Maps on statistical manifolds exactly reduced from the
Perron-Frobenius equations for solvable chaotic maps

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Abstract

Maps on a parameter space for expressing distribution functions are exactly derived from the Perron-Frobenius equations for a generalized Boole transform family. Here the generalized Boole transform family is a one-parameter family of maps where it is defined on a subset of the real line and its probability distribution function is the Cauchy distribution with some parameters. With this reduction, some relations between the statistical picture and the orbital one are shown. From the viewpoint of information geometry, the parameter space can be identified with a statistical manifold, and then it is shown that the derived maps can be characterized. Also, with an induced symplectic structure from a statistical structure, symplectic and information geometric aspects of the derived maps are discussed.

1 Introduction

Solvable chaotic maps are maps whose invariant measures are analytically expressed, and these maps play various roles in physics, applied mathematics and its engineering applications, since they provide analytic formulae for correlation functions and some average quantities [1]. With the solvable features one can investigate mathematical properties analytically. Aside from its purely academic interest, its resolution has some implications [2]. In Ref. [3], a one-parameter family of maps called the generalized Boole transform family was proposed, and the Lyapunov exponent for this family was analytically obtained [4]. In Ref. [5] the generalized Boole transform with a particular parameter was used as a toy model for clarifying mechanism of a class of synchronizations. Since the generalized Boole transform family has a mixing property, the long-time limit of distribution functions can be estimated. Thus, the next interest for us is to describe relaxation processes of these maps by analyzing the Perron-Frobenius equations. Here the Perron-Frobenius equation describes dynamics of distribution functions, and it is not necessary to give a finite dimensional description of dynamics of distribution functions even for solvable chaotic systems. If the Perron-Frobenius equation reduces to a finite dimensional map, then one can easily handle some relaxation processes. Thus, it is expected that such a reduction gives us various benefits.

Information geometry is a geometrization of mathematical statistics [6], and various mathematical statements have been found. This geometry gives tools to study statistical quantities defined on statistical manifolds, where statistical manifolds are identified as parameter spaces for parametric distribution functions. Examples of applications of information geometry include statistical interference, quantum information, and thermodynamics [6, 7, 8]. From these examples, one sees that the application of information geometry to sciences and engineering enables one to visualize theories and to utilize differential geometric tools for their analysis. Thus it is expected that enlarging the application area of information geometry can brings various benefits. Moreover, if the dimension of a statistical manifold is even, then symplectic geometry is of interest since symplectic geometry provides a set of comprehensive tools to understand dynamical systems. As an example, Darboux’s theorem guarantees the existence of canonical coordinates. Thus, a compatibility of
symplectic geometry and information geometry is of interest [9]. To state such a compatibility, the conditions that a statistical manifold admits a symplectic structure have been studied [10]. Also it should be noted that dynamical systems on statistical manifolds have been studied in the literature [11, 12, 13, 14].

In this paper, dynamics of distribution functions for the generalized Boole transform family is focused with the Perron-Frobenius equations, and a family of maps is derived, where this family is defined on a parameter space for parametric distribution functions associated with the transform family. Then it is shown that these derived maps are characterized with tools in information and symplectic geometries. In this way, the derived family of maps is geometrically formulated.

2 Generalized Boole transform

In this section, the generalized Boole transform is introduced and some basic properties are summarized.

The following one-parameter family of maps is focused in this paper.

**Definition 2.1. (The Generalized Boole transform, [15]).** Let $\mathbb{R} = \mathbb{R} \setminus \mathbb{R}^\infty$ be a subset of the real line $\mathbb{R}$ with $\mathbb{R}^\infty$ specified later, $\alpha > 0$ a real number, and $F_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ the map such that

$$F_\alpha(\xi) = \alpha \left( \xi - \frac{1}{\xi} \right).$$

The set $\mathbb{R}^\infty$ is a collection of points so that $F_\alpha(\xi)$ is finite for all points of $\mathbb{R}$. Then the map with $\alpha$, $F_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is referred to as the generalized Boole transform. Also, the one-parameter family of maps $\{F_\alpha\}$ is referred to as the generalized Boole transform family.

When treating $F_\alpha$ as a map of a dynamical system, we write it as $\xi \mapsto F_\alpha(\xi)$ or $\xi_{n+1} = F_\alpha(\xi_n)$ where $\xi_n \in \mathbb{R}$ with $n \in \mathbb{Z}$.

One can generalize this family of maps further. One possible generalization is $F_{\alpha,\beta}(\xi) = \alpha \xi - \beta / \xi$ with some $\beta \in \mathbb{R}$. In this case after introducing changes of variables one can show that $F_{\alpha,\beta}$ reduces to $F_\alpha$. Another generalization is found in Ref. [3].

With $\{F_\alpha\}$, one can have a family of one-dimensional dynamical systems on $\mathbb{R}$. This is stated as follows.

**Proposition 2.1. (Invariant measure of the generalized Boole transform, [4]).** The dynamical system $(\mathbb{R}, \mu_\alpha, F_\alpha)$ with $0 < \alpha < 1$ has a mixing property with the invariant measure

$$\mu_\alpha(d\xi) = C \left( \xi; 0, \sqrt{\frac{\alpha}{1 - \alpha}} \right) d\xi,$$

where

$$C(\xi; \nu, \gamma) := \frac{1}{\pi} \frac{\gamma}{((\xi - \nu)^2 + \gamma^2)}, \quad (\nu, \gamma) \in H$$

with $H := \mathbb{R} \times \mathbb{R}_{>0} \subset \mathbb{R}^2$.

Since it is known that a mixing property leads to an ergodic property in general, it follows from this Proposition that the dynamical system associated with $F_\alpha$ has an ergodic property.

The function $C(\xi; \nu, \gamma)$ is known as the Cauchy distribution, where $\gamma$ and $\nu$ are referred to as the scale parameter and the location parameter, respectively. This $C(\xi; \nu, \gamma)$ belongs to the stable distribution family, and $C(\xi; 0, 1)$ is referred to as the standard Cauchy distribution. The space $H$ is used as a parameter space for expressing Cauchy distributions, and is referred to as the upper half-plane.
3 Parameter maps from Perron-Frobenius equations

In this section, a family of maps on $H$ is exactly derived and its basic properties are discussed, where the derived maps describe dynamics of distribution functions of the generalized Boole transform family. Such maps are obtained by reducing the Perron-Frobenius equations, $\rho_n \mapsto \rho_{n+1}$ with $n \in \mathbb{Z}$

$$\rho_{n+1} (\xi') = \sum_{\xi = F_n(\xi')} \frac{1}{\frac{dF_n}{d\xi}} \rho_n (\xi),$$  \hspace{1cm} (3)

where $\xi$ in the sum denotes the set of all the preimages of a given point $\xi'$, and $\rho_n$ the distribution function. By analyzing (3), one has the time-evolution of distribution functions of the dynamical system.

In this paper, the way of viewing dynamical systems with the Perron-Frobenius equations is referred to as the statistical picture. In addition, the way of viewing dynamical systems with $\{F_\alpha\}$ is referred to as the orbital picture.

One then has the following in the statistical picture. This statement is the departure of the discussions below.

**Proposition 3.1.** (Parameter maps). Consider (3), where $F_\alpha$ is given by (2) with $0 < \alpha < 1$. If $\rho_n(\xi) = C(\xi; \nu, \gamma)$, then $\rho_{n+1}(\xi') = C(\xi'; \nu', \gamma')$ with $\xi' = F_\alpha(\xi)$,

$$\nu' = F_{\alpha,-}(\nu, \gamma) := \alpha \nu \frac{\gamma^2 + \nu^2 - 1}{\gamma^2 + \nu^2}, \hspace{1cm} \text{and} \hspace{1cm} \gamma' = F_{\alpha,+}(\gamma, \nu) := \alpha \frac{\gamma^2 + \nu^2 + 1}{\gamma^2 + \nu^2}. \hspace{1cm} (4)$$

**Proof.** Substituting $\rho_n(\xi) = C(\xi; \nu, \gamma)$ into (3), one can complete the proof. The details of this calculation with a fixed $\alpha$ are as follows.

Since the number of points that give a point $\xi'$ under the map iteration $\xi \mapsto \xi' = F_\alpha(\xi)$ is two, the Perron-Frobenius equation is of the form

$$\rho_{n+1}(\xi') = \sum_{j=1,2} \frac{1}{\frac{dF_j}{d\xi}} C(\xi_j; \nu, \gamma), \hspace{1cm} \xi' = F_\alpha(\xi_j).$$

To have an explicit form of the equation above, one needs the preimage of a given point $\xi' \in \mathcal{R}_-$. The preimage is $\xi - \cup \xi +$, where $\xi \pm$ are such that $\xi' = F_\alpha(\xi +)$ and $\xi' = F_\alpha(\xi -)$. They are obtained by solving

$$\xi' = \alpha \left( \frac{\xi - 1}{\xi} \right),$$

for $\xi$ as

$$\xi_+(\xi') = \frac{1}{2\alpha} \left[ \xi' + \sqrt{\xi'^2 + 4\alpha^2} \right], \hspace{1cm} \text{and} \hspace{1cm} \xi_-(\xi') = \frac{1}{2\alpha} \left[ \xi' - \sqrt{\xi'^2 + 4\alpha^2} \right].$$

From these explicit expressions one has the following relations:

$$\xi_- + \xi_+ = \frac{\xi'}{\alpha}, \hspace{1cm} \text{and} \hspace{1cm} \xi_- \xi_+ = -1,$$

from which

$$\xi_-^2 + \xi_+^2 = (\xi_- + \xi_+)^2 - 2\xi_- \xi_+ = \left( \frac{\xi'}{\alpha} \right)^2 + 2.$$

In addition it follows from

$$4 + \left( \frac{\xi'}{\alpha} \right)^2 = \left( \frac{\xi + 1}{\xi} \right)^2,$$

that

$$\frac{4 + (\xi'/\alpha)^2}{\xi_+^2 + 1} = \frac{\xi_+^2 + 1}{\xi_+^2}.$$
With these, one has
\[
\frac{d\xi'}{d\xi} \bigg|_{\xi_{\pm}} = \alpha \frac{1 + \xi^2}{\xi_{\pm}^2} = \alpha \frac{4 + (\xi' / \alpha)^2}{1 + \xi_{\pm}^2} = \left| \frac{d\xi'}{d\xi} \bigg|_{\xi_{\pm}} \right|
\]
Thus the Perron-Frobenius equations read
\[
\rho_{n+1}(\xi') = \frac{1}{4 + (\xi' / \alpha)^2} \frac{\gamma}{\alpha \pi} \left( \frac{1 + \xi_{\pm}^2}{\gamma^2 + (\xi_{\pm} - \nu)^2} + \frac{1 + \xi_{\pm}^2}{\gamma^2 + (\xi_{\pm} - \nu)^2} \right).
\]
Then it follows that
\[
\rho_{n+1}(\xi') = \frac{\gamma}{\alpha \pi [4 + (\xi' / \alpha)^2]} \left[ (1 + \xi_{\pm}^2)(\xi_{\pm} - \nu)^2 + \gamma^2 \right] + (1 + \xi_{\pm}^2)(\xi_{\pm} - \nu)^2 + \gamma^2.
\]
where \(\Upsilon_n\) and \(\Upsilon_d\) are given by
\[
\Upsilon_n = (1 + \xi_{\pm}^2)(\xi_{\pm} - \nu)^2 + \gamma^2, \quad \Upsilon_d = \gamma^4 + \gamma^2 \left\{ (\xi_{\pm} - \nu)^2 + (\xi_{\pm} - \nu)^2 \right\} + (\xi_{\pm} - \nu)^2(\xi_{\pm} - \nu)^2,
\]
respectively. The expression of \(\Upsilon_d\) and that of \(\Upsilon_n\) reduce to
\[
\Upsilon_n = (\gamma^2 + \nu^2 + 1) \left[ 4 + \frac{(\xi')^2}{\alpha^2} \right],
\]
\[
\Upsilon_d = (\gamma^2 + \nu^2) \left( \frac{\xi'}{\alpha} \right)^2 - 2 \nu(\gamma^2 + \nu^2 - 1) \frac{\xi'}{\alpha^2} + \gamma^4 + 2 \gamma^2(\nu^2 + 1) + (\nu^2 - 1)^2.
\]
The expression of \(\Upsilon_d\) reduces further by introducing
\[
\nu' = \alpha \frac{\nu(\gamma^2 + \nu^2 - 1)}{\gamma^2 + \nu^2},
\]
as
\[
\Upsilon_d = \frac{\gamma^2 + \nu^2}{\alpha^2} (\xi' - \nu')^2 + 4 \gamma^2 \left[ 1 + \frac{\gamma^2 + \nu^2}{4 \alpha^2} \left( \frac{\nu'}{\nu} \right)^2 \right],
\]
where the relation
\[
\gamma^4 + 2 \gamma^2(\nu^2 + 1) = 4 \gamma^2 \left[ 1 + \frac{\gamma^2 + \nu^2}{4 \alpha^2} \left( \frac{\nu'}{\nu} \right)^2 \right] - \frac{\gamma^2}{\gamma^2 + \nu^2} + \gamma^2 \nu^2,
\]
has been used. Combining \(\Upsilon_n\) and \(\Upsilon_d\), one has
\[
\rho_{n+1}(\xi') = \frac{\gamma}{\alpha} \frac{1}{\alpha \pi [4 + (\xi' / \alpha)^2]} \left[ \frac{4 + (\xi')^2}{\gamma^2 + (\xi' - \nu')^2} \right] \left( \gamma^2 + \nu^2 + 1 \right)
\]
from which
\[
\rho_{n+1}(\xi') = \frac{1}{\pi (\xi' - \nu')^2 + \left( \frac{\nu'}{\nu} \right)^2 + \frac{4 \alpha^2 \gamma^2}{\gamma^2 + \nu^2}}.
\]
The term in the bracket [· · ·] above can be written by introducing
\[
\gamma' = \alpha \frac{\gamma (\gamma^2 + \nu^2 + 1)}{\gamma^2 + \nu^2},
\]
as
\[
\left( \frac{\nu'}{\nu} \right)^2 + \frac{4\alpha^2 \gamma^2}{\gamma^2 + \nu^2} = (\gamma')^2.
\]
Thus, one arrives at
\[
\rho_{n+1} (\xi') = \frac{1}{\pi} \left[ \frac{\gamma'}{(\xi' - \nu')^2 + (\gamma')^2} \right],
\]
from which one concludes that
\[
\rho_{n+1} (\xi') = C(\xi'; \nu', \gamma').
\]

In this paper, the family of maps $H \to H$, $(\nu, \gamma) \mapsto (\nu', \gamma')$ is referred to as the parameter maps or the parameter map family, and $H$ the phase space of the parameter maps. For a fixed $\alpha$, the parameter map is denoted $F_\alpha$. When one needs to avoid the infinite points in $H$, one restricts $H$ to a subspace of $H$. Remarks on this family of parameter maps are listed below.

1. The case of $\alpha = 1/2$ was considered in Ref. [5], and the parameter map for $\alpha = 1/2$ was obtained. That previously derived map is consistent with (4).

2. When $\gamma^2, \nu^2 \gg 1$, the parameter maps (4) are approximately written as
\[
\gamma_{n+1} = \alpha \gamma_n, \quad \text{and} \quad \nu_{n+1} = \alpha \nu_n.
\]
The solution set is immediately obtained as
\[
\gamma_n = \alpha^n \gamma_0, \quad \text{and} \quad \nu_n = \alpha^n \nu_0.
\]

3. There exist at least two invariant manifolds when the phase space is extended. They are
\[
H_{\nu=0} := \{ (\nu, \gamma) \in H \mid \nu = 0 \}, \quad \text{and} \quad \Pi_{\gamma=0} := \{ (\nu, \gamma) \in \Pi \mid \gamma = 0 \},
\]
where $\Pi := \mathbb{R} \times \mathbb{R}_{\geq 0} \subset \mathbb{R}^2$. The dynamical system on each invariant manifold is
\[
H_{\nu=0} : \gamma' = G_\alpha (\gamma), \quad \text{and} \quad \Pi_{\gamma=0} : \nu' = F_\alpha (\nu),
\]
where $F_\alpha$ has been defined in (1) and $G_\alpha : \mathcal{R}_+ \to \mathcal{R}_+$ with $\mathcal{R}_+ = \mathbb{R} \setminus \mathcal{R}_+^\infty$ is such that
\[
G_\alpha (\gamma) := \alpha \left( \gamma + \frac{1}{\gamma} \right).
\]
Here the set $\mathcal{R}_+^\infty$ is a collection of points of $\mathbb{R}$ so that $G_\alpha (\gamma)$ is finite for all points of $\mathcal{R}_+$. On the boundary of the extended phase space $\Pi$, one can consider Dirac’s delta function as the limiting distribution function,
\[
\delta(\xi - \nu) = C(\xi; \nu, 0).
\]
The dynamics of $\nu$ takes place on $\Pi_{\gamma=0}$ and is exactly the same as that of the orbital picture. Note that the function $G_{1/2}$ is the same as $G$ introduced in Ref. [5].

4. A fixed point for the map with a fixed $\alpha$ is $(\overline{\nu}, \overline{\gamma}) \in H$, where
\[
\overline{\nu} = 0 \quad \text{and} \quad \overline{\gamma} = \sqrt{\frac{\alpha}{1 - \alpha}}.
\]
This fixed point is unique for a fixed $\alpha$. 

5
**Lemma 3.1.** Fix \( \{s, w, s'\} \in \mathbb{C} \) to be

\[
\begin{align*}
s &:= \nu - i \gamma, \\
w &:= \nu + i \gamma, \\
s' &:= \nu' - i \gamma', \\
w' &:= \nu' + i \gamma', \\
i &:= \sqrt{-1}.
\end{align*}
\]

Then the map \((s, w) \mapsto (s', w')\) under \((\nu, \gamma) \mapsto (\nu', \gamma')\) is the following set of maps:

\[
s' = F_\alpha(s), \quad \text{and} \quad w' = F_\alpha(w).
\]

**Proof.** To prove this, the statement is split into the following two.

(a) There is no fixed point on the set \(\{ (\nu, \gamma) | \nu \neq 0, \gamma > 0 \} \subset H\).

(b) The fixed point on \(H\) is given by \(0\), and is unique.

As a notational convenience introduce \(\overline{A} = \overline{\nu}^2 + \overline{\gamma}^2\).

(Proof of (a)) : Assume that there exists a fixed point \((\overline{\nu}, \overline{\gamma})\) with \(\overline{\nu} \neq 0\) and \(\overline{\gamma} > 0\). Then the fixed point are given by the solutions to

\[
(1 - \alpha) \overline{A} = -\alpha, \quad \text{and} \quad (1 - \alpha) \overline{A} = \alpha.
\]

Taking into account \(0 < \alpha < 1\), one has that \(\overline{A} = 0\), from which \((\overline{\nu}, \overline{\gamma}) = (0, 0)\). This is in contradiction to the assumption on \((\overline{\nu}, \overline{\gamma})\). This completes the proof of (a).

(Proof of (b)) : Taking into account (a), one looks for fixed points on \(H_{\nu=0} = \{ (\nu, \gamma) | \nu = 0, \gamma > 0 \}\).

The equation for determining fixed points on \(H_{\nu=0}\) is derived by substituting \(\nu = 0\) into \(4\) as

\[
(1 - \alpha) \overline{\gamma}^2 = \alpha.
\]

From this equation, there exists only one solution on \(H\) as \(\overline{\gamma} = \sqrt{\alpha/(1 - \alpha)}\).

When \(\alpha = 1/2\), one has \((\overline{\nu}, \overline{\gamma}) = (0, 1)\) (see Ref. [5]). This fixed point corresponds to the standard Cauchy distribution.

5. The linearized map around \((\overline{\nu}, \overline{\gamma})\), denoted \((\delta \nu, \delta \gamma) \mapsto (\delta \nu', \delta \gamma')\), is obtained from

\[
\delta \gamma' = \left( \frac{\partial F_{\alpha,+}}{\partial \gamma} \right)_{(\overline{\nu}, \overline{\gamma})} \delta \gamma + \left( \frac{\partial F_{\alpha,+}}{\partial \nu} \right)_{(\overline{\nu}, \overline{\gamma})} \delta \nu, \quad \delta \nu' = \left( \frac{\partial F_{\alpha,-}}{\partial \nu} \right)_{(\overline{\nu}, \overline{\gamma})} \delta \gamma + \left( \frac{\partial F_{\alpha,-}}{\partial \gamma} \right)_{(\overline{\nu}, \overline{\gamma})} \delta \nu,
\]

as

\[
\delta \gamma' = (2\alpha - 1) \delta \gamma, \quad \text{and} \quad \delta \nu' = (2\alpha - 1) \delta \nu.
\]

This shows that this fixed point is linearly stable for \(0 < \alpha < 1\) except for \(\alpha = 1/2\). Notice that the linear stability analysis fails when \(\alpha = 1/2\), and sequences converge in a quadratic order [10].

6. An estimate of the convergence of sequences on \(H_{\nu=0}\) is given as follows. For \(n \geq 2\),

\[
\begin{align*}
|\gamma_{n+1} - \overline{\gamma}| &\leq \alpha |\gamma_n - \overline{\gamma}|, \quad \text{for} \quad \frac{1}{2} \leq \alpha < 1 \\
|\gamma_{n+1} - \overline{\gamma}| &\leq (1 - \alpha) |\gamma_n - \overline{\gamma}|, \quad \text{for} \quad 0 < \alpha \leq \frac{1}{2}
\end{align*}
\]

where an initial point is specified with \(n = 0\).

7. Fix \(\alpha\). Then let \(F_\alpha : H \to H\), \((\nu, \gamma) \mapsto (\nu', \gamma')\) be the parameter map, and \(I : H \to H\), \((\nu, \gamma) \mapsto (-\nu, \gamma)\) the reflection operator. Then it follows from

\[
IF_\alpha(\nu, \gamma) = I(\nu', \gamma') = (-\nu', \gamma'), \quad \text{and} \quad F_\alpha I(\nu, \gamma) = F_\alpha (-\nu, \gamma) = (-\nu', \gamma')
\]

that \(F_\alpha\) and \(I\) are commute, \(F_\alpha I = IF_\alpha\).

When introducing a set of complex variables, one can formally rewrite the parameter maps \(\{F_\alpha\}\) with \(\{F_\alpha\}\) as follows.

**Lemma 3.1.** Fix \(\alpha\). Then let \((\nu, \gamma)\) and \((\nu', \gamma')\) be points of \(H\) that satisfy \([4]\) Define the complex variables \(s, w, s', w' \in \mathbb{C}\) to be

\[
\begin{align*}
s &:= \nu - i \gamma, \\
w &:= \nu + i \gamma, \\
s' &:= \nu' - i \gamma', \\
w' &:= \nu' + i \gamma', \\
i &:= \sqrt{-1}.
\end{align*}
\]

Then the map \((s, w) \mapsto (s', w')\) under \((\nu, \gamma) \mapsto (\nu', \gamma')\) is the following set of maps:

\[
s' = F_\alpha(s), \quad \text{and} \quad w' = F_\alpha(w).
\]
Proof. One completes the proof by substituting \( s, w, s', w' \) defined above into (4). The details are as follows. Introducing \( A = \nu^2 + \gamma^2 = sw \), one has

\[
\nu' = \alpha \nu \left( 1 - \frac{1}{sw} \right), \quad \text{and} \quad \gamma' = \alpha \gamma \left( 1 + \frac{1}{sw} \right).
\]

With these, it follows that

\[
s' = \nu' - i \gamma' = \alpha \left( s - \frac{1}{s} \right), \quad \text{and} \quad w' = \nu' + i \gamma' = \alpha \left( w - \frac{1}{w} \right).
\]

Similar to the lemma above, one has the following.

**Lemma 3.2.** Consider Lemma 3.1. Define \( \hat{s}, \hat{w}, \tilde{s}, \tilde{w} \in \mathbb{C} \) to be

\[
\hat{s} := is = \gamma + i\nu, \quad \hat{w} := iw = -\gamma + i\nu, \quad \tilde{s} := is' = \gamma' + i\nu', \quad \tilde{w} := iw' = -\gamma' + i\nu'.
\]

Then the map \((\hat{s}, \hat{w}) \mapsto (\tilde{s}, \tilde{w})\) under \((\nu, \gamma) \mapsto (\nu', \gamma')\) is the following set of maps:

\[
\tilde{s}' = \alpha \left( \hat{s} + \frac{1}{\hat{s}} \right) = G_\alpha(\hat{s}), \quad \text{and} \quad \tilde{w}' = \alpha \left( \hat{w} + \frac{1}{\hat{w}} \right) = G_\alpha(\hat{w}),
\]

where \( G_\alpha(\hat{s}) = \alpha(\hat{s} + 1/\hat{s}) \) has been defined by (4).

**Proof.** A way to prove this is analogous to that of Lemma 3.1.

The role of the complex variables \( s, w, s', w' \) is to decompose (4) into a set of \( F_\alpha \) being extended for complex-variables. Each decomposed map is used in the orbital picture. Thus Lemma 3.1 gives a relation between the statistical picture and orbital one. The next one plays the same role.

**Lemma 3.3.** Let \( F_\alpha : H \to H \) be the the parameter map defined by (4):

\[
\nu' := F_{\alpha,-}(\gamma, \nu), \quad \gamma' := F_{\alpha,+}(\gamma, \nu).
\]

In addition, let \((\xi^1, \xi^2) \in \mathbb{R}_- \times \mathbb{R}_-\) be a point. Then, introducing the complex variables

\[
\tilde{\xi} := \xi^1 + i\xi^2 \in \mathbb{C}, \quad \text{and} \quad \tilde{\xi}' := F_\alpha(\tilde{\xi}) = \xi^{1'} + i\xi^{2'} \in \mathbb{C},
\]

with \( \xi^{1'} = \text{Re}(\tilde{\xi}') \) and \( \xi^{2'} = \text{Im}(\tilde{\xi}') \), one has \( \tilde{\xi}' = F_{\alpha,-}(\xi^1, \xi^2) + iF_{\alpha,+}(\xi^1, \xi^2) : \)

\[
\xi^{1'} = F_{\alpha,-}(\xi^1, \xi^2), \quad \text{and} \quad \xi^{2'} = F_{\alpha,+}(\xi^1, \xi^2).
\]

**Proof.** Calculating \( F_\alpha(\tilde{\xi}) \), one can complete the proof. Substituting \( \tilde{\xi} = \xi^1 + i\xi^2 \) into \( F_\alpha(\tilde{\xi}) \), one has

\[
F_\alpha(\tilde{\xi}) = \alpha \left( \xi - \frac{1}{\xi} \right) = \alpha \left( \xi^1 + i\xi^2 - \frac{1}{\xi^1 + i\xi^2} \right) = \alpha \xi^1 \left( \frac{(\xi^1)^2 + (\xi^2)^2 - 1}{(\xi^1)^2 + (\xi^2)^2} \right) + i\alpha \xi^2 \left( \frac{(\xi^1)^2 + (\xi^2)^2 + 1}{(\xi^1)^2 + (\xi^2)^2} \right) = F_{\alpha,-}(\xi^1, \xi^2) + iF_{\alpha,+}(\xi^1, \xi^2).
\]

On the other hand, the left hand side of the equation above is \( F_\alpha(\tilde{\xi}) = \tilde{\xi}' = \xi^{1'} + i\xi^{2'} \).

Due to this lemma, the maps \( \{F_\alpha\} \) used in the orbital picture can be written with \( \{F_\alpha\} \) used in the statistical picture.

Combining Lemmas 3.1 and 3.3 one has the following.
Theorem 3.1. (Relation between statistical picture and orbital one): Fix $\alpha$. Then the parameter map $F_{\alpha}$ can be written in terms of $F_{\alpha}$ with some complex variables. On the other hand, the map $\xi' = F_{\alpha}(\xi)$ in the orbital picture is written in terms of the parameter map with some complex variables.

The reason why the finite-dimensional parameter maps have successfully been obtained is to choose a particular family of distribution functions for the Perron-Frobenius equations. In our case, the Cauchy distribution with a parameter set is chosen, and it has turned out that the distribution function with $(\nu, \gamma)$ approaches to the one with $(\nu, \gamma)$ after a long-time evolution if an initial parameter set is close enough to $(\nu, \gamma)$. Also, notice that the dimension of the phase space $H$ of the parameter map family $\{F_{\alpha}\}$ is the number of the parameters of the Cauchy distribution. This dimension of the parameter map family is not directly related to the dimension of the phase space of $\{F_{\alpha}\}$. One can generalize this notice as follows. Consider the case where a map in an $n$-dimensional phase space is solvable, and its invariant measure is written with a distribution function having $k$-parameters. Assume that a parameter map is obtained by reducing the Perron-Frobenius equation. Then the dimension of the phase space of a parameter map is $k$, and $n$ is not directly related to $k$ in general.

4 Information and symplectic geometric characterizations

In Section 3, the parameter maps have been derived and some relations between the orbital picture and statistical one have been obtained.

In this section, dynamics of the parameter maps in the phase space is characterized with information geometry and symplectic geometry. Since information geometry gives differential geometric tools for analyzing distribution functions on parameter spaces (or phase spaces), it is expected that the parameter maps can be characterized with such tools. Also, since it has been known how a symplectic structure is induced from a statistical manifold, one can apply symplectic geometry to the parameter maps. In particular, it is shown that the parameter maps are conformal on a Riemannian manifold and that the parameter maps are written with canonical coordinates. To this end, some connections and manifolds are introduced first.

4.1 Information geometric characterization of parameter maps

The following connection plays roles in information geometry.

Definition 4.1. (Dual connection, [2]). Let $(M, g)$ be a (pseudo) Riemannian manifold, and $\nabla$ a connection. The dual connection $\nabla^*$ associated with $g$ is a connection that satisfies $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)$ for all $X, Y, Z \in TM$.

It can be verified that the dual connection is determined uniquely from $\nabla$ and $g$, and that $(\nabla^*)^* = \nabla$.

The following is our definition of statistical manifold.

Definition 4.2. (Statistical manifold, [17]). Let $(M, g)$ be a (pseudo) Riemannian manifold, $\nabla$ a connection, and $\nabla^*$ the dual connection associated with $g$. If $\nabla$ and $\nabla^*$ are torsion-free connections, then the triplet $(M, g, \nabla)$ is referred to as the statistical manifold.

Note that any flatness condition is not imposed in the definition above. The torsion-free condition is written as $T^\nabla (X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0$, for all $X, Y \in TM$ with $[X, Y] = XY - YX$. Here $T^\nabla$ is referred to as torsion tensor.

A role of the Levi-Civita connection on statistical manifolds is as follows.

Lemma 4.1. Let $(M, g, \nabla)$ is a statistical manifold. Then,

$$\nabla = \nabla^* \iff \nabla \text{ is the Levi-Civita connection}$$
Proof. (Proof of \(\Rightarrow\)) : From the definition of statistical manifold, one has that \(T^\nabla = 0\). Then from the assumption, one has that \(X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\), which is equivalent to \(\nabla g = 0\). These two conditions guarantee that \(\nabla\) is the Levi-Civita connection.

(Proof of \(\Leftarrow\)) : Since \(\nabla\) is the Levi-Civita connection, one has \(\nabla g = 0\) :

\[ X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \]

Comparing this with the definition of dual connection Definition. 4.1, one has that \(\nabla = \nabla^*\).

In the standard information geometry, parametric distribution functions are considered. They form the following set.

**Definition 4.3.** (Statistical model, [6]). Let \(\mathcal{S}\) be a set of probability distribution functions that are parameterized by \(n\) real-valued variables \(\zeta = (\zeta^1, \ldots, \zeta^n)\) so that

\[ \mathcal{S} = \{ p_\zeta = p(\xi; \zeta) | \zeta = (\zeta^1, \ldots, \zeta^n) \in \mathbb{Z} \}, \]

where \(\mathbb{Z}\) is a subset of \(\mathbb{R}^n\) and the map \(\zeta \mapsto p_\zeta\) is injective. This \(\mathcal{S}\) is referred to as an \(n\)-dimensional statistical model.

It is assumed that the space of parameters for parametric distribution functions form a manifold.

**Postulate 4.1.** Any \(n\)-dimensional statistical model forms an \(n\)-dimensional manifold.

For a given probability distribution function, the Fisher metric tensor field is defined below.

**Definition 4.4.** (Fisher information matrix, Fisher metric tensor field, [6]). Let \(\mathcal{S}\), be a set of probability distribution functions that are parameterized by \(n\) real-valued variables \(\zeta = (\zeta^1, \ldots, \zeta^n)\) so that

\[ g^F_{ab}(\zeta) := \int \frac{\partial \ln p(\xi; \zeta)}{\partial \zeta^a} \frac{\partial \ln p(\xi; \zeta)}{\partial \zeta^b} p(\xi; \zeta) d\xi \]

is referred to as the Fisher information matrix. In addition, the metric tensor field

\[ g^F(\zeta) := g^F_{ab}(\zeta) d\zeta^a \otimes d\zeta^b, \]

is referred to as the Fisher metric tensor field.

The following is a relevant example for the parameter maps discussed in Section 3.

**Example 4.1.** (Fisher metric tensor field for the Cauchy distribution). For the Cauchy distribution \(p_\zeta = p(\xi; \zeta) = C(\xi; \nu, \gamma)\) given in [6] with \(\zeta = (\nu, \gamma)\), the explicit form of the Fisher metric tensor field is calculated as follows. It can be shown that \(g^F_{\nu\nu} = g^F_{\gamma\gamma} = 1/(\gamma^2)\) and \(g^F_{\nu\gamma} = g^F_{\gamma\nu} = 0\), from which

\[ g^F(\nu, \gamma) = \frac{d\nu \otimes d\nu + d\gamma \otimes d\gamma}{2\gamma^2}. \]

The Killing vector fields \(\{ K_a \} \in \mathcal{H}^T\), defined to be vector fields satisfying \(\mathcal{L}_{K_a} g^F = 0\) with \(\mathcal{L}_{K_a}\) being the Lie derivative along \(K_a\), are found as \(\{ K_1, K_2, K_3 \}\) where

\[ K_1 = (\nu^2 - \gamma^2) \frac{\partial}{\partial \nu} + 2\nu \gamma \frac{\partial}{\partial \gamma}, \quad K_2 = \nu \frac{\partial}{\partial \nu} + \gamma \frac{\partial}{\partial \gamma}, \quad K_3 = \frac{\partial}{\partial \nu}. \]

In addition, the Gaussian curvature for \(H\) is known to be negative.

Consider an \(n\)-dimensional (pseudo) Riemannian manifold \((\mathcal{M}, g)\). If the number of Killing vector fields on \(\mathcal{M}\) is \(n(n+1)/2\), then the manifold is referred to as a maximally symmetric space. Thus the manifold \(H\) is a maximally symmetric space.

The following is our main claim in this subsection and it characterizes our parameter maps with the Fisher metric tensor field.
**Proposition 4.1.** (Parameter map as a conformal map). Let \((\mathbb{H}, g^F)\) be the two-dimensional Riemannian manifold where \(\mathbb{H}\) is the phase space of the parameter maps (the upper half-plane), and \(g^F\) the Fisher metric tensor field given by \(\nu\). Then the parameter maps are conformal \([18]\) if \((\gamma^2 + \nu^2)^2 + 1 + 2(\nu^2 - \gamma^2) > 0\) in the sense that

\[
g^{F'} := \frac{d\nu' \otimes d\nu' + d\gamma' \otimes d\gamma'}{2(\gamma')^2} = \frac{(\gamma^2 + \nu^2)^2 + 1 + 2(\nu^2 - \gamma^2)}{\gamma^2 + \nu^2 + 1}g^F,
\]

where \(\nu' = F_{\alpha,-}(\gamma, \nu)\) and \(\gamma' = F_{\alpha,+}(\gamma, \nu)\).

**Proof.** One can complete this proof by the following straightforward calculations. As a notational convenience introduce \(A = \nu^2 + \gamma^2\). Then one starts with

\[
d\nu' = \alpha \left[ \left( 1 - \frac{1}{A} \right) d\nu + \frac{\nu}{A} dA \right], \quad \text{and} \quad d\gamma' = \alpha \left[ \left( 1 + \frac{1}{A} \right) d\gamma - \frac{\nu}{A} dA \right].
\]

After some tedious calculations, one has from

\[
\left( 1 - \frac{1}{A} \right)^2 + \frac{4\nu^2}{A^2} = \left( 1 + \frac{1}{A} \right)^2 - \frac{4\gamma^2}{A^2},
\]

that

\[
\frac{1}{\alpha^2} \left( d\nu' \otimes d\nu' + d\gamma' \otimes d\gamma' \right) = \left[ \left( 1 + \frac{1}{A} \right)^2 - \frac{4\gamma^2}{A^2} \right] \left( d\nu \otimes d\nu + d\gamma \otimes d\gamma \right).
\]

With this and \(\gamma' = \alpha \gamma \left( 1 + 1/A \right)\), one has that

\[
\frac{d\nu' \otimes d\nu' + d\gamma' \otimes d\gamma'}{2(\gamma')^2} = \left[ 1 - \frac{4\gamma^2}{(1 + A)^2} \right] \frac{d\nu \otimes d\nu + d\gamma \otimes d\gamma}{2\gamma^2} = \left[ 1 - \frac{4\gamma^2}{(1 + A)^2} \right] g^F.
\]

Substituting \(A = \nu^2 + \gamma^2\) into the equation above, one completes the proof. \(\square\)

Notice that the \(\alpha\) does not appear in this Proposition.

### 4.2 Symplectic information geometric characterization of parameter maps

Since the dimension of the phase space of our parameter maps is even and the complex variables play roles as seen in Section 3, one is interested in almost complex manifolds and its sub-classes. The definition of almost complex manifold is as follows.

**Definition 4.5.** (Almost complex structure and almost complex manifold, \([19, 17]\)). Let \(\mathcal{M}\) be a manifold. Almost complex structure is a type \((1,1)\) tensor field such that \(J \circ J = -\text{Id}\) with \(\text{Id}\) being the identical operator. In addition, the pair \((\mathcal{M}, J)\) is referred to as an almost complex manifold.

It is known that the dimension of almost complex manifolds is even and such manifolds are orientable.

Some classes of almost complex manifolds are known as follows.

**Definition 4.6.** (Almost Hermite manifold, \([19, 17]\)). Let \((\mathcal{M}, g)\) be a (pseudo) Riemannian manifold, and \(J\) an almost complex structure. If \(g(JX, JY) = g(X, Y)\) is satisfied for all \(X, Y \in TM\), then the triplet \((\mathcal{M}, g, J)\) is referred to as an almost Hermite manifold.

**Definition 4.7.** (Almost Kähler manifold, \([19]\)). Let \((\mathcal{M}, g, J)\) be an almost Hermite manifold, and \(\omega\) the two-form defined by

\[
\omega(X, Y) = g(JX, Y),
\]

for all \(X, Y \in TM\). If \(d\omega = 0\), then \((\mathcal{M}, g, J)\) is referred to as an almost Kähler manifold.

Since Killing vector fields on Riemannian manifolds play roles, one is interested in roles of Killing vector fields on almost Kähler manifolds.
Lemma 4.2. Let \((\mathcal{M}, g, J)\) be an almost Kähler manifold, and \(K\) a Killing vector field \((L_K g = 0)\). If \(L_K J = 0\), then \(L_K \omega = 0\).

**Proof.** Applying \(L_K\) on the both side of (9), one completes the proof. \(\square\)

To discuss Kähler manifolds, one needs the following.

**Definition 4.8.** (Integrable, [20]). An almost complex structure \(J\) on a manifold \(\mathcal{M}\) is referred to as integrable if and only if \(J\) is induced by a structure of complex manifold on \(\mathcal{M}\).

**Definition 4.9.** (Kähler manifold, [19]). Let \((\mathcal{M}, g, J)\) be an almost Kähler manifold. If \(J\) is integrable, then \((\mathcal{M}, g, J)\) is referred to as a Kähler manifold.

Lemma 4.3. Let \((\mathcal{M}, g, J)\) be an almost Hermite manifold, and \(\nabla\) the Levi-Civita connection. Then it follows that [21]

\[ \nabla J = 0 \iff (\mathcal{M}, g, J) \text{ is a Kähler manifold.} \]

Symplectic manifolds are of interest in even dimensional manifolds. They are defined as follows.

**Definition 4.10.** (Symplectic structure and symplectic manifold, [22]). Let \(M\) be a manifold, and \(\omega\) a two-form such that it is closed \((d\omega = 0)\) and non-degenerate. Then \(\omega\) is referred to as a symplectic (two-) form or a symplectic structure. In addition \((M, \omega)\) is referred to as a symplectic manifold.

The dimension of symplectic manifolds is even, and there exist special coordinates.

**Definition 4.11.** (Canonical coordinates, [22]). Let \((\mathcal{M}, \omega)\) be a \(2n\)-dimensional symplectic manifold. If the coordinates \((q, p)\) with \(q = \{q^1, \ldots, q^n\}\) and \(p = \{p^1, \ldots, p_n\}\) are such that

\[ \omega = \sum_{a=1}^{n} dp^a \wedge dq^a, \]

then \((q, p)\) are referred to as the canonical coordinates.

Symplectic vector fields on symplectic manifolds play roles. They are defined as follows.

**Definition 4.12.** (Symplectic vector field, [20]). Let \((\mathcal{M}, \omega)\) be a symplectic manifold, and \(X\) a vector field. If \(L_X \omega = 0\), then \(X\) is referred to as a symplectic vector field.

Symplectic connections on symplectic manifolds play roles. They are defined as follows.

**Definition 4.13.** (Symplectic connection, [23]). Let \((\mathcal{M}, \omega)\) be a symplectic manifold, and \(\nabla\) a connection. If the two conditions (i) \(\nabla \omega = 0\) and (ii) \(T^\nabla = 0\) are satisfied, then \(\nabla\) is referred to as a symplectic connection.

It is known a sufficient condition for statistical manifolds to admit a symplectic two-form.

**Lemma 4.4.** (Noda, [10]). Let \((\mathcal{M}, g, \nabla)\) be a statistical manifold, and \(J\) an almost complex structure such that \((\mathcal{M}, g, J)\) is an almost Hermite manifold, \(g(JX, JY) = g(X, Y)\). If the following conditions:

\[ \nabla^*_X Y = \nabla_X Y - J(\nabla_X J)Y, \quad \text{and} \quad (\nabla_X J)Y = (\nabla_Y J)X, \]

for all \(X, Y \in \mathcal{T} \mathcal{M}\) are satisfied, then the two-form \(\omega\) defined by (9) is a symplectic two-form, and \(\nabla\) a symplectic connection.

**Remark 4.1.** If \(\nabla\) is the Levi-Civita connection and \(\nabla J = 0\), then applying this Lemma, one has that \(\omega\) is a symplectic two-form and \(\nabla\) a symplectic connection.

One defines the following manifold, where it admits a symplectic structure and a symplectic connection.
Definition 4.14. (Symplectic statistical manifold, [11]). Let \((M, g, \nabla)\) be a statistical manifold, \((M, g, J)\) an almost Kähler manifold. If \(\nabla \omega = 0\) with \(\omega\) defined by (8), then \((M, g, J, \nabla)\) is referred to as a symplectic statistical manifold.

Remark 4.2. If a vector field \(K\) satisfies \(\mathcal{L}_K g = 0\) and \(\mathcal{L}_K J = 0\), then it follows from Lemma 4.1 that \(K\) is a symplectic vector field.

Similar to symplectic statistical manifold defined in Definition 4.14, some other manifolds have been proposed. These include holomorphic statistical manifold [24].

The following is a relevant example of symplectic statistical manifold.

Example 4.2. (A generalized Poincaré upper half-plane model as a symplectic statistical manifold). Let \(H\) be the upper half-plane introduced in Section 3, \((x, y)\) its coordinates, and \(g_0\) a function on \(H\). Put \(g\) and \(J\) to be

\[
g = g_0(x, y)dx \otimes dx + g_0(x, y)dy \otimes dy, \quad J = dy \otimes \frac{\partial}{\partial x} - dx \otimes \frac{\partial}{\partial y},
\]

so that \((M, g, J)\) is an almost Hermite manifold. Let \(\{ \Gamma_{a\bar{c}} \}, (a, b, c \in \{ x, y \})\) be a set of connection coefficients such that \(\nabla_{a\bar{c}} \partial / \partial \zeta = \Gamma_{a\bar{c}} \partial / \partial \zeta^c\), where \(\zeta^x = x\) and \(\zeta^y = y\). Choose the connection to be the Levi-Civita one. In this case one has

\[
\begin{align*}
\Gamma_{xx}^x &= \Gamma_x, & \Gamma_{xx}^y &= \Gamma_{xy} = -\Gamma_y, & \Gamma_{yx}^x &= -\Gamma_x, \\
\Gamma_{xy}^y &= \Gamma_y, & \Gamma_{xy}^y &= \Gamma_{yx}^y = \Gamma_x, & \Gamma_{yy}^y &= -\Gamma_y,
\end{align*}
\]

where \(\Gamma_x\) and \(\Gamma_y\) are the following functions on \(H\)

\[
\Gamma_x = \frac{1}{2g_0} \frac{\partial g_0}{\partial x}, \quad \text{and} \quad \Gamma_y = -\frac{1}{2g_0} \frac{\partial g_0}{\partial y}.
\]

Applying Lemma 4.1 one has that \(\nabla^* = \nabla\). It is straightforward to verify that \(\nabla J = 0\). By applying Lemma 4.3 with \(\nabla J = 0\), one has that \((H, g, J, \nabla)\) is a Kähler manifold. Also, by noticing Remark 4.1 one has that \((H, g, J, \nabla)\) is a symplectic statistical manifold. Thus the two-form

\[
\omega = -g_0 \, dx \wedge dy,
\]

constructed such that \(\omega(X, Y) = g(JX, Y)\) for all \(X, Y \in TH\), is a symplectic form. In addition, \((q, p)\) with \(q = x, p = G_0(x, y)\) is the set of canonical coordinates, where \(G_0\) is such that \(\partial G_0 / \partial y = g_0(x, y)\). To verify that all the Killing vector fields are symplectic vector fields, let \(K = K^x \partial / \partial x + K^y \partial / \partial y\) be a Killing vector field, and \(Z = Z^x \partial / \partial x + Z^y \partial / \partial y\) a vector field such that \(\mathcal{L}_Z J = 0\). Then the functions \(K^x\) and \(K^y\) should satisfy

\[
\frac{\partial K^x}{\partial y} + 2g_0 \frac{\partial K^x}{\partial x} = 0, \quad \frac{\partial K^y}{\partial x} + \frac{\partial K^x}{\partial y} = 0, \quad \frac{\partial K^y}{\partial x} - \frac{\partial K^y}{\partial y} = 0,
\]

and \(Z^x, Z^y\) should satisfy

\[
\frac{\partial Z^x}{\partial y} + \frac{\partial Z^y}{\partial x} = 0, \quad \frac{\partial Z^x}{\partial x} - \frac{\partial Z^y}{\partial y} = 0.
\]

Thus all the Killing vector fields satisfy \(\mathcal{L}_K J = 0\). Combining this with Remark 4.2 one has that \(\mathcal{L}_K \omega = 0\).

Then one has the following main theorem in this paper on the phase space of the parameter maps.

Theorem 4.1. (Phase space of the parameter maps). Let \(H\) be the upper half-plane, \(\{ \mathcal{F}_\alpha \}\) the parameter map family on \(H\), \(g^F\) the Fisher metric tensor field for the Cauchy distribution \(C(\xi; \nu, \gamma)\) given in (7), \(\nabla\) the Levi-Civita connection and \(J\) the almost complex structure defined in Example 4.2. Then \((H, g^F, J, \nabla)\) is a symplectic statistical manifold, \(\omega = -1/(2\gamma^2) \, dv \wedge d\gamma\) a symplectic form, and \((q, p)\) with \(q = \nu\) and \(p = 1/(2\gamma)\) a set of canonical coordinates. The parameter maps \(\{ \mathcal{F}_\alpha \}\) are dynamical systems on this symplectic statistical manifold. Also, it follows that \(\mathcal{L}_{K_1} \omega = \mathcal{L}_{K_2} \omega = \mathcal{L}_{K_3} \omega = 0\), where \(K_1, K_2\) and \(K_3\) have been defined in (8).
Proof. In Example 4.2 choose \( x = \nu, \ y = \gamma \), and \( g_0(x, y) = 1/(2y^2) \). Then one completes the proof.

With the canonical coordinates \((q, p)\), one can write the parameter map family \((q, p) \mapsto (q', p')\) as

\[
q' = \alpha q \frac{\left(\frac{1}{2p}\right)^2 + q^2 - 1}{\left(\frac{1}{2p}\right)^2 + q^2}, \quad p' = \frac{p}{\alpha} \frac{\left(\frac{1}{2p}\right)^2 + q^2}{\left(\frac{1}{2p}\right)^2 + q^2 + 1}.
\]

In these coordinates, the non-symplectic property of the parameter maps is clear: \( dq' \wedge dp' \neq dq \wedge dp \).

Note that when symplectic statistical manifolds are dually flat, the corresponding affine coordinates can be treated as canonical coordinates [10].

5 Conclusions

In this paper, the parameter maps have been derived by reducing the Perron-Frobenius equations for the generalized Boole transform family without any approximation. For the parameter maps, it has been found that the statistical picture and the orbital picture are related in terms of complex variables. Since the dimension of the parameter space expressing distribution functions is even, it has been natural to discuss information geometry and symplectic geometry. Then the derived parameter map family has geometrically been characterized.

There are some potential future studies that follow from this paper. One is to apply the present approach to other solvable chaotic systems. Since this study has been restricted to a particular family of maps, it is interesting to see if this approach can be extended to other maps, including higher dimensional maps. To this end, one needs parameter maps for given (solvable) maps. However, as it can be guessed from this work, it may not be straightforward to obtain such parameter maps from given maps. Thus, to develop a general theory, a sophisticated and systematic manner is demanded for obtaining parameter maps. After establishing such a systematic manner, one will investigate a parameter map on a statistical manifold. Also, a combination of this statistical manifold and symplectic or contact manifolds is expect to be a basis for discussions. If the dimension of a derived parameter map is even, approaches based on this work on symplectic and related structures are expected to give fruitful benefits, since known tools in symplectic and related geometries will be useful for characterizing parameter maps on statistical manifolds. These structures include conformal symplectic one [25, 26]. Besides, if the dimension of the parameter map is odd, then contact geometry will be expected to play a role, since contact geometry is an odd-dimensional counterpart of symplectic geometry [20, 22].

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