Calibration of local volatility model with stochastic interest rates by efficient numerical PDE methods

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Abstract
Long-maturity options or a wide class of hybrid products are evaluated using a local volatility-type modelling for the asset price \( S(t) \) with a stochastic interest rate \( r(t) \). The calibration of the local volatility function is challenging and time-consuming because of the multi-dimensional nature of the problem. A key requirement of any equity hybrid derivatives pricing model is the ability to rapidly and accurately calibrate to vanilla option prices. In this paper, we develop a calibration technique based on a partial differential equation (PDE) approach which allows an accurate calibration and provides an efficient implementation algorithm. The essential idea is based on solving the derived forward equation satisfied by \( P(t, S, r)Z(t, S, r) \), where \( P(t, S, r) \) represents the risk-neutral probability density of \( (S(t), r(t)) \) and \( Z(t, S, r) \) the projection of the stochastic discounting factor in the state variables \( (S(t), r(t)) \). The solution provides effective and sufficient information for the calibration and pricing. The PDE solver is constructed by using ADI (alternative direction implicit) method based on an extension of the Peaceman–Rachford scheme. Furthermore, an efficient algorithm to compute all the corrective terms in the local volatility function due to the stochastic interest rates is proposed by using the PDE solutions and grid points. It reduces by one order the computations costs and then allows to speed up significantly the calibration procedure. Different numerical experiments are examined and compared to demonstrate the results of our theoretical analysis.

Keywords Local volatility model · Stochastic interest rates · Hybrid · Calibration · Forward Fokker–Planck-type equation · Alternating direction implicit (ADI) method

JEL Classification C02

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1 Introduction

In quantitative finance, the local volatility-type model as introduced in Dupire (1994), Derman and Kani (1998), Rubinstein (1994) is widely used to model the price of underlying in order to capture the market volatility skew or smile in equity or foreign exchange market. It is known that with deterministic rates, the local volatility function can be obtained with the Dupire formula (see Eq. (13)) by using the European call and put option prices.

For long-maturity options (e.g. pure equity autocall derivative) or some hybrid products with a pay-off involving interest rate and underlying asset like a best-of interest rate-equity which pays coupons of the form

$$\max \left[ \text{LIBOR}, a \left( \frac{S_t}{S_0} - 1 \right) \right],$$

where the interest rates are potentially needed to be modelled as stochastic. It is then natural to extend the local volatility model to incorporate stochastic interest rates. This modelling framework is widely used in the financial industry (see, e.g. Benhamou et al. 2012; Clark 2011; Atlan 2006; Overhaus et al. 2007). To perfectly match the market implied volatility, the local volatility-type formula can be derived and is given by Eq. (12). We observe on top of the Dupire local volatility function, there is an additional corrective term taking into account the covariance between the equity price and short rate. Unfortunately, this extension of the Dupire formula is not easily applicable for calibration over the market since there seems no immediate way to link the expectation term with the European option prices or other liquid products.

The main challenge for the implementation of the model consists in the calibration (see also discussion in Piterbarg 2006). First and foremost, it corresponds to a two-factor model. Also, formula (12) requires a correction term on top of Dupire expression for each strike point \(K\) per maturity in the definition of the local volatility function. Accuracy, robustness in the calibration and pricing associated with an efficient implementation are required for execution in real time. Indeed, the model will be used not only for the pricing but also to compute all sensitivity factors for hedging purpose (e.g. delta, gamma and vega).

Many research results appeared in the last decade in different areas of quantitative finance on the local volatility model with stochastic rates. A brief introduction is given as follows:

- Theoretical results about the local volatility function and its calibration were exposed and discussed in Deelstra and Rayee (2012), Atlan (2006), Ren et al. (2007).
- For the option pricing, in Benhamou et al. (2012), Gobet and Hok (2014), the authors developed expansion formulas by applying the perturbation method using a proxy introduced by Benhamou et al. (2009). A pricing framework via partial differential equation (PDE) approach was studied in Dang et al. (2010), and the Crank–Nicolson scheme and also the alternating direction implicit (ADI) method were applied to build the PDE solver.
In terms of model calibration, for pricing the power reverse dual currency (PRDC) derivatives, Piterbarg in Piterbarg (2006) modelled the local volatility function for the forward foreign exchange rate using the constant elasticity variance (CEV) dynamic as a parametric form. A fast calibration procedure based on the so-called Markovian projection method was developed, and the skew averaging technique was discussed in Piterbarg (2005a, b). The authors in Benhamou et al. (2008) used Malliavin calculus, and this method relies on an approximation for the covariance between the short rate and the logarithmic stock price to derive an equation in the local volatility function, and it is then solved iteratively by using a fixed-point method. More recently in Joshi and Ranasinghe (2016), the authors use a mixture models approach to correlate the joint distribution between the stock price and the short rate. It allows then to derive an analytical approximation formula for the local volatility function. Finally, calibration by Monte Carlo approach using McKean’s particle method was studied in Guyon and Labordere (2011).

For models calibration in quantitative finance, different numerical solvers were constructed to solve similar types of high-dimensional PDEs. For example, an idea of using operator splitting method was provided by Ren et al. (2007) for the calibration of local volatility with stochastic volatility. A Heston-like term-structure model in FX market was examined by Tian et al. (2015) with a modified Douglas scheme and Maarten and in’t Hout (2017) with a modified Craig–Sneyd scheme. These ADI-type schemes were discussed in detail for dealing with convection–diffusion equations in terms of the stability and second-order convergence in [24].

There are certain limitations for these described calibration methodologies. In Benhamou et al. (2012), Gobet and Hok (2014), unlike in the traditional model, volatility is assumed to be the function of the stock price divided by the bank account. The main limitations of the approach are that it only works for Gaussian interest rates and that vanilla option prices must be calculated using complex expansion formulas. In Piterbarg (2006), the CEV parametric form for the local volatility function is able to capture the slope of the implied volatility surface, but cannot fit the smile. In Benhamou et al. (2008), Joshi and Ranasinghe (2016), their fitting method relies on approximation formulas, which can result in loss of accuracy in the calibration as discussed in Guyon and Labordere (2011), Joshi and Ranasinghe (2016). Finally, calibrating a model by Monte Carlo method as in Guyon and Labordere (2011) is intensive in computations and time-consuming.

To overcome the limitations above, here, our contribution is to provide a simple, accurate and efficient calibration procedure based on PDE approach which is able to be applied for a more generic model. More precisely, in this paper, we assume a Markovian setting with a stochastic differential equation (SDE) given by (2), i.e. the underlying $S$ follows a local volatility-type diffusion and the short rate $r(t)$ is represented by a general form dynamic. This model covers as a particular case, a local volatility diffusion for $S$ associated with a Gaussian Hull–White dynamic for $r$ (see Hull and White 1993), which is widely used for pricing hybrid products (see, e.g. Benhamou et al. 2012; Gobet and Hok 2014; Overhaus et al. 2007). Here, we focus on the model calibration which corresponds to the challenging part of the model implementation. The idea starts by introducing the projection of the stochastic discounting factor.
in the state variable \((S(t), r(t))\), quantity defined by the functional \(Z(t, S(t), r(t))\) in Eq. (18). Using a martingale argument, we derive the forward partial differential equation satisfied by the product \(P(t, S, r)Z(t, S, r)\), where \(P(t, S, r)\) is the probability density of \((S(t), r(t))\) (see our main result in Proposition 2). Once this PDE is solved, the adjustment due to stochastic rate effects to the Dupire formula can be evaluated in Eq. 12. For getting the numerical PDE solution efficiently, a simple and intuitive ADI scheme based on an extension of the Peaceman–Rachford scheme in Peaceman and Jr Rachford (1955) is proposed which leads to solve the discretized linear system with a reasonable matrix size. Moreover, we suggest an efficient algorithm to compute all the corrective terms sequentially in strike per maturity using the solutions and points in the PDE grid. This method reduces by one order the computations costs and then allows to speed up significantly the calibration procedure (see Sect. 4.3). Generally in the two-dimensional case, we believe our methodology is able to make PDE approach very efficient w.r.t. Monte Carlo method (see discussion in Guyon and Labordere 2011) and provides the required calibration accuracy in realistic or stressed market conditions, compared to approximations formulas proposed in Benhamou et al. (2008), Joshi and Ranasinghe (2016).

It should be mentioned that our approach is comparable to the one proposed in Overhaus et al. (2007). The authors derived a forward PDE for \(\phi(t, S, r)\), which represents the distribution of the state variable \((S(t), r(t))\) under the forward measure \(Q^f\) [see equation (8.10) in Overhaus et al. 2007], and also provided the method to calibrate the local volatility function by solving a minimization problem. Their method of derivation is different, but the results are consistent. Indeed, a simple computation shows that our quantity of interest \(P(t, S, r)Z(t, S, r)\) is equal to \(ZC(0, t)\phi(t, S, r)\) where \(ZC(0, t)\) represents the zero coupon price with maturity \(t\). Making this link provides an interpretation to \(P(t, S, r)Z(t, S, r)\), which can be read as the discounted probability distribution of the state variable at time \(t\) under \(Q^f\).

The paper is organized as follows: Sect. 2 describes the hybrid equity-interest rates modelling, and the derivation of the local volatility function is explained. The calibration framework is described in Sect. 3 to obtain the forward equation satisfied by \(P(t, S, r)Z(t, S, r)\). In Sect. 4, we construct an ADI-type method extending Peaceman–Rachford scheme for its resolution and then provide a numerical procedure to compute the corrective terms. Section 5 is devoted to the numerical tests. Finally, the conclusions and discussions are given in Sect. 6.

## 2 Hybrid equity-interest rates model

Let’s consider the 2-d stochastic differential equations, describing the spot price \(S(t)\) and short rates \(r(t)\) under the risk-neutral probability \(Q\), defined by

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= r(t)dt + \sigma(t, S(t))dW^1(t), \quad S(0) = S_0, \\
\,dr(t) &= \mu(t, r(t))dt + \alpha(t, r(t))(\rho dW^1(t) + \sqrt{1 - \rho^2}dW^2(t)), \quad r(0) = r_0,
\end{align*}
\]

(2)
where \((W(t))_{t \geq 0}\) is a standard Brownian motion in \(\mathbb{R}^2\) on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})\) with the usual assumptions on the filtration \((\mathcal{F}_t)_{t \geq 0}\). We assume the existence and uniqueness of the solution for (2) (see, e.g. Theorem 3.5.5 in Lamberton and Lapeyre 2012).

This model corresponds to an extension of the Dupire local volatility model (e.g. Dupire 1994; Derman and Kani 1998; Rubinstein 1994) which allows stochastic interest rates. The local volatility function \(\sigma(t, S)\) allows the model to calibrate to the surface of European call price \(C(T, K)\) where \(K\) represents the strike and \(T\) is the maturity. For the ease of reading and the manuscript to be self-contained, we provide its derivation by following the methodology in Musiela and Rutkowski (2005). Let’s note by

\[
Z(t) := e^{-\int_0^t r(u)du},
\]

and applying Tanaka’s formula to the convex but non-differentiable function \(Z(t)(S(t) - K)_{+}\) leads to

\[
Z(t)(S(t) - K)_+ = (S(0) - K)_+ - \int_0^T r(u)Z(u)(S(u) - K)_+ du
\]

\[
+ \int_0^T Z(u)1_{S(u) \geq K} dS(u) + \frac{1}{2} \int_0^T Z(u)dL^K_u(S),
\]

where \(L^K_u(S)\) is the local time of \(S\), and \((S(0) - K)_+ = \max(S(0) - K, 0)\). Since \(S\) is a continuous semimartingale, then almost surely (see, e.g. Revuz and Yor 2001)

\[
L^K(t)(S) = \lim_{\epsilon \searrow 0} \int_0^t \frac{1}{\epsilon} 1_{[K, K+\epsilon]}(S(u))d[S,S]_u.
\]

Using (2), it comes

\[
Z(t)(S(t) - K)_+ = (S(0) - K)_+ - \int_0^T r(u)Z(u)(S(u) - K)_+ du
\]

\[
+ \int_0^T Z(u)1_{S(u) \geq K} r(u) S(u) du
\]

\[
+ \int_0^T Z(u)1_{S(u) \geq K} S(u) \sigma(u, S(u)) dW^1_u + \frac{1}{2} \int_0^T Z(u)dL^K_u(S)
\]

\[
= (S(0) - K)_+ + \int_0^T KZ(u)1_{S(u) \geq K} r(u) du
\]

\[
+ \int_0^T Z(u)1_{S(u) \geq K} S(u) \sigma(u, S(u)) dW^1_u
\]

\[
+ \frac{1}{2} \int_0^T Z(u)dL^K_u(S)
\]

\[
= (S(0) - K)_+ + \int_0^T KZ(u)1_{S(u) \geq K} (r(u) - f(0, u)) du
\]

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\begin{align*}
&+ \int_0^T K Z(u) 1_{S(u) \geq K} f(0, u) du \\
&+ \int_0^T Z(u) 1_{S(u) \geq K} S(u) \sigma(u, S(u)) dW_u^1 + \frac{1}{2} \int_0^T Z(u) dL_u^K(S),
\end{align*}
\tag{7}

where \( f(0, u) = -\frac{\partial}{\partial u} \log(ZC(0, u)) \) means the forward rate at time 0 for investing at time \( u \) and \( ZC(t, T) \) is the zero coupon price at \( t \) for maturity \( T \). We introduce the forward rate in the last equality in order to compare below the expression of the local volatility when interest rate is stochastic, \( \sigma^2(T, K) \), and when interest rate is deterministic \( \sigma_{Dup}^2(T, K) \).

Assuming that the function \( Z(u) 1_{S(u) \geq K} S(u) \sigma(u, S(u)) \) is a member of the class \( \mathcal{V} \) (see def 3.1.4 in Oksendal 2003), namely the measurable and adapted functions \( f \) s.t. \( E\left[ \int_0^t f^2(s) ds \right] < \infty \), and by taking the expectation

\[
C(T, K) = C(0, K) + K \int_0^T E[Z(u) 1_{S(u) \geq K}(r(u) - f(0, u))] du \\
+ K \int_0^T E[Z(u) 1_{S(u) \geq K} f(0, u)] du \\
+ \frac{1}{2} K^2 \int_0^T E[Z(u) \delta(S(u) - K)\sigma^2(u, K)],
\tag{8}
\]

with \( \delta(.) \) the delta function at 0. Differentiating w.r.t. \( T \) leads to

\[
C_T(T, K) = KE[Z(T) 1_{S(T) \geq K}(r(T) - f(0, T))] + KE[Z(T) 1_{S(T) \geq K} f(0, T)] \\
+ \frac{1}{2} K^2 E[Z(T) \delta(S(T) - K)\sigma^2(T, K)].
\tag{9}
\]

Standard computations give

\[
E[Z(T) 1_{S(T) \geq K}] = -\frac{\partial C(T, K)}{\partial K},
\tag{10}
\]

\[
E[Z(T) \delta(S(T) - K)] = \frac{\partial^2 C(T, K)}{\partial K^2}.
\tag{11}
\]

Using the last two equations in (9), we obtain the expression of the local volatility \( \sigma^2(T, K) \) in terms of call prices \( C(T, K) \)

\[
\sigma^2(T, K) = \sigma_{Dup}^2(T, K) - \frac{E[Z(T)(r(T) - f(0, T)) 1_{S(T) > K}]}{\frac{1}{2} K^2 \frac{\partial^2 C(T, K)}{\partial K^2}},
\tag{12}
\]
with

\[
\sigma_{\text{Dup}}^2(T, K) = \frac{\frac{\partial C(T, K)}{\partial T} + K f(0, T) \frac{\partial C(T, K)}{\partial K}}{\frac{1}{2} K^2 \frac{\partial^2 C(T, K)}{\partial K^2}}.
\] (13)

\(\sigma_{\text{Dup}}(T, K)\) represents the Dupire local volatility function when interest rates are deterministic. Equation (12) shows the corrections to employ on the tractable Dupire local volatility surface in order to obtain the local volatility surface which takes into account the effect of stochastic interest rates. This expression is equivalent to other representations used in the literature (see, e.g. Atlan 2006; Deelstra and Rayee 2012; Ren et al. 2007).

\[
E[Z(T)(r(T) - f(0, T))1_{S(T) > K}]\]

represents the numerator of the extra term in the local volatility expression. No closed-form solution exists, and it is not directly related to European call prices or other liquid products. Its calculations need to be estimated. For sanity check, when interest rate becomes deterministic, we have \(r(T) = f(0, T)\) (see Eq. (16)) and \(\sigma^2(T, K)\) reduces to \(\sigma_{\text{Dup}}^2(T, K)\) in Eq. (12).

**Remark 1** Under \(T\)–forward measure \(\mathbb{Q}^T\), the extra term can be written as

\[
E[Z(T)(r(T) - f(0, T))1_{S(T) > K}] = ZC(0, T)E^T[(r(T) - f(0, T))1_{S(T) > K}].
\] (14)

In the Heath–Jarrow–Morton (HJM) framework under \(\mathbb{Q}^T\) (see, e.g. Brigo and Mercurio 2006), the forward rate \(f(t, T)\) is a martingale, with a vector of volatility \(\sigma(s, T)\), which can be written as

\[
f(t, T) = f(0, T) + \int_0^t \sigma(s, T) \cdot dW^T_s.
\] (15)

Using the fact that \(r(T) = f(T, T)\), it becomes

\[
r(T) = f(0, T) + \int_0^T \sigma(s, T) \cdot dW^T_s,
\] (16)

and with (16), the expression of the extra term in (14) shows clearly the impact of stochastic rates.

By assuming \(\int_0^T \sigma^2(s, T)ds < +\infty, \int_0^T \sigma(s, T) \cdot dW^T_s\) is a \(\mathbb{Q}^T\) martingale and

\[
E^T[(r(T) - f(0, T))1_{S(T) > K}] = \text{Cov}^T[(r(T) - f(0, T)), 1_{S(T) > K}],
\] (17)

which provides an interpretation to the corrective term as a covariance, under \(\mathbb{Q}^T\), between \(r(T) - f(0, T)\) and \(1_{S(T) > K}\).
3 Calibration

Before using a model to price any derivatives, we usually calibrate it on the vanilla market which means that it is able to price vanilla options with the concerned model and the resulting implied volatilities match the market-quoted ones. More precisely, it is necessary to determine all parameters presenting in the different stochastic processes which define the model. In such a way, all the European option prices derived in the model are as consistent as possible with the corresponding market ones.

The calibration procedure for the two-factor model with local volatility can be decomposed into three steps:

– Parameters present in the one-factor dynamic for the interest rates, $\mu(t, r(t))$ and $\alpha(t, r(t))$, are chosen to match European swaptions and caps, floors values. Methods for doing so are well developed in the literature (see, e.g. Brigo and Mercurio 2006).

– The correlation parameter $\rho$ is typically chosen either by historical estimation or from occasionally observed prices of hybrid product involving interest rate and the underlying spot (see discussion in Gobet and Hok 2014).

– After these two steps, the calibration problem consists in finding the local volatility function $\sigma(t, S(t))$ which is consistent with its associated implied volatility surface.

Here we focus on the third step of the calibration and propose to use some martingale properties of the model to allow an efficient implementation.

Let’s introduce the projection of the discount factor on the state variables $(S(t), r(t))$, defined as

$$Z(t) := E[Z(t)|S(t), r(t)] = Z(t, S(t), r(t)), \quad (18)$$

with $Z(t)$ given in (3) and assume that $Z \in C^{1,2}$ on $[0, T] \times \mathbb{R}^2$.

Our objective is to determine the product functional $P(t, S, r)Z(t, S, r)$ where $P(t, S, r)$ represents the joint distribution of $(S(t), r(t))$. Indeed, we write

$$E[e^{-\int_0^T r(s)ds} (r(T) - f(0, T))1_{S(T) > K}]$$

$$= E[E[e^{-\int_0^T r(s)ds} (r(T) - f(0, T))1_{S(T) > K} / S(T), r(T)]] \quad (19)$$

$$= E[Z(T, S(T), r(T))(r(T) - f(0, T))1_{S(T) > K}] \quad (20)$$

$$= \int (r - f(0, T)) 1_{S > K} (PZ)(T, S, r) dS dr. \quad (21)$$

We can then compute the corrective term (12) at least numerically. As $(S(t), r(t))$ and $(S(t), r(T))$ are the model state variables, $P(t, S, r)Z(t, S, r)$ can also be used for option pricing.

For any fix $T > 0$ and $h(S, r)$ a Borel-measurable function, let’s define the function

$$f(t, S, r) = E^{t,S,r} [e^{-\int_t^T r(s)ds} h(S(T), r(T))], \quad (22)$$

where we assume $E^{t,S,r} |h(S(T), r(T))| < +\infty$ for all $t, S, r$. 

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Using martingale argument as for the discounted Feynman–Kac theorem (see Theorem 6.4.3 in Shreve 2004), we derive the partial differential equation satisfied by \( f(\tau, S, r) \):

\[
\frac{\partial}{\partial \tau} f(t, S, r) + r S f_S + \mu f_r + \frac{1}{2} S^2 \sigma^2 f_{ss} + \frac{1}{2} \alpha^2 f_{rr} + \rho \sigma \alpha S f_{sr} - rf = 0, \tag{23}
\]

with the terminal condition

\[
f(T, S, r) = h(S, r) \quad \forall (S, r). \tag{24}
\]

In the risk-neutral pricing framework, we have the following result

**Proposition 1** For \( Z(t) \) and \( f \) defined, respectively, as in (18) and in (22), \( \tilde{Z}(t) f(t, S(t), r(t)) \) is a martingale.

**Proof** Using the martingality of \( Z(t) f(t, S(t), r(t)) \) and the Markov property of solutions in (2), for \( t \geq s \), we write

\[
E[Z(t) f(t, S(t), r(t)) | \mathcal{F}_s] = Z(s) f(s, S(s), r(s)), \quad (25)
\]

\[
E[E[Z(t) f(t, S(t), r(t)) | \mathcal{F}_t] | \mathcal{F}_s] = E[Z(s) f(s, S(s), r(s)) | \mathcal{F}_s], \quad (26)
\]

\[
E[E[Z(t) f(t, S(t), r(t)) | S(t), r(t)] | \mathcal{F}_s] = E[Z(s) f(s, S(s), r(s)) | S(s), r(s)], \quad (27)
\]

\[
E[Z(t) f(t, S(t), r(t)) | \mathcal{F}_s] = Z_s f(s, S(s), r(s)). \quad (28)
\]

Since the martingale \( Z(t, S(t), r(t)) f(t, S(t), r(t)) \) is an Itô process, it must have zero drift. Calculating the drift term using Itô formula and setting it to zero give the following corollary:

**Corollary 1** \( Z(t, S, r) f(t, S, r) \) is a solution of the following partial differential equation

\[
(Z f)_t + r S (Z f)_S + \mu (Z f)_r + \frac{1}{2} S^2 \sigma^2 (Z f)_{SS} + \frac{1}{2} \alpha^2 (Z f)_{RR} + \rho \sigma \alpha S (Z f)_{SR} = 0. \tag{29}
\]

Next proposition provides the forward equation satisfied by \( P(t, S, r) Z(t, S, r) \).

**Proposition 2** Assume the probability density \( P(t, S, r) \) and its derivatives decay fast enough to 0 for large \( |(S, r)| \) to preclude boundary terms, i.e. for \( |(S, r)| \rightarrow \infty \)

\[
\frac{\partial}{\partial S} (\sigma^2 S^2 P Z) = \frac{\partial}{\partial r} (\sigma^2 P Z) = \frac{\partial}{\partial S} (\sigma \alpha P Z) = \frac{\partial}{\partial r} (\sigma \alpha P Z) = 0. \tag{30}
\]
Then, $P Z$ satisfies the following forward equation with the Dirac delta function as the initial condition

\[
\begin{cases}
(P Z)_t + (r S P Z)_s + (\mu P Z)_r - \frac{1}{2} (s^2 \sigma^2 P Z)_{ss} - \frac{1}{2} (\alpha^2 P Z)_{rr} - \rho (\alpha \sigma S P Z)_{sr} + r (P Z) = 0, \\
(P Z)(0, S, r) = \delta(S - S_0, r - r_0).
\end{cases}
\] (31)

or

\[
\begin{cases}
(P Z)_t + [S(r - 2 \sigma^2 - \sigma \alpha r, - 2 \sigma \alpha S^2)] (P Z)_s + [\mu - 2 \sigma \alpha r - \alpha \rho (\sigma + S \sigma)] (P Z)_r \\
- \frac{1}{2} S^2 \sigma^2 (P Z)_{ss} - \frac{1}{2} \alpha^2 (P Z)_{rr} - \rho \alpha S (P Z)_{sr} \\
+ [2r + \mu - (\sigma^2 + 4 \sigma \alpha S + S^2 (\sigma^2 S + \sigma^2))] - (\alpha \sigma r + \alpha^2) - \rho \alpha (\sigma + S \sigma)] (P Z) = 0, \\
(P Z)(0, S, r) = \delta(S - S_0, r - r_0).
\end{cases}
\] (32)

**Proof** Let’s note by

\[
C^2_0(\mathbb{R}^2) = \{ h \in C^2(\mathbb{R}^2) \text{ with compact support} \}
\] (33)

and consider $f(t, S, r)$ defined as in (22) for $h \in C^2_0(\mathbb{R}^2)$.

Using the martingale property of $Z(t) f(t, S(t), r(t))$, we write for any $t \geq 0$

\[
f(0, S(0), r(0)) = E[Z(t) f(t, S(t), r(t))] = \int [P(t, S, r)Z(t, S, r)] f(t, S, r)dSdr.
\] (35)

By taking derivative w.r.t. $t$ in (35), using (23), performing integration by parts and using zero boundary conditions (30), we get

\[
\int f[(P Z)_t + (r S P Z)_s + (\mu P Z)_r - \frac{1}{2} (s^2 \sigma^2 P Z)_{ss} \\
- \frac{1}{2} (\alpha^2 P Z)_{rr} - \rho (\alpha \sigma S P Z)_{sr} + r (P Z)]dSdr = 0.
\] (36)

Note that when $t$ approaches $T$, the function $f(t, S, r)$ approaches $h(S, r)$ and the last equation becomes

\[
\int h[(P Z)_t + (r S P Z)_s + (\mu P Z)_r - \frac{1}{2} (s^2 \sigma^2 P Z)_{ss} \\
- \frac{1}{2} (\alpha^2 P Z)_{rr} - \rho (\alpha \sigma S P Z)_{sr} + r (P Z)]dSdr = 0.
\] (37)

Since $h \in C^2_0(\mathbb{R}^2)$ is arbitrary, we obtain the forward Eq. (31).

The Dirac delta initial condition means that at time $t = 0$ we are sure that the spot $S(0)$ equals $S_0$, the interest rate $r(0)$ equals $r_0$ and $Z(0) = 1$. □
Remark 2  We know (see, e.g. proposition 11.5 in Avellaneda and Laurence 2000) that $P(t, S, r)$ satisfies the forward Fokker–Planck equation given by

$$\begin{cases}
(P)_t + (rSP)_s + (\mu P)_r - \frac{1}{2}(s^2\sigma^2 P)_{ss} - \frac{1}{2}(\alpha^2 P)_{rr} - \rho(\alpha \sigma SP)_{sr} = 0, \\
(PZ)(0, S, r) = \delta(S - S_0, r - r_0).
\end{cases} \quad (38)$$

In comparison, the equation for $PZ$ in (31) has an additional reaction term. Also by definition, we have for all $t \geq 0$

$$\int P(t, S, r)dSdr = 1, \quad (39)$$

$$\int (PZ)(t, S, r)dSdr = E[Z(t, S(t), r(t))] = ZC(0, t). \quad (40)$$

4 ADI finite difference method

The targeted 2-d PDE to solve is (32) with a Dirac delta function as initial condition. Constructing fully implicit scheme for solving Eq. (32) may result in some difficulties due to the dimensionality as the discretized linear system becomes banded structure and requires efficient linear solvers. Besides if the employed grid numbers are important, the program initialization can cause problem as the size of matrix is too large. Therefore, we propose to use ADI (alternating direction implicit) finite difference method scheme which can resolve these issues. In the following, the details for temporal discretization in solving Eq. (32) are explained by using an extension of Peaceman–Rachford scheme.

4.1 Truncated domain, boundary and initial conditions

The spatial variables for $(S, r)$ are potentially defined in an unbounded domain typically $S \geq 0$ and $r \in \mathbb{R}$ (see Sect. 5 in numerical examples). In order to apply finite difference scheme, the computational domain is truncated as a finite interval $D := [S_{\text{min}}, S_{\text{max}}] \times [r_{\text{min}}, r_{\text{max}}]$ where $D$ must contain the most probable values of $(S(t), r(t))$ between $t = 0$ and the calibration horizon $t = T$. Thus, $S_{\text{min}}, S_{\text{max}}$ and $r_{\text{min}}, r_{\text{max}}$ should be, respectively, the extreme values of the asset price and the interest rate, and $D$ is chosen such that

$$\mathbb{Q}(\exists t \in [0, T], (S(t), r(t)) \notin D) < \epsilon, \quad (41)$$

for a small real number $\epsilon$. The truncated domain $D := [S_{\text{min}}, S_{\text{max}}] \times [r_{\text{min}}, r_{\text{max}}]$ used in this paper is given by:

$$r_{\text{max}} = r_0 + k_r \times \alpha \times \sqrt{T} \quad (42)$$

$$r_{\text{min}} = r_0 - k_r \times \alpha \times \sqrt{T} \quad (43)$$

$$y_{\text{max}} = \log(S_0) + k_s \times \sigma \times \sqrt{T} \quad (44)$$
where \( k_r \) and \( k_s \) represent the number of standard deviation and they are set to be 5. It corresponds to a standard approach. For further details, the reader can complement with the discussion in Windcliff et al. (2004).

The boundary conditions usually depend on the considered model, and we opt for Dirichlet boundary conditions. In this paper, the example provided in the numerical experiments, the interest rate \( r_t \) has Gaussian distribution for all \( t \) and \( \mathbb{Q}(S_t > 0, \forall t \geq 0) = 1 \). Then, we have \( P(t, S, r) \to 0 \) for \( |r| \to \infty \) and \( S \to 0 \) or \( S \to \infty \). The values for \( Z(t, S, r) \) are expected to be around 1 (see Fig. 1). Therefore reasonably, the boundary values for \( P \) are set to be 0 (see also discussions in Lamberton and Lapeyre (2012), Windcliff et al. (2004).

In order to calculate \( P Z(t, S, r) \) at initial time \( t = 0 \), we propose to use a Gaussian kernel to approximate the Dirac delta function, i.e. a family of Gaussian functions

\[
\gamma_N^r(S, r) = \frac{1}{2\pi \sqrt{\det \Sigma}} \exp \left\{ -\frac{1}{2} \left( \begin{array}{c} S - S_0 \\ r - r_0 \end{array} \right)' \Sigma^{-1} \left( \begin{array}{c} S - S_0 \\ r - r_0 \end{array} \right) \right\}
\]

parametrized by the parameter \( N > 0 \) with \( \Sigma = \left( \begin{array}{cc} \frac{1}{N} & 0 \\ 0 & \frac{1}{N} \end{array} \right) \) (see, e.g. Saichev and Woyczynski 2013). Using this approximation enhances the stability in solving the PDE in Eq. (32). Alternative approximation for Dirac delta function can be found in Haddou (2010).

### 4.2 ADI method

Let \( N_S \), \( N_r \) be the number of grid points for doing spatial discretizations on the direction \( S \) and \( r \), respectively, and \( N_t \) be the number of grid points employed on temporal discretization. The essential idea of Peaceman–Rachford scheme is to split the calcu-
lation in the time-marching scheme into several steps with respect to different spatial variables. More precisely, the scheme used for evaluating \((P\mathcal{Z})^{n+1}\) from the known value \((P\mathcal{Z})^n\) can be specified as the following two steps.

**ADI Step 1**
In the first step of ADI, the finite difference spatial discretization on \(S\) is treated implicitly and the rest terms are explicitly for time step from \(n\) to \(n + \frac{1}{2}\), namely

\[
\frac{(P\mathcal{Z})^{n+\frac{1}{2}}_{ij} - (P\mathcal{Z})_{ij}^{n}}{\Delta t/2} + C_1 \frac{(P\mathcal{Z})^{n+\frac{1}{2}}_{i+1,j} - (P\mathcal{Z})^{n+\frac{1}{2}}_{i-1,j}}{2\Delta S} + C_2 \frac{(P\mathcal{Z})_{i,j+1}^{n} - (P\mathcal{Z})_{i,j-1}^{n}}{2\Delta r}
\]

\[
+ C_3 \frac{(P\mathcal{Z})_{i+1,j}^{n+\frac{1}{2}} - 2(P\mathcal{Z})_{ij}^{n+\frac{1}{2}} + (P\mathcal{Z})_{i-1,j}^{n+\frac{1}{2}}}{(\Delta S)^2}
\]

\[
+ C_4 \frac{(P\mathcal{Z})_{i,j+1}^{n} - 2(P\mathcal{Z})_{ij}^{n} + (P\mathcal{Z})_{i,j-1}^{n}}{(\Delta r)^2}
\]

\[
+ C_5 \frac{(P\mathcal{Z})_{i+1,j+1}^{n} + (P\mathcal{Z})_{i-1,j-1}^{n} - (P\mathcal{Z})_{i-1,j+1}^{n} - (P\mathcal{Z})_{i+1,j-1}^{n}}{4\Delta S\Delta r}
\]

\[
+ C_6 (P\mathcal{Z})_{ij}^{n+\frac{1}{2}} = 0, \quad (49)
\]

where

\[
C_1 = (r_j S_i - 2S_i \sigma_i^2 - 2(S_i)^2 \sigma_i (\sigma_S)_i - \rho \sigma_i S_i (\alpha_r)_j),
\]

\[
C_2 = (\mu_j - \rho \sigma_i \alpha_j - \rho (\sigma_S)_i S_i \alpha_j - 2 \alpha_j (\alpha_r)_j),
\]

\[
C_3 = \left(\frac{-S_i^2 \sigma_i^2}{2}\right), \quad C_4 = \left(\frac{-\alpha_j^2}{2}\right), \quad C_5 = (-\rho \sigma_i S_i \alpha_j),
\]

\[
C_6 = (2r_j + (\mu)_j - \sigma_i^2 - 4S_i \sigma_i (\sigma_S)_i - ((\sigma_S)_i)^2 S_i^2 - \sigma_i (\sigma_{SS})_i S_i^2 - ((\alpha_r)_j)^2 - \alpha_j (\alpha_{rr})_j - \rho (\sigma_S)_i S_i (\alpha_r)_j - \rho \sigma_i (\alpha_r)_j),
\]

here \((P\mathcal{Z})_{ij}^{n}\) is the discretized notation of \((P\mathcal{Z})(n \Delta t, i \Delta S, j \Delta r)\) for \(i = 1, \ldots, N_S\), \(j = 1, \ldots, N_r\), and \(n + \frac{1}{2}\) is a dummy time step. The notations \((\sigma_S)_i\), \((\sigma_{SS})_i\) correspond to the evaluation of the sensitivities at the discretized point \(S_i\). Similarly, the notations \((\alpha_r)_j\), \((\alpha_{rr})_j\) show the values evaluated with the discretized point \(r_j\). Then, it leads to solve

\[
H_1(P\mathcal{Z})^{n+\frac{1}{2}} = f_1((P\mathcal{Z})^n) \quad (50)
\]

for each \(j\), where \(H_1\) is a tri-diagonal matrix with size \((N_S \times N_S)\) and entries equal to

\[
a_i = \frac{-C_1}{2\Delta S} + \frac{C_3}{(\Delta S)^2}, \quad b_i = \frac{2}{\Delta t} - \frac{2C_3}{(\Delta S)^2} + C_6, \quad c_i = \frac{C_1}{2\Delta S} + \frac{C_3}{(\Delta S)^2}.
\]
where $a_i$, $b_i$, $c_i$ are the entries for lower, main and upper diagonal for $i = 1, \ldots, N_S$. The right-hand-side value $f_1((P \mathcal{Z})^n)$ can be evaluated with the known value $(P \mathcal{Z})^n$ at current time step and is given by

$$f_1((P \mathcal{Z})^n) = (P \mathcal{Z})^n_{ij} \left( \frac{2}{\Delta t} + \frac{2C_4}{(\Delta r)^2} \right) - (P \mathcal{Z})^n_{i,j+1} \left( \frac{C_2}{2\Delta r} + \frac{C_4}{(\Delta r)^2} \right)$$

$$+ (P \mathcal{Z})^n_{i,j-1} \left( \frac{C_2}{2\Delta r} - \frac{C_4}{(\Delta r)^2} \right)$$

$$- C_5 \frac{(P \mathcal{Z})^n_{i+1,j+1} + (P \mathcal{Z})^n_{i-1,j-1} - (P \mathcal{Z})^n_{i-1,j+1} - (P \mathcal{Z})^n_{i+1,j-1}}{4\Delta S \Delta r}.$$  

(51)

**ADI Step 2**

Once the solution at the dummy time step $(P \mathcal{Z})^{n+\frac{1}{2}}$ is obtained, the second step of ADI is to consider the finite difference spatial discretization on $r$ which is treated implicitly and the rest terms are explicitly for time step from $n + \frac{1}{2}$ to $n + 1$ as follows:

$$\frac{(P \mathcal{Z})_{ij}^{n+1} - (P \mathcal{Z})_{ij}^{n+\frac{1}{2}}}{\Delta t/2} + C_1 \frac{(P \mathcal{Z})_{i+1,j}^{n+1} - (P \mathcal{Z})_{i-1,j}^{n+1}}{2\Delta S}$$

$$+ C_2 \frac{(P \mathcal{Z})_{i,j+1}^{n+\frac{1}{2}} - (P \mathcal{Z})_{i,j-1}^{n+\frac{1}{2}}}{2\Delta r}$$

$$+ C_3 \frac{(P \mathcal{Z})_{i+1,j}^{n+\frac{1}{2}} - 2(P \mathcal{Z})_{ij}^{n+\frac{1}{2}} + (P \mathcal{Z})_{i-1,j}^{n+\frac{1}{2}}}{(\Delta S)^2}$$

$$+ C_4 \frac{(P \mathcal{Z})_{i,j+1}^{n+\frac{1}{2}} - 2(P \mathcal{Z})_{ij}^{n+\frac{1}{2}} + (P \mathcal{Z})_{i,j-1}^{n+\frac{1}{2}}}{(\Delta r)^2}$$

$$+ C_5 \frac{(P \mathcal{Z})_{i+1,j+1}^{n+\frac{1}{2}} + (P \mathcal{Z})_{i-1,j-1}^{n+\frac{1}{2}} - (P \mathcal{Z})_{i-1,j+1}^{n+\frac{1}{2}} - (P \mathcal{Z})_{i+1,j-1}^{n+\frac{1}{2}}}{4\Delta S \Delta r}$$

$$+ C_6 (P \mathcal{Z})_{ij}^{n+1} = 0,$$

for $i = 1, \ldots, N_S$, $j = 1, \ldots, N_r$ and $n + 1$ is the time step of the pursued solution. Then it leads to solve

$$H_2(P \mathcal{Z})^{n+1} = f_2((P \mathcal{Z})^{n+\frac{1}{2}}),$$

(53)

for each $i$, where $H_2$ is a tri-diagonal matrix with size $(N_r \times N_r)$ and entries equal to

$$d_j = -\frac{C_2}{2\Delta r} + \frac{C_4}{(\Delta r)^2}, \quad e_j = \frac{2}{\Delta t} - \frac{2C_4}{(\Delta r)^2} + C_6, \quad f_j = \frac{C_2}{2\Delta r} + \frac{C_4}{(\Delta r)^2}.$$

where $d_j$, $e_j$, $f_j$ are the entries for lower, main and upper diagonal for $j = 1, \ldots, N_r$. The right-hand-side value $f_2((P \mathcal{Z})^{n+\frac{1}{2}})$ again can be evaluated with the calculated
solution \((PZ)^{n+\frac{1}{2}}\) at the dummy time step and is given by

\[
f_2((PZ)^{n+\frac{1}{2}}) = (PZ)^{n+\frac{1}{2}}_i j \left( \frac{2}{\Delta t} + \frac{2C_3}{(\Delta S)^2} \right) - (PZ)^{n+\frac{1}{2}}_{i, i+1, j} \left( \frac{C_1}{2\Delta S} + \frac{C_3}{(\Delta S)^2} \right) \\
+ (PZ)^{n+\frac{1}{2}}_{i-1, j} \left( \frac{C_1}{2\Delta S} - \frac{C_3}{(\Delta S)^2} \right) \\
- C_5 \cdot \left( \frac{(PZ)^{n+\frac{1}{2}}_{i, i+1, j+1} + (PZ)^{n+\frac{1}{2}}_{i-1, j, j+1} - (PZ)^{n+\frac{1}{2}}_{i, i-1, j-1} - (PZ)^{n+\frac{1}{2}}_{i+1, i, j-1}}{4\Delta S\Delta r} \right).
\]

\((54)\)

**Remark 3** – In the ADI steps 1 and 2, the coefficients \(C_i, i = 1, \ldots, 6\) are evaluated at time \(t_n\). It corresponds to some approximations and allows simplifications.

– Certain PDE schemes for solving the forward Fokker–Planck equation for \(P(t, S, r)\) lead to some negative probabilities and are source of instability (see, e.g. Itkin 2017; Sepp 2010). In our numerical examples in Sect. 5, we have not seen these issues when solving Eq. (32) for \(PZ\).

– Using the concept of generator and \(M\)-matrix in Itkin (2017), Itkin proposed PDE splitting scheme s.t. the integral of the discrete solution \(P(t, S, r)\) is equal to 1. Here, our solution for \((PZ)(t, S, r)\) should theoretically satisfy relation (40). In our experiments in Sect. 5, integrating the numerical solution can give slightly different results. In that case, we normalize the numerical solution for \((PZ)(t, S, r)\) s.t. relation (40) is satisfied.

**Algorithm 1**: Alternative direction implicit scheme

**Input**: initial condition \((PZ)^0 = \delta(S - S_0, r - r_0)\), parameters

**Output**: \((PZ)^N_t\)

**for** \(n = 0 : N_t - 1\) **do**

1. ADI step 1:

   **for** \(j = 1 : N_r - 1\) **do**
   
   Solve \(H_1(PZ)^{n+\frac{1}{2}} = f_1((PZ)^n)\) to get \((PZ)^{n+\frac{1}{2}}\);
   
   **end**

2. ADI step 2:

   **for** \(i = 1 : N_S - 1\) **do**
   
   Solve \(H_2(PZ)^{n+1} = f_2((PZ)^{n+\frac{1}{2}})\) to get \((PZ)^{n+1}\);
   
   **end**

**end**

### 4.3 Efficient implementation of the corrective terms

Let’s note by \(K_1 < K_2 < \cdots < K_N\) the strike grid where we want to compute the corrective terms \(E[Z(T)(r(T) - f(0, T))1_{S(T) > K}]\) at each maturity date \(T\). We
observe the corrective terms for two consecutive strikes are highly correlated. Indeed for \( K_i < K_{i+1} \), we have

\[
\text{Adj}(T, K_i) = \text{Adj}(T, K_{i+1}) + E[Z(T)(r(T) - f(0, T))1_{K_{i+1} \geq S(T) > K_i}], \tag{55}
\]

with \( \text{Adj}(T, K) = E[Z(T)(r(T) - f(0, T))1_{S(T) > K}] \).

So an efficient algorithm consists to calculate \( \text{Adj}(K_{N_s}) \) first and then to use the recurrence relation (55) to compute sequentially the other terms. With this method, computing all corrective terms for a given maturity consists essentially to perform one numerical integration in the whole domain of discretization for \( S \) and \( r \), which allows to speed up significantly the computation time. In our numerical experiments, the integrals for calibration and pricing are estimated numerically using the PDE solutions and grid points.

As an illustration, we compare the computation costs using 2D midpoint rule for the numerical integration. We begin by dividing the integration region into a set of little rectangular subdomains, each with length \( \Delta S \) and width \( \Delta r \) (see Fig. 2). The \( S \)-dimension is divided into \( N_s \) subdomains, while the \( r \)-dimension is divided into \( N_r \), with the centre point of each rectangle given by \((S_i, r_j)\). If we want to compute the corrective term

\[
\text{Adj}(T, K) = \int (r - f(0, T)) 1_{S > K}(PZ(T, S, r))dSdr
\]

for a given \((T, K)\), the volume of the box at this subdomain is then simply given by

\[
(r_j - f(0, T)) 1_{S_i > K}(PZ)(T, S_i, r_j)\Delta S\Delta r
\]

and \( \text{Adj}(T, K) \) is approximated by adding up all the little box elements, i.e.

\[
\text{Adj}(T, K) \approx \sum_{i, S_i > K}^{N_s} \sum_{j=1}^{N_r} (r_j - f(0, T)) (PZ)(T, S_i, r_j)\Delta S\Delta r \tag{56}
\]

Then for a given maturity \( T \), if we consider there are \( N_s \) corrective terms with \( K = S_1, \ldots, S_{N_s} \), the operation cost \( \text{Cost}(N_S, N_r) \) of a direct computation using (56) is given by

\[
\text{Cost}(N_S, N_r) = \sum_{i=1}^{N_s - 1} N_r(N_s - 1) \tag{57}
\]

\[
= \frac{N_rN_s(N_s - 1)}{2} \tag{58}
\]

\[
= O(N_rN_s^2). \tag{59}
\]

Instead if we use relation (55) with the efficient algorithm discussed above, the operation cost corresponds to

\[
\text{Cost}(N_S, N_r) = N_r(N_s - 1) = O(N_rN_s). \tag{60}
\]
So the algorithm allows to save one order in the computation cost, which is significant for a calibration procedure.

**Algorithm 2:** Calibration algorithm

**Input:** Necessary parameters for using the ADI method.

**Output:** \( \sigma(T_i, K_j), i = 1, \ldots, N_T, j = 1, \ldots, N_K \).

for \( i = 1 : N_T \) do

Solve Eq. (31) to get \((PZ)\);

for \( j = 1 : N_K \) do

1. Do numerical integration using Eq. (55) to calibrate the extra term;
2. Calculate the sensitivities to get \( \sigma_{Dup} \);
3. Evaluate \( \sigma(T_i, K_j) \);
end

end

5 Numericalexperiments

5.1 Black–Scholes Hull–White hybrid model

Let’s consider the Black–Scholes economy with Hull–white stochastic interest rates model:

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= r(t)dt + \sigma_1 dW^1(t), \quad S(0) = S_0, \\
\frac{dr(t)}{r(t)} &= a(\theta - r(t))dt + \sigma_2 (\rho dW^1(t) + \sqrt{1 - \rho^2} dW^2(t)), \quad r(0) = r_0.
\end{align*}
\]

\((61)\)

\(a\) represents the speed of reversion, \(\theta\) the long-term mean level. From Brigo and Mercurio (2006), the zero coupon \(ZC(0, T)\) price and the forward rates are given, respectively, by

\[
ZC(0, T) = A(0, T) e^{-B(0,T)r(0)},
\]

\((62)\)

\[
f(0, T) = -\frac{\sigma_1^2}{4a^2} + \theta - \left( \theta - \frac{\sigma_1^2}{a^2} - r(0) \right) e^{-aT} - \frac{\sigma_2^2}{2a^2} e^{-2aT},
\]

\((63)\)

with \(B(0, T) = \frac{1}{a} (1 - e^{-aT})\) and \(A(0, T) = e^{\left[ (\theta - \frac{\sigma_1^2}{2a^2})(B(0,T)-T) - \frac{\sigma_2^2}{4a^2} B(0,T)^2 \right]}\).

**Remark 4** We can exactly fit the term structure of interest rates being observed in the market by considering \(\theta = \theta(t)\). From Brigo and Mercurio (2006), the formula is given by

\[
\theta(t) = \frac{1}{a} \frac{\partial f(0, t)}{\partial t} + f(0, t) + \frac{1}{2} \left( \frac{\sigma_2^2}{a} \right)^2 \left( 1 - e^{-2at} \right).
\]

\((64)\)
Using the martingale method, the European call option price $C(T,K)$ with maturity $T$ and strike $K$ is derived in Haowen (2012) which we summarize in the next proposition.

**Proposition 3**

$$C(T,K) = S_0 N(d_1) - K ZC(0,T) N(d_2),$$

where $n(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$, $N(x) = \int_{-\infty}^{x} n(t) dt$, $k = \ln(K)$,

$$d_1 = \frac{\log \left( \frac{S_0}{K} \right) - \log(ZC(0,T)) + \frac{1}{2} \int_0^T \hat{\sigma}^2(t) dt}{\sqrt{\int_0^T \hat{\sigma}^2(t) dt}}, \quad d_2 = d_1 - \sqrt{\int_0^T \hat{\sigma}^2(t) dt},$$

$$\hat{\sigma}(t) = \sqrt{\sigma_1^2 + 2 \rho \sigma_1 \sigma_2 X(t) + \sigma_2^2 X^2(t)} \quad \text{and} \quad X(t) = -\frac{1}{ZC(0,T)} \frac{\partial ZC(0,T)}{\partial r}.$$  

From (62), we have $X(t) = B(0,t)$ and

$$\int_0^T \hat{\sigma}^2(t) dt = \sigma_1^2 T + \frac{2 \rho \sigma_1 \sigma_2}{a} \left[ T + \frac{e^{-\alpha T} - 1}{\alpha} \right] + \frac{\sigma_2^2}{a^2} \left[ T - \frac{1}{2a} \left( 3 - 4e^{-\alpha T} + e^{-2\alpha T} \right) \right].$$

The following corollary provides analytical formulas for $C_T$, $C_K$ and $C_{KK}$ used in the local volatility calibration expression (12).

**Corollary 2** Using the definitions in proposition (3) and $g(T) = \int_0^T \hat{\sigma}^2(t) dt$, we have

$$\frac{\partial C}{\partial T}(T,K) := C_T(T,K) = \frac{S_0 n(d_1)}{2} \frac{\hat{\sigma}^2(T)}{\sqrt{g(T)}} + K ZC(0,T) f(0,T) N(d_2),$$

$$\frac{\partial C}{\partial K}(T,K) := C_K(T,K) = -ZC(0,T) f(0,T) N(d_2),$$

$$\frac{\partial^2 C}{\partial K^2}(T,K) := C_{KK}(T,K) = \frac{ZC(0,T) n(d_2)}{K \sqrt{g(T)}}.$$  

The proof is given in “Appendix”.

This model offers tractability as we can compute the joint distribution $P(t,S,r)$ and the conditional discount factor $Z(t,y,r)$ analytically whose expressions are given in the next proposition.

**Proposition 4** Let’s define $Y(T) := \log(S(T))$, $R(T) := \int_0^T r(s) ds$ and assume the matrix $\Sigma_{yr}$, defined in (78), invertible. Then we have

$$P(T,S,r) = \frac{1}{2S\pi \sqrt{det(\Sigma_{yr})}} \exp \left\{ -\frac{\log(S) - \mu_y}{r(T) - \mu_r} \Sigma_{yr}^{-1} \frac{\log(S) - \mu_y}{r(T) - \mu_r} \right\}$$

$$= \frac{1}{2S\pi \sqrt{det(\Sigma_{yr})}} \exp \left\{ -\frac{1}{2} \Sigma_{yr}^{-1} \left( \begin{array}{c} \log(S) - \mu_y \\ r(T) - \mu_r \end{array} \right)^T \left( \begin{array}{c} \log(S) - \mu_y \\ r(T) - \mu_r \end{array} \right) \right\}.$$
\[ E[e^{-R(T)}|Y(T), r(T)] = \exp \left\{ -\mu_y - \frac{\Sigma_y}{T_y^R} \right\} = \exp \left\{ \frac{\Sigma_y}{T_y^R} \right\} \]  

(71) 

with 

\[ \mu_y = \log(S(0)) + \mu_R - \frac{1}{2} \sigma_1^2 T, \]  

(72) 

\[ \mu_r = r(0)e^{-aT} + \theta(1 - e^{-aT}), \]  

(73) 

\[ \mu_R = r(0) \left( 1 - e^{-aT} \right) + \theta T - \frac{\theta}{a} \left( 1 - e^{-aT} \right), \]  

(74) 

\[ \Sigma_y = \Sigma_R + \sigma_1^2 T + \frac{2\sigma_1 \sigma_2 \rho}{a} \left[ T - \frac{1}{a} \left( 1 - e^{-aT} \right) \right], \]  

(75) 

\[ \Sigma_R = \frac{\sigma_2^2}{2a} \left( 1 - e^{-2aT} \right), \]  

(76) 

\[ \Sigma_y = \left( \frac{\sigma_2}{a} \right)^2 \left[ T + \frac{1}{2a} \left( 1 - e^{-2aT} \right) - \frac{2}{a} \left( 1 - e^{-aT} \right) \right], \]  

(77) 

\[ \Sigma_{yr} = \left( \frac{\sigma_2}{a} \right)^2 \left( 1 - e^{-aT} \right), \]  

(78) 

\[ \Sigma_{yr,R} = \left( \frac{\sigma_2}{a} \right)^2 \left( \frac{1}{2} - e^{-aT} + \frac{1}{2} e^{-2aT} \right). \]  

(79) 

Proof Standard computations give 

\[ Y(T) = \log(S(0)) + R(T) - \frac{1}{2} \sigma_1^2 T + \sigma_1 W^1(T), \]  

(80) 

\[ r(T) = r(0)e^{-aT} + \theta(1 - e^{-aT}) + \sigma_2 \int_0^T e^{-a(T-t)} \left( \rho dW^1(t) + \sqrt{1-\rho^2} dW^2(t) \right), \]  

(81) 

\[ R(T) = \frac{1}{a} \left( 1 - e^{-aT} \right) (r(0) - \theta) + \theta T \]  

\[ + \frac{\sigma_2}{a} \int_0^T \left( 1 - e^{-a(T-t)} \right) \left( \rho dW^1(t) + \sqrt{1-\rho^2} dW^2(t) \right). \]  

(82) 

Then 

\[ \begin{pmatrix} Y(T) \\ r(T) \\ R(T) \end{pmatrix} \]  

is a Gaussian vector with mean \( \mu = \begin{pmatrix} \mu_y \\ \mu_r \\ \mu_R \end{pmatrix} \) and covariance matrix 

\[ \begin{pmatrix} \Sigma_{yr} & \Sigma_{yr,R} \\ \Sigma_{yr,R}^T & \Sigma_R \end{pmatrix}. \]  

From the joint Gaussian distribution \( \begin{pmatrix} Y(T) \\ r(T) \end{pmatrix} \) and by a change of
Table 1  Model parameters for the Black–Scholes Hull–white hybrid model

| Parameters | Set 1       | Set 2       |
|------------|-------------|-------------|
|            | 1           | 1           |
| $S_0$      | 2%          | 2%          |
| $r_0$      | 20%         | 20%         |
| $\sigma_1$| 4%          | 4%          |
| $\sigma_2$| 40%         | -40%        |
| $\rho$     | 0.5         | 0.5         |
| $\theta$  | 2%          | 2%          |
| $T$        | 1           | 2           |

variable $Y = \log(S)$, we deduce the joint distribution of $\left( \frac{S(T)}{r(T)} \right)$ which is given in expression (70).

It is well known that (see, e.g. Chapter 2.3 in Glasserman 2003) the conditional distribution of $R(T)$ given by $\left( \frac{Y(T)}{r(T)} \right)$ is a normal random variable with mean $\mu_{R|y,r}$ and variance $\sigma_{R|y,r}^2$ given, respectively, by

$$
\mu_{R|y,r} = \mu_R + \sum_{yr}^t \sum_{yr}^{-1} \left( \frac{Y(T) - \mu_y}{r(T) - \mu_r} \right),
$$

$$
\sigma_{R|y,r}^2 = \sum_{R} - \sum_{yr}^t \sum_{yr}^{-1} \sum_{yr,R}.
$$

Using the moment-generating function for a standard normal random variable $N$,

$$
E[e^{uN}] = e^{\frac{u^2}{2}}, \forall u \in \mathbb{R},
$$

then we obtain the expression in Eq. (71).

5.1.1 Tests

For the numerical tests, we consider two sets of parameters in Table 1. These model parameters correspond to the order of magnitude usually used in the equity and interest rates derivatives pricing. Also they are in line with the statistical estimates of model parameters performed by Kim in Kim (2002). Here we consider higher volatility $\sigma_2$ for the dynamic of interest rates and correlations values $\rho$ to better measure the impact of stochastic rates process.

For each set of model parameters, we solve PDE (32) in an uniform grid using the ADI method explained in the previous section. For set 1 and set 2, respectively, the sizes of doing temporal and spatial discretizations are given by $\Delta s = 0.0156$, $\Delta r = 0.0026$, $\Delta t = 0.0099$, and $\Delta s = 0.025$, $\Delta r = 0.0037$, $\Delta t = 0.019$.

For set 1, the analytic and numerical solutions for $PZ$ are shown in Fig. 3 and their discrepancy in Fig. 4. For set 2, similar results are illustrated in Figs. 5 and 6.
For both sets, we observe the discrepancies between analytic formula and numerical solutions on the PDE grid are reasonable small. For a more accurate assessment, we perform option pricing with the numerical solution $P Z$ at maturity $T$ for each set. We evaluate numerically the European call option prices for various strikes. Numerical and analytical prices solutions and also their discrepancies are illustrated, respectively, in Figs. 7 and 8 for sets 1 and 2. For both cases, we observe very good pricing accuracy as the differences are within the couple of basis points ($10^{-4}$).

Also we quantify the impact of the corrective term in expression (12) for stochastic interest rates. To avoid any scaling, we show the analytic formula in the numerator, i.e. $E[Z(T)(r(T) - f(0, T))1_{S(T) > K}]$ on the grid $(T, K)$. The results are illustrated in Figs. 9 and 10, respectively, for set 1 and set 2. In the first case with positive correlation parameter $\rho = 0.4$, we observe positive corrective terms where higher values are concentrated around the moneyness 1. It is expected as discussed in the remark of section 2 where the corrective term measures the covariance between interest rates and equity spot. Similarly, for set 2 where the correlation parameter is negative $\rho = -0.4$, we obtain negative values with higher levels concentrated around the moneyness 1.

Finally, we draw the projected discounted factor $Z(T, S, r)$ in Fig. 1 for sets 1 and 2, respectively. In both cases, the forms are similar and their levels vary around 1.

### 5.2 Hyperbolic local volatility Hull–White model

In our second example, we consider a skew model given by
Fig. 3 Parameters $S_0 = 1.0$, $r_0 = 0.02$, $\sigma_1 = 0.2$, $\sigma_2 = 0.04$, $\rho = 0.4$, $a = 0.5$, $\theta = 0.02$, $T = 1.0$

Fig. 4 Parameters $S_0 = 1.0$, $r_0 = 0.02$, $\sigma_1 = 0.2$, $\sigma_2 = 0.04$, $\rho = 0.4$, $a = 0.5$, $\theta = 0.02$, $T = 1.0$

Fig. 5 Parameters $S_0 = 1.0$, $r_0 = 0.02$, $\sigma_1 = 0.2$, $\sigma_2 = 0.04$, $\rho = -0.4$, $a = 0.5$, $\theta = 0.02$, $T = 2.0$
Fig. 6 Parameters $S_0 = 1.0$, $r_0 = 0.02$, $\sigma_1 = 0.2$, $\sigma_2 = 0.04$, $\rho = -0.4$, $a = 0.5$, $\theta = 0.02$, $T = 2.0$

Fig. 7 Parameters $S_0 = 1.0$, $r_0 = 0.02$, $\sigma_1 = 0.2$, $\sigma_2 = 0.04$, $\rho = 0.4$, $a = 0.5$, $\theta = 0.02$, $T = 1.0$

Fig. 8 Parameters $S_0 = 1.0$, $r_0 = 0.02$, $\sigma_1 = 0.2$, $\sigma_2 = 0.04$, $\rho = -0.4$, $a = 0.5$, $\theta = 0.02$, $T = 2.0$
where

\[
\begin{aligned}
\frac{dS(t)}{S(t)} &= r(t)dt + \sigma_H(S_t) dW^1(t), \quad S(0) = s_0, \\
dr(t) &= a(\theta(t) - r(t))dt + \sigma_2(\rho dW^1(t) + \sqrt{1 - \rho^2} dW^2(t)), \quad r(0) = r_0,
\end{aligned}
\]  

(86)

with \( v > 0 \) representing the level of volatility and \( \beta \in [0, 1] \) the skew parameter.

This model introduced in Jackel (2010) behaves closely to the CEV model and has been used for numerical experiments as in Bompis and Hok (2014), Hok et al. (2018). It presents the advantage to avoid zero to be an attainable boundary and then allows to avoid some numerical instabilities as seen in the CEV model when the
underlying $S$ is close to 0 (see, e.g. Andreasen and Andersen 2000). It corresponds to the Black–Scholes model for $\beta = 1$ and exhibits a skew for the volatility surface when $\beta \neq 1$. Figure 11 illustrates the impact of the parameter $\beta$ on the skew of the volatility surface. We observe that the skew increases significantly with decreasing value of $\beta$. For example with $\nu = 0.2$, $\beta = 0.2$, the difference in volatility between strikes at 50% and at 100% is about 20%.

We run two family of tests by considering, respectively, negative correlation $\rho = -30\%$ and positive correlation $\rho = 30\%$. Other model parameters are chosen in Table 2. For both tests, we solve the PDE in Eq. (31) for $PZ$ up to maturity $T$ in an uniform grid with $\Delta s = 0.012$, $\Delta r = 0.002$, $\Delta t = 0.0099$. Then, we perform the European call pricing with maturity $T$ for various strikes by numerical integration using PDE grid points and solution. We also run a Monte Carlo simulation for doing pricing by using Euler discretization for SDE (86) with $\Delta t = \frac{1}{300}$ and one million of paths (see, e.g. Glasserman 2003; Kloeden and Platen 1992).

For set 1 (respectively, for set 2), the pricing results and their discrepancies are shown in Fig. 12 (respectively, in Fig. 13). We obtain very accurate results as the differences of prices given by using PDE and Monte Carlo methods are all within a couple of basis points for all strikes in the range [0, 2].

![Figure 11 Impact of the value $\beta$ on the hyperbolic local volatility $\sigma_H$ for fixed volatility level $\nu = 0.2$.](image-url)

**Table 2** Model parameters for the hyperbolic local volatility Hull–White model

| Parameter | Value |
|-----------|-------|
| $S_0$     | 1     |
| $r_0$     | 3.75% |
| $\nu$     | 20%   |
| $\beta$   | 0.5   |
| $\sigma_2$| 4%    |
| $a$       | 0.5   |
| $T$       | 1     |
6 Conclusions and discussions

In this paper, we have proposed a new PDE-based technique for calibration on the local volatility model with stochastic interest rate. The main results are the derivation of the forward equation satisfied by $P(t, S, r)Z(t, S, r)$ and the constructed PDE solver based on ADI scheme which leads to a more efficient calibration method. Besides, some techniques of accelerating the calibration algorithm are also introduced which make the model practical for real-time execution. The numerical experiments complement our theoretical analysis and show consistent tests results. Furthermore, the discussed model is actually a general case which can cover most of the well-known extension of local volatility models used these days. Therefore, this calibration framework is useful for various problems in different asset class markets like equity, exchange rates or inflation. We suggest a couple of interesting avenues of research:

- Here, we have focused our numerical experiments on the resolution of the forward Eq. (31) with two widely used hybrid models in quantitative finance: the Black–Scholes Hull–White and the hyperbolic local volatility Hull–White models. It
would be interesting to complement the testing of the PDE calibration procedure with live market data.

– Through the numerical tests, our simple ADI scheme gives good convergence results and shows robustness w.r.t. strong skew and high correlation parameters. Performing numerical analysis and providing comparisons w.r.t. modern ADI schemes are parts of our future research.

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7 Appendix

For the proof of Corollary 2, we provide the following useful lemma

**Lemma 1** Using the definitions in Proposition 3 and Corollary 2, we have

\[ S_0 n(d_1) = K ZC(0, T) n(d_2) \]  \hspace{1cm} (88)

**Proof**

\[ d_2^2 - d_1^2 = (d_2 - d_1)(d_2 + d_1) = -\sqrt{g(T)}(d_2 + d_1) \]

\[ = -\sqrt{g(T)}(2d_1 - \sqrt{g(T)}) \]

\[ = -2 \left( \log \left( \frac{S_0}{K} \right) - \log ZC(0, T) \right) \]

\[ \log \left( \frac{n(d_1)}{n(d_2)} \right) = \log \left( \frac{K \log ZC(0, T)}{S_0} \right) \]  \hspace{1cm} (93)

From the last expression, we deduce directly (88). \( \square \)

For expression (67), we write

\[ C_T(T, K) = S_0 n(d_1) d_{1,T} - K \left[ ZC_T(0, T) N(d_2) + ZC(0, T) n(d_2)d_{2,T} \right] \]

\[ = S_0 n(d_1) (d_{1,T} - d_{2,T}) - KZC_T(0, T) N(d_2) \]

\[ = \frac{S_0 n(d_1)}{2} \hat{\sigma}^2(T) \sqrt{g(T)} + KZC(0, T) f(0, T) N(d_2) \]  \hspace{1cm} (96)

where we have used (88) in the second equality, \( d_{1,T} - d_{2,T} = \frac{1}{2} \hat{\sigma}^2(T) \sqrt{g(T)} \) and \( ZC_T(0, T) = -ZC(0, T) f(0, T) \) to obtain the third expression.

\[ C_K(T, K) = S_0 n(d_1) d_{1,K} - ZC(0, T) \left[ N(d_2) + Kn(d_2)d_{2,K} \right] \]  \hspace{1cm} (97)
Using $d_{1,K} = d_{2,K}$ and result of lemma (88), we get expression (68). Finally, we obtain formula (69) by deriving (68) w.r.t. $K$ and using $d_{1,K} = -\frac{1}{K\sqrt{g(T)}}$.

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