In this work we study the well known contact term, which is the key element in resolving the so-called $U(1)_A$ problem in QCD. We study this term using the dual Holographic Description. We argue that in the dual picture the contact term is saturated by the D2 branes which can be interpreted as the tunnelling events in Minkowski space-time. We quote a number of direct lattice results supporting this identification. We also argue that the contact term receives a Casimir -like correction $\sim (\Lambda_{QCD} R)^{-4}$ rather than naively expected $\exp(-\Lambda_{QCD} R)$ when the Minkowski space-time $R_{4,1}$ is replaced by a large but finite manifold with a size $R$. Such a behaviour is consistent with other QFT-based computations when power like corrections are due to nontrivial properties of topological sectors of the theory. In holographic description such a behaviour is due to massless Ramond-Ramond (RR) field living in the bulk of multidimensional space when power like corrections is a natural outcome of massless RR field. In many respects the phenomenon is similar to the Aharonov -Casher effect when the “modular electric field” can penetrate into a superconductor where the electric field is exponentially screened. The role of “modular operator” from Aharonov -Casher effect is played by large gauge transformation operator $T$ in 4d QCD, resulting the transparency of the system to topologically nontrivial pure gauge configurations. We discuss some profound consequences of our findings. In particular, we speculate that a slow variation of the contact term in expanding universe might be the main source of the observed Dark Energy.
I. INTRODUCTION

The holographic picture of QCD is known to provide some deep insights into the strongly coupled dynamics. The most well known example is the dual model of gluodynamics at zero and nonzero temperatures [1], and its generalization [2] when light fermions are included into the system.

The goal of this work is to study the well known contact term which is the key element in in resolving the so-called $U(1)_A$ problem in QCD [3-5], see also [6-8]. We want to understand some deep properties of the contact term from the dual perspective.

The basic tools in this study will be the D branes which are the crucial elements of the dual description. It is well known that D0 brane extended along $x_4$ can be identified with conventional instanton [9]. At the same, the D2 brane wrapped around both periodic coordinates was identified with magnetic string [10], while D2 brane wrapped around $x_4$ was identified with D2 domain wall [11]. These two branes will play an important role in our studies. For completeness, we should also mention the configuration of the D6 brane wrapped around the compact $S^4$ part of the dual geometry was interpreted as the domain wall separating two vacua in pure gluodynamics [12]. The corresponding generalization with quarks was given in [13, 14]. A large number of other D defects from this Zoo have been discussed in [15].

The crucial element which helps to make the identifications is the study of the confinement- deconfinement phase transition which is interpreted in the dual picture as the Hawking-Page phase transition [1]. In the dual picture the wrapping around $x_4$ is stable above the phase transition $T > T_c$ as a result of cylinder geometry, while it is unstable below the critical temperature $T < T_c$ as a result of cigar type geometry, see definitions and details below. The wrapping around $\tau$ does the opposite: the corresponding D brane is stable at small temperatures $T < T_c$ and unstable at high temperatures, $T > T_c$. Since the contact term is sensitive to the $\theta$-parameter we shall also discuss the $\theta$ dependence from the dual perspective.

To formulate the problem we start from the conventional definition of the topological susceptibility $\chi$ which plays a crucial role in resolution of the $U(1)$ problem\[3-8\]

$$\chi(\theta = 0) = \frac{\partial^2 E_{\text{vac}}(\theta)}{\partial \theta^2} \bigg|_{\theta=0} = \lim_{k \to 0} \int d^4x e^{ikx} \langle T\{q(x), q(0)\}\rangle, \quad (1)$$

where $\theta$ is the $\theta$ parameter which enters the Lagrangian along with topological density operator $q(x)$, see precise definitions below.

It is important that the topological susceptibility $\chi$ does not vanish in spite of the fact that $q = \partial_{\mu}K^{\mu}$ is total divergence. Furthermore, any physical asymptotic states gives a negative contribution\(^{1}\) to this diagonal correlation function

$$\chi_{\text{dispersive}} \sim \lim_{k \to 0} \int d^4x e^{ikx} \langle T\{q(x), q(0)\}\rangle \sim \lim_{k \to 0} \frac{\langle 0 \vert q \vert G \rangle \langle G \vert q \vert 0 \rangle}{-k^2 - m_G^2} \sim -\frac{|c_G|^2}{m_G^2} \leq 0. \quad (2)$$

where $m_G$ is the mass of asymptotic state, $k \to 0$ is its momentum, and $\langle 0 \vert q \vert G \rangle = c_G$ is its coupling to topological density operator $q(x)$. At the same time the resolution of the $U(1)_A$ problem requires a positive sign for the topological susceptibility (3), see the original reference [5] for a thorough discussion,

$$\chi_{\text{non-dispersive}} = \lim_{k \to 0} \int d^4x e^{ikx} \langle T\{q(x), q(0)\}\rangle > 0. \quad (3)$$

Therefore, there must be a contact contribution to $\chi$, which is not related to any propagating physical degrees of freedom, and it must have the “wrong” sign. The “wrong sign” in this paper implies a sign which is opposite to any contributions related to the physical propagating degrees of freedom. In the framework [3] the contact term with the “wrong sign” has been simply postulated, while in refs.[4, 5] the Veneziano ghost had been introduced to saturate the required property (3). These two descriptions are equivalent as they describe the same physics.

It interesting to note that the “wrong” sign in topological susceptibility (3) is not the only manifestation of this “weird” unphysical degree of freedom. In particular, one can argue that this term also contributes with a wrong sign into the entropy. While the entropy itself is a positively defined entity, the corresponding gauge invariant contribution related to the topological susceptibility contribute with the negative sign\ [16, 17].

The goal of this paper is to investigate this “weird term” using the holographic description. We also want to study the behaviour of the system when size of then system is large but not infinitely large. The last property, as we

\(^{1}\) We use the Euclidean notations where path integral computations are normally performed.
argue below, might be relevant for cosmological applications, see section V. We also want to study the correlation between drastic changes in $\theta$ behaviour when the phase transition is crossed at $T_c$. This point has been emphasized in [9, 10, 15, 18] from holographic perspective as well as from quantum field theory (QFT) viewpoint. This correlation is also supported by direct lattice studies, see review [19] and references on the original papers therein.

The paper is organized as follows. In section II we overview the nature of the contact term from QFT viewpoint. In Section III we describe our model based on the $N_c$ D4 branes with one compact world-volume coordinate. We also overview some D- defects relevant for our analysis. We treat D defects in the probe approximation when they do not deform the dual geometry. In Section IV we formulate our proposal on contact term from holographic perspective. We present a number of arguments supporting our construction, including the comparison with the direct lattice measurements. We also argue in section IV B that the contact term receives a Casimir -like correction rather than naively expected $\exp(-\Lambda_{QCD} R)$ when the Minkowski space-time $R_{3,1}$ is replaced by a large but finite manifold with a size $R$. In holographic description such a behaviour is due to massless Ramond-Ramond (RR) field living in the bulk of multidimensional space when power like corrections is a natural outcome of massless RR field. In many respects the phenomenon is similar to the Aharonov -Casher effect when the “modular electric field” can penetrate into a superconductor where the electric field is exponentially screened. In Section V we outline some profound consequences of our findings. In particular, we speculate that a slow variation of the contact term in expanding universe might be the main source of the so called Dark Energy (DE).

II. CONTACT TERM IN QCD FROM QFT VIEWPOINT.

In this section we want to provide some intuition on the nature of the contact term using some model computations. The simplest possible example is two dimensional $QED_2$ in the Kogut-Susskind formulation [20] where all essential elements related to the contact term can be explicitly worked out as discussed in refs. [16, 21]. However, for purposes of the present work we present the relevant exact results for four dimensional “deformed QCD” formulated in [22] where all computations can be explicitly performed as the model is in a weak coupling regime, see next subsection II A. We reformulate the same results for strongly coupled 4d QCD using effective description in terms of the Veneziano ghost as originally developed in [4, 5], see section II B.

A. Contact term in weakly coupled “deformed QCD”

In the deformed theory an extra term is put into the Lagrangian in order to prevent the center symmetry breaking[22]. It can be arranged in such a way the constructed theory remains confined at high temperature in a weak coupling regime, such that there is no order parameter to differentiate the low temperature confined regime from the high temperature confined regime. Therefore, one can use this weakly coupled gauge theory to test some deep theoretical ideas about confined phase.

The topological susceptibility in this model can be explicitly computed and is given by [23]

$$\chi_{QCD} = \int d^4x \langle q(x)q(0) \rangle = \frac{\zeta}{N_c L} \int d^4x \left[ \delta(x) - m^2 \frac{e^{-m q^2 r}}{4\pi r} \right], \quad (4)$$

where the monopole fugacity $\zeta$ can be explicitly computed in this model [22]. The monopoles are pseudoparticles in this model. Therefore, the monopole fugacity $\zeta$ should be understood as number of tunnelling events per unit time per unit volume. There are two terms in formula (4). The first term represents the non-dispersive contribution related to any physical propagating degrees of freedom, while the second terms represents the conventional $q^2$ contribution. As one can see the first term contributes with sign plus while the second term contributes with sign minus in accordance with general equation (2). Another important lesson from this equation: the Ward Identity (WI) expressed as $\chi_{QCD}(m_q = 0) = 0$ is automatically satisfied as a result of cancellation between the two terms.

It is important to note that the number of tunnelling events per unit time per unit volume (4) in pure gauge theory in this model (with no quarks) is related with the absolute value of the energy density of the system. Indeed,

$$E_{YM}(\theta) = -\frac{N_c \zeta}{L} \cos \left( \frac{\theta}{N_c} \right), \quad \chi_{YM}(\theta = 0) = \frac{\partial^2 E_{YM}(\theta)}{\partial \theta^2} \bigg|_{\theta = 0} = \frac{\zeta}{N_c L}, \quad (5)$$

where we keep only the lowest branch $l = 0$ in expression for $\cos \left( \frac{\theta + 2\pi l}{N_c} \right)$ to simplify formula (5), see detail discussions with complete set of references on this matter in [23]. In different words, the contact term in pure gauge theory $\chi_{YM} = \frac{\zeta}{N_c L}$ can be interpreted in terms of number tunnelling events between different topological sectors in the
system. Therefore, there is no surprise that it has “wrong sign” as the relevant physics cannot be described in terms of propagating physical degrees of freedom, but rather, is described in terms of the tunnelling events between different (but physically equivalent) topological sectors in the system.

B. Contact term in terms of the Veneziano ghost in strongly coupled QCD

Our discussions thus far were limited to the model where all computations are justified as a result of the weak coupling regime enforced by deformation [22]. Now we want to explain the same physics, the same contact term with “wrong sign” but in different way. We want to formulate this physics in terms of the Veneziano ghost which effectively describes the dynamics of the topological sectors in strongly coupled regime. Such an alternative description in terms of the Veneziano ghost [4, 5] will play an important role in our identification of the relevant configurations in next section when we will discuss the contact term using the dual holographic description.

The topological susceptibility in the chiral limit can be easily computed in the model [4, 5] using the Veneziano ghost and it is given by [24, 25]:

$$\chi_{QCD} \equiv \int d^4x \langle T\{q(x), q(0)\} \rangle = \frac{f^2 n^2 \eta'}{4} \cdot \int d^4x \left[ \delta^4(x) - \frac{m^2_{\eta'}}{m^2_{\eta'}} D^c(m^2_{\eta'} x) \right].$$

(6)

The Green’s function $D^c(m^2_{\eta'} x)$ in this expression describes free (in the chiral limit) massive $\eta'$ field and satisfies standard normalization $\int d^4x m^2_{\eta'} D^c(m^2_{\eta'} x) = 1$.

The structure of this expression (6) is identical to our formula (4) computed in weakly coupled “deformed QCD”. In particular, the term proportional $-D^c(m^2_{\eta'} x)$ with negative sign in eq. (6) is resulted from the lightest physical $\eta'$ state of mass $m_{\eta'}$. At the same time, the contribution with “wrong sign” expressed by $\delta^4(x)$ in eq. (6) is nothing else but the contact term

$$\chi_{YM} \equiv \int d^4x \langle T\{\bar{q}(x)q(0)\} \rangle_{YM} = \frac{f^2 n^2 \eta'}{4} \cdot \int d^4x \left[ \delta^4(x) \right].$$

(7)

In the model [4, 5] this term is saturated by the Veneziano ghost contribution. The required “wrong sign” for this contribution is due to the negative sign for the kinetic term for the Veneziano ghost. In our weakly coupled “deformed QCD” when all tunnelling processes can be explicitly computed this contact term in eq. (4) is expressed in terms of the monopole’s fugacity $N_c \zeta/L$, and describes tunnelling transition between the topological sectors of the theory. It is natural to expect that the same interpretation should be also applied to dimensional factor $\frac{f^2 n^2 \eta'}{4}$ which enters (7). In different words, this factor should be also interpreted in terms of tunnelling of some “objects”. However, the nature of these objects can not be easily identified in contrast with weakly coupled “deformed QCD” where the relevant pseudo-particles are obviously the monopoles saturating the contact term. We shall see however in next section that such an interpretation still can be advocated even in strongly coupled gauge theory. However, our arguments will be based on dynamics of “dual objects” rather than on dynamics of the original gauge fields.

One should say that the main features represented by eqs. (6), (7) such as singular behaviour of the contact term with positive sign, and a smooth behaviour of conventional dispersive contribution are explicitly seen in lattice simulations [26–29]. A similar structure has been also observed in QCD by different groups [30–34] and also in two dimensional $CP^{N-1}$ model [35]. Most important part for us from these numerical studies is that the singular behaviour of the contact term is not an artifact of any approximation, but an inherent feature of underlying gauge theory. Furthermore, there are no any physical scale factors (such as $\Lambda_{QCD}$) which would determine the singular behaviour of this term. In different words, the Veneziano ghost does model the crucial property of the topological susceptibility related to summation over topological sectors in gauge theories. This feature can not be accommodated by any physical asymptotic states as it is related to non-dispersive contribution with “wrong sign”.

This concludes our discussions of the contact term within QFT approach. The corresponding computations give us a hint on the nature of the contact term, and quantum configurations which saturate it. These insights will play an important role in our analysis of the contact term in holographic description, where we attempt to identify the relevant configurations which saturate $\chi_{YM}$ in the dual picture.

III. DESCRIPTION OF THE MODEL

We start with the Witten’s background [1] for $SU(N_c)$ Yang-Mills theory in large $N_c$ limit, which is the gravity dual to the theory on $N_c$ D4 branes wrapped around a compact dimension $S^3$. In the supergravity approximation
the geometry looks as $M_{10} = R_{3,1} \times D \times S^4$ and the corresponding metric in notations [9] reads as

$$ds^2 = \left(\frac{u}{R_0}\right)^{3/2}(-dt^2 + \delta_{ij}dx^i dx^j + f(u)dx_4^2) + \left(\frac{u}{R_0}\right)^{-3/2}(\frac{du^2}{f(u)} + u^2 d\Omega_4^2)$$

$$e^\Phi = \left(\frac{u}{R_0}\right)^4, \quad f(u) = 1 - \left(\frac{u_A}{u}\right)^3, \quad R_0 = (\pi g_s N_c)^{1/3}, \quad R = \frac{4\pi}{3}\left(\frac{R_0^3}{u_A}\right)^{1/2}. \quad (8)$$

The coupling constant of Yang-Mills theory and the radius of the compact dimension $R$ are related as follows

$$g_Y^2 = \frac{8\pi^2 g_s^4 s^4}{R}, \quad \lambda = g_M^2 N_c \frac{2\pi}{l_s^4} \quad (9)$$

In formula (8) the confined phase corresponds to the geometry of a cigar in the $(u, x_4)$ coordinates with the tip at $u = u_A$. At the same time, in de-confined phase the $(u, x_4)$ coordinates exhibit the cylinder geometry. The situation is reversed when one considers $(\tau, u)$ plane instead of $(x_4, u)$ plane, where $\tau$ is the Wick-rotated time coordinate $\tau = it$, $\tau \propto \tau + \beta$. It has been argued in [1] that the phase transition occurs exactly when one geometry replaces another.

Important feature of this transition is as follows. The wrapping around the internal $x_4$ circle is topologically stable, while the wrapping around the Euclidean time coordinate $\tau$ is unstable above the phase transition point $T > T_c$. The opposite pattern occurs of the two wrappings below the phase transition at $T < T_c$. Namely, the wrapping around the internal $x_4$ circle is topologically unstable, while the wrapping around the Euclidean time coordinate $\tau$ becomes stable at $T < T_c$.

We want to discuss the topological features of the theory related to the $\theta$-dependence. The corresponding bulk field that couples to the topological density operator is the Ramond-Ramond (RR) field. The precise relation between $\theta$ parameter and RR field is known,

$$\theta = \frac{1}{l_s^4} \int dx_4 \ C_1 \quad (10)$$

where $C_1$ is the RR 1-form field. Important consequence of the relation (10) is that the $\theta$ dependence of the world-volume theories on the D branes correlates with the wrapping around $x_4$ coordinate. In different words, D brane configurations might be relevant for study the $\theta$ dependence if the corresponding construction includes the wrapping around $x_4$ coordinate. Otherwise, the D brane configurations can not be sensitive to the $\theta$ parameter.

We conclude this section with short description of D branes which will play an important role in our discussions which follow, see [15] with description of other members of this Zoo:

D0 instantons. The D0 brane extended along $x_4$ was identified as the YM instanton [9]. At $T > T_c$ this wrapping is topologically stable, while at $T < T_c$ the D0 brane tends to shrink to the tip where its tension vanishes. Nevertheless, it has been argued in [15, 18] that the D0 branes might be still important configurations at $T < T_c$.

D2 string. The magnetic string is the probe D2 brane wrapped around $S_1$ parameterized by $x_4$ and its tension is therefore proportional to the effective radius $R(u)$ [10]. At small temperatures this wrapping is topologically unstable and the D2 brane tends to shrink to the tip where its tension vanishes. The magnetic string carries nontrivial 4d topological charge.

D2 domain wall. The D2 brane wrapped around $x_4$ was identified as D2 domain wall [11]. At small temperatures this wrapping is topologically unstable and the D2 branes tend to shrink to the tip where their tension vanish. At $T > T_c$ the D2 branes are topologically stable objects and have been identified with $Z_N$ domain walls which are present in the system in high temperature phase.

D6 domain wall. If D6 brane wraps around $\tau$ it behaves as the domain wall which is a source of the corresponding RR-form. Such a configuration has been interpreted in ref. [12] as the domain wall which separates different metastable vacua known to exist in gluodynamics at large $N_c$. Its worldvolume theory on the domain wall has no $\theta$ dependence as D6 brane wraps around $\tau$ rather than around $x_4$ which was the case for D0, D2 branes mentioned above.

IV. HOLOGRAPHIC DESCRIPTION OF THE CONTACT TERM

The computation of the contact term in holographic picture is known [12, 36], see also review paper [37]. The contact term has a positive sign and has expected $N_c$ behaviour in accordance with QFT expression (7). The final
formula in holographic computations is expressed as a surface integral in 5-th dimension and in notations of section III is given by [12, 36, 37]:

$$\chi_{YM} \sim \int_{\mu_{\Lambda}}^{\infty} du \partial_u [\sqrt{g}g^{\mu\nu}h(u)\partial_u h(u)] \sim M^4 \lambda^3 N_0^0, \quad h(u) \sim f(u) = \left[1 - \left(\frac{uA}{u}\right)^3\right],$$

(11)

where \( M = \left[4\int dx_4\right]^{-1} \) is the only dimensional parameter of the problem, to be identified with \( \Lambda_{QCD} \). Function \( h(u) \) in (11) is \( 4-th \) component of the R-R field \( C_1 \) which satisfies classical equation of motion

$$\partial_\mu [\sqrt{g}g^{\mu\nu}\partial_\mu h(u)] = 0.$$  

(12)

The solution of this equation is given by expression (11), see [12, 36, 37] for the technical details.

We have nothing new to add to those computations. The goal of this section is in fact quite different: we want to understand the physics which is hidden behind these computations. In different words, we want to understand the relation between the holographic formula (11) and the QFT based computations (6, 7). As we mentioned previously, the QFT based computations (6, 7) are supported by the lattice studies [28, 30–32], including the singular behaviour of the contact term with vanishing size in the continuum limit. This singular behaviour, as it is known, leads to a finite integral contribution to \( \chi_{YM} \) with the “wrong sign” in agreement with eqs. (6, 7). The understanding of this singularity in the dual description will be the crucial element in our analysis on a possible variations of the contact term when the background slightly varies, and when formulae similar to (11, 12) are not presently available. Our findings regarding these possible variations with slight variation of a background will play a key role in the application considered in section V where we interpret a tiny deviation of the \( \theta \)-dependent portion of the vacuum energy (coded by \( \chi_{YM} \)) in expanding universe as a main source of the observed dark energy.

A. Emergence of the contact term as a tunnelling process. Percolation of D2 branes.

The presence of non-dispersive contact term (7) in topological susceptibility in pure gluodynamics obviously implies that there must be unphysical pole in gauge variant correlation function

$$\chi_{YM}^{\mu\nu}(k) \sim \lim_{k \to 0} \int d^4 x e^{ikx} \langle TK^\mu(x), K^\nu(0) \rangle \sim k^{\mu}k^{\nu} \chi_{YM}^{\mu\nu} \sim k^{\mu}k^{\nu}.$$  

(13)

While the correlation function \( \chi_{YM}^{\mu\nu}(k) \) is gauge variant itself, the merely presence of the pole, its position, its residue with “wrong sign” are physically observable parameters as corresponding correlation function (1), (7) is gauge invariant object. What is the nature of this pole? What it could be? How it can be interpreted from the holographic perspective? Normally, a pole at zero mass corresponds to a massless gauge boson. Or it might be a result of spontaneous symmetry breaking effect. What is a symmetry which could be responsible for behaviour (13)? Furthermore, as explained in section II this pole must have a residue with a “wrong sign” such that it can not be identified with any physical propagating massless degree of freedom. In QFT description this pole corresponds to the contact term which emerges due to the presence of different topological sectors in the system as argued in section II. In particular, in weakly coupled “deformed QCD” the contact term is saturated by monopoles which describe the tunnelling between those topological sectors as discussed in II A. In strongly coupled regime analytical computations based on underlying gauge fields are not reliable. However, one can argue that the contact term is saturated by the Veneziano ghost as reviewed in section II B. In this framework the Veneziano ghost should be treated as an auxiliary unphysical degree of freedom which does not belong to the physical Hilbert space. Nevertheless, it saturates topological susceptibility (6) and describes the dynamics of these degenerate topological sectors of the theory.

The QFT-based insights presented above suggest that the origin for the contact term in holographic description should be also interpreted as a result of tunnelling events rather than presence of any massless propagating degrees of freedom. The tunnelling events in path integral approach in four dimensional Euclidean space should be manifested as a presence of some “objects” which live in four dimensions. We argue below that we can identify these objects as D2 branes wrapped around \( S_1 \) parametrized by \( x_4 \). As we mentioned in section III the tension for the D2 brane vanishes as a result of cigar geometry such that the D2 brane configuration shrinks to the tip. Therefore, an arbitrary large number of D2 branes could be produced, resulting in their condensation. The term “condensation” we use here is actually a jargon, as D2 branes are 4d time-dependent objects rather than 3d static objects when term “condensation” is normally used. The “tunnelling” or “4d-percolation” probably are more appropriate terms to describe the relevant physics in this case. However, we keep the term “condensate” as it is an appropriate term when the system is viewed from 5d perspective, which is precisely what holographic description is about. We shall present below a number of arguments supporting this identification.

To begin with our arguments, we remind the reader the Bloch’s theorem in condensed matter physics which states the following. Consider a system (such as a perfect lattice) which is invariant under displacement \( x \to x + a \) symmetry.
The system in such circumstances can be described by the bands of allowed and forbidden energies classified by a quasi-momentum. Furthermore, a quasi-electron in allowed band is characterized by ungapped dispersion relation $\epsilon(k) = \frac{k^2}{2m}$ such that the corresponding correlation function has a pole at $[\epsilon(k) - \frac{k^2}{2m}] = 0$ while all information about tunnelling properties are coded by parameter $m^*$ and by residue of the Green’s function. The presence of such pole implies that the corresponding quasi-particle can freely propagate to arbitrary large distances without any disturbance from the strongly interacting lattice as long as the symmetry $x \rightarrow x + a$ is respected and no any lattice’s defects are included into the consideration. It should be contrasted with a case when an electron is placed into a system with a conventional potential barrier, in which case the correlation function will be characterized by a gapped dispersion relation with a pole at $[\epsilon(k) = \omega_n + \frac{k^2}{2m}]$ such that a particle can propagate only for a short time $\sim \omega_n^{-1}$.

Our system with winding states $|n\rangle$, quasi-momentum $\theta$ and large gauge transformation operator $T$ with property $T|n\rangle = |n + 1\rangle$ is analogous to the Bloch’s displacement symmetry $x \rightarrow x + a$ with the “only” difference that the states $|n\rangle$ are not physically distinct states, and there is no any real “physical” degeneracy in the system. The physical vacuum state in QCD is unique and constructed as a superposition of $|n\rangle$ states. Nevertheless, the presence of the pole in eq. (13) can be interpreted as a result of tunnelling between topological sectors $|n\rangle$, similar to the tunnelling events in the Bloch’s system. It also explains the “wrong” sign in residues of the correlation function (13) as we describe the tunnelling of the Euclidean D2 objects between winding states $|n\rangle$ rather than the tunnelling of conventional physical degree of freedom between distinct vacuum states in condensed matter physics in the Bloch’s case, when the residue in the Green’s function (coefficient in front of the pole) is precisely the probability of the tunnelling. It further supports our interpretation of the coefficient $f_T^2 m^2/4$ in eq. (7) as a probability of tunnelling of “some objects” which we identify now with D2 branes.

Now we can answer the question formulated in the beginning of this section: what is a symmetry which could be responsible for behaviour (13)? It is the invariance under large gauge transformations defined by the operator $T$ such that $[T, H] = 0$, see Appendix A for a technical details. We come back to this interpretation at the end of this section when we discuss analogy with Aharonov-Casher effect and similarity between the large gauge transformation operator $T$ and their “modular operator” commuting with the hamiltonian.

The tunnel-based interpretation unambiguously implies that the corresponding D-defect which could be potentially responsible for tunnelling must be constructed with wrapping around $S_1$ parametrized by $x_4$ rather than by $\tau$ coordinate. In this case the corresponding D defect becomes the $\theta$- dependent object (and therefore becomes sensitive to the transitions $|n\rangle \rightarrow |n + 1\rangle$). This wrapping around $S_1$ parametrized by $x_4$ leads to vanishing action at the tip of the cigar geometry at low temperature. The D2 branes responsible for the transitions $|n\rangle \rightarrow |n + 1\rangle$ precisely satisfy both these properties, which explains our choice for a relevant configuration in the dual description. As we mentioned at the end of section III one type of the D2 branes was identified earlier in ref. [10] as magnetic strings. Another type of the D2 branes was identified as D2 domain walls [11]. The D2 magnetic strings carry 4d topological charge by construction, while D2 domain walls carry $Z_N$ charge [11]. Nevertheless, these D2 domain walls in combination with D0 branes in strong coupling regime at $T < T_c$ are capable to carry 4d topological charge. Indeed, the corresponding D0-D2 configurations can be explicitly constructed in some supersymmetric models [38] where analytical QFT based computations can be also performed. Similar D0-D2 configurations may also emerge in strongly coupled QCD at $T < T_c$ as argued in [15].

Therefore, in what follows we shall not discriminate between these two options in our discussions as lattice studies can not presently precisely determine the dimensionality of fluctuating quantum objects. In particular, in ref. [34] it has been argued that these long range configurations actually might be characterized by Hausdorff dimension which gradually varies with cooling procedure. Therefore, we shall use a single term “D2 brane” to describe these objects in spite of the fact that these configurations are physically different. We expect that some future, more refine lattice studies will allow to discriminate between those two options (or rule out both of them).2

The recent Monte Carlo studies of Yang Mills theory have revealed a laminar structure in the vacuum consisting of extended, thin, coherent, locally low-dimensional sheets of topological charge embedded in 4d space, with opposite sign sheets interleaved, see original QCD lattice results [26-29]. One should add that a similar structure has been also observed in QCD by a different groups [30-34]. It has been also observed in two dimensional $CP^{N-1}$ model [35]. We do not make any attempts in this work to cover this subject which has a number of subtle points. Instead, we concentrate on very few key properties of the gauge configurations which apparently make crucial contributions to the $\theta$ dependent portion of the energy expressed in terms of the topological susceptibility (7). One can argue that such a complex structure observed on the lattice can be thought as a result of dynamics of the D2 branes described above3.

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2 I am thankful to Ivan Horvath for his comments on present measurements of dimensionality of the objects observed on the lattice simulations, and on future progress which should be expected as a result of these measurements.

3 Similar idea with interpretation of the D6 branes in holographic description being responsible for topological susceptibility was re-
Indeed, it has been observed that the tension of the “low dimensional objects” vanishes below the critical temperature and these objects percolate through the vacuum, forming a kind of a vacuum condensate. Furthermore, these “objects” do not percolate through the whole 4d volume, but rather, lie on low dimensional surfaces $1 \leq d < 4$ [27]. The total area of these surfaces is dominated by a single percolating cluster of “low dimensional object”. From holographic perspective these Monte Carlo results fits very nicely with conjecture that the observed structure can be identified with the D2 branes described above. In particular the tension of the D2 branes vanishes in confined phase as a result of cigar geometry in the holographic 5d description such that the D2 brane shrinks to the tip as we already mentioned. Therefore, an arbitrary large number of D2 branes could be produced, resulting in their condensation. The observed percolation (condensation) of these objects in lattice simulations unambiguously imply that they must have vanishing effective tension. Otherwise, only finite, not infinite percolating clusters could be observed, in contrast with observations.

Another measurement suggests that the contribution of the percolating objects to the topological susceptibility $\chi_{YM}$ has the same sign compared to its total value. Furthermore, both contributions have opposite sign in comparison with any propagating physical states (2), (6). These lattice studies are consistent with non-dispersive nature of D2 branes, which can not be related to any propagating physical degrees of freedom. It unambiguously implies that the D2 branes should be thought as configurations describing the tunnelling events in Minkowski space, rather than as a real physical domain walls.

Furthermore, the width of the “objects” apparently vanishes in the continuum limit. These measurements, again, support the tunnelling interpretation of the contact term. Indeed, it is well known property of conventional quantum mechanics that the individual photons penetrate an optical tunnel barrier with an effective group velocity considerably greater than the vacuum speed of light [42], see also review [43]. In different words, the tunnelling time can not be distinguished (experimentally) from zero. In quantum field theory context it means that the configurations responsible for tunnelling events must have vanishing size (in continuum), which is precisely what has been measured on the lattices. The same feature can be also seen on Fig 5 of ref.[32] where the contact term with vanishing size (in continuum) has a “wrong sign” in Euclidean lattice simulations. This again supports the tunnelling interpretation of this term in Minkowski space.

### B. Variation of the contact term with small variation of the background

Our previous section IV A was devoted to the discussions of the contact term in QCD from holographic perspective. Essentially, we have not produced any new results in that section as the presence of the contact term is well established phenomena (supported by the lattice studies). We simply suggested a new interpretation of this term. However, the new interpretation may lead to some new physical results as we shall argue below.

Main question we address here is related to a possible variation of the contact term when the background is slightly changed. For example, we wish to know what happens when infinite time coordinate $\tau \in (-\infty, +\infty)$ is replaced by a finite size ring $\tau \in (0, 2\pi R)$, or what happens when the Minkowski space-time $R_{3,1}$ is replaced by FRW metric characterized by dimensional parameter $R \sim H^{-1}$ describing the size of the visible universe with $H$ being the Hubble constant.

Normally, one should not expect any strong dependence on very large distances as QCD has a mass gap $\sim \Lambda_{QCD}$. Therefore, one should expect the exponential suppression $\sim \exp(-\Lambda_{QCD}R)$ for any local observables. Formally, this expectation follows from the dispersion relation similar to (2) written in coordinate space which explicitly shows an exponentially weak sensitivity $\sim \exp(-\Lambda_{QCD}R)$ to large distances. The main point of this paper is that along with conventional dispersive contribution (2) there is also non-dispersive contribution (3) which is not related to any physical propagating degrees of freedom, and which can not be computed using the dispersion relations. This contribution as we discussed above emerges as a result of topologically nontrivial sectors in four dimensional QCD, and it may lead to a power like corrections $R^{-p}$ with power $p \geq 1$ rather than exponential like $\exp(-\Lambda_{QCD}R)$. In fact, this term in our framework is described by topologically protected massless ghost field (13) as discussed in section IV A. Therefore, in this framework, it is indeed quite natural to expect a power like corrections $R^{-p}$.

We want to get some insights on this problem from the holographic perspective. We start our discussions with few general remarks. In principle, when we slightly change the geometry, one should compute the corresponding changes in the vacuum energy as a result of changes of the boundary conditions. The corresponding changes represent the

[39] One should remark that these D6 domain walls are in fact $\theta$-independent configurations according to the classification presented in section III, and they can not interpolate between different topological sectors $|n\rangle$ and $|n+1\rangle$. In different words, these D6 domain walls interpolate between the ground state and an excited local minimum characterized by a small extra energy density $\Delta E \sim 1/N_c$. These domain walls are similar to the physical static domain walls in ferromagnetic systems interpolating between physically distinct vacua. These D6 domain walls have finite action and finite width in contrast with thin tensionless D2 branes. Such domain walls in QCD within QFT approach in the presence of light quarks in large $N_c$ limit when the $\eta'$ is light $m_{\eta'}^2 \sim 1/N_c$ had been previously studied in [40].
Casimir vacuum energy in multidimensional space due to new specific boundary conditions which were trivial ones in previous computations (11), (12) with infinite Minkowski space-time background. Technically, it could be very challenging problem as it requires the computation of a corresponding multidimensional Green’s function in curved background which satisfies specific boundary conditions. However, on a general ground we expect that the contact term (11) receives a power-like correction \( \sim \epsilon^p \), not exponentially small corrections \( \sim \exp(-1/\epsilon) \) if deformation of the geometry parametrized by small parameter \( \epsilon \sim (R\Lambda_{\text{QCD}})^{-1} \ll 1 \).

Indeed, the computations of the Casimir vacuum energy in 5-dimensional space \((x_\mu, u)\) can be accomplished by using effective Lagrangian describing the massless RR (10) or axion field living in the bulk of multidimensional space. This field \(C_1\) is responsible for the contact term (11). More importantly, the axion field is massless in multidimensional space as a consequence of corresponding gauge invariance. Therefore, while a computation of the Casimir vacuum energy in 5-dimensional space \((x_\mu, u)\) could be very challenging technical problem, an estimation of this energy is simple: we expect a power-like correction to the vacuum energy as a natural consequence of the dynamics of a massless axion field living in the bulk of multidimensional space, which is responsible for the contact term.

We will test this argument (on power like correction) below by replacing infinite Euclidean time coordinate \(\tau\) \((-\infty, +\infty)\) by a finite size ring \(\tau \in (0, 2\pi R)\). As is known, sufficiently small parameter \(R\) corresponds to large temperatures \(T \gg T_c\) where QFT based computations are justified. After that we apply the same argument in strong coupling regime where QFT based analytical computations are not available. Nevertheless, we will quote some recent lattice numerical results supporting our argument.

First, we consider de-confined phase when the length of the ring \(S^1\) is quite small and order of \(T^{-1}\). The contact term is expected to be exponentially suppressed at \(T > T_c\) in de-confined regime as D0 brane wrapping around \(x_4\) has a finite tension, saturates the topological susceptibility, and leads to the exponentially small contact term [9, 18].

This picture can also be easily understood from QFT viewpoint [15, 18]. Indeed, the wrapping around \(x_4\) corresponds to the well defined small instanton and one can use the standard instanton calculus to estimate the critical temperature \(T_c\) and the \(\theta\) behaviour above \(T_c\):

\[
V_{\text{inst}}(\theta, T > T_c) \sim \cos \theta \cdot e^{-\alpha N_c \left(\frac{T - T_c}{T_c}\right)}, \quad 1 \gg \left(\frac{T - T_c}{T_c}\right) \gg 1/N_c, \\
\chi_{YM}(T > T_c) \sim \frac{\partial^2 V_{\text{inst}}(\theta, T)}{\partial \theta^2} \sim e^{-\alpha N_c \left(\frac{T - T_c}{T_c}\right)} \rightarrow 0, \quad \alpha \sim 1, \quad N_c \gg 1. \tag{14}
\]

Such a behaviour implies that the dilute instanton gas approximation at large \(N_c\) is justified even in close vicinity of \(T_c\) as long as \(\frac{T - T_c}{T_c} \gg \frac{1}{N_c}\). Such a sharp drop of the topological susceptibility \(\chi_{YM}(T)\) is supported by numerical lattice results [19] which unambiguously suggest that the topological fluctuations are strongly suppressed in de-confined phase, and this suppression becomes more severe with increasing \(N_c\). The behaviour (14) in de-confined phase should be contrasted with formula (11) corresponding to the confined phase when \(\chi_{YM}(T = 0) \sim N_c^0\).

Now we want to address the following question: how does the contact term \(\chi_{YM}\) vary when the radius of the ring \(S^1\) slightly changes? We want to study these changes by remaining in a deep de-confined phase \((T \gg T_c)\), such that no drastic variations are expected to occur. One can easily answer this question by noticing that small variation in \(\tau \rightarrow \tau + \Delta \tau\) corresponds to a small variation of the temperature \(T \rightarrow T - \Delta T\) with \(\Delta T = (2\pi \Delta \tau)^{-1}\) while keeping \(T_c\) unchanged. Therefore, a small variation of the background \(\tau \rightarrow \tau + \Delta \tau\) results in a small (power-like) variation of the contact term

\[
\chi_{YM}(T - \Delta T) \sim e^{-\alpha N_c \left(\frac{T - \Delta T - T_c}{T_c}\right)} \sim \chi_{YM}(T) \cdot \left[1 + \alpha N_c \frac{\Delta T}{T_c}\right], \quad \Delta T/T_c \ll 1, \quad T_c \ll T. \tag{15}
\]

Therefore, a small increase of the the radius of the ring \(S^1\) leads to a small increase of the contact term reflected by eq. (15). The contact term obviously receives a power-like correction \(\sim \epsilon\) when the geometry is slightly changed \(\sim \epsilon\),

\[
\frac{\chi_{YM}(T - \Delta T) - \chi_{YM}(T)}{\chi_{YM}(T)} \sim \epsilon, \quad \text{where} \quad \epsilon \equiv \frac{\Delta \tau}{\tau} \sim \frac{\Delta T}{T}, \quad \Delta \tau \ll \tau. \tag{16}
\]

The correction (16) obviously supports our general argument presented above on power like corrections \(\sim \epsilon^p\) when manifold slightly changes as the variation of the manifold is explicitly translated in changes of the temperature in conventional Euclidean formulation.

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4 We use term “axion” which is a jargon here. There is no real new dynamical degree of freedom in 4d space such as axion, see reviews [41] about the physical 4d axion which is as real degree of freedom and the dark matter candidate. However, the \(\theta\) dependent portion of the vacuum energy in holographic picture is determined by the dynamics of the axion field living in multidimensional space. Precisely the dynamics of the axion field determines the magnitude of the integral (11) which corresponds to the physically observable topological susceptibility \(\chi_{YM}\). This relation of the axion field from multidimensional space with \(\chi_{YM}\) justifies our terminology.
Now, we want to study a similar variations of the contact term $\chi_{YM}(T - \Delta T)$ with slight change of radius of the ring $S^1$ but in confined phase with $T \ll T_c$. This corresponds to a strong coupling computations when no analytical formula similar to (14) exists in this regime. From the holographic perspective, however, the the result must be very similar to the previously discussed case (16), as the contact term (11) and corresponding vacuum energy, receives the power like corrections $\sim \sum_{p \geq 1} c_p \epsilon^p$, not exponentially small corrections $\sim \exp(-1/\epsilon)$ as all changes are governed by the massless axion field field living in the bulk of multidimensional space. This argument is consistent with the direct lattice computation [44] which indeed finds a power like corrections $R^{-p}$ with $p \approx 2.34$ for smallest lattice size $R = 12$ and $p \approx 1.33$ for the largest available lattice size $R = 24$ in lattice units. Such a behaviour should be contrasted with naive expectation $\exp(-\Lambda_{QCD} R)$ based on conventional dispersion relations without an accounting for the non-dispersive term.

We should emphasize that our general argument does not specify the power $p$ for the Casimir-like correction $\sim \epsilon^p$. In general, one could expect any integer number $p \geq 1$ as QFT based computations would suggest. Indeed, if 2d QED model is defined on a generic torus determined by Teichmuller complex parameter $\tau = \tau_1 + i\tau_0$ the correction behaves linearly in inverse size, $\sim R^{-1}$. The suppression becomes much stronger $\sim R^{-2}$ (but still, not exponential) in special symmetric case when metric $g_{\mu\nu} \sim \text{diag}(1,1)$ [45]. Also, computations of the vacuum energy in 2d $CP^{N-1}$ model defined on a finite interval of length $R$ with Dirishlet boundary conditions at large $N$ also exhibits the correction $\sim R^{-1}$ [46]. A similar effect occurs in “deformed QCD” model discussed in section II A when the theory is formulated on $S^1 \times R^3$ and infinitely large space $R^3$ is replaced by a sphere $S^3$ with radius R. The correction behaves in this case as $1/R$ [47]. One should expect $R^{-2}$ correction for a symmetric infrared regularization when infinitely large space $R^4$ is replaced by a sphere $S^4$ [47]. We suspect that a similar variation of power $p$ would also emerge in holographic description, however the corresponding study is beyond the scope of the present work. The experience with QFT computations suggest that a generic asymmetric background leads to a minimal $p = 1$ suppression, while more symmetric cases lead to a higher power of suppression $p \geq 2$. To conclude: all these simple calculable models with a gap exhibit the power like corrections, in contrast with naive expectation that there should exponentially weak $\sim \exp(-R)$ sensitivity to a size of the system $R$.

**Few more comments:**

We want to avoid any confusion with the terminology. In what follows, we shall use the term “topological Casimir effect” to discriminate it from conventional Casimir effect when power like behaviour is due to the real massless propagating degrees of freedom living in 4 dimensional space, such as $E\&M$ photons, in huge contrast with our case when there are no massless asymptotic states in QCD. As we mentioned previously, this “topological Casimir effect” occurs in gauge theories when nontrivial “degenerate” topological sectors are present in the system. For the present work it is important that the same “topological Casimir effect” in 4d can be thought from the dual holographic description as conventional Casimir effect in 5 dimensional space as a result of massless axion field living in the bulk of multidimensional space.

The “topological Casimir effect” obviously is very unnatural and unexpected effect which begs for a simple intuitive explanation. A formal explanation was given in the text and was formulated in terms of the topologically protected pole at $k^2 = 0$, see eq. (13) within QFT framework. A similar formal explanation in dual holographic description is given in terms of the massless axion field field living in the bulk of multidimensional space. As this effect plays a crucial role in the application considered in section V, we want to present here few other systems where a similar phenomena occurs, and where it has precisely the same nature. Furthermore, in these systems a similar problem can be exactly solved (in drastic contrast with strongly coupled 4d QCD). Most importantly, analogous effects in these systems have been experimentally observed.

Our first example is from quantum mechanics (QM) and it is the well known Aharonov -Casher effect as formulated in [48]. The relevant part of this work can be stated as follows. If one inserts an external charge into superconductor when the electric field is exponentially suppressed $\sim \exp(-r/\lambda)$ with $\lambda$ being the penetration depth, a neutral magnetic fluxon will be still sensitive to an inserted external charge at arbitrary large distance. The effect is pure topological and non-local in nature. The crucial element why this phenomenon occurs in spite of the fact that the system is gapped is very similar to what we discussed in the present work. First of all, it is the presence of different topological states $u_n$ (number of Cooper pairs) in the system and “tunnelling” between them (non-vanishing matrix elements between $u_n$ and $u_{n+1}$ states) as described in [48]. Those states are analogous to the topological sectors $|n\rangle$ in our work. As a result of the “tunnelling”, an appropriate ground state $U(\theta)$ must be constructed as discussed in [48], analogous to the $|\theta\rangle$ vacuum construction in gauge theories. This state $U(\theta)$ is an eigenstate of the so-called “modular operator” which commutes with the hamiltonian. In our work an analogous role plays the large gauge transformation operator $T$ such that $T|\theta\rangle = \exp(-i\theta)|\theta\rangle$. An explicit construction of the operator $T$ is known: it is non-local operator similar to non-local “modular operator” from ref. [48], see Appendix A for some technical details. The crucial element of the construction of ref. [48] is that the induced charges in presence of the gap can not screen the “modular charge” as a result of commuting the “modular operator” with hamiltonian. This eventually leads to a non-vanishing effect, i.e. collecting a non-zero phase at arbitrary large distance. We do not have a luxury to resolve the problem using
the hamiltonian description in strongly coupled four dimensional QCD. However, one can argue that the role of "modular operator" is played by large gauge transformation operator $T$ which also commutes with the hamiltonian $[T, H] = 0$, such that our system must be transparent to topologically nontrivial pure gauge configurations, similar to transparency of the superconductor to the "modular electric field" from ref. [48]. Such a behaviour of our system can be thought as a non-local topological effect similar to the non-local Aharonov-Casher effect as formulated in [48], see also few comments in Appendix A on this similarity. The last word whether this analogy can be extended to the strongly coupled four dimensional QCD remains, of course, the prerogative of the direct lattice computations similar to recent studies [44].

Our second example is description of a superconductor using the so-called topological "BF" action as presented in [49]. This QFT description deals exactly with the same question on how does a moving vortex detect a stationary charge, given that the electric field is exponentially screened. Paper [49] explicitly shows how pure gauge (but topologically nontrivial) field may lead to long range effects. In many respects this description is very similar to our description in terms of unphysical ghost degrees of freedom when the topological features of the system are represented by auxiliary fields which are responsible for the dynamics of the degenerate ground state. Our final example is description of topological insulators in terms of topological "BF" action as presented in [50]. In this case, again, some pure gauge, but topologically nontrivial auxiliary fields may penetrate into gapped insulator to produce an effective massless mode on the surface of a sample. Such a behaviour of the system can be thought as a non-local topological effect similar to the previously considered example [48].

We conclude this section with the following comment. Our interpretation of dynamics of the system, when it is described in terms of the different topological sectors $|n⟩$ and large gauge transformation operator $T$ commuting with the hamiltonian $[T, H] = 0$, is obviously a gauge variant interpretation, see Appendix A with some technical details. As we already mentioned previously, there is no any physical degeneracy in QCD, in contrast with condensed matter examples when distinct physical degenerate quantum states are present in the system. Nevertheless, such gauge-dependent interpretation helps us to understand a number of very nontrivial features of the contact term which are known to be present in lattice Monte Carlo simulations.

If, instead, we consider a physical Coulomb gauge when all unphysical degrees of freedom are removed from the system, the corresponding physics related to nontrivial topological structure of the gauge fields does not go away. Instead, it will reappear in terms of the so-called Gribov copies leading to a strong infrared (IR) singularity. Precisely this IR behaviour due to the Gribov copies was the crucial element in the approach advocated in ref.[44] where power-like correction $R^{-p}$ has been observed in lattice simulations. Similarly, the construction of the ground state in exactly solvable 2d QED in physical Coulomb gauge, as originally discussed in [20], is characterized by long range forces. This long range force prevents distant regions from acting independently. This is essentially the same manifestation of long range forces in physical Coulomb gauge in the absence of unphysical ghost. We opted to interpret the results in gauge-dependent framework in terms of different topological sectors $|n⟩$ and large gauge transformation operator $T$ where one can use analogy with condensed matter systems. Alternative option is lattice numerical simulations where entire notion of vacuum winding states $|n⟩$ and large gauge transformation operator $T$ do not even exist as there is a unique ground state. However, irrespectively to the interpretation, the last word in strongly coupled QCD is expected from the direct lattice simulations which eventually should confirm or rule out our arguments suggesting that the sensitivity to arbitrary large distances should be power like $R^{-p}$ rather than exponential like $\exp(-Λ_{QCD}R)$ as a consequence of nontrivial topological properties of strongly coupled QCD.

The most important result of this section which plays a crucial role in the application considered in next section can be formulated as follows. The contact term (and corresponding $θ$ dependent portion of the vacuum energy density) may receive power like corrections $\sim R^{-p}$ in QCD despite the presence of a mass gap in the system. From $4d$ viewpoint this is the "topological Casimir effect" when no massless degrees of freedom are present in the system, and effect within QFT framework can be explained as a result of degenerate topological sectors in the theory represented by topologically protected unphysical pole at $k^2 = 0$, see eq. (13). The same effect from $5d$ holographic description can be explained in terms of the massless axion field living in the bulk of multidimensional space. From $5d$ viewpoint this is a "conventional Casimir effect" due to a physical (in multidimensional space) massless axion.

V. CONTACT INTERACTION IN QCD AND PROFOUND CONSEQUENCES FOR EXPANDING UNIVERSE

This portion of the paper which is the direct manifestation of the “topological Casimir effect” discussed in previous section is much more speculative in nature than the previous sections. The corresponding speculation on profound consequences for cosmology of this effect have been already formulated in author’s previous works. However, there are two new elements on application of the “topological Casimir effect” to cosmology which were not discussed previously, and will be elaborated here. First of all, however, we need to make a short overview of this proposal.
The $\theta-$ dependent portion of the energy which is not related to any physical propagating degrees of freedom, is well established effect. It has been tested a numerous number of times in the lattice simulations. How does this energy change when the background slightly varies? The main motivation for this question is as follows. We adopt the paradigm that the relevant definition of the energy which enters the Einstein equations is the difference $\Delta E \equiv (E - E_{\text{Mink}})$, similar to the well known Casimir effect. Such a definition of the “physical energy” in fact is used in description of the horizon’s thermodynamics [51, 52] as well as in a course of computations of different Green’s function in a curved background [53]. In the present context such a definition $\Delta E \equiv (E - E_{\text{Mink}})$ for the vacuum energy for the first time was advocated in 1967 by Zeldovich [54], see also [16] with a large number of related references.

As we mentioned previously, a naive expectation suggests that $\Delta E \sim \exp(-\Lambda_{\text{QCD}}/H) \sim \exp(-10^{41})$ as QCD has a mass- gap $\sim \Lambda_{\text{QCD}}$, and therefore, $\Delta E$ must not be sensitive to size of our universe. In this estimate we do not distinguish the size of the visible universe $\sim H^{-1}$ from size $R$ of a compact manifold we used in our analysis in previous section. A crucial distinct feature which is important for our discussions is the presence of dimensional parameter $R \sim H^{-1}$ in a system which discriminates it from infinitely large Minkowski space-time.

This naive argument, however, may fail as a result of nontrivial topological properties of QCD as we discussed above. It may lead to a power like scaling $\Delta E \sim H^p$ rather than exponential like $\Delta E \sim \exp(-\Lambda_{\text{QCD}}/H)$. From holographic viewpoint such power like scaling $\Delta E \sim H^p$ follows from the fact that the contact term (or, what is the same, the $\theta$ dependent portion of the energy) is determined by massless axion field. It is naturally to expect that massless axion field produces power like corrections in holographic description as discussed in section IV B such that surface term (11) which determines the magnitude of the contact term receives a power like corrections when the background is slight modified. From holographic perspective, one can view this power like correction as a conventional Casimir effect in multidimensional space. However, in our four dimensional space this correction should be interpreted as the “topological Casimir effect” as there are no any physical massless asymptotic states in gapped QCD.

The power like scaling $\Delta E \sim H^p$ with $p = 1$ (as some explicit QFT computations suggest, see section IV B) may have some profound consequences for evolution of our universe. If true, the difference between two metrics (expanding universe and Minkowski space-time) would lead to an estimate

$$\Delta E \sim H \Lambda_{\text{QCD}}^3 \sim (10^{-3}\text{eV})^4,$$

(17)

which is amazingly close to the observed DE value today. Furthermore, the power like scaling (17) with $p = 1$ would imply that our universe approaches the de-Sitter state with constant expansion rate $H_{\infty} \simeq G \Lambda_{\text{QCD}}^3$ in asymptotic future [16]. Such a behaviour $\Delta E \sim H + \mathcal{O}(H)^2$ was postulated in [24]. This proposal has received some theoretical QFT based support in [16, 21, 47] where it was argued that power like scaling indeed may emerge as a result of non-dispersive nature of contact term (3), in contrast with conventional dispersive term (2) related to physical propagating degrees of freedom. This correction does not violate unitarity, causality and other important QFT properties.

We arrive to the same conclusion in the present work using the dual holographic arguments. In the dual picture the DE in this framework is due to the Casimir effect in multidimensional space as a result of the dynamics of the massless axion field in expanding universe. From four dimensional view point this effect should be interpreted as a “topological Casimir effect” when no massless asymptotic states are present in the system. A comprehensive phenomenological analysis of this model has been recently performed in [55] where comparison with current observational data including Sula, BAO, CMB, BBN has been presented, see also [56–61] with related discussions. The conclusion was that the model (17) is consistent with all presently available data, and we refer the reader to these papers on analysis of the observational data.

**CONCLUSIONS**

The main result of this work can be formulated as follows. We studied a number of different ingredients related to $\theta$ dependence, the non-dispersive contribution in topological susceptibility with the “wrong sign”, topological sectors in gauge theories, and many related subjects. We argued that the corresponding physics in holographic description should be interpreted as a result of tunnelling events governed by the dynamics of the D2 branes. We quoted a number of theoretical as well as numerical results supporting this interpretation. Essentially, we interpreted the holographic formula (11) in terms of a singular behaviour of the contact term (7). The diverging nature of the core and vanishing width (in the continuum limit) of this non-dispersive contribution to $\chi$ is supported by recent lattice studies. After all, it is not really a huge surprise that a vanishing width of the contact term (confirmed by the lattice studies) is saturated by the objects (also observed on the lattices) which have vanishing sizes and which lead to a superluminal behaviour (when interpreted in Minkowski space-time), see original lattice results in [26–34].

However, we believe that the most important result of our work is not just a new interpretation of old results [3–5] related to the resolution of the $U(1)_A$ problem in QCD. Rather, the holographic description allows us to test the
sensitivity of the gauge theory with non-trivial topological features to arbitrary large distances. A naive expectation based on dispersion relations dictates that a sensitivity to very large distances must be exponentially suppressed when the mass gap is present in the system. However, we argued that along with conventional dispersive contribution there exists a non-dispersive contribution, not related to any physical propagating degrees of freedom. This non-dispersive (contact) term with the “wrong sign” emerges in QFT-based framework as a presence of topologically nontrivial sectors and tunnelling events between them. Technically, it is formulated in terms of unphysical massless ghost field \( \lambda \) effectively describing the dynamics of these tunnelling transitions. In dual holographic description the same dynamics is described by the massless axion field living in the bulk of multidimensional space. From 4d viewpoint, the variation of this contact term with variation of the background leads to a power like “topological Casimir effect”. The same effect, but viewed from 5 dimensional perspective can be thought as conventional Casimir effect which is a consequence of dynamics of the axion massless field.

From 4d viewpoint the transparency of the QCD vacuum to topologically nontrivial gauge configurations is similar to transparency of a superconductor to the “modular electric field” in the Aharonov-Casher effect as discussed at the end of section IV B, see Appendix A with some technical details on this similarity. In both cases an exponential suppression of the effects is avoided as a result of topological properties of gauge theories, and can be interpreted as a nonlocal effect. We suspect that there should be a holographic description of the Aharonov-Casher effect when its “modular electric field” is represented by the axion-like massless field living in the bulk of multidimensional space. We leave this subject for the future studies.

The “topological Casimir effect” in QCD, if confirmed by future analytical and numerical studies, may have profound consequences for understanding of the expanding universe we live in. Finally, what is perhaps more remarkable is the fact that some elements of this framework can be, in principle, experimentally tested in heavy ion collisions, including the Casimir like \( L^{-\nu} \) scaling behaviour. The parameter \( L \) in heavy ion collisions describes a size of the region where the QCD vacuum is disturbed as a result of collision. Dependence on \( L \), in principle, can be experimentally studied as \( L \) is related to the physical size of the colliding ions, see [25, 62] for the details.

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Appendix A: The large gauge transformation operator \( \mathcal{T} \).

The goal of this Appendix is to give a short overview of properties of the \( \mathcal{T} \) operator which was essential element in our discussions of the physical interpretation of the contact term. We also want to present some additional arguments demonstrating striking similarity between \( \mathcal{T} \) operator and “modular operator” in the Aharonov-Casher effect as mentioned at the end of section IV B.

The basic properties of the \( \mathcal{T} \) operator were analyzed in late seventies and we follow classical paper [63] in description of some features of this operator in the context of the present work. The starting point is the construction of the unitary operator \( \mathcal{T} \),

\[
\mathcal{T} = e^{iq\lambda}, \quad Q\lambda \equiv \int d^3x \lambda^a \left\{ \nabla_i \lambda^a + f_{abc} A_i^b \lambda^c \right\}, \quad (A1)
\]

where we literally use notations of ref. [63]. An arbitrary function \( \lambda^a(x) \) in this formula is a c-function which is a parameter of the gauge transformations. In particular, if \( \lambda^a(x) \) vanishes at spatial infinity one can represent the operator \( Q\lambda \) in form

\[
Q\lambda = -\int d^3x \lambda^a(x) \left( \nabla_i \dot{A}_i^a + f_{abc} A_i^b \dot{A}_i^c \right) = -\int d^3x \lambda^a(x) C^a(A), \quad (A2)
\]

which obviously annihilates all physical states as the term in parentheses is just Gauss’s law \( C(A)|\mathcal{H}_{phys} \rangle = 0 \). However, if \( \lambda^a(x) \) describes the large gauge transformation, the operator \( \mathcal{T} \) becomes nontrivial operator and describes
the transition between topologically different sectors of the theory:

$$\mathcal{T}|n\rangle = |n+1\rangle.$$  \hspace{1cm} (A3)

In this case one should construct the so-called $|\theta\rangle$ vacuum state which is an eigenstate of the $\mathcal{T}$ operator:

$$|\theta\rangle = \sum_n e^{i n \theta}|n\rangle, \hspace{1cm} \mathcal{T}|\theta\rangle = e^{-i \theta}|\theta\rangle.$$  \hspace{1cm} (A4)

It is important to emphasize that while operator $\mathcal{T}$ formally constructed as an operator of gauge transformations, this operator does change the state (A3) as a result of global effect. Therefore, one should treat $\mathcal{T}$ as “improper” gauge transformation (the “large gauge transformation”). Still, $\mathcal{T}$ commutes with the hamiltonian $[\mathcal{T}, H] = 0$. As we mentioned in the text, there is no any physical degeneracy in QCD, in contrast with condensed matter examples when distinct physical degenerate quantum states are present in the system. Nevertheless, the gauge-dependent interpretation in terms of the tunnelling between $|n\rangle$ and $|n+1\rangle$ states is quite useful to develop an intuitive picture of relevant physics, which otherwise hidden in the numerical lattice Monte Carlo simulations with such puzzling elements as the “wrong sign” contributions with vanishing width as discussed in section IV.

While this paper is mainly devoted to 4d QCD it might be instructive to present the large gauge transformation operator $\mathcal{T}_{2d}$ for 2d QED as well. In this case, as it is known, the nontrivial topological structure emerges as a result of nontrivial mapping $\pi_1(U(1)) = \mathbb{Z}$ which replaces the classification of the topological sectors in QCD which was based on $\pi_3(SU(N)) = \mathbb{Z}$ mapping. In 2d case a similarity between the “modular electric field” in the Aharonov-Casher effect and operator $\mathcal{T}_{2d}$ becomes much more apparent. Indeed, the large gauge transformation operator $\mathcal{T}_{2d}$ for 2d QED can be represented as follows

$$\mathcal{T}_{2d} = e^{-2\pi ie E(\infty)} \cdot \frac{1}{\theta} \int_{\infty}^{-\infty} dx \lambda(x) C(A)$$  \hspace{1cm} (A5)

where $g$ is the coupling constant in 2d QED, and $E = -A_x$ is electric field. Formula (A5) was obtained by integrating by parts the two-dimensional analog of eq. (A1) and taking into account that the large gauge transformation in this case is characterized by $\lambda(x = \infty) - \lambda(x = -\infty) = 2\pi$. One can construct $|\theta\rangle$ state similar to (A4) with result that the expectation value of large gauge transformation operator $\mathcal{T}_{2d}$ is given by

$$\langle \theta | \mathcal{T}_{2d} | \theta \rangle = e^{-i \theta}, \hspace{1cm} \theta = 2\pi g E_x(\infty).$$  \hspace{1cm} (A6)

The fact that in 2d QED the electric field at infinity plays the role of the $\theta$ parameter of course is well known classical result [64]. A similarity of the expectation value (A6) with the Aharonov-Casher effect as presented in ref.[48] is striking as $\theta$ parameter in this case is almost identical to the “modular electric field” from ref.[48], with the “only” difference that $|n\rangle$ states in 2d QED are not physically distinct degenerate states as we already emphasized, in huge contrast with physical degeneracy in condensed matter systems discussed at the end of section IV.B. Eventually, this difference leads to the “wrong sign” in topological susceptibility as we argued in this work. The last word whether this analogy can be extended to the strongly coupled four dimensional QCD remains, of course, the prerogative of the direct lattice computations similar to recent studies [44].

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