On global location-domination in bipartite graphs

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Abstract

A dominating set $S$ of a graph $G$ is called locating-dominating, LD-set for short, if every vertex $v$ not in $S$ is uniquely determined by the set of neighbors of $v$ belonging to $S$. Locating-dominating sets of minimum cardinality are called LD-codes and the cardinality of an LD-code is the location-domination number $\lambda(G)$. An LD-set $S$ of a graph $G$ is global if it is an LD-set of both $G$ and its complement $\overline{G}$. The global location-domination number $\lambda_g(G)$ is the minimum cardinality of a global LD-set of $G$.

For any LD-set $S$ of a given graph $G$, the so-called $S$-associated graph $G^S$ is introduced. This edge-labeled bipartite graph turns out to be very helpful to approach the study of LD-sets in graphs, particularly when $G$ is bipartite.

This paper is mainly devoted to the study of relationships between global LD-sets, LD-codes and the location-domination number in a graph $G$ and its complement $\overline{G}$, when $G$ is bipartite.

Keywords: Domination, Global domination, Locating domination, Complement graph, Bipartite graph.

1 Introduction

Let $G = (V,E)$ be a simple, finite graph. The open neighborhood of a vertex $v \in V$ is $N_G(v) = \{u \in V : uv \in E\}$. The complement of a graph $G$, denoted by $\overline{G}$, is the graph on the same vertices such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. The distance between vertices $v,w \in V$ is denoted by $d_G(v,w)$. We write $N(u)$ or $d(v,w)$ if the graph $G$ is clear from the context. Given any pair of sets $A$ and $B$, $A \triangle B$ denotes its symmetric difference, that is, $(A \setminus B) \cup (B \setminus A)$. For further notation and terminology, we refer the reader to [6].

A set $D \subseteq V$ is a dominating set if for every vertex $v \in V \setminus D$, $N(v) \cap D \neq \emptyset$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$ [8]. A dominating set is global if it is a dominating set of both $G$ and its complement graph, $\overline{G}$. The minimum cardinality of a global dominating set of $G$, denoted by $\gamma_g(G)$, is the global domination number of $G$ [3,4,14]. If $D$ is a subset of $V$ and $v \in V \setminus D$, we say that $v$ dominates $D$ if $D \subseteq N(v)$.

A dominating set $S \subseteq V$ is a locating-dominating set, LD-set for short, if for every two different vertices $u,v \in V \setminus S$, $N(u) \cap S \neq N(v) \cap S$. The location-domination number of $G$, denoted by $\lambda(G)$, is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality $\lambda(G)$ is called an LD-code [13,15]. Certainly, every LD-set of a non-connected graph $G$ is the union of LD-sets of its connected components and the location-domination number is the sum of the location-domination number of its connected components. LD-codes and the location-domination parameter have been intensively studied during the last decade; see [1,2,5,7,11,9]. A complete and regularly updated list of papers on locating-dominating codes is to be found in [12].
The remaining part of this paper is organized as follows. In Section 2, we deal with the problem of approaching the relationship between $\lambda(G)$ and $\lambda(\overline{G})$, for any arbitrary graph $G$. In Section 3, we introduce the so-called $LD$-set-associated graph $G^S$, which is an edge-labeled bipartite graph constructed from an arbitrary $LD$-set $S$ of a given graph $G$, and show some basic properties of this graph. Finally, Section 4 is concerned with the study of relationships between the location-domination number $\lambda(G)$ of a bipartite graph $G$ and the location-domination number $\lambda(\overline{G})$ of its complement $\overline{G}$.

2 General case

This section is devoted to approach the relationship between $\lambda(G)$ and $\lambda(\overline{G})$, for any arbitrary graph $G$. Some of the results we present were previously shown in [9, 10] and we include them for the sake of completeness.

Notice that $N_G(x) \cap S = S \setminus N_G(x)$ for any set $S \subseteq V$ and any vertex $x \in V \setminus S$. A straightforward consequence of this fact are the following results.

**Proposition 1 ([10]).** If $S \subseteq V$ is an $LD$-set of a graph $G = (V, E)$, then $S$ is an $LD$-set of $\overline{G}$ if and only if $S$ is a dominating set of $\overline{G}$.

**Proposition 2 ([9]).** Let $S \subseteq V$ be an $LD$-set of a graph $G = (V, E)$. Then, the following holds.

(a) There is at most one vertex $u \in V \setminus S$ dominating $S$, and in the case it exists, $S \cup \{u\}$ is an $LD$-set of $\overline{G}$.

(b) $S$ is an $LD$-set of $\overline{G}$ if and only if there is no vertex in $V \setminus S$ dominating $S$ in $G$.

The following theorem is a consequence of the preceding propositions.

**Theorem 1 ([9]).** For every graph $G$, $|\lambda(G) - \lambda(\overline{G})| \leq 1$.

According to the preceding inequality, for every graph $G$, $\lambda(\overline{G}) \in \{\lambda(G) - 1, \lambda(G), \lambda(G) + 1\}$, all cases being feasible for some connected graph $G$. See Table 1 for some basic examples covering all possible cases.

We intend to obtain either necessary or sufficient conditions for a graph $G$ to satisfy $\lambda(\overline{G}) > \lambda(G)$, i.e., $\lambda(\overline{G}) = \lambda(G) + 1$. This problem was approached and completely solved in [10] for the family of block-cactus. In this work, we carry out a similar study for bipartite graphs. After noticing that solving the equality $\lambda(\overline{G}) = \lambda(G) + 1$ is closely related to analyzing the existence or not of sets that are simultaneously locating-dominating sets in both $G$ and its complement $\overline{G}$, the following definitions were introduced in [10].

**Definition 1 ([10]).** A set $S$ of vertices of a graph $G$ is a global $LD$-set if $S$ is an $LD$-set of both $G$ and its complement $\overline{G}$. The global location-domination number of a graph $G$, denoted by $\lambda_g(G)$, is defined as the minimum cardinality of a global $LD$-set of $G$. 

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According to Proposition [2], an LD-set \( S \) of a graph \( G \) is non-global if and only if there exists a (unique) vertex \( u \in V(G) \setminus S \) which dominates \( S \), i.e., such that \( S \subseteq N(u) \). Notice that, for every graph \( G \), \( \lambda_g(G) = \lambda_g(\overline{G}) \), since for every set of vertices \( S \subset V(G) = V(\overline{G}) \), \( S \) is a global LD-set of \( G \) if and only if it is a global LD-set of \( \overline{G} \). Observe also that an LD-code \( S \) of \( G \) is a global LD-set if and only if it is both an LD-code of \( G \) and an LD-set of \( \overline{G} \).

**Theorem 2** ([10]). For any graph \( G = (V, E) \), \( \max\{\lambda(G), \lambda(\overline{G})\} \leq \lambda_g(G) \leq \min\{\lambda(G) + 1, \lambda(\overline{G}) + 1\} \). Moreover,

(a) If \( \lambda(G) \neq \lambda(\overline{G}) \), then \( \lambda_g(G) = \max\{\lambda(G), \lambda(\overline{G})\} \).

(b) If \( \lambda(G) = \lambda(\overline{G}) \), then \( \lambda_g(G) \in \{\lambda(G), \lambda(G) + 1\} \), and both possibilities are feasible.

(c) \( \lambda_g(G) = \lambda(G) + 1 \) if and only if every LD-code of \( G \) is non-global.

**Corollary 1.** If \( G \) is a graph with a global LD-code, then \( \lambda(\overline{G}) \leq \lambda(G) \).

In Table 1, the location-domination number of some families of graphs is displayed, along with the location-domination number of its complement graphs and the global location-domination number. Concretely, we consider the path \( P_n \) of order \( n \geq 7 \); the cycle \( C_n \) of order \( n \geq 7 \); the wheel \( W_n \) of order \( n \geq 8 \), obtained by joining a new vertex to all vertices of a cycle of order \( n - 1 \); the complete graph \( K_n \) of order \( n \geq 2 \); the complete bipartite graph \( K_{r,n-r} \) of order \( n \geq 4 \), with \( 2 \leq r \leq n-r \) and stable sets of order \( r \) and \( n-r \), respectively; the star \( K_{1,n-1} \) of order \( n \geq 4 \), obtained by joining a new vertex to \( n-1 \) isolated vertices; and finally, the bi-star \( K_2(r,s) \) of order \( n \geq 6 \) with \( 3 \leq r \leq s = n-r \), obtained by joining the central vertices of two stars \( K_{1,r-1} \) and \( K_{1,s-1} \) respectively.

**Proposition 3** ([10]). Let \( G \) be a graph of order \( n \). If \( G \in \{ P_n, C_n, W_n, K_n, K_{1,n-1}, K_{r,n-r}, K_2(r,s) \} \), then the values of \( \lambda(G), \lambda(\overline{G}) \) and \( \lambda_g(G) \) are known and they are displayed in Table 1.

| \( G \) | \( P_n \) | \( C_n \) | \( W_n \) | \( K_n \) | \( K_{1,n-1} \) | \( K_{r,n-r} \) | \( K_2(r,s) \) |
|---|---|---|---|---|---|---|---|
| \( n \) | \( n \geq 7 \) | \( n \geq 7 \) | \( n \geq 8 \) | \( n \geq 2 \) | \( n \geq 4 \) | \( 2 \leq r \leq n-r \) | \( 3 \leq r \leq s \) |
| \( \lambda(G) \) | \( \lfloor \frac{2n}{3} \rfloor \) | \( \lfloor \frac{2n}{3} \rfloor \) | \( \lfloor \frac{2n-2}{5} \rfloor \) | \( n-1 \) | \( n-1 \) | \( n-2 \) | \( n-2 \) |
| \( \lambda(\overline{G}) \) | \( [\frac{2n}{3}] \) | \( [\frac{2n}{3}] \) | \( [\frac{2n-2}{5}] \) | \( n \) | \( n-1 \) | \( n-2 \) | \( n-3 \) |
| \( \lambda_g(G) \) | \( \lfloor \frac{2n}{3} \rfloor \) | \( \lfloor \frac{2n}{3} \rfloor \) | \( \lfloor \frac{2n+1}{5} \rfloor \) | \( n \) | \( n-1 \) | \( n-2 \) | \( n-2 \) |

Table 1: The values of \( \lambda(G), \lambda(\overline{G}) \) and \( \lambda_g(G) \) for some families of graphs.

### 3 The LD-set-associated graph

Let \( S \) be an LD-set of a graph \( G \). We introduce in this section a labeled graph associated to \( S \) and study some general properties. This graph will allow us to derive some properties related to LD-sets and the location-domination number of \( G \).

**Definition 2.** Let \( S \) be an LD-set with exactly \( k \) vertices of a connected graph \( G = (V, E) \) of order \( n \). Consider \( z \notin V(G) \) and define \( N_G(z) = \emptyset \). The so-called \( S \)-associated graph, denoted by \( G^S \), is the edge-labeled graph defined as follows.
(1) \( V(G^S) = (V \setminus S) \cup \{z\} \);

(2) For every pair of vertices \( x, y \in V(G^S) \), \( xy \in E(G^S) \) if and only if \( |(N_G(x) \cap S) \triangle (N_G(y) \cap S)| = 1 \);

(3) The label \( \ell(xy) \) of edge \( xy \in E(G^S) \) is the only element of \( (N_G(x) \cap S) \triangle (N_G(y) \cap S) \in S \).

Figure 1: Left: a graph \( G \). Right: the LD-set-associated graph \( G^S \), where \( S = \{1, 2, 3, 4, 5\} \).

Notice that two vertices of \( V \setminus S \) are adjacent in \( G^S \) if their neighborhood in \( S \) differ in exactly one vertex, the label of the edge, and \( z \) is adjacent to vertices of \( V \setminus S \) with exactly a neighbor in \( S \). Therefore, we can represent the graph \( G^S \) with the vertices lying on \( |S| + 1 \) levels, from bottom (level 0) to top (level \( |S| \)), in such a way that vertices with exactly \( k \) neighbors in \( S \) are at level \( k \). There is at most one vertex at level \( |S| \) and, if it is so, this vertex is adjacent to all vertices of \( S \). The vertices at level 1 are those with exactly one neighbor in \( S \) and \( z \) is the unique vertex at level 0. An edge of \( G^S \) has its endpoints at consecutive levels. Moreover, if \( e = xy \in E(G^S) \), with \( \ell(e) = u \in S \), and \( x \) is at exactly one level higher than \( y \), then \( N(x) \cap S = (N(y) \cap S) \cup \{u\} \), i.e., \( x \) and \( y \) have the same neighborhood in \( S \setminus \{u\} \). Therefore, the existence of an edge in \( G^S \) with label \( u \in S \) means that \( S \setminus \{u\} \) is not an LD-set. Hence, if \( S \) is an LD-code, then for every \( u \in S \) there exists at least an edge in \( G^S \) with label \( u \). See Figure 1 for an example of an LD-set-associated graph.

The following proposition states some properties of LD-set-associated graphs.

**Proposition 4.** Let \( S \) be an LD-set with exactly \( k \) vertices of a connected graph \( G = (V, E) \) of order \( n \). Let \( G^S \) be its \( S \)-associated graph. Then the following holds.

1. \( |V(G^S)| = n - k + 1 \).
2. \( G^S \) is bipartite.
3. Incident edges have different labels.
4. Every cycle of \( G^S \) contains an even number of edges labeled \( v \), for all \( v \in S \).
5. Let $\rho$ be a walk with no repeated edges in $G^S$. If $\rho$ contains an even number of edges labeled $v$ for every $v \in S$, then $\rho$ is a closed walk.

6. If $\rho = x_0x_1x_2 \ldots x_{i+h}$ is a path satisfying that vertex $x_{i+h}$ lies at level $i + h$, for any $h \in \{0, 1, \ldots, h\}$, then

(a) the edges of $\rho$ have different labels;
(b) for all $j \in \{i + 1, i + 2, \ldots, i + h\}$, $N(x_j) \cap S$ contains the vertex $\ell(x_kx_{k+1})$, for any $k \in \{i, i + 1, \ldots, j - 1\}$.

Proof. 1. It is a direct consequence from the definition of $G^S$.

2. Consider the sets $V_1 = \{x \in V(G^S) : |N(x) \cap S| \text{ is odd}\}$ and $V_2 = \{x \in V(G^S) : |N(x) \cap S| \text{ is even}\}$. Then $V(G^S) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Since $|N(x) \cap S| - |N(y) \cap S| = 1$ for any $xy \in E(G^S)$, it is clear that the vertices $x, y$ are not in the same subset $V_i$, $i = 1, 2$.

3. Suppose that the sets $N(x) \cap S$ and $N(y) \cap S$ differ only in element $v$ and the sets $N(y) \cap S$ and $N(z) \cap S$ differ only in element $v \in S$. It is only possible if $N(x) \cap S = N(z) \cap S$, implying that $x = z$.

4. Let $\rho$ be a cycle such that $E(\rho) = \{x_0x_1, x_1x_2, \ldots, x_{k+1}x_0\}$. The set of neighbors in $S$ of two consecutive vertices differ exactly in one vertex. If we begin with $N(x_0) \cap S$, each time we add (remove) the vertex of the label of the corresponding edge, we have to remove (add) it later in order to obtain finally the same neighborhood, $N(x_0) \cap S$. Therefore, $\rho$ contains an even number of edges with label $v$.

5. Consider the vertices $x_0, x_1, x_2, x_3, \ldots, x_{2k}$ of the walk $\rho$. In this case, $N(x_{2k}) \cap S$ is obtained from $N(x_0) \cap S$ by adding or removing the labels of all the edges of the walk. Since every label appears an even number of times, for each element $v \in S$ we can match its appearances in pairs, and each pair means that we add and remove (or remove and add) it from the neighborhood in $S$. Therefore, $N(x_{2k}) \cap S = N(x_0) \cap S$, and hence $x_0 = x_{2k}$.

6. It straightforward follows from the fact that $N(x_j) \cap S = N(x_{j-1} \cap S) \cup \{\ell(x_{j-1}x_j)\}$, for any $j \in \{i + 1, \ldots, i + h\}$. \hfill $\Box$

4 The bipartite case

In the sequel, $G = (V, E)$ stands for a bipartite connected graph of order $n = r + s \geq 4$, such that $V = U \cup W$, being $U, W$ their stable sets and $1 \leq |U| = r \leq |W| = s$.

This section is devoted to solving the equation $\lambda(G) = \lambda(G) + 1$ when we restrict ourselves to bipartite graphs. According to Corollary 1, this equality is feasible only for graphs without global LD-codes.

Lemma 1. Let $S$ be an LD-code of $G$. Then, $\lambda(G) \leq \lambda(G)$ if any of the following conditions holds.

1. $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$.
2. \( r < s \) and \( S = W \).
3. \( 2^r \leq s \).

**Proof.** If \( S \) satisfies item 1., then there is no vertex dominating \( S \) and, by Proposition 2, \( S \) is a global LD-code of \( G \), which, according to Corollary 1, means that \( \lambda(G) \leq \lambda(G) \). Next, assume that \( r < s \) and \( S = W \). In this case, \( U \) is not an LD-set, but is a dominating set since \( G \) is connected. Therefore, there exists a pair of vertices \( w_1, w_2 \in W \) such that \( N(w_1) = N(w_2) \). Hence, \( W - \{w_1\} \) is an LD-set of \( G - w_1 \). Let \( u \in U \) be a vertex adjacent to \( w_1 \) (it exists since \( G \) is connected), and notice that \( (W \setminus \{w_1\}) \cup \{u\} \) is an LD-code of \( G \) with vertices in both stable sets, which, by the preceding item, means that \( \lambda(G) \leq \lambda(G) \). Finally, if \( 2^r \leq s \) then \( S \neq U \), which means that \( S \) satisfies either item 1. or item 2. \( \square \)

**Corollary 2.** If \( \lambda(G) = \lambda(G) + 1 \), then \( r \leq s \leq 2^r - 1 \). Moreover, if \( r < s \) then \( U \) is the unique LD-code of \( G \), and if \( r = s \) we may assume that \( U \) is a non-global LD-code of \( G \).

**Proposition 5.** If \( G \) has order at least 3 and \( 1 \leq r \leq 2 \), then \( \lambda(G) \leq \lambda(G) \).

**Proof.** If \( r = 1 \), then \( G \) is the star \( K_{1,n-1} \) and \( \lambda(G) = \lambda(G) = n - 1 \).

| \( r \)   | \( \lambda(G) \) | \( \lambda(G) - 1 \) |
|---------|-----------------|-----------------|
| 1       | \( \lambda(G) \) | \( \lambda(G) - 1 \) |
| 2       | \( \lambda(G) \) | \( \lambda(G) - 1 \) |
|         | \( \lambda(G) \) | \( \lambda(G) - 1 \) |

Figure 2: Some bipartite graphs with \( 1 \leq r \leq 2 \).

Suppose that \( r = 2 \). If \( s \geq 2^2 = 4 \) then, by Lemma 1, \( \lambda(G) \leq \lambda(G) \).

If \( s = 2 \), then \( G \) is either \( P_4 \) and \( \lambda(P_4) = \lambda(P_4) = 2 \), or \( G \) is \( C_4 \) and \( \lambda(C_4) = \lambda(C_4) = 2 \).

If \( s = 3 \), then \( G \) is \( P_5, K_{2,3}, K_2(1,2), \) or a banner \( P \), and \( \lambda(P_5) = \lambda(P_5) = 2, \lambda(K_{2,3}) = \lambda(K_{2,3}) = 3, 2 = \lambda(K_2(1,2)) < \lambda(K_2(1,3)) = 3, \) and \( 2 = \lambda(P) < \lambda(P) = 3. \) \( \square \)

Notice that the only bipartite graphs \( G \) such that \( \lambda(G) = 2 \) are \( P_3, P_4, C_4 \) and \( P_5 \). Observe also that every bipartite graph \( G \) such that \( \lambda(G) = \lambda(G) + 1 \) satisfies \( \lambda(G) \geq r \), being \( r \) the order of its smallest stable set.

Next, we approach the case \( \lambda(G) \geq 3 \). That is to say, from now on we assume that \( r \geq 3 \).

**Lemma 2.** If \( \lambda(G) = \lambda(G) + 1 \) and \( U \) is an LD-code of \( G \), then \( G^U \) contains, for every vertex \( u \in U \), at least two edges with label \( u \).
Lemma 3. Let $\lambda(\mathcal{G}) = \lambda(G) + 1$ and assume that $U$ is an LD-code of $G$. Consider a subgraph $H$ of $G^U$ induced by a set of edges containing exactly two edges with label $u$, for each $u \in U$. Then, all connected components of $H$ are cactus.

Proof. We will prove that there is no edge lying on two different cycles of $H$. Suppose on the contrary that there is an edge $e_1$ contained in two different cycles $C_1$ and $C_2$ of $H$. If the label of $e_1$ is $u \in U$, by Proposition 4 both cycles $C_1$ and $C_2$ contain the other edge $e_2$ of $H$ labeled with $u$. Suppose that $e_1 = x_1y_1$ and $e_2 = x_2y_2$ and assume w.l.o.g. that there exist $x_1 - x_2$ and $y_1 - y_2$ paths in $C_1$ not containing edges $e_1, e_2$. Let $P_1$ and $P_1'$ denote respectively those paths (see Figure 4(a)).
We have two possibilities for $C_2$: (i) there are $x_1 - x_2$ and $y_1 - y_2$ paths in $C_2$ not containing neither $e_1$ nor $e_2$. Let $P_2$ denote the $x_1 - x_2$ path in $C_2$ in that case (see Figure 4(b); (ii) there are $x_1 - y_2$ and $y_1 - x_2$ paths in $C_2$ not containing neither $e_1$ nor $e_2$ (see Figure 4(c)).

In case (ii), the closed walk formed with the path $P_1$, $e_1$ and the $y_1 - x_2$ path in $C_2$ would contain a cycle with exactly an edge labeled with $u$, which is a contradiction (see Figure 4(d)).

In case (i), at least one the following cases holds: the $x_1 - x_2$ paths in $C_1$ and in $C_2$, $P_1$ and $P_2$, are different or the $y_1 - y_2$ paths in $C_1$ and in $C_2$ are different (otherwise, $C_1 = C_2$).

![Figure 4](image)

Figure 4: All connected components of the subgraph $H$ are cactus.

Assume that $P_1$ and $P_2$ are different. Let $z_1$ be the last vertex shared by $P_1$ and $P_2$ advancing from $x_1$ and let $z_2$ be the first vertex shared by $P_1$ and $P_2$ advancing from $z_1$ in $P_2$. Notice that $z_1 \neq z_2$. Consider the cycle $C_3$ formed with the $z_1 - z_2$ paths in $P_1$ and $P_2$. Let $P_1^*$ and $P_2^*$ be respectively the $z_1 - z_2$ subpaths of $P_1$ and $P_2$ (see Figure 4(e)). We claim that the internal vertices of $P_2^*$ do not lie in $P_1^*$. Otherwise, consider the first vertex $t$ of $P_1^*$ lying also in $P_2^*$. The cycle beginning in $x_1$, formed by the edge $e_1$, the $y_1 - t$ path contained in $P_1^*$, the $t - z_1$ path contained in $P_2^*$ and the $z_1 - x_1$ path contained in $P_1$ has exactly one appearance of an edge with label $u$, which is a contradiction (see Figure 4(f)). By Proposition 3, the labels of edges belonging to $P_1^*$ appear exactly two times in cycle $C_3$, but they also appear exactly two times in cycle $C_1$. But this is only possible if they appear exactly two times in $P_1^*$, since $H$ contains exactly to edges with the same label. By Proposition 4, $P_1^*$ must be a closed path, which is a contradiction.

We present next some properties relating parameters of bipartite graphs having cactus as connected components.

**Lemma 4.** Let $H$ be a bipartite graph of order at least 4 such that all its connected components are cactus. If $H$ has $cc(H)$ connected components and $cy(H)$ cycles, then the following holds.
1. \(|V(H)| = |E(H)| - \text{cy}(H) + \text{cc}(H)|.

2. If \(\text{ex}(H) = |E(H)| - 4\text{cy}(H)\), then \(\text{ex}(H) \geq 0\) and \(|V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\text{ex}(H) + \text{cc}(H)|.

3. \(|V(H)| \geq \frac{3}{4}|E(H)| + 1|.

4. \(|V(H)| = \frac{3}{4}|E(H)| + 1| \text{ if and only if } H \text{ is connected and all blocks are cycles of order } 4|.

Proof. 1. Since \(H\) is a planar graph with \(\text{cy}(H) + 1\) faces and \(\text{cc}(H)\) connected components, the equality follows from the generalization of Euler’s Formula:

\[
(\text{cy}(H) + 1) + |V(H)| = |E(H)| + (\text{cc}(H) + 1)\]

2. All cycles of a bipartite graph have at least 4 edges, hence \(\text{ex}(H) \geq 0\). By the preceding item,

\[
|V(H)| = |E(H)| - \text{cy}(H) + \text{cc}(H) = |E(H)| - \frac{1}{4}(|E(H)| - \text{ex}(H)) + \text{cc}(H) = \frac{3}{4}|E(H)| + \frac{1}{4}\text{ex}(H) + \text{cc}(H).
\]

3. It immediately follows from the preceding item.

4. Observe first that if \(H\) is connected and all blocks are cycles of order 4, then \(\text{cc}(H) = 1\) and \(|E(H)| = 4\text{cy}(H)|. Hence, \(\text{ex}(H) = |E(H)| - 4\text{cy}(H) = 0\) and by item 2, \(|V(H)| = \frac{3}{4}|E(H)| + 1|.

Conversely, suppose that \(|V(H)| = \frac{3}{4}|E(H)| + 1|\. The graph \(H\) must be connected, since otherwise \(|V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\text{ex}(H) + \text{cc}(H) \geq \frac{3}{4}|E(H)| + 2|\). On the other hand, if \(H\) contains a cycle of order at least 6 or a bridge, then \(\text{ex}(H) = |E(H)| - 4\text{cy}(H) > 0|\) implying that \(|V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\text{ex}(H) + \text{cc}(H) > \frac{3}{4}|E(H)| + \text{cc}(H) = \frac{3}{4}|E(H)| + 1|\. \]

Proposition 6. If \(r \geq 3\) and \(\lambda(G) = \lambda(G) + 1|\), then \(\frac{3r}{2} \leq s \leq 2r - 1|\).

Proof. By Corollary 2, we have that \(s \leq 2^r - 1\), and we may assume that \(U\) is a non-global LD-code and there is no LD-code with vertices in both stable sets.

Consider a subgraph \(H\) of \(G^U\) with exactly two edges with label \(u\) for any \(u \in U\). The graph \(H\) is bipartite since it is a subgraph of \(G^U\) and by Lemma 4,

\[
s + 1 = |V(G^U)| \geq |V(H)| \geq \frac{3}{4}|E(H)| + 1 = \frac{3}{4}(2r) + 1 = \frac{3r}{2} + 1
\]

and consequently \(s \geq \frac{3r}{2}\). \]

Lemma 5. If \(\lambda(G) = \lambda(G) + 1|\) and \(U\) is an LD-code of \(G\), let \(z\) be the vertex of \(G^U\) introduced in Definition 2 and let \(H\) be a subgraph of \(G^U\) with exactly two edges with label \(u\), for each \(u \in U\). Then the following holds.

1. If \(H\) has at least two connected components, then \(s \geq \frac{3r}{2} + 1\).

2. If \(\text{deg}_{G^U}(z) = 0\), then \(s \geq \frac{3r}{2} + 1|\).

3. \(\text{deg}_{G^U}(z) \neq 0\) if and only if there is at least a vertex in \(W\) of degree 1 in \(G|\).

4. If \(G\) has no vertex of degree 1 in \(W\), then \(s \geq \frac{3r}{2} + 1\).
Proof. 1. By Lemma 4, \( s+1 \geq |V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\text{ex}(H) + \text{cc}(H) \geq \frac{3}{4}|E(H)| + \frac{1}{4}\text{ex}(H) + \frac{1}{4} \), and thus, \( s \geq \frac{3r}{2} + 1 \).

2. If \( \text{deg}_{G'}(z) = 0 \), then \( z \) is not a vertex of \( H \). Hence, \( s \geq |V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\text{ex}(H) + \text{cc}(H) \geq \frac{3}{4}|E(H)| + 1 = \frac{3r}{2} + 1 \).

3. We know that \( \text{deg}_{G'}(z) \neq 0 \) if and only if there is a vertex \( w \in W \) satisfying \( N(w) \bigtriangleup N(z) = 0 \), i.e. if and only if \( \text{deg}_{G'}(w) = 1 \).

4. It is a straight consequence of items 2 and 3.

Proposition 7. There are no bipartite graphs \( G \) satisfying \( \lambda(G) = \lambda(G) + 1 \) if \( \frac{3r}{2} \leq s < \frac{3r}{2} + 1 \).

Proof. Suppose on the contrary that \( G \) is a bipartite graph satisfying the conditions of the proposition. Condition \( \lambda(G) = \lambda(G) + 1 \) implies that we may assume that \( U \) is an LD-code of \( G \), there is no LD-code with vertices in both stable sets and \( U \) is not an LD-set of \( \bar{G} \). Consider a subgraph \( H \) of \( G' \) with exactly two edges with label \( u \), for each \( u \in U \) (it exists by Lemma 2).

Observe that the inequality is only possible for \( s = \frac{3r}{2} \), whenever \( r \) is even, and for \( s = \frac{3r+1}{2} \), whenever \( r \) is odd. If \( r \) is even and \( s = \frac{3r}{2} \), then

\[
\frac{3r}{2} + 1 = s + 1 = |V(G')| \geq |V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\text{ex}(H) + \text{cc}(H) = \frac{3r}{2} + \frac{1}{4}\text{ex}(H) + \text{cc}(H).
\]

Since \( \text{ex}(H) \geq 0 \) and \( \text{cc}(H) \geq 1 \), this is only possible for \( \text{ex}(H) = 0 \), \( \text{cc}(H) = 1 \), and \( V(G') = V(H) \). By Lemma 4, \( H \) is a cactus with all blocks cycles of order 4, concretely, \( \frac{r}{2} \) cycles. If \( r \) is odd and \( s = \frac{3r+1}{2} \), then

\[
\frac{3r+1}{2} + 1 = s+1 = |V(G')| \geq |V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\text{ex}(H) + \text{cc}(H) = \frac{3r}{2} + \frac{1}{4}\text{ex}(H) + \text{cc}(H).
\]

This is only possible for \( \text{ex}(H) = 2 \), \( \text{cc}(H) = 1 \), and \( V(G') = V(H) \). By Lemma 4, \( H \) is a cactus with exactly \( \frac{r-1}{2} \) cycles: \( \frac{r-1}{2} - 1 \) cycles of order 4 and a cycle of order 6, or \( \frac{r-1}{2} \) cycles of order 4 and two bridges.

We also know that condition \( \lambda(G) = \lambda(G) \) implies the existence of a vertex \( w^* \in V(G) \subseteq V(G') = V(H) \) such that \( N_G(w^*) = U \), i.e., \( H \) has a vertex at the highest level. Lemma 5 allows us to conclude that \( H \) is connected and \( z \in V(H) \). Thus, \( H \) must be a chain of cycles of order 4, or a chain of a cycle of order 6 and cycles of order 4, or a chain of a bridge and cycles of order 4, plus another bridge hanging from a vertex of this chain, with both bridges having the same label and, by Proposition 4, not lying in a path with all vertices at different levels (see Figure 5).

In consequence, one of the following cases holds in \( H \): (i) \( z \) belongs to a cycle \( C \) of order 4; (ii) \( z \) belongs to a cycle \( C \) of order 6; (iii) \( z \) belongs to a bridge, \( e \). In this case, there is no \( x - z \) path of length \( i \) in \( H \) with consecutive vertices in levels \( i, i-1, \ldots, 1, 0 \) respectively containing both edges of \( H \) with label \( \ell(e) \). We may assume w.l.o.g. that the labels \( a, b, c \in U \) of the edges of \( C \) and \( e \) are those of Figure 6. Let \( w_0 \) be the vertex of \( G \) indicated in the same figure.

We claim that the set \( S = (U \setminus \{a\}) \cup \{w_0\} \) is an LD-set of \( \bar{G} \) with exactly \( r \) vertices. Indeed, if \( w_0 \neq w^* \), then \( N_G(a) \cap S = S \setminus \{w_0\} \), \( N_G(w^*) \cap S = \{w_0\} \) and for any \( x \in W \setminus \{w^*, w_0\}, N_G(x) \cap S = \{w_0\} \cup S' \), where \( S' = U \setminus (N_G(x) \cup \{a\}) \neq \emptyset \), since \( N_G(x) \neq U \setminus \{a\} \). Moreover, for
Figure 5: Examples of subgraphs $H$.

Figure 6: Possible cases for vertex $z$ in subgraph $H$.

any pair of different vertices $x, y \in W \setminus \{w^*, w_0\}$, $N_G(x) \cap (U \setminus \{a\}) \neq N_G(y) \cap (U \setminus \{a\})$, implies that $N_{\overline{G}}(x) \cap S \neq N_{\overline{G}}(y) \cap S$. If $w_0 = w^*$, then $N_{\overline{G}}(a) \cap S = S \setminus \{w^*\}$, and for any $x \in W \setminus \{w^*\}$, $N_{\overline{G}}(x) \cap S = \{w^*\} \cup S'$, where $S' = U \setminus (N_G(x) \cup \{a\})$. Moreover, for any pair of different vertices $x, y \in W \setminus \{w^*\}$, $N_G(x) \cap (U \setminus \{a\}) \neq N_G(y) \cap (U \setminus \{a\})$, implies that $N_{\overline{G}}(x) \cap S \neq N_{\overline{G}}(y) \cap S$.

**Proposition 8.** For every pair $(r, s)$, $r, s \in \mathbb{N}$, such that $3 \leq r$ and $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$, there exists a bipartite graph $G(r, s)$ such that $\lambda(G) = \lambda(G) + 1$.

**Proof.** Let $s = \left\lceil \frac{3r}{2} + 1 \right\rceil$. Consider the bipartite graph $G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ such that $V = U \cup W$, $U = [r] = \{1, 2, \ldots, r\}$, and $W \subseteq \mathcal{P}([r]) \setminus \{\emptyset\}$ is defined as follows. For $r = 2k$ even:

$$W = \{[r]\} \cup \{[r] \setminus \{i\} : i \in [r]\} \cup \{[r] \setminus \{2i - 1, 2i\} : 1 \leq i \leq k\}$$

and for $r = 2k + 1$ odd:

$$W = \{[r]\} \cup \{[r] \setminus \{i\} : i \in [r]\} \cup \{[r] \setminus \{2i - 1, 2i\} : 1 \leq i \leq k - 1\}$$

$$\cup \{[r] \setminus \{r - 2, r - 1\}, [r] \setminus \{r - 1, r\}, [r] \setminus \{r - 2, r - 1, r\}\}$$
By construction, $U$ is an LD-set of $G$ with $r$ vertices and by Corollary 2, $U$ is not an LD-set of $G$ (see in Figure 7 the $U$-associated graph, $G^U$). We claim that there is no LD-set in $G$ with at most $r$ vertices.

Suppose that $S$ is an LD-set of $G$. We already know that $S \neq U$. Let us assume that $|S \cap U| = r - k$, $k \geq 1$. Consider the subgraph $H$ of $G^U$ induced by $2k$ edges of $G^U$ with label $u \in U \setminus S$. Notice that, by definition, this subgraph exists and $z \notin V(H)$. Moreover, by Lemma 3 all connected components of $H$ are cactus. Observe that, by definition of the associated graph $G^U$, the vertices lying at the same connected component of $H$ have the same neighborhood in $S \cap U$. We know also that $W$ induces a complete graph in $G$. Therefore, at least all but one vertex of each connected component of $H$ must be in $S$. By Lemma 3, this value is

$$|V(H)| - cc(H) = \frac{3}{4} |E(H)| + \frac{1}{4} ex(H) = \frac{3}{4} 2k + \frac{1}{4} ex(H) = \frac{3}{2} k + \frac{1}{4} ex(H) \geq \frac{3}{2} k.$$ 

Hence, $|S| \geq (r - k) + \frac{3}{2} k = r + \frac{1}{2} k > r$.

**Remark.** We derive from this result that $\lambda(G) = r$. Nevertheless, a direct proof of this fact can be given: it can be proved in a similar way that there is no LD-set of $G$ with less than $r$ vertices.

For $s > \lceil \frac{3r}{2} + 1 \rceil$, we can add up to $2^r - 1 - r$ vertices to the set $W$ of the graph $G(r, \lceil \frac{3r}{2} + 1 \rceil)$ taking into account that the neighborhoods in $U$ of the vertices of $W$ must be different and non-empty.

**Theorem 3.** Let $r, s$ be a pair of integers such that $3 \leq r \leq s$.

1. There exists a bipartite graph $V(G) = U \cup W$ such that $|U| = r$, $|W| = s$ and $\lambda(G) = \lambda(G) - 1$.
2. There exists a bipartite graph $V(G) = U \cup W$ such that $|U| = r$, $|W| = s$ and $\lambda(G) = \lambda(G)$.
3. There exist a bipartite graph $V(G) = U \cup W$ such that $|U| = r$, $|W| = s$ and $\lambda(G) = \lambda(G) + 1$ if and only if $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$.

**Proof.** To prove item (1), take the bi-star $K_2(r, s)$ and check that $\lambda(K_2(r, s)) = r + s - 2$ and $\lambda(K_2(r, s)) = r + s - 3$. To prove item (2), take the biclique $K_{r,s}$ and check that $\lambda(K_{r,s}) = \lambda(K_{r,s}) = r + s - 2$. Finally, observe that item (3) is a corollary of Propositions 3, 7 and Proposition 8.
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