Solving a two-electron quantum dot model in terms of polynomial solutions of a Biconfluent Heun Equation

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Abstract

The effects on the non-relativistic dynamics of a system compound by two electrons interacting by a Coulomb potential and with an external harmonic oscillator potential, confined to move in a two dimensional Euclidean space, are investigated. In particular, it is shown that it is possible to determine exactly and in a closed form a finite portion of the energy spectrum and the associated eigenfunctions for the Schrödinger equation describing the relative motion of the electrons, by putting it into the form of a biconfluent Heun equation. In the same framework, another set of solutions of this type can be straightforwardly obtained for the case when the two electrons are submitted also to an external constant magnetic field.

Keywords: Schrödinger equation, harmonic oscillator potential, two-electron system, quantum dot model

1. Introduction

Let us consider the problem of two interacting electrons confined to move in an external harmonic oscillator potential of frequency $\Omega$. As an example of such system one can quote electrons confined in a nanometer-scale semiconductor structure, often called a quantum dot [1], where the electrons have been shown to exhibit a two dimensional behavior [2]. Previous numerical calculations suggest that the harmonic oscillator potential can be successfully employed to describe two-electron quantum dots [3]. Thus, this kind of
system can be described by the time independent Schrödinger equation, in atomic units ($\hbar = m = e = 1$),

$$\left\{ -\frac{1}{2}\nabla_1^2 + \nabla_2^2 + \frac{\Omega^2}{2}(r_1^2 + r_2^2) + \frac{1}{|\vec{r}_1 - \vec{r}_2|} \right\} \Psi(\vec{r}_1, \vec{r}_2) = E_T \Psi(\vec{r}_1, \vec{r}_2)$$  \hspace{1cm} (1)

where the subscripts 1 and 2 refer to each one of the electrons. The $\vec{r}_i$ are two-dimensional vectors with length $r_i = |\vec{r}_i|$. Introducing the usual relative and center of mass coordinates, $\vec{r} = \vec{r}_1 - \vec{r}_2$ and $\vec{R} = (\vec{r}_1 + \vec{r}_2)/2$, Eq. (1), with the choice $\Psi(\vec{r}_1, \vec{r}_2) = \chi(\vec{R})\psi(\vec{r})$, gives rise to the following pair of equations:

$$\left[-\frac{1}{2}\nabla^2_{\vec{R}} + \frac{\omega^2_R}{2} \vec{R}^2\right] \chi(\vec{R}) = 2\epsilon \chi(\vec{R})$$ \hspace{1cm} (2)

$$\left[-\frac{1}{2}\nabla^2_{\vec{r}} + \frac{\omega^2}{2} r^2 + \frac{1}{2r}\right] \psi(\vec{r}) = \frac{1}{2} \eta \psi(\vec{r})$$ \hspace{1cm} (3)

where one has defined the frequencies $\omega_R = 2\Omega$ and $\omega = \Omega/2$. The total energy is given by

$$E_T = \epsilon + \eta$$

The equation for the center of mass coordinate is the usual non relativistic equation for the harmonic oscillator in two dimensions, which means that the energy eigenvalues are

$$\epsilon = \omega_R (n_R + 1)$$

with $n_R = 0, 1, 2, 3, \cdots$. The 2D radial Schrödinger equation for the relative coordinate can be obtained by introducing the polar coordinates $(r, \theta)$ and putting its solution in the form

$$\psi(\vec{r}) = r^{-1/2} u(r) e^{\pm il\theta} = R(r) e^{\pm il\theta}$$ \hspace{1cm} (4)

where $l$ is the integer angular momentum quantum number of the two-dimensional system. The radial function $u(r)$ should satisfy the following equation:

$$\frac{d^2 u(r)}{dr^2} + \left[2\epsilon - \frac{1}{r} - \omega^2 r^2 - \left(\frac{l^2 - 1/4}{r^2}\right) \right] u(r) = 0$$ \hspace{1cm} (5)

In Ref. [4], where the problem of two electrons in an external oscillator potential is studied in three dimensions, it is shown that the above radial equation is quasi-exactly solvable [5, 6, 7], which means that it is possible to
find exact simple solutions for some, but not all, eigenfunctions, corresponding to a certain infinite set of discrete oscillator frequencies. In Refs. [8, 9, 10], the same problem was considered in the presence of a homogeneous magnetic field $B$. In this case, the Schrödinger equation can be written, in the notation of Ref. [10], as

$$\left\{ -\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2} \frac{(m^2 - 1/4)}{r^2} + \frac{1}{2} \tilde{\omega}_r^2 r^2 + \frac{1}{2r} \right\} u(r) = \tilde{\varepsilon} r u(r) \quad (6)$$

where $\tilde{\omega}_r = \bar{\omega}/2$, $\tilde{\varepsilon}_r = \varepsilon - m\omega_c/4$, with $\omega_c = B/c$ and $\bar{\omega} = \sqrt{\omega_0^2 + (\omega_c/2)^2}$; $\omega_0$ is the frequency of the harmonic confinement potential and $\omega_c$ is the cyclotron frequency of the magnetic field. It is straightforward to see that, putting $m = l$, Eq. (6) is the same as our Eq. (5) and thus it is sufficient to discuss here the simplest case of Eq. (5).

The main scope of this paper is to propose a different theoretical approach to find analytical solutions for the two-dimensional problem of quantum dots that, we hope, can be applied to other simple two-body physical systems and, at the same time, provide us a way to confirm or not the previous results [8, 9, 10]. The starting point is to transform Eq. (5) into a particular Heun’s differential equation, since it is well known that the solution for some quantum-mechanical systems described by a radial Schrödinger equation can be obtained in terms of those of the Biconfluent Heun Equations (BHE) [11]. The Eq. (5) is indeed a particular case of a Schrödinger equation for a kind of confining potential energy given by

$$U(r) = \pm a \frac{1}{r} + br + cr^2, \quad c > 0 \quad (7)$$

which was formally studied in the literature [12, 13, 14].

We cannot neglect a priori the possibility of finding other particular solutions for the quantum dot states within this new technique.

2. The Biconfluent Heun’s Equation

The radial Schrödinger equation for the confinement potential (with a repulsive Coulombian potential between the two electrons), given by Eq. (7), can be written, in atomic units, as

$$u'' + \left[ 2 \left( E - \frac{a}{r} - br - cr^2 \right) - \frac{\ell(\ell + 1)}{r^2} \right] u(r) = 0 \quad (8)$$
and can be straightforward transformed into the canonical form of BHE [15], namely,

\[ xy'' + (1 + \alpha - \beta x - 2x^2)y' + \left( (\gamma - \alpha - 2)x - \frac{1}{2}(\delta + (1 + \alpha)\beta) \right) y(x) = 0 \] (9)

A simple comparison between Eqs. (5) and (8) allow to define the parameters:

\[ E = \eta; \quad 2c = \omega^2; \quad a = \frac{1}{2}; \quad b = 0; \quad \ell = \pm l - 1/2 \]

Now, putting [16]

\[ u(r) = r^{\ell+1}e^{-\beta_F r + \frac{1}{2}a_F r^2} y(r) \] (10)

where \( x = \sqrt{\alpha_F} \) and

\[ \alpha_F = \sqrt{2c} = \omega; \quad \beta_F = \sqrt{\frac{2}{c}b} = 0; \quad \epsilon_F = \beta_F^2 + 2E = 2\eta, \]

it follows that Eq. (8) becomes

\[ xy'' + \left[ 2(\ell + 1)\sqrt{\alpha_F} - \frac{2\beta_F}{\sqrt{\alpha_F}} x - 2x^2 \right] y' + \]

\[ + \left\{ \frac{\epsilon_F}{\alpha_F} - (2\ell + 3) \right\} x + \frac{a - 2\beta_F(\ell + 1)}{\sqrt{\alpha_F}} \}

\[ y(x) = 0 \] (11)

This is exactly Eq. (9) provide one defines the four Heun parameters as

\[ \alpha = 2(\ell + 1)\sqrt{\alpha_F} - 1; \quad \beta = \frac{2\beta_F}{\sqrt{\alpha_F}}; \quad \gamma = \frac{\epsilon_F}{\alpha_F} + 2(\ell + 1)(\sqrt{\alpha_F} - 1) \]

and

\[ \delta = \frac{2}{\sqrt{\alpha_F}} \left[ -a + 2\beta_F(\ell + 1)(1 - \sqrt{\alpha_F}) \right] \]

Turning back to the original parameters (\( \omega, \eta, l \)) (and keeping the choice \( \ell + 1 = l + 1/2 \), which guarantees that the radial part of the wave function \( \psi \) will be regular at \( r = 0 \), for all values of \( l \)), we get

\[ \alpha = (2l + 1)\sqrt{\omega} - 1; \quad \beta = 0; \quad \gamma = \frac{2\eta}{\omega} + (2l + 1)(\sqrt{\omega} - 1); \quad \delta = -\frac{1}{\sqrt{\omega}} \]

and, therefore, solving Eq. (5) is equivalent to solve the equation

\[ xy''(x) + [1 + \alpha - 2x^2] y' + [-\delta/2 + (\gamma - \alpha - 2)x] y(x) = 0 \] (12)
3. Polynomial solutions

The polynomial solutions of Eq. (12), a particular case of Eq. (9), have been analysed by many authors [17, 18, 19, 20]. When $\alpha$ is not a negative integer, one can denote by $N(\alpha, \beta, \gamma, \delta; x)$ the series solution that can be written as

$$y_1(x) = N(\alpha, \beta, \gamma, \delta; x) = \sum_{p=0}^{\infty} \frac{A_p}{(1+\alpha)_p} x^p$$

with

$$(\alpha)_p = \frac{\Gamma(\alpha + p)}{\Gamma(\alpha)}, \quad p \geq 0$$

The two first coefficients are $A_0 = 1$ and $A_1 = [\delta + \beta(1+\alpha)]/2 = \delta/2$ (for $\beta = 0$). The substituting of $y_1(x)$, given by Eq. (13), as well as its first and second derivatives into Eq. (12), given by Eq. (13), leads to

$$(\alpha + 1) \left[ A_1 - \frac{\delta}{2} A_0 \right] + \left[ A_2 - \frac{\delta}{2} A_1 + (\gamma - \alpha - 2)(\alpha + 1)A_0 \right] x +$$

$$+ \sum_{p=1}^{\infty} \frac{x^{p+1}}{(p+1)!(\alpha + 1)_{p+1}} \left[ A_{p+2} - \frac{\delta}{2} A_{p+1} + (\gamma - \alpha - 2 - 2p)(p+1)(p + \alpha + 1)A_p \right] = 0$$

Therefore, defining $\delta' \equiv -\delta/2$, one should have

$$\delta' A_0 + A_1 = 0$$

$$(\gamma - \alpha - 2)(\alpha + 1)A_0 + \delta' A_1 + A_2 = 0$$

$$(\gamma - \alpha - 2 - 2p)(p+1)(p + \alpha + 1)A_p + \delta' A_{p+1} + A_{p+2} = 0$$

or, in matrix notation,
From the above recurrence expressions, the function $N$, defined in Eq. 13, becomes a polynomial of degree $n$ if and only if

$$\gamma - \alpha - 2 = 2n; \quad n = 0, 1, 2, 3, ... \quad \text{and} \quad A_{n+1} = 0$$

and so the $\gamma_p$ factors of the determinant are given by

$$\gamma_p = 2(n - p)(p + 1)(p + 1 + \alpha), \quad p \geq 2$$

In terms of the original parameters of the problem, this relation gives rise to a quantum condition between the energy $\eta$ and the frequency $\omega$, since

$$\gamma - \alpha - 2 = \left(2 \frac{\eta}{\omega} - 2l - 2\right) = 2n \quad \Rightarrow \quad \eta_{nl} = (n + l + 1) \omega \quad (14)$$

Note that in the case of the polynomial solutions, although they are not the most general one, it is straightforward to find the relation between the energy spectrum and a given oscillator frequency $\Omega = 2\omega$, diversely form the approach followed in Ref. [4].

Each coefficient $A_{n+1}$ is a polynomial of degree $n + 1$ in $\delta'$ and there are at most $n + 1$ suitable values of $\delta'$ [15].

Thus, when $p = n$, the $n$ roots can be determined by solving the $p \times p$ determinant $\Delta$ of the matrix $\mathbb{M}$ defined below:

$$\Delta = \det \mathbb{M} = \begin{vmatrix} \delta' & 1 & 0 & \ldots & \ldots & 0 \\ \gamma_1 & \delta' & 1 & \ldots & \ldots & 0 \\ 0 & \gamma_2 & \delta' & 1 & \ldots & 0 \\ \vdots & 0 & \gamma_3 & \delta' & 1 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \gamma_{n-1} & \delta' \end{vmatrix} = 0$$
This is the determinant of a tridiagonal matrix and can be calculated by
the decomposition \( M = LV \), such that

\[
\begin{vmatrix}
1 & 0 & 0 & \ldots & 0 \\
\ell_2 & 1 & 0 & \ldots & 0 \\
0 & \ell_3 & \ddots & \ddots & 0 \\
0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & 0 & \ell_n & 1
\end{vmatrix} \quad \text{and} \quad
\begin{vmatrix}
v_1 & c_1 & 0 & \ldots & 0 \\
0 & v_2 & c_2 & \ldots & 0 \\
0 & 0 & v_{n-1} & \ddots & 0 \\
0 & 0 & 0 & \ddots & v_n \\
0 & 0 & 0 & 0 & v_n
\end{vmatrix}
\]

where

\[
v_1 = \delta' \\
c_1 = c_2 = \ldots = c_{n-1} = 1 \\
\ell_i = \frac{\gamma_{i-1}}{v_{i-1}} \\
v_i = \delta' - \ell_i = \left( \delta' - \frac{\gamma_{i-1}}{v_{i-1}} \right), \quad i = 2, \ldots, n
\]

Since both matrices in Eq. (15) are triangular matrices, each determinant is equal to the product of the elements of the principal diagonal. Therefore,

\[
det M = det L \cdot det V = 1 \cdot v_1 \cdot v_2 \ldots v_{n-1} \cdot v_n \tag{16}
\]
or

\[
det M = \delta' \times \left[ \delta' - \frac{\gamma_1}{\delta'} \right] \times \left[ \delta' - \frac{\gamma_2}{\delta' - \frac{\gamma_1}{\delta'}} \right] \times \left[ \delta' - \frac{\gamma_3}{\delta' - \frac{\gamma_2}{\delta' - \frac{\gamma_1}{\delta'}}} \right] \times \ldots \times \\
\times \left[ \delta' - \frac{\gamma_{n-1}}{\delta' - \frac{\gamma_{n-2}}{\delta' - \frac{\gamma_{n-3}}{\delta' - \frac{\gamma_2}{\delta' - \frac{\gamma_1}{\delta'}}}}} \right]
\]
Let us now define $d_1 \equiv \delta'$ and
\[
\begin{align*}
    d_2 &= \delta' d_1 - \gamma_1 \\
    d_3 &= \delta' d_2 - \gamma_2 d_1 \\
    d_4 &= \delta' d_3 - \gamma_3 d_2 \\
    &\vdots \\
    d_n &= \delta' d_{n-1} - \gamma_{n-1} d_{n-2} \quad \text{for } n \geq 2.
\end{align*}
\]

Thus, it follows from Eq. (16) that
\[
\det \mathbb{M} = d_1 \times \left( \frac{d_2}{d_1} \right) \times \left( \frac{d_3}{d_2} \right) \times \left( \frac{d_4}{d_3} \right) \times \ldots \times \left( \frac{d_{n-1}}{d_{n-2}} \right) \times \left( \frac{d_n}{d_{n-1}} \right)
\]
or, simply, for each $n \times n$ matrix $\mathbb{M}$,
\[
\det \mathbb{M} = d_n \quad \text{(17)}
\]

The condition $\det \mathbb{M} = 0$ will give rise to a polynomial in $\delta'$ and its independent roots will be denoted by $\delta'_\mu$ ($0 \leq \mu \leq n$). From this procedure one gets the roots of $1/\sqrt{\omega}$ for which the physical polynomial solutions can be constructed. Once these roots are determined, the explicit polynomial form that depends on two quantum numbers $(n, l)$ can be written as:
\[
y_{nl}(x) = N(\alpha, 0, \alpha + 2(n + 1), \delta'_n; x) = \sum_{p=0}^{n} \frac{A_p}{(1 + \alpha)_p} \frac{x^p}{p!}
\]
remembering that $A_0 = 1$, $A_1 = -\delta'$ and, for $p \geq 2$, $A_p$ is a polynomial in $\delta'$ and $\alpha = (2l + 1)\sqrt{\omega} - 1$.

4. Numerical results

Let us now proceed to the explicit determination of the wave functions and respective energy levels for the ground state and other few lower states, namely $n = 1, 2, \ldots, 5$.

The first step is to determine the roots of $1/\sqrt{\omega}$ for each values of $n$ and $l$. Some values corresponding to $l = 0$ states are given in Table 1.

We have also calculated the roots for $l = 1$, as shown in Table 2.
Table 1: Some roots of $1/\sqrt{\omega}$ for $l = 0$ and different values of $n$, compared to those found in Ref. [4].

| $n$ | $1/\sqrt{\omega}$ | $1/\sqrt{\omega}$ (from [4]) |
|-----|------------------|-------------------------------|
| 2   | 0; 2.5198        | 2                             |
| 3   | 0; 7.3825        | 4.4721                        |
| 4   | 2.47047; 14.1004 | 2.2938; 7.3985                |
| 5   | 0; 7.65075       | 4.97; 10.738                  |

Table 2: Some roots of $1/\sqrt{\omega}$ for $l = 1$ and different values of $n$.

| $n$ | $1/\sqrt{\omega}$ |
|-----|-------------------|
| 2   | 0; 3.6342         |
| 3   | 0; 8.1091         |
| 4   | 3.5028; 14.5564   |
| 5   | 0; 8.3627; 21.6111|

Once the roots are determined, one can compute the energy levels and write down, from Eq. (18) the explicit form of the functions $y_{nl}$ ($1 \leq n \leq 5$). Some values for the low states are given in Table 3.

Similarly to Ref. [10], for $n = 2, 3, 5$, we find solutions corresponding to the limit $\omega \to \infty$ ($1/\sqrt{\omega} \to 0$), which was called an asymptotic solution. Notice, however, that for $n = 2$ such kind of solution was not reported in Ref. [10].
Table 3: Comparison between our results and those of Refs. [4, 8] for the lower energy values of different low $n$ states and for $l = 0$. The first column presents the energy calculated by the Hellman-Feynman theorem (in 3D) [4]. The second column displays the results of Ref. [8] and the third one displays our equivalent results, both in two-dimensions.

| $n$ | $\epsilon'$ | $\epsilon'_{int}$ | $\eta_{n0}$ |
|-----|-------------|-------------------|-------------|
| 2   | 0.6250      | 1.000             | 0.4725      |
| 3   | 0.1715      | 0.250             | 0.0734      |
| 4   | 0.8553      | 0.108             | 0.8192      |
| 5   | 0.2228      | 0.592             | 0.1025      |

The polynomials are given by:

\[
y_{1t} = A_0 + \frac{A_1}{(2l + 1)} r
\]

\[
y_{2t} = y_{1t} + \frac{A_2 \sqrt{\omega}}{2(2l + 1)[(2l + 1)\sqrt{\omega} + 1]} r^2
\]

\[
y_{3t} = y_{2t} + \frac{A_3 (\sqrt{\omega})^2}{6(2l + 1)[(2l + 1)\sqrt{\omega} + 1][(2l + 1)\sqrt{\omega} + 2]} r^3
\]

\[
y_{4t} = y_{3t} + \frac{A_4 (\sqrt{\omega})^3}{24(2l + 1)[(2l + 1)\sqrt{\omega} + 1]\ldots[(2l + 1)\sqrt{\omega} + 3]} r^4
\]

\[
y_{5t} = y_{4t} + \frac{A_5 (\sqrt{\omega})^4}{120(2l + 1)[(2l + 1)\sqrt{\omega} + 1]\ldots[(2l + 1)\sqrt{\omega} + 4]} r^5
\]  (19)
where
\[ A_0 = 1 \]
\[ A_1 = -\left(\frac{1}{2\sqrt{\omega}}\right) \]
\[ A_2 = \left(\frac{1}{2\sqrt{\omega}}\right)^2 - 4(2l + 1)\sqrt{\omega} \]
\[ A_3 = -\left(\frac{1}{2\sqrt{\omega}}\right)^3 + \frac{4}{\sqrt{\omega}} + 6(2l + 1) \]
\[ A_4 = \left(\frac{1}{2\sqrt{\omega}}\right)^4 - \frac{8}{(\sqrt{\omega})^2} - 6(2l + 1) \left[ \frac{1}{\sqrt{\omega}} - 16\sqrt{\omega} + 8 \right] \]
\[ A_5 = -\left(\frac{1}{2\sqrt{\omega}}\right)^5 + \frac{10}{(\sqrt{\omega})^3} - \frac{192}{\sqrt{\omega}} + (2l + 1) \left[ \frac{5}{(\sqrt{\omega})^2} - 24[1+4(2l+1)]\sqrt{\omega} - 400 \right] \]

Knowing the polynomials \( y_{nl} \), the non-normalized wave functions \( u_{nl}(r) \) are given by Eq. (10), namely
\[ u_{nl}(r) = r^{l+1/2} e^{-\omega_n r^2/2} y_{nl}(r) \]  
(21)
and
\[ \psi(\vec{r}) = r^l e^{-\omega_n r^2/2} y_{nl}(r) e^{\pm i\theta} = R_{nl}(r, \omega_n) e^{\pm i\theta} \]  
(22)
where each \( \omega_n \) is a root of the respective \( A_n \).

Let us now determine the wave function normalization. The highest polynomial we are considering here is of the type \( y = 1 + Ar + Br^2 + Cr^3 + Dr^4 + Er^5 \). In such case, its square can be written as
\[ y^2 = \sum_{n=0}^{10} c_n r^n \]
where the coefficients \( c_n \) are defined by: \( c_0 = 1 \), \( c_1 = 2A \), \( c_2 = A^2 + 2B \), \( c_3 = 2(C + AB) \); \( c_4 = B^2 + 2(D + AC) \); \( c_5 = 2(E + AD + BC) \); \( c_6 = C^2 + 2(AE + BD) \); \( c_7 = 2(BE + CD) \); \( c_8 = D^2 + 2CE \); \( c_9 = 2DE \) and \( c_{10} = E^2 \). Therefore, the normalization factor for this radial state is \( N_{nl} = 1/\sqrt{I} \), where
\[ I = \sum_{n=0}^{10} c_n \int_0^{\infty} e^{-\omega r^2} r^{(2l+1+n)} \, dr \]
All the integrals are of the type

\[ \int_0^\infty \exp(-\mu r^p) r^{\nu-1} \, dr = \frac{1}{p} \mu^{-\nu/p} \Gamma\left(\frac{\nu}{p}\right) \]

So,

\[ N_{nl} = \left[ \frac{1}{2} \sum_{k=0}^{10} c_k \left( \frac{1}{\sqrt{\omega}} \right)^{2(l+1)+k} \Gamma\left(l + 1 + \frac{k}{2}\right) \right]^{-1/2} \]

For the normalization of the state \( n = 4 \) we have just to put \( E = 0 \) in the previous formula; for \( n = 3, E = D = 0 \) and so on. The normalization of each \((n, l)\) state is shown in Table 4.

| \((n, l)\) | \(N_{nl}\) |
|------------|---------|
| (2,0)      | 0.240372 |
| (2,1)      | 0.0486791 |
| (3,0)      | 0.010507 |
| (3,1)      | 0.00186614 |
| (4,0)      | 0.00124923 |
| (4,1)      | 0.0413502 |
| (5,0)      | 0.017836 |
| (5,1)      | 0.00139954 |

Let us examine the system response for a microwave external excitation, which corresponds to the choice \( \Omega = 0.06 - 0.6 \) THz or \( 0.01 \text{ Ha} < \Omega < 0.1 \) Ha (remember that \( \omega = \Omega/2 \)).

A specific program was developed by the authors in C ++ language and both calculations and graphics shown below were done by using the CERN/ROOT package.

Fig. 1 and Fig. 2 present a plot of how the radial wave function \( R_{nl} \), defined by Eq. (22), depends on \( r \) and \( \omega \), respectively, for \( n = 1 \) and \( n = 5 \)
(both for the s state). In the case \( l = 0 \), the behavior of the states \( n = 2, 3 \) are similar to \( n = 1 \) while the case \( n = 4 \) is more like \( n = 5 \). These graphics lead us to fix from now on the value of \( \omega = 0.01 \) Ha in all the evaluations that follow.

Figure 1: Radial wave function \( R_{nl} \), for \( l = 1 \) and \( n = 2, 3, 4, 5 \).

Figure 2: Radial wave function \( R_{nl} \), for \( l = 1 \) and \( n = 2, 3, 4, 5 \).
In the next figure we plot the wave functions $R_{40}(r)$ for two possible frequencies.

![Graph showing wave functions $R_{40}(r)$ for two frequencies.](image)

Figure 3: The behaviour of the radial wave function $R_{40}$, for all the possible $l$ values.

The mean values $< r >$ of the distance between the electrons were calculated for the $(n = 2, 3, 4, 5; l = 0)$ states discussed in the paper, and are shown in Fig. 4 and in Table 5.

![Graph showing mean distance $< r >$ for different $n$ states.](image)

Figure 4: The mean value of the distance between the electrons, for different $n$ states. The legend of the figure is: triangle ($l = 0$) and square ($l = 1$).
Table 5: The mean values $< r >$ for $n = 2, 3, 4, 5$ and $l = 0, 1$ states.

| $(n, l)$ | $\langle r_{nl} \rangle \text{ [Bohr]}$ |
|---------|----------------------------------|
| (2,0)   | 3.66754                          |
| (2,1)   | 6.48012                          |
| (3,0)   | 12.8597                          |
| (3,1)   | 16.1769                          |
| (4,0)   | 3.79963                          |
| (4,1)   | 8.26335                          |
| (5,0)   | 15.3347                          |
| (5,1)   | 18.6872                          |

5. Discussions

Let us start remembering that the approach followed in this paper allows us to get a quantization relation between the energy of the quantum dot and the external frequency, namely, $\eta_{nl} = (n + l + 1)\Omega/2$. Indeed, there is no need to recourse to the Hellman-Feynman theorem to calculate the energy corresponding to the frequencies for which analytical solutions have been found [4].

So far the energy values are considered, we can see that there is a significant difference between our values and those found in Ref. [8] for all states from $n = 2$ to $n = 5$.

The smaller distance between the two electrons is found to be $< r > \simeq 3.7$ Bohr (for the state $n = 2$ and $l = 0$) (Fig. 4). The length scale of all the values shown in Table 5 is compatible with the semiconductor lattice parameter as should be expected for quantum dots, i.e., electrons confined in a nanometer-scale semiconductor structure. Therefore, for the set of analytical solutions found here, all the configurations have a mean characteristic distance of the electrons in a quantum dot, $< r >$, comprehend in the range of 3.7 to 18.7 Bohr, in the case of external oscillations in the microwave frequency range.
We hope that the technique developed here can be used to find solutions of the Schrödinger equation for other complex potentials like, for example, the Kratzer potential, which contains both a repulsive part and a long-range attraction and is relevant to describe some molecular systems, and for the charmonium and bottomonium states submitted to a confining phenomenological Cornell potential. Also, the cases of quasi-exactly solvable potentials for the Klein-Gordon and the Dirac equations can be analyzed within the framework of the present paper.

We are now investigating numerical solutions for quantum dots [21], following the technique of Ref. [22], considering the $1/r$ and the $\ln r$ Coulombian potentials.

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References

[1] S.M. Reimann, M. Manninen, Reviews of Modern Physics 74 (2002) 1283.

[2] Ch. Sikorski, U. Merkt, Physical Review Letters 62 (1989) 2164.

[3] U. Merkt, J. Huser, M. Wagner, Physical Review B 43 (1991) 7320.

[4] M. Taut, Physical Review A 48 (1993) 3561.

[5] A. Turbiner, Sov. Phys., JETP 67 (1988) 230.

[6] A. Turbiner, Commun. Math. Phys. 118 (1988) 467.

[7] A.G. Usheridze, Quasi-Exactly Solvable Models in Quantum Mechanics, Bristol: Institute of Physics (1993).

[8] M. Taut, J. Phys. A 27 (1994) 1045, and corrigendum J. Phys. A 27 (1994) 4723.

[9] M. Taut, J. Phys.: Condens. Matter 12 (2000) 3689.
[10] M. Taut, H. Eschrig, Z. Phys. Chem. 224 (2010) 631.

[11] A. Lemieux, A.K. Bose, Annales de L’Institut Henri Poincaré 10 (1969) 259.

[12] R.N. Chaudhuri, B. Mukerjee, J. Phys. A: Math. Gen. 16 (1983) 209.

[13] R.N. Chaudhuri, B. Mukerjee, J. Phys. A: Math. Gen. 17 (1984) 3327.

[14] E.R. Arriola, A. Zarzo, J.S. Dehesa, J. Comput. Appl. Math. 37 (1991) 161.

[15] A. Ronveaux (Ed.), Heun’s Differential Equations, Oxford: Oxford University Press (1995).

[16] B. Léauté, G. Marcilhacy, J. Phys. A: Math. Gen. 19 (1986) 3527.

[17] A. Hautot, Bull. Soc. Roy. Sci. Liège 38 (1969) 654; 660.

[18] A. Hautot, Bull. Soc. Roy. Sci. Liège 40 (1971) 13.

[19] J. Rovder, Math. Čas. 24 (1974) 15.

[20] K.M. Urwin, Bull. Soc. Roy. Sci. Liège 44 (1975) 169.

[21] F. Caruso, J. Martins, V. Oguri, in preparation.

[22] F. Caruso, J.A. Helayël-Neto, J. Martins, V. Oguri, Eur. Phys. J. B 86 (2013) a.n. 324.