Abstract

A series of conjectures is obtained as further investigation of the integral transformation \( I(\alpha) \) introduced in the previous paper. A Macdonald-type difference operator \( D \) is introduced. It is conjectured that \( D \) and \( I(\alpha) \) are commutative with each other. Studying the series for the eigenfunctions under termination conditions, it is observed that a deformed Weyl group action appears as a hidden symmetry. An infinite product formula for the eigenfunction is found for a spatial case of parameters. A one parameter family of hypergeometric-type series \( F(\alpha) \) is introduced. The series \( F(\alpha) \) is characterized by a covariant transformation property \( I(\alpha q^{-1}t) \cdot F(\alpha) = F(\alpha q^{-1}t) \) and a certain initial condition given at \( \alpha = t^{1/2} \). We call \( F(\alpha) \) the ‘quasi-eigenfunction’ for short. A class of infinite product-type expressions are conjectured for \( F(\alpha) \) at the special points \( \alpha = -t^{1/2} \), \( \alpha = t \), \( \alpha = \pm q^{1/2}t^{1/2} \), and \( \alpha = \pm qt^{1/2} \) (\( \ell = 1, 2, 3, \cdots \)).

1 Introduction

This is the second paper of a series in which a commutative family generated by an integral transformation is studied. In the last paper \([1]\), we have introduced the integral transformation \( I(\alpha) \) and conjectured the commutativity
\[ [I(\alpha), I(\beta)] = 0. \] Our arguments there were based on the basic properties for the eigenfunctions and the explicit formulas. The main purpose of the present paper is to state a series of conjectures which gives us several useful descriptions for the structure of the operator \( I(\alpha) \) and its eigenfunctions. We consider these as preparations for our application of \( I(\alpha) \) to the investigation of the vertex operators for the eiget-vertex model given in the next paper [2]. As for the eight-vertex model and the corner transfer matrix approach to the correlation functions, we refer the readers to [3, 4, 5] and [6, 7].

The integral transformation \( I(\alpha) \) acting on the space of formal power series \( F_n = \mathbb{C}[[\zeta_2/\zeta_1, \ldots, \zeta_n/\zeta_{n-1}]] \) was introduced in the first paper [1] (see Definition 1.1 of [1]) as follows:

\[
I(\alpha) f(\zeta_1, \ldots, \zeta_n) = \prod_{i=1}^{n} \left( \frac{(qt^{-1}; q)_{\infty}}{(\alpha s_i^{-1}qt^{-1}; q)_{\infty} (\alpha^{-1}s_i; q)_{\infty}} \right) h(\zeta_j/\zeta_i) \prod_{i<j} \frac{d\xi_i}{2\pi i \xi_i} \prod_{j \geq k} \frac{d\xi_j}{2\pi i \xi_j} \prod_{i=1}^{n} \frac{\Theta_q(\alpha s_i^{-1}qt^{-1/2} \zeta_i/\xi_i)}{\Theta_q(q^{1/2}t^{-1/2} \zeta_i/\xi_i)} \prod_{k=1}^{n} \left[ \prod_{j<k} g(\zeta_k/\xi_i) \prod_{j \geq k} g(\zeta_j/\zeta_k) \right] f(\xi_1, \ldots, \xi_n),
\]

where the integration contours \( C_i \) are given by the conditions \( |\zeta_i/\xi_i| = 1 \), and the functions \( h(\zeta) \) and \( g(\zeta) \) are given by

\[
h(\zeta) = (1 - \zeta) \frac{(qt^{-1}; q)_{\infty}}{(t\zeta; q)_{\infty}}, \quad g(\zeta) = \frac{(\sqrt{t} \zeta^{1/2}; q)_{\infty}}{(\sqrt{t} - \zeta; q)_{\infty}}.
\]

The main statement there was the conjecture for the commutativity (see Conjecture 1.2 in [1]):

\[ I(\alpha) I(\beta) = I(\beta) I(\alpha). \]

A proof of this for the case \( n = 2 \) was obtained by using some summation and transformation formulas for the basic hypergeometric series. In this paper, we continue our study on the operator \( I(\alpha) \) and try to obtain better understanding.

This paper consist of several conjectures (proved for some cases) and definitions summarized as follows.
1. Commutativity between the integral operator \( I(\alpha) \) and a Macdonald-type difference operator \( D \) acting on \( \mathcal{F}_n \).

2. Termination of the eigenfunctions of \( I(\alpha) \) and a hidden symmetry of a Weyl group action.

3. Infinite product formulas for the eigenfunctions of \( I(\alpha) \) for a particular case of parameters.

4. Introduction of the ‘quasi-eigenfunction’ \( F(\alpha) \) which is defined by a certain transformation property with respect to the action of \( I(\alpha) \) and a certain initial condition.

5. A class of infinite product(-type) formulas for \( F(\alpha) \) for particular values of \( \alpha \).

The Jack and the Macdonald symmetric polynomials \([8]\) and the raising operators were studied in \([9] [10]\). The raising operator can be characterized as an eigenfunction of a Macdonald-type difference operator \( D \). As for the detail, we refer the readers to Appendix A and the lecture note \([11]\). Note that the solution can be regarded as a basic analogue of the Heckman-Opdam hypergeometric function \([12]\). While obtaining the explicit formulas for the eigenfunctions of \( I(\alpha) \), the author realized that \( I(\alpha) \) and \( D \) share exactly the same eigenfunctions, at least for small \( n \). This coincidence seems quite mysterious and meaningful, because it suggests that there exists a profound relationship between the eight-vertex model and the Macdonald polynomials.

In \([11]\), explicit formulas for \( I(\alpha) \) for small \( n \) were studied. (See Theorem 2.1, Conjecture 3.3 and Proposition 3.4 in the first paper \([11]\).) We observed that the hypergeometric-type series expressions obtained there gives us an efficient way to organize the series for the eigenfunctions. These explicit expressions allow us to find the termination conditions for the series. One may observe that the resulting polynomials have a good symmetry. We will introduce a certain Weyl group action on the space of polynomials, to explain this hidden symmetry for the eigenfunctions.

We observe that the eigenfunction of \( I(\alpha) \) can be written as an infinite product, if we specialize the parameter \( s_i \)’s as \((s_1, s_2, \cdots, s_n) = (1, t, \cdots, t^{n-1})\).
This observation comes from the explicit formulas for small $n$. If we assume the above stated commutativity $[I(\alpha), D] = 0$, this factorization can be shown for general $n$.

Next, we introduce another class of ‘multi variable hypergeometric series’ depending on a parameter $\alpha$. Setting the parameters as $(s_1, s_2, \ldots, s_n) = (1, 1, \cdots, 1)$, we characterize a continuous family of series $F(\alpha)$ by imposing the following two conditions:

\begin{align*}
(I) \quad & I(\alpha q^{-1} t) \cdot F(\alpha) = F(\alpha q^{-1} t), \\
(II) \quad & F(t^{1/2}) = \prod_{1 \leq i < j \leq n} (1 - \frac{\zeta_j}{\zeta_i}) \frac{(qt^{-1/2} \zeta_j/\zeta_i ; q)^{\infty}}{(t^{1/2} \zeta_j/\zeta_i ; q)^{\infty}}.
\end{align*}

We call this function $F(\alpha)$ ‘quasi-eigenfunction’ for short. In the next paper [2], it will be explained that this definition for $F(\alpha)$ naturally emerges in the context of the study of the eight-vertex model based on the corner transfer matrix method.

It will be argued that a class of infinite product(-type) expressions can be found for the quasi-eigenfunctions $F(\alpha)$ at particular values for $\alpha$. Note that some of these product formulas were observed in the previous paper [13], while we studied the eight-vertex model for the parameters $p_{8v}^{1/2} = q_{8v}^{3/2}$, $-q_{8v}^{3}$ and $q_{8v}^3$. (Note that the basic parameters $p_{8v}$ and $q_{8v}$ used in [13] are related with the ones in this paper as $p_{8v}^{1/2} = q$ and $q_{8v} = t$.) The infinite product-type formulas obtained in this paper will be one of the essential ingredients in our study of the eight-vertex model in the next paper [2].

The plan of the paper is as follows. In Section 2, the Macdonald-type difference operator $D$ is introduced, and the commutativity $[I(\alpha), D] = 0$ is argued. In Section 3, the Weyl group action (denoted by $\pi_m$) on the eigenfunctions of $I(\alpha)$ is studied, under the termination condition for the series $t = q^m$ ($m = 1, 2, 3, \cdots$). In Section 4, it is observed that we have an infinite product formula for the eigenfunction of $I(\alpha)$ for the case $(s_1, s_2, \ldots, s_n) = (1, t, \cdots, t^{n-1})$. Comments for the non diagonalizable case (which occurs if $s_i = s_j$ is satisfied) are given Section 5. A conjecture for the structure of the Jordan blocks is obtained. In Section 6 and Section 7, we work with the homogeneous condition $(s_1, s_2, \cdots, s_n) = (1, 1, \cdots, 1)$. (Hence $I(\alpha)$ is not diagonalizable.) The quasi-eigenfunction $F(\alpha)$ is introduced. Explicit formulas for $F(\alpha)$ are studied for small $n$. A class of infinite product(-type)
formulas for $F(\alpha)$ are conjectured, which takes place at $\alpha = -t^{1/2} \pm t^{1/2}$, $\alpha = \pm q^{t^{1/2}} (\ell = 1, 2, 3, \cdots)$. Concluding remarks are given in Section 8. Appendix A is devoted to the explanation for the Fock realization of the Macdonald difference operators, and difference operators for the raising operators.

As was in the last paper, we use the standard notations for the $q$-shifted factorials and the basic hypergeometric series used in Gasper and Rahman [14] (hereafter referred to as GR). The notation for the elliptic theta function $\Theta_q(z) = (z; q)_\infty (q/z; q)_\infty (q; q)_\infty$ is used.

2 Macdonald-type Difference Operator $D$

In the first paper [1], it was conjectured that the integral transformation $I(\alpha)$ generates a commutative family of operators acting on the space $\mathcal{F}_n$. In this section, we present another supporting argument for this conjecture. A Macdonald-type difference operator will be introduced. By examining the eigenfunctions for $D$, we conjecture that $I(\alpha)$ and $D$ are commutative.

2.1 definition of the difference operator $D$

Let us introduce a difference operator which is acting on the space of power series $\mathcal{F}_n$.

**Definition 2.1** Let $s_1, s_2, \cdots, s_n$, $q$ and $t$ be parameters. Define a difference operator acting on the space $\mathcal{F}_n$ by

$$D(s_1, \cdots, s_n; q, t) = \sum_{i=1}^{n} s_i \prod_{j<i} \theta_-(\frac{\zeta_j}{\zeta_i}) \prod_{j>i} \theta_+\left(\frac{\zeta_j}{\zeta_i}\right) \cdot T_{q^{-1}, \zeta_i}. \quad (6)$$

Here $\theta_{\pm}(\zeta)$ are the series

$$\theta_{\pm}(\zeta) = \frac{1 - q^{\pm 1} t^{\pm 1} \zeta}{1 - q^{\pm 1} \zeta} = 1 + \sum_{n=1}^{\infty} (1 - t^{\pm 1}) q^{\pm n} \zeta^n, \quad (7)$$

and the difference operator $T_{x, \zeta}$ is defined by

$$T_{x, \zeta} \cdot g(\zeta_1, \zeta_2, \cdots, \zeta_n) = g(\zeta_1, \cdots, x \zeta_i, \cdots, \zeta_n). \quad (8)$$
Note that this difference operator $D$ was derived from the Macdonald difference operator $D^1_n$ \cite{Macdonald}, in the context of the raising operators. This is explained in Appendix A.

### 2.2 existence of the eigenfunctions of $D$

Let us study the existence of the eigenfunctions of the difference operator $D$.

**Proposition 2.2** Let the parameters $(s_1, s_2, \cdots, s_n)$ and $q$ be generic. Let $j_1, j_2, \cdots, j_{n-1}$ be nonnegative integers. In the space $\mathcal{F}_n$, there exist a unique solution to the equation

$$Df_{j_1, j_2, \cdots, j_{n-1}}(\zeta_1, \cdots, \zeta_n) = \lambda_{j_1, j_2, \cdots, j_{n-1}} f_{j_1, j_2, \cdots, j_{n-1}}(\zeta_1, \cdots, \zeta_n),$$

with the conditions

$$f_{j_1, j_2, \cdots, j_{n-1}}(\zeta_1, \cdots, \zeta_n) = \sum_{i_1 \geq j_1, \cdots, i_{n-1} \geq j_{n-1}} c_{i_1, i_2, \cdots, i_{n-1}} \left( \frac{\zeta_2}{\zeta_1} \right)^{i_1} \left( \frac{\zeta_3}{\zeta_2} \right)^{i_2} \cdots \left( \frac{\zeta_n}{\zeta_{n-1}} \right)^{i_{n-1}},$$

and $c_{j_1, j_2, \cdots, j_{n-1}} = 1$, if and only if

$$\lambda_{j_1, j_2, \cdots, j_{n-1}} = \sum_{i=1}^{n} s_i q^{-j_{i-1}+j_i},$$

is satisfied. Here $j_0 = 0$ and $j_n = 0$ are assumed.

**Proof.** We have

$$D \left( \frac{\zeta_2}{\zeta_1} \right)^{j_1} \left( \frac{\zeta_3}{\zeta_2} \right)^{j_2} \cdots \left( \frac{\zeta_n}{\zeta_{n-1}} \right)^{j_{n-1}} = \left( \frac{\zeta_2}{\zeta_1} \right)^{j_1} \left( \frac{\zeta_3}{\zeta_2} \right)^{j_2} \cdots \left( \frac{\zeta_n}{\zeta_{n-1}} \right)^{j_{n-1}}$$

$$\times \sum_{i=1}^{n} s_i q^{-j_{i-1}+j_i} \prod_{j<i} \theta_- \left( \frac{\zeta_i}{\zeta_j} \right) \prod_{j>i} \theta_+ \left( \frac{\zeta_j}{\zeta_i} \right),$$

and it is explicitly seen here that $D$ is lower triangular in the monomial basis with respect to the dominance order, or the lexicographic order. The diagonal elements are given by Eq. (11). If the parameters are generic, all the diagonal entries are distinct and we can construct the eigenfunction. □

By examining the matrix elements of $D$ given by Eq. (12), one finds that all the eigenfunctions are related by shifting the parameters $s_i$. 

6
Proposition 2.3 The eigenfunctions of $D$ satisfy
\[
f_{j_1,j_2,\ldots,j_{n-1}}(\zeta_1,\cdots,\zeta_n) = \prod_{i=1}^{n} \zeta_i^{j_{i-1}-j_i}(T_{q,s_i})^{-j_i-j_i+1} \cdot f_{0,0,\ldots,0}(\zeta_1,\cdots,\zeta_n).
\]
Here, $j_0 = 0, j_n = 0$ are assumed, and $T_{q,s_i}$ denotes the shift operator acting on the variable $s_i$.

Note that we have exactly the same property for the eigenfunctions of the integral transformation $I(\alpha)$. See Proposition 3.2 of [1].

2.3 commutativity between $I(\alpha)$ and $D$

Our aim here is to investigate the relationship between the integral transformation $I(\alpha)$ and the difference operator $D$, to claim

Conjecture 2.4 The integral transformation $I(\alpha; s_1, \cdots, s_n, q, t)$ and the difference operator $D(s_1, \cdots, s_n, q, t)$ are commutative with each other
\[
[I(\alpha; s_1, \cdots, s_n, q, t), D(s_1, \cdots, s_n, q, t)] = 0,
\]
for general $n \geq 2$.

In the first paper [1], we obtained the conjecture that the integral transformation $I(\alpha)$ generates a commutative family of operators acting on the space of series $\mathcal{F}_n$ (see Conjecture 1.2 of [1]). In Appendix A, it is observed that another commutative family of Macdonald-type difference operators exists, which are acting on $\mathcal{F}_n$ and containing $D(s_1, \cdots, s_n, q, t)$ (see Conjecture A.7). Therefore, Conjecture 2.4 gives us a complementary understanding of the two conjectures for these families of commuting operators.

In what follow, we argue that the eigenfunctions for $I(\alpha)$ and $D$ exactly coincide, at least for small $n$ up to certain degree in $\zeta$.

Let us start from the case $n = 2$.

Proposition 2.5 The first eigenfunction of $D$ is given by
\[
f_0(\zeta_1,\zeta_2) = (1 - \zeta_2/\zeta_1) \varphi_1 \left( \frac{qt^{-1}s_1/s_2}{qs_1/s_2}; q, t\zeta_2/\zeta_1 \right).
\]
All the other eigenfunctions $f_i(\zeta_1,\zeta_2)$ are given by Proposition 2.3.
Proof. Set \( f_0 = (1 - \zeta)g(\zeta) \) and \( g(\zeta) = \sum_{n=0}^{\infty} g_n \zeta^n \), where \( \zeta = \zeta_2/\zeta_1 \). The equation for \( f_0 \) gives us the difference equation for \( g \) as

\[
s_1(1 - qt^{-1}\zeta)g(q\zeta) + s_2(1 - q^{-1}t\zeta)g(q^{-1}\zeta) = (s_1 + s_2)(1 - \zeta)g(\zeta).
\]

Solving this with the condition \( g_0 = 1 \), we have

\[
g_n = \left(\frac{qt^{-1}, qt^{-1}s_1/s_2; q}{q, qs_1/s_2; q}\right)_n t^n.
\]

Hence we see that all the eigenfunctions of \( D \) for \( n = 2 \) are completely identical to the ones for the integral transformation \( I(\alpha) \). (See Theorem 2.1 of \([1]\).)

**Proposition 2.6** The integral transformation \( I(\alpha; s_1, s_2, q, t) \) and the difference operator \( D(s_1, s_2, q, t) \) are commutative on the space \( \mathcal{F}_2 \)

\[
[I(\alpha; s_1, s_2, q, t), D(s_1, s_2, q, t)] = 0. \tag{16}
\]

Now we proceed to looking at the case \( n \geq 3 \). By a brute force calculation, one can observe the following.

**Conjecture 2.7** The first eigenfunction of the difference operator \( D \) for the case \( n = 3 \) is given by

\[
f_{0,0}(\zeta_1, \zeta_2, \zeta_3) = \sum_{k=0}^{\infty} \frac{(qt^{-1}, qt^{-1}, t, t; q)_k}{(q, qs_1/s_2, qs_2/s_3, qs_1/s_3; q)_k} (qs_1/s_3)_k (\zeta_3/\zeta_1)_k \times \prod_{1 \leq i < j \leq 3} (1 - \zeta_j/\zeta_i) \phi_1 \left(\frac{q^{k+1}t^{-1}, qt^{-1}s_i/s_j}{q^{k+1}s_i/s_j; q, t\zeta_j/\zeta_i}\right).
\]

We realize the complete coincidence of the eigenfunctions for \( I(\alpha) \) and \( D \) for \( n = 3 \) also, at least up to certain degrees in \( \zeta \). See Conjecture 3.3 of \([1]\).

By using the partial result for \( n = 4 \) given in Proposition 3.4 of \([1]\), one can observe that the eigenfunctions for \( I(\alpha) \) and \( D \) are the same for small degrees in \( \zeta \). Hence Conjecture 2.4 is likely correct for general \( n \).
3 Weyl Group Symmetry

In this section, we study a hidden symmetry of the eigenfunctions of $I(\alpha)$ or $D$ in terms of the Weyl group of type $A_{n-1}$. This Weyl group symmetry appears when the series for the eigenfunctions are terminating, at the special points $t = q^m$ ($m = 1, 2, 3, \cdots$).

Let us fix our notations for the Weyl group action. Let $P_n$ be the space of polynomials in $\zeta_1, \zeta_2, \cdots, \zeta_n$ with coefficients in the field of rational functions in $s_i$’s. We introduce a representation of the Weyl group of type $A_{n-1}$ on the space $P_n$ as follows.

**Definition 3.1** Let $W(A_{n-1})$ be the Weyl group of type $A_{n-1}$ generated by $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ with the braid relations $\sigma_i^2 = \text{id}$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

Let $m$ be a positive integer. Define the action of $W(A_{n-1})$ on $P_n$ by

$$\pi_m(\sigma_i) f(\zeta_1, \cdots, \zeta_i, \zeta_{i+1}, \cdots, \zeta_n, s_1, \cdots, s_i, s_{i+1}, \cdots, s_n)$$

$$= \prod_{k=1}^{m-1} \frac{s_i - q^k s_{i+1}}{s_{i+1} - q^k s_i} f(\zeta_1, \cdots, \zeta_i, \zeta_{i+1}, \cdots, \zeta_n, s_1, \cdots, s_i, s_{i+1}, \cdots, s_n).$$

Now we are ready to state the main conjecture in this section.

**Conjecture 3.2** If the condition $t = q^m$ ($m = 1, 2, 3, \cdots$) is satisfied, the first eigenfunction of $I(\alpha)$ or $D$ for general $n$ becomes terminating

$$\prod_{k=1}^{n} \zeta_k^{(n-k)m} f_{0,0,\cdots,0}(\zeta_1, \zeta_2, \cdots, \zeta_n) \in P_n.$$  

This polynomial is antisymmetric with respect to the Weyl group action $\pi_m$

$$\pi_m(\sigma) \cdot \prod_{k=1}^{n} \zeta_k^{(n-k)m} f_{0,0,\cdots,0}(\zeta_1, \zeta_2, \cdots, \zeta_n)$$

$$= - \prod_{k=1}^{n} \zeta_k^{(n-k)m} f_{0,0,\cdots,0}(\zeta_1, \zeta_2, \cdots, \zeta_n) \quad (\forall \sigma \in W(A_{n-1})).$$

In what follows, we present several evidences which support the conjecture. We will check the termination of the eigenfunctions and the Weyl group symmetry under the specialization $t = q^m$ ($m = 1, 2, 3, \cdots$), by using the explicit formulas for small $n$. (See Proposition 2.5, Conjecture 2.7 in this paper, and Proposition 3.4 of [1].)
Let us consider the case $n = 2$. One finds that the following terminating $2\phi_1$ series is $W(A_1)$ symmetric.

**Lemma 3.3** For $m = 1, 2, 3, \cdots$, the equality

$$
\pi_m(\sigma_1) \cdot \left( \zeta_1^{m-1} 2\phi_1 \left( \frac{qt^{-1}, qt^{-1} s_1 / s_2}{qs_1 / s_2}; q, t \zeta_2 / \zeta_1 \right) \right)_{|t=q^m}
= \zeta_1^{m-1} 2\phi_1 \left( \frac{qt^{-1}, qt^{-1} s_1 / s_2}{qs_1 / s_2}; q, t \zeta_2 / \zeta_1 \right)_{|t=q^m},
$$

(21)

holds for $\sigma_1 \in W(A_1)$.

Hence we have

**Proposition 3.4** Conjecture 3.2 is true for $n = 2$.

Next, let us study the case $n = 3$. One may find a family of $W(A_2)$ symmetric expressions by using terminating $2\phi_1$ series.

**Lemma 3.5** Let $m$ be a positive integer and $k$ be a nonnegative integer. Set

$$
\varphi_k = (\zeta_3 / \zeta_1)^k (qs_1 / s_3)^k \frac{(qt^{-1}, qt^{-1}, t, t; q)_k}{(q, qs_1 / s_2, qs_2 / s_3, qs_1 / s_3; q)_k}
\times \prod_{1 \leq i < j \leq 3} 2\phi_1 \left( \frac{q^{k+1} t^{-1}, qt^{-1} s_i / s_j}{q^{k+1} s_i / s_j}; q, t \zeta_j / \zeta_i \right)_{|t=q^m}.
$$

(22)

Then, $\zeta_1^{2m-2} \zeta_2^{m-1} \varphi_k$ is terminating and $W(A_2)$ symmetric. Namely, we have $\zeta_1^{2m-2} \zeta_2^{m-1} \varphi_k \in P_3$ and

$$
\pi_m(\sigma) \cdot \zeta_1^{2m-2} \zeta_2^{m-1} \varphi_k = \zeta_1^{2m-2} \zeta_2^{m-1} \varphi_k.
$$

(23)

for all $\sigma \in W(A_2)$.

Since the first eigenfunction for $n = 3$ can be written as a sum

$$
f_{0,0} = \prod_{i<j} (1 - \zeta_j / \zeta_i) \sum_k \varphi_k
$$

(see Conjecture 2.7), we have

**Proposition 3.6** Conjecture 3.2 is true for $n = 3$, under the assumption that the formula for $f_{0,0}(\zeta_1, \zeta_2, \zeta_3)$ given in Conjecture 2.7 is correct.
By looking at the partial result for \( n = 4 \) given in Proposition 3.4 of [1], one can also observe that the first eigenfunctions for \( I(\alpha) \) or \( D \) can be decomposed by using \( W(A_3) \) antisymmetric terminating series for \( t = q^m \). For example, we can check the \( W(A_3) \) antisymmetry by setting \( t = q^m \) and multiplying \( \zeta_1 \zeta_2 \zeta_3 \zeta_4 \) to the combination

\[
\frac{\zeta_3}{\zeta_1} \frac{(q s_1)}{s_3} \frac{(qt^{-1})_1(t)_1}{(q)_1(q s_{12})_1(q s_{23})_1(q s_{13})_1} \varphi(1, 1, 0, 1, 0, 0) \\
+ \frac{\zeta_4}{\zeta_2} \frac{(q s_2)}{s_4} \frac{(qt^{-1})_1(t)_1}{(q)_1(q s_{23})_1(q s_{34})_1(q s_{24})_1} \varphi(0, 1, 1, 0, 1, 0) \\
+ \frac{\zeta_4}{\zeta_1} \frac{(q s_1)}{s_4} \frac{(qt^{-1})_1(t)_1}{(q)_1(q s_{12})_1(q s_{24})_1(q s_{14})_1} \varphi(1, 0, 0, 1, 1, 0) \\
+ \frac{\zeta_4}{\zeta_1} \frac{(q s_1)}{s_4} \frac{(qt^{-1})_1(t)_1}{(q)_1(q s_{13})_1(q s_{34})_1(q s_{14})_1} \varphi(0, 0, 1, 0, 1, 1).
\]

(As for the notations, see Eq.(66) in [1].) Thus, we expect that Conjecture 3.2 is correct for \( n = 4 \).

4 Product Formula for the Eigenfunction

In this section, we study the special case

\[
(s_1, s_2, \cdots, s_n) = (1, t, \cdots, t^{n-1}),
\]

and obtain an infinite product formula for the first eigenfunction. We claim the following.

**Conjecture 4.1** Under the specialization Eq. (24), the first eigenfunction of \( I(\alpha) \) can be written as the infinite product

\[
f_{0,0,\ldots,0}(\zeta_1, \zeta_2, \cdots, \zeta_n) |_{(s_1, s_2, \cdots, s_n) = (1, t, \cdots, t^{n-1})} = \prod_{1 \leq i < j \leq n} (1 - \zeta_j/\zeta_i) \frac{(qt^{-1}\zeta_j/\zeta_i; q)_\infty}{(t\zeta_j/\zeta_i; q)_\infty}.
\]

We give several arguments for this conjecture below.

For \( n = 2 \), we can easily see that the infinite series for the eigenfunction \( f_0(\zeta_1, \zeta_2) \) reduces into the infinite product.
Proposition 4.2  Conjecture 4.1 is true for $n = 2$.

**Proof.** Setting $s_1 = 1, s_2 = t$ in Eq. (15), and using the $q$-binomial theorem (Eq. (1.3.2) of GR [14]), we have

$$f_0(\zeta_1, \zeta_2) \bigg|_{(s_1, s_2) = (1, t)} = (1 - \zeta) \times _2\phi_1 \left( \begin{array}{c} qt^{-1}, qt^{-1}s \\ qs \end{array} ; q, t\zeta \right) \bigg|_{s = t^{-1}}$$

$$= (1 - \zeta) \frac{(qt^{-1}\zeta; q)_{\infty}}{(t\zeta; q)_{\infty}},$$

where $\zeta = \zeta_2 / \zeta_1$ and $s = s_1 / s_2$. \qed

Next, for $n = 3$ we have

**Proposition 4.3**  Conjecture 4.1 is true for $n = 3$, under the assumption that the formula for $f_{0,0}(\zeta_1, \zeta_2, \zeta_3)$ given in Conjecture 2.7 is correct.

**Proof.** Changing the order of the summation, and using the $q$-Pfaff-Saalschütz formula (Eq. (1.7.2) of GR [14]), we have the equality,

$$\sum_{k=0}^{\infty} \frac{(t; q)_k(t; q)_k}{(q; q)_k(qt^{-2}; q)_k} (qt^{-2}\zeta)^k _2\phi_1 \left( \begin{array}{c} q^{k+1}t^{-1}, qt^{-3} \\ q^{k+1}t^{-2} \end{array} ; q, t\zeta \right)$$

$$= \sum_{m=0}^{\infty} \frac{(qt^{-1}; q)_m(qt^{-3}; q)_m}{(qt^{-2}; q)_m(q; q)_m} t^m \zeta^m _3\phi_2 \left( t, t, q^{-m} \begin{array}{c} qt^{-1}, qt^{-3} \\ q^{-m}t^{-2} \end{array} ; q, q \right)$$

$$= \frac{(qt^{-1}\zeta; q)_{\infty}}{(t\zeta; q)_{\infty}}.$$ 

Then the product formula for the first eigenfunction

$$f_{0,0}(\zeta_1, \zeta_2, \zeta_3) \bigg|_{(s_1, s_2, s_3) = (1, t, t^2)} = \prod_{1 \leq i < j \leq 3} (1 - \zeta_j / \zeta_i) \frac{(qt^{-1}\zeta_j / \zeta_i; q)_{\infty}}{(t\zeta_j / \zeta_i; q)_{\infty}},$$

(26)

is derived from Eq. (17) by using the above identity and the $q$-binomial theorem. \qed

For the case $n = 4$, we can observe that the degeneration into a product expression occurs by using the partial result given in Proposition 3.4 of [1].

If we assume the commutativity $[I(\alpha), D] = 0$, we have
Proposition 4.4 Conjecture 4.1 is true for general $n$, under the assumption that Conjecture 2.4 is correct.

Proof. It suffices to show

$$D(1, t, \cdots, t^{n-1}; q, t) \prod_{1 \leq i < j \leq n} (1 - \zeta_j / \zeta_i) \frac{(qt^{-1}\zeta_j / \zeta_i; q)_{\infty}}{(t\zeta_j / \zeta_i; q)_{\infty}}$$

$$= (1 + t + \cdots + t^{n-1}) \prod_{1 \leq i < j \leq n} (1 - \zeta_j / \zeta_i) \frac{(qt^{-1}\zeta_j / \zeta_i; q)_{\infty}}{(t\zeta_j / \zeta_i; q)_{\infty}},$$

for general $n$. Using the eigenvalue of the Macdonald difference operator $D_n^1$ (see [8]), we have

$$\text{LHS} = t^{n-1} \prod_{1 \leq i < j \leq n} (1 - \zeta_j / \zeta_i) \frac{(qt^{-1}\zeta_j / \zeta_i; q)_{\infty}}{(t\zeta_j / \zeta_i; q)_{\infty}} \sum_{i=1}^{n} \prod_{j \neq i} \frac{t^{-1}\zeta_i - \zeta_j}{\zeta_i - \zeta_j} = \text{RHS}.$$ 

Thus, it is expected that Conjecture 4.1 is true for general $n$.

5 Generalized Eigenfunctions

So far, we have been studying the properties of the integral transformation $I(\alpha)$ based on the eigenfunctions. Here in this section, we will make several remarks for the case $s_i = s_j$. As we will see, the operator $I(\alpha)$ or $D$ acting on $\mathcal{F}_n$ becomes non diagonalizable, when $n \geq 3$ and $s_i = s_j$ is satisfied. Therefore, we need some treatments for constructing the generalized eigenfunctions. In this section, the structure of the eigenspace for the operator $I(\alpha)$ or $D$ at the homogeneous limit

$$s_1 = s_2 = \cdots = s_n = 1,$$  \hspace{1cm} (27)

will be conjectured.

By looking at the eigenvalues of $I(\alpha)$ given in Eq.(50) of [11]

$$\lambda_{j_1, j_2, \cdots, j_{n-1}}(\alpha) = \prod_{i=1}^{n} \frac{(\alpha s_i^{-1}; q)_{j_{i-1} - j_i}}{(\alpha s_i^{-1} qt^{-1}; q)_{j_{i-1} - j_i}},$$  \hspace{1cm} (28)
we realize that some of the eigenvalues become degenerate under the condition: \( n \geq 3 \) and \( s_i = s_j \) for some \( i \) and \( j \). Note, however, that for the case \( n = 2 \), we do not have any degeneration of the eigenvalues on the space \( F_2 \). (So, \( I(\alpha) \) remains diagonalizable, even if \( s_1 = s_2 \).) It can be seen from the explicit formulas that some of the eigenfunctions become divergent, if we have degenerate eigenvalues. (See Proposition 3.2, Conjecture 3.3 and Proposition 3.4 in the first paper \([1]\).) In this situation, the integral transformation \( I(\alpha) \) (and also for the difference operator \( D \)) has Jordan blocks.

The generalized eigenfunctions can be constructed as follows. Let \( X = X(u) \) be an operator (acting on \( F_n \)) depending on a parameter \( u \). Assume we have the eigenvalues \( \lambda(u), \mu(u) \) and the eigenfunctions \( f(u), g(u) \)

\[
X(u)f(u) = \lambda(u)f(u), \quad X(u)g(u) = \mu(u)g(u),
\]

and the expansions

\[
X(u) = X(0) + X(1)u + \cdots, \quad \lambda(u) = \lambda(0) + \lambda(1)u + \cdots, \quad \mu(u) = \mu(0) + \mu(1)u + \cdots, \quad f(u) = f(0) + f(1)u + \cdots, \quad g(u) = g(0) + g(1)u + \cdots,
\]

together with the degeneration condition \( \lambda(0) = \mu(0) \). We also assume that the dimension of the space of solution to the equation \( X(0)f = \lambda(0)f \) is one. Comparing the coefficients of the above equations in \( u \), we have

\[
X(0)f[-1] = \lambda(0)f[-1], \quad X(0)f[0] + X(1)f[-1] = \lambda(0)f[0] + \lambda(1)f[-1], \quad X(0)g[0] = \mu(0)g[0], \quad X(0)g[1] + X(1)g[0] = \mu(0)g[1] + \mu(1)g[0],
\]

and so on. Hence we have the proportionality \( f[-1] = cg[0] \) and the generalized eigenfunctions as

\[
X(0)(f[0] - cg[1]) = \lambda(0)(f[0] - cg[1]) + (\lambda[1] - \mu[1])cg[0].
\]

The generalized eigenfunctions of \( I(\alpha) \) (or \( D \)) can be obtained in the above manner. More degenerate cases such as \( s_i = s_j = \cdots = s_k \) can be treated in a similar manner.

Let us introduce some notations. Let \( \Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\} \) be the set of simple roots for \( A_{n-1} \), and \( Q_+ \) be the positive cone of the root lattice. We
identify the index for the eigenfunctions with the element in $Q_+$ in the natural way. We allow to use the same symbol also for the generalized eigenfunctions, for simplicity. We write $f_{i,j}(\zeta_1, \zeta_2, \zeta_3) = f_\alpha(\zeta_1, \zeta_2, \zeta_3)$ for $\alpha = i\alpha_1 + j\alpha_2$, for example, and we have the generalized eigenfunction $f_\alpha$ satisfying

$$I(\alpha)f_\alpha = \lambda_\alpha f_\alpha + \nu_\alpha(\alpha)f_{\alpha + \alpha_2},$$

when $s_2 = s_3$ (thus $\lambda_{\alpha_1} = \lambda_{\alpha_1 + \alpha_2}$), and so on.

Let us denote the Weyl chamber by $C(\Delta) = \{x | (x, \alpha_i) \geq 0, \alpha_i \in \Delta\}$. From the explicit formulas for the eigenfunctions (see Proposition 3.2, Conjecture 3.3 and Proposition 3.4 of [1]), we observe the following structures for the eigenfunctions of $I(\alpha)$ (and also for $D$) on the space of series $\mathcal{F}_n$.

**Conjecture 5.1** Let $\alpha \in Q_+$. If and only if $\alpha \in C(\Delta)$, the eigenfunction $f_\alpha$ remains finite at the homogeneous limit $s_1 = s_2 = \cdots = s_n = 1$. Otherwise, $f_\alpha$ becomes divergent at the homogeneous limit, and a generalized eigenfunction occurs.

**Conjecture 5.2** For the homogeneous case $s_1 = s_2 = \cdots = s_n = 1$, every eigenspace $V_\lambda$ of $I(\alpha)$ or $D$ is uniquely characterized by an element $\alpha \in Q_+$ which is in the Weyl chamber $\alpha \in C(\Delta)$ as

$$V_\lambda = \bigoplus_{\sigma \in W(\Delta_{n-1}), \sigma(\alpha) \in Q_+} C f_{\sigma(\alpha)},$$

where $f_\alpha$ denote the (generalized) eigenfunctions.

**Conjecture 5.3** Even in the non diagonalizable cases, the commutativity among the integral transformations $[I(\alpha), I(\beta)] = 0$ still holds.

### 5.1 example

To show an example for the explicit formula of the generalized eigenfunctions, we treat the simplest case. As we have noted, the integral transformation $I(\alpha)$ and the difference operator $D$ are diagonalizable on the space of series $\mathcal{F}_2$, even if we specialise as $s_1 = s_2$. However, if we allow negative powers in
\( \zeta_2/\zeta_1 \) and consider the action of \( I(\alpha) \) or \( D \) on \( \mathcal{F}_2[\zeta_1/\zeta_2] \), the situation changes and we have to consider the generalized eigenfunctions at the limit \( s_1 = s_2 \).

The eigenfunctions of \( I(\alpha) \) or \( D \) on the extended space \( \mathcal{F}_2[\zeta_1/\zeta_2] \) is given by \((\zeta_2/\zeta_1)^\alpha T_{q^s,s_2}T_{q^{-s},s_2}f_0(\zeta_1, \zeta_2) \) \( (i \in \mathbb{Z}) \), where \( f_0 \) is given in Eq. (15). The explicit formulas for the generalized eigenfunctions can be obtained by using the above stated method and the following.

**Lemma 5.4** For \( i = 0, 1, 2, \cdots \),

\[
2\phi_1 \left( \frac{qt^{-1}, q^{1+2i}t^{-1}s}{q^{1+2i}}; q, t\zeta \right) = 2\phi_1 \left( \frac{qt^{-1}, q^{1+2i}t^{-1}}{q^{1+2i}}; q, t\zeta \right) + \sum_{n=0}^{\infty} \binom{qt^{-1}; q}{q^{1+2i}; q} \binom{qt^{-1}; q}{q^{1+2i}; q} \binom{qt^{-1}; q}{q^{1+2i}; q} (t\zeta)^n
\]

\[
\times \sum_{k=1}^{n} \left[ \frac{1}{1-q^{-2i-k}} - \frac{1}{1-q^{-2i-k}} \right] + O((1-s)^2) \tag{30}
\]

and for \( i = 1, 2, \cdots \),

\[
2\phi_1 \left( \frac{qt^{-1}, q^{1-2i}t^{-1}s}{q^{1-2i}}; q, t\zeta \right) = \frac{1}{1-s} \binom{qt^{-1}; q}{q^{1-2i}; q} \binom{qt^{-1}; q}{q^{1-2i}; q} \binom{qt^{-1}; q}{q^{1-2i}; q} (t\zeta)^n
\]

\[
\times \sum_{n=0}^{\infty} \binom{qt^{-1}; q}{q^{1-2i}; q} \binom{qt^{-1}; q}{q^{1-2i}; q} \binom{qt^{-1}; q}{q^{1-2i}; q} (t\zeta)^n
\]

\[
\times \sum_{k=1}^{n} \left[ \frac{1}{1-q^{2i-k}} - \sum_{k=1}^{n} \frac{1}{1-q^{2i-k}} \right] + O(1-s) \tag{31}
\]

hold.
6 Quasi-Eigenfunction of the Integral Transformation $I(\alpha)$

One of the important characteristics of the basic hypergeometric series $_2\phi_1$ is the existence of various infinite product expressions

\[ _2\phi_1 \left( \frac{a, b}{b}; q, z \right) = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad (32) \]
\[ _2\phi_1 \left( \frac{a^2, aq}{a}; q, z \right) = (1 + az)\frac{(a^2 qz; q)_\infty}{(z; q)_\infty}, \quad (33) \]
\[ _2\phi_1 \left( \frac{a, -a}{-q}; q, z \right) = \frac{(a^2 z; q^2)_\infty}{(z; q^2)_\infty}, \quad (34) \]

and so on. We have studied a product formula for the eigenfunctions of $I(\alpha)$ in Section 4. One may find, however, that possible variety of infinite product formulas for the eigenfunctions is not rich enough compared with that of $_2\phi_1$.

In this section, we propose a one parameter family of 'quasi-eigenfunctions' of the integral transformation $I(\alpha)$, and study its properties. It will be pointed out in the next section that a class of infinite product(-type) formulas exists for the quasi-eigenfunctions.

6.1 definition of the quasi-eigenfunction $F(\alpha)$

From now on, we work with the homogeneous condition

\[ s_1 = s_2 = \cdots = s_n = 1, \quad (35) \]

and we work with the operator $I(\alpha) = I(\alpha; 1, 1, \cdots, 1, q, t)$.

We introduce the ‘quasi-eigenfunction $F(\alpha)$’ as follows.

**Definition 6.1** Set the parameters as $s_1 = s_2 = \cdots = s_n = 1$. The quasi-eigenfunction $F(\alpha) = F(\zeta_1, \zeta_2, \cdots, \zeta_n; \alpha, q, t)$ is defined by the ‘covariant transformation property’

\[ (I) \quad I(\alpha q^{-1} t) \cdot F(\alpha) = F(\alpha q^{-1} t). \quad (36) \]

and the ‘initial condition’

\[ (II) \quad F(t^{1/2}) = \prod_{1 \leq i < j \leq n} \frac{(1 - \zeta_j / \zeta_i)(qt^{-1/2} \zeta_j / \zeta_i; q)_\infty}{(t^{1/2} \zeta_j / \zeta_i; q)_\infty}. \quad (37) \]
Here, an explanation is in order. First, $F(\alpha)$ can be constructed at $\alpha = t^{1/2}, q^{-1}t^{3/2}, q^{-2}t^{5/2}, \cdots$ by the iterative action of $I(\alpha)$ as

$$F(q^{-1}t^{3/2}) = I(q^{-1}t^{3/2}) \cdot F(t^{1/2}),$$

and so on. Then the function $F(\alpha)$ is obtained by the analytic continuation with respect to the parameter $\alpha$ from these discrete points. One can check that the above definition of $F(\alpha)$ is well defined for small degrees in $\zeta$, by performing explicit analytic continuation of the coefficients.

It is expected that the function $F(\alpha)$ satisfies another transformation property which is similar to the condition (I).

**Conjecture 6.2** The function $F(\alpha)$ satisfies the transformation property

$$(I') \quad I(\alpha^{-1}q) \cdot F(\alpha) = F(\alpha q^{-1}t).$$

This can be checked up to certain order. Note that, the both transformation properties (I) and (I') will be needed in the next paper, while we study the vertex operator of the eight-vertex model [2].

### 6.2 explicit formula of $F(\alpha)$ for $n = 2$

For the case $n = 2$, one can easily obtain an explicit formula of $F(\alpha)$.

**Proposition 6.3** Let $F(\alpha)$ be the following $2\phi_1$ (or $4\phi_3$) series:

$$F(\alpha) = F(\zeta_1, \zeta_2; \alpha, q, t)$$

$$= (1 - \zeta_2/\zeta_1)_{2\phi_1} \left( \frac{qt^{-1}, \alpha qt^{-1}}{\alpha^{-1} q}; q, \alpha^{-1} t \zeta_2/\zeta_1 \right)$$

$$= 4\phi_3 \left( \frac{qt^{-\frac{1}{2}}, -qt^{-\frac{1}{2}}, t^{-1}, \alpha t^{-1}}{t^{-\frac{1}{2}}, -t^{-\frac{1}{2}}, \alpha^{-1} q}; q, \alpha^{-1} t \zeta_2/\zeta_1 \right).$$

Then this $F(\alpha)$ satisfies the transformation property (I) and the initial condition (II) in Definition 6.1 for $n = 2$. 

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\textbf{Proof.} The initial condition (II) can be checked by using the \(q\)-binomial theorem (Eq. (1.3.2) of GR [14]):

\[
F(t^{1/2}) = (1 - \zeta_2/\zeta_1) \frac{(qt^{-1/2}\zeta_2/\zeta_1; q)_{\infty}}{(t^{1/2}\zeta_2/\zeta_1; q)_{\infty}}.
\]

Next, let \(f_i(\zeta_1, \zeta_2)\)'s be the eigenfunctions of \(I(\alpha)\) given in Eq.(17) of [1], and

\[
\lambda_i(\alpha) = \frac{(\alpha^{-1}s_1t; q)_i}{(\alpha^{-1}s_1q; q)_i} \frac{(\alpha s_2^{-1}; q)_i}{(\alpha s_2^{-1}qt^{-1}; q)_i} (qt^{-1})^i,
\]

be the corresponding eigenvalues (see Eq.(16) of [1]). One finds that the series Eq.(41) can be neatly expanded in terms of the eigenfunctions \(f_i(\zeta_1, \zeta_2)\) (see Lemma 6.4 given below). Therefore the transformation property (I) can be shown as

\[
I(\alpha q^{-1}t) \cdot F(\zeta_1, \zeta_2; \alpha, q, t) = \sum_{i=0}^{\infty} f_i(\zeta_1, \zeta_2) \lambda_i(\alpha q^{-1}t) \frac{\alpha; q)_i}{(\alpha^{-1}q; q)_i} \frac{(qt^{-1}, q^{i+1}t^{-1}; q)_i}{(q, q^i; q)_i} \alpha^{-i} t^i.
\]

Note that the other transformation property (I') given in Conjecture 6.2 can also be checked.

It remains to examine the the eigenfunction expansion of \(F(\alpha)\). This is accomplished by using a summation formula for \(\phi_5\) series.

\textbf{Lemma 6.4} Set \(s_1 = s_2 = 1\). With respect to the eigenfunctions \(f_i(\zeta_1, \zeta_2)\) of \(I(\alpha)\), the function \(F(\zeta_1, \zeta_2; \alpha, q, t)\) given in Eq.(41) is expanded as

\[
F(\zeta_1, \zeta_2; \alpha, q, t) = \sum_{i=0}^{\infty} f_i(\zeta_1, \zeta_2) \frac{(\alpha; q)_i}{(\alpha^{-1}q; q)_i} \frac{(qt^{-1}, q^{i+1}t^{-1}; q)_i}{(q, q^i; q)_i} \alpha^{-i} t^i.
\]

\textbf{Proof.} From the explicit formula (given in Eq.(17) of [1]) for \(s_1 = s_2 = 1\)

\[
f_j(\zeta_1, \zeta_2) = \zeta^j \times 4\phi_3 \left( \begin{array}{c} q^{2jt^{-1}}, q^{j+1}t^{-1}, -q^{j+1}t^{-1}, t^{-1} \\ q^{jt^{-1}}, q^{j+1}t^{-1}, q^{2j+1} \\ \end{array} ; q, t \zeta \right),
\]

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we have the expansion with respect to the monomials
\[ f_j(\zeta_1, \zeta_2) = \sum_{i=0}^{\infty} c_{ij}(\zeta_2/\zeta_1)^i, \]
\[ c_{ij} = \begin{cases} 
\frac{(q^{2j+1}t^{-1}, q^{j+1}t^{-\frac{1}{2}}, -q^{j+1}t^{-\frac{1}{2}}, t^{-1}; q)_{i-j}t^{i-j}}{(q^{2j+1}, q^{j+1}t^{-\frac{1}{2}}, -q^{j+1}t^{-\frac{1}{2}}, q; q)_{i-j}} (i \geq j), \\
0 (i < j).
\end{cases} \]
where we have denoted \( \zeta = \zeta_2/\zeta_1 \) (see Eq.(30) of [1]). Writing
\[ F(\zeta_1, \zeta_2; \alpha, q, t) = \sum_{i=0}^{\infty} a_i(\zeta_2/\zeta_1)^i = \sum_{i=0}^{\infty} f_i(\zeta_1, \zeta_2)b_i, \]
we have the matrix equation \( a = Cb \), where \( a = \{a_0, a_1, \ldots\} \), \( b = \{b_0, b_1, \ldots\} \) and \( C = (c_{ij})_{0 \leq i, j < \infty} \) is the lower triangular matrix defined as above. Therefore we have \( b = C^{-1}a \).

In Proposition 2.7 of [1], we have obtained the entries for the inverse \( C^{-1} = (d_{ij})_{0 \leq i, j, \infty} \) as
\[ d_{ij} = \begin{cases} 
\frac{(q^{i+j+1}t^{-1}, t; q)_{i-j}}{(q^{i+j}, q; q)_{i-j}} (i \geq j), \\
0 (i < j). \end{cases} \] (44)
From the expression in Eq.(11), the coefficients \( a_i \)'s read
\[ a_j = \frac{(qt^{-\frac{1}{2}}, -qt^{-\frac{1}{2}}, t^{-1}, \alpha t^{-1}; q)_j}{(t^{-\frac{1}{2}}, -t^{-\frac{1}{2}}, \alpha^{-1}q, q; q)_j} (\alpha^{-1}t)^j. \]
Hence we arrive at the formula for \( b_i \)'s as follows
\[ b_i = \sum_{j=0}^{i} d_{ij}a_j = \frac{(t, q^{i+1}t^{-1}; q)_i}{(q, q^i; q)_i} \sum_{j=0}^{i} \frac{(q^{-i}, q^j; q)_j}{(q^{-i+1}t^{-1}, q^{i+1}t^{-1}; q)_j} (qt^{-1})^j a_j = \frac{(t, q^{i+1}t^{-1}; q)_i}{(q, q^i; q)_i} \times \phi_5 \left( \frac{qt^{-\frac{1}{2}}, -qt^{-\frac{1}{2}}, q^{-i}, q^i, t^{-1}, \alpha t^{-1}}{t^{-\frac{1}{2}}, -t^{-\frac{1}{2}}, q^{i+1}t^{-1}, q^{-i+1}t^{-1}, \alpha^{-1}q; \alpha^{-1}q} \right) \]
Here we have used the summation formula for the $\phi_5$ (Eq. (2.4.2) of GR [14]).

\section{Conjectural Form of $F(\alpha)$ for $n = 3$}

Working with the monomial basis, one can perform the iterative actions of $I(\alpha)$ (at least for small degrees in $\zeta$) as in Eq. (38), (39) etc., and study the analytic continuation with respect to the variable $\alpha$. Then an explicit expression of the quasi-eigenfunction $F(\alpha)$ for $n = 3$ is guessed as follows.

**Conjecture 6.5** The quasi-eigenfunction for $n = 3$ is written as

$$F(\zeta_1, \zeta_2, \zeta_3; \alpha, q, t) = \sum_{k=0}^{\infty} \frac{(\alpha^{2}t, qt^{-1}, q; \alpha^{-1}q^{-1}; q)_k}{(q, \alpha^{-1}q, \alpha^{-1}q; q)_k} (q \zeta_3/\zeta_1)_k \frac{\phi_1(\alpha^{-1}, q^{-k} ; \alpha t^{-1}; q, \alpha t)}{\alpha q^{-k+1} \zeta_3/\zeta_1}.$$

When $F(\alpha)$ is expanded in terms of the generalized eigenfunctions, we observe the following.

**Conjecture 6.6** We have the embedding

$$F(\alpha) \in \bigoplus_{k=0}^{\infty} V_{\lambda_{k\theta}},$$

where $\theta = \alpha_1 + \alpha_2$ denotes the maximal root for $A_2$.

This means that the initial condition (II) given in Definition 6.1 is quite restrictively chosen.
6.4 partial result for $n = 4$

For $n = 4$, a brute force calculation gives us the following observation.

**Conjecture 6.7** The quas-eigenfunction $F(\zeta_1, \zeta_2, \zeta_3, \zeta_4; \alpha, q, t)$ is given by the series

$$F(\zeta_1, \zeta_2, \zeta_3, \zeta_4; \alpha, q, t) = \sum_{k=0}^{\infty} Y_{k,0} + \sum_{k=0}^{\infty} Y_{k+1,1} + \sum_{k=0}^{\infty} Y_{k+1,1} + \cdots,$$

on the subspace of $\mathcal{F}_4$ spanned by the monomials

$$\left(\frac{\zeta_2}{\zeta_1}\right)^{i_1} \left(\frac{\zeta_3}{\zeta_2}\right)^{i_2} \left(\frac{\zeta_4}{\zeta_3}\right)^{i_3} \quad (0 \leq i_1 < \infty, 0 \leq i_2 < \infty, 0 \leq i_3 \leq 1).$$

Here the series $Y_{i,j,k}$ are defined as follows:

$$Y_{k,0} = q^k \left(\frac{\zeta_3}{\zeta_1}\right)^k \frac{(\alpha^{-2}t)k(qt^{-1})k(qt^{-1})k}{(q)_k(\alpha^{-1}q)_k(\alpha^{-1}q)_k} 2\phi_1\left(\frac{\alpha^{-1}, q^{-k}}{\alpha q^{-k+1}; q, \alpha t}\right) \times \phi(k, k, k, 0, 0), \quad (for \ k = 0, 1, 2, \cdots),$$

$$Y_{0,1,1} = q \left(\frac{\zeta_4}{\zeta_2}\right) \frac{(\alpha^{-2}t)1(qt^{-1})_1(qt^{-1})_1}{(q)_1(\alpha^{-1}q)_1(\alpha^{-1}q)_1} 2\phi_1\left(\frac{\alpha^{-1}, q^{-1}}{\alpha; q, \alpha t}\right) \times \phi(0, 1, 1, 0, 1, 0),$$

$$Y_{1,1,1} = q \left(\frac{\zeta_4}{\zeta_1}\right) \frac{(\alpha^{-2}t)1(qt^{-1})_1(qt^{-1})_1}{(q)_1(\alpha^{-1}q)_1(\alpha^{-1}q)_1} 2\phi_1\left(\frac{\alpha^{-1}, q^{-1}}{\alpha; q, \alpha t}\right) \times \phi(1, 1, 1, 0, 1, 1, 1),$$

$$Y_{k,k+1,1} = -q \left(\frac{\zeta_3}{\zeta_1}\right)^k \left(\frac{\zeta_4}{\zeta_2}\right) \frac{(\alpha^{-2}t)k(qt^{-1})k(qt^{-1})k}{(q)_{k-1}(q)_1(\alpha^{-1}q)_k(\alpha^{-1}q)_k(\alpha^{-1}q)_1}$$

$$\times \phi(1, 1, 1, 1, 1, 1, 1).$$

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\[ \times 2\phi_1 \left( \frac{\alpha^{-1}, q^{-1}}{\alpha}; q, \alpha t \right) \left[ \frac{(1 - \alpha)q^k}{(1 - q^k)\alpha} 2\phi_1 \left( \frac{\alpha^{-1}, q^{-k}}{\alpha q^{-k+1}}; q, \alpha qt \right) \right] \\
+ \frac{\alpha - q^k}{(1 - q^k)\alpha} 2\phi_1 \left( \frac{\alpha^{-1}, q^{-k}}{\alpha q^{-k+1}}; q, \alpha qt \right) \times \phi(k, k, 1, k, 1, 1) \]

\[ + q \left( \frac{\zeta_3}{\zeta_1} \right)^k \left( \frac{\zeta_4}{\zeta_1} \right)^k \left( \frac{\alpha^{-2} t}{\zeta_1} \right)^k \left( \frac{(q^{-2} t)_{k+1}(q t^{-1})_k(q t^{-1})_k}{(q)_{k+1}(q^{-1})_k(q^{-1})_k(q^{-1})_k} \right) \times 2\phi_1 \left( \frac{\alpha^{-1}, q^{-1}}{\alpha}; q, \alpha t \right) \left[ \frac{(1 - \alpha)q^{k+1}}{(1 - q^{k+1})\alpha} 2\phi_1 \left( \frac{\alpha^{-1}, q^{-k+1}}{\alpha q^{-k+1}}; q, \alpha t \right) \right] \\
+ \frac{\alpha - q^{k+1}}{(1 - q^{k+1})\alpha} 2\phi_1 \left( \frac{\alpha^{-1}, q^{-k+1}}{\alpha q^{-k+1}}; q, \alpha t \right) \times \phi(k + 1, k + 1, k + 1, 1, 1, 1) \]

\[ = -q \left( \frac{\zeta_3}{\zeta_1} \right)^k \left( \frac{\zeta_4}{\zeta_1} \right)^k \left( \frac{\alpha^{-2} t}{\zeta_1} \right)^k \left( \frac{(q^{-2} t)_{k+1}(q t^{-1})_k(q t^{-1})_k}{(q^{-1})_{k+1}(q^{-1})_k(q^{-1})_k(q^{-1})_k} \right) \times 2\phi_1 \left( \frac{\alpha^{-1}, q^{-1}}{\alpha}; q, \alpha t \right) \left[ \frac{(1 - \alpha)q^k}{(1 - q^k)\alpha} 2\phi_1 \left( \frac{\alpha^{-1}, q^{-k}}{\alpha q^{-k+1}}; q, \alpha t \right) \right] \\
+ \frac{\alpha - q^k}{(1 - q^k)\alpha} 2\phi_1 \left( \frac{\alpha^{-1}, q^{-k}}{\alpha q^{-k+1}}; q, \alpha t \right) \times \phi(k, k, 1, k, 1, 1) \]

\[ + q \left( \frac{\zeta_3}{\zeta_1} \right)^k \left( \frac{\zeta_4}{\zeta_1} \right)^k \left( \frac{\alpha^{-2} t}{\zeta_1} \right)^k \left( \frac{(q^{-2} t)_{k+1}(q t^{-1})_k(q t^{-1})_k}{(q)_{k+1}(q^{-1})_k(q^{-1})_k(q^{-1})_k} \right) \times 2\phi_1 \left( \frac{\alpha^{-1}, q^{-1}}{\alpha}; q, \alpha t \right) \left[ \frac{(1 - \alpha)q^{k+1}}{(1 - q^{k+1})\alpha} 2\phi_1 \left( \frac{\alpha^{-1}, q^{-k+1}}{\alpha q^{-k+1}}; q, \alpha t \right) \right] \\
\times \phi(k, k, 1, k + 1, 1, 1) + \phi(k + 1, k, 1, k, 1, 1), \quad (\text{for } k = 1, 2, 3, \cdots), \]
were, we have used the notations \((a)_k = (a; q)_k\) and
\[
\phi(k_{12}, k_{23}, k_{34}, k_{13}, k_{24}, k_{14}) = \prod_{1 \leq i < j \leq 4} (1 - \zeta_j / \zeta_i) \cdot 2\phi_1\left(q^{k_{ij}+1}t^{-1}, \alpha qt^{-1} \alpha^{-1}q^{k_{ij}+1}q; \alpha^{-1}t \zeta_j / \zeta_i\right).
\]

### 7 Product Formulas for the Quasi-Eigenfunction \(F(\alpha)\)

The aim of this section is to present several conjectures for infinite product(-type) formulas for the quasi-eigenfunction \(F(\alpha)\) at some particular values of \(\alpha\). One realizes that these can be regarded as multi variable generalizations of the product formulas for \(2\phi_1\) given in Eqs. (32)-(34).

#### 7.1 case \(\alpha = -t^{1/2}\) and \(\alpha = t\)

We observe that our quasi-eigenfunction \(F(\alpha)\) neatly factorizes into product expressions if we specialize the parameter \(\alpha\) to \(-t^{1/2}\) or \(t\).

**Conjecture 7.1** The infinite product formulas

\[
F(-t^{1/2}) = \prod_{1 \leq i < j \leq n} (1 - \zeta_j / \zeta_i) \frac{(-qt^{-1/2}\zeta_j / \zeta_i; q)_\infty}{(-t^{1/2}\zeta_j / \zeta_i; q)_\infty},
\]

\[
F(t) = \prod_{1 \leq i < j \leq n} (1 - \zeta_j / \zeta_i) \frac{(qt^{-1}\zeta_j / \zeta_i; q)_\infty}{(t\zeta_j / \zeta_i; q)_\infty},
\]

hold. Here, we have used the notation \(\prod_{1 \leq i < j \leq n} f_{ij} = f_{13}f_{15} \cdots f_{24}f_{26} \cdots\).

For the case \(n = 2\), we can prove the factorization.

**Proposition 7.2** Conjecture 7.1 is true for \(n = 2\).

**Proof.** From the explicit formula Eq. (11) and the \(q\)-binomial theorem, we have

\[
F(-t^{1/2}) = (1 - \zeta_2 / \zeta_1) \frac{(-qt^{-1/2}\zeta_2 / \zeta_1; q)_\infty}{(-t^{1/2}\zeta_2 / \zeta_1; q)_\infty}, \quad F(t) = 1.
\]
Next, we prove the factorization for the conjectural expression of $F(\alpha)$ for $n = 3$.

**Proposition 7.3** Conjecture [6.5] is true for $n = 3$, under the assumption that the formula for $F(\alpha)$ given in Conjecture [6.5] is correct.

**Proof.** Setting $\alpha = -t^{1/2}$ in Eq.(45), the summand becomes zero for $k > 0$. Then using the $q$-binomial theorem, we have

$$F(-t^{1/2}) = \prod_{1 \leq i < j \leq 3} \frac{(1 - \zeta_j/\zeta_i)(-qt^{-1/2}\zeta_j/\zeta_i; q)_\infty}{(-t^{1/2}\zeta_j/\zeta_i; q)_\infty}.$$

We proceed to proving the case $\alpha = t$. First, note that we have

$$2\phi_1\left(q^{k+1}t^{-1}, \alpha qt^{-1} \alpha^{-1}q^{k+1}; q, \alpha^{-1}t\zeta \right) \bigg|_{\alpha = t} = \frac{1}{1 - \zeta}.$$

Next, using Jackson’s transformation formula (Eq.(iii) in Exercise 1.15 of GR [14])

$$2\phi_1\left(q^{-n}, b \frac{c}{c}; q, z \right) = \frac{(c/b; q)_n}{(c; q)_n} 3\phi_2\left(q^{-n}, b, bq^{-n}/c; bq^{1-n}/c, 0 ; q, q \right),$$

we have

$$2\phi_1\left(\alpha^{-1}, q^{-k} \alpha q^{-k+1}; q, \alpha t \right) \bigg|_{\alpha = t} = q^{-k}t^{k} \frac{(t^{-2}; q)_k 1 + t^{-1}q^k}{(t^{-1}; q)_k 1 + t^{-1}}.$$

Therefore from Eq. (45) we have

$$F(\alpha) \bigg|_{\alpha = t} = \sum_{k=0}^{\infty} \frac{(t^{-1}; q)_k (t^{-2}; q)_k 1 + t^{-1}q^k}{(q; q)_k (t^{-1}; q)_k 1 + t^{-1}} (t\zeta_3/\zeta_1)_k$$

$$= \frac{1}{1 + t^{-1}} \frac{(t^{-1}\zeta_3/\zeta_1; q)_\infty + t^{-1}(qt^{-1}\zeta_3/\zeta_1; q)_\infty}{(t\zeta_3/\zeta_1; q)_\infty}$$

$$= (1 - \zeta_3/\zeta_1) \frac{(qt^{-1}\zeta_3/\zeta_1; q)_\infty}{(t\zeta_3/\zeta_1; q)_\infty}.$$

For the case $n = 4$, one can check the factorization by using the partial result given in Conjecture [6.7]. We expect that Conjecture [7.1] is true for general $n$. □
7.2 case $\alpha = \pm q^{1/2}t^{1/2}$

In the case $\alpha = \pm q^{1/2}t^{1/2}$, we observe that $F(\alpha)$ can be written as a multiple of a Pfaffian and an infinite product.

**Conjecture 7.4** Let $n$ be a positive even integer. At $\alpha = q^{1/2}t^{1/2}$, the expression

$$
F(q^{1/2}t^{1/2}) = \text{Pfaffian} \left( \frac{1 - \zeta_i^2 / \zeta_i^2}{(1 - q^{-1/2}t^{1/2} \zeta_j / \zeta_i)(1 - q^{1/2}t^{-1/2} \zeta_j / \zeta_i)} \right)_{1 \leq i, j \leq n}
$$

\begin{equation}
\times \prod_{1 \leq i < j \leq n} \frac{(q^{1/2}t^{-1/2} \zeta_j / \zeta_i; q)_{\infty}}{(q^{1/2}t^{1/2} \zeta_j / \zeta_i; q)_{\infty}},
\end{equation}

hold. The formula for odd $n$ is obtained by taking the limit $\zeta_n \to 0$.

Note the case $\alpha = -q^{1/2}t^{1/2}$ is obtained by negating $t^{1/2}$ in the above formula.

Let us examine the above statement for small $n$.

**Proposition 7.5** Conjecture 7.4 is true for $n = 2$.

**Proof.** From the explicit formula Eq.(41) and the product formula Eq.(33), we have

$$
F(q^{1/2}t^{1/2}) = \frac{1 - \zeta_i^2 / \zeta_i^2}{(1 - q^{-1/2}t^{1/2} \zeta_j / \zeta_i)(1 - q^{1/2}t^{-1/2} \zeta_j / \zeta_i)}
$$

\begin{equation}
\times \frac{(q^{1/2}t^{-1/2} \zeta_j / \zeta_i; q)_{\infty}}{(q^{1/2}t^{1/2} \zeta_j / \zeta_i; q)_{\infty}}.
\end{equation}

Next, we prove the product-Pfaffian formula for the conjectural expression of $F(\alpha)$ for $n = 3$.

**Proposition 7.6** Conjecture 7.4 is true for $n = 3$, under the assumption that the formula for $F(\alpha)$ given in Conjecture 6.5 is correct.
**Proof.** Setting $\alpha = q^{1/2}t^{1/2}$ in Eq.(45), the summand becomes zero for $k \geq 2$. Note that we have

$$
2\phi_1 \left( \left( \frac{q^{k+1}t^{-1}}{\alpha^{-1}q^{k+1}} : q, \alpha^{-1}t\zeta \right) \bigg|_{\alpha=q^{1/2}t^{1/2}} = (1 + \zeta)_{\left( q^{3/2}t^{-1/2} \zeta; q \right)_\infty},
$$

$$
2\phi_1 \left( \left( \frac{q^{k+1}t^{-1}}{\alpha^{-1}q^{k+1}} : q, \alpha^{-1}t\zeta \right) \bigg|_{\alpha=q^{1/2}t^{1/2}} = \frac{(q^{3/2}t^{-1/2} \zeta; q)_\infty}{(q^{-1/2}t^{-1/2} \zeta; q)_\infty},
$$

from the $q$-binomial theorem and the product formula Eq.(33). Using these and simplifying the rational factor in front of the infinite product, we have

$$F(q^{1/2}t^{1/2}) = \left( 1 - \frac{\zeta_2^2}{\zeta_1^2} \right) \cdot \left( 1 - \frac{\zeta_3^2}{\zeta_2^2} \right) \cdot \left( 1 - \frac{\zeta_4^2}{\zeta_3^2} \right) \cdot \ldots

\times \prod_{1 \leq i < j \leq 3} \left( \frac{(q^{1/2}t^{-1/2} \zeta_j \zeta_i; q)_\infty}{(q^{1/2}t^{1/2} \zeta_j \zeta_i; q)_\infty} \right).

$$

The case $n = 4$ can be examined by using the partial result given in Conjecture 6.7.

**7.3 case $\alpha = \pm q^\ell t^{1/2}$**

Let us move on to the case $\alpha = \pm q^\ell t^{1/2}$ ($\ell = 1, 2, \ldots$). For $\alpha = \pm q^{1/2}t^{1/2}$, it was argued that the Pfaffian appears in front of the infinite product. For $\alpha = \pm q^\ell t^{1/2}$ ($\ell = 1, 2, \ldots$), we have another rational expression.

Introduce some notations to describe the structure of the rational factor. Let $J$ be the index set $J = \{ \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_\ell) | \sigma_i = \pm \}$, and introduce $2^\ell \times 2^\ell$ matrices

$$G_\ell(\zeta) = (\gamma_{\ell,\sigma,\sigma'}(\zeta))_{\sigma,\sigma' \in J},

$$

by the following recursive rule.
Definition 7.7 Set the entries of the matrix $G_1(z)$ as

\[
\begin{align*}
\gamma_{1,+}(\zeta) &= 1, \\
\gamma_{1,-}(\zeta) &= \frac{1 - qt^{-1}\zeta}{1 - \zeta} \cdot \frac{1 + t^{1/2}\zeta}{1 + q^{-1}t^{1/2}\zeta}, \\
\gamma_{1,-,+}(\zeta) &= \frac{1 - qt^{-1}\zeta}{1 - \zeta} \cdot \frac{1 + t^{1/2}\zeta}{1 + q^{-1}t^{1/2}\zeta}, \\
\gamma_{1,-,-}(\zeta) &= 1.
\end{align*}
\]

Then define the matrices $G_\ell(z)$ for $\ell \geq 2$ recursively by

\[
G_\ell(\zeta) = \begin{pmatrix}
G_{\ell-1}(\zeta) & \gamma_{1,+}(\zeta) G_{\ell-1}(\zeta q) \\
\gamma_{1,-,+}(\zeta) G_{\ell-1}(\zeta q^{-1}) & G_{\ell-1}(\zeta)
\end{pmatrix}.
\]

Using this notation, our observation can be stated as follows.

Conjecture 7.8 Let $n \geq 2$ denote the number of variables for the quasi-eigenfunction $F(\alpha)$. Let $\ell$ be a positive integer, and $G_\ell(\zeta)$ be the matrix of rational functions in $\zeta$ given as above. Set

\[
\mu_\sigma = \prod_{i=1}^\ell \left(q^{(i-1)/2}t^{1/4}\right)^{\sigma_i},
\]

for $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_\ell) \in J$. Then the following expression for the quasi-eigenfunction for $\alpha = -q^\ell t^{1/2}$ holds:

\[
F(-q^\ell t^{1/2}) = \left(\sum_{\sigma \in J} \mu_\sigma\right)^{-n} \prod_{1 \leq i < j \leq n} \frac{(1 - \zeta_j/\zeta_i)\left(-qt^{-1/2}\zeta_j/\zeta_i; q\right)_\infty}{(-t^{1/2}\zeta_j/\zeta_i; q)_\infty} \times \prod_{\sigma_1, \sigma_2, \cdots, \sigma_n \in J} \prod_{1 \leq i < j \leq n} \gamma_{t,\sigma_i,\sigma_j}(\zeta_j/\zeta_i).
\]

Note the case $\alpha = q^\ell t^{1/2}$ is obtained by negating $t^{1/2}$ in the above formula.

By a tedious explicit calculation, one can observe the validity of the above conjecture for small $n$. 

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7.4 case $\alpha = -1$

Finally, let us make a short comment on the case $\alpha = -1$. By looking at the explicit formula of $F(\alpha)$ given in Eq.(41), we have

**Proposition 7.9** For $n = 2$, we have the infinite product expression

$$F(-1) = (1 - \frac{\zeta_2}{\zeta_1})\frac{(-q^2t^{-1}t_2/\zeta_1; q^2)\infty}{(-t\zeta_2/\zeta_1; q^2)\infty}.$$  \hspace{1cm} (67)

**Proof.** This follows from the product formula Eq.(34).  \(\square\)

A simple product formula may ‘not’ easily be found for $\alpha = -1$ if $n \geq 3$. Nevertheless, we are interested in the series $F(-1)$ for general $n$, since it is related with the study of the eight-vertex model in a nontrivial manner. We will argue this in some detail in the next paper \[2\].

8 Concluding Remarks

In this paper, we have studied several properties for the integral transformation $I(\alpha)$, which was introduced in the first paper \[1\].

It was observed that $I(\alpha)$ and the Macdonald-type difference operator $D$ are commutative with each other (Conjecture 2.4). The commutativity $[I(\alpha), D] = 0$ was rather unexpected, at least to the present author. We hope that the relationship between the eight-vertex model and the Macdonald polynomials will be investigated further.

Using the explicit formulas for the eigenfunctions, it was observed that the series for the eigenfunction becomes terminating under the conditions $t = q^m$ ($m = 1, 2, \cdots$). For such cases, we found the Weyl group symmetry as a hidden symmetry (Conjecture 3.2). This Weyl group symmetry helps us when we try to construct hypergeometric-type formula for the eigenfunction for general $q$ and $t$.

We have introduced the quasi-eigenfunction $F(\alpha)$ (Definition 6.1), using the action of $I(\alpha)$ for the homogeneous limit $s_1 = \cdots = s_n = 1$. The series $F(\alpha)$ has some resemblance to the first eigenfunction $f_0$. First, if we disregard the dependence on the ‘dynamical’ parameter $\alpha$ in the transformation property $I(\alpha q^{-1}t) \cdot F(\alpha) = F(\alpha q^{-1}t)$, one can compare this with the
equation $I(\alpha) \cdot f_0 = \lambda_0 f_0$. Next, we have similar hypergeometric-type series expressions for $f_0$ and $F(\alpha)$ (for $n = 3$, see Conjecture 6.7 and Conjecture 6.5). Note also that the infinite product formula for the first eigenfunction (Conjecture 6.1) and the initial condition (II) for $F(\alpha)$ given in Definition 6.1 look similar to each other.

For $n = 2$, we have studied $F(\alpha)$ in terms of the eigenfunctions $f_i$ (Proposition 6.3). For $n \geq 3$ (and $s_1 = \cdots = s_n = 1$), however, $I(\alpha)$ is non diagonalizable and our study based on the generalized eigenfunctions becomes very much complicated. Nevertheless, one may find some nontrivial observation as Conjecture 6.6.

In Section 7, we have obtained a variety of infinite product-type formulas for the quasi-eigenfunction $F(\alpha)$. These were checked for small $n$ by using the explicit formulas and several transformation and summation formulas for the basic hypergeometric series.

In the next paper [2], the matrix elements of the vertex operators for the eight-vertex model will be studied in some detail, based on the product(-type) formulas for $F(\alpha)$ obtained in this paper. Our aim there is to construct a class of Heisenberg representations which gives us the following description for the vertex operators:

$$\langle \Phi(\zeta_1)\Phi(\zeta_2)\cdots\Phi(\zeta_n) \rangle = \prod_{1 \leq i < j \leq n} \frac{\xi(z_j/z_i; p, q)}{1 - z_j/z_i} \cdot F(-1, p^{1/2}, q),$$

(68)

where

$$\xi(z; p, q) = \frac{(q^2 z; p, q^4)_\infty (pq^2 z; p, q^4)_\infty}{(q^4 z; p, q^4)_\infty (pz; p, q^4)_\infty}.$$  

(69)

Note that the basic parameters here have been switched to the ones for the eight-vertex model as $q \to p^{1/2} = p^{1/2}_{8v}$ and $t \to q = q_{8v}$.

Let us give some examples. Set $p^{1/2} = q^{3/2}$. From the transformation property (I) in Definition 6.1 and the product formula given in Conjecture 7.1, we have the $n$-fold integral representation for $F(-1, q^{3/2}, q)$ as

$$F(-1, q^{3/2}, q) = I(-1, q^{3/2}, q) \cdot F(-q^{1/2}, q^{3/2}, q).$$

(70)

In the same way, we have the integral representations $F(-1, -q^2, q) = I(-1, -q^2, q) \cdot F(q, -q^2, q)$ for $p^{1/2} = -q^2$, and $F(-1, q^3, q) = I(-1, q^3, q) \cdot F(-p^{1/4} q^{1/2}, q^3, q)$.
for \( p^{1/2} = q^3 \) (see product formulas in Conjecture 7.1 and Conjecture 7.4). These three cases have already been discussed in the previous paper [13].

If we set \( p^{1/2} = q^{(2\ell+1)/2\ell} \) \((\ell = 1, 2, \ldots)\), we obtain the \( \ell \times n \)-fold integral representation of \( F(-1) = F(-1, q^{(2\ell+1)/2\ell}, q) \) by \( I(\alpha) = I(\alpha, q^{(2\ell+1)/2\ell}, q) \) as

\[
F(-1) = I(-1) \cdots I(-pq^{-3/2}) \cdot I(-p^{1/2}q^{-1/2}) \cdot F(-q^{1/2}).
\]

(71)

In this way, one may construct various integral representations for \( F(-1) \) by using the infinite product(-type) formulas given in Conjecture 7.1, Conjecture 7.3, and Conjecture 7.8. In the next paper, we will construct a class of Heisenberg representations of the vertex operator \( \Phi(\zeta) \), which are consistent with the conjectures for \( F(\alpha) \) obtained in this article.

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### A Macdonald Difference Operators

In this appendix, we revisit the Heisenberg representation of the Macdonald difference operators \( D^r_n \) and the Macdonald symmetric function \( Q_\lambda(x; q, t) \). In this appendix, we use the standard notations for the Macdonald polynomials [8]. Note that \( q \) and \( t \) in this appendix are the ordinary parameters for the Macdonald polynomials, and should not be confused with the ones for the eight-vertex model \( p_{8v} \) and \( q_{8v} \).

#### A.1 commuting operators on the Fock space

The Macdonald difference operators (see [8]) acting on the ring of symmetric polynomials \( \Lambda_{n,F} \) (\( F = Q(q, t) \)) are defined by

\[
D^r_n = \sum_{I \subseteq \{1, 2, \ldots, n\}, \sum I = r} A_I(x; t) \prod_{i \in I} T_{q, x_i},
\]

(72)

\[
A_I(x; t) = t^{r(r-1)/2} \prod_{i \in I} \frac{tx_i - x_j}{x_i - x_j}.
\]

(73)

These operators are commutative with each other \([D^r_n, D^s_n] = 0\). Our aim in this appendix is to treat the commutative family generated by the Macdonald
difference operators over the space of symmetric functions $\Lambda_F$, namely, in the infinitely many variable situation. One finds that a use of the Heisenberg algebra will make our argument simple.

Introduce the Heisenberg algebra generated by $a_n$ ($n \in \mathbb{Z} \neq 0$) satisfying the commutation relations

$$[a_m, a_n] = m \frac{1 - q^{|m|}}{1 - t^{|m|}} \delta_{m+n,0}. \quad (74)$$

Let $|0\rangle$ be the vacuum vector satisfying $a_n|0\rangle = 0$ ($n = 1, 2, \cdots$), and $\mathcal{F}$ be the Fock space

$$\mathcal{F} = F[a_{-1}, a_{-2}, \cdots]|0\rangle. \quad (75)$$

We have the natural identification between the Fock space $\mathcal{F}$ and the ring of the symmetric functions $\Lambda_F$ by the rule:

$$\mathcal{F} \simeq \Lambda_F : \quad a_{-n} \longleftrightarrow p_n, \quad |0\rangle \longleftrightarrow 1. \quad (76)$$

Here $p_n$ denotes the power sum function $p_n = \sum_i x_i^n$.

In [9] (see also [11]), it was shown that the Macdonald difference operator $E$ defined by

$$E = \lim_{n \to -} E_n : \Lambda_F \longrightarrow \Lambda_F, \quad (77)$$

$$E_n = t^{-n} D_n^1 - \sum_{i=1}^n t^{-i} : \Lambda_{n,F} \longrightarrow \Lambda_{n,F}, \quad (78)$$

can be realized in terms of the Heisenberg algebra. It reads,

$$H_1 \equiv (t - 1) E + 1 = \eta_0, \quad (79)$$

$$\eta(z) = \sum_{n \in \mathbb{Z}} \eta_n z^{-n} = : \exp \left( - \sum_{n \neq 0} \frac{1 - t^n}{n} a_n z^{-n} \right) :, \quad (80)$$

where the symbol $:\cdots:$ means the usual normal ordering of the oscillators.

The above construction can be extended, and one can obtain a commutative family of operators $H_r$ acting on the Fock space $\mathcal{F}$. Let us introduce the $H_r$’s as follows.
**Definition A.1** Define the operators $H_r$ for $r = 1, 2, 3, \cdots$ by

$$H_r = \frac{[r]_t^{-1}!}{n!} \oint_{C_1} \cdots \oint_{C_r} \prod_{1 \leq i < j \leq r} \omega(z_j/z_i) : \eta(z_1) \eta(z_2) \cdots \eta(z_r) :,$$

(81)

where the contours $C_i$ are circles $|z_i| = 1$, and $\omega(z)$ is defined by

$$\omega(z) = \frac{(1 - z)(1 - z^{-1})}{(1 - t^{-1}z)(1 - t^{-1}z^{-1})} = \frac{2}{1 + t^{-1}} + \sum_{k=1}^{\infty} \frac{t^{-k} - t^{-k+1}}{1 + t^{-1}} (z^k + z^{-k}).$$

(82)

Note that we have normalized $H_r$ to simplify our later discussion, using the notations

$$[n]_x = \frac{1 - x^n}{1 - x}, \quad [n]_x! = \prod_{k=1}^{n} [k]_x.$$

(83)

Then we claim the following.

**Proposition A.2** On the bosonic Fock space $\mathcal{F}$, the commutation relations

$$[H_r, H_s] = 0,$$

(84)

hold, for all $r$ and $s$.

In the next section, we give a proof based on the commutativity of the Macdonald difference operators $D^r_n$.

**A.2 proof of $[H_r, H_s] = 0$.**

We prove Proposition [A.2] by using several operator product formulas and the commutation relation $[D^r_n, D^s_n] = 0$.

Let us introduce the generating function for the Macdonald symmetric function of length one,

$$\phi(x) = \sum_{n=0}^{\infty} \phi_{-n} x^n = : \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1 - t^n}{1 - q^n a_{-n} x^n} \right) :.$$

(85)

Namely we have $\phi_{-n}|_{a_{-n} \rightarrow p_n} = Q_{(n)}(q, t)$. The following formulas will be used for studying the action of $H_r$. 
Proposition A.3 We have the operator product formula
\[
\eta(z)\phi(x) = \mu(x/z) :\eta(z)\phi(x) :,
\]
(86)
\[
\mu(z) = \frac{1-z}{1-tz} = 1 + \sum_{k=1}^{\infty} (t^k - t^{k-1})z^k,
\]
(87)
the difference property
\[
: \eta(tx)\phi(x) : |0\rangle = \phi(qx)|0\rangle,
\]
(88)
and the expansion in terms of symmetric polynomials in \(x_i\)’s
\[
\phi(x_1)\phi(x_2) \cdots \phi(x_n)|0\rangle = \sum_{\lambda, \ell(\lambda) \leq n} \phi_{-\lambda} m_{\lambda}(x)|0\rangle.
\]
(89)
In Eq. (89), the summation is taken over the partitions, the notation \(\phi_{-\lambda} = \phi_{-\lambda_1} \phi_{-\lambda_2} \cdots \phi_{-\lambda_n}\) is used for \(\lambda = (\lambda_1, \cdots, \lambda_n)\), and the monomial symmetric polynomial in \(x_1, \cdots, x_n\) is denoted by \(m_{\lambda}(x)\).

By using Proposition A.3, one can study the action of the operators \(H_r\) on the vector \(\phi(x_1) \cdots \phi(x_n)|0\rangle\).

Proposition A.4 The equality
\[
H_r \cdot \phi(x_1) \cdots \phi(x_n)|0\rangle = t^{-rn} \sum_{k=0}^{r} (t-1)^k \binom{r}{k}_t [k]_t! D_n^k \cdot \phi(x_1) \cdots \phi(x_n)|0\rangle,
\]
holds, where \([n]_m = [n]_t!/[m]_t! [n-m]_t!\), and \(D_n^k\) is the Macdonald difference operator acting on the variable \(x_i\)’s given in Eq. (72).

In the usual notation, we have \(\phi_{-\lambda}|0\rangle = g_{\lambda}(x; q, t)\) (under the identification \(F \simeq \Lambda_F\)). Since the functions \(g_{\lambda}(x; q, t)\) form the dual basis to the monomial basis \((m_{\lambda})\), the vectors \(\phi_{-\lambda}\) form a basis of the Fock space \(F\). Therefore, from the commutation relations \([D_n^r, D_n^s] = 0\), Proposition A.4 means that the operators \(H_r\) are mutually commutative on the Fock space. This proves Proposition A.2. It remains to show Proposition A.4.

Proof. First, we note the identity
\[
\text{Symm} \prod_{1 \leq i < j \leq n} \frac{1 - x_j/x_i}{1 - t^{-1}x_j/x_i} = \frac{[n]_{t^{-1}}!}{n!} \prod_{1 \leq i < j \leq n} \omega(x_j/x_i),
\]
(91)
where the symbol ‘Symm’ means the symmetrization in the variable $x_i$’s. By using this, we have the expression

$$H_r \cdot \phi(x_1) \cdots \phi(x_n)|0\rangle = \oint_{C_1} \cdots \oint_{C_r} \frac{dw_1}{2\pi i w_1} \cdots \frac{dw_r}{2\pi i w_r} \prod_{i=1}^{n} \frac{w_i - x_j}{w_i - tx_j} \times \prod_{1 \leq i < j \leq n} \frac{w_i - w_j}{w_i - t^{-1}w_j} : \eta(w_1) \cdots \eta(w_r) \phi(x_1) \cdots \phi(x_n) : |0\rangle,$$

were the integration contours $C_i$ enclose poles at $w_i = tx_1, tx_2, \cdots, tx_n$, $w_i = t^{-1}w_{i+1}, t^{-1}w_{i+2}, \cdots, t^{-1}w_r$ and $w_i = 0$. This multiple integral can be easily performed and written as a multiple summation. Then we symmetrize the summation by using the identity Eq.(91) again in the form

$$\text{Symm} \prod_{1 \leq i < j \leq m} \frac{tx_i - x_j}{x_i - x_j} = \frac{[m]!}{m!}.$$

After the symmetrization, we can organize the result in terms of the Macdonald difference operators $D^*_n$, and obtain Eq.(90).

A.3 difference equation for the raising operator

Using the Heisenberg representation of the Macdonald difference operators, we obtain a difference equation for the raising operator of the Macdonald Polynomial.

Let $s_1, s_2, \cdots, s_n$ be parameters, and consider the space of formal power series $F[[x_2/x_1, x_3/x_2, \cdots, x_n/x_{n-1}]]$. Here and hereafter, we will work with $F = Q(q, t, s_1, s_2, \cdots, s_n)$. We introduce modified Macdonald difference operators as follows.

**Definition A.5** Define the difference operators $D^r(s_1, s_2, \cdots, s_n, q, t)$ acting on $F[[x_2/x_1, x_3/x_2, \cdots, x_n/x_{n-1}]]$ by

$$D^r(s_1, s_2, \cdots, s_n, q, t) = \sum_{I \subset \{1, 2, \cdots, n\}} \prod_{i \in I} \theta - \left( \frac{x_i}{x_j} \right) \prod_{i \in I} \theta_+ \left( \frac{x_j}{x_i} \right) \prod_{i \in I} s_i T_{q^{-1}x_i},$$
where $\theta_{\pm}(x)$ are the series
\begin{equation}
\theta_{\pm}(x) = \frac{1 - q^{\pm 1}t^{\pm 1}x}{1 - q^{\pm 1}x} = 1 + \sum_{n=1}^{\infty} (1 - t^{\pm 1})q^{\pm n}x^n.
\end{equation}

In Section 2, we have studied $D_1(s_1, s_2, \ldots, s_n, q, t)$ (note we have denoted $D = D_1$ for simplicity), and proved the existence of the eigenfunctions. Let us take the eigenfunction
\begin{equation}
D_1(s_1, s_2, \ldots, s_n, q, t) \cdot f = \sum_{i=1}^{n} s_i f,
\end{equation}
with the normalization $f = 1 + \cdots$. Then our claim for the integral representation or the raising operator for the Macdonald symmetric function is stated as follows.

**Proposition A.6** Let $f$ be as above. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition and $Q_\lambda(q, t)$ the Macdonald symmetric function (given as a vector in the Fock space). Then we have
\begin{equation}
Q_\lambda(q, t) = \oint \cdots \oint \frac{dx_1}{2\pi i x_1} \cdots \frac{dx_n}{2\pi i x_n} x_1^{-\lambda_1} \cdots x_n^{-\lambda_n} f(x_1, \ldots, x_n) \phi(x_1) \cdots \phi(x_n)|0\rangle,
\end{equation}
by specializing the parameters in $f$ as $s_i = t^{n-i}q^{\lambda_i}$. Here the integrals $\oint \frac{dx_k}{2\pi i x_k}$ mean to take the constant term in $x_k$.

**Proof.** By using Proposition [A.4] we have
\begin{align}
H_1 \cdot \text{(RHS of Eq.}\ (93)\text{)}
&= \mathcal{F} \cdots \mathcal{F} \frac{dx_1}{2\pi i x_1} \cdots \frac{dx_n}{2\pi i x_n} x_1^{-\lambda_1} \cdots x_n^{-\lambda_n} f(x_1, \ldots, x_n) \\
&\quad \times (t^{-n}(t - 1)D_1^n + t^{-n})\phi(x_1) \cdots \phi(x_n)|0\rangle \\
&= \mathcal{F} \cdots \mathcal{F} \frac{dx_1}{2\pi i x_1} \cdots \frac{dx_n}{2\pi i x_n} x_1^{-\lambda_1} \cdots x_n^{-\lambda_n} \phi(x_1) \cdots \phi(x_n)|0\rangle \\
&\quad \times \left[t^{-n}(t - 1)D_1(t^{n-1}q^{\lambda_1}, t^{n-2}q^{\lambda_2}, \ldots, q^{\lambda_n}, q, t) + t^{-n}\right] \\
&\quad \times f(x_1, \ldots, x_n) \\
&= \left[(t - 1) \sum_{i=0}^{\infty} t^{-i}(q^{\lambda_i} - 1) + 1\right] \text{(RHS of Eq.}\ (95)\text{)}.
\end{align}
Here we have made suitable $q^{-1}$-shifts in $x_i$ to transform the action of the operator $D_n^1$ on the vector $\phi(x_1) \cdots \phi(x_n)|0\rangle$ to the one on the series $f(x_1, \cdots, x_n)$. Note that one has to reinterpret the rational factors in $D_n^1$ as series acting on the space of symmetric polynomials $\Lambda_n, Q(q,t)$ by using the series Eq. (93). Then we arrive at the difference operator $D_n^1(t^{n-1}q^{\lambda_1}, t^{n-2}q^{\lambda_2}, \cdots, q^{\lambda_n}, q, t)$ which acts on the series $f(x_1, \cdots, x_n)$. Since $f(x_1, \cdots, x_n)$ satisfies the difference equation Eq. (94), we obtain RHS $\propto Q_\lambda(q,t)$. The proportionality can be argued by using the Pieri formula [8].

For example, if $n = 2$, the difference equation Eq. (94) gives us

$$f(x_1, x_2) = (1 - x_2/x_1)_{2} \phi_1 \left( q^{t-1}, q^{t-1}s_1/s_2; q, tx_2/x_1 \right).$$

(96)

By setting $s_1 = t q^{\lambda_1}$, $s_2 = t q^{\lambda_2}$, we recover Jing and Józefiak’s result [15]. For the case $n = 3$, the solution to the difference equation Eq. (94) can be guessed as (see Conjecture 2.7)

$$f(x_1, x_2, x_3) = \sum_{k=0}^{\infty} \frac{(qt^{-1}, q^{-1}t, t, q)_k}{(q, qs_1/s_2, qs_2/s_3, qs_1/s_3; q)_k} (qs_1/s_3)^k (x_3/x_1)^k \times \prod_{1 \leq i < j \leq 3} (1 - x_j/x_i)_{2} \phi_1 \left( q^{k+1}, q^{-1}s_1/s_j; q, tx_j/x_i \right).$$

(97)

We set $s_1 = t^2 q^{\lambda_1}$, $s_2 = t q^{\lambda_2}$, $s_3 = q^{\lambda_3}$, for the calculation of $Q_\lambda(q,t)$. Recently, Lassalle and Schlosser found a general formula for the raising operator of the Macdonald polynomials [16]. They found a way to invert the Pieri formula, and derived the raising operator in terms of certain determinant expressions. It was checked that the series in Eq. (97) for $n = 3$ agrees with their determinant formula up to certain degree in $x_i$'s.

It is not a straightforward task to derive the commutativity of the operator $D^r(s_1, s_2, \cdots, s_n, q, t)$’s from $[H_r, H_s] = 0$, since the expression Eq. (94) has a kernel and does not specify $f$ uniquely. Nevertheless, it is naturally expected that the $D^r(s_1, s_2, \cdots, s_n, q, t)$’s are commutative with each other.

**Conjecture A.7** On the space $F[[x_2/x_1, x_3/x_2, \cdots, x_n/x_{n-1}]]$, we have

$$[D^r(s_1, s_2, \cdots, s_n, q, t), D^s(s_1, s_2, \cdots, s_n, q, t)] = 0,$$

(98)

for all $r$ and $s$.

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References

[1] J. Shiraishi, A Commutative Family of Integral Transformations and Basic Hypergeometric Series. I. Eigenfunctions, [math.QA/0501251]

[2] J. Shiraishi, A Commutative Family of Integral Transformations and Basic Hypergeometric Series. III. Quasi-Eigenfunctions and the Eight-Vertex Model, in preparation.

[3] R.J. Baxter, Eight-Vertex Model in Lattice Statistics, Phys. Rev. Lett. 26 832-834 (1971).

[4] R.J. Baxter, Partition function of the Eight-Vertex Lattice Model, Ann. Phys. 70, 193-228 (1972).

[5] R.J. Baxter, Exactly Solved Models in Statistical Mechanics, Academic Press, London, (1982).

[6] O. Foda, K. Iohara, M. Jimbo, R. Kedem, T. Miwa and H. Yan, An elliptic quantum algebra for $\hat{sl}_2$, Lett. Math. Phys. 32, 259-268 (1994).

[7] O. Foda, K. Iohara, M. Jimbo, R. Kedem, T. Miwa and H. Yan, Notes on highest weight modules of the elliptic algebra $A_{q,p}(\hat{sl}_2)$, Prog. Theor. Phys. Suppl. 118, 1-34 (1995).

[8] I. G. Macdonald, Symmetric Functions and Hall Polynomials (2nd ed.), Oxford University Press, (1995).

[9] H. Awata, Y. Matsuo, S. Odake and J. Shiraishi, Collective field theory, Calogero-Sutherland model and generalized matrix models, Phys. Lett. B 347, 49-55 (1995).

[10] H. Awata, S. Odake and J. Shiraishi, Integral representations of the Macdonald symmetric polynomials, Comm. math. Phys. 179, 647-666 (1996).

[11] J. Shiraishi, Lectures on Quantum Integrable Systems, SGC Library vol. 28 (in Japanese), Saiensu-Sha co.,ltd, (2003).

[12] G. J. Heckman and E. M. Opdam, Root systems and hypergeometric function. I, Compositio Math. 64 329-352 (1987).
[13] J. Shiraishi, Free field constructions for the elliptic algebra $A_{q,p}(\hat{sl}_2)$ and Baxter’s eight-vertex model, *Int. J. Mod. Phys. A* **19**, 363-380 (2004).

[14] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, (1990).

[15] N. H. Jing and T. Joţefiak, A formula for two-row Macdonald functions, *Duke Math. J.* **67**, 377-385 (1992).

[16] M. Lassalle and M. Schlosser, Inversion of the Pieri formula for Macdonald polynomials, [math.CO/0402127](http://arxiv.org/abs/math.CO/0402127).