Numerical schemes for the optimal input flow of a supply-chain

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Abstract  An innovative numerical technique is presented to adjust the inflow to a supply chain in order to achieve a desired outflow, reducing the costs of inventory, or the goods timing in warehouses. The supply chain is modelled by a conservation law for the density of processed parts coupled to an ODE for the queue buffer occupancy. The control problem is stated as the minimization of a cost functional $J$ measuring the queue size and the quadratic difference between the outflow and the expected one. The main novelty is the extensive use of generalized tangent vectors to a piecewise constant control, which represent time shifts of discontinuity points. Such method allows convergence results and error estimates for an Upwind-Euler steepest descent algorithm, which is also tested by numerical simulations.

Keywords  Supply chains · ODE-PDE models · Optimal control · Upwind-Euler scheme · Tangent vectors

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1 Introduction

The mathematical modeling of industrial production, as well as the development of techniques for simulation and optimization purposes, is of great interest in order to reduce unwanted phenomena (bottlenecks, dead times at queues, and so on). Depending on the scale, one can distinguish different modeling approaches, for instance discrete (Discrete Event Simulations, [14]) or continuous (Differential Equations, [1, 2, 3, 21, 22]). The latter class includes models based on partial differential equations ([8, 11, 12, 16]). For a recent review see [10].

In this paper, we focus the attention on the continuous model for supply chains proposed by Göttlich, Herty and Klar in [16], briefly GHK model. A supply chain consists of processors with constant processing rate and a queue in front of each processor. The dynamics of parts on a processor is described by a conservation law, while the evolution of the queue buffer occupancy is given by an ordinary differential equation. The latter is simply determined by the difference of fluxes between the preceding and following processors. The complete model consists of a coupled PDE-ODE system.

Various optimal problems, corresponding to different types of controls, have been analysed for the GHK model (see [13, 17, 18, 19, 23]). Typically one may consider the input flow to the whole supply chain as a control as well as production rates and distribution coefficients in case of supply networks. These papers provide a number of results and, in particular, numerical algorithms to find the optimal control. In [23] two discretization techniques are compared: one consisting in first writing the adjoint system and then discretizing, while the second consists in inverting the order, that is first discretizing and then writing the adjoint system. All these methods compete with Discrete Events ones for numerical accuracy and computational times, see also [10].

In this paper we focus on the optimal control problem, where the control is given by the input flow to the supply chain and the cost functional $J$ is the sum of time-integral of queues and quadratic distance from a preassigned desired outflow. In [13], piecewise constant controls are considered together with generalized tangent vectors, which represent time shifts of discontinuities of the control. The technique of such generalized tangent vectors was extensively used for conservation laws, see [7], and for the case of network models, see [15, 20]. The main result of [13] is the existence of optimal controls.

The aim of this work is to introduce an innovative numerical approach, which builds up on the idea of generalized tangent vectors. Let us first explain in rough words the core idea of the approach. A good numerical method, in theory and practice, for an optimization problem is often based on a suitable choice of “perturbations”. Then one can define critical points, according to the chosen perturbations, and numerical algorithms stemming from such def-
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We base our method on perturbations of piecewise constant controls, obtained by time shifting the discontinuity points. The advantage of this method is twofold: on one side generalized tangent vectors are well suited for Wave Front Tracking (briefly WFT, see [6,15]) algorithms, which allow theoretical estimates on convergence rate. On the other side, generalized tangent vectors have a particularly simple evolution in time, which can be conveniently adapted to easily implementable methods as Upwind-Euler (briefly UE, see [9]). Moreover, in [9] convergence of the UE algorithm was proved by measuring the distance with WFT solutions by generalized tangent vectors.

The discretization of the evolution of generalized tangent vectors allows the numerical computation of the cost functional gradient. This can be combined with classical steepest descent or more advanced Newton methods for the optimization procedure by iterations.

Now, the results which complete the picture is the following. To estimate the gradient of the cost functional we can use alternatively WFT or UE algorithms for the evolution of generalized tangent vectors. Even more, we can measure the distance among the two, by again using suitably defined tangent vectors. This, in turn, allows to provide convergence results and error estimates.

Finally simulations are performed to show results for the proposed numerical algorithm in some case studies.

The outline of the paper is the following. In Section 2 we describe the GHK model and introduce the optimal control problem. The WFT algorithm to construct approximate solutions to the model is illustrated in Section 3 together with the definition and evolution of generalized tangent vectors. Then Section 4 describes the UE numerical algorithm for solutions to the coupled ODE-PDE system and also the numerics for generalized tangent vectors and cost functional derivative. Convergence results and rates for our method are then given in Section 5. Finally some numerical tests are discussed in Section 6.

2 An optimal control problem for supply chains

A supply chain consists of consecutive suppliers. Each supplier is composed of a processor for parts assembling and construction and a queue, located in front of the processor, for unprocessed parts. Formally we have the following definition.

Definition 1 A supply chain consists of a finite sequence of consecutive processors \( I_j, j \in J = \{1, \ldots, P\} \) and queues in front of each processor, except the first. Thus the supply chain is given by a graph \( G = (V, J) \) with arcs representing processors and vertices, in \( V = \{1, \ldots, P - 1\} \), representing queues. Each processor is parametrized by a bounded closed interval \( I_j = [a_j, b_j] \), with \( b_{j-1} = a_j, j = 2, \ldots, P \).

The maximal processing rate \( \mu_j \), and the processing velocity, given by \( v_j = L_j/T_j \) with \( T_j \) and \( L_j = b_j - a_j \) the processing time and the length of
the \( j \)-th processor, are constant parameters for each arc. The dynamics of the \( j \)-th processor is given by a conservation law

\[
\partial_t \rho_j (x,t) + \partial_x \min \{ \mu_j, v_j \rho_j (x,t) \} = 0, \quad \forall x \in [a_j, b_j], t \in \mathbb{R}^+,
\]

where \( \rho_j \in [0, \rho_j^{\text{max}}] \) is the unknown function, representing the density of parts, while the initial datum \( \rho_j(0) \) and the inflow \( f_{j,\text{inc}}(t) \) are to be assigned. An input profile \( u(t) \) on the left boundary \( \{ (a_1, t) : t \in \mathbb{R} \} \) is given for the first arc of the supply chain. Each queue buffer occupancy is modelled as a time-dependent function \( t \rightarrow q_j(t) \), satisfying the following equation:

\[
\dot{q}_j(t) = f_{j-1}(\rho_{j-1}(b_{j-1}, t)) - f_{j,\text{inc}}, \quad j = 2, \ldots, P,
\]

where the first term is defined by the trace of \( \rho_{j-1} \) (which is assumed to be of bounded variation on the \( x \) variable), while the second is defined by:

\[
f_{j,\text{inc}} = \begin{cases} 
\min \{ f_{j-1}(\rho_{j-1}(b_{j-1}, t)), \mu_j \} & \text{if } q_j(t) = 0, \\
\mu_j & \text{if } q_j(t) > 0.
\end{cases}
\]

This allows for the following interpretation: If the outgoing buffer is empty, we process as many parts as possible but at most \( \mu_j \). If the buffering queue contains parts, then we process at the maximal possible rate, namely again \( \mu_j \). Finally, the supply chain model is a coupled system of partial and ordinary differential equations given by

\[
\begin{aligned}
\partial_t \rho_j (x,t) + \partial_x \min \{ \mu_j, v_j \rho_j (x,t) \} &= 0, \quad j = 1, \ldots, P, \\
\dot{q}_j(t) &= f_{j-1}(\rho_{j-1}(b_{j-1}, t)) - f_{j,\text{inc}}, \quad j = 2, \ldots, P, \\
q_j(0) &= q_{j,0}, \\
\rho_j (x,0) &= \rho_{j,0}(x), \\
\rho_j (a_j,t) &= f_{j,\text{inc}}(t), \\
f_{1,\text{inc}}(t) &= u(t)
\end{aligned}
\]

where \( f_{j,\text{inc}} \) is given by \( u(t) \), for \( j = 2, \ldots, P \).

Fixed a time horizon \([0, T]\), define the cost functional:

\[
J(u) = \sum_{j=2}^P \int_0^T \alpha_1(t) q_j(t) dt + \int_0^T \alpha_2(t) \left[ v_P \cdot \rho_P(b_P,t) - \psi(t) \right]^2 dt
\]

\[
\hat{J}(u) = J_1(u) + J_2(u),
\]

where \( \alpha_1 \in L^1([0,T],[0,\infty)) \), \( \alpha_2 \in \text{Lip}([0,T],[0,\infty)) \) (the space of Lipschitz continuous functions) are weight functions, \( (\rho_j, q_j) \) is the solution to \( \text{(4)} \) for the control \( u \), \( v_P \cdot \rho_P(b_P,t) \) represents the outflow of the supply chain (assuming the density level is below \( \mu_P \)), while \( \psi(t) \in \text{Lip}([0,T],[0,\infty)) \) is
Numerical schemes for the optimal input flow of a supply-chain a pre-assigned desired outflow. Given $C > 0$, we consider the minimization problem

$$\min_{u \in \mathcal{U}_C} J(u)$$

where $\mathcal{U}_C = \{ u : [0, T] \to [0, \mu_1] ; u \text{ measurable}, T.V.(u) \leq C \}$ (with $T.V.$ indicating the total variation). In other words, we want to minimize the queues length and the distance between the exiting flow and the pre-assigned flow $\psi(t)$, using the supply chain input $u$ as control.

In [13], the existence of an optimal control was proved for a general problem, which includes the case (4)-(6):

**Theorem 1** (see [13]) Consider the optimal control problem (4), (6). If $J$ is lower semicontinuous for the $L^1$ norm, then there exists an optimal control.

Our aim is now to provide a new approach to solve (4)-(6) numerically. The key idea is to focus on piecewise constant controls and perturb the position of discontinuity points. The procedure corresponds to define (generalized) tangent vectors to $u$ (in the spirit of [7]). We can then take advantage of the knowledge of time evolution of such tangent vectors, developed in [20]. More precisely, we start giving the following:

**Definition 2** We indicate by $\tilde{\mathcal{U}} \subset \mathcal{U}_C$ the set of Piecewise Constant controls. For every $u \in \tilde{\mathcal{U}}$ we indicate by $\tau_k = \tau_k(u)$, $k = 1, \ldots, \delta(u)$, the discontinuity points of $u$.

We are now ready to define the perturbation to a piecewise constant control:

**Definition 3** Given $u \in \tilde{\mathcal{U}}$, a tangent vector to $u$ is a vector $\xi = (\xi_1, \ldots, \xi_{\delta(u)}) \in \mathbb{R}^{\delta(u)}$ representing shifts of discontinuities. The norm of the tangent vector is defined as:

$$\|\xi\| = \sum_{k=1}^{\delta(u)} |\xi_k| \cdot |u(\tau_k^+) - u(\tau_k^-)|.$$

Assume now for simplicity that $\tau_1 > 0$, $\tau_{\delta(u)} < T$, and set $\tau_0 = 0$, $\xi_0 = 0$, $\tau_{\delta(u)+1} = T$, $\xi_{\delta(u)+1} = 0$. Then given a tangent vector $\xi$ to $u$, for every $\varepsilon$ sufficiently small we define the infinitesimal displacement as:

$$u_\varepsilon = \sum_{k=0}^{\delta(u)} \chi_{[\tau_k + \varepsilon \xi_k, \tau_{k+1} + \varepsilon \xi_{k+1}]} u(\tau_k^+),$$

where $\chi$ is the indicator function. In other words $u_\varepsilon$ is obtained from $u$ by shifting the discontinuity points of $\varepsilon \xi$ (see Figure 1).

**Remark 1** The notion of tangent vector to piecewise constant function was introduced in [5] and used in [7] to prove uniqueness and continuous dependence of solutions to systems of conservation laws.
More precisely, one defines a distance of Finsler type among piecewise constant functions by considering paths which admit tangent vectors. Then, by density and using the usual $L^1$ metric one can extend the metric to the whole $L^1$. Finally one may study the evolution of tangent vectors and, in particular, estimates on their norms.

This same technique was generalized to flows on networks, see [15], and to the GHK supply chain model, see [20].

In next sections we are going to define numerical schemes for the solution of (4) and for the evolution of tangent vectors. The latter, in turn, will provide the information for the computation of numerical gradient of the cost functional $J$.

The evolution of tangent vectors is particularly clear for the theoretical numerical scheme given by the WFT algorithm. This was used in [9] to prove convergence of an Upwind-Euler scheme and in [13] to obtain the existence of an optimal control for (4)-(6). We thus recall briefly the WFT algorithm and the evolution of tangent vectors along approximate solutions constructed via the WFT algorithm.

### 3 The Wave Front Tracking algorithm

In this section we explain how to construct piecewise constant approximate solutions to (4) by WFT method, see [6] for details.

Given a discretization parameter $\sigma$ and initial conditions $\rho_{j,0}$ in $BV$, the space of bounded variation functions, a WFT approximate solution is constructed by a procedure sketched by the following steps:

- Approximate the initial datum by a piecewise constant function (with discretization parameter $\sigma$) and solve the Riemann Problems (RPs) corresponding to discontinuities of the approximation. In RPs solutions approximate rarefactions by rarefaction shocks of size $\sigma$.
Use the piecewise constant solution obtained piecing together the solutions to RPs up to the first time of interaction of two shocks; then solve the new RP created by interaction of waves and prolong the solution up to next interaction time, and so on.

To ensure the feasibility of such construction and the convergence of WFT approximate solutions to a weak solution as $\sigma \to 0$, it is enough to control the number of waves and interactions and the $BV$ norm. This is easily done in scalar case since both the number of waves and the $BV$ norm are decreasing in time (see [6] for details). For our system, we need also to approximate the queue evolution. For this we compute the exact solutions to (2) (see [20] for BV estimates for the complete PDE-ODE model [4]).

Notice that, as soon as a boundary datum will achieve a value below $\mu_j$, then in finite time all values above $\mu_j$ will disappear from the $j$-th processor, see also [20]. Therefore, for simplicity, we will assume

\[(H1) \quad \rho_{j,0}(x) \leq \mu_j \text{ for all } j \in J.\]

Then the same inequality will be satisfied for all times. In this case solutions to RPs are particularly simple, indeed the conservation law is linear, thus given some Riemann data $(\rho^-, \rho^+)$ on the $j$-th processor, the solution is always given by a shock travelling with velocity $v_j$.

### 3.1 Tangent vectors evolution

The infinitesimal displacement of each discontinuity of the control $u$ produces changes in the whole supply chain, whose effects are visible both on processors and on queues. In fact, every shift $\xi$ generates shifts on the densities and shifts on the queues, which spread along the whole supply chain.

Since the control $u$ is piecewise constant, the solution $(\rho_j, q_j)$ to (4) is such that $\rho_j$ is piecewise constant and $q_j$ is piecewise linear. A tangent vector to the solution $(\rho_j, q_j)$ is given by:

\[ (\beta \xi_j, \eta_j), \]

where $\beta$ runs over the set of discontinuities of $\rho_j$, $\beta \xi_j$ are the shifts of the discontinuities, while $\eta_j$ is the shift of the queue buffer occupancy $q_j$. The norm of a tangent vector is given by:

\[ \| (\beta \xi_j, \eta_j) \| = \sum_{\beta} |\beta \xi_j| |\Delta^\beta \rho_j| + \sum_j |\eta_j|, \]

(8)

where $\Delta^\beta \rho_j = \beta \rho^L_j - \beta \rho^R_j$ is the jump in $\rho$ of the discontinuity $\beta$ (where $\beta \rho^L_j$, respectively $\beta \rho^R_j$, is the value on the left, respectively right, of the discontinuity $\beta$).

Notice that this is compatible with the definition of norm of tangent vector
to a control. Because of assumption (H1), we have no wave interaction inside the processors. Therefore, densities and queues shifts remain constant for almost all times and change only at those times at which one of the following interactions occurs:

a) interaction of a density wave with a queue;

b) emptying of the queue.

We now provide formulas for such changes. Assume a wave with shift $\beta_{j-1}$ interact with the $j$-th queue and let $\ell$ be the interaction time. We use the letters $+$ and $-$ to indicate quantities before and after $\ell$, respectively. So, we indicate with $\rho^+_j$ and $\rho^-_j$ the densities on the processor $I_j$ before and after an interaction occurs and similarly for $I_{j-1}$. Also we use $\beta_j$ to denote the shift on the processor $I_j$ and with $\beta^-_j$ and $\beta^+_j$ the shifts on the queue $q_j$, respectively before and after the interaction. Consider the case a) and distinguish two subcases:

a.1) $q_j(\ell) = 0$;

a.2) $q_j(\ell) > 0$.

In case a.1) we have to further distinguish two subcases:

a.1.1) if $v_{j-1} \rho^+_j < v_{j-1} \rho^-_j < \mu_j$, then $\beta_{j} = \frac{v_{j-1}}{v_{j-1} \beta_{j-1}} + \beta_{j-1}$ and $\beta_{j} = \frac{v_{j-1}}{v_{j-1} \beta_{j-1}} - \beta_{j-1}$;

a.1.2) if $v_{j-1} \rho^+_j > \mu_j$, then $\beta_{j} = \frac{v_{j-1}}{v_{j-1} \beta_{j-1}} + \beta_{j-1}$ and $\beta_{j} = \frac{v_{j-1}}{v_{j-1} \beta_{j-1}} - \beta_{j-1}$.

In case a.2) we have: $\beta_{j} = 0$, $\beta_{j} = \frac{v_{j-1}}{v_{j-1} \beta_{j-1}} - \beta_{j-1}$.

Finally in case b) we get: $\beta_{j} = 0$, $\beta_{j-1} = 0$ and $\beta_{j} = \frac{v_{j-1}}{v_{j-1} \beta_{j-1}} - \beta_{j-1}$.

Using the above notations, indicate by $\beta_{\rho_P}$ the shift to a generic discontinuity of $\rho_P$ and by $\beta_{\rho_P^+}$, respectively $\beta_{\rho_P^-}$, the value of $\rho_P$ on the right, respectively left, of the discontinuity. The following holds:

**Proposition 1.** Consider a control $u \in \tilde{U}$ and a tangent vector $\xi \in \mathbb{R}^{\delta(u)}$ to $u$. The gradient of the cost functional $J$ with respect to $\xi$ is given by:

$$\nabla_\xi J(u) = \sum_j \int_0^T \alpha_1(t) \eta_j(t) dt + \sum_j \int_0^T \alpha_2(t) v_P \left( \beta_{\rho_P^+} + \beta_{\rho_P^-} - 2\psi(t^3) \right) \beta_{\rho_P} \Delta(\beta_{\rho_P})$$

$$\leq Y^{WFT}_1 + Y^{WFT}_2,$$

where $\Delta(\beta_{\rho_P}) = \beta_{\rho_P^+} - \beta_{\rho_P^-}$ and $t^3$ is the interaction time of the discontinuity indexed by $\beta$ with $\rho_P$, the right extreme of the supply chain.

**Proof** We have $\nabla_\xi J(u) = \lim_{\varepsilon \to 0} \frac{J((u) + \varepsilon \xi)}{\varepsilon \xi}$. By definition of the functional $J$, the infinitesimal change $Y^{WFT}_1$ in $J_1$ due to the infinitesimal displacement $\varepsilon \xi$ of the control $u$ is

$$Y^{WFT}_1 = \varepsilon \sum_j \int_0^t \alpha_1(t) \eta_j(t) dt,$$

(10)
while the infinitesimal change $Y_2^{WFT}$ in $J_2$ is

$$Y_2^{WFT} = \varepsilon \sum_{\beta} \alpha_2(t^3) v_P \left[ (\beta p_P)^2 - (\beta \rho_P)^2 - 2\psi(t^3) (\beta p_P - \beta \rho_P) \right] \beta \xi_P,$$

(11)

thus we conclude.

4 Steepest descent for the Upwind-Euler scheme

In this section we introduce first an Upwind-Euler scheme for the system and then a numerical scheme for the evolution of the tangent vectors to a solution to the PDE-ODE model. From the latter we will be able to compute numerically the derivative of the cost functional with respect to the discontinuities of the input flow. This, in turn, will be used in steepest descent methods to find the optimal control.

For a general introduction to numerical schemes for conservation laws we refer to [24], while for optimization algorithms to [4].

For simplicity we assume:

(H2) The lengths $L_j$ are rationally dependent.

Assumption (H2) allows us to use a unique space mesh for all processors $I_j$, $j = 1, \ldots, P$. Indeed there exists $\Delta$ so that all $L_j$ are multiple of $\Delta$ and we will always use time and space meshes dividing $\Delta$.

Remark 2 It is possible to choose different space and/or time grid meshes for different processors. This is necessary in the general case in which the lengths of arcs have not rational ratios. Details can be found in [9].

In next section we report briefly the Upwind-Euler method, analysed in [9] to construct numerical solutions to the supply chain model [4].

4.1 Upwind-Euler scheme for supply chains

Given a space mesh $\Delta x$, for each processor $I_j$, we set $\Delta t_j = \Delta x/v_j$ and define a numerical grid of $[0, L_j] \times [0, T]$ by:

- $(x_i, t^n)_j = (i \Delta x, n \Delta t_j)$, $i = 0, ..., N_j$, $n = 0, ..., M_j$ are the grid points;
- $j \rho_i^n$ is the value taken by the approximated density at the point $(x_i, t^n)_j$;
- $q_j^n$ is the value taken by the approximate queue buffer occupancy at time $t^n$.

The Upwind method reads:

$$j \rho_{i+1}^n = j \rho_i^n - \frac{\Delta t_j}{\Delta x} v_j \left( j \rho_i^n - j \rho_{i-1}^n \right) = j \rho_{i-1}^n,$$

(12)
where \( j \in \mathcal{J}, i = 0, ..., N_j \) and \( n = 0, ..., M_j \). Notice that the CFL condition is given by \( \Delta t_j \leq \frac{\Delta x_j}{v_j} \), and thus holds true. The explicit Euler method is given by:

\[
q^{n+1}_j = q^n_j + \Delta t_j \left( f^n_{j-1} - f^n_{j,\text{inc}} \right), \quad j \in \mathcal{J} \setminus \{1\}, \quad n = 0, ..., M_j,
\]

(13)

where \( f^n_{j,\text{inc}} \) needs to be defined and

\[
f^n_{j,\text{inc}} = \left\{ \begin{array}{ll}
\min \{ f^n_{j-1}(j-1) \rho^n_{N_{j-1}} \}, & q^n_j(t) = 0,
\mu^n_j(t) > 0.
\end{array} \right.
\]

(14)

Now, if \( \Delta t_{j-1} \leq \Delta t_j \) we set:

\[
f^n_{j-1} = \sum_{l=1}^{M(n) - m(n) - 1} \Delta t_{j-1} f^n_{j-1}(j-1) \rho^n_{N_{j-1}} = \sum_{l=1}^{\gamma} \Delta t_{j-1} f^n_{j-1}(j-1) \rho^n_{N_{j-1}}.
\]

(15)

where \( m(n) \) and \( M(n) \) are defined as:

\[
m(n) = \sup \{ m : m \Delta x_j \leq n \Delta t_j \},
\]

\[
M(n) = \inf \{ M : M \Delta x_j \geq (n+1) \Delta t_j \}.
\]

Otherwise, that is if \( \Delta t_{j-1} > \Delta t_j \), we set:

\[
f^n_{j-1} = f^n_{j-1} \left( j^{-1} \rho^n_{N_{j-1}} \right),
\]

(16)

where \( \lfloor \cdot \rfloor \) indicates the floor function. Boundary data are treated using ghost cells and the expression of inflows given by (14). The convergence of the scheme has been proved in [9] using a comparison with WFT approximate solutions.

### 4.2 Numerics for tangent vectors and cost functional

We first completely discretize the control space via the time mesh \( \Delta t \):

\[
\tilde{U}_{\Delta t} = \{ u \in \tilde{U} : \tau_k(u) = n(u, k) \Delta t, \quad n(k, u) \in \mathbb{N}, \quad k = 1, \ldots, \delta(u) \}.
\]

(17)

Now for every \( u \in \tilde{U}_{\Delta t} \), we consider shifts \( \xi \) so that the obtained time-shifted control is still in \( \tilde{U}_{\Delta t} \). Then every \( \xi_k \) is necessarily a multiple of \( \Delta t \). Hence from now on we will restrict to the case:

\[
\xi_k = \pm \Delta t, \quad k = 1, \ldots, \delta(u).
\]

(18)

For a generic processor \( I_j \) and a discontinuity point \( \tau_k \) of the control, we denote by \( ^k_j \xi_i^n \) and \( ^k_j \eta^n \) the approximations of \( ^k_j \xi_i^n(x_i, t^n) \), and \( ^k_j \eta^n(t^n) \), respectively. We define such approximations by a recursive procedure explained
in the following.
We initialize the tangent vector approximations by setting:
\[
\begin{align*}
    k,j \xi_n^1 &= 0, \quad \text{for } n = 1, \ldots, n(k-1,u), \quad j = 1, \ldots, P, \\
    k,j \xi_1^{n(k,u)} &= v_1(\pm \Delta t), \\
    k,j \eta^0 &= 0, \quad j = 1, \ldots, P.
\end{align*}
\]
(19)

The definition of \( k,j \xi_1^{n(k,u)} \) reflects the fact that the shift \( \xi_k \) provokes a shift of the wave generated on the first processor.

Now, the evolution of approximations of tangent vectors to \( \rho \) inside processors is simply given by:
\[
k,j \xi_{n+1}^1 = k,j \xi_n^{1_{\Omega_{n-1}}},
\]
On the other side, the approximation of \( \xi \) and \( \eta \) influence each other at interaction times with queues. More precisely, we consider the four cases described in Section 3.1 and get:

a.1.1): \( k,j \eta^{n+1} = 0, \quad k,j \xi^{n+1}_1 = \frac{v_j}{v_j - 1} k,j - 1 \xi_n^{N_j - 1} ; \)

a.1.2): \( k,j \xi^{n+1}_1 = \frac{v_j}{v_j - 1} k,j - 1 \xi_{N_j - 1}^n, \quad k,j \eta^{n+1} = k,j - 1 \xi_{N_j - 1}^n \left( \frac{v_j - 1 \rho_{N_j - 1}^{n+1} - \rho_j}{v_j - 1} \right) + k,j \eta^n ; \)

a.2): \( k,j \xi^{n+1}_{N_j - 1} = 0, \quad k,j \eta^{n+1} = k,j \xi^{n+1}_1 = -\frac{v_j}{v_j - 1} k,j \eta^n ; \)

b): \( k,j - 1 \xi_{N_j - 1} = 0, \quad k,j \eta^{n+1} = 0, \quad k,j \xi^{n+1}_1 = -\frac{v_j}{v_j - 1} k,j \eta^n ; \)

Notice that this is compatible with the evolution of tangent vectors along WFT approximate solutions and also the norm of approximate tangent vectors, in the sense of [8], is conserved.

Now we are ready to compute numerical approximations for \( \nabla \xi J \). We denote by \( k,j Y_1^n \), respectively \( k,j Y_2^n \), the numerical approximations of the \( k \)-th component of \( Y_1^{\text{WFT}} \), respectively \( Y_2^{\text{WFT}} \), as defined in [6] on processor \( I_j \) at time \( t^n \).

We initialize such approximation by setting:
\[
k,j Y_{10}^0 = 0, \quad k,j Y_{20}^0 = 0, \quad j = 1, \ldots, P, \quad k = 1, \ldots, \delta(u).
\]
(20)

Now the evolution is determined by the following simple rules. For \( Y_1 \), if \( q_j^{n+1} > 0 \), then we set
\[
k,j Y_1^{n+1} = k,j Y_1^n + \alpha_1(t^n) k,j \eta^n \Delta t,
\]
while for \( q_j^{n+1} = 0 \) we distinguish two subcases:

- if \( q_j^n > 0 \), then \( k,j Y_1^{n+1} = k,j Y_1^n ; \)
- if \( q_j^n > 0 \), then \( k,j Y_1^{n+1} = k,j Y_1^n + \frac{1}{2} \alpha_1(t^n) k,j \xi_1^{n+1} k,j \eta^n. \)

For \( Y_2 \) we set:
\[
k,j Y_2^{n+1} = k,j Y_2^n + \alpha_2(t^n) v_P \left( P \rho_{N_P}^n - \psi(t^n) \right)^2 - \left( P \rho_{N_P}^{n+1} - \psi(t^n) \right)^2 \right) k,j \xi_n^{N_P}.
\]
A steepest descent algorithm, denoting with $\vartheta$ the iteration step, is defined by setting

$$\tau_k^{\vartheta+1} = \tau_k^{\vartheta} + \left\lfloor \frac{h_\vartheta}{\Delta t} \left( \sum_j \sum_n k_j Y_{1,n}^k + \sum_n k Y_{2,n} \right) \right\rfloor \Delta t,$$

where $h_\vartheta$ is a coefficient to be suitably chosen. More precisely the parameter $h_\vartheta$ may be chosen to solve an optimization problem to get specific schemes, see [3].

5 Convergence and error estimates

In this section we will provide convergence results and error estimates for the Upwind-Euler steepest descent scheme illustrated in Section 4. We will make use of two natural parameters $\nu \in \mathbb{N}$ and $\theta \in \mathbb{N}$, the first referring to the Upwind-Euler (and WFT) discretization, while the second indicating the iterative step of the steepest descent algorithm.

We fix an initial space mesh $\Delta x$ (of which $\Delta$ is a multiple, see (H2)) and define $\Delta x_\nu = 2^{-\nu} \Delta x$. On each processor $I_j$ the time mesh is set as:

$$\Delta t_{j,\nu} = \frac{\Delta x_\nu}{v_j},$$

thus granting the CFL condition. Obviously $\Delta t_{j,\nu} \to 0$, $\Delta x_\nu \to 0$ as $\nu$ tends to $+\infty$.

The initial datum $\rho_{j,0}$ (see (4)) is sampled in the following way:

$$\nu,j \rho_{0}^i = \rho_{j,0}((a_j + i2^{-\nu} \Delta x) + ).$$

(21)

A control function $u_{\nu,\theta}$ will be defined by the iteration step of the steepest descent algorithm starting from a fixed $u_{\nu,0} \in \tilde{U}_{\Delta t_{1,\nu}}$. We will denote the discontinuity points of $u_{\nu,\theta}$ by $\tau_{k,\nu}^{\nu,\theta}$, for $k = 1, \ldots, \delta(u_{\nu,\theta})$.

Notice approximations can be constructed in such a way that $\delta(u_{\nu,\theta}) \to +\infty$ as $\nu \to +\infty$.

For simplicity $\nu,\theta(\rho, q)^{UE}$ will denote the numerical solution generated by the Upwind-Euler scheme, i.e. the collection $\nu,\theta(\rho_j^n, q_j^n)$ for $j = 1, \ldots, P$, $n = 1, \ldots, M_j^\nu$, $i = 1, \ldots, N_j^\nu$. Similarly $\nu,\theta(\xi, \eta)^{UE}$ will denote the numerical tangent vectors computed as in Section 4, i.e. the collection $\nu,\theta(\xi_j^n, \eta_j^n)$ for $j = 1, \ldots, P$, $n = 1, \ldots, M_j^\nu$, $i = 1, \ldots, N_j^\nu$, $k = 1, \ldots, \delta(u_{\nu,\theta})$.

We will also use the symbols $\nu,\theta(\rho, q)^{WFT}$, respectively $\nu,\theta(\xi, \eta)^{WFT}$, to indicate the solutions, respectively tangent vectors, produced by the Wave Front Tracking algorithm.

Now define:

$$\pi_{PC}(\nu,j \rho^n) = \frac{L_j/2^{-\nu} \Delta x_j - 1}{1} \sum_{i=0}^{\nu,j \rho^n} \chi_{[a_j + i2^{-\nu} \Delta x_j, a_j + (i+1)2^{-\nu} \Delta x_j]},$$

(22)
Numerical schemes for the optimal input flow of a supply-chain

\[ \pi_{PL}(\nu q_j(t)) = \nu q_j^n + (t - n\Delta t_{j,\nu} - n\Delta t_{j,\nu}) \]

for \( t \in [n\Delta t_{j,\nu}, (n+1)\Delta t_{j,\nu}] \).

Notice that \( \pi_{PC} \), respectively \( \pi_{PL} \), are operators taking values on the space of piecewise continuous, respectively piecewise linear, functions.

In [9] it was proved the following:

**Theorem 2** Assume that (H1), (H2), (21) hold true and that \( \rho_{j,0} \) are of bounded variation. Then \( \nu,\theta(\rho, q) \) is approximated by \( \nu,\theta(\pi_{PC}(\rho), \pi_{PL}(q)) \), more precisely:

\[ \| \nu,\theta(\rho) - \nu,\theta(\pi_{PC}(\rho)) \|_{L^1} + \| \nu,\theta(q) - \nu,\theta(\pi_{PL}(q)) \|_{L^1} \leq 2^{-\nu} K (T.V. (\rho_{j,0}), \mu_j), \]

(23)

where \( K \) is a suitable constant depending only on the total variation of \( \rho_{j,0} \) and the values \( \mu_j, j = 1, \ldots, P \).

Moreover \( \nu,\theta(\pi_{PC}(\rho), \pi_{PL}(q)) \) converges to a solution of (4) with a convergence rate as in (23).

The main idea behind Theorem 2 is to use tangent vectors to estimate the distance among WFT and UE approximate solutions. In the same fashion we are now going to estimate the distance among tangent vectors computed via WFT and UE schemes.

In each processor \( I_j \), the WFT wave shifts \( \beta \xi_j(t^n) \) are approximated by the UE shifts \( k_j \xi_n \) in the following sense. Each shift \( \beta \xi_j(t^n) \) corresponds to one or more non-vanishing values of \( k_j \xi_n \), whose position is possibly shifted by some amount. By splitting the shift \( \beta \xi_j(t^n) \) in one or more pieces, whose sum is equal to \( \beta \xi_j(t^n) \), we can define the difference with the UE generated shifts by tangent vectors \( \beta \zeta_j(t^n) \), which represent the space distance among the location of the discontinuity \( \beta \) of \( \rho_j \) at time \( t^n \) and the location of the non-vanishing values of \( k_j \xi_n \), i.e. \( a_j + i\Delta x_{\nu} \).

For instance in the WFT algorithm a shift \( \xi_k \) of the time discontinuity \( \tau_k \), gives rise to a shift \( \xi = v_1 \xi_k \) on processor \( I_1 \) at time \( \tau_k \) of the wave generated by the discontinuity \( \tau_k \). Similarly the UE algorithm will generate a non vanishing value \( k_1 \xi_n^{(k,u)} = v_1 \xi_k \). Therefore, we have \( \zeta_1(\tau_k) = \zeta_1(n(k,u)\Delta t_{1,\nu}) = 0 \), indeed the WFT and UE shifts coincide.

Analogously we define the tangent vectors \( t_j \) to the queue shifts \( \eta_j \) generated by WFT to recover those generated by UE.

We define the norm of the tangent vectors \( \zeta, \iota \) in the following way:

\[ \| (\zeta, \iota) \| = \sum_j \sum_{\beta} |\beta \zeta_j| |\beta \eta_j| |\Delta \rho_j\| + \sum_j |t_j| |\eta_j|. \]

(24)

In particular we have:

\[ \| (\zeta, \iota)(0) \| = 0. \]

(25)

Such tangent vectors \( (\zeta, \iota) \) evolve using the same rules of \( (\xi, \eta) \) for the Wave Front Tracking algorithm, see Section 3.1. In particular their norm may
increase because of interactions with queues. Moreover, the norm may also increase because of errors produced by the use of different time meshes among different processors, see formula (15) and (16). Summarizing, to estimate the increase in \((\zeta, \iota)\) norm we have to consider:

i) approximation errors occurring at interaction of waves with queues and at time in which a queue empties;

ii) the approximation errors due to different time discretizations \(\Delta t_{j,\nu}\).

Let us start by considering the case i) assuming there is no error approximation because of different time meshes. Since the evolution of \((\zeta, \iota)\) follows the same rules as those of \((\xi, \eta)\) for WFT, we can use the same estimates established in [9]. Referring to formulas (29), (30) and (31) of [9], and using the symbols \(\pm\) to indicate quantities before and after the interaction, we get:

\[
\| (\zeta, \iota)_+ \| \leq \left( \max \left\{ \frac{v_j}{v_{j-1}}, 1 \right\} \right) \| (\zeta, \iota)_- \| + v_j |\eta_j^-| \Delta t_{j,\nu} |\Delta \rho_j|,
\]

where \(j\) is the queue involved in the interaction and \(\Delta \rho_j\) is the jump in \(\rho\) of the wave exiting to processor \(I_j\). More precisely, the first term on the right of (26) is sufficient for the case of interaction of a wave with a queue, see also (29) and (30) of [9]. In case of emptying of the queue \(q_j\), the first term takes into account the evolution of \((\eta, \iota)\), as described in Section 3.1. The second term takes into account the additional shifts, provoked by the fact that the WFT produces a wave in \(I_j\) at a time which may be not a multiple of \(\Delta t_{j,\nu}\) (as it happens for all waves of the UE algorithm), see also formula (31) of [9].

Regarding ii), we refer to formulas (15) and (16). In case of \(\Delta t_{j-1,\nu} > \Delta t_{j,\nu}\), then no additional error occurs. Otherwise, the error is estimated by:

\[
\| (\zeta, \iota)_+ \| \leq \| (\zeta, \iota)_- \| + v_j \Delta t_{j-1,\nu} |\xi_{j-1}| |\Delta \rho_{j-1}|,
\]

where \(\xi_{j-1}\) is the shift of the interacting wave and \(\Delta \rho_{j-1}\) is the jump in \(\rho\) of the interacting wave.

Recalling (25), from the above estimates we get the following:

\[
\sup_{t \in [0, T]} \| (\zeta, \iota)(t) \| \leq \prod_{j=2}^P \max \left\{ \frac{v_j}{v_{j-1}}, 1 \right\} \cdot \max_{\beta} v_j \Delta t_{j,\nu} \cdot \sup_{t \in [0, T]} \| ^{\nu, \theta} (\xi, \eta)_{WFT}(t) \| \cdot \sup_{t \in [0, T]} T.V. (^{\nu, \theta} \rho_{WFT}(t)).
\]

Now in [20] it was proved (Lemma 2.7) that the norm of tangent vectors are decreasing along WFT solutions. Moreover, the tangent vectors for the WFT solutions satisfy the same initial estimate as (19), thus we get:

\[
\sup_{t \in [0, T]} \| ^{\nu, \theta} (\xi, \eta)_{WFT}(t) \| \leq \| ^{\nu, \theta} (\xi, \eta)_{WFT}(0) \| = v_1 \Delta t_{1,\nu} = \Delta x_{\nu} = 2^{-\nu} \Delta x.
\]
From (2.10a) of [20] and (21), we get:

\[
\sup_{t \in [0, T]} \text{T.V.}(\nu^{\theta} p^{\text{WFT}}(t)) \leq \text{T.V.}(\nu^{\theta} p^{\text{WFT}}(0)) + \|\partial_t (\nu^{\theta} q^{\text{WFT}}(t))\| \\
\leq \sum_j \text{T.V.}(\rho_j, 0) + \sum_j |\mu_j - \mu_{j-1}|. \tag{29}
\]

Since \( \max_j v_j \Delta t_j, \nu = \Delta x = 2^{-\nu} \Delta x \), we get:

**Theorem 3** Assume that \((H1), (H2), (21)\) hold true and that \(\rho_j, 0\) are of bounded variation. Then the following estimate holds true:

\[
\sup_{t \in [0, T]} \|\nu, \theta(\xi, \eta)\|_{W^1_T} \leq (2^{-\nu} \Delta x)^2 \prod_{j=2}^P \max \left\{ \frac{v_j}{v_{j-1}}, 1 \right\} \left( \sum_j \text{T.V.}(\rho_j, 0) + \sum_j |\mu_j - \mu_{j-1}| \right).
\]

We are now ready to prove the following:

**Proposition 2** Assume that \((H1), (H2), (21)\) hold true and that \(\rho_j, 0\) are of bounded variation. Let \(Y_{1,UE}\), \(i = 1, 2\), indicate the numerical approximation of \(Y_i\) via the Upwind-Euler steepest descent scheme of Section 4. Then there exists a constant \(K_1\) depending only on the data of the problem:

\[
K_1 = K_1 \left( \|\alpha_1\|_{L^1}, \text{Lip}(\alpha_2), \|\alpha_2\|_{\infty}, \text{Lip}(\psi), \|\psi\|_{\infty}, {v}_P, \mu_j, T.V.(\rho_j, 0) \right),
\]

where \(\text{Lip}(\cdot)\) indicates the Lipschitz constant of a function, \(\|\cdot\|_{\infty}\) the \(L^\infty\) norm and \(j = 1, \ldots, P\), such that the following estimates hold:

\[
\|Y^{WFT}_1 - \pi_{PC}(Y^{UE}_1)\| \leq K_1 \sup_{t \in [0, T]} \|\nu, \theta(\xi, \eta)\|_{W^1_T} \\
\|Y^{WFT}_2 - Y^{UE}_2\| \leq K_1 \sup_{t \in [0, T]} \|\nu, \theta(\xi, \eta)\|_{W^1_T}.
\]

**Proof** From (28) we have:

\[
\|Y^{WFT}_1 - \pi_{PC}(Y^{UE}_1)\| \leq \|\alpha_1\|_{L^1} \sup_{t \in [0, T]} \|\nu, \theta(\xi, \eta)\|_{W^1_T},
\]

and

\[
\|Y^{WFT}_2 - Y^{UE}_2\| \leq \left( \text{Lip}(\alpha_2) v_P \|\psi\|_{\infty} + 2\mu_P \right) \sup_{t \in [0, T]} \|\nu, \theta(\xi, \eta)\|_{W^1_T} \sup_{t \in [0, T]} \|\nu, \theta(q^{\text{WFT}}(t))\| \\
+ \|\alpha_2\|_{\infty} v_P \text{Lip}(\psi) \sup_{t \in [0, T]} \|\nu, \theta(q^{\text{WFT}}(t))\| \sup_{t \in [0, T]} \|\nu, \theta(p^{\text{WFT}}(t))\| \\
\cdot \sup_{t \in [0, T]} \|\nu, \theta(\xi, \eta)\|_{W^1_T} \|\nu, \theta(\xi, \eta)\|_{W^1_T}.
\]

By (28) and (29) we conclude.
We are now ready to state our main convergence result:

**Theorem 4** Assume that $(H1)$, $(H2)$, (21) hold true and that $\rho_{j,0}$ are of bounded variation. Fixing $\nu$, if $\delta(u_{\nu,\theta}) = \delta$, $h_0 \geq \bar{h} > 0$, for all $\theta$, and $\bar{\tau}_k^{\nu,\theta} \rightarrow \bar{\tau}_k$ as $\theta \rightarrow +\infty$, then $u_{\nu,\theta}$ strongly converges in $L^1$ to some $\bar{u} \in \mathcal{U}_C$ as $\theta \rightarrow +\infty$ and

$$\nabla \xi J(\bar{u}) \leq K_2 2^{-\nu} \Delta x$$

where $K_2$ depends only on the data of the problem as for $K_1$ (of Proposition 2) and on $v_j$, $j = 1, \ldots, P$.

**Proof** Clearly $u_{\nu,\theta}$ strongly converges in $L^1$ to some $\bar{u}$, moreover by Helly theorem $\bar{u} \in \mathcal{U}_C$.

Now from Proposition 1 we have that $\nabla \xi J(u_{\nu,\theta}) = Y_1^{WFT} + Y_2^{WFT}$. The tangent vectors $\nu,\theta(\xi,\eta)^{WFT}$ converge as $\theta \rightarrow +\infty$, in the sense that the positions and values of shifts converge. Then we get that $\nabla \xi J(u_{\nu,\theta})$ converges to $\nabla \xi J(\bar{u})$.

Now since $\bar{\tau}_k^{\nu,\theta}$ we have that $Y_1^{UE}$ and $Y_2^{UE}$ are bounded by $\Delta t_{1,\nu}$ (times a constant depending only on the data of the problem as for $K_1$). Thus we conclude by Theorem 3 and Proposition 2.

### 6 Simulations

In this Section we test the Euler-Upwind steepest descent algorithm on two test cases.

Consider first a supply chain characterized by 11 arcs with the following input function:

$$u(t) = \begin{cases} 
  u_1 & 0 \leq t \leq \tau_1, \\
  u_2 & \tau_1 < t \leq \tau_2, \\
  u_3 & \tau_2 < t \leq T.
\end{cases}$$

We assume that the supply chain is initially empty and set $v_j = 1 \forall j = 1, \ldots, 11$ and

$$\begin{align*}
  \mu_1 &= 200, \mu_2 = 75, \mu_3 = 100, \mu_4 = 65, \mu_5 = 150, \\
  \mu_6 &= 75, \mu_7 = 30, \mu_8 = 100, \mu_9 = 80, \mu_{10} = 100, \mu_{11} = 120.
\end{align*}$$

For simplicity we also set $\alpha_1 \equiv 1$ and $\alpha_2 \equiv 0$ and analyze two different cases:

Case a) $u_1 = 90, u_2 = 100, u_3 = 125, T = 10$, initial values $(\tau_1, \tau_2) = (1,3)$;

Case b) $u_1 = 100, u_2 = 90, u_3 = 125, T = 10$, initial values $(\tau_1, \tau_2) = (4,5)$.

As time and space grid meshes we choose $\Delta x = 0.02$ and $\Delta t = 0.016$ so as to satisfy the CFL condition. We use the condition that $J$ remains unchanged for five runnings of the algorithm as forced stop criterion. The times $\tau_1$ and $\tau_2$ found by the algorithm are:

Case a) $\tau_1 \simeq 8.98$, $\tau_2 \simeq 9.14$: as expected both $\tau_1$ and $\tau_2$ run toward $T$; in fact, in order to minimize the queues, the optimization algorithm tends to reduce the inflow levels which increase the queues (i.e. $u_2$ and $u_3$).
Case b) \( \tau_1 = 0, \tau_2 \simeq 9.05 \): \( \tau_1 \) runs to zero and \( \tau_2 \) runs toward \( T \); as in the previous case, the optimization algorithm works to reduce the inflow levels which lead to queues increasing (i.e. \( u_1 \) and \( u_3 \)).

In Table 1 we report the numerical values of \( \tau_1, \tau_2 \) and \( J \) at each iteration of the steepest descent algorithm for Case a).

Table 1  Case a: \( J \) versus \( \tau_1, \tau_2 \) in 42 iterations of the steepest descent algorithm.

| Iteration | \( \tau_1 \) | \( \tau_2 \) | \( J \)  |
|-----------|---------|---------|--------|
| 1         | 1.00    | 3.00    | 1117799.059 |
| 2         | 2.595   | 5.988   | 89474.759  |
| 3         | 3.870   | 7.480   | 79616.859  |
| 4         | 4.893   | 8.228   | 75291.519  |
| 5         | 5.710   | 8.600   | 73053.659  |
| 6         | 6.365   | 8.788   | 71737.058  |
| 7         | 6.888   | 8.880   | 70923.172  |
| 8         | 7.306   | 8.926   | 70419.523  |
| 9         | 7.640   | 8.957   | 70092.102  |
| 10        | 7.907   | 8.977   | 69879.287  |
| 11        | 8.120   | 8.987   | 69749.807  |
| 12        | 8.291   | 8.998   | 69660.449  |
| 13        | 8.427   | 9.005   | 69608.289  |
| 14        | 8.538   | 9.012   | 69570.370  |
| 15        | 8.625   | 9.017   | 69544.690  |
| 16        | 8.694   | 9.022   | 69530.770  |
| 17        | 8.748   | 9.028   | 69521.531  |
| 18        | 8.790   | 9.035   | 69514.210  |
| 19        | 8.818   | 9.047   | 69509.809  |
| 20        | 8.839   | 9.058   | 69507.807  |
| 21        | 8.858   | 9.066   | 69506.164  |

Figure 2 respectively 3 depicts the values assumed by \( J \), respectively \( (\tau_1, \tau_2) \), at each iteration step for Case a).

![Fig. 2 Supply chain with 11 arcs, Case a. J versus iteration steps.](image-url)
The behaviour of the cost functional $J$ in the plane $(\tau_1, \tau_2)$, is reported for Case b) in Figure 4 to confirm the goodness of the steepest descent algorithm. Notice that since $J$ decreases when the number of iteration increases, the updating of $\tau_1$ and $\tau_2$ values allows an effective decrement of queues at supply chain nodes.

We now analyze now a supply chain with 2 arcs, maximal processing rates $\mu_1 = 200, \mu_2 = 75$ and lengths and processing rates of each processor equal to 1. We assume that processors and queues are empty at $t = 0$, i.e. $\rho_{j,0}(x) = 0$, $\forall x \in [0,1], j = 1, 2$, $q_{3}(0) = 0$. The levels of the input flow are $u_1 = 100, u_2 = 80, u_3 = 50$. The total simulation time is $T = 20$ and numerical approximations are made with $\Delta x = 0.02, \Delta t = 0.016$. The aim is to optimize $J = J_1 + J_2$, with $\alpha_1 \equiv \alpha_2 \equiv 0.5$, i.e. minimize the queue handling a pre-assigned piecewise constant outflow:
\[ \tilde{\psi} = \psi(t) = \begin{cases} 100 & 0 \leq t \leq 10, \\ 75 & 10 < t \leq T. \end{cases} \]

Figures 5 and 6 show the values assumed by \( J \) at each iteration step and the “path” followed by the steepest descent algorithm in the plane \((\tau_1, \tau_2)\), starting with initial searching point \((\tau_1, \tau_2) = (5, 12)\). We observe that according to the aim of minimizing the queue, \( J \) is a decreasing function and, moreover, as expected the flux on the last arc is equal to the final outflow value. Finally \( J \) is minimized (its value is zero) by \((t_1, t_2) = (0, 0)\).

![Fig. 5 Supply chain with 2 arcs: \( J \) versus iteration steps.](image)

![Fig. 6 Supply chain with 2 arcs: “path” followed by the steepest descent algorithm in the plane \((\tau_1, \tau_2)\).](image)
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