Breakdown of the Mechanism of Forming Wakes by a Current-Carrying String

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March 27, 2022

Abstract

In this letter we emphasize the effect that the inclusion of electromagnetic properties for a string brings logarithmic divergences to the accretion problem and the mechanism of formation and evolution of wakes can break down.

1 Introduction

Topological defects are predicted in many gauge models as solitonic solutions resulting from spontaneous breaking of gauge or global symmetries. Among all these solutions, cosmic strings have attracted attention because they may
be the source of large-scale structure in the Universe [1]. A relevant mechanism to understand the structure formation by cosmic strings involves long strings moving with relativistic speed in the normal plane, giving rise to velocity perturbations in their wake [2]. The mechanism of forming wakes has been considered by many authors in the General Relativity theory [3, 4]. More recently, Masalskiene and Guimarães [5] have considered the formation of wakes by long strings in the context of scalar-tensor theories. They have shown that the presence of a gravitational scalar field - which from now on we will call generically as “dilaton” - induces a very similar structure as in the case of a wiggly cosmic string [4]. Further, Bezerra and Ferreira [6] showed that, if a torsion is presented, this effect is amplified.

From the point of view of purely gravitational physics, it was shown that the inclusion of a current hardly affects the metric outside the string [9]. However, it is well-known that inclusion of such an internal structure can change the predictions of cosmic string models in the microwave background anisotropies [10, 11]. This is the main goal of this work. Namely, we analyse the formation and evolution of wakes in this spacetime and we show explicitly how the current affects this mechanism. For this purpose, we consider a model in which non-baryonic cold dark matter propagates around the current-carrying string. The Zel’dovich approximation is carried out in order to treat this motion. We anticipate that our main result is to show that the inclusion of a current brings logarithmic divergences and can actually break down the accretion mechanism by wakes. We take here a slightly different approach from papers [12], although our results are very similar of them.

This work is outlined as follows. In section 2, after setting the relevant
microscopic model which describes a superconducting string carrying a current of timelike type, we present its gravitational field. In section 3, we consider the mechanism of formation and evolution of wakes in this framework by means of the Zel’dovich approximation. We obtained our results in the weak-field approximation. Finally, in section 4, we end up with some conclusions and remarks.

2 Timelike Current-Carrying Strings:

In this section we will study the gravitational field generated by a string carrying a current of timelike-type. We start with the action in the Jordan-Fierz frame

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-\tilde{g}} [\tilde{\Phi} \tilde{R} - \frac{\omega(\tilde{\Phi})}{\Phi} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\Phi} \partial_\nu \tilde{\Phi}] + S_m[\psi_m, \tilde{g}_{\mu\nu}] \quad (1)
\]

\(\tilde{g}_{\mu\nu}\) is the physical metric which contains both scalar and tensor degrees of freedom, \(\tilde{R}\) is the curvature scalar associated to it and \(S_m\) is the action for general matter fields which, at this point, is left arbitrary. The metric signature is assumed to be \((-,+,+,+\)).

In what follows, we will concentrate our attention to superconducting vortex configurations which arise from the spontaneous breaking of the symmetry \(U(1) \times U_{em}(1)\). Therefore, the action for the matter fields will be composed by two pairs of coupled complex scalar and gauge fields \((\varphi,B_\mu)\) and \((\sigma,A_\mu)\). Also, for technical purposes, it is preferable to work in the so-called Einstein (or conformal) frame, in which the scalar and tensor degrees
of freedom do not mix.

\[ S = \frac{1}{16\pi G_s} \int d^4x \sqrt{-g} \left[ R - 2\partial_\mu \phi \partial^\mu \phi \right] \]

+ \int d^4x \sqrt{-g} \left[ \Omega^2(\phi) \left( (D_\mu \varphi)^* D^\mu \varphi + (D_\mu \sigma)^* D^\mu \sigma \right) \right]

- \frac{1}{16\pi} (F_{\mu\nu} F^{\mu\nu} + H_{\mu\nu} H^{\mu\nu}) - \Omega^2(\phi) V(|\varphi|, |\sigma|), \quad (2)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), \( H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \) and the potential is suitably chosen in order that the pair \((\varphi, B_\mu)\) breaks one symmetry \(U(1)\) in vacuum (giving rise to the vortex configuration) and the second pair \((\sigma, A_\mu)\) breaks the symmetry \(U_{em}(1)\) in the core of the vortex (giving rise to the superconducting properties)

\[ V(|\varphi|, |\sigma|) = \frac{\lambda}{8} (|\varphi|^2 - \eta^2)^2 + f(|\varphi|^2 - \eta^2) |\sigma|^2 + \frac{\lambda_\sigma}{4} |\sigma|^4 + \frac{m_\sigma^2}{2} |\sigma|^2 \quad (3) \]

We restrict, then, ourselves to the configurations corresponding to an isolated, static current-carrying vortex lying in the \(z\)-axis. In a cylindrical coordinate system \((t, r, z, \theta)\) such that \(r \geq 0\) and \(0 \leq \theta < 2\pi\), we make the following ansatz:

\[ \varphi = \varphi(r) e^{i\theta} \quad B_\mu = \frac{1}{q} [Q(r) - 1] \quad (4) \]

The pair \((\sigma, A_\mu)\), which is responsible for the superconducting properties of the vortex, is set in the form

\[ \sigma = \sigma(r) e^{i\psi(t)} \quad A_t = \frac{1}{e} [P_t(r) - \partial_r \psi] \quad (5) \]

where \(P_t\) corresponds to the electric field which leads to a timelike current in the vortex. We also require that the functions \(\varphi, Q(r), \sigma(r)\) and \(P_t\) must be regular everywhere and must satisfy the usual boundary conditions of vortex [13] and superconducting configurations [14, 15].
The action (2) is obtained from (1) by a conformal transformation

\[ \tilde{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu} \]

and by the redefinition of the quantity

\[ G_* \Omega^2(\phi) = \tilde{\Phi}^{-1} \]

which makes evident the feature that any gravitational phenomena will be affected by the variation of the gravitation constant \( G_* \) in the scalar-tensor gravity, and by introducing a new parameter

\[ \alpha^2 = \left( \frac{\partial \ln \Omega(\phi)}{\partial \phi} \right)^2 = [2\omega(\tilde{\Phi}) + 3]^{-1} \]

Variation of the action (2) with respect to the metric \( g_{\mu\nu} \) and to the dilaton field \( \phi \) gives the modified Einstein’s equations and a wave equation for the dilaton, respectively. Namely,

\[
\begin{align*}
G_{\mu\nu} &= 2\partial_{\mu}\phi\partial_{\nu}\phi - g_{\mu\nu}g^{\alpha\beta}\partial_{\alpha}\phi\partial_{\beta}\phi + 8\pi G_* T_{\mu\nu} \\
\Box_{g}\phi &= -4\pi G_* \alpha(\phi)T
\end{align*}
\]

Where \( T_{\mu\nu} \) is the energy-momentum tensor which is obtained by

\[ T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_{mat}}{\delta g_{\mu\nu}}. \]  \( \tag{7} \)

We note, in passing, that, in the conformal frame, this tensor is not conserved providing us with an additional equation: \( \nabla_{\mu}T_{\nu}^{\mu} = \alpha(\phi)T\nabla_{\nu}\phi. \)

In what follows, we will write the general static metric with cylindrical symmetry corresponding to the electric case in the form

\[ ds^2 = -e^{2\psi}dt^2 + e^{2(\gamma-\psi)}(dr^2 + dz^2) + \beta^2 e^{-2\psi}d\theta^2 \]  \( \tag{8} \)
where $\Psi, \gamma, \beta$ are functions of $r$ only.

The non-vanishing components of the energy-momentum tensor using (7) are

\[
\begin{align*}
T_t^t & = -\frac{1}{2} \Omega^2(\phi) \left\{ -g^{tt} \sigma^2 P_t^2 + g^{rr} \left[ \varphi'^2 + \sigma'^2 \right] + g^{\theta\theta} \varphi^2 Q^2 \right\} \\
& - \frac{1}{8\pi} g^{rr} \left\{ -g^{tt} \frac{P_t^2}{e^2} + g^{\theta\theta} \frac{Q^2}{q^2} \right\} - \Omega^4(\phi)V(\sigma, \varphi) \\
T_r^r & = -\frac{1}{2} \Omega^2(\phi) \left\{ g^{tt} \sigma^2 P_t^2 - g^{rr} \left[ \varphi'^2 + \sigma'^2 \right] + g^{\theta\theta} \varphi^2 Q^2 \right\} \\
& + \frac{1}{8\pi} g^{rr} \left\{ g^{tt} \frac{P_t^2}{e^2} + g^{\theta\theta} \frac{Q^2}{q^2} \right\} - \Omega^4(\phi)V(\sigma, \varphi) \\
T_\theta^\theta & = -\frac{1}{2} \Omega^2(\phi) \left\{ g^{tt} \sigma^2 P_t^2 + g^{rr} \left[ \varphi'^2 + \sigma'^2 \right] - g^{\theta\theta} \varphi^2 Q^2 \right\} \\
& - \frac{1}{8\pi} g^{rr} \left\{ g^{tt} \frac{P_t^2}{e^2} - g^{\theta\theta} \frac{Q^2}{q^2} \right\} - \Omega^4(\phi)V(\sigma, \varphi) \\
T_z^z & = -\frac{1}{2} \Omega^2(\phi) \left\{ g^{tt} \sigma^2 P_t^2 + g^{rr} \left[ \varphi'^2 + \sigma'^2 \right] + g^{\theta\theta} \varphi^2 Q^2 \right\} \\
& - \frac{1}{8\pi} g^{rr} \left\{ g^{tt} \frac{P_t^2}{e^2} + g^{\theta\theta} \frac{Q^2}{q^2} \right\} - \Omega^4(\phi)V(\sigma, \varphi).
\end{align*}
\]

Therefore, for the electric case, eqs. (6) are written as

\[
\begin{align*}
\beta'' & = 8\pi G_s e^{2(\gamma-\psi)} \left[ T_1^1 + T_3^3 \right] \\
(\beta\psi')' & = 4\pi G_s e^{2(\gamma-\psi)} \left[ -T_0^0 + T_1^1 + T_2^2 + T_3^3 \right] \\
\beta' \gamma' & = \beta(\psi')^2 + \beta(\phi')^2 + 8\pi G_s \beta e^{2(\gamma-\psi)} T_1^1 \\
(\beta\phi')' & = -4\pi G_s \beta e^{2(\gamma-\psi)} T
\end{align*}
\]

In order to solve the above equations we will divide the space in two regions: the exterior region, $r \geq r_0$, in which only the electric component of the Maxwell tensor contributes to the energy-momentum tensor and the
internal region, \(0 \leq r < r_0\), where all matter fields survive. \(r_0\) is the string thickness. In the paper [16], the metric of such a configuration has been obtained to first order in the parameter \(G_0\). In order to obtain this metric, we used a method applied first by Linet [17] which consists in re-writing the components of the energy-momentum tensor (9) as \(\delta\)-functions. We have then

\[
\begin{align*}
\text{ds}^2 &= \left\{ 1 - 4G_0 \left[ \left( U - T + I^2 \right) \ln \left( \frac{r}{r_0} \right) + I^2 \ln^2 \left( \frac{r}{r_0} \right) \right] \right\} \left( dr^2 + dz^2 \right) \\
&\quad + r^2 \left[ 1 - 8G_0 \left( T - \frac{I^2}{2} \right) - 4G_0 \left( U - T - I^2 \right) \ln \left( \frac{r}{r_0} \right) - 4G_0 I^2 \ln^2 \left( \frac{r}{r_0} \right) \right] d\theta^2 \\
&\quad - \left\{ 1 + 4G_0 \left[ I^2 \ln^2 \left( \frac{r}{r_0} \right) + \left( U - T - I^2 \right) \ln \left( \frac{r}{r_0} \right) \right] \right\} dt^2
\end{align*}
\]

where \(U, T\) and \(I\) are macroscopic quantities which are defined as the energy per unit length, the tension per unit length and the string’s current, respectively.

3 Formation and Evolution of the Wakes and the Zel’dovich Approximation:

A relevant mechanism to understand the structure formation by cosmic strings involves long strings moving with relativistic speed in the normal plane, giving rise to velocity perturbations in their wake [2]. Matter through which a long string moves acquires a boost in the direction of the surface swept out by the string. Matter moves toward this surface by gravitational attraction and, as a consequence, a wake is formed behind the string. In what follows, we will study the implications of a timelike current on the formation
and evolution of a wake behind the string which generates the metric (11). For this purpose, we will mimic this situation with a simple model consisting of cold dark matter composed by non-relativistic collisionless particles moving past a long string. In order to make a quantitative description of accretion onto wakes, we will use the Zel’dovich approximation, which consists in considering the Newtonian accretion problem in an expanding universe using the method of linear perturbations.

To start with, we first compute the velocity perturbation of massive particles moving past the string. If we consider that the string is moving with normal velocity $v_s$ through matter, the velocity perturbation can be calculated with the help of the gravitational force due to metric (11):

$$u = 8\pi G_0 U v_s \gamma + \frac{\pi G_0}{v_s \gamma} \left[ 2\alpha (\phi_0)^2 \left( U + T + I^2 \right) - \left( U - T - I^2 \right) - 2 \ln \left( \frac{r}{r_0} \right) I^2 \right]$$

(12)

with $\gamma = (1 - v_s^2)^{-1/2}$. The first term in (12) is equivalent to the relative velocity of particles flowing past a string in general relativity. The other terms come as a consequence of the scalar-tensor coupling of the gravitational interaction and the superconducting properties of the string.

Let us suppose now that the wake was formed at $t_i > t_{eq}$. The physical trajectory of a dark particle can be written as

$$h(\vec{x}, t) = a(t) [\vec{x} + \psi(\vec{x}, t)]$$

(13)

where $\vec{x}$ is the unperturbed comoving position of the particle and $\psi(\vec{x}, t)$ is the comoving displacement developed as a consequence of the gravitational attraction induced by the wake on the particle. Suppose, for simplification,
that the wake is perpendicular to the $x$-axis (assuming that $dz = 0$ in the metric (11) and $r = \sqrt{x^2 + y^2}$) in such a way that the only non-vanishing component of $\psi$ is $\psi_x$. Therefore, the equation of motion for a dark particle in the Newtonian limit is

$$\ddot{h} = -\nabla_h \Phi$$  \hspace{1cm} (14)

where the Newtonian potential $\Phi$ satisfies the Poisson equation

$$\nabla_h^2 \Phi = 4\pi G_0 \rho$$  \hspace{1cm} (15)

where $\rho(t)$ is the dark matter density in a cold dark matter universe. For a flat universe in the matter-dominated era, $a(t) \sim t^{2/3}$. Therefore, the linearised equation for $\psi_x$ is

$$\ddot{\psi} + \frac{4}{3t} \dot{\psi} - \frac{2}{3t^2} \psi = 0$$  \hspace{1cm} (16)

with appropriated initial conditions: $\psi(t_i) = 0$ and $\dot{\psi}(t_i) = -u_i$. Eq. (16) is the Euler equation whose solution is easily found

$$\psi(x, t) = \frac{3}{5} \left[ \frac{u_i t_i^2}{t} - u_i t_i \left( \frac{t}{t_i} \right)^{2/3} \right]$$

Calculating the comoving coordinate $x(t)$ using the fact that $\dot{h} = 0$ in the “turn around”\(^1\), we get

$$x(t) = -\frac{6}{5} \left[ \frac{u_i t_i^2}{t} - u_i t_i \left( \frac{t}{t_i} \right)^{2/3} \right]$$  \hspace{1cm} (17)

With the help of (17) we can compute both the thickness $d(t)$ and the surface density $\sigma(t)$ of the wake [1]. We have, then, respectively (to first order in

\(^1\)The moment when the dark particle stops expanding with the Hubble flow and starts to collapse onto the wake.
where we have used the fact that $\rho(t) = \frac{1}{6\pi G_0 t^2}$ for a flat universe in the matter-dominated era and that the wake was formed at $t_i \sim t_{eq}$. Clearly, from eq. (18) we see that the presence of the current makes the accretion mechanism by wakes to diverge.

### 4 Conclusion:

Inclusion of a current in the internal structure of a cosmic string could drastically change the predictions of such models in the microwave background anisotropies. In particular, a current of a timelike-type could bring some divergences leading to some unbounded gravitational effects [9]. In this work we studied the effects of a timelike-current string in the weak-field approximation for the mechanism of wakes formation. For this purpose, we carried out an investigation of the mechanism of formation and evolution of wakes in this framework, showing the explicit contribution of the current to this effect.

Wakes produced by moving strings can provide an explanation for filamentary and sheetlike structures observed in the universe. A wake produced by the string in one Hubble time has the shape of a strip of width $\sim v_s t_i$. With the help of the surface density (18) we can compute the wake’s linear
mass density, say \( \bar{\sigma} \),

\[
\sigma(t) \approx \frac{2}{5} \frac{1}{\gamma t} \left( \frac{t}{t_i} \right)^{1/3} \left[ 8U(v_\gamma)\gamma^2 + 2\alpha(\phi_0)^2 (U + T + I^2) - (U - T - I^2) - 2 \ln \left( \frac{r}{r_0} \right) I^2 \right]
\]  

(19)

If the string moves slower or if we extrapolate our results to earlier epochs, we see that the logarithmic term would bring divergences and the mechanism of forming wakes, at least for this model, would break down.

Acknowledgements

ALNO would like to thank CAPES for a PhD grant. This work was partially supported by PROCAD/CAPES.

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