A NOTE ON $\mathcal{F}_n$-MULTIPLE ZETA VALUES

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Abstract. For several evaluations of special values and several relations known only in $A_n$-multiple zeta values or $S_n$-multiple zeta values, we prove that they are uniformly valid in $\mathcal{F}_n$-multiple zeta values for both the case where $\mathcal{F} = A$ and $\mathcal{F} = S$. In particular, the Bowman–Bradley type theorem and sum formulas for $S_2$-multiple zeta values are proved.

1. Introduction

We call a tuple of positive integers $k = (k_1, \ldots, k_r)$ an index. We call $\text{wt}(k) := k_1 + \cdots + k_r$ (resp. $\text{dep}(k) := r$) the weight (resp. depth) of $k$. If the condition $k_r \geq 2$ is satisfied, then we state that the index $k = (k_1, \ldots, k_r)$ is admissible. For an admissible index $k = (k_1, \ldots, k_r)$, the multiple zeta value (MZV) $\zeta(k)$ and the multiple zeta-star value (MZSV) $\zeta^\star(k)$ are defined by

$$\zeta(k) := \sum_{0 < n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}, \quad \zeta^\star(k) := \sum_{1 \leq n_1 \leq \cdots \leq n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$ 

These series are convergent. We set $\zeta(\emptyset) = \zeta^\star(\emptyset) = 1$ for the empty index $\emptyset (= \text{the empty tuple}).$

First we recall the definition of $A_n$-multiple zeta(-star) values ($A_n$-MZ(S)Vs) introduced by Rosen; see [Ro, Se]. For a positive integer $n$, set

$$A_n := \prod_p \frac{\mathbb{Z}/p^n\mathbb{Z}}{\bigoplus_p \mathbb{Z}/p^n\mathbb{Z}},$$

where $p$ runs over all prime numbers. For an index $k = (k_1, \ldots, k_r)$, the $A_n$-MZV $\zeta_{A_n}(k)$ and the $A_n$-MZSV $\zeta^\star_{A_n}(k)$ are defined by

$$\zeta_{A_n}(k) := \left( \sum_{0 < n_1 < \cdots < n_r < p} \frac{1}{n_1^{k_1} \cdots n_r^{k_r} \mod p^n} \right)_p,$$

$$\zeta^\star_{A_n}(k) := \left( \sum_{1 \leq n_1 \leq \cdots \leq n_r \leq p-1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r} \mod p^n} \right)_p.$$ 

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as elements of $\mathcal{A}_n$. We also set $\zeta_{\mathcal{A}_n}(\emptyset) = \zeta_{\mathcal{A}_n}(\emptyset) = 1$.

Next we recall the definition of $t$-adic symmetric multiple zeta values ($\hat{S}$-MZVs) introduced by Jarossary [J2]. Let $t$ be an indeterminate. For $\bullet \in \{\ast, \mathfrak{m}\}$ and an index $k = (k_1, \ldots, k_r)$, set

$$\zeta^\bullet_S(k) = \sum_{i=0}^r (-1)^{k_i+1+\cdots+k_r} \zeta^\bullet(k_1, \ldots, k_i) \sum_{l_{i+1}, \ldots, l_r \geq 0} \prod_{j=i+1}^r \left( \binom{k_j+l_j-1}{l_j} \right) \zeta^\bullet(k_r+l_r, \ldots, k_{i+1}+l_{i+1}) t^{l_{i+1}+\cdots+l_r} \in \mathcal{Z}[t].$$

Here, $\mathcal{Z}$ is the $\mathbb{Q}$-subalgebra of $\mathbb{R}$ generated by all MZVs and $\zeta^\ast(k) \in \mathcal{Z}$ (resp. $\zeta^\mathfrak{m}(k) \in \mathcal{Z}$) is the harmonic (resp. shuffle) regularized MZV. See Subsection 2.1 for details. It is known that $\zeta_S^\ast(k) - \zeta_S^\mathfrak{m}(k) \in (\zeta(2)[t]) \hat{\mathcal{Z}}$ for any index $k$ ([J2, Proposition 3.2.4] and [OSY, Proposition 2.1]). Thus, $\zeta_S^\hat{g}(k) := \zeta_S^\ast(k) \mod \zeta(2)$ is independent of the choice of the regularization $\bullet \in \{\ast, \mathfrak{m}\}$ and defines a well-defined element of $\mathcal{Z}[t] := (\mathcal{Z}/\zeta(2)[t]) \hat{\mathcal{Z}}$. We call $\zeta_S^\hat{g}(k)$ the $\hat{S}$-MZV. We also define the $t$-adic symmetric multiple zeta-star value ($\hat{S}$-MZSV) $\hat{\zeta}_S^\ast(k)$ by

$$\hat{\zeta}_S^\ast(k_1, \ldots, k_r) = \sum_{\square \text{ is either a comma }, \text{ or a plus \,'}+\text{'}} \hat{\zeta}_S(k_1 \square \cdots \square k_r).$$

See [HMO] Definition 1.1 for another equivalent definition of the $\hat{S}$-MZSV. For a positive integer $n$, let $\pi_n : \mathcal{Z}[t] \rightarrow \mathcal{Z}[t]/(t^n)$ be the natural projection.

**Definition 1.1.** For an index $k = (k_1, \ldots, k_r)$, we define the $S_n$-multiple zeta(-star) value ($S_n$-MZ(S)V) by

$$\zeta_{S_n}(k) := \pi_n(\zeta_S^\hat{g}(k)), \quad \hat{\zeta}_{S_n}^\ast(k) := \pi_n(\hat{\zeta}_S^\ast(k)) = \sum_{\square \text{ is either a comma }, \text{ or a plus \,'}+\text{'}} \zeta_{S_n}(k_1 \square \cdots \square k_r).$$

Note that $\hat{\zeta}_{S_1}(k)$ coincides with the usual symmetric multiple zeta value (SMZV) $\zeta_S(k)$ defined by Kaneko and Zagier [KZ].

$\mathcal{A}_n$-MZ(S)Vs and $S_n$-MZ(S)Vs are the main objects of this article and together they are called $\mathcal{F}_n$-MZ(S)Vs; $\mathcal{F}$ derives from the first letter of the word “finite”. Similar to the conjecture [OSY, Conjecture 4.3], it is conjectured that $\mathcal{A}_n$-MZVs and $S_n$-MZVs satisfy relations of the same form. Hence, a relation among $\mathcal{A}_n$-MZVs or $S_n$-MZVs is always described collectively as a relation of $\mathcal{F}_n$-MZVs, at least conjecturally. The purpose of
this paper is to confirm that several evaluations of special values and several relations
known only in \( A_n \)-MZVs or \( S_n \)-MZVs are uniformly valid in \( F_n \)-MZVs. In some cases, we
only deal with \( n = 1, 2, 3 \).

The remainder of the paper is structured as follows. In Section 2, we prepare relevant
tools including Zagier’s formula for MZVs, the double shuffle relation for \( F_n \)-MZVs and
the relation for \( F_n \)-MZVs derived from the antipode. In Section 3 we put forward some
explicit evaluations of \( F_n \)-MZ(S)Vs. In Section 4 we prove the Bowman–Bradley type
theorem for \( F_2 \)-MZ(S)Vs. In Section 5 we prove sum formulas for \( F_n \)-MZ(S)Vs with re-
spect to specific \( n \). Some complicated but elementary calculations for binomial coefficients
(= the proof of Proposition A.1) are proved in the Appendix.

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2. Preliminaries

In this section, we prepare tools which are used in the following sections.

2.1. Algebraic setup. First we recall the notion of the harmonic algebra introduced in
[HI]. Let \( H^1 := \mathbb{Q} + e_1 \mathbb{Q}\langle e_0, e_1 \rangle \supset H^0 := \mathbb{Q} + e_1 \mathbb{Q}\langle e_0, e_1 \rangle e_0 \), where \( \mathbb{Q}\langle e_0, e_1 \rangle \) is a non-
commutative polynomial algebra in two variables \( e_0 \) and \( e_1 \). For a positive integer \( k \),
we set \( e_k := e_1 e_0^{k-1} \). We define the harmonic product \( \ast \) on \( H^1 \) by
\( w \ast 1 = 1 \ast w = w, \)
\( e_k w_1 \ast e_k w_2 = e_k (w_1 \ast e_k w_2) + e_k (e_k w_1 \ast w_2) + e_k (e_k w_1 \ast e_k w_2) \)
\( (w, w_1, w_2 \text{ are words in } H^1, k \in \mathbb{Z}_{>0}) \) with \( \mathbb{Q} \)-bilinearity. We also define the shuffle product \( \mathfrak{m} \) on \( \mathbb{Q}\langle e_0, e_1 \rangle \)
by \( w \mathfrak{m} 1 = 1 \mathfrak{m} w = w, u_1 w_1 \mathfrak{m} u_2 w_2 = u_1 (w_1 \mathfrak{m} u_2 w_2) + u_2 (u_1 w_1 \mathfrak{m} w_2) \)
\( (w, w_1, w_2 \text{ are words in } \mathbb{Q}\langle e_0, e_1 \rangle, u_1, u_2 \in \{e_0, e_1\}) \) with \( \mathbb{Q} \)-bilinearity. Let \( \bullet \in \{\ast, \mathfrak{m}\} \). It is known
that \( H^1 \) becomes a commutative \( \mathbb{Q} \)-algebra with respect to the multiplication \( \bullet \), which is
denoted by \( H^1 \). The subspace \( H^0 \) of \( H^1 \) is closed under \( \bullet \) and becomes a \( \mathbb{Q} \)-subalgebra of
\( H^1 \), which is denoted by \( H^0 \). We define Muneta’s shuffle product \( \mathfrak{m} \) on \( H^1 \) ([Mun §3]) by
\( w \mathfrak{m} 1 = 1 \mathfrak{m} w = w, e_k w_1 \mathfrak{m} e_k w_2 = e_k (w_1 \mathfrak{m} e_k w_2) + e_k (e_k w_1 \mathfrak{m} w_2) \)
\( (w, w_1, w_2 \text{ are words in } H^1, k \in \mathbb{Z}_{>0}) \) with \( \mathbb{Q} \)-bilinearity.

Next, we recall the harmonic (resp. shuffle) regularized MZV introduced in [IKZ]. It
is known that \( H^1 \) \( \cong H^0 \langle e_1 \rangle \) as a \( \mathbb{Q} \)-algebra (see [HI] for \( \bullet = \ast \) and [Re] for \( \bullet = \mathfrak{m} \)).
Therefore, for \( \bullet \in \{\ast, \mathfrak{m}\} \), any \( a \in H^1 \) has a unique expression \( a = \sum_{i=0}^{n} a_i \bullet e_1^i \), where
\( n \in \mathbb{Z}_{>0}, a_i \in H^0 \) (\( 0 \leq i \leq n \)) and \( e_1^{i} := e_1 \bullet \cdots \bullet e_1 \). By this expression, we define a
Theorem 2.2

\[ \zeta \] results to evaluate some Zagier’s formulas for MZVs.

Lemma 2.1

\[ k := \zeta \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \] (Regularization formula, \cite{IKZ, Proposition 8}).

Theorem 2.3

Here \( \delta \) is Kronecker’s delta, and we understand \( x, y \) and \( \{2\}^a \) being odd. Define a positive integer \( k = n + m \) and \( k := m + n \) being odd. Define a positive integer \( K \) as \( k = 2K + 1 \). Then we have

\[ \zeta(m, n) = (-1)^m \sum_{s=0}^{K-1} \left\{ \binom{k-2s-1}{m-1} + \binom{k-2s-1}{n-1} - \delta_{n, 2s} + (-1)^m \delta_{s, 0} \right\} \zeta(2s) \zeta(k-2s). \]

Here \( \delta_{x, y} \) is Kronecker’s delta, and we understand \( \zeta(0) = -\frac{1}{2} \). In particular, we have

\[ \zeta(m, n) \equiv (1 - 1)^{m+1} \frac{1}{2} \left\{ \binom{k}{m} + (-1)^m \right\} \zeta(k) \mod \zeta(2). \]

2.2. Zagier’s formulas for MZVs. We quote some results on MZVs. We use these results to evaluate some \( S_1 \)-MZ(S)Vs and \( S_2 \)-MZ(S)Vs.

Theorem 2.2 (\cite{Z} Theorem 1). For non-negative integers \( a \) and \( b \), we have

\[ \zeta(\{2\}^a, 3, \{2\}^b) = 2 \sum_{r=1}^{a+b+1} (-1)^r \left\{ \binom{2r}{2a+2} - \left(1 - \frac{1}{2^{2r}}\right) \binom{2r}{2b+1} \right\} \zeta(\{2\}^{a+b-r+1}) \zeta(2r+1), \]

where \( \{2\}^a \) denotes \( a \) repetitions \( 2, \ldots, 2 \). In particular, we have

\[ \zeta(\{2\}^a, 3, \{2\}^b) \equiv 2(-1)^{a+b+1} \left\{ \binom{2a+2b+2}{2a+2} - \left(1 - \frac{1}{4^{a+b+1}}\right) \binom{2a+2b+2}{2b+1} \right\} \zeta(2a+2b+3) \mod \zeta(2). \]

Theorem 2.3 (\cite{Z} Proposition 7). Let \( m \) and \( n \) be positive integers with \( n \geq 2 \) and \( k := m + n \) being odd. Define a positive integer \( K \) as \( k = 2K + 1 \). Then we have

\[ \zeta(m, n) = (-1)^m \sum_{s=0}^{K-1} \left\{ \binom{k-2s-1}{m-1} + \binom{k-2s-1}{n-1} - \delta_{n, 2s} + (-1)^m \delta_{s, 0} \right\} \zeta(2s) \zeta(k-2s). \]

Here \( \delta_{x, y} \) is Kronecker’s delta, and we understand \( \zeta(0) = -\frac{1}{2} \). In particular, we have

\[ \zeta(m, n) \equiv (1 - 1)^{m+1} \frac{1}{2} \left\{ \binom{k}{m} + (-1)^m \right\} \zeta(k) \mod \zeta(2). \]

2.3. Double shuffle relation for \( \mathcal{F}_n \)-MZVs. The double shuffle relation (DSR) for \( \mathcal{F}_n \)-MZVs with \( \mathcal{F} \in \{ \mathcal{A}, \mathcal{S} \} \) established by Jarossay is a key tool in this paper. We define \( \mathbb{Q} \)-linear maps \( Z_{\mathcal{A}_n} : \mathfrak{S}^1 \to \mathcal{A}_n \) and \( Z_{\mathcal{S}_n} : \mathfrak{S}^1 \to \mathfrak{S}^1(t^1)/(t^n) \) by

\[ Z_{\mathcal{A}_n}(e_k) := \zeta_{\mathcal{A}_n}(k), \quad Z_{\mathcal{S}_n}(e_k) := \zeta_{\mathcal{S}_n}(k) \]

for any index \( k \).
Theorem 2.4 (DSR for \( F_n \)-MZVs, [J2], cf. [OSY] Theorems 1.3 and 1.9). For indices \( k \) and \( l = (l_1, \ldots, l_s) \) and a positive integer \( n \), we have the harmonic relation for \( F_n \)-MZVs

\[
Z_{F_n}(e_k \ast e_l) = Z_{F_n}(e_k)Z_{F_n}(e_l)
\]

and the shuffle relation for \( F_n \)-MZVs

\[
Z_{F_n}(e_k \boxtimes e_l) = (-1)^{wt(l)} \sum_{l'=(l'_1, \ldots, l'_s) \in \Z^s_{\leq n-1}} \left[ \prod_{j=1}^s \left( \frac{l_j + l'_j - 1}{l'_j} \right) \right] Z_{F_n}(e_k e_{l+l'}) x_{F_n}^{wt(l')}.\]

Here, we set \( wt(l') := l'_1 + \cdots + l'_s \), \( l + l' := (l_1 + l'_s, \ldots, l_s + l'_1) \) and

\[
x_{F_n} := \begin{cases} 
(p \mod p^n) & \text{if } F = A, \\
\hat{B}_j(p \mod t^n) & \text{if } F = S.
\end{cases}
\]

We refer to the case \( k = \emptyset \) of the shuffle relation as the reversal formula.

We also use the following relation for \( F_n \)-MZVs.

Proposition 2.5. For an index \( k = (k_1, \ldots, k_r) \), \( n \in \Z_{\geq 1} \) and \( F \in \{A, S\} \), we have

\[
\sum_{i=0}^{r} (-1)^i \zeta_{F_n}(k_1, \ldots, k_i) \zeta^{\ast}_{F_n}(k_r, \ldots, k_{i+1}) = 0.
\]

Proof. This follows from the harmonic relation and [IKOO] Proposition 6] (Note that the sign of [IKOO] Proposition 6] is mistaken). The case \( F = A \) was first mentioned in [SS, Corollary 3.16 (42)]. \( \square \)

3. Special values

In this section, we explicitly evaluate some \( F_n \)-MZ(S)Vs. For positive integers \( n \) and \( k \), set

\[
3_{F_n}(k) := \begin{cases} 
\left( \frac{B_{p^n-1(p-1)-k+1}}{k-1 + p^{n-1}} \mod p^n \right)_p & \text{if } F = A, \\
\zeta(k) \mod \zeta(2) & \text{if } F = S.
\end{cases}
\]

Here, \( B_j \) is the \( j \)-th Seki–Bernoulli number and \( \hat{B}_j \) denotes \( \frac{B_j}{j} \).

Proposition 3.1. Let \( n \) and \( k \) be positive integers. For \( 1 \leq l \leq n-1 \),

\[
3_{A_n}(k + l)p^l_n = \sum_{j=1}^{n-l} (-1)^j \binom{n-l}{j} \left( \hat{B}_{j(p-1)-k-l+1} \cdot p^l \mod p^n \right)_p \in A_n
\]
holds. In particular,

\[ 3A_2(k + 1)p_2 = \left(\frac{B_{p-k-1}}{k+1} \cdot p \mod p^2\right) \in A_2. \]

**Proof.** Let \( p \) be a sufficiently large prime number. Then, by using the Kummer-type congruence proved by Zhi-Hong Sun [Su, Corollary 4.1], we have

\[ \frac{B_{p^n-1}(p-1)-k-l+1}{k+l-1+p^{n-1}} \cdot p^l \]

\[ \equiv (-1)^{n+l} \sum_{j=1}^{n-l} (-1)^{j-1} \left( \frac{p^{n-1} - 1 - j}{n - l - j} \right) \left( \frac{p^{n-1} - 1}{j - 1} \right) \hat{B}_{j(p-1)-k-l+1} \cdot p^l \pmod{p^n}. \]

Since

\[ (-1)^{n+l-1} \left( \frac{p^{n-1} - 1 - j}{n - l - j} \right) \left( \frac{p^{n-1} - 1}{j - 1} \right) \equiv \left( \frac{n - l}{j} \right) \pmod{p^{n-1}}, \]

we have the desired formula. \( \square \)

### 3.1. Depth 1 case.

**Theorem 3.2.** For positive integers \( n \) and \( k \), we have

\[ \zeta_{F_n}(k) = (-1)^k \sum_{l=1}^{n-1} \left( \frac{k + l - 1}{l} \right) 3F_n(k + l)x_jF_n. \]

**Proof.** The case \( F = A \) is a special case of [W, Theorem 1]. Nevertheless, we can state the direct proof as follows. Let \( p \) be a sufficiently large prime number. By Euler’s formula and Faulhaber’s formula, we have

\[ \sum_{m=1}^{p-1} \frac{1}{m^k} \equiv \sum_{m=1}^{p-1} m^{\varphi(p^n) - k} \]

\[ \equiv \frac{1}{\varphi(p^n) - k + 1} \sum_{l=1}^{n-1} \left( \frac{\varphi(p^n) - k + 1}{l} \right) B_{\varphi(p^n) - k-l+1} \cdot p^l \]

\[ = - \sum_{l=1}^{n-1} \left( \frac{\varphi(p^n) - k}{l} \right) \frac{B_{\varphi(p^n) - k-l+1}}{k+l-1+p^{n-1}} \cdot p^l \pmod{p^n}, \]

where \( \varphi \) is Euler’s totient function. By a simple congruence

\[ \left( \frac{\varphi(p^n) - k}{l} \right) \equiv (-1)^l \left( \frac{k+l-1}{l} \right) \pmod{p^{n-1}} \]

and the fact that \( B_j \) vanishes for odd \( j \geq 3 \), we have the desired equality in \( A_n \). Since the case \( F = S \) is clear by definition, this completes the proof. \( \square \)
**Remark 3.3.** By combining the case $\mathcal{F} = \mathcal{A}$ of Theorem 3.2 and Proposition 3.1, we have

$$\sum_{m=1}^{p-1} \frac{1}{m^k} \equiv (-1)^k \sum_{l=1}^{n-1} \binom{k+l-1}{l} \sum_{j=1}^{n-l} (-1)^j \binom{n-l}{j} \widehat{B}_{j(p-1) - k - l + 1} p^l \pmod{p^n}$$

for a sufficiently large prime $p$. We can check that this holds for $p \geq n+k+1$. This congruence is a generalization of [Sun, Theorem 5.1 (a) and Remark 5.1] and [Tao, Theorem 2.1]. However, the proof is identical to that put forward by Sun.

**3.2. Depth 2 case.** Let $\tau_n : \mathbb{Z}[t] \to \mathbb{Z}[t]$ be the truncation map defined by $\tau_n(\sum_{l=0}^{\infty} z_l t^l) := \sum_{l=0}^{n-1} z_l t^l$ for a positive integer $n$. In the following argument, we often identify $\zeta^*_n(k)$ with $\tau_n(\zeta^*_S(k))$, where $\bullet \in \{\emptyset, \star\}$. Furthermore, we often abbreviate $\zeta(k) \pmod{\zeta(2)}$ (resp. $\zeta^w(k) \pmod{\zeta(2)}$) to $\zeta(k)$ (resp. $\zeta^w(k)$) in $\mathbb{Z}$.

**Theorem 3.4.** Let $k_1$ and $k_2$ be positive integers. Assume that $k := k_1 + k_2$ is even. Then we have

$$\zeta_{\mathcal{F}_2}(k_1, k_2) = \frac{1}{2} \left\{ (-1)^{k_1} k_2 \left( \frac{k+1}{k_1} \right) - (-1)^{k_2} k_1 \left( \frac{k+1}{k_2} \right) - k \right\} 3_{\mathcal{F}_2}(k+1)x_{\mathcal{F}_2},$$

(3.2) $$\zeta^*_\mathcal{F}_2(k_1, k_2) = \frac{1}{2} \left\{ (-1)^{k_1} k_2 \left( \frac{k+1}{k_1} \right) - (-1)^{k_2} k_1 \left( \frac{k+1}{k_2} \right) + k \right\} 3_{\mathcal{F}_2}(k+1)x_{\mathcal{F}_2}.$$ 

**Proof.** The case $\mathcal{F} = \mathcal{A}$ was proved by Zhao, see [Z1] Theorem 3.2]. Hereafter, we consider the case $\mathcal{F} = \mathcal{S}$. First, we prove (3.1) for the case $k_1 \geq 2$. By the definition of $\zeta_{\mathcal{S}_2}(k_1, k_2)$, we have

$$\zeta_{\mathcal{S}_2}(k_1, k_2) = \zeta_{\mathcal{S}_1}(k_1, k_2) + \{ k_2 \zeta(k_2 + 1, k_1) + k_1 \zeta(k_2, k_1 + 1) \} t.$$ 

Since $k_1 + k_2$ is even, we have $\zeta_{\mathcal{S}_1}(k_1, k_2) = 0$ by definition. Therefore, by using (2.2), we obtain (3.1) for the case $k_1 \geq 2$. Next, we prove (3.1) for the case $k_1 = 1$ (then $k_2$ is odd). We have

$$\zeta_{\mathcal{S}_2}(1, k_2) = \{ k_2 \zeta^w(k_2 + 1, 1) + \zeta(k_2, 2) \} t.$$ 

By applying Theorem 2.1 for $w = e_{1} e_{0}^{k_2}$ and the sum formula for MZVs of depth 2, we have

(3.4) $$\zeta^w(k_2 + 1, 1) = -\zeta(k_2, 2) - \cdots - \zeta(2, 2) - 2\zeta(1, k_2 + 1) = -\zeta(k_2 + 2) - \zeta(1, k_2 + 1).$$ 

By (2.2), we have

(3.5) $$\zeta(1, k_2 + 1) = \frac{k_2 + 1}{2} \zeta(k_2 + 2), \quad \zeta(k_2, 2) = \frac{1}{2} \left\{ \frac{(k_2 + 2)(k_2 + 1)}{2} - 1 \right\} \zeta(k_2 + 2).$$
From (3.3), (3.4), (3.5), we obtain (3.1) for the case \( k_1 = 1 \). The formula (3.2) follows from (3.1), the fact \( \zeta_{S_2}^*(k_1, k_2) = \zeta_{S_2}(k_1, k_2) + \zeta_{S_2}(k_1 + k_2) \), and \( \zeta_{S_2}(k) = (-1)^k k \zeta(k + 1)t \) (Theorem 3.2 with \( F_n = S_2 \)).

3.3. Depth 3 case.

**Theorem 3.5.** Let \( k_1, k_2, k_3 \) be positive integers. Suppose that \( k := k_1 + k_2 + k_3 \) is odd. Then we have

\[
\zeta_{F_1}(k_1, k_2, k_3) = -\zeta_{F_1}^*(k_1, k_2, k_3) = \frac{1}{2} \left\{ (-1)^{k_1} \left( \frac{k}{k_1} \right) - (-1)^{k_2} \left( \frac{k}{k_2} \right) \right\} 3_{F_1}(k).
\]

**Proof.** The case \( F = A \) was proved by Hoffman and Zhao; see [H2, Theorem 6.2] or [Zh, Theorem 3.5]. Hereafter, we consider the case \( F = S \). By Proposition 2.5 and the reversal formula for \( F_1 \)-MZVs, we have

\[
\zeta_{F_1}(k_1, k_2, k_3) = (-1)^{k_1+k_2+k_3} \zeta_{F_1}(k_1, k_2, k_3) = -\zeta_{F_1}(k_1, k_2, k_3).
\]

From

\[
\zeta_{F_1}(k_1, k_2, k_3) = \zeta_{F_1}(k_1, k_2, k_3) + \zeta_{F_1}(k_1+k_2, k_3) + \zeta_{F_1}(k_1, k_2+k_3)
\]

and the explicit formula for \( F_1 \)-double zeta values [Kan, (7.2), Example 9.4 (2)], we have

\[
\zeta_{F_1}(k_1, k_2, k_3) = \frac{-\zeta_{F_1}(k_1+k_2, k_3) + \zeta_{F_1}(k_1, k_2+k_3)}{2}
\]

\[
= -\frac{1}{2} \left\{ (-1)^{k_1} \left( \frac{k}{k_1+k_2} \right) + (-1)^{k_2+k_3} \left( \frac{k}{k_1} \right) \right\} 3_{F_1}(k)
\]

\[
= \frac{1}{2} \left\{ (-1)^{k_1} \left( \frac{k}{k_1} \right) - (-1)^{k_3} \left( \frac{k}{k_3} \right) \right\} 3_{F_1}(k).
\]

The formula for \( \zeta_{F_1}^*(k_1, k_2, k_3) \) is obtained by (3.6) and (3.7). \( \Box \)

3.4. General depth case.

**Theorem 3.6.** For positive integers \( r, k \) and for \( F \in \{ A, S \} \), we have

\[
\zeta_{F_2}(\{k\}^r) = (-1)^{r-1}k3_{F_2}(rk+1)x_{F_2},
\]

\[
\zeta_{F_2}^*(\{k\}^r) = k3_{F_2}(rk+1)x_{F_2}.
\]

Moreover, we have

\[
\zeta_{F_3}(\{k\}^r) = (-1)^{r+k+r-1} \left[ k3_{F_3}(rk+1)x_{F_3} + \frac{k(rk+1)}{2}3_{F_3}(rk+2) - k^2 \sum_{l=1}^{r-1} 3_{F_3}(lk+1)3_{F_3}((r-l)k+1) \right] x_{F_3}^2.
\]
and

\[(3.11) \quad \zeta_{\mathcal{F}_3}^F(\{k\}^r) = (-1)^rk \left[ \frac{k(3k+1)}{2} \mathfrak{F}_3(rk+2) + k^2 \sum_{l=1}^{r-1} \mathfrak{F}_3(lk+1) \mathfrak{F}_3((r-l)k+1) \right] x_{\mathcal{F}_3}^2. \]

**Remark 3.7.** If \(rk\) is odd, then \(\mathfrak{F}_3(rk+1)\) and \(\mathfrak{F}_3(lk+1)\mathfrak{F}_3((r-l)k+1)\) are 0, and we have

\[
\zeta_{\mathcal{F}_3}^F(\{k\}^r) = (-1)^r \frac{k(rk+1)}{2} \mathfrak{F}_3(rk+2)x_{\mathcal{F}_3}^2, \quad \zeta_{\mathcal{F}_3}^F(\{k\}^r) = -\frac{k(rk+1)}{2} \mathfrak{F}_3(rk+2)x_{\mathcal{F}_3}^2.
\]

These formulas for the case \(\mathcal{F} = \mathcal{A}\) were first proved by Zhou and Cai in the last remark of [ZC] but our proof differs from theirs.

**Proof.** Since (3.8) and (3.9) follows from (3.10) and (3.11) by taking modulo \(x_{\mathcal{F}_3}^2\), it is sufficient to prove (3.10) and (3.11). Note that \((-1)^r \mathfrak{F}_3(\mathfrak{F}_2(rk+1)x_{\mathcal{F}_3}^2 = \mathfrak{F}_3(rk+1)x_{\mathcal{F}_3}^2\) holds because if \(rk\) is odd, then \(\mathfrak{F}_3(rk+1)x_{\mathcal{F}_3}^2 = 0\).

By Theorem 3.2 and the symmetric sum formula (5.1) proved in Section 5 with \(k = (\{k\}^r)\), we have

\[
(3.12) \quad r! \zeta_{\mathcal{F}_3}^F(\{k\}^r) = (-1)^{rk+r-1}(r-1)! \left\{ r \mathfrak{F}_3(rk+1)x_{\mathcal{F}_3} + \binom{rk+1}{2} \mathfrak{F}_3(rk+2)x_{\mathcal{F}_3}^2 \right\}
+ (-1)^{rk+r-2} \sum_{B_1 \cup B_2 = \{1, \ldots, r\} \atop B_1, B_2 \neq \emptyset} (#B_1 - 1)! (#B_2 - 1)! b_1 b_2 \mathfrak{F}_3(b_1 + 1) \mathfrak{F}_3(b_2 + 1)x_{\mathcal{F}_3}^2,
\]

where \(b_1 = b_1(\{k\}^r)\) and \(b_2 = b_2(\{k\}^r)\) are defined as in Theorem 5.1. Set \(l := \#B_1\). Then we see that \(1 \leq l \leq r - 1\), \(\#B_2 = r - l\), \(b_1 = lk\) and \(b_2 = (r - l)k\). Moreover, the number of ways of dividing \(\{1, \ldots, r\}\) into two non-empty subsets \(B_1\) and \(B_2\) with \(\#B_1 = l\) is just \(\binom{r}{l}\). Therefore, the summation for the partition in the right-hand side of (3.12) coincides with

\[
\sum_{l=1}^{r-1} \binom{r}{l} (l-1)!(r-l-1)! \cdot l(r-l)k^2 \mathfrak{F}_3(lk+1) \mathfrak{F}_3((r-l)k+1)x_{\mathcal{F}_3}^2
= k^2 \cdot r! \sum_{l=1}^{r-1} \mathfrak{F}_3(lk+1) \mathfrak{F}_3((r-l)k+1)x_{\mathcal{F}_3}^2.
\]

Thus we obtained (3.10). The formula (3.11) is obtained in the same manner. \(\square\)
\textbf{Theorem 3.8.} For non-negative integers $a$ and $b$, we have

\begin{align*}
(3.13) & \quad \zeta_{\mathcal{F}_1}(\{1\}^a, 2, \{1\}^b) = (-1)^b \binom{a + b + 2}{a + 1} \, 3_{\mathcal{F}_1}(a + b + 2), \\
(3.14) & \quad \zeta^{\star}_{\mathcal{F}_1}(\{1\}^a, 2, \{1\}^b) = (-1)^b \binom{a + b + 2}{a + 1} \, 3_{\mathcal{F}_1}(a + b + 2).
\end{align*}

\textit{Proof.} The case $\mathcal{F} = \mathcal{A}$ was proved by Hessami-Pilehrood–Hessami-Pilehrood–Tauraso; see \cite{HHT} Theorem 4.1. Hereafter, we consider the case $\mathcal{F} = \mathcal{S}$. By the definition of $\zeta_{\mathcal{S}_1}(k)$ and the fact that $\zeta^m(\{1\}^k) = 0$ for $k \geq 1$, we have

\begin{equation}
(3.15) \quad \zeta_{\mathcal{S}_1}(\{1\}^a, 2, \{1\}^b) = \zeta^m(\{1\}^a, 2, \{1\}^b) + (-1)^{a + b} \zeta^m(\{1\}^b, 2, \{1\}^a).
\end{equation}

Applying Lemma \ref{lem:binomial} for $w = e_1^{a+1} e_0$ and $m = b$, and using the duality for MZVs, we have

\begin{equation}
(3.16) \quad \zeta^m(\{1\}^a, 2, \{1\}^b) = (-1)^b \binom{a + b + 1}{b} \zeta(a + b + 2).
\end{equation}

From (3.15) and (3.16), we obtain (3.13). The formula (3.14) follows from (3.13), Proposition \ref{prop:binomial} and the fact that $\zeta_{\mathcal{S}_1}(\{1\}^r) = \zeta^\star_{\mathcal{S}_1}(\{1\}^r) = 0$ for $r \geq 1$. The last fact is well-known and a special case of Theorem 3.6. \hfill \Box

\textbf{Remark 3.9.} We can also prove (3.14) using the Hoffman duality (\cite{H2} Theorem 4.6 and \cite{JT} Corollary 1.12)) and the explicit formula for $\zeta_{\mathcal{F}_1}(a + 1, b + 1)$.

\textbf{Theorem 3.10.} For non-negative integers $a$ and $b$, we have

\begin{align*}
(3.17) & \quad \zeta_{\mathcal{F}_1}(\{2\}^a, 3, \{2\}^b) = \frac{(-1)^{a + b} 2(a - b)}{a + 1} \binom{2a + 2b + 3}{2b + 2} \, 3_{\mathcal{F}_1}(2a + 2b + 3), \\
(3.18) & \quad \zeta^{\star}_{\mathcal{F}_1}(\{2\}^a, 3, \{2\}^b) = \frac{2(b - a)}{a + 1} \binom{2a + 2b + 3}{2b + 2} \, 3_{\mathcal{F}_1}(2a + 2b + 3).
\end{align*}

\textit{Proof.} The case $\mathcal{F} = \mathcal{A}$ was proved by Hessami-Pilehrood–Hessami-Pilehrood–Tauraso; see \cite{HHT} Theorem 4.1. Hereafter, we consider the case $\mathcal{F} = \mathcal{S}$. By the definition of the $\mathcal{S}_1$-MZV and the fact that $\zeta(\{2\}^r) \equiv 0 \pmod{\zeta(2)}$ for $r \geq 1$, we have

\[ \zeta_{\mathcal{S}_1}(\{2\}^a, 3, \{2\}^b) = \zeta(\{2\}^a, 3, \{2\}^b) - \zeta(\{2\}^b, 3, \{2\}^a). \]

Thus we obtain (3.17) by the formula (2.11) and straightforward calculation of binomial coefficients. The formula (3.18) is obtained by (3.17), Proposition 2.5 and a special case of Theorem 3.6 that is, the fact that $\zeta_{\mathcal{S}_1}(\{2\}^r) = \zeta^\star_{\mathcal{S}_1}(\{2\}^r) = 0$ for $r \geq 1$. \hfill \Box
Theorem 3.11. For non-negative integers \( a \) and \( b \), we have

\[
\zeta_{\mathcal{F}_1}(\{2\}^a, 1, \{2\}^b) = 4(-1)^{a+b} \frac{a - b}{2a+1} \left( 1 - \frac{1}{4^{a+b}} \right) \left( 2a + 2b + 1 \right) \mathcal{F}_1(2a + 2b + 1),
\]

\[
\zeta_{\mathcal{F}_1}(\{2\}^a, 1, \{2\}^b) = \frac{4(b-a)}{2a+1} \left( 1 - \frac{1}{4^{a+b}} \right) \left( 2a + 2b + 1 \right) \mathcal{F}_1(2a + 2b + 1).
\]

Proof. The case \( \mathcal{F} = \mathcal{A} \) was proved by Hessami-Pilehrood–Hessami-Pilehrood–Tauraso; see [HHT, Theorem 4.2]. Hereafter, we consider the case \( \mathcal{F} = \mathcal{S} \). First, we prove (3.19) for the case \( a, b \geq 1 \). By the duality for MZVs and the duality for MZVs, we obtain

\[
\zeta_{\mathcal{S}_1}(\{2\}^a, 1, \{2\}^b) = \zeta(\{2\}^{b-1}, 3, \{2\}^a) - \zeta(\{2\}^{a-1}, 3, \{2\}^b).
\]

Then we obtain (3.19) by a similar calculation in Theorem 3.10 using (2.1).

Next we prove (3.19) for the case \( a \geq 1 \) and \( b = 0 \). We have

\[
\zeta_{\mathcal{S}_1}(\{2\}^a, 1) = \zeta^\text{w}(\{2\}^a, 1) - \zeta(1, \{2\}^a).
\]

Applying Lemma 2.1 for \( w = (e_1 e_0)^a \) and \( m = 1 \), we obtain

\[
\zeta^\text{w}(\{2\}^a, 1) = -2 \sum_{j=0}^{a-1} \zeta(\{2\}^j, 1, \{2\}^{a-j}).
\]

Thus, by the duality for MZVs, we have

\[
\zeta_{\mathcal{S}_1}(\{2\}^a, 1) = -\zeta(1, \{2\}^a) - 2 \sum_{j=0}^{a-1} \zeta(\{2\}^j, 1, \{2\}^{a-j}) = -\zeta(\{2\}^{a-1}, 3) - 2 \sum_{j=0}^{a-1} \zeta(\{2\}^{a-j-1}, 3, \{2\}^j).
\]

By (2.1), we obtain

\[
\zeta(\{2\}^{a-j-1}, 3, \{2\}^j) = 2(-1)^{a} \left( \binom{2a}{2j} - \binom{2a+1}{2j+1} \right) \zeta(2a+1)
\]

for \( 0 \leq j \leq a - 1 \). Therefore, from (3.22) and (3.23), we obtain (3.19) for the case \( a \geq 1 \) and \( b = 0 \). The case \( a = 0 \) and \( b \geq 1 \) follows easily from the reversal formula and (3.22) with \( a \geq 1 \). This completes the proof of (3.19). The formula (3.20) is obtained by (3.19), Proposition 2.5 and the fact that \( \zeta_{\mathcal{S}_1}(\{2\}^r) = \zeta_{\mathcal{S}_1}(\{2\}^r) = 0 \) for \( r \geq 1 \).

Remark 3.12. Tasaka and Yamamoto proved an analogous formula of Theorem 2.2 for \( \zeta^*(\{2\}^a, 1, \{2\}^b) \); see [TY, Theorem 1.6]. We can also obtain Theorem 3.11 by a
similar approach using [TY, Theorem 1.6] and Proposition 2.5 instead of Zagier’s formula (Theorem 2.2).

The following theorem is a refinement of the even weight case in Theorem 3.8.

**Theorem 3.13.** Let $a$ and $b$ be non-negative integers. Assume that $a + b$ is even. Then we have

\[
\zeta_{\mathcal{F}_2}(\{1\}^a, 2, \{1\}^b) = \frac{1}{2} \left\{ 1 + (-1)^a \frac{(a + b + 3)}{b + 2} \right\} 3_{\mathcal{F}_2}(a + b + 3)x_{\mathcal{F}_2},
\]

(3.24)

\[
\zeta_{\mathcal{F}_2}(\{1\}^a, 2, \{1\}^b) = \frac{1}{2} \left\{ 1 + (-1)^a \frac{(a + b + 3)}{a + 2} \right\} 3_{\mathcal{F}_2}(a + b + 3)x_{\mathcal{F}_2}.
\]

(3.25)

**Proof.** The case $\mathcal{F} = \mathcal{A}$ was proved by Hessami-Pilehrood–Hessami-Pilehrood–Tauraso [HHT Theorem 4.5]; there is also another proof by Sakugawa and the third author [SS, Theorem 3.18]. Hereafter, we consider the case $\mathcal{F} = \mathcal{S}$. We first prove the formula (3.24). Set $d := a + b + 1$ and $k = (k_1, \ldots, k_d) := (\{1\}^a, 2, \{1\}^b)$. For $0 \leq i \leq d$, we set

\[
P_i(t) := (-1)^{k_{i+1} + \cdots + k_d} \zeta^a(k_1, \ldots, k_i)
\]

\[
\times \sum_{l_{i+1} + \cdots + l_d \geq 0 \atop l_{i+1} + \cdots + l_d \leq 1} \left[ \prod_{j=i+1}^d \left( k_j + l_j - 1 \right) \right] \zeta^a(k_d + l_d, \ldots, k_{i+1} + l_{i+1}) t^{l_{i+1} + \cdots + l_d}.
\]

Note that this expression for the case $i = d$ means $P_d(t) = \zeta^a(k)$. Then, by the definition of $\zeta_{\mathcal{S}_2}(k)$, we have

\[
\zeta_{\mathcal{S}_2}(k) = \sum_{i=0}^d P_i(t).
\]

We prove that for $1 \leq i \leq a + b + 1$, $P_i(t) = 0$ in $\overline{\mathbb{Z}}[t]$. Since $\zeta^a(\{1\}^k) = 0$ for a positive integer $k$, we obtain $P_1(t) = \cdots = P_a(t) = 0$. For $0 \leq j \leq b$, we calculate $P_{a+j+1}(t)$. By the definition of $P_i(t)$, we have

\[
P_{a+j+1}(t) = (-1)^{b-j} \zeta^a(\{1\}^a, 2, \{1\}^j) \left( \zeta^a(\{1\}^{b-j}) + \sum_{i=1}^{b-j} \zeta^a(\{1\}^{i-1}, 2, \{1\}^{b-j-i}) t \right).
\]

By (3.16), we have $\zeta^a(\{1\}^a, 2, \{1\}^j) = (-1)^j (a+j+1) \zeta(a+j+2)$. Since $\zeta(a+b+2) = 0$ in $\overline{\mathbb{Z}}$, we have $P_{a+b+1}(t) = 0$ in $\overline{\mathbb{Z}}[t]$. Assume that $j < b$. In this case, $\zeta^a(\{1\}^{b-j}) = 0$ holds. By the shuffle-regularized sum formula [Li, Lemma 3.3] (or [KS, Theorem 1.2]), and the duality for MZVs, we have

\[
\sum_{i=1}^{b-j} \zeta^a(\{1\}^{i-1}, 2, \{1\}^{b-j-i}) = (-1)^{b-j-1} \zeta(b-j+1).
\]
Since $a + b$ is even, $a + j + 2 \not\equiv b - j + 1 \pmod{2}$ and thus we have $P_{a+j+1}(t) = 0$ in $\mathbb{Z}$.

The calculation of $P_0(t)$ remains. State $P_0(t) = A + Bt$ with $A, B \in \mathbb{Z}$. Then from (3.16), we have $A = \zeta^m((1)^b, 2, \{1\}^a) = 0$ in $\mathbb{Z}$. By definition, $B$ is expressed as follows:

$$B = \sum_{l+m=b-1 \atop l,m \geq 0} \zeta^m(\{1\}^l, 2, \{1\}^m, 2, \{1\}^a)$$

$$+ 2\zeta^m(\{1\}^b, 3, \{1\}^a) + \sum_{m+n=a-1 \atop m,n \geq 0} \zeta^m(\{1\}^b, 2, \{1\}^m, 2, \{1\}^n).$$

For general non-negative integers $l, m$ and $n$, by the regularization formula (Lemma 2.1), we have

$$\zeta^m(\{1\}^l, 2, \{1\}^m, 2, \{1\}^n) = (-1)^n \sum_{r+s=n \atop r,s \geq 0} \binom{r+l+1}{r} \binom{s+m+1}{s} \zeta(\{1\}^{r+l}, 2, \{1\}^{s+m}, 2),$$

$$\zeta^m(\{1\}^l, 3, \{1\}^n) = (-1)^n \sum_{r+s=n \atop r,s \geq 0} \binom{r+l+1}{r} \zeta(\{1\}^{r+l}, 2, \{1\}^{s-1}, 2),$$

where $\zeta(\{1\}^{r+l}, 2, \{1\}^{-1}, 2)$ means $\zeta(\{1\}^{r+l}, 3)$. Thus, with the duality for MZVs, we obtain

$$B = \sum_{l+m=b-1 \atop l,m \geq 0} (-1)^a \sum_{r+s=a \atop r,s \geq 0} \binom{r+l+1}{r} \binom{s+m+1}{s} \zeta(s+m+2, r+l+2)$$

$$+ 2(-1)^a \sum_{r+s=a \atop r,s \geq 0} \binom{r+b+1}{r} \zeta(s+1, r+b+2)$$

$$+ \sum_{m+n=a-1 \atop m,n \geq 0} (-1)^n \sum_{r+s=n \atop r,s \geq 0} \binom{r+b+1}{r} \binom{s+m+1}{s} \zeta(s+m+2, r+b+2).$$
Since $a + b + 3$ is odd, by using (2.2), we can rewrite $B$ as a rational multiple of the Riemann zeta value $\zeta(a + b + 3)$. Specifically, we have $B = \frac{1}{2}C\zeta(a + b + 3)$ with

$$C = \sum_{l+m+n=s=0}^{l, m, n} (-1)^a \sum_{r+s=n}^{r, s=0} \left(\begin{array}{c} r + l + 1 \\ r \end{array}\right) \left(\begin{array}{c} s + m + 1 \\ s \end{array}\right) (-1)^{s+m+1} \left\{ \left(\begin{array}{c} a + b + 3 \\ s + m + 2 \end{array}\right) + (-1)^{s+m}\right\}$$

$$+ 2(-1)^a \sum_{r+s=n}^{r, s=0} \left(\begin{array}{c} r + b + 1 \\ r \end{array}\right) (-1)^{s} \left\{ \left(\begin{array}{c} a + b + 3 \\ s + 1 \end{array}\right) + (-1)^{s+1}\right\}$$

$$+ \sum_{m+n=a-1}^{m+n, n} (-1)^{n} \sum_{r+s=n}^{r, s=0} \left(\begin{array}{c} r + b + 1 \\ r \end{array}\right) \left(\begin{array}{c} s + m + 1 \\ s \end{array}\right) (-1)^{s+m+1} \left\{ \left(\begin{array}{c} a + b + 3 \\ s + m + 2 \end{array}\right) + (-1)^{s+m}\right\}.$$

Therefore, it suffices to prove the following:

$$(3.26) \quad C = 1 + (-1)^a \left(\begin{array}{c} a + b + 3 \\ b + 2 \end{array}\right).$$

We prove this in the Appendix. From this, we obtain the desired formula for $\zeta_{S_2}({1}^a, 2, {1}^b)$. Next, we prove (3.25). By Proposition 2.5, we have

$$\zeta_{S_2}^*({1}^a, 2, {1}^b) - \zeta_{S_2}({1}^b, 2, {1})^a = \sum_{j=1}^a (-1)^j \zeta_{S_2}({1}^b, 2, {1}^a-j) \zeta_{S_2}^*({1}^j)$$

$$+ \sum_{i=1}^b (-1)^{b-i} \zeta_{S_2}({1}^{b+1-i}) \zeta_{S_2}^*({1}^a, 2, {1}^{i-1}).$$

It is sufficient to show that the right-hand side vanishes. If $j$ is odd, then we have $\zeta_{S_2}^*({1}^j) = 0$ by (3.9). If $j$ is even, then both $\zeta_{S_2}^*({1}^j)$ and $\zeta_{S_2}({1}^b, 2, {1}^a-j)$ can be seen as elements of $t\overline{Z}[t]$ by (3.9) and (3.13). Thus the first summation vanishes in $\overline{Z}[t]/(t^2)$. Similarly, if $b - i$ is even, then we have $\zeta_{S_2}({1}^{b+1-i}) = 0$ by (3.8). If $b - i$ is odd, then both $\zeta_{S_2}({1}^{b+1-i})$ and $\zeta_{S_2}^*({1}^a, 2, {1}^{i-1})$ can be seen as elements of $t\overline{Z}[t]$ by (3.8) and (3.14). Therefore, the second summation also vanishes.

**Remark 3.14.** The proof of the case $\mathcal{F} = \mathcal{A}$ of Theorem 3.13 by Sakugawa and the third author is based on the ‘$A_2$-duality’ [SS, Remark 3.14 (40)]. If the ‘$S_2$-duality’ is established, then we can obtain another proof of the case $\mathcal{F} = \mathcal{S}$ of Theorem 3.13. When we were writing this paper, a preprint [TT] by Takeyama and Tasaka appeared on arXiv. Their [TT, Corollary 6.8] contains the $S_2$-duality as a special case.
4. Bowman–Bradley type theorem

Murahara, Onozuka and the third author [MOS] proved the Bowman–Bradley type theorem for $A_2$-$\text{MZ}(S)V_s (= \text{the case } F = A$ of Theorem 4.1). In this section, we prove the $S_2$-counterpart of their theorem. By combining these two theorems, we have the following.

**Theorem 4.1** (Bowman–Bradley type theorem for $F_2$-$\text{MZ}(S)V$). For non-negative integers $l$ and $m$ with $(l, m) \neq (0, 0)$, we have

$$
\sum_{m_0+\ldots+m_{2l}=m \atop m_0, \ldots, m_{2l} \geq 0} \zeta_{F_2}(\{2\}^{m_0}, 1, \{2\}^{m_1}, 3, \{2\}^{m_2}, \ldots, \{2\}^{m_{2l-2}}, 1, \{2\}^{m_{2l-1}}, 3, \{2\}^{m_{2l}}) = (-1)^m \left\{ (-1)^{l+2l} \binom{l+m}{l} - 4 \binom{2l+m}{2l} \right\} 3_{F_2}(4l+2m+1)x_{F_2},
$$

(4.1)

This gives a partial lift of the Bowman–Bradley type theorem for $F_1$-$\text{MZ}(S)V$ proved by Saito and Wakabayashi [SW2]. Note that the proof of the $F = S$ case in Theorem 4.1 presented here is essentially the same as the proof of the $F = A$ case by Murahara, Onozuka and the third author [MOS]. In contrast, proofs of some sum formulas which will be given in the next section are different from those in the previous study.

We prepare two lemmas for the proof.

**Lemma 4.2.** For non-negative integers $l$ and $m$ with $(l, m) \neq (0, 0)$, we have

$$
Z_{S_2}(e_2^{l+m} \in e_2^l) = (-1)^m 2 \left\{ 1 - 2 \binom{4l+2m}{2l} \right\} \zeta(4l+2m+1)t.
$$

Proof. This lemma is the $S_2$-counterpart of [MOS, Lemma 2.5] and is proved from the same argument in [MOS] by using the explicit evaluation of $\zeta_{S_2}(\{2\}^r)$ (3.8) with $k = 2$, (3.17) and (2.4) with $F_n = S_2$. □

For a positive integer $n$, define a $\mathbb{Q}$-linear map $Z_{S_n} : \mathcal{S}^1 \to \overline{\mathbb{Z}}[t]/(t^n)$ by $Z_{S_n}(e_k) := \zeta_{S_n}^*(k)$ for any index $k$. 

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Lemma 4.3. For non-negative integers \( l \) and \( m \), we have

\[
Z_{S_2}^*( (e_1 e_3)^l \ \tilde{m} \ e_2^m ) = \sum_{2i + k + u = 2l \atop j + n + v = m} \ (-1)^{i+k} \binom{k+n}{k} \binom{u+v}{u} Z_{S_2}^*( (e_1 e_3)^i \tilde{m} \ e_2^i ) Z_{S_2}^*( (e_2^{k+n}) ) Z_{S_2}^*( (e_2^{u+v}) ).
\]

Proof. This follows immediately from [Y, equation (3.1)] and (2.3) with \( F_n = S_2 \).

Proof of the \( F = S \) case in Theorem 4.1. We prove (4.1) by induction on \( l \geq 0 \). The case \( l = 0 \) holds by the explicit evaluation of \( \zeta_{S_2} (\{2\}^r) \) (3.8) with \( k = 2 \). Let \( l \) be a positive integer and \( m \) a non-negative integer. By [MOS, Lemma 2.1], we have

\[
Z_{S_2} ( (e_1 e_3)^l \ \tilde{m} \ e_2^m ) = 4^{-l} Z_{S_2} ( (e_2^{l+m} \ \tilde{m} \ e_2^l ) - \sum_{k=0}^{l-1} 4^{k-l} \binom{l+m-2k}{l-k} Z_{S_2} ( (e_1 e_3)^k \ \tilde{m} \ e_2^{l+m-2k} ) ) Z_{S_2} ( (e_1 e_3)^{k} \ \tilde{m} \ e_2^{2l+m-2k} ).
\]

Hence, by Lemma 4.2 and the induction hypothesis, we have

\[
Z_{S_2} ( (e_1 e_3)^l \ \tilde{m} \ e_2^m ) = (-1)^m 2^{1-2l} \left( 1 - 2^{4l+2m} \right) \zeta(4l+2m+1) t
- \sum_{k=0}^{l-1} 4^{k-l} \binom{2l+m-2k}{l-k} \cdot (-1)^m \left( (-1)^{2l+2k} \binom{2l+m-k}{k} - 4^{2l+m-2k} \right) \zeta(4l+2m+1) t \mod \zeta(2).
\]

We see that this coincides with the desired formula by using [MOS, Lemma 2.6]. We also obtain (4.2) by the same argument in [MOS] using (4.1), (3.9) with \( k = 2 \) and Lemma 4.3.

5. Sum formulas

In this section, we prove the \( F_n \)-symmetric sum formula (= Theorem 5.1), the \( F_n \)-sum formula over \( I_{k,r} \) for \( n = 2, 3 \) (= Theorem 5.2), and the \( F_2 \)-sum formula over \( I_{k,r,i} \) (= Theorem 5.4).

5.1. \( F_n \)-symmetric sum formula. We first state the \( F_n \)-symmetric sum formula.
Theorem 5.1 ($\mathcal{F}_n$-symmetric sum formula). Let $n$ and $r$ be positive integers and $k = (k_1, \ldots, k_r)$ an index. Then, we have

\begin{align}
(5.1) \quad & \sum_{\sigma \in \mathfrak{S}_r} \zeta_{\mathcal{F}_n}(\sigma(k)) = \sum_{\mathcal{B} = \{B_1, \ldots, B_l\}} (-1)^{r-l} c(\mathcal{B}) \zeta_{\mathcal{F}_n}(b_1(k)) \cdots \zeta_{\mathcal{F}_n}(b_l(k)), \\
(5.2) \quad & \sum_{\sigma \in \mathfrak{S}_r} \zeta_{\mathcal{F}_n}^*(\sigma(k)) = \sum_{\mathcal{B} = \{B_1, \ldots, B_l\}} c(\mathcal{B}) \zeta_{\mathcal{F}_n}(b_1(k)) \cdots \zeta_{\mathcal{F}_n}(b_l(k)).
\end{align}

Here, $\mathfrak{S}_r$ denotes the symmetric group of degree $r$. For $\sigma \in \mathfrak{S}_r$, set $\sigma(k) := (k_{\sigma(1)}, \ldots, k_{\sigma(r)})$. $\mathcal{B} = \{B_1, \ldots, B_l\}$ runs all partitions of $\{1, \ldots, r\}$, that is, $B = \{B_1, \ldots, B_l\}$ satisfies that $\{1, \ldots, r\} = \bigcup_{i=1}^l B_i$ and $B_i \neq \emptyset$ ($1 \leq i \leq l$). Moreover, we set $c(\mathcal{B}) := (#B_1 - 1)! \cdots (#B_l - 1)!$ and $b_l(k) := \sum_{j \in B_l} k_j$.

Proof. Since $\mathcal{F}_n$-MZVs satisfy the harmonic relation (2.3), we see that the desired formulas hold by the same argument as [12, Theorem 4.1]. \hfill \square

5.2. $\mathcal{F}_n$-sum formula over $I_{k,r}$ for $n = 2, 3$. Next, we prove the $\mathcal{F}_n$-sum formula over $I_{k,r}$ for $n = 2, 3$. For positive integers $n, r$ and $k$ with $r \leq k$, $\mathcal{F} \in \{\mathcal{A}, \mathcal{S}\}$ and $\bullet \in \{\otimes, \star\}$, set

$$S_{\mathcal{F}_n; k,r} := \sum_{k \in I_{k,r}} \zeta_{\mathcal{F}_n}^*(k),$$

where $I_{k,r}$ denotes the set of all indices $k$ with $\text{wt}(k) = k$ and $\text{dep}(k) = r$.

Theorem 5.2 ($\mathcal{F}_n$-sum formula over $I_{k,r}$ for $n = 2, 3$). For positive integers $r$ and $k$ with $r \leq k$, we have

\begin{align}
(5.3) \quad & S_{\mathcal{F}_2; k,r} = (-1)^{r-1} \binom{k}{r} 3_{\mathcal{F}_2}(k+1)x_{\mathcal{F}_2}, \quad S_{\mathcal{F}_2; k,r}^* = \binom{k}{r} 3_{\mathcal{F}_2}(k+1)x_{\mathcal{F}_2}, \\
(5.4) \quad & S_{\mathcal{F}_3; k,r} = (-1)^{k+r-1} \left[ \binom{k}{r} 3_{\mathcal{F}_3}(k+1)x_{\mathcal{F}_3} + \frac{k+1}{2} \binom{k}{r} 3_{\mathcal{F}_3}(k+2) - \frac{1}{r!} \cdot T_{k,r} \right] x_{\mathcal{F}_3}^2
\end{align}

and

\begin{align}
(5.5) \quad & S_{\mathcal{F}_3; k,r}^* = (-1)^k \left[ \binom{k}{r} 3_{\mathcal{F}_3}(k+1)x_{\mathcal{F}_3} + \frac{k+1}{2} \binom{k}{r} 3_{\mathcal{F}_3}(k+2) + \frac{1}{r!} \cdot T_{k,r} \right] x_{\mathcal{F}_3}^2,
\end{align}

where

$$T_{k,r} = \sum_{B_1 \cup B_2 = \{1, \ldots, r\}} \sum_{B_1, B_2 \neq \emptyset} (b_1)_{\# B_1} (b_2)_{\# B_2} \cdot 3_{\mathcal{F}_3}(b_1 + 1) 3_{\mathcal{F}_3}(b_2 + 1)$$

and the symbol $(n)_m$ denotes $n(n-1) \cdots (n-m+1)$. 17
Remark 5.3. If \( k = b_1 + b_2 \) is odd, since \( 3_{F_3}(k+1) \) and \( 3_{F_3}(b_1+1)3_{F_3}(b_2+1) \) are 0, we have

\[
S_{F_3;k,r} = (-1)^r \frac{k+1}{2} \binom{k}{r} 3_{F_3}(k+2)x^2_{F_3}, \quad S^*_{F_3;k,r} = -\frac{k+1}{2} \binom{k}{r} 3_{F_3}(k+2)x^2_{F_3},
\]

which were first proved by the third author and Yamamoto [SY] Theorem 2.5] for \( F = A \).

Proof of Theorem 5.2. Since (5.3) is obtained from (5.4) and (5.5) by taking modulo \( x^2_{F_3} \), it is sufficient to prove (5.4) and (5.5). Note that \((-1)^k 3_{F_2}(k+1)x_{F_3} = 3_{F_2}(k+1)x_{F_2} \) holds because if \( k \) is odd, then \( 3_{F_2}(k+1)x_{F_2} = 0 \).

Let us prove (5.4). By Theorem 3.2, we have

\[
(5.6) \quad \zeta_{F_3}(k) = (-1)^k \left\{ k3_{F_3}(k+1)x_{F_3} + \binom{k+1}{2} 3_{F_3}(k+2)x^2_{F_3} \right\}.
\]

Since \( x^l_{F_3} \) with \( l \geq 3 \) vanishes, by (5.1), we have

\[
S_{F_3;k,r} = \frac{1}{r!} \sum_{k \in I_{k,r}} \sum_{\sigma \in S_r} \zeta_{F_3}(\sigma(k))
\]

\[
= \frac{1}{r!} \sum_{k \in I_{k,r}} \left\{ (-1)^{r-1}(r-1)! \zeta_{F_3}(k)
\right.

\[
+ (-1)^{r-2} \sum_{B_1 \cup B_2 = \{1, \ldots, r\}} \sum_{B_1, B_2 \neq \emptyset} (#B_1 - 1)! (#B_2 - 1)! \zeta_{F_3}(b_1(k)) \zeta_{F_3}(b_2(k)) \right\}.
\]

We calculate the right-hand side. Since \( \#I_{k,r} = \binom{k-1}{r-1} \), by (5.6), we have

\[
\sum_{k \in I_{k,r}} \frac{(-1)^{r-1}}{r} \zeta_{F_3}(k) = (-1)^{k+r-1} \left\{ \binom{k}{r} 3_{F_3}(k+1)x_{F_3} + \frac{k+1}{2} \binom{k}{r} 3_{F_3}(k+2)x^2_{F_3} \right\}.
\]

Furthermore, since \( \#(k_1, \ldots, k_r) \in I_{k,r} \mid \sum_{i \in B_1} k_i = b_1 \) = \( \#I_{b_1, b_1} \cdot \#I_{b_2, b_2} \) for \( B_1, B_2 \neq \emptyset \) with \( B_1 \cup B_2 = \{1, \ldots, r\} \) and \( b_1, b_2 \) with \( b_1 + b_2 = k, b_1 \geq \#B_1, b_2 \geq \#B_2 \),
we have

$$\sum_{k \in I_{k,r}} \sum_{B_1 \cup B_2 = \{1, \ldots, r\}} \sum_{B_1, B_2 \neq \emptyset} \frac{(-1)^k}{\#B_1 - 1}!(\#B_2 - 1)!\zeta_{\mathcal{F}_3}(b_1(k))\zeta_{\mathcal{F}_3}(b_2(k))$$

$$= \sum_{B_1 \cup B_2 = \{1, \ldots, r\}} \sum_{B_1, B_2 \neq \emptyset} \sum_{b_1 + b_2 = k} \sum_{b_1 \geq \#B_1, b_2 \geq \#B_2} \frac{\zeta_{\mathcal{F}_3}(b_1(k))\zeta_{\mathcal{F}_3}(b_2)}{(\#B_1 - 1)! \cdot \#B_2 \cdot 3 \mathcal{F}_3(b_1 + 1) \mathcal{F}_3(b_2 + 1) \mathcal{F}_3^2 \sum_{b = \#B_2} \frac{1}{2}}.$$ 

Note that all terms of $x_{\mathcal{F}_3}$ with $l \geq 3$ for $\mathcal{F}_3$-MZVs vanish. This completes the calculation for (5.4). The formula (5.5) is obtained by a similar calculation using (5.2).

5.3. $\mathcal{F}_2$-sum formula over $I_{k,r,i}$. In this subsection, we prove the $\mathcal{F}_2$-sum formula over $I_{k,r,i}$. For positive integers $k$, $r$, $i$ with $1 \leq i \leq r < k$, let $I_{k,r,i}$ denote the set of indices $k = (k_1, \ldots, k_r)$ with $\text{wt}(k) = k$, $\text{dep}(k) = r$ and $k_i \geq 2$. For $\bullet \in \{\emptyset, \ast\}$ and a positive integer $n$, set

$$S_{\mathcal{F}_n;k,r,i}^\bullet := \sum_{k \in I_{k,r,i}} \zeta_{\mathcal{F}_n}(k).$$

Saito and Wakabayashi [SW1] ($\mathcal{F} = \mathcal{A}$) and Murahara [Mur] ($\mathcal{F} = \mathcal{S}$) proved that

$$S_{\mathcal{F}_1;k,r,i} = (-1)^i \left\{ \binom{k - 1}{i - 1} + (-1)^r \binom{k - 1}{r - i} \right\} 3 \mathcal{F}_1(k),$$

$$S_{\mathcal{F}_1;k,r,i}^\ast = (-1)^i \left\{ \binom{k - 1}{r - i} + (-1)^r \binom{k - 1}{i - 1} \right\} 3 \mathcal{F}_1(k).$$

If $k$ is even, then we have $S_{\mathcal{F}_1;k,r,i} = S_{\mathcal{F}_1;k,r,i}^\ast = 0$ by $3 \mathcal{F}_1(k) = 0$. Thus it is a natural question what is a lifting of $S_{\mathcal{F}_1;k,r,i}^\ast$ to $\mathcal{F}_2$, that is, $S_{\mathcal{F}_2;k,r,i}^\ast$. We give the answer in the following form.

Theorem 5.4 ($\mathcal{F}_2$-sum formula for $I_{k,r,i}$). Let $k$, $r$, $i$ be positive integers with $1 \leq i \leq r < k$ and suppose that $k$ is even. Then we have

$$S_{\mathcal{F}_2;k,r,i} = (-1)^r \frac{b_{k,r,i}}{2} \cdot 3 \mathcal{F}_2(k + 1) \mathcal{F}_2, \quad S_{\mathcal{F}_2;k,r,i}^\ast = \frac{b_{k,r,i}^\ast}{2} \cdot 3 \mathcal{F}_2(k + 1) \mathcal{F}_2,$$

where

$$b_{k,r,i} := \binom{k - 1}{r} + (-1)^{r - i} \left\{ (k - r) \binom{k}{i - 1} + \binom{k - 1}{r - i} + (-1)^{r - 1} \binom{k - 1}{r - i} \right\}.$$
and
\[ b_{k,r,i}^* := \left( \frac{k-1}{r} \right) + (-1)^{i-1} \left\{ (k-r) \left( \frac{k}{r-i} \right) + \left( \frac{k-1}{r-i} \right) + (-1)^{r-1} \left( \frac{k-1}{i-1} \right) \right\}. \]

The case \( \mathcal{F} = \mathcal{A} \) of Theorem 5.4 was proved by the third author and Yamamoto [SY]. In this subsection, we reprove their result and prove the case \( \mathcal{F} = \mathcal{S} \) simultaneously by a different method.

**Lemma 5.5** (Recurrence relations). For positive integers \( k, r, i \) with \( 2 \leq i+1 \leq r \leq k-1 \), we have
\[
(r-i)S_{\mathcal{F}2;k,r,i} + iS_{\mathcal{F}2;k,r,i+1} + (k-r)S_{\mathcal{F}2;k,r-1,i} = \sum_{l=1}^{k-r} \zeta_{\mathcal{F}2}(l)S_{\mathcal{F}2;k-l,r-1,i},
\]
\[
(r-i)S_{\mathcal{F}2;k,r,i}^* + iS_{\mathcal{F}2;k,r,i+1}^* - (k-r)S_{\mathcal{F}2;k,r-1,i}^* = \sum_{l=1}^{k-r} \zeta_{\mathcal{F}2}(l)S_{\mathcal{F}2;k-l,r-1,i}^*.
\]

**Proof.** Let \( \bullet \in \{ \emptyset, \ast \} \). From the same argument in [SW1 Lemma 2.1, Proposition 2.2], we see that the sum of the product
\[
\sum_{(k_1, \ldots, k_{r-1}, l) \in I_{k,r,i}} \zeta_{\mathcal{F}2}(l)\zeta_{\mathcal{F}2}^*(k_1, \ldots, k_{r-1}) = \sum_{l=1}^{k-r} \zeta_{\mathcal{F}2}(l)S_{\mathcal{F}2;k-l,r-1,i}^*
\]
coincides with the left-hand side of the desired recurrence relation by the harmonic relation for \( \mathcal{F}_2 \)-MZVs. \( \square \)

**Corollary 5.6.** If \( k \) is even, then we have
\[
(r-i)S_{\mathcal{F}2;k,r,i} + iS_{\mathcal{F}2;k,r,i+1} + (k-r)S_{\mathcal{F}2;k,r-1,i} = 0,
\]
\[
(r-i)S_{\mathcal{F}2;k,r,i}^* + iS_{\mathcal{F}2;k,r,i+1}^* - (k-r)S_{\mathcal{F}2;k,r-1,i}^* = 0
\]
for positive integers \( k, r, i \) with \( 2 \leq i+1 \leq r \leq k-1 \).

**Proof.** If \( l \) is odd, then \( \zeta_{\mathcal{F}2}(l) = 0 \) by Theorem 3.2 or (3.8). If \( l \) is even, then \( \zeta_{\mathcal{F}2}(l) \) is a multiple of \( x \mathcal{F} \) by Theorem 3.2 or (3.8) and \( S_{\mathcal{F}2;k-l,r-1,i}^* \) is also a multiple of \( x \mathcal{F} \) by Saito–Wakabayashi and Murahara’s sum formulas. \( \square \)

**Proof of Theorem 5.4.** We prove the non-star case by backward induction on \( r \leq k-1 \). Since
\[
b_{k,k-1,i} = \left( \frac{k-1}{k-1} \right) + (-1)^{k-1-i} \left\{ (k-k+1) \left( \frac{k}{i-1} \right) + \left( \frac{k-1}{i-1} \right) + (-1)^{k-2} \left( \frac{k-1}{k-1-i} \right) \right\}
\]
\[= 1 + (-1)^{i-1} \left( \frac{k+1}{i} \right),\]
we have
\[ S_{F_2;k,k-1,i} = \zeta_{F_2} \binom{1}{1^i} 2, \{1\}^{k-i-1} = \frac{1}{2} \left\{ 1 + (-1)^{i-1} \binom{k+1}{i} \right\} 3_{F_2}(k+1)x_{F_2} \]
\[ = \frac{b_{k,k-1,i}}{2} \cdot 3_{F_2}(k+1)x_{F_2}, \]
by the definition of \( S_{F_2;k,r,i} \) and \( \text{[5.22]} \). Hence, the case \( r = k - 1 \) is true. To complete the induction step, by Corollary \( \text{[5.6]} \) it suffices to prove that
\[ (5.7) \]
\[ (r - i)b_{k,r,i} + ib_{k,r,i+1} - (k-r)b_{k,r-1,i} = 0 \]
holds for \( 2 \leq r \leq k - 1 \). The left-hand side of \( \text{(5.7)} \) is
\[ (r - i) \left\{ \binom{k-1}{r} + (-1)^{r-i} \left\{ \binom{k-r}{i} \frac{k}{i-1} \right\} + \binom{k-1}{i-1} \right\} + i \left\{ \binom{k-1}{r} + (-1)^{r-i} \left\{ \binom{k-r}{i} \frac{k}{i} \right\} + \binom{k-1}{i} \right\} \\
- (k-r) \left\{ \binom{k-1}{r-1} + (-1)^{r-i} \left\{ \binom{k-r+1}{i} \frac{k}{i-1} \right\} + \binom{k-1}{i} \right\} \]
by definition. By \( \binom{k-1}{r-1} = \frac{r-k}{r} \binom{k-1}{r} \), we have
\[ (5.8) \]
\[ (r - i) \left( \frac{k-1}{r} \right) + i \left( \frac{k-1}{r} \right) - (k-r) \left( \frac{k-1}{r-1} \right) = 0. \]
By \( \binom{k}{i} = \frac{k-i+1}{i} \binom{k}{i-1} \), we have
\[ (5.9) \]
\[ (r - i)(k-r) \binom{k}{i} - i(k-r)(\frac{k}{i}) + (k-r)(k-r+1)(\frac{k}{i-1}) = 0. \]
By \( \binom{k-1}{i} = \frac{k-i}{i} \binom{k}{i-1} \), we have
\[ (5.10) \]
\[ (r - i) \left( \frac{k-1}{i-1} \right) - i \left( \frac{k-1}{i} \right) + (k-r) \left( \frac{k-1}{i-1} \right) = 0. \]
By \( \binom{k-1}{r-i} = \frac{k-r+i}{r-i} \binom{k-1}{r-i-1} \), we have
\[ (5.11) \]
\[ (r - i) \left( \frac{k-1}{r-i} \right) - i \left( \frac{k-1}{r-i-1} \right) - (k-r) \left( \frac{k-1}{r-i-1} \right) = 0. \]
From \( \text{(5.8)} \), \( \text{(5.9)} \), \( \text{(5.10)} \) and \( \text{(5.11)} \), we obtain \( \text{(5.7)} \) and we complete the proof of the formula for \( S_{F_2;k,r,i} \). In the star case, we should prove
\[ S_{F_2;k,k-1,i}^* = \frac{b_{k,k-1,i}^*}{2} \cdot 3_{F_2}(k+1)x_{F_2} \]
and the recurrence relation

\[(r - i)b^*_k, r, i + ib^*_k, r, i + 1 - (k - r)b^*_k, r - 1, i = 0.\]

These are proved similarly to the non-star case. \[\square\]

**Remark 5.7.** We can also prove the star case by connecting \(S_{F_2; k, r, i}\) and \(S^*_{F_2; k, r, i}\) directly using Proposition 2.5. This is the method used in [SY].

**Appendix A. Proof of equality** [3.26]

In this appendix, we prove the following proposition.

**Proposition A.1.** For non-negative integers \(a\) and \(b\), we have

\[C = 1 + (-1)^a \binom{a + b + 3}{b + 2};\]

see the proof of Theorem 3.13 for the definition of \(C\).

We divide \(C\) into six parts. Set

\[I := (-1)^{a+1} \sum_{l+m=b-1} \sum_{l,m \geq 0} \sum_{r+s=a} (-1)^{s+m} \binom{r + l + 1}{r} \binom{s + m + 1}{s} \binom{a + b + 3}{s + m + 2};\]

\[II := (-1)^{a+1} \sum_{l+m=b-1} \sum_{l,m \geq 0} \sum_{r+s=a} \binom{r + l + 1}{r} \binom{s + m + 1}{s};\]

\[III := 2(-1)^a \sum_{r+s=a} (-1)^s \binom{r + b + 1}{r} \binom{a + b + 3}{s + 1};\]

\[IV := 2(-1)^{a+1} \sum_{r+s=a} \binom{r + b + 1}{r};\]

\[V := \sum_{m+n=a-1} \sum_{m,n \geq 0} (-1)^n \sum_{r+s=n} (-1)^{s+m+1} \binom{r + b + 1}{r} \binom{s + m + 1}{s} \binom{a + b + 3}{s + m + 2};\]

\[VI := \sum_{m+n=a-1} \sum_{m,n \geq 0} (-1)^{n+1} \sum_{r+s=n} \binom{r + b + 1}{r} \binom{s + m + 1}{s} \binom{a + b + 2}{a};\]

Note that we easily obtain

\[(A.1) \quad IV = 2(-1)^{a+1} \binom{a + b + 2}{a}.\]
By the definition of negative binomial coefficients and the Chu–Vandermonde identity, we also have

\[(A.2) \quad II = (-1)^{a+1} \binom{a+b+2}{a}, \quad VI = (-1)^a \binom{a+b+1}{a-1}.\]

Next, we calculate I, III and V. We use the following equality repeatedly:

\[(A.3) \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{x+k} = \frac{n!}{x(x+1) \cdots (x+n)}.\]

Here, \(n\) is a non-negative integer and \(x\) is an indeterminate.

**Lemma A.2.**

\[I = 0.\]

**Proof.** By elementary calculations, we have

\[(A.4) \quad I = (-1)^{a+1} \sum_{m=0}^{b-1} \binom{a+b+2}{a+b-m-1} \binom{s+m+2}{s} = (-1)^a (a+1) \binom{a+b+2}{a+1} \sum_{s=0}^{a} (-1)^s \binom{a}{s} \times \sum_{m=0}^{b-1} (-1)^m \binom{b+1}{m+1} \left( \frac{1}{a+b+1-s-m} + \frac{1}{s+m+2} \right).\]

Applying (A.3) with \(x = -(a+b+2-s)\) and \(x = s+1\), we have

\[(A.5) \quad \sum_{m=0}^{b-1} (-1)^m \binom{b+1}{m+1} \left( \frac{1}{a+b+1-s-m} + \frac{1}{s+m+2} \right) = \sum_{m=0}^{b+1} (-1)^{m-1} \binom{b+1}{m} \left( \frac{1}{a+b+2-s-m} + \frac{1}{s+m+1} \right) + \frac{1}{a+b+2-s} + \frac{1}{s+1} + (-1)^{b+1} \left( \frac{1}{a+1-s} + \frac{1}{s+b+2} \right) = \frac{(-1)^b}{a+b+2-s} \left( \frac{a+b+1-s}{b+1} \right)^{-1} - \frac{1}{s+1} \left( \frac{b+s+2}{b+1} \right)^{-1} + \frac{1}{a+b+2-s} + \frac{1}{s+1} + \frac{(-1)^{b+1}}{a+1-s} + \frac{(-1)^{b+1}}{s+b+2}.\]
By substituting (A.5) into (A.4) and then using (A.3) again, we have

\[
(-1)^{a+b+1} \sum_{s=0}^{a} (-1)^{s} \binom{a+b+2}{a-s} = (-1)^{b+1} \binom{a+b+1}{a},
\]

\[
(-1)^{a} \sum_{s=0}^{a} (-1)^{s} \binom{a+b+2}{a-s} = \sum_{s=0}^{a} (-1)^{s} \binom{a+b+2}{s} = (-1)^{a} \binom{a+b+1}{a},
\]

\[
(-1)^{a+1}(a+1) \binom{a+b+2}{a+1} \sum_{s=0}^{a} (-1)^{s} \binom{a}{s} \frac{1}{a+b+2-s} = -1,
\]

\[
(-1)^{a+1}(a+1) \binom{a+b+2}{a+1} \sum_{s=0}^{a} (-1)^{s} \binom{a}{s} \frac{1}{s+1} = (-1)^{a+1} \binom{a+b+2}{a+1},
\]

\[
(-1)^{a+b}(a+1) \binom{a+b+2}{a+1} \sum_{s=0}^{a} (-1)^{s} \binom{a}{s} \frac{1}{a+1-s} = (-1)^{b} \binom{a+b+2}{a+1},
\]

\[
(-1)^{a+b}(a+1) \binom{a+b+2}{a+1} \sum_{s=0}^{a} (-1)^{s} \binom{a}{s} \frac{1}{s+b+2} = (-1)^{a+b}.
\]

Since \(a + b\) is even, we have the conclusion. \(\square\)

**Lemma A.3.**

\[\text{III} = 2 + 2(-1)^{a} \binom{a+b+2}{a+1}.\]

**Proof.** By (A.3), we have

\[
\text{III} = 2(-1)^{a} \sum_{s=0}^{a} (-1)^{s} \binom{a+b+1-s}{a-s} \binom{a+b+3}{s+1}
\]

\[
= 2(-1)^{a-1}(a+b+3) \binom{a+b+2}{a+1} \left\{ \sum_{s=0}^{a+1} (-1)^{s} \binom{a+1}{s} \frac{1}{a+b+3-s} - \frac{1}{a+b+3} \right\}
\]

\[
= 2(-1)^{a-1}(a+b+3) \binom{a+b+2}{a+1} \left\{ (-1)^{a-1} \binom{a+b+2}{a+1}^{-1} - \frac{1}{a+b+3} \right\}
\]

\[
= 2 + 2(-1)^{a} \binom{a+b+2}{a+1},
\]

which completes the proof. \(\square\)

**Lemma A.4.**

\[V = (-1)^{a} a \binom{a+b+2}{a+1} + (-1)^{a} \binom{a+b+1}{a} - 1.\]
Proof. Since

\[ V = (-1)^a \sum_{n=0}^{a-1} \sum_{s=0}^{n} (-1)^s \left( \frac{b+n+1-s}{b+1} \right) \left( \frac{a+s-n}{s} \right) \left( \frac{a+b+3}{a+s+1-n} \right) \]

\[ = (-1)^a (a+1) \left( \frac{a+b+2}{a+1} \right) \sum_{n=0}^{a-1} \binom{a}{n} \]

\[ \times \sum_{s=0}^{n} (-1)^s \binom{n}{s} \left( \frac{1}{a+s+1-n} + \frac{1}{b+2+n-s} \right) \]

and

\[ \sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{1}{a+s+1-n} = \frac{1}{a+1-n} \frac{(a+1)^{-1}}{n} , \]

\[ \sum_{s=0}^{n} (-1)^s \binom{n}{s} \frac{1}{b+2+n-s} = \frac{(-1)^n}{b+2+n} \frac{(b+n+1)^{-1}}{b+1} \]

hold by (A.3), we obtain the desired formula. □

Proof of Proposition 3.26. From (A.1), (A.2), Lemmas A.2, A.3, and A.4, we obtain the desired formula. □

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