Tverberg plus constraints

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Abstract

Many of the strengthenings and extensions of the topological Tverberg theorem can be derived with surprising ease directly from the original theorem: For this, we introduce a proof technique that combines a concept of ‘Tverberg unavoidable subcomplexes’ with the observation that Tverberg points that equalize the distance from such a subcomplex can be obtained from maps to an extended target space.

Thus, we obtain simple proofs for many variants of the topological Tverberg theorem, such as the colored Tverberg theorem of Živaljević and Vrecica (‘The colored Tverberg’s problem and complexes of injective functions’, J. Combin. Theory, Ser. A 61 (1992) 309–318). We also get a new strengthened version of the generalized van Kampen–Flores theorem by Sarkaria (‘A generalized van Kampen–Flores theorem’, Proc. Amer. Math. Soc. 11 (1991) 559–565) and Volovikov (‘On the van Kampen–Flores theorem’, Math. Notes (5) 59 (1996) 477–481), an affine version of their ‘j-wise disjoint’ Tverberg theorem, and a topological version of Soberón’s result (‘Equal coefficients and tolerance in coloured Tverberg partitions’, Proceeding of the 29th Annual Symposium on Computational Geometry (SoCG) (ACM, Rio de Janeiro, 2013), 91–96) on Tverberg points with equal barycentric coordinates.

1. Introduction

Tverberg’s 1966 theorem [20], which states that any set of \((r-1)(d+1)+1\) points in \(\mathbb{R}^d\) can be partitioned into \(r\) subsets whose convex hulls intersect, is a seminal result that has inspired many interesting variations, extensions and analogs, starting with the ‘topological Tverberg theorem’ of Bárány, Shlosman and Szűcs [5] and Özaydin [15]. See Section 2 for a brief review and Matoušek [14] for a friendly textbook treatment.

Here, we propose a new, simple and elementary, proof technique that combines a concept of ‘Tverberg unavoidable subcomplexes’ (which contain at least one simplex from each Tverberg partition), see Section 4, with the observation that Tverberg points that equalize the distance from such a subcomplex can be obtained from maps to an extended target space, see Section 3. This technique allows us to derive many of the variations and analogs directly from the original topological Tverberg theorem with surprising ease.

For example, our ansatz produces directly from the topological Tverberg theorem a colored version that is stronger than Živaljević and Vrecica’s 1992 ‘colored Tverberg’s theorem’ [26] (but weaker than the optimal colored Tverberg theorem [7] from 2009); see Section 5. Similarly, we obtain directly from the topological Tverberg theorem a strengthened version of the ‘generalized van Kampen–Flores theorem’ of Sarkaria [18] and Volovikov [22] from 1991/1996; see Section 6. As another example, our machinery reproves Soberón’s 2013 ‘Tverberg theorem...
with equal barycentric coordinates’ and at the same time produces a new topological version of this result; see Section 8.

Our machinery uses the topological Tverberg theorem as a black box: It provides the core for our proofs, but the proof technique relies only on the result, not on its proof. Thus, for example, an extension of the topological Tverberg theorem to some \( r \) that are not prime powers would immediately yield the same extension for our derived results. In place of the black box result, we can also use the optimal colored Tverberg theorem; in Section 9, we thus obtain further results of the ‘colored Tverberg’ type. The black box result can also be replaced by Tverberg’s original theorem if the constraint functions are affine; in this case, no prime power restriction on the number of parts is needed. Thus, we obtain a new affine version of the topological Tverberg theorem for \( j \)-wise disjoint simplices; see Section 7. Similarly, from the Tverberg theorem for maps to manifolds by Volovikov [21], one ‘automatically’ gets extensions of our results for maps to manifolds.

2. A brief history of Tverberg type results

For every drawing of a tetrahedron in the plane either a vertex will end up on top of the opposite face or two opposite edges intersect in the drawing. This is a first instance of a Tverberg-type result; it holds in any dimension, as was proved by Radon [16] in 1921:

**Radon’s theorem.** For any subset \( X \subseteq \mathbb{R}^d \) with (at least) \( d + 2 \) elements there are disjoint subsets \( S \) and \( T \) of \( X \) with the property that \( \text{conv}(S) \cap \text{conv}(T) \neq \emptyset \).

Radon’s theorem has an equivalent reformulation in terms of affine maps of the \((d+1)\)-dimensional standard simplex \( \Delta_{d+1} \) to \( \mathbb{R}^d \):

**Affine Radon theorem.** For every affine map \( f : \Delta_{d+1} \to \mathbb{R}^d \), there are disjoint faces \( \sigma \) and \( \tau \) of \( \Delta_{d+1} \) with the property that \( f(\sigma) \cap f(\tau) \neq \emptyset \).

Now, we can see many possible ways to extend this basic result. First, we can ask whether the assumption on \( f \) to be an affine map is essential. Is it enough to assume that \( f \) is only continuous? This question was answered by Bajmóczy and Bárány [2] in 1979 via a clever use of Borsuk–Ulam theorem.

**Topological Radon theorem.** For any continuous map \( f : \Delta_{d+1} \to \mathbb{R}^d \), there are disjoint faces \( \sigma \) and \( \tau \) of \( \Delta_{d+1} \) with the property that \( f(\sigma) \cap f(\tau) \neq \emptyset \).

It is also natural to ask for more than two pairwise disjoint subsets of a sufficiently large set of points \( X \subseteq \mathbb{R}^d \). Such a result was first achieved by Birch [6] in 1959 for \( d = 2 \), who also conjectured the tight result for \( d > 2 \), which was eventually proved by Tverberg [20] in 1966. In the equivalent affine version, his result reads as follows:

**Affine Tverberg theorem.** Let \( d \geq 1 \) and \( r \geq 2 \) be integers, and \( N = (r - 1)(d + 1) \). For any affine map \( f : \Delta_N \to \mathbb{R}^d \), there are \( r \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) of \( \Delta_N \) such that \( f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset \).

The set \( \{\sigma_1, \ldots, \sigma_r\} \) of disjoint faces of \( \Delta_N \) whose images intersect is called a Tverberg partition for \( f : \Delta_N \to \mathbb{R}^d \). The dimension of the simplex \( \Delta_N \) in the theorem is minimal.

Will the statement of Tverberg’s theorem still be true if the map \( f \) is assumed only to be continuous? Surprisingly, the available answers to this question depend on divisibility properties
of the parameter \( r \). Bárány, Shlosman and Szücs [5] in 1981 formulated the corresponding ‘topological Tverberg theorem’ for continuous maps and proved it for the case \( r \) is a prime number. This was extended to all prime powers \( r \) by Özaydin [15] in 1987.

**Topological Tverberg theorem.** Let \( r \geq 2, d \geq 1, and N = (r - 1)(d + 1) \). If \( r \) is a prime power, then, for every continuous map \( f : \Delta_N \to \mathbb{R}^d \), there are \( r \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) of \( \Delta_N \) such that \( f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset \).

The topological Tverberg theorem is derived from the nonexistence of an \( \mathfrak{S}_r \)-equivariant map from the join \([r]^{(N + 1)}\) into the sphere \( S(W_{r, d}^{d+1}) \) when \( r \) is a prime power. Here, \([r] = \{1, \ldots, r\}\) denotes a 0-dimension simplicial complex of \( r \) distinct points with the obvious action of the symmetric group \( \mathfrak{S}_r \), and \( W_r := \{(x_1, \ldots, x_r) : x_1 + \ldots + x_r = 0\}\) is an \( \mathfrak{S}_r \)-representation with an action given by permutation of coordinates. The fact that an \( \mathfrak{S}_r \)-equivariant map \([r]^{(N + 1)} \to S(W_{r, d}^{d+1})\) exists if \( r \) is not a prime power, established in [8], makes the extension of topological Tverberg theorem to non-primepowers into ‘one of the most challenging problems in this field’ [14, Notes to Section 6.4].

What is next? Can we say something about dimensions of simplexes in a Tverberg partition? What is the minimal dimension of a simplex \( \Delta \) such that for every mapping to the plane there are two disjoint faces whose images intersect? (The answer to this is given by Kuratowski’s theorem from graph theory: \( K_5 \) is not planar, so the 4-simplex will do.) How about higher-dimensional versions of this result? The classical van Kampen–Flores theorem [10, 12] from the 1930s provides a first answer to such questions.

**Van Kampen–Flores Theorem.** For \( d \geq 2 \) even, and a continuous map \( f : \Delta_{d+2} \to \mathbb{R}^d \), there are two disjoint faces \( \sigma_1, \sigma_2 \) of \( \Delta_{d+2} \) of dimension at most \( \frac{d}{2} \) in \( \Delta_{d+2} \) such that \( f(\sigma_1) \cap f(\sigma_2) \neq \emptyset \).

The generalized van Kampen–Flores theorems of Sarkaria [18] and Volovikov [22] extend this by providing conditions that guarantee the existence of multiple overlaps. These theorems are classically obtained via topological methods that are considerably more involved than those needed for the proof of topological Tverberg theorem in cases of primes or of prime powers.

Can we put any other restrictions on a Tverberg partition? Bárány, Füredi andLovász [3] in 1989 realized a need for ‘a colored version of Tverberg’s theorem’. They proved the first instance of such a result for three triangles in the plane. Extending these ideas, Bárány and Larman [4] in 1992 formulated the following general problem.

**The colored Tverberg problem.** Determine the smallest natural number \( n = n(d, r) \) such that for every collection \( \mathcal{C} = C_0 \cup \ldots \cup C_d \) of \( n \) points in \( \mathbb{R}^d \), where each ‘color class’ \( C_i \) satisfies \( |C_i| \geq r \), there are \( r \) disjoint subcollections \( F_1, \ldots, F_r \) of \( \mathcal{C} \) such that

\[ \text{(A) } |C_i \cap F_j| \leq 1 \text{ for every } i \in \{0, \ldots, d\}, j \in \{1, \ldots, r\}, \text{ and} \]

\[ \text{(B) } \text{conv}(F_1) \cap \ldots \cap \text{conv}(F_r) \neq \emptyset. \]

They proved that \( n(1, r) = 2r, n(2, r) = 3r \), presented a proof by Lovász for \( n(d, 2) = 2(d + 1) \), and conjectured that \( n(d, r) = r(d + 1) \). In the same year Živaljević and Vrećica [26] formulated the following modified colored Tverberg problem:

**Modified colored Tverberg problem.** Determine the smallest number \( t = t(d, r) \) such that for every simplex \( \Delta \) with \( (d + 1) \)-colored vertex set \( \mathcal{C} = C_0 \cup \ldots \cup C_d \), with \( |C_i| \geq t \)
for all $i$, and for every continuous map $f : \Delta \to \mathbb{R}^d$, there are $r$ disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta$ satisfying

(A) $|C_i \cap \sigma_j| \leq 1$ for every $i \in \{0, \ldots, d\}$, $j \in \{1, \ldots, r\}$, and

(B) $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$.

Živaljević and Vrećica obtained the upper bound for the function $tt(d, r) \leq 2r - 1$ in the case when $r$ is a prime; this yields the upper bound $tt(d, r) \leq 4r - 3$ for any $r$. (Živaljević later extended this to prime powers $r$.) These results were obtained by proving the nonexistence of an $\mathcal{S}_r$-equivariant map from the join $\Delta_r^{(d+1)}$ into the sphere $S(W_r^{(d+1)})$ when $r$ is a prime power. Here, $\Delta_r,2r-1$ denotes the $r \times (2r - 1)$ chessboard complex. The symmetric group $\mathcal{S}_r$ acts on $\Delta_r,2r-1$ by permuting the rows. The connectivity of the chessboard $\Delta_r,2r-1$ plays the central role in the proof of the nonexistence of an $\mathcal{S}_r$-equivariant map $\Delta_r,2r-1 \to S(W_r^{(d+1)})$.

Note, however, that the upper bound of Živaljević on the function $tt(d, r)$ does not provide any information about the function $n(d, r)$ of Bárány and Larman.

Only recently Blagojević, Matschke and Ziegler [7], as a consequence of their ‘optimal colored Tverberg theorem’, established that $n(d, r) = r(d + 1)$ when $r + 1$ is a prime and $n(d, r) \leq 2(r - 1)(d + 1) + 1$ when $r$ is a prime. Furthermore, they established $tt(d, r) \leq 2r - 2$ for any $r$. See [25] for an exposition.

Can we ask for even more? What about preimages of the intersection point of a Tverberg partition? Can we say something about the barycentric coordinates of these preimages? This question was addressed by Soberón [19] in 2013. His result is discussed in more detail in Section 8.

3. The topological Tverberg theorem with a constraining function

The original version of Tverberg’s theorem [20] asserts that any $(r - 1)(d + 1) + 1$ points in $\mathbb{R}^d$ can be partitioned into $r$ sets whose convex hulls intersect. As discussed, this can be phrased in terms of affine maps: Set $N := (r - 1)(d + 1)$ and denote by $\Delta_N$ the $N$-dimensional simplex; then Tverberg’s theorem says that, for every affine map $f : \Delta_N \to \mathbb{R}^d$, there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$. The set of faces $\{\sigma_1, \ldots, \sigma_r\}$ is called a Tverberg $r$-partition for $f$, or simply a Tverberg partition if it is clear which $f$ and $r$ we refer to. Points $x_i \in \sigma_i$ for $i = 1, \ldots, r$ with $f(x_1) = \ldots = f(x_r)$ are points of Tverberg coincidence for $f$. A first extension of this theorem to continuous maps was proved by Bárány, Shlosman and Szűcs [5] for a prime number $r$ of intersecting faces. This was later generalized to prime powers $r$ by Özaydin [15].

**Theorem 3.1** (Topological Tverberg; Bárány, Shlosman and Szűcs [5], Özaydin [15]). *Let $r \geq 2$ be a prime power, $d \geq 1$, and $N \geq (r - 1)(d + 1)$. Then, for every continuous map $f : \Delta_N \to \mathbb{R}^d$, there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$."

We will consider an additional constraining function on $\Delta_N$, and ask that the points of coincidence of $f$ lie in one fiber of this function. The proof of the following lemma shows that the topological Tverberg theorem itself yields that this can be achieved if we are willing to increase the size of the original set of points in $\mathbb{R}^d$; for every continuous constraint we must provide $r - 1$ additional points in order to obtain that there is a Tverberg partition equalizing the constraint.
Lemma 3.2 (Key lemma #1). Let \( r \geq 2 \) be a prime power, \( d \geq 1 \), and \( c \geq 0 \). Let \( N \geq N_c := (r-1)(d+1+c) \) and let \( f : \Delta_N \to \mathbb{R}^d \) and \( g : \Delta_N \to \mathbb{R}^c \) be continuous. Then, there are \( r \) points \( x_i \in \sigma_i \), where \( \sigma_1, \ldots, \sigma_r \) are pairwise disjoint faces of \( \Delta_N \) with \( g(x_1) = \ldots = g(x_r) \) and \( f(x_1) = \ldots = f(x_r) \).

Proof. Apply the topological Tverberg theorem to the continuous map \( \Delta_N \to \mathbb{R}^{d+c} \) given by \( x \mapsto (f(x), g(x)) \). \( \square \)

In this lemma, \( c \) is the number of additional constraints; thus, the special case \( c = 0 \) is the topological Tverberg theorem. Suitable choices of constraining functions will enable us to obtain from this the existence of various kinds of special Tverberg partitions.

Remark 3.3. Note that Lemma 3.2 remains true for \( r \) an arbitrary positive integer in the setting where the map \( f \) as well as the constraint function \( g \) are affine.

4. Tverberg unavoidable subcomplexes

Definition 4.1 (Tverberg unavoidable subcomplexes). Let \( r \geq 2 \), \( d \geq 1 \), \( N \geq r-1 \) be integers and \( f : \Delta_N \to \mathbb{R}^d \) a continuous map with at least one Tverberg \( r \)-partition. Then, a subcomplex \( \Sigma \subseteq \Delta_N \) is Tverberg unavoidable if for every Tverberg partition \( \{\sigma_1, \ldots, \sigma_r\} \) for \( f \), there is at least one face \( \sigma_j \) that lies in \( \Sigma \).

According to this definition, whether a subcomplex is Tverberg unavoidable depends on the parameters \( r \), \( d \) and \( N \), but also on the map \( f \). However, we will be interested only in subcomplexes that are large enough to be unavoidable for any map \( f \).

Lemma 4.2 (Key examples). Let \( d \geq 1 \), \( r \geq 2 \) and \( N \geq r-1 \), and assume that the continuous map \( f : \Delta_N \to \mathbb{R}^d \) has a Tverberg \( r \)-partition.

(i) The induced subcomplex (simplex) \( \Delta_{N-(r-1)} \) on \( N-r+2 \) vertices of \( \Delta_N \) is Tverberg unavoidable.

(ii) For any set \( S \) of at most \( 2r-1 \) vertices in \( \Delta_N \) the subcomplex of faces with at most one vertex in \( S \) is Tverberg unavoidable.

(iii) If \( k \) is an integer such that \( r(k+2) > N+1 \), then the \( k \)-skeleton \( \Delta_N^{(k)} \) of \( \Delta_N \) is Tverberg unavoidable.

(iv) If \( k \geq 0 \) and \( s \) are integers such that \( r(k+1) + s > N+1 \) with \( 0 \leq s \leq r \), then the subcomplex \( \Delta_N^{(k-1)} \cup \Delta_N^{(k)} \) of \( \Delta_N \) is Tverberg unavoidable.

Proof. All these are easy consequences of the pigeonhole principle:

(i) The simplex \( \Delta_{N-(r-1)} \) contains all but \( r-1 \) of the vertices of \( \Delta_N \), so for any Tverberg partition \( \sigma_1, \ldots, \sigma_r \) at most \( r-1 \) of the faces \( \sigma_i \) can have a vertex outside of \( \Delta_{N-(r-1)} \), so at least one face \( \sigma_j \) has all vertices in \( \Delta_{N-(r-1)} \).

(ii) If all faces \( \sigma_1, \ldots, \sigma_r \) had at least two vertices in \( S \), then we would have \( |S| \geq \sum_{i=1}^{r} |\sigma_i \cap S| \geq 2r \).

(iii) If all faces \( \sigma_1, \ldots, \sigma_r \) of a Tverberg partition had dimension at least \( k+1 \), then this would involve at least \( r(k+2) \) vertices.
(iv) If none of the faces $\sigma_1, \ldots, \sigma_r$ lies in $\Delta_N^{(k-1)} \cup \Delta_N^{(k)}$, then they all have dimension at least $k$, and since they are disjoint only $r-s$ of them can involve one of the last $r-s$ vertices, so $s$ of them must have dimension at least $k+1$. For this $r(k+1) + s$ vertices are needed. □

Parts (i), (ii) and (iii) of the lemma follow from the following more general statement: If $S$ is a set of at most $(s+1)r-1$ vertices, $s \geq 0$, then the subcomplex of faces with at most $s$ vertices in $S$ is Tverberg unavoidable. (Again, this immediately follows from the pigeonhole principle.) For $s = 0$, this gives that for a set $S$ of at most $r-1$ vertices there is a face not intersecting $S$ (having 0 vertices in common with $S$) in every Tverberg $r$-partition; this is part (i). For $s = 1$, this yields part (ii): Here, $S$ has size at most $2r-1$ and there is a face with at most one vertex in $S$. For $s$ large enough that $S$ may contain all $N+1$ vertices, this gives a bound on the skeleton. Namely, if $(s+1)r - 1 \geq N + 1$, then there is a face with at most $s$ vertices, that is, a face in the $(s-1)$-skeleton: This yields part (iii) for $k = s - 1$. We thank the anonymous referee for this generalization.

If $r$ is a prime power and $N \geq N_0 = (r-1)(d+1)$, then the topological Tverberg theorem guarantees the existence of a Tverberg $r$-partition for any continuous $f : \Delta_N \to \mathbb{R}^d$. We now show that for $N \geq N_1 = (r-1)(d+2)$, every subcomplex $\Sigma \subseteq \Delta_N$ that is Tverberg unavoidable for $f$ necessarily contains all the faces of some Tverberg $r$-partition.

**Lemma 4.3** (Key lemma #2). Let $r \geq 2$ be a prime power, $d \geq 1$, and $N \geq N_1 = (r-1)(d+1)$. Assume that $f : \Delta_N \to \mathbb{R}^d$ is continuous and that the subcomplex $\Sigma \subseteq \Delta_N$ is Tverberg unavoidable for $f$. Then, there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$, all of them contained in $\Sigma$, such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$.

**Proof.** Let $g : \Delta_N \to \mathbb{R}$ assign to each point $x \in \Delta_N$ its distance to $\Sigma$. This map $g$ is continuous. We have $g(x) = 0$ if and only if $x$ belongs to $\Sigma$, since $\Sigma \subseteq \Delta_N$ is closed.

By Lemma 3.2, there are points $x_1, \ldots, x_r$ in pairwise disjoint faces $\sigma_1, \ldots, \sigma_r \subset \Delta_N$ with $f(x_1) = \ldots = f(x_r)$ and $g(x_1) = \ldots = g(x_r)$. We may assume that the $\sigma_i$ are inclusion-minimal with $x_i \in \sigma_i$, that is, $\sigma_i$ is the unique face with $x_i$ in its relative interior. Since $\Sigma$ is Tverberg unavoidable for $f$, at least one of these faces, say $\sigma_j$, is contained in $\Sigma$. Thus $g(x_j) = 0$, which implies that $g(x_i) = 0$ for all $i = 1, \ldots, r$. Thus, all points $x_i$ lie in $\Sigma$ and since the $\sigma_i$ are inclusion-minimal and $g$ vanishes at all points in the relative interior of a face if and only if it vanishes at some such point, the $\sigma_i$ belong to $\Sigma$. □

With Lemma 4.2(i), this shows that the topological Tverberg theorem for maps to $\mathbb{R}^{d+1}$ immediately yields the lower-dimensional version for maps to $\mathbb{R}^d$. Thus, for some $r$ (not necessarily a prime power), it suffices to show the topological Tverberg theorem for arbitrarily large dimensions. (See de Longueville [13, Proposition 2.5] for this observation: his proof uses an extended target space and the observation that the induced subcomplex of $N-r+2$ vertices is unavoidable.) Lemma 4.2(ii) in the next section gets us colored versions of the topological Tverberg theorem. From the parts (iii) and (iv) of Lemma 4.2, in Section 6 we get versions of the topological Tverberg theorem with a bound on the dimension of the intersecting faces.

The following is a straightforward extension of Lemma 4.3.

**Theorem 4.4.** Let $r \geq 2$ be a prime power, $d \geq 1$, $c \geq 1$ and $N \geq N_c = (r-1)(d+1+c)$. Let $f : \Delta_N \to \mathbb{R}^d$ be continuous and let $\Sigma_1, \Sigma_2, \ldots, \Sigma_c \subseteq \Delta_N$ be Tverberg unavoidable subcomplexes for $f$. Then, there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ in $\Sigma_1 \cap \ldots \cap \Sigma_c$ such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$. 

Proof. We may assume that $N = N_\ell$. Let $g_i : \Delta_{N_\ell} \to \mathbb{R}$ assign to $x \in \Delta_N$ the distance to the subcomplex $\Sigma_i$, and consider $g : \Delta_{N_\ell} \to \mathbb{R}^\ell$, $x \mapsto (g_1(x), \ldots, g_\ell(x))$. Now, use Lemma 3.2 and that the $\Sigma_i$ are Tverberg unavoidable. \hfill \square

Question 4.5. Does Theorem 4.4 remain true for $r$ an arbitrary positive integer and $f$ affine?

5. Weak colored versions of the Tverberg theorem

An interesting way to constrain Tverberg partitions is to color the vertices of the simplex $\Delta_N$ and to require that the faces in the Tverberg partition have no two vertices of the same color: Bárány and Larman [4] asked for the minimal $N$ such that for any affine map and for any coloring of the $N + 1$ vertices of $\Delta_N$ by $d + 1$ colors, where each color class should have size at least $r$, a Tverberg partition of $r$ faces, each without repeated colors, exists.

Suppose that the vertices of $\Delta_N$ are partitioned into color classes. Denote by $R \subseteq \Delta_N$ the rainbow complex, that is, the subcomplex of faces that have at most one vertex of each color class. These faces are called rainbow faces. Lemma 3.2 implies that for any coloring of the vertices of $\Delta_N$ with $N \geq N_1 = (r - 1)(d + 2)$ and for any continuous map $f : \Delta_N \to \mathbb{R}^d$ there is a Tverberg partition such that the points of Tverberg coincidence for $f$ have the same distance to the rainbow complex. We will determine some conditions that a coloring must fulfill such that the rainbow complex is Tverberg unavoidable or is an intersection of few Tverberg unavoidable subcomplexes.

Lemma 5.1. Let $r \geq 2$ be a prime power, $d \geq 1$, and $N \geq N_1 = (r - 1)(d + 2)$. Let $S \subseteq \Delta_N$ be any set of at most $2r - 1$ vertices, and let $f : \Delta_N \to \mathbb{R}^d$ be any continuous function. Then, there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ with $|\sigma_i \cap S| \leq 1$ such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$.

Proof. The subcomplex $\Sigma$ of faces with at most one vertex in $S$ is Tverberg unavoidable by Lemma 4.2(ii). Thus, $\Sigma$ contains a Tverberg partition. \hfill \square

This is a colored version of the topological Tverberg theorem, where the vertices in $S$ are in one color class, and all other vertices are colored with distinct colors. If we assume that $|S| \geq r$, then we can also state this theorem with equalities $|\sigma_i \cap S| = 1$ by adding a point in $S$ to every face $\sigma_i$ that is disjoint from $S$.

The following ‘colored Radon theorem’ (a restatement of the Borsuk–Ulam theorem) is a direct consequence for $r = 2$.

Corollary 5.2 (Colored Radon: Vrečica and Živaljević [24, Corollary 7]). Let $d \geq 1$, let the map $f : \Delta_{d+2} \to \mathbb{R}^d$ be continuous and let $S \subseteq \Delta_{d+2}$ be a set of three vertices. Then, there are disjoint faces $\sigma_1, \sigma_2$ in $\Delta_{d+2}$ with $|\sigma_1 \cap S| \leq 1$ and $|\sigma_2 \cap S| \leq 1$ such that $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$.

Theorem 5.3 (Weak colored Tverberg). Let $r \geq 2$ be a prime power, $d \geq 1$, $N \geq N_{d+1} = (r - 1)(2d + 2)$ and let $f : \Delta_N \to \mathbb{R}^d$ be continuous. If the vertices of $\Delta_N$ are colored by $d + 1$ colors, where each color class has cardinality at most $2r - 1$, then there are $r$ pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$.
Proof. For each fixed color $i$, the subcomplex $\Sigma_i$ of faces that have at most one vertex of color $i$ is Tverberg unavoidable by Lemma 4.2(ii). The complex of rainbow faces is $\Sigma_1 \cap \ldots \cap \Sigma_{d+1}$. Now, use Theorem 4.4.

In Theorem 5.3 the fact that all vertices can be colored with $d+1$ colors of size at most $2r-1$ implies that $N+1 \leq (2r-1)(d+1)$. The theorem is ‘weak’ as it needs a large number of points/vertices to reach its conclusion, namely $N+1 \geq N_{d+1} + 1 = (r-1) \cdot (2d+2)+1$, while the optimal result requires only $N+1 \geq N_0 + 1 = (r-1)(d+1)+1$ of them, as in Theorem 9.1. The special case of Theorem 5.3 when all color classes have the same cardinality $2r-1$, and thus $N+1 = (d+1)(2r-1)$, is the colored Tverberg theorem of Živaljević and Vrećica [26]. As we do not need to require all color classes to have the same size (a simple observation that apparently was first made in [7]), we need $d$ points/vertices less to force a colored Tverberg partition.

Theorem 5.3 leaves some flexibility in the choice of color classes: For example, we could consider $d$ colors of cardinality $2r-2$ and one color class of size $2r-1$. Instead of shrinking the size of color classes, we can also allow fewer color classes. This gives a colored Tverberg theorem ‘of type B’ (in terminology of Vrećica and Živaljević introduced in [23]), that is, fewer than $d+1$ colors are possible.

We proceed using the method provided in Sections 3 and 4, and thus obtain the following result, which in the special case where all color classes have the same size is the main result of [23]. It also implies Theorem 5.3.

**Theorem 5.4.** Let $r \geq 2$ be a prime power, $d \geq 1$, $c \geq \lceil \frac{(r-1)d}{r} \rceil + 1$ and $N \geq N_c = (r-1)(d+1+c)$. Let $f : \Delta_N \to \mathbb{R}^d$ be continuous. If the vertices of $\Delta_N$ are divided into $c$ color classes, each of them of cardinality at most $2r-1$, then there are $r$ pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$.

**Proof.** Again, we may assume that $N = N_c$. We need that all $N_c + 1$ vertices can be colored by $c$ colors of cardinality at most $2r-1$ to use Theorem 4.4, that is, we need $c(2r-1) \geq N_c + 1$, which is equivalent to $c \geq \lceil \frac{(r-1)d}{r} \rceil + 1$.

6. Prescribing dimensions for Tverberg simplices

If $N$ is sufficiently large, can we set an upper bound $\dim(\sigma_i) \leq k$ for the dimensions of the faces in a Tverberg $r$-partition for $f : \Delta_N \to \mathbb{R}^d$? For the case $r = 2$, such a result is due to van Kampen [12] and independently Flores [10].

**Theorem 6.1** (van Kampen–Flores). Let $d \geq 2$ be even. Then, for every continuous map $f : \Delta_{d+2} \to \mathbb{R}^d$, there are two disjoint faces $\sigma_1, \sigma_2 \subset \Delta_{d+2}$ of dimension at most $\frac{d}{2}$ in $\Delta_{d+2}$ with $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$.

This was generalized to a prime number $r$ of faces by Sarkaria [18] and later to prime powers $r$ by Volovikov [22]. In fact, Volovikov’s result holds for maps to manifolds that induce a trivial homomorphism on cohomology in dimension $k$, where $k$ is the desired bound on the dimension of faces. We restate Volovikov’s result here for the case that the target space is Euclidean space. A collection of sets $S_1, \ldots, S_r$ is called $j$-wise disjoint if the intersection of any $j$ of these sets is empty.
Theorem 6.2 (Generalized van Kampen–Flores: Sarkaria [18], Volovikov [22]). Let \( r \geq 2 \) be a prime power, \( 2 \leq j \leq r, d \geq 1 \), and \( k < d \) such that there is an integer \( m \geq 0 \) that satisfies
\[
(r - 1)(m + 1) + r(k + 1) \geq (N + 1)(j - 1) > (r - 1)(m + d + 2).
\] (6.1)
Then, for every continuous map \( f : \Delta_N \to \mathbb{R}^d \), there are \( j \)-wise disjoint faces \( \sigma_1, \ldots, \sigma_r \) of \( \Delta_N \) with \( \dim \sigma_i \leq k \) for \( 1 \leq i \leq r \), such that \( f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset \).

Let us discuss which of the conditions posed by (6.1) are necessary. First, the left-hand side has to be strictly larger than the right-hand side, which yields the condition \( (m + 1)(r - 1) + r(k + 1) - (m + d + 2)(r - 1) > 0 \), that is,
\[
k \geq \frac{r - 1}{r}d.
\] (6.2)
This lower bound on \( k \) is necessary, as we see by looking at a generic affine map \( f \), which does not have the desired Tverberg \( r \)-partition unless the sum of the codimensions of the \( \sigma_i \) is at most \( d \), that is, \( r(d - k) \leq d \).

If the second inequality in (6.1) is satisfied for some \( m \geq 0 \), then it is, in particular, satisfied for \( m = 0 \), which gives \( (N + 1)(j - 1) > (r - 1)(d + 2) \), that is,
\[
N + 1 > \frac{r - 1}{j - 1}(d + 2).
\] (6.3)
It is not clear whether this lower bound on \( N \) is necessary, in general;
\[
N + 1 > \left\lfloor \frac{r - 1}{j - 1} \right\rfloor (d + 2)
\]
is necessary for \( k < d \), as one can see from an affine map \( \Delta_N \to \Delta_d \) that maps at most \( \left\lfloor \frac{r - 1}{j - 1} \right\rfloor \) vertices of \( \Delta_N \) to each of the vertices and to the barycenter of a \( d \)-simplex. This example is suggested by Sarkaria [18]. Note that for \( j = 2 \) the lower bound of (6.3) reads \( N + 1 > (r - 1)(d + 2) \), which is optimal, despite a mistaken claim in [18, Theorem 1.5 and the sentence after this] that the bound \( N \geq r(s + 1) - 2 \) is optimal, where \( s = k + 1 \) in Sarkaria’s notation. In the example he gives, the two bounds coincide.

Even if both conditions (6.2) and (6.3) hold the integer \( m \) that should satisfy (6.1) may not exist. This requirement is nontrivial, see Example 2. It is not necessary, as we shall see.

We will get our strengthened version of Theorem 6.2 as a direct consequence of the topological Tverberg theorem. For this, we first establish the case \( j = 2 \) as a corollary of Lemma 4.3.

Theorem 6.3. Let \( r \geq 2 \) be a prime power, \( d \geq 1 \), \( N \geq N_1 = (r - 1)(d + 2) \), and \( k \geq \left\lceil \frac{r - 1}{d} \right\rceil \), then, for every continuous map \( f : \Delta_N \to \mathbb{R}^d \), there are \( r \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) of \( \Delta_N \), with \( \dim \sigma_i \leq k \) for \( 1 \leq i \leq r \), such that \( f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset \).

Proof. It is sufficient to prove this for \( N = N_1 \). The \( k \)-skeleton \( \Delta^{(k)}_{N_1} \) of \( \Delta_{N_1} \) is Tverberg unavoidable by Lemma 4.2(iii).

Example 1. For \( d = r = 3 \) and \( f \) an affine map, this theorem asserts that given eleven points in \( \mathbb{R}^3 \), one can find three pairwise disjoint sets of three points whose convex hulls intersect. Ten points are not sufficient for this, as by the discussion above one needs more than \( (r - 1)(d + 2) = 10 \) points. (This solves a problem discussed by Matoušek [14, Example 6.7.4].)

We now obtain our strengthening of Theorem 6.2 as a corollary of Theorem 6.3, the special case \( j = 2 \).
THEOREM 6.4 (Generalized van Kampen–Flores, sharpened). Let \( r \geq 2 \) be a prime power, \( 2 \leq j \leq r, d \geq 1, \) and \( k \leq N \) such that

\[
k \geq \frac{r - 1}{r} d \quad \text{and} \quad N + 1 > \frac{r - 1}{j - 1} (d + 2).
\]

(6.4)

Then, for every continuous map \( f : \Delta_N \to \mathbb{R}^d \), there are \( r \) \( j \)-wise disjoint faces \( \sigma_1, \ldots, \sigma_r \) of \( \Delta_N \), with \( \dim \sigma_i \leq k \) for \( 1 \leq i \leq r \), such that \( f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset \).

Proof. Let \( N' := (N + 1)(j - 1) - 1 \), let \( p \) be the natural simplicial projection \( \Delta_{N'} \cong \Delta_{N(j - 1)} \to \Delta_N \) that maps each of the \( j - 1 \) copies of a vertex \( v \in \Delta_N \) in the join \( \Delta_{N(j - 1)}^* \) to the vertex \( v \), and set \( f' := f \circ p : \Delta_{N'} \to \mathbb{R}^d \).

We have \( N' \geq (r - 1)(d + 2) \). Thus, by Theorem 6.3 there are pairwise disjoint faces \( \sigma'_1, \ldots, \sigma'_r \subseteq \Delta_{N'} \), with \( \dim \sigma_i \leq k \) for all \( i \), such that \( f'(\sigma'_1) \cap \ldots \cap f'(\sigma'_r) \neq \emptyset \). By definition of \( f' \) this is equivalent to \( f(p(\sigma'_1)) \cap \ldots \cap f(p(\sigma'_r)) \neq \emptyset \). Now, let \( \sigma_1 = p(\sigma'_1), \ldots, \sigma_r = p(\sigma'_r) \). The faces \( p(\sigma'_i) \) of \( \Delta_N \) still satisfy the dimension bound \( \dim p(\sigma'_i) \leq k \), and they are \( j \)-wise disjoint. \( \square \)

EXAMPLE 2. To see that Theorem 6.4 is stronger than Theorem 6.2, let \( d = j = r = 3 \) and \( k = 2 \). Then, the prerequisites of Theorem 6.4 are satisfied for \( N = 5 \). Thus, for any continuous map \( f : \Delta_5 \to \mathbb{R}^3 \), there are three \( 3 \)-wise disjoint faces of dimension at most 2 whose images intersect. However, inequality (6.1) of Theorem 6.2 asks that \( (m + 1) \cdot 2 + 3 \cdot 3 \geq (N + 1) \cdot 2 \geq (m + 5) \cdot 2 \), that is, \( 2m + 11 \geq 2N + 2 \geq 2m + 10 \). Such an integer \( m \) exists for no \( N \), as \( 2N + 2 \) is even.

One can ask for a further strengthening of Theorem 6.4 where we would not put the same dimension bound on all simplices \( \sigma_i \). From Lemma 4.2(iv), we obtain the following theorem.

THEOREM 6.5 (Generalized van Kampen–Flores, sharpened further). Let \( r \geq 2 \) be a prime power, \( d \geq 1, N \geq (r - 1)(d + 2) \), and

\[
r(k + 1) + s > N + 1 \quad \text{for integers} \quad k \geq 0, 0 \leq s < r.
\]

Then, for every continuous map \( f : \Delta_N \to \mathbb{R}^d \), there are \( r \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) of \( \Delta_N \) such that \( f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset \), \( \dim(\sigma_i) \leq k \) for all \( i \), and the number \( \ell \) of simplices \( \sigma_i \) with \( \dim \sigma_i = k \) satisfies \( \ell(k + 1) \leq N - (r - s) + 1 \).

Proof. The complex \( \Delta_N^{(k-1)} \cup \Delta_N^{(k)} \) is Tverberg unavoidable by Lemma 4.2(iv). Thus, there is a Tverberg partition \( \sigma_1, \ldots, \sigma_r \) with all simplices in the unavoidable subcomplex, so we get \( \dim \sigma_i \leq k \). Moreover, the simplices \( \sigma_i \), altogether, can take up only \( N + 1 \leq r(k + 1) + s \) vertices. \( \square \)

We leave it to the reader to state and prove a \( j \)-wise disjoint version of Theorem 6.5. On the other hand, even for \( j = 2 \) we do not, up to now, seem to get the full result that one could hope for:

CONJECTURE 6.6. Let \( r \geq 2 \) be a prime power, \( d \geq 1, N \geq (r - 1)(d + 2) \), and

\[
r(k + 1) + s > N + 1 \quad \text{for integers} \quad k \geq 0, 0 \leq s < r.
\]
Then, for every continuous map \( f : \Delta_N \to \mathbb{R}^d \), there are \( r \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) of \( \Delta_N \) such that \( f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset \), with \( \dim \sigma_i \leq k + 1 \) for \( 1 \leq i \leq s \) and \( \dim \sigma_i \leq k \) for \( s < i \leq r \).

More generally, Roland Bacher has asked on mathoverflow [1] which dimensions \( d_i = \dim \sigma_i \) could be prescribed for a Tverberg partition if the number of points \( N \) is sufficiently large. We have already noted that a Tverberg \( r \)-partition in which the codimensions of the \( \sigma_i \) add to more than \( d \) will not exist for an affine general position map, so we need to assume that \( \sum_i (d - d_i) \leq d \). Also arbitrarily large families of \( N \) points on the moment curve, whose convex hulls are neighborly polytopes, show that we cannot force that \( \dim \sigma_i < \left\lfloor \frac{d}{2} \right\rfloor \) for any \( i \).

**Definition 6.7** (Admissible, Tverberg prescribable). For \( d \geq 1 \) and \( r \geq 2 \), an \( r \)-tuple \((d_1, \ldots, d_r)\) of integers is admissible for \( d \) if \( \left\lfloor \frac{d}{2} \right\rfloor \leq d_i \leq d \) and \( \sum_{i=1}^r (d - d_i) \leq d \). An admissible \( r \)-tuple \((d_1, \ldots, d_r)\) is Tverberg prescribable if there is an \( N \) such that for every continuous \( f : \Delta_N \to \mathbb{R}^d \) there is a Tverberg partition \( \{\sigma_1, \ldots, \sigma_r\} \) for \( f \) with \( \dim \sigma_i = d_i \).

Theorem 6.3 shows that every admissible \( r \)-tuple of equal integers is Tverberg prescribable; see also Haase [11]. Also, in the case \( r = 2 \), all admissible tuples \((d_1, d_2)\) are Tverberg prescribable. Indeed, for even \( d \) this is given by the van Kampen–Flores Theorem 6.1. However, for odd \( d \) we need a statement that is stronger than what you get directly from Theorems 6.2 or 6.3, namely that there are two disjoint faces of \( \Delta_{d+2} \), both of them of dimension at most \( \left\lfloor \frac{d}{2} \right\rfloor \), whose images intersect. The version that we need will be obtained from Theorem 6.5.

**Theorem 6.8** (van Kampen–Flores, sharpened). Let \( d \geq 1 \). Then, for every continuous map \( f : \Delta_{d+2} \to \mathbb{R}^d \), there are two disjoint faces \( \sigma_1, \sigma_2 \) of \( \Delta_{d+2} \) such that \( \dim \sigma_1 = \left\lfloor \frac{d}{2} \right\rfloor \), \( \dim \sigma_2 = \left\lfloor \frac{d}{2} \right\rfloor \), and \( f(\sigma_1) \cap f(\sigma_2) \neq \emptyset \).

**Proof.** It remains to settle the case when \( d \) is odd, with \( \dim \sigma_1 = \frac{d-1}{2} \) and \( \dim \sigma_2 = \frac{d+1}{2} \). In terms of Theorem 6.5, in this situation we have \( r = 2 \), \( k = \frac{d+1}{2} \), \( s = 1 \), and thus there is a Tverberg 2-partition \( \sigma_1, \sigma_2 \) with \( \dim \sigma_i \leq \frac{d+1}{2} \), where at most \( \ell \leq \left\lfloor \frac{N - (r-s)}{k+1} \right\rfloor = \left\lfloor \frac{(d+2)-(2-1)+1}{\frac{d+1}{2}+1} \right\rfloor = 1 \) of the \( \sigma_i \) have dimension \( k = \frac{d+1}{2} \).

For \( d = 3 \), this says that for every continuous map \( \Delta_5 \to \mathbb{R}^3 \), the images of a triangle and an edge of \( \Delta_5 \) intersect. This is equivalent to the Conway–Gordon–Sachs theorem [9, 17] from graph theory, which says that the complete graph \( K_6 \) (that is, the 1-skeleton of \( \Delta_5 \)) is ‘intrinsically linked’.

**Question 6.9.** Is every admissible \( r \)-tuple Tverberg prescribable?

### 7. \( j \)-wise disjoint Tverberg partitions

The following result is a version of Theorem 6.2, without a bound on the dimensions of the simplices \( \sigma_i \) in the Tverberg partition. We state it here, and give a simpler proof, since our methods will also yield a new affine version of this, which holds for all \( r \geq 2 \).
THEOREM 7.1 ($r$-wise disjoint topological Tverberg: Sarkaria [18] and Volovikov [22]). Let $r \geq 2$ be a prime power, $d \geq 1$, $2 \leq j \leq r$, and
\[ N + 1 > \frac{r - 1}{j - 1}(d + 1). \tag{7.1} \]
Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$ there are $j$-wise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$.

Proof. For this, we repeat the proof of Theorem 6.4 and use the topological Tverberg theorem, Theorem 3.1, in place of Theorem 6.3. Let $N : = (N + 1)(j - 1) - 1$, and $p$ be the natural simplicial projection $\Delta_N' \cong \Delta_N^{(j-1)} \to \Delta_N$ that maps each of the $j - 1$ copies of a vertex $v \in \Delta_N$ in the join $\Delta_N^{(j-1)}$ to the vertex $v$, and set $f' := f \circ p : \Delta_N' \to \mathbb{R}^d$.

We have $N' \geq (r - 1)(d + 1)$. Thus, by the topological Tverberg theorem, Theorem 3.1, there are pairwise disjoint faces $\sigma'_1, \ldots, \sigma'_r \subseteq \Delta_N'$ such that $f'(\sigma'_1) \cap \ldots \cap f'(\sigma'_r) \neq \emptyset$. By definition of $f'$ this is equivalent to $f(p(\sigma'_1)) \cap \ldots \cap f(p(\sigma'_r)) \neq \emptyset$. Now, let $\sigma_1 = p(\sigma'_1), \ldots, \sigma_r = p(\sigma'_r)$. The faces $p(\sigma'_i)$ of $\Delta_N$ are $j$-wise disjoint. $\square$

THEOREM 7.2 ($r$-wise disjoint Tverberg). Let $r \geq 2$, $d \geq 1$, $2 \leq j \leq r$, and
\[ N + 1 > \frac{r - 1}{j - 1}(d + 1). \tag{7.2} \]
Then, for every affine map $f : \Delta_N \to \mathbb{R}^d$, there are $j$-wise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$.

Proof. For affine maps $f : \Delta_N \to \mathbb{R}^d$ we need not even assume that $r$ is a prime power, if we use Tverberg’s original theorem as the black box result. This is possible since the projection map $p : \Delta_N^{(j-1)} \to \Delta_N$ as in the proof of Theorem 7.1 is affine. $\square$

8. Tverberg partitions with equal barycentric coordinates

Let the vertices of $\Delta_N$ be partitioned into $\ell$ color classes $C_0, \ldots, C_{\ell - 1}$. Every point $x \in R$ in the corresponding rainbow complex $R$ has unique barycentric coordinates $x = \sum_{i=0}^{\ell - 1} \alpha_i v_i$ with $0 \leq \alpha_i \leq 1$ and $v_i$ a vertex in the color class $C_i$ for $0 \leq i \leq \ell - 1$. We say that two points $x, y$ in the rainbow complex have equal barycentric coordinates if $x = \sum_{i=0}^{\ell - 1} \alpha_i v_i$ and $y = \sum_{i=0}^{\ell - 1} \alpha_i w_i$, where $v_i$ and $w_i$ are vertices in the color class $C_i$. The following theorem is a topological version of Soberón’s [19] ‘Tverberg’s theorem with equal barycentric coordinates’.

THEOREM 8.1 (Topological Tverberg with equal barycentric coordinates). Let $r \geq 2$ be a prime power, $d \geq 1$, and $N = N_{(r-1)d} = r(r - 1)d + 1 - 1$. Let $f : \Delta_N \to \mathbb{R}^d$ be continuous. If the vertices of $\Delta_N$ are partitioned into $(r - 1)d + 1$ color classes of size $r$, then there are points $x_1, \ldots, x_r$ with equal barycentric coordinates in $r$ pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ whose images intersect, with $f(x_1) = \ldots = f(x_r)$.

Proof. Let the color classes be $C_0, \ldots, C_{(r-1)d}$. Every point $x \in \Delta_N$ is a unique convex combination $x = \sum \alpha_i v_i$ of the vertices of $\Delta_N$. For $0 \leq k \leq (r - 1)d$, let $g_k : \Delta_N \to \mathbb{R}$ be given by $\sum \alpha_i v_i \mapsto \sum_{v_i \in C_k} \alpha_i$. Each $g_k$ is an affine function that is equal to 1 on the simplex $\text{conv}(C_k) \subset \Delta_N$ with vertex set $C_k$ and 0 on all other vertices of $\Delta_N$. 
By Lemma 3.2, there are \( x_1, \ldots, x_r \in \Delta_N \) with \( x_i \in \sigma_i \), where the \( \sigma_i \subset \Delta_N \) are pairwise disjoint and \( f(x_1) = \ldots = f(x_r) \) as well as \( g_k(x_1) = \ldots = g_k(x_r) \) for \( 1 \leq k \leq (r-1)d \); that is, the lemma does not guarantee equality for \( g_0 \). However, as \( g_0 + \ldots + g(r-1)d = 1 \) we also obtain \( g_0(x_1) = \ldots = g_0(x_r) \).

Suppose that for some \( k \), the face \( \sigma_j \) has at least one vertex in \( C_k \). As we may again assume that \( \sigma_j \) is the minimal face of \( \Delta_N \) that contains \( x_j \), this implies that \( g_k(x_j) \neq 0 \) and hence \( g_k(x_i) \neq 0 \) for \( 1 \leq i \leq r \). Thus, all \( r \) faces \( \sigma_i \) have at least one vertex in \( C_k \). However, as \( |C_k| = r \) and the \( \sigma_i \) are pairwise disjoint, every \( \sigma_i \) has exactly one vertex in \( C_k \). Since this is true for every color, the \( \sigma_i \) belong to the rainbow complex.

Thus, the numbers \( g_k(x_i) \) for \( 0 \leq k \leq (r-1)d \) are exactly the barycentric coordinates of \( x_i \). These are equal for all the \( x_i \) since \( g_k(x_1) = \ldots = g_k(x_r) \) for all \( k \).

Soberón [19, Section 4] suggests an alternative idea for how to obtain the topological analog of his result that we have obtained using our ansatz.

The special case \( r = 2 \) of Theorem 8.1, which also establishes the Bárany–Larman conjecture [4] for \( r = 2 \), is equivalent to the Borsuk–Ulam theorem.

**Corollary 8.2 (Borsuk–Ulam).** For any continuous map \( f : \partial \Diamond_{d+1} \rightarrow \mathbb{R}^d \) from the boundary of the \((d+1)\)-dimensional crosspolytope \( \Diamond_{d+1} = \{ x \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} |x_i| \leq 1 \} \) to \( \mathbb{R}^d \), there are two points \( x_1, x_2 \in \Diamond_{d+1} \) with \( f(x_1) = f(x_2) \) that lie in opposite faces of \( \Diamond_{d+1} \) with equal barycentric coordinates, that is, \( x_1 = -x_2 \).

**Proof.** For \( r = 2 \), Theorem 8.1 treats a simplex with \( 2d+2 \) vertices in \( d+1 \) color classes of cardinality 2. Thus, here the rainbow complex is exactly the boundary of the \((d+1)\)-dimensional crosspolytope.

We also obtain Soberón’s original result in the same way.

**Theorem 8.3 (Tverberg with equal barycentric coordinates: Soberón [19]).** Let \( r \geq 2 \), \( d \geq 1 \) and \( N = N_{(r-1)d} = r((r-1)d+1) - 1 \). Let \( f : \Delta_N \rightarrow \mathbb{R}^d \) be affine. If the vertices of \( \Delta_N \) are partitioned into \((r-1)d+1\) color classes of size \( r \), then there are points \( x_1, \ldots, x_r \) with equal barycentric coordinates in \( r \) pairwise disjoint rainbow faces \( \sigma_1, \ldots, \sigma_r \) of \( \Delta_N \) whose images intersect, with \( f(x_1) = \ldots = f(x_r) \).

**Proof.** The proof is the same as for Theorem 8.1. Here, the constraint functions are affine, so the proof follows by Remark 3.3.

Soberón shows that the number of color classes and the number of points per color class are optimal for the theorem to hold. Thus, the topological version, Theorem 8.1, is also optimal in that sense.

9. **Optimal colored versions of the topological Tverberg theorem**

The following theorem is a strengthening of the topological Tverberg Theorem 3.1 in the case when \( r \) is a prime.

**Theorem 9.1 (Optimal colored Tverberg; Blagojević, Matschke and Ziegler [7]).** Let \( r \geq 2 \) be a prime, \( d \geq 1 \), and \( N \geq N_0 = (r-1)(d+1) \). Let the vertices of \( \Delta_N \) be colored by \( m+1 \)
colors $C_0, \ldots, C_m$ with $|C_i| \leq r - 1$ for all $i$. Then for every continuous map $f : \Delta_N \rightarrow \mathbb{R}^d$ there are $r$ pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$, such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$.

As mentioned in the introduction, this theorem can as well be used as a black box result, to which the method provided in Sections 3 and 4 can be applied. In this section, we will give two examples.

**Theorem 9.2.** Let $r \geq 2$ be a prime, $d \geq 1$, $\ell \geq 0$ and $k \geq 0$. Let the vertices of $\Delta_N$ be colored by $\ell + k$ colors $C_0, \ldots, C_{\ell+k-1}$ with $|C_0| \leq r - 1, \ldots, |C_{\ell-1}| \leq r - 1$ and $|C_\ell| \geq 2r - 1, \ldots, |C_{\ell+k-1}| \geq 2r - 1$, where $|C_0| + \ldots + |C_{\ell-1}| > (r - 1)(d - k + 1) - k$. Then, for every continuous map $f : \Delta_N \rightarrow \mathbb{R}^d$, there are $r$ pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$.

**Proof.** Without loss of generality, we can assume that $|C_\ell| = \ldots = |C_{\ell+k-1}| = 2r - 1$, by deleting any additional vertices. Then, the simplex $\Delta_N$ has still $N + 1 = |C_0| + \ldots + |C_{\ell-1}| + k(2r - 1)$ vertices, so

$$N = |C_0| + \ldots + |C_{\ell-1}| + k(2r - 1) - 1 \geq (r - 1)(d + k + 1) = N_k.$$

Now, we split each of the color classes $C_\ell, \ldots, C_{\ell+k-1}$ into new color sub-classes of cardinality at most $r - 1$. (For example, singletons will do.) Let $\Sigma_i$ be the subcomplex of all faces of $\Delta_N$ with at most one vertex in $C_i$. Thus, Theorem 9.1 together with the proof technique of Theorem 4.4 yields that there is a Tverberg $r$-partition $\sigma_1, \ldots, \sigma_r$, where each of the simplices $\sigma_i$ is a rainbow simplex with respect to the refined coloring where the large color classes have been split into sub-classes, and it also lies in $\Sigma_\ell \cap \ldots \cap \Sigma_{\ell+k-1}$, that is, it uses at most one of the color sub-classes of each of $C_\ell, \ldots, C_{\ell+k-1}$ and thus respects the original coloring.

This Theorem 9.2 contains Theorem 9.1 as the special case $k = 0$, and also Vrećica and Živaljević’s [24, Proposition 5] as the special case $|C_0| = \ldots = |C_{\ell-1}| = r - 1$ and $|C_\ell| = \ldots = |C_{\ell+k-1}| = 2r - 1$. If we further specialize to $r = 2$ and $k = 1$, this, in turn, reduces to the ‘colored Radon’ Corollary 5.2, as noted in [24, Corollary 7]. For $\ell = 0$, we get Vrećica and Živaljević’s colored Tverberg theorem ‘of type B’, see [23] and [24, Corollary 8].

As a second instance of combining Theorem 9.1 with the proof technique of Theorem 4.4, we finally obtain from our method the following new result about colored Tverberg partitions with restricted dimensions.

**Theorem 9.3.** Let $r \geq 2$ be a prime, $d \geq 1$, $N \geq N_1 = (r - 1)(d + 2)$, and $k \geq \lceil \frac{r-1}{d} \rceil$. Let the vertices of $\Delta_N$ be colored by $m + 1$ colors $C_0, \ldots, C_m$ with $|C_i| \leq r - 1$ for all $i$. Then, for every continuous map $f : \Delta_N \rightarrow \mathbb{R}^d$, there are $r$ pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ with $\dim \sigma_i \leq k$ for $1 \leq i \leq r$, such that $f(\sigma_1) \cap \ldots \cap f(\sigma_r) \neq \emptyset$.

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