RADON TRANSFORMS FOR MUTUALLY ORTHOGONAL AFFINE PLANES

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Abstract. We study a Radon-like transform that takes functions on the Grassmannian of $j$-dimensional affine planes in $\mathbb{R}^n$ to functions on a similar manifold of $k$-dimensional planes by integration over the set of all $j$-planes that meet a given $k$-plane at a right angle. The case $j = 0$ gives the classical Radon-John $k$-plane transform. For any $j$ and $k$, our transform has a mixed structure combining the $k$-plane transform and the dual $j$-plane transform. The main results include action of such transforms on rotation invariant functions, sharp existence conditions, intertwining properties, connection with Riesz potentials and inversion formulas in a large class of functions. The consideration is inspired by the previous works of F. Gonzalez and S. Helgason who studied the case $j + k = n - 1$, $n$ odd, on smooth compactly supported functions.

1. Introduction

Let $G(n, j)$ and $G(n, k)$ be a pair of Grassmannian bundles of affine $j$-dimensional and $k$-dimensional unoriented planes in $\mathbb{R}^n$, respectively. In the present paper we study a Radon-like transform that takes a function $f$ on $G(n, j)$ to a function $\mathcal{R}_{j,k}f$ on $G(n, k)$ when the value $(\mathcal{R}_{j,k}f)(\zeta)$ for $\zeta \in G(n, k)$ is defined as an integral of $f$ over the set of all planes $\tau \in G(n, j)$, which meet $\zeta$ at a right angle. Our aim is to study properties of this transform and obtain explicit inversion formulas.

In the limiting case $j = 0$, when $G(n, j)$ is identified with $\mathbb{R}^n$, our transform is the well-known Radon-John $k$-plane transform, which was studied in many books and papers; see, e.g., [4, 14, 16, 20] and references therein. Another limiting case $k = 0$ corresponds to the dual Radon-John transform, which averages a given function on $G(n, j)$ over all $j$-dimensional planes passing through a fixed point $x \in \mathbb{R}^n$. These transforms are usually studied in parallel with the $j$-plane transforms, but have special features; see [8, 14, 20, 31]. Thus $\mathcal{R}_{j,k}f$ is a kind of

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mixture of these limiting cases. An inversion formula for the Radon transform $R_{j,k}f$, when $j + k = n - 1$, $n$ is odd and $f \in C_c^\infty(G(n,j))$, was obtained by Gonzalez [7, 8] in the form

$$f = c R_{k,j}(-\Delta_{n-k})^{(n-1)/2} R_{j,k}f,$$

where $R_{k,j}$ is the dual of $R_{j,k}$, $\Delta_{n-k}$ is the Laplace operator on the fiber of the Grassmannian bundle $G(n,k)$, and $c$ is a constant, which is explicitly evaluated; see also Helgason [14, p. 90], where the results from [7, 8] are announced.

The present paper contains new results for $R_{j,k}f$ for all $j + k < n$ and a large class of functions $f$. In particular, we establish sharp conditions of convergence of the integrals $R_{j,k}f$, their relation to Riesz potentials and Radon-John transforms, and obtain new inversion formulas.

A great deal has been written about Radon transforms on Grassmann manifolds; see, e.g., [2, 3, 5, 6, 10–13, 15, 17, 21, 25, 27, 28, 32, 33]. The classes of Radon transforms and the methods of these works differ from those in the present paper. More information about Radon transforms and their applications can be found, e.g., in the books [4, 14, 26] and references therein.

**Plan of the Paper and Main Results.** Section 2 contains necessary preliminaries. Besides the notation, it includes basic facts about Erdélyi–Kober fractional integrals and derivatives, Radon-John $k$-plane transforms and Riesz potentials. In Section 3 we give precise definition of the mixed $j$-plane to $k$-plane Radon transforms and prove the corresponding duality relation. In Section 4 we show that these transforms on radial (i.e., rotation invariant) functions are represented as compositions of Erdélyi–Kober fractional integrals and give some examples. The results of Section 4 are used in Section 5 to establish sharp conditions, under which the integral $R_{j,k}f$ exists in the Lebesgue sense. In Section 6 we derive new formulas connecting Radon transforms $R_{j,k}f$ with Riesz potentials. Section 7 is devoted to inversion formulas for $R_{j,k}f$. Here, by the dimensionality argument, the natural setting of the problem corresponds to $j + k < n$. The most complete information is obtained in the following cases:

1. any $j + k < n$, when $f$ is radial;
2. $j + k = n - 1$;
3. any $j + k < n$, when $f$ belongs to the range of the $j$-plane Radon-John transform.

Regarding other cases, we have the following

**Conjecture.** If $j + k < n - 1$, then $R_{j,k}$ is non-injective on the set of all infinitely smooth rapidly decreasing functions.
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2. Preliminaries

2.1. Notation. Let $G_{n,j}$ and $G(n,j)$ be the sets of all $j$-dimensional linear subspaces and $j$-dimensional unoriented affine planes in $\mathbb{R}^n$, respectively. Each “point” in $G_{n,j}$ represents a $j$-plane passing through the origin. Every $j$-plane $\tau \in G(n,j)$ is naturally parameterized by the pair $(\xi, u)$, where $\xi \in G_{n,j}$ and $u \in \xi^\perp$, the orthogonal complement of $\xi$ in $\mathbb{R}^n$. Under this parametrization, the manifold $G(n,j)$ is a fiber bundle with the base $G_{n,j}$ and the canonical projection $\pi : \tau(\xi, u) \rightarrow \xi$. The fiber $\pi^{-1}\xi$ over the point $\xi \in G_{n,j}$ is the set of all $j$-dimensional planes parallel to $\xi$. This set is $(n-j)$-dimensional and indexed by $u \in \xi^\perp$. We equip $G(n,j)$ with the product measure $d\tau = d\xi du$, where $d\xi$ is the standard probability measure on $G_{n,j}$ and $du$ is the Euclidean volume element on $\xi^\perp$. Abusing notation, we write $|\tau|$ for the Euclidean distance between $\tau \in G(n,j)$ and the origin. If $\tau \equiv \tau(\xi, u)$, then, clearly, $|\tau| = |u| = (u_1^2 + \cdots + u_j^2)^{1/2}$. Given a subspace $X$ of $\mathbb{R}^n$, we write $G_j(X)$ and $G(j, X)$ for the Grassmannians of all $j$-dimensional linear subspaces and $j$-dimensional unoriented affine planes in $X$, respectively.

In the following, $S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$. For $\theta \in S^{n-1}$, $d\theta$ stands for the Riemannian measure on $S^{n-1}$ so that the area of $S^{n-1}$ is $\sigma_{n-1} = \int_{S^{n-1}} d\theta = 2\pi^{n/2}/\Gamma(n/2)$.

Let $e_1, \ldots, e_n$ be the coordinate unit vectors in $\mathbb{R}^n$. We set

$$E_j = Re_1 \oplus \cdots \oplus Re_j, \quad E_k = Re_{n-k+1} \oplus \cdots \oplus Re_n; \quad (2.1)$$

$$E_\ell = Re_{j+1} \oplus \cdots \oplus Re_{j+\ell}, \quad \ell = n-j-k; \quad (2.2)$$

$$\mathbb{R}^{n-j} = Re_{j+1} \oplus \cdots \oplus Re_n, \quad \mathbb{R}^{n-k} = Re_1 \oplus \cdots \oplus Re_{n-k}. \quad (2.3)$$

The notation $O(n)$ for the orthogonal group of $\mathbb{R}^n$ is standard. For $\rho \in O(n)$, $d\rho$ stands for the $O(n)$-invariant probability measure on $O(n)$; $M(n) = \mathbb{R}^n \rtimes O(n)$ is the group of rigid motions in $\mathbb{R}^n$.

All integrals are understood in the Lebesgue sense. The letter $c$ (sometimes with subscripts) stands for an unessential positive constant that may be different at each occurrence.
2.2. Erdélyi–Kober Fractional Integrals. The following Erdélyi–Kober type fractional integrals on \( \mathbb{R}_+ = (0, \infty) \) arise in numerous integral-geometric problems:

\[
(I_{\alpha,2}^+)f(t) = \frac{2}{\Gamma(\alpha)} \int_0^t (t^2 - r^2)^{\alpha-1} f(r) r \, dr, \quad (2.4)
\]

\[
(I_{\alpha,2}^-)f(t) = \frac{2}{\Gamma(\alpha)} \int_t^\infty (r^2 - t^2)^{\alpha-1} f(r) r \, dr. \quad (2.5)
\]

We review basic facts about these integrals. More information can be found in [26, Subsection 2.6.2].

Lemma 2.1.

(i) The integral \((I_{\alpha,2}^+)f(t)\) is absolutely convergent for almost all \(t > 0\) whenever \(r \to rf(r)\) is a locally integrable function on \(\mathbb{R}_+\).

(ii) If

\[
\int_0^\infty |f(r)| r^{2\alpha-1} \, dr < \infty, \quad a > 0,
\]

then \((I_{\alpha,2}^-)f(t)\) is finite for almost all \(t > a\). If \(f\) is non-negative, locally integrable on \([a, \infty)\), and (2.6) fails, then \((I_{\alpha,2}^-)f(t) = \infty\) for every \(t \geq a\).

Fractional derivatives of the Erdélyi–Kober type are defined as the left inverses \(\mathcal{D}_{\pm,2}^\alpha = (I_{\pm,2}^\alpha)^{-1}\) and have different analytic expressions. For example, if \(\alpha = m + \alpha_0, \ m = [\alpha], \ 0 \leq \alpha_0 < 1\), then, formally,

\[
\mathcal{D}_{\pm,2}^\alpha \varphi = (\pm D)^{m+1} I_{\pm,2}^{1-\alpha_0} \varphi, \quad D = \frac{1}{2t} \frac{d}{dt}. \quad (2.7)
\]

More precisely, the following statements hold.

Theorem 2.2. Let \(\varphi = I_{\pm,2}^\alpha f\), where \(rf(r)\) is locally integrable on \(\mathbb{R}_+\). Then \(f(t) = (\mathcal{D}_{\pm,2}^\alpha \varphi)(t)\) for almost all \(t \in \mathbb{R}_+\), as in (2.7).

Theorem 2.3. If \(f\) satisfies (2.6) for every \(a > 0\) and \(\varphi = I_{-2}^\alpha f\), then \(f(t) = (\mathcal{D}_{-2}^\alpha \varphi)(t)\) for almost all \(t \in \mathbb{R}_+\), where \(\mathcal{D}_{-2}^\alpha \varphi\) can be represented as follows.

(i) If \(\alpha = m\) is an integer, then

\[
\mathcal{D}_{-2}^\alpha \varphi = (-D)^m \varphi, \quad D = \frac{1}{2t} \frac{d}{dt}. \quad (2.8)
\]
(ii) If $\alpha = m + \alpha_0$, $m = [\alpha]$, $0 < \alpha_0 < 1$, then

$$D_{-2}^{\alpha} \psi = t^2(1 - \alpha)D^{m+1}t^{2\alpha} \psi, \quad \psi = I_{-2}^{1-\alpha} t^{2m-2} \varphi. \quad (2.9)$$

In particular, for $\alpha = k/2$, $k$ odd,

$$D_{-2}^{k/2} \varphi = t (-D)^{(k+1)/2} t^{1/2} I_{-2}^{k/2} t^{-k-1} \varphi. \quad (2.10)$$

Fractional integrals and derivatives of the Erdélyi–Kober type possess the semi-group property

$$D^{\alpha \pm \beta}_{\pm 2} D^{\pm \beta}_{\pm 2} = D^{\alpha \pm \beta}_{\pm 2}, \quad I^{\alpha \pm \beta}_{\pm 2} = I^{\alpha \pm \beta}_{\pm 2} \quad (2.11)$$

in suitable classes of functions, which guarantee the existence of the corresponding expressions.

### 2.3. Radon-John $k$-Plane Transforms and Riesz Potentials

We recall some facts from [20, 24]; see also [4, 14]. The $k$-plane transform of a function $f$ on $\mathbb{R}^n$ is defined by the formula

$$(R_k f)(\zeta) = \int_{\zeta} f(x) d_\zeta x, \quad \zeta \in G(n, k), \quad 1 \leq k \leq n - 1, \quad (2.12)$$

where $d_\zeta x$ stands for the Euclidean measure on $\zeta$. Using parametrization $\zeta \equiv (\eta, v)$, $\eta \in G_{n,k}$, $v \in \eta^\perp$, we have

$$(R_k f)(\zeta) \equiv (R_k f)(\eta, v) = \int_{\eta} f(v + y) d_\eta y, \quad (2.13)$$

where $d_\eta y$ is the Euclidean volume element on $\eta$. The dual $k$-plane transform $R_k^* \varphi$ averages a function $\varphi$ on $G(n, k)$ over all $k$-planes passing through the fixed point $x \in \mathbb{R}^n$. Specifically,

$$(R_k^* \varphi)(x) = \int_{O(n)} \varphi(\gamma \eta_0 + x) d\gamma, \quad (2.14)$$

where $\eta_0$ is an arbitrary fixed $k$-plane through the origin. The duality relation

$$\int_{G(n,k)} (R_k f)(\zeta) \varphi(\zeta) d\zeta = \int_{\mathbb{R}^n} f(x) (R_k^* \varphi)(x) dx \quad (2.15)$$

holds provided that either side of this equality exists in the Lebesgue sense.

The following inequality, which is a particular case of Lemma 2.6 from [20], shows for which $f$ the integral $(R_k f)(\zeta)$ is well defined for
almost all $\zeta \in G(n, k)$. We have

$$\int_{G(n, k)} \frac{|(R_k f)(\zeta)|}{1 + |\zeta|} |\zeta|^{k-n+1} d\zeta \leq c \int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|} \log(2 + |x|) \, dx. \quad (2.16)$$

If $f$ and $\varphi$ are radial functions, then $R_k f$ and $R_k^* \varphi$ can be expressed through the Erdélyi–Kober fractional integrals as follows.

**Lemma 2.4.** (cf. [24, Lemma 3.1, Theorem 3.2], [20, Lemma 2.1])

(i) If $f(x) = f_0(|x|)$, then $(R_k f)(\zeta) = F_0(|\zeta|)$, where

$$F_0(s) = \pi^{k/2} \left( I_{\frac{k}{2}} f_0 \right)(s). \quad (2.17)$$

(ii) If $\varphi(\zeta) = \varphi_0(|\zeta|)$, then $(R_k^* \varphi)(x) = \Phi_0(|x|)$, where

$$\Phi_0(r) = \frac{\Gamma(n/2)}{\Gamma((n-k)/2)} r^{2-n} \left( I_{\frac{k}{2}}^{n-k-2} \varphi_0 \right)(r). \quad (2.18)$$

The formulas (2.17) and (2.18) hold provided that either side of the corresponding equality exists in the Lebesgue sense.

The duality (2.15) yields the following existence result.

**Lemma 2.5.** (cf. [24, Theorem 3.2])

(i) If $f$ is locally integrable in $\mathbb{R}^n \setminus \{0\}$ and satisfies

$$\int_{|x| > a} \frac{|f(x)|}{|x|^{n-k}} \, dx < \infty \quad \text{for some} \quad a > 0, \quad (2.19)$$

then $(R_k f)(\zeta)$ is finite for almost all $\zeta \in G(n, k)$. If $f$ is nonnegative, radial, and (2.19) fails, then $(R_k f)(\zeta) \equiv \infty$.

(ii) If $\varphi \in L^1_{\text{loc}}(G(n, k))$, then $R_k^* \varphi \in L^1_{\text{loc}}(\mathbb{R}^n)$.

There is a remarkable connection between the operators $R_k$, $R_k^*$ and the Riesz potential

$$(I_n^\alpha f)(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{f(y) \, dy}{|x-y|^{n-\alpha}}, \quad \gamma_n(\alpha) = \frac{2^\alpha \pi^{n/2} \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}, \quad (2.20)$$

$$\alpha > 0, \quad \alpha \neq n, n+2, n+4, \ldots .$$

If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < n/\alpha$, then $(I_n^\alpha f)(x) < \infty$ for almost all $x$, and the bounds for $p$ are sharp.

We recall that formally $I_n^\alpha f = (-\Delta_n)^{-\alpha/2} f$, where $\Delta_n$ is the Laplace operator in $\mathbb{R}^n$. The corresponding Riesz’s fractional derivative is defined as the left inverse

$$\mathbb{D}_n^\alpha = (I_n^\alpha)^{-1} \sim (-\Delta_n)^{\alpha/2} \quad (2.21)$$
and has many different analytic expressions, depending on the class of functions; see [26, Section 3.5] for details. For example, if \( \varphi = I_n^\alpha f \), \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p < n/\alpha \), then, by Theorem 3.44 from [26], \( D_n^\alpha \varphi \) can be represented as a hypersingular integral

\[
(D_n^\alpha \varphi)(x) = \frac{1}{d_{n,\ell}(\alpha)} \int_{\mathbb{R}^n} \frac{(\Delta_y^\varphi)(x)}{|y|^{n+\alpha}} dy \tag{2.22}
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{d_{n,\ell}(\alpha)} \int_{|y| > \varepsilon} \frac{(\Delta_y^\varphi)(x)}{|y|^{n+\alpha}} dy, \tag{2.23}
\]

where

\[
(\Delta_y^\varphi)(x) = \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \varphi(x - ky)
\]

is the finite difference of \( \varphi \) of order \( \ell \) with step \( y \) at the point \( x \),

\[
d_{n,\ell}(\alpha) = \frac{\pi^{n/2}}{2^{n} \Gamma((n + \alpha)/2)} \left\{ \begin{array}{ll}
\Gamma(-\alpha/2) B_l(\alpha) & \text{if } \alpha \neq 2, 4, 6, \ldots, \\
2(-1)^{\alpha/2-1} \frac{d}{d\alpha} B_l(\alpha), & \text{if } \alpha = 2, 4, 6, \ldots,
\end{array} \right.
\]

\[
B_l(\alpha) = \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} k^\alpha.
\]

The integer \( \ell \) is arbitrary with the choice \( \ell = \alpha \) if \( \alpha = 1, 3, 5, \ldots \) and any \( \ell > 2[\alpha/2] \) (the integer part of \( \alpha/2 \)), otherwise. The limit in (2.23) exists in the \( L^p \)-norm and in the almost everywhere sense. If, moreover, \( f \) is continuous, then the convergence in (2.23) is uniform on \( \mathbb{R}^n \).

The inversion formula (2.22) is non-local. If \( \alpha \) is an even integer, then a local inversion formula \(( -\Delta_n)^{\alpha/2} I_n^\alpha f = f \) is available under additional smoothness assumptions for \( f \); see [26, Theorem 3.32] for details.

The following theorem establishes remarkable connection between the Riesz potentials and the \( k \)-plane transform.

**Theorem 2.6.** For any \( 1 \leq k \leq n - 1 \),

\[
R_k^* R_k f = c_{k,n} I_n^k f, \quad c_{k,n} = \frac{2^{k} n^{k/2} \Gamma(n/2)}{\Gamma((n-k)/2)}, \tag{2.24}
\]

provided that either side of this equality exists in the Lebesgue sense.
The formula (2.24) is due to Fuglede [1]. Its derivation is straightforward and relies on the Fubini theorem. The following formulas can be proved in a similar way (cf. [26, Proposition 4.38, Lemma 4.100], [20, Section 3]):

\[ R_k I_n^\alpha f = I_n^{\alpha - k} R_k f, \quad I_n^\alpha R_k^* \varphi = R_k^* I_n^{\alpha - k} \varphi. \]  

\(2.25\)

Here \(I_n^{\alpha - k}\) stands for the Riesz potential on the \((n - k)\)-dimensional fiber of the Grassmannian bundle \(G(n, k)\). As above, it is assumed that either side of the corresponding equality exists in the Lebesgue sense. If \(f\) and \(\varphi\) are good enough, these formulas extend by analyticity to all complex \(\alpha\); cf. [19, Theorem 2.6]. It suffices to assume that \(f\) belongs to the Semyanistyi-Lizorkin space \(\Phi(\mathbb{R}^n)\) of Schwartz functions orthogonal to all polynomials and \(\varphi \in R_k(\Phi(\mathbb{R}^n))\).

**Corollary 2.7.** The inversion formula for the dual k-plane transform

\[ \varphi = c_{k,n}^{-1} R_k \mathbb{D}_n^k R_k^* \varphi, \quad c_{k,n} = \frac{2^k \pi^{k/2} \Gamma(n/2)}{\Gamma((n - k)/2)}, \]  

\(2.26\)

holds provided that \(\varphi = R_k f\) for some \(f \in L^p(\mathbb{R}^n), 1 \leq p < n/k\).

**Proof.** By (2.24),

\[ R_k \mathbb{D}_n^k R_k^* \varphi = R_k \mathbb{D}_n^k R_k^* R_k f = c_{k,n} R_k \mathbb{D}_n^k I_n^\alpha f = c_{k,n} R_k f = c_{k,n} \varphi. \]

\[\square\]

For further purposes, we also mention the following factorization formula, which connects the Riesz potential (2.20) with the \(\text{Erdélyi-Kober}\) fractional integrals. Specifically, if \(f\) is a radial function on \(\mathbb{R}^n\), \(f(x) = f_0(|x|)\), then \((I_n^\alpha f)(x) = F_0(|x|)\), where

\[ F_0(r) = 2^{-\alpha r^2 - n} (I_{+,2}^\alpha f_0 + I_{-,2}^\alpha f_0)(r); \]  

\(2.27\)

see [18], [26, formula (3.4.11)]. It is assumed that either side of (2.27) exists in the Lebesgue sense.

### 2.4. The Hyperplane Radon Transform, Its Dual, and the Funk Transform

We will need explicit relations connecting the hyperplane Radon transform, its dual, the Funk transform, and their inverses. Some of these facts are new. Others are scattered in the literature in different forms.

Every hyperplane \(\tau \in G(n, n - 1)\) can be parametrized by the pair \((\eta, t) \in S^{n-1} \times \mathbb{R}\), so that

\[ \tau = \tau(\eta, t) = \{ x \in \mathbb{R}^n : x \cdot \eta = t \}, \quad \tau(-\eta, -t) = \tau(\eta, t). \]  

\(2.28\)
The hyperplane Radon transform $g \rightarrow Rg$ takes a function $g$ on $\mathbb{R}^n$ to a function $Rg$ on $G(n, n - 1)$ (or on $\mathbb{S}^{n-1} \times \mathbb{R}$) by the formula

$$(Rg)(\tau) \equiv (Rg)(\eta, t) = \int_{\eta^\perp} g(t\eta + y) \, d_\eta y, \quad (2.29)$$

where $d_\eta y$ denotes the Euclidean measure on $\eta^\perp$. The dual transform $h \rightarrow R^*h$ averages a function $h$ on $G(n, n - 1)$ over the set of all hyperplanes passing through a fixed point $x \in \mathbb{R}^n$. Specifically,

$$(R^*h)(x) = \frac{1}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} h(\eta, x \cdot \eta) \, d\eta. \quad (2.30)$$

The Funk transform of an even function $f$ on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, $n \geq 2$, is defined by the formula

$$(Ff)(\theta) = \int_{\mathbb{S}^{n} \cap \theta^\perp} f(\sigma) \, d_\sigma \sigma, \quad \theta \in \mathbb{S}^n, \quad (2.31)$$

where $d_\sigma \sigma$ stands for the standard probability measure on the $(n-1)$-dimensional sphere $\mathbb{S}^n \cap \theta^\perp$. The integral operator $F$ is bounded from $L^1(\mathbb{S}^n)$ to $L^1(\mathbb{S}^n)$ and every integrable even function $f$ can be explicitly reconstructed from $Ff$. A variety of different inversion formulas depending on the class of functions can be found in [26, Section 5.1]; see also [4, 14]. For example, the following statement holds.

**Theorem 2.8.** (cf. [26, Theorem 5.40]) Let $\varphi = Ff$, $f \in L^1_{\text{even}}(\mathbb{S}^n)$,

$$\Phi_\theta(s) = \frac{1}{\sigma_{n-1}} \int_{\mathbb{S}^n \cap \theta^\perp} \varphi(s \sigma + \sqrt{1 - s^2} \theta) \, d\sigma, \quad -1 \leq s \leq 1. \quad (2.32)$$

Then

$$f(\theta) = \lim_{t \to 1} \left( \frac{1}{2t} \frac{\partial}{\partial t} \right)^{n-1} \left[ \frac{1}{(n-2)!} \int_0^t (t^2 - s^2)^{(n-3)/2} \Phi_\theta(s) \, s^{n-1} \, ds \right]. \quad (2.33)$$

In particular, for $n$ odd,

$$f(\theta) = \lim_{t \to 1} \pi^{1/2} \Gamma(n/2) \left( \frac{1}{2t} \frac{\partial}{\partial t} \right)^{(n-1)/2} [t^{n-2} \Phi_\theta(t)]. \quad (2.34)$$

Alternatively, for all $n \geq 2$ we have

$$f(\theta) = \lim_{t \to 1} \left( \frac{\partial}{\partial t} \right)^{n-1} \left[ \frac{1}{(n-2)!} \int_0^t (t^2 - s^2)^{(n-3)/2} \Phi_\theta(s) \, s \, ds \right]. \quad (2.35)$$
The limit in these formulas is understood in the $L^1$-norm. If $f \in C_{\text{even}}(\mathbb{S}^n)$, it can be interpreted in the sup-norm.

Numerous properties of the hyperplane Radon transform, its dual, and the Funk transform are described in the literature; see, e.g., [4, 14, 26] and references therein. Our main concern is explicit inversion formulas for these transforms under possibly minimal assumptions for functions.

We set $\mathbb{R}^n = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n,$ 

$$\mathbb{S}^n_+ = \{\theta = (\theta_1, \ldots, \theta_{n+1}) \in \mathbb{S}^n : 0 < \theta_{n+1} \leq 1\}. \quad (2.35)$$

Consider the projection map

$$\mathbb{R}^n \ni x \mapsto \theta = \frac{x + e_{n+1}}{|x + e_{n+1}|}, \quad \theta = \mu(x) \quad \theta = \mu(x). \quad (2.36)$$

for which

$$x = \mu^{-1}(\theta) = \frac{\theta'}{\theta_{n+1}}, \quad \theta' = (\theta_1, \ldots, \theta_n). \quad (2.37)$$

The map $\mu$ extends to the bijection $\tilde{\mu}$ from the affine Grassmannian $G(n, n - 1)$ onto the set

$$\tilde{\mathbb{S}}^n_+ = \{\omega = (\omega_1, \ldots, \omega_{n+1}) \in \mathbb{S}^n : 0 \leq \omega_{n+1} < 1\}. \quad (2.38)$$

cf. (2.35). Specifically, if $\tau = \tau(\eta, t) \in G(n, n - 1)$, $\eta \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, $t \geq 0$, and $\tilde{\tau}$ is the $n$-dimensional subspace containing the lifted plane $\tau + e_{n+1}$, then $\omega$ is a normal vector to $\tilde{\tau}$, so that

$$\omega \equiv \tilde{\mu}(\tau) = -\eta \cos \alpha + e_{n+1} \sin \alpha, \quad \tan \alpha = t. \quad (2.39)$$

**Theorem 2.9. Let**

$$\begin{align*}
(Af)(\tau) &= \frac{\sigma_{n-1}}{2\sqrt{1 + |\tau|^2}} f(\tilde{\mu}(\tau)), \quad \tau \in G(n, n - 1), \quad (2.40) \\
(Bg)(\theta) &= \frac{1}{|\theta_{n+1}|} g\left(\frac{\theta'}{\theta_{n+1}}\right), \quad \theta \in \mathbb{S}^n, \quad \theta_{n+1} \neq 0. \quad (2.41)
\end{align*}$$

If

$$\int_{\mathbb{R}^n} \frac{|g(x)| \, dx}{\sqrt{1 + |x|^2}} < \infty, \quad (2.42)$$

then $Bg \in L^1_{\text{even}}(\mathbb{S}^n)$ and

$$\begin{align*}
(Rg)(\tau) &= (AFBg)(\tau), \quad (2.43)
\end{align*}$$

where $F$ is the Funk transform (2.31).
Proof. We make use of Theorem 5.48 from [26], according to which
\[(Ff)(\omega) = (A_0 R B_0 f)(\omega), \quad \omega \in \tilde{S}^n_+\],
(2.44)
where the operators \(A_0\) and \(B_0\) have the form
\[(A_0 h)(\omega) = \frac{2}{\sigma_{n-1} |\omega'|} h(\tilde{\mu}^{-1}(\omega)), \quad \omega = (\omega', \omega_{n+1}), \quad \omega' = (\omega_1, \ldots, \omega_n),\]
\[(B_0 f)(x) = (1 + |x|^2)^{-n/2} f(\mu(x)), \quad x \in \mathbb{R}^n.\]
Extending (2.44) by the evenness to all \(\omega \in S^n\) and passing to inverses, we obtain
\[(Rg)(\tau) = (A_0^{-1} F B_0^{-1} g)(\tau), \quad \tau \in G(n, n-1),\]
where \(A_0^{-1}\) and \(B_0^{-1}\) are defined by the formulas
\[(A_0^{-1} f)(\tau) = \frac{\sigma_{n-1}}{2 \sqrt{1 + |\tau|^2}} f(\tilde{\mu}(\tau)) \quad (\equiv (Af)(\tau)),\]
(2.45)
(here we use (2.39), so that \(|\omega'|^2 = \cos^2 \alpha = (1 + t^2)^{-1} = (1 + |\tau|^2)^{-1})),
\[(B_0^{-1} g)(\theta) = \frac{1}{|\theta_{n+1}|} g \left( \frac{\theta'}{\theta_{n+1}} \right) \quad (\equiv (Bg)(\theta)),\]
(2.46)
(here we use (2.37) and extend the right-hand side as an even function of \(\theta\)). This gives (2.43).

The relation \(B_0 g \in L^1_{\text{even}}(S^n)\) that guarantees the existence of the Funk transform in (2.43) and justifies the above reasoning, is a consequence of (2.42). Indeed, using, e.g., [26, formula (1.12.17)], and assuming \(g \geq 0\), we have
\[\int_{S^n} (Bg)(\theta) d\theta = 2 \int_{S^n} g \left( \frac{\theta'}{\theta_{n+1}} \right) \frac{d\theta}{\theta_{n+1}} = 2 \int_{\mathbb{R}^n} \frac{g(x) dx}{\sqrt{1 + |x|^2}} < \infty.\]
\[\square\]

Corollary 2.10. If \(g\) satisfies (2.42), then the Radon transform \(h(\tau) = (Rg)(\tau)\) exists in the Lebesgue sense for almost all \(\tau \in G(n, n-1)\) and can be inverted by the formula
\[g(x) = (B^{-1} F^{-1} A^{-1} h)(x),\]
(2.47)
where the inverse Funk transform \(F^{-1}\) is evaluated by Theorem 2.8 and the operators \(A^{-1}\) and \(B^{-1}\) are defined by the formulas
\[(A^{-1} h)(\omega) = \frac{2}{\sigma_{n-1} |\omega'|} h(\tilde{\mu}^{-1}(\omega)), \quad |\omega'| \neq 0,\]
(2.48)
\[(B^{-1} f)(x) = (1 + |x|^2)^{-n/2} f(\mu(x)).\]
(2.49)
Our next aim is to obtain similar statements for the dual Radon transform (2.30).

**Theorem 2.11.** Let
\[
(A^*f)(x) = \frac{1}{\sqrt{1 + |x|^2}} f\left(\frac{-x + e_{n+1}}{\sqrt{1 + |x|^2}}\right), \quad x \in \mathbb{R}^n, \tag{2.50}
\]
\[
(B^*h)(\theta) = \frac{1}{|\theta'|^n} h\left(\frac{\theta'}{|\theta'|}, \frac{\theta_{n+1}}{|\theta'|}\right), \quad \theta \in \mathbb{S}^n, \quad |\theta'| \neq 0. \tag{2.51}
\]

If
\[
\int_{G(n,n-1)} \frac{|h(\tau)| \, d\tau}{\sqrt{1 + |\tau|^2}} < \infty, \tag{2.52}
\]
then $B^*h \in L^1_{\text{even}}(\mathbb{S}^n)$ and
\[
(R^*h)(x) = (A^*FB^*h)(x), \tag{2.53}
\]
where $F$ is the Funk transform (2.31).

**Proof.** We make use Lemma 4.16 from [26], according to which
\[
(R^*h)(x) = (\tilde{A}R\tilde{B}h)(x), \tag{2.54}
\]
\[
(\tilde{A}h)(x) = \frac{2}{|x| \sigma_{n-1}} \tilde{h}\left(\frac{x}{|x|}, \frac{1}{|x|}\right), \tag{2.55}
\]
\[
(\tilde{B}h)(x) = \frac{1}{|x|^n} h\left(\frac{x}{|x|}, \frac{1}{|x|}\right), \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{2.56}
\]

Here $\tilde{h}$ is a function on $G(n,n-1)$ parametrized by the pair $(\eta,t) \in \mathbb{S}^{n-1} \times \mathbb{R}_+$ and extended to $G(n,n-1) \sim \mathbb{S}^{n-1} \times \mathbb{R}$, so that $\tilde{h}(-\eta,-t) = \tilde{h}(\eta,t)$. In our case, $\eta = x/|x|$, $t = 1/|x|$. Combining (2.54) with (2.43), we obtain $R^* = AAFFBB$. This formally gives (2.53) with $A^* = \tilde{A}A$, $B^* = \tilde{B}B$.

Let us write $\tilde{A}A$ and $\tilde{B}B$ in the desired form. By (2.55) and (2.40),
\[
(A^*_f)(x) = (\tilde{A}Af)(x) = \frac{2}{|x| \sigma_{n-1}} (Af)\left(\frac{x}{|x|}, \frac{1}{|x|}\right) = \frac{1}{\sqrt{1 + |x|^2}} f(\tilde{\mu}(\tau)),
\]
where $\tau = \tau(\eta,t)$ with $\eta = x/|x|$, $t = 1/|x|$. By (2.39) with $\tan \alpha = t = 1/|x|$, we have
\[
\tilde{\mu}(\tau) = -\eta \cos \alpha + e_{n+1} \sin \alpha = \frac{-x + e_{n+1}}{\sqrt{1 + |x|^2}}.
\]
This gives (2.50). Further, by (2.41) and (2.56),
\[ (B_*h)(\theta) = (B\tilde{B}h)(\theta) = \frac{1}{|\theta_{n+1}|^n} (\tilde{B}h) \left( \frac{\theta'}{\theta_{n+1}} \right) = \frac{1}{|\theta'|^n} h \left( \frac{\theta'}{|\theta'|}, \frac{\theta_{n+1}}{|\theta'|} \right) \]
provided that \( \theta_{n+1} > 0 \) and \( |\theta'| \neq 0 \). Here \( h(\cdot, \cdot) \equiv h(\eta, t) \) with \( \eta = \theta' / |\theta'| \), \( t = \theta_{n+1} / |\theta'| \). Because \( h(\eta, t) = h(-\eta, -t) \), the last expression gives (2.51).

To complete the proof, it remains to show that \( g = \tilde{B}h \) satisfies (2.42). For \( h \geq 0 \), passing to polar coordinated and changing variables, we have
\[
\int_{\mathbb{R}^n} \frac{(\tilde{B}h)(x)}{\sqrt{1 + |x|^2}} \, dx = \int_{\mathbb{R}^n} \frac{1}{|x|^n} h \left( \frac{x}{|x|} \right) \frac{1}{\sqrt{1 + |x|^2}} \, dx = \int_0^\infty \frac{dr}{r \sqrt{1 + r^2}} \int_{G(n,n-1)} h(\eta, 1/r) \, d\eta = \int_0^\infty \frac{dt}{\sqrt{1 + t^2}} \int_{G(n,n-1)} h(\eta, t) \, d\eta
\]
\[
= \int_{G(n,n-1)} \frac{h(\tau)}{\sqrt{1 + |\tau|^2}} \, d\tau < \infty.
\]

\[ \square \]

**Corollary 2.12.** If \( h \) satisfies (2.52), then the dual Radon transform \( g(x) = (R^*h)(x) \) can be inverted by the formula
\[ h(\tau) = (B_*^{-1} F^{-1} A_*^{-1} g)(\tau), \quad \tau \in G(n, n-1), \quad (2.57) \]
where the inverse Funk transform \( F^{-1} \) is evaluated by Theorem 2.8 and the operators \( A_*^{-1} \) and \( B_*^{-1} \) are defined by the formulas
\[ (A_*^{-1} g)(\omega) = \frac{1}{|\theta_{n+1}|} g \left( -\frac{\theta'}{\theta_{n+1}} \right), \quad \theta \in S^n, \quad \theta_{n+1} \neq 0, \quad (2.58) \]
\[ (B_*^{-1} f)(\tau) = (1 + |\tau|^2)^{-n/2} f(\tilde{\mu}(-\tau)), \quad -\tau = \{ x \in \mathbb{R}^n : -x \in \tau \}. \quad (2.59) \]

The analytic expressions for \( A_*^{-1} \) and \( B_*^{-1} \) are consequences of the formulas (2.50) and (2.51) for \( A_* \) and \( B_* \) and the definitions of the maps \( \mu \) and \( \tilde{\mu} \).

**Remark 2.13.** If \( h \) is infinitely differentiable and rapidly decreasing, then the inversion formula for \( R^* \) can be written in the form (2.26). Specifically,
\[ h = c_n R \mathbb{D}_n^{-1} R^* h, \quad c_n = 2^{1-n} \pi^{1-n/2} / \Gamma(n/2). \quad (2.60) \]
Here \( \mathbb{D}_n^{-1} = (-\Delta_n)^{(n-1)/2} \) if \( n \) is odd. If \( n \) is even, then \( \mathbb{D}_n^{-1} = \Lambda_n^{-1} \), where \( \Lambda = \sum_{j=1}^n H_j \partial_j \) is the Calderon-Zygmund operator, \( H_j \), being some singular integral operators, called the Riesz transforms. In
this case, $D_n^{-1}R^*h \in C^\infty(\mathbb{R}^n)$ and has the order $O(|x|^{-n})$; see Solomon [31, p. 340] for details. The modification of (2.60) in the form $h = c_n D_n^{-1}R^*h$ was proved by Helgason [13, Theorem 4.5] in the framework of the Semyanistyi-Lizorkin spaces of Schwartz functions orthogonal to polynomials; see also [26, formula (4.6.38)]. When $n$ is odd, the formula (2.60) was obtained by Gonzalez in [7] (see also [8, Theorem 2.1]). More information on this subject, including further references, can be found in [26, p. 275, Notes 4.6].

Our formula (2.57) does not contain singular integral operators and is applicable to a much larger class of functions.

3. Mixed $j$-Plane to $k$-Plane Radon Transforms

3.1. Setting of the Problem. Let $\tau \in G(n, j), \zeta \in G(n, k)$, so that

\[ \tau \equiv \tau(\xi, u) = \xi + u, \quad \xi \in G_{n,j}, \quad u \in \xi^\perp, \]

\[ \zeta \equiv \zeta(\eta, v) = \eta + v, \quad \eta \in G_{n,k}, \quad v \in \eta^\perp. \]

The planes $\tau(\xi, u)$ and $\zeta(\eta, v)$ are called perpendicular (we write $\tau \perp \zeta$) if $a \cdot b = 0$ for all $a \in \xi$ and $b \in \eta$. We define the set of incidence

\[ \mathcal{I} = \{ (\tau, \zeta) \in G(n, j) \times G(n, k) : \tau \perp \zeta, \quad \tau \cap \zeta \neq \emptyset \} \]

and denote

\[ \hat{\mathcal{I}} = \{ \tau \in G(n, j) : (\tau, \zeta) \in \mathcal{I} \}, \quad \check{\mathcal{I}} = \{ \zeta \in G(n, k) : (\tau, \zeta) \in \mathcal{I} \}. \]

The corresponding mixed $j$-plane to $k$-plane Radon transform has the form

\[ (R_{j,k}f)(\zeta) = \int_{\hat{\mathcal{I}}} f(\tau) d\zeta, \quad (R_{k,j}\varphi)(\tau) = \int_{\check{\mathcal{I}}} \varphi(\zeta) d\tau. \quad (3.1) \]

Remark 3.1. It is clear that if $(\tau, \zeta) \in \mathcal{I}$, then, necessarily, $j + k \leq n$, because otherwise, there exist $a \in \xi$ and $b \in \eta$ such that $a \cdot b \neq 0$. If $j + k = n$, then $\xi = \eta^\perp$ and

\[ (R_{j,k}f)(\zeta) \equiv (R_{j,k}f)(\eta, v) = \int_{\eta} f(\eta^\perp + u) d\eta u \quad \forall v \in \eta^\perp. \]

This integral operator is in general non-injective. Indeed, if $f$ is a radial function, that is, $f(\xi, u) \equiv \tilde{f}(|u|)$ for some single-variable function $\tilde{f}$, then $R_{j,k}f \equiv \text{const}$ on the set of all functions $f$ of the form $f(\xi, u) =$
\( \tilde{f}_\lambda(|u|) = \lambda^{-k} \tilde{f}(|u|/\lambda), \lambda > 0 \). Specifically, by rotation and dilation invariance,

\[
(\mathcal{R}_{j,k}f)(\zeta) = \int_{\mathbb{R}^k} \tilde{f}_\lambda(|u|) \, du = \int_{\mathbb{R}^k} \tilde{f}_\lambda(|u|) \, du = \int_{\mathbb{R}^k} \tilde{f}(|u|) \, du \equiv \text{const},
\]

\( \mathbb{R}^k = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_k \), provided that this integral converges. Because of the lack of injectivity, the case \( j + k = n \) will be excluded from our consideration and we always assume \( j + k < n \).

The operators (3.1) can be explicitly written as

\[
(\mathcal{R}_{j,k}f)(\zeta) \equiv (\mathcal{R}_{j,k}f)(\eta,v) = \int_{G_j(\eta^\perp)} d_{\eta^\perp} \xi \int f(\xi + u + v) \, d\eta u, \quad (3.2)
\]

\[
(\mathcal{R}_{k,j}\varphi)(\tau) = (\mathcal{R}_{k,j}\varphi)(\xi,u) = \int_{G_k(\xi^\perp)} d_{\xi^\perp} \eta \int f(\eta + u + v) \, d\xi v. \quad (3.3)
\]

Here \( d_{\eta^\perp} \xi \) and \( d_{\xi^\perp} \eta \) denote the canonical probability measures on the corresponding Grassmannians \( G_j(\eta^\perp) \) and \( G_k(\xi^\perp) \), \( d\eta u \) and \( d\xi v \) stand for the Euclidean measures on \( \eta \) and \( \xi \), respectively.

If \( j = 0 \) and \( k \geq 1 \), then \( \mathcal{R}_{j,k} \) is the Radon-John transform (2.13). If \( j \geq 1 \) and \( k = 0 \), then \( \mathcal{R}_{j,k} \) is the dual \( j \)-plane transform; cf. (2.14) with \( k \) replaced by \( j \).

Our aim is to investigate the operators (3.2) and (3.3) under the assumption

\[
j > 0, \quad k > 0, \quad j + k < n.
\]

3.2. Duality.

Lemma 3.2. The duality relation

\[
\int_{G(n,k)} (\mathcal{R}_{j,k}f)(\zeta) \varphi(\zeta) \, d\zeta = \int_{G(n,j)} f(\tau)(\mathcal{R}_{k,j}\varphi)(\tau) \, d\tau \quad (3.4)
\]

holds provided that either integral exists in the Lebesgue sense.

Proof. Let \( I_r \) and \( I_l \) denote the right-hand side and the left-hand side of (3.4), respectively;

\[
\mathbb{E}_j = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_j, \quad \mathbb{E}_k = \mathbb{R}e_{n-k+1} \oplus \cdots \oplus \mathbb{R}e_n. \quad (3.5)
\]
We set
\[ I = \int_{M(n)} f(gE_j) \varphi(gE_k) \, dg \]
\[ = \int_{O(n)} \int_{\mathbb{R}^n} f(a + \rho E_j) \varphi(a + \rho E_k) \, da. \] (3.6)

It suffices to show that \( I = I_r = I_l \). Let \( O_j \) and \( O_k \) be the stationary subgroups of \( O(n) \) at \( E_j \) and \( E_k \), respectively. We replace \( \rho \) by \( \rho \rho_j \) (\( \rho_j \in O_j \)), then integrate in \( \rho_j \in O_j \), and, after that, replace \( a \) by \( \rho a \). Changing the order of integration and taking into account that \( \rho_j E_j = E_j \), we obtain
\[ I = \int_{O(n)} \int_{O_j} \int_{\mathbb{R}^n} f(\rho(a + E_j)) \varphi(\rho(a + \rho_j E_k)) \, da. \] (3.7)

Then we set \( a = t + b \), where \( t \in E_j \), \( b \in E_j^\perp \). Because \( t + E_j = E_j \), (3.7) gives
\[ I = \int d\rho \int_{E_j^\perp} f(\rho(b + E_j)) \, db \int_{E_j} \varphi(\rho b + \rho(t + \rho_j E_k)) \, d\rho \]
\[ = \int_{O(n)} \int_{E_j^\perp} f(\rho(b + E_j)) (R_{k,j} \varphi)(\rho(b + E_j)) \, db = I_r. \]

The proof of \( I = I_l \) is similar. \( \square \)

4. Mixed Radon Transforms of Radial Functions

We recall that a function \( f \) on \( G(n, j) \) is radial, if there is a function \( f_0 \) on \( \mathbb{R}^+ \), such that \( f(\tau) = f_0(|\tau|) \). If \( f \) is radial, then, by (3.2),
\[(R_{j,k} f)(\eta, v) = \int_{G_k(\eta^\perp)} d\eta \cdot \xi \int_{G_j(\eta^\perp)} f_0(|u + Pr_{\xi^\perp}v|) \, du, \] (4.1)

where \( Pr_{\xi^\perp}v \) denotes the orthogonal projection of \( v \) onto \( \xi^\perp \). The right-hand side of this equality is the \( k \)-plane transform of a radial function restricted to the \((n-j)\)-dimensional subspace \( \xi^\perp \) combined with the dual \( j \)-plane transform restricted to the \((n-k)\)-dimensional space \( \eta^\perp \).

**Lemma 4.1.** If \( f(\tau) \equiv f_0(|\tau|) \) satisfies the conditions
\[ \int_0^a |f_0(t)| \, t^{n-j-1} \, dt < \infty \quad \text{and} \quad \int_a^\infty |f_0(t)| \, t^{k-1} \, dt < \infty \] (4.2)
for some \( a > 0 \), then

\[
(R_{j,k}f)(\zeta) = (I_{j,k}f_0)(|\zeta|), \tag{4.3}
\]

where

\[
(I_{j,k}f_0)(s) = \frac{c_1}{s^{n-k-2}} \int_0^s (s^2-r^2)^{j/2-1} r^{\ell-1} dr \int f_0(t) (t^2-r^2)^{k/2-1} dt
\]

\[
= \frac{\tilde{c}_1}{s^{n-k-2}} (I_{j+2}^{\ell-2} f_{-2}^{k/2} f_0)(s), \quad l = n - j - k \geq 1, \tag{4.4}
\]

\[
c_1 = \frac{\sigma_{j-1} \sigma_{k-1} \sigma_{\ell-1}}{\sigma_{n-k-1}}, \quad \tilde{c}_1 = \frac{\pi^{k/2} \Gamma((n-k)/2)}{\Gamma(\ell/2)}. \tag{4.5}
\]

Moreover,

\[
\int_\alpha^\beta |(I_{j,k}f_0)(s)| ds < \infty \quad \text{for all} \quad 0 < \alpha < \beta < \infty. \tag{4.6}
\]

\textbf{Proof.} It is straightforward to show that the assumptions in (4.2) imply (4.6); see Appendix. Then \((I_{j,k}f_0)(s) < \infty\) for almost all \( s > 0 \), and therefore the proof of (4.3) presented below is well-justified.

We transform the integral

\[
(R_{j,k}f)(\zeta) \equiv (R_{j,k}f)(\eta, v) = \int_{G_j(\eta^+)} d_\eta^+ \xi \int f(\xi + u + v) d_\eta u
\]

by changing variable \( \xi = \gamma \xi' \), where \( \gamma \in O(n) \) is an orthogonal transformation satisfying \( \gamma \mathbb{E}_k = \eta \). Then, by (2.3),

\[
\xi' \subset \mathbb{E}_k^\perp = \mathbb{R}^{n-k} = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{n-k},
\]
so that \( \xi' \in G_{n-k,j} = G_j(\mathbb{R}^{n-k}) \). We also set \( u = \gamma u', v = \gamma v' \), where \( u' \in E_k, v' \in E_k^\perp = \mathbb{R}^{n-k} \). This gives

\[
(\mathcal{R}_{j,k} f)(\eta, v) = \int_{G_j(\mathbb{R}^{n-k})} \int_{\mathbb{R}_v} \int_{\mathbb{R}_u} (f \circ \gamma)(\xi' + u' + v') \, du'\, dv'\, d\xi',
\]

where \( \mathcal{R}_{j,k} f \) denotes the orthogonal projection of \( v' \) onto \( \xi'^\perp \). We set \( \xi' = \alpha E_j, \alpha \in O(n-k) \). Then (cf. (2.3))

\[
\xi'^\perp = \alpha E_j^\perp, \quad E_j^\perp = \mathbb{R}^{n-j} = \mathbb{R}e_{j+1} \oplus \cdots \oplus \mathbb{R}e_n,
\]

and therefore

\[
(\mathcal{R}_{j,k} f)(\eta, v) = \sigma_{k-1} \int_0^\infty r^{k-1} \, dr \int_{O(n-k)} \int_{\mathbb{R}_v} f_0(\sqrt{r^2 + |\text{Pr}_{E^\perp} v'|^2}) \, d\alpha.
\]

Because \( v' \in \mathbb{R}^{n-k} = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{n-k} \), we have

\[
\text{Pr}_{\alpha \mathbb{R}^{n-j}} v' = \text{Pr}_{\alpha \mathbb{R}^{n-j} \cap \mathbb{R}^{n-k}} v' = \text{Pr}_{\alpha(\mathbb{R}^{n-j} \cap \mathbb{R}^{n-k})} v' = \text{Pr}_{\alpha \mathbb{E}_j} v',
\]

where

\[
\mathbb{E}_\ell = \mathbb{R}e_{j+1} \oplus \cdots \oplus \mathbb{R}e_{n-k}, \quad \ell = n-k-j.
\]

Thus

\[
(\mathcal{R}_{j,k} f)(\eta, v) = \sigma_{k-1} \int_0^\infty r^{k-1} \, dr \int_{O(n-k)} \int_{\mathbb{R}_v} f_0(\sqrt{r^2 + |\text{Pr}_{\mathbb{E}_j} v'|^2}) \, d\alpha.
\]

Keeping in mind that \( |\text{Pr}_{\mathbb{E}_j} v'| = |\text{Pr}_{\mathbb{E}_j} \alpha^{-1} v'| \) and setting \( s = |v| \), we can write the last expression as

\[
(\mathcal{R}_{j,k} f)(\eta, v) = \frac{\sigma_{k-1}}{\sigma_{n-k-1}} \int_0^\infty r^{k-1} \, dr \int_{\mathbb{S}^{n-k-1}} f_0(\sqrt{r^2 + s^2|\text{Pr}_{\mathbb{E}_j} v'|^2}) \, d\theta,
\]
where $S^{n-k-1}$ is the unit sphere in $\mathbb{R}^{n-k}$. The inner integral can be transformed by making use of the bi-spherical coordinates
\[ \theta = a \cos \psi + b \sin \psi, \quad a \in S^{n-1} \cap \mathbb{E}, \quad b \in S^{n-1} \cap \mathbb{E}_j, \quad 0 < \psi < \pi/2, \]
\[ d\theta = \sin^{j-1} \psi \cos^{\ell-1} \psi \, da \, db \, d\psi, \]
see, e.g., [26, p. 31]. Setting $c_1 = \sigma_{k-1} \sigma_{\ell-1} \sigma_{j-1}/\sigma_{n-k-1}$, we obtain
\[
(\mathcal{R}_{j,k} f)(\eta, v) = c_1 \int_0^\infty r^{k-1}dr \int_0^{\pi/2} f_0 \left( \sqrt{r^2 + s^2 \cos^2 \psi} \right) \sin^{j-1} \psi \cos^{\ell-1} \psi \, d\psi
\]
\[
= c_1 \int_0^\infty r^{k-1}dr \int_0^1 f_0 \left( \sqrt{r^2 + s^2 \lambda^2} \right) \left( 1 - \lambda^2 \right)^{j/2-1} \lambda^{\ell-1} \, d\lambda
\]
\[
= \frac{c_1}{s^{n-k-2}} \int_0^\infty r^{k-1}dr \int_0^s f_0 \left( \sqrt{r^2 + t^2} \right) \left( s^2 - t^2 \right)^{j/2-1} t^{\ell-1} \, dt
\]
\[
= \frac{c_1}{s^{n-k-2}} \int_0^s \left( s^2 - r^2 \right)^{j/2-1} r^{\ell-1}dr \int_r^\infty f_0(t) \left( t^2 - r^2 \right)^{k/2-1} t \, dt.
\]
\[ \Box \]

The following analogue of Lemma 4.1 for the dual transform $\mathcal{R}_{k,j} \varphi$ follows from Lemma 4.1 by the symmetry.

**Lemma 4.2.** If $\varphi(\zeta) \equiv \varphi_0(|\zeta|)$ satisfies the conditions
\[
\int_0^a |\varphi_0(s)| s^{n-k-1}ds < \infty \quad \text{and} \quad \int_a^\infty |\varphi_0(s)| s^{j-1}ds < \infty \quad (4.8)
\]
for some $a > 0$, then
\[
(\mathcal{R}_{k,j} \varphi)(\tau) = (I_{j,k}^* \varphi_0)(|\tau|), \quad (4.9)
\]
where
\[
(I_{j,k}^* \varphi_0)(t) = \frac{c_2}{t^{n-j-2}} \int_0^t (t^2 - r^2)^{k/2-1} r^{\ell-1}dr \int_r^\infty \varphi_0(s)(s^2 - r^2)^{j/2-1} s \, ds
\]
\[
= \frac{\tilde{c}_2}{t^{n-j-2}} (I_{j,k/2}^{1/2} \varphi_0)(t), \quad l = n - j - k \geq 1, (4.10)
\]
\[
c_2 = \frac{\sigma_{j-1} \sigma_{k-1} \sigma_{\ell-1}}{\sigma_{n-j-1}}, \quad \tilde{c}_2 = \frac{\pi j/2 \Gamma((n-j)/2)}{\Gamma(\ell/2)}. \quad (4.11)
\]
Moreover,
\[
\int_{\alpha}^{\beta} |(I_{j,k}^* \varphi_0)(t)| \, dt < \infty \quad \text{for all } \ 0 < \alpha < \beta < \infty.
\]  
(4.12)

Remark 4.3. By Lemma 2.1 (ii), the finiteness of the second integrals in (4.2) and (4.8) is necessary for the existence of the corresponding integrals \((I_{j,k} f_0)(s)\) and \((I_{j,k}^* \varphi_0)(t)\).

Example 4.4. The following formulas can be easily obtained from (4.4) and (4.10) using tables of integrals (see, e.g., [9]):

(i) If \(f(\tau) = |\tau|^{-\lambda}, \ k < \lambda < n - j\), then \((R_{j,k} f)(\zeta) = c_{j,k} |\zeta|^{k-\lambda}\), where
\[
c_{j,k} = \frac{\pi^{k/2} \Gamma \left(\frac{n-k}{2}\right) \Gamma \left(\frac{j-k}{2}\right) \Gamma \left(\frac{n-j}{2}\right)}{\Gamma \left(\frac{n-j-k}{2}\right) \Gamma \left(\frac{j}{2}\right) \Gamma \left(\frac{n}{2}\right)}.
\]  
(4.13)

(ii) If \(\varphi(\zeta) = |\zeta|^{-\lambda}, \ j < \lambda < n - k\), then \((R_{k,j} \varphi)(\tau) = c_{k,j} |\tau|^{j-\lambda}\), where
\[
c_{k,j} = \frac{\pi^{j/2} \Gamma \left(\frac{n-j}{2}\right) \Gamma \left(\frac{\lambda-j}{2}\right) \Gamma \left(\frac{n-\lambda}{2}\right)}{\Gamma \left(\frac{n-j-k}{2}\right) \Gamma \left(\frac{j}{2}\right) \Gamma \left(\frac{n}{2}\right)}.
\]  
(4.14)

(iii) If \(f(\tau) = (1 + |\tau|^2)^{-n/2}\), then \((R_{j,k} f)(\zeta) = c_k (1 + |\zeta|^2)^{(j-k-n)/2}\), where
\[
c_k = \frac{\pi^{k/2} \Gamma \left(\frac{n-k}{2}\right)}{\Gamma \left(\frac{n}{2}\right)}.
\]  
(4.15)

(iv) If \(\varphi(\zeta) = (1 + |\zeta|^2)^{-n/2}\), then \((R_{k,j} \varphi)(\tau) = c_j (1 + |\tau|^2)^{(j-k-n)/2}\), where
\[
c_j = \frac{\pi^{j/2} \Gamma \left(\frac{n-j}{2}\right)}{\Gamma \left(\frac{n}{2}\right)}.
\]  
(4.16)

5. Existence of the Mixed Radon Transforms

Example 4.4 in conjunction with duality (3.4) gives information about the existence in the Lebesgue sense of the corresponding Radon transforms \(R_{j,k} f\) and \(R_{k,j} \varphi\).

Theorem 5.1. The following formulas hold provided that the integral in either side of the corresponding equality exists in the Lebesgue sense.

\[
\int_{G(n,k)} (R_{j,k} f)(\zeta) \, d\zeta = c_{k,j} \int_{G(n,j)} f(\tau) \, |\tau|^{\lambda-j} \, d\tau, \quad j < \lambda < n - k;
\]  
(5.1)

\[
\int_{G(n,k)} (R_{j,k} f)(\zeta) \, d\zeta = c_j \int_{G(n,j)} f(\tau) \, \frac{1}{(1 + |\tau|^2)^{(n-j-k)/2}} \, d\tau;
\]  
(5.2)
\[ \int_{G(n,j)} \frac{(\mathcal{R}_{k,j}\varphi)(\tau)}{|\tau|^\lambda} d\tau = c_{j,k} \int_{G(n,k)} \frac{\varphi(\zeta)}{|\zeta|^{\lambda-k}} d\zeta, \quad k < \lambda < n-j; \quad (5.3) \]

\[ \int_{G(n,j)} \frac{(\mathcal{R}_{k,j}\varphi)(\tau)}{(1 + |\tau|^2)^{n/2}} d\tau = c_k \int_{G(n,k)} \frac{\varphi(\zeta)}{(1 + |\zeta|^2)^{(n-j-k)/2}} d\zeta. \quad (5.4) \]

Here \( c_{k,j}, c_j, c_{j,k}, \) and \( c_k \) have the same meaning as in Example 4.4.

The next theorems characterize the existence of the Radon transforms \( \mathcal{R}_{j,k}f \) and \( \mathcal{R}_{k,j}\varphi \) in different terms.

**Theorem 5.2.** Let \( j + k < n \). If \( f \) is a locally integrable function on \( G(n,j) \) satisfying

\[ \int_{|\tau|>a} |f(\tau)| |\tau|^{k-j-n} d\tau < \infty \quad (5.5) \]

for some \( a > 0 \), then \( (\mathcal{R}_{j,k}f)(\zeta) \) is finite for almost all \( \zeta \in G(n,k) \). If for a nonnegative, radial, locally integrable function \( f \), the condition (5.5) fails, then \( (\mathcal{R}_{j,k}f)(\zeta) = \infty \) for all \( \zeta \in G(n,k) \).

**Proof.** It suffices to show that

\[ I_{\alpha,\beta} \equiv \int_{\alpha<|\zeta|<\beta} |(\mathcal{R}_{j,k}f)(\zeta)| d\zeta < \infty \quad \text{for all} \quad 0 < \alpha < \beta < \infty. \quad (5.6) \]

Because \( \mathcal{R}_{j,k} \) commutes with orthogonal transformations,

\[ I_{\alpha,\beta} \equiv \int_{\mathcal{O}(n)} d\gamma \int_{\alpha<|\zeta|<\beta} |(\mathcal{R}_{j,k}\tilde{f})(\gamma\zeta)| d\zeta \leq \int_{\alpha<|\zeta|<\beta} (\mathcal{R}_{j,k}\tilde{f})(\zeta) d\zeta, \]
where \( \tilde{f}(\tau) = \int_{O(n)} |f(\gamma \tau)| \, d\gamma \) is a radial function. We set \( \tilde{f}(\tau) = f_0(|\tau|) \). The assumptions for \( f \) in the lemma imply (4.2) for \( f_0 \). Indeed,

\[
\int_0^a f_0(t) t^{n-j-1} \, dt = \int_0^a t^{n-j-1} \, dt \int_{O(n)} |f(\mathbb{E}_j + te_{j+1})| \, d\gamma
\]

\[
= \int_0^a t^{n-j-1} \, dt \int_{O(n-j)} d\omega \int_{O(n)} |f(\mathbb{E}_j + t\omega e_{j+1})| \, d\gamma
\]

\[
= \frac{1}{\sigma_{n-j-1}} \int_0^a t^{n-j-1} \, dt \int_{O(n)} d\gamma \int_{\mathbb{R}^{n-j-1}} |f(\mathbb{E}_j + t\gamma)| \, d\gamma
\]

\[
= \frac{1}{\sigma_{n-j-1}} \int_{\mathbb{R}^n} d\gamma \int_{y \in \mathbb{R}^n} |f(\mathbb{E}_j + y)| \, dy
\]

\[
= \frac{1}{\sigma_{n-j-1}} \int_{\mathbb{R}^n} d\xi \int_{u \in \mathbb{R}^n} |f(\xi + u)| \, dy = \frac{1}{\sigma_{n-j-1}} \int_{|\tau| < a} |f(\tau)| \, d\tau < \infty,
\]

because \( f \) is locally integrable. Similarly,

\[
\int_0^\infty f_0(t) t^{k-1} \, dt = \frac{1}{\sigma_{n-j-1}} \int_{|\tau| > a} |f(\tau)| |\tau|^{k+j-n} \, d\tau < \infty.
\]

Hence, by (4.3) and (4.6),

\[
I_{\alpha,\beta} \leq \int_{\alpha < |\zeta| < \beta} (\mathcal{R}_{j,k,\tilde{f}}(\zeta)) \, d\zeta = \int_{\alpha < |\zeta| < \beta} (I_{j,k,f_0})(|\zeta|) \, d\zeta
\]

\[
= \sigma_{n-j-1} \int_{\alpha}^\beta (I_{j,k,f_0})(s) s^{n-j-1} \, ds \leq c \int_{\alpha}^\beta (I_{j,k,f_0})(s) \, ds < \infty,
\]

as desired.

To complete the proof, suppose that (5.5) fails for some nonnegative, radial, locally integrable function \( f \). If \( f(\tau) \equiv f_0(|\tau|) \) then the inner
integral in (3.2) becomes
\[ I(\xi, \eta, v) \equiv \int_\eta f(\xi + u + v) \, d_\eta u = \int_\eta f_0 \left( \sqrt{|u|^2 + |\text{Pr}_{\xi} v|^2} \right) \, d_\eta u \]
\[ = \sigma_{k-1} \int_s^\infty f_0(t)(t^2 - s^2)^{k/2-1} \, dt, \quad s = |\text{Pr}_{\xi} v|. \]

The condition (5.5) is equivalent to
\[ \int_a^\infty f_0(t)t^{k-1} \, dt = \infty \quad \text{for all} \quad a > 0. \]

By Lemma 2.1(ii), it follows that \( I(\xi, \eta, v) = \infty \) for all triples \((\xi, \eta, v)\), and therefore \((R_{j,k} f)(\zeta) = \infty \) for all \( \zeta \in G(n,k) \). \(\square\)

The following statement is an analogue of Theorem 5.2 for the dual transform \( R_{k,j} \varphi \) and holds by the symmetry.

**Theorem 5.3.** Let \( j + k < n \). If \( \varphi \) is a locally integrable function on \( G(n,k) \) satisfying
\[ \int_{|\zeta|>a} |\varphi(\zeta)| |\zeta|^{k+j-n} \, d\zeta < \infty \quad (5.7) \]
for some \( a > 0 \), then \((R_{k,j} \varphi)(\tau)\) is finite for almost all \( \tau \in G(n,j) \). If for a nonnegative, radial, locally integrable function \( \varphi \), the condition (5.7) fails, then \((R_{k,j} \varphi)(\tau) = \infty \) for all \( \tau \in G(n,j) \).

**Corollary 5.4.**

(i) If \( f \in L^p(G(n,j)) \), \( 1 \leq p < (n-j)/k \), then \((R_{j,k} f)(\zeta)\) is finite for almost all \( \zeta \in G(n,k) \).

(ii) If \( \varphi \in L^q(G(n,k)) \), \( 1 \leq q < (n-k)/j \), then \((R_{k,j} \varphi)(\tau)\) is finite for almost all \( \tau \in G(n,j) \).

The bounds \( p < (n-j)/k \) and \( q < (n-k)/j \) in these statements are sharp.

**Proof.** (i) By Theorem 5.2, it suffices to check (5.5). For any \( a > 0 \), the Hölder inequality yields
\[ \int_{|\tau|>a} |f(\tau)| |\tau|^{k+j-n} \, d\tau \leq ||f||_p \left( \int_{|\tau|>a} |\tau|^{(k+j-n)p'} \, d\tau \right)^{1/p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \]
Because the integral in bracket is finite whenever $1 \leq p < (n-j)/k$, the result follows. The proof of (ii) is similar. If $p \geq (n-j)/k$, the function
\[
f(\tau) = (2 + |\tau|)^{(j-n)/p}(\log(2 + |\tau|))^{-1}
\]
provides a counter-example. Indeed, this function belongs to $L^p(G(n,j))$ and does not obey (5.5). The case $q \geq (n-k)/j$ is similar. □

Remark 5.5. It is interesting to note that the same bounds for $p$ and $q$ can be obtained from (5.2) and (5.4) if we apply Hölder's inequality to the right-hand sides.

6. MIXED RADON TRANSFORMS AND RIESZ POTENTIALS

It is known that the classical hyperplane Radon transform, its $k$-plane generalization, and their duals intertwine Laplace operators on the source space and the target space. More general intertwining formulas can be obtained if we replace the Laplace operators by the corresponding Riesz potentials; cf. (2.25). Our aim in this section is to extend these formulas to the mixed Radon transforms (3.1). We also obtain certain Grassmannian analogues of Fuglede’s formula (2.24) and its generalizations (2.25). Throughout this section, we keep the notation from Subsection 2.3.

6.1. Intertwining Formulas.

Theorem 6.1. If $0 < \alpha < n - k - j$, then
\[
(I_{n-k}^\alpha R_{j,k} f)(\zeta) = (R_{j,k} I_{n-j}^\alpha f)(\zeta), \quad \zeta \in G(n,k),
\]
provided that either side of this equality exists in the Lebesgue sense.

Remark 6.2. Before we prove this theorem, some comments are in order.

1. In the limiting cases $j = 0$ (the $k$-plane transform) and $k = 0$ (the dual $j$-plane transform), the formula (6.1) agrees with (2.25). The corresponding formulas for the hyperplane Radon transform and its dual can be found in [26, Proposition 4.38].

2. It is natural to conjecture that for sufficiently good $f$, (6.1) extends by analyticity to all complex $\alpha$. If $\alpha = -2m, m \in \{1, 2, \ldots\}$, then (6.1) reads
\[
(-\Delta_{n-k})^m R_{j,k} f = R_{j,k} (-\Delta_{n-j})^m f,
\]
where $\Delta_{n-k}$ and $\Delta_{n-j}$ stand for the Laplace operators on the corresponding fibers.
3. In the case \( j + k = n - 1 \), \( m = 1 \), the formula (6.2) was proved by Gonzalez under the assumption \( f \in C_\infty^0(G(n, k)) \); cf. [8, Lemma 3.3]. Similar formulas for the Radon-John transform and its dual are due to Helgason [13, Lemma 8.1].

4. For the hyperplane Radon transform on \( \mathbb{R}^n \) and its dual, an analogue of (6.1) for all \( \alpha \in \mathbb{C} \) in the corresponding Semyanisty-Lizorkin spaces can be found in [26, Proposition 4.58].

**Proof of Theorem 6.1.** We assume \( \zeta \equiv \zeta(\mathbb{E}_k, 0) \) and set

\[
I = (I_{n-k}^\alpha R_{j,k} f)(\mathbb{E}_k, 0), \quad J = (R_{j,k} I_{n-j}^\alpha f)(\mathbb{E}_k, 0).
\]

Because the operators in (6.1) commute with rigid motions, it suffices to show that \( I = J \). We recall the notation for the subspaces:

\[
\mathbb{E}_j = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_j, \quad \mathbb{E}_k = \mathbb{R}e_{n-k+1} \oplus \cdots \oplus \mathbb{R}e_n;
\]

\[
\mathbb{R}^{n-j} = \mathbb{R}e_{j+1} \oplus \cdots \oplus \mathbb{R}e_n, \quad \mathbb{R}^{n-k} = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{n-k}.
\]

Then

\[
I = \frac{1}{\gamma_{n-k}(\alpha)} \int_{\mathbb{R}^{n-k}} |v|^{\alpha-n+k} dv \int_{G_j(\mathbb{R}^{n-k})} d\xi \int_{\mathbb{E}_k \cap \xi} f(\xi + u + v) du
\]

\[
= \frac{1}{\gamma_{n-k}(\alpha)} \int_{\mathbb{R}^{n-k}} |v|^{\alpha-n+k} dv \int_{O(n-k)} d\gamma \int_{\mathbb{E}_k} f(\gamma(\mathbb{E}_j + u + v)) du
\]

\[
= \frac{1}{\gamma_{n-k}(\alpha)} \int_{\mathbb{E}_k} du \int_0^\infty r^{\alpha-1} dr \int_0^{\pi/2} d\theta \int_{O(n-k)} f(\gamma(\mathbb{E}_j + u + r\theta)) d\gamma.
\]

This expression can be transformed by making use of the bi-spherical coordinates [26, p. 31]

\[
\theta = \varphi \cos \omega + \psi \sin \omega, \quad \varphi \in \mathbb{S}^{j-1}, \quad \psi \in \mathbb{S}^{n-k-j-1}, \quad 0 < \omega < \pi/2,
\]

\[
d\theta = \cos^{j-1} \omega \sin^{n-k-j-1} \omega \, d\varphi \, d\psi \, d\omega.
\]

We obtain

\[
I = \frac{1}{\gamma_{n-k}(\alpha)} \int_{\mathbb{E}_k} du \int_0^\infty r^{\alpha-1} dr \int_0^{\pi/2} \cos^{j-1} \omega \sin^{n-k-j-1} \omega \, d\omega
\]

\[
\times \int_{\mathbb{S}^{j-1}} d\varphi \int_{\mathbb{S}^{n-k-j-1}} d\psi \int_{O(n-k)} f(\gamma(\mathbb{E}_j + u + r\psi \sin \omega)) d\gamma.
\]
The integral over $O(n - k)$ depends only on $r \sin \omega$ and $u$. We denote it by $	ilde{f}(r \sin \omega, u)$ and continue:

$$
I = \sigma_{j-1} \frac{\sigma_{n-k-j-1}}{\gamma_{n-k}(\alpha)} \int_{E_k} \int_0^\pi \frac{\cos^{j-1} \omega \sin^{n-k-j-1} \omega}{\sin \omega} \int_0^\infty r^{\alpha-1} \tilde{f}(r \sin \omega, u) \, dr d\omega
$$

$$
= c_1 \int_{E_k} \int_0^\infty s^{\alpha-1} \tilde{f}(s, u) \, ds,
$$

where

$$
c_1 = \sigma_{j-1} \frac{\sigma_{n-k-j-1}}{\gamma_{n-k}(\alpha)} \int_0^{\pi/2} \cos^{j-1} \omega \sin^{n-k-j-1} \omega \sin \omega \, d\omega
$$

$$
= \frac{2^{1-a} \Gamma((n - k - j - \alpha)/2)}{\Gamma(\alpha/2) \Gamma((n - k - j)/2)}.
$$

Let us show that $J$ has the form (6.3) too. We have

$$
J = \int_{G_j(\mathbb{R}^{n-k})} d\xi \int_{\mathbb{E}_k \cap \xi^\perp} (I_{n-j}^\alpha f)(\xi, u) \, du \quad \text{(note that $\mathbb{E}_k \cap \xi^\perp = \mathbb{E}_k$)}
$$

$$
= \frac{1}{\gamma_{n-j}(\alpha)} \int_{O(n-k)} d\gamma \int_{\mathbb{E}_k} \int_{\mathbb{R}^{n-j}} |y|^{\alpha-n+j} f(\gamma(\mathbb{E}_j + u + y)) \, dy
$$

$$
= \frac{1}{\gamma_{n-j}(\alpha)} \int_{E_k} \int_0^\infty r^{\alpha-1} \, dr \int_{\mathbb{S}^{n-j-1}} d\theta \int_{O(n-k)} f(\gamma(\mathbb{E}_j + u + r\theta)) \, d\gamma.
$$

Setting

$$
\theta = \varphi \cos \omega + \psi \sin \omega, \quad \varphi \in S^{n-k-j-1}, \quad \psi \in S^{k-1}, \quad 0 < \omega < \pi/2,
$$

(cf. the first part of the proof), we continue

$$
J = \frac{1}{\gamma_{n-j}(\alpha)} \int_{E_k} \int_0^\infty \frac{\pi/2}{\cos^{n-j-k-1} \omega \sin^{k-1} \omega \sin \omega} \, d\omega
$$

$$
\times \int_{S^{n-k-j-1}} d\varphi \int_{S^{k-1}} d\psi \int_{O(n-k)} f(\gamma(\mathbb{E}_j + u + r\varphi \cos \omega + r\psi \sin \omega)) \, d\gamma.
$$
The integral over $O(n - k)$ depends only on $r \cos \omega$ and $u$. We denote it by $\tilde{f}(r \cos \omega, u)$. Then

$$J = \frac{\sigma_{k-1} \sigma_{n-k-j-1}}{\gamma_{n-j}(\alpha)} \int_{E_k} \int_{0}^{\pi/2} \cos^{n-j-k-1} \omega \sin^{k-1} \omega \, d\omega \int_{0}^{\infty} r^{\alpha-1} \tilde{f}(r \cos \omega, u) \, dr$$

$$= c_2 \int_{E_k} \int_{0}^{\infty} s^{\alpha-1} \tilde{f}(s, u) \, ds,$$  \hspace{1cm} (6.4)

where

$$c_2 = \frac{\sigma_{k-1} \sigma_{n-k-j-1}}{\gamma_{n-j}(\alpha)} \int_{0}^{\pi/2} \cos^{n-j-k-\alpha-1} \omega \sin^{k-1} \omega \, d\omega$$

$$= \frac{2^{1-\alpha} \Gamma((n-k-j-\alpha)/2)}{\Gamma(\alpha/2) \Gamma((n-k-j)/2)} = c_1.$$  \hspace{1cm}(6.5)

Comparing (6.3) and (6.4), we complete the proof.

6.2. **Fuglede Type Formulas.** We introduce the following integral operators acting on functions $h : \mathbb{R}^n \to \mathbb{C}$:

$$\Lambda_{j,k} h = R_k^* R_{j,k} R_j h, \quad \Lambda_{k,j} h = R_j^* R_{k,j} R_k h.$$  \hspace{1cm} (6.5)

If $h$ is a radial function, $h(x) = h_0(|x|)$, then (4.3) together with (2.17) and (2.18) gives

$$\Lambda_{j,k} h = \frac{\tilde{c}_1 \pi^{j/2} \Gamma(n/2)}{\Gamma((n-k)/2)} r^{2-n} I_{-2}^{j/2} R_{j,k} h_0 = \frac{\tilde{c}_1 \pi^{j/2} \Gamma(n/2)}{\Gamma((n-k)/2)} r^{2-n} I_{-2}^{(j+k)/2} R_{j,k} h_0.$$  \hspace{1cm}(6.6)

By (2.27), the last expression is a constant multiple of the Riesz potential $I_{n}^{j+k} h$.

This observation paves the way to the following general result.

**Theorem 6.3.** If $j + k < n$, then

$$R_k^* R_{j,k} R_j h = R_j^* R_{k,j} R_k h = c I_{n}^{j+k} h, \quad c = \frac{2^{j+k} \pi^{(j+k)/2} \Gamma(n/2)}{\Gamma((n-j-k)/2)},$$

provided that the Riesz potential $I_{n}^{j+k} h$ exists in the Lebesgue sense.
Proof. Because all operators in (6.6) commute with rigid motions and \(j\) and \(k\) are interchangeable, it suffices to show that \((R_k^* \mathcal{R}_{j,k} R_j h)(0) = c (I_n^{j+k}) h(0)\). By (2.14) and (3.2),

\[
(R_k^* \mathcal{R}_{j,k} R_j h)(0) = \int_{O(n)} (\mathcal{R}_{j,k} R_j h) (\gamma E_k) d\gamma
\]

\[
= \int_{O(n)} d\gamma \int_{G_j(\mathbb{R}^{n-k})} d\xi \int_{E_k} (R_j h)(\gamma \xi + \gamma y) dy
\]

\[
= \int_{O(n)} d\gamma \int_{O(n-k)} d\alpha \int_{E_k} (R_j h)(\gamma \alpha E_j + \gamma y) dy
\]

\[
= \int_{O(n)} d\gamma \int_{E_k} (R_j h)(\gamma (E_j + y)) dy.
\]

Using (2.13) (with \(k\) replaced by \(j\)), we write the last expression as follows.

\[
\int_{O(n)} d\gamma \int_{E_k} d y \int_{E_j} h(\gamma (y + z)) dz = \int_{O(n)} d\gamma \int_{E_j \oplus E_k} h(\gamma \bar{y}) d\bar{y}
\]

\[
= \frac{\sigma_{j+k-1}}{\sigma_{n-1}} \int_0^\infty r^{j+k-1} dr \int_{\mathbb{S}^{n-1}} h(r\theta) d\theta
\]

\[
= \frac{\sigma_{j+k-1}}{\sigma_{n-1}} \int_{\mathbb{R}^n} h(x) |x|^{j+k-n} dx = c (I_n^{j+k}) h(0),
\]

as desired. \(\square\)

The formula (6.6) is a generalization of Fuglede’s formula (2.24). The latter can be obtained from (6.6) if we formally set \(j = 0\) or \(k = 0\).

7. Inversion Formulas for \(\mathcal{R}_{j,k}f\)

The following preliminary discussion explains the essence of the matter and the plan of the section. It might be natural to expect that \(\mathcal{R}_{j,k}\) is injective on some standard function space, like \(C_c^\infty(G(n,j))\), if \(\dim G(n,j) \leq \dim G(n,k)\), which is equivalent to \((j+1)(n-j) \leq (k+1)(n-k)\). The latter splits in two cases:

(a) \(j + k = n - 1\) for all \(j\) and \(k\);
(b) \(j + k < n - 1\) for \(j \leq k\).
In the present paper we do not investigate both (a) and (b) in full
generality and proceed as follows. We first consider $R_{j,k}f$ for radial
$f$, when an explicit inversion formula is available for all $j + k < n$.
Then we address to the case (a) and obtain an inversion formula in a
sufficiently large class of functions $f$, including functions in Lebesgue
spaces and continuous functions. The case of all $j + k < n$ for such
functions remains open. Some progress can be achieved if we restrict
the class of functions $f$ to the range of the $j$-plane transform. Under
this assumption, $R_{j,k}f$ can be explicitly inverted, no matter whether
$j \leq k$ or vice versa.

7.1. The Radial Case. If $f$ is radial, then $R_{j,k}f$ is radial too and we
have the following result.

**Theorem 7.1.** Let $f(\tau) \equiv f_0(|\tau|)$ be a locally integrable radial function
on $G(n,j)$ satisfying (5.5). The function $f_0$ can be recovered from the
Radon transform $(R_{j,k}f)(\zeta) \equiv (I_{j,k}f_0)(|\zeta|)$ for all $j + k < n$ by the
formula

$$f_0(t) = \tilde{c}_1^{-1} (D_{-2}^{k/2} t^{n-2-n+k} D_{+2}^{j/2} s^{n-k-2} I_{j,k}f_0)(t),$$

(7.1)

where $\tilde{c}_1 = \pi^{k/2} \Gamma((n-k)/2)/\Gamma((n-j-k)/2)$ and the Erdélyi–Kober
fractional derivatives $D_{-2}^{k/2}$ and $D_{+2}^{j/2}$ are defined by (2.7)-(2.10).

**Proof.** By (4.4),

$$(I_{j,k}f_0)(s) = \tilde{c}_1 s^{k+2-n} (I_{+2}^{j/2} t^{n-j-k-2} I_{-2}^{k/2} f_0)(s).$$

(7.2)

The assumption (5.5) is equivalent to $\int_a^\infty |f_0(t)| t^{k-1} dt < \infty$ for some
$a > 0$. The local integrability of $f$ implies $r^{n-j-k-1} I_{-2}^{k/2} f_0 \in L^1_{loc}(\mathbb{R}^n)$
(a simple calculation is left to the reader). Hence the conditions of
Theorems 2.2 and 2.3 are satisfied and both fractional integrals in (7.2)
can be inverted to give (7.1). \hfill \Box

Interchanging $j$ and $k$, the reader can easily obtain a similar state-
ment for the dual transform $R_{k,j}\varphi$.

7.2. The Case $j + k = n - 1$.

7.2.1. The Structure of $R_{j,k}f$. We consider the flag

$\mathcal{F} = \{(\xi, \eta, v) : \xi \in G_{n,j}, \eta \in G_k(\xi^\perp), \ v \in \xi^\perp \cap \eta^\perp\}$.

Given a function $f$ on $G(n,j)$, we define a function $\tilde{f}$ on $\mathcal{F}$ by the
formula

$$\tilde{f}(\xi, \eta, v) = \int_{\eta} f(\xi + u + v) d_\eta u.$$
This function is the inner integral in the definition (3.2) of $R_{j,k}f$ and has two interpretations. On the one hand, for every fixed $\xi \in G_{n,j}$, $\tilde{f}$ is the $k$-plane transform of the function $u \rightarrow f(\xi,u)$ in the $(n-j)$-space $\xi^\perp$:

$$\tilde{f}_\xi(\zeta) \equiv \tilde{f}(\xi,\eta,v) = (R_{k,\xi^\perp}[f(\xi,\cdot)])(\zeta), \quad \zeta = \zeta(\eta,v) \in G(k,\xi^\perp). \quad (7.3)$$

On the other hand, for every fixed $\eta \in G_{n,k}$, $\tilde{f}$ is a function on the affine Grassmannian $G(j,\eta^\perp)$:

$$\tilde{f}_\eta(\tau') \equiv \tilde{f}(\xi,\eta,v) = \int_{\eta} f(\tau' + u) \, du, \quad \tau' = \tau'(\xi,\eta,v) \in G(j,\eta^\perp). \quad (7.4)$$

The dual $j$-plane transform of $\tilde{f}_\eta(\cdot)$ in the $(n-k)$-space $\eta^\perp$ at the point $v \in \eta^\perp$ has the form

$$(R_{j,\eta^\perp}^* \tilde{f}_\eta)(v) = \int_{O(\eta^\perp)} \tilde{f}_\eta(\gamma \xi + v) \, d\gamma \quad (7.5)$$

$$= \int_{G(j,\eta^\perp)} d_{\eta^\perp} \xi \int_{\eta} f(\xi + u + v) \, d_{\eta}u = (R_{j,k}f)(\eta,v),$$

where $O(\eta^\perp)$ is the subgroup of $O(n)$, which consists of orthogonal transformations in $\eta^\perp$.

The above reasoning shows that $R_{j,k}$ is a certain mixture (but not a composition) of the $k$-plane transform and the dual $j$-plane transform.

7.2.2. **Inversion Procedure.** According to the structure of the operator $R_{j,k}$, to reconstruct $f$ from $R_{j,k}f$, we first invert the dual $j$-plane transform (7.5) in the $(n-k)$-space $\eta^\perp$ and then the $k$-plane transform (7.3) in the $(n-j)$-space $\xi^\perp$. Because, in general, the dual $j$-plane transform is injective only in the co-dimension one case (cf. [21, Theorem 4.4]), we restrict to the case $j = n-k-1$, when the injectivity can be proved on a pretty large class of functions.

**STEP 1.** We make use of Corollary 2.12 with $R^n$ replaced by $\eta^\perp$, when the condition (2.52) becomes

$$\int_{G(j,\eta^\perp)} \frac{|\tilde{f}_\eta(\tau')|}{1 + |\tau'|} \, d\tau' < \infty \quad \text{for almost all } \eta \in G_{n,k}. \quad (7.6)$$

Under this condition,

$$\tilde{f}_\eta(\tau') = (R_{\eta^\perp}^*)^{-1}[(R_{j,k}f)(\eta,\cdot)](\tau'), \quad \tau' = \xi + v \in G(j,\eta^\perp), \quad (7.7)$$
where \((R_{\eta^+}^*)^{-1}\) stands for the inverse dual \(j\)-plane transform in the \((j + 1)\)-subspace \(\eta^+\). An explicit formula for \((R_{\eta^+}^*)^{-1}\) can be obtained from the equality (2.57) adapted for our case.

Our next aim is to find sufficient conditions for (7.6) in terms of \(f\).

We observe that (7.6) will be proved if we show that

\[
I = \int_{\mathcal{G}_{n,k}} d\eta \int_{\mathcal{G}_{j}(\eta^+)} d\eta \int_{\xi^+ \cap \eta^+} \frac{|\tilde{f}_\eta(\xi, v)|}{1 + |v|} dv < \infty.
\]

Changing the order of integration and using (7.3), we can write \(I\) as

\[
I = \int_{\mathcal{G}_{n,j}} d\xi \int_{\mathcal{G}_{k}(\xi^+)} d\xi \int_{\xi^+ \cap \eta^+} \frac{|(R_{k,\xi}[f(\xi, \cdot)])(\eta, v)|}{1 + |v|} dv,
\]

where \(R_{k,\xi^+}\) stands for the \(k\)-plane transform in the \((k + 1)\)-space \(\xi^+\).

Using (2.16) with \(n = k + 1\), we have

\[
\int_{\mathcal{G}_{k}(\xi^+)} \int_{\xi^+ \cap \eta^+} \frac{|(R_{k,\xi}[f(\xi, \cdot)])(\eta, v)|}{1 + |v|} dv \leq c \int_{\xi^+} \frac{|f(\xi, u)|}{1 + |u|} \log(2 + |u|) du,
\]

whence

\[
I \leq c \int_{\mathcal{G}_{n,j}} \int_{\xi^+} \frac{|f(\xi, u)|}{1 + |u|} \log(2 + |u|) du = c \int_{\mathcal{G}(n,j)} \frac{|f(\tau)|}{1 + |\tau|} \log(2 + |\tau|) d\tau.
\]

Thus the inversion formula (7.6) is valid if

\[
\int_{\mathcal{G}(n,j)} \frac{|f(\tau)|}{1 + |\tau|} \log(2 + |\tau|) d\tau < \infty. \tag{7.8}
\]

**Definition 7.2.** The class of all functions \(f\) satisfying (7.8) will be denoted by \(L_{\log(G(n,j))}\).

**STEP 2.** By Corollary 2.10, the function \(f(\xi, \cdot)\) on \(\xi^+\) can be reconstructed from its \(k\)-plane transform \(\tilde{f}_\xi(\zeta) = (R_{k,\xi^+}[f(\xi, \cdot)])(\zeta)\) if

\[
\int_{\xi^+} \frac{|f(\xi, u)|}{1 + |u|} du < \infty.
\]

The latter is guaranteed for almost all \(\xi\) if

\[
\int_{\mathcal{G}(n,j)} \frac{|f(\tau)|}{1 + |\tau|} d\tau < \infty. \tag{7.9}
\]
Thus Step 2 gives one more assumption for \( f \) which is, however, weaker than (7.8).

Combining Step 1 and Step 2, we arrive at the following statement.

**Theorem 7.3.** If \( j + k = n - 1 \), then every function \( f \in L^p_{\log}(G(n, j)) \) can be reconstructed from \( \varphi = R_{j,k}f \) by the formula

\[
    f(\xi, u) = (R_{k,\xi}^{-1}[\tilde{f}_\xi])(u),
\]

where

\[
    \tilde{f}_\xi(\eta, v) \equiv \tilde{f}_\eta(\xi, v) = (R_{\eta,+}^*)^{-1}[\varphi(\eta, \cdot)](\xi, v),
\]

with the inverse transforms \( R_{k,\xi}^{-1} \) and \( (R_{\eta,+}^*)^{-1} \) being defined according to Corollaries 2.10 and 2.12, respectively.

**Remark 7.4.** The condition \( f \in L^p_{\log}(G(n, j)) \) in Theorem 7.3 falls into the scope of the Existence Theorem 5.2 and differs from the latter only by the logarithmic factor. Thus Theorem 7.3 provides inversion of \( R_{j,k}f \) under almost minimal assumptions. Note also that by H"older’s inequality, any function in \( L^p(G(n, j)) \), \( 1 \leq p < (n-j)/k \), and any continuous function of order \( O(|\tau|^{-\mu}) \), \( \mu > k \), belong to \( L^p_{\log}(G(n, j)) \), where the bounds for \( p \) and \( \mu \) are sharp; cf. (5.8).

### 7.3. Inversion of \( R_{j,k}f \) on the Range of the \( j \)-Plane Transform.

Theorem 6.3 implies the following inversion result, which resembles Corollary 2.7 for the dual \( k \)-plane transform.

**Theorem 7.5.** Let \( f = R_j h, h \in L^p(\mathbb{R}^n) \). If \( 1 \leq p < n/(j + k) \), then

\[
    f = c^{-1} R_j \mathbb{D}_n^{j+k} R_k^* R_{j,k} f, \quad c = \frac{2^{j+k} \pi^{(j+k)/2} \Gamma(n/2)}{\Gamma((n-j-k)/2)},
\]

where \( \mathbb{D}_n^{j+k} \) is the Riesz fractional derivative (2.21). More generally, if \( 0 < \alpha < n - j - k \) and \( 1 \leq p < n/(j + k + \alpha) \), then

\[
    f = c^{-1} R_j \mathbb{D}_n^{j+k+\alpha} R_k^* I_{n-k}^\alpha R_{j,k} f = c^{-1} R_j \mathbb{D}_n^{j+k+\alpha} R_k^* R_{j,k} I_{n-j}^\alpha f
\]

with the same constant \( c \).

**Proof.** By (6.6),

\[
    c^{-1} R_j \mathbb{D}_n^{j+k} R_k^* R_{j,k} f = c^{-1} R_j \mathbb{D}_n^{j+k} R_k^* R_{j,k} R_j h = R_j \mathbb{D}_n^{j+k} I_{n-k}^\alpha h = R_j h = f.
\]

Further, combining (6.6) with the semigroup property of Riesz potentials, we obtain

\[
    I_{n-k}^\alpha R_k^* R_{j,k} f = I_{n-k}^\alpha R_k^* R_{j,k} R_j h = I_{n-k}^\alpha I_{n-k}^{j+k} h = I_{n-k}^{j+k+\alpha} h.
\]

However, by (2.25) and (6.1),

\[
    I_{n-k}^\alpha R_k^* R_{j,k} f = R_k^* I_{n-k}^\alpha R_{j,k} f = R_k^* R_{j,k} I_{n-k}^\alpha f.
\]

This gives (7.13).
The formula (7.13) can be used if we want to replace the nonlocal Riesz fractional derivative by the local one. For example, if \( j + k \) is odd and \( j + k < n - 1 \), we can apply (7.13) with \( \alpha = 1 \).

**Remark 7.6.** If \( f \) is good enough, then (7.12) formally agrees with the inversion formula of Gonzalez [8, Theorem 3.4]. In our notation his formula reads

\[
f = \tilde{c}^{-1} \mathcal{R}_{k,j} \mathbb{D}^{n-1}_{n-k} \mathcal{R}_{j,k} f, \quad \tilde{c} = 2^{n-1} \pi^{(n-3)/2} \Gamma \left( \frac{j+1}{2} \right) \Gamma \left( \frac{k+1}{2} \right).
\]

(7.14)

The following non-rigorous reasoning shows the consistency of (7.12) and (7.14). It suffices to show that

\[
c^{-1} R_j \mathbb{D}^{n-1} R_k \varphi = \tilde{c}^{-1} \mathcal{R}_{k,j} \mathbb{D}^{n-1}_{n-k} \varphi,
\]

(7.15)

where \( c \) is the constant from (7.12) with \( j + k = n - 1 \). We set \( \varphi = R_k h \) and apply \( R_j^* \) to both sides of (7.15). For the left-hand side, (2.24) yields

\[
c^{-1} R_j^* R_j \mathbb{D}^{n-1} R_k^* R_k h = c^{-1} c_{j,n} c_{k,n} I_j^n I_k^n h
\]

\[
= \frac{\pi^{1/2} \Gamma(n/2)}{\Gamma((n-j)/2) \Gamma((n-k)/2)} h.
\]

(7.16)

For the right-hand side, the formula \( \mathbb{D}^{n-1}_{n-k} R_k h = R_k \mathbb{D}^{n-1}_n h \) (cf. (2.25)) in conjunction with (6.6) gives

\[
\tilde{c}^{-1} R_j^* \mathcal{R}_{k,j} \mathbb{D}^{n-1}_{n-k} R_k h = \tilde{c}^{-1} R_j^* \mathcal{R}_{k,j} R_k \mathbb{D}^{n-1}_n h = c_1 I^n_{n-1} \mathbb{D}^{n-1}_n h = c_1 h,
\]

where \( c_1 \) is exactly the same as in (7.16). Thus, if \( R_j^* \) is injective and the class of functions \( f \) is good enough, we are done.

Note that the proof of convergence of the expression on right-hand side of (7.14) is rather nontrivial, even for \( f \in C_c^\infty(G(n,j)) \).

8. Appendix

**Proof of (4.6).** Let us show that (4.6) follows from (4.2). It suffices to assume \( f_0 \geq 0 \). We recall that \( \ell = n - j - k \geq 1 \) and the letter \( c \) stands for a constant that can be different at each occurrence. Let

\[
F(r) = \int_r^\infty f_0(t) (t^2 - r^2)^{k/2 - 1} t \, dt.
\]
For any $0 < \alpha < \beta < \infty$ we have

$$
\int_0^\beta (I_{j,k}f_0)(s) \, ds \leq c \int_0^\beta ds \int_0^s (s^2-r^2)^{j/2-1}r^{\ell-1}F(r) \, dr
$$

$$
\leq c \int_0^\beta ds \int_0^s (s-r)^{j/2-1}r^{\ell-1}F(r) \, dr
$$

$$
= c \int_0^{\beta} r^{\ell-1}F(r) \, dr \int_0^\beta (s-r)^{j/2-1}ds + c \int_0^{\beta} r^{\ell-1}F(r)dr \int_0^\beta (s-r)^{j/2-1}ds
$$

$$
\leq c \int_0^{\beta} r^{\ell-1}(\beta-r)^{j/2}F(r) \, dr - c \int_0^{\beta} r^{\ell-1}(\alpha-r)^{j/2}F(r) \, dr. \quad (8.1)
$$

Because the integrals in (8.1) have the same form, it suffices to show that

$$
I(\alpha) \equiv \int_0^\alpha r^{\ell-1}(\alpha-r)^{j/2}F(r) \, dr < \infty \quad \forall \ \alpha > 0.
$$

We have

$$
I(\alpha) = \int_0^\alpha r^{\ell-1}(\alpha-r)^{j/2}dr \int_0^\infty f_0(t)(t-r)^{k/2-1}t^{k/2}dt
$$

$$
\leq c \int_0^\alpha f_0(t) t^{k/2+\ell-1}dt \int_0^t (t-r)^{k/2-1}dr + c \int_0^\infty f_0(t) t^{k/2}dt \int_0^\alpha (t-r)^{k/2-1}dr
$$

$$
= c \int_0^\alpha f_0(t) t^{k+\ell-1}dt + c \int_0^\alpha f_0(t) t^{k/2}(t^{k/2} - (t-\alpha)^{k/2}) dt
$$

$$
\leq c \int_0^\alpha f_0(t) t^{n-j-1}dt + c \int_0^\alpha f_0(t) t^{k-1}dt < \infty.
$$

The last expression is finite by (4.2).

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