Abstract. The CD equalities were introduced to imply the gradient estimate of laplace operator on graphs. This article is based on the unbounded Laplacians, and finally concludes some equivalent properties of the CD(K,∞) and CD(K,n).

1. Introduction

Graph theory is the basic theory of the study of graphs and networks. The spectral graph theory, which is used for describing the structure and characteristic of graphs by adjacency matrix or the spectral density of Laplacian matrix, is the classic method for studying graphs described in [9].

We already know that we can find the curvature by solving a partial differential equation, and there are more examples for the geometric analysis such as the famous Li-Yau gradient estimate. Moreover, we can use some data to describe the graphs and optimize it such as the Cheeger constant on graphs.

The Laplacians on graph have always been an important research topic. In fact, Laplacians can be seen as the discrete analog form of the Schrodinger operator, or as the generator of symmetric Markov process. Laplacians always appear in the topics on the research of discrete structure for heat equations like in [2].

As for the Laplacians on graphs, the properties are different on different occasions, such as finite graphs, local finite graphs and infinite graphs. If we assume the graph is finite, then the properties of Laplacians are simple and good. But for some problems, the assumption of finite graph is obviously too narrow, so local finite graph on infinite graphs can be a better research object. We still can get good enough properties on it. In recent years, some research topics are as follows on Laplacians on infinite graph:

(a) Definition of the operators and essential selfadjointness.
(b) Absence of essential spectrum.
(c) Stochastic incompleteness.

Metric space has a relationship with the manifold which can be seen in [6] and [11]: if we admit that there exist singular points in space, then metric space can be seen as a natural extension of the manifold. Meanwhile, it also has a similar geometric structure as the manifold. Obviously, the graph can also be seen as a kind of metric space. We can define the metric between
two vertices of the graph as the natural metric, which is the number of the minimum edges connecting them. Then, we should consider whether the theories of Riemannian manifold can be extended to the graph, especially those about Ricci curvature. Many results in geometry analysis come from the Ricci curvature, especially the lower bound of Ricci curvature, such as heat kernel estimation, Harnack inequalities and Soblev inequalities. These conclusions have been made in [7].

There exists an equation like this:

\[ \frac{1}{2} \Delta |\nabla f|^2 = \langle \nabla f, \nabla \Delta f \rangle + \|Hess f\|^2 + \text{Ric} \langle \nabla f, \nabla f \rangle. \]

It is an identical equation on the n-dimension Riemannian manifold. When its Ricci curvature has a lower bound, we can make a conclusion that for any \( \eta \in TM \) there exists a \( K \in \mathbb{R} \) that satisfies \( \text{Ric}(\eta, \eta) \geq K |\eta|^2 \). Unfortunately in the discrete situation we can not define \( \|Hess f\|_2 \). But we can make use of Cauchy-Schwarz inequalities to get this inequality \( \|Hess f\|^2 \geq \frac{1}{n}(\nabla f)^2 \). Then the Bochner inequality can be rewrite into:

\[ \frac{1}{2} \Delta |\nabla f|^2 \geq \langle \nabla f, \nabla \Delta f \rangle + \frac{1}{n}(\Delta f)^2 + K |\nabla f|^2. \]

The inequality above is the Curvature-Dimension inequality on Riemannian manifold, and we call it CD inequality for short. According to it, we can easily find that if the lower bound of the curvature is already known in the space, the "Ricci curvature" in the discrete situation can be defined. Bakery and Emery have already proved that if the chain rule is satisfied, the CD inequalities can be extended to the Markov operators on a general metric space. Yet obviously the chain rule doesn’t true usually for discrete functions. Fortunately when \( p = \frac{1}{2} \), \( u^p \) satisfies the chain rule even on the discrete condition. So, [7] introduced an improved CD inequality- CDE inequality. This definitely is a key for the research of the discrete geometry analysis.

This paper gives an introducton of the CD inequality and several equivalent conditions of the CD inequality for the unbounded Laplacians on graph. The paper is organized into four parts:

Chapter 1 is the introduction of the graph, the Laplacians and CD inequalities on it.

Chapter 2 introduces some basic conclusions in order to get the main result. These conclusions include some definitions such as local finite graph, weighted graph and the domain of the operators.

Chapter 3 is the main conclusion of this thesis which includes some equivalent conditions of the CD inequalities.

2. GRAPHS, LAPLACIANS AND CD INEQUALITIES

Given a graph \( G = (V, E) \), for an \( x \in V \), if there exists another \( y \in V \) that satisfies \( (x, y) \in E \), we call them are neighbors, and written as \( x \sim y \).
If there exists an \( x \in V \) satisfying \( (x, x) \in E \), we call it a self-loop. In this paper we allow graphs have self-loops.

Now we will introduce some basic definitions and theorems before we get the main results.

**Definition 2.1.** (locally finite graph) We call a graph \( G \) is a locally finite graph if for any \( x \in V \), it satisfies \# \{ \( y \in V \mid y \sim x \) \} \( < \infty \). Moreover, it is called connected if there exists a sequence \( \{x_i\}_{i=0}^n \) satisfying: \( x = x_0 \sim x_1 \sim \cdots \sim x_n = y \).

**Definition 2.2.** (weighted graph) Given a graph \( G = (V, E) \), and two mappings \( \mu : E \to [0, +\infty) \) and \( m : V \to [0, +\infty) \) on it. \( m \) is symmetric on \( V \). For convenience, we extend \( \mu \) onto \( E \), that is to say, for any \( x, y \in V \), if \( x \not\sim y \) or \( (x, y) \notin E \), \( \mu(x, y) = 0 \).

**Definition 2.3.** (\( l^p(V, m) \) space) Let \( m \) be a measure defined as above. Then \( (V, m) \) is a measure space. We define \( l^p(V, m) \), \( 0 < p < +\infty \) space as follows:

\[
\{ u : V \to \mathbb{R} : \sum_{x \in V} m(x)|u(x)|^p < \infty \}
\]

Obviously, \( l^2(V, m) \) is a Hilbert space, the inner product is naturally defined as: \( \langle u, v \rangle = \sum_{x \in V} m(x)u(x)v(x) \). And the norm on it is defined as: \( \|u\| = \langle u, u \rangle^{\frac{1}{2}} \).

In addition, we use \( l^\infty(V) \) to define a set including all the bounded functions on \( V \), and we can easily know that this space is not influenced by the measure \( m \). The norm on it is defined as: \( \|u\| := \sup_{x \in V} |u(x)| \).

**Definition 2.4.** (finitely supported function) For a graph \( G = (V, E) \), we call \( C_0(V) \) the set of finitely supported functions if it is defined as: \( C_0(V) := \{ f : V \to \mathbb{R} \mid \# \{ x \in V \mid f(x) \neq 0 \} < \infty \} \).

Let \( D \) is a dense subspace of \( l^2(V, m) \). We define a symmetric nonnegative double mapping \( Q \) on \( D \times D \) to \( \mathbb{R} \). \( D \) is called the domain of \( Q \), and it is written as \( D(Q) \).

In fact, this mapping is determined by its values on the diagonal line. Then if we want to define such a mapping \( Q \), we can just define the values on the diagonal line like this:

\[
Q(u) := \begin{cases} Q(u, u) : u \in D, \\ \infty : u \notin D. \end{cases}
\]

If \( Q \) is lower semicontinuous, we call it closed. If \( Q \) has a closed extension it is called closable and the smallest extension is called the closure of \( Q \) as defined in \cite{3}.

**Definition 2.5.** (Dirichlet form) \( Q \) is called a Dirichlet form if it is closed and for all the contractions \( C \) and \( u \in l^2(V, m) \), it satisfies \( Q(Cu) \leq Q(u) \).
The more detailed definition can be seen in [4].

On the graph the Dirichlet form has a special form as follows:

\[ f \mapsto Q(f) := \frac{1}{2} \sum_{x,y \in V} \mu_{xy}(f(y) - f(x))^2. \]

Then we will introduce some kinds of operators on graphs.

**Definition 2.6.** (Laplacians on locally finite graphs) On a locally finite graph \( G = (V, E, \mu, m) \) the Laplacian has a form as follows:

\[ \Delta f(x) = \frac{1}{m(x)} \sum_{y \in V} \mu_{xy}(f(y) - f(x)), \quad \forall f \in C_0(V). \]

**Definition 2.7.** (gradient operator \( \Gamma \)) The operator \( \Gamma \) is defined as follows:

\[ \Gamma(f, g)(x) = \frac{1}{2}(\Delta(fg) - f\Delta g - g\Delta f)(x). \]

Always we write \( \Gamma(f, f) \) as \( \Gamma(f) \).

**Definition 2.8.** (gradient operator \( \Gamma_2 \)) The operator \( \Gamma_2 \) is defined as follows:

\[ \Gamma_2(f, g) = \frac{1}{2}(\Delta \Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(g, \Gamma f)). \]

Also we have \( \Gamma_2(f) = \Gamma_2(f, f) = \frac{1}{2}\Delta \Gamma(f) - \Gamma(f, \Delta f) \).

**Definition 2.9.** (nondegenerate measure) A measure \( m \) is called nondegenerate if it satisfies \( \delta := \inf_{x \in V} m(x) > 0 \).

Now we can introduce some results we need to get our main conclusions.

**Lemma 2.10.** For any \( f \in l^p(V, m), p \in [1, \infty) \), we have \( P_t f \in l^p(V, m) \) and

\[ \|P_t f\|_{l^p} \leq \|f\|_{l^p}. \]

And for any \( f \in l^2(V, m) \), we have \( P_t f \in D(\Delta) \).

**Lemma 2.11.** For any \( f \in D(\Delta) \) we have \( \Delta P_t f = P_t \Delta f \).

**Theorem 2.12.** Let \( m \) be a nondegenerate measure on \( V \). Then for any \( f \in l^p(V, m), p \in [1, \infty) \),

\[ |f(x)| \leq \delta \frac{1}{p} \|f\|_{l^p}, \quad \forall x \in V. \]

Moreover, for any \( p < q \leq \infty \), \( l^p(V, m) \hookrightarrow l^q(V, m) \).

The proof of Lemma 2.10, Lemma 2.11, and Theorem 2.12 are given by Bobo Hua and Yong Lin in [1].

Now we introduce the definition of the completeness of the graph.
Definition 2.13. (complete graph) A weighted graph \((V, E, \mu, m)\) is called complete if there is a nondecreasing sequence of finitely supported functions \(\{\eta\}_{k=1}^{\infty}\) such that
\[
\lim_{k \to \infty} \eta_k = 1, \text{and} \quad \Gamma(\eta_k) \leq \frac{1}{k}.
\]

Next we will introduce two important lemmas as follows.

Lemma 2.14. Let \((V, E, m, \mu)\) be a complete weighted graph. Then for any \(f \in D(Q)\) and \(g \in D(\Delta)\),
\[
\sum_{x \in V} f(x) \Delta g(x)m(x) = -\sum_{x \in V} \Gamma(f, g)(x)m(x).
\]

Lemma 2.15. Let \((V, E, m, \mu)\) be a complete graph. Then for any \(f \in C_0(V)\) and \(T > 0\), we have \(\max_{[0,T]} \Gamma(P_t f) \in \ell^1_m\) and
\[
\max_{[0,T]} \|\Gamma(P_t f, \frac{d}{dt} P_t f)\|_{\ell^1_m} \leq C_1(T, f),
\]
where \(C_1(T, f)\) is a constant depending on \(T\) and \(f\). Moreover,
\[
\max_{[0,T]} \|\Gamma(P_t f, \frac{d}{dt} P_t f)\|_{\ell^1_m} \leq C_2(T, f).
\]

These two lemmas are proved in [1].

Now we will introduce some basic CD inequalities (see also [7] and [8]).

Definition 2.16. (CD\((K, \infty)\) condition) We call a graph satisfies CD\((K, \infty)\) condition if for any \(x \in V\), we have
\[
\Gamma_2(f)(x) \geq K \Gamma(f)(x), \quad K \in \mathbb{R}.
\]

For finite dimensions of curvature, we have the CD\((K, n)\) condition.

Definition 2.17. (CD\((K, n)\) condition) We call a graph satisfies CD\((K, n)\) condition if for any \(x \in V\), we have
\[
\Gamma_2(f)(x) \geq \frac{1}{n} (\Delta f)^2 + K \Gamma(f). \quad K \in \mathbb{R}.
\]

Moreover, we have another condition called CDE\((x, K, n)\).

Definition 2.18. (CD\((K, n)\) condition) Let \(f : V \to \mathbb{R}^+\) satisfy \(f(x) > 0, \Delta f(x) < 0\). We call a graph satisfies CD\((x, K, n)\) condition if for any \(x \in V\), we have
\[
\Gamma_2(f)(x) - \Gamma \left( f, \frac{\Gamma(f)}{f} \right)(x) \geq \frac{1}{n} (\Delta f(x))^2 + K \Gamma(f)(x). \quad K \in \mathbb{R}.
\]
There some relations among these conditions as follows:
1. If semigroup $P_t = e^{t\Delta}$ is a diffusion semigroup, $CD(K,n)$ and $CDE'(K,n)$ are consistent.
2. On graphs, $CDE'(x,K,n)$ implies $CDE(x,K,n)$ and $CD(x,K,n)$.

Then all the preparations we need have be done. Now we will introduce the mains results of this paper.

3. MAIN RESULTS

When we look for the equivalent properties of CD inequalities, we often set a condition: $D_\mu := \max_{x \in V} \frac{\text{deg}(x)}{\mu(x)} < \infty$. And the equivalent properties have already been proved in [5] for these bounded Laplace operator on graphs. For unbounded Laplace operator, the following equivalent properties under the condition of nondegenerate measure were proved in [1] by Bobo HUA and Yong Lin.

Remark 3.1. Let $G = (V,E,m,\mu)$ be a complete graph and $m$ is nondegenerate, i.e. $\inf_{x \in V} m(x) > 0$. Then the following are equivalent:
(a) $G$ satisfies $CD(K,\infty)$.
(b) For any finitely supported function $f$,
$$\Gamma(P_t f) \leq e^{-2Kt} P_t(\Gamma(f)).$$
(c) For any $f \in D(Q)$,
$$\Gamma(P_t f) \leq e^{-2Kt} P_t(\Gamma(f)).$$

In this section, similarly in [1] we will give some equivalent properties of $CD(K,\infty)$ and $CD(K,n)$.

Theorem 3.2. Let $G = (V,E,m,\mu)$ be a complete graph and $m$ is nondegenerate. Then the following are equivalent:
(a) $G$ satisfies $CD(K,\infty)$.
(b) For any finitely supported function $f$,
$$P_t(f)^2 - (P_t f)^2 \leq \frac{1 - e^{-2Kt}}{K} P_t(\Gamma(f)).$$
(c) For any $f \in D(Q)$,
$$P_t(f)^2 - (P_t f)^2 \leq \frac{1 - e^{-2Kt}}{K} P_t(\Gamma(f)).$$

Proof First, for any $f, \xi \in C_0(V)$, we make this equation
$$G(s) = \sum_{x \in V} (P_{t-s} f)^2(x) P_s \xi(x) m(x).$$
Taking formal derivative of $G(s)$ in $s$, we get
$$G'(s) = \sum_{x \in V} (-2P_{t-s} f \triangle P_{t-s} f P_s(\xi(x)) m(x) + (P_{t-s} f)^2(x) \triangle P_s \xi(x) m(x)).$$
Now we have to show that $G(s)$ is differentiable in $s$. For the first part

$$2 \sum_{x \in V} \left| P_{t-s} f \Delta P_{t-s} f \right| |P_s(\xi(x))| m(x)$$

$$\leq 2 \| P_s(\xi(x)) \|_{L^\infty} \| \Delta P_{t-s} f \|_{L^\infty} \left( \sum_{x \in V} |P_{t-s} f| m(x) \right)$$

For $f, \xi \in C_0(V)$, from lemma 2.10 we can get

$$\| P_s \xi(x) \|_{L^\infty} \leq \| \xi \|_{L^\infty} < \infty$$

For $f \in C_0(V)$, we know $P_{t-s} f \in D(\Delta)$ and $\| \Delta P_{t-s} f \|_{L^\infty} = \| P_{t-s} \Delta f \|_{L^\infty} \leq \| \Delta f \|_{L^\infty} < \infty$

So we have

$$2 \sum_{x \in V} \left| P_{t-s} f \Delta P_{t-s} f \right| |P_s(\xi(x))| m(x)$$

$$\leq 2 \| \xi \|_{L^\infty} \| \Delta f \|_{L^\infty} \| P_{t-s} f \|_{L^2_{\xi}}$$

$$\leq 2 \| \xi \|_{L^\infty} \| \Delta f \|_{L^\infty} \| f \|_{L^2_{\xi}} < \infty$$

For the second part, notice that $f, \xi \in C_0(V)$ and $\xi(x) \in D(\Delta)$,

$$\sum_{x \in V} \left( P_{t-s} f \right)^2(x) \Delta P_s(\xi(x)) m(x)$$

$$\leq \| \Delta P_s(\xi(x)) \|_{L^\infty} \| P_{t-s} f \|_{L^2_{\xi}}^2$$

$$\leq \| P_s \Delta \xi(x) \|_{L^\infty} \| f \|_{L^2_{\xi}}^2$$

$$\leq \| \Delta \xi \|_{L^\infty} \| f \|_{L^2_{\xi}}^2 < \infty$$

Then we know that $G(s)$ can be differentiable in $s$, and

$$G'(s) = \sum_{x \in V} (-2 P_{t-s} f(x) \Delta P_{t-s} f(x) P_s \xi(x)m(x) + (P_{t-s} f)^2(x) \Delta P_s \xi(x)m(x))$$

For $f \in C_0(V)$, from lemma 2.10 and Theorem 2.12 we can easily get $(P_{t-s} f)^2 \in D(Q)$.

Then from Lemma 2.14, we get

$$G'(s) = \sum_{x \in V} (-2 P_{t-s} f \Delta P_{t-s} f P_s \xi m(x) - \Gamma((P_{t-s} f)^2, P_s \xi)m(x))$$

Now we replace $P_s \xi$ of $h$, and $h$ satisfies $0 < h \in C_0(V)$, that is to say, $h$ is a finitely supported function.
Then
\[
\sum_{x \in V} \left( -2P_{t-s} f \Delta P_{t-s} fh(x)m(x) - \Gamma(P_{t-s}f)^2, h(x) \right) m(x) \\
= \sum_{x \in V} \left( -2P_{t-s} f \Delta P_{t-s} fh(x)m(x) + \Delta(P_{t-s}f)^2 h(x)m(x) \right) \\
= \sum_{x \in V} 2\Gamma(P_{t-s}f)h(x)m(x)
\]

Then for \(0 < h \in D(Q)\), let \(h_k = h\eta_k\), and \(\eta_k\) satisfies
\[
\lim_{k \to \infty} \eta_k = 1, \Gamma(\eta_k) \leq \frac{1}{k}, k \in \mathbb{N}.
\]
we can get \(0 < h_k \in C_0(V)\). Then let \(k \to \infty\), and for any \(0 < h \in D(Q)\), we have
\[
\sum_{x \in V} \left( -2P_{t-s} f \Delta P_{t-s} fh(x)m(x) - \Gamma((P_{t-s}f)^2, h(x)) m(x) \right) \\
= \sum_{x \in V} 2\Gamma(P_{t-s}f)h(x)m(x).
\]

For \(\xi \in C_0(V)\), we easily know \(P_s \xi \in D(Q)\). Then let \(h = P_s \xi\), we have
\[
G'(s) = \sum_{x \in V} 2\Gamma(P_{t-s}f)P_s \xi m(x)
\]

Integrate the equation from 0 to \(t\) by both sides
\[
\int_0^t \left( \sum_{x \in V} 2\Gamma(P_{t-s}f)P_s \xi m(x) \right) = \int_0^t G'(s) \\
= G(t) - G(0) \\
= \sum_{x \in V} f^2(x)P_s \xi (x)m(x) - \sum_{x \in V} (P_s f)^2 \xi (x)m(x)
\]

Since \(P_t\) is a self-adjoint operator on \(l_2^m\), the right hand side of the equation can be changed into:
\[
\int_0^t \left( \sum_{x \in V} 2\Gamma(P_{t-s}f)P_s \xi m(x) \right) = \int_0^t G'(s) \\
= \int_0^t \sum_{x \in V} 2P_s \Gamma(P_{t-s}f) \xi (x)m(x) \\
= \sum_{x \in V} P_t(f)^2 \xi (x)m(x) - \sum_{x \in V} (P_t f)^2 \xi (x)m(x)
\]

For \(\xi (x) \in C_0(V)\), let \(\xi (x) = \delta_y (x)\) (when \(y = x\), \(\delta_y (x) = 1\) otherwise \(\delta_y (x) = 0\)).
Then, the equation is changed into:

$$P_t(f)^2(y) - (P_t f)^2(y) = 2 \int_0^t P_s \Gamma(P_{t-s} f)(y).$$

From Remark 3.1 we now have

$$P_t(f^2) - (P_t f)^2 \leq 2 \int_0^t e^{2Ks} \Gamma(P_s \circ P_{t-s} f) ds$$

$$= 2 \int_0^t e^{2Ks} ds \cdot \Gamma(P_t f)$$

$$= \frac{e^{2Kt - 1}}{K} \Gamma(P_t f)$$

As the change in the proof is equivalent, the properties of $CD(K, \infty)$ are still equivalent properties.

Also we can get equivalent properties of $CD(K,\infty)$.

**Theorem 3.3.** Let $G = (V, E, m, \mu)$ be a complete graph and $m$ is nondegenerate. Then the following are equivalent:

(a) $G$ satisfies $CD(K, \infty)$.

(b) For any finitely supported function $f$,

$$\Gamma(P_t f) \leq e^{-2Kt} P_t \Gamma(f) - \frac{2}{n} \int_0^t e^{-2Ks} P_s (\Delta P_{t-s} f)^2 ds, \quad 0 < s < t.$$

(c) For any $f \in D(Q)$,

$$\Gamma(P_t f) \leq e^{-2Kt} P_t \Gamma(f) - \frac{2}{n} \int_0^t e^{-2Ks} P_s (\Delta P_{t-s} f)^2 ds, \quad 0 < s < t.$$  

**Proof** First, for any $f, \xi \in C_0(V)$, we build this functional equation

$$G(s) = e^{-2Ks} \sum_{x \in V} \Gamma(P_{t-s} f)(x) P_s \xi(x) m(x).$$

Taking formal derivative of $G(s)$, we define the function as $A$. Then

$$A = -2K e^{-2Ks} \sum_{x \in V} \Gamma(P_{t-s} f)(x) P_s \xi(x) m(x) + e^{-2Ks} \sum_{x \in V} (-2 \Gamma(P_{t-s} f, \Delta P_{t-s} f)(x)$$

$$P_s \xi(x) m(x) + e^{-2Ks} \sum_{x \in V} \Gamma(P_{t-s} f) \Delta P_s \xi(x) m(x).$$

Now we will show that $G(s)$ is differentiable in $s$.

Without loss of generality, we assume that $\epsilon < s < t - \epsilon$ for some $\epsilon > 0$.

For the first part, from Lemma 2.15 we have

$$| -2K e^{-2Ks} \sum_{x \in V} \Gamma(P_{t-s} f)(x) P_s \xi(x) m(x) |$$

$$\leq |C_1\sum_{x \in V} |\Gamma(P_{t-s} f)(x)||P_s \xi||m(x)$$

$$\leq |C_1||P_s \xi||_{l^\infty}||\Gamma(P_{t-s} f)||_{l^1_m}$$

$$< \infty$$
$C_1$ is a constant, satisfying $| - 2 Ke^{-2Ks} | \leq C_1$. For the second part, from Lemma 2.15 we have

\[
|e^{-2Ks} \sum_{x \in V} (-2\Gamma(P_{t-s}f, \Delta P_{t-s}f)(x)P_s\xi(x)m(x))| \\
\leq |C_2| \sum_{x \in V} |(-2\Gamma(P_{t-s}f, \Delta P_{t-s}f)(x)||P_s\xi(x)|m(x)| \\
\leq |C_2||P_s\xi||_{\infty} ||\Gamma(P_{t-s}f, \Delta P_{t-s}f)||_{l_{\infty}} \\
< \infty
\]

$C_2$ is some constant, satisfying $|e^{-2Ks}| \leq C_2$. For the last part, from Lemma 2.15 we have

\[
|e^{-2Ks} \sum_{x \in V} \Gamma(P_{t-s}f)\Delta P_s\xi(x)m(x)| \\
\leq |C_2| \sum_{x \in V} |\Gamma(P_{t-s}f)||\Delta P_s\xi(x)|m(x) \\
= |C_2| \sum_{x \in V} |\Gamma(P_{t-s}f)||P_s\Delta\xi(x)|m(x) \\
\leq |C_2||\Delta\xi||_{\infty} ||\Gamma(P_{t-s}f)||_{l_{\infty}} \\
< \infty
\]

$C_2$ is defined as above.

Then we can know that $G(s)$ is differentiable in $s$, and

$$G'(s) = -2Ke^{-2Ks} \sum_{x \in V} \Gamma(P_{t-s}f)(x)P_s\xi(x)m(x) + e^{-2Ks} \sum_{x \in V} (-2\Gamma(P_{t-s}f, \Delta P_{t-s}f)(x)P_s\xi(x)m(x) + e^{-2Ks} \sum_{x \in V} \Gamma(P_{t-s}f)\Delta P_s\xi(x)m(x).$$

From Lemma 2.14 we get

$$G'(s) = -2Ke^{-2Ks} \sum_{x \in V} \Gamma(P_{t-s}f)(x)P_s\xi(x)m(x) + e^{-2Ks} \sum_{x \in V} (-2\Gamma(P_{t-s}f, \Delta P_{t-s}f)(x)P_s\xi(x)m(x) + e^{-2Ks} \sum_{x \in V} \Gamma(\Gamma(P_{t-s}f), P_s\xi(x)m(x).$$

Now we need to show that for all $h \in D(Q)$, we have

$$-2 \sum_{x \in V} \Gamma(P_{t-s}f, \Delta P_{t-s}f)(x)h(x)m(x) + \sum_{x \in V} \Gamma(\Gamma(P_{t-s}f), h(x))m(x) \\
= \sum_{x \in V} \Gamma_2(P_{t-s}f)h(x)m(x).$$

Obviously, this equation holds for all the finitely supported functions.

Now taking a series of functions $\{\eta_k\}$ in $C_0(V)$ defined as definition 2.10.
Let \( h_k = h\eta_k \), obviously \( h_k \in C_0(V) \), then

\[
-2 \sum_{x \in V} \Gamma(P_{t-s}f, \triangle P_{t-s}f)(x) h_k(x) m(x) + \sum_{x \in V} \Gamma(\Gamma(P_{t-s}f), h_k(x)) m(x)
\]

\[
= \sum_{x \in V} \Gamma_2(P_{t-s}f) h_k(x) m(x).
\]

Let \( k \to \infty \), then for all \( h \in D(Q) \), we can get

\[
-2 \sum_{x \in V} \Gamma(P_{t-s}f, \triangle P_{t-s}f)(x) h(x) m(x) + \sum_{x \in V} \Gamma(\Gamma(P_{t-s}f), h(x)) m(x)
\]

\[
= \sum_{x \in V} \Gamma_2(P_{t-s}f) h(x) m(x).
\]

For \( \xi \in V \), \( P_s \xi \in D(Q) \), then let \( h = P_s \xi \), we get

\[
-2 \sum_{x \in V} \Gamma(P_{t-s}f, \triangle P_{t-s}f)(x) P_s \xi(x) m(x) + \sum_{x \in V} \Gamma(\Gamma(P_{t-s}f), P_s \xi(x)) m(x)
\]

\[
= \sum_{x \in V} \Gamma_2(P_{t-s}f) P_s \xi(x) m(x).
\]

Then \( G'(s) \) can be rewritten as

\[
G'(s) = e^{-2Ks} \sum_{x \in V} (\Gamma_2(P_{t-s}f) - K \Gamma(P_{t-s}f)) P_s \xi(x) m(x)
\]

By use of the equivalent properties of \( CD(K, n) \), we get

\[
G'(s) \geq e^{-2Ks} \sum_{x \in V} \frac{2}{n} (\triangle P_{t-s}f)^2(x) P_s \xi(x) m(x)
\]

Now integrate the equation from 0 to \( t \) in \( s \) by both sides, we can get

\[
\int_0^t G'(s) = G(t) - G(0)
\]

\[
= e^{-2Kt} \sum_{x \in V} \Gamma(f)(x) P_t \xi(x) m(x) - \sum_{x \in V} \Gamma(P_t f)(x) \xi(x) m(x)
\]

\[
\geq \frac{2}{n} \int_0^t e^{-2Ks} \sum_{x \in V} (\triangle P_{t-s}f)^2 P_s \xi(x) m(x) ds
\]

Since \( P_t \) is a self-adjoint operator on \( l^2_m \), we can get

\[
e^{-2Kt} \sum_{x \in V} P_t \Gamma(f)(x) \xi(x) m(x) - \sum_{x \in V} \Gamma(P_t f)(x) \xi(x) m(x)
\]

\[
\geq \frac{2}{n} \int_0^t e^{-2Ks} \sum_{x \in V} P_s (\triangle P_{t-s}f)^2 \xi(x) m(x) ds
\]
Let $\xi(x) = \delta_y(x)$, then
\[
e^{-2Kt}P_t\Gamma(f)(y)m(y) - \Gamma(P_tf)(y)m(y)
\geq \frac{1}{2n} \int_0^t e^{-2Ks}P_s(\triangle P_{t-s}f)^2 m(y) ds
\]
For $m(y) > 0$,
\[
e^{-2Kt}P_t\Gamma(f) - \Gamma(P_tf)
\geq \frac{2}{n} \int_0^t e^{-2Ks}P_s(\triangle P_{t-s}f)^2 ds
\]
that is to say,
\[
\Gamma(P_tf) \leq e^{-2Kt}P_t(\Gamma(f)) - \frac{2}{n} \int_0^t e^{-2Ks}P_s(\triangle P_{t-s}f)^2 ds.
\]
As the change in the proof is equivalent, the properties of $CD(K, n)$ in the theorem are still equivalent properties.

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