Abstract

The theory of motives concerns the enrichment of (co)homology groups of varieties with extra structure, i.e., the structure of a motive. The problem of similarly enriching the topological fundamental group has a long history. For example, Hain proved in [Hai98] that, given a variation of Hodge structures satisfying certain assumptions, the Malcev completion of the fundamental group relative the associated monodromy representation carries a canonical mixed Hodge structure. Our main result (Thm. 7.3) is a motivic version of Hain’s theorem. More precisely, given a “motivic local system” (on the generic point of a smooth variety in characteristic zero) we show that the Malcev completion of the fundamental group relative the associated monodromy representation is canonically a (Nori) motive.

In fact, we prove something stronger, namely that the homomorphism from the topological fundamental group to the motivic fundamental group is Malcev complete, i.e., isomorphic to its Malcev completion. Our main tool (Thm. M) is an abstract criterion for homomorphisms of affine group schemes to be Malcev complete, and most of our work goes into proving this result. We also show how this criterion gives a new proof of the main theorem of [DE20], i.e., that the homomorphism from the topological fundamental group to the Hodge fundamental group is Malcev complete, and how to deduce from this a strengthening of Hain’s theorem (Thm. 6.14), valid for arbitrary admissible variations of mixed Hodge structures.
1 Introduction

The singular homology functors $H_i(-, \mathbb{Q})$ on algebraic varieties over $\mathbb{C}$ factor through the category of mixed Hodge structures, by the seminal work of Deligne. Conjecturally, they factor further, through a (Tannakian) category of motives, enjoying many useful properties. By Hurewicz, the functor $\pi_1(-, -)_{ab} \otimes \mathbb{Q}$ on pointed algebraic $\mathbb{C}$-varieties is isomorphic to $H_1(-, \mathbb{Q})$, so it too should factor through motives. Put differently, the abelianisation of the (topological) fundamental group of an algebraic variety is canonically a motive. It’s natural to investigate whether a larger part of $\pi_1(-, -)$ is a motive, as passing to the abelianisation is rather destructive. In particular, one can ask to what extent Hodge theory can be extended to the fundamental group, i.e., what part of $\pi_1(-, -)$ carries a natural mixed Hodge structure.

In this direction, Morgan ([Mor78]) (and Hain in [Hai87] for singular varieties) showed that the pro-unipotent completion of the fundamental group carries a natural mixed Hodge structure. Hain went further in [Hai98], and proved that certain relative pro-unipotent completions (called Malcev completions in the present document) of fundamental groups carry natural mixed Hodge structures.

In order to be more precise, suppose $P \rightarrow K$ is a homomorphism of affine group schemes over a field of characteristic zero. Then, the Malcev completion of this homomorphism is the initial group $\mathcal{G}$ equipped with a morphism from $P$ and a morphism to $K$, such that the latter has pro-unipotent kernel and such that everything commutes (see Section 4).

$$
1 \rightarrow U \rightarrow \mathcal{G} \rightarrow K \rightarrow 0
$$

If $P$ is an abstract group with a homomorphism to the rational points of $K$, we can still talk about the Malcev completion, by replacing $P$ with its pro-algebraic completion. Hain proved the following, of which Morgan’s theorem is a special case.
Theorem 1.1 ([Hai98, Thm. 13.1]). Let $X$ be a smooth connected variety over $\mathbb{C}$, and let $x$ be a complex point of $X$. Let $V$ be a polarisable variation of Hodge structure on $X$, such that the image of the monodromy representation $\pi_1(X, x) \to \text{Aut}(V_x)$ is Zariski dense in $\text{Aut}(V_x, \langle \cdot, \cdot \rangle)$. Then the Malcev completion of this monodromy representation carries a canonical (real) mixed Hodge structure.

Remark 1.2. What we mean by an affine group scheme $G$ over $\mathbb{Q}$ carrying a mixed Hodge structure, is that $\mathcal{O}(G)$ is a Hopf algebra object in the category of (infinite-dimensional, rational) mixed Hodge structures.

Motivic strengthenings of Morgan’s theorem have been established in some cases. For instance, Deligne ([De89]) equipped the pro-unipotent completion of the fundamental group of a rational variety with the structure of a compatible system of realisations, which is not too far from that of a motive. This was strengthened in [DG05], where the pro-unipotent completion of the fundamental group of a unirational variety defined over a number field was equipped with the structure of a mixed Tate motive, in Voevodsky’s sense.

The goal of this project is to prove a motivic version of Hain’s theorem on Malcev completions of the fundamental group. This is achieved in Thm. 7.3, with the caveat that our result is over the generic point of the variety. The strategy used is to axiomatise the situation and prove an abstract, group-theoretic result (Thm. M). A further application of this abstract theorem also provides a new proof of one of the main results of [DE20] (see [DE20, Thm. 4.3]), and in Section 6 we show how to use this result to weaken the assumptions of Hain’s theorem (see Thm. 6.14).

1.1 Main results

Let $X$ be a smooth connected variety over a subfield $k$ of $\mathbb{C}$, and let $\eta$ be the generic point of $X$. For a Zariski open subset $U$ of $X$, let $\text{LS}(U)$ denote the category of locally constant sheaves of finite-dimensional $\mathbb{Q}$-vector spaces on $U^\text{an}$, and let $\text{LS}(\eta) := 2\text{-colim}_U \text{LS}(U)$. Moreover, let $\text{PM}(U)$ denote the $\mathbb{Q}$-linear category of perverse motives over $U$ defined in [LM99], and let $\text{MLS}(\eta)$ denote $2\text{-colim}_U \text{PM}(U)$. Then $\text{LS}(\eta)$ and $\text{MLS}(\eta)$ are both neutral Tannakian categories, and we denote their Tannaka dual groups by $\pi_1(\eta)$ and $\mathcal{G}(\text{MLS}(\eta))$, respectively. There is a homomorphism $\pi_1(\eta) \to \mathcal{G}(\text{MLS}(\eta))$, whose image is denoted by $\pi_1^\text{mot}(\eta)$: it is the motivic fundamental group of $\eta$. We arrive at our main theorem, the second part of which is a motivic version of Hain’s theorem (over $\eta$).

Theorem 1.3 (Thm. 7.3). The homomorphism $\pi_1(\eta) \to \pi_1^\text{mot}(\eta)$ is Malcev complete. Moreover, given an object $M$ in $\text{MLS}(\eta)$ (a motivic local system on $\eta$), the Malcev completion of the monodromy representation of $\pi_1(\eta)$ associated to $M$ is canonically a (Nori) motive (i.e., carries a canonical action by the motivic Galois group).

Remark 1.4. Taking $M = 1$ in the second part yields a motivic version of Morgan’s theorem.

Fixing a $\mathbb{C}$-point $x$ of $X$, our methods also give the following theorem, where $\pi_1^{\text{Hdg}}(X, x)$ denotes the Hodge fundamental group of $X$ (see Section 6).

Theorem 1.5 (Thm. 6.14). The map $\pi_1(X, x) \to \pi_1^{\text{Hdg}}(X, x)$ is Malcev complete. Moreover, given an admissible variation of mixed Hodge structures $V$ on $X$, the Malcev completion of the monodromy representation $\pi_1(X, x) \to \text{Aut}(V_x)$ carries a canonical (rational) mixed Hodge structure (i.e., an action by the Hodge group).

The Malcev completeness of $\pi_1(X, x) \to \pi_1^{\text{Hdg}}(X, x)$ has already been proved by D’Addezio and Esnault by a different method in [DE20, Thm. 4.3] (their statement looks different, but it is in fact equivalent). The second part of the theorem is a corollary of the first (see Cor. 6.10), which did not appear in [DE20], and is a strengthening of Hain’s theorem [Hai98, Thm. 13.1]. Hain assumed that the variation $V$ was polarisable and that the image of the monodromy representation was Zariski dense in $\text{Aut}(V_x, \langle \cdot, \cdot \rangle)$, whereas we allow arbitrary admissible variations of mixed Hodge structures (with no density assumption).
Our proofs of these two theorems are based on a general criterion for a homomorphism to be Malcev complete, which we now explain. All the groups we consider are affine group schemes over a fixed field of characteristic zero. Let $P \to K$ be a group homomorphism, and denote restriction of (finite-dimensional) $K$-representations along this morphism by $\text{Res}_K^P : \text{Rep} K \to \text{Rep} P$. One can take the following as a definition: $P \to K$ is Malcev complete if and only if

(i) $P \to K$ is surjective, and

(ii) the image of $\text{Res}_K^P$ is stable under extension in $\text{Rep} P$.

Equivalently, a homomorphism is Malcev complete if it’s isomorphic to its own Malcev completion.

**Example 1.6.** Malcev completeness of $\pi_1(X, x) \to \pi_{\text{mot}}^0(X, x)$ (where we leave it to the reader to define the right hand side) is equivalent to the statement that local systems “coming from geometry” are stable under extension in the category of all local systems.

Let us write $S$ for the maximal pro-reductive quotient of $K$, and $\Gamma(K, -)$ and $\Gamma(P, -)$ for $K$- and $P$-invariants respectively. It’s not difficult to see that Malcev completeness of $P \to K$ is equivalent to

(iii) the composition $P \to K \to S$ is surjective, and

(iv) $R^i \Gamma(K, -) = R^i \Gamma(P, -) \circ \text{Res}_K^P$, for $i = 0, 1.\footnote{The surjectivity of $P \to S$ actually implies the $i = 0$ part of the second condition.}$

In applications, the following theorem offers a condition which is easier to check than (iv).

**Theorem 1.7 (Thm. M).** Let $G = K \rtimes H$ be a semi-direct product and consider a homomorphism $P \to K$. Then, writing $S$ for the maximal pro-reductive quotient of $K$, we have that $P \to K$ is Malcev complete if $P \to S$ is surjective and $R^i(\Gamma(K, -) \circ \text{Res}_K^G) = R^i \Gamma(P, -) \circ \text{Res}_K^P$ for $i = 0, 1$.

Note that $R^i(\Gamma(K, -) \circ \text{Res}_K^G)$ is not the same as $R^i \Gamma(K, -) \circ \text{Res}_K^G$, and that the assumption in the theorem is a priori weaker than (iv). In a nutshell, the former is about injective resolutions in $\text{Rep} G$ and the latter about injective resolutions in $\text{Rep} K$. The proof of Thm. 1.7 is by reducing to the following special case.

**Theorem 1.8 (Thm. U).** Let $G = K \rtimes H$ be a semi-direct product and consider a homomorphism $P \to K$. Then $P \to K$ is an isomorphism if $P$ is pro-unipotent and $R^i(\Gamma(K, -) \circ \text{Res}_K^G) = R^i \Gamma(P, -) \circ \text{Res}_K^P$ for $i = 0, 1$.

**Remark 1.9.** Using Tannaka duality, it’s easy to see that, when $P$ is pro-unipotent, $P \to K$ is an isomorphism if and only if condition (iv) above holds. Moreover, the assumptions in Thm. 1.8 easily imply that $K$ too is pro-unipotent, so that $S$ is trivial and the surjectivity of $K \to S$ is automatic, really making it a special case of Thm. 1.7.

### 1.2 A sketch of the proofs

Keep the notation as in Thm. 1.8. The choice of a section $H \to G$ makes $\mathcal{O}(K)$ into an algebra object in $\text{Ind Rep} G$. To prove Thm. 1.8, we need to show that $P \to K$ is an isomorphism, or equivalently that $\mathcal{O}(K) \to \mathcal{O}(P)$ is an isomorphism. We do this by explicitly constructing $\mathcal{O}(P)$ and $\mathcal{O}(K)$ in $\text{Ind Rep} P$ and $\text{Ind Rep} G$, respectively, and then showing that the natural map between them in $\text{Ind Rep} P$ is an isomorphism. They are constructed as certain universal iterated extensions starting with the trivial object 1, and this is carried out in Sections 3.1 and 3.2. The rest of Section 3 is then dedicated to showing that these constructions work as intended, and finally deducing the result.

This construction of the regular representation of a pro-unipotent group as a universal iterated extension has appeared in the literature in several special cases, for example in [Had11; AIK15; Laz15; CPS20]. As far as we know, the general form of Thm. 1.8 proved in the present paper has not appeared before, though it might have been known to experts.
To prove Thm. 1.7, we reduce to the unipotent case as follows. Roughly, we replace the trivial representation \( 1 \) of \( P \) by \( O(S) \), and then repeat the universal iterated extension construction that gave us \( O(P) \) in the unipotent case, except in this case it gives us the regular representation of the Malcev completion of \( P \to K \). In a similar way, \( O(K) \) is once again constructed in IndRep\( G \). The key tool in being able to port the universal iterated extension construction to this non-unipotent context is Prop. 5.3. This proposition makes precise the vague idea of increasing unipotency by replacing \( 1 \) with an ind-object containing all the non-unipotency.

In order for the abstract Thm. 1.7 to be applicable in the Hodge-theoretic context of Thm. 1.5, the two assumptions need to be checked, but let us first provide a dictionary between the groups appearing in the two theorems. The role of \( P \) is played by (the pro-algebraic completion of) \( \pi_1(X, x) \), while that of \( G \) is played by the Tannaka dual of the category of admissible variations on \( X \). The group \( H \) is then taken to be the Hodge group, i.e., the Tannaka dual of the category of rational mixed Hodge structures. Finally, \( K \) is in this context the kernel of the natural homomorphism \( G \to H \), and the map from \( P \to G \) is easily seen to factor through \( K \). The group-cohomological assumption is then established using the theory of Hodge modules, in Prop. 6.8, and the surjectivity assumption is established using Deligne’s semi-simplicity theorem, in Prop. 6.12. This gives the Malcev completeness of the relevant map, and the second part follows as a corollary, as is explained in the proof of Cor. 6.10.

In the motivic context of Thm. 1.3 we work over the generic point of \( X \) because classical Nori motives over its residue field supply us with a good (i.e., neutral Tannakian) category of motivic local systems. The category of local systems over the generic point is defined as the 2-colimit of local systems (or equivalently, of perverse sheaves) on the dense (Zariski) opens of \( X \). In Section 7.2, it is shown that motivic local systems over the generic point have the same relationship with perverse motives, as local systems over the generic point have with perverse sheaves. This lets us use a result from [IM19] to check the cohomological assumption of Thm. 1.7. The surjectivity assumption is dealt with by a result from [Ayo14b]. It is interesting to note, that results from both the world of Nori motives and that of Voevodsky motives were needed to obtain this motivic Hain’s theorem. A key tool in this story, is therefore the bridge between worlds given by Choudhury and Gallauer in [CG17], which says that the motivic Galois groups defined by Ayoub and Nori coincide.

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\(^3\)It’s induced under Tannaka duality by the functor sending a mixed Hodge structure to the associated constant variation.
Part I

Affine group schemes over a field of characteristic zero

2 Preliminaries

We fix a field $F$ of characteristic 0 and assume all additive categories to be $F$-linear. The category of finite-dimensional $F$-vector spaces will be denoted by Vec. We write $1$ for the unit object in a monoidal category. When there are several monoidal categories around, it will be understood from context which unit object we mean.

Convention. Given a short exact sequence
\[ 0 \to A \to B \to C \to 0, \]
we will say that $B$ is an extension of $A$, by $C$. This is opposite to common usage.

By pro-algebraic group we shall mean an affine group scheme over $F$, and such a group will be algebraic if it is of finite type. Any pro-algebraic group is a cofiltered limit of algebraic groups.

We will use the term tensor category to mean an $F$-linear symmetric monoidal category. That is, an $F$-linear category $C$ equipped with an $F$-bilinear functor $(\cdot \otimes \cdot): C \times C \to C$, as well as compatible natural isomorphisms encoding associativity and commutativity, that satisfy the usual axioms. We will say a functor $f$ between tensor categories is monoidal when it's equipped with an isomorphism $f(1) \cong 1$ and a natural isomorphism $f(X \otimes Y) \cong f(X) \otimes f(Y)$, both compatible with the associator and commutator isomorphisms of the two categories. Finally, a natural transformation $\phi: f \to g$ between monoidal functors $f$ and $g$ is monoidal when it's compatible with the monoidal structures of $f$ and $g$.

An object $X$ in a tensor category $C$ is (strongly) dualisable if there exists an object $X^\vee$ (called a dual of $X$) such that $X^\vee \otimes -$ is right adjoint to $X \otimes -$. Moreover, $C$ is rigid if every object in $C$ is dualisable. Duality is reflexive, in the sense that $X$ is a dual of $X^\vee$. Monoidal functors preserve duals.

2.1 Unipotency

Definition 2.1. Let $\mathcal{A}$ be an abelian category and $f: \mathcal{C} \to \mathcal{A}$ a functor. Say an object $U$ in $\mathcal{A}$ is $f$-unipotent if it can be built in a finite number of steps from the essential image of $f$, using extensions. More precisely, if there exists a filtration
\[ 0 = U_{-1} \subseteq U_0 \subseteq \ldots \subseteq U_n = U, \]
such that the quotients $U_i/U_{i-1}$ are in the essential image of $f$. We call such a filtration an $f$-unipotent filtration of $U$, of length $n$. If every object of $\mathcal{A}$ is $f$-unipotent, then $\mathcal{A}$ is said to be $f$-unipotent (or sometimes relatively unipotent over $\mathcal{C}$).

Definition 2.2. Let $U$ be an $f$-unipotent object, as above. Let $n$ be the smallest integer such that $U$ admits an $f$-unipotent filtration of length $n$. We say the $f$-unipotency index of $U$ is $n$.

Remark 2.3. The usefulness of unipotency is that it allows for unipotent induction, that is, induction on the unipotency index. This technique lets us prove many statements, by checking that they work for a smaller class of objects, and that they behave well with respect to extension.

Lemma 2.4. Let $f: \mathcal{C} \to \mathcal{A}$ be as above. Then $f$-unipotency is stable under extension.
**Proof.** Consider an extension

$$0 \to U \to A \to W \to 0,$$

where $U$ and $W$ are both $f$-unipotent. In the case when $W = f(C)$, $A$ is unipotent by definition. We now proceed by induction on the unipotency index of $W$. Let

$$0 \to W' \to W \to f(C) \to 0,$$

such that $W'$ is of a lower unipotency index than $W$. By induction, any extension of something $f$-unipotent by $W'$ is unipotent. The pullback $A'$ of $A \to W$ and $W' \to W$ is therefore unipotent, and $A$ is an extension

$$0 \to A' \to A \to f(C) \to 0,$$

by standard arguments. \hfill \Box

**Definition 2.5.** With $f : C \to A$ as above, we say that $A$ is **weakly $f$-unipotent** if every non-zero object of $A$ admits a non-zero subobject in the essential image of $f$.

Unipotency immediately implies weak unipotency, but the converse is only true under a finiteness condition.

**Proposition 2.6.** Let $f : C \to A$ be as above.

(i) If $A$ is $f$-unipotent, then it is weakly $f$-unipotent.

(ii) If $A$ is weakly $f$-unipotent, then any object in $A$ of finite length is $f$-unipotent.

**Proof.** (i) Let $A \in A$ be non-zero. Fix an $f$-unipotent filtration $A_i$ of $A$ as above. Then the minimal $i$ such that $A_i$ is non-zero gives the desired subobject.

(ii) Let $A$ be non-zero of finite length. We get a short exact sequence

$$0 \to f(C) \to A \to A_1 \to 0.$$ 

Then $A_1$ has a non-zero subobject $f(C_1)$, and we iterate this until, by finiteness of the length of $A$, we get $A_n = f(C_n)$. Using that unipotency is closed under extension (Lemma 2.4), we can travel back along the $A_n$, concluding that they, and $A$, are all $f$-unipotent. \hfill \Box

**Remark 2.7.** When $A$ is monoidal, $C = \text{Vec}$, and $f : \text{Vec} \to A$ is monoidal, then $f$ is uniquely determined, up to monoidal natural isomorphism (cf. Rmk. 2.9). In this case, we usually just say that $A$ is **unipotent**, without reference to $f$.

### 2.2 Neutral Tannakian categories

The notion of a (neutral) Tannakian category is central to this paper. A **neutral Tannakian category** (over $F$) is a rigid abelian tensor category $(A, \otimes, 1)$ such that $\text{End}(1) = F$, and which admits a faithful, exact, monoidal functor $A \to \text{Vec}$: such a functor is called a **(neutral) fibre functor**. A neutralised Tannakian category is a Tannakian category together with a neutral fibre functor.

**Remark 2.8.** The tensor product is required to be $F$-bilinear, and it is automatically exact in both arguments ([DM82, Prop. 1.17]). The rigidity implies existence of an internal Hom functor, which is (right and left) adjoint to the tensor product, and which is hence also exact in both arguments. We denote it by $\text{Hom}(-, -)$.

By morphism of neutral Tannakian categories $A$ and $B$ we shall mean an exact, faithful, monoidal functor. We will always write such a functor decorated with a raised asterisk, e.g. $p^* : A \to B$. These functors always have ind-right adjoints, i.e., functors $p_* : \text{Ind} B \to \text{Ind} A$, for which we use the same letters but
lowered asterisks. By morphism of neutralised Tannakian categories \((A, f^*)\) and \((B, \varphi^*)\) we shall mean a morphism \(p^* : A \to B\) together with a monoidal isomorphism \(\varphi^* \circ p^* \simeq f^*\).

This gives 2-categories \(n\Tan\) and \(n\Tan^*\) of neutral and neutralised Tannakian categories, respectively.

**Remark 2.9** (On sections of fibre functors.). A fibre functor \(f^*\) of a Tannakian category \(B\) always admits a section in the 2-category above. Fix an equivalence of \(\text{Vec}\) with its skeleton, consisting of objects \(1_1^n, n \geq 0\). On these objects, we simply define the section \(s^*\) in the only way we can: \(1\) has to go to \(1\), and direct sums must be respected. This is clearly a section, since \(f^*\) is monoidal and respects direct sums. Moreover, we can explicitly describe the ind-right adjoint \(s_\ast\), which turns out to restrict to an honest right adjoint of \(s^*\). Indeed, there’s a natural isomorphism \(s_\ast \simeq \text{Hom}_A(1, -)\). This follows immediately from the fact that \(\text{id}_\text{Vec}\) equals \(\text{Hom}_\text{Vec}(1, -)\).

Given a neutralised Tannakian category \((B, f^*)\), Tannaka duality produces a pro-algebraic group, which we denote \(\mathcal{G}(f^*)\), \(\mathcal{G}(B, f^*)\), or \(\mathcal{G}(B)\), depending on what we want to emphasise. We call this group the Tannaka dual of \((B, f^*)\). This group is such that \(f^*\) naturally factors through the forgetful functor \(\text{Rep}\mathcal{G}(f^*) \to \text{Vec}\), and such that \(B \to \text{Rep}\mathcal{G}(f^*)\) is a (monoidal) \(F\)-linear equivalence. We also have a canonical isomorphism \(\mathcal{G}(\text{Rep}(G)) \simeq G\) for all \(G\).

**Remark 2.10** (On Tannakian categories and unipotency).

- A pro-algebraic group \(G\) is pro-unipotent if and only if \(\text{Rep} G\) is unipotent.
- All objects in a neutral Tannakian category are of finite length, so for these categories, weak unipotency and unipotency coincide.
- If \(V\) is a unipotent \(G\)-representation, then any subquotient of \(V\) in \(\text{Rep} G\) is unipotent. (Proof: Enough to check for subobjects of \(V\), by rigidity. Given a unipotent filtration on \(V\), the induced filtration on a subobject is also a unipotent filtration, by the fact that any subobject of \(1^m\) is of the form \(1^n\).)

Tannaka duality is functorial: given a morphism \(p^*\) of neutralised Tannakian categories as above, we get an induced group-scheme homomorphism \(p : \mathcal{G}(\varphi^*) \to \mathcal{G}(f^*)\), which we always write using the same symbol, but without the asterisk. Conversely, given a group homomorphism \(p : G \to H\), we get a morphism of Tannakian categories \(p^* : \text{Rep} H \to \text{Rep} G\), namely restriction along \(p\). The ind-right adjoint \(p_\ast\) is in this context given by induction along \(p\). Cf. Section 2.3.

The following proposition is an example of how properties of group homomorphisms are reflected on the Tannakian side.

**Proposition 2.11** ([DM82, Prop. 2.21]). Let \(p : G \to H\) be a homomorphism of pro-algebraic groups.

1. \(p\) is a closed immersion (injective) if and only if every object in \(\text{Rep} G\) is isomorphic to a subquotient of an object in the image of \(p^*\).
2. \(p\) is faithfully flat (surjective) if and only if \(p^*\) is full and for every subobject \(X \hookrightarrow p^* Y\) in \(\text{Rep} G\), there exists a subobject \(Y'\) of \(Y\) such that \(p^* Y' \simeq X\).

With the second part of this proposition in mind, we make the following definition.

**Definition 2.12.** A Tannakian subcategory of a Tannakian category is a full abelian subcategory closed under direct sums, tensor products, taking duals, and taking subquotients.

Given a homomorphism \(G \to H\) of pro-algebraic groups, we sometimes denote by \(\text{Res}_G^H : \text{Rep} H \to \text{Rep} G\) the restriction functor from \(H\)-representations to \(G\)-representations.

**Lemma 2.13.** Given a homomorphism \(p : G \to H\), the smallest Tannakian subcategory of \(\text{Rep} G\) containing \(\text{im}(p^*)\) is the essential image of \(\text{Res}_G^{\text{im}(p)}\).
Proof. Let $\mathcal{T} \subseteq \text{Rep}(G)$ denote the smallest Tannakian subcategory containing $\text{im}(\rho^*)$. Note that we have a factorisation $G \to \mathcal{G} (\mathcal{T}) \to H$ of $\rho$, and that the first map is surjective by Prop. 2.11.2. Next, we check that $\mathcal{G} (\mathcal{T}) \to H$ is injective by Prop. 2.11.1. Thus, we need to show that every object in $\mathcal{T}$ is a subquotient of an object in the essential image of $\rho^*$. But this follows from the following elementary fact: if you take a full, monoidal subcategory of a Tannakian category, closed under taking duals, then the closure of this subcategory under taking subquotients is a Tannakian category. Using this fact for the monoidal subcategory $\text{im}(\rho^*)$ of $\text{Rep} G$, we get that $\mathcal{T}$ is its closure under forming subquotients, by the minimality of $\mathcal{T}$.

Lemma 2.14. A homomorphism $\rho: G \to H$ has pro-unipotent kernel if and only if the category $\text{Rep} G$ is generated under extensions and subquotients by its full subcategory $\text{im}(\rho^*)$.

Proof. Assume that $\text{Rep} G$ is generated under extensions and subquotients by its full subcategory $\text{im}(\rho^*)$. Then it is clear that whatever is in the image of $\text{Res} G^K$, is (absolutely) unipotent. By Prop. 2.11.1, every $K$-representation is then a subquotient of a unipotent $K$-representation. But such a representation must also be unipotent (Rmk. 2.10).

Now assume that $\text{ker}(\rho) := K$ is pro-unipotent. Then, given a non-zero $G$-representation $V$, the $K$-invariants $V^K$ are non-zero and form a sub-$G$-representation (since $K$ is normal in $G$). Moreover, $V^K$ is actually the restriction of an $\text{im} (\rho)$-representation, since $\text{im}(\rho) = \text{coker}(K \to G)$. This shows $\text{Res} G^\text{im}(\rho)$ to be a subcategory of $\text{Rep} G$, by Rmk. 2.10.

One perspective on Tannaka duality is as follows. Let $f^*$ be a fibre functor on a neutral Tannakian category $B$ as above. Then, using the 2-section $s^*$ of $f^*$ described in Rmk. 2.9, one can prove that $f^* f_1$ is naturally a Hopf algebra, and we denote it by $\mathcal{H}(f^*) := f^* f_1$. The Tannaka dual group is then $\mathcal{G}(f^*) := \text{Spec } \mathcal{H}(f^*)$. Note that $\mathcal{G}(f^*)$-representations are the same as $\mathcal{H}(f^*)$-comodules.

Remark 2.15. It’s worth noting at this point, that if $f^*$ denotes the canonical fibre functor on $\text{Rep} G$, then $f^* f_1$ is $\mathcal{O}(G)$, and $f_1$ is the regular representation of $G$. Moreover, $f_1$ is naturally an algebra object in $\text{Ind Rep} G$.

2.3 Restriction, induction, and invariants

Let $\rho: H \to G$ be any morphism of groups. We denote the restriction functor $\text{Rep} G \to \text{Rep} H$ by $\rho^* := \text{Res} H^G$. In the literature, induction is often only defined for $\rho$ injective. We depart from this convention and call the ind-right adjoint $\rho_* : \text{Ind Rep} H \to \text{Ind Rep} G$ of $\rho^*$ induction for any $\rho$. Note that we can decompose any $\rho$ as a surjection $p_0 : H \to \text{im}(\rho)$ followed by an injection $p_1 : \text{im}(\rho) \to G$. Our induction functor $\rho_*$ is then given by taking invariants $\Gamma(\text{ker}(p_0), -) = (-)^{\text{ker}(p_0)}$ under the action of the kernel of $p_0$, followed by classical induction $\text{Ind}_{\text{im}(\rho)}^G$ along the injective homomorphism $p_1$.

Remark 2.16. The invariants functor $\Gamma(G, -) = (-)^G$ on $\text{Rep} G$ is given by “induction” along $G \to 1$, and the trivial representation functor $\text{Vec} \to \text{Rep} G$ is given by restriction along $G \to 1$.

An important fact is that the projection morphism for the restriction-induction adjunction is an isomorphism.

Proposition 2.17 (The restriction-induction projection formula). Let $\rho$ be a homomorphism of pro-algebraic groups. Then there is an isomorphism $p_* (A \otimes \rho^* B) \simeq p_* A \otimes B$, natural in $A$ and $B$. In particular, setting $A = 1$ gives a natural isomorphism $p_* \rho^* \simeq p_* 1 \otimes (-)$.

Proof. For $\rho$ injective, this is [Jan03, I.3.6]. For $\rho$ surjective, it is obvious. Combining these two cases, we can conclude for arbitrary $\rho$.

We give two trivial consequences of this result, since they will be useful in the sequel. They both concern morphisms $\rho$ that admit a retraction $r$, in particular, morphisms that are injective. The first result gives
a description of $p^*$ in terms of $p_*$ and $r_*$. The second says that, when tensoring with $p_1$, one may first change the ker$(r)$-action to be trivial, and then tensor with $p_*1$, and get the same result.

**Lemma 2.18.** Let $p: H \to G$ be a homomorphism of pro-algebraic groups, and let $r: G \to H$ be a retraction of $p$. Then we have a natural isomorphism $p^* \simeq r_*(p_1 \otimes -)$.

**Proof.** By Prop. 2.17, we have an isomorphism $p_*p^* \simeq p_*1 \otimes (-)$. To this isomorphism, we apply $r_*$, then use that $r_*p_* = \text{id}$, and get $p^* \simeq r_*(p_1 \otimes -)$. □

**Lemma 2.19.** Let $p: H \to G$ be a homomorphism of pro-algebraic groups, and let $r: G \to H$ be a retraction of $p$. Then we have a natural isomorphism $p_1 \otimes (-) \simeq p_1 \otimes r^*p^*(-)$.

**Proof.** Using the projection formula, 2.17, we compute

$$p_*1 \otimes (-) \simeq p_*p^*(-) \simeq p_*(p_*r^*)p^*(-) \simeq p_*1 \otimes r^*p^*(-).$$ □

### 2.4 Group isomorphisms

Consider a homomorphism of pro-algebraic groups $p: G \to H$. We have a corresponding morphism of Tannakian categories $p^*: \text{Rep} H \to \text{Rep} G$ (restriction along $p$). Let $f^*$ and $\varphi^*$ be the canonical fibre functors of Rep $H$ and Rep $G$, respectively. Then $p^*$ induces a morphism of algebra objects $p_{\text{alg}}: p^*f_1 \to \varphi_1$, from the restriction along $p$ of the regular representation of $H$ to the regular representation of $G$ (cf. Rmk. 2.15). This map is simply the algebra homomorphism $\mathcal{O}(H) \to \mathcal{O}(G)$ induced by $p$, which respects the $\mathcal{O}(G)$-coactions, making it a morphism of algebra objects in Ind Rep $G$.

Another way of viewing this morphism, is via the universal property of $\varphi_1$. We have a natural isomorphism $\text{Hom}(-, \varphi_1) \simeq \text{Hom}(\varphi^*(-), 1)$. So, to give a map $p^*f_1 \to \varphi_1$, is the same as to give a map from $\varphi^*p^*f_1$ to 1. But $\varphi^*p^*f_1 \simeq f^*f_1$ has a natural map to 1: the counit it’s equipped with as a Hopf algebra. In this way, we recover the algebra homomorphism $p_{\text{alg}}: p^*f_1 \to \varphi_1$, above. We can see this by applying $\varphi^*$ to $p_{\text{alg}}$, and noting that the result is a morphism of Hopf algebras, and is hence compatible with the two counits. We get the following proposition.

**Proposition 2.20.** A homomorphism $p: G \to H$ of pro-algebraic groups is an isomorphism if and only if the algebra morphism $p^*\mathcal{O}(H) \to \mathcal{O}(G)$ in Ind Rep $G$, induced by the counit of $\mathcal{O}(H)$ under the universal property of $\mathcal{O}(G)$, is an isomorphism.

### 2.5 Categories of modules

Consider an abelian tensor category $\mathcal{A}$ with all filtered colimits, in which the tensor product commutes with finite colimits, and let $\mathcal{A}^{\text{cpt}}$ denote the full subcategory of compact objects: objects $C \in \mathcal{A}$ such that $\text{Hom}(C, -)$ commutes with filtered colimits. Take an algebra object $A$ in $\mathcal{A}$, and consider the category $\text{Mod}(A)$ of (left) $A$-modules in $\mathcal{A}$, i.e., objects $M \in \mathcal{A}$ together with an $A$-action $A \otimes M \to M$ satisfying the usual axioms. The category $\text{Mod}(A)$ is an abelian tensor category (under $\otimes_A$).

**Definition 2.21.** Let $M$ be an $A$-module, as above. Then,

- $M$ is strictly free of finite rank if $M \simeq A^\oplus n$ for some $n$;
- $M$ is free if $M \simeq A \otimes X$ for some $X$ in $\mathcal{A}$;
- $M$ is finitely generated free if $M \simeq A \otimes X$ for some $X$ in $\mathcal{A}^{\text{cpt}}$;
- $M$ is finitely generated if $M$ is a quotient $A \otimes X$ of some finitely generated free module.

**Remark 2.22.** If $\mathcal{B}$ is an abelian category, then $(\text{Ind} \mathcal{B})^{\text{cpt}}$ is exactly the essential image of $\mathcal{B}$ in Ind $\mathcal{B}$. This is the situation in which we will use these notions, i.e., $\mathcal{B}$ will be an abelian tensor category, $\mathcal{A}$ will be Ind $\mathcal{B}$, and $A$ will be an algebra object in Ind $\mathcal{B}$.
2.6 The Standard Situation

We now describe the Tannaka dual of the group theoretic setup from Theorems 1.7 and 1.8 from the introduction. Consider the following diagram in nTan:

\[ \begin{array}{c}
M \xrightarrow{B^*} T \\
\downarrow f^* \quad \downarrow \phi^* \quad \downarrow \sigma^* \\
\mathcal{E} \xrightarrow{b^*} \text{Vec}
\end{array} \] (1)

We sometimes refer to the left hand side as the relative side, the right hand side as the absolute side, and the pair of functors \((B^*, b^*)\) as a realisation from the former to the latter. We assume we’re given monoidal isomorphisms \(b^* f^* \simeq \phi^* B^*\), \(\phi^* \sigma^* \simeq \text{id}\), and \(f^* s^* \simeq \text{id}\). These combine to give various other isomorphisms, such as \(b^* \simeq \phi^* B^* s^*\). Moreover, \(\sigma^*\) and \(s^*\) are automatically full, since they are 2-sections of faithful functors. We denote the Tannaka duals as follows:

\[ H := \mathcal{G}(\mathcal{E}), \quad G := \mathcal{G}(M), \quad P := \mathcal{G}(T). \]

Let us extract from diagram (1) the sequence

\[ \mathcal{E} \xrightarrow{s^*} M \xrightarrow{B^*} T, \]

of morphisms of neutral Tannakian categories. We have a dual sequence of groups

\[ P \xrightarrow{B} G \xrightarrow{s} H. \]

Note that \(s\) is surjective, since the 2-retraction \(f^*\) of \(s^*\) in nTan produces a section \(f\) of \(s\).\(^5\)

Writing \(\iota: K \to G\) for the kernel of \(s\), we get a split exact sequence of pro-algebraic groups

\[ 1 \to K \xrightarrow{\iota} G \xrightarrow{f} H \to 1, \]

and an isomorphism \(G \simeq K \rtimes H\).

The splitting just described induces an action by \(H\) on \(K\), telling us that \(O(K)\) should naturally come from \(\text{Ind}\mathcal{E}\). Indeed, the fact that \(f^*\) has a nice section \(s^*\), gives rise to a natural Hopf algebra structure on \(f^* f_1\) in \(\text{Ind}\mathcal{E}\) (see [Ayo14a, Thm. 1.45]), and this Hopf algebra then describes the kernel group \(K\). More precisely, \(\text{Spec}(b^*(f^* f_1)) \simeq K\) (see [Ayo14b, Thm. 2.7]).

In analogy with the notation introduced in Section 2.2, we write \(f^* f_1 =: \mathcal{H}(M/E, f^*, s^*)\), omitting some of the arguments from the notation depending on what we want to emphasise. Likewise, we write \(\mathcal{G}(M/E, f^*, s^*)\) for its pro-dual in \((\text{Ind}\mathcal{E})^{op} \simeq \text{Pro}(\mathcal{E})^{op}\). Then one can make sense of the formula \(b^* \mathcal{G}(M/E) \simeq K\).

Note that the composition \(B^* \circ s^*\) factors through \(\sigma^*\), up to a natural isomorphism. Thus the composition \(s \circ B\) is trivial, and \(B\) factors as \(B = \iota \circ g\), through the kernel \(K\) of \(s\). The following diagrams summarise the situation:

\[ \begin{array}{ccc}
1 \to K \xrightarrow{\iota} G \xrightarrow{f} H \to 1 & & \mathcal{M} \xrightarrow{\iota^*} \text{Rep} K \xrightarrow{\varphi^*} T \\
\downarrow \varphi \quad \downarrow \iota & & \downarrow \varphi \quad \downarrow \sigma^* \\
\mathcal{E} \xrightarrow{b^*} \text{Vec} & & \mathcal{E} \xrightarrow{b^*} \text{Vec}
\end{array} \]

\(^4\)If the choice of symbols looks strange, it is because it has been made with certain applications in mind. The letters \(M\) and \(T\) were chosen with the words motivic and topological in mind, while \(B\) and \(b\) were taken from Betti. The letters \(f\) and \(\varphi\) are meant to evoke fibre, while \(s\) and \(\sigma\) simply stand for section.

\(^5\)It is somewhat unfortunate that \(f\) denotes the section of \(s\), instead of the other way around. But since the bulk of this paper takes place on the Tannakian side of things, it seemed reasonable to give priority in the notation to the fact that \(s^*\) is a section of \(f^*\).
Finally, there is a canonical map

\[ b^* R s_* \to R \sigma_* B^* \]  

which will be important to us. It corresponds under adjunction to the map \( \sigma^* b^* R s_* \to B^* \) which, via the isomorphism \( \sigma^* b^* \simeq B^* s^* \), is induced by the counit of the \((s^*, R s_*)\)-adjunction. In our two main theorems, the key assumption is that the degree 0 and 1 parts of this canonical map (2) are isomorphisms.

3 Theorem U

The goal of this section is to prove Thm. U. The methods are Tannakian, and the first step is to reformulate the result under Tannakian duality, via the Standard Situation (Section 2.6):

\[
\begin{array}{c}
\mathcal{M} \xrightarrow{B^*} \mathcal{T} \\
\downarrow^{f^*} \\
\mathcal{E} \xrightarrow{b^*} \operatorname{Vec}
\end{array}
\]

We may now state the first key result.

**Theorem 3.1.** Suppose \( \mathcal{T} \) is \( \sigma^* \)-unipotent. If the canonical maps \( b^* R^i s_* \to R^i \sigma_* B^* \), for \( i = 0, 1 \), are isomorphisms, then the natural algebra homomorphism \( B^* f_* 1 \to \varphi_* 1 \) (see Section 2.4) is an isomorphism.

**Remark 3.2.** The assumptions of Thm. 3.1 imply that \( \mathcal{M} \) is \( s^* \)-unipotent, as follows. By Rmk. 2.10, it’s enough to show weak unipotency. But given a non-zero object \( X \) of \( \mathcal{M} \), the subobject \( s^* s_* X \) must be non-zero. (The fact that it’s a subobject follows from the surjectivity of \( \mathcal{G}(\mathcal{M}) \to \mathcal{G}(\mathcal{E}) \).) To check this, we apply \( B^* \), yielding \( \sigma^* \sigma_* B^* X \). Now the first assumption gives that this is non-zero, since \( B^* X \) is non-zero and \( \sigma^* \sigma_* B^* X \) is the largest subobject of \( B^* X \) in the essential image of \( \sigma^* \).

Recall, that in the Standard Situation (Section 2.6), there is an induced group homomorphism

\[ \varrho: P = \mathcal{G}(\varphi^*) \to b^* \mathcal{G}(f^*) = K. \]

Using Thm. 3.1 and Prop. 2.20, we get Thm. 1.8 from the introduction, which in the present context reads as follows.

**Theorem U.** Suppose \( \mathcal{T} \) is \( \sigma^* \)-unipotent. If the canonical maps \( b^* R^i s_* \to R^i \sigma_* B^* \), for \( i = 0, 1 \), are isomorphisms, then \( \varrho \) is an isomorphism. \( \square \)

We prove Thm. 3.1 by constructing \( U \simeq \varphi_* 1 \) in \( \operatorname{Ind} \mathcal{T} \) (Section 3.1 and Prop. 3.7) and \( W \simeq f_* 1 \) in \( \operatorname{Ind} \mathcal{M} \) (Section 3.2 and Prop. 3.11) as well as an isomorphism \( B^* W \simeq U \) (Prop. 3.4). After all of that, we only have to check some compatibilities, which we do in Section 3.4.

3.1 Construction of \( U \)

The relevant context in this section is that of a Tannakian category \( \mathcal{T} \) with fibre functor \( \varphi^* \), which has a section \( \sigma^* \) as in Rmk. 2.9. We need no further assumptions for the construction.

We begin with some motivation. When \( \mathcal{T} \) is assumed to be unipotent, every object is (by definition) a successive extension by trivial objects, starting from \( 1 \). The ind-object \( \varphi_* 1 \) is also unipotent in such a situation. It is moreover universal in some sense, e.g. it contains all objects of \( \mathcal{T} \), in the sense that each of them appear as a subquotient of some \( (\varphi_* 1)^{\oplus n} \). The idea, when constructing \( U \), is therefore to start from \( 1 \), and iterate a kind of universal extension by a trivial object.

Let \( U_0 := 1_\mathcal{T} \) and construct \( U_{n+1} \) inductively:

\[ 0 \to U_n \to U_{n+1} \to \sigma^* V \to 0, \]  

which will be important to us. It corresponds under adjunction to the map \( \sigma^* b^* R s_* \to B^* \) which, via the isomorphism \( \sigma^* b^* \simeq B^* s^* \), is induced by the counit of the \((s^*, R s_*)\)-adjunction. In our two main theorems, the key assumption is that the degree 0 and 1 parts of this canonical map (2) are isomorphisms.
i.e., define $U_{n+1}$ as an extension of $U_n$ by a trivial object. We have to say what $V$ is and what the extension class is.

Note that

$$\text{Ext}^1(\sigma^* V, U_n) = \text{Hom}(\sigma^* V, U_n[1]) = \text{Hom}(V, R\sigma_* U_n[1]) = \text{Hom}(V, R^1\sigma_* U_n),$$

where the last equality is due to Vec being semi-simple. We therefore take $V$ to be $R^1\sigma_* U_n$ and the extension class to be given by the identity in $\text{End}(R^1\sigma_* U_n)$. Set $U := \text{colim}_{n \geq 0} U_n$ in $\text{Ind } T$.

We now record two key properties of this construction. Apply $\sigma_* \simeq \text{Hom}(1, -)$ to the defining sequence (3) of $U_{n+1}$ to get a long exact sequence

$$0 \to \text{Hom}(1, U_n) \to \text{Hom}(1, U_{n+1}) \to R^1\sigma_* U_n \to \text{Ext}^1(1, U_n) \to \text{Ext}^1(1, U_{n+1}) \to \cdots$$

We have that $\delta$ is an isomorphism, by definition. Therefore $a$ is an isomorphism and $b$ is the zero map. This establishes the following lemma.

**Lemma 3.3.** We have a canonical isomorphism $1 \simeq \sigma_* U_0$, and the canonical maps $\sigma_* U_n \to \sigma_* U_{n+1}$, $n \geq 0$, are isomorphisms. Moreover, the pushout extension along $U_n \to U_{n+1}$ of any extension of $U_n$ by a trivial object is split. \hfill \Box

### 3.2 Construction of $W$

For this construction, we work with the Tannakian categories $\mathcal{M}$ and $\mathcal{E}$ with a morphism $f^*$ and its section $s^*$, as introduced above. We need no further assumptions for the construction.

The idea for this construction is simply to implement the construction of $U$ in this more general context. The difficulty arises from the failure of $\mathcal{E}$ to be semi-simple (in contrast to Vec). While not every distinguished triangle in the derived category $D(\mathcal{E})$ of $\mathcal{E}$ splits, the ones we need to consider, it turns out, do split. The functor $f^*$ plays a key role in finding these splittings.

Let $W_0 := 1_{\mathcal{M}}$. We define $W_{n+1}$ inductively from $W_n$ as an extension

$$0 \to W_n \to W_{n+1} \to s^* R^1s_* W_n \to 0. \quad (4)$$

Such an extension is given by a map

$$s^* R^1s_* W_n \to W_n[1],$$

in the derived category. The counit of the $s$-adjunction gives us a map $s^* Rs_* W_n[1] \to W_n[1]$, and what’s missing is a map $s^* R^1s_* W_n \to s^* R_s W_n[1]$. This map will be the image under $s^*$ of a map $R^1s_* W_n \to Rs_* W_n[1]$ which we now spend some time constructing.

We shall make two auxiliary induction hypotheses.

(IH-1) We have a canonical isomorphism $1 \simeq s_* W_0$, and the canonical maps

$$s_* W_0 \to s_* W_1 \to \cdots \to s_* W_n,$$

are isomorphisms.

(IH-2) We have a map $\alpha_n : f^* W_n \to 1$, such that the composition

$$1 \simeq s_* W_n \simeq f^* s^* s_* W_n \to f^* W_n \xrightarrow{\alpha_n} 1,$$

is the identity map.
These hypotheses are satisfied for the base case \( n = 0 \): \( s_1 \simeq s_1^* s_1 \simeq 1 \), since \( s^* \) is monoidal and fully faithful; and we let \( \alpha_0 : f^* 1 \simeq 1 \) be the map coming from the monoidal structure of \( f^* \).

We are now ready to construct the missing map \( R^1 s_* W_n \to Rs_* W_n[1] \), using the auxiliary assumptions on \( W_n \). They allow us to split off \( s_* W_n \) from \( R s_* W_n \).

Note that (IH-1) gives us a map \( 1 \to Rs_* W_n \), which is an isomorphism on the zeroth cohomology. Now consider the cone \( C \) of this map and the resulting distinguished triangle

\[
1 \to Rs_* W_n \to C \to
\]

Moreover, consider the map \( \beta_n : Rs_* W_n \to f^* W_n \) defined as the composition

\[
Rs_* W_n \simeq f^* s^* Rs_* W_n \to f^* W_n.
\]

By (IH-2), the triangle (5) is split by retracting the left map, \( 1 \to Rs_* W_n \), with

\[
Rs_* W_n \xrightarrow{\beta_n} f^* W_n \xrightarrow{\alpha_n} 1,
\]

and we hence get a section \( s : C \to Rs_* W_n \). By construction, the first non-zero cohomology of \( C \) is \( H^1 C = R^1 s^* W_n \) and we get a map \( : R^1 s^* W_n \to C[1] \). Composing with \( s[1] : C[1] \to Rs_* W_n[1] \) gives the map \( R^1 s_* W_n \to R s_* W_n[1] \) we are looking for. The following diagram may clarify the situation.

\[
\begin{array}{ccc}
1 & \xrightarrow{\alpha_n} & Rs_* W_n \\
& \searrow & \downarrow s \\
& & C \\
& & \alpha_n \\
\end{array}
\]

\[
\begin{array}{ccc}
Rs_* W_n & \xrightarrow{\beta_n} & f^* W_n \\
\downarrow & \downarrow & \downarrow \\
R^1 s_* W_n[1] & & \end{array}
\]

What is left is to establish the induction hypotheses (IH-1) and (IH-2) for \( W_{n+1} \). We start with lifting \( \alpha_n \) to \( W_{n+1} \). Applying \( f^* \) and then the contravariant \( \text{Hom}(-, 1) \) to the defining sequence (4) of \( W_{n+1} \) yields an exact sequence

\[
\text{Hom}(f^* W_{n+1}, 1) \to \text{Hom}(f^* W_n, 1) \to \text{Ext}^1(R^1 s_* W_n, 1) \to a(\alpha_n)
\]

and hence the existence of \( \alpha_{n+1} \) is equivalent to \( a(\alpha_n) \) being zero. But \( a(\alpha_n) = (\alpha_n[1]) \circ \omega_n \), where \( \omega_n \) is the extension class of \( f^* W_{n+1} \), i.e.,

\[
\omega_n : R^1 s_* W_n \to C[1] \to Rs_* W_n[1] \to f^* W_n[1].
\]

Composing this with \( \alpha_n[1] \) gives zero, because \( \alpha_n \circ \beta_n \circ s = 0 \), as this was how we split the triangle (5) defining \( C \). Finally, consider the following diagram.

\[
\begin{array}{ccc}
1 & \xrightarrow{\sim} & s_* W_n & \xrightarrow{\sim} & f^* s^* s_* W_n & \to & f^* W_n \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \to & s_* W_{n+1} & \to & f^* s^* s_* W_{n+1} & \to & f^* W_{n+1} \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 1 & \xrightarrow{\sim} & 1 & \xrightarrow{\sim} & 1
\end{array}
\]

It is commutative, and hence the bottom row is the identity as required, establishing (IH-2) for \( W_{n+1} \).

Lastly, as the full faithfulness of \( s^* \) implies \( s_* s^* \simeq \text{id} \), the long exact sequence after applying \( s_* \) to (4) is:

\[
0 \to s_* W_n \xrightarrow{a} s_* W_{n+1} \to R^1 s_* W_n \to \cdots
\]

Because \( \delta \) is the identity map by construction, we get that \( a : s_* W_n \simeq s_* W_{n+1} \) is an isomorphism, establishing (IH-1) for \( W_{n+1} \). This finishes the construction of \( W : = \text{colim}_{n \geq 0} W_n \).

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3.3 Properties

In this section, we establish some key properties of $U$ and $W$. Most importantly, their isomorphisms with $\varphi_* \mathbf{1}$ and $f_* \mathbf{1}$, respectively, are proved in Prop. 3.7 and Prop. 3.11.

3.3.1 $W$ realises to $U$

We begin with establishing that $B^*W$ is canonically isomorphic to $U$.

**Proposition 3.4.** If the canonical map $b^* R^1 s_* \to R^1 \sigma_* B^*$ is an isomorphism, then $B^*W \simeq U$.

**Proof.** We give an isomorphism of filtered systems $(B^*W_n)_n \simeq (U_n)_n$. Clearly $B^*W_0 \simeq U_0$, since $B^*$ is monoidal. We now proceed by induction, and assume that $B^*W_n \simeq U_n$. Applying $B^*$ to the defining sequence (4) for $W_{n+1}$ yields an isomorphism of extensions

$$
0 \longrightarrow B^*W_n \longrightarrow B^*W_{n+1} \longrightarrow B^*(s^* R^1 s_* W_n) \longrightarrow 0
$$

where we've used that realisation commutes with pushforwards on the rightmost term. We have to check that the extension class is correct. Applying $B^*$ to the composition

$$
s^* R^1 s_* W_n \to s^* C[1] \to s^* R s_* W_n[1] \to W_n[1],
$$

gives

$$
\sigma^* R^1 \sigma_* U_n \to \sigma^* b^* C[1] \to \sigma^* b^* R s_* W_n[1] \to U_n[1].
$$

Under adjunction this corresponds to

$$
R^1 \sigma_* U_n \to b^* C[1] \to b^* R s_* W_n[1] \to R \sigma_* U_n[1].
$$

This map is in the semi-simple category $\mathcal{D}(\text{Vec})$, and the source is concentrated in degree zero, so it is equivalent to its zeroth cohomology

$$
R^1 \sigma_* U_n \to H^1(b^* C) \to b^* R^1 s_* W_n \to R^1 \sigma_* U_n,
$$

and, looking at the construction of $U$, we have to see that this composition is the identity. This is easy to see, using the definition of $C$ together with the definition of the first two maps, as well as the assumption of the proposition (to deal with the last map).

3.3.2 Characterisation of $U$

The universal property of $\varphi_* \mathbf{1}$ is that the natural transformation

$$
\text{Hom}(-, \varphi_* \mathbf{1}) \to \text{Hom}(\varphi^*(-), \mathbf{1}),
$$

is an isomorphism. This transformation takes a map on the left hand side, applies $\varphi^*$ to it, and then composes with $\varphi^* \varphi_* \mathbf{1} \to \mathbf{1}$. In order to show the isomorphism $U \simeq \varphi_* \mathbf{1}$, we simply have to establish the same universal property for $U$.

A key property of $\varphi_* \mathbf{1}$ (which we won’t use) is the following.

**Proposition 3.5.** Every short exact sequence sequence

$$
0 \to \varphi_* \mathbf{1} \to E \to X \to 0,
$$

in $\text{Ind } \mathcal{T}$ splits.
Proof. Since $\varphi^*$ is exact, $\varphi_*1$ is injective.

**Lemma 3.3** implies that $U$ has a weaker variant of this property, given by the following proposition.

**Proposition 3.6** (Splitting Property). Every short exact sequence of the form

$$0 \to U \to E \to \sigma^*V \to 0,$$

in $\text{Ind } T$, where $V$ is a finite-dimensional vector space, splits.

**Proof.** Without loss of generality, we may replace $\sigma^*V$ by $1$. We have that $E$ is the filtered colimit of its subobjects lying in $T$. Let $L$ be a subobject of $E$, in $T$, which is not contained in (the image of) $U$ (it is clearly possible to find such an $L$). Then, let $L' := L \cap U$, and denote the cokernel of $L' \to L$ by $Q$.

We obtain an inclusion of short exact sequences,

$$0 \to L' \to L \to Q \to 0$$

$$0 \to U \to E \to 1 \to 0$$

Thus $Q$ has to be either 0 or 1, and it can’t be 0 since $L \neq L'$ by assumption. We therefore have our sequence as a pushout along $L' \to U$ of this subsequence lying in $T$. The map $L' \to U$ factors through $U \to U_{n+1}$ for some $n$, and we conclude by **Lemma 3.3**.

Consider the map $u: \varphi^*U \to 1$, obtained from $b^*\alpha_n$:

$$b^*f^*W_n \xrightarrow{\sim} \varphi^*U_n$$

$$\downarrow b^*\alpha_n \downarrow u_n$$

$$1 \xleftarrow{\varphi^*}$$

By definition, $u_0: \varphi^*1 = \varphi^*U_0 \to 1$ is the map coming from the monoidal structure of $\varphi^*$. Using the splitting property (Prop. 3.6) and the fact that $u_0: \varphi^*U_0 \to 1$ is an isomorphism, we can now prove that $U \simeq \varphi_*1$.

**Proposition 3.7.** Assume that $T$ is unipotent. Then, the natural transformation

$$\nu: \text{Hom}(-, U) \to \text{Hom}(\varphi^*(-), 1)$$

$$\psi \mapsto u \circ \varphi^*(\psi)$$

is an isomorphism. In particular, the maps $U \to \varphi_*1$ and $\varphi_*1 \to U$ induced by $u: \varphi^*U \to 1$ and $\varphi^*\varphi_*1 \to 1$, respectively, are mutually inverse isomorphisms.

**Proof.** We need to prove that $\nu_L$ is bijective for every $L \in T$. The method of proof is induction on the unipotency index $n$ of $L$. More precisely, in the induction step we assume that for an $L'$ with unipotency index strictly lower than $n$, the map $\text{Hom}(L', U) \to \text{Hom}(\varphi^*L', 1)$, sending $\psi'$ to $u \circ \varphi^*(\psi')$, is a bijection.

**Surjectivity:** Base case: $L$ is isomorphic to some $\sigma^*V$. We suppose we’re given a map $\bar{\psi}: V \simeq \varphi^*\sigma^*V \to 1$. Applying $\sigma^*$, we get a map $\psi: \sigma^*V \to \sigma^*1 = U_0 \to U$. Then $\nu(\psi) = u \circ \varphi^*(\psi) = \bar{\psi}$.

**Induction step:** Now suppose that $L$ has unipotency index $n > 0$ and that

$$0 \to L' \xrightarrow{a} L \xrightarrow{b} \sigma^*V \to 0$$

where $L'$ has a unipotency index of at most $n - 1$. We suppose we’re given a map $\bar{\psi}: \varphi^*L \to 1$, which we need to lift to a map $\psi: L \to U$ such that $\nu(\psi) = \bar{\psi}$.
Composing \( \varphi^*(a) \) with \( \tilde{\psi} \) yields \( \tilde{\psi}^*: \varphi^* L' \to 1 \). By the induction hypothesis, \( \tilde{\psi}^* \) can be lifted to \( \psi^*: L' \to U \), such that \( \nu(\psi^*) = \tilde{\psi}^* \). We push out the short exact sequence (6) via \( \psi^* \):

\[
\begin{array}{c}
0 \to L' \to L \xrightarrow{b} \sigma^* V \to 0 \\
\downarrow_{\psi^*} \downarrow \ \\
0 \to U \xrightarrow{\tau} E \to \sigma^* V \to 0
\end{array}
\]

By Prop. 3.6, the lower sequence must split: choose a splitting. Denote the composed map \( L \to U \) by \( \psi_0 \). This is not quite the lift we’re looking for (it depends on the choice of splitting). We have \( \psi_0 a = \psi^* \), so that

\[
u(\psi) = u \circ \varphi^*(\psi_0) \circ \varphi^*(a) = u \circ \varphi^*(\psi^*) = \tilde{\psi} = \psi^* = \varphi^*(a)
\]

This implies that \( u \circ \varphi^*(\psi_0) - \tilde{\psi} \) factors through \( \varphi^*(b) \), i.e., that there exists \( \tilde{\psi}'': V \to 1 \) such that

\[
u(\psi) = u \circ \varphi^*(\psi) = u \circ \varphi^*(\psi_0 - \psi'' \circ b) = u \circ \varphi^*(\psi_0) - \tilde{\psi}'' \circ \varphi^*(b) = \tilde{\psi}.
\]

**Injectivity:** Base case: \( \varphi^* \) is fully faithful on the essential image of \( \sigma^* \), and \( u_0 \) is an isomorphism.

**Induction step:** Let \( \psi: L \to U \) be such that \( \nu(\psi) := u \circ \varphi^*(\psi) = 0 \). Then, with \( L' \) as before, the composition \( L' \to L \to U \) is zero, by induction. Therefore \( \psi = \psi'' \circ b \). Since \( b \) is epic and \( \nu(\psi) = 0 \), we get \( \nu(\psi'') = 0 \), and by induction \( \psi'' = 0 \), implying \( \psi = 0 \).

We have the following useful property of \( \varphi_1 \), which is also enjoyed by \( U \), given the previous result.

**Lemma 3.8.** We have that \( \sigma_*(\varphi_1 \otimes -) \simeq \varphi^*(-) \).

**Proof.** Apply Lemma 2.18.

### 3.3.3 Characterisation of \( W \)

In this section, we prove that \( W \simeq f_1 \) under the assumptions of Thm. 3.1. We are able to do this just using the fact that \( W \) realises to \( U \) (Prop. 3.4), and the characterisation of \( U \) (Prop. 3.7).

**Lemma 3.9.** We have that \( s_*(f_1 \otimes -) \simeq f^*(-) \).

**Proof.** Apply Lemma 2.18.

By the universal property of \( f_1 \), the map \( \alpha: f^* W \to 1 \) induces a map \( W \to f_1 \). This then yields a natural transformation

\[
s_*(W \otimes -) \to s_*(f_1 \otimes -).
\]

**Lemma 3.10.** Assume that \( \mathcal{T} \) is unipotent and that the canonical maps \( b^* R^i s_* \to R^i \sigma_* B^* \), for \( i \leq 1 \), are isomorphisms. Then the natural transformation (7) is an isomorphism.

**Proof.** After applying \( b^* \) we may view the transformation as the composition

\[
b^* s_*(W \otimes -) \simeq \sigma_*(U \otimes B^*(-)) \simeq \sigma_*(\varphi_1 \otimes B^*(-)) \simeq \varphi^* B^* \simeq b^* f^* \simeq b^* s_*(f_1 \otimes -),
\]

using \( b^* s_* \simeq \sigma_* B^* \), Prop. 3.4, Prop. 3.7, Lemma 3.8, and Lemma 3.9. We conclude by the conservativity of \( b^* \).
Proposition 3.11. Assume that $T$ is unipotent and that the canonical maps $b^* R^i s_* \to R^i \sigma_* B^*$, for $i \leq 1$, are isomorphisms. We have that $\alpha: f^* W \to 1$ induces an isomorphism $W \simeq f_* 1$.

Proof. For every $A \in M$, we have natural isomorphisms
\[
\text{Hom}(A, W) \simeq \text{Hom}(1, W \otimes A^\vee) \\
\simeq \text{Hom}(1, s_*(W \otimes A^\vee)) \\
\simeq \text{Hom}(1, s_*(f_* 1 \otimes A^\vee)) \quad \text{by Lemma 3.10.} \\
\simeq \text{Hom}(1, f_* 1 \otimes A^\vee) \\
\simeq \text{Hom}(A, f_* 1),
\]
and we conclude by the Yoneda lemma. □

3.4 Proof of Theorem 3.1

With the properties above established for $U$ and $W$, we can now prove the main result of this section.

Proof of Theorem 3.1. We have that
\[
B^* f_* 1 \simeq B^* W \simeq U \simeq \varphi_1,
\]
by Propositions 3.11, 3.4 and 3.7. We claim that this isomorphism is the natural algebra morphism $B^* f_* 1 \to \varphi_1$ induced by the counit $b^* f^* f_* 1 \to 1$. This follows from the commutativity of the following diagram.

```
\[\begin{array}{ccccccccc}
\varphi^* B^* f_* 1 & \to & \varphi^* B^* W & \to & \varphi^* U & \to & \varphi^* \varphi_1 \\
\downarrow b^* (\text{coun}) & & \downarrow b^* (\alpha) & & \downarrow \text{coun} & & \downarrow 1 & & \downarrow 1 \\
1 & \to & 1 & \to & 1 & \to & 1 & \to & 1 \\
\end{array}\]
```

4 Malcev completions

Let $\varrho: P \to K$ be a homomorphism of pro-algebraic groups.

Definition 4.1. Let $\text{Un}(\varrho^*)$ be the smallest Tannakian subcategory (see Def. 2.12) of $\text{Rep} P$ that contains the image of $\varrho^*$ and is stable under extensions. The Tannaka dual of $\text{Un}(\varrho^*)$ is called the Malcev completion of $P$ with respect to $\varrho: P \to K$.

Lemma 4.2. The category $\text{Un}(\varrho^*)$ is the full subcategory of $\text{Res}_{P}^{\text{im}(\varrho)}$-unipotent objects in $\text{Rep} P$.

Proof. By Lemma 2.13, $\text{Un}(\varrho^*)$ contains the image of $\text{Res}_{P}^{\text{im}(\varrho)}$. Thus, it also contains all $\text{Res}_{P}^{\text{im}(\varrho)}$-unipotent objects, since it’s stable under extensions. To conclude, note that the class of $\text{Res}_{P}^{\text{im}(\varrho)}$-unipotent objects is stable under extension (Lemma 2.4), and apply Lemma 4.3. □

Lemma 4.3 (Extension-closure of a Tannakian subcategory). Let $\mathcal{T}$ be a neutral Tannakian category and $S \subseteq \mathcal{T}$ a Tannakian subcategory. The smallest subcategory of $\mathcal{T}$ which contains $S$ and is closed under extensions is also a Tannakian subcategory.

Replacing $K$ with $\text{im}(\varrho)$, we may thus assume that $\varrho$ is surjective, without changing the Malcev completion.
Remark 4.4 (The universal property of the Malcev completion). Let $G$ be the Malcev completion of $\varrho$. We get an exact sequence of groups

$$1 \to U \to G \to K,$$

and the kernel $U$ is pro-unipotent (Rmk. 2.14 and Lemma 4.2). The map $G \to K$ is given dually, by sending an $K$-representation $X$ to $\varrho^*X$, which is an object in the full subcategory $\text{Rep}G$ of $\text{Rep}P$. There is a canonical map $\hat{\varrho}: P \to G$ which by definition factors $\varrho$. It’s defined via $\tilde{\varrho}^*$, which is just the full inclusion of $\text{Rep}G$ into $\text{Rep}P$.

The Malcev completion $G$ is universal among groups factoring $\varrho$ in this way. Let $G$ be another pro-algebraic group equipped with a map $G \to K$, which has a pro-unipotent kernel $U$, and a map $\tilde{\varrho}: P \to G$ factoring $\varrho$.

There is then a unique map $\phi: G \to G$ such that $\phi \circ \hat{\varrho} = \tilde{\varrho}$. To see this, we need to check that $\tilde{\varrho}^*$ lands in the full subcategory $\text{Rep}G$ of $\text{Rep}P$. Since $\text{Rep}G$ is closed under extensions in $\text{Rep}P$, we only need to check that $\tilde{\varrho}^*$ lands in $\text{Rep}G$ when restricted to those representations coming from $K$, by Prop. 2.11.2 and Lemma 2.14. But on those representations, $\tilde{\varrho}^*$ coincides with $\varrho^*$, which we know lands in $\text{Rep}G$, and we are done.

Lemma 4.5. Given a homomorphism $\varrho: P \to K$, its Malcev completion $\hat{\varrho}: P \to G$ is surjective.

Proof. Follows from Prop. 2.11.2.

Definition 4.6. Let $P \to K$ be a homomorphism of pro-algebraic groups. We say that it’s Malcev complete if the homomorphism $G \to K$ from the Malcev completion to $K$ is an isomorphism.

Example 4.7. As a trivial example, note that $\text{id}: P \to P$, and more generally any automorphism of $P$, is always Malcev complete. More generally, the Malcev completion of an injective homomorphism is the image of the homomorphism.

Lemma 4.8. A homomorphism $P \to K$ is Malcev complete if and only if it’s surjective and the essential image of $\text{Rep}K$ in $\text{Rep}P$ is closed under extensions.

Lemma 4.9. Let $P \to G \to K$ be homomorphisms such that $P \to G$ is Malcev complete. Then, the Malcev completions $G$ and $G'$ of $P \to K$ and $G \to K$, respectively, coincide.

Proof. Note that $\text{Rep}G$ is a full Tannakian subcategory of $\text{Rep}P$, and that $\text{Rep}G$ and $\text{Rep}G'$ are full Tannakian subcategories of $P$ and $G$, respectively. We check that $\text{Rep}G$ and $\text{Rep}G'$ coincide as collections of objects in $\text{Rep}P$. The former is constructed by successive extensions in $\text{Rep}P$ of objects coming from $K$. The latter is constructed by successive extensions in $\text{Rep}G$ of objects coming from $K$. But since $\text{Rep}G$ is closed under extensions in $\text{Rep}P$ by the previous lemma, we are done.

Definition 4.10. Let $P$ be a pro-algebraic group. The unipotent completion of $P$ is the Malcev completion of $P \to 1$. We denote it by $P_{\text{un}}$. 19
The unipotent completion of $P$ is the initial pro-unipotent group equipped with a homomorphism from $P$, and $P \to 1$ is Malcev complete if and only if $P$ is pro-unipotent. Moreover, given a homomorphism $P \to K$, the unipotent completion of $\ker(P \to K)$ is exactly the kernel $U$ of the Malcev completion $\hat{G} \to K$.

**Remark 4.11.** Our context for defining Malcev completions differs a bit from what is often found in the literature. Usually, $P$ is taken to be a discrete group $\pi$, $K$ is taken to be (pro-)reductive, and $\varrho: \pi \to K(F)$ is assumed to have Zariski-dense image. We round out this section by discussing these three differences, as well as functoriality.

**Definition 4.12.** Given a discrete group $\pi$, the Tannaka dual of $\text{Rep} \pi$ is a pro-algebraic group. We denote it by $\hat{\pi}$ and call it the pro-algebraic completion of $\pi$. (Here, $\text{Rep} \pi$ is the category of finite-dimensional $F$-linear $\pi$-representations.)

The pro-algebraic completion $\hat{\pi}$ comes equipped with a homomorphism $\pi \to \hat{\pi}(F)$. Given a pro-algebraic group $K$, composing homomorphisms $\hat{\pi} \to K$ with $\pi \to \hat{\pi}(F)$ gives a bijection $\text{Hom}(\hat{\pi}, K) \simeq \text{Hom}(\pi, K(F))$, which is natural in $K$. More precisely:

**Lemma 4.13.** The functor $\pi \mapsto \hat{\pi}$ from discrete groups to pro-algebraic groups given by pro-algebraic completion is left adjoint to the functor $P \mapsto P(F)$.

Thus, given a map from a discrete group $\pi$ to a pro-algebraic group $K$, our definition of the Malcev completion the corresponding map $\hat{\pi} \to K$, coincides with definition of the Malcev completion of $\pi \to K$ found in the literature. This takes care of the first difference in Rmk. 4.11.

Next, let $\varrho: P \to K$ be a homomorphism as before. As we’ve seen, replacing $K$ by $\text{im} \varrho$, we still get the same Malcev completion. Therefore we can assume that $\varrho$ is surjective (or that the image is Zariski-dense, when $P$ is discrete).

Finally, replacing $K$ with a quotient by a pro-unipotent subgroup, still gives the same Malcev completion (the most important example of this being the maximal (pro-)reductive quotient of $K$):

**Lemma 4.14.** Let $\varrho: P \to K$ be a homomorphism and $q: K \to S$ a quotient of $K$ by a pro-unipotent subgroup. Then the Malcev completions of $P \to K$ and $P \to S$ coincide.

**Proof.** We check it on the side of the Tannakian categories. Trivially, $\text{Un}(\varrho^*)$ contains $\text{Un}(\varrho^* q^*)$ (both are full subcategories of $\text{Rep} P$). For the other direction, note that every object in $\text{Rep} K$ is obtained by successive extension from objects coming from $q^*$ (Lemma 2.14). Thus, everything in the image of $\varrho^*$ is obtained by successive extension from objects coming from $\varrho^* q^*$. This gives the reverse inclusion, using the extension-stability of $\text{Un}(\varrho^* q^*)$ in $\text{Rep} P$. 

**Remark 4.15.** By Lemma 4.5, if $\varrho: P \to K$ is Malcev complete, then $\varrho$ is surjective. Moreover, the group $K$ and the homomorphism $P \to K$ are determined by the maximal pro-reductive quotient $S$ of $K$ and the homomorphism $P \to S$. We recover $K$ as the Malcev completion of $P \to S$.

**Proposition 4.16 (Functoriality).** Consider a commutative diagram of pro-algebraic groups,

$$
\begin{array}{ccc}
K & \longrightarrow & K' \\
\downarrow & & \downarrow \\
\hat{G} & \longrightarrow & \hat{G}' \\
\downarrow & & \downarrow \\
P & \longrightarrow & P'
\end{array}
$$

where $G$ and $G'$ are the Malcev completions of $P \to K$ and $P' \to K'$, respectively. Then, there is a unique group homomorphism $G \to G'$ making everything commute. Moreover, this construction is functorial.
Proof. To give such a map is the same as to give a map \( G \to G' \times_{K'} K \), which when composed with the projection to \( K \) gives the canonical map \( G \to K \), and which when precomposed with \( P \to G \) gives the obvious arrow \( P \to G' \times_{K'} K \). But the kernel

\[
\ker (G' \times_{K'} K \to K) = \ker (G' \to K')
\]

is pro-unipotent, so the universal property of \( G \) gives us a unique map of the form we wanted. Functoriality follows easily from the uniqueness.

**Proposition 4.17.** Malcev completion gives a functor from the category of homomorphisms between pro-algebraic groups, to the category of homomorphisms of pro-algebraic groups with pro-unipotent kernels. Moreover, this functor is left adjoint to the inclusion functor.

Proof. We already proved the functoriality, what is left to show is the adjunction. This follows easily from the universal property of Malcev completion, and the functoriality above (including the uniqueness statement).

**Variant 4.18.** Let \( K \) be a fixed pro-algebraic group. Malcev completion provides us with a functor from pro-algebraic groups with a map to \( K \), to pro-algebraic groups with a map to \( K \) with pro-unipotent kernel. Combining Lemma 4.13 and Prop. 4.17, we get a functor from discrete groups with a map to \( K(F) \) to pro-algebraic groups with a map to \( K \) with pro-unipotent kernel. Moreover, this functor is a left adjoint and hence commutes with colimits.

## 5 Theorem M

We once again place ourselves in the Standard Situation, keeping all the notation from Section 2.6:

Let us introduce some more notation that will be helpful in discussing the Malcev completeness of \( \bar{\varrho} \). Write \( q_1 : K \to S := K/R_u(K) \) for the maximal pro-reductive quotient of \( K \). Its representations are the semi-simple representations of \( K \). Denote the composition of \( \varrho \) with \( q_1 \) by \( \bar{\varrho} : P \to S \). We’ve seen, in Lemma 4.14, that the Malcev completions of \( \varrho \) and \( \bar{\varrho} \) coincide.

Let mlc be the full subcategory \( \text{Un}(\bar{\varrho}^*) \) of \( \mathcal{T} \), as in Def. 4.1. Its objects are essentially those built by successive extension from objects in the image of \( \bar{\varrho}^* \). In the following diagram, I omit the (systematically named) sections \((s^*, t^*, \sigma^*) \) of \((f^*, g^*, \varphi^*) \), plus variants with tilde) in order to reduce clutter. Let \( \mathcal{M} \) be the (full) preimage of \( \text{Rep}S \) in \( \mathcal{M} \).

Let \( \mathcal{K} \) for the Malcev completion of \( \bar{\varrho} : P \to S \), i.e., the Tannaka dual \( \mathcal{G}(\text{mlc}) \) of mlc, our goal becomes to prove the following theorem.
Theorem U. Assume that $\rho$ is surjective and that the canonical maps $b^* R^i \sigma_* \to R^i\sigma_* B^*$, for $i = 0, 1$, are isomorphisms. Then $\phi : K \to K$ is an isomorphism (or equivalently, $\rho : P \to K$ is Malcev complete).

Remark 5.1. This is the main technical result of this paper, and it has two key assumptions: the surjectivity of $\rho$ we refer to as the surjectivity assumption, and the other assumption we call the cohomological assumption. Note also, that the conclusion of the theorem implies the (stronger) surjectivity of $\rho$.

Our general strategy is, as it was for Thm. U, to utilise Prop. 2.20. That means we construct models of the regular representations of the groups and show that the natural map between them is an isomorphism.

Since both $K$ and $\mathcal{K}$ are semi-direct products of unipotent groups with $S$, it’s not too surprising that the unipotent theorem (Thm. U) becomes very useful. In Section 5.1, we show how, given a relatively unipotent neutral Tannakian category, one can produce a new one, which is unipotent over a deeper base (Prop. 5.3). In the rest of Section 5, we use this result to apply the unipotent theorem, and finally prove Thm. M.

5.1 On tensor products of Tannakian categories

Consider a 2-commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\tilde{s}^*} & \mathcal{B} \\
\downarrow{p^*} & & \downarrow{f^*} \\
C & \xleftarrow{f^* \circ p^* = f} & B
\end{array}
\]

in $\text{nTan}$. We assume that $\tilde{s}^*$ is a monoidal 2-section of $\tilde{f}^*$. This gives a monoidal 2-section $s^* := p^* \tilde{s}^*$ of $f^*$. Before we move on to the main topic of this subsection, we state a basic lemma which becomes useful in the sequel.

As in Section 2.4, we have an algebra morphism $p_{\text{alg}} : p^* \tilde{f}_* 1 \to f_* 1$. It is uniquely characterised by a compatibility with Hopf algebra counits after applying $f^*$. Under adjunction it corresponds to a morphism $\tilde{f}_* 1 \to p_* f_* 1$, and this morphism is also a morphism of algebras. On the other hand, there is an isomorphism $f_* 1 \cong p_* f_* 1$ of objects in $\text{Ind} A$, induced by the 2-commutativity of the above diagram.

Lemma 5.2. The two morphisms $\tilde{f}_* 1 \to p_* f_* 1$, described above, coincide. □

Let $A := p^*(\tilde{f}_* 1)$: it’s an algebra object in $\text{Ind} B$. We consider the tensor category $\text{Mod}^{\text{fil}}(A)$ of finitely generated $A$-modules in $\text{Ind} B$, as in Section 2.5, sitting as a full subcategory in the abelian tensor category $\text{Mod}(A)$ of $A$-modules in $\text{Ind} B$.

As in Section 2.6, it’s a fact that $f^* A = \tilde{f}^* \tilde{f}_* 1$ is a Hopf algebra object in $\mathcal{C}$, which thus has a counit map $f^* A \to 1$. We may therefore define a monoidal, right-exact functor $w^* : \text{Mod}^{\text{fil}}(A) \to \mathcal{C}$ by $w^*(-) := 1 \otimes f^* A f^*(-)$. (A finitely generated $1$-module in $\text{Ind} \mathcal{C}$ is just an object of $\mathcal{C}$.) It has a monoidal $\text{fil}$ 2-section $e^*$ defined by $e^*(-) := A \otimes s^*(-)$.

Proposition 5.3. Assume $\mathcal{B}$ is $p^*$-unipotent, that $p^*$ is fully faithful, and that the essential image of $p^*$ is closed under taking subobjects. Then $\text{Mod}^{\text{fil}}(A)$ is (i) $e^*$-unipotent and (ii) neutral Tannakian. Moreover, $w^*$ is faithful and exact.

Remark 5.4. In fact, $\text{Mod}^{\text{fil}}(A)$ is the tensor product $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}$ of $\mathcal{B}$ and $\mathcal{C}$ over $\mathcal{A}$, once one has defined what that means. Group theoretically, it means that $\mathcal{G}(\text{Mod}^{\text{fil}}(A))$ is isomorphic to $\mathcal{G}(\mathcal{B}) \times_{\mathcal{G}(\mathcal{A})} \mathcal{G}(\mathcal{C})$, and in particular, that $\mathcal{G}(\mathcal{B}) \to \mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\text{Mod}^{\text{fil}}(A)) \to \mathcal{G}(\mathcal{C})$ have the same (pro-unipotent) kernel.

We’ll prove this proposition using the following result due to Deligne.

\[\text{In particular, the isomorphism } w^* e^* \cong \text{id} \text{ is monoidal.}\]
Lemma 5.5 ([Del90, Cor. 2.10]). Let \( \mathcal{T} \) be a rigid abelian tensor category satisfying \( \text{End} \mathbf{1} = F \), and let \( \mathcal{M} \) be a non-zero abelian monoidal category with a right exact tensor product. Then any right exact functor \( \mathcal{T} \to \mathcal{M} \) is left exact and faithful.

The above lemma gives, in particular, that both \( e^* : \mathcal{C} \to \text{Mod}^\text{fg}(A) \) and \( A \otimes (-) : \mathcal{B} \to \text{Mod}^\text{fg}(A) \) are exact, which is useful. We now spend some time establishing the assumptions of Deligne’s result for the functor \( w^* \). More precisely, we prove that \( \text{Mod}^\text{fg}(A) \) is rigid and abelian, and that \( \text{End} \mathbf{1} = F \). To prove Prop. 5.3, it is then essentially enough to show the \( e^* \)-unipotency.

Lemma 5.6. Assume that \( \mathcal{B} \) is \( p^* \)-unipotent, that \( p^* \) is fully faithful, and that the essential image of \( p^* \) is closed under taking subobjects. Then \( \text{Mod}^\text{fg}(A) \) is Noetherian, i.e., any ascending chain of subobjects of an object in \( \text{Mod}^\text{fg}(A) \) must stabilise.

*Proof.* Since a quotient of a Noetherian object is Noetherian, it’s enough to consider finitely generated free modules \( A \otimes Y \). We proceed by induction on the \( p^* \)-unipotency index of \( Y \).

**Base case:** We have that \( A \otimes p^* X \simeq p^*(\widetilde{f}_r A \otimes X) \), so we reduce to studying sub-\( \widetilde{f}_r A \)-modules of \( \widetilde{f}_r A \otimes X \), by the assumption on the essential image of \( p^* \). But \( \text{Mod}(\widetilde{f}_r A) \) is equivalent to \( \mathcal{C} \) (e.g. by the fundamental theorem of Hopf modules and Tannakian duality), and \( \mathcal{C} \) is Noetherian (since it’s Tannakian).

**Induction step:** We assume that \( Y \) is an extension of \( Y’ \) by \( p^* X \) such that \( A \otimes Y’ \) is a Noetherian \( A \)-module. By the exactness of \( A \otimes (-) \), we get \( A \otimes Y \) as an extension of \( A \otimes Y’ \) by \( A \otimes p^* X \). A submodule \( N \) of \( A \otimes Y \) gives rise to a submodule \( N’ \) of \( A \otimes Y’ \) by intersecting with \( A \otimes Y’ \) (pullback), and to a submodule \( N'' \) of \( A \otimes p^* X \) by taking the quotient by \( N’ \). If a larger submodule \( \bar{N} \supseteq N \) gives rise, in this way, to the same \( N’ \) and \( N'' \), then it must coincide with \( N \). Since \( A \otimes Y’ \) and \( A \otimes p^* X \) are Noetherian, we are done. \( \square \)

Lemma 5.7. Assume that all finitely generated free \( A \)-modules are Noetherian. Then, \( \text{Mod}^\text{fg}(A) \) is closed under taking subobjects in \( \text{Mod}(A) \). In particular, \( \text{Mod}^\text{fg}(A) \) is abelian.

*Proof.* Let \( M \) be an \( A \)-module which is finitely generated by \( Y \in \mathcal{B} \), and let \( N \) be a submodule of \( M \). Then \( N \) is a quotient of a submodule \( N’ \) of \( A \otimes Y \) (namely the pullback of \( N \) in \( A \otimes Y \)), and we can thus replace \( M \) by \( A \otimes Y \), without loss of generality.

We now show that for every \( Y \in \mathcal{B} \), every sub-\( A \)-module of \( A \otimes Y \) is finitely generated. Let \( N \) be such a submodule, generated by \( \bar{Y} \) in \( \text{Ind} \mathcal{B} \), i.e., we have a map \( A \otimes \bar{Y} \to N \). Let the kernel of this map be denoted by \( K \), and let \( (\bar{Y}_a)_{a \in I} \) be a filtered system of \( \bar{Y}_a \in \mathcal{B} \) such that \( \bar{Y} = \text{colim}_a \bar{Y}_a \). We may assume that the \( \bar{Y}_a \) are subobjects of \( \bar{Y} \) and that all transition maps are monomorphisms [Del89, Lemme 4.2.1]. Note that \( A \otimes \bar{Y} \) is then isomorphic to \( \text{colim}_a A \otimes \bar{Y}_a \), a filtered colimit of sub-\( A \)-modules, by the exactness of \( A \otimes (-) \).

Let \( K_\alpha \) be the intersection of \( K \) and \( A \otimes \bar{Y}_\alpha \) in \( A \otimes \bar{Y} \), and let \( N_\alpha \) be the cokernel of \( K_\alpha \to A \otimes \bar{Y}_\alpha \). We have short exact sequences

\[
0 \to K_\alpha \to A \otimes \bar{Y}_\alpha \to N_\alpha \to 0,
\]

and since colimits commute with colimits, we get that \( \text{colim}_\alpha N_\alpha \) is the cokernel of \( K \to A \otimes \bar{Y} \), i.e., that it’s \( N \). The Noetherianity of \( A \otimes Y \) implies that the partially ordered system \( N_\alpha \) of subobjects of \( N \) has a maximal element \( N_\beta \) (since they are also subobjects of \( A \otimes Y \)). The fact that it’s filtered implies that there is a unique such maximal subobject, and it must be equal to all of \( N \), since the colimit equals \( N \). All in all, we get

\[
N = N_\beta = (A \otimes \bar{Y}_\beta)/K_\beta,
\]

so that \( N \) is finitely generated by \( \bar{Y}_\beta \), and we’re done. \( \square \)

Lemma 5.8. We have that \( A \otimes p^* X \simeq e^* \tilde{f}^* X \).
Lemma 2.19. Then and in mind, it’s enough to show that (a) Mod \rightarrow \text{Mod}_C \rightarrow \text{Mod}_B.

Lemma 5.8, it is therefore exact and faithful. (b) Clearly, \text{Mod}_C \rightarrow \text{Mod}_B.

Lemma 5.9. Assume that B is \text{p}^*\text{-unipotent}, that \text{p}^* is fully faithful, and that the essential image of \text{p}^* is closed under taking subobjects. Let M be a quotient of e^*Z in \text{Mod}^{\text{fg}}(A). Then M is in the essential image of e^*.

Proof. First, replace Z by Z/Z_0, where Z_0 is the largest subobject of Z such that e^*Z_0 \rightarrow M is zero.

By Lemmas 5.6 and 5.7, the kernel K of e^*Z \rightarrow M is finitely generated, i.e., it’s the quotient of some A \otimes Y. If Y is non-zero, the \text{p}^*\text{-unipotency of B implies that Y has a non-zero subobject of the form p^*X.}

But A \otimes p^*X \simeq e^*f^*X =: e^*Z', by Lemma 5.8. We thus have a map e^*Z' \rightarrow e^*Z, and its composition with e^*Z \rightarrow M is zero. By the fullness of e^*, we have an underlying map Z' \rightarrow Z, and it must be zero, because of the simplification of Z we started with. Therefore, we may replace Y by Y/e^*Z, and in the end assume Y is zero, so that K is zero and e^*Z \simeq M.

Lemma 5.10. End_{\text{Mod}(A)}(A) \simeq F.

Proof. Since \text{p}^* is fully faithful, we have End_{\text{Mod}(A)}(A) \simeq End_{\text{Mod}_C}(\hat{f}, 1), and the latter is F:

\text{Hom}_{\text{Mod}_C}(\hat{f}, 1 \otimes 1, \hat{f}, 1) \simeq \text{Hom}_{\text{Mod}_C}(1, \hat{f}, 1) \simeq \text{Hom}(1, 1) \simeq F

under the adjunction \hat{f}, 1 \otimes (-): \text{Ind} A \equiv \text{Mod}(\hat{f}, 1):\text{Forgetful}.

We’re now ready to prove the main result of this section.

Proof of Proposition 5.3. (i) Take N in \text{Mod}^{\text{fg}}(A), a quotient of A \otimes Y for some Y in B. We want to establish that N is an extension of something e^*\text{-unipotent by e^*Z for some Z \in C. This is done by induction on the p^*\text{-unipotency index of Y.}}

Base case: Assume Y = p^*X. Then A \otimes Y = e^*(\hat{f}, X), by Lemma 5.8. Then Lemma 5.9, says that N must be of the form A \otimes s^*(Z) = e^*(Z), for some Z \in C. In particular, it’s e^*\text{-unipotent.}

Induction step: We write Y as an extension

0 \rightarrow Y' \rightarrow Y \rightarrow p^*X \rightarrow 0,

where Y' is of a lower p^*\text{-unipotency index. The induction hypothesis tells us that any quotient of A \otimes Y' in \text{Mod}^{\text{fg}}(A) is e^*\text{-unipotent. Thus,}

\begin{align*}
0 & \rightarrow A \otimes Y' \rightarrow A \otimes Y \rightarrow A \otimes p^*(X) \rightarrow 0 \\
0 & \rightarrow U \rightarrow N \rightarrow e^*(Z) \rightarrow 0
\end{align*}

where U is e^*\text{-unipotent, and we’re done.}

(ii) With Lemmas 5.7 and 5.10 in mind, it’s enough to show that (a) \text{Mod}^{\text{fg}}(A) is rigid, and (b) w^* is faithful and exact. The existence of a fibre functor then follows from (b) and the neutrality of C. (See Section 2.2.)

(a) Rigidity is stable under extension, and everything in the image of e^* is rigid, since C is rigid and e^* is monoidal.

(b) Clearly, w^* is right exact. By Lemma 5.5, it is therefore exact and faithful.
5.2 Proof of the main theorem

We now go back to the situation described in the introduction to Section 5.

5.2.1 Applying the unipotent theorem

We write $\tilde{\varphi}^* := \varphi^* \tilde{g}^*$: $\text{mlc} \to \text{Vec}$. We also write $\tilde{\sigma}^*$ for a section to $\tilde{\varphi}^*$ such that $\tilde{g}^* \tilde{\sigma}^* = \sigma^*$.

Apply Prop. 5.3 to $(\text{Rep} S, \text{mlc}, \text{Vec})$ as $(A, B, C)$, and write $T$ for $q_s^* \tilde{g}_s 1$ (playing the role of the algebra $A$). The necessary assumptions on $\text{Rep} S \to \text{mlc}$ follow from the surjectivity of $\tilde{g}$. The unipotency assumption needed follows from the pro-unipotency of the kernel of $K \to S$ and the surjectivity of $\tilde{g}$ (by Lemma 2.14). The proposition then gives us a unipotent Tannakian category

$$\text{Mod}^\text{fg}(T)$$

with a forgetful functor to $\text{Ind mlc}$.

It is not hard to see that $M$ is unipotent over $\hat{M}$. In fact, $\mathcal{G}(\hat{M})$ is the pushout of $G \leftarrow K \to S$, so that $G \to \mathcal{G}(\hat{M})$ is surjective with the same pro-unipotent kernel, $R_s(K)$, as $K \to S$. We can thus apply Prop. 5.3 to $(\hat{M}, M, E)$ as $(A, B, C)$, with $M := p^* \tilde{f}_* 1$ playing the role of $A$. We get a relatively unipotent Tannakian category

$$\text{Mod}^\text{fg}(M)$$

over $E$, which has a forgetful functor to $\text{Ind mlc}$.

The composition of $t^*$ and $\phi^*$ induces a functor $\text{Mod}^\text{fg}(M) \to \text{Mod}^\text{fg}(T)$, since $\phi^* t^* M = \phi^* t^* p^* \tilde{f}_* 1 \simeq q_s^* \tilde{f}_* 1 \simeq T$.

Here, we’ve used the fact that $\tilde{t}^* \tilde{f}_* \simeq \tilde{g}_* 1$, which is true by definition. To summarise the situation, we have a diagram

\[
\begin{array}{ccc}
\text{Mod}^\text{fg}(M) & \xrightarrow{\phi^* t^*} & \text{Mod}^\text{fg}(T) \\
\downarrow{\omega^*} & \downarrow{\omega^*} & \\
E & \xrightarrow{\cdot b^*} & \text{Vec} \\
\end{array}
\]

in $\text{nTan}$, where $\text{Mod}^\text{fg}(M)$ is relatively unipotent over $E$ and $\text{Mod}^\text{fg}(T)$ is unipotent. In order to apply the unipotent theorem, we need the following result.

**Lemma 5.11.** If $b^* \circ R^i s_* \simeq R^i \sigma_* \circ B^*$ for $i \leq 1$, then $b^* \circ R^i e_* \simeq R^i \epsilon_* \circ (\phi^* t^*)$ for $i \leq 1$.

**Proof.** Note that $R e_*$ is the right adjoint of $\epsilon^*$ (on the derived level, of course). By definition, $\epsilon^*$ is the composition $(T \otimes -) \circ \tilde{\sigma}^*$. The (ind-)right adjoints of $(T \otimes -)$ and $\tilde{\sigma}^*$ are $\text{Forget}: \text{Mod}^\text{fg}(T) \to \text{Ind mlc}$, and $\tilde{\sigma}_*$, respectively. The former is exact and takes injectives to injectives (since $(T \otimes -)$ is exact), and we get $R e_* = (R \tilde{\sigma}_*) \circ \text{Forget'}$, where $\text{Forget'}: \text{Mod}^\text{fg}(M) \to \text{Ind mlc}$.

Thus for $i \leq 1$,

$$b^* \circ (R^i e_*) = b^* \circ (R^i s_*) \circ \text{Forget'}$$

$$= (R^i \sigma_*) \circ B^* \circ \text{Forget'}$$

$$= (R^i \sigma_*) \circ (\tilde{g}^* \phi^* t^*) \circ \text{Forget'}$$

$$= (R^i \sigma_*) \circ \tilde{g}^* \circ \text{Forget} \circ (\phi^* t^*)$$,

and we’ve reduced to showing $(R^i \sigma_*) \circ \tilde{g}^* = R^i \tilde{\sigma}_*$. For $i = 0$, this follows from the surjectivity of $\tilde{g}$. For $i = 1$, it follows from the stability of mlc in $\mathcal{T}$ under extensions. 

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We may thus apply Thm. U, yielding the following theorem.

**Theorem 5.12.** Assume that $\bar{\varrho}$ is surjective and that $b^* R^i s_* \to R^i \sigma_* B^*$ is an isomorphism for $i = 0, 1$.
Then the natural algebra morphism $\phi^* i^* \omega, 1 \to \omega, 1$ in $\text{Mod}^\mathbb{G}(T)$ is an isomorphism.

### 5.2.2 The final argument

**Proof of Theorem M.** From the previous section, we get that $\text{Forget}(\phi^* i^* \omega, 1) \to \text{Forget}(\omega, 1)$ in $\text{Ind mlc}$ is an isomorphism of algebras (Thm. 5.12). Note that, writing $\text{Forget}'' : \text{Mod}^\mathbb{G}(\iota^* M) \to \text{Ind Rep} K$, we have

$$\text{Forget}(\phi^* i^* \omega, 1) = \phi^* \circ \text{Forget}'' \circ i^* \omega, 1 = \phi^* i^* \circ \text{Forget}' \circ \omega, 1$$

**Lemma 5.2** gives us algebra isomorphisms $f_* 1 \simeq \text{Forget}' \circ \omega, 1$ (note that $M \otimes (-)$ is a morphism in $n\text{Tan}$, by **Lemma 5.5**, and that $\text{Forget}'$ is its right ind-adjoint), and $\hat{\varphi}_* 1 \simeq \text{Forget} \circ \omega, 1$. Finally, $i^* f_* 1 = g_* 1$, by definition. Putting everything together we thus get that the natural algebra morphism

$$\phi^* g_* 1 \to \hat{\varphi}_* 1$$

is an isomorphism. By **Prop. 2.20**, we are done.
6 Enriched local systems and Hodge theory

In this section, we establish one of our main applications of Thm. M and draw some consequences. The main result is Thm. 6.14, which reproves [DE20, Thm. 4.3] and provides a generalisation of a theorem of Hain.

6.1 Enriched local systems

Let $k$ be a fixed subfield of $\mathbb{C}$. We usually write $\text{pt}$ for $\text{Spec} \ k$. For a $k$-variety $X$, let $X^{an}$ denote $X(\mathbb{C})$ equipped with the complex analytic topology, and let $\text{LS}(X)$ denote the category of locally constant sheaves of finite-dimensional $F$-vector spaces (local systems) on $X^{an}$. Given a $k$-point $x$ of $X$, denote by $\pi_1(X, x)$ the (topological) fundamental group of $X^{an}$. Recall that $\text{LS}(X)$ is equivalent to the category of (finite-dimensional) $F$-representations of $\pi_1(X, x)$. Lastly, $\text{SmVar}_k$ is the category of smooth separated schemes of finite type over $k$.

Definition 6.1. Following [Ara10], we say that a weak theory of enriched local systems is a contravariant 2-functor

$$\mathcal{E} : \text{SmVar}_k \to \text{nTan}$$

$X \mapsto \mathcal{E}(X)$

$(f : Y \to X) \mapsto (f^*_E : \mathcal{E}(X) \to \mathcal{E}(Y))$

together with a monoidal natural transformation $b^* = b^*_E : \mathcal{E}(-) \to \text{LS}(-)$. Let $\mathcal{E} : = \mathcal{E}(\text{pt})$.

For any $X$ in $\text{SmVar}_k$ with a point $x$, we then get an instance of the Standard Situation (Section 2.6):

$$\mathcal{E} \begin{array}{c} b^*_E \downarrow \mathcal{E}(X) \rightarrow \text{Rep} K \rightarrow \text{LS}(X) \\ \downarrow f^* \quad \quad \quad \downarrow s^* \quad \quad \quad \downarrow \sigma^* \\ \mathcal{E} \rightarrow \text{Vec} \rightarrow \text{Vec} \quad \end{array}$$

Here, $s^* = p^*_E$ and $\sigma^* = p^*_L$ are induced by the structure morphism $p : X \to \text{pt}$, while $f^* = x^*_E$ and $\varphi^* = x^*_L$ are induced by the point $x : \text{pt} \to X$. We've denoted by $K = K^E(X)$ the kernel of $s : \mathcal{G}(\mathcal{E}(X)) \to \mathcal{G}(\mathcal{E})$.

We briefly translate the current setting to group theory.

Definition 6.2. We denote by $\text{LS}_E(X)$ the smallest abelian subcategory of $\text{LS}(X)$ containing the image of $b^*$. It is Tannakian, and we define the $E$-fundamental group of $(X, x)$ to be the Tannaka dual group

$$\pi_1^E(X, x) : = \mathcal{G}(\text{LS}_E(X), x^*)$$

of $\text{LS}_E(X)$, with fibre functor $x^* = \varphi^*$.
By construction, \( \varphi: \pi_1(X, x) \to K^\ell(X) \) factors through \( \pi_1^\ell(X, x) \), and in fact the latter is the Zariski closure of the image \( \text{im}(\varphi) \) of the former (by Lemma 2.13). In the end, we get an exact sequence

\[
\begin{array}{c}
S^\ell(X) \\
\uparrow \\
1 \longrightarrow K^\ell(X) \longrightarrow \mathcal{G}(\mathcal{E}(X), x^+) \longrightarrow \mathcal{G}(\mathcal{E}) \longrightarrow 1
\end{array}
\]

and in order to apply Thm. M, we need surjectivity of the vertical composition (here, as in the general case, \( S \) is used to denote the maximal pro-reductive quotient of \( K \)).

### 6.1.1 Checking the cohomological assumption

Our ultimate goal is to apply Thm. M, and we thus need a way to establish the cohomological assumption, namely, that the canonical maps \( b^* R^i s_* \to R^i \sigma_* B^* \), for \( i = 0, 1 \), are isomorphisms. The idea is that a theory of enriched local systems usually lives inside some ambient theory of (perverse or constructible) sheaves, where an analogue of the cohomological assumption holds. A motivating example to have in mind is the way admissible variations of mixed Hodge structures live in the larger category of mixed Hodge modules.

**Notation 6.3.** Let \( \mathcal{D}(X) \) be the derived category \( \mathcal{D}^b(X^{an}, F) \) of bounded complexes of sheaves of \( F \)-vector spaces on \( X^{an} \) with constructible cohomology. Denote by \( \mathcal{C}(X) \) the category of either constructible sheaves or perverse sheaves.

**Remark 6.4.** Given a morphism \( f: Y \to X \), we write \( Rf_* \) and \( f^* \) for the pushforward and pullback on \( \mathcal{D}(\_ \_ \_) \). Note that \( f^* \) takes local systems to local systems, and so restricts to a functor \( f^*_{LS}: \mathcal{LS}(X) \to \mathcal{LS}(Y) \). Moreover, when \( f \) is such that the restriction of \( R^i f_* \) to local systems lands in (ind-)local systems (e.g. when \( X = \text{pt} \)), then this restriction gives the right adjoint \( f^*_{LS} \) of \( f^*_{LS} \). However, for \( i > 0 \), the restriction of \( R^i f_* \) will generally not agree with \( R^i f^*_{LS} \), even when it takes values in (ind-)local systems.

**Definition 6.5.** Let \( \mathcal{E} \) be a weak theory of enriched local systems, and \( X \in \text{SmVar}_k \) connected. An ambient theory \( \mathcal{A}(X) \) for \( \mathcal{E} \) over \( X \) is an abelian category \( \mathcal{A}(X) \), together with

- a fully faithful and exact pullback functor \( p^*_A: \mathcal{A}(\text{pt}) \to \mathcal{A}(X) \) with right adjoints \( p^*_A \);
- a fully faithful and exact functor \( \text{inc}_A = \text{inc}: \mathcal{E}(X) \to \mathcal{A}(X) \);
- a faithful and exact realisation functor \( b^*_A = b^*: \mathcal{A}(X) \to \mathcal{C}(X) \);
- isomorphisms \( b^*_A \circ \text{inc}_A \simeq \text{inc}_C \circ b^*_C \);

such that

- (i) \( \text{inc}: \mathcal{E} := \mathcal{E}(\text{pt}) \to \mathcal{A}(\text{pt}) \) is an equivalence compatible with \( b^*_A \circ \text{inc}_A \simeq \text{inc}_C \circ b^*_C \);
- (ii) an object \( M \) in \( \mathcal{A}(X) \) lies in the full subcategory \( \mathcal{E}(X) \) if and only if \( b^* M \) lies in \( \mathcal{LS}(X) \);
- (iii) if \( p: X \to \text{pt} \) is in \( \text{SmVar}_k \), then the canonical maps \( b^* R^i p^*_A \to R^i p_* b^*_A \), for \( i = 0, 1 \), are isomorphisms.

**Example 6.6.** The trivial example is given by \( \mathcal{E}(X) := \mathcal{LS}(X) \) and \( \mathcal{A}(X) := \mathcal{C}(X) \) (see Notation 6.3). When \( \mathcal{C}(X) = \mathcal{P}(X) \) denotes the category of perverse sheaves on \( X \), the pullback functors \( p^*_p \) are given by the usual pullbacks \( p^* \), shifted: \( p^*_p := p^*[\dim X] \) (here, \( p: X \to \text{pt} \) denotes the structure map of \( X \)). These are fully faithful and \( t \)-exact by [BBD, Prop. 4.2.5]. The inclusion functors \( \text{inc}_\mathcal{P}: \mathcal{LS}(X) \to \mathcal{P}(X) \) are given by shifting by \( \dim X \) (in \( \mathcal{D}(X) \)).

For the rest of the section, we fix a weak theory \( \mathcal{E} \) of enriched local systems, \( X \in \text{SmVar}_k \) connected, and an ambient theory \( \mathcal{A} \) of \( \mathcal{E} \) over \( X \).
Lemma 6.7. The full subcategory $\mathcal{E}(X)$ is closed under extensions in $\mathcal{A}(X)$.

Proof. Let $E$ be an extension in $\mathcal{A}$ of $M$ and $N$, both lying in the full subcategory $\mathcal{E}$. Then $b^*E$ is an extension in $\mathcal{C}(X)$ of local systems $b^*M$ and $b^*N$, so $b^*E$ is a local system. Conclude by (ii) in the definition above.

Proposition 6.8. For a connected $X$ in $\text{SmVar}_k$, we have that the natural transformation

$$\text{inc} \circ R^i p^E_* \rightarrow R^i p^A_* \circ \text{inc},$$

is an isomorphism for $i = 0, 1$.

We get the following theorem as a corollary.

Theorem 6.9. Let $X$ in $\text{SmVar}_k$ be connected, with a $k$-point $x$. Assume that $\pi_1(X, x) \rightarrow S^\mathcal{E}(X)$ has Zariski dense image. Then Thm. M applies, i.e., $\pi_1(X, x) \rightarrow \tilde{\pi}_1^X(X, x)$ is Malcev complete.

Proof of Proposition 6.8. Let us suppress the functor inc from the notation for the duration of the proof, to reduce clutter; it can easily be reinserted. We begin with the $i = 0$ case. Note that, naturally in $E \in \mathcal{E}$ and $M \in \mathcal{E}(X)$, we have

$$\text{Hom}(E, p^E_* M) = \text{Hom}(p^E_* E, M) = \text{Hom}(p^A_* E, M) = \text{Hom}(E, p^A_* M)$$

which implies that $p^E_* \simeq p^A_*$.

Next, we deal with the more interesting $i = 1$ case. Applying $\text{Hom}(-, -)$ (in the second argument) to the triangles $R^0 p^E_* M \rightarrow R p^E_* M \rightarrow \tau^{\geq 1} R p^E_* M$, where $* \in \{\mathcal{E}, \mathcal{A}\}$, yields long exact sequences

$$\cdots \rightarrow \text{Hom}(-, R^0 p^E_* M[1]) \rightarrow \text{Hom}(-, R^1 p^E_* M[1]) \rightarrow \text{Hom}(-, \tau^{\geq 1} R p^E_* M[1]) \rightarrow \text{Hom}(-, R^0 p^E_* M[2]) \rightarrow \cdots$$

Denote the regular representation $b_*1$ in $\text{Ind} \mathcal{E}$ by $\mathcal{O}$. Evaluating these long exact sequences at the pro-object $\mathcal{O}^\vee$, the first and last term displayed both vanish, since $\mathcal{O}^\vee$ doesn’t have any non-trivial extensions (cf. Prop. 3.5). We thus have a commutative diagram

$$\begin{array}{ccc}
\text{Hom}(p^E_* \mathcal{O}^\vee, M[1]) & \text{Hom}(\mathcal{O}^\vee, R p^E_* M[1]) & \text{Hom}(\mathcal{O}^\vee, R^1 p^E_* M) \\
\downarrow & \downarrow & \downarrow \\
\text{Hom}(p^A_* \mathcal{O}^\vee, M[1]) & \text{Hom}(\mathcal{O}^\vee, R p^A_* M[1]) & \text{Hom}(\mathcal{O}^\vee, R^1 p^A_* M)
\end{array}$$

Here, the leftmost vertical arrow is an isomorphism, by Lemma 6.7. Thus, applying $\text{Hom}(\mathcal{O}^\vee, -)$ to the morphism $R^1 p^E_* M \rightarrow R^1 p^A_* M$, yields an isomorphism. We conclude by the conservativity of $\text{Hom}(\mathcal{O}^\vee, -) = \text{Hom}(-, \mathcal{O}) \circ (-)^\vee \simeq b^*$. \qed

Theorem 6.9 says in particular that (under the surjectivity assumption) the Malcev completion of $\pi_1(X, x)$ relative $\tilde{\pi}_1^X(X, x)$ is equipped with an $\mathcal{E}$-enrichment, i.e., an action by $\mathcal{G}(\mathcal{E})$. We’ll now show that a Malcev completion relative a single object of $\mathcal{E}(X)$ is $\mathcal{E}$-enriched in the same way. Fix an enriched local system $M$ in $\mathcal{E}(X)$ on a smooth connected variety $X$. Denote by $M_x$ the vector space $\varphi^*b^\chi M$. We may consider the associated monodromy representation $\pi_1(X, x) \rightarrow \text{Aut}(M_x)$. Take the Malcev completion of this morphism, and call it $\hat{\pi}_1^M(X, x)$.

Corollary 6.10. Under the assumptions of Thm. 6.9, the group $\hat{\pi}_1^M(X, x)$ is canonically enriched in $\mathcal{E}$, in the sense that $\mathcal{O}(\hat{\pi}_1^M(X, x))$ is naturally a Hopf algebra object in $\text{Ind} \mathcal{E}$.
Lemma 4.2

Prop. 4.16

Prop. 2.11

we need to prove (i) that $PS08$.

Thm. 6.9.

Lemma 4.9

the Malcev completion of $\mathcal{M}$. Note

that mixed Hodge modules provides an ambient theory in the sense of

We want to deduce the results from the previous section in this context. By

We have a factorisation

of the monodromy representation. By Lemma 4.2, the Malcev completion of $\pi(X, x) \to Aut(M_+)$ only depends on the image in $Aut(M_+)$, which by this factorisation lies in the subgroup $\mathcal{G}(\langle i^* M \rangle^\otimes)$. Thus, $\hat{\pi}_M^i(X, x)$ is the Malcev completion of $\pi_1(X, x) \to \mathcal{G}(\langle i^* M \rangle^\otimes)$.

Next, $\hat{\pi}_M^i(X, x)$ can be obtained as the Malcev completion of $\pi_1^\mathcal{E}(X, x) \to \mathcal{G}(i^* M)$, by Lemma 4.9. Note that $\mathcal{G}(E)$ acts on this morphism, and by functoriality of Malcev completions (Prop. 4.16), it acts on the Malcev completion, $\hat{\pi}_M^i(X, x)$. This finishes the proof. 

6.2 Variations of Hodge structures

We now move on to the weak theory of enriched local systems $\mathcal{E}(-) = \text{MHS}(-)$ given by admissible variations of mixed Hodge structures. (See, for example, [PS08, Ch. 10, 14].) In this context, we denote the corresponding fundamental group (previously $\pi_1^\mathcal{E}(X, x)$) by $\pi_1^\text{Hdg}(X, x)$ and call it the Hodge fundamental group. Over the point, we get the category MHS of graded-polarisable mixed Hodge structures.

We call the Tannaka dual group

We have a factorisation

$$\pi_1(X, x) \to \mathcal{G}(\langle i^* M \rangle^\otimes) \hookrightarrow Aut(M_+),$$

of the monodromy representation. By Lemma 4.2, the Malcev completion of $\pi(X, x) \to Aut(M_+)$ only depends on the image in $Aut(M_+)$, which by this factorisation lies in the subgroup $\mathcal{G}(\langle i^* M \rangle^\otimes)$. Thus, $\hat{\pi}_M^i(X, x)$ is the Malcev completion of $\pi_1(X, x) \to \mathcal{G}(\langle i^* M \rangle^\otimes)$.

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6.2 Variations of Hodge structures

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We call the Tannaka dual group $H$ of MHS the Hodge group, and its maximal pro-reductive quotient $H_{\text{red}}$ the pure Hodge group. The semi-simple objects in MHS($X$) are the polarisable variations of (non-mixed) Hodge structures, and they form the full subcategory HS($X$) of what we’ll call pure variations.

In particular, the pure Hodge group is the Tannaka dual of the category HS of polarisable pure Hodge structures.

Proposition 6.11 ([Sai90, Thm. 0.1, 0.2, 3.27]). We have that MHS($-)$ is a weak theory of enriched local systems, and that mixed Hodge modules provides an ambient theory in the sense of Def. 6.5. 

We want to deduce the results from the previous section in this context. By Thm. 6.9, the only thing we have to do is to establish the surjectivity assumption.

Proposition 6.12. The map $\tilde{\varrho}: \pi_1^\text{Hdg}(X, x) \to S^\mathcal{E}(X)$ is surjective.

Proof. We show it on the Tannakian side, i.e., we work with $\tilde{\varrho}^*$. By Prop. 2.11, we need to prove (i) that $\tilde{\varrho}^*$ is full, and (ii) that $\text{im} \tilde{\varrho}^*$ is stable under taking subobjects in LS($X$).

(i) Take two objects $M$ and $N$ in MHS($X$). Then, using the notation $\iota^*: \text{MHS}(X) \to \text{Rep } K$, as in previous sections,

$${\text{Hom}}(\iota^* M, \iota^* N) = b^* s_\ast \text{Hom}(M, N) = \sigma_\ast \text{Hom}(b^\ast M, b^\ast N) = \text{Hom}(\varrho^\ast \iota^* M, \varrho^\ast \iota^* N),$$

using the $i = 0$ part of Prop. 6.8, so we have fullness (for $\varrho^*$ and $\tilde{\varrho}^*$) on objects coming from MHS($X$).

Next, every object in Rep $S$ is a subquotient of a (semi-simple) object coming from MHS($X$), and by the semi-simplicity of Rep $S$, it’s actually a direct summand. Let two $S$-representations $A$ and $B$ be direct summands of $\iota^* M$ and $\iota^* N$, and cut out by projectors $e$ and $f$, respectively. Then,

$${\text{Hom}}(A, B) = f \circ \text{Hom}(\iota^* M, \iota^* N) \circ e = \varrho^*(f) \circ \text{Hom}(\varrho^* \iota^* M, \varrho^* \iota^* N) \circ \varrho^*(e) = \text{Hom}(\varrho^* A, \varrho^* B).$$

(ii) First, we note that by fullness of $\tilde{\varrho}^*$, the image of $\tilde{\varrho}^*$ is stable under taking direct summands (indeed, fullness lets us lift projectors).
Since $K$ is a normal subgroup of $\mathcal{G}(\text{MHS}(X))$, and the (pro-)unipotent radical of $K$ is a characteristic subgroup, we get a commutative diagram

$$
\begin{array}{ccc}
K & \rightarrow & \mathcal{G}(\text{MHS}(X)) \\
\downarrow & & \downarrow \\
S & \rightarrow & \mathcal{G}(\text{HS}(X))
\end{array}
$$

which tells us that pure (polarisable) variations are sent to $\text{Rep } S$ under $\iota^\ast$.

Next, note that $\text{Rep } S$ is generated under taking subquotients by pure variations. Indeed, it’s generated by objects coming from $\text{MHS}(X)$, and by semi-simplicity of $\text{Rep } S$, it’s generated by objects coming from the semi-simple part of $\text{MHS}(X)$, i.e., from pure variations.

Finally, if $A \in \text{Rep } S$ is a subquotient of a pure variation $\iota^\ast M$, and $L$ a subquotient of $\varrho^\ast A$, then $L$ is a subquotient of $\varrho^\ast \iota^\ast M$. But Deligne’s semi-simplicity theorem ([Del71, Thm. 4.2.6]) tells us that $\varrho^\ast \iota^\ast M$ stay semi-simple in $\text{LS}(X)$. Thus, $L$ is a direct summand of $\varrho^\ast \iota^\ast M = b_X M$ and hence in the image of $\overline{\varrho^\ast}$, as we saw above.

**Remark 6.13.** The above proof only relied on Deligne’s semi-simplicity theorem, so we could have axiomatised this too and had a general result for weak theories of enriched local systems “satisfying semi-simplicity”.

Having established both the surjectivity assumption, the results of the previous section are summarised in the following theorem.

**Theorem 6.14** (Generalised Hain’s Theorem).

1. The morphism $\pi_1(X,x) \rightarrow \pi_1^{\text{Hdg}}(X,x)$ is Malcev complete.
2. For $V$ an admissible variation of mixed Hodge structures, the Malcev completion $\hat{\pi}_1(V,X,x) \rightarrow \text{Aut}(V_x)$ is canonically equipped with a mixed Hodge structure.

**Remark 6.15.** Part 1 of Thm. 6.14 has been previously shown by D’Addezio and Esnault in [DE20, Thm. 4.3], by a different method (and stated in a different language). Part 2 is a generalisation of [Hai98, Thm. 13.1], where $V$ was assumed to be pure and polarisable, and $\pi_1(X,x)$ was assumed to map Zariski-densely onto $\text{Aut}(V_x,\langle , \rangle)$. In particular, we prove Conjecture 5.5 from [Ara10].

## 7 Motivic local systems

We would like to have a (sufficiently developed) theory of enriched local systems given by *motivic local systems*, and an ambient theory given by (for example) perverse motives. Then the theorems of the previous section would all apply. As we don’t quite have access to such a theory, we make due with working over a generic point. As before, $k$ is a subfield of $\mathbb{C}$. The role played by the Hodge group in the previous section is played by the motivic Galois group in the present section. The motivic Galois group $\mathcal{G}^{\text{mot}}(k)$ is the Tannaka dual of the category of Nori motives $\text{NM}(k)$ (with rational coefficients).

### 7.1 Local systems on the generic point

Let $X$ be a smooth, connected $k$-variety and let $\eta$ be its generic point. Suppose we have an embedding $\kappa(\eta) \subseteq \mathbb{C}$. This induces an embedding $k \subseteq \mathbb{C}$, and this is the way in which we consider $k$ as a subfield of $\mathbb{C}$ in this section, and analytify our varieties.\(^7\) We may view $\eta$ as a pro-scheme, namely the limit of all

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\(^7\)We do it this way for simplicity. If we instead simply fix an embedding of $k$ in $\mathbb{C}$, without worrying about $\kappa(\eta)$, the resulting category $\text{LS}(\eta)$ is still neutral Tannakian. A neutral fibre functor can be produced by iterating nearby cycles functors.
the dense opens in $X$, 
\[ \eta = \lim_{X \supseteq U \ni \eta} U. \]

The category of local systems on $\eta$ is defined as 
\[ \text{LS}(\eta) := \lim_{X \supseteq U \ni \eta} \text{LS}(U). \]

Concretely, a local system on $\eta$ is a local system on $U$ ($C$), for some dense open $U$ in $X$, and local systems $L$ and $L'$ on $U$ and $U'$ become isomorphic if their restrictions to $U \cap U'$ are isomorphic. We have an obvious functor 
\[ \text{LS}(X) \to \text{LS}(\eta), \]
and it’s not hard to see that this is fully faithful. The category $\text{LS}(\eta)$ is Tannakian, and is neutralised by the embedding $\kappa(\eta) \subseteq C$. We denote this fibre functor by $\varphi^\eta_* : \text{LS}(\eta) \to \text{Vec}$, and the Tannaka dual group by $\pi_1(\eta)$. Lastly, $\sigma^\eta_*$ is the section of $\varphi^\eta_*$ given by the constant sheaf functor.

**Remark 7.1.** If we denote the category of perverse sheaves on $U$ by $P(U)$, then we have an equivalence 
\[ \text{LS}(\eta) := \lim_{U \ni \eta} \text{LS}(U) \simeq \lim_{U \ni \eta} P(U). \]

### 7.2 Motivic local systems on the generic point

Ivorra and Morel have constructed and studied categories of perverse (Nori) motives in [IM19], and we will briefly review their construction. Given an additive category $Q$ and an additive functor $T : Q \to A$ to an abelian category $A$, they construct a universal abelian category $A^{ad}(Q,T)$ factoring this functor along an exact and faithful functor to $A$. Below, this universal category $A^{ad}(Q,T)$ will be called the Nori category of $T : Q \to A$.

Given a $k$-variety (separated and of finite type) $X$, they consider the additive functor 
\[ \mathcal{P}H^0 \circ \mathcal{B}ti^* : \text{DA}_{ct}(X) \to \mathcal{P}(X) \]
from the (triangulated) category of constructible étale motivic sheaves (with rational coefficients) on $X$ to perverse sheaves on $X$. We denote the Nori category of this functor by $\text{PM}(X)$: it is the category of perverse motives considered in [IM19].\(^8\) Here, $\mathcal{B}ti^*$ is the Betti realisation functor constructed by Ayoub in [Ayo10]. An important fact is that perverse motives $\text{PM}(k)$ over the field $k$ is equivalent to classical Nori motives $\text{NM}(k)$ ([IM19, Prop. 2.11]).

Perverse motives come equipped with a realisation functor $\text{PM}(X) \to \mathcal{P}(X)$ to perverse sheaves which we denote by $b_X^\ast$. Ivorra and Morel develop a full four-functor formalism for perverse motives, and in particular, we have pullback morphisms along inclusions of open subsets. This lets us make the next definition, in analogy with Rmk. 7.1.

**Definition 7.2.** Given $X$ smooth and connected with generic point $\eta$, we define the category of motivic local systems on $\eta$ as 
\[ \text{MLS}(\eta) := \lim_{U \ni \eta} \text{PM}(U). \]

The $b_U^\ast$ assemble into a faithful and exact Betti realisation functor $b_\eta^* : \text{MLS}(\eta) \to \text{LS}(\eta)$. There are also pushforward functors, in particular along the structure maps $U \to \text{pt}$, and we denote these by $s_U^\ast : \text{PM}(U) \to \text{NM}(k)$. They have left adjoints $s_U^! : \text{NM}(k) \to \text{PM}(U)$ (given by pullback along the same maps), and the triangle

\[ \begin{array}{ccc} 
\text{PM}(V) & \xrightarrow{s_V^!} & \text{PM}(U) \\
\text{NM}(k) & \xleftarrow{s_U^*} & \text{PM}(U) 
\end{array} \]

\(^8\)They denote it by $\mathcal{M}(X)$. 

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commutes, for an inclusion $U \subseteq V$. The composition of $s^*_X$ and PM($X$) → MLS($\eta$) defines a faithful and exact functor $s^*_k$: NM($k$) → MLS($\eta$).

By Prop. 8.1 from the appendix, MLS($\eta$) is the Nori category of

$$b^*_\eta: \text{colim}_U DA_{ct}(U) \to \text{colim}_U \mathcal{P}(U) \simeq \text{LS}(\eta).$$

Since LS($\eta$) → Vec is faithful and exact, replacing the former by the latter doesn’t change the Nori category MLS($\eta$), by the proof of [Ivo17, Lemma 5.1]. Moreover, by [Ayo14c, Cor. 3.22], $\text{colim}_U DA_{ct}(U) \simeq DA_{ct}(\eta)$. In the end, we get that MLS($\eta$) is equivalent to the category of Nori motives over the field $\kappa(\eta)$.

In particular, motivic local systems over $\eta$ form a Tannakian category, neutralised by the Betti realisation and $\eta$. Under these equivalences, the composition $s^*_k$: NM($k$) → MLS($\eta$) ≃ NM($\kappa(\eta)$) coincides the usual functor arising from the field extension $k \subseteq \kappa(\eta)$.

In summary, there is a commutative diagram

$$\begin{array}{ccc}
\text{MLS}(\eta) & \xrightarrow{b^*_\eta} & \text{LS}(\eta) \\
\uparrow^{\nu^*_\eta} & & \uparrow^{\gamma^*_\eta} \\
\text{NM}(k) & \xrightarrow{b^*_s} & \text{Vec}
\end{array}$$

in nTan. This corresponds, on the group side-theoretic side, to

$$1 \rightarrow K \rightarrow G^{\text{mot}}(\kappa(\eta)) \rightarrow G^{\text{mot}}(k) \rightarrow \pi_1(\eta)$$

By [CG17, Thm. 9.1], we may use results from [Ayo14b] in this setting. Firstly, Cor. 2.56 from [Ayo14b] gives us a section $G^{\text{mot}}(k) \rightarrow G^{\text{mot}}(\kappa(\eta))$, which puts us in the Standard Situation. In order to apply Thm. M, we thus need to verify the surjectivity assumption and the cohomological assumption. The surjectivity assumption is taken care of by [Ayo14b, Thm. 2.57], and we call the kernel group $K$ the motivic fundamental group of $\eta$, and denote it by $\pi_1^{\text{mot}}(\eta)$. The cohomological assumption follows from [IM19, Prop. 5.2(2)], which says that $b^*_s R\sigma^U_2 \simeq \tilde{R}\sigma^U_2 b^*_\eta$. Indeed, it holds that $s^U_2 \simeq \text{colim}_U s^U_2$ and $\sigma^U_2 \simeq \text{colim}_U \sigma^U_2$. We thus get the following theorem.

**Theorem 7.3 (Motivic Hain’s Theorem).**

1. The homomorphism $\pi_1(\eta) \rightarrow \pi_1^{\text{mot}}(\eta)$ is Malcev complete.

2. Given a motivic local system $M$ over $\eta$, the Malcev completion $\hat{\pi}_1^{\text{mot}}(\eta)$ of the associated monodromy representation $\pi_1(\eta) \rightarrow \text{Aut}(M_\eta)$, is motivic, in the sense that it carries a canonical action by the motivic Galois group.

The second part of the theorem is proved using the same techniques as its analogue in Section 6.1. Theorem 7.3 is a motivic refinement, albeit only over the generic point, of Hain’s theorem (Thm. 6.14).

### 8 Appendix: colimits of Nori categories

Let $I$ be a filtered category. Consider functors $i \mapsto Q_i$ and $i \mapsto \mathcal{P}_i$ from $I$ to the 2-categories of additive categories and of finite and hom-finite abelian categories, respectively. Denote their 2-colimits by $Q$ and $\mathcal{P}$, respectively, and assume that $\mathcal{P}$ is finite and hom-finite. Let $Q_i \rightarrow \mathcal{P}_i$ be a natural transformation between these functors, and let $\mathcal{N}_i$ be the associated Nori categories. This natural transformation assembles into a functor $Q \rightarrow \mathcal{P}$ and we denote the associated Nori category by $\mathcal{N}$. The $\mathcal{N}_i$ assemble into a filtered system.

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[A priori, we would define the motivic fundamental group as the image of $\pi_1(\eta)$ in $K$, but since $\pi_1(\eta) \rightarrow K$ is a posteriori surjective, it doesn’t make a difference in the end.]
Proposition 8.1. There is a canonical equivalence $\text{2-colim}_i N_i \simeq N$.

Proof. We have that $\mathcal{P}_i \to \mathcal{P}$ is exact, so by functoriality ([Ivo17, Prop. 6.6]), we get exact functors $N_i \to N'$, that assemble into an exact functor $\text{2-colim}_i N_i \to N$. This functor is moreover faithful, since the diagram

\[ \text{2-colim}_i N_i \longrightarrow N \quad \text{commutes, and it's easy to see that the diagonal arrow (assembled from } N_i \to \mathcal{P}_i \text{) is faithful (for example, using the concrete description of filtered 2-colimits found in [Was04, Appendix A]).} \]

Conversely, the exact and faithful functor $\text{2-colim}_i N_i \to \mathcal{P}$ together with $Q \to \text{2-colim}_i N_i$ induce a functor $N \to \text{2-colim}_i N_i$, by the universal property of $N$.

Finally, the universal property of $N$ gives that the composition

\[ N \to \text{2-colim}_i N_i \to N', \]

is isomorphic to the identity, from which we can conclude that the second arrow is essentially surjective and full. We already know that it’s faithful, so this concludes the proof. \qed

Remark 8.2. In [Ivo17], Ivorra works with Nori categories associated to representations of quivers taking values in finite and hom-finite abelian categories. However, [IM19, Lemma 1.4] ensures that, given an additive functor $T$ from an additive category to a finite and hom-finite abelian one, one obtains the same Nori category whether one proceeds in the way described in Section 7, or by forgetting that the source of $T$ is anything but a quiver, and proceeding as in [Ivo17].

References

[AIK15] Fabrizio Andreatta, Adrian Iovita, and Minhyong Kim. A $p$-adic nonabelian criterion for good reduction of curves. Duke Math. J., 164(13):2597–2642, 2015.

[Ara10] Donu Arapura. The Hodge theoretic fundamental group and its cohomology. In The geometry of algebraic cycles. Proceedings of the conference, Columbus, OH, USA, March 25–29, 2008, pages 3–22. Providence, RI: American Mathematical Society (AMS), 2010.

[Ayo10] Joseph Ayoub. Note sur les opérations de Grothendieck et la réalisation de Betti. J. Inst. Math. Jussieu, 9(2):225–263, 2010.

[Ayo14a] Joseph Ayoub. L’algèbre de Hopf et le groupe de Galois motiviques d’un corps de caractéristique nulle, I. J. Reine Angew. Math., 693:1–149, 2014.

[Ayo14b] Joseph Ayoub. L’algèbre de Hopf et le groupe de Galois motiviques d’un corps de caractéristique nulle, II. J. Reine Angew. Math., 693:151–226, 2014.

[Ayo14c] Joseph Ayoub. La réalisation étale et les opérations de Grothendieck. Ann. Sci. Éc. Norm. Supér. (4), 47(1):1–145, 2014.

[BBD] A. A. Beilinson, J. Bernstein, and P. Deligne. Perverse sheaves. Astérisque 100, 172 p. 1982.

[CPS20] Bruno Chiarellotto, Valentina Di Proietto, and Atsushi Shiho. Comparison of relatively unipotent log de rham fundamental groups, 2020. arXiv: 1903.03361 [math.NT].

[CG17] Utsav Choudhury and Martin Gallauer Alves de Souza. An isomorphism of motivic Galois groups. Adv. Math., 313:470–536, 2017.

[DE20] Marco D’Addezio and Hélène Esnault. On the universal extensions in tannakian categories, 2020. arXiv: 2009.14170 [math.AG].
[Del89] P. Deligne. Le groupe fondamental de la droite projective moins trois points. Galois groups over $\mathbb{Q}$, Proc. Workshop, Berkeley/CA (USA) 1987, Publ., Math. Sci. Res. Inst. 16, 79-297, 1989.

[Del90] P. Deligne. Catégories tannakiennes. The Grothendieck Festschrift, Collect. Artic. in Honor of the 60th Birthday of A. Grothendieck. Vol. II, Prog. Math. 87, 111-195, 1990.

[DM82] P. Deligne and J. S. Milne. Tannakian categories. In Hodge Cycles, Motives, and Shimura Varieties. Springer Berlin Heidelberg, Berlin, Heidelberg, 1982, pages 101-228.

[Del71] Pierre Deligne. Théorie de Hodge. II. Publ. Math., Inst. Hautes Étud. Sci., 40:5–57, 1971.

[DG05] Pierre Deligne and Alexander B. Goncharov. Groupes fondamentaux motiviques de Tate mixte. Ann. Sci. Éc. Norm. Supér. (4), 38(1):1–56, 2005.

[Had11] Majid Hadian. Motivic fundamental groups and integral points. Duke Math. J., 160(3):503–565, 2011.

[Hai87] Richard M. Hain. The de Rham homotopy theory of complex algebraic varieties. I. $K$-Theory, 1(3):271–324, 1987.

[Hai98] Richard M. Hain. The hodge de rham theory of relative malcev completion. Annales scientifiques de l'École Normale Supérieure, Ser. 4, 31(1):47–92, 1998.

[Ivo17] Florian Ivorra. Perverse Nori motives. Math. Res. Lett., 24(4):1097–1131, 2017.

[IM19] Florian Ivorra and Sophie Morel. The four operations on perverse motives, 2019. arXiv: 1901.02096 [math.AG].

[Jan03] Jens Carsten Jantzen. Representations of algebraic groups. 2nd ed. Providence, RI: American Mathematical Society (AMS), 2003, pages xiii + 576.

[Laz15] Christopher Lazda. Relative fundamental groups and rational points. Rend. Semin. Mat. Univ. Padova, 134:1–45, 2015.

[Mor78] John W. Morgan. The algebraic topology of smooth algebraic varieties. Publ. Math., Inst. Hautes Étud. Sci., 48:137–204, 1978.

[PS08] Chris A. M. Peters and Joseph H. M. Steenbrink. Mixed Hodge structures. Berlin: Springer, 2008, pages xiii + 470.

[Sai90] Morihiko Saito. Mixed Hodge modules. Publ. Res. Inst. Math. Sci., 26(2):221–333, 1990.

[Was04] Ingo Waschkies. The stack of microlocal perverse sheaves. Bull. Soc. Math. Fr., 132(3):397–462, 2004.