On the Green’s formula for a Stokes type problem

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Abstract. A time-periodic Stokes problem is studied in the domain with cylindrical outlets to infinity. Using the Fourier series the problem is reduced to a sequence of elliptic problem. For each of these elliptic boundary value problems a generalized Green’s formula is constructed. The analogous Green’s formula for the steady Stokes problem was obtained in [1].

Keywords: cylindrical outlets to infinity, time-periodic Stokes problem, generalized Green’s formula.

1. Formulation of the problem

Let \( \Omega \subset \mathbb{R}^3 \) be a domain with cylindrical outlets to infinity, i.e., outside the ball \( B_R = \{ x \in \mathbb{R}^3 : \| x \| \leq R \} \) the domain \( \Omega \) coincides with a system of \( J \) semi-infinite cylinders \( \Pi^j_\pm \) of a constant cross section \( \omega^j \). Let \( \Pi^j_\pm \cap \Pi^k_\pm = \emptyset, j \neq k \) and let the boundary \( \partial \Omega \) be smooth. We consider in \( \Omega \) the time-periodic Stokes problem

\[
\begin{align*}
\mathbf{v}_t - \nu \Delta \mathbf{v} + \nabla p &= \mathbf{f}, \quad (x, t) \in \Omega \times (0, 2\pi), \\
-\nabla \cdot \mathbf{v} &= 0, \quad (x, t) \in \Omega \times (0, 2\pi), \\
\mathbf{v} &= 0, \quad (x, t) \in \partial \Omega \times (0, 2\pi), \\
\mathbf{v}(x, 0) &= \mathbf{v}(x, 2\pi), \quad x \in \Omega.
\end{align*}
\]

We assume that the external force \( \mathbf{f} = (f_1, f_2, f_3)^T \) is \( 2\pi \)-time-periodic function. Problem (1)–(4) could be decomposed into a sequence of elliptic problems. Indeed, we can look for the solution to problem (1)–(4) in the form

\[
\mathbf{v}(x, t) = \frac{\mathbf{v}_0}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \left\{ \mathbf{v}_{ck}(x) \cos kt + \mathbf{v}_{sk}(x) \sin kt \right\},
\]

\[
\mathbf{p}(x, t) = \frac{\mathbf{p}_0}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \left\{ \mathbf{p}_{ck}(x) \cos kt + \mathbf{p}_{sk}(x) \sin kt \right\}.
\]

*This work was partly supported by the Lithuanian Science Council Student Research Fellowship Award.
Inserting series (5), (6) into equations and boundary conditions we get for coefficients $v_{ck}$, $v_{sk}$, $p_{ck}$, $p_{sk}$ series of the systems of elliptic problems

$$\begin{align*}
\begin{cases}
kv_{ck} - \nu \Delta v_{ck} + \nabla p_{ck} = f_{ck}, & x \in \Omega, \\
-kv_{sk} - \nu \Delta v_{sk} + \nabla p_{sk} = f_{sk}, & x \in \Omega, \\
-\nabla \cdot v_{ck} = 0, & x \in \partial \Omega, \\
-v_{sk} = 0, & x \in \partial \Omega.
\end{cases}
(7)
\end{align*}$$

Here $f_{00}/(2\pi), f_{ck}/\pi, f_{sk}/\pi$ are Fourier coefficients of the function $f = f(x, t)$.

In this paper we derive so-called generalized Green’s formula for problem (7). The analogous Green’s formula for the steady Stokes problem was obtained in [1]. The obtained below results are important for the construction of correct asymptotic conditions at infinity which describe real time-periodic physical processes (for example bloodstream).

2. The asymptotics of the solution to problem (7)

Let $x^j = (x^j_1, x^j_2, x^j_3)$ be the local coordinate system related to the cylinder $\Pi^j_+$ such that the axis $x^j_3$ is directed along cylinder axis. We consider problem (7) in a weighted Sobolev space $W^l_\beta(\Omega)$ which is a closure of $C^\infty_0(\Omega)$ (a class of infinitely differentiable functions with compact supports in $\Omega$) with respect to the norm

$$
\|u; W^l_\beta(\Omega)\|^2 = \sum_{|\alpha| \leq l} \int_\Omega \rho_\beta(x) \left| D^\alpha x u(x) \right|^2 dx,
$$

where $\rho_\beta$ is a smooth positive function on $\Omega$ such that $\rho_\beta(x) = \exp(\beta x^j_3)$ on $\Omega \setminus B_R, j = 1, \ldots, J$. If $\beta > 0$, elements of this space exponentially vanish as $x^j_3$ tends to infinity, and they may exponentially grow, if $\beta < 0$.

Consider problem (7) in the cylinder $\Pi^j_+$. Using the methods of the book [2] and arguing in the same way as in [1] we obtain four special solutions of the homogeneous problem (7):

$$
\begin{align*}
\begin{cases}
u_{ck}^0 = (0, 0, 0, 0, 0, 0, 0, 0)' , & \quad \nu_{ck}^1 = (0, 0, 0, 0, 0, 0, 0, 0)' , \\
\nu_{sk}^0 = (0, 0, 0, 0, 0, 0, 0, 0)' , & \quad \nu_{sk}^1 = (0, 0, 0, 0, 0, 0, 0, 0)' ,
\end{cases}
(8)
\end{align*}
$$

where the pair of functions $(\phi^j_k, \psi^j_k)$ is the unique solution of the problem

$$
\begin{align*}
\begin{cases}
k\phi^j_k + \nu \Delta \phi^j_k = 0, & x^j' \in \omega^j, \\
k\psi^j_k + \nu \Delta \psi^j_k = 0, & x^j' \in \partial \omega^j.
\end{cases}
(10)
\end{align*}
$$

According to Theorem 3.1.4 in [2] the sum of linear combinations of these solutions gives the main term (up to an exponentially vanishing term) of the asymptotic decom-
position of the "growing" at infinity solution. Let \( \chi_j(x) \) be a smooth cut-off function such that \( \text{supp}(\chi_j) \subseteq \Pi_j^j \) and \( \chi_j(x) = 1 \) if \( x_j > L \) for \( j = 1, \ldots, J \).

**Theorem 1.** Let \( \beta > 0 \). If \( u_k = (v_{ck}, p_{ck}, v_{sk}, p_{sk}) \in D^J_{-\beta} \mathcal{W} \) is the solution to problem (7) with the right-hand side \( f_k = (f_{ck}, f_{sk}) \in W^J_{-\beta} \), then

\[
\mathbf{u}_k(x) = \sum_{j=1}^{J} \chi_j(x) \left\{ a_j^{ij} v_{ck}^{kj} + a_j^{0} a_j^{0} u_{sk}^{ij} + b_j^{0} u_{sk}^{ij} + b_j^{0} u_{sk}^{ij} \right\} + \bar{u}_k, \tag{11}
\]

where \( \bar{u}_k \in D^J_{\beta} \mathcal{W}, a_j^{ij}, a_j^{0}, b_j^{0}, b_j^{0} \in \mathbb{C} \). Here \( D^J_{\beta} \mathcal{W} = W^J_{\beta+1} \mathcal{W} \times W^J_{\beta} \mathcal{W}^2 \).

### 3. Generalized Green’s formula

Let \( \mathbf{u}_k = (v_{ck}, p_{ck}, v_{sk}, p_{sk}), U_k = (V_{ck}, P_{ck}, V_{sk}, P_{sk}) \in C^\infty(\Omega) \). Integrating twice by parts in \( \Omega \) one gets the standard Green’s formula (see [3])

\[
\begin{align*}
    &-v \Delta v_{ck} + \nabla p_{ck} + k v_{sk}, v_{ck} \Delta + (-\nabla \cdot v_{ck}, P_{ck})_{\Omega} \\
    &+(-v \Delta v_{ck} + \nabla p_{sk} - k v_{ck}, V_{sk})_{\Omega} + (-\nabla \cdot v_{sk}, P_{sk})_{\Omega} \\
    &+ (v_{ck}, n \sigma_{ck} - \nabla u_{ck})_{\Omega} + (v_{sk}, n \sigma_{sk} - \nabla u_{sk})_{\Omega} \\
    &- (v_{ck} - \Delta v_{ck} + \nabla P_{ck}, k V_{sk})_{\Omega} - (p_{ck} - \nabla \cdot v_{ck})_{\Omega} \\
    &- (v_{ck} - \Delta v_{sk} + \nabla P_{sk}, k V_{ck})_{\Omega} - (p_{sk} - \nabla \cdot v_{sk})_{\Omega} \\
    &- (n p_{ck} - \nabla u_{ck}, V_{ck})_{\Omega} - (n p_{sk} - \nabla u_{sk}, V_{sk})_{\Omega} = 0,
\end{align*}
\]

(12)

here \( (, )_{\Omega} \) stands for a scalar product in \( L^2(\Omega) \). Denoting by \( q(\mathbf{u}_k, U_k) \) the left-hand side of the above formula we get

\[ q(\mathbf{u}, \mathbf{u}) = q(U, U) = 0 \]

for any \( \mathbf{u} \in D^J_{\beta} \mathcal{W} \) and \( \mathbf{U} \in D^J_{-\beta} \mathcal{W} \). Let \( S \) be an operator of problem (7) and \( S^* \) be an operator of the problem

\[
\begin{align*}
    &-k V_{sk} - \nabla ^2 V_{ck} + \nabla P_{ck} = F_{ck}, & x \in \Omega, \\
    &k V_{ck} - \nabla ^2 V_{sk} + \nabla P_{sk} = F_{sk}, & x \in \Omega, \\
    &-\nabla \cdot V_{ck} = 0, & x \in \Omega, \\
    &V_{ck} = 0, V_{sk} = 0, & x \in \partial \Omega.
\end{align*}
\]

(13)

It is clear that \( S^* \) is an adjoint operator to \( S \) with respect to the Green’s formula (12). Note that \( S \) is not self-adjoint operator. Homogeneous problem (13) in the cylinder \( \Pi_j^j \) has four special solutions

\[
\begin{align*}
    &U^{ij}_{ck} = (0, 0, 0, 1, 0, 0, 0, 0, 0)^t, & U^{0}_{ck} = (0, 0, \varphi^{ij}_k, x^j_k, 0, 0, \psi^{ij}_k, 0)^t, & (14) \\
    &U^{ij}_{sk} = (0, 0, 0, 0, 0, 1, 0, 0, 0)^t, & U^{0}_{sk} = (0, 0, -\psi^{ij}_k, 0, 0, \varphi^{ij}_k, x^j_k, 0)^t, & (15)
\end{align*}
\]

where functions \( \varphi^{ij}_k \) and \( \psi^{ij}_k \) are defined by formula (10). We denote by \( \mathbb{M}_{\pm \beta} \mathcal{W} \) the subset of functions \( u_k \in D^J_{-\beta} \mathcal{W} \) having expansion (11) and by \( \mathbb{M}^J_{\pm \beta} \mathcal{W} \) the
Inserting representations (11) and (16) into problems (7) and (13), respectively, we get, after cumbersome computation, that

\[
U_k = \sum_{j=1}^{J} \chi_j \left[ A'_{ck} U_{ck}^0 + A'_{sk} U_{sk}^0 + B'_{ck} U_{ck}^1 + B'_{sk} U_{sk}^1 \right] + U_k,
\]

(16)

where \( U_j^{h} \), \( h \in \{0, 1\} \), \( \varnothing \in \{c, s\} \), are defined by (14) and (15), \( \tilde{U}_k \in D'_\rho W(\Omega) \), \( A'_c, A'_s, B'_c, B'_s \in \mathbb{C} \).

Since \( \text{supp} \chi_j \cap \text{supp} \chi_l = \emptyset \), \( j \neq l \), we have

\[
q(\chi_j U_{\varnothing, k}^h, \chi_j U_{\varnothing, k}^m) = 0, \quad h, m \in \{0, 1\}, \quad \varnothing, \varnothing \in \{c, s\}.
\]

Using the fact that functions (8), (9) and (14), (15) are exact solutions to homogeneous problems (7) and (13), respectively, we get, after cumbersome computation, that

\[
q(\chi_j U_{\varnothing, k}^h, \chi_j U_{\varnothing, k}^\infty) = 0, \quad h = 0, 1, \quad \varnothing, \varnothing \in \{c, s\}.
\]

Inserting representations (11) and (16) into \( q(u_k, U_k) \) we get that a number of terms in \( q(u_k, U_k) \) vanishes and, finally, we find

\[
q(u_k, U_k) = \sum_{j=1}^{J} \left\{ a'_{ck} B'_{ck} q(\chi_j U_{ck}^0, \chi_j U_{ck}^1) + a'_{sk} B'_{sk} q(\chi_j U_{sk}^0, \chi_j U_{sk}^1) + b'_{ck} A'_{ck} q(\chi_j U_{ck}^1, \chi_j U_{ck}^0) + b'_{sk} A'_{sk} q(\chi_j U_{sk}^1, \chi_j U_{sk}^0) \right\}.
\]

Let us calculate the term \( q(\chi_j U_{ck}^0, \chi_j U_{ck}^1) \). We note, firstly, that the cut-off function \( \chi_j \) restricts all considerations to the cylinder \( \Pi_{L}^j \), secondly, that \( S(\chi_j U_{\varnothing, k}^0) \) and \( S^*(\chi_j U_{\varnothing, k}^j) \) have compact supports. Applying the Green’s formula (12) in the domain \( \Omega_L = \{ x \in \Omega : \text{if} \, x \in \Pi_{L}^j \text{ then } x_j < L, \, j = 1, \ldots, J \} \) we get an additional integral over the cross-section \( \omega^j \). Let \( n = (0, 0, 1)^t \) be the outward normal to \( \partial \Omega_L \) on \( \omega^j \) and \( \partial_3 = \partial / \partial x_j^3 \). Taking into account (8), (9) and (14), (15) we get

\[
q(\chi_j U_{ck}^0, \chi_j U_{ck}^1) = \left( v_{ck}^0, n P_{ck}^j - v \partial_3 v_{ck}^1 \right)_{\omega^j} + \left( v_{sk}^0, n P_{sk}^j - v \partial_3 v_{sk}^1 \right)_{\omega^j} - \left( n P_{ck}^j - v \partial_3 v_{ck}^0, v_{ck}^1 \right)_{\omega^j} - \left( n P_{sk}^j - v \partial_3 v_{sk}^0, v_{sk}^1 \right)_{\omega^j} = -(1, \psi^j)_{\omega^j}.
\]
The rest terms in the Green’s formula could be computed in the same way. Finally, we arrive at
\[
q(u_k, U_k) = \sum_{j=1}^{J} \left\{ \left( b_{ck}^{j} \overline{A}_{ck}^{j} + b_{sk}^{j} \overline{A}_{sk}^{j} - a_{ck}^{j} \overline{B}_{ck}^{j} - a_{sk}^{j} \overline{B}_{sk}^{j} \right) (\psi_{j}^{1}, 1)_{\omega} \right. \\
+ \left. \left( a_{ck}^{j} \overline{B}_{sk}^{j} + b_{sk}^{j} \overline{A}_{ck}^{j} - b_{ck}^{j} \overline{A}_{sk}^{j} - a_{sk}^{j} \overline{B}_{ck}^{j} \right) (\psi_{j}^{1}, 1)_{\omega} \right\}.
\]
Now we define operators \( \pi_{0}^{c}, \pi_{0}^{s}, \pi_{1}^{c}, \pi_{1}^{s} : D_{\pm}^{\beta}W(\Omega) \rightarrow C^{J} \) (operators \( \pi_{0}^{c}, \pi_{0}^{s}, \pi_{1}^{c}, \pi_{1}^{s} : D_{\pm}^{\beta}W(\Omega)^{*} \rightarrow C^{J} \) are defined in the same way) as follows:
\[
\pi_{0}^{c} u = (a_{1}^{c}, a_{2}^{c}, \ldots, a_{J}^{c})^{t}, \quad \pi_{0}^{s} u = (a_{1}^{s}, a_{2}^{s}, \ldots, a_{J}^{s})^{t}, \\
\pi_{1}^{c} u = (b_{1}^{c}, b_{2}^{c}, \ldots, b_{J}^{c})^{t}, \quad \pi_{1}^{s} u = (b_{1}^{s}, b_{2}^{s}, \ldots, b_{J}^{s})^{t},
\]
where the numbers \( a_{1}^{c}, \ldots, a_{J}^{c}, b_{1}^{c}, \ldots, b_{J}^{c} \) are the coefficients in expansion (11) of the function \( u \in D_{\pm}^{\beta}W(\Omega) \) (in expansion (16) for \( U \in D_{\pm}^{\beta}W(\Omega)^{*} \)). Let
\[
c_{k}^{j} = \int_{\omega} \psi_{j}^{1} d x^{j'}, \quad d_{k}^{j} = -\int_{\omega} \psi_{j}^{1} d x^{j'}, \quad x'^{j'} = (x_{1}^{j'}, x_{2}^{j'}),
\]
and
\[
C_{k} = \text{diag}\{c_{k}^{1}, c_{k}^{2}, \ldots, c_{k}^{J} \}, \quad D_{k} = \text{diag}\{d_{k}^{1}, d_{k}^{2}, \ldots, d_{k}^{J} \}
\]
be the \( J \times J \) matrices. Taking into account previous results and notations we get the following formula
\[
q(u_k, U_k) = (C_{k} \pi_{1}^{c} u_k - D_{k} \pi_{1}^{s} u_k, \pi_{0}^{c} U_k)_{J} + (C_{k} \pi_{1}^{s} u_k + D_{k} \pi_{1}^{c} u_k, \pi_{0}^{s} U_k)_{J},
\]
\[
- (\pi_{0}^{c} u_k, C_{k} \pi_{1}^{c} U_k + D_{k} \pi_{1}^{s} U_k)_{J} - (\pi_{0}^{s} u_k, C_{k} \pi_{1}^{s} U_k - D_{k} \pi_{1}^{c} U_k)_{J},
\]
where \( \langle \cdot, \cdot \rangle_{J} \) stands for a scalar product in \( C^{J} \). We call (17) the generalized Green’s formula.

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REZIUME

M. Skujus. Apie Gryno formulę vienam Stokso tipo uždavinui

Laiko atžvilgiu periodinis Stokso uždavinys begalinių cilindrų sistemoje Furjė eilučių pagalba suvedamas į elipsinių uždaviniiu seka. Šiems Stokso tipo kraštiniam uždaviniamis įvedama apibendrintoji Gryno formulė.

Raktiniai žodžiai: begalinių cilindrų sistema, laiko atžvilgiu periodinis Stokso uždavinys, apibendrintoji Gryno formule.