Generalized Property-Directed Reachability for Hybrid Systems

Kohei Suenaga¹,²[0000–0002–7466–8798] and Takuya Ishizawa¹

¹ Kyoto University, Kyoto, Japan
² JST PRESTO, Tokyo, Japan

Abstract. Generalized property-directed reachability (GPDR) belongs to the family of the model-checking techniques called IC3/PDR. It has been successfully applied to software verification; for example, it is the core of Spacer, a state-of-the-art Horn-clause solver bundled with Z3. However, it has yet to be applied to hybrid systems, which involve a continuous evolution of values over time. As the first step towards GPDR-based model checking for hybrid systems, this paper formalizes HGPDR, an adaptation of GPDR to hybrid systems, and proves its soundness. We also implemented a semi-automated proof-of-concept verifier, which allows a user to provide hints to guide verification steps.

Keywords: hybrid systems · property-directed reachability · IC3 · model checking · verification

1 Introduction

A hybrid system is a dynamical system that exhibits both continuous-time dynamics (called a flow) and discrete-time dynamics (called a jump). This combination of flows and jumps is an essential feature of cyber-physical systems (CPS), a physical system governed by software. In the modern world where safety-critical CPS are prevalent, their correctness is an important issue.

Model checking [14, 19] is an approach to guaranteeing hybrid system safety. It tries to prove that a given hybrid system does not violate a specification by abstracting its behavior and by exhaustively checking that the abstracted model conforms to the specification.

In the area of software model checking, an algorithm called property-directed reachability (PDR), also known as IC3, is attracting interest [5, 7, 12]. IC3/PDR was initially proposed in the area of hardware verification; it was then transferred to software model checking by Cimatti et al. [10]. Its effectiveness for software model checking is now widely appreciated. For example, the SMT solver Z3 [29] comes with a Horn-clause solver Spacer [21] that uses PDR internally; Horn-clause solving is one of the cutting-edge techniques to verify functional programs [6, 8, 17] and programs with loops [6].

We propose a model checking method for hybrid automata [3] based on the idea of PDR; the application of PDR to hybrid automata is less investigated compared to its application to software systems. Concretely, we propose an adaptation of a variant of PDR called generalized property-directed reachability (GPDR)
proposed by Hoder and Bjørner [20]. Unlike the original PDR, which is specialized to jump-only automata-based systems, GPDR is parametrized over a map over predicates on states (i.e., a forward predicate transformer); the detail of the underlying dynamic semantics of a verified system is encapsulated into the forward predicate transformer. This generality of GDPR enables the application of PDR to systems outside the scope of the original PDR by itself; for example, Hoder et al. [20] show how to apply GPDR to programs with recursive function calls.

An obvious challenge in an adaptation of GPDR to hybrid automata is how to deal with flow dynamics that do not exist in software systems. To this end, we extend the logic on which the forward predicate transformer is defined so that it can express flow dynamics specified by an ordinary differential equation (ODE). Our extension, inspired by the differential dynamic logic (dL) proposed by Platzer [32], is to introduce continuous reachability predicates (CRP) of the form \( \langle D | \phi_I \rangle \phi \) where \( D \) is an ODE and \( \phi_I \) and \( \phi \) are predicates. This CRP is defined to hold under valuation \( \sigma \) if there is a continuous transition from \( \sigma \) to certain valuation \( \sigma' \) that satisfies the following conditions: (1) the continuous transition is a solution of \( D \), (2) the valuation \( \sigma' \) makes \( \phi \) true, and (3) \( \phi_I \) is true at every point on the continuous transition. With this extended logic, we define a forward predicate transformer that faithfully encodes the behavior of a hybrid automaton. We find that we can naturally extend GPDR to hybrid automata by our predicate transformer.

We formalize our adaptation of GPDR to hybrid automata, which we call HGPDR. In the formalization, we define a forward predicate transformer that precisely expresses the behavior of hybrid automata [3] using dL. We prove the soundness of HGPDR. We also describe our proof-of-concept implementation of HGPDR and show how it verifies a simple hybrid automaton with human intervention.

In order to make this paper self-contained, we detail GPDR for discrete-time systems before describing our adaptation to hybrid automata. After fixing the notations that we use in Section 2, we define a discrete-time transition system and hybrid automata in Section 3. Section 4 then reviews the GPDR procedure. Section 5 presents HGPDR, our adaptation of GPDR to hybrid automata, and states the soundness of the procedure. We describe a proof-of-concept implementation in Section 6. After discussing related work in Section 7, we conclude in Section 8.

For readability, several definitions and proofs are presented in the appendices.

2 Preliminary

We write \( \mathbb{R} \) for the set of reals. We fix a finite set \( V := \{x_1, \ldots, x_N\} \) of variables. We often use primed variables \( x' \) and \( x'' \). The prime notation also applies to a set of variables; for example, we write \( V' \) for \( \{x'_1, \ldots, x'_N\} \). We use metavariable \( x \) for a finite sequence of variables. We write \( \text{Fml} \) for the set of quantifier-free
first-order formulas over \( V \cup V' \cup V'' \); its elements are ranged over by \( \varphi \). We call elements of the set \( \Sigma := (V \cup V' \cup V'') \to \mathbb{R} \) a valuations; they are represented by metavariable \( \sigma \). We use the prime notation for valuations. For example, if \( \sigma \in V \to \mathbb{R} \), then we write \( \sigma' \) for \( \{x'_1 \mapsto \sigma(x_1), \ldots, x'_N \mapsto \sigma(x_N)\} \). We write \( \sigma[x \mapsto r] \) for the valuation obtained by updating the entry for \( x \) in \( \sigma \) with \( r \). We write \( \sigma \models \varphi \) if \( \sigma \) is a model of \( \varphi \); \( \sigma \not\models \varphi \) if \( \sigma \models \varphi \) does not hold; \( \models \varphi \) if \( \sigma \models \varphi \) for any \( \sigma \); and \( \not\models \varphi \) if there exists \( \sigma \) such that \( \sigma \not\models \varphi \). We sometimes identify a valuation \( \sigma \) with a logical formula \( \bigwedge_{x \in V} x = \sigma(x) \).

3 State-transition systems and verification problem

We review the original GPDR for discrete-time systems \([20]\) in Section 4 before presenting our adaptation for hybrid systems in Section 5. This section defines the models used in these explanations (Section 3.1 and 3.2) and formally states the verification problem that we tackle (Section 3.3).

3.1 Discrete-time state-transition systems (DTSTS)

We model a discrete-time program by a state-transition system.

**Definition 3.1.** A discrete-time state-transition system (DTSTS) is a tuple \( \langle Q, q_0, \varphi_0, \delta \rangle \). We use metavariable \( S_D \) for DTSTS. \( Q = \{q_0, q_1, q_2, \ldots\} \) is a set of locations, \( q_0 \) is the initial location, \( \varphi_0 \) is the formula that has to be satisfied by the initial valuation, \( \delta \subseteq Q \times \text{Fml} \times \text{Fml} \times Q \) is the transition relation. We write \( (q, \sigma_1) \rightarrow_\delta (q', \sigma_2) \) if \( (q, \varphi, \varphi', q') \in \delta \) where \( \sigma_1 \models \varphi \) and \( \sigma_1 \cup \sigma_2' \models \varphi' \); we call relation \( \rightarrow_\delta \) the jump transition. A run of a DTSTS \( \langle Q, q_0, \varphi_0, \delta \rangle \) is a finite sequence \( \langle q^0, \sigma_0 \rangle , \langle q^1, \sigma_1 \rangle , \ldots , \langle q^N, \sigma_N \rangle \) where (1) \( q^0 = q_0 \), (2) \( \sigma_0 \models \varphi_0 \), and (3) \( \langle q^i, \sigma_i \rangle \rightarrow_\delta \langle q^{i+1}, \sigma_{i+1} \rangle \) for any \( i \in [0, N - 1] \).

\( \langle q, \varphi, \varphi', q' \rangle \in \delta \) intuitively means that, if the system is at the location \( q \) with valuation \( \sigma_1 \) and \( \sigma_1 \models \varphi \), then the system can make a transition to the location \( q' \) and change its valuation to \( \sigma_2' \) such that \( \sigma_1 \cup \sigma_2' \models \varphi' \). We call \( \varphi \) the guard of the transition. \( \varphi_0 \) is a predicate over \( V \cup V' \) that defines the command of the transition; it defines how the value of the variables may change in this transition. The elements of \( V \) represent the values before the transition whereas those of \( V'' \) represent the values after the transition.

**Example 3.2.** Figure 1 is an example of a DTSTS that models a program to compute the value of \( 1 + \cdots + x \); \( Q := \{q_0, q_1\} \) and \( \varphi_0 := x \geq 0 \land \text{sum} = 0 \). In the transition from \( q_0 \) to \( q_0 \), the guard is \( x > 0 \); the command is \( \text{sum}' = \text{sum} + x \land x' = x - 1 \). In the transition from \( q_0 \) to \( q_1 \), the guard is \( x \leq 0 \); the command is \( x' = x \land \text{sum}' = \text{sum} \) because this transition does not change the value of \( x \) and \( \text{sum} \). Therefore, the transition relation \( \delta = \{(q_0, x > 0, \text{sum}' = \text{sum} + x \land x' = x - 1, q_0) , (q_0, x \leq 0, x' = x \land \text{sum}' = \text{sum}, q_1)\} \).

The finite sequence \( \langle q_0, [x \mapsto 3, \text{sum} \mapsto 0] \rangle , \langle q_0, [x \mapsto 2, \text{sum} \mapsto 3] \rangle , \langle q_0, [x \mapsto 1, \text{sum} \mapsto 5] \rangle , \langle q_0, [x \mapsto 0, \text{sum} \mapsto 6] \rangle , \langle q_1, [x \mapsto 0, \text{sum} \mapsto 6] \rangle \) is a run of the DTSTS figure 1.
\[ x > 0 \land \text{sum}' = \text{sum} + x \land x' = x - 1 \]

\[ x \geq 0 \land \text{sum} = 0 \]

![Fig. 1. An example of DTSTS](image)

### 3.2 Hybrid automaton (HA)

We model a hybrid system by a hybrid automaton (HA) [3]. We define an HA as an extension of DTSTS as follows.

**Definition 3.3.** A hybrid automaton (HA) is a tuple \( \langle Q, q_0, \varphi_0, F, \text{inv}, \delta \rangle \). The components \( Q, q_0, \varphi_0, \) and \( \delta \) are the same as Definition 3.1. We use metavariable \( S_H \) for HA. \( F \) is a map from \( Q \) to ODE on \( V \) that specifies the flow dynamics at each location; \( \text{inv} \) is a map from \( Q \) to \( \text{Fml} \) that specifies the stay condition\(^3\) at each state.

A state of a hybrid automaton is a tuple \( \langle q, \sigma \rangle \). A run of \( \langle Q, q_0, \varphi_0, F, \text{inv}, \delta \rangle \) is a sequence of states \( \langle q_0, \sigma_0 \rangle \langle q_1, \sigma_1 \rangle \cdots \langle q_n, \sigma_n \rangle \) where \( \sigma_0 \models \varphi_0 \). The system is allowed to make a transition from \( \langle q_i, \sigma_i \rangle \) to \( \langle q_{i+1}, \sigma_{i+1} \rangle \) if (1) \( \sigma_i \) reaches a valuation \( \sigma' \) along with the flow dynamics specified by \( F(q_i) \), (2) \( \text{inv}(q_i) \) holds at every point on the flow, and (3) \( \langle q_i, \sigma' \rangle \) can jump to \( \langle q_{i+1}, \sigma_{i+1} \rangle \) under the transition relation \( \delta \). In order to define the set of runs formally, we need to define the continuous-time dynamics that happens within each location.

**Definition 3.4.** Let \( D \) be an ordinary differential equation (ODE) on \( V \) and let \( x_1(t), \ldots, x_n(t) \) be a solution of \( D \) where \( t \) is the time. Let us write \( \sigma^{(t)} \) for the valuation \( \{ x_1 \mapsto x_1(t), \ldots, x_n \mapsto x_n(t) \} \). We write \( \sigma \rightarrow_{D, \varphi} \sigma' \) if (1) \( \sigma = \sigma^{(0)} \) and (2) there exists \( t' \geq t \) such that \( \sigma' = \sigma^{(t')} \) and \( \sigma^{(t')} \models \varphi \) for any \( t'' \in (0, t'] \). We call relation \( \rightarrow_{D, \varphi} \) the flow transition.

![Fig. 2. An example of a hybrid automaton](image)

Intuitively, the relation \( \sigma \rightarrow_{D, \varphi} \sigma' \) means that there is a trajectory from the state represented by \( \sigma \) to that represented by \( \sigma' \) such that (1) the trajectory is a solution of \( D \) and (2) \( \varphi \) holds at any point on the trajectory. For example, let \( D \) be \( \dot{x} = v, \dot{v} = 1 \), where \( x \) and \( v \) are time-dependent variables; \( \dot{x} \) and \( \dot{v} \) are their time derivative. The solution of \( D \) is \( v = t + v_0 \) and \( x \equiv \frac{1}{2} + v_0 t + x_0 \) where \( t \) is the elapsed time, \( x_0 \) is the initial value of \( x \), and \( v_0 \) is the initial value of \( v \). Therefore, \( \{ x \mapsto 0, v \mapsto 0 \} \rightarrow_{D, \text{true}} \{ x \mapsto \frac{1}{2}, v \mapsto 1 \} \) holds because \( (x, v) = (\frac{1}{2}, 1) \) is the state at \( t = 1 \) on the above solution with \( x_0 = 0 \).

\(^3\) We use the word "stay condition" instead of the standard terminology "invariant" following Kapur et al. [23]
and $v_0 = 0$. \((x \mapsto 0, v \mapsto 0) \rightarrow_{D,x \geq 0} (x \mapsto \frac{1}{2}, v \mapsto 1)\) also holds because the condition $x \geq 0$ continues to hold along with the trajectory from \((x,v) = (0,0)\) to \((\frac{1}{2},1)\). However, \(\{x \mapsto 0, v \mapsto 0\} \rightarrow_{D,x \geq \frac{1}{2}} \{x \mapsto \frac{1}{2}, v \mapsto 1\}\) does not hold because the condition $x \geq \frac{1}{2}$ does not hold for the initial $\frac{1}{2}$ seconds in this trajectory.

Using this relation, we can define a run of an HA as follows.

**Definition 3.5.** A finite sequence \(\langle q^0, \sigma_0 \rangle, \langle q^1, \sigma_1 \rangle, \ldots, \langle q^N, \sigma_N \rangle\) is called a run of an HA \((Q, q_0, \varphi_0, F, \text{inv}, \delta)\) if (1) $q^0 = q_0$, (2) $\sigma_0 \models \varphi_0$, (3) for any $i$, if $0 \leq i \leq N - 2$, there exists $\langle q^i, \varphi_i, \varphi_{i+1}, q^{i+1} \rangle \in \delta$ and $\sigma^i$ such that $\sigma_i \rightarrow_{F(q^i), \text{inv}(q^i)} \sigma^{i+1}$ and $\sigma^i \models \varphi_i$ and $\langle q^{i+1}, \sigma^{i+1} \rangle \rightarrow_\delta \langle q^{i+1}, \sigma_{i+1} \rangle$, and (4) $\sigma_{N-1} \rightarrow_{F(q^{N-1}), \text{inv}(q^{N-1})} \sigma_N$.

**Remark 3.6.** This definition is more complicated than that of runs of DTSTS because we need to treat the last transition from \(\langle q^{N-1}, \sigma_{N-1} \rangle\) to \(\langle q^N, \sigma_N \rangle\) differently than the other transitions. Each transition from \(\langle q^i, \sigma_i \rangle\) to \(\langle q^{i+1}, \sigma_{i+1} \rangle\), if $0 \leq i \leq N - 2$, is a flow transition followed by a jump transition; however, the last transition consists only of a flow transition.

**Example 3.7.** Figure 2 shows a hybrid automaton with $Q := \{q_0, q_1\}$ schematically. Each circle represents a location $q$; we write $F(q)$ for the ODE associated with each circle. Each edge between circles represents a transition; we present the guard of the transition on each edge. We omit the $\varphi_c$ part; it is assumed to be the do-nothing command represented by $\wedge_{x \in V} x' = x$.

Both locations are equipped with the same flow that is the anticlockwise circle around the point $(x, y) = (0, 0)$ on the $xy$ plane. The system can stay at $q_0$ as long as $y \geq 0$ and at $q_1$ as long as $y \leq 0$. $y = 0$ holds whenever a transition is invoked. Indeed, for example, $\text{inv}(q_0) = y \geq 0$ and the guard from $q_0$ to $q_1$ is $y \leq 0$; therefore, when the transition is invoked, $\text{inv}(q_0) \wedge y \leq 0$ holds, which is equivalent to $y = 0$.

Starting from the valuation $\sigma_0 := \{x \mapsto 1, y \mapsto 0\}$ at location $q_0$, the system reaches $\sigma_1 := \{x \mapsto -1, y \mapsto 0\}$ by the flow $F(q_0)$ along which $\text{inv}(q_0) \equiv y \geq 0$ continues to hold; then the transition from $q_0$ to $q_1$ is invoked. After that, the system reaches $\sigma_2 := \{x \mapsto 0, y \mapsto -1\}$ by $F(q_1)$. Therefore, $\langle q_0, \sigma_0 \rangle \langle q_1, \sigma_1 \rangle \langle q_1, \sigma_2 \rangle$ is a run of this HA.

### 3.3 Safety verification problem

**Definition 3.8.** We say that $\sigma$ is reachable in DTSTS $S_D$ (resp., HA $S_H$) if there is a run of $S_D$ (resp., $S_H$) that reaches $(q, \sigma)$ for some $q$. A safety verification problem (SVP) for a DTSTS $(S_D, \varphi)$ (resp., HA $(S_H, \varphi)$) is the problem to decide whether $\sigma' \models \varphi$ holds for all the reachable valuation $\sigma'$ of the given $S_D$ (resp., $S_H$).

If an SVP is affirmatively solved, then the system is said to be safe; otherwise, the system is said to be unsafe. One of the major strategies for proving the safety of a system is discovering its inductive invariant.
Definition 3.9. Let \( \langle S_D, \varphi_P \rangle \) be an SVP for DTSTS where \( S_D = \langle Q, q_0, \varphi_0, \delta \rangle \). Then, a function \( R : Q \rightarrow \text{Fml} \) is called an inductive invariant if (1) \( \models \varphi_0 \Rightarrow R(q_0) \); (2) if \( \models \varphi(q) \) and \( q \xrightarrow{x} q' \), then \( \models \varphi(q') \); and (3) \( \models R(q) \Rightarrow \varphi_P \) for any \( q \).

Let \( \langle S_H, \varphi_P \rangle \) be an SVP for HA where \( S_H = \langle Q, q_0, \varphi_0, F, \text{inv}, \delta \rangle \). Then, a function \( R : Q \rightarrow \text{Fml} \) is called an inductive invariant if (1) \( \models \varphi_0 \Rightarrow R(q_0) \); (2) if \( \models \varphi(q) \) and \( q \xrightarrow{F(q), \text{inv}(q)} q'' \) and \( q'' \xrightarrow{\delta} q', \sigma' \), then \( \models R(q') \); and (3) \( \models R(q) \Rightarrow \varphi_P \) for any \( q \).

Unsafety can be proved by discovering a counterexample.

Definition 3.10. Define \( S_D, \varphi_P \), and \( S_H \) as in Definition 3.9. A run \( \langle \sigma_0, q_0 \rangle \ldots \langle \sigma_N, q_N \rangle \) of \( S_D \) (resp. \( S_H \)) is called a counterexample to the SVP \( \langle S_D, \varphi_P \rangle \) (resp. \( \langle S_H, \varphi_P \rangle \)) if \( \sigma_N \models \neg \varphi_P \).

GPDR is a procedure that tries to find an inductive invariant or a counterexample to a given SVP. SVP is in general undecidable. Therefore, the original GPDR approach [20] and our extension with hybrid systems presented in Section 5 do not terminate for every input.

4 GPDR for DTSTS

Before presenting our extension of GPDR with hybrid systems, we present the original GPDR procedure by Hoder and Bjørner [20] in this section. (The GPDR presented here, however, is slightly modified from the original one; see Remark 4.4.)

Given a safety verification problem \( \langle S_D, \varphi_P \rangle \) where \( S_D = \langle Q, q_0, \varphi_0, \delta \rangle \), GPDR tries to find (1) an inductive invariant to prove the safety of \( S_D \), or (2) a counterexample to refute the safety. To this end, GPDR (nondeterministically) manipulates a data structure called configurations. A configuration is either \textbf{Valid}, \textbf{Model} \( M \), or an expression of the form \( M \parallel R_0, \ldots, R_N ; N \). We explain each component of the expression \( M \parallel R_0, \ldots, R_N ; N \) in the following.

(\textbf{Valid} and \textbf{Model} \( M \) are explained later.)

- \( R_0, \ldots, R_N \) is a finite sequence of maps from \( Q \) to \text{Fml} (i.e., elements of \text{Fml}). Each \( R_i \) is called a frame. The frames are updated during an execution of GPDR so that \( R_i(q_j) \) is an overapproximation of the states that are reachable within \( i \) steps from the initial state in \( S_D \) and whose location is \( q_j \).
- \( N \) is the index of the last frame.
- \( M \) is a finite sequence of the form \( \langle \sigma_0, q_0, i \rangle, \langle \sigma_0, q_0, i + 1 \rangle, \ldots, \langle \sigma_N, q_N, N \rangle \).

This sequence is a candidate partial counterexample that starts from the one that is \( i \)-step reachable from the initial state and that ends up with a state \( \langle \sigma_N, q_N \rangle \) such that \( \sigma_N \models \neg \varphi_P \). Therefore, in order to prove the safety of \( S_D \), a GPDR procedure needs to prove that \( \langle q_i, \sigma_i \rangle \) is unreachable within \( i \) steps from an initial state.
In order to formalize the above intuition, GPDR uses a forward predicate transformer determined by $S_D$. In the following, we fix an SVP $\langle S_D, \varphi_P \rangle$.

**Definition 4.1.** $F(R)(q')$, where $F$ is called the forward predicate transformer determined by $S_D$, is the following formula:

$$(q' = q_0 \land \varphi_0) \lor \bigvee_{(q, \varphi, q', \varphi') \in \delta} \exists x''. \left(\left[\left[x''/x\right]R(q) \land \left[\left[x''/x\right]\varphi \land \left[x/x', x''/x\right]\varphi_c\right]\right)\right),$$

where $x''$ is the sequence $x_1'', \ldots, x_N''$.

Notice that $F(\lambda q.\text{false})$ is equivalent to $\varphi_0$. Intuitively, $\sigma' \models F(R)(q')$ holds if $(q', \sigma')$ is an initial state (i.e., $q' = q_0$ and $\sigma' \models \varphi_0$) or $(q', \sigma')$ is reachable in 1-step transition from a state that satisfies $R$. The latter case is encoded by the second disjunct of the above definition: The valuation $\sigma'$ satisfies the second disjunct if there are $q, \varphi, \varphi_c$ such that $(q, \varphi, q', \varphi') \in \delta$ (i.e., $q'$ is 1-step after $q$ in $\delta$) and there is a valuation $\sigma$ such that $\sigma \models R(q) \land \varphi$ (i.e., $\sigma$ satisfies the precondition $R(q)$ and the guard $\varphi$) and $\sigma'$ is a result of executing command $c$ under $\sigma$.

The following lemma guarantees that $F$ soundly approximates the transition of an DTSTS.

**Lemma 4.2.** If $\sigma_1 \models R(q_1)$ and $\langle q_1, \sigma_1 \rangle \rightarrow_\delta \langle q_2, \sigma_2 \rangle$, then $\sigma_2 \models F(R)(q_2)$.

*Proof.* Assume $\sigma_1 \models R(q_1)$ and $\langle q_1, \sigma_1 \rangle \rightarrow_\delta \langle q_2, \sigma_2 \rangle$. Then, by definition, $(q_1, \varphi, q_2) \in \delta$ and $\sigma_1 \models \varphi$ and $\sigma_1 \cup \sigma_2' \models \varphi_c$ for some $\varphi$ and $\varphi_c$. $\sigma_1' \cup \sigma_2' \models \left[\left[x''/x\right]R(q_1)\right]$ follows from $\sigma_1 \models R(q_1)$. $\sigma_1' \cup \sigma_2' \models \left[\left[x''/x\right]\varphi\right]$ follows from $\sigma_1 \models \varphi$. $\sigma_1' \cup \sigma_2' \models \left[\left[x''/x\right]\varphi\right]$ follows from $\sigma_1 \models \varphi$. Therefore, $\sigma_1' \cup \sigma_2' \models \left[\left[x''/x\right]R(q_1)\right] \land \left[\left[x''/x\right]\varphi\right] \land \left[\left[x/x', x''/x\right]\varphi_c\right]$. Hence, we have $\sigma_2 \models \exists x''. \left[\left[x''/x\right]R(q) \land \left[\left[x''/x\right]\varphi\right] \land \left[\left[x/x', x''/x\right]\varphi_c\right]\right]$ as required.

By using the forward predicate transformer $F$, we can formalize the intuition about configuration $M \parallel R_0, \ldots, R_N; N$ explained so far as follows.

**Definition 4.3.** Let $S_D$ be $\langle Q, q_0, \varphi_0, \delta \rangle$, $F$ be the forward predicate transformer determined by $S_D$, and $\varphi_P$ be the safety condition to be verified. A configuration $C$ is said to be consistent if it is (1) of the form $\text{Valid}$, (2) of the form $\text{Model}$ $(\sigma, q_0, 0) M$, or (3) of the form $M \parallel R_0, \ldots, R_N; N$ that satisfies all of the following conditions:

- $(\text{Con-A})$ $R_0(q_0) = \varphi_0$ and $R_0(q_i) = \text{false}$ if $q_i \neq q_0$;
- $(\text{Con-B})$ $\models R_i(q) \implies R_{i+1}(q)$ for any $q$;
- $(\text{Con-C})$ $\models R_i(q) \implies \varphi_P$ for any $q$ and $i < N$;
- $(\text{Con-D})$ $F(R_i)(q) \implies R_{i+1}(q)$ for any $i < N$ and $q$;
- $(\text{Con-E})$ if $(\sigma, q, N) \in M$, then $\models R_N(q) \land \neg \varphi^4_P$; and
- $(\text{Con-F})$ if $(\sigma_1, q_1, i), (q_2, q_2, i+1) \in M$ and $i < N$, then $(q_1, \varphi, \varphi_c, q_2) \in \delta$ and $\sigma_1, \sigma_2' \models R_i(q_1) \land \varphi \land \varphi_c$.

If $C$ is consistent, we write $\text{Con}(C)$. 
The GPDR procedure rewrites a configuration following the (nondeterministic) rewriting rules in Figure 3. We add a brief explanation below; for more detailed exposition, see [20]. Although the order of the applications of the rules in Figure 3 is arbitrary, we fix one scenario of the rule applications in the following for explanation.

1. The procedure initializes $M$ to $\emptyset$, $R_0$ to $\mathcal{F}(\lambda q.\text{false})$, and $N$ to 0 (Initialize).

2. If there is a valuation $\sigma$ and a location $q$ such that $\sigma \models R_N(q) \land \neg \varphi_P$ (Candidate), then the procedure adds $\langle \sigma, q, N \rangle$ to $M$. The condition $\sigma \models R_N(q) \land \neg \varphi_P$ guarantees that the state $\langle q, \sigma \rangle$ violates the safety condition $\varphi_P$; therefore, the candidate $\langle \sigma, q, N \rangle$ needs to be refuted. If not, then the frame sequence is extended by setting $N$ to $N + 1$ and $R_{N+1}$ to $\lambda q.\text{true}$ (Unfold); this is allowed since $\forall q \in Q, \models R_N(q) \implies \varphi_P$ in this case.

3. The discovered $\langle q, \sigma \rangle$ is backpropagated by successive applications of Decide:

   (a) If this backpropagation reaches $R_0$ (the rule Model), then it reports the trace of the backpropagation returning Model $M$.

   (b) If it does not reach $R_0$, in which case there exists $i$ such that $\sigma' \land \mathcal{F}(R_i)(q')$ is not satisfiable, then we pick a frame $R$ such that $\models R(q') \implies \neg \sigma'$ and $\models R(R_i)(q) \implies R(q)$ for any $q$ (the rule Conflict). Intuitively, $R$ is a frame that separates (1) the union of the initial states denoted by $\varphi_0$ and the states that are one-step reachable from a state denoted by $R_0(q')$ and

   \[ \neg \sigma \]

   We hereafter write $\langle \sigma, q, i \rangle \in M$ to express that the element $\langle \sigma, q, i \rangle$ exists in the sequence $M$ although $M$ is a sequence, not a set.
(2) the state denoted by \( \langle q', \sigma' \rangle \). In a GPDR term, \( R \) is a generalization of \( \neg \sigma' \). This formula is used to strengthen \( R_j \) for \( j \in \{1, \ldots, i + 1\} \).

4. The frame \( R \) obtained in the application of the rule CONFLICT is propagated forward by applying the rule INDUCTION. The condition \( \forall q \in Q. \models \mathcal{F}(\lambda q. R_i(q) \land R(q))(q) \Rightarrow R(q) \) forces that \( R \) holds in the one-step transition from a states that satisfies \( R_i \). If this condition holds, then \( R \) holds for \( i + 1 \) steps (Theorem 4.5); therefore, we conjoin \( R \) to \( R_1(q), \ldots, R_{i+1}(q) \). In order to maintain the consistency conditions (Con-E) and (Con-F), this rule clears \( M \) to the empty set to keep its consistency to the updated frames.\(^5\)

5. If \( \forall q \in Q. \models R_i(q) \Rightarrow R_{i-1}(q) \) for some \( i < N \), then the verification succeeds and \( R_i \) is an inductive invariant (Valid). If such \( i \) does not exist, then we go back to Step 2.

Remark 4.4. One of the differences of the above GPDR from the original one [20] is that ours deals with the locations of a given DTSTS explicitly. In the original GPDR, information about locations are assumed to be encoded using a variable that represents the program counter. Although such extension was proposed for IC3 by Lange et al. [26], we are not aware of a variant of GPDR that treats locations explicitly.

Soundness. We fix one DTSTS \( \langle Q, q_0, \phi_0, \delta \rangle \) in this section. The correctness of the GPDR procedure relies on the following lemmas.

Lemma 4.5. Con is invariant to any rule application of Figure 3.

Theorem 4.6. If the GPDR procedure is started from the rule INITIALIZE and leads to Valid, then the system is safe. If the GPDR procedure is started from the rule INITIALIZE and leads to Model \( \langle \sigma_0, q_0, 0 \rangle \ldots \langle \sigma_N, q_N, N \rangle \), then the system is unsafe.

5 HGPDR

We now present our procedure HGPDR that is an adaptation of the original GPDR to hybrid systems. An adaptation of GPDR to hybrid systems requires the following two challenges to be addressed.

1. The original definition of \( \mathcal{F} \) (Definition 4.1) captures only a discrete-time transition. In our extension of GPDR, we need a forward predicate transformer that can mention a flow transition.

2. A run of an HA (Definition 3.5) differs from that of DTSTS in that its last transition consists only of flow dynamics; see Remark 3.6.

In order to address the first challenge, we extend the logic on which \( \mathcal{F} \) is defined to be able to mention flow dynamics and define \( \mathcal{F} \) on the extended logic (Section 5.1). To address the second challenge, we extend the configuration used by

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\(^5\) We could filter \( M \) so that it is consistent for the updated frame. We instead discard \( M \) here for simplicity.
GPDR so that it carries an overapproximation of the states that are reachable from the last frame by a flow transition; the GPDR procedure is also extended to maintain this information correctly (Section 5.2).

5.1 Extension of forward predicate transformer

In order to extend \( F \) to accommodate flow dynamics, we extend the logic on which \( F \) is defined with continuous reachability predicates (CRP) inspired by the differential dynamic logic (d\( \mathcal{L} \)) proposed by Platzer [33].

**Definition 5.1.** Let \( D \) be an ODE over \( Y := \{y_1, \ldots, y_k\} \subseteq V \). Let us write \( \sigma \) for \( \{y_1 \mapsto e_1, \ldots, y_k \mapsto e_k\} \) and \( \sigma' \) for \( \{y_1 \mapsto e'_1, \ldots, y_k \mapsto e'_k\} \). We define a predicate \( (D \mid \varphi)_{\sigma} \) by: \( \sigma \models (D \mid \varphi)_{\sigma} \) iff. \( \exists \sigma'. \sigma \rightarrow_{D, \varphi} \sigma' \land \sigma' \models \varphi \). We call a predicate of the form \( (D \mid \varphi)_{\sigma} \) a continuous reachability predicate (CRP).

Using the above predicate, we extend \( F \) as follows.

**Definition 5.2.** For an HA \( (Q, q_0, \varphi_0, F, \text{inv}, \delta) \), the forward predicate transformer \( F_\mathcal{H}(R)(q') \) is the following formula:

\[
(q' = q_0 \land \varphi_0) \lor \bigvee_{(q, \varphi, q', q') \in \delta} \exists x'' \cdot \left( [x''/x]R(q) \land ([x''/x]F(q')) \land [x''/x]\text{inv}(q')([x''/x]\varphi \land [x/x', x''/x]\varphi_c) \right).
\]

In the above definition, \([x''/x]F(q)\) is the ODE obtained by renaming the variables \( x \) that occur in ODE \( F(q) \) with \( x'' \).

We also define predicate \( F_\mathcal{C}(R)(q') \) as follows:

\[
\exists x''. ([x''/x]R(q') \land ([x''/x]F(q') \land [x''/x]\text{inv}(q'))x = x'').
\]

Intuitively, \( \sigma' \models F_\mathcal{H}(\varphi)(q') \) holds if either (1) \( \langle q', \sigma' \rangle \) is an initial state or (2) it is reachable from \( R \) by a flow transition followed a jump transition. Similarly, \( \sigma' \models F_\mathcal{C}(R)(q') \) holds if \( \sigma' \) is reachable in a flow transition (not followed by a jump transition) from a state denoted by \( R(q') \). This definition of \( F_\mathcal{H} \) is an extension of Definition 4.1 in that it encodes the "flow-transition" part of the above intuition by the CRP. In the case of \( F_\mathcal{C} \), the postcondition part of the CRP is \( x = x'' \) because we do not need a jump transition in this case.

**Lemma 5.3.** If \( \sigma_1 \models R(q_1) \) and \( \sigma_1 \rightarrow_{F(q_1), \text{inv}(q_1)} \sigma_1' \) and \( \langle q_1, \sigma_1' \rangle \rightarrow_{\delta} \langle q_2, \sigma_2 \rangle \), then \( \sigma_2 \models F_\mathcal{H}(R)(q_2) \).

**Proof.** Assume (1) \( \sigma_1 \models R(q_1) \), (2) \( \sigma_1 \rightarrow_{F(q_1), \text{inv}(q_1)} \sigma_1' \), and (3) \( \langle q_1, \sigma_1' \rangle \rightarrow_{\delta} \langle q_2, \sigma_2 \rangle \). Then, by definition, (4) \( \langle q_1, \varphi, \varphi_c, \delta \rangle \in \delta \) and (5) \( \sigma_1' \models \varphi \) and (6) \( \sigma_1'' \models \varphi_c \) for some \( \varphi \) and \( \varphi_c \). We show \( \exists x''. ([x''/x]R(q) \land ([x''/x]F(q')) \land [x''/x]\text{inv}(q')([x''/x]\varphi \land [x/x', x''/x]\varphi_c)) \).

(5) implies (7) \( \sigma_1'' \models \varphi \). (6) implies (8) \( \sigma_1'' \models \varphi \land [x''/x]\varphi_c \).

(2) implies (9) \( \sigma_1'' \models [x''/x]F(q_1) \land [x''/x]\text{inv}(q_1)\sigma_1'' \). Therefore, from (8) and (9), we
We present our adaptation of GPDR for hybrid systems, which we call Proof.

Almost the same argument as the proof of Lemma 5.3.

Lemma 5.4. If $\sigma_1 \models R(q_1)$ and $\sigma_1 \rightarrow_{F(q_1),inv(q_1)} \sigma_2$, then $\sigma_2 \models F_C(R(q_1))$.

Proof. Almost the same argument as the proof of Lemma 5.3.

5.2 Extension of GPDR

We present our adaptation of GPDR for hybrid systems, which we call HGPD. Recall that the original GPDR in Section 4 maintains a configuration of the form $M \parallel R_0, \ldots, R_N; N$. HGPD uses a configuration of the form $M \parallel R_0, \ldots, R_N; R_{rem}; N$. In addition to the information in the original configurations, we add $R_{rem}$, which we call remainder frame. $R_{rem}$ overapproximates the states that are reachable from $R_N$ within one flow transition.

Figure 4 presents the rules for HGPD. The rules from INITIALIZE to CONFLICT are the same as Figure 3 except that (1) INITIALIZE and UNFOLD are adapted so that they set the remainder frame to $\lambda q.true$ and (2) CANDIDATE is dropped. We explain the newly added rules.

- PROPAGATE CONT discovers a fact that holds in $R_{rem}$. The side condition $\models R_N(q) \lor F_C(R_N)(q) \implies R(q)$ for any $q$ guarantees that $R(q)$ is true at the remainder frame; hence $R$ is conjoined to $R_{rem}$.
– **CANDIDATE** replaces **CANDIDATE** in the original procedure. It tries to find a candidate from the frame $R_{\text{rem}}$. The candidate $\langle q, \sigma \rangle$ found here is added to $M$ in the form $\langle \sigma, q, \text{rem} \rangle$ to denote that $\langle q, \sigma \rangle$ is found at $R_{\text{rem}}$.

– **DECIDE** propagates a counterexample $\langle \sigma', q', \text{rem} \rangle$ found at $R_{\text{rem}}$ to the previous frame $R_N$. This rule computes the candidate to be added to $M$ by deciding $\sigma \cup \sigma' \models R_N(q) \land \langle F(q) \mid inv(q) \rangle(x = x')$, which guarantees that $\sigma$ evolves to $\sigma'$ under the flow dynamics determined by $F(q)$ and $inv(q)$.

– **CONFLICT** uses $F_H$ instead of $F$ in the original GPDR. As in the rule **CONFLICT** in GPDR, the frame $R$ in this rule is a generalization of $\neg \sigma'$ which is not backward reachable to $R_i$.

– **CONFLICT** is the counterpart of **CONFLICT** for the frame $R_{\text{rem}}$. This rule is the same as **CONFLICT** except that it uses $F_C$ instead of $F_H$; hence, $R$ separates $\sigma'$ from both the states denoted by $\phi_0$ and the states that are reachable from $R_i$ in a flow transition (not followed by a jump transition).

### 5.3 Soundness

In order to prove the soundness of HGPDR, we adapt the definition of **Con** in Definition 4.3 for HGPDR.

**Definition 5.5.** Let $S_H$ be $(Q, \varphi_0, F, \text{inv}, \delta)$, $F_H$ and $F_C$ be the forward predicate transformers determined by $S_H$, and $\varphi_P$ be the safety condition to be verified. A configuration $C$ is said to be consistent if it is **Valid, Model** $(\sigma, q_0, 0)$ $M$, or **$\text{Con}_H(M \mid R_0, \ldots, R_N; R_{\text{rem}}; N)$** that satisfies all of the following:

- (Con-A) $R_0(q_0) = \varphi_0$ and $R_0(q_i) = \text{false}$ if $q_i \neq q_0$;
- (Con-B-1) $\models R_i(q) \implies R_{i+1}(q)$ for any $q$ and $i < N$;
- (Con-B-2) $\models R_N(q) \implies R_{\text{rem}}(q)$ for any $q$;
- (Con-C) $\models R_i(q) \implies \varphi_P$ if $i < N$;
- (Con-D-1) $\models F_H(R_i)(q) \implies R_{i+1}(q)$ for any $i < N$ and $q$;
- (Con-D-2) $\models F_C(R_N)(q) \implies R_{\text{rem}}(q)$ for any $q$;
- (Con-E) if $(\sigma, q, \text{rem}) \in M$, then $\sigma \models R_{\text{rem}}(q) \land \neg \varphi_P$;
- (Con-F-1) if $(\sigma_1, q_1, i), (\sigma_2, q_2, i + 1) \in M$ and $i < N$, then $(q_1, \varphi, \varphi_c, q_2) \in \delta$ and $\sigma_1, \sigma_2' \models R_i(q_1) \land \varphi \land \varphi_c$; and
- (Con-F-2) if $(\sigma_1, q_1, N), (\sigma_2, q_2, \text{rem}) \in M$, then $(q_1, \varphi, \varphi_c, q_2) \in \delta$ and $\sigma_1, \sigma_2' \models R_i(q_1) \land \varphi \land \varphi_c$.

The soundness proof follows the same strategy as that of the original GPDR.

**Lemma 5.6.** **$\text{Con}_H$** is invariant to any rule application of Figure 4.

**Theorem 5.7.** If HGPDR is started from the rule **INITIALIZE** and leads to **VALID**, then the system is safe. If HGPDR is started from the rule **INITIALIZE** and leads to **Model** $(\sigma_0, q_0, 0) \ldots (\sigma_N, q_N, N) (\sigma_{\text{rem}}, q_{\text{rem}}, \text{rem})$, then the system is unsafe.
Input: Hybrid automaton $S_H := (Q, q_0, \varphi_0, F, \text{inv}, \delta)$
Output: Model($M$) if $S_H$ is unsafe; $M$ is a witnessing trace. Valid($R$) if $S_H$ is safe; $R$ is an inductive invariant.

// Initialize
1 $N := 0$; $R_0 := \lambda q. (\text{if } q = q_0 \text{ then } \varphi_0 \text{ else false})$
2 $R_1 := \text{true}; R_{\text{rem}} := \text{true}; M := \emptyset$
3 while true do
4 for $q \in Q$ do
5 switch querySat($R_{\text{rem}}(q) \land \neg \varphi_I$) do
6 case Sat($\sigma'$) do
7 // CandidateCont
8 $M := (q, \sigma, \text{rem})$
9 switch RemoveTrace($M, R_0, \ldots, R_N, R_{\text{rem}}, N$) do
10 case Valid($R$) do
11 return Valid($R$)
12 case Cont($R_0, \ldots, R_N, R_{\text{rem}}$) do
13 $M := \emptyset$
14 Update $R_0, \ldots, R_N, R_{\text{rem}}$ to the returned frames
15 case Model($M$) do
16 return Model($M$)
17 end
18 case Unsat do
19 // Unfold
20 $M := \emptyset$; $R_{N+1} := \lambda q. \text{true}; R_{\text{rem}} := \lambda q. \text{true}; N := N + 1$
21 end
22 end
23 end

Algorithm 1: Definition of DetHybridPDR.

5.4 Operational presentation of HGPDR

The definition of HGPDR in Figure 4 is declarative and nondeterministic. For the sake of convenience of implementation, we derive an operational procedure from HGPDR; we call the operational version DetHybridPDR, whose definition is in Algorithm 1.

Discharging verification conditions. An implementation of HGPDR needs to discharge verification conditions during verification. In addition to verification conditions expressed as a satisfiability problem of a first-order predicate, which can be discharged by a standard SMT solver, DetHybridPDR needs to discharge conditions including a CRP predicate. Specifically, DetHybridPDR needs to deal with the following three types of problems.

- Checking whether $\delta := \psi \land (\mathcal{D} \mid \varphi_I)(\land_{x \in V} x = \sigma'(x))$ is satisfiable or not for given first-order predicates $\psi$ and $\varphi_I$, an ODE $\mathcal{D}$, and a valuation $\sigma'$. DetHybridPDR needs to discharge this type of predicates when it decides which of DecideCont and ConflictCont should be applied if the top
Input: Hybrid automaton $S_H := (Q, q_0, \phi_0, F, \text{inv}, \delta)$; Trace of counterexamples $M$; Frames $R_0, \ldots, R_N, R_{\text{rem}}$; Natural number $N$.

Output: while $M \neq \emptyset$ do
  if $M = (q', \sigma', \text{rem})$ then
    switch $\text{querySat}_C(R_N(q') \land (F(q') \lor \text{inv}(q'))(x = \sigma'(x)))$ do
      // Decision
      case $\text{Sat}(\sigma)$ do
        $M := (q', \sigma, N) M$
      // Conflict
      case $\text{Unsat}(R)$ do
        $M := \emptyset; R_{\text{rem}} := \lambda q. R_{\text{rem}}(q) \land R(q)$
        // Propagation
        for $\psi \in \text{Formulas}(R_N(q'))$ do
          switch $\text{querySat}_C(R_N(q') \land (F(q') \lor \text{inv}(q')) \neg \psi$ do
            case $\text{Sat}(\sigma)$ do
              $M := (q, \sigma) M$
            // Conflict
            case $\text{Unsat}(R)$ do
              for $j \in [1, i + 1]$ do
                $R_j := \lambda q. R_j(q) \land R(q); M := \emptyset$
              // Induction
              for $i \in [1, N - 1], \psi \in \text{Formulas}(R_i(q'))$ do
                switch $\text{querySat}_C(R_i(q') \land \psi \land (F(q') \lor \text{inv}(q')) \neg \psi$ do
                  case $\text{Unsat}(R)$ do
                    $R_j(q') := R_j(q') \land \psi$ for $j \in [1, i + 1]$
                end
              end
            end
          end
        end
      end
    end
  else if $M = (q', \sigma', 0)$ $M'$ then
    // Model
    return Model($M'$)
  else if $M = (q', \sigma', i + 1)$ $M'$ and $0 < i \neq \text{rem}$ then
    for $(q, \phi, \phi_c, q') \in \delta$ do
      switch $\text{querySat}_C(R_{i - 1}(q) \land (F(q) \lor \text{inv}(q)) (\phi \land \phi_c \land x = \sigma'(x)))$ do
        // Decision
        case $\text{Sat}(\sigma)$ do
          $M := (q, \sigma) M$
        // Conflict
        case $\text{Unsat}(R)$ do
          for $j \in [1, i + 1]$ do
            $R_j := \lambda q. R_j(q) \land R(q); M := \emptyset$
          // Induction
          for $i \in [1, N - 1], \psi \in \text{Formulas}(R_i(q'))$ do
            switch $\text{querySat}_C(R_i(q') \land \psi \land (F(q') \lor \text{inv}(q')) \neg \psi$ do
              case $\text{Unsat}(R)$ do
                $R_j(q') := R_j(q') \land \psi$ for $j \in [1, i + 1]$
            end
          end
        end
    end
  end
  else if $\exists i$ such that $\forall q. R_{i+1}(q) \implies R_i(q)$ then
    // Valid
    return Valid($R_i$)
  else
    // Inductive invariant is not reached yet.
    return Cont($R_0, \ldots, R_N, R_{\text{rem}}$)
end

Algorithm 2: Definition of RemoveTrace.
Algorithm 3: Algorithm for discharging $\delta := \psi \land (D \mid \varphi_I)(\land_{x \in V} x = \sigma'(x))$.

of $M$ is $(\sigma', q', \text{rem})$. We use Algorithm 3 for discharging $\delta$. This algorithm searches for a valuation $\sigma_i$ that witnesses the satisfiability of $\sigma$ by using a time-inverted simulation of $D$ as follows. Concretely, this algorithm numerically simulates $D^{-1}$, the time-inverted ODE of $D$, starting from the point $\{x \mapsto \sigma'(x)\}$. If it reaches a point $\sigma_i$ that satisfies $\psi$ and if all $\sigma_{i+1} \ldots \sigma'$ in the obtained solution satisfy $\varphi_I$, then $\sigma_i$ witnesses the satisfiability of $\delta$. If such $\sigma_i$ does not exist but there is $\sigma_i$ such that $\sigma_i \not\models \varphi_I$, then $\psi$ is not backward reachable from $\sigma'$ and hence $\delta$ is unsatisfiable. In this case, Algorithm 3 needs to return a predicate that can be used as $\psi'$ in the rule CONFLICTCONT in Figure 4. Currently, we assume that the user provides this predicate. We expect that we can help this step of discovering $\psi'$ by using techniques for analyzing continuous dynamics (e.g., automated synthesizer of barrier certificates [34] and Flow* [9] in combination with Craig interpolant synthesis procedures [2, 31]). If neither holds, then we give up the verification by aborting; this may happen if, for example, the value of $T$ is too small.

- Checking whether $\delta' := \psi \land (F(q) \mid \text{inv}(q))(\varphi \land \varphi_c \land x = \sigma'(x))$ is satisfiable or not. DEThybridPDR needs to solve this problem in the choice between DECIDE and CONFLICT. This query is different from the previous case in that the formula that appears after $(F(q) \mid \text{inv}(q))$ in $\delta'$ is $\varphi \land \varphi_c \land x = \sigma'(x)$, not $x = \sigma'(x)$; therefore, we cannot use numerical simulation to discharge $\delta'$. Although it is possible to adapt Algorithm 3 to maintain the sequence of predicates $\alpha_0 \alpha_1 \ldots \alpha_T$ instead of valuations so that each $\alpha_i$ becomes the preimage of $\alpha_{i-1}$ by $D$, the preimage computation at each step is pro-
Input: Formula $\varphi_1 \land (\dot{x} = f(x) | \varphi_I) \neg \varphi_2$ to be discharged; Number $r > 0$.
Output: Unsat or Otherwise; if Unsat is returned then the input formula is unsatisfiable.

1 if $\varphi_1 \land \varphi_2$ is satisfiable then
2 return Otherwise
3 end
4 Let $dt$ be a fresh symbol;
5 if $r > dt > 0 \land \varphi_1 \land \varphi_I \land [x + f(x) dt / x] \varphi_1$ and $\varphi_1 \land \varphi_2$ are unsatisfiable then
6 return Unsat
7 end
8 if $r > dt > 0 \land \neg \varphi_2 \land \varphi_I \land [x + f(x) dt / x] \varphi_2$ and $\varphi_1 \land \varphi_2$ are unsatisfiable then
9 return Unsat
10 end
11 return Otherwise

Algorithm 4: Algorithm for discharging $\varphi_1 \land (\dot{x} = f(x) | \varphi_I) \neg \varphi_2$. 

hibitively expensive. Instead, the current implementation restricts the input system so that there exists at most one $\sigma$ such that $\sigma \models \varphi \land \varphi_I \land x = \sigma'(x)$ for any $\sigma'$; if this is met, then one can safely use Algorithm 3 for discharging $\delta'$. Concretely, we allow only $\varphi_c$ that corresponds to the command whose syntax is given by $c ::= \text{skip} \mid x := r_1 x + r_2 \mid x := r_1 x - r_2$ where $\text{skip}$ is a command that does nothing; $r_1$ and $r_2$ are real constants.

– Checking whether $\varphi_1 \land (D \mid \varphi_I) \neg \varphi_2$ is unsatisfiable. $\text{DetHybridPDR}$ needs to discharge this type of queries when it applies $\text{Induction}$ or $\text{PropagateCont}$. This case is different from the previous case in that (1) $\text{DetHybridPDR}$ may answer Otherwise without aborting the entire verification if unsatisfiability nor satisfiability is proved, and (2) $\text{DetHybridPDR}$ does not need to return a generalization if the given predicate is unsatisfiable. We use Algorithm 4 to discharge this type of queries. This algorithm first checks the satisfiability of $\varphi_1 \land \varphi_2$ in Step 1; if it is satisfiable, then so is the entire formula. Then, Step 5 tries to prove that the entire formula is unsatisfiable by proving (1) $\varphi_1$ is invariant with respect to the dynamics specified by $D$ and $\varphi_I$ and (2) $\varphi_1 \land \varphi_2$ is unsatisfiable. In order to prove the former, the algorithm tries the following sufficient condition: For any positive $dt$ that is smaller than a positive real number $r$, $\models \varphi_1 \land \varphi_I \implies [x + f(x) dt / x] \varphi_1$, where $D \equiv \dot{x} = f(x)$.\(^6\) Step 8 tries the same strategy but tries to prove that $\neg \varphi_2$ is invariant. If both at-

\(^6\) This strategy is inspired by the previous work by one of the authors on nonstandard programming [18, 30, 36, 37].
tempts fail, then the algorithm returns Otherwise. This algorithm could be further enhanced by incorporating automated invariant-synthesis procedures [15, 28, 35]; exploration of this possibilities is left as future work.

6 Proof-of-concept implementation

We implemented DETHYBRIDPDR as a semi-automated verifier. We note that the current implementation is intended to be a proof of concept; extensive experiments are left as future work. The snapshot of the source code as of writing can be found at https://github.com/ksuenaga/HybridPDR/tree/master/src.

The verifier takes a hybrid automaton $\mathcal{H}$ specified with SPACEEX modeling language [27], the initial location $q_0$, the initial condition $\phi_0$, and the safety condition $\varphi_P$ as input; then, it applies DETHYBRIDPDR to discover an inductive invariant or a counterexample. The frontend of the verifier is implemented with OCaml; in the backend, the verifier uses Z3 [29] and ODEPACK [1] to discharge verification conditions.

As we mentioned in Section 5.4, when a candidate counterexample $(q', \sigma', i+1)$ turns out to be backward unreachable to $R_i$, then our verifier asks for a generalization of $\sigma'$ to the user; concretely, for example in an application of the rule CONFLICT, the user is required to give $\psi$ such that $\models \psi \implies \neg \sigma' \land \models (q, \phi, \varphi, q') \in \delta \land |x_0| R_i(q) \land |\{x_0|F(q) \mid |x_0| inv(q)\}|x_0|x_0| (\varphi \land \varphi_c) \implies \psi \land \models R_i(q') \implies \psi$. Instead of throwing this query at the user in this form, the verifier asks the following question in order to make this process easier for the user for each $(q, \phi, \varphi, q') \in \delta$:

\[
\text{Pre: } R_i(q); \text{ Flow: } F(q); \text{ Stay: } inv(q); \text{ Guard: } \varphi; \text{ Cmd: } \varphi_c; \text{ CE: } \sigma'; \text{ Init: } R_i(q').
\]

In applying CONFLICTCONT, the verifier omits the fields Guard and Cmd.

We applied the verifier to the hybrid automaton in Figure 2 with several initial conditions and the safety condition $\varphi_P := x \leq 1$. We remark that the outputs from the verifier presented here are post-processed for readability. We explain how verification is conducted in each setting; we write $D$ for the ODE $\dot{x} = -y, \dot{y} = x$.

- Initial condition $x = 0 \land y = 0$ at location $q_0$: The verifier finds the inductive invariant $\{q_0 \Rightarrow x = 0 \land y = 0, q_1 \Rightarrow x = 0 \land y = 0\}$ after asking for proofs of unsatisfiability to the user 5 times.

- Initial condition $x = \frac{1}{2}$ at location $q_0$: The verifier finds a counterexample $\{x \Rightarrow 0.490533, y \Rightarrow 1.93995\}$, from which the system reaches $\{x \Rightarrow 2.00100, y \Rightarrow 0\}$. The verifier asks 5 questions, one of which is the following:

\[
\text{Pre: } (x \leq 1 \land y \geq 0) \lor x \leq 0.5; \text{ Flow: } D; \text{ Stay: } y \geq 0; \\
\text{Guard: } y \leq 0; \text{ Cmd: skip; CE: } \{x \Rightarrow 0.998516; y \Rightarrow -1.889365\}; \\
\text{Init: } x \leq 0.5.
\]

\footnote{If the flow specified by $D$ is a linear or a polynomial, then we can apply the procedure proposed by Liu et al. [28], which is proved to be sound and complete for such a flow.}
Notice that the stay condition is $y \geq 0$ and the guard is $y \leq 0$; therefore the predicate $y = 0$ holds when a jump transition happens. Since the flow specified by $D$ is an anticlockwise circle whose center is $\{x \mapsto 0, y \mapsto 0\}$ with the stay condition $y \geq 0$, the states after the flow dynamics followed by a jump transition is $x \leq 0.5 \land y = 0$, which indeed does not intersect with $x = 0.998516 \land y = -1.889365$. The verification proceeds by giving $y \geq 0$ as a generalization in this case.

Initial condition $0 \leq x \leq \frac{1}{2} \land 0 \leq y \leq \frac{1}{2}$ at location $q_0$: The verifier finds an inductive invariant

$$R := \{q_0 \mapsto (y = 0 \land 0 \leq x \leq 0.707107) \lor (0 \leq x \leq 0.5 \land 0 \leq y \leq 0.5),
q_1 \mapsto y = 0 \land -0.707107 \leq x \leq 0\}$$

after asking for 8 generalizations to the user. This is indeed an inductive invariant. Noting $0.707107 \approx \frac{1}{\sqrt{2}}$, we can confirm that (1) the states that are reachable by flow dynamics followed by a jump transition is the set denoted by $R(q_0)$; the same holds for the transition from $R(q_1)$; (2) it contains the initial condition $0 \leq x \leq 0.5 \land 0 \leq y \leq 0.5$ at location $q_0$; and (3) it does not intersect with the unsafe region $x > 1$. The following is one of the questions that are asked by the verifier:

- **Pre:** $(y = 0 \land -0.707107 \leq x \leq 0) \lor (0 \leq x \leq 0.5 \land 0 \leq y \leq 0.5)$;
- **Flow:** $D$; **Stay:** $y \leq 0$; **CE:** $\{x \mapsto 0.998516; y \mapsto -1.889365\}$;
- **Init:** false.

Instead of a precise overapproximation $(x^2 + y^2 = 0.5 \land y \leq 0) \lor (0 \leq x \leq 0.5 \land 0 \leq y \leq 0.5)$ of the reachable states, we give $(-0.707107 \leq y \leq 0 \land -0.707107 \leq x \leq 0.707107) \lor (0 \leq x \leq 0.5 \land 0 \leq y \leq 0.5)$, which progresses the verification.

7 Related work

Compared to its success in software verification [5, 10, 12, 20, 21], IC3/PDR for hybrid systems is less investigated. HyComp [11, 13] is a model checker that can use several techniques (e.g., IC3, bounded model checking, and $k$-induction) in its backend. Before verifying a hybrid system, HyComp discretizes its flows so that the verification can be conducted using existing SMT solvers that do not directly deal with continuous-time dynamics. Compared to HyComp, HGPDR does not necessarily require prior discretization for verification. We are not aware of an IC3/PDR-based model checking algorithm for hybrid systems that does not require prior discretization.

Kindermann et al. [24, 25] propose an application of PDR for a timed system—a system that is equipped with clock variables; the flow dynamics of a clock variable $c$ is limited to $\dot{c} = 1$. A clock variable may be also reset to a constant in a jump transition. Kindermann et al. finitely abstract the state space of clock variables by using region abstraction [38]. The abstracted system is then verified using the standard PDR procedure. Later Isenberg et al. [22] propose a method...
that abstracts clock variables by using zone abstraction [4]. They do not deal with a hybrid system whose flow behavior at each location cannot be described by $\dot{c} = 1$; the system in Figure 2 is out of the scope of their work.

Our continuous-reachability predicates (CRP) are inspired by Platzer’s $d\mathcal{L}$ [33]. We may be able to use the theorem prover KeYmaera X for $d\mathcal{L}$ predicates [16] for our purpose of discharging CRP.

8 Conclusion

We proposed an adaptation of GPDR to hybrid systems. For this adaptation, we extended the logic on which the forward predicate transformer is defined with the continuous reachability predicates $\langle D | \varphi \rangle \varphi$ inspired by the differential dynamic logic $d\mathcal{L}$. The extended forward predicate transformer can precisely express the behavior of hybrid systems. We formalized our procedure HGPDR and proved its soundness. We also implemented it as a semi-automated procedure, which proves the safety of a simple hybrid system in Figure 2.

On top of the current proof-of-concept implementation, we plan to implement a GPDR-based model checker for hybrid systems. We expect that we need to improve the heuristic used in the application of the rule INDUCTION, where we currently check sufficient conditions of the verification condition. We are also looking at automating part of the work currently done by human in verification; this is essential when we apply our method to a system with complex continuous-time dynamics.

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A Proof

A.1 Soundness of Vanilla PDR

In the remainder of this section, we fix a DTSTS $S_D := (Q,q_0,\varphi_0,\delta)$; a predicate transformer $F$ determined by $S_D$; and a safety condition $\varphi_P$ to be verified.

**Definition A.1.** For a frame $R$, we write $[R]_q$ for

$$\exists q' \in Q. (q = q' \land R(q')).$$

**Lemma A.2.** Let $R_0 := F(\lambda q.\text{false})$. Then, $R_0(q_0) = \varphi_0$ and $R_0(q_i) = \text{false}$ if $q_i \neq q_0$.

**Proof.** By the definition of $F$. The frame $R_0$ is equivalent to $\lambda q'.(q' = q_0 \land \varphi_0)$ by Definition 4.1. Therefore, $R_0(q_0) = \varphi_0$ and $R_0(q_i) = \text{false}$ if $q_i \neq q_0$ as required.

**Lemma A.3.** $\models F(R')(q) \Rightarrow F(R)(q)$ for any $R$ and $q$ if $\models R'(q) \Rightarrow R(q)$ for any $q$. 
Proof. Recall the definition of $\mathcal{F}(R)(q')$:

$$(q' = q_0 \land \varphi_0) \lor \bigvee_{(q', \varphi, q'' \varphi) \in \delta} \exists x''. \left( \left[ x''/x \right] R(q) \land \left[ x''/x \right] \varphi \land \left[ x/x', x''/x \right] \varphi_c \right).$$

In the above definition, $R(q)$ appears in the position where the strength of $\mathcal{F}(R)(q')$ is monotonic with respect to the strength of $R(q)$. Therefore, changing $R$ to a pointwise-stronger frame $R'$ strengthens the entire formula as required.

**Lemma A.4.** If $\models \mathcal{F}(R')(q) \implies R(q)$, then $\models q = q_0 \land \varphi_0 \implies R(q)$.

**Proof.** From the definition of $\mathcal{F}$.

**Proof of Lemma 4.5:** Case analysis on the rules in Figure 3.

**Initialize** The condition (Con-A) follows from Lemma A.2. The condition (Con-B) follows from the side condition of INITIALIZE. The other conditions hold vacuously.

**Valid** The resulting configuration is consistent as required since it is Valid.

**Unfold** Let $C := M \parallel R_0, \ldots, R_N; N, \text{Con}(C)$, and UNFOLD is applied to this configuration. Let the resulting configuration $C' := \emptyset \parallel A[R_{N+1} := \lambda q. \text{true}; N := N + 1]$. From the side condition of UNFOLD, we have $\forall q \in Q. \models R_N(q) \implies \varphi_p$. We show $\text{Con}(C')$.

- (Con-A) holds since $R_0$ is not unchanged.
- For (Con-B), it is sufficient to show $\models [R_{N}]_q \implies [R_{N+1}]_q$ for any $q \in Q$. By definition, $[R_{N+1}]_q$ is equivalent to $\exists q' \in Q. (q = q')$, which is true for any $q \in Q$.
- (Con-C) holds from the above side condition.
- For (Con-D), it is sufficient to prove $\models \mathcal{F}(R_N)(q) \implies [R_{N+1}]_q$ for any $q$, which holds since $[R_{N+1}]_q$ is equivalent to $\exists q' \in Q. (q = q')$.

(Con-E) and (Con-F) hold vacuously.

**Induction** Let $C := M \parallel R_0, \ldots, R_N; N, \text{Con}(C)$, and INDUCTION is applied to this configuration. Let $A$ be $R_0, \ldots, R_N$. Then, the resulting configuration $C'$ is $M \parallel A[R := \lambda q. R_q(q) \land R(q)]_{j=1}^{N+1}$, where $\models \mathcal{F}(Aq[R_q(q) \land R(q)](q)) \implies R(q)$ for any $q \in Q$. We show $\text{Con}(C')$.

- (Con-A) holds since $R_0$ is unchanged.
- To prove (Con-B), fix $q'' \in Q$ and $i' \in \{0, \ldots, N-1\}$ arbitrarily. We prove $\models [R_i]_{q''} \implies [R_{i+1}]_{q''}$. If $i' > 0$, then (Con-B) immediately follows from (Con-B) for $C$. Only interesting case is $i' = 0$, in which we must show $\models [R_0]_{q''} \implies [\lambda q. R_1(q \land R(q))]_{q''}$. Since $\models [R_0]_{q''} \implies [R_1]_{q''}$, follows from the condition (Con-B) for $C$, we show $\models [R_0]_{q''} \implies R(q''')$, which follows from Lemma A.4 and (Con-A) for $C$.
- (Con-C) is trivial since INDUCTION strengthen each $[R_i]_{q''}$.
- To prove (Con-D), fix $j < N$ and $q \in Q$ arbitrarily; we show $\models \mathcal{F}(R'_j)(q) \implies [R_{j+1}]_q$ for $R'_j$ and $R_{j+1}$ in $C'$, where $R'_k$ is defined as follows:

$$R'_k := \begin{cases} R_0 & (k = 0) \\ \lambda q. R_k(q) \land R(q) & (1 \leq k \leq i + 1) \\ R_k & (i + 1 < k \leq N). \end{cases}$$
• If \( j > i + 1 \), then (Con-D) for \( C' \) follows from (Con-D) for \( C \).
• If \( j = i + 1 \), then we are to prove \( \models F(\lambda q.R_{i+1}(q) \land R(q))(q) \implies [R_{i+2}]_q \). This follows from (Con-D) for \( C \) and Lemma A.3.
• If \( 1 \leq j < i + 1 \), then we are to prove \( \models F(\lambda q.R_{i}(q) \land R(q))(q) \implies [\lambda q.R_{i+1}(q) \land R(q)]_q \), which is equivalent to (1) \( \models (q = q_0 \land \varphi_0) \implies R_{i+1}(q) \land R(q) \) and (2) \( \models [x''/x]R_j(q) \land [x''/x]R(q) \land [x''/x]\varphi/\varphi_c \implies R_{i+1}(q) \land R(q) \) for any \( (q', \varphi', \varphi_c, q) \in \delta \) and \( x'' \). From (Con-D) for \( C \), we already have \( \models q = q_0 \land \varphi_0 \implies R_{i+1}(q) \) and \( \models [x''/x]R_j(q') \land [x''/x]\varphi/\varphi_c \implies R_{i+1}(q) \) for any \( (q', \varphi', \varphi_c, q) \in \delta \) and \( x'' \). Therefore, it suffices to show that (1') \( \models (q = q_0 \land \varphi_0) \implies R(q) \) and (2') \( \models [x''/x]R_j(q) \land [x''/x]R(q) \land [x''/x]\varphi/\varphi_c \implies R(q) \) for any \( (q', \varphi', \varphi_c, q) \in \delta \) and \( x'' \). The combination of (1') and (2') is equivalent to \( \models F(\lambda q.R_i(q) \land R(q))(q) \implies R(q) \), which follows from the side condition of INDUCTION, (Con-B) for \( C \), and Lemma A.3.
• If \( j = 0 \), then we are to prove \( \models F(R_0)(q) \implies [\lambda q.R_1(q) \land R(q)]_q \), which is equivalent to \( \models F(R_0)(q) \implies (R_1(q) \land R(q)) \). By (Con-D) for \( C \), it suffices to show \( \models F(R_0)(q) \implies R(q) \). We show (1) \( \models F(\lambda q.R_0(q) \land R(q))(q) \implies R(q) \) and (2) \( \models R_0(q) \implies R_0(q) \land R(q) \); then (Con-D) for \( C' \) follows from Lemma A.3. (1) follows from the side condition of INDUCTION, (Con-B) for \( C \), and from Lemma A.3. (2) follows from Lemma A.4.

(Con-E) and (Con-F) hold vacuously.

Candidate (Con-A), (Con-B), (Con-C), and (Con-D) trivially hold because \( A \) is unchanged. (Con-E) is trivial. (Con-F) holds vacuously.

Decide (Con-A), (Con-B), (Con-C), and (Con-D) trivially hold because \( A \) is unchanged. (Con-E) holds because the \( N \)-th element of \( M \) is unchanged. (Con-F) is trivial.

Model Model \((\sigma, q_0, 0) \) \( M \) is consistent by definition.

Conflict \( C = (\sigma', q', i + 1) \) \( M \mid A \) and \( C' = \emptyset \mid A[R_j \leftarrow \lambda q.R_j(q) \land R(q)]_{j=1}^{i+1} \), where \( \models R(q) \implies \neg \sigma' \) and \( \forall q \in Q. \models F(R_1)(q) \implies R(q) \). From the condition \( \forall q \in Q. \models F(R_1)(q) \implies R(q) \) and Lemma A.3, we have \( \forall q \in Q. \models F(\lambda q.R_i(q) \land R(q))(q) \implies R(q) \), which is the same as the side condition for INDUCTION. Notice that the rewriting to a configuration by CONFLICT is the same as that of INDUCTION; therefore, the soundness for this case follows by the same reasoning as the case for INDUCTION.

\[ \square \]

Proof of Theorem 4.6: Suppose that an execution of GPDR starts from INITIALIZE and ends at VALID. By Lemma 4.5 and mathematical induction on the length of the execution, the configuration \( C \) just before it reaches VALID is consistent. Let \( C \) be \( M \mid A \); then, from the side condition of VALID, there exists \( i < N \) such that \( \forall q \in Q. \models R_i(q) \implies R_{i-1}(q) \). For such \( R_{i-1} \), we have the following three facts:

\[ \models R_0(q) \implies R_{i-1}(q) \text{ for any } q \in Q \text{ from (Con-A) and (Con-B);} \]
\[ F(R_{i-1})(q) \Rightarrow R_{i}(q) \text{ for any } q \text{ from (Con-B) and (Con-D)}; \]
\[ R_{i-1}(q) \Rightarrow \varphi_P \text{ for any } q \in Q \text{ from (Con-C)}. \]

Therefore, \( R_{i-1} \) is a fixed point that proves unreachability of \( \neg \varphi_P \) from \( R_0 \) via \( F \). This leads to the safety of the system since Lemma 4.2 asserts that \( F \) soundly approximates the dynamics of the DTSTS \( S_D \).

On the contrary, suppose an execution of GPDR leads to \( \text{Model} \langle \sigma_0, q_0, 0 \rangle \ldots \langle \sigma_N, q_N, N \rangle \) from \text{Initialize}. Let the configuration one step before the final one be \( C \). By the same discussion as the previous case, \( C \) is consistent. We have the following facts about \( \langle \sigma_0, q_0, 0 \rangle \ldots \langle \sigma_N, q_N, N \rangle \):

- \( \sigma_0 \models \varphi_0 \) by (Con-A) and (Con-F);
- From (Con-F), for each \( \langle \sigma^{(1)}, q^{(1)}, i \rangle \) and \( \langle \sigma^{(2)}, q^{(2)}, i + 1 \rangle \), there is a transition from the former to the latter; and
- \( \sigma_N \models \neg \varphi_P \) by (Con-E).

Therefore, \( \langle q_0, \sigma_0 \rangle \rightarrow_\delta \ldots \rightarrow_\delta \langle q_N, \sigma_N \rangle \) is a valid trace of \( S_D \), which witnesses that \( S_D \) is unsafe. \( \square \)

### A.2 Soundness of HGPDR

In the remainder of this section, set \( S_H \) to \( \langle Q, q_0, \varphi_0, F, \text{inv}, \delta \rangle \), and \( F_H \) and \( F_C \) to the forward predicate transformers determined by \( S_H \), and \( \varphi_P \) to a safety condition to be verified.

#### Lemma A.5.

Let \( R_0 := F_H(\lambda q.\text{false}) \). Then, \( R_0(q_0) = \varphi_0 \) and \( R_0(q_i) = \text{false} \) if \( q_i \neq q_0 \).

**Proof.** By the definition of \( F_H \). The proof is the same argument as that of Lemma A.2.

#### Lemma A.6.

\( \models F_H(R')(q) \Rightarrow F_H(R)(q) \) for any \( R \) and \( q \) if \( \models R'(q) \Rightarrow R(q) \) for any \( q \).

**Proof.** By the definition of \( F_H \). The proof is the same argument as that of Lemma A.3.

#### Lemma A.7.

If \( \models F_H(R')(q) \Rightarrow R(q) \), then \( \models q = q_0 \wedge \varphi_0 \Rightarrow R(q) \).

**Proof.** From the definition of \( F_H \).

**Proof of Lemma 5.6:** Case analysis on the rules in Figure 4. We omit the cases whose proof is almost the same as that of Lemma 4.5.

**Unfold** Let \( C := M \ || \ R_0, \ldots, R_N, R_{\text{rem}}; N, \ \text{Con}_H(C) \), and **Unfold** is applied to this configuration. Let the resulting configuration \( C' := \emptyset \ || \ A[R_{N+1} := \lambda q.\text{true}, \ R_{\text{rem}} := \lambda q.\text{true}; N := N + 1] \). From the side condition of **Unfold**, we have \( \forall q \in Q. \models R_{\text{rem}}(q) \Rightarrow \varphi_P \). We show \( \text{Con}_H(C') \).
Induction

Let \( q \in Q \). By definition, \( [R_N + 1]_q \) is equivalent to \( \exists q' \in Q, (q = q') \), which is true for any \( q \in Q \).

- (Con-C) holds from the above side condition and (Con-B-1) and (Con-B-2).
- (Con-D-1) and (Con-D-2) are similar to the proof of Lemma 4.5. 
  (Con-E), (Con-F-1), and (Con-F-2) hold vacuously.

**Conflict** Same as the proof of Lemma 4.5 wherein we use Lemma A.6 instead of Lemma A.3.

**Induction** Let \( C := M \parallel R_0, \ldots, R_N, R_{rem}; N, \text{Con}_H(C) \), and INDUCTION is applied to this configuration. Let \( A \) be \( R_0, \ldots, R_N, R_{rem} \). Then, the resulting configuration \( C' \) is \( M \parallel A[R_j := \lambda q.R_j(q) \wedge R(q)]_{j+1}^N \). We show \( \text{Con}_H(C') \). 

- (Con-A) holds since \( R_0 \) is unchanged.
- (Con-B) holds since \( \text{Con}_C \) is trivial since \( \text{Con}_A \) holds since \( \text{Con}_B \).
- (Con-C) is trivial since INDUCTION strengthens each \( [R_i]_q \).
- For (Con-D-1), fix \( j < N \) and \( q \in Q \) arbitrarily: we show \( I = \text{Con}_H \)(\( R_j(q) \)) \( \Longrightarrow [R_{j+1}]_q \) for \( R_j \) and \( R_{j+1} \). Define

\[
R'_k := \begin{cases} 
R_0 & (k = 0) \\
\lambda_q.R_k(q) \wedge R(q) & (1 \leq k \leq i + 1) \\
R_k & (i + 1 < k \leq N).
\end{cases}
\]

- If \( j > i + 1 \), then (Con-D-1) follows from (Con-D-1) for \( C' \).
- If \( j = i + 1 \), then we are to prove \( I = \text{Con}_H(\lambda q.R_{i+1}(q) \wedge R(q))(q) \) \( \Longrightarrow [R_{i+2}]_q \). This follows from (Con-D-1) for \( C' \) and Lemma A.6.
- If \( 1 \leq j < i + 1 \), then we are to prove \( I = \text{Con}_H(\lambda q.R_j(q) \wedge R(q))(q) \) \( \Longrightarrow [\lambda_q.R_{j+1}(q) \wedge R(q)]_q \), which is equivalent to (1) \( I = \langle q = q_0 \wedge \varphi_0 \rangle \) \( \Longrightarrow R_{j+1}(q) \wedge R(q) \) and (2) \( |x''/x|R(q) \wedge ((x''/x)F(q) \wedge (x''/x)\text{in}v(q)) \wedge |x''/x, x''/x|x|\varphi \). From (Con-D-1) for \( C' \), we already have \( I = \langle q = q_0 \wedge \varphi_0 \rangle \) \( \Longrightarrow R_{j+1}(q) \) and \( |x''/x|R(q) \wedge ((x''/x)F(q) \wedge (x''/x)\text{in}v(q)) \wedge |x''/x, x''/x|x|\varphi \). Therefore, it suffices to show that (1) \( I = \langle q = q_0 \wedge \varphi_0 \rangle \) \( \Longrightarrow R(q) \) and (2) \( |x''/x|R(q) \wedge ((x''/x)F(q) \wedge (x''/x)\text{in}v(q)) \wedge |x''/x, x''/x|x|\varphi \). The combination of (1) and (2) is equivalent to \( I = \text{Con}(\lambda q.R_j(q) \wedge R(q))(q) \) \( \Longrightarrow R(q) \), which follows from the side condition of INDUCTION, (Con-B-1) for \( C' \), and Lemma A.6.
If \( j = 0 \), then we are to prove \( \models F_H(R_0)(q) \implies [\lambda q. R_I(q) \land R(q)]_q \), which is equivalent to \( \models F_H(R_0)(q) \implies (R_I(q) \land R(q)) \). By (Con-B-1) for \( C \), it suffices to show \( \models F_H(R_0)(q) \implies R(q) \). We show (1) \( \models F_H(\lambda q. R_0(q) \land R(q))(q) \implies R(q) \) and (2) \( \models R_0(q) \implies R_0(q) \land R(q) \); then (Con-D-1) for \( C' \) follows from Lemma A.6. (1) follows from the side condition of induction, (Con-B-1) for \( C \), and from Lemma A.6. (2) follows from Lemma A.7.

(Con-E) and (Con-F) hold vacuously.

**PropagateCont**: Let \( C := M \mid R_0, \ldots, R_N, R_{rem}; N, \text{Con}_H(C) \), and PropagateCont is applied to this configuration. Let \( A \) be \( R_0, \ldots, R_N, R_{rem} \). Then, the resulting configuration \( C' \) is \( M \mid A[R_{rem} := \lambda q. R_{rem}(q) \land R(q)] \), where \( \models R_N(q) \lor F_C(R_N)(q) \implies R(q) \) for any \( q \in Q \). We show \( \text{Con}_H(C') \).

- (Con-A) holds since \( R_0 \) is unchanged.
- (Con-B-1), (Con-C), and (Con-D-1) trivially hold since PropagateCont changes only \( R_{rem} \).

- To prove (Con-B-2), fix \( q'' \in Q \) arbitrarily. We are to prove \( \models R_N(q'') \implies R_{rem}(q'') \land R(q'') \). From (Con-B-2), it suffices to show \( \models R_N(q'') \implies R(q'') \), which follows from the side condition of PropagateCont.

- To prove (Con-D-2), fix \( q \in Q \) arbitrarily. We are to show \( \models F_C(R_N)(q) \implies R_{rem}(q) \land R(q) \), which follows from (Con-D-2) for \( C \) and the side condition for PropagateCont.

(Con-E) and (Con-F) hold vacuously.

**ConflictCont**: The argument for this case is almost the same as that of Conflict.

\[ \square \]

**Proof of Theorem 5.7**: Suppose that an execution of GPDR starts from Initialize and ends at Valid. By Lemma 5.6 and mathematical induction on the length of the execution, the configuration \( C \) just before it reaches Valid is consistent. Let \( C \) be \( M \mid A \); then, from the side condition of Valid, there exists \( i < N \) such that \( \forall q \in Q. \models R_i(q) \implies R_{i-1}(q) \). By the same argument as the proof of Theorem 4.6, we can show that \( R_{i-1} \) is a fixed point that proves unreachability of \( \neg \varphi_F \) from \( R_0 \) via \( F_H \). This leads to the safety of the system since Lemma 5.3 asserts that \( F_H \) soundly approximates the dynamics of the HA \( S_H \).

On the contrary, suppose an execution of GPDR leads to

\[ \text{Model}(\sigma_0, q_0, 0) \ldots \langle \sigma_N, q_N, N \rangle \langle \sigma_{rem}, q_N, \text{rem} \rangle \]

from Initialize. Let the configuration one step before the final one be \( C \). By the same discussion as the previous case, \( C \) is consistent. We have the following facts about \( \langle \sigma_0, q_0, 0 \rangle \ldots \langle \sigma_N, q_N, N \rangle \langle \sigma_{rem}, q_N, \text{rem} \rangle \):

- \( \sigma_0 \models \varphi_0 \) by (Con-A) and (Con-F);
- From (Con-F-1), for each \( \langle \sigma^{(1)}, q^{(1)}, i \rangle \) and \( \langle \sigma^{(2)}, q^{(2)}, i + 1 \rangle \), there is a transition from the former to the latter;
- From (Con-F-2), for each \( \langle \sigma^{(1)}, q^{(1)}, N \rangle \) and \( \langle \sigma^{(2)}, q^{(2)}, \text{rem} \rangle \), there is a continuous transition from the former to the latter; and
− \( \sigma_N \models \neg \varphi \) by (Con-E).

Therefore, the run that consists of \( \langle q_0, \sigma_0 \rangle \ldots \langle q_N, \sigma_N \rangle \langle q_{\text{rem}}, \sigma_{\text{rem}} \rangle \) is a valid trace of \( S_H \), which witnesses that \( S_H \) is unsafe. \( \square \)
\( x > 0 \land sum' = sum + x \land x' = x - 1 \)

\( x \geq 0 \land sum = 0 \rightarrow q_0 \xrightarrow{x \leq 0} q_1 \)
$x > 0$

$q_0 \& \forall n0: \forall \sigma \sigma_0 \vdash x (\sigma)$

$x := x - 1$