Approximation properties of the Stancu type Dunkl generalization of the Kantorovich-Szász-Mirakjan-operators via q-calculus

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Abstract

In this paper we construct Stancu type q-Kantrovich-Szász-Mirakjan operators generated by Dunkl generalization of the exponential function. We obtain some approximation results using the Korovkin approximation theorem and the weighted Korovkin-type theorem for these operators. We also study convergence properties by using the modulus of continuity and the rate of convergence of these operators for functions belonging to the Lipschitz class. Furthermore, we obtain the rate of convergence in terms of the classical, second order, and weighted modulus of continuity.

Keywords and phrases: q-integers; Dunkl analogue; generating functions; generalization of exponential function; Szász operator; modulus of continuity; weighted modulus of continuity.

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1. INTRODUCTION AND PRELIMINARIES

In 1912, S.N Bernstein [3] introduced the following sequence of operators $B_n : C[0, 1] \to C[0, 1]$ defined by

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right), \quad x \in [0, 1]. \quad (1.1)$$

for $n \in \mathbb{N}$ and $f \in C[0, 1]$.

In 1950, for $x \geq 0$, Szász [20] introduced the operators

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f \left( \frac{k}{n} \right), \quad f \in C[0, \infty). \quad (1.2)$$

In the field of approximation theory, the application of q-calculus emerged as a new area in the field of approximation theory. The first q-analogue of the well-known Bernstein polynomials was introduced by Lupaş by applying the idea of q-integers [10]. In 1997 Phillips [16] considered another q-analogue of the classical Bernstein polynomials. Later on, many authors introduced q-generalizations of various operators and investigated several approximation properties [15, 11, 17, 12, 14].
The $q$-integer $[n]_q$, the $q$-factorial $[n]_q!$, and the $q$-binomial coefficient are defined by (see [2])

$$
[n]_q := \begin{cases} 
\frac{1-q^n}{1-q}, & \text{if } q \in \mathbb{R}^+ \setminus \{1\} \\
n, & \text{if } q = 1,
\end{cases}
$$

for $n \in \mathbb{N}$ and $[0]_q = 0$,

$$
[n]_q! := \begin{cases} 
[n]_q[n-1]_q \cdots [1]_q, & n \geq 1, \\
1, & n = 0,
\end{cases}
$$

respectively.

The $q$-analogue of $(1 + x)^n$ is the polynomial

$$(1 + x)_q^n := \begin{cases} 
(1 + x)(1 + qx) \cdots (1 + q^{n-1}x), & n = 1, 2, 3, \cdots \\
1, & n = 0.
\end{cases}$$

A $q$-analogue of the common Pochhammer symbol also called a $q$-shifted factorial is defined as

$$(x; q)_0 = 1, \quad (x; q)_n = \prod_{j=0}^{n-1} (1 - qx^j), \quad (x; q)_\infty = \prod_{j=0}^{\infty} (1 - q^jx).$$

The Gauss binomial formula is given by

$$(x + a)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q a^k x^{n-k}.$$

The $q$–analogue of Bernstein operators [16] is defined as

$$B_{n,q}(f; x) = \sum_{k=0}^{n} \binom{n}{k}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x) f\left(\frac{[k]_q}{[m]_q}\right), \quad x \in [0, 1], n \in \mathbb{N}. \quad (1.3)$$

There are two $q$-analogue of the exponential function $e^z$, defined as follows:

For $|z| < \frac{1}{1-q}$ and $|q| < 1$,

$$e(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \frac{1}{1 - ((1-q)z)_q^\infty}, \quad (1.4)$$

and for $|q| < 1$,

$$E(z) = \prod_{j=0}^{\infty} (1 + (1-q)q^j z)_q^\infty = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{z^k}{k!} = (1 + (1-q)z)_q^\infty, \quad (1.5)$$

where $(1-x)_q^\infty = \prod_{j=0}^{\infty} (1 - q^j x)$.

Sucu [19] defined a Dunkl analogue of Szász operators via a generalization of the exponential function [18] as follows:

$$S_{n,k}^{\ast}(f; x) := \frac{1}{e_{\mu}(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_n(k)} f\left(\frac{k + 2\mu \theta_k}{n}\right), \quad (1.6)$$
where \( x \geq 0, \ f \in C[0, \infty), \mu \geq 0, \ n \in \mathbb{N} \)

and

\[
e_{\mu}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_{\mu}(n)}.
\]

Here

\[
\gamma_{\mu}(2k) = \frac{2^{2k} k! \Gamma \left( k + \mu + \frac{1}{2} \right)}{\Gamma \left( \mu + \frac{1}{2} \right)},
\]

and

\[
\gamma_{\mu}(2k + 1) = \frac{2^{2k+1} k! \Gamma \left( k + \mu + \frac{3}{2} \right)}{\Gamma \left( \mu + \frac{3}{2} \right)}.
\]

There is given a recursion for \( \gamma_{\mu} \)

\[
\gamma_{\mu}(k + 1) = (k + 1 + 2\mu \theta_{k+1})\gamma_{\mu}(k), \ k = 0, 1, 2, \ldots,
\]

where

\[
\theta_{k} = \begin{cases} 
0 & \text{if } k \in 2\mathbb{N} \\
1 & \text{if } k \in 2\mathbb{N} + 1.
\end{cases}
\]

Cheikh et al. [4] stated the \( q \)-Dunkl classical \( q \)-Hermite type polynomials and gave definitions of \( q \)-Dunkl analogues of exponential functions and recursion relations for \( \mu > -\frac{1}{2} \) and \( 0 < q < 1 \).

\[
e_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_{\mu,q}(n)}, \ x \in [0, \infty)
\]  

(1.7)

\[
E_{\mu,q}(x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{\gamma_{\mu,q}(n)}, \ x \in [0, \infty)
\]  

(1.8)

\[
\gamma_{\mu,q}(n + 1) = \left( \frac{1 - q^{2\mu\theta_{n+1}+n+1}}{1 - q} \right) \gamma_{\mu,q}(n), \ n \in \mathbb{N},
\]  

(1.9)

\[
\theta_{n} = \begin{cases} 
0 & \text{if } n \in 2\mathbb{N}, \\
1 & \text{if } n \in 2\mathbb{N} + 1.
\end{cases}
\]

An explicit formula for \( \gamma_{\mu,q}(n) \) is

\[
\gamma_{\mu,q}(n) = \frac{(q^{2\mu+1}, q^{2})_{\frac{n+1}{2}}(q^2, q^2)_{\frac{n}{2}}}{(1 - q)^n} \gamma_{\mu,q}(n), \ n \in \mathbb{N}.
\]

And some of the special cases of \( \gamma_{\mu,q}(n) \) defined as:

\[
\gamma_{\mu,q}(0) = 1, \quad \gamma_{\mu,q}(1) = \frac{1 - q^{2\mu+1}}{1 - q}, \quad \gamma_{\mu,q}(2) = \left( \frac{1 - q^{2\mu+1}}{1 - q} \right) \left( \frac{1 - q^2}{1 - q} \right),
\]

\[
\gamma_{\mu,q}(3) = \left( \frac{1 - q^{2\mu+1}}{1 - q} \right) \left( \frac{1 - q^2}{1 - q} \right) \left( \frac{1 - q^{2\mu+3}}{1 - q} \right),
\]

\[
\gamma_{\mu,q}(4) = \left( \frac{1 - q^{2\mu+1}}{1 - q} \right) \left( \frac{1 - q^2}{1 - q} \right) \left( \frac{1 - q^{2\mu+3}}{1 - q} \right) \left( \frac{1 - q^4}{1 - q} \right).
\]
In [8], Gürhan Içöz gave a Dunkl generalization of Kantrovich type integral generalization of Szász operators. In [9], they gave a Dunkl generalization of Szász operators via q-calculus as:

\[
D_{n,q}(f; x) = \frac{1}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q} (k)} f \left( \frac{1 - q^{2\mu \theta_k + k}}{1 - q^n} \right),
\]

for \( \mu > \frac{1}{2} \), \( x \geq 0 \), \( 0 < q < 1 \) and \( f \in C[0, \infty) \).

**Lemma 1.1.**

1. \( D_{n,q}(1; x) = 1 \),
2. \( D_{n,q}(t; x) = x \),
3. \( x^2 + [1 - 2\mu]q^{2n} e_{\mu,q}(q[n]_q x) \frac{x}{[n]_q} \leq D_{n,q}(t^2; x) \leq x^2 + [1 + 2\mu]q^n \)
4. \( D_{n,q}(t^3; x) \geq x^3 + (2q + 1)[1 - 2\mu]q^{5n} e_{\mu,q}(q[n]_q x) \frac{x^2}{[n]_q} + q^4 [1 - 2\mu]q^{2n} e_{\mu,q}(q^2[n]_q x) \frac{x}{[n]_q} \)
5. \( D_{n,q}(t^4; x) \leq x^4 + 6[1 + 2\mu]q^n \frac{x^3}{[n]_q} + 7[1 + 2\mu]q^{2n} \frac{x^2}{[n]_q} + [1 + 2\mu]^3 \frac{x}{[n]_q} \)

In this paper we construct Kantrovich type Szász-Mirakjan operators generated by Dunkl generalization of the exponential function via q-integers. We obtain some approximation results via well known Korovkin’s type theorem and weighted Korovkin’s type theorem for these operators. We also study convergence properties by the modulus of continuity and the rate of convergence of the operators for functions belonging to the Lipschitz class. Furthermore, we obtain the rate of convergence in terms of the classical, second order, and weighted modulus of continuity.

**2. AUXILIARY RESULTS**

We define a Dunkl generalization of Szász-Mirakjan-Kantrovich operators via q-calculus as follows:

For any \( x \in [0, \infty) \), \( n \in \mathbb{N} \), \( 0 < q < 1 \), and \( \mu > \frac{1}{2} \), we define

\[
T_{n,q}^{*}(f; x) = \frac{[n]_q}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q} (k)} \int_{q^{k+2\mu \theta_k}[n]_q}^{\frac{[k+1+2\mu \theta_k][n]_q}{[n]_q}} f \left( \frac{nt + \alpha}{n + \beta} \right) dq^t,
\]

where \( f \) is a continuous and nondecreasing function on \([0, \infty)\).

**Lemma 2.1.** Let \( T_{n,q}^{*}(\cdot; \cdot) \) be the operators given by (2.1). Then we have the following identities and inequalities:

1. \( T_{n,q}^{*}(1; x) = 1 \),
2. \( T_{n,q}^{*}(t; x) = \frac{2n}{(n + \beta)^2 [2]_q} x + \frac{n}{(n + \beta)^2 [2]_q} + \alpha \frac{n}{n + \beta} \)
3. \( \frac{n^2}{(n + \beta)^2 [3]_q [2]_q} + \frac{2n^2}{(n + \beta)^2 [2]_q [n]_q} + \frac{\alpha^2}{(n + \beta)^2 [2]_q} + \left( \frac{n^2}{(n + \beta)^2 [3]_q [2]_q} + \frac{2n^2}{(n + \beta)^2 [2]_q} + \frac{3q^{2(n + 1)}}{[3]_q [n]_q} \right) \)
\times [1 - 2\mu] \frac{e_{\mu,q}(q[n]_q x)}{e_{\mu,q}(q[n]_q x)} \frac{x^2}{[n]_q} \leq T_{n,q}^{*}(t^2; x) \leq \frac{n^2}{(n + \beta)^2 [2]_q [n]_q} + \frac{2n^2}{(n + \beta)^2 [2]_q} + \frac{3q^{2(n + 1)}}{[3]_q [n]_q} \)
we have the following results:

From the Lemma

\[
\int \left[ \left. \frac{\left( n^2 + 3n\alpha + 3n^2 \right)}{4q[n]^4} \right| q \right] \frac{\left( \alpha^3 + \frac{1}{(n+\beta)^2} \right)^3}{2q} + \frac{\left( \frac{3n\alpha}{4q[n]^4} + \frac{2n}{2q} \right)}{2q} \right] \left[ \frac{1}{2q} \right] \left( 1 - 2\mu \right) \left[ e_{\mu,q}(nq(x)) \left| q \right] \right] \frac{\left( \frac{1}{2q} \right)^2}{2q} \right] \left( 1 - 2\mu \right) \left[ e_{\mu,q}(q^2[nq(x)) \right] x + \left[ \frac{1}{2q} \right] \left( \frac{1}{2q} \right)^2 \right] \left( 2q + 1 \right) \left[ 1 - 2\mu \right] \left[ e_{\mu,q}(q^2[nq(x)) \right] x^2 + \left[ \frac{1}{2q} \right] \left( \frac{1}{2q} \right)^2 x^3 \right] \leqq T^q \left( t^3, x \right)
\]

(5) \( T^q \left( t^4, x \right) \)

\[
= \left[ \frac{n}{(n+\beta)^4} \right] \left[ \frac{1}{2q} \right] \left( 1 + 2\mu \right) \left[ e_{\mu,q}(q^2[nq(x)) \right] x + \left[ \frac{1}{2q} \right] \left( \frac{1}{2q} \right)^2 \right] \left( 2q + 1 \right) \left[ 1 - 2\mu \right] \left[ e_{\mu,q}(q^2[nq(x)) \right] x^2 + \left[ \frac{1}{2q} \right] \left( \frac{1}{2q} \right)^2 x^3 \right] \leqq T^q \left( t^4, x \right)
\]

Proof. It is easily seen that

\[
k + 1 + 2\mu \theta \right] q = q \left[ k + 2\mu \theta \right] q + 1.
\]

so we get the followings

\[
\int \left[ q \left[ k + 2\mu \theta \right] q \right] \frac{1}{[n]^q} \right] 1 \right] d_q t = \frac{1}{[n]^q},
\]

\[
\int \left[ q \left[ k + 2\mu \theta \right] q \right] \frac{1}{[n]^q} \right] t \right] d_q t = \frac{1}{[2q]^2} \left[ 1 + 2q \left[ k + 2\mu \theta \right] q \right],
\]

\[
\int \left[ q \left[ k + 2\mu \theta \right] q \right] \frac{1}{[n]^q} \right] t^2 \right] d_q t = \frac{1}{[3q]^3} (1 + 3q \left[ k + 2\mu \theta \right] q + 3q^2 \left[ k + 2\mu \theta \right] q^2),
\]

\[
\int \left[ q \left[ k + 2\mu \theta \right] q \right] \frac{1}{[n]^q} \right] t^3 \right] d_q t = \frac{1}{[4q]^4} (1 + 4q \left[ k + 2\mu \theta \right] q + 6q^2 \left[ k + 2\mu \theta \right] q^2 + 4q^3 \left[ k + 2\mu \theta \right] q^3),
\]

and

\[
\int \left[ q \left[ k + 2\mu \theta \right] q \right] \frac{1}{[n]^q} \right] t^4 \right] d_q t = \frac{1}{[5q]^5} (1 + 5q \left[ k + 2\mu \theta \right] q + 10q^2 \left[ k + 2\mu \theta \right] q^2 + 10q^3 \left[ k + 2\mu \theta \right] q^3 + 5q^4 \left[ k + 2\mu \theta \right] q^4).
\]

From the Lemma 1.1 we have the following results:

\[
\sum_{k=0}^{\infty} \frac{\left( \frac{n}{q} \right)^k}{k!} \frac{1}{k!} \sum_{k=0}^{\infty} \frac{\left( \frac{n}{q} \right)^k}{k!} \left[ k + 2\mu \theta \right] q = x.
\]
\[ x^2 + q^{2n}[1 - 2\mu_q] \sum_{n \in [n_q]} x^n \leq \frac{1}{[n_q]^2} \sum_{k=0}^{\infty} \frac{([n_q]^{x})^k}{\gamma_{\mu_q}(k)} \left( k + 2\mu_k \right)_q^2 \leq x^2 + x, \] (2.9)

\[ \frac{1}{[n_q]^2} \sum_{k=0}^{\infty} \frac{([n_q]^{x})^k}{\gamma_{\mu_q}(k)} \left( k + 2\mu_k \right)_q^3 \leq x^3 + 3[1 + 2\mu]_q x^2 + [1 + 2\mu]_q x, \] (2.10)

\[ \frac{1}{[n_q]^2} \sum_{k=0}^{\infty} \frac{([n_q]^{x})^k}{\gamma_{\mu_q}(k)} \left( k + 2\mu_k \right)_q^3 \geq x^3 + (2q + 1)[1 - 2\mu_q] \frac{([n_q]^{x})^k}{\gamma_{\mu_q}(k)} \left( k + 2\mu_k \right)_q^3 + \frac{q^4[1 - 2\mu]^2 \frac{([n_q]^{x})^k}{\gamma_{\mu_q}(k)} x}{[n_q]^2}, \] (2.11)

and

\[ \frac{1}{[n_q]^4} \sum_{k=0}^{\infty} \frac{([n_q]^{x})^k}{\gamma_{\mu_q}(k)} \left( k + 2\mu_k \right)_q^3 \leq x^4 + [1 + 2\mu]_q x^3 + [1 + 2\mu]_q x^2 + [1 + 2\mu]_q x. \] (2.12)

(1) From (2.3) we have \( T_{n,q}^*(1; x) = \frac{[n_q]}{e_{\mu_q}([n_q]x)} \sum_{k=0}^{\infty} \frac{([n_q]^{x})^k}{\gamma_{\mu_q}(k)} \frac{1}{[n_q]} = 1. \)

(2) If \( f(t) = t \) then (2.1), (2.4) and (2.8) imply that

\[ T_{n,q}^*(t; x) = \frac{2n^n}{(n + \beta)[2]_q} x + \frac{n}{(n + \beta)[2]_q} \frac{\alpha}{n + \beta}. \]

(3) If \( f(t) = t^2 \) then from (2.1), (2.5), (2.8) and (2.9) we get (3).

(4) If \( f(t) = t^3 \) then from (2.1), (2.6), (2.8), (2.9), (2.10) and (2.11) we get (4)

(5) If \( f(t) = t^4 \) then from (2.1), (2.7), (2.8), (2.9), (2.10) and (2.12) we have (4)

Lemma 2.2. Let the operators \( T_{n,q}^*(\cdot; \cdot) \) be given by (2.1). Then

1. \( T_{n,q}^*(t - x; x) = \left( \frac{2n^n}{(n + \beta)[2]_q} - 1 \right) x + \frac{n}{(n + \beta)[2]_q} + \frac{\alpha}{n + \beta}. \)

2. \( T_{n,q}^*((t - x)^2; x) \leq \frac{n^n}{(n + \beta)[2]_q} \left( \frac{3n}{n + \beta} \right)_q + \frac{(n + \beta)^2}{(n + \beta)[2]_q} + \frac{\alpha^2}{n + \beta} + \frac{\alpha^2}{n + \beta} \left( \frac{n}{n + \beta} \right)_q - \frac{\alpha^2}{n + \beta} \left( \frac{3n}{n + \beta} \right)_q. \)

3. \( T_{n,q}^*((t - x)^4; x) \leq \frac{n^n}{(n + \beta)[2]_q} \left( \frac{5n}{n + \beta} \right)_q + \frac{2n^n}{(n + \beta)[2]_q} \left( \frac{5n}{n + \beta} \right)_q + \frac{(n + \beta)^2}{(n + \beta)[2]_q} + \frac{\alpha^2}{n + \beta} \left( \frac{n}{n + \beta} \right)_q + \frac{\alpha^2}{n + \beta} \left( \frac{3n}{n + \beta} \right)_q + \frac{\alpha^2}{n + \beta} \left( \frac{5n}{n + \beta} \right)_q + \frac{\alpha^2}{n + \beta} \left( \frac{5n}{n + \beta} \right)_q + \frac{\alpha^2}{n + \beta} \left( \frac{5n}{n + \beta} \right)_q + \frac{\alpha^2}{n + \beta} \left( \frac{5n}{n + \beta} \right)_q + \frac{\alpha^2}{n + \beta} \left( \frac{5n}{n + \beta} \right)_q. \)
Clearly from \((\ref{2.1}), \) uniformly on \([0, \infty)\), the convergence of a sequence of linear and positive operators, so it is enough to prove

\[
\frac{4}{n+\beta}^\frac{n}{2} + \frac{6n^2}{n+\beta}^\frac{1}{2} + \frac{6n^2}{n+\beta}^\frac{1}{2} + \frac{6n^2}{n+\beta}^\frac{1}{2} + \frac{6n^2}{n+\beta}^\frac{1}{2} + \frac{6n^2}{n+\beta}^\frac{1}{2} + \frac{6n^2}{n+\beta}^\frac{1}{2} + \frac{6n^2}{n+\beta}^\frac{1}{2}
\]

\[
\lim_{n \to \infty} T_{n,q_n}(f; x) = f(x)
\]

is uniformly on each compact subset of \([0, \infty)\).

**Proof.** The proof is based on the well known Korovkin’s theorem regarding the convergence of a sequence of linear and positive operators, so it is enough to prove the conditions

\[
\lim_{n \to \infty} T_{n,q_n}(f; x) = f(x)
\]

uniformly on \([0, 1]\).

Clearly from (3.1) and \(\frac{1}{n, q_n} \to 0 (n \to \infty)\) we have

\[
\lim_{n \to \infty} T_{n,q_n}(t; x) = x, \quad \lim_{n \to \infty} T_{n,q_n}(t^2; x) = x^2.
\]

Which completeS the proof. \(\square\)

We recall the weighted spaces of the functions on \(\mathbb{R}^+\), which are defined as follows:

\[
\begin{align*}
P_{\rho}(\mathbb{R}^+) &= \{ f : f(x) \leq Mf(x) \}, \\
Q_{\rho}(\mathbb{R}^+) &= \{ f : f \in P_{\rho}(\mathbb{R}^+) \cap C[0, \infty) \}, \\
Q_{\rho}^k(\mathbb{R}^+) &= \left\{ f : f \in Q_{\rho}(\mathbb{R}^+) \text{ and } \lim_{x \to \infty} \frac{f(x)}{\rho(x)} = k (k \text{ is a constant)} \right\},
\end{align*}
\]
where \( \rho(x) = 1 + x^2 \) is a weight function and \( M_f \) is a constant depending only on \( f \). Note that \( Q_\rho(\mathbb{R}^+) \) is a normed space with the norm \( \| f \|_\rho = \sup_{x \geq 0} \frac{f(x)}{\rho(x)} \).

**Theorem 3.2.** Let \( q = q_n \) satisfying (3.1), for \( 0 < q_n < 1 \) and if \( T_{n,q_n}^*(\cdot,\cdot) \) be the operators given by (2.1). Then for any function \( f \in Q^k_\rho(\mathbb{R}^+) \) we have

\[
\lim_{n \to \infty} \| T_{n,q_n}^*(f;\cdot) - f \|_\rho = 0.
\]

**Proof.** From Lemma 2.1, the first condition of (1) is fulfilled for \( \tau = 0 \). Now for \( \tau = 1, 2 \) it is easy to see that from (2), (3) of Lemma 2.1 by using (3.1)

\[
\| T_{n,q_n}^* (t^\tau; \cdot) - t^\tau \|_\rho = 0.
\]

This completes the proof. \( \square \)

### 4. Rate of Convergence

Here we calculate the rate of convergence of operators (2.1) by means of modulus of continuity and Lipschitz type maximal functions.

Let \( f \in C[0,\infty] \). The modulus of continuity of \( f \) denoted by \( \omega(f,\delta) \) gives the maximum oscillation of \( f \) in any interval of length not exceeding \( \delta > 0 \) and it is given by

\[
\omega(f,\delta) = \sup_{|y-x| \leq \delta} | f(y) - f(x) |, \quad x,y \in [0,\infty). \tag{4.1}
\]

It is known that \( \lim_{\delta \to 0^+} \omega(f,\delta) = 0 \) for \( f \in C[0,\infty) \) and for any \( \delta > 0 \) one has

\[
| f(y) - f(x) | \leq \left( \frac{|y-x|}{\delta} + 1 \right) \omega(f,\delta). \tag{4.2}
\]

**Theorem 4.1.** Let \( T_{n,q}^*(\cdot,\cdot) \) be the operators defined by (2.1). Then for \( f \in \tilde{C}[0,\infty), \ x \geq 0 \), \( 0 < q < 1 \) we have

\[
T_{n,q}^*(f;\cdot) - f(\cdot) \leq \left\{ 1 + \sqrt{\phi_n(x)} \right\} \omega \left( f; \frac{1}{\sqrt{|n|q}} \right),
\]

where

\[
\phi_n(x) = \frac{n}{(n+\beta)^2 |n|q} \left( \frac{n}{3|n|q} + \frac{2\alpha}{2|q|} \right) + \frac{\alpha^2}{(n+\beta)^2} + \frac{n^2}{(n+\beta)^2} \left( \frac{3}{3|q|n|q} - \frac{4n}{n+\beta} \right) + 1 \}
\]

and \( \tilde{C}[0,\infty) \) is the space of uniformly continuous functions on \( \mathbb{R}^+ \) and \( \omega(f,\delta) \) is the modulus of continuity of the function \( f \in \tilde{C}[0,\infty) \) defined in (4.1).
Proof. We prove it by using (4.1), (4.2) and Cauchy-Schwarz inequality. 
\[ |T_{n,q}^*(f; x) - f(x)| \]
\[
\leq \frac{[n]_q}{e_{μ, q}([n]_q x)} \sum_{k=0}^{∞} \frac{([n]_q x)^k}{\gamma_{μ, q}(k)} \int_{q[k+1+2μθ_k]_q}^{[k+1+2μθ_k]_q} |f(t) - f(x)| d_q(t) \\
\leq \frac{[n]_q}{e_{μ, q}([n]_q x)} \sum_{k=0}^{∞} \frac{([n]_q x)^k}{\gamma_{μ, q}(k)} \int_{q[k+1+2μθ_k]_q}^{[k+1+2μθ_k]_q} \left(1 + \frac{1}{δ} |t - x|\right) d_q(t) \omega(f; δ) \\
= \left\{1 + \frac{1}{δ} \left(\frac{[n]_q}{e_{μ, q}([n]_q x)} \sum_{k=0}^{∞} \frac{([n]_q x)^k}{\gamma_{μ, q}(k)} \int_{q[k+1+2μθ_k]_q}^{[k+1+2μθ_k]_q} (t - x)^2 d_q(t)\right)^{\frac{1}{2}} \left(T_{n,q}^*(1; x)\right)^{\frac{1}{2}} \right\} \omega(f; δ) \\
= \left\{1 + \frac{1}{δ} (T_{n,q}^*(t - x)^2; x)^{\frac{1}{2}} \right\} \omega(f; δ)
\]

if we choose \(δ = δ_n = \sqrt{\frac{1}{[n]_q}}\), then we get our result. \(\square\)

Now we give the rate of convergence of the operators \(T_{n,q}^*(f; x)\) defined in (2.1) in terms of the elements of the usual Lipschitz class \(\text{Lip}_M(ν)\).

Let \(f \in C[0, ∞), M > 0\) and \(0 < ν ≤ 1\). The class \(\text{Lip}_M(ν)\) is defined as
\[
\text{Lip}_M(ν) = \{f : |f(ζ_1) - f(ζ_2)| ≤ M |ζ_1 - ζ_2|^{ν} (ζ_1, ζ_2 \in [0, ∞))\} \quad(4.3)
\]

Theorem 4.2. Let \(T_{n,q}^*(\cdot; \cdot)\) be the operator defined in (2.1). Then for each \(f \in \text{Lip}_M(ν), (M > 0, \ 0 < ν ≤ 1)\) satisfying (4.3) we have
\[
|T_{n,q}^*(f; x) - f(x)| ≤ M (λ_n(x))^{\frac{ν}{2}}
\]

where \(λ_n(x) = T_{n,q}^*((t - x)^2; x)\).

Proof. We prove it by using (4.3) and Hölder inequality.
\[
|T_{n,q}^*(f; x) - f(x)| \leq |T_{n,q}^*(f(t) - f(x); x)| \\
≤ T_{n,q}^*(|f(t) - f(x)|; x) \\
≤ |MT_{n,q}^*(|t - x|^ν; x)|.
\]

Therefore
\[ | T_{n,q}^* (f; x) - f(x) | \]
\[ \leq M \left[ \frac{[n]_q}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \oint_{q^{[k+2\theta_{k}]}_q |q^n_q[|}} | t - x |^\nu \, d_q(t) \right] \]
\[ \leq M \left( \frac{[n]_q}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \oint_{q^{[k+2\theta_{k}]}_q |q^n_q[|}} | t - x |^\nu \, d_q(t) \right)^{\frac{1}{\nu}} \]
\[ = M \left( T_{n,q}^* (t - x)^2 ; x \right)^{\frac{1}{2}}. \]

Which completes the proof. \[ \square \]

Let \( C_B[0, \infty) \) denote the space of all bounded and continuous functions on \( \mathbb{R}^+ = [0, \infty) \) and
\[ C_B^2(\mathbb{R}^+) = \{ g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+) \}, \] (4.4)
with the norm
\[ \| g \|_{C_B^2(\mathbb{R}^+)} = \| g \|_{C_B(\mathbb{R}^+)} + \| g' \|_{C_B(\mathbb{R}^+)} + \| g'' \|_{C_B(\mathbb{R}^+)}, \] (4.5)
also
\[ \| g \|_{C_B(\mathbb{R}^+)} = \sup_{x \in \mathbb{R}^+} | g(x) |. \] (4.6)

**Theorem 4.3.** Let \( T_{n,q}^*(\cdot ; \cdot) \) be the operator defined in (2.1). Then for any \( g \in C_B^2(\mathbb{R}^+) \) we have
\[ | T_{n,q}^* (f; x) - f(x) | \leq \left( \frac{2g}{(n + \beta)[2]_q} - 1 \right) x + \frac{n}{(n + \beta)[2]_q[n]_q} + \frac{\alpha}{n + \beta} + \frac{\lambda_n(x)}{2} \| g \|_{C_B^2(\mathbb{R}^+)} \]
where \( \lambda_n(x) \) is given in Theorem 4.2.

**Proof.** Let \( g \in C_B^2(\mathbb{R}^+) \), then by using the generalized mean value theorem in the Taylor series expansion we have
\[ g(t) = g(x) + g'(x)(t - x) + g''(\psi) \frac{(t - x)^2}{2}, \psi \in (x, t). \]
By applying linearity property on \( T_{n,q}^* \), we have
\[ T_{n,q}^* (g; x) - g(x) = g'(x)T_{n,q}^* ((t - x); x) + \frac{g''(\psi)}{2} T_{n,q}^* ((t - x)^2; x), \]
which implies
\[ |T_{n,q}(g; x) - g(x)| \]
\[ \leq \left( \frac{2qn}{(n + \beta)[2]_q} - 1 \right) x + \frac{n}{(n + \beta)[2]_q[n]_q} + \frac{\alpha}{n + \beta} \| g' \|_{C_B(\mathbb{R}^+)} \]
\[ + \frac{n}{(n + \beta)[2]_q[n]_q} + \frac{2\alpha}{(n + \beta)^2} + \frac{\alpha^2}{(n + \beta)^3} + \frac{n^2}{(n + \beta)[3]_q[n]_q} \times (1 + [1 + 2\mu]_q) \]
\[ + \frac{2n}{(n + \beta)[2]_q} \left( 2\alpha - \frac{1}{[n]_q} - \frac{2\alpha}{(n + \beta)} \right) x + \left\{ \frac{n}{(n + \beta)[2]_q[n]_q} - \frac{4n}{(n + \beta)[2]_q} \right\} x^2 \times \frac{\| g'' \|_{C_B(\mathbb{R}^+)}^2}{2}. \]

On using (4.5) \( \| g' \|_{C_B[0, \infty)} \leq \| g \|_{C^2_B[0, \infty)} \) completes the proof from 2 of Lemma 2.2.

The Peetre’s \( K \)-functional is defined by
\[ K_2(f, \delta) = \inf_{C^2_B(\mathbb{R}^+)} \left\{ \| f - g \|_{C_B(\mathbb{R}^+)} + \delta \| g'' \|_{C^2_B(\mathbb{R}^+)} : g \in \mathcal{W}^2 \right\}, \tag{4.7} \]
where
\[ \mathcal{W}^2 = \left\{ g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+) \right\}. \tag{4.8} \]

There exists a positive constant \( C > 0 \) such that \( K_2(f, \delta) \leq C \omega_2(f, \delta^\frac{1}{3}), \delta > 0, \)
where the second order modulus of continuity is given by
\[ \omega_2(f, \delta^\frac{1}{3}) = \sup_{0 < h < \delta^\frac{1}{3}} \sup_{x \in \mathbb{R}^+} |f(x + 2h) - 2f(x + h) + f(x)|. \tag{4.9} \]

**Theorem 4.4.** Let \( T_{n,q}(\cdot, \cdot) \) be the operator defined in (2.1) and \( C_B[0, \infty) \) be the space of all bounded and continuous functions on \( \mathbb{R}^+ \). Then for \( x \in \mathbb{R}^+, \)
\[ f \in C_B(\mathbb{R}^+) \) we have
\[ |T_{n,q}(f; x) - f(x)| \]
\[ \leq 2M \left\{ \omega_2 \left( f; \sqrt{\left( \frac{4qn}{(n + \beta)[2]_q} - 2 \right)x + \frac{2\alpha}{4} + \frac{\alpha^2}{4(n + \beta)} + \lambda_n(x)} \right) + \min \left( 1, \frac{\left( \frac{4qn}{(n + \beta)[2]_q} - 2 \right)x + \frac{2\alpha}{4} + \frac{\alpha^2}{4(n + \beta)} + \lambda_n(x)}{1 + \frac{2\alpha}{4} + \frac{\alpha^2}{4(n + \beta)}} \right) \right\} \]
where \( M \) is a positive constant, \( \lambda_n(x) \) is given in Theorem 4.2 and \( \omega_2(f; \delta) \) is the second order modulus of continuity of the function \( f \) defined in (4.9).

**Proof.** We prove this by using the Theorem (4.3)
\[ |T_{n,q}(f; x) - f(x)| \leq |T_{n,q}(f - g; x)| + |T_{n,q}(g; x) - g(x)| + |f(x) - g(x)| \]
\[ \leq 2 \| f - g \|_{C_B(\mathbb{R}^+)} + \frac{\lambda_n(x)}{2} \| g \|_{C^2_B(\mathbb{R}^+)} \]
\[ + \left( \frac{2qn}{(n + \beta)[2]_q} - 1 \right) x + \frac{n}{(n + \beta)[2]_q[n]_q} + \frac{\alpha}{n + \beta} \| g \|_{C_B(\mathbb{R}^+)} \]
From (4.5) clearly we have \( \| g \|_{C_B[0, \infty)} \leq \| g \|_{C^2_B[0, \infty)} \).
Therefore,
\[ |T_{n,q}(f; x) - f(x)| \leq 2 \left( \| f - g \|_{C_B(\mathbb{R}^+)} + \frac{\left( \frac{4qn}{(n + \beta)[2]_q} - 2 \right)x + \frac{2\alpha}{4} + \frac{\alpha^2}{4(n + \beta)} + \lambda_n(x)}{1 + \frac{2\alpha}{4} + \frac{\alpha^2}{4(n + \beta)}} \right) \| g \|_{C^2_B(\mathbb{R}^+)} \).
where \( \lambda_n(x) \) is given in Theorem 4.2.

By taking infimum over all \( g \in C_B^2(\mathbb{R}^+) \) and by using (4.7), we get

\[
| T_{n,q}^*(f; x) - f(x) | \leq 2K_2 \left( f; \frac{4n}{(n+\beta)[2]_q} - 2 \right) x + \frac{2n}{(n+\beta)[2]_q [n]_q} + \frac{2n}{n+\beta} + \lambda_n(x)
\]

Now for an absolute constant \( D > 0 \) in [5] we use the relation

\[
K_2(f; \delta) \leq D \{ \omega_2(f; \sqrt{\delta}) + \min(1, \delta) \| f \| \}.
\]

This complete the proof. \( \square \)

Atakut and Ispir [1] introduced the weighted modulus of continuity and defined as, for an arbitrary \( f \in Q^k_\rho(\mathbb{R}^+) \)

\[
\Omega(f, \delta) = \sup_{x \in [0, \infty), |h| \leq \delta} \frac{|f(x + h) - f(x)|}{(1 + h^2)(1 + x^2)}.
\] (4.10)

The two main properties of this modulus of continuity are \( \lim_{\delta \to 0} \Omega(f, \delta) \to 0 \) and

\[
| f(t) - f(x) | \leq 2 \left( 1 + \frac{|t - x|}{\delta} \right) (1 + \delta^2)(1 + x^2)(1 + (t - x)^2) \Omega(f, \delta),
\] (4.11)

where \( f \in Q^k_\rho(\mathbb{R}^+) \) and \( t, x \in [0, \infty) \).

**Theorem 4.5.** Let \( T_{n,q}^*(\cdot; \cdot) \) be the operators defined by (2.1). Then for \( f \in Q^k_\rho(\mathbb{R}^+) \), \( 0 < q < 1 \) and \( x \geq 0 \) we have

\[
\sup_{x \in [0, \infty)} \frac{| T_{n,q}^*(f; x) - f(x) |}{(1 + x^2)} \leq C_\mu \left( 1 + \frac{1}{[n]_q} \right) \Omega(f, \frac{1}{[n]_q}/),
\]

where \( C_\mu \) is constant independent of \( n \).
Proof. We prove it by using (4.10), (4.11) and Cauchy-Schwarz inequality.

\[
|T_{n,q}(f;x) - f(x)| \leq \left| \frac{[n]_q}{e_{\mu,q}([n]_q)x} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{[k+1+2\mu\theta_q]}{[n]_q}}^{\frac{[k+1+2\mu\theta_q]}{[n]_q}} |f(t) - f(x)| \, d_q(t) \right|
\]

\[
\leq 2(1 + \delta^2)(1 + x^2) \Omega(f;\delta) \left| \frac{[n]_q}{e_{\mu,q}([n]_q)x} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{[k+1+2\mu\theta_q]}{[n]_q}}^{\frac{[k+1+2\mu\theta_q]}{[n]_q}} (1 + \frac{1}{\delta} |t - x|) (1 + (t - x)^2) \, d_q(t) \right|
\]

\[
= 2(1 + \delta^2)(1 + x^2) \Omega(f;\delta) \left| \frac{[n]_q}{e_{\mu,q}([n]_q)x} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{[k+1+2\mu\theta_q]}{[n]_q}}^{\frac{[k+1+2\mu\theta_q]}{[n]_q}} (t - x)^2 \, d_q(t) \right|
\]

\[
\leq 2(1 + \delta^2)(1 + x^2) \Omega(f;\delta) \left( 1 + T_{n,q}^*((t-x)^2;x) + \frac{1}{\delta} \sqrt{T_{n,q}^*((t-x)^2;x) T_{n,q}^*((t-x)^4;x)} \right)
\]

where \(T_{n,q}^* ((t-x)^2;x)\) and \(T_{n,q}^* ((t-x)^4;x)\) is defined in (2) and (3) of Lemma 2.2.

If we choose \(\delta = \delta_n = \sqrt{\frac{1}{[n]_q}}\), then we get our result. \(\square\)

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