Lexical and Derivational Meaning in Vector-Based Models of Relativisation

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Abstract

Sadrzadeh et al (2013) present a compositional distributional analysis of relative clauses in English in terms of the Frobenius algebraic structure of finite dimensional vector spaces. The analysis relies on distinct type assignments and lexical recipes for subject vs object relativisation. The situation for Dutch is different: because of the verb final nature of Dutch, relative clauses are ambiguous between a subject vs object relativisation reading. Using an extended version of Lambek calculus, we present a compositional distributional framework that accounts for this derivational ambiguity, and that allows us to give a single meaning recipe for the relative pronoun reconciling the Frobenius semantics with the demands of Dutch derivational syntax.

1 Introduction

Compositionality, as a structure-preserving mapping from a syntactic source to a target interpretation, is a fundamental design principle both for the set-theoretic models of formal semantics and for syntax-sensitive vector-based accounts of natural language meaning, see [1] for discussion. For typelogical grammar formalisms, to obtain a compositional interpretation, we have to specify how the Syn-Sem homomorphism acts on types (basic and complex) and on proofs (derivations, again basic (axioms) or compound, obtained by inference steps). There is a tension here between lexical and derivational aspects of meaning: the derivational aspects relate to the composition operations associated with the inference steps that put together phrases out of more elementary parts; the atoms for this composition process are the meanings of the lexical constants associated with the axioms of a derivation.

Relative clause structures form a suitable testbed to study the interaction between these two aspects of meaning, and they have been well-studied in the formal and in the distributional settings. Informally, a restrictive relative clause (‘books that Alice read’) has an intersective interpretation. In the formal semantics account, this interpretation is obtained by modeling both the head noun (‘books’) and the relative clause body (‘Alice read’) as (characteristic functions of) sets (type $e \rightarrow t$); the relative pronoun can then be interpreted as the intersection operation. In distributional accounts such as [2], full noun phrases and simple common nouns are interpreted in the same semantic space, say $N$, distinct from the sentence space $S$. In this setting, element-wise multiplication, which preserves non-null context features, is a natural candidate for an intersective interpretation; in the case at hand this means element-wise multiplication of a vector in $N$ interpreting the head noun, with a vector interpretation obtained from the relative clause body. To achieve this effect, [9] rely on the Frobenius algebraic structure of $\text{FVect}$, which provides operations for (un)copying, insertion and deletion of vector information. A key feature of their account is that it relies on structure-specific solutions of the lexical equation: subject and object relative clauses are obtained from distinct type assignments to the relative
pronoun (Lambek types \((n\backslash n)/(np\backslash s)\) vs \((n\backslash n)/(s/np)\)), associated with distinct instructions for meaning assembly.

For a language like Dutch, such an account is problematic. Dutch subordinate clause order has the SOV pattern Subj–Obj–TV, i.e. a transitive verb is typed as \(np\backslash (np\backslash s)\), selecting its arguments uniformly to the left. As a result, example (1)(a) is ambiguous between a subject vs object relativisation interpretation: it can be translated as either (b) or (c). The challenge here is twofold: at the syntactic level, we have to provide a single type assignment to the relative pronoun that can withdraw either a subject or an object hypothesis from the relative clause body; at the semantic level, we need a uniform meaning recipe for the relative pronoun that will properly interact with the derivational semantics.

\[
\begin{align*}
\text{a} & \quad \text{männer_{n} die_{7} frauen_{np} haten_{np\backslash (np\backslash s)}} \quad \text{(ambiguous)} \\
\text{b} & \quad \text{men who hate women} \quad \text{(subject rel)} \\
\text{c} & \quad \text{men who(m) women hate} \quad \text{(object rel)}
\end{align*}
\]

Figure 1: NL\(_{\diamond}\). Residuation rules; extraction postulates.

The paper is structured as follows. In §2, we present an extended version of Lambek calculus, and show how it accounts for the derivational ambiguity of Dutch relative clauses. In §3.1, we define the interpretation homomorphism that associates syntactic derivations with composition operations in a vector-based semantic model. The derivational semantics thus obtained is formulated at the type level, i.e. it abstracts from the contribution of individual lexical items. In §3.2, we bring in the lexical semantics, and show how the Dutch relative pronoun can be given a uniform interpretation that properly interacts with the derivational semantics. The discussion in §4 compares the distributional and formal semantics accounts of relativisation.

### Syntax

Our syntactic engine is NL\(_{\diamond}\) [6]: the extension of Lambek’s [3] Syntactic Calculus with an adjoint pair of control modalities \(\diamond, \Box\). The modalities play a role similar to that of the exponentials of linear logic: they allow one to introduce controlled, rather than global, forms of reordering and restructuring. In this paper, we consider the controlled associativity and commutativity postulates of [7]. One pair, \(\alpha_{\diamond}, \sigma_{\diamond}\), allows a \(\diamond\)-marked formula to reposition itself on left branches of a constituent tree; we use it to model the SOV extraction patterns in Dutch. A
symmetric pair $\alpha^*_c, \sigma^*_c$ would capture the non-local extraction dependencies in an SVO language such as English. Lambeck [4] has shown how deductions in a syntactic calculus can be viewed as arrows in a category. Figure 1 presents $\text{NL}_c$ in this format.

For parsing, we want a proof search procedure that doesn’t rely on cut. Consider the rules in Figure 2, expressing the monotonicity properties of the type-forming operations, and recasting the postulates in rule form. It is routine to show that these are derived rules of inference of $\text{NL}_c$. In [8] it is shown that by adding them to the residuation rules of Figure 1, one obtains a system equivalent to a display sequent calculus enjoying cut-elimination. By further restricting to focused derivations, proof search is free of spurious ambiguity.

We are ready to return to our example (1)(a). A type assignment $(n\n)\langle \square\Box np\langle s\rangle \rangle$ to the relative pronoun ‘die’ accounts for the derivational ambiguity of the phrase. The derivations agree on the initial steps

\[
\begin{align*}
\frac{n \rightarrow n}{(n\n) \rightarrow (n\n)} & \\
\frac{n \rightarrow n}{np \rightarrow (np\langle (np\langle s\rangle) \rangle)} & \\
\frac{(np\langle (np\langle s\rangle) \rangle) \rightarrow (n\n)}{np \rightarrow (np\langle (np\langle s\rangle) \rangle)}
\end{align*}
\]

but then diverge in how the relative clause body is derived:

\[
\begin{align*}
\frac{np \rightarrow np}{\square np \rightarrow \square np} & \\
\frac{\square np \rightarrow \square np}{\Box np \rightarrow \Box np} & \\
\frac{np \rightarrow np}{np\langle s\rangle \rightarrow \Box np\langle s\rangle} & \\
\frac{np \rightarrow (np\langle (np\langle s\rangle) \rangle)}{np \rightarrow (np\langle (np\langle s\rangle) \rangle)} & \\
\frac{np \rightarrow (np\langle (np\langle s\rangle) \rangle)}{np \rightarrow (np\langle (np\langle s\rangle) \rangle)}
\end{align*}
\]

In the derivation on the left, the $\Box np$ hypothesis is linked to the subject argument of the verb; in the derivation on the right to the object argument, reached via the $\hat{\sigma}^t_o$ reordering step.

\[
\begin{align*}
f : A \rightarrow B & \\
\Box f : \Box A \rightarrow \Box B & \\
\Box f : \Box A \rightarrow \Box B & \\
\Box f : \Box A \rightarrow \Box B & \\
\Box f : \Box A \rightarrow \Box B & \\
\Box f : \Box A \rightarrow \Box B & \\
\Box f : \Box A \rightarrow \Box B &
\end{align*}
\]

Figure 2: $\text{NL}_c$. Monotonicity; leftward extraction (rule version).
3 From source to target

3.1 Derivational semantics

Compositional distributional models are obtained by defining a homomorphism sending types and derivations of a syntactic source system to their counterparts in a symmetric compact closed category (sCCC); the concrete model for this sCCC then being finite dimensional vector spaces (\(FV\text{ect}\)) and (multi)linear maps. Such interpretation homomorphisms have been defined for pregrou grammars, Lambek calculus and CCG in \([2, 5]\). We here define the interpretation for \(\text{NL}_0\), starting out from \([10]\).

Recall first that a compact closed category (CCC) is monoidal, i.e. it has an associative \(\otimes\) with unit \(I\); and for every object there is a left and a right adjoint satisfying

\[
A^l \otimes A \xrightarrow{\eta^l} I \xrightarrow{\iota} A \otimes A^l \quad A \otimes A^r \xrightarrow{\eta^r} I \xrightarrow{\iota} A \otimes A^r
\]

In a symmetric CCC, the tensor moreover is commutative, and we can write \(A^*\) for the collapsed left and right adjoints.

In the concrete instance of \(FV\text{ect}\), the unit \(I\) stands for the field \(\mathbb{R}\); identity maps, composition and tensor product are defined as usual. Since bases of vector spaces are fixed in concrete models, there is only one natural way of defining a basis for a dual space, so that \(V^* \cong V\). In concrete models we may collapse the adjoints completely.

The \(\epsilon\) map takes inner products, whereas the \(\eta\) map (with \(\lambda = 1\)) introduces an identity tensor as follows:

\[
\epsilon_V : V \otimes V \to \mathbb{R} \quad \text{given by} \quad \sum_{ij} v_{ij} (\vec{e}_i \otimes \vec{e}_j) \mapsto \sum_i v_{ii}
\]

\[
\eta_V : \mathbb{R} \to V \otimes V \quad \text{given by} \quad \lambda \mapsto \sum_i \lambda (\vec{e}_i \otimes \vec{e}_i)
\]

**Interpretation: types** At the type level, the interpretation function \([\cdot]\) assigns a vector space to the atomic types of \(\text{NL}_0\); for complex types we set \([\otimes A] = [\Box A] = [A]\), i.e. the syntactic control operators are transparent for the interpretation; the binary type-forming operators are interpreted as

\[
[A \otimes B] = [A] \otimes [B] \quad \frac{A}{B} = [A] \otimes [B]^* \quad \frac{A\backslash B} = [A]^* \otimes [B]
\]

**Interpretation: proofs** From the linear maps interpreting the premises of the \(\text{NL}_0\) inference rules, we want to compute the linear map interpreting the conclusion. Identity and composition are immediate: \([1_A] = 1_{[A]}\), \([g \circ f] = [g] \circ [f]\). For the residuation inferences, from the map \([f] : [A] \otimes [B] \to [C]\) interpreting the premise, we obtain

\[
[\triangleright f] = [A] \xrightarrow{1_{[A]} \otimes \eta_{[B]}} [A] \otimes [B] \otimes [B]^* \xrightarrow{[f] \otimes 1_{[B]^*}} [C] \otimes [B]^*
\]

\[
[\triangleleft f] = [B] \xrightarrow{\eta_{[A]} \otimes 1_{[B]}} [A]^* \otimes [A] \otimes [B] \xrightarrow{1_{[A]^*} \otimes [f]} [A]^* \otimes [C]
\]

For the inverses, from maps \([g] : [A] \to [C/B]\), \([h] : [B] \to [A\backslash C]\) for the premises, we obtain

\[
[\triangleright^{-1} g] = [A] \otimes [B] \xrightarrow{[g] \otimes 1_{[B]}} [C] \otimes [B]^* \otimes [B] \xrightarrow{1_{[C]} \otimes \epsilon_{[B]}} [C]
\]

\[
[\triangleleft^{-1} h] = [A] \otimes [B] \xrightarrow{1_{[A]} \otimes [h]} [A] \otimes [A]^* \otimes [C] \xrightarrow{\epsilon_{[A]} \otimes 1_{[C]}} [C]
\]
Monotonicity. The case of parallel composition is immediate: \([f \otimes g] = [f] \otimes [g]\). For the slash cases, from \([f] : [A] \to [B]\) and \([g] : [C] \to [D]\), we obtain

\[
\begin{align*}
[f/g] & = [A] \otimes [D]^* \\
[f\backslash g] & = [B]^* \otimes [C]
\end{align*}
\]

\[
\begin{align*}
[f] \otimes \eta_{[C]} \otimes 1_{[D]^*} & \Rightarrow [B] \otimes [C]^* \otimes [C] \otimes [D]^* \\
1_{[B] \otimes [C]^*} \otimes [g] \otimes 1_{[D]^*} & \Rightarrow [B]^* \otimes [A] \otimes [A]^* \otimes [D] \\
1_{[B] \otimes [C]^*} \otimes \epsilon_{[D]} & \Rightarrow [B]^* \otimes [B] \otimes [A]^* \otimes [D] \\
\end{align*}
\]

Interpretation for the extraction structural rules is obtained via the standard associativity and symmetry maps of \(\text{FVect}\): \([\alpha_\bigotimes^\bigotimes f] = f \circ \alpha\) and \([\alpha_\bigotimes^\bigotimes j] = f \circ \alpha^{-1} \circ (\sigma \otimes 1_A) \circ \alpha\) and similarly for the rightward extraction rules.

**Simplifying the interpretation** Whereas the syntactic derivations of \(\text{NL}_c\) proceed in cut-free fashion, the interpretation of the inference rules given above introduces detours (sequential composition of maps) that can be removed. We use a generalised notion of Kronecker delta, together with Einstein summation notation, to concisely express the fact that the interpretation of a derivation is fully determined by the identity maps that interpret its axiom leaves, realised as the \(\epsilon\) or \(\eta\) identity matrices depending on their (co)domain signature.

Recall that vectors and linear maps over the real numbers can be equivalently expressed as (multi-dimensional) arrays of numbers. The essential information one needs to keep track of are the coefficients of the tensor: for a vector \(v \in \mathbb{R}^n\) we write \(v_i\) (with \(i\) ranging from 1 to \(n\)), an \(n \times m\) matrix \(A\) is expressed as \(A_{ij}\), an \(n \times m \times p\) cube \(B\) as \(B_{ijk}\), with the indices each time ranging over the dimensions. The Einstein summation convention on indices then states that in an expression involving multiple tensors, indices occurring once give rise to a tensor product, whereas indices occurring twice are contracted. Without explicitly writing a tensor product \(\otimes\), the tensor product of a vector \(a\) and a matrix \(A\) thus can be written as \(a_i A_{jk}\); the inner product between vectors \(a, b\) is \(a_i b_j\). Matrix application \(Aa\) is rendered as \(A_{ij} a_j\), i.e. the contraction happens over the second dimension of \(A\) and \(a\). For tensors of arbitrary rank we use uppercase to refer to lists of indices: we write a tensor \(T\) as \(T_i\). Tensor application then becomes \(T_i J, R_J\), for some tensor \(R\) of lower rank.

The identity matrix is given by the Kronecker delta (left), the identity tensor by its generalisation (right):

\[
\delta^j_j = \begin{cases} 
1 & i = j \\
0 & \text{otherwise}
\end{cases} \quad \delta^i_j = \begin{cases} 
1 & I_k = J_k \text{ for all } k \\
0 & \text{otherwise}
\end{cases}
\]

The attractive property of the (generalised) Kronecker delta is that it expresses unification of indices: \(\delta^j_j a_i = a_j\), which is simply a renaming of the index; the inner product can be computed by \(\delta^j_j a_i b_j = a_j b_j\). Left on its own, it is simply an identity matrix/tensor.
With the Kronecker delta, the composition of matrices $B \circ A$ is expressible as $\delta^i_j A_{ij} B_{kl}$, which is the same as $A_{ij} B_{jl}$ (or $A_{ik} B_{kl}$). We can show that order of composition is irrelevant:

$$\delta^i_j A_{ij} \delta^m_n B_{kl} C_{mn} = A_{ij} B_{jl} C_{ln} = \delta^i_m \delta^j_n A_{ij} B_{kl} C_{mn}$$

The special cases of tensor product of generalised Kronecker deltas is given by concatenating the index lists:

$$\delta^i_j \otimes \delta^k_l = \delta^i_j \delta^k_l$$

expressing the fact that $1_A \otimes 1_B = 1_{A \otimes B}$.

Since the generalised Kronecker delta is able to do renaming, take inner product, and insert an identity tensor, depending on the number of arguments placed behind it, it will represent pre- and we can label the proof system (with formulas already interpreted) with these generalised Kronecker deltas. The effect of the residuation rules and the structural rules is to only change the (co)domain signature of a Kronecker delta, whereas the rules for axioms and monotonicity also act on the Kronecker delta itself:

$$\begin{align*}
A & \xrightarrow{A_i} B \xrightarrow{C_{ij}} C \xrightarrow{D_{ijkl}} \delta^i_j \otimes \delta^k_l \\
A \otimes C & \xrightarrow{\delta^i_j} B \otimes D \xrightarrow{\delta^m_n} C \otimes D
\end{align*}$$

In Appendix A we show that this labelling is correct for the general interpretation of proofs in §3.1.

### 3.2 Lexical semantics

For the general interpretation of types and proofs given above, a proof $f : A \rightarrow B$ is interpreted as a linear map $[f]$ sending an element belonging to $[A]$, the semantic space interpreting $A$, to an element of $[B]$. The map is expressed at the general level of types, and completely abstracts from lexical semantics. For the computation of concrete interpretations, we have to bring in the meaning of the lexical items. For $A = A_1 \otimes \cdots \otimes A_n$, this means applying the map $[f]$ to $w_1 \otimes \cdots \otimes w_n$, the tensor product of the word meanings making up the phrase under consideration, to obtain a meaning $M \in [B]$, the semantic space interpreting the goal formula.

With the index notation introduced above, $[f]$ is expressed in the form of a generalised Kronecker delta, which is applied to the tensor product of the word meanings in index notation to produce the final meaning in $[B]$. In (4) we illustrate with the interpretation of some proofs derived from the same axiom leaves, $np \rightarrow np$ and $s \rightarrow s$. Assuming $[np] = N$ and $[s] = S$, these correspond to identity maps on $N$ and $S$. We use the convention that the formula components of the endsequent are labelled in alphabetic order; the correct indexing for the Kronecker delta is obtained by working back to the axiom leaves.

\[
\begin{align*}
\text{a: } & \text{dream}^{np} \xrightarrow{a} np \xrightarrow{a} np \\
& \text{dream}_{i,j}^{N \otimes S} \xrightarrow{\delta^i_j} T_{k,l}^{N \otimes S} \\
\text{b: } & \text{poets}^{np} \otimes \text{dream}^{np} \xrightarrow{b} s \\
& \text{poets}_{i}^{N} \otimes \text{dream}_{j,k}^{N \otimes S} \xrightarrow{\delta^i_j} V_{l}^{S} \quad (4) \\
\text{c: } & \text{poets}^{np} \xrightarrow{c} s/np \xrightarrow{c} s/np \xrightarrow{c} s/np \\
& \text{poets}_{i}^{N} \xrightarrow{\delta^i_j} R_{j,k,l}^{S \otimes N \otimes S}
\end{align*}
\]
(4)(a) expresses the linear map from \( \text{dream} \in \mathbb{N} \otimes S \) to a tensor \( T \in \mathbb{N} \otimes S \). Because we have \( T = \delta^{i,j}_{i,j} \text{dream}_{i,j} = \text{dream} \), this is in fact the identity map. (4)(b) computes a vector \( V \in S \) with \( V = \delta^{i,j}_{i,j} \text{poets} \otimes \text{dream} \). In (4)(c) we arrive at an interpretation \( R \in S \otimes \mathbb{N} \otimes S \) with \( R = \delta^{i,j}_{i,j} \text{poets} \). Note that we wrote the tensor product symbol \( \otimes \) explicitly.

In the case of our relative clause example (1), the derivational ambiguity of (3) gives rise to two ways of obtaining a vector \( v \in \mathbb{N} \). They differ in whether \( l \), the index of the \( \diamond \) hypothesis in the relative pronoun type, contracts with index \( p \) for the subject argument of the verb (5) or with the direct object index \( o \).(6).

\[
\begin{align*}
\text{mannen} \otimes \text{die}_{ijklm} \otimes \text{vrouwen}_{n} \otimes \text{haten}_{npq} \delta^{i,j,k,m,n}_{i,j,k,m,n} \rightarrow v^{subj}_{p} \in \mathbb{N} \\
\text{mannen} \otimes \text{die}_{ijklm} \otimes \text{vrouwen}_{n} \otimes \text{haten}_{npq} \delta^{i,j,k,m,n}_{i,j,k,m,n} \rightarrow v^{obj}_{p} \in \mathbb{N}
\end{align*}
\]

The picture in Figure 3 expresses this graphically.

\[
\begin{align*}
\text{mannen} \otimes \text{die}_{ijklm} \otimes \text{vrouwen}_{n} \otimes \text{haten}_{npq} \delta^{i,j,k,m,n}_{i,j,k,m,n} \rightarrow v^{subj}_{p} \in \mathbb{N} \\
\text{mannen} \otimes \text{die}_{ijklm} \otimes \text{vrouwen}_{n} \otimes \text{haten}_{npq} \delta^{i,j,k,m,n}_{i,j,k,m,n} \rightarrow v^{obj}_{p} \in \mathbb{N}
\end{align*}
\]

Open class items vs function words For open class lexical items, concrete meanings are obtained distributionally. For function words, the relative pronoun in this case, it makes more sense to assign them an interpretation independent of distributions. To capture the intersective interpretation of restrictive relative clauses, Sadrzadeh et al. [9] propose to interpret the relative pronoun with a map that extracts a vector in the noun space from the relative clause body, and then combines this by elementwise multiplication with the vector for the head noun. Their account depends on the identification \([np] = [n] = \mathbb{N}\): noun phrases and simple common nouns are interpreted in the same space; it expresses the desired meaning recipe for the relative pronoun with the aid of (some of) the Frobenius operations that are available in a compact closed category:

\[
\begin{align*}
\Delta : A \rightarrow A \otimes A \\
\mu : A \otimes A \rightarrow A \\
\iota : A \rightarrow I \\
\zeta : I \rightarrow A
\end{align*}
\]
In the case of \( \mathbf{FVec} \), \( \Delta \) takes a vector and places its values on the diagonal of a square matrix, whereas \( \mu \) extracts the diagonal from a square matrix. The \( \iota \) and \( \zeta \) maps respectively sum the coefficients of a vector or introduce a vector with the value 1 for all of its coefficients.

\[
\begin{align*}
\Delta_V : V &\rightarrow V \otimes V \text{ given by } \sum_i v_i \vec{e}_i \mapsto \sum_i v_i (\vec{e}_i \otimes \vec{e}_i), \\
\iota_V : V &\rightarrow \mathbb{R} \text{ given by } \sum_i v_i \vec{e}_i \mapsto \sum_i v_i, \\
\mu_V : V \otimes V &\rightarrow V \text{ given by } \sum_{ij} v_{ij} (\vec{e}_i \otimes \vec{e}_j) \mapsto \sum_i v_{ii} \vec{e}_i, \\
\zeta_V : \mathbb{R} &\rightarrow V \text{ given by } \lambda \mapsto \sum_i \lambda \vec{e}_i.
\end{align*}
\]

The analysis of [9] uses a pregroup syntax and addresses relative clauses in English. It relies on distinct pronoun types for subject and object relativisation. In the subject relativisation case, the pronoun lives in the space \( N \otimes N \otimes S \otimes N \), corresponding to \( n^s \cdot n^p \cdot s^l \cdot n^p \), the pregroup translation of a Lambek type \( (n \cdot n) / (n \cdot np) \); for object relativisation, the pronoun lives in \( N \otimes N \otimes N \otimes S \), corresponding to \( n^s \cdot n^p \cdot n^p \cdot np^l \), the pregroup translation of \( (n \cdot n) / (s \cdot np) \).

For the case of Dutch, the homomorphism \( \lceil \cdot \rceil \) of §3.1 sends the relative pronoun type \( (n \cdot n) / (s^l \cdot np) \) to the space \( N \otimes N \otimes N \otimes S \). This means we can import the pronoun interpretation for that space from [9], which now will produce both the subject and object relativisation interpretations through its interaction with the derivational semantics.

\[
\text{die} = (1_N \otimes \mu_N \otimes 1_N \otimes \zeta_S) \circ (\eta_N \otimes \eta_N)
\]  

(8)

Intuitively, the recipe (8) says that the pronoun consists of a cube (in \( N \otimes N \otimes N \)) which has 1 on its diagonal and 0 elsewhere, together with a vector in the sentence space \( S \) with all its entries 1. Substituting this lexical recipe in the tensor contraction equations of (5) and (6) yields the desired final semantic values (9) and (10) for subject and object relativisation respectively. We write \( \odot \) for elementwise multiplication; the summation over the \( S \) dimension reduces the rank-3 \( N \otimes N \otimes S \) interpretation of the verb to a rank-2 matrix in \( N \otimes N \), with rows for the verb’s object, columns for the subject. This matrix is applied to the vector \( \text{vrouwen} \) either forward in (10), where ‘vrouwen’ plays the subject role, or backward in (9) before being elementwise multiplied with the vector for \( \text{mannen} \).

\[
(5) \quad \text{mannen} \odot \left[ \left( \sum_S \text{haten} \right)^T \text{vrouwen} \right]
\]  

(9)

\[
(6) \quad \text{mannen} \odot \left[ \left( \sum_S \text{haten} \right)\text{vrouwen} \right]
\]  

(10)

Returning to English, notice that the pregroup type assignment \( n^s \cdot n^p \cdot n^p^l \cdot s^l \) for object relativisation in [9] is restricted to cases where the ‘gap’ in the relative clause body occupies the final position. To cover these non-subject relativisation patterns in general, also with respect to positions internal to the relative clause body, we would use an \( \mathbf{NL} \) type \( (n \cdot n) / (s / np \cdot np) \) for the pronoun, together with the rightward extraction postulates \( \alpha_{np}^+, \sigma_{np}^+ \) of Figure 1. For English subject relativisation, the simple pronoun type \( (n \cdot n) / (np \cdot s) \) will do, as this pattern doesn’t require any structural reasoning.
4 Discussion

We briefly compare the distributional and the formal semantics accounts, highlighting their similarities. In the formal semantics account, the interpretation homomorphism sends syntactic types to their semantic counterparts. Syntactic types are built from atoms, for example $s$, $np$, $n$ for sentences, noun phrases and common nouns; assuming semantic atoms $e$, $t$ and function types built from them, one can set $[s] = t$, $[np] = e$, $[n] = e \to t$, and $[A/B] = [B\setminus A] = [B] \to [A]$. Each semantic type $A$ is assigned an interpretation domain $D_A$, with $D_e = E$, for some non-empty set $E$ (the discussion domain), $D_t = \{0, 1\}$ (truth values), and $D_{A\to B}$ functions from $D_A$ to $D_B$.

In this setup, a syntactic derivation $A_1, \ldots, A_n \Rightarrow B$ is interpreted by means of a linear lambda term $M$ of type $[B]$, with parameters $x_i$ of type $[A_i]$. Linearity resulting from the fact that the syntactic source doesn’t provide the copying/deletion operations associated with the structural rules of Contraction and Weakening.

As in the distributional model discussed here, the proof term $M$ is an instruction for meaning assembly that abstracts from lexical semantics. In (11) below, one finds the proof terms for English subject (a) and object (b) relativisation. The parameter $w$ stands for the head noun, $f$ for the verb, $y$ and $z$ for its object and subject arguments; parameter $x$ for the relative pronoun has type $(e \to t) \to (e \to t) \to e \to t$.

\[
\begin{align*}
(a) \quad & n, (n\setminus n)/\langle np\rangle s, (np\setminus s)/np, np \Rightarrow n \quad (x_{\text{who}} \lambda z^e. (f^{e\to e\to t} y^e z^e) w^{e\to t}) \\
(b) \quad & n, (n\setminus n)/\langle s/np\rangle, (np\setminus s)/np \Rightarrow n \quad (x_{\text{who}} \lambda y^e. (f^{e\to e\to t} y^e z^e) w^{e\to t})
\end{align*}
\]

To obtain the interpretation of ‘men who hate women’ vs ‘men who(m) women hate’, one substitutes lexical meanings for the parameters of the proof terms. In the case of the open class items ‘men’, ‘hate’, ‘women’, these will be non-logical constants with an interpretation depending on the model. For the relative pronoun, we substitute an interpretation independent of the model, expressed in terms of the logical constant $\land$, leading to the final interpretations of (13), after normalisation.

\[
\begin{align*}
(x_{\text{who}}) := & \lambda x^e. (f^{e\to e\to t} y^e z^e). ((x z) \land ((y z))) \\
(a) \quad & \lambda x. ((\text{MEN } x) \land (\text{HATE } \text{WOMEN } x)) \\
(b) \quad & \lambda x. ((\text{MEN } x) \land (\text{HATE } x \text{ WOMEN}))
\end{align*}
\]

Notice that the lexical meaning recipe for the relative pronoun goes beyond linearity: to express the set intersection interpretation, the bound $z$ variable is copied over the conjuncts of $\land$. By encapsulating this copying operation in the lexical semantics, one avoids compromising the derivational semantics. In this respect, the formal semantics account makes the same design choice regarding the division of labour between derivational and lexical semantics as the distributional account, where the extra expressivity of the Frobenius operations is called upon for specifying the lexical meaning recipe for the relative pronoun.

5 Acknowledgments

We thank Giuseppe Greco for comments on an earlier version. The second author would also like to thank Mehrnoosh Sadrzadeh for the many discussions on compositional distributional
semantics and Frobenius operations, and Rob Klabbers for his interesting remarks on index notation. The second author gratefully acknowledges support by a Queen Mary Principal’s Research Studentship, the first author the support of the Netherlands Organisation for Scientific Research (NWO, Project 360-89-070, A composition calculus for vector-based semantic modelling with a localization for Dutch).

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A Simplifying the Interpretation

The simplification of section 3.1 uses generalised Kronecker deltas to interpret the proof terms of the proof system, leading to a relabelling of the proof system with formulas interpreted. The rules that change the generalised Kronecker delta are shown in section 3.1, the full system is shown in Figure 4. In this appendix we show that the simplification holds.

\[
\begin{align*}
A \xrightarrow{\delta^I_J} B \\
\rightarrow
A \xrightarrow{\delta^I_J} B \\
\leftarrow
A \xrightarrow{\delta^I_J} C \otimes B \\
\uparrow
A \xrightarrow{\delta^I_J} C \otimes B \\
\downarrow
A \xrightarrow{\delta^I_J} A \otimes C \\
\end{align*}
\]

\[
\begin{align*}
A \xrightarrow{\delta^I_J} B \\
\vdash
A \xrightarrow{\delta^I_J} A \otimes C \\
\end{align*}
\]

\[
\begin{align*}
A \xrightarrow{\delta^I_J} B \\
\vdash
A \xrightarrow{\delta^I_J} A \otimes C \\
\end{align*}
\]

\[
\begin{align*}
(A \otimes B) \otimes C \xrightarrow{\delta^I_J} D \\
\vdash
(A \otimes B) \otimes C \xrightarrow{\delta^I_J} D \\
\end{align*}
\]

\[
\begin{align*}
B \otimes (A \otimes C) \xrightarrow{\delta^I_J} D \\
\vdash
B \otimes (A \otimes C) \xrightarrow{\delta^I_J} D \\
\end{align*}
\]

Figure 4: NL\textsubscript{o}. Rules annotated with their generalised Kronecker deltas.

Analogous to \([f]\), we write \([f]_G\) for the generalised Kronecker delta associated with proof term \(f\). We define the expressions of a compact closed category for generalised Kronecker deltas. Then we show that for any proof \(f : A \rightarrow B\) we have that \([f] = [f]_G\). Proving this is done by induction over the size of proofs. The crucial point is that composition of two generalised Kronecker deltas is determined by their domain and codomain.

The CCC structure of generalised Kronecker deltas To give a generalised Kronecker delta an interpretation as a map, we need to give its domain and codomain. We will write for a generalised Kronecker delta, \(\delta^I_J : A_M \rightarrow B_N\) to indicate that \(A\) is the domain, \(B\) the codomain, and moreover that concrete tensors in \(A\) will be labelled with list \(M\), and output tensors will
be labelled with $N$. Writing $+$ for list concatenation and $\pi(L)$ for any permutation of list $L$, we then assume that $I + J = \pi(M + N)$. We can now go on and define the maps of a compact closed category in the generalised Kronecker delta form.

Note the way generalised Kronecker deltas are rewritten: a generalised Kronecker delta $\delta^{(I+K)}_J$ has pairs of indices on top and bottom that are linked. Whenever $\delta^{(I)}_J$ has an index occurring twice, a rewrite is done: let $(a, b), (c, d)$ be two pairs of indices, $a, c$ on top and $b, d$ on the bottom, that have an index in common, say $a = c$. Then we remove from $\delta^{(I)}_J$ the pair $(a, b)$, and replace the pair $(c, d)$ by $(b, d)$. This lowers the rank of the generalised Kronecker delta with 2, which is in line with the idea of the tensor contraction that is to be performed by the common index. This generalises to lists of indices: writing $(A, B)$ and $(C, D)$ for lists of indices such that pairs $(a, b)$ come from $A, B$ and pairs $(c, d)$ come from $C, D$, if we have that $A = C$ we can immediately remove $(A, B)$ and replace $(C, D)$ by $(B, D)$. The whole rewriting continues until there are only unique indices left. Write $(\delta^{(I)}_J)^*$ (or $\delta^{(I)}_{J^*}$) for the generalised Kronecker delta obtained by rewrites from the original $\delta^{(I)}_J$. Then, for two generalised Kronecker deltas, we get the relation

$$\delta^{(I)}_J \delta^{(K)}_L = \delta^{(I+K)}_J.$$ 

In particular this means for $\delta^{(I)}_J$ and $\delta^{(K)}_L$ with no indices in common that $\delta^{(I+K)}_{(J+L)} = \delta^{(I)}_J \delta^{(K)}_L$.

Another special case is when we have $\delta^{(A)}_{B+J}$ where $A$ occurs in $I + J$, but $B$ does not. Then we have that we remove $A$ and $B$ and, for each index $a$ in $I$ and $J$, we substitute the corresponding index $b$ in $B$:

$$\delta^{(A+I)}_{(B+J)^*} = (\delta^{(A)}_I \{A \mapsto B\})^*$$

When $I$ and $J$ have no elements in common, then the right hand side is already fully rewritten since $B$ is unique to $I$ and $J$, allowing us to drop the asterisk. We use these properties in the below definition for tensor product and composition of generalised Kronecker deltas, and in the proof in the next paragraph.

**Definition 1.** The maps in $FVect$ are defined in terms of generalised Kronecker deltas according the the list below:

1. For any vector space $V$ of rank $n$, the identity map $I_V : V \to V$ is given by the generalised Kronecker delta

$$\delta^{i_1, i_2, \ldots, i_n}_{j_1, j_2, \ldots, j_n} : V_{i_1, i_2, \ldots, i_n} \to V_{j_1, j_2, \ldots, j_n}$$

On an element $v \in V$, represented in index notation by $v_{i_1, \ldots, i_n}$, we get simply a renaming because

$$\delta^{i_1, i_2, \ldots, i_n}_{j_1, j_2, \ldots, j_n} v_{i_1, \ldots, i_n} = v_{j_1, \ldots, j_n}$$

2. For any vector space $V$ of rank $n$, the $\epsilon_V : V \otimes V \to \mathbb{R}$ map is given by the generalised Kronecker delta

$$\delta^{i_1, i_2, \ldots, i_n}_{j_1, j_2, \ldots, j_n} : V_{i_1, i_2, \ldots, i_n} \otimes V_{j_1, j_2, \ldots, j_n} \to \mathbb{R}$$

For two elements $v \in V$ and $w \in V$ represented by $v_{i_1, \ldots, i_n}$ and $w_{j_1, \ldots, j_n}$ we get the inner product between $v$ and $w$:

$$\delta^{i_1, i_2, \ldots, i_n}_{j_1, j_2, \ldots, j_n} v_{i_1, \ldots, i_n} w_{j_1, \ldots, j_n} = v_{j_1, \ldots, j_n} w_{j_1, \ldots, j_n}$$

3. For any vector space $V$ of rank $n$, the $\eta_V : \mathbb{R} \to V \otimes V$ map is given by

$$\delta^{i_1, i_2, \ldots, i_n}_{j_1, j_2, \ldots, j_n} : \mathbb{R} \to V_{i_1, i_2, \ldots, i_n} \otimes V_{j_1, j_2, \ldots, j_n}$$

It is given no elements to juxtapose with and thus simply gives the identity matrix on $V$.  

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4. Composition. Given two maps (left) and their generalised Kronecker delta representation (right)
\[ f : A \to B \quad \delta^I_f : A_M \to B_{N_1} \]
\[ g : B \to C \quad \delta^K_g : B_{N_2} \to C_O \]
their composition \( g \circ f : A \to C \) is represented by
\[ \delta^{(N_1+I+K)}_{(N_2+J+L)} : A_M \to C_O \]

We give the expression \( \delta^{N_2}_{N_1} \delta^I_f \delta^K_g \) for the composition, exactly to identify the indices in the codomain of \( f \) (\( N_1 \)) with the indices in the domain of \( g \) (\( N_2 \)). Since \( \delta^I_f \) and \( \delta^K_g \) have no indices in common (this may be assumed without loss of generality), we have \( \delta^I_f \delta^K_g = \delta^{I+K}_{J+L} \), but since \( N_2 \) occurs in \( K + L \) and \( N_1 \) occurs in \( I + J \), we will have a sequence of rewrites to do and so we get \( \delta^{(N_1+I+K)}_{(N_2+J+L)} \).

5. Tensor product. Given two maps (left) and their generalised Kronecker delta representation (right)
\[ f : A \to B \quad \delta^I_f : A_M \to B_N \]
\[ g : C \to D \quad \delta^K_g : C_O \to D_P \]
their tensor product \( f \otimes g : A \otimes C \to B \otimes D \) is represented by
\[ \delta^{I+J}_{K+L} : A_M \otimes C_O \to B_N \otimes D_P \]

Without loss of generality we may assume that \( \delta^I_f \) and \( \delta^K_g \) have no indices in common (if they had, we could rename them). Since \( I + J = \pi(M + N) \) and \( K + L = \pi(O + P) \), we also have that \( I + J + K + L = \pi(M + N + O + P) \). And since \( \delta^I_f, \delta^K_g \) have no indices in common, we have that juxtaposing them gives \( \delta^I_f \delta^K_g = \delta^{I+J}_{K+L} \).

6. Associativity. Since the tensor product is associative on vectors, the associativity maps disappear in index notation. For vector spaces \( A, B, C \) of rank \( k, l, m \) respectively, the associativity map \( \alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \) is represented as
\[ \delta^{M_1N_1O_1}_{M_2N_2O_2} : A_{M_1} \otimes B_{N_1} \otimes C_{O_1} \to A_{M_2} \otimes B_{N_2} \otimes C_{O_2} \]
where \( M_1, M_2 \) have length \( k \), \( N_1, N_2 \) have length \( l \) and \( O_1, O_2 \) have length \( m \). This acts simply as an identity map: on elements \( a \in A, b \in B, c \in C \), we get
\[ \delta^{M_1N_1O_1}_{M_2N_2O_2} a_{M_1} b_{N_1} c_{O_1} = a_{M_2} b_{N_2} c_{O_2} \]
which is simply a renaming of the input. The inverse associativity map \( \alpha^{-1} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C \) is represented exactly the same and again works as an identity map.

7. Symmetry. The tensor product is not commutative on vectors, but the generalised Kronecker delta for the symmetry map \( \sigma_{A,B} : A \otimes B \to B \otimes A \) on vector spaces \( A, B \) with rank \( k, l \) respectively, performs an identity. The order of evaluation is given by the switch in indices in input and output. So \( \sigma \) is represented by
\[ \delta^{M_1N_1}_{M_2N_2} : A_{M_1} \otimes B_{N_1} \to B_{N_2} \otimes A_{M_2} \]
On an input \( a \in A, b \in B \), we get
\[ \delta^{M_1N_1}_{M_2N_2} a_{M_1} b_{N_1} = a_{M_2} b_{N_2} \]
but here the order of the indices in the codomain dictates that the elements of \( a \) are placed after the elements of \( b \).
Reducing the interpretation. With the translation of maps of \( \textbf{FVect} \) in terms of generalised Kronecker deltas, we are ready to state our claim.

**Theorem 1.** For any proof \( f : A \rightarrow B \), we have \( [f] = [f] \).

**Proof.** By induction over the size of proofs. The base case is the case of the axiom, for which we have \( [1_A] = 1_{[A]} \). The identity map is represented by \( \delta^I_M : [A]_M \rightarrow [A]_M \). The formula above, for the inductive step, we proceed by cases. Just as in the main text, for brevity we will suppress the + symbol in list concatenation. We moreover write the composition symbol \( \circ \) in between generalised Kronecker deltas when its is clear what is intended.

1. **Residuation.** There are six subcases to consider:

   (a) \( \prec f \): assuming that \( [f] : [A \otimes B] \rightarrow [C] \) is equal to \( \delta^I_J : [A]_M \otimes [B]_N \rightarrow [C]_O \), we need to show that \( [\prec f] : [B] \rightarrow [A \setminus C] \) is equal to \( \delta^I_J : [B]_N \rightarrow [A]_M \otimes [C]_O \). We compute
   \[
   [\prec f] = (1_{[A]} \otimes [f]) \circ ([A] \otimes 1_{[B]})
   \]
   which, by the codomain of \( [A] \otimes 1_{[B]} \) and domain of \( 1_{[A]} \otimes [f] \) \( (A \otimes A \otimes B) \) means replacing as follows:
   \[
   = \delta_M^N \delta_N^M \delta_M^N \delta_M^N
   \]
   \[
   = \delta_M^N \delta_N^M \delta_M^N \delta_M^N
   \]
   \[
   = \delta_M^N \delta_N^M \delta_M^N \delta_M^N
   \]
   So this gives a renaming of the original map, hence we need to rename the domain and codomain to \( [B]_N, [A]_M, [C]_O \) but we can then also rename back to \( \delta^I_J : [B]_N \rightarrow [A]_M \otimes [C]_O \).

   (b) \( \prec^{-1} g \): assuming that \( [g] : [B] \rightarrow [A \setminus C] \) is equal to \( \delta^I_J : [B]_N \rightarrow [A]_M \otimes [C]_O \) we need to show that \( [\prec^{-1} g] : [A \otimes B] \rightarrow [C] \) is equal to \( \delta^I_J : [A]_M \otimes [B]_N \rightarrow [C]_O \). We compute
   \[
   [\prec^{-1} g] = (\epsilon_{[A]} \otimes 1_{[C]}) \circ ([A] \otimes [g])
   \]
   which, by the codomain of \( 1_{[A]} \otimes [g] \) and domain of \( \epsilon_{[A]} \otimes 1_{[C]} \) \( (A \otimes A \otimes C) \) means replacing as follows:
   \[
   = \delta_M^N \delta_N^M \delta_M^N \delta_M^N
   \]
   \[
   = \delta_M^N \delta_N^M \delta_M^N \delta_M^N
   \]
   \[
   = \delta_M^N \delta_N^M \delta_M^N \delta_M^N
   \]

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2. Monotonicity. Here we need to consider five subcases.

(a) $\Diamond f, \Box f$. These cases are again immediate since $\Diamond f = [f] = [\Box f]$.

(b) $f \otimes g$: since $[f \otimes g] = [f] \otimes [g]$ this case is immediate since the tensor product of generalised Kronecker deltas is performed by simply concatenating the upper and lower index lists together.

(c) $f \backslash g$: assuming $[f] : [A] \to [B]$ and $[g] : [C] \to [D]$ are equal to $\delta f_j : [A]_M \to [B]_N$ and $\delta g_k : [C]_O \to [D]_P$ we need to show that $[f \backslash g] : [B \backslash C] \to [A \backslash D]$ is equal to $\delta f_k L : [B]_N \otimes [C]_O \to [A]_M \otimes [D]_P$. We compute

$$[f \backslash g] = (\epsilon_{[B]} \otimes 1_{[A]_M}) \circ (1_{[B]} \otimes [f] \otimes 1_{[A]_M}) \circ (1_{[B]} \otimes \eta_{[A]} \otimes [g])$$

with (co)domain signature

$$[B] \otimes [C] \to [B] \otimes [A] \otimes [A] \otimes [D] \to [B] \otimes [B] \otimes [A] \otimes [D] \to [A] \otimes [D]$$

So writing the composition with generalised Kronecker deltas, we get

$$= (\delta P \otimes \delta N) \circ (\delta P \otimes \delta N) \circ (\delta P \otimes \delta N)$$

The composition is

$$= (\delta P \otimes \delta N) \circ (\delta P \otimes \delta N) \circ (\delta P \otimes \delta N)$$

which gives just a renaming of $\delta f_j$.

(c) The cases of $\triangleright f, \triangleright^{-1} g$ are exactly symmetric to the cases of $\triangleleft, \triangleleft^{-1}$.

(d) $\nabla f, \nabla^{-1} f$: since $[\nabla f] = [f] = [\nabla^{-1} f]$, these cases are immediate.

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$$[f \backslash g] = (\epsilon_{[B]} \otimes 1_{[A]_M}) \circ (1_{[B]} \otimes [f] \otimes 1_{[A]_M}) \circ (1_{[B]} \otimes \eta_{[A]} \otimes [g])$$

with (co)domain signature

$$[B] \otimes [C] \to [B] \otimes [A] \otimes [A] \otimes [D] \to [B] \otimes [B] \otimes [A] \otimes [D] \to [A] \otimes [D]$$

So writing the composition with generalised Kronecker deltas, we get

$$= (\delta P \otimes \delta N) \circ (\delta P \otimes \delta N) \circ (\delta P \otimes \delta N)$$

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which gives just a renaming of $\delta f_j$.

(c) The cases of $\triangleright f, \triangleright^{-1} g$ are exactly symmetric to the cases of $\triangleleft, \triangleleft^{-1}$.

(d) $\nabla f, \nabla^{-1} f$: since $[\nabla f] = [f] = [\nabla^{-1} f]$, these cases are immediate.
\[\begin{align*}
\delta_{P_2N_2N_4A_1A_3A_2A_4A_5A_6A_7A_8}^P &= \delta_{P_2N_2N_4A_1A_3A_2A_4A_5A_6A_7A_8}^P \\
&= \delta_{P_2N_2N_4A_1A_3A_2A_4A_5A_6A_7A_8}^P \\
&= \delta_{P_2N_2N_4A_1A_3A_2A_4A_5A_6A_7A_8}^P \\
&= \delta_{P_2N_2N_4A_1A_3A_2A_4A_5A_6A_7A_8}^P \\
&= \delta_{P_2N_2N_4A_1A_3A_2A_4A_5A_6A_7A_8}^P \\
&= \delta_{P_2N_2N_4A_1A_3A_2A_4A_5A_6A_7A_8}^P \\
&= \delta_{P_2N_2N_4A_1A_3A_2A_4A_5A_6A_7A_8}^P \\
&= \delta_{P_2N_2N_4A_1A_3A_2A_4A_5A_6A_7A_8}^P \\
\end{align*}\]

(d) \(f/g\): assuming \([f] : [A] \to [B]\) and \([g] : [C] \to [D]\) are equal to \(\delta^f_{f} : [A]_M \to [B]_N\) and \(\delta^g_{g} : [C]_O \to [D]_P\) we need to show that \([f/g] : [A/D] \to [B/C]\) is equal to \(\delta^{f/g}_{f/g} : [A]_M \otimes [D]_P \to [B]_N \otimes [C]_O\). We have the composition

\[\begin{align*}
[f/g] &= (1_{[B \otimes [C] \otimes [D]} \circ (1_{[B \otimes [C] \otimes [D]} \circ (1_{[B \otimes [C] \otimes [D]} \circ ([f] \otimes \eta_{[C] \otimes [D]}))
\end{align*}\]

with (co)domain signature

\[\begin{align*}
[A] \otimes [D] &\to [B] \otimes [C] \otimes [D] \\
&\to [B] \otimes [C] \otimes [D] \\
&\to [B] \otimes [C]
\end{align*}\]

which gets interpreted as

\[\begin{align*}
\delta^{f/g}_{f/g} &= \delta^{f/g}_{f/g} \\
&= \delta^{f/g}_{f/g} \\
&= \delta^{f/g}_{f/g} \\
&= \delta^{f/g}_{f/g} \\
&= \delta^{f/g}_{f/g} \\
&= \delta^{f/g}_{f/g} \\
&= \delta^{f/g}_{f/g} \\
&= \delta^{f/g}_{f/g}
\end{align*}\]
\[
\delta_{IPN OPJ}\left(\delta_{IKN OPJL}O \Rightarrow O_T, P \Rightarrow P_2\right)
\]

3. Structural Rules. All four cases are immediate: for the \(\hat{\alpha}_l^f\) and \(\hat{\alpha}_r^f\) rules we have that associativity is already built into generalised Kronecker deltas so precomposing with an associativity map does not change the generalised Kronecker delta. For the slightly more complicated \(\hat{\sigma}_l^f\) and \(\hat{\sigma}_r^f\) rules we have that the associativity maps do not change anything, but moreover the symmetry map is effectuated by switching the order of the arguments to the generalised Kronecker delta (the domain), so the generalised Kronecker delta itself does not change.