BOUNDEDNESS OF AVERAGING OPERATORS ON GEOMETRICALLY DOUBLING METRIC SPACES

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Abstract. We prove that averaging operators are uniformly bounded on $L^1$ for all geometrically doubling metric measure spaces, with bounds independent of the measure. From this result, the $L^1$ convergence of averages as $r \to 0$ immediately follows.

1. Introduction

It is a well known consequence of translation invariance, that for Lebesgue measure on $\mathbb{R}^d$, the averages $A_r f$ (also known as Steklov means) converge to $f$ in $L^1$ as $r \downarrow 0$. Rather surprisingly, the corresponding approximation question regarding arbitrary locally finite Borel measures $\mu$ on $\mathbb{R}^d$ (i.e., whether $\lim_{r \to 0} A_r,\mu f = f$ in $L^1(\mu)$) does not seem to have been studied in the literature.

Here we give an affirmative answer not just for $\mathbb{R}^d$, but for every geometrically doubling metric measure space (every metric space of homogeneous type, in the terminology of [CoWe], cf. Definition 2.4 below). We do this by proving that averaging operators are uniformly bounded on $L^1$, something that was previously unknown even for $\mathbb{R}^d$ (though two special cases appear in [Al1]). From this, the $L^1$ convergence immediately follows by a standard argument, since the result holds for a dense class in $L^1$.

We do not deal here with the question of a.e. convergence. While the Lebesgue differentiation theorem (with centered balls) holds for every locally finite Borel measure $\mu$ on $\mathbb{R}^d$, by the Besicovitch covering theorem, we mention that the analogous question in arbitrary geometrically doubling metric spaces is open, even for such well studied instances as the Heisenberg groups.

2. Definitions and notation

We will use $B(x, r) := \{y \in X : d(x, y) < r\}$ to denote open balls, and $B^d(x, r) := \{y \in X : d(x, y) \leq r\}$ to refer to metrically closed balls. It is always assumed that measures are not identically 0.

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Definition 2.1. A Borel measure is \(\tau\)-additive or \(\tau\)-smooth, if for every collection \(\{U_\alpha : \alpha \in \Lambda\}\) of open sets,

\[
\mu(\bigcup \alpha U_\alpha) = \sup \mu(\bigcup_{i=1}^n U_{\alpha_i}),
\]

where the supremum is taken over all finite subcollections \(\mathcal{F} = \{U_{\alpha_1}, \ldots, U_{\alpha_n}\}\) of \(\{U_\alpha : \alpha \in \Lambda\}\). We say that \((X, d, \mu)\) is a metric measure space if \(\mu\) is a \(\tau\)-additive Borel measure on the metric space \((X, d)\), such that \(\mu\) assigns finite measure to bounded Borel sets.

From now on we always assume that measures are locally finite (finite on bounded sets). For motivation regarding the definition of metric measure spaces using \(\tau\)-additivity, cf. [Al3]. Note that in separable metric spaces all Borel measures are \(\tau\)-additive (so the spaces considered above are more general than those given by some commonly used alternative definitions, cf. [HKST] for instance) and the same happens with all Radon measures in arbitrary metric spaces.

Recall that the complement of the support \((\text{supp} \mu)^c := \bigcup\{B(x, r) : x \in X, \mu B(x, r) = 0\}\) of a Borel measure \(\mu\), is an open set, and hence measurable.

Definition 2.2. Let \((X, d)\) be a metric space and let \(\mu\) be a locally finite Borel measure on \(X\). If \(\mu(X \setminus \text{supp} \mu) = 0\), we say that \(\mu\) has full support.

By \(\tau\)-additivity, if \((X, d, \mu)\) is a metric measure space, then \(\mu\) has full support, since \(X \setminus \text{supp} \mu\) is a union of open balls of measure zero. Actually, the other implication also holds, for the support is always separable, so having full support is equivalent to \(\tau\)-additivity (cf. [Bo] Proposition 7.2.10 for more details).

Definition 2.3. Let \((X, d, \mu)\) be a metric measure space and let \(g\) be a locally integrable function on \(X\). For each fixed \(r > 0\) and each \(x \in \text{supp} \mu\), the averaging operator \(A_{r,\mu}\) is defined as

\[
A_{r,\mu}g(x) := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} g \, d\mu.
\]

Averaging operators in metric measure spaces are defined almost everywhere, by \(\tau\)-additivity. Sometimes it is convenient to use closed balls in the definition, instead of open balls. In this case we use the following notation:

\[
A_{r,\mu}^{cl}g(x) := \frac{1}{\mu(B^{cl}(x, r))} \int_{B^{cl}(x, r)} g \, d\mu.
\]

Furthermore, when we are considering only one measure \(\mu\) we often omit it, writing \(A_r\) instead of the longer \(A_{r,\mu}\). While the rest of the paper is written for \(A_r\), we mention that the results are also valid for \(A_r^{cl}\).

Definition 2.4. A metric space is geometrically doubling if there exists a positive integer \(D\) such that every ball of radius \(r\) can be covered with no more than \(D\) balls of radius \(r/2\). We call the smallest such \(D\) the doubling constant of the space.
3. BOUNDEDNESS OF AVERAGING OPERATORS ON GEOMETRICALLY DOUBLING SPACES

The following proof reminds the reader of the fact that bounded continuous functions with bounded support are dense in $L^1$, and for these functions, averages converge in norm as $r \downarrow 0$.

**Theorem 3.1. $L^1$-Lebesgue differentiation.** Let $(X, d, \mu)$ be a metric measure space. If there is a constant $C > 0$ such that $\sup_{r > 0} ||A_r||_{L^1(\mu) \to L^1(\mu)} \leq C$, then for every $f \in L^1(\mu)$, $\lim_{r \to 0} A_rf = f$ in $L^1$.

**Proof.** Given any sequence $\{r_n\}_{n \geq 1}$ satisfying $r_n \downarrow 0$, we may suppose that $r_n \leq 1$, by disregarding a finite number of terms, if needed. We may also suppose that $C \geq 1$ (else, replace it by 1).

Recall that in metric measure spaces the integrable continuous functions are dense in $L^1$, by the standard argument whereby the case of integrable real valued functions is reduced to the case of non-negative functions, which by successive approximations is reduced first, to the case of simple functions, then to indicator functions of measurable sets, and finally, to indicator functions of closed sets $F$ (with finite measure); for these functions the result is true by the Tietze-Urysohn extension theorem: Given $\varepsilon > 0$ we choose $O$ open such that $F \subset O$ and $\mu(O \setminus F) < \varepsilon$; then we extend $g : O^c \cup F \to \{0, 1\}$ given by $g = 1$ on $F$, $g = 0$ on $O^c$, to a continuous function $G : X \to [0, 1]$.

Let $z \in X$, let $\varepsilon > 0$, and let $f \in L^1(\mu)$. Then there exist $t, R >> 0$ such that

$$\|f\|_1 < \|f \ 1_{B(z,R) \cap \{|f| \leq t\}}\|_1 + \frac{\varepsilon}{6C}.$$ 

Next we choose a continuous function $g$ such that $-t \leq g \leq t$, supp $g \subset B(z, R + 1)$ and

$$\|f \ 1_{B(z,R) \cap \{|f| \leq t\}} - g\|_1 < \frac{\varepsilon}{6C}.$$ 

Thus $\|f - g\|_1 < \frac{\varepsilon}{3C}$. Now $g$ is continuous, so for all $x \in X$, $\lim_n A_{r_n}g(x) = g(x)$. Since for all $n \geq 1$, $|A_{r_n}g(x) - g(x)| \leq 2t \ 1_{B(z,R+2)}(x) \in L^1(\mu)$, by the dominated convergence theorem, $\lim_n \|A_{r_n}g - g\|_1 = 0$. Hence,

$$\limsup_n \|A_{r_n}f - f\|_1 \leq \|A_{r_n}f - A_{r_n}g\|_1 + \lim_n \|A_{r_n}g - g\|_1 + \|g - f\|_1 < \frac{\varepsilon}{3} + \frac{\varepsilon}{3C} < \varepsilon.$$

$\Box$

**Definition 3.2.** We call

$$(3) \quad a_s(y) := \int_X \frac{1_B(y,s)(x)}{\mu B(x,s)} \ d\mu(x)$$

the conjugate function to the averaging operator $A_s$.

**Theorem 3.3.** Let $(X, d, \mu)$ be a metric measure space. The averaging operator $A_s$ is bounded on $L^1(\mu)$ if and only if $a_s \in L^\infty(\mu)$, in which case $\|A_s\|_{L^1(\mu) \to L^1(\mu)} = \|a_s\|_\infty$. 


Proof. Let \(0 \leq f \in L^1(\mu)\), and suppose \(a_s \in L^\infty(\mu)\). Since by Fubini-Tonelli

\[
\|A_s f\|_{L^1} = \int_X A_s f(x) \, d\mu(x) = \int_X \int_X \frac{1_{B(x,s)}(y)}{\mu B(x,s)} f(y) \, d\mu(y) \, d\mu(x)
\]

(4)

\[
= \int_X f(y) \int_X \frac{1_{B(y,s)}(x)}{\mu B(x,s)} \, d\mu(x) \, d\mu(y) = \int_X f(y) \, a_s(y) \, d\mu(y),
\]

(5)

it follows from Hölder’s inequality that \(\|A_s\|_{L^1(\mu) \rightarrow L^1(\mu)} \leq \|a_s\|_\infty\).

On the other hand, we claim that if \(\|A_s\|_{L^1(\mu) \rightarrow L^1(\mu)} \leq C\) then \(\|a_s\|_\infty \leq C\).

Towards a contradiction, suppose \(C < \|a_s\|_\infty\) (including the case \(\|a_s\|_\infty = \infty\)). Then there is a \(t > C\) and a measurable set \(A_t\) such that \(A_t \subset \{a_s > t\}\) and \(0 < \mu A_t < \infty\). Let \(f := 1_{A_t} \in L^1(\mu)\). Then

\[
\|A_s f\|_{L^1} = \int_X f(y) \, a_s(y) \, d\mu(y) > \int_{A_t} t \, d\mu(y) = t \mu(A_t) > C\|f\|_{L^1}.
\]

(6)

\[\square\]

**Definition 3.4.** We say that a measure \(\mu\) satisfies a local comparability condition if there exists a constant \(C \in [1, \infty)\) such that for all pairs of points \(x, y \in X\) and every \(r > 0\), whenever \(d(x, y) < r\), we have \(\mu(B(x, r)) \leq C \mu(B(y, r))\).

The preceding definition comes from [NaTa, p. 737]. There, local comparability is called a “mild uniformity assumption”; the term “local comparability” was introduced in [Al1], and used also in [Al2]. As indicated in [NaTa, p. 737], if \(\mu\) satisfies a \(C\) local comparability condition, then \(a_s(y) \leq C\), so \(\|A_s\|_{L^1 \rightarrow L^1} \leq C\).

It is natural to ask under which conditions one can have uniform boundedness of \(A_r\) without local comparability. In [Al1] Example 4.1] a metric measure space is exhibited where the measure lacks local comparability, and \(\|A_s\|_{L^p \rightarrow L^p}\) is unbounded for all \(p \in [1, \infty)\). Also, it is shown in [Al1] Theorem 4.8] that on \(\mathbb{R}\), there is a measure \(\mu\) such that the right directional averaging operator

\[
A^r_{s,\mu} f(x) := \frac{1}{\mu([x, x + s])} \int_{[x,x+s]} f(y) \, d\mu(y)
\]

is unbounded on \(L^1(\mu)\) for \(s = 1\).

On the other hand, the uniform \(L^1\) boundedness of the operators \(A_r\) was shown to hold for \(\mathbb{R}^d\) in two special cases, despite the lack of local comparability: The exponential density in dimension one, and the standard gaussian measures in every dimension (cf. [Al1 Theorems 4.2 and 4.3]).

Here we obtain the general result: Averaging operators are \(L^1\) bounded, uniformly on \(r\), for every locally finite measure in any geometrically doubling metric measure space, not just in \(\mathbb{R}^d\).
Theorem 3.5. Let \((X, d, \mu)\) be a geometrically doubling metric measure space, with doubling constant \(D\), and let \(M\) be the maximum number of points that can be placed at distance \(\geq r\) inside a ball of radius \(r\). Then for all \(s > 0\), \(\|A_s\|_{L^1(\mu) \rightarrow L^1(\mu)} \leq M \leq D\). Since \(\|A_s\|_{L^\infty(\mu) \rightarrow L^\infty(\mu)} = 1\), for \(1 < p < \infty\) we have \(\|A_s\|_{L^p(\mu) \rightarrow L^p(\mu)} \leq M^{1/p}\) by interpolation.

Proof. Let \(X\) be geometrically doubling with constant \(D\), and let \(M\) be the maximum size of an \(r\)-net in \(B(x, r)\), where the maximum is taken over all \(x \in X\) and all \(r > 0\). Note that \(M \leq D\), since every point in a maximal \(r\)-net inside \(B(x, r)\) is contained in one of the covering balls of radius \(r/2\), and each such ball can contain at most one point from the \(r\)-net.

By disregarding a set of measure zero if needed, we suppose that \(X = \text{supp } \mu\), so every ball has positive measure. Fix \(y \in X\). We want to show that \(a_s(y) \leq M\), and then the result follows from Theorem 3.3.

First we claim that \(b_1 := \inf \{\mu B(x, s) : x \in B(y, s)\} > 0\). To see why, select a sequence \(\{x_n\}_{n \geq 1}\) of points in \(B(y, s)\) so that \(\lim_{n} \mu B(x_n, s) = b_1\). Since \(X\) is geometrically doubling, \(B(y, s)\) can be covered by at most \(D\) balls of radius \(s/2\), so at least one of these balls, say, \(B(w, s/2)\), contains an infinite subsequence from \(\{x_n\}_{n \geq 1}\), which after relabelling, we also denote by \(\{x_n\}_{n \geq 1}\). Then \(b_1 \geq \mu B(w, s/2) > 0\), since for all \(n \geq 1\), \(B(w, s/2) \subset B(x_n, s)\). Now take \(0 < \varepsilon << 1\), and choose \(u_1 \in B(y, s)\) so that \(\mu B(u_1, s) < (1 + \varepsilon)b_1\); let \(b_2 := \inf \{\mu B(x, s) : x \in B(y, s) \setminus B(u_1, s)\}\), and select \(u_2 \in B(y, s) \setminus B(u_1, s)\) so that \(\mu B(u_2, s) < (1 + \varepsilon)b_2\); repeat, with \(b_{k+1} := \inf \{\mu B(x, s) : x \in B(y, s) \setminus \bigcup_k B(u_i, s)\}\), \(u_{k+1} \in B(y, s) \setminus \bigcup_k B(u_i, s)\), and \(\mu B(u_{k+1}, s) < (1 + \varepsilon)b_{k+1}\). Since the points \(u_i\) form an \(s\)-net in \(B(y, s)\), there is an \(m \leq M\) such that \(B(y, s) \setminus \bigcup_m B(u_i, s) = \emptyset\), and then the process stops.

Next, fix \(x \in B(y, s)\), and let \(i\) be the first index such that \(x \in B(u_i, s)\). Then

\[
\frac{1_{B(y,s)}(x)}{\mu B(x, s)} \leq (1 + \varepsilon) \frac{1_{B(y,s) \cap B(u_i, s)}(x)}{\mu B(u_i, s)} \leq (1 + \varepsilon) \sum_{j=1}^{m} \frac{1_{B(y,s) \cap B(u_j, s)}(x)}{\mu B(u_j, s)},
\]

so

\[
a_s(y) = \int_X \frac{1_{B(y,s)}(x)}{\mu B(x, s)} \ d\mu(x) \leq \int_X (1 + \varepsilon) \sum_{j=1}^{m} \frac{1_{B(y,s) \cap B(u_j, s)}(x)}{\mu B(u_j, s)} \ d\mu(x)
\]

\[
\leq (1 + \varepsilon) \sum_{j=1}^{m} \frac{1_{B(u_j, s)}(x)}{\mu B(u_j, s)} \ d\mu(x) \leq (1 + \varepsilon)M,
\]

and \(\|A_s\|_{L^1(\mu) \rightarrow L^1(\mu)} \leq M\) follows by letting \(\varepsilon \downarrow 0\).

It is obvious that for all \(s > 0\), \(\|A_s\|_{L^\infty(\mu) \rightarrow L^\infty(\mu)} = 1\) (we always have \(\|A_s\|_{L^\infty(\mu) \rightarrow L^\infty(\mu)} \leq 1\), since averages never exceed a supremum; for the other inequality, just take \(f = 1_X\)). Thus, by a standard interpolation argument, or simply by Jensen’s inequality (cf. [AI] Theorem 2.10) for all \(1 < p < \infty\) we have \(\|A_s\|_{L^p(\mu) \rightarrow L^p(\mu)} \leq M^{1/p}\). \(\square\)

While the bounds given in the preceding theorem do not seem very tight, in fact the dependency on the dimension of the space is unavoidable: It is shown in [AI] Theorem 4.3 that for the standard gaussian measure \(\gamma^d\) on \(\mathbb{R}^d\) and for every \(d\) sufficiently large, the weak type \((1, 1)\) constants satisfy \(\|A_{\gamma^d}\|_{L^1 \rightarrow L^{1,\infty}} > 1.019^d\).
Putting together Theorems 3.1, 3.3, and 3.5, we obtain the following

**Corollary 3.6.** Suppose \((X, d, \mu)\) is either a geometrically doubling metric measure space, or \(\mu\) satisfies a local comparability condition. Then for every \(f \in L^1(\mu)\), we have \(\lim_{r \downarrow 0} A_r f = f\) in \(L^1\).

The preceding corollary appears to be optimal or close to optimal, since in the infinite dimensional case, it already fails for a certain gaussian measure on a separable Hilbert space (cf. Corollary 3.8 below).

**Corollary 3.7.** Suppose \((X, d, \mu)\) is either a geometrically doubling metric measure space, or \(\mu\) satisfies a local comparability condition. Then for all \(f \in L^1_{\text{loc}}(\mu)\), \(Mf(x) \geq |f|(x)\) almost everywhere.

**Proof.** This follows from Corollary 3.6, since from \(L^1\) convergent sequences one can always extract subsequences converging a.e., and for every \(r > 0\), \(Mf(x) \geq A_r |f|(x)\).

The part of the preceding corollary dealing with geometrically doubling metric spaces had been originally obtained (cf. [Al3, Corollary 2.10]) by using a result of Hytönen (cf. [Hy, Lemma 3.3]) on the existence of arbitrarily small doubling balls, for general measures in geometrically doubling metric measure spaces.

In view of the exponential increase of the bounds, for the standard gaussian measure \(\gamma^d\) on \(\mathbb{R}^d\) ([Al1, Theorem 4.3]) one might suspect that in the infinite dimensional case the uniform boundedness of the averaging operators can fail. This is indeed the case, as it follows from a result of D. Preiss used next.

**Corollary 3.8.** There is a gaussian measure \(\gamma\) on an infinite dimensional separable Hilbert space \(H\), for which \(\sup_{r > 0} \|A_r\|_{L^1(\gamma) \to L^1(\gamma)} = \infty\).

**Proof.** By [Pr], there exists a gaussian probability \(\gamma\) on a separable Hilbert space \(H\) and a Borel subset \(C\) with \(0 < \gamma C < 1\), such that \(\gamma\)-a.e. \(x\),

\[
\lim_{r \downarrow 0} \frac{\gamma(C \cap B(x, r))}{\gamma(B(x, r))} = 0.
\]

By Theorem 3.1, if \(\sup_{r > 0} \|A_r\|_{L^1(\mu) \to L^1(\mu)} < \infty\), then we have convergence in \(L^1\) of the averages as \(r \downarrow 0\), so it is possible to extract a sequence \(r_n \downarrow 0\) such that \(\gamma\) a.e. \(x\),

\[
\lim_{n} \frac{\gamma(C \cap B(x, r_n))}{\gamma(B(x, r_n))} = 1_C(x),
\]

which is a contradiction.

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