Investment in the common good: free rider effect and the stability of mixed strategy equilibria

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Abstract

In the game of investment in the common good, the free rider problem can delay the stakeholders’ actions in the form of a mixed strategy equilibrium. However, it has been recently shown that the mixed strategy equilibria of the stochastic war of attrition are destabilized by even the slightest degree of asymmetry between the players. Such extreme instability is contrary to the widely accepted notion that a mixed strategy equilibrium is the hallmark of the war of attrition. Motivated by this quandary, we search for a mixed strategy equilibrium in a stochastic game of investment in the common good. Our results show that, despite asymmetry, a mixed strategy equilibrium exists if the model takes into account the repeated investment opportunities. The mixed strategy equilibrium disappears only if the asymmetry is sufficiently high. Since the mixed strategy equilibrium is less efficient than pure strategy equilibria, it behooves policymakers to prevent it by promoting a sufficiently high degree of asymmetry between the stakeholders through, for example, asymmetric subsidy.

Keywords: Investment in the common good, impulse control game, the war of attrition, free rider problem, mixed strategy equilibrium

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1 Introduction

A free rider problem commonly arises in a game of investment in the common good. For example, if an electronics retailer such as Best Buy offers corporate social responsibility (CSR) programs such as recycling, tech training, and supplier audit at its own expense, the other firms in the industry also benefit from them: recycling reduces material costs across the industry; technology training for the disadvantaged expands the market coverage for the industry; investments in CSR increase the industry-wide stock returns (Gunther, 2015; Serafeim, 2017). A free rider problem also arises in commodity advertising in the orange juice industry (Lee and Fairchild, 1988) and the salmon industry (Kinnucan and Myrland, 2003): one firm’s investment in an advertising campaign can benefit its competitors. The free rider problem can delay the stakeholders’ investment actions and hence diminish the level of the common good. Therefore, it behooves the policymakers to mitigate the free rider effect. The goal of this paper is to study a stochastic game of investment in the common good and examine the impact of asymmetry on the free rider effect.

The game of investment in the common good is a special case of a concession game. The free rider effect in a concession game often manifests itself as a mixed strategy equilibrium in which the players delay their actions by a random time. However, the mixed strategy equilibrium may be destabilized by even the smallest degree of asymmetry between the players in a stochastic environment (Georgiadis et al., 2022). This suggests that even an extremely small degree of asymmetry may mitigate the free rider effect. This result is surprising because it is contrary to the commonly accepted notion that a mixed strategy equilibrium is the hallmark of the games of concession. Furthermore, from the practical point of view, such extreme sensitivity to asymmetry may not be an accurate depiction of real life decision making. Motivated by this quandary, this paper searches for mixed strategy equilibria in the context of investment in the common good. In particular, we ask the following questions: Is there a mixed strategy equilibrium in the model that we study? If so, under what conditions do they exist? Lastly, what is the managerial implication to the policymakers?

To answer the proposed questions, we analyze a parsimonious model of a game between two players whose payoffs are under the influence of a state variable that models a stochastic stock of the common good. Each player has an infinite number of opportunities to invest in the common good. The timing and the size of the investment are both under the player’s discretion, but each investment requires a fixed upfront cost as well as a variable cost. Such a cost structure is typical in the investment in the common good; for example, a CSR program and an advertising campaign generally require a fixed upfront cost which stems from organizing a new team, drawing the plans for execution of the project, and purchasing equipment.
The game-theoretic model that we examine has a broad class of applications. In the example of commodity advertising, the state variable is the stock of goodwill on the commodity in question, and the investment is on advertising campaign to boost the goodwill (Nerlove and Arrow, 1962; Lon and Zervos, 2011; Reddy et al., 2016). In the example of investments in CSR, the players are firms in the electronics retail industry, and their industry-wide profitability can be considered as the state variable (Gunther, 2015; Serafeim, 2017).

The paper makes both managerial and methodological contributions. First, the paper sheds light on the impact of the repeated opportunities of investment. In contrast to the one-shot stochastic war of attrition, our model with repeated opportunities of investment possesses a mixed strategy equilibrium under a moderate degree of asymmetry. We therefore resolve the discrepancy between the results from the stochastic war of attrition and those from the canonical war of attrition. The result provides useful insights to policymakers: a sufficiently high degree of asymmetry between the investors can mitigate the free rider effect by destabilizing the mixed strategy equilibrium.

Second, the paper presents an equilibrium solution to a novel class of impulse control games possessing a free rider effect. In particular, it provides a verification theorem for this class of impulse control games and constructs a mixed strategy subgame perfect equilibrium for the first time in the literature on impulse control games.

The paper is organized as follows. We provide the context of our paper in the literature and its contributions to the relevant research domains in Section 2. As a benchmark, we first examine the single investment game in Section 3 and reproduce the result of Georgiadis et al. (2022) that the mixed strategy Markov perfect equilibrium (MPE) is destabilized by even the smallest degree of asymmetry. The central results of the paper are presented in Section 4: we examine the game allowing for an infinite number of investments, formulate a verification theorem for the equilibrium, and obtain mixed strategy equilibria in asymmetric games. We conclude the paper in Section 5. All mathematical proofs can be found in Electronic Companion (EC).

2 Related Literature

The current paper contributes to the literature on games of concession. One domain of this literature is on the war of attrition and its mixed strategy equilibrium. In the seminal paper by Maynard Smith (1974), a game of concession is modeled as the war of attrition which is shown to possess a mixed strategy equilibrium. Hendricks et al. (1988) obtains all possible mixed strategy equilibria in the continuous-time deterministic war of attrition. In the stochastic extensions of the war of attrition, Murto (2004) characterizes pure
strategy Markov perfect equilibria, and Steg (2015) obtains a mixed strategy equilibrium. The mixed strategy equilibrium has been also empirically found and examined (Wang, 2009; Takahashi, 2015). However, Georgiadis et al. (2022) shows that even the slightest degree of asymmetry between the players destabilizes a mixed strategy Markov perfect equilibrium in the stochastic war of attrition. Kwon (2020) also shows a similar absence of inefficient equilibrium in the game of contribution to the common good under asymmetry. In stark contrast, our paper shows that the mixed strategy equilibrium is recovered despite asymmetry if there are repeated investment opportunities.

The control game model which we study is suitable for modeling commodity advertising decisions, which is known to suffer from the free rider problem (Lee and Fairchild 1988, Kinnucan and Myrland 2003). Advertising for a product can be modeled as investment in the stock of goodwill (Nerlove and Arrow, 1962). Expenditure in advertising under a stochastic environment has been modeled as a stochastic control problem (Sethi, 1977). Specifically, it has been modeled as a singular control problem (Jack et al., 2008), a singular control combined with a discretionary timing of the product launch (Lon and Zervos, 2011), or as a singular control game in the context of a market share competition (Kwon and Zhang, 2015). Just as in our paper, advertising decision has also been modeled as an impulse control problem (Reddy et al., 2016) because an advertising campaign requires substantial upfront investment.

The model that we examine is also applicable to the investment decisions in CSR. The literature views CSR as private provision of public goods by firms and has attempted to delineate the conditions under which firms contribute to the production of public goods through CSR despite free-riding incentives. Bagnoli and Watts (2003) shows that firms may strategically engage in CSR activities to outperform their competition in the product market. Kotchen (2006) studies the green markets where products with different levels of greenness are available and examines the conditions under which the production of green products increases or decreases. Morgan and Tumlinson (2019) finds that the shareholders’ preference for public goods can encourage the firms to be more active in CSR. We complement this stream of literature by focusing on the free rider effect from the mixed strategy equilibria and identifying the conditions under which the provision of public goods such as CSR is hastened or delayed.

Lastly, the theory of stochastic impulse control has long been applied to various operations research problems including cash management (Constantinides and Richard, 1978), inventory management (Ormeci et al., 2008; Cadenillas et al., 2010), interest rate intervention (Mitchell et al., 2014), and production capacity expansion (Bensoussan and Chevalier-Roignant, 2019). The impulse control framework has also been applied to game-theoretic models. Amongst others, Stettner (1982), Cosso (2013), and El Asri and Mazid (2018)
examine and solve zero-sum impulse control games. Dutta and Rustichini (1995) solves a two-sided \((s,S)\) game in the context of a duopolist competition. Basei et al. (2019) and Guo and Xu (2019) examine impulse/singular control games with a large number of players using the mean-field approach. Ferrari and Koch (2019) models the pollution control as a nonzero-sum impulse control game between the government and the firm and constructs an equilibrium under certain conditions. Zabaljauregui (2020) proposes a simple and efficient algorithm for solving symmetric nonzero-sum impulse control games. Campi and De Santis (2020) considers an impulse controller-stopper nonzero-sum game and characterizes two qualitatively different types of equilibria. Aid et al. (2020) examines a class of nonzero-sum impulse control games and obtains a general verification theorem. Our paper contributes to this domain of research by examining a novel class of impulse control games possessing a free rider effect which yields a mixed strategy equilibrium.

### 3 Single Investment Game

In this section, we examine a single investment game as a benchmark model. Specifically, we assume that the common good allows for only one opportunity of investment. The single-investment assumption is unrealistic for the examples of the CSR program or advertising campaign because the firms naturally possess multiple investment opportunities. However, it is a convenient assumption as it considerably simplifies the analysis. A particularly convenient feature of the single investment game is that it reduces to a stopping game also known as a Dynkin game; see Proposition 1. Therefore, most of the results obtained for stopping games (e.g., Georgiadis et al. 2022, Touzi and Vieille 2002) translate to our model. One of the main goals of this paper is to show that this simplification may not be so innocuous after all; Section 4 shows that we obtain a markedly different result if this assumption is relaxed.

The central goal of this section is two-fold. First, we illustrate mixed strategy equilibria of a symmetric game in Section 3.3. Second, in Section 3.4, we turn our attention to an asymmetric game and search for a mixed strategy equilibrium that shares the same simple structures as those illustrated in Section 3.3.

#### 3.1 The Model and the Equilibrium Concept

This section introduces a model of two players whose payoffs are under the influence of the common good. At any point in time, each player can make costly investment to boost the stock of the common good by any amount, but the common good allows for only one opportunity of investment. Thus, if an investment is made by one of the players, the game is terminated immediately.
3.1.1 State Variable

We model the level of the common good (state variable) as a stochastic process. We let \((\Omega, \mathcal{F}, \mathbb{P})\) denote a probability space with the filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\), which satisfies the usual condition (p. 172, Rogers and Williams 2000). In the absence of the players’ control, the uncontrolled state variable \(X\) is modeled as a regular diffusion process (p. 13, Borodin and Salminen 1996) satisfying the following stochastic differential equation (SDE):

\[
dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.
\]  

(1)

Here \(W = (W_t)_{t \geq 0}\) is a Wiener process adapted to \(\mathbb{F}\). We assume that \(\mu(\cdot) < 0\) and \(\sigma(\cdot) > 0\) to model a stochastically declining state of the common good in the absence of investments. The process \(X\) takes values within the interval \(I = (a, b) \subseteq \mathbb{R}\), where \(a > -\infty\) and \(b \leq \infty\). For example, in the case of a geometric Brownian motion, \(a = 0\) and \(b = \infty\). We assume that the boundary points \(a\) and \(b\) are natural so that \(X\) never reaches \(a\) or \(b\) in finite time (II.6 of Borodin and Salminen, 1996). To ensure that the strong solution to the SDE (1) exists, we make the usual assumption of Lipschitz continuity of \(\mu(\cdot)\) and \(\sigma(\cdot)\) as well as the inequality 

\[
|\mu(x)| + |\sigma(x)| \leq \delta(1 + |x|)
\]

for some constant \(\delta > 0\) (Section 5.2, Øksendal 2003). Lastly, we impose the following transversality assumption (p.330, Alvarez 2001) so that the payoff functions are well-behaved in the limit \(t \to \infty\):

\[
\lim_{t \to \infty} \mathbb{E}_x [X_t e^{-rt}] = 0 \quad \forall x \in \mathcal{I}.
\]  

(2)

We now define the \(r\)-excessive characteristic differential operator for the diffusion process \(X\) (Øksendal, 2003):

\[
\mathcal{A} := \frac{1}{2} \sigma^2(x) \frac{d^2}{dx^2} + \mu(x) \frac{d}{dx} - r.
\]  

(3)

By the theory of diffusion (Chapter II of Borodin and Salminen, 1996), there are two solutions to the homogeneous differential equation \(\mathcal{A} f(x) = 0\). As a matter of notational convention, we let \(\phi(\cdot) > 0\) and \(\psi(\cdot) > 0\) respectively denote the decreasing and the increasing solution to the equation \(\mathcal{A} f(x) = 0\).

3.1.2 Strategy and Payoff

As a matter of convention, we let \(i\) and \(j\) denote the indices of the two players such that \(i \neq j\). Given the single-investment assumption, each player’s strategy can be written as \(\nu_i = (\tau_i, \zeta_i)\) where \(\tau_i\) is player \(i\)’s stopping time of investment and \(\zeta_i \geq 0\) is the boost in the state variable measurable with respect to \(\mathcal{F}_{\tau_i}\). Since the common good is assumed to allow for only one investment, the game is terminated at
the time $\tau^{(1)} \land \tau^{(2)}$. Following the convention from the literature (e.g., Dutta et al., 1995; Grenadier, 1996; Hoppe and Lehmann-Grube, 2001), we assume that, if both players attempt to invest simultaneously, i.e., $\tau^{(i)} = \tau^{(j)}$, then only one player's investment goes through with a probability of 50%; this accounts for the factor of 1/2 in the third term in (4) below.

The strategy profile $\nu = (\nu_1, \nu_2)$ and the tie-breaking rule described above determine a stopping time $\tau^{\nu} := \tau^{(i)} \land \tau^{(j)}$ at which an investment is made and the boost $\zeta^{\nu}$ made by one of the players. If $\tau^{(i)} < \tau^{(j)}$, then $\zeta^{\nu} = \zeta^{(i)}$; if $\tau^{(i)} = \tau^{(j)}$, then $\zeta^{\nu}$ can be either $\zeta^{(i)}$ or $\zeta^{(j)}$ with 50% probability each. Hence, we consider a new binary random variable $\zeta^{\nu}$, taking values of $\zeta^{(i)}$ or $\zeta^{(j)}$ with 50% probabilities. Under the strategy profile $\nu$, the controlled state variable is given by

$$X^{\nu}_t = X_0 + \int_0^t \mu(X^{\nu}_s)ds + \int_0^t \sigma(X^{\nu}_s)dW_s + \zeta^{\nu}1_{\{\tau^{\nu} \leq t\}}.$$ 

Next, we formulate the payoff to the players with a common discount rate $r > 0$. We let $\pi(X^{\nu}_t)$ denote the profit flow per unit time to each player. We assume that $\pi(\cdot)$ is a continuous and increasing function. When player $i$ makes an investment at time $\tau^{(i)}$ to boost $X^\nu$ by $\zeta^{(i)}$, it costs $c_i + k\zeta^{(i)}$ to the player, where $c_i > 0$ is the upfront cost and $k > 0$ is the variable cost of the boost. Based on this specification, we can write player $i$'s payoff at time $t \leq \tau^{\nu}$ conditional on $\mathcal{F}_t$ as follows:

$$V^{\nu}_{ij}(t) = e^{rt}\mathbb{E}^X\left[\int_t^\infty \pi(X^{\nu}_s)e^{-rs}ds - e^{-r\tau^{(i)}}(k\zeta^{(i)} + c_i)1_{\{\tau^{(i)} < \tau^{(j)}\}} - \frac{1}{2}e^{-r\tau^{(i)}}(k\zeta^{(i)} + c_i)1_{\{\tau^{(i)} = \tau^{(j)}\}}\right].$$

The first term is the cumulative discounted profit, and the second and third terms are the costs of investment. Note that $X^{\nu}_t$ coincides with the uncontrolled process $X_t$ for $t < \tau^{\nu} = \tau^{(i)} \land \tau^{(j)}$. Note also that, although $X^\nu$ undergoes a boost at time $\tau^\nu$ so that $X^\nu_t \neq X_t$ for $t \geq \tau^\nu$, the controlled state variable $X^\nu$ is not boosted anymore after time $\tau^\nu$ because of the single-investment assumption. This implies that the single investment game is fundamentally a one-shot stopping game, which will be established in the following subsection.

### 3.1.3 Transformation into a Stopping Game in a Subgame Perfect Equilibrium

We now introduce the equilibrium solution concept that will be used throughout this paper.

**Definition.** A strategy profile $\nu$ is a *subgame perfect equilibrium* (SPE) if it is a Nash equilibrium in any subgame. Formally, it means that $V^{(\nu_i, \nu_j)}_{i, \tau} \geq V^{(\nu'_i, \nu_j)}_{i, \tau}$ at any stopping time $\tau$ and any admissible strategy $\nu'_i$.

The definition of SPE stipulates that each player should optimize their payoff at each point in time. Thus,
if player $i$ invests at time $\tau^{(i)}$, the magnitude of the impulse $\zeta^{(i)}_{\tau^{(i)}}$ should be the one that maximizes player $i$’s payoff at time $\tau^{(i)}$.

To facilitate the analysis, we make the following assumption for $\pi(\cdot)$ (Alvarez and Lempa, 2008):

**Assumption 1** (i) $\lim_{x \downarrow a} \pi(x) = \pi_L > -\infty$ and (ii) $\mathbb{E}^x[\int_0^\infty |\pi(X_t)|e^{-rt}dt] < \infty$ for any $x \in \mathcal{I}$.

Here $\mathbb{E}^x[\cdot] = \mathbb{E}[\cdot|X_0 = x]$ represents the conditional expected value given the initial condition $X_0 = x$. The implication of the integrability condition $\mathbb{E}^x[\int_0^\infty |\pi(X_t)|e^{-rt}dt] < \infty$ for any $x \in \mathcal{I}$ is that the function

$$ (R, \pi)(x) := \mathbb{E}^x[\int_0^\infty \pi(X_t)e^{-rt}dt] $$

is well-defined (Alvarez, 2001). In addition, we make the following assumption to ensure a unique optimal value of $\zeta^{(i)}$:

**Assumption 2** There is a unique $z^* \in \mathcal{I}$ such that $(R, \pi)'(x) - k > 0$ if $x < z^*$ and $(R, \pi)'(x) - k < 0$ if $x > z^*$.

Here $z^*$ has the meaning of the optimal end-point of the boost in case the initial value of $x$ is less than $z^*$. If player $i$ invests at time $t$ with the current value of the state variable $X_t$, the player’s reward from investment is $(R, \pi)(z) - k(z - X_t) - c_i$ if the player boosts the state variable up to some value $z$. By Assumption 2, there is a well-defined unique value of $z$ that maximizes the reward from investment.

Under the assumptions made thus far, we arrive at the following lemma:

**Lemma 1** Under an SPE, the equilibrium magnitude of the investment in the single investment game is

$$ \zeta^{(i)}_{\tau^{(i)}} = \max\{z^* - X_t, 0\}. \quad (5) $$

Intuitively, under an SPE, each player’s optimal boost must maximize the reward from investment. Because $\zeta^{(i)}_{\tau^{(i)}}$ is uniquely determined by Lemma 1, the single investment game can be transformed into a stopping game as established by Proposition 1 below, where the *reward from stopping* is given by

$$ g_i(x) := \begin{cases} (R, \pi)(z^*) - k(z^* - x) - c_i & \text{if } x < z^* \\ (R, \pi)(x) - c_i & \text{if } x \geq z^* \end{cases}, \quad (6) $$

and the reward from the opponent’s stopping is given by

$$ m_i(x) := (R, \pi)(\max\{z^*, x\}). \quad (7) $$
Proposition 1 Under an SPE ν, the payoff function can be transformed into one of a stopping game:

\[ V^\nu_{ij} = e^{rt}E_x\left[\int_t^{\tau(i)\wedge\tau(j)} \pi(X_s)e^{-rs}ds + g_i(X_{\tau(i)})e^{-r\tau(i)}1_{\{\tau(i)<\tau(j)\}} + m_i(X_{\tau(j)})e^{-r\tau(j)}1_{\{\tau(j)<\tau(i)\}}\right] + \frac{1}{2}[g_i(X_{\tau(i)}) + m_i(X_{\tau(j)})]e^{-r\tau(i)}1_{\{\tau(i)\wedge\tau(j)\}}|F_t]. \] (8)

By virtue of Proposition 1, the single investment game completely transforms into a stopping game with the uncontrolled state variable X. For the remainder of this section, we analyze the game as if it is a stopping game. We therefore succinctly write ν ≡ τ(i) whenever necessary throughout this section.

Remark: For simplicity of the presentation, the probability space Ω and the measure P do not explicitly include the probabilistic mixture of stopping times, which is necessary to include mixed strategies in the strategy space. However, it is straightforward to verify that the expression for the payoff function (8) continues to be valid even if the probability space and the measure are replaced by Ω × L(1) × L(2) and \( \hat{P} \) that incorporate the randomizers of mixed strategies; see Sections 3.3.1 and 3.3.2 for the details.

3.2 Benchmark: Single-Player Problem

In Section 3.3, it will be shown that the payoff associated with a mixed strategy MPE is identical to the optimal payoff function from the single-player problem. Thus, it is imperative to study the single-player optimal stopping problem first. Towards this goal, we assume that player j never invests in the common good (τ(j) = ∞) and solve for player i’s best response and payoff. By virtue of Proposition 1, the single-player problem effectively reduces to the optimal stopping problem of maximizing

\[ E^i[\int_0^\tau \pi(X_t)e^{-rt}dt + e^{-r\tau}g_i(X_\tau)], \] (9)

with respect to stopping time τ.

To ensure the existence and uniqueness of the optimal stopping policy, we assume the following:

Assumption 3 (i) There exists \( \theta_i < z^* \) such that

\[ \alpha_i(x) := \frac{g_i(x) - (R_i\pi)(x)}{\phi(x)} \] (10)

achieves a unique global maximum at \( \theta_i \). Furthermore, \( \alpha_i(x) \) increases for \( x < \theta_i \) and decreases for \( x > \theta_i \).

(ii) There exists \( x^*_i \in (a, z^*) \) such that \( \alpha_i(x) + \pi(x) < 0 \) if and only if \( x < x^*_i \).
Assumptions 3(i) and (ii) are conventionally made in the optimal stopping problem literature (see, for example, Theorem 3 of Alvarez, 2001). The function $\alpha_i(\theta_i)$ has the meaning of the coefficient of $\phi(x)$ in the single player’s payoff function when the threshold of investment is $\theta_i$. Thus, Assumption 3(i) ensures that there exists a unique optimal threshold of investment. We make Assumption 3(ii) to ensure that $\pi(x)$ takes sufficiently low values for $x < x^c_i$ so that it is always optimal for the players to invest if the state variable $X$ is sufficiently low. Under the assumptions made so far, we can obtain the following optimal solution:

**Proposition 2** (i) The optimal time of investment for player $i$ is $\tau^* = \inf\{t \geq 0 : X_t \leq \theta_i\}$, and the optimal boost is $\zeta^* = z^* - X_{\tau^*}$.

(ii) Player $i$’s optimal payoff given the current value $x$ of the state variable is

$$V^*_i(x) = \begin{cases} 
\alpha_i(\theta_i)\phi(x) + (R, \pi)(x) & \text{if } x > \theta_i \\
g_i(x) & \text{otherwise}
\end{cases},$$

where $\alpha_i(\cdot)$ and $g_i(\cdot)$ are defined by (10) and (6).

(iii) $\theta_i$ decreases in $c_i$.

The threshold stopping rule of Proposition 2(i) is an intuitive result: an investment is made if the value of $X$ is sufficiently low. Proposition 2(ii) follows from the well-established optimal stopping theory (Alvarez, 2001). Lastly, Proposition 2(iii) holds because a player with a higher cost of investment $c_i$ would have lower incentive to invest, leading to a lower threshold of investment.

### 3.3 Mixed Strategy SPE of Symmetric Games

Next, we return to the two-player game and obtain mixed strategy SPEs. We assume that the players are identical, i.e., $c := c_1 = c_2$, in which case we have $\theta := \theta_1 = \theta_2$, and construct a mixed strategy MPE and asymmetric two-stage SPE (TSSPE).

#### 3.3.1 Formulation of Mixed Strategies

We first expand the strategy space to encompass probabilistic mixtures of stopping times. The canonical definition of a mixed strategy requires an additional random variable for each player which is realized and revealed to each player in the beginning of the game but unknown to the opponent. To incorporate the additional random variable into a player’s strategy, we follow the recipe provided by Section 7.1 of Touzi and Vieille (2002).
We first re-define \((\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})\) as the probability space of the sample paths of \(X\) (a Wiener space). By this definition, \(\mathcal{F}\) is the canonical filtration of the Brownian motion \(W\) introduced in (1), and \(\mathbb{P}\) is the Wiener measure. Following Touzi and Vieille (2002), we then augment \((\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})\) with the probability space of player \(i\)'s randomizer defined on \(L(i) = [0, 1]\) with a Borel \(\sigma\)-algebra \(\mathcal{L}(i)\) on \(L(i)\) and the Lebesgue measure \(\mathbb{L}(i)\) on \(L(i)\). By this recipe, the expanded probability space is \((\Omega \times L(1) \times L(2), (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}_t, \mathbb{P})\), where the new probability measure is defined as \(\mathbb{P} := \mathbb{P} \otimes \mathbb{L}(1) \otimes \mathbb{L}(2)\), and the new \(\sigma\)-algebra is given by \(\mathcal{F} = \mathcal{F} \otimes \mathcal{L}(1) \otimes \mathcal{L}(2)\). Similarly, the new filtration is defined as the product \(\sigma\)-algebra \(\mathcal{F}_t = \mathcal{F}_t \otimes \mathcal{L}(1) \otimes \mathcal{L}(2)\).

Next, we formulate the mixed strategy \(\nu\). Following Touzi and Vieille (2002), we can construct a mixed strategy by mapping \(L(i) = [0, 1]\) to the space of stopping times adapted to \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\). To construct such a mapping, we introduce a strategy-specific survival probability \(M(i) = \{M(i)_t : t \geq 0, M(i)_0 = 1\}\), which is a non-negative, non-increasing, and right-continuous process adapted to \(\mathcal{F}\). The survival probability \(M(i)_t\) has the interpretation of the probability that player \(i\) has not yet invested by time \(t\), and hence, player \(i\) invests at a rate of \(-dM(i)_t/M(i)_t\) at any given time \(t\). Then we parameterize player \(i\)'s investment strategy \(\nu\) as \(\tilde{\nu}(i) = \inf\{t \geq 0 : M(i)_t \leq \tilde{f}(i)\}\), where \(\tilde{f}(i)\) is player \(i\)'s randomizer that has a uniform distribution over \(L(i) = [0, 1]\).

Note that for any specific realized value of \(l(i) \in L(i)\), the random time \(\inf\{t \geq 0 : M(i)_t \leq l(i)\}\) is a stopping time adapted to \(\mathcal{F}\) because \(M(i)\) is adapted to \(\mathcal{F}\). Hence, because \(\tilde{f}(i)\) is a random variable of measure \(\mathbb{L}(i)\), the stopping time \(\tilde{\nu}(i) = \inf\{t \geq 0 : M(i)_t \leq \tilde{f}(i)\}\) is a stopping time adapted to \((\mathcal{F}_t)_{t \geq 0} := (\mathcal{F}_t \otimes \mathcal{L}(i))_{t \geq 0}\). Note also that the probability that player \(i\) has not invested by time \(t\) is given by \(\mathbb{L}(i)\{\tilde{\nu}(i) > t\} = M(i)_t\), which leads to the required rate of investment \(-dM(i)_t/M(i)_t\). Finally, we remark that this construction is equivalent to directly mixing stopping times through a functional analysis method (Section 7 of Touzi and Vieille 2002).

### 3.3.2 Characteristics of Mixed Strategy MPE

We are now in a position to provide the definition of MPE. Below we let \(\mathbb{E}_{L(i)}\) denote the expectation with respect to the measure \(\mathbb{L}(i)\).

**Definition.** An SPE \(\nu = (\nu_1, \nu_2)\) is a Markov perfect equilibrium (MPE) if each player’s strategy \(\nu_i\) of investment at time \(\tilde{\nu}(i)\) satisfies the following condition:

\[
\mathbb{E}_{L(i)}[1_{\{\tilde{\nu}(i) \in [\tau, \tau + s]\}} | \mathcal{F}_\tau, \tilde{\nu}(i) > \tau] = \mathbb{E}_{L(i)}[1_{\{\tilde{\nu}(i) \in [\tau, \tau + s]\}} | X_\tau, \tilde{\nu}(i) > \tau],
\]

for any stopping time \(\tau\) and \(s > 0\). If \(\nu\) is an MPE, we call \(\nu_1\) and \(\nu_2\) Markov strategies.
The Markov property (12) implies that the probability distribution of the investment time $\hat{\tau}^{(i)}$ depends only on the current value of the state variable rather than its full history up to time $\hat{\tau}^{(i)}$ (Maskin and Tirole, 2001).

For the time being, we are solely concerned with MPE, so we focus on characterizing Markov strategies. A Markov strategy of stopping is mathematically equivalent to a Markovian killing of a Markov process, which can be equivalently formulated in terms of a multiplicative functional and an additive functional. (For the definitions of multiplicative functionals and additive functionals, see, for example, Chapters III and IV, Blumenthal and Getoor 2007). Thus, following Chapter IV of Blumenthal and Getoor (2007), we can conveniently express a Markovian survival probability as a multiplicative functional given by

$$M^{(i)}_t = \exp(-A^{(i)}_t)N^{(i)}_t,$$  \hspace{1cm} (13)

where $A^{(i)}_t$ is an additive functional of the process $(X_t)_{t\in[0,1]}$, and $N^{(i)}_t$ is an auxiliary multiplicative functional of the form $N^{(i)}_t = \mathbf{1}_{\{t<\tau^{(i)}_E\}}$. By the definition of an additive functional, $A^{(i)}_t$ is a non-negative, non-decreasing, and right-continuous process adapted to $\mathbb{F}$ with the initial value $A^{(i)}_0 = 0$. Here $\tau^{(i)}_E$ is a stopping time at which player $i$ invests with probability one. According to this survival probability, player $i$ will invest at the probabilistic rate of $dA^{(i)}_t$ for $t < \tau^{(i)}_E$, but he will invest at time $\tau^{(i)}_E$ with probability one.

By the stipulation that the survival probability (13) must be Markov, the time of definite investment $\tau^{(i)}_E$ must be a hitting time $\tau^{(i)}_E = \inf\{t \geq 0 : X_t \in E_i\}$ of some closed set $E_i$ that has an interpretation of pure strategy investment region; by the diffusive property of $X$, $\tau^{(i)}_E$ is equal to the hitting time of the closure of $E_i$ even if $E_i$ is not closed (3.22 on p. 312, Revuz and Yor 1999). We also remark that the definition of additive functionals (II.21, Borodin and Salminen 1996) dictates that the probability distribution of $A^{(i)}_{t+s} - A^{(i)}_t$ for some $s > 0$ conditional on $\mathcal{F}_t$ depends solely on the current value $X_t$ and the time $s$, which naturally renders the survival probability (13) Markovian. Thus, a Markov strategy can be completely and unambiguously represented by the set $E_i$ and the additive functional $A^{(i)}$.

There are two well-established properties of an additive functional $A^{(i)}$ of a regular diffusion process $X$: (a) $A^{(i)}_t$ is continuous almost surely (II.21, Borodin and Salminen 1996), and (b) $A^{(i)}$ can be expressed as $A^{(i)}_t = \int_0^t \lambda_i(X_s)ds$, a time-integral of a non-negative Radon-Nikodym derivative $\lambda_i(X_t)$ (II.23 and II.24, Borodin and Salminen 1996). In addition to these convenient properties, we impose two regularity conditions to rule out unrealistic and impractical strategies as well as to ensure analytical tractability. First, we stipulate that the Radon-Nikodym derivative $\lambda_i(X_t)$ of $A^{(i)}_t$ is without singularities so that $A^{(i)}_t$ is differentiable with respect to time $t$. Secondly, we stipulate that the mixed strategy region $\Gamma_i := \{x \in \mathcal{I} : \lambda_i(x) > 0\}$ is a union
of disjoint intervals. This condition rules out any pathological topology of mixed strategy regions such as in a Cantor set, which is unrealistic and impractical to implement for real-life decision makers. Under this stipulation, we can always represent \( \Gamma_i \) as a union of disjoint open intervals because adding a countable set of points to \( \Gamma_i \) (or subtracting one from \( \Gamma_i \)) does not alter the outcome of the game almost surely. Hence, we will subsequently assume that \( \Gamma_i \) is an open set.

The next proposition provides a convenient alternative expression for the equilibrium payoff function. To this end, we first re-express (8) in terms of the expanded probability space and the strategy profile \( \nu = (\tilde{\tau}^{(i)}, \tilde{\tau}^{(j)}) \), where \( \tilde{\tau}^{(i)} \) is indeed enough to consider the payoff from a player's perspective while player \( j \) employs a mixed strategy from player \( i \)'s perspective. As we will show in the remainder of Section 3 below, it is indeed enough to consider the payoff from a \( \mathbb{F} \)-stopping time for player \( i \). This is because if a probabilistic mixture of stopping times is a best response to a given rival’s strategy, then each pure strategy stopping time in the mixture must also be a best response to the rival’s strategy by the definition of the mixed strategy equilibrium (p.665, Hendricks et al. 1988; p.12, Steg 2015). Another difference from (8) is that \( \mathbb{E}^x \) is replaced by \( \mathbb{E}_x^\nu \)[·] := \( \mathbb{E}_{\mathbb{P} \otimes \lambda_i, \mathbb{P}}[·|X_0 = x] \) representing the expectation over \( \mathbb{P} \otimes \lambda_i \) conditional on \( X_0 = x \).

The following proposition establishes an alternative expression for (14).

Proposition 3 Given player \( j \)'s Markov strategy \( \nu_j \) determined by \( \lambda_j(\cdot) \) and \( E_j \), player \( i \)'s payoff from investing at a \( \mathbb{F} \)-stopping time \( \tilde{\tau}^{(i)} \) is given by

\[
V_i^\nu(x) = \mathbb{E}^x \left[ \int_0^{\tilde{\tau}_x^{(i)}} \pi(X_t) e^{-rt} dt + g_i(X_{\tilde{\tau}_x^{(i)}}) e^{-r\tilde{\tau}_x^{(i)}} 1_{\{\tilde{\tau}_x^{(i)} < \bar{\tau}^{(i)}\}} + m_i(X_{\tilde{\tau}_x^{(i)}}) e^{-r\tilde{\tau}_x^{(i)}} 1_{\{\tilde{\tau}_x^{(i)} = \bar{\tau}^{(i)}\}} \right] + \frac{1}{2} \left[ g_i(X_{\tilde{\tau}_x^{(i)}}) + m_i(X_{\tilde{\tau}_x^{(i)}}) \right] e^{-r\tilde{\tau}_x^{(i)}} 1_{\{\tilde{\tau}_x^{(i)} = \bar{\tau}^{(i)}\}} .
\]

By virtue of Proposition 3, the payoff (14) is conveniently re-expressed as an expectation operator \( \mathbb{E}[·] \) over a new expression for the payoff that incorporates player \( j \)'s strategy through \( \lambda_j(\cdot) \) and \( \tau^{(j)} \).
3.3.3 Mixed strategy MPE

We now construct an MPE strategy profile \( \nu^M = (\nu^M_1, \nu^M_2) \). As per Section 3.3.2, we only need to specify \( \lambda_i(\cdot) \) and \( E_i \) to determine \( \nu^M_i \). Specifically, we choose \( E_i = \emptyset \) for both \( i \in \{1, 2\} \) and stipulate the form of \( \lambda_i(\cdot) \) with respect to \( g(\cdot) := g_1(\cdot) = g_2(\cdot) \) and \( \pi(\cdot) \) as follows:

\[
\lambda_i(x) := 1_{\{x \in (a, b]\}} \frac{-\mathcal{A} g(x) - \pi(x)}{(R, \pi)(z^*) - g(x)}.
\]

Recall that \( \mathcal{A} \) is the characteristic differential operator of \( X \) defined by (3). Note that the rate of investment is non-zero only if \( X_i \in (a, \theta) \), where \( \theta := \theta_1 = \theta_2 \). Hence, we call \((a, \theta)\) the common mixed strategy investment region. The function \( \lambda_i(x) \) is non-negative because both the numerator and the denominator are positive for \( x < \theta \). First, \( \mathcal{A} g(x) + \pi(x) < 0 \) holds for \( x < \theta \) because of the inequality \( \theta \leq x^*_i \) by the theory of optimal stopping (Chapter 10 of Øksendal, 2003) where \( x^*_i \) is given in Assumption 3(ii). Secondly, \((R, \pi)(z^*) > g(x)\) for all \( x \in (a, \theta) \) by the definition of \( g(\cdot) \) given in (6) and \( \theta_i < z^* \) in Assumption 3(i).

Below we let \( \nu^M_i(x) \) denote the payoff function associated with \( \nu^M \) when the current state variable is \( x \).

**Proposition 4** The strategy profile \( \nu^M \) defined above is an MPE. Furthermore, \( \nu^M_i(x) = V^*_i(x) \) given by (11).

Proposition 4 establishes that the strategy profile \( \nu^M \) is a mixed strategy MPE. It also establishes that the payoff under \( \nu^M \) is identical to the payoff from the single-player optimal stopping solution. The equality \( \nu^M_i(x) = V^*_i(x) \) has an heuristic explanation. According to \( \nu^M \), none of the players invests until the hitting time of \((a, \theta)\), where \( \theta \) is identical to the threshold of investment in the single-player solution, so \((\theta, b)\) takes the role of a continuation region for both players. Furthermore, the equilibrium payoff to player \( i \) within \((a, \theta)\) is equal to the payoff from immediate investment, i.e., \( g_i(x) \); this is because whenever \( X \) is in \((a, \theta)\), which is the common mixed strategy region under \( \nu^M \), an immediate investment must be one of player \( i \)'s best responses as per the definition of a mixed strategy equilibrium (Hendricks et al., 1988; Georgiadis et al., 2022). Thus, the payoff function has to satisfy the boundary condition \( \nu^M_i(\theta) = g_i(\theta) \). Since \((\theta, b)\) is a continuation region for player \( i \), \( \nu^M_i(x) \) can be obtained as the solution to a boundary value problem in the interval \([\theta, b]\) with the boundary condition \( \nu^M_i(\theta) = g_i(\theta) \). Note that \( V^*_i(x) \) is also the solution to a boundary value problem in the interval \([\theta, b]\) with the boundary condition \( V^*_i(\theta) = g_i(\theta) \). Hence, we have \( \nu^M_i(x) = V^*_i(x) \).
3.3.4 Two-stage SPE

Next, we construct a TSSPE $v^S$. As the name suggests, the equilibrium path of $v^S$ consists of two stages. In the first stage, each player $i$’s strategy is to invest with probability $q_i$ upon reaching $\tau_i := \inf\{t \geq 0 : X_t \in \Gamma\}$, the hitting time of the interval $\Gamma := (a, \theta)$, which we identified as the common mixed strategy region for the MPE in Section 3.3.3. In the second stage, the players employ the mixed strategy MPE profile presented in Section 3.3.3. We can thus define the first stage as the time interval $[0, \tau_i]$ and the second stage as $(\tau_i, \infty)$. We also stipulate that $q_1 q_2 = 0$; if $q_1$ and $q_2$ are both positive, then there is positive probability of simultaneous investment at $\tau_i$, and hence, one of the players can improve his payoff by forgoing with investment at time $\tau_i$. Without loss of generality, we set $q_1 = 0$ and $q_2 > 0$. Note that player 2’s strategy is non-Markov, and hence, $v^S$ is not an MPE; player 2’s plan of action depends not only on the current value of $X$ but also on whether $X$ has hit $\Gamma$ in the past history.

Based on the strategy profile described above, we can construct survival probability $M_i^{(i)}$ as follows:

$$M_i^{(1)} = \exp[-1_{(t > \tau_i)} \int_{\tau_i}^t \lambda_1(X_t) dt],$$  \hspace{1cm} (17)

$$M_i^{(2)} = \begin{cases} 
1 & \text{if } t < \tau_i, \\
(1 - q_2) \exp[- \int_{\tau_i}^t \lambda_2(X_t) dt] & \text{if } t \geq \tau_i.
\end{cases}$$  \hspace{1cm} (18)

The expression (17) reflects player 1’s strategy of investment at the hazard rate of $\lambda_1(X_t)$ in the second stage when $t > \tau_i$. The expression (18) is also based on player 2’s strategy. Player 2 does nothing until $\tau_i$ during stage 1, so $M_i^{(2)}$ is invariant in time for $t < \tau_i$. At time $\tau_i$, he invests with a probability of $q_2$, which results in the discontinuity of $M_i^{(2)}$. In the second stage, player 2 invests at the hazard rate of $\lambda_2(X_t)$, and hence, the exponential factor $\exp[- \int_{\tau_i}^t \lambda_2(X_t) dt]$.

We let $V^{v^S}_i(x, s)$ denote the payoff to player $i$ associated with $v^S$ in stage $s \in \{1, 2\}$ when the current state variable is $x$. Since stage 2 reduces to the mixed strategy MPE, we obtain $V^{v^S}_i(x, 2) = V^{v^M}_i(x)$ where $v^M$ is the mixed strategy MPE shown in Section 3.3.3. The first-stage payoff functions are given as follows:

$$V^{v^S}_1(x, 1) = \mathbb{E}^x \left\{ \int_0^{\tau_i} \pi(X_t) e^{-\tau_i} dt + e^{-\tau_i} \left[ (1 - q_2) V^{v^M}_1(X_{\tau_i}) + q_2 R(X_{\tau_i}) \right] \right\},$$  \hspace{1cm} (19)

$$V^{v^S}_2(x, 1) = \mathbb{E}^x \left\{ \int_0^{\tau_i} \pi(X_t) e^{-\tau_i} dt + e^{-\tau_i} \left[ (1 - q_2) V^{v^M}_2(X_{\tau_i}) + q_2 g(X_{\tau_i}) \right] \right\}$$

$$= \mathbb{E}^x \left\{ \int_0^{\tau_i} \pi(X_t) e^{-\tau_i} dt + e^{-\tau_i} g(X_{\tau_i}) \right\}.$$  \hspace{1cm} (20)
The expression for $V_1^{ν_S}(x, 1)$ in (19) indicates that player 2 invests at $τ_Γ$ with a probability $q_2 > 0$, resulting in the immediate payoff $(R, π)(z^*)$ to player 1; with a probability $1 - q_2$, the game enters stage 2, and player 1 earns the payoff $V_1^{ν_M}(X_{τ_Γ})$ at time $τ_Γ$. Similarly, player 2’s payoff at $τ_Γ$ in (20) is $g(X_{τ_Γ})$ if he invests at $τ_Γ$ and $V_2^{ν_M}(X_{τ_Γ})$ if he does not. The payoff $V_2^{ν_S}(x, 1)$ reduces to (21) because $V_2^{ν_M}(X_{τ_Γ}) = g(X_{τ_Γ})$ by (11) and Proposition 4. The following proposition establishes that $ν_S$ is an equilibrium.

**Proposition 5** The strategy profile $ν_S$ defined above is an SPE.

The mixed strategy TSSPE $ν_S$ is the stochastic analog of the mixed strategy SPE in Theorem 3 of Hendricks et al. (1988). In fact, all SPEs of deterministic wars of attrition take the form of the TSSPE (Hendricks et al., 1988). This is one of the simplest classes of SPE, and thus the most likely to be observed in practice. For this reason, we search for TSSPE in asymmetric games in the remainder of this paper.

### 3.4 Absence of Two-stage SPE in Asymmetric Games

We now establish the central result of this section: an asymmetric game has no TSSPE that can be represented by survival probability of the form given in (17) and (18). This result is a slight extension of Theorem 1 of Georgiadis et al. (2022); the only difference between the model examined by Georgiadis et al. (2022) and our current model is in the functional form of $g_i(·)$ and $m_i(·)$. Nevertheless, for the sake of completeness, we provide the complete proof in EC Appendix B.

**Theorem 1** In a single-investment game with $c_1 ≠ c_2$, there exists no MPE or TSSPE.

Theorem 1 holds because the two players must have the common mixed strategy region. If the two players have differing costs $c_1 ≠ c_2$, then the boundary points of their mixed strategy region cannot coincide due to the asymmetry, and hence, $Γ_1 = Γ_2$ cannot be satisfied.

Note, however, that Theorem 1 does not claim to have proved that a mixed strategy equilibrium of any kind is impossible under asymmetry. Instead, the main point of Theorem 1 is that an asymmetric single investment game does not possess a mixed strategy equilibrium that shares the same characteristics with known mixed strategy equilibria obtained in Sections 3.3.3 and 3.3.4 as well as in Hendricks et al. (1988) and Steg (2015). In contrast, the next section demonstrates that there is such an equilibrium in the infinite investment game despite asymmetry, which is the central goal of this paper.
4 Impulse Control Game

We now turn to a game with an infinite number of investment opportunities. The goal of this section is to prove that a moderately asymmetric game possesses a mixed strategy TSSPE. This result is in stark contrast to the result from the single investment model (Theorem 1) or the stochastic concession games examined by Georgiadis et al. (2022). Since we focus on asymmetric games, we assume $c_1 > c_2$ without loss of generality. We first derive a verification theorem (Theorem 2) and use it to construct a mixed strategy SPE (Theorem 3). We then illustrate an example of a mixed strategy TSSPE in Section 4.4. Lastly, we compare the mixed strategy TSSPE to a pure strategy SPE from the perspective of a social planner in Section 4.5.

4.1 The Model

As a first step, we extend the model introduced in Section 3.1 to allow for an infinite number of investment opportunities. We assume the same properties of the uncontrolled process $X$ as in Section 3.1.1, but the payoff and the strategy space differ.

We let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ denote the probability space of the sample paths of a Wiener process $W$ that satisfies the usual condition. Then we let $\hat{\mathcal{F}} = (\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t \geq 0}, \hat{\mathbb{P}})$ denote the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ augmented by the randomizers of mixed strategies, which we will specify later in this subsection, and we let $v_i = (\eta_n^{(i)}, \xi_n^{(i)})_{n \in \mathbb{N}}$ denote player $i$’s strategy that stipulates a series of strictly increasing $(\hat{\mathcal{F}}_t)_{t \geq 0}$-stopping times of investment $(\eta_n^{(i)})_{n \in \mathbb{N}}$ and the boosts $(\xi_n^{(i)})_{n \in \mathbb{N}}$. Each boost $\xi_n^{(i)}$ is measurable with respect to $\hat{\mathcal{F}}_{\eta_n^{(i)}}$. To avoid the pathological scenario of infinitely many investments within an infinitesimal time interval, we impose the condition $\lim_{n \to \infty} \eta_n^{(i)} = \infty$ (Alvarez and Lempa, 2008).

For convenience, we partition the timeline into periods of investment and specify the player’s strategy for each period. We let $m$ denote the period index which keeps track of the total number of investments made on the common good. We let $(T_m)_{m \in \{0,1,2,\ldots\}}$ denote the set of stopping times of investment made by either player such that $T_{m+1} \geq T_m$. As a matter of convention, we define $T_0 = 0$. By this convention, $T_m$ indicates the timing of the $m$-th investment, and the $m$-th period is defined as $[T_m, T_{m+1})$. We also note that $\lim_{m \to \infty} T_m = \infty$ by the stipulation that $\lim_{n \to \infty} \eta_n^{(i)} = \infty$ for each $i$. We focus on the strategy profiles which have no dependence on $m$ because this is the simplest class of strategy profiles and easiest to implement for decision makers in practice.

Next, we construct the probability space. Just as in Section 3.3.1, we utilize the recipe provided by Section 7.1 of Touzi and Vieille (2002) for each period and construct an expanded probability space $\hat{\mathcal{F}}$. Specifically,
the new probability space is given by

$$\mathcal{F} = (\Omega \times \prod_{m \in \mathbb{N}_0} L_m^{(i)} \times \prod_{m \in \mathbb{N}_0} L_m^{(j)}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, \mathbb{F})$$

where $L_m^{(i)} = [0, 1]$, $\mathbb{P} := \mathbb{P} \otimes_{m \in \mathbb{N}_0} \mathbb{L}_m^{(i)} \otimes_{m \in \mathbb{N}_0} \mathbb{L}_m^{(j)}$, where $\mathbb{L}_m^{(i)}$ is the Lebesgue measure on $L_m^{(i)}$. We let $\mathcal{L}_m^{(i)}$ denote the Borel $\sigma$-algebra on $L_m^{(i)}$, and we define $\hat{\mathcal{F}} = \mathcal{F} \otimes_{m \in \mathbb{N}_0} \mathcal{L}_m^{(i)} \otimes_{m \in \mathbb{N}_0} \mathcal{L}_m^{(j)}$. Likewise, $\hat{\mathcal{F}}$ is a product $\sigma$-algebra given by $\mathcal{F}_t \otimes_{m \in \mathbb{N}_0} \mathcal{L}_m^{(i)} \otimes_{m \in \mathbb{N}_0} \mathcal{L}_m^{(j)}$.

Following Touzi and Vieille (2002), we let player $i$’s $m$-th period mixed strategy be characterized by an $m$-th period survival probability $M_m^{(i)} = (M_{m,t}^{(i)})_{t \in [T_m, T_{m+1})}$ with the initial condition $M_{m,T_m}^{(i)} = 1$ and an $m$-th period randomizer $\hat{\tau}_m^{(i)} \in [0, 1]$ with a Lebesgue measure $\mathbb{L}_m^{(i)}$ on $[0, 1]$. The investment stopping time is then defined as $\tau_m^{(i)} := \inf\{t \geq T_m : M_{m,t}^{(i)} \leq \hat{\tau}_m^{(i)}\}$, and the period $m$ begins at the stopping time $T_m = \tau_m^{(i)} \wedge \tau_m^{(j)}$. We also let $\xi_{m+1}^{(i)}$ denote the $\hat{\mathcal{F}}_{m+1}^{(i)}$-measurable boost at time $\tau_m^{(i)}$ executed by player $i$. Just as in Section 3.3.1, utilizing the expanded probability space $\hat{\mathcal{F}}$ for each period is equivalent to directly mixing stopping times by virtue of the proof provided in Section 7 of Touzi and Vieille (2002). Furthermore we assume the same tie-breaking rule as in Section 3: if $\tau_m^{(i)} = \tau_m^{(j)}$, then only one of the two players goes through with the investment with a probability of 50%. Finally, under the strategy profile $v = (v_1, v_2)$, the controlled state variable satisfies the following stochastic integral equation:

$$X_t^v = X_0 + \int_0^t \mu(X_s^v)ds + \int_0^t \sigma(X_s^v)dW_s + \sum_{m=1}^{\infty} \xi_m^{(i)} 1_{\{T_m \leq t\}},$$

where $\xi_m^{(i)} = \xi_m^{(j)}$ if $\tau_m^{(i)} < \tau_m^{(j)}$, and $\xi_m^{(i)}$ can be either $\xi_m^{(i)}$ or $\xi_m^{(j)}$ with 50% probability each if $\tau_m^{(i)} = \tau_m^{(j)}$.

We now specify the payoff function. If player $i$ makes an investment at time $\tau_m^{(i)}$ to boost $X^v$ by $\xi_m^{(i)} \geq 0$, it costs $c_i + k\xi_m^{(i)}$. Based on this cost structure, we can express the payoff to player $i$ at time $t$ as follows:

$$V_t^{v,i} := e^{rt}\hat{\mathbb{E}}_t \left[ \int_t^{\infty} \pi(X_s^v)e^{-rs}ds - \sum_{m=1}^{\infty} e^{-r\tau_m^{(i)}}(k\xi_m^{(i)} + c_i) 1_{\{\tau_m^{(i)} < \tau_m^{(j)}\}} - \frac{1}{2} \sum_{m=1}^{\infty} e^{-r\tau_m^{(i)}}(k\xi_m^{(i)} + c_i) 1_{\{\tau_m^{(i)} = \tau_m^{(j)}\}} | \hat{\mathcal{F}}_t \right].$$

(22)

Here we let $\hat{\mathbb{E}}_t$ denote the expectation over the product measure $\mathbb{P} \otimes_{m \in \mathbb{N}_0} \mathbb{L}_m^{(j)}$. We remark that the expression implicitly assumes a pure strategy of player $i$ for simplicity of presentation. However, the expression (22) will equally well apply even if $v$ is a mixed strategy equilibrium because the payoff to player $i$ can be computed using one of the pure-strategy best responses (Hendricks et al. 1988, Steg 2015).
In general, it is tricky to account for the possibility of simultaneous moves by two players in impulse control games. This is because infinitely many impulse control events may take place between two players at a single point in time; for instance, the state variable could be trapped in perpetuity within a bounded interval because the two players push $X$ back and forth forever within an infinitesimal duration of time. The prior work in the literature circumvents this problem in various ways. Guo and Xu (2019) excludes simultaneous jump controls by multiple players from the admissible control set. Dutta and Rustichini (1995) argues that an infinite number of moves at a single instant would incur an arbitrarily large cost to the players and illustrates a set of restrictions on the strategy profile to exclude such a possibility. Aïd et al. (2020) prevents the players from accumulating impulse controls at some finite time by stipulating a condition that is similar to the assumption $\lim_{n \to \infty} \eta_i^{(n)} = \infty$ in our paper. We note that we can also impose a set of restrictions on the equilibrium strategy profile as Dutta and Rustichini (1995) to exclude this pathology. However, we do not explicitly impose these conditions because, in a mixed strategy TSSPE we consider in our paper, a player’s investment always moves the state variable to the common continuation region, and hence, such a complication does not arise.

4.2 Verification Theorem

In this subsection, we construct a specific strategy profile and payoff functions, and we prove that they constitute a mixed strategy equilibrium if they satisfy a set of conditions.

**Strategy profile:** In the proposed strategy profile $\nu^*$, each period is divided into two stages as in Section 3.3.4. Let $\tau_{m^*} := \inf \{ t \in [T_m, T_{m+1}) : X_t \leq \theta^* \}$ denote the hitting time of $(a, \theta^*)$, where $\theta^*$ is some threshold that is different from $\theta_i$ or $\theta_j$ defined in Section 3. Then we define the first stage as the time interval $[T_m, \tau_{m^*})$ and the second stage as $[\tau_{m^*}, T_{m+1})$.

In the first stage, if $X_{T_m}^\nu \geq \theta^*$, then player 2’s strategy is to invest at time $\tau_{m^*}$ with a probability of $q \in (0, 1)$. On the other hand, player 1’s strategy is not to invest at all in the first stage. If player 2 invests at $\tau_{m^*}$, then period $m$ ends, and the next period $m + 1$ begins. With a probability of $1 - q$, player 2 does not invest at $\tau_{m^*}$, in which case the second stage of period $m$ begins. In case $X_{T_m}^\nu < \theta^*$ in the beginning of the period, player 2’s probability of immediate investment is set to 0. In other words, if $X_{T_m}^\nu < \theta^*$, then the period immediately enters its second stage with probability one.

In the second stage, the investment strategies of $\nu^*$ is characterized by a common mixed strategy region of $\Gamma := (a, \theta^*)$ with the threshold $\theta^*$. Each player $i$ invests with a rate of $\lambda_i(\cdot) = \lambda_i(X_t)$ where $\lambda_i(\cdot)$ will be defined in (29) below. In either stage of $\nu^*$, player $i$’s magnitude of investment is $\max \{ z_i - x, 0 \}$, where $x$ is the value...
of $X$ at the time of investment. In general, $z_1$ and $z_2$ may differ.

See Figure 1 for a simulated sample path of $X^{v^*}$ in periods $0-2$, where player 2 invests at the end of periods 0 and 2 while player 1 invests at the end of period 1. In period 0, player 2 invests exactly at $\tau_0^{m^*}$, while in periods 1 and 2, the players invest in the second stage within the mixed strategy region $\Gamma$.

**Payoff functions:** Next, we construct functions $F_i(x) := \{F_{i,t}(x)\}_{t \geq 0}$ for each $i \in \{1, 2\}$, $x \in \mathcal{S}$, and $t \in \mathbb{R}^+$. $F_i(x)$ is the candidate for the payoff function for player $i$ associated with $v^*$. Below we list a number of conditions that these functions satisfy.

First, we assume that the function $F_i(\cdot)$ is expressible in terms of some functions $U_1(\cdot)$ and $V_i(\cdot)$ as follows:

$$
F_{1,t}(x) = \begin{cases} 
U_1(x) & \text{for } t \in [T_{m^*}, \tau_{0^*}^m), \\
V_1(x) & \text{for } t \in [\tau_{0^*}^m, T_{m+1}) 
\end{cases},
$$

(23)

$$
F_{2,t}(x) = V_2(x).
$$

We assume that the game begins at $t = 0$ in the first stage of the 0-th period so that $F_{1,0}(x) := U_1(x)$ and $F_{2,0}(x) := V_2(x)$. Here we stipulate that $U_1(\cdot)$ and $V_i(\cdot)$ satisfy $U_1(\cdot) \in C^2(\mathcal{S}\setminus\{\theta^*\}) \cap C^1(\mathcal{S}\setminus\{\theta^*\}) \cap C(\mathcal{S}\setminus\{\theta^*\})$, $V_i(\cdot) \in C^2(\mathcal{S}\setminus\{\theta^*\}) \cap C^1(\mathcal{S}) \cap C(\mathcal{S})$, and $V_i(\cdot) > (R, \pi)(\cdot)$. We also stipulate that $U_1(x) > V_1(x)$ for $x \geq \theta^*$ and $U_1(x) = V_1(x)$ for $x < \theta^*$. Note that $F_{1,t}(x)$ is discontinuous in time at $\tau_{0^*}^m$ if $x \geq \theta^*$; this is to reflect that the first and second stage payoffs to player 1 differ from each other because player 2 invests at $\tau_{0^*}^m$ with a probability of $q$.

Figure 1: A simulated sample path of $X^{v^*}$ under the mixed strategy profile $v^*$. 


Second, we stipulate that the functions $U_i(\cdot)$ and $V_i(\cdot)$ satisfy variational inequalities given by

\[ \mathcal{A}_i V_i(x) + \pi(x) = 0 \text{ for } x > \theta^*, \]  
\[ \mathcal{A}_i V_i(x) + \pi(x) < 0 \text{ for } x < \theta^*, \]  
\[ \mathcal{A}_i U_i(x) + \pi(x) = 0 \text{ for } x > \theta^*. \]  

In addition, we impose the quasi-variational inequalities for all $x \in \mathcal{S}$: $F_{i,0}(x) \geq \sup_{\zeta \geq 0}[F_{i,0}(x + \zeta) - k\zeta - c_i]$. Since $F_{i,0}(\cdot)$ is the first-stage functions, this inequality translates into the following set of inequalities:

\[ V_2(x) \geq \sup_{\zeta \geq 0}[V_2(x + \zeta) - k\zeta - c_2], \]  
\[ V_1(x) \geq \sup_{\zeta \geq 0}[U_1(x + \zeta) - k\zeta - c_1], \]  
\[ U_1(x) \geq \sup_{\zeta \geq 0}[U_1(x + \zeta) - k\zeta - c_1]. \]  

We also assume that $|U_i''(x)| < \infty$ and $|V_i''(x)| < \infty$ are satisfied for $x \neq \theta^*$.

Third, we relate $z_i$ and the investment rate $\lambda_i(\cdot)$ with the functions $F_{i,0}(\cdot)$. We assume that $z_i$ is the unique value that satisfies $z_i = \arg \max_z [F_{i,0}(z) - kz]$, which has the meaning of the optimal end point of the boost for player $i$. Then we can define

\[ g_i(x) := \begin{cases} F_{i,0}(z_i) - k(z_i - x) - c_i & \text{for } x < z_i, \\ F_{i,0}(x) - c_i & \text{for } x \geq z_i, \end{cases} \]  

which has the interpretation of the reward from investment. We also stipulate that $V_i(x) = g_i(x)$ for $x \in \Gamma = (a, \theta^*)$, which is reminiscent of the property of a mixed strategy equilibrium. Finally, we define the investment rate as:

\[ \lambda_i(x) = \mathbf{1}_{\{x \in (a, \theta^*)\}} \frac{-\mathcal{A}_i g_j(x) - \pi(x)}{F_{i,0}(z_i) - g_j(x)}. \]  

Based on the imposed conditions, we can construct an explicit functional forms of $U_1(\cdot)$ and $V_1(\cdot)$. The
equations $\mathcal{A}V_i(x) + \pi(x) = 0$ in (24) and $\mathcal{A}U_1(x) + \pi(x) = 0$ in (26) imply the following functional forms:

$$V_i(x) = \begin{cases} (R_i \pi)(x) + w_i \phi(x) & \text{for } x \geq \theta^* \\ g_i(x) & \text{for } x < \theta^* \end{cases}$$ \hfill (30)

$$U_1(x) = \begin{cases} (R_i \pi)(x) + u_1 \phi(x) & \text{for } x \geq \theta^* \\ g_1(x) & \text{for } x < \theta^* \end{cases}$$

for some positive coefficients $w_i$ and $u_1$. The coefficients $w_i$ and $u_1$ must satisfy the condition $u_1 > w_1$ (from $U_1(x) > V_1(x)$ for $x \geq \theta^*$). Lastly, we impose two more conditions that relate $\nu^*$ to $U_1$ and $V_i$. The first is the boundary condition $U_1(\theta^*) = qU_1(z_2) + (1-q)V_1(\theta^*)$ which we stipulate because player 2 invests at the hitting time of $\theta^*$ with a probability of $q$. The second condition is $U_1(z_2) \geq V_1(\theta^*) = g_1(\theta^*)$, which has the interpretation that player 1 earns higher payoff when player 2 invests than when player 1 invests at $\theta^*$.

**Survival probability:** Based on the strategy profile $\nu^*$, we can construct per-period survival probability $M_{m,i}^{(i)}$ as follows:

$$M_{m,i}^{(1)} = \exp\left[-1_{(t > \tau^m_{\nu^*})} \int_{\tau^m_{\nu^*}}^t \lambda_i(X_t) dt \right],$$ \hfill (31)

$$M_{m,i}^{(2)} = \begin{cases} 1 & \text{if } t < \tau^m_{\nu^*} \\ (1 - 1_{(X_{\tau^m_{\nu^*}} = \theta^*)} q) \exp\left[-\int_{\tau^m_{\nu^*}}^t \lambda_2(X_t) dt \right] & \text{if } t \geq \tau^m_{\nu^*} \end{cases}.$$ \hfill (32)

The expressions $M_{m,i}^{(i)}$ are analogous to (17) and (18) from the single investment game. The expression (31) clearly follows the strategy $\nu_1^*$ of investment at the hazard rate of $\lambda_i(X_t)$. The expression (32) requires some explanation. Player 2 does not invest until $\tau^m_{\nu^*}$, so $M_{m,i}^{(2)}$ does not vary in time for $t < \tau^m_{\nu^*}$. At the hitting time of $\theta^*$, if $X_{\tau^m_{\nu^*}} = \theta^*$, player 2 invests with a probability of $q$, so $M_{m,i}^{(2)}$ is reduced by $q$ at time $\tau^m_{\nu^*}$. However, if $X_{\tau^m_{\nu^*}} < \theta^*$, then the game immediately enters the second stage, so there is no immediate discontinuous reduction in $M_{m,i}^{(2)}$. After $\tau^m_{\nu^*}$, player 2’s rate of investment is $\lambda_2(X_t)$, which explains the factor $\exp\left[-\int_{\tau^m_{\nu^*}}^t \lambda_2(X_t) dt \right]$.

In summary, we have specified a strategy profile $\nu^*$ and hypothesized a pair of functions $F_1(\cdot)$ and $F_2(\cdot)$ that satisfy a set of conditions. We are now ready to present Theorem 2 which establishes that $\nu^*$ is an equilibrium given the existence of $F_1(\cdot)$ and $F_2(\cdot)$.

**Theorem 2** *Under Assumptions 1 and 2, if $F_1(\cdot)$ and $F_2(\cdot)$ that satisfy the conditions above exist, then $\nu^*$ is an SPE.*
Theorem 2 establishes a sufficient condition for a mixed strategy SPE. To our knowledge, this is the first verification theorem in the literature for a mixed strategy equilibrium of an impulse control game.

4.3 Existence of Mixed Strategy TSSPE

In this section, we utilize Theorem 2 and show that a mixed strategy TSSPE exists in our model under certain conditions. Closely following the conventions used in Alvarez and Lempa (2008), we define the following auxiliary functions associated with the uncontrolled process $X$ and the flow profit $\pi(\cdot)$:

$$S'(x) := \exp \left[ - \int_{x_0}^x \frac{2\mu(y)}{\sigma^2(y)} dy \right], m'(x) := \frac{2}{\sigma^2(x)S'(x)};$$

$$\rho(x) := \pi(x) + k[\mu(x) - rx], L(x) := -\frac{\rho(x)\phi'(x)}{S'(x)} - r \int_x^b \phi(y)\rho(y)m'(y)dy,$$

where $x_0 \in \mathcal{I}$ is an arbitrarily chosen reference point, $S(\cdot)$ is the scale function, and $m(\cdot)$ is the speed measure (II.4 in Borodin and Salminen, 1996).

We now make the following assumptions to ensure that the desired equilibrium exists.

**Assumption 4**

(i) $\rho(\cdot) \in C^1(\mathcal{I})$, and there exists $x^* \in \mathcal{I}$ such that $\rho'(x) > 0$ for $x < x^*$ and $\rho'(x) < 0$ for $x > x^*$. (ii) $\lim_{x \to b} \frac{\rho(x)\phi'(x)}{S'(x)} = 0$. (iii) $L(x) < 0$ for some $x \in \mathcal{I}$.

Assumption 4 can be found to be satisfied in many examples of stochastic impulse control problems. Assumption 4(i) is made by Alvarez and Lempa (2008), and Assumption 4(ii) is only slightly stronger than the condition $\lim_{x \to b} \frac{\rho(x)\phi'(x)}{S'(x)} = 0$ which must be satisfied by a natural boundary $b$. Assumption 4(iii) is needed to ensure that the function $L(\cdot)$ has a unique zero in $\mathcal{I}$, which is used to derive an optimal impulse control in Alvarez and Lempa (2008).

In addition, we define two more auxiliary functions:

$$I(x) := \frac{(R,\pi)'(x) - k}{\phi'(x)};$$

$$J(x) := (R,\pi)(x) - kx - I(x)\phi(x).$$

Based on Assumptions 1, 2, and 4, we obtain the following useful properties of $I(\cdot)$ and $J(\cdot)$:

**Lemma 2**

(i) There exists $\hat{x} < \min\{x^*, z^*\}$ such that $I'(x) < 0$ and $J'(x) > 0$ for $x < \hat{x}$, and $I'(x) > 0$ and $J'(x) < 0$ for $x > \hat{x}$.

(ii) $I(x) < 0$ for $x < z^*$ and $I(x) > 0$ for $x > z^*$. 

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Lastly, we make the following additional assumption:

**Assumption 5** There exist $\theta^*$ and $z_2$ that satisfy

\[
I(z_2) = I(\theta^*) \quad (35)
\]

\[
J(z_2) - J(\theta^*) = c_2. \quad (36)
\]

In the context of our model, Assumption 5 can be shown to ensure the optimal solution to the single-player impulse control problem (Section 4.2 of Alvarez and Lempa 2008; Lemma 3 of our paper). Thus, this is not a particularly stringent assumption because we are essentially stipulating that the optimal solution to the single-player impulse control problem exists.

Finally, under Assumptions 1, 2, 4, and 5, we establish the following theorem:

**Theorem 3** A mixed strategy TSSPE $\nu^*$ characterized by $\theta^*$, $z_1$, $z_2$, and $q \in (0, 1)$ exists if $c_1 \in (c_2, \hat{c})$ for some $\hat{c} > c_2$. Furthermore, the parameters satisfy $\theta^* < \hat{x} < z_1 < z_2$.

Theorem 3 stands in stark contrast to the result from the single investment game. In the model that allows for an infinite number of investment opportunities, a moderate amount of asymmetry does not destabilize the mixed strategy SPE. In our impulse control game, the mixed strategy equilibrium can be sustained despite asymmetry because player 2’s active investment strategy with $q > 0$ in the future periods increases player 1’s payoff in the current period. To elaborate, recall that in the single investment game, a mixed strategy MPE is destabilized by asymmetry because the mixed strategy regions $\Gamma_1$ and $\Gamma_2$ cannot coincide due to asymmetry. Namely, because $c_1 > c_2$, player 1 naturally has lower threshold of investment. In the repeated game of investment, however, player 2 can allocate a sufficiently high probability $q$ of investment at the hitting time of $\theta^*$, which then drives player 1’s investment threshold to $\theta^*$. For the same reason, $q$ increases in $c_1$ because, as $c_1$ increases, $q$ needs to increase to compensate for player 1’s disincentive to invest.

Theorem 3 also establishes that player 1’s magnitude of the investment is comparatively less than that of player 2, i.e., $z_1 < z_2$. Because of the probability $q > 0$ that player 2 would invest at the hitting time of $\theta^*$, player 1’s payoff function is particularly high when $X^{\nu^*}$ is close to the threshold $\theta^*$, but the effect of player 2’s investment at $\theta^*$ tapers off as $X^{\nu^*}$ moves far above $\theta^*$. Thus, player 1’s incentive to boost $X^{\nu^*}$ to a high value is not as strong as that of player 2, and hence, $z_1$ is less than $z_2$. 

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4.4 Example

We now illustrate an example of the mixed strategy TSSPE. As the uncontrolled state variable process, we consider a geometric Brownian motion with SDE $dX_t = \mu X_t dt + \sigma X_t dW_t$ for some constants $\mu < 0$ and $\sigma > 0$. In this case, $I = (0, \infty)$, and the solutions to the homogeneous differential equation $Af = 0$ are given by $\phi(x) = x^{Y_-}$ and $\psi(x) = x^{Y_+}$, where the power indices are given by

$$Y_\pm = \frac{1}{2} - \frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.$$ 

We assume a flow profit function of $\pi(x) = x^\alpha$ for some $\alpha \in (0, 1)$. Then it follows that $(R, \pi)(x) = x^\alpha/(r - \delta(\alpha))$ where $\delta(\alpha) = \alpha\mu + \sigma^2\alpha(\alpha - 1)/2 < 0$ (Alvarez and Lempa, 2008).

It is straightforward to verify that this example satisfies Assumptions 1, 2, and 4. In addition, the following proposition ensures that Assumption 5 holds if $c_2$ is not too large.

**Proposition 6** The example above satisfies Assumption 5 if $c_2 \in (0, J(z^*))$.

By Proposition 6 and Theorem 3, this example possesses a mixed strategy TSSPE if $c_2 \in (0, J(z^*))$ and the difference $c_1 - c_2$ is not too large. If we set $r = 1, \alpha = 0.5, \mu = -0.5, \sigma = 0.25, k = 1, c_2 = 0.015$, then we obtain $\theta^* = 0.0439$ and $z_2 = 0.1480$. Under this parameter set, a mixed strategy TSSPE $\nu^*$ exists as long as $c_1 \leq 0.01729$.

4.5 Comparison to Pure Strategy Equilibrium

In this subsection, we demonstrate that there exists a pure strategy equilibrium which is more efficient than the mixed strategy SPE $\nu^*$. Letting $V_1^\nu(x)$ denote the payoff function associated with a strategy profile $\nu$ conditional on $X_0 = x$, we say that an equilibrium $\nu$ is *more efficient* than another equilibrium $\nu'$ if $V_1^\nu(x) + V_2^\nu(x) > V_1^{\nu'}(x) + V_2^{\nu'}(x)$ for all $x \in \mathcal{I}$.

As a preliminary step, we establish the solution to the single-player model, which reduces to the standard impulse control problem (Alvarez and Lempa, 2008). For convenience, we let $c$ denote the upfront cost of investment by the single player. The following lemma directly follows from Alvarez and Lempa (2008):

**Lemma 3** Under Assumptions 1, 2, and 4, suppose that there exist $\theta^*$ and $z$ such that $\theta^* < z$, $I(\theta^*) = I(z)$, and $J(z) - J(\theta^*) = c$. In each period, the optimal policy $\nu$ of the single player is to invest at $\tau_\theta = \inf\{t \geq 0:$
\(X_t^\gamma \leq \theta^*\) to boost \(X^\gamma\) up to \(z\). Furthermore, the payoff function is given by

\[
V_s(x) = \begin{cases} 
-I(\theta^*)\phi(x) + (R, \pi)(x) & x > \theta^* \\
V_s(z) - k(z - x) - c & x \leq \theta^* 
\end{cases}
\]  

(37)

Next, we establish a pure strategy equilibrium which is more efficient than a mixed strategy equilibrium. We again assume that \(c_1 \geq c_2\) and let \(z_2\) and \(\theta^*\) denote the solution to \(I(\theta^*) = I(z_2)\) and \(J(z_2) - J(\theta^*) = c_2\). Recall that \(\hat{x}\) is the global minimizer of \(I(\cdot)\).

**Proposition 7** Suppose that

\[
\beta := \frac{(R, \pi)(z_2) - (R, \pi)(\theta^*)}{\phi(\theta^*) - \phi(z_2)} > -I(\hat{x})
\]

(38)

holds. Then there exists a pure strategy equilibrium \(\nu^p\) in which only player 2 invests. Furthermore, \(\nu^p\) is more efficient than the mixed strategy SPE \(\nu^*\) obtained in Theorem 3.

The condition (38) ensures that the variational inequality holds for the pure strategy equilibrium in which player 1 never invests and player 2 is the only player who invests. In the numerical example illustrated in Section 4.3, we confirm that (38) holds: we obtain \(\beta = 7.60 \times 10^{-4} > -I(\hat{x}) = 2.95 \times 10^{-4}\). It is not easy to verify whether (38) generally holds, but it is beyond the scope of this paper to conduct an exhaustive search for pure strategy equilibria.

Proposition 7 establishes that the pure strategy equilibrium is relatively more efficient. Intuitively, the investments are never delayed in a pure strategy equilibrium, so it is natural that this equilibrium is more efficient than the mixed strategy equilibrium. Thus, it behooves the policymaker or the social planner to attempt to thwart a mixed strategy equilibrium in favor of a pure strategy equilibrium.

To further elaborate on this point, we compare the payoffs associated with the mixed strategy equilibrium, the pure strategy equilibrium, and the social planner’s solution. For this purpose, we need to formulate and solve the social planner’s problem whose objective is to maximize the sum of the two players’ payoff functions. We first observe that the social planner should always have player 2 invest because \(c_2 < c_1\). Thus, the upfront cost of investment is effectively \(c_2\). Secondly, the total profit flow per unit time is \(2\pi(\cdot)\) because it is the sum of the two player’s profit flows. Hence, given a social planner’s optimal policy \(\nu = (\tau_n, \zeta_n)_{n \in \mathbb{N}}\) of impulse control on \(X\), the total payoff function is defined as follows:

\[
V^S(x) := \sup_{\nu} \mathbb{E}^X \left[ \int_0^\infty 2\pi(X_t^\nu)e^{-rt}dt - \sum_{n=1}^\infty e^{-r\tau_n}(k\zeta_n + c_2) \right].
\]

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Figure 2: The total payoffs associated with the mixed strategy equilibrium, the pure strategy equilibrium, and the social planner’s solution.

Now suppose that the conditions for Lemma 3 hold with the modified profit flow function $2\pi(\cdot)$ in place of $\pi(\cdot)$ and $c_2$ in place of $c$. We let $z^S$ and $\theta^S$ denote the solutions to $I(\theta^S) = I(z^S)$, and $J(z^S) - J(\theta^S) = c$ where $I(\cdot)$ and $J(\cdot)$ are appropriately modified to accommodate the profit flow $2\pi(\cdot)$. Then the social planner’s optimal policy is to boost $X$ up to $z^S$ whenever $X$ falls below $\theta^S$.

We also define $V^P(x) := V_{1}^{P}(x) + V_{2}^{P}(x)$ and $V^{M}(x) := V_{1}(x) + V_{2}(x)$ respectively as the total payoffs (to the social planner) associated with the pure strategy equilibrium and the mixed strategy equilibrium. For simplicity, we define $V^{M}$ as the total payoff of the second stage, but the qualitative features of the first-stage payoff $U_{1}(x) + V_{2}(x)$ are similar.

To compare the three kinds of total payoffs, we revisit the numerical example given in Section 4.3 with $c_1 = 0.0165$ and illustrate the payoffs in Figure 2. Not surprisingly, the social planner’s optimal total payoff is higher than the payoffs associated with either equilibrium. Most importantly, $V^{M}$ is strictly less than $V^{P}$, which confirms Proposition 7.

5 Conclusions

The recent theoretical result establishes that asymmetry destabilizes mixed strategy MPE in stochastic wars of attrition. This result is vexing because the extant literature views the mixed strategy equilibrium as the hallmark of the war of attrition. From the practical point of view, the mixed strategy equilibrium is regarded as the realistic outcome of a concession game. Thus, the non-existence result begs the question of whether the war of attrition model is an adequate framework for games with a free rider problem. In this paper, we establish that repeated opportunities of concession stabilizes the mixed strategy equilibrium despite asymme-
try. Therefore, we offer one possible resolution of the apparent discrepancy between the non-existence result from the stochastic war of attrition and the conventional view.

The result of this paper has practical implications to stakeholders of common goods. First, the free rider effect from the mixed strategy equilibrium is much more robust to asymmetry than suggested by the recent theoretical results on the stochastic war of attrition. Second, in light of our result that the mixed strategy equilibrium disappears under a sufficiently high degree of asymmetry, it may behoove policymakers to promote a sufficiently high degree of asymmetry to mitigate the inefficiency arising from the mixed strategy equilibrium. For instance, in the context of CSR, the government may induce sufficiently high asymmetry between the firms through asymmetric incentives such as selective subsidies.\(^4\)

This paper is potentially useful to future research on free rider problems. First, it will be fruitful to investigate whether the mixed strategy equilibrium can be stabilized by any other realistic features of a stochastic concession game. Second, the verification theorem constructed in this paper may be extended to other stochastic control games with free rider problems which might possess mixed strategy equilibria.

Notes

\(^1\) A singularity in \(\lambda_i(X_t)\) implies a singularly high rate of investment at time \(t\). For instance, a local time process does not possess a finite Radon-Nikodym derivative, but it can be approximated as one with extremely high values of finite Radon-Nikodym derivatives within a very small region; EC Appendix A illustrates this point. However, such an excessively large values of \(\lambda_i(\cdot)\) is not possible in a mixed strategy MPE because the equilibrium rate \(\lambda_i(\cdot)\) must possess a specific functional form given by (16). Thus, we exclude the possibility of non-differentiable \(A_i^{(l)}\).

\(^2\) In all of the known mixed strategy equilibria, the mixed strategy regions have simple topologies (unions of disjoint intervals), and \(A_i^{(l)}\)s possess a Radon-Nikodym derivative \(\lambda_i^{(l)}\).

\(^3\) If \(c_1 > 0.01729\), condition (B.23) of EC Appendix fails to hold, so the mixed strategy TSSPE does not exist. Note that condition (B.23) of EC Appendix assures that there is a unique optimal value of the end point \(z_1\) of the boost; if this condition is violated, then player 1 may improve his payoff by choosing a different point \(z_1'\) closer to \(\theta^*\) as the end point, which thus destabilizes the equilibrium.

\(^4\) For the body of work that examines the efficacy of selective subsidy to facilitate investments in public goods, see Ayres and Cramton (1996), Rothkopf et al. (2003), and Kotowski (2018).

References

Aïd, R., M. Basei, G. Callegaro, L. Campi, T. Vargiolu. 2020. Nonzero-sum stochastic differential games with impulse controls: A verification theorem with applications. \textit{Mathematics of Operations Research} \textbf{45}(1) 205–232.

Alvarez, L. H. R. 2001. Reward functionals, salvage values and optimal stopping. \textit{Mathematical Methods of Operations Research} \textbf{54}(2) 315–337.
Alvarez, L. H. R., J. Lempa. 2008. On the optimal stochastic impulse control of linear diffusions. *SIAM Journal on Control and Optimization* **47**(2) 703–732.

Ayres, I., P. Cramton. 1996. Deficit reduction through diversity: How affirmative action at the fcc increased auction competition. *Stanford Law Review* **48** 761–815.

Bagnoli, M., S. G. Watts. 2003. Selling to socially responsible consumers: Competition and the private provision of public goods. *Journal of Economics & Management Strategy* **12**(3) 419–445.

Basei, M., H. Cao, X. Guo. 2019. Nonzero-sum stochastic games and mean-field games with impulse controls.

Bensoussan, A., B. Chevalier-Roignant. 2019. Sequential capacity expansion options. *Operations Research* **67**(1) 33–57.

Blumenthal, R. M., R. K. Getoor. 2007. *Markov processes and potential theory*. Dover Publications, Mineola, N.Y.

Borodin, A., P. Salminen. 1996. *Handbook of Brownian motion - Facts and Formulae*. Birkhauser, Basel.

Cadenillas, A., P. Lakner, M. Pinedo. 2010. Optimal control of a mean-reverting inventory. *Operations Research* **58**(6) 1697–1710.

Campi, L., D. De Santis. 2020. Nonzero-sum stochastic differential games between an impulse controller and a stopper. *Journal of Optimization Theory and Applications* **186** 688–724.

Constantinides, G. M., S. F. Richard. 1978. Existence of optimal simple policies for discounted-cost inventory and cash management in continuous time. *Operations Research* **26**(4) 620–636.

Cosso, A. 2013. Stochastic differential games involving impulse controls and double-obstacle quasi-variational inequalities. *SIAM Journal on Control and Optimization* **51**(3) 2102–2131.

Dutta, P. K., S. Lach, A. Rustichini. 1995. Better late than early: Vertical differentiation in the adoption of a new technology. *Journal of Economics & Management Strategy* **4**(4) 563–589.

Dutta, P. K., A. Rustichini. 1995. (s, S) equilibria in stochastic games. *Journal of Economic Theory* **67**(1) 1–39.

El Asri, B., S. Mazid. 2018. Zero-sum stochastic differential game in finite horizon involving impulse controls. *Applied Mathematics & Optimization* **81**(3) 1055–1087.

Ferrari, G., T. Koch. 2019. On a strategic model of pollution control. *Annals of Operations Research* **275** 297–319.

Georgiadis, G., Y. Kim, H. D. Kwon. 2022. The absence of attrition in a war of attrition under complete information. *Games and Economic Behavior* **131** 171–185.
Grenadier, S. R. 1996. The strategic exercise of options: Development cascades and overbuilding in real estate markets. *The Journal of Finance* 51(5) 1653–1679.

Gunther, M. 2015. Amazon, best buy and the free rider problem. https://www.theguardian.com/sustainable-business/2015/aug/05/amazon-best-buy-electronic-waste-walmart-recyling. Accessed 2020-06-29.

Guo, X., R. Xu. 2019. Stochastic games for fuel follower problem: n versus mean field game. *SIAM J. Control Optim.* 57(1) 659–692.

Hendricks, K., A. Weiss, C. Wilson. 1988. The war of attrition in continuous time with complete information. *International Economic Review* 29(4) 663–680.

Hoppe, H. C., U. Lehmann-Grube. 2001. Second-mover advantages in dynamic quality competition. *Journal of Economics & Management Strategy* 10(3) 419–433.

Jack, A., T. C. Johnson, M. Zervos. 2008. A singular control model with application to the goodwill problem. *Stochastic Processes and their Applications* 118(11) 2098–2124.

Kinnucan, H. W., Ā. Myrland. 2003. Free-rider effects of generic advertising: The case of salmon. *Agribusiness* 19(3) 315–324.

Kotchen, M. J. 2006. Green markets and private provision of public goods. *Journal of Political Economy* 114(4) 816–834.

Kotowski, M. H. 2018. On asymmetric reserve prices. *Theoretical Economics* 13(1) 205–238.

Kwon, H. D. 2020. Game of variable contributions to the common good under uncertainty. *Operations Research (Forthcoming)*.

Kwon, H. D., H. Zhang. 2015. Game of singular stochastic control and strategic exit. *Mathematics of Operations Research* 40(4) 869–887.

Lee, J.-Y., G. F. Fairchild. 1988. Commodity advertising, imports and the free rider problem. *Journal of Food Distribution Research* 19(Number 2) 36–42.

Lon, P. C., M. Zervos. 2011. A model for optimally advertising and launching a product. *Mathematics of Operations Research* 36(2) 363–376.

Maskin, E., J. Tirole. 2001. Markov perfect equilibrium: I. Observable actions. *Journal of Economic Theory* 100(2) 191–219.

Maynard Smith, J. 1974. Theory of games and the evolution of animal conflicts. *Journal of Theoretical Biology* 47 209–221.
Mitchell, D., H. Feng, K. Muthuraman. 2014. Impulse control of interest rates. *Operations Research* **62**(3) 602–615.

Morgan, J., J. Tumlinson. 2019. Corporate provision of public goods. *Management Science* **65**(10) 4489–4504.

Murto, P. 2004. Exit in duopoly under uncertainty. *The RAND Journal of Economics* **35** 111–127.

Nerlove, M., K. J. Arrow. 1962. Optimal advertising policy under dynamic conditions. *Economica* **29**(114) 129–142.

Øksendal, B. 2003. *Stochastic Differential Equations: An Introduction with Applications*. 6th ed. Springer, Berlin.

Ormeci, M., J. G. Dai, J. V. Vate. 2008. Impulse control of brownian motion: The constrained average cost case. *Operations Research* **56**(3) 618–629.

Reddy, P. V., S. Wrzaczek, G. Zaccour. 2016. Quality effects in different advertising models - an impulse control approach. *European Journal of Operational Research* **255**(3) 984–995.

Revuz, D., M. Yor. 1999. *Continuous martingales and Brownian motion*, vol. 293. 3rd ed. Springer-Verlag, Berlin.

Rogers, L. C. G., D. Williams. 2000. *Diffusions, Markov Processes and Martingales*, vol. 1. 2nd ed. Cambridge University Press, Cambridge, UK.

Rothkopf, M. H., R. M. Harstad, Y. Fu. 2003. Is subsidizing inefficient bidders actually costly? *Management Science* **49** 71–84.

Serafeim, G. 2017. Can index funds be a force for sustainable capitalism? *Harvard Business Review*.

Sethi, S. P. 1977. Dynamic optimal control models in advertising: a survey. *SIAM Review* **19**(4) 685–725.

Steg, J.-H. 2015. Symmetric equilibria in stochastic timing games. *arXiv preprint arXiv:1507.04797*.

Stettner, L. 1982. Zero-sum markov games with stopping and impulsive strategies. *Applied Mathematics & Optimization* **9**(1) 1–24.

Takahashi, Y. 2015. Estimating a war of attrition: The case of the us movie theater industry. *American Economic Review* **105**(7) 2204–2241.

Touzi, N., N. Vieille. 2002. Continuous-time dynkin games with mixed strategies. *SIAM Journal on Control and Optimization* **41**(4) 1073–1088.

Wang, Z. 2009. (Mixed) strategy in oligopoly pricing: Evidence from gasoline price cycles before and under a timing regulation. *Journal of Political Economy* **117**(6) 987–1030.

Zabaljauregui, D. 2020. A fixed-point policy-iteration-type algorithm for symmetric nonzero-sum stochastic impulse control games. *Applied Mathematics and Optimization*.
Electronic Companion

A Case of Singular Rates of Investment

In this Appendix, we consider the possibility that $A_t^{(i)}$ is not differentiable with respect to time because its Radon-Nikodym derivative $\lambda_i(\cdot)$ has singularities, and we argue that this case can be approximated by finite but arbitrarily high rates of investment. One such example is a local time process denoted by $L_t(x)$ at some point $x \in \mathcal{F}$ because it is well-known that $L_t(x)$ is continuous but does not possess a time-derivative at every point in time.

By (6.2) on p. 202 of Karatzas and Shreve (1998),

$$L_t(x) = \lim_{\lambda \to \infty} \frac{\lambda}{4} \mathbb{L} \left( \{ 0 \leq s \leq t : |W_s - x| \leq \lambda^{-1} \} \right),$$

where $\mathbb{L}$ is the Lebesgue measure over time. The right hand side is the total duration of time that a Brownian motion $W_s$ stays within the interval $[x - \lambda^{-1}, x + \lambda^{-1}]$ as $\lambda \to \infty$.

Assuming that $X = W$, if this process were to be added to $A_t^{(i)}$, it implies that player $i$ invests at a hazard rate of $\lambda$ within the interval $[x - \lambda^{-1}, x + \lambda^{-1}]$ for a very large value of $\lambda$. Therefore, $L_t(\cdot)$ is a limiting case of arbitrarily large values of the rate of investment within an arbitrarily small interval.

We remark that this argument generally applies to any additive functionals of regular diffusion processes. Specifically, an arbitrary additive functional $A_t$ of a regular diffusion process has the following representation:

$$A_t = \int_0^t \lambda(X_s)ds,$$

where $\lambda(\cdot)$ is the killing measure density that characterizes $A_t$ (II.23 and II.24, Borodin and Salminen 1996). The killing measure density $\lambda(\cdot)$ may or may not have singularities, but if it is finite everywhere, then $A_t$ is time-differentiable. Even if $\lambda(\cdot)$ has a singularity (such as in a Dirac measure) at some point $x_0$, one can approximate it as a finite function taking very high values near $x_0$.

B Mathematical Proofs

Proof of Lemma 1: By the definition of SPE, the equilibrium value of $\zeta_t^{(i)}$ must be the one that maximizes the payoff to player $i$ at the time of investment $t$. We thus examine player $i$'s payoff at its time of investment. Towards this goal, we first formulate the time of investment in case $\tau^{(i)} = \tau^{(j)}$ by utilizing the following
the payoff (B.1) is uniquely maximized by $z$, thus follows that where $\tau$ invests at the stopping time $\tilde{\tau}(i)$, in which case player $i$ becomes an investor if and only if $Z_i = 1$, which occurs with probability 50%. Then we define an augmented probability space $(\Omega \times \{0, 1\}, \mathcal{F}, \tilde{\mathbb{P}}, \mathbb{P})$ where $\tilde{\mathbb{P}}$ is the filtration generated by $X$ and $Z_i$. Furthermore, the new probability measure is given by $\tilde{\mathbb{P}} = \mathbb{P} \otimes Z_i$, where $Z_i$ is the probability measure of $Z_i$. Under the new probability space, we extend the stopping time $\tau(i)$, which is adapted to $\mathbb{F}$, to a stopping time $\tilde{\tau}(i)$ adapted to $\tilde{\mathbb{P}}$ as follows:

$$
\tilde{\tau}(i) = \tau(i) \text{ if } \tau(i) \neq \tau(j)
= \tau(i) \text{ if } \tau(i) = \tau(j) \text{ and } Z_i = 1
= \infty \text{ if } \tau(i) = \tau(j) \text{ and } Z_i = 0.
$$

By this formulation, player $i$ invests at the stopping time $\tilde{\tau}(i)$ only if $\tilde{\tau}(i) < \tau(j)$; the event $\tilde{\tau}(i) = \tau(j)$ never takes place by construction.

Next, we examine player $i$’s payoff at its investment time $\tilde{\tau}(i)$ conditional on that player $i$ is an investor at time $\tilde{\tau}(i) < \infty$:

$$
\mathbb{E}^i \left[ \int_{\tilde{\tau}(i)}^{\infty} \pi(X_s^i) e^{-rs} ds - (k \zeta_{\tilde{\tau}(i)} + c_i) e^{-r \tilde{\tau}(i)} | \mathcal{F}_{\tilde{\tau}(i)} \right] (\tilde{\tau}(i) < \tau(j))
= e^{-r \tilde{\tau}(i)} \{ \mathbb{E}^i \left( \int_{\tilde{\tau}(i)}^{\infty} \pi(X_s^i) e^{-rs} ds - (k \zeta_{\tilde{\tau}(i)} + c_i) \right) \} = e^{-r \tilde{\tau}(i)} \{ (R, \pi)(z) - k(z - X_{\tilde{\tau}(i)}) - c_i \},
$$

(B.1)

where $z \equiv \zeta_{\tilde{\tau}(i)} + X_{\tilde{\tau}(i)}$. Here we use the fact that $X^i_s$ satisfies the SDE (1) for all $s \geq \tilde{\tau}(i)$ with the condition $X^i_{\tilde{\tau}(i)} = X_{\tilde{\tau}(i)} + \zeta_{\tilde{\tau}(i)}$ in the single investment game. Then by Assumption 2, because $\zeta_{\tilde{\tau}(i)} \geq 0$ must be satisfied, the payoff (B.1) is uniquely maximized by $z = z^*$ if $X_{\tilde{\tau}(i)} < z^*$, and it is maximized by $z = X_{\tilde{\tau}(i)}$ if $X_{\tilde{\tau}(i)} \geq z^*$. It thus follows that $\zeta_{\tilde{\tau}(i)} = z^* - X_t$ if $X_t < z^*$ and $\zeta_{\tilde{\tau}(i)} = 0$ if $X_t \geq z^*$ for any time $t$. ■

**Proof of Proposition 1:** We inspect the payoff function given by (4) and replace $\zeta^i_{\tau(i)}$ and $\zeta^j_{\tau(j)}$ with the SPE investment strategies given by (5). In the right-hand side of (4), we can decompose $\int_{\tau(i)}^{\infty} \pi(X^i_s) e^{-rs} ds$ as $\int_{\tau(i)}^{\tau(i) \wedge \tau(j)} \pi(X_s) e^{-rs} ds + \int_{\tau(i) \wedge \tau(j)}^{\infty} \pi(X^i_s) e^{-rs} ds$ where we use the fact that the controlled process $X^i_s$ coincides with the uncontrolled process $X$ for $s < \tau^i = \tau(i) \wedge \tau(j)$. Furthermore, $X^i_s$ satisfies the SDE (1) for $s \geq \tau^i$ with the condition that $X^i_{\tau^i} = \max\{z^*, X_{\tau^i}\}$. Hence, we can utilize the strong Markov property of the uncontrolled state variable $X$ and replace $\mathbb{E}^i \left[ \int_{\tau^i}^{\infty} \pi(X^i_s) e^{-rs} ds | \mathcal{F}_{\tau^i} \right]$ with $e^{-r \tau^i} (R, \pi)(X^i_{\tau^i})$. Then we can use the definitions of $g_i(\cdot)$ and $m_i(\cdot)$ given in (6) and (7) to rewrite (4) as (8). ■
**Proof of Proposition 2.** (i) By Proposition 1, it suffices to consider an optimal stopping problem of the form (9). We first note that it is never optimal to invest when \( X_t \geq z^* \). If a player invests when \( X_t \geq z^* \), the optimal boost is zero, and hence, the player would only waste the upfront cost \( c_i \). Thus, we limit our attention to the policy of investment when \( X_t < z^* \).

Note that we can re-write the payoff function (9) as follows:

\[
\mathbb{E}^x \{(R, \pi)(x) + e^{-rT}[g_i(X_T) - (R, \pi)(X_T)]\}.
\]

Thus, we have transformed the problem into an optimal stopping problem with a bequest function \( g_i(x) - (R, \pi)(x) \). By definition of \( g_i(\cdot) \) in (6), \( g_i(x) - (R, \pi)(x) \) decreases in \( x \) for \( x < z^* \) and stays constant for \( x \geq z^* \). Under Assumption 3, we can invoke Theorem 3(B) of Alvarez (2001) to conclude that \( \tau^* = \inf\{t \geq 0 : X_t \leq \theta_t\} \).

(ii) This statement directly follows from Theorem 3(B) of Alvarez (2001).

(iii) Lastly, we prove that \( \theta_t \) decreases in \( c_i \). Define \( f(x, c_i) = [g_i(x) - (R, \pi)(x)]/\phi(x) \). For \( x < z^* \), \( f(x, c_i) \) has well-defined derivatives with respect to \( x \). We find that

\[
\frac{\partial^2 f(x, c_i)}{\partial x \partial c_i} = \frac{\phi'(x)}{\phi^2(x)} < 0.
\]

It implies that \( \partial_x f(x, c_i) \) decreases in \( c_i \). We also know that \( \partial_{\theta_t} f(\theta_t, c_i) = 0 \) by the definition of \( \theta_t \) and that \( \partial_{\theta_t}^2 f(\theta_t, c_i) < 0 \) since \( \theta_t \) is the maximizer of \( f(\theta_t, c_i) \). Therefore, we have

\[
\frac{d\theta_t}{dc_i} = \frac{\partial^2 f(\theta_t, c_i)}{\partial \theta_t^2} \frac{d\theta_t}{dc_i} + \frac{\partial^2 f(\theta_t, c_i)}{\partial \theta_t \partial c_i} = 0,
\]

which implies that \( d\theta_t/dc_i < 0 \).

**Proof of Proposition 3:** Recall that the survival probability is given by \( M_t^{(j)} = \exp(-A_t^{(j)})N_t^{(j)} \) where \( N_t^{(j)} = 1_{\{t < \tau_t^{(j)}\}} \). For analytical simplicity, we introduce an associated stopping time \( \hat{\tau} := \inf\{t \geq 0 : \exp(-A_t^{(j)}) \leq \hat{f}^{(j)}\} \) and re-formulate \( \hat{\tau}^{(j)} \) as \( \hat{\tau}^{(j)} = \hat{\tau} \wedge \tau_t^{(j)} \); it is straightforward to verify that this expression is equivalent to \( \inf\{t \geq 0 : M_t^{(j)} \leq \hat{f}^{(j)}\} \).

To derive (15), we apply Tonelli’s theorem (2.37, Folland 1999) to the right-hand side of (14) and take the integral over the measure \( \mathbb{L}^{(j)} \), leaving behind the dependence on the sample path of \( X \). Tonelli’s theorem is applicable because all the terms in (14) are bounded from below on account of the fact that \( \pi(x) \geq \pi_L, m_i(x) \geq (R, \pi)(z^*) \), and \( g_i(x) \geq g_i(a) > -\infty \). Thus, we fix an arbitrary sample path \( X(\omega) \) for some \( \omega \in \Omega \) and proceed to integrate out \( \hat{f}^{(j)} \). Since \( \hat{f}^{(j)} \) has a uniform distribution over \([0, 1]\), the cumulative distribution function for
the stopping time $\hat{v}$ is of the form $F(v) = 1 - \exp(-A_{i,j}^{(j)})$ and $F'(v) = \lambda_j(X_t)\exp(-A_{i,j}^{(j)})$. Note that $F(0) = 0$ and $F(\infty) = 1$ because $\lim_{t \to \infty} \exp(-A_{i,j}^{(j)}) = 0$. (For the sample paths in which $\lim_{t \to \infty} \exp(-A_{i,j}^{(j)}) > 0$, see the remark at the end of this proof). Using the notation $\hat{A}(\cdot) := \hat{t}(\cdot) \land \tau_{\hat{E}}^{(j)}$, we obtain

$$E_{L,(j)}[\int_0^{\hat{t}(\cdot)} \pi(X_t)e^{-rt}dt] = E_{L,(j)}[\int_0^{\hat{v}} \pi(X_t)e^{-rt}dt] = \int_0^{\hat{v}} (\int_0^{\hat{v}} \pi(X_t)e^{-rt}dt)F'(v)dv$$

$$= \int_0^{\hat{v}} (1 - F(v))\pi(X_t)e^{-rv}dv = \int_0^{\hat{t}(\cdot) \land \tau_{\hat{E}}^{(j)}} \pi(X_t)e^{-rt-A_{i,j}^{(j)}}dt,$$

where the third equality is obtained via partial integration. Next, we obtain

$$E_{L,(j)}[m_i(X_{\hat{t}(\cdot)})e^{-r\hat{t}(\cdot)}1_{\{\hat{t}(\cdot) < \hat{t}(\cdot)\}} = \int_0^{\hat{t}(\cdot)} m_i(X_t)e^{-rv}F'(v)dv + E_{L,(j)}(\hat{v} > \tau_{\hat{E}}^{(j)})m_i(X_{\tau_{\hat{E}}^{(j)}})e^{-r\tau_{\hat{E}}^{(j)}}1_{\{\tau_{\hat{E}}^{(j)} < \hat{t}(\cdot)\}}$$

$$= \int_0^{\hat{t}(\cdot) \land \tau_{\hat{E}}^{(j)}} \lambda_j(X_t)m_i(X_t)e^{-rv-rt-A_{i,j}^{(j)}}dt + 1_{\{\hat{v} > \tau_{\hat{E}}^{(j)}\}}m_i(X_{\tau_{\hat{E}}^{(j)}})e^{-r\tau_{\hat{E}}^{(j)}-A_{i,j}^{(j)}}.$$

Next, we note that $1_{\{\hat{v} < \hat{t}(\cdot)\}} = 1_{\{\hat{t}(\cdot) < v\}}1_{\{\tau_{\hat{E}}^{(j)} < \hat{t}(\cdot)\}}$, so we obtain

$$E_{L,(j)}[g_i(X_{\hat{t}(\cdot)})e^{-r\hat{t}(\cdot)}1_{\{\hat{t}(\cdot) < \hat{t}(\cdot)\}} = E_{L,(j)}[1_{\{\hat{t}(\cdot) < v\}}g_i(X_{\hat{t}(\cdot)})e^{-r\hat{t}(\cdot)}1_{\{\tau_{\hat{E}}^{(j)} < \hat{t}(\cdot)\}}$$

$$= 1_{\{\hat{v} < \tau_{\hat{E}}^{(j)}\}}g_i(X_{\hat{t}(\cdot)})e^{-r\hat{t}(\cdot)-A_{i,j}^{(j)}}.$$

Lastly,

$$E_{L,(j)}[1_{\{\hat{t}(\cdot) = \hat{t}(\cdot)\}} = E_{L,(j)}[1_{\{\hat{t}(\cdot) = \tau_{\hat{E}}^{(j)}\}}1_{\{\hat{t}(\cdot) = \tau_{\hat{E}}^{(j)}\}} + 1_{\{\hat{t}(\cdot) = v\}}1_{\{\hat{t}(\cdot) = \tau_{\hat{E}}^{(j)}\}}$$

$$= 1_{\{\hat{t}(\cdot) = \tau_{\hat{E}}^{(j)}\}}e^{-A_{i,j}^{(j)}} + \int_0^{\tau_{\hat{E}}^{(j)}} 1_{\{\hat{t}(\cdot) = v\}}F'(v)dv = 1_{\{\hat{t}(\cdot) = \tau_{\hat{E}}^{(j)}\}}e^{-A_{i,j}^{(j)}}.$$
conditions originate from (2) because both $|(R, \pi)(x)|$ and $|m_i(x)| = |(R, \pi)(z^+ \lor x)|$ are bounded by $A + B|x|$ for some $A > 0$ and $B > 0$ due to Assumption 2. Thus, for the purpose of evaluating the payoff function, we can safely assume that $\lim_{t \to \infty} \exp(-A(j)^{(j)}) = 0$ for each sample path of $X$.

Proof of Proposition 4: Since $V^M$ is already a Markov strategy profile, it suffices to prove that $V^M_i$ and $V^M_j$ are best responses to each other. We achieve this task in two steps: We first show that $V^*_i(x)$ dominates any $V^i_{(V_i, V^M_j)}(x)$ for any $V_i$, and then we show that $V^M_i(x) = V^*_i(x)$.

(i) To prove that $V^*_i(x)$ dominates $V^i_{(V_i, V^M_j)}(x)$ for any $V_i$, it suffices to prove that $V^*_i(\cdot)$ satisfies the conditions of the verification theorem of optimal stopping problem for player $i$ given player $j$’s strategy $V^M_j$.

By Proposition 3, player $i$’s payoff can be expressed as follows:

$$V^i_{(V_i, V^M_j)}(x) = \mathbb{E}^i \{ \int_0^{\tau^j} [\pi(X_t) + \lambda_j(X_t)(R, \pi)(z^*)]e^{-rt - A(j)^{(j)}} dt + g(X_{\tau^j})e^{-r\tau^j - A(j)^{(j)}} \}.$$  

(B.2)

Recall that we set $E_1 = E_2 = 0$. Thus, as shown in equation III.18.13 of Rogers and Williams (2000), we can define a modified characteristic differential operator as

$$\tilde{\mathcal{A}}_j := \frac{1}{2} \sigma^2(x) \frac{d^2}{dx^2} + \mu(x) \frac{d}{dx} - (r + \lambda_j(x)) = \mathcal{A} - \lambda_j(x).$$

Notice that player $i$’s payoff (B.2) is equivalent to that of an optimal stopping problem with a modified operator $\tilde{\mathcal{A}}_j$ and a modified flow profit of $\pi(x) + \lambda_j(x)(R, \pi)(z^*)$.

We now verify that $V^*_i(x)$ satisfies the conditions of the verification theorem (Theorem 10.4.1 of Øksendal, 2003) and therefore, $V^*_i(x) \geq V^i_{(V_i, V^M_j)}(x)$ for any arbitrary $V_i$. From the expression (11), $V^*_i(\cdot)$ is twice continuously differentiable except at $x = \theta$ ($\neq \theta_i$) and continuously differentiable everywhere (Alvarez and Lempa, 2008). Since $V^*_i(\cdot)$ is the optimal value function of a stopping problem, $V^*_i(x) \geq g_i(x)$ for all $x \in \mathcal{F}$. Finally, note that $V^*_i(x)$ satisfies the following differential equation:

$$\tilde{\mathcal{A}}_j V^*_i(x) + \pi(x) + \lambda_j(x)(R, \pi)(z^*) = \begin{cases} \mathcal{A} V^*_i(x) + \pi(x) & \text{for } x > \theta, \\
\mathcal{A} V^*_i(x) + \pi(x) + \lambda_j(x)(R, \pi)(z^*) - V^*_i(x) & \text{for } x \leq \theta.\end{cases}$$

The first line of the right hand side results because $\lambda_j(x) = 0$ for $x > \theta$. By the expression (11), note that $\mathcal{A} V^*_i(x) + \pi(x) = 0$ for $x > \theta$. The second line also vanishes by the functional form of $\lambda_j(\cdot)$ and because $V^*_i(x) = g(x)$ for $x \leq \theta$. It follows that $\tilde{\mathcal{A}}_j V^*_i(x) + \pi(x) + \lambda_j(x)(R, \pi)(z^*) = 0$ for all $x$ except at $\theta$. Therefore, the conditions (i)–(vi) of Theorem 10.4.1 from Øksendal (2003) are all satisfied.
(ii) It remains to prove that $V_i^{x_i}(x) = V_i^x(x)$. Given that player $j$ employs $v_j^M$, suppose that player $i$ adopts a strategy $v_i$ of investing at some arbitrary stopping time $\hat{\tau}^{(i)}$ such that $X_{\hat{\tau}^{(i)}} \leq \theta$. (Note that $\hat{\tau}^{(i)}$ is not necessarily a hitting time). Our goal is to prove that $V_i^x(x)$ has exactly the same expression for $V_i^{(v_i,v_j^M)}(x)$ given by (B.2) for any such stopping time $\hat{\tau}^{(i)}$.

We note that $x\hat{\tau}^{(i)} V_i^x(x) + \pi(x) + \lambda_j(x)(R,\pi)(z^*) = 0$ for all $x$ except at $\theta$ and that $V_i^x \in C^2(\mathcal{F}\setminus\{\theta\}) \cap C(\mathcal{F})$. Closely following the proof of Theorem 10.4.1 of Øksendal (2003), we introduce a sequence of stopping times $\{u_n\}$ where $u_n = \min\{n, \inf \tau > 0 \{X_{\tau} \not\in G_n\}\}$ where $\{G_n\}$ is an increasing sequence of compact subsets of $\mathcal{F}$ such that $\lim_{n \to \infty} G_n = \mathcal{F}$. Theorem 10.4.1 of Øksendal (2003) establishes that

$$V_i^x(x) = \mathbb{E}^x\{ \int_0^{u_n} [\pi(X_t) + \lambda_j(X_t)(R,\pi)(z^*)]e^{-rt-A(t)}dt + V_i^x(X_{u_n})e^{-ru_n-A(u_n)} \}$$

for any $n$. From the functional form of $V_i^x(\cdot)$ given in (11) and by Assumption 2, we note that $|V_i^x(x)| < A|x| + B$ for some $A > 0$ and $B > 0$. Hence, by the condition (2) that $\lim_{\tau \to \infty} \mathbb{E}^x[X_{\tau} e^{-\tau r}] = 0$, we have $\lim_{n \to \infty} \mathbb{E}^x[e^{-ru_n-A(u_n)} V_i^x(X_{u_n})] = 0$. Furthermore, by Assumption 1, $\pi(\cdot)$ is absolutely integrable, so we can take the limit $n \to \infty$ and use the dominated convergence theorem to obtain

$$V_i^x(x) = \mathbb{E}^x\{ \int_0^{\infty} [\pi(X_t) + \lambda_j(X_t)(R,\pi)(z^*)]e^{-rt-A(t)}dt \}.$$ 

It follows that the following Dynkin’s formula holds

$$V_i^x(x) = \mathbb{E}^x\{ \int_0^{\hat{\tau}} [\pi(X_t) + \lambda_j(X_t)(R,\pi)(z^*)]e^{-rt-A(t)}dt + V_i^x(X_{\hat{\tau}})e^{-r\hat{\tau}} \}.$$ 

for any arbitrary stopping time $\tau > 0$.

We now set $\tau = \hat{\tau}^{(i)}$, in which case $V_i^x(X_{\hat{\tau}}) = g(X_{\hat{\tau}})$ by the functional form of $V_i^x(\cdot)$ because $X_{\hat{\tau}^{(i)}} \leq \theta$. Then we obtain

$$V_i^x(x) = \mathbb{E}^x\{ \int_0^{\hat{\tau}^{(i)}} [\pi(X_t) + \lambda_j(X_t)(R,\pi)(z^*)]e^{-rt-A(t)}dt + g(X_{\hat{\tau}^{(i)}})e^{-r\hat{\tau}^{(i)}-A^{(i)}} \},$$

which coincides with $V_i^{(v_i,v_j^M)}(x)$. Hence, $V_i^{(v_i,v_j^M)}(x) = V_i^x(x)$ as long as $X_{\hat{\tau}^{(i)}} \leq \theta$. Finally, even if $\hat{\tau}^{(i)}$ is a probabilistic mixture of $\mathbb{F}$-stopping times, the same argument holds as long as $X_{\hat{\tau}^{(i)}} \leq \theta$. 

**Proof of Proposition 5:** In the second stage subgame, the strategy profile reduces to the MPE of Proposition 4. Hence, it is enough to show that $v^S$ is an SPE in the first stage.

Suppose that $X_0 = x > \theta$ so that $\tau^* > 0$. Our goal is to prove that $v_1$ and $v_2$ are best responses to each other.
We first inspect player 2’s best response to \( v_1 \). Let \( f_i(x;\tau) \) denote player \( i \)'s first-stage payoff conditional on the current state variable \( x \) associated with the investment time \( \tau \leq \tau_1 \) given that player \( j \) employs strategy \( v_j \). As a matter of convention, if player \( i \)'s strategy is to never invest in the first stage, we say \( \tau > \tau_1 \).

(i) We first examine player 2’s best response to \( v_1^S \). For \( \tau \leq \tau_1 \), player 2’s payoff is given by

\[
V_2^{\text{S}} = \mathbb{E}^x \left[ \int_0^{\tau_1} \pi(x_t) e^{-r t} dt + e^{-r \tau_1} g(X_{\tau_1}) \right]
\]

Note that the reward from investment exactly at time \( \tau_1 \) (at the commencement of stage 2) is still \( g(X_{\tau_1}) \) because \( V_1^{\text{S}}(x) = V_1^*(x) \) by Proposition 4, where \( V_1^*(x) \) is given by (11). Upon comparing (B.3) to (9), we note that \( f_2(x;\tau) \) coincides with the payoff from a single-player investment problem. By Proposition 2, therefore, we conclude that \( f_2(x;\tau) \) is optimized when \( \tau = \tau_1 \), i.e., \( \sup_{\tau \leq \tau_1} f_2(x;\tau) = f_2(x;\tau_1) = V_2^*(x) \).

Next, suppose that player 2 never invests in the first stage of the strategy profile. Note that the mixed strategy MPE commences at time \( \tau_1 \) and that \( X_{\tau_1} \leq \theta \). Hence, the payoff to player 2 at time \( \tau_1 \) is given by \( V_2^*(X_{\tau_1}) \) by Proposition 4. In turn, because \( X_{\tau_1} \leq \theta \), we have \( V_2^*(X_{\tau_1}) = g(X_{\tau_1}) \) by the form of \( V_1^*(X_{\tau_1}) \). Thus, player 2’s payoff is given by

\[
\mathbb{E}^x \left[ \int_0^{\tau_1} \pi(x_t) e^{-r t} dt + e^{-r \tau_1} V_2^*(X_{\tau_1}) \right] = \mathbb{E}^x \left[ \int_0^{\tau_1} \pi(x_t) e^{-r t} dt + e^{-r \tau_1} g(X_{\tau_1}) \right] = f_2(x;\tau_1) = V_2^*(x).
\]

Recall that \( V_2^*(x) \) is player 2’s payoff function if he invests at \( \tau_1 \), which dominates the payoff from investing at any time \( \tau < \tau_1 \) within stage 1. This implies that investment at time \( \tau_1 \) and no investment in stage 1 are both best responses of player 2 to \( v_1^S \). Therefore, any probabilistic mix of investment at \( \tau_1 \) and entering stage 2 MPE without investing at \( \tau_1 \) is a best response for player 2. It follows that \( v_2^S \) is a best response to \( v_1^S \).

(ii) Next, we examine player 1’s best response to \( v_2^S \). We first consider the payoff \( f_1(x;\tau) \) associated with an investment time \( \tau < \tau_1 \). Note that \( f_1(x;\tau) \) coincides with the expression of \( f_2(x;\tau) \) when \( \tau < \tau_1 \). On the other hand, player 1’s payoff associated with \( v_1^S \) where player 1 invests only in stage 2 is given by (19), which is better than \( f_2(x;\tau) \) for any \( \tau < \tau_1 \) because of Proposition 2 for a single-player problem. We conclude that investment at \( \tau < \tau_1 \) is not a best response for player 1 to \( v_2^S \).

Lastly, we show that investment at \( \tau_1 \) is also not a best response for player 1. The payoff from investment
at $\tau_\Gamma$ is given by
\[
E^\pi_x \left\{ \int_0^{\tau_\Gamma} \pi(X_t)e^{-rt} dt + e^{-r\tau_\Gamma}(1 - q_2)g(X_{\tau_\Gamma}) + \frac{1}{2}e^{-r\tau_\Gamma}q_2[(R,\pi)(z^*) + g(X_{\tau_\Gamma})] \right\}.
\]
Because $g(X_{\tau_\Gamma}) = V^{\nu^M}(X_{\tau_\Gamma}) < (R,\pi)(z^*)$ and $\mathbb{P}(e^{-r\tau_\Gamma} > 0) > 0$, we conclude that this payoff is worse than (19). Therefore, player 1’s best response is to never invest in the first stage.

**Proof of Theorem 1**: Because TSSPE reduces to a mixed strategy MPE in the second stage subgame, the goal of this proof is to prove the non-existence of an MPE. For this purpose, we utilize Proposition 3 and the expression (15) for the payoff function. In order to obtain Theorem 1, we first need to prove five lemmas. As a first lemma, we establish that $E_i$ does not overlap with $\Gamma_j$ or with $E_j$.

**Lemma B.1** If $\nu$ is an MPE, then $E_i \cap \Gamma_j = \emptyset$ and $E_i \cap E_j = \emptyset$.

**Proof of Lemma B.1**. (i) First, we prove $E_i \cap E_j = \emptyset$. Suppose, on the contrary, that there exists a point $x \in E_i \cap E_j$. Then the payoff to player $i$ at $x$ is given by $\frac{1}{2}[(R,\pi)(\max\{z^*,x\}) + g_i(x)]$. However, if player $i$ deviates from the strategy and takes up an alternative strategy of not investing at $X_t = x$, then he earns $(R,\pi)(\max\{z^*,x\})$ instead, which is a higher payoff than $\frac{1}{2}[(R,\pi)(\max\{z^*,x\}) + g_i(x)]$ because $(R,\pi)(\max\{z^*,x\}) > g_i(x)$. This contradicts the assumption that $\nu$ is an equilibrium. We conclude that $E_i \cap E_j = \emptyset$.

(ii) Next, we prove $E_i \cap \Gamma_j = \emptyset$. For the sake of contradiction, we suppose that there exists $x \in E_i \cap \Gamma_j$. Under this assumption, one of the best responses of player $j$ is to invest immediately at $x$ because it belongs to the mixed strategy region $\Gamma_j$ for player $j$. Then, given the current value $x$ of the state variable, the reward from immediate investment is
\[
V^\nu_j(x) = \frac{1}{2}[g_j(x) + (R,\pi)(\max\{z^*,x\})].
\]
However, if player $j$ does not invest, player $i$ is the only player who invests, so player $j$’s reward will be $(R,\pi)(\max\{z^*,x\}) > V^\nu_j(x)$ because $(R,\pi)(\max\{z^*,x\}) > g_j(x)$. This contradicts the assumption that immediate investment is one of the best responses of player $j$. □

Lemma B.1 holds because a player has no incentive to invest when his opponent invests with probability one. Next, we establish a lemma regarding the mixed strategy regions and the equilibrium investment rate.
Lemma B.2 If \( \nu \) is an MPE, then \( \Gamma_1 = \Gamma_2 \subset (a, \min\{x'_i, x'_j\}) \), and

\[
\lambda_i^{(i)} = \lambda_i(X_t) := I_{\{X_t \in \Gamma_i\}} \frac{\mathcal{A}_j(X_t) - \pi(X_t)}{(R, \pi)(z^*) - g_j(X_t)}.
\]

(B.4)

**Proof of Lemma B.2.** (i) We first show that \( \Gamma_i \cap (x'_i, b) = \emptyset \). For the purpose of leading to a contradiction, suppose that \( \Gamma_i \cap (x'_i, b) \neq \emptyset \). Then because \( \Gamma_i \) is an open set, for any \( x_0 \in \Gamma_i \cap (x'_i, b) \), there is an open neighborhood \( N \subset \Gamma_i \cap (x'_i, b) \) of \( x_0 \). We let \( \tau_N := \{ t \geq 0 : X_t \not\in N \} \) be the escape time from \( N \).

Now, we claim that \( V_i^N(x_0) = g_i(x_0) \). To see this, note that whenever \( X_t \in \Gamma_i \), which is player \( i \)'s mixed strategy region, an immediate investment is one of the best responses for player \( i \). Recall also that \( \Gamma_i \cap E_j = \emptyset \) by Lemma B.1. Therefore, player \( i \)'s equilibrium payoff at \( x_0 \) must be equal to the reward from immediate investment, which is \( g_i(x_0) \).

Furthermore, player \( i \) should earn the same payoff, \( g_i(x_0) \), by investing at time \( \tau_N \). This is because \( X_{\tau_N} \in \Gamma_i \), which implies that investing at \( \tau_N \) must be one of the best responses for player \( i \) and should yield the same equilibrium payoff to player \( i \) as an immediate investment so that player \( i \) mixes those pure strategies in equilibrium. Therefore, we can obtain

\[
V_i^N(x_0) = g_i(x_0) = E_x \left[ \int_0^{\tau_N} \left[ \pi(X_t) + \lambda_j(X_t)m_i(X_t) \right] e^{-\eta - A_i^{(i)}} dt + g_i(X_{\tau_N}) e^{-r\tau_N - A_i^{(i)}} \right],
\]

(B.5)

where the expected value in equation (B.5) is player \( i \)'s payoff from investing at time \( \tau_N \) because \( \Gamma_i \cap E_j = \emptyset \) by Lemma B.1.

Finally, because \( N \) is an arbitrary neighborhood of \( x_0 \), we can obtain the following HJB equation by letting \( N \to \{ x_0 \} \) along with equation (7.5.1) of Øksendal (2003):

\[
(\mathcal{A} - \lambda_j(x_0))g_i(x_0) + \pi(x_0) + \lambda_j(x_0)m_i(x_0) = 0.
\]

(B.6)

However, because \( \lambda_j(\cdot) \geq 0, m_i(\cdot) > g_i(\cdot), \) and \( \mathcal{A}g_i(x_0) + \pi(x_0) > 0 \) for \( x_0 > x'_i \) by Assumption 3(ii), equation (B.6) can never be satisfied for any \( x_0 \in \Gamma_i \cap (x'_i, b) \), which leads to a contradiction. Therefore, we can conclude that \( \Gamma_i \cap (x'_i, b) = \emptyset \).

(ii) We now prove \( \Gamma_1 = \Gamma_2 \) and derive the functional form of \( \lambda_i(\cdot) \). First, choose any \( x_0 \in \Gamma_i \). Then because \( \Gamma_i \cap (x'_i, b) = \emptyset \) by part (i) above and \( \Gamma_i \) is an open set, it must be the case that \( x_0 < x'_i \). Next, similarly to part (i) above, we can pick an open neighborhood \( N \subset \Gamma_i \cap (a, x'_i) \) of \( x_0 \) and use an identical argument as in part (i) to arrive at the same expression (B.5). Hence, if we take the limit \( N \to \{ x_0 \} \) use equation (7.5.1) of Øksendal
(2003), we again obtain the expression \( (B.6) \), leading to

\[
\lambda_j(x_0) = -\frac{\mathcal{A} g_i(x_0) - \pi(x_0)}{m_i(x_0) - g_i(x_0)},
\]

which is positive because \( \mathcal{A} g_i(x_0) + \pi(x_0) < 0 \) for \( x_0 < x_1^* \) by Assumption 3(ii) and \( m_i(\cdot) > g_i(\cdot) \). Then because \( x_0 \) is an arbitrary point of \( \Gamma_i \), it follows that \( \lambda_j(x) > 0 \) whenever \( x \in \Gamma_i \), which implies that \( \Gamma_i \subseteq \Gamma_j \) because \( \lambda_j(x) > 0 \) if and only if \( x \in \Gamma_j \) by the definition of \( \Gamma_j \). Moreover, we can use a symmetric argument to obtain \( \Gamma_j \subseteq \Gamma_i \). Therefore, we can conclude that \( \Gamma := \Gamma_i = \Gamma_j \). Note that this also implies that \( \Gamma_1 = \Gamma_2 \subseteq (a, \min\{x_1^*, x_2^*\}) \) because \( \Gamma_i \cap (x_1^*, b) = \emptyset \) by part (i) above and \( \Gamma_i \) is an open set.

Finally, by using (a) our derivation of \( \lambda_j(\cdot) \) above, (b) \( \Gamma = \Gamma_i = \Gamma_j \subseteq (a, \min\{x_1^*, x_2^*\}) \), (c) \( m_i(x) = (R, \pi)(\max\{z^+, x\}) \) by definition, and (d) \( x_j^* \leq z^+ \) by Assumption 3(ii), we can obtain

\[
\lambda_i(x) = -\frac{\mathcal{A} g_j(x) - \pi(x)}{(R, \pi)(z^+) - g_j(x)} 1_{\{x \in \Gamma\}},
\]

which completes the derivation of the functional form of \( \lambda_i(\cdot) \).

We now establish that the mixed strategy region \( \Gamma := \Gamma_1 = \Gamma_2 \) and the investment region \( E_i \) must not share boundary points. This statement is different from \( \Gamma \cap E_i = \emptyset \), which is already established by Lemma B.1.

**Lemma B.3** \( \partial E_i \cap \partial \Gamma = \emptyset \) in an MPE.

**Proof of Lemma B.3:** Suppose that \( y \in \partial E_i \cap \partial \Gamma \). Without loss of generality, suppose that \( (c, y) \subseteq \Gamma \) for some \( c < y \). Then \( V_j^y(x) = g_j(x) \) for all \( x \in (c, y) \). Since \( y \) is at the boundary of \( E_i \), we also have \( V_j^y(y) = m_j(y) \) because player \( i \) would immediately invest when \( X_i = y \) due to the diffusive nature of \( X \) and by the definition of \( E_i \) being a closed set.

Suppose that player \( j \) adopts an alternative strategy \( v_j' \) which is identical to \( v_j \) except that \( (d, y) \) is his continuation region for some \( d > c \). We let \( v' = (v_i, v_j') \) denote this new strategy profile. Because \( \lambda_i(\cdot) \) is a continuous function within \( \Gamma \), player \( j \)'s payoff function under \( v' \) within the interval \( (d, y) \) can be obtained as the solution to the boundary value problem (Chapter 9, Øksendal 2003) which satisfies the differential equation

\[
\mathcal{A} V_j^{v'}(x) + \pi(x) + \lambda_i(x)[(m_j(x) - V_j^{v'}(x))] = 0,
\]

and the boundary conditions \( V_j^{v'}(d) = g_j(d) \) and \( V_j^{v'}(y) = m_j(y) \). Since the solution should be continuous, there is a subinterval of \( (d, y) \) within which \( V_j^{v'}(x) > g_j(x) \) holds because \( V_j^{v'}(y) = m_j(y) > g_j(y) \). This implies
that player $j$’s payoff is higher than the equilibrium payoff for some subinterval of $(d, y)$, and it contradicts the assumption that $v$ is an equilibrium. We can construct an identical argument for the case $c > y$. We conclude that $\partial E_i \cap \partial \Gamma = \emptyset$ must hold. 

The next lemma establishes that no player invests when $X$ is above $x_i^c$.

**Lemma B.4** $E_i \cap (x_i^c, b) = \emptyset$ in an MPE.

**Proof of Lemma B.4:** Suppose, on the contrary, there exists $y \in E_i \cap (x_i^c, b)$. Because player $i$ immediately invests at $y$, it follows that $V_i^\nu(y) = g_i(y)$. We consider a bounded stopping time $\tau$ such that $\tau < \inf \{ t \geq 0 : X_t \leq x_i^c \} \wedge \tau_E^{(i)}$. In other words, $X$ does not hit either $x_i^c$ or $E_j$ before $\tau$. Because $E_i \cap E_j = \emptyset$ and $y > x_i^c$, we have $\tau > 0$ almost surely. We let $V'_i$ denote player $i$’s alternative strategy of investment at time $\tau$ instead. Then the payoff associated with the new strategy profile $\nu' = (V'_i, V_j)$ is given by

$$V_i^{\nu'}(y) = E_y^{\nu'}[\int_0^\tau \pi(X_t) e^{-\gamma t} dt + \int_0^\tau \pi(X_t) e^{-\gamma t} g_i(X_t)]$$

$$= E_y^{\nu'}[\int_0^\tau (\pi(X_t) + \alpha g_i(X_t)) e^{-\gamma t} dt] + g_i(y) > g_i(y) = V_i^{\nu}(y)$$

Note that the process $X$ is an uncontrolled state variable for $t < \tau$ because $\Gamma$ does not intersect with $(x_i^c, b)$ by virtue of Lemma B.2. The inequality is due to the fact that $\pi(X_t) + \alpha g_i(X_t) > 0$ for $t \in (0, \tau)$ and that $\tau > 0$. This contradicts the assumption that $v$ is an equilibrium. We conclude that $E_i \cap (x_i^c, b) = \emptyset$. 

In the final lemma before completing the proof, we establish that $E_1$ and $E_2$ are absent in a mixed strategy MPE.

**Lemma B.5** $E_1 = E_2 = \emptyset$ in a mixed strategy MPE.

**Proof of Lemma B.5:** For the purpose of leading to a contradiction, suppose that $E_1 \cup E_2 \neq \emptyset$. By virtue of Lemma B.1, there exists an interval $(c, d)$ which is adjacent to $\Gamma$ and $E_i$ for one of $i \in \{1, 2\}$ but which does not overlap with $E_j$. Without loss of generality, suppose that $c$ belongs to the boundary of $\Gamma$ and $d$ belongs to the boundary of $E_i$. It follows that $V_i^{\gamma}(c) = g_i(c)$ and $V_i^{\gamma}(d) = g_i(d)$.

By virtue of Lemma B.2, $\alpha g_i(x) + \pi(x) < 0$ for all $x \in (c, d)$. For any $y \in (c, d)$ as the initial point, we consider $\tau_c := \inf \{ t \geq 0 : X_t \notin (c, d) \}$, the time of escape from $(c, d)$. Because $V_i^{\gamma}(c) = g_i(c)$ and $V_i^{\gamma}(d) = ...
implies that we can consider \( V_i(\nu) \) equivalent strategy of player \( i \). Then \( V_i(\nu) - V_i(\nu') \) is to never invest within the interval because it is one of the best responses. This implies that player \( i \) is better off investing immediately at \( y \), which contradicts the assumption that \( v \) is an equilibrium. The contradiction results from the assumption that \( E_1 \cup E_2 \neq \emptyset \), and therefore, we conclude \( E_1 = E_2 = \emptyset \).

We finally complete the proof of Theorem 1 by showing that sup \( \Gamma = \theta_i \); since sup \( \Gamma \) cannot take two different values, this is possible only if \( \theta_1 = \theta_2 \), thus proving the theorem. To lead to a contradiction, suppose that \( \sup \Gamma := \gamma \neq \theta_i \) for one player \( i \). Since \( \Gamma \) is union of disjoint intervals, there is some \( \epsilon < \gamma \) such that \( [\epsilon, \gamma] \subset \Gamma \). Furthermore, \( (\gamma, b) \) is a continuation region for both players, i.e., no player invests in the interval \( (\gamma, b) \). Since the interval \( (\epsilon, \gamma) \) is a mixed strategy region, by the definition of a mixed strategy, one possible payoff-equivalent strategy of player \( i \) is to never invest within the interval because it is one of the best responses. This implies that we can consider \( (\epsilon, b) \) a subset of a continuation region for the purpose of computing the payoff function. Then \( V_i^\nu(\cdot) \) must be a solution to a boundary value problem (Chapter 9, Øksendal 2003) within the interval \( (\epsilon, b) \) satisfying the equation (B.6) and the boundary conditions \( V_i^\nu(\epsilon) = g_i(\epsilon) \) and \( \lim_{x \to b} [V_i^\nu(x) - (R, \pi)(x)] = 0 \), and hence, it must be continuously differentiable within \( (\epsilon, b) \). In particular, the first derivative of \( V_i^\nu(\cdot) \) must be continuous at \( x = \gamma \).

We formally prove this statement below. We let \( \tau_\epsilon := \inf \{ t \geq 0 : X_t \notin (\epsilon, b) \} \) denote the escape time from the interval \( (\epsilon, b) \). By virtue of Lon and Zervos (2011), there exists a well-defined solution to the second-order differential equation

\[
\sigma f(x) + \pi(x) + \lambda_j(x)[(m_j(x) - f(x)] = 0
\]

within a bounded interval \( x \in (\epsilon, \gamma) \). Suppose that \( f(x) \) is a continuously differentiable and continuous function in the interval \( [\epsilon, b] \) that is constructed to satisfy the equation above for \( x \in (\epsilon, \gamma) \) as well as \( \sigma f(x) + \pi(x) = 0 \) for \( x \in (\gamma, b) \). Furthermore, we suppose that \( f(\epsilon) = g_i(\epsilon) \) and \( \lim_{x \to b} [f(x) - (R, \pi)(x)] = 0 \). Since \( [\epsilon, \gamma] \) is compact and \( f(x) = \beta \phi(x) + (R, \pi)(x) \) for \( x \in (\gamma, b) \) where \( \beta \) is some coefficient, we find that \( f(x) - (R, \pi)(x) \) is a bounded function of \( x \) in the interval \( [\epsilon, b] \).
Following the same line of argument as in the proof part (ii) of Proposition 4, we find that

$$f(x) = E^x \left\{ \int_0^{u_n \wedge \tau_e} \left[ \pi(X_t) + \lambda_j(X_t)(R, \pi)(z^*) \right] e^{-rt - A^{(j)}_{\pi}} dt + f(X_u \wedge \tau_e) e^{-ru_n \wedge \tau_e - A^{(j)}_{\pi}} \right\},$$

where we employ the same notation $u_n$ as in the proof of Proposition 4. Note that $f(x)$ is bounded within the interval $[\varepsilon, \gamma]$ and that $f(x) - (R, \pi)(x) \to 0$ in the limit $x \to \infty$. Hence, in the limit $n \to \infty$, $f(X_{u_n \wedge \tau_e}) \to f(\varepsilon)$ if $X_{\tau_e} = \varepsilon$ and $f(X_{u_n \wedge \tau_e}) - (R, \pi)(X_{u_n \wedge \tau_e}) \to 0$ if $X_{\tau_e} = \infty$ in case $\tau_e = \infty$. (By the definition of $\tau_e$, these are the only two possibilities). Since $\lim_{\tau \to \infty} E^\pi [e^{-rt}(R, \pi)(X_\tau)] = 0$ from the transversality assumption that we made in (2), we can employ the bounded convergence theorem to obtain

$$\lim_{n \to \infty} E^x [f(X_{u_n \wedge \tau_e}) e^{-ru_n \wedge \tau_e - A^{(j)}_{\pi}}] = f(\varepsilon) e^{-r\tau_e - A^{(j)}_{\pi}}.$$

Using the fact that $f(\varepsilon) = g_t(\varepsilon)$ we finally obtain the following:

$$f(x) = E^x \left\{ \int_0^{\tau_e} \left[ \pi(X_t) + \lambda_j(X_t)(R, \pi)(z^*) \right] e^{-rt - A^{(j)}_{\pi}} dt + g_t(X_{\tau_e}) e^{-r\tau_e - A^{(j)}_{\pi}} \right\}.$$

The right-hand-side of the equation coincides with the expression for $V_t^{\gamma}(x)$. Since $f(\cdot)$ was constructed to be continuously differentiable, so is $V_t^{\gamma}(x)$.

We now compute the discontinuity in the first derivative $(V_t^{\gamma})'(\gamma^+) - (V_t^{\gamma})'(\gamma^-)$ at $\gamma$ and see if it is non-zero. The payoff function $V_t^{\gamma}(\cdot)$ in the interval $[\gamma, b)$ can be obtained from the solution to the boundary value problem (Chapter 9, Øksendal 2003):

$$V_t^{\gamma}(x) = \alpha_t(\gamma) \phi(x) + (R, \pi)(x),$$

while $V_t^{\gamma}(x) = g_t(x)$ for $x \in (\varepsilon, \gamma)$. From the definition of $\alpha_t(\cdot)$ in (10), it is straightforward to verify the following:

$$(V_t^{\gamma})'(\gamma^+) - (V_t^{\gamma})'(\gamma^-) = \alpha_t(\gamma) \phi'(\gamma) + (R, \pi)'(\gamma) - g_t'(\gamma) = -\phi(\gamma) \alpha_t'(\gamma).$$

By Assumption 3(i), $\alpha_t'(\gamma) = 0$ only at $\gamma = \theta$, and therefore, the first derivative of $V_t^{\gamma}(\cdot)$ is discontinuous if $\gamma \neq \theta$. It follows that $V_t^{\gamma}(\cdot)$ cannot be continuously differentiable if $\gamma \neq \theta$. By Lemma B.2, this implies that $\gamma = \theta_i = \theta_j$ must hold, which is not possible if $\theta_i \neq \theta_j$. We conclude that there is no mixed strategy MPE if $c_1 \neq c_2$.

**Proof of Theorem 2**: To complete the proof of the theorem, we establish two lemmas. First, we show that
Lemma B.6  If the strategy profile $\nu^*$ given in Section 4.2, then it is an SPE. 

Proof of Lemma B.6: Our goal is to show that $F_i(\cdot)$ dominates any payoff to player $i$ given player $j$’s strategy $\nu_j$.

(i) We first prove the optimality of player 2’s payoff function $F_2(\cdot)$ given player 1’s strategy of $\nu_1^*$. From the form of $\lambda_i(\cdot)$ in (29) and the fact that $V_2(x) = g_2(x)$ for $x \in \Gamma = (a, \theta^*)$ as well as (24), it is straightforward to verify that

$$\mathcal{A}V_2(x) + \pi(x) + \lambda_1(x)[V_2(z_1) - V_2(x)] = 0,$$

for $x \in \mathcal{I} \setminus \{\theta^*\}$.

Next, we will show that $V_2(\cdot) = R_2(\cdot)$, where $R_2(\cdot)$ is defined as follows:

$$R_2(x) := \mathbb{E}^x \left\{ \int_0^\infty e^{-s-A_1^{(1)}} [\pi(X_t) + \lambda_1(X_t)V_2(z_1)]dt \right\},$$

where $A_1^{(1)} := \int_0^t \lambda_1(X_s)ds$ and $X$ is the uncontrolled state variable. Note that $\lambda_1(\cdot)$ is a continuous function well-defined in the interval $[a, \theta^*]$ and zero elsewhere, so $|\lambda_i(x)|$ is bounded. Furthermore, Assumption 1 ensures that the integral above is well-defined.

To show $V_2(\cdot) = R_2(\cdot)$, we closely follow the proof of Theorem 10.4.1 of Øksendal (2003) and introduce a sequence of stopping times $\{u_n\}$ defined as $u_n = \min\{n, \inf_{t \geq 0} \{X_t \not\in G_n\}\}$, where $(G_n)_{n \in \mathbb{N}}$ is an increasing sequence of compact subsets of $\mathcal{I}$ such that $\lim_{n \to \infty} G_n = \mathcal{I}$. Then because $V_2(\cdot)$ satisfies (B.7), the arguments in the proof of Theorem 10.4.1 of Øksendal (2003) establish that

$$V_2(x) = \mathbb{E}^x \left\{ \int_0^{u_n} e^{-\tau - A_1^{(1)}} [\pi(X_t) + \lambda_1(X_t)V_2(z_1)]dt + e^{-ru_n - A_1^{(1)}} V_2(X_{u_n}) \right\},$$

for any $n$. From the functional form of $V_2(x)$ in (30) and the fact that $(R, \pi)'(x) < k$ for sufficiently large $x$ by Assumption 2, we note that $|V_2(x)| < A|x| + B$ for some $A > 0$ and $B > 0$. By the transversality condition (2) and the dominated convergence theorem, we have $\lim_{n \to \infty} \mathbb{E}^x[e^{-ru_n - A_1^{(1)}} V_2(X_{u_n})] = 0$. Thus, if we take the limit $n \to \infty$ of (B.9), using the dominated convergence theorem once again, the right hand side converges to the expression for $R_2(\cdot)$, so we have $V_2(x) = R_2(x)$. 


Next, we let \( \nu := (\nu_1, \nu_2) \) denote the strategy profile of player 1 employing \( \nu_1 \) and player 2 employing some arbitrary strategy \( \nu_2 \). Our goal is to show that the payoff to player 2 is dominated by \( V_2(\cdot) \). We let \( \tau_{m+1} \) denote player 2’s stopping time of investment after \( T_m \) under the strategy \( \nu_2 \), and we let \( \nu_{m+1} \) denote player 1’s time of investment after \( T_m \) under the strategy \( \nu_1 \). For notational convenience, we let \( \xi^{(2)}(\tau_m) \) denote player 2’s boost in \( X^\nu \) at the time \( \tau_m \). Under this notation, we note \( T_{m+1} = \tau_{m+1} \wedge \nu_{m+1} \).

Because \( V_2(x) = R_2(x) \), we can apply the Dynkin’s theorem to \( V_2(\cdot) \) for any arbitrary stopping time \( (\nu_1, \nu_2) \) (Alvarez and Lempa, 2008) and obtain the following:

\[
e^{-rT_m} V_2(X_{T_m}) - \mathbb{E}^{X_m} \left[ e^{-r\tau_{m+1} - A^{(1)}_{m+1}} V_2(X_{\tau_{m+1}}) \right] = \mathbb{E}^{X_m} \left\{ \int_{T_m}^{\tau_{m+1}} e^{-rt - A^{(1)}_{m+1}} [\pi(X_t) + \lambda_1(X_t) V_2(z_1)] dt \right\}, \tag{B.10}
\]

where we used the notation \( A^{(1)}_{m+1} := \int_{T_m}^{\tau_{m+1}} \lambda_1(X_s) ds \). In the expression above, \( X \) is an *uncontrolled* state variable which evolves within period \( m \) with the initial value \( X_{T_m} \). We define \( \hat{T}_{m+1} = \tau_{m+1} \wedge \nu_{m+1} \). Since \( \nu_{m+1} \) is the random time with an arrival rate of \( \lambda_1(X^\nu) \), we can re-express it in terms of the *controlled* state variable \( X^\nu \) and \( \nu_{m+1} \) as follows:

\[
V_2(X^\nu_{T_m}) e^{-rT_m} = \mathbb{E}^{X_m}_{\hat{\nu}_1} \left\{ \int_{T_m}^{\tau_{m+1} \wedge \nu_{m+1}} e^{-rt} \pi(X^\nu_t) dt + \mathbbm{1}_{\{\tau_{m+1} \leq \nu_{m+1}\}} e^{-r\nu_{m+1}} V_2(z_1) \right\}
+ \mathbb{E}^{X_m}_{\hat{\nu}_1} \left[ \int_{T_m}^{\hat{T}_{m+1}} e^{-rt} \pi(X^\nu_t) dt + e^{-r\hat{T}_{m+1}} V_2(X^\nu_{\hat{T}_{m+1}}) \right], \tag{B.11}
\]

where \( X^\nu_{\hat{T}_{m+1}} = z_1 \) if \( \hat{T}_{m+1} = \nu_{m+1} \). Recall that \( \hat{\nu}_1 \) is the expectation over the measure \( \mathbb{P} \otimes \nu_{m+1} \mathbb{I}^{(1)}_{m,t} \) when player 1’s survival probability \( M^{(1)}_{m,t} \) is given by (31). Note that the expression (B.11) reduces to (B.10) by the same argument that derived Proposition 3.

From (B.11), we derive the following expression:

\[
\hat{\mathbb{E}}^{\nu}_1 \left[ V_2(X^\nu_{T_m}) e^{-rT_m} - e^{-r\hat{T}_{m+1}} V_2(X^\nu_{\hat{T}_{m+1}}) \right] = \mathbb{E}^{X_m}_{\hat{\nu}_1} \left[ \mathbb{E}^{X_m}_{\nu_{m+1}} \left\{ \int_{T_m}^{\hat{T}_{m+1}} V_2(X^\nu_t) e^{-rt} - e^{-r\hat{T}_{m+1}} V_2(X^\nu_{\hat{T}_{m+1}}) \right\} dt \right]
+ \mathbb{E}^{X_m}_{\hat{\nu}_1} \left\{ \int_{T_m}^{\hat{T}_{m+1}} e^{-rt} \pi(X^\nu_t) dt \right\}.
\]
Then we sum the expression above for \( m \) from 0 up to some finite integer \( M \):

\[
V_2(x) + \hat{E}_1^m \{ \sum_{m=1}^{M} e^{-\tau_m} [V_2(X_{\tau_m}^X) - V_2(X_{\tau_m}^X)] - e^{-\tau_{m+1}}V_2(X_{\tau_{m+1}}^X) \} = \hat{E}_1^m \left[ \int_{0}^{T_{m+1}} e^{-\tau} \pi(X_{\tau}^Y) d\tau \right],
\]

where we exploit the fact that \( e^{-\tau_m} = e^{-\hat{\tau}_m} \) and that \( \int_{0}^{T_{m+1}}(\ldots) d\tau = \int_{0}^{\hat{T}_{m+1}}(\ldots) d\tau \). We note that \( V_2(X_{\tau_m}^X) = V_2(X_{\hat{\tau}_m}^X) \) whenever \( \hat{\tau}_m = \tau_m \), i.e., whenever \( m \)-th period begins with player 1’s investment. Thus, we can re-express \( V_2(x) \) as follows:

\[
V_2(x) = \hat{E}_1^m \{ \sum_{m=1}^{M} e^{-\tau_m} 1_{\{\tau_m < \tau_m\}} [V_2(X_{\tau_m}^X) - V_2(X_{\tau_m}^X)] + \int_{0}^{T_{m+1}} e^{-\tau} \pi(X_{\tau}^Y) d\tau \} + \hat{E}_1^m [e^{-\tau_{m+1}}V_2(X_{\tau_{m+1}}^X)] \quad (B.12)
\]

\[
\geq \hat{E}_1^m \{ \sum_{m=1}^{M} e^{-\tau_m} 1_{\{\tau_m < \tau_m\}} \left[-k\xi^2(\tau_m) - c_2 \right] + \int_{0}^{T_{m+1}} e^{-\tau} \pi(X_{\tau}^Y) d\tau \} + \hat{E}_1^m [e^{-\tau_{m+1}}\pi_L/r]. \quad (B.13)
\]

The inequality holds because of the quasi-variational inequality \( V_2(X_{\tau_m}^X) \geq \sup_{\zeta > 0}[V_2(X_{\tau_m}^X + \zeta) - k\xi - c_2] \) from (27) and Assumption 1(i) which leads to \( V_2(x) \geq (R, \pi)(x) \geq \pi_L/r \).

Now we take the limit \( M \to \infty \) on (B.13). Noting that \( -k\xi^2(\tau_m) - c_2 < 0 \), we employ the monotone convergence theorem to find that

\[
\lim_{M \to \infty} \hat{E}_1^m \{ \sum_{m=1}^{M} e^{-\tau_m} 1_{\{\tau_m < \tau_m\}} \left[-k\xi^2(\tau_m) - c_2 \right] \} = \hat{E}_1^m \{ \sum_{m=1}^{\infty} e^{-\tau_m} 1_{\{\tau_m < \tau_m\}} \left[-k\xi^2(\tau_m) - c_2 \right] \}.
\]

Similarly, because \( \pi(x) > \pi_L \) and \( \lim_{m \to \infty} T_m = \infty \), we can employ the monotone convergence theorem and find

\[
\hat{E}_1^m [\int_{0}^{T_{m+1}} e^{-\tau} \pi(X_{\tau}^Y) d\tau] \text{ converges to } \hat{E}_1^m [\int_{0}^{\infty} e^{-\tau} \pi(X_{\tau}^Y) d\tau] \text{ in the same limit. Lastly, because } \lim_{M \to \infty} \hat{E}_1^m [e^{-\tau_{m+1}}\pi_L/r] = 0, \text{ we find the following inequality:}
\]

\[
V_2(x) \geq \hat{E}_1^m \{ \sum_{m=1}^{\infty} e^{-\tau_m} 1_{\{\tau_m < \tau_m\}} \left[-k\xi^2(\tau_m) - c_2 \right] + \int_{0}^{\infty} e^{-\tau} \pi(X_{\tau}^Y) d\tau \}.
\]

Note that the right-hand-side of the inequality is the payoff to player 2 under the strategy profile \( \nu := (\nu_1^*, \nu_2) \). Since \( \nu_2 \) is an arbitrary strategy of player 2, it follows that \( V_2(\cdot) \) dominates all possible payoffs associated with any investment strategies of player 2 conditional on player 1’s strategy \( \nu_1^* \). Since we assume that \( V_2(\cdot) \) is player 2’s payoff from \( \nu^* \), we conclude that \( \nu_2^* \) is a best response to \( \nu_1^* \).

(ii) Next, we prove that \( \nu_1^* \) is player 1’s best response to \( \nu_2^* \). Using the same notational convention as in (i), we let \( \nu := (\nu_1, \nu_2) \) denote the strategy profile of player 1 employing an arbitrary investment strategy \( \nu_1 \) and player 2 employing \( \nu_2^* \). We let \( \tau_{m+1} \) denote player 1’s stopping time of investment after \( T_m \) under the strategy \( \nu_1 \), and we let \( \nu_{m+1} \) denote player 2’s time of investment after \( T_m \) under the strategy \( \nu_2^* \). We also let \( \xi^1(\tau_m) \)
denote player 1’s boost in $X^\tau_t$ at the time $\tau_m$.

We now define the following:

$$R_1(x) := E^x \left\{ \int_0^\infty e^{-rt-A_1^{[2]}} [\pi(x) + \lambda_2(x)U_1(z)] dt \right\}.$$ 

Just as in (i), it is straightforward to verify that $V_1(\cdot)$ satisfies

$$\mathcal{A}V_1(x) + \pi(x) + \lambda_2(x)[U_1(z_2) - V_1(x)] = 0. \quad (B.14)$$

Furthermore, we can utilize the same line of arguments as in (i) to derive $V_1(x) = R_1(x)$. We now define $\tilde{T}_{m+1} = \tau_m + \nu_{m+1}$ just as in (i), where $\tau_{m+1}$ is player 1’s timing of investment. From equation (26), which reduces to $\mathcal{A}U_1(x) + \pi(x) = 0$ for $x > \theta^\ast$, we obtain the following if $X_{T_m} > \theta^\ast$:

$$U_1(X_{T_m})e^{-rT_m} = E^{X_{T_m}} \left[ e^{-r_{m+1} \wedge \nu_{m+1}} U_1(X_{\tau_{m+1}}) + \int_{\tau_{m+1} \wedge \nu_{m+1}}^\tau e^{-rT} \pi(X_t) dt \right].$$

Since $U_1(\theta^\ast) = qU_1(z_2) + (1-q)V_1(\theta^\ast)$, we have

$$e^{-r_{m+1} \wedge \nu_{m+1}} U_1(X_{\tau_{m+1} \wedge \nu_{m+1}}) = 1_{\{\tau_{m+1} < \nu_{m+1}\}} e^{-r_{m+1}} U_1(X_{\tau_{m+1}}) + 1_{\{\tau_{m+1} > \nu_{m+1}\}} q e^{-r_{m+1}} U_1(z_2)$$

$$+ 1_{\{\tau_{m+1} = \nu_{m+1}\}} e^{-r_{m+1}} [qU_1(z_2) + (1-q)V_1(\theta^\ast)]$$

$$+ 1_{\{\tau_{m+1} > \nu_{m+1}\}} (1-q) E^{X_{\nu_{m+1}}} \left\{ e^{-r_{m+1} - A_m^{[2]}_{m+1}} V_1(X_{\tau_{m+1}}) \right\}$$

$$+ \int_{\tau_{m+1} \wedge \nu_{m+1}}^\tau e^{-rT} \pi(X_t) dt$$

where we use the notation $A_m^{[2]} := \int_0^{\tau_m} \lambda_2(X_t) dt$. In the last equality, we applied the Dynkin’s theorem to $V_1(\cdot)$.

As in (i), $X$ is an uncontrolled state variable which evolves within period $m$.

From the inequality $U_1(z_2) \geq V_1(\theta^\ast)$, we have the following:

$$1_{\{\tau_{m+1} = \nu_{m+1}\}} e^{-r_{m+1}} [qU_1(z_2) + (1-q)V_1(\theta^\ast)] \geq 1_{\{\tau_{m+1} = \nu_{m+1}\}} e^{-r_{m+1}} \left[ \frac{1}{2} q U_1(z_2) + (1 - \frac{1}{2}) q V_1(\theta^\ast) \right]$$

$$= 1_{\{\tau_{m+1} = \nu_{m+1}\}} e^{-r_{m+1}} \left[ \frac{1}{2} q U_1(z_2) + (1 - \frac{1}{2}) q (U_1(z_1) - k(z_1 - \theta^\ast) - c_1) \right]$$

which emulates what takes place in case $\tau_{m+1} = \nu_{m+1}$: there is a probability of $q/2$ that player 2 invests which
yields $U_1(z_2)$, and a probability of $1 - q/2$ that player 1 invests, resulting in $U_1(z_1) - k(z_1 - \theta^*) - c_1$.

By the same argument as in (i), switching to the controlled state variable $X^\gamma$, $V_1(\cdot)$ satisfies the following:

$$V_1(X^\gamma_{\hat{T}_{m+1}}) e^{-rT_m} = \mathbb{E}_2^X \left[ \int_{T_m}^{\hat{T}_{m+1}} e^{-rT} \pi(X^\gamma_T) dt + 1_{\{\nu_{m+1} < \tau_{m+1}\}} e^{-r\nu_{m+1}} U_1(z_2) \right]$$

$$+ 1_{\{\tau_{m+1} \leq \nu_{m+1}\}} e^{-r\nu_{m+1}} V_1 (X^\gamma_{\hat{T}_{m+1}})$$

$$= \mathbb{E}_2^X \left[ \int_{T_m}^{\hat{T}_{m+1}} e^{-rT} \pi(X^\gamma_T) dt + e^{-r\hat{T}_{m+1}} F_1,\hat{\nu}_{m+1} (X^\gamma_{\hat{T}_{m+1}}) \right],$$

because we can identify $F_1,\hat{\nu}_{m+1} (X^\gamma_{\hat{T}_{m+1}}) = U_1(z_2)$ if $\nu_{m+1} < \tau_{m+1}$ and $F_1,\hat{\nu}_{m+1} (X^\gamma_{\hat{T}_{m+1}}) = V_1 (X^\gamma_{\hat{T}_{m+1}}) = V_1 (X^\gamma_{\tau_{m+1}})$ if $\tau_{m+1} \leq \nu_{m+1}$. Here, we have used the notation $\mathbb{E}_2^X$ which represents the expectation over the measure $\mathbb{P} \otimes_{m \in \mathbb{N}} \mathbb{I}_m^{(2)}$ conditional on the value of $X^\gamma_{\tau_m}$ with the survival probability given by $M_{m+1}^{(2)}$ in (32). Recall that player 2's strategy $\nu_2^m$ is to invest at $\tau_{m}^0$, with a probability of $q$, and to invest at the rate of $\lambda_2(X_t)$ for $t \in (\tau_{m}^0, T_{m+1})$. Hence, from (B.15), we obtain a similar expression for $U_1(X^\gamma_{\tau_m})$ for $X^\gamma_{\tau_m} \geq \theta^*$:

$$U_1(X^\gamma_{\tau_m}) e^{-rT_m} \geq \mathbb{E}_2^X \left[ \int_{T_m}^{\hat{T}_{m+1}} e^{-rT} \pi(X^\gamma_T) dt + e^{-r\hat{T}_{m+1}} F_1,\hat{T}_{m+1} (X^\gamma_{\hat{T}_{m+1}}) 1_{\{\nu_{m+1} \neq \tau_{m}^0\}} \right]$$

$$+ 1_{\{\nu_{m+1} = \tau_{m}^0\}} e^{-r\tau_{m}^0} \left[ (1 - \chi_m) U_1(z_2) + \chi_m (U_1(z_1) - k(z_1 - \theta^*) - c_1) \right].$$

Here it is understood that $F_1,\hat{T}_{m+1} (\cdot) = U_1 (\cdot)$ for $\hat{T}_{m+1} < \tau_{m}^0$ and $F_1,\hat{T}_{m+1} (\cdot) = V_1 (\cdot)$ for $\hat{T}_{m+1} \geq \tau_{m}^0$. Furthermore, $F_1,\hat{T}_{m+1} (X^\gamma_{\hat{T}_{m+1}}) = U_1(z_2)$ if $\nu_{m+1} < \tau_{m+1}$.

The last term following $1_{\{\nu_{m+1} = \tau_{m}^0\}}$ requires some explanation. We define $\chi_m$ a Bernoulli random variable measurable with respect to $\mathcal{F}_m$, but independent of the Wiener process $W$, such that $\mathbb{P}(\chi_m = 0) = q/2$ and $\mathbb{P}(\chi_m = 1) = 1 - q/2$. We interpret $\chi_m = 0$ as the event in which player 2 invests in case $\nu_{m+1} = \tau_{m}^0$. Similarly, $\chi_m = 1$ is the event in which player 1 invests in case $\tau_{m+1} = \tau_{m}^0$. We note that $F_1,\hat{T}_{m+1} (X^\gamma_{\hat{T}_{m+1}}) = U_1(z_2)$ in case player 2 invests at $\tau_{m}^0$, and $F_1,\hat{T}_{m+1} (X^\gamma_{\hat{T}_{m+1}}) = U_1(z_1)$ in case player 1 invests at $\tau_{m}^0$. Thus, we can re-express the last line of (B.16) as

$$1_{\{\nu_{m+1} = \tau_{m}^0\}} e^{-r\tau_{m}^0} \left[ F_1,\hat{T}_{m+1} (X^\gamma_{\hat{T}_{m+1}}) - \chi_m (k(z_1 - \theta^*) + c_1) \right].$$

Now we can combine the two expressions to obtain the following:
\begin{align*}
F_{1,Tm}(X^Y_{Tm})e^{-rTm} & \geq \mathbb{E}_2^{X^Y_m} \left[ \int_{Tm}^{\hat{T}_{m+1}} e^{-rt} \pi(X^Y_t) dt + e^{-r\hat{T}_{m+1}}F_{1,\hat{T}_{m+1}}(X^Y_{\hat{T}_{m+1}}) \mathbf{1}_{\{\tau_{m+1} = \tau^m_0 > Tm \text{ or } \tau^m_0 = Tm\}} \right] \\
& \quad + \mathbf{1}_{\{\tau_{m+1} = \tau^m_0 > Tm\}} e^{-r\tau^m_0} [F_{1,Tm}(X^Y_{\tau^m_0}) - \chi_m(k(z_1 - \theta^*) + c_1)] .
\end{align*}

which is analogous to (B.11) except for the case of \( \tau_{m+1} = \tau^m_0 \). We now define the following random variable:

\[
\hat{F}_{\hat{T}_{m+1}} := F_{1,\hat{T}_{m+1}}(X^Y_{\hat{T}_{m+1}}) \mathbf{1}_{\{\tau_{m+1} = \tau^m_0 > Tm \text{ or } \tau^m_0 = Tm\}} + \mathbf{1}_{\{\tau_{m+1} = \tau^m_0 > Tm\}} [F_{1,Tm}(X^Y_{\tau^m_0}) - \chi_m(k(z_1 - \theta^*) + c_1)] ,
\]

so that we can re-express (B.17) as

\[
F_{1,Tm}(X^Y_{Tm})e^{-rTm} \geq \mathbb{E}_2^{X^Y_m} \left[ \int_{Tm}^{\hat{T}_{m+1}} e^{-rt} \pi(X^Y_t) dt + e^{-r\hat{T}_{m+1}}\hat{F}_{\hat{T}_{m+1}} \right] .
\]

Then we can employ the same line of logic as in (i) and sum over the terms \( \mathbb{E}_2^{X^Y_m} [e^{-rTm}[\hat{F}_{\hat{T}_{m}} - F_{1,Tm}(X^Y_{Tm})]] \) to arrive at the following expression:

\[
F_{1,\nu}(x)e^{-rt} \geq \mathbb{E}_2 \left\{ \sum_{m=1}^{\infty} e^{-rTm} \left( \mathbf{1}_{\{\tau_m < \nu_m\}} + \mathbf{1}_{\{\tau_m = \nu_m > Tm-1\}} \chi_m \left[ -k_2^{(2)}(\tau_m) - c_2 \right] + \int_T^{\tau_m} e^{-rt} \pi(X^Y_t) dt \right) \right\} .
\]

Note that the right-hand side is the payoff to player 1 associated with the strategy profile \((\nu_1, \nu^*_2)\). We conclude that \( F_{1,\nu}(x) \) dominates the payoff from any arbitrary strategy of player 1 against \( \nu^*_2 \). Thus, \( \nu^*_1 \) is the best response to \( \nu^*_2 \).

Under the same set of assumptions, the next lemma establishes that the payoff functions associated with \( \nu^* \) are \( F_{1,\nu}(\cdot) \) if it satisfies all the stipulated conditions.

**Lemma B.7** The functions \( F_{1,\nu}(\cdot) \) and \( F_{2,\nu}(\cdot) \), which satisfy the conditions given in Section 4.2, are the payoff functions associated with the strategy profile \( \nu^* \).

**Proof of Lemma B.7.** We prove the statement of the lemma for \( F_{2,\nu}(\cdot) \) in part (i) and for \( F_{1,\nu}(\cdot) \) in part (ii).

(i) We show that \( F_{2,\nu}(x) = V_2(x) \) is player 2’s payoff associated with \( \nu^* \). Adopting the notational convention used in the proof of Lemma B.6, we let \( \nu_m \) and \( \tau_m \) respectively denote the stopping time of investment by players 1 and 2 after time \( T_{m-1} \) under the strategy profile \( \nu^* \). We also use the definition \( \hat{T}_m = \tau^-_m \wedge \nu_m \).
Under strategy $v_2^*$, if player 2 invests at time $\tau_m$, then $X^v_{\tau_m} \in (a, \theta^*)$ and $X^v_{\tau_m} = z_2$ must be satisfied, which implies that $V_2(X^v_{\tau_m}) - V_2(X^v_{\tau_m}) = -k(z_2 - X^v_{\tau_m}) - c_2$ from the functional form of $V_2(\cdot)$ because $V_2(x) = g_2(x) = V_2(z_2) - k(z_2 - x) - c_2$ for $x \in (a, \theta^*)$. From (B.12), we then obtain the following relation:

$$V_2(x) = \hat{E}^x \{ \sum_{m=1}^{M} e^{-rT_m} 1_{\{\tau_m < \theta_m\}} [-k(z_2 - X^v_{\tau_m}) - c_2] + \int_0^{T_{m+1}} e^{-rt} \pi(X^v_{t}) dt \} + \hat{E}^x[e^{-rT_{M+1}} V_2(X^v_{T_{M+1}})] .$$

Here $\hat{E} := E_{P \otimes L(1) \otimes L(2)}$ is an expectation over $\mathbb{P}$ and the two players’ mixed strategies. Note that

$$\lim_{M \to \infty} \hat{E}^x[e^{-rT_{M+1}} V_2(X^v_{T_{M+1}})] = 0$$

because the times between each period $(T_{m+1} - T_m)$ are mutually independent random variables so that $\hat{E}[e^{-r(T_{m+1} - T_m)}] = ... = \hat{E}[e^{-r(T_2 - T_1)}] < 1$. From Assumption 1 (i) and (ii) and using the monotone convergence theorem, we arrive at the following expression in the limit $M \to \infty$:

$$V_2(x) = \hat{E}^x \{ \sum_{m=1}^{\infty} e^{-rT_m} 1_{\{\tau_m < \theta_m\}} [-k(z_2 - X^v_{\tau_m}) - c_2] + \int_0^{\infty} e^{-rt} \pi(X^v_{t}) dt \} .$$

It follows that $F_{2,r}(x) = V_2(x)$ is the payoff induced by $v_2^*$ given that player 1 employs $v_1^*$.

(ii) Next, we prove that $F_{1}(\cdot)$ is player 1’s payoff induced by $v^*$. In this proof, we do not need to be concerned about the event $\tau_m = \tau_m^{m-1}$ because player 1’s strategy $v_1^*$ does not allow it. Note that $F_{1,r}(\cdot)$ satisfies (B.17) which is analogous to (B.11). By the same line of arguments as in (i), it follows that $F_{1}(\cdot)$ satisfies

$$e^{-rt} F_{1,r}(x) = \hat{E}^x \{ \sum_{m=1}^{\infty} e^{-rT_m} 1_{\{\tau_m < \theta_m\}} [-k(z_2 - X^v_{\tau_m}) - c_2] + \int_t^{\infty} e^{-rs} \pi(X^v_{s}) ds \},$$

which implies that $F_{1,r}(\cdot)$ is player 1’s payoff associated with $v^*$. □

Finally, by virtue of Lemmas B.6 and B.7, we immediately obtain Theorem 2.

**Proof of Lemma 2.** (i) Because of the relationship

$$J'(x) = (R, \pi)'(x) - k - \phi'(x)I(x) - I'(x)\phi(x) = -I'(x)\phi(x),$$

it suffices to prove the statement for $I'(\cdot)$.

From the property of diffusion processes, there exists a constant parameter $B$ (p. 706, Alvarez and Lempa, 2008) given by

$$B = \frac{\psi'(x)\phi(x) - \phi'(x)\psi(x)}{S'(x)}, \quad (B.18)$$
and \((R_r f)(x) := \mathbb{E}^X [\int_0^\infty e^{-rt} f(X_t) dt]\) can be expressed as follows (Alvarez and Lempa, 2008):

\[
(R_r f)(x) = B^{-1} \phi(x) \int_a^x \psi(y) f(y) m'(y) dy + B^{-1} \psi(x) \int_x^b \phi(y) f(y) m'(y) dy,
\]

(B.19)

for a given integrable function \(f(\cdot)\). From (B.18) and (B.19), the following expression can be derived:

\[
I'(x) = \frac{2S'(x)}{\theta^2(x) [\theta'(x)]^2} L(x).
\]

(B.20)

From the definition of \(L(\cdot)\), it is also straightforward to obtain

\[
L'(x) = -p'(x) \frac{\phi'(x)}{S'(x)}.
\]

By Assumption 4(i), we note that \(L'(x) > 0\) for \(x \in (a,x^*)\) and \(L'(x) < 0\) for \(x \in (x^*, b)\).

Next, we examine the limiting forms of \(L(\cdot)\). By Assumption 4(ii), we find that \(\lim_{x \to b} L(x) = 0\). Furthermore, we obtained above that \(L'(x) > 0\) for \(x \in (a,x^*)\) and \(L'(x) < 0\) for \(x \in (x^*, b)\). From Assumption 4(iii), we deduce that there exists a unique point \(\hat{x} \in (a,x^*)\) such that \(L(x) < 0\) for all \(x \in (a,\hat{x})\) and \(L(x) > 0\) for all \(x \in (\hat{x}, b)\). From (B.20), it follows that \(I'(x) < 0\) for all \(x \in (a,\hat{x})\) and \(I'(x) > 0\) for all \(x \in (\hat{x}, b)\).

Lastly, we note that \(I(\cdot)\) achieves its global minimum value at \(\hat{x}\). From the form of \(I(\cdot)\) in (33), its minimum value must be negative, which means that \(\hat{x} < z^*\) by Assumption 2. Thus, \(\hat{x} < \min\{z^*, x^*\}\).

(ii) This statement immediately follows from Assumption 2, the expression (33), and the fact that \(\phi'(x) < 0\).

Proof of Theorem 3: Before we prove this theorem, we state and prove the following lemma which lays out the sufficient conditions for the verification theorem (Theorem 2). In practice, we only need to consult the following lemma to see if a mixed strategy equilibrium exists. For the purpose of this lemma, we continue to suppose Assumptions 1, 2, 4, and 5.

Lemma B.8 Consider the strategy profile \(\nu^*\) given in Section 4.2 which is characterized by \(q, \theta^*, z_1,\) and \(z_2\). Suppose that \(q \in (0,1), \theta^*, z_1,\) and \(z_2\) satisfy \(\theta^* < \hat{x} < z_1 < z_2\) and the following conditions:
It remains to show that \( A \) function \( \text{continuously differentiable everywhere except at } \theta \). Thus, the first derivative of \( \hat{\theta} \) we have \( \hat{\theta} \). Next, we note that \( \hat{\theta} \) everywhere. From \( \alpha \) we have \( \alpha \). Similarly, from \( \alpha \) we have \( \alpha \). Similarly, from \( \alpha \) we have \( \alpha \). From the definition of \( \text{continuous everywhere in } \mathcal{S} \). Next, we note that \( \hat{\theta} \). Thus, the first derivative of \( \hat{\theta} \) is continuous at \( \theta \) and thus continuous everywhere in \( \mathcal{S} \).

We also prove that \( \hat{\theta} \) satisfies (24) and (25). By construction, \( \hat{\theta} \) holds for \( x > \theta \). It remains to show that \( \hat{\theta} \). From the definition of \( I(\cdot) \) in (33) and the fact that

\[
J(z_1) - J(\theta^*) = c_1 \tag{B.21}
\]

\[
\phi(\theta^*)[I(\theta^*) - I(z_1)]
- \frac{I(z_1)\phi(z_2) + (R,\pi)(z_2) + I(\theta^*)\phi(\theta^*) - (R,\pi)(\theta^*)}{(R,\pi)(\theta^*)} = q. \tag{B.22}
\]

\[
-\frac{I(z_1)\phi(\theta^*) + (R,\pi)(\theta^*) - k\theta^*}{J(z_1)} < 0. \tag{B.23}
\]

Then \( \nu^* \) is a mixed strategy TSSPE.

**Proof of Lemma B.8:** Our goal is to prove that \( F_{ij}(\cdot) \) satisfying all the conditions specified in Section 4.2 exists if equations (35) – (B.22) are satisfied. Then the statement of the proposition follows from Theorem 2.

(i) We first prove that the function \( F_{ij}(x) = V_2(x) \) characterized in Section 4.2 exists if (35) and (36) are satisfied. Our plan is to construct a function which satisfies all the conditions of \( V_2(x) \).

We define a coefficient \( \alpha = -I(\theta^*) = -I(z_2) \), which is positive from Lemma 2(ii), and we construct a function

\[
\hat{V}_2(x) := \begin{cases} 
\hat{g}_2(x) & \text{for } x \in (a, \theta^*], \\
\alpha \phi(x) + (R,\pi)(x) & \text{for } x \in (\theta^*, b),
\end{cases}
\]

where \( \hat{g}_2(x) := \alpha \phi(z_2) + (R,\pi)(z_2) - k(z_2 - x) - c_2 = \hat{V}_2(z_2) - k(z_2 - x) - c_2 \).

We now show that \( \hat{V}_2(\cdot) \in C^2(\mathcal{S}\setminus\{\theta^*\}) \cap C^1(\mathcal{S}) \cap C(\mathcal{S}) \). By construction, \( \hat{V}_2(\cdot) \) is already twice continuously differentiable everywhere except at \( \theta^* \), so it remains to prove that it is continuously differentiable everywhere. From \( \alpha = -I(\theta^*) \), we have \( \lim_{x \downarrow \theta^*} \hat{V}_2(x) = -I(\theta^*)\phi(\theta^*) + (R,\pi)(\theta^*) \). Similarly, from \( \alpha = -I(z_2) \), we have \( \hat{V}_2(\theta^*) = \hat{g}_2(\theta^*) = -I(z_2)\phi(z_2) + (R,\pi)(z_2) - k(z_2 - \theta^*) - c_2 \). From the definition of \( J(\cdot) \) in (34) and the assumed relation (36), we arrive at \( \lim_{x \uparrow \theta^*} \hat{V}_2(x) = \hat{V}_2(\theta^*) \), and hence, \( \hat{V}_2(\cdot) \) is continuous everywhere in \( \mathcal{S} \). Next, we note that \( \hat{V}_2(\theta^*) = k \) and, by the definition of \( I(\cdot) \) from (33),

\[
\hat{V}_2'(\theta^*) = \alpha \phi(\theta^*) + (R,\pi)'(\theta^*) = -I(\theta^*)\phi(\theta^*) + (R,\pi)'(\theta^*) = k.
\]

Thus, the first derivative of \( \hat{V}_2(\cdot) \) is continuous at \( \theta^* \) and thus continuous everywhere in \( \mathcal{S} \).

We also prove that \( \hat{V}_2(\cdot) \) satisfies (24) and (25). By construction, \( \hat{\mathcal{A}} \hat{V}_2(x) + \pi(x) = 0 \) holds for \( x > \theta^* \).
\( \alpha = -I(z_2) \), the first derivative of \( \hat{V}_2(\cdot) \) is given by
\[
\hat{V}_2'(x) = \begin{cases} 
  k & \text{for } x \in (a, \theta^*) \\
  k + \varphi'(x)(I(x) - I(z_2)) & \text{for } x \in [\theta^*, b) 
\end{cases}.
\] (B.24)

Because \( \theta^* < \hat{x} < z_2 \) by assumption, we have \( I(x) < I(z_2) \) for \( x \in (\theta^*, z_2) \) and \( I(x) > I(z_2) \) for \( x \in (z_2, b) \). Thus, \( \hat{V}_2'(x) > k \) for \( x \in (\theta^*, z_2) \). Because \( \hat{V}_2''(\theta^*) = k \), this implies that \( \hat{V}_2''(\theta^*) \geq 0 \). Because \( \varphi \hat{V}_2(x) + \pi(x) = 0 \) for \( x > \theta^* \), we have
\[
\varphi \hat{V}_2(\theta^*) + \pi(\theta^*) = \frac{1}{2} \sigma^2(\theta^*) \hat{V}_2''(\theta^*) + \mu(\theta^*)k - r \hat{V}_2(\theta^*) + \pi(\theta^*) = 0,
\]
which implies that \( \mu(\theta^*)k - r \hat{V}_2(\theta^*) + \pi(\theta^*) \leq 0 \). For \( x < \theta^* \), we have \( \hat{V}_2(x) = k(x - \theta^*) + \hat{V}_2(\theta^*) \) by the assumed relation (36), so the following holds:
\[
\varphi \hat{V}_2(x) + \pi(x) = \rho(x) + rk\theta^* - r \hat{V}_2(\theta^*).
\]

Recall that \( \rho(x) \) is increasing for \( x < \theta^* \) by Assumption 4(i) because \( \theta^* < \hat{x} < x^* \) by Lemma 2. Therefore, we find \( \varphi \hat{V}_2(x) + \pi(x) < \varphi \hat{V}_2(\theta^*) + \pi(\theta^*) \leq 0 \) for all \( x < \theta^* \) where the last inequality holds because we showed \( \mu(\theta^*)k - r \hat{V}_2(\theta^*) + \pi(\theta^*) \leq 0 \) above. Therefore, \( \hat{V}_2(\cdot) \) satisfies the differential properties (24) and (25).

Next, we show that \( \hat{V}_2(x) > (R, \pi)(x) \). By construction, it is clear that \( \hat{V}_2(x) > (R, \pi)(x) \) for \( x \geq \theta^* \) because \( \alpha = -I(\theta^*) > 0 \) by Lemma 2(ii). Recall that \( \hat{V}_2'(x) = k \) for \( x < \theta^* \) and that \( (R, \pi)'(x) > k \) for \( x < \theta^* \) by Assumption 2 and the fact that \( \theta^* < \hat{x} < z^* \) by Lemma 2(i). Thus, \( \hat{V}_2'(x) < (R, \pi)'(x) \) for all \( x < \theta^* \), which implies that \( \hat{V}_2(x) > (R, \pi)(x) \) for \( x < \theta^* \) as well because both \( \hat{V}_2(\cdot) \) and \( (R, \pi)(\cdot) \) are in \( C(\mathcal{F}) \).

Now it remains to prove the quasi-variational inequality \( \hat{V}_2(x) \geq \sup_{\xi \geq 0} [\hat{V}_2(x + \xi) - k\xi - c_2] \). We first prove that \( z_2 \) maximizes \( \hat{V}_2(x) - kx \). From (B.24), we have \( \hat{V}_2(x) = k \) for \( x \in (a, \theta^*) \), \( \hat{V}_2'(x) > k \) for \( x \in (\theta^*, z_2) \), and \( \hat{V}_2'(x) < k \) for \( x \in (z_2, b) \). It follows that \( \hat{V}_2(z) - k\xi \) has a unique global maximum exactly at \( \xi = z_2 \). Thus, we obtain the following expression:
\[
\hat{V}(x) := \sup_{\xi \geq 0} [\hat{V}_2(x + \xi) - k\xi - c_2] = \begin{cases} 
  \hat{V}_2(z_2) - k(z_2 - x) - c_2 & \text{for } x < z_2 \\
  \hat{V}_2(x) - c_2 & \text{for } x \geq z_2 
\end{cases}.
\]
Then the difference $\hat{V}_2(x) - \tilde{V}(x)$ is given as follows:

$$
\hat{V}_2(x) - \tilde{V}(x) = \begin{cases} 
0 & x \in (a, \theta^*) \\
\alpha \phi(x) + (R, \pi)(x) - kx - J(z_2) + c_2 & x \in [\theta^*, z_2] \\
c_2 & x \in (z_2, b) 
\end{cases}
$$

Recall from (36) that $J(\theta^*) - J(z_2) + c_2 = \alpha \phi(\theta^*) + (R, \pi)(\theta^*) - k\theta^* - J(z_2) + c_2 = 0$ and note that $\hat{V}_2'(x) \geq k = \hat{V}'(x)$ for $x \in (\theta^*, z_2)$. Hence, we have $\alpha \phi(x) + (R, \pi)(x) - kx - J(z_2) + c_2 \geq 0$ for all $x \in (\theta^*, z_2)$. We conclude that $\hat{V}_2(x) \geq \tilde{V}(x)$ for all $x$, so the variational inequality is satisfied.

(ii) We now prove that there exists $F_{1,t}(\cdot)$ which satisfies all the conditions of Section 4.2. We define $\alpha := -I(\theta^*)$ and $\beta := -I(z_1)$ and construct $\hat{F}_{1,t}(\cdot)$ as follows: $\hat{F}_{1,t}(x) = \hat{U}_1(x)$ if $t \in [T_m, T_{m+1})$ and $\hat{F}_{1,t}(x) = \hat{V}_1(x)$ if $t \in [T_{m+1}, T_{m+2})$, where $\hat{U}_1(\cdot)$ and $\hat{V}_1(\cdot)$ are given by

$$
\hat{V}_1(x) = \alpha \phi(x) + (R, \pi)(x),
$$

$$
\hat{U}_1(x) = \beta \phi(x) + (R, \pi)(x),
$$

for $x \in [\theta^*, b)$, and

$$
\hat{V}_1(x) = \hat{g}_1(x) := \hat{U}_1(z_1) - k(z_1 - x) - c_1,
$$

$$
\hat{U}_1(x) = \hat{V}_1(x),
$$

for $x \in (a, \theta^*)$. We also impose a boundary condition $\hat{U}_1(\theta^*) = q \hat{U}_1(z_2) + (1 - q) \hat{V}_1(\theta^*)$ which is derived from the condition that player 2 boosts $X$ to $z_2$ with a probability of $q$ when $X$ reaches $\theta^*$ from above. These conditions ensure that $\hat{V}_1(\cdot)$ and $\hat{U}_1(\cdot)$ satisfy the same differential equations as $V_1(\cdot)$ and $U_1(\cdot)$ as well as the condition of investment for $x \in (a, \theta^*)$ up to $z_1$.

We first prove that $\hat{V}_1(\cdot)$ is continuous at $\theta^*$. We compute $\hat{V}_1(\theta^{*+}) - \hat{g}_1(\theta^{*-})$ and see if it equals zero:

$$
\hat{V}_1(\theta^{*+}) - \hat{g}_1(\theta^{*-}) = -I(\theta^*) \phi(\theta^*) + (R, \pi)(\theta^*) - I(z_1) \phi(z_1) - (R, \pi)(z_1) + k(z_1 - \theta^*) + c_1
$$

$$
= J(\theta^*) - J(z_1) + c_1 = 0,
$$

where the last equality is due to (B.21). Thus, $\hat{V}_1(\cdot)$ is continuous everywhere. We then note that $\hat{V}_1(\cdot)$ coincides with $\hat{V}_2(\cdot)$ constructed in (i); it follows from the fact that $\alpha = -I(\theta^*)$, which leads to $\hat{V}_1(x) = \hat{V}_2(x)$.
for \( x \geq \theta^* \) and that \( \hat{V}'(x) = \hat{V}_2'(x) = k \) for \( x < \theta^* \). Hence, we immediately have \( \hat{V}_1(\cdot) \in C^2(\mathcal{S}\setminus\{\theta^*\}) \cap C^1(\mathcal{S}) \cap C(\mathcal{S}) \), which must be satisfied by \( V_1(\cdot) \). It also follows that \( \hat{V}_1(x) > (R, \pi)(x) \) for all \( x \in \mathcal{S} \).

Next, we show that the boundary conditions of \( F_{1,1}(\cdot) \) at \( \theta^* \) are satisfied by \( \hat{F}_{1,1}(\cdot) \) if (B.22) holds. It is straightforward to verify that the boundary condition \( \hat{U}_1(\theta^*) = q\hat{U}_1(z_2) + (1-q)\hat{V}_1(\theta^*) \) is equivalent to (B.22) because

\[
q = \frac{\hat{U}_1(\theta^*) - \hat{V}_1(\theta^*)}{\hat{U}_1(z_2) - \hat{V}_1(\theta^*)},
\]

leads to (B.22).

Thus far, we have proved that \( \hat{F}_1(\cdot) \) satisfies the same differential equations and boundary conditions as \( F_1(\cdot) \). It remains to prove that \( \hat{U}_1(x) > \check{V}_1(x) \) for \( x \geq \theta^* \) and that \( \hat{F}_1(\cdot) \) satisfies the quasi-variational inequality \( \hat{V}_1(x) \geq \sup_{\zeta \geq 0} \{ \hat{U}_1(x + \zeta) - k\zeta - c_1 \} \).

We first derive \( \hat{U}_1(x) > \check{V}_1(x) \) for \( x \geq \theta^* \) from the equality \( J(z_1) - J(\theta^*) = c_1 \). Since \( J'(x) < 0 \) for \( x > \check{x} \) and \( J(z_2) - J(\theta^*) = c_2 \), we have \( \check{x} < z_1 < z_2 \) because \( c_1 < c_2 \). Furthermore, \( J'(x) > 0 \) for \( x > \check{x} \), so that \( -I(z_1) > -I(z_2) = -I(\theta^*) \), which means that \( \beta > \alpha \). It follows that \( \hat{U}_1(x) > \check{V}_1(x) \) for \( x \geq \theta^* \).

Next, we examine the derivative of \( \hat{U}_1(z) - kz \) for \( z > \theta^* \):

\[
\hat{U}_1'(z) - k = -I(z_1)\phi'(z) + (R, \pi)'(z) - k = \phi'(z)[I(z) - I(z_1)].
\]

For \( z \in (\theta^*, z_1), I(z) \) starts out higher than \( I(z_1) \) for \( z \) sufficiently close to \( \theta^* \) because (35), but \( I(z) \) reaches its global minimum at \( \check{x} \) and then increases in \( z \). Then, because \( \phi'(\cdot) < 0, \hat{U}_1'(z) - k \) crosses zero from positive to negative across \( z_1 \). Thus, \( \hat{U}_1(z) - kz \) has its local interior maximum at \( z_1 \). However, \( \hat{U}_1(z) - kz \) may have its global maximum at the boundary \( \theta^* \) because \( \hat{U}_1'(z) - k \) is negative for \( z \) close to \( \theta^* \). Hence, if \( \hat{U}_1(\theta^*) - k\theta^* < \hat{U}_1(z_1) - kz_1 \), then \( \hat{U}_1(z) - kz \) has its global maximum at \( z_1 \). It can be verified that \( \hat{U}_1(\theta^*) - k\theta^* < \hat{U}_1(z_1) - kz_1 \) holds if and only if (B.23) is satisfied. (We do not need to consider \( z < \theta^* \) because \( \hat{U}_1'(x) = \hat{V}_1'(x) = k \) for \( x < \theta^* \) so that \( \hat{U}_1(z) - kz \) cannot have a global maximum in the interval \( (a, \theta^*) \) unless \( \hat{U}_1(\theta^*) - k\theta^* \geq \hat{U}_1(z_1) - kz_1 \).

Lastly, we define \( \check{U}(x) = \sup_{\zeta \geq 0} \{ \check{U}_1(x + \zeta) - k\zeta - c_1 \} \) for any \( x > \theta^* \). Then \( \check{U}(x) = \check{U}_1(z_1) - k(z_1 - x) - c_1 \) for \( x \in (\theta^*, z_1) \) and \( \check{U}(x) = \check{U}_1(x) - c_1 \) for \( x > z_1 \). Therefore,

\[
\check{U}_1(x) - \check{U}(x) = \begin{cases} 
0 & x \leq \theta^* \\
-I(\theta^*)\phi(x) + (R, \pi)(x) - kx - J(z_1) + c_1 & x \in (\theta^*, z_1) \\
-I(\theta^*)\phi(x) + I(z_1)\phi(x) + c_1 & x > z_1
\end{cases}
\]
For $x \leq \theta^*$, $\hat{U}(x) = \hat{g}_1(x)$, so it follows that $\hat{V}_1(x) - \hat{U}(x) = 0$. Note also that $-I(\theta^*)\phi(\theta^*) + (R, \pi)(\theta^*) - k\theta^* - J(z_1) + c_1 = J(\theta^*) - J(z_1) + c_1 = 0$ from (B.21). We note that the first derivative of $-I(\theta^*)\phi(x) + (R, \pi)(x) - kx$ is non-negative for $x \leq z_2$ because this function coincides with $\hat{V}_2(x) - kx$ which is proved to have non-negative first derivative for $x \leq z_2$ in (i). Then because $z_1 < z_2$, $-I(\theta^*)\phi(x) + (R, \pi)(x) - kx - J(z_1) + c_1$ is non-decreasing in $(\theta^*, z_1)$, and hence non-negative in this interval. Since $\hat{U}(x)$ is continuous at $z_1$, it follows that $-I(\theta^*)\phi(z_1) + (z_1)\phi(z_1) + c_1 \geq 0$. We also note that $I(z_1) < I(\theta^*)$ because (35), $\theta^* < \hat{x} < z_1 < z_2$, and Lemma 2, and that $\phi(\cdot)$ is decreasing. We conclude that $[I(z_1) - I(\theta^*)]\phi(x) + c_1$ is non-negative for all $x > z_1$, and hence, the quasi-variational inequality is satisfied.

Lastly, note that we already proved above that $\hat{V}_1(x) > \hat{V}_1(x)$ for $x \geq \theta^*$ and we have $\hat{U}_1(x) = \hat{V}_1(x)$ for $x < \theta^*$ by construction. Furthermore, the inequality $\hat{U}_1(z_2) - \hat{V}_1(\theta^*) > 0$ must be satisfied because otherwise $q$ in (B.25) cannot be a positive quantity. Hence, the requisite inequality $\hat{U}_1(z_2) \geq \hat{V}_1(\theta^*)$ is satisfied. We conclude that $\hat{U}_1(\cdot)$ satisfies all the conditions of $F_1(\cdot)$. \hfill \Box

Now we proceed to prove the theorem. It suffices to prove that there exist $z_1$ and $q$ such that $\theta^* < \hat{x} < z_1 < z_2$ which satisfy (i) (B.21) and (B.22), and (ii) (B.23) if $c_1$ is sufficiently close to $c_2$.

(i) We prove that given $\theta^*$ and $z_2$ satisfying (35) and (36), there exist $z_1$ and $q$ which satisfy (B.21) and (B.22). Recall that $J(\cdot)$ is globally maximized at $\hat{x}$. If $c_1$ is sufficiently close to $c_2$ so that $J(\hat{x}) - J(\theta^*) \geq c_1$, then there exists a unique value of $z_1 \in (\hat{x}, z_2)$ which satisfies $J(z_1) = J(\theta^*) + c_1$ because $J(\cdot)$ is monotone in the interval $(\hat{x}, b)$. (If $c_1 > J(\hat{x}) - J(\theta^*)$, then there is no such $z_1$ which satisfies $J(z_1) = J(\theta^*) + c_1$).

Next, from the functional forms $\hat{U}_1(\cdot)$ and $\hat{V}_1(\cdot)$ defined in the proof of Lemma B.8, (B.22) is equivalent to the relation $q = [\hat{U}(\theta^*) - \hat{V}_1(\theta^*)]/[\hat{U}_1(z_2) - \hat{V}_1(\theta^*)]$ from (B.25). We only need to prove that this quantity $q$ is within the interval $(0, 1)$. First, because $-I(\theta^*) < -I(z_1)$, we have $\hat{U}_1(x) > \hat{V}_1(x)$ for all $x \geq \theta^*$, which implies that $\hat{U}_1(\theta^*) - \hat{V}_1(\theta^*) > 0$. Furthermore, from (B.21), for $c_1$ sufficiently close to $c_2$, $z_1$ can be made sufficiently close to $z_2$ in which case $\hat{U}_1(x) = -I(z_1)\phi(x) + (R, \pi)(x)$ and $\hat{V}_1(x) = -I(z_2)\phi(x) + (R, \pi)(x)$ can be made arbitrarily close to each other in the interval $(\theta^*, z_2)$. More specifically, $\hat{U}_1(x)$ uniformly converges to $\hat{V}_1(x)$ in the compact interval $[\theta^*, z_2]$ in the limit $c_1 \downarrow c_2$. Since $\hat{V}_1(\cdot)$ is an increasing function in the interval $(\theta^*, z_2)$, we have $\hat{V}_1(z_2) > \hat{V}_1(\theta^*)$, which implies that $\hat{U}_1(z_2) > \hat{U}_1(\theta^*)$ for $c_1$ sufficiently close to $c_2$. Because $\hat{U}_1(z_2) > \hat{U}_1(\theta^*) > \hat{V}_1(\theta^*)$, we conclude that $q \in (0, 1)$ for $c_1$ sufficiently close to $c_2$.

(ii) Next, we prove that (B.23) holds if $c_1$ is sufficiently close to $c_2$. From the functional form of $\hat{U}_1(\cdot)$ in the proof of Lemma B.8, the inequality $\hat{U}_1(\theta^*) - k\theta^* < \hat{U}_1(z_1) - kz_1$ is equivalent to (B.23). We already established in (i) above that $z_1 \rightarrow z_2$ in the limit $c_1 \rightarrow c_2$. Furthermore, since $\hat{U}_1(x)$ uniformly converges to $\hat{V}_1(x)$ in the compact interval $[\theta^*, z_2]$ in the limit $c_1 \rightarrow c_2$, we arrive at $\hat{U}_1(\theta^*) - k\theta^* < \hat{U}_1(z_1) - kz_1$ from the
inequality $\hat{V}_1(\theta^*) - k\theta^* < \hat{V}_1(z_2) - kz_2$.

**Proof of Proposition 6:** From (33) and (34), we have

$$I(x) = \frac{1}{\gamma_-} \left( \frac{\alpha x^{\alpha - \gamma_-}}{r - \delta(\alpha)} - kx^{1 - \gamma_-} \right)$$

$$J(x) = (1 - \frac{\alpha}{\gamma_-}) \frac{x^\alpha}{r - \delta(\alpha)} - kx(1 - \frac{1}{\gamma_-}).$$

It follows that we have $I(0) = J(0)$ and $J(z^*) > 0$. In addition, by the definition of $z^*$, we have $I(x) < 0$ for $x < z^*$ and $I(x) > 0$ for $x > z^*$.

By Lemma 2, $I(\cdot)$ and $J(\cdot)$ are strictly monotone functions in each of the intervals $(0, \hat{x})$ and $(\hat{x}, z^*)$. Then we can define bijective continuous functions $I_1 : (0, \hat{x}) \mapsto (I(\hat{x}), 0)$ and $I_2 : (\hat{x}, z^*) \mapsto (I(\hat{x}), 0)$ such that $I_1(x) = I(x)$ for $x \in (0, \hat{x})$ and $I_2(x) = I(x)$ for $(\hat{x}, z^*)$. We now define another continuous function $Z(x) := I_2^{-1}(I_1(x))$ which maps $(0, \hat{x})$ to $(\hat{x}, z^*)$, where $I_2^{-1}$ is the inverse function of $I_2$. Thus defined, $x$ and $Z(x)$ satisfies $I(x) = I(Z(x))$ where $x \in (0, \hat{x})$ and $Z(x) \in (\hat{x}, z^*)$.

Next, we define yet another continuous function $C(x) := J(Z(x)) - J(x)$. Since $\hat{x}$ is the point of the global minimum of $I(\cdot)$, $\lim_{x \to \hat{x}} Z(x) = \hat{x}$. Hence, $\lim_{x \to \hat{x}} C(x) = 0$. Furthermore, $\lim_{x \to 0} Z(x) = z^*$, so $\lim_{x \to 0} C(x) = J(z^*) - J(0) = J(z^*)$. Since $C(\cdot)$ continuously varies from 0 to $J(z^*)$ as $x$ varies from $\hat{x}$ to 0, there is a value $\theta^* \in (0, \hat{x})$ that satisfies $C(\theta^*) = c_2$ as long as $c_2 \in (0, J(z^*))$. Thus, if we set $z_2 = Z(\theta^*)$, Assumption 5 is satisfied.

**Proof of Lemma 3:** We only need to verify that the function $V_s(x; c)$ satisfies the conditions of Lemma 2.1 from Alvarez and Lempa (2008). Note that $V_s(x; c)$ coincides with the function $\hat{V}_2(x)$ defined in the proof of Lemma B.8 if we identify $c = c_2$ and $z = z_2$. In the proof of Lemma B.8, it is established that $\hat{V}_2(\cdot)$ satisfies all the conditions of Lemma 2.1 from Alvarez and Lempa (2008). In particular, we proved that (i) $(\hat{V}_2 - (R, \pi))(x) > 0$ (non-negativity), (ii) $\mathcal{A}(\hat{V}_2 - (R, \pi))(x) = \mathcal{A}\hat{V}_2(x) + \pi(x) \leq 0$ for all $x \neq \theta^*$ (r-superharmonicity for $X$), and (iii) $\hat{V}_2(x) \geq \sup_{\xi \geq 0} [\hat{V}_2(x + \xi) - k\xi - c_2]$ (quasi-variational inequality). Finally, from its functional form, it is evident that $V_s(x; c)$ is the payoff from the stated policy of a single player.

**Proof of Proposition 7:** (i) We first construct a pure strategy MPE. We let $v^P_1$ be player 1’s strategy of no investment at all and $v^P_2$ be player 2’s strategy of investment at $\tau_{v^P} = \inf\{t \geq 0 : X^v_t \leq \theta^*\}$ to boost $X^v$ up to $z_2$. We only need to prove that the two players’ strategies are best responses to each other. Since the strategy profile $v^P$ is Markov, we can let $V^v_i(x)$ denote the payoff function to players $i$ with the current state variable $x$.

First, we prove that $v^P_1$ is the best response to $v^P_2$. Note that $\mathcal{A}V^v_1(x) + \pi(x) = 0$ for $x > \theta^*$ and $V^v_1(x) = \infty$ for $x \leq \theta^*$.
$V^\nu_1(x)$ for $x \leq \theta^*$ with the boundary condition $V^\nu_1(\theta^*) = V^\nu_1(z_2)$ from player 2’s strategy of boosting $X^\nu$ up to $z_2$ at the hitting time of $\theta^*$. Based on the differential equation and the boundary condition, we immediately obtain the following:

$$V^\nu_1(x) = \beta \phi(x) + (R, \pi)(x) \text{ for } x \geq \theta^*$$

$$= V^\nu_1(z_2) \text{ for } x < \theta^*$$,

where $\beta$ is given in (38).

Next, we prove the variational inequality $V^\nu_1(x) \geq \sup_{z \geq 1} \{V^\nu_1(z) - k(z - x) - c_1\}$. We note that this inequality is always satisfied if $(V^\nu_1)'(x) - k < 0$ for all $x > \theta^*$. From the definition of $I(\cdot)$, we find that the inequality $(V^\nu_1)'(x) - k < 0$ can be re-expressed as $\beta > -I(x)$ for all $x > \theta^*$. This inequality is always satisfied if $\beta > -I(\hat{x})$ because $\hat{x}$ is the global minimizer of $I(\cdot)$. Thus, (38) is the sufficient condition for the variational inequality.

In sum, $V^\nu_1(x)$ satisfies the variational inequality and the differential equation $\mathcal{A}V^\nu_1(x) + \pi(x) = 0$ for $x > \theta^*$ as well as the inequality $V^\nu_1(x) > (R, \pi)(x)$ because $\beta > 0$. Then we can employ the same line of argument used to prove Lemma B.6 (also similarly to Lemma 2.1 of Alvarez and Lempa, 2008) and conclude that $V^\nu_1(x)$ dominates player 1’s payoff from any arbitrary strategy against $v^\nu_2$. Therefore, $v^\nu_1$ is the best response to $v^\nu_2$.

Next, we show that $v^\nu_2$ is the best response to $v^\nu_1$. If player 1 never invests, then the investment decision for player 2 reduces to that of a single player. By Lemma 3, $v^\nu_2$ is the optimal policy for a single player with cost $c_2$. Therefore, $v^\nu_2$ is the best response to $v^\nu_1$.

(ii) Let $v^*$ denote the mixed strategy SPE obtained in Theorem 3. We show that $v^\nu$ is more efficient than $v^*$. We first note that $V^\nu_2(x) = V^\nu_2(x)$ for all $x$, so it remains to show that $V^\nu_1(x) > F_1(x)$ for all $x$ and $t$, where $F_1(\cdot)$ is given by (23).

First, we consider $F_1(x) = V_1(x)$ for $t \geq \tau^m_0$. Recall that $V_1(x) = V_2(x) = V_3(x; c_2) = -I(\theta^*)\phi(x) + (R, \pi)(x)$ for $x \geq \theta^*$. In (i), we already established that $V^\nu_1(x) > V_3(x; c_2)$ because $\beta > -I(\hat{x})$, so it follows that $V^\nu_1(x) > F_1(x; t)$ if $t > \tau^m_0$.

Next, we consider $F_1(x) = U_1(x)$ for $t < \tau^m_0$. Because $V_1(x) = V_3(x; c_2)$ and $U_1(\theta^*) = qU_1(z_2) + (1 - q)V_1(\theta^*)$, we can construct $U_1(\cdot)$ as a payoff to a single player with a policy characterized by $\theta^*$ and $z_2$ with a dynamic upfront cost. To be specific, his upfront cost is initially zero, but it permanently increases to $c_2$ with a probability of $1 - q$ at each investment event; once it increases to $c_2$, the cost does not change anymore. In
comparison, \( V_1^{\nu'}(\cdot) \) can be obtained as a payoff associated with a policy of \( \theta^* \) and \( z_2 \) but with zero upfront cost. Therefore, \( V_1^{\nu'}(x) > U_1(x) \).

References

Alvarez, L. H. R. 2001. Reward functionals, salvage values and optimal stopping. *Mathematical Methods of Operations Research* **54**(2) 315–337.

Alvarez, L. H. R., J. Lempa. 2008. On the optimal stochastic impulse control of linear diffusions. *SIAM Journal on Control and Optimization* **47**(2) 703–732.

Borodin, A., P. Salminen. 1996. *Handbook of Brownian motion - Facts and Formulae*. Birkhauser, Basel.

Folland, G. B. 1999. *Real Analysis*. Wiley, New York.

Karatzas, I., S. E. Shreve. 1998. *Brownian Motion and Stochastic Calculus*. 2nd ed. Springer, New York.

Lon, P. C., M. Zervos. 2011. A model for optimally advertising and launching a product. *Mathematics of Operations Research* **36**(2) 363–376.

Øksendal, B. 2003. *Stochastic Differential Equations: An Introduction with Applications*. 6th ed. Springer, Berlin.

Rogers, L. C. G., D. Williams. 2000. *Diffusions, Markov Processes and Martingales*, vol. 1. 2nd ed. Cambridge University Press, Cambridge, UK.