Beurling densities and frames of exponentials on the union of small balls

Jean-Pierre Gabardo∗
Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario, L8S 4K1, Canada
gabardo@mcmaster.ca

Chun-Kit Lai†
Department of Mathematics
San Francisco State University
San Francisco, CA 94132, USA
cklai@sfsu.edu

Abstract
If $x_1, \ldots, x_m$ are finitely many points in $\mathbb{R}^d$, let $E_\epsilon = \bigcup_{i=1}^m x_i + Q_\epsilon$, where $Q_\epsilon = \{x \in \mathbb{R}^d, |x_i| \leq \epsilon/2, i = 1, \ldots, d\}$ and let $\hat{f}$ denote the Fourier transform of $f$. Given a positive Borel measure $\mu$ on $\mathbb{R}^d$, we provide a necessary and sufficient condition for the frame inequalities

$$A \|f\|_2^2 \leq \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \, d\mu(\xi) \leq B \|f\|_2^2, \quad f \in L^2(E_\epsilon),$$

to hold for some $A, B > 0$ and for some $\epsilon > 0$ sufficiently small. If $m = 1$, we show that the limits of the optimal lower and upper frame bounds as $\epsilon \to 0$ are equal, respectively, to the lower and upper Beurling density of $\mu$. When $m > 1$, we extend this result by defining a matrix version of Beurling density. Given a (possibly dense) subgroup $G$ of $\mathbb{R}$, we then consider the problem of characterizing those measures $\mu$ for which the inequalities above hold whenever $x_1, \ldots, x_m$ are finitely many points in $G$ (with $\epsilon$ depending on those points, but not $A$ or $B$). We point out an interesting connection between this problem and the notion of well-distributed sequence when $G = a\mathbb{Z}$ for some $a > 0$. Finally, we show the existence of a discrete set $\Lambda$ such that the measure $\mu = \sum_\lambda \delta_\lambda$ satisfy the property above for the whole group $\mathbb{R}$.

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1 Introduction

If $E \subset \mathbb{R}^d$ is measurable with $|E| < \infty$, where $|E|$ denotes the Lebesgue measure of $E$, let $L^2(E)$ be the space of (complex-valued) square-integrable functions on $E$. Given a discrete subset $\Lambda \subset \mathbb{R}^d$, consider the collection of exponentials $\mathcal{E}(\Lambda) = \{e^{2\pi i \lambda \cdot x}, \lambda \in \Lambda\}$. This collection forms a Fourier frame for $L^2(E)$ if there exist constants $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{\lambda \in \Lambda} \left| \int_E f(x) e^{-2\pi i \lambda \cdot x} \, dx \right|^2 \leq B \|f\|^2, \quad f \in L^2(E). \quad (1.1)$$

If we allow the constant $A$ to be zero as well, $\mathcal{E}(\Lambda)$ is then called a Bessel collection in $L^2(E)$. Defining the Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ by the formula

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} \, dx, \quad \xi \in \mathbb{R}^d,$$

and the measure $\mu = \delta_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$, we can rewrite the frame inequalities (1.1) as

$$A \|f\|^2 \leq \int_{\mathbb{R}^d} |\hat{f}(\lambda)|^2 \, d\mu(\lambda) \leq B \|f\|^2, \quad f \in L^2(E). \quad (1.2)$$

If the inequalities (1.2) hold for a general positive Borel $\mu$ on $\mathbb{R}^d$, we call the measure $\mu$ an exponential frame measure (abbr. $\mathcal{F}$-measure) for $L^2(E)$. Similarly, we call $\mu$ an exponential Bessel measure (abbr. $\mathcal{B}$-measure) for $L^2(E)$ if $A$ is allowed to be 0 in (1.2).

The notion of frame was first introduced by Duffin and Schaeffer [DS]. This area of research has been developing rapidly in recent years, both in theory and applications, and has become one of the main tools in applied harmonic analysis, including Gabor analysis, wavelet theory, sampling theory and signal processing. Readers may refer to [Chr] for general background on the theory of frames. Fourier frames were first introduced in [DS] under the name of non-harmonic Fourier series. They are theoretically attractive since in contrast to orthonormal bases, Fourier frames are easy to construct on bounded sets and are robust to small perturbation of the set of frequencies. They are also valuable in applications since Fourier frames on $L^2(E)$ allow for the reconstruction of signals whose frequency band is supported on $E$. We refer the reader to [Yo] for classical results concerning frames of exponentials. The concept of $\mathcal{F}$-measure as defined in (1.2) is a particular case of “generalized frame” associated with a measure introduced in [GH]. In addition to making our results more general, it allows us to simplify notations and provide further flexibility when considering problems about Fourier frames [DHW, GL].

One of the main goal of this paper is to provide necessary and sufficient conditions for a measure $\mu$ (and a discrete set $\Lambda$) to be an $\mathcal{F}$-measure (resp. a $\mathcal{B}$–measure) for $L^2(E)$ when $E$ is a union of finitely many sufficiently “small” balls. As in the case with many results related to sampling [GR, Ja, Lan], the notions of upper and lower Beurling density appear naturally in the solution of our problems. We first recall that the upper and lower Beurling density of a positive Borel measure $\mu$ on $\mathbb{R}^d$ are defined, respectively, as

$$\mathcal{D}^+(\mu) = \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\mu(x + Q_h)}{h^d},$$
and
\[ D^-(\mu) = \liminf_{h \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\mu(x + Q_h)}{h^d}, \]
where
\[ Q_h = \{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d, |x_i| \leq d/2, i = 1, \ldots, d \} \]
is the hypercube of side length \( h \) centered at the origin. If \( D^-(\mu) = D^+(\mu) < \infty \), the common value of both densities, denoted by \( D(\mu) \), is called the the Beurling density of \( \mu \). Note that, if \( x \in \mathbb{R}^d \) and \( A, B \) are subsets of \( \mathbb{R}^d \), we use the notation \( x + A \) for the set \( \{ x + a, a \in A \} \) and \( A + B \) for the set \( \{ a + b, a \in A, b \in B \} \). If \( \Lambda \) is a countable set contained in \( \mathbb{R}^d \), we define \( D^-(\Lambda) = D^-(\delta_\Lambda) \), \( D^+(\Lambda) = D^+(\delta_\Lambda) \) and \( D(\Lambda) = D(\delta_\Lambda) \), where \( \delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda \). A positive Borel measure \( \mu \) is called translation-bounded if there exists a constant \( C > 0 \) such that
\[ \mu(x + [0, 1]^d) \leq C, \quad \forall x \in \mathbb{R}^d. \]

It is known that \( \mu \) is translation-bounded if and only if \( D^+(\mu) < \infty \) (see also Proposition 6 in Section 2).

Suppose that \( B(a, \epsilon) \) is a ball of radius \( \epsilon \) centered at \( a \). It was shown in \cite{Lai}, using a perturbation argument, that if \( D^-(\Lambda) > 0 \), then for sufficiently small \( \epsilon > 0 \), \( \mathcal{E}(\Lambda) \) is a Fourier frame for \( L^2(B(a, \epsilon)) \) for any \( a \in \mathbb{R}^d \). The converse clearly holds by the density result of Landau \cite{Landau} (i.e. if \( \mathcal{E}(\Lambda) \) is a frame for \( L^2(E) \), then \( D^-(\Lambda) \geq |E| \)). A similar result was also obtained by Beurling \cite{Beurling} who showed that if \( \Lambda \subset \mathbb{R}^d \) is a uniformly discrete set satisfying the covering property
\[ \bigcup_{\lambda \in \Lambda} (B(0, 1/r) + \lambda) = \mathbb{R}^d, \]
then, \( \Lambda \) is a Fourier frame for \( L^2(\epsilon B(0, r)) \) if \( \epsilon < 1/4 \). This is now known as the Beurling covering theorem and has found application in MRI reconstruction \cite{BW}. Our first theorem complements these results by providing a precise relation between the Beurling densities and the frame bounds in the case where \( E \) is a small neighborhood of a single point in \( \mathbb{R}^d \) (which we take to be a cube for convenience).

**Theorem 1.** Let \( \mu \) be positive, locally finite Borel measure on \( \mathbb{R}^d \). Then the following are equivalent.

(a) There exists constants \( A, B > 0 \) and \( \epsilon > 0 \) such that
\[ A \| f \|_2^2 \leq \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\mu(\xi) \leq B \| f \|_2^2, \quad f \in L^2(Q_\epsilon). \]

(b) We have \( 0 < D^-(\mu) \leq D^+(\mu) < \infty \).

Moreover, if (a) holds, we have \( A \leq D^-(\mu) \leq D^+(\mu) \leq B \) and if (b) holds we can find, for any \( \rho > 0 \) a corresponding \( \epsilon > 0 \) such that the inequalities in (a) hold with \( A = D^-(\mu) - \rho \) and \( B = D^+(\mu) + \rho \).
Remark 2. Note that the theorem also has a “Bessel” version, in which $A = 0$ and only $D^+(\mu)$ plays a role, (See Theorem [13]).

This theorem leads to the following corollary, showing that Beurling densities as limit of optimal frame bounds of small ball.

**Corollary 3.** Under the assumptions of Theorem [1], suppose that $D^+(\mu) < \infty$. Define $A_\epsilon$ and $B_\epsilon$ to be the optimal bounds for the inequalities

$$A_\epsilon \| f \|^2 \leq \int_{\mathbb{R}^d} |\hat{f}(\lambda)|^2 d\mu(\lambda) \leq B_\epsilon \| f \|^2, \; f \in L^2(B(a, \epsilon)),$$

(1.3)

Then, $\lim_{\epsilon \to 0} A_\epsilon = D^-(\mu)$ and $\lim_{\epsilon \to 0} B_\epsilon = D^+(\mu)$.

(Theorem [1] will not be proved here as they are particular cases of Theorem [13] and Corollary [16], respectively, which will be proved in section 2.)

If $\Omega = \bigcup_{i=1}^{N} (a_i + Q_i)$ is a finite union of disjoint cubes with side length $\epsilon > 0$, Theorem [1] is no longer true since the lower-frame bound inequality might fail under the conditions $0 < D^-(\mu) \leq D^+(\mu) < \infty$ even if $\epsilon$ is small. For example, $\mu = \sum_{n \in \mathbb{Z}} \delta_n$ is not an $F$-measure for the set $\Omega = [-\epsilon, \epsilon] \cup [1 - \epsilon, 1 + \epsilon]$. To see this, we consider $\hat{f} = \chi_{[\epsilon, \epsilon]} - \chi_{[-\epsilon, 1+\epsilon]}$. Then, for all $n \in \mathbb{Z}$, we have

$$\hat{f}(n) = \int_{-\epsilon}^{\epsilon} e^{-2\pi inx} dx - e^{2\pi in} \int_{-\epsilon}^{\epsilon} e^{-2\pi inx} dx = 0.$$

This means that the collection $\{e^{-2\pi inx}\}_{n \in \mathbb{Z}}$ is not even complete in $L^2(\Omega)$. By introducing notions of lower and upper density for Borel measures on $\mathbb{R}^d$ taking values in the cone of positive-definite matrices, we characterize $B$-measures and $F$-measures for $L^2(\Omega)$ in the case where $\Omega = \bigcup_{i=1}^{N} (a_i + Q_i)$ and $\epsilon$ is small enough. This provides thus analogues of Theorem [1] and Corollary [3] for this more general situation.

After establishing these results, we will consider a related problem which involves a uniformity condition on the frame bounds with respect to a subgroup $G$ of $\mathbb{R}$.

**Definition 4.** Let $\mu$ positive Borel measure on $\mathbb{R}$ and let $G$ be a subgroup of $\mathbb{R}$. If $A, B > 0$, we say that $\mu$ is a uniform $F$-measure for $G$ with limiting lower bound larger than or equal to $A$ and limiting upper frame bound less than or equal to $B$, if given any $x_1, \ldots, x_M \in G$ and any $\delta > 0$, there exists $\epsilon > 0$ such that

$$(A - \delta) \| f \|^2 \leq \int_{\mathbb{R}^d} |\hat{f}(\lambda)|^2 d\mu(\lambda) \leq (B + \delta) \| f \|^2, \; f \in L^2(\Omega),$$

(1.4)

for $\Omega = \bigcup_{j=1}^{N} (x_j + Q_j)$. We denote the collection of such measures by $\mathcal{F}(G, A, B)$. The notion of uniform $B$-measure for $G$ with limiting upper frame bound less than or equal to $B$ is defined in a similar way and the collection of such measures is denoted by $\mathcal{B}(G, B)$. Finally, the measures in the collection $\mathcal{F}(G, A, A)$ are called uniform tight $F$-measures with limiting tight frame bound $A$ for $G$. 
The construction of Borel measures $\mu$ in $\mathcal{F}(G, A, B)$ can be viewed as a continuous version of a compressed sensing problem (See [FR] for details about compressed sensing). A vector $v$ is a finite-dimensional space is $s$-sparse (where $s \geq 1$ is an integer) if it has at most $s$ non-zero components. In general, the indices corresponding to the non-zero components of $v$ are unknown. In its discrete and finite-dimensional setting, the compressed sensing problem consist in trying to recover an $s$-sparse vector $v$ in $\mathbb{C}^d$ by computing the inner products $\langle v, u_i \rangle$ with some fixed vectors $u_i$, $i \in I$. As can be expected, the smaller $s$ is, the fewer vectors $u_i$ are needed for the recovery of the data. If we think of a function $f$ in $L^2(E)$ as a vector with non-zero components concentrated on the set $E$ and if $\mu = \delta_\Lambda$, for some discrete set $\Lambda$, the fact that $\mu \in \mathcal{F}(G, A, B)$ allows for the recovery of any such function from the knowledge of the inner products $\langle f, e_\lambda \rangle_{L^2(E)}$ where $e_\lambda(x) = e^{-2\pi i \lambda \cdot x} \chi_E(x)$. This will be possible if $f$ is sparse enough in the sense that it should be supported in a small enough neighborhood of a finite subset of the group $G$. The fact that the constants $A, B$ are independent of the points chosen in the group $G$ implies that the robustness of the reconstruction formula is also independent of the exact location of this neighborhood.

One trivial element inside $\mathcal{F}(G, 1, 1)$ for any subgroup $G$ is the Lebesgue measure on $\mathbb{R}$. We are particularly interested in the existence of discrete measures inside these collections. We first consider the problem with $G$ being a finitely-generated subgroup of $\mathbb{R}$ and completely solve this problem with the help of Theorem 1. It turns out that, interestingly, in the case of measures of the form $\mu = \delta_\Lambda$, $\Lambda \subset \mathbb{R}$, the answer to these questions is related to the probabilistic notion of “equidistributed sequence” or, more specifically, that of “well-distributed sequence” (Theorem 21). We will show, in particular, that if $G = a\mathbb{Z}$, $\delta_\Lambda \in \mathcal{F}(G, A, A)$ if and only if $D(\Lambda) = A$ and $\Lambda$ is a well-distributed sequence (mod $a^{-1}$) (Corollary 25). Our results can also be interpreted as characterizations of certain inequalities satisfied by almost-periodic functions with spectrum in the group $G$ (Theorem 27).

Finally, we consider the problem for the whole group $\mathbb{R}$. Using a recent result of S. Nitzan, A. Olevskii and A. Ulanovskii ([NOU]) about the existence of Fourier frames on any unbounded set of finite measure, which is based on the solution of the Kadison-Singer problem, we deduce the existence of a discrete $\Lambda$ such that $\delta_\Lambda \in \mathcal{F}(\mathbb{R}, A, B)$ for some $A, B$ ($A < B$). It would be reasonable to think that the measure associated with a simple quasicrystal in the sense of Meyer ([Me1]) may belong to some space $\mathcal{F}(\mathbb{R}, A, B)$ in view of the results on universal sampling obtained in [MM], but we will show that this is never the case. Since the solution to the celebrated Kadison-Singer conjecture in [MSS] is a probabilistic result, it would be interesting, in line with the current research, to find some deterministic discrete sets with associated measure belonging to some space $\mathcal{F}(\mathbb{R}, A, B)$.

We organize our paper as follows: in Section 2, we provide the basic preliminary results on Beurling density and introduce the Beurling densities of Borel measures on $\mathbb{R}^d$ taking values in the cone of positive-definite matrices. We will prove the matrix version of Theorem 1 and Corollary 3 in Section 3. In Section 4, we characterize the measures in $\mathcal{F}(G, A, B)$ and $\mathcal{B}(G, B)$. We study the case $G = \mathbb{R}$ in the last section.

**Local square integrability of the Fourier transform.**

Before we develop our theory in the next section, we mention an additional consequence of Theorem 1. Note that by the implication (b) $\implies$ (a) in Theorem 1 if $\mu$ is translation-
bounded (i.e. $D^+(\mu) < \infty$), then the “analysis” operator

$$T : L^2(B(a, \epsilon)) \to L^2(\mu) : f \mapsto \hat{f}$$

is bounded for all $a \in \mathbb{R}^d$ and $\epsilon > 0$ small enough, where $B(a, \epsilon)$ is the Euclidean ball of radius $\epsilon$ centered at $a$. and thus so is its adjoint, the “synthesis” operator $T^* : L^2(\mu) \to L^2(B(a, \epsilon))$. It is easy to see that if $\mu$ is translation bounded, and $F \in L^2(\mu)$, then $Fd\mu$ defines a tempered distribution on $\mathbb{R}^d$. In that case, taking $g \in C^\infty_0(B(a, \epsilon))$, we have

$$(T^*F, g)_2 = (F, Tg)_{L^2(\mu)} = \int_{\mathbb{R}^d} F(\lambda) \overline{g(\lambda)} \, d\mu(\lambda) = \langle \mathcal{F}^{-1}(F \, d\mu), g \rangle,$$

where the bracket $\langle \cdot, \cdot \rangle$ represents the duality between tempered distributions in $\mathcal{S}'(\mathbb{R}^d)$ and the test functions in the Schwartz space $\mathcal{S}'(\mathbb{R}^d)$ and where $\mathcal{F}^{-1}$ denote the (distributional) inverse Fourier transform. It follows that, if the positive Borel measure $\mu$ is translation bounded and $F \in L^2(\mu)$, then the synthesis operator $T^*$ is defined by

$$T^* : L^2(\mu) \to L^2(B(a, \epsilon)) : F \mapsto \mathcal{F}^{-1}(F \, d\mu)|_{B(a, \epsilon)}.$$

(1.5)

In particular, this implies that the distribution defined by $\mathcal{F}^{-1}(F \, d\mu)$ is locally square-integrable on $\mathbb{R}^d$. This property was actually proved in 1990 by R. Strichartz ([Str, Lemma 4.2]) using a different method. We now show that the converse of this statement is also true: if $\mu$ is a positive tempered measure on $\mathbb{R}^d$ and $\mathcal{F}^{-1}(F \, d\mu)$ is locally square-integrable on $\mathbb{R}^d$ for every $F \in L^2(\mu)$, the $\mu$ must be translation-bounded. Indeed it is easily checked that the mapping defined in (1.5) is closed and it is thus bounded by the closed graph theorem. Since the boundedness of $T^*$ is equivalent to the boundedness of $T$, it follows that $\mu$ is translation bounded using the implication (a) $\implies$ (b) in Theorem 1. We note also, that since the property of being square-integrable on every ball $B(a, \epsilon)$, with center $a \in \mathbb{R}^d$ and fixed radius $\epsilon > 0$, is clearly equivalent to being square-integrable on every ball $B(a, r)$, where $r > 0$ is arbitrary, it follows that if $\mu$ is a $\mathcal{B}$-measure for $L^2(B(a, \epsilon))$ for some $a \in \mathbb{R}^d$ and some $\epsilon > 0$, then $\mu$ is a $\mathcal{B}$-measure for $L^2(B(x, r))$, for any $x \in \mathbb{R}^d$ and any $r > 0$ (but with the Bessel constant dependent on $r$). We summarize these conclusions in the following theorem.

**Theorem 5.** Let $\mu$ be a positive tempered measure on $\mathbb{R}^d$. Then $\mathcal{F}^{-1}(F \, d\mu)$ is locally square-integrable on $\mathbb{R}^d$ for every $F \in L^2(\mu)$ if and only if $\mu$ must be translation-bounded.

More generally, if $E = \Omega$ where $\Omega$ is an open subset of $\mathbb{R}^d$ with $|\Omega| < \infty$, the Bessel inequality

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \, d\mu(\xi) \leq B \|f\|^2_2, \quad f \in L^2(\Omega),$$

is equivalent to the property

$$\mathcal{F}^{-1}(F \, d\mu)|_{\Omega} \in L^2(\Omega), \quad F \in L^2(\mu).$$

If $\Omega$ is bounded, this property will hold if and only if $\mu$ is translation bounded by Theorem 5. On the other hand, if $\Omega$ is unbounded but with finite Lebesgue measure, the Bessel
inequality might fail for translation bounded measures. For example, if \( \mu = \delta_Z = \sum_{n \in \mathbb{Z}} \delta_n \), and, if \( F \in L^2(\mu) \), we have
\[
F d\mu = \sum_{n \in \mathbb{Z}} c_n \delta_n \quad \text{with} \quad \sum_{n \in \mathbb{Z}} |c_n|^2 < \infty.
\]

It follows that the set \( \{ \mathcal{F}^{-1}(F d\mu), F \in L^2(\mu) \} \) is exactly the collection of locally square-integrable 1-periodic function on the real line. If we take
\[
\Omega = \bigcup_{j \geq 1} (j, j + 1/j^2)
\]
and define \( H(x) \) to be the 1-periodic function with
\[
H(x) = x^{-\alpha}, \quad 0 < x \leq 1,
\]

it is easily checked that \( H \) is locally square-integrable if and only if \( \alpha < 1/2 \). In that case, we have thus \( H = \mathcal{F}^{-1}(F d\mu) \) for some \( F \in L^2(\mu) \). However, the restriction of \( H \) to \( \Omega \) is square-integrable if and only if
\[
\sum_{j=1}^{\infty} \int_0^{1/j^2} x^{-2\alpha} \, dx = \frac{1}{1-2\alpha} \sum_{j=1}^{\infty} \frac{1}{j^{2-4\alpha}} < \infty
\]
i.e. \( \alpha < 1/4 \). Hence, if we take \( \alpha \) with \( 1/4 \leq \alpha < 1/2 \), \( \mathcal{F}^{-1}(F d\mu)|_\Omega \notin L^2(\Omega) \) and the Bessel property fails.

2 Densities of positive matrix-valued measures

We start by mentioning some properties equivalent to “translation-boundedness”.

**Proposition 6** ([Ga]). Let \( \mu \) be a positive Borel measure on \( \mathbb{R}^d \). Then, the following are equivalent:

(a) \( \mu \) is translation bounded.

(b) \( D^+(\mu) < \infty \).

(c) There exists \( f \in L^1(\mathbb{R}^d) \) with \( f \geq 0 \), \( \int f \, dx = 1 \) and a constant \( C > 0 \) such that \( \mu \ast f \leq C \) a.e. on \( \mathbb{R}^d \).

As the last condition in the previous proposition shows, the notion of upper Beurling density is related to certain convolution inequalities satisfied by the measure \( \mu \). More generally, we have the following result, which will be useful later on.

**Theorem 7** ([Ga]). Let \( \mu \) be a positive Borel measure on \( \mathbb{R}^d \) and let \( h \in L^1(\mathbb{R}^d) \) with \( h \geq 0 \). Let \( A, B > 0 \) be constants. Then

(a) If \( \mu \ast h \leq B \) a.e. on \( \mathbb{R}^d \), then \( D^+(\mu) \int h \, dx \leq B \).
(b) If $\mu$ is translation-bounded and $A \leq \mu \ast h$ a.e. on $\mathbb{R}^d$, then $A \leq \mathcal{D}^-(\mu) \int h \, dx$

If we assume now that $\mu$ is an $\mathcal{F}$-measure for $L^2(E)$, then applying the frame inequalities \((1.2)\) to the function $g(x) = e^{2\pi i \xi \cdot x} f(x)$, where $\xi \in \mathbb{R}^d$ and $f \in L^2(E)$, we obtain that

$$A \|f\|^2 \leq \int_{\mathbb{R}^d} |\hat{f}(\xi - \lambda)|^2 \, d\mu(\lambda) \leq B \|f\|^2, \quad f \in L^2(E).$$

which can also be written as

$$A \|f\|^2 \leq (\mu \ast |\hat{f}|^2)(\xi) \leq B \|f\|^2, \quad \xi \in \mathbb{R}^d, \ f \in L^2(E).$$

Since $\int_{\mathbb{R}^d} |\hat{f}|^2 \, d\lambda = \|f\|^2$ by Plancherel’s theorem, we can apply Theorem 7 to the function $h := |\hat{f}|^2$ to obtain that

$$0 < A \leq \mathcal{D}^-(\mu) \leq \mathcal{D}^+(\mu) \leq B < \infty.$$ 

Of course, the same argument show that, if $\mu$ is a $\mathcal{B}$-measure for $L^2(E)$ with Bessel constant $B$, then

$$\mathcal{D}^+(\mu) \leq B < \infty.$$ 

This gives a proof for one of the implications in Theorem 1.

We now define appropriate notions of densities for positive matrix-valued measures generalizing the known notions of Beurling densities defined in the introduction.

**Definition 8.** We will denote by $\mathcal{MP}_N(\mathbb{R}^d)$ the set of $N \times N$ matrices $\hat{\mu} = (\mu_{i,j})$ whose entries $\mu_{i,j}, 1 \leq i, j \leq N$, are complex, locally finite Borel measures on $\mathbb{R}^d$ and are positive-definite in the sense that, for any $v = (v_1, \ldots, v_N) \in \mathbb{C}^N$, we have

$$\hat{\mu}_v := \sum_{i,j=1}^N \mu_{i,j} v_i \overline{v_j} \geq 0, \quad (2.1)$$

i.e. the left-hand side of the previous inequality defines a positive measure on $\mathbb{R}^d$.

The lower and upper Beurling densities of an element $\hat{\mu}$ of $\mathcal{MP}_N(\mathbb{R}^d)$ can then be defined respectively as

$$\mathcal{D}_N^-(\hat{\mu}) := \inf \{ \mathcal{D}^-(\hat{\mu}_v) : v \in \mathbb{C}^N, \ |v|^2 = 1 \}$$

and

$$\mathcal{D}_N^+(\hat{\mu}) := \sup \{ \mathcal{D}^+(\hat{\mu}_v) : v \in \mathbb{C}^N, \ |v|^2 = 1 \}.$$ 

If $E$ is a bounded Borel subset of $\mathbb{R}^d$ and $\mu \in \mathcal{MP}_N(\mathbb{R}^d)$, the $N \times N$ matrix $\hat{\mu}(E)$ with (complex) entries $\mu_{i,j}(E)$ is positive-definite by definition. Its eigenvalues are thus real and non-negative and we can define $\lambda_{\max}(\hat{\mu}, E)$ and $\lambda_{\min}(\hat{\mu}, E)$, to be the largest and smallest eigenvalue of $\hat{\mu}(E)$, respectively. The following lemma provides an alternative definition of the lower and upper Beurling densities for elements of $\mathcal{MP}_N(\mathbb{R}^d)$. 
Lemma 9. Let $\tilde{\mu} \in \mathcal{MP}_N(\mathbb{R}^d)$. Then, we have

$$\mathcal{D}_N^+(\tilde{\mu}) = \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\lambda_{\max}(\tilde{\mu}, x + Q_h)}{h^d}$$

and, if $\mathcal{D}_N^+(\tilde{\mu}) < \infty$, we have also

$$\mathcal{D}_N^-(\tilde{\mu}) = \liminf_{h \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\lambda_{\min}(\tilde{\mu}, x + Q_h)}{h^d}.$$  

Proof. Using standard properties of positive-definite matrices, we have, for any bounded Borel subset $E$ of $\mathbb{R}^d$ and any $v \in \mathbb{C}^N$, that

$$\lambda_{\min}(\tilde{\mu}, E) \|v\|_2 \leq \sum_{i,j=1}^N \mu_{i,j}(E) v_i \overline{v_j} \leq \lambda_{\max}(\tilde{\mu}, E) \|v\|_2^2.$$  

In particular, if $\|v\|_2 = 1$, we have

$$\mathcal{D}_N^+(\tilde{\mu}_v) = \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\tilde{\mu}_v(x + Q_h)}{h^d} \leq \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\lambda_{\max}(\tilde{\mu}, x + Q_h)}{h^d}$$

and

$$\mathcal{D}_N^-(\tilde{\mu}_v) = \liminf_{h \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\tilde{\mu}_v(x + Q_h)}{h^d} \geq \liminf_{h \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\lambda_{\min}(\tilde{\mu}, x + Q_h)}{h^d}$$

which show that

$$\mathcal{D}_N^+(\tilde{\mu}) \leq \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\lambda_{\max}(\tilde{\mu}, x + Q_h)}{h^d} \quad \text{and} \quad \mathcal{D}_N^-(\tilde{\mu}) \geq \liminf_{h \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\lambda_{\min}(\tilde{\mu}, x + Q_h)}{h^d}.$$  

To prove the reverse inequality

$$\limsup_{h \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\lambda_{\max}(\tilde{\mu}, x + Q_h)}{h^d} \leq \mathcal{D}_N^+(\tilde{\mu}),$$  \hspace{1cm} (2.2)

we can assume that $\mathcal{D}_N^+(\tilde{\mu}) < \infty$. Consider sequences $\{h_n\}$ and $\{x_n\}$, with $h_n > 0$, $x_n \in \mathbb{R}^d$ and $h_n \to \infty$ as $n \to \infty$, such that

$$\lim_{n \to \infty} \frac{\lambda_{\max}(\tilde{\mu}, x_n + Q_{h_n})}{h_n^d} = \limsup_{h \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\lambda_{\max}(\tilde{\mu}, x + Q_h)}{h^d}.$$  

Let $v_n \in \mathbb{C}^N$ be an eigenvector of norm 1 associated with the largest eigenvalue of the matrix $\tilde{\mu}(x_n + Q_{h_n})$. We have then

$$\lambda_{\max}(\tilde{\mu}, x_n + Q_{h_n}) = \tilde{\mu}_{v_n}(x_n + Q_{h_n}).$$  

Letting $e_i$, $i = 1, \ldots, N$, denote the vectors in the standard basis of $\mathbb{C}^N$, it follows that $\tilde{\mu}_{e_i} = \mu_{ii}$, $i = 1, \ldots, N$, and, in particular, $\mathcal{D}_N^+(\mu_{ii}) < \infty$ since we assume that $\mathcal{D}_N^+(\tilde{\mu}) < \infty$. We can thus find constants $K, h_0 > 0$ such that

$$\frac{\mu_{ii}(x + Q_h)}{h^d} \leq K, \quad 1 \leq i \leq N, \quad x \in \mathbb{R}^d, \quad h \geq h_0.$$
Using another well-known property of positive-definite matrices, we have also that
\[
\frac{|\mu_{ij}(x + Q_h)|}{h^d} \leq \left( \frac{\mu_{ii}(x + Q_h)}{h^d} \right)^{1/2} \left( \frac{\mu_{jj}(x + Q_h)}{h^d} \right)^{1/2} \leq K, \quad 1 \leq i, j \leq N, \ x \in \mathbb{R}^d,
\]
for \( h \geq h_0 \). The entries of the matrices \( G_n := \hat{\mu}(x_n + Q_{h_n})/h_n^d \) are thus uniformly bounded, and we can assume, by compactness, after passing to a subsequence if necessary that
\[
G_n \to G \quad \text{and} \quad v_n \to v, \quad n \to \infty,
\]
where \( G \) is a positive-definite \( N \times N \) matrix and \( v \) a unit vector in \( \mathbb{C}^N \). We have then,
\[
D_N^+(\hat{\mu}) \geq \lim_{n \to \infty} \frac{\hat{\mu}_v(x_n + Q_{h_n})}{h_n^d} = \lim_{n \to \infty} \langle G_n v, v \rangle = \lim_{n \to \infty} \langle G_n v_n, v_n \rangle = \lim_{n \to \infty} \frac{\max_{n} \lambda(\hat{\mu}_n, x_n + Q_{h_n})}{h_n^d}
\]
which proves the inequality (2.2). It remains to prove the inequality
\[
\liminf_{h \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\lambda_{\min}(\hat{\mu}, x + Q_h)}{h^d} \geq D_N^-(\hat{\mu}) \tag{2.3}
\]
under the additional assumption that \( D_N^+(\hat{\mu}) < \infty \). Consider sequences \( \{k_n\} \) and \( \{y_n\} \), with \( k_n > 0, y_n \in \mathbb{R}^d \) and \( k_n \to \infty \) as \( n \to \infty \), such that
\[
\lim_{n \to \infty} \frac{\lambda_{\min}(\hat{\mu}, y_n + Q_{k_n})}{k_n^d} = \liminf_{h \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\lambda_{\min}(\hat{\mu}, x + Q_h)}{h^d}.
\]
Let \( u_n \in \mathbb{C}^N \) be an eigenvector of norm 1 associated with the smallest eigenvalue of the matrix \( \mu(y_n + Q_{k_n}) \). Since \( D_N^+(\hat{\mu}) < \infty \), the entries of the matrix \( H_n := \hat{\mu}(y_n + Q_{k_n})/k_n^d \) are uniformly bounded and, similarly, as above we can assume that
\[
H_n \to H \quad \text{and} \quad u_n \to u, \quad n \to \infty,
\]
where \( H \) is a positive-definite \( N \times N \) matrix and \( u \) a unit vector in \( \mathbb{C}^N \). We have then,
\[
D_N^-(\hat{\mu}) \leq \lim_{n \to \infty} \frac{\hat{\mu}_u(y_n + Q_{k_n})}{k_n^d} = \lim_{n \to \infty} \langle H_n u, u \rangle = \lim_{n \to \infty} \langle H_n u_n, u_n \rangle = \liminf_{h \to \infty} \inf_{x \in \mathbb{R}^d} \frac{\lambda_{\min}(\hat{\mu}, x + Q_h)}{h^d},
\]
proving our claim. \( \Box \)
3 Fourier Frames on the union of small cubes

In this section, we will be exclusively interested in elements \( \hat{\mu} \) of \( \mathcal{MP}_N(\mathbb{R}^d) \) constructed starting from a positive, locally finite Borel measure \( \mu \) on \( \mathbb{R}^d \) and \( N \) points \( x_1, \ldots, x_N \in \mathbb{R}^d \). We define \( \hat{\mu} = (\mu_{ij}) \) by the formula

\[
d\mu_{ij}(\xi) = e^{-2\pi i (x_i - x_j) \cdot \xi} d\mu(\xi), \quad 1 \leq i, j \leq N. \tag{3.1}
\]

If \( E \) is a bounded subset of \( \mathbb{R}^d \), the associated matrix \( (\mu_{ij}(E)) \) has thus entries

\[
\mu_{ij}(E) = \int_E e^{-2\pi i (x_i - x_j) \cdot \xi} d\mu(\xi), \quad i, j = 1, \ldots N.
\]

Furthermore, if \( v \) a vector in \( \mathbb{C}^N \), we have

\[
d\hat{\mu}_v = \sum_{i,j=1}^N d\mu_{i,j} v_i \overline{v_j} = \left| \sum_{j=1}^N v_j e^{-2\pi i x_j \cdot \xi} \right|^2 d\mu(\xi), \tag{3.2}
\]

showing that \( \hat{\mu} \) is a positive matrix-valued measure.

**Example 10.** If \( N = 2 \), we can use Lemma 9 to obtain an explicit formula for the densities \( D_{-2}(\hat{\mu}) \) and \( D_{+2}(\hat{\mu}) \), defined at the end of the previous section, which are associated with a measure \( \mu \) and two distinct points \( x_1, x_2 \in \mathbb{R}^d \) using (3.1). For any bounded Borel set \( E \subset \mathbb{R}^d \), we have

\[
\hat{\mu}(E) = \begin{bmatrix}
\int_E 1 d\mu(\xi) & \int_E e^{-2\pi i (x_1 - x_2) \cdot \xi} d\mu(\xi) \\
\int_E e^{-2\pi i (x_2 - x_1) \cdot \xi} d\mu(\xi) & \int_E 1 d\mu(\xi)
\end{bmatrix}
\]

and the eigenvalues of \( \hat{\mu}(E) \) are given by

\[
\lambda_{\max}(\hat{\mu}, E) = \int_E 1 d\mu(\xi) + \left| \int_E e^{-2\pi i (x_1 - x_2) \cdot \xi} d\mu(\xi) \right|
\]

and

\[
\lambda_{\min}(\hat{\mu}, E) = \int_E 1 d\mu(\xi) - \left| \int_E e^{-2\pi i (x_1 - x_2) \cdot \xi} d\mu(\xi) \right|.
\]

The densities are then computed as

\[
D_{-2}(\hat{\mu}) = \liminf_{R \to \infty} \inf_{t \in \mathbb{R}^d} \frac{1}{R^d} \left\{ \int_{t+Q_R} 1 d\mu(\xi) - \left| \int_{t+Q_R} e^{-2\pi i (x_1 - x_2) \cdot \xi} d\mu(\xi) \right| \right\},
\]

and

\[
D_{+2}(\hat{\mu}) = \limsup_{R \to \infty} \sup_{t \in \mathbb{R}^d} \frac{1}{R^d} \left\{ \int_{t+Q_R} 1 d\mu(\xi) + \left| \int_{t+Q_R} e^{-2\pi i (x_1 - x_2) \cdot \xi} d\mu(\xi) \right| \right\}.
\]

We need to prove a few technical lemmas before getting to the main result of this section,
Lemma 11. Let \( \psi \in \mathcal{S}(\mathbb{R}^d) \). Then, there exists \( C > 0 \) such that

\[
\delta^d \sum_{k \in \mathbb{Z}^d} \sup_{\gamma \in Q_\delta} |\psi(\xi - k\delta - \gamma)| \leq C, \quad \xi \in \mathbb{R}^d, \quad 0 < \delta \leq 1.
\]

Proof. Let us define

\[
g(\gamma) = \frac{1}{1 + \gamma^2}, \quad \gamma \in \mathbb{R},
\]

and suppose that \( 0 < \delta \leq 1 \). If \( \xi \in [-\delta/2, \delta/2] \) and \( k \in \mathbb{Z} \setminus \{0\} \),

\[
\inf_{|\gamma| \leq \delta/2} |\xi - \delta k - \gamma| = \min\{|\xi - k\delta - \delta/2|, |\xi - k\delta + \delta/2|\} \geq \delta (|k| - 1).
\]

Hence,

\[
\delta \sum_{k \in \mathbb{Z}} \sup_{|\gamma| \leq \delta/2} g(\xi - k\delta - \gamma) \leq \left( \delta + \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\delta}{1 + \delta^2 (|k| - 1)^2} \right) = 3 \delta + 2 \sum_{n=1}^{\infty} \frac{\delta}{1 + \delta^2 n^2} \leq 3 + 2 \int_{0}^{\infty} \frac{1}{1 + x^2} dx = c < \infty.
\]

Since the left-hand side of the previous expression is \( \delta \)-periodic, it follows that the inequality holds for all \( \xi \in \mathbb{R} \). If \( \psi \in \mathcal{S}(\mathbb{R}^d) \), we have the estimate

\[
|\psi(\gamma)| \leq C_1 \prod_{i=1}^{d} g(\gamma_i), \quad \gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{R}^d.
\]

Therefore, for any \( \xi \in \mathbb{R}^d \), we obtain

\[
\delta^d \sum_{k \in \mathbb{Z}^d} \sup_{\gamma \in Q_\delta} |\psi(\xi - k\delta - \gamma)| \leq \sum_{k \in \mathbb{Z}^d} C_1 \prod_{i=1}^{d} \delta \sup_{|\gamma_i| \leq \delta/2} g(\xi_i - \delta k_i - \gamma_i)
\]

\[
= C_1 \prod_{i=1}^{d} \delta \sum_{k_i \in \mathbb{Z}} \sup_{|\gamma_i| \leq \delta/2} g(\xi_i - \delta k_i - \gamma_i) \leq C_1 c^d = C < \infty.
\]

Lemma 12. Let \( \mu \) be a locally finite, positive Borel measure on \( \mathbb{R}^d \) and let \( \delta > 0 \). Suppose that \( F_1, \ldots, F_N \in \mathbb{R}^d \) are compactly supported. Then, for any \( \epsilon > 0 \), we have the inequalities

\[
G(\delta, \epsilon) - I(\delta, \epsilon) \leq \left( \int_{\mathbb{R}^d} e^d \left| \sum_{i=1}^{N} \hat{F}_i(\epsilon \lambda) e^{-2\pi i x_i \lambda} \right|^2 d\mu(\lambda) \right)^{1/2} \leq G(\delta, \epsilon) + I(\delta, \epsilon)
\]

if \( I(\delta, \epsilon) < \infty \), where

\[
G(\delta, \epsilon) = \left( \sum_{k \in \mathbb{Z}^d} \int_{\delta k / \epsilon + Q_{\delta/\epsilon}} e^d \left| \sum_{i=1}^{N} \hat{F}_i(\delta k) e^{-2\pi i x_i \lambda} \right|^2 d\mu(\lambda) \right)^{1/2}
\]
Applying Minkowski’s inequality twice, we have

\[ I(\delta, \epsilon) = \left( \sum_{k \in \mathbb{Z}^d} \int_{\delta k / \epsilon + Q_{\delta / \epsilon}} \epsilon^d \left| \sum_{i=1}^N \left( \hat{F}_i(\epsilon\lambda) - \hat{F}_i(\delta k) \right) e^{-2\pi i x \cdot \lambda} \right|^2 d\mu(\lambda) \right)^{1/2}. \]

**Proof.** If \( k \in \mathbb{Z}^d \), define

\[ S_k(\gamma) = \epsilon^{d/2} \sum_{i=1}^N \hat{F}_i(\delta k) e^{-2\pi i x \cdot \gamma} \quad \text{and} \quad R_k(\gamma) = \epsilon^{d/2} \sum_{i=1}^N \left( \hat{F}_i(\epsilon\lambda) - \hat{F}_i(\delta k) \right) e^{-2\pi i x \cdot \gamma} \]

Applying Minkowski’s inequality twice, we have

\[
\left( \int_{\mathbb{R}^d} \epsilon^d \left| \sum_{i=1}^N \hat{F}_i(\epsilon\lambda) e^{-2\pi i x \cdot \lambda} \right|^2 d\mu(\lambda) \right)^{1/2} = \left( \sum_{k \in \mathbb{Z}^d} \int_{\delta k / \epsilon + Q_{\delta / \epsilon}} \left| S_k(\gamma) + R_k(\gamma) \right|^2 d\mu(\lambda) \right)^{1/2}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}^d} \left[ \left( \int_{\delta k / \epsilon + Q_{\delta / \epsilon}} |S_k(\gamma)|^2 d\mu(\lambda) \right)^{1/2} + \left( \int_{\delta k / \epsilon + Q_{\delta / \epsilon}} |R_k(\gamma)|^2 d\mu(\lambda) \right)^{1/2} \right] \right)^2 \]

\[
\leq \left( \sum_{k \in \mathbb{Z}^d} \int_{\delta k / \epsilon + Q_{\delta / \epsilon}} |S_k(\gamma)|^2 d\mu(\lambda) \right)^{1/2} + \left( \sum_{k \in \mathbb{Z}^d} \int_{\delta k / \epsilon + Q_{\delta / \epsilon}} |R_k(\gamma)|^2 d\mu(\lambda) \right)^{1/2}
\]

\[ = G(\delta, \epsilon) + I(\delta, \epsilon). \]

Similarly, reversing the role of \( \delta k \) and \( \lambda \) in the above computation, we obtain the inequality

\[ G(\delta, \epsilon) \leq \left( \int_{\mathbb{R}^d} \epsilon^d \left| \sum_{i=1}^N \hat{F}_i(\epsilon\lambda) e^{-2\pi i x \cdot \lambda} \right|^2 d\mu(\lambda) \right)^{1/2} + I(\delta, \epsilon). \]

This completes the proof of this lemma. \( \square \)

**Lemma 13.** Suppose that \( \mu \) is a locally finite, positive Borel measure on \( \mathbb{R}^d \) and consider \( N \) functions \( F_1, \ldots, F_N \in L^2(Q_1) \). If \( 0 < \delta \leq 1 \) and \( \gamma \in \mathbb{R}^d \), let \( \upsilon_{\gamma} = (\hat{F}_1(\gamma), \cdots, \hat{F}_N(\gamma)) \in \mathbb{C}^N \). Then, there exists a constant \( C > 0 \) such that

\[ |I(\delta, \epsilon)|^2 \leq C \delta \int_{\mathbb{R}^d} \sup_{\zeta \in \mathbb{R}^d} \frac{\mu_{\upsilon_{\gamma}}(\zeta + Q_{\delta / \epsilon})}{(\delta / \epsilon)^d} d\gamma, \quad \epsilon > 0, \]

where

\[ I(\delta, \epsilon) = \left( \sum_{k \in \mathbb{Z}^d} \int_{\delta k / \epsilon + Q_{\delta / \epsilon}} \epsilon^d \left| \sum_{i=1}^N \left( \hat{F}_i(\epsilon\lambda) - \hat{F}_i(\delta k) \right) e^{-2\pi i x \cdot \lambda} \right|^2 d\mu(\lambda) \right)^{1/2}. \]

**Proof.** Let \( \beta \in C_0^\infty(\mathbb{R}^d) \) with \( \beta = 1 \) on a neighborhood of \( Q_1 \) and let \( \psi = \hat{\beta} \). Then \( \psi \in \mathcal{S}(\mathbb{R}^d) \) and \( F\beta = F \) for any \( F \in L^2(Q_1) \) which implies that \( \hat{F} \ast \psi = \hat{F} \). Using this last identity
together with the Cauchy-Schwarz inequality, we have

\[
\left| \sum_{i=1}^{N} \left( \hat{F}_i(\epsilon \lambda) - \hat{F}_i(\delta k) \right) e^{-2\pi i x_i \lambda} \right|^2 = \int_{\mathbb{R}^d} \left( \psi(\epsilon \lambda - \gamma) - \psi(\delta k - \gamma) \right) \left( \sum_{i=1}^{N} \hat{F}_i(\gamma) e^{-2\pi i x_i \lambda} \right) d\gamma \leq \left( \int_{\mathbb{R}^d} \left| \psi(\epsilon \lambda - \gamma) - \psi(\delta k - \gamma) \right| \left| \sum_{i=1}^{N} \hat{F}_i(\gamma) e^{-2\pi i x_i \lambda} \right|^2 d\gamma \right)^{1/2} \times \left( \int_{\mathbb{R}^d} \left| \psi(\epsilon \lambda - \gamma) - \psi(\delta k - \gamma) \right| d\gamma \right)^{1/2} \leq 2 \left\| \psi \right\|_{L^1} \int_{\mathbb{R}^d} \left| \psi(\epsilon \lambda - \gamma) - \psi(\delta k - \gamma) \right| \left| \sum_{i=1}^{N} \hat{F}_i(\gamma) e^{-2\pi i x_i \lambda} \right|^2 d\gamma.
\]

Given \( \epsilon > 0 \), we consider the element \( \tilde{\mu}^\epsilon \) of \( \mathcal{MP}_N(\mathbb{R}^d) \) associated via formula (3.3) to the measure \( \mu^\epsilon \) defined by

\[
\int_{\mathbb{R}^d} \phi(\xi) d\mu^\epsilon(\xi) = \int_{\mathbb{R}^d} \epsilon^d \phi(\epsilon \xi) d\mu(\xi), \quad \phi \in C_c(\mathbb{R}^d).
\]

In particular, we have, for any \( \nu \in \mathbb{C}^N \)

\[
\tilde{\mu}^\epsilon(\xi + Q\delta) = \epsilon^d \sum_{i,j=1}^{N} v_i \overline{v}_j \mu_{i,j}(\xi/\epsilon + Q\delta/\epsilon) = \delta^d \tilde{\mu}_\nu(\xi/\epsilon + Q\delta/\epsilon) d\xi, \quad \xi \in \mathbb{R}^d, \epsilon, \delta > 0,
\]

which yields the inequalities

\[
\delta^d \inf_{\xi \in \mathbb{R}^d} \frac{\tilde{\mu}_\nu(\xi' + Q\delta/\epsilon)}{(\delta/\epsilon)^d} \leq \tilde{\mu}^\epsilon(\xi + Q\delta) \leq \delta^d \sup_{\xi \in \mathbb{R}^d} \frac{\tilde{\mu}_\nu(\xi' + Q\delta/\epsilon)}{(\delta/\epsilon)^d}.
\]

Hence, using Fubini’s theorem and letting \( C_0 = 2 \left\| \psi \right\|_{L^1} \), we have

\[
\left| I(\delta, \epsilon) \right|^2 \leq C_0 \sum_{k \in \mathbb{Z}^d} \int_{\delta k + Q\delta \cap \mathbb{Z}^d} \int_{\mathbb{R}^d} \epsilon^d \left| \psi(\epsilon \lambda - \gamma) - \psi(\delta k - \gamma) \right| \left| \sum_{i=1}^{N} \hat{F}_i(\gamma) e^{-2\pi i x_i \lambda} \right|^2 d\gamma d\mu(\lambda)
\]

\[
= C_0 \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \left( \int_{\delta k + Q\delta} \sum_{i,j=1}^{N} \left| \psi(\lambda - \gamma) - \psi(\delta k - \gamma) \right| d\hat{F}_i(\gamma) \hat{F}_j(\gamma) d\mu_{i,j}(\lambda) d\gamma \right) d\mu^\epsilon(\lambda)
\]

Using the mean-value theorem, we have the estimate

\[
\left| \psi(\lambda - \gamma) - \psi(\delta k - \gamma) \right| \leq \delta \sqrt{d} \sum_{i=1}^{d} \sup_{\xi' \in \delta k + Q\delta} \left| \frac{\partial \psi}{\partial \xi_i}(\xi' - \gamma) \right|, \quad \lambda \in \delta k + Q\delta.
\]
Hence, by (3.3),

\[ |I(\delta, \epsilon)|^2 \leq \int_{\mathbb{R}^d} C_0 \delta \sqrt{d} \left( \sup_{\zeta \in \mathbb{R}^d} \frac{\mu_{\nu_\epsilon}(\zeta + Q_{\delta/\epsilon})}{(\delta/\epsilon)^d} \right) \sum_{i=1}^d \sum_{k \in \mathbb{Z}} \sup_{\xi' \in \delta k + Q_{\delta}} \delta^d \left| \frac{\partial \psi}{\partial \xi_i}(\xi' - \gamma) \right| d\gamma \]

Applying Lemma 11 to each function \( \frac{\partial \psi}{\partial \xi_i} \) in \( \mathcal{S}(\mathbb{R}^d) \), we can thus find a constant \( C > 0 \) such that

\[ |I(\delta, \epsilon)|^2 \leq C \delta \int_{\mathbb{R}^d} \sup_{\zeta \in \mathbb{R}^d} \frac{\mu_{\nu_\epsilon}(\zeta + Q_{\delta/\epsilon})}{(\delta/\epsilon)^d} d\gamma, \quad \epsilon > 0, \]

as claimed.

We now state the main result of this section.

**Theorem 14.** Let \( x_1, \cdots, x_N \in \mathbb{R}^d \) be distinct and let \( \mu \) be a locally finite, positive Borel measure on \( \mathbb{R}^d \). Define the associate positive matrix-valued measure \( \tilde{\mu} \) using formula (3.1). Then the following are equivalent.

(a) There exist constants \( A, B > 0 \) and \( \epsilon > 0 \) such that the sets \( x_j + Q_{\epsilon}, j = 1, \ldots, N \), are disjoint and such that the frame inequalities

\[
A \| f \|_2^2 \leq \int_{\mathbb{R}^d} |\hat{f}(\lambda)|^2 \, d\mu(\lambda) \leq B \| f \|_2^2, \quad f \in L^2(\Omega), \tag{3.4}
\]

are satisfied for \( \Omega = \bigcup_{j=1}^N (x_j + Q_{\epsilon}) \).

(b) We have \( 0 < \mathcal{D}_N^- (\tilde{\mu}) \leq \mathcal{D}_N^+ (\tilde{\mu}) < \infty \).

Moreover, if (a) holds, we have the inequalities \( A \leq \mathcal{D}_N^- (\tilde{\mu}) \leq \mathcal{D}_N^+ (\tilde{\mu}) \leq B \). Conversely, if (b) holds, then, for any \( \rho > 0 \) such that \( \mathcal{D}_N^- (\tilde{\mu}) - \rho > 0 \), there exists \( \epsilon > 0 \) such that (a) holds with \( A = \mathcal{D}_N^- (\tilde{\mu}) - \rho \) and \( B = \mathcal{D}_N^- (\tilde{\mu}) + \rho \).

**Proof.** If \( \xi \in \mathbb{R}^d \), define the modulation operator \( M_\xi \) acting on \( L^2(\mathbb{R}^d) \), by

\[ M_\xi f(x) = e^{2\pi i \xi \cdot x} f(x), \quad f \in L^2(\mathbb{R}^d). \]

If (a) holds, let \( \mathbf{v} = (v_1, \ldots, v_N) \in \mathbb{C}^N \) with \( \|\mathbf{v}\|_2 = 1 \), let \( \xi \in \mathbb{R}^d \) and, if \( f \in L^2(Q_{\epsilon}) \), consider the function

\[ h := \sum_{i=1}^N v_i \delta_{x_i} \ast (M_\xi \overline{f}) \in L^2(\Omega). \]

Then,

\[
\int_{\Omega} |h(x)|^2 \, dx = \left( \sum_{i=1}^N |v_i|^2 \right) \left( \int_{Q_{\epsilon}} |f(x)|^2 \, dx \right) = \int_{Q_{\epsilon}} |f(x)|^2 \, dx.
\]
On the other hand, we have also
\[
\int_{\mathbb{R}^d} |\hat{h}(\lambda)|^2 \, d\mu(\lambda) = \int_{\mathbb{R}^d} \left| \sum_{i=1}^{N} v_i \hat{f}(\xi - \lambda) e^{-2\pi i x \cdot \lambda} \right|^2 \, d\mu(\lambda) = \sum_{i,j=1}^{N} \int_{\mathbb{R}^d} v_i \overline{v_j} |\hat{f}(\xi - \lambda)|^2 \, d\mu_{i,j}(\lambda)
\]
\[
= \int_{\mathbb{R}^d} |\hat{f}(\xi - \lambda)|^2 \, d\hat{\mu}_\nu(\lambda) = (\hat{\mu}_\nu * |\hat{f}|^2)(\xi).
\]

Using the frame inequalities \((3.4)\), we have thus
\[
A \int_{Q_\epsilon} |f(x)|^2 \, dx \leq \hat{\mu}_\nu * |\hat{f}|^2 \leq B \int_{Q_\epsilon} |f(x)|^2 \, dx.
\]

As \(\hat{\mu}_\nu\) is a positive measure, Theorem \([7]\) shows that \(0 < A \leq D^- (\hat{\mu}_\nu) \leq D^+ (\hat{\mu}_\nu) \leq B < \infty\) for any vector \(\mathbf{v}\) of norm 1. Taking infimum and supremum, respectively, over the unit ball of \(CN\), we obtain (b).

We now prove that (b) implies (a). Let \(\Omega = \bigcup_{i=1}^{N} (x_i + Q_\epsilon)\) with \(\epsilon > 0\) chosen small enough so the sets in the previous union are pairwise disjoint. A function \(f \in L^2(\Omega)\) can then be uniquely written as
\[
f = \sum_{i=1}^{N} \delta_{x_i} * f_i, \quad f_i \in L^2(Q_\epsilon).
\]

For each of the functions \(f_i\) above, let \(F_i \in L^2(Q_1)\) be defined by \(f_i(x) = \epsilon^{-d/2} F_i(x/\epsilon)\). Clearly \(\|F_i\|_2 = \|f_i\|_2\) and
\[
\int_{\Omega} |f(x)|^2 \, dx = \sum_{i=1}^{N} \int_{\mathbb{R}^d} |\hat{F}_i(\lambda)|^2 \, d\lambda \quad \text{with} \quad \hat{f}(\xi) = \epsilon^{d/2} \sum_{i=1}^{N} \hat{F}_i(\epsilon \xi) e^{-2\pi i x \cdot \xi}.
\]

Hence, proving \((3.4)\) is equivalent to showing that for any \(F_1, \ldots, F_N \in L^2(Q_1)\) and for \(\epsilon > 0\) sufficiently small, we have the inequalities
\[
A \sum_{i=1}^{N} \int_{\mathbb{R}^d} |\hat{F}_i(\lambda)|^2 \, d\lambda \leq \int_{\mathbb{R}^d} \epsilon^d \left| \sum_{i=1}^{N} \hat{F}_i(\epsilon \lambda) e^{-2\pi i x \cdot \lambda} \right|^2 \, d\mu(\lambda) \leq B \sum_{i=1}^{N} \int_{\mathbb{R}^d} |\hat{F}_i(\lambda)|^2 \, d\lambda. \quad (3.6)
\]

Given any number \(\rho > 0\) with \(D_N^- (\hat{\mu}) - \rho > 0\), we choose \(\rho'\) with \(0 < \rho' \leq 1\) and small enough so that
\[
(\sqrt{D_N^+ (\hat{\mu})} + \rho' + \sqrt{\rho'})^2 \leq (D_N^+ (\hat{\mu}) + \rho) \quad \text{and} \quad (\sqrt{D_N^- (\hat{\mu})} - \rho' - \sqrt{\rho'})^2 \geq D_N^- (\hat{\mu}) - \rho.
\]

With this particular chosen \(\rho' > 0\), we can find a number \(\delta\) with \(0 < \delta \leq 1\) and small enough so that
\[
4 C \delta (D_N^+ (\hat{\mu})) + D_N^- (\hat{\mu}) + 1 < \rho',
\]
where $C$ is the constant obtained in Lemma \[13\]. With that value of $\delta$ fixed, we use the assumption (b) and the definition of $D^+_N(\hat{\mu})$ to obtain the existence of a number $\epsilon > 0$ small enough so that

$$
\sup_{\zeta \in \mathbb{R}^d} \frac{\hat{\mu}_\nu(\zeta + Q_{\delta/\epsilon})}{(\delta/\epsilon)^d} \leq (D^+_N(\hat{\mu})) + \rho') \|v\|^2.
$$

and

$$
\inf_{\zeta \in \mathbb{R}^d} \frac{\hat{\mu}_\nu(\zeta + Q_{\delta/\epsilon})}{(\delta/\epsilon)^d} \geq (D^-_N(\hat{\mu}) - \rho') \|v\|^2
$$

for any $\|v\| = 1$. If $\gamma \in \mathbb{R}^d$, let us denote by $v_\gamma$ the vector $(\hat{F}_1(\gamma), \ldots, \hat{F}_N(\gamma)) \in \mathbb{C}^N$.

Applying Lemma \[13\] to the functions $F_1, \ldots, F_N \in L^2(Q_1)$ defined above and letting

$$
I(\delta, \epsilon) = \left( \sum_{k \in \mathbb{Z}^d} \int_{\delta k/\epsilon + Q_{\delta/\epsilon}} e^d \left( \sum_{i=1}^N (\hat{F}_i(\epsilon \lambda) - \hat{F}_i(\delta k)) e^{-2\pi i x \cdot \lambda} \right)^2 d\mu(\lambda) \right)^{1/2},
$$

we obtain that

$$
|I(\delta, \epsilon)|^2 \leq C \delta \int_{\mathbb{R}^d} \sup_{\zeta \in \mathbb{R}^d} \frac{\hat{\mu}_\nu(\zeta + Q_{\delta/\epsilon})}{(\delta/\epsilon)^d} d\gamma \leq C \delta (D^+_N(\hat{\mu}) + \rho') \int_{\mathbb{R}^d} \|v_\gamma\|^2 d\gamma
$$

$$
\leq \rho' \int_{\mathbb{R}^d} \sum_{i=1}^N |\hat{F}_i(\gamma)|^2 d\gamma = \rho' \|f\|_2^2.
$$

On the other hand, letting

$$
|G(\delta, \epsilon)|^2 = \sum_{k \in \mathbb{Z}^d} \int_{\delta k/\epsilon + Q_{\delta/\epsilon}} e^d \left( \sum_{i=1}^N \hat{F}_i(\delta k) e^{-2\pi i x \cdot \lambda} \right)^2 d\mu(\lambda)
$$

we have

$$
|G(\delta, \epsilon)|^2 = \sum_{k \in \mathbb{Z}^d} \sum_{i,j=1}^N \hat{F}_i(\delta k) \overline{F_j(\delta k)} e^{d \mu_{i,j}(\delta k/\epsilon + Q_{\delta/\epsilon})} = \sum_{k \in \mathbb{Z}^d} e^d \hat{\mu}_{\nu\delta k}(\delta/\epsilon + Q_{\delta/\epsilon})
$$

$$
\leq \delta^d \sup_{k \in \mathbb{Z}^d, \zeta \in \mathbb{R}^d} \frac{\hat{\mu}_{\nu\delta k}(\zeta + Q_{\delta/\epsilon})}{(\delta/\epsilon)^d} \leq \delta^d \sum_{k \in \mathbb{Z}^d} (D^+_N(\hat{\mu}) + \rho') \|v_{\delta k}\|_2^2
$$

$$
= (D^+_N(\hat{\mu}) + \rho') \sum_{i=1}^N \sum_{k \in \mathbb{Z}^d} \delta^d |\hat{F}_i(\delta k)|^2.
$$

Since each function $F_i$ is supported in $Q_1$ and $\delta \leq 1$, the Shannon sampling theorem shows that

$$
\sum_{k \in \mathbb{Z}^d} \delta^d |\hat{F}_i(\delta k)|^2 = \|F_i\|_2^2, \quad i = 1, \ldots, N.
$$

Hence, we obtain the inequality

$$
|G(\delta, \epsilon)|^2 \leq (D^+_N(\hat{\mu}) + \rho') \|f\|_2^2.
$$
A similar computation shows also that

$$|G(\delta, \epsilon)|^2 \geq (D_N^+ (\hat{\mu})) - \rho) \|f\|_2^2.$$  

Using the estimates for $I(\delta, \epsilon)$ and $G(\delta, \epsilon)$ just obtained, we deduce from Lemma 12 that

$$\int_{\mathbb{R}^d} e^d \left| \sum_{i=1}^N \hat{f}_i(\epsilon \lambda) e^{-2\pi i x_i \cdot \lambda} \right|^2 d\mu(\lambda) \leq \left( \sqrt{D_N^+ (\hat{\mu}) + \rho'} + \sqrt{\rho'} \right)^2 \|f\|_2^2 \leq (D_N^+ (\hat{\mu}) + \rho) \|f\|_2^2$$

and

$$\int_{\mathbb{R}^d} e^d \left| \sum_{i=1}^N \hat{f}_i(\epsilon \lambda) e^{-2\pi i x_i \cdot \lambda} \right|^2 d\mu(\lambda) \geq \left( \sqrt{D_N^- (\hat{\mu}) - \rho'} - \sqrt{\rho'} \right)^2 \|f\|_2^2 \geq (D_N^- (\hat{\mu}) - \rho) \|f\|_2^2.$$  

This completes the proof.

Of course, ignoring the lower-bound estimates in the proof just given, we can also prove the Bessel version of the previous theorem.

**Theorem 15.** Let $x_1, \ldots, x_N \in \mathbb{R}^d$ be distinct and let $\mu$ be a locally finite, positive Borel measure on $\mathbb{R}^d$. Define the associate positive matrix-valued measure $\hat{\mu}$ using formula (3.1). Then the following are equivalent.

(a) There exist constants $A > 0$ and $\epsilon > 0$ such that the sets $x_j + Q_\epsilon$, $j = 1, \ldots, N$, are disjoint and such that the Bessel inequality

$$\int_{\mathbb{R}^d} |\hat{f}(\lambda)|^2 d\mu(\lambda) \leq B \|f\|_2^2, \quad f \in L^2 (\Omega)$$

holds for $\Omega = \bigcup_{j=1}^N (x_j + Q_\epsilon)$.

(b) We have $D_N^+ (\hat{\mu}) < \infty$.

Moreover, if (a) holds, we have the inequality $D_N^+ (\hat{\mu}) \leq B$. Conversely, if (b) holds, then, for any $\rho > 0$, there exists $\epsilon > 0$ such that (a) holds with $B = D_N^+ (\hat{\mu}) + \rho$.

The following gives an interpretation of the lower and upper density of the positive matrix-valued measure $\hat{\mu}$ defined (3.1) as limiting lower and upper frame bounds, respectively.
Therefore, our main focus will be to deal with finitely generated subgroups of $G$ with respect to a given group $G$ independent of the points $x$. The form $\bigcup_{j=1}^{N} (x_j + Q_\epsilon)$. Then $\lim_{\epsilon \to 0} A_\epsilon = D^-_N(\hat{\mu})$ and $\lim_{\epsilon \to 0} B_\epsilon = D^+_N(\hat{\mu})$.

Proof. If $D^-_N(\hat{\mu}) > 0$, the inequalities obtained in Theorem 14 show, for any $\rho > 0$, that

$$D^-_N(\hat{\mu}) - \rho \leq A_\epsilon \leq D^-_N(\hat{\mu}) \quad \text{and} \quad D^+_N(\hat{\mu}) \leq B_\epsilon \leq D^+_N(\hat{\mu}) + \rho,$$

if $\epsilon > 0$ is small enough, proving our statement in that case. If $D^-_N(\hat{\mu}) = 0$ and $\rho > 0$, we have $A_\epsilon = 0$ and the inequalities $D^+_N(\hat{\mu}) \leq B_\epsilon \leq D^+_N(\hat{\mu}) + \rho$ if $\epsilon > 0$ is small enough, from which our claim follows immediately.

**Remark 17.** We note that using (3.3), given a positive Borel measure $\mu$ on $\mathbb{R}^d$, the quantity $D^+_N(\hat{\mu})$, where $\hat{\mu}$ is defined in (3.7), can also be defined as the smallest constant $B \geq 0$ such that

$$\limsup_{h \to \infty} \sup_{t \in \mathbb{R}^d} \frac{1}{h^d} \int_{t+Q_h} \left| \sum_{i=1}^{m} a_i e^{2\piix_i \xi} \right|^2 d\mu(\xi) \leq B \sum_{i=1}^{m} |a_i|^2,$$

for any $a_1, \ldots, a_m \in \mathbb{C}$. Similarly, if $D^-_N(\hat{\mu}) < \infty$, the quantity $D^-_N(\hat{\mu})$ can also be defined as the largest constant $A \geq 0$ such that

$$\liminf_{h \to \infty} \inf_{t \in \mathbb{R}^d} \frac{1}{h^d} \int_{t+Q_h} \left| \sum_{i=1}^{m} a_i e^{2\piix_i \xi} \right|^2 d\mu(\xi) \geq A \sum_{i=1}^{m} |a_i|^2,$$

for any $a_1, \ldots, a_m \in \mathbb{C}$. We will use these alternate definitions in the next section.

### 4 Uniform limiting frame bounds for subgroups of $\mathbb{R}$.

This section will be devoted to the characterization of uniform $\mathcal{F}$-measures and $\mathcal{B}$-measures over subgroups of $\mathbb{R}$. These are the elements of the sets $\mathcal{F}(G, A, B)$ and $\mathcal{B}(G, B)$, respectively, defined in Definition 4 of the introduction. Recall that our goal will be to characterize the measures having the property of being a common $\mathcal{F}$-measure (resp. $\mathcal{B}$-measure) for $L^2(\Omega)$, where $\Omega$ is any set of the form $\Omega = \bigcup_{j=1}^{N} (x_j + Q_\epsilon)$, for $\epsilon > 0$ small enough and dependent on the points $x_i$, where the points $x_j$ belong to a given subgroup $G$ of $\mathbb{R}$, but with the limiting frame bounds (resp. Bessel bounds), as defined in the end of the last section, being independent of the points $x_i$, $i = 1, \ldots, N$.

In the problems stated, it is clear that a measure will satisfy one of the properties mentioned with respect to a given group $G$ if and only if it will do so for any one of its finitely generated subgroup, as we need only to check those properties on each finite subset of $G$. Therefore, our main focus will be to deal with finitely generated subgroup of $\mathbb{R}$, i.e. those of the form

$$G = \left\{ \sum_{i=1}^{s} m_i a_i, \ m_i \in \mathbb{Z} \right\}.$$
where $a_1, \ldots, a_s$ can be assumed to be linearly independent over $\mathbb{Q}$.

The notion of weak-$*$ convergence of measures defined below and the property of weak-$*$ compactness will play an important role in the proofs of our main results. Let $C_c(\mathbb{R}^s)$ denote the space of complex-valued continuous functions with compact support defined on $\mathbb{R}^s$.

**Definition 18.** Let $\sigma_i, i \geq 1$, and $\sigma$ be locally finite, positive Borel measures on $\mathbb{R}^s$. We say that $\sigma_i$ converges to $\sigma$ in the weak-$*$ topology as $i \to \infty$ if for any $\varphi \in C_c(\mathbb{R}^s)$, we have

$$
\lim_{i \to \infty} \int_{\mathbb{R}^s} \varphi(\xi) \, d\sigma_i(\xi) = \int_{\mathbb{R}^s} \varphi(\xi) \, d\sigma(\xi).
$$

We have the following criterion for weak-$*$ compactness: if $\{\sigma_i\}_{i \geq 1}$ is a sequence of locally finite, positive Borel measures on $\mathbb{R}^s$ which is locally uniformly bounded, i.e. for any compact $K \subset \mathbb{R}^s$, there exists a constant $C(K)$ such that

$$
\sup_{i \geq 1} \sigma_i(K) \leq C(K),
$$

then the sequence $\{\sigma_i\}_{i \geq 1}$ admits a subsequence which is convergent in the weak-$*$ topology. We need some preliminary lemmas.

**Lemma 19.** Let $\{\tau_j\}_{j \geq 1}$ be a sequence of positive Borel measures on $\mathbb{R}^s$ such that $\tau_j \to \tau$ in the weak-$*$ topology. If $\tau$ is absolutely continuous with respect to the Lebesgue measure with Radon-Nikodym derivative $G \in L^\infty(\mathbb{R}^s)$, then we have

$$
\lim_{j \to \infty} \int_F H(\xi) \, d\tau_j(\xi) = \int_F H(\xi) \, d\tau(\xi)
$$

for any continuous function $H \geq 0$ on $\mathbb{R}^s$ where $F = I_1 \times \cdots \times I_s$ and $I_s \subset \mathbb{R}, \ j = 1, \ldots, s$, are bounded intervals.

*Proof.* Let $B = \|G\|_\infty$. Given $\epsilon > 0$, choose real-valued functions $\phi_1, \phi_2 \in C_c(\mathbb{R}^s)$ such that $\phi_1(\xi) \leq H(\xi) \chi_F(\xi) \leq \phi_2(\xi)$, for $\xi \in \mathbb{R}$, and with

$$
\int_{\mathbb{R}^s} \phi_2(\xi) - \phi_1(\xi) \, d\xi \leq \epsilon.
$$

Then,

$$
\limsup_{j \to \infty} \int_{\mathbb{R}^s} \phi_2(\xi) - H(\xi) \chi_F(\xi) \, d\tau_j(\xi) \leq \lim_{j \to \infty} \int_{\mathbb{R}^s} \phi_2(\xi) - \phi_1(\xi) \, d\tau_j(\xi)
$$

$$
= \int_{\mathbb{R}^s} G(\xi) \, (\phi_2(\xi) - \phi_1(\xi)) \, d\xi \leq B \int_{\mathbb{R}^s} (\phi_2(\xi) - \phi_1(\xi)) \, d\xi \leq B \epsilon
$$

and thus

$$
\liminf_{j \to \infty} \int_F H(\xi) \, d\tau_j(\xi) \geq \int_{\mathbb{R}^s} G(\xi) \phi_2(\xi) \, d\xi - B \epsilon \geq \int_{\mathbb{R}^s} G(\xi) H(\xi) \chi_F(\xi) \, d\xi - B \epsilon.
$$
Similarly,

\[
\limsup_{j \to \infty} \int_{\mathbb{R}^s} H(\xi) \chi_F(\xi) - \phi_1(\xi) \, d\tau_j(\xi) \leq \lim_{j \to \infty} \int_{\mathbb{R}^s} \phi_2(\xi) - \phi_1(\xi) \, d\tau_j(\xi) \leq B \epsilon
\]

which shows that

\[
\limsup_{j \to \infty} \int_{\mathbb{F}} H(\xi) \, d\tau_j(\xi) \leq \int_{\mathbb{R}^s} G(\xi) \phi_1(\xi) \, d\xi + B \epsilon \leq \int_{\mathbb{R}^s} G(\xi) H(\xi) \chi_F(\xi) \, d\xi + B \epsilon.
\]

We have thus

\[
\int_{\mathbb{F}} H(\xi) G(\xi) \, d\xi - B \epsilon \leq \liminf_{j \to \infty} \int_{\mathbb{F}} H(\xi) \, d\tau_j(\xi) \leq \limsup_{j \to \infty} \int_{\mathbb{F}} H(\xi) \, d\tau_j(\xi)
\]

\[
\leq \int_{\mathbb{F}} H(\xi) G(\xi) \, d\xi + B \epsilon,
\]

and the result follows since \(\epsilon > 0\) is arbitrary and \(d\tau = G(\xi) \, d\xi\).

\[\square\]

If \(\rho\) is a signed or complex measure on \(\mathbb{R}\), we will denote by \(|\rho|\) its total variation.

**Lemma 20.** Let \(a_1, \ldots, a_s\) be \(s\) positive real numbers and let \(\mu\) be a positive translation-bounded Borel measure on \(\mathbb{R}\). Consider the positive Borel measure \(\nu_\mu\) on \(\mathbb{R}^s\) defined by

\[
\int_{\mathbb{R}^s} \varphi(\xi_1, \ldots, \xi_s) \, d\nu_\mu(\xi_1, \ldots, \xi_s) = \int_{\mathbb{R}^s} \varphi(\lambda_1, \ldots, \lambda) \, d\mu(\lambda), \quad \varphi \in C_c(\mathbb{R}^s),
\]

and define

\[
\sigma_{c,R} := \delta_c * \frac{1}{R} \sum_{0 \leq k_1 \leq a_1 R^{-1}} \cdots \sum_{0 \leq k_s \leq a_s R^{-1}} \delta_{(-k_1/a_1, \ldots, -k_s/a_s)} * \nu_\mu
\]

where \(c \in \mathbb{R}^s\) and \(R > 0\). Let \(\mathcal{L}\) denote the lattice \(\prod_{k=1}^s a_k^{-1} \mathbb{Z}\). Then, the following properties hold.

(a) For any compact \(K \subset \mathbb{R}^s\), the set \(\{\sigma_{c,R}(K) \mid c \in \mathbb{R}^s, \ R \geq 1\}\) is bounded.

(b) Let \(\sigma_j := \sigma_{c_j,R_j}\) where \(R_j \to \infty\) and \(c_j \in \mathbb{R}^s\). If the sequence \(\{\sigma_j\}_{j \geq 1}\) converges to the measure \(\sigma\) in the weak-* topology, then \(\sigma\) is \(\mathcal{L}\)-periodic.

**Proof.** If \(r > 0\) and \(c = (c_1, \ldots, c_s)\), we have

\[
\sigma_{c,R}([-r,r]^s) = \int_{\mathbb{R}^s} \chi_{[-r,r]^s}(\xi) \, d\sigma_{c,R}(\xi)
\]

\[
= \frac{1}{R} \sum_{0 \leq k_1 \leq a_1 R^{-1}} \cdots \sum_{0 \leq k_s \leq a_s R^{-1}} \int_{\mathbb{R}^s} \chi_{[-r,r]^s}(\xi + c + (-k_1/a_1, -k_2/a_2, \ldots, -k_1/a_s)) \, d\nu_\mu(\xi)
\]

\[
= \frac{1}{R} \sum_{0 \leq k_1 \leq a_1 R^{-1}} \cdots \sum_{0 \leq k_s \leq a_s R^{-1}} \int_{\mathbb{R}} \prod_{m=1}^s \chi_{[-r,r]}(\lambda - k_m/a_m + c_m) \, d\mu(\lambda)
\]
Note that if for some integer \( k_1 \) and some \( m \geq 2 \) and some integer \( k \), we have

\[
(-r, r] + k_1/a_1 - c_1) \cap (-r, r] + k/a_m - c_m) \neq \emptyset,
\]
then \( |k/a_m - b_m| \leq 2r \), where \( b_m = c_m - c_1 + k_1/a_1 \). Hence, \( |k - a_m b_m| \leq 2r a_m \) and the number of integers \( k \) satisfying this inequality can be at most \( 4r a_m + 1 \). It follows that, if \( C = \sup_{t \in \mathbb{R}} \mu([-r, r] + t) < \infty \), we have

\[
\sigma_{c,R}([-r, r]^s) \leq \frac{C}{R} \sum_{0 \leq k_1 \leq a_1 R - 1} \prod_{m=2}^s (4r a_m + 1) \leq C a_1 \prod_{m=2}^s (4r a_m + 1),
\]
which proves (a). To prove (b), it is enough to show that if \( \nu_{l,j} := (\delta_t - \delta_0) * \sigma_j \) and, for any \( l \in \mathcal{L} \), \( |\nu_{l,j}|(K) \to 0 \) as \( j \to \infty \) for any compact set \( K \subset \mathbb{R}^s \). Indeed, if it is the case, then \( \nu_{l,j} \to 0 \) in the weak-* topology and, in particular, if \( \sigma_j \to \sigma \) in the weak-* topology, then

\[
(\delta_t - \delta_0) * \sigma = \lim_{j \to \infty} \nu_{l,j} = 0,
\]
showing that \( \sigma \) is \( \mathcal{L} \)-periodic. We consider first the case \( l = a_i^{-1} e_i \), where \( e_i, i = 1, \ldots, s \), is the standard basis in \( \mathbb{R}^s \). We only deal with the case \( i = 1 \) since the other cases are similar. We have then

\[
(\delta_{e_1/a_1} - \delta_0) * \sigma_j = \gamma_{e_1} * (\delta_{e_1 N_1/a_1} - \delta_0) * \frac{1}{R_j} \sum_{0 \leq k_2 \leq a_2 R_j - 1} \cdots \sum_{0 \leq k_s \leq a_s R_j - 1} \delta(0, -k_2/a_2, \ldots, -k_1/a_1) * \nu_{l,j},
\]
where \( N_1 = [a_1 R_1] \) with \( [x] \) being the largest integer less than or equal to \( x \). Let

\[
\rho_j := \gamma_{e_1} * \frac{1}{R_j} \sum_{0 \leq k_2 \leq a_2 R_j - 1} \cdots \sum_{0 \leq k_s \leq a_s R_j - 1} \delta(0, -k_2/a_2, \ldots, -k_1/a_1) * \nu_{l,j}.
\]

As in the proof of (a), if \( r > 0 \), we have, letting \( C = \sup_{t \in \mathbb{R}} \mu([-r, r] + t) < \infty \), that

\[
\rho_j([-r, r]^s) \leq \frac{C}{R_j} \prod_{m=2}^s (4r a_m + 1) \to 0, \quad j \to \infty.
\]

For the same reason, \( (\delta_{e_1 N_j/a_1} * \rho_j)([-r, r]^s) \to 0 \) as \( j \to \infty \). Hence,

\[
\left| (\delta_{e_1^{-1} e_1} - \delta_0) * \sigma_j \right|([-r, r]^s) = \left| (\delta_{N_j a_1^{-1} e_1} - \delta_0) * \rho_j \right|([-r, r]^s) \leq (\delta_{E_1 N_j a_1^{-1} e_1} + \delta_0) * \rho_j([-r, r]^s) \to 0,
\]
as \( j \to \infty \) and our claim follows when \( l = a_i^{-1} e_i, i = 1, \ldots, s \), since \( r > 0 \) is arbitrary. In general, if \( l \in \mathcal{L} \), we can write

\[
\delta_l - \delta_0 = \sum_{m=1}^s \tau_m * (\delta_{a_m^{-1} e_m} - \delta_0)
\]
where each $\tau_m$ is a finite sum of Dirac masses. Hence,

$$
| (\delta_l - \delta_0) \ast \sigma_j (K) | = \left| \sum_{m=1}^{s} \tau_m \ast (\delta_{\epsilon_m} - \delta_0) \ast \sigma_j \right| (K) \to 0,
$$
as $j \to \infty$, for any compact set $K \subset \mathbb{R}^s$, which proves our claim. \hfill \square

If $c > 0$ and $x$ is a real number, we denote by $x \pmod{c}$ the unique real number $y$ in the interval $[0, c)$ such that $x - y \in c \mathbb{Z}$. The following theorem will be the key result to answer the questions raised at the beginning of this section. We will only prove the equivalence of the upper-bounds inequalities (i.e. those involving the constant $B$) as the lower-bound ones (involving the constant $A$ if $A > 0$) can be obtained by very similar techniques. We leave the details to the interested reader.

**Theorem 21.** Let $a_1, \ldots, a_s$ be $s$ positive real numbers linearly independent over $\mathbb{Q}$, with $s \geq 1$, and let $G$ be the subgroup of $\mathbb{R}$ generated by $a_1, \ldots, a_s$, i.e.

$$
G = \left\{ \sum_{i=1}^{s} m_i a_i, \ m_i \in \mathbb{Z} \right\}.
$$

Let $\mu$ be a positive translation-bounded Borel measure on $\mathbb{R}$ and associate with it the positive Borel measure $\nu_\mu$ on $\mathbb{R}^s$ defined by (4.1). Let $A, B$ be real constants with $A \geq 0$ and $B > 0$. Then, the following are equivalent:

(a) For any distinct $x_1, \ldots, x_m \in G$ and any $c_1, \ldots, c_m \in \mathbb{C}$, we have

$$
\limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{R} \int_{[t, t+R]} \left| \sum_{i=1}^{m} c_i e^{-2\pi i x_i \lambda} \right|^2 d\mu (\lambda) \leq B \sum_{i=1}^{m} |c_i|^2.
$$

and

$$
\liminf_{R \to \infty} \inf_{t \in \mathbb{R}} \frac{1}{R} \int_{[t, t+R]} \left| \sum_{i=1}^{m} c_i e^{-2\pi i x_i \lambda} \right|^2 d\mu (\lambda) \geq A \sum_{i=1}^{m} |c_i|^2.
$$

(b) Any weak-$*$ limit $\sigma$ of a sequence extracted from the collection

$$
\delta(-t, \ldots, -t) \ast \frac{1}{a_1 \ldots a_s R} \sum_{0 \leq k_1 \leq a_1 R-1} \ldots \sum_{0 \leq k_s \leq a_s R-1} \delta(\frac{-k_1}{a_1}, \ldots, \frac{-k_s}{a_s}) \ast \nu_\mu
$$

where $t \in \mathbb{R}$ and $R \to \infty$ is absolutely continuous with respect to the Lebesgue measure and with Radon-Nikodym derivative

$$
\frac{d\sigma}{d\xi} = G \text{ holding } A \leq G \leq B \text{ a.e. on } \mathbb{R}^s.$$

(c) For any intervals $I_1 \subset [0, 1/a_1), \ldots, I_s \subset [0, 1/a_s)$, we have
\[
\limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{R a_1 \ldots a_s} \mu(E(t, R, I_1, \ldots, I_s)) \leq |I_1| \ldots |I_s| B
\]
and
\[
\liminf_{R \to \infty} \inf_{t \in \mathbb{R}} \frac{1}{R a_1 \ldots a_s} \mu(E(t, R, I_1, \ldots, I_s)) \geq |I_1| \ldots |I_s| A,
\]
where
\[
E(t, R, I_1, \ldots, I_s) = \{ \lambda \in \mathbb{R} : t \leq \lambda \leq t + R, \lambda (\text{mod } a_1^{-1}) \in I_1, \ldots, \lambda (\text{mod } a_s^{-1}) \in I_s \}.
\]

Proof. As mentioned above we will only prove the case $A = 0$ of the theorem.

Fix $x_1, \ldots, x_m \in G$, with $x_i = \sum_{j=1}^s n_{ij} a_j$, $n_{ij} \in \mathbb{Z}$, and let $c_1, \ldots, c_m \in C$. Note that
\[
\limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{R} \int_{[t,t+R]^s} \left| \sum_{i=1}^m c_i e^{-2\pi i (\sum_{j=1}^s n_{ij} a_j \cdot) \xi_i} \right|^2 d\nu_\mu(\xi_1, \ldots, \xi_s)
\]
\[
= \limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{R} \int_{[t,t+R]} \left| \sum_{i=1}^m c_i e^{-2\pi i x_i} \right|^2 d\mu(\lambda) := L.
\]
Furthermore, replacing the set $[t, t + R]^s$ by the smaller set
\[
[t, t + N_1(R) a_1^{-1}] \times \cdots \times [t, t + N_s(R) a_s^{-1})
\]
where $N_k(R)$ are the unique integers satisfying
\[
N_k(R) a_k^{-1} \leq R < (N_k(R) + 1) a_k^{-1}, \quad k = 1, \ldots, s,
\]
does not change the first of the limits above. Indeed, letting $I = [t, t + R]$ and $J_k = [t, t + N_k(R) a_k^{-1})$ for $k = 1, \ldots, s$, we have
\[
I^s \setminus \prod_{k=1}^s J_k \subset \left( (I \setminus J_1) \times I^{s-1} \right) \cup \left( I \times (I \setminus J_2) \times I^{s-2} \right) \cup \cdots \cup \left( I^{s-1} \times (I \setminus J_s) \right) := \bigcup_{k=1}^s (I \setminus J_k).
\]
Letting $Q(\xi) = Q(\xi_1, \ldots, \xi_s) = \sum_{i=1}^m c_i e^{-2\pi i (\sum_{j=1}^s n_{ij} a_j \cdot) \xi_i}$ we have
\[
\frac{1}{R} \int_{I^s \setminus \prod_{k=1}^s J_k} |Q|^2 d\nu_\mu \leq \sum_{k=1}^s \frac{1}{R} \int_{C_k} |Q|^2 d\nu_\mu
\]
\[
\leq \|Q\|_\infty^2 \sum_{k=1}^s \frac{1}{R} \int_{C_k} 1 d\nu_\mu = \|Q\|_\infty^2 \sum_{k=1}^s \frac{1}{R} \int_{I \setminus J_k} 1 d\mu
\]
\[
\leq \|Q\|_\infty^2 \frac{1}{R} \sum_{k=1}^s \mu([t + N_k(R) a_k^{-1}, t + (N_k(R) + 1) a_k^{-1}]) \to 0, \text{ as } R \to \infty.
\]
since $\mu$ is translation bounded. We have thus

$$L = \limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{R} \int_{\prod_{k=1}^{s}[t,N_k(R) \ a_k^{-1})] \ |Q(\xi)|^2 \ d\nu(\xi)$$

$$= \limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{R} \sum_{0 \leq k_1 \leq a_1 R^{-1}} \ldots \sum_{0 \leq k_s \leq a_s R^{-1}} \int_{I(t,k_1,\ldots,k_s)} |Q(\xi)|^2 \ d\nu(\xi).$$

where

$$I(t, k_1, \ldots, k_s) := [t + k_1 \ a_1^{-1}, t + (k_1 + 1) \ a_1^{-1}) \times \cdots \times [t + k_s \ a_s^{-1}, t + (k_s + 1) \ a_s^{-1})].$$

Letting

$$\sigma_{t,R} = \delta(-t, \ldots, -t) \ast \frac{1}{a_1 \ldots a_s R} \sum_{0 \leq k_1 \leq a_1 R^{-1}} \ldots \sum_{0 \leq k_s \leq a_s R^{-1}} \delta(-k_1/a_1, \ldots, -k_s/a_s) \ast \nu,$$ (4.3)

and, using the periodicity of $Q$ with respect to the lattice $\mathcal{L} := \prod_{k=1}^{s} a_k^{-1} \mathbb{Z}$, we have

$$L = \limsup_{R \to \infty} \sup_{t \in \mathbb{R}} a_1 \ldots a_s \int_{[0,a_1^{-1}) \times \cdots \times [0,a_s^{-1})} |Q(\xi)|^2 \ d\sigma_{t,R}(\xi).$$ (4.4)

Using part (a) of Lemma 20 with $c = (-t, \ldots, -t)$, it follows that the set

$$\{\sigma_{t,R}(K), \ t \in \mathbb{R}, \ R \geq 1\}$$

is bounded for any compact subset $K$ of $\mathbb{R}^s$. Hence, any sequence extracted from the collection of measures $\{\sigma_{t,R}, \ t \in \mathbb{R}, \ R \geq 1\}$ must have a weak-* convergent subsequence. Furthermore, by part (b) of Lemma 20 any weak-* limit of a sequence $\sigma_{t_j,R_j}$, where $R_j \to \infty$, must be periodic with respect to the lattice $\mathcal{L}$ defined above.

If (b) holds, consider sequences $\{t_j\}$ and $\{R_j\}$ with $R_j \to \infty$ such that

$$L = \lim_{j \to \infty} a_1 \ldots a_s \int_{[0,a_1^{-1}) \times \cdots \times [0,a_s^{-1})} |Q(\xi)|^2 \ d\sigma(t_j,R_j)(\xi).$$

By weak-* compactness, we can assume, by passing to a subsequence if necessary, that $\{\sigma_{t_j,R_j}\}$ is weak-* convergent to a measure $\sigma$ as $j \to \infty$. Using our hypothesis, $\sigma$ is absolutely continuous with respect to the Lebesgue measure and with Radon-Nikodym derivative
\[ d\sigma/d\xi = G \text{ and } \|G\|_\infty \leq B. \] Using Lemma \([9]\) with \(F = |Q|^2\), it follows that

\[
L = a_1 \ldots a_s \int_{[0,a_1^{-1}] \times \ldots \times [0,a_s^{-1}]} |Q(\xi)|^2 G(\xi) \, d\xi
\]

\[
\leq B a_1 \ldots a_s \int_{[0,a_1^{-1}] \times \ldots \times [0,a_s^{-1}]} |Q(\xi)|^2 \, d\xi
\]

\[
= B a_1 \ldots a_s \sum_{i,j=1}^m c_i c_j \int_{[0,a_1^{-1}] \times \ldots \times [0,a_s^{-1}]} e^{-2\pi i (\sum_{j=1}^s n_{ij} a_j \xi_j)} \, d\xi
\]

\[
= B a_1 \ldots a_s \sum_{i,j=1}^m c_i c_j \prod_{j=1}^s \int_{[0,a_j^{-1}]} e^{-2\pi i (n_{ij} - n_{ij}) a_j \xi_j} \, d\xi
\]

\[
= B \sum_{i=1}^m |c_i|^2,
\]

showing that (a) holds. Note that, in the last step of the previous computation, we used the fact that \(n_{ij} = n_{ij}\) for all \(j = 1, \ldots, m\), implies that \(x_i = x_i\) (using the linear independence of \(a_1, \ldots, a_s\) over \(Q\)) and thus that \(i = l\), since the \(x_i\)'s are assumed to be distinct.

Conversely, if (a) holds and \(\sigma\) is a weak-* limit of a sequence \(\{\sigma_j, R_j\}\), with \(R_j \rightarrow \infty\) as \(j \rightarrow \infty\), then \(\sigma\) is periodic with respect to the lattice \(L\) and we have by the computation above, that

\[
a_1 \ldots a_s \int_{[0,a_1^{-1}] \times \ldots \times [0,a_s^{-1}]} |Q(\xi)|^2 \, d\sigma(\xi) \leq B a_1 \ldots a_s \int_{[0,a_1^{-1}] \times \ldots \times [0,a_s^{-1}]} |Q(\xi)|^2 \, d\xi
\]

for any trigonometric polynomial \(Q(\xi) = \sum_{i=1}^m c_i e^{-2\pi i (\sum_{j=1}^s n_{ij} a_j \xi_j)}\). Since the space of such trigonometric polynomials is dense (with respect to the sup-norm) in the space of continuous functions which are periodic with respect to the lattice \(L\), we have

\[
\int_{[0,a_1^{-1}] \times \ldots \times [0,a_s^{-1}]} \sum_{l \in L} \phi(\xi + l) \, d\sigma(\xi) \leq B \int_{[0,a_1^{-1}] \times \ldots \times [0,a_s^{-1}]} \sum_{l \in L} \phi(\xi + l) \, d\xi
\]

for any compactly supported continuous function \(\phi \geq 0\) on \(\mathbb{R}^s\) and thus, since \(\sigma\) is \(L\)-periodic,

\[
\int_{\mathbb{R}^s} \phi(\xi) \, d\sigma(\xi) \leq B \int_{\mathbb{R}^s} \phi(\xi) \, d\xi.
\]

Standard arguments show that \(\sigma\) must absolutely continuous with respect to the Lebesgue measure and with a Radon-Nikodym derivative \(G \in L^\infty(\mathbb{R}^s)\) satisfying \(\|G\|_\infty \leq B\), which shows that (b) holds. Thus the statements (a) and (b) are equivalent. Now, consider intervals \(I_j \subset [0, 1/a_j)\) for \(1 \leq j \leq s\) and define the sets \(F_j = \bigcup_{k \in \mathbb{Z}} I_j + k/a_j\) and let \(F = \bigcap_{j=1}^s F_j \subset \mathbb{R}^s\). Note that

\[
\{ \lambda \in \mathbb{R} : t \leq \lambda \leq t + R, \lambda \, (\text{mod } a_1^{-1}) \in I_1, \ldots, \lambda \, (\text{mod } a_s^{-1}) \in I_s \} = [t, t + R] \cap F_1 \cap \cdots \cap F_s.
\]
and that the function
\[ \chi_F(\xi) = \chi_{F_1}(\xi_1) \cdots \chi_{F_s}(\xi_s), \quad \xi = (\xi_1, \ldots, \xi_s) \in \mathbb{R}^s \]
is \( \mathcal{L} \)-periodic. Let \( M := \limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{R} \mu \left( \left\{ \lambda \in \mathbb{R} : t \leq \lambda \leq t + R, \lambda (\text{mod} \ a_1^{-1}) \in I_1, \ldots, \lambda (\text{mod} \ a_s^{-1}) \in I_s \right\} \right) \)

\[ = \limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{R} \int_{[t,t+R]} \chi_{F_1}(\lambda) \cdots \chi_{F_s}(\lambda) \, d\mu(\lambda) \]
\[ = \limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{R} \int_{[t,t+R]^s} \chi_{F_1}(\xi_1) \cdots \chi_{F_s}(\xi_s) \, d\nu(\xi_1, \ldots, \xi_s) \]
\[ = \limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{R} \int_{[t,t+R]^s} \chi_F(\xi) \, d\nu(\xi). \]

By a computation similar to the one done to obtain (4.4) (with \( \chi_F(\xi) \) replacing \( |Q(\xi)|^2 \)) we obtain, using the \( \mathcal{L} \)-periodicity of \( \chi_F(\xi) \), that

\[ M = \limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \, a_1 \cdots a_s \int_{[0,a_1^{-1}] \times \cdots \times [0,a_s^{-1}]} \chi_F(\xi) \, d\sigma_{t,R}(\xi) \]

with \( \sigma_{t,R} \) as in (4.3). Let \( \sigma_{t_j,R_j} \) a sequence with \( R_j \to \infty \) such that

\[ M = \lim_{j \to \infty} \, a_1 \cdots a_s \int_{[0,a_1^{-1}] \times \cdots \times [0,a_s^{-1}]} \chi_F(\xi) \, d\sigma_{t_j,R_j}(\xi). \]

By a weak-* compactness argument, we can assume that \( \sigma_{t_j,R_j} \) converges in the weak-* topology to a \( \mathcal{L} \)-periodic measure \( \sigma \). If (b) holds, we can use Lemma \( \text{[19]} \) applied to the sequence \( \{\sigma_{t_j,R_j}\} \) to show that

\[ M = a_1 \cdots a_s \int_{[0,a_1^{-1}] \times \cdots \times [0,a_s^{-1}]} \chi_F(\xi) \, G(\xi) \, d\xi. \]

We have thus

\[ M \leq B a_1 \cdots a_s |I_1| \cdots |I_s|, \]

which shows that (c) holds. Conversely, if (c) holds, then for any intervals \( I_j \) with \( I_j \subset [0, 1/a_j] \) for \( 1 \leq j \leq s \), we have

\[ \limsup \sup_{R \to \infty} \int_{I_1 \times \cdots \times I_s} 1 \, d\sigma_{t,R}(\xi) \leq B |I_1| \cdots |I_s|. \]

Let \( H_r = I_{r}^r \times \cdots \times I_{s}^r \), for \( 1 \leq r \leq R \), where, for each \( r \), \( I_r^r \) is an interval contained in \([0, 1/a_j]\). If \( H_r \cap H_s = \emptyset \) when \( r \neq s \) and \( h = \sum_{r=1}^{R} c_r \chi_{H_r} \) with \( c_r \geq 0 \), we have

\[ \limsup \sup_{R \to \infty} \int_{R^r} h(\xi) \, d\sigma_{t,R}(\xi) \leq B \sum_{r=1}^{R} c_r |I_1^r| \cdots |I_s^r| = B \int_{R^r} h(\xi) \, d\xi. \]
Suppose that $\sigma$ is a weak-* limit of a sequence $\sigma_{t_j, R_j}$ with $R_j \to \infty$ and let $\phi \geq 0$ be a compactly supported continuous function of $\mathbb{R}^s$. We have

$$
\int_{\mathbb{R}^s} \phi(\xi) \, d\sigma(\xi) = \lim_{j \to \infty} \int_{\mathbb{R}^s} \phi(\xi) \, d\sigma_{t_j, R_j}(\xi) = \lim_{j \to \infty} \sum_{l \in \mathcal{L}} \int_{[0,a_1^{-1}) \times \cdots \times [0,a_s^{-1})} \tilde{\phi}(\xi) \, d\sigma_{t_j, R_j}(\xi)
$$

where only a finite number of terms are non-zero in the last series, since $\phi$ is compactly supported. Hence,

$$
\int_{\mathbb{R}^s} \phi(\xi) \, d\sigma(\xi) = \lim_{j \to \infty} \sum_{l \in \mathcal{L}} \int_{[0,a_1^{-1}) \times \cdots \times [0,a_s^{-1})} \phi(\xi) \, d\sigma_{t_j, R_j}(\xi) - \int_{[0,a_1^{-1}) \times \cdots \times [0,a_s^{-1})} \phi(\xi) \, d\sigma_{\delta_0}(\xi) + \int_{[0,a_1^{-1}) \times \cdots \times [0,a_s^{-1})} \phi(\xi) \, d\nu_j(\xi)
$$

where $\nu_j := (\delta_l - \delta_0) \star \sigma_{t_j, R_j}$. Using part (b) of Lemma 20 and Lemma 19, the second limit above must be zero.

Hence,

$$
\int_{\mathbb{R}^s} \phi(\xi) \, d\sigma(\xi) = \lim_{j \to \infty} \int_{[0,a_1^{-1}) \times \cdots \times [0,a_s^{-1})} \psi(\xi) \, d\sigma_{t_j, R_j}(\xi),
$$

where

$$
\psi(\xi) = \sum_{l \in \mathcal{L}} \phi(\xi + l), \quad \xi \in \mathbb{R}^s,
$$

is continuous and $\mathcal{L}$-periodic. Since the restriction of $\psi$ to the set $[0, a_1^{-1}) \times \cdots \times [0, a_s^{-1})$ is uniformly continuous, we can find, for any $\epsilon > 0$, disjoint sets $H_r, r = 1, \ldots, R$, as above and constants $c_r, d_r \geq 0$ such that if $h_1 = \sum_{r=1}^{R} c_r \chi_{H_r}$ and $h_2 = \sum_{r=1}^{R} d_r \chi_{H_r}$, we have $h_1 \leq \psi \leq h_2$ and

$$
h_2 - h_1 \leq \epsilon
$$

on the set $[0, a_1^{-1}) \times \cdots \times [0, a_s^{-1})$. We have thus,

$$
\int_{\mathbb{R}^s} \phi(\xi) \, d\sigma(\xi) \leq \lim_{j \to \infty} \int_{[0,a_1^{-1}) \times \cdots \times [0,a_s^{-1})} \psi(\xi) \, d\sigma_{t_j, R_j}(\xi)
$$

$$
\leq \limsup_{j \to \infty} \int_{[0,a_1^{-1}) \times \cdots \times [0,a_s^{-1})} h_2(\xi) \, d\sigma_{t_j, R_j}(\xi) \leq B \int_{[0,a_1^{-1}) \times \cdots \times [0,a_s^{-1})} h_2(\xi) \, d\xi
$$

$$
\leq B \int_{[0,a_1^{-1}) \times \cdots \times [0,a_s^{-1})} (h_1(\xi) + \epsilon) \, d\xi \leq B \int_{[0,a_1^{-1}) \times \cdots \times [0,a_s^{-1})} \psi(\xi) \, d\xi + B a_1^{-1} \cdots a_s^{-1} \epsilon.
$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$
\int_{\mathbb{R}^s} \phi(\xi) \, d\sigma(\xi) \leq B \int_{[0,a_1^{-1}) \times \cdots \times [0,a_s^{-1})} \psi(\xi) \, d\xi = B \int_{\mathbb{R}^s} \phi(\xi) \, d\xi.
$$
In particular, we have that, for any complex-valued compactly supported continuous function on \( \mathbb{R}^s \),

\[
\int_{\mathbb{R}^s} |\phi(\xi)| \, d\sigma(\xi) \leq B \int_{\mathbb{R}^s} |\phi(\xi)| \, d\xi.
\]

Standard arguments show that \( \sigma \) must be absolutely continuous with respect to the Lebesgue measure and with a Radon-Nikodym derivative \( G \) satisfying \( \|G\|_{\infty} \leq B \), proving (b).

**Corollary 22.** Under the same assumptions as Theorem 21, the following statements are equivalent:

(a) There exists a positive constant \( A \) such that, for any distinct \( x_1, \ldots, x_m \in G \) and any \( c_1, \ldots, c_m \in \mathbb{C} \), we have

\[
\lim_{R \to \infty} \frac{1}{R} \int_{[t, t+R]} \left| \sum_{i=1}^{m} c_i e^{-2\pi i x_i \lambda} \right|^2 d\mu(\lambda) = A \sum_{i=1}^{m} |c_i|^2,
\]

uniformly for \( t \in \mathbb{R} \).

(b) There exists a constant \( A > 0 \) such that any weak-\(*\) limit \( \sigma \) of a sequence extracted from the collection

\[
\delta(-t, \ldots, -t) \ast \frac{1}{a_1 \ldots a_s R} \sum_{0 \leq k_1 \leq a_1 R-1} \ldots \sum_{0 \leq k_s \leq a_s R-1} \delta(-k_1/a_1, \ldots, -k_s/a_s) \ast \nu \mu
\]

is equal to the absolutely continuous measure \( d\sigma = A \, d\xi \), where \( d\xi \) represents the Lebesgue measure on \( \mathbb{R}^s \).

(c) There exists a constant \( A > 0 \) such that, for any intervals \( I_1 \subset [0, 1/a_1), \ldots, I_s \subset [0, 1/a_s) \), we have

\[
\lim_{R \to \infty} \frac{1}{R a_1 \ldots a_s} \mu( E(t, R, I_1, \ldots, I_s) ) = |I_1| \ldots |I_s| A,
\]

uniformly for \( t \in \mathbb{R} \), where

\[
E(t, R, I_1, \ldots, I_s) = \{ \lambda \in \mathbb{R} : t \leq \lambda \leq t + R, \lambda (\text{mod } a_1^{-1}) \in I_1, \ldots, \lambda (\text{mod } a_s^{-1}) \in I_s \}.
\]

If \( G \) is a subgroup of \( \mathbb{R} \), we will denote by \( \Pi_G \) the set of trigonometric polynomials \( P(\lambda) \) on \( \mathbb{R} \) with spectrum in \( G \), i.e. those of the form

\[
P(\lambda) = \sum_{i=1}^{m} c_i e^{-2\pi i x_i \lambda}, \quad x_i \in G, \ c_i \in \mathbb{C}.
\]

The mean of \( P \) is defined to be

\[
\mathcal{M}(P) = \lim_{R \to \infty} \frac{1}{R} \int_{[-R/2, R/2]} P(\lambda) \, d\lambda.
\]

Note that \( \mathcal{M}(|P|^2)| = \sum_{i=1}^{m} |c_i|^2 \), if \( P(\lambda) \) is as above. Combining these results with those of the previous sections, we obtain the following characterizations.
Theorem 23. Let \( a_1, \ldots, a_s \) be \( s \) positive real numbers linearly independent over \( \mathbb{Q} \), with \( s \geq 1 \), and let \( G \) be the subgroup of \( \mathbb{R} \) generated by \( a_1, \ldots, a_s \), i.e.

\[
G = \left\{ \sum_{i=1}^{s} m_i a_i, \ m_i \in \mathbb{Z} \right\}.
\]

Let \( \mu \) be a positive translation-bounded Borel measure on \( \mathbb{R} \). If \( I_k \) are intervals with \( I_k \subset [0, 1/a_k), k = 1, \ldots, s \), let \( E(t, R, I_1, \ldots, I_s) \) denote the set

\[
\left\{ \lambda \in \mathbb{R} : t \leq \lambda \leq t + R, \lambda (\mod a_1^{-1}) \in I_1, \ldots, \lambda (\mod a_s^{-1}) \in I_s \right\}.
\]

(a) The measure \( \mu \in B(G, B) \) if and only if any of the two following statement holds:

(i) For any intervals \( I_k \subset [0, 1/a_k), k = 1, \ldots, s \), we have

\[
\limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{R a_1 \ldots a_s} \mu(E(t, R, I_1, \ldots, I_s)) \leq |I_1| \ldots |I_s| B.
\]

(ii) We have

\[
D^+(|P|^2 \mu) \leq B \mathcal{M}(|P|^2), \quad P \in \Pi_G.
\]

(b) The measure \( \mu \in F(G, A, B) \) if and only if any of the two following statement holds:

(i) For any intervals \( I_k \subset [0, 1/a_k), k = 1, \ldots, s \), we have

\[
\liminf_{R \to \infty} \inf_{t \in \mathbb{R}} \frac{1}{R a_1 \ldots a_s} \mu(E(t, R, I_1, \ldots, I_s)) \geq |I_1| \ldots |I_s| A.
\]

and

\[
\limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{R a_1 \ldots a_s} \mu(E(t, R, I_1, \ldots, I_s)) \leq |I_1| \ldots |I_s| B.
\]

(ii) We have

\[
A \mathcal{M}(|P|^2) \leq D^{-}(|P|^2 \mu) \leq D^{+}(|P|^2 \mu) \leq B \mathcal{M}(|P|^2), \quad P \in \Pi_G.
\]

(c) The measure \( \mu \in F(G, A, A) \) if and only if any of the two following statement holds:

(i) For any intervals \( I_k \subset [0, 1/a_k), k = 1, \ldots, s \), we have

\[
\lim_{R \to \infty} \frac{1}{R a_1 \ldots a_s} \mu(E(t, R, I_1, \ldots, I_s)) = |I_1| \ldots |I_s| A.
\]

uniformly for \( t \in \mathbb{R} \).

(ii) We have

\[
D(|P|^2 \mu) = A \mathcal{M}(|P|^2), \quad P \in \Pi_G.
\]

Proof. The proof of (a) and (b) follow immediately from the equivalence of conditions (a) and (c) in Theorem 21 together with Theorem 15 and Theorem 14, respectively. The statement in (c) is an immediate consequence of Corollary 22 and the case \( A = B \) of Theorem 14. \( \Box \)
Definition 24. If $b > 0$, we call a sequence of real numbers $\{\lambda_n\}_{n \in \mathbb{Z}}$ well-distributed modulo $b$ if, for every interval $I \subset [0, b)$ and every integer $M \in \mathbb{Z}$,

$$\lim_{N \to \infty} \frac{\#\{n, \lambda_n \pmod{b} \in I, M \leq n \leq M + N - 1\}}{N} = |I|/b$$

uniformly for $M \in \mathbb{Z}$.

The condition (c) of Theorem 23 can be rephrased using the notion of well-distributed sequences when dealing with measures of the form $\mu = \delta_{\Lambda}$, where $\Lambda$ is a discrete subset of $\mathbb{R}$ (i.e. the intersection of $\Lambda$ with any compact set is finite).

Corollary 25. Let $G = a\mathbb{Z}$ where $a > 0$. Suppose that $\Lambda$ is a discrete subset of $\mathbb{R}$ and consider a sequence $\{\lambda_n\}_{n \in \mathbb{Z}}$ enumerating the elements of $\Lambda$ in such a way that

$$\lambda_n < \lambda_{n+1}, \quad n \geq 1.$$

Then, $\delta_{\Lambda} \in \mathcal{F}(G, A, A)$ if and only if

(i) $\mathcal{D}(\Lambda) = A$.

(ii) The sequence $\{\lambda_n\}_{n \in \mathbb{Z}}$ is well-distributed modulo $a^{-1}$.

Proof. Using part (c) of Theorem 23 we easily see that $\delta_{\Lambda} \in \mathcal{F}(G, A, A)$ if and only if, for any interval $I \subset [0, a^{-1})$, we have

$$\lim_{R \to \infty} \frac{1}{R} \# \{\lambda \in \Lambda, t \leq \lambda \leq t + R, \lambda (\text{mod } a^{-1}) \in I\} = a |I| A,$$

uniformly for $t \in \mathbb{R}$. In particular, if (i) holds, we obtain, taking $I = [0, a^{-1})$, that $\mathcal{D}(\Lambda) = A$.

If $M \in \mathbb{Z}$, the set $A_{M,N} := \{\lambda_n, M \leq n \leq M + N - 1\} = [\lambda_M, \lambda_{M+N-1}] \cap \Lambda$. Note that for each $M$, $\lambda_{M+N-1} - \lambda_{M} \to \infty$ as $N \to \infty$, uniformly for $M \in \mathbb{Z}$. Otherwise, we could find a number $L > 0$ and intervals $I_N$ of length bounded by $L$ containing at least $N$ elements of $\Lambda$, for any $N \geq 1$, which would imply that $\mathcal{D}^{+}(\Lambda) = \infty$. Hence, for any interval $I \subset [0, a^{-1})$, we have

$$\frac{\#\{n, \lambda_n (\text{mod } a^{-1}) \in I, M \leq n \leq M + N\}}{N} = \frac{\#\{\lambda \in \Lambda, \lambda_M \leq \lambda \leq \lambda_{M+N-1}, \lambda (\text{mod } a^{-1}) \in I\}}{\lambda_{M+N-1} - \lambda_{M}} \left(\frac{\lambda_{M+N-1} - \lambda_{M}}{\#A_{M,N}}\right)$$

Since $\mathcal{D}(\Lambda) = A$, we have $\lim_{N \to \infty} \frac{\#A_{M,N}}{\lambda_{M+N-1} - \lambda_{M}} = A$, uniformly for $M \in \mathbb{Z}$. Furthermore, using (4.5), we obtain

$$\lim_{N \to \infty} \frac{\#\{\lambda \in \Lambda, \lambda_M \leq \lambda \leq \lambda_{M+N-1}, \lambda (\text{mod } a^{-1}) \in I\}}{\lambda_{M+N-1} - \lambda_{M}} = a |I| A.$$
uniformly for $M \in \mathbb{Z}$ as $N \to \infty$. We deduce that
\[
\lim_{N \to \infty} \frac{\# \{ n, \lambda_n (\text{mod} \ a^{-1}) \in I, \ M \leq n \leq M + N \}}{N} = a |I|,
\]
uniformly for $M \in \mathbb{Z}$, proving (ii). Conversely, if (i) and (ii) hold, we have, for any $t \in \mathbb{R}$ and any $R > 0$ large enough, that
\[
\{ \lambda \in \Lambda, \ t \leq \lambda \leq t + R \} = \{ \lambda_n, \ M(t, R) \leq n \leq M(t, R) + N(t, R) - 1 \},
\]
for some $M(t, R) \in \mathbb{Z}$ and $N(t, R) \geq 1$. Furthermore, using (i), we have
\[
N(t, R)/R \to A, \quad R \to \infty,
\]
uniformly for $t \in \mathbb{R}$ and using (ii), we have
\[
\lim_{R \to \infty} \frac{1}{R} \# \{ \lambda_n, \ M(t, R) \leq n \leq M(t, R) + N(t, R) - 1, \ \lambda_n (\text{mod} \ a^{-1}) \in I \} = |I| a,
\]
uniformly for $t \in \mathbb{R}$. Hence, for any interval $I \subset [0, a^{-1})$, we have
\[
\lim_{R \to \infty} \frac{1}{R} \# \{ \lambda \in \Lambda, \ t \leq \lambda \leq t + R, \ \lambda (\text{mod} \ a^{-1}) \in I \}
= \lim_{R \to \infty} \frac{1}{R} \# \{ \lambda_n, \ M(t, R) \leq n \leq M(t, R) + N(t, R) - 1, \ \lambda_n (\text{mod} \ a^{-1}) \in I \}
= |I| a A,
\]
showing that (156) holds and thus that $\delta_\Lambda \in \mathcal{F}(G, A, A)$.

Note that the condition (i) in the previous result is essential. For example, we can easily construct a discrete set $\Lambda$ with $D^+(\Lambda) = 0$ such that the associated sequence $\{ \lambda_n \}$ defined in the previous corollary is well-distributed modulo 1.

When the group $G$ is generated by at least two linearly independent (over $\mathbb{Q}$) elements, it must be dense in $\mathbb{R}$ and the conditions given in Theorem 23 become more difficult to satisfy. However, it is easy to check that the statement (a) in Corollary 22 holds for any finitely generated subgroup $G$ if $d\mu = d\lambda$, the Lebesgue measure on $\mathbb{R}$. It follows therefore that if $a_1, \ldots, a_s$ are real numbers linearly independent over $\mathbb{Q}$ and $I_1, \ldots, I_s$ are intervals with $I_j \subset [0, 1/a_j)$, $j = 1, \ldots, s$, then
\[
\lim_{R \to \infty} \frac{1}{R a_1 \ldots a_s} |E(t, R, I_1, \ldots, I_s)| = |I_1| \ldots |I_s|
\]
uniformly for $t \in \mathbb{R}$. This can be interpreted, in the language of probability theory, as saying that the events of belonging to the intervals $I_j$ modulo $1/a_j$, $j = 1, \ldots, s$, are asymptotically independent. For any finitely generated subgroup $G$, one can also construct discrete measures in $\mathcal{F}(G, A, A)$. In fact, lattices will yield such measures as long as $A$ does not belong to the $\mathbb{Q}$-linear span of $G$. 
Proposition 26. Let \(a_1, \ldots, a_s\) be \(s\) positive real numbers linearly independent over \(\mathbb{Q}\), with \(s \geq 1\), and let \(G\) be the subgroup of \(\mathbb{R}\) generated by \(a_1, \ldots, a_s\), i.e.

\[
G = \left\{ \sum_{i=1}^{s} m_i a_i, \ m_i \in \mathbb{Z} \right\}.
\]

Let \(b > 0\) and \(\mu = \delta_\Lambda\), where \(\Lambda = b \mathbb{Z}\). Then, there exist constants \(A, B > 0\) such that \(\mu \in \mathcal{F}(G, A, B)\) if and only if \(1/b \notin \text{span}_\mathbb{Q}(a_1, \ldots, a_m)\) and, in that case, we can take \(A = B = 1/b\).

Proof. If \(b^{-1} \in \text{span}_\mathbb{Q}(a_1, \ldots, a_m)\), there exists an integer \(l \geq 1\) such that \(l b^{-1} \in \mathbb{G}\). Hence \(k l b^{-1} \in G\) for any integer \(k\), and if \(c_0, \ldots, c_M \in \mathbb{C}\), we have

\[
\frac{1}{R} \int_{[t, t+R]} \left| \sum_{k=0}^{M} c_k e^{-2\pi i k t - 1} \right|^2 d\mu(\Lambda) = \left| \sum_{k=0}^{M} c_k \right|^2 \mu([t, t+R]) / R \to 1/b \left| \sum_{k=0}^{M} c_k \right|^2
\]

uniformly for \(t \in \mathbb{R}\) as \(R \to \infty\). Taking \(c_k = 1\), for all \(k\), yields

\[
\left| \sum_{k=0}^{M} c_k \right|^2 = (M + 1) \sum_{k=0}^{M} |c_k|^2,
\]

which shows that \(\mu \notin \mathcal{B}(G, B)\) for any \(B > 0\).

If \(1/b \notin \text{span}_\mathbb{Q}(a_1, \ldots, a_m)\), then \(x k b \notin \mathbb{Z}\) for any \(k \in \mathbb{Z} \setminus \{0\}\) and any \(x \in G \setminus \{0\}\). Hence if \(x_1, \ldots, x_m\) are distinct elements of \(G\) and \(c_1, \ldots, c_m \in \mathbb{C}\), we have

\[
\frac{1}{R} \int_{[t, t+R]} \left| \sum_{j=1}^{m} c_j e^{-2\pi i x_j t} \right|^2 d\mu(\Lambda) = \sum_{j,l=1}^{m} c_j c_l \frac{1}{R} \sum_{t/b \leq k \leq (t+R)/b} e^{-2\pi i (x_j - x_l) k b} \to \frac{1}{b} \sum_{j=1}^{m} |c_j|^2,
\]

uniformly for \(t \in \mathbb{R}\), as \(R \to \infty\), showing that \(\mu \in \mathcal{F}(G, 1/b, 1/b)\). \(\square\)

Recall (see [Ka, Me1]) that a function \(F\) defined on the real line is called (Bohr) almost-periodic if it is continuous and for every \(\epsilon > 0\) there exists a number \(\Lambda = \Lambda(\epsilon, F) > 0\) such that every interval of length \(\Lambda\) contains a number \(\tau\) such that

\[
\sup_{x \in \mathbb{R}} |F(x - \tau) - F(x)| < \epsilon.
\]

(4.6)

A number \(\tau\) such that (4.6) holds is called an \(\epsilon\)-almost period of \(F\). The space of almost-periodic functions on \(\mathbb{R}\) can be characterized as the sup-norm closure of the space of trigonometric polynomials \(P(\lambda)\) associated with arbitrary real frequencies, i.e. functions of the form

\[
P(\lambda) = \sum_{i=1}^{m} c_i e^{-2\pi i x_i \lambda}, \ c_i \in \mathbb{C}, \ x_i \in \mathbb{R}.
\]

If \(F(\lambda)\) is almost-periodic, the mean-value of \(F\), \(\mathcal{M}(F)\), defined by

\[
\mathcal{M}(F) = \lim_{R \to \infty} \frac{1}{R} \int_{[-R/2, R/2]} F(\lambda) \, d\lambda
\]
exists and the spectrum of $F$ consists of all the real numbers $x$ such that
\[ \mathcal{M}(F(\lambda) e^{-2\pi i x \lambda}) \neq 0. \]
If the spectrum of $F$ is contained in the subgroup $G$, then $F$ can be uniformly approximated arbitrary closely by trigonometric polynomials with spectrum in $G$ (see [Me1]). Note that if $F(\lambda)$ is almost-periodic and a trigonometric polynomial $P(\lambda)$ satisfies $\|F - P\|_\infty \leq \epsilon$, then for any positive translation-bounded Borel measure $\mu$ on $\mathbb{R}$, we have
\[ \frac{1}{R} \left[ \frac{1}{R} \left( |F(\lambda) - P(\lambda)|^2 d\mu(\lambda) \right) \right] \leq \epsilon^2 D^+(\mu). \]
and, in particular,
\[ \limsup_{R \to \infty} \sup_{t \in \mathbb{R}} \frac{1}{R} \left[ \frac{1}{R} \left( |F(\lambda) - P(\lambda)|^2 d\mu(\lambda) \right) \right] \leq \epsilon^2 D^+(\mu). \]
It follows immediately that any of the inequalities satisfied by the class of trigonometric polynomials with spectrum in the subgroup $G$ and used to characterize the measures in $\mathcal{B}(G, B)$ or $\mathcal{F}(G, A, B)$ must also be satisfied by the functions in $\mathcal{A}(G)$, the collection of almost-periodic functions on $\mathbb{R}$ having spectrum contained in $G$. We have thus the following.

**Theorem 27.** Let $G$ be a subgroup of $\mathbb{R}$ and let $\mu$ be a positive Borel measure on $\mathbb{R}$. Then,

(a) $\mu$ belongs to $\mathcal{B}(G, B)$ if and only if
\[ D^+(|F|^2 \mu) \leq B \mathcal{M}(|F|^2), \quad F \in \mathcal{A}(G). \]

(b) $\mu$ belongs to $\mathcal{F}(G, A, B)$ if and only if
\[ A \mathcal{M}(|F|^2) \leq D^-(|F|^2 \mu) \leq D^+(|F|^2 \mu) \leq B \mathcal{M}(|F|^2), \quad F \in \mathcal{A}(G). \]

(c) $\mu$ belongs to $\mathcal{F}(G, A, A)$ if and only if
\[ \mathcal{D}(|F|^2 \mu) = A \mathcal{M}(|F|^2), \quad F \in \mathcal{A}(G). \]

## 5 Existence of discrete measures in $\mathcal{F}(\mathbb{R}, A, B)$

As we mentioned earlier, the Lebesgue measure on $\mathbb{R}$, $d\mu = d\lambda$, belongs to $\mathcal{F}(\mathbb{R}, 1, 1)$. On the other hand, we do not know a single explicit example of a discrete set $\Lambda \subset \mathbb{R}$ such that the associated measure $\mu = \delta_\Lambda$ belongs to $\mathcal{F}(\mathbb{R}, A, B)$ for some constants $0 < A \leq B < \infty$. The fact that the construction of such a set should be extremely difficult is pretty clear by considering the conditions that the sets $E(t, R, I_1, \ldots, I_s)$ need to satisfy in part (b) of Theorem 23 and this for any choice of numbers $a_1, \ldots, a_s$ linearly independent over $\mathbb{Q}$. In this last section, our goal will be to prove the existence of such a set.

It might seem, a priori, that simple quasicrystals could yield an answer to the problem above as they have been shown to be universal sampling sets my B. Matei and Y. Meyer in
for any compact set $K \subset \mathbb{R}^d$ is called a universal sampling set if $D(\Lambda)$ exists and $\Lambda$ is a set of stable sampling for any compact set $K \subset \mathbb{R}^d$ with $|K| < D(\Lambda)$, which means, in the terminology used in this paper, that $\delta_\Lambda$ is an $\mathcal{F}$-measure for $L^2(K)$ if $|K| < D(\Lambda)$. A universal sampling set $\Lambda$ will thus yield a frame for $L^2(K)$ where $K = \bigcup_{i=1}^{N} [x_i - \epsilon/2, x_i + \epsilon/2]$ for any $x_1, \ldots, x_N \in \mathbb{R}$ if $\epsilon > 0$ is small enough and dependent on the $x_i$’s. However, the associated frame constants are dependent on the points $x_i$’s as well and it might not be possible to find frame bounds compatible with all the finite subsets $X = \{x_i, \ i = 1, \ldots, N\}$ of real numbers. In fact, this will be the case for simple quasi-crystals as the following result shows. The proof is based on an idea used by B. Matei to show us that simple quasi-crystals cannot yield frames for $L^2(F)$ if $F$ is unbounded (\cite{Ma}).

**Proposition 28.** Let $\Lambda \subset \mathbb{R}$ be a simple quasicrystal. Then, for any $B > 0$, the measure $\delta_\Lambda$ cannot belong to $\mathcal{B}(\mathbb{R}, B)$.

**Proof.** The main ingredient of this proof is that any simple quasicrystal $\Lambda$ is an harmonious set (see \cite{Me1, Me2}), which implies the existence of a sequence $\{x_j\}_{j \geq 1}$ of real numbers having the property that

$$\sup_{\lambda \in \Lambda} |e^{-2\pi i \lambda x_j} - 1| \to 0 \quad \text{as} \quad j \to \infty.$$ 

Given any integer $N \geq 1$ and any $\epsilon > 0$, we can thus find some elements $y_j = x_{n_j}$, $j = 1, \ldots, N$, of our sequence such that

$$\sup_{\lambda \in \Lambda} |e^{-2\pi i \lambda y_j} - 1| \leq \epsilon.$$ 

By Minkowski’s inequality, if $t \in \mathbb{R}$, $R > 0$ and $c_1, \ldots, c_N \in \mathbb{C}$, we have, letting $\mu = \delta_\Lambda$, that

$$\left(\sum_{j=1}^{N} |c_j|^2\right)^{1/2} \left(\frac{\mu([t,t+R])}{R}\right)^{1/2} = \left(\frac{1}{R} \int_{[t,t+R]} |\sum_{j=1}^{N} c_j|^2 d\mu(\lambda)\right)^{1/2} \leq \left(\frac{1}{R} \int_{[t,t+R]} \sum_{j=1}^{N} c_j (1 - e^{-2\pi i \lambda y_j})^2 d\mu(\lambda)\right)^{1/2} + \left(\frac{1}{R} \int_{[t,t+R]} \sum_{j=1}^{N} c_j e^{-2\pi i \lambda y_j}^2 d\mu(\lambda)\right)^{1/2} \leq \epsilon \left(\frac{1}{R} \int_{[t,t+R]} \left(\sum_{j=1}^{N} |c_j|^2\right)^2 d\mu(\lambda)\right)^{1/2} + \left(\frac{1}{R} \int_{[t,t+R]} \sum_{j=1}^{N} c_j e^{-2\pi i \lambda y_j}^2 d\mu(\lambda)\right)^{1/2}$$

Choosing $c_j = 1$ for all $j$ and taking limits as $R \to \infty$, we obtain that

$$N D(\mu)^{1/2} \leq \epsilon N D(\mu)^{1/2} + \lim_{R \to \infty} \left(\frac{1}{R} \int_{[t,t+R]} \sum_{j=1}^{N} e^{-2\pi i \lambda y_j}^2 d\mu(\lambda)\right)^{1/2}$$

If $\mu \in \mathcal{B}(\mathbb{R}, B)$, it would then follow from part (a) of Theorem 28 that, for any $\epsilon > 0$ and any $N \geq 1$, 

$$N (1 - \epsilon) D(\mu)^{1/2} \leq B N^{1/2}$$

which yields a contradiction since $D(\mu) = D(\Lambda) > 0$ if $\Lambda$ is a simple quasicrystal. \hfill $\square$
Despite the previous negative result, we will to show the existence of a discrete set \( \Lambda \) such that the measure \( \delta_\Lambda \in \mathcal{B}(\mathbb{R}, A, B) \) for some constants \( 0 < A < B < \infty \). In doing so, our task will be greatly simplified by the following powerful recent result of S. Nitzan, A. Olevskii and A. Ulanovskii.

**Theorem 29** ([NOU]). Every measurable set \( E \subset \mathbb{R} \) with \( |E| < \infty \) admits a discrete set \( \Lambda \) such that \( \{ e^{2\pi i \lambda x} \}_{\lambda \in \Lambda} \) is a frame for \( L^2(E) \).

The proof of this last theorem itself is far from trivial if \( E \) is an unbounded set. It uses a result in [MSS] in which the authors solve the long standing Kadison-Singer conjecture. The idea to reach our goal is then to use Theorem 29 for a particular unbounded set \( E \) satisfying the properties in the following lemma.

**Lemma 30.** There exists a set an open set \( E \subset \mathbb{R} \) with \( |E| < \infty \) such that, for any \( x_1, \ldots, x_m \in \mathbb{R} \), we have 

\[
E \cap (E + x_1) \cap \ldots (E + x_m) \neq \emptyset
\]

**Proof.** Define for any integer \( j \geq 0 \) the set

\[
E_j = \bigcup_{k \in \mathbb{Z}} (k/2^j - 1/2^{j+|k|}+1, k/2^j + 1/2^{j+|k|}+1)
\]

and let

\[
E = \bigcup_{j \geq 0} E_j.
\]

Note that \( |E_j| = \sum_{k \in \mathbb{Z}} 1/2^{j+|k|} = 3/2^j \) and thus \( |E| \leq 3 \sum_{j=0}^{\infty} 2^{-j} = 6 < \infty \). Let \( x_1, \ldots, x_m \in \mathbb{R} \). Note that

\[
E \cap (E + x_1) \neq \emptyset \iff x_1 \in E - E.
\]

Since \( (-1/2, 1/2) \subset E \) and \( E \) contains a neighborhood of each integer, we have \( E - E = \mathbb{R} \), so our claim is true for \( m = 1 \). To prove our claim for arbitrary \( m \), we use an induction argument. If we have

\[
E \cap (E + x_1) \cap \ldots (E + x_{m-1}) \neq \emptyset,
\]

let \( J \) be a non-empty open interval contained in the intersection above. The intersection \( (E + x_m) \cap J \) is non-empty if and only if \( x_m \in J - E \). If \( j \geq 0 \) is chosen so that \( 1/2^j < |J| \), we have \( J - E_j = \mathbb{R} \) since \( E_j \) contains a neighborhood of each point in \( 2^{-j} \mathbb{Z} \). Hence, \( J - E = \mathbb{R} \) and our claim follows.

**Theorem 31.** There exists a discrete set \( \Lambda \) such that \( \delta_\Lambda \in \mathcal{B}(\mathbb{R}, A, B) \) for some constants \( 0 < A < B < \infty \).

**Proof.** Let \( E \) be the set constructed in the previous lemma and let \( \mathcal{F} \subset \mathbb{R} \) be a discrete set such that \( \delta_\Lambda \) is an \( \mathcal{F} \)-measure for \( L^2(E) \) and whose existence follows from Theorem 29. We claim that, for any \( x_1, \ldots, x_m \in \mathbb{R} \), there exists \( \epsilon > 0 \) such that the collection \( \{ e^{2\pi i \lambda x} \}_{\lambda \in \Lambda} \) is a frame for \( L^2(\bigcup_{i=1}^{m} x_i + Q_\epsilon) \) with frame bounds independent of \( x_1, \ldots, x_m \) and \( \epsilon \). Indeed, if \( E \) is as above, there exists, by Lemma 30, \( z \in E \cap (E + x_1) \cap \ldots (E + x_m) \neq \emptyset \). Then \( z = z_1 + x_1 = \cdots = z_m + x_m \) with \( z_i \in E \) or \( x_i - z = -z_i \in -E = E \). For \( \epsilon > 0 \) small enough, we have \( \bigcup_{i=1}^{m} (x_i - z) + Q_\epsilon \subset E \) and \( \{ e^{2\pi i \lambda x} \}_{\lambda \in \Lambda} \) is a frame for \( L^2(\bigcup_{i=1}^{m} (x_i - z) + Q_\epsilon) \) and thus also for \( L^2(\bigcup_{i=1}^{m} x_i + Q_\epsilon) \).

\( \square \)
We note that the constants \( A, B \) found in Theorem 31 are not equal since \( \mathcal{E}(\Lambda) \) is a Fourier frame for the unbounded set \( E \) with \( A, B \) being the frame bounds. It was shown in [GL] that such set cannot admit any tight Fourier frames, so \( A, B \) cannot be equal. It is unknown whether we can make \( A, B \) equal for some other discrete set \( \Lambda \).

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