Confinement-Deconfinement Phase Transition in Hot and Dense QCD at Large N

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We conjecture that the confinement-deconfinement phase transition in QCD at large number of colors $N$ and $N_f \ll N$ at $T \neq 0$ and $\mu \neq 0$ is triggered by the drastic change in $\theta$ behavior. The conjecture is motivated by the holographic model of QCD where confinement-deconfinement phase transition indeed happens precisely at the value of temperature $T = T_c$ where $\theta$ dependence experiences a sudden change in behavior[1]. The conjecture is also supported by quantum field theory arguments when the instanton calculations (which trigger the $\theta$ dependence) are under complete theoretical control for $T > T_c$, suddenly break down immediately below $T < T_c$ with sharp changes in the $\theta$ dependence. Finally, the conjecture is supported by a number of numerical lattice results. We employ this conjecture to study confinement-deconfinement phase transition of dense QCD at large $\mu$ in large $N$ limit by analyzing the $\theta$ dependence. We find that the confinement-deconfinement phase transition at $N_f \ll N$ happens at very large quark chemical potential $\mu \sim \sqrt{N} \Lambda_{QCD}$. This result agrees with recent findings by McLerran and Pisarski[2]. We also speculate on case when $N_f \sim N$.

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I. INTRODUCTION

Understanding the phase diagram at nonzero external parameters $T, \mu$ is one of the most difficult problem in QCD. Obviously, this area is a prerogative of numerical lattice computations. However, some insights about the basic features of the phase diagram may be inferred by using some analytical approaches. In particular, some qualitative questions can be formulated and answered by considering a theory with large number of colors $N$ or/and flavors $N_f$, see recent papers on then subject [1, 2, 3] and references on previous works therein. Generically, to study a phase transition one should find an appropriate order parameter. It is easy to find an order parameter for gluodynamics when light quarks are not present in the system. If massless quarks are introduced into the system, one can study a chiral phase transition and use the chiral condensate as an order parameter. For massive, but light quarks this is not an option. However, in the limit of very large $N$ one can consider the free energy as an order parameter. In confined phase it is order of one, while in deconfined phase it is order of $\sim N^2$. Small number of flavors $N_f \ll N$ (massless or massive quarks) does not change the basic picture.

We formulate a different criteria for confinement-deconfinement phase transition, and therefore we use a different order parameter to analyze the phase transition. The new criteria is based on observation that the deconfined phase transition is always accompanied by very sharp changes in $\theta$ behavior which represents our basic conjecture. Therefore, in principle, if our conjecture is correct, one can use any order parameter which nontrivially depends on $\theta$ and study this dependence on two sides of the phase transition line. Very natural question immediately comes into mind: why and how these two different things (phase transition vs sharp $\theta$ changes ) could be linked? What is the basic motivation for this proposal? First of all, this criteria is motivated by the observation that in holographic model of QCD the confinement-deconfinement phase transition happens precisely at the value of temperature $T = T_c$ where $\theta$ dependence experiences a sudden change in behavior[1]. Secondly, the proposal is supported by the numerical lattice results [4]-[8], see also a review article [9], which unambiguously suggest that the topological fluctuations are strongly suppressed in deconfined phase, and this suppression becomes more severe with increasing $N$. These general features observed in the lattice simulations have very simple explanation within our proposal on the origin of the confinement-deconfinement phase transition, see next section for details. Finally, our new criteria is based on a physical picture which can be shortly summarized as follows.

For sufficiently high temperatures $T > T_c$ the instanton gas is dilute with density $\sim e^{-\gamma(T)N}$ which implies a strong suppression1 of the topological fluctuations at large $N$ where $\gamma(T) > 0$, see below for details on structure of $\gamma(T)$ function. The calculations in this region are under complete field theoretical control and the vacuum energy has a nice analytic behavior $\sim \cos \theta e^{-\gamma(T)N}$ as function of $\theta$. At the critical value of temperature, $T = T_c$ where $\gamma(T)$ changes the sign, the instanton expansion breaks down and one should naturally expect that at $T = T_c$ there should be a sharp transition in $\theta$ behavior as simple formula $\sim \cos \theta$ can only be valid when the instanton gas is dilute and semiclassical.

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1 See [11] and references therein for earlier discussions on the subject.
calculations are justified which is obviously not the case for $T < T_c$. Therefore, it is naturally to associate sharp changes in $\theta$ behavior with confinement-deconfinement transition, just as in the holographic model\cite{1}. There is a very narrow window of temperatures in deconfined phase, $0 < (T - T_c)/T_c \leq 1/N$ when the instanton expansion is not valid. This vicinity of $T_c$ is extremely interesting, see our comments about physics in this region in conclusion. This region shrinks to a point at $N = \infty$.

The main goal of this paper is to apply this criteria to the region with large chemical potential at large $N$ and $N_f \ll N$ and make a specific prediction on magnitude $\mu_c(T)$ for confinement-deconfinement transition line at large $\mu$ and sufficiently small $T \ll \mu$. The corresponding estimation of $\mu_c(T)$ is based on well-developed instanton calculus in deconfined phase where dilute gas approximation is justified.

The plan of the paper is as follows. We start in Section II by reviewing recent work\cite{1} on estimation $T_c$ using instanton calculus. We also present a picture explaining how and why two apparently different phenomena (sharp changes in $\theta$ and confinement-deconfinement transition) may in fact be tightly linked. In section III we apply the same technique to argue that the confinement-deconfinement phase transition happens at very large quark chemical potential $\mu_c \sim \sqrt{N} \Lambda_{QCD}$, where $\mu = \mu_B/N$ is already properly scaled quark chemical potential. This result agrees with recent analysis by McLerran and Pisarski\cite{2} which was based on fundamentally different starting point. Finally, in section IV we make few comments for the case when number of flavors $N_f \approx N$.

II. CONFINEMENT-DECONFINEMENT PHASE TRANSITION IN HOT QCD AT LARGE $N$.

We start with a short review of ref.\cite{1} where the conjecture (that the confinement-deconfinement phase transition happens precisely where $\theta$ behavior sharply changes) was implemented for large $N$ QCD at $T \neq 0$. Such a sharp transition is indeed observed in the holographic model of QCD. From quantum field theory viewpoint such a transition can be understood as follows. Instanton calculations are under complete theoretical control in the region $T > T_c$ as the instanton density is parametrically suppressed at large $N$ in deconfined region\cite{1}:

$$V_{inst}(\theta) \sim e^{-\gamma N} \cos \theta, \quad \gamma = \left[ \frac{11}{3} \ln \left( \frac{\pi T}{\Lambda_{QCD}} \right) - 1.86 \right].$$

(1)

It is assumed that a higher order corrections may change the numerical coefficients in $\gamma(T)$, but they do not change the structure of eq. (1). The critical temperature is determined by condition $\gamma = 0$ where exponentially small expansion parameter $e^{-\gamma N}$ suddenly blows up and becomes exponentially large. Numerically, it happens at

$$\gamma = \left[ \frac{11}{3} \ln \left( \frac{\pi T_c}{\Lambda_{QCD}} \right) - 1.86 \right] = 0 \Rightarrow T_c(N = \infty) \simeq 0.53 \Lambda_{QCD},$$

(2)

where $\Lambda_{QCD}$ is defined in the Pauli-Villars scheme. Our computations are carried out in the regime where the instanton density $\sim \exp(-\gamma N)$ is parametrically suppressed at any small but finite $\gamma(T) = \epsilon > 0$ when $N = \infty$. From eq. (1) one can obtain the following expression for instanton density in vicinity of $T > T_c$:

$$V_{inst}(\theta) \sim \cos \theta \cdot e^{-\gamma N \left( \frac{T - T_c}{T_c} \right)}, \quad 1 \gg \left( \frac{T - T_c}{T_c} \right) \gg 1/N.$$  

(3)

where $\alpha = \frac{11}{3}$ and $T_c(N = \infty) \simeq 0.53 \Lambda_{QCD}$ are estimated at one loop level. Such a behavior does imply that the dilute gas approximation is justified even in close vicinity of $T_c$ as long as $\frac{T - T_c}{T_c} \gg \frac{1}{N}$. Therefore, the $\theta$ dependence, which is sensitive to the topological fluctuations is determined by \cite{3} all the way down to the temperatures very close to the phase transition point from above, $T = T_c + O(1/N)$. The topological susceptibility is order of one for $T < T_c$ in confined phase while it vanishes $\sim e^{-\gamma N} \rightarrow 0$ for $T > T_c$ in deconfined phase. Non topological quantum fluctuations on the other hand could be quite large in this region, but they do not effect the structure of eq. (3). We do anticipate, of course, that the perturbative corrections in the instanton background may change our numerical estimate for $T_c$ and $\alpha$. However, we do not expect that a qualitative picture of the phase transition may be affected as a result of these corrections. We note that the lattice numerical computations\cite{4,5} do suggest that the topological fluctuations are strongly suppressed in deconfined phase immediately above $T_c$, and this suppression becomes more severe with increasing $N$ starting from physically relevant case $N = 3$. Holographic QCD also supports this picture\cite{1}. We do not expect any changes in the phase transition when small number of flavors $N_f \ll N$ are introduced into the system\cite{2}.

\textsuperscript{2} We have to make the following remark here in order to avoid any confusions later in the text. In the presence of the massless chiral fermions the $\theta$ dependence goes away in QCD in both phases: confined as well as deconfined. It is a simple reflection of the fact that
There are three basic reasons for a generic structure \( \Box \) to emerge:

1. The presence of the exponentially large “\( T \)- independent” contribution (e.g. \( e^{+1.86N} \) in eq. \([1]\)). This term basically describes the entropy of the configuration. It is due to a number of contributions such as a number of embedding \( SU(2) \) into \( SU(N) \) etc;

2. The presence of the “\( T \)- dependent” contribution to \( V_{\text{inst}}(\theta) \) which comes from \( \int n(\rho) d\rho \) integration, see below \([8]\). It is proportional to

\[
\left( \frac{\Lambda_{QCD}}{\pi T} \right)^{\frac{12N}{3}} = \exp \left[ -\frac{11}{3} N \cdot \ln \left( \frac{\pi T}{\Lambda_{QCD}} \right) \right].
\]

3. The fermion related contributions such as a chiral condensate, diquark condensate or non-vanishing mass term enter the instanton density as follows \( \sim \langle \bar{\psi} \psi \rangle_{N_f} \sim e^{N \cdot (c \ln |\langle \bar{\psi} \psi \rangle|)} \). For \( N \equiv \frac{N_f}{N} \rightarrow 0 \) this term obviously leads to a sub leading effects \( 1/N \) in comparison with two main terms in the exponent \([11]\). Therefore, such terms can be neglected as they do not change any estimates at \( N = \infty \). It is in accordance with the general arguments suggesting that the fundamental fermions can not change the dynamics of the relevant gluon configurations as long as \( N_f \ll N \).

The crucial element in this analysis is that both leading contributions (items 1 and 2 above) have exponential \( e^N \) dependence, and therefore at \( N \rightarrow \infty \) for \( T > T_c \) the instanton gas is dilute with density \( e^{-\gamma N} \), \( \gamma > 0 \) which ensures a nice \( \cos \theta \) dependence \([6]\), while for \( T < T_c \) the expansion breaks down, and \( \theta \) dependence must sharply change at \( T < T_c \). We have identified such sharp changes with first order phase transition.

Once \( T_c \) is fixed one can compute the entire line of the phase transition \( T_c(\mu) \) for relatively small \( \mu \ll T_c \) for large but finite \( N \gg N_f \). The result in the leading loop order can be presented as follows\([1]\),

\[
T_c(\mu) = T_c(\mu = 0) \left[ 1 - \frac{3N_f \mu^2}{4N \pi^2 T_c^2 (\mu = 0)} \right], \quad \mu \ll \pi T, \quad N_f \ll N.
\]

As expected, \( \mu \) dependence goes away in large \( N \) limit in agreement with general large \( N \) arguments\([11]\). This formula is in excellent agreement with numerical computations \([12, 13, 14]\) which show very little changes of the critical temperature \( T_c \) with \( \mu \) for sufficiently small chemical potential. In particular, even for the case \( N_f = 2, N = 3 \) where the expression \([5]\) is not expected to give a good numerical estimate, it still works amazingly well even for \( N = 3 \). Indeed, the result quoted in \([12]\) can be written as

\[
T_c(\mu)_{\text{lat}} = T_c(\mu = 0)_{\text{lat}} \left[ 1 - 0.500(67) \frac{\mu^2}{\pi^2 T_c^2 (\mu = 0)_{\text{lat}}} \right], \quad N_f = 2, \quad N = 3.
\]

It should be compared with our theoretical prediction \([5]\) for this case

\[
T_c(\mu)^{\text{th}} = T_c(\mu = 0)^{\text{th}} \left[ 1 - \frac{1}{2} \frac{\mu^2}{\pi^2 T_c^2 (\mu = 0)^{\text{th}}} \right].
\]

The eq.\([6]\) suggests very slow change of \( T_c \) with \( \mu \) at large \( N \). Such slow variation implies that a sufficiently large changes of order one \( \Delta T_c \sim N_{QCD} \) may occur only when chemical potential changes are very large, \( \Delta \mu \sim \sqrt{N} \Lambda_{QCD} \). In next section we confirm this expectation by a direct computations of \( \mu_c(T = 0) \) where we predict that the confinement-deconfinement phase transition happens at very large \( \mu_c(T = 0) \approx \sqrt{N} \Lambda_{QCD} \) if \( N_f \ll N \).

One more comment on this proposal. Our conjecture (that the confinement-deconfinement phase transition in QCD is triggered by the drastic change in \( \theta \) at the same point \( T = T_c \)) implicitly implies that the configurations which are responsible for sharp \( \theta \) changes must also play a significant role in confined phase at \( T < T_c \). On the other hand, at \( T > T_c \) the dilute instantons completely determine the \( \theta \) dependence \([3]\), while at \( T < T_c \) the small size instantons obviously can not provide confinement \([15]\). How can this be consistent with our conjecture that these two things must

one can redefine the fermi fields in the chiral limit such that \( \theta \) parameter completely disappears from the partition function. To avoid the identical vanishing of \( V_{\text{inst}}(\theta) \) one can introduce a non-zero quark mass \( m_q \neq 0 \). It does not effect any of our estimates as long as \( N_f \ll N \) as all such changes lead to a sub leading \( 1/N \) corrections, see item 3 below. Our goal here is to study the coefficient in front of \( \cos \theta \) in deconfined phase. By such an analysis we trigger the point when this coefficient is suddenly blows up, and the \( \theta \) dependence must drastically change. The sharp changes of this coefficient \( \sim V_{\text{inst}}(\theta) \) we identify with complete reconstruction of the ground state, drastic changes of the relevant gluon configurations, and finally, with confinement-deconfinement phase transition. One should also remark here that the assumption made in \([1]\) on non-vanishing chiral condensate in vicinity \( T > T_c \) as a holographic model of QCD suggests, is not crucial for our arguments to hold as it leads to a sub leading \( 1/N \) correction, see item 3 below.
be linked? We note that quark confinement can not be described in the dilute gas approximation, when the instantons and anti-instantons are well separated and maintain their individual properties (sizes, positions, orientations), as it happens at large $T > T_c$. However, in strongly coupled theories the instantons and anti-instantons lose their individual properties (instantons will “dissociate”) their sizes become very large and they overlap. The relevant description is that of instanton-quarks\(^3\), the quantum objects with fractional topological charges $\pm 1/N$ which become the dominant quasi-particles. The instanton quarks carry, along with fractional topological charges, the fractional magnetic charges which are capable to propagate far away from instantons- anti-instantons parents being strongly correlated with each other. For such configurations the confinement is a possible outcome of the dynamics. It makes the instanton quarks to become the perfect candidates to serve as the dynamical magnetic monopoles, the crucial element of the standard model of fundamental forces.

We follow the same logic as in\(^3\), and study the $\theta$ dependence in order to make a prediction about the phase transition point $\mu_c$. In the regime $\mu > \mu_c$ the $\theta$ dependence is determined by the dilute instanton gas approximation. We expect that the expansion breaks down only in close vicinity of $\mu_c$ at large $N$ as it happens in our previous analysis with phase transition at $T = T_c$. According to the conjecture this point will be identified with confinement-deconfinement phase transition point $\mu_c$. In the present case of analyzing $\mu_c(N, N_f)$ as a function of $N, N_f$ at very large $N$ and finite $N_f \ll N$ in order to compare with results of refs.\(^2, 3\) where the authors presented a very strong argument suggesting a very large $\mu_c \sim \sqrt{N}$ where the phase transition could happen.

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### III. CONFINEMENT-DECONFINEMENT PHASE TRANSITION IN DENSE QCD AT LARGE N.

In this section we estimate the value of $\mu_c$ where the instanton expansion breaks down and therefore, the $\theta$ dependence should experiences a sharp change. According to our conjecture we should identify this place with the phase transition point. Similar arguments have been put forward previously\(^20\) for numerical estimation of $\mu_c$ for small $N, N_f = 2, 3$, see also review talk on this subject\(^20\). Our goal here is quite different: we want to analyze the confinement-deconfinement phase transition at very large chemical potential $\mu \geq \Lambda_{QCD}$ when available technique does not allow to perform the lattice computations. One should also note that presently available holographic models of QCD also can not address this question. With this motivation in mind we want to analyze confinement-deconfinement phase transition at large chemical potential and compare the obtained results with corresponding analysis\(^2, 3\) which is based on fundamentally different starting point.

\(^3\) Instanton quarks originally appeared in 2d models. Namely, using an exact accounting and resummation of the $n$-instanton solutions in $2d CP^{N-1}$ models, the original problem of a statistical instanton ensemble was mapped unto a 2d-Coulomb Gas (CG) system of pseudo-particles with fractional topological charges $\sim 1/N\(^16\). This picture leads to the elegant explanation of the confinement phase and other important properties of the $2d CP^{N-1}$ models\(^16\). Unfortunately, similar calculations in 4d gauge theories is proven to be much more difficult to carry out\(^17\).
In our estimates below we assume that the color superconducting phase is realized in deconfined phase for all \( N \), see e.g. recent review [27]. It is known though that for extremely large \( N = \infty \) one could expect that another phase is more energetically favorable [28]. Still, for all reasonably large \( N \) the color superconducting phase prevails [29]. In any case, the difference between the two options would lead to a sub-leading \( 1/N \) corrections as explained in item 3 above.

As we shall see below, the instanton density in deconfined phase has the following generic behavior, \( \sim \cos \theta \exp [-N \gamma(\mu)] \), where \( \gamma(\mu) \sim \text{const.} + o(1/N) \) in large \( N \) limit. Such a behavior implies that for any small (but finite) positive \( \gamma > 0 \) the instanton density is exponentially suppressed and our calculations are under complete theoretical control. In contrast: at arbitrary small and negative \( \gamma < 0 \) the instanton expansion obviously breaks down, theoretical control is lost as an exponential growth \( \sim \exp(\gamma N) \) for the instanton density makes no sense. The \( \theta \) behavior must drastically change at this point. Therefore, the value of \( \mu_c \) is determined by the following condition,

\[
\gamma(\mu = \mu_c) = 0 \implies \mu_c = c \Lambda_{QCD}.
\]

Our goal is to compute the coefficient \( c \) by approaching the critical point \( \mu_c \) from deconfined side of phase boundary. Therefore, we will be interested in the instanton density in the dilute gas regime at \( \mu > \mu_c \) where analytical instanton calculations are under control.

As we already mentioned in footnote 2, pg.2 the \( \theta \) dependence goes away in full QCD in both phases: confined as well as deconfined in the presence of the massless chiral fermions. However we are interested in the magnitude of the instanton contribution \( \sim V_{\text{inst}}(\theta) \) in deconfined phase rather than in \( \theta \) dependence of full QCD. Precisely this coefficient triggers the point where the instanton expansion suddenly blows up. The sharp changes in \( V_{\text{inst}}(\theta) \) we identify with complete reconstruction of the ground state, drastic changes of the relevant gluon configurations, and finally, with confinement-deconfinement phase transition. To avoid identical vanishing of \( V_{\text{inst}}(\theta) \) in the presence of massless fermions one can assume a non zero chiral condensate in deconfined phase, as it has been done for hot matter in [30] with motivation from holographic model of QCD. Also, one can assume a non-vanishing masses \( m_q \neq 0 \) for the fermions, or non-vanishing diquark condensate \( \langle \bar{\psi} \psi \rangle \neq 0 \) to avoid identical vanishing of \( V_{\text{inst}}(\theta) \) for dense matter at large \( \mu \). None of these assumptions effects any numerical estimates given below in the limit \( N \to \infty, \; N_f \ll N \), as all these assumptions lead to a sub-leading effects \( \sim 1/N \) which will be ignored in what follows. We shall see in section IV that such kind of assumptions indeed play a crucial role but only when \( N_f \sim N \).

To be definite, we assume that the non-vanishing diquark condensate \( \langle \bar{\psi} \psi \rangle \neq 0 \) develops for \( \mu > \mu_c \). A precise magnitude of the diquark condensate is not essential for our calculations as it effects only sub-leading terms \( \sim 1/N \) which will be consistently ignored in what follows. The instanton-induced effective action for \( N_f \) massless fermions can be easily constructed. In particular, for \( N_f = 2 \) flavors, \( u, d \) the corresponding expression takes the following form, [30, 31, 32, 33, 34, 35],

\[
L_{\text{inst}} = e^{-i\theta} \int d\rho n(\rho) \left( \frac{4}{3} \pi^2 \rho^3 \right)^{N_f} \left\{ \left( \bar{u}_R u_L \right)(\bar{d}_R d_L) + \frac{3}{32} \left[ \left( \bar{u}_R \lambda^a u_L \right)(\bar{d}_R \lambda^a d_L) - \frac{3}{4} \left( \bar{u}_R \sigma_{\mu \nu} \lambda^a u_L \right)(\bar{d}_R \sigma_{\mu \nu} \lambda^a d_L) \right] \right\} + \text{H.c.}
\]

We wish to study this problem at nonzero chemical potential \( \mu \) and nonzero small temperature \( T \ll \mu \) (to be discussed later in the text). We use the standard formula for the instanton density at two-loop order [30, 31, 32, 33, 34, 35],

\[
n(\rho) = C_N(\beta_1(\rho))^{2N} \rho^{-5} \exp[-\beta_{II}(\rho)] \times \exp[-(N_f \mu^2 + \frac{1}{3}(2N + N_f)^2 T^2)\rho^2],
\]

where

\[
C_N = \frac{0.466 e^{-1.679} N^{1.34} N_f}{(N - 1)!(N - 2)!}, \quad \beta_1(\rho) = -b \log(\rho \Lambda_{QCD}), \quad \beta_{II}(\rho) = \beta_1(\rho) + \frac{b'}{2b} \log \left( \frac{2 \beta_1(\rho)}{b} \right), \quad b = \frac{11}{3} N - 2 N_f, \quad b' = \frac{34}{3} N^2 - \frac{13}{3} N_f N + \frac{N_f}{N}.
\]

This formula contains, of course, the standard instanton classical action \( \exp(-8\pi^2/g^2(\rho)) \sim \exp[-\beta_1(\rho)] \) which however is hidden as it is expressed in terms of \( \Lambda_{QCD} \) rather than in terms of coupling constant \( g^2(\rho) \). The chemical potential \( \mu = \mu_B/N \) in this expression is already properly normalized quark chemical potential (rather than baryon chemical potential). By taking the average of eq. (4) over the state with nonzero vacuum expectation value for the diquark condensate \( \langle \bar{\psi} \psi \rangle \neq 0 \) as described in [28, 29], integrating over \( \rho \), and taking large \( N \) limit using the standard Stirling formula

\[
\Gamma(N + 1) = \sqrt{2\pi} N N_e^{-N} \left( 1 + \frac{1}{12N} + O \left( \frac{1}{N^2} \right) \right)
\]
one finds the following expression for the instanton induced potential\textsuperscript{4},

$$V_{\text{inst}}(\theta) \sim e^{-\gamma N} \cos \theta, \quad \gamma = \left[ \frac{11}{6} \ln \left( \frac{N_f \bar{\mu}^2}{\Lambda_{\text{QCD}}^2} \right) \right] \sim \frac{1}{1.1}, \quad \mu^2 \equiv N \bar{\mu}^2,$$

(10)

where we introduced reduced chemical potential $\bar{\mu} \equiv \mu/\sqrt{N}$ and neglected all powers $N^p$ in front of $e^{-\gamma N}$. The crucial difference in comparison with similar computation at nonzero temperature \textsuperscript{1} is emerging of parameter $\bar{\mu}$ instead of the original quark chemical potential $\mu \equiv \sqrt{N} \bar{\mu}$. It implies that the critical chemical potential where $\gamma$ changes the sign (and therefore where the phase transition is expected) is parametrically large $\mu_c \sim \sqrt{N}$ because $\bar{\mu}_c \sim 1$, see below for numerical estimates. The origin for this phenomenon can be traced from eq. \textsuperscript{8} where temperature dependent factor in the instanton density is proportional to $\sim N$ while chemical potential enters this expression with factor $\sim N_f \ll N$. Therefore, a very large chemical potential $\mu \sim \sqrt{N} \Lambda_{\text{QCD}}$ is required in order to achieve the same effect as temperature $T \sim \Lambda_{\text{QCD}}$. The physics of this phenomenon can be explained as follows: at $T \sim 1$ a large number of gluons $N^2$ can get excited while at $\mu \sim 1$ only a relatively small number of quarks in fundamental representation $N$ can get excited. Therefore, it requires a very large chemical potential $\mu^2 \sim N$ in order for fundamental quarks play the same role as gluons do at $T \sim 1$. As explained above, the critical chemical potential is determined by condition $\gamma = 0$ where exponentially small expansion parameter $e^{-\gamma N}$ at $\mu > \mu_c$ suddenly blows up at $\mu < \mu_c$. Numerically, it happens at

$$\gamma = \left[ \frac{11}{6} \ln \left( \frac{N_f \bar{\mu}^2}{\Lambda_{\text{QCD}}^2} \right) \right] \sim \frac{1}{1.1} = 0 \quad \Rightarrow \quad \mu_c(N = \infty) \approx 1.4 \cdot \Lambda_{\text{QCD}} \sqrt{\frac{N}{N_f}}, \quad N_f \ll N,$$

(11)

where $\Lambda_{\text{QCD}}$ is defined in the Pauli - Villars scheme. The topological susceptibility vanishes $\sim e^{-\gamma N} \rightarrow 0$ for $\mu > \mu_c$ while it must be drastically different for $\mu < \mu_c$ as $\theta$ dependence must experience some drastic changes in this region as the instanton expansion breaks down, and therefore simple $\cos \theta$ dependence must be replaced by something else. It is very likely that the standard Witten’s arguments (valid for the confined phase) still hold in this region $\mu < \mu_c$ in which case the topological susceptibility is order of one.

The $\Lambda_{\text{QCD}}$ in the Pauli - Villars scheme which enters our formula \textsuperscript{11} is not well-known numerically. Therefore, for numerical estimates one can trade $\Lambda_{\text{QCD}}$ in favor of $T_c(N = \infty)$ at $\mu = 0$ estimated in\textsuperscript{1}, see eq. \textsuperscript{2}. Therefore, our final numerical estimate for $\mu_c(N = \infty)$ can be presented as follows,

$$\mu_c(N = \infty) \approx 2.6 \cdot \sqrt{\frac{N}{N_f}} \cdot T_c(N = \infty, \mu = 0), \quad N_f \ll N.$$

(12)

If one uses the numerical value for $T_c(N = 3) \approx 260$ MeV \textsuperscript{6,8}, one arrives to $\mu_c(N = \infty) \approx 690 \sqrt{N/N_f}$ MeV which is our final numerical estimate for the critical chemical potential where deconfined phase transition is predicted for very large $N$. Few remarks are in order:

a. The most important result of the present studies is the observation that the confinement- deconfinement phase transition according to \textsuperscript{11} happens at very large $\mu_c \sim \sqrt{N}$ if $N_f \ll N$. This is consistent with the results of \textsuperscript{2} where parametrically large scale for $\mu_c \sim \sqrt{N}$ had been predicted. However, the technique of ref. \textsuperscript{2} does not allow to answer the question whether the transition would be the first order or it would be a crossover. Within our framework at $N \gg 1$ and $N_f \ll N$ the entire phase transition line (which starts at $T = T_c \sim \Lambda_{\text{QCD}}$ at $\mu = 0$ and ends at $\mu = \mu_c \sim \sqrt{N} \Lambda_{\text{QCD}}$ at $T = 0$) is predicted to be the first order phase transition at large $N$ and $N_f \ll N$. This is because the nature for the phase transition along the entire line is one and the same: it is drastic changes of $\theta$ dependence when the phase transition line is crossed.

b. Our computations are carried out in the regime where the instanton density $\sim \exp(-\gamma N)$ is parametrically suppressed at $N = \infty$. From eq. \textsuperscript{10} one can obtain the following expression for instanton density in vicinity of $\mu > \mu_c$,

$$V_{\text{inst}}(\theta) \sim \cos \theta \cdot e^{-\alpha N \left( \frac{\mu - \mu_c}{\mu_c} \right)}, \quad \frac{1}{N} \ll \left( \frac{\mu - \mu_c}{\mu_c} \right) \ll 1,$$

(13)

\textsuperscript{4} The diquark condensate in large $N$ limit has behavior $\langle \psi \psi \rangle \sim \exp(-\frac{\mu}{2}) \sim \exp(-\sqrt{N})$, see e.g. review \textsuperscript{21}. It is still a sub-leading $1/\sqrt{N}$ effect in comparison with the main terms \textsuperscript{10}. Author thanks an anonymous referee for pointing out on this, potentially large, correction.
where $\alpha$ is $11/3$ at one loop level, but the perturbative corrections could be large and they may considerably change this numerical coefficient. Such a behavior \cite{14} does imply that the dilute gas approximation is justified even in close vicinity of $\mu_c$ as long as $\frac{\mu_c}{\Lambda_{QCD}} \gg \frac{1}{\sqrt{T}}$. In this case the diluteness parameter remains small. We can not rule out, of course, the possibility that the perturbative corrections may change our numerical estimate for $\mu_c$. However, we expect that a qualitative picture of the phase transition advocated in this paper remains unaffected as a result of these perturbative corrections in dilute gas regime.

**c.** In our estimate for $\mu_c$ we neglected $(\log \rho \Lambda_{QCD})^4$ in evaluating of the $\int d\rho$ integral. The corresponding correction changes our estimate \cite{11} very slightly, and it will be ignored in what follows. Numerical smallness of correction is due to the strong cancellation between the second loop contribution in the exponent (term proportional to $b/b$) and the first loop contribution in the pre-exponent in eq. (8).

**d.** Once $\mu_c$ is fixed one can compute the entire segment of the phase transition line $\mu_c(T)$ for relatively small $T$. Indeed, in the dilute gas regime at $\mu > \mu_c$ the $T$ dependence of the instanton density is determined by a simple insertion $\sim \exp[-2/3N\pi^2T^2\rho^2]$ in the expression for the density \cite{3}. In the leading loop order $\mu_c(T)$ varies as follows,

$$
\mu_c(T) = \mu_c(T = 0) \left[ 1 - \frac{N\pi^2T^2}{3NF\mu_c^2(T = 0)} \right], \quad \sqrt{NT} \ll \mu_c.
$$

One should remark that a variation of the critical chemical potential $\Delta \mu_c(T)$ is very large $\sim \sqrt{N}$ when the temperature variation $\Delta T \sim 1$ is order of one in units of $\Lambda_{QCD}$. This is in huge contrast with a similar expression \cite{9} which shows very little change $\sim 1/N$ of the critical temperature $\Delta T_c(\mu) \sim 1/N$ with variation of chemical potential of order one, $\Delta \mu \sim 1$. The nature of this difference between $\mu_c$ and $T_c$ was already mentioned before and can be explained by the fact that at $T \sim 1$ a large number of quarks $\sim N^2$ can get excited while at $\mu \sim 1$ only a relatively small number of quarks in fundamental representation $\sim N$ can get excited. Therefore, it requires a very large chemical potential $\mu^2 \sim N$ when quarks can play the same role as gluons do at $\mu > \mu_c$ as a typical value of $\mu$.

**IV. DECONFINEMENT TRANSITION IN HOT AND DENSE QCD AT $N_f \sim N$. SPECULATIONS.**

Our estimations \cite{11,12,13,14} have been derived under assumption that $N \to \infty$ while $N_f$ is kept fixed such that $\kappa \equiv N_f/N \to 0$. In particular, for the case of hot matter with $T \neq 0$ studied in \cite{1} the fermi fields and the chiral condensate were introduced exclusively with a single purpose to elucidate the physical interpretation of the phase transition for pure gluodynamics rather than for full QCD. The physical results in that case did not depend on $N_f$ nor they depend on a magnitude of the chiral condensate in deconfined phase or a value of the quark’s mass if it would be nonzero. The same remark also applies for analysis of dense matter with $\mu \neq 0$ discussed in the previous section as long as $\kappa \equiv N_f/N \to 0$.

In this section we want to speculate what happens when $\kappa \equiv N_f/N \sim 1$ by considering hot matter $T \neq 0$, $\mu \simeq 0$. In contrast with previous analysis we anticipate a very strong dependence from all (previously unessential) parameters such as quark’s mass $m_q$, number of flavors $N_f = \kappa N$, magnitude of the chiral condensate $\langle \bar{\psi} \psi \rangle$, etc. We start by considering variation of transition properties on quark’s masses. For simplicity, consider the limit in the system in the limit $m_q \gg \Lambda_{QCD}$. When $m_q$ is getting smaller but still sufficiently large $m_q \geq \Lambda_{QCD}$ one can easily demonstrate that the structure of $\gamma(T)$ remains the same, but the corresponding critical temperature will be slowly decreasing with $m_q$ as follows

$$
T_c(\kappa \neq 0, m_q) = T_c(\kappa = 0) \left( 1 - \kappa \frac{3}{11} \cdot \frac{C_2(T_c)}{m_q^2} - \kappa \frac{3}{11} \cdot \frac{C_4(T_c)}{m_q^4} + \ldots \right), \quad m_q \gg \Lambda_{QCD}, \quad \kappa = \frac{N_f}{N} \sim 1 \quad (15)
$$

where the first few coefficients of the expansion $1/(m_q \rho)^k$ in the instanton background have been explicitly calculated long ago \cite{37}, see also recent paper \cite{38}. For our estimates \cite{15} we replaced $\rho \to (\pi T_c)^{-1}$ as a typical value of $\rho$ where the integral $\int d\rho$ converges. The expansion \cite{15} can be trusted starting from $(m_q \rho) \sim m_q/(\pi T_c) > 1$. The fermion contribution is a sub leading effect $\sim 1/N$; it becomes of order one when $\kappa = \frac{N_f}{N} \sim 1$, as expected.

Now, we want to demonstrate a strong dependence of the transition as a function of the chiral condensate $\langle \bar{\psi} \psi \rangle$ magnitude. If the chiral condensate does not vanish identically in close vicinity of the phase transition, $T > T_c$ as holographic model of QCD suggests \cite{1}, one can repeat the corresponding calculations with the following result: the structure of $\gamma(T)$ function as defined in \cite{1} remains the same while its coefficients would now depend on dimensionless
parameters $\kappa$ and the value of the chiral condensate$^5$. The numerical values of the critical temperature $T_c$ and coefficient $\alpha$ would change, however the sharp changes of $\theta$ dependence which is a consequence of a generic structure of $\gamma(T)$ remain the same. Therefore, we expect the first order phase transition to hold in this case in complete analogy to previously considered case $\kappa \to 0$.

However, we think it is very unlikely for the chiral condensate to remain finite at $T > T_c$ when $\kappa \equiv N_f/N \sim 1$. It is much more likely that the chiral condensate vanishes at $T > T_c$ when $\kappa \sim 1$. In this case our analysis based on the dilute instanton approximation [5] will be obscured due to the long range interactions between instantons and anti-instantons induced by massless quarks, [29, 40]. This induced interaction becomes crucial even when instantons are still far away from each other, and the instanton gas is still dilute. The corresponding estimations for $T_c$ and studying the properties of the transition in the case $\kappa \sim 1$ become very model dependent analysis, and we shall not elaborate on this issue in the present paper. We anticipate that the transition properties will be very sensitive to the quark’s masses as the instanton interactions drastically depend on the quark’s features in this case. Such a sensitivity is consistent with the lattice results which suggest that for vanishing quark masses there will be first order phase transition while for physical masses it becomes a smooth crossover, see e.g. recent reviews [41, 42]. We should emphasize here that our basic principle which relates sharp changes in $\theta$ dependence and transition properties still holds for $\kappa \sim 1$ case. This principle is a simple reflection of the fact that the point where the confinement sets in corresponds to the regime where the instanton density suddenly blows up and the instantons dissociate into the instanton quarks as mentioned in chapter II and discussed in a more details in [1] and references therein. The large number of fermions when $\kappa \sim 1$ obscures a simple analysis when the critical point can be estimated by approaching from deconfined phase where the instanton density is parametrically suppressed [3] and the system remains under theoretical control up to a close vicinity of $T_c$. In the case of $\kappa \sim 1$ the corresponding analysis becomes much more involved due to the reasons mentioned above.

Therefore, the main lesson from estimates presented above is as follows. In the case when $N_f \sim N$ we observe a great sensitivity of the transition properties on specific details of the system such as quark’s masses, magnitude of the chiral condensate, value of $\kappa$. It is very difficult to make any solid predictions in this situation as they would largely depend on underlying assumptions. Such a sensitivity of the transition to quark’s properties at $\kappa \sim 1$ is in a huge contrast with our previous estimates when there is unambiguous prediction for the first order phase transition at $T \sim 1$ and small chemical potential [23, 5] and very large $\mu \sim \sqrt{N}$ and small temperature [12, 3, 14] when $\kappa \ll 1$ and when all specific quark’s details are irrelevant as their contribution is suppressed at least by factor $1/\sqrt{N}$.

V. SUMMARY. FUTURE DIRECTIONS

We explore the consequences of the assumption that in the large $N$ QCD at $N_f \ll N$ confinement-deconfinement phase transition takes place exactly at the point where the dilute instanton calculation breaks down, and therefore where $\theta$ dependence must drastically change. This conjecture for $T \neq 0$ is supported by lattice computations and holographic arguments. At large chemical potential we do not have such independent support. However, the basic governing principle remains the same, and therefore, our results [12, 3, 14] can be considered as a prediction. The most important consequence of our conjecture is observation that the critical $\mu_c$ is very large, $\mu_c \sim \sqrt{N}$ which is consistent with fundamentally different arguments presented in ref. [2]. Another important observation is the fact that the first order phase transition at $\kappa \ll 1$ holds all the way down from $T_c \sim 1$, $\mu = 0$ to $\mu_c \sim \sqrt{N}, T = 0$ as a consequence of the same nature of the transition. In different words, the $\theta$ behavior experience sharp changes whenever the phase transition line is crossed. This feature is very robust consequence of our conjecture, not sensitive to details of quark’s properties such as masses, chiral condensation etc, as they may influence the sub-leading terms only. Situation becomes drastically different when $N_f \sim N$ in which case everything becomes very sensitive to details: quark masses, chiral condensate, precise value of $\kappa$, etc.

A general comment on this proposal can be formulated as follows. Our conjecture which relates two apparently unrelated phenomena (phase transition vs sharp changes in $\theta$ behavior) implicitly implies that topological configurations which are linked to $\theta$ must play a crucial role in the dynamics of the phase transition. For $T > T_c$ such configurations are well-known: they are dilute instantons with density $\sim e^{-\gamma N} \cos \theta$. We presented arguments in [1] (see also earlier references therein) suggesting that at $T < T_c$ the instantons do not disappear from the system, but rather dissociate

$^5$ Indeed, in our previous analysis the chiral condensate enters the instanton density as follows $\sim \langle \bar{\psi} \psi \rangle^{N_f} \sim e^{N (\kappa \ln |\langle \bar{\psi} \psi \rangle|)}$. For $\kappa = \frac{N_f}{N} \to 0$ this term obviously leads to a sub leading effects $1/N$ in comparison with the main terms in the exponent [1]. For $\kappa \sim 1$ this terms becomes the same order of magnitude as other contributions in [1].
into fractionally charged constituents, the so-called instanton quarks. In this sense the phase transition can be understood as a phase transition between molecular phase (deconfined) and plasma phase (confined) of these fractionally charged constituents. The same arguments still hold for the entire phase transition line in \((T,\mu)\) plane. A similar conclusion on sharp changes in \(\theta\) behavior at \(T = T_c\) was also observed in ref.\cite{24} where the authors studied the D2 branes in confined and deconfined phases at \(T \neq 0\). The topological objects (sensitive to \(\theta\)) were identified in ref.\cite{24} as magnetic strings.

Our final remark here is as follows. If the picture advocated in the present work about the nature of the transition turns out to be correct, it would strongly suggest that fractionally charged constituents (which carry the magnetic charges as discussed in\cite{1} and references therein) may play a very important role in dynamics in deconfined phase in close vicinity of the transition \(0 < (T - T_c) \leq 1/N\). For large \(N\) this region shrinks to a point, however for finite \(N\) it could be an extended region in temperatures. In this region the instantons are not formed yet, and our semiclassical analysis is not justified yet as eq.\cite{3} suggests. However, the constituents in this region are already not in condensed form. Therefore they may become an important magnetic degrees of freedom which may contribute to the equation of state, similar to analysis on wrapped monopoles in ref.\cite{19,20}. The role of these fractional magnetic constituents could be even more profound if \(N_f \sim N\) where smooth crossover likely to take place \cite{21,22}. In this case the region of interests is order of one \((T - T_c) \sim 1\) in \(\Lambda_{QCD}\) units in contrast with a narrow region \(0 < (T - T_c) \leq 1/N\) if the first order phase transition takes place. The region above \(T_c\) is also very interesting from phenomenological viewpoint as reviewed in \cite{25}.

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