A note on wavemap-tensor cosmologies*

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Abstract

We examine theories of gravity which include finitely many coupled scalar fields with arbitrary couplings to the curvature (wavemaps). We show that the most general scalar-tensor $\sigma$-model action is conformally equivalent to general relativity with a minimally coupled wavemap with a particular target metric. Inflation on the source manifold is then shown to occur in a novel way due to the combined effect of arbitrary curvature couplings and wavemap self-interactions. A new interpretation of the conformal equivalence theorem proved for such ‘wavemap-tensor’ theories through brane-bulk dynamics is also discussed.

1 Introduction

Scalar fields currently play a prominent role in the construction of cosmological scenarios aiming at describing the structure and evolution of the early universe. The standard inflationary idea requires that there is a period of slow-roll evolution of a scalar field (the inflaton) during which its potential energy dominates the kinetic energy and drives the universe in a quasi-exponential, accelerated expansion [1, 2]. In string cosmology, there is another rolling scalar field (the dilaton) whose kinetic energy drives a stage of accelerated contraction [3]. The main dynamical role of the existence of scalar fields in such contexts lies in driving periods of accelerated evolution of the scale factor. During such phases of evolution the curvature

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grows (as in a string or a Kaluza-Klein general relativistic context), or is constant or slightly decreasing (as in standard inflationary dynamics) leading through the behaviour of the Hubble radius $|H|^{-1}$ to shrinking or growing horizons. The close interplay of the time behaviours of scale factors and horizon scales in such models leads to fundamental properties characterizing such cosmologies and to important current ideas for the behaviour of spacetime near the Planck scale $\mathbb{P}$.

Scalar fields also arise naturally in alternative theories of gravity which aim at extending general relativity, e.g., higher order gravity theories, scalar-tensor and string theories. In higher derivative gravity, due to its conformal relation with general relativity $\mathbb{S}$, scalar fields appear in the Einstein frame with a self-interaction, nonlinear potential term which mimics the higher order curvature properties of the original (Jordan) frame. In a scalar-tensor theory, there are scalar fields which are typically coupled nonminimally to the curvature leading to interesting nonsingular cosmologies even in the isotropic category $\mathbb{I}$. (The simplest scalar-tensor theory developed by Brans and Dicke $\mathbb{B}$ involves a massless scalar field with constant coupling to matter. Generalizations of the Brans-Dicke theory lead to scalar-tensor theories with scalar field self-interactions and dynamical couplings to matter. Further generalizations can be achieved by considering multiple scalar fields (see e.g., $\mathbb{8}$)). Similar results can be proved for simple bosonic string theories $\mathbb{S}$.

All problems above may be considered in the Einstein frame for an Einstein-scalarfield system defined on spacetime for we know that due to their conformal properties all couplings of a scalar field to the curvature are equivalent $\mathbb{E}$. In this paper we prove, among other things, a scalar-tensor generalization of this result for any self-interacting system of a finite number of scalar fields. Such objects are known as wavemaps to mathematicians $\mathbb{1}$, $\mathbb{2}$ and as nonlinear-$\sigma$ models to physicists. We shall also see that brane cosmologies or, in general, the classical dynamics of ‘brane objects’ (see, for example, $\mathbb{13}$) can be obviously described through (in fact they are completely equivalent to) wavemaps.

The plan of this paper is as follows. In the next Section, we present the basic equations of our theory. Section 3 proves the conformal equivalence theorem for our wavemap-tensor theory to an Einstein-wavemap system thus showing the generalization that all couplings of a wavemap to the curvature are equivalent. In Section 4, we discuss a new way to implement an inflationary phase in this theory through a mechanism completely distinct from others that have appeared in the literature and show that the theory allows in a natural way to built a cosmological constant as remnant of the early state of the universe. We conclude in Section 5 giving some connections of the present set of problems with those of classical brane dynamics.
2 Wavemaps and multiscalar-tensor gravity theories

A wavemap is a map from a spacetime manifold to any (semi-) Riemannian manifold. More precisely, consider a spacetime \((M^m, g_{\mu\nu})\), a Riemannian manifold \((N^n, h_{ab})\) and a \(C^\infty\) map \(\phi : M \rightarrow N\). We call \(M\) the source manifold and \(N\) the target manifold

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For instance, if \(N = \mathbb{R}\) then \(\phi\) is simply a real scalar field on the source \(M\). If, on the other hand, we choose \(N = \mathbb{R}^n\) then the wavemap \(\phi = (\phi^1, \ldots, \phi^n)\) may be thought of as \(n\) uncoupled scalar fields on \(M\). We may think of the scalar fields \(\phi^a\), \(a = 1, \ldots, n\), as coordinates parametrizing the Riemannian target. The metric \(h_{ab}\) (which in general is not flat) expresses the possible couplings of the scalar fields.

A \(C^\infty\) map \(\phi : M \rightarrow N\) as above is called a wavemap if it is a critical point of the action,

\[
S = \int_M L \, dv_g, \quad dv_g = \sqrt{-g} dx,
\]

with,

\[
L = -g^{\mu\nu} h_{ab} \partial_{\mu} \phi^a \partial_{\nu} \phi^b.
\]

The Euler-Lagrange equations for this action are,

\[
\Box_g \phi^a + \Gamma^a_{bc}(h) g^{\mu\nu} \partial_{\mu} \phi^b \partial_{\nu} \phi^c = 0,
\]

and constitute a quasi-linear system of hyperbolic PDEs for \(\phi^a\).

We can consider a wavemap coupled to the curvature that is, regard wavemaps as ‘sources’ of the gravitational field and start with the Hilbert action,

\[
S = \int_M \left( R - g^{\mu\nu} h_{ab} \partial_{\mu} \phi^a \partial_{\nu} \phi^b \right) \, dv_g,
\]

where \(R\) is the scalar curvature of the source \(M\). Varying this action with respect to the metric \(g\) and the fields \(\phi^a\) we arrive at the so-called Einstein-wavemap system, namely,

\[
\Box_g \phi^a + \Gamma^a_{bc}(h) g^{\mu\nu} \partial_{\mu} \phi^b \partial_{\nu} \phi^c = 0.
\]

The stress-energy tensor of the wavemap is defined in the standard way through the basic wavemap lagrangian \(\Box^2\) and is given by,

\[
T_{\mu\nu} = h_{ab} \left( \partial_{\mu} \phi^a \partial_{\nu} \phi^b - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_{\rho} \phi^a \partial_{\sigma} \phi^b \right).
\]

1 We use greek indices for tensorfields defined on the source manifold \(M\) and latin indices for those on the target manifold \(N\). \(\Gamma^\rho_{\mu\nu}\) denotes the metric connection of the spacetime \(M\) while \(\Gamma^a_{bc}\) is the metric connection of the target manifold \(N\).
This has the nice properties of being a symmetric, divergence-free tensorfield and satisfies $T_{\mu\nu}u^\mu u^\nu \geq 0$, for all future-directed timelike vector fields $u^\mu$ on the source spacetime. In the simple case where $\mathcal{N} = \mathbb{R}$, we see that $T_{\mu\nu}$ is reduced to the stress-energy tensor of a massless scalar field.

Our starting point is the general scalar-tensor action functional,

$$S = \int_{\mathcal{M}} L_g dv_g, \quad dv_g = \sqrt{-g} dx,$$

with,

$$L_g = A(\phi)R - B(\phi)g^{\alpha\beta}h_{ab} \partial_\alpha \phi^a \partial_\beta \phi^b,$$

where $A, B$ are arbitrary $C^\infty$ functions of $\phi$. This class of gravity theories which we call wavemap-tensor theories, includes as special cases many of the scalar field models considered in the literature e.g., [8, 14, 15]. Choosing $A(\phi) = \phi$ and $B(\phi) = \omega/\phi$, with $\omega = \text{const.}$, we recognize the standard Brans-Dicke theory. Secondly, taking $\mathcal{N} = \mathbb{R}$, with $A(\phi) = B(\phi) = 1$, we obtain General Relativity with a massless scalar field as the matter source. In the case of an arbitrary Riemannian manifold $\mathcal{N}$, setting $A(\phi) = B(\phi) = 1$, the field equations derived upon variation of the corresponding action (8) reduce to the Einstein-wavemap system (5, 6).

In the general case, $S$ as given in (8)-(9) has arbitrary couplings to the curvature and kinetic terms. Varying it with respect to the metric $g$ and the scalar fields $\phi$ we obtain the system,

$$G_{\mu\nu} = \frac{B}{A} h_{ab} \left( \partial_\mu \phi^a \partial_\nu \phi^b - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi^a \partial_\beta \phi^b \right) + \frac{1}{A} \left( \nabla_\mu \nabla_\nu A - g_{\mu\nu} \Box_g A \right),$$

$$\Box_g \phi^a + \Gamma^a_{bc}g^{\mu\nu} \partial_\mu \phi^b \partial_\nu \phi^c + \frac{1}{2} RA^a = 0, \quad \Gamma^a_{bc} = \Gamma^a_{bc}(h) + C^a_{bc},$$

where we have set $A_a = \partial A/\partial \phi^a$, $C^a_{bc} = (1/2) \left( \delta^a_b B_c + \delta^a_c B_b - h_{bc} B^a \right)$ and $B_a = \partial \ln B/\partial \phi^a$.

We see that $\phi$ satisfies a wavemap-type equation with the connection coefficients $\Gamma^a_{bc}$ defining a Weyl geometry in $\mathcal{N}$. \footnote{However, the Weyl vector $B_a$ is a gradient and can be gauged away by the conformal transformation of the target metric, $\hat{h}_{ab} = B(\phi) h_{ab}$ (see, for example, Schouten [16]). We find $\hat{\Gamma}^a_{bc} = \Gamma^a_{bc}(\hat{h})$ that is, the connection is the Levi-Civita connection of the metric $\hat{h}$. This result is already clear from the form of the general wavemap-tensor action and we could have absorbed from the beginning the function $B(\phi)$ into the metric of the target manifold.} In the following, without loss of generality, we set $B = 1$ in Eq. (11) and drop the tilde on $h$.  

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3 Conformal structure of wavemap-tensor theories

The right-hand side of the field equation (10), defines an ‘energy-momentum tensor’ and splits into two parts. The first term, apart from a proportionality function, is exactly the energy-momentum tensor of the wavemap, namely,

\[ h_{ab} \left( \partial_\mu \phi^a \partial_\nu \phi^b - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \phi^a \partial_\sigma \phi^b \right). \tag{12} \]

The second part is the well-known combination,

\[ \nabla_\mu \nabla_\nu A - g_{\mu\nu} \Box A, \tag{13} \]

which contains second order covariant derivatives of the fields. Therefore the total energy-momentum tensor does not necessarily satisfy the strong energy condition, i.e. the energy density of the fields cannot always be made nonnegative. Furthermore, it contains terms proportional to the connection \( \Gamma^c_{\mu\nu} \) responsible for the dynamical evolution of the gravitational field and therefore we could ascribe to these terms a physical meaning as properly ‘belonging’ to the LHS of the field equations (10) and not be part of the ‘material content’ of the theory.

These difficulties as well as many others may be overcome by performing a suitable conformal transformation on the source manifold and redefining the fields in suitable ways. Defining a new metric by,

\[ \tilde{g}_{\mu\nu} = A(\phi) g_{\mu\nu}, \tag{14} \]

the scalar curvature transforms as,

\[ R = A \left( \tilde{R} + 3 \Box g \ln A - \frac{3}{2} \tilde{g}_{\mu\nu} \partial_\mu A \partial_\nu A \frac{A}{A^2} \right). \tag{15} \]

Dropping a total divergence and noting that, \( \partial_\mu A = (\partial A/\partial \phi^a) \partial_\mu \phi^a \), the action transforms into,

\[ \tilde{S} = \int_M d\nu_{\tilde{g}} \left( \tilde{R} - \tilde{g}^{\mu\nu} \left( \frac{3}{2A^2} A_a A_b + \frac{1}{A} h_{ab} \right) \partial_\mu \phi^a \partial_\nu \phi^b \right). \]

In general, the quadratic form

\[ \pi_{ab} := \frac{3}{2A^2} A_a A_b + \frac{1}{A} h_{ab} \tag{16} \]

is not positive definite (unless one imposes further conditions on \( A[\phi]\)). Assuming that rank \( \pi_{ab} = \dim N \), we may define the reciprocal tensor \( \pi^{ab} \), i.e., \( \pi^{ab} \pi_{bc} = \delta^a_c \) and endow the target manifold with the new metric \( \pi_{ab} \) and use this metric to raise and lower indices in \( \mathcal{N} \).

\[^3\text{For example, in simple scalar-tensor theories with } A(\phi) = \phi \text{ and } B(\phi) = \omega(\phi)/\phi \text{ one has to impose the condition that } \omega(\phi) \geq -3/2 \text{ in order that the energy density of the scalar field be nonnegative.} \]
Using this metric defined by Eq. (16), the original wavemap-tensor theory (1)-(2) is conformally equivalent to a wavemap minimally coupled to general relativity,

\[ \tilde{S} = \int_{\mathcal{M}} \tilde{L}_\sigma d\tilde{v}_3, \quad \tilde{L}_\sigma = \sqrt{\tilde{g}} \left( \tilde{R} - \tilde{g}^{\mu\nu} \pi_{ab} \partial_\mu \phi^a \partial_\nu \phi^b \right). \] (17)

This result shows that all couplings of the wavemap to the curvature are equivalent. Varying this conformally related action, \[ \dot{\tilde{S}} = 0, \] we find the Einstein-wavemap system field equations for the \[ \tilde{g} \] metric and involving the \( \pi_{ab} \) metric, namely,

\[ \tilde{G}_{\mu\nu} = \pi_{ab} \left( \partial_\mu \phi^a \partial_\nu \phi^b - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}^{\rho\sigma} \partial_\rho \phi^a \partial_\sigma \phi^b \right), \] (18)

\[ \square \tilde{g}^a + \tilde{D}_a^{\mu\nu} \tilde{g}^{b\nu} \partial_\mu \phi^b \partial_\nu \phi^a = 0, \quad \tilde{D}_a^{\mu\nu} = \Gamma_{bc}^a (\tilde{h}) + T_{abc}; \] (19)

where \( T_{abc} := \partial_c Q_{ab} + \partial_b Q_{ac} - \partial_a Q_{bc} \) and \( Q_{ab} := Q_a Q_b \) with \( Q_a = \sqrt{3/2} A^{-1} A_a \).

We shall return to these equations in the last Section where we shall give a new physical interpretation of their conformal properties.

### 4 Inflation in wavemap-tensor theories

We now present a way of generating a cosmological constant term without using a potential. Let us assume that the source manifold \((\mathcal{M}^m, g_{\mu\nu})\) is a 4-dimensional flat FRW model in the original wavemap-tensor theory. The Friedman equation in the Einstein frame reads,

\[ H^2 = T_{WM}^{00}, \] (20)

where the 00-component of the wavemap energy-momentum tensor is given by,

\[ T_{WM}^{00} = \pi_{ab} \dot{\phi}^a \dot{\phi}^b. \] (21)

In this frame, scalar fields evolve according to (19). It then follows that at critical points of \( T_{WM}^{00} \), the universe inflates exponentially. This is the simplest example of a general procedure, which we call \( \sigma \)-inflation, in which inflation is driven both by the coupling \( A(\phi) \) and the self-interacting (target manifold is curved!) scalar fields \( (\phi^a) \) which, however, have no potentials.

This mechanism reduces to the so-called hyperextended inflation mechanism \[ [13] \] when the target space is the real line. On the other hand, when the curvature coupling \( A(\phi) \) is equal to one, \( T_{WM}^{00} \) can have no critical points and so we have no inflationary solutions. In this case we obtain the so-called tensor-multiscalar models \[ [17] \]. Inflationary solutions become possible in this case by adding ‘by hand’ extra potential terms and models of this sort abound.
From (21) we see that specific forms of the function $A(\phi)$ lead to a series of critical points of $T^{00}_{WM}$ corresponding to different values of the cosmological constant. Further investigation of the stability of these points is expected to single out a preferred family of functions $A(\phi)$ and target manifolds $(N^n, h_{ab})$ allowing a slow evolution of $T^{00}_{WM}$ (necessary for an adequate amount of inflation). Hence, large values of the energy density of the wavemap drive the early universe to an inflationary stage that lasts until $T^{00}_{WM}$ reaches a critical point leaving us with a cosmological constant which must be compatible with observational parameters and other constraints (see [18] for a recent review of the problems associated to the cosmological constant). This problem is currently under study.

5 Wavemap-tensor theories as brane worlds

We showed in Section 3 the (con)formal equivalence between the original wavemap-tensor theory (8), (9) and the Einstein-wavemap system (17). Suppose now, for the sake of illustration, that $\dim N > \dim M$ so that $\phi(M) \subset N$ can be considered as an $(1 + (m - 1))$-dimensional subset in the $(1 + (m - 1) + (n - m))$-dimensional target manifold.

A cursory look at the conformal system (19) reveals that the 'fields' are constrained to propagate only on the 'brane' manifold $(M, \tilde{g})$ since their derivatives are taken only with respect to the metric $\tilde{g}$. On the other hand, the gravitational field defined from Eq. (18) propagates freely on the 'bulk' manifold $(N, \pi_{ab})$ where the bulk metric is now the $\pi_{ab}$ metric defined by Eq. (16). As an example, we can consider a Randall-Sundrum type model [19] where the brane $(M, \tilde{g})$ has minimal codimension $(n - m = 1)$.

We believe that this is an interesting point which requires further careful analysis for it provides a new interpretation of the conformal equivalence result for wavemap-tensor theories. Hence, a wavemap in the original frame could perhaps be considered as a matter field there since when conformally transformed becomes a set of scalar fields which live on the brane $(M, \tilde{g})$ while gravity propagates off this subset and into the bulk space $(N, \pi_{ab})$.

A more detailed analysis of the results presented here will be given elsewhere.

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4 Inclusion of matterfields in the original wavemap-tensor theory (for instance a perfect fluid) does not seem to alter this result. In the field equations for the conformally transformed matter, derivatives will be again taken only with respect to the conformal 'brane' metric $\tilde{g}$.
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