CONSERVATIVE ANOSOV Diffeomorphisms of $T^2$ WITHOUT AN ABSOLUTELY CONTINUOUS INVARIANT MEASURE.

ZEMER KOSLOFF

Abstract. We construct examples of $C^1$ Anosov diffeomorphisms on $T^2$ which are of Kreiger type $\text{III}_1$ with respect to Lebesgue measure. This shows that the Gurevic Oseledec phenomena that conservative $C^{1+\alpha}$ Anosov diffeomorphisms have a smooth invariant measure does not hold true in the $C^1$ setting.

1. Introduction

This paper provides the first examples of Anosov diffeomorphisms of $T^2$ which are conservative and ergodic yet there is no Lebesgue absolutely continuous invariant measure.

Let $M$ be a compact, boundaryless smooth manifold and $f : M \to M$ be a diffeomorphism. A natural question which arises is whether $f$ preserves a measure which is absolutely continuous with respect to the volume measure on $M$. In order to avoid confusion in what follows, we would like to stress out that in this paper, the term conservative means the definition from ergodic theory which is non existence of wandering sets of positive measure. That is $f$ is conservative if and only for every $W \subset M$ so that $\{ f^n W \}_{n \in \mathbb{Z}}$ are disjoint (modulo the volume measure), $\text{vol}(W) = 0$.

It follows from [LS] that for a generic $C^2$ Anosov diffeomorphism there exists no absolutely continuous invariant measure (a.c.i.m.), [B, p. 72, Corollary 4.15.]. On the other hand, Gurevic and Oseledec [GO] have shown that if $f$ is a conservative $C^2$ Anosov (hyperbolic) diffeomorphism, then $f$ preserves a probability measure in the measure class of the volume measure. Thus in this case the two definitions of conservativity (smooth dynamical vs. ergodic theoretical) are the same and a generic $C^2$ diffeomorphism is dissipative. The proof in [GO] uses the absolute continuity of the foliations and existence of SRB measures to show that if the SRB measure for $f$ is not equal to the SRB measure for $f^{-1}$ then there exists a continuous function $g : M \to \mathbb{R}$ and a set $A \subset M$ of positive volume so that,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g\left(f^k(x)\right) \neq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g\left(f^{-k}(x)\right), \forall x \in A.$$

It is then a straightforward argument to construct a set $B \subset A$ of positive volume measure so that for almost every $x \in B$, the set $\{ k \in \mathbb{N} : f^k x \in B \}$ is finite, in contradiction with Halmos Recurrence Theorem [Aar1].

This result remains true for $C^{1+\alpha}$, $\alpha > 0$ Anosov diffeomorphisms. However, since there exists $C^1$ Anosov diffeomorphisms whose stable and unstable foliations are not absolutely continuous [RY], this proof can not be generalized for the $C^1$ setting. The question arises whether every conservative Anosov diffeomorphism has a volume absolutely continuous invariant measure.

Since there are not many explicit Anosov transformations which are not linear toral automorphisms, a natural approach in finding $C^1$ examples with a certain property is to
prove that the property is generic in the $C^1$ topology. For example, Avila and Bochi in the case of expanding maps AB2 AB1 have shown that a generic $C^1$ expanding map has no a.c.i.m. However since a generic $C^1$ Anosov map is dissipative, it is not clear to us how to use this approach to find a conservative example without an a.c.i.m. Nonetheless we prove the following.

**Theorem 1.** There exists a $C^1$- Anosov diffeomorphism of the two torus $T^2$ which is ergodic, conservative and there exists no $\sigma$-finite invariant measure which is absolutely continuous with respect to the Lebesgue measure on $T^2$.

In fact, the ergodic type III transformations (a transformation without an a.c.i.m.) can be further decomposed into the Krieger Araki-Woods classes $\text{III}_\lambda$, $0 \leq \lambda \leq 1$ [Kri], see Section 2 and our examples are of type $\text{III}_1$.

The examples are constructed by modifying a linear Anosov diffeomorphism to obtain a change of coordinates which takes the Lebesgue measure to a measure which is equivalent to a type $\text{III}_1$ Markovian measure (on a Markov partition of the linear diffeomorphism). These examples are greatly inspired by the ideas of Bruin and Hawkins [BH] where they modify the map $f(x) = 2x \mod 1$ using the push forward (with respect to the dyadic representation) of a Hamachi measure on $\{0, 1\}^\mathbb{N}$ to the circle. Since by embedding a horseshoe in a linear transformation one loses explicit formula for the Radon Nykodym derivatives of the modified transformations we couldn’t use measures on a full shift space. That is why in Section 3 we start with a Markov measure (corresponding to the Lebesgue measure) supported on a topological Markov shift $\Sigma$ and change the measure on the future to obtain type $\text{III}_1$ Markov measure (for the shift on $\Sigma$). We think that this section has many more possibilities for future applications and is of independent interest.

This paper is organized as follows. In Section 2 we start by introducing the definitions and background material from nonsingular ergodic theory and smooth dynamics which are used in this paper. We end this section with a discussion on the method of the construction. Section 3 deals with the construction of the type $\text{III}_1$ Markov shift examples. In Section 4 we show how to use the one sided Markov measures from the previous section to obtain a modification of the golden mean shift. Finally, in section 5 we show how to embed and modify smartly the one dimensional perturbations of the previous sections to obtain homeomorphisms of the two torus, which when applied as conjugation to a certain total automorphism (the natural extension of the golden mean shift) are examples of type $\text{III}_1$ Anosov diffeomorphisms.

2. Preliminary definitions and a discussion on the method of construction

2.1. Basics of non singular ergodic theory. This section is a short introduction to non singular ergodic theory. For more details and explanations please see [Aar1].

Let $(X, B, \mu)$ be a standard probability space. In what follows equalities (and inclusions) of sets are modulo the measure $\mu$ on the space. A measurable map $T : X \to X$ is non singular if $T_* \mu := \mu \circ T^{-1}$ is equivalent to $\mu$ meaning that they have the same collection of negligible sets. If $T$ is invertible one has the Radon Nykodym derivatives

$$(T^n)'(x) := \frac{d\mu \circ T^n}{d\mu}(x) : X \to \mathbb{R}_+,$$

A set $W \subset X$ is wandering if $\{T^nW\}_{n \in \mathbb{Z}}$ are pairwise disjoint and as was stated before we say that $T$ is conservative if there exists no wandering set of positive measure. By the Halmos’ Recurrence Theorem a transformation is conservative if and only if it satisfies
Poincaré recurrence, that is given a set of positive measure $A \in \mathcal{B}$, almost every $x \in A$ returns to itself infinitely often. In order to prove that a transformation is conservative we will use Hopf’s criteria which says that $T$ is conservative (w.r.t $\mu$) if and only if

$$\sum_{n=1}^{\infty} (T^n)'(x) = \infty \mu - a.e.$$  

A transformation $T$ is **ergodic** if there are no non trivial $T$ invariant sets. That is $T^{-1}A = A$ implies $A \in \{\emptyset, X\}$. If $(X, \mathcal{B}, \mu)$ is a non atomic measure space and $T$ is invertible and ergodic then $T$ is also conservative. The converse implication is not true in general as there are conservative non-ergodic transformations. Since proving ergodicity is usually a harder task then proving conservativity we would like to concentrate on a class of transformations for which conservativity implies ergodicity (as happens by a theorem of Anosov in the case of $C^2$ Anosov diffeomorphisms). These transformations are the $K$-automorphisms.

A transformation is a **$K$-automorphism** if there exists a $\sigma$-algebra $\mathcal{F} \subset \mathcal{B}$ such that:

- $T^{-1}\mathcal{F} \subset \mathcal{B}$ meaning that $\mathcal{F}$ is a factor of $\mathcal{B}$.
- $\cap_{n=1}^{\infty} T^{-n} \mathcal{F} = \{\emptyset, X\}$ (exactness) and $\cup_{n\in\mathbb{Z}} T^n \mathcal{F} = \mathcal{B}$ (exhaustiveness).
- $T'(x)$ is $\mathcal{F}$ measurable.

The first two properties are the standard definition of a $K$-automorphism in the case of measure preserving automorphisms. The condition on the measurability of the Radon-Nykodym derivative comes to ensure that $(X, \mathcal{B}, \mu, T)$ is the **unique** natural extension of the non invertible exact transformation $(\mathcal{X}/\mathcal{F}, \mathcal{F}, \mu|_{\mathcal{F}}, T)$. It was shown in [Kre, ST] that a conservative and $K$ transformation is necessarily ergodic.

We end this subsection with the definition of the **Krieger ratio set $R(T)$**. We say that $r \geq 0$ is in $R(T)$ if for every $A \in \mathcal{B}$ of positive $\mu$ measure and for every $\epsilon > 0$ there exists an $n \in \mathbb{Z}$ such that

$$\mu(A \cap T^{-n} A \cap \{x \in X : |(T^n)'(x) - r| < \epsilon\}) > 0.$$  

The ratio set of an ergodic measure preserving transformation is a closed multiplicative subgroup of $[0, \infty)$ and hence it is of the form $\{0\}, \{1\}, \{0,1\}, \{0\} \cap \{\lambda^n : n \in \mathbb{Z}\}$ for $0 < \lambda < 1$ or $[0, \infty)$. Several ergodic theoretic properties can be seen from the ratio set. One of them is that $0 \in R(T)$ if and only if there exists no $\sigma$-finite $T$-invariant $\mu$-a.c.i.m. Another interesting relation is that $1 \in R(T)$ if and only if $T$ is conservative (Maharam’s Theorem). If $R(T) = [0, \infty)$ we say that $T$ is of type $\text{III}_1$.

### 2.2. Smooth ergodic theory of Anosov diffeomorphisms.

Smooth dynamics deals with the case where $M$ is a Riemannian manifold and $f : M \to M$ is a diffeomorphism on $M$. In this paper we would only talk about the class of Anosov (uniformly hyperbolic) automorphisms. A diffeomorphism $f$ is Anosov if for every $x \in M$ there is a decomposition of the tangent bundle at $x$, $T_x M = E^s_x \oplus E^u_x$, such that

- The decomposition is $Df$- equivariant, here $Df$ denotes the differential of $f$. That is $Df(E^s_x) = E^s_{f(x)}$ and $Df(E^u_x) = E^u_{f(x)}$.
- There exists $0 < \lambda < 1$ so that

$$\|Df^n(v)\| \leq \lambda^n \|v\|, \text{ for every } v \in E^s_x, \ n \geq 0$$

and

$$\|Df^{-n}(u)\| \leq \lambda^n \|u\|, \text{ for every } u \in E^u_x, \ n \geq 0.$$
A topological Markov shift (TMS) on $S$ is the shift on a shift invariant subset $\Sigma \subset S^\mathbb{Z}$ of the form

$$\Sigma_A := \left\{ x \in S^\mathbb{Z} : A_{x_i,x_{i+1}} = 1 \right\},$$

where $A = \{A_{s,t}\}_{s,t} \in S$ is a $\{0,1\}$ valued matrix on $S$. A TMS is mixing if there exists $n \in \mathbb{N}$ such that $A_{s,t}^n > 0$ for every $s, t \in S$.

Markov partitions of the manifold $M$ as in [AW, SI, B, Adl] are an important tool in the study of $C^{1+\alpha}$ Anosov diffeomorphisms. They provide a semiconjugacy between a TMS and the Anosov transformation $f$. One of the important contribution of this paper is that it uses a connection between inhomogenous Markov chains supported on a TMS to the Anosov diffeomorphism with the the push forward of the Markov measure.

**Example 2.** Consider $f : \mathbb{T}^2 \to \mathbb{T}^2$ the Toral automorphism defined by

$$f(x, y) = \left( \{x + y\}, x \right) = \left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) \mod 1,$$

where $\{t\}$ is the fractional part of $t$. Since $\det \left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right) = 1$, $f$ preserves the Lebesgue measure on $\mathbb{T}^2$. In addition for every $z \in \mathbb{T}^2$, the tangent space can be decomposed as $\text{span} \{v_s\} \oplus \text{span} \{v_u\}$ where $v_u = (1, 1/\varphi)$ and $v_s = (1, -\varphi)$. Here and throughout the paper $\varphi$ denotes the golden mean ($\varphi := \frac{1 + \sqrt{5}}{2}$).

For every $w \in V_u := \text{span} \{v_u\}$ and $z \in \mathbb{T}^2$

$$D_f(z)w = \left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right)w = \varphi w,$$

For every $u \in V_s := \text{span} \{v_s\}$ and $z \in \mathbb{T}^2$, $D_f(z)u = \left( -\frac{1}{\varphi} \right) u$. These facts can be used [Adl, AW] to construct the Markov partition for $f$ with three elements $\{R_1, R_2, R_3\}$, see figure 2.1.
The adjacency Matrix of the Markov partition is then defined by $A_{i,j} = 1$ if and only if $R_i \cap f^{-1}(R_j) \neq \emptyset$. Here the adjacency Matrix is

$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Let $\Phi : \Sigma_A \to \mathbb{T}^2$ be the map defined by

$\Phi(x) := \bigcap_{n=-\infty}^{\infty} f^{-n} R_{x_n}$,

note that $\{\bigcap_{n=-N}^{N} f^{-n} R_{x_n}\}_{N=1}^{\infty}$ is a decreasing sequence of compact sets hence by the Baire Category Theorem $\Phi(x)$ is well defined. The map $\Phi : \Sigma_A \to \mathbb{T}^2$ is continuous, finite to one, and for every $x \in \Sigma_A$,

$\Phi \circ T(x) = f \circ \Phi(x)$.

In addition, for every $x \in \mathbb{T}^2 \setminus \bigcup_{n \in \mathbb{Z}} \cup_{i=1}^{3} f^{-n} (\partial R_i)$ there exists a unique $w \in \Sigma_A$ so that $w = \Phi^{-1}(x)$. Thus $\Phi$ is a semi-conjugacy (topological factor map) between $(\Sigma_A, T)$ to $(\mathbb{T}^2, f)$. The Lebesgue measure $\lambda$ on $\mathbb{T}^2$ is conservative and invariant under $f$. One can check that $\Phi$ defines an isomorphism between $(\mathbb{T}^2, \lambda, f)$ and $(\Sigma_A, \mu_{\pi Q}, T)$ where $\mu_{\pi Q}$ is the stationary Markov measure with

$P_j \equiv Q := \begin{pmatrix} \phi^{1/\sqrt{5}} & 0 & 1/\sqrt{\phi} \\ \phi^{1/\sqrt{5}} & 0 & 1/\sqrt{\phi} \\ 0 & 1 & 0 \end{pmatrix}$.

and

$\pi_j = \pi_Q := \begin{pmatrix} \phi^{1/\sqrt{5}} \\ \phi^{1/\sqrt{5}} \\ 1/\sqrt{\phi} \end{pmatrix} = \begin{pmatrix} \lambda(R_1) \\ \lambda(R_2) \\ \lambda(R_3) \end{pmatrix}$.

2.2.1. Non singular Markov shifts: Let $S$ be a finite set. An inhomogeneous Markov Chain is a stochastic process $\{X_n\}_{n \in \mathbb{Z}}$ such that for each times $t_1, \ldots, t_l \in \mathbb{Z}$, and $s_1, \ldots, s_l \in S$,

$\mathbb{P}(X_{t_1} = s_1, X_{t_2} = s_2, \ldots, X_{t_l} = s_l) = \mathbb{P}(X_{t_1} = s_1) \prod_{k=1}^{l-1} \mathbb{P}(X_{t_{k+1}} = s_{k+1} | X_{t_k} = s_k)$.

Note that unlike in the classical setting of Markov Chains $\mathbb{P}(X_{t_{k+1}} = s_{k+1} | X_{t_k} = s_k)$ can depend on $t_k$.

The ergodic theoretical formulation is as follows. Let $\{P_n\}_{n=-\infty}^{\infty} \subset M_{S \times S}$ be a sequence of aperiodic and irreducible stochastic matrices on $S$. In addition let $\{\pi_n\}_{n=-\infty}^{\infty}$ be a sequence of probability distributions on $S$ so that for every $s \in S$ and $n \in \mathbb{Z}$,

$\sum_{t \in S} \pi_{n-1}(t) \cdot P_n(t, s) = \pi_n(s)$.

Then one can define a measure on the collection of cylinder sets,

$[b]_k^l := \{ x \in S^\mathbb{Z} : x_j = b_j \ \forall j \in [k, l] \cap \mathbb{Z} \}$.
by
\[ \mu \left( [b]_k^l \right) := \pi_k(b_k) \prod_{j=k}^{l-1} P_j(b_j, b_{j+1}). \]

Since the equation (2.3) is satisfied, \( \mu \) satisfies the consistency condition and therefore by Kolmogorov’s extension theorem \( \mu \) defines a measure on \( S^2 \). In this case we say that \( \mu \) is the Markov measure generated by \( \{ \pi_n, P_n \}_{n \in \mathbb{Z}} \) and denote \( \mu = M \{ \pi_n, P_n : n \in \mathbb{Z} \} \). By \( M \{ \pi, P \} \) we mean the measure generated by \( P_n \equiv P \) and \( \pi_n \equiv \pi \). We say that \( \mu \) is non singular for the shift \( T \) on \( S^2 \) if \( T_* \mu \sim \mu \). See subsection 3.1.2 for an extension of Kakutani’s Theorem for product measures which is the tool we use to show non singularity of the measures we construct.

2.3. Sections Overview.

- **Section 3**: The first step is to build type \( III_1 \), inhomogenous Markov measures for the shift on \( \Sigma_A \) with the additional property that for every \( k \leq 0 \) the transition matrices of \( \mu \) at \( k \) are the same as the ones arising from the Lebesgue measure. That is
  \[ \forall k \leq 0, \; P_k = Q. \]

This will say that (after a rotation of the coordinates to the \( v_u, v_s \) coordinates) with \( \Phi : \Sigma_A \to \mathbb{T} \) being the semiconjugacy map arising from the Markov partition we have
\[ d\Phi_*\mu(x,y) = d\mu^+(x)dy. \]

Here \( \mu^+ \) is the image of the push forward on the stable manifold of the Markov measure on \( \{ 1, 2, 3 \}^\mathbb{N} \) given by \( \{ \pi_k, P_k \}_{k=1}^\infty \). See the beginning of Section 4 for a more precise statement. The measure \( \Phi_*\mu \) is nonatomic and gives zero measure to the images of the boundaries of the rectangles of the Markov partition, hence \( \Phi \) is an isomorphism of \( (\mathbb{T}^2, \Phi_*\mu, f) \) and \( (\Sigma_A, \mu, Shift) \) and thus \((\mathbb{T}^2, \Phi_*\mu, f)\) is a type \( III_1 \) dynamical system.

- **The structure of \( \Phi_*\mu \)** means that the map which changes the Lebesgue measure on \( \mathbb{T}^2 \) to \( \Phi_*\mu \) can be realised by conditioning on the unstable manifold. More precisely if \( h(x) = \mu^+[0,x] \) then the map \( H : (x,y) \mapsto (h(x), y) \) takes the Lebesgue measure on \( \mathbb{T}^2 \) to \( \Phi_*\mu \). That means that the systems \((\mathbb{T}^2, Leb, H \circ f \circ H^{-1})\) and \((\mathbb{T}^2, \Phi_*\mu, f)\) are isomorphic and consequently \( H \circ f \circ H^{-1} \) is a type \( III_1 \) renormalization for the Lebesgue measure on \( \mathbb{T}^2 \). The problem is that it is not necessarily differentiable. The aim of Section 3 is to construct a map \( H_\epsilon : \mathbb{T}^2 \to \mathbb{T}^2 \) such that \( H_\epsilon \circ f \circ H_\epsilon^{-1} \) is an Anosov diffeomorphism and \( H_\epsilon \) takes the Lebesgue measure to a measure \( \eta \) which is equivalent to \( \Phi_*\mu \).

- The first step in constructing \( H_\epsilon \) is to work on the action of \( f \) on the unstable manifold which is the Golden mean shift and to apply perturbations there. That is the content of Section 4.

3. Type \( III_1 \) Markov shifts supported on topological Markov shifts

In this part we consider inhomogeneous Markov measures supported on the topological Markov shift \( \Sigma_A \) of Example 2 and show examples of such type \( III_1 \) shifts.

This section is organized in three subsections. In the first subsection we recall the basics of stationary Markov chains that we need for our construction and proofs. In Subsection 3.1 we give a condition for non singularity of the Markov measure which is an application
of ideas of Shiryaev and al. [Shi, LM] and prove a necessary condition for exactness of a one sided shift. In subsection 3.2 we construct the aforementioned examples and prove that they are indeed type III$_1$ measures for the shift.

3.1. Markov Chains.

3.1.1. Basics of Stationary (homogenous) Chains. Let $S$ be a finite set which we regard as the state space of the chain, $\pi = \{\pi(s)\}_{s \in S}$ a probability vector on $S$ and $P = (P_{s,t})_{s,t \in S}$ a stochastic matrix. The vector $\pi$ and $P$ define a Markov chain $\{X_n\}$ on $S$ by

$$P_{s,t} := \pi(t),$$

and $P$ is irreducible if for every $s, t \in S$, there exists $n \in \mathbb{N}$ such that

$$P_{s,t}^n := P(X_n = t | X_0 = s) > 0,$$

and $P$ is aperiodic if for every $s \in S$

$$\gcd\{n : P_{s,s}^n > 0\} = 1.$$ 

Given an irreducible and aperiodic $P$, there exists a unique stationary probability $\pi_P$, that is $\pi_P P = \pi_P$. In addition for every $s, t \in S$,

$$P_{s,t}^n \xrightarrow{n \to \infty} \pi_P(t).$$

Since $S$ is a finite state space, it follows that for any initial distribution $\pi$ on $S$,

$$\mathbb{P}_\pi (X_n = t) = \sum_{s \in S} \pi(s) P_{s,t}^n \xrightarrow{n \to \infty} \pi_P(t).$$

An important fact which will be used in the sequel is that the stationary distribution is continuous with respect to the stochastic matrix. That is if $\{P_n\}_{n=1}^\infty$ is a sequence of irreducible and aperiodic stochastic matrices such that

$$\|P_n - P\|_\infty := \max_{s,t \in S} |(P_n)_{s,t} - P_{s,t}| \xrightarrow{n \to \infty} 0$$

and $P$ is irreducible and aperiodic then

$$\|\pi_P - \pi P\|_\infty \to 0.$$

3.1.2. Non singular Markov shifts:

Non Singularity criteria for Markov shifts. In order to check if a measure is shift non singular we apply the following reasoning of [Shi], see also [LM] and [JOP].

Definition 3. Given a filtration $\{F_n\}$, we say that $\nu \ll_{loc} \mu$ ($\nu$ is locally absolutely continuous with respect to $\mu$) if for every $n \in \mathbb{N}$

$$\nu_n \ll \mu_n,$$

where

$$\nu_n = \nu |_{F_n}.$$
Suppose that $\nu \ll_{\text{loc}} \mu$ w.r.t. $\{F_n\}$, set

$$z_n := \frac{d\nu_n}{d\mu_n},$$

and

$$\alpha_n(x) := z_n(x) \cdot z_{n-1}^\oplus(x),$$

where $z_{n-1}^\oplus = \frac{1}{z_{n-1}} \cdot 1_{[z_{n-1} \neq 0]}$. The question is when $\nu \ll_{\text{loc}} \mu$ implies $\nu \ll \mu$.

**Theorem 4.** [Shi, Thm. 4, p. 528]. If $\nu \ll_{\text{loc}} \mu$ then $\nu \ll \mu$ if and only if

$$\sum_{k=1}^{\infty} \left[ 1 - E_\mu \left( \sqrt{\alpha_n} | F_k \right) \right] < \infty \quad \nu \text{ a.s.}$$

If $\nu \ll \mu$ then

$$\frac{d\nu}{d\mu} = \lim_{n \to \infty} z_n.$$

Given a a markovian measure $\mu = \mathcal{M} \{\pi_n, P_n : n \in \mathbb{Z}\}$ on $S^\mathbb{Z}$, we consider the algebra

$$F_n := \sigma \left\{ \left(b_n \right)^- b \in S^{\mathbb{Z}([-n,n])} \right\}. $$

We want to know when $\mu \sim \mu \circ T$.

In our examples, we will consider measures supported on the TMS $\Sigma_A$ and for every $n \in \mathbb{N}$,

$$\text{supp}P_n := \{(s,t) \in S \times S : P_n(s,t) > 0\} = \text{supp}A.$$

This implies that $\mu \circ T \ll_{\text{loc}} \mu$.

In addition the measure $\nu$ will be half stationary in the sense that for every $j \leq 0$

$$P_j := Q,$$

where $Q$ is the matrix from Example 2. This property implies that for every $j \leq 0$,

$$\pi_j = \pi Q,$$

and

$$\alpha_n(x) = \frac{P_{n-2}(x_n, x_{n-1})}{P_{n-1}(x_{n-1}, x_n)}, \quad \forall n > 0.$$  

So the shift will be $\mu$-non singular if and only if

$$\sum_{n=-\infty}^{\infty} \left[ 1 - E_\nu \left( \sqrt{\alpha_n} | F_{n-1} \right) \right] (x) = \sum_{n=-\infty}^{\infty} \left[ 1 - \sum_{s \in S} \sqrt{P_{n-2}(x_{n-1}, s)} P_{n-1}(x_{n-1}, s) \right] < \infty \quad \nu \circ T \text{ a.s. } x. $$

The following corollary concludes our discussion.

**Corollary 5.** Let $\nu = \mathcal{M} \{\pi_n, P_n : n \in \mathbb{Z}\}$, where $\{P_n\}$ are fully supported on a TMS $\Sigma_A$ and there exists an aperiodic and irreducible $P \in \mathcal{M}_{S \times S}$ such that for all $n \leq 0$, $P_n \equiv P$

- $\nu \circ T \sim \nu$ if and only if

$$\sum_{n=-\infty}^{\infty} \sum_{s \in S} \left( \sqrt{P_n(x_n, s)} - \sqrt{P_{n-1}(x_n, s)} \right)^2 < \infty, \quad \nu \circ T \text{ a.s. } x.$$
• If $\nu \circ T \sim \nu$ then for all $n \in \mathbb{N}$,

$$\left( T^n \right)'(x) = \prod_{k=0}^{\infty} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})}.$$ 

A condition for exactness of the one sided shift. Let $S$ be a countable set and $\{(\pi_n, P_n)\}_{n=1}^{\infty} \subset \mathcal{P}(S) \times \mathcal{M}_{S \times S}$. Denote the one sided shift on $S^\mathbb{N}$ by $\sigma$. The following is a sufficient condition for exactness (trivial tail $\sigma$-field) of the one sided shift. In the theory of non-homogenous Markov chains it is well known. We include a simple ergodic theoretic proof for the sake of completeness.

**Proposition 6.** Let $S$ be a countable set and $\mu$ be a Markovian measure on $S^\mathbb{N}$ which is defined by $\{\pi_k, \mu_k : k \in \mathbb{N}\}$. If there exists $C > 0$ and $N_0 \in \mathbb{N}$ so that for every $s, t \in S$, and $k \in \mathbb{N}$,

$$\left( P_k P_{k+1} \cdots P_{k+N_0} \right)_{s, t} \geq C$$

then the one sided shift $(S^\mathbb{N}, \mathcal{B}, \mu, \sigma)$ is exact.

**Remark.** In the setting of Markov maps, exactness was proved under various distortion properties [see [Aar1, Th]]. Their conditions guarantees the existence of an absolutely continuous $\sigma$-finite invariant measure.

**Proof.** The measure $\mu \circ T^{-n}$ is the Markov measure generated by $Q_k := P_{k+n}$ and $\pi_k := \pi_{k+n}$. Let $\alpha_n$ be the collection of cylinders of the form $[d]^n \cup \alpha^* = \bigcup_n \alpha_n$.

For every $D \in \alpha_n$ and $B \in \alpha^*$,

$$\mu \left( D \cap T^{-(n+N_0)} B \right) = \mu(D) \left( P_n P_{n+1} \cdots P_{n+N_0} \right)_{a_n, b_1} \prod_{j=1}^{\text{length}(B)-1} P_{N_0+n+j} (b_j, b_{j+1}).$$

Since

$$\prod_{j=1}^{\text{length}(B)-1} P_{N_0+n+j} (b_j, b_{j+1}) \geq \pi_{n+N_0+1} (b_1) \prod_{j=1}^{\text{length}(B)-1} P_{N_0+n+j} (b_j, b_{j+1}) = \mu \circ T^{n+N_0} (B),$$

it follows that

$$\mu \left( D \cap T^{-(n+N_0)} B \right) \geq \mu(D) \mu \circ T^{n+N_0} (B) \left( P_n P_{n+1} \cdots P_{n+N_0} \right)_{a_n, b_1} \geq C \cdot \mu(D) \mu \circ T^{n+N_0} (B).$$

Consequently for all $B \in \mathcal{B}$ and $D \in \alpha_n$,

$$\mu \left( D \cap T^{-(n+N_0)} B \right) \geq C \cdot \mu(D) \mu \circ T^{n+N_0} (B).$$

Now let $B \in \bigcap_{n=1}^{\infty} \sigma^{-n} \mathcal{B}(S^\mathbb{N})$ of positive $\mu$ measure. Then there exists $B_n \in B$ such that $B = T^{-n} B_n$. For every $D \in \alpha_n$,

$$\mu \left( D \cap B \right) = \mu \left( D \cap T^{-(n+N)} B_n \right) \geq C \cdot \mu(D) \mu \circ T^{n+N} (B_n) = C \mu(D) \mu(B).$$
Therefore for every $n \in \mathbb{N}$,
\[ \mu(B|\alpha_n) \geq C\mu(B) > 0 \]
It follows from the Martingale convergence theorem that
\[ \mu(B|\alpha_n)(x) \xrightarrow{n \to \infty} 1_B(x) \mu - a.s. \]
Therefore
\[ 1_B(x) \geq C\mu(B) > 0 \mu - a.s. \]
Whence $\mu(B) = 1$ (since $1_B = 1$ a.s.) and the shift is exact. \( \square \)

3.2. **Type III Markov Shifts.** We will construct a Markov measure supported on $\Sigma_A$ from Example\([2]\) In the end of this subsection we will explain what needs to be altered in the case of a general mixing TMS and conclude with some open questions.

In this subsection, let $\Omega := \Sigma_A$, $B := B_{\Sigma_A}$ and $T$ is the two sided shift on $\Omega$. For two integers $k < l$, write $F(k,l)$ for the algebra of sets generated by cylinders of the form $[b]_{l,k}$, $b \in \{1, 2, 3\}^{l-k}$. That is the smallest $\sigma$-algebra which makes the coordinate mappings $\{w_i(x) := x_i : i \in [k,l]\}$ measurable.

3.2.1. **Idea of the construction of the type III Markov measure.** The construction uses the ideas in [Kos]. For every $j \leq 0$
\[ P_j \equiv Q \text{ and } \pi_j \equiv \pi_Q, \]
where $Q$ and $\pi_Q$ are as in (2.1) and (2.2) respectively. On the positive axis one defines on larger and larger chunks the stochastic matrices which depend on a distortion parameter $\lambda_k \geq 1$ where 1 means no distortion. Now a cylinder set $[b]_{l,k}$ fixes the values of the first $n$ terms in the product form of the Radon Nykodym derivatives. We would like to be able to correct the values in order that we can enforce a given number to be in the ratio set. This corresponds to a lattice condition on $\lambda_k$ which is less straightforward then the one in [Kos]. However this is not enough for a Markov measure, since the states are not independent, this forces us to utilize strongly the convergence to the stationary distribution and the mixing property for stationary chains.

Another difficulty is that the measure of the set $[b]_{l,k} \cap T^{-N}[b]_{l,k} \cap \{(T^N)' \approx a\}$ could be very small which forces us to look for many approximately independent such events so that their union covers at least a fixed proportion of $[b]_{l,k}$.

More specifically the construction goes as follows. We define inductively 5 sequences $\{\lambda_j\}$, $\{m_j\}$, $\{n_j\}$, $\{N_j\}$ and $\{M_j\}$ where
\[ M_0 = 1 \]
\[ N_j := N_{j-1} + n_j \]
\[ M_j := N_j + m_j. \]
This defines a partition of $\mathbb{N}$ into segments $\{[M_{j-1}, N_j), [N_j, M_j)\}_{j=1}^{\infty}$. The sequence $\{P_n\}$ equals $Q$ on the $[N_j, M_j)$ segments while on the $[M_{j-1}, N_j)$ segments we have $P_n \equiv Q_{\lambda_j}$, the $\lambda_j$ perturbed stochastic matrix. The $Q$ segments facilitate the form of some of the Radon Nykodym derivatives while the perturbed segments come to ensure that $\mu \perp M\{\pi_Q, Q\}$ and that the ratio set condition is satisfied for cylinder sets.

Notation: By $x = a \pm b$ we mean $a - b \leq x \leq a + b$. 

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3.2.2. The construction. Choice of the base of induction: Let \( M_0 = 1 \), \( \lambda_1 > 1 \), \( n_1 = 2 \), \( N_1 = 3 \) and

\[
Q_1 := \begin{pmatrix}
\frac{\lambda_1 \varphi}{1 + \lambda_1 \varphi} & 0 & \frac{1}{1 + \lambda_1 \varphi} \\
0 & 1 & 0 \\
\frac{1}{1 + \varphi} & 0 & 1 \\
\end{pmatrix}
\]

be the \( \lambda_1 \) perturbed matrix. Set \( P_1 = P_2 = Q_1 \) and \( \pi_0 = \pi Q \). The measures \( \pi_1, \pi_2 \) are then defined by equation (2.3). Let \( m_1 = 3 \) and thus \( M_1 = 6 \). Set \( P_j = Q \) for \( j \in [N_1, M_1) = [3, 6) \) and \( \pi_3, \pi_4, \pi_5 \) be defined by equation (2.3).

Assume that \( \{\lambda_j, m_j, n_j, N_j, M_j\}_j=1^{l-1} \) have been chosen.

**Choice of \( \lambda_l \):** Notice that the function

\[
f(x) := \frac{1 + \varphi}{1 + \varphi x}
\]

is monotone increasing and continuous in the segment \([1, \infty)\). Therefore we can choose \( \lambda_n > 1 \) which satisfies the following three conditions:

1. **Finite approximation of the Radon-Nykodym derivatives condition:**

   \[
   (\lambda_l)^{2m_{l-1}} < e^{\frac{1}{2^l}}.
   \]

   This condition ensures an approximation of the derivatives by a finite product.

2. **Lattice condition:**

   \[
   \lambda_{l-1} \cdot \frac{1 + \varphi}{1 + \varphi \lambda_{l-1}} \in \left( \lambda_l \cdot \frac{1 + \varphi}{1 + \varphi \lambda_l} \right)^N,
   \]

   where \( a^N := \{a^n : n \in \mathbb{N}\} \).

3. **Let**

\[
Q_l := \begin{pmatrix}
\frac{\varphi \lambda_l}{1 + \varphi \lambda_l} & 0 & \frac{1}{1 + \varphi \lambda_l} \\
0 & 1 & 0 \\
\frac{1}{1 + \varphi} & 0 & 1 \\
\end{pmatrix},
\]

and \( \pi Q_l \) it’s unique stationary probability. Notice that when \( \lambda_l \) is close to 1, then \( Q_l \) is close to \( Q \) in the \( L_\infty \) sense. Therefore by continuity of the stationary distribution we can demand that

\[
\|\pi Q - \pi Q_l\|_\infty < \frac{1}{2^l}.
\]

**Choice of \( n_l \):** It follows from the Lattice condition, equation (3.4), that for each \( k \leq l-1 \),

\[
\left( \lambda_k \cdot \frac{1 + \varphi}{1 + \varphi \lambda_k} \right) \in \left( \lambda_l \cdot \frac{1 + \varphi}{1 + \varphi \lambda_l} \right)^N.
\]

Choose \( n_l \) large enough so that for every \( k \leq l-1 \) (notice that the demand on \( k = 1 \) is enough) there exists \( N \ni p = p(k, l) \leq \frac{n_l}{20} \) so that

\[
\left( \lambda_l \cdot \frac{1 + \varphi}{1 + \varphi \lambda_l} \right)^p = \left( \lambda_k \cdot \frac{1 + \varphi}{1 + \varphi \lambda_k} \right).
\]
Till now we have defined \( \{ P_j, \pi_j \}_{j=-\infty}^{M_{l-1}} \). By the mean ergodic theorem for Markov chains [LPW, Th. 4.16] and (3.5), one can demand by enlarging \( n_l \) if necessary that in addition

\[
\nu_{\pi_{M_{l-1}}, Q_l} \left( x : \frac{1}{n_l} \sum_{j=1}^{n_l} 1_{[x_j=1]} \in \left( \frac{1}{\sqrt{5}} - \frac{1}{2}, \frac{1}{\sqrt{5}} + \frac{1}{2} \right) \right) > 1 - \frac{1}{l},
\]

and

\[
\nu_{\pi_{M_{l-1}}, Q_l} \left( x : \frac{1}{n_l} \sum_{j=1}^{n_l} 1_{[x_j=2, x_{j+1}=3]} > \frac{1}{15} \right) > 1 - \frac{1}{l},
\]

where \( \nu \) is the Markov measure on \( \{1, 2, 3\}^N \) defined by and \( Q_l \) and \( \pi_{M_{l-1}} \). The numbers inside the set were chosen since

\[
\nu_{\pi_{M_{l-1}}, Q_l}(1) \in \left( \frac{1}{\sqrt{5}} - \frac{1}{2}, \frac{1}{\sqrt{5}} + \frac{1}{2} \right),
\]

and similarly for \( l \) large enough

\[
\int 1_{[x_0=2, x_1=3]}(x) d\nu_{\pi_{Q_l}, Q_l} = \nu_{\pi_{Q_l}(2)}(Q_l)_{2,3} = \left( \frac{1}{\sqrt{5}} \pm \frac{1}{2} \right) \frac{1}{\varphi + 1} > \frac{1}{15}.
\]

**Choice of \( N_l \):** Let \( N_l := M_{l-1} + n_l \). Now set for all \( j \in [M_{l-1}, N_l) \),

\[
P_j = Q_l
\]

and \( \{ \pi_j \}_{j=M_{l-1}+1}^{N_l} \) be defined by equation (2.3).

**Choice of \( m_l \):** Let \( k_l \) be the \( \left( 1 \pm \left( \frac{1}{3} \right)^{3N_l} \right) \) mixing time of \( Q \). That is for every \( n > k_l \), \( A \in \mathcal{F}(0, l) \), \( B \in \mathcal{F}(l + n, \infty) \) and initial distribution \( \tilde{\pi} \),

\[
\nu_{\tilde{\pi}, Q}(A \cap B) = \left( 1 \pm 3^{-3N_l} \right) \nu_{\tilde{\pi}, Q}(A) \nu_{\pi_{Q_l}}(T^n B).
\]

Demand in addition that \( k_l > N_l \) and

\[
\left\| \pi_{M_l} Q^{k_l} - \pi_{Q_l} \right\|_\infty < 3^{-3N_l}.
\]

To explain the last condition notice that equation (2.3) together with the fact that \( P_j \) is constant on blocks means that

\[
\pi_{N_l} Q^m = \pi_{N_l+m}.
\]

Let \( m_l \) be large enough so that

\[
(1 - 9^{-3N_l}) ^{m_l/4k_l} \leq \frac{1}{l},
\]

and

\[
(m_l - N_l) \lambda_1^{-2N_l} \geq 1.
\]

To summarize the construction. We have defined inductively sequences \( \{ n_l \}, \{ N_l \}, \{ m_l \}, \{ M_l \} \) of integers which satisfy

\[
M_l < N_{l+1} = M_l + n_l < M_{l+1} = N_{l+1} + m_{l+1}.
\]
In addition we have defined a monotone decreasing sequence \( \{ \lambda_j \} \) which decreases to 1 and using that sequence we defined new stochastic matrices \( \{ Q_j \} \). Now we set

\[
P_j := \begin{cases} Q, & j \leq 0 \\ Q_i, & M_{l-1} \leq j < N_l, \\ Q, & N_l \leq j < M_l 
\end{cases}
\]

and \( \pi_j = \pi_p \) for \( j \leq 0 \). The rest of the \( \pi_j \)'s are defined by the consistency condition, equation (2.3). Finally let \( \mu \) be the Markovian measure on \( \{1, 2, 3\}^\mathbb{Z} \) defined by \( \{ \pi_j, P_j \}_{j=-\infty}^{\infty} \).

Notice that for all \( j \in \mathbb{N} \), \( \text{supp} P_j \equiv \text{supp} A = \text{supp} Q \).

### 3.2.3. Statement of the Theorem and the proof of non singularity and conservativity.

**Theorem 7.** The shift \( \{1, 2, 3\}^\mathbb{Z}, \mu, T \) is non singular, conservative, ergodic and of type III₁.

**Proof.** [Non Singularity and \( K \) property]

Since \( \mu \circ T \) is the markovian measure generated by \( \tilde{P}_j = P_{j-1} \) and \( \tilde{\pi}_j = \pi_{j-1} \), it follows from (3.1) and the block structure of \( P_j \) that the shift is non singular if and only if

\[
\sum_{t=1}^{\infty} \sum_{s \in S} \left\{ \left( \sqrt{P_{N_t}(x_{N_t}, s)} - \sqrt{P_{N_{t-1}}(x_{N_{t-1}}, s)} \right)^2 \\
+ \left( \sqrt{P_{M_t}(x_{M_t}, s)} - \sqrt{P_{M_{t-1}}(x_{M_{t-1}}, s)} \right)^2 \right\} < \infty, \quad \mu \circ T \text{ a.s. } x.
\]

Since for all \( j \in \mathbb{Z} \), \( P_j(3, 2) \equiv 1 \), \( P_j(2, 1) = 1 - P_j(2, 3) \equiv \frac{x}{1+\varphi} \), the sum is dominated by

\[
\sum_{k=1}^{\infty} \sum_{s \in S} \left\{ 2 \left( \sqrt{P_{N_k}(1, s)} - \sqrt{P_{N_{k-1}}(1, s)} \right)^2 \right\} = \sum_{k=1}^{\infty} \left\{ \left( \frac{\lambda_k \varphi}{1 + \lambda_k \varphi} - \sqrt{\frac{\varphi}{1 + \varphi}} \right)^2 \\
+ \left( \sqrt{\frac{1}{1 + \lambda_k \varphi}} - \sqrt{\frac{1}{1 + \varphi}} \right)^2 \right\}.
\]

This sum converges or diverges together with \( \sum_{k=1}^{\infty} |\lambda_k - 1|^2 \). As a consequence of condition (3.3) on \( \{ \lambda_j \} \), this sum is finite. Since \( P_j \equiv Q \) for all \( j \leq 0 \),

\[
T'(x) = \frac{d \mu \circ T}{d \mu}(x) = \prod_{k=1}^{\infty} \frac{P_{k-1}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})}.
\]

The sequence \( \{ P_j \}_{j \in \mathbb{Z}} \) satisfies

\[
\inf \{ (P_j P_{j+1} P_{j+2})(s, t) : s, t \in \{1, 2, 3\}, j \in \mathbb{Z} \} := c > 0,
\]

thus by Proposition 6 the one sided shift \( \{1, 2, 3\}^\mathbb{N}, \mathcal{F}, \mu_+, \sigma \) is an exact factor and \( T' \) is \( \mathcal{F} \) measurable and thus the shift is a \( K \) automorphism. Here \( \mu_+ \) denotes the measure on the one sided shift space defined by \( \{ \pi_j, P_j \}_{j \geq 1} \).

In order to show the other properties of the Markov Shift, we will need a more concrete expression of the Radon Nykodym derivatives. The measure \( \mu \), or more concretely it's
transition matrices, differs from the stationary \( \{ \pi_Q, Q \} \) measure only when one moves inside state 1 in the segments \([M_j, N_{j+1}]\). Denote by

\[ L_j(x) := \# \{ k \in [M_{j-1}, N_j) : x_k = 1 \} \]

and

\[ V_j(x) = \# \{ k \in [M_{j-1}, N_j) : x_k = x_{k+1} = 1 \}. \]

**Lemma 8.** For every \( \epsilon > 0 \), there exists \( t_0 \in \mathbb{N} \) s.t for every \( t > t_0, N_t \leq n < m_t \) and \( x \in \{1, 2, 3\}^\mathbb{Z} \),

\[ (T^n)'(x) = (1 + \epsilon) \prod_{k=1}^t \left[ \left( \frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k \circ T^n(x) - L_k(x)} \lambda_k^{V_k \circ T^n(x) - V_k(x)} \right]. \]

**Proof.** Let \( \epsilon > 0, t \in \mathbb{N} \) and \( N_t \leq n < m_t \). Canceling out all the \( k \)'s such that \( P_{k-n} = P_k \) one can see that

\[ (T^n)'(x) = I_t \cdot \tilde{I}_t \]

where

\[ I_t = \prod_{u=1}^t \left[ \left( \prod_{k=M_{u-1}}^{N_u} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \right) \cdot \left( \prod_{k=M_{u-1} + n}^{N_u+n-1} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \right) \right] \]

and

\[ \tilde{I}_t = \prod_{u=t+1}^\infty \left[ \left( \prod_{k=M_{u-1}}^{N_{u-1}+n-1} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \right) \cdot \left( \prod_{k=N_u}^{N_u+n-1} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \right) \right]. \]

We will analyze the two terms separately. Since for every \( M_{u-1} \leq k < M_{u-1} + n, P_k = Q_u \) and \( P_{k-n} = Q \),

\[ \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \leq \frac{Q_{1,1}}{(Q_u)_{1,1}} = \frac{1 + \varphi \lambda_u}{1 + \varphi} \leq \lambda_u. \]

Similarly for \( N_u \leq k < N_u + n, P^{(k)} = Q \) and \( P_{k-n} = Q_u \). Therefore

\[ \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \leq \frac{(Q_u)_{1,1}}{Q_{1,1}} \leq \lambda_u. \]

and

\[ \lambda_u^{-2n} \leq \left( \prod_{k=M_{u-1}}^{N_{u-1}+n-1} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \right) \cdot \left( \prod_{k=N_u}^{N_u+n-1} \frac{P_{k-n}(x_k, x_{k+1})}{P_k(x_k, x_{k+1})} \right) \leq \lambda_u^{2n}, \]

here the lower bound is achieved by a similar analysis. This gives

\[ \tilde{I}_t = \prod_{u=t+1}^\infty \left[ \lambda_u^{-2n} \right] = \prod_{u=t+1}^\infty \left[ \lambda_u^{-2m_n} \right] \quad \text{(since } \forall u > t, n < m_t < m_u) \]

\[ e^{\sum_{n=t+1}^\infty \frac{1}{n}} \xrightarrow{t \to \infty} 1. \]
Consequently there exists \( t_0 \in \mathbb{N} \) so that for all \( x \in \Sigma_A \), for all \( t > t_0 \) and \( N_t \leq n \leq m_t \),

\[
(T^n)'(x) = (1 \pm \epsilon)I_t.
\]

By noticing that for \( k \in \bigcup_{j=1}^k ([M_{j-1}, N_j) \cup [M_{j-1} + n, N_j + n]) \),

\[
P_{k-n}(x_k, x_{k+1}) \neq P_k(x_k, x_{k+1})
\]

if and only if \( x_k = 1 \) one can check that

\[
I_t = \prod_{k=1}^{t} \left( \frac{1 + \varphi}{1 + \varphi \lambda_k} \right) \frac{L_k \circ T^n(x) - L_k(x)}{\lambda_k} \frac{V_k \circ T^n(x) - V_k(x)}{\lambda_k}.
\]

\[\square\]

**Corollary 9.** The shift \((\{1, 2, 3\}^\mathbb{Z}, \mu, T)\) is conservative and ergodic.

**Proof.** Since the shift is a \( K \)-automorphism it is enough to prove conservativity.

For every \( j \in \mathbb{N} \), \( 0 \leq L_k(x), D_k(x) \leq n_k \). Whence

\[
\left( \frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k \circ T^n(x) - L_k(x)} \frac{V_k \circ T^n(x) - V_k(x)}{\lambda_k} \geq \left( \frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k \circ T^n(x)} \frac{V_k \circ T^n(x)}{\lambda_k} \geq \lambda_k^{-2n_k} \geq \lambda_1^{-2n_k}.
\]

Consequently for every \( t \in \mathbb{N} \),

\[
\prod_{k=1}^{t} \left( \frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k \circ T^n(x) - L_k(x)} \frac{V_k \circ T^n(x) - V_k(x)}{\lambda_k} \geq \lambda_1^{-2} \sum_{k=1}^{n_k} \lambda_k^{-2N_t} \geq \lambda_1^{-2N_t}.
\]

It follows form Lemma 8 that there exists \( t_0 \in \mathbb{N} \) such that for all \( t \geq t_0 \), \( N_t \leq n \leq m_t \) and \( x \in \Sigma_A \),

\[
(T^n)'(x) \geq \frac{\lambda_1^{-2N_t}}{2}.
\]

Therefore for all \( x \in \Sigma_A \),

\[
\sum_{n=1}^{\infty} (T^n)'(x) \geq \sum_{t=1}^{\infty} \sum_{n=N_t}^{m_t} (T^n)'(x) \geq \sum_{t=1}^{\infty} \frac{1}{2} (m_t - N_t) \lambda_1^{-2N_t}.
\]

\[\text{\ref{eq:3.12}}\]

By Hopf’s criteria the shift is conservative. \[\square\]

**3.2.4. Proof of the type \( \text{III}_1 \) property.** In order to prove that the ratio set is \([0, \infty)\) we are going to use the following principle: since \( R(T) \) is a multiplicative subset it is enough to show that there exists \( y_n \in R(T) \setminus \{1\} \) with \( y_n \to 1 \) as \( n \to \infty \).

**Theorem 10.** Let \( \mu \) be the Markovian measure constructed in Subsection 3.2.2. For every \( n \in \mathbb{N} \), \( \lambda_n \cdot \frac{1 + \varphi}{1 + \varphi \lambda_n} \in R(T) \) and therefore the shift is type \( \text{III}_1 \).
Fix $n \in \mathbb{N}$. The first stage in proving that $\lambda_n \cdot \frac{1+\varphi}{1+\varphi \lambda_n} \in R(T)$ is to show that the ratio set condition is satisfied for all cylinders with a positive proportion of the measure of the cylinder set. Then for a general $A \in \mathcal{B}_+$, we use the density of cylinder sets in $\mathcal{B}$.

Given $t \in \mathbb{N}$, denote by $\mathcal{C}(t)$ the collection of all $[c_0^N]$ cylinder sets such that

$$L_t(c) = \sum_{k=M_t-1}^{N_t-1} 1_{[c_k=1]} \leq \frac{n_t}{4} : \frac{n_t}{2}$$

and

$$(3.13) \sum_{k=M_t-1}^{N_t-1} 1_{[c_k=2, c_{k+1}=3]} \geq \frac{n_t}{15}.$$  

Since

$$\mu ([c]^N_{M_t-1}) = \nu_{\pi_{M_t-1}} Q_\pi ([c]^N_{0}),$$

it follows from (3.7) and (3.8) that for all $t$ large enough,

$$\mu \left( \bigcup_{C \in \mathcal{C}(t)} C \right) \geq 1 - \frac{1}{2^t}.$$  

In order to shorten the notation, given $M, j \in \mathbb{N}, B \in \mathcal{B}$ and $\epsilon > 0$, let

$$\mathcal{R}_B \subseteq (M, B, j, \epsilon) := B \cap T^{-M} B \cap \left( T^M \right)' = \lambda_j \cdot \frac{1+\varphi}{1+\varphi \lambda_j} \cdot (1+\epsilon),$$

and for $M \in \mathbb{N}$,

$$\Sigma_M (M) := \{1, 2, 3\}^M \cap \Sigma_A.$$  

**Lemma 11.** For every $[b]^n_{-n}$ cylinder set, $\epsilon > 0$ and $j \in \mathbb{N}$, there exists a $t_0 \in \mathbb{N}$ so that for all $t > t_0$ the following holds:

For every $C = [c]^N_{-N} \in \mathcal{C}(t)$ there exists $d = d(b, C) \in \Sigma_M (N_t + n)$ such that for every $N \geq l \leq m_t/k_t$,

$$(3.14) \quad [d]_{lk_l-1}^{lk_l+N_t-1} \subset T^{-lk_l} [b]^n_{-n} \cap \left( T^{lk_l} \right)' = \lambda_j \cdot \frac{1+\varphi}{1+\varphi \lambda_j} \cdot (1+\epsilon).$$

Recall that $k_l > N_t$ is defined as a $(1+3^{-3N_t})$ mixing time for $Q$.

**Proof.** Let $[b]^n_{-n}, \epsilon > 0$ and $j \in \mathbb{N}$ be given. By Lemma 8 there exists $\tau$ such that for every $t \geq \tau$ and $1 \leq l \leq m_t/k_t$ (here $lk_l \in [N_t, m_t)$),

$$\left( T^{lk_l} \right)' (x) = (1+\epsilon) \prod_{k=1}^{l} \left( \frac{1+\varphi}{1+\varphi \lambda_k} \right)^{L_k(x) - L_k^{lk_l}(x)} V_k^{\varphi T^{lk_l}(x)-V_k(x)}. $$

Choose $T_0$ to be any integer which satisfies $t_0 > \max (\tau, j)$ and $M_{t_0} > n$.

Let $t > t_0$ and choose a cylinder set $[c]^N_{0} \in \mathcal{C}(t)$ which intersects $[b]^n_{-n}$. That is $c_i = b_i$ for $i \in [0, n]$. We need now to choose $d \in \Sigma_M (N_t + n)$ which satisfies (3.14). Notice that
for \( x \in \lbrack d_{lk_t-n}^{lk_t+N_t} \cap [c]_0^{N_t} \),
\[
\prod_{k=1}^t \left( \frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k \circ T^{lk_t}(x) - L_k(x)} \cdot \lambda_k^{V_k \circ T^{lk_t}(x) - V_k(x)}
\]
\[
= \prod_{k=1}^t \left( \frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k(d) - L_k(c)} \cdot \lambda_k^{V_k(d) - V_k(c)}
\]
in this representation we look at \([d]_{-n}^{N_t}\). For all \( k \in [0, M_t-1] \), let
\[
d_k = c_k
\]
and for all \( k \in [-n, 0) \),
\[
d_k = b_k.
\]
Notice that this means that for \( k \in [-n, n] \), \( d_k = b_k \) and thus
\[
[d]_{lk_t-n}^{lk_t+n} \subset T^{-lk_t} [b]_{-n}^n.
\]
Let \( p(j, t) \leq \frac{n}{3\vartheta} \) be the integer (condition (3.6)) such that
\[
\left( \lambda_t \cdot \frac{1 + \varphi}{1 + \varphi \lambda_t} \right)^{p(j, t)} = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j}.
\]
Set \( d_k = 1 \) for all \( k \in [M_t-1, M_t-1 + V_{t}(c) + p(j, t)] \) and then continue repeatedly with the sequence ”321” \( L_t(c) - V_t(c) \) times. Since \( c \) satisfies (3.13), this construction is well defined (e.g. we have not reached yet \( k = N_t - 1 \)). Continue with sequences of 32 till \( k = N_t - 1 \).

Thus we have defined \( d \) in such a way so that
\[
L_t(d) - L_t(c) = p(j, t)
\]
and
\[
V_t(d) - V_t(c) = p(j, t).
\]
In addition for all \( 0 \leq k < t \),
\[
L_k(d) = L_k(c) \text{ and } V_k(c) = V_k(d).
\]
Thus for all \( x \in \lbrack d_{lk_t-n}^{lk_t+n} \cap [c]_0^{N_t} \),
\[
\left( T^{lk_t} \right)'(x) = (1 + \varepsilon) \prod_{k=1}^t \left( \frac{1 + \varphi}{1 + \varphi \lambda_k} \right)^{L_k(d) - L_k(c)} \cdot \lambda_k^{V_k(d) - V_k(c)}\]
\[
= (1 + \varepsilon) \left( \lambda_t \cdot \frac{1 + \varphi}{1 + \varphi \lambda_t} \right)^{p(j, t)}
\]
\[
= (1 + \varepsilon) \left( \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \right).
\]
This proves the lemma. \( \square \)

In the course of the proof one sees that the event
\[
\left( [b]_{-n}^n \cap [c]_0^{N_t} \right) \cap \left( T^{-lk_t} [b]_{-n}^n \cap \left\{ \left( T^{lk_t} \right)' = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 + \varepsilon) \right\} \right)^c
\]
is \( \mathcal{F} (lk_t - n, lk_t + N_t) \) measurable and does not depend on \( \mathcal{F} (lk_t + N_t, lk_t + 2N_t) \).
Remark 12. Given \([c]_0^{N_t} \in \mathcal{C}_t\) we have defined \(d = d(c) \in \Sigma_A (N_t + n)\). The definition of \(d\) is not necessarily one to one. This is because if \([c]_0^{M_{t-1}} = [\hat{c}]_0^{M_{t-1}}, V_t(c) = V_t(\hat{c})\) and \(L_t(c) = L_t(\hat{c})\) then \(d(c) = d(\hat{c})\). In order to make it one to one we will use

\[
[d(c), c]_{lk_t + 2N_t}
\]

instead of \([d(c)]_{lk_t + N_t}\) where by \([a, b]_l^{l + \text{length}(a) + \text{length}(b)}\) we mean the concatenation of \(a\) and \(b\). This can be thought of as putting a Marker on \(d(c)\). In order that the concatenation will be in \(\Sigma_A\) we need that

\[
Q(d(c)_{N_t-1}, c_0) > 0.
\]

This can be done by possibly changing the last two coordinates of \(d(c)\). This will change the value of \((T^{4k_t})'\) by at most a factor of \(\lambda_t^{24}\), which is close enough to one. We will denote by \(\mathbf{d}(c) := (d(c), c)\). We still have

\[
[d(c), c]_{lk_t + 2N_t} \subset T^{-lk_t}[b]_{l - n}^n \cap \left((T^{4k_t})' = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} (1 + \epsilon)\right),
\]

but now the map \(c \mapsto \mathbf{d}(c)\) is one to one.

In the proof of the next lemma we will make use of the fact that for every cylinder set \(([a]_{m})^c\) is \(\mathcal{F}(m, l)\) measurable.

Lemma 13. For every \([b]_{m-n}^n\) cylinder set, \(\epsilon > 0\) and \(j \in \mathbb{N}\) there exists \(t_0 \in \mathbb{N}\) such that for all \(t > t_0\),

\[
\mu \left( \bigcup_{l=1}^{m_t/4k_t} \mathcal{R}\mathcal{C} (4l_k, [b]_{m-n}^n, j, \epsilon) \right) \geq 0.8 \mu ([b]_{m-n}^n).
\]

Proof. Let \([b]_{m-n}^n\) be a cylinder set and \(t_0\) be as in Lemma 11. For all \(t \geq t_0\), \([c]_{0}^{N_t} \in \mathcal{C}(t)\) which intersects \([b]_{m-n}^n\) and \(1 \leq l \leq m_t/4k_t\),

\[
([c]_0^{N_t} \cap [b]_{m-n}^n) \cap (\mathcal{R}\mathcal{C} (4l_k, [b]_{m-n}^n, j, \epsilon))^c \subset ([c]_0^{N_t} \cap [b]_{m-n}^n \cap (\mathbf{d}(c))_{4l_k - n}^{4l_k + N_t})^c.
\]

The fact that \(P_j \equiv Q\) for \(j \in [N_t, M_t)\) implies that

\[
\mu \left( [d(c)]_{2l_k - n}^{4l_k + 2N_t} \right) \geq \nu_{\pi_{4l_k - n}} Q \left( [d(c)]_{0}^{4l_k + 2N_t + n} \right) \geq \frac{\pi_{4l_k - n}}{\pi_{4l_k - n} (d_{-n})} \frac{Q_{l_k + n - 1}}{Q_{l_k + n - 1}} \geq \frac{1}{3^N} \frac{Q_{l_k + n - 1}}{Q_{l_k + n - 1}} \quad (\text{by } (3.10))
\]

and

\[
\frac{1}{3^N}.
\]
Therefore, since $k_t$ is a $(1 \pm \left(\frac{1}{3}\right)^{3N_l})$ mixing time, one has by many application of (3.9)

$$
\mu \left( \left( [b]_{-n} \cap [c]_0^{N_l} \right) \cap \left( \bigcup_{l=1}^{m_t/4k_t} \mathcal{RSE} \left( l k_t, [b]_{-n}, j, \epsilon \right) \right) \right) 
\leq \mu \left( \left( [b]_{-n} \cap [c]_0^{N_l} \right) \cap \left( \bigcup_{l=1}^{m_t/4k_t} \left( [d_c]_{4l k_t+N_l} \right)^c \right) \right) 
\leq \mu \left( \left( [b]_{-n} \cap [c]_0^{N_l} \right) \cap \left( \bigcup_{l=1}^{m_t/4k_t} \left( 1 + 3^{-3N_l} \right) \left( 1 - \nu_{\mathcal{Q}, \mathcal{Q}} \left( [d_c]_{4l k_t+N_l} \right) \right) \right) \right) 
\leq \mu \left( \left( [b]_{-n} \cap [c]_0^{N_l} \right) \cap \left( 1 - \frac{t}{4} \right) \mu \left( [b]_{-n} \cap [c]_0^{N_l} \right) \right) 
\leq 0.9 \mu \left( [b]_{-n} \cap [c]_0^{N_l} \right).
$$

Notice that we used the fact that

$$
(4 (l + 1) k_t - n) - (4l k_t + 2N_l) > (4l + 3) k_t - (4l + 2) k_t = k_t.
$$

and $k_t$ is a mixing time. If $t$ is large enough then

$$
\mu \left( \bigcup_{C \in \mathcal{C}(t)} \Sigma_A \setminus C \right) < 0.1 \mu \left( [b]_{-n} \right),
$$

and for all $[c]_0^{N_l} = C \in \mathcal{C}(t),$

$$
\mu \left( [b]_{-n} \cap [c]_0^{N_l} \cap \left( \bigcup_{l=1}^{m_t/4k_t} \mathcal{RSE} \left( l k_t, [b]_{-n}, j, \epsilon \right) \right) \right) > \left( 1 - \frac{t}{4} \right) \mu \left( [b]_{-n} \cap [c]_0^{N_l} \right)
\geq 0.9 \mu \left( [b]_{-n} \cap [c]_0^{N_l} \right).
$$

The Lemma follows from

$$
\mu \left( \bigcup_{l=1}^{m_t/4k_t} \mathcal{RSE} \left( l k_t, [b]_{-n}, j, \epsilon \right) \right)
\geq \mu \left( \bigcup_{[c]_0^{N_t} \in \mathcal{C}(t)} \left[ b \cap [c]_0^{N_t} \cap \bigcup_{l=1}^{m_t/4k_t} \mathcal{RSE} \left( l k_t, [b]_{-n}, j, \epsilon \right) \right] \right)
\geq 0.9 \sum_{[c]_0^{N_t} \in \mathcal{C}(t)} \mu \left( [b]_{-n} \cap [c]_0^{N_t} \right)
\geq 0.8 \mu \left( [b]_{-n} \right).
$$

\(\square\)

**Proof of Theorem 10.** This is a standard approximation technique. Let $j \in \mathbb{N}, A \in \mathcal{B},$ \(\mu(A) > 0\) and \(\epsilon > 0\). Since the ratio set condition on the derivative is monotone with respect to \(\epsilon\) and
\[1 < \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} < 2,\]

we can assume that
\[(3.15) \quad 1 \leq \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} (1 + \epsilon) \leq 2.\]

Since \(\mathcal{F}(-n, n) \uparrow \mathcal{B}\) as \(n \to \infty\), there exists a cylinder set \(b = [b]_n^m\) such that
\[
\mu(A \cap b) > 0.99\mu(b).
\]

By Lemma 13 there exists \(t \in \mathbb{N}\) for which
\[
\mu\left( b \cap \left( \bigcup_{l=1}^{m_t/4k_t} T^{-4lk_t} b \cap \left[ \left(T^{4lk_t} \right) = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 + \epsilon) \right] \right) \right) > 0.8\mu(b).
\]

Denote by
\[
B = b \cap \left( \bigcup_{l=1}^{m_t/4k_t} T^{-4lk_t} b \cap \left[ \left(T^{4lk_t} \right) = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 + \epsilon) \right] \right).
\]

We can assume that for \(x \in B\), there exists \(C(x) = [c]^{N_t}_0 \in \mathcal{C}_t\) so that \(x \in C(x)\). Then by the proof of Lemma 11 there exists \(d(C(x)) \in \Sigma_{\mathbf{A}} (2N_t + n)\) such that if \(x \in [d(C(x))]_{4lk_t-n}^{4lk_t+2N_t}\), then
\[(3.16) \quad \left(T^{4lk_t} \right)(x) = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 + \epsilon) \quad \text{and} \quad x \in T^{-4lk_t} b.
\]

Define \(\phi : B \to \mathbb{N}\)
\[
\phi(x) := \inf \left\{ l \leq m_t/4k_t : [c]^{4lk_t+2N_t}_{4lk_t-n} = [d(C(x))]_{4lk_t-n}^{4lk_t+2N_t} \right\}
\]
and \(S = T^\phi : B \to S(B) \subset b\). We claim that \(S\) is one to one. Indeed, since the map \([c]^{N_t}_0 \to d(c)\) is one to one, for every \(x, y \in B\) such that \(C(x) \neq C(y)\),
\[
[Sy]^{2N_t} = [d(C(y))]^{2N_t} \neq [d(C(x))]^{2N_t} = [Sx]^{2N_t},
\]
consequently \(Sx \neq Sy\). In addition, by the definition of \(\phi\), if \(x \neq y\) and \(C(x) = C(y)\) then \(Sx \neq Sy\).

It follows from (3.16) and (3.15), that for all \(x \in B\),
\[
S'(x) := \frac{d\mu \circ S}{d\mu}(x) = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 + \epsilon) \in [1, 2].
\]

Therefore \(\frac{d\mu \circ S}{d\mu}(y) \geq \frac{1}{2}\) for all \(y \in S(B)\). A calculation shows that
\[
\mu(S(B) \cap A) > \mu(S(B)) - \mu(b \setminus A) > \mu(B) - \mu(b \setminus A) = 0.79\mu(b),
\]
and
\[
\mu\left( S^{-1} (S(B) \cap A) \right) > \frac{\mu(S(B) \cap A)}{2} > 0.39\mu(b).
\]
So

\[
\sum_{l=1}^{m_l/4k_1} \mu \left( A \cap \bigcup_{l=1}^{m_l/4k_1} T^{-4lk_1} A \cap \left[ \left( T^{4lk_1} \right)' = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 \pm \epsilon) \right] \right) \cap [\phi = 4lk_1] \geq \mu \left( (B \cap A) \cap S^{-1} (S(B) \cap A) \right) \quad \text{\{Notice that } B, S(B) \subset b\} \\
\geq \mu (B \cap A) - \mu (b \backslash S^{-1} (S(B) \cap A)) \\
\geq 0.18 \mu (b) ,
\]

and thus there exists \( l \in \mathbb{N} \) such that

\[
\mu \left( A \cap T^{-4lk_1} A \cap \left[ \left( T^{4lk_1} \right)' = \lambda_j \cdot \frac{1 + \varphi}{1 + \varphi \lambda_j} \cdot (1 \pm \epsilon) \right] \right) > 0.
\]

This proves the Theorem. \( \square \)

**Remark 14.** One feature of this construction is that if \( f(x, y) = (x + y, x) \), \( \Phi : \Sigma_A \rightarrow \mathbb{T}^2 \) is the semi conjugacy map constructed from the Markov partition \( \{R_1, R_2, R_3\} \) and \( \nu = \mu \circ \Phi^{-1} \), then:

- \( \Phi : (\Sigma_A, \mu, T) \rightarrow (\mathbb{T}^2, \nu, f) \) is one to one, hence a measure theoretic isomorphism.
- For every \( n_k < n_{k-1} < \cdots < n_1 < 0 \) and \( i_1, \ldots, i_k \in \{1, 2, 3\}, \)

\[
\nu \left( \bigwedge_{j=1}^{k} f^{-n_j} R_{i_k} \right) = \operatorname{Leb} \left( \bigwedge_{j=1}^{k} f^{-n_j} R_{i_k} \right).
\]

**Remark 15.** Given a general topologically mixing TMS \( \Sigma_A \), one can construct a type III\(_1\) Markov shift supported on \( \Sigma_A \) as follows. Define \( Q \) to be the matrix with

\[
Q_{i,j} = \begin{cases} 
\frac{1}{\sum_{l=1}^{n} A_{i,l}}, & A_{i,j} = 1 \\
0 & A_{i,j} = 0.
\end{cases}
\]

It is easy to prove that in that case \( A \) has a row \( i \in \{1, \ldots, |S|\} \) with at least two 1’s. Then one can proceed as in our example to define \( Q_i \) to be the matrix \( Q \) perturbed in the \( i \)-th row between two non zero coordinates. The rest of the proof remains the same.

We end this section with the following open question: Given a mixing TMS \( \Sigma \subset F^\mathbb{Z} \) with \( F \) finite. Does there exist a shift non singular Markov measure \( \nu = M \{ \pi_k, R_k \} \) so that:

- \( \nu \) is fully supported on \( \Sigma \) and the shift is ergodic with respect to \( \nu \).
- The non singular Markov shift \( (\Sigma, B, \nu, T) \) is of type II\(_{\infty}\) (preserves an a.c.i.m but no a.c.i.p.) or III\(_{\lambda} \), \( 0 \leq \lambda < 1 \).
- Preferably \( \nu \) is half stationary meaning that there exists an irreducible and aperiodic stochastic Matrix \( R \) so that for all \( k < 0 \) \( R_k = R \) and \( \pi_{\lambda} = \pi_R \).

If in addition \( \Sigma \) is a TMS arising from a hyperbolic Toral automorphism and \( R \) is the stochastic matrix representing the Lebesgue measure then a positive answer to this question may give new examples of Anosov diffeomorphisms by the methods of the next section. In the case where \( \Sigma = \{0,1\}^\mathbb{Z} \) and \( \nu \) is a non stationary product measure it was shown in [?]? that the shift is either of type III\(_1\) or is equivalent to a classical Bernoulli shift (\( \nu \) is equivalent to a product measure with i.i.d entries).
4. Type III Perturbation of the Golden Mean Shift Arising from Markovian Measures

In this section we will do the first step towards a construction of type $\text{III}_1$ Anosov diffeomorphism by using the type $\text{III}_1$ Markov shifts from the previous section. The idea is as follows, let $f(x,y) = (x + y, x) \mod 1$, $\{R_1, R_2, R_3\}$ be the corresponding Markov partition for $f$ and $\Phi : (\Sigma A, \mu, T) \to (T^2, \Phi_*\mu, f)$. We remark that since $\mu$ is conservative, $\Phi$ is one to one on the support of $\mu$ (since it gives zero measure to the sequences with non unique expansion), hence $(T^2, \Phi_*\mu, f)$ is a type $\text{III}_1$ dynamical system. Let $G : T^2 \to T^2$ be the homeomorphism (it is an homeomorphism because of the structure of $\mu$) which takes the Lebesgue measure to $\Phi_*\mu$. The transformation $G \circ f \circ G^{-1} : (T^2, m = \text{Leb}) \circ$ is measure theoretically isomorphic to $(T^2, \Phi_*\mu, f)$, hence a type $\text{III}_1$ system. The goal of this section’s construction is to find a homeomorphism $G : T^2 \to T^2$ so that

(1) $(G_\epsilon)_* m \sim \Phi_*\mu = G_*m$. Consequently the system $(T^2, B_{T^2}, m, G \circ f \circ G^{-1})$ is of type $\text{III}_1$ since it is measure theoretically isomorphic to $(T^2, B_{T^2}, (G_\epsilon)_* m, f)$ and the fact that the type $\text{III}_1$ property is invariant upon changing the measure to an equivalent measure.

(2) $G_\epsilon \circ f \circ G_\epsilon^{-1}$ is $C^1$ and Anosov.

In order to obtain this goal it is easier for us to build $f$ as the natural extension of (the non invertible) golden mean shift $Sx = \varphi x \mod 1$. The partition $\{J_1 = [0, 1/\varphi^2], J_2 = [1/\varphi, 1], J_3 = [1/\varphi^2, 1/\varphi]\}$ is a Markov partition for the golden mean shift with $A$ (the same matrix as the one for $f$) as its adjacency matrix. See figure 4.1.

![Figure 4.1. The Markov partition of $\varphi x \mod 1$](image)

Denote by $\sigma$ the one sided shift on $\Sigma_A^+$. It can be verified that $(\Sigma_A^+, \nu_{\pi Q}, \sigma)$ is isomorphic to $(T, m_{\text{Leb}}, S)$ where $m_{\text{Leb}}$ is the Lebesgue measure on $T$. The natural extension of $(\Sigma_A^+, \nu_{\pi Q}, Q, \sigma)$ is $(\Sigma_A, M \{Q, Q\}, \sigma)$ which is isomorphic to $(T^2, m_{\text{Leb}}, f)$. This shows that $f$ is indeed the natural extension of the Golden mean shift. In this section we are first going to do a perturbation of $S$ in the spirit of the one done for $2x \mod 1$ in [BH]. Our perturbation has an additional property that the Markov partition $\{J_1, J_2, J_3\}$ is invariant under all the perturbation maps. This will enable us to use this perturbation to build a two dimensional one. For this purpose we would like to see how $S$ and $f$ are related. This is of course done by looking at the Markov partitions and moving to the $V_\varphi, V_{-1/\varphi}$
coordinates. On those coordinates \( f \) acts almost as
\[
(u, v) \mapsto (\varphi u \mod 1, -\varphi^{-1}v) = (Su, -\varphi^{-1}v),
\]
where the mistake is in the second coordinate. To make it precise let \( M = [0, 1/\varphi] \times \left[ -\frac{\varphi^2}{\varphi^2 + 2}, \frac{\varphi^2}{\varphi^2 + 2} \right] \cup \left[ \frac{1}{\varphi}, 1 \right] \times \left[ -\frac{\varphi}{\varphi^2 + 2}, \frac{1}{\varphi^2 + 2} \right]. \) Define \( \tilde{f} : M \to M \) by
\[
\tilde{f}(x, y) = \begin{cases} 
(\varphi x, -\varphi^{-1}y), & 0 \leq x \leq 1/\varphi \\
(\varphi x - 1, -\varphi^{-1} \left( y - \frac{\varphi^2}{\varphi^2 + 2} \right)), & 1/\varphi \leq x \leq 1.
\end{cases}
\]
See Figure 4.2 for the way \( \tilde{f} \) maps its 3 rectangles, as can be seen by this picture the action of \( \tilde{f} \) is the same as how \( f \) acts on its Markov partition.

![Figure 4.2. Action of \( \tilde{f} \) on its (soon to be) Markov partition](image)

In order that \( \tilde{f} \) be a diffeomorphism of \( M \) we identify by orientation preserving piecewise translations the following intervals (for a geometric understanding one can see that this identification comes from the way the Markov partition of \( f \) tiles the plane):
\[
\{0\} \times \left[ 0, \frac{\varphi^2}{\varphi + 2} \right] \simeq \{1\} \times \left[ -\frac{\varphi}{\varphi + 2}, \frac{1}{\varphi + 2} \right]
\]
\[
\{0\} \times \left[ -\frac{\varphi}{\varphi^2 + 2}, 0 \right] \simeq \left\{ \frac{1}{\varphi} \right\} \times \left[ \frac{1}{\varphi + 2}, \frac{\varphi^2}{\varphi + 2} \right]
\]
\[
\left[ 0, \frac{1}{\varphi^3} \right] \times \left\{ \frac{\varphi^2}{\varphi + 2} \right\} \simeq \left[ \frac{1}{\varphi^2}, \frac{1}{\varphi} \right] \times \left\{ -\frac{\varphi}{\varphi + 2} \right\}
\]
\[
\left[ \frac{1}{\varphi^2}, \frac{1}{\varphi} \right] \times \left\{ \frac{\varphi^2}{\varphi + 2} \right\} \simeq \left[ \frac{1}{\varphi}, 1 \right] \times \left\{ -\frac{\varphi}{\varphi + 2} \right\}
\]
\[
\left[ 0, \frac{1}{\varphi} \right] \times \left\{ -\frac{\varphi}{\varphi^2 + 2} \right\} \simeq \left[ \frac{1}{\varphi}, 1 \right] \times \left\{ \frac{1}{\varphi + 2} \right\}.
\]
The resulting Manifold will be denoted by \( M_\infty \).

### 4.1. A perturbation of the golden mean shift:
Let \( \nu = M \{ \pi_k, P_k \}_{k=-\infty}^{\infty} \) be the type III\(_1\) (for the shift on \( \{1, 2, 3\}^\mathbb{Z} \)) Markov measure from Section \( 3 \) for the two sided shift. It follows from [ST, Thm. 4.4.] that the one sided Markov measure \( \nu^+ = M \{ \pi_k, P_k \}_{k=1}^{\infty} \) on \( \{1, 2, 3\}^\mathbb{N} \) is a type III measure for the (one sided) shift.
Let $Sx = \varphi x \mod 1$ and $J_1 := (0, 1/\varphi^2)$, $J_2 := (1/\varphi, 1)$ and $J_3 := (1/\varphi^2, 1/\varphi)$ be a Markov partition for $S$. Denote by

$$\text{Bd} (S) := \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{3} \partial (S^{-n}J_i).$$

The map $\Theta : \Sigma^+_A \rightarrow [0, 1], \Theta (w) = \bigcap_{n=0}^{\infty} S^{-n}J_{w_n}$ is a semiconjugacy of $(\Sigma^+_A, \sigma)$ and $(\mathbb{T}, S)$ and for each $x \notin \text{Bd} (S)$, $\Theta^{-1}(x)$ consists of one point (point of uniqueness for the $\Theta$ representation). Since the support of $\nu^+$ is contained in $G (\sigma) := \Theta^{-1} (\mathbb{T}\setminus \text{Bd} (S))$, the map $\Theta$ is a metric isomorphism between $(\Sigma^+_A, \nu^+, \sigma)$ and $(\mathbb{T}, \Theta, (\nu^+)\circ S)$ and therefore the measure $\mu^+ := \Theta_* (\nu^+)$ is a type III measure for $S$. Since $\mu^+$ is a continuous measure, its cumulative distribution function $\nu (x) = \mu^+ ([0, x])$ is a homeomorphism of $\mathbb{T}$ such that $\mu^+ \circ h^{-1}$ is Lebesgue measure. It follows that the map $(\mathbb{T}, m_{\text{Leb}}, h \circ S \circ h^{-1})$ is a type III transformation, where $m_{\text{Leb}}$ denotes the Lebesgue measure. The problem is that $h \circ S \circ h^{-1}$ is not necessarily smooth, we construct $h_\epsilon$ close to $h$ in the $C^0$ norm such that

- $h_\epsilon \circ S \circ h^{-1}_\epsilon$ is $C^1$ and uniformly expanding.
- $m_{\text{Leb}} \circ h_\epsilon \sim \mu^+$.
- We will have in addition that $h_\epsilon J_i = J_i$ for every $i \in \{1, 2, 3\}$, this extra property is crucial for the extension to two dimensions.

Before we go through the construction we would like the reader to recall that the Lebesgue measure on $\mathbb{T}$ is the measure arising from $M \{\pi, Q\}$. The main idea is to approximate the change of measure between Lebesgue measure and $\mu^+$ on the semi algebras

$$\mathcal{R}(n) := \left\{ C_{[w]} := \bigcap_{k=0}^{n-1} S^{-k}J_{w_k} : x \in \Sigma_A \right\} = \left\{ C_w : w \in \Sigma_A (n) \right\}.$$

The construction goes as follows: We first assume that we are given a type III Markovian measure defined by $\{\lambda_k, M_k, N_k\}_{k=1}^{\infty}$. Then we would like to choose inductively, mostly by continuity arguments a sequence $\epsilon = \{\epsilon_k\}$ that will give us the perturbation. However in the end we arrive at a problem that we need that the size of $M_k$ is relatively large with respect to $1/\epsilon_{k-1}$. This problem will be solved by modifying the induction process of Section 3 and adding the choice of the sequence $\epsilon$ to the induction. The new induction will be explained in Subsection 4.2.2.

**Remark 16.** Before we continue with the construction we would like to remind the reader that at each stage in the inductive construction of the Markovian measure in Section 3 we can take $\lambda_t$ to be as close to 1 as we like and $n_t, M_t/N_t$ to be as large as we want. This is because the conditions on $\lambda_t$ ((3.3), (3.4) and (3.5)) are that $\lambda_t$ is small enough whilst the conditions on $n_t$ ((3.6), (3.7) and (3.8)) and $M_t/n_t \sim m_t/n_t ((3.11) and (3.12))$ are to be large enough.

**Special interpolation functions:** Given $\alpha > 0$ we would like to define a Lipschitz function $g_\alpha$ so that $g_\alpha (0) = 1, g_\alpha (1) = \alpha$ and $\int_0^1 g_\alpha (x) dx = \alpha$. We will use the functions $g_\alpha : [0, 1] \rightarrow [0, 1]$ defined by $g_\alpha (0) = 0$ and

$$g_\alpha' (x) = \begin{cases} 1 + 3x, & \frac{5\alpha - 5}{4} \leq x \leq \frac{1}{3} \\ \frac{5\alpha - 1}{4}, & \frac{1}{3} \leq x \leq \frac{2}{3} \\ \frac{5\alpha - 1}{4} - (3x - 2) \frac{\alpha - 1}{4}, & \frac{2}{3} \leq x \leq 1 \end{cases}$$
which have the additional property that if $\alpha > 1$ then 
\[ 1 = \inf_{x \in [0,1]} g_\alpha(x) < \sup_{x \in [0,1]} g_\alpha(x) = \frac{5\alpha - 1}{4} < \alpha^2 \]
and if $\frac{1}{4} \leq \alpha < 1$ then 
\[ \alpha^2 < \frac{5\alpha - 1}{4} = \inf_{x \in [0,1]} g_\alpha(x) < \sup_{x \in [0,1]} g_\alpha(x) = 1. \]

4.2. Realization of the homeomorphism of change of measures. For $\epsilon > 0$ and $\lambda > 1$, let $\psi_{\epsilon, \lambda} : [0, 1] \to [0, 1]$ be the function defined by $\psi_{\epsilon, \lambda}(0) = 0$ and

\[ \psi_{\epsilon, \lambda}'(x) := \begin{cases} 
  g\left( \frac{\lambda^2}{1 + \lambda \varphi} \right)(\frac{x}{\epsilon}), & 0 \leq x \leq \epsilon \\
  \lambda \varphi^2 \left( \frac{1}{1 + \lambda \varphi} \right)(\frac{x}{\epsilon}), & \frac{\epsilon}{\varphi} < x \leq \frac{1}{\varphi} - \epsilon \leq x \leq \frac{1}{\varphi} \\
  g\left( \frac{\lambda^2}{1 + \lambda \varphi} \right)\left( \frac{x-1/\varphi}{\epsilon} \right), & \frac{1}{\varphi} - \epsilon < x \leq \frac{1}{\varphi} + \epsilon \\
  \varphi^2 \left( \frac{1}{1 + \lambda \varphi} \right)\left( \frac{x-1/\varphi}{\epsilon} \right), & \frac{1}{\varphi} + \epsilon < x \leq 1 - \epsilon \\
  g\left( \frac{\varphi^2}{1 + \lambda \varphi} \right)\left( \frac{1-x}{\epsilon} \right), & 1 - \epsilon < x \leq 1 
\end{cases} \]

If $\epsilon = 0$ then by a rescaling procedure one can use these functions to define the cumulative distribution function of $\Theta_\alpha\left( \nu_{\pi \mathcal{Q}_\lambda} \mathcal{Q}_\lambda \right)$. The function $\psi_{\epsilon, \lambda}$ is basically an interpolation of a piecewise constant function in order to make it continuous so that the following properties hold:

1. $\psi'(0) = \psi'(1) = 1$. This is needed in order to glue $\psi_{\epsilon, \lambda}$ with the identity function and still have a $C^3$ function.
2. For every $\epsilon, \lambda$,

\[ \psi_{\epsilon, \lambda}(1) := \int_0^1 \psi_{\epsilon, \lambda}'(s)ds = 1, \quad \psi_{\epsilon, \lambda}(1/\varphi) = \frac{\lambda \varphi}{1 + \lambda \varphi}. \]
(3) For every $\epsilon < x < \frac{1}{2} - \epsilon$, 
\[
\frac{\psi_{\epsilon, \lambda}(x)}{\psi_{\epsilon, \lambda}(1/\varphi)} = \varphi x,
\]
and for every $\frac{1}{2} + \epsilon < x < 1 - \epsilon$, 
\[
\frac{\psi_{\epsilon, \lambda}(x) - \psi_{\epsilon, \lambda}(1/\varphi)}{\psi_{\epsilon, \lambda}(1) - \psi_{\epsilon, \lambda}(1/\varphi)} = \frac{x - 1/\varphi}{1 - 1/\varphi}.
\]
This follows from the fact that for every $\alpha > 0$, $\int_{0}^{\epsilon} g_{\alpha}(x)dx = \alpha\epsilon$.

(4) $\psi'_{\epsilon, \lambda}$ is Lipschitz with Lipschitz constant of the order $\frac{1}{\epsilon}$ when $\epsilon \to 0$ and for every $x \in \mathbb{T}$,
\[
\lambda^{-2} < \psi'_{\epsilon, \lambda}(x) < \lambda^2.
\]
Given two sequences $\epsilon_{k} \geq 0$ and $\lambda_{k} \geq 1$, denote by $\psi_{k} = \psi_{\lambda_{k}, \epsilon_{k}}$.

Define an order on $\Sigma_{\mathbb{A}}^{+}$ in the following way. For $w, z \in \Sigma_{\mathbb{A}}^{+}$, let 
\[
j(w, z) := \inf \{ n \in \mathbb{N} : w_{n} \neq z_{n} \}.
\]
Then $w < z$ if either $y_{j(w, z)} = 1$ or $y_{j(w, z)} = 3$ and $x_{j(w, z)} = 2$. This order has the following property. If $[w]_{1}^{n_{1}} \neq [y]_{1}^{n_{2}}$ for some $n_{1} \in \mathbb{N}$, then $C_{[w]_{1}^{n_{1}}}^{-1}$ is to the left of $C_{[y]_{1}^{n_{2}}}$ if and only if $w < y$.

In addition for $w \in \Sigma_{\mathbb{A}}$ and $n \in \mathbb{N}$ we write $\bar{x}_{n}, \underline{x}_{n} : \Sigma_{\mathbb{A}} \to \mathbb{T}$ to be defined by
\[
C_{[w]_{1}^{n}} := [\underline{x}_{n}(w), \bar{x}_{n}(w)].
\]
We will define inductively a sequence $\{h_{n}\}_{n=1}^{\infty}$ of diffeomorphisms of $\mathbb{T}$. Notice that since $\mathbb{T} = \bigcup_{w \in \Sigma_{\mathbb{A}}(n)} C_{[w]_{1}^{n}}$ and each $h_{k}, k < n$ is onto $\mathbb{T}$,
\[
\mathbb{T} = \bigcup_{w \in \Sigma_{\mathbb{A}}(n)} H_{n-1} \left( C_{[w]_{1}^{n}} \right),
\]
where $H_{n-1} := h_{n-1} \circ h_{n-2} \circ \cdots \circ h_{1}$.

- If $\psi_{k} \neq \psi_{l}$ for some $k, l < n$, then $h_{n}$ is the identity.
- If $M_{k} - 1 < n < M_{t}$ for some $t \in \mathbb{N}$, then $h_{n}$ is made from $\#(\Sigma(n))$ scalings of $\psi_{k}$ or the identity function. Let $w(n, 1), \ldots, w(n, \#(\Sigma(n)))$ be an enumeration of $\Sigma_{\mathbb{A}}(n)$ with respect to $\prec$. Set $h_{n}(0) = 0$. Assume we have defined $h_{n}$ on $\bigcup_{k=1}^{n-1} H_{n-1} \left( C_{w(n,k)} \right)$, we will now define $h_{n}$ on $H_{n-1} \left( C_{w(n,l)} \right)$.

- If $w(n, l) = 1$, we define for $z \in H_{n-1} \left( C_{w(n,l)} \right)$,
\[
h_{n}(z) := H_{n-1} \left( \bar{x}_{n}(w) + l(n, w) \psi_{l} \left( \frac{z - H_{n-1} \left( \bar{x}_{n}(w) \right)}{l(n, w)} \right) \right),
\]
where $w = w(n, l)$ and
\[
l(n, w) := m_{\text{Leb}} \left( H_{n-1} \left( w \right) \right) = H_{n-1} \left( \bar{x}_{n}(w) \right) - H_{n-1} \left( \underline{x}_{n}(w) \right).
\]
- If $w(n, l) \neq 1$ then for all $z \in [z_{l}, z_{l+1})$,
\[
h_{n}(z) = z.
\]
- Notice that since $\psi_{\eta, \eta}(1) = 1$ for all $\eta$, $z_{l} = \lim_{z \to z_{l}} h_{n}(z)$, this shows that $h_{n}$ is continuous. The differentiability at points $z_{l}, l \leq \#\Sigma_{\mathbb{A}}(n)$ follows from $\psi'_{l}(1) = \psi'_{l}(1) = 1$. 

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We need to define \( h_n \) for all \( n \in \{ M_t \}_{t=1}^\infty \). Here we apply a statistical correction procedure which we will now proceed to describe. In what follows we assume that \( \epsilon_t \) is small enough so that

\[
m_{\text{Leb}} \left( \psi_t \left( C_{\|w\|_1}^{N_t} \right) \right) = m_{\text{Leb}} \left( \psi_t \left( C_{\|w\|_1^2} \right) \right) m_{\text{Leb}} \left( C_{\|w\|_1^{N_t}} \right) C_{\|w\|_1^2}.
\]

The first equality follows from property 3 of \( \psi_t \) as long as epsilon is small enough so that for every \( w \in \Sigma_\mathcal{A}^+ \), the end points of \( C_{\|w\|_1^2} \) are in \([\epsilon, \varphi^{-1} - \epsilon] \cup [\varphi^{-1} + \epsilon, 1 - \epsilon] \cup \{0, \varphi^{-1}, 1\} \).

The equality then follows from \( \psi_t(1/\varphi) = \frac{\lambda \varphi}{t + \lambda \varphi} \). This relation gives for example that

\[
m_{\text{Leb}} \left( H_{N_t} \left( C_{\|w\|_1^{N_t}} \right) \right) = \mu^+ \left( C_{\|w\|_1^{N_t}} \right),
\]

and we have good knowledge of where the point in \( \frac{1}{\varphi} \) proportion in \( H_{n-1} \left( C_{\|w\|_1^n} \right) \) travels. However, since \( M_t \) is generally much larger then \( N_t \) we loose this control and the useful equality

\[
(4.1) \quad m_{\text{Leb}} \left( H_{M_t} \left( C_{\|w\|_1^{N_t}} \right) \right) = m_{\text{Leb}} \left( H_{M_t} \left( C_{\|w\|_1^{M_t}} \right) \right) m_{\text{Leb}} \left( C_{\|w\|_1^{N_t}} \right) C_{\|w\|_1^{M_t}}
\]

need no longer hold true. The role of \( h_{M_t} \) is to take care that equality (4.1) holds true.

The function \( H_{N_t} \) is a product of bounded Lipschitz functions and so is also a bounded Lipschitz function. Therefore if \( M_t \) is large enough with respect to \( N_t \), then (here we use the fact that \( h_{n} = Id \) for \( N_t < n < M_t \) \( H_{N_t}' = H_{M_t-1}' \) is almost constant on \( H_{M_t-1} \left( C_{\|w\|_1^{M_t}} \right) \).

That means that for every \( x \in H_{M_t-1} \left( C_{\|w\|_1^{M_t}} \right) \),

\[
\left| H_{M_t-1}'(x) - \frac{1}{m_{\text{Leb}} \left( C_{\|w\|_1^{M_t}} \right) C_{\|w\|_1^{M_t}}} \int H_{M_t-1}'(s) \, ds \right| \ll 1.
\]

By using a similar idea as in the construction of \( \psi \) with the \( g_a \) we define \( h_{M_t} \) restricted to \( H_{M_t-1} \left( C_{\|w\|_1^{M_t}} \right) \) so that equality (4.1) holds. This is done as follows: For \( \alpha_1, \alpha_2 \in \mathbb{R} \), let \( g_{\alpha_1, \alpha_2} : [0, 1] \to [0, 1] \) be defined by \( g_{\alpha_1, \alpha_2}(0) = 0 \) and

\[
(4.2) \quad G'_{\alpha_1, \alpha_2}(x) := \begin{cases} \alpha_1 + \frac{5(\alpha_2 - \alpha_1)}{4} x, & 0 \leq x \leq 1/3 \\ \frac{5\alpha_2 - \alpha_1}{4}, & 1/3 \leq x \leq 2/3 \\ \frac{5\alpha_2 - \alpha_1}{4} + \frac{\alpha_2 - \alpha_1}{4} (3x - 2) & 2/3 \leq x \leq 1 \end{cases}
\]

This function is a \( C^1 \) function which satisfies \( g_{\alpha_1, \alpha_2}(1) = \alpha_2, g'_{\alpha_1, \alpha_2}(0) = \alpha_1 \) and \( g'_{\alpha_1, \alpha_2}(1) = \alpha_2 \).

Define \( \alpha : \mathbb{N} \times \Sigma_\mathcal{A} \to (0, \infty) \) by

\[
\alpha(t, w) := \frac{1}{m_{\text{Leb}} \left( C_{\|w\|_1^{M_t}} \right) C_{\|w\|_1^{M_t}}} \int H_{M_t-1}'(s) \, ds.
\]

In addition for a finite word \( w \in \Sigma_\mathcal{A} (M_t) \) we denote by \( w^- \) the predecessor of \( w \) with respect to \( \prec \) restricted on \( \Sigma_\mathcal{A} (M_t) \). We define \( h_{M_t}' \circ H_{M_t-1}(x) \) on \( C_{\|w\|_1^{M_t}} \) to be equal to \( \frac{\alpha(t, w)}{H_{M_t-1}'(x)} \) off an \( \epsilon_t + 1 \) neighborhood of the left endpoint of the segment
\[ C_{[w_1^M_1]}, \frac{\alpha(t,w^-)}{H_{M_1-1}(x)} \] on the left endpoint (which is in the boundary of \( H_{M_1-1} \left( C_{[w_1^M_1]} \right) \)) and an interpolation in between by using \( G_{\alpha_1,\alpha_2} \). Here \( \epsilon_{t+1} \) has to be small enough so that the end points of \( \{ H_{M_1-1} \left( C_{[w_1^N_1]} \right) : [w_1^N_1]_{M_1} \in \Sigma_{A}(n_t) \} \) are not in an \( \epsilon_{t+1} \) neighborhood of the left end point of \( H_{M_1-1} \left( C_{[w_1^M]} \right) \). Formally \( h_{M_1} \circ H_{M_1-1} |_{C_{[w_1^M]}^1} \) is defined by \( h_{M_1} \circ H_{M_1-1} \left( \mathcal{E}_{M_1}(w) \right) = H_{M_1-1} \left( \mathcal{E}_{M_1}(w) \right) \) and

\[
h'_{M_1} \circ H_{M_1-1}(x) := \begin{cases} \frac{G'}{\beta}(\mathcal{E}_{M_1}(w)) \mathcal{E}(w,t) (x), & 0 \leq x - \mathcal{E}_{M_1}(w) < \epsilon_{t+1} m_{\text{Leb}} \left( C_{[w_1^M]} \right) \\ \beta(x), & \mathcal{E}_{M_1}(w) + \epsilon_{t+1} m_{\text{Leb}} \left( C_{[w_1^M]} \right) \leq x < \mathcal{E}_{M_1}(w) \end{cases}
\]

where \( \beta(x) := \frac{\alpha(t,w)}{H_{M_1-1}(x)} \).

\[ \mathcal{E}(w,t) := \beta \left( \mathcal{E}_{M_1}(w) + \epsilon_{t+1} m_{\text{Leb}} \left( C_{[w_1^M]} \right) \right) \]

and \( \beta \left( \mathcal{E}_{M_1}(w) \right) := \frac{\alpha(t,w^-)}{H'_{M_1-1}(\mathcal{E}_{M_1}(w))} \).

If \( H_{M_1-1}(x) \) is in the right part of \( H_{M_1-1} \left( C_{[w_1^M]} \right) \) then by the chain rule

\[
H'_{M_1}(x) = h'_{M_1} \circ H_{M_1-1}(x) \cdot H'_{M_1-1}(x) = \alpha(t,w)
\]

and if \( x \) is the left end point endpoint of \( C_{[w_1^M]} \) then

\[
H'_{M_1}(x) = \alpha(t,w^-).
\]

Figure 4.4 depicts the graph of \( H'_{M_1} \). One consequence of this definition is that

\[
h_{M_1} \left( H_{M_1-1} \left( C_{[w_1^M]} \right) \right) = H_{M_1-1} \left( C_{[w_1^M]} \right)
\]

for all \( w \in \Sigma_A \).

\[ \text{Figure 4.4.} \text{ The graph of } H'_{M_1} \text{ restricted to } C_{[w_1^M]} \]
Remark 17. An important feature of this construction that will be used in the extension to two dimensions is that for any $1 \leq l \leq \# \Sigma_A(n)$,

$$h_n \left( H_{n-1} \left( C_w(n,l) \right) \right) = H_{n-1} \left( C_w(n,l) \right).$$

This in turn implies that for every $n \in \mathbb{N}$, $H_n(x,y) := (H_n(x),y)$ is a diffeomorphism of $M_\omega$ and the Markov partition $\{R_1, R_2, R_3\}$ for $\bar{f}$ defined by

$$R_i := \begin{cases} J_i \times \left[ -\frac{e^2}{2}, \frac{e^2}{2} \right], & i \in \{1,3\} \\ J_2 \times \left[ -\frac{e^2}{2}, \frac{1}{2} \right], & i = 2 \end{cases}$$

is preserved by $H_n$, meaning that $H_n^{-1} R_i = R_i$ for $i \in \{1,2,3\}$.

Theorem 18. There exists a choice of $\lambda_k \downarrow 1$, $\{n_k, m_k, N_k, M_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ and $\varepsilon = \{\varepsilon_k\}_{k \in \mathbb{N}}$ so that:

(i). The Markov measure from the construction of Section 3 is a type III_1 measure for the shift on $\Sigma_A$.

(ii). The function $h_\varepsilon$ is a circle homeomorphism and $m_{Leb} \circ h_\varepsilon \sim \mu^+$ where $\mu^+ = \Theta \cdot M(P_{k}, \pi_k)_{k=1}^\infty$.

(iii) The function $g = h_\varepsilon \circ S \circ h_\varepsilon^{-1}$ is $C^1$, and for every $x \in T$,

$$1.6 \leq g'(x) \leq 1.7.$$ 

The proof of this Theorem is by showing that we can use the inductive construction of Section 3 (with three extra conditions) and include a new sequence $\{\varepsilon_k\}$ in it so that the following properties hold:

(1) $h_\varepsilon := \lim_{n \to \infty} H_n$ is a homeomorphism of $\mathbb{T}$.

(2) $\{g_n\}$ is a convergent subsequence in the $C^1$ topology, here $g_n := H_n \circ S \circ H_n^{-1}$.

(3) The limit function $g = \lim_{n \to \infty} g_n = h_\varepsilon \circ S \circ h_\varepsilon^{-1}$ satisfies $1.6 \leq g'(x) \leq 1.7$.

(4) $m \circ h_\varepsilon \sim \mu$.

Lemma 19. If $M_t$ is sufficiently large with respect to $N_t$ and $\varepsilon_{t+1}$ is small enough then the following two properties hold:

(i) For all $x \in T$,

$$h^t_{M_t}(x) = e^{2^{-N_t}}.$$ 

(ii) Let $w \in \Sigma_A$ and $M_t < n \leq M_t$. Denote by $\xi(n,w) = \frac{1}{\varphi} (x_n(w) - x_n(w))$ the point in $\frac{1}{\varphi}$ proportion in $C_{[w]^n_t}$. Then

$$\frac{H_{n-1}(\xi_n(w)) - H_{n-1}(x_n(w))}{H_{n-1}(\bar{x}_n(w)) - H_{n-1}(\bar{x}_n(w))} = \frac{1}{\varphi}.$$ 

That is the (reference) point in $\frac{1}{\varphi}$ proportion in $C_{[w]}^n$ travels under $H_{n-1}$ to the reference point in $H_{n-1} \left( C_{[w]}^n \right)$.

Proof. Since $h_n$ is the identity for $N_t < n < M_t$, then $H_{M_t-1} = H_{N_t}$. The function $H_{M_t}$ is a product of $N_t$ bounded Lipschitz functions. Therefore there exists $K(t) > 0$, which depends only on $\{\lambda_s, N_s, M_s, \varepsilon_{s} \}_{s=1}^{t-1}$ and $\{N_t, \lambda_t, \varepsilon_t\}$, such that for every $x, y \in T$,

$$|H_{M_t-1}(x) - H_{M_t-1}(y)| = |H_{N_t}(x) - H_{N_t}(y)| \leq K(t)|x - y|$$

and for every $x \in T$,

$$K(t)^{-1} \leq |H_{M_t-1}(x)| < K(t).$$
By uniform expansion of $S$, if $M_t$ is sufficiently large then
\[
\sup_{w \in \Sigma_A} m_{\text{Leb}} \left( H_{M_t-1} \left( C_{[w]_1^{M_t}} \right) \right) \leq \varphi^{-(M_t-1)}K(t) < \frac{\delta}{K(t)}.
\]
This implies that for every $w \in \Sigma_A$ and $x, y \in H_{M_t-1} \left( C_{[w]_1^{M_t}} \right)$,
\[
|H_{M_t}'(x) - H_{M_t}'(y)| \leq K(t)|x - y| \leq K(t)m_{\text{Leb}} \left( H_{M_t-1} \left( C_{[w]_1^{M_t}} \right) \right) < \delta.
\]
Part (i) follows by choosing an appropriate $\delta$ and the definition of $h_{M_t}$ since for every $x \in C_{[w]_1^{M_t}}$
\[
\left| \frac{1}{m_{\text{Leb}} \left( C_{[w]_1^{M_t}} \right) C_{[w]_1^{M_t}}} \int_{C_{[w]_1^{M_t}}} H_{M_t-1}'(s)ds - H_{M_t-1}'(x) \right| < \delta,
\]
and (4.4). By choosing $x = \bar{x}_{M_t}(w) = \bar{x}_{M_t}(w^{-})$ in the previous inequality one it follows that
\[
\forall w \in \Sigma_A \ (M_t), \ |\alpha(t, w) - \alpha (t, w^{-})| < 2\delta.
\]
The first part follows by choosing a small enough $\delta$.
(ii) By the definition of $h_{M_t}$, if $\alpha_{t+1}$ is small enough then equation (1.1) holds. Then a simple induction using property 3 of $\psi_{t_i, \lambda_i}$ gives that for all $M_t < n < N_t$,
\[
(4.5) \quad m_{\text{Leb}} \left( H_n \left( C_{[w]_1^{n+1}} \right) \right) = m_{\text{Leb}} \left( H_n \left( C_{[w]_1^{n+1}} \right) \right) m_{\text{Leb}} \left( C_{[w]_1^{n+1}} \right) C_{[w]_1^{n+1}}.
\]
The conclusion then follows from
\[
\frac{m_{\text{Leb}} \left( H_n \left( C_{[\tilde{w}]_1^{n+2}} \right) \right)}{m_{\text{Leb}} \left( H_n \left( C_{[\tilde{w}]_1^{n+1}} \right) \right)} = m_{\text{Leb}} \left( C_{[\tilde{w}]_1^{n+2}} \right) C_{[w]_1^{n+1}} = \frac{1}{\varphi}.
\]
\[
4.2.1. \quad \text{The inductive choice of } \{\epsilon_l\}_{l=1}^\infty. \quad \text{Before we continue we would like to set up some notation which will be used.}
\]

- Given $\xi = \{\epsilon_{k}\}_{k=1}^{l}$ and $n \leq N_t$ we denote by $h_{\xi, n}$ the function in the construction with the sequence $\xi$ at level $n$.
- For $j \leq N_t$, $H_{\xi, j} := h_{\xi, j} \circ h_{\xi, j-1} \circ \cdots \circ h_{\xi, 1}$.
- $H_{0, 0}$ will denote the function with $\xi = \emptyset$.
- We write $\check{\xi}_n(x) := h_{\xi, n} \left( H_{\xi, n-1}(x) \right)$
Proposition 20. Assume that for every \( t \in \mathbb{N} \), and all \( x \in \mathbb{T} \),
\[
h'_{M_t}(x) = \pm 2^{-N_t}
\]
then:
(1) If \( \epsilon_t = 0 \) and \( \varphi/\lambda^2_1 > 1.6 \) then for all \( M_t \leq n < M_{t+1} \),
\[
\sup_{x \in \mathbb{T}} |h_n(x) - x| \leq e^{1.6} - n.
\]
(2) There exists a sequence \( \delta_k > 0 \) so that for all \( k \in \mathbb{N} \), \( \epsilon_k < \delta_k \),
\[
\sup_{n \in \mathbb{N}} |h_{\epsilon,n}(x) - x| \leq 3(1.5)^{-n}
\]
and consequently
\[
\lim_{n \to \infty} H_{\epsilon,n}(x) = h_\epsilon(x)
\]
is a homeomorphism of \( \mathbb{T} \).

Proof. (1) If for some \( \tau \leq t+1 \), \( M_\tau < k \leq M_{\tau+1} \) then since the sequence \{\( \lambda_k \)\} is decreasing for every \( k \in \mathbb{N} \),
\[
\sup_{x \in \mathbb{T}} h'_k(x) = \sup_{s \in \mathbb{T}} |\psi_k(s)| \leq \lambda^2_\tau \leq \lambda^2_1.
\]
Therefore for every \( M_t < n < M_{t+1} \),
\[
m_{\text{Leb}} \left( H_{n-1} \left( C_{[w]_1^t} \right) \right) \leq \left\{ \prod_{k=1}^t \sup_{x \in \mathbb{T}} h'_{M_k}(x_k) \right\} \lambda^2_n m_{\text{Leb}} \left( C_{[x]_1^t} \right)
\]
\[
\leq \prod_{k=1}^\infty e^{2^{-N_t}} \left( \frac{\lambda^2_1}{\varphi} \right)^n
\]
\[
\leq e^{1.6} - n.
\]
The invariance of \( H_{n-1} \left( C_{[w]_1^t} \right) \) under \( h_n \) implies that
\[
\sup_{x \in \mathbb{T}} |h_n(x) - x| \leq \sup_{w \in \Sigma^A} m_{\text{Leb}} \left( H_{n-1} \left( C_{[w]_1^t} \right) \right) \leq (1.6)^{-n}.
\]
(2) One can choose \( \{\delta_k\}_{k=1}^\infty \) by a simple inductive argument which uses the fact that \( \psi_{\epsilon,\lambda} \) converges uniformly to \( \psi_{0,\lambda} \) as \( \epsilon \to 0+ \). It then follows that if \( \epsilon_k < \delta_k \) for all \( k \in \mathbb{N} \) the for every \( n < m \),
\[
|H_{\epsilon,m}(z) - H_{\epsilon,n}(z)| \leq \sum_{k=n}^m |H_{\epsilon,k+1}(z) - H_{\epsilon,k}(z)|
\]
\[
\leq 3 \sum_{k=n}^m \sup_{z \in \mathbb{T}} |h_{\epsilon,k+1}(z) - z|
\]
\[
\leq 3 \sum_{k=n}^m (1.5)^{-k}.
\]
This shows that \( \{H_{\epsilon,m}\}_{m=1}^\infty \) is a Cauchy sequence in \( C(\mathbb{T}) \), its limit being a continuous and strictly increasing function is a homeomorphism of \( \mathbb{T} \). \qed
Lemma 21. Assume \( \{\epsilon_k\}_{k=1}^t \) are already chosen and \( M_t \) is large enough in order that for all \( w \in \Sigma_k \) and \( x, y \in C_{[w_1]}^{M_t} \)

\[
\frac{H^\prime_{x,N_t}(x)}{H^\prime_{y,N_t}(y)} = e^{\pm 2^{-N_t}}.
\]

There exists \( \delta_{t+1} > 0 \) so that if \( \epsilon_{t+1} < \delta_{t+1} \) then

\[
(4.7) \quad g^\prime_{N+1}(x) = \lambda_{t+1}^{\pm M_t} e^{\pm 2^{-N_t+2}} g^\prime_{N_t}(x)
\]

Here \( g_{N_t} = H_{x,N_t} \circ S \circ H_{y,N_t}^{-1} \).

Proof. Assume first that \( \epsilon_{t+1} = 0 \) and since we are not going to vary \( \xi \) we write \( H_{x,n} \) to denote \( H_{x,n} \). Let \( z \in \mathbb{T} \), there exists a unique \( y = y(z) \) such that \( z = H_{N_t+1}(y) \). By a repeated use of the chain rule and the fact that for a diffeomorphism \( H \)

\[
\left(h^{-1}\right)^\prime(x) = \frac{1}{h^\prime \circ h^{-1}(x)},
\]

it follows that (where \( g_{N_t} \) is differentiable!)

\[
g^\prime_{N+1}(z) = g^\prime_{N_t}(H_{N_t}(y)) \cdot \prod_{j=M_t}^{N_t+1} \frac{h^\prime_j(H_{j-1}(Sy))}{h^\prime_j(H_{j-1}(y))}.
\]

To understand the validity of the last inequality notice that since for \( N_t \leq m < M_t \), \( h_m \) is the identity then \( H_{N_t} = H_{M_t-1} \).

Fix \( j \in [M_t, N_{t+1}] \). Notice that \( H_{j-1}(y) \in H_{j-1} \left(C_{[w_1]^j}\right) \) if and only if

\[
H_{j-2}(Sy) \in H_{j-2} \left(C_{[w_2 \cdots w_j]}\right),
\]

and that by Lemma 19 (ii), \( H_{j-1}(y) \) and \( H_{j-2}(Sy) \) are to the right of \( \xi \left(C_{[w_1]^j}\right) \) and \( \xi \left(C_{[w_2 \cdots w_j]}\right) \) respectively if and only if \( y \) is to the right of the reference point in \( C_{[w_1]^j} \). In addition by the requirement on \( M_t \) and Lemma 19

\[
h^\prime_{M_t}(x) = e^{\pm 2^{-N_t}}.
\]

Thus under the assumption that \( \epsilon_{t+1} = 0 \) (by the property of the construction), for all \( j \in (M_t + 1, N_{t+1}] \),

\[
\frac{h^\prime_{j-1}(H_{j-2}(Sy))}{h^\prime_{j-1}(H_{j-1}(y))} = 1.
\]

The last equality yields

\[
g^\prime_{N+1}(z) = g^\prime_{N_t}(H_{M_t-1}(y)) \cdot \frac{h^\prime_{M_t}(H_{M_t-1}(Sy))}{h^\prime_{M_t}(H_{M_t-1}(y))} \cdot \frac{h^\prime_{M_t+1}(H_{M_t}(Sy))}{h^\prime_{M_t+1}(H_{M_t}(y))} \cdot \prod_{j=M_t+2}^{N_t+1} \frac{h^\prime_{j-1}(H_{j-2}(y))}{h^\prime_{j-1}(H_{j-1}(y))}.
\]

\[
= \left( \lambda_{t+1}^{\pm 2^{-N_t+1}} \right) \cdot \frac{g^\prime_{N_t}(H_{M_t-1}(y))}{g^\prime_{N_t}(z)} \cdot g^\prime_{N_t}(z).
\]

\[
= \left( \lambda_{t+1}^{\pm 2^{-N_t+2}} \right) \cdot g^\prime_{N_t}(z).
\]
Here the last inequality follows from the conditions of the Lemma on $M_t$ since $H_{M_t-1}^{-1}(z) = H_{N_t}^{-1}(z)$, $y \in C_{[w]_{m}^{n}}$ and
\[
\frac{g'_{N_t} (H_{M_t-1}(y))}{g'_{N_t} (z)} = \frac{H'_{N_t} (H_{N_t}^{-1}(z))}{H'_{N_t} (S y)} \frac{H'_{N_t} (S y)}{H'_{N_t} (SH_{N_t}^{-1}(z))}.
\]

In [BH] they argue that the estimate on the derivative is continuous (uniformly) with respect to $\epsilon_{t+1}$ since $\psi'_{t+1} \epsilon_{t+1}$ converges pointwise to $\psi'_{0, \lambda_{t+1}}$ when $\epsilon \to 0$. However this convergence is not uniform (and it can’t be as it converges to a step function) and therefore this argument is not sufficient for a uniform statement (which is needed for convergence in the $C^1$-norm).

We proceed as follows. For $n \in (M_t, N_{t+1})$ and $w \in \Sigma_\Lambda$ denote by
\[
\text{BS}(n, w) := \left\{ y \in C_{[w]_n^m} : h'_{t} \circ H_{n-1}(z) \neq \frac{\lambda_t+1 \varphi^2}{1+\lambda_t+1 \varphi} \text{ or } \frac{\varphi^2}{1+\lambda_t+1 \varphi} \text{ in a punctured neighb. of } y \right\}
\]
(the Bad Set at stage $n$ for $w$) to be the pull back under $H_{n-1}$ of the union of the 4-segments of length $\epsilon_{t+1} \cdot m_{\text{Leb}}(H_{n-1}(C_{[w]_n^m}))$ where the derivative of $\psi_{t+1, \lambda_{t+1}}$ (scaled to $H_{n-1}(C_{[w]_n^m})$ as in the definition of $h_{t+1,n}(z)$) is interpolated.

First we demand that $\delta_{t+1}$ is small enough so that the conclusion of Lemma 19 and equation (1.5) hold for all $\epsilon_{t+1} < \delta_{t+1}$.

Secondly we demand that $\delta_{t+1}$ is small enough so that for $M_t < n < m \leq N_{t+1}$, if $\text{BS}(m, w) \cap \text{BS}(n, w) \neq \emptyset$ then one of the end points of $H_{m-1}(C_{[w]_n^m})$ is either an end point of $H_{n-1}(C_{[w]_n^m})$ or the point in $\frac{1}{\varphi}$ proportion in $H_{n-1}(C_{[w]_n^m})$; see the following figure where the small intervals are the possibilities of $\text{BS}(m, w)$ (the scale is of course incorrect and for demonstration purposes).

\[\text{BS}(n, w) \cap \text{BS}(m, w)\]

To understand why we choose these points, notice that in those marked endpoints $h'_{t}(x) = 1$.

This can be done if for example
\[
\frac{m_{\text{Leb}}(H_{n-1}(C_{[w]_n^m}))}{m_{\text{Leb}}(H_{n-1}(C_{[w]_n^m}))} \geq \left( \frac{1}{\varphi^2 \lambda_t^2} \right)^{n-m} \geq \frac{1}{5^{-m+1}} \gg \delta_{t+1},
\]
because then the bad sets are pull backs of unions of 4 intervals of considerably smaller length so if there is intersection we know that the smaller interval is one of the 4 subintervals which contains one of the proposed points. In fact we can demand by taking perhaps a smaller $\delta_{t+1}$ that for all $w \in \Sigma_\Lambda$ and $M_t < n < m \leq N_{t+1},$
\[
\text{BS}(n, w) \cap \text{BS}(m, w)
\]
is always one interval for which one of its end points satisfies
\[
h'_{t}(H_{n-1}(x)) = 1
\]
In addition, 
\[
\frac{m_{\text{Leb}}(\text{BS}(n, w) \cap \text{BS}(m, w))}{m_{\text{Leb}}(\text{BS}(n, w))} \leq \sup_{x \in \mathbb{T}} (h_n^{-1} \circ \cdots \circ h_1^{-1})' (x) \frac{m_{\text{Leb}}(C_{[\omega_1]}^n)}{m_{\text{Leb}}(C_{[\omega_1]}^n)} \\
\leq \left( \frac{\lambda_1^2}{\varphi} \right)^{n-m} \leq (1.6)^{n-m}.
\]

By the definition of \( h_{\xi,n} \), \( h_{\xi,n}' \circ H_{n-1} \) is a Lipschitz function with a Lipschitz constant of order 
\[
\text{Const.}
\]

It follows from the above consideration on the size of \( \text{BS}(n, w) \cap \text{BS}(m, w) \) inside \( \text{BS}(n, w) \) and the fact that for one point on the boundary of \( (n, w) \cap \text{BS}(m, w), h_n'(H_{n-1}(x)) = 1 \), there exists a constant \( B > 0 \) such that for all \( y \in \text{BS}(n, w) \cap \text{BS}(m, w), \)
\[
h_n'(H_n(y)) = e^{\pm B(1.6)^{n-m}}.
\]

The final argument is as follows: Given \( x \in \mathbb{T} \) there is a unique \( y \in \mathbb{T} \) such that \( x = H_{N_t}(y) \). Let \( w \) be such that \( y \in C_{[\omega_1]}^{N_t+1} \). If \( y \notin \cup_{n=M_t+1}^{N_t} \text{BS}(n, w) \) then a similar analysis as in the case \( \epsilon_{t+1} = 0 \) yields the conclusion. Otherwise there exists a maximal \( J = J(y) \leq N_t+1 \) such that \( y \in \text{BS}(J, w) \). A similar analysis as in the case \( \epsilon_{t+1} = 0 \) yields
\[
g'_{N_t+1}(z) = \lambda_{t+1}^{\pm 4} g'_{J} \circ H_{J-1}(y).
\]
For \( M_t + 2 \leq n < J - M_t/4 \), either \( y \notin \text{BS}(k, w) \) for all \( k \leq n \) and then we proceed as in the case \( \epsilon_{t+1} = 0 \) or \( y \in \text{BS}(n, w) \cap \text{BS}(J, w) \) and then,
\[
h_n' \circ H_{n-1}(y) = e^{\pm B(1.6)^{n-J}}.
\]
In addition, \( S(\text{BS}(n, w) \cap \text{BS}(J, w)) \) is an interval of size \( \varphi m_{\text{Leb}}(\text{BS}(n, w) \cap \text{BS}(J, w)) \) with one point (either end point or \( \frac{1}{\varphi} \) point of \( C_{[\omega_1, \ldots, \omega_{n+1}]} \)) for which \( h_n' \circ H_{n-1}(x) = 1 \). Therefore as before,
\[
h_n' \circ H_{n-1}(Sy) = e^{\pm \varphi B(1.6)^{n-J}}.
\]
Thus
\[
g'_{J} \circ H_{J-1}(y) \leq \left( g'_{N_t}(H_{M_t-1}(y)) h'_{M_t}(H_{M_t-1}(y)) \right)^{J-M_t/4} \left( \prod_{n=M_t+1}^{J-M_t/4} e^{3B(1.6)^{n-J}} \right) \left( \prod_{k=J-M_t/4}^{J} h'_{k}(H_{k-1}(Sy)) \right)
\leq \left( g'_{N_t}(z) e^{2-N_t+2} \right) e^{C(1.6)^{-M_t/4} \lambda_{t+1}^{M_t}}.
\]
The upper bound follows from the last equation together with (4.8) since \( M_t/4 \gg N_t \). The lower bound is similar.

A consequence of this Lemma is that we can choose \( \epsilon = \{ \epsilon_k \}_{k=1}^\infty \) so that \( g_{N_t} \) and \( D_{g_{N_t}} \) converge uniformly to a map \( g \) with
\[
D_g(x) = \varphi \cdot \prod_{t \in \mathbb{N}} \lambda_t^{M_t-1} \cdot e^{\sum_{t=1}^\infty 2^{-N_t+4}}.
\]

By taking care that for each \( t \in \mathbb{N} \), \( \lambda_t^{M_t-1} \) is small enough and the \( N_t \) are large enough (which is possible in the inductive construction of the type III_1 Markov measure) we can
take care that the right hand side is bigger than 1 and arbitrarily close to \( \varphi \). The condition we get here on \( \lambda_k, N_k \text{ and } M_k \) is
\[
1.6 \leq \varphi \cdot \left( \prod_{t \in \mathbb{N}} \lambda_t^{\pm M_{t-1}} \right) \cdot \exp \left( \sum_{t=1}^{\infty} 2^{-N_t+4} \right) \leq 1.7.
\]
What remains to be shown before we can explain the modified inductive construction of \( \{\lambda_k, M_k, N_k, \epsilon_k\}_{k=1}^{\infty} \) is that we can choose \( \epsilon \) so that
\[
m_{\text{Leb}} \circ h_\epsilon \sim \mu^+.
\]

**Lemma 22.** Assume that \( \mu^+ \) is a push forward (via the map defined by the Markov partition for \( S \)) of the Markovian type III\(_1\) measure for the shift defined by \( \{\lambda_k, m_k, n_k, M_k, N_k\}_{k=1}^{\infty} \). Then there exists a sequence \( \epsilon = (\epsilon_k)_{k=1}^{\infty} \) such that for every \( \epsilon = (\epsilon_k)_{k=1}^{\infty} \) which satisfies \( \forall k \in \mathbb{N}, \epsilon_k \leq \epsilon \), the function \( h_\epsilon \) defined previously satisfies
\[
m_{\text{Leb}} \circ h_\epsilon \sim \mu^+.
\]

**Proof.** The proof of the Lemma will be done by applying the theory of local absolute continuity of Shiryaev with \( \mathcal{F}_t \) the sigma algebra generated by \( \{C_{[w]}_{\mu^t} : w \in \Sigma_A\} \).

Given \( \epsilon = (\epsilon_k)_{k=1}^{\infty} \),
\[
(m_{\text{Leb}} \circ h_\epsilon)_t := m_{\text{Leb}} \circ h_\epsilon |_{\mathcal{F}_t} = m_{\text{Leb}} \circ H_{\epsilon,N_t},
\]
and
\[
(\mu^+)_t := \mu |_{\mathcal{F}_t} = m_{\text{Leb}} \circ H_{0,N_t}.
\]

A calculation shows that
\[
z_t(x) := \frac{d (m_{\text{Leb}} \circ h_\epsilon)_t(x)}{d (\mu^+)_t(x)} = \frac{H_{\epsilon,N_t}(x)}{H_{0,N_t}(x)}.
\]

Writing \( \tilde{H}_{E,k,t} \) for the function \( H_{\delta(\epsilon,k),N_t} \) with
\[
\delta(\epsilon,k) := \begin{cases} \epsilon_j, & 1 \leq j \leq k \\ 0, & j > k \end{cases},
\]
and noticing that \( \tilde{H}_{\delta(\epsilon,0),N_t} = H_{0,N_t} \) we get
\[
z_t(x) = \prod_{k=1}^{t} \frac{\tilde{H}_{E,k,t}(x)}{\tilde{H}_{E,k-1,t}(x)}.
\]

By [Shl p. 527 remark 2] it remains to show that we can choose \( \epsilon \) such that if for all \( k \in \mathbb{N}, \epsilon_k < \epsilon \), then \( \{z_t\}_{t=1}^{\infty} \) is uniformly integrable with respect to \( \mu \). We proceed to show how to choose \( \epsilon \). Let \( x \in T \setminus \text{Bd}(S) \).

Fix \( k \in \mathbb{N} \). By the chain rule and the fact that \( H_{\delta(\epsilon,k),M_{t-1}} = H_{\delta(\epsilon,k-1),M_{t-1}} \) one sees that
\[
\frac{\tilde{H}_{E,k,t}(x)}{\tilde{H}_{E,k-1,t}(x)} = \left( \prod_{l=M_{k-1}+1}^{N_k} \frac{\rho_{\delta(\epsilon,k),l}(x)}{\rho_{\delta(\epsilon,k-1),l}(x)} \right) \cdot \left( \prod_{l=N_k}^{N_t} \frac{\rho_{\delta(\epsilon,k),l}(x)}{\rho_{\delta(\epsilon,k-1),l}(x)} \right).
\]

First we will want to prove that if \( \epsilon_k \) is small enough, then
\[
\prod_{l=N_k}^{N_t} \frac{\rho_{\delta(\epsilon,k),l}(x)}{\rho_{\delta(\epsilon,k-1),l}(x)} \leq e^{3(1.6)^{-N_k}}.
\]
To see (4.11), first notice that since for every $s \geq k$ and $N_s < n < M_s$,

$$
\|h_{\delta(x),n} = h_{\delta(x,k),n} = id,
$$
then for all $k \leq s \leq t - 1$,

$$
\prod_{l = M_s}^{M_{s-1}} \frac{\rho_{\delta(x,k),l}(x)}{\rho_{\delta(x,k),l}(x)} = 1.
$$

Secondly, for $s \geq k$ and $M_s < n \leq N_{s+1}$ there exists $w \in \Sigma_A$ such that $x \in C_{[w]_l}$. If $w_n \neq 1$ then $\rho_{\delta(x,k),l}(x) = \rho_{\delta(x,k),l}(x) = 1$. Otherwise notice that for $\delta \in \{ \delta(x), \delta(x,k) \}$, $H_{\delta,n-1}(x)$ is to the right of the point in $\frac{1}{p}$ proportion in $H_{\delta,n-1}(C_{[w]_l}^1)$ if and only if $x$ is to the right of the point in $\frac{1}{p}$ in $C_{[w]_l}$. Since $(\delta)_s + 1 = 0$ then $\rho_{\delta(x,k),l}(x) = \rho_{\delta(x,k-1),l}(x)$ and consequently

$$
\prod_{l = M_s}^{N_{s+1}} \frac{\rho_{\delta(x,k),l}(x)}{\rho_{\delta(x,k),l}(x)} = 1.
$$

Therefore by Lemma 19(i),

$$
\prod_{s = k}^{t-1} \prod_{l = M_s}^{N_{s+1}} \frac{\rho_{\delta(x,k),l}(x)}{\rho_{\delta(x,k),l}(x)} \leq \prod_{s = k}^{t-1} \frac{\rho_{\delta(x,k),M_s}(x)}{\rho_{\delta(x,k-1),M_s}(x)} \leq e^{-3(1.6)^{-N_t}} \leq e^{3(1.6)^{-N_k}}.
$$

We remark here that similarly one can get that

$$
\prod_{s = k}^{t-1} \prod_{l = M_s}^{N_{s+1}} \frac{\rho_{\delta(x,k),l}(x)}{\rho_{\delta(x,k),l}(x)} \geq e^{-3(1.6)^{-N_k}},
$$

which in turn shows that there exists $c > 1$ such that

(4.12)

$$
z_t(x) = e^{+1} \prod_{k = 1}^{t} \prod_{l = M_{k-1}}^{N_k} \frac{\rho_{\delta(x,k),l}(x)}{\rho_{\delta(x,k),l}(x)},
$$

here one can notice that the product in the right hand side is similar to the formula of the Radon Nykodym derivatives for the non homogenous Markov shifts. If for every $t < k$, $M_t$ is large enough such that Lemma 19(i) holds then there exists $c > 0$ such that

$$
z_t(x) = e^{+1} \prod_{k = 1}^{t} \prod_{l = M_{k-1}+1}^{N_k} \frac{\rho_{\delta(x,k),l}(x)}{\rho_{\delta(x,k),l}(x)}.
$$

Next when $\epsilon_k \to 0$,

$$
\psi_k(x) := \psi_{\epsilon_k,\lambda_k}(x) \to \psi_{0,\lambda_k}(x) \mu \text{ a.e. } x,
$$
(in fact the convergence is on all but a finite collection of points). It follows that

$$
\frac{\rho_{\delta(x,k),l}(x)}{\rho_{\delta(x,k),l}(x)} \epsilon_k \to 1 \mu \text{ a.e. } x.
$$
By Egorov there exists $A_k \in \mathcal{B}_T$, with $\mu (A_k) > 1 - \frac{1}{2^n \prod_{l=1}^{M_n} (\lambda_n)^{n_l}}$ such that

$$\prod_{l=M_{k-1}+1}^{N_k} \frac{\rho \delta_{(x,k),l}(x)}{\rho \delta_{(x,k-1),l}(x)} \xrightarrow{\epsilon_k \to 0} 1,$$

uniformly in $x \in A_k$.

The lower bound on the measure of $A_k$ is chosen because for every $\epsilon_k > 0$

$$\max_{x,y \in T} \prod_{l=M_{k-1}+1}^{N_k} \frac{\rho \delta_{(x,k),l}(x)}{\rho \delta_{(x,k-1),l}(y)} \leq \left( \max_{x,y \in T} \frac{\psi_{A_k,\epsilon_k}(x)}{\psi_{A_k,\epsilon_k}(y)} \right)^{n_k} = (\lambda_k)^{n_k}.$$

Now we are finally in a position to define the sequence $\epsilon$. Let $\epsilon_k$ be small enough so that for every $\epsilon$ with $\epsilon_k < \epsilon$ and $x \in A_k$,

$$1 - \frac{1}{k^2} \leq \prod_{l=M_{k-1}+1}^{N_k} \frac{\rho \delta_{(x,k),l}(x)}{\rho \delta_{(x,k-1),l}(x)} \leq 1 + \frac{1}{k^2}.$$

Let $\epsilon$ which satisfies for every $k \in \mathbb{N}$, $\epsilon_k < \epsilon$. For large $M$, if for some $n \in \mathbb{N}$ and $x \in T$, $z_n(x) > M$, then there exists $q = q(M) \leq n$ such that $x \in \bigcup_{r=q}^{\alpha c_r} J_r$. Therefore by (4.12) and decomposing the set $[z_n > M]$ by the last $r \leq n$ for which $x \in A_r$,

$$\int_{[z_n > M]} z_n(x) \, dx \leq c \int_{[z_n > M]} \left( \prod_{k=1}^{t} \prod_{l=M_{k-1}+1}^{N_k} \frac{\rho \delta_{(x,k),l}(x)}{\rho \delta_{(x,k-1),l}(x)} \right) d\mu(x)$$

$$\leq c \sum_{r=q(M)}^{n} \int_{A_r \cap \bigcap_{j=r+1}^{r} J_j} \left( \prod_{k=1}^{t} \prod_{l=M_{k-1}+1}^{N_k} \frac{\rho \delta_{(x,k),l}(x)}{\rho \delta_{(x,k-1),l}(x)} \right) d\mu(x)$$

Since for $x \in A_j$, $\prod_{l=M_{j-1}+1}^{N_j} \frac{\rho \delta_{(x,j),l}(x)}{\rho \delta_{(x,j-1),l}(x)} \leq 1 + \frac{1}{j^2}$,

$$\sum_{r=q(M)}^{n} \int_{A_r \cap \bigcap_{j=r+1}^{r} J_j} \left( \prod_{k=1}^{t} \prod_{l=M_{k-1}+1}^{N_k} \frac{\rho \delta_{(x,k),l}(x)}{\rho \delta_{(x,k-1),l}(x)} \right) d\mu(x) \leq k \prod_{j=1}^{\infty} \left( 1 + \frac{1}{j^2} \right) \sum_{r=q(M)}^{n} \mu (A_r) \max_{x \in T} \prod_{k=1}^{t} \prod_{l=M_{k-1}+1}^{N_k} \frac{\rho \delta_{(x,k),l}(x)}{\rho \delta_{(x,k-1),l}(x)}$$

When $M \to \infty$ then $q(M) \to \infty$ and therefore

$$\sup_{n \in \mathbb{N}} \int_{[z_n > M]} z_n(x) \, d\mu(x) \leq 2q(M-1) \prod_{j=1}^{\infty} \left( 1 + \frac{1}{j^2} \right) \to 0 \text{ as } M \to \infty.$$

This shows that $\{z_n\}$ is uniformly integrable and hence

$$m_{\text{Leb}} \circ \mathcal{H}_\epsilon \sim \mu^+$$
4.2.2. The modified induction process for choosing \( \{\lambda_k, N_k, M_k, n_k, m_k, \epsilon_k\} \) and the proof of Theorem \([18]\). In the course of the construction here we arrived at 2 conditions on \( \{\epsilon_k\} \) and 2 extra conditions on \( \{\lambda_k, M_k\} \). In order to show the existence of these sequences hence the type III expanding map of \( T \) arising from the Golden Mean shift one has to modify the induction process of Section 3 as follows and insert the choice of \( \{\epsilon_t\} \) in the induction.

In the proof of the previous Lemmas we have an extra condition on the size of \( M_t \) (or \( m_t = M_t - N_t \)) which is determined by \( \{N_s, \lambda_s, M_{s-1}, \epsilon_s\}_{s=1}^{t} \).

The choice of \( \epsilon_{t+1} \) in Lemma \([22]\) \( \delta_{t+1} \) in Lemma \([21]\) and \( \epsilon_{t+1} \) in Proposition \([19]\) is determined by \( \{N_s, \lambda_s, M_{s-1}, \epsilon_s\}_{s=1}^{t} \) and \( \{N_{t+1}, M_t\}, \epsilon_{t+1} \). We also need to take care that

\[
1.6 \leq \phi \cdot \left( \prod_{t \in \mathbb{N}} \lambda_t^{\pm 2M_{t-1}} \right) \cdot \exp \left( \pm \sum_{t=1}^{\infty} 2^{-N_{t+4}} \right) \leq 1.7.
\]

This shows now that the order of choice in the induction is as follows

\[
\{\lambda_s, n_s, N_s, m_s, M_s, \epsilon_s\}_{s=1}^{t} \Rightarrow \lambda_{t+1} \Rightarrow \{n_{t+1}, N_{t+1}\} \Rightarrow \epsilon_{t+1} \Rightarrow \{m_{t+1}, M_{t+1}\}.
\]

The modifications needed to be done in the inductive construction are: First change the condition \((3.3)\) on \( \lambda_{t+1} \) with the condition

\[
\lambda_{t+1}^{2M_t} \leq \exp \left( 2^{-N_t} \right),
\]

as this involves making \( \lambda_{t+1} \) smaller there is no contradiction. This gives that

\[
\phi \cdot \left( \prod_{t \in \mathbb{N}} \lambda_t^{\pm 2M_t} \right) \cdot \exp \left( \pm \sum_{t=1}^{\infty} 2^{-N_{t+4}} \right) = \phi \cdot \exp \left( \pm \sum_{t=1}^{\infty} 2^{-N_{t+5}} \right).
\]

By demanding now that \( N_1 > 20 \), we get

\[
\phi \cdot \left( \prod_{t \in \mathbb{N}} \lambda_t^{\pm 2M_t} \right) \cdot \exp \left( \pm \sum_{t=1}^{\infty} 2^{-N_{t+4}} \right) = \phi \cdot \exp \left( \pm \sum_{t=1}^{\infty} 2^{-N_{t+5}} \right) \in (1.6, 1.7)
\]

as we required. There is no further change in the inductive choice of \( \lambda_t, n_t, N_t \) as they won’t depend on \( \epsilon_t \).

Given \( \{\lambda_s, n_s, N_s, m_s, M_s, \epsilon_s\}_{s=1}^{t} \) and \( N_{t+1} \) we choose \( \epsilon_{t+1} \) to be small enough so that the conclusions of Proposition \([19]\) (ii), Lemma \([21]\) and Lemma \([22]\) hold true.

Then we choose \( m_{t+1} \) based on the original constraints from Section 3 together with the restriction that \( M_{t+1} = m_{t+1} + N_{t+1} \) is large enough so that the conclusion of Proposition \([19]\) (i) is true. Since this involves perhaps enlarging \( m_{t+1} \) it is consistent with the other constraints of the induction.

Proof. of Theorem \([18]\).

Choose \( \{\lambda_k, N_k, M_k, n_k, m_k, \epsilon_k\}_{k=1}^{\infty} \) as in the inductive construction. Build the Markovian measure \( \eta = M \{P_k, \pi_k : k \in \mathbb{Z}\} \) the Markovian measure determined by \( \{\lambda_k, N_k, M_k, n_k, m_k\}_{k=1}^{\infty}, \mu := \Phi_\pm (\eta) \) and \( \mu^+ = \Theta_\pm (M \{P_k, \pi_k : k \in \mathbb{Z}\}) \).

Part (i) follows from Theorem \([10]\) since \( \{\lambda_k, N_k, M_k, n_k, m_k\}_{k=1}^{\infty} \) satisfy the constraints of the inductive construction in Section 3 hence it is a type III measure for the shift.

(ii) and (iii): By the extra condition on the \( M_t \)-s and the choice of \( \epsilon = \{\epsilon_k\} \) we have by Lemma \([22]\) that

\[
m_{\text{Leb}} \circ \mathbb{H}_\epsilon \sim \mu^+.
\]
and by Lemma \[21\] that for all \( t \in \mathbb{N} \) and \( x \in \mathbb{T}, \)
\[
g_{N_{t+1}}'(x) = \exp \left( \pm 2^{-N_{t+1}} \right) \lambda_{t+1}^\pm g_{N_t}'(x).
\]
Therefore \( \{g_{N_t}'\}_{t=1}^\infty \) is a Cauchy sequence in \( C(\mathbb{T}) \), its limit function satisfies
\[
1.6 \leq g'(x) = \varphi \cdot \left( \prod_{t \in \mathbb{N}} \lambda_{t}^\pm \right) \cdot \exp \left( \pm \sum_{t=1}^\infty 2^{-N_{t+1}} \right) \leq 1.7.
\]
\( \square \)

5. **Type III Anosov Diffeomorphisms**

Let \( \{\lambda_k, m_k, n_k, M_k, N_k\}_{k=1}^\infty \) and \( \epsilon = \{\epsilon_k\}_{k=1}^\infty \) as in Theorem \[18\] and let \( h_\epsilon \) be the resulting function. Set \( \tilde{\mathcal{F}}_\epsilon(x, y) := (h_\epsilon(x), y) \) and
\[
\mathcal{G}(x, y) := \lambda_\epsilon \circ \tilde{f} \circ \mathcal{F}_\epsilon^{-1}(x, y) = \begin{cases} (g(x), -\varphi^{-1}y), & 0 \leq x \leq 1/\varphi \\ (g(x), -\varphi^{-1}(y - \frac{\varphi^2}{\varphi+2})), & 1/\varphi \leq x \leq 1 \end{cases}
\]
In the construction of Section \[3\] \( P_k = Q \) for all \( k < 0 \). Writing \( m_\mathcal{M} \) for the Lebesgue measure on \( \mathcal{M}_\sim \) one then has
\[
dm \circ \mathcal{F}_\epsilon(x, y) = d\mu^+(x)dy = d\mu(x, y),
\]
or in other words \( m_\mathcal{M} \circ \mathcal{F}_\epsilon = \Phi_\epsilon \mathcal{M} \{ P_k, \pi_k : k \in \mathbb{Z} \} \). Therefore since \( m_\mathcal{T} \circ h_\mathcal{L} \sim \mu^+ \),
\[
\mathcal{M}_\sim, \mathcal{B}_\mathcal{M} \sim, \mathcal{M}_\mathcal{M}, \mathcal{G}
\]
is a type III transformation. This is because \( (\mathcal{M}_\sim, \mathcal{B}_\sim, m_\mathcal{M}, \mathcal{G}) \) is measure theoretically equivalent to \( (\mathcal{M}_\sim, \mathcal{B}_\sim, m_\mathcal{M} \circ \mathcal{F}_\epsilon, \tilde{f}) \) which is orbit equivalent to \( (\mathcal{M}_\sim, \mathcal{B}_\sim, \mu, \tilde{f}) \).

By Remark \[17\] \( \mathcal{G}_\epsilon \) is one to one and onto. In addition, for every \( (x, y) \notin \partial \mathcal{M} \), \( \mathcal{G} \) is differentiable in a neighborhood of \( (x, y) \) as all the partial derivatives are continuous in \( \mathcal{M} \setminus \partial \mathcal{M} \), and
\[
D_{\mathcal{G}}(x, y) = \begin{pmatrix} g'(x) & 0 \\ 0 & -\varphi^{-1} \end{pmatrix}
\]
The problem is that \( \mathcal{G} \) when viewed as a transformation of \( \mathcal{M}_\sim \) is not even continuous on the horizontal lines of \( \partial \mathcal{M} \). Our first attempt was to have \( g|_{\partial \mathcal{M}} = \varphi x \mod 1 \) and \( g|_{\mathcal{M} \setminus \{ x : d(x, \partial \mathcal{M}) > \delta \}} = h_\mathcal{L} \circ S \circ h_\mathcal{L}^{-1} \) and to apply a specific family of homotopies to the identity. This resulted with a family of homeomorphisms of \( \mathbb{T} \), \( \{ \tilde{\mathcal{R}}_y : y \in \left[ -\frac{\varphi^2}{\varphi+2}, \frac{\varphi^2}{\varphi+2} \right] \} \) and \( \{ \tilde{\mathcal{R}}_y : y \in \left[ -\frac{\varphi^2}{\varphi+2}, \frac{1}{\varphi} \right] \} \) preserving the Markov partition \( \{ [0, 1/\varphi^2], [1/\varphi^2, 1/\varphi], [1/\varphi, 1] \} \) such that the map
\[
Z(x, y) := \begin{cases} \left( \tilde{\mathcal{R}}_y \circ S \circ \tilde{\mathcal{R}}_y^{-1}(x), -\varphi^{-1}y \right), & 0 \leq x \leq 1/\varphi \\ \left( \tilde{\mathcal{R}}_y \circ S \circ \tilde{\mathcal{R}}_y^{-1}(x), -\varphi^{-1}(y - \frac{\varphi^2}{\varphi+2}) \right), & 1/\varphi \leq x \leq 1 \end{cases}
\]
is a \( C^1 \) \( (\mathcal{M}_\sim) \) Anosov diffeomorphism (and not \( C^{1+\alpha} \) for any \( \alpha > 0 \)) and for every \( (x, y) \) with distance from \( \partial \mathcal{M} \) greater or equal to \( \delta \) (where \( \delta > 0 \) is arbitrarily small) \( Z(x, y) := \mathcal{G}(x, y) \). In addition for any \( y \) except at the horizontal boundary segments,
\[
m_\mathcal{T} \circ \tilde{\mathcal{R}}_y \sim m_\mathcal{T} \circ h_\mathcal{L} \sim \mu^+.
\]
However, we found it quite hard to work with the map $Z$, a thing which resulted in using a more clever map on $\partial M$ then the $\varphi x \mod 1$. This map is done by decomposing the horizontal lines of $\partial M$ into identified segments. On each subpart we apply a coupling of the symbolic orbits (to times where 1 and 3 can appear w.r.t. Lebesgue measures on both parts) which tells us when to start to change the measure $m_T$ to the measure $m_T \circ h_z$.

5.1. A further perturbation of $\mathcal{G}$. For each $y \in \left[ -\frac{\varphi}{\varphi^2+2}, \frac{\varphi^2}{\varphi^2+2} \right]$ we will define a perturbation map $h_y : T \to T$ in the spirit of Section 4 and then setting

$$Z(x,y) := \begin{cases} (h_{-\varphi y} \circ S \circ h_y^{-1}(x), -y/\varphi), & x \leq 1/\varphi \\ (h_{-\varphi y} \circ S \circ h_y^{-1}(x), -y/\varphi + \frac{\varphi}{\varphi^2+2}), & 1/\varphi \leq x \leq 1. \end{cases}$$

$M_{\sim} \to M_{\sim}$.

For a fixed $y$, we define a sequence of functions $r_n(x,y) : T \times \left[ -\frac{\varphi}{\varphi^2+2}, \frac{\varphi^2}{\varphi^2+2} \right] \to T$, $n \in \mathbb{N}$. This defines a sequence $h_{n,y}(\cdot) := r_n (\cdot, y) : (T \text{ or } [0,1/\varphi]) \to T$ and (we will take care that this limit exists)

$$h_y(x) := \lim_{n \to \infty} h_{n,y} \circ h_{n-1,y} \circ \cdots \circ h_{1,y}(x).$$

Particular care is taken in order to ensure that if $(x,y) \sim (\hat{x}, \hat{y})$ then

$$h_y(x) = h_y(\hat{x}),$$

which is needed for the continuity of $Z$ on $M_{\sim}$. To obtain this we first couple segments on the horizontal lines of $\partial M$.

5.2. Definition of the coupling time on the horizontal boundary of $\mathcal{M}$. Denote by

$$U_1 := [0,1/\varphi] \times \left( \frac{\varphi^2}{\varphi+2} - \frac{1}{\varphi^{10}}, \frac{\varphi^2}{\varphi+2} \right) \cup [1/\varphi, 1] \times \left[ -\frac{\varphi}{\varphi+2}, -\frac{\varphi}{\varphi+2} + \frac{1}{\varphi^{10}} \right],$$

$$U_2 := (1/\varphi, 1) \times \left( \frac{1}{\varphi+2} - \frac{1}{\varphi^{10}}, \frac{1}{\varphi+2} \right) \cup [1/\varphi, 1] \times \left[ -\frac{\varphi}{\varphi+2}, -\frac{\varphi}{\varphi+2} + \frac{1}{\varphi^{10}} \right]$$

and $U := U_1 \cap U_2$. Then $U^c$ is a neighborhood of the horizontal lines of $\partial M$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure51.png}
\caption{The darker color denotes $U_{\varepsilon_n}$}
\end{figure}
In our construction for any \((x, y) \in U\),
\[
\tau _n (x, y) = h_n (x),
\]
with \(h_n\) the functions in the one dimensional example in Section 3. This means that for any \((x, y) \in U\),
\[
h_y (x) = h_z (x).
\]
We now will proceed to specify the construction of \(h_y (x)\) for \((x, y) \in \mathbb{M} \setminus U\).

The first step is to do a coupling on the horizontal lines which roughly tells us where is the break that causes for example the problem that although \((x, y)\) is the same point in \(\mathbb{M}_n\) as \((\hat{x}, \hat{y})\),
\[
(h_z (x), y) \neq (h_z (\hat{x}), \hat{y}).
\]
For example consider the case \(x = 1/\varphi^3\), \(y = \frac{\varphi^2}{\varphi + 2}\) and \(\hat{x} = 1/\varphi\), \(\hat{y} = -\frac{\varphi}{\varphi + 2}\). The point \(\frac{1}{\varphi}\) is a fixed point for \(h_z\) meaning \(h_z (1/\varphi) = 1/\varphi\). Since \(h_z (1/\varphi^3) = \frac{\lambda_1}{\varphi (1 + \lambda_1 \varphi)} \neq \frac{1}{\varphi}\) we get
\[
(h_z (x), y) \neq (h_z (\hat{x}), \hat{y}) = (\hat{x}, \hat{y}).
\]
However if we took care that \(x = \frac{1}{\varphi^3}\) is a fixed point for \(h_y\) then we will have the desired equality. It turns out that the correct way to do this will be by setting \(h_{1, y} | _{[0, 1/\varphi^3]} = h_{2, y} | _{[0, 1/\varphi^3]} = Id | _{[0, 1/\varphi^3]}\) and to start perturbing (similarly as in the definition of \(h_n\) from the previous Section) from \(n \geq 3\). In general we will have a decomposition of the horizontal lines of \(\partial \mathbb{M}\) to \(\{V_i\}_{i=1}^\infty\) and we will start perturbing at \(V_i\) from \(n \geq i + 1\).

To be more precise the horizontal boundary consists of the lines \([0, 1/\varphi] \times \left\{ \frac{\varphi^2}{\varphi + 2} \right\}\), \([1/\varphi] \times \left\{ \frac{1}{\varphi^2 + 2} \right\}\) and \(T \times \left\{ \frac{-\varphi}{\varphi + 2} \right\}\). We look at a countable partition of the horizontal lines \(\partial \mathbb{M}\) which are identified by \(\sim\) and couple them in a time \(T \in \mathbb{N}\) such that in the symbolic space on \(T\), the move \(x_T \to 1\) is possible for both pieces identified.

The segments and their coupling time:

1. \(V_1 := \left[0, 1/\varphi^2\right] \times \left\{ \frac{-\varphi^2}{\varphi + 2} \right\} \sim \left[1/\varphi, 1\right] \times \left\{ \frac{1}{\varphi^2 + 2} \right\}\). In this case \([0, 1/\varphi^2] = C_{[1]}\) and \([1/\varphi, 1] = C_{[2]}\). We choose \(T (V_1) = 2\). One can check that \(m_T \left( C_{[1]} \right) = m_T \left( C_{[2]} \right)\) and \(m_T \left( C_{[2]} \right) = m_T \left( C_{[2]} \right)\). We will define the map \(\mathfrak{M}_{Q_1} := \lim_{n \to \infty} \tilde{H}_{n, V_1} \circ \cdots \tilde{H}_{1, V_1}\) where for \(j < T\), \(\tilde{H}_{j, V_1} (x, y) = Id (x, y)\) and for \(j > T\), \(\tilde{H}_{j, V_1} (x, y) = (\tilde{h}_{j, V_1} (x), y)\), where \(\tilde{h}_{j, Q_1} : [0, 1/\varphi^3] \cup [1/\varphi^2, 1/\varphi] \xrightarrow{\text{onto}} [0, 1/\varphi^3] \cup [1/\varphi^2, 1/\varphi]\) is built as in the perturbation of Section 4, with the parameters \(\{N_s, \lambda_s, M_{s-1}, \epsilon_s\}\).

2. \(V_2 := \left[0, 1/\varphi^3\right] \times \left\{ \frac{\varphi^2}{\varphi + 2} \right\} \sim \left[1/\varphi^2, 1/\varphi\right] \times \left\{ -\frac{\varphi}{\varphi + 2} \right\}\). In this case \([0, 1/\varphi^3] = C_{[1]}\) and \([1/\varphi^2, 1/\varphi] = C_{[3]}\). We choose \(T (V_2) = 3\). In this case \(m_T \left( C_{[1]} \right) = m_T \left( C_{[3]} \right)\) and \(m_T \left( C_{[3]} \right) = m_T \left( C_{[3]} \right)\).

3. \(V_3 := \left[1/\varphi^3, 1/\varphi^2\right] \times \left\{ \frac{\varphi^2}{\varphi + 2} \right\} \sim \left[1/\varphi, 1/\varphi + 1/\varphi^4\right] \times \left\{ -\frac{\varphi}{\varphi + 2} \right\}\). In this case \([1/\varphi, 1/\varphi^2] = C_{[1]}\) and \([1/\varphi, 1/\varphi + 1/\varphi^4] = C_{[2]}\). We choose \(T (V_3) = 4\).

4. \(V_4 := \left[1/\varphi^2, 1/\varphi^2 + 1/\varphi^3\right] \times \left\{ \frac{\varphi^2}{\varphi + 2} \right\} \sim \left[1/\varphi + 1/\varphi^4, 1/\varphi + 1/\varphi^3\right] \times \left\{ -\frac{\varphi}{\varphi + 2} \right\}\). In this case \(C_{[3]} = \left[1/\varphi^2, 1/\varphi^2 + 1/\varphi^3\right]\) and \(C_{[3]} = \left[1/\varphi + 1/\varphi^4, 1/\varphi + 1/\varphi^3\right]\). We define \(T (V_4) = 5\).
(5) For general $j > 4$, $V_j := C_{[w(j)]_1^1} \times \left\{ \frac{x^2}{\varphi+2} \right\} \sim C_{[w(j)]_1^1} \times \left\{ \frac{-x}{\varphi+2} \right\}$ where $w(j), \tilde{w}(j)$ are the following words of length $j$,

\[
\begin{align*}
    w(j) &= \begin{cases} 
        32 \cdots 32132 & j \text{ odd} \\
        32 \cdots 3211, & j \text{ even}
    \end{cases} \\
    \tilde{w}(j) &= \begin{cases} 
        23 \cdots 23211, & j \text{ odd} \\
        23 \cdots 232132, & j \text{ even}.
    \end{cases}
\end{align*}
\]

For convenience we set

\[
w(1) = 1 \text{ and } \tilde{w}(1) = 2.
\]

As is expected $T(V_j) = j + 1$.

The following is immediate from our construction.

**Claim 23.** For any $j \geq 2$,

\[
\tilde{f}(V_j) = V_{j-1},
\]

and

\[
\tilde{f}(V_1) = [0, 1/\varphi] \times \{1\}.
\]

5.2.1. **Definition of the perturbation maps** $h_n,y$. For $w \in \Sigma_A$ and $n \in \mathbb{N}$, we write

\[
C_{[w]}_{1}^{1} := [x_n(w), \tilde{x}_n(w)].
\]

Let

\[
\underline{u}(x, y) := \begin{cases} 
    \min \left\{ \frac{x^2}{\varphi+2} - y, y + \frac{\varphi}{\varphi+2} \right\}, & 0 \leq x \leq \frac{1}{\varphi} \\
    \min \left\{ \frac{1}{\varphi+2} - y, y + \frac{\varphi}{\varphi+2} \right\}, & \frac{1}{\varphi} \leq x \leq 1
\end{cases}
\]

be the minimal distance of $(x, y)$ to the horizontal lines of $\partial \mathcal{M}$. In addition we will write $y(x, y) : \mathcal{M} \to \left\{ \frac{-\varphi}{\varphi+2}, \frac{1}{\varphi+2}, \frac{\varphi}{\varphi+2} \right\}$ to be the value so that

\[
\underline{u}(x, y) = |y(x, y) - y|.
\]

Under that notation $(x, y(x, y))$ is the closest point to $(x, y)$ in the horizontal boundary.

Let $(x, y) \in \mathcal{M}_\infty$.

**Case 1** $(x, y) \in U$: we do the regular construction as in Section 4. That is for any $N_t \leq n < M_t$, $h_n$ is the identity. For any $M_t < n \leq N_t$, if $w_n = 1$ then $h_n|_{H_{n-1}(C_{[w]}_{1}^{1})}$ is a rescaling of $\psi_{\lambda_t, \epsilon_t}$ to the interval $H_{n-1}(C_{[w]}_{1}^{1})$ and if $w_n \neq 1$ then $h_n|_{H_{n-1}(C_{[w]}_{1}^{1})}$ is the identity. If for some $t$, $n = M_t$ then $h_{M_t}|_{C_{[1]}_{M_t}}$ is the distribution correction function in the construction.

Finally we set

\[
r_n(x, y) = h_{n,y}(x) := h_n(x)
\]

and

\[
K_{n,y}(x) := h_{n,y} \circ h_{n-1,y} \circ \cdots \circ h_{1,y}(x) = H_n(x).
\]

**Case 2**, $(x, y) \notin U$: In this case $\underline{u}(x, y) < \frac{1}{\varphi \sqrt{2}}$. Let $(x, y(x, y)) \in \partial \mathcal{M}$ be the closest point on the horizontal lines of $\partial \mathcal{M}$ to $(x, y)$. Let $j(x, y) \in \mathbb{N}$ be the integer so that

\[
(x, y(x, y)) \in V_j(x, y).
\]
The idea behind this construction can be summarized as follows: For a fixed point \( x \in C_{\bar{M},\bar{M}}[\varphi] \) (if \( x \leq 1/\varphi \)) or \( x \in C_{\bar{M},\bar{M}}[\varphi] \) (1/\( \varphi \) \(< x \leq 1 \)). We will define the construction for \( x \in C_{\bar{M},\bar{M}}[\varphi] \), the other case being similar (just with reverse orientation in \( y \)). First we define for any \( x \in C_{\bar{M},\bar{M}}[\varphi] \),

\[
K_{j(x,y),y(x,y)}(x) = x.
\]

Then for any \((x,y) \in \mathbb{M} \setminus U\) such that \( x \in C_{\bar{M},\bar{M}}[\varphi] \) and \((x,y(x,y)) \in V_j\) we set

\[
K_{j(x,y),y(x,y)}(x) := \left( \frac{H_j(x) - x}{\varphi^{10}} \right) \left[ 3\varphi^{-10}(u(x,y))^2 - 2u(x,y)^3 \right] + x
\]

For \( n > j(x,y) \), assume we have defined for all \( j < k < n \) \( h_{k,y} := \varepsilon_k(\cdot,y) \mid_{\bar{M}_{\bar{M}}[\varphi]} \) and \( x \in C_{\bar{M},\bar{M}}[\varphi] \subseteq C_{\bar{M},\bar{M}}[\varphi] \). We set

\[
K_{n-1,y} \mid C_{\bar{M},\bar{M}}[\varphi] = h_{n-1,y} \circ \ldots \circ h_{j+1,y} \circ K_{j,y}(x),
\]

and

\[
t_n(y,w) := m_{\text{Leb}} (K_{n-1,y}( C_{\bar{M},\bar{M}}[\varphi])) = K_{n-1,y}(\varepsilon_n(w)) - K_{n-1,y}(\varepsilon_n(w)).
\]

If \( w_n \neq 1 \) then for all \( \bar{x} \in C_{\bar{M},\bar{M}}[\varphi] \),

\[
\varepsilon_n(x,y) := x.
\]

Else if \( w_n = 1 \) then

\[
\varepsilon_n(x,y) := K_{n-1,y}(\varepsilon_n(w)) + t_n(y,w)\psi_t \left( \frac{x - K_{n-1,y}(\varepsilon_n(w))}{t_n(y,w)} \right),
\]

Remark 24. The 2 variable function

\[
q_j(x,u) := \left( \frac{H_j(x) - x}{\varphi^{10}} \right) \left[ 3\varphi^{-10}u^2 - 2u^3 \right] + x
\]

was chosen because of it’s following properties:

1. \( q_j(x,\varphi^{-10}) = H_j(x) \) and \( q_j(x,0) = x \). This means that for \( y \in \partial \mathbb{M} \), \( \varepsilon_n(x,y) = x \) and therefore \( K_{j,y} \) interpolates between the identity map and \( H_j \mid_{\bar{M}_{\bar{M}}[\varphi]} \).

2. \( \frac{\partial q_j}{\partial x} (x,0) = 1 \) and \( \frac{\partial q_j}{\partial x} (x,\varphi^{-10}) = \frac{\partial H_j}{\partial x} (x) \). This is needed in order that \( \frac{\partial K_{j,y}}{\partial x} \) will be continuous in \( y \).

3. \( \frac{\partial q_j}{\partial u} (x,0) = 0 \) which is necessary for continuity of \( \frac{\partial K_{j,y}}{\partial y} \).

4. \( \sup_{0 \leq z \leq \varphi^{-10}} \left| \frac{\partial q_j}{\partial z}(x,z) \right| = \frac{3}{2} \varphi^{10} |H_j(x) - x| \). We will show that the right hand side is uniformly exponentially small when \( j \to \infty \). The control of the derivatives in the \( y \)-direction is to our opinion the hardest part in (and the heart of) this section.

The idea behind this construction can be summarized as follows: For a fixed \((x,y)\) which is close enough to the horizontal segment on the boundary we first look at the coupling time of the interval which contains the point closest to \((x,y)\) on the boundary. On the boundary we start to apply the rescaling after the coupling time to ensure that the resulting map will be a map of \( \mathbb{M}_{\mathbb{M}} \) (respects the equivalence relation). Inside \( \mathcal{U} \) we just start perturbing from the start and in what remains we do a specific interpolation of the boundary map and the map in \( \mathcal{U} \).
5.2.2. Definition of $\mathcal{F}_N$ and the new examples of Anosov diffeomorphisms: Define $\mathcal{F}_N : M_\infty \to M_\infty$,

$$\mathcal{F}_N(x, y) = \begin{cases} (K_{N,-y/\varphi} \circ S \circ K_{N,y}^{-1}(x), -y/\varphi), & (x, y) \in R_1 \cup R_3 \\ (K_{N,-y/\varphi + \varphi/\varphi+2} \circ S \circ K_{N,y}^{-1}(x), -y/\varphi + \varphi/\varphi+2), & (x, y) \in R_2 \end{cases}.$$

**Remark 25.** In the construction of the previous subsection for every $n \in \mathbb{N}, x \in \{0, 1/\varphi, 1\}$ are fixed points for $h_{\epsilon,n}$ (Remark [17]). This remains true for $x_n$ and $\tilde{x}_n$ in the sense that for every $y$ and $x \in \{0, 1/\varphi, 1\}$

$$x_n(x, y) = \tilde{x}_n(x, y) = x.$$

This shows that $\mathcal{F}_N$ is continuous. In addition if $x$ is an endpoint of the segment $K_{n,y} \left( C_{[w][1]} \right)$ for some $w$, then

$$\frac{\partial x_n}{\partial x}(x, y) = 1.$$ 

Therefore for every $n \in \mathbb{N}$ and $x \in \{0, 1/\varphi, 1\}$,

$$\frac{\partial x_n}{\partial x}(x, y) = \frac{\partial \tilde{x}_n}{\partial x}(x, y) = 1.$$ 

This gives that $\mathcal{F}_N$ is $C^1$. The invariance of the Markov partition $\{J_1, J_2, J_3\}$ of $S$ under $K_{y,n}$ gives that $\mathcal{F}_N$ is one to one and onto and

$$\mathcal{F}_N^{-1}(x, y) = \begin{cases} K_{n,y}^{-1} \circ (S_{|J_1 \cup J_3})^{-1} \circ K_{n,-y}(x), -y), & -\frac{\varphi}{\varphi+2} \leq y \leq \frac{1}{\varphi+2} \\ K_{n,y}^{-1} \circ (S_{|J_2})^{-1} \circ K_{n,-y+\varphi^2/(\varphi+2)}(x), -y + \varphi^2/\varphi+2), & 1/\varphi < y \leq \frac{\varphi^2}{\varphi+2} \end{cases}.$$

Here $(S_{|J_1 \cup J_3})^{-1}(x) := \frac{x}{\varphi} : [0, 1] \to [1, 1/\varphi]$ is the inverse branch of $S$ to the segment $[0, 1/\varphi]$ and $(S_{|J_2})^{-1}(x) := \frac{x+1}{\varphi} : [1/\varphi, 1] \to [1/\varphi, 1]$ is the inverse branch of $S$ to the segment $[1/\varphi, 1]$. This in fact shows that $\mathcal{F}_N$ is a diffeomorphism.

**Theorem 26.** The sequence $\mathcal{F}_N$, converges in the $C^1$ topology to a type $\text{III}_1$ Anosov diffeomorphism.

The proof of this Theorem of this paper is by a series of Lemma’s. The first step is to show that $K_{n,y}(x)$ converges uniformly in $M$ as $t \to \infty$.

**Lemma 27.** If $x \in C_{[w(J)]_1^T} \cup C_{[\tilde{w}(J)]_1^T}$ and $J \in [M_{t-1}, N_t]$,

$$|H_J(x) - x| \leq m_T \left( C_{[w(x)]_1^{J-3}} \right) \leq \varphi^{-(J-3)},$$

where $w(x) \in \{w(J), \hat{w}(J)\}$ is such that $x \in C_{[w(x)]_1^T}$.

**Proof.** By the form of $w(J)$ and $\tilde{w}(J)$ one has that for all $l \leq J - 3$,

$$w(x)_l \in \{2, 3\}$$

hence

$$H_{J-3}(x)|C_{[w(x)]_1^{J-3}} = x.$$
Since \( x_{J-3}(w(x)) \), \( \bar{x}_{J-3}(w(x)) \) are fixed points of \( h_{j-2}, h_{j-1} \) and \( h_j \) one has that
\[
H_j \left( C_{[w(x)]_1^{J-3}} \right) = C_{[w(x)]_1^{J-3}},
\]
the lemma follows. \( \square \)

**Corollary 28.** The limit
\[
\lim_{n \to \infty} K_{n,y}(x) =: h_y(x)
\]
eexists uniformly in \( \mathcal{M} \) and is a continuos function and the function \( \mathcal{H}(x,y) = (h_y(x), y) \) is a homeomorphism of \( \mathcal{M}_\sim \).

**Proof.** First we claim that for every \( n \in \mathbb{N} \),
\[
(5.1) \quad \sup_{(x,y) \in \mathcal{M}_\sim} |r_n(x,y) - x| \leq (1.5)^{-n}.
\]
For every \( w \in \Sigma_A \),
\[
r_n \left( K_{n-1,y} \left( C_{[w]_n} \right), y \right) = K_{n-1,y} \left( C_{[w]_n} \right),
\]
consequently
\[
|r_n(x,y) - x| \leq m_{\text{Leb}} \left( K_{n-1,y} \left( C_{[w]_n} \right) \right).
\]
Since \( \left| \frac{\partial \psi}{\partial x} \right| \leq \lambda_1^2 \) for every \( l \in \mathbb{N} \), it follows that
\[
\sup_{x \in T} \left| \frac{\partial r_k}{\partial x} (x,y) \right| \leq \lambda_1^2.
\]
The previous inequality is true since \( r_k \) is either the identity, defined piecewise on intervals by \( \psi_t \) for some \( t \in \mathbb{N} \) or an interpolation of the two. Proceeding as in Lemma 20 one has
\[
m_{\text{Leb}} \left( K_{n-1,y} \left( C_{[w]_n} \right) \right) \leq \sup_{x \in T} \left| \frac{\partial K_{n-1,y}}{\partial x} \right| m_{\text{Leb}} \left( C_{[w]_n} \right)
\]
\[
\leq \left( \prod_{k=1}^{n-1} \sup_{x \in T} \left| \frac{\partial r_k}{\partial x} (x,y) \right| \right) \varphi^{-n}
\]
\[
\leq \left( \frac{\lambda_1^2}{\varphi} \right)^{-n}. \quad \Box (5.1)
\]
Now we can show that \( \{K_{n,y}(x)\}_{n=1}^\infty \) is a Cauchy sequence in the uniform topology. Indeed for every \( n < m \) and \( (x,y) \in T \times \left[ -\frac{\varphi}{\varphi+2}, \frac{\varphi}{\varphi+2} \right] \),
\[
|K_{m,y}(x) - K_{n,y}(x)| \leq \sum_{j=n+1}^{m} |K_{j,y}(x) - K_{j-1,y}(x)|
\]
\[
= \sum_{j=n+1}^{m} |r_j (K_{j-1,y}(x), y) - K_{j-1,y}(x)|
\]
\[
\overset{\text{5.1}}{\leq} \sum_{j=n+1}^{m} (1.5)^{-j}.
\]
As in the proof of Lemma 21, we write $h_y(x)$ is a continuous function in $\mathbb{M}$ as it is a uniform limit of continuous functions. Notice that for a fixed $y$, $h_y$ is a homeomorphism of the circle for $y \in \left[\frac{1}{\phi+2}, \frac{1}{\phi+2}\right]$ or of $[1, 1/\phi]$ if $y \in \left[\frac{1}{\phi+2}, \frac{\phi^2}{\phi+2}\right]$.

It remains to show that if $(x, y) \sim (\hat{x}, \hat{y})$ (for points on $\partial \mathbb{M}$) then $(h_y(x), y) \sim (h_y(\hat{x}), \hat{y})$. Let $(x, y), (\hat{x}, \hat{y}) \in \partial \mathbb{M}$ with $(x, y) \sim (\hat{x}, \hat{y})$. There exists $j(x, y) \in \mathbb{N}$ such that $(x, y), (\hat{x}, \hat{y}) \in V_j$. Since $(x, y) \sim (\hat{x}, \hat{y})$, it follows that for every $n > j$ and a word $w \in \Sigma_A(n)$,

$$x \in C_{[w(j)|w|_n]_{n+j}} \iff \hat{x} \in C_{[w(j)|w|_n]_{n+j}}.$$ 

This property, since for all $n \leq j$,

$$r_n|_{V_j} = Id_{Z},$$

yields that for all $n \in \mathbb{N}$,

$$(K_{n,y}(x), y) \sim (K_{n,y}(\hat{x}), \hat{y}),$$

and the Lemma follows by taking $n \to \infty$. $\square$

Denote the function of the first coordinate by $\mathfrak{z}_n(x, y)$. Our goal is to prove that the limit

$$\mathfrak{z}(x, y) := \lim_{t \to \infty} \mathfrak{z}_n(x, y)$$

exists for all $(x, y)$ and $\mathfrak{z}$ is a $C^1 (\mathbb{M}_\infty)$ function with

$$1.6 \leq \frac{\partial \mathfrak{z}}{\partial x}(x, y) \leq 1.7.$$

The conclusion of hyperbolicity of $\mathfrak{z}$ will follow from a standard Lemma in the theory of Lyapunov exponents.

**Lemma 29.** If in addition

$$1.6 \leq \phi \cdot \lambda_{\phi}^6 \left( \prod_{t \in \mathbb{N}} \lambda_t^{\pm 2M_t-1} \right) \cdot \exp \left( \pm \sum_{t=1}^{\infty} 2^{-N_t+1} \right) \leq 1.7$$
then $\frac{\partial \mathfrak{z}}{\partial x}$ is a continuous function in $\mathbb{M}_\infty$ and

$$1.6 \leq \frac{\partial \mathfrak{z}}{\partial x}(x, y) \leq 1.7.$$

**Remark.** The extra condition in this Lemma can easily be inserted into the inductive construction of the sequence $\{\lambda_k, M_k, N_k, m_k, n_k, \epsilon_k\}_{k=1}^\infty$.

**Proof.** Let $(x, y) \in \mathbb{M}_\infty$. For convenience to the reader, we will first show that $\frac{\partial N_t}{\partial x}(x, y)$ converges pointwise and $1.6 \leq \frac{\partial \mathfrak{z}}{\partial x}(x, y) \leq 1.7$ and then argue that the convergence is in fact uniform.

Let $t \in \mathbb{N}$ and $(x, y) \in \mathbb{M}_\infty$ be fixed. There exists a $w \in \Sigma_A$ such that $x \in K_{N_t,y}(C_{[w]\|k]}_{N_t})$. As in the proof of Lemma 21, we write $z_y(x)$ to be the unique point in $C_{[w]\|k]}_{N_t}$ such that $x = H_{N_t,y}(z_y(x))$. Recall that

$$\mathfrak{z}_{N_t}(x, y) = \begin{cases} K_{N_t,y/\phi} (\varphi K_{N_t,y}(x)), & 0 \leq x \leq 1/\phi \\ K_{N_t,y/\phi+\varphi/\phi+2} (\varphi K_{N_t,y}(x)-1), & 1/\phi \leq x \leq 1 \\ \end{cases}$$

$$= \begin{cases} K_{N_t,y/\phi} (S_{z_y}(x)), & 0 \leq x \leq 1/\phi \\ K_{N_t,y/\phi+\varphi/\phi+2} (S_{z_y}(x)), & 1/\phi \leq x \leq 1 \end{cases}.$$
By the chain rule, the Lemma will follow once we show that uniformly in \((x, y) \in \mathbb{M}_\infty\) with \(0 \leq x \leq 1/\varphi\),

\[
\lim_{t \to \infty} \left( \frac{\partial K_{N_t, -y/\varphi}}{\partial x} (S_z(x)) \right) \cdot \left( \frac{\partial K_{N_t, y}}{\partial x} (z_y(x)) \right)^{-1} \in \left( \frac{1.6}{\varphi}, \frac{1.7}{\varphi} \right),
\]

and for every \((x, y) \in \mathbb{M}_\infty\) with \(1/\varphi \leq x \leq 1/2\),

\[
\lim_{t \to \infty} \left( \frac{\partial K_{N_t, -y/\varphi + \varphi/\varphi + 2}}{\partial x} (S_z(x)) \right) \cdot \left( \frac{\partial K_{N_t, y}}{\partial x} (z_y(x)) \right) \in \left( \frac{1.6}{\varphi}, \frac{1.7}{\varphi} \right).
\]

We will separate the proof for three cases: We assume that \((x, y) \in \mathbb{R}_1 \cup \mathbb{R}_3\), equivalently \(0 \leq x \leq 1/\varphi\), the proof when \((x, y) \in \mathbb{R}_2\) is similar and just involves changing the appearance of \(-y/\varphi\) by \(-y/\varphi + \varphi/(\varphi + 2)\).

**Case 1:** \((x, y) \in \mathbb{R} \cap \mathbb{R}_3 \cup 1\). In this case

\[
\frac{J}{N_t} (x, y) = H_{N_t} \circ S \circ H_{N_t} (x),
\]

and the conclusion is true by Lemma 21.

**Case 2:** \((x, y) \in \mathbb{R} \cap \mathbb{R}_3 \cup 1\). Firstly since \((x, y) \in \mathbb{R}\) then \(K_{n, y} (x) = H_n (x)\). In this case for \(1 \leq l \leq N_t\),

\[
h_{l, y} = \cdots h_{N_t, y} = H_{n-1} (z_y(x)).
\]

Secondly if \(j(S_z(y), -y/\varphi) = J\) then

\[
[w_2, ..., w_{J+1}]_J \in [w(J)]_1 \text{ or } [w(J)]_1,
\]

consequently since \(w(J)_1 \neq 1\) for all \(l \leq J - 3\) we have that for all \(2 \leq l \leq J - 2\),

\[
\frac{\partial h_{l, y}}{\partial x} (h_{l, y} \cdots h_{N_t, y} (x)) = \frac{\partial h_{l, y}}{\partial x} (H_{n-1} (z_y(x))) = 1.
\]

By the preceding equation and the chain rule

\[
\frac{\partial K_{N_t, y}}{\partial x} (z_y(x)) = \frac{\partial h_{1, y}}{\partial x} (z_y(x)) \cdot \prod_{k=1}^{N_t} \frac{\partial h_{1, y}}{\partial x} (H_{n-1} (z_y(x))).
\]

In addition by the definition of the construction

\[
\frac{\partial K_{N_t, -y/\varphi}}{\partial x} (S_z(x)) = \prod_{l=1}^{N_t} \frac{\partial h_{l, -y/\varphi}}{\partial x} (K_{l-1, -y/\varphi} (S_z(x)));
\]

and, here \(\lambda_{l,J} = \lambda_k\) if \(N_k - 1 \leq J \leq N_k\),

\[
\left( \frac{\partial K_{N_t, -y/\varphi}}{\partial x} (S_z(x)) \right) \cdot \left( \frac{\partial K_{N_t, y}}{\partial x} (z_y(x)) \right)^{-1} = \lambda_{l,J} \left( \frac{\partial h_{l, y}}{\partial x} (z_y(x)) \right)^{-1} \cdot I.
\]

Where

\[
I := \left( \prod_{l=1}^{N_t} \frac{\partial h_{l, -y/\varphi}}{\partial x} (K_{l-1, -y/\varphi} (S_z(x))) \right) \cdot \left( \prod_{l=1}^{N_t} \frac{\partial h_{l, y}}{\partial x} (H_{l} (z_y(x))) \right)^{-1}.
\]

As in the proof of Lemma 21 assuming that \(\varepsilon \equiv 0\), one has that for \(l \geq J + 2\), \(K_{l-2, -y/\varphi} (S_z(x))\) is to the right of the point in \(1/\varphi\) proportion in \(K_{l-2, -y/\varphi} (C_{[w_2, ..., w_1]}^{-1})\)
if and only if \( K_{1-1,y} (z(y)) \) is to the right of the point in \( 1/\varphi \) proportion in \( K_{t-1,y/\varphi} (C[w]_1') \).
This means that in this case \( (\xi = 0) \)
\[
\left( \frac{\partial h_{1-1,y}}{\partial x} (K_{1-2,y/\varphi} (Sz(y))) \right) \cdot \left( \frac{\partial h_{1,y}}{\partial x} (K_{1-1,y} (z(y))) \right)^{-1} = 1.
\]
By proceeding with the analysis of the the bad sets as in Lemma \( 21 \) one proves that
\[
I = \left( \prod_{k=t(J)}^t \lambda_k^{+2M_{k-1}} \right) \exp \left( \pm \sum_{k=t(J)}^t 2^{-N_{k+4}} \right),
\]
and thus
\[
\left( \frac{\partial K_{N_1,y/\varphi}}{\partial x} (Sz_y(x)) \right) \cdot \left( \frac{\partial K_{N_1,y}}{\partial x} (z_y(x)) \right)^{-1} = \left[ \left( \frac{\partial h_1}{\partial x} (z_y(x)) \right)^{-1} \left( \prod_{k=t(J)}^t \lambda_k^{+2M_{k-1}} \right) \right]
\cdot \exp \left( \pm \sum_{k=t(J)}^t 2^{-N_{k+4}} \right).
\]
This shows \( 5.2 \). In fact, because
\[
\lim_{s \to \infty} \left( \prod_{k=s}^{\infty} \lambda_k^{+2M_{k-1}} \right) \exp \left( \pm \sum_{k=s}^{\infty} 2^{-N_{k+4}} \right) = 1
\]
one has that the convergence is uniform.

**Case 3:** \((x,y) \in U^c\). In this case let \( \tilde{J} \in \mathbb{N} \) be such that the closest point to \((x,y)\) on the Horizontal segments of \( \partial \mathbb{M} \) is in \( V_{\tilde{J}} \). If \( \tilde{J} = 1 \) then \((Sz_y(x),-y/\varphi) \in U^c\). Otherwise \((Sz_y(x),-y/\varphi) \in U^c\) and the closest point to it on the Horizontal segments of \( \partial \mathbb{M} \) is in \( V_{\tilde{J}-1} \). That shows that
\[
\left( \frac{\partial K_{N_1,y/\varphi}}{\partial x} (Sz_y(x)) \right) \cdot \left( \frac{\partial K_{N_1,y}}{\partial x} (z_y(x)) \right)^{-1} = \left( \prod_{l=\tilde{J}-1}^{N_1} \frac{\partial h_{l,-y/\varphi}}{\partial x} (K_{l-1,y/\varphi} (Sz_y(x))) \right)
\cdot \left( \prod_{l=\tilde{J}}^{N_1} \frac{\partial h_{l,y}}{\partial x} (K_{l-1,y} (z_y(x))) \right)^{-1}.
\]
Similarly as in case 2, one has
\[
\left( \frac{\partial K_{N_1,y/\varphi}}{\partial x} (Sz_y(x)) \right) \cdot \left( \frac{\partial K_{N_1,y}}{\partial x} (z_y(x)) \right)^{-1} = \lambda_{\tilde{J}}^{+2} \left( \prod_{k=\tilde{J}(J)}^t \lambda_k^{+2M_{k-1}} \right) \exp \left( \pm \sum_{k=t(J)}^t 2^{-N_{k+4}} \right)
\]
and the convergence is uniform.
\( \square \)

5.2.3. **Proving differentiability in the y-direction.** 3. Again we will prove differentiability in the \( y \) direction for \((x,y) \in R_1 \cup R_3 \). The idea of the proof here is as follows. If \((x,y) \in U \) then \( K_{n,\tilde{y}}(x) = H_{\tilde{y},n}(x) \) for all \( \tilde{y} \) in a neighborhood of \((x,y)\), hence \( \frac{\partial K_{n,\tilde{y}}}{\partial y} (\cdot) \equiv 0 \). Otherwise, for \((x,y) \in U^c \), \( K_{j(x,y)-1,\tilde{y}}(x) = x \) and the first change between \( K_{n,\tilde{y}}(x) \) and \( K_{n,\tilde{y}}(x) \) appears at time \( n = j(x,y) \). We will show that for our construction the \( y \) derivative
of $K_{n,y}(x)$ can be bounded up by a (bounded) constant times $\frac{\partial K_{[x,y],y}(x)}{\partial y}$, the uniform convergence of $\frac{\partial h_n}{\partial y}$ will follow from the chain rule and simple arithmetic.

The following notation will be used in this subsection. Usually we will consider $x \in [0,1/\varphi]$ and work constantly with a fixed $w \in \Sigma_A$ such that $x \in C_{[w]}$ for all $n \in \mathbb{N}$. If that is the case we will write $[x_n, \tilde{x}_n]$ to denote $C_{[w]}$.

For $-\frac{\varphi^2}{\varphi + 2} \leq y \leq \frac{\varphi^2}{\varphi + 2}$ and $n \geq N(y)$, let $BS(n, w, y) \subset C_{[w]}$ to be the bad set as in the proof of Lemma 21 with $H_n$ replaced by $K_{n,y}$.

For an $n \in \Sigma_A$, we denote by $w^n = w_1w_2 \cdots w_n$ the finite word derived by $w$ up to time $n$. Given a finite word $w^n$, $\bar{w}$ denotes the $n$-periodic word defined by $w^n$. Finally given two words $w$ and $\bar{w}$ (in which case $w$ is a finite word), the word $\bar{w} w$ denotes the concatenation of $w$ and $\bar{w}$.

Recall the definition of $K_{[x,y],y}(x) = r_{[x,y]}(x, y)$ which is defined by

$$r_{[x,y]}(x, y) := (H_{[x,y]}(x) - x) \mathcal{P}(x, y) + x,$$

where

$$\mathcal{P}(x, y) = \left[3\varphi^{-10} (u(x,y))^2 - 2u(x,y)^3\right] / \varphi^{10}.$$

In the following proof if $[x,y] = J$ we will need a different definition of the bad set for $n = J$. Let

$$BS(J) := \begin{cases} BS(J - 2, w(J)), & \text{J odd} \\ BS(J - 1, w(J)) \cap BS(J, w(J)) & \text{J even} \end{cases}$$

if $(x, y) \in R_1 \cup \mathbb{R}_3$ (For $(x, y) \in R_2$ change the odd to even and even to odd). To understand why this set appears. If $x \notin BS(J)$ and $[x,y] = J$ then $H_{[x,y]}(x) - H_{[x(y)]}(\bar{w} w(J))$ is a linear function.

**Lemma 30.** Assume that $M_1 \leq j \leq N_{t+1}$, and $x \in C_{[w]}^{N_{t+1}}$. If $x \notin BS(J)$, for every $j \leq n \leq N_{t+1}$, there exists $0 \leq \beta_n(x) \leq \varphi$ so that for every $(x, y)$ so that $[x,y] = J$, $K_{n,y}(x) = K_{n,y}(\bar{w} n(w)) + \beta_n(x)l_0(y, w)$, in addition $\beta_{N_{t+1}}(x)$ is continuous in $x$.

**Proof.** Let $(x, y) \in U^c \cup (R_1 \cup \mathbb{R}_3)$ so that $[x,y] = J$ (the case $(x, y) \in R_2$ is similar. The proof is an induction on $n$. Since $x \notin BS(J)$, $\bar{w}_{J+1}(w) \in \{x_{J}, \bar{x}_{J} + \varphi^{-1}(\bar{x}_{J} - x_{J})\}$ and

$$w(J) = \begin{cases} 32 \cdots 32132, & \text{J odd} \\ 32 \cdots 32111, & \text{J even} \end{cases},$$

it follows that if $J$ is even then

$$H_J(x) - H_J(\bar{w}_{J+1}(w)) = h_J \circ h_{J-1}(x) - h_J \circ h_{J-1}(x_{J+1})$$

$$= \left\{\left[\varphi \psi_{J+1}(1/\varphi)\right]^2 [x - \bar{x}_{J+1}], \right\} + \left\{\left[\varphi \psi_{J+1}(1/\varphi)\right] \left[\varphi^2 [1 - \psi_{J+1}(1/\varphi)] \right] [x - \bar{x}_{J+1}(w)], \right\}$$

$$= \frac{P_{J-1}(w_{J-1}, w_{J})}{Q(w_{J}, w_{J+1})} \left[P_{J}(w_{J}, w_{J+1}) \left[32 \cdots 32132, \bar{x}_{J} + \varphi^{-1}(\bar{x}_{J} - x_{J})\right]\right]$$

$$:= b \left[\min(\bar{w}_{J+1}(w)), x - \bar{x}_{J+1}(w)\right].$$
and if \( J \) is odd then
\[
H_J (x) - H_J (x_{J+1}(w)) = h_J (x) - h_J (x_{J+1}) \\
= \varphi^2 \left[ 1 - \psi_{t+1} \left[ 1/\varphi \right] \right] \left[ x - x_{J+1}(w) \right] \\
= \frac{P_{J-2} (w_{J-2}, w_{J-1})}{Q (w_{J-2}, w_{J-1})} \left[ x - x_{J+1}(w) \right] \\
:= b \left( x - x_{J+1}(w) \right).
\]

It then follows that
\[
K_{J,y} (x) - K_{J,y} (x_{J+1}(w)) : = r_J (x, y) - r_J (x_{J+1}(w), y) \\
= \left[ x - x_{J+1}(w) \right] \left[ (b-1) \mathcal{P} (x, y) + 1 \right].
\]
and
\[
l_{J+1} (y, w) : = K_{J,y} (x_{N+1}(w)) - K_{J,y} (x_{J+1}(w)) \\
= \left[ x_{J+1}(w) - x_{J+1}(w) \right] \left[ (b-1) \mathcal{P} (x, y) + 1 \right].
\]
This implies that
\[
\frac{K_{J,y} (x) - K_{J,y} (x_{J+1}(w))}{l_{J+1} (y, w)} = \frac{b \left( x - x_{J+1}(w) \right)}{x_{J+1}(w) - x_{J+1}(w)}.
\]
\( K_{J+1,y} (x) = K_{J+1,y} (x_{J+1}(w)) + \psi_{t+1} \left( \frac{x - x_{J+1}(w)}{x_{N+1}(w) - x_{J+1}(w)} \right) l_{N+1} (y, w) \)
and the base of induction is proved.

For the inductive step notice that if the conclusion of the Lemma is true for \( n \in \mathbb{N} \), then
\[
\frac{K_{n,y} (x) - K_{n,y} (x_{n+1}(w))}{l_{n+1} (y, w)} = \frac{\beta_n (x) - \beta_n (x_{n+1}(w))}{\beta_n (x_{n+1}(w)) - \beta_n (x_{n+1}(w))}
\]
does not depend on \( y \). The conclusion then follows for \( n+1 \) with
\[
\beta_{n+1} (x) : = \psi_{t+1} \left( \frac{\beta_n (x) - \beta_n (x_{n+1}(w))}{\beta_n (x_{n+1}(w)) - \beta_n (x_{n+1}(w))} \right)
\]
and the continuity of \( \beta_{n+1} \) follows from the continuity of \( \beta_n \) and \( \psi_{t+1} \).

The last lemma shows the importance of knowing how \( \frac{\partial \mathcal{N}}{\partial y} \) decays when \( \mathcal{N}(y) < n \leq N_{t+1} \). We will now show that it is exponential in \( n \).

**Lemma 31.** Let \( M_t \leq j(x, y) < n \leq N_{t+1} \) then
\[
\left| \frac{\partial \mathcal{N}}{\partial y} (y, w) \right| \leq (1.6)^{j(x,y) - n} \left| \frac{\partial \mathcal{N}(y)}{\partial y} (y, w) \right|
\]
and
\[
\left| \frac{\partial l_j (x,y)}{\partial y} (y, w) \right| \leq (1.6)^{-j(x,y)}.
\]
Proof. We assume \((x, y) \notin \partial U\), the proof for the case \((x, y) \in \partial U\) is similar. In this case for small \(|h|\), \((x, y + h) \in U^c\). Let \(w \in \Sigma_A\).

Since \(\bar{x}_{N+1}(w)\) is not in the bad set \(\mathbb{B}(j, x, y)\), it follows from \(5.3\) that for small \(|h|\),

\[
m_{\text{Leb}}\left( K_{j, (x, y), y+h} \left( C_{[w]^j}^{N+1} \right) \right) = \frac{K_{j, (x, y), y+h} (\bar{x}_{N+1}(w)) - K_{j, (x, y), y+h} (\bar{x}_{N+1}(w))}{\bar{x}_{N+1}(w) - \bar{x}_{j, (x, y)+1}(w)} = \frac{\bar{x}_{N+1}(w) - \bar{x}_{j, (x, y)+1}(w)}{\bar{x}_{j, (x, y)+1}(w)}
\]

It then follows by definition of \(h_{n, y}\) for \(n > j, x, y\) that for \(|h|\) small,

\[
I_{N+1}(w, y + h) = I_{j, (x, y) + 1}(w, y + h) \prod_{k=j, (x, y) + 1}^{N+1} P_k \left( w_k, w_{k+1} \right),
\]

hence

\[
\frac{I_{N+1}(y + h, w)}{I_{N+1}(y, w)} = \frac{I_{j, (x, y) + 1}(y + h, w)}{I_{j, (x, y) + 1}(y, w)}.
\]

This yields that

\[
I_{N+1}(y + h, w) - I_{N+1}(y, w) = \frac{I_{N+1}(y, w)}{I_{j, (x, y) + 1}(y, w)} \left[ I_{j, (x, y) + 1}(y + h, w) - I_{j, (x, y) + 1}(y, w) \right],
\]

dividing by \(h\) and taking limit \(h \to 0\) we get

\[
\left| \frac{\partial I_{N+1}(y, w)}{\partial y} \right| = \frac{I_{N+1}(y, w)}{I_{j, (x, y) + 1}(y, w)} \left| \frac{\partial I_{j, (x, y) + 1}(y, w)}{\partial y} \right| \leq (1.6)^{j, (x, y) - N+1} \left| \frac{\partial I_{j, (x, y) + 1}(y, w)}{\partial y} \right|.
\]

The last inequality follows from

\[
\prod_{k=j, (x, y) + 1}^{N+1} P_k \left( w_k, w_{k+1} \right) \leq \left( \frac{\lambda_{t+1} \varphi}{1 + \lambda_{t+1} \varphi} \right)^{j, (x, y) - N+1} \leq \left( \frac{\lambda_{t+1} \varphi}{1 + \lambda_{t+1} \varphi} \right)^{j, (x, y) - N+1},
\]

and \(\lambda_{t+1} \varphi < 1.6\).

For the proof of the second part notice that for \(x \in \left\{ \mathbb{B}(j, x, y)(w), \bar{x}_{N}(y)(w) \right\}\),

\[
\left| \frac{\partial I_{j, (x, y)}(x, y)}{\partial y} \right| = \left| \frac{\partial \varphi}{\partial y}(x, y) \right| H_{j, (x, y)}(x) \leq \frac{3}{2} \varphi^{10} \left| m \left( C_{[w]_{1}^{j, (x, y) - 2}} \right) \right| \leq \frac{3}{2} \varphi^{12} \varphi^{-j, (x, y)}.
\]

Here the second inequality follows from

\[
H_{j, (x, y)} \left( C_{[w]_{1}^{j, (x, y) - 2}} \right) = C_{[w]_{1}^{j, (x, y) - 2}}.
\]
Consequently,
\[
\left| \frac{\partial y(x,y+1)}{\partial y} (y,w) \right| = 3 \varphi^2 \varphi^{-j(x,y)} \\
\leq (1.6)^{-j(x,y)}, \text{ for large } j.
\]

\[\square\]

Lemma 32 shows that if \( x \) is not in a bad set of an interval \( C_{[n]} (x,y) \) then the \( y \)-derivative of \( K_{N_{t+1},y}(x) \) (here \( t \) is the number such that \( N_t < j(x,y) < N_{t+1} \)) is controlled by the derivative on a finite collection of points plus the evolution of the lengths of the intervals. We would like to point out that there is actually no bad set if \( N_t < j(x,y) \leq M_t \) because then \( h_{j(x,y),y} C_{[n]} = \id_T \). This idea will be reiterated with a slight modification for the derivatives \( \partial K_{n,y}/\partial y \) for \( j(x,y) < N_t < M_n < n < N_n \).

For points in the bad set we will apply a correction point procedure which we call the \( x \)-delta method. Assume that \( x \in \mathbb{B}_S (j(x,y)) \cap C_{[n]} N_t \). For \( \Delta \)-small (so that \( (x,y+\Delta) \in U^c \)) there exists a unique \( x (\Delta) \) such that
\[
K_{j(x,y),y+\Delta} (x (\Delta)) - K_{j(x,y),y+\Delta} (E_{j(x,y)+1} (w)) = K_{j(x,y),y} (x) - K_{j(x,y),y} (E_{j(x,y)+1} (w)) \]
\[
\frac{l_{j(x,y)+1} (y + \Delta, w)}{l_{j(x,y)+1} (y, w)}.
\]

We will use Lemma 31 to obtain a first order approximation for \( x (\Delta) \) when \( \Delta \) is small.

In the next Lemma \( \beta_{j+1} (x) := \psi \left( \frac{K_{j,y} (x) - K_{j,y} (E_{j+1} (w))}{l_{j+1} (y,w)} \right) \) and for \( j+1 < n < N_t \)
\[
\beta_{n+1} (x) := \psi_{n+1} \left( \frac{\beta_n (x) - \beta_n (E_{n+1} (w))}{\beta_n (E_{n+1} (w)) - \beta_n (E_{n+1} (w))} \right).
\]

Lemma 32. Assume that \( M_t \leq j \leq N_{t+1} \) and \( x \in C_{[n]} N_{t+1} \). The following holds:
(i) For every \( y \) so that \( j(x,y) = j \) and \( \Delta \) so that \( (x,y+\Delta) \in U^c \),
\[
K_{N_t+1,y+\Delta} (x (\Delta)) = K_{j(x,y),y+\Delta} (E_{N_t+1} (w)) + \beta_{N_t+1} (x) l_{N_t+1} (y + \Delta, w),
\]
(ii) \( |x (\Delta) - x| \leq 2 (1.6)^{-N_{t+1} \Delta + o (\Delta)} \) as \( \Delta \to 0 \).

Proof. (i) This is the same as the proof of Lemma 30 by using (5.4) as the starting point.
(ii) If \( x \notin \mathbb{B}_S (j) \cap C_{[n]} N_t \) then by equation (5.3), \( x (\Delta) = x \). Since \( E_{N_t+1} (w) \notin \mathbb{B}_S (j) \),
\[
\frac{K_{j,y+\Delta} (E_{N_t+1} (w)) - K_{j,y+\Delta} (E_{j(x,y)+1} (w))}{l_{j+1} (y + \Delta, w)} = \frac{x_{N_t+1} (w) - x_{j(x,y)+1} (w)}{x_{j+1} (w) - x_{j+1} (w)}.
\]
\[
= \frac{K_{j,y} (E_{N_t+1} (w)) - K_{j,y} (E_{j+1} (w))}{l_{j+1} (y, w)}.
\]

Therefore by adding and subtracting a term one has that equation (5.4) is equivalent to (5.5)
\[
K_{j,y+\Delta} (x (\Delta)) - K_{j,y+\Delta} (E_{N_t+1} (w)) = \frac{l_{j+1} (y + \Delta, w)}{l_{j+1} (y, w)} \left( K_{j,y} (x) - K_{j,y} (E_{N_t+1} (w)) \right).
\]
Corollary 33. (i) For every \((x, y) \in R_1 \cup R_3\), if \(N_t < j(x, y) \leq N_{t+1}\) then
\[
\left| \frac{\partial K_{N_{t+1}, j}(x)}{\partial y} \right| \leq (1.6)^{-j(x, y)},
\]
in addition if \((x, y) \in \partial U^c \cup \partial M\) then
\[
\frac{\partial K_{N_{t+1}, j}(x)}{\partial y} = 0.
\]
(ii) \[
\left| \frac{\partial M_{n+1}(y, w)}{\partial y} \right| \leq 3(1.6)^{-N_{t+1}}.
\]

**Proof.** (i) First we claim that for all \( w \in \Sigma_A \) such that \([w]_1^{j(x,y)} = w(j(x,y))\),
\[
(5.6) \quad \frac{\partial K_{N_{t+1},y}(x_{N_{t+1}}(w))}{\partial y} = \pm 4(1.6)^{-j(x,y)}.
\]

This is true because of the following argument. For each \( j(x,y) < n \leq N_{t+1} \), either \( x_{n-1}(w) = x_n(w) \) and then
\[
K_{n,y}(x_n(w)) = K_{n-1,y}(x_{n-1}(w))
\]
or \( x_n(w) = x_{n-1}(w) + \varphi^{-1}(\bar{x}_{n-1}(w) - x_{n-1}(w)) \) and then
\[
K_{n,y}(x_n(w)) = K_{n-1,y}(x_n(w)) = K_{n-1,y}(x_{n-1}(w)) + \frac{\lambda_{t+1}\varphi}{1 + \lambda_{t+1}\varphi}r_n(y,w).
\]

This equality remains true in a neighborhood of \( y \). Therefore for all \( j(x,y) < n \leq N_{t+1} \),
\[
\left| \frac{\partial K_{n,y}(x_n(w))}{\partial y} \right| \leq \left| \frac{\partial K_{n-1,y}(x_{n-1}(w))}{\partial y} \right| + \frac{2}{3} \max_{[w]_1^{N_{t+1}} \subset [w(j(x,y))]} \left| \frac{\partial r_n(y,w)}{\partial y} \right|
\]
\[
\leq \left| \frac{\partial K_{n-1,y}(x_{n-1}(w))}{\partial y} \right| + \frac{2}{3} (1.6)^{-n}
\]

and so
\[
\left| \frac{\partial K_{N_{t+1},y}(x_{N_{t+1}}(w))}{\partial y} \right| \leq \left| \frac{\partial K_{j(x,y),y}(x_{j(x,y)}(w))}{\partial y} \right| + \frac{2}{3} \sum_{n=j(x,y)+1}^{N_{t+1}} (1.6)^{-n}
\]
\[
\leq 4(1.6)^{-j(x,y)}.
\]

Now for a general \( 0 \leq x \leq 1/\varphi \),
\[
\frac{\partial K_{N_{t+1},y}(x)}{\partial y} = \lim_{\Delta \to 0} \frac{K_{N_{t+1},y+\Delta}(x) - K_{N_{t+1},y}(x)}{\Delta}
\]
\[
= \lim_{\Delta \to 0} \frac{K_{N_{t+1},y+\Delta}(x(\Delta)) - K_{N_{t+1},y}(x)}{\Delta} + \lim_{\Delta \to 0} \frac{K_{N_{t+1},y+\Delta}(x(\Delta)) - K_{N_{t+1},y}(x)}{\Delta}.
\]

Since
\[
\left| \frac{\partial K_{N_{t+1},y+\Delta}(x)}{\partial x} \right| \leq \lambda_{t+1}^{2(N_{t+1} - j(x,y))},
\]
it follows that
\[
(5.7) \lim_{\Delta \to 0} \left| \frac{K_{N_{t+1},y+\Delta}(x(\Delta)) - K_{N_{t+1},y+\Delta}(x)}{\Delta} \right| \leq \lambda_{t+1}^{2(N_{t+1} - j(x,y))} \lim_{\Delta \to 0} \left| \frac{x(\Delta) - x}{\Delta} \right|
\]
\[
\leq (1.6)^{-N_{t+1}}.
\]
By Lemma 32(i) if $x \in C_{|w|_1}^{N_{t+1}}$ then,

$$\lim_{\Delta \to 0} \left| K_{N_{t+1},y+\Delta}(x(\Delta)) - K_{N_{t+1},y}(x) \right| \leq \lim_{\Delta \to 0} \left| K_{N_{t+1},y+\Delta}(\bar{z}_{N_{t+1}}(w)) - K_{N_{t+1},y}(\bar{z}_{N_{t+1}}(w)) \right|$$

$$+ \beta_n(x) \lim_{\Delta \to 0} \left| l_{N_{t+1}}(y + \Delta, w) - l_{N_{t+1}}(y, w) \right|$$

$$\leq \left| \frac{\partial K_{N_{t+1},y}(\bar{z}_{N_{t+1}}(w))}{\partial y} \right| + \left| \frac{\partial l_{N_{t+1}}(y, w)}{\partial y} \right|$$

$$\leq 4(1.6)^{-J(x,y)} + (1.6)^{-N_{t+1}}.$$ 

and the conclusion follows.

For the second part of (i) in the Corollary, notice that if $(x, y) \in \partial U^c \cup \partial M$, then

$$\frac{\partial f}{\partial y}(x, y) = 0$$

and therefore

$$\frac{\partial f(x,y)}{\partial y}(y, w) = 0$$

and $x(\Delta) = x + o(\Delta)$.

(ii) Let $w \in \Sigma_\Lambda(M_{t+1})$. Since for all $y$, $K_{N_{t+1},y}(\bar{z}_{M_{t}}(w))$, $K_{N_{t+1},y}(\bar{x}_{M_{t}}(w)) : w \in \Sigma_\Lambda(M_{t+1})$ are fixed points for $h_{N_{t+1},y}$ it follows that for all $\Delta$,

$$l_{M_{t+1}}(y + \Delta, w) = K_{M_{t+1},y+\Delta}(\bar{x}_{M_{t}}(w)) - K_{M_{t+1},y}(\bar{x}_{M_{t}}(w))$$

$$= K_{N_{t+1},y+\Delta}(\bar{x}_{M_{t}}(w)) - K_{N_{t+1},y}(\bar{x}_{M_{t}}(w)).$$

The last line follows from $h_{N_{t+1},y+\Delta} = id_T$ for $N_{t+1} < n < M_{t+1}$. Writing $\bar{x}(\Delta)$ (respectively $\bar{x}(\Delta)$) for the x-delta point of $\bar{x}_{M_{t+1}}(w)$ (respectively $\bar{x}_{M_{t+1}}(w)$). By Lemma 32 for $|\Delta|$ small,

$$K_{N_{t+1},y+\Delta}(\bar{x}(\Delta)) = K_{N_{t+1},y+\Delta}(\bar{z}_{N_{t+1}}(w)) + \beta_{N_{t+1}}(\bar{z}_{M_{t}}(w)) l_{N_{t+1}}(y + \Delta, w),$$

and

$$K_{N_{t+1},y+\Delta}(\bar{x}(\Delta)) = K_{N_{t+1},y+\Delta}(\bar{z}_{N_{t+1}}(w)) + \beta_{N_{t+1}}(\bar{x}_{M_{t}}(w)) l_{N_{t+1}}(y + \Delta, w).$$

It then follows that for $|\Delta|$ small,

$$l_{M_{t+1}}(y + \Delta, w) = \{\beta_{N_{t+1}}(\bar{x}_{M_{t}}(w)) - \beta_{N_{t+1}}(\bar{x}_{M_{t}}(w))\} l_{N_{t+1}}(y + \Delta, w) + \bar{T}_{\Delta} + \bar{I}_{\Delta},$$

where by (5.7),

$$|\bar{T}_{\Delta}| := |K_{N_{t+1},y+\Delta}(\bar{x}(\Delta)) - K_{N_{t+1},y+\Delta}(\bar{x}_{M_{t}}(w))| \leq \Delta(1.6)^{-N_{t+1}}$$

and

$$|\bar{I}_{\Delta}| := |K_{N_{t+1},y+\Delta}(\bar{x}(\Delta)) - K_{N_{t+1},y+\Delta}(\bar{x}_{M_{t}}(w))| \leq \Delta(1.6)^{-N_{t+1}}.$$ 

It then follows that

$$\left| \frac{\partial l_{M_{t+1}}(y, w)}{\partial y} \right| \leq \left\{ \beta_{N_{t+1}}(\bar{x}_{M_{t}}(w)) - \beta_{N_{t+1}}(\bar{x}_{M_{t}}(w)) \right\} \left| \frac{\partial l_{N_{t+1}}(y, w)}{\partial y} \right| + \lim_{\Delta \to 0} \left| \frac{\bar{T}_{\Delta} + |\bar{I}_{\Delta}|}{\Delta} \right|$$

$$\leq 3(1.6)^{-N_{t+1}}.$$ 

\[\square\]
So far we have managed to show to control $\frac{\partial K_{N+1,y}(x)}{\partial y}$ by a constant time the derivative at level $j(x,y)$ where $t = t(y) = \min \{ t \in \mathbb{N} : N_{t+1} \geq j(x,y) \}$. The next step is for $s > t(y)$, to obtain a relation between $\frac{\partial K_{s,y}(x)}{\partial y}$ and $\frac{\partial K_{N,y}}{\partial y}$ (at maybe different point then $x$). The idea is similar and uses the fact that $h_{M,y}$ is almost a linearization of $K_{M-1,y}$ and enables us to do that via a similar analysis. The easier part is the control of scaling functions.

**Definition 34.** For $M_s < n \leq N_{s+1}$, $x \in C_{[w]_{1}^{N_{s+1}}}$ and $|\Delta|$ small we define $x_s(\Delta)$ to be the unique point such that

$$\frac{K_{M_s,y+\Delta}(x_s(\Delta)) - K_{M_s,y+\Delta}(x_{M_s+1}(w))}{l_{M_s+1}(y + \Delta, w)} = \frac{K_{M_s,y}(x) - K_{M_s,y}(x_{M_s+1}(w))}{l_{M_s+1}(y, w)}.$$ 

Setting similarly to before for $0 \leq x \leq 1/\varphi$ and $y$ such that $j(x,y) < M_s$,

$$\beta_{M_s+1}(x) = \beta_{M_s+1}(x) := \psi_{s+1}\left(\frac{K_{M_s,y}(x) - K_{M_s,y}(x_{M_s+1}(w))}{l_{M_s+1}(y, w)}\right)$$

and for $M_s + 1 < n \leq N_{s+1},$

$$\beta_n(x) := \psi_{s+1}\left(\frac{\beta_{n-1}(x) - \beta_{n-1}(x_{n}(w))}{\beta_{n-1}(x_{n}(w)) - \beta_{n-1}(x_{n}(w))}\right).$$

**Lemma 35.** For all $(x,y) \in R_1 \cup R_3$ with $j(x,y) < N_s$ the following holds:

1. for every $|\Delta|$ small,

$$K_{N_s+1,y+\Delta}(x_s(\Delta)) = K_{N_s+1,y+\Delta}(x_{N_s+1}(w)) + \beta_{N_s+1}(x)l_{N_s+1}(y + \Delta, w).$$

2. (i) For every $M_s < n \leq N_{s+1},$

$$\left|\frac{\partial l_n}{\partial y}(y, w)\right| \leq (1.6)^{M_s-n} \left|\frac{\partial l_{M_s}}{\partial y}(y, w)\right|.$$

(ii) $\left|\frac{\partial d_{M_s}}{\partial y}(y, w)\right| \leq (1.6)^{-n_s}.$

(iii) Assume that $\{k, M_{k-1}, \varepsilon, \lambda_k, k = 1\}$ are chosen, there exists a choice of $M_s, \lambda_{s+1}, N_{s+1}$ and $\epsilon_{s+1}$ (compatible with the inductive procedure) such that

$$|x_s(\Delta) - x| \leq \varphi^{-N_{s+1} \Delta + o(\Delta)}$$

as $\Delta \rightarrow 0$.

**Proof.** This is done by induction on $s$. The base of induction is the first $s \in \mathbb{N}$ such that $N_s > j(x,y)$.

(i) This is similar to the proof of Lemma 32(i).

2. (i) Let $w \in \Sigma_A$. The starting point is that by the definition of $h_{M,y}$ (as a distribution correcting function), equation (4.1) holds for $K_{M,y}$. Therefore for all $w \in \Sigma_A$ and $h > 0$ small,

$$m_{\text{Leb}}\left(K_{M,y+h}\left(C_{[w]_1^{N_{s+1}}}\right)\right) = l_{M}(y + h, w) \frac{m_{\text{Leb}}\left(C_{[w]_1^{N_{s+1}}}\right)}{m_{\text{Leb}}\left(C_{[w]_1^{N_{s+1}}}\right)}.$$ 

The rest is similar to the proof of the first part of Lemma 31.
2. (ii) Since for all \( y \), \( \{ K_{n,y} (\bar{x}_M(w)) , K_{n,y} (\bar{x}_M(w)) : w \in \Sigma_M(M_n) \} \) are fixed points for \( h_{M,y} \), it follows that

\[
I_{M_t}(y + \Delta, w) = K_{n,y+\Delta}(\bar{x}_M(w)) - K_{n,y+\Delta}(\bar{x}_M(w)).
\]

The base of induction is Corollary 33 (ii).

The proof of the inductive step is the same as the proof of the base of induction where we use the induction hypothesis that

\[
|\mathbf{x}_{\Delta} - x| \leq \varphi^{-n_{s+1}}\Delta + o(\Delta).
\]

Therefore,

\[
|I_{\Delta}(s)| := |K_{n+1,y+\Delta}(\mathbf{x}_{\Delta}) - K_{n+1,y+\Delta}(\bar{x}_M(w))| \leq (1.6)^{-n_{s+1}}.
\]

It then follows that

\[
|\frac{\partial I_{M}(y, w)}{\partial y}| \leq \frac{1}{\rho(y, w)} \left| \frac{\partial I_{M}(y, w)}{\partial y} \right| + \lim_{\Delta \to 0} \frac{|I_{\Delta}(s)| + |I_{\Delta}(s)|}{\Delta} \left| \frac{\partial I_{M}(y, w)}{\partial y} \right|
\]

\[
\leq (1.6)^{-n_{s+1}}(1.6)^{-n_{s+1}} + 2(1.6)^{-n_{s+1}} \leq (1.6)^{-n_{s}}.
\]

(iii) We first recall the definition of \( K_{M_t,y} \). Define \( \alpha : \mathbb{N} \times \Sigma_M \times \left[ \frac{-\varphi}{\varphi + 2}, \frac{\varphi^2}{\varphi + 2} \right] \) by

\[
\alpha(t, w, y) := \int_{C_{[w]_1^{M_t}}} \frac{\partial K_{M_t-1,y} (x)}{\partial y} \frac{\partial K_{M_t-1,y} (x)}{\partial y} dx = \frac{I_{M_t}(y, w)}{m(C_{[w]_1^{M_t}})}.
\]

It follows that for \( t \in \mathbb{N} \) such that \( N_t > j(x, y) \), the definition of \( K_{M_t,y} \) restricted to \( C_{[w]_1^{M_t}} \) is the function defined by \( K_{M_t,y} (\bar{x}_M(w)) = K_{N_t,y} (\bar{x}_M(w)) \) and \( x \)-derivative

\[
\frac{\partial K_{M_t,y}}{\partial y}(x) := \begin{cases} 
\frac{\partial G_{\alpha(t,w,y)}}{\partial x} \left( x - \bar{x}_M(w) \right) \left( C_{[w]_1^{M_t}} \right), & 0 \leq x - \bar{x}_M(w) \leq \epsilon_{t+1} m \left( C_{[w]_1^{M_t}} \right), \\
\alpha(t, w, y), & x - \bar{x}_M(w) \geq \epsilon_{t+1} m \left( C_{[w]_1^{M_t}} \right)
\end{cases}
\]

where \( w^- \) is the predecessor of \( w \) in \( \Sigma_M(M_t) \) and \( G_{\alpha_1,\alpha_2} : [0, 1] \to [0, 1] \) is the function from \( \ldots \).

Therefore the function

\[
\mathcal{K}_{t,y}(x) := \frac{K_{M_t,y}(x) - K_{M_t,y}(\bar{x}_M(w))}{I_{M_t}(y, w)} = m(C_{[w]_1^{M_t}}) \cdot \frac{K_{M_t,y}(x) - K_{M_t,y}(\bar{x}_M(w))}{\alpha(t, y, w)}
\]
satisfies $\mathcal{K}_{t,y}(x^{\mathcal{M}}(w)) = 0$ and

$$
\frac{\partial \mathcal{K}_{t,y}}{\partial y}(x) := m\left(C_{\lfloor w \rfloor_1^{M_t}}\right) \times \begin{cases} 
\frac{\partial g_{\Upsilon(t,w,y)}}{\partial x} \left(\frac{x - x^{\mathcal{M}}(w)}{\epsilon_{t+1} m(C_{\lfloor w \rfloor_1^{M_t}})}\right), & 0 \leq x - x^{\mathcal{M}}(w) \leq \epsilon_{t+1} m(C_{\lfloor w \rfloor_1^{M_t}}), \\
1, & x - x^{\mathcal{M}}(w) \geq \epsilon_{t+1} m(C_{\lfloor w \rfloor_1^{M_t}}) 
\end{cases}
$$

where $\Upsilon(t,w,y) = \frac{a(t,w,y)}{a(t,w,y)}$. The function $\mathcal{K}_{t,y}$ is important since the definition of $x_t(\Delta)$ is as the unique point so that

$$
\mathcal{K}_{t,y+\Delta}(x_t(\Delta)) = \mathcal{K}_{t,y}(x).
$$

The reader can easily prove that if $x - x^{\mathcal{M}}(w) \leq \epsilon_{t+1} m(C_{\lfloor w \rfloor_1^{M_t}})$,

$$
\mathcal{K}_{t,y+\Delta}(x) = \mathcal{K}_{t,y}(x) + |\Upsilon(t,w,y+\Delta) - \Upsilon(t,w,y)| \cdot (x - x^{\mathcal{M}}(w))
$$

and if $x - x^{\mathcal{M}}(w) \geq \epsilon_{t+1} m(C_{\lfloor w \rfloor_1^{M_t}})$ then

$$
\mathcal{K}_{t,y+\Delta}(x) = \mathcal{K}_{t,y}(x).
$$

Indeed this follows from the fact that for all $0 \leq x \leq 1$

$$
|G_{a,1}(x) - G_{b,1}(x)| \leq x |b - a|,
$$

and that for all $a$, $g_{a,1}(1) = 1$.

In addition,

$$
|\Upsilon(t,w,y+\Delta) - \Upsilon(t,w,y)| = \frac{m(C_{\lfloor w \rfloor_1^{M_t}})}{m(C_{\lfloor w \rfloor_1^{M_t}})} \cdot \left|\frac{\partial M_t}{\partial y}(y+\Delta,w^-) - \frac{\partial M_t}{\partial y}(y^-)\right| \leq \Delta \frac{\varphi\left(|\partial_{\partial y}^t M_t(y,w)| + |\partial_{\partial y}^t M_t(y,w^-)|\right)}{\min\left\{|\partial M_t(y,w),\partial M_t(y,w^-)|\right\}} + o(\Delta).
$$

Thus if $\epsilon_{t+1}$ is sufficiently small (this choice depends on $\{N_s, M_s, \lambda_s : s \leq t\}$ and $N_{t+1}$) then

$$
\mathcal{K}_{t,y+\Delta}(x) = \mathcal{K}_{t,y}(x) + \frac{1}{2} \Delta \varphi^{-N_{t+1}} + o(\Delta),
$$

and for all $\Delta$ sufficiently small

$$
\mathcal{K}_{t,y}(x) = \mathcal{K}_{t,y+\Delta}(x_t(\Delta)) = \mathcal{K}_{t,y}(x_t(\Delta)) + \frac{1}{2} \Delta \varphi^{-N_{t+1}} + o(\Delta).
$$

Since $\frac{\partial \mathcal{K}_{t,y}}{\partial x}(x) \geq \frac{1}{2}$, it follows that

$$
|x - x_t(\Delta)| \leq 2|\mathcal{K}_{t,y}(x) - \mathcal{K}_{t,y+\Delta}(x_t(\Delta))| = \Delta \varphi^{-N_{t+1}} + o(\Delta)
$$

as required. \(\square\)

The next corollary is the final ingredient for the proof of Theorem 26.
Corollary 36. There exists a choice of \( \{\lambda_t, n_t, N_t, m_t, M_t, \epsilon_t\}_{t \in \mathbb{N}} \) such that:

(i) For all \((x, y) \in \mathbb{M} \) and for all \( t \in \mathbb{N} \) such that \( j(x, y) < N_t \),

\[
\left| \frac{\partial K_{N_{t+1},y}(x)}{\partial y} \right| \leq \left| \frac{\partial K_{N_t,y}(\varepsilon M_t(w))}{\partial y} \right| + 2(1.6)^{-n_t}.
\]

(ii) \( \frac{\partial K_{N_{t+1},y}(x)}{\partial y} \) converges uniformly in \( R_1 \cup R_3 \) as \( t \to \infty \). For every \((x, y) \in \partial (R_1 \cup R_3) \) or \((x, y) \in U\),

\[
\lim_{t \to \infty} \left| \frac{\partial K_{N_{t+1},y}(x)}{\partial y} \right| = 0
\]

Proof. We assume that \( \{\lambda_t, n_t, N_t, m_t, M_t, \epsilon_t\} \) are chosen so that Lemma 35 holds.

(i) Similarly as in the proof of Corollary 33(i) one can use the facts that

\[
\left| \frac{\partial K_{N_{t+1},y}(\varepsilon M_t(w))}{\partial y} \right| \leq \left| \frac{\partial K_{M_t,y}(\varepsilon M_t(w))}{\partial y} \right| + \sum_{n=M_t+1}^{N_t} \left| \frac{\partial n}{\partial y}(y, w) \right|
\]

and Lemma 35.2.(i)(ii), to show that

\[
\left| \frac{\partial K_{N_{t+1},y}(\varepsilon M_t(w))}{\partial y} \right| \leq \left| \frac{\partial K_{N_t,y}(\varepsilon M_t(w))}{\partial y} \right| + \sum_{n=M_t+1}^{N_t} \left| \frac{\partial n}{\partial y}(y, w) \right| + \frac{5}{3}(1.6)^{-n_t}.
\]

Therefore by Lemma 35.1 and 2.(i)+(ii),

\[
\lim_{\Delta \to 0} \frac{K_{N_{t+1},y+\Delta}(x_t(\Delta)) - K_{N_{t+1},y}(x)}{\Delta} \leq \left| \frac{\partial K_{N_t,y}(\varepsilon M_t(w))}{\partial y} \right| + \frac{5}{3}(1.6)^{-n_t} + \beta_n(1.6)^{-n_t+1}
\]

and by Lemma 35.2.(iii),

\[
\lim_{\Delta \to 0} \frac{K_{N_{t+1},y+\Delta}(x_t(\Delta)) - K_{N_{t+1},y+\Delta}(x)}{\Delta} \leq \lambda_1^{2N_t} \varphi^{-N_t+1} \leq (1.6)^{-N_t+1}.
\]

Part (i) of this corollary follows from those two estimates and \( n_t = o(N_t) \), \( N_t = o(n_t+1) \).

(ii) Let \( t \in \mathbb{N} \), and \((x, y) \in R_1 \cup R_3 \) By applying part (i) of this corollary repeatedly one has that with \( t(x, y) = \min \{t : N_t > j(x, y)\} \) and \( w \) such that \( x \in C_{[w]}^{N_{t+1}} \),

\[
\left| \frac{\partial K_{N_t,y}(x)}{\partial y} \right| \leq \left| \frac{\partial K_{N_{t+1},y}(\varepsilon M_t(x, y))}{\partial y} \right| + 2\sum_{k=t(x, y)+1}^{t} \frac{1}{1.6} \leq (1.6)^{-j(x, y)} + 2(1.6)^{-n_t(x, y)}.
\]
This is enough to show that \( \left\{ \frac{\partial K_{N_{l+1},y}(x)}{\partial y} \right\} \) is a Cauchy sequence in the uniform topology. Indeed if \( s, t > t(x,y) \), then
\[
\left| \frac{\partial K_{N_{l+1},y}(x)}{\partial y} - \frac{\partial K_{N_{l},y}(x)}{\partial y} \right| \leq \sum_{k=t}^{s} (1.6)^{-n_k} \leq 2(1.6)^{-n_l}.
\]
If \( N_t < j(x,y) - 3 \leq N_{s+1} \) then \( K_{N_{l+1},y}(x) = x \) in a neighborhood of \( y \) and hence
\[
\left| \frac{\partial K_{N_{l+1},y}(x)}{\partial y} - \frac{\partial K_{N_{l+1},y}(x)}{\partial y} \right| = \left| \frac{\partial K_{N_{l+1},y}(x)}{\partial y} \right|
\leq (1.6)^{-j(x,y)} + 2(1.6)^{-n_l}
\leq 3(1.6)^{-n_l}
\]
We leave the bound on the easier cases \( t = t(x,y) - 1 < s, s, t < t(x,y) - 1 \) to the reader. □

**Proof of Theorem 26** By the Chain rule writing
\[
B_N(x,y) = \begin{pmatrix}
\frac{\partial K_{N,y}(x)}{\partial x} & \frac{\partial K_{N,y}(x)}{\partial y}
\end{pmatrix}
\]
\[
D_{3N_t}(x,y) = B_{N_t} \left( SK_{N_{l},y}^{-1}(x), -y/\phi \right) \begin{pmatrix}
\phi & 0 \\
0 & -1/\phi
\end{pmatrix} B_{N_t}^{-1}(K_{N_{l},y}(x), y).
\]
This yields that
\[
\frac{\partial D_{3N_t}}{\partial y}(x,y) = \frac{\partial K_{N_{l} - y/\phi}(SK_{N_{l},y}^{-1}(x))}{\partial x} \cdot \left( -\frac{\partial K_{N_{l} - y/\phi}(K_{N_{l},y}^{-1}(x))}{\partial x} \right)^{-1} \left( \frac{\partial K_{N_{l},y}(K_{N_{l},y}^{-1}(x))}{\partial y} \right)
\]
\[
-\frac{1}{\phi} \frac{\partial K_{N_{l} - y/\phi}(SK_{N_{l},y}^{-1}(x))}{\partial y} - \frac{1}{\phi} \frac{\partial K_{N_{l} - y/\phi}(SK_{N_{l},y}^{-1}(x))}{\partial y}
\]
Since all the terms on the right hand side converge uniformly as \( t \to \infty \), the Theorem is proved.

5.3. **Proof of the Anosov property for \( T \).** So far we have shown that \( D_{3N_t} \) converges uniformly to \( T \) and estimated the derivatives. We are going to use the following well known Lemma, it’s proof can be found in [VI]. A function \( A : \mathbb{M}_\omega \times \mathbb{Z} \to SL(2, \mathbb{R}) \) is **linear cocycle** over a homomorphism \( f : \mathbb{M}_\omega \to \mathbb{M}_\omega \) if for any \( m, n \in \mathbb{Z} \) and \( x \in \mathbb{T}^2 \),
\[
A_{m+n}(x) = A_m \circ f^n(x) A_n(x).
\]
We say that the cocycle is Hyperbolic if there are \( \sigma > 1 \) and \( C > 0 \) so that for every \( x \in \mathbb{M}_\omega \) there exists transverse lines \( E_x^s \) and \( E_x^u \) in \( \mathbb{R}^2 \) such that
1. \( A'(x) E_x^s = E_x^s \) and \( A'(x) E_x^u = E_x^u \).
2. \( |A_n(x) v^s| \leq C \sigma^n |v^s| \) and \( |A_{-n}(x) v^u| \leq C \sigma^n |v^u| \) for every \( v^s \in E_x^s, v^u \in E_x^u \) and \( n \geq 1 \).
Proposition. [Vi] Prop. 2.1 |Let $A : \mathbb{M}_\sim \times \mathbb{Z} \to SL(2, \mathbb{R})$ be a linear cocycle over a homeomorphism $f : \mathbb{M}_\sim \to \mathbb{M}_\sim$. If there exists $v \in \mathbb{R}^2$, constants $c > 0$ and $\sigma > 1$ such that $|A_n(x)v| \geq c\sigma^n$ then $A$ is hyperbolic. The transverse lines $E^s_x, E^u_x$ in $\mathbb{R}^2$ satisfy that for any $\sigma_0 < \sigma$, there exists $C > 0$ so that for any $v^s \in E^s_x$ and $v^u \in E^u_x$ and $n \geq 1$,
\[ |A_n(x)v^s| \leq C\sigma_0^n |v^s| \quad \text{and} \quad |A_{-n}(x)v^u| \leq C\sigma_0^n |v^u| . \]

Proof that $\mathcal{Z}$ is Anosov. Define $A : \mathbb{M}_\sim \times \mathbb{Z} \to SL(2, \mathbb{R})$ by
\[ A(x) = \frac{1}{\sqrt{\det(D_3(x,y))}} D_3(x,y) . \]
Since $D_3$ is of the form
\[ \begin{pmatrix} \frac{\partial z}{\partial x}(x,y) & * \\ 0 & -1/\varphi \end{pmatrix} \]
and
\[ 1.6 \leq \frac{\partial}{\partial y}(x,y) \leq 1.7 \]
one has that for all $(x,y) \in \mathbb{M}_\sim$,
\[ \frac{1.6}{\varphi} \leq |\det(D_3(x,y))| \leq \frac{1.7}{\varphi} . \]
and
\[ |A_n(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix}| = \left| \left( \frac{\sqrt{\varphi^n \prod_{k=0}^{n-1} \frac{\partial}{\partial x} \mathcal{Z}^k(x,y)}}{0} \right) \right| \geq (1.6)^n \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| . \]
It then follows that there exists transverse lines $E^s_x$ and $E^u_x$ in $\mathbb{R}^2 \cong T_x\mathbb{M}_\sim$ and $C > 0$ so that for any $v^u \in E^u_x$,
\[ |A_n(x)v^s| \geq (1.58)^n |v^s| . \]
It follows that
\[ |D_3^n(x,y)v^u| = |A_n(x)v^u| \prod_{k=0}^{n-1} \left| \det(D_3(\mathcal{Z}^k(x,y))) \right| \geq C(1.58)^n \left( \frac{1.6}{\varphi} \right)^n |v^u| \geq C(1.5)^n |v^u| . \]
Similarly one has for every $v^s \in E^s_x$,
\[ |D_3^n(x,y)v^s| \leq C(1.58)^{-n} \left( \frac{1.7}{\varphi} \right)^n |v^s| \leq C \left( \frac{2}{3} \right)^n \]
and so $\mathcal{Z} : \mathbb{M}_\sim \to \mathbb{M}_\sim$ is Anosov. \( \square \)
5.4. Proof of the type III₁ property for 3. For \(-\frac{\varphi}{\varphi+2} < y < \frac{1}{\varphi+2}\) let,
\[
R_y(x) := \lim_{n \to \infty} K_{n,y}(x) : \mathbb{T} \to \mathbb{T}
\]
and for \(\frac{1}{\varphi+2} \leq y \leq \frac{\varphi^2}{\varphi+2}\),
\[
R_y(x) := \lim_{n \to \infty} K_{n,y}(x) : [0, 1/\varphi] \to [0, 1/\varphi].
\]
In both cases it is an orientation preserving homeomorphism.
We will show that the measures \(m_{\text{Leb}(\mathbb{T})} \circ R_y\) and \(m_{\text{Leb}(0,1/\varphi)} \circ \tilde{R}_y\) are equivalent measures to \(\mu^+\), the measure on \(\mathbb{T}\) arising from \(\{\lambda_k, m_k, M_k, n_k, N_k\}\) in the previous section.
In addition the Radon Nykodym derivatives
\[
\frac{d\eta_y}{d\mu^+}(x) : \mathbb{M} \to [0, \infty)
\]
defined by
\[
\frac{d\eta_y}{d\mu^+}(x) = \frac{dm_{\text{Leb}(\mathbb{T})} \circ \tilde{R}_y}{dm_{\mu^+}}(x),
\]
is a \((\mathbb{M}_\sim, \mathcal{B}(\mathbb{M}_\sim), \mu)\) measurable function.
This means that the measure \(\eta\) on \(\mathbb{M}_\sim\) defined by
\[
\int_{\mathbb{M}_\sim} u(x,y)d\eta = \int_{\mathbb{M}_\sim} u(x,y)d\eta_y(x)dy = \int_{\mathbb{M}_\sim} u(x,y)d\eta_y(x)d\mu^+(x)dy
\]
is equivalent to \(\mu = m_{\mathbb{M}} \circ \tilde{\theta}_0 = \mu^+ \otimes dy\).
Since \((\mathbb{M}_\sim, \mathcal{B}_\mathbb{M}, \mu, \tilde{f})\) is a type III₁ transformation and \(\mu \sim \eta\), \((\mathbb{M}, \mathcal{B}_\mathbb{M}, \eta, \tilde{f})\) is a type III₁ transformation. Thus \(\mathbb{M}, \mathcal{B}_\mathbb{M}, m_{\mathbb{M}}, 3\) is a type III₂ transformation since it is metric equivalent to \((\mathbb{M}, \mathcal{B}_\mathbb{M}, \eta, \tilde{f})\) with the isomorphism defined by
\[
\pi(x, y) := (\tilde{R}_y(x), y).
\]
Therefore what is left to prove is the following.

**Lemma 37.** (i) For all \(-\frac{\varphi}{\varphi+2} < y < \frac{1}{\varphi+2}\), \((\tilde{R}_y)^{-1}\), \(m_{\text{Leb}(\mathbb{T})}\) is an equivalent measure to \(\mu^+\) (the measure on \(\mathbb{T}\) arising from \(\{\lambda_k, m_k, M_k, n_k, N_k, \epsilon_k\}\) in the previous section).
(ii) For all \(\frac{1}{\varphi+2} < y < \frac{\varphi^2}{\varphi+2}\), \((\tilde{R}_y)\), \(m_{\text{Leb}(0,1/\varphi)}\) is an equivalent measure to \(\mu^+|_{[0,1/\varphi]}\).
(iii) The Radon Nykodym derivatives
\[
\frac{d\eta_y}{d\mu^+}(x) : \mathbb{M} \to [0, \infty)
\]
defined by
\[
\frac{d\eta_y}{d\mu^+}(x) = \frac{dm_{\text{Leb}(\mathbb{T})} \circ \tilde{R}_y}{dm_{\mu^+}}(x),
\]
are measurable in \((\mathbb{M}_\sim, \mathcal{B}_{\mathbb{M}_\sim}, \mu)\).

**Proof.** Fix \(-\frac{\varphi}{\varphi+2} < y < \frac{1}{\varphi+2}\). The proof is the same as in Lemma 22 by using the theory of local absolute continuity of Shiryaev with \(\mathcal{F}_t := \{C_{[w]}^\gamma \mid w \in \Sigma_k\}\). By the construction
\[
(m_{\text{Leb}(\mathbb{T})} \circ \tilde{R}_y)_t := m_{\text{Leb}(\mathbb{T})} \circ \tilde{R}_y|_{\mathcal{F}_t} = m_{\text{Leb}(\mathbb{T})} \circ K_{N_1,y},
\]
and \( (\mu^+) = m_{\text{Leb}(T)} \circ H_{0,N_t} \).

Therefore,
\[
Z_{t,y}(x) := \frac{d (m_{\text{Leb}(T)} \circ R_y)_t}{d (\mu^+)_t}(x) = \frac{\frac{\partial K_{N_t,y}}{\partial x}}{\frac{\partial H_0}{\partial x}}(x).
\]

The rest of the proof that \( Z_{t,y}(x) \) is uniformly integrable and hence converges a.s. as \( t \to \infty \) is the same as in Lemma 22. This proves (i) and (ii).

To see (iii), notice that the function \((x,y) \mapsto \frac{d\eta_y}{d\mu^+}(x) = \lim_{t \to \infty} Z_{t,y}(x)\) is a pointwise limit of continuous functions (on a set of \( \mu \) measure 1), hence measurable. \(\square\)

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Current address: Mathematics institute, University of Warwick, Gibbet Hill Road, Coventry, CV47AL.

E-mail address: z.kosloff@warwick.ac.uk