ON HYPERSURFACES IN A LOCALLY AFFINE RIEMANNIAN BANACH MANIFOLD II

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In our previous work (2002), we proved that an essential second-order hypersurface in an infinite-dimensional locally affine Riemannian Banach manifold is a Riemannian manifold of constant nonzero curvature. In this note, we prove the converse; in other words, we prove that a hypersurface of constant nonzero Riemannian curvature in a locally affine (flat) semi-Riemannian Banach space is an essential hypersurface of second order.

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1. Introduction. Let \( M \) be an infinite-dimensional Banach manifold of class \( C^k \), \( k \geq 1 \), modelled on a Banach space \( E \), and let \( \bar{g} \) be a symmetric bilinear form defined on \( M \), that is, \( \bar{g} \in L^2(M; \mathbb{R}) \). The metric \( \bar{g} \) is said to be strongly nonsingular if \( \bar{g} \) associates a mapping \( \bar{g}^* : x \in M \to \bar{g}^* (x, \cdot) \in L(M; \mathbb{R}) \) which is bijective [2]. Let \( \bar{\Gamma} \) be the linear connection on \( M \). A \( C^k \) Banach manifold \( (M, \bar{\Gamma}) \), \( k \geq 3 \), is called locally affine if its curvature and torsion tensors are zero. In general, it is proved in [2] that a Banach manifold \( (M, \bar{\Gamma}) \) is locally affine if and only if there exists an atlas \( \mathcal{A} \) on \( M \) such that for any chart \( c = (U, \Phi, E) \) at \( \bar{x} \in M \), the model of the linear connection \( \bar{\Gamma} \) is the chart \( \Phi \) at \( \bar{x} \) with respect to the chart \( c \), and \( i \) is the model of \( i \) with respect to the charts \( c \) and \( d \). Then we have an inclusion

\[
i : x = \Psi(\bar{x}) \in \Psi(V) \subset F \quad \text{and} \quad i(x) = z = \Phi(\bar{x}) \in \Phi(V) \subset E
\]

of a hypersurface of a semi-Riemannian Banach space \( E \).
In this case, (2.1) is called the local equation of the submanifold \( N \subset M \) with respect to the charts \( c \) and \( d \). Also \( N \) will be a Riemannian submanifold of \( M \) with induced metric \( \tilde{g} \), which is defined by the rule

\[
\bar{g}(\bar{X}_1, \bar{X}_2) = \frac{1}{\bar{g}_{ji}(x)}(T_{x_i}(\bar{X}_1), T_{x_i}(\bar{X}_2)),
\]

for all \( \bar{x} \in N \) and \( \bar{X}_1, \bar{X}_2 \in T_{\bar{x}}N \), where \( T_{\bar{x}}\bar{i}: T_{\bar{x}}N \to T_{\bar{x}}M \) is the tangent mapping of \( \bar{i} \) at the point \( \bar{x} \in N \) (see [1]).

Assume that \( \bar{g} \) is a strongly nonsingular metric on \( N \). Also we have that \( M \) and \( N \) are Riemannian manifolds with free-torsion connections \( \bar{\Gamma} \) and \( \Gamma \), respectively, such that \( \bar{\nabla} \bar{g} = 0 \) and \( \nabla \bar{g} = 0 \) (see [3, 4]). Let \( X_1, X_2 \in F \) be the models of \( \bar{X}_1, \bar{X}_2 \in T_{\bar{x}}N \) with respect to the chart \( d \) on \( N \). Then \( Y_1 = D_{ix}(X_1) \) and \( Y_2 = D_{ix}(X_2) \) are the models of \( \bar{X}_1 \) and \( \bar{X}_2 \) with respect to the chart \( c \) on \( M \).

In this case, the local equation of (2.2) takes the form

\[
\bar{g}(X_1, X_2) = \frac{1}{\bar{g}_x}(D_{ix}(X_1), D_{ix}(X_2)).
\]

**Theorem 2.1.** A local hypersurface of constant nonzero Riemannian curvature in a locally affine (flat) semi-Riemannian Banach space is an essential hypersurface of second order.

**Proof.** Let \( N \) be a local hypersurface of constant curvature \( K_0 \) of the Banach type in the Riemannian manifold \( (M, \bar{g}) \) such that \( \dim N > 2 \). We know that the first differential equation of the hypersurface \( N \subset M \) has the form (see [5])

\[
\bar{\nabla}D_{ix}(X, Y) = eA_x(X, Y)\bar{\xi}_x,
\]

where \( \bar{\xi}_x \in T_{0+0}(M) = T_0^1(M) \) is a unit vector in \( M \) orthogonal to \( N \) at the point \( \bar{x} \in M \), that is,

\[
\frac{1}{\bar{g}}(\bar{\xi}_x, \bar{\xi}_x) = e, \quad \frac{1}{\bar{g}}(\bar{\xi}_x, \bar{X}) = 0,
\]

for all \( \bar{x} \in N \subset M \) and all \( \bar{X} \in T_{\bar{x}}N \), and \( A_x \) is the second fundamental form for the hypersurface \( N \) which is defined by the equality (see [5])

\[
A_x(X, Y) = \frac{1}{\bar{g}_x}(D^2_{ix}(X, Y), \bar{\xi}_x) = -\frac{1}{\bar{g}_x}(D_{ix}(X), D_{ix}(Y)).
\]

Taking into account that \( T_{\bar{x}}\bar{i} \in T_{0+0+1}(N) \) is a mixed tensor of type \( (1+0,0+1) \) on the submanifold \( N \) (see [7]), \( \bar{\xi}_x \in T_0^1(M) \), and (2.6), we conclude that \( A_x \) is a symmetric tensor of type \( (0,2) \) on \( N \) at the point \( \bar{x} \in N \).

Now let \( \bar{\xi}: \bar{x} = \Psi(\bar{x}) \in \Psi(V) \subset F \to \bar{\xi}_x \in E \) be the model of the vector field

\[
\bar{\xi}: \bar{x} \in N \to \bar{\xi}_x \in T_{\bar{x}}M,
\]
with respect to the charts $c$ and $d$ at the point $\bar{x}$. Then the local equations of equalities (2.5) take the form

$$\frac{1}{g}(\xi_x, \xi_x) = e, \quad \frac{1}{g}(Di_x(X), \xi_x) = 0,$$

(2.8)

for all $x \in \Psi(V) \subset F$ and all $X \in F$. Furthermore, the integral condition for (2.4) takes the form

$$\frac{1}{g}(Di_x(R_x(Y;Z,X),Di_x(S))) = \frac{2}{g}(R_x(Y;Z,X),S) = eA_x(Z,Y)A_x(X,S).$$

(2.9)

**Remark 2.2.** In formula (2.9), there exists an alternation with respect to the underlined vectors without division by 2. This convention will be used henceforth.

Similarly, the second differential equation of the hypersurface $N \subset M$ will be (see [5])

$$D\xi_x(X) = Di_x(H_x(X)),$$

(2.10)

where $H_x \in L(F;F)$. Also by using (2.6), we find that

$$A_x(X,Y) = -\frac{1}{g_x}(Di_x(X),D\xi_x(Y)) = -\frac{1}{g_x}(Di_x(X),Di_x(H_x(Y))) = -\frac{2}{g_x}(X,H_x(Y)),$$

(2.11)

that is,

$$\frac{2}{g_x}(X,H_x(Y)) = -A_x(X,Y),$$

(2.12)

for all $x = \Psi(\bar{x}) \in \Psi(V) \subset F$ and all $X,Y \in F$. Furthermore, the integral condition for (2.10) has the form (see [5])

$$\nabla A_x(X;Z,Y) = 0,$$

(2.13)

for all $x = \Psi(\bar{x}) \in \Psi(V) \subset F$ and all $X,Y,Z \in F$.

Now we find that

$$\frac{2}{g_x}(\frac{2}{g_x}(R_x(Y;Z,X),S) = \frac{1}{g_x}(Di_x(\frac{2}{g_x}(R_x(Y;Z,X)),Di_x(S)) = eA_x(Z,Y)A_x(X,S).$$

(2.14)

Since $N$ is a hypersurface of constant curvature, then (2.14) takes the form (see [2])

$$\frac{2}{g_x}(\frac{2}{g_x}(K_0\xi_x(Z,Y),X,S) = eA_x(Z,Y)A_x(X,S),$$

(2.15)
where $K_0 \in \mathbb{R}$ is a constant independent of the choice of the point, and is called the curvature of the hypersurface $N$. Then, we obtain

$$A_x(Z,Y)A_x(X,S) - A_x(X,Y)A_x(Z,S)$$

$$= K \left( \frac{2}{g_x(Z,Y)} \frac{2}{g_x(X,S)} - \frac{2}{g_x(X,Y)} \frac{2}{g_x(Z,S)} \right), \quad (2.16)$$

for all $x = \Psi(\bar{x}) \in \Psi(V) \subset F$ and all $X, Y, Z, S \in F$, where $K = K_0/e$.

Now we prove that $A_x$ is a weakly nonsingular form. Let $X$ be a fixed vector and $A_x(X,Y) = 0$, for all $Y \in F$. Then, from (2.16) we obtain

$$\frac{2}{g_x(Z,Y)} \frac{2}{g_x(X,S)} - \frac{2}{g_x(X,Y)} \frac{2}{g_x(Z,S)} = 0, \quad (2.17)$$

for all $Y \in F$, that is, $\frac{2}{g_x(Y)} \frac{2}{g_x(X,S)} \cdot Z - \frac{2}{g_x(Z,S)} \cdot X = 0$. By using that $\frac{2}{g_x}$ is nonsingular, we obtain $\frac{2}{g_x}(X,S) \cdot Z - \frac{2}{g_x}(Z,S) \cdot X = 0$, for all $x = \Psi(\bar{x}) \in \Psi(V) \subset F$ and all $X, Z, S \in F$. Since $\dim E > 2$, then, for any $S$, we can choose $Z$ which is not a multiple of $X$ and thus $\frac{2}{g_x}(X,S) = 0$, for all $S \in F$. But $\frac{2}{g_x}$ is nonsingular, hence, $X = 0$ and this proves that $A_x$ is a weakly nonsingular form.

Now from (2.12) and (2.16), we obtain

$$\frac{2}{g_x(Z,H_x(Y))} \frac{2}{g_x(X,H_x(S))} = K \left( \frac{2}{g_x(Z,Y)} \frac{2}{g_x(X,S)} \right), \quad (2.18)$$

and then we have

$$\frac{2}{g_x}(Z,\frac{2}{g_x}(X,H_x(S)) \cdot H_x(Y) - \frac{2}{g_x}(X,H_x(Y)) \cdot H_x(S)$$

$$- K \left( \frac{2}{g_x}(X,S) \cdot Y - \frac{2}{g_x}(X,Y) \cdot S \right) \right) = 0, \quad \forall Z \in F. \quad (2.19)$$

Taking into account that the metric tensor $\frac{2}{g_x}$ is nonsingular, we obtain

$$\frac{2}{g_x}(X,H_x(S)) \cdot H_x(Y) - \frac{2}{g_x}(X,H_x(Y)) \cdot H_x(S)$$

$$- K \frac{2}{g_x}(X,S) \cdot Y + K \frac{2}{g_x}(X,Y) \cdot S = 0. \quad (2.20)$$

Furthermore, we find

$$\frac{2}{g_x}(X,H_x(Y)) = A_x(X,Y) = A_x(Y,X) = \frac{2}{g_x}(Y,H_x(X)) = \frac{2}{g_x}(H_x(X),Y), \quad (2.21)$$

that is,

$$\frac{2}{g_x}(X,H_x(Y)) = \frac{2}{g_x}(H_x(X),Y), \quad (2.22)$$
and then from (2.20) and (2.22), we obtain
\[
\begin{align*}
\frac{2}{\varrho_x}(H_x(X),S) & \cdot H_x(Y) - \frac{2}{\varrho_x}(H_x(X),Y) \cdot H_x(S) \\
- K & \frac{2}{\varrho_x}(X,S) \cdot Y + K \frac{2}{\varrho_x}(X,Y) \cdot S = 0,
\end{align*}
\] (2.23)
for all \( x = \Psi(\bar{x}) \in \Psi(V) \subset F \) and all \( X, Y, S \in F \).

Since \( \dim F > 2 \), then, for every \( X, Y \in F \) such that \( \frac{2}{\varrho_x}(X,Y) = 0 \), there exists a vector \( S \in F \) orthogonal to each \( X \) and \( H_x(X) \) [2]. Using this fact in (2.23) and taking into account (2.12), we obtain \( A_x(X,Y) \cdot H_x(S) = 0 \). By using the nonsingularity of the tensor \( A_x \), we conclude that \( A_x(X,Y) = 0 \). Since, for any pair of vectors \( X, Y \in F, \frac{2}{\varrho_x}(X,Y) = 0 \) implies that \( A_x(X,Y) = 0 \), then there exists a real number \( \lambda \) such that (see [2])
\[
A_x(X,Y) = \frac{\lambda}{2} \varrho_x(X,Y).
\] (2.24)
Substituting (2.24) into (2.16), we obtain
\[
\lambda \frac{2}{\varrho_x}(Z,Y) \frac{2}{\varrho_x}(X,S) = K \frac{2}{\varrho_x}(Z,Y) \frac{2}{\varrho_x}(X,S),
\] (2.25)
for all \( x = \Psi(\bar{x}) \in \Psi(V) \subset F \) and all \( X, Y, Z, S \in F \). Taking into account the nonsingularity of \( \frac{2}{\varrho_x} \), we obtain \( \lambda^2 = K = K_0/e \). It is convenient to put \( K_0 = e/r^2 \), where \( r \) is a nonzero real number and \( e = \pm 1 \), then we have \( \lambda = \pm 1/r \). We find that in our case, it is convenient to take \( \lambda = -1/r \). Substituting \( \lambda \) in (2.24), we obtain
\[
A_x(X,Y) = -\frac{1}{r} \varrho_x(X,Y),
\] (2.26)
and in fact this equation is the unique solution, up to sign, of (2.9) and (2.13). Substituting this solution in (2.12), we have
\[
\frac{2}{\varrho_x}(X,H_x(Y)) = \frac{1}{r} \varrho_x(X,Y), \quad \forall x \in \Psi(V) \subset F, \forall X,Y \in F,
\] (2.27)
which implies that \( H_x(Y) = (1/r)Y \). Hence (2.10) will be
\[
D \xi_x(X) = \frac{1}{r} D i_x(X).
\] (2.28)
Integrating this equation gives us \( \xi_x = (1/r)i(x) \). Then
\[
\frac{1}{r} \varrho(i(x),i(x)) = r^2 \frac{1}{r} \varrho(\xi_x,\xi_x).
\] (2.29)
Letting \( y = i(x) \) and using equalities (2.8), the above equation takes the form
\[
\frac{1}{r} \varrho(y,y) = er^2, \quad \forall x \in \Psi(V) \subset F, \quad e = \pm 1.
\] (2.30)
This last equation shows that the hypersurface \( N \subset M \) of constant nonzero Riemannian curvature will be locally an essential hypersurface of second order, and this completes the proof. □
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