THE STRUCTURE OF DARK MATTER HALOS IN HIERARCHICAL CLUSTERING THEORIES
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ABSTRACT

During hierarchical clustering, smaller masses generally collapse earlier than larger masses and so are denser on the average. The core of a small-mass halo could be dense enough to resist disruption and survive undigested when it is incorporated into a larger object. We explore the possibility that a nested sequence of undigested cores in the center of the halo that have survived the hierarchical, inhomogeneous collapse to form larger and larger objects determines the halo structure in the inner regions. For a flat universe with \( P(k) \propto k^2 \), scaling arguments then suggest that the core density profile is \( \rho \propto r^{-\alpha} \), with \( \alpha = (9 + 3n)/(5 + n) \). For any \( n < 1 \), the signature of undigested cores is a core density profile shallower than \( \rho \propto 1/r^2 \) and dependent on the power spectrum. For typical objects formed from a cold dark matter (CDM)–like power spectrum, the effective value of \( n \) is close to \(-2\), and thus \( \alpha \) could typically be near \( 1 \), the Navarro, Frenk, & White (NFW) value. Velocity dispersions should also decrease with decreasing radius within the core. However, whether such behavior holds depends on detailed dynamics. We first examine the dynamics using a fluid approach to the self-similar collapse solutions for the dark matter phase-space density, including the effect of velocity dispersions. We highlight the importance of tangential velocity dispersions to obtain density profiles shallower than \( 1/r^2 \) in the core regions. If tangential velocity dispersions in the core are constrained to be less than the radial dispersion, a cuspy core density profile shallower than \( 1/r \) cannot hold in self-similar collapse. We then look at the profiles of the outer halos in low-density cosmological models in which the total halo mass is convergent. We find a limiting \( r^{-4} \) outer profile for the open case and a limiting outer profile for the \( \Lambda \)-dominated case, which approaches the form \( \left[ 1 - \left( r/r_\lambda \right)^{-3} \right]^{1/2} \), where \( 3e \) is the logarithmic slope of the initial density profile. Finally, we analyze a suite of dark halo density and velocity dispersion profiles obtained in cosmological \( N \)-body simulations of models with \( n = 0, -1, \) and \(-2\). The core-density profiles show considerable scatter in their properties, but nevertheless do appear to reflect a memory of the initial power spectrum, with steeper initial spectra producing flatter core profiles. These results apply as well for low-density cosmological models \( (\Omega_{\text{matter}} = 0.2–0.3) \), since high-density cores were formed early, where \( \Omega_{\text{matter}} \approx 1 \).

Subject headings: dark matter — galaxies: clusters: general — galaxies: formation — galaxies: halos — large-scale structure of universe

1. INTRODUCTION

Galaxies and clusters are thought to originate from the growth of small density fluctuations due to gravitational instability, in a universe dominated by dark matter. In hierarchical clustering models, such as the cold dark matter (CDM) models, small-mass clumps of dark matter form first, and subsequently gather into larger and larger masses. The structure of these dark matter clumps, or “halos,” is likely to be related to how the halos formed, the initial spectrum of the density fluctuations, and the underlying cosmology.

Early work on the structure of dark halos concentrated on their outer density profiles, especially in trying to understand flat rotation curves of disk galaxies. The secondary infall paradigm (Gunn & Gott 1972; Fillmore & Goldreich 1984, hereafter FG84; Bertschinger 1985, hereafter B85), suggests that gravitational collapse around a seed perturbation will generically lead to divergent extended halos, which produce a nearly flat rotation curve in the outer regions for the case of \( \Omega_{\text{matter}} \equiv \Omega_m = 1 \). The Gunn-Gott picture would lead to steep convergent profiles in the outer regions for low-density \( (\Omega_m < 1) \) universes. If current estimates for \( \Omega_m \) are correct \( (\Omega_m = 0.3 \pm 0.1) \), then the high-density cores \( (\rho_{\text{core}}/\langle \rho \rangle > 10^3) \) were formed early enough so that the \( \Omega_m = 1 \) picture effectively applies. However, the outer halos represent the current, low-density universe.

The density profiles of dark halos in the inner regions are also of considerable importance. The structure of dark halo cores determines the efficiency of gravitational lensing by galactic and cluster halos, the X-ray emissivity of clusters, and galactic rotation curves in the inner regions. These properties can be well constrained by observations. Therefore, if the core density profiles of dark halos are fossils that do depend on some of the properties of structure formation models, such as their initial power spectrum, one would have a useful observational handle on these properties. It is therefore necessary to understand what determines the nature of the density profiles of dark matter halos, and their cores, ab initio. We discuss this issue here.

Navarro, Frenk, & White (1995, 1996, 1997; collectively NFW) have proposed from their \( N \)-body simulations that dark matter halos in hierarchical clustering scenarios develop a universal density profile, regardless of the scenario for structure formation or cosmology. The NFW profile has an inner cuspy form with density \( \rho \propto r^{-1} \) and an outer envelope of the form \( \rho \propto r^{-3} \). Several other investiga-
tors have also found that the NFW profile provides a moderately good fit to numerical simulations (Cole & Lacey 1996; Tormen, Bouchet, & White 1997; Huss, Jain, & Steinmetz 1999a, 1999b; Thomas et al. 1998). Recently, however, high-resolution simulations of cluster formation in a CDM model by Moore et al. (1998) yielded a core density profile of \( \rho(r) \propto r^{-1.4} \), shallower than \( r^{-2} \), but steeper than the \( r^{-1} \) form preferred by NFW, and consistent with the earlier high-resolution work of Xu (1995). A similar result was also found earlier by Fukushige & Makino (1997). For small-mass halo cores, on the other hand, Kravtsov et al. (1998) find an average core density profile shallower than the NFW form. Xu (1995) also found that there was a large scatter in the logarithmic slope of halo density profiles in both the core and outer regions. One motivation of our work is to examine this issue on general theoretical grounds, while at the same time checking with some of our own numerical experiments the properties of dark halo density and velocity dispersion profiles.

In the next section we discuss the processes that may determine the halo density profile and consider the role of undigested cores in setting the structure of dark halos' cores. For a flat universe with \( P(k) \propto k^n \), scaling arguments suggest that \( \rho \propto r^{-\alpha} \), where \( \alpha = \alpha_{n} = (9 + 3n)/(5 + n) \). As an aside, we note here that for popular cosmological models \( n \approx -2 \) in the appropriate range of wavelengths, giving \( \alpha = 1 \), the NFW value. However, whether such a scaling law indeed holds depends on the detailed dynamics.

In order to explore the dynamical issues, we first consider the self-similar collapse of a single spherically symmetric density perturbation in a scale-free universe. We introduce a fluid approach for analyzing this problem in §3. We highlight the importance of tangential velocity dispersions in obtaining density laws shallower than \( 1/r^2 \) in the core regions. In a companion paper (Subramanian 2000, hereafter KS00), one of us considers these self-similar collapse solutions in greater detail, by deriving and solving numerically the scaled moment equations for such a collapse, including the effect of tangential velocity dispersions.

In §4 we analyze, following the Gunn-Gott paradigm, the outer profiles expected in low-density universes, where an outer profile steeper than \( r^{-3} \) must exist. In §5, we analyze dark halo density and velocity dispersion profiles obtained in cosmological N-body simulations of models with \( n = 0, -1, \) and \(-2 \). We show that the core-density profiles of dark halos show significant scatter in their properties, but nevertheless do appear to reflect a memory of the initial power spectrum. Finally, §6 discusses the results and presents our conclusions.

2. THE DENSITY PROFILES OF DARK HALOS

To fix ideas, consider initially an Einstein–de Sitter universe with \( \Omega = 1 \). This is almost certainly not the correct cosmological model, but it provides a convenient context within which to discuss the formation of structure, and it is likely to be a very good approximation at the epochs \((1 + z) > \Omega_{n}^{-1} \) at which the cores of familiar objects have formed. Also assume that the power spectrum of density fluctuations is a power law, \( P(k) = A k^n \), where the spectral index \( n \) lies between the limits \(-3 < n < 1 \). Structure then grows hierarchically in a self-similar fashion, with small scales going nonlinear first and larger and larger mass scales going nonlinear at progressively later times. What would determine the density profile of a dark matter halo in such a scale-free universe?

First, during the inhomogeneous collapse to form a dark halo, the changing gravitational potential and phase mixing will cause some amount of violent relaxation or “virialization” to occur. A general constraint on the resulting halo will be set by energy and mass conservation, together with the scaling laws that pertain in a hierarchical clustering scenario. Second, in the cosmological context, a collapsed mass is not isolated and will therefore continue to accrete surrounding material as long as matter dominates the energy density. This will alter and/or determine the halo structure in the outer regions.

We here emphasize a third process: when any mass scale collapses, in a hierarchical theory, it will already contain a dominant smaller mass dark halo that collapsed earlier and is therefore denser on average. Typically, the core of such a smaller mass halo is dense enough to resist disruption and survive undigested when it is incorporated into the larger object. A nested sequence of undigested cores in the center of the halo, which have survived the hierarchical inhomogeneous collapse to form larger and larger objects, could thus determine the halo structure in the inner regions. We illustrate this idea schematically in Figure 1.

Suppose a halo of mass \( M \) collapses to form a “virialized” object with a characteristic density \( \rho_0 \) and core radius \( r_c \). For \( P(k) \propto k^n \), simple standard scaling arguments using linear theory (cf. Peebles 1980; Padmanabhan & Subramanian 1992; Padmanabhan 1993) predict

\[
\rho_0(M) \propto M^{-(n + 3)/2}, \quad r_c(M) \propto M^{(n + 5)/6}, \quad \rho_0 \propto r_c^{-(9 + 3n)/(5 + n)}. \tag{1}
\]

Therefore, in the above sequential collapse to form larger and larger objects, the undigested core of each member of the sequence typically contributes a density \( \rho_0 \) at a scale \( r_c \) satisfying the relation \( \rho_0 = c_1 r_c^{-(9 + 3n)/(5 + n)} \), with some constant \( c_1 \). The inner density profile of the larger halo, which is the envelope of the profiles of the nested sequence of smaller mass cores, would then have a form

\[
\rho(r) \propto r^{-\alpha}, \quad \alpha = \alpha_n = \frac{9 + 3n}{5 + n}. \tag{2}
\]

Note that for any \( n < 1 \), we have \( \alpha < 2 \). This form for the density profile (as against the correlation function) is also argued for by Peebles (1980; § 26). Syer & White (1998) use the same form for the case when bigger halos form purely through mergers of smaller halos. Our argument (concluding with eq. [2]) of course neglects both previous generations of undigested cores and secondary infall and is only designed to model the innermost part of a currently virializing object, where their effects on the energetics should be minimal. In addition, since the initial density is a Gaussian random field, there will be a scatter in the subhalo properties, and thus a scatter in the values of \( \alpha \), for different halos (cf. Nusser & Sheth 1999 and §5 below).

One can also state the above argument in terms of the velocity dispersion profile, \( \sigma(r) \). Since \( \sigma \propto (M/r_c)^{1/2} \) and \( r_c \propto M^{(n + 5)/6} \), we therefore have \( \sigma^2 \propto r_c^{-(1 - n)/(5 + n)} \). Thus, for any \( n < 1 \), smaller mass objects have a smaller velocity dispersion and higher phase space densities than larger mass objects. The survival of a nested sequence of cores during
the inhomogeneous collapse to form larger and larger objects then suggests that the velocity dispersion profile in the core regions will scale as \( \sigma^2(r) \propto r^{(1-n)/(5+n)} \). For any \( n < 1 \), an alternate signature of undigested cores is then a velocity dispersion that decreases with decreasing radius, as above. Indeed, the cluster-scale halo core in the Moore et al. (1998) simulation does show the expected decrease of \( \sigma \) with decreasing \( r \) (B. Moore 1998, private communication).

The arguments so far have been semiquantitative but general. We consider below an alternate approach to the density profiles of halo cores via spherically symmetric, self-similar collapse solutions to the Vlasov equation. This model will allow us to examine, in a simple setting, the dynamical constraints on obtaining core density profiles of the form given by equation (2).

3. SELF-SIMILAR COLLAPSE AND HALO DENSITY PROFILES: A FLUID APPROACH

Consider the collapse of a single spherically symmetric density perturbation in a flat-background universe, with the scale factor \( a(t) \propto t^{2/3} \). We examine the dynamics by directly studying the evolution of the distribution function of the dark matter. During the course of this work, we learned that a number of authors (T. Padmanabhan, in preparation; Padmanabhan 1996a; Chieze, Teyssier, & Alimi 1997; Henriksen & Widrow 1997, 1999) have also adopted this approach to the purely radial self-similar collapse problem considered by FG84 and B85. We also emphasize here the role of nonradial motions in self-similar collapse solutions.

The evolution of dark matter phase-space density, \( f(r, v, t) \), is governed by the Vlasov-Poisson system,

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} + a \cdot \frac{\partial f}{\partial v} = 0 ,
\]

\[
\nabla^2 \Phi = 4\pi G \rho = 4\pi G \int fd^3v ,
\]

where \( r \) is the proper coordinate, \( v = \dot{r} \) is the velocity, and \( a = \ddot{v} = -\nabla \Phi \) is the acceleration of the particles. These equations admit self-similar solutions of the form

\[
f(r, v, t) = k_2 k_1^{-3} t^{-q-2p} F \left( \frac{r}{k_1 t^p}, \frac{v}{k_1 t^q} \right) \quad p = q + 1 ,
\]

where \( k_1 \) and \( k_2 \) are constants that we fix to convenient values below. The same solution in comoving coordinates for the density is given by Padmanabhan (1996a), and an elegant derivation from symmetry principles is given by Henriksen & Widrow (1997, 1999). It is convenient to define new scaled coordinates, \( y = r/(k_1 t^p) \), \( w = v/(k_1 t^q) \). For self-similar evolution, using \( p = q + 1 \), the density is given by

\[
\rho(r,t) = \int fd^3v = k_2 t^{-2} \int F(y, w)d^3w \equiv k_2 t^{-2} \psi(y) ,
\]

where \( r = |r| \) and \( y = |y| \). In a flat universe, the background matter density evolves as \( \rho_b(t) = 1/(6\pi G t^2) \). Thus, the density contrast \( \rho(r, t)/\rho_b(t) = \psi(y) \) [with \( k_2 = 1/(6\pi G) \)], and is a function of \( y \) alone.

3.1 Linear and Nonlinear Limits

Let the initial excess density contrast averaged over a sphere of comoving radius \( x = r/a(t) \propto rt^{-2/3} \) be a power law, \( \delta(x, t) \propto x^{-3\epsilon} \). Since \( \rho/\rho_b = \psi(y) \), the \( \delta(x, t) \) will also be a function only of \( y \). Note that in the linear regime, it is the excess density contrast averaged over a comoving sphere that grows as the scale factor \( a(t) \). Therefore, in the linear
Substituting into equation (9), and using

\[ \delta(r, t) = \delta_0 x^{-3\epsilon t^{2/3}} = \delta_0 r^{-3\epsilon t^{2/3} + 2\epsilon} = \delta_0 y^{-3\epsilon r^{-3\epsilon p} + 2/3 + 2\epsilon}. \]  

This is a function of \( y \) alone if and only if \( -3\epsilon p + 2/3 + 2\epsilon = 0 \), which gives

\[ p = \frac{2 + 6\epsilon}{9\epsilon}. \]  

Consider now what happens in the nonlinear limit. The zeroth moment of the Vlasov equation gives

\[ \frac{\partial \rho}{\partial t} + \nabla_r \cdot (\rho v) = 0. \]  

Here \( \bar{v} \) is the mean velocity (first moment of \( f \) over the velocity). In the nonlinear regime, one expects each shell of dark matter, which was initially expanding, to turn around and collapse after reaching a maximum radius. Subsequently, the shell would oscillate between a minimum radius and collapse after reaching a maximum radius. Subsequently, the shell would oscillate between a minimum radius, which depends on how much nonradial velocity the shell particles have, and a maximum radius, which depends on how the mass within the shell grows with time. In regions that have had a large number of shell crossings, the halo particles have settled to nearly zero average infall velocity, that is, \( \bar{v}_r = 0 \) (although they could of course still have velocity dispersions). Using \( \bar{v}_r = 0 \) in equation (8), we have \( \langle \partial \rho / \partial t \rangle = 0 \), and therefore

\[ \rho(r, t) = Q(r) = Q(y^{\epsilon}) = \frac{1}{6\pi G^2} y(\epsilon) . \]  

Substituting \( Q(r) = q_0 r^{-\alpha} \) into equation (9), and using \( r \propto yt^\epsilon \), we obtain \( y^{-3\epsilon t^{2/3}} \propto t^{-2} D(y) \), which implies \( \alpha x = 2 \). Therefore, the density in the nonlinear regime is given by \( \rho(r) \propto r^{-\alpha} \), where

\[ \alpha = \frac{2}{p} = \frac{9\epsilon}{3\epsilon + 1}. \]  

The above result, obtained by following the similar particle trajectory by B85 (for \( \epsilon = 1 \)) and FG84 (for \( 2/3 \leq \epsilon < 1 \)), can therefore be simply obtained by just combining the self-similar solution \( f / \rho \) and the static core condition. Both of these authors also restricted themselves to purely radial orbits. In this case, FG84 argued that while equation (10) holds for \( 2/3 \leq \epsilon < 1 \), for \( \epsilon < 2/3 \), one goes to the limiting value close to \( \alpha = 2 \). However, this is only true for purely radial trajectories (cf. White & Zaritsky 1992; Sikivie, Tkachev, & Wang 1997). We also see below, by considering the higher moments of the Vlasov equation, that \( \alpha < 2 \) can only hold if the system has nonradial velocity dispersions.

What should we choose for the value of \( \epsilon \)? For a power law \( P(k) \propto k^\alpha \), the fractional density contrast averaged over a comoving sphere of radius \( x \) is distributed as a Gaussian, with a variance \( \propto x^{-(3+n)/2} \) (cf. Peebles 1980). This suggests a "typical" spherically averaged initial density law for a halo collapsing around a randomly placed point of the form \( \delta(x, t_0) \propto x^{-(3+n)/2} \), or \( 3\epsilon = (3 + n)/2 \). For such an \( \epsilon \), in the static core regions, \( \rho(r, t) \propto r^{-\alpha} \), where

\[ \alpha = \alpha_n = \frac{9 + 3n}{5 + n}. \]  

This result should apply for collapses from a power-law initial power spectrum.\(^1\) Remarkably, this is the same law we derived earlier for the core of a collapsed halo, assuming that the cores of a sequence of subhalos are left undigested during the formation of the larger halo.

Note that for \( n < 1 \), the density law given by equation (11) is shallower than \( 1/r^2 \), which was claimed to be a limiting form by FG84 in case of radial collapse. To see how such a restriction comes about and when one can obtain a slope shallower than \( r^{-2} \) for the halo cores, it is interesting to consider the higher moments of the Vlasov equation (the Jeans equations) for the spherical self-similar solution.

### 3.2. A Fluid Approach to Collisionless Dynamics

Suppose we multiply the Vlasov equation by the components of \( v \) and integrate over all \( v \). In regions where large amounts of shell crossing have occurred, one can assume that a "quasi-equilibrium" state exists, whereby all odd moments of the distribution function over \( (v - \bar{v}) \) can be neglected. Assume that there is no mean rotation to the halo, that is, \( \bar{v}_r = 0 \) and \( \bar{v}_\phi = 0 \). Then we get

\[ \frac{\partial (\rho \bar{v}_r)}{\partial t} + \frac{\partial (\rho \bar{v}_r^2)}{\partial r} + \frac{\rho}{r} (2\bar{v}_r^2 - \bar{v}_\phi^2 - \bar{v}_r^2) + \frac{GM(r)\rho}{r^2} = 0, \]  

(12)

\[ \bar{v}_\phi^2 = \bar{v}_r^2, \]  

(13)

where \( M(r) \) is the mass contained in a sphere of radius \( r \). For a purely radial collapse, we can set \( \bar{v}_\phi^2 = \bar{v}_r^2 = 0 \). Let us also assume to begin with that one can set \( \bar{v}_r = 0 \) in the inner parts. Then integrating the Jeans equation (12) with \( \rho = q_0 r^{-\alpha} \) gives

\[ \bar{v}_r^2 = r^2 \left[ \frac{4\pi G q_0}{2(\alpha - 2)(3 - \alpha)} \right] ^{1/2} = \frac{1}{2\alpha} \frac{GM(r)}{r}. \]  

(14)

Thus, purely radial self-similar collapse with no tangential velocities, and with \( \alpha > 2 \), leads to a radial velocity dispersion in the core that scales as \( \bar{v}_r^2 \propto r^{-1/2} \). This agrees with the radial velocity dispersion scaling as \( r^{-1/2} \) for the B85 gaseous collapse solution (\( \alpha = 2 \) needs to be treated separately). Furthermore, the right-hand side of equation (14) should necessarily be nonnegative, which is violated when \( \alpha < 2 \). If one has a purely spherically symmetric collapse and zero tangential velocities, then the density law cannot become shallower than \( \alpha = 2 \) and maintain a static core with \( \bar{v}_r = 0 \). This agrees with FG84. Our example illustrates a point we mentioned earlier. Even if simple scaling arguments suggest a \( \alpha < 2 \) possibility, there could be dynamical restrictions on realizing such core profiles.

Let us now include the effect of tangential velocity dispersions. The Jeans equation gives two equations for the three unknown velocity dispersions, even for a static core. To see whether one can close the system, let us look at the second moments of the Vlasov equation (the energy equations).\(^1\)

\(^1\) An alternate choice, \( 3\epsilon = (3 + n) \), would be relevant if one were considering the collapse around an isolated high-density peak, since in this case the initial density profile would be proportional to the correlation function to lowest order (cf. Bardeen et al. 1986). In this case one gets \( \alpha = (9 + 3n)(4 + n) \) (Hofmann & Shaham 1985; Padmanabhan 1996b). (Since \( \epsilon < 1 \) for overdense perturbations, we can use this choice only for \( n < 0 \).)
We get
\[
\frac{\partial (\rho \bar{v}_\theta^2)}{\partial t} + \frac{1}{r^2} \frac{\partial (\rho r^4 \langle v_r v_\theta^2 \rangle)}{\partial r} - \frac{2\rho \langle v_r v_\theta^2 \rangle}{r} \cot \theta + \frac{\rho \bar{v}_\theta^2}{r} \cot \theta = 0 ,
\]
\[\text{equation (15)}
\]
\[\frac{\partial (\rho \bar{v}_\phi^2)}{\partial t} + \frac{1}{r^2} \frac{\partial (\rho r^4 \langle v_r v_\phi^2 \rangle)}{\partial r} + \rho \langle v_r v_\phi^2 \rangle \cot \theta = 0 ,
\]
\[\text{equation (16)}
\]
\[\frac{\partial (\rho \bar{v}_z^2)}{\partial t} + \frac{1}{r^2} \frac{\partial (\rho r^4 \langle v_r v_z^2 \rangle)}{\partial r} - 2\rho \langle v_r (v_\theta^2 + v_\phi^2) \rangle}{r} + 2\bar{v}_r \rho GM}{r^2} = 0 ,
\]
\[\text{equation (17)}
\]
where \(M = \int 4\pi r^2 \rho \) is the mass within \(r \) and angle brackets or a bar over a variable denote a normalized moment over \(f \).

Consistent with our statistical assumption for the core regions, we assume that initially the tangential velocity dispersions have zero skewness. Then in purely spherically symmetric evolution they would not develop any skewness, that is, \(\bar{v}_\theta = \bar{v}_\phi = \langle v_r v_\theta \rangle = 0 \) for all times. In addition, if the initial velocity ellipsoid had one of its principle axes pointing radially, we do not expect this axis to become misaligned in purely spherical evolution. This means that we can assume \(\langle v_r v_\theta \rangle = \bar{v}_\theta = 0 \) in the static core. Equation (15) then implies \(\frac{\partial (\rho \bar{v}_\theta^2)}{\partial t} = 0 \) or \(\rho \bar{v}_\theta^2 = K(r) \) independent of \(t \).

For the scaling solution, we then have
\[
\rho \bar{v}_\theta^2 = K(r) = K(yr^p) = k_2 \epsilon^{4/\epsilon} 2^{-2p} \int w_0 F(y, w) d^3 w .
\]
\[\text{equation (18)}
\]

Once again substituting a power-law solution, \(K(r) = K_0 r^s \), to this functional equation, we get the constraint from matching the power of \(t \) on both sides, \(ps = 4q - 2p \). Using \(p = q + 1 \), we then get \(s = 2 - 4/p = 2 - 2\alpha \), and so
\[
\rho \bar{v}_\theta^2 = K_0 r^{2 - 2\alpha} .
\]
\[\text{equation (19)}
\]

Integrating the radial momentum equation using equations (12), (13), and (19) and using \(p = q_0 r^{-\alpha} \), equation (14) for the radial velocity dispersion is now altered to
\[
\bar{v}_r^2 = r^2 \frac{1}{(2 - \alpha)q_0} \left[ \frac{K_0}{(2 - \alpha)q_0} - \frac{4\pi Gq_0}{2(2 - \alpha)(3 - \alpha)} \right]
\]
\[
\equiv \frac{1}{(2 - \alpha)}\left[ \frac{\bar{v}_r^2(r) - GM(r)}{2r} \right] .
\]
\[\text{equation (20)}
\]

Several important points are to be noted from the above equation. A crucial one is that when \(\alpha < 2 \), the right-hand side of equation (20) can remain positive only provided that one has a nonzero tangential velocity dispersion. In fact, for any \(\alpha < 2 \), one needs the tangential velocity dispersion to be at least as large as \(GM/2r \), comparable to the gravitational potential energy per unit mass. One can also see that to obtain static cores with \(\alpha < 1 \), the required tangential dispersion must be larger than the radial velocity dispersion. Therefore, if in halo cores the tangential velocity dispersions are constrained to be smaller than the radial velocity dispersions, then a core density profile shallower than \(1/r \) cannot hold in the self-similar case. Note also that for \(\alpha < 2 \), all the components of velocity dispersions decrease with decreasing radius, as suggested by the simple scaling arguments of the previous section.

In a realistic collapsing halo, it is quite likely that particles develop nonradial velocities. Tidal forces by mass concentrations outside the halo and the presence of substructure within the collapsing halo will lead to nonradial motion of particles. More generally, the process of violent relaxation during the inhomogeneous collapse to form the halo will lead to a more isotropic velocity dispersion.

The above results for the halo core arise simply from the properties of the self-similar solution and the assumption of a static core. From the energy equation (17), we note that a time-independent radial-velocity dispersion can only apply if the radial velocity skewness \(\langle (v_r - \bar{v}_r) \rangle \) is also zero. Note that in the core regions, where large amounts of shell crossing has occurred, as we stated earlier, the radial skewness is indeed expected to be small. Thus, for the core regions one can in fact make this statistical assumption. Such a treatment will correspond to considering a fluid-like limit to the Vlasov equation.

However, the radial skewness will become important near the radius at which infalling matter meets the outermost reexpanding shell of matter. This region will appear like a shock front in the fluid limit. A possible treatment of the full problem in the fluid approach to the Vlasov equation then suggests itself. This is to take the radial skewness to be zero both inside and outside a “shock” or “caustic” radius, whose location is to be determined as an eigenvalue, so as to match the inner core solution that we determine in this section with an outer spherical infall solution. One must also match various quantities across this “shock,” using jump conditions derived from the equations themselves. To do this requires a numerical solution of the self-consistent set of moment equations, to the scaled Vlasov equation. The details of such a treatment are given in a companion paper (KS00). Here we summarize the general conclusions of this work.

The numerical results in KS00 show the importance of tangential velocity dispersions in deciding whether the self-similar solution with an initial density profile shallower than \(1/r^2 \) \((\epsilon < 2/3) \) retains a memory of this initial profile or whether the density profile tends to a universal \(1/r^2 \) form. The set of solutions show that for a large enough \(\bar{v}_\theta^2/\bar{v}_r^2 > 1 \), the core density profile is indeed close to the form \(\rho \propto r^{-\alpha} \), with \(\alpha = 9\epsilon/(1 + 3\epsilon) \). For \(\bar{v}_\theta^2/\bar{v}_r^2 \sim 1 \), some memory of the initial density profile is always retained; the density profile has an asymptotic form \(\rho \propto r^{-\alpha} \), with \(\alpha < 2 \). When \(\bar{v}_\theta^2/\bar{v}_r^2 \ll 1 \), the density profile converts to the \(1/r^3 \) form derived by FG84. In addition, for very shallow initial density profiles with \(\alpha < 1 \), one must necessarily have a tangential dispersion much larger than the radial dispersion to get a static core region retaining the memory of the initial density profile.

The spherical self-similar collapse solutions provide a useful means of examining the dynamics of dark halo formation and its implications for the core-density profiles, although limited by the spherical symmetry assumption. A complementary approach would be direct cosmological \(N\)-body simulations of halo formation, which we consider in § 5. Before this, we briefly consider the outer profiles of dark halos, especially in low-density universes.

4. OUTER PROFILES OF DARK HALOS

In the cosmological context, any collapsed mass will continue to accrete surrounding material, as long as the matter density dominates the energy density. We now analyze how
such secondary infall alters the outer profile of halos, following the Gunn-Gott paradigm, relaxing the restriction of a flat universe.

Consider the collapse of a spherically symmetric density perturbation in a universe with present matter density parameter $\Omega_m$ and cosmological constant $\Lambda$. Let the initial density be $\rho(r, t_0) = \rho_0(t_0)[1 + \delta(r)]$ and the initial velocity be the Hubble velocity. Consider a spherical shell initially at a radius $r_i$. The evolution of the proper radius $r(t)$ of any such shell before shell crossing is governed by

$$\frac{1}{2} \left( \frac{dr}{dt} \right)^2 - \frac{GM}{r} - \frac{\Lambda r^2}{6} = E(M), \quad (21)$$

where $M = \rho_0(t_0)(4\pi r_i^3/3)(1 + \hat{\delta})$ is the mass enclosed by the shell, and

$$\hat{\delta}(r_i) = \frac{3}{2r_i^2} \int_0^{r_i} \delta_0(u)u^2 du \quad (22)$$

is the spherically averaged value of $\delta(r)$ within $r_i$. The “energy,” $E(M)$, can be fixed by evaluating the left-hand side of equation (21) at the initial time. The shell will turn around at a time, say $t_m$, when $dr/dt = 0$, and its radius is, say, $r(t_m, r_i) \equiv R(r_i)$. Setting $dr/dt = 0$ in equation (21) gives

$$\frac{1 + \hat{\delta}}{Y} + Y^2 \lambda = \frac{\Omega_m - 1}{\Omega_m} - \frac{\Omega_m}{\Lambda} \quad (23)$$

where the total value of $\Omega$ (including the cosmological constant) is $\Omega = \Omega_m(1 + \lambda)$. Here $Y = R(r_i)/r_i$ and $\lambda = \Lambda(3H_i^2\Omega_m)$, where $H_i$ is the Hubble parameter and $\Omega_m$ is the matter density parameter at the initial time $t_0$.

Let us begin with the case $\Lambda = 0$. Then equation (23) gives

$$Y = \frac{R(r_i)}{r_i} = \frac{1 + \hat{\delta}}{\delta_i - \hat{\delta}} \quad (24)$$

where we define $\delta_i \equiv \Omega_m^{-1} - 1$. In an open universe with $\Omega_m < 1$, one needs $\delta_i > \delta$ for a shell to turn around and collapse. For a monotonically decreasing initial density profile, there will then be an outermost shell with an initial radius $r_j$, satisfying $\delta(r_j) = \delta_j$, such that only shells with $r_i < r_j$ can collapse onto the density perturbation. To work out the outer profile after collapse, we need to know the final effective radius, $r(R)$, of a shell turning around at radius $R$. Following the Gunn-Gott picture, we assume that these two radii can be related by $r(R) = f_0 R$, where $f_0$ is some constant. Note that $f_0$ is indeed a constant if the collapse is exactly self-similar and the initial density profile is sufficiently steep ($\delta > 2/3$). For deriving the dominant scaling of the outer profile with radius, even in an open universe, it should also suffice to treat it to be approximately constant (see below). Consider an initial density profile $\delta(r) = \delta_0(r/r_i)^{-3\epsilon}$. Then mass conservation, $4\pi r^2 dr \rho(r) = 4\pi r_i^2 dr \rho_0(t_i)(1 + \delta_0)$, gives a final, postcollapse density profile, $\rho(r)$:

$$\rho(r) = \rho_0(t_i) \frac{\delta(r_i)[1 - (r/r_i)^{3\epsilon}]}{f_0^{3\epsilon} \left[ 1 + 3\epsilon - (r/r_i)^{3\epsilon} \right]} \quad (25)$$

with

$$r_i = r \frac{\delta_i [1 - (r_i/r_i)^{3\epsilon}]}{f_0} \quad (26)$$

(Here we have also assumed that the initial $\delta_i \ll 1$.)

Two limits are of interest. First, for a $\Omega_m = 1$ universe, $\delta_c = 0$, $r_f \rightarrow \infty$, and $[1 - (r_f/r_i)^{3\epsilon}] \rightarrow 1$. Then we recover the standard result

$$\rho(r) = \frac{\rho_0(t_i)}{1 + 3\epsilon} \frac{f_0^3}{r_0^3 \left( r_0 - r_f \right)^{9\epsilon/(1 + 3\epsilon)}} \quad (27)$$

For $\epsilon = 1$, which is the steepest possible value for an initially localized overdense perturbation, we recover a halo profile $\rho(r) \propto r^{-9/4}$ (BB5 and §3).

Now we return to the case of an open universe, but examine the outermost profile, where $r_i \rightarrow r_j$, the critical radius. From equations (25) and (26), the outer profile at large radii is given by

$$\rho(r) \propto \frac{\rho_0(t_i)f_0}{3\epsilon \delta_c} \left( \frac{r}{r_j} \right)^{-4} \quad (28)$$

the slope being independent of the initial power-law slope $\epsilon$. This interesting result seems to have been already known to Gunn (1977), but not much emphasized since then. It is close to the profile obtained observationally for the lowest metallicity stars in our galaxy. Since the outer profile is also a pure power law, our assumption of a constant $f_0$ is likely to be valid in this case. It would be of interest to find an exact similarity solution of the form given in §3, and valid in an open universe that recovers this outer profile.

We now turn to flat cosmological models with $\Lambda \neq 0$. First, Einstein’s equation gives $\lambda - \Omega_m^2 + 1 = 0$ (or $\Omega_i = 1$). Using this and taking $\delta_i \ll 1$, equation (23) for $R(r_i)$ becomes

$$\frac{1}{Y} + Y^2 \lambda = \delta_i \quad (29)$$

For a monotonically decreasing $\delta(r_i)$, the right-hand side of equation (29) monotonically decreases. However, the left-hand side as a function of $Y$ has a minimum value at $Y = Y_c = (2\lambda)^{-1/3}$, which is given by $\delta_j = (3/2)(2\lambda)^{-1/3}$. There again exists a critical radius $r_j$, defined by $\delta(r_j) = \delta_j$, such that only those shells with $r_i < r_j$ will be able to turn around and collapse. For shells with initial radii $r_i > r_j$, the repulsion due to the cosmological constant overcomes the attractive gravitational force, and so they expand forever. (The critical value of $Y$ is $Y_c$ can also be written as $Y_c = 3/2\delta_j$, a limit found by Barrow & Saich 1993, eq. [26].) Although this feature is similar to the open universe case, there is a major difference between the open model and the flat universe with a cosmological constant. In the $\Lambda$-dominated model, even for the limiting case $r_i \rightarrow r_j$, the turnaround radius tends to a finite limit, $R(r_i) \rightarrow R_j Y_c$. In the open model, on the other hand, as $r_i \rightarrow r_j$ (the limiting critical radius beyond which shells expand forever), $\delta_i \rightarrow \delta_c$ and $R(r_i) \sim (\delta_i - \delta_c)^{-1} \rightarrow \infty$. However, in both cases the accreted mass is finite and equal to that initially within $r_i$ or $r_j$.

Consider now the limiting density profile about this outermost cutoff radius for the $\Lambda$-dominated model. For this, it is sufficient to consider values of $r_i$ close to but less than $r_j$ and expand the left-hand side of equation (29) about $Y = Y_c$, retaining up to quadratic terms in $(Y - Y_c)$. We then have for the outermost shells $\delta_j + 3\lambda(Y - Y_c)^3 = \delta_j(r_j)$ or

$$R(r_i; r_i \rightarrow r_j) = r_i Y_c (3\lambda)^{-1/2}[\delta_j(r_i) - \delta_j]^{1/2} r_i \quad (30)$$

where the negative square root must be taken as the turnaround radius, since the collapsing shells are smaller than the maximum value of $R = r_j Y_c$. For computing the collapsed
In each simulation, the center of each halo is selected as the local maximum of the mass distribution within spheres of comoving radius $10^{-1} h^{-1}$ kpc. The density profile of each halo is calculated using spherical averaging with a logarithmic bin size of 0.02 dex. The velocity dispersions (both tangential and radial) are computed in the rest frame of each spherical shell. We have used the density and velocity dispersion profiles of 20 halos in each model for the analysis described below. In particular, we would like to examine whether the halo density profiles show evidence for the scaling laws of §§ 2 and 3 and a dependence on the power spectrum.

In our analysis of the halo density profiles, for each halo we first fitted $\rho(r)$ by a double power law model of the form given by

$$\rho(r) = \frac{r_0}{r} \frac{\rho_0 2^{\beta_0}}{[1 + (r/r_c)]^{\beta_0}}.$$  

(32)

We used the log density versus log radius data, taking all radii within a nominal “virial radius,” say $r_c$, where the density dropped to about 200 times the background density. We made these fits using an interactive data language (IDL) routine (CURVEFIT), which takes a trial model set of parameters and iterates them to minimize the squared sum of the deviations of the fit from the actual density profile. By judicious choice of the starting values of the parameters, it is relatively simple to obtain good convergence in the set of model parameters. The density profile of the 20 halos in each model, $n = 0$, $n = -1$, and $n = -2$, are shown as dotted lines in Figures 2, 3, and 4, respectively.

In our analysis of the halo density profiles, for each halo we first calculated the local power-law slope of the density profile by evaluating $s(r) = (\ln \rho)/d(\ln r)$ from the model fit. We plot this local slope, $s(r)$, for every halo as a thick solid line in the same plot as the density profile plot in Figures 2, 3, and 4. These $s(r)$ plots give the most detailed information regarding the slope of the halo density profiles. If the density profile has a power-law regime, then for this radius range, $s(r)$ would be a straight horizontal line. Some general conclusions can be drawn from the figures themselves. First, for almost all the halos, $s(r)$ keeps increasing monotonically with radius, showing that the halo density profiles are in general curved and that the density profile keeps steepening with radius.

Previous researchers, such as NFW, have in general adopted model double power law profiles to fit halo density profiles, with specific values of the inner slope, $\alpha_0 = 1$, and outer slope, $\beta_0 + \alpha_0 = 3$. We find the heterogeneity of the density profiles to be striking. No simple formulae with fixed $(\alpha_0, \beta_0)$ can fit this data. Indeed, for an unbiased...
Fig. 2.—Density profiles of the 20 halos in $N$-body simulations with $n = 0$, shown as dotted lines in each panel. The halos are numbered in each panel for easy identification. A double power-law fit to each density profile data set is superposed as a light solid line in these figures. For each halo, the local logarithmic slope of the density profile, $s(r) = d(\ln \rho)/d(\ln r)$, calculated from the model fit, is shown as a thick solid line, in the same plot as the density profile.

double power law fit, the innermost value of $s(r)$ lies between 1 and 2, shows a general increase with increasing $n$, and is in general not equal to 1. In addition, the outermost value is generally not equal to 3.

In order to quantify these conclusions better, and since we have a moderately large number of halos (20) in each model, we can look at the statistics of the inner core and outer slopes for each model of structure formation (given by the power spectrum index $n$). We do this by looking at the distribution of $s(r_0)$ and $s(r_v)$ for the 20 halos in each model with a given value of $n$. Here $s(r)$ is the slope function of the $i$th halo in a given model, calculated from the model fit, and $r_0$ is some fiducial characteristic inner radius of a halo.

In Figure 5, we show a histogram of the distribution of the inner core slopes, $s(r_0)$, for the different models of structure formation with $n = 0, -1, \text{ and } -2$, adopting three
different values for \( r_0 \). In the left-hand side of this plot, we have taken \( r_0 \) to be the innermost radius, \( r_i \), of the halo; in the middle histograms we have taken it to be a fixed percentage (10\%, say) of the virial radius, with \( r_0 = 0.1 r_v \), and in the rightmost histograms we have taken a larger values, \( r_0 = 0.15 r_v \). For all the halos in the \( n = -1 \) or \( n = 0 \) models, these three choices correspond to progressively larger and larger value of \( r_0 \). For the \( n = -2 \) case for about half the halos, the innermost radius, \( r_i \), is of the order of \( 0.1 r_v \); hence the close similarity of the histograms for these two cases. The filled arrow in each of these histogram plots shows the location of the median value of the distribution. We also show for comparison, by a thin arrow, the location of the core-slope \( z_n = 3(3 + n)/(5 + n) \) expected on the basis of the scaling arguments of §2.

From this figure we see first that all halos do have a cuspy inner density profile, mostly with \( s(r_0) > 1 \). Furthermore, the core slopes show a clear spectral index dependence, although the inner power laws are all somewhat steeper than the \( z_n = 3(3 + n)/(5 + n) \) form predicted by the scaling.
laws. For example, the median value of the core slope for the $n = -2$ models is $s(r) \sim 1.3$ (leftmost histogram; compared to $s_a = 1$), while for the $n = -1$ and $n = 0$ models, the corresponding median value of the core slope shifts to $s(r) \sim 1.6$ ($s_a = 1.5$) and $s(r) = 1.8$ ($s_a = 1.8$), respectively. For any fixed $n$, there is also a systematic increase of the median slope as one increases $r_0$ and goes from the leftmost to the rightmost histogram. This is to be expected, since we have a curved and continuously steepening density profile.

However, the trends between different models remain (a steeper inner profile for larger $n$). This can be seen already, for example, by comparing the left and right sides of Figure 5.

Furthermore, we checked using the Kolmogorov-Smirnov two-sample test whether the distribution of core slopes, $s(r)$, for different values of $n$ are drawn from the same population or not. We used the one-tailed test to decide whether the values in one sample (say, $n = 0$) are
stochastically larger than the values of the other sample (say, \(n = -2\)). In this test, one computes the two-sample statistic \(D_{M,N} = \max [S_M(X) - S_N(X)]\), where \(S_M(X)\) and \(S_N(X)\) are the cumulative probability distributions of the two samples with \(M\) and \(N\) number of points, respectively, in each sample (in our case, \(M = N = 20\)). The value of \(NMD_{M,N}\) being larger than a given number is then used to rule out the hypothesis (that the samples are drawn from the same population) at various levels of confidence (see Siegel & Castellan 1988, p. 144). This test shows that the distribution of core slopes of the \(n = 0\) model is stochastically larger than the core slopes of the \(n = -2\) model, and not drawn from the same population, at a 99% confidence level. The hypothesis that the core slopes of the \(n = -1\) and \(n = -2\) models are drawn from the same population is ruled out at a weaker 90% confidence level, and the hypothesis of the core slopes of \(n = 0\) and \(n = -1\) models being drawn from the same population is ruled out at a 90%–95% confidence level, depending on the binning used for the data.

Our preliminary conclusion, therefore, from analyzing these cosmological N-body simulations (see Figs. 2–5), is that the core density profiles of dark matter halos do depend on the initial power spectrum of density fluctuations, becoming steeper as the spectral index increases from \(n = -2\) to \(n = 0\).

In Figure 6, we have given the corresponding histogram for the distribution of the outer slopes, \(s(r_0)\), for different models of structure formation. We see from the figure that the distribution of outer profiles is fairly broad. For the models with \(n = 0\) and \(n = -1\), they are spread, with large deviations, about a median value of \(s(r_0) = 3.06\) or \(s(r_0) = 3.02\), respectively. For the halos in the \(n = -2\) simulation, the outer profile is somewhat shallower, being spread around a median value of \(s(r_0) = 2.55\). These results for the outer profile suggest a large scatter about the favored NFW
value of $\beta_0 + \alpha_0 = 3$, even though for some models ($n = 0$ and $n = -1$) the median is near the NFW result.

Figure 7 summarizes the information on the core and outer slopes of dark matter halos, in different models, as a scatter plot. Each point in these scatter plots marks the value of the inner core and outer slope for a particular halo. We also show as a solid cross the location of the median value of these slopes, with the extent of the cross giving the $\pm \sigma_m$ error on the median. (We adopt an error on the median of $\sigma_m = c_N \sigma_N / N^{1/2}$, where $\sigma_N$ is the standard deviation of the $N$ values of the slope distribution and $c_N = 1.214$; see Kendall & Stewart 1963, p. 327). This figure further illustrates the result that the distributions of the core and outer slopes have a large scatter but appear to display systematic trends as one goes from $n = 0$ to $n = -2$.

At this point, we should add a note of caution regarding the determination of the inner slopes. Ideally, to determine the inner core properties, one requires as small a resolution as possible and as many particles within the virial radius as possible, although it is not at present clear what these numbers should be. In the largest halos, we have a few times $10^5$ particles, and our resolution in these halos is about 5%–7.5% of the virial radius. The larger resolution scale is because we have extracted halos from a cosmological PM simulation, although it is one of the best-resolved PM simulation (with a $768^3$ mesh). Of course, one advantage of the present work is that we have a large number of halos (20) in each model, and so can look at the halo properties in a statistical fashion as well. We saw above that the statistical analysis reveals the trend in core slopes more clearly. At the spatial resolution of the current simulations, the overdensity is about $10^4$–$10^5$, which is about the overdensity of real galaxy or cluster halos on the same scale. Therefore,
our simulations may not be severely spatial resolution limited for the present purpose of examining the properties of halos on these scales and larger. Still, it would be very useful to see whether the trends we have found here for the core slope hold up with further high-resolution studies of a large number of halo density profiles. In particular, it will be of great interest to go to even higher overdensity, more inner regions of halos using higher mass and spatial resolution simulations to further test the present findings from simulations as well as our analytic results.

Apart from the density profiles, it is also of interest to study the velocity dispersion profiles of the halos, to see if there is any evidence for undigested cores. As we argued in §§ 2 and 3, this will lead to rising velocity dispersions in the core regions, of the form \( \sigma \propto r^{\frac{1}{2}} \), which also implies a mild but systematic spectral dependence. It is also of interest to check the relative importance of tangential and radial dispersions in the halo. Recall from § 3 that tangential dispersions are needed to get cuspy density profiles shallower than \( 1/r^2 \) in self-similar collapse from power-law initial density profiles, and that a cuspy profile narrower than \( 1/r \) required tangential dispersions to dominate radial dispersion. Our data on the velocity dispersion profiles are too noisy to draw very firm conclusions. However, we do find that most halos show a rise in the radial velocity dispersion with increasing radius in the core regions. Furthermore, the tangential dispersions are smaller than radial and also in general show much weaker (sometimes no) rise with increasing radius in the core.

One may wonder about the importance of two-body relaxation effects in determining the properties of the halo cores in the simulation; for example, significant two-body relaxation could lead to an artificial steepening of the density profiles of the halos in the core regions. We can use the halo properties in the simulation itself to check this, using the standard estimate for the two-body relaxation timescale, \( t_{\text{rel}} \) (cf. Binney & Tremaine 1987, eq. [8.71]; see also Steinmetz & White 1997), and comparing it with the Hubble time, \( t_{\text{H}} \). We obtain

\[
\frac{t_{\text{rel}}}{t_{\text{H}}} = 11.8 h^{-1} \left( \frac{\sigma_s}{\sigma_{\ast}} \right)^{\frac{3}{2}} \left( \frac{m_u}{m} \right) \left( \frac{\rho / \rho_{\ast}}{d_{\ast}} \right)^{-1} \left( \frac{\ln \Lambda}{10} \right)^{-1},
\]

where we have taken the fiducial values \( \sigma_s = 200 \text{ km s}^{-1}, m_u = 1.65 \times 10^7 M_{\odot}, d_{\ast} = 4 \times 10^3, \) and \( \ln \Lambda \) is the usual “Coulomb” logarithm, which is of the order of 10. In general, we find that this number is much larger than unity for all halos, even in the innermost regions. Therefore, two-body relaxation is not expected to be important in the halo cores.

6. DISCUSSION AND CONCLUSIONS

We have concentrated here on the structure of the cores of dark halos in hierarchical clustering theories of galaxy formation. In such theories, cores of dark matter halos could well harbor undigested earlier generation material. Their density structure, in physical as well as phase space, will reflect the times and the cosmological densities when the core material was gathered.

In a flat universe with a power spectrum \( P(k) \propto k^n \), one consequence of undigested cores could be a cuspy core density profile, shallower than that of a singular isothermal profile, having a velocity dispersion profile and rotation curve that rise with increasing radius. Scaling arguments incorporating energy and mass conservation suggest a form for the core density profile of \( \rho(r) \propto r^{-\alpha_h} \), with \( \alpha_h = (9 + 3n)/(5 + n) \). However, they do not tell us how and in fact whether this form will be realized dynamically. To explore this dynamical issue, we first study a simple tractable model: the spherical self-similar collapse of dark matter density perturbations in a flat universe. In § 5, we analyzed the properties of halos obtained in some cosmological N-body simulations, with a power-law spectrum of density fluctuations.

The spherical self-similar collapse has been studied using a fluid approach, in which we directly examine the evolution of the moments of the phase space density. For a purely radial collapse, with initial density profile \( \rho \propto r^{-3}e^{-r} \) and steeper than \( r^{-2} \), by demanding that the core be static, we recover the asymptotic form of the nonlinear density profile: \( \rho \propto r^{-\alpha} \propto r^{-3e(1 + 3e)} \). This agrees with B85 and FG84, who followed particle trajectories to solve the problem. For initial density profiles shallower than \( 1/r^2 \), with \( \epsilon < 2/3 \), we showed that nonradial velocities are required to obtain a static core. In fact, one needs \( \tilde{v}_\theta > GM/2r \). In addition, if a static core must have a cuspy density profile shallower than \( 1/r \) (with \( \epsilon < 1 \)), this requires \( \tilde{v}_\theta > \tilde{v}_r \). Importantly, for the case in which \( 3\epsilon = (3 + n)/2 \) (as would be relevant for collapse around a typical point in the universe), we recover the simple result \( \alpha = \alpha_n = (9 + 3n)/(5 + n) \), with \( \alpha_n < 2 \), for \( n < 1 \).

The results of introducing nonradial velocity dispersions in this approach have been derived by adopting a closure approximation to the moment equations, whereby the skewness of the \( (\bar{v} - \bar{v}) \), is neglected in the core regions. While this will be a good approximation in the multi-streaming regions of the core, the radial peculiar velocity will necessarily have a nonzero skewness (nonzero third moment) near a caustic radius, where collapsing dark matter particles meet the outermost shell of re-expanding matter. In a companion paper (KS00), to take this into account we have idealized this transition region as a “shock,” and numerically integrated the full set of moment equations. The location of the caustic or shock, \( y_o \), in scaled coordinates, is found as an eigenvalue to the problem of matching the single-stream collapse solution at \( y = y_o \) with a core solution, adopting \( v_c = M = 0 \) as a boundary condition at \( y = 0 \). The results of KS00 largely bear out the expectations of § 3 regarding the importance of tangential velocity dispersions. They also illustrate features that may be relevant in more realistic collapse: if newly collapsing material is constrained to mostly contribute to the density at larger and larger radii, then the memory of initial conditions can be retained.

The density profiles of the outer regions of dark halos are briefly studied in § 4, relaxing the restriction to a flat universe and following the Gunn-Gott paradigm. For an open universe and at late times, the outer density profile converts to a limiting form, \( \rho(r) \propto r^{-4} \), where the slope is independent of the power-law slope of the initial density profile. The corresponding limiting outer profile for a \( \Lambda \)-dominated universe was shown to have a form \( \rho(r) \propto [1 - (r/R_c)^{-3e}]^{1/2} \), where \( R_c \) is a characteristic cutoff radius. Therefore, in open and \( \Lambda \)-dominated models the halo mass is convergent and halos have characteristic density cutoffs, which may be observationally testable.

We then turned to a complementary approach, looking at halo properties in numerical simulations of structure for-
mation models with a power spectrum \( P(k) \propto k^n \), with three different values of the spectral index, \( n = -2, -1, \) and 0, and with \( \Omega_m = 1 \). Perfect simulations with these characteristics should evolve in a self-similar fashion. The results are summarized in Figures 2–7. One preliminary conclusion is that the core density profiles of dark matter halos are in general curved, but do depend on the initial power spectrum of density fluctuations, with the local core slope becoming steeper as the spectral index increases from \( n = -2 \) to \( n = 0 \). For example, the median value of the inner core slope, \( s(r_0) \), for the \( n = -2 \) models is \( 1.3 \pm 0.07 \), while for the \( n = -1 \) and \( n = 0 \) models it shifts to \( 1.6 \pm 0.09 \) and \( 1.8 \pm 0.09 \), respectively. These values are generally steeper than \( s_a \), which scaling arguments predict. Furthermore, the Kolmogorov-Smirnov two-sample test shows that the distribution of core slopes of the \( n = 0 \) model is stochastically larger than that of the \( n = -2 \) model, at a 99% confidence level. The NFW value of \( s_a \) is not favored, most halos having a steeper core density profile. Some recent higher resolution simulations of cluster and galaxy scale dark halo formation in the CDM model by Moore et al. (1998, 1999), which resolve the core very well (but only for a few halos), have also obtained a core profile of \( \rho \propto r^{-1.5} \), steeper than the NFW profile.

The velocity dispersion profiles of halo cores in the \( N \)-body simulations are somewhat noisy, but do indicate for most halos a rise in the radial velocity dispersion in the core regions. The tangential dispersions are smaller than radial and show much weaker (sometime no) such rise. The Moore et al. (1998) cluster halo also shows a rise in the velocity dispersions in the core (B. Moore 1998, private communication). Indeed, they point out that a significant fraction of the core material is made from high-density material that collapsed at higher redshift.

An understanding of the core density profiles of dark halos is important for several reasons. For example, the multiple-imaging lensing cross sections will be very different depending on whether one has \( x = 0, -1, \) or \(-2 \) (compare Subramanian, Rees, & Chitre 1987, where \( x = 0 \) was assumed, with Narayan & White 1988, where \( x = -2 \) is adopted). More specifically, a given system is capable of producing multiple images if its surface density exceeds a critical value \( \Sigma_c \) (Turner, Ostriker, & Gott 1984; Subramanian & Cowling 1986). For \( x \geq 1 \), the surface density is divergent in the central regions. Thus, generically, if \( x \geq 1 \), all clusters are capable of producing multiple images. For \( x < 1 \), some clusters can produce multiple images and some cannot. The NFW value of \( x = 1 \) at the boundary with the surface density is logarithmically singular. Thus, the actual values of \( x \) for real systems is quite relevant to the frequency of strong lensing. Since the frequency of multiple images from gravitational lensing also provides a powerful independent test of cosmological models (see, e.g., Cen et al. 1994; Wambsganss, Cen, & Ostriker 1998; Cen 1998), it is important to determine the structure of halo cores. Furthermore, the existence and properties of radial arcs, which have been observed in some cluster-scale lenses, depends on the slope of the inner cusp (see, e.g., Mellier, Fort, & Kneib 1993; Miralda-Escude 1995; Bartelmann 1996; Evans & Wilkinson 1998). For the singular isothermal profile and stronger cusps, radial arcs do not form.

The rotation curves of disk galaxies also hold clues to the core density profile of dark halos. However, it is more difficult to decompose the observed rotation curve unambiguously into contributions from the luminous stellar disk/bulge and the dark halo. The fluid approach adopted in §3 and in KS00 proposes a new way of exploring nonlinear dynamics, which can extend analytic approximations such as the Zeldovich approximation, valid in a single-stream flow, to the multistreaming regime. In the fluid approach, multistreaming regions would correspond to regions with velocity dispersions, generated by the Zeldovich-type caustics. Note that the adhesion approximation is one extreme in which the multistreaming regions are collapsed onto a caustic. It would be interesting to explore this issue further. In this work we have not included the dynamics of the gaseous (baryonic) component, which will be in fact relevant for the interpretation of X-ray observations of clusters. The gas necessarily has an isotropic velocity dispersion, and so will have a different dynamical evolution from that of the dark matter. We hope to return to some of these issues in the future.

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