Global analysis of fragmentation functions and their application to polarized top quark decays considering new BABAR and Belle experimental data

S. M. Moosavi Nejad\textsuperscript{1,3}, M. Soleymaninia\textsuperscript{3}, A. N. Khorramian\textsuperscript{2,3}

\textsuperscript{1} Faculty of Physics, Yazd University, P.O. Box 89195-741, Yazd, Iran
\textsuperscript{2} Faculty of Physics, Semnan University, 35131-19111 Semnan, Iran
\textsuperscript{3} School of Particles and Accelerators, Institute for Research in Fundamental Sciences, P.O.Box 19395-5531, Tehran, Iran

March 3, 2014

Abstract. Recently, the Belle and BABAR Collaborations published the single-inclusive electron-positron annihilation data at the center of mass energies (\sqrt{s}) of 10.52 GeV and 10.54 GeV, respectively. These new data offer one the possibility to determine the nonperturbative initial conditions of fragmentation functions much more accurately. In our previous work \cite{1}, we extracted the fragmentation functions for \(\pi^{\pm}\) and \(K^{\pm}\) particles at next-to-leading order (NLO) including these new data, for the first time. These new data are in the regions of larger scaled-energy \(z\) and lower \(\sqrt{s}\). Our main purpose is to show that adding these new data in our analysis how much improve the fragmentation functions of \(\pi^{\pm}\) and \(K^{\pm}\) at NLO. We hope this analysis can obvious the effects of these recent data on FFs for whom is studying the behavior of light meson FFs. We also apply, for the first time, the extracted fragmentation functions to make our predictions for the scaled-energy distributions of \(\pi^{\pm}\) and \(K^{\pm}\) inclusively produced in polarized top quark decays at NLO.

1 Introduction

Fragmentation functions (FFs) \(D_{i}^{h}(z, \mu_{F}^{2})\), which can be interpreted as the probability for a parton \(i\) at the factorization scale \(\mu_{F}\) to fragment to a hadron \(h\) carrying away a fraction \(z\) of its momentum, are the key quantities for calculating the hadron production cross section, investigating the properties of quarks in heavy ion collisions and spin physics. Specially, to study the properties of top quark at LHC, one of the proposed channels is to consider the energy spectrum of outgoing mesons from top decays, in which by having parton-level differential decay rates \cite{2,3,4} and the FFs of partons into hadrons, one can calculate the energy distribution of observed mesons.

Generally, there are two main approaches to evaluate the FFs. The first approach is based on the fact that the FFs for mesons containing a heavy quark can be computed theoretically using perturbative QCD (pQCD) \cite{16,17,18,19}. The first theoretical attempt to explain the procedure of hadron production from a heavy quark was made by Bjorken \cite{10} by using a naive quark-parton model (QPM). He deduced that the inclusive distribution of heavy hadron should peak almost at \(z = 1\), where \(z\) refers to the scaled-energy variable. The pQCD scheme was followed by Peterson \cite{11}, Suzuki \cite{12}, Amiri and Ji \cite{13}, while in this scheme Suzuki calculates the heavy FFs using a convenient Feynman diagram. One of us, using the Suzuki approach has calculated the FF for c-quark to split into S-wave \(D^{0}/D^{+}\) meson \cite{14} and the initial FF of gluon to split into S-wave charmonium state (\(J/\psi\)) \cite{15} to leading order in the QCD coupling constant \(\alpha_{s}\).

In the second approach, which is frequently used to obtain the FFs, these functions are extracted from experimental data analysis using the data from \(e^{+}e^{-}\) reactions, lepton-hadron and hadron-hadron scattering processes. This situation is very similar to the determination of the parton distribution functions (PDFs). Among all scattering processes, the best processes which provide a clean environment to determine the FFs are \(e^{+}e^{-}\) annihilation processes \cite{16,17}. There are several theoretical studies on QCD analysis of FFs which used special parametrization and different experimental data in their global analysis. Recent extracted FFs are related to SKMA \cite{1}, AKK \cite{18}, DSSV \cite{19} and HKNS \cite{20}, using different phenomenological models. Since the hadronization mechanism is universal and independent of the perturbative process which produces partons, one can exploit, for example, the existing data on \(e^{+}e^{-}\to bb\) events to fit such models and describe the b-quark non-perturbative fragmentation in other processes, such as top decay.

In our previous work \cite{1}, we determined the nonperturbative \(\pi^{\pm}\) and \(K^{\pm}\) FFs, both at Leading Order (LO) and NLO in the modified Minimal-Subtraction (\(\overline{MS}\)) factorization scheme, by global fitting the fractional-energy spectra of these hadrons obtained from the single-inclusive \(e^{+}e^{-}\) annihilation (SIA) and the semi-inclusive deep in-
elastic scattering (SIDIS) data from HERMES and COMPASS. However, data for the production of $\pi^\pm$ is generally more accurate and plentiful than for the production of other particles at the initial hadron. New data on $\pi^+/K^+$ production with much higher accuracy at larger $z$ and lower $\sqrt{s}$ have been presented by the Belle Collaboration at $\sqrt{s} = 10.52$ GeV [18] and BABAR Collaboration at $\sqrt{s} = 10.54$ GeV [19]. These new data offer us the possibility to determine the nonperturbative initial conditions of the FFs much more accurately. Note that large $z$, low $\sqrt{s}$ impose more constraints on the gluon FF than the smaller $z$, higher $\sqrt{s}$ ones do. Furthermore, the large span in center-of-mass (c.m.) energy ($\sqrt{s}$) ranging from 10.52 GeV way up to 91.2 GeV [21, 22, 23, 24, 25, 26, 27, 28] provides us with a powerful lever arm to test the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution of the FFs.

In the Standard Model (SM) of particle physics top quark has short lifetime and it decays before hadronization takes place, then its full polarization content is retained when it decays. Therefore we can study the top spin state using the angular distributions of its decay products [30]. In this work we make our predictions for the scaled-energy distribution of $\pi^+/K^+$ inclusively produced in polarized top quark decays, $t(\uparrow) \rightarrow W^+ + b(\rightarrow \pi^+/K^+ + X)$, using the extracted FFs from new data. In particular, these predictions will enable us to deepen our understanding of the nonperturbative aspects of $\pi^+/K^+$ formation by hadronization and to pin down the $b \rightarrow \pi^+/K^+$ FFs. This paper is organized as follows. In section 2 we describe our formalism and parametrization form for pion and kaon fragmentation densities. In section 3 we explain the effect of Belle and BABEL data on FFs determination. Our predictions for the energy spectrum of pions and kaons produced in polarized top quark decays are presented in section 4. In section 5 global minimization and error calculation method are described. Our conclusion is summarized in section 6.

2 QCD analysis of fragmentation functions

The FFs are the nonperturbative part of the hadronization processes and they have an important role in the calculation of single-inclusive hadron production in any reaction. According to the factorization theorem, the leading twist component of any single hadron inclusive production measurement can be expressed as the convolution of FFs with the equivalent productions of real partons, which are perturbatively calculable, up to possible PDFs to account for any hadrons in the initial state. As an example, the production of hadron $H$ in the typical scattering process of $A + B \rightarrow H + X$, can be expressed as

$$d\sigma = \sum_{a,b,c} \int_0^1 dx_a \int_0^1 dx_b \int_0^1 dz f_{a/A}(x_a, \mu) f_{b/B}(x_b, \mu) \times$$

$$\frac{1}{\sigma_{tot}} dz \int dz \sigma(e^+e^{-} \rightarrow HX) = \sum_i C_i(z, \alpha_s) \otimes D_i^H(z, Q^2),$$

where, the function $D_i^H(z, Q^2)$ indicates the probability to find the hadron $H$ from a parton $i(z = g, u, d, s, \ldots)$ with the energy fraction $z = 2E_H/\sqrt{s}$ and $C_i(z, \alpha_s)$ is the Wilson coefficient function based on the partonic cross section $e^+e^- \rightarrow q\bar{q}$ which is calculated in the perturbative QCD [31, 32, 33], and the convolution integral is defined as $f(z) \otimes g(z) = \int_0^1 dy f(y)g(z/y)$. In the equation above, $X$ stands for the unobserved jets and $\sigma_{tot}$ is the total hadronic cross section [34].

There are several different strategies to extract the FFs from data analysis so in the present analysis we adapt the zero-mass variable-flavor-number (ZM-VFN) scheme [35]. This scheme works best for high energy scales, where the mass of heavy quarks are set to zero from the start and the non-zero values of the $c$- and $b$-quark masses only enter through the initial conditions of the FFs, and the mass of the heavy hadron sets the lower bound on the scaling variable $z$. In the phenomenological approach, the FFs are parameterized in a convenient functional form at the initial scale $\mu_0$ in each order, i.e. LO and NLO. Various phenomenological models like Peterson model [11], Power model [35], Cascade model [36] etc., have been developed to describe the FFs. Here, we apply very flexible parameterization form for the $\pi$ and $K$ FFs at NLO, considering SIDIS data from LEP (ALEPH [24], DELPHI [22, 23] and OPAL [24] Collaborations), SLAC (BABAR [17], SLD [25] and TPC [27] Collaborations), DESY (TASSO [26] Collaboration) and KKE (Belle [18] and TOPAZ [25] Collaborations) and SIDIS data from HERMES05 [37] and COMPASS [38, 39]. At the initial scale $\mu_0$ this parametrization contains a functional form as

$$D_i^H(z, \mu_0^2) = N_i e^{\alpha_i(1 - z)^3[1 - e^{-\gamma_i z}]},$$

which is an appropriate form for the light hadrons. To control medium $z$ region and to improve the accuracy of the global fit the term $[1 - e^{-\gamma_i z}]$ is considered. The free parameters $N_i$, $\alpha_i$, $\beta_i$, and $\gamma_i$ are determined by global fitting $\chi^2$ using the SIDIS and JETSCAPE data and their $\mu$ evolution is determined by the DGLAP equations. Our results are listed in Tables 1 and 2 for $\pi$ and $K$ FFs. The initial scale $\mu_0$ is different for partons so that the value of $\mu_0^2 = 1$ GeV$^2$ is chosen for splitting of the light-quirks
\( (u, d, s) \) and gluon into the \( \pi^\pm/K^\pm \) mesons and for the \( c- \) and \( b- \) quarks it is taken to be \( \mu_0^u = m_c^2 \) and \( \mu_0^b = m_b^2 \), respectively. According to the partonic structure of \( \pi^- (\bar{u}d) \) and \( K^- (\bar{u}s) \), the following assumptions are considered during our calculations:

\[
D_i^\pi^- (z, \mu_0^2) = D_i^{\bar{p}} (z, \mu_0^2), \quad D_i^K^- (z, \mu_0^2) = D_i^K (z, \mu_0^2),
\]

where \( i = u, d, s, c, b \) and for the gluon FFs, it reads

\[
D_g^\pi^- (z, \mu_0^2) = D_g^{\bar{p}} (z, \mu_0^2), \quad D_g^K^- (z, \mu_0^2) = D_g^K (z, \mu_0^2).
\]

### 3 The impact of Belle and BABAR data on FFs

Recently the Belle \[16\] and BABAR \[17\] Collaborations published inclusive hadron production cross sections at the c.m. energies of 10.52 GeV and 10.54 GeV, respectively. These new data contain a purely \( e^+e^- \rightarrow q\bar{q} \) sample, where \( q = u, d, s, c \), and the c.m. energies are below the threshold of \( b\bar{b} \) pair production.

The large amounts of data from Belle and BABAR Collaborations are available at the lower scales of \( Q = 10.52 \) GeV and \( Q = 10.54 \) GeV, while the energy scales of the other SIA experimental data extracted from 29 GeV to 91.2 GeV and most of them are limited to results from experiments at LEP and SLAC at \( Q = M_Z \). In addition, these new data include differential cross sections at larger \( z \) values, i.e. \( z > 0.7 \). Since the cross section measurements at the small \( z \) depend on the FFs at all larger \( z \) values, the inclusion of these measurements will also lead to improved constraints on the FFs at the \( z \) values currently determined in global fits. Unfortunately, while this procedure is simple to any order, explicit results for a full small \( z \) resummed NLO calculation do not exist yet.

After adding these new data in analysis, our results for \( \pi^\pm \) and \( K^\pm \) at \( Q = 10.52 \) GeV and \( Q = 10.54 \) GeV are compared with experimental data in Figs. 1-3 and 4. Other FF models are also compared with the new data and these comparisons show a nice agreement between our model and these data.

In Figs. 3 and 4, we present the extracted NLO FFs of \( \pi^+ \) and \( K^+ \) in the initial scale \( \mu_0 \). To show that adding these new data how much modify \( \pi^+ \) and \( K^+ \) FFs in our analysis, the ratio of obtained FFs by including these new data (scenario 1) to ones without containing them (scenario 2) are also shown in Figs. 3 and 4. According to these figures, the differences between these two scenarios are considerable at some regions of \( z \) for most of FFs and adding Belle and BABAR data change the light quark FFs more than the gluon and heavy quark FFs. Since the c.m. energies of Belle and BABAR data are below the threshold of \( b \) quark production, it could be expected that the \( b \) quark FFs do not change considerably. This is confirmed in the figures.

### 4 Energy spectrum of the inclusive \( \pi \) and \( K \) in top quark decays

In this section, we apply the extracted nonperturbative FFs to make our phenomenological predictions for the energy spectrum of the light mesons \( \pi^\pm \) and \( K^\pm \) produced through polarized top decays

\[
t(\uparrow) \rightarrow b + W^+(g) \rightarrow \pi^\pm/K^\pm + X,
\]

where \( X \) stands for the unobserved final state. Both the \( b \)-quark and the gluon may hadronize to the outgoing light mesons whereas the gluon contributes to the real radiation at NLO. To obtain the energy distribution of the hadron \( H \), we employ the factorization theorem of the QCD improved parton model where the energy distribution of a hadron can be expressed as the convolution of the nonperturbative FFs \( D_i^H (z, \mu_F) \) with the parton-level spectrum as

\[
\frac{d\Gamma}{dx_H} = \sum_{i=b,g} \int_{x_i^{\text{min}}}^{x_i^{\text{max}}} \frac{dx_i}{x_i} \frac{d\Gamma_i^{\text{pol}}}{dx_i} (\mu_R, \mu_F) D_i^H \left( \frac{x_H}{x_i}, \mu_F \right).
\]

Here, we define the scaled-energy fraction of hadron as \( x_H = 2E_H/(m_t^2 - m_W^2) \) and \( d\Gamma_i^{\text{pol}}/dx_i \) is the parton-level differential rates of the process \( t(\uparrow) \rightarrow i + W^+ (i = b, g) \). The analytical expressions for the parton-level differential decay widths \( d\Gamma_i^{\text{pol}}/dx_i \) at NLO are presented in Ref. 30. Here, the factorization and the renormalization scales are set to \( \mu_R = \mu_F = m_t \).

In Figs. 5 and 6, our predictions for the pion and kaon mesons are shown by studying the contributions of the \( b \rightarrow \pi^+/K^+ \) (dashed lines) and \( g \rightarrow \pi^+/K^+ \) (dot-dashed lines) fragmentation channels at NLO. As seen, the gluon contribution is negative and appreciable only in the low \( x_H \) region. Note that the contribution of the gluon FF cannot be discriminated. It is calculated to see where it contributes to \( d\Gamma/dx_H (H = \pi^+, K^+) \). So, this part of the plot is of more theoretical relevance. In the scaled energy of mesons as an experimental quantity, all contributions including the \( b \) quark, gluon, and light quarks contribute. The total contribution (solid line) at \( \mu_F = m_t \) is presented too. In these figures, the scaled-energy \( (x_H) \) distribution of light mesons produced in unpolarized (dot-dot-dashed line) and polarized (solid line) top quark decays are also studied. As is seen, in the unpolarized top decay the partial decay width at the hadron level is higher than the one in the polarized top decay.

### 5 Global minimization of \( \chi^2 \) and error calculation

The free parameters in the proposed functional forms of \( \pi/K \) FFs are determined by minimizing \( \chi^2 \) for differential cross section and asymmetry data in \( x \) space. Results are reported in Tables. 1 and 2. The \( \chi^2 \) for \( k \) data points is defined as

\[
\chi^2 = \sum_{j=1}^{k} \left( \frac{E_j - T_j}{\sigma_j^2} \right)^2.
\]
After finding the appropriate parameters which minimize the likelihood function, we can determine the behavior of $\Delta \chi^2$ by moving away the parameters from their obtained values.

$$\Delta \chi^2 \equiv \chi^2 - \chi^2_{\text{min}} = \sum_{i,j=1}^{n} H_{ij}(a_i - a_i^0)(a_j - a_j^0), \quad (9)$$

where, $H_{ij}$ are the elements of the Hessian matrix ($H_{ij} = \frac{\partial^2 \chi^2}{\partial a_i \partial a_j}|_{\text{min}}$) and $n$ is the number of free parameters. According to the linear propagation of error, one can use the following formula for calculation of error on any quantity $F$

$$\langle \Delta F \rangle^2 = \Delta \chi^2 \sum_{j,k} \frac{\partial F}{\partial a_j} C_{jk}(a) \frac{\partial F}{\partial a_k}. \quad (10)$$

Here, $T_j$ and $E_j$ stand for the theoretical results and experimental values of data and $\sigma^E_j$ is the error of corresponding experimental value. The obtained values of $\chi^2/d.o.f$ for pion and kaon are 1.47 and 1.54 in our global fit, respectively.

After finding the appropriate parameters which minimize $\chi^2$, we can determine the behavior of $\Delta \chi^2$ by moving away the parameters from their obtained values.

$$\Delta \chi^2 \equiv \chi^2 - \chi^2_{\text{min}} = \sum_{i,j=1}^{n} H_{ij}(a_i - a_i^0)(a_j - a_j^0), \quad (9)$$

where, $H_{ij}$ are the elements of the Hessian matrix ($H_{ij} = \frac{\partial^2 \chi^2}{\partial a_i \partial a_j}|_{\text{min}}$) and $n$ is the number of free parameters. According to the linear propagation of error, one can use the following formula for calculation of error on any quantity $F$

$$\langle \Delta F \rangle^2 = \Delta \chi^2 \sum_{j,k} \frac{\partial F}{\partial a_j} C_{jk}(a) \frac{\partial F}{\partial a_k}. \quad (10)$$
(11) reduces to

$$\sum_{j=1}^{n} C_{ij}(a) v_{jk} = \lambda_{k} v_{ik}.$$  

We can expanded the parameter variation around the global minimum in a basis of eigenvectors and eigenvalues, that is,

$$\Delta a_i \equiv (a_i - a_i^0) = \sum_{k=1}^{n} e_{ik} z_k,$$  

where $e_{ik} \equiv \sqrt{\lambda_{k}} v_{ik}$. It can be shown that the expansion of the $\chi^2$ in the fit parameters $a_i$ near the global minimum (Eq. 9) reduces to

$$\Delta \chi^2 = \sum_{k=1}^{n} z_k^2,$$  

where $\sum_{k=1}^{n} z_k^2 \leq T^2$ is the interior of a sphere of radius $T$. In order to investigate that whether $\Delta \chi^2$ shows the assumed quadratic behavior of the parameters from the best fit, we present the dependence of the global $\Delta \chi^2$ along some random samples of eigenvector directions in Figs. 7 and 8.

To obtain the standard linear errors of FFs, we use

$$[\Delta D_i^H(z)]^2 = \Delta \chi^2 \sum_{j,k} \frac{\partial D_i^H(z, a_j)}{\partial a_j} C_{jk}(a) \frac{\partial D_i^H(z, a_k)}{\partial a_k},$$  

where $D_i^H(z; Q^2)$ is the evolved fragmentation density at $Q^2$ and $n$ is the number of parameters in the global fit.

Finally we can compute the uncertainties of any FFs at any value of $Q^2$ by the QCD evolution. The $\pi^+$ and $K^+$ FFs and their uncertainties based on this method are presented in Figs. 9 and 10 at NLO. More information and detailed discussions can be found in Refs. [20, 40, 41].

6 Conclusion and results

In the present work we determined the nonperturbative FFs of partons into the pion and kaon from global analysis on SIA and SIDIS data at NLO. Our main aim was to show that adding the recent SIA data from Belle and BABAR Collaborations at $\sqrt{s} = 10.52$ GeV and $\sqrt{s} = 10.54$ GeV, respectively, how much improve the results obtained for partonic FFs. Our analysis showed that these new data thus far the $\pi^+\pi^-$ FFs at the large-$z$ region while the $s \to \pi^+$ FF is affected at low-$z$. As Fig. 3 shows, these new data do not change the FFs of gluon and heavy quarks into the pion.

Concerning the effects of new data on kaon FFs, as is seen from Fig. 9 the $u \to K^+$ FF is affected at low-$z$ ($z < 0.2$) more than large-$z$, but the $d \to K^+$ FF affected at $z > 0.07$. The FF of $g \to K^+$ is decreased everywhere, e.g. about 25% at $z = 0.01$. The $c \to K^+$ FF is increased at large-$z$ when we consider new data. We hope our results can obvious the effects of the recent new data on FFs for whom is studying the behavior of light meson FFs. In [1], using the computed FFs we have studied the scaled-energy $(x_H)$ distribution of the light mesons in unpolarized top quark decays and in the present work we made our predictions for the scaled-energy $(x_H)$ distributions of
the pion and kaon in polarized top decays. The scaled-energy distribution of hadrons in polarized/unpolarized top quark decays at LHC enables us to deepen our knowledge of the hadronization process. The universality and scaling violations of the pion and kaon FFs will be able to test at LHC by comparing our NLO predictions with future measurements of $dΓ/dx_H$ and $dΓ(↑)/dx_H$.

Note, the FORTRAN package containing our unpolarized fragmentation functions for pion and kaon at LO and NLO can be obtained via e-mail from the authors.

7 Acknowledgments

We warmly acknowledge G. Corcella for valuable discussions, critical remarks and reading the manuscript. A. N. K. and S. M. M. N. thank the CERN TH-PH division for its hospitality where a portion of this work was performed. We thank the School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM) for financial support.

References

1. M. Soleymaninia, A. N. Khorramian, S. M. Moosavi Nejad and F. Arbabifar, Phys. Rev. D 88, 054019 (2013), [arXiv:1306.1612] [hep-ph].
2. B. A. Kniehl, G. Kramer and S. M. Moosavi Nejad, Nucl. Phys. B 862 (2012) 720 [arXiv:1206.2528] [hep-ph].
3. M. Cacciari, G. Corcella and A. D. Mitov, JHEP 0212, 015 (2002) [hep-ph/0209204].
4. G. Corcella and A. D. Mitov, Nucl. Phys. B 623, 247 (2002) [hep-ph/0110319].
5. J. P. Ma, Nucl. Phys. B 506 (1997) 329.
6. E. Braaten and T. C. Yuan, Phys. Rev. Lett. 71 (1993) 1673.
7. C. -H. Chang and Y. -Q. Chen, Phys. Lett. B 284 (1992) 127.
8. E. Braaten, K. -m. Cheung and T. C. Yuan, Phys. Rev. D 48 (1993) 4230.
9. D. M. Scott, Phys. Rev. D 18 (1978) 210.
10. J. D. Bjorken, Phys. Rev. D 17 (1978) 171.
11. C. Peterson, D. Schlatter, I. Schmitt and P. M. Zerwas, Phys. Rev. D 27 (1983) 105.
12. M. Suzuki, Phys. Lett. B 71 (1979) 139.
13. F. Amiri and C. -R. Ji, Phys. Lett. B 195 (1987) 593.
14. S. M. M. Nejad and A. Armat, Eur. Phys. J. Plus 128 (2013) 121 [arXiv:1307.0351] [hep-ph].
15. S. M. M. Nejad and D. Mahdi [Belle Collaboration], Phys. Rev. Lett. 111, 062002 (2013).
16. J. P. Lees et al. [BaBar Collaboration], Phys. Rev. D 88, 032011 (2013) [arXiv:1306.2895] [hep-ex].
17. S. Albino, B. A. Kniehl and G. Kramer, Nucl. Phys. B 803, 42 (2008) [arXiv:0803.2768] [hep-ph].
18. D. de Florian, R. Sassot and M. Stratmann, Phys. Rev. D 75, 114010 (2007) [hep-ph/0703242] [HEP-PH].
19. M. Hirai, S. Kumano, T. -H. Nagai and K. Sudoh, Phys. Rev. D 75, 094009 (2007) [hep-ph/0702250].
20. D. Buskulic et al. [ALEPH collaboration], Z. Phys. C66, 355 (1995); R. Barate et al., Phys. Rep. 294, 1 (1998).
21. P. Abreu et al. (DELPHI collaboration), Eur. Phys. J. C5, 585 (1998).
22. P. Abreu et al. (DELPHI collaboration), Nucl. Phys. B444, 3 (1995).
Fig. 4. Upper panels: fragmentation functions for $K^+$ at $Q^2_0 = 1$ GeV$^2$, $m^2_c$ and $m^2_b$ at NLO. Rest panels: ratios of our fragmentation functions from scenario 1 to the ones of scenario 2.
Fig. 5. $\frac{d\Gamma(t\uparrow \rightarrow \pi^+ + X)}{dx_{\pi^+}}$ as a function of $x_{\pi^+}$ (solid line) at $\mu_F = m_t$. Left panel: The NLO result is broken up into the contributions due to $b \rightarrow B$ (dashed line) and $g \rightarrow B$ (dot-dashed line) fragmentation. Right panel: The unpolarized (dot-dot-dashed line) and polarized (solid line) partial decay rates at NLO.

Fig. 6. As in Fig. 5 but for $K^+$ at NLO.
Fig. 7. Examples of pion $\Delta \chi^2$ deviations from the expected quadratic behavior $\Delta \chi^2 = T^2$ for random sample eigenvector directions.

Fig. 8. As in Fig. 7 but for kaon, considering some random sample eigenvector directions.
Fig. 9. Fragmentation densities and their uncertainties are shown for $\pi^+$ at $Q^2_0 = 1\text{ GeV}^2$, $m_c^2$ and $m_b^2$ at NLO.

Fig. 10. As in Fig. 9 but fragmentation densities and their uncertainties for $K^+$ at NLO.