A NOTE ON COVERINGS OF VIRTUAL KNOTS

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Abstract. For a virtual knot $K$ and an integer $r \geq 0$, the $r$-covering $K^{(r)}$ is defined by using the indices of chords on a Gauss diagram of $K$. In this paper, we prove that for any finite set of virtual knots $J_0, J_2, J_3, \ldots, J_m$, there is a virtual knot $K$ such that $K^{(r)} = J_r$ ($r = 0$ and $2 \leq r \leq m$), $K^{(1)} = K$, and otherwise $K^{(r)} = J_0$.

1. Introduction

Odd crossings are first introduced for constructing a simple invariant called the odd writhe of a virtual knot by Kauffman [8]. By using odd crossings, Manturov defines a map from the set of virtual knots to itself by replacing the odd crossings with virtual crossings [10].

Later the notion of index is introduced in [2, 4, 6, 11] which assigns an integer to each real crossing such that the parity of the index coincides with the original parity. The $n$-writhe is defined as a refinement of the odd writhe. Jeong defines an invariant called the zero polynomial of a virtual knot by using real crossings of index 0 [7]. In fact, Im and Kim prove that the zero polynomial is coincident with the writhe polynomial of the virtual knot obtained by replacing the real crossings whose indices are non-zero with virtual crossings [5]. They also study the operation replacing the real crossings whose indices are not divisible by $r$ for a positive integer $r$. This operation is originally considered for flat virtual knots by Turaev [12] where he calls the obtained knot the $r$-covering.

The writhe polynomial $W_K(t)$ of a virtual knot $K$ is the polynomial such that the coefficient of $t^n$ is equal to the $n$-writhe of $K$. A characterization of $W_K(t)$ is given as follows.

Theorem 1.1 ([11]). For a Laurent polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$, the following are equivalent.

(i) There is a virtual knot $K$ with $W_K(t) = f(t)$.

(ii) $f(1) = f'(1) = 0$.

For an integer $r \geq 0$, we denote by $K^{(r)}$ the $r$-covering of a virtual knot $K$. By definition, we have $K^{(1)} = K$ and $K^{(r)} = K^{(0)}$ for a sufficiently large $r$. The aim of this note is to prove the following.

The first author is partially supported by JSPS Grants-in-Aid for Scientific Research (C), 17K05265. The third author is partially supported by JSPS Grants-in-Aid for Scientific Research (C), 16K05147.

2010 Mathematics Subject Classification. Primary 57M25; Secondary 57M27.

Key words and phrases. virtual knot, covering, Gauss diagram, index, Möbius function.
Theorem 1.2. Let $m \geq 1$ be an integer, $J_n$ $(0 \leq n \leq m, n \neq 1)$ $m$ virtual knots, and $f(t)$ a Laurent polynomial with $f(1) = f'(1) = 0$. Then there is a virtual knot $K$ such that

$$K^{(r)} = \begin{cases} J_0 & \text{for } r = 0 \text{ and } r \geq m + 1, \\ K & \text{for } r = 1, \\ J_r & \text{otherwise,} \end{cases}$$

and $W_K(t) = f(t)$.

This paper is organized as follows. In Section 2 we define the $r$-covering $K^{(r)}$ of a (long) virtual knot $K$. We also introduce an anklet of a chord in a Gauss diagram which will be used in the consecutive sections. In Sections 3 and 4 we study the 0-covering $K^{(0)}$ and $r$-covering $K^{(r)}$ for $r \geq 2$ of a long virtual knot, respectively. In Section 5 we review the writhe polynomial of a virtual knot, and prove Theorem 1.1.

2. Gauss diagrams

A circular or linear Gauss diagram is an oriented circle or line equipped with a finite number of oriented and signed chords spanning the circle or line, respectively. The closure of a linear Gauss diagram is the circular Gauss diagram obtained by taking the one-point compactification of the line.

Let $c$ be a chord of a Gauss diagram $G$ with sign $\varepsilon = \varepsilon(c)$. We give signs $-\varepsilon$ and $\varepsilon$ to the initial and terminal endpoints of $c$, respectively. We consider the case that $G$ is circular. The endpoints of $c$ divide the circle into two arcs. Let $\alpha$ be the arc oriented from the initial endpoint of $c$ to the terminal. See Figure 1. The index of $c$ is the sum of signs of endpoints of chords on $\alpha$, and denoted by $\text{Ind}_G(c)$ (cf. [1, 9, 11]). In the case that $G$ is linear, the index of $c$ is defined as that of $c$ in the closure of $G$.

Figure 1. The orientation and signs of a chord and its endpoints.

Let $G$ be a circular or linear Gauss diagram. For a positive integer $r$, we denote by $G^{(r)}$ the Gauss diagram obtained from $G$ by removing all the chords $c$ with $\text{Ind}_G(c) \not\equiv 0 \pmod{r}$ (cf. [12]). In particular, we have $G^{(1)} = G$. For $r = 0$, we denote by $G^{(0)}$ the Gauss diagram obtained from $G$ by removing all the chords $c$ with $\text{Ind}_G(c) \neq 0$. Since the number of chords of $G$ is finite, we have $G^{(r)} = G^{(0)}$ for sufficiently large $r$.

Two circular Gauss diagrams $G$ and $H$ are equivalent, denoted by $G \sim H$, if $G$ is related to $H$ by a finite sequence of Reidemeister moves I–III as shown in Figure 2. A virtual knot is an equivalence class of circular Gauss diagrams up to this equivalence relation (cf. [3,8]). Similarly, the equivalence relation among linear Gauss diagrams are defined, and an equivalence class is called a long virtual knot. The trivial (long) virtual knot is presented by a Gauss diagram with no chord.
Lemma 2.1 (cf. [5,12]). Let $G$ and $H$ be circular or linear Gauss diagrams such that $G \sim H$. Then it holds that $G^{(r)} \sim H^{(r)}$ for any integer $r \geq 0$. \hfill \Box

Although only the case of a circular Gauss diagram is studied in [5,12], Lemma 2.1 for a linear Gauss diagram can be proved similarly.

Definition 2.2. Let $K$ be a (long) virtual knot, and $r \geq 0$ an integer. The $r$-covering of $K$ is the (long) virtual knot presented by $G^{(r)}$ for some Gauss diagram $G$ of $K$. We denote it by $K^{(r)}$.

The well-definedness of $K^{(r)}$ follows from Lemma 2.1. We have $K^{(1)} = K$ and $K^{(r)} = K^{(0)}$ for sufficiently large $r$.

Let $c(G)$ denote the number of chords of a Gauss diagram $G$. The real crossing number of a (long) virtual knot $K$ is the minimal number of $c(G)$ for all Gauss diagrams $G$ of $K$, and denoted by $c(K)$.

Lemma 2.3. Let $K$ be a (long) virtual knot, and $r \geq 0$ an integer. Then it holds that $c(K^{(r)}) \leq c(K)$. In particular, $c(K^{(r)}) = c(K)$ holds if and only if $K^{(r)} = K$.

Proof. Let $G$ be a Gauss diagram of $K$ with $c(G) = c(K)$. Since $G^{(r)}$ is obtained from $G$ by removing some chords, we have

$$c(K^{(r)}) \leq c(G^{(r)}) \leq c(G) = c(K).$$

In particular, if the equality holds, then $G^{(r)} = G$ and $K^{(r)} = K$. \hfill \Box
Let $c_1, \ldots, c_n$ be chords of a Gauss diagram $G$. We add several parallel chords near an endpoint of $c_i$ ($i = 1, \ldots, n$) to obtain a Gauss diagram $G'$. Here, the orientations and signs of the added chords are chosen arbitrarily. See Figure 3(i). The parallel chords added to $c_i$ are called anklets of $c_i$. We remark that the index of an anklet is equal to $\pm 1$.

![Figure 3. Anklets.](image)

**Lemma 2.4.** Let $c_1, \ldots, c_n$ be chords of a Gauss diagram $G$. For any integers $a_1, \ldots, a_n$, by adding several anklets to each $c_i$ near its initial endpoint suitably, we obtain a Gauss diagram $G'$ such that $\text{Ind}_{G'}(c_i) = a_i$ for any $i = 1, \ldots, n$, and $\text{Ind}_{G'}(c) = \text{Ind}_G(c)$ for any $c \neq c_1, \ldots, c_n$.

**Proof.** Put $d_i = a_i - \text{Ind}_G(c_i)$ for $i = 1, \ldots, n$. We add $|d_i|$ anklets to $c_i$ near its initial endpoint such that the signs of right endpoints of the anklets are equal to $\varepsilon_i$, where $\varepsilon_i$ is the sign of $d_i$. See Figure 3(ii).

Let $G'$ be the obtained Gauss diagram. Then we have

$$\text{Ind}_{G'}(c_i) = \text{Ind}_G(c_i) + \varepsilon_i|d_i| = \text{Ind}_G(c_i) + d_i = a_i.$$ 

Furthermore the index of a chord other than $c_1, \ldots, c_n$ does not change. \hfill $\square$

### 3. The $0$-covering $K^{(0)}$

For an integer $n \geq 2$, we define a map $f_n : \{2, 3, \ldots, n\} \to \mathbb{Z}$ which satisfies

$$\sum_{c = 0 \mod r, \ c \leq n} f_n(i) = -1$$

for any integer $r$ with $2 \leq r \leq n$. The map $f_n$ exists uniquely. Put

$$P_n = \{i \mid 2 \leq i \leq n, f_n(i) \neq 0\}.$$

**Example 3.1.** For $n = 10$, we have

$$\begin{align*}
f_{10}(2) + f_{10}(4) + f_{10}(6) + f_{10}(8) + f_{10}(10) &= -1 \quad \text{for } r = 2, \\
f_{10}(3) + f_{10}(6) + f_{10}(9) &= -1 \quad \text{for } r = 3, \\
f_{10}(4) + f_{10}(8) &= -1 \quad \text{for } r = 4, \\
f_{10}(5) + f_{10}(10) &= -1 \quad \text{for } r = 5, \\
f_{10}(6) &= -1 \quad \text{for } r = 6, \\
f_{10}(7) &= -1 \quad \text{for } r = 7, \\
f_{10}(8) &= -1 \quad \text{for } r = 8, \\
f_{10}(9) &= -1 \quad \text{for } r = 9, \\
f_{10}(10) &= -1 \quad \text{for } r = 10.
\end{align*}$$
Therefore we have
\[
f_{10}(i) = \begin{cases} 
2 & (i = 2), \\
1 & (i = 3), \\
0 & (i = 4, 5), \\
-1 & (6 \leq i \leq 10) 
\end{cases}
\]
and \(P_{10} = \{2, 3, 6, 7, 8, 9, 10\}\).

**Theorem 3.2.** For any integer \(n \geq 1\) and long virtual knot \(J\), there is a long virtual knot \(K\) such that
\[
K^{(r)} = \begin{cases} 
J & \text{for } r = 0 \text{ and } r \geq n + 1, \\
K & \text{for } r = 1, \text{ and} \\
O & \text{otherwise}.
\end{cases}
\]
Here, \(O\) denotes the trivial long virtual knot.

**Proof.** Let \(H\) be a linear Gauss diagram of \(J\). We construct a linear Gauss diagram \(G\) as follows: First, we replace each chord \(c\) of \(H\) by \(1 + \sum_{i \in P_n} |f_n(i)|\) parallel chords labeled \(c_0\) and \(c_{ij}\) for \(i \in P_n\) and \(1 \leq j \leq |f_n(i)|\). The orientations of \(c_0\) and \(c_{ij}\)'s are the same as that of \(c\). The signs of them are given such that
\[
\begin{align*}
(i) & \quad \varepsilon(c_0) = \varepsilon(c), \\
(ii) & \quad \varepsilon(c_{ij}) = \varepsilon(c) \delta_i \text{ for } i \in P_n, \text{ where } \delta_i \text{ is the sign of } f_n(i) \neq 0.
\end{align*}
\]
Next, we add several anklets to each of \(c_0\) and \(c_{ij}\)'s such that
\[
\begin{align*}
(iii) & \quad \text{Ind}_{G}(c_0) = 0, \text{ and} \\
(iv) & \quad \text{Ind}_{G}(c_{ij}) = i \text{ for } i \in P_n.
\end{align*}
\]
The obtained Gauss diagram is denoted by \(G\).

Figure 4 shows the case \(n = 10\) replacing each chord \(c\) of \(H\) with nine chords \(c_0, c_{21}, c_{22}, c_{31}, \ldots, c_{10, 1}\) and several anklets. The boxed numbers of the chords indicate their indices.

![Figure 4. The case \(n = 10\).](image_url)

By the conditions (i)–(iv), we have \(G^{(0)} = H\); in fact, we remove the chords whose indices are non-zero from \(G\) to obtain \(G^{(0)}\). Similarly, we have \(G^{(r)} = H\) for any \(r \geq n + 1\). Therefore it holds that \(K^{(0)} = K^{(r)} = J\) for \(r \geq n + 1\).

For \(2 \leq r \leq n\), \(G^{(r)}\) is obtained from \(G\) by removing the chords whose indices are not divisible by \(r\). In particular, all the anklets are removed. Among the chords \(c_0\) and \(c_{ij}\)'s, the sum of signs of chords whose indices are divisible by \(r\) is equal to
\[
\varepsilon(c) + \varepsilon(c) \sum_{\substack{r \leq i \leq n \\ i \equiv 0 \pmod{r}}} \delta_i |f_n(i)| = \varepsilon(c) + \varepsilon(c) \sum_{\substack{r \leq i \leq n \\ i \equiv 0 \pmod{r}}} f_n(i) = 0.
\]
Therefore all the chords of \(G^{(r)}\) can be canceled by Reidemeister moves II so that \(K^{(r)}\) is the trivial long virtual knot. \(\square\)
4. The $r$-covering $K^{(r)}$ for $r \geq 2$

For an integer $n \geq 2$, we define a map $g_n : \{2, 3, \ldots, n\} \rightarrow \mathbb{Z}$ which satisfies

$$g_n(n) = 1 \quad \text{and} \quad \sum_{i \equiv 0 \pmod{r}} g_n(i) = 0$$

for any integer $r$ with $2 \leq r < n$. The map $g_n$ exists uniquely. Put

$$Q_n = \{ i \mid 2 \leq i \leq n, g_n(i) \neq 0 \}.$$

**Example 4.1.** (i) For $n = 10$, it holds that

$$\begin{align*}
g_{10}(2) + g_{10}(4) + g_{10}(6) + g_{10}(8) + g_{10}(10) &= 0 \quad \text{for } r = 2, \\
g_{10}(3) + g_{10}(6) + g_{10}(9) &= 0 \quad \text{for } r = 3, \\
g_{10}(4) + g_{10}(8) &= 0 \quad \text{for } r = 4, \\
g_{10}(5) + g_{10}(10) &= 0 \quad \text{for } r = 5, \\
g_{10}(6) &= 0 \quad \text{for } r = 6, \\
g_{10}(7) &= 0 \quad \text{for } r = 7, \\
g_{10}(8) &= 0 \quad \text{for } r = 8, \\
g_{10}(9) &= 0 \quad \text{for } r = 9, \\
g_{10}(10) &= 1 \quad \text{for } r = 10.
\end{align*}$$

Therefore we have

$$g_{10}(i) = \begin{cases} 
1 & (i = 10), \\
-1 & (i = 2, 5), \\
0 & \text{(otherwise),}
\end{cases} \quad \text{and } Q_{10} = \{2, 5, 10\}.$$

(ii) For $n = 12$, we have

$$g_{12}(i) = \begin{cases} 
1 & (i = 2, 12), \\
-1 & (i = 4, 6), \\
0 & \text{(otherwise),}
\end{cases} \quad \text{and } Q_{12} = \{2, 4, 6, 12\}.$$

**Theorem 4.2.** For any integer $n \geq 2$ and long virtual knot $J$, there is a long virtual knot $K$ such that

$$K^{(r)} = \begin{cases} 
J & \text{for } r = n, \\
K & \text{for } r = 1, \text{ and} \\
O & \text{otherwise.}
\end{cases}$$

**Proof.** The proof is similar to that of Theorem 3.2 by using $g_n$ instead of $f_n$. Let $H$ be a linear Gauss diagram of $J$. We construct a linear Gauss diagram $G$ of $K$ as follows: First, we replace each chord $c$ of $H$ by $\sum_{i \in Q_n} |g_n(i)|$ parallel chords labeled $c_i$ for $i \in Q_n$. The orientations of $c_i$’s are the same as that of $c$. The signs of them are given such that $\varepsilon(c_i) = \varepsilon(c)\delta_i$, where $\delta_i$ is the sign of $g_n(i) \neq 0$. Next, we add several anklets to each of $c_i$’s such that $\text{Ind}_G(c_i) = i$ for $i \in Q_n$. The obtained Gauss diagram is denoted by $G$.

In the left of Figure 5, we shows the case $n = 10$ replacing each chord $c$ of $H$ with three chords $c_2, c_5, c_{10}$ and several anklets. In the right figure, the case of $n = 12$ is given.

Since any chord $c$ of $G$ satisfies $1 \leq |\text{Ind}_G(c)| \leq n$, we obtain $G^{(0)}$ and $G^{(r)}$ for $r \geq n + 1$ by removing all the chords from $G$. 
For $2 \leq r < n$, $G^{(r)}$ is obtained from $G$ by removing the chords whose indices are not divisible by $r$. Among the chords $c_i$’s, the sum of signs of chords whose indices are divisible by $r$ is equal to

$$\varepsilon(c) \sum_{r \leq i \leq n \atop i \equiv 0 \mod r} \delta_i |g_n(i)| = \varepsilon(c) \sum_{r \leq i \leq n \atop i \equiv 0 \mod r} g_n(i) = 0.$$ 

Therefore all the chords of $G^{(r)}$ can be canceled by Reidemeister moves II so that $K^{(r)}$ is the trivial long virtual knot.

Finally, for $r = n$, we have $G^{(n)} = H$ by definition immediately. □

We see that $g_n$ is coincident with a famous function as follows.

**Proposition 4.3.** Let $\mu$ be the Möbius function. Then we have

$$g_n(i) = \begin{cases} \mu\left(\frac{n}{i}\right) & \text{if } n \text{ is divisible by } i, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Let $h_n(i)$ be the right hand side of the equation in the proposition. Since $h_n(n) = \mu(1) = 1 = g_n(n)$, it is sufficient to prove that

$$\sum_{r \leq i \leq n \atop i \equiv 0 \mod r} h_n(i) = 0$$

for any integer $r$ with $2 \leq r < n$.

Assume that $n$ is not divisible by $r$. Then $n$ is not divisible by any $i$ such that $r \leq i \leq n$ and $i \equiv 0 \mod r$. Therefore we have

$$\sum_{r \leq i \leq n \atop i \equiv 0 \mod r} h_n(i) = 0.$$ 

Assume that $n$ is divisible by $r$. By the property of the Möbius function, it holds that

$$\sum_{r \leq i \leq n \atop i \equiv 0 \mod r} h_n(i) = \sum_{r \leq d \leq n \atop d \equiv 0 \mod r} \mu\left(\frac{n}{i}\right) = \sum_{d \equiv 0 \mod r} \mu(d) = 0.$$ 

Therefore we have $g_n = h_n$. □

### 5. The writhe polynomial

For an integer $n \neq 0$ and a sign $\varepsilon = \pm 1$, the $(n, \varepsilon)$-snail is a linear Gauss diagram consisting of a chord $c$ with $\varepsilon(c) = \varepsilon$ and $|n|$ anklets such that the indices of $c$ and each anklet are equal to $n$ and 1, respectively. See Figure [6].

Let $G$ be a Gauss diagram of a (long) virtual knot $K$. For an integer $n \neq 0$, we denote by $w_n(G)$ the sum of signs of all chords $c$ of $G$ with $\text{Ind}_G(c) = n$. Then
$w_n(G)$ does not depend on a particular choice of $G$ of $K$; that is, $w_n(G)$ is an invariant of $K$. In [11], the proof is given for a virtual knot, and the case of a long virtual knot is similarly proved. It is called the $n$-writhe of $K$ and denoted by $w_n(K)$. The writhe polynomial of $K$ is defined by

$$W_K(t) = \sum_{n \neq 0} w_n(K)t^n - \sum_{n \neq 0} w_n(K) \in \mathbb{Z}[t,t^{-1}].$$

This invariant was introduced in several papers [2, 9, 11] independently.

**Theorem 5.1.** Let $J$ be a long virtual knot, and $f(t) \in \mathbb{Z}[t,t^{-1}]$ a Laurent polynomial with $f(1) = f'(1) = 0$. Then there is a long virtual knot $K$ such that

(i) $K^{(r)} = J^{(r)}$ for any integer $r = 0$ and $r \geq 2$, and

(ii) $W_K(t) = f(t)$.

**Proof.** Put $g(t) = f(t) - W_J(t) = \sum_{n \in \mathbb{Z}} a_n t^n$. By Theorem 1.1, we have $g(1) = g'(1) = 0$. Therefore it holds that $a_0 = \sum_{n \neq 0,1} (n-1)a_n$ and $a_1 = -\sum_{n \neq 0,1} na_n$.

Let $H$ be a linear Gauss diagram of $J$. We construct a linear Gauss diagram $G$ by juxtaposing $H$ and $|a_n|$ copies of $S(n,\varepsilon_n)$ for every integer $n$ with $n \neq 0,1$ and $a_n \neq 0$. Here, $\varepsilon_n$ is the sign of $a_n$.

Let $K$ be the long virtual knot presented by $G$. The contribution of each snail $S(n,\varepsilon_n)$ to the writhe polynomial $W_K(t)$ is equal to $\varepsilon_n t^n - \varepsilon_n nt + \varepsilon_n(n-1)$. Therefore it holds that

$$W_K(t) = W_J(t) + \sum_{n \neq 0,1} |a_n|(\varepsilon_n t^n - \varepsilon_n nt + \varepsilon_n(n-1))$$

$$= W_J(t) + \sum_{n \neq 0,1} a_n (t^n - nt + (n-1))$$

$$= W_J(t) + g(t) = f(t).$$

By definition, $S(n,\varepsilon_n)^{(r)}$ has the only chord $c$ if $n$ is divisible by $r$. Otherwise it has no chord. Therefore $G^{(r)}$ is equivalent to $H^{(r)}$, and hence $K^{(r)} = J^{(r)}$ for any integer $r = 0$ and $r \geq 2$. 

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**Figure 6.** The snail $S(n,\varepsilon)$. 

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\[
\begin{align*}
\text{Figure 6.} \quad & \begin{array}{c}
\text{The snail } S(n,\varepsilon).
\end{array}
\end{align*}
\]
Theorem 5.2. Let \( m \geq 1 \) be an integer, \( J_n \) \((0 \leq n \leq m, n \neq 1)\) \( m \) long virtual knots, and \( f(t) \) a Laurent polynomial with \( f(1) = f'(1) = 0 \). Then there is a long virtual knot \( K \) such that

\[
K^{(r)} = \begin{cases} 
J_0 & \text{for } r = 0 \text{ and } r \geq m + 1, \\
K & \text{for } r = 1, \\
J_r & \text{otherwise,}
\end{cases}
\]

and \( W_K(t) = f(t) \).

Proof. Let \( K_0 \) be a long virtual knot obtained by applying Theorem 5.2 to the pair of \( m \) and \( J_0 \). Let \( K_n \) be a long virtual knot obtained by applying Theorem 4.2 to each pair of \( n \) and \( J_n \) \((2 \leq n \leq m)\). We juxtapose \( K_0, K_2, \ldots, K_m \) to have a long virtual knot \( K' \). Let \( K \) be a long virtual knot obtained by applying Theorem 5.1 to the pair of \( K' \) and \( f(t) \). Then we see that \( K \) is a desired long virtual knot. \( \square \)

Proof of Theorem 1.2. Let \( J^\circ_n \) be a long virtual knot whose closure is \( J_n \) \((0 \leq n \leq m, n \neq 1)\). Let \( K^\circ \) be a long virtual knot obtained by applying Theorem 5.2. Then we see that the closure of \( K^\circ \) is a desired virtual knot. \( \square \)

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