A note on scheduling with low rank processing times

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Abstract

We consider the classical minimum makespan scheduling problem, where the processing time of job $j$ on machine $i$ is $p_{ij}$, and the matrix $P = (p_{ij})_{m \times n}$ is of a low rank. It is proved in [1] that rank 7 scheduling is NP-hard to approximate to a factor of $3/2 - \epsilon$, and rank 4 scheduling is APX-hard (NP-hard to approximate within a factor of $1.03 - \epsilon$). We improve this result by showing that rank 4 scheduling is already NP-hard to approximate within a factor of $3/2 - \epsilon$, and meanwhile rank 3 scheduling is APX-hard.

Keywords. Complexity, APX-hardness, Scheduling

1 Introduction

Recently Bhaskara et al. [1] study the minimum makespan scheduling problem in which the processing time of job $j$ on machine $i$ is $p_{ij}$, and the matrix formed by the processing times $P = (p_{ij})_{m \times n}$ is of a low rank. Formally speaking, in this problem the matrix of processing times could be expressed as $P = M J$, where $M$ is an $m \times D$ matrix in which the row vector $u_i$ represents the D-dimensional speed vector of machine $i$, and $J$ is a $D \times n$ matrix in which the column vector $v_j^T$ represents the D-dimensional size vector of job $j$. The processing time of job $j$ on machine $i$ is defined by $u_i \cdot v_j^T$. We adopt the notations of [1] by denoting the above problem as $LRS(D)$. It is easy to see that in $LRS(D)$, the rank of the matrix $P$ is at most $D$.

It is a new way of studying the traditional scheduling problem. From this point of view the unrelated machine scheduling problem is a scheduling problem where the matrix of job processing times could be of arbitrary rank, while the related machine scheduling problem is a scheduling problem where the matrix is of rank one.

In 1988, Hochbaum and Shmoys [4] gave a PTAS (polynomial time approximation scheme) for the related machine scheduling problem, i.e., $LRS(1)$. Later, Lenstra et al. [5] provided a 2-approximation algorithm for the unrelated machine scheduling problem, i.e., $LRS(D)$ for arbitrary $D$. Such a result was improved to a $(2 - 1/m)$-approximation algorithm by Shchepin and Vakhania [6]. It remains open whether there exists a polynomial time algorithm with approximation ratio strictly less than 2 for the unrelated machine scheduling problem.

Bhaskara et al. [1] prove that $LRS(D)$ is APX-hard (NP-hard to approximate within a factor of $1.03 - \epsilon$) when $D = 4$, and NP-hard to approximate within a factor of $3/2 - \epsilon$ when $D = 7$. 
In this paper we improve the results in [1] by showing that \( LRS(4) \) is already NP-hard to approximate within a factor of \( \frac{3}{2} - \epsilon \), and \( LRS(3) \) is APX-hard.

Roughly speaking, the overall structure of our reduction for \( LRS(4) \) is similar to that of [1]. The key ingredient to our stronger result is that we construct the reduction from a variation of the 3-dimensional matching problem (instead of the standard 3-dimensional matching problem, as is used in [1]), and design the job processing times in a more delicate way. For the APX-hardness of \( LRS(3) \), we make use of the idea from [2] to design the reduction from the one-in-three 3SAT problem.

## 2 Inapproximability of Rank 4 scheduling

In this section, we prove that \( LRS(4) \) is already NP-hard to approximate within a factor of \( \frac{3}{2} - \epsilon \) for any \( \epsilon > 0 \) via a reduction from a variation of the 3-dimensional matching problem, as is shown in the following.

### 2.1 A variation of the 3-Dimensional Matching problem

The standard 3DM problem contains three disjoint element sets \( W \cup X \cup Y \) where \( |W| = |X| = |Y| \), and a set of triples \( T \subseteq \{(w_i, x_j, y_k) | w_i \in W, x_j \in X, y_k \in Y\} \) where every triple of \( T \) is called as a match. A perfect matching for 3DM is a subset \( T' \subseteq T \) in which every element of \( W \cup X \cup Y \) appears exactly once. Deciding whether there exists a perfect matching for 3DM is NP-complete [3].

In the standard 3DM, the subscripts of elements in a match could be arbitrary. In this paper, however, we focus on the following restricted form of 3DM.

- **Elements:** there are three disjoint sets of elements \( W = \{w_i, \bar{w}_i | i = 1, \cdots, 3n\} \), \( X = \{s_i, a_i | i = 1, \cdots, 3n\} \) and \( Y = \{s'_i, b_i | i = 1, \cdots, 3n\} \) where \( |W| = |X| = |Y| = 6n \)

- **Matches:** there are two sets of matches \( T_1 \subseteq \{(w_i, s_j, s'_j) | w_i \in W, s_j \in X, s'_j \in Y\} \), \( T_2 = \{(w_i, a_i, b_{\zeta(i)}) | i = 1, \cdots, 3n\} \) where \( \zeta \) is defined as \( \zeta(3k+1) = 3k+2, \zeta(3k+2) = 3k+3 \) and \( \zeta(3k+3) = 3k+1 \) for \( k = 0, \cdots, n - 1 \)

We remark that in the special form of 3DM above, \( T_2 \) is already fixed. Similarly, a subset of \( T_1 \cup T_2 \) is called a perfect matching if among its matches every element of \( W \cup X \cup Y \) appears exactly once. For simplicity, we denote the problem of determining whether the set of matches in the above form admits a perfect matching as \( 3DM' \). We prove that the \( 3DM' \) problem is NP-complete in the following theorem. It turns out that the idea of the proof is similar to that of [2].

**Theorem 1** \( 3DM' \) is NP-complete.

**Proof.** We reduce 3SAT to \( 3DM' \). Given an instance of 3SAT, say, \( I_{sat} \), we first apply Tovey’s method [7] to alter it into \( I'_{sat} \) so that every variable appears exactly three times. It is simple,
if a variable, say, \( z \), only appear once, then we add a dummy clause \( (z \lor \neg z) \). Otherwise it appears \( d \geq 2 \) times, and we replace all its occurrences by new variables \( z_1, z_2, \ldots, z_d \), one for each, and meanwhile add clauses \( (z_1 \lor \neg z_2), (z_2 \lor \neg z_3), \ldots, (z_d \lor \neg z_1) \) to enforce that \( z_1 \) to \( z_d \) are taking the same truth value. Let \( I'_{\text{sat}} \) be the new SAT instance, then we have

- Every variable appears exactly three times in \( I'_{\text{sat}} \).
- \( I'_{\text{sat}} \) is satisfiable if and only if \( I_{\text{sat}} \) is satisfiable.

We apply Tovey’s method for a second time to transform \( I'_{\text{sat}} \) to \( I''_{\text{sat}} \). Since every variable, say, \( z_i \), appears three times in \( I'_{\text{sat}} \), we replace its three occurrences with \( \hat{z}_{3i-2}, \hat{z}_{3i-1} \) and \( \hat{z}_{3i} \), and meanwhile add \( (\hat{z}_{3i-2} \lor \neg \hat{z}_{3i-1}), (\hat{z}_{3i-1} \lor \neg \hat{z}_{3i}), (\hat{z}_{3i} \lor \neg \hat{z}_{3i-2}) \). It is not difficult to verify that \( I''_{\text{sat}} \) satisfies the following conditions:

- Clauses of \( I''_{\text{sat}} \) could be divided into \( C_1 \) and \( C_2 \) such that
  - either \( z_i \) or \( \neg z_i \) appears in \( C_1 \), and it appears once
  - all the clauses of \( C_2 \) could be listed as \( (z_{3i-2} \lor \neg z_{3i-1}), (z_{3i-1} \lor \neg z_{3i}), (z_{3i} \lor \neg z_{3i-2}) \) for \( i = 1, \ldots, n \).
- \( I''_{\text{sat}} \) is satisfiable if and only if \( I'_{\text{sat}} \) is satisfiable.

We construct an instance \( I_{3dn} \) (of 3DM’) such that it admits a perfect matching if and only if \( I''_{\text{sat}} \) is satisfiable, and thus if and only if \( I_{\text{sat}} \) is satisfiable.

**Real elements:** we construct \( w_i \in W \) for every positive literal \( z_i \), and \( \bar{w}_i \in W \) for negative literal \( \neg z_i \). We construct \( s_j \in X \) and \( s'_j \in Y \) for every clause \( \beta_j \in C_1 \).

**Dummy elements:** we construct \( a_i \in X, b_i \in Y \) for \( 1 \leq i \leq 3n \), and \( u_j \in X, u'_j \in Y \) for \( 1 \leq j \leq 3n - |C_1| \) (here \( |C_1| \) is the number of clauses in \( C_1 \)). Thus in all, \( |W| = |X| = |Y| = 6n \).

**Real matches:** if the positive literal \( z_i \) is in clause \( \beta_j \in C_1 \), we construct \( (w_i, s_j, s'_j) \). Else if \( \neg z_i \) is in \( \beta_j \), we construct \( (\bar{w}_i, s_j, s'_j) \).

**Dummy matches:** we construct \( (w_i, a_i, b_i) \) and \( (\bar{w}_i, a_i, b_{\beta(i)}^\prime) \) for \( i = 1, \ldots, 3n \). We also construct \( (w_i, u_j, u'_j) \) and \( (\bar{w}_i, u_j, u'_j) \) for every \( 1 \leq i \leq 3n \) and \( 1 \leq j \leq 3n - |C_1| \).

It is easy to see that the above instance is an instance of 3DM’ where \( W = \{w_i, \bar{w}_i | 1 \leq i \leq 3n\}, X = \{a_i, s_j, u_k | 1 \leq i \leq 3n, 1 \leq j \leq |C_1|, 1 \leq k \leq 3n - |C_1|\} \), \( Y = \{b_i, s'_j, u'_k | 1 \leq i \leq 3n, 1 \leq j \leq |C_1|, 1 \leq k \leq 3n - |C_1|\} \), \( T_2 \) is a subset of dummy matches (i.e., \( (w_i, a_i, b_i) \) and \( (\bar{w}_i, a_i, b_{\beta(i)}^\prime) \)), and \( T_1 \) is the set of remaining dummy matches (i.e., \( (w_i, u_j, u'_j) \) and \( (\bar{w}_i, u_j, u'_j) \)) together with all the real matches.

**Completeness.** Suppose \( I''_{\text{sat}} \) is satisfiable, we prove that \( I_{3dn} \) admits a perfect matching by selecting them out from \( T_1 \cup T_2 \). Since clauses \( (z_{3i+1} \lor \neg z_{3i+2}), (z_{3i+2} \lor \neg z_{3i+3}) \) and \( (z_{3i+3} \lor \neg z_{3i}) \) are all satisfied for \( 0 \leq i \leq n - 1 \), the variables \( z_{3i+1}, z_{3i+2}, z_{3i+3} \) should be all true or all false. If they are all false, we take out \( (w_{3i+1}, a_{3i+1}, b_{3i+1}), (w_{3i+2}, a_{3i+2}, b_{3i+2}) \) and \( (w_{3i+3}, a_{3i+3}, b_{3i+3}) \) from \( T_2 \). Otherwise we take out \( (w_{3i+1}, a_{3i+1}, b_{3i+2}), (w_{3i+2}, a_{3i+2}, b_{3i+3}) \) and \( (w_{3i+3}, a_{3i+3}, b_{3i+1}) \) from \( T_2 \) instead. Since each clause \( \beta_j \in C_1 \) is satisfied, it is satisfied by at least one literal in
it. Suppose it is satisfied by the literal $z_i$ (or $\neg z_i$), then the variable $z_i$ is true (or false), and we select $(u_i, s_j, s'_j)$ (or $(\bar{w}_i, s_j, s'_j)$) from $T_1$. Now consider all the matches we select out so far. We have selected $3n + |C_1|$ matches, and among them every $s_j, s'_j$, $a_i, b_i$ appear once, and every $u_i$ (or $\bar{w}_i$) appears at most once. Thus, there are $3n - |C_1|$ elements of $W$ that do not appear in these matches, and we select $3n - |C_1|$ dummy matches from $T_1$ so that every $u_j, u'_j$ and $w_i, \bar{w}_i$ appear once.

**Soundness.** Suppose there exists a perfect matching of $I_{3dm}$, say, $T' \subseteq T_1 \cup T_2$, we prove that $I''_{sat}$ is satisfiable.

Consider elements of $X$ and $Y$. For each $0 \leq i \leq n - 1$, to ensure that $a_{3i+1}, b_{3i+1}, a_{3i+2}, b_{3i+2}$ and $a_{3i+3}, b_{3i+3}$ appear once respectively, in the perfect matching $T'$ we have to choose either $(\bar{w}_{3i+1}, a_{3i+1}, b_{3i+2}), (\bar{w}_{3i+2}, a_{3i+2}, b_{3i+3}), (\bar{w}_{3i+3}, a_{3i+3}, b_{3i+1})$, or choose $(w_{3i+1}, a_{3i+1}, b_{3i+1}), (w_{3i+2}, a_{3i+2}, b_{3i+2}), (w_{3i+3}, a_{3i+3}, b_{3i+3})$.

If $(\bar{w}_{3i+1}, a_{3i+1}, b_{3i+2}), (\bar{w}_{3i+2}, a_{3i+2}, b_{3i+3}), (\bar{w}_{3i+3}, a_{3i+3}, b_{3i+1})$ are in $T'$, we let variables $z_{3i+1}, z_{3i+2} + z_{3i+3}$ be true. Otherwise we let variables $z_{3i+1}, z_{3i+2}$ and $z_{3i+3}$ be false. It can be easily verified that every clause of $C_2$ is satisfied.

We consider $\beta_j \in C_1$. Notice that $s_j \in X$ appears once in $T'$. Suppose the match containing $s_j$ is $(u_i, s_j, s'_j)$ for some $i$, then it follows that the positive literal $z_i \in \beta_j$. The fact that $w_i$ can only appear once in $T'$ implies that $(w_i, a_i, b_i)$ is not in $T'$, and thus the variable $z_i$ is true and $\beta_j$ is satisfied. Otherwise the matching containing $s_j$ is $(\bar{w}_i, s_j, s'_j)$ for some $i$, and similar arguments show that the negative literal $\neg z_i \in \beta_j$ and variable $z_i$ is false, again $\beta_j$ is satisfied.

\[\square\]

### 2.2 Factor 3/2 hardness for rank 4 scheduling

We prove the following theorem.

**Theorem 2** For every $\epsilon > 0$, there is no $(3/2 - \epsilon)$-approximation algorithm for LRS(4), assuming $P \neq NP$.

**Proof.** Given an instance $I_{3dm}$ of 3DM', we construct an instance $I_{sch}$ of LRS(4) such that if $I_{3dm}$ admits a perfect matching, then there exists a feasible schedule of $I_{sch}$ with makespan $2 + O(\epsilon)$ (where $\epsilon < 1/6$ is an arbitrary positive number), otherwise there is no feasible schedule of makespan less than 3.

We construct $I_{sch}$ that consists of the following parts.

- **Machines:** there are $|T| = |T_1 \cup T_2|$ machines, one for every match.
- **Real jobs:** there is one job for each element of $X \cup Y$.
- **Dummy jobs:** if $w_i$ (or $\bar{w}_i$) appears $d(w_i)$ (or $d(\bar{w}_i)$) times in all the matches, then there are $d(w_i) - 1$ (or $d(\bar{w}_i) - 1$) jobs for $w_i$ (or $\bar{w}_i$).

Let $N = O(n/\epsilon^2)$. We aim to design the speeds of machines and the size (workload) of jobs such that if an element job is put on a match machine whose corresponding match contains
this element, then its processing time is $1 + O(\epsilon)$, otherwise the processing time is at least $1/(2\epsilon) > 3$.

See Table 1 as the speed vectors of machines, and Table 2 as the size vector of jobs. Here for simplicity we use a match to denote its corresponding machine, and an element to denote its corresponding job. Recall that the processing time of a job on a machine is defined to be the inner product of their corresponding vectors.

### Table 1: Speed Vectors of Machines

| Machines | Speeds |
|----------|--------|
| $(w_i, s_i, s_j')$ | $(N^1_i, N^{-1}_i, N^{j+1}_i, N^{-j-N}_i)$ |
| $(w_i, s_i, s_j)$ | $(N^{-i}i, N^1_i, N^{j+N}_i, N^{-j-N}_i)$ |
| $(w_{3i}, a_{3i}, b_{3i})$ | $(N^{3i}_i, N^{-3i}_i, N^{-3i-1}_i, N^{-3i-1}_i)$ |
| $(w_{3i+1}, a_{3i+1}, b_{3i+1})$ | $(N^{3i+1}_i, N^{-3i-1}_i, N^{-3i-1}_i, 2N^{-3i})$ |
| $(w_{3i+2}, a_{3i+2}, b_{3i+2})$ | $(N^{3i+2}_i, N^{-3i-2}_i, N^{-3i-2}_i, 2/N^{-3i-1})$ |
| $(w_{3i+1}, a_{3i+1}, b_{3i+2})$ | $(N^{-3i-1}_i, N^{3i+1}_i, N^{-3i-1}_i, 1/N^{-3i-1})$ |
| $(w_{3i+2}, a_{3i+2}, b_{3i})$ | $(N^{-3i-2}_i, N^{3i+2}_i, N^{-3i-2}_i, 2N^{-3i-1})$ |

### Table 2: Size Vectors of Jobs

| Jobs | Sizes |
|------|-------|
| $w_i$ | $(N^{-1}, N^0, 0, 0)$ |
| $\bar{w}_i$ | $(N^0, N^{-1}, 0, 0)$ |
| $s_j$ | $(0, 0, 1/2N^{-j-N}, 1/2N^{j+N})$ |
| $s'_j$ | $(0, 0, 1/2N^{-j-N}, 1/2N^{j+N})$ |
| $a_i$ | $(N^{-1}_i, N^{0}_i, 0, 0)$ |
| $b_{3i}$ | $(\epsilon N^{-3i}, \epsilon N^{-3i-1}, 1/2N^{3i}, 1/2N^{3i+1})$ |
| $b_{3i+1}$ | $(\epsilon N^{-3i-1}, 1/\epsilon N^{-3i-1}, 1/(2\epsilon)N^{3i}, 1/2N^{3i})$ |
| $b_{3i+2}$ | $(\epsilon N^{-3i-2}, \epsilon N^{-3i-1}, 1/2N^{3i+1}, \epsilon/2N^{3i+1})$ |

We check the processing times of jobs on different machines. The following observation is easy to verify (by focusing on the first three coordinates of vectors).

- A $w_i$-job (or $\bar{w}_i$-job) has a processing time of 2 on a machine whose corresponding match contains $w_i$ (or $\bar{w}_i$), and has a processing time of $\Omega(N)$ on other machines.
- An $s_j$-job ($s'_j$-job) has a processing time of 1 on a machine whose corresponding match contains $s_j$ ($s'_j$), and has a processing time of $\Omega(N)$ on other machines.
- An $a_i$-job has a processing time of $1 + O(\epsilon)$ on a machine whose corresponding match contains $a_i$, and has a processing time of $\Omega(\epsilon N)$ on other machines.
For $b_i$-jobs, the reader may refer to Table 3 for their processing times.

**Completeness.** Suppose $I_{3dm}$ admits a perfect matching, we prove that $I_{sch}$ admits a feasible schedule of makespan $2 + O(\epsilon)$. We let $M_1$ be the set of machines corresponding to the matches of the perfect matching, and $M_2$ be the set of remaining machines. We provide a schedule in which all the real jobs are put onto machines of $M_1$, while all the dummy jobs are put onto machines of $M_2$. Since every element of $X \cup Y$ appears once in the perfect matching, we put an $s_j$-job onto a machine of $M_1$ whose corresponding match contains $s_j$. $s'_j$-jobs, $a_1$-jobs and $b_i$-jobs are scheduled in the same way. It is easy to see that the load of each machine in $M_1$ is $2 + O(\epsilon)$. Meanwhile, every $w_i$ (or $\bar{w}_i$) also appears once in the perfect matching, thus it appears $d(w_i) - 1$ (or $d(\bar{w}_i) - 1$) times in the remaining matches. Notice that the number of $w_i$-jobs (or $\bar{w}_i$-jobs) is $d(w_i) - 1$ (or $d(\bar{w}_i) - 1$), thus we can put one $w_i$-job (or $\bar{w}_i$-job) onto a machine in $M_2$ whose corresponding match contains $w_i$ (or $\bar{w}_i$), and again it is easy to see that the load of each machine in $M_2$ is at most 2.

**Soundness.** Suppose $I_{sch}$ admits a feasible schedule of makespan strictly less than 3, we prove that $I_{3dm}$ admits a perfect matching.

As $1/(2\epsilon) > 3$, according to our discussion on the processing times we know every element job should be on a machine whose corresponding match contains this element. Notice that the processing time of a $w_i$-job (or $\bar{w}_i$-job) is at least 2, while the processing time of an $a_1$-job, $b_i$-job, $s_j$-job or $s'_j$-job is at least 1, thus every $w_i$-job (or $\bar{w}_i$-job) occupies one machine, and there are no other jobs on this machine. Let $M_2$ be the set of machines where $w_i$-jobs and $\bar{w}_i$-jobs are scheduled, and let $M_1$ be the set of remaining machines, we show that the matches corresponding to machines in $M_1$ forms a perfect matching. Let $T'$ be the set of these matches. Notice that the number of $w_i$-jobs (or $\bar{w}_i$-jobs) is $d(w_i) - 1$ (or $d(\bar{w}_i) - 1$), thus in $T'$ every $w_i$ (or $\bar{w}_i$) appears exactly once, which implies that $|T'| = 6n$. Furthermore, all the jobs corresponding to elements of $X \cup Y$ are on machines of $M_1$, thus every element of $X \cup Y$ appears at least once in $T'$. Notice that $|W| = |X| = |Y| = 6n$, since every element of $W \cup X \cup Y$ appears at least once in $T'$ and $|T'| = 6n$, we conclude that every element of $W \cup X \cup Y$ appears exactly once in $T'$, implying that $T'$ is a perfect matching. □

### Table 3: Processing times of $b_i$-jobs

| Machines/Jobs                  | $b_{3i}$ | $b_{3i+1}$ | $b_{3i+2}$ |
|-------------------------------|----------|------------|------------|
| $(w_{3i}, a_{3i}, b_{3i})$    | $1 + O(\epsilon)$ | $1/(2\epsilon) + O(\epsilon)$ | $\Omega(N)$ |
| $(w_{3i+1}, a_{3i+1}, b_{3i+1})$ | $\Omega(N)$ | $1 + O(\epsilon)$ | $\Omega(\epsilon N)$ |
| $(w_{3i+2}, a_{3i+2}, b_{3i+2})$ | $\Omega(N)$ | $\Omega(\epsilon N)$ | $1 + O(\epsilon)$ |
| $(\bar{w}_{3i}, a_{3i}, b_{3i+1})$ | $\Omega(N)$ | $1 + O(\epsilon)$ | $\Omega(\epsilon N)$ |
| $(\bar{w}_{3i+1}, a_{3i+1}, b_{3i+2})$ | $\Omega(N)$ | $1/(2\epsilon) + O(\epsilon)$ | $1 + O(\epsilon)$ |
| $(\bar{w}_{3i+2}, a_{3i+2}, b_{3i})$ | $1 + O(\epsilon)$ | $\Omega(N)$ | $\Omega(\epsilon N)$ |
| Other machines                | $\Omega(\epsilon N)$ | $\Omega(\epsilon N)$ | $\Omega(\epsilon N)$ |
3 APX-hardness for rank 3 scheduling

We start with the one-in-three 3SAT problem. It is a variation of the 3SAT problem. Precisely speaking, an input of the one-in-three 3SAT is a collection of clauses where each clause consists of exactly three literals, and the problem is to determine whether there exists a truth assignment of the variables such that each clause is satisfied by exactly one literal (i.e., one literal is true and two other literals are false).

It is proved in \[8\] that the one-in-three 3SAT problem is NP-complete.

Given an instance of the one-in-three 3SAT problem, say, \(\text{I}_{\text{sat}}\), we can apply Tovey’s method to transform it into \(\text{I}'_{\text{sat}}\) such that

- Each clause of \(\text{I}_{\text{sat}}\) contains two or three literals
- Each variable appears three times in clauses, among the three occurrence there are either two positive literals and one negative literal, or one positive literal and two negative literals
- There exists a truth assignment for \(\text{I}'_{\text{sat}}\) where every clause is satisfied by exactly one literal if and only if there is a truth assignment for \(\text{I}_{\text{sat}}\) where every clause is satisfied by exactly one literal

The transformation is straightforward. For any variable \(z\), if it only appears once in the clauses, then we add a dummy clause as \((z \lor \neg z)\). Otherwise suppose it appears \(d \geq 2\) times in the clauses, then we replace its \(d\) occurrences with \(d\) new variables as \(z_1, z_2, \ldots, z_d\), and meanwhile add \(d\) clauses as \((z_1 \lor \neg z_2), (z_2 \lor \neg z_3), \ldots, (z_d \lor \neg z_1)\) to enforce that these new variables should take the same truth assignment. It is not difficult to verify that the constructed instance satisfies the above requirements.

Let \(\epsilon\) be an arbitrary small positive. Throughout the following part of this section we assume that \(\text{I}'_{\text{sat}}\) contains \(n\) variables and \(m\) clauses, and let \(\xi = 2^3, r = 2^{10} \cdot 2^{13}, N = n/\epsilon^2\).

We will construct a scheduling instance \(\text{I}_{\text{sch}}\) in the following part of this section such that if there exists a truth assignment for \(\text{I}'_{\text{sat}}\) where every clause is satisfied by exactly one literal, then \(\text{I}_{\text{sch}}\) admits a feasible schedule whose makespan is \(r + O(\epsilon)\). On the other hand if \(\text{I}_{\text{sch}}\) admits a feasible schedule whose makespan is strictly less than \(r + 1\), then there exists a truth assignment for \(\text{I}'_{\text{sat}}\) where every clause is satisfied by exactly one literal. This would be enough to prove that an algorithm of approximation ratio strictly less than \(1 + 1/(r+1) = 1 + 1/(2^{13} + 1)\) implies that \(P = \text{NP}\).

3.1 Construction of the scheduling instance

We construct jobs. For every variable \(z_i\), eight variable jobs are constructed, namely \(v^\gamma_{i,k}\) for \(k = 1, 2, 3, 4\) and \(\gamma = T, F\). The size vectors are (for simplicity, we use \(s(j)\) to denote the size vector of job \(j\)): 

\[s(j) = (s_1(j), s_2(j), \ldots, s_m(j))\]
\[ s(v_{i,1}^T) = (\epsilon N^{4i+1}, 0, 1/8r - 10\xi - 2), s(v_{i,2}^T) = (\epsilon N^{4i+2}, 0, 1/8r - 20\xi - 2), \]
\[ s(v_{i,3}^T) = (\epsilon N^{4i+3}, 0, 1/8r - 18\xi - 2), s(v_{i,4}^T) = (\epsilon N^{4i+4}, 0, 1/8r - 12\xi - 2). \]
\[ s(v_{i,k}^F) = s(v_{i,k}^T) - (0, 0, 2), k = 1, 2, 3, 4 \]

For every variable \( z_i \), eight truth-assignment jobs are constructed, namely \( a_i^T, b_i^T, c_i^T, d_i^T \) with \( \gamma = T, F \). The size vectors are:

\[ s(a_i^T) = (0, \epsilon N^i, 2\xi + 1), s(b_i^T) = (0, \epsilon N^i, 4\xi + 1), \]
\[ s(c_i^T) = (0, \epsilon N^i, 8\xi + 1), s(d_i^T) = (0, \epsilon N^i, 16\xi + 1). \]
\[ s(\tau_i^F) = s(\tau_i^T) + (0, 0, 1), \tau = a, b, c, d \]

For every clause \( \beta_j \), if it contains two literals, then we construct two clause jobs, namely \( u_j^T \) and \( u_j^F \). Otherwise it contains three literals, and we construct three clause jobs, namely one \( u_j^T \) and two \( u_j^F \). The size vectors are:

\[ s(u_j^T) = (0, \epsilon N^{N+j}, 1/4r + 2), s(u_j^F) = (0, \epsilon N^{N+j}, 1/4r + 4). \]

We construct \( 2n - m \) true dummy jobs \( \phi^T = (0, 0, 1/16r + 2) \), and \( m - n \) false dummy jobs \( \phi^F = (0, 0, 1/16r + 4) \) (here it is not difficult to verify that \( n \leq m \)).

Finally we construct huge jobs. Indeed, there is a one-to-one correspondence between huge jobs and machines. For ease of description we first construct machines, and then construct those huge jobs.

We construct \( 8n \) machines.

For every variable \( z_i \), we construct \( 4n \) truth assignment machines, and they are denoted as \((v_{i,1}, a_i, c_i), (v_{i,2}, b_i, d_i), (v_{i,3}, a_i, d_i), (v_{i,4}, b_i, c_i)\). The symbol of a machine indicates the jobs on it (except the huge jobs) in the solution with makespan at most \( r + 2\epsilon \). The speed vectors are (For simplicity the speed vector of a machine, say, \((v_{i,1}, a_i, c_i)\), is denoted as \(g(v_{i,1}, a_i, c_i)\)):

\[ g(v_{i,1}, a_i, c_i) = (N^{-4i-1}, N^{-i}, 1), g(v_{i,2}, b_i, d_i) = (N^{-4i-2}, N^{-i}, 1), \]
\[ g(v_{i,3}, a_i, d_i) = (N^{-4i-3}, N^{-i}, 1), g(v_{i,4}, b_i, c_i) = (N^{-4i-4}, N^{-i}, 1). \]

For every clause \( \beta_j \), if the positive (or negative) literal \( z_i \) (or \( \neg z_i \)) appears in it for the first time (i.e., it does not appear in \( \beta_k \) for \( k < j \)), then we construct a clause machine \((v_{i,1}, u_j)\) (or \((v_{i,3}, u_j)\)). Else if it appears for the second time, then we construct a clause machine \((v_{i,2}, u_j)\) (or \((v_{i,4}, u_j)\)). The speed vectors are:

\[ g(v_{i,k}, u_j) = (N^{-4i-k}, N^{-N-j}, 1). \]

Recall that for every variable, in all the clauses there are either one positive literal and two negative literals, or two positive literals and one negative literal. If \( z_i \) appears once and \( \neg z_i \)
appears twice, then we construct a dummy machine \((v_{i,2}, \phi)\), otherwise we construct a dummy machine \((v_{i,4}, \phi)\). The speed vectors are:

\[
g(v_{i,2}, \phi) = (N^{-4i-2}, 0, 1), g(v_{i,4}, \phi) = (N^{-4i-4}, 0, 1).
\]

According to our construction, it is not difficult to verify that if \(z_i\) appears once and \(\neg z_i\) appears twice, then we construct machines \((v_{i,k}, u_{jk})\) for \(k = 1, 3, 4, 1 \leq j_k \leq m\), and machine \((v_{i,2}, \phi)\). Otherwise we construct machines \((v_{i,k}, u_{jk})\) for \(k = 1, 2, 3, 1 \leq j_k \leq m\), and machine \((v_{i,4}, \phi)\).

We now describe the huge jobs. There is one huge job for each machine and for simplicity, we also use the symbol of a machine to denote its corresponding huge job. The size vectors are:

\[
s(v_{i,1}, a_i, c_i) = (\epsilon N^{4i+1}, \epsilon N^i, 7/8r), s(v_{i,2}, b_i, d_i) = (\epsilon N^{4i+2}, \epsilon N^i, 7/8r),
\]

\[
s(v_{i,3}, a_i, d_i) = (\epsilon N^{4i+3}, \epsilon N^i, 7/8r), s(v_{i,4}, b_i, c_i) = (\epsilon N^{4i+4}, \epsilon N^i, 7/8r).
\]

\[
s(v_{i,1}, u_j) = (0, \epsilon N^{N^j}, 5/8r + 10\xi), s(v_{i,2}, u_j) = (0, \epsilon N^{N^j}, 5/8r + 20\xi),
\]

\[
s(v_{i,3}, u_j) = (0, \epsilon N^{N^j}, 5/8r + 18\xi), s(v_{i,1}, u_j) = (0, \epsilon N^{N^j}, 5/8r + 12\xi),
\]

\[
s(v_{i,2}, \phi) = (0, N^{2N}, 13/16r + 20\xi), s(v_{i,4}, \phi) = (0, N^{2N}, 13/16r + 12\xi).
\]

### 3.2 From 3SAT to Scheduling

Given a truth assignment of \(I_{sat}'\), we schedule jobs according to Table 4.

| Table 4: Overview of jobs |
|---------------------------|
| machines                  | jobs                          |
| \((v_{i,1}, a_i, c_i)\)   | \((v_{i,1}, a_i, c_i, (v_{i,1}, a_i, c_i)\) |
| \((v_{i,2}, b_i, d_i)\)   | \((v_{i,2}, b_i, d_i, (v_{i,2}, b_i, d_i)\) |
| \((v_{i,3}, a_i, d_i)\)   | \((v_{i,3}, a_i, d_i, (v_{i,3}, a_i, d_i)\) |
| \((v_{i,4}, b_i, c_i)\)   | \((v_{i,4}, b_i, c_i, (v_{i,4}, b_i, c_i)\) |
| \((v_{i,k}, u_j)\)        | \((v_{i,k}, u_j, (v_{i,k}, u_j)\) |
| \((v_{i,k}, \phi)\)       | \((v_{i,k}, \phi, (v_{i,k}, \phi)\) |

Recall that except for the huge jobs, the symbol of a job, say, \(a_i\), may represent either \(a_i^T\) or \(a_i^F\), we determine whether each job in the above table is true or false according to the truth assignment of variables.

If variable \(z_i\) is false, then we schedule jobs on truth assignment machines as \((v_{i,1}^T, a_i^T, c_i^T)\), \((v_{i,2}^T, b_i^T, d_i^T)\), \((v_{i,3}^T, a_i^T, d_i^T)\), \((v_{i,4}^T, b_i^T, c_i^T)\), otherwise we schedule jobs as \((v_{i,1}^F, a_i^F, c_i^F)\), \((v_{i,2}^F, b_i^F, d_i^F)\), \((v_{i,3}^F, a_i^F, d_i^F)\), \((v_{i,4}^F, b_i^F, c_i^F)\).

Notice that every clause, say, \(\beta_j\), is satisfied by exactly one literal. Suppose it contains three variables (the argument is the same if it contains two literals), namely, \(z_{i_1}\), \(z_{i_2}\) and \(z_{i_3}\) and is satisfied by the first variable.
Consider variable $z_i$. According to the construction of machines if $\beta_j$ contains its positive literal then machine $(v_{i_1,k_1}, u_j)$ is constructed for $k_1 \in \{1, 2\}$, and we schedule $u_j^T$ and $v_{i_1,k_1}^T$ on this machine. This is possible since variable $z_i$ is true, and $v_{i_1,k_1}^T$ is thus not scheduled with truth assignment jobs. Similarly if $\beta_j$ contains the negative literal $\neg z_i$, then the satisfaction of $\beta_j$ by variable $z_i$ implies that this variable is false. Furthermore, machine $(v_{i_1,k_1}, u_j)$ is constructed for $k_1 \in \{3, 4\}$ and again we schedule jobs $u_j^T$ and $v_{i_1,k_1}^T$ on this machine.

Consider variable $z_{i_2}$ (for variable $z_{i_3}$ the argument is the same). Again if $\beta_j$ contains its positive literal then machine $(v_{i_2,k_2}, u_j)$ is constructed for $k_2 \in \{1, 2\}$, and we schedule $u_j^F$ and $v_{i_2,k_1}^F$ on it. This is possible since $\beta_j$ is not satisfied by literal $z_{i_2}$ and the variable $z_{i_2}$ is thus false, meaning that $v_{i_2,k_1}^F$ is not scheduled with truth assignment jobs. Else if $\beta_j$ contains the negative literal $\neg z_{i_2}$, then machine $(v_{i_2,k_2}, u_j)$ for $k_2 \in \{3, 4\}$ is constructed and the variable $z_{i_2}$ is true, we schedule jobs $u_j^F$ and $v_{i_2,k_2}^F$ on this machine.

It is not difficult to verify that by scheduling in the above way, the load of every truth assignment machine and clause machine is $r + O(\epsilon)$, and furthermore, for every $i$, 7 jobs out of $v_{i,k}^\gamma$ are scheduled on these machines where $k = 1, 2, 3, 4$ and $\gamma = T, F$. If the positive literal $z_i$ appears in clauses for once and $\neg z_i$ for twice, then the job $v_{i,2}$ is not scheduled. Otherwise if $z_i$ appears for twice while $\neg z_i$ for once, the job $v_{i,4}$ is left. These jobs are scheduled on dummy machines according to Table 4. Notice that there are in all $4n$ true variable jobs, among them $2n$ ones are on truth assignment machines, $m$ are on clause machines (as $u_j^T$ is with a true variable job, and $u_j^F$ is with with a false one), thus $2n - m$ true ones are on dummy machines. Recall that there are $2n - m$ true dummy jobs $\phi^T$ and $m - n$ false dummy jobs, we always schedule a true dummy job with a true variable job, and a false dummy job with a false variable job. It is easy to see that in this way, the load of every dummy machine is $r + O(\epsilon)$.

Thus in all, if there exists a truth assignment for $I'_{sat}$ in which every clause is satisfied by exactly one literal, then there exists a feasible schedule for $I_{sch}$ whose makespan is $r + O(\epsilon)$.

### 3.3 From Scheduling to 3SAT

The whole subsection is devoted to proving the following theorem.

**Theorem 3** If there is a solution for $I_{sch}$ whose makespan is strictly less than $r + 1$, then there exists a truth assignment for $I'_{sat}$ where every clause is satisfied by exactly one literal.

To prove the theorem, we start with the following simple observation.

**Observation:** The processing time of a job on every machine is greater than or equal to the third coordinate of its size vector.

Using the above observation, it is not difficult to calculate that the total processing time of all the jobs is at least $8nr$. Let $Sol^*$ be the solution whose makespan is strictly less than $r + 1$, then the load of every machine is in $[r, r + 1)$. We check the scheduling of jobs in this solution.

**Lemma 1** In $Sol^*$, there is one huge job on each machine, furthermore

- A huge job corresponds to a dummy machine is on a dummy machine
• A huge job corresponds to a clause machine is on a clause machine
• A huge job corresponds to a truth assignment machine is on a truth assignment machine

Proof. According to the observation, the processing time of a huge job is at least $5/8r - 20\xi > 1/2r + 1$, thus there is at most one huge job on each machine. Given the fact that there are $8n$ machines and $8n$ huge jobs, we know that there is one huge job on each machine in $Sol^*$. Consider any huge job corresponding to a dummy machine. Notice that the second coordinate of its size vector is always $N^{2N}$, implying that the processing time of this job is $\Omega(N)$ on clause machines and truth assignment machines, thus this job is on a dummy machine. Recall that there are $n$ dummy machines and $n$ huge jobs corresponding to dummy machines, these huge jobs must be on these dummy machines, one for each.

Consider any huge job corresponding to a clause machine. The second coordinate of its size vector is at least $N^N$, implying that its processing time is at least $\Omega(N)$ if it is put on a truth assignment machine. On the other hand it could not be put on a dummy machine either, thus it must be on a clause machine.

Similar arguments show that a huge job corresponding to a truth assignment machine must be on a truth assignment machine.

Lemma 2 In $Sol^*$, except for the huge jobs,

• There is a variable job on each machine
• There is a clause job on each clause machine
• There is a dummy job on each dummy machine

Proof. Consider a clause job. Its processing time is greater than $1/4r$. If it is put on a truth assignment machine, then the load of this machine becomes larger than $1/4r + 7/8r > r + 1$, which is contradiction. Else if it is put on a dummy machine, then the load of this machine becomes larger than $1/4r + 13/16r + 20\xi > r + 1$, which is also a contradiction. Hence a clause job could only be on a clause machine. Meanwhile if there are two clause jobs on one clause machine, then the load of this machine also becomes larger than $5/8r + 20\xi + 1/2r > r + 1$. As there are $n$ clause jobs and clause machine, there is exactly one clause job on one clause machine.

Consider a variable job. It is not difficult to verify that there could not be two variable jobs on one machine since the total processing time of two variable job is at least $1/4r - 40\xi - 8 > 3/16r$ (due to the fact that $r = 2^{10}\xi$). Given that there are $8n$ variable jobs and $8n$ machines, there is one variable job on each machine. Now a dummy job could only be on a dummy machine, and similar arguments show that there could be at most one dummy job on a dummy machine, hence there is one dummy job on one dummy machine.
A machine is called variable-satisfied, if the variable job on this machine coincide with the symbol of this machine, i.e., for any machine denoted as \((v_{i,k}, *)\) or \((v_{i,k}, *, *)\), the variable job on it is \(v_{i,k}\) where \(k = 1, 2, 3, 4\). We have the following lemma.

**Lemma 3** Every machine is variable-satisfied.

**Proof.** Consider the eight jobs \(v^\gamma_{n,k}\) where \(\gamma = T, F, k = 1, 2, 3, 4\). For any machine denoted as \((v_{j,k}, *)\) or \((v_{j,k}, *, *)\), the first coordinate of its speed vector is \(N^{-4j-k}\), thus the processing time of \(v_{n,k}\) on this machine becomes \(\Omega(\epsilon N)\) if \(j < n\). Furthermore, it can be easily seen that \(v_{n,4}\) could only be on machines denoted as \((v_{n,4}, *)\) or \((v_{n,4}, *, *)\). Since there are two jobs \(v_{n,4}\) (one true and one false), and two machines denoted as \((v_{n,4}, *)\) or \((v_{n,4}, *, *)\) (either machines \((v_{n,4}, b_n, c_n)\) and \((v_{n,4}, \phi)\), or machines \((v_{n,4}, b_n, c_n)\) and \((v_{i,4}, u_{j})\) for some \(j\)), thus the two machines are satisfied. Iteratively applying the above arguments we can prove that every machine is satisfied. \(\square\)

Using similar arguments as the proof the above lemma, we can also prove that the huge job on every machine also coincide with the symbol of this machine.

A machine is called satisfied, if all the jobs on this machine coincide with the symbol of this machine, i.e., jobs are scheduled according to Table 4.

**Lemma 4** Every machine is satisfied.

**Proof.** Notice that the second coordinate of a clause job \(u_j\) is \(\epsilon N^{j+i}\), and there is one clause job on every clause machine, thus using similar arguments as the proof of Lemma 2 we can show that the clause job \(u_j\) is on a the clause machine \((v_{i,k}, u_j)\). Now adding up the processing times of the huge job, clause job and variable job on a clause machine, the sum is at least \(r - 2\), implying that there is no truth assignment jobs on clause machines, and thus every clause machine is satisfied.

Consider a dummy machine. According to Lemma 2 the total processing time of a dummy job and a variable job on a machine is at least \(r - 2\), thus again there is no truth assignment jobs on it and every dummy machine is satisfied.

Consider truth assignment machines. The above analysis implies that all the truth assignment jobs are on these machines. We check machines \((v_{1,1,1}, a_1, c_1)\), \((v_{1,2,1}, b_1, d_1)\), \((v_{1,3,1}, a_1, d_1)\), \((v_{1,4,1}, b_1, c_1)\). The total load of the four machines falls in \([4r, 4r + 4]\), and the amount contributed by variable and huge jobs is among \([4r - 60\xi - 16, 4r - 60\xi - 8]\), thus the amount contributed by truth assignment jobs is in \([60\xi + 8, 60\xi + 20]\). Notice that for any \(i \geq 2\), the processing time of job \(a_i, b_i, c_i\) or \(d_i\) is at least \(\Omega(\epsilon N)\) on the four machines we consider, thus there are at most 8 truth assignment jobs on these machines, namely \(a_1, b_1, c_1\) and \(d_1\). The total processing time of the \(8\) jobs is \(60\xi + 12\), while each of them has a processing time of at least \(2\xi + 1\), implying that all these jobs are on the four machines. Consider the two jobs \(d_1\) (one true and one false), either has a processing time at least \(16\xi\), implying that they can only be on machine \((v_{1,2,1}, b_1, d_1)\) and \((v_{1,3,1}, a_1, d_1)\). Furthermore, they can not be on the same machine, thus there is
one \( d_1 \) on machine \((v_{1,2}, b_1, d_1)\) and \((v_{1,3}, a_1, d_1)\). Using the same argument we can prove that \( a_1 \) and \( c_1 \) are on machine \((v_{1,1}, a_1, c_1)\), \( b_1 \) and \( d_1 \) are on machine \((v_{1,2}, b_1, d_1)\), \( a_1 \) and \( d_1 \) are on machine \((v_{1,3}, a_1, d_1)\), and \( b_1 \) and \( c_1 \) are on machine \((v_{1,4}, b_1, c_1)\). In all, the four machines \((v_{1,1}, a_1, c_1), (v_{1,2}, b_1, d_1), (v_{1,3}, a_1, d_1), (v_{1,4}, b_1, c_1)\) are all satisfied. Iteratively using the above arguments, we can prove that every truth assignment machine is satisfied.

\(\square\)

Notice that except for a huge job, the symbol of a job, say, \( a_i \), may represent either \( a_i^T \) or \( a_i^F \). A machine is called truth benevolent, if except the huge job, all the jobs on it are either all true or all false.

**Lemma 5** Every machine is truth benevolent.

*Proof.* Consider a truth assignment machine, say, \((v_{i,1}, a_i, c_i)\). If \( v_{i,1}^T \) is on this machine, then \( a_i \) and \( c_i \) are both true, for otherwise one of them is false, and the total processing time of the three jobs is at least \( 1/8r + 1 \), implying that the load of this machine is at least \( r + 1 \), which is a contradiction. Similarly, if \( v_{i,1}^F \) is on this machine, then \( a_i \) and \( c_i \) are both false, for otherwise one of them is true, and the total processing time of the three jobs plus the huge job is at most \( r - 1 + O(\epsilon) < r \), which is a contradiction. Iteratively applying the above arguments we can show that every truth assignment machine is truth benevolent.

Consider a clause machine, say, \((v_{i,k}, u_j)\) for \( k = 1, 2, 3, 4 \). If \( v_{i,k}^T \) is on this machine, then \( u_j \) is true for otherwise the load of this machine is at least \( r + 2 \), which is a contradiction. If \( v_{i,k}^F \) is on this machine, then \( u_j \) is false, for otherwise the load of this machine is at most \( r - 2 + O(\epsilon) < r \), which is also a contradiction. Thus every clause machine is truth benevolent. Using the same argument we can also prove that every dummy machine is truth benevolent. \(\square\)

Now we come to the proof of Theorem 3. It is easy to see that for every \( i \), jobs are either scheduled as \((v_{i,1}^T, a_i^T, c_i^T)\), \((v_{i,2}^T, b_i^T, d_i^T)\), \((v_{i,3}^F, a_i^F, d_i^F)\), \((v_{i,4}^F, c_i^F, b_i^F)\) or \((v_{i,1}^F, a_i^F, c_i^F)\), \((v_{i,2}^F, b_i^F, d_i^F)\), \((v_{i,3}^F, a_i^T, d_i^T)\), \((v_{i,4}^T, a_i^T, c_i^T)\). If the former case happens, we let the variable \( z_i \) be false, otherwise we let \( z_i \) be true. We prove that, by assigning the truth value in this way, every clause of \( I_{sat} \) is satisfied by exactly one literal.

Consider any clause, say, \( \beta_j \), and suppose it contains three literals, say, \( v_{i_1,k_1}, v_{i_2,k_2} \) and \( v_{i_3,k_3} \) where \( k_1, k_2, k_3 \in \{1, 2, 3, 4\} \). Since there is one \( u_{i,j}^T \) and two \( u_{i,j}^F \), we assume that \( u_{i,j}^T \) is scheduled with \( v_{i_1,k_1} \).

We prove that \( \beta_j \) is satisfied by variable \( z_{i_1} \). There are two possibilities. If \( k_1 \in \{1, 2\} \), then \( u_{i,j}^T \) and \( v_{i_1,k_1}^T \) are on the machine \((v_{i_1,k_1}, u_j)\), and according to the construction of machines, such a machine is constructed as the positive literal \( z_i \) appears in clause \( \beta_j \) for the first or second time. According to our truth assignment, variable \( z_i \) is true, for otherwise \( v_{i_1,k_1}^T \) is scheduled with \( a_i^T \), \( c_i^T \) or \( b_i^T \), \( d_i^T \). Otherwise \( k_1 \in \{3, 4\} \), and machine \((v_{i_1,k_1}, u_j)\) is constructed as the negative literal \( \neg z_i \) appears in clause \( \beta_j \) for the first or second time. Again according to the truth assignment now the variable \( z_i \) is false, thus in both cases \( \beta_j \) is satisfied by variable \( z_{i_1} \).
We prove that $\beta_j$ is not satisfied by either variable $z_{i_2}$ or $z_{i_3}$. Consider $z_{i_2}$, again there are two possibilities. If $k_2 \in \{1, 2\}$, then the positive literal $z_{i_2}$ appears in $\beta_j$ for the first or second time, and meanwhile variable $z_{i_2}$ is false because otherwise $v_{i_2,k_2}^F$ is scheduled with $a_i^F$, $c_i^F$ or $b_i^F$, $d_i^F$, rather than $u_j^F$. Thus $\beta_j$ is not satisfied by variable $z_{i_2}$. Using the same argument we can prove that if $k_2 \in \{3, 4\}$, $\beta_j$ is not satisfied by variable $z_{i_2}$, either. The proof is the same for variable $z_{i_3}$.

Thus in all, $\beta_j$ is satisfied by exactly one literal when it contains three literals. The same result also holds when $\beta_j$ contains two literals via the same proof.

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