Driven tracer dynamics in a one dimensional quiescent bath

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Abstract
The dynamics of a driven tracer in a quiescent bath subject to geometric confinement models a broad range of phenomena. We explore this dynamics in a 1D lattice model, where geometric confinement is tuned by varying the rate of particle overtaking. Previous studies of the model’s stationary properties on a ring of \(L\) sites have revealed a phase in which the bath density profile extends over an \(\sim \mathcal{O}(L)\) distance from the tracer and the tracer’s velocity vanishes as \(\sim 1/L\). Here, we study the model’s long time dynamics in this phase for \(L \to \infty\).

We show that the bath density profile evolves on a \(\sim \sqrt{t}\) time-scale and, correspondingly, that the tracer’s velocity decays as \(\sim 1/\sqrt{t}\). Unlike the well-studied case of a non-driven tracer, whose dynamics becomes diffusive whenever overtaking is allowed, we here find that driving the tracer preserves its hallmark sub-diffusive single-file dynamics, even in the presence of overtaking.

Keywords: single-file, nonequilibrium transport, anomalous-diffusion

1. Introduction

The motion of a passive tracer, or tagged particle, in a bath of identical hard-core particles confined to a narrow environment models a broad range of phenomena. Such scenarios are abundant in biological systems, with examples including transport in porins [1], through the nuclear pores in eukaryotic cells [2–4] and along microtubules [5]. For a sufficiently narrow channel, where particles cannot overtake one another, the bath’s correlated dynamics strongly restrict the tracer. Extensive studies of its motion in such settings, termed ‘single-file’ (SF) dynamics, has led to the celebrated sub-diffusive \(\sim \sqrt{t}\) scaling of the tracer’s mean-square displacement (MSD) \(\langle \Delta X(t)^2 \rangle \equiv \langle (X(t))^2 \rangle - \langle X(t) \rangle^2\) [6–8]. Evidently, this behavior becomes
very fragile as the degree of geometric confinement is reduced to the point where particles can overtake one-another. In fact, it was shown that any finite overtaking rate ultimately yields a diffusive scaling of the tracer’s MSD, i.e. \( \langle \Delta X(t)^2 \rangle \sim t \) for large \( t \) [9–12].

Motivated by a variety of physical, biological and chemical setups [13–15], as well as applications in micro rheology [16] and in nanotechnology and microfluidics [17], recent years have seen a growing interest in geometrically constrained systems in which the tracer is driven or biased by an external force [18–27]. When confinement is strong enough to prevent particle overtaking, the driven tracer’s motion generates a blockade of bath particles that, in turn, restricts its propagation. This is manifested in the tracer’s stationary velocity, which was shown to vanish as \( v \sim L^{-1} \) on a long 1D ring of length \( L \) [18–20, 23, 28].

The stationary behavior of a driven tracer model with finite overtaking rates was recently studied on a ring of \( L \) sites, occupied by \( N \) symmetric hard-core bath particles of mean density \( \bar{\rho} = N / (L - 1) \) [29]. Two distinct phases were identified and the model’s phase diagram was determined in terms of the dynamical rates and \( \bar{\rho} \). Each of the two phases, called ‘localized’ and ‘extended’ respectively, was characterized by the corresponding stationary tracer velocity \( v \) and bath particle density profile \( \rho_\ell \), as seen in the tracer’s frame of reference, where \( \ell \) denotes the site label. In the localized phase, for large \( L \), the tracer attains a finite, \( L \)-independent velocity and the deviation of the bath density profile \( \rho_\ell \) from its mean value \( \bar{\rho} \) remains localized around the tracer. On the other hand, in the extended phase the tracer’s velocity vanishes as \( \sim L^{-1} \) and the density profile \( \rho_\ell \) continues to vary throughout the entire system.

It is hard to avoid drawing a naive correspondence between the non-driven tracer’s and the driven tracer’s behaviors. The non-driven tracer’s unrestricted diffusive behavior, which arises whenever overtaking is possible, is consistent with the finite velocity and localized bath density profile found in the driven tracer’s localized phase. Similarly, the non-driven tracer’s sub-diffusive dynamics, which arises in the absence of overtaking, is consistent with the vanishing velocity and extended bath density profile found in the driven tracer’s extended phase. Yet, defying naive intuition, the extended phase was recently surprisingly shown to persist for a range of finite overtaking rates in [29].

In light of the intriguing stationary behavior found in the driven tracer’s extended phase, one is left to wonder how the model’s dynamical properties are affected by overtaking. For example, how does the bath density’s non-local profile evolve in time? How does the tracer’s velocity vanish as \( t \to \infty \)? Moreover, in the absence of overtaking, the sub-diffusive scaling which characterizes the non-driven tracer’s MSD is well-known to extend to the driven case [25]. Can this behavior also persist in the presence of overtaking?

In this paper, we explore the dynamical properties of a driven tracer, propagating in a crowded bath of symmetric bath particles subject to varying geometric confinement. This is carried out by studying the dynamics of the 1D lattice model introduced in [29], where geometric confinement is incorporated by allowing the driven tracer to overtake neighboring bath particles at fixed rates. Focusing on the model’s ‘extended’ phase, we use the mean-field (MF) approximation to compute the bath density profile’s long-time, asymptotic evolution starting from the flat initial profile \( \rho_\ell(0) = \bar{\rho} \). This is then used to show that in an infinitely long system, the tracer’s velocity \( v(t) \) decays as \( \sim 1/\sqrt{t} \) for asymptotically large \( t \). Using extensive numerical simulations we demonstrate that, in this limit, the classical result \( \langle \Delta X(t)^2 \rangle \propto \sqrt{t} \), which was obtained for the non-driven tracer in the absence of overtaking, remarkably persists in the driven tracer’s extended phase, even in the presence of finite overtaking rates. Finally, we argue that the tracer’s dynamics can be reduced to that of a biased random walker with time-dependent hopping rates. Calculating these rates from the MF expression for the time-dependent bath density profile yields the correct sub-diffusive scaling in a specific sub-region of this phase.
The paper is organized as follows: in section 2 we introduce the model. The main results are presented in section 3. In section 4 we carry out a MF analysis of the system’s dynamics in the extended phase, which yields the temporal evolution of the bath density profile and the tracer’s mean displacement (MD). The insight gained from the MF analysis is then used to construct a corresponding biased random walker dynamics which is used to calculate the tracer’s MSD. The results derived in this section are compared with direct numerical simulations of the model. Concluding remarks are given in section 5.

2. The model

Consider a 1D lattice of \( L \) sites labeled \( \ell = 0, 1, 2, \ldots, L - 1 \). Working in the tracer’s reference frame, site \( \ell = 0 \) is set to be the tracer’s position at all times, while at \( t = 0 \) the remaining sites are uniformly occupied by bath particles of average density \( \overline{\rho} \equiv N/(L - 1) \). Correspondingly, whenever the tracer moves, site \( \ell = 0 \) moves along with it, thus shifting all of the remaining lattice indices accordingly. In the moving tracer’s reference frame, the system ultimately reaches a steady state whose dynamical properties are the main objective of this study. All particles interact via hard-core exclusion, whereby each site may be occupied by one particle at most. The model evolves by random sequential updates which correspond to continuous time dynamics. Here a particle, either the tracer or a bath particle, is randomly selected and attempts to move with the rates specified below. Bath particles attempt to hop to a vacant neighboring site, on either side, with rate 1, while the tracer attempts to hop to right and left neighboring sites with respective rates \( p \) and \( q \). The varying degree of geometric confinement is incorporated by allowing the tracer to attempt to overtake, or exchange places with, a neighboring bath particle to its right, at rate \( p' \), and to its left, with rate \( q' \). We stress that hops only occur if the target neighboring site is vacant, whereas exchanges only occur if the target neighboring site is occupied by a bath particle. This dynamics is schematically illustrated in figure 1.

The model’s stationary behavior was thoroughly explored in [29]. Its phase diagram was computed in the MF approximation and shown to feature a non-equilibrium phase transition between ‘localized’ and ‘extended’ phases. In the localized phase, the tracer attains a finite velocity as \( L \to \infty \) and the bath density profile deviates from \( \overline{\rho} \) only in an \( \sim \mathcal{O}(1) \) region around the tracer. On the other hand, in the extended phase the tracer’s velocity decays as \( \sim L^{-1} \) and the bath density profile extends over an \( \sim \mathcal{O}(L) \) region.

The model’s phase diagram is most conveniently presented when the hopping and exchange rates are expressed as

\[
\begin{align*}
\{ p &= r (1 + \delta) ; q = r (1 - \delta) \\
\{ p' &= r' (1 + \delta') ; q' = r' (1 - \delta')
\end{align*}
\]  

Here \( r \geq 0 \) and \( r' \geq 0 \) are the respective average hopping and exchange rates, while \( r\delta \) and \( r'\delta' \) are the biases, with \(-1 \leq \delta, \delta' \leq 1 \). Two critical manifolds, separating the extended and localized phases, were identified at the critical mean bath densities \( \overline{\rho}_L \) and \( \overline{\rho}_L' \):

\[
\begin{align*}
\overline{\rho}_L &= \frac{q' (p - q)}{pq' - qp'} \equiv \frac{\delta (1 - \delta')}{\delta - \delta'} \\
\overline{\rho}_L' &= \frac{p' (p - q)}{pq' - qp'} \equiv \frac{\delta (1 + \delta')}{\delta - \delta'}
\end{align*}
\]  

Since these two manifolds are independent of the average rates \( r \) and \( r' \), the phase diagram may be represented in the 3D parameter space \( \{ \overline{\rho}, \delta, \delta' \} \). The phase diagram in the \( (\delta', \delta) \) plane is
Figure 1. Schematic illustration of the model dynamics. Bath particles are depicted by empty circles and the driven tracer is depicted by a red circle. Arrows represent the allowed moves, with their respective attempt rates.

Figure 2. The MF phase diagram for average bath density $\bar{\rho} = 1/4$. The localized and extended phases are respectively denoted by $L$ and $E$. Red circles mark the ‘extreme’ points of the extended phase.

depicted in figure 2 for $\bar{\rho} = 1/4$. Although these results were derived using the uncontrolled MF approximation, excellent agreement was demonstrated in [29] between the MF predictions and the model’s numerical simulation results.

The focus of this study lies on the model’s dynamical properties in the extended phase. To this end, we distinguish between two sub-regions within the extended phase. The first region consists of the ‘extreme’ points, marked in red in figure 2, i.e. $\delta = -\delta' = 1$ and $\delta = -\delta' = -1$. For $\delta = -\delta' = 1$ the dynamical rates are $q = p' = 0$ and $p, q' > 0$, implying that the hopping process is fully biased to the right and the exchange process is fully biased to the left. A symmetric and opposite picture arises for $\delta = -\delta' = -1$, where $p = q' = 0$ and $p, q' > 0$. The second region is the complementary ‘bulk’ domain of the extended phase, where the hopping and exchange rates are only partially counter-biased. While the characteristic dynamical features of the model, namely the long-time relaxation of the bath density profile and the tracer’s MD and MSD, are found to be qualitatively the same in both sub-regions, it is convenient to consider them separately. The reason is that the simplifying feature of the fully biased dynamics makes it possible to analytically analyze the MSD of the tracer and demonstrate its sub-diffusive behavior in the ‘extreme’ region beyond numerical simulations.
Figure 3. The density profile \( \rho(\ell, t) \) versus \( \ell \) in a region of 540 sites around the tracer. Different times are denoted by different markers and MF predictions are depicted by black lines. The extreme region is represented by the rates \( q = q' = 0 \) and \( p = p' = 1 \), while for the complementary bulk region we consider \( p = q' = 1 \) and \( q = p' = 0.1 \).

Figure 4. The density profile \( \rho(z) \) versus the scaling variable \( z \) for different times. The different times are denoted by different markers and MF predictions are depicted by black lines. The dynamical rates are the same as in figure 3.

3. Main results

The model’s dynamical properties are studied in the extended phase using MF analysis and in numerical simulations, using the Gillespie algorithm [30]. For convenience, and without loss of generality, we shall hereafter explicitly consider \( \delta > 0 \) (i.e., \( p > q \)). Moreover, the MF analysis is carried out in the limit of infinite system-size \( L \to \infty \). We thus replace the finite-\( L \) lattice site indexing \( \ell = 0, 1, \ldots, L - 1 \) by \( \ell = -\infty, \ldots, -1, 0, 1, \ldots, \infty \), identifying site \( \ell = L - 1 \) of the finite system with site \( \ell = -1 \) of the infinite system.

Starting from a flat initial condition, the bath density profile at long times is shown in equations (12) and (24) to become a scaling function of the variables

\[
z_{\pm} = \frac{c_{\pm} + \ell}{\sqrt{t}},
\]

where \( z_+ \) and \( z_- \) respectively correspond to \( \ell > 0 \) and \( \ell < 0 \). The parameters \( c_{\pm} \) are then determined by the appropriate boundary conditions in each of the two regions. The resulting profile is used to compute the tracer’s velocity \( v(t) \), which is shown in equation (18) to decay as \( \sim 1/\sqrt{t} \) for large \( t \).

The MF approach applied in the present study does not, in itself, yield the MSD of the tracer. We have thus carried out extensive numerical simulations of the model’s dynamics...
which shows that in the extended phase the tracer’s MSD retains the classical sub-diffusive scaling $\langle \Delta X(t)^2 \rangle \propto \sqrt{t}$, even for finite overtaking rates. This stands in contrast with the results of numerous non-driven tracer models, which have established that diffusion asymptotically arises for any finite overtaking rates.

To go beyond numerical simulations of the MSD we restrict our analysis to the ‘extreme’ region, where we model the tracer’s dynamics as a biased random walk with time-dependent rates. In section 4 we show that for the ‘extreme’ parameters, the bath density near the tracer satisfies $\rho_1 \approx 1 - a/\sqrt{t}$ and $\rho_{-1} \approx b/\sqrt{t}$, where $a$ and $b$ may be read off equation (25). Thus, both the density of vacancies to the right of the tracer and the density of bath particles to its left approach $0$ as $\sim 1/\sqrt{t}$ in the long time limit. As such, due to the fully-biased nature of the dynamical rates, the tracer carries out right and left moves with rates which decrease in time as $\sim 1/\sqrt{t}$. When correlations between consecutive moves are neglected, we obtain $\langle \Delta X(t)^2 \rangle \propto \sqrt{t}$.

The main results of the present study are summarized by the following figures, produced from $\sim 10,000$ realizations for each parameter set. In figure 3 we plot the bath density profile $\rho_\ell(t)$ near the tracer at different times. The left panel shows the extreme parameter results, for the ‘extreme rates’ $q = q' = 0$ and $p = q' = 1$, while the right panel shows the bulk parameter results for representative ‘bulk rates’ $p = q' = 1$ and $q = q' = 0.1$. These rates, as well as the mean bath density $\bar{\rho} = 1/4$ and system-size $L = 2048$, are used throughout the paper. In both cases the profile appears to approach the mean bath density $\bar{\rho} = 1/4$ far from the tracer. An excellent fit to the MF prediction in equation (24) is noted in the extreme region, while a reasonable fit is found for the bulk region, which is slightly affected by correlations that are neglected from our analysis. Figure 4 shows a data collapse of the density profile as a function of the scaling variables $z_\pm$ of equation (3). Note that the collapse is equally convincing for both the bulk and extreme parameters, suggesting that the correlations observed in the bulk region do not qualitatively change the density profile’s scaling form. Figure 5 shows simulations results for the tracer’s velocity as a function of time in both the extreme and bulk regions, demonstrating a $\sim 1/\sqrt{t}$ scaling in the long time limit. The corresponding results for the MSD are given in figure 6, which shows that the MSD scales sub-diffusively with time as $\sim \sqrt{t}$. Finally, we remark that the results depicted in these figures are obtained in the long-time limit, for times much shorter than the diffusive time-scale set by the simulated system’s finite size $L$, i.e. for $0 \ll t \ll L^2$.
Figure 6. Log–log plot of the tracer’s MSD versus time for short times. Blue stars depict simulation results for $L = 2048$ and the dashed black curves depict a linear fit to $\text{const} + \frac{1}{2} \log t$. The left panel shows simulation results for the extreme parameters while the right panel shows results for the bulk parameters. The extreme region is represented by the rates $q = p' = 0$ and $p = q' = 1$, while for the bulk region we consider $p = q = q' = 1$ and $q = p' = 0.1$.

4. Dynamics

Our analysis begins with formulating the equations describing the evolution of the bath occupation variable $\tau_\ell(t)$, which takes the value 1 if site $\ell$ is occupied at time $t$ and 0 otherwise. Analyzing the contribution of each possible process to $\tau_\ell(t)$ yields

$$\tau_\ell(t + dt) - \tau_\ell(t) = \Gamma_\ell(t),$$

where, at the boundary sites $\ell = \pm 1$, $\Gamma_\ell(t)$ is given by

$$\Gamma_1(t) = \begin{cases} 
\tau_2 \text{ w.p. } (1 + p)(1 - \tau_1)dt \\
-\tau_1 \text{ w.p. } [(1 + p')(1 - \tau_2) + q(1 - \tau_{-1})]dt \\
\tau_{-1} \text{ w.p. } q'(1 - \tau_1)dt
\end{cases}$$

(5)

and

$$\Gamma_{-1}(t) = \begin{cases} 
\tau_{-2} \text{ w.p. } (1 + q)(1 - \tau_{-1})dt \\
-\tau_{-1} \text{ w.p. } [(1 + q')(1 - \tau_{-2}) + p(1 - \tau_1)]dt \\
\tau_1 \text{ w.p. } p'(1 - \tau_{-1})dt
\end{cases}$$

(6)

and for the remaining sites $\ell = -\infty, \ldots, -2, 2, \ldots, \infty$, $\Gamma_\ell(t)$ is given by

$$\Gamma_\ell(t) = \begin{cases} 
\tau_{\ell+1} \text{ w.p. } (1 - \tau_\ell)dt \\
\tau_{\ell-1} \text{ w.p. } (1 - \tau_\ell)dt \\
-\tau_\ell \text{ w.p. } (1 - \tau_{\ell+1})dt \\
\tau_{\ell+1} - \tau_\ell \text{ w.p. } [p(1 - \tau_\ell) + p'\tau_\ell]dt \\
\tau_{\ell-1} - \tau_\ell \text{ w.p. } [q(1 - \tau_{-1}) + q'\tau_{-1}]dt
\end{cases}$$

(7)

Here w.p. is used to abbreviate ‘with probability’. 
4.1. MF approximation

The MF equations for the bath density profile $\rho_\ell (t)$ at site $\ell$ and time $t$ are obtained by averaging equations (4)–(7) over different initial configurations and realizations of the dynamics, such that $\rho_\ell (t) = \langle \tau_\ell (t) \rangle$. In the MF approximation, correlations between the occupation of two different sites are neglected, thus replacing terms of the form $\langle \tau_m \tau_n \rangle$ by the product $\rho_m \rho_n$. Although the MF approximation is inherently uncontrolled, it still often provides qualitatively correct predictions in lattice models with exclusion interactions. The boundary equations at sites $\ell = \pm 1$ provide

$$
\begin{align*}
\partial_t \rho_1 &= (1 - \rho_1) \left[ q' \rho_{-1} + (1 + p) \rho_2 \right] \\
&\quad - (1 + p') \rho_1 (1 - \rho_2) - q \rho_1 (1 - \rho_{-1}) \\
\partial_t \rho_{-1} &= (1 - \rho_{-1}) \left[ p' \rho_1 + (1 + q) \rho_{-2} \right] \\
&\quad - (1 + q') \rho_{-1} (1 - \rho_{-2}) - p \rho_{-1} (1 - \rho_1)
\end{align*}
$$

and the general equation for $\ell \neq \pm 1$ becomes

$$
\partial_t \rho_\ell = \rho_{\ell + 1} - 2 \rho_\ell + \rho_{\ell - 1} + v_+ (\rho_{\ell + 1} - \rho_\ell) - v_- (\rho_\ell - \rho_{\ell - 1}),
$$

where

$$
v_+ = p (1 - \rho_1) + p' \rho_1,
$$

is the rate at which the tracer moves (either by hopping or exchange) to the right and

$$
v_- = q (1 - \rho_{-1}) + q' \rho_{-1},
$$

is the rate at which it similarly moves to the left. Note that, unlike the occupation variable $\tau_\ell$, the density profile $\rho_\ell$ takes continuous values in $[0, 1]$.

4.1.1. Asymptotic scaling form. Since the tracer permanently occupies site $\ell = 0$, the bath density profile $\rho_\ell (t)$ is expected to be discontinuous at $\ell = 0$. As such, we must separately treat equations (8) and (9) for $\rho_\ell (t)$ in each of the two domains $\ell = -\infty, \ldots, -2, -1$ and $\ell = 1, 2, \ldots, \infty$, respectively denoting the density profile in each region by $\rho^+ (t)$ and $\rho^- (t)$. We are interested in studying the long-time behavior of $\rho^\pm (t)$, starting from an initially flat density profile $\rho_\ell (0) = \overline{\rho}$. Given that equation (9) for the general dynamics at $\ell \neq \pm 1$ is simply a diffusion equation with drive, it is natural to seek a solution where $\ell$ scales as $\sim \sqrt{t}$. In particular, we consider a large-time ansatz of the form

$$
\rho^\pm (t) = \overline{\rho} + \delta \rho^\pm (z^\pm); \quad z^\pm = \frac{\ell + c^\pm}{\sqrt{t}},
$$

where $\delta \rho^\pm (z^\pm)$ is the deviation from the initially uniform density profile $\overline{\rho}$ and $c^\pm$ are yet-unknown parameters. These parameters may be neglected in the scaling regime, where $|\ell|$ is large and scales as $\sim \sqrt{t}$, but they must be considered when solving the boundary equations for $\ell = \pm 1$. At long times, $z^\pm$ approaches a continuous variable, such that $z_- \in (-\infty, 0)$ and $z_+ \in (0, +\infty)$. In this limit, equation (9) takes the continuous form

$$
0 = \left( 1 + \frac{u}{2} \right) \delta \rho^\pm (z^\pm) + \left( v \sqrt{t} + \frac{z^\pm}{2} \right) \delta \rho^\pm (z^\pm),
$$

where we define the respective tracer velocity $v (t)$ and moving rate $u (t)$ as
\( v(t) = v_+(t) - v_-(t), \quad u(t) = v_+(t) + v_-(t). \) (14)

### 4.1.2. Velocity and moving rate.

In [29], the model’s stationary behavior was studied on a finite chain of \( L \) sites and the stationary density profile \( \rho_{\pm}^s \) was recovered in the limit of large \( L \). In the extended phase, where sub-diffusive dynamical behavior might be expected, it was shown that

\[
\rho_{\pm}^s \approx \frac{q'(p - q)}{pq' - qp'} \quad \text{and} \quad \rho_{L-1}^s \approx \frac{p'(p - q)}{pq' - qp'}.
\] (15)

In our current scenario of an infinite chain, where \( \rho_{L-1}^s \to \rho_{L-1}^s \), one can show that substituting \( \rho_{\pm}^s \) into equations (10), (11) and (14) yields a stationary tracer velocity and moving rate that behave as

\[
\lim_{t \to \infty} v(t) \to 0 \quad \text{and} \quad \lim_{t \to \infty} u(t) \to u_0, \quad (16)
\]

where

\[
u_0 = p \left(1 - \rho_{\pm}^s\right) + p' \rho_{\pm}^s + q \left(1 - \rho_{\pm}^s\right) + q' \rho_{L-1}^s. \] (17)

However, here we are interested in the model’s temporal behavior. In light of equation (13), self-consistency requires that the leading long-time behavior of \( v(t) \) be

\[
v(t) \approx \frac{\omega}{\sqrt{t}}, \quad (18)
\]

where \( \omega \) is an unknown constant. Since \( v(t) = v_+(t) - v_-(t) \) with \( v_\pm(t) \geq 0 \), it is reasonable to guess a similar asymptotic behavior for the moving rate \( u(t) \), which we assume to be of the form

\[
u(t) \approx u_0 + \frac{\eta}{\sqrt{t}}. \quad (19)
\]

Simulation results for the tracer’s velocity are shown in figure 5 in natural log–log scale. Results for both the extreme and bulk regions, which have been defined in terms of the model parameters in section 3, are provided alongside their respective MF predictions, showing an excellent fit. As in figures 3 and 4, these results are obtained in the long-time limit \( 0 \ll t \ll L^2 \).

### 4.1.3. Density profile.

Substituting \( \rho_{\pm}^s(t) \) of equation (12) and \( v(t) \) of equation (18) into equation (9) for \( \rho_{\pm}(t) \), and taking the continuum limit as \( t \to \infty \), yields the ordinary differential equation

\[
\delta \rho_{\pm}(z_{\pm}) = -\frac{2\omega + z_{\pm}}{2 + u_0} \delta \rho_{\pm}(z_{\pm}), \quad (20)
\]

whose solution is

\[
\delta \rho_{\pm}(z_{\pm}) = A_{\pm} + B_{\pm} \text{Erf} \left[ \frac{2\omega + z_{\pm}}{\sqrt{2(2 + u_0)}} \right]. \quad (21)
\]

The parameters \( A_{\pm} \) and \( B_{\pm} \) will next be determined by imposing the appropriate boundary conditions. The first boundary condition is obtained by noting that, at a sufficiently large distance from the tracer, the density profile decays to \( \overline{\rho} \), implying that

\[
\delta \rho_{\pm}(z_{\pm} \to \pm\infty) = 0. \quad (22)
\]
The second boundary condition is deduced using the known stationary densities
\[ \lim_{t \to \infty} \rho_{\pm1}(t) = \rho_{\pm1}^0 \] in equation (15), giving
\[ \delta \rho_{\pm}(z_{\pm} \to 0) \equiv \rho_{\pm1}^0 - \rho. \] (23)

Accounting for both boundary conditions in equations (22) and (23), \( \delta \rho_{\pm}(z_{\pm}) \) assumes its final form
\[ \delta \rho_{\pm}(z_{\pm}) = \left( \rho_{\pm1}^0 - \rho \right) \frac{1 \mp \text{Erf} \left[ \frac{\sqrt{2} \pm z_{\pm}}{2 (2 + u_0)} \right]}{1 \mp \text{Erf} \left[ \frac{1}{\sqrt{2 + u_0}} \omega \right]} \] (24)

The tracer’s asymptotic velocity and moving rates are now within reach. Using the bath density deviation \( \delta \rho_{\pm}(z_{\pm}) \), we easily compute the long-time behavior of the bath density near the tracer (i.e. for \( |\ell| \sim O(1) \)) as
\[ \rho_{\pm}(t) \approx \rho_{\pm1}^0 + D_{\pm} \frac{c_{\pm} + \ell}{\sqrt{t}}, \] (25)
where
\[ D_{\pm} = \pm \sqrt{\frac{2}{\pi (2 + u_0)}} \left( \frac{\rho_{\pm1}^0 - \rho}{1 \mp \text{Erf} \left[ \frac{1}{\sqrt{2 + u_0}} \omega \right]} \right). \] (26)

The only remaining parameters which must be set to uniquely determine \( \rho_{\pm}(t) \) are \( c_{\pm} \) of equation (12) and \( \omega \) of equation (18). The parameters \( c_{\pm} \) are set by the boundary equation (8) by substituting \( \rho_{\pm1}(t) \) and \( \rho_{\pm2}(t) \) of equation (25). Their cumbersome explicit expressions provide little physical insight and are thus not presented. For the extreme rates \( q = p' = 0 \) and \( p = q' = 1 \) these reduce to \( c_{+} = c_{-} = 0 \), while for the bulk parameters \( p = q' = 1 \) and \( q = p' = 0.1 \) we find \( c_{+} \approx 0.336 \) and \( c_{-} \approx 19.825 \). Finally, a transcendental equation for \( \omega \) is obtained by demanding that the tracer’s velocity \( v(t) = v_{+} - v_{-} \), which is an explicit function of \( \rho_{\pm1}(t) \), be consistent with the form \( v \approx \frac{\omega}{\sqrt{t}} \), assumed in equation (18). We find
\[ \omega = \left( 1 + \frac{u_0}{2} \right) \frac{D_{-} - D_{+}}{\rho_{\pm1}^0 - \rho_{\pm1}^0}. \] (27)

Figures 3 and 4 compare the model’s density profile, as obtained in numerical simulations of its microscopic dynamics, to the MF expression in equation (24). Specifically, figure 3 presents the profile’s \( \rho_{i}(t) \) temporal evolution near the tracer, while figure 4 shows a collapse of the profile, at different times, as a function of the scaling variable \( z_{\pm} \) in equation (3).

With this we conclude our MF analysis of the bath density profile’s evolution and the tracer’s velocity in the limit of large \( t \).

4.2. Mean-square displacement

In this section we first show that, in the extreme region, the tracer’s dynamics can be represented by a random walker with time-dependent hopping rates. We then show that this description readily yields the sub-diffusive nature of the MSD. Finally, we present results of extensive numerical simulations showing that the MSD of the tracer grows sub-diffusively with time in both the extreme and bulk regions.
4.2.1. A biased random walk with time-dependent rates. While our analysis has thus-far mostly concerned with the bath density profile, let us shift our focus to the tracer’s dynamics. In general, the tracer may be viewed as a walker which makes right and left moves with some probability. We use the bath density profile to calculate these moving probabilities and show that, at the extreme points, these probabilities decay in time. This decay, in turn, is responsible for the observed subdiffusive behavior of the tracer, which is recovered even when correlations between its moves are neglected.

Within this framework, we model the tracer’s dynamics as a random walker with time dependent moving rates, whose $n$th discrete step $Z_n$ satisfies

$$Z_n = \begin{cases} +1 & \text{w.p. } P_n = \frac{v_n^+}{u_n} \\ -1 & \text{w.p. } Q_n = \frac{v_n^-}{u_n} \end{cases}$$

(28)

where

$$P_n + Q_n = 1 \quad \text{and} \quad P_n - Q_n = \frac{v_n}{u_n}.$$  

(29)

Denoting the tracer’s position after $N$ steps by $X_N = \sum_{n=1}^{N} Z_n$ and the difference between the tracer’s $n$th step $Z_n$ and its mean value $\langle Z_n \rangle$ by

$$Y_n = Z_n - \langle Z_n \rangle,$$

(30)

the tracer’s MSD becomes

$$\langle \Delta X_N^2 \rangle = \left\langle \left( \sum_{n=1}^{N} Y_n \right)^2 \right\rangle = \sum_{n=1}^{N} \langle Y_n^2 \rangle + \sum_{m \neq n} \langle Y_m Y_n \rangle.$$  

(31)

Neglecting correlations between consecutive steps then allows factorizing $\langle Y_m Y_n \rangle \approx \langle Y_m \rangle \langle Y_n \rangle$, which identically vanish by definition since $\langle Y_n \rangle = 0$. Using $\langle Z_n^2 \rangle = 1$ and $\langle Z_n \rangle = P_n - Q_n$, which follow directly from equation (29), a straightforward calculation yields

$$\langle \Delta X_N^2 \rangle \approx \sum_{n=1}^{N} \left[ \langle Z_n^2 \rangle - \langle Z_n \rangle^2 \right] = \sum_{n=1}^{N} \left[ 1 - \left( \frac{v_n}{u_n} \right)^2 \right].$$  

(32)

To proceed one needs to express the number of moves $N(t)$ that take-place during the time interval $(0, t)$, as a function of $t$. Noting that $u(t)dt$ is the probability that the tracer makes a move during the time interval $dt$, the number of moves $N(t)$ satisfies

$$\frac{dN}{dt} = u(t),$$

(33)

and thus

$$N(t) = \int_0^t u(\tau)d\tau.$$  

(34)

Approximating the sum in equation (32) by an integral, it may be expressed as

$$\langle \Delta X(t)^2 \rangle \approx \int_0^t u(\tau) \left( 1 - \left( \frac{v(\tau)}{u(\tau)} \right)^2 \right).$$  

(35)
We have seen in section 4.1 that at the extreme points of the extended phase, where \( u_0 = 0 \), the tracer’s moving rate and velocity respectively vanish as \( u(t) \approx \eta/\sqrt{t} \) and \( v(t) \approx \omega/\sqrt{t} \) at large \( t \), for \( 1 \ll t \ll L^2 \). On the other hand, performing the analysis described in [29] at the extreme points shows that in the steady state, i.e. for \( t \gg L^2 \gg 1 \), the tracer’s moving rate and velocity respectively vanish as \( u^{\infty}(L) \approx \mu/L \) and \( v^{\infty}(L) \approx \nu/L \). Using these asymptotic limits in equation (35) yields the following expression for the MSD

\[
\langle \Delta X(t)^2 \rangle \approx \begin{cases} 
\frac{2(\eta^2 - \omega^2)}{\sqrt{t}} & t/L^2 \ll 1 \\
\frac{\eta^2}{\mu^2 - \nu^2} \frac{1}{L^2} & t/L^2 \gg 1 
\end{cases},
\]

(36)

where the parameters \( \eta, \omega, \mu \) and \( \nu \) can be expressed in terms of the average bath density \( \bar{\rho} \) and the dynamical rates \( p, q, p' \) and \( q' \). Note that the large \( t \) expression is valid only as long as \( t \ll L^3 \): in a finite system of length \( L \), the MSD is bounded from above by \( \sim O(L^2) \). At \( t \sim O(L^3) \) the MSD saturates this bound and stops growing.

We thus conclude that the MSD exhibits three distinct types of behavior, separated by two crossover regimes. For \( t \ll L^2 \) the tracer’s dynamics is sub-diffusive, as is the case for SF dynamics. At \( t \sim O(L^2) \), a crossover to ordinary diffusive behavior takes place with a diffusion constant \( D(L) \) which decays to zero as \( \sim 1/L \) at large \( L \). Finally, at \( t \sim O(L^3) \), the MSD saturates at a value of \( \sim O(L^2) \) and reaches a constant value. In the first crossover regime, the MSD can be described by a scaling function of the form

\[
\langle \Delta X(t)^2 \rangle \approx L^2 \chi \left( \frac{\sqrt{t}}{L} \right),
\]

(37)

with

\[
\chi(x) \approx \begin{cases} 
x & x \ll 1 \\
x^2 & x \gg 1 
\end{cases}.
\]

(38)

The main assumption in the derivation of the scaling behavior of the MSD using the random walk approach, is that the tracer’s steps are uncorrelated in time. This assumption seems to be valid in the extreme region, where the bath density near the tracer reaches 1 to its right and 0 to its left. Repeating the same analysis in the bulk of the extended phase yields ordinary diffusive behavior of the tracer. The reason is that, in this region, the moving rate of the tracer in the long time limit, \( u_0 \), does not vanish, resulting in a linear growth in time of the MSD (see equation (35)). However, numerical simulations suggest that the scaling form in equations (37) and (38) remains valid in this region as well (see figures 6–9 and discussion in section 4.2.2). We thus conclude that, in the bulk region, the tracer’s moves are strongly correlated in time, implying that the random walk approach cannot be applied.

### 4.2.2. Numerical simulations

The scaling form obtained within the random walk picture, qualitatively agrees with the simulation results for the MSD and the effective diffusion coefficient presented in figures 6–9. While the random walk picture applies only to the extreme region, the numerical simulations suggest that results of this analysis for the MSD remain valid also for the bulk region. In figure 6 we zoom-in on the short-time behavior of the tracer’s MSD in a large system of size \( L = 2048 \). This figure shows that, for times satisfying \( 0 \ll t \ll L^2 \), the MSD scales as \( \sim \sqrt{t} \) in both the extreme and bulk regions. It is then shown in figures 7 and 8 that, for \( L^2 \ll t \ll L^3 \), the dynamics is diffusive with \( \langle \Delta X(t)^2 \rangle \approx D(L)t \), where the diffusion coefficient \( D(L) \) is found to decay as \( \sim 1/L \) in the large \( L \) limit at both the extreme and bulk
Figure 7. Plot of the tracer’s MSD versus time for different values of $L$. The left panel shows simulation results for the extreme parameters while the right panel shows results for the bulk parameters (same parameters as in figure 6).

Figure 8. The diffusion constant versus $1/L$ at long times satisfying $L^2 \ll t \ll L^3$. Blue stars show simulation data in the bulk while the orange dots depict the extreme points. The dashed and dotted curves simply serve as a guide for the eye.

regions of the phase diagram. In figures 7 and 9 both the sub-diffusive and diffusive domains of the tracer’s dynamics are observed, separated by a crossover regime which takes place at $t \sim O(L^2)$. In the long time limit, $t \gg L^2$ (but with $t \ll L^3$) the system reaches a steady state. The vanishing of $D(L)$ in the limit $L \to \infty$ is consistent with the observed sub-diffusive $\sim \sqrt{t}$ scaling of the MSD for $t \ll L^2$. Figure 9 further supports this picture, providing a data collapse of $\langle \Delta x(t)^2 \rangle / L$ versus the scaling variable $t/L^2$ for different values of $L$. Note that at even larger $t \sim O(L^3)$ (not shown in the figures), another crossover takes place whereby the MSD stops growing with time, reaching its maximal value of $\sim O(L^2)$, as imposed by the system’s finite size.

An important remark, concerning the neglected correlations between the walker’s steps in equation (32), must be made to properly frame these results. At the extreme points one recovers the correct scaling exponent of the MSD, but not the correct coefficient. To illustrate this point for the extreme parameters used to generate figures 6 and 7, one has $\omega \approx 0.87$ and $\eta \approx 0.95$, which give $\langle \Delta X(t)^2 \rangle \approx 0.28\sqrt{t}$. Yet the fit in figure 6 instead shows $\langle \Delta X(t)^2 \rangle \approx 6.05\sqrt{t}$. Moreover, repeating this analysis in the bulk regime, where the moving rate of the tracer does not vanish in the large-$L$ limit, incorrectly predicts a diffusive scaling, i.e. $\langle \Delta X(t)^2 \rangle \propto t$. This dis-
crepancy and the fact that we observe sub-diffusive scaling of the MSD in the bulk region directly follow from the neglected correlations of the random walker moves. At the extreme points, the neglect of correlations is less severe due to the tracer’s fully biased dynamics, hopping only to the right and exchanging only to the left for $\delta = -\delta' = 1$. As such, the MSD’s scaling form remains correct, with the correlations merely modifying the amplitude of the MSD temporal growth. However, in the bulk regime these correlations are more significant and cannot be ignored. Nevertheless, obtaining the correct sub-diffusive scaling from such a robust MF mechanism provides important physical intuition into such correlated dynamics and is expected to apply to many additional scenarios.

5. Conclusions

In this paper we have studied the dynamics of a 1D driven tracer moving through a geometrically-confined quiescent bath. For the non-driven tracer, when strong geometric confinement prevents particles from overtaking one another, one famously recovers the sub-diffusive behavior which characterizes SF dynamics. Our main result is that for the driven tracer, this sub-diffusive behavior remarkably persists even when the degree of confinement is reduced and overtaking is allowed. This stands in contrast with the non-driven tracer dynamics, which are well-known to become diffusive at any finite overtaking rate.

In [29], the model’s steady state was studied and found to exhibit an extended phase, where the bath density profile extends throughout the entire system and the tracer’s velocity vanishes in the thermodynamic limit. Here we have focused on the model’s dynamical properties in this phase. Using the MF approximation, we have computed the bath density profile’s temporal evolution $\rho_\ell(t)$, as viewed from the tracer’s reference frame. This was shown to approach a scaling function $\rho(t/\xi)$ with a characteristic length $\xi$ which grows in time as $\sim \sqrt{t}$. Moreover, we have shown that the tracer’s velocity $v(t)$ asymptotically scales as $\sim 1/\sqrt{t}$ for large $t$. These results have allowed us to model the tracer’s dynamics as a biased random walk with time-dependent rates and compute its MSD at specific regions of the extended phase. Using this approach, we show that the tracer’s dynamics remains sub-diffusive with $\langle \Delta X(t)^2 \rangle \sim \sqrt{t}$ even for finite overtaking rates, and confirm this picture using extensive numerical simulations. While the MF approximation is uncontrolled, simulations show that it effectively captures the model’s qualitative features. We hope these results motivate a discussion which might lead to an exact derivation of these interesting features.
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