COMPACTIFICATION OF THE MODULI OF POLARIZED ABELIAN VARIETIES AND MIRROR SYMMETRY

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Abstract. We construct a toroidal compactification $\overline{\mathcal{A}}_{g,δ}$ of the moduli space of polarized abelian varieties $\mathcal{A}_{g,δ}$ for an arbitrary polarization $δ$. The data needed for a toroidal compactification is a collection of fans. We construct the mirror family for each 0-cusp of $\mathcal{A}_{g,δ}$ and obtain a collection of fans from the Mori fans of the minimal models of the mirror families. Furthermore, we extend the universal family over the boundary. There is a local moduli meaning for our extended family. Associate a canonical set of divisors $S(K_2)$ to each cusp. Near the cusp, a polarized semiabelic scheme $(X,G,L)$ is from our family, if and only if $(X,G,Θ)$ is an object in $\mathcal{A}P_{g,d}$ for any $Θ ∈ S(K_2)$.

Contents

1. Introduction 2
   1.1. Motivation and Results 2
   1.2. Outline of the Paper 5
   1.3. Acknowledgement 5
   1.4. Conventions and Notations 5
2. The Toroidal Compactification and Mirror Symmetry 7
   2.1. The Toroidal Compactification over Complex Numbers 7
   2.2. A Tropical Interpretation of the Cones 10
   2.3. Mirror Symmetry for Abelian Varieties 17
3. The Compactification of the Moduli of Polarized Abelian Varieties 25
   3.1. Toroidal Compactifications 25
   3.2. AN Families 26
   3.3. Standard Data 35
   3.4. Construction of the Stack 40
4. Representations of the Theta Group and the Stable Pairs 50
   4.1. The Heisenberg Relation and the Fourier Decomposition 50
   4.2. The Stable Pairs 52
   Appendix A. the quasiperiodic functions 58
   Appendix B. Degenerations over one-parameter family 59
   B.1. Maximal Degeneration 59
   B.2. General Degeneration 63
References 67
1. Introduction

1.1. Motivation and Results. The main problem we concern in this paper is the compactification of the moduli spaces of polarized abelian varieties $\mathcal{A}_{g,\delta}$. Given a non-compact moduli space $\mathcal{M}$ of higher dimensions, the compactification, if exists, is not unique, as we can always do some birational transforms to the boundary. Therefore, we are looking for a good compactification $\overline{\mathcal{M}}$. Following Deligne–Mumford’s compactification of the moduli of curves, by a good compactification $\overline{\mathcal{M}}$, we mean the following:

1. The space $\overline{\mathcal{M}}$ itself is not too singular, and the boundary $\overline{\mathcal{M}}\setminus \mathcal{M}$ is a divisor of $\overline{\mathcal{M}}$.
2. The universal family over $\mathcal{M}$ is extended over $\overline{\mathcal{M}}$. Furthermore, the geometric objects added to the boundary should be reasonable, e.g. the fibers are reduced, with mild singularities, and the families are flat over the base.
3. The family over $\overline{\mathcal{M}}$ can be defined by a reasonable moduli functor extending the moduli functor for $\mathcal{M}$. Ideally, the extended functor is characterized by geometric properties.

We construct a compactification $\overline{\mathcal{A}}_{g,\delta}^m$ over $k = \mathbb{Z}[1/d, \zeta_M]$ of the moduli stack $\mathcal{A}_{g,\delta}$. For the notations, see Definition 1.4. We have achieved (1) and (2) on the above list, but not quite (3). More precisely, the stack $\overline{\mathcal{A}}_{g,\delta}^m$ is a log smooth, proper Deligne-Mumford stack with the coarse moduli space $\mathcal{A}_X$. The algebraic space $\mathcal{A}_X$ is normal, and over $\mathbb{C}$, $\mathcal{A}_X \otimes \mathbb{C}$ is a projective toroidal compactification. See Theorem 3.48, Proposition 3.49, Lemma 3.50. There is a family over $\overline{\mathcal{A}}_{g,\delta}^m$ extending the universal family over $\mathcal{A}_{g,\delta}$. The geometric fibers $(X,G,L)$ are polarized stable semiabelic varieties ([Ale02] Definition 1.1.15). In particular, the geometric objects are reduced and semi-normal (Lemma 3.19).

Although we can’t achieve (3), we have an intrinsic local description of the families near each cusp. For each cusp, we define a canonical set of divisors $S(K_2)$ (Definition 4.8) that only depends on an isotropic subgroup $K_2 \subset K(L)$ associated to the cusp. Assume $(X,G,L)/S$ is a family of polarized stable semiabelic scheme over an integral normal complete local base $S$ near the cusp, then it is the pull-back family along a morphism $S \to \overline{\mathcal{A}}_{g,\delta}^m$, if and only if $(X,G,\Theta)$ is a stable pair in $\mathcal{A}_{g,d}$ for one (equivalently any) divisor $\Theta$ from $S(K_2)$. For a precise statement, see Theorem 4.11. Notice that this description is very much like by a functor except that we can’t extend it to a family over a general base. Moreover, this description is geometric, because $\overline{\mathcal{A}}_{g,d}$ is a special version of the moduli of stable pairs in the sense of KSBA$^1$ and is characterized in terms of geometry. Therefore this is a geometric description of our compactification $\overline{\mathcal{A}}_{g,\delta}^m$.

The balanced set $S(K_2)$ is defined as follows: Let $K(L)$ the kernel of the polarization $\chi \to \chi^d$. We define a group scheme $G(M)$ which is a finite version of the theta group $G(L)$, and is a central extension of $K(L)$ by the group of roots of unity $\mu_M$. We have a representation $S_g^*: g \in G(M)$ of $G(M)$ on the locally free sheaf $\pi_*L$. If $K_2$ is a well-defined isotropic subgroup of $K(L)$ over $S$, the restriction of the Weil pairing defines a

\footnote{It stands for Kollár–Shepherd–Barron–Alexeev. See [Ale96].}
homomorphism $K(L) \to \hat{K}_2$. Let the composition $G(M) \to \hat{K}_2$ be denoted by $w$ and the kernel of $w$ be denoted by $K_w$. After an étale base change, decompose $\pi_*\mathcal{L}$ into $K_w$-irreducible representations $\pi_*\mathcal{L} = \oplus_{\alpha \in I} \mathcal{V}_\alpha$, where the index set $I$ is a $\hat{K}_2$-torsor. For any $g \in G(M)$, the action of $g$ translates $\mathcal{V}_\alpha$ to $\mathcal{V}_{\alpha + w(g)}$. Pick an element in $I$ and denote it by 0. After another possible base change, we can choose a section $\vartheta_0 \in \mathcal{V}_0$ such that $\vartheta_0$ does not vanish along any geometric fiber $X_s$ for $s \in S$. Choose a lift $\sigma : \hat{K}_2 \to G(M)$ of $w$ and define a divisor $\Theta$ to be the zero locus of the section

$$\vartheta = \sum_{\alpha \in \hat{K}_2} S_{\sigma(\alpha)}^* \vartheta_0.$$

The set $S(K_2)$ is defined to be the set of all divisors obtained this way. It only depends on the isotropic subgroup $K_2$. In a sense, $S(K_2)$ is the set of most “symmetric” divisors, since it is an average over the lift $\sigma(K_2)$. If $K_2$ is a maximal isotropic subgroup and admits an isotropic complement $K_1$ in $K(L)$, and we require $\sigma$ to be a group homomorphism, then $S_{\sigma(\alpha)}^* \vartheta_0$ are classical theta functions parametrized by $K_1$. The heuristic picture is that, locally near the boundary, $\mathcal{A}^m_{g,\delta}$ should be the normalization of a slice of the moduli space $\mathcal{M}_{g,d}$, and the slice is defined by a choice of a “balanced” divisor from $S(K_2)$. The reason why we can’t get a moduli functor this way is that $K_2$ is not well-defined over the whole moduli space.

There is a compactification defined over $\mathbb{Z}$ that satisfies all (1)-(3) on the above list, and that is $\mathcal{A}_{g,d}$ constructed in [Ols08]. One of the motivations of our work is to understand this compactification. We find that it is almost impossible to find a better compactification than $\mathcal{A}_{g,d}$. However, the construction of $\mathcal{A}_{g,d}$ uses the rigidification, an abstract procedure involving stackification, while our construction of $\mathcal{A}^m_{g,\delta}$ is more elementary and explicit. For example, near each cusp, we have the geometric characterization of the families in terms of stable pairs. Moreover, our stack is a toroidal compactification, which is easier to understand. On the other hand, the two compactifications are closely related. In fact, over $k = \mathbb{Z}[1/d, \mu_M]$, there is a proper, surjective representable morphism $\mathcal{A}^m_{g,\delta} \to \mathcal{A}_{g,d}$ such that the coarse moduli space $\mathcal{A}^m_{g,\delta}$ is the normalization of the coarse moduli space for $\mathcal{A}_{g,\delta}$ (Proposition 3.51). Therefore, our compactification might serve as a simplification of Olsson’s moduli compactification $\mathcal{A}_{g,\delta}$. However, we do not know if the two compactifications are isomorphic.

Our compactification is motivated by ideas from mirror symmetry, as is advertised in the title. The problem with the toroidal compactification is that it depends on a choice of a collection of fans and there are infinitely many choices one can make. According to the picture in [Mor93], the cone associated to each 0-cusp can be identified with the Kähler

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\[^2\text{In this case, the polarization is separable. $\mathcal{A}_{g,\delta}$ is the correspondent connected component of $\mathcal{A}_{g,d}$.} \]
cone of the mirror family (associated to each 0-cusp). Our fans are induced from the Mori fans for the minimal models of the mirror families. The Mori fan is canonical for each mirror family (independent of the choice of a minimal model). The compactification $\mathcal{A}_{g,\delta}^m$ is thus constructed by using the data from the mirror families.

The success of using mirror symmetry in this case should be regarded as something non-trivial. The amazing thing is that all the locally constructed families are compatible, and can be glued globally. When the moduli space has more than one 0-cusp, the compatibility issue is nontrivial. All the toroidal compactifications known to the author assume that there is only one 0-cusp. We believe that, when the space $\mathcal{A}_{g,\delta}$ has more than one 0-cusps, the construction of an admissible collection of fans in Theorem 3.1 is new. Even in the case of principal polarization, we get the interesting fact that the second Voronoi fan is equal to the Mori fan of the mirror (Theorem 2.39).

The drawback of this approach of compactification is that it doesn’t provide a moduli functor. Characterizing the families in terms of some intrinsic properties is usually a much harder problem. Nevertheless, this approach of using mirror symmetry can be applied to many compactification problems. In fact, one of the projects of Gross–Hacking–Keel is to compactify the moduli of polarized K3 surfaces by using similar ideas. Therefore, the compactification $\mathcal{A}_{g,\delta}$ should be regarded as an example of this very general approach.

The compactification problem for moduli of polarized abelian varieties has a long history and has been studied by many people. Most of the work focuses on the moduli of principally polarized abelian varieties (PPAV) $\mathcal{A}_g$. For a good summary of the history, see the introduction of [Nak]. We only mention the most relevant works here. A good compactification of $\mathcal{A}_g$ was not found until [Ale02]. Building on the work of Namikawa [Nam80], Alexeev had the simple, but ingenious observation that for PPAV, the data $(A, \mathcal{L})$ is equivalent to the data $(A, P, \Theta)$, where $P$ is an $A$-torsor. If we reinterpret $\mathcal{A}_g$ as the moduli space of pairs $(A, P, \Theta)$, we can map it into the moduli of stable pairs, which is proper, and then take the closure. See [Ale96]. This idea was developed in [Ale02], and gave rise to the construction of the moduli space $\mathcal{A}_{P,g,d}$. In particular, when $d = 1$, the main irreducible component of $\mathcal{A}_{P,g}$ is a moduli compactification of $\mathcal{A}_g$, and is closely related to the second Voronoi compactification.

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3Here we follow the basic philosophy of mirror symmetry that the modular parameters of a Calabi–Yau space correspond to the Kähler parameters of the mirror, near each maximal degeneration.

4The superscript “m” is for the mirror symmetry.

5Although, in [Ale02], the main component of $\mathcal{A}_{P,g,1}$ is claimed to be isomorphic to the second Voronoi compactification, the proof is incomplete. The author thanks Iku Nakamura for pointing this out. See [Nak Subsection 14.3].
For higher polarizations, the linear system of $L$ is not a point any more. So new ideas are required for the generalization to higher polarizations. The generalization is achieved in [Ols08]. Another work that inspires this paper is [Nak10].

1.2. Outline of the Paper. Now we make a summary of the content. In Sect. 2 we first explain the cones needed for the toroidal compactification and interpret them as cones of polarized tropical abelian varieties. Then we study the mirror symmetry for abelian varieties and get the fans needed for the toroidal compactification. Sect. 3 is the main part, where we construct the families and the algebraic stack $\mathcal{M}_{g,\delta}$. Finally, in Sect. 4 we explain how to extend the family over the boundary by using the group scheme $G(M)$.

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1.4. Conventions and Notations. If $X$ is a finitely generated free $\mathbb{Z}$-module, $X_k := X \otimes \mathbb{Z} k$ for a field $k$. $X \subset X_k$ is an integral structure on the affine space $X_k$. A polytope $Q \subset X_R$ is the convex hull of finite points in $X_R$ and is always bounded. If all the points can be chosen from the lattice $X$, then $Q$ is called a lattice polytope. A polyhedron is the intersection of (not necessarily finite) closed half spaces. For an arbitrary subset $S$ of an affine space, the cone generated by $S$ is denoted by $C(S)$, and the convex hull is denoted by $\text{Conv}(S)$. We denote $X \oplus \mathbb{Z}$ by $X$, and regard $X \subset X$ as the hyperplane of height 1. If $\sigma$ is a polytope in $X_R$, the cone $C(\sigma)$ is inside $X_R$. Any piecewise affine function $\varphi$ over $X_R$ has a unique linear extension over $X_R$, denoted by $\tilde{\varphi}$.

A paving $\mathcal{P}$ is a set of polyhedrons of $X_R$, such that

1. For any two elements $\sigma, \tau \in \mathcal{P}$, the intersection $\sigma \cap \tau$ is a proper face of both $\sigma$ and $\tau$.
2. Any face of a polytope $\sigma \in \mathcal{P}$ is again an element of $\mathcal{P}$.
3. The union $\bigcup_{\sigma \in \mathcal{P}} \sigma$ is a polyhedron $Q$ of $X_R$.
4. For any bounded subset $W \subset X_R$, there exists only finitely many $\sigma \in \mathcal{P}$ with $W \cap \sigma \neq \emptyset$.

A paving is called bounded, if all cells are polytopes. A paving is called a triangulation, if all cells are simplices. A paving or a triangulation is called integral, if all cells are lattice polytopes. We also assume that all pavings and triangulations are regular, that is, they are obtained as the affine regions of piecewise affine functions.

A function $f$ is convex, if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$. Notice that it is different from the usual convention in the literature of toric geometry. For a Cartier divisor $\{m_\sigma\}$ on a toric variety, we define the associated piecewise linear function to be $\varphi = -m_\sigma$. 

For a Weil divisor $D = \sum a_\omega D_\omega$, the associated function is $\psi(\omega) = a_\omega$. $\text{Aff}$ stands for the vector space of affine functions, $\text{PA}$ the space of piecewise affine functions, and $\text{PL}$ the space of piecewise linear functions. Usually there are other decorations in the notations.

If $M$ is a monoid, then $M^{\text{gp}}$ denotes the group associated to the monoid, $M^{\text{sat}}$ the saturation, and $M^*$ the set of invertible elements.

The real valued functions are generalized to functions with values in vector spaces in [GHK11]. Let $P$ be a toric monoid, i.e. $P$ is fine and saturated and the associated group $P^{\text{gp}}$ is torsion free. Assume $\sigma_P := \text{Conv}(P)$ in $P^{\text{gp}}_R$. $\sigma_P$ is a polyhedral cone. Introduce a partial order on $P^{\text{gp}}_R$.

**Definition 1.1.** For $u, v \in P^{\text{gp}}_R$, we say $u$ is $P$-above $v$, and denote it by $u \overset{P}{\geq} v$, if $u - v \in \sigma_P$. We say $u$ is strictly $P$-above $v$, if in addition, $u - v \in \sigma_P \setminus P^*_R$.

Let $\varphi$ be a piecewise affine function over $X_R$ with values in $P^{\text{gp}}_R$. Assume the region where $\varphi$ is affine gives a paving $\mathcal{P}$, i.e. for each $\sigma \in \mathcal{P}_{\text{max}}$, $\varphi|_\sigma$ is an element in $X^*_R \otimes P^{\text{gp}}_R$. $\varphi$ is called integral, if all $\varphi|_\sigma \in X^* \otimes P^{\text{gp}}_R$.

**Definition 1.2 (bending parameters).** For each codimension 1 cell $\rho \in \mathcal{P}$ contained in maximal cells $\sigma_+, \sigma_- \in \mathcal{P}$, we can write

$$\varphi|_{\sigma_+} - \varphi|_{\sigma_-} = n_\rho \otimes p_\rho,$$

where $n_\rho$ is the unique primitive element that defines $\rho$ and is positive on $\sigma_+$, and $p_\rho \in P^{\text{gp}}_R$ is called the bending parameter.

**Definition 1.3.** A piecewise affine function $\varphi$ is $P$-convex, if for every codimension one cell $\rho \in \mathcal{P}$, $p_\rho \in P$. It is strictly $P$-convex, if all $p_\rho \in P \setminus P^*$. For toric varieties, we use the definitions and notations in [CLS11] unless specified otherwise. However, for the fan (the second Voronoi fan) for the toroidal compactification, $\sigma$ is the open cone following the convention of [FC90]. Instead we define the cones $C(\mathcal{P})$ to be closed.

Assume $\mathcal{P}$ is a paving of a lattice polytope $Q \subset X_R$, and $\varphi : Q \to P^{\text{gp}}_R$ is a $\mathcal{P}$-piecewise affine, $P$-convex, integral function. Define

$$Q_\varphi := \left\{ (\alpha, h) \in Q \times P^{\text{gp}}_R : h \overset{P}{\geq} \varphi(\alpha) \right\}.$$

Define $R_\varphi := k[S(Q_\varphi)]$ and $S := \text{Spec } k[P]$. Notice that $R_{\varphi, 0} = k[P]$. $R_\varphi$ is of finite type over $k[P]$. Define $X_\varphi = \text{Proj } R_\varphi$ and we have a projective morphism $\pi : X_\varphi \to S$. The line bundle $\mathcal{L} := \mathcal{O}(1)$ is $\pi$-ample. The $X$-grading on $k[S(Q_\varphi)]$ induces an action $\varrho$ of $\mathbb{T}$ on $R_\varphi$.

For a normal scheme $S$, $\text{Div } S$ is the group of Cartier divisors of $S$. For other notations in birational geometry, we follow [KM98].
Definition 1.4. Fix $\delta = (\delta_1, \ldots, \delta_g)$ a sequence of positive integers $\delta_i$ such that $\delta_i \mid \delta_{i+1}$. A polarization $\lambda : X \to X^t$ of an abelian variety is of type $\delta$, if the kernel of $\lambda$ is isomorphic to $K(\delta) = H(\delta) \times \widehat{H(\delta)}$ for $H(\delta) = \bigoplus_{i=1}^g \mathbb{Z}/\delta_i \mathbb{Z}$. $d := \prod_{i=1}^g \delta_i$ is called the degree of the polarization. Let $M = 2d$.

2. The Toroidal Compactification and Mirror Symmetry

2.1. The Toroidal Compactification over Complex Numbers. For a non-degenerate skew-symmetric bilinear form $E$ over $\mathbb{R}^{2g}$, define

$$\text{Sp}(E, \mathbb{R}) := \{ M \in M(2g, \mathbb{R}) ; MEM^T = E \}.$$ 

We consider the following situation. The bilinear form $E$ takes integral values on a lattice $\Lambda \cong \mathbb{Z}^{2g} \subset \mathbb{R}^{2g}$. By the elementary divisor theorem, we can always choose a symplectic basis $\{ \lambda_1, \lambda_2, \ldots, \lambda_g, \mu_1, \mu_2, \ldots, \mu_g \}$ of $\Lambda$, such that with respect to this basis,

$$E = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}, \text{ where } \delta = \begin{pmatrix} \delta_1 & \cdots \\ & \ddots \\ & & \delta_g \end{pmatrix}.$$ 

The subgroup $\text{Sp}(E, \mathbb{R})$ is conjugate to the symplectic group $\text{Sp}(2g, \mathbb{R})$ in $\text{GL}(2g, \mathbb{R})$ by the following element

$$M \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}^{-1} M \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}.$$ 

There are various arithmetic subgroups of $\text{Sp}(2g, \mathbb{R})$. We denote $\text{Sp}(E, \mathbb{Z})$ by $\Gamma(\delta)$. $\Gamma(\delta)$ is an arithmetic subgroup of $\text{Sp}(2g, \mathbb{R})$ through the map $\mathbb{Z}$.

The following lemma is a direct generalization of ([Mum06], Chap.II Lemma 4.1) to higher polarizations, so we omit the proof. We choose the convention that a vector in $H^1(A, C)$ be a column vector, and a period matrix is thus a $2g \times g$-matrix.

Fix a symplectic basis, and identify $\Lambda$ (resp. $\Lambda \otimes \mathbb{R}$) with $\mathbb{Z}^{2g}$ (resp. $\mathbb{R}^{2g}$). Let $\Lambda_1$ be the sublattice generated by $\{ \lambda_i \}$ and $\Lambda_2$ be the sublattice generated by $\{ \mu_j \}$. We use $\lambda$ to denote a vector in $\Lambda_1 \otimes \mathbb{R}$ and $\mu$ a vector in $\Lambda_2 \otimes \mathbb{R}$. The linear actions on the vectors are right actions.

Lemma 2.1. The following data on $\mathbb{R}^{2g}$ are equivalent,

a) a complex structure $J : \mathbb{R}^{2g} \to \mathbb{R}^{2g}, J^2 = -1$ such that $E = \Im(H)$, $H$ a positive definite Hermitian form for this complex structure. The existence of $H$ is equivalent to:

$$E(uJ, vJ) = E(u, v), \forall u, v \in \mathbb{R}^{2g},$$

$$E(vJ, v) > 0, \forall v \in \mathbb{R}^{2g} - \{0 \}.$$ 

In this case, $H(u, v) = E(uJ, v) + iE(u, v)$. 


b) a homomorphism $\iota : \mathbb{Z}^{2g} \rightarrow V$, where $V$ a complex vector space of dimension $g$, plus a positive definite Hermitian $H$ on $V$ such that
$$\Im(H)(iu, iv) = E(u, v)$$

c) a $g$-dimensional complex subspace $P \subset \mathbb{C}^{2g}$ such that
$$EC(u, v) = 0, \quad \forall u, v \in P,$$
$$iEC(u, \bar{u}) < 0, \quad \forall u \in P - \{0\},$$

d) a $g \times g$ complex symmetric matrix $\tau$ with $\Im(\tau)$ positive definite.

For a), we denote the set of $J$ in a) by $C_0(\text{Sp}(E, \mathbb{R}))$. It is a tube domain in the real Lie group $\text{Sp}(E, \mathbb{R})$. $\text{Sp}(E, \mathbb{R})$ is acting on it from the left by conjugation $J \mapsto MJM^{-1}$. Therefore, $C_0(\text{Sp}(E, \mathbb{R}))$ is a homogeneous space for $\text{Sp}(E, \mathbb{R})$. Because of the following lemma ([BL04] Lemma 17.2.3), we identify $C_0(\text{Sp}(E, \mathbb{R}))$ with the Siegel upper half space $\mathcal{S}_g = \{ \tau \in \text{M}(g, \mathbb{C}); \tau = \tau^T, \Im(\tau) > 0 \}$.

**Lemma 2.2.** The map from a) to d) is equivariant with respect to the action of Lie groups if we identify $\text{Sp}(E, \mathbb{R})$ and $\text{Sp}(2g, \mathbb{R})$ by the map (2).

The group $\Gamma(\delta)$ is acting on the set of symplectic basis. Therefore, we have the action on $\mathcal{S}_g$. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(\delta)$, the action is
$$M(\tau) = (a\tau + b\delta)(c\tau + d\delta)^{-1}\delta.$$ 

Suppose $\Lambda = H_1(A, \mathbb{Z})$ and $V = (H^{1,0}(A))^*$ for an abelian variety $A$. Over $\mathbb{C}$, the coarse moduli space of abelian varieties with polarization type $\delta$ is the quotient $\Gamma(\delta)\backslash \mathcal{S}_g$ ([BL04] Theorem 8.2.6, and Remark 8.10.4). As an arithmetic quotient of a Hermitian symmetric domain, $\Gamma(\delta)\backslash \mathcal{S}_g$ admits a type of compactifications called the toroidal compactifications. We won’t get into the details of the constructions, but we need to explain the cones. Our basic reference is [HKW93]. We have a minimal compactification by adding rational boundary components to $\mathcal{S}_g$. We also call the rational boundary components cusps. Each cusp $F$ corresponds to a rational isotropic subspace $U(F)$ of $V$. The basic result is

**Proposition 2.3** ([HKW93] Proposition. 3.16, Proposition. 3.19). $U$ is a bijection between the following objects:

1. a) rational boundary components $F$,
   b) rational $E$-isotropic subspaces $U \subset \mathbb{R}^{2g}$.
2. a) pairs of adjacent boundary components $F' \succ F$,
   b) pairs of $E$-isotropic subspaces $U' \subsetneq U$.

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6Notice that our convention is different from theirs by a transposition.
7Although the book concerns the moduli of abelian surfaces, the statements of the toroidal compactification is true for all degrees. For general statements, see [AMRT10]
8Better known as the Baily–Borel–Satake compactification.
Moreover, the correspondence is $\text{Sp}(2g, \mathbb{R})$-equivariant. We have $U(M(F')) = U(F) \cdot M^{-1}$, for $M \in \Gamma(\delta)$. Each rational boundary component is equivalent under $\text{Sp}(E, Q)$ (left action) to one of the following boundary components,

$$F(g') = \left\{ \begin{pmatrix} Z' & 0 \\ 0 & I_{g'-g} \end{pmatrix} ; Z' \in \mathcal{D}_{g'} \right\}, \ 0 \leq g' \leq g.$$

$$U(g') = U(F(g')) = \left\{ (0 \ x) ; 0 \in \mathbb{R}^{g'+g'}, x \in \mathbb{R}^{g-g'} \right\}.$$

**Remark 2.4.** Notice that the map $U$ depends on the choice of $\delta$. If we denote the correspondent isotropic subspace in the principal polarized case by $U^0$, then $U(U^0) = U^0$.

For any cusp $F$, the stabilizer $\mathcal{P}(F)$ is a parabolic subgroup. If $F' = M(F)$, for $M \in \text{Sp}(2g, \mathbb{R})$, we have $U(F') = U(F) \cdot M^{-1}$ and $\mathcal{P}(F') = M\mathcal{P}(F)M^{-1}$. The center of the unipotent radical $\mathcal{P}(F)$ is a real vector space. The normalizer of $\mathcal{P}'(F)$ in $\text{Sp}(2g, \mathbb{R})$ is $\mathcal{P}(F)$, and $\mathcal{P}(F)$ is acting on $\mathcal{P}'(F)$ by the conjugation. Let the centralizer of $\mathcal{P}'(F)$ in $\mathcal{P}(F)$ be $Z(\mathcal{P}')$ and $\mathcal{P}(F) = G_t(F) \cong Z(\mathcal{P}')$. So it is essentially the action of $G_t$. Denote $\Gamma(\delta) \cap G_t(F)$ by $G_t(F)$ and the image of $G_t(F)$ in $\text{GL}(\mathcal{P}')$ by $\overline{\mathcal{P}}(F)$.

The cone $\mathcal{C}(F)$ that supports the fan is a special orbit of $\mathcal{P}$-action in $\mathcal{P}'$, which is given by the Harish-Chandra map. While it is hard to describe the Harish-Chandra map in general, it is easy to write down everything for the special case $F(g')$.

**Example 2.5.** Over $F(g')$, we have,

1. $U(g') = \left\{ (0 \ x) ; 0 \in \mathbb{R}^{g'+g'}, x \in \mathbb{R}^{g-g'} \right\}$.

2. $\mathcal{P}(g') = \left\{ [Q] : 0 < Q \in \text{Sym}_{g-g'}(\mathbb{R}) \right\} \cong \text{Sym}_{g-g'}(\mathbb{R})$.

3. $\mathcal{C}(g') = \left\{ [Q] ; 0 < Q \in \text{Sym}_{g-g'}(\mathbb{R}) \right\}$.

4. $G_t(g') = \left\{ \begin{pmatrix} I_{g'} & 0 & 0 \\ 0 & (u^T)^{-1} & 0 \\ 0 & 0 & I_{g'} \end{pmatrix} ; u \in \text{GL}(g-g', \mathbb{R}) \right\} \cong \text{GL}(g-g', \mathbb{R})$.

5. $\delta_g = \begin{pmatrix} \delta_{g'+1} \\ \vdots \\ \delta_g \end{pmatrix}$.

6. $\overline{\mathcal{P}}(g') \cong \text{GL}(\delta_g) := \left\{ u \in \text{GL}(g-g', \mathbb{Z}) ; \delta_g u \delta_{g+1}^{-1} \in \text{GL}(g-g', \mathbb{Z}) \right\}$.

The action of $\overline{\mathcal{P}}(g')$ on the vector space $\mathcal{P}(g')$ is

$$Q \mapsto (u^T)^{-1}Qu^{-1}.$$
Definition 2.6. For an arbitrary boundary component $F$, let $M \in \text{Sp}(2g,\mathbb{R})$ such that $F = M(F^{(g')})$, define $\mathcal{C}(F) := MC^{(g')}M^{-1}$.

Remark 2.7. The definition of $\mathcal{C}(F)$ is independent of the choice of $M$.

For each cone $\mathcal{C}(F)$, we need a fan $\Sigma(F)$ whose support is the rational closure of $\mathcal{C}(F)$. If the collection of fans is admissible ([HKW93] Definition 3.61, Definition 3.66), then we can construct a proper algebraic space, called the toroidal compactification of $\Gamma(\delta) \setminus \mathcal{O}_g$. If $g = 2$, this is ([HKW93] Theorem 3.82). The general case is the main theorem ([AMRT10] Theorem 5.2).

Since it is difficult to deal with the cones and various discrete groups in a Lie group, we interpret the cones $\mathcal{C}(F)$ as the moduli of tropical abelian varieties.

2.2. A Tropical Interpretation of the Cones. Fix a rational isotropic subspace $U$, we define the tropicalization in the direction of $U$ abstractly. Recall that an abelian variety with a polarization and a symplectic basis $(A, E, \{\lambda_i, \mu_j\})$ is equivalent to the data $J \in C_0(\text{Sp}(E, \mathbb{R}))$ in Lemma 2.1.

Lemma 2.8. $V = UJ \oplus U^\perp$.

Proof. If $u \in UJ \cap U^\perp$, then $uJ \in U$. Since $u \in U^\perp$, $E(uJ, u) = 0$. $E(\cdot, \cdot)$ is nondegenerate, so $u = 0$, and $UJ \cap U^\perp = \{0\}$. Since $\dim UJ + \dim U^\perp = 2g$, $V = UJ \oplus U^\perp$. □

Definition 2.9. For $J \in C_0(\text{Sp}(E, \mathbb{R}))$, define $g_J$ to be the isomorphism $UJ \rightarrow V/U^\perp$, $\lambda \mapsto g_J(\lambda J)$.

By Lemma 2.8, $f_J$ is an isomorphism. Denote the inverse by $\tilde{\phi} : V/U^\perp \rightarrow U$. Define $Y := \Lambda/\Lambda \cap U^\perp$. It is a full rank lattice in $V/U^\perp$. Consider the real torus $\tilde{B} := U/\tilde{\phi}(Y)$. Denote the lattice $U \cap \Lambda$ by $X^*$, and $\text{Hom}(\Lambda \cap U, \mathbb{Z})$ by $X$. $\tilde{B}$ is a tropical torus with the lattice $X^*$ in the tangent plane of $\tilde{B}$. The non-degenerate pairing $E(\cdot, \cdot)$ induces an injection $\phi : Y \rightarrow X$. The data $\phi$ is equivalent to the positive symmetric pairing $\tilde{g} = E(\cdot, \cdot)$ on $U$. Therefore $\phi$ is a polarization for the tropical torus $\tilde{B}$, and $(\tilde{B}, \tilde{\phi}, \phi)$ is a polarized tropical abelian variety in the sense of [MZ08]. Denote the image of $\phi(Y)$ by $\tilde{Y}$. Assume $X/\tilde{Y} \cong \mathbb{Z}/d_1 \times \mathbb{Z}/d_2 \times \ldots \times \mathbb{Z}/d_r$ for $d_i | d_{i+1}$. Let $\mathfrak{d}$ be the diagonal matrix

$$
\mathfrak{d} := \begin{pmatrix}
d_1 & & \\
& \ddots & \\
& & d_r
\end{pmatrix}.
$$

$\mathfrak{d}$ is called the type of $\phi$. 
Corollary 2.10. \((\bar{B}, \bar{\phi}, \phi)\) is a \(r\)-dimensional polarized abelian variety of type \(\delta\).

Definition 2.11. Let \(X, \bar{Y}\) be two lattices such that \(\bar{Y} \subset X\) and \(X/\bar{Y} \cong \mathbb{Z}/d_1 \times \mathbb{Z}/d_2 \times \cdots \times \mathbb{Z}/d_r\). A basis of \(X\) is called a compatible basis, if they are sent to the generators of \(\mathbb{Z}/d_i\) under the above isomorphism.

Proposition 2.12. Up to the action of \(\Gamma(\delta)\), there is a unique rational boundary component that corresponds to the polarized tropical abelian varieties of type \(\delta\), and this is the orbit of \(F^{(0)}\). For any 0-cusp, we have \(d_1d_2\cdots d_g = \delta_1\delta_2\cdots \delta_g\).

Proof. For the first statement, it suffices to find a symplectic basis \(\{\lambda_i, \mu_j\}\) such that \(\{\mu_j\}\) is a basis of \(\Lambda\cap U\). Choose a compatible basis \(\{y_j\}_{j \in 1, \ldots, g}\) of \(X\) such that \(x_i = \delta_i y_i\) is a basis of \(\bar{Y}\). Let \(\xi\) denote the sublattice generated by \(\{y_j\}\) by \(\xi\). Let \(\mu_j\) be the dual basis of \(\{y_j\}\). Consider the map \(E(\cdot, \cdot)|_U : \Lambda \to X\), and lift \(x_i\) to an element \(\lambda_i \in \Lambda\). So we have \(E(\lambda_1, \mu_1) = \delta_1\) and \(E(\lambda_1, \mu_j) = 0\) for \(j \neq 1\). Now recall how \(\delta_1\) is defined \((\text{[GH94]} \text{ pp. 304-305.})\). The set of values \(E(\Lambda, \Lambda)\) is an ideal in \(\mathbb{Z}\) generated by \(\delta_1\).

For the second statement, choose a compatible basis \(\{y_1, \ldots, y_g\}\) of \(X\). Then there exists \(\{u_1, \ldots, u_g\} \subset \Lambda\) that are lifts of \(d_i y_i\). Let \(\{v_1, \ldots, v_g\} \subset U\) be the dual basis of \(\{y_1, \ldots, y_g\}\). Then \(\{u_1', \ldots, u_g', v_1, \ldots, v_g\}\) is a basis of \(\Lambda\). With respect to this new basis, \(E\) is the matrix

\[
\begin{pmatrix}
S & \delta \\
-S & 0
\end{pmatrix},
\]

where \(\delta\) is the matrix \((2.2)\), and \(S\) is some skew symmetric integral matrix (nonzero if \(d \neq \delta\)).

Because the transformation matrix between two basis \(\{u_1', \ldots, u_g', v_1, \ldots, v_g\}\) and \(\{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\}\) is in \(\text{GL}(2g, \mathbb{Z})\), and has determinant \(\pm 1\), we have \(d_1d_2\cdots d_g = \delta_1\delta_2\cdots \delta_g\) by computing the determinant. \(\square\)

Corollary 2.13. For any maximal isotropic subspace \(U \subset \mathbb{R}^{2g}\), if \(U \cap \Lambda\) has an isotropic complement, i.e. there is an isotropic subspace \(U' \subset \mathbb{R}^{2g}\) such that \((U \cap \Lambda) \oplus (U' \cap \Lambda) = \Lambda\), then \(U = U^{(0)}M^{-1}\) for some \(M \in \Gamma(\delta)\).

Proof. If \(U \cap \Lambda\) has an isotropic complement \(U' \cap \Lambda\) in \(\Lambda\), then choose a basis from \(U \cap \Lambda\) and \(U' \cap \Lambda\), we get a symplectic basis. \(\square\)

Definition 2.14. A maximal rational boundary component is called splitting if it is congruent to \(F^{(0)}\). Otherwise it is called non-splitting.

Lemma 2.15. With \(X\) and \(\bar{Y} \subset X\) fixed, the set of tropical abelian varieties is identified with the set of positive definite quadratic forms on \(X_R\), and is denoted by \(\mathcal{C}(X)\).

Proof. Since \(\bar{B}\) and \(\phi : Y \to X\) are fixed, a polarized tropical abelian variety is equivalent to the data \(\phi : Y \to U\), which is equivalent to the positive symmetric bilinear form \(\langle \cdot, \phi^{-1}(\cdot) \rangle\) on \(X_R\). \(\square\)
Lemma 2.19. The intersection of this lattice and \( M \) is a linear isomorphism. It induces an isomorphism \( M \). A map \( C \) satisfies \( GL(\cdot) \). Consider \( X \) as a lattice in \( U^* \). We identify \( GL(\cdot) \) with \( \mathbb{L}^* \) in \( \Gamma^2(U) \). \( M \) is acting on \( X \). The set of integral elements is a lattice, denoted by \( \mathbb{Z} \). We have

**Definition 2.18.** A tropical abelian variety \( \tilde{B}, \phi, \phi \) is called integral, if \( \tilde{Y} \subset X^* \). A quadratic form \( Q \in \Gamma^2(U) \) is called integral if the associated symmetric bilinear form \( B \) satisfies \( B(Y, X) \in \mathbb{Z} \). The set of integral elements is a lattice, denoted by \( \mathbb{L}^* \), in \( \Gamma^2(U) \). The intersection of this lattice and \( C(X) \) is the set of integral tropical abelian varieties.

Note that \( \Gamma^2 X \subset \mathbb{L}^* \). It is convenient to write down everything in terms of a fixed basis. Define

\[
GL(\delta) := \{ u \in GL(r, \mathbb{Z}) : \delta u \delta^{-1} \in GL(r, \mathbb{Z}) \}. 
\]

Choose a compatible basis of \( X \), we see that \( GL(X, Y) = GL(\delta)^T \). Or with respect to the corresponding basis of \( \tilde{Y} \), \( GL(X, Y) \cong GL(\delta) \). With respect to a compatible basis of \( X \), \( C(X) \cong \{ Q \in \text{Sym}_Y(\mathbb{R}) ; Q > 0 \} \). The action is \( Q \mapsto (u^T)^{-1}Qu^{-1} \). The set of integral tropical abelian varieties corresponds to \( \{ Q \in C(X) ; Q \phi \in M(r, \mathbb{Z}) \} \).

Given two cusps \( F \) and \( F' \), and an element \( M \in \text{Sp}(E, \mathbb{Q}) \) such that \( F' = M(F) \). \( M \) gives a map \( C(X) \to C(X') \), still denoted by \( M \), as follows: Since \( U' = UM^{-1} \), \( M^{-1}|_U : U \to U' \) is a linear isomorphism. It induces an isomorphism \( M^{-1}|_U \otimes M^{-1}|_U : \Gamma^2(U) \to \Gamma^2(U') \). This linear isomorphism sends \( C(X) \) to \( C(X') \).

**Lemma 2.19.** We have the following commutative diagram

\[
\begin{array}{ccc}
C_0(\text{Sp}(E, \mathbb{R})) & \xrightarrow{M} & C_0(\text{Sp}(E, \mathbb{R})) \\
\uparrow{\text{Tr}} & & \downarrow{\text{Tr}} \\
C(X) & \xrightarrow{M} & C(X')
\end{array}
\]

The first line is the conjugation by \( M \). Furthermore, if \( M \in \Gamma(\delta) \), then \( \text{Tr}(J) \) is isomorphic to \( \text{Tr}(M J) \) as polarized tropical abelian varieties.

**Proof.** First \( M \in \text{Sp}(E, \mathbb{Q}) \). Notice that \( E(.J, \cdot)|_U = (\phi \phi^{-1}(\cdot), \cdot) \) on \( U \). For any \( J \in C_0(\text{Sp}(E, \mathbb{R})) \), \( M(J) = MJM^{-1} \). \( \text{Tr}(J) \) is represented by \( E(.J, \cdot)|_U \). The bilinear form \( \text{Tr}(M(J)) \) is

\[
E(.M(J), .)|_{UM^{-1}} = E(.MJM^{-1}, .)|_{UM^{-1}} = E(.M, .)|_{UM^{-1}}
\]
This is the same with the bilinear form induced by $M^{-1}$ in the definition.

Now $M \in \Gamma(\delta)$. Because $\cdot M^{-1}$ is a bijection $\mathbb{R}^{2g} \to \mathbb{R}^{2g}$, $(S \cap U)M^{-1} = SM^{-1} \cap UM^{-1}$ for any subset $S \subset \mathbb{R}^{2g}$. Since $M \in \Gamma(\delta)$, $(\Lambda \cap U)M^{-1} = \Lambda M^{-1} \cap UM^{-1} = \Lambda \cap U'$ and $(\Lambda \cap U')M^{-1} = \Lambda \cap U''\perp$. Consider the following commutative diagram for $J' = MJM^{-1}$,

$$
\begin{array}{ccc}
\Lambda \cap U & \xrightarrow{J} & UJ \oplus U' \perp \\
\cdot M^{-1} & \downarrow & \cdot M^{-1} \\
\Lambda \cap U' & \xrightarrow{J'} & UJ' \oplus U'' \perp \\
\end{array}
$$

It shows that $f_{J'}(\Lambda \cap U') = (f_J(\Lambda \cap U))M^{-1}$ and $\tilde{B} \cong \tilde{B}'$ as tropical tori. The polarization is preserved because of the first part. \qed

If $F = F'$, we can apply the above proof to $M \in \mathcal{P}(F)$, and get

**Corollary 2.20.** Fix a cusp $F$. The action of $\mathcal{P}(F)$ on $\mathcal{C}(X)$ is denoted by $\rho_X$. Then $\text{Tr} : \mathcal{C}_0(\text{Sp}(E, \mathbb{R})) \to \mathcal{C}(X)$ is $\mathcal{P}(F)$-equivariant. The image $\rho_X(P(F))$ is inside $\text{GL}(X, Y)$.

Let’s consider the map $\text{Tr} : \mathfrak{S}_g \to \mathcal{C}(X)$ for the cusps $F^{(g')}$. For $F^{(g')}, U^{(g')} = \{(0, x); 0 \in \mathbb{R}^{y+g'}, x \in \mathbb{R}^{y-g'}\}$. We have the natural basis for every lattice. Use the basis of $X$, that is $\{y_i\}_{i \leq g'}$, we regard $\mathcal{C}(X)$ as an open cone in $\text{Sym}_g(\mathbb{R})$. Assume $\tau \in \mathfrak{S}_g$ corresponds to $J \in \mathcal{C}_0(\text{Sp}(E, \mathbb{R}))$. Since $E(\cdot J, \cdot) = \Re H$, and with respect to the coordinates $z_i = (x \tau + y \delta)_i$, $H$ is $(\Im \tau)^{-1}$, the restriction of $E(\cdot J, \cdot)$ to $U^{(g')}$ is

$$
E(\cdot J, \cdot)|_{U^{(g')}} = H|_{U^{(g')}} = \sum_{g' < i, j \leq g} \delta_i y_i ((\Im \tau)^{-1})_{ij} \delta_j y_j.
$$

It follows that the matrix for $\text{Tr}(\tau)$ is the inverse matrix of $T = ((\Im \tau)^{-1})_{g' < i, j \leq g}$ with respect to the basis of $\tilde{Y}$. We write $\tau$ in blocks, where $\tau_1$ is a $g' \times g'$ matrix,

$$
\tau = \begin{pmatrix}
\tau_1 & \tau_2 \\
\tau_3 & \tau_2
\end{pmatrix},
$$

**Lemma 2.21.** Given an invertible matrix and its inverse, both in blocks,

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^{-1}.
$$

Assume that $A$ and $D$ are both invertible, then

$$
D^{-1} = d - ca^{-1}b
$$

**Proof.** We have

$$
\begin{align*}
&Aa + Bc = \text{Id}, \\
&Ab + Bd = 0, \\
&Ca + Dc = 0, \\
&Cb + Dd = \text{Id}.
\end{align*}
$$
Therefore
\[ D(d - ca^{-1}b) = Dd - Dca^{-1}b = Id - Cb + Caa^{-1}b = Id - Cb + Cb = Id. \]

\[ \Box \]

**Proposition 2.22.** With respect to the basis \( \{ \delta_i y_i \}_{i > g'} \) of \( \check{Y} \),
\[ \text{Tr}(\tau) = 3\tau_2 - (3\tau_3^T)(3\tau_1)^{-1}\tau_3. \]

**Proof.** Since \( \tau \) is positive definite, \( \tau_1 \) and \( \tau_2 \) are positive definite. So we can apply the lemma. \( \Box \)

Now we can compare \( C(X) \) with \( C(F) \) for \( F = F'(g') \).

**Proposition 2.23.** For \( F = F'(g') \), there exists an isomorphism \( h(F) : P'(F) \to \Gamma^2(U) \) restricting to a bijection \( h(F) : C(F) \to C(X) \), such that the following diagram commutes,
\[ \begin{array}{ccc}
\mathcal{G}_g & \xrightarrow{=} & C_0(\text{Sp}(E, \mathbb{R})) \\
\Phi(F) \downarrow & & \downarrow \text{Tr}(F) \\
C(F) & \xrightarrow{h(F)} & C(F')
\end{array} \]

Moreover, the following statements are true.

a) Every map in the diagram is \( P(F) \)-equivariant.

b) \( \text{Tr} \) is surjective.

c) Denote the induced isomorphism \( \text{Aut}(\Gamma^2(U)) \to \text{Aut}(P'(F)) \) by \( \rho_h \), then \( \rho_h(\text{GL}(X, Y)) = P'(F) \).

d) Thus \( \rho_X(P(F)) = \text{GL}(X, Y) \).

e) \( P'(F) \cap C(F) \) is identified with the integral tropical abelian varieties by \( h(F) \).

**Proof.** We identify \( \Gamma^2(U) \) with \( \text{Sym}_r(\mathbb{R}) \) by the above basis \( \{ \delta_i y_i \}_{i > g'} \). We have also identified \( P'(F'(g')) \) with \( \text{Sym}_r(\mathbb{R}) \) by \( [\cdot] \) in Example 2.5. \( h(F) \) is defined to be the composition of these two isomorphisms. By (Nam80, p. 31 ii), \( \Phi(\tau) = 3\tau_2 - (3\tau_3^T)(3\tau_1)^{-1}\tau_3 = \text{Tr}(\tau) \). Thus the diagram commutes.

Since \( \Phi \) and \( \text{Tr} \) are both \( P(F) \)-equivariant, and are both surjective, \( h \) is also \( P(F) \)-equivariant, and we get a). For c), we have computed \( P'(F) = \text{GL}(\delta_g') = \text{GL}(X, Y) \) under the above identification. For e), \( Q \in P'(F) \) if and only if \( \delta_g^{-1}Q \in M(g - g', \mathbb{Z}) \). Remember we are using the basis of \( \check{Y} \). \( \Box \)

Assume \( F' = M(F) \), for \( M \in \text{Sp}(E, \mathbb{Q}) \).

**Lemma 2.24.** The following diagram
\[ \begin{array}{ccc}
\mathcal{G}_g & \xrightarrow{M} & \mathcal{G}_g \\
\Phi \downarrow & & \Phi \\
C(F) & \xrightarrow{M} & C(F')
\end{array} \]
commutes.

Proof. Recall the definition of $\Phi$ in [Nam80]. Embed $S_g \cong \mathcal{P}(F)/(\mathcal{P}(F) \cap K)$ into $S(\mathcal{P}(F)) \cong \mathcal{P}'(F)/\mathcal{P}(F)$. The map $\Phi : S(\mathcal{P}(F)) \to \mathcal{P}'(F)$ is the composition

$$
P'(F)C\mathcal{P}(F)/(\mathcal{P}(F) \cap K) \longrightarrow P'(F) = u \cdot p \mod \mathcal{P}(F) \cap K \mapsto u \cdot p \mod \mathcal{P}(F) = u \mapsto \exists u

Assume $\tau = u \cdot p \mod \mathcal{P}(F) \cap K \in S_g$,

$$
M(\tau) = M(u \cdot p \mod \mathcal{P}(F) \cap K) = MuM^{-1} \cdot MpM^{-1} \mod M(\mathcal{P}(F) \cap K)M^{-1}.
$$

Because $M$ is a real matrix,

$$
\Phi(M\tau) = \exists MuM^{-1} = M\exists uM^{-1} = M\Phi(\tau).
$$

For any cusp $F$, there exists $M \in \text{Sp}(E, \mathbb{Q})$ such that $F = M(F'(g'))$ for some $g'$. By definition, $\mathcal{P}'(F) = M\mathcal{P}'(F(g'))M^{-1}$. We can now define $h(F) : \mathcal{P}'(F) \to \Gamma^2(U)$ to be the unique map that makes the following diagram commute,

$$
\mathcal{P}'(F(g')) \overset{M}{\longrightarrow} \mathcal{P}'(F) \\
\downarrow\downarrow\downarrow \downarrow \downarrow \downarrow \\
\Gamma^2(U(g')) \overset{M}{\longrightarrow} \Gamma^2(U).
$$

The definition of $h(F)$ is independent of the choice of $M$, because $h(F(g'))$ is $\mathcal{P}(F(0))$-equivariant. $h(F)$ is an isomorphism. Most of the statements for $h(F(g'))$ can be generalized to a general $h(F)$.

**Proposition 2.25.** The following diagram commutes for any cusp $F$.

$$
\begin{array}{ccc}
S_g & \overset{=}\longrightarrow & C_0(\text{Sp}(E, \mathbb{R})) \\
\Phi(F) & \downarrow\downarrow & \text{Tr}(F) \\
C(F) & \overset{h(F)}\longrightarrow & C(X)
\end{array}
$$

Moreover, the following statements are true.

a) Every map in the diagram is $\mathcal{P}(F)$-equivariant.
b) $\text{Tr}$ is surjective.
c) $\rho_X(F) : \mathcal{P}(F) \to GL(X_{\mathbb{R}})$ is surjective.
d) Denote the induced isomorphism $\text{Aut}(\Gamma^2(U)) \to \text{Aut}(\mathcal{P}'(F))$ by $\rho_h$, then $\overline{\mathcal{P}}(F) \subset \rho_h(GL(X, \overline{Y}))$ is a subgroup of finite index.

Proof. We have the following diagram
Imagine the diagram as a pyramid with five faces. The diagram concerning us is the triangular face on the right. It commutes because all the other faces commute and the arrows in the bottom square are all bijections. Then a), b) and c) follow.

For d). Since \( \rho_X \) is surjective and defined over \( \mathbb{Q} \), by ([Bor69] 8.11), the image of the arithmetic subgroup \( P(F) \) is an arithmetic subgroup of \( \text{GL}(X,\mathbb{R}) \). So \( \rho_X(P(F)) \) is commensurable with \( \text{GL}(X,Y) \). Because \( \rho_X(P(F)) \) is a subgroup of \( \text{GL}(X,Y) \), it is a subgroup of finite index. \( \square \)

Let \( F \) be a 0-cusp that corresponds to a maximal isotropic space \( U \). Fix a basis \( \{v_1, \ldots, v_g\} \) of \( U \cap \Lambda = X^* \). Take any \( M \in \text{Sp}(E,\mathbb{Q}) \), such that \( U = M \cdot U(0) = U(0)M^{-1} \), and \( M^{-1} \) maps the basis \( \{\mu_1, \ldots, \mu_g\} \) of \( \Lambda_2 \) in \( U(0) \) to the basis \( \{v_1, \ldots, v_g\} \). Such \( M \) always exists. It suffices to find a rational Lagrangian complement \( L \) of \( U \). Extend the basis \( \{v_1, \ldots, v_g\} \) of \( U \cap \Lambda \) to a new basis \( \{u'_1, \ldots, u'_g, v_1, \ldots, v_g\} \) of \( \Lambda \). Assume under this basis,

\[
E = \begin{pmatrix} S & \delta \\ -\delta & 0 \end{pmatrix}.
\]

Here \( S \) is an integral skew-symmetric matrix. Denote the matrix that maps \( \{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\} \) to this new basis by \( M'^{-1} \). \( M' \in \text{GL}(2g,\mathbb{Z}) \). Assume

\[
\begin{pmatrix} A & B \\ 0 & I_g \end{pmatrix} M'^{-1} = M^{-1}
\]

Since

\[
M'^{-1} \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} (M'^{-1})^T = \begin{pmatrix} S & \delta \\ -\delta & 0 \end{pmatrix},
\]

we have

\[
(4) \quad S = A^{-1}B\delta - (A^{-1}B\delta)^T
\]

(5) \quad \begin{pmatrix} A \delta\delta^{-1} \\ 0 \end{pmatrix}

We can always change \( M \) by

\[
\begin{pmatrix} I_g & \delta Q' \\ 0 & I_g \end{pmatrix} M,
\]
where $Q'$ is a symmetric rational matrix. Therefore we can always choose $M$ such that

\[ A^{-1}B\delta = \frac{1}{2} S \tag{6} \]

\[ A = \delta\delta^{-1} \tag{7} \]

**Corollary 2.26.** The lattices $P'(F)$ and $L^*$ agree. In particular, the set $C(F) \cap P'(F)$ is the set of integral tropical abelian varieties for every $F$.

**Proof.** It suffices to prove it for maximal corank boundary components. Let $U$ be a rational maximal isotropic subspace. $Q' \in GL(2g, \mathbb{Z})$ corresponds to an element in $C(F) \cap P'(F)$ if and only if $M \left( I_g \delta Q I_g \right) M^{-1} \in GL(2g, \mathbb{Z})$.

This is true, if and only if $\delta Q \in \mathbb{Z}$, which means $Q$ represents an integral tropical abelian variety in $C(X)$. \(\square\)

### 2.3. Mirror Symmetry for Abelian Varieties. **Warning:** We use $X$ (resp. $Y$) to denote both the complex torus and the lattice. We hope there is no confusion. Fix a complex number $t \in \mathbb{H}$, and the polarization $E$ on an abelian variety $X$, we get a symplectic manifold $(X, \Omega)$, with $\Omega = tE$ a complexified Kähler form. Any maximal rational isotropic subspace $U$ is a linear Lagrangian for $(X, \Omega)$. Recall $Y := \Lambda/\Lambda \cap U$ is a lattice in $V/U$. Define the affine torus $B := (V/U)/Y$. Any maximal isotropic subspace $U$ is a linear Lagrangian for $(X, \Omega)$. We use the construction in [Pol03] to define the mirror torus $(Y_t, J_t)$. For $t = a + bi$, define $\Omega' = (ai - b)E$. Define $\phi : V \to V^*$ by $E(v, w) = \phi(v)(w)$. Define a complex structure $J_\Omega$ on $V \oplus V^*$ by requiring that the function

\[ I_x(v, f) := \Omega'(x, v) + if(x) \]

is complex linear for each $x \in V$. It can be checked that

\[ J_\Omega = \begin{pmatrix} b^{-1}a & -b^{-1}\phi^{-1} \\ (b + a^2b^{-1})\phi & -ab^{-1} \end{pmatrix} \]

satisfies the condition. Denote the space of functions that vanish on $U$ by $\text{ann}(U)$. $J_\Omega$ preserves $U \oplus \text{ann}(U)$, and thus descends to a complex structure $J_t$ on

\[ \tilde{V} := (V \oplus V^*)/(U \oplus \text{ann}(U)) = V/U \oplus U^*. \]

Define the lattice $\Gamma := Y \oplus X \subset V/U \oplus U^*$. The complex torus $(\tilde{V}/\Gamma, J_t)$ is defined to be the mirror $Y_t$. There is a natural dual fibration $\tilde{f} : Y_t \to B$. Let $u'_i \in Y$ be the image of $-u'_i$, and $v_i \in X$ be the dual of $v_i \in X^*$. Then

\[ J_t u'_i = b^{-1}a u'_i - (b + a^2b^{-1})d_i v_i. \]
Define $\tilde{e}_i = d_i \tilde{v}_i$, then $\tilde{u}'_i = (a + b J_t) \tilde{e}_i$. Use the basis $\{\tilde{e}_i\}, \{\tilde{u}'_i, \tilde{v}_j\}$, the period matrix is

$$
\begin{pmatrix}
\frac{tI_2}{\delta^{-1}} \\
\end{pmatrix}.
$$

Therefore $Y_t = E_1 \times E_2 \times \ldots \times E_g$, where $E_i := \mathbb{C}/d_i^{-1} \mathbb{Z} + t \mathbb{Z}$ is an elliptic curve. We write $E_i$ as $\mathbb{C}^*/(q^\delta)^\mathbb{Z}$, for $q = e^{2\pi i t}$. Take $\Delta^*$ small enough, so that $E_i$ has no complex multiplication for $q \in \Delta^*$. The fiber product of elliptic curves $Y^* = E_1 \times E_2 \times \ldots \times E_g \to \Delta^*$ is defined to be the mirror family for the maximal degeneration in the direction $U$.

While we can define such a mirror family for any 0-cusp $F$, for general $F$, the mirror is not a space, but a gerbe. If the mirror is a space, then there should exist a Lagrangian section for the maximal degeneration in the direction $U$.

2.3.1. the Splitting boundary component. We consider the mirror symmetry for the splitting cusp $F^{(0)}$ first. Let $L$ be the subspace generated by $\{\lambda_1, \ldots, \lambda_g\}$. Therefore $L$ is a Lagrangian section for the fiberation $X \to B$, and we can identify $V$ with $U \oplus V/U$. We use $\{\lambda_i, \mu_j\}$ as a basis for $V$.

**Remark 2.27.** By ([GLO01] proposition 9.6.1), in this case, for each $\tau \in \mathfrak{S}_g$, there exists $\omega_\tau$, such that $(Y_t, \omega_\tau)$ is mirror symmetric to the algebraic pairs $(X_\tau, tE)$ as defined in loc. cit. So our definition of the mirror agrees with the mirror in [GLO01].

A Chern class of a line bundle is given either by a hermitian form $H$ on $\tilde{V}$, or equivalently, by a skew symmetric integral form of $\omega = \Im(H)$ on $\Gamma$. In terms of $\omega$, the Riemann Conditions are

\begin{align}
\omega(J_t x, J_t y) &= \omega(x, y), \\
\omega(J_t x, x) &> 0, \forall x \neq 0.
\end{align}

Assume, with respect to the basis $\{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_g, \tilde{\mu}_1, \ldots, \tilde{\mu}_g\}$ of the lattice $\Gamma$,

$$
\omega = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} \in M(2g, \mathbb{Z}).
$$

Since $E_i$ are elliptic curves without complex multiplication and $\Im(t) > 0$, by the Riemann conditions, we have

- $Q_1 = Q_4 = 0$;
- $Q_2 \delta = -\delta Q_3$;
- $Q_2 = -Q_3^T$;
- $Q_3 > 0$.

Let $Q = Q_3 \delta^{-1}$ be a positive definite symmetric matrix.

$$
\omega = \begin{pmatrix} 0 & -\delta Q \\ Q \delta & 0 \end{pmatrix},
$$
\[ \omega = \sum_{i,j=1}^{9} -\delta_i Q_{ij} \, d\bar{x}_i \wedge d\bar{y}_j, \]

where \( \{\bar{x}_i, \bar{y}_j\} \) are the dual of \( \{\lambda_i, \mu_j\} \), and \( d\bar{x}_i \wedge d\bar{y}_j = d\bar{x}_i \otimes d\bar{y}_j - d\bar{y}_j \otimes d\bar{x}_i \).

It follows that \( X = \mathbb{Z} \mu_i \) is a maximally isotropic subgroup with respect to any \( \omega \). Define \( \phi : Y \rightarrow \bar{Y} \) by \( \lambda_i \mapsto \delta_i \mu_i \) and identify \( Y \) and \( \bar{Y} \) by \( \phi \). \( \omega \) descends to a bilinear form \( X \times Y \rightarrow \mathbb{Z} \). This bilinear form is symmetric and positive definite over \( U_{\mathbb{R}} \), and thus corresponds to a positive quadratic form \( Q \). We see that the Kähler cone \( \mathcal{K}(Y_t) \) is the set of all positive quadratic forms on \( X \). Therefore, we have the mirror map \( \mathcal{C}(0) \cong \mathcal{C}(X) \cong \mathcal{K}(Y_t) \).

Remark 2.28. We can check that the mirror map \( \mathcal{C}(X) \rightarrow \mathcal{K}(Y_t), \) identifying \( Q \), is given by the Fourier transform \( \text{Four}(L_\phi, \mathcal{C}) \) defined in [Pol03], where \( L_\phi \) is the Lagrangian determined by the graph of \( \dot{\phi} \).

Now we study the automorphism group of \( Y_t \). Let \( u \in \text{GL}(g, \mathbb{C}) \) be the representation of an automorphism under the basis \( \{\bar{e}_i\} \). Under the basis \( \{\lambda_i, \mu_j\} \), the matrix is

\[
\begin{pmatrix}
u & 0 \\
0 & \delta u \delta^{-1}
\end{pmatrix}.
\]

Therefore \( u \in \text{GL}(\delta) \) under the basis of \( Y \). The group of automorphisms of \( Y_t \) is \( \text{GL}(X, Y) \). The action on the quadratic form \( Q \) is \( Q' = (u^T)^{-1} Q u^{-1} \). It is identified with the action of \( \text{GL}(X, Y) \) on \( \mathcal{C}(X) \).

We want to construct the relative minimal models \( \mathcal{Y} / \Delta \) of the mirror family \( \mathcal{Y}^* / \Delta^* \). By relative, we mean every map is over \( \Delta \). A projective morphism \( \bar{\pi} : \mathcal{Y} \rightarrow \Delta \) is a relative minimal model, if \( \mathcal{Y} \) is \( \mathbb{Q} \)-factorial, \( \mathbb{Q} \)-Gorestein, terminal, and \( K_{\mathcal{Y}} \) is \( \bar{\pi} \)-nef. It is well known that the relative minimal model for \( \mathcal{E}_t \rightarrow \Delta^* \) is a Tate curve, whose relative complete model is described by a fan \( \Sigma_t \) over \( \mathbb{R}_{\geq 0} \). The product will give the relative complete model for \( \mathcal{Y}^* \rightarrow \Delta^* \). To get a relative minimal model, we need to take a partial resolution of singularities. This has been worked out in ([Mum72] Sect. 6).

To use Mumford’s construction, we replace \( \Delta^* \) by a complete discrete valuation ring \( (R, \mathfrak{m}) \). Let \( K \) be the quotient field, and \( \kappa = R / \mathfrak{m} \) the residue field. The base is \( S = \text{Spec} \, R \). The closed point is \( s = \text{Spec} \, \kappa \), and the generic point is \( \eta = \text{Spec} \, K \). Consider \( X_\mathbb{R} = U^* \) as an affine plane of height 1 in the vector space \( X_\mathbb{R} \).

**Definition 2.29.** An integral paving \( \mathcal{P} \) of \( U^* \) is called \( y \)-invariant, if \( \mathcal{P} \) is invariant under the translation action of \( \bar{Y} \) on \( U^* \). Assume \( \mathcal{P} \) is further a triangulation. It is called minimal, if each vertex of \( X \) is a cell of \( \mathcal{P} \), i.e. \( X \subset \mathcal{P} \).
By (CLSI Exercise 8.2.14. & Proposition 11.4.12.), the minimal condition is necessary and sufficient for the relative complete model to be normal, \( \mathbb{Q} \)-factorial, \( \mathbb{Q} \)-Gorenstein, and terminal. Fix a \( Y \)-invariant integral paving \( \mathcal{P} \). For each cell \( \sigma \in \mathcal{P} \), construct the cone \( C(\sigma) \) in \( \mathbb{X}_{\mathbb{R}} \). The collection \( \{ C(\sigma) \}_{\sigma \in \mathcal{P}} \) form a fan denoted by \( \Sigma_{\mathcal{P}} \). Denote the infinite toric embedding \( X_{\Sigma_{\mathcal{P}}} \) by \( \tilde{\mathcal{Y}}_{\mathcal{P}} \). Since the fan \( \Sigma_{\mathcal{P}} \) contains a basis of \( \mathbb{X} \), \( \tilde{\mathcal{Y}}_{\mathcal{P}} \) is simply connected as a complex analytic space. An embedding of \( \tilde{\mathbb{T}} := \mathbb{G}_{m}/S \) into \( \tilde{\mathcal{Y}}_{\mathcal{P}} \) is given by the vertical ray \( C(\{0\}) \) for \( 0 \in \mathbb{X} \). To define a line bundle over \( \tilde{\mathcal{Y}}_{\mathcal{P}} \), with actions of \( Y \) and \( \tilde{\mathbb{T}} \), we need additional data: a piecewise affine function \( \varphi \) on \( U^* \) such that

1) For each cell \( \sigma \in \mathcal{P} \), \( \varphi|_{\sigma} \) is affine. We say that \( \varphi \) is compatible with \( \mathcal{P} \).

2) \( \varphi \) is strictly convex. That means the lower boundary of the graph of \( \varphi \) gives \( \mathcal{P} \).

3) \( \varphi \) is \( Y \)-quasiperiodic.

4) \( \varphi \) is integral. That means for each \( \sigma \in \mathcal{P} \), \( \varphi|_{\sigma} \) is an element in \( \text{Aff}(X, \mathbb{Z}) \).

We study quasiperiodic functions in Appendix \[.\] See Definition \[\text{A.1}\] The set of functions which satisfy 1), 2) and 3) is denoted by \( \text{CPA}^Y(\mathcal{P}, \mathbb{R}) \). The set of functions which satisfy all 4 conditions is denoted by \( \text{CPA}^Y(\mathcal{P}, \mathbb{Z}) \).

By Lemma \[\text{A.3}\] in Appendix \[A\] \( \varphi \) should satisfy

\[
\varphi(y + \phi(\lambda)) = \varphi(y) + A(\lambda) + \langle y, \hat{\phi}(\lambda) \rangle,
\]

where \( A \) is a quadratic function on \( Y \) such that

\[
\langle \phi(\lambda), \hat{\phi}(\mu) \rangle = A(\lambda + \mu) - A(\lambda) - A(\mu).
\]

If \( \varphi \in \text{CPA}^Y(\mathcal{P}, \mathbb{Z}) \), it defines an ample line bundle \( \mathcal{L} \) on \( X_{\Sigma_{\mathcal{P}}} \), with actions of \( Y \) and \( \tilde{\mathbb{T}} \). The data \(( \mathcal{P}, \varphi, \phi, \hat{\phi}, A )\) gives rise to a relative complete model for the family \( \mathcal{Y}^* \to \text{Spec} \mathbb{K} \).

We claim that an element in \( \text{Pic} \mathcal{Y}_{\mathcal{P}} \) is represented by such an element \( \varphi \), and the restriction map \( r : \text{Pic} \mathcal{Y}_{\mathcal{P}} \to \text{NS} \mathcal{Y} \) is given by \( r(\varphi) \mapsto Q = \hat{\phi} \circ \phi^{-1} \). The relation between the data is more explicit, if we use the discrete Legendra transform, which, according to Gross–Siebert program, is the mirror symmetry on the tropical level.

Define \( B := U^*/\phi(Y) \), and call it the dual intersection complex. Define \( \tilde{B} := U/\hat{\phi}(Y) \), and call it the intersection complex for \( \mathcal{Y}^* \). We need to perform the discrete Legendre transform of \((B, \mathcal{P}, \varphi)\) to get \((\tilde{B}, \mathcal{P}, \hat{\varphi})\). See [Gro12] for the local definition. Since there exists the universal cover \( U \), we have a global construction. Take the tangent cone of \( \varphi \) at each vertex \((x, \varphi(x)) \) for \( x \in \mathcal{P} \). Since the subset above the graph of \( \varphi \) is a polyhedron \( Q_\varphi \), the dual cones in \( U \times \mathbb{R} \) form a fan in \( U \times \mathbb{R} \). Take the intersection of the fan with \( U \times \{1\} \), and get a paving \( \mathcal{P} \). \( \mathcal{P} \) is \( Y \)-invariant, and descends to a paving of \( \tilde{B} \).
**Definition 2.30.** Define the function over $U$,

\[ \tilde{\varphi}(\mu) := - \inf_{y \in U^*} \{ \varphi(y) + \langle y, \mu \rangle \}. \]

**Lemma 2.31.**

\[ \tilde{\varphi}(\mu + \tilde{\phi}(\lambda)) = \tilde{\varphi}(\mu) + \langle \phi(\lambda), \mu \rangle + A(\lambda), \forall \lambda \in Y. \]

**Proof.**

\[
\begin{align*}
\tilde{\varphi}(\mu + \tilde{\phi}(\lambda)) &= - \inf_{y \in U^*} \{ \varphi(y) + \langle y, \mu + \tilde{\phi}(\lambda) \rangle \} \\
&= - \inf_{y \in U^*} \{ \varphi(y) + \langle y, \tilde{\phi}(\lambda) \rangle + A(\lambda) + \langle y, \mu \rangle - A(\lambda) \} \\
&= - \inf_{y \in U^*} \{ \varphi(y + \phi(\lambda)) + \langle y + \phi(\lambda), \mu \rangle + \langle \phi(\lambda), \mu \rangle + A(\lambda) \} \\
&= \tilde{\varphi}(\mu) + \langle \phi(\lambda), \mu \rangle + A(\lambda).
\end{align*}
\]

By toric geometry, the line bundle $\mathcal{L}$ corresponds to a polyhedron in $X^*_R$. The vertices of the polyhedron are $(\mu, \tilde{\varphi}(\mu))$. Therefore, we have

**Corollary 2.32.** The $Y$-action on the homogeneous coordinate ring is given as follows. Define $\zeta_\mu := X^\mu q^{\tilde{\varphi}(\mu)} \theta$ for $\mu \in X^*$ to be a monomial section of $\mathcal{L}$. Then

\[ S^a_\lambda(\zeta_\mu) = \zeta_{\mu + \tilde{\phi}(\lambda)} = X^{\mu + \tilde{\phi}(\lambda)} q^{\tilde{\varphi}(\mu + \tilde{\phi}(\lambda))} \theta = X^\mu(\lambda) q^{A(\lambda)} X^{\tilde{\phi}(\lambda)} \zeta_\mu. \]

**Corollary 2.33.** If $\psi = \varphi - \varphi'$ is an integral affine function, then the induced line bundles $\mathcal{L}$ and $\mathcal{L}'$ by $\varphi$ and $\varphi'$ are isomorphic on $Y$.

**Proof.** $\psi$ defines an isomorphism between the homogeneous coordinate rings of $X_{\Sigma_{\varphi'}}$. If $\psi(y) = \langle y, a \rangle + h$, then $\tilde{\varphi}(\mu) = \tilde{\varphi}'(\mu + a) - h$. The isomorphism is a translations by $- (a, h)$. It is $Y$-equivariant, and induces an isomorphism between $\mathcal{L}$ and $\mathcal{L}'$.

A real valued function $\psi$ on $X$ is called a $Y$-quasiperiodic, if it is a sum of a quadratic functions and a $Y$-periodic function ($\tilde{\varphi} \subset X$). The set of $Y$-quasiperiodic functions over $X$ is a vector space of finite dimension. The quotient of this vector space by $\text{Aff}$ is identified with $\Gamma(B, \mathcal{P}, \mathcal{P}A/\text{Aff})$, where $\mathcal{P}$ is a minimal triangulation, $\mathcal{P}A$ is the sheaf of $\mathcal{P}$-piecewise affine functions, and $\text{Aff}$ is the sheaf of affine functions. Fix a $Y$-invariant integral triangulation $\mathcal{T}$ of $X_R$, for any function $\psi$ over $X$, we can define a piecewise affine function $g_{\psi, \mathcal{T}}$. For each $\alpha \in X$, if it is an element of $\mathcal{T}$, define $g_{\psi, \mathcal{T}}(\alpha) = \psi(\alpha)$. Then $g_{\psi, \mathcal{T}}$ is obtained by affine interpolation over each simplex of $\mathcal{T}$.

**Definition 2.34.** Let $\mathcal{T}$ be a $Y$-invariant integral triangulation. We shall denote by $C^Y(\mathcal{T})$ the cone consisting of $Y$-quasiperiodic functions $\psi$ over $X$ with the following two properties:

a) The function $g_{\psi, \mathcal{T}}$ is convex.

b) For any $\alpha \in X$ but not a vertex of any simplex from $\mathcal{T}$, we have $g_{\psi, \mathcal{T}}(\alpha) \leq \psi(\alpha)$. 

Define $C^Y(\mathcal{P})$ to be $\tilde{C}^Y(\mathcal{P})/\text{Aff}$, a cone inside $\Gamma(B,PA/\text{Aff})$. If $\mathcal{P}$ is a minimal triangulation, $C^Y(\mathcal{P}) = CPA^Y(\mathcal{P},R)/\text{Aff}$. It follows from Corollary 2.33 that we have a map $p : C^Y(\mathcal{P}) \to K(\mathcal{P})$. If we use complex geometry, we can talk about the universal covering map $\Upsilon : \tilde{Y}_{\mathcal{P}} \to Y_{\mathcal{P}}$. Since the Cartier divisors on $\tilde{Y}_{\mathcal{P}}$ are described by piecewise affine functions, the pull back of line bundles from $Y_{\mathcal{P}}$ to $\tilde{Y}_{\mathcal{P}}$ defines $\Upsilon^* : K(Y) \to C^Y(\mathcal{P})$, such that $\Upsilon^* \circ p = \text{Id}$. Therefore, the map $p : C^Y(\mathcal{P}) \to K(\mathcal{P})$ is an injection.

**Corollary 2.35.** Regard $C^Y(\mathcal{P})$ as a subset of $K(\mathcal{P})$. The restriction map $r : C^Y(\mathcal{P}) \to \text{NS}(Y_i)$ is given by,

$$
(12) \quad \varphi \mapsto \omega = Q = \hat{\varphi} \circ \phi^{-1}.
$$

**Proof.** It follows from the computation in Corollary 2.32.

Assume that the set of the irreducible components of $Y_s$ is indexed by $I = B(Z)$ and $|I| = \prod_i \delta_i = d$. Each irreducible component is a prime divisor $D_i$, $i \in I$.

**Lemma 2.36.** We have the following exact sequence,

$$
(13) \quad 0 \longrightarrow \mathbb{Z}^d/\mathbb{Z} \longrightarrow \text{Pic}(\mathcal{P}) \longrightarrow \text{NS}(Y_i) \longrightarrow 0.
$$

The mod $\mathbb{Z}$ is because $\sum_{i \in I} D_i = \pi^{-1}(0) \equiv 0$.

In particular, $p : C^Y(\mathcal{P}) \to K(\mathcal{P})$ is a bijection, and $\text{Pic}(\mathcal{P})$ is identified with the space $\Gamma(B,PA/\text{Aff})$.

**Proof.** For the exactness of the sequence (13), the only thing left to check is that $\sum_{i \in I} D_i = 0$ is the only relation in $\text{Pic}(\mathcal{P})$ for $\{D_i\}$. If $\mathcal{P}$ is a surface, this follows from ([Sha13](#)) Theorem 4.14. For the general case, assume we have a relation $D_r$. Intersect $D_r$ with a generic hypersurface $S'$ that is flat over $\Delta$. Then, by induction, $D_r \cdot S'$ is a multiple of $\sum_{i \in I} D_i \cdot S'$. It follows that $D_r$ is a multiple of $\sum_{i \in I} D_i$.

Since the sequence (13) is exact, $p \circ \Upsilon^* = \text{Id}$, and $\text{Pic}(\mathcal{P}) \cong \Gamma(B,PA/\text{Aff})$ by the 5-lemma.

The relative minimal model $\tilde{\pi} : \mathcal{P} \to \Delta$ is not unique, but any two models are isomorphic up to codimension 1, and are connected by a sequence of flops. See [Kaw08](#). We need to study the Mori fan of the minimal model $\mathcal{P}$. This fan is independent of the choice of the relative minimal model. The support is the rational closure of the big cone.

**Lemma 2.37.** The pseudo-effective cone $\text{Eff}(\mathcal{P})$ is the closure of $r^{-1}(K(Y_i))$.

**Proof.** The restriction $r(D)$ of an effective divisor $D$ is effective. Since $Y_i$ is an abelian variety, $r(D)$ is in the closure of $K(Y_i)$. Therefore, $D$ is in the closure of $r^{-1}(K(Y_i))$.

On the other hand, $\ker(r)$ are all effective, by adding $\sum_{i \in I} D_i$. Moreover, for any rational class $Q \in K(Y_i)$, it is easy to construct an effective divisor $\varphi_Q$ such that $r(\varphi_Q) = Q$. For example, define $A_X = 1/2Q|_X$ and make it integral by multiplying a positive integer. $A_X$
determines a Delaunay decomposition of \((U^*, X)\). Define \(\varphi_Q\) to be the affine interpolation of \(A_X\).

**Proposition 2.38.** Every Mori chamber is of the form \(C^Y(\mathcal{T})\) for some triangulation \(\mathcal{T}\). Every relative minimal model is isomorphic to \(Y_{\mathcal{P}'}\) for some minimal triangulation \(\mathcal{P}'\).

**Proof.** Fix \(\mathcal{P}\) a minimal triangulation. For each \(Y\)-invariant triangulation \(\mathcal{T}\), the rational map \(Y_P/\mathcal{P}\) is a contraction. It is a small contraction, if and only if \(\mathcal{T}\) is also minimal.

We claim that each \(C^Y(\mathcal{T})\) is contained in one Mori chamber. For a \(Q\)-Cartier \(D \in C^Y(\mathcal{T})\) corresponding to a rational function \(\psi_D\) over \(X\), decompose \(\psi_D = \psi_A + \psi_E\), where \(\psi_A = g_{\psi, \mathcal{T}}\) and \(\psi_E = \psi_D - g_{\psi, \mathcal{T}}\). Since \(g_{\psi, \mathcal{T}}\) is strictly convex, \(A\) is ample on \(Y_{\mathcal{T}}\). Then \(D\) defines the rational map \(f_D : Y_P/\mathcal{P}\) and \(D = f_D^*A + E\) for \(E\) exceptional. It proves the claim.

By Lemma 2.36 and Lemma 2.37, we can identify \(\text{Eff}(Y_P)\) with the closed cone \(C^+\) in \(\Gamma(B, P_A \text{Aff})\), where the associated quadratic form \(Q\) is semi-positive definite. Denote the interior of \(C^+\) by \(C^+\), which corresponds to the big cone. The support of the Mori fan is the rational closure of \(C^+\), denoted by \(C^{rc}\). However, by the same argument as in ([GKZ94] Chap.7, Proposition 1.5), the cones \(C^Y(\mathcal{T})\) already form a fan supported on \(C^{rc}\).

Therefore, each cone in the Mori fan is of the form \(C^Y(\mathcal{T})\), and the contraction is small, if and only if \(\mathcal{T}\) is minimal.

In the case of principally polarized abelian varieties, we have \(\text{Pic}(Y) \cong \text{NS}(Y_t)\) for any minimal model \(Y\).

**Theorem 2.39.** Let \(Y^{0, *} / \Delta^*\) be the mirror family for the principally polarized abelian varieties. For any relative minimal model \(Y / \Delta\), we have \(\text{Eff}(Y) = C(X)\). Moreover, the Mori fan of \(Y\) agrees with the second Voronoi fan.

In general, \(\text{Eff}(Y)\) is much bigger than \(C(X)\). We need a section to get a canonical fan on \(C(X)^{rc}\).

**Proposition 2.40.** For any irreducible component \(D_\alpha\) (for \(\alpha \in I\)) of \(Y_s\), the complement \(Y_s \setminus D_\alpha\) is contractible in \(Y\), and the contraction is denoted by \(p_\alpha : Y \to Y_\alpha\). We call \(Y_\alpha\) a cusp model. It is not unique, but they are all isomorphic up to codimension 1. They are \(\mathbb{Q}\)-factorial, normal, and Gorenstein.

**Proof.** We construct the cusp model \(Y_\alpha\) directly by the Mumford’s construction. Fix \(\alpha\), consider the lattice \(\alpha + Y\) in \(U^*\). Construct a fan from the cones over a Delaunay decomposition \(\mathcal{P}_D\) with respect to \(\alpha + Y\). Take a \(Y\)-quasiperiodic, convex, piecewise affine function \(\varphi\), we can get a relative complete model as before. By Mumford’s construction, we get one of the models \(Y_\alpha\).

By a similar argument as Proposition 2.38, the Mori fan of \(Y_\alpha\) is identified with the second Voronoi fan with respect to the lattice \(\alpha + Y\). Furthermore, since the central fiber is irreducible, the restriction map \(\text{Pic}(Y_\alpha) \to \text{NS}(Y_t)\) is an isomorphism. The Mori fans are
identified in $\text{NS}(Y_t)$ because the second Voronoi fan doesn’t depend on the translation of the lattice on $U^\ast$.

Now fix a cusp $\alpha$. Consider the common Mori fan in $\text{Pic}(Y'_\alpha)$. Each cone is an ample cone $K(Y'_\alpha) \to \text{Pic}(Y)$

That means, on each ample cone, we just pull back the Cartier divisor $\sigma_\alpha$ is not linear, but piecewise linear and convex. Regard the function $\sigma_\alpha$ as a piecewise linear section $\text{NS}(Y_t) \to \text{Pic}(Y)$. Take the average on $\text{NS}(Y_t)$, and define

(14) $\sigma = \frac{1}{d} \sum_\alpha \sigma_\alpha$.

**Proposition 2.41.** $\sigma$ is linear.

*Proof.* Consider the bending parameters of $\sigma$. Since all different linear pieces of $\sigma$ are sections of $r$, all bending parameters live in the kernel $\ker(r) = \mathbb{Z}^d/\mathbb{Z}$. On the other hand, $X$ is acting on $X$ by translation. It induces an action of $X/Y$ on $\text{Pic}(Y)$ since elements of $\text{Pic}(Y)$ are functions on $X$. The set $\{\sigma_\alpha\}$ is a torsor under the action of $X/Y$. It follows that the image of $\sigma$ is invariant under $X/Y$. Thus the bending parameters are also $X/Y$-invariant. It implies that all the coefficients are the same, and the bending parameters are $0$ in $\mathbb{Z}^d/\mathbb{Z}$.

Since $\sigma$ is a linear section, it is easy to compute that $\sigma : \text{NS}(Y_t) \to \text{Pic}(Y)$ is

$\sigma : Q \mapsto \varphi = \text{affine interpolation of } 1/2Q|_X$.

The image of $\sigma$ is characterized by being invariant under the action of $X/Y$. We denote it by $\text{Pic}^X(Y)$.

**Remark 2.42.** This choice of degeneration data for higher polarizations has been considered in [ABH02] as well as in [Nak10].

Identify $\text{Pic}^X(Y)$ with the subspace of $X$-quasiperiodic functions in $\text{Pic}(Y) = \Gamma(B, \mathcal{P}A/\mathcal{A}f)$, i.e., the sections are the pull-backs from $U^\ast/X$. Consider the Mori fan of $\text{Pic}(Y)$, and pull it back through $\sigma$. Each cone is of the form $\sigma^{-1}(C(\mathcal{P})) = \text{Pic}^X(Y) \cap C(\mathcal{P})$, and is denoted by $C(\mathcal{P})$. Since every element in $C(\mathcal{P})$ is $X$-quasiperiodic, it can be identified with the associated quadratic form. Therefore, the pull back fan is the second Voronoi fan with respect to $X$ for $C(X)$. This is the fan we are going to use for the toroidal compactification, and is denoted by $\Sigma(X)$. 

2.3.2. Non-splitting boundary components. Now we consider the general case. Assume \( \phi: Y \to X \) is of type \( \mathfrak{d} \), the mirror family is \( Y_t = E_1 \times E_2 \times \ldots \times E_g \), where \( E_i = C/(d_i^{-1}Z + tZ) \). There is no Lagrangian section for the fibration \( X \to B \) but local Lagrangian sections. Therefore, if we use Fourier–Mukai transform to define the mirror map, the images are twisted sheaves by a gerbe. However, the only influence here is that \( \mathcal{P}(X) \) is not the whole automorphism group of \( \mathcal{Y}^* / \Delta^* \). The rest goes the same as in the splitting case.

In sum, if \( F \) is a 0-cusp, we get a canonical fan supported on \( \mathcal{C}(X)^c \), which is the second Voronoi fan with respect to the lattice \( X \). If \( F_\xi \) is an arbitrary cusp, and \( F_\xi > F \) for \( F \) a 0-cusp, we need \( \Sigma(F_\xi) = \Sigma(F) \cap \mathcal{P}(F) \) because of c) in ([HKW93] Definition 3.66). Therefore, \( \Sigma(F) \xi \) is the second Voronoi fan for \( X_{\xi,R} \) with respect to \( X_\xi \).

**Definition 2.43.** For each cusp \( F \), we define the fan \( \Sigma(F) \) (or \( \Sigma(X) \)) to be second Voronoi fan with respect to \( X \), supported on \( \mathcal{C}(X)^c \). Define \( \Sigma := \{ \Sigma(F) \} \).

3. The Compactification of the Moduli of Polarized Abelian Varieties

3.1. Toroidal Compactifications.

**Proposition 3.1.** The collection of fans \( \bar{\Sigma} = \{ \Sigma(F) \} \) in Definition 2.43 is an admissible collection.

**Proof.** It suffices to check the conditions in ([HKW93] Definition 3.66). For a), we check the conditions in ([HKW93] Definition 3.61), and prove that \( \Sigma(F) \) is an admissible fan for every \( F \). Recall that \( \Sigma(F) \) is an admissible fan with respect to \( \text{GL}(X) \) by [Nam80] Theorem 9.9. Since \( \mathcal{P}(F) \subset \text{GL}(X,Y) \subset \text{GL}(X) \) is of finite index (Proposition 2.25), \( \Sigma(F) \) is also admissible with respect to \( \mathcal{P}(F) \).

For b). If \( M \in \Gamma(\delta) \), \( M : \Gamma^2U \to \Gamma^2U' \) is induced from the isomorphism \( M^{-1}|_U : U \to U' \). And \( M(\Lambda \cap U) = \Lambda \cap U' \). So the dual map \( (M^{-1}|_U)^* : X'_R \to X_R \) maps \( X \) onto \( X' \). Therefore \( M : \mathcal{C}(X) \to \mathcal{C}(X') \) maps the second Voronoi fan \( \Sigma(X) \) to \( \Sigma(X') \).

For c). If \( F' > F \), we have the quotient \( q_R : X_R \to X'_R \) induced from a quotient map of the lattices \( q : X \to X' \). The pull back \( \Gamma^2U' \to \Gamma^2U \) identifies \( \mathcal{C}(X') \) with the positive semi-definite quadratic forms with nullspaces \( \ker(q_R) \). Let \( \mathcal{P}_{D'} \) be a Delaunay decomposition of \( X'R \) and \( \sigma' \in \Sigma(F') \) be the cone associated to \( \mathcal{P}_{D'} \). The pull back \( \mathcal{P} = q_{R}^{-1}(\mathcal{P}_{D'}) \) is a Delaunay decomposition of \( X_R \). (The cells are infinite in the direction of \( \ker(q_{R}) \).) The cone \( \sigma \) associated to \( \mathcal{P} \) is a cone in \( \Sigma(F) \), and it contains \( \sigma' \). On the other hand, the supports of \( \Sigma(F') \) and \( \Sigma(F) \cap \Gamma^2U' \) are the same. Therefore \( \Sigma(F') = \Sigma(F) \cap \Gamma^2U' \). \( \square \)

**Theorem 3.2.** Over \( C \), we have a toroidal compactification \( \overline{\mathcal{A}}_{\Sigma} \) of \( \mathcal{A}_{g,\mathfrak{d}} \). Furthermore, \( \overline{\mathcal{A}}_{\Sigma} \) is projective.

\(^9\)For the proof, see part c) of the proof of Theorem 3.1 below.
Proof. The construction of $\overline{\mathfrak{A}}_{\Sigma}$ for the admissible collection of fans $\tilde{\Sigma}$ is ([HKW93] Theorem 3.82), or ([AMRT10] Theorem 5.2). To prove that it is projective, we apply Tai’s criterion ([FC90] Definition 2.4) or ([AMRT10] Chap. IV Definition 2.1, Corollary 2.3). For each cone $\mathcal{C}(F)$, the fan $\Sigma(F)$ is the second Voronoi fan with respect to $X$, we can use the polarization function provided by $\Sigma(X_{R/X})$ in [Ale02]. □

Following [FC90], we are going to construct an arithmetic toroidal compactification from the admissible collection of fans $\tilde{\Sigma}$. In order to avoid the reduction of bad primes, we work over $k = \mathbb{Z}[1/d]$. For some technical reasons, we have to work over $k = \mathbb{Z}[1/d, \zeta_M]$. Over $k$, the polarizations are separable and the stack $\mathcal{A}g,\delta$ is a connected component of $\mathcal{A}g,d$.

Although the algebraic stack $\mathcal{A}FCg,\delta$ constructed in [FC90] admits a family over it, the family is not proper, but a semiabelian group scheme. Basically, we replace the families of semiabelian schemes in [FC90] by the AN families defined below.

3.2. AN Families.

3.2.1. The AN Construction. For this section, $k = \mathbb{Z}$. Recall the modified Mumford’s construction in [AN99]. Fix a base scheme $S = \text{Spec } R$, for $R$ a Noetherian, excellent, and normal integral domain, complete with respect to an ideal $I = \sqrt{I}$. Denote the residue ring $R/I$ by $\kappa$, and the field of fractions by $K$. The closed subscheme is denoted by $S_0 = \text{Spec } \kappa$, and the generic point by $\eta = \text{Spec } K$. Recall the categories $\text{DEG}$ and $\text{DD}$ from [FC90]. The objects of the category $\text{DEG}_{\text{ample}}$ are pairs $(G, L)$, where $G$ is semiabelian over $S$, with $G_\eta$ abelian over $K$, and $L$ is an invertible sheaf on $G$, with $L_\eta$ ample. The morphisms are group morphisms that respect the invertible sheaves. An object of $\text{DD}_{\text{ample}}$ is the following degeneration data:

1. An abelian scheme $A/S$ of relative dimension $g'$, a split torus $T/S$ defined by the character group $X \cong \mathbb{Z}^r$, with $r = g - g'$, and a semiabelian group scheme $\tilde{G}$ defined by $c : X \to A^t$. We use the same letter to denote the group $X$ and the correspondent constant sheaf $X$ on $S$.

\[ 1 \longrightarrow T \longrightarrow \tilde{G} \overset{\pi}{\longrightarrow} A \longrightarrow 0. \]

2. A rank $r$ free abelian group $Y \cong \mathbb{Z}^r$ and the constant sheaf $Y$ over $S$.

3. A homomorphism between group schemes $c^t : Y \to A$. This is equivalent to an extension

\[ 1 \longrightarrow T^t \longrightarrow \tilde{G}^t \overset{\pi^t}{\longrightarrow} A^t \longrightarrow 0. \]

4. An injection $\phi : Y \to X$ of type $d_1$.

5. A homomorphism $\iota : Y_\eta \to \tilde{G}_\eta$ over $S_\eta$ lying over $c^t_\eta$. This is equivalent to a trivialization of the biextension $\tau : 1_{Y \times X} \to (c^t \times c)^* P_{A,\eta}^{-1}$. Here $P_A$ is the Poincaré sheaf on $A \times A^t$ which has a canonical biextension structure. We require that the induced trivialization $\tau \circ (\text{Id} \times \phi)$ of $(c^t \times c \circ \phi)^* P_{A,\eta}^{-1}$ over $Y \times Y$ is symmetric. The trivialization $\tau$ is required to satisfy the following positivity condition: $\tau(\lambda, \phi(\lambda))$ for all $\lambda$ extends to a section of $P_{A,\eta}^{-1}$ on $A \times S A^t$, and is $0$ modulo $I$ if $\lambda \neq 0$. 


(6) An ample sheaf \( \mathcal{M} \) on \( A \) inducing a polarization \( \lambda_A : A \to A' \) of type \( \phi' \) such that 
\[ \lambda_A c' = c \phi. \] This is equivalent to giving a \( T \)-linearized sheaf \( \mathcal{L} = \pi^* \mathcal{M} \) on \( \mathcal{G}. \)

(7) An action of \( Y \) on \( \mathcal{L}_\eta \) compatible with \( \phi \). This is equivalent to a cubical trivialization \( \psi : 1_Y \to (c')^* \mathcal{M}_\eta^{-1} \), which is compatible with the trivialization \( \tau \circ (\Id \times \phi) \).

Denote the rigidified line bundle \( c(\alpha) \) by \( \mathcal{O}_\alpha \), and the rigidified line bundle \( \mathcal{M} \otimes \mathcal{O}_\alpha \) by \( \mathcal{M}_\alpha \). For any \( S \)-point \( a : S \to A \), and any line bundle \( \mathcal{L} \) on \( A \), the pull back \( a^* \mathcal{L} \) is denoted by \( \mathcal{L}(a) \). For any \( \lambda \in Y \), \( \alpha \in X \), \( \tau(\lambda, \alpha) \in \mathcal{O}_\alpha(c'(\lambda))_\eta = \mathcal{P}_A^{-1}(c'(\lambda), c(\alpha))_\eta \), and \( \psi(\lambda) \in \mathcal{M}(c'(\lambda))^{-1}_\eta \) are \( K \)-sections. Therefore the data \( \tau \) and \( \psi \) gives trivializations 
\[ \psi(\lambda)^d(\tau(\lambda, \alpha)) : (\mathcal{M}^d \otimes \mathcal{O}_\alpha)_\eta^{-1}(c'(\lambda)) \cong \mathcal{O}_K \] for every \( \lambda \in Y \), \( \alpha \in X \). By (\textit{Ols08} Lemma 5.2.2.), the data \( \psi(\lambda)^d \tau(\lambda, \alpha) \) is equivalent to an isomorphism of line bundles 
\[ \psi(\lambda)^d \tau(\lambda, \alpha) : T_{c'(\lambda)}(\mathcal{M}^d \otimes \mathcal{O}_\alpha)_\eta \to \mathcal{M}^d \otimes \mathcal{O}_{\alpha+d(\phi(\lambda))} \eta. \]

By the property of biextensions, (\textit{Ols08} Proposition 2.2.13), it defines an action of the group \( Y \) on the graded algebra
\[ S = \left( \prod_{d \geq 0} \left( \bigoplus_{\alpha \in X} \mathcal{M}^d \mathcal{O}_\alpha \right) \otimes_R \mathcal{O}_K \right), \]

denoted by \( S^Y_\lambda \).

Now assume \( P \) is a toric monoid, \( P = \sigma'_P \cap P^{gp} \), and \( \alpha : P \to R \) is a prelog structure which maps the toric maximal ideal \( P \setminus \{0\} \) to \( I \). Fix data: an integral, \( Y \)-quasiperiodic, \( P \)-convex piecewise affine function \( \varphi : X_R \to P^{gp}_R \) decided by a collection of bending parameters \( \{ p_x \in P \setminus \{0\} \} \). For simplicity, the composition \( \alpha \circ \varphi : X_R \to R \) is also denoted by \( \varphi \). Suppose the associated quadratic form \( Q \) is positive definite, i.e. \( Q(x) \in \sigma'_P \setminus \{0\} \) for \( x \neq 0 \), and the associated paving \( \mathcal{P} \) is bounded. Construct the following functions 
\[ a_t : Y \to \mathbb{G}_m(K) \text{ and } b_t : Y \times X \to \mathbb{G}_m(K) \] by
\[ (15) \quad a_t(\lambda) = X^{\varphi(\phi(\lambda))}, \quad \forall \lambda \in Y; \]
\[ (16) \quad X^{\varphi(\alpha + \phi(\lambda))} = a_t(\lambda) b_t(\lambda, \alpha) X^{\varphi(\alpha)}, \quad \forall \alpha \in X. \]

By the properties of \( \varphi \), \( a_t \) is quadratic, \( b_t \) is bilinear and \( b_t(\lambda, \phi(\mu)) \) is symmetric on \( Y \times Y \). Recall that, by (\textit{Ols08} 2.2.8), for any biextension over \( Y \times X \), the automorphism is classified by \( \text{Hom}(Y \otimes X, \mathbb{G}_m). \) If we require that the extension is symmetric on \( Y \times \phi(Y) \), then the bilinear form \( b \in \text{Hom}(Y \otimes X, \mathbb{G}_m) \) should satisfy that \( b(\lambda, \phi(\mu)) \) be symmetric. Furthermore, the automorphism of a central extension over \( Y \) is classified by quadratic functions over \( Y \). Fix a biextension (resp, central extension), regard the set of trivializations as a torsor over \( \text{Hom}(Y \times X, \mathbb{G}_m) \) (resp, quadratic forms). For any trivialization \( \tau \) of a biextension, (reps, any trivialization \( \psi \) of a central extension), denote the trivialization obtained by the action of \( b_t^{-1} \) (resp, \( a_t^{-1} \)) by \( b_t^{-1} \tau \) (resp, \( a_t^{-1} \psi \)).

**Definition 3.3.** Fix the data \( b_t, a_t \) obtained from \( \varphi \). We say the combinatorial data \( \varphi \) is compatible with the trivializations \( \tau \) and \( \psi \), if \( b_t^{-1} \tau \) can be extended to a trivialization of
the biextension \((c^! \times c)^*P_{A^{-1}}\) over \(S\), and \(a_t^{-1} \psi\) can be extended to a trivialization of the central extension \((c^!)^*M_{A^{-1}}\) over \(S\).

The data \(\varphi\) gives the choice of embeddings \(M_\alpha \to M_{\alpha, P}\), therefore we can use constructions in [AN99] to construct the family of varieties over \(S\). More explicitly, consider the piecewise linear function \(\tilde{\varphi} : X_R \to P_{R^{GP}}^{\text{gp}}\) and its graph in \(X_R \times P_{R^{GP}}^{\text{gp}}\). The piecewise linear function \(\tilde{\varphi}\) is integral because the bending parameters \(p_\rho\) are all in \(P\). Denote the convex polyhedron \(P\)-above the graph by \(Q_{\tilde{\varphi}}\). Consider the graded \(O_A\)-algebra

\[
\mathcal{R} := \bigoplus_{(\alpha, d, \theta) \in Q_{\tilde{\varphi}}} X^p \otimes O_\alpha \otimes M^d \theta^d.
\]

Since \(\varphi\) and \(\tau, \psi\) are compatible, the action \(S^*_\lambda, (\lambda \in Y)\) is

\[(17) \quad \psi'(\lambda)^d \tau'(\lambda, \alpha) : T_{c^!}^* (X^p \otimes O_\alpha \otimes M^d \theta^d) \cong X^p \otimes O_\alpha \otimes M^d \theta^d.
\]

for \(\psi'(\lambda)^d \tau'(\lambda, \alpha)\) \(R\)-sections of \(M^d \otimes O_\alpha (c^!(\lambda))\). Therefore the \(Y\)-action preserves the subalgebra \(\mathcal{R}\). Let \(\tilde{X}\) denote the scheme \(\text{Proj} \mathcal{R}\). Construct the infinite toric embedding \(X_{\varphi}\) from the convex polyhedron \(Q_{\tilde{\varphi}}\). This is a toric degeneration over the toric variety \(\text{Spec} k[P]\). Denote the pull back of \(X_{\varphi}\) along \(k[P] \to R\) by \(\tilde{P}^r\). As in ([AN99], 3B, 3.22), the total space \(\tilde{X}\) is isomorphic to contracted product \(\tilde{P}^r \times^T \tilde{G}\) over \(S\).

**Lemma 3.4.** Every irreducible component of \(\tilde{X}_0\) is reduced, and proper over \(S_0\).

**Proof.** Use the isomorphism \(\tilde{X} \cong \tilde{P}^r \times^T \tilde{G}\) over \(S\). The total space \(\tilde{X}\) is covered by \(U(\omega) \times^T \tilde{G}\), where \(U(\omega)\) is the affine toric variety corresponding to the vertex \(\omega\) of \(Q_{\tilde{\varphi}}\). The action \(S^*_\lambda\) maps \(U(\omega) \times^T \tilde{G}\) to \(U(\omega + \phi(\lambda)) \times^T \tilde{G}\). Reduce to \(k\), \(\tilde{X}_0\) is the contracted product of \(\tilde{P}^r_0 := \tilde{P}^r \times k[P] \times \tilde{G}_0\). Since \(\varphi\) is integral, \(\tilde{P}^r_0\) is reduced. Since \(P \setminus \{0\}\) is mapped into \(I\), \(\tilde{P}^r_0\) is \(\text{lim} X_{\sigma}\). The inductive limit is over \(\sigma \in \mathcal{P}\). Each irreducible component of \(\tilde{P}^r_0\) is the toric variety \(X_{\sigma}\) associated to the maximal cell \(\sigma \in \mathcal{P}\). As a result, each irreducible component of \(\tilde{X}_0\) is a fiber bundle over \(A_0\) with the fiber \(X_{\sigma}\), and is thus reduced and proper over \(S_0\). \(\square\)

**Lemma 3.5.** The data \((\tilde{X}, O(1), S^*_\lambda, G)\) is a relative complete model as defined in ([FC90], III. Definition 3.1).

**Proof.** The proof is similar to ([AN99], Lemma 3.24). It is not necessary to check the complete condition, since we have shown that every irreducible component of \(\tilde{X}_0\) is proper over \(S_0\). To construct an embedding \(\tilde{G} \to \tilde{X}\), consider the dual fan \(\Sigma(\tilde{\varphi})\) of \(Q_{\tilde{\varphi}}\) in \(X^*_R \times (P^{\text{gp}}_R)^*\). It defines a toric degeneration \(\tilde{P}^r\) over the affine toric variety \(\text{Spec} k[P]\). It follows that we can lift the fan \(\sigma_P\) as a subfan in \(\Sigma(\tilde{\varphi})\). Since \(\sigma_P\) is a fan defined by faces of one rational polyhedral cone, the subfan induces an embedding of the trivial torus bundle \(T\) into \(\tilde{P}^r\). By ([AN99], B 3.22), the embedding \(T \to \tilde{P}^r\) induces the embedding \(\tilde{G} \cong T \times^T \tilde{G} \to \tilde{P}^r \times^T \tilde{G} \cong \tilde{X}\). \(\square\)
As in [AN99], reduce to \( I^n \) for various \( n \in \mathbb{N} \), and take the quotient by \( Y \), then by Grothendieck’s existence theorem (EGA III 5.4.5), we get an algebraic family \( \pi : (\mathcal{X}, \mathcal{L}) \to S \).

3.2.2. convergence problem. Notice that in the construction above, the only place we use the condition that \( P \setminus \{0\} \) is mapped to \( I \) is in the proof of Lemma 3.4. In this case, closed subscheme \( S_0 \) is sent to the 0-strata in the affine toric variety \( \text{Spec} \, k[P] \). However, we may also consider other closed strata.

Suppose \( P = \sigma_P^\vee \cap P^{\text{gp}} \) is a toric monoid obtained from a rational polyhedral cone \( \sigma_P \), and \( \varphi : X_R \to P^{\text{gp}}_R \) is a function as above. Let \( m \) be the maximal toric ideal \( P \setminus \{0\} \), \( J \) be a prime toric ideal of \( P \), and \( F \) be the face \( P \setminus J \). Assume \( F = \tau^\perp \cap \sigma_P^\vee \) for \( \tau \) a face of \( \sigma_P \). Consider the monoid \( P' = P/F \) and the piecewise affine function \( \varphi' \) defined as the composition of \( \varphi \) with \( P^{\text{gp}}_R \to (P')^{\text{gp}}_R \). The associated paving \( \mathcal{P}' \) of \( \varphi' \) is coarser than the associated paving \( \mathcal{P} \) of \( \varphi \). Suppose that \( \mathcal{P}' \) is still a bounded paving. Let \( P_F \) be the localization of \( P \) with respect to the face \( F \), and \( J_F \) be the toric ideal of \( P_F \) generated by \( J \).

**Definition 3.6.** All the notations as above. We call the data \( (P', \varphi', \mathcal{P}') \), the data associated to the face \( F \).

**Lemma 3.7.** Every irreducible component of \( \bar{P}' \times_{k[P]} \text{Spec} \, k[P]/J \) is proper over \( \text{Spec} \, k[P]/J \).

**Proof.** The family \( \bar{P}' \to \text{Spec} \, k[P] \) is an infinite toric degeneration constructed from the infinite piecewise affine function \( \varphi \). The restriction to \( \text{Spec} \, k[P]/J \) is still a toric degeneration. Since \( \text{Spec} \, k[P_F]/J_F \) is open and dense in \( \text{Spec} \, k[P]/J \), and the closure of an irreducible space is irreducible, each irreducible component of \( \bar{P}' \times_{k[P]} \text{Spec} \, k[P]/J \) is a toric degeneration of an irreducible component of \( \bar{P}' \times_{k[P]} \text{Spec} \, k[P]/J \). Therefore, it suffices to prove the statement for \( \bar{P}' \times_{k[P]} \text{Spec} \, k[P_F]/J_F \).

Since \( P \) is toric monoid from a rational polyhedral cone \( \sigma_P^\vee \), we can choose a splitting \( P_F = P'/F^{\text{gp}} \). Consider the function \( \varphi' : X_R \to (P')^{\text{gp}}_R \) and the convex polyhedron \( Q_{\varphi'} \) over the graph. Let \( \bar{P}' \) be the toric embedding associated to the convex polyhedron \( Q_{\varphi'} \). The localization \( P'_F := \bar{P}' \times_{k[P]} \text{Spec} \, k[P_F] \) is isomorphic to \( \bar{P}' \times \text{Spec} \, k[F^{\text{gp}}] \) over \( \text{Spec} \, k[P']/\text{Spec} \, k[F^{\text{gp}}] \). Since \( J \) corresponds to the maximal ideal \( m' = P' \setminus \{0\} \), it reduces to the case \( \bar{P}' \times_{k[P]} \text{Spec} \, k[P']/m' \). Again this is \( \lim_{\longrightarrow \sigma'} \text{Spec} \, k[\sigma'] \) for a maximal cell \( \sigma' \in \mathcal{P}' \), with each irreducible component the proper toric variety \( X_{\sigma'} \) for a maximal cell \( \sigma' \in \mathcal{P}' \).

**Definition 3.8.** For any prime toric ideal \( J \), associate the face \( F \), the quotient monoid \( P' \), the function \( \varphi' : X_R \to (P')^{\text{gp}}_R \), and the paving \( \mathcal{P}' \) as above. A toric prime ideal \( J \) is called admissible for \( \varphi \), if the paving \( \mathcal{P}' \) is bounded.

**Lemma 3.9.** Given a toric ideal \( J \) such that \( J = \bigcap_i J_i \) for each \( J_i \) an admissible prime toric ideal. Then every irreducible component of \( \bar{P}' \times_{k[P]} \text{Spec} \, k[P]/J \) is proper over \( \text{Spec} \, k[P]/J \).
Proof. The irreducible components of $\text{Spec } k[P]/J$ are $\text{Spec } k[P]/J_i$. Each irreducible component of $\tilde{P}^r \times_{k[P]} \text{Spec } k[P]/J$ is contained in the family over some $\text{Spec } k[P]/J_i$ for some $J_i$. It reduces to Lemma 3.7.

The set of all admissible prime toric ideals for $\varphi$ is finite. Take the intersection of all these admissible prime ideals, we get a toric ideal denoted by $I_\varphi$. The correspondent ideal in the ring $k[P]$ is also denoted by $I_\varphi$.

Definition 3.10. Let $F$ be a face of the toric monoid $P$, and $J = P\setminus F$ is the prime toric ideal. Let $\mathcal{P}'$ be the paving associated to the face $F$. A prime ideal $p \subset R$ is said to be in the interior of the $\mathcal{P}'$-strata, if $J$ is the maximal prime toric ideal such that the correspondent ideal $J \subset \alpha^{-1}(p)$.

Definition 3.11. An ideal $I \subset R$ is called admissible for $\varphi$, if its radical $\sqrt{I}$ contains $\alpha(I_\varphi)$. The set of admissible ideal is denoted by $\mathfrak{A}_\varphi$.

Remark 3.12. Both of the definitions depend only on the log structure not the chart $\alpha : P \to R$, because the ideals $\alpha^{-1}(I)$ and $\alpha^{-1}(p)$ do not depend on the choice of the chart $\alpha$.

Lemma 3.13. If $I \subset I'$ and $I \in \mathfrak{A}_\varphi$, then $I' \in \mathfrak{A}_\varphi$. If $\sqrt{I} \in \mathfrak{A}_\varphi$, then $I \in \mathfrak{A}_\varphi$. The collection $\mathfrak{A}_\varphi$ is closed under finite intersection.

Proof. The first and the second statements follow directly from the definition. The third statement follows from the fact $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

Fix an ideal $I \subset R$, admissible for $\varphi$. Take the $I$-adic completion $\widehat{R}_I$.

Lemma 3.14. The complete ring $\widehat{R}_I$ is a Noetherian, excellent, normal, integral domain, provided that the closed subscheme $\text{Spec } R/I$ is connected.

Proof. Since $R$ is excellent, the $I$-adic completion $\widehat{R}_I$ is regular over $R$. ([GD67] EGA IV 2 7.8.3 (v)). In particular, since $R$ is normal, $\widehat{R}_I$ is normal. If $\text{Spec } R/I$ is connected, $\widehat{R}_I$ is a normal integral domain. Moreover, $\widehat{R}_I$ remains excellent by ([Val76] Theorem 9).

Proposition 3.15. Assume there is an object of $\text{DD}_{\text{ample}}$ over $R$ that is compatible with $\varphi$, and $I$ is an admissible ideal for $\varphi$, then there exists a projective family $(\mathcal{X}, \mathcal{L})$ over $S = \text{Spec } \widehat{R}_I$ such that the generic fiber $\mathcal{X}_\eta \cong \tilde{G}_\eta$ is abelian with an ample polarization.

Proof. Check the construction. The only thing we need to proof is a statement similar to Lemma 3.4. This follows from Lemma 3.9.

Lemma 3.16 ([FC90] III. Proposition 4.11). The total space $\mathcal{X}$ is irreducible.

Remark 3.17. We call this construction AN construction because it is introduced in the paper [AN99] when the base is a complete discrete valuation ring.
3.2.3. **Properties.** First we point out the relation between AN construction and the standard construction in [Ols08]. Recall $S(X) = X \otimes \mathbb{N}$. Introduce the monoid $S(X) \rtimes P$ on the underlying set $S(X) \times P$ with the addition law

$$(\alpha, p) + (\beta, q) = (\alpha + \beta, p + q + \tilde{\varphi}_p(\alpha) + \tilde{\varphi}_q(\beta) - \tilde{\varphi}_p(\alpha + \beta)).$$

**Lemma 3.18.**

$$S(Q_{\varphi}) \cong Q_{\tilde{\varphi}} \cong S(X) \rtimes P.$$ 

**Proof.** Notice that $P \to S(Q_{\varphi})$ is an integral morphism of integral monoids. $S(Q_{\varphi})^{gp} = \underline{X} \times P^{gp} \rtimes \mathbb{R} \cong \underline{X} \times P^{gp}$. Under this isomorphism, the morphism $P^{gp} \to S(Q_{\varphi})^{gp}$ is the homomorphism

$$P^{gp} \to \underline{X} \times P^{gp},$$

$$p \mapsto (0, p).$$

Then the cokernel of $\tilde{f}^\#: P \to S(Q_{\varphi})$ is equal to the image of $S(Q_{\varphi})$ in $S(Q_{\varphi})^{gp}/P^{gp} = \underline{X}$, which is $S(Q)$. Denote the projection $S(Q_{\varphi}) \to S(Q)$ by $b$. Since $P$ is sharp, by Lemma 3.1.32, for any $q \in S(Q)$, there exists a unique element $\tilde{q} \in b^{-1}(q)$ such that $\tilde{q} \leq x$ for all $x \in b^{-1}(q)$. For any $q \in S(Q)$, define $\vartheta_q := X^\tilde{q}$. By definition, for any $x \in b^{-1}(q)$, $x - \tilde{q} \in P$. Therefore $R_{\tilde{\varphi}}$ is a free $k[P]$-module generated by $\{\vartheta_q\}$ for all $q \in S(Q)$. We map the lowest elements $(\alpha, p, d)$ in $Q_{\tilde{\varphi}}$ to $(\alpha, d, 0)$ in $S(X) \rtimes P$. This induces an isomorphism of monoids. \hfill \square

As a result, $\tilde{P}^r \cong \text{Proj } k[S(X) \rtimes P] \otimes_{k[P]} R$.

Reduction to $I^n$ for each $n$, this is the standard construction in [Ols08] 5.2, except that we have a toric monoid $P$ instead of $H_{\varphi}$. Under this isomorphism, the data $\tau, \psi$ in [Ols08] corresponds to $b^{-1}_r \tau$ and $a^{-1}_q \psi$ in our case. The constructions of group actions and log structures in [Ols08] for the standard constructions can be applied to our case directly. In particular, we have the action $g_n$ of $\tilde{G}$ on $\mathcal{X}_n$ over $S_n := \text{Spec } R/I^n$. If we do the algebraization, we get a $G$-action $\varrho$ on $\mathcal{X}$ over $S$, where $G$ is the semiabelian scheme obtained by Mumford’s original construction ([Mum72] pp. 297 or [FC90] pp. 66).

The choice of $\mathcal{M}$ gives a $T$-linearization of the line bundle $\tilde{\mathcal{L}}$ on $\tilde{X}$. The action of $T$ gives the Fourier decomposition of $H^0(\tilde{X}, \tilde{\mathcal{L}})$. By the Heisenberg relation of the action of $Y$ and $\tilde{G}$, the decomposition is preserved by the action of $Y$. Therefore, we have the Fourier decomposition

$$H^0(\mathcal{X}, \mathcal{L}) = \bigoplus_{\alpha \in X/\phi(Y)} H^0(A, \mathcal{M}_\alpha)$$

(18)

For every $\alpha \in X/\phi(Y)$, choose $\vartheta_{A,\alpha} \in H^0(A, \mathcal{M}_\alpha)$ such that the residue $\vartheta_{A,\alpha,s} \neq 0$ for any $s \in S$. Define a section $\vartheta_n \in H^0(\mathcal{X}_n, \mathcal{L}_n)$ as a descent from

$$\tilde{\vartheta} = \sum_{\lambda \in Y} \sum_{\alpha \in X/\phi(Y)} S^\lambda_n(\vartheta_{A,\alpha}).$$

(19)
This is a finite sum for every admissible ideal \( I^n \). Let \( \Theta_n \) denote the divisor defined by \( \vartheta_n \). The collection \( \{ \Theta_n \} \) gives a divisor \( \Theta \) on \( X \) such that \( (X, \epsilon \Theta) \) is a stable pair.

**Lemma 3.19.** For any admissible ideal \( I \), and any geometric point \( \bar{x} \to \text{Spec } R/I \), the geometric fiber \((X_{\bar{x}}, \mathcal{L}_{\bar{x}}, \varrho_{\bar{x}}, \Theta_{\bar{x}})\) is a stable semiaflexible pair as defined in [Ale02]. In particular, if the image of \( x \) in \( \mathcal{P}' \)-strata, for a bounded paving \( \mathcal{P}' \) of \( X_R \), then \((X_{\bar{x}}, \mathcal{L}_{\bar{x}}, \varrho_{\bar{x}})\) is an element in \( M^P[\Delta_{\mathcal{P}'}, c, \mathcal{E}, \mathcal{M}](\kappa(\bar{x})) \) defined by \( \tilde{\vartheta}_0, \psi^{(\bar{x})}_0(\cdot)\tau_0(\cdot) \). Here \( \bar{x} \) is the residue field for \( x \), and \( \Delta_{\mathcal{P}'} \) is the complex defined by the paving \( \mathcal{P}' \) on \( X_R/\bar{Y} \). The notations \( \tilde{\vartheta}_0, \psi_0, \bar{\vartheta}_0 \) is defined in the proof. For other notations, see [Ale02].

**Proof.** Let \( J \) and \( F \) be the prime toric ideal and the face associated to \( \mathcal{P}' \). We can localize with respect to \( F \) first, and replace \( \varphi \) by \( \varphi' \), which gives the paving \( \mathcal{P}' \). Consider the geometric fiber \( \tilde{X}_{\bar{x}} \). Since \( \alpha(J) \subseteq \mathfrak{m}_x \), if \( (\alpha, m) \) and \( (\beta, n) \) are not in the same cell in \( C(\mathcal{P}', \varphi'_{x, m} + \varphi'_{x, n} - \varphi'_{x, \alpha + \beta, m + n}) \) is mapped to zero in \( \kappa(\bar{x}) \). Therefore, denote \( \mathcal{X}(\alpha, m) \otimes \mathcal{O}_{\alpha} \otimes \mathcal{M}g^m \) \( \mathcal{M}_{\alpha, m}, \tilde{X}_{\bar{x}}, \tilde{\mathcal{L}}_{\bar{x}} \) is the gluing of \( (P, L)[\delta, c, \mathcal{M}] \) for each \( \delta \in \mathcal{P}' \). Here \( (P, L)[\delta, c, \mathcal{M}] \) is defined in [Ale02] Definition 5.2.5. The action of \( Y \) by \( \psi^{(\bar{x})}(\cdot) \tau(\cdot) \), i.e. the residue of \( \psi', \tau' \) in \( \kappa(\bar{x}) \) for \( \psi', \tau' \) in Equation (17). Let the residues be denoted by \( \psi_0, \tau_0 \). It is an element in \( Z^1(\Delta_{\mathcal{P}'}, \tilde{\mathcal{X}}) \).

The residue of \( \tilde{\vartheta} \) in \( \kappa(\bar{x}) \) is denoted by \( \tilde{\vartheta}_0 \). By the definition of \( \vartheta_n \), the residue \( \vartheta_0 \) is in \( C^0(\Delta_{\mathcal{P}'}, \mathcal{M}_{\alpha, m}) \), and \( \Theta_{\bar{x}} \) does not contain any \( G_x \)-strata entirely. The condition that \( \tilde{\vartheta} \) is \( Y \)-invariant means that \( \tilde{\vartheta}_0, \psi^{(\bar{x})}_0(\cdot)\tau_0(\cdot) \) is an element in \( Z^1(\Delta_{\mathcal{P}'}, \tilde{\mathcal{X}}) \). \( \square \)

**Lemma 3.20.** For any admissible ideal \( I \) and any \( n \), the family \( (X_n, \mathcal{L}_n, \varrho_n, \Theta_n) \) over \( \text{Spec } R/I^n \) is an object in \( \mathcal{A} \mathcal{T}_{G, d}(\text{Spec } R/I^n) \).

**Proof.** By Lemma 3.19 it suffices to prove that \((X_n, \mathcal{L}_n)\) is flat over \( S_n := \text{Spec } R/I^n \). First, \( \tilde{X} \) is flat over \( A \), since \( R \) is locally free as \( \mathcal{O}_A \)-module. Secondly, when we take the quotient by \( Y \), we first take a quotient by \( Y_n \), that \( Y_n \subset Y \) is of finite index, and the action of \( Y_n \) on \( \tilde{X}_n \) is free. So this quotient preserves the flatness. Finally, the quotient by the finite group \( Y/Y_n \), the \( Y/Y_n \)-invariant part is a direct summand. Therefore \((X_n, \mathcal{L}_n)\) is flat over \( S_n \). \( \square \)

By Grothendieck’s existence theorem ([GD63] EGA III 1.5.4.5), we get a family \((\mathcal{X}, \mathcal{L}, G, \varrho, \Theta) \) in \( \mathcal{A} \mathcal{T}_{G, d}(S) \).

**Corollary 3.21.** The coherent sheaves \( \pi_* \mathcal{L}^m \) are locally free of rank \( dm^g \) for all \( m \geq 0 \). The coherent sheaves \( R^1 \pi_* \mathcal{L}^m = 0 \) for all \( m > 0, i > 0 \). In particular, we have \( \pi_* \mathcal{L}^m \otimes \kappa(x) \cong H^0(X_x, \mathcal{L}_x^m) \).

**Proof.** For any \( x \in S \), consider \( x \to S \) over the image \( x \), with \( \kappa(\bar{x}) \) the algebraic closure of \( \kappa(x) \). Consider the fiber \((X_x, \mathcal{L}_x) \). By [Ale02] Theorem 5.4.1, \( H^1(X_x, \mathcal{L}_x^m) = 0 \) for any \( i, m \geq 1 \). Since \( \kappa(x) \to \kappa(\bar{x}) \) is flat, \( H^1(X_x, \mathcal{L}_x^m) = 0 \) for any \( i, m \geq 1 \) by ([Hart77] Chap. III Proposition 9.3). Since the base \( S \) is still Noetherian ([AM69] Theorem 10.26) and
Moreover, the degeneration data over $S$ with charts $\pi_*\mathcal{L}^m$ is flat over $S$, we can apply (**Har77** Chap. III Theorem 12.11). We have $R\pi_*\mathcal{L}^m = 0$ for all $i, m > 0$, and $\pi_*\mathcal{L}^m \otimes \kappa(x) \to H^0(X, \mathcal{L}^m_x)$ is surjective. Apply (**Har77** Chap. III Theorem 12.11) again, $\pi_*\mathcal{L}^m$ are locally free, and $\pi_*\mathcal{L}^m \otimes \kappa(x) \cong H^0(X, \mathcal{L}^m_x)$. For the generic $\eta$, $X_\eta$ is an abelian variety and $H^0(X, \mathcal{L}^m_\eta) = \dim^\eta$, so $\pi_*\mathcal{L}^m$ is of rank $\dim^\eta$. \qed

By abusing of notations, the morphism $A \to S$ is also denoted by $\pi$. Then $\pi_*\mathcal{M}$ is also locally free. The Fourier decomposition is a decomposition of locally free sheaves over $S$.

$$\pi_*\mathcal{L} \cong \bigoplus_{\alpha \in \mathcal{X}/\phi(Y)} \pi_*\mathcal{M}_\alpha.$$ \hfill (20)

For any coherent sheaf $\mathcal{F}$ over $S$, define $\mathcal{F}^*$ to be the subsheaf

$$\mathcal{F}^*(U) := \{f \in \mathcal{F}(U) : \text{Supp } f = \emptyset \} \quad \forall U \subset S.$$ 

If we choose a section $\vartheta \in \oplus_{\alpha \in \mathcal{X}/\phi(Y)} \pi_*\mathcal{M}_\alpha^*$, then $(\mathcal{X}, \mathcal{L}, G, g, \Theta)$ is an object in $\mathcal{A}D_{g,d}(\mathcal{R}_I)$.

The AN construction is functorial. Assume $S, S'$ are as above, with degeneration data, charts $P \to R$ and $P' \to R'$, and compatible functions $\varphi : X_R \to P^\text{gp}_R$, $\varphi' : X_R \to (P')^\text{gp}_R$, with charts $\alpha : P \to R$ and $\alpha' : P' \to R'$. Suppose $(f, f')$ is a strict log morphism $S' \to S$. Moreover, the degeneration data over $S'$ is isomorphic to the pull back of degeneration data along $f$, and $\psi := f^\sharp \alpha \circ \varphi/\alpha' \varphi'$ takes values in $(R')^*$. **Proposition 3.22** (functorial). The pull back of the AN family $(\mathcal{X}, \mathcal{L}, G, g)$ over $S$ is isomorphic to the AN family $(\mathcal{X}', \mathcal{L}', G', g')$ over $S'$.

**Proof.** Consider the following map

$$\Pi : k[S(X) \times \varphi P] \otimes k[P] \otimes R' \to k[S(X') \times \varphi' P'] \otimes k[P'] \otimes R'$$ 

$$\zeta^{(\alpha, p)} \mapsto f^\sharp(\zeta^{(\alpha, p)} \varphi(\alpha)^{\deg(\alpha)}).$$ \hfill (21) \hfill (22)

We need to check that $\Pi$ is homomorphism of algebras, i.e.

$$\Pi((\zeta^{(\alpha, p)} \zeta^{(\beta, q)})) = \Pi(\zeta^{(\alpha, p)}) \Pi(\zeta^{(\beta, q)}).$$

Since the morphism is strict, we can write $f^\sharp(\zeta^{(\alpha, p)})$ as $\zeta^{(\alpha', \beta')}_{r'}$ for $r' \in (R')^*$. Then it reduces to the following identity

$$f^\sharp(\tilde{\varphi}(\alpha) + \tilde{\varphi}(\beta) - \tilde{\varphi}(\alpha + \beta)) \varphi(\alpha + \beta)^{\deg(\alpha + \beta)} = (\tilde{\varphi}'(\alpha) + \tilde{\varphi}'(\beta) - \tilde{\varphi}'(\alpha + \beta)) \varphi(\alpha)^{\deg(\alpha)} \varphi(\beta)^{\deg(\beta)}.$$ 

Since $\psi$’s are invertible in $R'$, we get an isomorphism $\tilde{P}^r \times_S S' \to (\tilde{P}^r)'$. Moreover, the pull back of $\tilde{G}$ is $G'$ by assumption. Since the contracted product commutes with the pull back, we have the isomorphism $\tilde{X}' \to \tilde{X} \times_S S'$. By assumption, the degeneration data also commutes with the pull back, the isomorphism commutes with $Y$-action. \qed

Given $S$ as above, with degeneration data, a function $\varphi : X_R \to P^\text{gp}_R$, and a chart $\alpha : P \to R$ such that $\alpha \circ \varphi$ is compatible with the degeneration data. We get the family $(\mathcal{X}_1, \mathcal{L}_1, G_1, g_1)$ over $R/I$. Suppose there is a different chart $\alpha' : P \to R$, then $\psi := \alpha \circ \varphi/\alpha' \circ \varphi$ takes values in $R^*$. Therefore $\alpha' \circ \varphi$ is also compatible with the degeneration data. Since the graded sheaf of algebra $\mathcal{R}$ as a subsheaf of $S$ is defined independent of the
choice of the chart \( \alpha : P \to R \), the set of admissible ideals are the same for the two charts. We get the same family \((X_1, L_1, G_1, g_1)\) over \( R/I \) by using \( \alpha' \) in the AN construction. Moreover the Fourier decomposition \([20]\) are the same for \( \alpha \) and \( \alpha' \).

**Lemma 3.23.** The AN construction \((X, L, G, g)\), and the Fourier decomposition \( \pi^* L = \bigoplus_{\alpha \in X/\phi(Y)} \pi_* M_\alpha \) only depends on the log structure induced by \( \alpha : P \to R \).

Since a log structure is an étale sheaf, we can glue the above local models and generalize the base \( S \).

**Assumption 3.24.** Assume the log scheme \((S, M_S)\) is log smooth over a base \((\overline{S}, \mathcal{O}^*_S)\). Moreover, assume that the base \( \overline{S} \) is Noetherian, excellent, and integral.

Denote the open subset where \( M_S \) is trivial by \( S^* \), and the boundary \( S \setminus S^* \) by \( \partial S \). We can define the degeneration data over \( S \). It is defined to be an étale sheaf of data \((A, A^t, G, G^t, T, T^t, c, c^t, X, Y, \phi, \psi, \tau, \mathcal{M})\), such that for each étale affine neighborhood \( U \) of \( S \), and the completion of \( U \) with respect to \( U \times_S \partial S \), it is the degeneration data defined before. For any point \( x \in S \), we can choose an affine étale neighborhood \( U \to S \) such that \( U = \text{Spec} \ R \) is isomorphic to a product \( \overline{S} \times \text{Spec} \ k[P] \) with some base scheme \( \overline{S} \). A choice of such isomorphism gives a chart \( \alpha : P \to R \). Fix a piecewise affine function \( \varphi : X_R \to I^R \) as above. By choosing the local chart, we can talk about whether \( \varphi \) is compatible with the degeneration data, and the set of admissible ideals.

**Lemma 3.25.** Whether \( \varphi \) is compatible with the degeneration data over \( U \), and the set of admissible ideals are both independent of the choice of the trivialization of \( U \).

**Proof.** This is different from different trivializations give the same log structure. \( \square \)

**Definition 3.26.** The function \( \varphi \) and the log structure on \( S \) is said to be compatible with the degeneration data over \( S \), if they are compatible under any local trivialization.

**Definition 3.27.** A closed subscheme \( Z \) with support inside \( \partial S \) is called admissible, if restricted to any affine étale open set \( U \), under any local trivialization of \( U \), \( Z \) corresponds to an admissible ideal.

**Theorem 3.28.** Assume \( S \) satisfies Assumption \([3.24]\). Assume there is degeneration data over \( S \) and a piecewise affine function \( \varphi : X_R \to I^R_{\varphi} \) compatible with the degeneration data through the log structure of \((S, M_S)\). Then for any admissible closed subscheme \( Z \), we construct a family \((X_Z, L_Z, G_Z, g_Z)\) over \( Z \). Furthermore, there is a subsheaf of \( \pi_* \mathcal{L} \) defined by the Fourier decomposition \([20]\) such that for any \( \Theta_Z \) defined by a section of this subsheaf, the family \((X_Z, L_Z, G_Z, g_Z, \Theta_Z)\) is an object in \( A \mathcal{P}_{g,d}(Z) \). This construction is functorial with respect to morphisms of admissible closed subschemes \( Z' \to Z \).

Let \((X, L, G, g)\)/\( S \) be an AN family. Then the relative ample line bundle \( L \) over \( X \) induces a polarization on the semiabelian scheme \( G/S \). We can associate constructible étale sheaves \( X, Y \) over \( S \), and a bilinear pairing \( B : Y \times X \to \text{Div} \ S \). Here \( \text{Div} \ S \) is the sheaf of Cartier divisors. See \([19]\) Chap. III Theorem 10.1). For \( s \in S \), the fiber \( X_s \) is the group of characters for the toric part of \( G_s \), and \( Y_s \) is the group of characters for the...
toric part of $G'_s$. The pairing $B$ agrees with $\tau$. Notice that the open locus where $X, Y, B$ vanishes is exactly the open locus $S^*$ where the log structure $M_S$ is trivial.

### 3.3. Standard Data.

Fix an $X$-invariant integral paving $\mathcal{P}$ of $X_R$. By definition, $\mathcal{P}$ is obtained as the set of affine domains of some $X$-quasiperiodic real-valued piecewise affine function $\psi$. Associate to $\mathcal{P}$ is a rational polyhedral cone $C(\mathcal{P})$ in the second Voronoi fan $\Sigma(X)$. The support of $\Sigma(X)$ is $C(X)^{rc} \subset \Gamma^2 U$. Let $CPA(\mathcal{P}, R)$ be the space of convex piecewise affine functions whose associated paving is coarser than $\mathcal{P}$. Let $CPA^X(\mathcal{P}, R)$ be the subspace of $X$-quasiperiodic functions. The map $\psi \to Q$, which maps $\psi$ to the associated quadratic form $Q$, identifies $CPA^X(\mathcal{P}, R)/Aff$ with $C(\mathcal{P})$. Let $C(\mathcal{P}, Z)$ be the image of integral convex functions in $CPA(\mathcal{P}, R)/Aff$ and $C^X(\mathcal{P}, Z)$ be the image of integral, $X$-quasiperiodic, convex functions in $C(\mathcal{P})$.

Recall the definition of $H_{\mathcal{P}}$ in [Ols08]. For any $\sigma_i \in \mathcal{P}$ define $N_i = S(\sigma_i)$. Define $N_{\mathcal{P}} := \lim_i N_i$ in the category of integral monoids. The cone $S(X) = \lim_i N_i$ in the category of sets. By universal property of $S(X)$ we have a natural map $\varphi' : S(X) \to N_{\mathcal{P}}$. Since $\text{Hom}(\lim_i N_i, Z) \cong \lim_i (\text{Hom}(N_i, Z))$, The group $N_{\mathcal{P}}^{gp}$ is equal to $PA(\mathcal{P}, Z)^{gp}$.

Let $\tilde{H}_{\mathcal{P}}$ be the submonoid of $N_{\mathcal{P}}^{gp}$ generated by

$$\alpha \ast \beta := \varphi'(\alpha) + \varphi'(\beta) - \varphi'(\alpha + \beta), \quad \forall \alpha, \beta \in S(X).$$

**Proposition 3.29.** Let $C(\mathcal{P}, Z)^{\vee}$ be the dual of the monoid $C(\mathcal{P}, Z)$ in $\text{Hom}(C(\mathcal{P}, Z), Z)$.

$$\tilde{H}_{\mathcal{P}}^{sat} = C(\mathcal{P}, Z)^{\vee}.$$ 

**Proof.** It is easy to see that $\alpha \ast \beta \in C(\mathcal{P}, Z)^{\vee}$ by evaluation. It follows that $\tilde{H}_{\mathcal{P}} \subset C(\mathcal{P}, Z)^{\vee}$. On the other hand, for any $\psi \in PA(\mathcal{P}, Z)$. Pick any lift $\tilde{\psi} \in PA(\mathcal{P}, Z)$. $\psi(\alpha \ast \beta) \geq 0$ for all $\alpha, \beta$, if and only if it is convex. Therefore $(\tilde{H}_{\mathcal{P}})^{\vee} = C(\mathcal{P}, Z)$. We claim that $\tilde{H}_{\mathcal{P}}^{gp} = \text{ann}(Aff) \cap N_{\mathcal{P}}^{gp}$. Assuming this, then, since $\tilde{H}_{\mathcal{P}}$ is sharp, $\tilde{H}_{\mathcal{P}}^{sat} = ((\tilde{H}_{\mathcal{P}})^{\vee})^{\vee} = C(\mathcal{P}, Z)^{\vee}$.

Now we prove the claim. In the proof of ([Ols08] Lemma 4.1.6), it is shown that the image of $\pi : SC_1(\mathbb{X}_{\geq 0})/B_1 \to \tilde{H}_{\mathcal{P}}$ is equal to $\tilde{H}_{\mathcal{P}}$. By the description of $SC_1(\mathbb{X}_{\geq 0})/B_1$, it is the monoid generated by images of all affine linear relations for $\mathbb{X}_R$. Therefore $\tilde{H}_{\mathcal{P}}^{gp}$ is saturated in $\text{ann}(Aff)$. 

The group $X$ is acting on $S(X) \subset \mathbb{X}$. It induces an action of $X$ on $N_{\mathcal{P}}$ and $\tilde{H}_{\mathcal{P}}$. The quotient of this action is denoted by $H_{\mathcal{P}}$. There is a natural map $H_{\mathcal{P}} \to C^X(\mathcal{P}, Z)^{\vee}$.

**Proposition 3.30.** We have

\begin{align*}
(23) & \quad \text{Hom}(H_{\mathcal{P}}, Z_{\geq 0}) = C^X(\mathcal{P}, Z), \\
(24) & \quad H_{\mathcal{P}}^{sat}/(H_{\mathcal{P}}^{sat})_{tor} = C^X(\mathcal{P}, Z)^{\vee}.
\end{align*}

\footnote{Notice we define $CPA(\mathcal{P}, R)$ to be closed.}
Proof. We prove $C^X(\mathcal{P}, \mathbb{Z}) \subset \text{Hom}(H_\mathcal{P}, \mathbb{Z}_{\geq 0})$ first. The inclusion $C^X(\mathcal{P}, \mathbb{Z}) \to C(\mathcal{P}) \subset \Gamma^2U$ induces a map $S^2X \to H^\text{gp}_\mathcal{P}$, and this is the map $s : S^2X \to H^\text{gp}_\mathcal{P}$ defined in ([Ols08] Lemma 5.8.2). By ([Ols08] Lemma 5.8.16), $C(\mathcal{P})^\text{gp}_Q \cong \text{Hom}(H_\mathcal{P}, \mathbb{Q})$. Therefore $C^X(\mathcal{P}, \mathbb{Z}) \subset \text{Hom}(H_\mathcal{P}, \mathbb{Z}_{\geq 0})$ is an inclusion. On the other hand, for each $\psi \in \text{Hom}(H_\mathcal{P}, \mathbb{Z}_{\geq 0})$, denote its image in $\text{Hom}(\tilde{H}_\mathcal{P}, \mathbb{Z}_{\geq 0})$ by $\tilde{\psi}$. Since $H^\text{sat}_\mathcal{P} = C(\mathcal{P}, \mathbb{Z})^\vee$, $\tilde{\psi}$ is an element in $C(\mathcal{P}, \mathbb{Z})$. Being invariant under the action of $X$ means exactly being $X$-quasiperiodic, therefore $\psi \in C^X(\mathcal{P}, \mathbb{Z})$.

Both $C^X(\mathcal{P}, \mathbb{Z})^\vee$ and $H^\text{sat}_\mathcal{P}/(H^\text{sat})_\text{tor}$ are toric, and $\text{Hom}(H_\mathcal{P}, \mathbb{Z}_{\geq 0}) = \text{Hom}(H^\text{sat}_\mathcal{P}/(H^\text{sat})_\text{tor}, \mathbb{Z}_{\geq 0})$. So the second statement follows from the first statement. □

Denote the monoid $C^X(\mathcal{P}, \mathbb{Z})^\vee$ by $P_\mathcal{P}$. It is a sharp toric monoid. The natural morphism $H^\text{sat}_\mathcal{P} \to P_\mathcal{P}$ is the quotient out the torsion part, since $s_Q : S^2X \otimes \mathbb{Q} \to H^\text{gp}_\mathcal{P}Q$ is an isomorphism ([Ols08] Proposition 5.8.15).

If the paving $\mathcal{P}$ is a triangulation $\mathcal{T}$, the cone $C(\mathcal{T})$ of maximal dimension in $\Gamma^2U$. The group $\text{Hom}(C^X(\mathcal{P}, \mathbb{Z})^\text{gp}, \mathbb{Z})$ is a lattice in $S^2U^*$, and is denoted by $L_\mathcal{T}$. We have $S^2X \subset L_\mathcal{P}$. For a general paving $\mathcal{P}$, let $I_\mathcal{P}$ denote the set of triangulations that refine $\mathcal{P}$. Define

$$L_\mathcal{P} := \sum_{\mathcal{T} \in I_\mathcal{P}} L_\mathcal{T}.$$ 

Let $C(\mathcal{T})^\vee$ be the dual cone of $C(\mathcal{T})$ in $S^2U^*$, and $S_\mathcal{P}$ be the toric monoid $C(\mathcal{T})^\vee \cap L_\mathcal{P}$. If $\mathcal{P}$ is a triangulation, $S_\mathcal{P} = P_\mathcal{P}$.

Introduce the monoid $S(X) \times H_\mathcal{P}$ on the set $S(X) \times H_\mathcal{P}$ with the addition law

$$(\alpha, p) + (\beta, q) = (\alpha + \beta, p + q + \alpha \ast \beta).$$

The morphism $H^\text{sat}_\mathcal{P} \to P_\mathcal{P}$ induces the natural morphism $S(X) \times H^\text{sat}_\mathcal{P} \to S(X) \times P_\mathcal{P}$.

Consider the minimal models of the mirror family $\mathcal{Y}$ in the principal polarized case. By Theorem [2.39], we can identify the second Voronoi fan $\Sigma(X)$ with the Mori fan of $\mathcal{Y}$. For any bounded paving $\mathcal{P}$, the nef cone $\overline{\text{Ker}}(\mathcal{Y}_\mathcal{P}) \cap \text{Pic}(\mathcal{Y}_\mathcal{P})$ is identified with $C^X(\mathcal{P}, \mathbb{Z})$. Therefore

$$P_\mathcal{P} = \overline{\text{NE}}(\mathcal{Y}_\mathcal{P}) \cap \text{Pic}(\mathcal{Y}_\mathcal{P})^*.$$ 

For an $X$-quasiperiodic $P^\text{gp}_{\mathcal{P} R}$-valued function $\varphi$ over $X_R$, the bending parameters are $X$-periodic: $p_\rho = p_{\rho + X}$. Define an $X$-quasiperiodic, $\text{NE}(\mathcal{Y}_\mathcal{P})$-convex function $\varphi : X_R \to N_1(\mathcal{Y}_\mathcal{P}) = P^\text{gp}_{\mathcal{P} R}$ by requiring that, at each codimension-1 cell $\rho \in \mathcal{P}$, the bending parameter $p_\rho \in \text{NE}(\mathcal{Y}_\mathcal{P})$ is the corresponding curve class $\Upsilon_*[V(C(\rho))]$, where $\Upsilon : \mathcal{Y}_\mathcal{P} \to \mathcal{Y}_\mathcal{P}$ is the universal covering, and $\mathcal{Y}_\mathcal{P}$ is the infinite toric variety for the fan $\{C(\sigma) \}_{\sigma \in \mathcal{P}}$.

Lemma 3.31. Let $\rho$ be a codimension-1 wall between maximal cells $\sigma_i$ and $\sigma_j$. Let $\omega \in X$ be a vector that maps to a primitive generator of $X/(\mathbb{R}_\rho \cap \mathbb{Z}) \cong \mathbb{Z}$. For any integral,
real-valued, X-quasiperiodic piecewise affine function ψ whose paving is coarser than \( \mathcal{P} \), define \( \psi_i, \psi_j \) to be the affine extension of \( \psi|_{\sigma_i} \) and \( \psi|_{\sigma_j} \). The bending parameter \( p_\rho \) for \( \varphi_\mathcal{P} \) is

\[
p_\rho(\psi) = \psi_i(\omega) - \psi_j(\omega).
\]

In particular, the function \( \varphi_\mathcal{P} \) is integral with respect to the integral structure \( P_\mathcal{P} \).

**Proof.** Since \( \psi \) is X quasi-periodic, the divisor \( D_\psi \) it represents is inside \( \text{Pic}^0(\check{Y}_\mathcal{P}) \). Therefore

\[
p_\rho(\psi) = (\Upsilon_*[V(C(\rho))]) \cdot D_\psi = \Upsilon_*([V(C(\rho))]) \cdot \Upsilon^*D_\psi = \psi_i(\omega) - \psi_j(\omega).
\]

The last equality is by the formula for toric varieties, since \( \check{Y}_\mathcal{P} \) is locally a toric variety.

If \( \psi \) is integral, since \( \omega \in X \), \( p_\rho(\psi) \) is an integer. In other words, \( p_\rho \in P_\mathcal{P} \), and \( \varphi_\mathcal{P} \) is integral.

**Corollary 3.32.** For any function \( \psi \in C(\mathcal{P}) \), the evaluation \( \varphi_\mathcal{P} \) on \( \psi \) is equal to \( \psi \) up to a linear function.

**Proof.** Choose a piecewise affine function \( \psi \) as the representative. By the Lemma 3.31, the bending parameters of \( \psi \circ \varphi_\mathcal{P} \) is equal to the bending parameters of \( \psi \).

If \( C(\mathcal{P}') \) is a face of a cone \( C(\mathcal{P}) \). Consider the dual \( C(\mathcal{P}')^\vee \) in the space \( \mathbf{N}_1(\check{Y}_\mathcal{P}) \). Since \( \mathcal{P} \) refines \( \mathcal{P}' \), there is a contraction \( f : \check{Y}_\mathcal{P} \to \check{Y}_\mathcal{P}' \), inducing \( f_* : \mathbf{N}_1(\check{Y}_\mathcal{P}) \to \mathbf{N}_1(\check{Y}_\mathcal{P}') \). We have an exact sequence of monoids

\[
0 \longrightarrow \mathcal{R} \longrightarrow C(\mathcal{P}')^\vee \xrightarrow{f_*} \mathbf{NE}(\check{Y}_\mathcal{P}') \longrightarrow 0,
\]

where \( \mathcal{R} \) is the \( \mathbf{R} \)-vector space generated by the \( f \)-contracted curve classes.

**Corollary 3.33.** The standard sections \( \varphi \) are compatible.

\[
\varphi_\mathcal{P} = f_* \circ \varphi_\mathcal{P} : B \to \mathbf{N}_1(\check{Y}_\mathcal{P}).
\]

**Proof.** Regard \( \mathbf{N}_1(\check{Y}_\mathcal{P}) \) as the dual space of \( \text{Pic}(\check{Y}_\mathcal{P}) \), and \( \text{Pic}(\check{Y}_\mathcal{P}) \to \text{Pic}(\check{Y}_\mathcal{P}') \) as an inclusion. The map \( f_* \) is the restriction of functions to the subspace \( C(\mathcal{P}') \). By Lemma 3.31 the bending parameters have the same description as functionals on \( C(\mathcal{P}') \).

Let \( (X, \mathcal{L}, G, \varrho) \) be an AN family over \( S = \text{Spec} R \) constructed from \( \varphi_\mathcal{P} \) and a chart \( \alpha : P_\mathcal{P} \to R \). We add a log structure to the family. Locally on the abelian scheme \( A/S \), choose a trivialization of \( M \) and compatible trivializations of \( \mathcal{O}_A \), the algebra \( \mathcal{R} \) is isomorphic to \( k[S(X) \times P_\mathcal{P}] \otimes_{k[P_\mathcal{P}]} \mathcal{O}_A \). Define the log structure \( \check{P} \to \mathcal{O}_X \) locally by the descent of the chart

\[
S(X) \times P_\mathcal{P} \to k[S(X) \times P_\mathcal{P}] \otimes_{k[P_\mathcal{P}]} \mathcal{O}_A.
\]

For any \( n \in \mathbf{N} \), do the reduction over \( S_n := \text{Spec} R/I^n \), the pull-back \( \check{P}_n \) on \( \check{X}_n \) descends to a log structure \( P_n \) on the quotient \( X_n \), by ([Ols08] 4.1.18, Lemma 4.1.19, & 4.1.22).
By ([Ols08] Lemma 4.1.11), \((\tilde{X}, \tilde{P})\) is integral and log smooth over \((S, M_S)\). Therefore \((X_n, P_n)\) is log smooth and integral over \((S_n, M_{S_n})\) for every \(n\).

The underlying topological space of \(S_n\) inherits a stratification from the toric stratification of \(\text{Spec} k[P_{\mathcal{P}}]\). Since the image of \(\text{Spec} R/I\) is in the union of the toric strata defined by \(I_{\mathcal{P}_B}\), each stratum corresponds to an admissible prime toric ideal \(J\). Let \(J\) be an admissible prime toric ideal of \(P_{\mathcal{P}}\), and \(F\) be the face \(P_{\mathcal{P}} \setminus J\). The monoid \(P_{\mathcal{P}}\) is the integral points in \(C(\mathcal{P})^\vee\), for \(C(\mathcal{P})\) a cone in the second Voronoi fan. Each face \(\tau\) of \(C(\mathcal{P})\) is a cone \(C(\mathcal{P}')\) for some coarser paving \(\mathcal{P}'\). Therefore \(F = C(\mathcal{P}')^\perp \cap C(\mathcal{P})^\vee\). Then the stratum of \(S_n\) is called the stratum associated to the paving \(\mathcal{P}'\), or the \(\mathcal{P}'\)-stratum.

We can associate a paving and a piecewise affine function to the face \(F\) as in Definition [3.3]. By Corollary [3.33] the piecewise affine function is \(\varphi_{\mathcal{P}'}\), and the paving is \(\mathcal{P}'\).

**Lemma 3.34.**

\[ P_{\mathcal{P}}/(F \cap P_{\mathcal{P}}) = P_{\mathcal{P}'}. \]

**Proof.** The monoid \(P_{\mathcal{P}}\) is given by the intersection \(\overline{\text{NE}}(Y_{\mathcal{P}}) \cap \text{Pic}(Y_{\mathcal{P}})^{\ast}\). The dual integral structure is \(C^X(\mathcal{P}, Z)^{\text{gp}}\). We claim that \(C^X(\mathcal{P}', Z) = C(\mathcal{P}') \cap C^X(\mathcal{P}, Z)^{\text{gp}}\). Assume \(\psi\) is a piecewise function and its image is in \(C^X(\mathcal{P}', Z)\). Since \(\mathcal{P}'\) is coarser than \(\mathcal{P}\), each top-dimensional cell \(\sigma \in \mathcal{P}\) is contained in some top-dimensional cell \(\sigma' \in \mathcal{P}'\) of \(\mathcal{P}'\). Therefore, the restriction of \(\psi\) to \(\sigma\) is integral, and \(\psi \in C(\mathcal{P}') \cap C^X(\mathcal{P}, Z)^{\text{gp}}\). On the other hand, each top-dimensional cell \(\sigma' \in \mathcal{P}'\) contains some top-dimensional cell \(\sigma \in \mathcal{P}\). If \(\psi\) is integral on \(\sigma\), it is integral on \(\sigma'\). Therefore if \(\psi \in C(\mathcal{P}') \cap C^X(\mathcal{P}, Z)^{\text{gp}}\), then \(\psi \in C^X(\mathcal{P}', Z)\). This proves the claim. It follows that the dual integral structures also agree. \(\square\)

**Corollary 3.35.** Assume \(C(\mathcal{P}')\) is a face of \(C(\mathcal{P})\). The morphism \(k[S_{\mathcal{P}}] \to k[S_{\mathcal{P}'}, \mathcal{P}']\) is étale.

**Proof.** Recall that \(S_{\mathcal{P}} = L_{\mathcal{P}} \cap C(\mathcal{P})^\vee\), and \(L_{\mathcal{P}} \subset L_{\mathcal{P}'}\) is a sublattice of finite index. The morphism \(k[S_{\mathcal{P}}] \to k[S_{\mathcal{P}'}, \mathcal{P}']\) can be decomposed into a localization and the inclusion \(f : k[C(\mathcal{P}')^\vee \cap L_{\mathcal{P}}] \to k[C(\mathcal{P}')^\vee \cap L_{\mathcal{P}'}]\). Denote \(C(\mathcal{P}')\) by \(\tau\), and \(C(\mathcal{P}')^\perp\) by \(F^{\text{gp}}\). Apply Lemma [3.33] to all \(\mathcal{P}\) that refines \(\mathcal{P}'\), we get exact sequences.

\[
\begin{array}{cccccc}
0 & F^{\text{gp}} \cap L_{\mathcal{P}} & \longrightarrow & L_{\mathcal{P}} \cap \tau^\vee & P_{\mathcal{P}'} & 0 \\
0 & F^{\text{gp}} \cap L_{\mathcal{P}'} & \longrightarrow & L_{\mathcal{P}'} \cap \tau^\vee & P_{\mathcal{P}'} & 0 \\
\end{array}
\]

It follows that the first square is the pushout ([Ogu06] Chap.I Proposition 1.1.4 part. 2). The associated morphism to \(f'\) is a group homomorphism of tori and is étale, the base change \(f\) is thus étale. \(\square\)

**Lemma 3.36.** For each geometric point \(\bar{x} \to S\) in interior of the \(\mathcal{P}'\)-stratum, the fiber \((X_{n,\bar{x}}, \mathcal{L}_{n,\bar{x}}, P_{n,\bar{x}}, G_{n,\bar{x}}) \to (\bar{x}, M_{\bar{x}})\) is isomorphic to the collection of data obtained from the saturation of the standard construction defined in [Ols08].
Proof. Denote $P_{\varphi}$ by $P$, and $P_{\varphi'}$ by $P'$ in this proof. Assume the admissible prime toric ideal associated to $\mathcal{P}'$-stratum is $J$. Use the notations $F$, $P_F$, and $J_F$ as in Lemma 3.34. By Lemma 3.34, $P_F \cong P' \oplus F^{\text{gp}}$. Denote the geometric point $\bar{x} \to S \to \text{Spec } k[P]$ by $\bar{y} \to \text{Spec } k[P]$.

The fiber $(\mathcal{X}_{n,\bar{x}}, \mathcal{L}_{n,\bar{x}}, G_{n,\bar{x}})$ is isomorphic to the fiber $(\mathcal{X}_{\bar{x}}, \mathcal{L}_{\bar{x}}, G_{\bar{x}})$. Since the image of $\bar{x}$ is contained in the $\mathcal{P}'$-stratum, $\bar{y}$ is contained in the localization $\text{Spec } k[P_F]$. Therefore, we can replace $R$ by the base change $R' = k[P_F] \otimes_{k[P]} R$. As in the proof of Lemma 3.37 after the localization $k[P_F]$, the function $\varphi_{\varphi'}$ is equal to $\varphi_{\varphi'} + \psi$, where $\psi$ has bending parameters in $F^{\text{gp}}$. Up to a global affine function, we can assume that $\varphi_{\varphi'} = \varphi_{\varphi'} + \psi$ with the values in $P_F = P' \oplus F^{\text{gp}}$. Since $F^{\text{gp}}$ are sent to $(R')^*$, the pull back of the degeneration data is compatible with $\varphi_{\varphi'}$. By Lemma 3.22 the base change of $\mathcal{X}$ to $R'$ is isomorphic to the AN construction by using $P' \to R'$ and $\varphi_{\varphi'}$.

Assume $\bar{x} = \text{Spec } \Omega(x)$. Since the prime ideal of the image of $\bar{y}$ contains $J_F$ which contains $P' \setminus \{0\}$, the prime ideal of the image of $\bar{x}$ contains $P' \setminus \{0\}$. Therefore, the map $H_{\varphi'}^{\text{sat}} \to P' \to \Omega(x)$ sends non-invertible elements to 0. Further pull back the degeneration data to $\Omega(x)$ . Use the pull back of $b_{t}^{-1}\tau$ and $a_{t}^{-1}\psi$ as the trivializations, we get the data in (Ols08 5.2.1). The fiber $(\mathcal{X}_{\bar{x}}, \mathcal{L}_{\bar{x}}, G_{\bar{x}})$ is isomorphic to the saturation of the standard construction in (Loc.cit. 5.2).

Over $S' = \text{Spec } R'$, we can replace the chart $P$ by the localization $P_F$. Since $P_F = P' \oplus F^{\text{gp}}$ and $F^{\text{gp}}$ are sent to $(R')^*$, we can use the chart $P' \to R'$. Pull back to $\Omega(k)$, $P' \to \Omega(x)$ is the chart. Consider the composition $\alpha' : H_{\varphi'}^{\text{sat}} \to P' \to \Omega(x)$. Since $(H_{\varphi'}^{\text{sat}})_{\text{tot}} \subset (H_{\varphi'}^{\text{sat}})^*$ and is sent to $1 \in \Omega(x)^*$ by $\alpha'$, it is quotient out in the associated log structure. Therefore $\alpha' : H_{\varphi'}^{\text{sat}} \to \Omega(x)$ is also a chart of the pull back of the log structure $M$. Similarly, over $\tilde{X}_{\bar{x}}$, the log structure $\tilde{P}_{\bar{x}}$ is locally defined by the chart $S(X) \times H_{\varphi'}^{\text{sat}}$. Therefore, the log structures are also obtained from the saturation of the standard construction. \hfill \Box

It follows that the family $(\mathcal{X}_n, \mathcal{L}_n, P_n, G_n, \theta_n) \to (S_n, M_n)$ is an object in $\mathcal{F}_{g,d}$. By the fact that $\mathcal{F}_{g,d}$ is an Artin stack, we get a family $(\mathcal{X}, \mathcal{L}, P, G, \theta) \to (S, M)$ in $\mathcal{F}_{g,d}(S)$, and call this the standard family.

**Theorem 3.37.** The same assumption as in Theorem 2.28 with $P = P_{\varphi}$ and $\varphi = \varphi_{\varphi'}$. If $Z$ is a closed subscheme admissible for $\varphi_{\varphi'}$, then there exists a family $\pi : (\mathcal{X}_Z, \mathcal{L}_Z, P_Z, G_Z, \theta_Z)$ over $(Z, M_Z)$, which is an object in $\mathcal{F}_{g,d}(Z)$. Moreover, this construction is functorial with respect to $Z' \to Z$.

**Remark 3.38 (The Log Structure).** When we construct the versal families, we will have $\mathcal{S}$ regular. Then both the base $(S, M_S)$ and $(\mathcal{X}, P)$ are log regular. Denote the open subset where $M_S(P)$ is trivial by $S^*$ (resp. $\mathcal{X}^*$), and the boundary by $\partial S$ (resp. $\partial \mathcal{X}$). By (Kat94 Theorem 11.6), the log structures $M_S$ and $P$ are isomorphic to the divisorial log structures. However, the divisorial log structures are not preserved by the finite base change, unless...
Choose a symmetric integral matrix $X$ of $M = 2\delta$, and $\delta$ is a primitive $M$-th root of unity. We also write the primitive root of unit $\delta$ as $\exp(2\pi i/M)$. As a localization of the Dedekind domain $\mathbb{Z}[\delta]$, $k$ is a Dedekind domain of characteristic 0. In particular, the bases below would satisfy Assumption 3.24.

3.4. Construction of the Stack. From now on, fix the base ring $k = \mathbb{Z}[1/d, \zeta_M]$, where $M = 2\delta$, and $\zeta_M$ is the minimal base change such that the fibers are all étale. The log structures $M_S$ and $P$ are minimal in the sense that, when we define the versal family, we take the minimal base change such that the fibers are all reduced.

3.4.1. Local Charts: Formal Theory. Fix a general cusp $F_\xi$, and we decorate every notation associated to this cusp by $\xi$. Assume the associated rational isotropic subspace is $U_\xi$ of dimension $r$. Let $X_\xi^* = U_\xi \cap \Lambda$ and $X_\xi = \text{Hom}(X_\xi^*, \mathbb{Z})$. Choose a basis $\{v_1, \ldots, v_r\}$ of $X_\xi^*$. Define $Y_\xi = \Lambda/U_\xi^+ \cap \Lambda$. The restriction of the pairing $E$ defines the polarization $\phi : Y_\xi \to X_\xi$ of type $\delta$. Lift a compatible basis of $Y_\xi$ to $\{u'_1, \ldots, u'_g\} \subset \Lambda$. The restriction of $E$ is represented by a skew-symmetric integral matrix $S_\xi$ under the basis $\{u'_1, \ldots, u'_r\}$. Choose a symmetric integral matrix $S'_\xi$ such that $S'_\xi \equiv S_\xi \pmod{2\mathbb{Z}}$. Define the twist data

$$b' : Y_\xi \times X_\xi \to k, \quad a' : Y_\xi \to k,$$

$$b'(\lambda, \alpha) = \exp \left( -\pi i(x'_1(\lambda), \ldots, x'_r(\lambda)) S_\xi \delta^{-1} \begin{pmatrix} v_1(\alpha) \\ \vdots \\ v_r(\alpha) \end{pmatrix} \right),$$

$$a'(\lambda) = \exp \left( -1/2\pi i(x'_1(\lambda), \ldots, x'_r(\lambda)) S'_\xi \begin{pmatrix} x'_1(\lambda) \\ \vdots \\ x'_r(\lambda) \end{pmatrix} \right),$$

where $\{x'_1, \ldots, x'_r\}$ are the coordinates on $Y_\xi$ with respect to the basis $\{u'_1, \ldots, u'_r\}$.

The pairing $E$ also induces a nondegenerate skew-symmetric pairing on $U_\xi^+/U_\xi$ of type $\delta'$. Denote $\mathcal{A}_g', \delta'$ by $\mathcal{F}_\xi$. Therefore over $\mathcal{F}_\xi$, we have the universal family $\mathcal{A} \times_{\mathcal{F}_\xi} \mathcal{A}'$, where $\mathcal{A}$ is the universal family of abelian varieties with polarization $\lambda_\xi : \mathcal{A} \to \mathcal{A}'$ of type $\delta'$. Let $X_\xi$ and $Y_\xi$ be the constant sheaves over $\mathcal{F}_\xi$.

From now on, choose a 0-cusp with associated maximal isotropic subspace $U \supset U_\xi$. Extend the basis of $X_\xi^*$ to a basis of $X^* = U \cap \Lambda$. Extend $\{u'_1, \ldots, u'_r\}$ to $\{u'_1, \ldots, u'_g\}$ such that it is a lift of the basis of $Y = \Lambda/U_\xi \cap \Lambda$. Let $Y'$ denote the lattice generated by $u'_{r+1}, \ldots, u'_g$, and $(X')^*$ denote the lattice generated by $u'_{r+1}, \ldots, u'_g$. Under this chosen basis, $E$ is

$$E = \begin{pmatrix} S & \delta \\ -\delta & 0 \end{pmatrix}.$$

Write the $g \times g$-matrix $S$ in blocks

$$S = \begin{pmatrix} S_\xi & S_2 \\ S_3 & S_4 \end{pmatrix}.$$
Denote the type of \( \phi : Y_\xi \to X_\xi \) by \( \mathfrak{d}_1 \) and the type of \( \phi : Y \to X \) by \( \mathfrak{d} \) such that

\[
\mathfrak{d} = \begin{pmatrix}
\mathfrak{d}_1 & 0 \\
0 & \mathfrak{d}_2
\end{pmatrix}.
\]

Consider the sheaves \( \mathcal{H}om(X_\xi, A^t) \) and \( \mathcal{H}om(Y_\xi, A) \) over \( F_\xi \). The bundle \( F_\xi \times \overline{V}_\xi \) is the subset of \( \mathcal{H}om(X_\xi, A^t) \times \mathcal{H}om(Y_\xi, A) \) such that the following diagram commutes

\[
\begin{CD}
Y_\xi @>\phi>> X_\xi \\
| @VVV | \\
A @>\lambda_\xi>> A^t
\end{CD}
\]

(29)

Introduce an automorphism \( \iota \) on the set of the data \((c', c)\).

(30) \[ c'(\lambda) \mapsto \exp(-\pi i(x'_1(\lambda), \ldots, x'_r(\lambda))S_2\mathfrak{d}_2^{-1})c'(\lambda) \quad \forall \lambda \in Y_\xi, \]

(31) \[ c(\alpha) \mapsto \exp(-\pi iS_3\mathfrak{d}_1^{-1}(v_1(\alpha), \ldots, v_r(\alpha))T)c(\alpha) \quad \forall \alpha \in X_\xi. \]

Let’s explain the notations. Recall over \( C \), \( A \) can be regarded as a quotient of \( \mathbb{G}_m \), \( g' = g-r \), by the periods \( Y' \equiv \mathbb{Z}^d \). In Equation (30), the row vector \( \exp(-\pi i(x'_1(\lambda), \ldots, x'_r(\lambda))S_2\mathfrak{d}_2^{-1}) \), as an element of \( \mathbb{G}_m \) is acting on the sections of \( A \). In general, notice that \( d \) is invertible in the base ring \( k \). The row vector is an element of \( \mu_{2d_{r+1}} \times \ldots \times \mu_{2d_1} \). The choice of the 0-cusp determines a maximal isotropic subgroup for every finite subgroup scheme of \( A \), and thus a morphism of \( \mu_{2d_{r+1}} \times \ldots \times \mu_{2d_1} \). In Equation (31), the column vector \( \exp(-\pi iS_3\mathfrak{d}_1^{-1}(v_1(\alpha), \ldots, v_r(\alpha))T) \) should be regarded as an element in \( \mu_M \times \ldots \times \mu_M \), acting on \( A^t \). The morphism of \( \mu_M \times \ldots \times \mu_M \) into \( A^t \) is determined by the choice of the 0-cusp and the polarization. The map \( \iota \) is an automorphism of the set of \((c', c)\) where the diagram (29) commutes, because \( S_2 = S_3^T \) and they are both integral matrices.

Therefore, after using \( \iota \), there are the tautological extensions over \( F_\xi \times \overline{V}_\xi \)

\[
\begin{array}{cccccc}
1 &\longrightarrow & T &\longrightarrow & \tilde{G} &\longrightarrow & \pi &\longrightarrow & A &\longrightarrow & 0, \\
1 &\longrightarrow & T^t &\longrightarrow & \tilde{G}^t &\longrightarrow & \pi^t &\longrightarrow & A^t &\longrightarrow & 0.
\end{array}
\]

Let \( L_\xi \subset \Gamma^2 U_\xi \) be the integral structure from the integral polarized tropical abelian varieties, and \( L_\xi \subset \mathcal{S}_2 U_\xi^* \) be its dual. Identify \( C(F_\xi) \) with \( C(X_\xi) \). The torus is \( T_\xi = L_\xi \otimes \mathbb{G}_m \).

The \( T_\xi \)-bundle \( \Xi_\xi \) over \( F_\xi \times \overline{V}_\xi \) is defined as follows: For each character \( \phi(\lambda) \otimes \alpha \in L_\xi \subset \mathcal{S}_2 U_\xi^* \), the push-out along \( \phi(\lambda) \otimes \alpha \subset T_\xi \rightarrow \mathbb{G}_m \) is defined to be the rigidified \( \mathbb{G}_m \)-torsor \( (c'(\lambda) \times c(\alpha))^* \mathcal{P}_A^{-1} \) over \( F_\xi \times \overline{V}_\xi \). Here \( \mathcal{P}_A \) is the pull-back of the Poincaré bundle \( \mathcal{P}_A \) over \( A \times \overline{V}_\xi \). Since \( (c'(\lambda) \times c(\alpha))^* \mathcal{P}_A^{-1} \) is defined to be the push-out, the pull-back of it over \( \Xi_\xi \) is canonically trivial. Denote this tautological trivialization by \( \tau_\xi \). The space \( \Xi_\xi \) is the moduli space of the trivializations of biextensions \((c' \times c)^* \mathcal{P}_A^{-1} \). Since \((\mathrm{Id} \times \phi)^* b' \) is

\footnote{For example, we can choose a level structure of \( A \) that is compatible with the 0-cusp.}
symmetric on $Y_{\xi} \times Y_{\xi}$, $b'$ is acting on the trivializations of trivial biextensions over $Y_{\xi} \times Y_{\xi}$. Define $\tau$ to be

$$\tau(\lambda, \alpha) = b'(\lambda, \alpha)\tau(\lambda, \alpha).$$

Étale locally, we can choose a line bundle $M$ over $A$ which gives the polarization $\lambda_{\xi}$. This gives a $T$-linearized sheaf $\mathcal{L} = \pi^*M$ over $\mathcal{G}$. Moreover, étale locally, there exists a trivialization $\psi$ compatible with the universal trivialization $\tau$. We make an étale base change so that $\psi$ and $M$ are defined over $\Xi_{\xi}$.

The standard data is the second Voronoi fan $\Sigma(X_{\xi})$ with the support $\mathcal{C}(X_{\xi})^{\text{rc}}$. For each cone $C(\mathcal{P}) \in \Sigma(X_{\xi})$ that is in the interior of $\mathcal{C}(X_{\xi})$, we have the lattice $\mathbb{L}_{\mathcal{P}}$. Let $T_{\mathcal{P}}$ denote the torus with character group $\mathbb{L}_{\mathcal{P}}$. Étale locally, pick a section of the $T_{\xi}$-torsor $\Xi_{\xi}$ and define the projection $\Xi_{\xi} \rightarrow T_{\xi}$. Make the étale base change $\Xi_{T, \mathcal{P}} := \Xi_{\xi} \times_{T_{\xi}} T_{\mathcal{P}}$ along the morphisms $T_{\mathcal{P}} \rightarrow T_{\xi}$. Define $\Xi_{\mathcal{P}} := \Xi_{T, \mathcal{P}} \times_{T_{\mathcal{P}}} U_{C(\mathcal{P})}$, and the closed subscheme $\mathfrak{Z}_{\mathcal{P}} = \Xi_{T, \mathcal{P}} \times_{T_{\mathcal{P}}} V(C(\mathcal{P}))$. Define $\widehat{\Xi}_{\mathcal{P}}$ to be the completion of $\Xi_{\mathcal{P}}$ along $\mathfrak{Z}_{\mathcal{P}}$. For each cone $C(\mathcal{P})$, the fiber bundle $\widehat{\Xi}_{\mathcal{P}}$ satisfies the Assumption 3.24 and serves as the formal base along the boundary. While the fiber bundle $\Xi_{\mathcal{P}}$ provides the étale neighborhoods near the boundary.

We extend the family over the formal base $\widehat{\Xi}_{\mathcal{P}}$. Choose a sharp chart $\alpha : P_{\mathcal{P}} \rightarrow S_{\mathcal{P}} = \mathbb{L}_{\mathcal{P}} \cap C(\mathcal{P})^V$ for the log structure on the affine toric variety $U_{C(\mathcal{P})}$. Since $\Xi_{\mathcal{P}}$ is locally a product of the toric variety $U_{C(\mathcal{P})}$ with a regular scheme, $\Xi_{\mathcal{P}}$ has a log structure with a chart $P_{\mathcal{P}}$. This is actually the divisorial log structure, by Remark 3.38. The tautological section $\varphi_{\mathcal{P}} : X_{\xi, \mathbb{R}} \rightarrow S^2U_{\xi}^*$, with the log structure on $(\Xi_{\mathcal{P}}, M)$ is compatible with the degeneration data $\tau, \psi$. By Theorem 3.37 plus Grothendieck’s existence theorem (EGA III 1.5.4.5), we have an algebraic family $\pi : (X_{\mathcal{P}}, L_{\mathcal{P}}, G_{\mathcal{P}}, P_{\mathcal{P}}, \varphi_{\mathcal{P}}) \rightarrow (\widehat{\Xi}_{\mathcal{P}}, M_{\mathcal{P}})$. The polarization of the generic fiber is of type $\delta$. Since, over $\mathbb{Z}[1/d]$ the type of polarization is constant over a connected base, it suffices to check the type on a $C$-point. By Appendix B, we know the degeneration data gives the polarization of type $\delta$. To simplify the notations and to follow the constructions in [FC90], when $\bar{\sigma} = C(\mathcal{P})$ is a cone in $\Sigma(F_{\xi})$, we also denote the formal base $\widehat{\Xi}_{\mathcal{P}}$ by $S_{\bar{\sigma}}$, and the family by $\pi : (X_{\bar{\sigma}}, L_{\bar{\sigma}}, G_{\bar{\sigma}}, P_{\bar{\sigma}}, \varphi_{\bar{\sigma}})/(S_{\bar{\sigma}}, M_{\bar{\sigma}})$. We call this universal family a good formal $\sigma$-model.

Remark 3.39. By ([FC90] Chap. I Proposition 2.7), the semiabelian group scheme $G_{\mathcal{P}}$ is unique up to a unique isomorphism, and $\tau$ can be intrinsically defined by $G_{\mathcal{P}}$. Therefore the definition of $\tau$ does not depend on the choice of $S_{\xi}$, the 0-cusp, and the matrix $S$. Moreover, the isomorphism class of the family $(X_{\mathcal{P}}, G_{\mathcal{P}}, P_{\mathcal{P}}, \varphi_{\mathcal{P}})$ does not depend on the choices of $\psi, M$. The choices of $\psi, M$ are part of the data called a framing in ([A] Defintion 5.3.7). It is possible that, if we start from different data, $L_{\mathcal{P}}$ is changed to $L_{\mathcal{P}} \otimes \pi^*N$ for some invertible sheaf $N$ over the base $\widehat{\Xi}_{\mathcal{P}}$. We allow this in our definition of an isomorphism of families.
If \( C(\mathcal{P}') \) is a face of \( C(\mathcal{P}) \), the gluing map is \( k[S_{\mathcal{P}}] \to k[S_{\mathcal{P}'},] \). By Corollary 3.35, this morphism is étale. Moreover, by Corollary 3.33, the difference \( f^2\alpha \varphi_{\mathcal{P}} / \alpha' \varphi_{\mathcal{P}} \) is invertible in \( k[S_{\mathcal{P}}] \) and thus invertible in \( \Xi_{\mathcal{P}} \), and the families can be glued by Proposition 3.22.

There is another type of étale pre-equivalence relation described by the finite group scheme \( k[L_{\mathcal{P}}/L_{\xi}] \).

For each \( \alpha \in B(\mathcal{Z}) := X/\phi(Y) \), choose \( \vartheta_{A,\alpha} \in H^0(A, M_\alpha) \) such that \( \vartheta_{A,\alpha} \) does not vanish along any fiber \( A_s, s \in S_\sigma \). Define \( \vartheta \) as the descent \((Y\)-action\) from \( \tilde{\vartheta} := \sum_{\alpha \in B(\mathcal{Z})} \sum_{\lambda \in Y} S^*_\lambda \vartheta_{A,\alpha} \).

Let \( \Theta \) be the zero locus of \( \vartheta \). By Lemma 3.20, the family \( \pi : (X_\sigma, L_\sigma, G_\sigma, \Theta, \tau_\sigma) \to S_\sigma \) is an object in \( \mathcal{A}P_{g,d} \).

**Remark 3.40.** When we apply the isomorphisms from \( \text{Spec} k[L_{\mathcal{P}}/L_{\xi}] \), each component \( \sum_{\lambda \in Y} S^*_\lambda \vartheta_{A,\alpha} \) will be changed by a constant in \( \mu_{M_\alpha} \), and this constant depends on \( \alpha \). Therefore \( \Theta \) only exists étale locally, and does not descend over the moduli stack.

If the cusp \( F_\xi \) is a 0-cusp, the abelian part \( A \) is trivial, \( \tilde{\vartheta} = \sum_{\alpha \in B(\mathcal{Z})} \sum_{\lambda \in Y} S^*_\lambda (X_\alpha \varphi_{\mathcal{P}}(\alpha) \theta) \).

The choices of \( \vartheta \) become finite. In general, the set of stable sections \( \vartheta \) can be characterized intrinsically by making use of a representation of a group scheme \( G(M) \). We will discuss this issue in Section 4.

The discrete group \( \mathcal{P}(F) \) is a subgroup of \( \text{GL}(X,Y) \) of finite index. The group \( \text{GL}(X,Y) \) is acting on both \( U^* \) and \( S^2U^* \). The sections \( \varphi_{\mathcal{P}} \) are \( \text{GL}(X,Y) \)-equivariant, and thus \( \mathcal{P}(F) \)-equivariant. That implies the construction is compatible with the action of \( \mathcal{P}(F) \). Moreover, since \( \text{GL}(X,Y) \) preserves \( X \) and \( Y \), it preserves the set \( B(\mathcal{Z}) \). It follows that the construction is \( \mathcal{P}(F_\xi) \)-equivariant.

**Remark 3.41.** Ideally, we would like to define the moduli space over \( k = \mathbb{Z}[1/d] \). Since \( \mathbb{Z}[1/d] \to \mathbb{Z}[1/d, \zeta_M] \) is a faithfully flat, the base change for \( \Xi_{\mathcal{P}} \) (resp. \( \hat{\Xi}_{\mathcal{P}} \)) is also faithfully flat. It is possible to construct the miniversal families by descent. Or equivalently, we hope to construct the degeneration data \((c^i, c, \tau, \psi)\) over \( \mathbb{Z}[1/d] \) from a faithfully flat descent.

3.4.2. **Algebraization and Glues.** After we have obtained the miniversal families over the formal bases (complete rings), we need to extend them to étale bases (rings of finite type over \( k \)). The étale neighborhoods of the boundary in \( \Xi_{\mathcal{P}} \) server as the étale bases.

We follow the procedure in [FC90]. Let \( \sigma \) denote the rational cone \( C(\mathcal{P}) \) associated to a bounded paving \( \mathcal{P} \). Since \( \Xi_{\mathcal{P}} \) is a fibered by the toric variety \( U_\sigma, \Xi_{\mathcal{P}} \) has the natural stratification induced by the toric stratification of \( U_\sigma \). Then we can define the étale constructible sheaf \( X_\xi \) (resp. \( Y_\xi \)). If \( \tau \) is a face of \( \sigma \), the elements in \( \tau \) are quadratic forms.
over $X_{\tau, R}$, and $X_{\tau}$ (resp. $Y_{\tau}$) is the quotient of $X_{\sigma}$ (resp. $Y_{\sigma}$). Then, over the $\tau$-stratum, define the sheaf $X_{\xi}$ (resp. $Y_{\xi}$) to be the constant sheaf $X_{\tau}$ (resp. $Y_{\tau}$). Moreover, over each $\tau$-stratum, we have the tautological bilinear pairing $B : Y_{\tau} \times X_{\tau} \to \mathbb{L}_{\tau}$. The elements in $L_{\tau}$ are sections of $K^* / O^*$ over the toric variety $U_{\tau} \subset U_{\sigma}$. Therefore, we get a pairing $B_{\xi} : Y_{\xi} \times X_{\xi} \to \text{Div} \Xi_{\varphi}$ from the toric data.

If $(X_{\sigma}, L_{\sigma}, G_{\sigma}, P_{\sigma}, \varrho_{\sigma})$ over $(S_{\sigma}, M_{\sigma})$ is a good formal $\sigma$-model, we can forget about the other data, and get a good formal $\sigma$-model $(G_{\sigma}, \lambda)$ over $S_{\sigma}$, with a general polarization $\lambda$ ([FC90] Chap. IV Definition 3.2 & Proposition 3.3 (i)). Notice that all the results in ([FC90] Chap. III Sect. 9 & 10) are proved for general polarizations, therefore (loc. cit. Chap. IV Definition 3.2 & Proposition 3.3 (i)) can be naturally generalized for the general polarizations. Since $(S_{\sigma}, M_{\sigma})$ is the divisorial log structure for the toroidal boundary $\partial S_{\sigma}$, let the log differential $\Omega^1_{(S_{\sigma}, M_{\sigma})}$ be denoted by $\Omega^1_{S_{\sigma}}[d \log \infty]$, the differential with log poles along the boundary. In particular for good formal $\sigma$-model, we have

**Proposition 3.42.** Suppose $(X_{\sigma}, L_{\sigma}, G_{\sigma}, P_{\sigma}, \varrho_{\sigma})$ over $(S_{\sigma}, M_{\sigma})$ is a good formal $\sigma$-model.

1. The sheaves and the pairing $(B, X, Y)$ obtained from the polarized semi-abelian scheme $(G_{\sigma}, \lambda)$ agree with the sheaves and the pairing $(B_{\xi}, X_{\xi}, Y_{\xi})$ obtained from the toric variety fibers.

2. Let $\Omega = \Omega(G_{\sigma} / S_{\sigma})$ be the dual of the relative invariant Lie algebra of $G_{\sigma}$ and $\Omega^t = \Omega(G_{\sigma}^t / S_{\sigma})$ be the dual of the relative invariant Lie algebra of $G_{\sigma}^t$. The Kodaira–Spencer map induces an isomorphism $S^2 \Omega^t \cong \Omega^1_{S_{\sigma}}[d \log \infty]$.

Notice that our formal base is the same with that in [FC90], we can do the same approximation to the rings of finite type over $\Xi_{\varphi}$. The point is the two properties in Proposition 3.42 are preserved in the process of approximation. Therefore we have (loc. cit. IV Proposition 4.3, 4.4) for our case.

**Proposition 3.43.** Let $\sigma = C(\mathcal{P})$, $R$ be the strict local ring of a geometric point $\mathfrak{p}$ of the $\sigma$-stratum of $\Xi_{\varphi}$, $I$ be the ideal defining the $\sigma$-stratum, and $\hat{R}$ be the $I$-adic completion of $R$. There exists an étale neighborhood $S' = \text{Spec} R'$ of $\mathfrak{p}$, a family $(X, L, P, G, \varrho)$ over $(S', M')$, and an embedding $R' \to \hat{R}$ close to the canonical inclusion, such that

1. The family $(X, L, P, G, \varrho)$ is an object in $\mathcal{F}_{g,d}(S')$. The log structure is the divisorial log structure.

2. The map $\text{Spf} \hat{R} \to \hat{\Xi}_{\varphi}$ coincides on the $\sigma$-stratum with the map induced by the inclusion $R' \to \hat{R}$.

3. Over $R/I$, $(X, L, P, G, \varrho)$ is isomorphic to the pull back of the good formal $\sigma$-model $(X_{\sigma}, L_{\sigma}, P_{\sigma}, G_{\sigma}, \varrho_{\sigma})$.

4. The étale sheaves and the pairing $(B, X, Y)$ obtained from the family $G$ coincide with the pull backs of the étale sheaves and the pairing $(B_{\xi}, X_{\xi}, Y_{\xi})$ over $\Xi_{\varphi}$.

5. The Kodaira–Spencer map induces an isomorphism $S^2 \Omega^t \cong \Omega^1_{R'}[d \log \infty]$.

**Proof.** For (2)–(5), the same proof as that of ([FC90] Chap. IV Proposition 4.4). The statement (1) follows from (3) and ([Ols08] Theorem 5.9.1).
We call the family obtained above a good algebraic $\sigma$-model. For any geometric point $\bar{x}$ over $x \in S$ in the $\tau$-stratum of a good algebraic $\sigma$-model over $S$, denote the strict henselization by $\widehat{R}_{\bar{x}}$. The polarized family $G$ over $\widehat{R}_{\bar{x}}$ gives the degeneration data. By Proposition 3.43 (4), we know $B$ agrees with $B_{\bar{x}}|_{\tau}$. By the universal property described in (at the end of the third paragraph [FC90] p. 106), this defines a morphism $f : \text{Spf} \widehat{R}_{\bar{x}} \to \Xi_{\tau}$. The family $G$ over $\widehat{R}_{\bar{x}}$ is a good formal $\tau$-model. The boundary divisors agree under $f$. It follows that $f$ is strict for the log structures. Moreover, by Proposition 3.43 (5), the log differentials $\Omega^1_{S,M} \cong f^*\Omega^1_{\Xi_{\tau},M_{\tau}}$. Therefore, $f$ is log étale. Openness of versality follows from the following lemma.

**Lemma 3.44.** If a log morphism $f : (X,M_X) \to (Y,M_Y)$ is strict, then $f$ being formally log étale implies that the underlying morphism $f$ between schemes is formally étale.

**Proof.** Consider the affine thickening diagram

\[
\begin{array}{ccc}
T_0 & \xrightarrow{a^0} & X \\
\downarrow j & & \downarrow f, \\
T & \xrightarrow{a} & Y
\end{array}
\]

where $T$ is affine, and $j$ is a closed immersion defined by a nilpotent ideal. Provide $T$ (resp. $T_0$) the log structure $a^*M_Y$ (resp. $a_0^*M_X$), the diagram is still commutative. Because $f$ is strict, $j$ is strict. Since $f$ is formally log étale, there is a unique lift $b : T \to X$. Thus $f$ is formally étale. \qed

The good algebraic models are the étale neighborhoods of the compactification. Define $U$ to be the disjoint union of finitely many good algebraic models that cover all strata $\mathfrak{Z}_\varphi$ up to the action of $\Gamma(\delta)$. This is possible because there are only finitely map 0-cusps up to $\Gamma(\delta)$-action, and for each 0-cusp $F$, the number of $\mathcal{P}(F)$-orbits of cones in $\Sigma(F)$ is finite. The cone $\{0\}$ corresponds to the moduli space $\mathfrak{A}_{\varphi,\delta}$. However, by ([Ols08] Proposition 5.1.4), the universal family is interpreted as the family $(X,\mathcal{L},\varrho,G)$ with $G$ abelian. Let $U_0$ denote the dense open stratum of $U$ where $G$ is abelian. Let $R_0$ denote the étale relation $R_0 := U_0 \times_{\mathfrak{A}_{\varphi,\delta}} U_0$. Define $R$ to be the normalization of the image $R_0 \to U \times U$, i.e. normalization of the closure $Z$ of the image of $R_0$ in $U \times U$ with respect to the finite extensions of the function fields at the maximal points of $Z$. We can extend the isomorphisms by an explicit construction. To save the work, we use the following analog of ([FC90] Chap. I Proposition 2.7).

**Proposition 3.45.** Let $S$ be a noetherian normal scheme, and $\pi_i : (\mathcal{X}_i,P_i,\mathcal{L}_i,G_i,\varrho_i)$ for $i = 1,2$ be two good algebraic models over $S$. Suppose that over a dense open subscheme $U$ of $S$, the log structures $P_i$ are trivial, and there is an isomorphism $f_U$ between $(\mathcal{X}_1,P_1,\mathcal{L}_1,G_1,\varrho_1)|U$. Then $f_U$ extends to a unique isomorphism $f : (\mathcal{X}_1,P_1,G_1,\varrho_1) \to (\mathcal{X}_2,P_2,G_2,\varrho_2)$ and there exists a line bundle $\mathcal{M}$ over $S$, such that $\mathcal{M}|_U$ is trivial, and $\mathcal{L}_1 \otimes \pi_1^*\mathcal{M} \cong f^*\mathcal{L}_2$. 
Proof. For the extension problem of morphisms between data $(X_i, G_i, g_i)$, by a standard reduction process ([FC90] Chap. I Proof of Proposition 2.7 Step (a)), it suffices to consider the case when $S$ is the spectrum of a discrete valuation ring. This case is proved by ([Ols08] Proposition 5.11.6). Let the unique extension be $f$. From this reduction to discrete valuation rings, we also know that $f^* L_2$ and $L_1$ are isomorphic on each fiber. Since $X_1$ is projective over $S$ and the fibers are all integral, by ([Har77] Chap. III Exercise 12.4), there exists $M$ over $S$ such that $L_1 \otimes \pi_1^* M \cong f^* L_2$. Finally, by ([Ols08] Proposition 5.10.2), $f$ preserves the log structures. \qed

By Proposition 3.45, the isomorphism between the universal families over $R_0$ is extended to the isomorphism between the good algebraic models over $R''$. Since morphisms in $\mathcal{M}_{g,d}$ is defined up to an action of $G_m$, we have

Corollary 3.46. Let $R' := U \times_{\overline{\mathcal{M}}_{g,d}} U$. We have defined a morphism $R \to R'$, and this map is injective.

Denote the two projections $R \to U$ by $s$ and $t$. Notice that the definitions of $U$ and $R$ only involve the collection of fans $\Sigma$ and the algebraization of the bases. Therefore we can use the same $U$ and $R$ as those in ([FC90] Chap. IV. Sect. 5.). We simply replace the semi-abelian schemes over $U$ and $R$ by AN families.

Proposition 3.47 ([FC90] Chap. IV Lemma 5.2). Suppose $R$ is a normal complete local ring which is strictly henselian, $K$ its field of fractions and $\kappa$ its residue field. Assume furthermore that $G_\delta$ is a semi-abelian scheme over $R$ whose generic fiber $G_\delta^0$ is an abelian variety with a polarization $\lambda_K$ of type $\delta$. Associated to this we have the character group $X_\delta$ (resp. $Y_\delta$) of the torus part of the special fiber $G_s$ (resp. $G_s^0$), a polarization $\phi : Y_\delta \to X_\delta$, and a bimultiplicative form $b$ on $Y_\delta \times X_\delta \to K^*$, defined up to units in $R$, such that $b(Y_\delta, \phi(Y_\delta))$ is symmetric. The following statements are equivalent

1. Write $X_\delta$ as the quotient lattice of $X = \mathbb{Z}^4$. If $v : K^* \to \mathbb{Z}$ is any discrete valuation defined by a prime ideal of $R$ of height one, $v \circ b$ defines a positive semi-definite symmetric bilinear form on $X$. There exists a (closed) cone $C(\mathcal{P})$ in the second Voronoi fan $\Sigma(X)$ which contains all $v \circ b$ obtained this way.

2. $(G_\delta^0, \lambda)$ is the semi-abelian scheme of the pull-back of a good formal $\sigma$-model $(X_\sigma, L_\sigma, G_\sigma, g_\sigma)/S_\sigma$ via a morphism $\text{Spec} R \to \Sigma_\sigma$ (equivalently, a map $\text{Spf} R \to S_\sigma$) for some (open) cone $\sigma$ in $\Sigma(X)$.

The cone $C(\mathcal{P})$ in (1) and the cone $\sigma$ in (2) are in the same $GL(X,Y)$-orbits, hence can be chosen to be the same.

Theorem 3.48. The morphism $(s,t) : R \to U \times U$ defines an étale groupoid in the category of $k$-schemes. Let $\mathcal{M}_{g,\delta}$ be the stack $[U/R]$ over $k$. It is a proper Deligne-Mumford stack, containing $\mathcal{M}_{g,\delta}$ as an open dense substack. It admits a coarse moduli space. Over $\mathcal{C}$, the coarse moduli space is the toroidal compactification $\mathcal{M}_\Sigma$.  

\footnote{Again we say $f$ preserves the line bundles, if $L_1 \otimes \pi_1^* M \cong f^* L_2$.}
Proof. Notice that none of the arguments in ([FC90] Chap. IV Proposition 5.4, Corollary 5.5 & Theorem 5.7 (1)) uses the principal polarization. By ([Ols08] Theorem 1.4.2), the coarse moduli space exists. By the construction, the coarse moduli space over $\mathbb{C}$ is $\mathcal{A}_\Sigma$. □

From now on, let $\mathcal{A}_\Sigma$ also denote the coarse moduli space of $\mathcal{A}_{g,\delta}$ over the ring $k$.

**Proposition 3.49.** The algebraic stack $\mathcal{A}_{g,\delta}$ has a log structure $M_\Sigma$ such that $(\mathcal{A}_{g,\delta}, M_\Sigma)$ is log smooth over $k = \mathbb{Z}[1/d, \zeta_M]$.

Proof. It follows from the description of a good algebraic model in Proposition 3.43 □

**Lemma 3.50.** The algebraic space $\mathcal{A}_\Sigma$ is normal.

Proof. Recall that the local chart for the algebraic stack $\mathcal{A}_{g,\delta}$ is a toric monoid. By ([Ogu06] Chap. I Proposition 3.3.1.), $\mathcal{A}_{g,\delta}$ is normal. Since $\mathcal{A}_{g,\delta}$ is a separated Deligne-Mumford stack, étale locally, it is presented as $[U/G]$ for $U$ a scheme étale over the stack, and $G$ a finite group ([AV02] Lemma 2.2.3). Étale locally, the coarse moduli space is the coarse moduli space $[U/G] \to U/G$. Now $U$ is normal, and $U/G$ is a quotient by a finite group, thus is also normal. □

Since $d$ is invertible in $k$, $\mathcal{A}_{g,d}$ is a disjoint union of $\mathcal{A}_{g,\delta}$ for all polarization type $\delta$ of degree $d$. Moreover, $\mathcal{A}_{g,d}$ is log smooth in this case. Therefore $\mathcal{A}_{g,d}$ is a disjoint union of connected components $\mathcal{A}_{g,\delta}$. The coarse moduli space $\mathcal{A}_{g,d}$ is a disjoint union of connected components denoted by $\mathcal{A}_{g,\delta}$.

**Proposition 3.51.** Over $k = \mathbb{Z}[1/d, \zeta_M]$, the morphism $F : \mathcal{A}_{g,\delta} \to \mathcal{A}_{g,\delta}$ induced by the AN family is proper, surjective, and representable. Furthermore, the coarse moduli space $\mathcal{A}_\Sigma$ is the normalization of $\mathcal{A}_{g,\delta}$.

Proof. There is a morphism $U \to \mathcal{A}_{g,d}$ induced from Proposition 3.43 (1), and an injection $R \to R'$. This induces a morphism between stacks $[U/R] \to [U/R']$. By ([LMR00] Proposition (3.8)), $[U/R']$ is a substack of $\mathcal{A}_{g,\delta}$. The composition is a morphism $\tilde{F} : \mathcal{A}_{g,\delta} \to \mathcal{A}_{g,\delta}$. Since both $\mathcal{A}_{g,\delta}$ and $\mathcal{A}_{g,d}$ are proper, by ([Ols13] Proposition 10.1.4 (iv)), $\tilde{F}$ is proper in the sense of ([Ols13] Definition 10.1.3). The morphism $\tilde{F}$ is identity on the dense open substack $\mathcal{A}_{g,\delta}$ of $\mathcal{A}_{g,\delta}$. Then it is surjective onto the closure $\mathcal{A}_{g,\delta}$.

By Lemma 3.52 and Corollary 3.46, $\tilde{F}$ is representable. In particular, by ([Ols13] Proposition 10.1.2), $\tilde{F}$ is proper as a representable morphism.

Let the morphism between the coarse moduli spaces be denoted by $f : \mathcal{A}_\Sigma \to \mathcal{A}_{g,\delta}$. As a morphism between proper spaces, $f$ is proper. Moreover, by Lemma 3.53, $f$ is quasi-finite, and thus finite. Now $f$ is also birational, and $\mathcal{A}_\Sigma$ is normal by Lemma 3.50, hence $f$ is the normalization of $\mathcal{A}_{g,\delta}$. Here the normalization of an algebraic space is defined in ([KM99] Appendix N.3). □
Lemma 3.52. If $F$ induces an injection $R \to R'$, then it is representable between algebraic stacks.

Proof. We claim that the functor $F$ is faithful. By the property of categories fibered in groupoids ([Ols13] Lemma 3.1.8), it suffices to prove that for any $S \in (\text{Sch}/k)$, and $u, u' \in \mathcal{T}_Q(S)$, the map $F_S: \text{Isom}_{\mathcal{X}_Q(S)}(u, u') \to \text{Isom}_{\mathcal{X}_Q(S)}(F(u), F(u'))$ is injective. Since $\text{Isom}_{\mathcal{X}_Q(S)}(u, u')$ is a torsor over $\text{Aut}_{\mathcal{X}_Q(S)}(u)$, if it is not empty, it can be reduced to the case $u = u'$. These are global sections of presheaves $\text{Isom}(u, u)$ and $\text{Isom}(F(u), F(u))$. Since both $\mathcal{T}_Q$ and $\mathcal{X}_Q$ are algebraic stacks, both of the presheaves are sheaves for the étale topology on $(\text{Sch}/S)$. So we can check the injectivity on some étale covering of $S$.

Let $\xi: S' \to S$ be the pullback of $U \to \mathcal{T}_Q$ along $u: S \to \mathcal{T}_Q$.

$$\begin{array}{ccc}
S' & \xrightarrow{\nu} & U \\
\xi \downarrow & & \downarrow \\
S & \xrightarrow{u} & \mathcal{T}_Q.
\end{array}$$

By definition, there is an isomorphism $\rho: \xi^*u \to v$ in $\mathcal{T}_Q(S')$. Using $\rho$, replace $\xi^*u$ by $v$. Consider the Cartesian diagrams

$$\begin{array}{ccc}
\text{Isom}(v, v) & \xrightarrow{} & S' \\
\downarrow & & \downarrow \\
\mathcal{T}_Q & \xrightarrow{\Delta} & \mathcal{T}_Q \times \mathcal{T}_Q, \\
\text{Isom}(F(v), F(v)) & \xrightarrow{} & S' \\
\downarrow & & \downarrow \\
\mathcal{T}_Q & \xrightarrow{\Delta} & \mathcal{T}_Q \times \mathcal{T}_Q.
\end{array}$$

By the universal property of $R$, there is a 1-morphism $H: \text{Isom}(v, v) \to R$ such that the following diagram commutes,

$$\begin{array}{ccc}
\text{Isom}(v, v) & \xrightarrow{} & S' \\
H \downarrow & & \downarrow \ \\
R & \xrightarrow{(s,t)} & U \times U.
\end{array}$$

This diagram is Cartesian. Similarly, after applying $F$, we have the Cartesian diagram

$$\begin{array}{ccc}
\text{Isom}(F(v), F(v)) & \xrightarrow{} & S' \\
\mathcal{F}(H) \downarrow & & \downarrow \ \\
R' & \xrightarrow{(s,t)} & U \times U.
\end{array}$$

Since the composition of two pullbacks is a pullback, the diagram

$$\begin{array}{ccc}
\text{Isom}(v, v) & \xrightarrow{} & \text{Isom}(F(v), F(v)) \\
H \downarrow & & \mathcal{F}(H) \downarrow \\
R & \xrightarrow{} & R'.
\end{array}$$
is Cartesian. Then $R \to R'$ is injective implies that the map $\overline{F}_{S'} : \text{Aut}_{\mathcal{X}(S')}(v) \to \text{Aut}_{\mathcal{X}(S')}(\overline{F}(v))$ is injective. The claim that $\overline{F}$ is faithful is thus proved.

By ([dJea] Lemma 67.15.2), $\overline{F}$ is representable. □

Lemma 3.53. The morphism $f : \mathcal{A}_g \to \mathcal{A}_{g,d}$ is quasi-finite.

Proof. It suffices to check that for any algebraic closed field $\kappa(\bar{x})$ and point $\bar{x} : \text{Spec} \kappa(\bar{x}) \to \mathcal{A}_{g,d}$, the fiber $\overline{F}^{-1}(\bar{x})$ is discrete, since $f$ is of finite type and thus quasi-compact. So we fix such a point $\bar{x}$ on the boundary of $\mathcal{A}_{g,d}$. Forget about the log structure, it represents a polarized stable semiabelic variety $(X,G,L,g)$ over $\kappa(\bar{x})$. By ([Ols08] 5.3.4), $(X,G,L,\bar{g})$ determines the decomposition $\mathcal{P}$, the subgroup $\phi : Y \to X$, the maps $c : X \to A$ and $c^t : Y \to A^t$. Since the $\mathcal{P}$-stratum in $\mathcal{A}_g$ is a quotient of the interior $\mathcal{Z}_{\mathcal{P}}$, and still denote it by $\overline{F}^{-1}(\mathcal{P})$. More precisely, $\overline{F}^{-1}(\mathcal{P})$ is contained in the fiber over $(A,A^t,\phi,\lambda,c,c^t) \in \mathcal{P}_\xi \times \mathcal{V}_\xi$. Let’s denote the fiber by $V(\mathcal{P})$. It is a torus and is isomorphic to $H^0(\Delta_\mathcal{P},\hat{\mathcal{L}})$ in [Ale02]. The claim is that there is no positive dimensional sub locus of $V(\mathcal{P})$ where the fibers are all isomorphic as polarized stable semiabelic varieties (SSAV).

Recall the framing for a polarized SSAV in ([Ale02] Definition 5.3.6). By ([Ale02] Theorem 5.3.8), the framed polarized SSAV are classified by the groupoid $M_{\text{fr}}[\Delta_\mathcal{P},c,c^t,M](\kappa(\mathcal{P}))$ which is equivalent to $[Z^1(\Delta_\mathcal{P},\hat{\mathcal{X}})/C^0(\Delta_\mathcal{P},\hat{\mathcal{X}})]$. Fix the framing, an element in $Z^1(\Delta_\mathcal{P},\hat{\mathcal{X}})$ is the gluing data denoted by $\psi_0\tau_0$. Fix $(A,A^t,\phi,\lambda,c,c^t)$, we can construct the unframed groupoid $M[\Delta_\mathcal{P},c,c^t](\kappa(\mathcal{P}))$, i.e. the groupoid of polarized SSAV as in (loc. cit. 5.4.4). Let $\text{Pic}^\lambda A$ be the component of $\text{Pic} A$ of polarization $\lambda$. Over $\text{Pic}^\lambda A$, take the family $Z^1(\Delta_\mathcal{P},\hat{\mathcal{X}})$, and denote the union by $M_{\text{fr}}$. Then quotient out the choice of the framing. We won’t write down all the equivalent relations, because it suffices to only consider the choices that are not discrete. There are two type of non-discrete choices we have to quotient out. One is the choice of the projection to $A$, i.e. a choice of a point in the minimal $A$-orbit. See (loc. cit. Definition 5.3.7 (4)). The other is the action of $C^0(\Delta_\mathcal{P},\hat{\mathcal{X}})$.

In our construction, we have chosen $M$ and $\psi$ étale locally. Therefore, there is a natural framing for our family over $V(\mathcal{P})$. Since we construct the base $\hat{\mathcal{X}}_\xi(\mathcal{P})$ as the moduli space for the degeneration data $\tau$. It defines a finite map $\overline{F}$ from $V(\mathcal{P})$ to the fiber $Z^1(\Delta_\mathcal{P},\hat{\mathcal{X}})$ over $\mathcal{M} \in \text{Pic}^\lambda A$. Consider the image of this map as a subset in $M_{\text{fr}}$ and denote it by $\mathcal{Y}$. It suffices to show that the intersection of $\mathcal{Y}$ with the group orbits is discrete.

First, consider the group orbits generated by changing the projections to $A$. If the projection to $A$ is changed by an element $a \in A$, then $\mathcal{M}$ is changed to $T_{-a}\mathcal{M}$. However, $\mathcal{M}$ is ample, there are only finitely many $a \in A$ such that $\mathcal{M} \cong T_{-a}\mathcal{M}$. Therefore, this

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13Here we mean every fiber of the map is finite.
group orbit intersects the fiber $Z^1(\Delta, \hat{X})$ over $\mathcal{M} \in \text{Pic}^λA$ at finitely many points, thus intersects $\mathcal{Z}$ at only finitely many points.

Secondly, the action of $C^0(\Delta, \hat{X})$ on the fiber $Z^1(\Delta, \hat{X})$ over $\mathcal{M} \in \text{Pic}^λA$ at finitely many points, thus intersects $\mathcal{Z}$ at only finitely many points.

In sum, there is no positive dimensional locus in $V(\mathcal{P})$ that is mapped to a point in the groupoid of polarized SSAV, and our claim is proved. □

Remark 3.54. Although the algebraic stack $\mathcal{A}_{g,d}$ is normal over $\mathbb{Z}[1/d]$, we are unable to prove that its coarse moduli space is also normal. The problem is we don’t know if $\mathcal{A}_{g,d}$ is Deligne–Mumford or not. Therefore, locally, we can only choose a finite flat groupoid representation $R → U \times U$, instead of an étale representation. Then we don’t know how to choose $U$ to be normal.

4. Representations of the Theta Group and the Stable Pairs

Fix the base field $k = \mathbb{C}$ in this section.

4.1. The Heisenberg Relation and the Fourier Decomposition. Let’s study a single abelian variety first. In order to avoid the confusion about the abelian variety $X$ and the group of characters $X$, we use $G$ to denote an abelian variety for now. Fix a polarized abelian variety $(G, \lambda)$ over $\mathbb{C}$. Suppose the polarization $\lambda : G → G^t$ is induced by an ample line bundle $L$. Let $K(L)$ be the kernel of $\lambda$. The automorphism group of the pair $(G, L)$ over the translations of $G$ is the theta group $\mathcal{G}(L)$. $\mathcal{G}(L)$ is a central extension of $K(L)$ by $\mathbb{C}^*$

$\begin{array}{c}
1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{G}(L) \xrightarrow{p} K(L) \longrightarrow 0.
\end{array}$

The space of global sections $\Gamma(G, L)$ is an irreducible $\mathcal{G}(L)$-module. For the reference of the facts about the theta group, see [Mum67].

Represent $G = V/\Lambda$, where $V$ is the universal cover and $\Lambda = H_1(G, \mathbb{Z})$. The polarization is represented by an integral skew symmetric form $E$ over $\Lambda$. Define $\Lambda := \{v \in V; E(v, v') \in \mathbb{Z}, \forall v' \in \Lambda\}.$

Definition 4.1. Define $\mathcal{H}(V, E)$ to be a central extension of $V$ by $\mathbb{C}^*$,

$\begin{array}{c}
1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{H}(V, E) \xrightarrow{p} V \longrightarrow 0.
\end{array}$

that induces the pairing $\exp(-2\pi iE)$ on $V$. In other words, it is the Heisenberg group with the commutator

$$
(v, t)(v', t')(v, t)^{-1}(v', t')^{-1} = (0, \exp(-2\pi iE(v, v'))) .
$$

14Use the notations in (loc. cit. Definition 5.3.4), $ψ_0$ is a function over $C_0(\Delta, \hat{X})$. 
Fix a Lagrangian $U \subset V$. Denote $U \cap \Lambda$ by $X^*$ and $\Lambda/X^*$ by $Y$. Choose a classical factor of automorphy $e_L \in Z^1(\Lambda, \Gamma(V, \mathcal{O}_V^*))$ that is trivial over $X^*$. Assume (BL01 p. 50)

$$e_L(\lambda, v) = \chi(\lambda) \exp \left( \pi(H-B)(v, \lambda) + \frac{\pi}{2}(H-B)(\lambda, \lambda) \right), \quad \lambda \in \Lambda, v \in V.$$ 

The semi-character $\chi$ defines a lift $\Lambda \rightarrow \mathcal{H}(V, E)$ via $\lambda \mapsto (\lambda, \chi^{-1}(\lambda))$. It is a group homomorphism since $\chi$ is a semi-character. From now on, we regard $\Lambda$ as a subgroup of $\mathcal{H}(V, E)$. Notice that $X^*$ is lifted to $(X^*, 1)$. Take the normalizer $N(\Lambda)$ of $\Lambda$ and the normalizer $N(X^*)$ of $X^*$ in $\mathcal{H}(V, E)$.

Consider $h : \Lambda \times V \rightarrow \mathbb{C}^*$ defined by

$$h(\lambda, v) := \exp \left( \pi(H-B)(v, \lambda) + \frac{\pi}{2}(H-B)(\lambda, \lambda) \right), \quad \lambda \in \Lambda, v \in V.$$ 

Extend $h$ to $V \times V \rightarrow \mathbb{C}^*$. Then $h$ satisfies

$$h(v_1 + v_2, v')e^{\pi i E(v_1, v_2)} = h(v_1, v' + v_2)h(v_2, v').$$ 

Define the action of $\mathcal{H}(V, E)$ on the trivial bundle $V \times \mathbb{C}$

$$U_{(v, t)}(v', t') = (v + v', t^{-1}t'h(v, v')).$$ 

This is a group representation of $\mathcal{H}(V, E)$ by Relation (33). Since $\Lambda$ (resp. $X^*$) is in the center of $N(\Lambda)$ (resp. $N(X^*)$), the action $U$ induces an action of $N(\Lambda)/\Lambda$ (resp. $N(X^*)/X^*$) on the quotient $(G, \mathcal{L}) = (V, V \times \mathbb{C})/\Lambda$ (resp. $(T, T \times \mathbb{C}) = (V, V \times \mathbb{C})/X^*$). Notice that if $\mathcal{L}$ is considered as the coherent sheaf of sections instead of the line bundle, then the action of the center $\mathbb{C}^* \subset N(\Lambda)/\Lambda$ is of weight 1. Therefore, the action of $N(\Lambda)/\Lambda$ on $\Gamma(G, \mathcal{L})$ is identified with the action of $G(\mathcal{L})$ on $\Gamma(G, \mathcal{L})$. On the other hand, $N(X^*)/X^*$ contains the subgroup $(U/X^*, 1)$. By (BL01 Lemma 3.2.2. a)), $h(U, V) = 1$. Then the action (34) of $a \in (U/X^*, 1)$ on $T \times \mathbb{C}$ is $T_a^* : (v, t) \mapsto (av, t)$. Therefore we also identify the action of $(U/X^*, 1)$ on $T \times \mathbb{C}$ with the action of the real subtorus $U/X^* \subset V/X^* = T$ on $T \times \mathbb{C}$. Define $\phi : \Lambda^\vee \rightarrow X$ by $\phi(\lambda)(u) = E(\lambda, u)$ for $\lambda \in \Lambda^\vee$, $u \in X^*$. We generalize the Heisenberg relation in [Mum72].

Lemma 4.2 (Heisenberg relation I). Consider the two groups $N(\Lambda)/X^*$ and $T$ acting on the trivial line bundle $T \times \mathbb{C}$. Denote the action of $N(\Lambda)/X^*$ by $S$, and the action of $T$ by $T$, we have

$$T_a^* S_g = X^{\phi(p(g))(a)} \cdot S_g^* T_a^*, \quad \forall g \in N(\Lambda)/X^*, a \in T,$$

where $p$ is the projection $N(\Lambda)/X^* \rightarrow \Lambda^\vee/X^*$.

Proof. It suffices to check (35) for the real points $a \in U/X^* \subset T$. In this case, we can identify it as an element in $N(X^*)/X^*$, and use the commutator relation (32) in the Heisenberg group $\mathcal{H}(V, E)$ for $N(\Lambda)$ and $(U, 1)$. 

We can generalize this relation to a general cusp $F_\xi$. Let $U_\xi$ be a rational isotropic subspace of dimension $r$. Represent $G$ as $\tilde{G}/Y_\xi$, where $\tilde{G} = V/\Lambda \cap U_\xi^\perp$ and $Y_\xi = \Lambda/\Lambda \cap U_\xi^\perp$. 


Denote $\Lambda \cap U_\xi$ by $X^*_\xi$ and its dual by $X_\xi$. $E$ induces the polarization $\phi : Y_\xi \to X_\xi$. The algebraic torus $T_\xi = \text{Hom}(X_\xi, C^*)$ is the torus part of the semi-abelian variety $\tilde{G}$.

Choose a Lagrangian $U \supset U_\xi$. We have inclusions $X^*_\xi \subset X^*$, $T_\xi \subset T$, and quotients $X \to X_\xi$, $Y \to Y_\xi$. The free group $Y' := U_{\xi}^\perp \cap \Lambda / X^*$ is the kernel of $Y \to Y_\xi$. Then $\Lambda' / U_{\xi}^\perp \cap \Lambda$ (resp. $(\tilde{G}, \tilde{\mathcal{L}})$) is the quotient of $\Lambda'/X^*$ (resp. $(T, T \times \mathbb{C})$) by $Y'$. $Y'$ is naturally considered as a subgroup of $N(\Lambda)/X^*$ by the lift $\chi^{-1}$. Then $N(\Lambda)/U_{\xi}^\perp \cap \Lambda$ is a quotient of $N(\Lambda)/X^*$ by $Y'$. The following lemma is just the $Y'$-quotient of Lemma 4.2. Notice that by the relation (35), $Y'$ commutes with $T_\xi$ on $T \times \mathbb{C}$. Therefore the action of $T_\xi$, and the relation (35) descends to $(\tilde{G}, \tilde{\mathcal{L}})$.

**Lemma 4.3** (Heisenberg relation II). Consider the two groups $N(\Lambda)/U_{\xi}^\perp \cap \Lambda$ and $T_\xi$ acting on $(\tilde{G}, \tilde{\mathcal{L}})$. Denote the action of $N(\Lambda)/U_{\xi}^\perp \cap \Lambda$ by $S$, and the action of $T_\xi$ by $T$, we have

$$T_{a} \cdot S_{g} = X^{\phi(p(g))}(a) \cdot S_{g} T_{a}, \quad \forall g \in N(\Lambda)/U_{\xi}^\perp \cap \Lambda, a \in T_\xi,$$

where $p$ is the projection $N(\Lambda)/(U_{\xi}^\perp \cap \Lambda) \to \Lambda'/U_{\xi}^\perp \cap \Lambda$, and $\phi$ is the induced map $\Lambda'/U_{\xi}^\perp \cap \Lambda \to X_\xi \subset E$ by $E$.

The theta group $\mathcal{G}(\mathcal{L}) = N(\Lambda)/\Lambda$ is the quotient of $N(\Lambda)/U_{\xi}^\perp \cap \Lambda$ by $Y_\xi$. Denote the subgroup $\Lambda' \cap U_\xi / \Lambda \cap U_\xi$ by $K_2$. The restriction of the Weil pairing to $K_2$ induces the surjection $K(\mathcal{L}) \to \tilde{K}_2$. Compose with $p : \mathcal{G}(\mathcal{L}) \to K(\mathcal{L})$, we have $w : \mathcal{G}(\mathcal{L}) \to \tilde{K}_2$. Denote the kernel of $w$ by $K_w$. Identify $\tilde{K}_2$ with $X_\xi / \phi(Y_\xi)$. Let $I$ be an $X_\xi / \phi(Y_\xi)$-torsor.

**Lemma 4.4.** Consider the canonical representation $H^0(G, \mathcal{L})$ of $\mathcal{G}(\mathcal{L})$. $H^0(G, \mathcal{L}) = \bigoplus_{\alpha \in I} V_{\alpha}$ is decomposed into irreducible representations of $K_w$, labelled by $I$. This is the same decomposition as the Fourier decomposition

$$H^0(G, \mathcal{L}) = \bigoplus_{\alpha \in X_\xi / \phi(Y_\xi)} H^0(A, M_\alpha)$$

in the AN construction. Moreover, for $g \in \mathcal{G}(\mathcal{L})$, the action of $g$ translates the space labelled by $\alpha$ to the space labelled by $\alpha + w(g)$.

**Proof.** Recall that the Fourier decomposition is obtained by the Fourier decomposition of $H^0(G, \mathcal{L})$ from the action of $T_\xi$. It follows from Lemma 4.3 that the action of $K_w$ preserves this decomposition. By (Ols08 Theorem 5.4.2.), the theta group $\mathcal{G}(\mathcal{M})$ is contained in the subgroup $K_w$. Since each $H^0(A, M_\alpha)$ is already an irreducible representation of the subgroup $\mathcal{G}(\mathcal{M})$, it is irreducible for $K_w$. Since $w = \phi \circ p$, the last sentence follows from Equation (36). \qed

4.2. The Stable Pairs. It is not hard to generalize the above picture to the relative case. Let $f : \mathcal{X} \to S$ be an abelian scheme over a locally Noetherian base $S$, with a relative ample invertible sheaf $\mathcal{L}$ over $\mathcal{X}$. Let $K(\mathcal{L})$ be the kernel of the polarization $\lambda_\mathcal{L} : \mathcal{X} \to \mathcal{X}^\vee$ given by $\mathcal{L}$. $K(\mathcal{L})$ is a finite flat group scheme. The line bundle $\mathcal{L}$, regarded as a $\mathbb{G}_m$-torsor,
restricted to $K(\mathcal{L})$ is a central extension by $\mathbb{G}_m$. This central extension is the finite theta group for $\mathcal{L}$, and denoted by $\mathcal{G}(\mathcal{L})$.

\begin{equation}
1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(\mathcal{L}) \overset{p}{\longrightarrow} K(\mathcal{L}) \longrightarrow 0.
\end{equation}

The commutators in $\mathcal{G}(\mathcal{L})$ give the Weil pairing $e^{\mathcal{L}}$ on $K(\mathcal{L})$. Recall $f_*\mathcal{L}$ is locally free of rank $d$ and $f_*\mathcal{L}$ is an irreducible representation of $\mathcal{G}(\mathcal{L})$. ($[\text{Shi12}]$ Proposition 2.12, 2.13).

Now we need to replace $H(\delta)$ by a group scheme. Let $M = 2\delta$. Consider the map

\begin{equation}
P_M : \mathcal{G}(\mathcal{L}) \rightarrow G(\mathcal{L}) \quad g \mapsto g^M \quad \forall g \in \mathcal{G}(\mathcal{L})(S).
\end{equation}

\textbf{Lemma 4.5.} The map $P_M$ is a group homomorphism.

\textbf{Proof.} Fix a scheme $S$, and consider $g, h \in \mathcal{G}(\mathcal{L})(S)$. Recall the commutator $ghg^{-1}h^{-1} = e^{\mathcal{L}}(g, h)$. By induction, we have

\[ g^n h^n = (e^{\mathcal{L}}(g, h))^{n-1} ghg^{-1}h^{-1} = (e^{\mathcal{L}}(g, h))^{n(n-1)/2}(gh)^n. \]

Since the order of $e^{\mathcal{L}}(g, h)$ divides $\delta$, $P_M(g)P_M(h) = P_M(gh)$. \qed

Let $G(M)$ denote the kernel of $P_M$. It is a subgroup scheme of $\mathcal{G}(\mathcal{L})$. Since $\mu_M$ is defined as the kernel of the group homomorphism

\begin{equation}
\mathbb{G}_m \rightarrow \mathbb{G}_m
\end{equation}

and

\begin{equation}
g \mapsto g^M \quad \forall g \in \mathbb{G}_m(S),
\end{equation}

we have

\begin{equation}
1 \longrightarrow \mu_M \longrightarrow G(M) \overset{p}{\longrightarrow} K(\mathcal{L}) \longrightarrow 0.
\end{equation}

Define $H(\delta, M)$ to be the central extension of $K(\delta)$ by $\mu_M$ such that the induced Weil pairing on $K(\delta)$ is the standard pairing $e_\delta$. Denote $\prod_{i=1}^n \mathbb{Z}/\delta_i \mathbb{Z}$ by $H$. Étale locally, $G(M)$ is isomorphic to the constant group scheme $H(\delta, M)$. Recall ($[\text{Nak99}]$ Lemma 7.11) that the Heisenberg group $H(\delta, M)$ has an irreducible representation of weight one, called the Schrödinger representation and denoted by $V(\delta)$.

\textbf{Assumption 4.6.} Suppose $S = \text{Spec} R$ for $R$ Noetherian, normal integral, complete, local ring, with generic point $\eta = S^*$, closed point $S_0$. Denote the field of fractions by $K$ and the residue field by $\kappa$.

Assume $\pi : (X, G, \mathcal{L}) \rightarrow S$ is the pull-back of the AN family over $\mathcal{A}^m_{g, \delta}$ along a morphism $g : S \rightarrow \mathcal{A}^m_g$, such that the generic fiber $\pi : (X^*, \mathcal{L}^*) \rightarrow S^*$ is a polarized abelian variety. By ([FC90] Chap. II Sect. 2, we have two Raynaud extensions $0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$ and $0 \rightarrow T^t \rightarrow G^t \rightarrow A^t \rightarrow 0$ from the polarized semi-abelian scheme $G$. Let $X_\xi$ (resp. $Y_\xi$) be the character group of $T$ (resp. $T^t$). The restriction of the polarization to the toric part is $\lambda_T : T \rightarrow T^t$ and is induced by $\phi : Y_\xi \rightarrow X_\xi$. Let the kernel of $\lambda_T$ be $K_2$. It
is an isotropic finite subgroup scheme of the kernel $K(L)$ and is denoted by $K_S(L_\eta)^m$ in [Nak99]. The Cartier dual is $\tilde{K}_2 = X/\phi(Y)$. The restriction of the Weil pairing defines a surjection $K(L^*) \to \tilde{K}_2$. Composing with $G(M) \to K(L^*)$, we get a homomorphism $w : G(M) \to \tilde{K}_2$. Let $K_w$ be the kernel of $w$. Consider the irreducible $G(M)$-representation $\pi_*L^*$. After a base change, étale over $S^*$, $G(M)$ and $\tilde{K}_2$ are constant, $\pi_*L^*$ decomposes into irreducible $K_w$-representations $\pi_*L^* = \bigoplus_{\alpha \in T} V_\alpha$, where $I$ is a torsor for the group $\tilde{K}_2$. Fix an element of $I$ and identify $I$ with the group $\tilde{K}_2$. A lift $\sigma : \tilde{K}_2 \to G(M)$ is a section for the map $w : G(M) \to \tilde{K}_2$. We do not require $\sigma$ to be a group homomorphism.

**Proposition 4.7.** If $S$ satisfies Assumption 4.6 and $\pi : (\mathcal{X}, G, L, g)/S$ is the pull-back family of a morphism $g : S \to \Xi\mathcal{P}$ as above, then after an étale base change, we can extend $G(M)$ and $K_w$-representation $\pi_*L = \bigoplus_{\alpha \in T} V_\alpha$ over $S$. Take any section $\vartheta_0 \in V_\alpha^*$, and any lift $\sigma : \tilde{K}_2 \to G(M)$. Define

$$\vartheta := \sum_{\alpha \in I} S^{*}_{\sigma(\alpha)} \vartheta_0.$$  (43)

Let $\Theta$ be the zero locus of $\vartheta$. Then $(\mathcal{X}, G, L, \Theta, g)/S$ is an object in $\overline{\mathcal{A}\mathcal{P}}_{g,d}$.

**Proof.** Make an étale base change if necessary, we can assume $g : S \to \Xi\mathcal{P}$ over the cusp $F_\xi$ for some bounded paving $\mathcal{P}$ on $U_\xi$. The family $(\mathcal{X}, L, G, g)/S$ is the pull-back of AN family along $g$. Since AN construction is functorial (Proposition 3.22), $(\mathcal{X}, L, G, g)/S$ is constructed from the following data. There is an exact sequence of abelian sheaves

$$1 \longrightarrow T^* \longrightarrow \tilde{G}^* \longrightarrow A^* \longrightarrow 0$$

(44)

$$1 \longrightarrow T \longrightarrow \tilde{G} \longrightarrow A \longrightarrow 0,$$

where the top line is the restriction to $S^*$. Recall $X_\xi = (U_\xi \cap \Lambda)^*, Y_\xi = \Lambda/\Lambda \cap U_\xi^\perp$ and $\phi : Y_\xi \to X_\xi$. Moreover, we have $c, c', \tau, \psi, M$. Over $S^*$, $\tilde{G}^* = \text{Spec } A^* \bigoplus_{\alpha \in X_\xi} O_\alpha$, and the line bundle $\tilde{L}^*$ is defined by

$$S := \prod_{d \geq 0} \left( \bigoplus_{\alpha \in X_\xi} O_\alpha \otimes M^{d}\theta^d \right) \otimes O_{A^*} = \bigoplus_{(d,\alpha) \in S(X_\xi)} S_{\alpha,d}\theta^d.$$

By Lemma 4.4 the pull back of the decomposition $\bigoplus V_\alpha$ is the same with the degree-1 part of the decomposition above. Over $S$, $(\tilde{X}, \tilde{L})$ is defined by the graded $O_{A}$-algebra

$$\mathcal{R} := \bigoplus_{(a,d) \in Q_\mathcal{P}} X^p \otimes O_\alpha \otimes M^{d}\theta^d = \bigoplus_{(d,\alpha) \in S(X_\xi)} R_{\alpha,d}\theta^d.$$

The degree-1 part is the Fourier decomposition. Therefore, we have $\pi_*L = \bigoplus_{\alpha \in I} V_\alpha$ over $S$ and $V_\alpha = \pi_*M_\alpha$. The inclusion $\mathcal{R} \to \mathcal{S}$ is defined by the graph of the $\mathcal{P}$-piecewise affine
function $\varphi$. As the pull-back of $\varphi_\varphi$, $\varphi$ is $X_\xi$-quasiperiodic.

For $\lambda \in Y_\xi$, the action $S^*_\lambda$ on $\tilde{G}^*$ is expressed as

$$\psi(\lambda)^m \tau(\lambda, \alpha) : T^*_\varphi(\lambda) (M^m \otimes \mathcal{O}_\alpha) \to M^m \otimes \mathcal{O}_{\alpha + m \phi(\lambda)}$$

for $\psi(\lambda) = a(\lambda) \psi'(\lambda)$, $\tau(\lambda, \alpha) = b(\lambda, \alpha) \tau'(\lambda, \alpha)$, with $\psi'$, $\tau'$ trivializations over $S$ and with $a, b$ having values in $\mathcal{O}_S$. Since $\{S^*_\lambda\}$ is a group action, $a, b$ satisfy relations

$$a(0) = 1 \quad (45)$$

$$a(\lambda + \mu) = b(\lambda, \phi(\mu)) a(\lambda) a(\mu). \quad (46)$$

Consider a discrete valuation $v$ associated to a height 1 prime ideal. Let $A_\varphi(v) : X_\xi \to \mathbb{Z}$ be the quadratic function associated to $v \circ \varphi$. Let $A(v) : Y_\xi \to \mathbb{Z}$ be the composition $v \circ a$ and $B(v) : Y_\xi \times X_\xi \to \mathbb{Z}$ be the composition $v \circ b$. Then $A(v)$ is quadratic over $Y_\xi$ with the associated bilinear form $B(v)(., \phi(\cdot))$. $A(v)_\varphi|_Y = A(v)$.

Over $\mathbb{C}$, we can regard $G(M)$ as a subgroup of $\mathcal{H}(\delta)$, and present it as $N'(\Lambda)/\Lambda$ for $N'(\Lambda)$ a subgroup of $N(\Lambda)$. The following diagram is Cartesian.

$$\begin{array}{ccc}
N'(\Lambda)/\Lambda \cap U_\xi^+ & \xrightarrow{w} & X_\xi \\
\downarrow & & \downarrow \\
G(M) & \xrightarrow{w} & \tilde{K}_2
\end{array}$$

$$\text{(47)}$$

The vertical arrows are quotients by $Y_\xi$. The group action of $G(M)$ on $(X^*, L^*)$ is represented by the group action of $N'(\Lambda)/(U_\xi^+ \cap \Lambda)$ on $(\tilde{G}^*, \tilde{L}^*)$ which extends the action of $Y$ on $(\tilde{G}^*, \tilde{L}^*)$. Write the action of $N'(\Lambda)/(U_\xi^+ \cap \Lambda)$ also in form of $a^b b^{\psi'} d^{\tau'}[15]$, and take the valuations $A_X(v) := v \circ a$ and $B_X(v) := v \circ b$. These two functions factor through $X_\xi$ and $X_\xi \times X_\xi$. This is because for different lifts of elements from $X_\xi$ to $N'(\Lambda)/(U_\xi^+ \cap \Lambda)$, the differences are in $\mu_M$, and they have the same order under $v$. Since $a, b$ are defined from group actions, they satisfy the same relations as Relation $\text{(45)}$. Therefore $A_X(v)$ is also a quadratic function over $X_\xi$ extending $A(v)$. We have $A_\varphi(v) = A_X(v)$ for all $v$.

Since $R$ is a normal Noetherian domain, $R = \cap_{ht_p = 1} R_p$ ([Mat86] Theorem 11.5). It follows that the difference between the quadratic part of $X^{\psi(\psi(\varphi))}$ and $a(g)$ is invertible in $R$ for all $g \in N'(\Lambda)/(U_\xi^+ \cap \Lambda)$. Therefore the values of $a, b$ for $N'(\Lambda)/(U_\xi^+ \cap \Lambda)$ are regular functions over $S$. The action of $G(M)$ is defined on $(X, L)$ over $S$ and $S^*_\varphi$ maps $R_{0,1}$ to $R_{\phi(g), 1}$ for any $g \in N'(\Lambda)/(U_\xi^+ \cap \Lambda)$. In particular, $S^*_g(\theta_0)$ is in $(\pi_* M_{\phi(g)})^*$ over $S$. Therefore, for any lift $\sigma : I \to G(M)$, the section

$$\vartheta := \sum_{\alpha \in I} S^*_\sigma(\alpha) \vartheta_0$$

$^{15}$In other words, $\psi'$, $\tau'$ are trivializations over $S$. 
is stable. The family \((\mathcal{X}, G, \Theta, \varrho)/S\) is in \(\mathcal{A}\mathcal{P}_{g,d}\) by Theorem 3.28.

**Definition 4.8.** If an isotropic subgroup \(K_2\) of the kernel of the polarization is well-defined for the family, call the set of divisors obtained in Theorem 4.7 the balanced set and denote it by \(S(K_2)\).

**Remark 4.9.** The isotropic subgroup \(K_2\) is well-defined for the family when the base is very local. It is also well-defined when the interior \(S^*\) is a punctured polydisc \((\Delta^*)^n\) (CCK79 Proposition 2.1) \(K_2 = \Lambda^+ \cap W_0/\Lambda \cap W_0\). Essentially, the base should be very close to some cusp \(F_\xi\).

**Corollary 4.10.** If the base \(S\) is a DVR, and we have a polarized abelian variety over the generic point, we can add the central fiber as follows. The monodromy defines an isotropic subgroup \(K_2\). Pick any divisor \(\Theta\) from the balanced set \(S(K_2)\), we get an object in \(\mathcal{A}\mathcal{P}_{d,g}\). Since \(\mathcal{A}\mathcal{P}_{d,g}\) is proper, we can uniquely extend the family over \(S\). Then forget about the divisor, we get a family \((\mathcal{X}, G, L, \varrho)\) over \(S\). This is the pull-back family from \(\mathcal{A}\mathcal{P}_{g,\delta}\), and is independent of the choice of \(\Theta \in S(K_2)\).

**Proof.** Since \(\mathcal{A}\mathcal{P}_{g,\delta}\) is also proper, we have a unique morphism \(S \to \mathcal{A}\mathcal{P}_{g,\delta}\) and we can pull back the AN family. By Proposition 4.7 the pull-back family coincides with \((\mathcal{X}, G, L, \varrho)\).

Let \((\mathcal{X}, G, L, \varrho)/S\) be a polarized stable semiabelic scheme over \(S\) ([Ale02 Definition 1.1.14]). Following [Ols08], define \(G(L)\) to be the group of automorphisms of \((\mathcal{X}, G, L, \varrho)\) that commute with the action \(\varrho\). Define \(G(M)\) to be the kernel of \(P_M\). The following is the geometric description of our extended families near a cusp \(F_\xi\).

**Theorem 4.11.** Suppose \(S\) satisfies Assumption 4.6 and in addition \(R\) strictly henselian. Let \((\mathcal{X}, G, L, \varrho)/S\) be a polarized stable semiabelic scheme over \(S\), with the generic fiber abelian. Suppose that there is an étale covering \(T \to S\) such that \(G(M)_{T^*} \cong H(\delta, M)_{T^*}\). The group subscheme \(K_2 \subset K(L)\) and the stable set of divisors \(S(K_2)\) are hence well-defined over \(S\). Then \((\mathcal{X}, G, L, \varrho)\) is the pull-back of the AN family along some morphism \(g: S \to \mathcal{A}\mathcal{P}_{g,\delta}\), if and only if for one (equivalently any) section \(\Theta\) from \(S(K_2)\), \((\mathcal{X}, G, \Theta, \varrho)\) is an object in \(\mathcal{A}\mathcal{P}_{d,g}\).

**Proof.** The “only if” part is Proposition 4.7. Suppose there exists \(\Theta \in S(K_2)\) such that \((\mathcal{X}, G, \Theta, \varrho)\) is an object in \(\mathcal{A}\mathcal{P}_{d,g}\). This is equivalent to a morphism \(f: S \to \mathcal{A}\mathcal{P}_{d,g}\). The semi-abelian scheme \(G/S\) gives rise to étale constructible sheaves \(X, Y, \varrho\) polarization \(\phi: Y \to X\) and pairing \(B: Y \times X \to \text{Div}\, S\). Over the closed point \(S_0\), the fiber is \(X_\xi, Y_\xi, \phi: Y_\xi \to X_\xi\) and \(b: Y_\xi \times X_\xi \to \mathcal{K}^*\). Over any \(s \in S\), \(X_\delta\) (resp. \(Y_\delta\)) is the quotient of \(X_\xi\) (resp. \(Y_\xi\)).

For each point \(s \in S\), choose a DVR \(T \to S\) with the generic point mapped to \(\eta\) and the closed point mapped to \(s\). By Corollary 4.10 the pull-back family over \(T\), denoted by \((\mathcal{X}_T, G_T, \Theta_T, \varrho_T)\), comes from AN family over \(\mathcal{A}\mathcal{P}_{g,\delta}\). In particular, \((\mathcal{X}_T, G_T, \Theta_T, \varrho_T)\) is constructed from an \(X_\delta\)-quasiperiodic piecewise affine function \(\varphi_T: X_{\delta,R} \to \mathbb{R}\). Therefore
the paving \( \mathcal{P} \) associated to \( \varphi_T \) is \( X_{\Delta} \)-invariant. Let \( b_s \) be \( B_s \) and \( v \) the discrete valuation of \( T \). The bilinear form \( v \circ b_s \) is inside the cone \( C(\mathcal{P}) \) of the second Voronoi fan \( \Sigma(X_{\Delta}) \). The paving \( \mathcal{P} \), regarded as a cell decomposition of \( \Delta_{n,R}/\phi(X_{\Delta}) \), is the cell complex \( \Delta_0 \) in (\cite{Ale02} Definition 5.7.2) and can be defined by the fiber \((X_s,L_s)\). Moreover, since \( s \) specializes to \( S_0 \), if we pull back the cell decomposition \( \mathcal{P} \) along \( X_s \to X_r \), it is coarser than the paving \( P \) associated to the central fiber \((X_0,L_0)\). Therefore the cone \( C(\mathcal{P}) \) is a face of \( C(P) \) in \( \Sigma(X_{\Delta}) \). Write \( X_s \) as a quotient of \( X = \mathbb{Z}^r \). In particular, if \( v : K^* \to \mathbb{Z} \) is any discrete valuation defined by a prime ideal of height one, then \( v \circ b \) is contained in the closed cone \( C(\mathcal{P}) \) in the second Voronoi fan \( \Sigma(X) \). By Proposition 4.17 there exists a morphism \( g : S \to \overline{\mathcal{A}}_{g,d} \) such that if \( (\mathcal{X}', \mathcal{L}', G', \Theta') \) is the pull-back of the AN family along \( g \), then \( G' \cong G \). Choose \( \Theta' \) from \( S(K_2) \) so that \( \Theta'|_{S^*} \) is identified with \( \Theta|_{S^*} \) through the isomorphism. By Proposition 4.17 we get another morphism \( g' : S \to \overline{\mathcal{A}}_{g,d} \). Consider the cartesian diagram

\[
\begin{array}{ccc}
S' & \longrightarrow & \overline{\mathcal{A}}_{g,d} \\
\downarrow & & \downarrow \Delta \\
S & \xrightarrow{\left( f,g' \right)} & \overline{\mathcal{A}}_{g,d} \times \overline{\mathcal{A}}_{g,d}
\end{array}
\]

Since \( \Delta \) is finite (\cite{Ale02} Theorem 5.10.1), \( h \) is finite. Moreover, \( S^* \to S \) factors through \( h \). Since the local charts for \( \overline{\mathcal{A}}_{g,d} \) are integral (\cite{Ale02} 5.9, they are semigroup \( k \)-algebras), \( S' \) is integral. By Lemma 4.11, \( S' = S \). The families \((\mathcal{X}, \mathcal{L}, G, \Theta, \varrho)\) and \((\mathcal{X}', \mathcal{L}', G', \Theta', \varrho')\) are isomorphic. \( \square \)

**Lemma 4.12.** Let \( h_1 : R \to R' \) be a finite extension of integral domains and \( h_2 : R' \to K \) be a morphism such that the composition \( h_2 \circ h_1 : R \to K \) is the inclusion of \( R \) into its field of fractions. If \( R \) is normal, then \( h \) is an isomorphism.

**Proof.** We claim that \( h_2 \) is injective. Suppose there is \( 0 \neq b \in R' \) that is mapped to zero in \( K \). Since \( b \) is integral over \( R \), we have \( b^n + a_1b^{n-1} + \ldots + a_n = 0 \) for \( a_i \in R \) and \( a_n \neq 0 \). However, it implies that \( a_n \) is mapped to 0 in \( K \), a contradiction. Therefore \( R \subset R' \subset K \). If \( R \) is normal, then \( R = R' \). \( \square \)

**Remark 4.13.** Heuristically, locally near the boundary, the compactification \( \overline{\mathcal{A}}_{g,d} \) should be the normalization of a slice of \( \overline{\mathcal{A}}_{g,d} \). The slice is defined by a choice of a divisor in the balanced set \( S(K_2) \). The divisors in \( S(K_2) \) should be regarded to be the most “symmetric” because it is almost invariant under a finite group action (not quite, but a lift of the finite group \( \hat{K}_2 \)). The tropical avatar of the slice is the linear section \( \sigma \) in Equation (14).

**Remark 4.14.** The families have been studied in \cite{Nak10} and \cite{Ols08}. Moreover, when the polarization is separable, it is proved in (\cite{Nak99} Definition 5.11, Lemma 5.12) and (\cite{Ols08}) that \( G(M) \) and its representation can be extended to the boundary. What we are suggesting here is, it is possible to characterize the extended families by using \( G(M) \) and \( \overline{\mathcal{A}}_{g,d} \).

**Remark 4.15.** However, we can not get a moduli functor this way, because the balanced set \( S(K_2) \) depends on \( K_2 \) and is not well defined over the whole moduli space. It can be
proved that there is no section stable for all $K_2$ unless the polarization is principal. To fix the issue, it is natural to consider the moduli problem with a $G(M)$-level structure (or theta structure) as considered in [Nak10]. In that case, we get a canonical divisor over the whole moduli space.

If $K_2$ is a Lagrangian, $A$ and $M$ are trivial and $\mathcal{V}_0 \cong \mathcal{O}_S$. The number of lifts $\sigma$ is finite. If the Lagrangian $K_2$ further splits, i.e., it admits an isotropic complement $K(L) = K_1 \oplus K_2$, we can identify $K_1$ and $I$, and require that the lifts are group homomorphisms $\sigma : K_1 \to G(M)$. In this case, the lift $\sigma(K_1)$ is a maximal level subgroup $\tilde{K}_1$ of $G(L)$. There are altogether $d$ choices, and each choice is equivalent to a choice of the descent data $h : \mathcal{X} \to \mathcal{X}/K_1$, with $\nu : h^*\mathcal{L}' \cong \mathcal{L}$ for a principal polarized line bundle $\mathcal{L}'$ over $\mathcal{X}/K_1$. The stable section $\vartheta$ is the pull back of the unique (up to a scaling) section of $\mathcal{L}'$.

**Appendix A. The quasiperiodic functions**

Let $Y \cong \mathbb{Z}^g$ be a finitely generated free abelian group acting on an affine space $V \cong \mathbb{R}^g$ by translations. Let $\psi$ be a real valued piecewise affine function on $V$.

**Definition A.1.** A piecewise affine function $\psi$ is called quasi-periodic with respect to $Y$, if

\[
\psi(x + \lambda) - \psi(x) = A_\lambda(x), \quad \forall \lambda \in Y, \forall x \in V,
\]

for $A_\lambda(x)$ an affine function on $V$ that depends on $\lambda \in Y$.

**Remark A.2.** A $Y$-quasiperiodic function $\psi$ can be regarded as an element in $\Gamma(V/Y, \mathcal{P}A/\mathcal{A}ff)$.

**Lemma A.3.** If $\psi$ is quasi-periodic with respect to $Y$, then there exists some quadratic function $A$ such that $\psi - A$ is a $Y$-periodic function on $V$.

**Proof.** Fix a point $x_0 \in V$ and regard $V$ as a vector space. Embed $Y$ as a subset of $V$. For $\lambda, \mu \in Y$, we have

\[
\begin{align}
\psi(x + \mu + \lambda) &= \psi(x + \mu) + A_\lambda(x + \mu) \\
&= \psi(x) + A_\mu(x) + A_\lambda(x + \mu) \\
&= \psi(x) + A_\lambda(x) + A_\mu(x + \lambda) \\
&= \psi(x) + A_{\mu+\lambda}(x).
\end{align}
\]

From (51) and (52)

\[
A_{\lambda+\mu}(x) = A_\mu(x + \lambda) + A_\lambda(x)
\]

Therefore $\{A_\lambda\}$ is a 1-cocycle for the $Y$-module $\mathcal{A}ff(X_{\mathbb{R}}, \mathbb{R})$.

Suppose $A_\lambda(x) = B(\lambda, x) + A(\lambda)$, for a linear function $B(\lambda, \cdot)$ on $V$ and a constant $A(\lambda)$. From (50) and (51), we have

\[
B(\mu, \lambda) = A_\mu(x + \lambda) - A_\mu(x) = A_\lambda(x + \mu) - A_\lambda(x) = B(\lambda, \mu).
\]
Applying (52),

\[ A_{\mu+\lambda}(x) - A_{\mu}(x) - A_{\lambda}(x) = B(\mu, \lambda), \]

It follows that

\[ B(\mu + \lambda, x) = B(\mu, x) + B(\lambda, x), \]

\[ A(\mu + \lambda) - A(\mu) - A(\lambda) = B(\mu, \lambda). \]

Therefore \( B(\mu, \lambda) \) is a symmetric bilinear form on \( Y \), and

\[ A(\gamma) = \frac{1}{2} B(\gamma, \gamma) + \frac{1}{2} L(\gamma) \]

is a quadratic function associated to \( B \). Here \( L \) is a linear function on \( Y \).

Extend \( A \) and \( B \) to functions on \( V \). Since \( A(x + y) - A(x) = B(y, x) + A(y) \), for all \( x, y \in V \),

\[ \psi(x + \lambda) - A(x + \lambda) = \psi(x) - A(x), \quad \forall \lambda \in Y, x \in V. \]

In other words, \( \psi - A \) is a periodic function with respect to \( Y \). \( \square \)

If we forget about \( x_0 \) initially chosen, the quadratic part of \( A \) is well-defined.

**Definition A.4.** The quadratic form \( \frac{1}{2} B(x, x) \) is called the quadratic form associated to \( \psi \).

If the piecewise affine function \( \varphi \) takes values in a vector space \( P_{\mathbb{R}}^{\text{gp}} \), we can define \( Y \)-quasiperiodic similarly. Recall a quadratic function is an element in \( \Gamma^2 V^* \otimes P_{\mathbb{R}}^{\text{gp}} \). We can generalize Lemma A.3.

**Corollary A.5.** If \( \psi \) is quasi-periodic with respect to \( Y \), then there exists some quadratic function \( A \) such that \( \psi - A \) is a \( Y \)-periodic function on \( V \).

**Appendix B. Degenerations over one-parameter family**

**B.1. Maximal Degeneration.** In this section, we compute the degeneration data for a general 0-cusp. Assume the base field \( k = \mathbb{C} \).

Let \( \pi : \mathcal{X} \to \Delta^* \) be a one-parameter family of polarized abelian varieties and \( X \) be a generic fiber. Let \( V := H_1(X, \mathbb{Z}) \) and \( \Lambda := H_1(X, \mathbb{Z}) \). The polarization is defined by a skew-symmetric integral bilinear form \( E \) over \( \Lambda \). The log monodromy defines a weight filtration \( 0 \subset W_0 \subset W_1 \subset W_2 = V \). In our usual notation, \( U = W_0, U^\perp = W_1 \). Define \( X^* := \Lambda \cap U \) and \( Y \cong \Lambda/U^\perp \cap \Lambda \). Fix a symplectic basis \( \{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\} \) of \( \Lambda \) for \( E \). We use \((x, y)\) to denote the coordinates of a vector \( v = x\lambda + y\mu \), where \( x \) and \( y \) are both row vectors of \( g \) elements. Now assume the degeneration is maximal and \( U \) is a rational Lagrangian of dimension \( g \). Suppose \( v_1, \ldots, v_g \) is a basis of \( X^* = \Lambda \cap U \) and

\[ v_i = \sum_{j=1}^g c_{ij} \lambda_j + d_{ij} \mu_j, \quad \forall i = 1, \ldots, g. \]
Since \( \{v_1, \ldots, v_g\} \) is a complex basis for \( V \), \( c\tau + d\delta \) is invertible, where \( c \) and \( d \) are \( g \times g \)-matrices. The coordinates of the vector \( x\lambda + y\mu \) with respect to the complex basis \( \{v_1, \ldots, v_g\} \) is \( (x\tau + y\delta)(c\tau + d\delta)^{-1} \).

Suppose the dual basis of \( \{v_1, \ldots, v_g\} \) of \( X^\ast \) is a compatible basis of \( X \). Then we can find rational vectors \( \{u_1, \ldots, u_g\} \) (not necessarily in \( \Lambda \)), such that, under \( \{u_1, \ldots, u_g, v_1, \ldots, v_g\} \), \( E \) is still \( \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} \).

Suppose the transformation matrix from \( \{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\} \) to \( \{u_1, \ldots, u_g, v_1, \ldots, v_g\} \) is by \( M \):
\[
M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(E, \mathbb{Q})
\]

Notice that \( c, d \) are integral matrices.

We fix a punctured holomorphic disk \( \Delta^* \) with coordinate \( q \). The universal covering is the upper half plane \( \mathbb{H} \) with coordinate \( t \), and \( q = e^{2\pi it} \). Fix \( \tau' \), and choose \( \tau'_0 \in \mathfrak{S}_g \).

Consider the family \( \pi : \mathcal{X}/\Delta^* \) defined by the holomorphic map \( \tau(t) : \mathbb{H} \to \mathfrak{S}_g \)
\[
\tau'_0 + \Im(\tau')t = (a\tau(t) + b\delta)(c\tau(t) + d\delta)^{-1}\delta.
\]

This family has a multiplicative uniformization, which is a trivial algebraic torus \( \widetilde{G} \) over \( \Delta^* \). For a point \( (x, y) \) in \( V \), consider the image in \( \widetilde{G}_t : = V_t/X^* \cong (\mathbb{C}^*)^g \) for different \( t \). We always use the dual basis of \( \{v_1, \ldots, v_g\} \) as the standard coordinates of \( (\mathbb{C}^*)^g \). The coordinates of \( (x, y) \) in \( (\mathbb{C}^*)^g \) is the row vector
\[
\exp \left( 2\pi i (x\tau(t) + y\delta)(c\tau(t) + d\delta)^{-1} \right).
\]

For the convenience of computation, we change to a new basis. Extend \( \{v_1, \ldots, v_g\} \) to a basis \( \{u'_1, \ldots, u'_g, v_1, \ldots, v_g\} \) of \( \Lambda \), and under this new basis,
\[
E = \begin{pmatrix} S & d \\ -d & 0 \end{pmatrix},
\]

where \( S \) is an integral, skew symmetric matrix. Let the transformation matrix be \( M' \in \text{GL}(2g, \mathbb{Z}) \). Recall we can always choose \( M \) and \( M' \) such that
\[
\begin{pmatrix} A & B \\ 0 & I_g \end{pmatrix} M'^{-1} = M^{-1}
\]

And
\[
A^{-1} = d\delta^{-1},
\]
\[
A^{-1} B d = \frac{1}{2} S.
\]
Let \((x', y')\) be the coordinates under the basis \(\{u'_1, \ldots, u'_g, v_1, \ldots, v_g\}\), and \((x, y)\) be the coordinates under the basis \(\{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\}\). The transformation of coordinates is

\[
(x' \ y') = (x \ y) M'.
\]

Therefore,

\[
(x' \ y') \begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & I_g \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (x \ y).
\]

Under this new basis the coordinates are

\[
\exp \left( 2\pi i \begin{pmatrix} x' \\ y' \end{pmatrix} \begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & I_g \end{pmatrix} \begin{pmatrix} (\tau'_0 + 3(\tau')t)\delta^{-1} \end{pmatrix} \right).
\]

We only have to consider the periods \(y' = 0\). Identify \(Y = \Lambda/\Lambda \cap U\) with the subgroup \(\langle u'_1, \ldots, u'_g \rangle\) in \(\Lambda\), and use the row vector \((x')\) as the coordinates. It defines a bilinear map \(b\) from \(Y \times X\) to holomorphic functions of \(t\).

\[
b(x', \alpha) = \exp (2\pi ix'(A^{-1}(\tau'_0 + 3(\tau')t)\delta^{-1} - A^{-1}B)\alpha).
\]

Here we use \(\alpha\) to denote a vector in \(X\) and also the column vector of the coordinates of \(\alpha\) with respect to the basis \(\{v_1, \ldots, v_g\}\). And \(x'\) means both the vector \(\sum_{i=1}^g x'u'_i\) and its coordinates. Decompose \(b\)

\[
b(\lambda, \alpha) = b_0(\lambda, \alpha) \cdot b(t)(\lambda, \alpha) \cdot b'(\lambda, \alpha),
\]

where

\[
b_0(\lambda, \alpha) = \exp (2\pi ix'(A^{-1}\tau'_0\delta^{-1})\alpha),
\]

\[
b_t(\lambda, \alpha) = b(x', \alpha) = \exp (2\pi ix'(A^{-1}3(\tau')t\delta^{-1})\alpha) = q^{(\alpha, \phi(\lambda))},
\]

\[
b'(\lambda, \alpha) = \exp (-\pi ix'S\delta^{-1}\alpha).
\]

Fix the notation. \(\phi(\lambda)(v) = E(\lambda, v)\). Therefore \(\phi(u'_i) = d_i v^*_i\). Notice that since \(A^{-1}(\tau'_0 + 3(\tau')t)(A^{-1})^T\) is a symmetric matrix and \(S\) is integral skew symmetric, \(b(\lambda_1, \phi(\lambda_2)) = b(\lambda_2, \phi(\lambda_1))\). Define the symmetric bilinear form \(b_S : Y \times Y \to \mu_2 \cong \mathbb{Z}/2\mathbb{Z}\)

\[
b_S := \exp (-\pi i E(\lambda, \lambda')) = \exp (-\pi ix'S(x')^T) = \exp (\pi ix'S(x')^T) \equiv S \pmod{2}.
\]

Notice that \(b_S(2\lambda_1, \lambda_2) = 1\) for any \(\lambda_1, \lambda_2 \in Y\). Therefore \(b_S\) is a symmetric bilinear form over \(Y/2Y\) which is a vector space over \(\mathbb{Z}/2\mathbb{Z}\). There always exists some quadratic form \(\chi\) whose associated bilinear form is \(b_S\). For example we can define a symmetric matrix \(S'\) by requiring that \(S'_{ij} = 1\) if \(S_{ij}\) is odd, and everywhere else is 0. Then

\[
\chi(\lambda) := \exp \left( \frac{1}{2} \pi ix'S'(x')^T \right)
\]

is such a quadratic form. There are maybe more than one choices of such quadratic form (e.g. over characteristic 2). We fix a choice and denote it by \(\chi\). Let \(\alpha' = \chi^{-1}\). We can even extend \(\chi\) to get a semi-character \(\chi : \Lambda \to \mathbb{S}^1\) by defining it to be trivial over \(\Lambda \cap U\).
Define the positive quadratic forms $Q_0$ and $Q$ by the symmetric matrices $\frac{1}{2}A^{-1}\tau_0'(A^{-1})^T$ and $\frac{1}{2}A^{-1}\Im(\tau')(A^{-1})^T$ respectively. Notice that $Q(\lambda) = \frac{1}{2}\langle \phi(\lambda), \check{\phi}(\lambda) \rangle$.

Define $a(\lambda) = \chi^{-1}(\lambda) \exp (2\pi i(Q_0(\lambda) + tQ(\lambda)))$

where

$a_0(\lambda) := \exp (2\pi iQ_0(\lambda))$, 
$a_t(\lambda) := \exp (2\pi itQ(\lambda)) = q^{Q(\lambda)}$, 
$a'(\lambda) := \chi^{-1}(\lambda)$.

Since $Q$ is not necessarily integral, we may need to make a base change of degree 2 to make $a_t(\lambda)$ a function over $\Delta^*$.

**Lemma B.1.** Use $a, b$ above to define the $Y$-linearization of the trivial line bundle $O_{\tilde{G}}$ over $\tilde{G}$, then $L$, the descent of $O_{\tilde{G}}$ over $X$, has the polarization of type $\delta$.

**Proof.** We do the computation explicitly. Use complex coordinates $z_\alpha = d_\alpha v_\alpha^*$. We can modify the 1-cocycle $\{e(\lambda, z)\}$ by a coboundary such that $e(u'_i, z) = \exp (-2\pi iz_i)$.

Denote the matrix of $2Q_0 + 2Qt$ by $\tau''(t)$, and $\tau''(t) = X(t) + iY(t)$. Let $W(t) = Y(t)^{-1}$. We have

$$dz_\alpha = \sum_i (\tau''(t) - S/2)_{i\alpha} dx'_i + d_\alpha dy'_\alpha$$

$$d\bar{z}_\beta = \sum_j (\tau''(t) - S/2)_{j\beta} dx'_j + d_\beta dy'_\beta$$

Following ([GH94] pp. 310-311), we can define the hermitian metric by a transition function

$$h(z) = \exp \left( \frac{\pi}{2} \sum W_{\alpha\beta} (z_\alpha - \bar{z}_\alpha)(z_\beta - \bar{z}_\beta - 2iY_{\beta\beta}) \right)$$

Do similar calculations as in [GH94], we can check that $h$ defines a hermitian metric for the 1-cocycle $\{e(\lambda, z)\}$. Moreover, the curvature of this hermitian metric is

$$\Theta_L = \pi \sum_{\alpha, \beta} W_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta$$

$$= 2\pi i \left( \sum_{\alpha > \beta} S_{\alpha\beta} dx'_\alpha \wedge dx'_\beta + \sum_{\alpha} d_\alpha dx'_\alpha \wedge dy'_\alpha \right).$$
It verifies that the degeneration data \( \{a, b\} \) defines a line bundle \( \mathcal{L} \) whose polarization is of type \( \delta \).

The limit Hodge filtration \( F_\infty \) is decided by \( b_0b' \). The log monodromy \( N = \tilde{\phi} : V \to U \) is given by \( b_t \). Although this family looks special, the general degenerating family with the same log monodromy \( N \) and limit Hodge filtration \( F_\infty \) is asymptotic to this family in a precise sense. This is the content of the nilpotent orbit theorem. Therefore we can use the data obtained from this family as a model for the general degeneration. Without loss of generality, we can choose \( \tau_0' \) to be 0, so that \( a_0 \) is 1, because we need to do approximation anyway. The conclusion is that \( a' \) is the twist necessary for the direction \( U \), with respect to the basis \( \{u'_i, v_j\} \).

**B.2. General Degeneration.** Consider a general one-parameter degeneration family \( \mathcal{X} \) over \( \Delta^* \), whose abelian part is non-trivial. Let the associated isotropic subspace be \( U_\xi \subset V \) of dimension \( r \leq g \). \( U_\xi \) is obtained as the space of vanishing cycles \( W_0^g \subset \Lambda_\mathbb{R} \). Let \( J \) be the complex structure on \( V \). Let \( \tilde{T} \) be the subspace of \( V \) generated by \( U_\xi \) and \( U_\xi J \). \( T := \tilde{T}/U_\xi \cap \Lambda \) is the toric part of dimension \( r \). Since \( U_\xi^\perp \cap U_\xi J = \{0\} \), \( V/\tilde{T} \cong U_\xi^\perp/U_\xi \) is a complex vector space of dimension \( g' := g - r \). We have a pure Hodge structure of weight \(-1\) on \( U_\xi^\perp/U_\xi = W_1^g/W_0^g \). This is the period map of the abelian part \( A \). The bilinear form \( E \) restricted to \( U_\xi^\perp/U_\xi \) is non-degenerate. This gives the polarization on \( A \).

Let \( \tilde{G} = V/U_\xi^\perp \cap \Lambda \), \( Y_\xi = \Lambda/U_\xi^\perp \cap \Lambda \). The family of abelian varieties \( \mathcal{X} \) is the quotient of the family of semi-abelian varieties \( \tilde{G} \) by periods \( Y_\xi \). We have the extension sequence of abelian group varieties over \( \Delta^* \)

\[
\begin{array}{cccc}
0 & \longrightarrow & T & \longrightarrow & \tilde{G} & \longrightarrow & A & \longrightarrow & 0.
\end{array}
\]

Let \( X_\xi \) be the group of characters of \( T \). It is the dual of the fundamental group \( U_\xi \cap \Lambda \). For any \( \alpha \in X_\xi \), \( \alpha \) is a group homomorphism \( \alpha : T \to \mathbb{G}_m \). The push-out of the short exact sequence \( \text{(55)} \) along \( \alpha \) is a \( \mathbb{G}_m \)-torsor over \( A \) whose associated invertible sheaf is denoted by \( \mathcal{O}_{-\alpha} \). Since the total space has a group structure, \( \mathcal{O}_{-\alpha} \) is in \( \text{Pic}^0(A/\Delta^*) \). This defines the map \( c : X_\xi \to A^1 \), where \( A^1 \) is the dual in the category of complex analytic spaces. Since \( Y_\xi \) is a group of \( \Delta^* \)-sections of \( \tilde{G} \), for any \( \lambda \in Y_\xi \), \( \alpha \in X_\xi \), the push-out of \( \lambda \) along \( \alpha \) is a \( \Delta^* \)-section of \( \mathcal{O}_{-\alpha} \). Denote the projection under \( \pi \) by \( c^\lambda(\lambda) \subset A \), and the section by \( \tau(\lambda, \alpha) \). This gives the trivialization \( \tau \) of the biextension \( (c^\lambda \times c^\alpha) \mathcal{P}^{-1} \).

The relative ample line bundle of type \( \delta \) on \( \mathcal{X} \) is represented as a line bundle \( \tilde{\mathcal{L}} \) over \( \tilde{G} \) with a \( Y_\xi \)-action. Since \( \tilde{G} \) is a \( T \)-torsor, \( \tilde{\mathcal{L}} \) is also equipped with a \( T \)-action, and we can do partial Fourier expansion. Restrict to any section of \( A \), \( \tilde{\mathcal{L}} \) is trivial. Therefore, suppose that \( \tilde{\mathcal{L}} \) descends to an ample line bundle \( \mathcal{M} \) of \( A \). This is a choice, and we fix this choice. Then \( \alpha \)-eigenspace of \( \Gamma(\tilde{G}, \tilde{\mathcal{L}}) \) is identified with \( \Gamma(A, \mathcal{M}_\alpha) \) and the partial Fourier expansion is
\[ \Gamma(\tilde{G}, \tilde{L}) = \bigoplus_{\alpha \in \mathcal{X}^r} \Gamma(A, \mathcal{M}_\alpha). \]

Denote the Fourier coefficients by the homomorphisms \( \sigma_\alpha : \Gamma(\tilde{G}, \tilde{L}) \to \Gamma(A, \mathcal{M}_\alpha) \). Restrict to the \( Y_\xi \)-invariant subspace \( \Gamma(\mathcal{X}, \mathcal{L}) \), there is a relation between the homomorphisms \( \sigma_\alpha \) and \( \sigma_{\alpha + \phi(\lambda)} \) for \( \lambda \in Y_\xi \). That is

\[ \sigma_{\alpha + \phi(\lambda)} = \psi(\lambda) \tau(\lambda, \alpha) T_{\tau(\lambda)} \circ \sigma_\alpha, \]

for \( \psi(y) \) a \( \Delta^* \)-section of \( \mathcal{M}(c^r(\lambda))^{-1} \). This defines the trivialization \( \psi \) of the central extension over \( Y_\xi \).

To get the explicit data \( \tau \) and \( \psi \), we choose a 0-cusp \( F(U) \) that is in the closure of \( F(U_\xi) \), i.e. a maximal rational isotropic subspace \( U \) that contains \( U_\xi \). Assume \( \Lambda \cap U_\xi \) be spanned by \( \{v_1, v_2, \ldots, v_r\} \), and \( U \cap \Lambda \) be spanned by \( \{v_1, \ldots, v_r, u_1+1, \ldots, v_g\} \). We choose the complement \( \{u_1', \ldots, u_g'\} \) as in the above section. Therefore \( U_\xi^\perp \cap \Lambda \) is spanned by

\[ \{v_1, \ldots, v_r, u_1+1, \ldots, v_g', u_1'+1, \ldots, u_g'\}. \]

For simplicity, assume the periods for the family is \( \tau_0' + t\mathfrak{I}(\tau') \) after the transformation by \( M \). Write \( \tau_0' \) in blocks.

\[ \tau_0' = \begin{pmatrix} \tau_1' & \tau_2' \\ \tau_3' & \tau_4' \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}, \]

where \( \tau_1' \) is a \( r \times r \)-matrix and \( \tau_4' \) is a positive-definite \( g' \times g' \)-matrix. Similarly, we write \( S \) in blocks. Assume \( \mathfrak{I}(\tau') \) is positive definite for the upper left \( r \times r \) block and vanishes anywhere else. As in the above model, we take the quotient of the group generated by \( \{v_1, v_2, \ldots, v_g\} \) and get an algebraic torus \( \mathbb{G}_{m}^g \) over \( \Delta^* \). Denote the group generated by \( \{u_1', \ldots, u_g'\} \) by \( Y_1 \) and the group generated by \( \{u_1, \ldots, u_g\} \) by \( Y_\xi \). Use \( \lambda \) for a vector in \( Y_\xi \) and \( z \) for a vector in \( Y_1 \). Use \( X^\alpha \) for coordinate functions of \( \mathbb{G}_{m}^g \), and \( W^\beta \) for coordinate functions on \( \mathbb{G}_{m}^r \). The family of abelian varieties \( A \) is the quotient of \( \mathbb{G}_{m}^r \) by \( Y_1 \). \( \tilde{G} \) is the quotient of \( \mathbb{G}_{m}^g \) by \( Y_\xi \). By this multiplicative uniformization \( \mathbb{G}_{m}^g \), all line bundles \( \mathcal{O}_\alpha \) are trivialized canonically after pull back over \( \mathbb{G}_{m}^g \). Of course there is ambiguity from the action of \( Y_1 \) for the trivialization of every fiber \( \mathcal{O}_\alpha(c^d(\lambda)) \). However, the lift of \( Y_\xi \) to points \( u_1', \ldots, u_g' \) fixes this ambiguity. The upshot is \( \mathcal{O}_{\alpha}^{-1}(c^d(\lambda)) \) is thus trivialized, and the section \( \tau(\lambda, \alpha) \) is equivalent to a function over \( \Delta^* \). Similarly, the pull back of \( \mathcal{M} \) to \( \mathbb{G}_{m}^g \) is trivial. A lift of \( Y_\xi \) gives a canonical trivialization of \( \mathcal{M}^{-1}(c^d(\lambda)) \). The section \( \psi(\lambda) \) should also be a function over \( \Delta^* \).

Regard \( \tilde{L} \) as the quotient of \( (\mathbb{C}^*)^g \times (\mathbb{C}^*)^{g'} \times \mathbb{C} \times \Delta^* \) by \( Y_1 \). The action is parametrized by \( q \in \Delta^* \). A section \( \vartheta \in \Gamma(\mathcal{X}, \mathcal{L}) \) is a \( Y \)-invariant function over \( (\mathbb{C}^*)^r \times (\mathbb{C}^*)^{g'} \). Do the partial Fourier decomposition
\[\vartheta = \sum_{\alpha \in \mathcal{X}} \left( \sum_{\beta} a_{\alpha \beta} W^\beta \right) X^\alpha.\]

The function over \((\mathbb{C}^*)^g\)
\[\sigma_\alpha(\vartheta) = \sum_{\beta} a_{\alpha \beta} W^\beta\]
is a section of \(\Gamma(A, \mathcal{M}_\alpha)\).

This can be easily verified. We use \(e, a, b\) from the above section. For \(\mu \in Y\), we have
\[\vartheta((W, X) + \mu) = e(\mu, (W, X))\vartheta(W, X) = \frac{1}{a(\mu)b((W, X), \phi(\mu))}\vartheta(W, X).\]

If \(\mu = z \in Y_1\),
\[b((W, X), \phi(z)) = W^{\phi(z)} .\]

On the one hand,
\[\vartheta((W, X) + z) = \sum_{\alpha \in \mathcal{X}} \left( \sum_{\beta} a_{\alpha \beta} b(z, \beta) W^\beta \right) b(z, \alpha) X^\alpha\]
(58)
\[= \sum_{\alpha \in \mathcal{X}} \sigma_\alpha(\vartheta)(W + z) b(z, \alpha) X^\alpha.\]
(59)

On the other hand,
\[e(z, (W, X))\vartheta(W, X) = \frac{1}{a(z)W^{\phi(z)}} \sum_{\alpha \in \mathcal{X}} \sigma_\alpha(\vartheta)(W) X^\alpha .\]
(60)

Compare the coefficients of \(X^\alpha\),
\[\sigma_\alpha(\vartheta)(W + z) = \frac{1}{b(z, \alpha)a(z)W^{\phi(z)}} \sigma_\alpha(\vartheta)(W)\]
(61)

Here \(1/b(z, \alpha)\) are the factors of automorphy for \(\mathcal{O}_\alpha\), and \(1/(a(z)W^{\phi(z)})\) are the factors of automorphy for \(\mathcal{M}\). Hence \(\sigma_\alpha(\vartheta)\) is a section of \(\mathcal{M}_\alpha\). The abelian part \(A\) is determined by \(b(z, \beta)\), i.e. the periods \(\tau'_4\) and the twist \(S_4\).

If \(\mu = \lambda \in Y_\xi\),
\[b(W, X, \phi(\lambda)) = X^{\phi(\lambda)} .\]

We have
\[ \vartheta((W, X) + \lambda) = \sum_{\alpha \in X} \left( \sum_{\beta} a_{\alpha\beta} b(\lambda, \beta) W^\beta \right) b(\lambda, \alpha) X^\alpha \]

\[ = \sum_{\alpha \in X} \sigma_\alpha(\vartheta)(W + c'(\lambda)) b(\lambda, \alpha) X^\alpha. \]

And

\[ e(\lambda, (W, X)) \vartheta(W, X) = \frac{1}{a(\lambda)} \sum_{\alpha \in X} \sigma_\alpha(\vartheta)(W) X^{\alpha - \phi(\lambda)}. \]

Compare the coefficients of \( X^\alpha \),

\[ \sigma_{\alpha + \phi(\lambda)}(\vartheta)(W) = a(\lambda) b(\lambda, \alpha) \sigma_\alpha(\vartheta)(W + c'(\lambda)), \]

\[ = a(\lambda) b(\lambda, \alpha) T_{c'\lambda} \circ \sigma_\alpha(\vartheta)(W). \]

Therefore, after using the canonical trivialization from the 0-cusp \( U \),

\[ \psi(\lambda) = a(\lambda) \]

\[ \tau(\lambda, \alpha) = b(\lambda, \alpha). \]

Use the coordinates with respect to \( \{u'_1, \ldots, u'_r\} \), the points are

\[ b(x, \alpha) = \exp \left( 2\pi i (x, 0) (A^{-1} (\tau'_0 + \Im(\tau') t) \delta^{-1} - 1/2S \delta^{-1}) \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right) \]

\[ = b_0 b'(x, \alpha) b_t(x, \alpha), \]

where \( \alpha \) is a column vector of \( r \) elements, \( b_0 b'(x, \alpha) \) is a constant in \( \mathbb{C} \), while \( b_t(x, \alpha) \) is a function over \( \Delta^* \). Write \( b_t \) as an \( r \times r \)-matrix of functions \( q^{2Q_{\alpha^{-1}}} \). Write \( a(x) = a_0 a'(x) a_t(x) \), where \( a_0 a'(x) \) is a constant, and \( a_t(x) = q^{Q(x)} \) is a function of \( q \) if we make a necessary base change.

Since the quadratic form \( Q \in C(X_\xi) \),

**Corollary B.2.** The trivializations \( \tau \) and \( \psi \) are compatible with \( Q \).

Since we will need to do approximation anyway, assume \( a_0 = b_0 = 1 \). Denote the function field on \( \Delta^* \) by \( K \). The explicit data for \( U_\xi \) is

**Proposition B.3.** If we make the choice of the 0-cusp \( U \), and use the data from \( U \), we can write \( \tau, \psi \) as follows.

\[ \tau = b' b_t : Y_\xi \times X_\xi \to K \]

\[ \psi = a' a_t : Y_\xi \to K, \]
where \( b_t = q^{2Q^{-1}} \) and \( a_t = q^{Q(x)} \) is defined in terms of \( Q \in \mathcal{C}(X_\xi) \), and \( b', a' \) are twists associated to the boundary component \( U_\xi \)

\[
\begin{align*}
    b'(x, \alpha) & = \exp(-\pi ixS_\xi \partial^{-1} \alpha), \\
    a'(x) & = \exp(-1/2\pi ixS'_\xi x^T),
\end{align*}
\]

where \( S'_\xi \) is a symmetric matrix such that \( S'_\xi \equiv S_\xi \pmod{2\mathbb{Z}}. \)

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