Numerical Analysis of Two-Dimensional Quantum Turbulence

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Abstract. This is the first numerical study of two-dimensional quantum turbulence by the Gross-Pitaevskii equation. We start from a uniform state and obtain quasi-steady quantum turbulence. Then the energy spectrum of the incompressible kinetic energy shows an indication of inverse cascade which is a remarkable phenomenon of two-dimensional classical turbulence. It is possible to make experiments of two-dimensional quantum turbulence in the film of superfluid helium and atomic Bose-Einstein condensates.

1. Introduction

Turbulence is one of the most important unresolved problems in classical physics. Many works have been piled up not only on three-dimensional classical turbulence (3DCT) but also on two-dimensional classical turbulence (2DCT). 2DCT has long been investigated in many fields like geophysics, magnetohydrodynamics mainly to apply it to the study of meso-scale meteorological phenomena [1]. Since there are two constants, kinetic energy and enstrophy, in the inviscid limit of 2D Navier-Stokes equation, 2DCT has some remarkable characters which lack in 3DCT. The most remarkable one is inverse cascade, which carries energy from small to large scales and leads to self-organizing character, for example, the formation of large-scale eddies. The energy spectrum of inverse cascade shows the power law $k^{-5/3}$ with the wave number $k$. This striking character has been confirmed by both numerical simulations and experiments [2-5].

Quantum turbulence (QT) consists of quantized vortices which have a definite core size of the healing length and the circulation around the core is quantized. On the other hand, in CT the definition of vortices is not clear and the circulation takes an arbitrary value. Inspite of the difference between CT and QT, the Kolmogolov’s -5/3 law, which is the most important statistical law in 3DCT, recently has been confirmed in 3DQT by numerical simulations [6,7].

These trends of the research of 3DQT make us expect the inverse cascade in 2DQT [8]. Making use of the feature that QT consists of quantized vortices, we would like to work out the relation between inverse cascade and the motion of quantized vortices. In this work, for the first time, the numerical simulation of 2DQT is made. We find an indication of inverse cascade in the energy spectrum.
2. Numerical Analysis

2.1. Numerical method

We analyze the dimensionless two-dimensional Gross-Pitaevskii (GP) equation with small-scale dissipation,

\[ [i - \gamma(k)] \frac{\partial}{\partial t} \tilde{\Phi}(k,t) = [k^2 - \mu(t)]\tilde{\Phi}(k,t) + \tilde{h}(k,t). \]  

Here \( \tilde{\Phi}(k,t) \) is the Fourier component of the wave function \( \Phi(r,t) \), \( \gamma(k) = \gamma_0 \theta(k - 2\pi/\xi) \) is the small-scale dissipation, \( \xi \) is the healing length, \( \mu \) is the chemical potential, \( g \) is the coupling constant, \( \tilde{h}(k,t) \) is the Fourier transformation of \( h(r,t) \), and \( V(r,t) \) is the external forcing potential. It should be noted that the dissipation term \( \gamma(k) \) does not conserve the particle number, so we make the chemical potential \( \mu \) change in time to conserve it.

We use \( 256^2 \) pseudo-spectral method with periodic boundary condition. In this simulation, we ignore the effect of aliasing error because the GP equation (1) does not include spatial differential in the nonlinear term and the dissipation term works at high wave numbers. The time development of the wave function is calculated by the fourth-order Runge-Kutta method. Resolutions of time and space are \( \Delta t = 0.0001 \), \( \Delta x = 0.125 \) respectively.

The external forcing potential is set following the method used in [9], though we improve the method a little. The concept of the method is shown in Fig. 1. First, \( F_0 e^{i\phi(k,t+\Delta t)} \) is made, where \( k \) is the wavenumber, \( F_0 = 1(k_{\text{min}} < k < k_{\text{max}}) \) or \( 0(k \leq k_{\text{min}}, k_{\text{max}} \leq k) \) and \( \phi(k,t) \) is randomly taken in the range of \( -\pi \leq \phi(k,t) \leq \pi \). Second, we obtain the Fourier transformation of \( F_0 e^{i\phi(k,t+\Delta t)} \), which is defined as \( V_{\text{new}}(r,t+\Delta t) \), and remain only \( \text{Re}\{V_{\text{new}}(r,t+\Delta t)\} \) to ensure to make \( V(r,t+\Delta t) \) real. Third, \( V_{\text{new}}(r,t+\Delta t) \) is normalized as \( 0 \leq V_{\text{new}}(r,t+\Delta t) \leq V_0 \). Finally, \( V(r,t+\Delta t) \) becomes

\[ V(r,t+\Delta t) = \text{Re}\{V_{\text{new}}(r,t+\Delta t)\} + R \times V(r,t), \]

where \( R \) is the time correlation factor and \( 0 \leq V(r,t+\Delta t) \leq V_0 \).

Let us explain some physical quantities relative to this simulation. The total kinetic energy \( E_{\text{kin}}(t) \) can be divided into two parts [10]. One is the compressible kinetic energy \( E_{\text{kin}}^c(t) \) and the other is the incompressible kinetic energy \( E_{\text{kin}}^i(t) \) which stems from quantized vortices. Through the Madelung transformation of the wave function \( \Phi(r,t) = f(r,t)e^{i\theta(r,t)} \), we can express \( E_{\text{kin}}(t) = \frac{1}{N} \int d^2r \{p(r,t)\}^2 \), \( E_{\text{kin}}^c(t) = \frac{1}{\mu} \int d^2r \left\{ \{p(r,t)\}^c \right\}^2 \) and \( E_{\text{kin}}^i(t) = \frac{1}{N} \int d^2r \left\{ \{p(r,t)\}^i \right\}^2 \), where \( N = \int d^2r \left\{ |\Phi(r,t)|^2 \right\} \) is the particle number, \( p(r,t) = f(r,t)\nabla\theta(r,t), \nabla \times \{p(r,t)\}^c = 0 \) and \( \nabla \cdot \{p(r,t)\}^i = 0 \). The total energy is described as \( E(t) = \frac{1}{N} \int d^2r \Phi^*(r,t)\left[ -\nabla^2 + \frac{\mu}{2} + \frac{\gamma_0}{\xi} \right] \Phi(r,t) \).
Since the phase $\theta(r, t)$ varies from 0 to $2\pi$ around the cores of quantized vortex, the number of vortices can also be counted by investigating the value of $\theta(r, t)$.

2.2. Results of numerical simulation

The calculation starts from the uniform state wave function $\Phi(r, 0) = 1$. The parameters are the following: $g = 1$, $f = 1$, $\gamma_0 = 1$, $V_0 = 50$, $k_{\text{in}} = (2\pi/\xi)^{-3}$, $k_{\text{max}} = (2\pi/\xi)^{-1}$ and $R = 1/e$, where $\xi = 1/f \sqrt{g} = 1$.

Physical quantities such as $E(t), E_{\text{kin}}(t), E_{\text{kin}}^c(t), E_{\text{kin}}^i(t)$, the number of quantized vortices and enstrophy increase in time under the external force as shown in Fig. 2 and Fig. 3. In Fig. 3, the time development of vortex number mimics that of enstrophy. Hence, quantized vortices generates enstrophy. These quantites become almost constant after $t \simeq 150$, i.e., we obtain quasi-steady 2DQT. Different from 3DQT, it is conjectured in 2DQT that the energy injected in small scale near $\xi$ should be carried not only to small scale $r < \xi$ but also to large scale $r > \xi$ if inverse cascade really occurs. Therefore, we think that the energy injection does not simply balance with the small scale dissipation. This is why we call the 2DQT not steady but “quasi-steady”. In the quasi-steady state, a lot of quantized vortex pairs are created and their average number is about 300. Vortex distribution at $t = 400$ is shown in Fig. 4. Since the system size of the numerical analysis is $(256 \times \Delta x)^2 = 1024\xi^2$, the average vortex density per unit area is about $0.29\xi^{-2}$.

Next, we show the incompressible kinetic energy spectrum $E_{\text{kin}}^i(k, t)$ in Fig. 5, which is expected to show inverse cascade. At $t = 1$ in the early stage, the energy is injected in $k_{\text{in}} < k_1 < k_{\text{max}}$. As time goes by, the injected energy is carried to both $k < k_1$ and $k_1 < k$. At $t = 400$ in the quasi-steady stage, we find some indication of inverse cascade. Once the quasi-steady state is achived, the energy spectrum keeps the form.

3. Discussion

In this work, we show some indication of inverse cascade. In Fig. 5, the energy spectrum of the quasi-steady state shows both inverse and direct cascades. The inverse cascade means that injected energy at $k_1$ is carried to $k < k_1$, the characteristic $k^{-5/3}$ law is not confirmed clearly though. The direct cascade comes from the following mechanism. In the quasi-steady state, many vortex pairs are certainly created and their mean distance between a vortex and an antivortex is found to be about $\xi$ estimated from the average vortex density. Once a pair is created...
at the scale $\xi$, their mean distance extends to $2\pi/k_f$ under the external forcing. Then, these pairs shrink to the order of $\xi$ and annihilate because of the dissipation. The ensemble average of the motion of such pairs lead to the direct cascade. In the quasi-steady state, direct cascade makes a peak in the incompressible kinetic energy spectrum between $k_f$ and $2\pi$ corresponding to scale $\sim \xi$.

We will discuss hereafter how to extend the inertial range of the inverse cascade. If inverse cascade occurs, large scale structure should appear that quantized vortices having the same circulation get together without merging into a single core. Such kind of large scale structure is not confirmed in our calculation. By applying the force which expands the intervals of pairs, i.e. electric field like, large scale structure may appear, which enlarges the inertial range.

The system size is also likely to affect the dynamics of quantized vortices. Extention of the system size is equal to thinning vortex core, i.e. increasing the coupling constant $g$ when the system size is unchanged. Then, the sound velocity of phonon $\propto \sqrt{g}$ increases to make the quantum fluid more incompressible. Inverse cascade is likely to occur in such a situation. Now, we are working with simulations with a larger system size to examine this effect.

Moreover, we are trying the vortex point model analogous to the three-dimensional vortex filament model. It is possible for this model to clearly focus on the motion of quantized vortices.

Acknowledgements
We acknowledge Michikazu Kobayashi for numerical skills and helpful discussions.

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