Transport properties and equation-of-state of hot and dense QGP matter near the critical end-point in the phenomenological dynamical quasiparticle model.

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(Dated: February 3, 2022)

We extend the effective dynamical quasiparticle model (DQPM) - constructed for the description of non-perturbative QCD phenomena of the strongly interacting quark-gluon plasma (QGP) - to large baryon chemical potentials, $\mu_B$, including a critical end-point and a 1st order phase transition. The DQPM description of quarks and gluons is based on partonic propagators with complex selfenergies where the real part of the selfenergies is related to the quasiparticle mass and the imaginary part to a finite width of their spectral functions (i.e. the imaginary parts of the propagators). In DQPM the determination of complex selfenergies for the partonic degrees of freedom at zero and finite $\mu_B$ has been performed by adjusting the entropy density to the lattice QCD (lQCD) data. The temperature-dependent effective coupling (squared) $g^2(T/T_c)$, as well as the effective masses and widths or the partons are based on this adjustment. The novel extended dynamical quasiparticle model, named "DQPM-CP", makes it possible to describe thermodynamical and transport properties of quarks and gluons in a wide range of temperature, $T$, and baryon chemical potential, $\mu_B$, and reproduces the equation-of-state (EoS) of lattice QCD calculations in the crossover region of finite $T, \mu_B$. We apply a scaling ansatz for the strong coupling constant near the critical endpoint CEP, located at $(T^{CEP}, \mu_B^{CEP}) = (0.100, 0.960)$ GeV. We show the equation-of-state as well as the speed of sound for $T > T_c$ and for a wide range of $\mu_B$, which can be of interest for hydrodynamical simulations. Furthermore, we consider two settings for the strange quark chemical potentials (I) $\mu_q = \mu_u = \mu_s = \mu_B/3$ and (II) $\mu_s = 0, \mu_u = \mu_d = \mu_B/3$. The isentropic trajectories of the QGP matter are compared for these two cases. The phase diagram of DQPM-CP is close to PNJL calculations. The leading order $p$QCD transport coefficients of both approaches differ. This elucidates that the knowledge of the phase diagram alone is not sufficient to describe the dynamical evolution of strongly interacting matter.

I. INTRODUCTION

The extension of the QCD phase diagram to a finite baryon chemical potential is a challenging task. It is believed that QCD matter undergoes a phase transition from the confined hadronic phase to the deconfined QGP phase if one increases the chemical potential at moderate temperatures and the transition line in $(T, \mu_B)$ is expected to terminate at a critical end-point. This is the least explored area of the QCD phase diagram but of particular interest for future experimental programs and theoretical studies (see recent review [1]). To realize these studies in a viscous hydrodynamical model one has to know the equation-of-state, EoS, of strongly interacting matter but also the transport coefficients. The time evolution of the QGP medium, produced in heavy-ion collisions (HICs), can also be addressed in microscopic transport approaches, which provide the time evolution of the degrees of freedom of the system. They require in addition the knowledge of the microscopic properties of the partonic degrees of freedom, such as effective masses, widths and cross sections, which all may depend on $\mu_B$ and $T$. The large value of the running coupling requires nonperturbative methods as lattice QCD (lQCD) calculations, or effective models with a phenomenological input. Therefore, it is notoriously difficult to estimate thermodynamic properties of the deconfined QCD matter, especially in the vicinity of a phase transition.

The lQCD calculations at vanishing baryon chemical potential are well established. However, due to the fermion sign problem, it is not at all easy to extend these calculations to a large baryon chemical potential. One possibility for exploring thermodynamic functions at $\mu_B > 0$ is to employ a Taylor expansion of the partition function in the vicinity of $\mu_B = 0$. It shares with other approaches, which were designed for $\mu_B = 0$, that the uncertainty of the predictions increases with the increase of $\mu_B$.

Here we present a new phenomenological model, the generalized quasiparticle model, DQPM-CP, for the description of non-perturbative features of the (strongly interacting) QCD. It reproduces the lQCD EoS for $\mu_B = 0$ as well as the first coefficient of the Taylor expansion

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towards finite $\mu_B$ but can be extended to a wide range of $\mu_B$. For this we combine the findings of DQPM, with a new parametrisation of the coupling constant. One of the important features of the DQPM-CP is the appearance of ‘critical’ scaling in the vicinity of the CEP. The main goal of the DQPM-CP is to provide the microscopic and macroscopic properties of the partonic degrees of freedom for the region of the phase diagram which is characterized by moderate $T$ and moderate or high $\mu_B$. Their knowledge allows subsequently to calculate the transport coefficients as well as the EoS, the ingredients of viscous hydrodynamic calculations.

In the present study we employ results from different methods such as IQCD calculations and results from the $N_f = 3$ PNJL model extended beyond mean field, because more rigorous approaches, such as Dyson-Schwinger equations (DSE) [2], functional renormalization group FRG [3] or pQCD/HTL calculations [4] can presently not cover the interesting observables in the full $(T, \mu_B)$-plane. There are no fully consistent calculations within a single approach, which includes the QGP thermodynamic observables and simultaneously the transport coefficients for the region of moderate and high $\mu_B$ or $\mu_q$. The presented results are model-dependent, however, qualitatively in agreement with the results from various effective models such as PNJL, NJL, LSM, and non-conformal holographic models. Only at moderate baryon density we can compare to the more rigorous methods such as IQCD, FRG, DSE. Nevertheless, the demand for an EoS and transport coefficients at moderate and high baryon chemical potentials, is high due to the ongoing investigation of heavy-ion collisions (HIC) by transport or/hydrodynamic models [11]. These investigations aim at the exploration of observables, which may carry information about this region of the phase diagrams. Therefore the presented results, even if they allow only for qualitative predictions, can be useful for transport studies, which have multiple issues to solve in the region of moderate baryon density while awaiting results from more rigorous approaches.

The main advantage of the use of quasiparticle models is the simple implementation in the transport framework for the evolution of the QGP matter. The DQPM has been implemented in the PHSD transport approach [5] [6], whereas the QPM [7] is implemented in the Catania transport approach [5]. Recently also results from approximate models of QCD, like that from NJL and PNJL models, have been implemented in the AMPT model [8] via scalar and vector potentials. The goal of the presented model is to interpolate the EoS and partonic properties such as effective masses, scalar potential and cross-sections between the region of high $T$ and $\mu_B = 0$ to the region of moderate $T$ and high $\mu_B$. In particular, we are currently working on the implementation of the DQPM-CP in the PHSD transport approach [5] [6].

The degrees of freedom of the DQPM are strongly interacting dynamical quasiparticles – quarks and gluons – with a broad spectral function, whose ‘thermal’ masses and widths increase with growing temperature. The knowledge of the $T$ and $\mu_B$ dependence of the mass of our degrees of freedom allows for the calculation of transport coefficients in lowest order in pQCD. They can be compared with the transport coefficients, calculated recently in the PNJL approach, which has a very similar phase diagram, but other degrees of freedom (interacting massless quarks and no gluons). The comparison of the transport coefficients shows that they depend indeed on the properties of the degrees of freedom and may be rather different in two theories with almost the same phase diagram.

The paper is organized as follows: In Sec. II we give a brief review of the basic ingredients of the dynamical quasiparticle model and its extension to the finite $\mu_B$ region. In Sec. III we discuss the thermodynamic observables for two setups of quark chemical potential: (I) $\mu_q = \mu_u = \mu_s = \mu_B/3$ and (II) $\mu_s = 0, \mu_u = \mu_d = \mu_B/3$. Furthermore, we study second order derivatives of the partition function, such as speed of sound and specific heat, and isentropic trajectories of the QGP matter. Further in Sec. IV we present transport coefficients of the DQPM-CP such as the specific shear viscosity and ratio of electric $\sigma_{QQ}/T$, baryon $\sigma_{BB}/T$ and strange $\sigma_{SS}/T$ conductivities to temperature based on the relaxation time approximation of the Boltzmann equation. In addition, we show the ratio of dimensionless transport coefficients for the full range of chemical potentials. We finalize our study with conclusions in Sec. V.

II. BASIC PROPERTIES OF THE QUASIPARTICLE MODEL

A. Main ingredients of the off-shell quasiparticle models

In the dynamical quasiparticle model, DQPM [6] [10]–[12], the QGP medium is described in terms of strongly interacting quasiparticles, quarks and gluons. The quasiparticles are massive and can be characterized by broad spectral functions $\rho_i (\omega, q, g)$, which are no longer $\delta$-functions in the invariant mass squared but given by

$$\rho_i (\omega, p) = \frac{\gamma_i}{E_{i,p}^2} \left( \frac{1}{(\omega - E_{i,p})^2 + \gamma_i^2} - \frac{1}{(\omega + E_{i,p})^2 + \gamma_i^2} \right) = \frac{4 \omega \gamma_i}{(\omega^2 - p^2 - m_i^2)^2 + 4 \gamma_i^2 \omega^2}. \tag{1}$$

Here, we introduced the off-shell energy $E_{i,p} = \sqrt{p^2 + m_i^2 - \gamma_i^2}$, with $m_i, \gamma_i$ are being the pole mass and width, which differ for quarks, antiquarks and gluons. Then the quasiparticle (retarded) propagators can be expressed in the Lehmann representation via the spectral
and the number of flavors, respectively. $C_q = \frac{N_c^2 - 1}{2N_c} = 4/3$ and $C_g = N_c = 3$ are the QCD color factors for quarks and for gluons, respectively. The strange quark has a larger bare mass which needs to be considered in its dynamical quasi particle pole mass. We fix $m_\pi(T, \mu_B) = m_\pi(T, 0) + \Delta m$ and $\Delta m \approx 30 \text{ MeV}$ [6].

Furthermore, the quasiparticles in DQPM have thermal widths, which are adopted in the form [14, 15]

$$\gamma_j(T, \mu_B) = \frac{1}{3} C_j g^2(T, \mu_B) T \ln \left( \frac{2c_m}{g^2(T, \mu_B)} + 1 \right).$$  \hspace{1cm} (5)

The parameter $c_m$, which is related to a magnetic cut-off, is fixed to $c_m = 14.4$.

In the DQPM the coupling constant at $\mu_B = 0$ is parameterized employing the entropy density $s(T, \mu_B = 0)$ from lattice QCD calculations of Refs. [16, 17] in the following way:

$$g^2(T, \mu_B = 0) = d \left( \left( s(T, 0)/s_{SB}^{QCD} \right)^e - 1 \right)^f,$$  \hspace{1cm} (6)

with the Stefan-Boltzmann entropy density $s_{SB}^{QCD}/T^3 = 19/9 \pi^2$ and the dimensionless parameters $d = 169.934$, $e = -0.178434$ and $f = 1.14631$.

We note that the DQPM has been used to explore the crossover region in the phase diagram by introducing an effective coupling constant which depends on the baryon chemical potential. In this region of a moderate baryon chemical potential the basic thermodynamic observables, computed in lQCD, show a smooth $\mu_B$ dependence. Therefore we expect a similar behaviour for the effective coupling.

The effective coupling at finite baryon chemical potential $\mu_B$ is obtained by applying the 'scaling hypothesis' introduced in [11]. It assumes that $g^2$ is a function of the ratio of the effective temperature

$$T^* = \sqrt{T^2 + \mu_B^2/\pi^2}$$  \hspace{1cm} (7)

(where the quark chemical potential is defined as $\mu_q = \mu_u = \mu_s = \mu_B/3$ ) and the $\mu_B$-dependent critical temperature $T_c(\mu_B)$ as [15]:

$$T_c(\mu_B) = T_c(0)(1 - \alpha \mu_B^2)^{1/2},$$  \hspace{1cm} (8)

where $T_c(0)$ is the critical temperature at vanishing chemical potential ($\approx 0.158 \text{ GeV}$) and $\alpha = 0.974 \text{ GeV}^{-2}$. Thus, the DQPM effective coupling $\alpha_S^{DQPM}(T, \mu_B)$ reads

$$\alpha_S^{DQPM}(T, \mu_B) \equiv \begin{cases} 
\mu_B = 0 : g^2(T, \mu_B = 0)/(4\pi) \\
\mu_B > 0 : g^2(T_{\text{scale}}(T, \mu_B))/(4\pi), \\
\text{with } T_{\text{scale}} = \frac{T_c(\mu_B)}{T_c(0)}.
\end{cases}$$  \hspace{1cm} (9)

Having fixed the quasiparticle properties (or propagators) as described above, one can evaluate
the basic thermodynamic observables: the entropy density
$s(T,\mu_B)$, the pressure $P(T,\mu_B)$ and energy
density $\epsilon(T,\mu_B)$ in a straight forward manner by
starting with the quasiparticle entropy density and
number density. The entropy density and the quark
number density follow from the same thermodynamic
potential $\Omega[\Delta, S_q]$ [18, 19], which is expressed as a
functional of the full quasiparticle propagators for
gluons and quarks($\Delta, S_q$) in a symmetry-conserving
($\Phi$-derivable) two-loop approximation:

\[ s_{\text{dqp}} = \] (10)

\[- \frac{d_q}{2\pi (2\pi)^3} \frac{\partial f_q}{\partial T} \left( \text{Im} \ln \Delta - \text{Im} \, \text{Re} \, \Delta \right) \]

- \frac{d_q}{2\pi (2\pi)^3} \frac{\partial f_q(\omega - \mu_q)}{\partial \mu_q} \left( \text{Im} \ln S_q - \text{Im} \Sigma_q \, \text{Re} \, S_q \right) \]

- \frac{d_q}{2\pi (2\pi)^3} \frac{\partial f_q(\omega + \mu_q)}{\partial \mu_q} \left( \text{Im} \ln S_q - \text{Im} \Sigma_q \, \text{Re} \, S_q \right) \]

where $f_q(\omega)$ and $f_q(\omega - \mu_q)$ denote the Bose-Einstein
and Fermi-Dirac distribution functions (see Eq. [25]),
respectively, while $\Delta = (p^2 - \Pi)^{-1}$, $S_q = (p^2 - \Sigma_q)^{-1}$
and $\Sigma_q = (p^2 - \Sigma_q)^{-1}$ stand for the full (scalar) quasiparticle
propagator of gluons $g$, quarks $q$ and antiquarks $\bar{q}$.
In Eq. (10) we consider for simplicity scalar
(retarded) quasiparticle self-energies $\Pi = \Sigma = \Sigma_q \approx \Sigma_q$,
which are expressed via dynamical masses and widths
as $\Pi = m_q^2 - 2i\gamma_q \omega_i$, where, for the off-shell case,
$\omega_i$ is an independent variable. Furthermore,
the number of transverse gluonic degrees of freedom is
d$q = 2 \times (N_c^2 - 1)$ while for the fermion degrees
of freedom we use $d_q = 2 \times N_c$ and $d_\bar{q} = 2 \times N_c$.

B. Extension of quasiparticle DQPM-CP effective
coupling constant for the inclusion of the CEP

Now we proceed with the extension of the DQPM to
the region of large $\mu_B$ where a possible critical end-point
is located. In order to extent the quasiparticle model to
the large $\mu_B$ region and to describe the critical behavior
near the CEP we depart from the ‘scaling hypothesis’,
used for the moderate baryon chemical potentials in
crossover region, and introduce a simple parametrization
of the coupling constant as a function of the scaled
temperature and the baryon chemical potential. To
simplify an extension of the effective coupling for finite
$\mu_B$ we parametrize $\alpha_s^{\text{DQPM-CP}}$ as a function of a
dimensionless scaled temperature $x_T = T/T_c$.
We determine first the parameters at vanishing quark or
baryon chemical potential by fitting extracted values
of $g_0^{\text{DQPM}}(T,\mu_B=0)$ from Eq. (6) as a function
of $f(T/T_c,\mu_B=0)$ ($T_c = 0.158$) with help of the nonlinear
least-squares (NLLS) Marquardt-Levenberg algorithm.
Later we use critical line values of $T_c$ for each value of
baryon chemical potential. The critical line of the present
model, which is an input parameter for our calculations,
reads:

\[ T_c(\mu_B) = T_c(0)[1 - \kappa_{PNJL}(\mu_B/T_c(0))^2], \] (12)

where $T_c(0) = 0.158$ GeV is fixed in accordance with
the results from IQCD [21,22], while $\kappa_{PNJL} = 0.00989$
corresponds to the estimates from the PNJL model [23].

Figure 2 shows a comparison of the critical lines of the
DQPM (green dashed line), of the DQPM-CP and the
predictions from the IQCD calculations. The DQPM-
CP phase boundary, given by Eq. (12), is shown as a
black dashed-dotted line in the crossover region, means
for moderate baryon chemical potentials. The critical
endpoint in the presented model is located at $(T^{\text{CEP}},
\mu_B^{\text{CEP}}) = (0.100, 0.960)$ GeV.

The exact location of the CEP is an open question and
there are many predictions from various methods
(for a compilation of theoretical predictions for $(T^{\text{CEP}},
\mu_B^{\text{CEP}})$,
\( \mu_B^{CEP} \) we refer the reader to Fig. 6 in Ref. [24], and to Fig. 19 in Ref. [25]. Current state-of-the-art lQCD results disfavor a critical point for \( \mu_B/T \leq 3 \) [16] 22 20 27. Furthermore, it has been found that the temperature of the hypothetical chiral critical end point should not exceed the critical temperature of the chiral phase transition (for \( m_u = m_d = 0 \) \( T_0^0 = 132^{+3}_{-6} \) MeV [28] 29). Recently, on approximate position of the chiral CEP (for vanishing external magnetic field) of \( \mu_B^{CEP} = 0.800(0.140) \) GeV has been conjectured by the lQCD simulations of finite density QGP under external magnetic fields [30]. It has been shown that the critical line of the PNJL model generally depends on the parameters of the model [31]. Many (P)NJL model predictions of the location of the CEP lie at high \( \mu_B \approx 0.8 - 1 \) GeV [24], for instance for the \( N_f = 2 \) NJL model in Ref. [32] \( (T^{CEP},\mu_B^{CEP}) = (0.081,0.987) \) GeV \( (\mu_B^{CEP} = 3\mu_q^{CEP}) \), while for the \( N_f = 3 \) PNJL model considered in Ref. [33] \( (T^{CEP},\mu_B^{CEP}) = (0.121,0.875) \) GeV.

We base this study on the predictions from the extended beyond mean field \( N_f = 3 \) PNJL model [23]. However, in order to fit lattice results at moderate \( \mu_B \leq 0.6 \) GeV we use the value of the pseudo-critical temperature at \( \mu_B = 0 \) from lQCD estimates. First, the value of the baryon chemical potential of the hypothetical CEP \( \mu_B^{CEP} \) is chosen in accord with predictions from the PNJL model, then the temperature of the CEP follows from the chosen critical line \( T_c(\mu_B) \). The chosen location of the hypothetical CEP is within the allowed range of the lQCD estimates. The first-order phase transition is shown as a solid black line. The DQPM phase boundary for moderate baryon chemical potentials, \( \mu_B \leq 0.6 \) GeV, given by Eq. (8), is shown as a dashed green line. The colored areas illustrate the predictions from the lQCD calculations for QCD with \( N_f = 2 + 1 \): grey area - from Ref. [29], red area - from Ref. [21], violet area - from Ref. [22].

To interpolate the EoS and microscopic properties of quarks and gluons between the region of the vanishing baryon chemical potential and the asymptotic behavior in the region of high baryon density \( \mu_B \gg T \) \( (T > T_c(\mu_B)) \) we employ a simple ansatz for the \( \mu_B \)-dependence, which reflects the decrease of the effective coupling with \( \mu_B \). We assume that the coupling constant does not depend explicitly on the temperature but only on the scaled temperature \( (x_T = T/T_c(\mu_B)) \) \( (T_c \text{ varies with } \mu_B \text{ according to the chosen critical line}) \) for all \( \mu_B \geq 0 \). Furthermore, we introduce an additional factor \( \sigma(\mu_B) = 1 \) at \( \mu_B = 0 \) to take into account the decrease of the coupling constant with \( \mu_B \) at moderate \( x_T \). At high \( T \) \( (T/T_c(\mu_B) \equiv x_T \gg 1) \) the effective coupling constant is not affected by the baryon chemical potential. Therefore, the DQPM-CP coupling constant can be parametrized as a function of a scaled temperature and \( \mu_B \):

\[ \alpha_{SC}^{cross} = a_0 + \frac{a_2}{x_T} - \frac{a_3}{x_T^3} + \frac{a_4}{x_T^4} + \frac{a_5 \cdot \sigma(\mu_B)}{x_T^6}. \]  

Here the coefficients \( a_i \) are fixed at \( \mu_B = 0 \) (where \( \sigma(\mu_B = 0) = 1 \)) by fitting the DQPM coupling constant \( g^2(T,\mu_B = 0) \) obtained from Eq. (10) (see comparison of the basic thermodynamic observables from DQPM-CP and lQCD predictions in Fig. 6): \( a_0 = 0.25, a_2 = 1.77, a_3 = 2.17, a_4 = 2.13, a_6 = 0.85 \).

The motivation to use the decrease in the effective coupling is based on the expectations of the QGP matter to approach the non-interacting Stefan-Boltzmann limit at large \( \mu_B \) on the order of a few GeV and small temperatures \( \mu_B \gg T \) (see recent pQCD results on the pressure in Ref. [34]). Therefore, it is reasonable to assume that the coupling constant also decreases at \( T_c \).
with increasing $\mu_B$. To describe decrease of the coupling constant near the $T_c$ with $\mu_B$ we introduced an additional factor, affecting the region near the phase transition:

$$\sigma(\mu_B) = 1 - \sigma_2 \mu_B^2 - \sigma_4 \mu_B^4,$$

where $\sigma_2 = 0.45\text{GeV}^{-2}$ and $\sigma_4 = 0.15\text{GeV}^{-4}$. We fixed the values of $\sigma_2$ and $\sigma_4$ by fitting the quasiparticle entropy from Eq. (10) to the IQCD data points of the entropy density from the BMW collaboration [16] at finite $\mu_B = 0.10, 0.2, 0.3, 0.4 \text{ GeV}$ for given temperature points $T_c(\mu_B) < T < T_{\text{max}}$, where $T_c(\mu_B)$ denotes the critical temperature and $T_{\text{max}} = 0.4 \text{ GeV}$.

We note that the adjustment of the effective coupling constant is made in order to interpolate results for thermodynamic observables between the region of vanishing baryon chemical potential and that of the high baryon chemical potential. High and moderate temperatures above the phase transition line $T > T_c(\mu_B)$ are considered. The aim of the model is to describe on the one side qualitatively the behavior of the thermodynamic observables in the regions of high/moderate baryon density for $T > T_c(\mu_B)$, and on the other side to reproduce the IQCD EoS the region of moderate baryon chemical potential. For quantitative results one has to refer to more rigorous approaches. To verify the $\mu_B$-dependence of the effective coupling we compare in Fig. 4 the ratio $p/p_{SB}(T = 0)$ ($p_{SB}(T = 0) = \frac{\mu_B^4}{108\pi^2}$) from DQPM-CP calculations for high $\mu_B$ with the pQCD calculations from Ref. [24]. Although the uncertainties of the pQCD results are quite large, we see that the resulting pressure from the DQPM-CP is compatible with the pQCD predictions.

Furthermore, to accumulate 'critical' behaviour near the CEP, where the phase transition is of second order, we use an additional 'critical' term for the coupling constant. The goal of this term is to describe the critical behaviour at the second order phase transition for the microscopic and thermodynamic quantities. To obtain the parametrization of the 'critical' coupling constant, we fit the entropy density to the results from the PNJL [28]. The resulting parametrization for the 'critical' coupling constant is given by

$$\alpha^{\text{crit}}_S = a \cdot (T/T_c)_{\text{CEP}}^{-12},$$

where $a = \alpha^{\text{cross}}_S(T = T_{\text{CEP}})$.

The total coupling constant $\alpha^{\text{DQPM-CP}}_S$ then reads

$$\alpha^{\text{DQPM-CP}}_S = \begin{cases} \mu_B = \mu_{\text{CEP}}: & \alpha^{\text{CEP}}_S = \frac{1 - F(T)}{2} \alpha^{\text{crit}}_S + \frac{1 + F(T)}{2} \alpha^{\text{cross}}_S, \\ \mu_B \neq \mu_{\text{CEP}}: & \alpha^{\text{cross}}_S \end{cases}$$

$\alpha^{\text{cross}}_S$ corresponds to the coupling constant for the crossover region defined by Eq. (13), while at $\mu_B = \mu_{\text{CEP}} = 0.960 \text{ GeV}$ the effective coupling $\alpha^{\text{DQPM-CP}}_S$ in Fig. 4 shows the effective coupling of the DQPM-CP at fixed $\mu_B = 0$ (black solid line) and $\mu_B = 0.96 \text{ GeV}$ (red points) as a function of the scaled temperature $T/T_c$. At $\mu_B = 0$ the coupling constant equals the DQPM effective coupling $g^2(T, \mu_B = 0)$ obtained from Eq. (13).

$\alpha^{\text{CEP}}_S$ includes the additional 'critical' contribution $\alpha^{\text{crit}}_S$, defined by Eq. (15). To match the two coupling constants we employ the smoothing function:

$$F(T) = \tanh \left[ \frac{T - 0.1004}{\delta T} \right],$$

where $\delta T = 0.002 \text{ GeV}$ is the region in the vicinity of the CEP, where the two coupling constants have to match. The values of $\delta T$ and $T$ are chosen in accordance with the $T/T_c$ - dependence of the PNJL entropy density. While the temperature $T_0 = 0.1004 \text{ GeV}$ regulates the size or temperature range $(T_{\text{CEP}}, T_0)$ of the critical contribution, $\delta T$ affects the derivatives of the pressure at $T_0$. However, a slight change of $T_0$ as well as $\delta T$ up to 20% will not change the qualitative results; the effect of an increase of the second-order derivatives will be less pronounced for smaller $T_0$.

The scaling behavior of the quasiparticle masses has been employed in condensed matter physics [37], where the interaction with bosonic fluctuations near the critical point causes a divergence in the effective masses of the quasiparticles. We note that one can include the scaling behavior of the thermodynamic observables in a more rigorous way as done in Ref. [35], where the EoS from the
The effective masses (a) and widths (b) of quarks and gluons in the DQPM-CP from Eqs. (4-5) along the critical line (given by Eq. (12)) as function of baryon chemical potential $\mu_B$. The dashed line represents the critical value of the baryon chemical potential $\mu_{CEP} = 0.96$ GeV.

\[ \frac{m_i}{\gamma_i} \propto a_i + b_i \frac{\mu^2_B}{T^2} g(T, \mu_B) \ln \left( \frac{2c_m/g^2(T, \mu_B)}{1} \right), \]

where we use the shorthand notation for constants $a_i = 1$ (for quarks), $1 + \frac{N_f}{2N_c}$ (for gluons), $b_i = 1$ (for quarks), $3/2$ (for gluons). For vanishing chemical potential the ratio $m_i/\gamma_i \approx 9$ for quarks and $\approx 6$ for gluons. The ratio increases with $\mu_B$ since the coupling constant decreases with $\mu_B$, for instance at $\mu_B = 1$ GeV: we set $m_i/\gamma_i \approx 16$ for quarks and $\approx 11$ for gluons.

Importantly, we see that in DQPM-CP the quark masses are larger than a third of the free proton mass. This means that the production of baryons across the critical line, the dominant process for large $\mu_B$ and small $T$, is an exothermic process in DQPM-CP.

Figure 6. The scaled pressure $P(T)/T^4$ (blue line), entropy density $s(T)/T^3$ (red line), scaled energy density $\epsilon(T)/T^4$ (orange line), and interaction measure $I(T)/T^4$ (green line), from the DQPM-CP in comparison to the lQCD results from Refs. 16, 17 (circles) for zero baryon chemical potential.

In this Section we consider the basic thermodynamic observables from the DQPM-CP for finite chemical potential. Starting point for the calculation of the thermodynamic functions in the dynamical quasiparticle models is the evaluation of the entropy density and the quark densities via the propagators as described in Eqs. (10) and (11).

Then it is straightforward to derive the pressure...
Figure 7. Scenario: $\mu_q = \mu_u = \mu_s = \mu_B/3$. From top to bottom: Scaled pressure $p/T^4$, entropy density $s/T^3$, and scaled energy density $\epsilon/T^4$ from the DQPM (lines) as a function of temperature $T$ at various values of $\mu_B$ [GeV]. The lQCD results obtained by the BMW group are taken from Refs. [16, 17] (circles) and from Ref. [27] (squares). The dashed line displays the critical temperature $T_{CEP} = 0.10$ GeV.

Figure 8. Scenario: $\mu_s = 0, \mu_u = \mu_B/3$. From top to bottom: Scaled pressure $p/T^4$, entropy density $s/T^3$, and scaled energy density $\epsilon/T^4$ from the DQPM-CP (lines) as a function of temperature $T$ at various values of $\mu_B$ [GeV]. The lQCD results obtained by the BMW group are taken from Refs. [16, 17] (circles). The PNJL results for the entropy density (colored area) are taken from Ref. [23].
$p$ and later the energy density, employing the Maxwell relation for a grand canonical ensemble:

$$p(T, \mu_B) = p_0(T, 0) + \int_0^{\mu_B} n_B(T, \mu_B') \, d\mu_B'. \quad (19)$$

For the pressure at $\mu_B = 0$ we use the lQCD parametrization of the pressure $p_0(T, 0)$ from Ref. [10,17]. The energy density $\epsilon$ then follows from the Euler relation

$$\epsilon = Ts - p + \sum_i \mu_i n_i. \quad (20)$$

Furthermore, the interaction measure is defined as:

$$I \equiv \epsilon - 3P = Ts - 4p + \sum_i \mu_i n_i, \quad (21)$$

which vanishes in the non-interacting limit of massless degrees of freedom at $\mu_B = 0$. The scaled pressure, entropy density and energy density of the QGP phase are supposed to increase with the temperature. However, lQCD calculations of the thermodynamic observables show [39], that the massless non-interacting limit can not be reached even at temperatures of $T \sim 1$ GeV.

We consider two setups for the quark chemical potentials: (I) $\mu_q = \mu_u = \mu_s = \mu_B/3$ and (II) $\mu_s = 0, \mu_u = \mu_B/3$. The quark chemical potential can be related to the strange, baryon and electric charge chemical potentials as $\mu_i = B_i \mu_B + Q_i \mu_Q + S_i \mu_S$, where $B_i, Q_i$ and $S_i$ are baryon number, electric charge and strangeness of the considered quark. Herein we fix $\mu_Q = 0$, therefore for the symmetric QGP matter (I) $\mu_q = \mu_u = \mu_s = \mu_B/3$ the strange and the electric charge potentials are vanishing $\mu_S = \mu_Q = 0$, while for (II) $\mu_s = 0, \mu_u = \mu_B/3$ the strange chemical potential is finite $\mu_S = \mu_B/3$.

The T-dependence of the thermodynamic quantities such as the scaled entropy density, the pressure and the energy density from the DQPM-CP for various baryon chemical potentials $0 \leq \mu_B \leq 0.99$ GeV is shown in Fig. 7 ($\mu_q = \mu_u = \mu_s = \mu_B/3$) and Fig. 8 ($\mu_s = 0, \mu_u = \mu_B/3$). For setup (I) we found a good agreement between the DQPM-CP results (lines) and results from lQCD, obtained by the BMW group [16,17] at $\mu_B = 0$ and $\mu_B = 400$ MeV. The thermodynamical quantities increase with $\mu_B$. When approaching the CEP at $\mu_B = 0.96$ GeV the values of the entropy density, of the energy density as well as of the quark or baryon density rise suddenly.

For setup (II) we compare the results for the entropy density to that of the Nantes PNJL approach [23]. The DQPM-CP results are in agreement with the PNJL results in the high temperature region $T \geq 0.3$ GeV, while in the vicinity of the phase transition there is a clear deviation from the PNJL results, which can be expected since the two models encompass different microscopic properties of the degrees of freedom. The resulting values of the thermodynamic observables for setup (II) is smaller than for setup (I) since the contribution from the strange quarks to the quasiparticle entropy density (see Eq. (10)) is smaller for $\mu_s = 0$ mainly due to the derivatives $\frac{\partial f_q(\omega - \mu_q)}{\partial T}$.

![Figure 9. Scenario: $\mu_q = \mu_u = \mu_s = \mu_B/3$. The speed of sound squared $c_s^2$ from the DQPM-CP for a crossover phase transition ($0 \leq \mu_B < 0.96$) as a function of $T$ and $\mu_B$.](image)

**A. Approaching the CEP from the deconfined phase**

To realize a critical behaviour of the thermodynamic observables in the vicinity of the CEP we introduce, as described in Sec. II B, the ‘critical’ contribution to the coupling constant that affects the microscopic and macroscopic quantities. At the CEP, where the transition is of second order, the entropy density and baryon density increase rapidly but remain finite, while the quark susceptibility and the specific heat $C_V/T^3 = \frac{d\epsilon}{dT} - \frac{dP}{dT}$ diverge. Therefore, the speed of sound (squared) vanishes as one approaches the CEP. We consider the speed of sound and the specific heat at fixed $\mu_B$. For fixed $\mu_B$ the speed of sound can be expressed as:

$$c_s^2 = \frac{dp}{d\epsilon} = \frac{dp/dT}{d\epsilon/dT} = \frac{s}{C_V}. \quad (22)$$

The speed of sound squared in the DQPM-CP is depicted in Fig. 9 as a function of temperature $T$ and baryon chemical potential $\mu_B$ in the crossover region, where $\mu_B \leq 0.95$ GeV. The resulting $c_s^2$ increases with temperature and decreases near the phase transition with increasing $\mu_B$. In Fig. 10(a) we show the comparison
of the DQPM-CP results for $c_s^2$ at vanishing $\mu_B$ with the available lQCD estimations from the Wuppertal-Budapest collaboration [17] (light red circles) and the HotQCD collaboration [26] (blue triangles). The DQPM-CP results are in agreement with the lattice QCD predictions within the estimated errors. Figure 10 (b) shows the speed of sound squared $c_s^2$ from the DQPM-CP as a function of the temperature for a wide range of baryon chemical potentials, including the region of the CEP. At high temperatures, values of $c_s^2$ are approaching the limit of a non-interacting gas of massless quarks and gluons (SB limit, black dash-dotted line) $c_s^2(SB) = 1/3$. When increasing the baryon chemical potential the speed of sound near the transition temperature decreases, while at the CEP the speed of sound undergoes a sharp decrease.

The DQPM-CP results for the scaled specific heat $C_V/T^3$ as a function of $T$ are presented in Fig. 11. As compared to the speed of sound, the specific heat shows an opposite tendency near the phase transition. For moderate values of the baryon chemical potential $\mu_B$ the scaled specific heat increases moderately with decreasing temperature. As it approaches the CEP, $C_V/T^3$ diverges as a function of $T$, which is consistent with the expectations for a second-order phase transition. The $T$-dependence of the specific heat for $\mu_B = 0.96$ GeV near the CEP enables us to estimate the value of the critical exponent for $T > T_{CEP}$:

$$\ln(C_V) = -\alpha \cdot \ln(T - T_{CEP}) + \text{const.}$$  \hspace{1cm} (23)

For the presented parametrization of the coupling constant we obtain the following values: $\alpha = 0.63 \pm 0.02$ and $\text{const.} = -5.48 \pm 0.01$. The value of the critical exponent $\alpha$ is in agreement with the predictions from the PNJL model for $T > T_{CEP}$ $\alpha_{PNJL} = 0.68 \pm 0.01$ [31] and the expectations from the universality argument $\alpha = 2/3$ in Ref. [10].

In the case of QCD with finite quark masses both the chiral and center symmetries are explicitly broken. What remains is the $Z(2)$ sign symmetry of the order parameter of the chiral phase transition. Therefore it has been assumed that the CEP of QCD with finite quark masses belongs to the three-dimensional $Z(2)$ Ising universality class [11][13]. The corresponding critical exponent in the $Z(2)$ universality class is $\alpha \approx 0.11$ [14]. However, it is known that the critical exponents in the PNJL(NJL) model and $Z(2)$ universality class differ [31][45][48].

To explore the high density region, it is essential for effective models to consider isentropic trajectories for which the ratio of entropy to baryon number is held

![Figure 10](image1.png)  \hspace{1cm} ![Figure 11](image2.png)

Figure 10. Scenario: $\mu_q = \mu_u = \mu_s = \mu_B/3$. The speed of sound squared $c_s^2$ from the DQPM-CP for (a) $\mu_B = 0$ and (b) $\mu_B \geq 0$ as a function of $T$ compared to lQCD results for $\mu_B = 0$ obtained by the Wuppertal-Budapest collaboration [17] (light red circles) and the HotQCD collaboration [26] (blue triangles).

Figure 11. Scenario: $\mu_q = \mu_u = \mu_s = \mu_B/3$. The specific heat $C_V/T^3$ from the DQPM-CP at fixed $\mu_B$ as a function of $T$ compared to lQCD results for $\mu_B = 0$ of the HotQCD collaboration [26].
The presence of the CEP can affect the isentropic trajectories, since the entropy density and baryon density undergo a rapid change as the phase transition is approached. It is supposed that the CEP acts as an attractor of isentropic trajectories \[ \text{[51]} \]. Moreover, a different choice for the strange quark chemical potential affects the trajectories as well. Therefore we compare the resulting isentropic trajectories for two setups of the strange chemical potential.

Figure 12 displays the isentropic trajectories from the DQPM-CP for (a) \( \mu_q = \mu_s = \mu_B/3 \) and (b) \( \mu_s = 0, \mu_{u} = \mu_{d} = \mu_B/3 \) in the phase diagram. Comparing (a) to (b) one can clearly see that the trajectories for the zero strange quark chemical potential are shifted towards higher \( \mu_B \) values. In the case of vanishing chemical potential of the strange quark \( \mu_s = 0, \mu_{u} = \mu_{d} = \mu_B/3 \), the entropy density, which has also contributions from the light (anti-)quarks and gluons, is less affected than the baryon density. Therefore, for finite \( \mu_B > 0 \) and \( \mu_s = 0 \) the ratio \( s/n_B \) is larger than in the case of a symmetric setup \( \mu_s = \mu_u = \mu_B/3 \) and the value of the baryon density decreases faster than the entropy density. This observation is in agreement with previous studies of the PNJL model \[33\] and the results from Ref. \[38, 52\], where the lQCD EoS from the WB collaboration \[16, 17, 53\] with a critical point in the 3D Ising model universality class is considered for moderate baryon chemical potentials \( \mu_B \leq 0.45 \text{ GeV} \). Thus, a ‘critical’ trajectory, which goes through the CEP, for (a) corresponds to \( s/n_B \approx 13.35 \), for (b) corresponds to \( s/n_B \approx 15 \). The comparison of isentropic trajectories in a vicinity of the CEP is presented in Fig. 12 (c). In the vicinity of the CEP, the trajectories with \( s/n_B = 15, 13.35, 12 \) shown in Fig. 12 (c) are focussed to the critical endpoint.

### IV. TRANSPORT COEFFICIENTS

We continue to investigate the transport properties of QGP matter using the DQPM-CP. We consider the specific shear \( \eta/s \) and bulk \( \zeta/s \) viscosities, the ratio of electric \( \sigma_{QQ}/T \), baryon \( \sigma_{BB}/T \) and strange \( \sigma_{SS}/T \) conductivities to temperature. At vanishing baryon chemical potential the DQPM-CP model equals the DQPM, therefore one can find the comparison of DQPM transport coefficients at \( \mu_B = 0 \) with the recent results from various approaches in previous papers \[0, 54, 55\].

All transport coefficients are calculated within the Relaxation Time Approximation (RTA) of the Boltzmann equation. In the relaxation time approximation (in first order in the deviation from
equilibrium) the collision term is given by [56]

$$\sum_{j=1}^{N_{\text{species}}} C_{ij}^{(1)}[f_i] = -\frac{E_i}{\tau_i} (f_i - f_i^{(0)}) = -\frac{E_i}{\tau_i} f_i^{(1)} + \mathcal{O}(\text{Kn}^2),$$

(24)

where $\tau_i$ is the relaxation time in the heat bath rest system for the particle species $i$, Kn $\sim l_{\text{micro}}/l_{\text{macro}}$ is the Knudsen number which denotes the ratio between the relevant microscopic scale (mean free path) over the characteristic length scale of the system. The equilibrium state of the system is described by the Bose-Einstein and Fermi-Dirac distribution functions

$$f_i^{(0)}(E_i, T, \mu_i) = \frac{1}{\exp((E_i - \mu_i)/T) - a_i},$$

(25)

where $\mu_i$ is the quark chemical potential, $E_i = \sqrt{p_i^2 + m_i^2}$ is the on-shell quark/gluon energy, $a_i = +1$(gluons), $-1$(anti-)quarks). In Eq. (24) $f_i^{(1)}(x, k, t)$ contains $\delta f_i(x, k, t)$, which is the nonequilibrium part to first order in gradients.

The first step in the calculation of the transport coefficients within the RTA framework is the estimation of relaxation times, which are supposed to depend on the momentum of the partons, on the temperature and on the baryon chemical potential. The momentum dependent relaxation time can be expressed through the on-shell interaction rate in the rest system of the medium, in which the incoming quark has a four-momentum $P_i = (E_i, p_i)$:

$$\tau_i^{-1}(p_i, T, \mu_q) = \Gamma_i(p_i, T, \mu_q)$$

$$= \frac{1}{2E_i} \sum_{j=1}^{N_{\text{species}}} \frac{1}{1 + \delta_{ij}} \int \frac{d^3p_j}{(2\pi)^32E_j} d_q f_j^{(0)}(E_j, T, \mu_q)$$

$$\times \int \frac{d^3p_c}{(2\pi)^32E_c} \int \frac{d^3p_d}{(2\pi)^32E_d} |\mathcal{M}|^2(p_i, p_j, p_c, p_d)$$

$$\times (2\pi)^4 \delta^{(4)}(p_i + p_j - p_c - p_d)(1 - f_c^{(0)})(1 - f_d^{(0)}),$$

(26)

where $|\mathcal{M}|^2$ denotes the matrix element squared averaged over the color and spin of the incoming partons, and summed over those of the final partons. The $|\mathcal{M}|^2$ is calculated by the use of the effective coupling and propagators in leading order (for further details see [6]). The notation $\sum_{j=q,\bar{q},g}$ includes the contribution from all possible partons which in our case are the gluons and the (anti-)quarks of three different flavors ($u, d, s$). The quark relaxation time is expected to become very large near CEP, since the correlation length increases rapidly close to the CEP.

A. Specific viscosities

We start with the most common transport coefficients for the hydrodynamical simulations - the shear and bulk viscosity. The viscosities of the QCD matter have been studied within a variety models in the confined and the deconfined phases. The shear viscosity reveals the strength of the interaction inside the QCD medium, in particular within the kinetic theory it can be related to the hadron or parton interaction rates, which is a challenge to evaluate on the basis of first principles. A plethora of theoretical model predictions show that the temperature dependence of the QCD shear viscosity over entropy density $\eta/s$ is qualitatively different for the two phases. Starting from the hadronic phase below the phase transition $T < T_c$, $\eta/s$ monotonically decreases with $T$ since the system is dominated by
pions with weaker interactions at lower $T$. While above the phase transition $T > T_c$, $\eta/s$ increases with temperature because the interaction attenuates at high $T$. Approaching the phase transition from hadronic to the QGP phase at vanishing chemical potential, $\eta/s$ has a wide dip followed by an increase with temperature. A similar property of the temperature dependence of the specific shear viscosity $\eta/s$ is seen for other fluids such as $H_2O, He$ and $N_2$.

The specific bulk viscosity of the QGP matter is predicted to be low, yet it is expected to be finite near the phase transition. The presence of the bulk viscosity reduces the speed of the fluid radial expansion and hence affects the mean momentum of the produced particles. For conformal fluids, the bulk viscosity is known to be identically zero, and the deconfined QCD medium is expected to adopt a conformal behaviour in the high-energy or temperature regime. Nevertheless, the lQCD results on the enhanced trace anomaly close to the energy or temperature regime. Nevertheless, the lQCD results on the enhanced trace anomaly close to $T_c$ have shown that it is probably not the case for the deconfined QCD medium in the vicinity of the phase transition.

The shear and bulk viscosity for quasiparticles with medium-dependent masses $m_i(T,\mu_B)$ can be derived using the Boltzmann equation in the RTA through the relaxation time:

$$\eta(T,\mu_B) = \frac{1}{15T} \sum_{i=q,g} \int \frac{d^3p}{(2\pi)^3} \frac{p^4}{E_i} \tau_i(p, T, \mu_B) \times d_i(1 \pm f_i)f_i,$$

$$\zeta(T,\mu_B) = \frac{1}{9T} \sum_{i=q,g} \int \frac{d^3p}{(2\pi)^3} \tau_i(p, T, \mu_B)$$

$$d_i(1 \pm f_i) \left[ \frac{c_s^2}{E_i^2} \left( \mathbf{p}^2 - 3c_s^2(E_i^2 - T^2 \frac{dm_i^2}{dT^2}) \right) \right]^{1/2},$$

where $d(q) = u, d, s(u, \bar{u}, \bar{d}, \bar{s})$, $d_q = 2N_c = 6$ and $d_g = 2(N_f^2 - 1) = 16$ are the degeneracy factors for spin and color for quarks and gluons, respectively, $\tau_i$ are the relaxation times. $c_s$ is the speed of sound for a fixed $\mu_B$ given by Eq. 22, $\frac{dm_i^2}{dT^2}$ is the derivative of the effective masses. As it was shown in the previous studies, we found previously that the DQPM results for specific shear and bulk viscosity are very close to the predictions from the gluodynamic IQCD calculations.

For moderate values of the baryon chemical potential, the specific shear viscosity $\eta/s$ of the QGP matter increases with temperature, while the specific bulk viscosity $\zeta/s$ decreases with temperature, independent of baryon chemical potential. However, the $T$ dependence of $\eta/s$ near the phase transition changes with increasing $\mu_B$. Fig. 13 shows the DQPM-CP results for $\eta/s$ as a function of scaled temperature $T/T_c(\mu_B)$, for different $\mu_B$ values. The specific shear viscosity for $\mu_B = 0$ (red line) shows a dip followed by an increase with temperature while for $\mu_B \geq 0.9$ GeV $\eta/s$ decreases with increasing temperature near the phase transition ($T \leq 2T_c$). The parton relaxation time $\tau_i$ decreases with increasing temperature at lower $T$, and remains approximately constant at high $T$ for moderate values of chemical potential $\mu_B \geq 0.6$ GeV. Therefore the shear viscosity $\eta \sim T^4$, while the entropy density grows as $s \sim T^3$. Thus, in the high temperature region the ratio $\eta/s$ increases as $\sim T$. It is important to note that transport coefficients rely on the microscopic properties of the degrees of freedom. While a variety of the models can reproduce the IQCD results of basic thermodynamic observables, the transport coefficients differ between the models.
Here we compare results of the specific shear viscosity and later of the electric conductivity for non-zero baryon or quark chemical potential with the RTA results from the PNJL model for $\mu_s = 0, \mu_u = \mu_B/3$ in Fig. 13 (a). The specific shear viscosity results from the DQPM-CP (solid lines) for $\mu_B = 0, 0.6$ GeV agree well with the predictions of the PNJL model (dashed lines) in the vicinity of the phase transition for temperatures $T \leq 1.5 T_c$. The DQPM-CP results for $\eta/s$ at $\mu_B = 0.96$ GeV for $T \leq 2T_c$ is higher than from the PNJL model, while the temperature dependence is similar. The discrepancy between the results is caused by the different treatment of the gluonic degrees of freedom, which has a pronounced critical behaviour of the thermal masses in the DQPM-CP model. Increasing the baryon chemical potential, one can see not only an increase in magnitude but also a change in the $T$-dependence of $\eta/s$ and $\zeta/s$ as shown in Fig. 13. In particular, in the vicinity of the phase transition $T < 1.5 T_c$ for moderate values of $\mu_B$ the specific shear viscosity shows a dip after the phase transition, which is vanishing at high values of $\mu_B$ as can be seen in Fig. 13.

As pointed out in Refs. [32, 67, 69] in the vicinity of the CEP, the divergences of bulk and shear viscosities of the QCD matter are determined by the dynamic and the static critical exponents. The dynamical universality class of the QCD critical endpoint is argued to be that of the H-model [67, 70] according to the classification of dynamical critical phenomena by Hohenberg and Halperin [71]. Whereas in the vicinity of the CEP the shear viscosity has a mild divergence in the critical region, the bulk viscosity has a more pronounced divergence [67, 71, 72]: $\eta \sim \xi_T^{Z_\eta}(Z_\eta \approx 1/19), \zeta \sim \xi_T^{Z_\zeta}(Z_\zeta \approx 3)$. The thermal correlation length is controlled by the static critical exponent $\xi_T \sim t^{-\nu}, t = (T_c/T)^{Z_\eta}$, with $\nu$ being the static critical exponent. Using the hyperscaling relation [73] for the static critical exponents we can estimate $\nu$:

$$2 - \alpha = d\nu,$$

(29)

where $d = 3$ denotes the number of the spatial dimensions, $\alpha \approx 0.63$. We obtain $\nu \approx 0.46$. Taking into account the dynamical and static exponents the divergence of the bulk viscosity is assumed to be $\zeta \approx t^{-Z_\zeta,\nu+\alpha}$ [69, 72, 73].

Here we consider small deviations from equilibrium where the quark relaxation times are not large: $\tau_q$ is about $4.5 - 2.5$ fm/c for the temperature range $T_c < T \leq 2T_c$, so that the slight divergence of the transport coefficients near the CEP is determined by the static exponents. The specific shear and bulk viscosities from the DQPM-CP increase rapidly when approaching the critical endpoint from the partonic phase. However, the increase near the CEP is more pronounced for the specific bulk viscosity which rises by a factor of five, while the specific shear viscosity rises only by $\leq 10\%$ for the same temperature range $1.07 - 1.01 T_c/2$. The increase of $\zeta/s$ is related to the rapid decrease in the speed of sound and corresponds to the static critical exponents, that affects the bulk viscosity. In terms of heavy-ion collisions observables, this increase in the bulk viscosity is expected to show up as the decrease of average transverse momentum of produced particles as well as in an increase of the charged particle multiplicity per unit momentum rapidity [61, 68]. However, this has to be checked by a transport simulations or by a hydrodynamical simulation of the expanding QGP. Such a substantial increase of the charged particle and net-baryon multiplicities per unit momentum rapidity due to the enhancement of the bulk viscosity near the CEP has been observed in a longitudinally expanding $1 + 1$
dimensional causal relativistic hydrodynamical evolution at non-zero baryon density [75].

We note that the specific bulk and shear viscosities have been considered near the CEP and the 1st order phase transition for the $N_f = 2$ NJL model in the previous study [32]. We found good qualitative agreement for the $T$-dependence of the shear and bulk viscosity of the NJL model from Ref. [32], while the numerical values differ due to the different quark relaxation times and the absence of gluonic degrees of freedom in case of the NJL model.

B. Electric, baryon and strange conductivities

Let us consider first the diagonal conductivities for electric, baryon and strange charge. The DQPM-CP results for $\sigma_{QQ}/T$, $\sigma_{BB}/T$ and $\sigma_{SS}/T$ are shown in Figs. 15, 16, 17 as a function of the scaled temperature $T/T_c$ for two setups of the strange quark chemical potential. The scaled electric, strange and baryon conductivity have a similar temperature dependence: at high $T$ the ratios increase with temperature increase as $\sim T$ which is mainly due to the quark density increasing with temperature. The most prominent difference between the conductivities is the $\mu_B$-dependence, which is shown in Figs. 15, 16, 17 the electric and strange conductivities increase with $\mu_B$, while the baryon conductivity decreases with $\mu_B$ for the symmetric setup $\mu_s = \mu_u = \mu_B/3$. With the increase of baryon chemical potential the net baryon density increases, which influences the baryon conductivity. A similar trend for the $\sigma_{QQ}/T$, $\sigma_{BB}/T$ and $\sigma_{SS}/T$ at moderate values of baryon chemical potential $\mu_B \lesssim 0.4$ GeV has been observed in the non-conformal Einstein-Maxwell-Dilaton (EMD) holographic model [77]. Furthermore, we compare the $\mu_B$ dependencies of the scaled conductivities for the two setups of strange quark chemical potential. We have found that in case of vanishing strange quark chemical potential (setup (II)) the scaled conductivities show a much less pronounced of $\mu_B$-dependence for the baryon and strange conductivities, which is expected due to the vanishing net strangeness density $n_s = 0$. While the electric conductivity has similar $\mu_B$-dependence for the two settings of strange quark chemical potential. Near the CEP, the electric conductivity decreases, but as for the PNJL results, there is no pronounced divergence behavior. The same behavior has been found for the baryon and strange conductivities.

V. CONCLUSIONS AND OUTLOOK

By extending the phenomenological dynamical quasiparticle model to a wide range of baryon chemical potentials we obtain an EoS, which is in agreement with
the lattice data at moderate baryon chemical potentials and can at the same time be extended to the whole \((T, \mu_B)\) plane. This extension allows for calculating the transport coefficients of the partonic phase. To mimic the \(T\)-dependence of the basic thermodynamic observables near the CEP we have adopted the critical behaviour of the effective coupling constant by using the entropy density from the PNJL model near the CEP. For moderate values of the chemical potential \(\mu_B \leq 0.4\) GeV the dependence of the thermodynamic quantities on \(\mu_B\) are in agreement with the previous results from the DQPM [6 15 78].

\begin{itemize}
  \item We presented the results for the thermodynamic observables \(p/T^4, \epsilon/T^4, s/T^3\), as well as for the speed of sound and the specific heat for a wide range of chemical potentials. We have shown that the ‘critical’ behaviour of the effective coupling affects the thermodynamic observables. Moreover, we have found that the resulting value of the critical exponent \(\alpha \approx 0.63\) is in good agreement with the predictions of the PNJL model and the expectations from the universality argument \(\alpha = 2/3\).

  \item To quantify the \(\mu_B\)-dependence of the bulk observables we have have studied isentropic trajectories of the deconfined QCD medium described by the DQPM-CP for a wide range of baryon chemical potential, including the vicinity of the CEP.

  \item We have evaluated transport properties of the deconfined QCD medium for a wide range of baryon chemical potential within the DQPM-CP: the specific shear \(\eta/s\) and bulk \(\zeta/s\) viscosity and the ratio of electric \(\sigma_{QQ}/T\), baryon \(\sigma_{BB}/T\) and strange \(\sigma_{SS}/T\) conductivities to temperature on the basis of the Boltzmann equation in the relaxation time approximation. We have found that the resulting \(\mu_B\)-dependence of \(\eta/s\) and \(\sigma_{QQ}/T\) for the PNJL model and the DQPM-CP are qualitatively the same in the vicinity of the phase transition, while there is a clear difference in the electric conductivity.

  \item We have found that the DQPM-CP estimates of the specific bulk viscosity show a rapid increase when approaching the CEP from the high-temperature region originating from the rapid decrease of the speed of sound \(c_s^2 \to 0\), whereas for the specific shear viscosity and the \(B, Q, S\) conductivities there is only a small enhancement \(\leq 10\%\), caused mainly by the ‘critical’ contribution of the effective coupling constant.
\end{itemize}

Although the extracted results for the transport coefficients are model-dependent, the qualitative picture of the \(T\) and \(\mu_B\) dependence is consistent with expectations from more rigorous approaches. Our results can be implemented in hydrodynamic simulations as well as be employed for the partonic phase of transport approaches.

**ACKNOWLEDGMENTS**

The authors thank Wolfgang Cassing, Juan Torres-Rincon, Claudia Ratti and Taesoo Song for useful discussions. O.S. acknowledge support from the “Helmholtz Graduate School for Heavy Ion research”. O.S. and E.B. acknowledge support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the grant CRC-TR 211 ‘Strong-interaction matter under extreme conditions’ - Project...
number 315477589 - TRR 211. This work is supported by the European Union’s Horizon 2020 research and innovation program under grant agreement No 824093 (STRONG-2020) and by the COST Action THOR, CA15213. Computational resources were provided by the Center for Scientific Computing (CSC) of the Goethe University.

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