Notes on Backward Stochastic Differential Equations for Computing XVA

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Abstract

The X-valuation adjustment (XVA) problem, which is a recent topic in mathematical finance, is considered and analyzed. First, the basic properties of backward stochastic differential equations (BSDEs) with a random horizon in a progressively enlarged filtration are reviewed. Next, the pricing/hedging problem for defaultable over-the-counter (OTC) derivative securities is described using such BSDEs. An explicit sufficient condition is given to ensure the non-existence of an arbitrage opportunity for both the seller and buyer of the derivative securities. Furthermore, an explicit pricing formula is presented in which XVA is interpreted as approximated correction terms of the theoretical fair price.

Keywords: BSDE, XVA, derivative pricing, defaultable security, arbitrage-free price

1 Introduction

Backward stochastic differential equations (BSDEs) have been studied intensively from both theoretical and application viewpoints. Bismut (1976, 1978) studied BSDEs related to stochastic control problems, and Pardoux and Peng (1990) introduced general nonlinear BSDEs driven by Brownian motion as a noise process. After those early pioneering studies and since the late 1990s, the field of mathematical finance has provided various interesting research topics to develop the theory and application of BSDEs (e.g., El Karoui et al., 2000). In the present paper, we are interested in one such recent research topic in mathematical finance, namely, the X-valuation adjustment (XVA) problem. The pricing and hedging methodology for over-the-counter (OTC) financial derivative securities for practitioners in financial institutions has been modified since the global financial crisis in 2008. The pre-crisis pricing was based on the Black–Scholes–Merton paradigm, and

\[
\text{PRN} := \mathbb{E}[DF_r(T)\xi_T]
\]

was regarded as the “fair” price of the derivative security \((T,\xi_T)\). Here, \(\xi_T\) is a random variable representing the payoff at the maturity date \(T \in \mathbb{R}_+:=(0,\infty)\) of the derivative security,
DF_r(T) := \exp \left\{ - \int_0^T r(u)du \right\} is a suitable discounting factor, where \( r := (r(t))_{t \geq 0} \) is a risk-free interest rate process, and \( \mathbb{E}[\cdot] \) represents the expectation with respect to the so-called risk-neutral probability measure. By contrast, the post-crisis pricing formula used by practitioners in financial institutions is now described as

\[ \bar{p}_{RN} + \sum_x xVA \] (1)

for the derivative security \((T, \xi_T)\). Here,

\[ \bar{p}_{RN} := \mathbb{E}[DF_r(T)\xi_T], \]

employing \( \bar{r} := (\bar{r}(t))_{t \geq 0} \) as a risk-free interest rate process, which is different from \( r \) used in the pre-crisis model, and

\[ \sum_x xVA = CVA - DVA + FVA + ColVA + \cdots \]

represents various valuation adjustments (e.g., credit valuation adjustment, debt valuation adjustment, funding valuation adjustment, collateral valuation adjustment). We may interpret the post-crisis modification as reflecting the following current situations.

(a) The credit risk (default risk) of investors and their counterparties and the liquidity risk (of assets and cash) are widely recognized and now considered seriously.

(b) As a consequence of (a), the differences in various interest rates (e.g., risk-free rate, repo rate, funding rate, collateral rate) can no longer be neglected.

In this paper, we aim to understand the post-crisis pricing formula (1) in a better way from a theoretical viewpoint. Using BSDEs, which model the value processes of hedging portfolios, we interpret (1) as an approximate value of the fair price (i.e., the replication cost) of a derivative security. Concretely, this paper is organized as follows.

- In Section 2, we prepare a BSDE with a random horizon, where two random times \( \tau_1, \tau_2 \) and the progressively enlarged filtration by these random times are introduced, and the horizon is set as \( \tau_1 \wedge \tau_2 \wedge T (T \in \mathbb{R}^{++}) \). We review some basic properties of such a BSDE, that is, the existence of a unique solution and its construction, using a reduced BSDE defined on a smaller filtration (see Theorems 1–3). These results are then used in Section 3.

- In Section 3, we construct a financial market model that generalizes the model given by Bichuch et al. (2018). On it, we derive BSDEs for pricing and hedging derivative securities, which express nonlinear dynamic hedging portfolio values of the seller and buyer. Here, we model the default time of the hedger (i.e., the seller of a derivative security) \( \tau_1 \) and that of her counterparty (i.e., the buyer of the derivative security) \( \tau_2 \), each of which are defined by random times. The contract between the hedger and her counterparty expires if the hedger or the counterparty defaults. Hence, \( \tau_1 \wedge \tau_2 \wedge T \) is interpreted as the (random) horizon of the contract, where \( T \) is the prescribed fixed maturity, and we naturally have BSDEs considered in Section 2.

{\footnotesize 1 The London Interbank Offered Rate (LIBOR) was a popular choice as the risk-free rate in pre-crisis models, whereas the Overnight Index Swap (OIS) rate is now recognized as a suitable candidate as the risk-free rate in post-crisis models.}
• In Section 4, working with the BSDEs introduced in Section 3, we obtain the following.

(i) An explicit sufficient condition is presented to ensure the non-existence of an arbitrage opportunity for both the seller and buyer of the derivative security (see Theorem 4). We note that a rather restrictive condition is necessary to ensure the existence of an arbitrage-free price (see Remark 14).

(ii) The pricing formula (1) used by practitioners is interpreted as an approximation of the theoretical fair price of the derivative security: XVA is regarded as certain “zero-th” order approximated correction terms. (see Theorem 5, Corollary 1, Proposition 3, and Remark 16). Furthermore, we mention a higher first-order approximation (see Subsection 4.3).

We intend to write this paper in an expository manner generally: Section 2 is devoted for reviewing known results and some results in Section 4 (that is, Theorem 4 and Proposition 1 and 2) are rather straightforward extensions of existing results of the closely related work by Bichuch et al. (2015, 2018) and Tanaka (2019). For other parts, we regard the following as being the contributions of the paper in comparison with Bichuch et al. (2015, 2018) and Tanaka (2019).

1) The market model is generalized: our model treats

(i) a multiple risky asset model, and

(ii) a stochastic factor model that includes a stochastic volatility, a stochastic interest rate, and a stochastic hazard rate.

2) Different definitions of arbitrages and admissible trading strategies are employed (see Subsection 3.5). Because we analyze the pricing/hedging problem of derivative securities by using BSDEs, our choices seem to be natural and clear.

3) For XVA, an interpretation of pricing formula (1) is given as well as its arbitrage-free property (see Theorem 5, Corollary 1, and Proposition 3 with the following Remark 16 in Subsection 4.2, and cf. the results in [24]).

4) Regarding the lending-borrowing spreads of interest rates as “small parameters”, the first order perturbed BSDEs are derived and the associated approximated valuation adjustment terms are computed (see Proposition 4 in Subsection 4.3).

2 BSDE with a Random Horizon in a Progressively Enlarged Filtration

2.1 Setup

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space and let \(W := (W(t))_{t \geq 0}, W(t) := (W_1(t), \ldots, W_n(t))^\top\) be an \(n\)-dimensional Brownian motion on it. Define the filtration by

\[ \mathcal{F}_t := \sigma(W(s); s \in [0, t]) \vee \mathcal{N}, \quad t \geq 0, \]
where $\mathcal{N}$ is the totality of null sets. Let $E_1, E_2$ be exponentially distributed random variables, assuming that $W, E_1,$ and $E_2$ are mutually independent. Using nonnegative $\mathcal{F}_t$-progressively measurable processes $h_i := (h_i(t))_{t \geq 0},$ $(i = 1, 2),$ define the random times $\tau_1, \tau_2$ by

$$
\tau_i := \inf \left\{ t \geq 0 \mid \int_0^t h_i(u)du \geq E_i \right\}.
$$

(2)

The indicator processes for $\tau_i$ $(i = 1, 2)$, namely

$$
N_i(t) := 1_{\{t \geq \tau_i\}}, \quad t \geq 0,
$$

are submartingales with respect to the filtration

$$
\mathcal{H}_t := \sigma (N_1(s), N_2(s); \ s \in [0, t]), \quad t \geq 0,
$$

and their Doob–Meyer decompositions are written as

$$
N_i(t) = M_i(t) + \int_0^t \{1 - N_i(s)\} h_i(s)ds, \quad t \geq 0
$$

for $i = 1, 2,$ where

$$
M_i(t) := N_i(t) - \int_0^t \{1 - N_i(s)\} h_i(s)ds, \quad t \geq 0
$$

$(i = 1, 2)$ are two independent martingales with respect to $(\mathcal{H}_t)_{t \geq 0}.$ Moreover, $(W, M_1, M_2)$ remain as martingales with respect to the progressively enlarged filtration,

$$
\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t, \quad t \geq 0
$$

(e.g., see Section 2.3 of Aksamit and Jeanblanc, 2017), which are mutually independent. Also, we deduce that for $0 \leq s \leq t,$

$$
P \left( \tau_i > s \mid \mathcal{F}_t \right) = P \left( \tau_i > s \mid \mathcal{F}_\infty \right) = \exp \left\{ - \int_0^s h_i(u)du \right\},
$$

where $\mathcal{F}_\infty := \sigma (\cup_{t \geq 0} \mathcal{F}_t).$ From this, we see that for $ds \ll 1,$

$$
P \left( \tau_i \leq s + ds \mid \tau_i > s, \mathcal{F}_\infty \right) \approx \frac{P \left( s < \tau_i \leq s + ds \mid \mathcal{F}_\infty \right)}{P \left( \tau_i > s \mid \mathcal{F}_\infty \right)}
$$

$$
= 1 - \exp \left\{ - \int_s^{s+ds} h_i(u)du \right\} \approx h_i(s)ds,
$$

and $h_i$ is called the hazard rate (or intensity) process for $\tau_i.$ Following Pham (2010), we employ the notation below.

**Notation 1.**

- $\mathcal{F} := (\mathcal{F}_t)_{t \geq 0}, \ \mathbb{G} := (\mathcal{G}_t)_{t \geq 0},$ and $\mathbb{H} := (\mathcal{H}_t)_{t \geq 0}.$

- $\mathcal{P}(\mathcal{F})$ (resp. $\mathcal{P}(\mathbb{G})$): $\sigma$-algebra generated by $\mathcal{F}$ (resp. $\mathbb{G}$)-predictable measurable subsets on $\mathbb{R}_+ \times \Omega.$ Equivalently, $\sigma$-algebra on $\mathbb{R}_+ \times \Omega$ generated by $\mathcal{F}$-adapted left-continuous processes.
• $\mathcal{O}(\mathbb{F})$ (resp. $\mathcal{O}(\mathbb{G})$): $\sigma$-algebra generated by $\mathbb{F}$ (resp. $\mathbb{G}$)-optional measurable subsets on $\mathbb{R}_+ \times \Omega$. Equivalently, $\sigma$-algebra on $\mathbb{R}_+ \times \Omega$ generated by $\mathbb{F}$-adapted right-continuous processes.

• $\mathcal{P}_\mathbb{F}$ (resp. $\mathcal{P}_\mathbb{G}$): the space of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable processes.

• $\mathcal{O}_\mathbb{F}$ (resp. $\mathcal{O}_\mathbb{G}$): the space of $\mathbb{F}$ (resp. $\mathbb{G}$)-optional processes.

• $\mathcal{P}_\mathbb{F}^{(k)}$: the space of the parametrized processes, $f : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+^k \ni (t, \omega, u) \mapsto f_t(\omega, u) \in \mathbb{R}$, which is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+^k)/\mathcal{B}(\mathbb{R})$-measurable.

• $\mathcal{O}_\mathbb{F}^{(k)}$: the space of the parametrized processes, $f : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+^k \ni (t, \omega, u) \mapsto f_t(\omega, u) \in \mathbb{R}$, which is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+^k)/\mathcal{B}(\mathbb{R})$-measurable.

• Denote by $\mathcal{P}_{\mathbb{G}, t} := \{ f 1_{[0, t]} | f \in \mathcal{P}_\mathbb{G} \}$, $\mathcal{O}_{\mathbb{G}, t} := \{ f 1_{[0, t]} | f \in \mathcal{O}_\mathbb{G} \}$, $\mathcal{P}_{\mathbb{G}, t}^{(k)} := \{ f(\cdot) 1_{[0, t]} | f \in \mathcal{P}_\mathbb{G}^{(k)} \}$, and $\mathcal{O}_{\mathbb{G}, t}^{(k)} := \{ f(\cdot) 1_{[0, t]} | f \in \mathcal{O}_\mathbb{G}^{(k)} \}$, for example.

We recall the following basic properties of stochastic processes under the progressively enlarged filtration $\mathbb{G}$.

**Lemma 1** (Lemmas 5.1 and 2.1 of Pham, 2010).

1. Any $\mathcal{G}_t$-predictable process $(P(t))_{t \geq 0}$ has the expression that

   $$P(t) = p_0(t)1 \{ t \leq \tau_1 \wedge \tau_2 \} + p_1^1(\tau_1)1 \{ \tau_1 < t \leq \tau_2 \} + p_1^2(\tau_2)1 \{ \tau_2 < t \leq \tau_1 \} + p_t^{1,2}(\tau_1, \tau_2)1 \{ t > \tau_1 \vee \tau_2 \},$$

   where $(p_0(t))_{t \geq 0} \in \mathcal{P}_\mathbb{F}$, $(p_i^j(t))_{t \geq 0} \in \mathcal{P}_{\mathbb{F}, t}^{(1)}$ $(i = 1, 2)$ and $(p_t^{1,2}(.,.))_{t \geq 0} \in \mathcal{P}_{\mathbb{F}}^{(2)}$.

2. Any $\mathcal{G}_t$-optional process $(P(t))_{t \geq 0}$ has the expression that

   $$P(t) = p_0(t)1 \{ t < \tau_1 \wedge \tau_2 \} + p_1^1(\tau_1)1 \{ \tau_1 < t < \tau_2 \} + p_2^1(\tau_2)1 \{ \tau_2 < t < \tau_1 \} + p_t^{1,2}(\tau_1, \tau_2)1 \{ t > \tau_1 \vee \tau_2 \},$$

   where $(p_0(t))_{t \geq 0} \in \mathcal{O}_\mathbb{F}$, $(p_i^j(t))_{t \geq 0} \in \mathcal{O}_{\mathbb{F}, t}^{(1)}$ $(i = 1, 2)$ and $(p_t^{1,2}(.,.))_{t \geq 0} \in \mathcal{O}_{\mathbb{F}}^{(2)}$.

3. Any $\mathcal{G}_t$-measurable random variable $G_t$ has the expression that

   $$G_t = g_0^01 \{ t < \tau_1 \wedge \tau_2 \} + g_1^1(\tau_1)1 \{ \tau_1 < t < \tau_2 \} + g_2^1(\tau_2)1 \{ \tau_2 < t < \tau_1 \} + g_t^{1,2}(\tau_1, \tau_2)1 \{ t > \tau_1 \vee \tau_2 \},$$

   where $g_0^0$ is an $\mathcal{F}_t$-measurable random variable, $(g_i^j(t))_{t \geq 0} \in \mathcal{O}_{\mathbb{F}, t}^{(1)}$ $(i = 1, 2)$, and $(g_t^{1,2}(.,.))_{t \geq 0} \in \mathcal{O}_{\mathbb{F}}^{(2)}$.

Now, on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G})$, we consider the BSDE

$$-dY(t) = f_t(t, Y(t), Z(t), U_1(t), U_2(t)) \, dt$$

$$-Z(t)^T \, dW(t) - U_1(t) \, dM_1(t) - U_2(t) \, dM_2(t),$$

$$t \in [0, \tau_1 \wedge \tau_2 \wedge T],$$

$$Y(\tau_1 \wedge \tau_2 \wedge T) = \phi_1(\tau_1)1 \{ \tau_1 < \tau_2 \wedge T \} + \phi_2(\tau_2)1 \{ \tau_2 < \tau_1 \wedge T \} + \xi_T1 \{ T < \tau_1 \wedge \tau_2 \},$$

where $T \in \mathbb{R}_{++} := (0, \infty)$ is a fixed terminal time, and the following conditions are imposed.
Assumption 1. (i) $\xi_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

(ii) For $i = 1, 2$, $\phi_i \in \mathcal{O}_T^\mathbb{P}$ so that $E \left[ \sup_{t \in [0, T]} |\phi_i(t)|^2 \right] < \infty$.

(iii) $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^2) / \mathcal{B}(\mathbb{R})$-measurable and satisfies, with some positive constant $K_f > 0$,

$$|f(t, y, z, u_1, u_2) - f(t, y', z', u'_1, u'_2)| \leq K_f (|y - y'| + |z - z'| + |u_1 - u'_1| + |u_2 - u'_2|)$$

for all $(y, z, u_1, u_2), (y', z', u'_1, u'_2)$ a.e. $(t, \omega) \in [0, T] \times \Omega$.

(iv) It holds that

$$E \left[ \int_0^T |f(t, 0, 0, 0)|^2 dt \right] < \infty.$$

2.2 Existence, Uniqueness, and Construction of Solution

A specific feature of BSDE (3) is that it has the random time horizon $\tau_1 \wedge \tau_2 \wedge T$, where $\tau_i$ is the (first) jump time for the martingale $M_i (i = 1, 2)$. As for the definition of the solution to such a BSDE, we employ the following (cf. Darling and Pardoux, 1997 as an example of related work).

Definition 1. We call the quadruplet $(Y, Z, U^1, U^2) : [0, T] \times \Omega \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ a solution to BSDE (3) if it satisfies the following conditions.

(a) $Y := (Y(t))_{t \in [0, T]}$ is a $\mathcal{G}$-adapted RCLL (i.e., right continuous and having left limit) process (which is an element of $\mathcal{O}_{\mathcal{G}, T}$), and $(Z, U^1, U^2) \in (\mathcal{P}_{\mathcal{G}, T})^{n+2}$.

(b) For $t \in [0, T]$, it holds that

$$Y(t)1_{\{\tau_1 \wedge \tau_2 \leq t\}} = \{\phi_1(\tau_1)1_{\{\tau_1 < \tau_2\}} + \phi_2(\tau_2)1_{\{\tau_2 < \tau_1\}}\}1_{\{\tau_1 \wedge \tau_2 \leq t\}},$$

$$Z(t)1_{\{\tau_1 \wedge \tau_2 \leq t\}} = 0,$$

$$U_i(t)1_{\{\tau_1 \wedge \tau_2 \leq t\}} = 0, \quad i = 1, 2.$$

(c) For $t \in [0, T]$, it holds that

$$Y(t) = \phi_1(\tau_1)1_{\{\tau_1 < \tau_2, \tau_1 \leq T\}} + \phi_2(\tau_2)1_{\{\tau_2 < \tau_1, \tau_2 \leq T\}} + \xi_T1_{\{\tau_1 \wedge \tau_2 > T\}}$$

$$+ \int_{t \wedge \tau_1 \wedge \tau_2}^T f(s, Y(s), Z(s), U_1(s), U_2(s)) ds$$

$$- \int_{t \wedge \tau_1 \wedge \tau_2}^T \{Z(s)^T dW(s) + U_1(s) dM_1(s) + U_2(s) dM_2(s)\}.$$
Furthermore, we define the following spaces of stochastic processes, namely,

\[ S^2_{\beta,T} := \{ Y \in \mathcal{O}_{G,T} \mid \| Y \|_{\beta,T}^2 < \infty \}, \]

\[ H^{2,d}_{\beta,T} := \{ Z \in (\mathcal{P}_{G,T})^d \mid \| Z \|_{\beta,T}^2 < \infty \}, \]

letting \( \beta \in \mathbb{R} \) and denoting

\[ \| Y \|_{\beta,T}^2 := \mathbb{E} \left[ \int_0^T e^{\beta t} |Y(t)|^2 dt \right]. \]

We then obtain the following.

**Theorem 1.** Under Assumption 1, BSDE \((3)\) admits a unique solution \((Y, Z, U_1, U_2) \in S^2_{\beta,T} \times H^{2,d}_{\beta,T}\) for any sufficiently large \(\beta > 0\).

**Sketch.** The method of proof is standard, although the horizon is random, which is rather “non-standard”. We consider a Picard-type iteration, that is, for a given \((\tilde{Y}, \tilde{Z}, \tilde{U}_1, \tilde{U}_2) \in S^2_{\beta,T} \times H^{2,d}_{\beta,T}\), we construct the solution to BSDE

\[
-dY(t) = f(t, \tilde{Y}(t), \tilde{Z}(t), \tilde{U}_1(t), \tilde{U}_2(t)) dt \\
- Z(t)^\top dW(t) - U_1(t) dM_1(t) - U_2(t) dM_2(t), \\
t \in [0, \tau],
\]

where we denote

\[ \tau_0 := \tau_1 \wedge \tau_2, \quad \tau := \tau_0 \wedge T, \]

\[ \zeta := \phi_1(\tau_1)1_{\{\tau_1 < \tau_2 \wedge T\}} + \phi_2(\tau_2)1_{\{\tau_2 < \tau_1 \wedge T\}} + \xi_T 1_{\{\tau < \tau_1 \wedge \tau_2\}}. \]

Indeed, using the \(G\)-martingale representation

\[
\mathcal{M}(t) := \mathbb{E} \left[ \zeta + \int_0^T f(u, \tilde{Y}(u), \tilde{Z}(u), \tilde{U}_1(u), \tilde{U}_2(u)) du \mid \mathcal{G}_t \right] \\
= \mathbb{E} \left[ \zeta + \int_0^T f(u, \tilde{Y}(u), \tilde{Z}(u), \tilde{U}_1(u), \tilde{U}_2(u)) du \right] \\
+ \int_0^T \phi(u)^\top dW(u) + \int_0^T \psi_1(u) dM_1(u) + \int_0^T \psi_2(u) dM_2(u), \quad t \in [0, T]
\]

for some \((\phi, \psi^1, \psi^2) \in H^{2,n+2}_{\beta,T}\) (e.g., see Section 5.2 of Bielecki and Rutkowski, 2004), we define

\[
\tilde{Y}_t := \mathbb{E} \left[ \zeta + \int_{t \wedge \tau}^T f(u, \tilde{Y}_u, \tilde{Z}_u, \tilde{U}^1_u, \tilde{U}^2_u) du \mid \mathcal{G}_t \right], \quad t \in [0, T],
\]

\[
\tilde{Z} := \phi, \quad \tilde{U}^1 := \psi^1, \quad \tilde{U}^2 := \psi^2.
\]

Note that the martingale \((\mathcal{M}_t)_{t \in [0,T]}\) with respect to the right-continuous filtration \(\mathcal{G}\) admits an RCLL modification. Hence,

\[
\tilde{Y}(t) = \mathcal{M}(t) - \int_0^{t \wedge \tau} f(u, \tilde{Y}(u), \tilde{Z}(u), \tilde{U}_1(u), \tilde{U}_2(u)) du
\]
also admits an RCLL modification, which is denoted by \( \tilde{Y}(t) \) again. Furthermore, we can check the integrability, \( \tilde{Y} \in \mathcal{S}^2_{\beta,T} \). Hence, \((\tilde{Y}, \tilde{Z}, \tilde{U}_1, \tilde{U}_2)\) is the solution to (4). Next, we show that the map
\[
\Psi : \mathcal{S}^2_{\beta,T} \times \mathbb{H}^{2,n+2}_{\beta,T} \ni (\tilde{Y}, \tilde{Z}, \tilde{U}_1, \tilde{U}_2) \mapsto (\tilde{Y}, \tilde{Z}, \tilde{U}_1, \tilde{U}_2) \in \mathcal{S}^2_{\beta,T} \times \mathbb{H}^{2,n+2}_{\beta,T}
\]
is a contraction for sufficiently large \( \beta > 0 \), and using the fixed point theorem for the contraction map, we conclude that the fixed point of the map \( \Psi \) is the solution.

**Remark 1.** We refer to Section 19 of Cohen and Elliott (2015) for the detail of such a Picard-type iteration argument, where a more general semimartingale BSDE (driven by Lévy noise) is treated with a fixed constant time horizon.

Actually, we can construct the solution to BSDE (3) on the filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G})\), using another reduced BSDE on the smaller filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\). Assuming

**Assumption 2.** \( h_i \) \((i = 1, 2)\) are bounded,

we obtain the following.

**Theorem 2.** Under Assumptions 1 and 2, the solution \((Y, Z, U_1, U_2) \in \mathcal{S}^2_{\beta,T} \times \mathbb{H}^{2,n+2}_{\beta,T}\) has the representation that
\[
\begin{align*}
Y(t) &= \tilde{Y}(t)1_{\{0 < t < \tau_1 \land \tau_2 \land T\}} \\
&\quad + \left\{ \phi_1(\tau_1)1_{\{\tau_1 < \tau_2 \land T\}} + \phi_2(\tau_2)1_{\{\tau_2 < \tau_1 \land T\}} + \xi_T1_{\{T < \tau_1 \land \tau_2\}} \right\}1_{\{t = \tau_1 \land \tau_2 \land T\}}, \\
Z(t) &= \tilde{Z}(t), \\
U_i(t) &= \phi_i(t) - \tilde{Y}(t), \quad i = 1, 2.
\end{align*}
\]

Here, \((\tilde{Y}, \tilde{Z}) \in \mathcal{S}^2_{\beta,T} \times \mathbb{H}^{2,n}_{\beta,T}\) is the solution to a BSDE on \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\), namely,
\[
\begin{align*}
d\tilde{Y}(t) &= \tilde{f}(t, \tilde{Y}(t), \tilde{Z}(t)) \, dt - \tilde{Z}(t)^\top \, dW(t), \quad t \in [0, T], \\
Y_T &= \xi_T,
\end{align*}
\]
\[
\tilde{f}(t, y, z) := f(t, y, z, \phi_1(t) - y, \phi_2(t) - y) + \{\phi_1(t) - y\} h_1(t) + \{\phi_2(t) - y\} h_2(t).
\]

**Remark 2.** Similar reduction results for BSDEs (into smaller filtrations) have been studied by Crépey and Song (2016) and Pham (2010) in more-general settings.

**Sketch.** Note that BSDE (3) is rewritten as
\[
\begin{align*}
dY(t) &= \tilde{f}(t, Y(t), Z(t), U_1(t), U_2(t)) \, dt - Z(t)^\top \, dW(t) \quad \text{on} \quad \{0 \leq t \leq \tau_1 \land \tau_2 \land T\}, \\
\Delta Y(t) &= U_1(\tau_1)1_{\{\tau_1 < \tau_2 \land T\}} + U_2(\tau_2)1_{\{\tau_2 < \tau_1 \land T\}}, \\
Y(t) &= \phi_1(\tau_1)1_{\{\tau_1 < \tau_2 \land T\}} + \phi_2(\tau_2)1_{\{\tau_2 < \tau_1 \land T\}} + F_T1_{\{T < \tau_1 \land \tau_2\}} \quad \text{on} \quad \{t = \tau_1 \land \tau_2 \land T\},
\end{align*}
\]
where we use $\Delta Y(t) := Y(t) - Y(t-)\,$ and

$$
\hat{f}(t, y, z, u_1, u_2) = f(t, y, z, u_1, u_2) + u_1 h_1(t) + u_2 h_2(t).
$$

We show that if we define $(Y, Z, U^1, U^2)$ by (5), then it actually satisfies (7). First, we see that BSDE (6) on $(\Omega, \mathcal{F}, \mathbb{P}, F)$ has a unique solution $(\hat{Y}, \hat{Z}) \in S^{2}_{\beta,T} \times H^{2, n}_{\beta, T}$ for any sufficiently large $\beta > 0$, recalling that $\hat{f}$ is a standard driver (e.g., $\hat{f}(t, y, z)$ satisfies a globally Lipschitz condition with respect to $(y, z)$). Next, we can check that (5) indeed satisfies (7); for example, on $\{t = \tau_1 \wedge \tau_2 \wedge T\}$,

$$
\Delta Y(t) = \phi_1(\tau_1) 1_{\{\tau_1 < \tau_2 \wedge T\}} + \phi_2(\tau_2) 1_{\{\tau_2 < \tau_1 \wedge T\}} + \xi_T 1_{\{T < \tau_1 \wedge \tau_2\}} - \hat{Y}(t-)
$$

$$
= \phi_1(\tau_1) 1_{\{\tau_1 < \tau_2 \wedge T\}} + \phi_2(\tau_2) 1_{\{\tau_2 < \tau_1 \wedge T\}} + \xi_T 1_{\{T < \tau_1 \wedge \tau_2\}}
$$

$$
- (\hat{Y}(\tau_1 \wedge \tau_2) 1_{\{\tau_1 \wedge \tau_2 \leq T\}} + \xi_T 1_{\{\tau_1 \wedge \tau_2 > T\}})
$$

$$
= U_1(\tau_1) 1_{\{\tau_1 < \tau_2 \wedge T\}} + U_2(\tau_1) 1_{\{\tau_2 < \tau_1 \wedge T\}}.
$$

Hence, the desired assertion follows as it is easy to see the integrabilities given by (5), $(Y, Z, U^1, U^2) \in S^{2}_{\beta,T} \times H^{2, n}_{\beta, T}$. \hfill \square

**Remark 3.** We impose Assumption 2 to simplify the statement of Theorem 2. We can relax it by employing a different solution space (from $S^{2}_{\beta,T} \times H^{2, n}_{\beta, T}$) associated with the so-called stochastic Lipschitz BSDEs. For the study of such BSDEs, see El Karoui and Huang (1997) and Nagayama (2019), for example.

### 2.3 Markovian Model

When we treat BSDE (3) in a practical application, more-concrete modeling is preferable: In this subsection, we consider BSDE (3) under Assumptions 1 and 2 and the following setting.

(i) There is a Markovian state variable process $X := (X(t))_{t \geq 0}$, which is governed by the following Markovian forward stochastic differential equation (FSDE), namely,

$$
dX(t) = b(t, X(t))dt + a(t, X(t))dW(t), \quad X(0) \in \mathbb{R}^d, \tag{8}
$$

on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$, where $a : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times n}$ and $b : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$.

(ii) $h_i(t) := \tilde{h}_i(X(t)), i = 1, 2$, where $\tilde{h}_i : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is bounded.

(iii) The driver $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^2 \rightarrow \mathbb{R}$ of BSDE (3) is written as

$$
f(t, \omega, y, z, u_1, u_2) := g(t, X(t, \omega), y, z, u_1, u_2),
$$

where $g : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

(iv) $\xi_T := \Xi(X(T)), \quad \Xi : \mathbb{R}^d \rightarrow \mathbb{R}$.

(v) $\phi_i(t) := \varphi_i(X(t)), \quad i = 1, 2$, where $\varphi_i : \mathbb{R}^d \rightarrow \mathbb{R}$.

In this case, the solution to BSDE (3) can be constructed as follows using the solution to a second-order parabolic semilinear partial differential equation (PDE).
Theorem 3. Consider the second-order parabolic semilinear PDE
\[-\partial_t V(t,x) = L_t V(t,x) + g(t,x,V(t,x),a(t,x)\nabla V(t,x)) , \quad (t,x) \in [0,T) \times \mathbb{R}^d,\]

\[V(T,x) = \Xi(x),\]  \hspace{1cm} (9)

where
\[L_t V := \frac{1}{2} \text{tr} \left(aa^\top (t,\cdot) \nabla \nabla V \right) + b^\top (t,\cdot) \nabla V \]  \hspace{1cm} (10)

is the infinitesimal generator for X with the gradient \(\nabla V := (\partial_{x_1} V, ..., \partial_{x_d} V)^\top\) and the Hessian matrix \(\nabla \nabla V := (\partial^2_{x_i x_j} V)_{1 \leq i,j \leq d}\), and
\[\bar{g}(t,x,y,z) := g(t,x,y,z,\varphi_1(x) - y, \varphi_2(x) - z) + \sum_{i=1}^{2} \{\varphi_i(x) - y\} \tilde{h}_i(x).\]

Suppose that there exists a unique classical solution \(V \in C^{1,2}([0,T] \times \mathbb{R}^d)\) to (9). Then, the solution to BSDE (3) is represented as
\[Y(t) = V(t,X(t)) 1_{0 \leq t < \tau_1 \land \tau_2 \land T} + \left\{ \varphi_1 (X(\tau_1)) 1_{\tau_1 < \tau_2 \land T} + \varphi_2 (X(\tau_2)) 1_{\tau_2 < \tau_1 \land T} + \Xi (X(T)) 1_{T \land \tau_1 \land \tau_2} \right\} 1_{t=\tau_1 \land \tau_2 \land T},\]
\[Z(t) = a(t,X(t))^\top \nabla V (t,X(t)) , \quad U_i(t) = \varphi_i (X(t)) - V (t,X(t)), \quad i = 1, 2.\]

Sketch. Associated with BSDE (6), we consider the (decoupled) forward-backward stochastic differential equation (FBSDE)
\[dX(t) = b(t,X(t)) dt + a(t,X(t)) dW(t), \quad X(0) \in \mathbb{R}^d,\]
\[-dY(t) = \bar{g}(t,X(t),\bar{Y}(t),\bar{Z}(t)) dt - \bar{Z}(t)^\top dW(t), \quad \bar{Y}(T) = \Xi (X(T)).\]  \hspace{1cm} (11)

By the nonlinear Feynman–Kac formula (e.g., see El Karoui et al., 2000 or Zhang, 2017), the solution to (11) is expressed as
\[\bar{Y}(t) := V(t,X(t)) , \quad \bar{Z}(t) := a(t,X(t))^\top \nabla V (t,X(t)), \quad t \in [0,T].\]

The desired assertion follows by using Theorem 2. \(\square\)

Remark 4. In the study of credit risk modeling in mathematical finance, similar techniques, namely the reduction of a BSDE (onto a Brownian filtration) combined with the (nonlinear) Feynman–Kac formula, have been utilized: see Bichuch et al. (2015), Bielecki et al. (2005), and Crépey (2015), for example.
3 XVA Calculation via BSDE

In this section, we introduce a “post-crisis” financial market model and a hedger’s model for pricing OTC financial derivative securities, which generalize those employed by Bichuch et al. (2015, 2018) and Tanaka (2019). We then derive BSDEs that describe the self-financing hedging portfolio values of the hedger (seller) and her counterparty (buyer). After preparing mathematical models of a financial market, a hedger, and her counterparty, we formulate hedging problems and give the definition of the arbitrage-free price of a derivative security. Throughout this section, we continue to use the mathematical setup introduced in Section 2.

3.1 Non-defaultable/Defaultable Risky Assets

Let \( T \in \mathbb{R}_{++} \) be a fixed time horizon, and consider a frictionless financial market model in continuous time. In it, there are price processes of \( n \) non-defaultable risky assets \( S := (S_1, \ldots, S_n)^\top \), \( S_i := (S_i(t))_{t \in [0,T]} \), one defaultable risky asset \( P_I := (P_I(t))_{t \in [0,T]} \) issued by an investor’s firm, and one defaultable risky asset \( P_C := (P_C(t))_{t \in [0,T]} \) issued by the firm of a counterparty of the investor. They are governed by the following stochastic differential equations (SDEs) on \((\Omega, \mathcal{F}, \mathbb{P})\):

\[
\begin{align*}
    dS(t) &= \text{diag}(S(t)) \{ \sigma(t)dW(t) + r_D(t)1dt \}, \quad S(0) \in \mathbb{R}^n_{++}, \\
    dP_I(t) &= P_I(t-1) \{ \sigma_I(t)dW(t) - dM_1(t) + r_D(t)dt \}, \quad P_I(0) \in \mathbb{R}_{++}, \\
    dP_C(t) &= P_C(t-1) \{ \sigma_C(t)dW(t) - dM_2(t) + r_D(t)dt \}, \quad P_C(0) \in \mathbb{R}_{++}.
\end{align*}
\]

Here, \( \sigma \in (\mathcal{P}_{\mathbb{F},T})^{n \times n}, \sigma_i \in (\mathcal{P}_{\mathbb{F},T})^{1 \times n}, i \in \{I, C\}, \) and \( r_D \in \mathcal{P}_{\mathbb{F},T}, \) which are assumed to be bounded, and \( \sigma(t, \omega) \) is invertible for a.e. \((t, \omega) \in [0,T] \times \Omega.\) Furthermore, we denote \( \text{diag}(x) = (x_i\delta_{ij})_{1 \leq i, j \leq n} \)

\[
\text{for } x := (x_1, \ldots, x_n)^\top \in \mathbb{R}^n \text{ and } 1 := (1, \ldots, 1)^\top \in \mathbb{R}^n.
\]

**Remark 5.** We regard the process \( r_D \) as the risk-free interest rate process in the market, which does not contain credit risk spread\(^2\). Define the cash account process \( B_D := (B_D(t))_{t \geq 0} \) associated with the risk-free rate \( r_D \) by

\[
dB_D(t) = B_D(t)r_D(t)dt, \quad B_D(0) = 1,
\]

or equivalently

\[
B_D(t) = \exp \left\{ \int_0^t r_D(u) du \right\}.
\]

We then see that

\[
\frac{S_i}{B_D}, \quad i = 1, \ldots, n, \quad \frac{P_j}{B_D}, \quad j = 1, 2
\]

are \( \mathcal{G} \)-local martingales. These mean that we are starting with the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a risk-neutral (pricing) probability \( \mathbb{P} \), not with the real-world (physical) probability.

\(^2\) A typical example of such an interest rate in a real financial market is the OIS rate.

\(^3\) More precisely, \( \mathbb{P} \) is an equivalent martingale measure (EMM). See Remark 13 in Subsection 3.5.
The random times $\tau_1$ and $\tau_2$ defined by (2) are interpreted as the default times of the investor who issues $P_I$ and the counterparty who issues $P_C$, respectively. We solve (13) as

$$P_I(t) = P_I(0) \exp \left[ \int_0^t \sigma_I(u) dW(u) + \int_0^t \left( r_D(u) + h_1(u) - \frac{1}{2} |\sigma_I(u)|^2 \right) du \right] \{ 1 - N_1(t) \},$$

for example. Recall that the price becomes zero when defaults occur, i.e., $P_I(\tau_1) = 0$.

**Remark 6.** As concrete examples of $P_I$ and $P_C$, we can consider T-maturity zero coupon bonds without recoveries, namely

$$P_I(t) = \mathbb{E} \left[ \exp \left\{ - \int_t^T (r_D(u) + h_1(u)) du \right\} \big| \mathcal{F}_t \right] \{ 1 - N_1(t) \},$$

$$P_C(t) = \mathbb{E} \left[ \exp \left\{ - \int_t^T (r_D(u) + h_2(u)) du \right\} \big| \mathcal{F}_t \right] \{ 1 - N_2(t) \}.$$
Remark 7. A typical example of the payoff \((\xi_T, \phi_1, \phi_2)\) is

\[
\xi_T := h \left( (S(t))_{t \in [0,T]} \right)
\]

with \(h : C([0, T], \mathbb{R}^n_{++}) \rightarrow \mathbb{R}\) and, for \(i = 1, 2,\)

\[
\phi_i(t) := \varphi_i \left( \tilde{V}(t) \right)
\]

with some nonlinear (piecewise-linear) \(\varphi_i : \mathbb{R} \rightarrow \mathbb{R}\) and

\[
\tilde{V}(t) := \mathbb{E} \left[ \exp \left\{ \int_t^T r_D(u) du \right\} \xi_T \mid \mathcal{F}_t \right], \quad t \in [0, T]. \quad (16)
\]

(16) is interpreted as the reference value process of the derivative \((T, \xi_T)\) with the payoff \(\xi_T\) at the maturity \(T\) in a default-free market. In Bichuch et al. (2018),

\[
\phi_1(v) := v - L_I (v - \alpha v)^+ \quad \text{and} \quad \phi_2(v) := v + L_C (v - \alpha v)^-
\]

are employed, where \(x^+ := \max(x, 0), \ x^- := \max(-x, 0) = -\min(x, 0), 0 \leq L_I, L_C, \alpha \leq 1.\) The constant \(L_I\) (resp. \(L_C\)) is called the loss rate upon default of the investor (resp. the counterparty), and \(\alpha\) is called the collateralization level. For a more detailed explanation, see Sections 3.2 and 3.4 of Bichuch et al. (2018).

### 3.3 Dynamic Portfolio Strategy

For hedging purposes, the writer (seller) of the derivative security given in Definition 2 constructs a dynamic portfolio, which is denoted by \((\pi, \pi_I, \pi_C, \pi_f, \pi_r, \pi_{col})\). Here,

\[
\pi := (\pi_1, \ldots, \pi_n) \in (\mathcal{P}_G, T)^n, \quad \pi_j := (\pi_j(t))_{t \in [0, T]}
\]

is an investment strategy for the risky assets \(S := (S_1, \ldots, S_n)\),

\[
\pi^j := (\pi^j(t))_{t \in [0, T]} \in \mathcal{P}_G, j \in \{I, C\}
\]

are investment strategies for the risky assets \(P_I\) and \(P_C\), respectively, and

\[
\pi^j := (\pi^j(t))_{t \in [0, T]} \in \mathcal{P}_G, j \in \{f, r, col\}
\]

are investment strategies for the cash accounts \(B_f, B_r,\) and \(B_{col}\), which are called the funding account, the repo account, and the collateral account, respectively. They are defined by

\[
dB_j(t) = B_j(t) \left\{ r^-_j(t) 1_{\{\pi^j(t) < 0\}} + r^+_j(t) 1_{\{\pi^j(t) > 0\}} \right\} dt, \quad B_j(0) = 1 \quad (18)
\]

with \(r^-_j := (r^-_j(t))_{t \in [0, T]} \in \mathcal{P}_{F, T}, r^+_j := (r^+_j(t))_{t \in [0, T]} \in \mathcal{P}_{F, T},\) and \(j \in \{f, r, col\},\) where \(r^-_f, r^+_r\) and \(r^\pm_{col}\) are called the funding rate, the repo rate, and the collateral rate, respectively.
Remark 8. The cash account process $B_f$ represents the cumulative amount of cash that the hedger borrows from (or lends to) her treasury desk. The rate $r_f^-$ is called the funding borrowing rate and the rate $r_f^+$ is called the funding lending rate. The cash account process $B_r$ represents the cumulative amount of cash that the investor borrows from (or lends to) a repo market. The rate $r_r^-$ is called the repo borrowing rate, which is applied when the hedger borrows money from the repo market and implements a long position for the non-defaultable risky assets $S$. The rate $r_r^+$ is called the repo lending rate, which is applied when the hedger lends money to the repo market and implements a short-selling position for the non-defaultable risky assets $S$. The cash account process $B_{col}$ represents the cumulative amount of cash that the investor receives from (or posts to) the counterparty as the collateral of the derivative security. The rate $r_{col}^-$ is paid by the hedger to the counterparty if he/she has received the collateral. The rates $r_{col}^+$ is received by the hedger if he/she has posted the collateral. These rates can differ because different markets may be used to determine the contractual rates earned by cash collateral.

For $r_j^\pm$ and $r_{col}^\pm$, it is natural and realistic to assume that

$$2\epsilon_j \equiv r_j^- - r_j^+ \geq 0 \text{ for } j \in \{f, r\}.$$  \hspace{1cm} (19)

For $j \in \{f, r\}$, denoting the “mid-rate” by

$$r_j^0 \equiv \frac{r_j^- + r_j^+}{2},$$

we see that

$$r_j^\pm \equiv r_j^0 \mp \epsilon_j.$$

The value process $Y := (Y(t))_{t \in [0, T]}$ associated with a given dynamic portfolio strategy $(\pi, \pi^I, \pi^C, \pi^f, \pi^r, \pi^{col})$ is governed by an SDE on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G})$, namely,

$$dY(t) = \pi(t)^\top dS(t) + \pi^I(t)dP_I(t) + \pi^C(t)dP_C(t)$$

$$+ \pi^f(t)dB_f(t) + \pi^r(t)dB_r(t) + \pi^{col}(t)dB_{col}(t),$$

$$Y(0) = y,$$ \hspace{1cm} (20)

subject to

$$Y(t) = \pi(t)^\top S(t) + \pi^I(t)P_I(t) + \pi^C(t)P_C(t)$$

$$+ \pi^f(t)B_f(t) + \pi^r(t)B_r(t) + \pi^{col}(t)B_{col}(t),$$ \hspace{1cm} (21)

$$\pi(t)^\top S(t) + \pi^I(t)B_r(t) = 0,$$ \hspace{1cm} (22)

$$\pi^{col}(t)B_{col}(t) - \alpha \hat{V}(t) = 0.$$ \hspace{1cm} (23)

Here, (21) corresponds to the so-called self-financing condition, (22) implies that the hedger accesses the repo market to purchase/sell non-defaultable risky assets (stocks), and (23) implies that $\alpha \hat{V}(t)$

\[\text{For example, the choice of currency (USD, Euro, etc.). We refer the interested reader to Fujii and Takahashi (2011), where the impact of the choice of currency of collateral is studied.}\]
is regarded as the collateral value at time \( t \), where \( \alpha \in [0, 1] \) is the collateral level, which is the same as the one given in Remark 7. From (21)–(23), recall that the relations

\[
\pi^r(t) = -B_r(t)^{-1}\pi(t)^\top S(t),
\]

\[
\pi^{col}(t) = B_{col}(t)^{-1} \alpha \hat{V}(t),
\]

\[
\pi^f(t) = B_f(t)^{-1} \left\{ Y(t) - \pi^l(t)P_I(t-) - \pi^C(t)PC(t-) - \alpha \hat{V}(t) \right\}
\]

hold. Hence, we can interpret that \((y, \Pi) \in \mathbb{R} \times (\mathcal{P}_{G,T})^{n+2} \), where \( \Pi := (\pi, \pi^l, \pi^C) \), is a portfolio strategy that determines the portfolio value process (20), and we sometimes write

\[
Y := Y^{(y,\Pi)},
\]

emphasizing the portfolio strategy \((y, \Pi)\). Combining (20) with (12)–(14), (18), and (24)–(26), we see that

\[
dY(t) = \pi(t)^\top \text{diag}(S(t)) \{ \sigma(t)dW(t) + \{ r_D(t) - r_r(t; \pi^r(t)) \} dt \}
\]

\[
+ \pi^l(t)P_I(t-) \{ \sigma_I(t)dW(t) - dM_1(t) + \{ r_D(t) - r_f(t; \pi^f(t)) \} dt \}
\]

\[
+ \pi^C(t)PC(t-) \{ \sigma_C(t)dW(t) - dM_2(t) + \{ r_D(t) - r_f(t; \pi^f(t)) \} dt \}
\]

\[
+ \left\{ Y(t) - \alpha \hat{V}(t) \right\} [ r_f(t; \pi^f(t))dt + \alpha \hat{V}(t)r_{col}(t; \pi^{col}(t))dt,
\]

where we denote

\[
 r_j(t; p) := r_j^-(t)1_{\{ p < 0 \}} + r_j^+(t)1_{\{ p > 0 \}}, \quad j \in \{ f, r, col \}.
\]

**Remark 9.** Suppose that \( r_D \equiv r^+_f \equiv r^+_r \equiv r^+_{col} \). Then (27) becomes

\[
dY(t) = \pi(t)^\top \text{diag}(S(t)) \{ \sigma(t)dW(t) + \pi^l(t)P_I(t-) \{ \sigma_I(t)dW(t) - dM_1(t) \}
\]

\[
+ \pi^C(t)PC(t-) \{ \sigma_C(t)dW(t) - dM_2(t) \} + r_D(t)Y(t)dt,
\]

which is solved as

\[
Y^{(y,\Pi)}(t) = B_D(t) \left[ y + \int_0^t B_D(s)^{-1} \pi(s)^\top \text{diag}(S(s))\sigma(s)\,dW(s)
\right.
\]

\[
+ \int_0^t B_D(s)^{-1} \pi^l(s)P_I(s-) \{ \sigma_I(s)dW(s) - dM_1(s) \}
\]

\[
+ \int_0^t B_D(s)^{-1} \pi^C(s)PC(s-) \{ \sigma_C(s)dW(s) - dM_2(s) \}
\]

\[
\left. + \int_0^t B_D(s)^{-1} \pi^C(s)PC(s-) \{ \sigma_C(s)dW(s) - dM_2(s) \} \right].
\]

That is, the discounted value process \( Y/B_D \) is a local martingale, which is a standard result shared in a classical framework with “one risk-free rate world.”

For the derivative security given in Definition 2, we call the portfolio strategy \((\hat{y}, \hat{\Pi}) \in \mathbb{R} \times (\mathcal{P}_{G,T})^{n+2} \) that satisfies

\[
Y^{(\hat{y},\hat{\Pi})}_{\tau_1 \land \tau_2 \land T} = H
\]

(29)
the replicating portfolio strategy for the hedger.

Furthermore, for pricing purposes, we next consider a dynamic portfolio strategy \((-\tilde{\pi}, -\tilde{\pi}^f, -\tilde{\pi}^C, \tilde{\pi}^r, \tilde{\pi}_{col})\) and the associated value process \(\hat{Y}\) of the buyer (counterparty). We define

\[
d\hat{Y}(t) = -\tilde{\pi}(t)\hat{S}(t) - \tilde{\pi}^f(t)\hat{P}_f(t) - \tilde{\pi}^C(t)\hat{P}_C(t)
+ \tilde{\pi}^r(t)\hat{B}_r(t) + \tilde{\pi}_{col}(t)\hat{B}_{col}(t),
\]

subject to

\[
\hat{Y}(0) = -\tilde{y}
\]
\[
\hat{Y}(t) = -\tilde{\pi}(t)\hat{S}(t) - \tilde{\pi}^f(t)\hat{P}_f(t) - \tilde{\pi}^C(t)\hat{P}_C(t)
+ \tilde{\pi}^r(t)\hat{B}_r(t) + \tilde{\pi}_{col}(t)\hat{B}_{col}(t),
\]

\[
-\tilde{\pi}(t)\hat{S}(t) + \tilde{\pi}^r(t)\hat{B}_r(t) = 0,
\]

\[
\tilde{\pi}_{col}(t)\hat{B}_{col}(t) + \alpha\hat{V}(t) = 0,
\]

where \(\tilde{\pi} \in (P_{G,T})^n\) and \(\tilde{\pi}^i \in P_{G,T}\) for \(i \in \{I, C, f, r, col\}\). Here, as we see in (32), the collateral value at time \(t\) is regarded as \(-\alpha\hat{V}(t)\), the opposite value of that for the writer (hedger). Because we see that

\[
\tilde{\pi}^r(t) = \hat{B}_r(t)^{-1}\tilde{\pi}(t)\hat{S}(t),
\]

\[
\tilde{\pi}_{col}(t) = -\hat{B}_{col}(t)^{-1}\alpha\hat{V}(t),
\]

\[
\tilde{\pi}^f(t) = \hat{B}_f(t)^{-1}\left\{\hat{Y}(t-\hat{B}_f(t)^{-1})(t-\hat{B}_f(t)^{-1}) + \tilde{\pi}^C(t)\hat{P}_C(t-\hat{B}_f(t)^{-1}) + \alpha\hat{V}(t)\right\}
\]

from (30)-(32), we regard \((-\tilde{y}, -\bar{\Pi}) \in \mathbb{R} \times (P_{G,T})^{n+2}\) with \(\bar{\Pi} := (\tilde{\pi}, \tilde{\pi}^f, \tilde{\pi}^C)\) as the portfolio strategy, and we rewrite the dynamics of \(\hat{Y} := \hat{Y}(-\tilde{y}, -\bar{\Pi})\) as

\[
d\hat{Y}(t) = -\tilde{\pi}(t)^\top\text{diag}(\hat{S}(t))\left[\sigma(t)dW(t) + \{r_D(t) - r_r(t; \pi^r(t))\} dt\right]
- \hat{\pi}^r(t)\hat{P}_f(t-\hat{B}_f(t)^{-1})\left[\sigma_i(t)dW(t) - dM_1(t) + \{r_D(t) - r_f(t; \pi^f(t))\} dt\right]
- \hat{\pi}^C(t)\hat{P}_C(t-\hat{B}_f(t)^{-1})\left[\sigma_C(t)dW(t) - dM_2(t) + \{r_D(t) - r_f(t; \pi^f(t))\} dt\right]
+ \left\{\hat{Y}(t) + \alpha\hat{V}(t)\right\} r_f(t; \pi^f(t)) dt - \alpha\hat{V}(t)r_{col}(t; \pi_{col}(t)) dt.
\]

**Remark 10.** We have assumed that the funding rate \(r_{f,1}^+\) for the investor (writer) and the funding rate \(r_{f,1}^\pm\) for the counterparty (buyer) are identical, i.e., \(r_f^+ \equiv r_{f,1}^+ \equiv r_{f,1}^\pm\), which is a restrictive situation. However, without such an assumption, it looks difficult and complicated to derive an explicit sufficient condition to ensure the no-arbitrage property (see Theorem 4 and its proof).

**Remark 11.** Suppose that \(r_D \equiv r_D^+ \equiv r_{D}^\pm\). Using a similar calculation to that in Remark 9, we solve (33) to see that \(\hat{Y}(-\tilde{y}, -\bar{\Pi}) = -Y(y, \bar{\Pi})\), where the right-hand side \(Y(y, \bar{\Pi})\) is given by (28) by letting \(y := y'\) and \(\Pi := \bar{\Pi}\).

If the portfolio strategy \((-\tilde{y}, -\bar{\Pi}) \in \mathbb{R} \times (P_{G,T})^{n+2}\) satisfies

\[
\hat{Y}(-\tilde{y}, -\bar{\Pi}) = -H
\]

for the derivative security given in Definition 2, then we call it the replicating portfolio strategy for the buyer.
3.4 Deriving BSDE

The replicating portfolio $\tilde{Y}, \tilde{I}$ that satisfies (29) is represented using the solution to a BSDE. Let

$$Y^+ := Y^{(\hat{y}, \hat{I})},$$
$$U^+_1(t) := - \pi^I(t)P_I(t-),$$
$$U^+_2(t) := - \pi^C(t)P_C(t-),$$
$$Z^+(t) := \sigma(t) \tau \text{diag}(S(t)) \pi(t) - U^+_1(t) \sigma_I(t) - U^+_2(t) \sigma_C(t).$$

Recalling

$$\pi^I(t)B_I(t) = Y^+(t) + U^+_1(t) + U^+_2(t) - \alpha \hat{V}(t),$$

we see that $\pi^I(t) \geq 0$ (resp. $\leq 0$) is equivalent to

$$Y^+(t) + U^+_1(t) + U^+_2(t) - \alpha \hat{V}(t) \geq 0, \text{ (resp. } \leq 0).$$

Also, recalling

$$-\pi^C(t)B_C(t) = \pi(t) \text{diag}(S(t)) 1$$
$$= \{Z^+(t) + U^+_1(t) \sigma_I(t) + U^+_2(t) \sigma_C(t)\} \sigma(t)^{-1} 1,$$

we see that $\pi^C(t) \geq 0$ (resp. $\leq 0$) is equivalent to

$$\{Z^+(t) + U^+_1(t) \sigma_I(t) + U^+_2(t) \sigma_C(t)\} \sigma(t)^{-1} 1 \leq 0 \text{ (resp. } \geq 0).$$

Using these relations, we then rewrite (27) as

$$dY^+(t) = -f^+(t, Y^+(t), Z^+(t), U^+_1(t), U^+_2(t); \hat{V}(t)) dt$$
$$+ Z^+(t) \tau dW(t) + U^+_1(t) dM_1(t) + U^+_2(t) dM_2(t),$$

where

$$f^+(t, y, z, u_1, u_2; \hat{v}) := f^0(t, y, z, u_1, u_2) + \alpha \left\{ r_0^I(t) \hat{v} - r_0^C(t) \hat{v} + r_0^1(t) \hat{v} + r_0^2(t) \hat{v} \right\}$$
$$+ \epsilon_f(t) \left\{ y + u_1 + u_2 - \alpha \hat{v} \right\}$$
$$+ \epsilon_r(t) \left\{ \{ z \tau + u_1 \sigma_I(t) + u_2 \sigma_C(t) \} \sigma(t)^{-1} 1 \right\},$$

with

$$f^0(t, y, z, u_1, u_2) := -r_0^0(t)y + \left\{ r_0^0(t) - r_0^D(t) \right\} z \tau \sigma(t)^{-1} 1$$
$$+ \left[ - \left\{ r_0^0(t) - r_0^D(t) \right\} + \left\{ r_0^0(t) - r_0^D(t) \right\} \sigma_I(t) \sigma(t)^{-1} 1 \right] u_1$$
$$+ \left[ - \left\{ r_0^0(t) - r_0^D(t) \right\} + \left\{ r_0^0(t) - r_0^D(t) \right\} \sigma_C(t) \sigma(t)^{-1} 1 \right] u_2.$$

So, we consider the BSDE on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G})$, namely

$$-dY^+(t) = f^+(t, Y^+(t), Z^+(t), U^+_1(t), U^+_2(t); \hat{V}(t)) dt$$
$$- Z^+(t) \tau dW(t) - U^+_1(t) dM_1(t) - U^+_2(t) dM_2(t)$$

for $0 \leq t \leq \tau_1 \wedge \tau_2 \wedge T,$

$$Y^+(\tau_1 \wedge \tau_2 \wedge T) = H.$$
Using the solution to (37), the replicating portfolio \( \hat{y}, \hat{\Pi} \) that satisfies (29) is constructed as

\[
\hat{y} := Y^+(0),
\]

\[
\hat{\pi}(t) := \text{diag}(S_t)^{-1} \left( \sigma(t)^\top \right)^{-1} \left\{ Z^+(t) + U_1^+(t)\sigma_1^+(t) + U_2^+(t)\sigma_C^+(t) \right\},
\]

\[
\hat{\pi}_I(t) := -P_I(t-)^{-1}U_1^+(t),
\]

\[
\hat{\pi}_C(t) := -P_C(t-)^{-1}U_2^+(t)
\]

for \( 0 \leq t \leq \tau_1 \wedge \tau_2 \wedge T \). Similarly, the replicating portfolio \((-\hat{y}, -\hat{\Pi})\) that satisfies (34) can be represented using the solution to a BSDE. Let

\[
Y^- := -\tilde{Y}(-\hat{y}, -\hat{\Pi}),
\]

\[
U_1^-(t) := -\hat{\pi}_I(t)P_I(t-),
\]

\[
U_2^-(t) := -\hat{\pi}_C(t)P_C(t-),
\]

\[
Z^-(t) := \sigma(t)^\top \text{diag}(S(t))\pi(t) - U_1^-(t)\sigma_1(t)^\top - U_2^-(t)\sigma_C(t)^\top.
\]

Recalling

\[-\hat{\pi}_I(t)B_f(t) = \tilde{Y}^-(t) + U_1^-(t) + U_2^-(t) - \alpha \tilde{V}(t),\]

we see that \( \pi_I(t) \geq 0 \) (resp. \( \leq 0 \)) is equivalent to

\[Y^-(t) + U_1^-(t) + U_2^-(t) - \alpha \tilde{V}(t) \leq 0, \text{ (resp. } \geq 0).\]

Also, recalling

\[\hat{\pi}_I(t)B_f(t) = \pi(t)^\top \text{diag}(S(t))1 = \left\{ Z^-(t)^\top + U_1^-(t)\sigma_1(t) + U_2^-(t)\sigma_C(t) \right\} \sigma(t)^{-1}1,\]

we see that \( \pi_I(t) \geq 0 \) (resp. \( \leq 0 \)) is equivalent to

\[\left\{ Z^-(t)^\top + U_1^-(t)\sigma_1(t) + U_2^-(t)\sigma_C(t) \right\} \sigma(t)^{-1}1 \geq 0 \text{ (resp. } \leq 0).\]

Using these relations, we then rewrite (33) as

\[dY^-(t) = -f^-(t, Y^-(t), Z^-(t), U_1^-(t), U_2^-(t), \tilde{V}(t)) dt + Z^-(t)^\top dW(t) + U_1^-(t)dM_1(t) + U_2^-(t)dM_2(t),\]

where

\[
f^-(t, y, z, u_1, u_2; \hat{v}) := -f^+(t, -y, -z, -u_1, -u_2; -\hat{v})
\]

\[
= f^0(t, y, z, u_1, u_2) + \alpha \left\{ r^0(t)\hat{v} + r^+_{col}(t)\hat{v}^+ - r^-_{col}(t)\hat{v}^- \right\}
\]

\[
- \epsilon_f(t) \left| y + u_1 + u_2 - \alpha \hat{v} \right|
\]

\[
- \epsilon_r(t) \left| \{ z^\top + u_1\sigma_1(t) + u_2\sigma_C(t) \} \sigma(t)^{-1}1 \right|.
\]

(38)
So, we consider the BSDE on the filtered probability space \((\Omega, F, P, \mathcal{G})\)

\[-dY^-(t) = f^-(t, Y^-(t), Z^-(t), U^-_1(t), U^-_2(t); \hat{V}(t)) dt \]

\[- Z^-(t) \hat{V}(t) dt - U^-_1(t) dM_1(t) - U^-_2(t) dM_2(t) \text{ for } 0 \leq t \leq \tau_1 \wedge \tau_2 \wedge T,\]

\[Y^-(\tau_1 \wedge \tau_2 \wedge T) = H.\]

The replicating portfolio \((-\bar{y}, -\bar{\Pi})\) that satisfies (34) is now constructed as

\[
\bar{y} := Y^-(0),
\]

\[
\hat{\pi}(t) := \text{diag} \left( S_t \right)^{-1} \left( \sigma(t)^\top \right)^{-1} \left\{ Z^-(t) + U^-_1(t) \sigma^\top_1(t) + U^-_2(t) \sigma^\top_C(t) \right\},
\]

\[
\hat{\pi}^I(t) := -P_I(t^-)^{-1} U^-_1(t),
\]

\[
\hat{\pi}^C(t) := -P_C(t^-)^{-1} U^-_2(t)
\]

for \(0 \leq t \leq \tau_1 \wedge \tau_2 \wedge T\), using the solution to (39).

**Remark 12.** BSDEs (37) and (39) with (15) and (16) can be seen as the system of BSDEs

\[-dY^\pm(t) = f^\pm(t, Y^\pm(t), Z^\pm(t), U^\pm_1(t), U^\pm_2(t); \hat{V}(t)) dt \]

\[- Z^\pm(t) \hat{V}(t) dt - U^\pm_1(t) dM_1(t) - U^\pm_2(t) dM_2(t), \text{ for } 0 \leq t \leq \tau_1 \wedge \tau_2 \wedge T,\]

\[Y^\pm(\tau_1 \wedge \tau_2 \wedge T) = H,\]

\[-d\hat{V}(t) = -r_D(t) \hat{V}(t) dt - \Delta(t)^\top dW(t) \text{ for } 0 \leq t \leq T,\]

\[\hat{V}(T) = \xi_T,\]

in which \((Y^\pm, Z^\pm, U^\pm_1, U^\pm_2, \hat{V}, \Delta)\) are solutions.

### 3.5 Hedging Problem

To study the hedging problem via BSDEs (37) and (39) with (15) and (16), it is natural to employ the following space of admissible hedging strategies

\[
\mathcal{A}_{\beta,T} := \left\{ \left( \pi, \pi^I, \pi^C \right) \in \left( \mathcal{P}_{\beta,T} \right)^{d+2} \mid \left( \sigma^\top \text{diag} \left( S_t \right) \pi, \pi^I P^-_1, \pi^C P^-_C \right) \in \mathbb{H}_{\beta,T}^{2,n+2} \right\},
\]

where \(\beta > 0\) is a fixed (sufficiently large) constant and we denote \(P^-_i(t) := P_i(t^-)\) for \(t > 0\) and \(P^-_i(0) := P_i(0)\). We then formulate the minimal superhedging price (i.e., the maximal price for the writer) and the maximal subhedging price (i.e., the minimal price for the buyer) as follows.

**Definition 3.** For the derivative security given in Definition 2,

\[
\bar{p} := \inf \left\{ y \in \mathbb{R} \mid -H + Y^{(y,\Pi)}(\tau_1 \wedge \tau_2 \wedge T) \geq 0 \text{ for some } (y, \Pi) \in \mathbb{R} \times \mathcal{A}_{\beta,T} \right\}
\]
is called the minimal superhedging price, which is the maximal price of the writer (seller), and
\[ p := \sup \{ y \in \mathbb{R} \mid H + \hat{Y}(y, -\Pi)(\tau_1 \land \tau_2 \land T) \geq 0 \text{ for some } (y, \Pi) \in \mathbb{R} \times \mathcal{A}_{\beta,T} \} \]
is called the maximal subhedging price, which is the minimal price of the buyer. If there exists \( \Pi \in \mathcal{A}_{\beta,T} \) such that
\[ -H + Y(\bar{p}, \bar{\Pi})(\tau_1 \land \tau_2 \land T) \geq 0, \]
then the pair \( (\bar{p}, \bar{\Pi}) \) is called the minimal superhedging strategy, and if there exists \( \Pi \in \mathcal{A}_{\beta,T} \) such that
\[ H + \hat{Y}(-\tilde{p}, -\Pi)(\tau_1 \land \tau_2 \land T) \geq 0, \]
then the pair \( (-\tilde{p}, -\Pi) \) is called the maximal subhedging strategy.

Associated with the hedging problem, we give the following definition.

**Definition 4.** Consider the derivative security given in Definition 2. Suppose that a writer sells the derivative security with price \( p \in \mathbb{R} \) at time 0. If it holds that
\[ -H + Y(p, \Pi)(\tau_1 \land \tau_2 \land T) \geq 0 \text{ and } \mathbb{P} \left( -H + Y(p, \Pi)(\tau_1 \land \tau_2 \land T) > 0 \right) > 0 \]
for some \( \Pi \in \mathcal{A}_{\beta,T} \), then we say that an arbitrage opportunity for the writer occurs. Similarly, suppose that a buyer purchases the derivative security with price \( p \in \mathbb{R} \) at time 0. If it holds that
\[ H + \hat{Y}(-p, -\Pi)(\tau_1 \land \tau_2 \land T) \geq 0 \text{ and } \mathbb{P} \left( H + \hat{Y}(-p, -\Pi)(\tau_1 \land \tau_2 \land T) > 0 \right) > 0 \]
for some \( \Pi \in \mathcal{A}_{\beta,T} \), then we say that an arbitrage opportunity for the buyer occurs. Moreover, if the price \( \tilde{p} \in \mathbb{R} \) at time 0 does not admit arbitrage opportunities for both writer and buyer, then \( \tilde{p} \) is called an arbitrage-free price.

**Remark 13.** In our financial market model, we assume implicitly that the probability measure \( \mathbb{P} \) is an EMM. Hence, \( \mathbb{P} \sim \mathbb{P}_0 \), where \( \mathbb{P}_0 \) is a real-world (physical) probability measure given in the same measurable space \((\Omega, \mathcal{F})\). Therefore, in Definition 3, the \( \mathbb{P} \)-a.s. statement can be replaced by the \( \mathbb{P}_0 \)-a.s. statement. Also, in Definition 4, \( \mathbb{P} \) can be replaced by \( \mathbb{P}_0 \) to claim that \( \mathbb{P}_0 (\cdots) > 0 \).

### 3.6 Markovian Model

The following Markovian model is typical and popularly treated in practice. Let the coefficients of the market model be described as
\[
\begin{align*}
\sigma(t) := & \tilde{\sigma}(t, F(t)), \quad r_D(t) := \hat{r}_D(t, F(t)), \\
\sigma_i(t) := & \tilde{\sigma}_i(t, F(t)), \quad i \in \{I, C\}, \\
h_j(t) := & \tilde{h}_j(t, F(t)), \quad j \in \{1, 2\}, \\
r^\alpha_k(t) := & \hat{r}^\alpha_k(t, F(t)), \quad \epsilon_k(t) := \hat{\epsilon}_k(t, F(t)), \quad k \in \{f, r\}, \\
\text{and} \quad r^\pm_{col}(t) := & r^\pm_{col}(t, F(t)),
\end{align*}
\]
Throughout this section, we always assume that
\[
\tilde{\sigma}_i, \tilde{h}_j, \tilde{r}_k, \tilde{\epsilon}_k, \tilde{r}_{col} : [0, T] \times \mathbb{R}^m \to \mathbb{R}, \quad \text{and} \quad (F(t))_{t \in [0, T]} \text{ is called the stochastic factor process, which can be interpreted as a model of economic factors and affects the market model through the coefficients } \sigma, \sigma_i (i \in \{I, C\}), h_j (j = 1, 2), r_0, \epsilon_k (k \in \{f, r\}), \text{ and } \tilde{r}_{col}. \text{ It is given by the solution to the SDE}
\]
\[
dF(t) = \mu_F(t, F(t)) \, dt + \sigma_F(t, F(t)) \, dW(t), \quad F(0) \in \mathbb{R}^m
\]
on \(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}\), where \(\mu_F : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m\) and \(\sigma_F : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{m \times n}\). Let
\[
X^\top := (X^\top_1, X^\top_2) := (S^\top, F^\top)
\]
and define, for \(x := (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m\),
\[
b(t, x) := \begin{pmatrix} \text{diag}(x_1) r_D(t, x_2) \\ \mu_F(t, x_2) \end{pmatrix}, \quad a(t, x) := \begin{pmatrix} \text{diag}(x_1) \sigma(t, x_2) \\ \sigma_F(t, x_2) \end{pmatrix}.
\]
Then, the SDE for \(X\) is written as (8) with \(d = n + m\). Furthermore, we set
\[
\xi_T := \Xi(X(T)) \quad \text{and} \quad \phi_i(t) := \varphi_i(\hat{V}(t)) \quad \text{for } i \in \{1, 2\},
\]
where \(\Xi : \mathbb{R}^{n+m} \to \mathbb{R}\) and \(\varphi_i : \mathbb{R} \to \mathbb{R}\). In this situation, we can apply Theorem 3 to represent the solution to BSDEs (40), using the solutions to the associated PDEs (see Proposition 2 in Section 4).

4 Results

Throughout this section, we always assume that \(\sigma_i (i \in \{I, C\}), \sigma, \sigma^{-1}, r_D, r_j^\pm (j \in \{f, r, col\})\), and \(h_k (k = 1, 2)\) are bounded. Applying the results in Section 2 and a comparison theorem for BSDEs, the following claims are straightforward to see.

**Proposition 1.** For any sufficiently large \(\beta > 0\), there exist unique solutions \((Y^\pm, Z^\pm, U^\pm_1, U^\pm_2) \in \mathcal{S}^2_{\beta, T} \times \mathcal{H}^2_{\beta, T}\) to BSDEs (37) and (39) with (15) and (16). Moreover, the solutions have the representations that
\[
Y^\pm(t) = \tilde{Y}^\pm(t) 1_{\{0 \leq t < r_1 \wedge \tau_2 \wedge T\}} + \begin{cases} \phi_1(\tau_1) 1_{\{r_1 \leq \tau_2 < \tau_1 \wedge T\}} + \phi_2(\tau_2) 1_{\{\tau_2 \leq \tau_1 \wedge T\}} + \xi_T 1_{\{T < \tau_1 \wedge \tau_2\}}; \\
\end{cases} \quad 1_{\{t = \tau_1 \wedge \tau_2 \wedge T\}};
\]
\[
Z^\pm(t) = \tilde{Z}^\pm(t),
\]
\[
U^\pm_1(t) = \phi_1(t) - \tilde{Y}^\pm(t), \quad i = 1, 2.
\]
Here, \((\tilde{Y}^\pm, \tilde{Z}^\pm) \in \mathcal{S}^2_{\beta, T} \times \mathcal{H}^2_{\beta, T}\) are the solutions to BSDEs on \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\), namely
\[
-d\tilde{Y}^\pm(t) = \tilde{f}^\pm \left(t, \tilde{Y}^\pm(t), \tilde{Z}^\pm(t); \tilde{V}(t), \phi_1(t), \phi_2(t)\right) dt - \tilde{Z}^\pm(t)^\top dW(t)
\]
for \(0 \leq t \leq T\),
\[
\tilde{Y}^\pm(T) = \xi_T,
\]
\[
-d\tilde{V}(t) = -r_D(t) \tilde{V}(t) dt - \Delta(t)^\top dW(t) \quad \text{for} \quad 0 \leq t \leq T, \quad \tilde{V}(T) = \xi_T,
\]
\[
\tilde{f}^\pm(t, \tilde{Y}_t, \tilde{Z}_t) := \begin{pmatrix} \phi_1(t) & \phi_2(t) \end{pmatrix}
\]
and \(\Delta(t) = \begin{pmatrix} \sigma(t) \end{pmatrix} \in \mathbb{R}^{m \times n}\) is the volatility matrix of the BSDEs. In this case, the representations also hold for \(Y^\pm, Z^\pm, U^\pm_1, U^\pm_2\).
where we define
\[
\tilde{f}^\pm (t, y, z; \hat{v}, p_1, p_2) := f^\pm (t, y, z, p_1, p_2 - y, \hat{v}) + (p_1 - y)h_1(t) + (p_2 - y)h_2(t).
\]
In addition to Condition (19), assume that
\[
r_{col}^- \geq r_{col}^+.
\]
Then, it always holds that
\[
Y^- \leq Y^+ \quad \text{and} \quad \tilde{Y}^- \leq \tilde{Y}^+.
\]

**Sketch.** Using (19) and (44), we see that
\[
\tilde{f}^+(t, y, z; \hat{v}, p_1, p_2) - \tilde{f}^-(t, y, z; \hat{v}, p_1, p_2)
= \alpha \left\{ r_{col}^-(t) - r_{col}^+(t) \right\} |\hat{v}| + 2\epsilon_f(t) |y + (p_1 - y) + (p_2 - y) - \alpha \hat{v}|
+ 2\epsilon_r(t) \left\{ z^T + (p_1 - y)\sigma_f(t) + (p_2 - y)\sigma_C(t) \right\} \sigma(t)^{-1} 1 \geq 0.
\]
Hence, (45) follows from a comparison theorem of BSDEs. Other assertions follow from the results in Section 2.

Next, consider the Markovian model given in Subsection 3.6. Then, corresponding to (42), we have the Markovian system of BSDEs (decoupled FBSDEs)
\[
dX(t) = b(t, X(t))dt + a(t, X(t))dW(t), \quad X(0) \in \mathbb{R}^{n+m},
-d\tilde{Y}^\pm(t) = \tilde{g}^\pm \left( t, X_2(t), \tilde{Y}^\pm(t), \tilde{Z}^\pm(t); \tilde{V}(t), \varphi_1(\tilde{V}(t)), \varphi_2(\tilde{V}(t)) \right) dt
- \tilde{Z}^\pm(t)^\top dW(t),
\tilde{Y}^\pm(T) = \Xi(X(T)),
-d\tilde{V}(t) = -\tilde{r}_D(t, X_2(t))\tilde{V}(t)dt - \Delta(t)^\top dW(t),
\tilde{V}(T) = \Xi(X(T)).
\]
Here, the relation
\[
\tilde{g}^\pm(t, X_2(t, \omega), y, z; \hat{v}, p_1, p_2) = \tilde{f}^\pm(t, \omega, y, z; \hat{v}, p_1, p_2)
\]
holds, and the functions \(\tilde{g}^\pm : [0, T] \times \mathbb{R}^{m} \times \mathbb{R} \times \mathbb{R}^{n} \times \mathbb{R}^{3} \rightarrow \mathbb{R}\) are written as
\[
\tilde{g}^\pm(t, x_2, y, z; \hat{v}, p_1, p_2) := \tilde{g}^0(t, x_2, y; \hat{v}, p_1, p_2)
+ \alpha \left\{ r_f^0(t, x_2)\hat{v} + \tilde{r}_{col}^+(t, x_2)\tilde{v}^\pm \pm \tilde{r}_{col}^-(t, x_2)\hat{v}^\mp \right\}
\pm \tilde{\epsilon}_f(t, x_2) |y + (p_1 - y) + (p_2 - y) - \alpha \hat{v}|
\pm \tilde{\epsilon}_r(t, x_2) \left\{ z^T + (p_1 - y)\tilde{\sigma}_1(t, x_2) + (p_2 - y)\tilde{\sigma}_C(t, x_2) \right\} \tilde{\sigma}(t, x_2)^{-1} 1
\]
with
\[
\tilde{g}^0(t, x_2, y; \hat{v}, p_1, p_2) := z^T \left\{ (\tilde{r}_r^0 - \tilde{r}_D)\tilde{\sigma}^{-1} 1 \right\} (t, x_2)
- \left\{ (2\tilde{r}_D - \tilde{r}_D) + \tilde{h}_1 + \tilde{h}_2 \right\} (\tilde{r}_r^0 - \tilde{r}_D)(\tilde{\sigma}_1 + \tilde{\sigma}_C)\tilde{\sigma}^{-1} 1 \right\} (t, x_2)y
+ \left\{ \tilde{h}_1 - (\tilde{r}_r^0 - \tilde{r}_D) + (\tilde{r}_r^0 - \tilde{r}_D)\tilde{\sigma}_1\tilde{\sigma}^{-1} 1 \right\} (t, x_2)p_1
+ \left\{ \tilde{h}_2 - (\tilde{r}_r^0 - \tilde{r}_D) + (\tilde{r}_r^0 - \tilde{r}_D)\tilde{\sigma}_C\tilde{\sigma}^{-1} 1 \right\} (t, x_2)p_2.
\]
Utilizing Theorem 3, we obtain the following.

**Proposition 2.** Denote \(d := n + m\) and consider the system of second-order parabolic semilinear PDEs

\[
-\partial_t V = \{L_t - \tilde{r}_D(t, x_2)\} V, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\
V(T, x) = \Xi(x), \\
-\partial_t U^{\pm} = L_t U^{\pm} + \tilde{g}^\pm(t, x_2, U, a^\top \nabla U^{\pm}; V, \varphi_1(V), \varphi_2(V)), \\
(t, x) \in [0, T] \times \mathbb{R}^d, \\
U^{\pm}(T, x) = \Xi(x),
\]

where \(L_t(\cdot)\) is the infinitesimal generator for \(X\) given by (10). Suppose that there exists a unique classical solution \((V, U^{\pm}) \in (C^{1,2}([0, T] \times \mathbb{R}^d))^2\) to (47). Then the solution to BSDE (46) is represented as

\[
\hat{Y}^{\pm}(t) = U^{\pm}(t, X(t)), \quad \hat{Z}^{\pm}(t) = (a \nabla U^{\pm})(t, X(t)), \quad t \in [0, T].
\]

**4.1 Results on Arbitrage**

**Theorem 4.** In addition to Conditions (19) and (44), assume the following:

\[
h_1 \geq r_f^r - r_D - (r_f^+ - r_D) (\sigma_I \sigma^{-1} \mathbf{1})^+ + (r_f^- - r_D) (\sigma_I \sigma^{-1} \mathbf{1})^-, \\
h_2 \geq r_f^r - r_D - (r_f^+ - r_D) (\sigma_C \sigma^{-1} \mathbf{1})^+ + (r_f^- - r_D) (\sigma_C \sigma^{-1} \mathbf{1})^-,
\]

and

\[
r_f^+ \geq r_{col}^-.
\]

Then it holds that \(p = Y^- (0) \leq Y^+(0) = \bar{p}\). Hence, for the derivative security given in Definition 2, any price \(p \in [Y^- (0), Y^+(0)]\) at time \(0\) is arbitrage-free.

**Remark 14.** The conditions imposed in Theorem 4 to ensure the arbitrage-free property look to be rather strong: violating (44), (48), or (49) seems to be realizable in real situations. Relaxing the arbitrage-free condition by admitting "certain" arbitrage opportunities might be an interesting research direction for this bilateral hedging scheme with collateralizations. We refer to Thoednithi (2015) and Nie and Rutkowski (2018) as related studies.

**Sketch.** Using (35), (36), (38), and (43), we see that

\[
\tilde{d}^{\pm} (t, y, z; \hat{v}, p_1, p_2) \\
= \frac{z^T \left\{ (r_f^0 - r_D) \sigma^{-1} \mathbf{1} \right\} (t)}{\partial_t} \\
- \left\{ (2r_D - r_f^0 + h_1 + h_2) + (r_f^0 - r_D) (\sigma_I + \sigma_C) \sigma^{-1} \mathbf{1} \right\} (t) y \\
+ \left\{ h_1 - (r_f^0 - r_D) + (r_f^0 - r_D) \sigma_I \sigma^{-1} \mathbf{1} \right\} (t) p_1 \\
+ \left\{ h_2 - (r_f^0 - r_D) + (r_f^0 - r_D) \sigma_C \sigma^{-1} \mathbf{1} \right\} (t) p_2 \\
+ \alpha \left\{ r_f^0 (t) \hat{v} \mp r_{col}^+ (t) \hat{v} \mp \pm r_{col}^- (t) \hat{v} \mp \right\} \\
\pm \epsilon_f (t) \left| y + (p_1 - y) + (p_2 - y) - \alpha \hat{v} \right| \\
\pm \epsilon_r (t) \left| \{ z^T + (p_1 - y) \sigma(t) + (p_2 - y) \sigma_C(t) \} \sigma(t)^{-1} \mathbf{1} \right|.
\]
So, for \( \delta_0, \delta_1, \delta_2 \geq 0 \), we see that
\[
\begin{align*}
\hat{f}^+(\cdot, y, z; \hat{v} + \delta_0, p_1 + \delta_1, p_2 + \delta_2) - \hat{f}^+(\cdot, y, z; \hat{v}, p_1, p_2) = & \left\{ h_1 - (r_f^0 - r_D) + (r_f^0 - r_D)\alpha \right\} \delta_1 \\
& + \left\{ h_2 - (r_f^0 - r_D) + (r_f^0 - r_D)\alpha \right\} \delta_2 \\
& + \alpha \left\{ r_f^0 \delta_0 - r_{\text{col}} \right\} \left\{ (\hat{v} + \delta_0)^+ - \hat{v}^+ \right\} + r_{\text{col}} \left\{ (\hat{v} + \delta_0)^- - \hat{v}^- \right\} \\
& + \epsilon_f \left\{ |p_1 + p_2 - \alpha \hat{v} - y + (\delta_1 + \delta_2 - \alpha \delta_0)| - |p_1 + p_2 - \alpha \hat{v} - y| \right\} \\
& + \epsilon_r \left\{ |z^T + (p_1 - y)\sigma_I + (p_2 - y)\sigma_C| \right\} \sigma^{-1} \right\} \delta_0,
\end{align*}
\]
(50)
Using the inequality \(|x + y| - |x| \geq -|y|\) and the relation
\[
\begin{align*}
r_{\text{col}} \left\{ (\hat{v} + \delta_0)^+ - \hat{v}^+ \right\} - r_{\text{col}} \left\{ (\hat{v} + \delta_0)^- - \hat{v}^- \right\} \leq (r_{\text{col}} \lor r_{\text{col}}) \delta_0,
\end{align*}
\]
we see that
\[
\begin{align*}
(50) \geq & \left\{ h_1 - (r_f^0 - r_D) + (r_f^0 - r_D)\alpha \right\} \delta_1 \\
& + \left\{ h_2 - (r_f^0 - r_D) + (r_f^0 - r_D)\alpha \right\} \delta_2 + \alpha (r_f^0 - r_{\text{col}}) \delta_0 \\
& - \epsilon_f (\delta_1 + \delta_2 + \alpha \delta_0) - \epsilon_r \left\{ |\sigma_I \sigma^{-1}| \delta_1 + |\sigma_C \sigma^{-1}| \delta_2 \right\} \\
& = \left\{ h_1 - r_f^+ + r_D + (r_f^0 - r_D)\sigma_I \sigma^{-1} \right\} \delta_1 \\
& + \left\{ h_2 - r_f^+ + r_D + (r_f^0 - r_D)\sigma_C \sigma^{-1} \right\} \delta_2 \\
& + \alpha (r_f^+ - r_{\text{col}}) \delta_0 \geq 0,
\end{align*}
\]
(51)
where we use (48) and (49). Consider the system of BSDEs (42) and write the solution as
\[
\begin{align*}
\hat{Y}^\pm(t; \xi_T, \phi_1, \phi_2), \quad \hat{Z}^\pm(t; \xi_T, \phi_1, \phi_2), \quad t \in [0, T]
\end{align*}
\]
by emphasizing the parameters \( (\xi_T, \phi_1, \phi_2) \). Take other payoff parameters \( (\hat{\xi}_T, \hat{\phi}_1, \hat{\phi}_2) \) such that \( \hat{\xi}_T \geq \xi_T, \hat{\phi}_1 \geq \phi_1, \text{ and } \hat{\phi}_2 \geq \phi_2 \). Using the comparison theorem for BSDEs twice (for \( \hat{Y} \) and \( \hat{Y}^+ \)), and using relations (50) and (51), we deduce that
\[
\hat{Y}^+ \left( \hat{\xi}_T, \hat{\phi}_1, \hat{\phi}_2 \right) \geq \hat{Y}^+ (\xi_T, \phi_1, \phi_2)
\]
and that
\[
Y^+ (\hat{\xi}_T, \hat{\phi}_1, \hat{\phi}_2) \geq Y^+(\xi_T, \phi_1, \phi_2).
\]
This implies the minimality of \( \hat{Y}^+ (\xi_T, \phi_1, \phi_2) \) and the equality,
\[
\bar{p} = Y^+(0; \xi_T, \phi_1, \phi_2).
\]
The equality,
\[
\bar{p} = Y^-(0; \xi_T, \phi_1, \phi_2),
\]
can be seen similarly.
We obtain the following.

\[ Y^\pm(t; k\xi_T, k\phi_1, k\phi_2) \equiv k Y^\pm(t; \xi_T, \phi_1, \phi_2) \quad \text{for } t \in [0, T]. \]

This positive homogeneity is seen from those of the drivers of BSDEs (42), namely

\[ f^\pm(t, ky, kz; k\bar{\nu}, k\phi_1, k\phi_2) = k f^\pm(t, y, z; \bar{\nu}, \phi_1, \phi_2), \]

\[ -r_D(t)(k\bar{\nu}) = k \{ -r_D(t)\bar{\nu} \}. \]

See Jiang (2008) for the details.

### 4.2 Results on XVA

In this subsection, we assume that

\[ \epsilon_f \vee \epsilon_r \leq \epsilon \]

with some (small) positive constant \( \epsilon \ll 1 \). Consider the system of BSDEs

\[ -dY^{0,\pm}(t) = f^{0,\pm}(t, Y^{0,\pm}(t), Z^{0,\pm}(t), U_1^{0,\pm}(t), U_2^{0,\pm}(t); \bar{V}(t)) dt \]

\[ -Z^{0,\pm}(t) \bar{\Delta}(t) dW(t) - U_1^{0,\pm}(t) dM_1(t) - U_2^{0,\pm}(t) dM_2(t), \]

for \( 0 \leq t \leq \tau_1 \wedge \tau_2 \wedge T \),

\[ Y^{0,\pm}(\tau_1 \wedge \tau_2 \wedge T) = H, \]

\[ -d\bar{V}(t) = -r_D(t)\bar{V}(t) dt - \Delta(t)^\top dW(t) \quad \text{for } 0 \leq t \leq T, \]

\[ \bar{V}(T) = \xi_T \]

on \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G})\), where

\[ f^{0,\pm}(t, y, z, u_1, u_2; \bar{\nu}) := f^0(t, y, z, u_1, u_2) + \alpha \left\{ r_0^0(t)\bar{\nu} \mp r_{col}^+(t)\bar{\nu}^+ \pm r_{col}^-(t)\bar{\nu}^- \right\}. \]

Associated with (53), consider the reduced system of BSDEs

\[ -dY^{0,\pm}(t) = f^{0,\pm}(t, Y^{0,\pm}(t), Z^{0,\pm}(t); \bar{V}(t), \phi_1(t), \phi_2(t)) dt \]

\[ -Z^{0,\pm}(t) \bar{\Delta}(t) dW(t) \quad \text{for } 0 \leq t \leq T, \]

\[ Y^{0,\pm}(T) = \xi_T, \]

\[ -d\bar{V}(t) = -r_D(t)\bar{V}(t) dt - \Delta(t)^\top dW(t) \quad \text{for } 0 \leq t \leq T, \]

\[ \bar{V}(T) = \xi_T \]

on \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\), where

\[ \bar{f}^{0,\pm}(t, y, z; \bar{\nu}, p_1, p_2) := f^{0,\pm}(t, y, z, p_1 - y, p_2 - y; \bar{\nu}) + (p_1 - y)h_1(t) + (p_2 - y)h_2(t). \]

We obtain the following.
Theorem 5. Assume Conditions (19) and (44). For \((Y^\pm, Z^\pm), (Y^{0,\pm}, Z^{0,\pm})\), which are solutions to BSDEs (42) and (54), respectively, it holds that

\[
    Y^- \leq Y^{0,-} \leq Y^{0,+} \leq Y^+
\]

and that

\[
    \|Y^\pm - Y^{0,\pm}\|_{\beta,T} + \|Z^\pm - Z^{0,\pm}\|_{\beta,T} = O(\epsilon)
\]

as \(\epsilon \to 0\) in both + and - cases.

Sketch. The relation (55) is easily seen from the comparison theorem of BSDEs. To see (56), we can apply the continuity (and the differentiability) results with their proofs with respect to parameterized BSDEs, shown in El Karoui et al. (2000) (see Proposition 2.4 and its proof in [15] for the details).

Combining Theorems 4 and 5, we see the following.

Corollary 1. Assume Conditions (19), (44), (48), and (49). Then \(Y^{0,-}(0)\) and \(Y^{0,+}(0)\) are arbitrage-free prices at time 0 for the derivative security given in Definition 2.

The above corollary implies that \(Y^{0,\pm}(0)\) may be regarded as approximated prices of the derivative security for the writer and her counterparty, which prohibit the existence of an arbitrage opportunity. Because BSDEs for \((Y^{0,\pm}, Z^{0,\pm})\) are linear, we obtain the closed-form expressions for \(Y^{0,\pm}\) as follows. Let us introduce the probability measure \(\tilde{P}_T\) on \((\Omega, \mathcal{F}_T)\) by

\[
d\tilde{P}_T \big|_{\mathcal{F}_t} = \mathcal{E}(t) d\tilde{P} \big|_{\mathcal{F}_t}, \quad t \in [0, T],
\]

where

\[
    \mathcal{E}(t) := \exp\left[ \int_0^t \left\{ r^0_r(u) - r_D(u) \right\} \mathbf{1}^\top (\sigma(u)^{-1})^\top dW(u) - \frac{1}{2} \int_0^t \left\{ r^0_r(u) - r_D(u) \right\}^2 |\sigma(u)^{-1}\mathbf{1}|^2 du \right].
\]

We denote the expectation with respect to \(\tilde{P}_T\) conditioned by \(\mathcal{F}_t\) by \(\tilde{E}_t[\cdots] = \tilde{E}[\cdots]|_{\mathcal{F}_t}\). Recall that

\[
    \tilde{W}(t) := W(t) - \int_0^t \left\{ r^0_r(u) - r_D(u) \right\} \sigma(u)^{-1}\mathbf{1} du, \quad t \in [0, T]
\]

is a \((\tilde{P}_T, \bar{\mathcal{F}})\)-Brownian motion by the Maruyama–Girsanov theorem, and on \((\Omega, \mathcal{F}, \tilde{P}_T, \bar{\mathcal{F}})\) the risky asset price process \(S\) has the dynamics

\[
    dS(t) = \text{diag}(S(t)) \left\{ \sigma(t) d\tilde{W}(t) + r^0_r(t)\mathbf{1} dt \right\}, \quad S(0) \in \mathbb{R}^n_{++}.
\]

Also, we denote

\[
    DF_r(t,u) := \exp\left\{ - \int_t^u r(s) ds \right\}
\]

for the process \(r := (r(t))_{t \in [0, T]}\). We then obtain the following.

\[\text{That is, the drivers } f^{0,\pm}(t, y, z, u_1, u_2; \hat{v}) \text{ are linear with respect to } (y, z, u_1, u_2).\]
Proposition 3. The following representation holds:

\[ \bar{Y}^{0,\pm}(t) = V(t) + VA_1(t) + VA_2(t) + VA_3(t) + VA_4(t) + VA_5^\pm(t). \]  

(57)

Here,

\[ V(t) := \tilde{E}_t \left[ DF_{r_0}(t,T)\xi_T \right], \]

\[ VA_1(t) := \tilde{E}_t \left[ \int_t^T DF_R(t,u)h_1(u)\hat{\phi}_1(u)du \right], \]

\[ VA_2(t) := \tilde{E}_t \left[ \int_t^T DF_R(t,u)h_2(u)\hat{\phi}_2(u)du \right], \]

\[ VA_3(t) := -\tilde{E}_t \left[ \int_t^T DF_R(t,u) \left\{ \left( r^+_f - r_D \right) \left( \hat{\phi}_1 + \hat{\phi}_2 \right) \right\} (u)du \right], \]

\[ VA_4(t) := \tilde{E}_t \left[ \int_t^T DF_R(t,u) \left\{ \left( r^0_f - r_D \right) \left( \hat{\phi}_1 \sigma_I + \hat{\phi}_2 \sigma_C \right) \sigma^{-1}1 \right\} (u)du \right], \]

\[ VA_5^\pm(t) := \alpha \tilde{E}_t \left[ \int_t^T DF_R(t,u) \left\{ \left( r^0_f - r^\pm_{col} \right) \hat{V}^+ - \left( r^0_f - r^\mp_{col} \right) \hat{V}^- \right\} (u)du \right], \]

where we define

\[ \hat{\phi}_i := \phi_i - V \quad \text{for} \quad i = 1, 2, \quad \text{and} \]

\[ R := r_D - \left( r^0_f - r_D \right) + \left\{ \left( r^0_f - r_D \right) (\sigma_I + \sigma_C) (\sigma)^{-1}1 \right\} + h_1 + h_2. \]

Proof. Using the representation formula for linear BSDE (e.g., see Proposition 2.2 of [15]), we see that

\[ \bar{Y}^{0,\pm}(t) = \tilde{V}(t) + \nabla A_1(t) + \nabla A_2(t) + \nabla A_3(t) + \nabla A_4(t) + \nabla A_5^\pm(t), \]

where

\[ \tilde{V}(t) := \tilde{E}_t \left[ DF_R(t,T)\xi_T \right], \]

\[ \nabla A_1(t) := \tilde{E}_t \left[ \int_t^T DF_R(t,u)h_1(u)\phi_1(u)du \right], \]

\[ \nabla A_2(t) := \tilde{E}_t \left[ \int_t^T DF_R(t,u)h_2(u)\phi_2(u)du \right], \]

\[ \nabla A_3(t) := -\tilde{E}_t \left[ \int_t^T DF_R(t,u) \left\{ \left( r^+_f - r_D \right) \left( \hat{\phi}_1 + \hat{\phi}_2 \right) \right\} (u)du \right], \]

\[ \nabla A_4(t) := \tilde{E}_t \left[ \int_t^T DF_R(t,u) \left\{ \left( r^0_f - r_D \right) \left( \hat{\phi}_1 \sigma_I + \hat{\phi}_2 \sigma_C \right) \sigma^{-1}1 \right\} (u)du \right]. \]
Furthermore, we see that

\[
[VA_1 + VA_2 + VA_3 + VA_4 - \nabla A_1 - \nabla A_2 - \nabla A_3 - \nabla A_4](t) \\
= -\mathbb{E}_t \left[ \int_t^T DF_R(t,u)V(u) \{ R(u) - r^0_i(u) \} \, du \right] \\
= -\mathbb{E}_t \left[ \int_t^T DF_R(t,u)\mathbb{E}_u \left[ DF_{r^0_j}(u,T)\xi_T \right] \{ R(u) - r^0_j(u) \} \, du \right] \\
= \mathbb{E}_t \left[ DF_{r^0_j}(t,T)\xi_T \int_t^T \frac{\partial}{\partial u} DF_{R-r^0_j}(t,u) \, du \right] \\
= \mathbb{E}_t \left[ DF_{r^0_j}(t,T) \left\{ DF_{R-r^0_j}(t,T) - 1 \right\} \xi_T \right] \\
= \mathbb{E}_t \left[ \left\{ DF_R(t,T) - DF_{r^0_j}(t,T) \right\} \xi_T \right] = \hat{V}(t) - V(t),
\]

hence the proof is complete. \(\square\)

**Remark 16.** Suppose that \(r^0_i \equiv r^0_j \equiv r_D \) holds. In this case, \(\hat{P}_T \equiv P \) and \(V \equiv \hat{V} \) follow. Furthermore, consider \(\phi_i(t) := \hat{\phi}_i \left( \hat{V}(t) \right) \), where (17) is employed for \(i = 1, 2\). Then, in (57), \(VA_3 \equiv VA_4 \equiv 0\), and \(-VA_1, VA_2, \) and \(VA^\pm_5\) are called the debt valuation adjustment (DVA), the credit valuation adjustment (CVA), and the collateral valuation adjustment (ColVA), respectively, which are popularly used XVA terms in practice for the valuation adjustment in the pricing of derivative securities. Concretely, DVA, CVA, and ColVA at time \(t\) are written as

\[
\text{DVA}(t) := -\mathbb{E}_t \left[ \int_t^T DF_{r_D+h_1+h_2}(t,u)h_1(u)\hat{\phi}_1(u) \, du \right],
\]

\[
\text{CVA}(t) := \mathbb{E}_t \left[ \int_t^T DF_{r_D+h_1+h_2}(t,u)h_2(u)\hat{\phi}_2(u) \, du \right],
\]

\[
\text{ColVA}^\pm(t) := \mathbb{E}_t \left[ \int_t^T DF_{r_D+h_1+h_2}(t,u) \left\{ (r_D-r^\pm_{\text{col}})\alpha\hat{V}^+ - (r_D-r^\pm_{\text{col}})\alpha\hat{V}^- \right\} (u) \, du \right],
\]

respectively, where we denote \(\mathbb{E}_t[(\cdots)] := \mathbb{E}[(\cdots)|\mathcal{F}_t]\). Further,

\[
\text{FVA}(t) := \mathbb{E}_t \left[ \int_t^T DF_{r_D+h_1+h_2}(t,u) \left\{ (r^0_i - r_D) (\phi_1 + \phi_2) \right\} (u) \, du \right],
\]

called the funding valuation adjustment (FVA) at time \(t\), is another popularly used adjustment term in practice, which reflects the funding cost of uncollateralised derivatives above the riskfree rate of return. We can roughly relate these XVA terms with the correction terms in Proposition 3 as follows: Let \(r^0_i \equiv r_D\) \(^\dagger\) which implies \(VA_4 \equiv 0\). Further, suppose \(r^0_j \approx r_D\). Then, we may

\(^\dagger\)In practice, the difference \(r^0_i - r_D\) seems to have been usually ignored.
interpret as

\[ \text{DVA} \approx - \text{VA}_1, \]
\[ \text{CVA} \approx \text{VA}_2, \]
\[ \text{ColVA}^\pm \approx \text{VA}^\pm_5, \]

and

\[ \text{FVA} \approx \text{VA}_3, \]

or

\[ \text{FVA} \approx \text{VA}_3 + (\text{VA}_1 + \text{DVA}) + (\text{VA}_2 - \text{CVA}) + (\text{ColVA}^\pm - \text{VA}^\pm_5). \]

For other theoretical studies on the valuation adjustments and related interpretation of XVA used in practice, we refer to Brigo et al. (2020) and the references therein. Also, for comprehensive information on XVA issue and expanding related issues (e.g., computational issue), see for example Gregory (2015) and Glau et al. (2016), and the references therein, which are still nonexhaustive.

### 4.3 Perturbed BSDEs

As we see in Theorem 5 and Corollary 1, under certain conditions, \( Y^{0,\pm}(t) < Y^+(t) \), which is a zeroth-order approximation of the minimal hedging cost \( Y^+(t) \), is an arbitrage-free price for the writer at time \( t \). In this subsection, we try to improve our hedging strategy by using a first-order approximation. Using the solution to BSDE (53), consider the linear BSDE

\[
-dY^{1,\pm}(t) = f^0(t, Y^{1,\pm}(t), Z^{1,\pm}(t), U^{1,\pm}_1(t), U^{1,\pm}_2(t)) dt \\
+ f^{1,\pm}(t, Y^{0,\pm}(t), Z^{0,\pm}(t), U^{0,\pm}_1(t), U^{0,\pm}_2(t), \hat{V}(t)) dt \\
- Z^{1,\pm}(t) dW(t) - U^{1,\pm}_1(t) dM_1(t) - U^{1,\pm}_2(t) dM_2(t),
\]

\( Y^{1,\pm}(\tau_1 \wedge \tau_2 \wedge T) = 0 \)

on \( (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G}) \), where

\[
f^{1,\pm}(t, y, z, u_1, u_2; \hat{v}) := \pm \epsilon_f(t) \left| y + u_1 + u_2 - \alpha \hat{v} \right| + \pm \epsilon_v(t) \left| \left\{ z^T + u_1 \sigma_1(t) + u_2 \sigma_C(t) \right\} \sigma(t)^{-1} \right|.
\]

Furthermore, using the solution to BSDE (54), consider the linear BSDE

\[
-d\hat{Y}^{1,\pm}(t) = \hat{f}^0(t, \hat{Y}^{1,\pm}(t), \hat{Z}^{1,\pm}(t); \phi_1(t), \phi_2(t)) dt \\
+ \hat{f}^{1,\pm}(t, \hat{Y}^{0,\pm}(t), \hat{Z}^{0,\pm}(t); \hat{V}(t), \phi_1(t), \phi_2(t)) dt \\
- \hat{Z}^{1,\pm}(t) dW(t),
\]

\( \hat{Y}^{1,\pm}(T) = 0 \)

on \( (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}) \), where

\[
\hat{f}^0(t, y, z; p_1, p_2) := f^0(t, y, z, p_1 - y, p_2 - y),
\]
\[
\hat{f}^{1,\pm}(t, y, z; \hat{v}, p_1, p_2) := \pm \epsilon_f(t) \left| y + (p_1 - y) + (p_2 - y) - \alpha \hat{v} \right| + \pm \epsilon_v(t) \left| \left\{ z^T + (p_1 - y) \sigma_1(t) + (p_2 - y) \sigma_C(t) \right\} \sigma(t)^{-1} \right|.
\]
Using a similar technique to that used in the proof of Theorem 5, we can show the following.

**Proposition 4.** It holds that for any sufficiently large \( \beta > 0 \),
\[
\| \tilde{Y}^\pm - (\tilde{Y}^0, \pm) \|_{\beta,T} + \| \tilde{Z}^\pm - (\tilde{Z}^0, \pm) \|_{\beta,T} = O(\epsilon^2)
\]
as \( \epsilon \to 0 \), where we assume (52).

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