Note on Trace Class Groups

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Abstract

A Lie group $G$ is called a trace class group if for every irreducible unitary representation $\pi$ of $G$ and every $C^\infty$ function $f$ with compact support the operator $\pi(f)$ is of trace class. In this note we prove that the semidirect product of $\mathbb{R}^n$ and a real semisimple algebraic subgroup $G$ of $\text{GL}(n,\mathbb{R})$ is a trace class group only if $G$ is compact. The converse has been shown elsewhere. We also make a descent start with the study of semidirect products with Heisenberg-type groups.

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1 Introduction

In this note we resume the study of trace class groups from [4]. An irreducible unitary representation $\pi$ of a Lie group is said to be of trace class if for every $C^\infty$ function $f$ with compact support the operator $\pi(f)$ is of trace class. A Lie group is said to be of trace class, or briefly, a trace class group, if every irreducible unitary representation is of trace class. Well-known examples of such groups are reductive Lie groups and unipotent Lie groups. In general each (real algebraic) Lie group is a semidirect product of a reductive and a unipotent Lie group. One of the highlights of [4] is the theorem that the semidirect product of a real algebraic semisimple Lie group and its Lie algebra is a trace class group if and only if the group is compact. In this note we prove a generalization of this theorem, provided by the case of a semisimple real algebraic group $G$ acting on a real finite-dimensional vector space $V$ by linear transformations and considering the semidirect product of $V$ and $G$. We also make a beginning with the study of real algebraic groups with unipotent radical equal to a Heisenberg group.
2 Formula for the character of an induced representation

Let \( G \) be a locally compact group and \( N \) a closed subgroup. Choose right Haar measures \( dg \) on \( G \) and \( dn \) on \( N \). We may find a strictly positive continuous function \( q \) on \( G \) satisfying

\[
q(e) = 1, \\
q(n) = \Delta_N(n) \Delta_G(n^{-1}) q(g) \quad (n \in N, g \in G),
\]

(1)

where \( \Delta_N, \Delta_G \) denote the modular functions on \( N, G \). For example

\[
\int_G f(a^{-1} g) dg = \Delta_G(a) \int_G f(g) dg
\]

for all \( a \in G \) and \( f \in C_c(G) \).

The function \( q \) defines a quasi-invariant measure \( d_q \) on \( G/N \) (the space of right cosets with respect to \( N \)) as follows. For \( f \in C_c(G) \) set \( T_N f(\hat{g}) = \int_N f(n g) dn, \hat{g} = N g \). Then \( d_q \) is defined by

\[
\int_{G/N} T_N f(\hat{g}) d_q(\hat{g}) = \int_G f(g) q(g) dg.
\]

Let \( \gamma \) be a unitary representation of \( N \) and set \( \pi = \text{Ind}_N^G \gamma \). We write down a formula for the character of \( \pi \) in terms of that for \( \gamma \). Let us give the definition of \( \pi \). Let \( \mathcal{H} \) be the Hilbert space of \( \gamma \). Then \( \pi \) acts on the space \( \mathcal{H}_\pi \) of function \( f: G \to \mathcal{H} \) satisfying

\[
f(n g) = \gamma(n) f(g) \quad \text{and} \quad \int_{G/N} \|f(\hat{g})\|^2 d_q \hat{g} < \infty.
\]

The action of \( \pi \) is

\[
\pi(g) f(x) = f(x g) [q(x g)/q(x)]^{1/2}.
\]

Theorem 2.1 ([2], Theorem 3.2). Let \( \varphi \in C_c(G) \), \( \varphi^* (g) = \overline{\varphi(g^{-1})} \Delta_G(g^{-1}) \) and set \( \psi = \varphi \ast_G \varphi^* \). Then

\[
\text{tr} \pi(\psi) = \int_{G/N} \Delta_G(g^{-1})^2 q(g^{-1}) \text{tr} [\int_N \psi(g^{-1} n g) \gamma(n) \Delta_G(n) \Delta_N(n^{-1/2} dn)] d_q \hat{g}
\]

(2)

in the sense that both sides are finite and equal or both \( +\infty \).
A group $G$ is called unimodular if $\Delta_G = 1$. If $G$ is a unimodular Lie group we have $\text{tr} \pi(\varphi)$ is finite for all functions $\varphi \in C_c^\infty(G)$ if and if $\text{tr} \pi(\psi)$ is for all functions $\psi$ of the form $\psi = \varphi * G \varphi^*$ with $\varphi \in C_c^\infty(G)$. This is because any $\varphi$ is a finite sum of functions of the form $\psi$ by [1].

3 Application to semidirect products

Let $V$ be a finite-dimensional real vector space and $H$ a closed subgroup of $\text{GL}(V)$. Set $G = V \rtimes H$. The product in $G$ is given by

$$(v, h)(v', h') = (v + h \cdot v', hh') \quad \text{for } v, v' \in V, h, h' \in H.$$ 

If $dh$ is a right Haar measure on $H$ and $dv$ one on $V$, then $dg = dv dh$ is a right haar measure on $G$. Denote by $\hat{V}$ the space of continuous unitary characters of $V$. For each $\chi \in \hat{V}$ and each $h \in H$ consider the function

$$v \mapsto \chi(h \cdot v) \quad (v \in V).$$

This is again a continuous unitary character of $V$, which we call $\chi \cdot h$. The set of all $\chi \cdot h$ is called the orbit of $\chi$ in $\hat{V}$ and

$$H_\chi = \{ h \in H : \chi \cdot h = \chi \}$$

the stability subgroup of $\chi$. Choose an irreducible unitary representation $\rho$ of $H_\chi$ and define $\chi \otimes \rho$ by

$$(v, h) \mapsto \chi(v) \rho(h) \quad (v \in V, h \in H_\chi).$$

Then $\chi \otimes \rho$ is an irreducible unitary representation of $V \rtimes H_\chi$ and the induced representation $\pi_{\chi, \rho}$ is an irreducible unitary representation of $V \rtimes H$, see [3], p. 43. Now apply (2) with $G = V \rtimes H$, $N = V \rtimes H_\chi$ and $\pi = \pi_{\chi, \rho}$. Choose as before $dv dh$ and similarly $dv dh_\chi$ as right Haar mesures on $G$ and $N$. Then $\Delta_G$ and $\Delta_N$ are given by

$$\Delta_G(v, h) = \det(h), \Delta_H(h) \quad (v \in V, h \in H)$$

and similarly

$$\Delta_N(v, h) = \det(h) \Delta_{H_\chi}(h) \quad (v \in V, h \in H_\chi).$$

Notice that $q$ is left $V$-invariant, so we may write $q(v, h) = Q(h)$. $Q$ satisfies

$$Q(h_0 h) = \Delta_{H_\chi}(h_0) \Delta_H(h_0^{-1})Q(h)$$

for $h \in H, h_0 \in H_\chi$. Let us now rewrite (2) for the above particular case.
Theorem 3.1 Let $\psi$ be as in Theorem 2.1. Then

$$\text{tr} \pi(\psi) = \int_{H/H_x} \Delta_H(h)^{-1} Q(h)^{-1}.$$

$$\text{tr} \left[ \int_V \int_{H_x} \psi(v, h^{-1} h_0 h)(\chi \cdot h)(v) \rho(h_0) \Delta_H(h_0)^{1/2} \Delta_{H_x}(h_0)^{-1/2} dv dh_0 \right] dQ(h).$$

(3)

4 A special case

Let $V$ be a finite-dimensional real vector space with inner product $\langle \cdot, \cdot \rangle$ and with complexification $V$. Denote by $G$ a connected, complex, semisimple, linear algebraic subgroup of $\text{GL}(V)$. Assume that $G$ is defined over $\mathbb{R}$ and set $G = G(\mathbb{R})$ for its group of real points. Then $G$ is a semisimple Lie group with finite center and finitely many connected components. For any $g \in \text{GL}(V)$ set $\langle g \cdot v, w \rangle = \langle v, g^{-1} \cdot w \rangle$ for all $v, w \in V$. Assume that $G$ is invariant under the Cartan involution $\theta$ defined by $\theta(g) = {}^t g^{-1}$ ($g \in G$).

Write $G_1 = V \rtimes G$. Notice that $G_1$ is unimodular. The purpose of this note is to show

Theorem 4.1 The group $G_1$ is a trace class group only if the group $G$ is compact.

The converse of this theorem has been proved in [4], Lemma 14.2.

Proof. Assume $G$ to be non-compact. The Cartan involution of $G$ gives rise to a Cartan involution of the Lie algebra $\mathfrak{g}$ of $G$, that we again denote by $\theta$. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the decomposition of $\mathfrak{g}$ into $\pm 1$-eigenspaces of $\theta$. Then $\mathfrak{k}$ consists of anti-symmetric and $\mathfrak{p}$ of symmetric elements. Set $K = G \cap O(n, \mathbb{R})$. Then $\mathfrak{k}$ is the Lie algebra of $K$. Select a non-trivial maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, which exists because $G$ is non-compact, and let $\Sigma$ denote the set of roots of $\langle \mathfrak{g}, \mathfrak{a} \rangle$. Then $\Sigma$ is root system (with multiplicities). Let $\Delta$ be a set of simple roots and $\Sigma^+$ the set of positive roots with respect to $\Delta$. Denote by $\mathfrak{n}$ the Lie subalgebra spanned by the positive root vectors and by $N$ the corresponding algebraic subgroup. Then one has $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ and similarly $G = KAN$, the Iwasawa decomposition of $G$. Let $\xi_0$ be a highest weight vector in $V$ with highest weight $\lambda \neq 1$ (with respect to $N$ and $A$). Such a vector exists. Indeed, if all highest weight vectors have weight equal to one, then $G = \{1\}$, which is not the case.
Set $G_0$ for the stabilizer of $\xi_0$ in $G$. Denote by $B$ the Killing form of $\mathfrak{g}$ and define $H_1$ by
\[
\lambda(H) = B(H, H_1) \quad (H \in \mathfrak{a}),
\]
and set $A_1 = \exp \mathbb{R} H_1$. Let us denote by $P$ the stabilizer of the half-line $\mathbb{R}_+^* \xi_0$. Then $P = A_1 G_0$, where $A_1 \cap G_0 = \{1\}$, $a_1 G_0 a_1^{-1} = G_0$ for all $a_1 \in A_1$. Let $da_1$ and $dg_0$ denote right Haar measures on $A_1$ and $G_0$ respectively. Then $dp = \delta(a_1) da_1 dg_0$, where $\delta(a_1) = \det \text{Ad}(a_1^{-1})|_{g_0}$ is a right Haar measure on $P$. Since $A \subset P$ and $N \subset P$ we have $G = PK$ and
\[
dg = d(g_0 a_1 k) = \Delta_{G_0} (g_0^{-1} a_1^{-1}) dg_0 da_1 dk.
\]
Define the character $\chi_0$ of $V$ by
\[
\chi_0(v) = e^{-2\pi i \langle \xi_0, v \rangle} \quad (v \in V).
\]
Then Stab $\chi_0 = \theta(G_0)$. Let us write $H_0 = \theta(G_0)$, and let $dh_0$ be a right Haar measure on $H_0$. In a similar way as above we have
\[
dg = d(h_0 a_1 k) = \Delta_{H_0} (h_0) \delta_0(a_1) dh_0 da_1 dk,
\]
where $\delta_0(a_1) = \det \text{Ad}(a_1)|_{h_0}$. We will consider the representation $\pi$ given by
\[
\pi = \text{Ind}_{V \times H_0}^{V \times G} \chi_0 \otimes 1
\]
and determine whether its trace exists. Let
\[
Q(h_0 a_1 k) = \Delta_{H_0} (h_0) \delta_0(a_1).
\]
Equation (3) then becomes:
\[
\text{tr} (\psi) = \int_{A_1} \hat{\psi}_1(a_1 \cdot \xi_0) \left[ \int_{H_0} \psi_2(h_0) \Delta_{H_0}(h_0)^{-1/2} dh_0 \right] da_1
\]
where we take $\psi \in C^\infty_c(G)$ of the form $\psi(v, g) = \psi_1(v) \psi_2(g)$ ($v \in V, g \in G$), both $\psi_1$ and $\psi_2$ $K$-invariant and $\hat{\psi}_1(w) = \int_V \psi_1(v) e^{-2\pi i \langle v, w \rangle} dv$ for $w \in V$. Clearly the integral
\[
\int_{A_1} \hat{\psi}_1(a_1 \cdot \xi_0) da_1 = \int_0^\infty \hat{\psi}_1(\mu \xi_0) \frac{d\mu}{\mu}
\]
diverges for suitable $\psi_1$. So we may conclude that $G$ cannot be non-compact. This concludes the proof of the theorem. \qed
5 Semidirect products with Heisenberg groups

In this section we extend our scope to semidirect products $G = V \rtimes H$ with $V$ a non-necessarily Abelian normal subgroup of $G$. Notice that any real algebraic group is of this form according to the Levi decomposition, see [4], Proposition 2.1. Let us begin with some preparations.

**Lemma 5.1** Let $G$ be a Lie group and $N$ a closed normal subgroup of $G$. If $G$ is a trace class group, then the quotient group $\hat{G} = G/H$ is

**Proof.** Let $\hat{\pi}$ be an irreducible unitary representation of $\hat{G}$ and $\pi$ the corresponding representation of $G$. Choose a right Haarmeasure $dn$ on $N$ and define for $f \in C_c^\infty(G)$

$$T_N f(\hat{x}) = \int_{G/N} f(nx)dn.$$ 

Then $f \mapsto T_N f$ is a continuous surjective linear map $C_c^\infty(G) \to C_c^\infty(\hat{G})$ and one has

$$\pi(f) = \hat{\pi}(T_N f) \quad (f \in C_c^\infty(G)).$$

So the result follows. $\square$

Let us consider a special case. Denote by $G$ a Lie group, being the semidirect product $G = V \rtimes H$ where $H$ is a closed subgroup of $G$ and $V$ a closed normal subgroup of $G$. Let $W$ be a closed normal subgroup of $G$ contained in $V$. Denote by $v \mapsto \hat{v}$ the canonical map $V \to V/W = \hat{V}$. The group $H$ acts on $\hat{V}$ by

$$h \cdot \hat{v} = \hat{h} \cdot v \quad (h \in H, v \in V).$$

Set $\hat{G} = \hat{V} \rtimes H$. Then we have

**Lemma 5.2** The map $(v, h) \mapsto (\hat{v}, h)$ is a surjective homomorphism from $G$ to $\hat{G}$ with kernel $W$.

We will now specialize to real algebraic groups $G$ of the form $G = V \rtimes H$ with $V$ a unipotent and $H$ a semisimple real algebraic group. Set

$$V_1 \supset V_2 \supset \cdots \supset V_k \supset \{1\}$$

for the descending series of $V$, where $V_1 = V$, $V_i = [V, V_{i-1}]$, $V_k = Z$, the (non-trivial center of $V$. Notice that each $V_i$ is normal in $G$, so in particular $H$-invariant. Define

$$L = \{h \in H : h = id \text{ on } V_i/V_{i+1} \text{ for all } i = 1, \ldots, k\}$$

6
Clearly $L$ is a closed real algebraic normal subgroup of $H$. We can now formulate a conjecture.

**Conjecture 5.3** Let $G = V \rtimes H$ with $V$ a unipotent and $H$ a semisimple real algebraic group. Then $G$ is a trace class group if and only if $H/L$ is compact.

Let us consider an example with $V$ of a special nature, namely a Heisenberg group. Such a group can be seen as the most simple choice for a non-Abelian group $V$. We shall show that the conjecture holds in this case.

**Example 5.4** Denote by $V$ the $(2n+1)$-dimensional Heisenberg group with Lie algebra basis $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ with $[x_i, y_i] = z$ and all other brackets equal to zero. Then $Z$ is one-dimensional and spanned by $z$. Let $H$ be any semisimple real algebraic group acting on $V$ algebraically and set $G = V \rtimes H$.

There are two kinds of irreducible unitary representations $\pi$ of $V$, depending on their behaviour on $Z$. By Schur’s Lemma we have $\pi(z) = \lambda(z) I$ ($z \in Z$) for some character $\lambda$ of $Z$.

If $\lambda \neq 1$, then $\pi$ is equivalent with an infinite-dimensional representation $\pi_{\lambda}$ satisfying $\pi_{\lambda}(z) = \lambda(z) I$ and $\pi_{\lambda}$ is square-integrable modulo $Z$.

If $\lambda = 1$, then $\pi$ is actually a one-dimensional representation of $V/Z \simeq \mathbb{R}^{2n}$, so a character $\chi$ of $V$.

Let us now perform the usual construction for the determination of the irreducible unitary representations of $G$, see [2], p. 470. Let us begin with $\pi_{\lambda}$. Since $H$ acts trivially on $Z$, one has $\text{Stab} \pi_{\lambda} = H$. If $\rho$ is an irreducible unitary representation of $H$ then $\pi_{\lambda} \otimes \rho$ in one of $G$, of trace class. Let now $\chi$ be a character of $V/Z$ (hence of $V$). By the usual construction (see [4]), we always obtain a trace class representation of $V/Z \rtimes H = G/Z$, so of $G$, if and only if $H/L_0$ is compact, where $L_0 = \{ h \in H : h = id \text{ on } V/Z \}$. Clearly $L_0 = L$, so if and only if $H/L$ is compact. Resuming, $G = V \rtimes H$ is trace class if and only if $H/L$ is compact.

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