THE HYPERBOLIC AX-LINDEMANN-WEIERSTRASS CONJECTURE

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1. Introduction.

1.1. The hyperbolic Ax-Lindemann-Weierstraß conjecture. Around 1885 Lindemann and Weierstraß proved that if $\alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly independent algebraic numbers then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$ ([12], [32]). This classical transcendence result has the following functional “flat” analogue, which is a particular case of a result of Ax [2]: Define $\pi = (\exp, \ldots, \exp) : \mathbb{C}^n \rightarrow (\mathbb{C}^*)^n$. Let $V \subset (\mathbb{C}^*)^n$ be an algebraic subvariety. Any maximal complex irreducible algebraic subvariety $Y \subset \pi^{-1}(V)$ is a translate of a rational linear subspace. Another “flat” Ax-Lindemann-Weierstraß theorem is obtained when studying the uniformizing map of an abelian variety: Let $\pi : \mathbb{C}^n \rightarrow A$ be the uniformizing map of a complex abelian variety of dimension $n$. Let $V \subset A$ be an algebraic subvariety. Any maximal complex irreducible algebraic subvariety $Y \subset \pi^{-1}(V)$ is the preimage of a translate of an abelian subvariety contained in $V$.

The main result of this paper is a proof of a similar statement, the hyperbolic Ax-Lindemann-Weierstraß conjecture, for any arithmetic variety $S := \Gamma \backslash X$. Here $X$ denotes a Hermitian symmetric domain and $\Gamma$ is any arithmetic subgroup of the real adjoint Lie group $G$ of biholomorphisms of $X$. This means that there exists a semisimple $\mathbb{Q}$-algebraic group $G$ and a surjective morphism with compact kernel $p : G(\mathbb{R}) \rightarrow G$ such that $\Gamma$ is commensurable with the projection $p(G(\mathbb{Z}))$ (cf. section 2.1 for the definition of $G(\mathbb{Z})$ and [13] for a general reference on arithmetic lattices).

While $X$ is not a complex algebraic variety it admits a canonical realisation as a bounded symmetric domain $\mathcal{D} \subset \mathbb{C}^N$ (with $N = \dim_{\mathbb{C}} X$) (cf. [27] §II.4]). We will say that a subset $Y \subset \mathcal{D}$ is an irreducible algebraic subvariety of $\mathcal{D}$ if $Y$ is an irreducible component of the analytic set $\mathcal{D} \cap \bar{Y}$ where $\bar{Y}$ is an algebraic subset of $\mathbb{C}^N$. An algebraic subvariety of $\mathcal{D}$ is then defined as a finite union of irreducible algebraic subvarieties. On the other hand the arithmetic variety $S$ admits a natural structure of complex quasi-projective variety via the Baily-Borel embedding $\mathbb{B}$. Recall that an irreducible algebraic subvariety of $S$ is said weakly special if its smooth locus is totally geodesic in $S$ endowed with its canonical Hermitian metric.

The uniformization map $\pi : \mathcal{D} \rightarrow S = \Gamma \backslash \mathcal{D}$ is highly transcendental with respect to these algebraic structures (in the simplest case where $\mathcal{D}$ is the Poincaré disk and $S$ is the modular curve, the map $\pi : \mathcal{D} \rightarrow S$ is the usual $j$-invariant seen on the disk). The hyperbolic Ax-Lindemann-Weierstraß conjecture is the following statement:
Theorem 1.1. (The hyperbolic Ax-Lindemann-Weierstraß conjecture.) Let $S = \Gamma \backslash \mathcal{D}$ be an arithmetic variety with uniformising map $\pi : \mathcal{D} \rightarrow S$. Let $V$ be an algebraic subvariety of $S$. Maximal irreducible algebraic subvarieties of $\pi^{-1}V$ are precisely the irreducible components of the preimages of maximal weakly special subvarieties contained in $V$.

Remarks 1.1. (a) In [31] Ullmo and Yafaev proved the theorem 1.1 in the special case where $S$ is compact. In [25] Pila and Tsimerman proved theorem 1.1 in the special case $S = \mathcal{A}_g$, the moduli space of principally polarised abelian varieties of dimension $g$.

(b) Mok has a very nice, entirely complex-analytic, approach to the hyperbolic Ax-Lindemann-Weierstraß conjecture. In the rank 1 case his approach should extend some of the results of this text to the non-arithmetic case. We refer to [15], [16] for partial results.

(c) We defined algebraic subvarieties of $X$ using the Harish-Chandra realisation $\mathcal{D}$ of $X$ but we could have used as well any other realisation of $X$ in the sense of [29, section 2.1]. Indeed morphisms of realisations are necessarily semi-algebraic, thus $X$ admits a canonical semi-algebraic structure and a canonical notion of algebraic subvarieties (cf. appendix B for details). Hence one can replace $\mathcal{D}$ in theorem 1.1 by any other realisation of $X$, for example the Borel realisation (cf. [14, p.52]).

1.2. Motivation: the André-Oort conjecture. Let $(G,X_G)$ be a Shimura datum. Let $X$ be a connected component of $X_G$ (hence $X$ is a Hermitian symmetric domain). We denote by $G(\mathbb{Q})_+$ the stabiliser of $X$ in $G(\mathbb{Q})$. Let $K_f$ be a compact open subgroup of $G(\mathbb{A}_f)$, where $\mathbb{A}_f$ denotes the finite adèles of $\mathbb{Q}$ and let $\Gamma := G(\mathbb{Q})_+ \cap K_f$ be the corresponding congruence arithmetic lattice of $G(\mathbb{Q})$.

Then the arithmetic variety $S := \Gamma \backslash X$ is a component of the complex quasi-projective Shimura variety

$$\text{Sh}_K(G,X) := G(\mathbb{Q})_+ \backslash X \times G(\mathbb{A}_f)/K_f.$$

The variety $S$ contains the so-called special points and special subvarieties (these are the weakly special subvarieties of $S$ containing one special point, we refer to [5] or [17] for the detailed definitions). One of the main motivations for studying the Ax-Lindemann-Weierstraß conjecture is the André-Oort conjecture predicting that irreducible subvarieties of $S$ containing Zariski dense sets of special points are precisely the special subvarieties. The André-Oort conjecture has been proved under the assumption of the Generalised Riemann Hypothesis (GRH) by the authors of this paper ([30], [11]). Recently Pila and Zannier [26] came up with a new proof of the Manin-Mumford conjecture for abelian varieties using the flat Ax-Lindemann-Weierstraß theorem. This gave hope to prove the André-Oort conjecture unconditionally with the same strategy. In [22] Pila succeeded in applying this strategy to the case where $S$ is a product of modular curves. Roughly speaking, the strategy consists of two main ingredients: the first is the problem of bounding
below the sizes of Galois orbits of special points and the second is the hyperbolic Ax-Lindemann-Weierstraß conjecture (cf. [28]).

We refer to [29] for details on how the hyperbolic Ax-Lindemann-Weierstraß conjecture and a good lower bound on the sizes of Galois orbits of special points imply the André-Oort conjecture. As a direct corollary of theorem 1.1 and the proof of [29, theor.5.1] one obtains:

**Corollary 1.2.** The André-Oort conjecture holds for $A^n_k$ for any positive integer $n$.

A new proof of the André-Oort conjecture under the GRH is a consequence of theorem 1.1 and an upper bound for the height of special points in Siegel sets. This last step is currently studied by C.Daw and M.Orr.

1.3. **Strategy of the proof of theorem 1.1.** Our strategy for proving theorem 1.1 is as follows:

(i) Let $S := \Gamma \setminus X$ and $\pi: X \to S$ be the uniformising map. Even though the map $\pi$ is transcendental, it still enables us to relate the semi-algebraic structures on $X$ and $S$ through a larger o-minimal structure. We refer to [31, section 3], [3], [7] for details on o-minimal structures. Recall that a fundamental set for the action of $\Gamma$ on $X$ is a connected open subset $\mathcal{F}$ of $X$ such that $1\mathcal{F} = X$ and such that the set $\{\gamma \in \Gamma : \gamma \mathcal{F} \cap \mathcal{F} \neq \emptyset\}$ is finite. We prove in section 4 the following result:

**Theorem 1.2.** There exists a semi-algebraic fundamental set $\mathcal{F}$ for the action of $\Gamma$ on $X$ such that the restriction $\pi|_{\mathcal{F}}: \mathcal{F} \to S$ is definable in the o-minimal structure $\mathbb{R}_{an,exp}$.

**Remarks 1.3.**

(a) The special case of theorem 1.2 when $S$ is compact is much easier and was proven in [31], Proposition 4.2. In this case, the map $\pi|_{\mathcal{F}}$ is even definable in $\mathbb{R}_{an}$. Theorem 1.2 in the case where $X = \mathcal{H}_g$ is the Siegel upper half plane of genus $g$ was proven by Peterzil and Starchenko (see [21] and [20]) and is a crucial ingredient in [25]. Notice that this particular case implies theorem 1.2 for any special subvariety $S$ of $A_g$ (see Proposition 2.5 of [29]).

(b) Our proof of theorem 1.2 does not use [21] or [20] but relies on the general theory of compactifications of arithmetic varieties (cf. [1]).

(ii) Choose a semi-algebraic fundamental set $\mathcal{F}$ for the action of $\Gamma$ as in the theorem 1.2 above. The choice of a reasonable representation $\rho: G \to GL(E)$ (cf. section 2.1) allows us to define a *height function* $H: \Gamma \to \mathbb{R}$ (cf. definition 2.2). In section 5 we show the following result:

**Theorem 1.3.** Let $Y$ be a positive dimensional irreducible algebraic subvariety of $X$. Define

$$N_Y(T) = |\{\gamma \in \Gamma : H(\gamma) \leq T, Y \cap \gamma \mathcal{F} \neq \emptyset\}|.$$

Then there exists a positive constant $c_1$ such that for all $T$ large enough:

$$N_Y(T) \geq T^{c_1}.$$
Remark 1.4. When $S$ is compact Ullmo and Yafaev proved (cf. [31]) that the length function grows exponentially and theorem 1.3 follows in this case. We were not able to obtain such a result on the length in the general case.

(iii) In section 6, applying the counting result above and some strong form of Pila-Wilkie’s theorem [23], we prove:

**Theorem 1.4.** Let $V$ be an algebraic subvariety of $S$ and $Y$ a maximal irreducible algebraic subvariety of $\pi^{-1}V$. Let $\Theta_Y$ denote the stabiliser of $Y$ in $G(\mathbb{R})$ and define $H_Y$ as the connected component of the identity of the Zariski closure of $G(\mathbb{Z}) \cap \Theta_Y$. Then $H_Y$ is a non-trivial $\mathbb{Q}$-subgroup of $G$, such that $H_Y(\mathbb{R})$ is non-compact.

(iv) Without loss of generality one can assume that $V$ is the smallest algebraic subvariety of $S$ containing $\pi(Y)$. With this assumption we show in section 7 that $\tilde{V}$ is invariant under $H_Y(\mathbb{Q})$, where $\tilde{V}$ is an analytic irreducible component of $\pi^{-1}V$ containing $Y$, and then conclude that $\pi(Y) = V$ is weakly special using monodromy arguments.

2. Preliminaries

2.1. Notations. In the rest of the text:

- $X$ denotes a Hermitian symmetric domain (not necessarily irreducible).
- $G$ is the adjoint semi-simple real algebraic group, whose set of real points, also denoted by $G$, is the group of biholomorphisms of $X$; hence $X = G/K$ where $K$ is a maximal compact subgroup of $G$.
- $\Gamma \subset G$ is an arithmetic lattice. This means (cf. [13]) that there exists a semi-simple linear algebraic group $G$ over $\mathbb{Q}$ and $p : G(\mathbb{R}) \rightarrow G$ a surjective morphism with compact kernel such that $\Gamma$ is commensurable with $p(G(\mathbb{Z}))$. Here we recall that two subgroups of a group are commensurable if their intersection is of finite index in both of them; moreover $G(\mathbb{Z})$ denotes $G(\mathbb{Q}) \cap \rho^{-1}(\text{GL}(E_\mathbb{Z}))$ for some faithful representation $\rho : G \rightarrow \text{GL}(E)$, where $E$ is a finite-dimensional $\mathbb{Q}$-vector space and $E_\mathbb{Z}$ is a $\mathbb{Z}$-lattice in $E$; the commensurability of $\Gamma$ and $p(G(\mathbb{Z}))$ is independant of the choice of $\rho$ and $E_\mathbb{Z}$.
- One easily checks that theorem [14] holds for $\Gamma$ if and only if it holds for any $\Gamma'$ commensurable with $\Gamma$. In particular without loss of generality one can and will assume that the group $G(\mathbb{Z})$ is neat (meaning that for any $\gamma \in G(\mathbb{Z})$ the group generated by the eigenvalues of $\rho(\gamma)$ is torsion-free) and the group $\Gamma$ coincides with $p(G(\mathbb{Z}))$ and is torsion-free.
- Without loss of generality we can and will assume that the group $G$ is of adjoint type. Indeed let $\lambda : G \rightarrow G^{\text{ad}}$ denotes the natural algebraic morphism to the adjoint group $G^{\text{ad}}$ of $G$ (quotient by the centre). As the Lie group $G$ is adjoint
the morphism \( p : G(\mathbb{R}) \rightarrow G \) factorises through

\[
\begin{array}{ccc}
G(\mathbb{R}) & \xrightarrow{\lambda} & G^{\text{ad}}(\mathbb{R}) \\
p \downarrow & & \downarrow p^{\text{ad}} \\
G & & G
\end{array}
\]

and \( \Gamma \) is commensurable with \( p^{\text{ad}}(G^{\text{ad}}(\mathbb{Z})) \).

- Without loss of generality we can and will assume that each \( \mathbb{Q} \)-simple factor of \( G \) is \( \mathbb{R} \)-isotropic. Indeed let \( H \) be the quotient of \( G \) by its \( \mathbb{R} \)-anisotropic \( \mathbb{Q} \)-factors. Again, the morphism \( p : G(\mathbb{R}) \rightarrow G \) factorises through \( H(\mathbb{R}) \) and \( \Gamma \) is commensurable with the projection of \( H(\mathbb{Z}) \).

- \( K_\infty = p^{-1}K \) is a maximal compact subgroup of \( G(\mathbb{R}) \). Hence \( X = G(\mathbb{R})/K_\infty \).

- We denote by \( \mathcal{P} \) the set of all places of \( \mathbb{Q} \).

- We denote by \( \mathcal{X} \) any realization of \( X \) (cf. appendix B).

2.2. **Norm, distance, height.** Let * be the adjunction on \( E_\mathbb{R} \) associated to the Hilbert structure \( \| \cdot \|_\infty \) on \( E_\mathbb{R} \). The restriction of the bilinear form \( (u,v) \mapsto \text{tr}(u^*v) \) to the Lie algebra \( \text{Lie}(G(\mathbb{R})) \) defines a \( G(\mathbb{R}) \)-invariant K\"ahler metric \( g_X \) on \( X \). We denote by \( d : X \times X \rightarrow \mathbb{R} \) the associated distance and by \( \omega \) the associated K\"ahler form.

For each place \( v \) of \( \mathbb{Q} \) we still denote by \( \| \cdot \|_v \) the operator norm on \( \text{End} E_v \) associated to \( \| \cdot \|_v \) on \( E_v \):

\[
\forall \varphi \in \text{End} E_v, \quad \| \varphi \|_v = \sup_{\| x \|_v \leq 1} \| \varphi(x) \|_v .
\]

By restriction we also denote by \( \| \cdot \|_v : G(\mathbb{Q}_v) \rightarrow \mathbb{R} \) the function \( \| \cdot \|_v \circ \rho \).

**Remark 2.1.** As \( K_\infty \) preserves the norm \( \| \cdot \|_\infty \) on \( E_\mathbb{R} \), the function \( \| \cdot \|_\infty : G(\mathbb{R}) \rightarrow \mathbb{R} \) is \( K_\infty \)-bi-invariant, in particular descends to a function \( \| \cdot \|_\infty : X \rightarrow \mathbb{R} \).

**Definition 2.2.** We define the (multiplicative) height function \( H : \text{End} E \rightarrow \mathbb{R} \) as

\[
\forall \varphi \in \text{End} E, \quad H(\varphi) = \prod_{v \in \mathcal{P}} \max(1, \| \varphi \|_v) .
\]

**Remark 2.3.** When \( \dim_{\mathbb{Q}} E = 1 \), this height function coincides with the usual multiplicative height function on rational numbers.
By restriction, we also denote by $H : \mathbf{G}(\mathbb{Q}) \to \mathbb{R}$ the function $H \circ \rho$. As usual the height is particularly simple on $\mathbf{G}(\mathbb{Z})$:

$$\forall \varphi \in \mathbf{G}(\mathbb{Z}), \quad H(\varphi) = \max(1, \|\varphi\|_\infty).$$

Notice that $\|\varphi\|_\infty$ is the square root of the largest eigenvalue of the positive definite matrix $\varphi^* \varphi$. As $\varphi$ is integral it follows that $\|\varphi\|_\infty$ is at least 1, hence

$$\forall \varphi \in \mathbf{G}(\mathbb{Z}), \quad H(\varphi) = \|\varphi\|_\infty.$$

It follows from remark 2.1 that the height function on $\mathbf{G}(\mathbb{Z})$ factorizes through $H : \Gamma \to \mathbb{R}$.

3. Compactification of arithmetic varieties

3.1. Siegel sets. First we recall the definition of Siegel sets for $\Gamma$. We refer to [4, § 12] for details. We follow Borel’s conventions, except that for us the group $G$ acts on $X$ on the left.

Let $P$ be a minimal $\mathbb{Q}$-parabolic subgroup of $G$ such that $K_\infty \cap P(\mathbb{R})$ is a maximal compact subgroup of $P(\mathbb{R})$. Let $U$ be the unipotent radical of $P$ and let $A$ be a maximal split torus of $P$. We denote by $S$ a maximal split torus of $\text{GL}(E)$ containing $\rho(A)$. We denote by $M$ the maximal anisotropic subgroup of the connected centralizer $Z(A)^0$ of $A$ in $P$ and by $\Delta$ the set of simple roots of $G$ with respect to $A$ and $P$. We denote by $A \subset S(\mathbb{R})$ the real torus $A(\mathbb{R})$. For any real number $t > 0$ we let

$$A_t := \{a \in A \mid a^\alpha \geq t \text{ for any } \alpha \in \Delta\}.$$

A Siegel set for $\mathbf{G}(\mathbb{R})$ for the data $(K_\infty, P, A)$ is a product:

$$\Sigma'_t, \Omega := \Omega \cdot A_t \cdot K_\infty \subset \mathbf{G}(\mathbb{R})$$

where $\Omega$ is a compact neighborhood of $e$ in $M^0(\mathbb{R}) \cdot U(\mathbb{R})$.

The image

$$\Sigma_{t, \Omega} := \Omega \cdot A_t \cdot x_o \subset \mathcal{X}$$

of $\Sigma'_{t, \Omega}$ in $\mathcal{X}$ is called a Siegel set in $\mathcal{X}$.

Theorem 3.1. [4, theor.13.1] Let $X$, $G$, $\mathbf{G}$, $\Gamma$, $P$, $A$, $K_\infty$, and $\mathcal{X}$ be as above. Then for any Siegel set $\Sigma_{t, \Omega}$, the set $\{\gamma \in \Gamma \mid \gamma \Sigma_{t, \Omega} \cap \Sigma_{t, \Omega} \neq \emptyset\}$ is finite. There exist a Siegel set (called a Siegel set for $\Gamma$) $\Sigma_{t_0, \Omega}$ and a finite subset $J$ of $\mathbf{G}(\mathbb{Q})$ such that $\mathcal{F} := J \cdot \Sigma_{t_0, \Omega}$ is a fundamental set for the action of $\Gamma$ on $\mathcal{X}$.

When $\Omega$ is chosen to be semi-algebraic the Siegel set $\Sigma_{t, \Omega}$ and the fundamental set $\mathcal{F}$ are semi-algebraic as by definition of a complex realisation (cf. appendix [3]) the action of $\mathbf{G}(\mathbb{R})$ on $\mathcal{X}$ is semi-algebraic and the subset $\Omega \cdot A_t$ of $\mathbf{G}(\mathbb{R})$ is semi-algebraic.

We will only consider semi-algebraic Siegel sets in the rest of the text.
3.2. Boundary components. General references for this section and the next one are [18] and [1].

Let $\mathcal{D} \hookrightarrow \mathbb{C}^N$ be the Harish-Chandra realisation of $X$ as a bounded symmetric domain. The action of $G$ extends to the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ in $\mathbb{C}^N$. The boundary $\partial \mathcal{D} := \overline{\mathcal{D}} \setminus \mathcal{D}$ is a smooth manifold which decomposes into a (continuous) union of boundary components, which are defined as maximal complex analytic submanifolds of $\partial \mathcal{D}$ (or alternatively as holomorphic path components of $\partial \mathcal{D}$). Explicitly, let us say that a real affine hyperplane $H \subset \mathbb{C}^N$ is a supporting hyperplane if $H \cap \mathcal{D}$ is nonempty but $H \cap \overline{\mathcal{D}}$ is empty. Let $H$ be a supporting hyperplane and let $\mathcal{F} = H \cap \mathcal{D} = H \cap \partial \mathcal{D}$. Let $L$ be the smallest affine subspace of $\mathbb{C}^N$ which contains $\mathcal{F}$. Then $\mathcal{F}$ is the closure of a nonempty open subset $\mathcal{F} \subset L$ which is then a single boundary component of $\mathcal{D}$ (cf. [27, §III.8.11]). The boundary component $\mathcal{F}$ turns out to be a bounded symmetric domain in $L$.

Fix a boundary component $\mathcal{F}$. The normaliser $N(\mathcal{F}) := \{ g \in G \mid g \mathcal{F} = \mathcal{F} \}$ turns out to be a proper parabolic subgroup of $G$. The Levi decomposition $N(\mathcal{F}) = R(\mathcal{F}) \cdot W(\mathcal{F})$ (where $W(\mathcal{F})$ denotes the unipotent radical of $N(\mathcal{F})$ and $R(\mathcal{F})$ is the unique reductive Levi factor stable under the Cartan involution corresponding to $K$) can be refined into

$$N(\mathcal{F}) = (G_h(\mathcal{F}) \cdot G_l(\mathcal{F}) \cdot M(\mathcal{F})) \cdot V(\mathcal{F}) \cdot U(\mathcal{F}),$$

where:
- $U(\mathcal{F})$ is the centre of $W(\mathcal{F})$. It is a real vector space;
- $V(\mathcal{F}) = W(\mathcal{F})/U(\mathcal{F})$ turns out to be abelian. It is a real vector space of even dimension $2l$, and we get a decomposition $W(\mathcal{F}) = V(\mathcal{F}) \cdot U(\mathcal{F})$ using “exp”;
- $G_l(\mathcal{F}) \cdot M(\mathcal{F}) \cdot V(\mathcal{F}) \cdot U(\mathcal{F})$ acts trivially on $\mathcal{F}$ and $G_h(\mathcal{F})$ modulo a finite center is $\text{Aut}^0(\mathcal{F})$;
- $G_h(\mathcal{F}) \cdot M(\mathcal{F}) \cdot V(\mathcal{F}) \cdot U(\mathcal{F})$ commutes with $U(\mathcal{F})$ and $G_l(\mathcal{F})$ modulo a finite central group acts faithfully on $U(\mathcal{F})$ by inner automorphisms;
- $M(\mathcal{F})$ is compact.

The boundary component $\mathcal{F}$ is said to be rational if $\Gamma_\mathcal{F} := \Gamma \cap N(\mathcal{F})$ is an arithmetic subgroup of $N(\mathcal{F})$. There are only finitely many $\Gamma$-orbits of rational boundary components, we choose representatives $\mathcal{F}_1, \ldots, \mathcal{F}_r$ for these $\Gamma$-orbits. Then the Baily-Borel compactification of $S$ is

$$\overline{S}^{BB} = S \cup \bigcup_{i=1}^r (\Gamma_{\mathcal{F}_i} \setminus \mathcal{F}_i)$$

with a suitable analytic structure.

3.3. Toroidal compactifications and local coordinates. Let $X^\vee$ be the compact dual of $X$ and $\mathcal{D} \hookrightarrow X^\vee$ be the Borel embedding. Recall that $X^\vee$ has an algebraic action by $G_{\mathbb{C}}$. Given a boundary component $\mathcal{F}$ of $\mathcal{D}$ we define, following [18, section 3], an open subset $\mathcal{D}_\mathcal{F}$ of $X^\vee$ containing $\mathcal{D}$ as follows:

$$\mathcal{D}_\mathcal{F} = \bigcup_{g \in U(\mathcal{F})_{\mathbb{C}}} g \cdot \mathcal{D}. $$
The embedding of $D$ in $D_F$ is Piatetskii-Shapiro’s realisation of $D$ as Siegel Domain of the third kind. In fact there is a canonical holomorphic isomorphism (we refer to the proof of lemma 4.2 for a precise description of this isomorphism):

$$D_F \overset{j}{\cong} U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F .$$

This biholomorphism defines complex coordinates $(x,y,t)$ on $D_F$, such that

$$D \overset{j}{\cong} \{(x,y,t) \in U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \mid \text{Im}(x) + l_t(y,y) \in C(F)\} \subset D_F$$

where $\text{Im}(x)$ is the imaginary part of $x$, $C(F) \subset U(F)$ is a self-adjoint convex cone homogeneous under the $G_t(F)$-action on $U(F)$ and $l_t : \mathbb{C}^l \times \mathbb{C}^l \to U(F)$ is a symmetric $\mathbb{R}$-bilinear form varying real-analytically with $t \in F$. The group $U(F)_{\mathbb{C}}$ acts on $D_F$ and in these coordinates the action of $a \in U(F)(\mathbb{C})$ is given by:

$$(x,y,t) \mapsto (x + a, y, t).$$

From now on we fix a $\Gamma$-admissible collection of polyhedra $\sigma = (\sigma_\alpha)$ (cf. [1, definition 5.1]) such that the associated toroidal compactification $\overline{S} = \overline{S}_\sigma$ constructed in [1] is smooth projective and the complement $\overline{S} \setminus S$ is a divisor with normal crossings. We refer to [1] for details and we just recall what is needed for our purposes.

The compactification $\overline{S}$ is covered by a finite set of coordinates charts constructed as follows (cf. [18] p.255-256):

(a) Take a rational boundary component $F$ of $D$;

(b) We may choose some complex coordinates $x = (x_1, \ldots, x_k)$ on $U(F)_{\mathbb{C}}$ (depending on the choice of $\Sigma$) such that the following diagram commutes:

$$\begin{array}{ccc}
D^c & \overset{j}{\cong} & U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \\
\downarrow \exp_F & & \downarrow \exp_F \\
\pi_F \quad \pi_F & & \quad \pi_F \\
S & \subset & \mathbb{C}^k \times \mathbb{C}^l \times F \\
\end{array}$$

where $\exp_F : U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \to \mathbb{C}^k \times \mathbb{C}^l \times F$ is given by

$$(x,y,t) \mapsto (\exp(2i\pi x), y, t), \quad \text{where} \quad \exp(2i\pi x) = (\exp(2i\pi x_1), \ldots, \exp(2i\pi x_k)).$$

(c) Define the “partial compactification of $\exp_F(D)$ in the direction $F$” to be the set $\exp_F(D)^\vee$ of points $P$ in $\mathbb{C}^k \times \mathbb{C}^l \times F$ having a neighborhood $\Theta$ such that

$$\Theta \cap \mathbb{C}^k \times \mathbb{C}^l \times F \subset \exp_F(D)^\vee .$$

Then there exists an integer $m$, $1 \leq m \leq k$, such that $\exp_F(D)^\vee$ contains

$$S(F, \sigma) = \bigcup_{i=1}^m \{(z,y,t) \mid z = (z_1, \ldots, z_k), z_i = 0\}.$$
(d) The basic property of $S$ is that the covering map $\pi_F : \exp_F(D) \to S$ extends to a local homeomorphism $\pi_F : \exp_F(D) \to S$ making the diagram

\[
\begin{array}{ccc}
D & \xrightarrow{\exp_F} & \exp_F(D) \\
\pi & \downarrow & \downarrow \pi_F \\
S & \xrightarrow{\exp_F} & S
\end{array}
\]

commutative. Moreover every point $P$ of $S - S$ is of the form $\pi_F((z, y, t))$ with $z_i = 0$ for some $i \leq m$, for some $F$.

The following proposition summarizes what we will need:

**Proposition 3.2.** Let $\Sigma = \Sigma_{t, \Omega} \subset D$ be a Siegel set for the action of $\Gamma$. Then $\Sigma$ is covered by a finite number of open subsets $\Theta$ having the following properties. For each $\Theta$ there is a rational boundary component $F$, a simplicial cone $\sigma \subset C(F)$, a point $a \in C(F)$, relatively compact subsets $U', Y'$ and $F'$ of $U(F)$, $C_l$ and $F$ respectively such that the set $\Theta$ is of the form

\[
\Theta \cong \{ (x, y, t) \in U(F)_C \times C_l \times F, \Re(x) \in U', y \in Y', t \in F' \mid \Im(x) + l_t(y, y) \in \sigma + a \}
\]

where $E \subset F$ and $\omega_W \subset W(F)$ are compact, $C_0 \subset C(F)$ is a rational core and $\sigma_0^F$ is one of the polyhedra in our decomposition of $C(F)$.

Considering $C(F)$ as a cone in $\sqrt{-1} \cdot U(F)$ and decomposing $W(F)$ as $U(F) \cdot V(F)$, the isomorphism $\Psi$ extends to the real-analytic isomorphism $D_F \cong U(F)_C \times V(F) \times F$ constructed in [11, p.235]. Hence the Siegel set $\Sigma$ is covered by a finite number of sets $\Theta$ of the form

\[
\Theta \cong \Psi(D) \cap \{ (x, s, t) \in U(F)_C \times V(F) \times F \mid \Re(x) \in U', s \in S', t \in F' \}
\]

where $F' \subset F$, $U' \subset U(F)$ and $S' \subset V(F)$ are relatively compact.

Using the definition of $j$ given in [33, §7] and recalled in the proof of lemma [42] below, it follows, as stated in [11, p.238], that the diffeomorphism $j \circ \Psi^{-1} : U(F)_C \times V(F) \times F \cong
$U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F$ is a change of trivialisation of the real-analytic bundle

\[
\begin{array}{c}
\xymatrix{
\mathcal{D}_F 
\ar@{->}[d]_{\pi_F'} 
\ar@{->}[rr]_{\pi_F} 
\ar@{->}[ddr]_{p_F} 
\ar@{->}[d]_{\pi_F} 
\ar@{->}[ddr]_{p_F} 
\ar@{->}[ddr]_{p_F} 

& 
\mathcal{D}_F' 
\ar@{->}[d]_{\pi_F'} 
\ar@{->}[rr]_{\pi_F} 
\ar@{->}[ddr]_{p_F} 
\ar@{->}[ddr]_{p_F} 
\ar@{->}[ddr]_{p_F} 

& 
F 
\end{array}
\]

studied in \cite[p.237]{1}. Here the map $\pi_F'$ is a $U(F)_{\mathbb{C}}$-principal homogeneous space, the map $p_F$ is a $V(F)$-principal homogeneous space, and the map $j \circ \Psi^{-1}$ is $U(F)_{\mathbb{C}}$-equivariant and respects the fibrations over $F$. These two properties ensure that $j \circ \Psi^{-1}$ identifies the set $\Psi(\Theta)$ of \eqref{3.5} to a set of the required form

\[
\Theta \xrightarrow{j} \{ (x,y,t) \in U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F, \, \text{Re}(x) \in U', \, y \in Y', \, t \in F' \mid \text{Im}(x) + l_t(y,y) \in \sigma + a \} 
\subset U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F.
\]

□

4. Definability of the uniformisation map: proof of theorem \ref{1.2}

First notice that, although the variety $S$ does not canonically embed into some $\mathbb{R}^n$, the statement of theorem \ref{1.2} makes sense as $S$ has a canonical structure of real algebraic manifold, hence of $\mathbb{R}_{\text{an,exp}}$-manifold: cf. appendix \ref{A}.

By theorem \ref{3.1} there exist a semi-algebraic Siegel set $\Sigma$ and a finite subset $J$ of $G(\mathbb{Q})$ such that $\mathcal{F} := J \cdot \Sigma$ is a (semi-algebraic) fundamental set for the action of $\Gamma$ on $\mathcal{D}$. Hence theorem \ref{1.2} follows from the following more precise result.

**Theorem 4.1.** The restriction $\pi|_{\Sigma} : \Sigma \to S$ of the uniformising map $\pi : \mathcal{D} \to S$ is definable in $\mathbb{R}_{\text{an,exp}}$.

**Proof.** By the proposition \ref{3.2} we know that $\Sigma$ is covered by a finite union of open subsets $\Theta$ with the following properties. For each $\Theta$ there is a rational boundary component $F$, a simplicial cone $\sigma \in \mathfrak{c}$ with $\sigma \subset C(F)$, a point $a \in C(F)$, relatively compact subsets $U'$, $Y'$ and $F'$ of $U(F)$, $\mathbb{C}^l$ and $F$ respectively such that the set $\Theta$ is of the form

\[
\Theta \xrightarrow{j} \{ (x,y,t) \in U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F, \, \text{Re}(x) \in U', \, y \in Y', \, t \in F' \mid \text{Im}(x) + l_t(y,y) \in \sigma + a \} 
\subset U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F.
\]

We first prove that the holomorphic coordinates we introduced on $\mathcal{D}_F$ are definable:

**Lemma 4.2.** The canonical isomorphism $j : \mathcal{D}_F \simeq U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F$ is semi-algebraic.

**Proof.** The isomorphism $j$ was studied in \cite{21} and in full generality in \cite[§7]{33} (cf. \cite[§1.6]{3} for a survey). To keep the amount of definitions at a reasonable level we follow in this
the composition of the semi-algebraic holomorphic maps $F$ associated to a rational boundary component $p$ bounded symmetric domain of $\pi p$ is proven in \[33, \text{theorem 6.8}\]. In particular it follows that $\exp(4.2)$

\[ \alpha \in \{33, \text{p.901}\}. \] For any form of the complexified group $G\alpha$ to $\theta$ partial Cayley transform associated with $U$ subsets respectively of $p$ algebraic: indeed in restriction to $p$ algebraic: indeed

\[ \xi \in \{33, \text{remark 3 p.932}\} \]

Following \[33, \text{§4.1}\]) the map $j : D \longrightarrow U(F)\subset \mathbb{C}^l \times F \subset U(F)\subset \mathbb{C}^l \times p^- \leftarrow$ is the composition of the semi-algebraic holomorphic maps

\[ D \xrightarrow{\xi^{-1}c_{\Delta-\theta}} p^- = p_{\Delta-\theta,1} \oplus p_{2}^{-} \oplus p_{\theta}^{-} \xrightarrow{\exp \circ \ad c_{\Delta-\theta}} U(F)\subset \mathbb{C}^l \times p_{\theta}^{-} \]

which finishes the proof of lemma 4.2

The previous lemma enables us to forget about the definable biholomorphism $j$. From now on and for simplicity of notations we simply write $D_F = U(F)\subset \mathbb{C}^l \times F$.

In the description (4.1) we may and do assume that $U'$, $Y'$ and $F'$ are semi-algebraic subsets respectively of $U(F)\subset \mathbb{C}^l$ and $F$. Then the set $\Theta$ is definable in $\mathbb{R}_{\text{an}}$, because:

- the function $\psi : \ Y' \times F' \to U(F)$ defined by $\psi(y,t) = l_t(y,y)$ is analytic and defined on a compact semi-algebraic set.
- the cone $\sigma$ is polyhedral, hence semi-algebraic.

Hence the restriction $\pi|_{\Sigma} : \Sigma \to S$ is definable in $\mathbb{R}_{\text{an,exp}}$ if and only if the restriction $\pi|_{\Theta} : \Theta \to S$ to any set $\Theta$ appearing in the proposition 3.2 is definable in $\mathbb{R}_{\text{an,exp}}$.

Fix such a set

\[ \Theta = \{(x,y,t), y \in Y', t \in F', \text{Re}(x) \in U'^l | \text{Im}(x) + l_t(y,y) \in \sigma + a\} \]

associated to a rational boundary component $F \in \{F_1, \ldots, F_r\}$. 
Consider the left-hand side of the diagram (3.4):

\[
\begin{array}{c}
D \xrightarrow{\exp_F} F \\
\downarrow \quad \downarrow \\
\pi_F \quad \pi_F \\
\exp_F(D) \xleftarrow{\pi} S \\
\end{array}
\]

Recall that \( \exp_F : F \to \mathbb{C}^* \times \mathbb{C}^l \times F \) is given by

\[
(x, y, t) \mapsto (\exp(2i\pi x, y, t), \ldots, \exp(2i\pi x_k))
\]

The function \( \Re(x_i), 1 \leq i \leq k \), is bounded on \( \Theta \) hence the restriction to \( \Theta \) of the map \( x \mapsto \exp(2i\pi \Re(x)) \) is definable in \( \mathbb{R}_{\text{an}}. \) On the other hand the restriction to \( \Theta \) of the function \( x \mapsto \exp(-2\pi \Im(x)) \) is definable in \( \mathbb{R}_{\exp} \) by definition of \( \mathbb{R}_{\exp}. \) Thus the restriction to \( \Theta \) of the map \( \pi_F : \exp_F(\Theta) \to S \) is definable in \( \mathbb{R}_{\text{an}, \exp}. \)

Consider the lower part of the diagram (3.4):

\[
\begin{array}{c}
\exp_F(D) \xleftarrow{\pi_F} \exp_F(D) \\
\downarrow \quad \downarrow \\
\pi_F \quad \pi_F \\
S \xrightarrow{\exp_F} S
\end{array}
\]

As \( U', V', F' \) are relatively compact and the imaginary part of \( x \) has a lower bound on \( \Theta, \) the closure \( \overline{\exp_F(\Theta)} \) of \( \exp_F(\Theta) \) is compact in \( \exp_F(D) \). Hence \( \pi_F : \exp_F(\Theta) \to S, \) which is the restriction of the analytic map \( \overline{\pi_F} : \exp_F(D) \to S \) to the relatively compact subset \( \overline{\exp_F(\Theta)} \) of \( \exp_F(D) \), is definable in \( \mathbb{R}_{\text{an}}. \)

\[ \square \]

5. Growth of the height: proof of theorem 1.3

In this section we prove theorem 1.3.

5.1. Comparing norm and distance.

**Lemma 5.1.** For any \( g \in G(\mathbb{R}) \) the following inequality holds:

\[
\log \|g\|_\infty \leq d(g \cdot x_0, x_0)
\]

**Proof.** Let \( G(\mathbb{R}) = K_\infty \cdot A_\infty \cdot K_\infty \) be a Cartan decomposition of \( G(\mathbb{R}) \) associated to \( K_\infty, \)

where \( A_\infty \) is a maximal split real torus of \( G \) containing \( A. \) Let \( g \in G(\mathbb{R}) \) and write

\[ g = k_1 \cdot a \cdot k_2 \]

its Cartan decomposition, with \( k_1, k_2 \in K_\infty \) and \( a \in A_\infty. \) As \( \| \cdot \|_\infty \)

is \( K_\infty \)-bi-invariant and \( d \) is \( G(\mathbb{R}) \)-equivariant the equalities \( \log \|g\|_\infty = \log \|a\|_\infty \) and

\[ d(g \cdot x_0, x_0) = d(a \cdot x_0, x_0) \]

do hold.
The torus $A_\infty$ is diagonalisable in an orthonormal basis $(f_1, \ldots, f_n)$ of $E_\mathbb{R}$. Write $a = \text{diag}(a_1, \ldots, a_n)$ in this basis, then:

$$\log \|a\|_\infty = \max_i \log |a_i| \quad \text{and} \quad d(a \cdot x_0, x_0) = \sqrt{\sum_{i=1}^{n} (\log |a_i|)^2}$$

hence the result. 

5.2. Comparing norm and height.

**Lemma 5.2.** There exists a constant $B > 0$ such that the following holds. Let $\gamma \in G(\mathbb{Z})$ and $u \in \gamma \mathcal{F}$. Then

$$H(\gamma) \leq B \cdot \|u\|_\infty .$$

**Proof.** Write $u = \gamma \cdot j \cdot x$ with $j \in J$ and $x = \omega \cdot a \cdot k \in \Sigma_{t_0, \Omega} = \Omega \cdot A_{t_0} \cdot K_\infty$. Hence $\gamma = u \cdot k^{-1} \cdot a^{-1} \cdot \omega^{-1} \cdot j^{-1}$. As the operator norm $\| \cdot \|$ is sub-multiplicative, one obtains:

$$H(\gamma) = \|\gamma\|_\infty \leq \|u\|_\infty \cdot \|a^{-1}\|_\infty \cdot \|\omega^{-1} \cdot j^{-1}\|_\infty ,$$

where we used that $\| \cdot \|_\infty$ is constant equal to 1 on $K_\infty$. As $\Omega$ is compact and $J$ is finite, the norm of $\omega^{-1} \cdot j^{-1}$ is bounded by a constant independant of $\gamma$. As $a$ belongs to $A_{t_0} \subset A(\mathbb{R})$ the inequality $(a^{-1})^\alpha \leq \max(1, 1/t_0)$ holds for any simple root $\alpha$ of $\text{GL}(E)$ associated to $A$ and a Borel subgroup of $\text{GL}(E)$ containing $P$. Hence $\|a^{-1}\|_\infty$, which is the maximal absolute value of an eigenvalue of $a^{-1}$, is bounded independently of $\gamma$. The result follows.

5.3. **Lower bound for the volume of an algebraic curve.** In [10, Corollary 3 p.1227], Hwang and To prove the following lower bound for the area of any complex analytic curve in $\mathcal{D}$:

**Theorem 5.1** (Hwang and To). Let $C$ be a complex analytic curve in $\mathcal{D}$. For any point $x_0 \in C$ there exist positive constants $a_1, c_1$ such that for any positive real number $R$ one has:

$$\text{Vol}_C(C \cap B(x_0, R)) \geq a_1 \exp(c_1 \cdot R) .$$

Here $\text{Vol}_C$ denotes the area for the Riemannian metric on $C$ restriction of the metric $g_X$ on $\mathcal{D}$ and $B(x_0, R)$ denotes the geodesic ball of $\mathcal{D}$ with center $x_0$ and radius $R$.

5.4. **Upper bound for the volume of algebraic curves on Siegel sets.**

**Lemma 5.3.** (i) There exists a constant $A_0 > 0$ such that for any algebraic curve $C \subset \mathcal{D}$ of degree $d$ we have the bound

$$\text{Vol}_C(C \cap \Sigma) \leq A_0 \cdot d .$$
(ii) There exists a constant $A > 0$ such that for any algebraic curve $C \subset D$ of degree $d$ we have the bound
\[ \text{Vol}_C(C \cap \mathcal{F}) \leq A \cdot d. \]

Proof. We first prove (i). Recall that $\Sigma$ is covered by a finite union of open subsets $\Theta$ described in proposition 3.2: there is a rational boundary component $F$, a simplicial cone $\sigma \in \Sigma$ with $\sigma \subset C(F)$, a point $a \in C(F)$, relatively compact subsets $U'$, $Y'$ and $F'$ of $U(F)$, $\mathbb{C}$ and $F$ respectively such that the set $\Theta$ is of the form
\[ \Theta = \{(x,y,t) \in D_F, y \in Y', t \in F', \Re(x) \in U', \Im(x) + l_t(y, y) \in \sigma + a\} \subset D_F = U(F) \times \mathbb{C} \times F'. \]
Recall that $\omega$ denotes the natural Kähler form on $X$. As $C \subset X$ is a complex analytic curve, one has:
\[ \text{Vol}_C(C \cap \Theta) = \int_{C \cap \Theta} \omega. \]
On the other hand let $\omega_{D_F}$ be the Poincaré metric on $D_F$ defined in the Siegel coordinates by:
\[ \omega_{D_F} = \sum \frac{dx_i \wedge d\overline{x}_i}{\Im(x_i)^2} + \sum dy_j \wedge d\overline{y}_j + \sum df_k \wedge d\overline{f}_k. \]
Mumford [18, Theor.3.1] proved that there exists a positive constant $c$ such that on $D$:
\[ \omega \leq c \cdot \omega_{D_F}. \]
Hence:
\[ \text{Vol}_C(C \cap \Theta) \leq c \int_{C \cap \Theta} \omega_{D_F}. \]
Let $p_{x_i}$, $p_{y_j}$ and $p_{f_k}$ be the projections on $D_F$ to the coordinates $x_i$, $y_j$ and $f_k$.
As the curve $C$ has degree $d$ the restriction of these maps to $C \cap \Theta$ are either constant or at most $d$ to 1, hence
\[ \text{Vol}_C(C \cap \Theta) \leq c \cdot d \cdot (\sum \int_{p_{x_i}(\Theta)} \frac{dx_i \wedge d\overline{x}_i}{\Im(x_i)^2} + \sum \int_{p_{y_j}(\Theta)} dy_j \wedge d\overline{y}_j + \sum \int_{p_{f_k}(\Theta)} df_k \wedge d\overline{f}_k). \]
Let $i$ be such that the map $p_{x_i}$ is not constant. In view of the description of $\Theta$ the projection $p_{x_i}(\Theta)$ is contained in a usual fundamental set of the upper-half plane, of finite hyperbolic area.
Let $w$ be a coordinate $y_j$, $f_k$ and $p_w$ be the associated projection on the $w$ axis. By the definition of $\Theta$ the projection $p_w(\Theta)$ is a relatively compact open set of the plane, hence of finite Euclidean area.
This finishes the proof of (i).

Let us prove (ii). As $C \cap \mathcal{F} = C \cap J \cdot \Sigma$, one has the inequality:
\[ \text{Vol}_C(C \cap \mathcal{F}) \leq \sum_{j \in J} \text{Vol}_C(C \cap j \cdot \Sigma) = \sum_{j \in J} \text{Vol}_{j^{-1}C}(j^{-1}C \cap \Sigma) \leq |J| \cdot A_0 \cdot d \]
where we used part (i) applied to the algebraic curves $j^{-1}C$ of $D$, $j \in J$, which are of degree $d$.
This finishes the proof of lemma 5.3. \qed
5.5. Proof of theorem 1.3. Choose $C \subseteq Y$ an irreducible algebraic curve. To prove theorem 1.3 for $Y$ it is enough to prove it for $C$.

Consider the set 
\[ C(T) := \{ z \in C \text{ and } \|z\|_\infty \leq T \} \]
As $F$ is a fundamental domain for the action of $\Gamma$ one has on the one hand:
\[ C(T) = \bigcup_{\gamma \in \Gamma} \{ u \in \gamma F \cap C \text{ and } \|u\|_\infty \leq T \} \]
\[ \subseteq \bigcup_{\gamma \in \Gamma} \{ u \in \gamma F \cap C \} \text{ by lemma 5.2} \]
Taking volumes:
\[ \text{Vol}_C(C(T)) \leq \sum_{\gamma \in \Gamma} \text{Vol}_C(\gamma^{-1}C) \]
hence
\[ (5.2) \quad \text{Vol}_C(C(T)) \leq (A \cdot d) \cdot N_C(B \cdot T) \]
where we applied lemma 5.3(ii) to the algebraic curves $\gamma^{-1}C$, $\gamma \in \Gamma$, which are all of degree $d$.

On the other hand if follows from lemma 5.1 that
\[ C \cap B(x_0, \log T) \subseteq C(T) \]
hence
\[ (5.3) \quad \text{Vol}_C(C \cap B(x_0, \log T)) \leq \text{Vol}_C(C(T)) \]
The result now follows from inequalities (5.2), (5.3) and theorem 6.1.

6. Stabilisers of a maximal algebraic subset: proof of theorem 1.4.

6.1. Pila-Wilkie theorem.

**Definition 6.1.** The classical height $H_{\text{class}}(x)$ of a point $x = (x_1, \ldots, x_m) \in \mathbb{Q}^m$ is defined as
\[ H_{\text{class}}(x) = \max(H(x_1), \ldots, H(x_m)) \]
where $H$ is the usual multiplicative height of a rational number.

Let $Z \subseteq \mathbb{R}^m$ be a subset and $T \geq 0$ a real number, we define:
\[ \Psi_{\text{class}}(Z, T) := \{ x \in Z \cap \mathbb{Q}^m : H(x) \leq T \} \]
and
\[ N_{\text{class}}(Z, T) := |\Psi(Z, T)| \]
For $Z \subset \mathbb{R}^m$ a definable set in a o-minimal structure we define the algebraic part $Z^{\text{alg}}$ of $Z$ to be the union of all positive dimensional semi-algebraic subsets of $Z$.

Recall (cf. definition 3.3 of [31]), that a semi-algebraic block of dimension $w$ in $\mathbb{R}^m$ is a connected definable set $W \subset \mathbb{R}^m$ of dimension $w$, regular at every point, such that there exists a semi-algebraic set $A \subset \mathbb{R}^m$ of dimension $w$, regular at every point with $W \subset A$.

The following result is a strong form, proven by Pila [22, theorem 3.6], of the original theorem of Pila and Wilkie [23]:

**Theorem 6.1** (Pila-Wilkie). Let $Z \subset \mathbb{R}^m$ be a definable set in a o-minimal structure. For every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$N_{\text{class}}(Z \setminus Z^{\text{alg}}, T) < C_\varepsilon T^\varepsilon$$

and the set $\Psi_{\text{class}}(Z, T)$ is contained in the union of at most $C_\varepsilon T^\varepsilon$ semi-algebraic blocks.

6.2. Comparison of heights.

**Lemma 6.2.** Let $H_{\text{class}}$ be the classical height defined on $\text{End } E$ using the basis $(e_i^* \otimes e_j)_{i,j}$. There exists a constant $C > 1$ such that, if $H$ is the height on $\text{End } E$ as in definition 2.2, then:

$$\forall \varphi \in \text{End } E, \quad \frac{1}{C} \cdot H_{\text{class}}(\varphi) \leq H(\varphi) \leq C \cdot H_{\text{class}}(\varphi).$$

**Proof.** Let $v \in \mathcal{P}$ and define the classical norm $|\cdot|_v$ on $\text{End } E_v$ as follows. Given $\varphi \in \text{End } E_v$ and $(\varphi)_{i,j}$ its matrix in the $\mathbb{Q}_v$-basis $(e_1, \ldots, e_r)$ or $E_v$, one defines $|\varphi|_v := \max_{i,j} |\varphi_{i,j}|_v$.

The lemma follows immediately from the classical fact that for any finite $v \in \mathcal{P}$ and any $\varphi \in \text{End } E_v$ one has:

$$|\varphi|_v \leq \|\varphi\|_v \leq (\dim_{\mathbb{Q}} E) \cdot |\varphi|_v.$$

□

As a corollary, theorem 6.1 still holds if one replaces $H_{\text{class}}$ by $H$:

**Corollary 6.2.** Let $Z \subset \text{End } E_\mathbb{R}$ be a definable set in a o-minimal structure. Define $\Psi(Z, T) := \{ x \in Z \cap \text{End } E : H(x) \leq T \}$ and $N(Z, T) := |\Psi(Z, T)|$. For every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$N(Z \setminus Z^{\text{alg}}, T) < C_\varepsilon T^\varepsilon$$

and the set $\Psi(Z, T)$ is contained in the union of at most $C_\varepsilon T^\varepsilon$ semi-algebraic blocks.

6.3. **Proof of theorem 1.4.** Let $V$ be an algebraic subvariety of $S$ and $Y$ a maximal irreducible algebraic subvariety of $\pi^{-1}V$. Let $\Theta_Y$ be the stabiliser of $Y$ in $G(\mathbb{R})$ and $H_Y$ be the neutral component of the Zariski-closure of $G(\mathbb{Z}) \cap \Theta_Y$ in $G$. We want to show that $H_Y$ is a non-trivial subgroup of $G$, acting non-trivially on $X$.

Via $\rho : G \rightarrow \text{GL}(E)$, we view $G(\mathbb{R})$ as a semi-algebraic (and hence definable) subset of $\text{End } E_\mathbb{R}$. As $\pi|_\mathcal{F} : \mathcal{F} \rightarrow S$ is definable by theorem 1.2 lemmas 5.1 and 5.2 of [31] show the following:
Proposition 6.3. Let us define
\[ \Sigma(Y) = \{ g \in G(\mathbb{R}) : \dim(gY \cap \pi^{-1}V \cap F) = \dim(Y) \} \]
and \[ \Sigma'(Y) = \{ g \in G(\mathbb{R}) : g^{-1}F \cap Y \neq \emptyset \}. \]

The following properties hold:
1. The set \( \Sigma(Y) \) is definable and for all \( g \in \Sigma(Y) \), \( gY \subset \pi^{-1}V \).
2. For all \( \gamma \in \Sigma(Y) \cap G(\mathbb{Z}) \), \( \gamma Y \) is a maximal algebraic subset of \( \pi^{-1}V \).
3. The following equality holds:
\[ \Sigma(Y) \cap G(\mathbb{Z}) = \Sigma'(Y) \cap G(\mathbb{Z}) . \]

It follows that the number \( N_Y(T) \) defined in theorem \ref{thm:1.3} coincide with \( |\Theta(Y, T)| \), where
\[ \Theta(Y, T) := G(\mathbb{Z}) \cap \Psi(\Sigma(Y), T) . \]

We can now finish the proof of the theorem \ref{thm:1.4} in exactly the same way as the proof
of theorem 5.4 of \cite{31}. For the sake of completeness, we reproduce it here. As \( \Theta(Y, T) \subset \Psi(\Sigma(Y), T) \) it follows from the version \ref{thm:6.2} of Pila-Wilkie’s theorem, that for \( T \) large
enough, the set \( \Theta(Y, T^{1/N}) \) is contained in at most \( T^{1/N} \) semi-algebraic blocks. As \( |\Theta(Y, T^{1/N})| = N_Y(T^{1/N}) \geq T^{1/N} \) by theorem \ref{thm:1.3} we see that there is a semi-algebraic block \( W \) in \( \Sigma(Y) \)
containing at least \( T^{1/N} \) elements \( \gamma \in \Sigma(Y) \cap G(\mathbb{Z}) \) such that \( H(\gamma) \leq T^{1/N} \).

Using lemma 5.5 of \cite{30} which applies verbatim in our case, we see that there exists an
element \( \sigma \) in \( \Sigma(Y) \) such that \( \sigma \Theta_Y \) contains at least \( T^{1/N} \) elements \( \gamma \in \Sigma(Y) \cap G(\mathbb{Z}) \) such that \( H(\gamma) \leq T^{1/N} \).

Let \( \gamma_1 \) and \( \gamma_2 \) be two elements of \( \sigma \Theta_Y \cap G(\mathbb{Z}) \) such that \( H(\gamma) \leq T^{1/N} \).

Let \( \gamma := \gamma_2^{-1} \gamma_1 \in G(\mathbb{Z}) \cap \Theta_Y \). Using elementary properties of heights, we see that
\( H(\gamma) \leq c_n T^{1/2} \) where \( c_n \) is a constant depending on \( n \) only. It follows that for all \( T \)
large enough, \( \Theta_Y \) contains at least \( T^{1/N} \) elements \( \gamma \in G(\mathbb{Z}) \) with \( H(\gamma) \leq T \). Hence the
connected component of the identity \( H_Y \) of the Zariski closure of \( G(\mathbb{Z}) \cap \Theta_Y \) in \( G \) is a
positive dimensional algebraic subgroup of \( G \) contained in \( \Theta_Y \). This finishes the proof of
the theorem \ref{thm:1.4}.

7. End of the proof of theorem \ref{thm:1.1}

Let \( V \) be an algebraic subvariety of \( S \). Our aim is to show that maximal irreducible
algebraic subvarieties \( Y \) of \( \pi^{-1}V \) are precisely the irreducible components of the preimages
of maximal weakly special subvarieties contained in \( V \).

Using Deligne’s interpretation of Hermitian symmetric spaces in terms of Hodge theory
the representation \( \rho : G \hookrightarrow GL(E) \) defines a polarized \( \mathbb{Z} \)-variation of Hodge structure on
\( S \). We refer to \cite{17} section 2] for the definition of the Hodge locus of \( X \) and \( S \). Recall
that an irreducible analytic subvariety \( M \) of \( X \) or \( S \) is said to be Hodge generic if it is not
contained in the Hodge locus. If \( M \) is not irreducible we say that \( M \) is Hodge generic if
all the irreducible components of \( M \) are Hodge generic.
Let \( V' \subset V \) be the Zariski closure of \( \pi(Y) \), as \( Y \) is analytically irreducible it easily follows that \( V' \) is irreducible. Replacing \( V \) by \( V' \) we can without loss of generality assume that \( \pi(Y) \) is not contained in a proper algebraic subvariety of \( V \). We now have to show that \( \pi(Y) = V \) and \( V \) is an arithmetic subvariety of \( S \).

Since the group \( G \) is adjoint, it is a direct product \( G = G_1 \times \cdots \times G_r \) where the \( G_i \)'s are the \( \mathbb{Q} \)-simple factors of \( G \). This induces decompositions

\[
G = \prod_{i=1}^r G_i, \quad X = \prod_{i=1}^r X_i, \quad G(\mathbb{Z}) = \prod_{i=1}^r G_i(\mathbb{Z}), \quad \Gamma = \prod_{i=1}^r \Gamma_i, \quad S = \prod_{i=1}^r S_i,
\]

where \( G_i \) is a group of Hermitian type, \( X_i \) its associated Hermitian symmetric domain, \( \Gamma_i \) is an arithmetic lattice in \( G_i \), \( S_i := \Gamma_i \setminus X_i \) is the associated arithmetic variety and \( \pi_i : X_i \to S_i \) the associated uniformization map.

Our main theorem 1.1 is then a consequence of the following:

**Theorem 7.1.** Let \( \tilde{V} \) be the an analytic irreducible component of \( \pi^{-1}V \) containing \( Y \). In the situation described above, after, if necessary, reordering the factors, one has

\[
\tilde{V} = X_1 \times \cdots \times X_t \times \tilde{V}_{>t}
\]

where \( \tilde{V}_{>t} \) is an analytic subvariety of \( X_{t+1} \times \cdots \times X_r \) (in particular if \( r = 1 \) then \( \tilde{V} = X_1 = X \)).

We first show:

**Proposition 7.1.** Theorem 7.1 implies the main theorem 1.1.

**Proof.** Let \( t, 1 \leq t \leq r \), be the largest integer such that, after reordering the factors if necessary, we have:

\[
\tilde{V} = X_1 \times \cdots \times X_t \times \tilde{V}_{>t}
\]

with \( \tilde{V}_{>t} \) an analytic irreducible subvariety of \( X_{t+1} \times \cdots \times X_r \) which does not (after reordering the factors if necessary) decompose into a product \( X_{t+1} \times \tilde{V}_{>t+1} \).

In this case necessarily one has:

\[
Y = X_1 \times \cdots \times X_t \times Y_{>t}
\]

where \( Y_{>t} \) is a maximal algebraic subset of \( \tilde{V}_{>t} \).

Suppose that \( \dim_{\mathbb{C}}(\tilde{V}_{>t}) > 0 \). Let \( x_{\leq t} \) be a special point on \( X_1 \times \cdots \times X_t \) and \( x_{>t} \) be a Hodge generic point of \( Y_{>t} \). Let \( H \subset G \) be the Mumford-Tate group of the point \( (x_{\leq t}, x_{>t}) \) of \( X \) and let \( X_H \subset X \) be the \( H(\mathbb{R}) \)-orbit of \( x \). Replace \( G \) by \( H \) the group of biholomorphisms of \( X_H \), \( X \) by \( X_H \), \( G \) by \( \text{H} \text{ad} \), \( \Gamma \) by \( \Gamma_H \) the projection of \( H(\mathbb{Z}) \) on \( H \), \( S \) by \( S_H := \Gamma_H \setminus X_H \), \( \pi : X \to S \) by \( \pi_H : X_H \to S_H \), \( V \) by \( V_H := \pi_H(x_{\leq t} \times \tilde{V}_{>t}) \) and \( Y \) by \( x_{\leq t} \times Y_{>t} \) and apply theorem 7.1 for these new data: this shows that there exists \( t' > t + 1 \) such that \( \tilde{V}_{>t'} = X_{t+1} \times \cdots \times X_{t'} \times \tilde{V}_{>t'} \). This contradicts the maximality of \( t \).
Hence $\tilde{V}_{>t}$ is a point $(x_{t+1}, \ldots, x_r)$. Thus

$$\tilde{V} = X_1 \times \cdots \times X_t \times (x_{t+1}, \ldots, x_r)$$

is weakly special, in particular algebraic, hence by maximality

$$Y = \tilde{V} = X_1 \times \cdots \times X_t \times (x_{t+1}, \ldots, x_r)$$

and $Y$ is weakly special.

Let us prove theorem 7.1. Let $H_Y$ be the maximal connected $\mathbb{Q}$-subgroup in the stabiliser of $Y$ in $G(\mathbb{R})$. By theorem 1.4 the group $H_Y$ is a non-trivial algebraic subgroup of $G$.

**Lemma 7.2.** The group $H_Y(\mathbb{Q})$ stabilises $\tilde{V}$.

**Proof.** Suppose there exists $h \in H_Y(\mathbb{Q})$ such that

$$\tilde{V} \neq h\tilde{V}.$$ As $Y$ is contained in $\tilde{V} \cap h\tilde{V}$ and $Y$ is irreducible, we can choose an analytic irreducible component $\tilde{V}'$ of $\tilde{V} \cap h\tilde{V}$ containing $Y$. Notice that $\pi(\tilde{V}')$ is an irreducible component, say $V'$, of $V \cap T_h(V)$. As $\dim_{\mathbb{C}}(\tilde{V}') < \dim_{\mathbb{C}}(\tilde{V})$, we have that $\dim_{\mathbb{C}}(V') < \dim_{\mathbb{C}}(V)$.

As $\pi(Y) \subset V'$, this contradicts the assumption that $\pi(Y)$ is Zariski dense in $V$. □

Choose a Hodge generic point $z$ of $V^{\text{sm}}$ (smooth locus of $V$) and a point $\tilde{z}$ of $\tilde{V}$ lying over $z$. Let

$$\rho^{\text{mon}}: \pi_1(V^{\text{sm}}, z) \longrightarrow \text{GL}(E_\mathbb{Z})$$

be the corresponding monodromy representation. We let $\Gamma_V \subset G(\mathbb{Z})$ be the image of $\rho$. By usual topological Galois theory the group $\Gamma_V$ is the subgroup of $G(\mathbb{Z})$ stabilising $\tilde{V}$ (cf. section 3 of [17]), in particular $\Gamma_V$ contains $H_Y(\mathbb{Z})$.

By Deligne’s monodromy theorem (see Theorem 1.4 of [17]), the connected component of the identity $H_{\text{mon}}$ of the Zariski closure $\Gamma_V^{\text{Zar}, \mathbb{Q}}$ of $\Gamma_V$ in $G$ is a normal subgroup of $G$. As $G$ is semi-simple of adjoint type, after reordering the factors we may assume that $H_{\text{mon}}$ coincides with $G_1 \times \cdots \times G_t \times \{1\}$ for some integer $t \geq 1$. In particular $H_Y \subset G_1 \times \cdots \times G_t \times \{1\}$.

We claim that $\Gamma_V$ normalises $H_Y$. Let $\gamma \in \Gamma_V$. Consider the $\mathbb{Q}$-algebraic group $F$ generated by $H_Y$ and $\gamma H_Y \gamma^{-1}$. Then $F(\mathbb{R})^+ \cdot \tilde{V} = \tilde{V}$, where $F(\mathbb{R})^+$ denotes the connected component of the identity of $F(\mathbb{R})$. Hence $F(\mathbb{R})^+ \cdot Y \subset \tilde{V}$. By lemma 7.2 there exists an irreducible (complex) algebraic subvariety $\tilde{Y}$ of $\tilde{V}$ containing $U$, hence $Y$. By maximality of $Y$ one has $\tilde{Y} = Y$ hence

$$F(\mathbb{R})^+ \cdot Y = Y.$$ By maximality of $H_Y$, we have $F = H_Y$. This proves the claim.

As $H_Y$ is normalised by $\Gamma_V$, it is normalised by $H_{\text{mon}} = G_1 \times \cdots \times G_t \times \{1\}$. It follows that (after possibly reordering factors) $H_Y$ contains $G_1 \times \{1\}$. 


The fact that $H_Y(\mathbb{R})$ stabilises $\tilde{V}$ shows (by taking the $H_Y(\mathbb{R})$-orbit of any point of $\tilde{V}$) that $\tilde{V} = X_1 \times \tilde{V}_{\geq 1}$. This concludes the proof of theorem 7.1 and hence of theorem 1.1.

**Appendix A. Definability**

A.1. About theorem 1.2 Let $\mathcal{R}$ be any fixed o-minimal expansion of $\mathbb{R}$ (in our case $\mathcal{R} = \mathbb{R}_{\text{an,exp}}$). Recall [6, chap.10] that a *definable manifold* of dimension $n$ is an equivalence class (for the usual relation) of triple $(X, X_i, \phi_i)_{i \in I}$ where $\{X_i : i \in I\}$ is a finite cover of the set $X$ and for each $i \in I$:

(i) we have injective maps $\phi_i : X_i \longrightarrow \mathbb{R}^n$ such that $\phi_i(X_i)$ is an open, definably connected, definable set.

(ii) each $\phi(X_i \cap X_j)$ is an open definable subset of $\phi_i(X_i)$.

(iii) the map $\phi_{ij} : \phi_i(X_i \cap X_j) \longrightarrow \phi_j(X_i \cap X_j)$ given by $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ is a definable homeomorphism for all $j \in I$ such that $X_i \cap X_j \neq \emptyset$.

We say that a subset $Z \subset X$ is definable (resp. open or closed) if $\phi_i(Z \cap X_i)$ is a definable (resp. open or closed) subset of $\phi_i(X_i)$ for all $i \in I$. A definable map between abstract definable manifolds is a map whose graph is a definable subset of the definable product manifold.

Notice in particular that $X = \mathbb{P}^n \mathbb{C}$ has a canonical structure of a definable manifold (for any $\mathcal{R}$): take $X_i = \mathbb{C}^n = \{[z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n] \in \mathbb{P}^n \mathbb{C}, 0 \leq i \leq n$ where we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. As a corollary any complex quasi-projective variety is canonically a definable manifold. This apply in particular to $S$. In particular the statement of theorem 1.2 has an intrinsic meaning.

**Appendix B. Algebraic subvarieties of $X$**

Recall from [29, section 2.1] that a realisation $\mathcal{X}$ of $X$ for $G$ is any analytic subset of a complex quasi-projective variety $\tilde{X}$, with a transitive holomorphic action of $G(\mathbb{R})$ on $\mathcal{X}$ such that for any $x_0 \in \mathcal{X}$ the orbit map $\psi_{x_0} : G(\mathbb{R}) \longrightarrow \mathcal{X}$ mapping $g$ to $g \cdot x_0$ is semi-algebraic and identifies $G(\mathbb{R})/K_\infty$ with $X$. A morphism of realisations is a $G(\mathbb{R})$-equivariant biholomorphism. By [29, lemma 2.1] any realisation of $X$ has a canonical semi-algebraic structure and any morphism of realisations is semi-algebraic. Hence $X$ has a canonical semi-algebraic structure.

Let $\mathcal{X}$ be a realisation of $X$ for $G$. A subset $Y \subset \mathcal{X}$ is called an *irreducible algebraic subvariety* of $\mathcal{X}$ if $Y$ is an irreducible component of the analytic set $\mathcal{X} \cap \tilde{Y}$ where $\tilde{Y}$ is an algebraic subset of $\tilde{X}$. By [9, section 2] the set $Y$ has only finitely many analytic irreducible components and these components are semi-algebraic. An algebraic subvariety of $\mathcal{X}$ is defined to be a finite union of irreducible algebraic subvarieties of $\mathcal{X}$.

**Lemma B.1.** A subset $Y$ of $\mathcal{X}$ is algebraic if and only if $Y$ is a closed complex analytic subvariety of $\mathcal{X}$ and semi-algebraic in $\mathcal{X}$. 

Proof. Let $Y \subset X$ be a closed complex analytic subvariety of $X$, semi-algebraic in $X$. Without loss of generality we can assume that $Y$ is irreducible as an analytic subvariety, of dimension $d$. Consider the real Zariski-closure $\tilde{Y}$ of $Y$ in the real algebraic variety $\text{Res}_{\mathbb{C}/\mathbb{R}}\tilde{X}$, where $\text{Res}_{\mathbb{C}/\mathbb{R}}\tilde{X}$ denotes the Weil restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$. Let us show that $\tilde{Y}$ has a canonical structure of a complex subvariety of $\tilde{X}$. Choose an affine open cover $(\tilde{X}_i)_{i \in I} \subset \mathbb{A}^n_i$ of $\tilde{X}$ and denote by $\tilde{Y}_i$ the intersection $\tilde{Y} \cap \tilde{X}_i$. Let $i \in I$ such that $\tilde{Y}_i$ is non-empty. As $Y$ is semi-algebraic, $Y$ is open in $\tilde{Y}$ for the Hausdorff topology, hence $Y_i := Y \cap \tilde{X}_i$ is non-empty and open in $\tilde{Y}_i$ for the Hausdorff topology. Consider the Gauss map $\varphi_i$ from the smooth part $\tilde{Y}_i^{sm}$ of $\tilde{Y}_i$ to the real Grassmannian $\text{Gr}^{2d,2n_i}$ of real $2d$-planes of $\text{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{A}^n_i$ associating to a point its tangent space. The map $\varphi_i$ is real analytic and its restriction to the open subset $Y_i^{sm}$ of $\tilde{Y}_i^{sm}$ takes values in the closed real analytic subvariety $\text{Gr}^{d,n_i}_{\mathbb{C}} \subset \text{Gr}^{2d,2n_i}$. By analytic continuation $\varphi_i$ takes values in $\text{Gr}^{d,n_i}_{\mathbb{C}}$. Hence $\tilde{Y}_i$ is a complex algebraic subvariety of $\mathbb{A}^n_i$. As this is true for all $i \in I$, $\tilde{Y}$ is a complex algebraic subvariety of $\tilde{X}$. As $Y \subset \tilde{Y}$ is open and $Y$ is closed analytically irreducible in $X$, it follows that $Y$ is an irreducible component of $X \cap \tilde{Y}$, hence algebraic.

The other implication is clear. \hfill $\Box$

As any morphism of realisations is an analytic biholomorphism and semi-algebraic the previous lemma implies immediately:

Corollary B.1. Let $\varphi : X_1 \longrightarrow X_2$ be a morphism of realisations of $X$. A subset $Y_1$ of $X_1$ is algebraic if and only if its image $Y_2 := \varphi(Y_1) \subset X_2$ is algebraic.

This defines the notion of algebraic subsets of $X$.

Lemma B.2. Let $X$ be a realisation of a Hermitian symmetric domain $X$. Let $Z \subset X \subset \mathbb{C}^n$ be a complex analytic subvariety and $W \subset Z$ a semi-algebraic set. There exists an irreducible complex algebraic subvariety $Y \subset \mathbb{C}^n$ such that

$$W \subset Y \cap X \subset Z$$

Proof. This is a consequence of the proof of [24, lemma 4.1]. \hfill $\Box$

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