A Dynamically Diluted Alignment Model Reveals the Impact of Cell Turnover on the Plasticity of Tissue Polarity Patterns

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### 1 Supplementary Movie

Trajectory of a typical simulation of the IPS model on a $100 \times 100$ lattice with periodic boundaries, $\epsilon_n = 4.5$, $\delta = 0.2$, $\beta = 1$, $\epsilon_s = 1$, $s = (1, 0)$. The movie covers 1 time unit in the dedimensionalised model time; time resolution of visualisation is 0.001. See main text fig. 2C for colour code. The bottom left detail of the same simulation is shown in main text fig. 3A for times 0.05, 0.2, 0.5, 0.8 (a-d), and its analysis is shown in main text fig. 3B.
2 Details of the mean-field analysis

Using mean-field approximation, we derive an ODE which approximates the temporal evolution of \( \mathbf{p}(t) = \mathbf{p}(\mathbf{\eta}(t)) \). Denote the fraction of nodes in states \( e_0, \ldots, e_8 \) at time \( t \) in the IPS model by

\[
a_i(t) := a_i(\mathbf{\eta}(t)) := \frac{\# \{ z \in S \colon \mathbf{\eta}_z(t) = e_i \}}{\# S}, \quad i = 0, \ldots, 8
\]

as in main text eq. (15). Writing shorthand \( \mathbf{a}(t) := (a_1(t), \ldots, a_8(t))^T \) where \( (\cdot)^T \) denotes matrix transposition, the mean polarisation vector is related to vector \( \mathbf{a} \) via main text eq. (16)

\[
\mathbf{p}(t) = (p_x(t), p_y(t))^T = \sum_{i=1}^{8} a_i(t) \cdot e_i = M \mathbf{a}(t)
\]

where

\[
M := \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & 0 & 0
\end{pmatrix}
\]

Note that \( a_0(t) = 1 - \sum_{i=1}^{8} a_i(t) \) is determined by \( \mathbf{a}(t) \) and that the fraction of polarised cells is \( p_p(t) = 1 - a_0(t) \).

The mean-field assumption (MFA) presumes that the local director field acting at a single node can be approximated by the average field of all nodes. Hence the local director \( \mathbf{\nu}_z(\mathbf{\eta}) \), see main text eq. (2), is approximated by

\[
\mathbf{\nu}_z(\mathbf{\eta}(t)) = \frac{1}{\# S_x} \sum_{x \in S_x} \mathbf{\eta}_x(t) \approx \frac{1}{\# S} \sum_{x \in S} \mathbf{\eta}_x(t) = \mathbf{p}(t).
\]

Then main text eq. (4) becomes

\[
c_z(\mathbf{\eta}, e_i) = \text{exp} \left( \langle e_i, e_n \cdot \mathbf{\nu}_z(\mathbf{\eta}) + e_s \cdot \mathbf{s} \rangle \right) \\
\approx \text{exp} \left( \langle e_i, e_n \cdot \mathbf{p} + e_s \cdot \mathbf{s} \rangle \right)
\]

\[
= \text{exp} \left( \langle e_i, e_n \cdot M \mathbf{a} + e_s \cdot \mathbf{s} \rangle \right) =: r_i(\mathbf{a}), \quad \text{if } \mathbf{\eta}_z \neq e_0, i = 1, \ldots, 8,
\]

therewith introducing the substitutes \( r_i(\mathbf{a}) \). Analogously, main text eq. (7) is approximated as

\[
c_z(\mathbf{\eta}, e_i) \approx \frac{\beta}{\sum_{k=1}^{8} r_k(\mathbf{a})} \cdot \frac{\text{exp} \left( \langle e_i, e_n \cdot \mathbf{\nu}_z(\mathbf{\eta}) + e_s \cdot \mathbf{s} \rangle \right)}{\sum_{k=1}^{8} \text{exp} \left( \langle e_k, e_n \cdot \mathbf{p} + e_s \cdot \mathbf{s} \rangle \right)}
\]

\[
= \frac{\beta}{\sum_{k=1}^{8} r_k(\mathbf{a})} \cdot \frac{r_i(\mathbf{a})}{\sum_{k=1}^{8} r_k(\mathbf{a})}, \quad \text{if } \mathbf{\eta}_z = e_0, i = 1, \ldots, 8.
\]

For convenience, we abbreviate

\[
R(\mathbf{a}) := \sum_{k=1}^{8} r_k(\mathbf{a}),
\]

such that

\[
c_z(\mathbf{\eta}, e_i) \approx \frac{\beta r_i(\mathbf{a})}{R(\mathbf{a})}, \quad \text{if } \mathbf{\eta}_z = e_0, i = 1, \ldots, 8
\]
as stated in main text eq. (17). Note that the approximations in eqs. (5), (6) and (8) are exact for \( \epsilon_n = 0 \) and that the approximation error increases with \( \epsilon_n \).

In the limit for increasing lattice size the \( a_i \)'s become continuous quantities and their dynamic behaviour can be described by an ODE system [references 50, 51, 52 in the main text]. For convenience, main text eq. (18) is reproduced here:

\[
\frac{d\hat{a}_i}{dt} = -\hat{a}_i \cdot \left( \delta + \sum_{k=1}^{s} r_k (\hat{a}) \right) + \hat{a}_0 \cdot \beta \cdot \frac{r_i (\hat{a})}{\sum_{k=1}^{s} r_k (\hat{a})} + \sum_{k=1}^{s} \hat{a}_k \cdot r_i (\hat{a})
\]

\[
= -\hat{a}_i \cdot (\delta + R (\hat{a})) + \left[ \hat{a}_0 \cdot \frac{\beta}{R (\hat{a})} + (1 - \hat{a}_0) \right] r_i (\hat{a}), \quad i = 1, \ldots, 8
\]

\[
\frac{d\hat{a}_0}{dt} = -\hat{a}_0 \cdot (\delta + (1 - \hat{a}_0)) = \delta - (\beta + \delta) \hat{a}_0.
\]

Here \( \hat{a} \) and \( \hat{a}_i \) denote the counterparts of \( a \) and \( a_i \) under mean-field approximation (MFA). We call eq. (9) the mean-field model and note that the overall error of approximation introduced in eqs. (5) and (6) increases with \( \epsilon_n, \epsilon_s \), and the fraction of polarised cells \( p_p \). Note that other, even irregular, lattice geometries, different neighbourhood templates and spatially asymmetrically weighted neighbour polarity information yield the same MFA eq. (9) as long as neighbour polarity information is weighted independently of the considered cell's polarity state.

The fraction of unpolarised cells \( \hat{a}_0 (t) \) decouples with unique solution

\[
\hat{a}_0 (t) = \frac{\delta}{\beta + \delta} + \left( \hat{a}_0 (0) - \frac{\delta}{\beta + \delta} \right) \cdot \exp \left( - (\beta + \delta) t \right), \quad (10)
\]

that tends to the unique, globally attracting equilibrium \( \hat{a}_0^* = \frac{\delta}{\beta + \delta} \). Hence the fraction of polarised cells converges, \( \hat{p}_p = 1 - \hat{a}_0 \left( \begin{array}{c} t \rightarrow \infty \end{array} \right) \frac{\beta}{\beta + \delta} \), in perfect agreement with the dynamic equilibrium \( p_{eq} = \frac{\beta}{\beta + \delta} \) of death and de-novo polarisation in the original IPS (eq. (9)). We will exploit this steady state expression for \( a_0 \) and \( \hat{a}_0 \) in various places. Inserting \( \hat{a}_0 = \hat{a}_0^* = \frac{\delta}{\beta + \delta} \), the ODE system (9) simplifies to main text eq. (19)

\[
\frac{d\hat{a}_i}{dt} = -\hat{a}_i \cdot (\delta + R (\hat{a})) + \left[ \hat{a}_0 \cdot \frac{\beta}{R (\hat{a})} + (1 - \hat{a}_0) \right] r_i (\hat{a}), \quad i = 1, \ldots, 8, \quad (11)
\]

An approximate solution for \( \mathbf{p} (t) = \mathbf{p} (\eta (t)) \), denoted \( \hat{\mathbf{p}} (t) \) in the following, can be obtained by solving the mean-field model ODE system (11) and using the mean-field analogue of eq. (2) (main text eq. (16))

\[
\hat{\mathbf{p}} (t) = \sum_{k=1}^{8} \hat{a}_k (t) \cdot e_k. \quad (12)
\]

Alternatively, eqs. (10) and (13) describe an ODE system. Equation

\[
\frac{d\hat{\mathbf{p}}}{dt} (t) = - (\delta + R (\hat{\mathbf{p}})) \cdot \hat{\mathbf{p}} + \left[ \hat{a}_0 \cdot \frac{\beta}{R (\hat{\mathbf{p}})} + (1 - \hat{a}_0) \right] \sum_{i=1}^{s} r_i (\hat{\mathbf{p}}) \cdot e_i \quad (13)
\]

follows from summing (11) for \( i = 1, \ldots, 8 \) and replacing \( M \hat{a} \) by \( \hat{\mathbf{p}} \) in (5) to define \( r_i (\hat{\mathbf{p}}) \) and \( R (\hat{\mathbf{p}}) \).

We observe that solutions of (11) preserve the symmetry of the initial condition with respect to the \( x \)-axis, that was imposed by setting \( s = (1, 0) \), \( a_1 (0) = a_7 (0) \),
\[ a_2(0) = a_0(0) \text{ and } a_3(0) = a_5(0) \text{ (cf. main text eq. (11)). For such a symmetric initial condition, it holds that } \dot{\hat{p}}(t) = 0 \text{ for all } t \geq 0 \text{ such that } \hat{p}(t) = (\hat{p}_x(t), 0). \]

We solve eq. (11) numerically in main text section 3.2 and calculate \( \|\hat{p}\| = |\hat{p}_x| \) and angular \( \hat{p}_y = \pi \cdot \text{sgn}(\dot{\hat{p}}_y). \)

The precise initial condition \( \hat{a}(0) \) used for numerical solving of eq. (11) (main text eq. (19)) follows from main text eq. (11) as

\[
\hat{a}_k(0) = \beta \frac{\exp((\epsilon_n + \epsilon_s)(\mathbf{e}_k, \mathbf{e}_i))}{\exp(\epsilon_n + \epsilon_s) + 2\exp(\sqrt{\delta/5}(\epsilon_n + \epsilon_s)) + 2 + 2\exp(-\sqrt{\delta/5}(\epsilon_n + \epsilon_s)) + \exp(-(\epsilon_n + \epsilon_s))}, \quad k = 1, \ldots, 8.
\]

### 3 Linearisation of mean-field model

We can approximate the non-linear mean-field model (main text eq. (19), or equivalently eq. (11)) further to obtain an analytically tractable ODE. Linearisation of \( \exp(\cdot) \) in eq. (5) by Taylor expansion around 0 yields

\[
r_i(\mathbf{a}) = 1 + \epsilon_n \sum_{k=1}^{8} a_k \langle \mathbf{e}_k, \mathbf{e}_i \rangle + \epsilon_s \langle \mathbf{s}, \mathbf{e}_i \rangle + \mathcal{O}((\epsilon_n + \epsilon_s)^2), \quad i = 1, \ldots, 8,
\]

where the remainder term has been estimated using the bounds \( |\langle \mathbf{e}_k, \mathbf{e}_i \rangle| \leq 1, |a_k| \leq 1 \) for \( i, k = 1, \ldots, 8 \). Inserting this result into eq. (7) one obtains

\[
R(\mathbf{a}) = \sum_{i=1}^{8} \left( 1 + \epsilon_n \sum_{k=1}^{8} a_k \langle \mathbf{e}_k, \mathbf{e}_i \rangle + \epsilon_s \langle \mathbf{s}, \mathbf{e}_i \rangle + \mathcal{O}((\epsilon_n + \epsilon_s)^2) \right)
\]

\[
= 8 + \epsilon_n \sum_{k=1}^{8} a_k \left( \sum_{i=1}^{8} \mathbf{e}_k \cdot \mathbf{e}_i + \epsilon_s \langle \mathbf{s}, \mathbf{e}_i \rangle \right) + \mathcal{O}((\epsilon_n + \epsilon_s)^2).
\]

To indicate this second approximation by linearisation, id est dropping \( \mathcal{O}((\epsilon_n + \epsilon_s)^2) \) terms for \( (\epsilon_n + \epsilon_s)^2 \ll 1 \), we add an overline to the approximated quantities. The ODE system (11) (or main text eq. (19)) with linearised \( r_i(\cdot) \)'s reads then

\[
\frac{d\bar{a}_i}{dt} = (\delta + 8) \left( -\bar{a}_i + \frac{1}{8} \frac{\beta}{\beta + \delta} \left[ 1 + \epsilon_n \sum_{k=1}^{8} \bar{a}_k \langle \mathbf{e}_k, \mathbf{e}_i \rangle + \epsilon_s \langle \mathbf{s}, \mathbf{e}_i \rangle \right] \right), \quad i = 1, \ldots, 8,
\]

which shows main text eq. (20).

Using \( \dot{\bar{p}}_x = (\mathbf{p}, \mathbf{e}_x) \) with \( \mathbf{p} = \sum_{k=1}^{8} \bar{a}_k \cdot \mathbf{e}_i \), we obtain (see extra suppl. 4 for detailed calculations)

\[
\frac{d\bar{p}_x}{dt} = (\delta + 8) \left[ \left( -1 + \frac{\epsilon_n}{2} \frac{\beta}{\beta + \delta} \right) \bar{p}_x + \frac{\beta}{\beta + \delta} \left( \frac{\epsilon_s}{2} \bar{p}_x \right) \right].
\]

The linear ODE (15) has the form \( \frac{d\bar{p}_x(t)}{dt} = A\bar{p}_x + B_x \) with \( A = (\delta + 8) \left( -1 + \frac{\epsilon_n}{2} \frac{\beta}{\beta + \delta} \right) \) and \( B_x = (\delta + 8) \frac{\beta}{\beta + \delta} \frac{\epsilon_s}{2} \bar{p}_x \) and hence has the solution

\[
\bar{p}_x(t) = \left[ \bar{p}_x(0) + \frac{B_x}{A} \right] \exp(At) - \frac{B_x}{A}.
\]
Since the initial condition and parameter $s$ are chosen symmetric with respect to the $x$-axis such that $\bar{p}_y(0) = 0$ and $s_y = 0$, it also holds that $\bar{p}_y(t) = 0$, $t \geq t_0$ and $\bar{p}(t) = (\bar{p}_x(t), 0)$ and the symmetry is preserved over time. Then equation (15) provides a simple ODE for $\bar{p}(t)$ compared to deriving $\hat{p}(t) = \sum_{k=1}^{8} \hat{a}_k(t) \bar{e}_k$ according to (2) from a solution of (11) (main text eq. (19)). For non-symmetric initial conditions and more general $s$, $\bar{p}_y(t)$ will differ from 0 and be governed by an ODE analogous to (15) with $x$ being replaced by $y$.

When $p_x(0)$ opposes $s_x$, then $p_x(\cdot)$ must change sign for polarity reorientation. This occurs if and only if $A < 0$ or equivalently $\epsilon_n/\sqrt{\gamma} < 2$. Then the time of minimal order in the linear MF model $T_{mo}$ is given as the unique root of $p_x(\cdot)$ as

$$T_{mo} = \frac{1}{A} \ln \left( \frac{B_x}{\bar{p}_x(0) + B_x} \right) = \frac{2}{(\delta + 8)} \ln \left( 1 + \frac{2 + \epsilon_n/\sqrt{\gamma}}{\epsilon_s} \cdot \bar{p}_x(0) \right),$$

which shows main text eq. (21). The last term is always positive due to our initial condition where $\bar{p}_x(0)$ and $s_x$ have opposite signs. Additionally, the term equals 1 in case of perfect alignment among polarised cells in the initial configuration. In the case $\epsilon_n/\sqrt{\gamma} > 2$ and hence $A > 0$, $\bar{p}_x$ does not change sign. To summarise, polarity reverses in the linearised mean-field model only for $\epsilon_n/\sqrt{\gamma} < 2$.

The precise initial condition $\bar{p}_x(0)$ in eq. (17) (main text eq. (21)) follows from main text eq. (11) as

$$\bar{p}_x(0) = \frac{\epsilon_{\beta + \delta}}{\beta + \delta} \frac{-\exp(\epsilon_n + \epsilon_s) - \sqrt{2} \exp\left(\sqrt{\frac{2}{3}}(\epsilon_n + \epsilon_s)\right) + \sqrt{2} \exp\left(-\sqrt{\frac{2}{3}}(\epsilon_n + \epsilon_s)\right) + \exp(-\epsilon_n + \epsilon_s)}{\exp(\epsilon_n + \epsilon_s) + 2 \exp\left(\sqrt{\frac{2}{3}}(\epsilon_n + \epsilon_s)\right) + 2 + 2 \exp\left(-\sqrt{\frac{2}{3}}(\epsilon_n + \epsilon_s)\right) + \exp(-\epsilon_n + \epsilon_s)}.$$

4 Detailed derivation of supplement equation (15)

Recall the preceding eq. (14). By an analog of eq. (2) (or main text eq. (16)) holds $\bar{p}(t) = (\bar{p}_x(t), \bar{p}_y(t))^T = \sum_{i=1}^{8} \bar{a}_i(t) \cdot \bar{e}_k$. Hence with $\bar{e}_8 = (1, 0)$

$$\bar{p}_x(t) = \langle \bar{p}(t), \bar{e}_8 \rangle$$

$$= \sum_{i=1}^{8} \bar{a}_i(t) \cdot \langle \bar{e}_i, \bar{e}_8 \rangle.$$
Plugging in eq. (14) yields
\[ \frac{d\bar{p}_x}{dt}(t) = \sum_{i=1}^{8} \frac{d\bar{a}_i}{dt}(t) \cdot \langle e_i, e_8 \rangle \]
\[ = (\delta + 8) \sum_{i=1}^{8} \left[ -\bar{a}_i(t) + \frac{1}{8} \frac{\beta}{\beta + \delta} \left[ 1 + \epsilon_n \sum_{k=1}^{8} \bar{a}_k(t) \langle e_k, e_i \rangle + \epsilon_s \langle s, e_i \rangle \right] \right] \cdot \langle e_i, e_8 \rangle \]
\[ = - (\delta + 8) \sum_{i=1}^{8} \bar{a}_i(t) \cdot \langle e_i, e_8 \rangle + (\delta + 8) \frac{1}{8} \frac{\beta}{\beta + \delta} \sum_{i=1}^{8} \langle e_i, e_8 \rangle \]
\[ + (\delta + 8) \frac{1}{8} \frac{\beta}{\beta + \delta} \epsilon_n \sum_{i=1}^{8} \bar{a}_k(t) \langle e_k, e_i \rangle \langle e_i, e_8 \rangle \]
\[ + (\delta + 8) \frac{1}{8} \frac{\beta}{\beta + \delta} \epsilon_s \sum_{i=1}^{8} \langle s, e_i \rangle \langle e_i, e_8 \rangle \]  
(18)

Note that \( \sum_{i=1}^{8} \langle e_i, e_8 \rangle = 0 \). Further,
\[ \sum_{i=1}^{8} \sum_{k=1}^{8} \bar{a}_k(t) \langle e_k, e_i \rangle \langle e_i, e_8 \rangle = \sum_{i=1}^{8} \left( \sum_{k=1}^{8} \bar{a}_k(t) \langle e_k, e_i \rangle \right) \langle e_i, e_8 \rangle = \sum_{i=1}^{8} \langle \mathbf{p}(t), e_i \rangle \langle e_i, e_8 \rangle. \]

Because of the choice of unit vectors \( \mathbf{e}_i = (\cos (i\pi/4), \sin (i\pi/4)), i = 1, 2, \ldots, 8, \) cf. main text eq. (1), holds for arbitrary vector \( \mathbf{v} = (v_x, v_y) \in \mathbb{R}^2 \)
\[ \sum_{i=1}^{8} \langle \mathbf{v}, e_i \rangle \langle e_i, e_8 \rangle = \sum_{i=1}^{8} \langle e_i, v_x e_8 + v_y e_2 \rangle \langle e_i, e_8 \rangle = \sum_{i=1}^{8} \langle e_i, e_8 \rangle^2 v_x + \sum_{i=1}^{8} \langle e_i, e_8 \rangle \langle e_i, e_2 \rangle v_y = 4v_x. \]

We employ this identity specifically for \( \mathbf{v} = (v_x, v_y) \in \{ \mathbf{s}, \mathbf{p}(t) \mid t \in [0, \infty) \} \). Together with the aforementioned relations this simplifies eq. (18) to
\[ \frac{d\bar{p}_x}{dt}(t) = - (\delta + 8) \langle \mathbf{p}(t), e_8 \rangle + (\delta + 8) \frac{1}{8} \frac{\beta}{\beta + \delta} \epsilon_n \cdot 4\bar{p}_x(t) + (\delta + 8) \frac{1}{8} \frac{\beta}{\beta + \delta} \epsilon_s \cdot 4s_x \]
\[ = (\delta + 8) \left[ -\bar{p}_x(t) + \frac{4}{8} \frac{\beta}{\beta + \delta} \epsilon_n \bar{p}_x(t) + \frac{4}{8} \frac{\beta}{\beta + \delta} \epsilon_s s_x \right] \]

and yields the desired equation (15)
\[ \frac{d\bar{p}_x}{dt} = (\delta + 8) \left[ \left( 1 + \frac{\epsilon_n}{2} \frac{\beta}{\beta + \delta} \right) \bar{p}_x + \frac{\beta}{\beta + \delta} \frac{\epsilon_s}{2} s_x \right] \]
as claimed.

5 Derivation of main text equation (22) for vanishing neighbour coupling strength \( \epsilon_n = 0 \)

For vanishing neighbour coupling strength \( \epsilon_n = 0 \), cells evolve independently as is evident from the definition of the IPS rates in eqs. (4), (6) and (7). In particular, rates
do only depend on the global signal $s$ and the state of the cell itself (polarised or not), but not on neighbour cells. Hence, mean-field approximation in eqs. (5), (6) and (8) (or main text eq. (17)) is exact. Moreover, the auxiliary rates $r_i (a), R (a)$ defined in eqs. (5) and (7), respectively, become independent of the actual fraction of cells in each of the nine states as

$$ r_i = \exp (\epsilon_s (e_i, s)), \quad i = 1, \ldots, 8, \quad (19) $$

$$ R = \sum_{i=1}^{8} \exp (\epsilon_s (e_i, s)). \quad (20) $$

Hence eq. (13) simplifies to

$$ \frac{d \hat{p}}{dt} (t) = - (\delta + R) \cdot \hat{p} + \left[ \hat{a}_0 \cdot \frac{\beta}{R} + (1 - \hat{a}_0) \right] \sum_{i=1}^{8} r_i \cdot e_i \quad (21) $$

and with steady state of cell death and de-novo polarisation ($\hat{a}_0 = \hat{a}_0^* = \frac{\delta}{\beta + \delta}$, confer eq. (10)) further to

$$ \frac{d \hat{p}}{dt} (t) = - (\delta + R) \cdot \hat{p} + \frac{\beta}{\beta + \delta} \frac{\delta + R}{R} \sum_{i=1}^{8} r_i \cdot e_i \quad (22) $$

which is a linear ODE for $\hat{p}$. Note that $(\delta + R)$ and $\frac{\beta}{\beta + \delta} \frac{\delta + R}{R}$ are scalar factors in this 2-dimensional ODE. For simpler notation write the right-most term as $(\delta + R) \cdot q$ where $q := \frac{\beta}{\beta + \delta} \frac{1}{R} \sum_{i=1}^{8} r_i \cdot e_i$. Then the general solution of (22) reads

$$ \hat{p} (t) = \exp \left\{ - (\delta + R) \left( t - t_0 \right) \right\} \hat{p} (t_0) - q \cdot q. \quad (23) $$

Let $t_0 = 0$. To find the minima of the order parameter $\|\hat{p} (t)\|$ one can equally consider its square, that can be expressed in terms of scalar products as

$$ \|\hat{p} (t)\|^2 = (\hat{p} (t), \hat{p} (t)) $n$,

$$ = \exp \left\{ - 2 \cdot (\delta + R) \cdot t \right\} \|\hat{p} (0) - q \cdot q\|^2 + 2 \exp \left\{ - (\delta + R) \cdot t \right\} (\hat{p} (0) - q, q) + \|q\|^2. \quad (24) $$

The first and last summands are non-negative. Hence, if $\langle \hat{p} (0) - q, q \rangle \geq 0$ the modulus $\|\hat{p} (t)\|$ decreases monotonically with time $t$ towards the limit value $\|q\|$. In this case no distinct time of minimal order exists. In contrast, if $\langle \hat{p} (0) - q, q \rangle < 0$ then there is a unique time of minimal order $\hat{T}_{mo} (\epsilon_n = 0)$ that we find from

$$ 0 = \frac{d}{dt} \left( \|\hat{p} (t)\|^2 \right) \left( \hat{T}_{mo} \right) $$

$$ = -2 \cdot (\delta + R) \exp \left\{ -2 \cdot (\delta + R) \cdot \hat{T}_{mo} \right\} \|\hat{p} (0) - q \cdot q\|^2 $$

$$ - 2 \cdot (\delta + R) \exp \left\{ - (\delta + R) \cdot \hat{T}_{mo} \right\} (\hat{p} (0) - q, q) $$

as

$$ \hat{T}_{mo} (\epsilon_n = 0) = \frac{1}{\delta + R} \cdot \log \left( - \frac{\|\hat{p} (0) - q \cdot q\|^2}{\langle \hat{p} (0) - q, q \rangle} \right). \quad (24) $$

We now determine $\hat{p} (0)$ for the initial condition described by main text eq. (11). Note that $e_k = -e_{(k+4) \mod 8}$ from the defining eq. (1) and rewrite the initial condition
(main text eq. (11)) as
\[
\hat{p}(0) = \frac{\beta}{(\beta + \delta)} \frac{1}{Z} \sum_{k=1}^{8} \exp \left( \epsilon_s \langle e_k, e_4 \rangle \right) e_k
\]
\[
= \frac{\beta}{(\beta + \delta)} \frac{1}{Z} \sum_{k=1}^{8} \exp \left( \epsilon_s \langle e_{(k+4) \mod 8}, e_8 \rangle \right) e_{(k+4) \mod 8}
\]
\[
= \frac{\beta}{(\beta + \delta)} \frac{1}{Z} \sum_{i=1}^{8} \exp \left( \epsilon_s \langle e_i, e_8 \rangle \right) \cdot (-e_i)
\]
\[
= -\frac{\beta}{(\beta + \delta)} \frac{1}{Z} \sum_{i=1}^{8} r_i e_i.
\]

The normalisation denominator is
\[
Z = \sum_{i=1}^{8} r_i = R,
\]
such that
\[
\hat{p}(0) = -\frac{\beta}{(\beta + \delta)} \frac{\sum_{i=1}^{8} r_i e_i}{R} = -q.
\]

Hence \( \langle \hat{p}(0) - q, q \rangle = (-q - q, q) = -2 \| q \|^2 < 0 \), so there is a uniquely determined time of minimal order \( \hat{T}_{mo} \). Plugging eq. (25) into eq. (24) yields
\[
\hat{T}_{mo}(\epsilon_n) = \frac{1}{\delta + R} \log \left( -\frac{\| q - q \|^2}{\langle q - q, q \rangle} \right) = \frac{\log 2}{\delta + R} = \frac{\log 2}{\delta + \sum_{k=1}^{8} \exp \left( \epsilon_s \langle s, e_k \rangle \right)}.
\]

which proves main text eq. (22). Note that by the specific choice of our initial condition, which in particular uses the equilibrium fraction of unpolarised cells, there is no dependence on the parameter \( \beta \). However, when comparing the effects of alignment dynamics to the effects of cell turnover it is natural to vary de-novo polarisation rate \( \beta \) together with the death rate \( \delta \) to keep their ratio constant. The time of minimal order \( \hat{T}_{mo}(\epsilon_n) \) decreases with increasing cell turnover, here apparent from \( \delta \), and with increasing sensitivity \( \epsilon_s \) to the global signal \( s \). For \( \delta \) in the order of magnitude 1, the latter has the bigger impact on \( \hat{T}_{mo}(\epsilon_n) \) because \( R \geq 8 \) and \( R \) grows exponentially with \( \epsilon_s \). In the limit of \( \delta \to 0 \), the largest time of minimal order is observed, yet it is still finite.

To further see the equality with the time of minimal order for the time series \( \langle p(t) \rangle \) claimed at the main text eq. (22), remember that the approximations in eqs. (5), (6) and (8) are equalities for \( \epsilon_n = 0 \). Hence the mean-field ODE (13) in \( \hat{p} \) is valid as well for \( \langle p(t) \rangle \), the order parameter in the IPS at time \( t \) averaged across a sufficient number of realisations of the stochastic system. Then the derivations shown above lead to an equation like (22) for the time of minimal order for the time series \( \langle p(t) \rangle \), finishing the proof of main text eq. (22).

Note that the time of minimal order for the mean order parameter \( \langle p(t) \rangle \) might differ from the mean time of minimal order \( T_{mo} \) in the IPS because the (in general non-linear) operator argmin_{\epsilon \in [0,1]} and averaging by \( \langle \cdot \rangle \) are interchanged. However, we observe close agreement between empirical \( T_{mo} \) from 25 simulated trajectories of the IPS, the time of minimal order \( \hat{T}_{mo} \) from numerical solution of the mean-field model (19) and the analytical expression of main text eq. (22) derived here, see suppl. fig. 5.

\[\text{We neglect the case } q = 0, \text{ which is only possible for } \epsilon_s = 0, \text{ in addition to the assumption of } \epsilon_n = 0.\]
6 Details on the numerical solution of the mean-field model

This section extends maintext section 3.2 by giving a more detailed description of the numerical solution of the mean-field model, see in particular maintext eq. (19) and maintext fig. 3.

The time courses of \( \| \mathbf{p} \| \) and \( \text{ang} (\mathbf{p}) \) for the turning cases exhibit several common characteristics independent of the specific parameter sets and with those for the original model, described as follows. Throughout, \( \hat{a}_0 = \hat{a}_0^* = \frac{\delta}{\pi + \delta} \), confirming the analytical prediction by eq. (10). According to the initialisation specified by eq. (11), the majority of polarised cells starts with polarisation direction \( \mathbf{e}_4 = (-1, 0) \), i.e. \( \hat{a}_4 \approx p_{eq} = \frac{\beta}{\pi + \beta} \), cf. fig. 4A-C. The fraction \( \hat{a}_4 \) declines in favour of the other polarisation directions, in the beginning especially in favour of \( \hat{a}_3 \) and \( \hat{a}_5 \). Then the fractions \( \hat{a}_2, \hat{a}_6 \) and to less extent \( \hat{a}_1, \hat{a}_7 \) and \( \hat{a}_8 \) increase as well while \( \hat{a}_4 \) declines further. After \( \hat{a}_3, \hat{a}_5 \) start to decrease again, all fractions are approximately equally abundant at the time of minimal order \( T_{mo} \). Decline in \( \hat{a}_2 \) to \( \hat{a}_6 \) in favour of further increase in \( \hat{a}_1, \hat{a}_7 \) plus strong increase in \( \hat{a}_8 \) leads into a plateau. Because of symmetry, \( \hat{p}_x (t) = (0, 0) \), and especially \( \hat{p}_x (T_{mo}) = (0, 0) \). The rates \( r_k \left( \hat{a} \left( T_{mo} \right) \right) \) are biased towards \( r_3 \) because of the global signal, changing \( \hat{p}_x \) from negative to positive sign and driving the polarity pattern towards a stable asymptotic state of dominant \( \hat{a}_8 \) accompanied by major fractions \( \hat{a}_1, \hat{a}_7 \). In this asymptotic state, \( \hat{p} \) is parallel to \( \mathbf{s} = \mathbf{e}_8 \) and \( \| \mathbf{p} \| \) is almost as large as \( p_{eq} \). Because of coherence between global signal \( \mathbf{s} = (1, 0) \) and strong local signal \( \hat{p} \) in exactly the same direction, the solution reaches a stable equilibrium there. The transition from alignment among cells conflicting with the global signal to alignment with the global signal happens via a disordered state when each polarisation direction is approximately equally abundant around \( T_{mo} \), see fig. 4A-C.

The dynamics described are well recapitulated in the time course of the order parameter \( \hat{p} \), see fig. 4C. Modulus \( \| \mathbf{p} \| \) starts at a high value \( \approx \frac{\beta}{\pi + \beta} \), decreases to a distinct minimum \( \| \mathbf{p} \left( T_{mo} \right) \| = 0 \) indicating complete disorder and then increases again to a plateau. The angle \( \text{ang} (\mathbf{p}) \) first remains equal to \( -\pi \), and switches to \( 0 \) at \( T_{mo} \).

However, there is no time of minimal order at which \( \| \mathbf{p} \left( T_{mo} \right) \| = 0 \) in the cases \( \delta = 0, \epsilon_n \in \{4.5, 5\} \). Instead of approaching dominant \( \hat{a}_8, \hat{a}_1, \hat{a}_7 \), the solution of the ODE system (19) remains trapped in a stable asymptotic state with high \( \hat{a}_4, \hat{a}_3, \hat{a}_5 \) and \( \hat{p}_x < 0 \), see suppl. fig. 4A,B. Still a stable asymptotic state with dominant \( \hat{a}_8, \hat{a}_1, \hat{a}_7 \) and \( \hat{p}_x > 0 \) does exist, see suppl. fig. 4C,D, but the initial condition is not within its domain of attraction. This trapping represents a phase transition to non-turning behaviour with diverging \( T_{mo} \) as \( \epsilon_n \) is increased and/or \( \delta \) is decreased towards the critical parameter values. However, this phase transition is only present in the mean-field approximation as the errors introduced by mean-field assumption grow with neighbour coupling strength \( \epsilon_n \), see main text sec. 2.4.2. With increasing neighbour coupling, each single cell in the dynamically diluted alignment model is less probable to deviate from the initially dominant polarisation direction. However, such rare events of spontaneous polarity change still can occur in the original IPS and can initiate progressive polarity reorientation, whereas in the mean-field description the influence of deviating cells is neglected by averaging. As increasing cell death rate \( \delta \) reduces the expected fraction of polarised cells \( p_{eq} = \frac{\beta}{\pi + \beta} \), cf. eq. (9). This latter approximation introduces the phase transition into the mean-field model. Note that a stable equilibrium of (19) with dominant \( \hat{a}_8, \hat{a}_1, \hat{a}_7 \) does still exist, but it is not reached from the initial state when another equilibrium with \( \hat{p}_x < 0 \) arises for high neighbour coupling \( \epsilon_n \), cf. suppl.

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fig. 4A, B versus C, D. Close to that transition and beyond, the mean-field description is no longer a valid approximation of the IPS model. Therefore we focus the discussion in the main text on the parameter range of lower $\epsilon_n$ and/or larger $\delta$. 
Supplementary Figure 1: Supplement to maintext fig. 3. Data shown here is analogous to that in maintext fig. 3, except for lattice size of $20 \times 20$ used here. Qualitative and quantitative similarity indicates that lattice size of $100 \times 100$ employed in the maintext is sufficient to avoid finite size effects. A. Snapshots of a typical simulation at times $0.05$, $0.2$, $0.5$, $0.8$ ($a$-$d$), see fig. 2C for colour code. Zoomed details of a $20 \times 20$ lattice with parameters as in maintext fig. 3A, id est periodic boundaries, $\epsilon_n = 4.5$, $\delta = 0.2$, $\beta = 1$, $\epsilon_s = 1$, $s = (1,0)$. B. Mean polarisation vector $p$ depicted as modulus $\|p\|$ (solid, left axis) and angle $\text{ang}(p)$ (dashed, right axis, colour code as in panel A and fig 2C). The distinctive minimum of the $\|p\|$ time course defines the time of minimal order $T_{mo}$. The fraction of polarised cells $p_p$ (dash-dotted, left axis) fluctuates around $p_{eq} = 0.83$. Fluctuations in $\|p\|$, $\text{ang}(p)$ and $p_p$ are stronger than in maintext fig. 3B, and the $T_{mo,i}$ from time courses $i \in \mathbb{N}$ for equal parameters are more scattered. Nevertheless, characteristics of the time courses are preserved, where $100 \times 100$ lattice yields even smoother trajectories. C,D. Simulation results for mean $T_{mo}$ of 25 repetitions shown as heatmap with contourlines (isotemporales at marked levels, C) and all data points (D), fixed parameters as in A,B and maintext fig. 3A,B. Asterisks denote parameter values of panels A,B and maintext fig. 3A,B. For the mean $T_{mo}$, differences between $20 \times 20$ and $100 \times 100$ lattice (C versus maintext fig. 3C) are marginal. Note inverted colour bar for $\delta$ in D.
Supplementary Figure 2: Supplement to maintext fig. 3. Effects of de-novo polarisation rate $\beta$ on the time of minimal order $T_{mo}$. The time of minimal order $T_{mo}$ in IPS simulations is statistically robust across the parameter space, and measured data collapse onto a linear dependence upon rescaling to effective neighbour coupling strength. A-C. Time of minimal order for $\beta = 0.1$(A), $\beta = 1$(B), and $\beta = 10$(C) with lattice size $100 \times 100$, $\epsilon_n = 1$, $s = (1,0)$. Heatmap and black isopotentes at marked levels are for mean, surrounding white isopotentes for mean ± SEM from 25 repetitions for each data point. Panel B is a reproduction of maintext fig. 3C with white isopotentes added. Colour code is common to A-F and the same as in maintext fig. 3C. D-F. Comparison of isopotentes for IPS model (solid), mean-field model (dashed) and linearised mean-field model (dash-dot). Shown levels are 0.07, 0.08, 0.09, 0.10, 0.11, 0.12, 0.14, 0.16, 0.18 (D, $\beta = 0.1$) and 0.04, 0.05, 0.06, 0.07, 0.08, 0.10, 0.12, 0.15, 0.20 (E, $\beta = 1$) and 0.01, 0.012, 0.015, 0.02, 0.03, 0.04, 0.07, 0.10, 0.20 (F, $\beta = 10$), respectively. For $\epsilon_n > 4.5$ the mean-field model undergoes a phase transition to non-turning dynamics (grey shaded region in bottom right of D-F) for all values $\beta = 0.1, 1, 10$ studied, see also maintext fig. 4E. G-I. All data of IPS simulation for the respective value of $\beta$ (A-C) collapse to approximately linear dependence (black curves). Replacement rate $\delta$ is colour-coded to have fixed ratio with $\beta = 0.1$ (G), $\beta = 1$ (H) and $\beta = 10$ (I), respectively (horizontal colour-bars). Panel H is equivalent to maintext fig. 6B. Additionally accounting for the time offset $T_{mo}$ ($\epsilon_n = 0$) from maintext eq. (22) reduces the scatter further, shown in fig. 6 for $\beta = 1$. J Colour-code for time of minimal order $T_{mo}$ and $\hat{T}_{mo}$ in A-F.
Supplementary Figure 3: Supplement to maintext fig. 3. **A.** Fractions $a_0, a_1, \ldots, a_8$ of nodes in state $e_0, e_1, \ldots, e_8$, respectively, of the example simulation shown in fig. 3A,B and suppl. movie 2. Parameters are $\epsilon_n = 4.5, \delta = 0.2, \beta = 1, \epsilon_s = 1, s = (1, 0)$. Observe high agreement with $\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_8$ in mean-field approximation, see fig. 4B using the same parameters. **B.** The time of minimal order $T_{mo}$ in IPS simulations is statistically robust for a fixed parameter set. Upper part: Example simulation with turn of mean polarisation vector clockwise, compared to counter-clockwise turn shown in maintext fig. 3B. Modulus solid, angle dashed. Note the starting angle of $\pi$ is equivalent to $-\pi$ shown in maintext fig. 3B as angular argument. Parameters as in fig. 3A,B and in suppl. movie 2 ($\epsilon_n = 4.5, \delta = 0.2, \beta = 1$). Lower part: statistics of $T_{mo}$ from 25 sampled trajectories for the same parameter set. Red and blue dots indicate the time of minimal order of the trajectories shown in the upper part and in maintext fig. 3A,B, respectively.
Supplementary Figure 4: The mean-field model ODE system (19) is bistable for neighbour coupling strength $\epsilon_n > 4.5$ and cell death rate $\delta < \delta_{crit}(\epsilon_n)$, exemplified here for $\epsilon_n = 4.5$, $\delta = 0$, $\epsilon_s = 1$, $\beta = 1$. A,B. The polarity does not reverse when starting from the initial condition used throughout the paper, given by main text eq. (11). Fraction $\hat{a}_4$ decreases initially, but none of the fractions $\hat{a}_8$, $\hat{a}_1$, $\hat{a}_7$ increases significantly. Instead, the system approaches a stable steady state with high $\hat{a}_4$, $\hat{a}_3$, $\hat{a}_5$ and $p_x < 0$, id est the mean polarisation $\hat{p}$ remains pointing left counter-directional to the global signal $s$. C,D. Using a different initial condition, namely $\hat{a}_0 (0) = \frac{\delta}{\beta}$, $\hat{a}_1 (0) = \ldots = \hat{a}_4 (0) = \frac{\beta}{3 \alpha}$, the system approaches a different stable steady state with $\hat{p}$ aligned to the global signal $s$. 
Supplementary Figure 5: Comparison of time of minimal order obtained by different approaches for vanishing neighbour coupling strength $\epsilon_n = 0$. The time of minimal order $\hat{T}_{mo}$ from numerical solutions of the mean-field ODE (19) (red) coincides with the analytical expression given in main text eq. (22) (green). The time of minimal order $T_{mo}$ of the IPS scatter closely follows them. Average of 25 simulated trajectories (blue line) and all data points (blue transparent dots, 25 for each value of $\delta$). The transparent circles partly lay on top of each other because the state of the simulation was logged at a frequency of 0.001.
Supplementary Figure 6: Supplement to maintext fig. 6, where measured data collapse onto a linear dependence upon rescaling to effective neighbour coupling strength, cf. maintext eq. (25). The small remaining scatter of the data points around the black line in maintext fig. 6B is largely given by a $\delta$-dependent offset as the ordered colours of the data points indicate. This $\delta$-dependent offset is analytically known for $\epsilon_n = 0$ by maintext eq. (22). Subtracting the offset at $\epsilon_n = 0$, where $T_{mo}(\epsilon_n = 0) = T_{mo}(\epsilon_n = 0) = \log 2^{\frac{\delta + \sum_{k=1}^8 \exp(\epsilon_{s}(s,e_k))}{2}}$ is taken from the analytical expression in maintext eq. (22), reduces the scatter further. **A.** All data of the mean-field model, $\beta = 1$. Colour code of cell death rate $\delta$ applies to both panels. **B.** All data of the IPS simulations, $\beta = 1$. The black line obeys

$$T_{mo} = T_{mo}(\epsilon_n^{eff}) = (0.0516 \pm 0.0001) \frac{\beta}{\beta + \delta} \epsilon_n + T_{mo}(\epsilon_n = 0)$$

where values in brackets denote (mean ± std) of an orthogonal distance regression.
Supplementary Figure 7: The time of minimal order $T_{mo}$ depends smoothly on the coupling strength $\epsilon_s$ to the global signal, and IPS and mean-field model agree closely. **A.** IPS simulation results for mean $T_{mo}$ of 25 repetitions shown as heatmap with contourlines (isotemporales at marked levels). **B.** Comparison of time of minimal order as contourlines for IPS model ($T_{mo}$, solid) and mean-field model ($\hat{T}_{mo}$, dashed). Colour code of the time of minimal order applies to both panels. For all values $\epsilon_s \in \{0.1, 0.2, 0.5, 1.0, 2.0, 3.0, 4.0, 5.0\}$ the tissue polarity pattern reorganises, both in the IPS simulations and in the mean-field model. The time of minimal order decreases for higher coupling strength $\epsilon_s$ to the global signal, as predicted. For $\epsilon_s = 0.0$, there is no influence of a global signal and no dominant polarisation direction develops (grey shaded areas in A,B). Note that by the choice of our initial condition, cf. main text eq. (11), no dominant polarisation direction exists for $\epsilon_s = 0.0$ in the initial state either. Other parameters $\epsilon_n = 1.0$, $\beta = 1.0$. **C.** Same data as in A, shown as a function of the ratio $\frac{\epsilon_n}{\epsilon_s}$ of the neighbour coupling strength (here $\epsilon_n = 1$) and coupling strength to the global signal, $\epsilon_s$. The observed qualitative and nearly quantitative agreement with main text fig. 3C, where $\epsilon_n$ is varied for fixed $\epsilon_s = 1$, is expected since the two coupling strengths act inversely in the superposition of global and local alignment cues. Because of the finite range of $\epsilon_s$ studied, there are no data available for $\frac{\epsilon_n}{\epsilon_s} < 0.2$ (grey shaded area). Colour bar applies to all three panels.