Safe Reinforcement Learning with Stability & Safety Guarantees Using Robust MPC

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Abstract

Reinforcement Learning offers tools to optimize policies based on the data obtained from the real system subject to the policy. While the potential of Reinforcement Learning is well understood, many critical aspects still need to be tackled. One crucial aspect is the issue of safety and stability. Recent publications suggest the use of Nonlinear Model Predictive Control techniques in combination with Reinforcement Learning as a viable and theoretically justified approach to tackle these problems. In particular, it has been suggested that robust MPC allows for making formal stability and safety claims in the context of Reinforcement Learning. However, a formal theory detailing how safety and stability can be enforced through the parameter updates delivered by the Reinforcement Learning tools is still lacking. This paper addresses this gap. The theory is developed for the generic robust MPC case, and further detailed in the robust tube-based linear MPC case, where the theory is fairly easy to deploy in practice.

Key words: Safe Reinforcement Learning, Robust Model Predictive Control, Stability

1 Introduction

Reinforcement Learning (RL) is a powerful tool for tackling Markov Decision Processes (MDP) without depending on a detailed model of the probability distributions underlying the state transitions. Indeed, most RL methods rely purely on observed state transitions, and realizations of the stage cost $L(s, a) \in \mathbb{R}$ assigning a performance to each state-input pair $s, a$ (the inputs are often labeled actions in the RL community). RL methods seek to increase the closed-loop performance of the control policy deployed on the MDP as observations are collected. RL has drawn an increasingly large attention thanks to its accomplishments, such as, e.g., making it possible for robots to learn to walk or fly without super-

Most RL methods are based on learning the optimal control policy for the real system either directly, or indirectly. Indirect methods typically rely on learning a good approximation of the optimal action-value function underlying the MDP. The optimal policy is then indirectly obtained as the minimizer of the value-function approximation over the inputs $a$. Direct RL methods, based on policy gradients, seek to adjust the parameters $\theta$ of a given policy $\pi_\theta$ such that it yields the best closed-loop performance when deployed on the real system. An attractive advantage of direct RL methods over indirect ones is that they are based on formal necessary conditions of optimality for the closed-loop performance of $\pi_\theta$, and therefore asymptotically (for a large enough data set) guarantee the (possibly local) optimality of the parameters $\theta$ [18,19].

RL methods often rely on Deep Neural Networks (DNN) to carry the policy approximation $\pi_\theta$. While effective in practice, control policies based on DNNs provide limited opportunities for formal verification of the resulting closed-loop behavior, and for imposing hard constraints on the evolution of the state of the real system. The development of safe RL methods, which aims at tackling this issue, is currently an open field of research.
Model Predictive Control (MPC) is a very successful tool for generating policies that minimize a certain cost under some state and input constraints. However, for computational reasons, simple models are usually preferred in the MPC scheme. As a result, MPC can deliver a reasonable approximation of the optimal policy, which, however, is usually suboptimal. Choosing the MPC model parameters that maximize the closed-loop performance of the MPC scheme is a difficult problem. Indeed, e.g. the parameters that best fit the MPC model to the real system are not guaranteed to yield the best MPC policy [7]. In [7], it is shown that adjusting the MPC model, cost and constraints can be beneficial to achieve the best closed-loop performances, and RL is proposed as a possible approach to perform that adjustment in practice. The use of learning techniques within control has been proposed in, e.g., [3,4,10–13,15]. To the best of our knowledge, [2,7,22,23] are the first works proposing to use MPC as a function approximator in RL.

In this paper, we are interested in using Reinforcement Learning (RL) techniques to find the optimal MPC parameters from the data collected on the real system. In [22], these ideas were extended to using robust Model Predictive Control to support safe policies in the sense detailed in Section 3. An aspect of safe RL, briefly invoked in [22] but not further investigated, is the problem of enforcing the safety and stability of the combination of RL and robust MPC through the updates of the parameters performed by RL. We address these questions in this paper.

The paper is organized as follows. Section 2 provides background material on MDPs. Section 3 proposes a definition of safe policies and Section 4 provides background material on robust MPC. Section 5 presents conditions for building a safe combination of RL and robust MPC, where the parameter updates provided by RL do not jeopardize the safety of the robust MPC scheme. Section 6 presents conditions for RL to update the MPC parameters while maintaining the system stability, discussed in an augmented parameter-state space.

2 Background

We consider real systems that can be described as continuous Markov Chains, having an underlying conditional transition probability density over states $s$ and actions $a$ labelled as:

$$
\varphi [s_{i+1} \mid s_i, a_i] : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+,
$$

(1) is not known, or only inaccurately known. We will assume in the following that a stage cost

$$
L(s_i, a_i) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}
$$

is provided, and that our goal is to find the parameters $\theta$ of a policy

$$
\pi_\theta(s_i) : \mathbb{R}^n \to \mathbb{R}^m
$$

that minimize the expected discounted cost:

$$
J(\pi_\theta) = \mathbb{E} \left[ \sum_{i=0}^{\infty} \gamma^i L(s_i, a_i) \mid a_i = \pi_\theta(s_i) \right],
$$

where $\mathbb{E}[\cdot]$ is the expected value operator applying to the real trajectories yielded by (1) in closed-loop with policy $\pi_\theta$, and $\gamma \in (0, 1]$ a discount factor. We will consider that the actions delivered by the policy are possibly restricted to a subset of $\mathbb{R}^m$, i.e., the minimization of $J$ is subject to

$$
\pi_\theta(s_i) \in \mathcal{U}, \quad \forall s_i.
$$

We additionally consider in this paper that we seek policies that keep the system safe in the sense of respecting some state constraints

$$
s_i \in \mathcal{X},
$$

for any time $i = 0, \ldots, \infty$. Note that for the sake of simplicity, we will not treat mixed state-input constraints here, even though the proposed results arguably readily extend to that case. Throughout the paper, we will assume that the real state transition (1) is imperfectly or only coarsely known, and difficult to capture.

3 Safe Policies

We consider a system as safe if (6) holds at all time $i$ with probability one. Unfortunately, guaranteeing this unitary probability is impossible without a perfect knowledge of the system dynamics (1). Hence any notion of safe policy ought to accommodate the imperfect knowledge we have of (1). In robust MPC, the real system stochasticity is typically represented via a structured model:

$$
s_{i+1} = F_\theta(s_i, a_i, w_i),
$$

where $F_\theta : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^n$, and $\theta$ includes the possible model parameters, together with

$$
w_i \in \mathcal{W}_\theta,
$$

an external, typically stochastic disturbance contained in a compact set $\mathcal{W}_\theta$, modelling the stochasticity of the real system.
Provided that for any state-action pair \( s, a \), the support of the real system dynamics (1) is (almost entirely) included in the set:

\[
\mathcal{D}_\theta(s, a) = \{ F_\theta(s, a, w) \mid \forall w \in \mathcal{W}_\theta \},
\]

i.e.,

\[
\int_{\mathcal{D}_\theta(s, a)} \varphi(s_+ | s, a) ds_+ = 1, \quad \forall s, a,
\]

then a policy guaranteeing that (6) is satisfied for all the possible trajectories resulting from (7)-(8) is safe by construction. Robust MPC techniques can be used to build such policies.

A fundamental issue is then to verify that condition (10) holds. Indeed, the validation of (10) is limited to our knowledge of the system supported by all data and prior knowledge available about it, labelled \( \mathcal{D} \) here. Such data can, e.g., be the set of all observed state transitions \( s, a, s_+ \), but also include some prior or structural knowledge of the system. In the context of Bayesian inference, (10) is then regarded as probabilistic, conditioned on the prior knowledge and data \( \mathcal{D} \). The notion of safety hence becomes a probabilistic one, which can be defined as:

\[
\sigma := \mathbb{P}[ (10) | \mathcal{D} ].
\]

The probability (11) is to be understood in the context of Bayesian hypothesis testing. One can easily verify that if a policy (5) ensures that the closed-loop trajectories of (7)-(10) respect the state constraints (6) at all time, then the probability that the policy is safe for the real system is at least \( \sigma \). In this paper, we adopt that operational notion of knowledge-based safety and will label such a policy \( \sigma \)-safe. In practice, a minimum requirement for \( \sigma > 0 \) to hold is to ensure that:

\[
s_+ \in \mathcal{D}_\theta(s, a)
\]

holds for all observed state transition triplets \( \{ s, a, s_+ \} \in \mathcal{D} \). Condition (12) then defines a set \( \Theta_{\mathcal{D}} \) in the parameter space to which the robust MPC parameters \( \theta \) should be restricted. In this paper, we will consider a set \( \Theta_{\mathcal{D}} \) possibly formed from (12), and possibly further restricted by prior or structural knowledge of the system. For a complete discussion on the definition of \( \Theta_{\mathcal{D}} \) based on (12) and its deployment within safe RL, we refer to [22].

### 4 Stability Guarantees for Robust MPC

Consider a sequence of parametrized policies:

\[
a_k = \pi^k_\theta(v, s_k),
\]

where \( v \) is a set of variables used to shape the policy sequence. In this paper, we consider robust MPC schemes of the form:

\[
\bar{V}_\theta(s) = \min_v \varphi_\theta(v, s) \quad \text{for all } s \in \mathcal{X}_0
\]

s.t. \( \mathcal{X}_{k+1} = F_\theta(\mathcal{X}_k, \pi^k_\theta(v, s_k), \mathcal{W}_\theta) \),

\[
\mathcal{X}_0 = s_i, \quad \mathcal{X}^0 = \mathcal{X}_0, \quad \forall k = 0, \ldots, N - 1,
\]

where the state propagation (14b), called tube, is the extension of (7) to set propagation, i.e., we read (14b) as:

\[
\mathcal{X}^k = \{ F_\theta(s, \pi^k_\theta(v, s), \mathcal{W}) | \forall s \in \mathcal{X}_k, \mathcal{W} \in \mathcal{W}_\theta \}.
\]

Set \( \mathcal{U} \) in (14) represents the possible limitations on the feasible actions (e.g., actuator limitations), and the terminal set \( \mathcal{X}^N \) must be constructed such that (14) being feasible entails that the the state constraint \( \mathcal{X}_N \subseteq \mathcal{X} \) can be enforced at all future time. This is typically achieved by resorting to a terminal control law which makes \( \lambda^t_\theta \) forward invariant. The cost function (14a) is left unspecified here, as it can take different forms such as, e.g., a worst-case cost (as in min-max robust MPC); a nominal cost (as in tube MPC); an expected cost, or more elaborate risk-adverse costs. Function \( \bar{V}_\theta(s) \) receives an infinite value for the states \( s \) for which problem (14) is infeasible. For the sake of simplicity, we do not consider mixed state-input constraints here, although the proposed theory readily applies to that case.

The parametrized policy (13) used in (14) is often selected as a nominal state-input reference sequence together with an additional linear state feedback, i.e.,

\[
v = \{ \bar{u}_0, \ldots, N-1, \bar{x}_0, \ldots, N-1 \},
\]

\[
\pi^k_\theta(v, s_i) = \bar{u}_k - K_\theta(x - \bar{x}_k).
\]

The robust MPC scheme (14) defines a policy \( \pi^k_\theta(s) \) given by the first control input of the policy sequence adopted in the robust MPC scheme, i.e.:

\[
\pi^k_\theta(s_i) = \pi^0_\theta(v^*, s_i) = \bar{u}_0,
\]

where \( v^* \) is the solution of (14).

Let us define \( \lambda^t_\theta \) the set of states \( s \) for which the robust MPC scheme (14) is feasible. We ought to remind here that, because of (11), one cannot obtain any safety guarantee better than the upper bound:

\[
\mathbb{P}[ s_{i+1} \notin \lambda^0_\theta | s_i \in \lambda^0_\theta ] \leq 1 - \sigma.
\]
We define next the following sets in the space of parameters $\theta$.

**Definition 1** Let us define the set $\Theta_F$ of parameters $\theta$ such that the robust MPC scheme (14) is recursively feasible for the dynamics (7)-(8) for some non-empty set $\mathcal{X}_0^\theta$ of initial conditions.

**Definition 2** Let us define the set $\Theta_L$, of parameters $\theta$ such that the value function $\hat{V}(s)$ defined by (14) is a Lyapunov function, i.e., $\hat{V}(s)$ is:

1. lower and upper bounded by $\mathcal{K}_\infty$ functions;
2. such that $\hat{V}(s_+) \leq \gamma \hat{V}(s) + \delta_\theta$ for all $s \in \mathcal{X}_0^\theta$ and $s_+ \in \{ F_\theta(s, \pi_\theta(v, s), w) \mid \forall w \in \mathcal{W}_\theta \},$ for some positive constants $\delta_\theta, \gamma < 1$.

We observe that for any parameter $\theta \in \Theta_L \cap \Theta_F$, the robust MPC scheme (14) is recursively feasible and stabilizes the state of the system to the sub-level set

$$\mathcal{L}_\theta = \left\{ s \mid \hat{V}(s) \leq \frac{\delta_\theta}{1 - \gamma} \right\}$$

(20)

for any initial condition $s \in \mathcal{X}_0^\theta$, see [17]. We note that establishing conditions on $\theta$ such that $\theta \in \Theta_F \cap \Theta_L$ is typically done in practice via min-max robust MPC or tube-based MPC [17].

**Assumption 1** In the remainder of the paper, we will use the following assumptions:

1. The set of feasible initial conditions $\mathcal{X}_0^\theta$ is compact and continuous in $\theta$.
2. The conditional density (1) underlying the real system is bounded for all state-action pairs.
3. The set $\mathcal{L}_\theta$ is compact.

These technical assumptions will be required in some parts of the theory proposed in this paper. We ought to note here that the compactness of set $\mathcal{X}_0^\theta$ is a mild assumption for a well-posed MPC scheme. The continuity of the set with respect to $\theta$ requires that all functions involved in the MPC scheme are sufficiently differentiable, and that the MPC formulation satisfies standard regularity assumptions for Nonlinear Programs. Assumption 1.2, e.g., excludes deterministic systems, and systems where the stochasticity impacts only some parts of the state space. Assumption 1.3 follows from Assumption 1.1 and the choice of assigning an infinite value to $\hat{V}(s)$ when (14) is infeasible.

4.1 Safety & Stability Constraints in RL

In the following, we will consider the safety and stability conditions detailed in this section as constraints applied to the RL steps. More specifically, similarly to [22], we consider that the RL steps taken on the robust MPC scheme are feasible steps $\Delta \theta$ taken on the constrained optimization problem:

$$\min_{\theta} \ J(\pi_\theta), \quad (21a)$$

s.t. $\theta \in \Theta_L \cap \Theta_F \cap \Theta_D$, (21b)

in the sense that each update $\theta_{p+1} = \theta_p + \Delta \theta$ satisfies (21b) and reduces the cost (21a). More specifically, the RL steps will be computed according to:

$$\min_{\theta_{p+1}} \frac{1}{2} ||\theta_{p+1} - \theta_p||^2_H + \alpha \nabla_\theta J(\pi_\theta) \ T (\theta_{p+1} - \theta_p), \quad (22a)$$

s.t. $\theta_{p+1} \in \Theta_L \cap \Theta_F \cap \Theta_D$, (22b)

for some positive-definite matrix $H \approx \nabla_\theta J(\pi_\theta)$, and some $\alpha \in (0, 1]$. We should recall here that for any $H > 0$ and $\alpha$ small enough, the sequence $\theta_{1, \infty}$ stemming from (22) converges to a (possibly local) solution of (21) [14].

We will assume here that the gradient $\nabla_\theta J$ in (22a) is either evaluated directly via actor-critic or policy search techniques, or replaced by a surrogate based on Q-learning techniques, all formed using data collected on the real system in closed-loop with policy $\pi_\theta$. The safety and stability constraints (21b) will then be built based on (6), (7), (8), and (12). In the remainder of the paper, a mild technical assumption on the Nonlinear Program (22) will be very helpful.

**Assumption 2** The solution $\theta_{p+1}$ of (22) is continuous with respect to $\alpha$ in a neighborhood of $\alpha = 0$.

Assumption 2 follows from technical assumptions on the set $\Theta_L \cap \Theta_F \cap \Theta_D$, which we propose to not discuss extensively here for the sake of brevity. In particular, we note that Assumption 2 naturally holds if the set $\Theta_L \cap \Theta_F \cap \Theta_D$ can be represented by a finite set of continuous inequality constraints, and if the resulting problem (22) fulfills classical regularity assumptions and sufficient second-order conditions (SOSC) [14].

An important question that needs to be addressed is how feasible parameter updates $\theta_{p+1} = \theta_p + \Delta \theta$ resulting from (22) can be implemented in the robust MPC scheme without jeopardizing the safety and stability of the closed-loop system. We discuss the safety question in the next section using two different possible approaches.

5 Recursive Feasibility with RL-Based Parameter Updates

Let us consider a sequence of parameters $\theta_{0, \infty}$ resulting from (22), and consider that each parameter $\theta_p$ of
that sequence is applied for a certain amount of time (i.e., at least one sampling time of the robust MPC scheme). This section provides conditions such that this sequence of parameter updates does not jeopardize the safety of the corresponding sequence of policies $\pi_{0,\ldots,\infty}$ resulting from the corresponding robust MPC schemes. Note that a parameter update $\theta_p \to \theta_{p+1}$ occurring at a time sample $i$ means here that $a_i = \pi_{\theta_p}(s_i)$ and the inputs $a_j = \pi_{\theta_{p+1}}(s_j)$ with $j > i$ are used until the next parameter update. The next Theorem provides a first set of conditions for ensuring the safety of the parameter updates.

**Theorem 1**  Assume that for all $p$, parameter $\theta_p$ satisfies $\theta_p \in \Theta_\mathcal{F} \cap \Theta_p$, and that the initial conditions $s_0$ are in the set $\lambda_{\theta_0}$. If each parameter update $\theta_p \to \theta_{p+1}$ takes place in a state $s_i$ such that

$$s_i \in \lambda_{\theta_{p+1}}^{0}$$

(23)

holds, then the closed-loop trajectories $s_0,\ldots,\infty$ resulting from applying the sequence of policies $\pi_{0,\ldots,\infty}$ is $\sigma$-safe.

**PROOF.** Assuming that (10) holds, then a standard result for robust MPC is that if $\theta_{p+1} \in \Theta_\mathcal{F}$ and if the initial state at which policy $\pi_{\theta_{p+1}}$ is deployed satisfies condition (23), then policy $\pi_{\theta_{p+1}}$ ensures that the state trajectories are feasible at all time with unitary probability. We then observe that if every parameter update $\theta_p \to \theta_{p+1}$ is applied under condition (23), then each policy keeps the state trajectories feasible. As a result, if (23) is ensured at every parameter update $\theta_p \to \theta_{p+1}$, then the entire state trajectory $s_0,\ldots,\infty$ resulting from the policy sequence $\pi_{0,\ldots,\infty}$ remains feasible at all time.

If statement (10) does not hold, then the closed-loop trajectories $s_0,\ldots,\infty$ may become infeasible, though not necessarily. Hence if statement (10) holds with a probability $\sigma$, then the closed-loop trajectories $s_0,\ldots,\infty$ have probability no smaller than $\sigma$ to be feasible.

The results of Theorem 1 can be leveraged in practice by solving the robust MPC schemes associated to both $\theta_p$ and $\theta_{p+1}$ in parallel at every sampling instant and selecting the control input associated to $\theta_{p+1}$ as soon as the MPC scheme associated to $\theta_{p+1}$ is feasible.

The proposed theorem ensures the recursive feasibility of the sequence of robust MPC schemes such that the closed-loop state trajectories are contained in the set $\mathcal{X}$, within the framework presented in Section 3. An important caveat, though, is that there is no guarantee that condition (23) can be met in finite time by the closed-loop trajectories under policy $\pi_{\theta_p}$. As a result, it might be possible that a parameter update $\theta_{p+1}$, though feasible for (21), yields an update condition (23) that never becomes satisfied, hence blocking the learning process. The remaining of this section proposes two different approaches to tackle that issue, using either backtracking or additional constraints in (21).

In order to support and simplify the coming argumentation, it is useful to introduce a technical Lemma. Let us consider the trivial locally compact measure on $\mathbb{R}^n$, associating to any compact set $\mathcal{A} \subset \mathbb{R}^n$ the bounded positive real number:

$$\mu(A) = \int_A ds.$$  \hspace{1cm} (24)

**Lemma 1** Consider a bounded state transition probability density (1), i.e.:

$$\varphi(s \mid s, a) \leq \overline{\varphi} < \infty, \quad \forall s, s, a,$$  \hspace{1cm} (25)

and an arbitrary policy $a = \pi(s)$, yielding a (continuous) Markov Chain $s_0,\ldots,\infty$. Then for any set $\mathcal{A}$ and initial condition $s_0 \in \mathcal{A}$, the following inequality holds:

$$P[s_0,\ldots,\infty \in \mathcal{A}] \leq \mu(\mathcal{A}) \overline{\varphi}. \hspace{1cm} (26)$$

**PROOF.** Denoting $\phi[s_k]$ the density of the Markov Chain at time $k$, we observe that:

$$\phi[s_k] := \int \varphi[s_0 \mid s_{k-1}, \pi(s_{k-1})] \phi[s_{k-1}] ds_{k-1} \leq \overline{\varphi}$$  \hspace{1cm} (27)

holds for any $k > 0$. It follows that

$$P[s_k \in \mathcal{A}] = \int_{\mathcal{A}} \phi[s_k] ds_k \leq \mu(\mathcal{A}) \overline{\varphi}. \hspace{1cm} (28)$$

Then (26) follows from the Fréchet inequalities, stating:

$$P[s_0,\ldots,\infty \in \mathcal{A}] \leq \inf_k P[s_k \in \mathcal{A}] \leq \mu(\mathcal{A}) \overline{\varphi}. \hspace{1cm} (29)$$

\[\square\]

5.1 Parameter Update via Backtracking

In this subsection, we consider the use of backtracking on the parameter updates computed from (22) to ensure the feasibility of updating the parameters in finite time. For the sake of simplicity in the following developments, rather than a line-search strategy [14], we adopt a gradient adaptation strategy in the cost (22a), by iteratively reducing parameter $\alpha$, therefore generating a step ranging from a full step ($\alpha = 1$) to $\Delta \theta = 0$ (with $\alpha = 0$). The following theorem then guarantees that there is some $\alpha > 0$ such that the probability that the parameter update condition (23) is not met in finite time is less than $1 - \sigma$. 

\[\square\]
Theorem 2 Consider the closed loop trajectory \( s_i, \ldots, \infty \) under policy \( \pi_{\theta_p} \), starting at the physical sampling time \( i \) with the initial state \( s_i \). Consider the parameter update \( \theta_{p+1}(\alpha) \) resulting from (22). Then the probability that the update condition (23) is not met in finite time can be made arbitrarily small by selecting \( \alpha \) small enough, i.e., the following limit holds:

\[
\lim_{\alpha \to 0} P \left[ s_i, \ldots, \infty \notin \mathcal{X}_{\theta_{p+1}(\alpha)}^0 \right] \leq 1 - \sigma. \tag{30}
\]

A simple interpretation of Theorem 2 is that it is always possible to backtrack to a short-enough parameter update (\( \alpha \) small enough) such that the update condition (23) becomes satisfied in finite time with probability arbitrarily close to \( 1 - \sigma \), which corresponds to the bound in (19).

The intuition behind this result is that, by continuity arguments, \( \mathcal{X}_{\theta_{p+1}(\alpha)}^0 \) tends to \( \mathcal{X}_{\theta_p}^0 \) as \( \alpha \) becomes small, such that the two sets match asymptotically. Moreover, we observe that under policy \( \pi_{\theta_p} \), the closed-loop trajectories evolve in set \( \mathcal{X}_{\theta_p}^0 \) and the update is infeasible if they are outside of set \( \mathcal{X}_{\theta_{p+1}(\alpha)}^0 \). It follows that for a parameter update to be blocked forever despite \( \alpha \) being arbitrarily small, if (10) holds, the closed-loop state trajectories under policy \( \pi_{\theta_p} \) need to evolve on an infinitely small set. This would require unbounded densities in the real closed-loop dynamics (1), which is excluded by Assumption 1. We formalize these explanations in the next proof.

PROOF. (of Theorem 2) If (10) holds, we first observe that \( \theta_{p+1}(0) = \theta_p \) trivially holds, such that

\[
s_i, \ldots, \infty \in \mathcal{X}_{\theta_{p+1}(0)}^0
\tag{31}
\]

holds by construction. Let us further define the set:

\[
\Delta \mathcal{X}^0(\theta_p, \theta_{p+1}) = \left\{ s \mid s \in \mathcal{X}_{\theta_p}^0 \text{ and } s \notin \mathcal{X}_{\theta_{p+1}}^0 \right\},
\tag{32}
\]

such that \( \Delta \mathcal{X}^0(\theta_p, \theta_p) = \emptyset \). A trajectory \( s_k, \ldots, \infty \) in closed-loop under policy \( \pi_{\theta_p} \), that never satisfies the update condition (23) must evolve in \( \Delta \mathcal{X}(\theta_p, \theta_{p+1}) \). We observe that by Assumption 2 \( \theta_{p+1}(\alpha) \) is continuous in a neighborhood of \( \alpha = 0 \), and by Assumption 1.1 we have that the set \( \Delta \mathcal{X}^0(\theta_p, \theta_{p+1}) \) is continuous in \( \theta_{p+1} \). It follows that

\[
\lim_{\alpha \to 0} \mu \left( \Delta \mathcal{X}^0(\theta_p, \theta_{p+1}(\alpha)) \right) = 0.
\tag{33}
\]

Algorithm 1: Safe and Stable learning - back-tracking

Input: MPC parameter \( \theta \), and \( q, n, H \)

1 while Learning do
2 Set update = false, \( \alpha = 1 \), fail = 0
3 Compute \( \theta_{+} \) from (22)
4 while not update do
5 Compute MPC solution \( u_0 \) from \( \theta \)
6 Compute MPC solution \( u^*_0 \) from \( \theta_{+} \)
7 if MPC solution from \( \theta_{+} \) is feasible then
8 \( \text{Set } u_0 \leftarrow u^*_0 \) and \( \theta \leftarrow \theta_{+} \)
9 update = true
10 else
11 fail = fail + 1
12 Apply input \( u_0 \) to the system
13 if fail \( \geq n \) then
14 \( \alpha = \alpha \sigma \) and recompute \( \theta_{+} \) from (22)

We can then conclude using Lemma 1, point 2:

\[
\lim_{\alpha \to 0} P \left[ s_i, \ldots, \infty \notin \mathcal{X}_{\theta_{p+1}(\alpha)}^0 \right] = 0
\tag{34}
\]

\[
\lim_{\alpha \to 0} P \left[ s_i, \ldots, \infty \in \Delta \mathcal{X}^0(\theta_p, \theta_{p+1}(\alpha)) \right] = 0.
\]

Since (10) holds with probability \( \sigma \), (30) readily follows.

A practical implementation of the backtracking approach is detailed in Algorithm 1. The implementation consists in reducing parameter \( \alpha \) if the update condition is not met for \( n \) time steps. We ought to stress here that lines 3 and 14 of Algorithm 1 can be performed off-line independently of the state of the system and of the robust MPC schemes. It follows that the online computational burden is limited to solving two independent robust MPC scheme, possibly in parallel.

5.2 Parameter Updates via Constrained Feasibility

As an alternative to backtracking, we propose next an approach imposing additional constraints in (22). We then no longer rely on taking short-enough steps in \( \theta \) to achieve the feasibility of the parameter updates, but rather form an update that is feasible by construction. This entails that updates can be performed at every time step \( i \) such that \( p = i \), hence we will use the notation \( \theta_i \) throughout this section.

We define as \( \mathcal{X}_{\theta_{i+1}}^0(s_i, \pi_{\theta_i}(s_i)) \) the 1-step dispersion set starting from state \( s_i \), applying the input \( a = \pi_{\theta_i}(s_i) \), and using set \( W_{\theta_{i+1}} \) to model the stochasticity of the system. Note the subtle but important difference with \( \mathcal{X}_{\theta_{i+1}}^0 = \mathcal{X}_{\theta_{i+1}}^0(s_i, \pi_{\theta_{i+1}}(s_i)) \), where we apply action \( a = \pi_{\theta_{i+1}}(s_i) \).
Theorem 3 The parameter update \( \theta_{t+1} \) given by

\[
\begin{align}
\min_{\theta_{t+1}} & \frac{1}{2} \| \theta_{t+1} - \theta_{t} \|^2_H + \nabla_\theta J(\pi_{\theta_{t}}) (\theta_{t+1} - \theta_{t}), \\
\text{s.t.} & \quad \theta_{t+1} \in \Theta_L \cap \Theta_F \cap \Theta_D, \\
& \quad \Delta \theta_{t+1} \supseteq \overline{X}^{\theta_{t+1}}(s_i, \pi_{\theta_{t}}(s_i)).
\end{align}
\]  

(35a-b-c)

satisfies (23) by construction with probability at least \( \sigma \), which corresponds to the bound (19).

**PROOF.** If (10) holds, then \( s_{i+1} \in \overline{X}^{\theta_{t+1}}(s_i, \pi_{\theta_{t}}(s_i)) \). Using (35c), \( \overline{X}^{\theta_{t+1}}(s_i, \pi_{\theta_{t}}(s_i)) \subseteq \Delta \theta_{t+1} \) holds, and robust MPC is feasible for all possible realizations of \( s_{i+1} \), i.e., \( s_{i+1} \in \Delta \theta_{t+1} \). Since (10) holds with probability \( \sigma \), (23) holds with probability at least \( \sigma \).

We elaborate next on how constraint (35c) can be formulated. We observe that recursive feasibility of MPC (14) implies that, if a given state is feasible, then the tube around the predicted trajectory is also feasible, such that

\[ \overline{X}^{\theta_{t+1}}(s_i, \pi_{\theta_{t}}(s_i)) \subseteq \Delta \theta_{t+1} \iff \Delta \theta_{t+1} \]  

(36)

where \( \Delta \theta_{t+1} \) defines the set of states \( s \) for which the robust MPC scheme (14) is feasible under the additional constraint \( u_0 = a \). The second condition in (36) is more easily written as a condition on the parameters and the nominal MPC trajectory: a detailed discussion on how this is done is provided in [22] for linear tube MPC. The main difference between that approach and the one used in this paper is that in that case the constraint takes the form \( h(v, \theta) \leq 0 \), while in this paper it takes the form \( h^*(v^*(\theta), \theta) \leq 0 \). Since the main idea is unchanged, we do not provide further details for the sake of brevity.

We prove next that constraint (35) is non-blocking, i.e., that the parameter update yielded by (35) cannot be \( \theta_{t+1} = \theta_t \) at all times, unless \( \theta_t = \theta^* \).

**Theorem 4** Consider the closed loop trajectory \( s_{i,...,\infty} \) under policy \( \pi_{\theta_{t}} \) starting at the physical sampling time \( i \) with the initial state \( s_i \). Assume that Assumption 1.1 holds also for \( \Delta \theta_{t+1} \), i.e., if the first action is fixed in the MPC scheme, the set of feasible initial conditions is compact and continuous in \( \theta \). Then, the probability that (35) yields \( \theta_{t+1} = \theta_t \) for all times is zero, i.e.,

\[ \mathbb{P} \{ \theta_{p+1} = \theta_p \neq \theta^*, \ p = i,...,\infty \} = 0. \]  

(37)

**PROOF.** In order to prove the result, we will prove that there does exist a parameter update \( \theta_{t+1} \neq \theta_t \), which does satisfy (35c). To that end, we consider \( \theta_{t+1} \) resulting from (22), and we prove next that

\[
\begin{align}
\lim_{\alpha \to 0} \mathbb{P} \left[ \lambda_{\theta_{t+1}}^0(s_j, \pi_{\theta_{t}}(s_j)) \right. \\
& \left. \supseteq \overline{X}_{\theta_{t+1}}^0(s_j, \pi_{\theta_{t}}(s_j)), \ j = i,...,\infty \right] = 0,
\end{align}
\]  

(38)

where we used (36) to obtain the inequality above. Because \( H > 0 \), any update \( \theta_{t+1} \) reducing (22a) must also be a descent direction for (35a). Consequently, if additionally \( \theta_{t+1} \) is feasible for (35c), then (35) cannot yield a 0 update.

Since by using \( \alpha = 0 \), (22) yields \( \theta_{t+1} = \theta_t \), we have

\[ s_{i,...,\infty} \in \lambda_{\theta_{t+1}}^0(\pi_{\theta_{t}}(s_i)) = \lambda_{\theta_t}^0(\pi_{\theta_{t}}(s_i)) = \lambda_{\theta_t}^0(\pi_{\theta_{t}}(s_i)), \]  

(39)

We use (36) to define the set:

\[ \Delta \lambda_{\pi_{\theta_{t}}(s_i)}(\theta_t, \theta_{t+1}) = \left\{ s \mid s \in \lambda_{\theta_t}^0, \text{ and } s \notin \lambda_{\theta_{t+1}}^0(\pi_{\theta_{t}}(s_i)) \right\}, \]  

(40)

i.e., the set for which \( \theta_{t+1} \) violates the constraint (35c).

Note that \( \Delta \lambda_{\pi_{\theta_{t}}(s_i)}(\theta_t, \theta_{t+1}) = 0 \). Consider a trajectory \( s_{i,...,\infty} \) in closed-loop under policy \( \pi_{\theta_{t}} \) such that parameter \( \theta_{t+1} \) solves (22) but never satisfies constraint (35c). By definition such trajectory must evolve in set \( \Delta \lambda_{\pi_{\theta_{t}}(s_i)}(\theta_t, \theta_{t+1}) \). We observe that, by Assumption 2, \( \theta_{t+1} \) is continuous in a neighborhood of \( \alpha = 0 \), and by assumption we have that the set \( \Delta \lambda_{\pi_{\theta_{t}}(s_i)}(\theta_t, \theta_{t+1}) \) is continuous in \( \theta_{t+1} \). It follows that

\[
\lim_{\alpha \to 0} \mu \left( \Delta \lambda_{\pi_{\theta_{t}}(s_i)}(\theta_t, \theta_{t+1}) \right) = 0.
\]  

(41)

We can then conclude using Lemma 1, point 2:

\[
\lim_{\alpha \to 0} \mathbb{P} \left[ s_{i,...,\infty} \notin \lambda_{\theta_{t+1}}^0(\pi_{\theta_{t}}(s_i)) \right] = 0.
\]  

(42)

\[
\lim_{\alpha \to 0} \mathbb{P} \left[ s_{i,...,\infty} \notin \Delta \lambda_{\pi_{\theta_{t}}(s_i)}(\theta_t, \theta_{t+1}) \right] = 0.
\]  

(43)

**6 Stability of MPC with Parameter Updates**

In the previous section, we investigated the recursive feasibility of performing RL-based parameter updates on the MPC scheme. In this section we will discuss the stability of a sequence of MPC schemes satisfying the recursive feasibility conditions discussed above. We will show that, assuming that the sequence of parameter converges linearly, the sequence of MPC policies stabilizes the system in the state-parameter space. If \( \theta_p \in \Theta_L \cap \Theta_F \cap \Theta_D \)
for all \( p \), the sequence of parameters \( \theta_0, \ldots, \infty \) yields a sequence of Lyapunov functions \( V_{\theta_p} \) on their respective feasible sets \( X_0^{\theta_p} \). Hence each MPC with parameter \( \theta_p \) is stabilizing the system trajectory to its specific level set \( L_{\theta_p} \). The stability of the system trajectories when updating the parameters \( \theta_p \) can then be investigated by piecing together the individual Lyapunov functions \( \hat{V}_{\theta_0, \ldots, \infty} \), and by assuming some regularity condition on the functions \( \hat{V}_{\theta_0, \ldots, \infty} \) as well as a sufficiently fast convergence of the parameter sequence \( \theta_0, \ldots, \infty \). This statement is formalized in the following Theorem.

**Theorem 5** Let us assume that at every parameter update \( \theta_p \rightarrow \theta_{p+1} \), the inequality:

\[
\| \theta_{p+1} - \theta_* \| \leq r \| \theta_p - \theta_* \| \tag{42}
\]

holds for some \( r \in [0, 1] \), where \( \theta_* \) is the solution of (21). Suppose, furthermore that

\[
\sup_{s \in X_0^{\theta_p} \cap \Theta_{p+1}} | \hat{V}_{\theta_{p+1}}(s) - \hat{V}_{\theta_p}(s) | \leq \alpha V \| \theta_{p+1} - \theta_p \| \tag{43}
\]

holds for all \( p \) for some \( \alpha V > 0 \). Let us additionally assume that each parameter is given by (22) or (35) such that \( \theta_p \in \Theta_L \cap \Theta_F \cap \Theta_D \), and the parameter updates satisfy the parameter update conditions (23) or (35c). Then the sequence of robust MPC schemes with parameters \( \theta_0, \ldots, \infty \) is asymptotically stable in the joint state-parameter update space for any \( s_0 \in X_0^{\theta_0} \), and steers the system trajectory to the level set \( L_{\theta_\infty} \).

**PROOF.** Consider an augmentation of the state \( s \) with the current parameters \( \theta_p \). Let us label the augmented state \( S \). We then propose the candidate Lyapunov function:

\[
W(S) = \hat{V}_{\theta_p}(s) + \sigma \Delta_p \tag{44}
\]

where we label \( \Delta_p := \| \theta_p - \theta_* \| \), and \( \sigma \) is a positive constant. Function (44) tactile the regular state space of the system jointly with the parameter update space, and will allow us to establish stability in that joint space for \( \sigma \) large enough. We first observe that since \( \hat{V}_{\theta_0, \ldots, \infty}(s) \) are Lyapunov functions, for any \( \theta_p \), \( W(S) \) is a Lyapunov function in between parameter updates, i.e., \( W \) is decreasing along the system trajectory. Moreover, since \( \Delta_p \) is a norm in the state-parameter space, \( W(S) \) is adequately lower and upper bounded if \( \hat{V}_{\theta_p} \) is. Finally, since by assumption \( \Delta_p \rightarrow 0 \), the system state \( s \) is eventually steered towards \( L_{\theta_\infty} \). We observe that between parameter updates the decrease of \( W(S) \) under a specific parameter \( \theta_p \) holds from:

\[
W(S_{i+1}) = \hat{V}_{\theta_p}(s_{i+1}) + \sigma \Delta_p \leq \gamma \hat{V}_{\theta_p}(s_i) + \delta \theta_p + \sigma \Delta_p < W(S_i), \tag{45}
\]

for any \( s_i \) outside of \( L_{\theta_p} \).

Upon updating the parameter \( \theta_p \rightarrow \theta_{p+1} \) at a specific time instant \( i \), we observe that:

\[
W(S_{i+1}) - W(S_i) = \hat{V}_{\theta_{p+1}}(s_{i+1}) - \hat{V}_{\theta_p}(s_i) \leq V_{\theta_p}(s_i, s_{i+1}, s_{i+1}) + \sigma (\Delta_{p+1} - \Delta_p). \tag{46}
\]

If the state at time \( s_i \) lies outside of the level set \( L_{\theta_{p+1}} \) such that:

\[
\hat{V}_{\theta_{p+1}}(s_{i+1}) < \hat{V}_{\theta_{p+1}}(s_i) \tag{48}
\]

holds under the MPC with parameter \( \theta_{p+1} \), then \( W \) is decreasing over the parameter updates at time \( i \) if:

\[
\hat{V}_{\theta_{p+1}}(s_{i+1}) - \hat{V}_{\theta_p}(s_i) + \sigma (\Delta_{p+1} - \Delta_p) < \hat{V}_{\theta_{p+1}}(s_i) - \hat{V}_{\theta_p}(s_i) + \sigma (\Delta_{p+1} - \Delta_p) \leq 0. \tag{49}
\]

Using (43), condition (49) holds if:

\[
\hat{V}_{\theta_{p+1}}(s_i) - \hat{V}_{\theta_p}(s_i) + \sigma (\Delta_{p+1} - \Delta_p) \leq \alpha V \| \theta_{p+1} - \theta_p \| + \sigma (\| \theta_{p+1} - \theta_* \| - \| \theta_p - \theta_* \|) \leq 0. \tag{50}
\]

Using (42) we observe that

\[
\| \theta_{p+1} - \theta_p \| \leq \| \theta_{p+1} - \theta_* \| + \| \theta_p - \theta_* \| \leq (r + 1) \| \theta_p - \theta_* \|. \tag{51}
\]

It follows that (50) holds if

\[
\alpha V (r + 1) + \sigma (r - 1) \leq 0, \tag{52}
\]

Hence \( W \) decreases when \( s_i \) lies outside of the level set \( L_{\theta_{p+1}} \) if:

\[
\alpha V (r + 1) + \sigma (r - 1) \leq 0, \tag{54}
\]

which can always be ensured by choosing:

\[
\sigma \geq \frac{\alpha V (r + 1)}{1 - r}. \tag{55}
\]

As a result at all time \( k \) whether a parameter update takes place or not, either function \( W \) is decreasing or the system trajectory is contained in the level set \( L_{\theta_p} \) corresponding to the MPC parameters in use. \( \Box \)
We elaborate in the next remarks on the assumptions and the stability claim made by Theorem 5.

**Remark 1** The decrease of function \( W \) at a time instant \( i \) with corresponding parameter index \( p \) is ensured for any \( \lambda^i \) outside the level sets \( \mathbb{L}_{\lambda^i} \), regardless of whether a parameter update has occurred or not. However, if \( \lambda^i \in \mathbb{L}_{\lambda^i} \), then \( W \) is not guaranteed to decrease, but the trajectory \( \lambda^{i+1} \ldots \) is guaranteed to remain in the level set \( \mathbb{L}_{\lambda^i} \) until the next parameter update occurs.

The stability described in Theorem 5 guarantees the stabilization of the system trajectory in the state-parameter space, and under the Lyapunov function \( W \), to the sequence of level sets \( \mathbb{L}_{\lambda_n} \) converging to \( \mathbb{L}_{\lambda_\infty} \). Theorem 5 hence guarantees the stability of the system trajectory under fast parameter updates (e.g., at every sampling time \( k \)), albeit the parameter updates entering in the Lyapunov function via the norm \( \Delta_p \) can temporarily drive the system trajectory away from the level sets (due to the fact that stability is guaranteed in the state-parameter update space, as opposed to the state space alone).

Hence a case covered by Theorem 5 and expected in practice is one where the parameter updates are fairly slow and small compared to the system dynamics, possibly yielding a situation where the system trajectory is stabilized to \( \mathbb{L}_{\lambda_\infty} \) mostly by moving from level set to level set, i.e., \( \mathbb{L}_{\lambda_p} \rightarrow \mathbb{L}_{\lambda_{p+1}} \rightarrow \ldots \), without \( W \) systematically decreasing. Theorem 5, however, guarantees that updating the parameters faster and more aggressively does not jeopardize the system stability.

Finally, we ought to stress that this result could also be interpreted in the framework of Input-to-State Stability (ISS), since the term \( \sigma(\Delta_{p+1} - \Delta_p) \) in (47) can be upper bounded by a class-K function, under the mild assumption of bounded parameter updates. Our analysis, however, yields a stronger result, since we additionally assume that the parameter updates are converging.

**Remark 2** Assumption (43) is not trivial, and we elaborate on it next. Indeed, the continuity of the value function of optimization problems is known to be intricate, with the notable exception of linear robust MPC with a quadratic cost and polyhedral uncertainty. Assuming that the parameter sequence \( \theta_0, \ldots, \theta_\infty \) belongs entirely to a bounded and connected (BC) set \( \Theta \) such that \( \lambda^0 \) is non-empty everywhere in \( \Theta \), we observe that (43) holds if \( V_\theta \) is Lipschitz continuous on \( \Theta \) with a bounded Lipschitz constant for all \( \lambda \) in the applicable domain of definition. This, in turn, holds if for all \( \lambda \) in the domain of definition, \( V_\theta \) is almost everywhere differentiable with respect to \( \lambda \) on \( \Theta \), with bounded and Lebesgue integrable derivatives. To our best knowledge, this requires in general fairly involved conditions on the constraints and model of the MPC scheme. However, if the BC set \( \Theta \) exists, we expect (43) to hold if the cost functions and constraints in (14) are sufficiently differentiable, and if some sufficiently strong constraint qualification holds for the MPC problem. A complete discussion of assumption (43) is beyond the scope of this paper.

**Remark 3** We ought to comment as well on assumption (42) here, i.e., convergence of the parameters sequence \( \theta_0, \ldots, \theta_\infty \) delivered by RL scheme. We observe that many RL methods deliver a sequence of parameters that is stochastic by nature, because they are based on measurements taken from a stochastic system. We then observe that assumption (42) is only satisfied asymptotically for large data sets. For RL methods based on very small data sets, such as, e.g., to the extreme those using basic stochastic gradient methods, one could consider an extension of Theorem 5 where the decrease of \( W \) holds only in a stochastic sense, or as practical or ISS stability. To that end, one can, e.g., assume

\[
\|\theta_{p+1} - \theta_p\| \leq r \|\theta_p - \theta_\star\| + q,
\]

which implies that the parameter updates converge to a neighborhood of zero. The main conclusions of the Theorem still hold, mutatis mutandis, with asymptotic stability replaced by practical stability. A formal discussion of this extension is avoided for the sake of simplicity.

### 7 Numerical Examples

In this section we provide numerical examples which illustrate the theoretical developments.

#### 7.1 Recursive Feasibility

We first discuss a simple academic example which is constructed, but allows us to discuss the theory in simple terms. Consider the scalar linear system

\[
s_{t+1} = As + Ba + w, \quad A = 1.1, \quad B = 0.1,
\]

with \( w \in [|w|, |w|] := [-0.1, 0.1] \). We construct MPC such that it delivers a policy as close as possible to \(-Ks + a^\star\), where \( \theta = \{K, a^\star\} \) are parameters to be adjusted by RL and the state and input must satisfy

\[
s \leq \pi := 0.1, \quad a \in [-10, 10 - 0.5K].
\]

One can verify that the robust MPC formulation

\[
\min_u (u - (a^\star - Ks))^2
\]

s.t. \( As + Bu + \bar{w} \leq \pi, \quad u \in [-10, 10 - 0.5K], \)

guarantees that the state constraint \( s \leq \pi \) is never violated with the given dynamics and process noise. The
stage cost \( \ell(s, a) = (s - 40)^2 + 10^{-4}a^2 \) should be minimized by RL, and the MPC region of attraction at convergence must include the interval \([s^0_b, s^1_b] = [0, 0.1] \).

We consider a discount factor \( \gamma = 0.9 \) and solve the problem by applying constrained policy gradient to the exact total expected cost \( J \). At each policy gradient iteration \( p \) we solve the problem

\[
\theta_{p+1} := \arg\min_{\theta} 0.5\|\theta - \theta_p\|_2^2 + \alpha \nabla_{\theta} J^\top (\theta - \theta_p)
\]

s.t. \( As^j_b + Ba^j_{b,j} + \bar{w} \leq \bar{\pi}, \quad j = 0, 1, \)

\[
a^j_i := \left[-Ks^j_b + a^i\right]^{10-0.5K}, \quad j = 0, 1,
\]

\[
A - BK \in [-1 + \epsilon, 1 - \epsilon],
\]

where \( \|\cdot\|_{\infty} := \max(a, \min(c, b)), \epsilon = 10^{-6} \) and \( \alpha = 1 \).

We initialize the problem with \( K = 2, a^s = 0 \). The problem converges in one iterate to the optimal solution \( K_\infty = 11, a^s = 0.9 \), but, depending on the initial state, the solution cannot be immediately applied to the system. Indeed, the region of attraction for the initial guess is the interval \( S_0 := [-8.9, 0.1] \), while for the optimal solution the region of attraction is the interval \( S_\infty := [-4.4, 0.1] \). We display in Figure 1 a simulation starting from \( s = -8 \) and using a backtracking strategy (Algorithm 1) with \( \alpha = 1 \), i.e., if the solution is not feasible, \( \alpha \) is reduced with \( \rho = 0.9 \). One can observe that, in the beginning, parameter \( \theta \) is not updated until: (a) \( \alpha \) becomes smaller and, consequently, the region of attraction becomes larger; and (b) the state approaches the region of attraction.

We also performed a simulation in which the update was done by trajectory-constrained parameter updates, where the following problem was solved

\[
\theta_{t+1} := \arg\min_{\theta} 0.5\|\theta - \theta_t\|_2^2 + \alpha \nabla_{\theta} J^\top (\theta - \theta_t)
\]

s.t. \( As^j_t + Ba^j_{b,j} + \bar{w} \leq \bar{\pi}, \quad j = 0, 1, \)

\[
a^j_i := \left[-K_is^j_t + a^i\right]^{10-0.5K_i}, \quad j = 0, 1,
\]

\[
A - BK \in [-1 + \epsilon, 1 - \epsilon],
\]

where \( S_\theta \) is the region of attraction given \( \theta \), and \( s^0_{wc} \) are the one-step worst-case state realizations which, in this specific case, are given by

\[
s^0_{wc} := As_t + B[-K_is_t + a^1_{\infty}]^{10-0.5K} + \bar{w},
\]

\[
s^1_{wc} := As_t + B[-K_is_t + a^1_{\infty}]^{10-0.5K} + \bar{w}.
\]

The closed-loop trajectories are shown in Figure 2, where one can see that the convergence is slower than with backtracking, since the parameters are updated at each time, which reduces the maximum implementable control and, therefore, makes the convergence to the optimal operating set slower. Nevertheless, also in this case we recover the optimal solution \( K_\infty = K_\star, a^s_{\infty} = a^s_\star \).
7.2 Value Function

We consider now the linear system with dynamics and stage cost

\[ s_+ = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix} a + w, \]

\[ \ell(s, a) = \begin{bmatrix} s - s^r \\ a - a^r \end{bmatrix}^\top \begin{bmatrix} 1 & 0 \\ 0 & 0.01 \end{bmatrix} \begin{bmatrix} s - s^r \\ a - a^r \end{bmatrix}, \]

where \( s = (p, v) \) and \( s^r = (-3, 0) \). We formulate a problem with prediction horizon \( N = 50 \) and introduce the state and control constraints \(-1 \leq s \leq 1, -1 \leq a \leq 10 \). The real noise set is selected as a regular octagon, and we parametrize \( \mathcal{W}_\omega \) as a polytope with 4 facets.

We formulate tube based MPC as

\[
Q_\theta(s, a) := \min_{z} \sum_{k=0}^{N-1} \left( \left\| x_k - x_t \right\|_H^2 + \left\| x_N - x_t \right\|_p^2 + \left\| x_0 \right\|_\Lambda^2 + \lambda^\top x_0 + l \right)
\]

\[
\text{s.t. } x_0 = s, \quad u_i = a, \quad x_{k+1} = Ax_k + Bu_k + b, \quad k \in I_0^{N-1},
\]

\[
C \bar{x}_k + D \bar{u}_k + \bar{c}_k \leq 0, \quad k \in I_0^{N-1},
\]

\[
Gx_N + \bar{g} \leq 0,
\]

where one must enforce that the system dynamics (57c) and a parametrized compact uncertainty set \( \mathcal{W}_\omega \) are such that \( s_+ - (As + Ba + b) \in \mathcal{W}_\omega \). This issue has been discussed in detail in [22], where the set is parametrized as the polyhedron \( \mathcal{W}_\omega := \{ w \mid Mw \leq m \} \) and the following set membership constraint is imposed on \( \omega = (M, m) \) for all past samples \( s_{i+1}, s_i, a_i, i \in I \):

\[
M(s_{i+1} - (As_i + Ba_i + b)) \leq m, \quad \forall i \in I.
\]

Then, \( c_k \) is computed by tightening the original constraints \( Cs + Da + \bar{c} \leq 0 \), so as to guarantee that, for any process noise \( w \in \mathcal{W}_\omega \), the constraints are satisfied. Moreover, parameters \( x_t, u_t \) must be a steady-state for the system dynamics (57c), i.e.,

\[
(A - I)x_t + Bu_t = 0.
\]

Finally, \( G \) and \( g \) must be selected such that they define a robust positively invariant terminal set for the feedback law \( u = -K(x - x_t) + u_t \), with \( K \) the solution to the LQR formulated with \( A, B, H, P \). The vector of MPC parameters is then defined as

\[
\theta = \{ \Lambda, \lambda, l, H, x_t, u_t, M \},
\]

and we consider \( K, P, c_k, G, g \) as functions of these parameters. Vector \( m \) can also be included in \( \theta \), but, as discussed in [22] this is not necessary. Matrices \( C, D \) and vector \( \bar{c} \) are assumed to be known. Finally, \( A, B, b \) could in principle also be included in the parameter vector \( \theta \). However, as discussed in [22] this makes the safe RL problem much harder to formulate and solve, since it obliges one to store very large amounts of data and formulate an equally large amount of constraints.

The set of parameters guaranteeing safety and stability then becomes

\[
\Theta := \{ \theta \mid H > 0, \quad M(s_{i+1} - (As_i + Ba_i + b)) \leq m, \quad \forall i \in I,
\]

\[
(A - I)x_t + Bu_t = 0, \quad \exists x \text{ s.t. } Gx \leq g \},
\]

i.e., the noise set must include all observed noise samples, the reference must be a steady-state of the system and the terminal set must be nonempty. This last condition also entails that the MPC domain is nonempty.

We update \( \theta \) using a batch \( Q \) learning approach with batches of horizon \( N_b = 20 \) with learning factor \( \alpha = 0.1 \), using the backtracking strategy.

We simulated the system starting from state \( s_0 = (0.8, 0) \). The backtracking strategy never rejected nor reduced any step. The resulting closed-loop trajectory is displayed in Figure 3, together with the reference, maximum robust positive invariant (MRPI) and terminal sets at the beginning and end of the simulation, as well as the minimum robust positive invariant (mRPI) sets throughout the simulation. We display the noise set approximation at the end of the simulation in Figure 4, and the evolution throughout the RL epochs of the parameter \( \theta \) and the average TD error in each batch in Figure 5. We display the MPC Lyapunov functions \( V_\theta \) and \( W \) in time in Figure 6. One can see that in the beginning \( V_\theta \) sometimes increases upon parameter updates, but decreases inside each batch. Note that this result is perfectly in line with Theorem 5 and Remark 1. After the displayed time interval, the Lyapunov function \( V_\theta \) was always 0, i.e., the state trajectory remained inside the mRPI set, even when this set was updated by a parameter change. Some words are due in order to discuss function \( \hat{W} \); as pointed out in Remark 3, in practice one can at best expect that the parameters converge to a neighborhood of the optimal ones. Therefore, we selected \( \hat{\theta}_\star \) as the average of \( \theta \) over the last 100 epochs, when, as shown in Figure 5, the parameter is at convergence. We observed that \( \sigma = 0.1 \) was sufficiently
Fig. 3. MRPI (red), terminal (cyan) sets and reference $x'$ (black and grey circle) at the beginning and end of the learning process; state trajectory (black line) and mRPI sets (yellow) at each time instant.

Fig. 4. True process noise set (transparent octagon), noise samples (black dots), their convex hull (red dots) and noise set parametrized by matrix $M$ (cyan).

Fig. 5. Top plot: parameter evolution through the epochs. Bottom plot: TD error through the epochs.

Fig. 6. Top figure: Lyapunov function $\hat{V}_\theta$ over the first epochs. Bottom plot: Lyapunov function $W$ over time.

Fig. 7. Performance $J$ in terms of total discounted cost for each batch. The dashed lines indicate the maximum and minimum of $J$ over all future times.

8 Conclusions

This paper discusses how to implement Learning-based adaptations of a robust MPC scheme in order to improve its closed-loop performance while maintaining the stability and safety of the control policy. We show in particular that these requirements can be treated via constraints on the learning steps, and parameter update conditions that are fairly simple to verify, and that can be implemented online in real time. We additionally establish that the proposed approach ensures that the update conditions are not blocking the learning process, in the sense that they are met in final time with probabil-
ity one. We finally show that under some conditions on the learning process, a form of stability of the resulting learning-based robust MPC scheme is guaranteed in the state-parameter space. The proposed approaches are illustrated in two simulated examples.

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