rightarrow (s : (A \rightarrow B) \land t : A). \]

From this and axiom TC2, it follows that

\[ r * L r' \rightarrow (s : (A \rightarrow B) \land t : A). \]
Moreover, from axiom Appl and BL5a we deduce that

\[(s : (A \to B) \& t : A) \to s \cdot t : B.\]

Therefore, \(r \cdot Lr' \to s \cdot t : B \).

For clause 3, suppose that \(s \vdash A\), i.e. \(r \to s : A\), is provable in \(RPLJ_{CS}\). Now from this and axiom BL1, we obtain \(r \to s + t : A\), or \(s + t \vdash A\).

From rules JGMP, Mon1, and Mon2, it is immediately follows that the following rules are also admissible in \(RPLJ_{CS}\):

\[
\frac{s : (A \to B)}{s \cdot t : r \cdot Lr' B \quad JGMP'},
\frac{s \vdash A}{s \cdot t : r \cdot r' A \quad Mon_1'},
\frac{s \vdash A}{t + s : r \cdot r' A \quad Mon_2'}.\]

The following form of the Deduction Theorem holds in \(RPLJ\). The proof is similar to that given in [10] for BL.

**Lemma 5 (Deduction Theorem).** \(T, A \vdash_{CS} B\) iff \(T \vdash_{CS} A^n \to B\) for some \(n\), where \(A^n := A \& \ldots \& A\), \(n\) times.

Note that the ordinary deduction theorem, \(T, A \vdash B\) iff \(T \vdash A \to B\), only holds in Gödel justification logic \(GJ\) (for more details cf. [10]).

**Lemma 6 (Lifting Lemma).** Given axiomatically appropriate constant specification \(CS\), if

\[A_1, \ldots, A_n \vdash_{CS} F,\]

then there is a justification term \(t(x_1, \ldots, x_n)\), for variables \(x_1, \ldots, x_n\), such that

\[x_1 : A_1, \ldots, x_n : A_n \vdash_{CS} t(x_1, \ldots, x_n) : F.\]

**Proof.** The proof is by induction on the derivation of \(F\). We have two base cases:

- If \(F\) is an axiom, then there is a justification constant \(c\) such that \(c : F \in CS\).
  
  Put \(t := c\), and use IAN to obtain \(c \vdash F\).

- If \(F = A_i\), then put \(t := x_i\).

For the induction step we have two cases.

- Suppose \(F\) is obtained by Modus Ponens from \(G \to F\) and \(G\). By the induction hypothesis, there are terms \(u(x_1, \ldots, x_n)\) and \(v(x_1, \ldots, x_n)\) such that

  \[x_1 : A_1, \ldots, x_n : A_n \vdash_{CS} u(x_1, \ldots, x_n) \vdash (G \to F),\]

  and

  \[x_1 : A_1, \ldots, x_n : A_n \vdash_{CS} v(x_1, \ldots, x_n) \vdash G.\]

  Then put \(t := u \cdot v\), and use the rule JGMP' to obtain

  \[x_1 : A_1, \ldots, x_n : A_n \vdash_{CS} u(x_1, \ldots, x_n) \cdot v(x_1, \ldots, x_n) : F,\]

  which, by Lemma 3 clause 6, implies that

  \[x_1 : A_1, \ldots, x_n : A_n \vdash_{CS} u(x_1, \ldots, x_n) \cdot v(x_1, \ldots, x_n) : F.\]
Suppose \( F \) is obtained from the rule GIAN, so \( F = c_{i_0} : c_{i_{m-1}} : \ldots : c_{i_1} : B \in CS \), for some axiom instance \( B \). Then since \( CS \) is axiomatically appropriate, there is a justification constant \( c \) such that \( c : c_{i_0} : c_{i_{m-1}} : \ldots : c_{i_1} : B \in CS \).

Put \( t := c \).

\[ \square \]

**Lemma 7 (Internalization Lemma).** Given an axiomatically appropriate constant specification \( CS \), if \( \vdash_{CS} F \), then there is a justification term \( t \) such that \( \vdash_{CS} t : F \).

**Proof.** Special case of Lemma 6. \[ \square \]

Next we introduce fuzzy Fitting models for RPLJ.

**Definition 17.** A structure \( M = (\mathcal{W}, R, E, V) \) is a fuzzy Fitting model for RPLJ\(_{CS} \) (or simply a RPLJ\(_{CS} \)-model), if it is a \( L \)-model such that \( E \) is a fuzzy admissible evidence function \( E : \mathcal{W} \times Tm \times Fm_{RPLJ} \to [0, 1] \) satisfying the following conditions:

- **FE1.** If \( E_w(s, A \rightarrow B) \ast_L E_w(t, A) \leq E_w(s \cdot t, B) \).
- **FE2.** \( E_w(s, A) \leq E_w(t + s, A), E_w(s, A) \leq E_w(s + t, A) \).
- **FE3.** If \( c : F \in CS \), then \( E_w(c, F) = 1 \), for every \( w \in \mathcal{W} \).

The truth valuation \( V \) extends uniquely to all formulas of BLJ as follows:

- **V1.** \( V_w(\overline{r}) = r \), for all \( r \in \mathbb{Q} \cap [0, 1] \).
- **V2.** \( V_w(A \rightarrow B) = V_w(A) \Rightarrow_L V_w(B) \).
- **V3.** \( V_w(A \& B) = V_w(A) \ast_L V_w(B) \).
- **V4.** \( V_w(t : A) = E_w(t, A) \ast_L V_w(A) \).

where

\[ V_w(A) = \inf \{ V_v(A) \mid wRv \} \]

**Definition 18.** – For a RPLJ\(_{CS} \)-model \( M = (\mathcal{W}, R, E, V) \) and a set of formulas \( T \), \( M \) is a model of \( T \) if \( V_w(F) = 1 \), for all formulas \( F \in T \) and all \( w \in \mathcal{W} \).

– We say that a formula \( F \) is a semantic consequence of \( T \) in RPLJ\(_{CS} \), written \( T \models_{CS} F \), if every RPLJ\(_{CS} \)-model of \( T \) is a RPLJ\(_{CS} \)-model of \( F \) (or more precisely of \( \{ F \} \)).

– A formula \( F \) is RPLJ\(_{CS} \)-valid if \( \models_{CS} F \).

For a model \( M = (\mathcal{W}, R, E, V) \) sometimes we write \( w \in M \) instead of \( w \in \mathcal{W} \).

The truth valuation of graded justification assertions are as follows:

\[ V_w(t : r : A) = r \Rightarrow_L V_w(t : A) \quad \text{(2)} \]
\[ V_w(t : \not r : A) = V_w(t : A) \Rightarrow_L r \quad \text{(3)} \]
\[ V_w(t : r : A) = \min(V_w(t : r : A), V_w(t : \not r : A)) \quad \text{(4)} \]

**Lemma 8.** Let \( M \) be an arbitrary RPLJ\(_{CS} \)-model and \( w \in M \) be an arbitrary world. Then, the following holds:
This completes the base case. The induction step is routine.

Since $A$ is an axiom, $\mathcal{V}_w(t : A) = 1$ for every $t$. Thus, the case where $A$ is an axiom or is obtained from MP is similar to the proof of Theorem 4 (in particular, it is easy to show that axioms TC1 and TC2 are RPLJCS-valid).

Suppose $F = c_{i_0} \mathbin{1} : c_{i_{n-1}} \mathbin{1} : \ldots : c_{i_1} \mathbin{1} : A$ is obtained by the rule GIAN. Thus $c_{i_0} \mathbin{1} : c_{i_{n-1}} \mathbin{1} : \ldots : c_{i_1} \mathbin{1} : A \in \text{CS}$. Since CS is downward closed we have

$$c_{i_j} \mathbin{1} : c_{i_{j-1}} \mathbin{1} : \ldots : c_{i_1} \mathbin{1} : A \in \text{CS},$$

for every $1 \leq j \leq n$. Let $\mathcal{M}$ be a BLJCS-model, and $w \in \mathcal{M}$. From $\mathcal{F}\mathcal{E}3$ we have

$$\mathcal{E}_w(c_{i_j}, c_{i_{j-1}} \mathbin{1} : \ldots : c_{i_1} \mathbin{1} : A) = 1,$$

for every $1 \leq j \leq n$.

It is not difficult to prove, by induction on $j$, that

$$\mathcal{V}_v(c_{i_j} : c_{i_{j-1}} \mathbin{1} : \ldots : c_{i_1} \mathbin{1} : A) = 1,$$

for every $1 \leq j \leq n$ and for every $v \in \mathcal{W}$. We only check the base case $j = 1$. Suppose $c \mathbin{1} : A \in \text{CS}$, and $v \in \mathcal{W}$. We want to show that $\mathcal{V}_v(c \mathbin{1} : A) = 1$. It suffices to prove that $\mathcal{V}_v(c \mathbin{1} : A) = \inf \{\mathcal{V}_w(A) : v \mathbin{R} u\}$. Since $A$ is an axiom, $\mathcal{V}_w(A) = 1$ for all $u \in \mathcal{W}$. Thus, $\mathcal{V}_v(c : A) = 1$. Hence,

$$\mathcal{V}_v(c : A) = 1 \Rightarrow \mathcal{V}_v(c : A) = 1 \Rightarrow L 1 = 1,$$

$$\mathcal{V}_v(c : A) = \mathcal{V}_v(c : A) \Rightarrow L 1 = 1 \Rightarrow L 1 = 1.$$
Next we show that all principles of Milnikel’s logic of uncertain justifications ([14]) are valid in RPLJ.

**Lemma 9.** Given an arbitrary constant specification CS, the following statements are RPLJCS-valid:

1. \( s : (A \rightarrow B) \rightarrow (t : r A \rightarrow s \cdot t : r s L r' B) \).
2. \( s : r A \rightarrow s + t : r A, s : r A \rightarrow t + s : r A \).
3. \( t : r A \rightarrow t : r A \), where \( r \leq r' \).

**Proof.** Let \( \mathcal{M} \) be an arbitrary RPLJCS-model and \( w \in \mathcal{M} \) be an arbitrary world.

1. From the proof of soundness theorem we have
   \[
   \mathcal{V}_w(s : (A \rightarrow B)) \ast_L \mathcal{V}_w(t : A) \leq \mathcal{V}_w(s \cdot t : B).
   \]
   By Lemma 2 (clause 2),
   \[
   (r \ast_L r') \Rightarrow_L (\mathcal{V}_w(s : (A \rightarrow B)) \ast_L \mathcal{V}_w(t : A)) \leq (r \ast_L r') \Rightarrow_L \mathcal{V}_w(s \cdot t : B).
   \]
   By Lemma 2 (clause 3),
   \[
   (r \Rightarrow_L \mathcal{V}_w(s : (A \rightarrow B)) \ast_L (r' \Rightarrow_L \mathcal{V}_w(t : A)) \leq (r \ast_L r') \Rightarrow_L \mathcal{V}_w(s \cdot t : B).
   \]
   That is
   \[
   \mathcal{V}_w(s : r (A \rightarrow B)) \ast_L \mathcal{V}_w(t : r' A) \leq \mathcal{V}_w(s \cdot t : r s L r' B).
   \]
   Hence,
   \[
   \mathcal{V}_w(s : r (A \rightarrow B)) \leq \mathcal{V}_w(t : r' A) \Rightarrow_L \mathcal{V}_w(s \cdot t : r s L r' B).
   \]
   Thus,
   \[
   \mathcal{V}_w(s : r (A \rightarrow B) \rightarrow (t : r' A \rightarrow s \cdot t : r s L r' B)) = 1.
   \]

2. From \( F \mathcal{E}2 \) we have
   \[
   \mathcal{E}_w(s, A) \leq \mathcal{E}_w(t + s, A),
   \]
   \[
   \mathcal{E}_w(s, A) \leq \mathcal{E}_w(s + t, A).
   \]
   From these and properties of t-norms we obtain
   \[
   \mathcal{E}_w(s, A) \ast_L \mathcal{V}_w^\square(A) \leq \mathcal{E}_w(t + s, A) \ast_L \mathcal{V}_w^\square(A),
   \]
   \[
   \mathcal{E}_w(s, A) \ast_L \mathcal{V}_w^\square(A) \leq \mathcal{E}_w(s + t, A) \ast_L \mathcal{V}_w^\square(A).
   \]
   Thus,
   \[
   \mathcal{V}_w(s : A) \leq \mathcal{V}_w(t + s : A),
   \]
   \[
   \mathcal{V}_w(s \cdot A) \leq \mathcal{V}_w(s + t : A).
   \]
   By Lemma 2 (clause 2),
   \[
   r \Rightarrow_L \mathcal{V}_w(s, A) \leq r \Rightarrow_L \mathcal{V}_w(t + s, A),
   \]
   \[
   r \Rightarrow_L \mathcal{V}_w(s, A) \leq r \Rightarrow_L \mathcal{V}_w(s + t, A).
   \]
   That are
   \[
   \mathcal{V}_w(s : r A) \leq \mathcal{V}_w(t + s : r A),
   \]
   \[
   \mathcal{V}_w(s \cdot r A) \leq \mathcal{V}_w(s + t : r A).
   \]
Lemma 12 (Truth Lemma). Let $F$ be a propositional variable. The case when $\neg F$ follows from Definition 21. If $F = \neg \neg F$ and $S \in W^c$, then $V^c_S(\neg F) = r$, and by definition of provability degree $|\neg F|^*_S = r$. Thus

$$V^c_S(\neg F) = |\neg F|^*_S = r.$$ 

Definition 19. Let $T$ be a set of formulas, and $A$ be a formula.

1. Truth degree of $A$ over $T$ respecting $CS$ is defined as follows:

$$\|A\|^*_T := \inf\{V_w(A)|M \text{ is a RPLJ}_{CS}-model of } T \text{ and } w \in M\}.\)

2. Provability degree of $A$ over $T$ respecting $CS$ is defined as follows:

$$|A|^*_T := \sup\{r|T \vdash^*_{CS} \bar{r} \rightarrow A\}.$$ 

Definition 20. Let $S$ be a set of formulas. $S$ is $CS$-consistent if $S \not\vdash^*_{CS} \bot$. $S$ is $CS$-complete if $S \vdash^*_{CS} A \rightarrow B$ or $S \vdash^*_{CS} B \rightarrow A$, for each pair of formulas $A, B$.

The proof of the following lemmas are similar to that given in [10] (Lemmas 3.3.7 and 3.3.8).

Lemma 10. If $S \vdash^*_{CS} \bar{r} \rightarrow A$, then $S \cup \{A \rightarrow \bar{r}\}$ is $CS$-consistent.

Lemma 11. Let $S$ be a $CS$-consistent and $CS$-complete set of formulas. The following properties hold:

1. $|A \rightarrow B|^*_S = |A|^*_S \Rightarrow L |B|^*_S$.
2. $|A \wedge B|^*_S = |A|^*_S \wedge L |B|^*_S$.

Moreover, it can be shown that every $CS$-consistent set of formulas can be extended to a $CS$-consistent and $CS$-complete set.

Definition 21 (Canonical model). The canonical model $M^c = (W^c, R^c, \mathcal{E}^c, \mathcal{V}^c)$ for RPLJ$_{CS}$ is defined as follows:

1. $W^c = \{S|S \text{ is a CS-consistent and CS-complete set}\}$.
2. $S_1 \mathcal{R}^c S_2$ iff $S_1^c = \{A|t : A \in S_1\} \subseteq S_2$, for $S_1, S_2 \in W^c$.
3. $\mathcal{E}^c_S(t, A) := \{t : A|^*_S\}$, for $S \in W^c$.
4. $\mathcal{V}^c_S(p) := |p|^*_S$, for $S \in W^c$ and $p \in \mathcal{P}$.

Lemma 12 (Truth Lemma). Let $M^c = (W^c, R^c, \mathcal{E}^c, \mathcal{V}^c)$ be the canonical model of RPLJ$_{CS}$. For every formula $F$ and every $S \in W^c$, if $F \in S$, then

$$\mathcal{V}^c_S(F) = |F|^*_S = 1.$$ 

Proof. First note that for every formula $F$ in $S$, we have $S \vdash^*_{CS} \bar{1} \rightarrow F$. Hence $|F|^*_S = 1$. Next by induction on the complexity of $F$ we show that for every $S \in W^c$ if $F \in S$, then

$$\mathcal{V}^c_S(F) = |F|^*_S.$$ 

The case when $F$ is a propositional variable $p$ follows from Definition 21. If $F = \bar{r}$ and $S \in W^c$, then $\mathcal{V}^c_S(\bar{r}) = r$, and by definition of provability degree $|\bar{r}|^*_S = r$. Thus

$$\mathcal{V}^c_S(\bar{r}) = |\bar{r}|^*_S = r.$$ 

20
The cases for which the main connective of \( F \) is \& or \( \to \) follows from Lemma 11. Finally, for \( S \in \mathcal{W}^c \) if \( F = t : A \) is in \( S \), then \( |t : A|_S^{\mathcal{CS}} = 1 \). Furthermore, for any \( S' \in \mathcal{W} \) such that \( SR^cS' \), by definition of \( R^c \), we have \( A \in S' \). By the induction hypothesis, \( \mathcal{V}_S^c(A) = |A|_S^{\mathcal{CS}} = 1 \). Thus,

\[
\mathcal{V}_S^c(A) = \inf \{ \mathcal{V}_{S'}^c(A) \mid SR^cS' \} = 1.
\]

Therefore,

\[
\mathcal{V}_S^c(t : A) = E_S^c(t, A) *_L \mathcal{V}_S^c(A) = |t : A|_S^{\mathcal{CS}} *_L \mathcal{V}_S^c(A) = 1 *_L 1 = 1.
\]

Hence, \( \mathcal{V}_S^c(t : A) = |t : A|_S^{\mathcal{CS}} \). \( \square \)

**Lemma 13.** The canonical model \( \mathcal{M}^c = (\mathcal{W}^c, R^c, \mathcal{E}^c, \mathcal{V}^c) \) is a model of RPLJcs.

**Proof.** It suffices to show that \( \mathcal{E}^c \) satisfies conditions \( F\mathcal{E}1 - F\mathcal{E}3 \). Suppose \( S \) is an arbitrary world in \( \mathcal{W}^c \).

- **Condition \( F\mathcal{E}1 \):** let

\[
X = \{ r \mid S \vdash_{\mathcal{CS}} \bar{r} \rightarrow s : (A \rightarrow B) \},
\]

\[
Y = \{ r \mid S \vdash_{\mathcal{CS}} \bar{r} \rightarrow t : A \},
\]

and

\[
Z = \{ r \mid S \vdash_{\mathcal{CS}} \bar{r} \rightarrow s \cdot t : B \}.
\]

It is easy to show that \( X \cap Y \subseteq Z \), and hence \( \sup(X \cap Y) \leq \sup(Z) \). On the other hand, one can verify that \( X \) and \( Y \) have the following property: if \( r \) is in \( X \) (or \( Y \)) and \( r' \leq r \), then \( r' \) is in \( X \) (or \( Y \)). Therefore, \( X \subseteq Y \) or \( Y \subseteq X \). If \( X \subseteq Y \), then \( \sup(X \cap Y) = \sup(X) \leq \sup(Z) \), which yields

\[
|s : (A \rightarrow B)|_S^{\mathcal{CS}} \leq |s \cdot t : B|_S^{\mathcal{CS}}.
\]

Since \( x *_L y \leq x \), for all \( x, y \in [0, 1] \), we have

\[
|s : (A \rightarrow B)|_S^{\mathcal{CS}} *_L |t : A|_S^{\mathcal{CS}} \leq |s \cdot t : B|_S^{\mathcal{CS}}.
\]

Hence

\[
E_S^c(s, A \rightarrow B) *_L E_S^c(t, A) \leq E_S^c(s \cdot t, B).
\]

If \( Y \subseteq X \), then \( \sup(X \cap Y) = \sup(Y) \leq \sup(Z) \), which yields

\[
|t : A|_S^{\mathcal{CS}} \leq |s \cdot t : B|_S^{\mathcal{CS}}.
\]

Since \( x *_L y \leq y \), for all \( x, y \in [0, 1] \), we have

\[
|s : (A \rightarrow B)|_S^{\mathcal{CS}} *_L |t : A|_S^{\mathcal{CS}} \leq |s \cdot t : B|_S^{\mathcal{CS}}.
\]

Hence

\[
E_S^c(s, A \rightarrow B) *_L E_S^c(t, A) \leq E_S^c(s \cdot t, B).
\]

21
- Condition $FE2$: it is easy to show that
\[
\{r\|S \vdash_{CS} \bar{r} \rightarrow s : A\} \subseteq \{r\|S \vdash_{CS} \bar{r} \rightarrow s + t : A\},
\]
\[
\{r\|S \vdash_{CS} \bar{r} \rightarrow s : A\} \subseteq \{r\|S \vdash_{CS} \bar{r} \rightarrow t + s : A\}.
\]
Thus
\[
|s : A|_S^{CS} \leq |s + t : A|_S^{CS},
\]
\[
|s : A|_S^{CS} \leq |t + s : A|_S^{CS}.
\]
Hence
\[
\mathcal{E}_S^c(s, A) \leq \mathcal{E}_S(s + t, A),
\]
\[
\mathcal{E}_S^c(s, A) \leq \mathcal{E}_S(t + s, A).
\]
- Condition $FE3$: Suppose $c \nvdash F \in CS$. Then, by GIAN, we have $\vdash_{CS} c \nvdash F$.
Thus, by Lemma 3 item 7, $\vdash_{CS} c : F$. Hence, $\vdash_{CS} \bar{1} : c : F$. Therefore,
\[
\mathcal{E}_S^c(c, F) = |c : F|_S^{CS} = \sup\{r\|S \vdash_{CS} \bar{r} \rightarrow c : F\} = 1.
\]
This completes the proof. \(\Box\)

**Theorem 9 (Graded-style completeness).** For each set of formulas $T$, formula $A$ and constant specification $CS$:
\[
\|A\|_T^{CS} = |A|_T^{CS}.
\]

**Proof.** We first prove that $|A|_T^{CS} < \|A\|_T^{CS}$. Suppose $|A|_T^{CS} = r$. Suppose $r'$ is an arbitrary element of $\{r\|T \vdash_{CS} \bar{r} \rightarrow A\}$, i.e.
\[
T \vdash_{CS} \bar{r'} \rightarrow A.
\]
By Theorem 8 we obtain $r' \leq \mathcal{V}_w(A)$, for every RPLJ$_{CS}$-model $M$ of $T$, and every $w \in M$. Thus $r' \leq \|A\|_T^{CS}$. It follows that $r \leq \|A\|_T^{CS}$, that is $|A|_T^{CS} < \|A\|_T^{CS}$.

For the converse, we show that for all $r \in Q \cap [0, 1]$ if $r < \|A\|_T^{CS}$ then $T \vdash_{CS} \bar{r} \rightarrow A$ (this implies that $\|A\|_T^{CS} < |A|_T^{CS}$), or rather its contrapositive. Suppose $T \nvdash_{CS} \bar{r} \rightarrow A$. Hence $T \cup \{A \rightarrow \bar{\bar{r}}\}$ is $CS$-consistent, and can be extended to a $CS$-consistent $CS$-complete set $S$. Now consider the canonical model $\mathcal{M}_c = (W^c, R^c, E^c, V^c)$ of RPLJ$_{CS}$. Clearly $S \in W^c$. By the Truth Lemma (Lemma 12), $\mathcal{M}_c$ is a model of $S$, and thus $\mathcal{V}_S^c(A \rightarrow \bar{\bar{r}}) = 1$. Therefore, $\mathcal{V}_S^c(A) \leq r$, and hence $r \leq \|A\|_T^{CS}$.
\(\Box\)

**Corollary 1 (Fuzzy completeness).** For each set of formulas $T$, formula $A$ and constant specification $CS$, if $T \vdash_{CS} A$ then for every $n \geq 1$:
\[
T \vdash_{CS} \left(1 - \frac{1}{n}\right) A.
\]
Proof. Note that $T \models_{CS} A$ implies that $\|A\|_{T}^{CS} = 1$. Thus, by the graded-style completeness, $\|A\|_{T}^{CS} = 1$. Therefore, for every $n \geq 1$ there is $r \in \mathbb{Q} \cap [0,1]$ such that $T \models_{CS} \bar{r} \rightarrow A$ and $1 - \frac{1}{n} \leq r < 1$. Since $\models_{CS} (1 - \frac{1}{n}) \rightarrow \bar{r}$, by axiom BL1 it follows that $T \models_{CS} (1 - \frac{1}{n}) \rightarrow A$. □

Lemma 14. Every CS-consistent set of RPLJ-formulas have a RPLJCS-model.

Proof. Suppose $T$ is a CS-consistent set of RPLJ-formulas. Then it can be extended to a CS-consistent and CS-complete set $S$. Now consider the canonical model $\mathcal{M}^{c} = (W^{c}, R^{c}, \mathcal{E}^{c}, V^{c})$ of RPLJCS. Clearly $S \in W^{c}$. By the Truth Lemma (Lemma 12), $\mathcal{M}^{c}$ is a model of $S$, and thus is a model of $T$. □

Lemma 15 (Compactness). For each set of RPLJ-formulas $T$, if each finite $T_{0} \subseteq T$ has a RPLJCS-model then $T$ has a RPLJCS-model.

Proof. Suppose that $T$ has no RPLJCS-model. Then, by Lemma 14, $T$ should be CS-inconsistent. Hence, a finite subset $T_{0} \subseteq T$ is CS-inconsistent, and thus has no RPLJCS-model. □

Lemma 16 (Conservativity). Let $A$ be a formula in the language of Rational Pavelka logic RPL. If $A$ is provable in RPLJCS, then it is provable in RPL.

Proof. Suppose $A$ is a formula in the language of RPL, and it is provable in RPLJCS. Now suppose that $V$ is an arbitrary truth valuation of RPL. It suffices to show that $V(A) = 1$. Then by completeness of RPL (Theorem 7), $\models_{RPL} A$.

Now I aim to construct an RPLJCS-model from the valuation $V$ of RPL. Let $\mathcal{M} = (W, R, \mathcal{E}, V')$ be defined follows:

- $W = \{w_{0}\}$.
- $R = \emptyset$.
- $\mathcal{E}_{w_{0}}(t, A) = 1$, for all $t \in Tm$ and RPLJ-formula $A$.
- $V'(p) = V(p)$, for all $p \in \mathcal{P}$.

Clearly, $\mathcal{E}$ satisfies the conditions $FE1 - FE3$, and thus $\mathcal{M}$ is an RPLJCS-model. By induction on the complexity of $A$ it can be easily shown that $V(A) = \mathcal{V}_{w_{0}}(A)$. Since $A$ is provable in RPLJCS, then by Lemma 8, $A$ is RPLJCS-valid, that is $\mathcal{V}_{w}(A) = \mathcal{1}$ for every RPLJCS-model $\mathcal{M} = (W, R, \mathcal{E}, V)$ and every $w \in W$. Thus, $\mathcal{V}_{w_{0}}(A) = \mathcal{1}$, and hence $V(A) = \mathcal{1}$. □

In fact the model constructed in the above proof is a Mkrtychev model of RPLJ. Similar to M-models of BLJ we can define M-models for RPLJ. All the results of this section holds for these M-models too.

6 Other justification principles

Another important principle in justification logics is the factivity axiom, referred to as the axiom jT:

$$t : A \rightarrow A.$$
In classical justification logics axiom jT states that justifications are factive: every justified statement is true. Let us see how the addition of axiom jT to our fuzzy justification logics BLJ, LJ, GJ, ΠJ, and RPLJ affects matter (specially the proof of soundness theorems).

In the framework of fuzzy justification logic, at first sight it seems that axiom jT is unacceptable. For weak justifications (justifications with low certainty degree) for a statement, would not necessarily imply the truth of that statement. But note that in the framework of t-norm based fuzzy logics, validity of jT means that the truth value of \( t : A \) is less than or equal to the truth value of \( A \), which holds in all fuzzy Fitting models that have a reflexive accessibility relation \( R \).

The proof is as follows. Let \( M = (W, R, E, V) \) be an arbitrary fuzzy Fitting model, \( w \in W \) be an arbitrary world, and \( R \) be reflexive. Then

\[
V^\square_w(A) = \inf\{V_v(A)|w \rightarrow v\} \leq V_w(A).
\]

Thus,

\[
V_w(t : A) = E_w(t, A) * \big( V^\square_w(A) \leq V_w(A) \big).
\]

By adding axiom jT to RPLJ, the followings are provable:

1. \( t : t \vdash A \rightarrow A \).
2. \( t : r \vdash A \rightarrow (\bar{r} \rightarrow A) \).
3. \( \neg t : \bar{0} \).

Instead of axiom jT, either of the above formulas can be added to our fuzzy justification logics. The last one is an instance of axiom jT, \( t : \bar{0} \rightarrow \bar{0} \), and is called axiom jD. This axiom is acceptable if we want to axiomatize consistent belief (i.e. it is impossible to hold contradictory belief), or normative belief (in which the notion of “ought to believe” is concerned).

It is easy to see that jD is valid in those fuzzy Fitting models of LJ, GJ, ΠJ, or RPLJ that have a serial accessibility relation \( R \). The proof is as follows. Let \( M = (W, R, E, V) \) be an arbitrary fuzzy Fitting model, \( w \in W \) be an arbitrary world, and \( R \) be serial. Then,

\[
V^\square_w(\bar{0}) = \inf\{V_v(\bar{0})|w \rightarrow v\} = 0.
\]

Thus, \( V_w(t : \bar{0}) = 0 \). Note that in all logics LJ, GJ, ΠJ, and RPLJ we have that \( V_w(\neg A) = 1 \), whenever \( V_w(A) = 0 \). Hence, \( V_w(\neg t : \bar{0}) = 1 \).

We leave the study of adding other justification principles to our fuzzy systems for future work.

7 Conclusion

We extended fuzzy logic and justification logic to fuzzy justification logic, a framework for reasoning on justifications under vagueness. We replaced the classical base of the justification logic J with some known fuzzy logics: Hajek’s basic

\footnote{It is worth noting that the modal axiom \( \square A \rightarrow A \) is admitted in many fuzzy modal logics, see e.g. [10, 12, 19].}
logic, Łukasiewicz logic, Gödel logic, product logic, and rational Pavelka logic. In all of the resulting systems we introduced fuzzy models (fuzzy possible world semantics with crisp accessibility relation) for our systems, and established the soundness theorems. In the extension of rational Pavelka logic we also proved a graded-style completeness theorem.

This paper is the first study of fuzzy justification logics, and there are many directions for further work. The main problem remain open in this paper is the proof of the standard completeness theorem (i.e. if a formula is valid, then it is provable). Another direction for further research is to study fuzzy versions of Fitting models with soft accessibility relations $R : W \times W \rightarrow [0, 1]$. Kripke models of this kind have been studied in the literature for fuzzy modal logics, initiating from the work of Fitting [6]. The relationships between our fuzzy versions of $J$ and fuzzy versions of modal logic $K$ is also interesting (the realization theorem).

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