SU(2) QCD IN THE PATH REPRESENTATION:
GENERAL FORMALISM AND MANDELSTAM IDENTITY

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Abstract

We introduce a path-dependent hamiltonian representation (the path representation) for SU(2) with fermions in 3 + 1 dimensions. The gauge-invariant operators and hamiltonian are realized in a Hilbert space of open path and loop functionals. We obtain two new types of Mandelstam identities one that connects open path operators with loop operators and other involving the end points of the paths.
1 Introduction

For many years, considerable effort has been made to understand the numerical aspects of lattice gauge theories. However, in spite of many remarkable achievements, these calculations have some inherent limitations and they do not help to understand many of the beautiful analytic features of the theory. The Montecarlo methods are probably not the most economical device to understand the real dynamics of the system. In fact, the highly symmetric nature of the action makes the numerical treatments of the Euclidean partition function somewhat inefficient. Furthermore, computational simulations including dynamical fermions are very expensive in computer time due to the Grassmannian character of the fermionic variables.

A first natural step towards the identification of the real degrees of freedom of the problem is to use the hamiltonian approach. The residual time independent gauge symmetry can be eliminated with the introduction of gauge invariant objects. Because of their simple behavior under gauge transformations, holonomies or phase factors may be used as the basic variables, without ever resorting to the use of gauge potentials or field strengths. The hamiltonian approach to pure lattice gauge theories in terms of a basis of gauge invariant loop states is known as the loop representation \[1\]. This approach has recently deserved much attention due to the work of Rovelli and Smolin \[2, 3\] on the \(SL(2, \mathbb{C})\) gauge theory formulation of quantum general relativity. That is why several alternative proposals to formulate the lattice gauge theory in a gauge invariant manner have recently appeared \[4, 5, 6, 7, 8\].

The use of the loop representation as a tool to extract information from a hamiltonian eigenvalue equation has been considered by many authors \[4, 5, 6, 7, 8\] mainly in the case of pure abelian and non abelian \(SU(2)\) and \(SU(3)\) gauge theories. The action of the hamiltonian operator typically results in geometric deformations or rearrangements (fusion, fission, rerouting, etc) of the loops. In order to introduce matter into the formalism, loops need to be generalized to open paths with charged fermions at the end points. The path representation has been recently applied to the study of quantum electrodynamics with light fermions \[13\] leading to very interesting results.
about the phase structure of the hamiltonian theory \cite{14, 15}.

In this paper, we extend the path representation to the case of lattice QCD with staggered fermions. As it was already noticed \cite{13} in the case of QED the open path representation presents clear advantages when fermions are present. In fact, the main features of the loop representation for pure gauge theories are preserved. The Grassmannian character of the fermionic variables is absorbed into the geometry of the gauge invariant path functionals and the introduction of fermions only implies a moderate increment in the dimensionality of the Hilbert space. The basis of the Hilbert space in the path representation is naturally associated with the physical excitations (gluons, mesons and barions) of the theory. The loop representation generally involve an overcomplete set of basis elements labeled by loops. In order to have a representation equivalent to the original connection representation, one has to impose appropriate conditions on the set of loop dependent functions. The conditions usually called the Mandelstam constraints are a set of group dependent relations among the loop variables \cite{16, 1, 5, 7}. In the case of gauge theories in presence of matter we have two new types of Mandelstam constraints, one that relates open path dependent functions with loop dependent functions and other that involves the end points of the paths. These constraints are related with the reconstruction properties of the spinorial variables and play a fundamental role at the physical level. In fact we shall see that in this language the QCD hamiltonian only involves mesonic operators and the barionic states only arise via the new Mandelstam constraints. Although the formulation is general in scope we have restricted ourselves in this paper to the $SU(2)$ QCD.

The paper is organized as follows. In section 2 we introduce the path representation for $SU(2)$ with fermions in $3 + 1$ dimensions and we study the realization of the operators in a Hilbert space. In section 3 we introduce the new Mandelstam identities for open paths and we show how this constraints may be used to reduce the redundancy of the basis of paths. In section 4 we examine the symmetries and action of the hamiltonian in the path representation and we sketch how the path dependent formalism allows to accomplish explicit calculations. Conclusions and some final remarks are given in section 5.
2 Algebra of the Gauge Invariant Path Functionals

The formulation of a quantum gauge theory in the Hilbert space of kets $|\Psi, A_\mu\rangle$ requires to work with a multiplicity of equivalent states under gauge transformations. This redundancy in the states may be eliminated if we work in a path-dependent hamiltonian representation, the $P$ representation, which offers a direct gauge-invariant characterization of the physical states by a basis of the kets $|P\rangle$, where $P$ labels a set of open paths and loops, associate to lines of flux and pure excitations. The gauge invariant objects that one can construct from the phase factor $U(P)$ and the fermionic fields are given by

$$W(L) = Tr[U(L)]$$
$$\Omega_{ij}^{fg}(P_x^y) = \Psi_{iA}^f(x)U_{AB}(P_x^y)\Psi_{jB}^g(y)$$
$$\Xi_{ij}^{fg}(P_x^y) = \Psi_{iA}^f(x)\epsilon_{AB}U_{BC}(P_x^y)\Psi_{jC}^g(y)$$

where $L$ is a loop and the fermionic fields are located at the ends of the open path $P$. The indices $f$ and $g$ label the flavours of the quarks, $i$ and $j$ are spinorial indices and $A, B = 1, 2$ are the internal color indices for the representation of SU(2). The difficulty in changing to the $P$ representation is that the scalar product between the former configuration vectors and $P$ vectors is not defined. In fact the gauge invariant object $\Omega$ is not a pure object of the configuration space because it includes the canonical conjugate moment of $\Psi$. The solution is to consider the Hilbert space of kets $|\Psi_u^\dagger, \Psi_d, A_\mu\rangle$, where $u$ corresponds to the upper part of the Dirac spinor and $d$ to the lower part, and to define the scalar inner product with a $P$ vector as

$$\Phi_{ij}^f(P_x^y f g) = \langle P_{x, i, j}^{y, f, g} | \Psi_{u}^\dagger, \Psi_d, A_\mu \rangle$$
$$= \Psi_{uiA}^f(x)U_{AB}(P_x^y)\Psi_{djB}^g(y)$$
$$= \Psi_{uiA}^f(x)\left\{ exp \left[ -i \int_x^y A_i(y, t) dy \right]_{ord} \right\}_{AB} \Psi_{djB}^g(y)$$

We have also the following gauge invariant path operators
\[
\Phi_1 \left( P_{x;i,j}^{y,f,g} \right) = \Psi_{uiA}^f(x) U_{AB}(P_x^y) \Psi_{ujB}(y)
\]
\[
\Phi_2 \left( P_{x;i,j}^{y,f,g} \right) = \Psi_{diA}^f(x) U_{AB}(P_x^y) \Psi_{djB}(y)
\]
\[
\Phi_3 \left( P_{x;i,j}^{y,f,g} \right) = \Psi_{diA}^f(x) U_{AB}(P_x^y) \Psi_{ujB}(y)
\]
\[W(L) = Tr \left[ U(L) \right] = Tr \left\{ \exp \left[ -ig \oint_L A_i(y, t) dy_i \right] \right\} \] (4)

In the lattice we consider the configuration basis \{ | \chi^\dagger(x_e), \chi(y_o), U(\ell) \rangle \}, where the \chi's are single-component lattice fields, or staggered fermions [17], and \( x_e \) and \( y_o \) labels even and odd sites in the net. The \( \chi \) field carries a color index and obeys the canonical anticommutation relations

\[
\left[ \chi^\dagger(\vec{r}), \chi(\vec{r}') \right]_+ = \delta_{\vec{r}, \vec{r}'} \quad \left[ \chi(\vec{r}), \chi(\vec{r}'') \right]_+ = 0
\] (5)

We define the path operator \( \tilde{U}(P_x^y) \) as the product of link operators \( U(\ell) \), along of the path \( P_x^y \)

\[
\tilde{U}(P_x^y) = \prod_{\ell \in P_x^y} U(\ell) \eta(\ell)
\] (6)

the \( \eta \)'s are

\[
\eta(x_1) = (-1)^{x_3} \quad \eta(x_2) = (-1)^{x_1} \quad \eta(x_3) = (-1)^{x_2} \quad \eta(-\hat{n}) = \eta(\hat{n})
\] (7)

where \( (x_1, x_2, x_3) \) are the origin coordinates of the link \( \ell \).

The factors \( \eta(\ell) \) arise when we work with staggered fermions [13], they have the information of the Dirac matrices in the lattice, and their introduction in the \( \tilde{U}(P_x^y) \) operator allows to simplify the expression of the hamiltonian and its symmetry properties.

Negative links are not independent since one demands

\[
U(\overline{\ell}) = U^\dagger(\ell)
\] (8)
The inner products between configuration space vectors and path dependent vectors are given by

\[
\langle \chi^\dagger(x_e), \chi(y_o), U(\ell) \mid L \rangle = \text{Tr} [U(L)]
\]

\[
\langle \chi^\dagger(x_e), \chi(y_o), U(\ell) \mid P_{y_o}^{y_e} \rangle = \chi^\dagger_A(x_e) \tilde{U}_{AB}(P_{x_e}^y) \chi_B(y_o)
\]

\[
\langle \chi^\dagger(x_e), \chi(y_o), U(\ell) \mid P_{x_o}^{y_e} \rangle = \chi_A(x_o) \varepsilon_{AB} \tilde{U}_{BC}(P_{x_e}^y) \chi_C(y_o)
\]

\[
\langle \chi^\dagger(x_e), \chi(y_o), U(\ell) \mid P_{x_e}^{y_e} \rangle = \chi^\dagger_A(x_e) \tilde{U}_{AB}(P_{x_e}^y) \varepsilon_{BC} \chi^\dagger_C(y_e)
\]

where $L$ is a loop and the paths $P_{x_e}^{y_o}$ connect one antiquark with one quark, $P_{x_e}^{y_o}$ two quarks, and $P_{x_e}^{y_e}$ two antiquarks. We choose the even sites, $x_e$, as the beginning and odd site, $y_o$, as the end of the paths between one antiquark and one quark, later we shall explain the reason for this choice. Then in this representation we introduce the following set of path dependent operators:

\[
\tilde{W}(L) = \text{Tr} \left[ \prod_{\ell \in L} U(\ell) \eta(\ell) \right]
\]

\[
\phi_0(P_{x}^{y}) = \chi^\dagger_A(x_e) \tilde{U}_{AB}(P_{x_e}^y) \chi_B(y_o)
\]

\[
\phi_1(P_{x}^{y}) = \chi^\dagger_A(x_e) \tilde{U}_{AB}(P_{x_e}^y) \chi_B(y_e)
\]

\[
\phi_2(P_{x}^{y}) = \chi_A(x_o) \tilde{U}_{AB}(P_{x_x}^y) \chi_B(y_o)
\]

\[
\phi_3(P_{x}^{y}) = \chi_A(x_o) \tilde{U}_{AB}(P_{x_x}^y) \chi_B(y_e)
\]

We can write the four operators $\phi_i$ like

\[
\phi(P_{x}^{y}) = \chi^\dagger_A(x) \tilde{U}_{AB}(P_{x_x}^y) \chi_B(y)
\]

where the sites may be even or odd. The operator $\tilde{W}(L)$ is associated with the Wilson operator through

\[
\tilde{W}(L) = \text{Tr} \left[ \prod_{\ell \in L} U(\ell) \eta(\ell) \right] = \left[ \prod_{\ell \in L} \eta(\ell) \right] W(L)
\]

and for the single plaquette we have

\[
\tilde{W}(\Box) = -W(\Box)
\]
We shall have another set of four gauge-invariant operators

\[ \Gamma(P_y x) = \chi_A(x) \varepsilon_{AB} \tilde{U}_{BC}(P_y x) \chi_C(y) \] (15)

The operator \( \tilde{W} \) will be associated to gluon excitations. The operator \( \phi_0 \) represents the quark-antiquark interaction, a meson, and

\[ \Gamma_2(P_y x) = \chi_A(x_o) \varepsilon_{AB} \tilde{U}_{BC}(P_y x) \chi_C(y_o) \] (16)
\[ \Gamma_1^\dagger(P_y x) = -\chi_A^\dagger(y_e) \tilde{U}_{AB}(P_y x) \varepsilon_{BC} \chi_C^\dagger(x_e) \] (17)

are respectively associated to quark-quark and antiquark-antiquark interactions, which are the analogous of barions and antibarions in SU(2) \[18\]. We shall call \( \phi_0 \) a mesonic operator and \( \Gamma_2 \) and \( \Gamma_1^\dagger \) barionic operators.

We have also the electric operator

\[ E_\ell = E_\ell^A X^A \] (18)

which satisfies the commutation relations

\[ [E_\ell^A, E_{\ell'}^B] = i \varepsilon^{ABC} \delta_{\ell\ell'} E_\ell^C \] (19)
\[ [E_\ell^A, \tilde{U}_{\ell'}] = -\delta_{\ell\ell'} X^A \tilde{U}_{\ell'} \] (20)

where \( \varepsilon^{ABC} \) are the structure constants and \( X^A \) the generators

\[ [X^A, X^B] = i \varepsilon^{ABC} X^C \] (21)

which are normalized by

\[ Tr(X^A X^B) = \delta^{AB} \] (22)

The algebra of this gauge invariant representation is obtained from the commutators (19) and (20), the anticommutators (5), and from the following equal time commutators of the bosonic and fermionic operators

\[ [E_\ell^A, \chi(x)] = [E_\ell^A, \chi^\dagger(x)] = [\tilde{U}_\ell, \chi(x)] = [\tilde{U}_\ell, \chi^\dagger(x)] = 0 \] (23)
Then, the algebra of the gauge invariant operators is given by

\[
\begin{align*}
[\phi(P_y^x), \phi(P_u^v)] &= 2 \delta_{yu} \phi(P_y^x P_u^v) - \delta_{vx} \phi(P_u^v P_y^x) \\
[\Gamma(P_y^x), \Gamma(P_u^v)] &= 0 \\
[\Gamma(P_y^x), \Gamma^+(P_u^v)] &= \delta_{yu} \delta_{xz} \bar{W}(P_y^x P_u^v) + \delta_{vu} \delta_{xz} \bar{W}(P_u^v P_y^x) \\
[\phi(P_y^x),\Gamma(P_u^v)] &= -\delta_{xv} \Gamma(P_u^v P_y^x) - \delta_{ux} \Gamma(P_x^z P_u^v) \\
[\phi(P_y^x),\Gamma^+(P_u^v)] &= \delta_{uy} \Gamma^+(P_x^y P_u^v) + \delta_{vy} \Gamma^+(P_y^y P_u^v) \\
\end{align*}
\]

They are

\[
\begin{align*}
[E_{op}, \bar{W}(L)] &= \sum_{\ell,\ell'} \delta_{\ell\ell'} \left\{ \bar{W}(L_{xz}) \bar{W}(L_{zz}) - \frac{1}{2} \bar{W}(L) \right\} - 2 \sum_{\ell \in L} \bar{W}(\ell, L) \\
[E_{op}, \phi(P_x^y)] &= \sum_{\ell,\ell' \in P_x^y} \delta_{\ell\ell'} \left\{ \phi(P_x^y) \bar{W}(L_{zz}) - \frac{1}{2} \phi(P_x^y) \right\} - 2 \sum_{\ell \in P_x^y} \Pi(\ell, P_x^y) \\
[E_{op}, \Gamma(P_x^y)] &= \sum_{\ell,\ell' \in P_x^y} \delta_{\ell\ell'} \left\{ \Gamma(P_x^y) \bar{W}(L_{zz}) - \frac{1}{2} \Gamma(P_x^y) \right\} - 2 \sum_{\ell \in P_x^y} \Lambda(\ell, P_x^y) \\
\end{align*}
\]

where \(z\) is the origin of the link \(\ell\),

\[
\delta_{\ell\ell'} = \begin{cases} 1 & \text{for } \ell = \ell' \\ -1 & \text{for } \ell = \ell' \\ 0 & \text{otherwise} \end{cases}
\]
and $\tilde{W}(\ell, L)$, $\Pi(\ell, P^y_x)$ and $\Lambda(\ell, P^y_x)$ respectively are the gauge invariant conjugate operators of $\tilde{W}(L)$, $\phi(P^y_x)$ and $\Gamma(P^y_x)$,

$$
\tilde{W}(\ell, L) = \text{Tr} \left[ \tilde{U}(L_{zz}) E_\ell \right]
$$

$$
\Pi(\ell, P^y_x) = \chi^{-1}_M(x) \tilde{U}_{MN}(P^y_x) X^A_{NS} \tilde{U}_{ST}(P^y_z) \chi_T(y) E^A_\ell \tag{35}
$$

$$
\Lambda(\ell, P^y_x) = \chi_M(x) \varepsilon_{MN} \tilde{U}_{NS}(P^y_x) X^A_{SP} \tilde{U}_{TR}(P^y_z) \chi_R(y) E^A_\ell \tag{36}
$$

where $z$ is the origin of the link $\ell$ and $x, y, z$ may be even or odd. These operators are the analogous to the $T^{(1)}$ in the language of Rovelli-Smolin [4].

Finally, in order to study the action of the path dependent operators we need the commutators

$$
[\tilde{W}(L), \tilde{W}(\ell, L')] = \sum_{\ell' \in L} \delta_{\ell \ell'} \left\{ \tilde{W}(L'_{zz} L_{zz}) - \frac{1}{2} \tilde{W}(L) \tilde{W}(L') \right\} \tag{37}
$$

$$
[\tilde{W}(L), \Pi_0(\ell, P^v_u)] = \sum_{\ell' \in L} \tilde{\delta}_{\ell \ell'} \left\{ \phi_0(P^v_u L_{zz}^z P^v_z) - \frac{1}{2} \phi_0(P^v_u) \tilde{W}(L) \right\} \tag{38}
$$

$$
[\tilde{W}(L), \Lambda_2(\ell, P^v_u)] = \sum_{\ell' \in L} \tilde{\delta}_{\ell \ell'} \left\{ \Gamma_2(P^v_u L_{zz}^z P^v_z) - \frac{1}{2} \Gamma_2(P^v_u) \tilde{W}(L) \right\} \tag{39}
$$

$$
[\phi_0(P^y_x), \tilde{W}(\ell, L)] = \sum_{\ell' \in P^y_x} \delta_{\ell \ell'} \left\{ \phi_0(P^y_x L_{zz}^z P^y_z) - \frac{1}{2} \phi_0(P^y_x) \tilde{W}(L_{zz}) \right\} \tag{40}
$$

$$
[\phi_0(P^y_x), \Pi_0(\ell, P^v_u)] = - \sum_{\ell' \in P^y_x} \delta_{\ell \ell'} \left\{ \phi_0(P^y_x L_{zz}^z P^y_z) \phi_0(P^v_u L_{zz}^z P^v_z) + \frac{1}{2} \phi_0(P^y_x) \phi_0(P^v_u) \right\} \tag{41}
$$

$$
[\phi_0(P^y_x), \Lambda_2(\ell, P^v_u)] = - \sum_{\ell' \in P^y_x} \delta_{\ell \ell'} \left\{ \Gamma_2(P^v_u L_{zz}^z P^y_z) \phi_0(P^y_x L_{zz}^z P^y_z) \right\} \tag{42}
$$
\[
\left[ \Gamma_2(P_y^x), \tilde{W}(L, L) \right] = \sum_{\ell' \in P_y^x} \delta_{\ell \ell'} \left\{ \Gamma_2(P_{x \to z}^z L_{zz} P_{z \to y}^y) - \frac{1}{2} \Gamma_2(P_y^x) \tilde{W}(L_{zz}) \right\}
\]

\[
\left[ \Gamma_2(P'_{u \to y}), \Pi_0(\ell, P_y^x) \right] = - \sum_{\ell' \in P'_{u \to y}} \delta_{\ell \ell'} \left\{ \phi_0(P_{x \to z}^z P_{z \to y}^y) \Gamma_2(P_{u \to y}^y) \right\} + \frac{1}{2} \phi_0(P_y^x) \Gamma_2(P_{u \to y}^y)
\]

\[
\left[ \Gamma_2(P_y^x), \Lambda_2(\ell, P'_{u \to y}) \right] = - \sum_{\ell' \in P_y^x} \delta_{\ell \ell'} \left\{ \Gamma_2(P_{x \to z}^z P'_{z \to y}^y) \Gamma_2(P'_{u \to y}^y) \right\} + \frac{1}{2} \Gamma_2(P_y^x) \Gamma_2(P_{u \to y}^y)
\]

where \(z\) is the origin of the link \(\ell\) and \(P_{x \to z}^z\) is the portion of path \(P_y^x\) from the site \(x\) to the site \(z\). The algebra displayed in the Eqs. (24-29), (31-33) and (36-42) generalizes the algebra of gauge invariant operators [1, 2] for the case of open paths. With the help of this algebra we shall study the realization of the operators in a Hilbert space spanned by the kets | \(P\rangle\).

The action of the \(\tilde{W}(L)\) operator is to add a loop to \(P\)
\[
\tilde{W}(L) | P \rangle = | P + L \rangle
\]

The operators \(\phi_0(P_y^x), \Gamma_2(P_y^x)\) and \(\Gamma_1^\dagger(P_y^x)\) add the open path \(P_y^x\) to the original collection of paths labeled by \(P\)

\[
\phi_0(P_{x_e}^y) | P \rangle = | P + P_{x_e}^y \rangle
\]

\[
\Gamma_2(P_{x_e}^y) | P \rangle = | P + P_{x_e}^y \rangle
\]

\[
\Gamma_1^\dagger(P_{x_e}^y) | P \rangle = | P + P_{x_e}^y \rangle
\]

The action of the operator \(\phi_3\) on a mesonic path is to annihilate the quarks of the ends of the path
\[
\phi_3(P'_{u \to y}^y) | P_{x_e}^y \rangle = \delta_{xu} \delta_{yu} \tilde{W}(P'_{u_o}^y P_{x_e}^y)
\]
on a single barionic path is null

\[ \phi_3(P_u' v) \mid P_{x_o} y o, P_{x_e} y e \rangle = 0 \] (48)

and on two open paths is to joint the paths

\[ \begin{aligned}
\phi_3(P_x y) \mid P_{u_o} v_o P_{t_o} s_o \rangle &= \\
\delta_{xt} \delta_{yu} \mid P'_{x_o} P_{s_e} y o, P'_{u_e} v_o \rangle - \delta_{xt} \delta_{yu} \mid P_{s_e} x_o P_{u_e} y o, P'_{v_o} v_o \rangle \\
- \delta_{xv} \delta_{ys} \mid P'_{u_e} P_{x_o} y o P''_{s_o} t_o \rangle
\end{aligned} \] (49)

\[ \begin{aligned}
\phi_3(P_x y) \mid P_{u_o} v_o P_{s_o} t_o \rangle &= \\
\delta_{yu} \delta_{xv} \mid P_{x_o} P_{u_e} v_o P'_{s_o} t_o \rangle - \delta_{yu} \delta_{xs} \mid P'_{v_o} P_{x_o} y e P''_{s_o} t_o \rangle \\
- \delta_{yu} \delta_{xt} \mid P'_{v_o} P_{y e} P''_{u_o} t_o \rangle
\end{aligned} \] (50)

\[ \begin{aligned}
\phi_3(P_x y) \mid P_{u_o} v_o P_{s_e} t_e \rangle &= \\
\delta_{yt} \delta_{xv} \mid P'_{s_e} x_o P_{y e} P'_{v_o} v_o \rangle + \delta_{xt} \delta_{ys} \mid P'_{s_e} x_o P_{u_e} v_o \rangle \\
+ \delta_{xt} \delta_{ys} \mid P'_{s_e} x_o P_{y e} P'_{t_o} t_o \rangle + \delta_{xs} \delta_{yt} \mid P'_{s_e} x_o P_{y e} P'_{u_o} u_o \rangle
\end{aligned} \] (51)

From the matricial element \( \langle \Psi^*_{u_o}, \Psi_d, A_\mu \mid E_\ell \mid 0 \rangle = 0 \) we see that the action of the \( E_\ell \) operator acting on the null path vanishes and, using the commutation relations of \( E_\ell \) with path dependent operators Eqs. (31-33), we may see that the action of \( E_{op} \) on the basis of path can be written as the sum of four terms with a well defined geometrical action

\[ E_{op} \mid P_{1 x_1} y_1, P_{2 x_2} y_2, \ldots, P_{m x_m} y_m \rangle = \{ E_L + E_\Lambda + E_{fus} + E_{fis} \} \mid P_{1 x_1} y_1, P_{2 x_2} y_2, \ldots, P_{m x_m} y_m \rangle \] (52)

The first and second terms are diagonal
\[ E_L | P_{1x_1}, \ldots, P_{mx_m} \rangle = 2 \sum_{i=1}^{m} L_i | P_{1x_1}, \ldots, P_{mx_m} \rangle \] (53)

\[ E_\Lambda | P_{1x_1}, \ldots, P_{mx_m} \rangle = -\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \Lambda_{ij} | P_{1x_1}, \ldots, P_{mx_m} \rangle \] (54)

\( L_i \) is the number of links of the \( i \)th path and

\[ \Lambda_{ij} = \sum_{\ell \in P_i} \sum_{\ell' \in P_j} \delta_{\ell\ell'} \] (55)

is a quadratic measure of the overlap between pairs of paths in the list. It will be called the quadratic length.

The other two terms are related with fusion and fission effects on paths. The fission effects are produced by the presence of equal or reversed links in different paths of the state [Fig. 1]. In general it is given by

\[ E_{fus} | P_{1x_1}, \ldots, P_{mx_m} \rangle = -2 \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{i<j} \sum_{\ell \in P_i} \sum_{\ell' \in P_j} \delta_{\ell\ell'} | P_{1x_1}, \ldots, P_{1x_{i-1}}, P_{i-1x_{i-1}}, P_{ix_i} P_{jx_j} y_j, \ldots, P_{jx_{j-1}}, P_{jx_j} P_{ix_i} P_{jx_{j-1}}, \ldots, P_{mx_m} \rangle \] (56)

but this expression will depend on the kind of paths involucrated in the fussion:

1. When the fusion is between two open paths the sum in \( P_i \) and \( P_j \) is given by

\[ \sum_{\ell \in P_i} \sum_{\ell' \in P_j} \delta_{\ell\ell'} | P_{1x_1}, \ldots, P_{1x_{i-1}}, P_{ix_i} P_{jx_j} y_j, \ldots, P_{jx_{j-1}}, P_{jx_j} P_{ix_i} P_{jx_{j-1}}, \ldots, P_{mx_m} \rangle \] (57)
2. If the fusion is between one loop \((P_j = L)\) and an open path \((P_i)\)

\[
- \sum_{\ell \in P_i} \sum_{\ell' \in L} \delta_{\ell \ell'} \left| P_{1, x_1}^{y_1}, \ldots, P_{i-1, x_{i-1}}^{y_{i-1}}; P_{i, z}^{x_i} ; P_{i, y_i} L_{zz}, \ldots, P_{j, x_j}^{y_j} ; P_{j+1, y_{j+1}}^{x_{j+1}} , \ldots, P_{m, y_m} \right> \quad (58)
\]

3. When the fusion is between two loops, \(L_i\) and \(L_j\), we have that

\[
- \sum_{\ell \in L_i} \sum_{\ell' \in L_j} \delta_{\ell \ell'} \left| P_{1, x_1}^{y_1}, \ldots, P_{i-1, x_{i-1}}^{y_{i-1}}; P_{i, z}^{x_i} ; L_{i, z} L_{j, z}, \ldots, P_{j, x_j}^{y_j} ; P_{j+1, y_{j+1}}^{x_{j+1}} , \ldots, P_{m, y_m} \right> \quad (59)
\]

In Eqs. (58-59) the site \(z\) is the origin of the link \(\ell\). Therefore, the fusion term creates one new path from the product of colliding paths. Decreasing the path number in the state by one.

Finally, the fission term contributes when there are multiple links inside one path, [Fig. 2]. In these cases the paths breaks into two parts increasing the number of paths by one. The action of \(E_{fis}\) is given by

\[
E_{fis} \left| P_{1, x_1}^{y_1}, \ldots, P_{m, y_m} \right> = \sum_{i=1}^{m} \sum_{\ell, \ell' \in P_i} \delta_{\ell \ell'} \left| P_{1, x_1}^{y_1}, \ldots, P_{i, x_i}^{y_i} P_{x_i}^{y_i} ; L_{i, z} L_{j, z}, \ldots, P_{m, y_m} \right> \quad (60)
\]

where \(z\) is the origin of the link \(\ell\).

3 Mandelstam Identities for Open Paths

It is important to realize that not all states \(|P\rangle\) are independent because the SU(2) group properties and the Mandelstam identities of the first and second kind [19, 16, 11, 5, 7]:

\[
\]
\[ \tilde{W}(L_1 \circ L_2) = \tilde{W}(L_2 \circ L_1) \quad (61) \]
\[ \tilde{W}(L_1) \tilde{W}(L_2) = \tilde{W}(L_1 \circ L_2) + \tilde{W}(L_1 \circ \overline{L_2}) \quad (62) \]

impose relations among them [Fig. 3a].

The usual Mandelstam identities connect only loops, we have now new identities that connect open paths and loops

\[ \tilde{W}(L) \phi_0(P_x^y) = \phi_0(P_x^z L_{zz} P_z^y) + \phi_0(P_x^z \overline{L}_{zz} P_z^y) \quad (63) \]
\[ \tilde{W}(L) \Gamma_2(P_x^y) = \Gamma_2(P_x^z L_{zz} P_z^y) + \Gamma_2(P_x^z \overline{L}_{zz} P_z^y) \quad (64) \]
\[ \tilde{W}(L) \Gamma_1^i(P_x^y) = \Gamma_1^i(P_x^z L_{zz} P_z^y) + \Gamma_1^i(P_x^z \overline{L}_{zz} P_z^y) \quad (65) \]

where \( z \) is a contact point between the loop \( L \) and the open paths \( P \) [Fig. 3b]. It is important to notice that it is not necessary that the loop \( L \) and the open path have a common point since using the cyclic property of the loop functional \( \tilde{W} \) we always may choose a loop

\[ L' = Q_z^u L_{ww} Q_w^z \]

where \( z \) is a point of the open path and, for instance, the Eq. (63) may be rewritten as

\[
\tilde{W}(L) \phi_0(P_x^y) = \tilde{W}(L') \phi_0(P_x^y) \\
= \phi_0(P_x^z Q_z^u L_{ww} Q_w^z P_z^y) + \phi_0(P_x^z Q_z^u \overline{L}_{ww} Q_w^z P_z^y) \quad (66)
\]

This identity holds for any path \( Q \).

For \( L_1 = L_0 \) (the null loop) we have from the Mandelstam identity of first kind, Eq. (61), that

\[ \tilde{W}(L) = \tilde{W}(\overline{L}) \quad (67) \]
that is, the global orientation of the loop does not matter.

We have also a new type of Mandelstam indentities that exclusively connect operators of open paths, this constraints involve the end points of the paths connecting mesonic and barionic operators. The fermionic character of the $\chi$ fields ensure that

$$\chi_A(y) \chi_B(y) = \frac{1}{2} \{ \chi_A(y) \chi_B(y) - \chi_B(y) \chi_A(y) \} = \frac{1}{2} \varepsilon_{AB} \varepsilon_{ST} \chi_S(y) \chi_T(y) \quad (68)$$

then, for two open paths with the same end points we have [Fig. 4]

$$\phi_0(P_x^y) \phi_0(P_x'^y) = \frac{1}{4} \tilde{W}(P_y^x P_x'^y) \Gamma_1^\dagger(P_o^{xx}) \Gamma_2(P_y^{yy}) \quad (69)$$

$$\Gamma_2(P_x^y) \Gamma_2(P_x'^y) = -\frac{1}{4} \tilde{W}(P_x'^y P_y^x) \Gamma_2(P_o^{xx}) \Gamma_2(P_y^{yy}) \quad (70)$$

$$\Gamma_1^\dagger(P_x^y) \Gamma_1^\dagger(P_x'^y) = -\frac{1}{4} \tilde{W}(P_x'^y P_y^x) \Gamma_1^\dagger(P_o^{xx}) \Gamma_1^\dagger(P_y^{yy}) \quad (71)$$

These identities together with the Mandelstam constraints (62) allow to obtain the identities (63-65). For two open paths with one equal end point [Fig. 5] we have

$$\phi_0(P_x^y) \phi_0(P_u^y) = \frac{1}{2} \Gamma_1^\dagger(P_u^y P_y^x) \Gamma_2(P_o^{yy}) \quad (72)$$

$$\phi_0(P_x^y) \phi_0(P_v^u) = \frac{1}{2} \Gamma_1^\dagger(P_o^{xx}) \Gamma_2(P_y^x P_v^u) \quad (73)$$

$$\phi_0(P_x^y) \Gamma_2(P_u^y) = -\frac{1}{2} \phi_0(P_x^y P_u^y) \Gamma_2(P_o^{yy}) \quad (74)$$

$$\phi_0(P_x^y) \Gamma_1^\dagger(P_v^u) = -\frac{1}{2} \Gamma_1^\dagger(P_o^{xx}) \phi_0(P_v^u P_y^x) \quad (75)$$

$$\Gamma_2(P_x^y) \Gamma_2(P_u^y) = -\frac{1}{2} \Gamma_2(P_x^y P_u^y) \Gamma_2(P_o^{yy}) \quad (76)$$

$$\Gamma_1^\dagger(P_x^y) \Gamma_1^\dagger(P_v^u) = -\frac{1}{2} \Gamma_1^\dagger(P_x^y P_v^u) \Gamma_1^\dagger(P_o^{yy}) \quad (77)$$

where $P_o$ is the null path.
Finally, we can see that the product $\chi_A(y_o) \chi_B(y_o) \chi_C(y_o)$ is null since it will have two identical indices. Therefore, two is the maximum number of quarks that we can put in a site, then, we have

$$\phi_0(P_x^y) \Gamma_2(P'_y^y) = \phi_0(P_x^y) \Gamma_1^\dagger(P'_x^x) = 0$$

(78)

and

$$\Gamma_2(P_x^y) \Gamma_2(P'_y^y) = \Gamma_1^\dagger(P_x^y) \Gamma_1^\dagger(P'_x^x) = 0$$

which are not independent constraints because they may be obtained from the constraints (72) and (76) with the (78).

The Mandelstam identities may be used to reduce the redundancy of the basis of open paths and loops. For instance, if we consider a loop $L$ containing a double link $\ell$, we may write $L$ as $A\ell B\ell$ with $A$ and $B$ closed parts of $L$, [Fig. 5a]. Then, from the Mandelstam identity of second kind, Eq. (62), we have

$$\tilde{W}(L) = \tilde{W}(L_1 L_2) = \tilde{W}(A\ell B\ell) = \tilde{W}(A\ell) \tilde{W}(B\ell) - \tilde{W}(A\ell B) = 0$$

(80)

therefore, a state containing loops with multiple links may be expressed as a linear combination of states where the links have reduced multiplicity. Then, in forming the basis, one has only to consider states with loops without multiple links. From the identities (63-65) we can obtain a similar expression for open paths with a double link [Fig. 5b]. Finally, the constraints for open paths (69-79) allow one to work with a basis where different open paths do not have the same end points.

4 Hamiltonian Dynamics

The hamiltonian is given by
\[ H = \frac{g^2}{2} \left( W_E + W_m + \lambda W_q + \lambda^2 W_M \right) \]  \hspace{1cm} (81)

with \( \lambda = 1/g^2 \) and

\[
    W_E = E_{op} \quad (82)
\]
\[
    W_m = m \sum_{\vec{r}} (-1)^{x+y+z} \chi^\dagger(\vec{r}) \chi(\vec{r}) \quad (83)
\]
\[
    W_q = \sum_{\vec{r}, \vec{n}} \chi^\dagger(\vec{r}) \tilde{U}(\vec{r}, \vec{n}) \chi(\vec{r} + \vec{n}) + H.c. \quad (84)
\]
\[
    W_M = -\sum_{\Box} \left( W(\Box) + W^\dagger(\Box) \right) \quad (85)
\]

where \( \vec{r} = (x, y, z) \) labels the sites and \( \Box \) the plaquettes [17].

The action of the electric term, \( W_E \), is given by the action of the electric operator \( E_{op} \), Eqs. (52-60). Then, \( W_E \) will give a measure of the length of the paths, the overlap between couples of paths (quadratic length) and the interaction effects among paths (fission and fussion terms).

We can see that the action of the mass term, \( W_m \), over a state \( |P\rangle \) is given by

\[
    W_m |P\rangle = m \left( 2\mathcal{N}(P) - \frac{1}{2} N_s \right) |P\rangle \quad (86)
\]

where \( \mathcal{N}(P) \) is the number of open paths in the state \( |P\rangle \) and \( N_s \) is the number of lattice sites. The action of \( W_m \) justifies our election of the even-odd orientation of the mesonic paths since for a state \( |P_{\text{even}}\rangle \) we have

\[
    W_m |P_{\text{even}}\rangle = m \left( 2 - \frac{1}{2} N_s \right) |P_{\text{even}}\rangle \quad (87)
\]

while if we choose the other representation, where we interchange the parity of the sites, we have

\[
    W_m |P_{\text{odd}}\rangle = -m \left( 2 - \frac{1}{2} N_s \right) |P_{\text{odd}}\rangle \quad (88)
\]
Then, the first representation (where the even sites are the starting points of the paths) has less energy and is the natural choice for the orientation of the mesonic paths. Thus, as it is well known, the mass term breaks the chiral symmetry.

The interaction term, \( W_q \), can be written in terms of the operators \( \phi_0 \) and \( \phi_3 \) as

\[
W_q = \sum_{\vec{r}_e, \hat{n}} \phi_0(\ell_{\vec{r}}) + \sum_{\vec{r}_o, \hat{n}} \phi_3(\ell_{\vec{r}}) + H.c. \tag{89}
\]

where the sum is over the six directions \( \hat{n} \); and \( \ell_{\vec{r}} \) is the link starting in \( \vec{r} \) and ending in \( \vec{r} + a\hat{n} \), with \( a \) equal to the lattice spacing. Therefore, through the action of \( \phi_0 \) the term \( W_q \) can create mesonic and barionic states. If we compute the perturbation expansion in \( \lambda = 1/g^2 \) in the strong-coupling region for the rescaled hamiltonian

\[
W = W_E + W_m + \lambda W_q + \lambda^2 W_M
\]

we can see that the mesonic states arise in the first order of the expansion and the barionic states in the second order. From the action of \( \phi_0 \), Eq. (46), the term \( W_q \) produces mesonic states when perturbs the zeroth order of the expansion

\[
W_q | 0 \rangle = \sum_{\vec{r}_e, \hat{n}} \phi_0(\ell_{\vec{r}}) | 0 \rangle = \sum_{\vec{r}_e, \hat{n}} | \ell_{\vec{r}} \rangle
\]

and, from the Mandelstam constraints (69), (72) and (73), \( W_q \) generates barionic states when acts over the mesonic states of the first order

\[
W_q | \ell_{\vec{r}}^{\vec{r}+\hat{n}} \rangle = \sum_{\vec{s}, \hat{m}} \phi_0(\ell_{\vec{s}+\hat{m}}) | \ell_{\vec{r}}^{\vec{r}+\hat{n}} \rangle = \sum_{\vec{s}, \hat{m}} \phi_0(\ell_{\vec{s}+\hat{m}}) \phi_0(\ell_{\vec{r}}^{\vec{r}+\hat{n}}) | 0 \rangle
\]
\[
\sum_{\bar{s}_e \neq \bar{r}_e, \bar{m} \neq \bar{n}} \phi_0(\ell_{x+\bar{m}}) \phi_0(\ell_{y+\bar{n}}) | 0 \rangle \\
+ \frac{1}{2} \sum_{\bar{m} \neq \bar{n}} \Gamma_1^\dagger(P_{\bar{0}}) \Gamma_2(\ell_{\bar{r}+\bar{n}}) | 0 \rangle \\
+ \frac{1}{4} \tilde{W}(\ell_{\bar{r}+\bar{n}}) \sum_{\bar{m} \neq \bar{n}} \Gamma_1^\dagger(P_{\bar{0}}) \Gamma_2(\ell_{\bar{r}+\bar{n}}) | 0 \rangle
\]

where \( P_0 \) is the null path. Finally, through the action of \( \phi_3 \) the term \( W_q \) can also join two open paths, Eqs. (47-51).

In according to (14) and (67) we can write the magnetic term as

\[
W_M = \sum_{\square} \left( \tilde{W}(\square) + \tilde{W}^\dagger(\square) \right) = 2 \sum_{\square} \tilde{W}(\square) = 4 \sum_{\square > 0} \tilde{W}(\square)
\]

then, the action of \( W_M \) is given by

\[
W_M | P \rangle = 4 \sum_{\square > 0} | P \square \rangle
\]

Now, we shall examine the symmetries of the hamiltonian (81) in this representation, in each case we describe the transformation laws of the gauge invariant operators. As we are going to show latter on, these six symmetries allow one to use the invariance of the vacuum state in order to reduce the redundancy of the basis elements.

**Lattice translation by even number of links**

The parity of the sites remain invariant under translation by an even number of links

\[
\chi(x, y, z) \rightarrow \chi(x + 2n_x, y + 2n_y, z + 2n_z)
\]

where \( \ell, m \) and \( n \) are integer numbers. The path dependent operators transform as

\[
\tilde{W}(L) \rightarrow \tilde{W}(L') \\
\phi(P_{\ell}) \rightarrow \phi(P_{\ell+2n}) \\
\Gamma_2(P_{\ell}) \rightarrow \Gamma_2(P_{\ell+2n}) \\
\Gamma_1^\dagger(P_{\ell}) \rightarrow \Gamma_1^\dagger(P_{\ell+2n})
\]
where $2 \hat{n} = (2n_x, 2n_y, 2n_z)$ and $L'$ is the translated loop.

**Lattice translation by odd number of links**

For translations of a link in the $\hat{z}$ direction the even sites transform into odd sites and vice versa, and the operators transform as

\[
\begin{align*}
\tilde{W}(L) &\rightarrow \tilde{W}(L') \\
\chi(\vec{r}) &\rightarrow -i(-1)^{x_r} \chi(\vec{r} + \hat{n}_z) \\
\phi(P_r \uparrow) &\rightarrow \phi(\uparrow (P_{r+n_z}) \\
\Gamma_2(P_r \uparrow) &\rightarrow -\Gamma_1(P_{r+n_z}) \\
\Gamma_1^\dagger(P_r \uparrow) &\rightarrow -\Gamma_2^\dagger(P_{r+n_z})
\end{align*}
\]

(94)

**Rotations by $\pi/2$**

For rotations around direction $\hat{n}_z$ we have

\[
\begin{align*}
\tilde{W}(L) &\rightarrow T(L) \tilde{W}(L') \\
\chi(\vec{r}) &\rightarrow R(x, y, z) \chi(\vec{r}') \\
\phi(P_r \uparrow) &\rightarrow R(\vec{r}) R(\vec{s}) T(P_r \uparrow) \phi(P_r \uparrow') \\
\Gamma_2(P_r \uparrow) &\rightarrow R(\vec{r}) R(\vec{s}) T(P_r \uparrow) \Gamma_2(P_r \uparrow') \\
\Gamma_1^\dagger(P_r \uparrow) &\rightarrow R(\vec{r}) R(\vec{s}) T(P_r \uparrow) \Gamma_1^\dagger(P_r \uparrow')
\end{align*}
\]

(95)

where

\[
R(\vec{r}) \equiv \frac{1}{2} \left( (-1)^{x_r} + (-1)^{y_r} + (-1)^{z_r} - (-1)^{x_r+y_r+z_r} \right)
\]

(96)

and

\[
T(P) \equiv \left( \prod_{\ell_x \in P} (-1)^{y_{\ell'} + z_{\ell'}} \right) \left( \prod_{\ell_y \in P} (-1)^{z_{\ell'} + x_{\ell'}} \right) \left( \prod_{\ell_z \in P} (-1)^{x_{\ell'} + y_{\ell'}} \right)
\]

(97)

For the case of a plaquette and a link we obtain

\[
\begin{align*}
\tilde{W}(\Box) &\rightarrow \tilde{W}(\Box') \\
\phi(\ell_x) &\rightarrow \phi(\ell_y) \\
\phi(\ell_y) &\rightarrow \phi(\ell_{-x}) \\
\phi(\ell_z) &\rightarrow \phi(\ell_z)
\end{align*}
\]

(98)
and for path with two links

\[ \Gamma_2(P_r^s) \rightarrow \Gamma_2(P_r^{s'}) \]
\[ \Gamma_1^\dagger(P_r^s) \rightarrow \Gamma_1^\dagger(P_r^{s'}) \]  

(99)

**Rotations by \( \pi \) about a lattice center**

These are the rotations around any of the three axes passing through the geometrical center of a lattice cube

\[ \tilde{W}(L) \rightarrow \tilde{W}(L') \]
\[ \chi(\vec{r}) \rightarrow \chi(\vec{r'}) \]
\[ \phi(P_r^s) \rightarrow \phi(P_r^{s'}) \]  

(100)

**Parity**

This is the reflection through the origin.

\[ \tilde{W}(L) \rightarrow \tilde{W}(L') \]
\[ \chi(\vec{r}) \rightarrow \chi(-\vec{r}) \]
\[ \phi(P_r^s) \rightarrow \phi(P_{-r}^{-s}) \]  

(101)

**G Parity**

This is the complex conjugation of the operators

\[ \tilde{W}(L) \rightarrow \tilde{W}^\dagger(L) = \tilde{W}(\overline{L}) = \tilde{W}(L) \]
\[ \phi_0(P_r^s) \rightarrow \phi_0^\dagger(P_r^s) = \phi_3(P_r^s) \]
\[ \phi_3(P_r^s) \rightarrow \phi_3^\dagger(P_r^s) = \phi_0(P_r^s) \]  

(102)
The identification of these symmetries in the continuum was studied by Banks, Kogut and Susskind in \cite{17, 20}.

This path-dependent formalism allows to accomplish explicit calculations of the vacuum energy density and the mass-gap solving the Schrödinger equation by means of cluster approximation \cite{11}. The idea of this approximation is to consider a Hilbert space restricted to a subspace of the entire space of paths in the lattice. This space is spanned by a finite basis of simple paths, without repeated links, or clusters. We define a cluster as a list of open paths and loops confined in a finite spatial region, under this approximation a generic path in the lattice is understood as an unordered list of clusters separated by formally infinite interdistances. Then, a physical state is described, specifying the occupation number of each nonequivalent cluster, as \( | n_1, n_2, \ldots, n_m, \ldots \rangle \) where \( n_i \) denotes the number of times that cluster \( i \) appears in the list of paths that describes the state.

We are mainly interested in a description of the quantum ground state of the gauge system. The invariance of this state under the symmetries of the hamiltonian allows one to eliminate equivalent elements of the path basis. For instance, by application of the symmetries (94), (98) and (102) we can show that all the mesonic states with only one link are equivalent, independently of its origin or direction, the same occurs for the plaquettes.

We need to work with a finite number of clusters, to do so that we introduce an order notion between clusters. This may be done in a recursive way, the null path is zeroth order, the path of one link and one plaquette constitute the first order, the \( N \)th order will include all the paths that are obtained from the \( (N-1) \)th order by addition of plaquettes or links, through application of the terms \( \tilde{W} \) and \( \phi_0 \) of the hamiltonian. The idea is to give a set of rules in order to build any element of the basis. To do that we apply the operators \( \phi_0 \) and \( \tilde{W} \) as follows: we only compute the action of the operator \( \phi_0(\ell) \) over a loop \( L \) when \( \ell \) is a link of \( L \), and the action of the operator \( \tilde{W}(\Box) \) over open paths and loops when some of the four links of the plaquette coincides with some of the path links. The action of \( \phi_0(\ell) \) over an open path is to add links in the neighborhood of the path. All other links and plaquettes do not generate other clusters, they only increase the occupation numbers of clusters associates to one link and one plaquette.
Independent elements of the basis are obtained when the electric operator and $\phi_3$ act on the given elements of order $N$. The operator $\phi_3(\ell)$ joins two open paths that are separated by one link. The electric part of the hamiltonian will also generate clusters of the $N$th order through the fusion and fission terms. The fusion produce clusters with multiple links and the fission term breaks loops and produces disconnected clusters. Finally, the Mandelstam constraints for open paths, Eqs. (69), (72) and (73), allow one to obtain the barionic states.

In the generation of the clusters we must check if the new cluster is not equivalent, under some of the hamiltonian symmetries, to any one of the already generated clusters or, through Mandelstam identities for loops, Eqs. (61-62), and the new Mandelstam constraints for open paths, Eqs. (63-65), to a combination of the already generates clusters.

The basis generated must be truncated for computational purposes. The generation is iterated up to a maximum order $N_{max}$. In Fig. 6 we show the basis generated with this rules for the order $N_{max} = 2$ (we have chosen the symbol • to represent one antiquark and a ◦ to represent one quark).

It is well known from previous applications [11, 12, 13, 14, 21] that the cluster approximation works quite well in the strong-coupling region, and by considering a big enough basis of clusters and introducing some collective variables it is possible to reach the weak-coupling regime.

In a following paper we will present the results obtained using this approximation, for the quantum ground state energy and the mass-gap for the SU(2) QCD.

5 Conclusion

The hamiltonian formalism of lattice QCD in the path representation has been introduced. Each term of the hamiltonian has a well defined geometrical effect inducing deformations or rearrangements over paths representing the physical states. We have restricted ourselves in this paper to the intro-
duction of the general formalism and the study of the Mandelstam identities for the $SU(2)$ QCD. Even though it is straightforward to extend this formalism to full QCD, there is a renewed interest in the study of the $SU(2)$ gauge theory with dynamical fermions [22]. Besides reasons of economy in computational time [23, 24, 25, 26] this study is also motivated by some recent developments in the area of High-$T_c$ superconductivity leading to a Heisenberg model written as a fermionic theory. A connection between this hamiltonian and a $SU(2)$ lattice gauge theory with dynamical fermions in $2 + 1$ dimensions has been recently made [22]. Dagotto, Kocić and Kogut have pointed out that a manifestly gauge invariant approach should be developed to guide the computational work in this problem. We hope that the present formulation could help to treat this problem.

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Figure captions

Figure 1:
Examples of fusions between two loops (a), between a loop and an open path (b) and between two open paths (c).

Figure 2:
(a) The fission term breaks the loop $L$ and gives rise to the cluster $(L_1, L_2)$.
(b) In this case it breaks the open path $P$ and gives rise to the cluster $(L, Q)$, where $L$ is a loop and $Q$ is an open path.

Figure 3:
The Mandelstam identities for two loops (a) and for one loop and one open path (b) with a common link.

Figure 4:
The Mandelstam identities for two mesonic paths (a) and two barionic paths (b) with the same end points. The black circle represents one antiquark and the white circle one quark, two circles near represent two quarks placed on the same site.

Figure 5:
The Mandelstam identities for two open paths with one equal end point.

Figure 6:
The basis of eleven clusters for $N_{\text{max}} = 2$. 
Fig. 1

(a)

(b)

(c)
Fig. 3

\[(A\ell B\ell)\]

(a)

(b)
Fig. 4

(a) $= \frac{1}{4}$

(b) $= -\frac{1}{4}$
Fig. 5

\[ \bullet \longrightarrow \circ = \frac{1}{2} \bullet \longrightarrow \circ \]

\[ \circ \longrightarrow \bullet = \frac{1}{2} \bullet \longrightarrow \circ \]

\[ \bullet \longrightarrow \circ = -\frac{1}{2} \bullet \longrightarrow \circ \]

\[ \bullet \longrightarrow \circ = -\frac{1}{2} \bullet \longrightarrow \circ \]

\[ \circ \longrightarrow \bullet = -\frac{1}{2} \bullet \longrightarrow \circ \]

\[ \circ \longrightarrow \bullet = -\frac{1}{2} \bullet \longrightarrow \circ \]
Fig. 6

Order 1:

Order 2: