SCALAR, ELECTROMAGNETIC AND WEYL
PERTURBATIONS OF BTZ BLACK HOLES:
QUASI NORMAL MODES

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Abstract

We calculate the quasinormal modes and associated frequencies of the Bañados, Zanelli and Teitelboim (BTZ) non-rotating black hole. This black hole lives in 2+1-dimensions in an asymptotically anti-de Sitter spacetime. We obtain exact results for the wavefunction and quasi normal frequencies of scalar, electromagnetic and Weyl (neutrino) perturbations.
1 Introduction

When one is describing the evolution of some conservative system, one often considers a small perturbation or a small departure from a known solution of the system, and one generally arrives at a wave equation describing it. For a system with no explicit time dependence, one finds the normal mode solutions of the wave equation, satisfying certain boundary conditions, and one can then specify completely the perturbation as a linear superposition of these normal modes. In this case the operator associated to the perturbation is self-adjoint, the frequencies are real and the modes are complete.

However, when one deals with open dissipative systems, as it is the case in this paper, such an expansion is not possible. Instead of normal modes, one considers quasi normal modes (QNM) for which the frequencies are no longer pure real, signalling that the system is losing energy. Although QNMs are in general not complete and therefore insufficient to fully describe the dynamics (see [1, 2] and references therein), they nevertheless dominate the signal during the intermediate stages of the perturbation, being therefore extremely important.

QNMs of black holes were first numerically computed by Chandrasekhar and Detweiler [3], and subsequent numerical simulations [4, 5, 6] showed that the amplitude is dominated, at intermediate times, by a ringing signal due to the QNMs. Aside from the pure mathematical interest, black hole’s QNM calculations have been a very active field, and new methods, both numerical and analytical have been developed (see [7] for a review).

Up until very recently, all these works dealt with asymptotically flat spacetimes. In the past few years there has been a growing interest in asymptotically AdS (anti-de Sitter) spacetimes. Indeed, the Bañados-Teitelboim-Zanelli (BTZ) black hole in 2+1-dimensions [8], as well as black holes in 3+1 dimensional AdS spacetimes with nontrivial topology (see, e.g. [10]), share with asymptotically flat spacetimes the common property of both having well defined charges at infinity, such as mass, angular momentum and electromagnetic charges, which makes them a good testing ground when one wants to go beyond asymptotic flatness. Another very interesting aspect of these black hole solutions is related to the AdS/CFT (Conformal Field Theory) conjecture [9]. For instance, due to this AdS/CFT duality, quasi-normal frequencies in the BTZ black hole spacetime yield a prediction for the
thermalization timescale in the dual two-dimensional CFT, which otherwise would be very difficult to compute directly. If one has, e.g., a 10-dimensional type IIB supergravity, compactified into a BTZ $\times S^3 \times T^4$ spacetime, the scalar field used to perturb the BTZ black hole, can be seen as a type IIB dilaton which couples to a CFT field operator $\mathcal{O}$. Now, the BTZ in the bulk corresponds to a thermal state in the boundary CFT, and thus the bulk scalar perturbation corresponds to a thermal perturbation with nonzero $\langle \mathcal{O} \rangle$ in the CFT.

There has been some recent work on perturbations of Schwarzschild AdS spacetimes: Horowitz and Hubeny [11] studied the QNM frequencies for scalar perturbations in 4, 5 and 7 dimensions. Wang et al [12, 13] studied scalar perturbations and QNMs on a Reissner-Nordstrom geometry, Chan and Mann [14] studied the QNM frequencies for a conformally coupled scalar field. For work on BTZ black holes such as entropy of scalar fields, see [15] and references therein.

In this paper we shall consider the QNMs of the 3D non-rotating BTZ black hole [8]. The non-rotating BTZ black hole metric for a spacetime with negative cosmological constant, $\Lambda = -\frac{1}{l^2}$, is given by

$$ds^2 = (-M + \frac{r^2}{l^2})dt^2 - (-M + \frac{r^2}{l^2})^{-1}dr^2 - r^2d\phi^2,$$

(1)

where $M$ is the black hole mass. The horizon radius is given by $r_+ = M^{1/2}l$. We shall in what follows suppose that the scalar, electromagnetic and Weyl (neutrino) fields are a perturbation, i.e., they propagate in a spacetime with a BTZ metric. We will find that all these fields obey a wave equation and the associated QNM are exactly soluble yielding certain hypergeometric functions. As for the frequencies one has exact and explicit results for scalar and electromagnetic perturbations and numerical results for Weyl perturbations. To our knowledge, this is the first exact solution of QNMs for a specific model (see [1]).

In section 2 we give the wave equation for scalar and electromagnetic perturbations, and find the QNMs themselves and their frequencies. In section 3 we find the wave equation for Dirac and Weyl (neutrino) perturbations and analyze their QNMs.
2 Perturbing a black hole with scalar and electromagnetic fields

2.1 The wave equation

In this subsection we shall analyze the scalar and electromagnetic perturbations, which as we shall see yield the same effective potential, and thus the same wave equation.

First, for scalar perturbations, we are interested in solutions to the minimally coupled scalar wave equation

$$\Phi^\mu;_\mu = 0,$$

(2)

where, a comma stands for ordinary derivative and a semi-colon stands for covariant derivative. We make the following ansatz for the field $\Phi$

$$\Phi = \frac{1}{r^{1/2}} f(r) e^{-i\omega t} e^{im\phi},$$

(3)

where $m$ is the angular quantum number. It is useful to use the tortoise coordinate $r_*$ defined by the equation $dr_* = \frac{dr}{-Ml^2/r^2}$, and given implicitly by

$$r = -M^{1/2} \coth(M^{1/2}r_*),$$

(4)

with $r_* \in ]-\infty, 0]$, ($r_* = -\infty$ corresponds to $r = r_+$, and $r_* = 0$ corresponds to $r = \infty$).

With the ansatz (3) and the tortoise coordinate $r_*$, equation (2) is given by,

$$\frac{d^2 f(r)}{dr_*^2} + (\omega - V(r)) f(r) = 0,$$

(5)

where,

$$V(r) = \frac{3r^2}{4l^4} - \frac{M}{2l^2} - \frac{M^2}{4l^2} + \frac{m^2}{l^2} - \frac{Mm^2}{r^2},$$

(6)

and it is implicit that $r = r(r_*)$. The rescaling to the radial coordinate $\hat{r} = \frac{r}{l}$ and to the frequency $\hat{\omega} = \omega l$ is equivalent to take $l = 1$ in (3) and (4), i.e., through this rescaling one measures the frequency and other quantities in terms of the AdS lengthscale $l$. 

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Now, the electromagnetic perturbations are governed by Maxwell’s equations
\[ F_{\mu\nu} ;_{\nu} = 0 \], with \( F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \), \( \text{(7)} \)
where \( F_{\mu\nu} \) is the Maxwell tensor and \( A_{\mu} \) is the electromagnetic potential. As the background is circularly symmetric, it would be advisable to expand \( A_{\mu} \) in 3-dimensional vector spherical harmonics (see [16] and [17]):
\[ A_{\mu}(t, r, \phi) = \begin{pmatrix} g_m(t, r) \\ h_m(t, r) \\ k_m(t, r) \end{pmatrix} e^{im\phi}, \] \( \text{(8)} \)
where \( m \) is again our angular quantum number, and this decomposition is similar to the one in eigenfunctions of the total angular momentum in flat space [16].

However, going through the same steps one finds that the equation for electromagnetic perturbations is the same as the one for scalars, equation (5). The reason is that in three dimensions the 2-form Maxwell field \( F = F_{\mu\nu} dx^\mu \wedge dx^\nu \) is dual to a 1-form \( d\Phi \).

2.2 QNMs for scalar and electromagnetic perturbations

Although a precise mathematical definition for a QNM can be given, as a pole in the Green’s function [7], we shall follow a more phenomenological point of view. A QNM describes the decay of the field in question. For the equation (5) it is defined as a corresponding solution which (i) near the horizon is purely ingoing, \( \sim e^{i\omega r_*} \), corresponding to the existence of a black hole, and (ii) near infinity is purely outgoing, \( \sim e^{-i\omega r_*} \), (no initial incoming wave from infinity is allowed). One can see that the potential \( V(r) \) diverges at infinity, so we require that the perturbation vanishes there (note that \( r = \infty \) corresponds to a finite value of \( r_* \), namely \( r_* = 0 \)). This vanishing of the solution at \( \infty \) will only be possible for a discrete set of complex frequencies \( \omega \) called quasinormal frequencies.
2.2.1 Exact calculation

Putting \( l = 1 \) and using the coordinate \( r_* \), the wave equation (9) takes the form

\[
\frac{\partial^2 a(r)}{\partial r_*^2} + \left[ \omega^2 - \frac{3M}{4\sinh(M/2r_*)^2} + \frac{M}{4\cosh(M/2r_*)^2} + \frac{L^2}{\cosh(M/2r_*)^2} \right] a(r) = 0 .
\] (9)

On going to a new variable \( x = \frac{1}{\cosh(M/2r_*)^2}, \) \( x \in [0,1] \) equation (9) can also be written as

\[
4x(1-x)\partial^2_x a + (4-6x)\partial_x \psi + \bar{V}(x)a = 0 ,
\] (10)

where

\[
\bar{V}(x) = \frac{1}{4x(1-x)} \left[ \frac{4\omega^2(1-x)}{M} - 3x - x(1-x) - \frac{4m^2x(1-x)}{M} \right] .
\] (11)

By changing to a new wavefunction \( y \) (see \[18\] for details), through

\[
\psi \rightarrow \frac{(x-1)^{3/4}}{x^{2M/2}}y ,
\] (12)

equation (10) can be put in the canonical form \[18, 19\]

\[
x(1-x)y'' + \left[ c - (a+b+1)x \right] y' - aby = 0 ,
\] (13)

with \( a = 1 + \frac{im}{2M^{1/2}} - \frac{i\omega}{2M^{1/2}}, b = 1 - \frac{im}{2M^{1/2}} - \frac{i\omega}{2M^{1/2}}, \) and \( c = 1 - \frac{i\omega}{M^{1/2}}, \) which is a standard hypergeometric equation. The hypergeometric equation has three regular singular points at \( x = 0, x = 1, x = \infty \), and has two independent solutions in the neighbourhood of each singular point. We are interested in solutions of (13) in the range \([0,1]\), satisfying the boundary conditions of ingoing waves near \( x = 0 \), and zero at \( x = 1 \). One solution may be taken to be

\[
y = (1-x)^{-a-b}F(c-a, c-b, c; x) ,
\] (14)

where \( F \) is a standard hypergeometric function of the second kind. Imposing \( y = 0 \) at \( x = 1 \), and recalling that \( F(a, b, c, 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \), we get

\[
a = -n , \text{ or } b = -n ,
\] (15)
with \( n = 0, 1, 2, \ldots \), so that the quasi normal frequencies are given by

\[
\omega = \pm m - 2iM^{1/2}(n + 1). \tag{16}
\]

The lowest frequencies, namely those with \( n = 0 \) and \( m = 0 \) had already been obtained by \([20]\) and agree with our results.

### 2.2.2 Numerical calculation of the frequencies

In order to check our results, we have also computed numerically the frequencies. By going to a new variable \( z = \frac{1}{r}, \ h = \frac{1}{r_+} \) one can put the wave equation \([3]\) in the form (see \([11]\) for further details)

\[
s(z) \frac{d^2}{dz^2} \Theta + t(z) \frac{d}{dz} \Theta + u(z) \Theta = 0, \tag{17}
\]

where \( \Theta = e^{i\omega r} a(r) \), \( s(z) = z^2 - Mz^4 \), \( t(z) = 2Mz^3 - 2i\omega z^2 \) and \( u(z) = \frac{V}{-Mz^2} \), with \( V \) given by \([3]\). Now, \( z \in [0, h] \) and one sees that in this range, the differential equation has only regular singularities at \( z = 0 \) and \( z = h \), so it has by Fuchs theorem a polynomial solution. We can now use Fröbenius method (see for example \([22]\)) and look for a solution of the form \( \Theta(z) = \sum_{n=0}^{\infty} \theta_n(\omega)(z - h)^n \), where \( \alpha \) is to be taken from the boundary conditions. Using the boundary condition of only ingoing waves at the horizon, one sees \([11]\) that \( \alpha = 0 \). So the final outcome is that \( \Theta \) can be expanded as

\[
\Theta(z) = \sum_{n=0}^{\infty} \theta_n(\omega)(z - h)^n. \tag{18}
\]

Imposing now the second boundary condition, \( \Theta = 0 \) at infinity (\( z = 0 \)) one gets

\[
\sum_{n=0}^{\infty} \theta_n(\omega)(-h)^n = 0. \tag{19}
\]

The problem is reduced to that of finding a numerical solution of the polynomial equation \([19]\). The numerical roots for \( \omega \) of equation \([19]\) can be evaluated resorting to numerical computation. Obviously, one cannot determine the full sum in expression \([14]\), so we have to determine a partial
sum from 0 to N, say, and find the roots $\omega$ of the resulting polynomial expression. We then move onto the next term N+1 and determine the roots. If the method is reliable, the roots should converge. We have stopped our search when a 3 decimal digit precision was achieved. We have computed the lowest frequencies for some parameters of the angular quantum number $m$ and horizon radius $r_+$. The frequency is written as $\omega = \omega_r + i\omega_i$, where $\omega_r$ is the real part of the frequency and $\omega_i$ is its imaginary part.

In tables 1 and 2 we list the numerical values of the lowest QNM frequencies, for $m = 0$ and $m = 1$, respectively, and for selected values of the black hole mass.

| $m = 0$ | Numerical | Exact |
|--------|-----------|-------|
| $M^{1/2}$ | $\omega_r$ | $-\omega_i$ | $\omega_r$ | $-\omega_i$ |
| $\frac{\pi}{2}$ | 0.000 | 1.000 | 0 | 1 |
| 1 | 0.000 | 2.000 | 0 | 2 |
| 5 | 0.000 | 10.000 | 0 | 10 |
| 10 | 0.000 | 20.000 | 0 | 20 |
| 50 | 0.000 | 100.000 | 0 | 100 |
| 100 | 0.000 | 200.000 | 0 | 200 |
| 1000 | 0.000 | 2000.000 | 0 | 2000 |

Table 1. Lowest ($n = 0$) QNM frequencies for $m = 0$.

| $m = 1$ | Numerical | Exact |
|--------|-----------|-------|
| $M^{1/2}$ | $\omega_r$ | $-\omega_i$ | $\omega_r$ | $-\omega_i$ |
| $\frac{\pi}{2}$ | 1.000 | 1.000 | 1 | 1 |
| 1 | 1.000 | 2.000 | 1 | 2 |
| 5 | 1.000 | 10.000 | 1 | 10 |
| 10 | 1.000 | 20.000 | 1 | 20 |
| 50 | 1.000 | 100.000 | 1 | 100 |
| 100 | 1.000 | 200.000 | 1 | 200 |
| 1000 | 1.000 | 2000.000 | 1 | 2000 |

Table 2. Lowest ($n = 0$) QNM frequencies for $m = 1$. 
The numerical results agree perfectly with (16), and one sees that the imaginary part of the frequency scales with the horizon whereas the real part depends only on the angular index \( m \).

3 Perturbing a black hole with Dirac and Weyl spinor fields

3.1 The wave equation

We shall develop Dirac’s equation for a massive spinor, and then specialize to the massless case. The two component massive spinor field \( \Psi \), with mass \( \mu_s \) obeys the covariant Dirac equation

\[ i\gamma^\mu \nabla_\mu \Psi - \mu_s \Psi = 0, \tag{20} \]

where \( \nabla_\mu \) is the spinor covariant derivative defined by

\[ \nabla_\mu = \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma[a \gamma_b], \]

and \( \omega_\mu^{ab} \) is the spin connection, which may be given in terms of the tryad \( e_\mu^{a} \).

As is well known there are two inequivalent two dimensional irreducible representations of the \( \gamma \) matrices in three spacetime dimensions. The first may be taken to be \( \gamma^0 = i\sigma^2, \gamma^1 = \sigma^1, \) and \( \gamma^2 = \sigma^3 \), where the matrices \( \sigma^k \) are the Pauli matrices. The second representation is given in terms of the first by a minus sign in front of the Pauli matrices. From equation (20), one sees that a Dirac particle with mass \( \mu_s \) in the first representation is equivalent to a Dirac particle with mass \(-\mu_s \) in the second representation. To be definitive, we will use the first representation, but the results can be interchanged to the second one, by substituting \( \mu_s \rightarrow -\mu_s \). For Weyl particles, \( \mu_s = 0 \), both representations yield the same results.

Again, one can separate variables by setting

\[ \Psi(t, r, \phi) = \begin{bmatrix} \Psi_1(t, r) \\ \Psi_2(t, r) \end{bmatrix} e^{-i\omega t} e^{im\phi}. \tag{21} \]

On substituting this decomposition into Dirac’s equation (20) we obtain

\[ \begin{align*}
- \frac{i(M - 2r^2)}{2\Delta^{1/2}} r \Psi_2 + i\Delta^{1/2} \partial_r \Psi_2 + \frac{r^2 \omega}{\Delta^{1/2}} \Psi_2 &= (m + \mu_s) \Psi_1, \\
- \frac{i(M - 2r^2)}{2\Delta^{1/2}} r \Psi_1 + i\Delta^{1/2} \partial_r \Psi_1 + \frac{r^2 \omega}{\Delta^{1/2}} \Psi_1 &= (m + \mu_s) \Psi_2,
\end{align*} \tag{22} \tag{23} \]
where we have put $\Delta = -Mr^2 + \frac{r^4}{l^2}$, we have restored the AdS lengthscale $l$, and in general we follow Chandrasekhar’s notation [22]. Defining $R_1$, $R_2$, and $\hat{m}$ through the relations

$$\Psi_1 = i\Delta^{-1/4}R_1,$$
$$\Psi_2 = \Delta^{-1/4}R_2,$$
$$m = i\hat{m},$$

we obtain,

$$(\partial_{\hat{r}_*} - i\omega)R_2 = \frac{i\Delta^{1/2}}{r^2}(\hat{m} - i\mu_s r)R_1,$$  \hspace{1cm} (27)

$$(\partial_{\hat{r}_*} + i\omega)R_1 = \frac{i\Delta^{1/2}}{r^2}(\hat{m} + i\mu_s r)R_2.$$  \hspace{1cm} (28)

Defining now $\nu$, $\Upsilon_1$, $\Upsilon_2$, and $\hat{r}_*$ through the relations

$$\nu = \arctan\left(\frac{\mu_s r}{\hat{m}}\right),$$
$$R_1 = e^{2i\nu} \Upsilon_1,$$  \hspace{1cm} (30)

$$R_2 = e^{-2i\nu} \Upsilon_2,$$  \hspace{1cm} (31)

$$\hat{r}_* = r_* + \frac{1}{2\omega} \arctan\left(\frac{\mu_s r}{\hat{m}}\right),$$  \hspace{1cm} (32)

we get

$$(\partial_{\hat{r}_*} - i\omega)\Upsilon_2 = W\Upsilon_1,$$  \hspace{1cm} (33)

$$(\partial_{\hat{r}_*} - i\omega)\Upsilon_2 = W\Upsilon_2.$$  \hspace{1cm} (34)

where,

$$W = \frac{i\Delta^{1/2}(\hat{m}^2 + \mu_s^2\hat{r}^2)^{3/2}}{r^2(\hat{m}^2 + \mu_s^2\hat{r}^2) + \frac{\hat{m}\Delta}{2\omega}}.$$  \hspace{1cm} (35)

Finally, putting $Z_\pm = \Upsilon_1 \pm \Upsilon_2$ we have

$$(\partial_{\hat{r}_*}^2 + \omega^2)Z_\pm = V_\pm Z_\pm,$$  \hspace{1cm} (36)

with

$$V_\pm = W^2 \pm \frac{dW}{d\hat{r}_*}. $$  \hspace{1cm} (37)
We shall be concerned with massless spinors ($\mu_s = 0$) for which $\hat{r}_s = r_s$, and $W = i\Delta^{1/2}\hat{m}$. Thus,

$$V_\pm = \frac{m^2}{r^2} \left( \frac{r^2}{l^2} - M \right) \pm \frac{Mm}{r^2} \left( \frac{r^2}{l^2} - M \right)^{1/2}. \quad (38)$$

In the form (37) one immediately recognizes that the two potentials $V_+$ and $V_-$ should yield the same spectrum. In fact they are, in SUSY language, superpartner potentials derived from a superpotential $W$ (see [23]). Once again, we can rescale $r$ and take $l = 1$, by measuring everything in terms of $l$.

### 3.2 QNMs for Weyl perturbations

Similarly, the wave equation (36) for Weyl (until recently also called neutrino) perturbations may be put in the form

$$\partial_r^2 Z_\pm + \left[ \omega^2 - m \left( \frac{m}{\cosh(M^{1/2}r_s)} \pm M^{1/2} \frac{\sinh(M^{1/2}r_s)^2}{\cosh(M^{1/2}r_s)^2} \right) \right] Z_\pm = 0. \quad (39)$$

Going to a new independent variable, $x = -\sinh(M^{1/2}r_s)$, $x \in [\infty, 0]$, we can write

$$(1 + x^2)Z'' + xZ' + \frac{\omega^2(1+x^2) - m^2 \pm mx}{1 + x^2} Z = 0. \quad (40)$$

By changing the wavefunction $Z$ to $\chi$

$$\chi = e^{\left(\frac{M^{1/2}x - 2m}{2M^{1/2}} - \frac{x}{2}\right) \text{arctan}(x)}, \quad (41)$$

we have

$$(1 + x^2)\chi'' + \left( \frac{2m}{M^{1/2}} + x \right) \chi' + \left( \frac{\omega^2}{M} \right) \chi = 0. \quad (42)$$

On putting $s = \frac{1 + iz}{2}$, $s \in \left[\frac{1}{2}, i\infty\right]$, we have again the hypergeometric equation (13), with $a = \frac{\omega}{M^{1/2}r_s}$, $b = -\frac{\omega}{M^{1/2}r_s}$, and $c = \frac{1}{2} \pm \frac{m}{M^{1/2}}$, so that the solution to the wave equation is again specified around each singular point, and is given by the analytic continuation of the standard hypergeometric function to the complex plane [13, 14].

Since infinity is located at $s = \frac{1}{2}$, there is no easy way to determine the QNM frequencies, so we have to resort to numerical calculations. If we put
in the form (14) one again sees that it has no essential singularities, so the numerical method just outlined in the previous section may be applied. Moreover, since $V_+$ and $V_-$ have the same spectrum [23] and the same QNM frequencies [22] we need only to workout the frequencies for one of them. In table 3 we present the numerical results for the QNM frequencies for neutrino perturbations and for selected values of the black hole mass.

| $m = 1$ | Numerical |
|---------|-----------|
| $M^{1/2}$ | $\omega_r$ | $-\omega_i$ |
| 2       | 0.378     | 2.174     |
| 5       | 0.316     | 5.027     |
| 10      | 0.224     | 10.006    |
| 50      | 0.099     | 50.001    |
| 100     | 0.071     | 100.000   |
| 500     | 0.0316    | 500.000   |

Table 3. Lowest QNM frequencies for $m = 1$.

For large black holes one can see that the imaginary part of the frequencies scale with the horizon ($r_+ = M^{1/2}$), just as in the scalar and electromagnetic case. We have also computed some higher modes, and the real part of the frequency $\omega_r$, does not seem to depend on which mode we are dealing with, just as in the scalar and electromagnetic case.

4 Conclusions

We have computed the scalar, electromagnetic and neutrino QNM of BTZ black holes. These modes dictate the late time behaviour of the fields. In all cases, these modes scale with the horizon radius, at least for large black holes and, since the decay of the perturbation has a timescale $\tau = \frac{1}{\omega_i}$, this means that the greater the mass, the less time it takes to approach equilibrium. We have also found that for large black holes, the QNM frequencies are proportional to the black hole radius. Since the temperature of a BTZ black is proportional to the black hole radius, the QNM frequencies scale with the temperature, as a simple argument indicates [11]. For the study of QNM
of 3+1-dimensional spherical, as well as the toroidal black holes found by Lemos [24] see [25].

Is there, for small black holes, any relation with these quasinormal modes and critical phenomena as speculated by Horowitz and Hubeny [11]? Though that would be an extremely interesting relation, we still are not able to answer that.

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