Communication Compression for Decentralized Nonconvex Optimization

Xinlei Yi, Shengjun Zhang, Tao Yang, Tianyou Chai, and Karl H. Johansson

Abstract

This paper considers decentralized nonconvex optimization with the cost functions being distributed over agents. Noting that information compression is a key tool to reduce the heavy communication load for decentralized algorithms as agents iteratively communicate with neighbors, we propose three decentralized primal–dual algorithms with compressed communication. The first two algorithms are applicable to a general class of compressors with bounded relative compression error and the third algorithm is suitable for two general classes of compressors with bounded absolute compression error. We show that the proposed decentralized algorithms with compressed communication have comparable convergence properties as state-of-the-art algorithms without communication compression. Specifically, we show that they can find first-order stationary points with sublinear convergence rate $O(1/T)$ when each local cost function is smooth, where $T$ is the total number of iterations, and find global optima with linear convergence rate under an additional condition that the global cost function satisfies the Polyak–Łojasiewicz condition. Numerical simulations are provided to illustrate the effectiveness of the theoretical results.

Index Terms—Communication compression, decentralized optimization, linear convergence, non-convex optimization, Polyak–Łojasiewicz condition

I. INTRODUCTION

We consider decentralized nonconvex optimization. Specifically, consider a network of $n$ agents, each of which has a private local (possibly nonconvex) cost function $f_i: \mathbb{R}^d \mapsto \mathbb{R}$.

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The whole network aims to solve the following optimization problem

\[
\min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x). \tag{1}
\]

Throughout this paper we assume each \( f_i \) is smooth. Note that each agent alone cannot solve the above optimization problem since it does not know other agents’ local cost functions. Therefore, agents need to communicate with each other through an underlying communication network. Decentralized nonconvex optimization has gained extensive attention in recent years due to its wide applications, such as power allocation in wireless adhoc networks [1], distributed clustering [2], dictionary learning [3], and empirical risk minimization [4].

The problem (1) has been extensively studied in the literature, e.g., [5]–[10], just to name a few. Due to nonconvexity, these studies typically showed that first-order stationary points can be found at a sublinear convergence rate. For example, [5]–[9] showed that first-order stationary points can be found with an \( O(1/T) \) convergence rate, where \( T \) is the total number of iterations. However, the algorithms proposed in these studies require significant amount of data exchange as agents iteratively communicate with neighbors. Noting that communication bandwidth and power are limited, it is vital to propose communication-efficient decentralized algorithms. In this paper, we propose decentralized algorithms with compressed communication to improve communication efficiency.

A. Related Works and Motivation

It is straightforward to combine existing decentralized algorithms and communication compression directly. However, such a simple strategy does not converge to the accurate solution due to the compression error, and even may leads to divergence as the compression error would accumulate. Examples have been provided in [11], [12] to illustrate this. Therefore, communication compression in decentralized algorithms has gained considerable attention recently.

When each local cost function is convex, various decentralized algorithms with compressed communication have been proposed. For example, [13], [14] used unbiased compressors with bounded relative compression error to design decentralized stochastic gradient descent (SGD) algorithms; [15] employed biased but contractive compressors to design a decentralized SGD algorithm; [16] and [17], [18] utilized unbiased compressors to respectively design decentralized gradient descent and primal–dual algorithms; [19] and [20] made use of the standard uniform
quantizer to respectively design decentralized subgradient methods and alternating direction method of multipliers approaches; [21], [22] and [23] respectively adopted the unbiased random quantization and the adaptive quantization to design decentralized projected subgradient algorithms; [24] and [25]–[28] exploited the standard uniform quantizer with dynamic quantization level to respectively design decentralized subgradient and primal–dual algorithms; and [29] applied the standard uniform quantizer with a fixed quantization level to design a decentralized gradient descent algorithm. The compressors mentioned above can be unified into three general classes. Specifically, [30] proposed a wider class of compressors with bounded relative compression error which covers the compressors used in [13]–[18]; [31] considered a general class of compressors with globally bounded absolute compression error which accommodates the compressors used in [21]–[23]; and [32] studied a general class of compressors with locally bounded absolute compression error which contains the compressors used in [24]–[29]. These studies also analyzed the convergence properties of the proposed algorithms. Especially, some of them showed that the achieved convergence rates under compressed communication are comparable to and even match those under accurate communication. For instance, linear convergence was achieved in [17], [18], [25]–[28], [30], [32] under the standard strong convexity assumption.

While various algorithms with compressed communication have been designed for decentralized convex optimization, communication compression for decentralized nonconvex optimization is relatively less studied because the analysis is more challenging due to the nonconvexity. Moreover, when considering decentralized nonconvex optimization, existing decentralized algorithms with compressed communication are all SGD algorithms although different types of compressors have been used. For instance, [11] used the modular arithmetic for communication quantization (Moniqua); [33] used unbiased compressors with bounded relative or absolute compression error; [34]–[37] used biased but contractive compressors; [38] used unbiased compressors with bounded absolute compression error. These studies also analyzed the convergence properties of the proposed algorithms. For instance, [11], [33]–[37] showed that the proposed SGD algorithms with compressed communication achieve linear speedup convergence rate \(O(1/\sqrt{nT})\), which is the same as that achieved by decentralized SGD algorithms with exact communication. However, when replacing the stochastic gradient by the precise gradient it is unclear whether these studies can recover the classic \(O(1/T)\) convergence rate or not. Observing this, one core theoretical question arises.

(Q1) Under compressed communication, can first-order stationary points be found with the
well-known $O(1/T)$ convergence rate?

On the other hand, noting that it has been shown in [9], [39], [40] that global optima of nonconvex optimization can be linearly found if the global cost function satisfies the Polyak–Łojasiewicz (P–Ł) condition, another core theoretical question arises.

(Q2) Under compressed communication, can global optima be linearly found when the global cost function satisfies the P–Ł condition?

**B. Main Contributions**

In this paper, we provide positive answers to the above questions. Specifically, we propose three decentralized primal–dual algorithms with compressed communication and each of them can provide positive answers to the above two questions. More specifically, the contributions of this paper are summarized as follows.

(C1) We first use a general class of compressors with bounded relative compression error, which incorporates various commonly used compressors including unbiased compressors and biased but contractive compressors, to design a communication-efficient decentralized algorithm (Algorithm 1). This algorithm only requires each agent to communicate one compressed variable with its neighbors per iteration. We show that this compressed communication algorithm has comparable convergence properties as state-of-the-art algorithms with accurate communication. Specifically, we show in Theorem 1 that it can find a first-order stationary point with the well-known $O(1/T)$ convergence rate, thus (Q1) is answered. Moreover, if the global cost function satisfies the P–Ł condition, we show in Theorem 2 that it can find a global optimum with linear convergence rate, thus (Q2) is answered.

(C2) We then propose an error feedback based compressed communication algorithm (Algorithm 2) for biased compressors particularly. This algorithm can correct the bias induced by biased compressors under the cost that it requires each agent to communicate two compressed variables with its neighbors per iteration. We show in Theorems 3 and 4 that this algorithm has similar convergence properties as the first algorithm, which respectively answer (Q1) and (Q2).

(C3) We finally use two general classes of compressors with globally and locally bounded absolute compression error, which cover various commonly used compressors including unbiased compressors with bounded variance, random/adaptive/ uniform quantization, and even 1-bit binary quantizer, to design a communication-efficient decentralized algorithm (Algorithm 3). This algorithm also only requires each agent to communicate one compressed variable with
its neighbors per iteration. When the compressors have globally bounded absolute compression error, we show in Theorems 5 and 6 that this algorithm has similar convergence properties as the first algorithm, which respectively answer (Q1) and (Q2). When the compressors have locally bounded absolute compression error, we show in Theorem 7 that this algorithm can find a global optimum with linear convergence rate if the global cost function satisfies the P–Ł condition and the corresponding P–Ł constant is known a priori, which answers (Q2).

In summary, the main contribution of this paper is to propose three decentralized primal–dual algorithms with compressed communication for decentralized nonconvex optimization, which have comparable convergence properties as state-of-the-art algorithms with accurate communication. This is a significant theoretical development and to the best of our knowledge, it is the first time to achieve this.

C. Outline

The rest of this paper is organized as follows. Section II introduces some preliminaries. Section III presents the problem formulation. Sections IV–VI provide three communication-efficient decentralized algorithms and analyze their convergence properties. Section VII gives numerical simulations. Finally, Section VIII concludes this paper. Proofs are given in the Appendix.

Notations: \( \mathbb{N}_0 \) denotes the set of nonnegative integers. \([n]\) denotes the set \(\{1, \ldots, n\}\) for any positive constant integer \(n\). \(\| \cdot \|_p\) represents the \(p\)-norm for vectors or the induced \(p\)-norm for matrices, and the subscript is omitted when \(p = 2\). Given a differentiable function \(f\), \(\nabla f\) denotes its gradient.

II. Preliminaries

In this section, we briefly introduce algebraic graph theory and the P–Ł condition.

A. Algebraic Graph Theory

Let \(G = (\mathcal{V}, \mathcal{E}, A)\) denote a weighted undirected graph with the set of vertices (nodes) \(\mathcal{V} = [n]\), the set of links (edges) \(\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}\), and the weighted adjacency matrix \(A = A^\top = (a_{ij})\) with nonnegative elements \(a_{ij}\). A link of \(G\) is denoted by \((i, j) \in \mathcal{E}\) if \(a_{ij} > 0\), i.e., if vertices \(i\) and \(j\) can communicate with each other. It is assumed that \(a_{ii} = 0\) for all \(i \in [n]\). Let \(\mathcal{N}_i = \{j \in [n] : a_{ij} > 0\}\) and \(\deg_i = \sum_{j=1}^{n} a_{ij}\) denote the neighbor set and weighted degree of vertex \(i\), respectively. The degree matrix of graph \(G\) is \(\text{Deg} = \text{diag}([\deg_1, \cdots, \deg_n])\). The
Laplacian matrix is $L = (L_{ij}) = \text{Deg} - A$. A path of length $k$ between vertices $i$ and $j$ is a subgraph with distinct vertices $i_0 = i, \ldots, i_k = j \in [n]$ and edges $(i_j, i_{j+1}) \in \mathcal{E}$, $j = 0, \ldots, k - 1$. An undirected graph is connected if there exists at least one path between any two distinct vertices.

B. Polyak–Łojasiewicz Condition

Let $f(x) : \mathbb{R}^d \mapsto \mathbb{R}$ be a differentiable function. Let $\mathbf{x}^* = \arg \min_{x \in \mathbb{R}^d} f(x)$ and $f^* = \min_{x \in \mathbb{R}^d} f(x)$. Moreover, we assume that $f^* > -\infty$.

**Definition 1.** The function $f$ satisfies the Polyak–Łojasiewicz (P–Ł) condition with constant $\nu > 0$ if

$$\frac{1}{2} \|\nabla f(x)\|^2 \geq \nu (f(x) - f^*), \quad \forall x \in \mathbb{R}^d. \quad (2)$$

It is straightforward to see that every (essentially or weakly) strongly convex function satisfies the P–Ł condition. The P–Ł condition implies that every stationary point is a global minimizer. But unlike the (essentially or weakly) strong convexity, the P–Ł condition alone does not imply convexity of $f$. Moreover, it does not imply that the global minimizer is unique either. In fact, P–Ł condition generalizes strong convexity to nonconvex functions. The function $f(x) = x^2 + 3\sin^2(x)$ given in [39] is an example of a nonconvex function satisfying the P–Ł condition with $\nu = 1/32$. Moreover, it was shown in [41] that the loss functions in some applications satisfy the P–Ł condition in the local region near a local minimum. Moreover, [42] proved that the cost function of the policy optimization for the linear quadratic regulator problem is nonconvex and satisfies the P–Ł condition.

### III. Problem Formulation

In this section, we introduce three general classes of compressors and provide the assumptions on the communication network and cost functions.

A. **Compressors**

To improve communication efficiency, we consider the scenario that the communication between agents is compressed. Specifically, we consider a class of compressors with bounded relative compression error, and two classes of compressors respectively with globally and locally bounded absolute compression error satisfying the following assumptions.
Assumption 1. The compressor \( C : \mathbb{R}^d \mapsto \mathbb{R}^d \) satisfies

\[
E_C \left[ \left\| \frac{C(x)}{r} - x \right\|^2 \right] \leq (1 - \varphi) \|x\|^2, \quad \forall x \in \mathbb{R}^d,
\]

(3)

for some constants \( \varphi \in (0, 1] \) and \( r > 0 \). Here \( E_C[\cdot] \) denotes the expectation over the internal randomness of the stochastic compression operator \( C \).

From (3), we have

\[
E_C[\|C(x) - x\|^2] = E_C \left[ \left\| r \left( \frac{C(x)}{r} - x \right) + (r - 1)x \right\|^2 \right]
\leq 2r^2 E_C \left[ \left\| \frac{C(x)}{r} - x \right\|^2 \right] + 2(1 - r)^2 \|x\|^2
\leq r_0 \|x\|^2, \quad \forall x \in \mathbb{R}^d,
\]

(4)

where \( r_0 = 2r^2(1 - \varphi) + 2(1 - r)^2 \). Therefore, the class of compressors satisfying Assumption 1 is the same as that used in [30]. As explained in [30], the class of compressors satisfying Assumption 1 is broad, which incorporates the commonly used unbiased compressors and biased but contractive compressors, such as random quantization and sparsification, e.g., [13]–[18], [33]–[37], and also includes biased and non-contractive compressors, such as the norm-sign compressor. Moreover, it is straightforward to check that the class of compressors satisfying Assumption 1 also covers the three classes of biased compressors considered in [12]. In other words, Assumption 1 is weaker than various commonly used assumptions for compressors in the literature.

Assumption 2. The compressor \( C : \mathbb{R}^d \mapsto \mathbb{R}^d \) satisfies

\[
E_C[\|C(x) - x\|^p] \leq C, \quad \forall x \in \mathbb{R}^d,
\]

(5)

for some real number \( p \geq 1 \) and constant \( C \geq 0 \).

The same class of compressors satisfying Assumption 2 has also been used in [31], which incorporates the deterministic quantization used in [19]–[21] and the unbiased random quantization used in [21], [33], [38].

Assumption 3. The compressor \( C : \mathbb{R}^d \mapsto \mathbb{R}^d \) satisfies

\[
\|C(x) - x\|_p \leq (1 - \varphi), \quad \forall x \in \{ x \in \mathbb{R}^d : \|x\|_p \leq 1 \},
\]

(6)
for some real number \( p \geq 1 \) and constant \( \varphi \in (0, 1] \).

The same class of compressors satisfying Assumption 3 has also been used in [32], which covers the standard uniform quantizer with dynamic and fixed quantization levels respectively used in [24]–[28] and [29], and the Moniqua used in [11]. Moreover, as pointed out in [32], the 1-bit binary quantizer satisfies Assumption 3. The difference between Assumptions 2 and 3 is that the former is a global assumption while the latter is a local assumption. It should be pointed out that all the Assumptions 1–3 do not require the compressors to be unbiased.

The above three general classes of compressors cover most of existing compressors used in machine learning and signal processing applications, which substantiate the generality of our results later in this paper.

B. Communication Network and Cost Functions

The following assumptions for the problem (1) are made.

**Assumption 4.** The underlying communication network is modeled by an undirected and connected graph \( G \).

**Assumption 5.** The minimum function value of the optimization problem (1) is finite.

**Assumption 6.** Each local cost function \( f_i(x) \) is smooth with constant \( L_f > 0 \), i.e., it is differentiable and

\[
\|\nabla f_i(x) - \nabla f_i(y)\| \leq L_f \|x - y\|, \quad \forall x, y \in \mathbb{R}^d.
\]

**Assumption 7.** The global cost function \( f(x) \) satisfies the P–Ł condition with constant \( \nu > 0 \).

Assumptions 4–6 are standard in the literature to guarantee the well-known \( O(1/T) \) convergence rate. Assumption 7 is weaker than the assumption that the global or each local cost function is strongly convex, but it still can guarantee linear convergence. Note that the convexity of the cost functions and the boundedness of their gradients are not assumed. We also make no assumptions on the boundedness of the deviation between the gradients of local cost functions. In other words, we do not assume that \( \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x) - \nabla f(x)\|^2 \) is bounded. Moreover, we do not assume that the optimal set is a singleton or finite set either.
Algorithm 1

1: **Input**: positive parameters $\alpha$, $\beta$, $\eta$, and $\psi$.
2: **Initialize**: $x_{i,0} \in \mathbb{R}^d$, $a_{i,0} = b_{i,0} = v_{i,0} = 0_d$, and $q_{i,0} = C(x_{i,0})$, $\forall i \in [n]$.
3: for $k = 0, 1, \ldots$ do
4:   for $i = 1, \ldots, n$ in parallel do
5:     Broadcast $q_{i,k}$ to $\mathcal{N}_i$ and receive $q_{j,k}$ from $j \in \mathcal{N}_i$.
6:     Update
7:       $a_{i,k+1} = a_{i,k} + \psi q_{i,k}$, \hspace{1cm} (8a)
8:       $b_{i,k+1} = b_{i,k} + \psi \left( q_{i,k} - \sum_{j=1}^{n} L_{ij} q_{j,k} \right)$, \hspace{1cm} (8b)
9:       $x_{i,k+1} = x_{i,k} - \eta \alpha \left( a_{i,k} - b_{i,k} + \sum_{j=1}^{n} L_{ij} q_{j,k} \right) - \eta \beta v_{i,k} + \nabla f_i(x_{i,k})$, \hspace{1cm} (8c)
10:      $v_{i,k+1} = v_{i,k} + \eta \beta \left( a_{i,k} - b_{i,k} + \sum_{j=1}^{n} L_{ij} q_{j,k} \right)$, \hspace{1cm} (8d)
11:      $q_{i,k+1} = C(x_{i,k+1} - a_{i,k+1})$. \hspace{1cm} (8e)
5:     end for
6:   end for
7: **Output**: $\{x_{i,k}\}$.

IV. COMPRESSED COMMUNICATION ALGORITHM: BOUNDED RELATIVE COMPRESSION ERROR

In this section, we use the compressors with bounded relative compression error to design a communication-efficient decentralized algorithm and analyze the convergence properties of the proposed algorithm.

A. Algorithm Description

The communication-efficient decentralized algorithm is presented in pseudo-code as Algorithm 1.

In Algorithm 1, each agent $i$ only communicates the compressed variable $q_{i,k}$ with its neighbors. Therefore, Algorithm 1 is a communication-efficient algorithm.

Noting that $a_{i,0} = b_{i,0} = 0_d$, by mathematical induction, it is straightforward to check that $b_{i,k} = a_{i,k} - \sum_{j=1}^{n} L_{ij} a_{j,k}$, $\forall i \in [n]$. Denote

$$\hat{x}_{i,k} = a_{i,k} + q_{i,k},$$

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then (8c) and (8d) respectively can be rewritten as

\[ x_{i,k+1} = x_{i,k} - \eta \left( \alpha \sum_{j=1}^{n} L_{ij} \hat{x}_{j,k} + \beta v_{i,k} + \nabla f_i(x_{i,k}) \right), \tag{10a} \]

\[ v_{i,k+1} = v_{i,k} + \eta \beta \sum_{j=1}^{n} L_{ij} \hat{x}_{j,k}. \tag{10b} \]

Note that replacing \( \hat{x}_{j,k} \) in (10) by \( x_{j,k} \) yields the decentralized primal–dual algorithm proposed in [9]. Therefore, Algorithm 1 is a communication-efficient extension of the decentralized primal–dual algorithm proposed in [9].

B. Convergence Analysis

In this section, we provide convergence analysis for both scenarios without and with Assumption 7. We first have the following convergence result for Algorithm 1 without Assumption 7.

**Theorem 1.** Suppose that Assumptions 2 and 4–6 hold. Let \( \{x_{i,k}\} \) be the sequence generated by Algorithm 1 with \( \alpha = \kappa_1 \beta, \beta > \kappa_2, \eta \in (0, \kappa_3), \) and \( \psi \in (0, 1/r], \) where \( \kappa_1, \kappa_2, \kappa_3 \) are positive constants given in Appendix B. Then, for any \( T \in \mathbb{N}_0, \)

\[ \sum_{k=0}^{T} \sum_{i=1}^{n} \mathbb{E}_C[\|x_{i,k} - \bar{x}_k\|^2 + \|\nabla f(\bar{x}_k)\|^2] = O(1), \tag{11a} \]

\[ \mathbb{E}_C[f(\bar{x}_T) - f^*] = O(1), \tag{11b} \]

where \( \bar{x}_k = \frac{1}{n} \sum_{i=1}^{n} x_{i,k}. \)

**Proof:** The explicit expressions of the right-hand sides of (11a)–(11b), and the proof are given in Appendix B.

We have several remarks on Theorem 1. Firstly, from (11a), we know that \( \min_{k \in [T]} \{ \sum_{i=1}^{n} \mathbb{E}_C[\|x_{i,k} - \bar{x}_k\|^2 + \|\nabla f(\bar{x}_k)\|^2] \} = O(1/T). \) In other words, Algorithm 1 finds a first-order stationary point with the well-known rate \( O(1/T), \) which is the same as that achieved by the decentralized algorithms with accurate communication in the literature, e.g., [5]–[9]. Secondly, from (11b), we know that the cost difference between the global optimum and the resulting stationary point is bounded. Thirdly, it should be pointed out that the settings on the parameters \( \alpha, \beta, \) and \( \eta \) are just sufficient conditions. With some modifications of the proofs, other forms of settings for these algorithm parameters still can guarantee the same type of convergence result. Fourthly,
observe that the definitions of $\kappa_1$ and $\kappa_2$ given in Appendix B are independent of the parameters related to the compressors. Therefore, the choice of the parameters $\alpha$ and $\beta$ is independent of the compressors. Finally, the proof of Theorem 1 is inspired by the proof of Theorem 1 in [9]. However, due to the compressed compression, a different Lyapunov function is appropriately designed and the details are also different.

Then, with Assumption 7 the following result states that Algorithm 1 can find a global optimum and the convergence rate is linear.

**Theorem 2.** Suppose that Assumptions 1 and 4–7 hold. Let $\{x_{i,k}\}$ be the sequence generated by Algorithm 1 with the same $\alpha$, $\beta$, $\eta$, and $\psi$ given in Theorem 1. Then, for any $k \in \mathbb{N}_0$,

$$\sum_{i=1}^{n} \mathbb{E}[\|x_{i,k} - \bar{x}_k\|^2 + f(\bar{x}_k) - f^*] = O((1 - \epsilon)^k),$$

where $\epsilon$ is a constant in $(0, 1)$ given in Appendix C.

**Proof:** The explicit expression of the right-hand side of (12), and the proof are given in Appendix C.

We have several remarks on Theorem 2. Firstly, observe that Algorithm 1 uses the same algorithm parameters for the cases without and with the P–Ł condition in Theorems 1 and 2, respectively. As a result, it is not needed to check the P–Ł condition before implementing Algorithm 1 which is important since it is normally difficult to check that condition. Secondly, compared to [30] which used the same type of compressors and established linear convergence under the condition that the global cost function is strongly convex, we show linear convergence under the weaker P–Ł condition and only use a half number of compression and communication operations per iteration since in the algorithm proposed in [30] each agent needs to communicate two compressed variables with its neighbors. Thirdly, compared to [17], [18] which used unbiased compressors with bounded relative compression error and established linear convergence under the condition that each local cost function is strongly convex, we use the more general compressors and the weaker P–Ł condition to show linear convergence. Lastly, compared to [16] which used unbiased compressors with bounded relative compression error but only achieved sublinear convergence under the condition that each local cost function is strongly convex, we not only use the more general compressors and the weaker P–Ł condition, but also show strictly faster convergence.
Algorithm 2

1: **Input**: positive parameters $\alpha$, $\beta$, $\eta$, $\psi$, and $\sigma$.
2: **Initialize**: $x_{i,0} \in \mathbb{R}^d$, $a_{i,0} = b_{i,0} = e_{i,0} = v_{i,0} = 0$, and $q_{i,0} = \hat{q}_{i,0} = C(x_{i,0})$, $\forall i \in [n]$.
3: **for** $k = 0, 1, \ldots$ **do**
4:   **for** $i = 1, \ldots, n$ in parallel **do**
5:     Broadcast $q_{i,k}$ and $\hat{q}_{i,k}$ to $\mathcal{N}_i$ and receive $q_{j,k}$ and $\hat{q}_{j,k}$ from $j \in \mathcal{N}_i$.
6:     Update
   \[
   \begin{align*}
   a_{i,k+1} &= a_{i,k} + \psi q_{i,k}, \\
   b_{i,k+1} &= b_{i,k} + \psi \left( q_{i,k} - \sum_{j=1}^{n} L_{ij} q_{j,k} \right), \\
   x_{i,k+1} &= x_{i,k} - \eta \alpha \left( a_{i,k} - b_{i,k} + \sum_{j=1}^{n} L_{ij} \hat{q}_{j,k} \right) \\
   &\quad - \eta \left( \beta v_{i,k} + \nabla f_i(x_{i,k}) \right), \\
   v_{i,k+1} &= v_{i,k} + \eta \beta \left( a_{i,k} - b_{i,k} + \sum_{j=1}^{n} L_{ij} \hat{q}_{j,k} \right), \\
   q_{i,k+1} &= C(x_{i,k+1} - a_{i,k+1}), \\
   e_{i,k+1} &= \sigma e_{i,k} + x_{i,k} - a_{i,k} - \hat{q}_{i,k}, \\
   \hat{q}_{i,k+1} &= C(\sigma e_{i,k+1} + x_{i,k+1} - a_{i,k+1}).
   \end{align*}
   \]
7:   **end for**
8: **end for**
9: **Output**: $\{x_{i,k}\}$.

V. ERROR FEEDBACK BASED COMPRESSED COMMUNICATION ALGORITHM: BOUNDED RELATIVE COMPRESSION ERROR

In this section, we extend Algorithm 1 to error feedback version for biased compressors particularly.

A. **Algorithm Description**

The error feedback based communication-efficient decentralized algorithm is presented in pseudo-code as Algorithm 2.

Without ambiguity, we denote
\[
\hat{x}_{i,k} = a_{i,k} + \hat{q}_{i,k},
\]
then (13c) and (13d) respectively can be written as (10a) and (10b) since \( b_{i,k} = a_{i,k} - \sum_{j=1}^{n} L_{ij} a_{j,k} \), \( \forall i \in [n] \). Therefore, Algorithm 2 also is a communication-efficient extension of the decentralized primal–dual algorithm proposed in [9].

Compared to Algorithm 1 in Algorithm 2 there are two new variables \( \hat{q}_{i,k} \) and \( e_{i,k} \) which are used to estimate the biased compression error and accumulate the biased compression errors, respectively. Then each agent can use \( \hat{q}_{i,k} \) to correct the bias induced by the biased compressors.

However, compared to Algorithm 1, there are twice number of compression and communication operations per iteration in Algorithm 2.

B. Convergence Analysis

Algorithm 2 has similar convergence properties as Algorithm 1. Similar to Theorem 1, we first have the following sublinear convergence result for Algorithm 2 without Assumption 7.

**Theorem 3.** Suppose that Assumptions 1 and 4–6 hold. Let \( \{x_{i,k}\} \) be the sequence generated by Algorithm 2 with \( \alpha = \kappa_1 \beta, \beta > \kappa_2, \eta \in (0, \tilde{\kappa}_3), \sigma \in (0, \kappa_0) \) and \( \psi \in (0, 1/r] \), where \( \kappa_1, \kappa_2 \) and \( \kappa_0, \tilde{\kappa}_3 \) are positive constants given in Appendices 3 and 7 respectively. Then, for any \( T \in \mathbb{N}_0 \),

\[
\sum_{k=0}^{T} \sum_{i=1}^{n} \mathbb{E}_c[\|x_{i,k} - \bar{x}_k\|^2 + \|\nabla f(\bar{x}_k)\|^2] = O(1), \tag{15a}
\]

\[
\mathbb{E}_c[f(\bar{x}_T) - f^*] = O(1). \tag{15b}
\]

**Proof:** The explicit expressions of the right-hand sides of (15a)–(15b), and the proof are given in Appendix 1.

Similar to Theorem 2, we then have the following linear convergence result for Algorithm 2 with Assumption 7.

**Theorem 4.** Suppose that Assumptions 1 and 4–7 hold. Let \( \{x_{i,k}\} \) be the sequence generated by Algorithm 2 with the same \( \alpha, \beta, \eta, \sigma, \) and \( \psi \) given in Theorem 3. Then, for any \( k \in \mathbb{N}_0 \),

\[
\sum_{i=1}^{n} \mathbb{E}_c[\|x_{i,k} - \bar{x}_k\|^2 + f(\bar{x}_k) - f^*] = O((1 - \bar{\epsilon})^k), \tag{16}
\]

where \( \bar{\epsilon} \) is a constant in \((0, 1)\) given in Appendix 4.

\(^1\)For unbiased compressors, it is unnecessary to consider error feedback since \( \mathbb{E}_c[e_{i,k}] = 0_d \).
Algorithm 3

1: Input: positive parameters $\alpha$, $\beta$, $\eta$, and a positive scaling sequence $\{s_k\}$.
2: Initialize: $x_{i,0} \in \mathbb{R}^d$, $\hat{x}_{i,-1} = y_{i,-1} = v_{i,0} = 0$, and $q_{i,0} = \mathcal{C}(x_{i,0}/s_0)$, $\forall i \in [n]$.
3: for $k = 0, 1, \ldots$ do
4: for $i = 1, \ldots, n$ in parallel do
5: Broadcast $q_{i,k}$ to $\mathcal{N}_i$ and receive $q_{j,k}$ from $j \in \mathcal{N}_i$.
6: Update
   \begin{align}
   \hat{x}_{i,k} &= \hat{x}_{i,k-1} + s_k q_{i,k}, \quad (17a) \\
   y_{i,k} &= y_{i,k-1} + s_k q_{i,k} - s_k \sum_{j=1}^{n} L_{ij} q_{j,k}, \quad (17b) \\
   x_{i,k+1} &= x_{i,k} - \eta \alpha(\hat{x}_{i,k} - y_{i,k}) - \eta \beta v_{i,k} + \nabla f_i(x_{i,k}), \quad (17c) \\
   v_{i,k+1} &= v_{i,k} + \eta \beta(\hat{x}_{i,k} - y_{i,k}), \quad (17d) \\
   q_{i,k+1} &= \mathcal{C}( (x_{i,k+1} - \hat{x}_{i,k})/s_{k+1} ). \quad (17e)
   \end{align}
7: end for
8: end for
9: Output: $\{x_{i,k}\}$.

Proof: The explicit expression of the right-hand side of (16), and the proof are given in Appendix E. ■

VI. COMPRESSED COMMUNICATION ALGORITHM: BOUNDED ABSOLUTE COMPRESSION ERROR

In this section, we use the compressors with bounded absolute compression error to design a communication-efficient decentralized algorithm and analyze the convergence properties of the proposed algorithm in various setups.

A. Algorithm Description

The communication-efficient decentralized algorithm is presented in pseudo-code as Algorithm 3.

By mathematical induction, it is straightforward to check that $y_{i,k} = \hat{x}_{i,k} - \sum_{j=1}^{n} L_{ij} \hat{x}_{j,k}$, $\forall i \in [n]$. Therefore, (17c) and (17d) can be rewritten as (10a) and (10b), respectively. Therefore, Algorithm 3 also is a communication-efficient extension of the decentralized primal–dual algorithm proposed in [9]. Moreover, same as Algorithm 1 in Algorithm 3 each agent only communicates
one compressed variable with its neighbors. The difference between Algorithms 1 and 3 is that they use different types of compressors.

B. Convergence Analysis

Similar to Theorem 1, we first have the following sublinear convergence result for Algorithm 3 when the class of compressors satisfying Assumption 2 is used.

**Theorem 5.** Suppose that Assumptions 2 and 4–6 hold. Let \( \{x_{i,k}\} \) be the sequence generated by Algorithm 3 with \( \alpha = \kappa_1 \beta, \beta > \kappa_2, \eta \in (0, \tilde{\kappa}_3), \) and \( s_k = s_0 \gamma^k, \) where \( \kappa_1 \) and \( \kappa_2 \) are positive constants given in Appendix B, \( \tilde{\kappa}_3 \) a positive constant given in Appendix F, \( s_0 \) is an arbitrary positive constant, and \( \gamma \) is an arbitrary constant in \( (0, 1) \). Then, for any \( T \in \mathbb{N}_0, \)

\[
\sum_{k=0}^{T} \sum_{i=1}^{n} \mathbb{E}_C[\|x_{i,k} - \bar{x}_k\|^2 + \|\nabla f(\bar{x}_k)\|^2] = \mathcal{O}(1),
\]

(18a)

\[
\mathbb{E}_C[f(\bar{x}_T) - f^*] = \mathcal{O}(1).
\]

(18b)

**Proof:** The explicit expressions of the right-hand sides of (18a)–(18b), and the proof are given in Appendix \( F \).

The remarks after Theorem 1 are still valid for Theorem 5. Moreover, we would like to point out that the choice of the parameter \( \gamma \) is also independent of the compressors since the definition of \( \tilde{\kappa}_3 \) given in Appendix F is independent of the parameters related to the compressors.

Similar to Theorem 2, we then have the following linear convergence result for Algorithm 3 when the class of compressors satisfying Assumption 2 is used.

**Theorem 6.** Suppose that Assumptions 2 and 4–7 hold. Let \( \{x_{i,k}\} \) be the sequence generated by Algorithm 3 with the same \( \alpha, \beta, \eta, \) and \( s_k \) given in Theorem 5. Then, for any \( k \in \mathbb{N}_0, \)

\[
\sum_{i=1}^{n} \mathbb{E}_C[\|x_{i,k} - \bar{x}_k\|^2 + f(\bar{x}_k) - f^*] = \mathcal{O}((1 - \tilde{\epsilon})^k),
\]

(19)

where \( \tilde{\epsilon} \) is a constant in \( (0, 1) \) given in Appendix G.

**Proof:** The explicit expression of the right-hand side of (19), and the proof are given in Appendix G.

Compared to [31] which used the same class of compressors satisfying Assumption 2, Theorem 6 shows that a global optimum can be precisely found with a linear convergence rate under
the P–Ł condition. In contrast, although [31] assumed the stronger strong convexity assumption and also showed that convergence rate is linear, the parallel algorithms proposed in [31] only converged to a neighbor of the unique optimal point.

We also have the following linear convergence result for Algorithm 3 when the class of compressors satisfying Assumption 3 is used.

**Theorem 7.** Suppose that Assumptions 3–7 hold and the P–Ł constant $\nu$ is known in advance. Let $\{x_{i,k}\}$ be the sequence generated by Algorithm 3 with $\alpha = \kappa_1 \beta$, $\beta > \kappa_2$, $\eta \in (0, \hat{\kappa}_3)$, and $s_k = s_0 \gamma^k$, where $\kappa_1$ and $\kappa_2$ are positive constants given in Appendix B, $\hat{\kappa}_3$, $s_0$, and $\gamma$ are positive constants given in Appendix H with $\gamma \in (0, 1)$. Then, for any $k \in \mathbb{N}_0$,

$$\sum_{i=1}^{n} (\|x_{i,k} - \bar{x}_k\|^2 + f(\bar{x}_k) - f^*) = \mathcal{O}(\gamma^k).$$

**Proof:** The explicit expression of the right-hand side of (20), and the proof are given in Appendix H.

We have several remarks on Theorem 7. Firstly, compared to Theorems 2, 4, and 6 in Theorem 7 the P–Ł constant $\nu$ needs to be known in advance, which is used to design the parameters $s_0$ and $\gamma$ as shown in Appendix H. This is a potential drawback since this constant is normally unknown due to the difficulty to check the P–Ł condition. However, for strongly convex cost functions, this is not a drawback since if a function is strongly convex with convex parameter $\nu$, then it also satisfies the P–Ł condition with the same constant $\nu$. Secondly, linear convergence has also been established in [32] which used the same type of compressors. However, [32] assumed that each local cost function is strongly convex, which is stronger than the condition that the global cost function satisfies the P–Ł condition as used in Theorem 7 and required that the absolute compression error satisfies an inequality determined by the number of agents and the communication network, which is not needed in Theorem 7. Moreover, [32] required an unpractical condition that the unique optimal point needs to be known a priori to design algorithm parameters, which is a drawback. Thirdly, compared to [25]–[28] which used the standard uniform quantizer with dynamic quantization level and established linear convergence under the condition that each local cost function is strongly convex, we use the more general compressors and the weaker P–Ł condition to show linear convergence. Finally, compared to [29] which used the standard uniform quantizer with fixed quantization level and established linear convergence under the assumption that each local cost function is quadratic and the global
cost function is strongly convex, we not only use the more general compressors but also consider the more general nonconvex functions satisfying the weaker P–Ł condition.

VII. SIMULATIONS

In this section, we verify and illustrate the theoretical results through numerical simulations.

We consider the nonconvex distributed binary classification problem studied in [8], [9], which is formulated as the optimization problem (1) with each component function \( f_i \) being given by

\[
 f_i(x) = \frac{n}{m} \sum_{l=1}^{m_i} \left( (1 - y_{il}) \log \left( 1 + e^{x^\top z_{il}} \right) + y_{il} \log \left( 1 + e^{-x^\top z_{il}} \right) \right) + \sum_{s=1}^{d} \frac{\lambda \mu [x]_s^2}{1 + \mu [x]_s^2},
\]

where \( m = \sum_{i=1}^n m_i, m_i \) is the number of observations held privately by agent \( i \), \( z_{il} \in \mathbb{R}^p \) is the \( l \)-th observation with label \( y_{il} \in \{0, 1\} \) owned by agent \( i \), \( \lambda \) and \( \mu \) are regularization parameters, and \( [x]_s \) is the \( s \)-th coordinate of \( x \in \mathbb{R}^d \). All settings for cost functions and the communication graph are the same as those described in [8], [9]. Specifically, \( n = 20, d = 50, m_i = 200, \lambda = 0.001 \), and \( \mu = 1 \). The graph used in the simulation is the random geometric graph and the graph parameter is set to be 0.5. We independently and randomly generate \( m \) data points.

We consider the following five compressors:

- **Unbiased \( l \)-bits quantizer** [17]

\[
 C_1(x) = \frac{\|x\|_\infty^{2l-1}}{2l-1} \text{sign}(x) \circ \left[ \frac{2^{l-1}\|x\|_\infty}{\|x\|_\infty} + \omega \right],
\]

where \( \text{sign}(\cdot), |\cdot|, \text{ and } [\cdot] \) are the element-wise sign, absolute, and floor functions, respectively, \( \circ \) denotes the Hadamard product, and \( \omega \) is a random perturbation vector uniformly sampled from \([0, 1]^d\). This compressor is unbiased and satisfies Assumption[1] with \( r = 1 + r_1, \varphi = 1/(1 + r_1) \), and \( r_1 = d/4^l \). As pointed out in [32], transmitting \( C_1(x) \) needs \((l+1)d + b_1 \) bits if a scalar can be transmitted with \( b_1 \) bits with sufficient precision, since only \( \|x\|_\infty, \text{sign}(x) \), and the positive integer in the bracket need to be transmitted. In this section, we choose \( l = 2 \) and \( b_1 = 64 \).

- **Greedy (Top-\( k \)) sparsifier** [12]

\[
 C_2(x) = \sum_{s=1}^{k} [x]_{i_s} e_{i_s},
\]

where \( \{e_1, \ldots, e_d\} \) is the standard basis of \( \mathbb{R}^d \) and \( i_1, \ldots, i_k \) are the indices of largest \( k \) coordinates in magnitude of \( x \). This compressor is biased but contractive. Moreover, it
satisfies Assumption 1 with \( r = 1 \) and \( \varphi = k/d \). Therefore, it also satisfies Assumption 3 with \( p \geq 2 \) and \( \varphi = k/d \). Transmitting \( C_2(x) \) needs \( kb_1 \) bits since only \( k \) scalars need to be transmitted. In this section, we choose \( k = 10 \).

- Norm-sign compressor

\[ C_3(x) = \frac{\|x\|_\infty}{2} \text{sign}(x). \]

This compressor is biased and non-contractive, but satisfies Assumption 1 with \( r = d/2 \) and \( \varphi = 1/d^2 \), see [30]. It also satisfies Assumption 3 with \( p = \infty \) and \( \varphi = 0.5 \). Transmitting \( C_3(x) \) needs \( 2d + b_1 \) bits since only \( \|x\|_\infty \) and \( \text{sign}(x) \) need to be transmitted.

- Standard uniform quantizer

\[ C_4(x) = \Delta \left\lfloor \frac{x}{\Delta} + \frac{1}{2} \right\rfloor, \]

where \( \Delta \) is a positive integer. This compressor satisfies Assumption 2 with \( p = \infty \) and \( C = \Delta^2/4 \). Therefore, it also satisfies Assumption 3 with \( p = \infty \) and \( \varphi = 0.5 \) when \( \Delta = 1 \). Transmitting \( C_4(x) \) needs \( db_2 \) bits if \( b_2 \) bits are allocated to transmit an integer. In this section, we choose \( \Delta = 1 \) and \( b_2 = 8 \).

- 1-bit binary quantizer [32]

\[ C_5(x) = \text{col}(Q_1([x]_1), \ldots, Q_1([x]_d)), \]

where \( Q_1([x]_s) = 0.5 \) for \( [x]_s \geq 0 \) and \( Q_1([x]_s) = -0.5 \) otherwise. This compressor satisfies Assumption 3 with \( p = \infty \) and \( \varphi = 0.5 \). Transmitting \( C_5(x) \) needs \( d \) bits since for each coordinate only two symbols needs to be transmitted.

We implement Algorithm 1 using \( C_1\text{–}C_3 \), Algorithm 2 using \( C_2 \) and \( C_3 \), and Algorithm 3 using \( C_2\text{–}C_5 \). Noting that to the best of our knowledge in the literature there are no other similar communication-efficient decentralized algorithms for decentralized nonconvex optimization as ours, we only compare the proposed communication-efficient decentralized algorithms with their uncompressed counterpart, i.e., the decentralized primal–dual algorithm (DPDA) proposed in [9]. It is straightforward to see that each agent sends \( db_1 \) bits per iteration when implementing DPDA. All the hyper-parameters used in the experiment are given in TABLE 1.
TABLE I: Parameter settings for different algorithm and compressor combinations.

| Algorithm | Compressor | $\alpha$ | $\beta$ | $\eta$ | $\psi$ | $\sigma$ | $s_0$ | $\gamma$ |
|-----------|------------|----------|---------|--------|--------|---------|--------|--------|
| DPDA      | —          | 85       | 5       | 1.4    | —      | —       | —      | —      |
| Algorithm 1 | $C_1$        | 85       | 5       | 1.4    | 0.2    | —       | —      | —      |
| Algorithm 1 | $C_2$        | 85       | 5       | 1.4    | 0.05   | —       | —      | —      |
| Algorithm 1 | $C_3$        | 85       | 5       | 1.3    | 0.05   | —       | —      | —      |
| Algorithm 2 | $C_2$        | 85       | 5       | 1.4    | 0.05   | 0.03    | —      | —      |
| Algorithm 2 | $C_3$        | 85       | 5       | 1.3    | 0.05   | 0.03    | —      | —      |
| Algorithm 3 | $C_2$        | 85       | 5       | 0.46   | —      | —       | 1      | 0.99   |
| Algorithm 3 | $C_3$        | 85       | 5       | 0.64   | —      | —       | 1      | 0.99   |
| Algorithm 3 | $C_4$        | 85       | 5       | 0.46   | —      | —       | 0.01   | 0.99   |
| Algorithm 3 | $C_5$        | 85       | 5       | 0.46   | —      | —       | 1      | 0.99   |

Fig. 1: Evolutions of $P(T)$ with respect to the number of iterations.

We use

$$P(T) = \min_{k \in [T]} \left\{ \| \nabla f(\bar{x}_k) \|^2 + \frac{1}{n} \sum_{i=1}^{n} \| x_{i,k} - \bar{x}_k \|^2 \right\}$$
Fig. 2: Evolutions of $P(T)$ with respect to the number of transmitted bits.

to measure the performance of each algorithm. We plot the convergence of $P(T)$ with respect to both number of iterations and bits transmitted between two neighbor agents for the above algorithm and compressor combinations with the same initial condition, as shown in Fig. 1 and Fig. 2 respectively. Moreover, the comparison of transmitted bits for different algorithm and compressor combinations to reach $P(T) \leq 10^{-30}$ is provided in Fig. 3. We highlight the following observations:

- From Fig. 1 we can see that all of the algorithm and compressor combinations have comparable convergence speeds as the corresponding algorithm with accurate communication, i.e., DPDA, which is consistent with our theoretical results. Especially, Algorithm 1-C$_1$ and DPDA have almost the same convergence speed.

- From Fig. 1 we can also see that Algorithm 1-C$_3$ (Algorithm 2-C$_3$) has almost the same convergence speed as Algorithm 1-C$_2$ (Algorithm 2-C$_2$), and Algorithm 3-C$_3$ has faster convergence speed than Algorithm 3-C$_2$. Therefore, non-contractive compressors, e.g., C$_3$, can converge faster than contractive compressors, e.g., C$_2$. 
Fig. 3: Transmitted bits for different algorithm and compressor combinations to reach $P(T) \leq 10^{-30}$.

- From the zoomed figure in Fig. 1 we can see that the error feedback method Algorithm 2-C (Algorithm 2-C3) has faster convergence speed than Algorithm 1-C (Algorithm 1-C3), which demonstrates the benefit of using the error feedback to correct the bias induced by the biased compressors.
- From Fig. 2 we can see that our communication-efficient algorithms converge faster than their exact-communication counterpart when comparing their performances based on the number of bits that each agents communicates, which shows the effectiveness of our proposed algorithms. Especially, Algorithm 3-C5, Algorithm 1-C1, Algorithm 1-C3, Algorithm 3-C3, and Algorithm 2-C3 converge significantly faster than DPDA. For example, it is illustrated in Fig. 3 that Algorithm 3-C5 only needs 6.24% of the bits used by DPDA to reach a specific level of error.
- From Fig. 2 we can also see that Algorithm 1-C (Algorithm 1-C3) converges faster than its error feedback version, i.e., Algorithm 2-C (Algorithm 2-C3) when comparing their performances based on the number of transmitted bits, which reveals the drawback of using
the error feedback to correct the bias induced by the biased compressors.

VIII. CONCLUSIONS

In this paper, we studied communication compression for decentralized nonconvex optimization. We used three general classes of compressors to design three communication-efficient decentralized primal–dual algorithms. We showed that the proposed algorithms can achieve comparable convergence results to state-of-the-art algorithms although the communication is compressed. Interesting directions for future work include considering more general network topologies and reducing communication complexity.

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**APPENDIX**

A. Notations and Useful Lemmas

1. $1_n$ ($0_n$) denotes the column one (zero) vector of dimension $n$. $I_n$ is the $n$-dimensional identity matrix. $\text{col}(z_1, \ldots, z_k)$ is the concatenated column vector of vectors $z_i \in \mathbb{R}^{d_i}$, $i \in [k]$. Given a vector $[x_1, \ldots, x_n]^{\top} \in \mathbb{R}^n$, $\text{diag}([x_1, \ldots, x_n])$ is a diagonal matrix with the $i$-th diagonal element being $x_i$. The notation $A \otimes B$ denotes the Kronecker product of matrices $A$ and $B$. Given two symmetric matrices $M, N$, $M \succeq N$ means that $M - N$ is positive semi-definite. $\text{null}(A)$ is the null space of matrix $A$. $\rho(\cdot)$ stands for the spectral radius for matrices and $\rho_2(\cdot)$ indicates the minimum positive eigenvalue for matrices having positive eigenvalues. For any square matrix $A$, denote $\|x\|_A^2 = x^{\top}A^2x$.

The following results are used in the proofs.

**Lemma 1.** For any $x \in \mathbb{R}^d$, it holds that $\|x\|_p \leq \tilde{d}\|x\|_2$ and $\|x\|_2 \leq \tilde{d}\|x\|_p$, where $\tilde{d} = d^\frac{1}{p} \cdot d - \frac{1}{2} \cdot d$ and $\tilde{d} = 1$ when $p \in [1, 2]$, and $\tilde{d} = 1$ and $\tilde{d} = d^\frac{1}{p} \cdot d - \frac{1}{2} \cdot d$ when $p > 2$. 

Lemma 2. Let $L$ be the Laplacian matrix of an undirected and connected graph $G$ with $n$ agents and $K_n = I_n - \frac{1}{n} 1_n 1_n^\top$. Then $L$ and $K_n$ are positive semi-definite, $\text{null}(L) = \text{null}(K_n) = \{1_n\}$, $L \leq \rho(L) I_n$, $\rho(K_n) = 1$,

\begin{align}
K_n L &= L K_n = L, \quad \text{(21a)} \\
0 &\leq \rho_2(L) K_n \leq L \leq \rho(L) K_n. \quad \text{(21b)}
\end{align}

Moreover, there exists an orthogonal matrix $[r \, R] \in \mathbb{R}^{n \times n}$ with $r = \frac{1}{\sqrt{n}} 1_n$ and $R \in \mathbb{R}^{n \times (n-1)}$ such that

\begin{align}
L &= \begin{bmatrix} r & R \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_1 \end{bmatrix} \begin{bmatrix} r^\top \\ R^\top \end{bmatrix}, \quad \text{(22a)} \\
PL &= LP = K_n, \quad \text{(22b)} \\
\rho^{-1}(L) I_n &\leq P \leq \rho_2^{-1}(L) I_n, \quad \text{(22c)}
\end{align}

where $\Lambda_1 = \text{diag}(\{\lambda_2, \ldots, \lambda_n\})$ with $0 < \lambda_2 \leq \cdots \leq \lambda_n$ being the nonzero eigenvalues of the Laplacian matrix $L$, $Q = R \Lambda_1^{-1} R^\top$, and

\begin{align}
P &= \begin{bmatrix} r & R \end{bmatrix} \begin{bmatrix} \lambda_1^{-1} & 0 \\ 0 & \Lambda_1^{-1} \end{bmatrix} \begin{bmatrix} r^\top \\ R^\top \end{bmatrix}.
\end{align}

Proof: From Lemmas 1 and 2 in the online version of [43], we know that all the results except (22b)–(22c) hold.

From $[r \, R]$ is an orthogonal matrix, (22a), and the definitions of $K_n$, $P$, $Q$, it is straightforward to check that (22b)–(22c) holds.

B. Proof of Theorem [1]

Denote $\mathbf{x} = \text{col}(x_1, \ldots, x_n)$, $\tilde{f}(\mathbf{x}) = \sum_{i=1}^{n} f_i(x_i)$, $L = L \otimes I_d$, $H = \frac{1}{n}(1_n 1_n^\top \otimes I_d)$, $K = K_n \otimes I_d = I_{nd} - H$, $P = P \otimes I_p$, $\bar{x}_k = 1_n \otimes \bar{x}_k = H \mathbf{x}_k$, $g_k = \nabla \tilde{f}(\mathbf{x}_k)$, $g_k = H g_k$, $g^0_k = \nabla \tilde{f}(\bar{x}_k)$, $\bar{g}^0_k = H \bar{g}^0_k = 1_n \otimes \nabla f(\bar{x}_k)$. Moreover, without ambiguity, we denote $C(x) = \text{col}(C(x_1), \ldots, C(x_n))$.

We also denote

\begin{align}
\mathbf{M}_1 &= \frac{\alpha}{2} L - \frac{1}{4}(\beta + 4 + 5L_f^2) K,
\end{align}
\[ M_2 = \left( \beta^2 + \frac{1}{2} \right) K + \frac{\beta(\beta - \alpha)}{2} L + \frac{3\alpha^2}{2} L^2, \]
\[ U_k = \frac{1}{2} \| x_k \|_K^2 + \frac{1}{2} \| v_k + \frac{1}{\beta} g_k^0 \|_{P_{ee}}^2 + x_k^\top KP \left( v_k + \frac{1}{\beta} g_k^0 \right) + n(f(\bar{x}_k) - f^*), \]
\[ V_k = U_k + \| x_k - a_k \|^2, \]
\[ \dot{U}_k = \| x_k \|_K^2 + \| v_k + \frac{1}{\beta} g_k^0 \|_{P_{ee}}^2 + n(f(\bar{x}_k) - f^*), \]
\[ \dot{V}_k = \dot{U}_k + \| x_k - a_k \|^2, \]
\[ \kappa_1 \geq \max \left\{ \frac{9 + \kappa_4}{2\rho_2(L)^{1}}, 1 \right\}, \quad \kappa_2 = \max \{ \kappa_5, \sqrt{\kappa_6} \}, \]
\[ \kappa_3 = \min \left\{ \frac{\epsilon_1}{\epsilon_2}, \frac{\epsilon_3}{\epsilon_4}, \frac{\epsilon_5}{\epsilon_6}, \frac{\sqrt{\epsilon_8^2 + 4\epsilon_7\epsilon_9} - \epsilon_8}{2\epsilon_9} \right\}, \]
\[ \kappa_4 > 0, \quad \kappa_5 = \max \left\{ \frac{4 + 5L_j^2}{\kappa_4}, \frac{6}{\rho_2(L)} \right\}, \]
\[ \kappa_6 = \frac{8(\kappa_1 + 1)^2 L_j^2}{\kappa_5 \rho_2(L)} + \frac{4L_j^2}{\rho_2(L)}, \]
\[ \epsilon_1 = \frac{\alpha}{2} \rho_2(L) - \frac{1}{4}(9\beta + 4 + 5L_j^2), \]
\[ \epsilon_2 = 3L_j^2 + \frac{4(2 + \varphi \psi r)}{\varphi \psi r} (\alpha^2 \rho^2(L) + L_j^2) + 2\beta^2 + 1 + 3\alpha^2 \rho^2(L), \]
\[ \epsilon_3 = \frac{\beta}{2} - 3\rho_2^{-1}(L), \]
\[ \epsilon_4 = 2\beta^2 \rho(L) + \frac{4(2 + \varphi \psi r)\beta^2 \rho^2(L)}{\varphi \psi r} + \rho_2^{-1}(L), \]
\[ \epsilon_5 = \frac{1}{8} - \frac{(\alpha + \beta)^2 L_j^2}{\beta^5 \rho_2(L)} - \frac{L_j^2}{2\beta^2 \rho_2^2(L)}, \]
\[ \epsilon_6 = \frac{(\alpha + \beta) L_j^2}{2\beta^3 \rho_2(L)} + \frac{3L_j^2}{4} + \frac{L_j^2}{2\beta^2 \rho_2^2(L)} + \frac{L_j}{2}, \]
\[ \epsilon_7 = \frac{\varphi \psi r}{2}, \]
\[ \epsilon_8 = \frac{1}{2}(\alpha + 2\beta) \rho(L) r_0 + 2\beta r_0, \]
\[ \epsilon_9 = \frac{(8 + 7\varphi \psi r)\alpha^2 \rho^2(L) r_0}{\varphi \psi r} + (2\beta^2 + 1) r_0, \]
\[ \epsilon_{10} = \frac{\alpha \rho_2(L) - \beta}{2\alpha \rho_2(L)}, \]
\[ c_1 = \left( \frac{(\alpha + \beta)^2}{\eta \beta^5} + \frac{\alpha + \beta}{2\beta^3} \right) \frac{1}{\rho_2(L)} + \frac{1}{2}, \]
\[
c_2 = \frac{\eta + 1}{2\eta^2 \rho^2_2(L)} + \frac{1}{4}, \quad c_3 = \frac{\varphi \psi r}{2},
\]
\[
c_4 = c_3 + 2c_3^2 - 4\eta^2 (1 + c_3^{-1}) \alpha^2 \rho^2(L)r_0,
\]
\[
c_5 = \frac{\alpha}{2} \rho_2(L) - \frac{1}{4} (\beta + 4 + 5L^2_j),
\]
\[
c_6 = 3L^2_j + 4(1 + c_3^{-1})(\alpha^2 \rho^2(L) + L^2),
\]
\[
c_7 = \beta^2 + \frac{1}{2} + \frac{3\alpha^2 \rho^2(L)}{2}.
\]

To prove Theorem 1, the following lemma is used, which presents a general relation between two consecutive outputs of Algorithm 1.

\textbf{Lemma 3.} Suppose Assumptions 1 and 4–6 hold. Let \( \{x_{i,k}\} \) be the sequence generated by Algorithm 1 with \( \alpha \geq \beta \) and \( \psi \in (0, 1/r] \). Then,

\[
E[\sigma_{k+1}] \leq E\left[ V_k - \eta \frac{1}{2} \left\| \hat{g}_k^0 \right\|^2 - \left\| x_k \right\|^2_{\eta(\epsilon_1 - \eta \epsilon_2)} - \left\| v_k + \frac{1}{\beta} \hat{g}_k^0 \right\|^2_{\eta(\epsilon_3 - \eta \epsilon_4)} - \eta(\epsilon_5 - \eta \epsilon_6) \left\| \tilde{g}_k \right\|^2 - (\epsilon_7 - \eta \epsilon_8 - \eta^2 \epsilon_9) \left\| x_k - a_k \right\|^2 \right].
\]

(23)

\textbf{Proof:} We first note that \( V_k \) is well defined since \( f^* > -\infty \) as assumed in Assumption 5.

The compact form of (8a), (8b), (10a), (10b), and (8e) is

\[
a_{k+1} = a_k + \psi q_k, \tag{24a}
\]
\[
b_{k+1} = b_k + \psi (I_{np} - L)q_k, \tag{24b}
\]
\[
x_{k+1} = x_k - \eta (\alpha L\hat{x}_k + \beta v_k + \nabla \tilde{f}(x_k)), \tag{24c}
\]
\[
v_{k+1} = v_k + \eta \beta L\hat{x}_k, \tag{24d}
\]
\[
q_{k+1} = C(x_{k+1} - a_{k+1}). \tag{24e}
\]

Denote \( \tilde{v}_k \) by \( \frac{1}{n}(1_n^\top \otimes I_p)\nu_k \). Then, from (24d) and \( \sum_{i=1}^n L_{ij} = 0 \), we know that \( \tilde{v}_{k+1} = \tilde{v}_k \).

This together with the fact that \( \sum_{i=1}^n \tilde{v}_{i,0} = 0_d \) implies

\[
\tilde{v}_k = 0_d. \tag{25}
\]

Then, from (25) and (24c), we know that

\[
x_{k+1} = x_k - \eta \hat{g}_k. \tag{26}
\]
Noting that $\nabla \tilde{f}$ is Lipschitz-continuous with constant $L_f > 0$ as assumed in Assumption 6, we have

$$
\|g^0_k - g_k\|^2 \leq L_f^2 \|\bar{x}_k - x_k\|^2 = L_f^2 \|x_k\|_K^2. \tag{27}
$$

Then, from \((27)\) and $\rho(H) = 1$, we have

$$
\|g^0_k - g_k\|^2 = \|H(g^0_k - g_k)\|^2 \leq \|g^0_k - g_k\|^2 \leq L_f^2 \|x_k\|_K^2. \tag{28}
$$

From $\nabla \tilde{f}$ is Lipschitz-continuous and \((26)\), we have

$$
\|g^0_{k+1} - g^0_k\|^2 \leq L_f^2 \|\bar{x}_{k+1} - \bar{x}_k\|^2 = \eta^2 L_f^2 \|g_k\|^2. \tag{29}
$$

From Lemma 1.2.3 in [44], we know that \((7)\) implies

$$
|f_i(y) - f_i(x) - (y - x)^\top \nabla f_i(x)| \leq \frac{L_f}{2} \|y - x\|^2, \ \forall x, y \in \mathbb{R}^d. \tag{30}
$$

From \((30)\) and \((26)\), we have

$$
\tilde{f}(\bar{x}_{k+1}) - \tilde{f}(\bar{x}_k) \leq -\eta g_k^\top g_k^0 + \frac{\eta^2 L_f}{2} \|g_k\|^2. \tag{31}
$$

We have

$$
\frac{1}{2} \|x_{k+1}\|_K^2 = \frac{1}{2} \|x_k - \eta(\alpha L \hat{x}_k + \beta v_k + g_k)\|_K^2
$$

$$
= \frac{1}{2} \|x_k\|_K^2 - \eta \alpha x_k^\top L \hat{x}_k + \|\hat{x}_k\|^2 \frac{\beta \alpha}{2} L^2
$$

$$
- \eta \beta (x_k^\top - \eta \alpha \hat{x}_k^\top L) K \left( v_k + \frac{1}{\beta} g_k \right) + \|v_k + \frac{1}{\beta} g_k\|^2 \frac{\beta \alpha}{2} K
$$

$$
= \frac{1}{2} \|x_k\|_K^2 - \eta \alpha x_k^\top L (x_k + \hat{x}_k - x_k) + \|\hat{x}_k\|^2 \frac{\beta \alpha}{2} L^2
$$

$$
- \eta \beta (x_k^\top - \eta \alpha \hat{x}_k^\top L) K \left( v_k + \frac{1}{\beta} g_k^0 + \frac{1}{\beta} g_k - \frac{1}{\beta} g_k^0 \right)
$$

$$
+ \left\|v_k + \frac{1}{\beta} g_k^0 + \frac{1}{\beta} g_k - \frac{1}{\beta} g_k^0\right\|^2 \frac{\beta \alpha}{2} K
$$

$$
\leq \frac{1}{2} \|x_k\|_K^2 - \|x_k\|^2 \frac{\beta \alpha}{2} L^2 + \|x_k\|^2 \frac{\beta \alpha}{2} L^2 + \|x_k - x_k\|^2 \frac{\beta \alpha}{2} L^2
$$

$$
+ \|\hat{x}_k\|^2 \frac{\beta \alpha}{2} L^2 - \eta \beta x_k^\top K \left( v_k + \frac{1}{\beta} g_k^0 \right) + \frac{\eta \beta}{2} \|x_k\|^2 K
$$

$$
+ \frac{\eta \beta}{2} \|g_k - g_k^0\|^2 + \|\hat{x}_k\|^2 \frac{\beta \alpha}{2} L^2 + \frac{\eta \beta^2}{2} \left\|v_k + \frac{1}{\beta} g_k^0\right\|^2
$$

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\begin{align*}
&+ \| \dot{x}_k \|^2 \frac{2\alpha}{2} + \frac{\eta^2}{2} \| g_k - g_k^0 \|^2 \\
&+ \eta^2 \beta^2 \| v_k + \frac{1}{\beta} g_k^0 \|^2 + \eta^2 \| g_k - g_k^0 \|^2 \\
= & - \frac{1}{2} \| x_k \|_K^2 - \| x_k \|^2 \frac{2\alpha}{2} L - \frac{\alpha}{2} K + \| \ddot{x}_k \|^2 \frac{2\alpha}{2} L^2 \\
&+ \frac{\eta}{2} (1 + 3\eta) \| g_k - g_k^0 \|^2 + \| \ddot{x}_k - x_k \|^2 \frac{2\alpha}{2} L \\
&- \eta \beta (\dot{x}_k + x_k - \dot{x}_k) \top K \left( v_k + \frac{1}{\beta} g_k^0 \right) \\
&+ \frac{3\eta^2 \beta^2}{2} \| v_k + \frac{1}{\beta} g_k^0 \|^2 \\
\leq & - \frac{1}{2} \| x_k \|_K^2 - \| x_k \|^2 \frac{2\alpha}{2} L - \frac{\alpha}{2} K - \frac{2\alpha}{2} (1 + 3\eta) L \bar{K} + \| \ddot{x}_k \|^2 \frac{2\alpha}{2} L^2 \\
&- \eta \beta \dot{x}_k \top K \left( v_k + \frac{1}{\beta} g_k^0 \right) + \| v_k + \frac{1}{\beta} g_k^0 \|^2 \frac{2\alpha}{2} L + \eta \beta^2 \| \ddot{x}_k \|^2 \frac{2\alpha}{2} L \\
&+ \frac{\eta}{2} (\alpha + 2\beta) \rho(L) \| x_k - \dot{x}_k \|^2, 
\end{align*}

where the first equality holds due to \((24c)\); the second equality holds due to \((21a)\); the first and second inequalities hold due to the Cauchy–Schwarz inequality and \(\rho(K) = 1\); and the last inequality holds due to \((22c)\) and \((27)\).

We have

\begin{align*}
\frac{1}{2} \| v_{k+1} + \frac{1}{\beta} g_{k+1}^0 \|^2 \frac{2\alpha}{2} P
&= \frac{1}{2} \| \dot{v}_k + \frac{1}{\beta} g_k^0 \|^2 + \eta \beta L \dot{x}_k + \frac{1}{\beta} (g_k^0 - g_k^0) \|^2 \frac{2\alpha}{2} P \\
&= \frac{1}{2} \| \dot{v}_k + \frac{1}{\beta} g_k^0 \|^2 + \eta (\alpha + \beta) \ddot{x}_k \top K \left( v_k + \frac{1}{\beta} g_k^0 \right) \\
&+ \| \ddot{x}_k \|^2 \frac{2\alpha}{2} (\alpha + \beta) L + \frac{1}{2\beta^2} \| g_k^0 - g_k^0 \|^2 + \frac{1}{\beta} (v_k + \frac{1}{\beta} g_k^0) \top \left( P + \frac{\alpha}{\beta} P \right) (g_{k+1}^0 - g_k^0) \\
&+ \eta \ddot{x}_k \top \left( K + \frac{\alpha}{\beta} K \right) (g_{k+1}^0 - g_k^0)
\end{align*}
where the first equality holds due to (24d); the second equality holds due to (22b); the first inequality holds since the Cauchy–Schwarz inequality; the second inequality holds due to (22c); the first equality holds due to (24d); the second equality holds due to (22b); the first equality holds due to (24d); the second equality holds due to (22b); and the last inequality holds due to (29).

We have

\[
\mathbf{x}_{k+1}^\top K P \left( \mathbf{v}_{k+1} + \frac{1}{\beta} \mathbf{g}_k^0 \right) = \left( \mathbf{x}_k - \eta (\alpha L \hat{\mathbf{x}}_k + \beta \mathbf{v}_k + \mathbf{g}_k^0 + \mathbf{g}_k - \mathbf{g}_k^0) \right)^\top K P \left( \mathbf{v}_k + \frac{1}{\beta} \mathbf{g}_k^0 \right) \\
+ \frac{1}{\beta} \mathbf{g}_k^0 + \eta \beta L \hat{\mathbf{x}}_k + \frac{1}{\beta} (\mathbf{g}_{k+1}^0 - \mathbf{g}_k^0) \\
= \left( \mathbf{x}_k^\top K P - \eta (\alpha + \eta \beta^2) \hat{\mathbf{x}}_k^\top K \right) \left( \mathbf{v}_k + \frac{1}{\beta} \mathbf{g}_k^0 \right) \\
+ \eta \beta \mathbf{x}_k^\top K \hat{\mathbf{x}}_k - \| \hat{\mathbf{x}}_k \|^2_{\eta^2 \alpha L}.
\]
\[
\begin{align*}
&+ \frac{1}{\beta} (x_k^\top KP - \eta \alpha \hat{x}_k^\top K)(g_{k+1}^0 - g_k^0) \\
&- \eta (\beta v_k + g_k^0 + g_k - g_k^0 - \hat{g}_k)\top P (v_k + \frac{1}{\beta} g_k^0) \\
&- \eta \left( v_k + \frac{1}{\beta} g_k^0 \right)^\top P K (g_{k+1}^0 - g_k^0) \\
&- \eta (g_k - g_k^0)^\top \left( \eta \beta K \hat{x}_k + \frac{1}{\beta} KP (g_{k+1}^0 - g_k^0) \right) \\
\leq (x_k^\top KP - \eta \alpha \hat{x}_k^\top K) \left( v_k + \frac{1}{\beta} g_k^0 \right) + \|x_k\|_2^2 + \|\hat{x}_k\|_2^2 + \|\hat{g}_k\|_2^2 \\
&+ \eta \beta \hat{x}_k \top K (g_{k+1}^0 - g_k^0) - \|v_k + \frac{1}{\beta} g_k^0\|_{\eta \beta}^2 \\
&+ \eta \left( g_k - g_k^0 \right)^2 + \eta \|g_k\|_2^2 + \|v_k + \frac{1}{\beta} g_k^0\|_{3\eta \beta}^2 \\
&+ \|v_k + \frac{1}{\beta} g_k^0\|_{\eta \beta}^2 + \|g_k - g_k^0\|_2^2 + \frac{\eta}{2} \|g_k - g_k^0\|_2^2 \\
&+ \|\hat{x}_k\|_2^2 + \eta \|g_k - g_k^0\|_2^2 + \|g_{k+1}^0 - g_k^0\|_{2\beta}^2 \\
&= (x_k^\top KP - \eta \alpha \hat{x}_k^\top K) \left( v_k + \frac{1}{\beta} g_k^0 \right) + \|x_k\|_{2(\beta+2)}^2 \\
&+ \|\hat{x}_k\|_{\eta \beta K + \eta^2 (\beta^2 K - \alpha \beta L)}^2 + \frac{\eta}{4} (1 + 6\eta) \|g_k - g_k^0\|_2^2 \\
&+ \|g_{k+1}^0 - g_k^0\|_{2\eta \beta}^2 + \|\hat{g}_k\|_{2\eta \beta}^2 + \|g_k - g_k^0\|_2^2 \\
&+ \eta \left( g_k - g_k^0 \right)^2 + \eta \|g_k\|_2^2 - \|v_k + \frac{1}{\beta} g_k^0\|_{\eta \beta P - 3\eta \beta P^2 - \eta^2 P^2 - \frac{\eta}{2} \beta^2}^2 \\
\leq (x_k^\top KP - \eta \alpha \hat{x}_k^\top K) \left( v_k + \frac{1}{\beta} g_k^0 \right) + \|x_k\|_{2(\beta+2)}^2 \\
&+ \|\hat{x}_k\|_{\eta \beta K + \eta^2 (\beta^2 K - \alpha \beta L)}^2 + \frac{\eta}{4} (1 + 6\eta) \|g_k - g_k^0\|_2^2 \\
&+ c_2 \|g_{k+1}^0 - g_k^0\|_2^2 + \frac{\eta \alpha}{\beta} \hat{x}_k^\top K (g_{k+1}^0 - g_k^0) + \frac{\eta}{8} \|\hat{g}_k\|_2^2 \\
&- \|v_k + \frac{1}{\beta} g_k^0\|_{\eta (\beta - \beta^2 \rho^{-1}(L)) P - \eta^2 (\rho^{-1}(L) + \frac{\rho^2}{\beta^2} \rho(L)) P}^2 \\
\leq x_k^\top KP \left( v_k + \frac{1}{\beta} g_k^0 \right) - \eta \alpha \hat{x}_k^\top K \left( v_k + \frac{1}{\beta} g_k^0 \right) \\
&+ \|\hat{x}_k\|_{\eta \beta K + \eta^2 (\beta^2 K - \alpha \beta L)}^2 + \|x_k\|_{2(\beta+2)}^2 + \frac{\eta}{8} (1 + 6\eta) \|L\|_2^2 \\
\end{align*}
\]
where the first equality holds due to (24c) and (24d); the second equality holds due to (21a), (22b), (25) and $K = I_{nd} - H$; the first inequality holds due to the Cauchy–Schwarz inequality and $\rho(K) = 1$; the second inequality holds due to (22c); and the last inequality holds due to (27) and (29).

We have

\[
n(f(\tilde{x}_{k+1}) - f^*) = \tilde{f}(\tilde{x}_{k+1}) - n f^* + \tilde{f}(\tilde{x}_{k+1}) - \tilde{f}(\tilde{x}_k) \\
\leq \tilde{f}(\tilde{x}_k) - n f^* - \eta g_k^\top g_k + \frac{\eta^2 L_f}{2} \|g_k\|^2 \\
= \tilde{f}(\tilde{x}_k) - n f^* - \eta g_k^\top g_k + \frac{\eta^2 L_f}{2} \|g_k\|^2 \\
= n(f(\tilde{x}_k) - f^*) - \eta g_k^\top (g_k + \rho_k^0 - g_k) \\
- \frac{\eta}{2} (g_k - g_k^0 + g_k^0)^\top \rho_k^0 + \frac{\eta^2 L_f}{2} \|g_k\|^2 \\
\leq n(f(\tilde{x}_k) - f^*) - \frac{\eta}{4} \|g_k\|^2 + \frac{\eta}{4} \|g_k^0 - g_k\|^2 - \frac{\eta}{4} \|g_k^0\|^2 \\
+ \frac{\eta}{4} \|g_k^0 - g_k\|^2 + \frac{\eta^2 L_f}{2} \|g_k\|^2 \\
= n(f(\tilde{x}_k) - f^*) - \frac{\eta}{4} (1 - 2\eta L_f) \|g_k\|^2 + \frac{\eta}{2} \|g_k^0 - g_k\|^2 - \frac{\eta}{4} \|g_k^0\|^2 \\
\leq n(f(\tilde{x}_k) - f^*) - \frac{\eta}{4} (1 - 2\eta L_f) \|g_k\|^2 + \|x_k\|^2 H g_k^\top K - \frac{\eta}{4} \|g_k^0\|^2, 
\]

(35)

where the first inequality holds due to (31); the third equality holds due to $g_k^\top g_k^0 = g_k^\top H g_k^0 = g_k^\top H g_k^0 = \tilde{g}_k^\top g_k^0$; the second inequality holds due to the Cauchy–Schwarz inequality; and the last inequality holds due to (28).

Denote $C_r(\cdot) = C(\cdot)/r$, then we have

\[
\|x_{k+1} - a_{k+1}\|^2 = \|x_{k+1} - x_k + x_k - a_k - \psi q_k\|^2 \\
= \|x_{k+1} - x_k + (1 - \psi r)(x_k - a_k) + \psi r(x_k - a_k - C_r(x_k - a_k))\|^2 \\
\leq (1 + c_3^{-1}) \|x_{k+1} - x_k\|^2 + (1 + c_3) \|1 - \psi r)(x_k - a_k) + \psi r(x_k - a_k - C_r(x_k - a_k))\|^2 \\
\]

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\[ \begin{align*}
&\leq (1 + c_3^{-1})\|x_{k+1} - x_k\|^2 + (1 + c_3)(1 - \psi r)\|x_k - a_k\|^2 \\
&\quad + (1 + c_3)\psi r\|x_k - a_k - C_r(x_k - a_k)\|^2 \\
&\leq (1 + c_3^{-1})\|x_{k+1} - x_k\|^2 + (1 + c_3)(1 - \psi r)\|x_k - a_k\|^2 \\
&\quad + (1 + c_3)\psi r(1 - \varphi)\|x_k - a_k\|^2 \\
&= (1 + c_3^{-1})\|x_{k+1} - x_k\|^2 + (1 - c_3 - 2c_3^2)\|x_k - a_k\|^2; \\
\end{align*} \tag{36} \]

where the first equality holds due to (24a); the second equality holds due to (24c); the first inequality holds due to the Cauchy–Schwarz inequality and \(c_3 > 0\); the second inequality holds due to the Cauchy–Schwarz inequality and \(\psi r \in (0, 1]\); and the last inequality holds due to (3).

We have

\[\begin{align*}
\|x_{k+1} - x_k\|^2 &= \eta^2\|\alpha L\hat{x}_k + \beta v_k + g_k\|^2 \\
&= \eta^2\|\alpha L(\hat{x}_k - x_k) + \alpha Lx_k + \beta v_k + g_k^0 + g_k - g_k^0\|^2 \\
&\leq 4\eta^2(\alpha^2\|\hat{x}_k - x_k\|^2 L^2 + \alpha^2\|x_k\|^2 L^2 + \|\beta v_k + g_k^0\|^2 + \|g_k - g_k^0\|^2) \\
&\leq 4\eta^2\left(\alpha^2\eta^2(L)\|x_k - x_k\|^2 + \|x_k\|^2(\alpha^2\eta^2(L) + L_2^2)K + \|v_k + \frac{1}{\beta}g_k^0\|^2 \beta^2\eta^2(L)P, \right) \tag{37}
\end{align*}\]

where the first equality holds due to (24c); the first inequality holds due to the Cauchy–Schwarz inequality; and the second inequality holds due to (21b), (22c), and (27).

From (9), (24c), and (4), we have

\[\begin{align*}
E_C[\|x_k - \hat{x}_k\|^2] = E_C[\|x_k - a_k - C(x_k - a_k)\|^2] \leq r_0\|x_k - a_k\|^2. \tag{38}
\end{align*}\]

From (36)–(38), we have

\[\begin{align*}
E_C[\|x_{k+1} - a_{k+1}\|^2] &\leq E_C\left[(1 - c_4)\|x_k - a_k\|^2 + \|x_k\|^2 4\eta^2(1 + c_3^{-1})(\alpha^2\eta^2(L) + L_2^2)K + \|v_k + \frac{1}{\beta}g_k^0\|^2 \beta^2\eta^2(L)P, \right]. \tag{39}
\end{align*}\]

Then, we have

\[\begin{align*}
E_C[V_{k+1}] &\leq E_C\left[V_k - \|x_k\|^2 \frac{1}{2\lambda}L - \frac{\eta}{4}K - \frac{\eta}{4}(1 + 3\eta)L_2^2K + \|\hat{x}_k\|^2 \frac{2\alpha^2\eta^2}{1 + \alpha^2\eta^2}L_2^2 \\
&\quad + \|v_k + \frac{1}{\beta}g_k^0\|^2 \beta^2\eta^2(L) + \eta^2L_2^2P + \eta^2(\alpha + 2\beta)\rho(L)r_0\|x_k - a_k\|^2 \right]
\end{align*}\]
From (38), (40) and (41), we know that (23) holds.

We first show that all of the used constants are positive.

We are now ready to prove Theorem 1. From \(\alpha\), we have

\[
\kappa \epsilon = 1 = \kappa
\]

\[
\epsilon \leq \beta \leq \alpha
\]

\[
\parallel \hat{x}_k \parallel^2 = \parallel x_k - x_k + x_k \parallel^2 \leq 2 \parallel x_k - x_k \parallel^2 + 2 \parallel x_k \parallel^2.
\]

From (38), (40) and (41), we know that (23) holds.

We are now ready to prove Theorem 1.

(i) We first show that all of the used constants are positive.

From \(\alpha = \kappa_1 \beta\), \(\kappa_1 \geq \frac{9 + \kappa_4}{2 \rho_2(L)}\), \(\kappa_4 > 0\), and \(\beta > \kappa_2 \geq \frac{4 + 5L_\beta^2}{\kappa_4}\), we have

\[
\epsilon_1 = \frac{\kappa_1 \beta}{2} \rho_2(L) - \frac{1}{4} (9 \beta + 4 + 5L_\beta^2) > \frac{\kappa_1 \beta}{2} \rho_2(L) - \frac{1}{4} (9 \beta + \kappa_4 \beta) \geq 0.
\]
From $\beta > \kappa_2 \geq \frac{6}{\rho_2(L)}$, we have

$$\epsilon_3 > 0. \quad (43)$$

From $\alpha = \kappa_1 \beta$ and $\beta > \kappa_2 = \max\{\kappa_5, \sqrt{\kappa_6}\}$, we have

$$\epsilon_5 = \frac{1}{8} - \frac{(\kappa_1 + 1)^2 L_f^2}{\beta \rho(L)} - \frac{L_f^2}{2 \beta^2 \rho_2^2(L)} > \frac{1}{8} - \frac{(\kappa_1 + 1)^2 L_f^2}{\kappa_5 \beta^2 \rho_2(L)} - \frac{L_f^2}{2 \beta^2 \rho_2^2(L)} > 0. \quad (44)$$

From (42)–(44), we have

$$\eta(\epsilon_1 - \eta \epsilon_2) > 0, \quad (45a)$$

$$\eta(\epsilon_3 - \eta \epsilon_4) > 0, \quad (45b)$$

$$\eta(\epsilon_5 - \epsilon_6) > 0, \quad (45c)$$

$$\epsilon_7 - \eta \epsilon_8 - \eta^2 \epsilon_9 > 0. \quad (45d)$$

(ii) We then show that (11a) and (11b) hold.

From the Cauchy–Schwarz inequality, we have

$$V_k \geq \frac{1}{2} \|x_k\|_K^2 + \frac{1}{2} \left(1 + \frac{\alpha}{\beta}\right) \left\|v_k + \frac{1}{\beta} g_k^0\right\|_P^2 - \frac{\beta}{2 \alpha \rho_2(L)} \|x_k\|_K^2$$

$$- \frac{\alpha}{2 \beta} \left\|v_k + \frac{1}{\beta} g_k^0\right\|_P^2 + n(f(\bar{x}_k) - f^*) + \|x_k - a_k\|^2 \quad (46a)$$

$$\geq \epsilon_{10} \left(\|x_k\|_K^2 + \left\|v_k + \frac{1}{\beta} g_k^0\right\|^2_P + n(f(\bar{x}_k) - f^*) + \|x_k - a_k\|^2 \right)$$

$$\geq \epsilon_{10} \hat{V}_k \geq 0. \quad (46b)$$

From (23) and (45b)–(45d), we have

$$E_C[V_{T+1}] \leq V_0 - \sum_{k=0}^{T} E_C[\|x_k\|_{\eta(\epsilon_1 - \eta \epsilon_2) K}] - \sum_{k=0}^{T} E_C \left[\frac{\eta}{4} \|g_k^0\|^2\right]. \quad (47)$$

From (47), (45a), and (46b), we have

$$\sum_{k=0}^{T} E_C[\|x_k\|_K^2 + \|g_k^0\|^2] \leq \frac{V_0}{\min\{\eta(\epsilon_1 - \eta \epsilon_2), \frac{\eta}{4}\}},$$

which yields (11a).
From (47), (45a), and (46a), we have

\[ E_C[n(f(\bar{x}_T) - f^*)] \leq E_C[V_T] \leq V_0, \]

which yields (47b).

C. Proof of Theorem 2

In this proof, in addition to the notations used in the proof of Theorem 1, we also denote

\[ \epsilon = \frac{\epsilon_{12}}{\epsilon_{11}}, \quad \epsilon_{11} = \max \left\{ \frac{1}{2} + \frac{\alpha}{\beta}, \frac{\alpha \rho_2(L) + \beta}{2 \alpha \rho_2(L)} \right\}, \]

\[ \epsilon_{12} = \eta \min \left\{ \epsilon_1 - \eta \epsilon_2, \epsilon_3 - \eta \epsilon_4, \frac{\nu}{\eta}, \frac{\nu}{\epsilon_7} \right\}. \]

(i) We first show that \( \epsilon \in (0, 1) \).

From (45a)–(45d), we have

\[ \epsilon_{12} > 0 \quad \text{and} \quad \epsilon = \frac{\epsilon_{12}}{\epsilon_{11}} > 0. \]  \hspace{1cm} (48)

Noting that \( \epsilon_{11} \geq \frac{1}{2} + \frac{\alpha}{\beta} \geq \frac{3}{2} \), and \( \epsilon_{12} < \epsilon_7 = \frac{\varphi \psi r}{1 + \varphi \psi r} < 1 \) due to \( \varphi \psi r \in (0, 1) \), we have

\[ 0 < \epsilon = \frac{\epsilon_{12}}{\epsilon_{11}} < 1. \]  \hspace{1cm} (49)

(ii) We then show that (12) holds.

From the Cauchy–Schwarz inequality, we have

\[ V_k \leq \epsilon_{11} \hat{V}_k. \]  \hspace{1cm} (50)

From Assumptions 5 and 7 as well as (2), we have that

\[ \| g_k^0 \|_2^2 = n \| \nabla f(\bar{x}_k) \|_2^2 \geq 2 \nu n (f(\bar{x}_k) - f^*). \]  \hspace{1cm} (51)

Then, from (23), (45c), (48), and (50)–(51), we have

\[ E_C[V_{k+1}] \leq E_C[V_k - \epsilon_{12} \hat{V}_k] \leq E_C[V_k - \frac{\epsilon_{12}}{\epsilon_{11}} \hat{V}_k]. \]  \hspace{1cm} (52)

Hence, from (52) and (49), we have

\[ E_C[V_{k+1}] \leq (1 - \epsilon) E_C[V_k] \leq (1 - \epsilon)^{k+1} V_0, \]
which yields (12).

D. Proof of Theorem 3

In this proof, in addition to the notations used in the proof of Theorem 1, we also denote

\[
\kappa_0 = \frac{1}{\sqrt{r_0 r_1}}, \quad r_1 > 1, \quad r_2 = \frac{r_1}{r_1 - 1},
\]

\[
\hat{\kappa}_3 = \min \left\{ \frac{\epsilon_1}{\epsilon_2}, \frac{\epsilon_3}{\epsilon_4}, \frac{\epsilon_5}{\epsilon_6}, \frac{\tau_1}{\tau_2} \right\},
\]

\[
\hat{\epsilon}_8 = ((\alpha + 2\beta)\rho(L) + 4\beta)(r_0 r_1 + 1)\sigma^2,
\]

\[
\hat{\epsilon}_9 = 2(7\alpha^2 \rho^2(L) + 2\beta^2 + 1)(r_0 r_1 + 1)\sigma^2,
\]

\[
\hat{\epsilon}_{10} = \alpha \rho_2(L) - \beta, \quad \hat{c}_4 = c_3 + 2\epsilon_3^2 - 16\eta^2(1 + c_3^{-1})\alpha^2 \rho^2(L)r_0 r_2,
\]

\[
\tau_0 \in (0, \frac{\epsilon_7}{r_0 r_2}),
\]

\[
\tau_1 = \sqrt{\frac{r_2^2 \epsilon_8^2 + 2r_2 \epsilon_9(\epsilon_7 - r_0 r_2 \tau_0) - r_2^2 \epsilon_8}{2r_2^2 \epsilon_9}},
\]

\[
\tau_2 = \frac{\sqrt{\epsilon_8^2 + 4(1 - r_0 r_1 \sigma^2)\tau_0 \epsilon_9 - \epsilon_8}}{2 \epsilon_9},
\]

\[
\tau_3 = \epsilon_7 - r_0 r_2 \tau_0 - 2\eta r_2 \epsilon_8 - 2\eta^2 r_2 \epsilon_9,
\]

\[
\tau_4 = (1 - r_0 r_1 \sigma^2)\tau_0 - \eta \epsilon_8 - \eta^2 \epsilon_9,
\]

\[
W_k = V_k + \tau_0 \|e_k\|^2, \quad \hat{W}_k = \hat{V}_k + \tau_0 \|e_k\|^2.
\]

(i) We first show that all of the used constants are positive.

Noting that the settings on \(\alpha\) and \(\beta\) in both Theorems 1 and 3 are the same, (42)–(44) still hold. From (42)–(44) and \(\eta \in (0, \hat{\kappa}_3)\), we know that (45a)–(45c) still hold.

From \(\tau_0 \in (0, \frac{\epsilon_7}{r_0 r_2})\), we have

\[
\epsilon_7 - r_0 r_2 \tau_0 > 0 \quad \text{and} \quad \tau_1 > 0.
\]

Then, from (53) and \(\eta \in (0, \tau_1)\), we have

\[
\tau_3 = \epsilon_7 - r_0 r_2 \tau_0 - 2\eta r_2 \epsilon_8 - 2\eta^2 r_2 \epsilon_9 > 0.
\]
From $\sigma \in (0, \kappa_0)$ and $\tau_0 > 0$ we have

$$(1 - r_0r_1\sigma^2)\tau_0 > 0 \text{ and } \tau_2 > 0.$$  \hfill (55)

Then, from (55) and $\eta \in (0, \tau_2)$, we have

$$\tau_4 = (1 - r_0r_1\sigma^2)\tau_0 - \eta \hat{c}_8 - \eta^2 \hat{c}_9 > 0.$$  \hfill (56)

(ii) We then show that (15a) and (15b) hold.

Noting that the compact form of (13c) and (13d) respectively can be rewritten as (24c) and (24d), we know that (32)–(35), and (37) still hold. Moreover, (36) still holds since the compact form of (13a) and (13e) is (24a) and (24e), respectively.

From (13g), (13f), and (4), we have

$$\mathbb{E}_{\mathcal{C}}[\|e_{k+1}\|^2] \leq r_0\|\sigma e_k + x_k - a_k\|^2 \leq r_0r_1\sigma^2\|e_k\|^2 + r_0r_2\|x_k - a_k\|^2.$$  \hfill (57)

We have

$$\mathbb{E}_{\mathcal{C}}[\|x_k - \hat{x}_k\|^2] = \mathbb{E}_{\mathcal{C}}[\|e_{k+1} - \sigma e_k\|^2]$$

$$\leq \mathbb{E}_{\mathcal{C}}[2\|e_{k+1}\|^2 + 2\sigma^2\|e_k\|^2]$$

$$\leq \mathbb{E}_{\mathcal{C}}[2(r_0r_1 + 1)\sigma^2\|e_k\|^2 + 2r_0r_2\|x_k - a_k\|^2],$$  \hfill (58)

where the first equality holds due to (14) and (13f); and the last inequality holds due to (57).

From (56), (37), (58), we have

$$\mathbb{E}_{\mathcal{C}}[\|x_k + 1 - a_{k+1}\|^2] \leq \mathbb{E}_{\mathcal{C}}[(1 - \hat{c}_4)\|x_k - a_k\|^2 + \|x_k\|_{4\eta^2(1+c_3^{-1})(\alpha^2\rho^2(L)+L^2)^3}K]$$

$$+ \left\|v_k + \frac{1}{\beta}g_k\right\|^2_{4\eta^2(1+c_3^{-1})(\alpha^2\rho^2(L)+L^2)^3P} + 8\eta^2\alpha^2\rho^2(L)(r_0r_1 + 1)\sigma^2\|e_k\|^2.$$  \hfill (59)

Similar to the way to get (40), from (32)–(35), (58), and (59), we have

$$\mathbb{E}_{\mathcal{C}}[V_{k+1}] \leq \mathbb{E}_{\mathcal{C}}\left[V_k - \|x_k\|^2_{\frac{2\alpha}{\alpha+2\beta}L - \frac{2}{\alpha+2\beta}K - \frac{\eta}{(1+3\eta)L^2}K} + \|\hat{x}_k\|^2_{\frac{2}{\alpha+2\beta}L^2} + \left\|v_k + \frac{1}{\beta}g_k\right\|^2_{\frac{2\eta^2\alpha^2\rho^2(L)+\eta^2\rho^2(L)}{4\eta^2(1+c_3^{-1})(\alpha^2\rho^2(L)+L^2)^3P}}$$

$$+ \eta(\alpha + 2\beta)\rho(L)((r_0r_1 + 1)\sigma^2\|e_k\|^2 + r_0r_2\|x_k - a_k\|^2)$$

$$+ \|\hat{x}\|^2_{\frac{2}{\alpha+2\beta}(\alpha+2\beta)L + \frac{2}{\alpha+2\beta}K} + \left\|v_k + \frac{1}{\beta}g_k\right\|^2_{\frac{2\eta^2\alpha^2\rho^2(L)+\eta^2\rho^2(L)}{4\eta^2(1+c_3^{-1})(\alpha^2\rho^2(L)+L^2)^3P}},$$

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+ η^2 c_1 L_2^2 \|\hat{g}_k\|^2 + \|\hat{x}_k\|^2_{\eta^2 K + \eta^2 (\beta^2 K - \alpha \beta L)}
+ \|x_k\|^2_{\frac{2(\beta+2)}{4} K + \frac{1}{4}(1+6\eta)L_2^2 K} + \left(\eta^2 c_2 L_2^2 + \frac{\eta}{8}\right)\|\bar{g}_k\|^2
- \left\|v_k + \beta g_k\right\|^2_{\eta^2 K\eta^2 (\beta - 3\rho^2(L))\rho^2(L) - \eta^2 (\beta^2 (\bar{g}_k k_0)^2 \|\rho(L)\|^2 P - \eta^2 \rho^2(\rho^2(L) + \frac{\beta^2}{2} \rho(L)) P
- \frac{\eta}{4}(1 - 2\eta L_L) \|\bar{g}_k\|^2 + \|x_k\|^2_{\frac{2}{4} L_2^2 K} - \frac{\eta}{4} \|g_k^0\|^2
- \bar{c}_4 \|x_k - a_k\|^2 + \|x_k\|^2_{\frac{2}{4} \eta^2 (1 + c_4^2) (\alpha^2 \rho^2(L) + L_2^2)} K
+ \|v_k + \beta g_k\|^2_{\frac{2}{4} \eta^2 (1 + c_4^2) \beta^2 \rho^2(L) P}
+ 8\eta^2 \alpha^2 \rho^2(L) (r_0 r_1 + 1) \sigma^2 \|e_k\|^2 \right]}
= E_c \left[ V_k - \left\|x_k\right\|^2_{\eta M_1 - \eta^2 c_6 K} + \|\hat{x}_k\|^2_{\eta^2 K + \eta^2 M_2}
- \left\|v_k + \beta g_k^0\right\|^2_{\eta^2 (\zeta - \eta c_4) P} - \eta(\epsilon_5 - \eta \epsilon_6) \|\bar{g}_k\|^2
- \left(\bar{c}_4 - 2\eta(\alpha + 2\beta) \rho(L) r_0 r_2\right) \|x_k - a_k\|^2 - \frac{\eta}{4} \|g_k^0\|^2
+ \eta(\alpha + 2\beta + 8\alpha^2 \rho(L) \eta) \rho(L) (r_0 r_1 + 1) \sigma^2 \|e_k\|^2 \right]\]
\leq E_c \left[ V_k - \left\|x_k\right\|^2_{\eta^2 (\zeta - \eta c_6) K} + \|\hat{x}_k\|^2_{\eta^2 (\beta + \eta \gamma) K}
- \left\|v_k + \beta g_k^0\right\|^2_{\eta^2 (\zeta - \eta c_4) P} - \eta(\epsilon_5 - \eta \epsilon_6) \|\bar{g}_k\|^2
- \left(\bar{c}_4 - 2\eta(\alpha + 2\beta) \rho(L) r_0 r_2\right) \|x_k - a_k\|^2 - \frac{\eta}{4} \|g_k^0\|^2
+ \eta(\alpha + 2\beta + 8\alpha^2 \rho(L) \eta) \rho(L) (r_0 r_1 + 1) \sigma^2 \|e_k\|^2 \right].
(60)

From (57), (58), (60) and (41), we have

\[ E_c[W_{k+1}] \leq E_c \left[ W_k - \frac{\eta}{4} \|g_k^0\|^2 - \left\|x_k\right\|^2_{\eta^2 (\zeta - \eta c_6) K} + \left\|v_k + \beta g_k^0\right\|^2_{\eta^2 (\zeta - \eta c_4) P}
- \eta(\epsilon_5 - \eta \epsilon_6) \|\bar{g}_k\|^2 - \tau_3 \|x_k - a_k\|^2 - \tau_4 \|e_k\|^2 \right].
(61)

Same as the way to get (46a) and (46b), we have

\[ W_k \geq \epsilon_{10} \left(\|x_k\|^2_K + \left\|v_k + \beta g_k^0\right\|^2_P + n(f(\bar{x}_k) - f^*) + \|x_k - a_k\|^2 + \tau_0 \|e_k\|^2 \right)
\geq \epsilon_{10} \hat{W}_k \geq 0.
(62a)
From (61), (45b)–(45c) (54), and (56), we have

\[ \mathbb{E}_C[W_{T+1}] \leq W_0 - \sum_{k=0}^{T} \mathbb{E}_C[\|x_k\|^2_\mathcal{K}] - \sum_{k=0}^{T} \mathbb{E}_C\left[\frac{\eta}{4}\|\tilde{g}_k\|^2\right]. \tag{63} \]

From (63), (45a), and (62b), we have

\[ \sum_{k=0}^{T} \mathbb{E}_C[\|x_k\|^2_\mathcal{K} + \|\tilde{g}_k\|^2] \leq \frac{W_0}{\min\{\eta(\epsilon_1 - \eta \epsilon_2), \frac{\eta}{4}\}}, \]

which yields (15a).

From (63), (45a), and (62a), we have

\[ \mathbb{E}_C[n\left(f(\bar{x}_T) - f^\star\right)] \leq \mathbb{E}_C[W_T] \leq W_0, \]

which yields (15b).

\textbf{E. Proof of Theorem 4}

In this proof, in addition to the notations used in the proofs of Theorems 1–3, we also denote

\[ \bar{\epsilon} = \frac{\epsilon_{12}}{\epsilon_{11}}, \quad \epsilon_{12} = \eta \min\left\{\epsilon_1 - \eta \epsilon_2, \epsilon_3 - \eta \epsilon_4, \frac{\nu}{2}, \frac{\tau_3}{\eta}, \frac{\tau_4}{\eta} \right\}. \]

(i) We first show that \( \bar{\epsilon} \in (0, 1) \).

From (45b)–(45c) (54), and (56), we have

\[ \bar{\epsilon}_{12} > 0 \text{ and } \bar{\epsilon} = \frac{\bar{\epsilon}_{12}}{\epsilon_{11}} > 0. \tag{64} \]

From \( \tau_2 < \epsilon_7 \) and (49), we have

\[ 0 < \bar{\epsilon} = \frac{\bar{\epsilon}_{12}}{\epsilon_{11}} \leq \frac{\epsilon_{12}}{\epsilon_{11}} < 1. \tag{65} \]

(ii) We then show that (16) holds.

From the Cauchy–Schwarz inequality, we have

\[ W_k \leq \epsilon_{11} \hat{W}_k. \tag{66} \]

Then, from (61), (45c), (64), (66), and (51), we have

\[ \mathbb{E}_C[W_{k+1}] \leq \mathbb{E}_C[W_k - \bar{\epsilon}_{12} \hat{W}_k] \leq \mathbb{E}_C\left[W_k - \frac{\bar{\epsilon}_{12}}{\epsilon_{11}} W_k\right]. \tag{67} \]
Hence, from (67) and (65), we have
\[ E_C[W_{k+1}] \leq (1 - \tilde{\epsilon})E_C[W_k] \leq (1 - \tilde{\epsilon})^{k+1}W_0, \]
which yields (16).

F. Proof of Theorem 5

In this proof, in addition to the notations used in the proof of Theorem 1, we also denote
\[ \tilde{\kappa}_3 = \min \left\{ \frac{\epsilon_1}{\tilde{\epsilon}_2}, \frac{\epsilon_3}{\tilde{\epsilon}_4}, \frac{\epsilon_5}{\tilde{\epsilon}_6} \right\}, \]
\[ \tilde{\epsilon}_2 = 3L_f^2 + 2\beta^2 + 1 + 3\alpha^2 \rho^2(L), \]
\[ \tilde{\epsilon}_4 = 2\beta^2 \rho(L) + \rho_2^{-1}(L), \]
\[ \tilde{\epsilon}_8 = \frac{1}{2}(\alpha + 2\beta)\rho(L) + 2\beta, \]
\[ c_8 = \eta(\tilde{\epsilon}_8 + 2c_7\eta)n\tilde{d}^2Cs_0^2. \]

(i) We first show that all of the used constants are positive.

Noting that the settings on \( \alpha \) and \( \beta \) in both Theorems 1 and 5 are the same, (42)–(44) still hold. From (42)–(44), and \( 0 < \eta < \tilde{\kappa}_3 \), we have
\[ \eta(\epsilon_1 - \eta \tilde{\epsilon}_2) > 0, \quad (68a) \]
\[ \eta(\epsilon_3 - \eta \tilde{\epsilon}_4) > 0, \quad (68b) \]
\[ \eta(\epsilon_5 - \eta \epsilon_6) > 0. \quad (68c) \]

(ii) We then show that (18a) and (18b) hold.

Noting that (17c) and (17d) can respectively be rewritten as (10a) and (10b), we know that (32)–(35), and (37) still hold.

Similar to the way to get (40), from (32)–(35), we have
\[ U_{k+1} \leq U_k - \| x_k \|^2_{\frac{\alpha_0}{2}L - \frac{4}{9}K - \frac{\alpha_0}{9}(1+3\eta)L_2^2K} \quad + \| \hat{x}_k \|^2_{\frac{2\alpha_2}{2}L_2^2} + \| v_k + \frac{1}{\beta}g_k \|^2_{\frac{1}{\alpha_0^2\beta^2 \rho(L)+\eta^2}P} \]
\[ + \frac{\eta}{2}(\alpha + 2\beta)\rho(L)\| x_k - \hat{x}_k \|^2 \]
\[ + \| \hat{x} \|^2_{\frac{\alpha_0}{2}(\alpha+\beta)L+\alpha^2} + \| v_k + \frac{1}{\beta}g_k \|^2_{\frac{1}{\alpha_0^2}P} \]
\[ + \eta^2 c_1 L_f^2 \| \tilde{g}_k \|^2 + \| \hat{x}_k \|^2_{\eta\beta K + \eta^2(\beta^2K-\alpha L)} \]
\[ + \| x_k \|^2 - \frac{\nu \bar{d}^2}{2} - \frac{1}{4}(1 + 6\eta) L^2_{12} K + \left( \eta^2 c_2 L^2_{12} + \frac{\eta}{8} \right) \| g_k \|^2 \\
- \| v_k + \frac{1}{\beta} g_k^0 \|^2_{\eta(1-3\nu_2^{-1}(L))P - \eta^2(\rho_2^{-1}(L) + \frac{\nu^2}{2} \rho(L))P} \\
- \frac{\eta}{4} (1 - 2\eta L_f) \| g_k \|^2_{\eta(1-3\nu_2^{-1}(L))P} + \| x_k \|^2_{\frac{1}{2} L^2_{12} K} - \frac{\eta}{4} \| g_k^0 \|^2 \\
= U_k - \| x_k \|^2_{\eta M_1 - 3\eta^2 L^2_{12} K} + \| \bar{x}_k \|^2_{\eta \beta K + \eta^2 M_2} \\
- \| v_k + \frac{1}{\beta} g_k^0 \|^2_{\eta(1-3\nu_2^{-1}(L))P} - \eta(\epsilon_5 - \eta \epsilon_6) \| g_k \|^2 \\
+ \frac{\eta}{2} (\alpha + 2\beta) \rho(L) \| x_k - \bar{x}_k \|^2 - \frac{\eta}{4} \| g_k^0 \|^2 \\
\leq U_k - \| x_k \|^2_{\eta(1-3\nu_2^{-1}(L))P} + \| \bar{x}_k \|^2_{\eta(\beta + \eta^2) K} \\
- \| v_k + \frac{1}{\beta} g_k^0 \|^2_{\eta(1-3\nu_2^{-1}(L))P} - \eta(\epsilon_5 - \eta \epsilon_6) \| g_k \|^2 \\
+ \frac{\eta}{2} (\alpha + 2\beta) \rho(L) \| x_k - \bar{x}_k \|^2 - \frac{\eta}{4} \| g_k^0 \|^2. \tag{69} \]

From (61) and (69), we have

\[ U_{k+1} \leq U_k - \| x_k \|^2_{\eta(1-3\nu_2^{-1}(L))P} - \| v_k + \frac{1}{\beta} g_k^0 \|^2_{\eta(1-3\nu_2^{-1}(L))P} - \eta(\epsilon_5 - \eta \epsilon_6) \| g_k \|^2 - \frac{\eta}{4} \| g_k^0 \|^2 + \eta(\epsilon_8 + 2\eta \gamma) \| x_k - \bar{x}_k \|^2. \tag{70} \]

From Lemma 1 we have

\[ \| x_k - \bar{x}_k \|^2 = \sum_{i=1}^n \| x_{i,k} - \hat{x}_{i,k} \|^2 \leq \sum_{i=1}^n d^2 \| x_{i,k} - \hat{x}_{i,k} \|_p^2 \leq \eta d^2 \max_{i \in [n]} \| x_{i,k} - \hat{x}_{i,k} \|_p^2. \tag{71} \]

From (17a), (17b), and (5), we have

\[ \mathbb{E}[\| x_{i,k} - \hat{x}_{i,k} \|_p^2] = \mathbb{E}[\| x_{i,k} - \hat{x}_{i,k} \|_p^2] = \mathbb{E}[s_k^2 \| x_{i,k} - \hat{x}_{i,k} \|_p^2] \\
\leq C s_k^2 \leq C s_0^2 \gamma^{2k}. \tag{72} \]

Same as the way to get (46a) and (46b), we have

\[ U_k \geq \epsilon_{10} \left( \| x_k \|^2_{K} + \| v_k + \frac{1}{\beta} g_k^0 \|^2_{P} \right) + n(f(\bar{x}_k) - f^*) \geq \epsilon_{10} \hat{U}_k \geq 0. \tag{73a} \]
From (70)–(72) and (68b)–(68c), we have
\[
E_C[U_{T+1}] \leq U_0 - \sum_{k=0}^{T} E_C[\|x_k\|_2^2 \eta(\epsilon_1 - \eta \bar{\epsilon}_2)\kappa + \frac{\eta}{4} \|g_k^0\|^2] + \frac{c_8}{1 - \gamma^2},
\] (74)

From (74), (68a), and (73b), we have
\[
\sum_{k=0}^{T} E_C[\|x_k\|_2^2 \kappa + \|g_k^0\|^2] \leq \frac{U_0 + \frac{c_8}{1 - \gamma^2}}{\min\{\eta(\epsilon_1 - \eta \bar{\epsilon}_2), \frac{\eta}{4}\}},
\]
which yields (18a).

From (74), (68a), and (73a), we have
\[
E_C[\eta(f(\bar{x}_T) - f^*)] \leq U_T \leq U_0 + \frac{c_8}{1 - \gamma^2},
\]
which yields (18b) holds.

G. Proof of Theorem 6

In this proof, in addition to the notations used in the proofs of Theorems 1–2 and 5, we also denote
\[
\bar{\epsilon} \in (0, \min\{\bar{\epsilon}_0, 1 - \gamma^2\}), \quad \bar{\epsilon}_0 = \frac{\epsilon_{12}}{\epsilon_{11}},
\]
\[
\bar{\epsilon}_{12} = \eta \min \left\{ \epsilon_1 - \eta \bar{\epsilon}_2, \ \epsilon_3 - \eta \bar{\epsilon}_4, \ \frac{U}{2} \right\}.
\]

(i) We first show that \(\bar{\epsilon} \in (0, 1)\).

From (68a)–(68c) and \(\gamma \in (0, 1)\), we have
\[
\bar{\epsilon}_{12} > 0 \quad \text{and} \quad \min\{\epsilon_0, 1 - \gamma^2\} > 0.
\] (75)

Noting that \(\epsilon_1 < \alpha \rho_2(L)/2, \ \bar{\epsilon}_2 > 3\alpha^2 \rho^2(L)\), and \(\epsilon_{11} \geq \frac{1}{2} + \frac{\alpha}{2} \geq \frac{3}{2}\), we have
\[
\bar{\epsilon} < \bar{\epsilon}_0 = \frac{\bar{\epsilon}_{12}}{\epsilon_{11}} \leq \frac{\eta (\epsilon_1 - \eta \bar{\epsilon}_2)}{\epsilon_{11}} \leq \frac{\epsilon_{12}^2}{4\epsilon_2 \epsilon_{11}} < 1.
\] (76)

(ii) We then show that (19) holds.

From the Cauchy–Schwarz inequality, we have
\[
U_k \leq \epsilon_{11} \bar{U}_k.
\] (77)
From (70)–(72), (51), and (75)–(77), we have

\[
E_c[U_{k+1}] \leq E_c[U_k - \tilde{c}_12U_k] + c_8\gamma^{2k}
\]

\[
\leq (1 - \tilde{\epsilon}_0)E_c[U_k] + c_8\gamma^{2k}
\]

\[
\leq (1 - \tilde{\epsilon}_0)k^{k+1}U_0 + \sum_{t=0}^{k} c_8(1 - \tilde{\epsilon}_0)^t\gamma^{2(k-t)}
\]

which yields (19).

**H. Proof of Theorem 7**

In this proof, in addition to the notations used in the proofs of Theorems 1–2 and 5–6, we also denote

\[
\hat{\epsilon}_0 = \min\left\{\frac{\hat{\epsilon}_1}{2\epsilon_2}, \frac{\epsilon_1}{\epsilon_0}, \sqrt{\kappa_{15}}\right\}, \gamma \in [\max\{\sqrt{\kappa_{11}}, \sqrt{\kappa_{12}}\}, 1)
\]

\[
s_0 \geq \max\{\sqrt{\kappa_8/\kappa_7}, \max_{i \in [n]} \|x_{i,0}\|\}, \kappa_7 > \frac{\kappa_{10}(1 - \varphi)^2}{\kappa_9},
\]

\[
\kappa_8 = \frac{1}{2}\|x_0\|_K^2 + \frac{1}{2}\|v_0 + \frac{1}{\beta}g_0^0\|^2_{\frac{\kappa_9}{\kappa_9}}
\]

\[+ x_0^\top KP \left(v_0 + \frac{1}{\beta}g_0^0\right) + \frac{1}{2\nu}\|g_0^0\|^2,
\]

\[
\kappa_9 = \min\left\{\frac{\hat{\epsilon}_1}{2\epsilon_{11}}, \frac{\nu}{2\epsilon_{11}}\right\}, \kappa_{10} = \left(\hat{\epsilon}_8 + \frac{\epsilon_7\hat{\epsilon}_1}{\hat{\epsilon}_2}\right)nd^2,
\]

\[
\kappa_{11} = 1 - \eta\kappa_9 + \frac{\eta\kappa_{10}(1 - \varphi)^2}{\kappa_7},
\]

\[
\kappa_{12} = (1 + \varphi + \eta^2\kappa_{13})(1 - \varphi)^2 + \frac{\eta^2\kappa_{14}}{\epsilon_{10}}\kappa_7,
\]

\[
\kappa_{13} = 4(1 + \varphi^{-1})d^2n^2d^2\alpha^2\rho^2(L),
\]

\[
\kappa_{14} = 4(1 + \varphi^{-1})d^2(\alpha^2\rho^2(L) + L_f^2),
\]

\[
\kappa_{15} = \frac{\left(\varphi + \varphi^{-2} - \varphi^3\right)\epsilon_{10}}{\kappa_{13}(1 - \varphi)^2\epsilon_{10} + \kappa_{14}\kappa_7},
\]

\[
\hat{\epsilon}_0 = \frac{\hat{\epsilon}_1}{\epsilon_{11}}, \hat{\epsilon}_1 = \min\{\epsilon_1, \epsilon_3\}, \hat{\epsilon}_2 = \max\{\tilde{\epsilon}_2, \tilde{\epsilon}_4\},
\]

\[
\hat{\epsilon}_1 = \eta \min\left\{\hat{\epsilon}_1 - \eta\hat{\epsilon}_2, \frac{\nu}{2}\right\}, \alpha_0 = \eta(\hat{\epsilon}_8 + 2c_7\eta)nd^2.
\]

(i) We first show that \(\kappa_{11}, \kappa_{12} \in (0, 1)\).
From (42)–(44) and $0 < \eta \leq \hat{\kappa}_3 \leq \frac{\dot{\epsilon}_1}{\hat{\epsilon}_2}$, we have

$$
\dot{\epsilon}_{12} \geq \eta \min \left\{ \frac{\dot{\epsilon}_1}{2}, \frac{\nu}{2} \right\} > 0, \quad (78a)
$$

$$
\dot{\epsilon}_0 = \frac{\dot{\epsilon}_{12}}{\epsilon_{11}} \geq \eta \kappa_9 > 0, \quad (78b)
$$

$$
0 < \epsilon_9 \leq \kappa_{10}. \quad (78c)
$$

From (76), we have

$$
\eta \kappa_9 \leq \dot{\epsilon}_0 \leq \ddot{\epsilon}_0 < 1. \quad (79)
$$

From (78b) and $\kappa_7 > \frac{\kappa_{10}(1-\varphi)^2}{\kappa_9}$, we have

$$
\kappa_7 > 0 \text{ and } \kappa_{11} = 1 - \eta \kappa_9 + \frac{\eta \kappa_{10}(1-\varphi)^2}{\kappa_7} < 1. \quad (80)
$$

From (78c), (79), and $\kappa_7 > 0$, we have

$$
\kappa_{11} > 0. \quad (81)
$$

From $\kappa_7 > 0$, $\varphi \in (0,1)$, and $\eta < \sqrt{\kappa_{15}}$, we have

$$
0 < \kappa_{12} = (1 + \varphi + \eta^2 \kappa_{13})(1-\varphi)^2 + \frac{\eta^2 \kappa_{14} \kappa_7}{\epsilon_{10}}
= 1 - (\varphi + \varphi^2 - \varphi^3) + \frac{\eta^2 (\kappa_{13}(1-\varphi)^2 \epsilon_{10} + \kappa_{14} \kappa_7)}{\epsilon_{10}}
< 1. \quad (82)
$$

(ii) We next show that (20) holds.

From (70), (71), (51), (77), and (78), we have

$$
U_{k+1} \leq U_k - \dot{\epsilon}_{12} \hat{U}_k + c_0 \max_{i \in [n]} \|x_{i,k} - \hat{x}_{i,k}\|_p^2
\leq (1 - \dot{\epsilon}_0)U_k + c_0 \max_{i \in [n]} \|x_{i,k} - \hat{x}_{i,k}\|_p^2
\leq (1 - \eta \kappa_9)U_k + \eta \kappa_{10} \max_{i \in [n]} \|x_{i,k} - \hat{x}_{i,k}\|_p^2. \quad (83)
$$

We have

$$
\|x_{i,k+1} - \hat{x}_{i,k}\|_p^2 = \|x_{i,k+1} - x_{i,k} + x_{i,k} - \hat{x}_{i,k}\|_p^2
$$
\[
\leq (\|x_{i,k+1} - x_{i,k}\|_p + \|x_{i,k} - \hat{x}_{i,k}\|_p)^2
\]
\[
\leq (1 + \varphi^{-1})\|x_{i,k+1} - x_{i,k}\|_p^2 + (1 + \varphi)\|x_{i,k} - \hat{x}_{i,k}\|_p^2
\]
\[
\leq (1 + \varphi^{-1})\beta^2\|x_{i,k+1} - x_{i,k}\|_p^2 + (1 + \varphi)\|x_{i,k} - \hat{x}_{i,k}\|_p^2
\]
\[
\leq (1 + \varphi^{-1})\frac{\beta^2}{\beta_{13}}\|x_{i,k+1} - x_k\|_p^2 + (1 + \varphi)\|x_{i,k} - \hat{x}_{i,k}\|_p^2, \quad (84)
\]

where the first inequality holds due to the Minkowski inequality; the second inequality holds due to the Cauchy–Schwarz inequality and \(\varphi > 0\); the third inequality holds due to Lemma 1.

We have

\[
\max_{i \in [n]} \|x_{i,k+1} - \hat{x}_{i,k}\|_p^2
\]
\[
\leq (1 + \varphi + \eta^2\kappa_{13})\max_{i \in [n]} \|x_{i,k} - \hat{x}_{i,k}\|_p^2 + \eta^2\kappa_{14}\left(\|x_k\|_K^2 + \|v_k + \frac{1}{\beta}g_k^0\|_p^2\right)
\]
\[
\leq (1 + \varphi + \eta^2\kappa_{13})\max_{i \in [n]} \|x_{i,k} - \hat{x}_{i,k}\|_p^2 + \eta^2\kappa_{14}\epsilon_{10}U_k, \quad (85)
\]

where the first inequality holds due to (37), (71), and (84); and the second inequality holds due to (73a).

In the following, we use mathematical induction to prove

\[
U_k \leq \kappa_7 s_k^2 \text{ and } \max_{i \in [n]} \|x_{i,k} - \hat{x}_{i,k-1}\|_p^2 \leq s_k^2. \quad (86)
\]

From (51) and \(s_0 \geq \sqrt{\kappa_8/\kappa_7}\), we have

\[
U_0 \leq \kappa_8 \leq \kappa_7 s_0^2. \quad (87)
\]

From \(s_0 \geq \max_{i \in [n]} \|x_{i,0}\| \text{ and } \hat{x}_{i,-1} = 0_d\), we have

\[
\max_{i \in [n]} \|x_{i,0} - \hat{x}_{i,-1}\|_p^2 \leq s_0^2. \quad (88)
\]

Therefore, from (87) and (88), we know that (86) holds at \(k = 0\). Suppose that (86) holds at \(k\). We next show that (86) holds at \(k + 1\).

We have

\[
\|x_{i,k} - \hat{x}_{i,k}\|_p = \|x_{i,k} - \hat{x}_{i,k-1} - s_kC((x_{i,k} - \hat{x}_{i,k-1})/s_k)\|_p
\]
\[
\quad = s_k\|((x_{i,k} - \hat{x}_{i,k-1})/s_k) - C((x_{i,k} - \hat{x}_{i,k-1})/s_k)\|_p
\]
where the first equality holds due to ($17_a$) and ($17_e$); and the inequality holds due to ($86$) and ($6$).

We have

$$U_{k+1} \leq (1 - \eta \kappa_9) \kappa_7 s_k^2 + \eta \kappa_{10} (1 - \varphi) s_k^2 = \frac{\kappa_{11}}{\gamma^2} \kappa_7 s_{k+1}^2 \leq \kappa_7 s_{k+1}^2,$$

(90)

where the first inequality holds due to ($83$), ($86$), and ($89$); and the last inequality holds due to $\gamma \geq \sqrt{\kappa_{11}}$.

We have

$$\max_{i \in [n]} \| x_{i,k+1} - \hat{x}_{i,k} \|^2_p \leq (1 + \varphi + \eta^2 \kappa_{13})(1 - \varphi) s_k^2 + \frac{\eta^2 \kappa_{14}}{\epsilon_{10}} \kappa_7 s_k^2 = \frac{\kappa_{12}}{\gamma^2} s_{k+1}^2 \leq s_{k+1}^2,$$

(91)

where the first inequality holds due to ($85$), ($86$), and ($89$); and the last inequality holds due to $\gamma \geq \sqrt{\kappa_{12}}$.

Therefore, from (90) and (91), we know that ($86$) holds at $k + 1$. Finally, by mathematical induction, we know that ($86$) holds for any $k \in \mathbb{N}_0$. Hence, (20) holds.