Well-Ordered Model Universes

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December 18, 2018

Abstract

In this paper we show how to build a model of ZFC such that all its inner models satisfying the Axiom of Choice are well-ordered with respect to inclusion, and that said ordering is of arbitrary height (including possibly Ord high). We do this by iterating $\kappa$-Sacks forcing for ever-increasing $\kappa$, while showing that such forcings do not add any unexpected intermediate inner models.

1 Introduction

In this section we aim to present and formalize the concept of a well-ordered model universe, and establish the proper axiomatic framework to work with it. We start with an informal presentation to explain the motivation behind this idea.

Definition 1.1. Let $M$ be a model of ZFC (ZF). We call a substructure $N$ of $M$ an inner model of ZFC (ZF) if:

1. $N$ is a model of ZFC (ZF);
2. The interpretation of $\in_N$ is $\in_M \cap N^2$;
3. The domain of $N$ is a transitive class of $M$;
4. $N$ has the same ordinals as $M$.

A model that satisfies only conditions 1 and 2 is called a standard model. Hence an inner model is a standard transitive model that has the same ordinals as the base model.

Unless otherwise stated, we reserve the term inner model to refer exclusively to inner models of ZFC. If we want to discuss an inner model of ZF we shall refer to it explicitly as such.

It was Kurt Gödel that proved in [8] that any model of ZF has a least inner model $L$, called the constructible universe, which is also a model of ZFC + GCH.
Thus, if $V \neq L$, there are non-trivial inner models of $V$, and we can then partially order them with respect to inclusion. This partial order has a unique least element $L$ and a unique greatest element $V$, and we may naturally enquire about its other order-theoretic properties. The aim of this paper is to explore the consistency and implications of the well-ordering property for this partial order, our main theorem being the construction of a model of ZFC where the inner models are not only well-ordered with respect to inclusion, but said ordering is in fact order-isomorphic to all ordinals.

In practice, this means we construct a model $M$ of ZFC and a “sequence” of inner models $\langle M_\alpha \mid \alpha \in \text{Ord} \rangle$ such that

1. $\alpha < \beta$ if and only if $M_\alpha \subsetneq M_\beta$;
2. For every inner model $N$ of $M$ there is an $\alpha \in \text{Ord}$ such that $N = M_\alpha$.

This is a sort of tower of inner models, and to construct it we will need to gradually extend the tower from its base, passing through the successor case, singular limits, regular limits and finally the class case. That will be the general path we follow, but first we need to address a tricky part of our definition - in ZF we cannot formally talk about a sequence of classes, and each inner model of $V$ is by definition a proper class. Normally, it is enough to use the class notation as a shorthand for formulas and ignore the particulars, but in this paper we’ll also be interested in the interplay between these classes. So in order to deal with these explicitly, we turn to Bernays-Gödel set theory, in short BG (or BGC if we add Global Choice), to serve as our axiomatic framework.

Bernays-Gödel set theory, sometimes known as Von Neumann–Bernays–Gödel set theory (NBG for short), has its origins in a 1925 paper by John von Neumann [13], which formally introduced classes into set theory for the first time. Von Neumann’s theory employed functions and arguments as its primitive notions, and used them to define sets and classes. However, in the 1930s Paul Bernays reformulated the theory by taking classes and sets as the primitive notions [2]. Later, while working on his proof for the relative consistency of the Axiom of Choice [8], Gödel significantly simplified Bernay’s theory, leading to what is now known as Bernays-Gödel set theory.

Unlike Zermelo-Fraenkel, Bernays-Gödel set theory allows for two types of objects: classes and sets. Every set is also considered a class, and if a class is a member of another class then it is also a set. Thus, a model $\langle V, \mathcal{V}, \in \rangle$ of BG consists of a collection $\mathcal{V}$ of classes together with a subcollection $V \subseteq \mathcal{V}$ of sets and a relation $\in \subseteq V \times V$. The axioms of this system are mostly very similar to those of ZF, and a formal exposition of them and their application in class forcing can be found in [15]. To avoid any ambiguity, we always denote a model of BG in the form of a $\langle V, \mathcal{V}, \in \rangle$ triplet, and a model of ZF as plain $V$.

In order to justify our use of BG, we quote the following facts about the link between it and ZF.

Fact 1.2. BG (BGC) is a conservative extension of ZF (ZFC).
This fact was proven by Paul Cohen in [3]. Furthermore, Mostowski [12] showed that every set-theoretical statement provable in ZF (ZFC), and that if a sentence involving only set variables is provable in BG (BGC), then it is provable in ZF (ZFC) as well.

**Fact 1.3.** Let \( \langle V, V', \in \rangle \) be a model of BG (BGC), where \( V \) is the collection of classes and \( V \subseteq V \) is the collection of sets. Then by taking \( V \) and \( \in \cap (V \times V) \), we get a model of ZF (ZFC).

So if we have a model of BG, by “throwing away” the classes, we are left with a model of ZF. What about the other way round?

**Fact 1.4.** Let \( V \) be a model of ZF. Take \( V' \) to be the collection of all classes definable in \( V \) with set parameters, and take \( \in \) to be the obvious extension of the membership relation to \( V' \). Then \( \langle V, V', \in \rangle \) is a model of BG.

Note that even if \( V \) is a model of ZFC then \( \langle V, V', \in \rangle \) as defined above might only satisfy BG, not BGC. But using class forcing one can add a uniform choice function that is \( \kappa \)-closed for each \( \kappa \), and so adds no new sets to the universe. The resulting model \( \langle V, V', \in \rangle \) extends \( \langle V, V, \in \rangle \), satisfies BGC and has the same sets and the same restriction of \( \in \) to sets as \( \langle V, V, \in \rangle \) (see A.1 in [15]).

Facts [1.3] and [1.4] establish a useful correspondence between models of ZF and BG.

**Corollary 1.5.** \( M \) is a definable proper inner model of \( V \) if and only if \( M \neq V \) and \( \langle M, M, \in \rangle \subseteq \langle V, V, \in \rangle \) (as defined in fact [1.4]).

**Proof.** \( \Rightarrow \) \( M \) is proper and so \( M \neq V \). \( M \) is a definable class of \( V \), therefore \( M \in V \), and also every class definable in \( M \) with set parameters is similarly definable in \( V \). Therefore \( M \subseteq V \), and \( \langle M, M, \in \rangle \subseteq \langle V, V, \in \rangle \).

\( \Leftarrow \) \( M \in \mathcal{M} \) and so \( M \in V \), so \( M \) is a definable class in \( V \). \( M \subseteq V \) and according to fact [1.3] is a model of ZF.

But moving to BG doesn’t quite work out all the kinks. In particular, BG doesn’t allow for class membership within another class. Therefore in order to speak of a sequence of classes we need to abandon our standard definition of a sequence, and instead use an alternative definition that is also suitable for classes.

**Definition 1.6.** Let \( I \) be a class. Then we call \( S \subseteq I \times V \) an \( I \)-indexed family of classes, and for each \( i \in I \) we denote \( S_i = \{ x \mid (i, x) \in S \} \). If \( I \) is well-ordered, then we say \( S \) is a sequence, and \( I \) is its underlying order.

Thus, instead of having a 'family of classes' which each class is a member of, each enumerated class is generated using straightforward class comprehension. This perspective allows us to speak of sequences of classes within BG.

Now that we’ve set up this notion, we finally come to the primary definition of this article:
**Definition 1.7.** Let $\langle V, V, \in \rangle$ be a model of $\text{BG}$. We call a model $N \subseteq V$ of $\text{ZFC}$ where all its inner models are well-ordered with respect to inclusion a *well-ordered model universe*. Formally, we postulate the existence of a class $\mathcal{M}$ in $\langle V, V, \in \rangle$, which is the sequence of all proper inner models of $N$ ordered by inclusion. This means:

1. $\mathcal{M} \subseteq I \times N$;
2. $M$ is a proper inner model of $N$ if and only if there exists a unique $a \in I$ such that $M = M_a = \{ x \mid (a, x) \in \mathcal{M} \}$;
3. $I$ is a well-ordered class;
4. If $a <_I b$ then $(a, x) \in \mathcal{M} \rightarrow (b, x) \in \mathcal{M}$.

In summary, applying the convention that lower-case letters indicate sets and upper-case letters indicates classes, we demand the following be true:

$$\exists \exists I \ (\mathcal{M} \subseteq I \times N \land (M \subseteq N \text{ is an inner model} \leftrightarrow \exists a \in I \ (M = \{ x \mid (a, x) \in \mathcal{M} \}) \land I \text{ is well-ordered} \land a <_I b \rightarrow ((a, x) \in \mathcal{M} \rightarrow (b, x) \in \mathcal{M})$$

Where there’s any ambiguity about the base model we denote class $M$ as $M(N)$.

A few important remarks are in order.

**Remark 1.8.** Class $M(N)$ belongs by definition to $\langle V, V, \in \rangle$. It need not be definable in $N$, nor even in $V$. However in the example that we build later in the article, $M(V)$ actually *will* be definable in $V$, and then instead of working with some background model of $\text{BG}$ we will take $V$ to be the collection of all classes definable in $V$ with set parameters, exactly as we did in fact 1.4.

**Remark 1.9.** Our demand that $M$ be a *proper* inner model is superfluous, and only used to simplify discussion of the case $I = \text{Ord}$, when all the proper inner models are in a bijection with the ordinals. This convention allows us to prove general theorems about all models $M_\alpha \alpha \in \text{Ord}$, without having to constantly special-case “$M_{\text{Ord}}$”.

**Remark 1.10.** It is natural to ask why when defining the sequence we only demand $I$ be well-ordered, instead of being equal to some ordinal or $\text{Ord}$. The reason for this is that we also can also consider sequences that are *longer* than the ordinals, and we don’t want to unnecessarily exclude them from the definition. We revisit this issue in the last section of this article, but in the meanwhile we define what it means for a well-ordered model universe to be *nice*.

**Definition 1.11.** We call a well-ordered model universe *nice* if the underlying order of $\mathcal{M}$ is equal to some ordinal or to $\text{Ord}$.

In essence, a well-ordered model universe is nice if its model tower isn’t ’too tall’. This restriction is not superficial. We shall later see an interesting property that fails if the well-ordered model universe isn’t nice.
Definition 1.12. Let $N$ be a nice well-ordered model universe. We define its \textit{height} to be the order-type of the underlying order of $M$, so $ht(N) = \text{otp}(I)$. In case $I = \text{Ord}$, we instead define $ht(N) = \infty$. Note that for convenience, we designate $M_{ht(N)} = N$, even though it is \textit{not} formally part of the sequence of proper inner models.

To summarize our notational conventions, throughout this article:

1. Models of $\text{BG}$ are always denoted as a triplet $\langle V, \mathcal{V}, \in \rangle$, whereas models of $\text{ZF}$ are denoted using plain letters $V$.
2. $M$ refers exclusively to the sequence of proper inner models as defined in 1.7.
3. $M_\alpha$ shall refer to the $\alpha$th inner model of sequence $M$.
4. The height of a well-ordered model universe, denoted $ht(V)$, is the order-type of the underlying order of $M$.
5. A well-ordered model universe is considered nice if the underlying order of $M$ isn’t longer than $\text{Ord}$.

2 Implications

Now, it is time to explore some of the implications of the inner model well-ordering property. For the rest of this section we assume $\langle V, \mathcal{V}, \in \rangle \models \text{BG}$, $V$ is a well-ordered model universe, and $M \subseteq I \times V$ is its sequence of proper inner models ordered by inclusion.

Lemma 2.1. $M_0 = L$.

Proof. We know from G"odel [8] that $L$ is the least inner model of $V$. Therefore, in order to be included in our hierarchy, we must have $M_0 = L$. \qed

This begs the question of how 'close' are $V$ and $L$, assuming our well-ordered model universe exists. One quick observation, resulting directly from the well-ordering of the inner models, is that $V$ cannot contain a measurable cardinal.

Theorem 2.2. $V \models \text{There is no measurable cardinal}$.

Proof. Suppose to the contrary, that there is a measurable cardinal $\kappa \in V$. Then there exists an elementary embedding $j : V \to M$, where $M$ is an inner model of $V$ [17]. Therefore $M = M_\alpha$ for some $\alpha \in I$. But because the embedding is elementary, $j(\kappa)$ is measurable in $M_\alpha$, so there exists an additional elementary embedding $j(j) : M_\alpha \to N$. But this means there is an elementary embedding $V \to N$, and so $N$ is itself an inner model of $V$ as well, so $N = M_\beta$ for some $\beta \in I$. However, $N \nsubseteq M_\alpha$, and therefore $\beta <_I \alpha$.

By induction, we can repeat this process and construct an infinite descending chain of inner models. But $V$ is a well-ordered model universe, so the inner models are well-ordered and this is impossible. Therefore, there is no measurable cardinal in $V$. \qed

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However, not only are there no measurable cardinals in a well-ordered model universe, but we can further show it has no $0^\#$ as well, although this is a bit less straightforward.

**Theorem 2.3.** $V \models \neg \exists 0^\#$ does not exist.

**Proof.** Suppose to the contrary, that $0^\#$ does exist. Then every uncountable cardinal in $V$ is inaccessible in $L$ (see corollary 18.3 in [9]).

Now remember Cohen forcing [3], where we use finite partial functions, and define $\mathbb{P} = \text{Fin}(\omega, 2)^L$. In $L$ we have $|\mathbb{P}| = \aleph_0$, and therefore $|\mathcal{P}(\mathbb{P})| = 2^{\aleph_0} = \aleph_1$, which is obviously smaller than the first inaccessible cardinal. Hence in $V$ we have $|\mathcal{P}(\mathbb{P})| = \aleph_0$, meaning there are at most countable many dense subsets of $\mathbb{P}$, and therefore by the Rasiowa-Sikorski lemma [14] there exists a generic set $G \in V \setminus L$ that intersects them all. Hence $V \models \exists G(L[G] \supseteq L \land L[G] \models \text{ZFC})$. Therefore $L[G]$ is an inner model of $V$.

However each Cohen forcing is isomorphic to the product of two separate Cohen forcings. Namely, for each $I_0 \subsetneq I$ we have $\text{Fin}(I, 2) \cong \text{Fin}(I_0, 2) \times \text{Fin}(I \setminus I_0, 2)$ (see Kunen [11] ch. VIII 2.1). So take $I = \omega$ and $I_0$ the set of even natural numbers. According to the theorem $G_0 = G \cap \text{Fin}(I_0, 2)$ is $\text{Fin}(I_0, 2)$-generic over $L$, $G_1 = G \cap \text{Fin}(\omega \setminus I_0, 2)$ is $\text{Fin}(\omega \setminus I_0, 2)$-generic over $L[G_0]$, and $L[G] = L[G_0][G_1]$. So $G_1 \notin L[G_0]$, and for the same reasoning we have $G_0 \notin L[G_1]$. But as both $L[G_0]$ and $L[G_1]$ are inner models of $L[G]$ and so of $V$, either $L[G_0] \subsetneq L[G_1]$ or $L[G_1] \subsetneq L[G_0]$. Either way we arrive at a contradiction. Therefore $0^\#$ does not exist. \qed

Now that we know $V$ cannot be too far off $L$, we may wonder if there is perhaps a deeper connection between the two. For this we turn to the notion of relative constructibility.

Note that there is a lot of confusion regarding its notation, so we shall now present the notation used by Jech in [9] and which we adhere to.

Constructibility can be generalized in two different ways. One way is to consider sets constructive relative to a given set $A$, resulting in the inner model $L[A]$.

This is done by defining $\text{def}_A(M) = \{X \subseteq M \mid X$ is definable over $(M, \in, A \cap M)\}$, where $A \cap M$ is a unary predicate, and then defining a cumulative hierarchy:

- $L_0[A] = \emptyset$
- $L_{\alpha+1}[A] = \text{def}_A(L_\alpha[A])$
- $L_\delta[A] = \bigcup_{\alpha < \delta} L_\alpha[A]$ for limit ordinals
- $L[A] = \bigcup_{\alpha \in \text{Ord}} L_\alpha[A]$

The resulting model $L[A]$ is a model of ZFC (see ch. 13 in [9]).

Another way, yields for every set $A$ the smallest inner model of ZF that contains it. However, in general, this model need not satisfy the Axiom of Choice.
Let $T = TC\{A\}$ be the transitive closure of $A$, and define the following cumulative hierarchy:

\[
L_0(A) = T \\
L_{\alpha+1}(A) = def\ (L_\alpha(A)) \\
L_\delta(A) = \bigcup_{\alpha<\delta} L_\alpha(A) \text{ for limit ordinals} \\
L(A) = \bigcup_{\alpha\in\text{Ord}} L_\alpha(A)
\]

The resulting model $L(A)$ is an inner model of ZF, contains $A$, and is the smallest such model.

**Theorem 2.4.** Let $V$ be a nice well-ordered model universe. For all $\alpha \in I$ $M_\alpha = L[A]$ for some $A \in V$.

**Proof.** We prove this theorem by induction on $I$. It is trivially true for $M_0 = L = L[\emptyset]$.

For the successor stage, note that $M_\alpha = V$. By a theorem of Vopěnka [19], this means there exists a set of ordinals $A \in M_{\alpha+1} \setminus M_\alpha$. Thus $M_{\alpha+1} \supseteq L(A) \supseteq M_\alpha$. However, note that for sets of ordinals $L(A) = L[A]$, because $A \subseteq L$. Therefore $M_{\alpha+1} \supseteq L[A] \supseteq M_\alpha$. However we know there is no model of ZFC strictly between $M_\alpha$ and $M_{\alpha+1}$. Therefore $M_{\alpha+1} = L[A]$.

We turn to the limit stage. Let $\delta$ be a limit ordinal, and assume the theorem was proven for all $\beta < \delta$. We want to show that even working in $M_\delta$ we can enumerate all the inner models preceding it on the model tower. Still working in $V$, for every $\beta < \delta$ there exists a set of ordinals $C_\beta$ such that $M_\beta = L[C_\beta]$ (see ex. 13.27 in [3]). $C_\beta \in M_\delta$ and so $C_\beta \in M_\delta$, meaning $L(C_\beta)^{M_\delta} = L[C_\beta] = M_\beta$ and therefore $M_\beta$ is a definable with set parameters in $M_\delta$. So, working in $\langle M_\delta, M_\delta, \in \rangle$ as defined using fact 1.4, for each $\beta < \delta$ $M_\beta \in M_\delta$.

Also, note that any inner model of the form $L[A]$ where $A \in M_\delta$ is definable using set parameters in $V$ as well, and therefore is equal to $M_\beta$ for some $\beta < \delta$.

Despite having each individual model definable with set parameters in $M_\delta$, we still can’t be sure we can actually enumerate all the inner models preceding $M_\delta$ within $\langle M_\delta, M_\delta, \in \rangle$. So next we define an equivalence relation on sets of ordinals: $A \sim B \iff L[A] = L[B]$. For each equivalence class $[A]$ let $r[A]$ be the sets of minimal rank in $[A]$. Obviously for each $A$ $r[A]$ is a set, and by the induction hypothesis and the note above there are at most $\delta$ different models of the form $L[A]$. So $\{r[A] \mid A \in V\}$ is a set of sets, and using the Axiom of Choice we can choose a representative from each $r[A]$.

Next, because all of said models are equal to some $M_\beta$ on the chain, we can sort the representatives according to the binary relation $A \leq B \iff A \in L[B]$. The models are well-ordered because as noted above they all belong on the tower. We also already established that each one is definable with set parameters in $M_\delta$, so what we get is a sequence of representatives $\langle A_\beta \mid \beta < \delta \rangle$, which completely
enumerates the $M_\alpha$’s for all $\beta < \delta$, and which is defined using set parameters within $M_\delta$.

We define inductively two sequences $\langle B_\beta \mid \beta < \delta \rangle$ and $\langle \gamma_\beta \mid \beta < \delta \rangle$. Let $B_0 = \emptyset$, $\gamma_0 = 0$. For each $\beta < \delta$ define $\gamma_\beta = \sup \left( \bigcup_{\alpha < \beta} B_\alpha \right)$ and $B_\beta = \{ \gamma_\beta + \epsilon \mid \epsilon \in A_\beta \}$. It is clear by the definitions that $\gamma$ is strictly monotonously increasing, and that all the $B$’s are mutually pairwise disjoint.

Let $B_\delta = \bigcup_{\beta < \delta} B_\beta$. Clearly $B_\delta$ is a set of ordinals. We claim $L[B_\delta] = M_\delta$.

First note that for each $\beta < \delta$ $B_\beta = (B_\beta \cap \gamma_{\beta+1}) \setminus \gamma_\beta$ and $A_\beta = \{ \epsilon \mid \gamma_\beta + \epsilon \in B_\beta \}$. Therefore $A_\beta \subseteq L[B_\delta]$ and so $M_\beta = L[A_\beta] \subseteq L[B_\delta]$. Therefore $L[B_\delta] \supseteq M_\delta$.

On the other hand, we’ve already shown that $\langle A_\beta \mid \beta < \delta \rangle \in M_\delta$. So $\langle B_\beta \mid \beta < \delta \rangle \in M_\delta$ and therefore $B_\delta \in M_\delta$, implying $L[B_\delta] = L(B_\delta) \subseteq M_\delta$.

We conclude that $L[B_\delta] = M_\delta$, and so the induction is complete. \hfill \Box

It is instructive to note that we used the niceness property exactly once, to justify how we could simultaneously choose a representative from each $r[A]$. To do this for class-many sets would have required the Axiom of Global Choice (see \cite{4}), which as noted could be false in $(M_\delta, M_\delta, \in)$. Moreover, if the underlying order was longer than $\text{Ord}$, this proof would fail because “$\gamma_{\text{Ord}}$” would be undefinable, as $\gamma$ is a strictly increasing sequence of ordinals.

**Corollary 2.5.** If $ht(V) < \infty$ then $V = L[A]$ for some $A \in V$.

*Proof.* Use the proof above, only substitute $M_{ht(V)}$ for $V$. \hfill \Box

**Corollary 2.6.** If $\langle V, V, \in \rangle \models \text{BGC}$ and $I = \text{Ord}$ then $V = L[A]$ for some class $A \subseteq \text{Ord}$.

*Proof.* Using Global Choice, we can choose in the limit stage class-many representatives from all the $r[A]$’s simultaneously. Then we take $B_\delta = \bigcup_{\alpha \in \text{Ord}} B_\alpha$.

By the same arguments as in the theorem, for all $\alpha \in \text{Ord}$ $L[B_\delta] \supseteq L[B_\alpha]$. But this means $L[B_\delta]$ contains all the proper inner models, hence $L[B_\delta] = V$. \hfill \Box

It now emerges that the models in our tower are not arbitrary at all. They are in fact the very familiar models of the form $L[A]$. We thus conclude that a nice well-ordered model universe $V$ is inherently quite ‘small’ and ‘close’ to $L$, especially if $ht(V) < \infty$.

Before proceeding to the next section, it is worth noting what would happen if instead of basing our model tower on $L$, we would base it on some arbitrary inner model $M$. Obviously $M$ wouldn’t be well-ordered anymore, so we would have to relax our definition. We will only require that all inner models containing $M$ be on a well-ordered chain, and that all other inner models be contained in $M$. So below $M$ everything could be completely chaotic, but above $M$ we would have a well-ordered tower. Now let’s consider the implications.

First of all, as for theorem 2.2, this alteration potentially allows for an infinite descending chain of models. So let’s assume that $V$ does have a measurable
cardinal and \( j : V \to N \) is the corresponding elementary embedding. Then \( N \)
must also contain a measurable cardinal, and repeating this process, due to the
well-ordering we arrive at a model \( N_0 \subseteq M \) after a finite number of steps. Thus
there is an elementary embedding \( k : V \to N_0 \), and so \( k \upharpoonright M \)
is an elementary embedding of \( M \) into some smaller inner model. Therefore \( M \)
also include a measurable cardinal.

Theorem 2.4 would still work as well, using \( M \) as the base for the induction. Accordingly, corollary 2.6 would still hold up as well.

After analyzing the structure of well-ordered model universes, we turn to
the problem of constructing one of arbitrary height.

3 Perfect set forcing

In lemma 2.1 we proved the base of our model tower is \( L \). In this section we
show how to build the first step in our tower. Unlike the previous section, from
here on we only assume that we’re working within a model of \( \text{ZFC} \), not \( \text{BG} \). Also
note that throughout this paper we follow the Israeli convention for forcing, i.e
if \( p > q \) are forcing conditions, then \( p \) is the stronger condition.

To construct the first floor in the tower, we call upon the notion of Sacks
forcing \cite{16}, first invented by Gerald Sacks, which is useful for creating minimal
generic extensions. In this section we present the original Sacks forcing
and some of its most important properties.

We assume the reader has a basic understanding of forcing. For a general
introduction to the technique of forcing, the reader may refer to ch. VII of
Kunen’s book \cite{11}. For a more thorough exposition and analysis of Sacks forcing,
the reader may consult Geschke and Quickert \cite{7}.

**Definition 3.1.** Let \( \text{Seq} \) denote the set of all finite binary sequences.

1. A tree is a set \( p \subseteq \text{Seq} \), such that for each \( s \in p \) if \( s \upharpoonright n \in p \) then for all
   \( m < n \) \( s \uparrow m \in p \).

2. If \( p \subseteq \text{Seq} \) and \( s \in p \), we say that \( s \) splits in \( p \) if \( s \uparrow 0 \in p \) and \( s \uparrow 1 \in p \).

3. If \( p \subseteq \text{Seq} \) and \( s \) splits in \( p \) then we say \( s \) is an order \( n \) splitting node if
   \( |\{t \subseteq s \mid t \text{ splits in } p \}| = n \).

4. If \( p \subseteq \text{Seq} \), we say \( s \) is a stem of \( p \) if \( s \) is a splitting node and for all \( t \subseteq s \)
   \( t \) is not a splitting node.

**Definition 3.2.** We say \( p \subseteq \text{Seq} \) is a perfect tree if:

1. \( p \) is a tree;

2. And for every \( s \in p \) there exists a splitting node \( t \in p \) such that \( t \supseteq s \).

**Definition 3.3.** If \( p \) is a perfect tree and \( s \in p \) we denote \( p \upharpoonright s = \{ t \in p \mid s \subseteq t \uparrow t \subseteq s \} \).
Plainly \( p \upharpoonright s \) is perfect as well.
Definition 3.4. We call \( P = \{ p \subseteq \text{Seq} \mid p \text{ is a perfect tree} \} \), where \( P \) is ordered by reverse inclusion: \( p \leq q \iff p \supseteq q \), Sacks forcing. Later, after we present the generalized form, we shall refer to it as \( \aleph_0 \)-Sacks forcing.

We can identify the generic set \( G \) with a function \( f : \omega \to 2 \). First, note that the set of perfect trees with a stem of height at least \( n \) is a dense set in \( P \). Thus there are trees of arbitrarily long finite stems in \( G \). Also, if two trees \( p, q \) both have stems of height greater or equal than \( n \), but the restrictions of the stems on \( n \) differ, then \( \text{ht}(p \cap q) < n \), and so there is no \( r \in P \) such that \( r > p, q \). Hence all trees belonging to the generic set \( G \) must agree on their stems. Thus we can define \( f(n) = s(n) \), where \( s \) is part of the stem of any \( p \in G \). Due to their agreement, the function \( s \) is well defined, and due to the arbitrary finite length of the stems \( f \) is defined on \( \omega \). For the other direction, we may define \( G = \{ p \in P \mid \forall n \in \omega \left( f \upharpoonright n \in p \right) \} \). So in essence \( G \) is equivalent to a new real number, called a Sacks real.

Lemma 3.5. CH implies that \( |P| = \aleph_1 \) and so \( P \) satisfies the \( \aleph_2 \)-antichain condition.

Proof. We simply count the number of possible conditions. There are at most \( \aleph_0 \) finite binary sequences, and therefore at most \( 2^{\aleph_0} \) possible trees. Assuming CH \( 2^{\aleph_0} = \aleph_1 \), and so there are at most \( \aleph_1 \) conditions and no antichains of cardinality \( \aleph_2 \).

We note that \( P \) does not offer much in way of closure. It is plainly not \( \aleph_1 \)-closed, as one may take any perfect tree \( p \) and build the following sequence: \( \langle p_n \mid n \in \omega \rangle \) where \( p_0 = p \) and \( p_{n+1} = p_n \upharpoonright s \rightarrow 0 \) where \( s \) is the single order 0 splitting node of \( p_n \). This is obviously a sequence of perfect trees such that for all \( n \) \( p_{n+1} > p_n \), however \( \bigcap_{n \in \omega} p_n \) has no splitting nodes at all, and therefore is not a perfect tree.

Luckily, perfect trees offer a slightly weaker form of closure, using the technique of fusion.

Definition 3.6. Suppose \( p, q \in P \). We say \( p \geq_n q \) if:

1. \( p \geq q \);
2. And \( s \in p \) is an order \( n \) splitting in \( p \) node if and only if \( s \in q \) is an order \( n \) splitting node in \( q \).

Lemma 3.7. Fusion: Let \( \langle p_n \in P \mid n \in \omega \rangle \) be a sequence of conditions such that for all \( n \) \( p_{n+1} \geq_n p_n \). Then \( \bigcap_{n \in \omega} p_n \in P \).

Proof. Define \( \bigcap_{n \in \omega} p_n = p_\omega \). We claim \( p_\omega \in P \), meaning it’s a perfect tree. Take \( s \in p_\omega \). Let \( |\{ t \subseteq s \mid t \text{ splits in } p_\omega \}| = m \). Take \( p_{m+1} \). By definition \( s \in p_{m+1} \). However \( p_{m+1} \) is a perfect tree, and so has a splitting node of order \( m + 1 \) above \( s \), which we denote \( t \supseteq s \). But because it is an order \( m + 1 \) splitting node and
$p_{n+2} \geq m+1 \ p_{m+1}$ we have $t \in p_{m+2}$. By induction we get $t \in p_\omega$, but $s \subseteq t$ so we found a splitting node in $p_\omega$ above our arbitrary $s$. Hence $p_\omega$ is indeed perfect.

Note that it is obvious from the chain condition that all cardinals greater than or equal to $\aleph_2$ are preserved, as is of course $\aleph_0$. We now complete the picture with showing $\aleph_1$ is preserved.

**Lemma 3.8.** $\aleph_1$ is preserved under $\aleph_0$-Sacks forcing.

**Proof.** Assume $X$ is a countable set of ordinals in $V[G]$. We show the existence of a set $A \in V$ countable in $V$ such that $X \subseteq A$. Let $\dot{F}$ be a name and let $p$ be a condition such that $p \Vdash \dot{F}$ witnesses that $X$ is countable, that is $p \Vdash \dot{F} : \omega \to \dot{X}$ is surjective.

We now build a fusion sequence $(p_\alpha \mid \alpha \in \omega)$ starting with $p_0 = p$. Assume we defined $p_n$. Let $S_n$ be the set of all order $n$ splitting nodes of $p_n$. For each $s \in S_n$ let $q_{s^\frown 0}, q_{s^\frown 1}$ and $a_{s^\frown 0}, a_{s^\frown 1}$ be such that $q_{s^\frown i} \geq p_n \upharpoonright s^\frown i$ and $q_{s^\frown i} \Vdash \dot{F}(n) = a_{s^\frown i}$. Let $p_{n+1} = \bigcup_{s \in S_n, i = 0, 1} q_{s^\frown i}$.

Note that the union of perfect trees is a perfect tree, and that all splitting nodes of order $\leq n$ are preserved: if $t$ is an order $m < n$ splitting node in $p_n$ then it is also a splitting node in $q_{s^\frown 0}$ for the $s \in S_n$ that is $s \supseteq t$, and so is in $p_{n+1}$; whereas if $t$ is an order $n$ splitting node in $p_n$ then $t$ is a splitting node in $q_{s^\frown 0} \cup q_{s^\frown 1}$ and so is in $p_{n+1}$. Thus all splitting nodes of order $n$ are preserved in $p_{n+1}$, and so $p_{n+1} \geq n \ p_n$. Using lemma 3.7 we get $q = \bigcap_{n \in \omega} p_n \in \mathbb{P}$.

Now define $A = \bigcup_{n \in \omega} \{a_{s^\frown i} \mid s \in S_n \wedge i = 0, 1\}$. Note that $A$ is a countable union of finite sets, hence $A$ is countable in $V$. Now observe that $q \Vdash \text{ran} (\dot{F}) \subseteq A$. As $\dot{F}$ is the name of the function that witnesses the countability of $X$ this means $q \Vdash X \subseteq A$.

In this process we built a specific $q \geq p$, so $q$ is not guaranteed to be in the generic set $G$. However, as we found a $q \Vdash X \subseteq A$ above any condition stronger or equal to $p$, due to density, there is some $r \geq p$ in $G$ such that $r \Vdash X \subseteq A$. Therefore $V[G] \vDash X \subseteq A$, where $A$ is countable in $V$, which implies that $\aleph_1$ is preserved.

**Theorem 3.9.** Sacks forcing produces a minimal extension of $V$, meaning that for every model $W$ of ZFC if $V \subseteq W \subseteq V[G]$, then either $W = V$ or $W = V[G]$.

**Proof.** According to theorem 15.43 of [10], every intermediate model of ZFC is equal to $V[A]$, where $A$ is a set of ordinals. Hence it is sufficient to show that for any set of ordinals $A$ in $V[G]$, either $V[A] = V$ or $V[A] = V[G]$.

Let $\dot{A}$ be the name of a set of ordinals in $V[G]$. There is an ordinal $\alpha$ such that $0 \Vdash \dot{A} \subseteq \alpha$, and let $\dot{z}$ be the name of the characteristic function of $A$, $z : \alpha \to 2$. If $A \in V$ then obviously $V[A] = V$. Assume then $p \in G$ is a condition that forces $A \notin V$.

For a condition $q \in \mathbb{P}$ let $\dot{z}_q$ be the longest initial segment of $\dot{z}$ that is decided by $q$, and $\gamma_q$ be the first ordinal for which $\dot{z}$ is undecided. For $q \geq p$ they must
be well-defined, because if \( q \) decides all of \( z \), it decides all of \( \dot{A} \), and then \( A \in V \), in contradiction to \( p \not\models A \notin V \). Plainly \( \gamma_q < \alpha \).

Mark \( p_0 = p \). Assume we’ve already chosen \( p_n \). For every splitting node \( s \in S_n \), where \( S_n \) is defined as in lemma 3.8, let’s look at \( \gamma_{p_n \upharpoonright s} \) and conditions \( p_n \upharpoonright s^{-i} \). Suppose that for both \( i = 0, 1 \) we have \( p_n \upharpoonright s^{-i} \models \dot{z} (\gamma_{p_n \upharpoonright s}) = j \).

Thus, if for a certain \( i \) there is a \( j \) such that \( p_n \upharpoonright s^{-i} \models \dot{z} (\gamma_{p_n \upharpoonright s}) = j \), then there is some \( q_{s^{-i-1-1}} \geq p_n \upharpoonright s^{-1} \) such that \( q_{s^{-1-1}} \models \dot{z} (\gamma_{p_n \upharpoonright s}) = 1 - j \), and we take \( q_{s^{-i}} = p_n \upharpoonright s^{-i} \) so that \( q_{s^{-i}} \models \dot{z} (\gamma_{p_n \upharpoonright s}) = j \). If there is no such \( i \), then we are free to take for both \( i = 0, 1 \) \( q_{s^{-i}} \geq p_n \upharpoonright s^{-i} \) such that \( q_{s^{-i}} \models \dot{z} (\gamma_{p_n \upharpoonright s}) = i \). The point is that in both cases we found \( q_{s^{-i}} \) that decide \( \dot{z} (\gamma_{p_n \upharpoonright s}) \) in conflicting ways for \( i = 0, 1 \).

We now take \( p_{n+1} = \bigcup_{s \in S_n, i = 0, 1} q_{s^{-i}} \). Again, exactly as in lemma 3.8 we recognize \( \langle p_n \mid n \in \omega \rangle \) is a fusion sequence. Thus we can take condition \( q = \bigcap_{n \in \omega} p_n \).

Let \( f = \{ s \in q \mid \dot{z}_{q \upharpoonright s} \subseteq \dot{z}_G \} \). This is a branch of \( q \), because if \( s \) is a splitting node of \( q \), then either for \( i = 0 \) or \( i = 1 \), but not both, \( \dot{z}_{q \upharpoonright s^{-i}} (\gamma_{q \upharpoonright s}) \neq \dot{z}_q (\gamma_{q \upharpoonright s}) \), hence for only one \( i \) we have \( s^{-i} \in f \). Thus \( f \) is a completely definable branch in \( V [A] \), and so \( f \in V [A] \).

We now note that given \( p \) we created a stronger condition \( q \) and so from density we can assume \( q \in G \). We claim that \( f \) is our Sacks real \( f \). Mark the Sacks real as \( g \). If \( f \) disagrees with \( g \), then because both are branches in \( q \), there must be a splitting node \( s \) of \( q \) where they diverge. But that would imply \( \dot{z}_f (\gamma_{q \upharpoonright s}) \neq \dot{z}_g (\gamma_{q \upharpoonright s}) \) in contradiction to the definition of \( f \). Thus \( f = g \), \( f \) is our Sacks real, and we get \( G \in V [A] \).

Therefore \( V [G] \subseteq V [A] \subseteq V [G] \) and we conclude \( V [A] = V [G] \).

**Corollary 3.10.** \( V \models CH \Rightarrow V [G] \models CH \).

**Proof.** In the proof of theorem 3.9 let’s assume \( A \) is a ’new’ subset of \( \mathbb{N}_0 \), meaning we have \( V [G] \models A \subseteq \mathbb{R}_0 \land A \notin V \). Using fusion, we generate a perfect tree \( q \in \mathbb{P} \) in the ground model that is used to interpret \( A \) according to the Sacks real \( G \).

Viewed another way, and taking \([q]\) to signify the branches of \( q \), \( q \) is in fact a continuous map \( q : [q] \rightarrow \mathcal{P} (\mathbb{N}_0) \) such that \( V [G] \models q (G) = A \). Therefore given \( G \), there can be no two subsets of \( \mathbb{N}_0 \) \( A_1 \neq A_2 \) that produce the same \( q \).

But according to lemma 3.9 there are at most \( \mathbb{N}_1 \) conditions in \( \mathbb{P} \), so there are at most \( \mathbb{N}_1 \) new subsets of \( \mathbb{N}_0 \) in \( V [G] \). Hence, \( (\mathbb{N}_0 = \mathbb{N}_1)^{V [G]} \).

**4 \( \kappa \)-Sacks forcing**

In this section we show how we can extend our model tower through the successor steps.
Naïvely we could try and repeat the Sacks forcing, hoping that no unexpected models 'pop up' along the way. However, ultimately we desire to iterate our forcing class-many times, so we need to be wary of preserving the Power Set Axiom. Because each application of classical Sacks forcing adds a real number, were we simply to iterate the forcing class-many times, \(2^{\aleph_0}\) would 'explode', and the resultant model would fail to satisfy ZF. Instead, what we need to do is find a way to build minimal models where the subsets of each cardinal eventually stabilize.

For this, we turn to perfect trees of height \(\kappa\), via Kanamori’s extension of Sacks forcing to uncountable cardinals \([10]\). The following definitions are an almost perfect analogue to the definitions of the previous section, except where noted otherwise.

**Definition 4.1.** Let \(\text{Seq} = \bigcup_{\alpha<\kappa} 2^\alpha\).

1. A tree is a set \(p \subseteq \text{Seq}\), such that for each \(s \in p\) if \(s \upharpoonright \alpha \in p\) then for all \(\beta < \alpha \ s \upharpoonright \beta \in p\).
2. If \(p \subseteq \text{Seq}\) and \(s \in p\), we say that \(s\) splits in \(p\) if \(s^\frown 0 \in p\) and \(s^\frown 1 \in p\).
3. If \(p \subseteq \text{Seq}\) and \(s\) splits in \(p\) then we say \(s\) is an order \(\alpha\) splitting node if when we order \(\{t \subseteq s \mid t \text{ splits in } p\}\) by inclusion it is the \(\alpha\)th node.
4. If \(p \subseteq \text{Seq}\), we say \(s\) is a stem of \(p\) if \(s\) is a splitting node and for all \(t \subseteq s\) \(t\) is not a splitting node.

**Definition 4.2.** We say \(p \subseteq \text{Seq}\) is a perfect tree if:

1. \(p\) is a tree.
2. For every \(s \in p\) there exists a splitting node \(t \in p\) such that \(t \supseteq s\).
3. If \(\delta < \kappa\) is a limit ordinal, \(s \in^\delta 2\) and \(s \upharpoonright \beta \in p\) for every \(\beta < \delta\), then \(s \in p\).
   Intuitively '\(p\) is closed'.
4. If \(\delta < \kappa\) is a limit ordinal, \(s \in^\delta 2\) and for arbitrarily large \(\beta < \alpha \ s \upharpoonright \beta\) splits in \(p\), then \(s\) splits in \(p\). Intuitively 'the splitting nodes of \(p\) are closed'.

The last two conditions are new, though it is easy to see that the original \(\aleph_0\)-Sacks forcing satisfies them by default. Conditions \(\text{3}\) and \(\text{4}\) are necessary to ensure the closure property in lemma \(\text{4.5}\). Without condition \(\text{3}\) the limit of \(\omega\) trees might be empty, and without condition \(\text{4}\) the limit might consist of just a branch without any splitting nodes.

**Definition 4.3.** If \(p\) is a perfect tree and \(s \in p\) we denote \(p \upharpoonright s = \{t \in p \mid s \subseteq t \land t \subseteq s\}\).
Plainly \(p \upharpoonright s\) is perfect as well.

**Definition 4.4.** We call \(\mathbb{P} = \{p \subseteq \text{Seq} \mid p\ \text{is a perfect tree}\}\), where \(\mathbb{P}\) is ordered by reverse inclusion \(p \leq q \iff p \supseteq q\), \(\kappa\)-Sacks forcing.
As before, we can identify the generic set \( G \) with a function \( f : \kappa \to 2 \). There are trees with arbitrarily long stems in \( G \), and these stems must coincide on their mutual domain. Thus we can define \( f(\alpha) = s(\alpha) \), where \( s \) is part of a stem for some \( p \in G \). This function is well-defined on \( \kappa \). For the other direction, we may define \( G = \{ p \in P \mid \forall \alpha < \kappa (f \upharpoonright \alpha \in p) \} \). So in essence \( G \) defines a new subset of \( \kappa \).

Next, to achieve maximal closure in \( P \), we require \( \kappa \) to be regular. So from here on it is assumed \( \kappa \) is a regular cardinal.

**Lemma 4.5.** \( P \) is \( \kappa \)-closed.

**Proof.** Let \( \delta < \kappa \), \( \langle p_\alpha \mid \alpha < \delta \rangle \) be a sequence of increasing conditions. We claim \( p = \bigcap_{\alpha < \delta} p_\alpha \in P \). Conditions 3 and 4 of definition 4.2 are trivially true in \( p \). It is left to show that each node in \( p \) has a splitting node above it.

Let \( S \) be the splitting nodes of \( p \) and for each \( \alpha < \delta \) let \( S_\alpha \) be the set of splitting nodes of \( p_\alpha \). \( p \) is not empty because \( \emptyset \in p_\alpha \) for all \( \alpha \).

Assume \( s \in p \). Then for each \( \alpha < \delta \) \( s \in p_\alpha \), and denote \( t_\alpha \) as the order 0 splitting node of \( p_\alpha \). If \( \alpha < \beta < \delta \) then \( S_\alpha \supseteq S_\beta \supseteq S \). Therefore either \( \langle t_\alpha \mid \alpha < \delta \rangle \) stabilizes, in which case for some \( \gamma \) \( t_\gamma \in S_\alpha \) for all \( \alpha < \delta \), and therefore \( s \subseteq t_\gamma \in S \). Or for each \( \beta < \delta \) \( \{ t_\alpha \mid \beta \leq \alpha < \delta \} \in p_\beta \) is an unbounded sequence of splitting nodes under \( \bigcup_{\alpha < \beta} t_\alpha \), and therefore due to definition 4.2 condition 4 holds \( \bigcup_{\alpha < \delta} t_\alpha \in S_\beta \). But that means \( s \subseteq \bigcup_{\alpha < \delta} t_\alpha \in S \).

Either way, we found a splitting node above an arbitrary \( s \in p \), and therefore \( p \) is perfect. \( \square \)

**Lemma 4.6.** If \( 2^{<\kappa} = \kappa \) and \( 2^\kappa = \kappa^+ \) then \( |P| = \kappa^+ \) and so \( P \) satisfies the \( \kappa^+ \)-antichain condition.

**Proof.** We simply count the number of possible conditions. There are at most \( \kappa \) binary sequences of length \( < \kappa \), and therefore at most \( 2^\kappa = \kappa^+ \) possible trees. So there are at most \( \kappa^+ \) conditions and no antichains of cardinality \( \kappa^+ \). \( \square \)

We now extend the technique of fusion to this forcing.

**Definition 4.7.** Suppose \( p, q \in P \). We say \( p \geq_\alpha q \) if:

1. \( p \geq q \);
2. And for all \( \beta \leq \alpha \), \( s \in p \) is an order \( \beta \) splitting in \( p \) node if and only if \( s \in q \) is an order \( \beta \) splitting node in \( q \).

Note that because of the closure of the splitting nodes (definition 4.2 condition 4), if \( \delta \) is a limit ordinal \( p \geq_\delta q \iff \forall \alpha < \delta (p \geq_\alpha q) \).

**Lemma 4.8.** Fusion: Let \( \langle p_\alpha \in P \mid \alpha < \kappa \rangle \) be a sequence of conditions such that for all \( \alpha \) \( p_{\alpha+1} \geq_\alpha p_\alpha \), and for \( \delta \) a limit ordinal \( p_\delta = \bigcap_{\alpha < \delta} p_\alpha \). Then \( \bigcap_{\alpha < \kappa} p_\alpha \in P \).
Proof. Define $\bigcap_{\alpha<\kappa} p_\alpha = p_\kappa$. We claim $p_\kappa \in \mathbb{P}$, meaning it is a perfect tree. Conditions 3 and 4 of definition 4.2 are trivially true in $p_\kappa$. It is left to show that each node in $p_\kappa$ has a splitting node above it.

Take $s \in p_\kappa$. Let $\beta$ be the order type of $\{t \subseteq s \mid t \text{ splits in } p_\kappa\}$ ordered by inclusion. Take $p_{\beta+1}$. By definition $s \in p_{\beta+1}$. However $p_{\beta+1}$ is a perfect tree, and so has a splitting node of order $\beta+1$ above $s$, which we denote $t \supseteq s$.

Now we proceed by transfinite induction. Assume $t$ is an order $\beta+1$ splitting node in $p_\kappa$, where $\epsilon > \beta$. Then $p_{\epsilon+1} \supseteq p_\kappa$, so $t$ is also an order $\beta+1$ splitting node in $p_{\epsilon+1}$. Let $\delta < \kappa$ be a limit ordinal, such that for all $\epsilon$ with $\beta < \epsilon < \delta$ $t$ is an order $\beta+1$ splitting node in $p_\epsilon$. Then by definition 4.2 condition 4 $t$ is an order $\beta+1$ splitting node in $p_\delta$.

Therefore by induction $t \supseteq s$ is a splitting node in $p_\kappa$. We found a splitting node in $p_\kappa$ above our arbitrary $s$. Hence $p_\kappa$ is indeed perfect.

Note that it is obvious from the chain condition that all cardinals greater than or equal to $\kappa^{++}$ are preserved. On the other hand, due to closure all cardinals less than or equal to $\kappa$ are preserved. So to complete the picture we must show $\kappa^+$ is preserved.

Lemma 4.9. If $2^{<\kappa} = \kappa$ then $\kappa^+$ is preserved under $\kappa$-Sacks forcing.

Proof. This proof closely mirrors the proof of lemma 3.8.

Assume $X$ is a set of ordinals in $V[G]$, such that $|X| = \kappa^{V[G]}$. We show the existence of a set $A \in V$ of cardinality $\kappa$ in $V$ such that $X \subseteq A$. Let $\dot{F}$ be a name and let $p$ be a condition such that $p \Vdash \dot{F} : \kappa \to \dot{X}$ is surjective.

We now build a fusion sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$ with $p_0 = p$. Assume we defined $p_\alpha$. Let $S_\alpha$ be the set of all order $\alpha$ splitting nodes of $p_\alpha$. For each $s \in S_\alpha$ let $q_{s-\alpha}, q_{s-\alpha}^1, s_{s-\alpha}^i$ be such that $q_{s-\alpha}^i \geq p_\alpha \upharpoonright s^i$ and $q_{s-\alpha}^i \Vdash \dot{F}(\alpha) = a_{s-\alpha}^i$. Let 

$p_{\alpha+1} = \bigcup_{s \in S_\alpha, i < 1} q_{s-\alpha}^i.$

All splitting nodes of order $\leq \alpha$ are preserved: if $t$ is an order $\beta < \alpha$ splitting node in $p_\alpha$ then it is also a splitting node in $q_{s-\alpha}^i$ for the $s \in S_\alpha$ that is $s \supseteq t$, and so in $p_{\alpha+1}$; whereas if $t$ is an order $\alpha$ splitting node in $p_\alpha$ then $t$ is a splitting node in $q_{s-\alpha}^i \cup q_{s-\alpha}^1$ and so is in $p_{\alpha+1}$. Thus all splitting nodes of order $\alpha$ are preserved in $p_{\alpha+1}$, and so $p_{\alpha+1} \geq p_\alpha$.

In the limit case we define $p_\delta = \bigcap_{\alpha<\delta} p_\alpha$. Thus we have a fusion sequence, and using lemma 4.8 we get $q = \bigcap_{\alpha<\delta} p_\alpha \in \mathbb{P}$.

Now define $A = \bigcup_{\alpha<\kappa} \{a_{s-\alpha}^i \mid s \in S_\alpha \land i = 0, 1\}$. Note that $A$ is a union of $\kappa$ sets of at most $2^{<\kappa} = \kappa$ cardinality, so $|A| = \kappa$. Now observe that $q \Vdash \text{ran} (\dot{F}) \subseteq A$. As $\dot{F}$ is the name of the function that witnesses the cardinality of $X$, this means $q \Vdash \dot{X} \subseteq A$.

Although we built a specific $q \geq p$, due to density there is some $r \geq p$ in $G$ such that $r \Vdash \dot{X} \subseteq A$. Therefore $V[G] \Vdash X \subseteq A$, where $A$ is of cardinality $\kappa$ in $V$, which implies that $\kappa^+$ is preserved. □
Theorem 4.10. \( \kappa \)-Sacks forcing produces a minimal extension of \( V \), such that for every model \( W \) of ZFC if \( V \subseteq W \subseteq V[G] \), then either \( W = V \) or \( W = V[G] \).

Proof. This proof closely mirrors the proof of theorem 3.10 and so is given here in a more concise form.

We show that for any set of ordinals \( A \in V[G] \) either \( V[A] = V \) or \( V[A] = V[G] \). Let \( A \) be the name of a set of ordinals in \( V[G] \), and let \( \dot{z} \) be the name of its characteristic function. Assume \( p \in G \) forces \( A \notin V \).

For a condition \( q \in P \) let \( \dot{z}_q \) be the longest initial segment of \( \dot{z} \) that is decided by \( q \), and \( \gamma_q \) be the first ordinal for which \( \dot{z} \) is undecided.

Mark \( p_0 = p \). Assume we’ve already chosen \( p_\alpha \). For every splitting node \( s \in S_\alpha \), where \( S_\alpha \) is defined as in lemma 4.9, let’s look at \( \gamma_{p_\alpha} | s \) and conditions \( p_\alpha \mid s \models i \). If for a certain \( i \) there is a \( j \) such that \( p_\alpha \mid s \models i \models \dot{z}(\gamma_{p_\alpha} | s) = j \) and we take \( q_{s \models i} = p_\alpha \mid s \models i \), there will be \( q_{s \models (1-i)} \models p_\alpha \mid s \models (1-i) \) such that \( q_{s \models (1-i)} \models \dot{z}(\gamma_{p_\alpha} | s) = 1 - j \). If there is no such \( i \), then we are free to take for both \( i = 0, 1 q_{s \models i} \models p_\alpha \mid s \models i \) such that \( q_{s \models i} \models \dot{z}(\gamma_{p_\alpha} | s) = i \). Either case we found \( q_{s \models i} \) that decide \( \dot{z}(\gamma_{p_\alpha} | s) \) in conflicting ways for \( i = 0, 1 \).

We now take \( p_{\alpha + 1} = \bigcup_{s \in S_\alpha, i = 0, 1} q_{s \models i} \), and for limit ordinals \( p_\delta = \bigcap_{\alpha < \delta} p_\alpha \).

Exactly as in lemma 4.9 \( \langle p_\alpha \mid \alpha < \kappa \rangle \) is a fusion sequence and we can take \( q = \bigcap_{\alpha < \kappa} p_\alpha \). Let \( f = \{ s \in q \mid \dot{z}_q | s \subseteq \dot{z}_G \} \). \( f \) is a branch of \( q \), and using \( A \) is completely definable, so \( f \in V[A] \).

Due to density we may assume \( q \in G \), in which case \( f \) is actually our new function \( \kappa \to 2 \), which we identify with \( G \). Thus we get \( G \in V[A] \), and so \( V[A] = V[G] \). \( \square \)

Corollary 4.11. \( V \models 2^\kappa = \kappa^+ \land 2^{<\kappa} = \kappa \Rightarrow V[G] \models 2^\kappa = \kappa^+ \).

Proof. Again, in direct analogy to corollary 3.10, In the proof of theorem 4.10 let’s assume \( A \) is a ‘new’ subset of \( \kappa \), meaning we have \( V[G] \models A \subseteq \kappa \land A \notin V \). Using fusion, we generate a perfect tree \( q \in P \) in the ground model that is used to interpret \( A \) according to \( G \).

Taking \( [q] \) to signify the branches of \( q \), we can view \( q \) as a mapping \( q : [q] \to \mathcal{P}(\kappa) \) between the branches of \( q \) and subsets of \( \kappa \). Our construction method for \( q \) implies that \( V[G] \models (q(G)) = A \). Therefore for a given \( G \), there cannot be two subsets of \( \kappa A_1 \neq A_2 \) that produce the same \( q \).

But according to lemma 4.6 there are at most \( \kappa^+ \) conditions in \( P \), so there can be at most \( \kappa^+ \) new subsets of \( \kappa \) in \( V[G] \). Hence, \( (2^\kappa = \kappa^+) \models V[G] \). \( \square \)

After proving minimality and preservation of cardinals we conclude this section with showing that \( \kappa \)-Sacks forcing preserves GCH above \( \kappa \).

Lemma 4.12. \( V \models \forall \lambda \geq \kappa (2^\lambda = \lambda^+) \Rightarrow V[G] \models \forall \lambda > \kappa (2^\lambda = \lambda^+) \).

Proof. For the preservation of GCH above \( \kappa \), we turn to the notion of a nice name (see ch. VII definition 5.11 in [11]). A name for a subset of \( \sigma \in V^P \) is considered nice if it is of the form \( \bigcup \{ \{ \pi \} \times A_\pi \mid \pi \in \text{dom}(\sigma) \} \), where each \( A_\pi \) is an antichain in \( P \). Every subset has a nice name.
According to lemma 4.6 there are at most $\kappa^+$ conditions, and therefore at most $\kappa^+$ elements in an antichain. Meaning, there are at most $(\kappa^+)\kappa^+ = \kappa^{++}$ different possible antichains.

Hence for a given cardinal $\lambda$ there are at most $(\kappa^{++})^\lambda$ different nice names for subsets of $\dot{\lambda}$. Thus for $\lambda \geq \kappa^{++}$ we have $(\kappa^{++})^\lambda = 2^\lambda = \lambda^+$, meaning $(2^\lambda \leq \lambda^+)^{V[G]}$. But of course this means $V[G] \models 2^\lambda = \lambda^+$. For the case $\lambda = \kappa^+$ we derive this instead from $\lambda^+ = \kappa^{++} = (\kappa^{++})^{\kappa^+} = (\kappa^{++})^\lambda$.

Thus $V[G] \models \forall \lambda > \kappa (2^\lambda = \lambda^+)$, as required.

We now briefly summarize the attributes we demanded from $\kappa$ for the forcing notion and the above theorems to make sense:

1. $\kappa$ needed to be regular, for the closure to work (lemma 4.5).
2. $2^{<\kappa} = \kappa$ is necessary to preserve $\kappa^+$ (lemma 4.9).
3. $2^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$ are necessary for the antichain condition (lemma 4.6).

All of these conditions are necessary to prove the preservation of cardinals by $\kappa$-Sacks forcing. All of them are automatically true if $\kappa$ strongly inaccessible. However, for our construction we don’t want to rely on the existence of large cardinals. Notably, both conditions 2 and 3 are also implied by GCH, so in the next section corollary 4.11 and lemma 4.12 will serve us in maintaining enough of GCH to make the forcing iteration work.

5. $\kappa$-Sacks iteration

After defining individual $\kappa$-Sacks forcing, it is time to stitch everything together. We now define the forcing iteration that will enable us to build a model tower through limit ordinals, and up to arbitrary height.

For a general introduction to iterated forcing the reader can refer to Shelah [18].

For the rest of this section, let $L$ be our base model, and let $\zeta$ be the height of the model tower that we wish to build.

**Definition 5.1.** For $\alpha \leq \zeta$, define the forcing iteration $\mathbb{P}_\alpha$ as follows:

1. Let $\dot{Q}_\alpha$ be trivial if $\alpha$ is a limit ordinal, and the name of $\aleph_\alpha$-Sacks forcing in $V^{P_\alpha}$ otherwise.
2. $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha \ast \dot{Q}_\alpha$.
3. At limit stages we use full support, i.e if $\delta$ is a limit ordinal then $p \in \mathbb{P}_\delta \iff \forall \alpha < \delta (p \upharpoonright \alpha \in \mathbb{P}_\alpha)$.

**Definition 5.2.** Denote:
1. $M_0 = L$.

2. $G_\alpha$ as the generic set in partial order $\mathbb{P}_\alpha$ over $M_0$.

3. $M_\alpha = M_0 [G_\alpha]$.

4. $G_{\alpha, \beta}$ as the generic set in partial order $\mathbb{P}_\beta / G_\alpha$ over $M_0 [G_\alpha]$, so that $M_0 [G_\alpha] [G_{\alpha, \beta}] = M_0 [G_\beta]$.

**Lemma 5.3.** For every $\alpha < \beta$ $\mathbb{P}_\beta / G_\alpha$ is $\aleph_\alpha$-closed. If $\alpha$ is a limit ordinal, then $\mathbb{P}_\beta / G_\alpha$ is $\aleph_{\alpha+1}$-closed.

**Proof.** Every coordinate of the forcing $\mathbb{P}_\beta / G_\alpha$ is either trivial or $\aleph_\gamma$-Sacks forcing for $\gamma \geq \alpha$. According to lemma 4.5 each coordinate is therefore $\aleph_\alpha$-closed.

By definition 5.1 we use full support, and therefore the iteration $\mathbb{P}_\beta / G_\alpha$ as a whole is $\aleph_\alpha$-closed.

If $\alpha$ is limit ordinal then $Q_\alpha$ is trivial, and so $\mathbb{P}_\beta / G_\alpha = \mathbb{P}_\beta / G_\alpha + 1$ which is $\aleph_{\alpha+1}$-closed.

We now show that all cardinals are preserved throughout the entire forcing iteration.

**Definition 5.4.** Let $OK (\alpha)$ denote that:

1. $M_\alpha$ has the same cardinals as $M_0$;
2. $M_\alpha \models 2^{<\aleph_\alpha} = \aleph_\alpha$;
3. $M_\alpha \models \forall \lambda \geq \aleph_\alpha (2^\lambda = \lambda^+)$.

**Lemma 5.5.** $OK (0)$.

**Proof.** $M_0 = L$ and so satisfies GCH.

**Lemma 5.6.** If $OK (\alpha)$ then $OK (\alpha + 1)$.

**Proof.** If $Q_\alpha$ is trivial then $M_{\alpha+1} = M_\alpha$ and conditions 1 and 3 are trivially true for $\alpha + 1$. As for condition 2 we know $M_\alpha \models 2^{<\aleph_\alpha} = \aleph_\alpha$ and so $M_{\alpha+1} \models 2^{<\aleph_{\alpha+1}} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}$ as required.

Otherwise, $OK (\alpha)$ implies $M_\alpha$ has the same cardinals as $M_0$, $M_\alpha \models 2^{<\aleph_\alpha} = \aleph_\alpha$ and $M_\alpha \models \forall \lambda \geq \aleph_\alpha (2^\lambda = \lambda^+)$. According to lemmas 4.3, 4.6 and 4.9 $M_{\alpha+1}$ has the same cardinals as $M_\alpha$, and therefore the same as $M_0$. According to corollary 4.11 $M_{\alpha+1} \models 2^{<\aleph_{\alpha+1}} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}$. And according to lemma 4.12 $M_{\alpha+1} \models \forall \lambda \geq \aleph_{\alpha+1} (2^\lambda = \lambda^+)$.

So $OK (\alpha + 1)$ is true.

**Lemma 5.7.** Let $\delta \leq \zeta$ be a limit ordinal. If for all $\alpha < \delta$ $OK (\alpha)$, then $M_\delta$ has the same cardinals as $M_0$. 

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Proof. For any $\alpha < \delta$ we have $M_\delta = M_\alpha [G_{\alpha, \delta}]$. But $^{p_\delta / G_\alpha}$ is $\aleph_\alpha$-closed by lemma 5.3. Thus, all cardinals less than or equal $\aleph_\alpha$ in $M_\alpha$ are preserved in $M_\delta$. But $OK(\alpha)$ implies $M_0$ has the same cardinals as $M_0$, so all cardinals less than or equal $\aleph_\alpha$ in $M_0$ are preserved in $M_\delta$.

As this is true for all cardinals less than $\aleph_\delta$, it is true for $\aleph_\delta$ itself.

Also note $|P_\delta| = \prod_{\alpha < \delta} 2^{\aleph_\alpha} = \prod_{\alpha < \delta} \aleph_{\alpha + 1} \leq 2^{\aleph_\delta} = \aleph_{\delta + 1}$. Therefore $P_\delta$ is $\aleph_{\delta + 2}$-c.c., and $M_\delta$ preserves all cardinals greater or equal to $\aleph_{\delta + 2}$.

It remains to be proven that $\aleph_{\delta + 1}$ is preserved.

First, assume $\aleph_\delta$ is singular and suppose that the iteration does collapse it. Denote $cf (\aleph_{\delta + 1}) = \aleph_\gamma$. The collapse of the cardinal implies $\aleph_\gamma \leq \aleph_\delta$, and due to the latter’s singularity $\aleph_\gamma < \aleph_\delta$, meaning there is a new set of ordinals $A \in M_\delta$, such that $|A| = \aleph_\gamma$. However according to lemma 5.3 $^{P_\delta / G_{\gamma + 1}}$ is $\aleph_{\gamma + 1}$-closed, so no sets of ordinals of cardinality $\aleph_\gamma$ are added when forcing $M_\delta = M_{\gamma + 1} [G_{\gamma + 1, \delta}]$. Thus $A \in M_{\gamma + 1}$, which implies $M_{\gamma + 1} \models cf (\aleph_{\delta + 1}) \leq \aleph_\gamma < \aleph_{\delta + 1}$, in contradiction to $\aleph_{\delta + 1}$ being preserved in $M_{\gamma + 1}$. Therefore $\aleph_{\delta + 1}$ must be preserved as well.

For the case $\aleph_\delta$ is regular we proceed with a variation of the argument used in lemma 4.9.

Assume $\aleph_\delta$ is regular, meaning $\aleph_\delta = \delta$, and that $F : \delta \rightarrow \text{Ord}$ is a function in $M_\delta$. We show the existence of a set $A \in M_0$ of cardinality $\delta$, such that in $M_\delta$ $\text{ran}(F) \subseteq A$. Let $p \in P_\delta$ be a condition such that $p \Vdash F : \delta \rightarrow \text{Ord}$.

For any condition $q \in P_\delta$, let $q^\alpha$ denote the $\alpha$ coordinate of $q$, and similarly let $q^{\leq \alpha}$ denote the first $\alpha$ coordinates, and $q^{> \alpha}$ denote the name of all higher coordinates. For consistency, if we discuss a condition $q \in P_\delta / H_\beta$ where $H_\beta$ is a generic set in $P_\beta$, then we fix the first coordinate to be $q^0$.

Inductively we are going to build an increasing sequence of conditions $\langle p(\alpha) \in P_\delta \mid \alpha < \delta \rangle$. Each coordinate is also going to be built inductively.

Start with the first coordinate. Let $p_0 = p$, and assume we’ve defined $p_n$ for $n < \omega$. Let $S_n$ be the set of all order $n$ splitting nodes of $p_n$. For each $s \in S_n$ let $q_s^{\leq 0}$, $q_s^{> 0}$, and $a_s$, be such that $q_s^{\leq 0}(\alpha, n) = a_s(\alpha, n)$ and $q_s^{> 0}(\alpha, n) = \langle p_n(\alpha, n) \mid \alpha < n \rangle$. Let $p_{n+1} = \bigcup_{s \in S_n} q_s^{> 0}$. Let $p_{n+1}$ be the amalgamation of the $q_s^{> 0}$'s, just like we did in the proof of lemma 4.9. For the rest of the coordinates, we define $p_{n+1}^{> 0}$ with accordance to the path taken in the first coordinate. Meaning that $q_s^{> 0}$ forces $p_{n+1}^{> 0} = q_s^{> 0}$.

From the way we defined the $q_s^{> 0}$'s it is clear $p_{n+1}^{> 0} \geq p_{n}^{> 0}$, and so $p_{n+1}^{> 0} \geq p_n$. Also, all splitting nodes of order $\leq n$ are preserved in the first coordinate, and therefore $p_{n+1}^{> 0} \geq p_n^{> 0}$. Thus what we have in the first coordinate is a classical fusion sequence, and in the rest of the coordinates an increasing sequence. So thanks to lemmas 3.7 and 5.3 we can conclude $p_\omega = \bigcap_{n < \omega} p_n \in P_\delta$.

We define $A_0 = \{ a_s \mid s \in S_n \land i = 0, 1 \}$. Obviously $|A_0| = \aleph_0$.

In essence, we used a fusion argument on the first coordinate to create $p_\omega$, which is a sort of ’decision tree’ for the first $\omega$ values of $F$. We set $p(0) = p_\omega$. Next, we are going to repeat this construction using the higher coordinates. In
each step $p(\beta)$ will be such a decision tree for the first $\omega_\beta$ values of $\hat{F}$.

So assume now that we’ve already defined $p(\beta)$, and we shall show how to define $p(\beta + 1)$. We use $Q_{\beta + 1}$ to decide the values of $\hat{F}$ up to $\omega_{\beta + 1}$. If $\beta$ is a limit ordinal we also set $A_\beta = \emptyset$.

Let $H_{\beta + 1} \subseteq P_{\beta + 1}$ be any generic set such that $\left< p(\beta)^0, \hat{p}(\beta)^1, ..., \hat{p}(\beta)^\alpha, ..., | \alpha < \beta \right> \in H_{\beta + 1}$.

Now, working in $M_0[H_{\beta + 1}]$ we repeat the construction. To start the induction, set $p_\omega = p(\beta)^{>\beta} \in P_\beta / H_{\beta + 1}$.

Assume $p_\alpha$ is defined, we are going to define $p_{\alpha + 1}$. Remember $Q_{\beta + 1}$ is $N_{\beta + 1}$-Sacks forcing. So let $S_\alpha$ be the set of all order $\alpha$ splitting nodes of $p_\alpha^{>\beta + 1}$. For each $s \in S_\alpha$ let $q_s \leftarrow q_s^{-0}, q_s^{-1}$ and $a_s \leftarrow a_s^{-0}, a_s^{-1}$ be such that $q_s^{-i} \geq \left< p_\alpha^{>\beta + 1} \cup q_s^{-i}, p_\alpha^{>\beta + 1} \right>$ and $q_s^{-i} \Vdash \hat{F}(\alpha) = a_s^{-i}$. Let $p_{\alpha + 1}^{\beta + 1} = \bigcup_{s \in S_\alpha, i = 0, 1} a_{s^{-i}}$ be the amalgamation of the $q_s^{-i}$’s first coordinate. As for the rest of the coordinates define $\hat{p}_{\alpha + 1}^{>\beta + 1}$ with accordance to the path taken in the first coordinate. Meaning that $q_s^{-i}$ forces $p_{\alpha + 1}^{>\beta + 1} = q_s^{-i}$.

From the way we defined the $q_s^{-i}$’s it is clear $\hat{p}_{\alpha + 1}^{\beta + 1} \geq \hat{p}_{\alpha}^{>\beta + 1}$, and so $p_{\alpha + 1}^{\beta + 1} \geq p_{\alpha}^{\beta + 1}$. Also, all splitting nodes of order $\leq \alpha$ are preserved in $p_{\alpha + 1}^{\beta + 1}$, and therefore $p_{\alpha + 1}^{\beta + 1} \geq \alpha p_{\alpha + 1}^{\beta + 1}$.

In limit stages $\tau$ we simply define $p_\tau = \bigcap_{\omega_\delta \leq \eta < \tau} p_\alpha$. In each coordinate of $p_\tau$, we have $N_{\beta + 1}$-closure according to lemma [\ref{5}]. So for $\tau < \omega_{\beta + 1}$ $p_\tau \in P_\beta / H_{\beta + 1}$. For the case $\tau = \omega_{\beta + 1}$ note that we have a fusion sequence in the first coordinate. So thanks to lemma [\ref{8}] and the $N_{\beta + 2}$-closure of the higher coordinates we have $p_{\omega_{\beta + 1}} \in P_\beta / H_{\beta + 1}$.

Now, for each $a_s^{-i}$ we used, we pick a $P_{\beta + 1}$-name $\check{a}_s^{-i}$ in $M_0$. Because $|P_{\beta + 1}| < \delta$ we can find in $M_0$ a set $A_\beta$ of cardinality $< \delta$ such that $p_\omega \Vdash \hat{A}_\beta \supseteq \{\check{a}_s^{-i} | s \in S_\alpha \land i = 0, 1 \land \omega_\beta \leq \alpha < \omega_{\beta + 1}\}$.

We also pick a name for $p_{\omega_{\beta + 1}}$ such that

$$p(\beta)^{\leq \delta} \Vdash P_{\beta + 1} \left< p_{\omega_{\beta + 1}}^{\beta + 2}, p_{\omega_{\beta + 1}}^{>\beta + 1}, ..., p_{\omega_{\beta + 1}}^0, ..., | \eta < \delta \right>$$

is as we constructed it.

We set $p(\beta + 1) = \left< p(0)^0, \check{p}(1)^1, ..., \check{p}(\beta)^\beta, p_{\omega_{\beta + 1}}^{\beta + 1}, ..., p_{\omega_{\beta + 1}}^0, ..., | \eta < \delta \right> \in P_\delta$. Obviously $p(\beta + 1) \geq p(\beta)$.

For limit stages $\tau$ we define $p(\tau) = \bigcap_{\beta < \tau} p(\beta)$. We claim $p(\tau) \in P_\delta$. For each coordinate $\beta < \tau$ note that the condition stabilizes, and so $p(\tau)^\beta = p(\beta)^\beta$. For coordinates $\geq \tau$ lemma [\ref{5}],[\ref{5}] provides at least $N_{\tau + 1}$-closure, and because we’re using full support this shows $p(\tau) \in P_\delta$.

Therefore by induction we can construct condition $p(\delta) = \left< p(0)^0, \check{p}(1)^1, ..., \check{p}(\eta)^\eta, ..., | \eta < \delta \right> \in P_\delta$.

Now define $A = \bigcup A_\alpha$. Because for all $\alpha < \delta OK(\alpha)$, we have $|A| = \delta$. 20
Observe that \( p(\delta) \vDash \text{ran}(F) \subseteq \hat{A} \). Now if \( \aleph_{\delta+1} \) is not preserved then in \( M_\delta \)
\[ |\aleph_{\delta+1} | = \delta, \] and we can take \( F \) to be the bijection between \( \aleph_{\delta+1} \) and \( \delta \). Applying the construction to this \( F \) we get \( p(\delta) \vDash \aleph_{\delta+1} \subseteq \hat{A} \). But then we get in \( M_\delta \) a surjection from \( \delta \) onto \( \aleph_{\delta+1} \), which is impossible. Therefore \( p(\delta) \vDash \aleph_{\delta+1} = \aleph_{\delta+1} \).

We know that \( p(\delta) \geq p(0) \geq p \), and so by density we can assume without loss of generality that \( p(\delta) \in G_\delta \). Therefore \( \aleph_{\delta+1} \) is indeed preserved.

Thus all cardinals are preserved in all cases, and overall.

\[ \square \]

Lemma 5.8. Let \( \delta \leq \zeta \) be a limit ordinal. If for all \( \alpha < \delta \) \( \text{OK}(\alpha) \), then \( M_\delta \vDash 2^{\aleph_\delta} = \aleph_\delta \).

Proof. Note that obviously \( M_\delta \vDash 2^{\aleph_\delta} \geq \aleph_\delta \).

Now assume \( M_\delta \vDash 2^{\aleph_\delta} > \aleph_\delta \). That means for some \( \alpha < \delta \) \( M_\delta \vDash 2^{\aleph_\delta} > \aleph_\delta \). However by lemma 5.3 \( p/_{G_{\alpha+1}} \) is \( \aleph_{\alpha+1} \)-closed, and so new subsets of \( \aleph_\delta \) are added when forcing \( M_\delta = M_{\alpha+1} \langle G_{\alpha+1, \delta} \rangle \). Therefore \( M_{\alpha+1} \vDash 2^{\aleph_\delta} > \aleph_\delta \) and so \( M_{\alpha+1} \vDash 2^{\aleph_\delta} > \aleph_{\alpha+2} \), in contradiction to \( \text{OK}(\alpha+1) \).

Therefore \( M_\delta \vDash 2^{\aleph_\delta} = \aleph_\delta \).

\[ \square \]

Lemma 5.9. Let \( \delta \leq \zeta \) be a limit ordinal. If for all \( \alpha < \delta \) \( \text{OK}(\alpha) \), then \( M_\alpha \vDash \forall \lambda \geq \aleph_\alpha (2^{\lambda} = \lambda^+) \).

Proof. Recall the proof of lemma 4.12 in the previous section. As already shown in lemma 5.7 \( |P_\delta| = \aleph_{\delta+1} \). Therefore there are at most \( \aleph_{\delta+1} = \aleph_{\delta+2} \) different antichains. With the rest of the proof identical, we get \( M_\delta \vDash \forall \lambda > \aleph_\delta (2^{\lambda} = \lambda^+) \).

We now show that \( M_\delta \vDash 2^{\aleph_\delta} = \aleph_\delta \).

Let \( A \subseteq \aleph_\delta \), and assume \( p \in P_\delta \) such that \( p \Vdash \hat{A} \subseteq \aleph_\delta \wedge \forall \beta < \delta (\hat{A} \notin M_\beta) \).

We denote individual coordinates like so: \( p = \langle p^0, p^1, ..., p^\alpha, ... | \alpha < \delta \rangle \).

Let \( \dot{\bar{\gamma}} \) be the name of the characteristic function of \( A \). For any condition \( q \in P_\delta \) stronger than \( p \) let \( \dot{\bar{\gamma}} \) be the longest initial segment of \( \dot{\bar{\gamma}} \) that is decided by \( q \), let \( \gamma_q \) be the first ordinal for which \( \dot{\bar{\gamma}} \) is undecided, let \( q^0 \) denote the first coordinate of \( q \), i.e. from \( \aleph_0 \)-Sacks forcing over \( M_0 \), and let \( \dot{q} \) denote the name of the rest of the coordinates. \( \gamma_q \) and \( \gamma_q \) must be well-defined, because if \( q \) decides all of \( \dot{\bar{\gamma}} \), it decides all of \( \hat{A} \), and then \( A \notin M_0 \), in contradiction to \( p \). Plainly \( \gamma_q < \omega_\delta \).

We now build a fusion sequence. Mark \( p_0 = p \). For the successor case, assume that we’ve already chosen \( p_i \). Note that \( p_0^0 \) is just a perfect tree in \( Q_0 \), and \( p_i^0 \) is a name in \( M_0 \). Let \( S_i \) denote the order \( \epsilon \) splitting nodes of \( p_i^0 \). For every splitting node \( s \in S_i \), let’s look at \( \gamma_{(p_i^0 | s, \bar{p}_i)} \) and conditions \( \langle p_i^0 | s^- i, \bar{p}_i \rangle \).

Suppose that for both \( i = 0, 1 \) we have \( \langle p_i^0 | s^- i, \bar{p}_i \rangle \vDash \dot{\bar{\gamma}} (\gamma_{(p_i^0 | s, \bar{p}_i)}) = j \).

\( \gamma_{(p_i^0 | s, \bar{p}_i)} \) is undecided, so take \( q \geq \langle p_i^0 | s, \bar{p}_i \rangle \) such that \( q \vDash \dot{\bar{\gamma}} (\gamma_{(p_i^0 | s, \bar{p}_i)}) = 1 - j \).

Either \( q \cap (p_i^0 | s^- 0, \bar{p}_i) \in P_\delta \) or \( q \cap (p_i^0 | s^- 1, \bar{p}_i) \in P_\delta \). But \( q \) and \( \langle p_i^0 | s^- i, \bar{p}_i \rangle \) are incompatible for \( i = 0, 1 \), therefore our supposition is impossible. Thus, if for a certain \( i \) there is a \( j \) such that \( \langle p_i^0 | s^- i, \bar{p}_i \rangle \vDash \dot{\bar{\gamma}} (\gamma_{(p_i^0 | s, \bar{p}_i)}) = j \), we can define \( q_{s^- i} = \langle p_i^0 | s^- i, \bar{p}_i \rangle \) so that \( q_{s^- i} \vDash \dot{\bar{\gamma}} (\gamma_{(p_i^0 | s, \bar{p}_i)}) = j \), and know there will be some \( q_{s^- (1-i)} \geq \langle p_i^0 | s^- (1-i), \bar{p}_i \rangle \) such that \( q_{s^- (1-i)} \vDash \dot{\bar{\gamma}} (\gamma_{(p_i^0 | s, \bar{p}_i)}) = 1 - j \).
Alternatively, there is no such \( i \), and we are free to select for both \( i = 0, 1 \) \( \gamma_{s^{-i}} \geq \langle p^0_{e} \upharpoonright s^{-i}, \vec{p}^0_e \rangle \) such that \( q_{s^{-i}} \models z \langle \gamma(p^{0}_{s^{-i}}) \rangle = i \).

We now define the first coordinate as an amalgamation of the \( q^0_{s^{-i}} \)'s: \( q^0_{s+1} = \bigcup_{s \in S, i = 0, 1} q^0_{s^{-i}} \), just like we did in the proof of theorem 3.4 of the original Sacks forcing. For the rest of the coordinates, we define \( p^0_{s+1} \) with accordance to the path taken in the first coordinate. Meaning that \( q^0_{s^{-i}} \) forces \( p^0_{s+1} \) is \( q^0_{s^{-i}} \)-i. From the way we defined the \( q_{s^{-i}} \)'s it is clear \( p^0_{s+1} \geq \vec{p}^e_s \), and so \( p_{s+1} \geq p_e \).

In the first coordinate we get a classical fusion sequence \( \langle p^0_e \upharpoonright \epsilon < \omega \rangle \). Therefore we can define \( p^0_e = \bigcap_{\epsilon < \omega} p_e^0 \in \mathbb{Q}_0 \). As for the higher coordinates, we can define \( \vec{p}_s = \bigcap_{\epsilon < \omega} p^0_{\epsilon} \) because of the \( \mathbb{N}_1 \)-closure, proven in lemma 5.3. So \( p_s \in \mathbb{P}_\delta \).

So we can now define \( p(0) = p_\omega \). Next we are going to repeat by induction the construction above, using the higher coordinates.

So for the successor case, assume that we’ve already defined \( p(\beta) \). For all coordinates \( 0 < \alpha < \beta \) define \( \hat{p}(\beta + 1)^\alpha = \hat{p}(\alpha)^\alpha \), and let \( p(\beta + 1)^0 = p(0)^0 \).

Let \( H_{\beta + 1} \subseteq \mathbb{P}_\beta \) be any generic set such that \( \langle p(\beta)^0, \hat{p}(\beta)^1, ..., \hat{p}(\beta)^\alpha, ..., | \alpha < \beta \rangle \rangle \in H_{\beta + 1} \). Note that \( M_0[H_{\beta + 1}] = \) the interpretation of \( \bar{M}_{\beta + 1} \) according to \( \bar{H}_{\beta + 1} \).

Working in \( M_0[H_{\beta + 1}] \), we know that \( \langle p(\beta)^{\beta + 1}, \hat{p}(\beta)^{\beta + 2}, ..., p(\beta)^\alpha, ..., | \alpha < \delta \rangle \rangle \models \hat{A} \notin M_0[H_{\beta + 1}] \). Just as before, let \( z \) be the name of the characteristic function of \( A \). For any condition \( q \in \mathbb{Z}_{H_{\beta + 1}} \) let \( z_q \) be the longest initial segment of \( z \) that is decided by \( q \), let \( \gamma_q \) be the first ordinal for which \( z \) is undecided, let \( q^{\beta + 1} \) denote the first coordinate of \( q \), i.e. from \( \mathbb{N}_{\beta + 1} \)-Sacks forcing over \( M_0[H_{\beta + 1}] \), and let \( \vec{q} \) denote the name of the rest of the coordinates. \( z_q \) and \( \gamma_q \) must be well-defined, because if \( q \) decides all of \( z \), it decides all of \( A \), and then \( A \in M_0[H_{\beta + 1}] \), in contradiction to \( p(\beta) \). Plainly \( \gamma_q < \omega_{\beta + 1} \).

Just as before, we again build a fusion sequence. This time mark \( p_0 = p(\beta) \). For the successor case, assume that we’ve already chosen \( p_e \). Note that \( p^{\beta + 1}_e \) is just a perfect tree in \( \mathbb{Q}_{\beta + 1} \), and \( \vec{p}_s \) is a name of a condition. Let \( S_i \) denote the order \( \epsilon \) splitting nodes of \( p^{\beta + 1}_e \). For every splitting node \( s \in S_i \), let’s look at \( \gamma_{\langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle} \) and conditions \( \langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle \).

Suppose that for both \( i = 0, 1 \) we have \( \langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle \models z \langle \gamma_{\langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle} \rangle = j \). \( \gamma_{\langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle} \) is undecided, so take \( q \geq \langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle \) such that \( q \models z \langle \gamma_{\langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle} \rangle = 1 - j \). Either \( q \cap \langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle \in \mathbb{Z}_{H_{\beta + 1}} \) or \( q \cap \langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle \in \mathbb{Z}_{H_{\beta + 1}} \). But \( q \) and \( \langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle \) are incompatible for \( i = 0, 1 \), therefore our supposition is impossible. Thus, if for a certain \( i \) there is a \( j \) such that \( \langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle \models z \langle \gamma_{\langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle} \rangle = 1 - j \), we can define \( q_{s^{-i}} = \langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle \) so that \( q_{s^{-i}} \models z \langle \gamma_{\langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle} \rangle = j \), and know there will be some \( q_{s^{-1 - i}} \geq \langle p^{\beta + 1}_e \upharpoonright s^{-1 - i}, \vec{p}^e_s \rangle \) such that \( q_{s^{-1 - i}} \models z \langle \gamma_{\langle p^{\beta + 1}_e \upharpoonright s^{-1 - i}, \vec{p}^e_s \rangle} \rangle = 1 - j \).

Alternatively, there is no such \( i \), and we are free to select for both \( i = 0, 1 \) \( q_{s^{-i}} \geq \langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle \) such that \( q_{s^{-i}} \models z \langle \gamma_{\langle p^{\beta + 1}_e \upharpoonright s^{-i}, \vec{p}^e_s \rangle} \rangle = i \).
We now define the first coordinate as an amalgamation of the \( q_{\beta+1}^{\beta+1} \): \( p^{\beta+1}_{\tau+1} = \bigcup_{s \in S_{\tau+1} = 0,1} q_{\beta+1}^{\beta+1} \). For the rest of the coordinates, we define \( p^{\beta+1}_+ \) with accordance to the path taken in the first coordinate. Meaning that \( q_{\beta+1}^{\beta+1} \) forces that \( p^{\beta+1}_+ = q_{\beta+1}^{\beta+1} \). From the way we defined the \( q_{\beta+1}^{\beta+1} \)’s it is clear \( p^{\beta+1}_+ \geq \vec{p}_{\tau} \), and so \( p^{\beta+1}_{+1} \geq p_{\tau} \).

In limit stages \( \tau < \omega_{\beta+1} \) we just use the \( \mathcal{N}_{\beta+1} \)-closure to define \( p_\tau = \bigcap_{\alpha < \tau} p_\alpha \).

Now, in order to define \( p_{\omega_{\beta+1}} \), note that in the first coordinate we again get a fusion sequence \( \langle p^{\beta+1}_\epsilon \mid \epsilon < \omega \rangle \). Therefore \( p^{\beta+1}_{\omega_{\beta+1}} = \bigcap_{\epsilon < \omega} p^{\beta+1}_\epsilon \in \mathcal{Q}_{\beta+1} \). As for the rest of the coordinates, we can use the \( \mathcal{N}_{\beta+1} \)-closure proven in lemma 5.3 to define \( p_{\omega_{\beta+1}} = \bigcap_{\epsilon < \omega} \vec{p}_\epsilon \).

For all \( \alpha > \beta \) we define \( \vec{p}(\beta+1)^\alpha = p^{\alpha}_{\omega_{\beta+1}} \).

We now pick a name \( \langle \vec{p}(\beta+1)^{\beta+1}, \vec{p}(\beta+1)^{\beta+2}, ..., \vec{p}(\beta+1)^\alpha, ... \mid \alpha < \delta \rangle \) for the \( \langle p(\beta+1)^{\beta+1}, p(\beta+1)^{\beta+2}, ..., p(\beta+1)^\alpha, ... \mid \alpha < \delta \rangle \) that we constructed in \( \mathcal{P}_{/\mathcal{H}_{\beta+1}} \), such that \( \langle p(\beta)^0, p(\beta)^1, ..., p(\beta)^\beta \rangle \) forces it to be the way it was defined.

Finally, set \( p(\beta+1) = \langle p(0)^0, p(1)^1, ..., p(\beta+1)^{\beta+1}, p(\beta+1)^{\beta+2}, ..., p(\beta+1)^\alpha \mid \alpha < \delta \rangle \).

For limit stages \( \delta \) we define \( p(\delta) = \bigcap_{\alpha < \delta} p(\alpha) \). As each coordinate lesser than \( \delta \) stabilizes, and by lemma 5.3 each coordinate \( \geq \delta \) is at least \( \mathcal{N}_{\delta+1} \)-closed, and because we’re using full support, \( p(\delta) \in \mathcal{P}_\delta \).

As we’ve constructed \( p(\delta) \) stronger than a general \( p \in \mathcal{P}_\delta \), then by density arguments, we may assume without loss of generality \( p(\delta) \in G_\delta \).

Just as in corollary 1.1 we can view \( p(\delta) \) as a mapping: \( p(\delta) \) takes as input a sequence of branches \( \langle \vec{h}_\alpha \mid \alpha < \delta \rangle \), where each \( \vec{h}_\alpha \) is a branch of \( \vec{p}(\delta)^\alpha \), and interprets \( A \).

For each \( G_{\alpha, \alpha+1} \), let \( g_{\alpha, \alpha+1} \) be the generic branch interdefinable with \( G_{\alpha, \alpha+1} \). Not that because of the way \( p(\delta) \) was defined, \( M_\delta \models p(\delta)(\langle g_{\alpha, \alpha+1} \mid \alpha < \delta \rangle) = A \). Therefore, for a given \( G_\delta \), there can’t be two different subsets of \( \mathcal{N}_\delta \) that produce the same \( p(\delta) \in \mathcal{P}_\delta \) in the construction above.

As \( |\mathcal{P}_\delta| = \mathcal{N}_{\delta+1} \), there are at most \( \mathcal{N}_{\delta+1} \) new subsets of \( \mathcal{N}_\delta \). Therefore, \( (2^{\mathcal{N}_\delta} = \mathcal{N}_{\delta+1}) \mathcal{M}_\delta \).

\[ \square \]

**Lemma 5.10.** For all \( \beta \leq \zeta \), \( M_\beta \) has the same cardinals as \( M_0 \).

**Proof.** By induction we prove \( OK(\beta) \). By lemma 5.5 \( OK(0) \). The successor case is handled in lemma 5.6.

For the limit case, lemmas 5.7, 5.8 and 5.9 show that if \( OK(\alpha) \) is true for all \( \alpha < \delta \), then \( OK(\delta) \).

Therefore by induction \( OK(\beta) \), and so \( M_\beta \) has the same cardinals as \( M_0 \). \[ \square \]

Next, we want to verify that during the iteration we don’t create any inner model of \( M_\epsilon \) other than the \( M_\alpha \’s \) for \( \alpha < \zeta \).
Note that while theorem 4.10 shows that applying $\mathfrak{S}_\alpha$-Sacks forcing doesn’t add any inner model between $M_\alpha$ and $M_{\alpha+1}$, it says nothing about limit stages. If $M_\delta$ is the limit model, then theoretically there might be another inner model lurking between $M_\delta$ and $\bigcup_{\alpha<\delta} M_\alpha$.

A second type of problem could arise even in the successor stages. Applying the forcing over $M_\alpha$ with $\alpha > 0$, one might inadvertently create some new inner model between $M_0 = L$ and $M_{\alpha+1}$ outside the chain. Therefore we need to prove our construction avoids creating both types of ‘accidental’ models.

**Lemma 5.11.** If $N$ is an inner model of $M_\zeta$ such that for all $\alpha \leq \zeta$ $N \neq M_\alpha$, and $\beta$ is the least ordinal such that $N \not\subseteq M_\beta$, then $\beta$ is a limit ordinal.

**Proof.** It is enough to show that there is no greatest $\beta$ such that $M_\beta \subseteq N$. Working to the contrary, assume $\beta < \zeta$ is such that $M_\beta \subseteq N$ but $M_{\beta+1} \not\subseteq N$.

Obviously if $Q_\beta$ is trivial then $M_{\beta+1} \not\subseteq N$ in contradiction to the assumption. Therefore we may assume $Q_\beta$ is $\aleph_\beta$-Sacks forcing.

$N$ is a model of ZFC between $M_\beta$ and $M_\zeta$. Therefore according to lemma 15.43 in Jech [4], $N = M_\beta [A]$ for some set of ordinals $A \in M_\zeta$.

$N \neq M_\beta$ so obviously $A \in N \setminus M_\beta$. Let $\dot{A}$ be its name in $M_\beta$ in the forcing $\mathbb{P}_{\zeta/G_\beta}$. There is an ordinal $\nu$ such that $0 \Vdash \dot{A} \subseteq \nu$, and let $\dot{z}$ be the name of the characteristic function of $A$, $z : \nu \to 2$. $A \not\in M_\beta$, so there is a condition $p \in G_\beta \zeta$ that forces $\dot{A} \not\in M_\beta$.

For any condition $q \in \mathbb{P}_{\zeta/G_\beta}$ stronger than $p$, let $\dot{z}_q$ be the longest initial segment of $\dot{z}$ that is decided by $q$, let $\gamma_q$ be the first ordinal for which $\dot{z}$ is undecided, let $q^0$ denote the first coordinate of $q$, i.e from $\aleph_\beta$-Sacks forcing over $M_\beta$, and let $\vec{q}$ denote the name of the rest of the coordinates. $\dot{z}_q$ and $\gamma_q$ must be well-defined, because if $q$ decides all of $\dot{z}$, it decides all of $\dot{A}$, and then $A \in M_\beta$, in contradiction to $p$. Plainly $\gamma_q < \nu$.

We’re now going to build a fusion sequence. Mark $p_0 = p$. For the successor case, assume that we’ve already chosen $p_i$. Note that $p^0$ is just a perfect tree in $Q_\beta$, and $\vec{p}_i$ is a name in $M_\beta$. Let $S_i$ denote the order $\epsilon$ splitting nodes of $p^i$. For every splitting node $s \in S_i$, let’s look at $\gamma(p^i|s,\vec{p}_i)$ and conditions $\langle p^i | s^{-i}, \vec{p}_i \rangle$.

Suppose that for both $i = 0, 1$ we have $\langle p^0 | s^{-i}, \vec{p}_i \rangle \Vdash \dot{z} (\gamma(p^i|s,\vec{p}_i)) = j$. $\gamma(p^i|s,\vec{p}_i)$ is undecided, so take $q \geq \langle p^0 | s, \vec{p}_i \rangle$ such that $q \Vdash \dot{z} (\gamma(p^0|s,\vec{p}_i)) = 1 - j$. Either $q \cap \langle p^0 | s^{-0}, \vec{p}_i \rangle \in \mathbb{P}_{\zeta/G_\beta}$ or $q \cap \langle p^0 | s^{-1}, \vec{p}_i \rangle \in \mathbb{P}_{\zeta/G_\beta}$. But $q$ is in the forcing over $M_\beta$, and $\vec{q}$ is incompatible for $i = 0, 1$, therefore our supposition is impossible.

Thus, if for a certain $i$ there is a $j$ such that $\langle p^0 | s^{-i}, \vec{p}_i \rangle \Vdash \dot{z} (\gamma(p^0|s,\vec{p}_i)) = j$, we can define $q_{s^{-i}} = \langle p^0 | s^{-i}, \vec{p}_i \rangle$ so that $q_{s^{-i}} \Vdash \dot{z} (\gamma(p^0|s,\vec{p}_i)) = j$, and know there will be some $q_{s^{-i}}(1-i) \geq \langle p^0 | s^{-i} \rightarrow (1 - i), \vec{p}_i \rangle$ such that $q_{s^{-i}}(1-i) \Vdash \dot{z} (\gamma(p^0|s,\vec{p}_i)) = 1 - j$.

Alternatively, there is no such $i$, and we are free to select for both $i = 0, 1$ $q_{s^{-i}} \geq \langle p^0 | s^{-i}, \vec{p}_i \rangle$ such that $q_{s^{-i}} \Vdash \dot{z} (\gamma(p^0|s,\vec{p}_i)) = i$.

We now define the first coordinate as an amalgamation of the $q_{s^{-i}}$‘s: $p^0_{\epsilon+1} = \bigcup_{s \in S_i, i = 0, 1} q^0_{s^{-i}}$. Just like we did in the proof of theorem 3.33 of the original Sacks forcing. For the rest of the coordinates, we define $p^i_{\epsilon+1}$ with accordance to
the path taken in the first coordinate. Meaning that if \( q^0 \in G_{\beta, \beta+1} \) then \( p^1 = q^{-1} \). From the way we defined the \( q^{-1} \)'s it is clear \( p^1 \geq p^0 \), and so \( p^1 \geq p^0 \).

In limit stages \( \delta \leq \omega_\beta \) we take \( p^0 = \bigcap_{\epsilon < \delta} p^\epsilon \), so in the first coordinate we get a fusion sequence \( \langle p^\epsilon \mid \epsilon < \delta \rangle \) just like in the proof of theorem 4.10. Therefore \( p^0 \in Q_\beta \). As for the rest of the coordinates, \( p^0 = T_{\xi}G_{\beta+1} \) because of the \( N_{\beta+1} \)-closure, as shown in lemma 5.3.

So we can now define \( q = p_{\omega_\beta} \in T_{\xi}G_\beta \). Note that we constructed such a \( q \geq p \) over any \( p \in G_{\beta, \xi} \), so due to density we may assume without loss of generality that \( q \in G_{\beta, \xi} \).

Now let \( f = \{ s \in q^0 \mid \hat{z}(q^0[s,0]) \subseteq \hat{z} G_{\beta, \xi} \} \). We claim \( f \) is a branch of \( q^0 \).

From density we know that for every \( \alpha < \omega_\beta \) there is an \( r \in G_{\beta, \xi} \) such that \( r^0 \) has a stem with length at least \( \alpha \). Because \( G_{\beta, \xi} \) is generic, there is some condition \( t \geq q \cap r \) in \( G_{\beta, \xi} \). This \( t^0 \) has a stem with length at least \( \alpha \), and so there is some node \( s \) in level \( \alpha \) of \( q^0 \) such that \( t^0 \geq q^1 \downarrow s \). Obviously \( \hat{z}(q^0[s,0]) \subseteq \hat{z}(q^1[s,0]) \subseteq \hat{z} G_{\beta, \xi} \), and so for every \( \alpha < \omega_\beta \) there is some \( s \) in that level of \( q^0 \) such that \( s \in f \). Also, if \( s \in f \), then it's trivial that for all \( \alpha \) \( s \downarrow \alpha \in f \).

Next, we show that \( f \) has no splitting nodes. Suppose \( s \) is a splitting node of \( q \), then for \( i = 0,1, \hat{z}(q^0[s,0]) \neq \hat{z}(q^1[s,0]) \) and therefore \( \hat{z}(g_{\beta, \xi}) = \hat{z}(q^0[s,0]) \neq \hat{z}(q^1[s,0]) \), or either \( \hat{z}(g_{\beta, \xi}) = \hat{z}(q^0[s,0]) \) and therefore \( s^0 \downarrow 0 \) is in \( f \), but not both. Therefore \( s \) is not a splitting node in \( f \), and so there are no splitting nodes in \( f \). We conclude that \( f \) is indeed a branch in \( q^0 \).

In fact, we claim that \( f \) is equal to the generic branch \( g_{\beta, \beta+1} \) derived from the generic set \( G_{\beta, \xi} \). Let \( s \in g_{\beta, \beta+1} \). Then due to density there is an \( r \in G_{\beta, \xi} \) such that \( s \) is part of the stem of \( r^0 \), and some condition \( t \geq q \cap r \) in \( G_{\beta, \xi} \). As above, \( t^0 \geq q^0 \downarrow s \), and so \( \hat{z}(q^0[s,0]) \subseteq \hat{z}(q^1[s,0]) \subseteq \hat{z} G_{\beta, \xi} \). Therefore \( s \in f \).

Hence \( f = g_{\beta, \beta+1} \) is a branch of \( q^0 \) that is definable in \( M_\beta \). Using \( A_{\beta, \xi} = A \), we managed to recover the generic branch \( g_{\beta, \beta+1} \). But remember, the generic branch \( g_{\beta, \beta+1} \) is in fact interdefinable with the generic set \( G_{\beta, \beta+1} \), and so \( g_{\beta, \beta+1} \subseteq M_\beta \). Therefore \( M_{\beta+1} = M_\beta [g_{\beta, \beta+1}] = M_\beta [g_{\beta, \beta+1}] \subseteq M_\beta [A] \subseteq N \), in contradiction to our assumption that \( M_{\beta+1} \not\subseteq N \).

We conclude that if \( N \) violates the theorem, there is no greatest \( \beta \) such that \( M_\beta \not\subseteq N \). Thus, the least inner model of the tower that isn’t included in \( N \) must be \( M_\delta \) for some limit ordinal \( \delta \).

\( \square \)

**Lemma 5.12.** If \( N \) is an inner model of \( M_\xi \), and \( \delta \) is a limit ordinal such that for all \( \beta < \delta \) \( N \supseteq M_{\beta} \), then \( N \supseteq M_\delta \).

**Proof.** We show this inductively. So let \( \delta \) be a limit ordinal, and assume the lemma is true for every limit ordinal \( \epsilon < \delta \). Let \( N \) be an inner model of \( M_\xi \) such that for all \( \beta < \delta \) \( M_\beta \not\subseteq N \). We aim to show that \( M_\delta \subseteq N \) by showing that \( \langle G_{\beta, \beta+1} \mid \beta < \delta \rangle \in N \).
To start things off we first want to define a sequence \( g = \langle g_\beta \mid \beta < \delta \rangle \in N \) such that for each \( \beta < \delta \) if \( \beta \) is not a limit ordinal then \( g_\beta \subseteq \aleph_\beta \) and \( g_\beta \notin M_\beta \). Note that while each \( M_\beta \) is by itself definable in \( N \) using set parameters, the sequence might not be, so we can’t simply define \( A_\beta = \{ a \subseteq \aleph_\beta \mid a \in N \setminus M_\beta \land \sup (a) = \aleph_\beta \} \) and then choose some \( g_\beta \in A_\beta \) whenever \( \beta \) is not a limit.

Instead, we build this sequence inductively, working in \( N \). Let \( N_0 = L \). Next, for all \( \beta < \delta \), assuming \( N_\beta \) is defined, let \( A_\beta = \{ a \subseteq \aleph_\beta \mid a \in N \setminus N_\beta \land \sup (a) = \aleph_\beta \} \), and if \( A_\beta \neq \emptyset \) choose some \( g_\beta \in A_\beta \), otherwise set \( g_\beta = \emptyset \). For the successor step, assuming \( N_\beta \) is defined, we define \( N_{\beta+1} = N_\beta [g_\beta] \). In the limit step, assuming \( N_\beta \) is defined for all \( \beta < \epsilon < \delta \), we define \( N_\epsilon \) as the least inner model that includes every \( N_\beta \).

We claim that for all \( \beta < \delta \) \( N_\beta \) is definable and equal to \( M_\beta \), and that if \( \beta \) is not a limit ordinal then \( g_\beta \subseteq \aleph_\beta \) and \( g_\beta \notin M_\beta \). For the base case, note that \( N_0 = L = M_0 \), which is of course definable in \( N \). Next, assuming \( N_\beta = M_\beta \), then \( A_\beta = \{ a \subseteq \aleph_\beta \mid a \in N \setminus M_\beta \land \sup (a) = \aleph_\beta \} \). If \( \beta \) is a limit, then \( Q_\beta \) is trivial, and so \( M_\beta = M_{\beta+1} \). On the other hand, \( P_\beta / G_\beta \) is \( \aleph_{\beta+1} \)-closed, as shown by lemma 5.3. Therefore \( M_\beta \) and \( M_\epsilon \) have the same subsets of \( \aleph_\beta \). Hence \( A_\beta = \emptyset \), and therefore \( g_\beta = \emptyset \). We get \( N_{\beta+1} = N_\beta [\emptyset] = N_\beta = M_\beta = M_{\beta+1} \) as required.

If \( \beta \) is not a limit, then \( Q_\beta \) is \( \aleph_\beta \)-Sacks forcing, and therefore there is a new subset of \( \aleph_\beta \) in \( M_{\beta+1} \setminus M_\beta \). Hence \( A_\beta \neq \emptyset \). On the other hand \( P_\beta / G_\beta \) is \( \aleph_{\beta+1} \)-closed, so \( M_{\beta+1} \) has the same subsets of \( \aleph_\beta \) as \( M_\epsilon \). Therefore \( M_\beta \subseteq M_\beta [g_\beta] \subseteq M_{\beta+1} \). But \( M_{\beta+1} \) is generated from \( M_\beta \) using \( \aleph_\beta \)-Sacks forcing, and so according to theorem 4.10 there is no intermediate model. Hence \( N_{\beta+1} = N_\beta [g_\beta] = M_\beta [g_\beta] = M_{\beta+1} \).

In the limit step, assume that for \( \epsilon \) a limit ordinal we’ve already shown that \( M_\beta = N_\beta \) for all \( \beta < \epsilon \). \( G_\epsilon \in N \), and therefore \( M_\epsilon \) is definable with set parameters in \( N \). Hence \( N \) recognizes that \( M_\epsilon \) is its inner model. Working towards a contradiction, assume \( K \) is an inner model of \( N \) such that for all \( \beta < \epsilon \) \( M_\beta \subseteq M_\beta \) but \( M_\epsilon \nsubseteq K \). \( K \) is definable with set parameters in \( N \), which is definable with set parameters in \( M_\epsilon \). Therefore \( K \) is definable with set parameters in \( M_\epsilon \), and therefore \( K \) is an inner model of \( M_\epsilon \) with said properties. But by the induction hypothesis the lemma is true for every \( \epsilon < \delta \), so \( K \supseteq M_\epsilon \) in contradiction to our assumption. Therefore there is no such inner model \( K \). So every inner model of \( N \) that includes all the \( M_\beta \)’s for \( \beta < \epsilon \) must necessarily include \( M_\epsilon \). Therefore \( M_\epsilon \) is the least inner model that includes every \( M_\beta \). But this exactly coincides with our definition of \( N_\epsilon \), and so \( N_\epsilon = M_\epsilon \).

Thus the induction is now complete and we’ve managed to define \( N_\beta \) and show that it is in fact equal to \( M_\beta \) for all \( \beta < \delta \). We’ve also shown that if \( \beta \) is not a limit then \( A_\beta \neq \emptyset \) and so \( g_\beta \subseteq \aleph_\beta \) and \( g_\beta \notin M_\beta \), as required. Therefore the set \( \langle g_\beta \mid \beta < \delta \rangle \in N \) is exactly the set which we set out to define.

The sequence \( \langle g_\beta \mid \beta < \delta \rangle \in M_\epsilon \). However, by lemma 6.3 \( P_\beta / G_\beta \) is \( \aleph_{\beta+1} \)-closed, and for all \( \beta < \delta \) \( g_\beta \in M_\beta \). Therefore \( \langle g_\beta \mid \beta < \delta \rangle \in M_\beta \).

Hence there exists a condition \( u \in G_\delta \) that forces

\[
\dot{g} = \langle g_\beta \mid \beta < \delta \rangle \land \forall \beta < \delta \ (\beta \text{ is not a limit } \rightarrow (g_\beta \subseteq \aleph_\beta \land g_\beta \in \dot{M}_{\beta+1} \setminus \dot{M}_\beta))
\]
By the definition of the forcing, we may assume that for each $\beta < \delta \dot{g}_\beta$ is a $P_\beta \ast Q_\beta$-name of $g_\beta$.

Assume now that we have a condition $p \in P_\delta$ stronger than $u$. We denote individual coordinates like so: $p = \langle p^0, \dot{p}^1, \ldots, \dot{p}^\alpha, \ldots \mid \alpha < \delta \rangle$.

We know that $p \models \dot{g}_0 \subseteq \aleph_0 \setminus \dot{g}_0 \notin M_0$. Let $\dot{z}$ be the name of the characteristic function of $\dot{g}_0$. For any condition $q \in P_\delta$ let $\dot{z}_q$ be the longest initial segment of $\dot{z}$ that is decided by $q$, let $\gamma_q$ be the first ordinal for which $\dot{z}$ is undecided, let $q^0$ denote the first coordinate of $q$, i.e from $\aleph_0$-Sacks forcing over $M_0$, and let $q$ denote the name of the rest of the coordinates. $\dot{z}_q$ and $\gamma_q$ must be well-defined, because if $q$ decides all of $\dot{z}$, it decides all of $\dot{g}_0$, and then $g_0 \in M_0$, in contradiction to $p$. Plainly $\gamma_q < \omega$.

We now build a fusion sequence. Mark $p_0 = p$. For the successor case, assume that we’ve already chosen $p_i$. Note that $p_i^0$ is just a perfect tree in $Q_0$, and $p_i^\epsilon$ is a name in $M_0$. Let $S_\epsilon$ denote the order $\epsilon$ splitting nodes of $p_i^0$. For every splitting node $s \in S_\epsilon$, let’s look at $\gamma_{\langle p_i^0 \upharpoonright s \upharpoonright i, p_i^\epsilon \rangle}$ and conditions $\langle p_i^0 \upharpoonright s^{-i}, p_i^\epsilon \rangle$.

Suppose that for both $i = 0, 1$ we have $\langle p_i^0 \upharpoonright s^{-i}, p_i^\epsilon \rangle \models \dot{z} \left( \gamma_{\langle p_i^0 \upharpoonright s \upharpoonright i, p_i^\epsilon \rangle} \right) = j$. If $\gamma_{\langle p_i^0 \upharpoonright s \upharpoonright i, p_i^\epsilon \rangle}$ is undecided, so take $q \geq \langle p_i^0 \upharpoonright s \upharpoonright i, p_i^\epsilon \rangle$ such that $q \models \dot{z} \left( \gamma_{\langle p_i^0 \upharpoonright s \upharpoonright i, p_i^\epsilon \rangle} \right) = 1 - j$. Either $q \cap \langle p_i^0 \upharpoonright s^{-i}, p_i^\epsilon \rangle \in P_\delta$ or $q \cap \langle p_i^0 \upharpoonright s^{-1}, p_i^\epsilon \rangle \in P_\delta$. But $q$ and $\langle p_i^0 \upharpoonright s^{-i}, p_i^\epsilon \rangle$ are incompatible for $i = 0, 1$, therefore our supposition is impossible. Thus, if for a certain $i$ there is a $j$ such that $\langle p_i^0 \upharpoonright s^{-i}, p_i^\epsilon \rangle \models \dot{z} \left( \gamma_{\langle p_i^0 \upharpoonright s \upharpoonright i, p_i^\epsilon \rangle} \right) = j$, we can define $q_{s^{-i}} = \langle p_i^0 \upharpoonright s^{-i}, p_i^\epsilon \rangle$ so that $q_{s^{-i}} \models \dot{z} \left( \gamma_{\langle p_i^0 \upharpoonright s \upharpoonright i, p_i^\epsilon \rangle} \right) = j$, and know there will be some $q_{s^{-1}(1-i)} \geq \langle p_i^0 \upharpoonright s^{-1}(1-i), p_i^\epsilon \rangle$ such that $q_{s^{-1}(1-i)} \models \dot{z} \left( \gamma_{\langle p_i^0 \upharpoonright s \upharpoonright i, p_i^\epsilon \rangle} \right) = 1 - j$.

Alternatively, there is no such $i$, and we are free to select for both $i = 0, 1 q_{s^{-i}} \geq \langle p_i^0 \upharpoonright s^{-i}, p_i^\epsilon \rangle$ such that $q_{s^{-i}} \models \dot{z} \left( \gamma_{\langle p_i^0 \upharpoonright s \upharpoonright i, p_i^\epsilon \rangle} \right) = i$.

We now define the first coordinate as an amalgamation of the $q_{s^{-i}}$’s: $p_{\epsilon+1} = \bigcup_{s \in S_\epsilon, i = 0, 1} q_{s^{-i}}^\epsilon$, just like we did in the proof of theorem 3.9 of the original Sacks forcing. For the rest of the coordinates, we define $p_{\epsilon+1}$ with accordance to the path taken in the first coordinate. Meaning that $q_{s^{-i}}^\epsilon$ forces $p_{\epsilon+1} = q_{s^{-i}}$. From the way we defined the $q_{s^{-i}}$’s it is clear $p_{\epsilon+1} \geq p_{\epsilon}$, and so $p_{\epsilon+1} \geq p_{\epsilon}$.

In the first coordinate we get a classical fusion sequence $\langle p_{\epsilon}^0 \mid \epsilon < \omega \rangle$. Therefore we can define $p_0^\omega = \bigcap_{\epsilon < \omega} p_{\epsilon}^\omega \in Q_0$. As for the higher coordinates, we can define $p_\omega = \bigcap_{\epsilon < \omega} p_{\epsilon}^\omega$ because of the $\aleph_1$-closure, proven in lemma 3.9. So $p_\omega \in P_\delta$.

So we can now define $p(0) = p_\omega$. Next we are going to repeat by induction the construction above, using the higher coordinates.

So for the successor case, assume that we’ve already defined $p(\beta)$. For all coordinates $0 < \alpha \leq \beta$ define $\dot{p}(\beta + 1)^\alpha = \dot{p}(\alpha)^\alpha$, and let $p(\beta + 1)^0 = p(0)^0$. We are now going to deal with $\dot{g}_{\beta+1}$. So let $H_{\beta+1} \subseteq P_{\beta+1}$ be any generic set such that $\langle p(\beta)^0, \dot{p}(\beta)^1, \ldots, \dot{p}(\beta)^\alpha, \ldots \mid \alpha \leq \beta \rangle \in H_{\beta+1}$.

Note that $M_0[H_{\beta+1}] = \text{the interpretation of } M_{\beta+1} \text{ according to } H_{\beta+1}$. So working in $M_0[H_{\beta+1}]$, we know that

$$\langle p(\beta)^{\beta+1}, \dot{p}(\beta)^{\beta+2}, \ldots, \dot{p}(\beta)^\alpha, \ldots \mid \alpha < \delta \rangle \models \dot{g}_{\beta+1} \subseteq \aleph_0 \setminus \dot{g}_{\beta+1} \notin M_0[H_{\beta+1}]$$
Just as before, let \( \dot{z} \) be the name of the characteristic function of \( \dot{g}_{\beta+1} \). For any condition \( q \in \mathcal{P}_{\delta}/H_{\beta+1} \) let \( \dot{z}_q \) be the longest initial segment of \( \dot{z} \) that is decided by \( q \), let \( \gamma_q \) be the first ordinal for which \( \dot{z} \) is undecided, let \( q^{\beta+1} \) denote the first coordinate of \( q \), i.e. from \( \mathcal{N}_{\beta+1} \)-Sacks forcing over \( M_0[H_{\beta+1}] \), and let \( \bar{q} \) denote the name of the rest of the coordinates. \( \dot{z}_q \) and \( \gamma_q \) must be well-defined, because if \( q \) decides all of \( \dot{z} \), it decides all of \( \dot{g}_{\beta+1} \), and then \( g_{\beta+1} \in M_0[H_{\beta+1}] \), in contradiction to \( p(\beta) \). Plainly \( \gamma_q < \omega_{\beta+1} \).

Just as before, we again build a fusion sequence. This time mark \( p_0 = p(\beta) \). For the successor case, assume that we’ve already chosen \( p_e \). Note that \( p^{\beta+1}_e \) is just a perfect tree in \( \mathcal{Q}_{\beta+1} \), and \( \bar{p}_e \) is a name of a condition. Let \( S_e \) denote the order \( \epsilon \) splitting nodes of \( p^{\beta+1}_e \). For every splitting node \( s \in S_e \), let’s look at \( \gamma_{\langle p^{\beta+1}_s, \bar{p}_e \rangle} \) and conditions \( \langle p^{\beta+1}_s \upharpoonright s^{-i}, \bar{p}_e \rangle \).

Suppose that for both \( i = 0, 1 \) we have \( \langle p^{\beta+1}_e \upharpoonright s^{-i}, \bar{p}_e \rangle \models \dot{z} \left( \gamma_{\langle p^{\beta+1}_s, \bar{p}_e \rangle} = j \right) \). \( \gamma_{\langle p^{\beta+1}_s, \bar{p}_e \rangle} \) is undecided, so take \( q \geq \langle p^{\beta+1}_e \upharpoonright s, \bar{p}_e \rangle \) such that \( q \models \dot{z} \left( \gamma_{\langle p^{\beta+1}_s, \bar{p}_e \rangle} = 1-j \right) \). Either \( q \cap (p^{\beta+1}_e \upharpoonright s^0, \bar{p}_e) \in \mathcal{P}_{\delta}/H_{\beta+1} \), or \( q \cap (p^{\beta+1}_e \upharpoonright s^{-1}, \bar{p}_e) \in \mathcal{P}_{\delta}/H_{\beta+1} \). But \( q \) and \( \langle p^{\beta+1}_s \upharpoonright s^{-i}, \bar{p}_e \rangle \) are incompatible for \( i = 0, 1 \), therefore our supposition is impossible. Thus, if for a certain \( i \) there is a \( j \) such that \( \langle p^{\beta+1}_e \upharpoonright s^{-i}, \bar{p}_e \rangle \models \dot{z} \left( \gamma_{\langle p^{\beta+1}_s, \bar{p}_e \rangle} = j \right) \), we can define \( q_{s^{-i}} = \langle p^{\beta+1}_e \upharpoonright s^{-i}, \bar{p}_e \rangle \) so that \( q_{s^{-i}} \models \dot{z} \left( \gamma_{\langle p^{\beta+1}_s, \bar{p}_e \rangle} = j \right) \), and now there will be some \( q_{s^{-i}} \upharpoonright (1-i) \geq \langle p^{\beta+1}_e \upharpoonright s^{-i}, \bar{p}_e \rangle \) such that \( q_{s^{-i}} \upharpoonright (1-i) \models \dot{z} \left( \gamma_{\langle p^{\beta+1}_s, \bar{p}_e \rangle} = 1-j \right) \).

Alternatively, there is no such \( i \), and we are free to select for both \( i = 0, 1 \) \( q_{s^{-i}} \geq \langle p^{\beta+1}_e \upharpoonright s^{-i}, \bar{p}_e \rangle \) such that \( q_{s^{-i}} \models \dot{z} \left( \gamma_{\langle p^{\beta+1}_s, \bar{p}_e \rangle} = i \right) \).

We now define the first coordinate as an amalgamation of the \( q_{s^{-i}} \)'s: \( p^{\beta+1}_{e+1} = \bigcup_{s \in S_e, i = 0, 1} q^{\beta+1}_{s^{-i}} \). For the rest of the coordinates, we define \( p^{\beta+1}_{e+1} \) with accordance to the path taken in the first coordinate. Meaning that \( q^{\beta+1}_{s^{-i}} \) forces that \( p^{\beta+1}_{e+1} = q_{s^{-i}} \). From the way we defined the \( q_{s^{-i}} \)'s it is clear \( p^{\tau+1}_{e+1} \geq p_e \), and so \( p^{\tau+1}_{e+1} \geq p_e \). In limit stages \( \tau < \omega_{\beta+1} \) we just use the \( \mathcal{N}_{\beta+1} \)-closure to define \( p_\tau = \bigcap_{\alpha < \tau} p_\alpha \).

Now, in order to define \( p_{\omega_{\beta+1}} \), note that in the first coordinate we again get a fusion sequence \( \langle p^{\beta+1}_e \upharpoonright \epsilon < \omega \rangle \). Therefore \( p^{\beta+1}_{\omega_{\beta+1}} = \bigcap_{\epsilon < \omega_{\beta+1}} p^{\beta+1}_e \in \mathcal{Q}_{\beta+1} \). As for the rest of the coordinates, we can use the \( \mathcal{N}_{\beta+1} \)-closure proven in lemma 4.3 to define \( p_{\omega_{\beta+1}} = \bigcap_{\epsilon < \omega_{\beta+1}} \bar{p}_e \).

For all \( \alpha > \beta \) we define \( \dot{p}(\beta+1)^\alpha = p^{\alpha}_{\omega_{\beta+1}} \).

We now pick a name \( \langle \dot{p}(\beta+1)^{\beta+1}, \dot{p}(\beta+1)^\beta, \ldots, \dot{p}(\beta+1)^0, ... \mid \epsilon < \delta \rangle \) for the \( \langle p(\beta+1)^{\beta+1}, \dot{p}(\beta+1)^\beta, ..., \dot{p}(\beta+1)^0, ... \mid \epsilon < \delta \rangle \) that we constructed in \( \mathcal{P}_{\delta}/H_{\beta+1} \), such that \( \langle p(\beta)^\alpha, \dot{p}(\beta)^1, ..., \dot{p}(\beta)^\beta \rangle \) forces it to be the way it was defined.

Finally, set \( p(\beta+1) = \langle p(0)^0, \dot{p}(1)^1, ..., \dot{p}(\beta+1)^{\beta+1}, \dot{p}(\beta+1)^\beta, ..., \dot{p}(\beta+1)^0 \mid \epsilon < \delta \rangle \).
For limit stages $\delta$ we define $p(\delta) = \bigcap_{\alpha < \delta} p(\alpha)$. As each coordinate lesser than $\delta$ stabilizes, and by lemma 5.3 each coordinate $\geq \delta$ is at least $\aleph_{\delta+1}$-closed, and because we’re using full support, $p(\delta) \in \mathbb{P}_{\delta}$.

As we’ve constructed $p(\delta)$ stronger than a general $p \in \mathbb{P}_{\delta}$, then by density arguments, we may assume without loss of generality $p(\delta) \in G_{\delta}$.

Now we’re going to use $p(\delta)$ and the sequence of $\langle g_\beta \mid \beta < \delta \rangle$ to recover $\langle G_{\beta,\beta+1} \mid \beta < \delta \rangle$.

By induction, assume that for some $\beta < \delta$ we already recovered $\langle G_{\alpha,\alpha+1} \mid \alpha < \beta \rangle$, and thus $M_\beta$. Therefore, working in $M_\beta$, let $\dot{z}$ be as before the name of the characteristic function of $\dot{y}_\beta$ and for each $r \in \mathbb{P}_\beta \cap G_\beta$ let $\dot{z}_r$ be the longest initial segment of $\dot{z}$ that is decided by $r$.

Let $q = p(\delta)$, and define $f = \{ s \in q^\beta \mid \dot{z}_{\langle q^\beta \rangle \langle t \rangle} \subseteq \dot{z}_{G_{\beta,s}} \}$. We claim that $f$ is a branch of $q^\beta$.

From density we know that for every $\alpha < \omega_\beta$ there is an $r \in G_{\beta,\delta}$ such that $r^\alpha$ has a stem with length at least $\alpha$. Because $G_{\beta,\delta}$ is generic, there is some condition $t \geq q \cap r$ in $G_{\beta,\delta}$. This $t^\beta$ has a stem with length at least $\alpha$, and so there is some node $s$ in level $\alpha$ of $q^\beta$ such that $t^\beta \triangleright q^\beta \upharpoonright s$. Obviously $\dot{z}_{\langle q^\beta \rangle \langle t \rangle} \subseteq \dot{z}_{\langle q^\beta \rangle \langle s \rangle}$, and so for every $\alpha < \omega_\beta$ there is some $s$ in that level of $q^\beta$ such that $s \in f$. Also, if $s \in f$, then it’s trivial that for all $\alpha \in f$.

Next, we show that $f$ has no splitting nodes. Suppose that $s$ is a splitting node of $q^\beta$, then for $i = 0, 1$ $\dot{z}_{\langle q^\beta \rangle \langle s \rangle} \langle \gamma_{\langle q^\beta \rangle \langle s \rangle} \rangle \neq \dot{z}_{\langle q^\beta \rangle \langle s-1 \rangle} \langle \gamma_{\langle q^\beta \rangle \langle s \rangle} \rangle$ and therefore either $\dot{z}_{G_{\beta,s}} = \dot{z}_{\langle q^\beta \rangle \langle s \rangle} \langle \gamma_{\langle q^\beta \rangle \langle s \rangle} \rangle$ or $\dot{z}_{G_{\beta,s}} = \dot{z}_{\langle q^\beta \rangle \langle s-1 \rangle} \langle \gamma_{\langle q^\beta \rangle \langle s \rangle} \rangle$, so either $s \leq 0$ or $s \leq 1$ is in $f$, but not both. Therefore $s$ is not a splitting node in $f$, and so there are no splitting nodes in $f$. We conclude that $f$ is in fact a branch in $q^\beta$.

Moreover, we claim that $f$ is equal to the generic branch $h_{\beta,\beta+1}$ derived from the first coordinate of the generic set $G_{\beta,\delta}$. Let $s \in h_{\beta,\beta+1}$. Then due to density there is an $r \in G_{\beta,\delta}$ such that $s$ is part of the stem of $r^\beta$, and some condition $t \geq q \cap r$ in $G_{\beta,\delta}$. As above, $t^\beta \triangleright q^\beta \upharpoonright s$, and so $\dot{z}_{\langle q^\beta \rangle \langle s \rangle} \subseteq \dot{z}_{\langle t \rangle} \subseteq \dot{z}_{G_{\beta,s}}$. Therefore $s \in f$.

Hence $f = h_{\beta,\beta+1}$ is a branch of $q^\beta$ that is definable in $M_\beta \left[ \dot{z}_{G_{\beta,s}} \right] = M_\beta \left[ \dot{g}_{G_{\beta,s}} \right]$, and therefore in $M_\beta \left[ \dot{g}_{G_{\beta,s}} \right]$. Meaning we managed to recover the generic branch $h_{\beta,\beta+1}$. But remember, the generic branch $h_{\beta,\beta+1}$ is in fact interdefinable with the generic set $G_{\beta,\beta+1}$, and so $G_{\beta,\beta+1} \in M_\beta \left[ \dot{g}_{G_{\beta,s}} \right]$. But by our inductive assumption $\langle G_{\alpha,\alpha+1} \mid \alpha < \beta \rangle \in M_0 \left[ \dot{g}_{G_{\delta}} \right]$. Therefore $G_{\beta,\beta+1} \in M_0 \left[ \dot{g}_{G_{\delta}} \right]$.

Completing the induction, $G_\delta \in M_0 \left[ \dot{g}_{G_{\delta}} \right]$, and so $M_\delta \subseteq M_0 \left[ \dot{g}_{G_{\delta,s}} \right]$.

But $M_0 \left[ \dot{g}_{G_{\delta}} \right] \subseteq N$, and so we conclude $M_\delta \subseteq N$ as required.

Looking at the proof of lemma 5.12 it becomes evident why we couldn’t have used bounded support even in regular limits: condition $p(\delta)$ that lies at the heart of the proof satisfies $\dot{p}(\delta)^\alpha \neq 0$ whenever $Q_\alpha$ is non-trivial. Moreover, had we used bounded support, we would have needed to construct a condition $p$ that is at once bounded, and so has at most than $\aleph_\beta < \aleph_\delta$ splitting nodes,
yet is still somehow able to distinguish the value of the generic branch in $Q_{β+1}$, even though in the standard forcing that task requires $R_{β+1}$ splitting nodes.

At last we arrive at the central theorem for the model tower construction:

**Theorem 5.13.** $K$ is an inner model of $M_ζ$ if and only if for some $α ≤ ζ$ $K = M_α$.

*Proof.* Suppose to the contrary that $K$ is an inner model of $M_ζ$ such that $K ≠ M_α$ for all $α ≤ ζ$. Then there is a minimal ordinal $β$ such that $K ⊈ M_β$.

By lemma 5.11 $β$ must be a limit ordinal. So $K ⊇ M_α$ for all $α < β$, but $K ⊈ M_β$. However, according to lemma 5.12 if $K ⊇ M_α$ for all $α < β$ then $K ⊇ M_β$. We arrived at a contradiction. Meaning that there is no such inner model $K$.

Thus $K$ is an inner model of $M_ζ$ if and only if $K = M_α$ for some $α < ζ$.

**Corollary 5.14.** There exists a well-ordered model universe of arbitrary height.

*Proof.* $M_ζ$ is a well-ordered model universe of height $ζ$.

### 6 Class forcing

In the previous section we defined the iterated forcing notion for sets, and we used it to construct a well-ordered model universe of arbitrary height. Because that iteration could successfully go through strongly inaccessible cardinals, we proved that the existence of well-ordered model universes with ordinal height is in fact consistent with $\text{ZFC}$. We could simply take $V \cup G_κ$, where $κ$ is a strongly inaccessible cardinal, and $G_κ$ is the generic set of $P_κ$ as defined in 5.1.

Now however we want to iterate our model tower 'all the way' by the use of class forcing. And to make formal use of class forcing, we return in this section to the axiomatic framework of $\text{BGC}$, as expounded upon in the introduction.

So for the rest of this section we shall assume to be working within $⟨V, V, ∈⟩$ a model of $\text{BGC}$, and we shall use forcing to extend a base model of $\text{BGC}$ to another model thereof.

A basic introduction of class forcing the reader may be found in Friedman [5]. For a more thorough presentation of class forcing within the context of $\text{BGC}$ the reader may refer to Reitz (appendix A of [15]).

Before going on, it is important to note the main difficulty with class forcing, which is that unlike set forcing, the generic extension of class forcing might actually fail to be a model of $\text{BGC}$ (and its sets a model of $\text{ZFC}$). Specifically, the Power Set Axiom and the Axiom of Replacement might fail (theorem 91 in [15]). For $\text{BGC}$ and $\text{ZFC}$ to be satisfied, we will need to prove that our forcing iteration is *progressively closed*, as will be defined later.

We fix the base of our forcing iteration to be $⟨L, L, ∈⟩$, where of course $L$ is the constructible universe, and $L$ is the collection of classes definable therein (remember fact 1.4).

**Definition 6.1.** Let $P ∈ L$ be a partially ordered class defined as follows:
1. Let \( \dot{Q}_\alpha \) be trivial if \( \alpha \) is a limit ordinal, and let it be the name of \( \aleph_{\alpha} \)-Sacks forcing in \( V^{P_{\alpha}} \) otherwise.

2. \( P_{\alpha+1} = P_\alpha \ast \dot{Q}_\alpha \).

3. At limit stages we use full support, i.e if \( \delta \) is a limit ordinal then \( p \in P_\delta \iff \forall \alpha < \delta \ (p \upharpoonright \alpha \in P_\alpha) \).

4. \( P = \bigcup_{\alpha \in \text{Ord}} P_\alpha \). That is, every condition in \( P \) is bounded in its coordinates.

It should be noted that unlike in the ordinal limit stages, where we use the indirect limit (i.e full support) all the way through, in the class limit we employ the direct limit instead.

As explained in the previous section, using indirect limits even for regular cardinals would have spoiled the construction of the condition used to simultaneously discover all the generic sets - which was necessary to prove that no inner model ‘squeezes in’ between the ascending chain of models and the limit model. But as will be shown later, unlike the ordinal limit stages, if we use a direct limit in the class stage the generic extension is simply the union of the ascending chain, and therefore automatically minimal over it. So the entire construction of theorem 5.13 is unnecessary for the class limit case.

**Definition 6.2.** Denote:

1. \( M_0 = L \).
2. \( G_\alpha \) as the generic set in partial order \( P_\alpha \) over \( M_0 \).
3. \( M_\alpha = M_0 [G_\alpha] \).
4. \( G \) as the generic class in partial order \( P \) over \( M_0 \).
5. \( M_0[G] \) the generic extension of \( \langle L, L, \in \rangle \) by \( G \).
6. \( M_\infty \) the restriction of \( M_0[G] \) to sets.

Note that we have yet to establish that \( M_0[G] \) is a model of BGC, or that \( M_\infty \) is a model of ZFC.

**Lemma 6.3.** The forcing \( P/G_\alpha \) is \( \aleph_\alpha \)-closed.

*Proof.* Each coordinate is \( \aleph_\alpha \)-closed, and the limit of a set of bounded conditions in \( P/G_\alpha \) is itself bounded. \(\square\)

**Lemma 6.4.** \( M_\infty \) has the same cardinals as \( M_0 \).

*Proof.* Let \( \aleph_\alpha \) be a cardinal in \( M_0 \). According to lemma 6.10 \( M_0 \) has the same cardinals as \( M_{\alpha+1} \).

But by lemma 6.3 the forcing \( P/G_{\alpha+1} \) is \( \aleph_{\alpha+1} \)-closed, and so adds no new subsets of \( \aleph_\alpha \). Therefore \( \aleph_\alpha^{M_0} = \aleph_\alpha^{M_{\alpha+1}} = \aleph_\alpha^{M_\infty} \), and so all cardinals are preserved. \(\square\)
Lemma 6.5. $M_\infty$ satisfies the Power Set Axiom.

Proof. It is enough to prove the Power Set Axiom for cardinals. Let $\aleph_\alpha$ be a cardinal in $M_\infty$. By lemma 6.3 it is also a cardinal in $M_{\alpha+1}$.

By lemma 6.3 the forcing $P/\mathcal{G}_{\alpha+1}$ is $\aleph_{\alpha+1}$-closed, and so adds no new subsets of $\aleph_\alpha$. Therefore $P(\mathcal{N}_\alpha)^{M_{\alpha+1}} = P(\mathcal{N}_\alpha)^{M_\infty}$.

Thus the power set of $\aleph_\alpha$ is also a set in $M_\infty$.

Definition 6.6. A partially ordered class $R$ is a chain of complete subposets if

$
R = \bigcup_{\alpha \in \text{Ord}} R_\alpha,
$

where each $R_\alpha$ is a partially ordered set, such that if $\alpha \leq \beta$ then $R_\alpha$ is a complete suborder of $R_\beta$.

Lemma 6.7. $P$ is a chain of complete subposets.

Proof. By definition $P = \bigcup_{\alpha \in \text{Ord}} P_\alpha$. This is an iterated forcing, and so the identity map $i_{\alpha,\beta} : P_\alpha \rightarrow P_\beta$ is a complete embedding (see ch. VIII lemma 5.11 in [11]). Therefore $P_\alpha$ is a complete suborder of $P_\beta$.

Definition 6.8. $P = \bigcup_{\alpha \in \text{Ord}} P_\alpha$ is a progressively closed iteration if $P$ is a chain of complete subposets, and for arbitrarily large regular cardinals $\delta$ there are arbitrarily large $\alpha$ such that there is a $P_\alpha$-name $\dot{P}_{[\alpha,\infty)}$ satisfying:

1. For every $\beta > \alpha$ the poset $P_\beta$ is isomorphic to the two-stage iteration $P_\beta \cong P_\alpha \star \dot{P}_{[\alpha,\beta]}$;

2. $P_\alpha \models \delta$ is a regular cardinal and $\dot{P}_{[\alpha,\beta]}$ is $< \delta$-closed;

3. For $\beta' > \beta > \alpha$ the isomorphisms at $\beta$ and $\beta'$ yield complete subposets $P_\alpha \star P_{[\alpha,\beta]} \subseteq P_\alpha \star P_{[\alpha,\beta']}$ such that the complete embeddings commute with the isomorphisms.

4. $P_\alpha \models \dot{P}_{[\alpha,\infty)}$ is a chain of complete subposets.

Lemma 6.9. $P$ is a progressively closed iteration.

Proof. By lemma 6.7 $P$ is a chain of complete subposets. Let $\aleph_\delta$ be a successor cardinal, and $\alpha = \delta + 1$.

1. By definition $P_\beta = P_\alpha \star \dot{P}_{[\alpha,\beta]}$.

2. By lemma 6.4 all the cardinals are preserved, and by lemma 6.3 $\dot{P}_{[\alpha,\beta]}$ is $\aleph_{\delta+1}$-closed.

3. Let $p \in P_\beta$. Then $p = \langle p^{<\alpha}, \dot{p}^\alpha, \dot{p}^{\alpha+1}, ..., \dot{p}^\gamma, ..., | \gamma < \beta \rangle$. Which embeds to $\langle p^{<\alpha}, \dot{p}^\alpha, \dot{p}^{\alpha+1}, ..., \dot{p}^\gamma, ..., 0, ... | \gamma < \beta \rangle \in P_\alpha \star \dot{P}_{[\alpha,\beta]}$. Similarly $p$ embeds to $\langle p^{<\beta}, 0, ... | \beta \rangle \in P_\beta$, which through the isomorphism is equal to $\langle p^{<\alpha}, \dot{p}^\alpha, \dot{p}^{\alpha+1}, ..., \dot{p}^\gamma, ..., 0, ... | \gamma < \beta \rangle \in P_\alpha \star \dot{P}_{[\alpha,\beta]}$. Hence the complete embeddings commute with the isomorphisms.
4. Lemma 6.7 applies to the tail of the forcing as well.

Lemma 6.10. \( M_0 [G] \models \text{BGC} \) and \( M_\infty \models \text{ZFC} \).

Proof. By theorem 98 of [15] a progressively closed iteration generates a generic extension that satisfies BGC.

So \( M_0 [G] \models \text{BGC} \) and by fact 15.3 \( M_\infty \models \text{ZFC} \).

Lemma 6.11. \( M_\infty = \bigcup_{\alpha \in \text{Ord}} M_\alpha \).

Proof. According to lemma 6.7 \( \mathbb{P} \) is a chain of complete subposets. So applying lemma 88 of [15] to the sets, we get \( M_\infty = \bigcup_{\alpha \in \text{Ord}} M_0 [G_\alpha] \).

Theorem 6.12. \( N \) is a proper inner model of \( M_\infty \) if and only if for some \( \alpha \in \text{Ord} \) \( N = M_\alpha \).

Proof. Working to the contrary, assume there exists \( N \) an inner model of \( M_\infty \) such that \( N \neq M_\alpha \) for all \( \alpha \in \text{Ord} \).

Suppose there exists a greatest ordinal \( \beta \) such that \( M_\beta \subseteq N. \) \( M_\beta \models \text{AC} \) so according to Vopěnka [19] there exists a set of ordinals \( A \in N \setminus M_\beta. \) By lemma 6.11 there exists an ordinal \( \alpha \) such that \( A \in M_\alpha. \) But that means \( M_\beta \subseteq M_\beta [A] \subseteq M_\alpha, \) so according to theorem 5.13 \( M_\beta [A] = M_\gamma \) for some \( \beta < \gamma \leq \alpha. \)

Therefore \( M_\beta [A] = M_\gamma \subseteq N, \) in contradiction to \( \beta \) being the greatest ordinal such that \( M_\beta \subseteq N. \) So there is no such greatest \( \beta. \)

Next, suppose that for a limit ordinal \( \delta, M_\alpha \subseteq N \) for all \( \alpha < \delta. \)

As shown in the proof of lemma 5.12 there is a set \( g = \langle g_\alpha \mid \alpha < \delta \rangle \in N \) such that if \( \alpha \) is not a limit ordinal \( g_\alpha \in M_{\alpha+1} \setminus M_\alpha \) and \( g_\alpha \subseteq \delta_\alpha, \) and if \( \alpha \) is a limit ordinal then \( g_\alpha = \emptyset. \) Define \( g'_\alpha = \{ \omega_\alpha + \beta \mid \beta \in g_\alpha \}. \) Obviously \( g'_\alpha \subseteq \omega_{\alpha+1} \setminus \omega_\alpha, \) Define \( g' = \bigcup_{\alpha < \delta} g'_\alpha. \)

We get \( g' \in N \) a set of ordinals. If \( g' \in M_\alpha \) for some \( \alpha < \delta \) then \( g'_\alpha = (g' \cap \omega_{\alpha+1}) \setminus \omega_\alpha \in M_\alpha \) and then \( g_\alpha \in M_\alpha \) in contradiction to its definition. Therefore \( g \not\in M_\alpha \) for all \( \alpha < \delta. \)

By lemma 6.11 there exists an ordinal \( \beta \) such that \( g' \in M_\beta, \) so \( M_0 [g'] \subseteq M_\beta. \)

According to theorem 5.13 this means \( M_0 [g'] = M_\delta \) for some \( \delta \leq \gamma \leq \beta, \) and so \( M_0 [g'] \supseteq M_\delta. \)

On the other hand, because \( g' \) is a set of ordinals and \( M_0 \models \text{AC}, \) \( M_0 [g'] \) is the smallest model of ZFC such that \( g' \in M_0 [g'], \) and so \( M_0 [g'] \subseteq N. \) Hence \( M_\delta \subseteq N. \)

As a result, by induction for all \( \alpha \in \text{Ord} \) \( M_\alpha \subseteq N, \) and so \( M_\infty = \bigcup_{\alpha \in \text{Ord}} M_\alpha \subseteq N, \) in contradiction of \( N \) being a proper inner model of \( M_\infty. \)

We conclude that \( N \) is a proper inner model of \( M_\infty \) if and only if \( N = M_\alpha \) for some \( \alpha \in \text{Ord}. \)
At last, we arrive at what we set out to prove:

**Corollary 6.13.** The existence of well-ordered model universes with the height of the ordinals is consistent with ZFC.

**Proof.** Theorem 6.12 shows that $M_\infty$ is a well-ordered model universe with $ht(M_\infty) = \infty$. \qed

We conclude this section with the observation that in $M_\infty$ the class of inner models $M(M_\infty)$ (as defined in 1.7) is in fact definable in $M_\infty$: $M_0 = L$, for all $\alpha M_{\alpha+1} = L \left( P(\aleph_\alpha)^{M_\infty} \right)$, and for all limit $\delta M_\delta = L \left( P(\aleph_\delta)^{M_\infty} \right)$. Thus $M_\infty$ in a sense ‘knows’ that it is a well-ordered model universe.

7 Open questions

In the previous section we constructed an example of a nice well-ordered model universe of height equal to $\text{Ord}$. We did this by an iteration of progressively increasing $\kappa$-Sacks forcing. In this section we discuss some remaining open questions regarding well-ordered model universes:

1. Can we construct a well-ordered model universe that isn’t nice?
2. What can we say about models when the inner models are just totally-ordered, not well-ordered by inclusion?
3. What if we consider all inner models of ZF, not just inner models of ZFC?

7.1 Non-nice well-ordered model universes

For the first question, recall definition 1.11. A well-ordered model universe is considered nice if its underlying order is equivalent to some ordinal or to $\text{Ord}$. This is essentially a limit on the length of the well-ordering. Any well-ordered set is order-isomorphic to some ordinal, so if a well-ordered model universe isn’t nice then the underlying order must be a proper class, but one which is not order-isomorphic to $\text{Ord}$.

Can we define such a well-ordering? Of course - just take $A = \{ \alpha \mid \alpha \in \text{Ord} \lor \alpha = \{1\} \}$, and extend the natural ordering by defining $\{1\} > \alpha$ for all $\alpha \in \text{Ord}$. It is easy to see that this is indeed a well-ordering: if $B \subseteq A$ is a non-empty class, then if it contains any ordinal, then the least ordinal it contains is its least element according to our extended ordering, and if not then $\{1\}$ is the least element. It is also obvious that our extended ordering is not order-isomorphic to $\text{Ord}$ - our ordering has a greatest element, whereas $\text{Ord}$ clearly does not.

So such a well-ordering is very much definable. Could we extend our construction further then we did in the previous section?

For the rest of the subsection, let $M_\infty$ be as defined in 6.2. In general, there is no obstacle to applying $\aleph_\beta$-Sacks forcing to $M_\infty$. The normal properties of Sacks-forcing would still hold, i.e there won’t be any intermediate model between
$M_{\infty}$ and $M_{\infty}[G]$. Moreover, there would be a chain of inner models of $M_{\infty}[G]$ that would be 'longer' than $\text{Ord}$. However, regardless of the $\aleph_{\beta}$-Sacks forcing we use, $M_{\infty}[G]$ would invariably contain some new inner model that is not on the chain.

**Lemma 7.1.** Let $\aleph_{\beta}$ be a regular cardinal, let $S$ be the $\aleph_{\beta}$-Sacks forcing notion over $M_{\infty}$, and let $H$ be a generic set in $S$. Then $M_{\infty}[H]$ is not a well-ordered model universe.

**Proof.** By lemma 6.11 $M_{\infty} = \bigcup_{\alpha \in \text{Ord}} M_{\alpha}$. Therefore $\mathcal{P}(S) \in M_{\alpha}$ for some $\alpha \in \text{Ord}$.

As all the dense sets of $S$ in $M_{\infty}$ are already in $M_{\alpha}$, $H$ is also a generic set of $S \in M_{\alpha}$, and so $M_{\alpha}[H]$ is a generic extension generated by $\aleph_{\beta}$-Sacks forcing over $M_{\alpha}$.

Obviously $M_{\alpha}[H] \subseteq M_{\infty}[H]$, but $M_{\alpha}[H] \not\subseteq M_{\infty}$. So for all $\gamma \in \text{Ord}$ $M_{\alpha}[H] \neq M_{\gamma}$ and $M_{\alpha}[H] \neq M_{\infty}$.

By theorem 4.10 there are no intermediate models between $M_{\infty}$ and $M_{\infty}[H]$. So either $M_{\alpha}[H] = M_{\infty}[H]$, or $M_{\alpha}[H]$ is 'off-chain'.

But $M_{\alpha}[H]$ is $\aleph_{\beta}$-Sacks forcing over $M_{\alpha}$, and so has no intermediate model between $M_{\alpha}$ and $M_{\alpha}[H]$, whereas $M_{\alpha} \subseteq M_{\infty} \subseteq M_{\infty}[H]$. Therefore $M_{\alpha}[H] \neq M_{\infty}[H]$, so $M_{\alpha}[H]$ is 'off-chain', and $M_{\infty}[H]$ is not a well-ordered model universe.

Okay, but lemma 7.1 only shows that we can’t use $\kappa$-Sacks forcing to produce the next step of the construction. Could some other set forcing notion do the trick for us?

Looking back at theorem 6.13 we proved that we could take any set created by the iteration and use it to completely recover all the preceding generic sets. So essentially, each generic set must code all the preceding generic sets. But because we used class-many generic sets to construct $M_{\infty}$, we need our new generic set to encode class-many previous generic sets, which is a tall order. In fact, it is impossible:

**Lemma 7.2.** Let $S$ be some minimal set forcing notion over $M_{\infty}$, and let $H$ be a generic set in $S$. Then $M_{\infty}[H]$ is not a well-ordered model universe.

**Proof.** Let $\aleph_{\beta} = |S|$. By lemma 6.11 there exists an ordinal $\alpha$ such that $\mathcal{P}(S) \in M_{\alpha}$. Take $\gamma = \max(\alpha, \beta + 1)$.

Obviously $S$ has the $\aleph_{\beta+1}$-c.c. property. By lemma 6.3, $\mathcal{P}/\mathcal{G}_{\gamma}$ is at least $\aleph_{\beta+1}$-closed, and by lemma 6.7 $\mathcal{P}/\mathcal{G}_{\gamma}$ is a chain of complete subposets.

Because $S \in M_{\gamma}$, $M_{\infty}[H]$ is actually the result of product forcing, where the first forcing is the tail of class forcing $\mathcal{P}$ and the second set forcing $S$, so

$M_{\infty}[H] = M_{\gamma}[\mathcal{P}/\mathcal{G}_{\gamma}][H].$

By lemma 121 in [13] we have that $\mathcal{P}/\mathcal{G}_{\gamma}$ is $\mathcal{P}/\mathcal{G}_{\gamma}$-generic over $M_{\gamma}[H]$. Therefore $M_{\gamma}[H] \neq M_{\infty}[H].$

Also, because $H \in M_{\gamma}[H]$ we have $M_{\gamma}[H] \not\subseteq M_{\infty}$. And because $S$ is minimal, so there is no intermediate model between $M_{\infty}$ and $M_{\infty}[H]$.

Therefore $M_{\gamma}[H]$ is a proper inner model of $M_{\infty}[H]$ that is off the chain, so $M_{\infty}[H]$ is not a well-ordered model universe.

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And what about class forcing? Could that be used to somehow lengthen our well-ordered model universe?

For class forcing of chain of complete subposets, this is again impossible.

**Lemma 7.3.** Let $\mathbb{S}$ be a chain of complete subposets over $M_\infty$, and let $\mathbb{H}$ be a generic class in $\mathbb{S}$ such that $M_\infty [\mathbb{H}] \models \text{ZFC}$. Then $M_\infty [\mathbb{H}]$ is not a well-ordered model universe.

**Proof.** According to lemma 88 of [15], $M_\infty [\mathbb{H}] = \bigcup_{\alpha \in \text{Ord}} M_\infty [H_\alpha]$, so $A \in M_\infty [H_\alpha]$ for some $\alpha \in \text{Ord}$.

But by lemma 7.2 $M_\infty [H_\alpha]$ is not a well-ordered model universe, so the inner models of $M_\infty [H_\alpha]$ are not well-ordered by inclusion.

As $M_\infty [H_\alpha]$ is definable with set parameters in $M_\infty [\mathbb{H}]$, so are the inner models of $M_\infty [H_\alpha]$ similarly definable, and so they are inner models of $M_\infty [\mathbb{H}]$.

Therefore the inner models of $M_\infty [\mathbb{H}]$ are not well-ordered by inclusion.

What about some more general form of class forcing?

At first thought this might also appear impossible, because even for class forcing to minimally extend $M_\infty$, we would still need every new set in $M_\infty [\mathbb{H}]$ to somehow encode the entire class of generic sets! However, the remarkable Jensen’s Coding Theorem [1] actually uses class forcing to achieve something similar: the existence of class forcing notion $\mathbb{P}$ such that if $G$ is $\mathbb{P}$-generic over $V$ then $V [G] \models \text{ZFC} + V = L[A] + A \subseteq \omega$. This set $A$ in effect ‘codes the universe’. Applying the theorem to $M_\infty$, all we really need is for every new set in $M_\infty [\mathbb{H}]$ to code this $A$, which sounds far more reasonable. So we are left with the following open question:

**Problem 7.4.** Is the existence of a well-ordered model universe with an underlying order longer than $\text{Ord}$ consistent with $\text{ZFC}$?

### 7.2 Totally-ordered model universes

So far in this article we focused exclusively on models where all the inner models are well-ordered by inclusion. However, a natural weakening of the definition is to demand the ordering to only be total, i.e for any two inner models of $V$, either $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$. We’ll call this a totally-ordered model universe.

Prima facie, this concept is far weaker than a well-ordered model universe. For one, our proof that $V$ has no measurable cardinals in theorem 2.2 immediately fails, because in theory there could be an infinite descending sequence of inner models. However, upon closer inspection we find that the proof of theorem 2.3 actually still holds, because it hinges on the fact if both $G_0$ is generic over $L[G_1]$ and vice-versa, then the inner models aren’t totally-ordered. Therefore:

**Theorem 7.5.** If $V$ is a totally-ordered model universe, $0^\sharp$ doesn’t exist.

**Proof.** Identical to theorem 2.3. □
Corollary 7.6. If \( V \) is a totally-ordered model universe, then \( V \) has no measurable cardinal.

Proof. By Gaifman [6], the existence of a measurable cardinal implies the existence of \( 0^\sharp \).

So some of the basic properties of well-ordered model universes extend to totally-ordered model universes, and the total-ordering property by itself is sufficient to prove that \( V \) is inherently small and quite 'close' to \( L \).

Therefore for each result proven about well-ordered model universes, we should ask ourselves whether it extends to totally-ordered model universes as well.

7.3 Inner models of \( ZF \)

Another natural extension of the definition of a well-ordered model universe is the consideration of general inner models of \( ZF \), not just inner models that satisfy Choice.

Returning to the framework we introduced at the beginning, we give the following expanded definition, which is almost verbatim definition [1.7]

Definition 7.7. Let \( \langle V, \mathcal{V}, \in \rangle \) be a model of \( BG \). We call a model \( N \subseteq V \) of \( ZF \) where all its inner \( ZF \) models are well-ordered with respect to inclusion a well-ordered \( ZF \) model universe. Formally, we postulate the existence of a class \( M \in \langle V, \mathcal{V}, \in \rangle \), which is the sequence of all proper inner \( ZF \) models of \( N \) ordered by inclusion. This means:

1. \( M \subseteq I \times N \);
2. \( M \) is a proper inner \( ZF \) model of \( N \) if and only if there exists a unique \( a \in I \) such that \( M = M_a = \{ x \mid (a, x) \in M \} \);
3. \( I \) is a well-ordered class;
4. If \( a <_I b \) then \( (a, x) \in M \rightarrow (b, x) \in M \).

In summary, applying the convention that lower-case letters indicate sets and upper-case letters indicates classes, we demand the following be true:

\[
\exists M \exists I \left( M \subseteq I \times N \wedge (M \subseteq N \text{ is an inner } ZF \text{ model } \leftrightarrow \exists a \in I \ (M = \{ x \mid (a, x) \in M \}) \wedge I \text{ is well-ordered } \wedge a <_I b \rightarrow ((a, x) \in M \rightarrow (b, x) \in M) \right)
\]

We define the height of \( V \) in exactly the same way we did for the original definition, using the order type of \( I \).

Obviously, if \( N \models ZFC \) is a well-ordered \( ZF \) model universe then it is also a well-ordered model universe. Given the sequence of proper \( ZF \) inner models we can directly define the sequence of proper \( ZFC \) inner models as defined in [1.7]. However, by our definition well-ordered \( ZF \) model universes are not required
themselves to satisfy $\text{AC}$, and therefore not every well-ordered $\text{ZF}$ model universe is necessarily a well-ordered model universe.

Next we outline a few of the basic properties of well-ordered $\text{ZF}$ model universes. For the rest of the subsection, assume $\langle V, \mathcal{V}, \in \rangle \models \text{BG}$, $V$ is a well-ordered $\text{ZF}$ model universe, and $\mathcal{M} \subseteq I \times V$ is its sequence of proper inner $\text{ZF}$ models ordered by inclusion.

**Lemma 7.8.** $M_0 = L$

*Proof.* Identical to lemma 2.1 \hfill \Box

**Theorem 7.9.** $V \models \text{There is no measurable cardinal.}$

*Proof.* Identical to theorem 2.2 \hfill \Box

The proof of theorem 2.4 doesn’t work for well-ordered $\text{ZF}$ model universes, as it involves heavy use of the Axiom of Choice. In general, it is very much possible to have an infinite chain of inner models that satisfy $\text{AC}$, but that the least inner model to include them all does not. So even though we can carry out the successor stage of the proof, we cannot prove $M_\omega \models \text{AC}$, meaning it is very possible $M_\omega \neq L[A]$ for all $A \in V$.

Note that because we can still carry out the successor stages of the induction, we have the following corollary:

**Corollary 7.10.** If $V \models \neg \text{AC}$ then $ht(V) \geq \omega$.

*Proof.* Applying the successor steps in the proof of theorem 2.4, we get that for all $n < \omega$ $M_n = L[A]$ for some $A \in V$. \hfill \Box

This result can actually be strengthened, even without the well-ordering property:

**Theorem 7.11.** If $V$ is not of the form $L[A]$ for some $A \in V$, then $V$ has an infinite number of inner models.

*Proof.* By induction we prove every model with a finite number of inner models is of the form $L[A]$. The case for 0 proper inner models is trivially true because $L = L[\emptyset]$.

Now assume that we’ve proven the induction for models with $n$ proper inner models. Assume $K$ is a model that has $n + 1$ proper inner models. Every inner model of an inner model of $K$ is an inner model of $K$ itself, so all inner models of $K$ have at most $n$ proper inner models, and so they are all of the form $L[A_m]$ for some $m \leq n$.

Let’s consider two possibilities: either $K$ has a greatest proper inner model $R \subsetneq K$, or it doesn’t. If $R$ exists, then by Vopěnka [19] there is a set of ordinals $A \in K \setminus R$, and so $K = L[A]$ as required.

Otherwise, just as we did in the proof of theorem 2.4, we can arrange a family of mutually disjoint sets of ordinals $B_m$ such that for all $m$ $L[A_m] = L[B_m]$. \hfill 38
Take $L \left[ \bigcup_{m \leq n} B_m \right]$. For all $m$, $L \left[ \bigcup_{m \leq n} B_m \right] \supseteq L[B_m]$. But the only model that includes all proper inner models of $K$ is $K$ itself. Therefore $L \left[ \bigcup_{\alpha} B_\alpha \right] = K$.

Therefore if $V$ is not a model of the form $L[A]$, $V$ must have an infinite number of inner models. 

**Corollary 7.12.** If $V$ is a well-ordered ZF model universe and $ht(V) < \omega$ then $V$ is a well-ordered model universe.

**Proof.** According to theorem 7.11 $V$ and all of its inner models must satisfy the Axiom of Choice. Therefore by definition 7.7 $V$ is a well-ordered model universe. 

Despite the failure of theorem 2.4 for well-ordered ZF model universes, we have a small consolation prize:

**Lemma 7.13.** Let $\alpha + 1 \in I$ denote the successor of $\alpha$ in the well-ordering of $I$. Then $M_{\alpha+1} = L(A)$ for some $A \in V$.

**Proof.** Take $A \in M_{\alpha+1} \setminus M_\alpha$. $L(A)$ is the smallest inner ZF model containing $A$, so obviously $L(A) \subseteq M_{\alpha+1}$. But because $V$ is a well-ordered universe $M_\alpha \subseteq L(A)$, and there are no intermediate inner models between $M_\alpha$ and $M_{\alpha+1}$. Therefore $L(A) = M_{\alpha+1}$.

**Corollary 7.14.** If $V$ has a greatest proper inner ZF model $K$, there exists $A \in V$ such that $V = L(A)$.

**Proof.** Take $A \in V \setminus K$. We get $K \subseteq L(A) \subseteq V$, and so for the same reasons as lemma 7.13 $L(A) = V$.

In conclusion, there isn’t much we know about well-ordered ZF model universes.

As for actually constructing a well-ordered ZF model universe, corollary 7.12 shows that using the iteration defined in 5.1 up to finite height, would generate a well-ordered ZF model universe.

A well-ordered ZF model universe of height $\omega$ is achievable by iterating the forcing up to $M_\omega$, and then taking $N = \text{HOD}(\langle G_n \mid n < \omega \rangle) \subseteq M_\omega$. This $N$ will be the minimal inner model of ZF that includes $M_n$ for all $n < \omega$.

However, we can’t use the same construction to build well-ordered ZF model universes of arbitrary height, because we can’t tell what’s going on between $N$ and $M_\omega$. The intermediate inner models there might not even be totally-ordered.

So we are left with one glaring open question:

**Problem 7.15.** Is the existence of a well-ordered ZF model universe the height of the ordinals consistent with ZF?
References

[1] Aharon Beller, Ronald Jensen and Philip Welch, *Coding the universe*, Cambridge University Press, Cambridge, Cambridge, 1982. MR0645538.

[2] Paul Bernays, *A System of Axiomatic Set Theory--Part II*, J. Symb. Log. 6 (1941), No. 1, 1-17. MR0003382.

[3] Paul J. Cohen, *Set Theory and the Continuum Hypothesis*, W.A Benjamin, New York, NY, 1966. MR232676.

[4] Abraham Fraenkel, Yehoshua Bar-Hillel and Azriel Levy, *Foundations of set theory*, Studies in Logic and the Foundations of Mathematics, Vol. 67, North-Holland Publishing Co., Amsterdam, 1973. MR0345816.

[5] Sy D. Friedman, *Fine Structure and Class Forcing*, Logic and Its Applications, 3, de Gruyter, Berlin, 2000. MR1780138.

[6] Haim Gaifman, *Elementary embeddings of models of set-theory and certain subtheories*, Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967), 33-101, Amer. Math. Soc., Providence, RI, 1974. MR0376347.

[7] Stefan Geschke and Sandra Quickert, *On Sacks Forcing and the Sacks Property*, Classical and new paradigms of computation and their complexity hierarchies, 95-139, Trends Log. Stud. Log. Libr., 23, Kluwer Acad. Publ., Dordrecht, 2004. MR2155534.

[8] Kurt Gödel, *The Consistency of the Continuum Hypothesis*, Ann. of Math. Studies, No. 3, Princeton University Press, Princeton, N.J., 1940. MR 2,66c.

[9] Thomas Jech, *Set Theory*, The third millennium edition, revised and expanded, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. MR1940513.

[10] Akihiro Kanamori, *Perfect-set forcing for uncountable cardinals*, Ann. Math. Logic 19 (1980), no. 1-2, 97-114. MR0593029.

[11] Kenneth Kunen, *Set theory, An introduction to independence proofs*, North-Holland Publishing Co., Amsterdam, 1980. MR82f:03001.

[12] Andrzej Mostowski, *Some Impredicative Definitions in the Axiomatic Set-Theory*, Fund. Math. 37 (1950), 111-124. MR0041083.

[13] John von Neumann, *Eine Axiomatisierung der Mengenlehre*, J. reine angew. Math. 154 (1925), 219-240. MR1581062.

[14] Helena Rasiowa and Roman Sikorski, *Mathematics of Metamathematics*, Monografie Matematyczne, Vol. 41, P.W.N. Polish Scientific Publishers, Warsaw, 1970. MR0344067.
[15] Jonas Reitz, *The Ground Axiom*, Ph.D. dissertation, City Univ. of New York, NY, 2008. MR2709224.

[16] Gerald E. Sacks, *Forcing with perfect closed sets*, in *Axiomatic Set Theory* (D. S. Scott, ed.), Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, CA, 1967. Amer. Math. Soc., Providence, RI, 1971, 331-355, MR0276079.

[17] Dana Scott, *Measurable cardinals and constructible sets*, Bull. Acad. Polon. Sci. Série. Sci. Math. Astronom. Phys. 9 (1961), 521-524. MR0143710.

[18] Saharon Shelah, *Proper and improper forcing*, Second edition, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1998. MR1623206.

[19] Petr Vopěnka and Bohuslav Balcar, *On complete models of the set theory*, Bull. Acad. Polon. Sci. Série. Sci. Math. Astronom. Phys. 15 (1967), 839-841. MR0242659.