ENDOSCOPY FOR AFFINE HECKE CATEGORIES

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Abstract. We show that the neutral block of the affine monodromic Hecke category for a reductive group is monoidally equivalent to the neutral block of the affine Hecke category for its endoscopic group. The semisimple complexes of both categories can be identified with the generalized Soergel bimodules via the Soergel functor. We extend this identification of semisimple complexes to the neutral blocks of the affine Hecke categories by the technical machinery developed by Bezrukavnikov and Yun.

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1. INTRODUCTION

1.1. Hecke category. Let $G$ be a connected split reductive group over $\mathbb{F}_q$, $T$ be a maximal torus inside a Borel subgroup $B$ and $W$ be the Weyl group of $G$. The Hecke algebra $\mathcal{H}_q(W)$ is the ring of $\mathbb{Q}_\ell$-valued functions on the double cosets $B(\mathbb{F}_q) \backslash G(\mathbb{F}_q) / B(\mathbb{F}_q)$, that is the bi-$B(\mathbb{F}_q)$-invariant functions on $G(\mathbb{F}_q)$. The convolution product gives the multiplication structure. The Hecke algebra is ubiquitous in representation theory. For instance, it is related to the representations of finite groups of Lie type [DL] and the characters of the Verma modules in the BGG category $\mathcal{O}$ via Kazhdan-Lusztig canonical basis [Hu].

The affine Hecke algebra (or the Iwahori-Hecke algebra) is a generalization of the Hecke algebra, where the loop group $G_k\mathbb{K}$ and the Iwahori subgroup $I$ take the roles of $G$ and $B$ respectively. This can be thought of as a quantization of the group algebra of the affine Weyl group $\hat{W}$. The affine Hecke algebra is a central object in the local geometric Langlands program. It governs the irreducible admissible representations of $G(\mathbb{F}_q((t)))$ with Iwahori-invariant vectors.
The affine Hecke category categorifies the affine Hecke algebra. We consider \textbf{I}-equivariant constructible sheaves instead of \textbf{I}-equivariant functions. Under the sheaf-function dictionary of Grothendieck, we can associate a sheaf with a function by taking the trace of Frobenius. To adapt the six functors formalism, it is better to consider the derived category of equivariant sheaves $D_m^b(\textbf{I}\backslash\textbf{G}_K/\textbf{I})$, which is a monoidal category. The affine Hecke category and its relatives play extremely important roles in the geometric representation theory. They are related to the representations of the quantum group at the root of unity and coherent sheaves on the Springer resolution $[\text{ABG}]$.

1.2. Results. The affine Hecke category has monodromic counterparts. Let $\textbf{I}_u$ be the pro-unipotent radical of $\textbf{I}$. Suppose $\mathcal{L}'$ and $\mathcal{L}$ are two character sheaves over $T$. We can consider the $\textbf{I}_u$-equivariant derived category of mixed complexes of sheaves on $\textbf{G}_K/\textbf{I}_u$ whose monodromy under left (resp. right) $T$-action is $\mathcal{L}'$ (resp. $\mathcal{L}^{-1}$). Denote the category as $\mathcal{L}'D_{\mathcal{L}}$. For any two sheaves in $\mathcal{L}'D_{\mathcal{L}}$ and $\mathcal{L}D_{\mathcal{L}}$, one can define their convolution product, which is a sheaf in $\mathcal{L}'D_{\mathcal{L}}$. In particular, $\mathcal{L}D_{\mathcal{L}}$ is a monoidal category. $\mathcal{L}D_{\mathcal{L}}$ can be decomposed into several smaller (not necessarily monoidal) subcategories. Denote the identity block of the decomposition as $\mathcal{L}D_{\mathcal{L}}^\circ$.

**Theorem 1.1.** Let $G$ be a connected split reductive group over $\mathbb{F}_q$, and $T$ be a maximal torus inside a Borel subgroup $B$. Assume that $\mathcal{L}$ is a character sheaf over $T$ and $H$ is the endoscopic group for $\mathcal{L}$. Let $\textbf{H}_K$ and $\textbf{I}_H$ be the loop group and the Iwahori subgroup of $H$ respectively.

Then there is a canonical monoidal equivalence of triangulated categories,

$$
\Phi_{\mathcal{L}}^\circ : D_m^b(\textbf{I}_H\backslash\textbf{H}_K/\textbf{I}_H)^\circ \cong \mathcal{L}D_{\mathcal{L}}^\circ
$$

preserving the standard objects, the IC sheaves and the costandard objects, i.e. $\Phi_{\mathcal{L}}^\circ$ sends $\Delta(w)_H, \text{IC}(w)_H, \nabla(w)_H$ to $\Delta(w)_{\mathcal{L}}, \text{IC}(w)_{\mathcal{L}}, \nabla(w)_{\mathcal{L}}$ respectively for all $w \in \overline{W}_\mathcal{L} = \overline{W}_H^\circ$. $D_m^b(\textbf{I}_H\backslash\textbf{H}_K/\textbf{I}_H)^\circ$ is the identity block of $D_m^b(\textbf{I}_H\backslash\textbf{H}_K/\textbf{I}_H)$. In other words, it contains the sheaves supported on the identity connected component of $\textbf{I}_H\backslash\textbf{H}_K/\textbf{I}_H$.

In fact $\Phi_{\mathcal{L}}^\circ$ satisfies more properties which we cover in Theorem 9.1. We also prove the analogous statement for the central extension of the loop group in Theorem 9.2. In this case, $\mathcal{L}$ is a character sheaf over $T$, but not over its central extension. The similar statement for the reductive group case was proved by Lusztig and Yun in [LY].

1.3. Potential application to local geometric Langlands program. In [Be], Bezrukavnikov proved the case of the unipotently ramified local geometric Langlands correspondence. More precisely, he proved the equivalence between the affine Hecke category associated with a reductive group $G$ over $\mathbb{F}_q$ and the DG category of equivariant coherent sheaves on the derived Steinberg variety of Langlands dual group $G^L$. He then conjectured the tamely ramified local geometric Langlands correspondence, which is a similar statement for the affine monodromic Hecke category and the DG category of equivariant coherent sheaves on the twisted derived Steinberg variety. See Conjecture 58 in loc.cit..

When $G$ is simple and of adjoint type, it can be shown that the twisted derived Steinberg variety for $G^L$ is isomorphic to the derived Steinberg variety for $H^L$. Therefore the conjecture is equivalent to the equivalence of non-neutral blocks via Bezrukavnikov’s equivalence. This will be discussed in a forthcoming paper.
1.4. Method of proof. To explain the proof, we recall how the reductive group case is proved in [LY]. Their proof makes extensive use of the monodromic Sorgel functor $\mathcal{M}$ which maps a semisimple sheaf $\mathcal{F}$ to the $H^\ast_T(pt)$-bimodule of homomorphisms from a maximal IC sheaf to $\mathcal{F}$. As a special case when $\mathcal{L}$ is trivial, $\mathcal{M}$ is the same as the global sections functor. Via this functor, the IC sheaves can be identified with generalized Soergel bimodules. The general DG model construction in [BY] is then used to extend this equivalence to the neutral blocks of derived categories.

The main difficulty in extending the above argument in [LY] to the loop group case is that there are no IC sheaves with maximal support, hence the above definition of $\mathcal{M}$ does not apply. The main contribution of this paper is to consider monodromic sheaves on $\text{Bun}_{0,\infty}$, where $\text{Bun}_{0,\infty}$ is the moduli stack classifying $G$-bundles on $\mathbb{P}^1$ along with a Borel reduction at zero and infinity. There is a unique maximal orbit in each block of $L\tilde{W}$. We can define the maximal IC sheaves as the IC sheaves which are supported on these orbits. This allows us to define $\mathcal{M}$ for the loop group case. When $\mathcal{L}$ is trivial, the maximal IC sheaf is the constant sheaf.

In this special case, H" arterich proved in [H" a] that the subcategory of semisimple complexes can be realized as the Soergel bimodules of the affine Weyl group via taking equivariant cohomology.

To prove that $\mathcal{M}$ is a monoidal functor, we construct a coalgebra structure on each maximal IC sheaf $\Theta$. Note that we cannot define the two-fold convolution product of $\Theta$ directly. Instead, we consider the convolution product $\Theta \star \text{Av}_! \Theta$, where $\text{Av}_!$ is the functor averaging sheaves along $\mathcal{I}_\lambda$-orbits. The key in constructing the coalgebra map from $\Theta$ to $\Theta \star \text{Av}_! \Theta$ boils down to computing stalks of $\text{Av}_! \Theta$. These are computed in Proposition 6.7 by applying Braden’s hyperbolic localization in [Br].

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2. Notations and conventions

We follow most of the notations and conventions in [LY].

2.1. Frobenius. Throughout the article, $k = \overline{F}_q$ is a fixed algebraic closure of $F_q$ and $\ell$ is a prime different from $p = \text{char}(k)$. Denote $pt$ to be $\text{Spec}(F_q)$.

For any Artin stack $X$ over $F_q$. We denote by $D^b_{m}(X)$ the derived category of étale $\overline{Q}_\ell$-complexes on $X$ whose cohomology sheaves are mixed. When $J$ is an algebraic group over $F_q$ acting on a scheme $Y$ over $F_q$, $D^b_{m,J}([J\backslash Y])$ is the $J$-equivariant derived category of étale $\overline{Q}_\ell$-complexes on $Y$ whose cohomology sheaves are mixed. In this case, we sometimes write $D^b_{m,J}(Y)$ instead.

For any Artin stack $X$ over $F_q$, denote $\text{Fr}_X$ to be the geometric Frobenius of $X$. The subscript will be suppressed whenever the context is clear.

For any Artin stack $X$ over $F_q$, the projection map $X_k := X \times_{pt} \text{Spec}(k) \to X$ induces a pullback functor $\omega : D^b_{m}(X) \to D^b_{m}(X_k)$. Let $\mathcal{F}, \mathcal{G} \in D^b_{m}(X)$. Define the following two types of homomorphisms between $\mathcal{F}$ and $\mathcal{G}$,

$$\text{ext}^i(\mathcal{F}, \mathcal{G}) := \text{Hom}_{D^b_{m}(X)}(\mathcal{F}, \mathcal{G}[i])$$

$$\text{Ext}^i(\mathcal{F}, \mathcal{G}) := \text{Hom}_{D^b_{m}(X_k)}(\omega \mathcal{F}, \omega \mathcal{G}[i])$$
Note that the latter module carries a Frobenius action. We distinguish the corresponding derived functors by \( \mathbf{R}\text{Hom}(F,G) \) always means \( \mathbf{R}\text{Hom}(\omega F,\omega G) \).

Let \( M \) be a graded module over a graded algebra \( A \). Denote \( M[1] \) to be the graded module with \( M[1]_i = M_{i+1} \).

Suppose Frobenius acts on \( A \) and \( M \) is a graded \( A \)-module with a compatible Frobenius action \( \text{Fr}_M \), that is \( \text{Fr}_A(a) \text{Fr}_M(m) = \text{Fr}_M(am) \). Define \( M(1) \) to be the same underlying \( A \)-module with the twisted Frobenius action \( \text{Fr}_M(1)(m) = q^{\text{deg}(m)} \text{Fr}_M(m) \).

Fix a square root of \( q \) in \( \mathbb{Q}_q \). We define \( M(1) = M[1](1/2) \). This convention applies to both mixed sheaves and graded modules with Frobenius action.

2.2. Rank one character sheaves. Recall the definition of rank one character sheaves.

**Definition 2.1.** Let \( J \) be an algebraic group over \( \mathbb{F}_q \). A rank one character sheaf over \( J \) is a triple \( (\mathcal{L}, \iota, \phi) \) where \( \mathcal{L} \) is a rank one local system on \( J \), \( \iota : \mathcal{L} \rightarrow \mathcal{L} \) is the rigidification of stalk at the identity element \( e \) of \( J \) and \( \phi : m^* \mathcal{L} \rightarrow \mathcal{L} \otimes \mathcal{L} \) is an identification between the pullback of \( \mathcal{L} \) along \( m \), the multiplication map of \( J \), and the exterior product of \( \mathcal{L} \), satisfying the associativity and unital axioms. Denote \( \text{Ch}(J) \) to be the isomorphism class of rank one character sheaves over \( J \).

By abuse of notation, we write \( \mathcal{L} \) instead of \( (\mathcal{L}, \iota, \phi) \). Details about rank one character sheaves can be found in [LY] and [Y2]. We mention two important properties of character sheaves here. Firstly, the automorphism group of \( \mathcal{L} \) is trivial when \( J \) is connected. Secondly, when \( J \) is a split torus over \( \mathbb{F}_q \), \( \text{Ch}(J) \) can be identified with \( \text{Hom}(J(\mathbb{F}_q), \mathbb{Q}_q) = J^L(\mathbb{Q}_q) \), where \( J^L \) is the Langlands dual of \( J \) over \( \mathbb{Q}_q \).

2.3. Group theory.

2.3.1. Loop group. Let \( G \) be a connected split reductive group over \( \mathbb{F}_q \). Fix a pair of opposite Borel subgroups \( B \) and \( B^- \). Denote \( U \) and \( U^- \) as the unipotent radical of \( B \) and \( B^- \) respectively. Recall the functor of points of some ind-group schemes associated with \( G, B, B^-, U, U^-, T \). Let \( A \) be a \( \mathbb{F}_q \)-algebra.

1. Loop group \( G_K \), which sends \( A \) to \( G(A[[t]]) \);
2. Positive loop group \( G_O \), which sends \( A \) to \( G(A[[t]]) \);
3. Standard Iwahori subgroup \( I \), which sends \( A \) to the elements of \( G(A[[t]]) \) whose image is in \( B(A) \) when evaluating \( t \) at zero.
4. Standard opposite Iwahori subgroup \( I^- \), which sends \( A \) to the elements of \( G(A[[t]]) \) whose image is in \( B^-(A) \) when evaluating \( t^{-1} \) at zero.
5. Pro-unipotent radical of standard Iwahori subgroup \( I_u \), which sends \( A \) to the elements of \( G(A[[t]]) \) whose image is in \( U(A) \) when evaluating \( t \) at zero.
6. Pro-unipotent radical of standard opposite Iwahori subgroup \( I_u^- \), which sends \( A \) to the elements of \( G(A[[t]]) \) whose image is in \( U^-(A) \) when evaluating \( t^{-1} \) at zero.

The affine flag variety \( G_K/I \) and the enhanced affine flag variety \( G_K/I_u \) play a central role in this paper. We will also work with the affine Grassmannian \( G_{K\mathbb{Q}} = G_K/G_O \) in Section \( \underline{2} \). Note that these ind-schemes may have a non-reduced structure.
2.3.2. Affine Weyl group. Let \( \widetilde{W} \) be the extended affine Weyl group associated to \( G_K \) and \( \leq \) be the standard Bruhat order of \( \widetilde{W} \). Define \( T_K = T \times_{G_q} F_q[[t]] \) and \( T_O = T \times_{G_q} F_q[[t]] \). Note that \( \widetilde{W} \) is isomorphic to \( N_{G_K}(T_K)/(F_q)/T_O(F_q) \) as an abstract group, where \( N_{G_K}(T_K) \) is the normalizer of \( T_K \) in \( G_K \). We fix a set of coset representatives \( \{w\} \subset N_{G_K}(T_K)(F_q) \), where \( w \) is the coset representative of \( w \in \widetilde{W} \).

For \( s \) a simple reflection in \( \widetilde{W} \), let \( P_s \) be the standard parahoric subgroup corresponding to \( s \). Denote the pro-unipotent radical of \( P_s \) by \( I_u \) and the Levi subgroup by \( L_s \).

Define \( M(w, w') \) to be the subset \( \{x \in \widetilde{W} : x \geq uw \text{ for any } u \leq w \text{ and } u \leq w' \} \) for any \( w, w' \).

2.4. Geometry. All varieties in this article refer to quasi-projective separated schemes of finite type over \( F_q \). In particular varieties can be non-reduced. For example, \( \text{Spec}(F_q[x]/(x^2)) \) is a variety in our sense. The fact that \( D^b(X) \) is equivalent to \( D^b(X_{\text{red}}) \) implies no information is lost when passing to the reduced structure.

2.4.1. Affine flag variety. The left multiplication of \( I \) induces a stratification on \( G_K/I \). The strata are indexed by \( \widetilde{W} \) by the Bruhat decomposition. Each stratum contains a unique \( w \). We denote \( wI \) as \( G_K, w \). It is well known that \( G_K, w = I \) is an affine space of dimension \( \ell(w) \) and \( G_K, w = I \in G_K, w = I \) if and only if \( w \leq w' \), where the overline stands for the closure. We denote \( G_K, w = I \) by \( G_K, w \).

2.4.2. Moduli space of \( G \)-bundles over \( \mathbb{P}^1 \). Let \( \text{Bun}_{G_0, \mathbb{P}^1} \) be the moduli stack classifying the data of \( (E, F_0, F_{\infty}) \), where \( E \) is a \( G \)-bundle on \( \mathbb{P}^1 \), \( F_0 \) is a \( B^- \) reduction at zero and \( F_{\infty} \) is a \( B^- \) reduction at infinity. For instance, when \( G = GL_n \), \( \text{Bun}_{G_0, \mathbb{P}^1} \) classifies rank \( n \) vector bundles on \( \mathbb{P}^1 \) together with a complete flag at zero and infinity respectively.

There is a natural map from \( \text{Bun}_{G_0, \mathbb{P}^1} \) to \( \text{Bun}_{G, \mathbb{P}^1} =: \text{Bun}_G \) by forgetting the two Borel reductions. Via this map we may view \( \text{Bun}_{G_0, \mathbb{P}^1} \) as a \( (G/B \times G/B^-) \)-fibration over \( \text{Bun}_G \). Since \( \text{Bun}_G \) is locally of finite type, so is \( \text{Bun}_{G_0, \mathbb{P}^1} \). Since we will be working with a fixed \( G \), in the remaining of the paper we will write \( \text{Bun}_{G_0, \mathbb{P}^1} \) instead of \( \text{Bun}_{G_0, \mathbb{P}^1} \).

\( G_K/I \) can be interpreted as the moduli space of \( G \)-bundles \( E \) on \( \mathbb{P}^1 \) with a Borel reduction at 0 and a trivialization of \( E \) on \( \mathbb{P}^1 \backslash \{0\} \) (see [1]). From this, we know that there is a natural map from \( G_K/I \) to \( \text{Bun}_{G_0, \mathbb{P}^1} \). This map induces a natural isomorphism \( [\Gamma \backslash G_K/I] \simeq \text{Bun}_{G_0, \mathbb{P}^1} \) (see [1] and [2] Theorem 2.3.7). From the Birkhoff decomposition, \( [\Gamma \backslash G_K/I] \) is naturally stratified and the strata are indexed by \( \widetilde{W} \). For \( w \in \widetilde{W} \), let \( \text{Bun}_w^{\mathbb{P}^1} \) be the locally closed substack of \( \text{Bun}_{G_0, \mathbb{P}^1} \) corresponding to the stratum \( w \). \( \text{Bun}_w^{\mathbb{P}^1} \) is of codimension \( \ell(w) \) in \( \text{Bun}_{G_0, \mathbb{P}^1} \). In particular, \( \text{Bun}_w^{\mathbb{P}^1} \) is an open substack of \( \text{Bun}_{G_0, \mathbb{P}^1} \).

It is well known that (e.g. see [1]) \( \text{Bun}_w^{\mathbb{P}^1} \) is in the closure of \( \text{Bun}_w^{\mathbb{P}^1} \) if and only if \( w \geq w' \). Let \( \text{Bun}_{\geq w}^{\mathbb{P}^1} \) be the union of \( \text{Bun}_w^{\mathbb{P}^1} \) for \( w \geq w' \). It is the closure of \( \text{Bun}_w^{\mathbb{P}^1} \). Let \( \text{Bun}_{\leq w}^{\mathbb{P}^1} \) be the union of \( \text{Bun}_w^{\mathbb{P}^1} \) for \( w' \leq w \). It is an open substack of \( \text{Bun}_{G_0, \mathbb{P}^1} \).

There are some variants of \( \text{Bun}_{G_0, \mathbb{P}^1} \). Let \( \text{Bun}_{0, \mathbb{P}^1} \) (resp. \( \text{Bun}_{0, \mathbb{P}^1}, \text{Bun}_{0, \mathbb{P}^1} \)) be the moduli stack of \( G \)-bundles on \( \mathbb{P}^1 \) with a \( U \)-reduction (resp. a \( B^- \) reduction, a \( U \)-reduction) at zero and a \( B^- \) reduction (resp. a \( U^- \) reduction, a \( U^- \) reduction)
at infinity. There are natural smooth maps from each variant to $\text{Bun}_{0,\infty}$ as a $U$-reduction gives a $B$-reduction naturally. In particular, $\text{Bun}_{0,\infty}$ is a $T \times T$-bundle over $\text{Bun}_{0,\infty}'$. The stratification of $\text{Bun}_{0,\infty}'$ gives rise to a stratification of each variant. Define $\text{Bun}_{0,\infty}^w$ as the strata indexed by $w$ of $\text{Bun}_{0,\infty}'$. Similarly, we can define $\text{Bun}_{0,\infty}^\infty$, $\text{Bun}_{0,\infty}^{\infty\infty}$ etc.

We will also consider bundles with parahoric level structures. Let $\text{Bun}_{0,\infty}$ be the moduli stack of $G$-bundle on $\mathbb{P}^1$ with a full level structure at zero and a $U'$-reduction at infinity. That is, $\text{Bun}_{0,\infty}$ classifies the data of $(E, F_{\infty}, \tau)$, where $E$ is a $G$-bundle on $\mathbb{P}^1$, $F_{\infty}$ is a $U'$-reduction at infinity and $\tau$ is a trivialization of the restriction of $E$ to the formal disc at zero, which is identified with Spec $\mathbb{F}_q((t))$. Notice that $G_K$ naturally acts on $\text{Bun}_{0,\infty}$ from the right (see [YL Construction 4.2.2]).

Example 2.2. When $G = GL_2$ and $s$ is the non-affine simple reflection, $\text{Bun}_{0,\infty}$ classifies the ideal sheaf of $\infty$ in $\mathbb{P}^1$, $E_0$ and $E_1$ are vector bundles of rank $2$ on $\mathbb{P}^1$ such that $E_0/E_1$ is of length $1$, $\gamma_i$ are isomorphisms $E_i/E_{i+1} \xrightarrow{\sim} \delta_i$ for $i = 0, 1$, $\gamma$ is an isomorphism $E_0/E_0 \xrightarrow{\sim} \delta_0 \rightarrow \delta_0^{\infty}$, $\delta_0$ and $\delta_{\infty}$ are the skyscraper sheaf at $0$ and $\infty$ respectively.

3. Affine monodromic Hecke categories

In this section, we introduce our main players in this paper $\mathcal{L}^* D_{\mathcal{L}}$ and $\mathcal{L}^* D_{\mathcal{L}'}$, the categories of positive sheaves and negative sheaves respectively. The former one is usually called affine monodromic Hecke categories. We give examples at the end. The monodromic sheaves are also discussed in [BFO] and [L].

3.1. Construction of $\mathcal{L}^* D_{\mathcal{L}}$. Let $X$ be a stack of finite type over $\mathbb{F}_q$, $J$ be a connected group acting on $X$ and $\mathcal{L}$ a character sheaf on $J$. Lusztig and Yun defined a $\mathbb{Q}_l$-linear triangulated category $D_{(J, \mathcal{L}), m}(X)$ in Section 2.5 in [LY].

For each $w \in \tilde{W}$, there exists a normal, finite codimension subgroup $J_w$ of $I$ such that it is contained in $I_u$ and acts trivially on $G_{K, \leq w}/I_u$. Replacing $J_w$ by their common intersection, we may assume $J_{w'} \subset J_w$ for $w \leq w'$. Hence $D_{m}^b(I_u \backslash G_{K, \leq w}/I_u)$ can be defined as $D_{m}^b((I_u/J_w) \backslash G_{K, \leq w}/I_u)$, which is independent of the choice of $J_w$. We apply the construction of Lusztig and Yun to $(I_u/J_w) \backslash G_{K, \leq w}/I_u$ and obtain $D_{(T \times T', \mathcal{L} \boxtimes \mathcal{L}')}, m(I_u \backslash G_{K, \leq w}/I_u)$ for any two character sheaves $\mathcal{L}$, $\mathcal{L}'$ on $T$. To simplify the notation, we define $\mathcal{L}^* D(\leq w)_{\mathcal{L}} := D_{(T \times T', \mathcal{L} \boxtimes \mathcal{L}^{-1})}, m(I_u \backslash G_{K, \leq w}/I_u)$ and $\mathcal{L}^* D(w)_{\mathcal{L}} := D_{(T \times T', \mathcal{L} \boxtimes \mathcal{L}^{-1})}, m(I_u \backslash G_{K, w}/I_u)$.

Once we define the monodromic equivariant derived categories for finitely many strata, we could define the monodromic equivariant derived categories for the affine flag variety

$$\mathcal{L}^* D_{\mathcal{L}} = 2\cdot \lim_{\substack{\rightarrow \quad w \in \tilde{W}}} \mathcal{L}^* D(\leq w)_{\mathcal{L}}.$$
to be the inductive 2-limit of $\mathcal{L} D(\leq w)_{\mathcal{L}}$ with respect to the inductive system of pushforward functor of closed embedding $i_{w,w'} : \mathcal{I}_{u} \backslash \mathcal{G}_{K,\leq w}/\mathcal{I}_{u} \to \mathcal{I}_{u} \backslash \mathcal{G}_{K,\leq w'}/\mathcal{I}_{u}$ for $w \leq w'$. Objects in this category are called positive sheaves.

When $\mathcal{L}$ and $\mathcal{L}'$ are trivial, $\mathcal{L}' D_{\mathcal{L}}$ coincides with the usual Hecke category $D^{b}_{m}(\Gamma(\mathcal{G}_{K}/\mathcal{I}))$ in [BY] up to the difference between Kac-Moody group and the loop group.

3.2. Construction of $\mathcal{L}' D_{\mathcal{L}}$. We study monodromic sheaves on $\text{Bun}_{0,\infty}^{\leq w}$.

Since $\text{Bun}_{0,\infty}^{\leq w}$ is a scheme of finite type quotient by an algebraic group [E] Lemma 6], $D^{b}_{m}(\text{Bun}_{0,\infty}^{\leq w})$ and its monodromic counterparts $\mathcal{L}' D(\leq w)_{\mathcal{L}} := D^{b}_{m}(\Gamma T \times T, \mathcal{L}_{\mathcal{L}})_{m}(\text{Bun}_{0,\infty}^{\leq w})$ can be defined via the construction by Lusztig and Yun.

For each pair $w' \leq w$, the open embedding $j_{w,w'} : \text{Bun}_{0,\infty}^{\leq w} \to \text{Bun}_{0,\infty}^{\leq w}$ induces a functor of derived categories $j_{w,w'}^{*} : \mathcal{L}' D(\leq w)_{\mathcal{L}} \to \mathcal{L}' D(\leq w')_{\mathcal{L}}$. These functors form a projective system of triangulated categories. Let

$$\mathcal{L}' D_{\mathcal{L}} = 2 - \lim_{\rightarrow} \mathcal{L}' D(\leq w)_{\mathcal{L}}$$

be the projective 2-limit of this system.

Concretely, objects in this category are $(\mathcal{F}_{w}, \chi_{w,w'})$, where $\mathcal{F}_{w}$ is a sheaf on $\text{Bun}_{0,\infty}^{\leq w}$ with the prescribed monodromy and $\chi_{w,w'}$ is an isomorphism from $\mathcal{F}_{w}$ to $j_{w,w'}^{*} \mathcal{F}_{w'}$ such that $j_{w,w'}^{*} \chi_{w,w'} \circ \chi_{w,w'} = \chi_{w,w''}$ whenever $w \leq w' \leq w''$. The morphism between $(\mathcal{F}_{w}, \chi_{w,w'})$ and $(\mathcal{G}_{w}, \psi_{w,w'})$ is a family of morphisms $\phi_{w} : \mathcal{F}_{w} \to \mathcal{G}_{w}$ such that $\psi_{w,w'} \circ \phi_{w} = \phi_{w'} \circ \chi_{w,w'}$ for any $w \leq w'$. We call objects in this category negative sheaves.

3.3. Examples. Let us mention some objects in the derived categories defined in the previous subsections.

3.3.1. Positive sheaves. It is clear that $\mathcal{L}' D(w)_{\mathcal{L}}$ is zero unless $\mathcal{L}' = w \mathcal{L}$. For $w \in \tilde{W}$ and its lifting $\dot{w}$, and taking stalks at $\dot{w}$ induces an equivalence from $w \mathcal{L} D(w)_{\mathcal{L}}$ to $D^{b}_{m}(T \times T, \mathcal{L}_{\mathcal{L}})_{m}(\dot{w}T) \simeq D^{b}_{m}(\Gamma(\dot{w}), m(\dot{w}))$, where $\Gamma(\dot{w}) = \{(w,t) \in T \times T\}$. Denote $C(\dot{w})_{\mathcal{L}}$ to be any sheaf corresponding to constant sheaf $\underline{\mathbb{C}}(\ell(w))$ on $\dot{w}$ under this equivalence. Note that $C(\dot{w})_{\mathcal{L}}$ is unique up to a scalar.

From the local system $C(\dot{w})_{\mathcal{L}}$, we can construct other related sheaves in $\mathcal{L}' D(\leq w)_{\mathcal{L}}$ such as the standard sheaf $\Delta(\dot{w})_{\mathcal{L}} := i_{w,*} C(\dot{w})_{\mathcal{L}}$, the costandard sheaf $\nabla(\dot{w})_{\mathcal{L}} := i_{w,*} C(\dot{w})_{\mathcal{L}}$, and the intersection cohomology sheaf $IC(\dot{w})_{\mathcal{L}} := i_{w,*} C(\dot{w})_{\mathcal{L}}$, where $i_{w,*}$ is the open embedding from $\mathcal{I}_{u} \backslash \mathcal{G}_{K,w}/\mathcal{I}_{u}$ to its closure $\mathcal{I}_{u} \backslash \mathcal{G}_{K,\leq w}/\mathcal{I}_{u}$. The same notations are used to denote their images in $\mathcal{L}' D_{\mathcal{L}}$, and they are called positive standard sheaf, positive costandard sheaf and positive IC sheaf respectively. Notice that $\omega(\Delta(\dot{w})_{\mathcal{L}})$, $\omega(\nabla(\dot{w})_{\mathcal{L}})$ and $\omega(\nabla(\dot{w})_{\mathcal{L}})$ are isomorphic for any lifting $\dot{w}$. We denote these isomorphism classes under $\omega$ by $\Delta(w)_{\mathcal{L}}$, $\nabla(w)_{\mathcal{L}}$ and $\nabla(w)_{\mathcal{L}}$ respectively.

3.3.2. Negative sheaves. Denote $D^{b}_{m}(T \times T, \mathcal{L}_{\mathcal{L}})_{m}(\text{Bun}_{0,\infty}^{\leq w})$ as $\mathcal{L}' D(w)_{\mathcal{L}}$. We can define the constant sheaf $C(\dot{w})_{\mathcal{L}} \in \mathcal{L}' D(w)_{\mathcal{L}}$, whose stalk at $\dot{w}$ is $\underline{\mathbb{C}}(-\ell(w))$. Note that $\mathcal{L}' D(w)_{\mathcal{L}}$ is zero unless $\mathcal{L}' = w \mathcal{L}$. Since $\ell$-extension, $\ast$-extension and IC-extension from open subsets are well-behaved with respect to open restriction, we can define the negative standard sheaf $\Delta(\dot{w})_{\mathcal{L}} := j_{w,*} C(\dot{w})_{\mathcal{L}}$, the negative costandard sheaf $\nabla(\dot{w})_{\mathcal{L}} := j_{w,*} C(\dot{w})_{\mathcal{L}}$. 

and the negative IC sheaf $\text{IC}(\dot{w})_{\mathcal{L}}^{-} := j_{w!*}C(\dot{w})_{\mathcal{L}}^{-}$ in $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}$, where $j_w$ is the embedding from $\text{Bun}_{w0,\infty}$ to $\text{Bun}_{0,\infty}$. Similar to positive sheaves, we denote the isomorphism classes of images of above sheaves under $\omega$ by $\Delta(w)_{\mathcal{L}}^{-}$, $\text{IC}(w)_{\mathcal{L}}^{-}$ and $\Sigma(w)_{\mathcal{L}}^{-}$ respectively.

As an illustration, we construct $\text{IC}(\dot{w})_{\mathcal{L}}^{-}$ explicitly and show that it is well-defined. For any $w' \geq w \geq w$, we have a chain of embeddings of stacks of finite type, $\text{Bun}_{w0,\infty} \xrightarrow{\bar{j}_w} \text{Bun}_{w',\infty} \xrightarrow{j_{w',w''}} \text{Bun}_{0,\infty}$. Since $j_{w',w''}$ is an open embedding, $(j_{w',w''})^*((\bar{j}_w)_{!*}(C(\dot{w})_{\mathcal{L}}^{-}))$ is canonically isomorphic to $(\bar{j}_w)_{!*}(C(\dot{w})_{\mathcal{L}}^{-})$ via the restriction map. Since the set $\{w': w' \geq w\}$ is cofinal, $\text{IC}(\dot{w})_{\mathcal{L}}^{-} \in \mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}$ can be defined as the projective system $(\bar{j}_w)_{!*}(C(\dot{w})_{\mathcal{L}}^{-})$.

3.4. Basic operations. The construction of sheaf hom $\mathbf{RHom}$ in [LY] can be easily generalized to affine case. By viewing negative sheaves as sheaves on $G_K/I_u$, $\mathbf{RHom}(F, G)$ is also well-defined for $F \in \mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}$, $G \in \mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}$.

We define the renormalized Verdier dualities $\mathbb{D}^+$ and $\mathbb{D}^-$ on $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}$ and $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}$ respectively as follows.

For $F \in \mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}$, suppose its support is contained in $G_K, I_u$. Define $\mathbb{D}^+(F) := D_{G_K, I_u}(F)[2\dim(T)]$, where $D_{G_K, I_u}$ is the usual Verdier duality for the scheme $G_K, I_u$. It is independent of the choice of $w$. Moreover, it is constructible along $I_u$-orbit and its monodromy is reversed. Hence $\mathbb{D}^+$ is a contravariant functor from $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}$ to $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}$. Let $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}^0$ (resp. $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}^{-0}$) be the full subcategory of objects $F \in \mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}$ such that $F[\dim(T)]$, as a complex on $G_K/I_u$, lies in $pD^{\leq 0}(G_K/I_u)$ (resp. $pD^{\geq 0}(G_K/I_u)$). Since closed embedding is perverse exact, it is justified to define $pD^{\leq 0}(G_K/I_u)$ and $pD^{\geq 0}(G_K/I_u)$.

For $F \in \mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}$, define $\mathbb{D}^-(F) := D_{\text{Bun}_{0,\infty}}(F)[-2\dim(T)]$, where $D_{\text{Bun}_{0,\infty}}$ is the standard Verdier duality on $\text{Bun}_{0,\infty}$. Note that it is well-defined because open restriction commutes with the standard Verdier duality.

Let $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}^{-0}$ (resp. $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}^{-0}$) be the full subcategory of objects $F \in \mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}$ such that $F[\dim(T)]$, as a complex on $\text{Bun}_{0,\infty}$, lies in $pD^{\leq 0}(\text{Bun}_{0,\infty})$ (resp. $pD^{\geq 0}(\text{Bun}_{0,\infty})$). Since taking restriction to an open subset is perverse exact, it makes sense to define $pD^{\leq 0}(\text{Bun}_{0,\infty})$ and $pD^{\geq 0}(\text{Bun}_{0,\infty})$. More precisely, $(F_w)_{w \in \tilde{W}} \in pD^{\leq 0}(\text{Bun}_{0,\infty})$ if and only if $F_w \in pD^{\leq 0}(\text{Bun}_{0,\infty})$.

Note that $\mathbb{D}^+$ (resp. $\mathbb{D}^-$) is an involutive anti-equivalence of categories between $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}$ and $\mathcal{L}^\vee \mathcal{-1}$ (resp. $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}^{-1}$ and $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}^{-1}$).

Starting from this point, we define perverse sheaves to be the objects in $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}^{0} \cap \mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}^{0}$ or $\mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}^{0} \cap \mathcal{L}^\vee \mathcal{D}_{\mathcal{L}}^{0}$. By our degree shift convention, $\mathbb{D}^+$ and $\mathbb{D}^-$ preserve perverse sheaves. Note that both positive and negative IC sheaves are perverse in this sense.

4. Convolution product

In this section, we give the definition of convolution products between the sheaves of different types.

4.1. Convolution product of two positive sheaves. The convolution product of the affine monodromic Hecke categories is defined in a similar way to [LY] and [MV] by taking both infinite-dimensional varieties and monodromies into account.
Let $\mathcal{F} \in \mathcal{L}(\leq w)_{\mathcal{L}}$ and $\mathcal{G} \in \mathcal{L}(\leq w')_{\mathcal{L}}$. Choose $w'' \in M(w, w')$. Consider the following maps

\[ G_{K, < w} / I_u \times G_{K, < w} / I_u \xrightarrow{p} G_{K, < w} / J_w \times G_{K, < w} / I_u \xrightarrow{\phi} G_{K, < w} / J_w \times G_{K, < w} / I_u \]

\[ G_{K, < w} / J_w \times I_u \times G_{K, < w} / I_u \xrightarrow{\phi^+} G_{K, < w} / J_w \times G_{K, < w} / I_u \]

Here $p, q$ are the canonical quotient maps and $\mu_{w, w'}^+$ is the multiplication map. Since $J_w$ is normal in $I$, there is a natural action of $T$ on $G_{K, < w} / J_w \times I_u / J_w$. $G_{K, < w} / I_u$ by setting $t(g_1, g_2) = (g_1 t^{-1}, g_2)$. Let $\phi^+$ be the quotient map of this action.

Since $p^*(F \boxtimes G)$ is $I_u / J_w$-equivariant, we can find a sheaf $F \boxtimes G$ on $G_{K, < w} / J_w \times I_u / J_w$ such that $p^*(F \boxtimes G) \simeq q^*(F \boxtimes G)$. Similarly, $F \boxtimes G$ has a natural $T$-equivariant structure under the aforementioned $T$ action, hence it descends to a sheaf $F \boxtimes G$ on $G_{K, < w} / J_w \times I_u / J_w$.

$(\mu_{w, w'}^+)^*(F \boxtimes G)$ is $I_w$-equivariant and hence can be viewed as an object in $\mathcal{L}(\leq w') D_{\mathcal{L}}$. One can check that it is independent of the choice of $w, w', w''$. Observe that both leftmost and rightmost $T$-monodromic structures are preserved under the above operations. Hence we can define a functor from $\ast_{+} : \mathcal{L}(\leq w') D_{\mathcal{L}} \times \mathcal{L}(\leq w) D_{\mathcal{L}}$ to $\mathcal{L}(\leq w') D_{\mathcal{L}}$ sending $(F, G)$ to $F \ast_{+} G := (\mu_{w, w'}^+)^*(F \boxtimes G)$.

Since all the above maps are of finite type, the base change theorem shows that $\ast_{+}$ is associative.

### 4.2. Convolution product of negative sheaves and positive sheaves

We also need to define the convolution product of negative sheaves and positive sheaves. In the case of trivial monodromies, it is known as the Hecke modification in the literature (see [G2]). Most properties of this special case can be generalized to monodromic cases without much difficulty.

Let $\mathcal{H}_{0, \infty}$ be the Hecke stack of $\text{Bun}_{0, \infty}$ with a modification at zero. That is, $\mathcal{H}_{0, \infty}$ classifies the data of $(E^i, F^i_0, E^i_\infty, F^i_0, F^i_\infty, \beta)$, where $(E^i, F^i_0, F^i_\infty) \in \text{Bun}_{0, \infty}$ for $i = 1, 2$ and $\beta$ is an isomorphism $E^1|_{\{t = 0\}} \simeq E^2|_{\{t = 0\}}$ sending $F^1_\infty$ to $F^2_\infty$.

Let $pr_1, pr_2$ be the two projections from $\mathcal{H}_{0, \infty}$ to $\text{Bun}_{0, \infty}$. The fibers of $pr_1$ (or $pr_2$) are isomorphic to $G_{K, I_u}$. Hence we can view $\mathcal{H}_{0, \infty}$ as a twisted product of $\text{Bun}_{0, \infty}$ and $G_{K, I_u}$. Let $\mathcal{H}_{w, \infty}$ (resp. $\mathcal{H}_{w, 0, \infty}$) be the inverse image $pr_1^{-1}(\text{Bun}_{0, \infty}^w)$ (resp. $pr_2^{-1}(\text{Bun}_{0, \infty}^w)$).

The stratification of $G_{K, I_u}$ induces a stratification of $\mathcal{H}_{0, \infty}$. Denote the substack corresponding to $G_{K, \leq w} / I_u$ (resp. $G_{K, w} / I_u$) by $\mathcal{H}_{0, \infty, \leq w}$ (resp. $\mathcal{H}_{0, \infty, w} / I_u$). $\mathcal{H}_{0, \infty, \leq w}$ can be realized as a twisted product of $\text{Bun}_{0, \infty}$ and $G_{K, \leq w} / I_u$. For any $w, w' \in \tilde{W}$, let $\mathcal{H}_{0, \infty, \leq w}$ be the intersection of $\mathcal{H}_{0, \infty, \leq w}$ and $\mathcal{H}_{0, \infty, \leq w'}$. Similarly, we can define $\mathcal{H}_{0, \infty, \leq w, w'}$ and $\mathcal{H}_{0, \infty, \leq w, w'}$.

Let $\mathcal{H}_{0, \infty}$ be the quotient of $\mathcal{H}_{0, \infty}$ by the $T$-action on the $U$-reduction at zero of the first $G$-bundle. That is, $\mathcal{H}_{0, \infty}$ classifies the data of $(E^i, F^i_0, E^i_\infty, F^i_0, F^i_\infty, \beta)$, where $F^1_0$ is a $B$-reduction of $E^1$ and the rest is same as $\mathcal{H}_{0, \infty}$. Since $\mathcal{H}_{0, \infty, \leq w}$ is stable under the $T$-action, we can define $\mathcal{H}_{0, \infty, \leq w'}$ as the quotient of $\mathcal{H}_{0, \infty, \leq w}$ by $T$, which is a substack of $\mathcal{H}_{0, \infty}$.

By abuse of notation, we denote the projections from $\mathcal{H}_{0, \infty}$ to $\text{Bun}_{0, \infty}$ (resp. $\text{Bun}_{0, \infty}$) by $pr_1$ (resp. $pr_2$) respectively. We also use the same notation when restricted to a substack.
The definition of convolution product of a negative sheaf $F$ and a positive sheaf $G$ is as follows.

Let $\mathcal{F} = (\mathcal{F}_w)_{w \in \tilde{w}} \in \mathcal{L}^\times D_{\mathcal{L}}$, $\mathcal{G} \in \mathcal{L}^\times D(\leq w')_{\mathcal{L}}$. For each $w \in \tilde{W}$, choose $w'' \in M(w, w'^{-1})$ and $w''' \in M(w'', w')$. Consider the following maps

$$\text{Bun}_{0,\infty,\leq w}'' \xrightarrow{pr_1} \mathcal{H}_{0,\infty,\leq w'} \xrightarrow{q} \mathcal{H}_{0,\infty,\leq w''} \xrightarrow{pr_2} \text{Bun}_{0,\infty,\leq w''}.$$ 

Since $\mathcal{G}$ is $\mathcal{L}_w$-equivariant, we can form the twisted product of $\mathcal{F}_{w''} \mathcal{G}$ on $\mathcal{H}_{0,\infty,\leq w''}$. It descends to a sheaf $\mathcal{F}_{w''} \boxtimes \mathcal{G}$ on $\mathcal{H}_{0,\infty,\leq w''}$. Hence we obtain a sheaf $pr_{2*}(\mathcal{F}_{w''} \boxtimes \mathcal{G}) \in \mathcal{L}^\times D(\leq w''')_{\mathcal{L}}$. It is not hard to check that its restriction on $\text{Bun}_{0,\infty,\leq w'''}$ is independent of the choice of $w'$, $w''$, $w'''$. They form a projective system when $w$ runs through $\tilde{W}$. From this, we obtain a negative sheaf and denote it as $\mathcal{F} \boxtimes \mathcal{G} \in \mathcal{L}^\times D_{\mathcal{L}}$. It is clear that $\boxtimes$ is a functor from $\mathcal{L}^\times D_{\mathcal{L}} \times \mathcal{L}^\times D_{\mathcal{L}}$ to $\mathcal{L}^\times D_{\mathcal{L}}$.

By the base change theorem, we have $\mathcal{F} \boxtimes (\mathcal{G} \boxtimes \mathcal{H}) \simeq (\mathcal{F} \boxtimes \mathcal{G}) \boxtimes \mathcal{H}$ for $\mathcal{F} \in \mathcal{L}^\times D_{\mathcal{L}}$, $\mathcal{G} \in \mathcal{L}^\times D_{\mathcal{L}}$, $\mathcal{H} \in \mathcal{L}^\times D_{\mathcal{L}}$.

4.3. **Definition of $\tilde{W}_{\mathcal{L}}$ and $\tilde{W}_{\mathcal{L}}^\circ$.** Lusztig defined $\tilde{W}_{\mathcal{L}}^\circ$ attached to any $\mathcal{L} \in \text{Ch}(T)$ in $[L^*]$. Since this group plays an important role in our paper, we recall its construction. First, we construct the root system attached to any $\mathcal{L} \in \text{Ch}(T)$.

Let $\Phi^\vee$ be the coroot lattice of $G$, $\Phi^\vee_{\mathcal{L}}$ is the subset of coroots in $\Phi^\vee(G, T)$ such that restriction of $\mathcal{L}$ to those coroots are trivial. Let $\Phi_{\mathcal{L}}$ be the corresponding subset of roots. We form the endoscopic group $H$ using the root datum $(X^\vee(T), X_*(T), \Phi_{\mathcal{L}}, \Phi^\vee_{\mathcal{L}})$. Note that $H$ and $G$ share the same maximal torus, but $H$ is not a subgroup of $G$ in general. $W_{\mathcal{L}}^\circ$ is the Weyl group of this root system. It is a normal subgroup of $W_{\mathcal{L}}$, the stabilizer of $\mathcal{L}$ in the finite Weyl group $W$ of $G$.

Similar to the reductive group case, we define $\tilde{\Phi}^\vee_{\mathcal{L}}$ to be the subset of affine coroots whose restrictions of $\mathcal{L}$ are trivial. Fix a non-degenerate $W$-invariant bilinear form on $X_*(T) \otimes_\mathbb{Z} \mathbb{Q}$. Let $\tilde{W}_{\mathcal{L}}^\circ$ be the group generated by the reflections of $X_*(T) \otimes_\mathbb{Z} \mathbb{Q}$ correspond to the coroots in $\tilde{\Phi}^\vee_{\mathcal{L}}$. The results in $[Bo]$ show that $\tilde{W}_{\mathcal{L}}^\circ$ is a Coxeter group. On the other hand, the affine Weyl group of $H_{\mathcal{L}'}$, $\tilde{W}_{\mathcal{H}} = \tilde{W}_{H} \ltimes Q_{\mathcal{H}}$, is a subgroup of $\tilde{W} = W \ltimes X_*(T)$. We now show that $\tilde{W}_{\mathcal{L}}^\circ = \tilde{W}_{\mathcal{H}}$. Notice that $(w, \phi) \in W \ltimes X_*(T)$ sends $-\phi$ to 0. Hence $(w, \phi)$ is an orthogonal reflection if and only if it is the reflection along the affine hyperplane passing through $-\phi/2$ and perpendicular to $\phi$. Therefore $(w, \phi)$ is a reflection if and only if $\phi$ is a multiple of a coroot $\alpha^\vee$ and $w$ is the associated reflection $s_\alpha$. For $m \in \mathbb{Z}$, the reflection $(s_\alpha, ma^\vee)$ is in $\tilde{W}_{\mathcal{L}}^\circ$ if and only if $s_\alpha \in W_{\mathcal{L}}$. It is now clear that both $\tilde{W}_{\mathcal{L}}^\circ$ and $\tilde{W}_{\mathcal{H}}$ are generated by same set of reflections.

It is also not hard to prove that $\tilde{W}_{\mathcal{L}}^\circ$ is a normal subgroup in $\tilde{W}_{\mathcal{L}}$, the stabilizer of $\mathcal{L}$ in $\tilde{W}$. Here the action of $\tilde{W}$ on $\text{Ch}(T)$ is defined to factor through $W$. Note that $\tilde{W}_{\mathcal{L}}^\circ$ is isomorphic to $W_{\mathcal{L}} \ltimes X_*(T)$. Henceforth we fix $\alpha$, a $W$-orbit of $\text{Ch}(T)$.

**Definition 4.1.** Let $\mathcal{L}, \mathcal{L}' \in \alpha$. Define $\mathcal{L} \cdot \tilde{W}_{\mathcal{L}} := \{w \in \tilde{W} : w(\mathcal{L}) = \mathcal{L}'\} = \mathcal{L} \cdot W_{\mathcal{L}} \ltimes X_*(T)$ and define $\mathcal{L} \cdot \tilde{W}_{\mathcal{L}}^\circ := \mathcal{L} \cdot \tilde{W}_{\mathcal{L}} \cap \tilde{W}_{\mathcal{L}}^\circ$. Elements in this coset space are called the **blocks** of $\alpha$.

**Example 4.2.** Let $G = \text{Sp}_{2n}$. The roots of $G$ are $\{ \pm L_i \pm L_j \} \cup \{ \pm 2L_i \}$ and the coroots are $\{ \pm e_i \pm e_j \} \cup \{ \pm e_i \}$, where $L_i$'s are orthonormal and $e_i$ is the dual basis.

---

1. Lusztig used $W^+$ instead of $\tilde{W}_{\mathcal{L}}^\circ$. Here we follow the notation in [LY].
The cocharacter lattice is spanned by the coroots and the character lattice is dual to the cocharacter lattice. There exists an order 2 character sheaf \( \mathcal{L} \), which is fixed by the Weyl group of \( G \), with \( \Phi^\vee_\mathcal{L} \) equal to the set of all long coroots \( \{ \pm e_i \pm e_j \} \).

Hence \( \Phi_\mathcal{L} \) is the set of all short roots \( \{ \pm L_i \pm L_j \} \). In this case, the endoscopic group \( H_\mathcal{L} \) is \( SO_{2n} \). Moreover \( \tilde{W}_\mathcal{L} \) is an index four normal subgroup of \( \tilde{W} = \tilde{W} \), that is there are four blocks of \( \mathcal{O} \).

5. Basic properties of Hecke categories

Most results in Section 3 and 4 of [LY] remain true for positive sheaves by replacing the groups with their affine analogs. The statements involve the finiteness of the Weyl group need to be modified. For instance, Lemma 4 in [LY] needs to be substituted by the results of Section 2 in [L]. This is because \( \tilde{W}_\mathcal{L} \) could be infinite and there are no maximal elements in each block. The arguments in [LY] work well after such modifications because the essence of the proof involves reducing the argument for reductive groups to the case of SL2, and the same argument applies to loop groups as well. Therefore we do not reproduce the arguments for positive sheaves and only give a sketch of the proof for negative sheaves. Let us first introduce the analogs of the lemmas used in [LY] for our cases.

**Lemma 5.1.** For \( F \in \mathcal{L}_\mathcal{L} \cdot D_\mathcal{L} \), \( G \in \mathcal{L}_\mathcal{L} \cdot D_\mathcal{L} \), and \( \mathcal{H} \in \mathcal{L}_\mathcal{L} \cdot D_\mathcal{L} \), there are canonical isomorphisms

\[
\mathbb{D}^+(F \ast_+ \mathcal{H}) \cong \mathbb{D}^+(F) \ast_+ \mathbb{D}^+(\mathcal{H}), \quad \mathbb{D}^-(G \ast_- \mathcal{H}) \cong \mathbb{D}^-(G) \ast_- \mathbb{D}^+(\mathcal{H}).
\]

**Proof.** Note that \( \mathcal{G} \mathcal{K} \cdot \mathcal{L} \mathcal{U} \) is proper. \( \square \)

**Lemma 5.2.**

1. If \( F \in \mathcal{L}_\mathcal{L} \cdot D_\mathcal{L} \), \( G \in \mathcal{L}_\mathcal{L} \cdot D_\mathcal{L} \), and \( \mathcal{H} \in \mathcal{L}_\mathcal{L} \cdot D_\mathcal{L} \) are semisimple complexes, that is their images under \( \omega \) are direct sums of shifted simple non-mixed perverse sheaves, then \( F \ast_+ \mathcal{H} \) and \( G \ast_- \mathcal{H} \) are also semisimple.

2. If \( F \in \mathcal{L}_\mathcal{L} \cdot D_\mathcal{L} \), \( G \in \mathcal{L}_\mathcal{L} \cdot D_\mathcal{L} \), and \( \mathcal{H} \in \mathcal{L}_\mathcal{L} \cdot D_\mathcal{L} \) are pure of weight zero, then so are \( F \ast_+ \mathcal{H} \) and \( G \ast_- \mathcal{H} \).

**Proof.** The same reason as the previous lemma. \( \square \)

**Lemma 5.3.** Let \( w_1, w_2, w_3, w_4 \in \tilde{W} \) such that \( \ell(w_1) + \ell(w_2) = \ell(w_3) = \ell(w_4) \) and \( \ell(w_3) - \ell(w_4) = \ell(w_3 \cdot w_4) \).

Then we have canonical isomorphisms

\[
\Delta(\hat{\omega}_1)_{w_2 \mathcal{L}}^\ast + \Delta(\hat{\omega}_2)_{w_2 \mathcal{L}}^\ast \cong \Delta(\hat{\omega}_1 \hat{\omega}_2)_{w_2 \mathcal{L}}^\ast, \quad \nabla(\hat{\omega}_1)_{w_2 \mathcal{L}}^\ast \ast_+ \nabla(\hat{\omega}_2)_{w_2 \mathcal{L}}^\ast \cong \nabla(\hat{\omega}_1 \hat{\omega}_2)_{w_2 \mathcal{L}}^\ast,
\]

\[
\Delta(\hat{\omega}_3)_{w_4 \mathcal{L}}^- \ast_- \Delta(\hat{\omega}_4)_{w_4 \mathcal{L}}^- \cong \Delta(\hat{\omega}_3 \hat{\omega}_4)_{w_4 \mathcal{L}}^-,
\]

\[
\nabla(\hat{\omega}_3)_{w_4 \mathcal{L}}^- \ast_- \nabla(\hat{\omega}_4)_{w_4 \mathcal{L}}^- \cong \nabla(\hat{\omega}_3 \hat{\omega}_4)_{w_4 \mathcal{L}}^-.
\]

**Proof.** It follows from the fact that \( \mathcal{H}_{w_3 w_4}^{\mathcal{L} w_3 w_4} \to \mathcal{B} \mathcal{U}_{w_3 w_4}^{\mathcal{L} w_3 w_4} \) is a biregular morphism if \( \ell(w_3) - \ell(w_4) = \ell(w_3 \cdot w_4) \). \( \square \)

**Lemma 5.4.** Let \( w \in \tilde{W} \), there are canonical isomorphisms

\[
\Delta(\hat{\omega}^{-1})_{w \mathcal{L}}^\ast \ast_+ \nabla(\hat{\omega})_{w \mathcal{L}}^\ast \cong \Delta(e)_{w \mathcal{L}}^\ast \cong \nabla(\hat{\omega}^{-1})_{w \mathcal{L}}^\ast \ast_+ \Delta(\hat{\omega})_{w \mathcal{L}}^\ast.
\]

Therefore the functor \(( - ) \ast_+ \Delta(\hat{\omega})_{w \mathcal{L}}^\ast \) is an equivalence from \( \mathcal{L}_w \mathcal{L} \) to \( \mathcal{L}_w \mathcal{L} \) with inverse given by \(( - ) \ast_- \nabla(\hat{\omega})_{w \mathcal{L}}^\ast \ast_+ \Delta(\hat{\omega})_{w \mathcal{L}}^\ast \) Similarly, the functor \(( - ) \ast_- \Delta(\hat{\omega})_{w \mathcal{L}}^\ast \) is an equivalence from \( \mathcal{L}_w \mathcal{L} \) to \( \mathcal{L}_w \mathcal{L} \) with the inverse given by \(( - ) \ast_- \nabla(\hat{\omega})_{w \mathcal{L}}^\ast \ast_+ \Delta(\hat{\omega})_{w \mathcal{L}}^\ast \).

**Proof.** See Lemma 3.5 in [LY]. \( \square \)
Lemma 5.5. Let \( s \in \widehat{W} \) be a simple reflection and \( s \notin \widehat{W}_0^\circ \).

1. The natural maps of \( \Delta(s)^{+} \to IC(s)^{+}_L \to \nabla(s)^{+}_L \) are isomorphisms.
2. The functor \((-) \star_{+} IC(s)^{+}_L : \mathcal{C}D_{sL} \to \mathcal{C}D_{L} \) is an equivalence and the inverse is given by \((-) \star_{+} IC(s^{-1})^{+}_L \).
3. The functor \((-) \star_{-} IC(s)^{+}_L : \mathcal{C}D_{sL} \to \mathcal{C}D_{L} \) is an equivalence and the inverse is given by \((-) \star_{-} IC(s^{-1})^{+}_L \).
4. For any \( w \in \mathcal{C}W_{sL} \), the equivalence \((-) \star_{+} IC(s)^{+}_L \) sends \( \Delta(w)^{+}_L, IC(w)^{+}_L \), \( \nabla(w)^{+}_L \) to the isomorphism classes \( \Delta(w^s)^{+}_L, IC(w^s)^{+}_L, \nabla(w^s)^{+}_L \) respectively.
5. For any \( w \in \mathcal{C}W_{sL} \), the equivalence \((-) \star_{-} IC(s)^{+}_L \) sends \( \Delta(w^s)^{+}_L, IC(w)^{+}_L, \nabla(w)^{+}_L \) to the isomorphism classes \( \Delta(w^s)^{+}_L, IC(w^s)^{+}_L, \nabla(w^s)^{+}_L \) respectively.

Proof. (3) follows from (1) and Lemma 5.3. For (5), if \( \ell(w) = \ell(ws) + 1 \), then \( \Delta(w)^{-}_{sL} \star IC(s)^{+}_L \cong \Delta(w)^{+}_{sL} \star IC(s)^{-}_L \) by Lemma 5.3. If \( \ell(w) = \ell(ws) - 1 \), then \( \Delta(w)^{-}_{sL} \star IC(s)^{+}_L \cong \Delta(w)^{+}_{sL} \star \nabla(s)^{+}_L \cong \Delta(w^s)^{+}_{sL} \) by the same lemma. The same argument works for costandard sheaves. Any object with finite type support in \( \mathcal{C}D_{sL} \) is a successive extension of \( \mathcal{C}D_{sL} \) and \( M \) for \( w' \in \mathcal{C}W_{sL}, n \geq 0 \) and finite-dimensional Frobenius module \( M \). As a result, the functor \((-) \star_{-} IC(s)^{+}_L \) sends \( \mathcal{C}D_{sL}^{<0} \to \mathcal{C}D_{sL}^{-]<0} \). By Verdier duality, it also sends \( \mathcal{C}D_{sL}^{-0} \to \mathcal{C}D_{sL}^{-0} \). Therefore it is perverse \( \ell \)-exact and preserves simple perverse sheaves. From the fact that there is a nonzero map from \( \Delta(w^s)^{+}_{sL} \cong \Delta(w^s)^{+}_{sL} \star IC(s)^{+}_L \) to IC\((w^s)^{+}_{sL} \star IC(s)^{+}_L \), the latter sheaf has to be IC\((w^s)^{+}_L \).

Let \( s \) be a simple reflection in \( \widehat{W} \) such that \( s \in \widehat{W}_0^\circ \). There is a canonical object \( IC(s)^{+}_L \in \mathcal{C}D_{sL} \) with an isomorphism from its stalk at \( s \) and \( \widehat{W}(1) \) (see [LY, Section 3.7]). \( L \) can be extended to \( \widehat{L} \) on \( L_s \). Similar to the construction of negative sheaves, we define \( \mathcal{C}D_{sL}^{-} \) to be \( D_{(T \times L_s, \mathcal{C}D_{sL}^{-1} \cdot \mathcal{C}D_{sL}^{<0})} \). There are two adjoint pairs \((\pi^{+}_s, \pi^{-}_s) \) and \((\pi^{+}_s, \pi^{-}_s) \). Here \( \pi^{-}_s \) sends \( \mathcal{C}D_{sL}^{-} \to \mathcal{C}D_{sL}^{<0} \) and \( \pi^{+}_s \cong \pi^{+}_s(2) : \mathcal{C}D_{sL}^{-} \to \mathcal{C}D_{sL}^{+} \).

Lemma 5.6. Let \( \mathcal{L}, \mathcal{L}' \in \mathcal{O} \) and \( s \) be a simple reflection in \( \widehat{W} \) such that \( s \in \widehat{W}_0^\circ \). There is a canonical isomorphism of endo-functors

\[
(-) \star_{+} IC(s)^{+}_L \cong \pi^{+}_s \pi^{-}_s (-)/(1) \cong \pi^{+}_s \pi^{-}_s (-)/(-1) \in \text{End}(\mathcal{C}D_{sL}^{-}).
\]

As a result, \((-) \star_{-} IC(s)^{+}_L \) is right adjoint to itself.

Proof. See Lemma 3.8 in [LY].

Fix \( w \in \{ w : ws < w \} \) and \( \bar{w} \) be a representative of \( NGK(T_K)(\mathbb{F}_q)/T_{O}(\mathbb{F}_q) \). We define \( C(\bar{w})^{-}_{L} \in D_{(T \times L_s, \mathcal{C}D_{sL}^{-1} \cdot \mathcal{C}D_{sL}^{<0})} \) whose stalk at \( \bar{w} \) is \( \sum_{\ell}(\text{max}\{-\ell(w), -\ell(ws)\}) \) as in Section 3.10 in [LY]. \( \Delta(\bar{w})^{-}_{L}, IC(\bar{w})^{-}_{L} \), and \( \nabla(\bar{w})^{-}_{L} \) can be defined similarly and isomorphism classes of their images under \( \omega \) are denoted by \( \Delta(\bar{w})^{-}_{L}, IC(\bar{w})^{-}_{L}, \nabla(\bar{w})^{-}_{L} \), and \( \nabla(\bar{w})^{-}_{L} \) respectively.

Lemma 5.7. Let \( s \) be a simple reflection in \( \widehat{W} \) such that \( s \in \widehat{W}_0^\circ \). Suppose \( \ell(w) > \ell(ws) \), then \( \pi^{+}_s IC(\bar{w})^{-}_{L} \cong IC(\bar{w})^{-}_{L} \).

Proof. See Lemma 3.10 in [LY]. Note that \( \pi^{-}_s \) is a smooth \( \mathbb{P}^{1} \)-fibration, hence \( \pi^{+}_s[1] \) sends (simple) perverse sheaves to (simple) perverse sheaves.

Proposition 5.8. Let \( w \in \widehat{W}, \mathcal{L} \in \mathcal{O} \) and \( v \in w_{eL} \). Let \( i_v : I_{u} \backslash G_{K,v}/I_{u} \to I_{u} \backslash G_{K}/I_{u} \).
Proof. See Proposition 3.11 in [LY].

For a block \( \beta \in \mathcal{LC} \), the restriction of the Bruhat order of \( \tilde{W} \) gives a partial order \( \leq \) in \( \beta \). Lusztig showed that there is a unique minimal element \( \omega^\beta \) in each block \( \beta \in \mathcal{LC} \) and it is characterized by sending every positive coroot in \( \Phi^\vee_\mathbb{C} \) to positive coroots in \( \Phi^\vee_\mathbb{C} \).

Lemma 5.9. Let \( \beta \in \mathcal{LC} \) and \( \gamma \in \mathcal{LC} \), then \( w^\gamma w^\beta = w^{\gamma + \beta} \).

Proof. See Corollary 4.3 in [LY].

Definition 5.10. Let \( \beta \in \mathcal{LC} \) and \( w \in \beta \). There is a unique \( v \in \tilde{W}^w \) such that \( w = \omega^\beta v \). Define \( \ell_\beta(w) := \ell_v(w) \), the length of \( v \) in the Coxeter group \( \tilde{W}^w \). Define the partial order \( \leq_\beta \) on the block \( \beta \) as follows. Suppose \( w' \in \beta \) and \( v' \in \tilde{W}^w \) be its corresponding element as above, define \( w \leq_\beta w' \) if and only if \( v \leq_\beta v' \).

Remark 5.11. \( \leq_\beta \) could be different from that in \( \tilde{W}^w \).

Proposition 5.12. For \( w, w' \in \beta \), if \( w \leq_\beta w' \), then \( w \leq w' \).

Proof. See Lemma 4.8 in [LY].

Definition 5.13. Let \( \beta \in \mathcal{LC} \).

1. Denote \( \mathcal{C} \mathcal{D}_\mathcal{C} \) to be the full subcategory generated by \( \{ \Delta(w)^\mathcal{C} \}_{w \in \beta} \). It consists of sheaves whose cohomology is supported on \( w \in \beta \). Denote \( \mathcal{C} \mathcal{D}_\mathcal{C}^\beta \subset \mathcal{C} \mathcal{D}_\mathcal{C} \) to be the preimage of \( \mathcal{C} \mathcal{D}_\mathcal{C}^\beta \) under \( \omega \). \( \mathcal{C} \mathcal{D}_\mathcal{C}^\beta \) (resp. \( \mathcal{C} \mathcal{D}_\mathcal{C}^\beta \)) is called a block of \( \mathcal{C} \mathcal{D}_\mathcal{C} \) (resp. \( \mathcal{C} \mathcal{D}_\mathcal{C}^\beta \)).

2. If \( \beta \) is the unit coset \( \tilde{W}^w \), we say \( \mathcal{C} \mathcal{D}_\mathcal{C}^\beta \) (resp. \( \mathcal{C} \mathcal{D}_\mathcal{C}^\beta \)) is the neutral block and denote it by \( \mathcal{C} \mathcal{D}_\mathcal{C}^\beta \) (resp. \( \mathcal{C} \mathcal{D}_\mathcal{C}^\beta \)).

Proposition 5.14. There is a direct sum decomposition of the triangulated category
\[
\mathcal{C} \mathcal{D}_\mathcal{C} = \bigoplus_{\beta \in \mathcal{LC}} \mathcal{C} \mathcal{D}_\mathcal{C}^\beta.
\]

Proof. See Proposition 4.11 in [LY].

Definition 5.15. For any \( \beta \in \mathcal{LC} \), we call \( \mathcal{F} \in \mathcal{C} \mathcal{D}_\mathcal{C} \) (resp. \( \mathcal{C} \mathcal{D}_\mathcal{C} \)) a maximal (resp. minimal) IC sheaf if \( \omega \mathcal{F} \) is isomorphic to \( \mathcal{IC}(w^\beta) \mathcal{C} \) (resp. \( \mathcal{IC}(w^\beta) \mathcal{C} \)).

A rigidified maximal IC sheaf is a pair \((\mathcal{C} \Theta_\mathcal{C}, \epsilon_\mathcal{C})\) where \( \mathcal{C} \Theta_\mathcal{C} \in \mathcal{C} \mathcal{D}_\mathcal{C} \) is a maximal IC sheaf and \( \epsilon_\mathcal{C} : \mathcal{C} \Theta_\mathcal{C} \rightarrow \delta_\mathcal{C} \) is a non-zero morphism.

Both \( \mathcal{C} \Theta_\mathcal{C} \) and \( \delta_\mathcal{C} \) are viewed as sheaves on \( \mathbf{G}_K/I_u \). Rigidified maximal sheaves clearly exist and are unique up to unique isomorphism.

Proposition 5.16. Let \( \beta \in \mathcal{LC} \) and \( \gamma \in \mathcal{LC} \). For any \( w \in \beta \).

1. The convolution \( \mathcal{IC}(w^\gamma)^\mathcal{C} \ast \mathcal{IC}(w^\beta)^\mathcal{C} \) is isomorphic to a direct sum of shifts of \( \mathcal{IC}(w^\gamma)^\mathcal{C} \).

2. The perverse cohomology \( \mathcal{H}^i(\mathcal{IC}(w^\gamma)^\mathcal{C} \ast \mathcal{IC}(w^\beta)^\mathcal{C}) \) vanishes unless \( i \leq \ell_\beta(w) \).
Proposition 5.18. Let \( \beta \in \hat{W}_\mathcal{L} \) and \( w \in \beta \). For \( j_w : \text{Bun}_{0, \infty}^w \hookrightarrow \text{Bun}_{0, \infty} \), we have \( j_w^* \mathcal{L}(w^\beta)_E \cong \mathcal{C}(w)_E[\ell_\beta(w)] \) and \( j_w^* \mathcal{L}(w^\beta)_E \cong \mathcal{C}(w)_E[\ell_\beta(w)] \).

Proof. The second statement follows from applying Verdier duality to the first statement.

We prove the first statement for all \( \mathcal{L} \) in \( \mathfrak{g} \) by induction on \( \ell(w) \). If \( w = e \), it is trivial. Suppose \( \ell(w) < n \), we are going to show it also holds for elements of length \( n \). Let \( w \in \hat{W} \) of length \( n \) and \( s \) be a simple reflection such that \( \ell(w) = \ell(w') + 1 \). The same argument in the proof of Lemma 6.3 in [LY] shows \( \mathcal{L}(w')_E \ast \mathcal{L}(s)_E \) is perverse. The decomposition theorem implies \( \mathcal{L}(w)_E \) is a direct summand of \( \mathcal{L}(w')_E \ast \mathcal{L}(s)_E \). Hence it suffices to show the analogous statements hold for \( \mathcal{L}(w')_E \ast \mathcal{L}(s)_E \), which follows from induction.

Proposition 5.17. Let \( \beta \in \hat{W}_\mathcal{L} \) and \( w \in \beta \). For \( j_w : \text{Bun}_{0, \infty}^w \hookrightarrow \text{Bun}_{0, \infty} \), we have \( j_w^* \mathcal{L}(w^\beta)_E \cong \mathcal{C}(w)_E[\ell_\beta(w)] \).

Proof. The second statement follows from applying Verdier duality to the first statement.

We prove the first statement for all \( \mathcal{L} \) in \( \mathfrak{g} \) by induction on \( \ell(w) \). If \( w = e \), it is trivial. Suppose \( \ell(w) \) holds for any \( w \) with \( \ell(w') < n \), we are going to show it also holds for elements of length \( n \). Let \( w \in \hat{W} \) of length \( n \) and \( s \) be a simple reflection such that \( \ell(w) = \ell(ws) + 1 \).

If \( s \notin \hat{W}_\mathcal{L} \), then \( w^\beta s \) is also a minimal element in a block. By Lemma 5.3, \( \mathcal{L}(w^\beta)_E \ast \mathcal{L}(s)_E \cong \mathcal{C}(w^\beta)_E \ast \mathcal{C}(s)_E \). Since the stalk can be computed from \( \text{Hom}(\mathcal{C}(w^\beta)_E, \mathcal{C}(s)_E) \) and right convolution of \( \mathcal{L}(s)_E \) is an equivalence, the statement follows from the induction hypothesis for \( ws \).

If \( s \in \hat{W}_\mathcal{L} \), Lemma 5.7 implies the stalks of \( \mathcal{L}(w^\beta)_E \) at \( \bar{w} \) and \( \bar{w} s \) are isomorphic. The latter one is known by the induction hypothesis. The fact that \( \ell_\beta(w) = \ell_\beta(ws) + 1 \) finishes the proof.

For restrictions of maximal IC sheaves on other strata, we have the following proposition.

Proposition 5.18. Let \( \beta \in \hat{W}_\mathcal{L} \) and \( w \notin \beta \). For \( j_w : \text{Bun}_{0, \infty}^w \hookrightarrow \text{Bun}_{0, \infty} \), we have \( j_w^* \mathcal{L}(w^\beta)_E \cong j_w^* \mathcal{L}(w^\beta)_E \cong 0 \).

Proof. The argument in 5.17 applies.

6. Maximal IC sheaves

In this section we introduce the averaging functor of maximal IC sheaf \( \Lambda_{\mathcal{L}} (\mathcal{C}/\Theta \mathcal{L}) \) and compute its stalks. After that, we give a characterization of the sheaf and construct maps between several-fold convolution products of \( \Lambda_{\mathcal{L}} (\mathcal{C}/\Theta \mathcal{L}) \).
6.1. **Averaging.** The goal in this subsection is to define an averaging functor, which is similar to that in [BY].

Let \( J_w \) be a normal subgroup of \( I \) chosen in Section 5.1 and \( m_{J_w} : I_u/J_w \times G_{K, \leq w}/I_u \to G_{K, \leq w}/I_u \) be the left multiplication map. It is a morphism between finite-dimensional varieties. We define the functor \( Av_{I_u, \leq w} : D_m^b(G_{K, \leq w}/I_u) \to D_m^b(G_{K, \leq w}/I_u) \) by sending \( F \) to \( m_{J_w}(\langle \mathbb{Q}_{T'} \boxtimes F \rangle (2 \dim(I_u/J_w))) \). It is independent of the choice of \( J_w \). Let \( J'_w \subset J_w \) be another choice. It follows from the fact that \( p \times id \) is an affine space fibration of dimension \( 2 \dim(J_w/J'_w) \) and the following diagram commutes:

\[
\begin{array}{ccc}
I_u/J'_w \times G_{K, \leq w}/I_u & \xrightarrow{m_{J'_w}} & G_{K, \leq w}/I_u \\
\downarrow{p \times id} & & \downarrow{m_{J_w}} \\
I_u/J_w \times G_{K, \leq w}/I_u & \xrightarrow{m_{J_w}} & G_{K, \leq w}/I_u
\end{array}
\]

\( Av_{I_u, \leq w} \) commutes with \(*\)-pullback. That is, \( i^*: Av_{I_u, \leq w} \) is canonically isomorphic to \( Av_{I_u, \leq w} \circ i^*: Av_{I_u, \leq w} \). where \( i^*: G_{K, \leq w}/I_u \to G_{K, \leq w}/I_u \) is the closed embedding. One way to see this is to apply the proper base change theorem to the following Cartesian diagram.

\[
\begin{array}{ccc}
I_u/J_w' \times G_{K, \leq w}/I_u & \xrightarrow{m_{J_w'}} & G_{K, \leq w}/I_u \\
\downarrow{id \times i_{w', w}} & & \downarrow{i_{w', w'}} \\
I_u/J_w' \times G_{K, \leq w}/I_u & \xrightarrow{m_{J_w'}} & G_{K, \leq w}/I_u.
\end{array}
\]

Notice that the independence of \( J_w \) means it can be taken as \( J_{w'} \). Moreover, the isomorphism is compatible with \( w \leq w' \leq w'' \).

As a result, \( \lim_{\leftarrow} Av_{I_u, \leq w} : D_m^b(T \times T', \mathcal{L}' \boxtimes \mathcal{L}^{-1}), m(G_{K}/I_u) \to D_m^b(T \times T', \mathcal{L}' \boxtimes \mathcal{L}^{-1}), m(G_{K}/I_u) \) is well defined. Here

\[
D_m^b(T \times T', \mathcal{L}' \boxtimes \mathcal{L}^{-1}), m(G_{K}/I_u) := 2 - \lim_{w \in \widehat{W}} D_m^b(T \times T', \mathcal{L}' \boxtimes \mathcal{L}^{-1}), m(G_{K, \leq w}/I_u)
\]

is a projective system of triangulated categories with pullback functors \( i^*: T \to T' \). Concretely, the objects in this category are \( (\mathcal{F}_w, \chi_{w, w'}) \), where \( \mathcal{F}_w \) is a sheaf on \( G_{K, \leq w}/I_u \) with the prescribed monodromy and \( \chi_{w, w'} \) is an isomorphism from \( \mathcal{F}_w \) to \( i^* \mathcal{F}_{w'} \) such that \( i^\ast w, w' \circ \chi_{w, w'} = \chi_{w, w'} \) whenever \( w \leq w' \leq w'' \). The morphism between \( (\mathcal{F}_w, \chi_{w, w'}) \) and \( (\mathcal{G}_w, \psi_{w, w'}) \) is a family of morphisms \( \phi_w : \mathcal{F}_w \to \mathcal{G}_w \) such that \( \psi_{w, w'} \circ \phi_w = \phi_{w'} \circ \chi_{w, w'} \) for any \( w \leq w' \).

Any object in the image of \( \lim_{\leftarrow} Av_{I_u, \leq w} \) has a natural \( I_u \)-equivariant structure. Therefore it makes sense to define the averaging functor as follows.

**Definition 6.1.** Define \( Av_{I_u} : D_m^b(T \times T', \mathcal{L}' \boxtimes \mathcal{L}^{-1}), m(G_{K}/I_u) \to D_m^b(T \times T', \mathcal{L}' \boxtimes \mathcal{L}^{-1}), m(G_{K}/I_u) \) by remembering the \( I_u \)-equivariant structure. Explicitly, it sends \( \langle \mathcal{F}_w \rangle_{w \in \widehat{W}} \) to \( (m_{J_w}(\langle \mathbb{Q}_{T'} \boxtimes \mathcal{F}_w \rangle (2 \dim(I_u/J_w))))_{w \in \widehat{W}} \). Here

\[
D_m^b(T \times T', \mathcal{L}' \boxtimes \mathcal{L}^{-1}), m(I_u \setminus G_{K, \leq w}/I_u).
\]
For \( w \leq w' \in \hat{W} \), the image of the composition map \( G_{K, \leq w}/I_u \to G_K/I_u \to \text{Bun}_{0, \infty} \) is contained in \( \text{Bun}_{0, \infty}^{\leq w'} \). Let \( i_{w'}^w : G_{K, \leq w} \to \text{Bun}_{0, \infty}^{\leq w} \) be its restriction map. Since \( i_{w'}^w \) is a map between two finite type stacks, it makes sense to \( * \)-pullback sheaves on \( \text{Bun}_{0, \infty}^{\leq w} \) to \( G_{K, \leq w}/I_u \) via \( i_{w'}^w \). Therefore, we can average objects in \( \mathcal{L}D_\mathcal{L}^- \) and \( \mathcal{L}D_\mathcal{L} \). In particular, \( \text{Av}_1 \mathcal{L}T \mathcal{L}^- \in \mathcal{L}D_\mathcal{L} \).

6.1.1. Basic properties of the averaging functor.

**Lemma 6.2.** Let \( \mathcal{G} \in \mathcal{L}D_\mathcal{L}^- \). \( \text{Av}_1 \mathcal{G} \) is canonically isomorphic to \( \mathcal{G} \) by viewing \( \mathcal{L}D_\mathcal{L}^- \) as a subcategory of \( \mathcal{L}D_\mathcal{L} \).

**Lemma 6.3.** \( \text{Av}_1 \) commutes with the convolution product. That is, \( \text{Av}_1(F \ast_* \mathcal{G}) \) is canonically isomorphic to \( \text{Av}_1 F \ast_\mathcal{G} \) for \( F \in \mathcal{L}D_\mathcal{L}^- \), \( \mathcal{G} \in \mathcal{L}D_\mathcal{L}^- \).

**Proof.** This follows from the base change theorem. \( \square \)

**Corollary 6.4.** \( \text{Av}_1(\mathcal{L} \Theta^- \ast_* \mathcal{G}) \) is canonically isomorphic to \( \text{Av}_1 \mathcal{L} \Theta^- \ast_\mathcal{G} \) for any \( \mathcal{G} \in \mathcal{L}D_\mathcal{L}^- \).

**Proof.** Let \( F = (F_w)_{w \in \hat{W}} \in \mathcal{L}D_\mathcal{L}^- \) and \( \mathcal{G} \in \mathcal{L}D(\leq w') \). For any \( w \in \hat{W} \), choose \( w' \in M(w, w'-1) \). From the definition of Hecke stack, we can see that \( i_{w'}^w(F_w) \ast_\mathcal{G} \cong i_{w'}^w(F_w) \ast_\mathcal{G} \). The rest follows from Lemma 6.3. \( \square \)

**Lemma 6.5.** \( \text{Av}_1 \) is left adjoint to forgetful functor. In particular, \( \text{Hom}_{G_K/I_u}(\mathcal{L} \Theta^- \mathcal{L} \ast_\mathcal{L} F) = \text{Hom}_{I_u / G_K / I_u}(\text{Av}_1 \mathcal{L} \Theta^- \mathcal{L} F) \) for \( F \in \mathcal{L}D_\mathcal{L}^- \).

**Corollary 6.6.** Recall that \( \delta_\mathcal{L} \) is the monoidal unit in \( \mathcal{L}D_\mathcal{L}^- \). There is a canonical map \( \epsilon_\mathcal{L} : \text{Av}_1 \mathcal{L} \Theta^- \mathcal{L} \to \delta_\mathcal{L} \) which induces the identity map at stalk \( e \).

6.2. Hyperbolic localization. The stalks of \( \text{Av}_1 \mathcal{L} \Theta^- \mathcal{L} \) can be computed via hyperbolic localization. To do so, we identify \( \text{Bun}_{0, \infty} \) with \( [I_u / G_K / I_u] \) as in 2.4.2 and make use of the loop rotation on \( G_K \).

6.2.1. Construction of the \( \mathbb{G}_m \)-action. We note that \( \mathbb{G}_m \) acts on \( G_K \) by the loop rotation and we denote this action by rot. Explicitly, when \( a \in \mathbb{G}_m \) and \( x \in G_K \), \( \text{rot}(a)(x) \) is replacing the uniformizer \( t \) in \( x \) with \( at \).

For any positive integer \( n \), let \( J^n \) be the unipotent group whose \( A \)-points are \( \ker(G(\mathbb{A}[t^{-1}]) \to G(\mathbb{A}[t^{-1}]/(t^n))). \) As in 2.4.2, we can define \( \text{Bun}_{0, J^n} \) the moduli stack of \( G \)-bundles on \( \mathbb{P}^1 \) with a \( J^n \)-level structure at infinity and a \( U \)-reduction at zero. There is a natural map from \( \text{Bun}_{0, J^n} \) to \( \text{Bun}_{0, \infty} \). Let \( \text{Bun}_{0, J^n}^{\leq w} \) (resp. \( \text{Bun}_{0, \infty}^{\leq w} \)) be the preimage of \( \text{Bun}_{0, \infty}^{\leq w} \) (resp. \( \text{Bun}_{0, \infty}^{\leq w} \)). \( \text{Bun}_{0, J^n} \) is smooth and locally of finite type. It can be identified with \( [J^n / G_K / I_u] \).

6.2.2. Attracting set and repelling set. Fix \( w \in \hat{W} \) and \( \hat{w} \) is a representative in the coset \( N_{G_K}(T_K)(\mathbb{F}_q) / T_D(\mathbb{F}_q) \) corresponding to \( w \) throughout this subsection. By \( [\mathcal{E}] \), there exists a positive integer \( n \) such that \( \text{Bun}_{0, J^n}^{\leq w} \) is a variety. Fix such an \( n \) and denote \( J^n \) by \( J_{\infty} \). We construct a \( \mathbb{G}_m \)-action \( \phi_w \) on \( G_K \) by declaring \( \phi_w(a)(x) := \chi^w(a) \text{rot}(a^{2h+1})/(x) \lambda(a), \) where \( \lambda(a) \) is defined to be \( \text{rot}(a^{2h+1})/(\hat{w})^{-1} \chi^w(a^{-1} \hat{w} \in T \), \( \chi^w \) is the sum of positive coorots and \( h \) is the Coxeter number of \( G \). It is easy to check that \( \phi_w \) descends to an action on \( \text{Bun}_{0, J_{\infty}}^{\leq w} \). We also denote the induced action by \( \phi_w \).
It is clear that \( \dot{w}T = [J_{\infty} \setminus J_{\infty} w I / I_u] \) is fixed by \( \phi_w \). We want to find the attracting set \( X_+ := \{ x \in \text{Bun}_{0, J_{\infty}} w : \lim_{u \to 0} \phi_w(a)(x) \in \dot{w}T \} \) and the repelling set \( X_- := \{ x \in \text{Bun}_{0, J_{\infty}} w : \lim_{u \to 0} \phi_w(a)(x) \in \dot{w}T \} \).

Rewrite \( \phi_w(a)(x) \) as \( \text{Ad}(\chi^\vee(a)) \circ \text{rot}(a^{2h+1})(\dot{x} w^{-1}) \dot{w} \). When \( a \) tends to zero, \( \text{Ad}(\chi^\vee(a)) \circ \text{rot}(a^{2h+1}) \) contracts \( I_u \) to \( e \). When \( a^{-1} \) tends to zero, \( \text{Ad}(\chi^\vee(a)) \circ \text{rot}(a^{2h+1}) \) contracts \( I_u \) to \( e \). It follows that \( X_+ \) contains \( [J_{\infty} \setminus J_{\infty} G_{K, w} / I_u] \) and \( X_- \) contains \( \text{Bun}_{0, J_{\infty}} w \). Notice that \( [J_{\infty} \setminus J_{\infty} G_{K, w} / I_u] \) can be naturally identified as \( G_{K, w} / I_u \) a \( T \)-torsor over \( I_u \dot{w} I_u / I_u \). Since \( I_u \dot{w} I_u / I_u \) and \( \text{Bun}_{0, J_{\infty}} w \) intersect transversally at \( \dot{w} \), we know that

1. \( \dot{w}T \) is a connected component of fixed points of \( \phi_w \);
2. \( X_+ = \text{Bun}_{0, J_{\infty}} w \);
3. \( X_- = [J_{\infty} \setminus J_{\infty} G_{K, w} / I_u] = G_{K, w} / I_u \).

6.2.3. Computation of stalks. Let \( \pi : \text{Bun}_{0, J_{\infty}} w \to \text{Bun}_{0, \infty} w \) be the projection map. Then \( \pi^* \mathcal{L} \Theta_{\mathcal{L}} \) is \( \phi_w \)-equivariant. Hence hyperbolic localization is applicable to \( \pi^* \mathcal{L} \Theta_{\mathcal{L}} \). It states that \( i_{w T}^* i_{X_-}^* (\pi^* \mathcal{L} \Theta_{\mathcal{L}}(\pi^* \mathcal{L} \Theta_{\mathcal{L}})) \cong i_{w T}^* i_{X_+}^* (\pi^* \mathcal{L} \Theta_{\mathcal{L}}(\pi^* \mathcal{L} \Theta_{\mathcal{L}})), \) where \( i_{w T}, i_{X_-}, i_{X_+} \) are the embeddings of \( w T, X_- w, X_+ w \) into \( \text{Bun}_{0, J_{\infty}} w \) respectively.

**Proposition 6.7.** Let \( i_w \) be the embedding of \( G_{K, w} / I_u \) into \( G_{K, w} / I_u \). When \( w \in \dot{W}_\mathcal{L}^\circ, i_w^* (\text{Av}_1 \mathcal{L} \Theta_{\mathcal{L}}) \) is isomorphic to \( \mathcal{C}(w)_{\mathcal{L}} [\ell_{\mathcal{L}}(w)] \). When \( w \notin \dot{W}_\mathcal{L}^\circ, i_w^* (\text{Av}_1 \mathcal{L} \Theta_{\mathcal{L}}) \) vanishes.

**Proof.** It suffices to show that the stalk of \( \dot{w} \) is \( \overline{\mathbb{Q}^e \ell_{\mathcal{L}}(w)} \) when \( w \in \dot{W}_\mathcal{L}^\circ \) and zero otherwise.

\[
\begin{align*}
i_w^* (\text{Av}_1 \mathcal{L} \Theta_{\mathcal{L}}) &= H^*_{\mathcal{L}}(I_u \dot{w} I_u / I_u, \mathcal{L} \Theta_{\mathcal{L}}) & \text{proper base change theorem} \\
&= i_w^* i_{\dot{w} I_u}^* i_{I_u}^* (\mathcal{L} \Theta_{\mathcal{L}}) & \text{because } \mathcal{L} \Theta_{\mathcal{L}} \text{ is } \mathcal{G}_m \text{-equivariant} \\
&= i_w^* i_{\dot{w} I_u}^* i_{I_u}^* i_{X_-}^* (\pi^* \mathcal{L} \Theta_{\mathcal{L}}) & \text{base change theorem} \\
&= i_w^* i_{\dot{w} I_u}^* i_{I_u}^* i_{X_+}^* (\pi^* \mathcal{L} \Theta_{\mathcal{L}}) & \text{base change theorem} \\
&= i_w^* i_{\dot{w} I_u}^* (\pi^* i_{X_-}^* \mathcal{L} \Theta_{\mathcal{L}}) & \text{base change theorem}
\end{align*}
\]

The last term is \( \overline{\mathbb{Q}^e \ell_{\mathcal{L}}(w)} \) when \( w \in \dot{W}_\mathcal{L}^\circ \) by Proposition 5.17 and vanishes when \( w \notin \dot{W}_\mathcal{L}^\circ \) by Proposition 6.18.

Here \( \pi' \) is the projection map from \( X_- \) to \( \text{Bun}_{0, \infty} w \), and \( i_A \) is used to denote an embedding from \( A \) to some space. It is abuse of notation as \( i_{I_u \dot{w} I_u / I_u} \) in second row is different from the third row, the meaning is clear though.

The same argument or Lemma 6.3 shows that we can generalize the proposition to other maximal IC sheaves.

**Proposition 6.8.** Let \( \beta \in \ell_{\mathcal{L}} \dot{W}_\mathcal{L} \). Let \( i_w \) be the embedding of \( G_{K, w} / I_u \) into \( G_{K, w} / I_u \). When \( w \in \beta, i_w^* (\text{Av}_1 \mathcal{L} \Theta_{\mathcal{L}}(w^\beta)) \) is isomorphic to \( \mathcal{C}(w)^{\beta}_{\mathcal{L}} [\ell_{\mathcal{L}}(w)] \). When \( w \notin \beta, i_w^* (\text{Av}_1 \mathcal{L} \Theta_{\mathcal{L}}(w^\beta)) \) vanishes.

**Definition 6.9.** For \( w \in \dot{W}_\mathcal{L}^\circ \), define \( C(w)^{\uparrow}_{\mathcal{L}} := i_w^* (\text{Av}_1 \mathcal{L} \Theta_{\mathcal{L}}(\ell_{\mathcal{L}}(w))) \). The above proposition shows that \( \omega C(w)^{\uparrow}_{\mathcal{L}} \) is isomorphic to \( \mathcal{C}(w)^{\uparrow}_{\mathcal{L}} \). Define \( \Delta(w)^{\uparrow}_{\mathcal{L}} := i_w^* (C(w)^{\uparrow}_{\mathcal{L}}), \) \( \nabla(w)^{\wedge}_{\mathcal{L}} := i_w^* (C(w)^{\wedge}_{\mathcal{L}}), \) and \( \text{IC}(w)^{\uparrow}_{\mathcal{L}} := i_w^* (C(w)^{\uparrow}_{\mathcal{L}}) \).
6.3. Characterization of averaging maximal IC sheaves. We give a characterization of averaging maximal IC sheaves in this subsection.

6.3.1. Facts about general Coxeter group. We first recall three facts about general Coxeter groups and proofs can be found in [BE]. Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter group and $\ell$ be the length function.

**Lemma 6.10.** Let $u > v$ be two elements in $\mathcal{W}$ such that $\ell(u) = \ell(v) + 2$. There exists exactly two elements $x_1$ and $x_2$ in $W$ such that $u > x_i > v$ for $i = 1, 2$.

**Lemma 6.11.** For any reflection $t$ in $\mathcal{W}$, there exists a palindromic reduced expression for $t$.

**Lemma 6.12.** Let $u \in \mathcal{W}$. Let $v_1$ and $v_2$ be two elements in $\mathcal{W}$ such that $u > v_i$ and $\ell(u) = \ell(v_i) + 1$ for $i = 1, 2$. There exists $y, z_1, z_2 \in W$ ($z_1$ and $z_2$ can be the same) such that $y^{-1}u$ is a simple reflection, $u > v_i > z_i$, $u > y > z_i$ and $\ell(u) = \ell(z_i) + 2$ for $i = 1, 2$.

**Proof.** Write $y$ in a reduced expression $s_{i_1}s_{i_2}\cdots s_{i_n}$. For $k = 1, 2$, write $v_k = s_{i_1}s_{i_2}\cdots s_{i_k}\cdots s_{i_n}$. Take $y = s_{i_1}u$ and $z_k = s_{i_1}\cdots s_{i_k}\cdots s_{i_n}$ for $k = 1, 2$. \hfill $\square$

6.3.2. Definition of glueable objects.

**Definition 6.13.** Let $\mathcal{F} \in \mathcal{E} \mathcal{D}_L$. Denote $\mathcal{F}_w$ to be the extension by zero of $\mathcal{F}|_{G_{K_w}/I_w}$. We call $\mathcal{F}$ glueable with respect to $\beta \in \widetilde{W}/\widetilde{W}_2^2$ if it satisfies following properties

1. $\mathcal{F}_w = 0$ for $w \notin \beta$;
2. $\omega_{\mathcal{F}_w} \cong \Delta_{(w)_{\mathcal{L}}}[-\ell_{\beta}(w)]$ for $w \in \beta$;
3. For any $w \in \beta$ and simple reflection $s \in \widetilde{W}_2^2$ (not necessarily a simple reflection in $\widetilde{W}$) such that $ws > \beta w$, we have hom$^0(\mathcal{F}_w[\ell_{\beta}(w)], \mathcal{F}_{ws}[\ell_{\beta}(ws)]) = 0$. Moreover, the first homomorphism in the open-closed triangle

$$\mathcal{F}_w[\ell_{\beta}(w)] \rightarrow \mathcal{F}_{ws}[\ell_{\beta}(ws)] \rightarrow \mathcal{F}|_{G_{K_w}/I_w \cup G_{K_w,ws}/I_w}[\ell_{\beta}(ws)]^{[1]}$$

is a non-zero homomorphism.

Similarly, we can define glueable objects in the non-mixed category by replacing condition 2 and 3 with

1. $\mathcal{F}_w \cong \Delta_{(w)_{\mathcal{L}}}[\ell_{\beta}(w)]$ for $w \in \beta$;
2. For any $w \in \beta$ and simple reflection $s \in \widetilde{W}_2^2$ such that $ws > \beta w$, the first homomorphism in the open-closed triangle

$$\mathcal{F}_w[\ell_{\beta}(w)] \rightarrow \mathcal{F}_{ws}[\ell_{\beta}(ws)] \rightarrow \mathcal{F}|_{G_{K_w}/I_w \cup G_{K_w,ws}/I_w}[\ell_{\beta}(ws)]^{[1]}$$

is a non-zero homomorphism.

It is clear that $\omega\mathcal{F}$ is glueable when $\mathcal{F}$ is glueable.

**Example 6.14.** When $\mathcal{L}$ is trivial, the constant sheaf is glueable.

6.3.3. Characterization of averaging maximal IC sheaves via glueable objects.

**Theorem 6.15.** $\omega\Pi_{\mathbb{E}}$ is the unique (up to tensoring with a one-dimensional Fr-module) glueable object and $\omega\mathbb{A}_{\Pi_{\mathbb{E}}}$ is the unique glueable non-mixed object with respect to $\beta$, where $\mathbb{E}$ is a maximal IC sheaf for $\beta$.

The proof of this theorem follows from Section 6.3.4 and 6.3.5.
6.3.4. Averaging maximal IC sheaves are glueable.

Proposition 6.16. For any maximal IC sheaf \( \Xi \) for \( \beta \), \( \text{Av}_1 \Xi \) is glueable with respect to \( \beta \).

Proof. The first two conditions are proved in Lemma 6.7. Let us now check condition 3. We know from Lemma 5.5 that the convolution product of a maximal IC sheaf and a minimal IC sheaf is a maximal IC sheaf. Since right convolution with a minimal IC sheaf is an equivalence, we know that condition 3 holds for the tuple \( (F = \text{Av}_1 \Xi, w, s) \) if and only if it holds for \( (F = \text{Av}_1 \Xi \ast \xi, wu, usu) \) for some \( u \) such that \( \xi \) is a minimal IC sheaf with \( \omega \xi \cong \mathbb{IC}(u) \). Choose \( u \) to be the \( x \) in Lemma 6.17 below, we may and will assume \( s \) is a simple reflection in \( \tilde{W} \). Condition 3 follows from Lemma 6.7. \( \square \)

Lemma 6.17. Let \( s \) be a simple reflection in \( \tilde{W}_2 \), there exists a minimal element \( x \) in \( \mathcal{L} \) such that \( x^{-1}sx \) is a simple reflection in \( \tilde{W} \) and \( x^{-1}sx \in \tilde{W}_2 \).

Proof. By Lemma 6.11 write \( s = s_{i_1}s_{i_2} \cdots s_{i_k}si_i \) in a palindromic reduced expression in \( \tilde{W} \). Let \( L_j = s_{i_1} \cdots s_{i_j}(\mathcal{L}) \) for \( j \geq 1 \). Since \( \ell_1(\tilde{W}) = 1 \), we know that \( s_{i_j} \notin \tilde{W}_{2j-1} \) for \( j < k \) and \( s_{i_k} \in \tilde{W}_{2k-1} \) by Lemma 4.6 in [LY]. We can choose \( x \) to be \( s_{i_{k-1}} \cdots s_{i_2}si_i \). \( \square \)

6.3.5. Glueable objects are averaging maximal IC sheaves. Let \( V_{n,\beta} \) be the set of \( (w_1, w_2) \in \beta^2 \) such that \( w_1 \leq_{\beta} w_2 \) and \( \ell_\beta(w_1) + 1 = \ell_\beta(w_2) = n \). Let \( \beta \) be the union of \( V_{n,\beta} \) for \( n \in \mathbb{N} \). When \( \beta \) is the neutral block, we write \( V_0^\beta \) and \( V^\beta \) instead of \( V_{n,\beta} \) and \( \beta \). Notice that there is a natural bijection from \( V_{n,\beta}^\beta \) to \( V_{n,\beta} \) by left multiplication of \( w_\beta \) (which is different from right multiplication of \( w_\beta \)).

Let \( \tilde{W}_{\leq n}^\beta \) be the elements \( w \) in \( \beta \) such that \( \ell_\beta(w) \leq n \). Let \( \text{cl}(\tilde{W}_{\leq n}^\beta) \) be the subset of \( w' \in \tilde{W} \) such that \( w' \leq w \) for some \( w \in \tilde{W}_{\leq n}^\beta \). Let \( \tilde{W}_{\leq n}^\beta \) be \( \text{cl}(\tilde{W}_{\leq n}^\beta) \). For any sheaf \( F \) with support contained in \( \beta \), denote \( F_{\leq n}^\beta \) (resp. \( F_{\leq n}^\beta \)) be the restriction of \( F \) on \( \bigcup_{w \in \text{cl}(\tilde{W}_{\leq n}^\beta)} G_{K,w}^\beta / I_w \) (resp. \( \bigcup_{w \in \tilde{W}_{\leq n}^\beta} G_{K,w}^\beta / I_w \)). We also denote \( F_w^\beta \) (resp. \( F_{\leq n}^\beta \)) be the extension by zero of \( F_{G_{K,w}^\beta} \) (resp. \( F_{G_{K,w}^\beta} \)).

Let \( F \) be a non-mixed glueable sheaf with respect to \( \beta \).

Lemma 6.18. Let \( w_1, w_2 \in \beta \) such that \( w_1 \leq_{\beta} w_2 \) and let \( k = \ell_\beta(w_2) - \ell_\beta(w_1) \). Then \( \text{Ext}^i(F_{w_1}, F_{w_2}) = 0 \) unless \( i \geq k \).

Proof. It follows from the fact that the perverse degrees of \( F_{w_i}^\beta \) is \( \ell_\beta(w_i) \) for \( i = 1, 2 \). \( \square \)

Proposition 6.19. Let \( w_1, w_2 \in \beta \) such that \( w_1 < w_2 \) and \( \text{Hom}^0(F_{w_1}, F_{w_2}[1]) \neq 0 \).

Then \( (w_1, w_2) \in V_{\beta}^\beta \).

Proof. For any \( x, y \in \tilde{W} \), the costalk of \( \text{IC}(x) \) at \( y \) is non-zero if and only if \( y \leq_{\beta'} x \), where \( \beta' \) is the block containing \( x \). Therefore \( F_{w}[1] \) can be written as the successive extension of \( \text{IC}(y)[n] \) for \( y \leq_{\beta} w \) and \( n \in \mathbb{Z} \). This implies that \( w_1 \leq_{\beta} w_2 \). The length relationship follows from Lemma 6.18.

The converse of Proposition 6.19 is true.

Lemma 6.20. Let \( (w_1, w_2) \in V_{\beta}^\beta \). Then \( \text{Hom}^0(F_{w_1}, F_{w_2}[1]) = \text{IC}(x) \).

In particular, the dimension of \( \text{hom}^0(\Delta(w_1), \Delta(w_2)) \) is either 0 or 1.
Proof. Let \( w_i := (w^\beta)^{-1} w_i \) for \( i = 1, 2 \). Let \( w' = s_{i_1} s_{i_2} \cdots s_{i_k} \) be a reduced expression in terms of simple reflections in \( \hat{W}^{\beta} \). Then \( w'_j = s_{i_1} s_{i_2} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_k} \) for some \( j \). Since left (right) convolution with \( \Delta(s_{i_j}) \) is an equivalence for \( 1 \leq l \leq k \), the lemma follows from \( \text{Hom}^0(\Delta(v), \Delta(s_{i_j})) = \Omega \).

By Proposition 6.19, we have a distinguished triangle \( \mathcal{F}_n \to \mathcal{F}_{\leq n} \to \mathcal{F}_{\leq n-1} \to [1] \), which gives an element in \( \mathcal{E}_n \in \text{Ext}^1(\mathcal{F}_{\leq n-1}, \mathcal{F}_n) \). Since \( \mathcal{F} \) is constructed by successive extensions of standard sheaves on \( \bigcup_{w \in \hat{W}^{\beta}} \mathbf{G}_{K, w}/I_u \), the data of extension classes \( \mathcal{E}_n \) determines the isomorphism class of \( \mathcal{F} \). We now examine the extension group \( \text{Ext}^1(\mathcal{F}_{\leq n-1}, \mathcal{F}_n) \).

Note that there is a long exact sequence

\[
\rightarrow \text{Ext}^1(\mathcal{F}_{\leq n-2}, \mathcal{F}_n) \rightarrow \text{Ext}^1(\mathcal{F}_{\leq n-1}, \mathcal{F}_n) \rightarrow \text{Ext}^1(\mathcal{F}_{n-1}, \mathcal{F}_n) \rightarrow \text{Ext}^2(\mathcal{F}_{\leq n-2}, \mathcal{F}_n) \rightarrow .
\]

The first term vanishes because of Lemma 6.18. Note that \( \text{Ext}^1(\mathcal{F}_{n-1}, \mathcal{F}_n) \) is the direct sum of \( \text{Ext}^1(\mathcal{F}_{n'}, \mathcal{F}_n) \) for \( (n', n) \in V_{n, \beta} \) and \( \text{Ext}^1(\mathcal{F}_{n'}, \mathcal{F}_n) \) is one-dimensional by Lemma 6.20. The boundary map is the cup product with \( \mathcal{E}_{n-1} \). Since the natural map \( \text{Ext}^2(\mathcal{F}_{\leq n-2}, \mathcal{F}_n) \rightarrow \text{Ext}^2(\mathcal{F}_{n-2}, \mathcal{F}_n) \) is injective, we have the following exact sequence,

\[
0 \rightarrow \text{Ext}^1(\mathcal{F}_{\leq n-1}, \mathcal{F}_n) \rightarrow \text{Ext}^1(\mathcal{F}_{n-1}, \mathcal{F}_n) \rightarrow \text{Ext}^2(\mathcal{F}_{n-2}, \mathcal{F}_n) \rightarrow ,
\]

where the last map is the cup product with the canonical class of \( \mathcal{F} \mid_{\bigcup_{w \in \hat{W}^{\beta} \cup \hat{W}^{\beta}_{n-1}} \mathbf{G}_{K, w}/I_u} \) in \( \text{Ext}^1(\mathcal{F}_{n-2}, \mathcal{F}_{n-1}) \).

For \( w \in \beta \), the automorphism group of \( \Delta(v) \) is \( \Omega \). We choose an identification of the stalk of \( \Delta(v) \) at any point in \( \mathbf{G}_{K, w}/I_u \) with \( \Omega \). Then we identify \( \text{Ext}^1(\mathcal{F}_{n-1}, \mathcal{F}_n) \) as the space of \( \Omega \)-valued functions on \( V_{n, \beta} \). Different identifications of stalks change the function by conjugation, that is \( f(w', w) \) transforms to \( g(w') f(w', w) g(w)^{-1} \). The function \( e_n \) corresponding to \( \mathcal{E}_n \) has a nice property.

**Definition 6.21.** A \( \Omega \)-valued function \( f \) defined on \( V_{n, \beta} \) (or \( V^\circ \)) is said to be **anti-commutative** if it satisfies the following condition. For any \( (u, v) \in V_{n, \beta} \) (or \( V^\circ \)), we have

\[
f(u, x_1) f(x_1, v) + f(u, x_2) f(x_2, v) = 0,
\]

where \( x_1 \) and \( x_2 \) are the two elements given in Lemma 6.18 (where the Coxeter group there is \( \hat{W}^{\beta} \)). For a \( \Omega \)-valued function \( f \) on \( V_{n, \beta} \) (or \( V^\circ \)), we call it anti-commutative if \( f \) is anti-commutative under the bijection of \( V_{n, \beta} \) and \( V^\circ \) (resp. \( V^\circ \) and \( V^\circ \)).

It is not hard to check that the definition is independent of whether \( V_{n, \beta} \) and \( V_{n, \beta} \) are identified by left or right multiplication of \( w^\beta \).

From the above discussion, we know that \( c_n \) is anti-commutative. Since \( \omega \mathbf{Av} \mathbf{\Xi} \) is glueable, its corresponding function \( c_n \) is also anti-commutative.

**Lemma 6.22.** Let \( f \) be an anti-commutative function of \( V_{\beta} \). If \( f(u, us) \neq 0 \) for any \( u \in \beta \) and simple reflection \( s \) in \( \hat{W}^{\beta} \) such that \( u < \beta \), then \( f(u, v) \neq 0 \) for any \( (u, v) \in V_{\beta} \).

**Proof.** It is obvious. \( \square \)

To prove \( \mathcal{F} \) is isomorphic to \( \omega \mathbf{Av} \mathbf{\Xi} \), we apply the following combinatorical proposition to \( e_n \) and \( c_n \).
There exists a $\mathbb{Q}_l$-valued function $g$ of $\mathcal{W}^\circ_\mathbb{L}$ such that
\[ f_1(u,v)g(v) = g(u)f_2(u,v) \]
for all $(u,v) \in V$.

**Proof.** We prove by induction. Suppose we construct $g_{n-1}$ satisfying the above property for all $(u,v) \in V_{\leq n-1}^\circ$. Let $f_3(u,v) = g_{n-1}(u)f_2(u,v)g_{n-1}(v)^{-1}$. Then $f_1$ and $f_2$ coincide on $V_{\leq n-1}^\circ$. For $(u,v_1), (u,v_2) \in V_n^\circ$, let $y, z_1, z_2$ be in Lemma 6.12. Then $f_j(u,v_1)f_j(v_1, z_1) + f_j(u,y)f_j(y, z_1) = 0$ for $i = 1, 2$ and $j = 1, 3$. This implies there exists a constant $h(u)$ such that $f_1(u,v) = h(u)f_3(u,v)$ for $(u,v) \in V_{\leq n}^\circ$. We let $g_n(u) = h(u)$ for $f_{\mathcal{W}_\mathbb{L}}(u) = n$ and $g_n(u) = g_{n-1}(u)$ for $f_{\mathcal{W}_\mathbb{L}}(u) < n$.

**Proposition 6.24.** Any two non-mixed glueable sheaves $\mathcal{F}$ and $\mathcal{G}$ in the block $\beta$ are isomorphic. In particular, $\mathcal{F}$ is isomorphic to $\omega \Lambda V_\mathbb{L}^{\geq 1}$, where $\Xi$ is a maximal $IC$ sheaf in $\beta$.

**Proof.** We construct a family of isomorphisms $\phi_{\leq n} : \mathcal{F}_{\leq n} \to \mathcal{G}_{\leq n}$ by induction. For $n = 0$, we choose an isomorphism from $\mathcal{F}_0$ to $\mathcal{G}_0$. Suppose now we have an isomorphism $\phi_{\leq n-1} : \mathcal{F}_{\leq n-1} \to \mathcal{G}_{\leq n-1}$. Consider the following diagram of distinguished triangles

\[
\begin{align*}
\mathcal{F}_{\leq n-1}[-1] & \longrightarrow \mathcal{F}_n & \longrightarrow \mathcal{F}_{\leq n} & \longrightarrow [1] \\
\phi_{\leq n-1}[-1] \downarrow & \quad & \downarrow & \downarrow \\
\mathcal{G}_{\leq n-1}[-1] & \longrightarrow \mathcal{G}_n & \longrightarrow \mathcal{G}_{\leq n} & \longrightarrow [1].
\end{align*}
\]

Notice that $\mathcal{F}_n$ and $\mathcal{G}_n$ are direct sum of standard sheaves. From the above discussion and Proposition 6.23, we know that there exists a morphism $\phi_n : \mathcal{F}_n \sim \mathcal{G}_n$ such that the first square is commutative. By the axiom of triangulated categories, we can construct an (non-canonical) isomorphism from $\phi_n : \mathcal{F}_{\leq n} \sim \mathcal{G}_{\leq n}$ extending $\phi_{\leq n-1}$.

The proof for glueable sheaves in the mixed categories is similar. We simply note the following two lemmas.

**Lemma 6.25.** Let $w_1, w_2 \in \mathcal{W}^\circ_\mathbb{L}$ such that $\ell_\mathcal{L}(w_2) = \ell_\mathcal{L}(w_1) + 1$ and $w_1 < w_\mathcal{W}_\mathbb{L} w_2$ (the Bruhat order for $\mathcal{W}^\circ_\mathbb{L}$). Let $M$ be a one-dimensioinal $Fr$-module. The dimension of $\text{hom}(\Delta(w_1)_\mathcal{L}, \Delta(w_2)_\mathcal{L} \otimes M)$ is at most one. Moreover, there exists a unique (up to isomorphism) $M$ such that the dimension of $\text{hom}^0(\Delta(w_1)_\mathcal{L}, \Delta(w_2)_\mathcal{L} \otimes M)$ is one.

**Lemma 6.26.** Suppose $\mathcal{F} \in \mathcal{L}^D_{\mathcal{L}}$ such that its Frobenius pullback $\omega \mathcal{F}$ is isomorphic to $\Delta(w)_\mathcal{L}$, there exists a one-dimensioinal Fr-module $M$ such that $\mathcal{F} \cong \Delta(w)_\mathcal{L} \otimes M$.

**Proof.** From the exact sequence (5.1.2.5) in [BBE], we have $\text{hom}(\mathcal{F}, \Delta(w)_\mathcal{L}) = \text{Hom}(\omega \mathcal{F}, \Delta(w)_\mathcal{L})^Fr = \mathbb{Q}_l^{\text{Fr}}$. Take $M$ to be the dual of $\text{hom}(\mathcal{F}, \Delta(w)_\mathcal{L})$. $\square$
6.4. Convolution products of maximal sheaves.

**Lemma 6.27.** For \( w \in \tilde{W}_L^p \), \( \mathcal{L}_\mathcal{O}_L^* \rightarrow \mathcal{L}(w)^+ \) is canonically isomorphic to \( \mathcal{L}_\mathcal{O}_L^*[-\ell_L(w)] \).

**Proof.** By expressing \( \mathcal{L}(w)^+ \) as the product of positive standard sheaves, it reduces to the case when \( w = s \in \tilde{W}_L^0 \). The first statement follows from Lemma 5.7.

The proper base change theorem and the fact that \( \mathcal{H}^s(\mathcal{A}^1) = \mathcal{O}_L[2] \). The second statement can be proved similarly.

In particular, when \( s \in \tilde{W}_L^0 \), taking the long exact sequence for the convolution of \( \mathcal{L}_\mathcal{O}_L^* \) with \( \mathcal{L}(s)^+ \rightarrow \mathcal{L}(s)^+ \rightarrow \mathcal{L}_\mathcal{O}_L^* \), \( \mathcal{L}(s)^+ \rightarrow \mathcal{L}_\mathcal{O}_L^* \rightarrow \mathcal{L}_\mathcal{O}_L^* \) shows that the restriction map \( \mathcal{H}^\ell(\mathcal{L}_\mathcal{O}_L^* \rightarrow \mathcal{L}(s)^+ \rightarrow \mathcal{L}_\mathcal{O}_L^*) \) is an isomorphism.

**Lemma 6.28.** For \( w \in \tilde{W}_L^0 \), there is a unique map \( \theta_w^u : \text{Av}_1 \mathcal{L}_\mathcal{O}_L^* \rightarrow \mathcal{I}(\mathcal{C}(w)^+ \rightarrow \mathcal{L}(w)^+ \rightarrow \mathcal{L}_\mathcal{O}_L^*) \), whose restriction to \( \mathcal{G}_{K,w}/I_w \) is the identity map of \( \mathcal{C}(w)^+ \rightarrow \mathcal{L}(w)^+ \rightarrow \mathcal{L}_\mathcal{O}_L^* \).

**Proof.** We mimic the proof of Theorem 3.4.1 in [BGS]. For any \( n \in \mathbb{N}_{\geq 0} \), let \( Y_n \) be the union of \( \mathcal{G}_{K,w} \) for all \( w \) such that \( w' \leq w \) and \( \ell(w') = n \). With respect to this stratification, \( \mathcal{C}(w)^+ \rightarrow \mathcal{L}(w)^+ \rightarrow \mathcal{L}_\mathcal{O}_L^* \) is filtered with the \( n \)-th subquotient \( \bigoplus_{\ell(w') = n} i_{w'}^! \mathcal{I}(\mathcal{C}(w') \rightarrow \mathcal{L}(w') \rightarrow \mathcal{L}_\mathcal{O}_L^*) \).

This gives rise to a spectral sequence whose \( E_1 \)-page is \( \bigoplus_{\ell(w') = n} \text{Hom}^u(i_{w'}^! \mathcal{C}(w') \rightarrow \mathcal{L}(w') \rightarrow \mathcal{L}_\mathcal{O}_L^*) \).

We claim that the spectral sequence degenerates "like a chessboard". It suffices to show the degeneracy after base change to \( k \). By Lemma 5.8 and Proposition 6.7, \( \text{Hom}^u(i_{w'}^! \mathcal{C}(w') \rightarrow \mathcal{L}(w') \rightarrow \mathcal{L}_\mathcal{O}_L^*) \) is the direct sum of \( \text{Hom}^u(i_{w'}^! \mathcal{C}(w') \rightarrow \mathcal{L}(w') \rightarrow \mathcal{L}_\mathcal{O}_L^*) \), where \( m = \ell(w) - \ell(w') \mod 2 \). The fact that \( \ell(w') = \ell(w') \mod 2 \) implies the degeneracy. As a result, \( M := \text{Hom}^u(\text{Av}_1 \mathcal{L}_\mathcal{O}_L^* \rightarrow \mathcal{C}(w)^+ \rightarrow \mathcal{L}(w)^+ \rightarrow \mathcal{L}_\mathcal{O}_L^*) \) admits an increasing filtration \( F_n \) by Fr-submodules whose associated graded piece \( \text{Gr}_n^F \) is \( \bigoplus_{\ell(w') = n} \text{Hom}^u(i_{w'}^! \mathcal{C}(w') \rightarrow \mathcal{L}(w') \rightarrow \mathcal{L}_\mathcal{O}_L^*) \).

After showing the existence of filtration, the argument in [LY] Lemma 6.8 implies both the existence and uniqueness of \( \theta_w^u \) by showing the quotient map \( M \rightarrow \text{Gr}_n^F \) is an isomorphism in degrees \( \leq 1 \).

**Lemma 6.29.**

1. \( \text{IC}(e)^+ \rightarrow \mathcal{L}_\mathcal{O}_L^* \) can be identified with \( \mathcal{L}_\mathcal{O}_L^* \) canonically so that \( \theta_w^u \) coincides with the rigidification \( \theta_w^u \).

2. When \( s \in \tilde{W}_L^0 \) is a simple reflection, there is a unique isomorphism \( \varphi_s : \text{IC}(s)^+ \rightarrow \text{IC}(s)^+ \) so that \( \varphi_s \circ \theta_w^u : \text{Av}_1 \mathcal{L}_\mathcal{O}_L^* \rightarrow \text{IC}(s)^+ \rightarrow \mathcal{L}_\mathcal{O}_L^* \) restricts to the identity map on the stalks at \( e \). Recall that \( \text{IC}(s)^+ \rightarrow \mathcal{L}_\mathcal{O}_L^* \) is introduced in Lemma 5.8 and both stalks of \( \text{Av}_1 \mathcal{L}_\mathcal{O}_L^* \rightarrow \mathcal{C}(s)^+ \rightarrow \mathcal{L}_\mathcal{O}_L^* \) at \( e \) are equipped with an isomorphism with trivial Fr-module \( \mathcal{O}_e \).

**Proof.** See Lemma 6.9 in [LY].

**Lemma 6.30.** There is a unique map \( \alpha \) from \( \mathcal{L}_\mathcal{O}_L^* \rightarrow \text{Av}_1 \mathcal{L}_\mathcal{O}_L^* \) making the following diagram commute.

\[
\begin{array}{ccc}
\mathcal{L}_\mathcal{O}_L^* & \xrightarrow{\alpha} & \text{Av}_1 \mathcal{L}_\mathcal{O}_L^* \\
\downarrow_{\mathcal{L}_\mathcal{O}_L^*} & & \downarrow_{\text{Av}_1 \mathcal{L}_\mathcal{O}_L^*} \\
\delta_L & \xrightarrow{=} & \delta_L
\end{array}
\]

**Proof.** By the definition of homomorphisms between limit objects, a map from \( \mathcal{L}_\mathcal{O}_L^* \rightarrow \text{Av}_1 \mathcal{L}_\mathcal{O}_L^* \) is equivalent to give a family of compatible maps from \( \mathcal{L}_\mathcal{O}_L^* \) to...
Define $i^*_v(Av_l Θ_{l^-})$ for each $v$, which is the same as maps from $Av_l Θ_{l^-}$ to $i^*_v(Av_l Θ_{l^-})$ by Lemma 6.5. The restriction map is the canonical family of maps in the last description and we let $α : Θ_{l^-} → Av_l Θ_{l^-}$ corresponding to that. It is easy to see $α$ completes the commutative diagram.

To prove uniqueness, it suffices to show that $\text{Hom}^0(Θ_{l^-}, Av_l Θ_{l^-}) = Q_l$. For $m = 0, 1$ and $w \neq e \in \hat{W}$,

$$\text{Hom}^m(Θ_{l^-}, i_w i^*_w Av_l Θ_{l^-}) = \text{Hom}^m(Θ_{l^-}, \Delta(w)[−ℓ_l(w)])$$

$$= \text{Hom}^m(Θ_{l^-} * \sqcap(w^{-1}), δ_l[−ℓ_l(w)])$$

$$= \text{Hom}^m(Θ_{l^-}[ℓ_l(w)], δ_l[−ℓ_l(w)]) = 0.$$

Using the iteration of open-closed distinguished triangles, we get

$$\text{Hom}^0(Θ_{l^-}, Av_l Θ_{l^-}) = \lim_v \text{Hom}^0(Θ_{l^-}, i^*_v Av_l Θ_{l^-})$$

$$= \text{Hom}^0(Θ_{l^-}, i^*_w Av_l Θ_{l^-})$$

$$= \text{Hom}^0(Θ_{l^-}, δ_l) = Q_l.$$

\[ \square \]

**Lemma 6.31.** There is a unique map $χ$ from $Θ_{l^-}$ to $Θ_{l^-} * Av_l Θ_{l^-}$ such that the composition $Θ_{l^-} ≥ l Θ_{l^-} * Av_l Θ_{l^-} \xrightarrow{id * - ℓ_l} Θ_{l^-} * \delta_l = Θ_{l^-}$ is the identity map. Moreover the composition $Θ_{l^-} ≥ l Θ_{l^-} * Av_l Θ_{l^-} \xrightarrow{id * - ℓ_l} Θ_{l^-} * Av_l Θ_{l^-}$ is the identity map.

**Proof.** Note that the stalks of $Θ_{l^-} * Av_l Θ_{l^-}$ are infinite-dimensional. Therefore it does not live in the category $D{\text{_{l^-}}}$, and we have to view all objects as pro-objects.

By the definition of pro-objects, it suffices to show for $w \in \hat{W}$, there is a unique map $χ_w : Θ_{l^-} → Θ_{l^-} * i^*_w Av_l Θ_{l^-}$ such that the composition $(id * - ℓ_l) \circ χ_w$ is the identity map.

For any $w'' \in \hat{W}$, let $j^{w''}_w$ be the restriction of $l D_{l^-}$ to $l (D(\leq w''))_{l^-}$. The perverse degree of $j^{w''}_w(Θ_{l^-} * Av_l Θ_{l^-})$ is concentrated in degree $2ℓ_l(w'')$ for any $w', w'' \in \hat{W}$. Since the associated piece of $i^*_w Av_l Θ_{l^-}$ with respect to the Schubert stratification is given by $\Delta(w')[−ℓ_l(w')]$, it follows that the lowest perverse cohomology of $j^{w''}_w(Θ_{l^-} * Av_l Θ_{l^-})$ is $j^{w''}_w(Θ_{l^-})$ in degree zero. We define $χ_w$ to be the limit of the embedding of perverse truncation at degree less than one.

Since the restriction of the first composition at $e$ is identity and $\text{Hom}^0(Θ_{l^-}, Θ_{l^-}) = Q_l$, it has to be the identity map.

To check the second composition is $α$, it suffices to show $\text{Hom}^0(Θ_{l^-}, i^*_w Av_l Θ_{l^-}) = Q_l$ for any $w \in \hat{W}$. Notice that $ℓ_l$ can be completed to a triangle $ηη'(Av_l Θ_{l^-}) → i^*_w Av_l Θ_{l^-} → δ_l$, where $η$ is the complement of the embedding $G_{K,(−w)l} / I_w$ into $G_{K,≤w}/I_w$. Apply the functor $\text{Hom}(Θ_{l^-}, −)$ to the convolution product of $Θ_{l^-}$ with this triangle. The above argument shows $\text{Hom}^m(Θ_{l^-}, Θ_{l^-} * ηη'(Av_l Θ_{l^-})) = 0$ for $m < 2$. Hence the composition of $ℓ_l$ induces $\text{Hom}^0(Θ_{l^-}, i^*_w Av_l Θ_{l^-}) \xrightarrow{ℓ_l} \text{Hom}^0(Θ_{l^-}, δ_l) = Q_l$. \[ \square \]
Proposition 6.32. There is a natural isomorphism $\gamma : (L\Theta^- \ast \ast_\beta_1 L\Theta^-) \ast_\beta_1 L\Theta^- \to L\Theta^- \ast_\beta_1 L\Theta^- \ast_\beta_1 L\Theta^- \ast_\beta_1 L\Theta^-)$ such that the following diagram commutes.

\[
\begin{array}{ccc}
L\Theta^- & \xrightarrow{\chi} & L\Theta^- \ast_\beta_1 L\Theta^- \\
\downarrow{id} & & \downarrow{\gamma} \\
L\Theta^- & \xrightarrow{\chi} & L\Theta^- \ast_\beta_1 L\Theta^- \ast_\beta_1 L\Theta^- \ast_\beta_1 L\Theta^- \\
\end{array}
\]

Here we view all objects as pro-objects of $L\mathcal{D}_C^-$ in $\mathcal{D}_C^\circ$. 

Proof. For each pair $(a, b) \in \mathcal{W} \times \mathcal{W}$, $\alpha$ restricts to a map from $L\Theta_\beta$ to $\ast_{a, b} L\Theta_\beta$. Hence we get $\alpha \ast_\beta \ast : L\Theta_\beta \ast_\beta_1 \ast_{a, b} L\Theta_\beta \to \ast_{a, b} L\Theta_\beta \ast_\beta_1 \ast_{a, b} L\Theta_\beta$. Now choose $a' \in M(a, b)$. $\alpha \ast_\beta \ast$ can be thought as a map from $\ast_{a, a'} L\Theta_\beta \ast_\beta_1 \ast_{b, b'} L\Theta_\beta$ to $\ast_{a, b} L\Theta_\beta \ast_\beta_1 \ast_{a, b} L\Theta_\beta$. One can check that it is well-behaved with respect to $a'$ and $(a, b)$, hence we get a map $\gamma : (L\Theta_\beta \ast_\beta_1 L\Theta^- \ast_\beta_1 L\Theta^- \ast_\beta_1 L\Theta^-) \to \ast_{a, b} L\Theta_\beta \ast_\beta_1 \ast_{a, b} L\Theta_\beta$.

The construction of the inverse of $\gamma$ is similar. Fix $(a, b) \in \mathcal{W} \times \mathcal{W}$ and choose $a' \in M(a, b)$. As in the previous paragraph, we have a map from $\ast_{a, a'} L\Theta_\beta \ast_\beta_1 \ast_{b, b'} L\Theta_\beta$ to $\ast_{a, b} L\Theta_\beta \ast_\beta_1 \ast_{b, b'} L\Theta_\beta$. It is an isomorphism and its inverse is well-behaved with respect to $a'$ and $(a, b)$. From this, we get a map from the opposite direction. One can check that it is the inverse of $\gamma$.

The proof is completed by noting that $\gamma$ preserves the lowest perverse cohomology, that is the perverse truncation at degree zero. 

After examining the properties of $\ast_{a, b} L\Theta_\beta$, we discuss the maximal IC sheaves in non-neutral blocks. Let $\mathcal{L}, \mathcal{L}' \in \mathcal{O}_C^\circ$, $\beta \in \mathcal{W}_C^\circ$, $\xi = IC(\mathcal{L}^\beta) \ast_\beta_1 L\Theta_\beta$ be a minimal IC sheaf for $\beta$. Let $\epsilon(\xi)$ be the canonical map $\mathcal{L}' \ast_\beta_1 L\Theta_\beta \to \mathcal{L}$. By abuse of notation, the same symbol $\epsilon(\xi)$ will also be used for the canonical map $\ast_{a, b} \mathcal{L}' \ast_\beta_1 L\Theta_\beta \to \mathcal{L}_a \ast_\beta_1 L\Theta_\beta \ast_\beta_1 L\Theta_\beta \ast_\beta_1 L\Theta_\beta$. 

Lemma 6.33. Let $\beta \in \mathcal{W}_C^\circ$ be a block, $\xi = IC(\mathcal{L}^\beta) \ast_\beta_1 L\Theta_\beta$ be a minimal IC sheaf for $\beta$. There exists a unique isomorphism $\tau(\xi) : \ast_\beta_1 L\Theta_\beta \to \ast_\beta_1 L\Theta_\beta$ such that the following diagram commutes.

\[
\begin{array}{ccc}
\xi \ast_\beta_1 L\Theta_\beta & \xrightarrow{\tau(\xi)} & \ast_\beta_1 L\Theta_\beta \\
\downarrow{id + \epsilon(\xi)} & & \downarrow{\epsilon(\xi)} \\
\xi \ast_\beta_1 \delta & = & \xi \ast_\beta_1 \delta
\end{array}
\]

Proof. It is easy to check that both $\xi \ast_\beta_1 L\Theta_\beta$ and $\ast_\beta_1 L\Theta_\beta \ast_\beta_1 L\Theta_\beta$ are glueable sheaves. Hence they are isomorphic by Theorem 6.3.4 Since right convolution with $\xi$ is an equivalence, we have $\text{Hom}^0(\xi \ast_\beta_1 L\Theta_\beta, L\Theta_\beta \ast_\beta_1 \ast_\beta_1 L\Theta_\beta) = \mathbb{Q}_l$. The existence and uniqueness of $\tau(\xi)$ follow from that. 

\]
Proof. (1) follows from the following commutative diagram.

\[ \begin{array}{ccc}
\Theta^- \ast \xi_2 \ast \xi_1 & \xrightarrow{\tau(\xi_2)} & \Theta^- \ast A \Theta^- \ast \xi_2 \ast \xi_1 \\
\Theta^- \ast \xi_2 \ast A \Theta^- \ast \xi_1 & \xrightarrow{\epsilon(\xi_2) \ast \epsilon(\xi_1)} & \xi_2 \ast \xi_1
\end{array} \]

(2) The following diagram commutes.

\[ \begin{array}{ccc}
\Theta^- \ast \xi_2 \ast \xi_1 & \xrightarrow{\alpha} & A \Theta^- \ast \xi_2 \ast \xi_1 \\
\Theta^- \ast A \Theta^- \ast \xi_2 \ast \xi_1 & \xrightarrow{\tau(\xi_2)} & \Theta^- \ast \xi_2 \ast A \Theta^- \ast \xi_1
\end{array} \]

(3) The following two compositions are the same.

(i) \[ \Theta^- \ast \xi_3 \ast \xi_2 \ast \xi_1 \xrightarrow{\chi} \Theta^- \ast A \Theta^- \ast \xi_3 \ast \xi_2 \ast \xi_1 \xrightarrow{\tau(\xi_3) \ast \xi_2 \ast \xi_1} \Theta^- \ast \xi_3 \ast \xi_2 \ast \xi_1 \]

(ii) \[ \Theta^- \ast \xi_3 \ast \xi_2 \ast \xi_1 \xrightarrow{\chi} \Theta^- \ast A \Theta^- \ast \xi_3 \ast \xi_2 \ast \xi_1 \xrightarrow{\tau(\xi_3) \ast \xi_2 \ast \xi_1} \Theta^- \ast \xi_3 \ast \xi_2 \ast \xi_1 \]

\[ \xrightarrow{A \chi} \Theta^- \ast \xi_3 \ast A \Theta^- \ast \xi_2 \ast \xi_1 \rightarrow \Theta^- \ast \xi_3 \ast \Theta^- \ast \xi_2 \ast \xi_1 \]

The second last map in (ii) is given by Lemma \[6.34\].

Proof. (1) follows from the following commutative diagram.

\[ \begin{array}{ccc}
\Theta^- \ast \xi_2 & \xrightarrow{\chi} & \Theta^- \ast A \Theta^- \ast \xi_2 \\
\Theta^- \ast \delta \ast \xi_2 & \xrightarrow{\epsilon} & \Theta^- \ast \xi_2 \ast \delta
\end{array} \]

(2) notice that \( \xi_2 \ast A \Theta^- \ast \xi_1 \) has only scalar endomorphisms. It suffices to check the diagram commutes after applying \( \epsilon \), which is clear.

Since the proof for (3) is very similar to Proposition \[6.32\], we will omit it. The crucial observation is that both maps compute the perverse truncation at degree zero.

\[ \square \]

7. Affine monodromic Soergel functor

We construct the affine monodromic Soergel functors \( M \) and \( \bar{M} \) in this section by introducing the extended affine monodromic Hecke category. We identify semisimple complexes (resp. in the extended category) with a subcategory of \( S \)-modules (resp. \( \bar{R} \)-bimodules) via \( M \) (resp. \( \bar{M} \)). See Theorem \[7.11\] for the precise statement. The proof is in parallel with Section 7 in [LY].

7.1. Central extension.
7.1.1. Central extension of loop group. We recall the construction of a central extension of the loop group $G_K$ in this subsection. We consider the loop group of the general linear group $GL_{n,k}$ first. It is well-known that there is a determinant line bundle $L_{det}$ on $GL_{n,k}$. Let us describe its $k$-valued points $L_{det}(k)$. Points for other rings can be described similarly (cf. [E] Section 2).

Let $O = k([t])$ and $K = k((t))$. For any $O$-lattices $\Lambda_1, \Lambda_2$ in the $n$-dimensional $K$-vector space $K^n$, there exists a third lattice $\Lambda_3 \subset \Lambda_1 \cap \Lambda_2$. The relative determinant line of $\Lambda_1$ with respect to $\Lambda_2$ is defined to be $\det(\Lambda_1 : \Lambda_2) = \det(\Lambda_1/\Lambda_3) \otimes_k \det(\Lambda_2/\Lambda_3)^{-1}$. Different choices of $\Lambda_3$ give canonically isomorphic lines. Let $\Lambda_0$ be the standard lattice $O^n \subset K^n$. $L_{det}(k)$ consists of pairs $(g, \gamma)$, where $g \in GL_n(k((t)))$ and $\gamma$ is a $k$-linear isomorphism $k \to \det(Ad(g)\Lambda_0 : \Lambda_0)$.

From the construction, we see that $L_{det}$ descends to $Gr_{GL_n}$, the affine Grassmannian for $GL_n$ naturally and also that the complement of the zero section in the total space of $L_{det}$ is a central $G_m$-extension of $GL_{n,k}$.

We return to the general case when $G$ is a split reductive group over $\mathbb{F}_q$. Let $\phi : G \to GL(V)$ be a representation of $G$. We pullback $L_{det}$ via $\phi$ and obtain a line bundle over $Gr_G$ and $G_K$. We denote the resulting line bundle over $G_K$ also as $L_{det}$. The complement of the zero section in the total space of $L_{det}$ is again a central $G_m$-extension of $G_K$, which we denote as $G_K$. Notice that $G_K$ depends on $\phi$. If $H$ is a subgroup of $G_K$, we denote the central extension of $H$ by $\tilde{H}$. The bi-$G_O$-equivariant structure of $L_{det}$ implies that it descends to $G_O\backslash G_K/G_O$ and moreover $G_O$ is a trivial $G_m$-extension of $G_O$. In particular, we can define $z \in H^2(I, G_K/I)$, the Chern class of $L_{det}$. In contrast with the reductive group case, $H^2(G/I, G_K/I)$ is strictly larger than $H^2(I, G_K/I)$. When $G$ is almost simple and $V$ is faithful, $H^2(G/I, G_K/I)$ is of codimension one and does not contain $z$. In this case, a different choice of $V$ will give the same $z$ up to a $Q$-multiple.

When $G$ is not almost simple, consider the isogeny $G \to G' := G/Z(G) \times G/[G,G]$. Suppose $G$ has $n$ simple factors $G_i$, that is $G/Z(G) = G_1 \times \cdots \times G_n$, we choose a faithful representation $V_i$ for each $G_i$. By taking the trivial extension on the $G_i$, we obtain an $G_i^m$-extension on the loop group $G_K$. Choose a character of $G_i^m$, which is non-trivial on each coordinate, equivalently, choose $n$ non-zero integers. We pushout the central $G_i^m$-extension to a central $G_m$-extension of $G_K$ via the chosen character. Finally we pullback this $G_m$-extension along $G_K \to G'_{K}$ to obtain a $G_m$-extension of $G_K$.

7.1.2. Compatible $G_m$-torsors over $Bun_{0,\infty}$. Let $\omega_{Bun}$ be the canonical bundle of $Bun_G$. Let $\omega_{Bun}$ (resp. $\omega_{Bun'}$) be the pullback of $\omega_{Bun}$ along the projection map $Bun_{G,0,\infty} \to Bun_G$. Let $Bun_{0,\infty}$ be the total space of the $G_m$-torsor associated to the pullback of $\omega_{Bun}^n$.

When $G$ is almost simple, we know that $\omega_{Bun}$ is a power of the determinant line bundle on $Bun_G$ as $\text{Pic}(Bun_G) \cong \mathbb{Z}$ (cf. [Z]). Therefore we can choose a faithful representation $V$ and an integer $n$ such that there is a surjection map from $G_K$. 


(depends on $V$) to $\text{Bun}^{\circ}_{0,\infty}$ making the following diagram Cartesian.

\[
\begin{array}{ccc}
\tilde{G}_K & \longrightarrow & \text{Bun}^{\circ}_{0,\infty} \\
\downarrow & & \downarrow \\
G_K & \longrightarrow & \text{Bun}_{0,\infty}
\end{array}
\]

Since $\text{Bun}_{0,\infty}$ is isomorphic to $[I_u \backslash G_K/I_u]$, we can identify $\text{Bun}^{\circ}_{0,\infty}$ with $[I_u \backslash G_K/I_u]$ via the above Cartesian diagram.

For general $G$, we choose $(V_i, n_i)$ for each simple factor $G_i$ such that the above diagram commutes. By taking the trivial line bundle on the $\text{Bun}_{G_i/[G,G_i],0,\infty}$ and the same chosen character of $G^n$ as the central extension of loop group, we have the following lemma.

**Lemma 7.1.** There exists a central $G_m$-extension $\tilde{G}_K \to G_K$ and a $G_m$-torsor $\text{Bun}_{0,\infty} \to \text{Bun}_{0,\infty}$ such that the following diagram Cartesian.

\[
\begin{array}{ccc}
\tilde{G}_K & \longrightarrow & [I_u \backslash \tilde{G}_K/I_u] \\
\downarrow & & \downarrow \\
G_K & \longrightarrow & [I_u \backslash G_K/I_u]
\end{array}
\]

From now on, we always fix a central $G_m$-extension $\tilde{G}_K \to G_K$ and a $G_m$-torsor $\text{Bun}_{0,\infty} \to \text{Bun}_{0,\infty}$ in Lemma 7.1.

There are natural stratifications of $\tilde{G}_K$ and $\text{Bun}_{0,\infty}$ induced from $G_K$ and $\text{Bun}_{0,\infty}$ respectively.

### 7.2. Soergel functor

Denote $R = H^\bullet_{T_k}(pt_k, \mathbb{C}^\ast(T))$, $\tilde{R} = H^\bullet_{\tilde{T}_k}(pt_k, \mathbb{C}^\ast(\tilde{T})) = \text{Sym}(X^\ast(T))_{\mathbb{C}^\ast(T)}$. $S = \text{Sym}(X^\ast(T) \oplus \mathbb{Z} \oplus X^\ast(T))_{\mathbb{C}^\ast(T)} \subset \text{Sym}(H^2(I\backslash G_K/I))_{\mathbb{C}^\ast(T)}$. These are all polynomial algebras with generators placed at degree 2. Moreover, they carry Frobenius actions which send each generator $x$ to $qx$. We may sometimes identify $\tilde{R}$ with $R \otimes \mathbb{C}[z]$ and $S$ with $R \otimes \mathbb{C}[z] \otimes R$. Note that $S$ is canonically a subring of $\tilde{R} \otimes \tilde{R}$.

One can define several categories associated to $\tilde{R}$ and $S$ in a similar way to [LY]. Denote $S$-gmod to be the category of graded $S$-modules; $(S, \text{Fr})$-gmod to be the category of graded $S$-modules with a compatible Fr action; $\tilde{R} \otimes \tilde{R}$-gmod to be the category of graded $\tilde{R} \otimes \tilde{R}$-modules; $(\tilde{R} \otimes \tilde{R}, \text{Fr})$-gmod to be the category of graded $(\tilde{R} \otimes \tilde{R}, \text{Fr})$-modules with a compatible Fr action.

There is a functor $\omega$ forgetting the Fr action. $[-], (-), (-)$ denote degree shift, weight twist, their composition respectively. Hom$^\ast(-, -)$ stands for the inner Hom. These categories carry natural monoidal structures. See Definition 7.2 below.

We denote Ind and Res be the adjoint pair of induction and restriction functors between $S$-mod and $\tilde{R} \otimes \tilde{R}$-mod. Same notations are also used for graded and with Frobenius action counterparts. If $M$ and $N$ are two $\tilde{R} \otimes \tilde{R}$-modules such that Ind$(M) \cong$ Ind$(N)$, then $M \cong N$.

$\tilde{W}$ acts on $\tilde{T}$ and $\tilde{R}$. For each $w \in \tilde{W}$, let $\tilde{R}(w)$ be the graded $\tilde{R}$-bimodule which is the quotient of $\tilde{R} \otimes \tilde{R}$ by the ideal generated by $w(a) \otimes 1 - 1 \otimes a$ for all $a \in \tilde{R}$. 
Let \( \overline{R}(w) \in (S, Fr) \)-gmod such that \( \text{Ind}(\overline{R}(w)) \cong \overline{R}(w) \). Denote 1 to be the unique element in \( \overline{R}(w) \) such that \( \text{Ind}(1) = 1 \in \overline{R}(w) \).

Denote \( \mathcal{X}_* (\overline{T}) \otimes_{\overline{R}} \overline{Q} \) to be \( \overline{t} \). By abuse of notation, let \( \Delta_{\text{alg}}^{\text{cen}} \) be the diagonal embedding of \( \mathcal{X}_*(\overline{G}^\text{cen}_m) \otimes_{\overline{R}} \overline{Q} \) into either \( \overline{T}^2 \) or \( \overline{T}^3 \). We identify \( S \)-modules with the quasi-coherent sheaves on \( \overline{t} \times \overline{t} / \Delta_{\text{alg}}^{\text{cen}} \). For \( 1 \leq i < j \leq 3 \), let \( p_{ij} : \overline{T}^3 / \Delta_{\text{alg}}^{\text{cen}} \to \overline{T}^2 / \Delta_{\text{alg}}^{\text{cen}} \) be the natural projections.

**Definition 7.2.** Given two quasi-coherent sheaves \( \mathcal{F} \) and \( \mathcal{G} \) on \( \overline{t} \times \overline{t} / \Delta_{\text{alg}}^{\text{cen}} \), define their convolution product \( \mathcal{F} \bullet \mathcal{G} \) as \( p_{13} (p_{12}^{*} \mathcal{F} \otimes p_{23}^{*} \mathcal{G}) \). In a similar fashion, we define the convolution product \( \overline{F} \bullet \overline{G} \) for quasi-coherent sheaves \( \overline{F} \) and \( \overline{G} \) on \( \overline{T}^2 \).

It is easy to check that the convolution product commutes with Ind. An \( S \)-module is the same as an \( R \)-bimodule with an action induced by \( z \). From this point of view, the convolution product \( M \bullet N \) is \( M \otimes_R N \) along with the action \( z \otimes 1 + 1 \otimes z \).

**Definition 7.3.** The extended affine monodromic Hecke category \( \mathcal{C}' \mathcal{D}_C \) is the 2-limit of \( D^{h}_{(\overline{T} \times \overline{T}, \mathcal{L} \boxtimes \mathcal{L}^{-1})} (\overline{I}_u \backslash \overline{G}_K / \overline{I}_u, \overline{I}_u \backslash \overline{G}_K / \overline{I}_u) \), where \( \mathcal{L}' \boxtimes \mathcal{L}^{-1} \) is viewed as a character sheaf on \( \overline{T} \times \overline{T} \) via pullback of the natural projection \( \overline{T} \times \overline{T} \to \overline{T} \times \overline{T} \).

There are no extra difficulties for defining the extended version of other categories and sheaves mentioned before. A tilde is added whenever to denote the extended analog, with the only exception is the convolution product. Statements in previous sections can be easily generalized to the extended case. As an illustration, the first statement of Lemma \( \ref{lem:convolution} \) becomes \( \overline{\Delta} (\overline{w}_1)_{\overline{T}^2}^{\pm} \star \overline{\Delta} (\overline{w}_2)_{\overline{T}^2}^{\pm} \cong \overline{\Delta} (\overline{w}_1 \overline{w}_2)_{\overline{T}^2}^{\pm} \) for \( \ell(w_1) + \ell(w_2) = \ell(w_1 w_2) \).

There is a natural map \( \pi : [\overline{I}_u \backslash \overline{G}_K / \overline{I}_u] \to [\overline{I}_u \backslash \overline{G}_K / \overline{I}_u] \) and the convolution product commutes with the pullback \( \pi^{*} \).

For \( \overline{F}, \overline{G} \in \mathcal{C}' \mathcal{D}_C, \text{Hom}^*(\overline{F}, \overline{G}) \) is a module over \( H^* (\overline{I} \backslash \overline{G}_K / \overline{I}) \). Hence it is a \( \overline{R} \)-bimodule. For example, \( \overline{R}(w) \) is isomorphic to \( \text{Hom}^*(\overline{C}(\overline{w})_{\overline{T}^2}^{\pm}, \overline{C}(\overline{w})_{\overline{T}^2}^{\pm}) \) in \( (\overline{R} \otimes \overline{R}, \text{Fr}) \)-gmod for any \( w \in \overline{W} \). Similarly, for \( \overline{F}, \overline{G} \in \mathcal{C}' \mathcal{D}_C, \text{Hom}^*(\overline{F}, \overline{G}) \) is a \( S \)-module and \( \overline{R}(w) \) is isomorphic to \( \text{Hom}^*(\overline{C}(\overline{w})_{\overline{T}^2}^{\pm}, \overline{C}(\overline{w})_{\overline{T}^2}^{\pm}) \) in \( (S, Fr) \)-gmod for any \( w \in \overline{W} \).

**Definition 7.4.** Let \( \beta \in \mathcal{C}' \mathcal{W}_C, \xi \) (resp. \( \xi \)) be a (resp. non-mixed) minimal IC sheaf for \( \beta \).

1. Define \( M_{\xi} \) the **mixed Soergel functor** associated with \( \xi \) to be the functor from \( \mathcal{C}' \mathcal{D}_C^\delta \) to \( (S, Fr) \)-gmod, which sends \( \overline{F} \) to
   \[ M_{\xi}(\overline{F}) := \text{Hom}^* (\overline{C}(\overline{w})_{\overline{T}^2}^{\pm}, \star \xi, \overline{F}). \]

2. Define \( \overline{M}_{\xi} \) the **non-mixed Soergel functor** associated with \( \xi \) to be the functor from \( \mathcal{C}' \mathcal{D}_C^\delta \) to \( S \)-gmod, which sends \( \overline{F} \) to
   \[ \overline{M}_{\xi}(\overline{F}) := \text{Hom}^* (\overline{C}(\overline{w})_{\overline{T}^2}^{\pm}, \star \xi, \overline{F}). \]

3. When \( \xi = \delta_C \) (resp. \( \xi = \delta_C \)) the corresponding Soergel functors are denoted as \( M_{\delta_C} \) (resp. \( \overline{M}_{\delta_C} \)).

It is straightforward to generalize the above definition to the extended case. The two Soergel functors \( M \) and \( \overline{M} \) are related by the following lemma.
Lemma 7.5. Let $\pi : [I_u \backslash \tilde{G}_K / I_u] \rightarrow [I_u \backslash G_K / I_u]$ and $F \in L^* D_L$. The natural homomorphism $\text{Ind}(M(F)) \cong M(F) \otimes_{S} (R \otimes R) \rightarrow M(\pi^* F)$ is an $(R \otimes R, Fr)$-gmod isomorphism.

Proof. The homomorphism exists because $\pi^* \Theta$ is isomorphic to $\tilde{\Theta}$. It suffices to show the statement for costandard sheaves, which are generators of $L^* D_L$. It is clear in this case. □

Here are some simple computations about Soergel functors.

Lemma 7.6. There is an unique isomorphism in $(S, Fr)$-gmod, $M_\xi(\xi) \sim \tilde{R}(w^\beta)$, which sends $\epsilon(\xi)$ to 1. The extended case holds similarly.

Proof. The argument in the proof of Lemma 7.3 in [LY] still works. □

Lemma 7.7. Let $s \in \tilde{W}_L$ be a simple reflection and $F \in L^* D_L$. There is an unique isomorphism in $(S, Fr)$-gmod, $M_\xi(F) \bullet \tilde{R}(s) \sim M_\xi(F *_{-} IC(s)^!_{L}(1))$ such that the composition

$M_\xi(F) \sim M_\xi(F) \bullet \tilde{R}(s) \sim M_\xi(F *_{+} IC(s)^!_{L}(1)) \xrightarrow{\text{Lemma 7.6}} M_\xi(\pi^* \pi_* s \cdot F) \xrightarrow{\text{adj}} M_\xi(F)$

is the identity.

Moreover, when taking $F = \delta_L$, this isomorphism sends $1 \otimes 1$ to $\theta_1^!$ after pre-composing the isomorphism in Lemma 7.6. The extended case holds similarly.

Proof. The proofs for a non-affine simple reflection $s$ and the extended case are similar to that in Lemma 7.4 in [LY]. We briefly explain the proof for the former case. Notice that $M_\xi(F) \bullet \tilde{R}(s) \cong M_\xi(F) \otimes_{\tilde{R}} \tilde{R}$. The key observation is that $L^* \tilde{\Theta}_{\tilde{L}^*} *_{-} \tilde{\xi}$ can be expressed as $\pi_s^{\tilde{\theta}} \tilde{\Theta}$ for some shifted perverse sheaf $\tilde{\Theta} \in L^* \tilde{D}_{\tilde{L}}$ by Lemma 5.7 and $s$ is the Chern class of a pullback of a line bundle on $G_K/P$. So the computation can be done in $L^* \tilde{D}_{\tilde{L}}$. The projection formula and the properties of $R\text{Hom}$ imply the composition is indeed the identity.

When $s$ is an affine simple reflection, $L_{det}$ is not the pullback of a line bundle on $G_K/P$. Nonetheless, there exists a character $\lambda \in X^*(T)$ such that the tensor product of its associated line bundle $\lambda_{\chi}$ on $G_K/L$ and $L_{det}$ descends to $G_K/P$, hence we can apply the previous argument to this tensor product. □

7.3. Monoidal structure. We construct the monoidal structure of $M$.

Let $\beta_1 \in L^* \tilde{W}_L$ and $\beta_2 \in L^* \tilde{W}_{L'}$. Let $\xi_1, \xi_2$ be two minimal sheaves for $\beta_1, \beta_2$ respectively. Suppose $F \in L^* D_L, \tilde{G} \in L^* D_{L'}$ and we are given maps $g : L^* \tilde{\Theta}_{\tilde{L}^*} *_{-} \xi_2 \rightarrow \tilde{G}[i]$ and $f : L^* \tilde{\Theta}_{\tilde{L}^*} *_{+} \xi_1 \rightarrow \tilde{F}[j]$. We can define $\lambda(g, f)$, the product of $g$ and $f$ in $\text{Hom}^{i+j}(L^* \tilde{\Theta}_{\tilde{L}^*} *_{-} \xi_2 *_{-} \xi_1, \tilde{G} *_{+} \tilde{F})$, as follows. $\lambda(g, f)$ is the composition

$\Theta * \xi_2 * \xi_1 \xrightarrow{\lambda} \Theta * Av_1 \Theta * \xi_2 * \xi_1 \xrightarrow{\tau} \Theta * \xi_2 * \xi_1 \xrightarrow{\theta \circ f} \tilde{G} * \tilde{F}[i + j]$.\nnote{Note that the map $\alpha$ (defined in Section 6.4) is in the last map implicitly. We have omitted certain superscripts and subscripts. We would do so henceforth as long as it causes no confusion.}

Taking the direct sum over $i, j \in \mathbb{Z}$ gives a pairing

$\langle \cdot, \cdot \rangle : M_{\xi_2}(\tilde{G}) \times M_{\xi_1}(\tilde{F}) \rightarrow M_{\xi_2 * \xi_1}(\tilde{G} * \tilde{F})$,

which descends to $M_{\xi_2}(\tilde{G}) \otimes_R M_{\xi_1}(\tilde{F}) \rightarrow M_{\xi_2 * \xi_1}(\tilde{G} * \tilde{F})$.\n
Furthermore, because of the multiplicative property of Chern class, it gives a functor

\[ c_{ξ_2,ξ_1}(G, F) : M_{ξ_2}(G) \cdot M_{ξ_1}(F) \to M_{ξ_2+ξ_1}(G \star F). \]

Lemma 6.33(3) gives an associativity constraint. The two ways of compositions from \( M_{ξ_2}(H) \cdot M_{ξ_2}(G) \cdot M_{ξ_1}(F) \) to \( M_{ξ_2+ξ_2+ξ_1}(H \star G \star F) \) are equal. All the constructions above are applicable to the extended case.

Lemma 7.8. Both \( c_{ξ_2,ξ_1}(G, ξ_1) \) and \( c_{ξ_2,ξ_1}(ξ_2, F) \) are isomorphisms in \((S, Fr)\)-gmod. The extended case holds similarly.

Proof. Lemma 6.33 shows that the composition

\[ M_{ξ_2}(G) \cdot R(w^{β_1}) \cong M_{ξ_2}(G) \cdot M_{ξ_1}(ξ_1) \xrightarrow{c_{ξ_2,ξ_1}(G, ξ_1)} M_{ξ_2+ξ_1}(G \star ξ_1) \xrightarrow{(ξ_1)^{-1}} M_{ξ_2}(G) \]

sends \( g \otimes 1 \) to \( g \). Hence \( c_{ξ_2,ξ_1}(G, ξ_1) \) is an isomorphism.

It requires slightly more effort to prove that \( c_{ξ_2,ξ_1}(ξ_2, F) \) is an isomorphism. Lemma 6.34(2) implies the composition

\[ M_{ξ_2}(G) \cdot R(w^{β_2}) \cong M_{ξ_2}(ξ_2) \cdot M_{ξ_1}(F) \xrightarrow{c_{ξ_2,ξ_1}(ξ_2, F)} M_{ξ_2+ξ_1}(ξ_2 \star F) \]

sends \( 1 \otimes f \) to \( f \).

Lemma 7.9. If \( s ∈ \tilde{W}_ξ^g \) is a simple reflection, then \( c_{ξ_2,ξ_1}(G, IC(s)) \) is an isomorphism in \((S, Fr)\)-gmod. The extended case holds similarly.

Proof. We use the same notation as Lemma 7.7. Let \( s : Θ → G \) be a morphism and \( g : Θ → G \) be its adjoint. Notice that the following two diagrams commute.

\[ \begin{array}{ccc}
\varepsilon & \xrightarrow{adj} & π_s Θ \\
\downarrow & & \downarrow
\\
π_s Θ & \xrightarrow{adj} & π_s π_s Θ
\end{array} \]

\[ \begin{array}{ccc}
Θ \star ξ_2 \star IC(s)⟨−1⟩ & \xrightarrow{Lemma 5.6} & π_s π_s(Θ \star ξ_2) \\
\downarrow & & \downarrow
\\
G \star IC(s)⟨−1⟩ & \xrightarrow{Lemma 5.6} & π_s π_s Θ
\end{array} \]

Hence, by diagram chasing, it suffices to check that the composition

\[ π_s Θ = Θ \star ξ_2 \xrightarrow{χ} Θ \star Av; Θ \star ξ_2 \xrightarrow{τ(ξ_2)} Θ \star ξ_2 \star Av; Θ \xrightarrow{β} Θ \star ξ_2 \star IC(s)(−1) = π_s π_s Θ \]

is equal to the one induced by the adjoint map \( Θ → π_s π_s Θ \). To see this, we compose it with the natural map \( Θ \star ξ_2 \star IC(s)(−1) → Θ \star ξ_2 δ = Θ \star ξ_2 \) and reduce to show that \( π_s Θ → π_s π_s π_s Θ → π_s Θ \) is the identity map. This is clear as its stalk at \( e \) is the identity and the degree zero part of the endomorphism ring of \( Θ \star ξ_2 \) is \( Θ \).

Corollary 7.10. The map \( c_{ξ_2,ξ_1}(G, F) \) is an isomorphism whenever \( F \) is a semisimple complex. The extended case holds similarly.
Note that the statement is asymmetric, unlike its reductive group analog. We do not claim that $c_{\xi_2, \xi_1}(G, F)$ is an isomorphism when $G$ is a semisimple complex.

**Proof.** By the decomposition theorem, $\omega F$ is a direct summand of certain $\omega(\mathrm{IC}(s_i_1)_L, \cdots \cdot \mathrm{IC}(s_i_n)_L)$ for some simple reflections $s_i_j$ and suitable $L_j$. Applying Lemma 7.8 and Lemma 7.9 $n$ times will give the result. $\square$

Now we are going to prove the main result in this section that the Soergel functor is fully faithful on semisimple complexes. The theorem is false if we only consider the $R$-bimodule structure instead of the $S$-module or $R$-bimodule structure. More precisely, $m(F, G)$ fails to be surjective. This indicates the consideration of the central torus is necessary.

**Theorem 7.11.** Let $\beta \in L, \widetilde{W}_L$ and $\xi$ be a minimal sheaf in the block $\beta$. Let $F$ and $G$ be two semisimple complexes in $L(D_L)$. Then the natural map

$$m(F, G) : \mathrm{Hom}^\bullet(F, G) \to \mathrm{Hom}^\bullet_{S*}(M_{\xi}(F), M_{\xi}(G))$$

is an isomorphism in $(S, Fr)$-gmod. The extended case holds similarly. That is,

$$m(F, G) : \mathrm{Hom}^\bullet(F, G) \to \mathrm{Hom}^\bullet_{R\otimes \widetilde{R}*, \mathrm{gmod}}(\widetilde{M}_{\xi}(F), \widetilde{M}_{\xi}(G))$$

is an isomorphism in $(\widetilde{R} \otimes \widetilde{R}, Fr)$-gmod for any extended minimal sheaf $\tilde{\xi}$ and semisimple complexes $\tilde{F}, \tilde{G}$ in $L(D_L)$.

**Proof.** It is clear that $m(F, G)$ is Fr-equivariant, hence it suffices to prove that the non-mixed map is an isomorphism in $S$-gmod. Therefore we may and will assume $F, G \in L(D_L)$. We induct on the dimension of the support of $F$ and do not fix a particular block, that is $L$ and $L'$ can change.

The base case is when $F$ is a direct sum of shifted $\delta_L$. Without loss of generality, assume $F$ is $\delta_L$ and $G$ is $\mathrm{IC}(w)_L$. Consider the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{Hom}^\bullet(\delta_L, i_{e*}i'_L(G)) & \overset{m(\delta_L, i_{e*}i'_L(G))}{\longrightarrow} & \mathrm{Hom}^\bullet_{S*}(\widetilde{R}(e), \mathrm{M}^\circ(i_{e*}i'_L(G))) \\
\downarrow \alpha & & \downarrow \mathrm{M}^\circ(a) \\
\mathrm{Hom}^\bullet(\delta_L, G) & \overset{m(\delta_L, G)}{\longrightarrow} & \mathrm{Hom}^\bullet_{S*}(\widetilde{R}(e), \mathrm{M}^\circ(G)),
\end{array}$$

where $a$ is induced by the adjunction map $i_{e*}i'_L(G) \to G$.

It is clear that $\alpha$ is an isomorphism. Since $i_{e*}i'_L(G)$ is a direct sum of shifted $\delta_L$, the top horizontal map is an isomorphism. To see $\mathrm{M}^\circ(a)$ is an isomorphism, notice that we have showed $\mathrm{M}^\circ(G)$ has an increasing filtration with associated graded $Gr^F_1 = \bigoplus_{\ell(w') = n} \mathrm{Hom}^\bullet(i_{w'}^*, \mathrm{Av}_{w'} \Theta_{w'} i_{w'}^* \mathrm{IC}(w')_L)$ in Lemma 6.28. Apply Hom$(\widetilde{R}(e), -$) to the exact sequence, $0 \to F^{\leq n-1}_e \to F^{\leq n}_e \to Gr^F_1 \to 0$ for $1 \leq n \leq \ell(e)$. Note that $Gr^F_1$ is direct sum of $\widetilde{R}(e')$ for $w' \leq w$ with length $n$. From the $\widetilde{W}$-action on $\widetilde{R}$, we know that $\widetilde{R}(w')$ is both left and right free $\widetilde{R}$-module. Since this action is faithful, there are no non-zero maps from $\widetilde{R}(e)$ to $\widetilde{R}(w')$ when $\ell(w') \geq 1$. The analogous statement holds for $\widetilde{R}(e)$ and $\widetilde{R}(w')$. In other words, Hom$(\widetilde{R}(e), F^{\leq n-1}_e) \to$ Hom$(\widetilde{R}(e), F^{\leq n}_e)$ is bijective for all $1 \leq n \leq \ell(e)$. Therefore $\mathrm{M}^\circ(a) : \mathrm{Hom}(\widetilde{R}(e), F_0) \to \mathrm{Hom}(\widetilde{R}(e), F^{\leq \ell(w)}_e)$ is bijective. This completes the base case.

Suppose now the dimension of the support of $F$ is $n$ and the statement holds for any $F$ of dimension of support smaller than $n$. By the decomposition theorem, $\mathrm{IC}(w)_L$ is a direct summand of $\mathrm{IC}(w)_L^+ = \mathrm{IC}(s_{i_1})_L^+ \cdot \mathrm{IC}(s_{i_2})_L^+ \cdot \cdots \cdot \mathrm{IC}(s_{i_n})_L^+$.
for some reduced word expression \( w = (s_{i_1}, s_{i_2}, \ldots, s_{i_n}) \) of \( w \) and suitable \( \mathcal{L} \). It suffices to show the statement holds for \( \mathcal{I}(w)^+_{\mathcal{L}} \). Let \( w' = (s_{i_1}, s_{i_2}, \ldots, s_{i_{n-1}}) \) be a reduced word expression of \( w' = s_i s_{i_1} \cdots s_{i_{n-1}} \) and \( s = s_{i_n} \), then \( \mathcal{I}(w)^+_{\mathcal{L}} \cong \mathcal{I}(w')^+_{\mathcal{L}} \star \mathcal{I}(s)^+_{\mathcal{L}} \). Consider the following diagram where each solid arrow is well-defined up to a nonzero scalar.

\[
\begin{array}{ccc}
\text{Hom}^\bullet(\mathcal{I}(w')^+_{\mathcal{L}} \star \mathcal{I}(s)^+_{\mathcal{L}}, G) & \overset{\text{adj}}{\longrightarrow} & \text{Hom}^\bullet(\mathcal{I}(w')^+_{\mathcal{L}}, G \star \mathcal{I}(s)^+_{\mathcal{L}}) \\
\downarrow m & & \downarrow m' \\
\text{Hom}^\bullet(\mathcal{M}(\mathcal{I}(w')^+_{\mathcal{L}} \star \mathcal{I}(s)^+_{\mathcal{L}}), \mathcal{M}(G)) & \overset{b}{\longrightarrow} & \text{Hom}^\bullet(\mathcal{M}(\mathcal{I}(w')^+_{\mathcal{L}}), \mathcal{M}(G) \star \mathcal{M}(\mathcal{I}(s)^+_{\mathcal{L}})) \\
\downarrow c & & \downarrow c' \\
\text{Hom}^\bullet(\mathcal{M}(\mathcal{I}(w')^+_{\mathcal{L}}), \mathcal{M}(\mathcal{I}(s)^+_{\mathcal{L}})) & \overset{\text{adj}}{\longrightarrow} & \text{Hom}^\bullet(\mathcal{M}(\mathcal{I}(w')^+_{\mathcal{L}}), \mathcal{M}(G) \star \mathcal{M}(\mathcal{I}(s)^+_{\mathcal{L}}))
\end{array}
\]

where \( \text{adj} \) is an isomorphism given by Lemma [5.5] if \( s \notin \widehat{W}^2_{\mathcal{L}} \) or Lemma [5.6] if \( s \in \widehat{W}^2_{\mathcal{L}} \); \( m \) and \( m' \) are induced by the Soergel functor; \( c \) and \( c' \) are given by Corollary [7.10].

Every solid arrow except for \( m \) is an isomorphism. If we can define an isomorphism \( b \) so that the diagram commutes up to a non-zero scalar, then \( m \) will be also an isomorphism.

When \( s \notin \widehat{W}^2_{\mathcal{L}} \), \( \mathcal{M}(\mathcal{I}(s)^+_{\mathcal{L}}) \) is isomorphic to \( \overline{R}(s) \). Notice that there exists a natural isomorphism \( R \cong \overline{R}(e) \rightarrow \overline{R}(s) \star \overline{R}(s) \cong R \) as \( S \)-module. We define the map \( b \) to be induced by this isomorphism. To check the commutativity, it suffices to check for the universal object, that is when \( G = \mathcal{I}(w)^+_{\mathcal{L}} \) and these two compositions send the identity to the same element. It is true because \( \text{adj} \) is also induced by an isomorphism \( \mathcal{I}(w)^+_{\mathcal{L}} \rightarrow \mathcal{I}(s)^+_{\mathcal{L}} \star \mathcal{I}(s)^+_{\mathcal{L}} \).

When \( s \in \widehat{W}^2_{\mathcal{L}} \), we prove the general statement: For \( M_1, M_2 \in S\text{-gmod} \), there is a bifunctorial isomorphism of \( S\text{-gmod} \)

\[
\text{Hom}^\bullet_{S\text{-gmod}}(M_1, M_2 \star \mathcal{M}(\mathcal{I}(s)^+_{\mathcal{L}})) \cong \text{Hom}^\bullet_{S\text{-gmod}}(M_1 \star \mathcal{M}(\mathcal{I}(s)^+_{\mathcal{L}}), M_2).
\]

It suffices to construct the unit and counit map. We use the \( R \)-bimodule with \( z \)-action description for \( S \)-module. From this point of view, \( \mathcal{M}(\mathcal{I}(s)^+_{\mathcal{L}}) \cong R \otimes_R R \otimes_R R \) as \( R \)-bimodule, with \( z \) acts by 0 when \( s \) is a non-affine simple reflection and non-trivially otherwise.

Let \( \alpha_s \) be a degree two element in \( R \) corresponding to \( s \). Here \( \alpha_s \) is unique up to a scalar. The unit map is the \( R \)-bilinear map \( \overline{R}(e) \rightarrow R \otimes_R R \) sending 1 to \( \alpha_s \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \alpha_s \). The counit map is the \( R \)-bilinear map \( R \otimes_R R \star R \otimes_R R \rightarrow \overline{R}(e) \), sending \( r_1 \otimes 1 \otimes r_2 \) to zero and \( r_1 \otimes \alpha_s \otimes r_2 \) to \( r_1 r_2 \). It is not hard to check the \( z \)-action compatibility directly. The map \( b' \) is obtained by specializing \( M_1 = \mathcal{M}(\mathcal{I}(w)^+_{\mathcal{L}}) \) and \( M_2 = \mathcal{M}(G) \). To check commutativity, it suffices to check when \( \mathcal{G} = \mathcal{I}(w)^+_{\mathcal{L}} \) and these two compositions send the identity to the same element. This is true because the tensor product and the forgetful functor correspond to \( \pi'_1 \) and \( \pi_{s*} \) respectively.

The proof for the extended case is similar (and easier), hence omitted. \( \square \)
8. Soergel bimodule

In this section, we recall some basics of Soergel bimodules without proofs. Details can be found in [LY] and [S]. The main result is to identify $\mathcal{M}^0(\mathcal{I}(w))_L$ and $\mathcal{M}^0(\mathcal{I}(w)_L)$ with certain extended Soergel bimodules.

Given a Coxeter system $(W_0, S_0)$ and a reflection faithful $\mathbb{Q}_r$-representation $V$, let $R_0 = \text{Sym}_{\mathbb{Q}_r}(V^*)$ with $\deg(V^*) = 2$. $\text{Spec}(R_0)$ is canonically isomorphic to $V$. Soergel associated each tuple $(W_0, S_0, V)$ to two families of graded $R_0$-bimodules, Bott-Samelson bimodules $S(s_{i_n}, s_{i_{n-1}}, \cdots, s_{i_1})$ and indecomposable Soergel bimodules $S(w)$. Up to degree shifts, the former one is indexed by words in $W_0$ while the latter one is indexed by elements in $W_0$. Let $\text{SB}(W_0) \subset R_0 \otimes R_0$-gmod be the full subcategory consisting of Soergel bimodules. $\text{SB}(W_0)$ carries a monoidal structure given by the tensor product $(-) \otimes_{R_0} (-)$.

It is easy to see that the $\widehat{W}_L^\circ$-action on $\mathfrak{i}$ is reflection faithful. Hence the above usual Soergel bimodule theory package is available. Moreover, for $\mathcal{L} \in \mathfrak{o}$ and $w \in \widehat{W}_L^\circ$, we define a graded $\check{R} \otimes \check{R}$-gmod $S(w)_\mathcal{L}$ following Lusztig and Yun. Let $\beta$ be the block containing $w$. Write $w = x w^\beta$ for $x \in \widehat{W}_L^\circ$. Define the extended Soergel bimodule

$$S(w)_\mathcal{L} := S(x) \check{W}_L^\circ \otimes_{\check{R}} \check{R}(w^\beta).$$

It carries a rigidification with the degree zero element $1 \otimes 1$. For any sequence $(s_{i_n}, s_{i_{n-1}}, \cdots, s_{i_1})$ of simple reflections in $\check{W}$, let $\mathcal{L}_j = s_{i_j} \cdots s_{i_1} \mathcal{L}$. Define the extended Bott-Samelson bimodule

$$S(s_{i_n}, s_{i_{n-1}}, \cdots, s_{i_1})_\mathcal{L} := S(s_{i_n})_{\mathcal{L}_{n-1}} \otimes_{\check{R}} S(s_{i_{n-1}})_{\mathcal{L}_{n-2}} \otimes_{\check{R}} \cdots \otimes_{\check{R}} S(s_{i_1})_{\mathcal{L}}.$$

When $\mathcal{L}$ is trivial and $w \in \widehat{W}_L^\circ$, $S(s_{i_n}, s_{i_{n-1}}, \cdots, s_{i_1})_\mathcal{L}$ and $S(w)_\mathcal{L}$ become the usual Bott-Samelson bimodules and Soergel bimodules for the affine Weyl group $\widehat{W}_L^\circ$.

Lemma 8.1. Let $\mathcal{L} \in \mathfrak{o}$ and $w \in \widehat{W}_L^\circ$. Let $M$ be an indecomposable graded $\check{R} \otimes \check{R}$-module such that

1. $\text{Supp}(M) \supset \Gamma(w)$ as a subset of $\text{Spec}(\check{R} \otimes \check{R})$.
2. For some reduced expression $w = s_{i_n} s_{i_{n-1}} \cdots s_{i_1}$ in $\check{W}$, $M$ is a direct summand of $S(s_{i_n}, s_{i_{n-1}}, \cdots, s_{i_1})_\mathcal{L}$.

Then $M \cong S(w)_\mathcal{L}$.

Proof. See Lemma 8.1 in [LY].

Let $(\check{R} \otimes \check{R}, \text{Fr})$-gmodpure be the full subcategory of $(\check{R} \otimes \check{R}, \text{Fr})$-gmod containing those $M = \bigoplus_{n \in \mathbb{Z}} M^n$ such that $M^n$ is pure of weight $n$ as Fr-module. The forgetful functor $\omega : (\check{R} \otimes \check{R}, \text{Fr})$-gmodpure $\to \check{R} \otimes \check{R}$-gmod has an one-side inverse

$$(-)^\circ : \check{R} \otimes \check{R}\text{-gmod} \to (\check{R} \otimes \check{R}, \text{Fr})\text{-gmodpure}$$

defined by declaring Fr to act on the $n$-th graded piece by $q^{n/2}$. Let $\text{SB}_m(\widehat{W}_L^\circ) \subset (\check{R} \otimes \check{R}, \text{Fr})$-gmodpure be the full subcategory consisting of those $M$ such that $\omega M \in \text{SB}(\widehat{W}_L^\circ)$. It inherits a monoidal structure from $(\check{R} \otimes \check{R}, \text{Fr})$-gmodpure.

Proposition 8.2. Let $\mathcal{L} \in \mathfrak{o}$ and $w \in \widehat{W}_L^\circ$. Then there is a unique isomorphism in $(\check{R} \otimes \check{R}, \text{Fr})$-gmod

$$\widetilde{\mathcal{M}}^0(\check{I}(w))_\mathcal{L}(-\ell_\mathcal{L}(w)) \cong S(w)_\mathcal{L}$$
which sends \( \tilde{\theta}_w^1 \) to \( 1 \in \mathcal{S}(w)_{\tilde{W}_L}^2 \).

**Proof.** Let \( M \) be \( \overline{\mathcal{M}(\mathcal{IC}(w)_{\mathcal{L}}[-\ell_{\beta}(w)])} \), where \( \beta \) is the unique block contains \( w \). Theorem 7.11 shows \( \text{End}(M) = \mathbb{Q}_f \), hence \( M \) is indecomposable. The support of the last piece of the filtration of \( M \) in Lemma 6.28 corresponds to \( \Gamma(w) = \text{Supp}(\tilde{R}(w)) \subset \text{Supp}(M) \). \( M \) is a direct summand of \( \overline{\mathcal{M}(\mathcal{IC}(s_{i_n}, \ldots, s_{i_1})_{\mathcal{L}}[-\ell_{\beta}(w)])} \) by the decomposition theorem combined with Lemma 7.6 and Lemma 7.7. Hence we can apply Lemma 8.1 to prove that \( \overline{\mathcal{M}(\mathcal{IC}(w)_{\mathcal{L}}[-\ell_{\beta}(w)])} \cong \mathcal{S}(w)_{\mathcal{L}} \). The rest of the proof is similar to Proposition 8.7 in [LY]. \( \square \)

**Proposition 8.3.** Let \( M \in \text{SB}_m(\tilde{W}_L^2) \). There exists a finite filtration \( 0 = F_0M \subset F_1M \subset \cdots \subset F_nM = M \) in \( \text{SB}_m(\tilde{W}_L^2) \) satisfying

1. For \( 1 \leq i \leq n \), \( \text{Gr}_i^M \cong \mathcal{S}(w_i)^2(n_i) \otimes V_i \) for some \( w_i \in \tilde{W}_L^2 \), \( n_i \in \mathbb{Z} \) and finite-dimensional Fr-module \( V_i \) pure of weight zero.
2. The filtration \( \omega F_i M \) of \( \omega M \) splits in \( R \otimes \tilde{R} \)-gmod.

**Proof.** The proof is similar to Proposition 8.10 in [LY]. By Theorem 7.11 and Proposition 8.2 the homomorphism space between two Soergel bimodules at certain degrees vanish due to perverse degree consideration. The filtration in the proposition is analogous to the perverse filtration. \( \square \)

There are analogous definitions and results for \( S \)-module case. Lemma 7.6 and Proposition 8.2 guarantee the existence of \( \mathcal{S}(w)_{\mathcal{L}} \), the Soergel bimodules for \( S \) and \( \mathcal{S}(s_{i_n}, s_{i_{n-1}}, \ldots, s_{i_1})_{\mathcal{L}} \), the Bott-Samelson bimodules for \( S \). They are sent to \( \mathcal{S}(w)_{\mathcal{L}} \) and \( \mathcal{S}(s_{i_n}, s_{i_{n-1}}, \ldots, s_{i_1})_{\mathcal{L}} \) under induction functor Ind. They are unique up to unique isomorphism after rigidification. After defining Soergel bimodules for \( S \), we can define the Frobenius version \((S, Fr)\)-gmodpure and its full subcategory \( \text{SB}_m(\tilde{W}_L^2) \). The above results and proofs can be generalized to \( S \)-modules by replacing \( \otimes \tilde{R} \) and \( \mathcal{S} \) by \( \otimes \mathcal{R} \) and \( \mathcal{S} \) respectively. For completeness, let us state the analog of Proposition 8.2 for \( S \)-module.

**Proposition 8.4.** Let \( \mathcal{L} \in \mathfrak{a} \) and \( w \in \tilde{W}_L^2 \). Then there is a unique isomorphism in \((S, Fr)\)-gmod

\[
\mathcal{M}^\circ(\mathcal{IC}(w)_{\mathcal{L}}[-\ell_{\mathcal{L}}(w)]) \cong \mathcal{S}(w)_{\tilde{W}_L^2}^2
\]

which sends \( \theta_w^1 \) to \( 1 \in \mathcal{S}(w)_{\tilde{W}_L^2}^2 \).

9. **Equivalence for the neutral block**

In this section we prove the main statement, which is the equivalence between neutral blocks of monodromic Hecke categories. It relies heavily on the machinery developed in [BY], Appendix B).

Let \( \mathcal{L} \in \mathfrak{a} \). Denote \( H_{\mathcal{L}} \) to be the endoscopic group of \( G \) corresponding to \( \mathcal{L} \). It is a reductive group sharing the same maximal torus with \( G \) (cf. [LY], Section 9.1). The subscript would be dropped whenever there is no ambiguity. Let \( H_K \) be the loop group of \( H \) and \( I_H \) be its standard Iwahori subgroup. The usual affine Hecke category for \( H_K \) is

\[
D_{H_K} := D^\mathfrak{b}_m(I_H \backslash H_K / I_H).
\]

It is the affine monodromic Hecke category for \( H \) with the trivial character sheaf on \( T \). Hence all constructions before are applicable to \( D_{H_K} \). Denote the IC sheaf,
standard sheaf, and costandard sheaf in $D_{H_K}$ by $IC(w)_{H_K}, \Delta(w)_{H_K}$, and $\nabla(w)_{H_K}$ respectively. Unlike its reductive group analog, $D_{H_K}$ can usually be decomposed into smaller categories. We define $D_{H_K}^0$ to be its neutral block, which is a special case in Definition 5.13. One can check that $D_{H_K}^0$ contains the sheaves supported on the neutral connected component of $I_H \setminus H_K/I_H$. The orbits in this component are those whose index is in $\tilde{W}_\omega^0$.

As in the previous sections, adding a tilde (resp. an underline) means its extended (resp. non-mixed) counterparts. Here are the main theorems of the paper.

**Theorem 9.1.** Let $L \in \text{Ch}(T)$ and $H_K$ be the loop group of the endoscopic group of $G$ associated with $L$. There is a natural monoidal equivalence of triangulated categories

$$\Psi^0_L : D_{H_K}^0 \cong \mathcal{E} D_{L}^0$$

such that

1. For all $w \in \tilde{W}_\omega^0$, 
   $$\Psi^0_L(\text{IC}(w)_{H_K}) \cong \text{IC}(w)_{L}, \quad \Psi^0_L(\Delta(w)_{H_K}) \cong \Delta(w)_{L}, \quad \Psi^0_L(\nabla(w)_{H_K}) \cong \nabla(w)_{L}.$$ 
   In particular, $\Psi^0_L$ is t-exact for the perverse t-structures.

2. There is a bi-functorial isomorphism of graded $(S, \text{Fr})$-modules for all $\mathcal{F}, \mathcal{G} \in D_{H_K}^0$
   $$\text{Hom}^*(\mathcal{F}, \mathcal{G}) \cong \text{Hom}^*(\Psi^0_L(\mathcal{F}), \Psi^0_L(\mathcal{G})).$$

**Theorem 9.2** (Extended version of Theorem 9.1). We use the same notation as Theorem 9.1. There is a natural monoidal equivalence of triangulated categories

$$\tilde{\Psi}^0_L : \tilde{D}_{H_K}^0 \cong \mathcal{E} \tilde{D}_{L}^0$$

such that

1. For all $w \in \tilde{W}_\omega^0$, 
   $$\tilde{\Psi}^0_L(\tilde{\text{IC}}(w)_{H_K}) \cong \tilde{\text{IC}}(w)_{L}, \quad \tilde{\Psi}^0_L(\tilde{\Delta}(w)_{H_K}) \cong \tilde{\Delta}(w)_{L}, \quad \tilde{\Psi}^0_L(\tilde{\nabla}(w)_{H_K}) \cong \tilde{\nabla}(w)_{L}.$$ 
   In particular, $\tilde{\Psi}^0_L$ is t-exact for the perverse t-structures.

2. There is a bi-functorial isomorphism of graded $(\mathcal{R} \otimes \mathcal{R}, \text{Fr})$-modules for all $\tilde{\mathcal{F}}, \tilde{\mathcal{G}} \in \tilde{D}_{H_K}^0$
   $$\text{Hom}^*(\tilde{\mathcal{F}}, \tilde{\mathcal{G}}) \cong \text{Hom}^*(\tilde{\Psi}^0_L(\tilde{\mathcal{F}}), \tilde{\Psi}^0_L(\tilde{\mathcal{G}})).$$

Their proofs are very similar. We only discuss the proof for the first theorem, which occupies the rest of the section.

Let us introduce several related categories as in Section 9.3 in [LY]. For $w \in \tilde{W}_\omega^0$, let $\mathcal{E} \mathcal{C}(\leq w)_{L} \subset \mathcal{E} D_{L}^0$ be the full subcategory consisting of sheaves whose support on $I_{u} \setminus G_K \subset H_K$ is a similar fashion. We also let $\mathcal{E} \mathcal{C}(\leq w)_{L} \subset \mathcal{E} \mathcal{D}(\leq w)_{L}$ (resp. $\leq \mathcal{C}(\leq w)_{L} \subset \mathcal{E} \mathcal{D}(\leq w)_{L}$) be the full subcategory consisting of objects that are pure of weight zero. Note that any object $\mathcal{F}$ in $\mathcal{E} \mathcal{C}_L(\leq w)$ is pure in the sense that $i_w^* \mathcal{F}$ and $i_w^! \mathcal{F}$ are pure of weight zero. Let $\mathcal{E} \mathcal{C}(\leq w)_{L} \subset \mathcal{E} \mathcal{D}(\leq w)_{L}$ (resp. $\leq \mathcal{C}(\leq w)_{L} \subset \mathcal{E} \mathcal{D}(\leq w)_{L}$) under $\omega : \mathcal{E} \mathcal{D}(\leq w)_{L} \rightarrow \mathcal{E} \mathcal{D}(\leq w)_{L}$ (resp. $\omega : \mathcal{E} \mathcal{D}(\leq w)_{L} \rightarrow \mathcal{E} \mathcal{D}(\leq w)_{L}$). Let $K^b(\mathcal{E} \mathcal{C}_L(\leq w))$ be the homotopy category of bounded complexes in $\mathcal{E} \mathcal{C}_L(\leq w)$ and $K^b(\mathcal{E} \mathcal{C}_L(\leq w))_0$ be the thick subcategory consisting of the complexes that are null-homotopic when mapped to $K^b(\mathcal{E} \mathcal{C}_L(\leq w))$. Similarly, one can define $K^b(\mathcal{E} \mathcal{C}(\leq w)_{L})$ and $K^b(\mathcal{E} \mathcal{C}(\leq w)_{L})_0$. 

It is clear that the above categories have extended counterparts, we denote them with a tilde.

Let $\text{SB}(\tilde{W}_c^\omega(\leq w))$ be the full subcategory of $\text{SB}(\tilde{W}_c^\omega)$ consisting of Soergel bimodules supported on the union of $\Gamma(v)$ for $v \leq w$. Let $\text{SB}_m(\tilde{W}_c^\omega(\leq w))$ be its mixed analog, which is the full subcategory of $\text{SB}_m(\tilde{W}_c^\omega)$ consists of those $M$ such that $\omega M \in \text{SB}(\tilde{W}_c^\omega(\leq w))$. Since the graph $\Gamma(v)$ is $\Delta_{\text{alg}}$-invariant, it makes sense to define $\text{SB}(\tilde{W}_c^\omega(\leq w))$ and $\text{SB}_m(\tilde{W}_c^\omega(\leq w))$ for $S$-module counterparts.

The formalism in [BY, Appendix B] shows that there is a triangulated functor (the realization functor) $\rho_{\leq w} : K^b_{\omega}(\mathcal{C}(\leq w)_c^\omega) \to \mathcal{D}(\leq w)^0_c$. The arguments in Section 9.4 in [LY] imply the following lemma.

**Lemma 9.3.** For any $w \in \tilde{W}_c^\omega$, the functor $\rho_{\leq w}$ descends to an equivalence

$$\rho_{\leq w} : K^b_{\omega}(\mathcal{C}(\leq w)_c^\omega)/K^b_{\omega}(\mathcal{C}(\leq w)_c^\omega)_0 \simeq \mathcal{D}(\leq w)^0_c.$$  

The extended analog holds similarly.

From the construction of $\rho_{\leq w}$, we know that $\rho_{\leq w}$ is compatible with $\rho_{\leq w'}$ whenever $w \leq w'$.

**Proposition 9.4.** The restriction of $\mathcal{M}^0$ gives a monoidal equivalence

$$\varphi_0 : \mathcal{C}^\omega _c \simeq \text{SB}_m(\tilde{W}_c^\omega)$$

such that for $\mathcal{F}, \mathcal{G} \in \mathcal{C}^\omega _c$, there is a canonical isomorphism in $(S, \text{Fr})$-gmod

$$\text{Hom}^*(\mathcal{F}, \mathcal{G}) \cong \text{Hom}^*_{S\text{-gmod}}(\varphi_0(\mathcal{F}), \varphi_0(\mathcal{G})).$$

Moreover $\varphi_0$ restricts to an equivalence $\varphi_{0, \leq w} : \mathcal{C}(\leq w)_c^\omega \simeq \text{SB}_m(\tilde{W}_c^\omega(\leq w))$ for any $w \in \tilde{W}_c^\omega$. The extended case holds similarly.

**Proof.** Let $\varphi_0 : \mathcal{C}^\omega _c \to S\text{-gmod}$ be the restriction of $\mathcal{M}^0$. Corollary 7.10 shows that $\varphi_0$ is a monoidal functor.

First, we show that the image of $\varphi_0$ lies in $\text{SB}_m(\tilde{W}_c^\omega)$. By Proposition 8.2

$$\varphi_0(\omega \mathcal{F}) \cong \omega \varphi_0(\mathcal{F}) \in \text{SB}(\tilde{W}_c^\omega).$$

It remains to show that $\text{Ext}^i_{\mathcal{C}^\omega _c}(\mathcal{F}, \mathcal{G}) = \text{Ext}^i_{A\text{v}_1 \mathcal{C}^\omega _c}(\mathcal{F}, \mathcal{G})$ is pure of weight $i$ for any $\mathcal{F} \in \mathcal{C}^\omega _c$. This is true if we can prove that $A\text{v}_1 \mathcal{C}^\omega _c$ is $*$-pure of weight zero and $\mathcal{F}$ is $!$-pure of weight zero (cf. [BY, Lemma 3.1.5]). The latter has already been shown in Lemma 5.8. For the former, we know that $\mathcal{F}^*_{A\text{v}_1 \mathcal{C}^\omega _c}$ is an one-dimensional local system by Proposition 6.7. Since the hyperbolic localization functor preserves the purity of weight (cf. [BH, Theorem 8]), the stalk at $\tilde{w}$ is pure of weight zero and hence $A\text{v}_1 \mathcal{C}^\omega _c$ is $*$-pure of weight zero. Moreover, it is clear that $\varphi_0$ sends $\mathcal{C}(\leq w)_c^\omega$ to $\text{SB}_m(\tilde{W}_c^\omega(\leq w))$.

Note that $\text{Ext}^i(\mathcal{F}, \mathcal{G})$ is pure of weight $i$ because of the $*$-purity of $\mathcal{F}$ and $!$-purity of $\mathcal{G}$ by Lemma 5.8. This shows $\text{hom}_{\mathcal{C}^\omega _c}(\mathcal{F}, \mathcal{G}) = \text{Hom}(\mathcal{F}, \mathcal{G})^F = \text{Hom}^*(\mathcal{F}, \mathcal{G})^F$. Meanwhile, $\text{hom}_{(S, \text{Fr})\text{-gmod}}(M, M') = \text{Hom}_{S\text{-gmod}}(M, M')^F$ for $M, M' \in \text{SB}_m(\tilde{W}_c^\omega)$. Theorem 7.11 states that the functor $\varphi_0$ sending $\text{Hom}^*(\mathcal{F}, \mathcal{G})$ to $\text{Hom}^*_{S\text{-gmod}}(\varphi_0(\mathcal{F}), \varphi_0(\mathcal{G}))$ is an isomorphism. Taking the Frobenius invariant of both sides implies $\varphi_0$ is fully faithful.

Finally, we show that if $M \in \text{SB}_m(\tilde{W}_c^\omega(\leq w))$, then there exists $\mathcal{F} \in \mathcal{C}(\leq w)_c^\omega$ such that $\varphi_0(\mathcal{F}) \cong M$. The argument for the reductive group case works perfectly well here. The idea is to utilize the filtration in Proposition 8.3 and to compare the extension classes of sheaves of pure weight zero and the Soergel bimodules by
Theorem 7.11. Notice that the support of an extension between two sheaves are contained in the union of that of the two sheaves. Hence \( \mathcal{F} \) satisfies the support condition. The extended case can be proved similarly.

We introduce some notation before stating the next theorem. For \( \mathcal{F}, \mathcal{G} \in \mathcal{C} \mathcal{D}_\rightarrow^0 \), let \( \text{Ext}^n(\mathcal{F}, \mathcal{G})_m \) be the weight \( m \) summand of the Fr-module \( \text{Ext}^n(\mathcal{F}, \mathcal{G}) \). For \( M, M' \in K^b(\mathcal{S} \text{-mod}) \), their morphism space \( \text{HOM}_{K^b(\mathcal{S} \text{-mod})}(M, M') \) is the space of homotopy classes of \( \mathcal{S} \)-linear chain maps \( M \to M' \). Denote the degree shift of complexes in \( K^b(\mathcal{S} \text{-mod}) \) by \( \{ \} \). Denote

\[
\text{HOM}_{K^b(\mathcal{S} \text{-mod})}(M, M') = \bigoplus_{n \in \mathbb{Z}} \text{HOM}_{K^b(\mathcal{S} \text{-mod})}(M, M'[n]).
\]

If \( M, M' \in K^b(\mathcal{S} \text{-gmod}) \), \( \text{HOM}_{K^b(\mathcal{S} \text{-mod})}(M, M') \) has a natural internal grading, whose \( m \)-th graded piece is denoted by \( \text{HOM}_{K^b(\mathcal{S} \text{-mod})}(M, M')_m \). Moreover, when \( M, M' \in K^b((\mathcal{S}, \text{Fr}) \text{-gmod}) \), \( \text{HOM}_{K^b(\mathcal{S} \text{-mod})}(M, M')_m \) carries a Frobenius action naturally. Replacing \( \mathcal{S} \) by \( \bar{\mathcal{R}} \) will give the extended analog.

**Theorem 9.5.**

1. Let \( K^b(\widehat{\mathcal{S}\mathcal{B}_{m}(\bar{W}_{\mathcal{Z}})(\leq w)}/K^b(\mathcal{C}(\leq w)_{\mathcal{Z}}) \rightarrow K^b(\mathcal{C}(\leq w)_{\mathcal{Z}}) \rightarrow K^b(\mathcal{B}_{m}(\bar{W}_{\mathcal{Z}})(\leq w)) \rightarrow K^b(\mathcal{B}_{m}(\bar{W}_{\mathcal{Z}})(\leq w))_0 \). Then \( \varphi_{\leq w} \) induces an equivalence of triangulated categories,

\[
\text{Hom}^*(\mathcal{F}, \mathcal{G}) \sim \text{HOM}_{K^b(\mathcal{S} \text{-mod})}(\varphi_{\leq w}(\mathcal{F}), \varphi_{\leq w}(\mathcal{G})),
\]

which sends \( \text{Ext}^n(\mathcal{F}, \mathcal{G})_m \) to \( \text{HOM}_{K^b(\mathcal{S} \text{-mod})}(\varphi_{\leq w}(\mathcal{F}), \varphi_{\leq w}(\mathcal{G}))[n - m]_m \) for all \( m, n \in \mathbb{Z} \).

**Proof.** Both statements can be proved by methods in the reductive group analog. For (1), it is easy to see that any null-homotopy maps for the objects in \( K^b(\mathcal{B}_{m}(\bar{W}_{\mathcal{Z}})(\leq w))_0 \) can be transported to the Soergel bimodule side via \( \varphi_0 \) and vice versa.

For (2), the crucial observation is the spectral sequence induced by stupid filtrations of \( \rho_{\leq w}^* \mathcal{F} \) and \( \rho_{\leq w}^* \mathcal{G} \), which abuts to \( \text{Ext}(\mathcal{F}, \mathcal{G}) \), degenerates on the \( E_2 \) page because differentials are weight preserving.

**Proof of Theorem 9.4.** Fix a representation of \( H \) and an \( \mathcal{G}_m \)-extension of \( \mathcal{H}_K \) as in Section 7.1. One can check that the Soergel bimodule categories for \( (\bar{W}_{\mathcal{H}}, \mathcal{I}) \) and \( (\bar{W}_{\mathcal{Y}, \mathcal{I}}) \) are isomorphic. Note that \( \mathcal{I} \) depends on the choice of the fixed representations of \( \mathcal{G} \) and \( \mathcal{H} \). The two categories are isomorphic because of the fact that a non-trivial central extension of the loop group of a simple group is unique up to a non-zero \( \mathbb{Q} \)-multiple. The condition on the pushout character guarantees that the central extension is non-trivial when restricted to each simple factor (see the discussion in Section 7.4).

Apply Theorem 9.3 to the endoscopic group \( H \) with the trivial character sheaf on \( T \). We obtain an equivalence

\[
\varphi_{H, \leq w} : D(\leq w)_{\mathcal{H}_K} \sim K^b(\widehat{\mathcal{S}\mathcal{B}_{m}(\bar{W}_{\mathcal{H}})(\leq w)}/K^b(\mathcal{B}_{m}(\bar{W}_{\mathcal{H}})(\leq w))_0.
\]
where \( \tilde{W}_G \) is the (non-extended) affine Weyl group associated to \( H \), which is the same as \( \tilde{W}_L \) as a subset in \( \tilde{W} \). However, \( \leq \) is the Bruhat order in \( \tilde{W} \) induced by this canonical isomorphism, rather than the Bruhat order of \( \tilde{W}_H \). The two Bruhat orders are different as stated in Remark 5.11.

For \( \mathcal{F}, \mathcal{G} \in D(\leq w)^0_{G} \), there is a natural isomorphism of \((S,Fr)\)-modules

\[
\text{Hom}^* (\mathcal{F}, \mathcal{G}) \sim \text{Hom}_{K^S(S\text{-mod})}(\omega \varphi_{H,\leq w}(\mathcal{F}), \omega \varphi_{H,\leq w}(\mathcal{G})).
\]

Let \( \Psi^\circ_{\leq w} = \varphi^{-1}_{\leq w} \circ \varphi_{H,\leq w} \). Then \( \Psi^\circ_{\leq w} \) is an equivalence of triangulated categories between \( D(\leq w)^0_H \) and \( \mathcal{E} D(\leq w)^0_G \). Moreover, \( \Psi^\circ_{\leq w} \) is compatible with natural inclusions of triangulated categories when \( w \) runs through \( \tilde{W}_G \) (cf. [BY, Proposition B3.1]). Hence the limit of \( \Psi^\circ_{\leq w} \) is well-defined, which we denote by \( \Psi^\circ \). It is an equivalence between \( D^G_H \) and \( \mathcal{E} D^G_G \). Furthermore, it is clear that from construction of \( \Psi^\circ \), the map in (2) is an isomorphism.

It is clear from [BY, Remark B3.2] that \( \Psi^\circ \) is a monoidal functor. By Proposition 8.2, \( \Psi^\circ (IC(w)^G_H) \cong IC(w)^G \). It remains to show \( \Psi^\circ (\Delta(w)^G_H) \cong \Delta(w)^G \) and \( \Psi^\circ (\nabla(w)^G_H) \cong \nabla(w)^G \). Section 9.8 in [LY] gives a criterion for a sheaf being the standard sheaf (or costandard sheaf). The criterion remains true for the affine case. This is because the information of knowing the homomorphism space from a sheaf to all IC sheaves is enough to determine whether the sheaf is the standard sheaf or not. Hence the proof is complete. \( \square \)

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