WEIGHTED EMBEDDINGS FOR FUNCTION SPACES ASSOCIATED WITH HERMITE EXPANSIONS

THE ANH BUI, JI LI, AND FU KEN LY

Abstract. We study weighted Besov and Triebel–Lizorkin spaces associated with Hermite expansions and obtain (i) frame decompositions, and (ii) characterizations of continuous Sobolev-type embeddings. The weights we consider generalize the Muckenhoupt weights.

1. Introduction

One of the most important features of smooth function spaces is the presence of Sobolev type embeddings. The ability to exchange regularity for increased integrability is a critical tool in the study of partial differential equations. In particular weighted embedding theorems for Besov and Triebel–Lizorkin spaces have found applications in a variety of situations (see [27, 29] and the references therein).

To facilitate the rest of this discussion let us recall such embeddings. For a non-negative and locally integrable function $w$ let $B_{\alpha,w}^{p,q}$ and $F_{\alpha,w}^{p,q}$ be the weighted inhomogeneous Besov and Triebel–Lizorkin spaces on $\mathbb{R}^n$ respectively, where $0 < p < \infty$ is the integrability index, $0 < q < \infty$ the fine index, and $\alpha \in \mathbb{R}$ the smoothness index. Suppose $\alpha_1 \geq \alpha_2$, $0 < p_1 \leq p_2 < \infty$, $0 < q_1, q_2 \leq \infty$ and $\alpha_1 - \gamma/p_1 = \alpha_2 - \gamma/p_2$ for some $\gamma > 0$. Then under suitable conditions on $w$ one has the continuous embedding

\begin{equation}
F_{\alpha_2,w}^{p_2,q_2} \hookrightarrow F_{\alpha_1,w}^{p_1,q_1},
\end{equation}

and for $0 < q_2 \leq q_1 \leq \infty$,

\begin{equation}
B_{\alpha_2,w}^{p_2,q_2} \hookrightarrow B_{\alpha_1,w}^{p_1,q_1}.
\end{equation}

Situations for which (1.1)–(1.2) holds include of course the unweighted case of $w = 1$ and $\gamma = n$, which can be found in [11, 12, 34]. Other situations include power weights [20, 27] and radial weights [7]. When $w$ is a Muckenhoupt weight (1.1)–(1.2) is known to hold if and only if $w$ satisfies a kind of (local) lower bound property, namely that

\begin{equation}
w(B(x,r)) \geq Cr^\gamma, \quad x \in \mathbb{R}^n, \quad 0 < r \leq 1
\end{equation}

for some $C > 0$ independent of $x$ and $r$. This was proved in [3, Theorem 2.6], [20, Prop 2.1(i)], [28, Theorem 1.2]. That latter two articles in fact consider two-weight embeddings of which (1.1)–(1.3) are a special case.

In recent times various works have been devoted to generalizations of the Besov and Triebel–Lizorkin spaces in a variety of directions. Such directions include generalizing the ambient space $\mathbb{R}^n$ [8, 19], replacing the Fourier system by other orthogonal expansions [9, 10, 22, 23, 30], spaces adapted to a general class of operators [11, 16, 21, 13] or hybrids of the above.

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It is thus natural to ask if embeddings similar to (1.1)–(1.2) can be retained in these generalizations. For the setting of a space of homogeneous type, a complete characterization of (1.1)–(1.2) was given in [18] by (1.3), where \( w = \mu \) is the underlying measure, unifying and completing several earlier results [14, 15, 16, 17].

In the present article we consider the case of Hermite function expansions [33], which are naturally associated with the harmonic oscillator
\[
\mathcal{L} = -\Delta + |x|^2
\]
on \( \mathbb{R}^n \) for \( n \geq 1 \). Besov and Triebel–Lizorkin spaces associated with Hermite expansions have been developed in [5, 9, 10, 30]. In this article we introduce weighted versions with the aim of characterizing Sobolev–type embeddings.

To describe our main results let us first define our function spaces. Let \( (\varphi_0, \varphi) \) be an admissible pair as defined in Definition 2.1. Let \( \{ h_\xi \}_{\xi \in \mathbb{N}^n_0} \) be the multi-dimensional Hermite functions (see Section 2) and consider, for \( j \geq 0 \), the Littlewood–Paley type operators defined by
\[
\varphi_j(\sqrt{\mathcal{L}}) f := \sum_{\xi \in \mathbb{N}^n_0} \varphi_j(\sqrt{2|\xi| + n}) (h_\xi, f) h_\xi, \quad f \in \mathcal{S}'
\]
where \( (h_\xi, f) := f(h_\xi) \). Then we introduced the following weighted spaces. Let \( \alpha \in \mathbb{R} \), \( 0 < q \leq \infty \) and \( w \) be a non-negative and locally integrable function. For \( 0 < p \leq \infty \), we define the weighted Hermite Besov space \( B_{\alpha,w}^{p,q} \) as the class of tempered distributions \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that
\[
\| f \|_{B_{\alpha,w}^{p,q}} = \left( \sum_{j \in \mathbb{N}_0} \left( 2^{j\alpha} \| \varphi_j(\sqrt{\mathcal{L}}) f \|_{L^p_w} \right)^q \right)^{1/q} < \infty,
\]
while for \( 0 < p < \infty \), the weighted Hermite Triebel–Lizorkin space \( F_{\alpha,w}^{p,q} \) consists of all those \( f \in \mathcal{S}'(\mathbb{R}^n) \) for which
\[
\| f \|_{F_{\alpha,w}^{p,q}} = \left( \left( \sum_{j \in \mathbb{N}_0} \left( 2^{j\alpha} |\varphi_j(\sqrt{\mathcal{L}}) f| \right)^q \right)^{1/q} \right)^{1/q} \|_{L^p_w} < \infty.
\]
We adopt the usual modifications to the norm when \( q = \infty \) (or \( p = \infty \) in the Besov case). The spaces can be shown to be independent of the choice of \( (\varphi_0, \varphi) \) (see Theorem 3.1).

Our aim in this paper is to characterize embedding theorems for such spaces for a suitable class of weights. The kinds of weights we consider are more general than (and include) the Muckenhoupt weights. Such weights, introduced in [2], have sparked some recent activity (see for example [11, 24, 25, 32]). They can be non-doubling and may even admit a certain degree of exponential growth (see Section 2.2). Note that embedding in the unweighted situation (with \( w = 1 \)) was given in [30].

Our main result, Theorem 1.1 characterizes (1.1)–(1.2) via a similar condition to (1.3).

**Theorem 1.1.** Let \( \gamma > 0 \) and \( w \in A_\infty(\mathcal{L}) \). Then one has the embeddings (1.1) or (1.2) if and only if \( w \) satisfies the following “Hermite lower bound property” of order \( \gamma \): there exists a constant \( C > 0 \) such that,
\[
w(B(x,r)) \geq C r^\gamma, \quad x \in \mathbb{R}^n, \quad 0 < r \leq \frac{1}{1 + |x|_\infty}.
\]
Examples of weights which satisfy \((1.6)\) include the power weights \(w(x) = |x|^{\epsilon}\), for \(-n < \epsilon < \infty\) provided that \(\gamma \geq n + \epsilon\). An example which is not a Muckenhoupt weight is the exponential weight \(w(x) = e^{\gamma|x|^2}\) with \(\gamma \geq n\).

Some remarks on our method of proof are in order. Our overall approach to Theorem 1.1 is to work with the associated sequence spaces (see Definition 2.7). To do so we firstly obtain suitable frame characterizations (Theorem 3.1) before obtaining embeddings for sequences (Theorem 4.1). For the actual embeddings in the Triebel–Lizorkin scale we employ the well known distribution function method which harks back to Jawerth [12] and refined in [34]. The Besov scale are relatively easier.

We give a final important remark on the relevance of condition \((1.6)\). Our proofs rely on the ability to pass from a lower bound estimate on frame elements to the estimate on balls (Proposition 4.2). This is a simple matter in the classical situation since the frame elements are dyadic cubes whose sizes depend only on the scale. In the Hermite setting however, the frame elements are so-called ‘tiles’ (see Section 2.3) which are dependent on both scale and location. This principle is quantified in \((1.6)\) and Proposition 4.2.

The organization of the article is as follows. In Section 2 we give the necessary background that will be used throughout the paper including facts concerning Hermite functions, kernel estimates for our Littlewood–Paley type operators, Hermite weights and Hermite tiles (frame elements). In particular the associated weighted sequence spaces are introduced here in Definition 2.7. In Section 3 we introduce frames, and give frame characterizations in Theorems 3.1. In Section 4 we prove our embedding results. Here we first develop embeddings for Hermite sequence spaces in Theorem 4.1 before finally giving the proof of Theorem 1.1.

Finally it would be interesting to extend our results to two-weights embeddings, as done in [20, 28]. However we do not pursue this here.

**Notation:** We conclude this introduction with some notational matters. We set \(|x|_\infty := \max_{1 \leq i \leq n} |x_i|\). We denote by \(Q(x, r) := \{ y \in \mathbb{R}^n : |x - y|_\infty < r \}\) to be the cube centred at \(x\) with sidelength \(2r\). By a ‘cube’ \(Q\) we mean the cube \(Q(x_Q, r_Q)\) with some fixed centre \(x_Q\) and sidelength \(2r_Q\). By a weight we mean a non-negative locally integrable function. We denote \(\lambda_k := 2^k + n\) for \(k \in \mathbb{N}_0\), where \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\). For two Banach spaces \(X_0\) and \(X_1\) the notation \(X_0 \hookrightarrow X_1\) means that \(X_0\) embeds continuously into \(X_1\). We denote by \(\mathcal{S}(\mathbb{R}^n)\) the space of Schwartz functions on \(\mathbb{R}^n\) and by \(\mathcal{S}'(\mathbb{R}^n)\) the space of tempered distributions on \(\mathbb{R}^n\). The letter \(n\) will always mean Euclidean dimension.

## 2. Preliminaries

In this section we give the necessary material that will be used throughout the rest of the paper. After detailing some basic facts concerning Hermite functions we introduce and give some kernel estimates for our Littlewood–Paley operators in Section 2.1. In Section 2.2 we define and give some useful estimates for the Hermite weight classes employed throughout the paper. Finally in Section 2.3 we detail the construction of dyadic ‘tiles’ and introduce weighted sequence spaces associated with Hermite expansions.

For each \(k \in \mathbb{N}\), the Hermite function of degree \(k\) is

\[
h_k(t) = (2^k k! \sqrt{\pi})^{-1/2} H_k(t) e^{-t^2/2}, \quad t \in \mathbb{R}
\]

where

\[
H_k(t) = (-1)^k e^{t^2} \partial_t^k (e^{-t^2})
\]
is the $k$th Hermite polynomial.

The $n$-dimensional Hermite functions $h_{\alpha}$ are defined over the multi-indices $\alpha$ by

$$h_{\xi}(x) = \prod_{j=1}^{n} h_{\xi_j}(x_j), \quad x \in \mathbb{R}^n, \quad \xi \in \mathbb{N}_0^n.$$ 

The Hermite functions are eigenfunctions of $\mathcal{L}$ in the sense that $\mathcal{L}(h_{\xi}) = \lambda_{|\xi|} h_{\xi}$, where

$$\lambda_k = 2k + n.$$ 

Furthermore they form an orthonormal basis for $L^2(\mathbb{R}^n)$.

Let $W_k = \text{span}\{h_{\xi} : |\xi| = k\}$ and $V_N = \bigoplus_{k=0}^{N} W_k$. We define the orthogonal projection of $f$ onto $W_k$ by

$$\mathbb{P}_k f = \sum_{|\xi| = k} \langle f, h_{\xi} \rangle h_{\xi} \quad \text{with kernel} \quad \mathbb{P}_k(x, y) = \sum_{|\xi| = k} h_{\xi}(x) h_{\xi}(y).$$

We also define the orthogonal projection of $f$ onto $V_N$ by

$$\mathbb{Q}_N f = \sum_{k=0}^{N} \mathbb{P}_k f \quad \text{with kernel} \quad \mathbb{Q}_N(x, y) = \sum_{k=0}^{N} \sum_{|\xi| = k} h_{\xi}(x) h_{\xi}(y).$$

The following bounds are known (see [30, p.376]): there exists $\vartheta > 0$ such that for any $N \geq 1$

$$\mathbb{Q}_N(x, x) \lesssim \begin{cases} N^{n/2} & \forall x \\ e^{-2\vartheta|x|^2} & \text{if} \quad |x| \geq \sqrt{4N + 2}. \end{cases} \tag{2.1}$$

We also set

$$e_N(x) := \begin{cases} 1 & \text{if} \quad |x|^2 < N \\ e^{-\vartheta|x|^2} & \text{if} \quad |x|^2 \geq N. \end{cases}$$

It follows from (2.1) that for any $\varepsilon > 4$ and $N \in \mathbb{N}$, we have

$$\mathbb{Q}_{\varepsilon N}(x, x) \lesssim 2^jN(e_{\varepsilon N}(x))^2 \quad \forall j \in \mathbb{N}, \tag{2.2}$$

where the implicit constant depends only on $N$, $n$, $\varepsilon$ and $\vartheta$.

2.1. Littlewood–Paley operators. The following band-limited functions will play a fundamental role in this paper. Recall that they were used to define our weighted function spaces in (1.4) and (1.5).

**Definition 2.1 (Admissible functions).** We say that $(\varphi_0, \varphi)$ are an admissible pair if $\varphi_0, \varphi \in C^\infty(\mathbb{R}^n)$ and

$$\text{supp } \varphi_0 \subset [0, \sqrt{2}], \quad |\varphi_0| > c > 0 \quad \text{on} \quad [0, 2^{-3/8}], \quad \varphi_0^{(m)}(0) = 0, \quad \forall m \in \mathbb{Z}_+$$

$$\text{supp } \varphi \subset [\sqrt{2}, \sqrt{2}], \quad |\varphi| > c > 0 \quad \text{on} \quad [2^{-3/8}, 23/8].$$

Given an admissible pair $(\varphi_0, \varphi)$, we set $\varphi_j(\lambda) := \varphi(2^{-j}\lambda)$ if $j \geq 1$ and call the resulting collection $\{\varphi_j\}_{j \geq 0}$ an admissible system.

Since the Hermite functions $\{h_{\xi}\}_{\xi \in \mathbb{N}_0^n}$ are members of $\mathcal{S}(\mathbb{R}^n)$, then for an admissible system $\{\varphi_j\}_{j \in \mathbb{N}_0}$ we may define the operators $\varphi_j(\sqrt{\mathcal{L}})$ on $\mathcal{S}'(\mathbb{R}^n)$ by

$$\varphi_j(\sqrt{\mathcal{L}}) f = \sum_{\xi \in \mathbb{N}_0} \varphi_j(\sqrt{\lambda_{\xi}}) \mathbb{P}_{\xi} f, \quad f \in \mathcal{S}'(\mathbb{R}^n),$$
where \( (f, \phi) = f(\phi) \) for \( f \in \mathcal{A}'(\mathbb{R}^n) \). The kernels of operators \( \varphi_j(\sqrt{L}) \) are given by

\[
\varphi_j(\sqrt{L})(x, y) = \sum_{\xi \in \mathbb{N}_0} \varphi_j(\sqrt{\lambda_\xi}) P_\xi(x, y) = \sum_{\xi \in \mathbb{N}_0} \varphi_j(\sqrt{\lambda_\xi}) \sum_{|\mu|=\xi} h_\mu(x) h_\mu(y).
\]

Let \( \{I_j\}_{j \in \mathbb{N}_0} \) denote the following subsets of \( \mathbb{N}_0 \): \( I_j = [4^{j-1} - \frac{n}{2^n}, 4^j - \frac{n}{2^n}] \cap \mathbb{N}_0 \) for \( j \in \mathbb{N} \), \( I_0 = \{0\} \) if \( n = 1, 2 \) and \( I_0 = \emptyset \) if \( n \geq 3 \). The support of \( \varphi_j \) implies that

\[
\varphi_j(\sqrt{L})(x, y) = \sum_{\xi \in I_j} \varphi_j(\sqrt{\lambda_\xi}) P_\xi(x, y).
\]

We give now some estimates on the kernels of the operators \( \varphi_j(\sqrt{L}) \). Some of these have appeared in [26, 30] but we give their proofs for the convenience of the reader.

**Proposition 2.2 (Kernel estimates).** Let \( \{\varphi_j\}_{j \geq 0} \) be an admissible system. Then for each \( N \geq 1 \) and \( \varepsilon > 4 \), we have

\[
|\varphi_j(\sqrt{L})(x, y)| \lesssim \frac{2^{jn}}{(1 + 2^j|x - y|)N^e_{4^j}(x)e_{4^j}(y)}, \quad \forall \ x, y \in \mathbb{R}^n.
\]

Suppose that \( \{\psi_j\}_{j \geq 0} \) is another admissible system. If \( |j - k| \geq 2 \) then

\[
\varphi_j(\sqrt{L})\psi_k(\sqrt{L})(x, y) = 0, \quad \forall \ x, y \in \mathbb{R}^n.
\]

If \( |j - k| \leq 1 \) then for any \( N \geq 1 \) and \( \varepsilon > 4 \) we have

\[
|\varphi_j(\sqrt{L})\psi_k(\sqrt{L})(x, y)| \lesssim \frac{2^{kn}}{(1 + 2^k|x - y|)N^e_{4^k+1}(x)e_{4^k}(y)}
\]

for every \( x, y \in \mathbb{R}^n \). Here the implicit constants depend on \( N, \vartheta, \varepsilon \) and max \( \{\|\varphi_0^{(N)}\|_\infty, \|\varphi^{(N)}\|_\infty\} \).

**Proof.** We begin with the proof of (2.4). If \( |x - y| \leq 2^{-j} \) then by the Cauchy–Schwarz inequality,

\[
|\varphi_j(\sqrt{L})(x, y)| \leq \|\varphi_j\|_\infty \sum_{\xi \in I_j} \sum_{|\mu|=\xi} |h_\mu(x)| |h_\mu(y)| \lesssim Q_{4^j+N}(x, x)^{1/2}Q_{4^j+N}(y, y)^{1/2}.
\]

If on the other hand, \( |x - y| > 2^{-j} \), then applying the identity in [30, Lemma 8] (see also [33, p.72] or [26 (B.1)]) to (2.3) we obtain

\[
|2^N(x - y)_N \varphi_j(\sqrt{L})(x, y)| = \sum_{N/2 \leq \ell \leq N} c_{\ell,N} \sum_{\xi \in I_j} |\Delta^\ell \varphi_j(\sqrt{\lambda_\xi})| |(A_i^{(y)} - A_i^{(x)})2^\ell - N|P_\xi(x, y)|,
\]

where \( c_{\ell,N} = (-4)^N \ell (2N - 2\ell - 1)! \left(\frac{N}{2\ell - N}\right) \). Using the mean value theorem for finite differences along with Hoppe’s chain rule formula we have

\[
|\Delta^\ell \varphi_j(\sqrt{\lambda_\xi})| = |\partial_\nu^\ell \varphi_j(\sqrt{\lambda_\nu})| = \left| \sum_{r=1}^\ell c_r \varphi_j^{(r)}(\sqrt{\lambda_\nu}) \lambda_\nu^{r/2 - \ell} 2^{-jr} \right|, \quad |c_r| \leq \ell!
\]

for some \( \xi < \nu < \xi + \ell \). Now our assumption \( \varphi_j^{(m)}(0) = 0 \) for every \( m \in \mathbb{Z}_+ \) in conjunction with Taylor’s remainder theorem gives \( |\varphi_j^{(r)}(x)| \lesssim \|\varphi_j^{(N)}\|_\infty |x|^{-r} \) for any \( N \geq 1 \) and \( x \in \mathbb{R}_+ \). Inserting this into (2.8) we arrive at

\[
|\Delta^\ell \varphi_j(\sqrt{\lambda_k})| \lesssim \lambda_k^{N/2 - \ell} 2^{-Nj}
\]
since \( k < \nu < k + \ell \). Next we recall the following estimate from [30, pp.397–398],

\[
(A_i^{(y)} - A_i^{(x)}) 2^{\ell - N} \mathcal{P}_\xi(x, y) \lesssim L^{\ell - N/2} \sum_{k=0}^{2\ell - N} \mathcal{P}_{\xi+k}(x, x)^{1/2} \left( \sum_{k=0}^{2\ell - N} \mathcal{P}_{\xi+k}(y, y)^{1/2} \right),
\]

which is obtained through the Binomial theorem and Cauchy–Schwarz inequality. Inserting (2.9) and (2.10) into (2.7) and applying Cauchy–Schwarz inequality we obtain

\[
|(x_i - y_i)^N \varphi_j(\sqrt{L})(x, y)| 
\lesssim 2^{-jN} \sum_{N/2 \leq \ell \leq N} \sum_{\xi \in I_j} \left( \sum_{k=0}^{2\ell - N} \mathcal{P}_{\xi+k}(x, x)^{1/2} \right) \left( \sum_{k=0}^{2\ell - N} \mathcal{P}_{\xi+k}(y, y)^{1/2} \right) 
\lesssim 2^{-jN} Q_{\nu+N}(x, x)^{1/2} Q_{\nu+N}(y, y)^{1/2}.
\]

Now combining both cases we arrive at

\[
|\varphi_j(\sqrt{L})(x, y)| \lesssim \frac{Q_{\nu+N}(x, x)^{1/2} Q_{\nu+N}(y, y)^{1/2}}{(1 + 2j|x - y|)^N},
\]

which, in view of (2.2), yields (2.4).

We turn to the proof of (2.5). By orthonormality of the Hermite functions

\[
\varphi_j(\sqrt{L}) \psi_k(\sqrt{L})(x, y) = \sum_{\xi \in \mathbb{N}_0} \varphi_j(\sqrt{\lambda_\xi}) \varphi_k(\sqrt{\lambda_\xi}) \sum_{|\mu| = \xi} h_\mu(x) h_\mu(y).
\]

From the supports of \( \varphi_j \) and \( \psi_k \) (see the comments after (2.3)), it can be easily seen that \( \varphi_j(\sqrt{\lambda_\xi}) \psi_k(\sqrt{\lambda_\xi}) = 0 \) whenever \( |j - k| \geq 2 \), which gives (2.5).

We now prove (2.6). From the bounds (2.4) and the fact that \( |j - k| \leq 1 \) we have

\[
|\varphi_j(\sqrt{L}) \psi_k(\sqrt{L})(x, y)| 
\leq \int |\varphi_j(\sqrt{L})(x, w)| \psi_k(\sqrt{L})(w, y) \, dw
\lesssim e_{\varepsilon^4}(x) e_{\varepsilon^4}(y) \int \frac{2^{kn}}{(1 + 2^{k}|x - w|)^N (1 + 2^{k}|w - y|)^M} \, dw
\]

for some \( M > N + n \). The triangle inequality then gives

\[
|\varphi_j(\sqrt{L}) \psi_k(\sqrt{L})(x, y)| \lesssim \frac{2^{kn} e_{\varepsilon^4}(x) e_{\varepsilon^4}(y)}{(1 + 2^{k}|x - y|)^N} \int \frac{2^{kn}}{(1 + 2^{k}|w - y|)^M} \, dw.
\]

The integral is finite since \( M > N + n \) and, on noting that \( e_{\varepsilon^4}(x) \leq e_{\varepsilon^{4k+1}}(x) \) (which follows from \( j - 1 \leq k \)) we complete our proof.

\[ \square \]

### 2.2. Hermite weights.

Here we define the classes of weights used in our function and sequence spaces. They generalize the Muckenhoupt classes and were introduced in [2]. These weights are based on the following ‘critical radius’ function.

\[
\rho(x) := \frac{1}{1 + |x|_\infty};
\]

which we used in our function and sequence spaces. They generalize the Muckenhoupt classes and were introduced in [2]. These weights are based on the following ‘critical radius’ function.
which was introduced in \cite{31} in a more general context. It can be easily seen that there exists \(C > 0\) and \(\kappa \geq 1\) such that
\[
C^{-1} \varrho(x) \left(1 + \frac{|x-y|}{\varrho(x)}\right)^{-\kappa} \leq \varrho(y) \leq C \varrho(x) \left(1 + \frac{|x-y|}{\varrho(x)}\right)^{\frac{\kappa}{\kappa+1}}
\]
for any \(x, y \in \mathbb{R}^n\) (see also \cite{31, Lemma 1.4}).

**Definition 2.3** (Hermite weights). For each cube \(Q\) and \(\theta \geq 0\) set
\[
\Psi_\theta(Q) := \left(1 + \frac{\ell(Q)}{\varrho(x_Q)}\right)^\theta.
\]

For each \(1 < p < \infty\) and \(\eta > 0\) we define the class \(A^\eta_p(\mathcal{L})\) to be the collection of all non-negative and locally integrable functions \(w\) such that for some \(C > 0\),
\[
\left(\frac{1}{|Q|} \int_Q w(y) dy\right)^{1/p} \left(\frac{1}{|Q|} \int_Q w^{1-p'}(y) dy\right)^{1/p'} \leq C \Psi_\eta(Q)
\]
for every cube \(Q\). For \(p = 1\) we require the following in place of (2.13)
\[
\frac{1}{|Q|} \int_Q w(y) dy \leq w(x) \quad \text{a.e. } x \in Q.
\]

We also set
\[
A_p(\mathcal{L}) = \bigcup_{\eta \geq 0} A^\eta_p(\mathcal{L}) \quad \text{and} \quad A_\infty(\mathcal{L}) := \bigcup_{p \geq 1} A_p(\mathcal{L}).
\]

**Remark 2.4.** (i) Throughout the rest of the article we shall drop the \(\mathcal{L}\) in the notation and just write \(A^\eta_p\) in place of \(A^\eta_p(\mathcal{L})\), and \(A_p\) in place of \(A_p(\mathcal{L})\).

(ii) We denote the usual class of Muckenhoupt weights by \(A_p\) for \(p \geq 1\). Then one may observe that the Hermite weights form a larger class than the Muckenhoupt weights in the sense that for each \(p \geq 1\) and \(\eta \geq 0\) we have \(A_p \subset A^\eta_p\).

(iii) We note that these weights are increasing in both parameters in the sense that for any \(\eta \geq 0\), then whenever \(1 \leq p_1 \leq p_2 < \infty\) we have
\[
A^\eta_{p_1} \subset A^\eta_{p_2}
\]
and for any \(p \geq 1\) and \(\eta_1 \leq \eta_2\) then
\[
A^\eta_{p_1} \subset A^{\eta_2}_{p_2}.
\]

(iv) An example of such a weight is the following. For each \(p > 1\) we have \(e^{\epsilon_1|x|^2}, e^{-\epsilon_2|x|^2} \in A_p\) for some \(\epsilon_1, \epsilon_2 > 0\). See \cite{24}.

(v) The class \(A_\infty\) is ‘open’ in the sense that whenever \(w \in A_p\) then \(w \in A_{p-\epsilon}\) for some \(\epsilon > 0\). Thus for each \(w \in A_\infty\) we may define
\[
r_w := \inf \{ r > 1 : w \in A_r \}.
\]

For each \(s > 0\) and \(\theta \geq 0\) we define a maximal function as follows:
\[
M_s^\theta f(x) := \sup_{Q \ni x} \left(\frac{1}{\Psi_\theta(Q)} \int_Q |f(y)|^s dy\right)^{1/s}.
\]
When \(s = 1\) we drop the \(s\) and write \(M^\theta\). These maximal functions satisfy a Fefferman–Stein inequality for vector valued functions.

**Lemma 2.5** (Fefferman–Stein type inequality). Assume one of the following two conditions.
Then there exists $\theta$ depending on $p, q, \eta, n, w$ and $\kappa$ (the constant from (2.12)) such that
\[
\left\| \left( \sum_j |\mathcal{M}_x^\theta(f_j)|^q \right)^{1/q} \right\|_{L_w^p} \leq C \left\| \left( \sum_j |f_j|^q \right)^{1/q} \right\|_{L_w^p}.
\]

Proof. The proof of part (a) can be found in the proof of [32, Theorem 2.7]. To prove (b) we first observe that since $w \in \mathbb{A}_\infty$ then $w \in \mathbb{A}_{\frac{p}{r_w}}^\eta$ for some $\eta \geq 0$. Now if $s < \frac{p}{r_w}$ then $w \in \mathbb{A}_{p/s}^\eta$ by Remark 2.4. We may then apply part (a) because $p/s > 1$ and $q/s > 1$. Thus we have
\[
\left\| \left( \sum_j |\mathcal{M}_x^\theta(f_j)|^q \right)^{\frac{1}{q}} \right\|_{L_w^p} \leq \left\| \left( \sum_j |f_j|^q \right)^{\frac{1}{q}} \right\|_{L_w^{ps}} = \left\| \left( \sum_j |f_j|^q \right)^{\frac{1}{q}} \right\|_{L_w^p}.
\]

2.3. Hermite tiles and sequence spaces. The aim of this section is to detail an appropriate notion of discretized sets for $\mathbb{R}^n$ and then use these sets to define weighted sequence spaces in Definition 2.7.

It is well known that the dyadic cubes of $\mathbb{R}^n$ play an important role in the development of frames for the classical distribution spaces. In the Hermite setting, the notion of ‘tiles’ or ‘rectangles’ constructed in [30] will play the role of dyadic cubes for $\mathcal{L}$. The construction of these tiles rely on the zeros of the Hermite polynomials.

Fix a constant $\delta_* \in (0, \frac{1}{3})$ and for each $j \geq 0$ we set
\[
N_j = [(1 + 11\delta_*)(\frac{1}{2})^j] + 3.
\]
Let $\{\xi_\nu\}$ with $\nu \in \{\pm 1, \pm 2, \ldots, \pm N_j\}$ be the zeros of the $H_{2N_j}(t)$, ordered so that
\[
(2.15) \quad \xi_{-N_j} < \cdots < \xi_{-1} < 0 < \xi_1 < \cdots < \xi_{N_j}.
\]
It will be useful to note that $\xi_{-\nu} = -\xi_\nu$.

The index sets $\mathcal{X}_j$ for $j \geq 0$ are now defined in the following manner. For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha_\nu| < N_j$ we write $\xi_\alpha = (\xi_{\alpha_1}, \ldots, \xi_{\alpha_n})$ where each $\xi_{\alpha_\nu}$ is a zero of $H_{2N_j}(t)$ from (2.15). Then $\mathcal{X}_j$ will be the collection of all such points or nodes $\xi_\alpha$ in $\mathbb{R}^n$. We write $\mathcal{X} = \bigcup_{j \geq 0} \mathcal{X}_j$.

The families of dyadic tiles $\mathcal{E}_j$ for $j \geq 0$ are defined as follows. For each $\xi_\alpha \in \mathcal{X}_j$ we set
\[
R_{\xi_\alpha} := I_{\alpha_1} \times I_{\alpha_2} \times \cdots \times I_{\alpha_n},
\]
where $I_{\alpha_\nu}$ are the intervals defined by
\[
I_\nu := \begin{cases} 
[0, (\xi_1 + \xi_2)/2], & \nu = 1 \\
[(\xi_{\nu-1} + \xi_\nu)/2, (\xi_{\nu} + \xi_{\nu+1})/2], & \nu \in \{2, 3, \ldots, N_j-1\} \\
[(\xi_{N_j-1} + \xi_{N_j})/2, \xi_{N_j} + 2^{-j/6}], & \nu = N_j
\end{cases},
\]
and $I_{-\nu} = -I_\nu$ for $\nu \in \{1, 2, \ldots, N_j\}$.

We denote by $\mathcal{E}_j$ the collection of tiles $\{R_\xi\}_{\xi \in \mathcal{X}_j}$ and by $\mathcal{E} := \bigcup_{j \geq 0} \mathcal{E}_j$ the collection of all tiles. Then $\mathcal{E}_j$ contains approximately $4^{jn}$ tiles (with constants depending on $\delta_*$ and $n$). By a rectangle (or tile) $R$ we mean a rectangle from $\mathcal{E}_j$ for some $j \geq 0$ and denote its node by $x_R$. We summarize the properties of these tiles below. For their proofs we direct the reader to [30, pp.379–380].

Lemma 2.6. There exist constants $c_0$, $c_1$, $c_2$, $c_3$ and $c_4$ depending only on $\delta_*$ and $n$ such that for each $j \geq 0$ and each tile $R \in \mathcal{E}_j$ the following holds.
(a) If $|x_R| \leq (1 + 4\delta_\ast)2^{j+1}$, it holds that
\begin{equation}
R = Q(x_R, c_0 2^{-j}).
\end{equation}

(b) In general, it holds that
\begin{equation}
Q(x_R, c_1 2^{-j}) \subset R \subset Q(x_R, c_2 2^{-j/3}).
\end{equation}

(c) Set $Q_j := \bigcup_{P \in \mathcal{E}_j} P = Q(0, \xi_{N_j} + 2^{-j/6})$; it holds that
\begin{equation}
Q(0, 2^j) \subset Q_j \subset Q(0, c_3 2^j).
\end{equation}

(d) $R$ can be subdivided into a disjoint union of subcubes of side length roughly equal to $2^{-j}$; more precisely each such subcube $Q$ satisfies
\begin{equation}
Q(x_Q, c_4 2^{-j-1}) \subset Q \subset Q(x_Q, c_4 2^{-j}).
\end{equation}

Denoting by $\hat{\mathcal{E}}_j$ the collection of all subdivided cubes from $\mathcal{E}_j$, then it holds that
\begin{equation}
\bigcup_{Q \in \hat{\mathcal{E}}_j} Q = Q_j.
\end{equation}

We may now introduce our weighted sequence spaces associated with Hermite expansions, which are spaces of sequences indexed over the collection of tiles just described.

**Definition 2.7 (Weighted Hermite sequence spaces).** Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$ and $0 \leq w \in L^1_{\text{loc}}(\mathbb{R}^n)$. For $0 < p \leq \infty$, we define the weighted Hermite Besov sequence space $b^{p,q,\alpha}_w$ as the set of all sequences of complex numbers $s = \{s_R\}_{R \in \mathcal{E}}$ such that
\begin{equation}
\|s\|_{b^{p,q,\alpha}_w} := \left\{ \sum_{j \geq 0} 2^{j\alpha q} \left( \sum_{R \in \mathcal{E}_j} \left( w(R)^{1/p} |R|^{-1/2} |s_R| \right)^p \right)^{q/p} \right\}^{1/q} < \infty;
\end{equation}
for $0 < p < \infty$, we define the weighted Hermite Triebel–Lizorkin sequence space $f^{p,q,\alpha}_w$ as the set of all sequences of complex numbers $s = \{s_R\}_{R \in \mathcal{E}}$ such that
\begin{equation}
\|s\|_{f^{p,q,\alpha}_w} := \left\| \left( \sum_{j \geq 0} 2^{j\alpha q} \sum_{R \in \mathcal{E}_j} (1_R(\cdot)|R|^{-1/2} |s_R|)^q \right)^{1/q} \right\|_{L^p_w} < \infty.
\end{equation}

We use the supremum norm $\ell^\infty$ when $q = \infty$ (or if $p = \infty$ in the Besov case).

These spaces are critical for our frame characterizations in Section 3 and the embedding results of Section 4.

Throughout the rest of this article, we will use the notation $A^{p,q,\alpha}_w(\mathcal{L})$ (or $b^{p,q,\alpha}_w$) to refer to $B^{p,q,\alpha}_w(\mathcal{L})$ or $F^{p,q,\alpha}_w(\mathcal{L})$, with the understanding that $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, and either $0 < p \leq \infty$ if $A = B$ or $0 < p < \infty$ if $A = F$. An analogous comment applies to the sequence spaces, denoted by $a^{p,q,\alpha}_w(\mathcal{L})$ (or $a^{p,q,\alpha}_w$).

### 3. Frame characterizations

In this section we obtain frame characterizations of our weighted Hermite spaces. Frames were constructed in [30] for the unweighted setting and here we show that the same notions can be used to characterize the weighted spaces. Our frames rely on a certain cubature
formula for functions in \( V_N \) (the spaces of projections defined in Section 2). Consider the well known ‘Christoffel function’

\[
\tau(N, x) := \frac{1}{Q_N(x, x)}, \quad x \in \mathbb{R}, \ N \in \mathbb{N}_0,
\]

which has certain useful asymptotic properties (see [30, p376]). Then the following cubature formula

\[
(3.1) \quad \int_{\mathbb{R}^n} f(x) g(x) \, dx \sim \sum_{\xi \in \mathcal{X}_j} \tau_\xi f(\xi) g(\xi), \quad \xi = (\xi_\alpha, \ldots, \xi_n), \ \tau_\xi = \prod_{k=1}^n \tau(2N_j, \xi_\alpha_k)
\]
is exact for all \( f \in V_k \) and \( g \in V_\ell \) with \( k + \ell \leq 4N_j - 1 \).

If \( \{\varphi_j\}_{j \geq 0} \) is an admissible system then for each tile \( R \in \mathcal{E}_j \) we set

\[
(3.2) \quad \varphi_R(x) := \tau_R^{1/2} \varphi_j(\sqrt{L})(x, x_R)
\]
where \( x_R \) is the node of \( R \) and \( \tau_R = \tau_{x_R} \), the coefficient in the cubature formula (3.1). They satisfy \( |\tau_R| \sim |R| \) for any tile \( R \) (see [30, (2.33)]). The functions \( \varphi_R \) were termed needlets in [30].

Given any admissible systems \( \{\varphi_j\}_{j \geq 0} \) and \( \{\psi_j\}_{j \geq 0} \) we define the synthesis \( S_\varphi \) and analysis \( T_\psi \) operators by

\[
S_\varphi : f \mapsto \{f, \varphi_R\}_{R \in \mathcal{E}} \quad \text{and} \quad T_\psi : \{s_R\}_{R \in \mathcal{E}} \mapsto \sum_{R \in \mathcal{E}} s_R \psi_R.
\]

The main result in this section is the following. For the unweighted case see [30 Theorems 3 and 5].

**Theorem 3.1** (Frame characterization). Let \( \alpha \in \mathbb{R}, \ 0 < q \leq \infty, \) and \( 0 < p < \infty \) if \( A^p_\alpha(q) = F^p_\alpha(q) \) or \( 0 < p \leq \infty \) if \( A^p_\alpha(q) = B^p_\alpha(q) \). Suppose that \( \{\varphi_j\}_{j \geq 0} \) and \( \{\psi_j\}_{j \geq 0} \) are two admissible systems. Then,

(a) the operator \( T_\psi : a^p_\alpha(q) \rightarrow A^p_\alpha(q) \) is bounded;

(b) the operator \( S_\varphi : A^p_\alpha(q) \rightarrow a^p_\alpha(q) \) is bounded;

(c) if \( \{\varphi_j\}_{j \geq 0} \) and \( \{\psi_j\}_{j \geq 0} \) satisfy

\[
(3.3) \quad \sum_{j \geq 0} \psi_j(\lambda) \varphi_j(\lambda) = 1 \quad \forall \lambda \geq 0,
\]

then \( T_\psi \circ S_\varphi = I \) on \( A^p_\alpha(q) \), with convergence in \( \mathcal{S}'(\mathbb{R}^n) \). Furthermore, the definitions of \( A^p_\alpha(q) \) are independent of the choice of \( (\varphi_0, \varphi) \).

The proof of Theorem 3.1 is given in Section 3.2. Before embarking on the proof we need to give some preparatory facts and estimates. Our first set of results deals with the analysis and synthesis operators, which adapts [30, Proposition 3] to our band-limited operators.

**Proposition 3.2.** Let \( \{\varphi_j\}_{j \geq 0} \) and \( \{\psi_j\}_{j \geq 0} \) be two admissible systems satisfying (3.3). Then we have the following:

\[
(3.4) \quad f = \sum_{j \geq 0} \psi_j(\sqrt{L}) \varphi_j(\sqrt{L}) f \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n)
\]

and

\[
(3.5) \quad f = \sum_{R \in \mathcal{E}} \{f, \varphi_R\} \psi_R \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n).
\]
Remark 3.3. It is worth pointing out that if (3.3) holds then Proposition 3.2 implies that $T_{\psi} \circ S_{\phi} = I$ on $\mathcal{S}'(\mathbb{R}^n)$. Another consequence of Proposition 3.2 is that our spaces $B^p_{\alpha,w}(\mathcal{L})$ and $F^p_{\alpha,w}(\mathcal{L})$ are quasi-Banach spaces embedded continuously into $\mathcal{S}'(\mathbb{R}^n)$. To see this one can reason as in [30, Proposition 4] and [30, Section 5.1].

Proof of Proposition 3.2. The reproducing formula (3.4) can be found in [21, Proposition 5.5(b)] and [30, Proposition 3], so we only address (3.5).

To prove (3.5) we firstly have

$$\psi_j(\sqrt{L}) \varphi_j(\sqrt{L})(x,y) = \int \psi_j(\sqrt{L})(x,u) \varphi_j(\sqrt{L})(u,y) \, du = \sum_{\xi \in \mathcal{X}_j} \tau_{\xi} \psi_j(\sqrt{L})(x,\xi) \varphi_j(\sqrt{L})(\xi,y).$$

Then in view of (3.2) we see that

$$\psi_j(\sqrt{L}) \varphi_j(\sqrt{L})(x,y) = \sum_{R \in \mathcal{E}_j} \tau_R^{1/2} \psi_j(\sqrt{L})(x,\xi) \tau_R^{1/2} \varphi_j(\sqrt{L})(y,\xi) = \sum_{R \in \mathcal{E}_j} \psi_R(x) \varphi_R(y).$$

The result now follows by invoking (3.4) and then utilising the above equality. Thus we have

$$f(x) = \sum_{j \geq 0} \int \psi_j(\sqrt{L}) \varphi_j(\sqrt{L})(x,y) f(y) \, dy = \sum_{j \geq 0} \sum_{R \in \mathcal{E}_j} \left( \int \varphi_R(y) f(y) \, dy \right) \psi_R(x),$$

which is (3.5).

The next estimate will be needed in the proofs of Theorems 3.1 and 4.1.

Lemma 3.4. Let $\{\varphi_j\}_{j \geq 0}, \{\psi_j\}_{j \geq 0}$ be any pair of admissible systems and let $\sigma \geq 1$. Then for any sequence of numbers $s = \{s_R\}_{R \in \mathcal{E}}$,

$$|\varphi_j(\sqrt{L})(T_{\psi}s)(x)| \lesssim \sum_{k=j-1}^{j+1} \sum_{R \in \mathcal{E}_k} \left| R \right|^{-1/2} |s_R| \left(1 + 2^k|x - x_R|\right)^{-\sigma}, \quad \forall x \in \mathbb{R}^n.$$

Proof of Lemma 3.4. Recall from (2.5) that $\varphi_j(\sqrt{L}) \varphi_R(x) = 0$ whenever $R \in \mathcal{E}_k$ with $|j - k| \geq 2$. It follows that

$$\varphi_j(\sqrt{L})(T_{\psi}s)(x) = \sum_{k=j-1}^{j+1} \sum_{R \in \mathcal{E}_k} s_R \tau_R^{1/2} \varphi_j(\sqrt{L}) \psi_k(\sqrt{L})(x,x_R).$$

Then by (2.6) with $\varepsilon = 4(1 + 4\delta_*^2)^2$ we obtain for any $\sigma \geq 1$,

$$|\varphi_j(\sqrt{L})(T_{\psi}s)(x)| \lesssim \sum_{k=j-1}^{j+1} \sum_{R \in \mathcal{E}_k} |s_R| |R|^{1/2} 2^{kn} e_{4^k}(x_R) e_{4^{k+1}}(x) \left(1 + 2^k|x - x_R|\right)^{-\sigma}.$$

The proof is concluded by invoking the following estimate:

$$|R|^{1/2} 2^{kn} e_{4^k}(x_R) \lesssim |R|^{-1/2} \quad \forall \, R \in \mathcal{E}_k, k \in \mathbb{N}_0.$$

We can see this estimate by considering the two types of tiles listed in Lemma 2.6. Indeed if $|x_R| \leq (1 + 4\delta_*) 2^{k+1}$ then $2^{kn} \sim |R|^{-1}$ by (2.16). Otherwise, from our choice of $\varepsilon$ we have $e_{4^k}(x_R) \lesssim 2^{-2kn/3}$, and so $2^{kn} e_{4^k}(x_R) \lesssim |R|^{-1}$ by (2.17).
3.1. **Maximal lemmas.** In this section we gather and prove some maximal lemmas for sequences (Lemmas 3.5, 3.6, and 3.7) that will be needed in the proof of Theorem 3.1. Recall that the maximal function \( M_s^\theta \) was defined in (2.14).

We first have a generalization of [30, Lemma 4].

**Lemma 3.5.** Let \( s > 0, \theta \geq 0, \sigma > \frac{\theta}{s} + \max\{n, \frac{n}{s}\} \) and \( j \geq 0 \). For a collection of numbers \( \{ a_R \}_{R \in \mathcal{E}_j} \) we set

\[
a_j^*(x) := \sum_{R \in \mathcal{E}_j} \frac{|a_R|}{(1 + 2^j |x - x_R|)^\sigma}, \quad x \in \mathbb{R}^n,
\]

and \( a_R^* := a_j^*(x_R) \). Then there exists \( C > 0 \) such that for any \( \tilde{c} \in (0, 2c_4] \) and \( x \in \mathbb{R}^n \) we have

\[
a_j^*(x) \leq C \inf_{y \in Q(x, \tilde{c}2^{-j})} \langle \mathcal{M}_s^\theta \rangle \left( \sum_{R \in \mathcal{E}_j} |a_R|1_R \right)(y),
\]

and

\[
a_R^*1_R(x) \leq C \inf_{y \in Q(x, \tilde{c}2^{-j})} \langle \mathcal{M}_s^\theta \rangle \left( \sum_{R \in \mathcal{E}_j} |a_R|1_R \right)(y).
\]

**Proof of Lemma 3.5.** We first set

\[
\tilde{a}_j(x) := \sum_{R \in \mathcal{E}_j} \frac{|a_R|}{(1 + 2^j |x - y|_\infty)^\sigma}
\]

where \( d(x, R) := \sup_{y \in R} |x - y|_\infty \). Since \( a_j^*(x) \lesssim \tilde{a}_j(x) \) and \( a_R^*1_R(x) \lesssim \tilde{a}_j(x) \) for every \( x \in \mathbb{R}^n \), then to obtain (3.7) and (3.8) it suffices to prove

\[
\tilde{a}_j(x) \lesssim \inf_{y \in Q(x, \tilde{c}2^{-j})} \langle \mathcal{M}_s^\theta \rangle \left( \sum_{R \in \mathcal{E}_j} |a_R|1_R \right)(y), \quad \forall x \in \mathbb{R}^n.
\]

Set \( \nu = 1 - \min\{1, 1/s\} \) and \( \bar{Q} := Q(x, \tilde{c}2^{-j}) \). Let \( c_3 \) and \( c_4 \) be the constants from Lemma 2.6. We split the proof of (3.9) into two cases.

Case 1: \( |x|_\infty > 2(c_3 + c_4)2^j \).

For each such \( x \) we have \( d(x, R) > \frac{|x|_\infty}{2} \) for any \( R \in \mathcal{E}_j \). Then by either Hölder’s inequality if \( s > 1 \) or the \( s \)-triangle inequality if \( s \leq 1 \), we have

\[
\tilde{a}_j(x) \lesssim \left( \frac{2^{-j}}{|x|_\infty} \right)^\sigma \sum_{R \in \mathcal{E}_j} |a_R| \lesssim \frac{2^{-j(\sigma - 2\nu)}}{|x|_\infty^\sigma} \left( \sum_{R \in \mathcal{E}_j} |a_R|^s \right)^{1/s}.
\]

Set \( Q_x := Q(0, 2|x|_\infty) \). Then Lemma 2.6 (c) ensures \( \mathcal{Q}_j \subseteq Q_x \). Using the \( 1/s \)-triangle inequality if \( s \geq 1 \), and the Hölder inequality if \( s \geq 1 \), along with the estimates \( \Psi_\theta(Q(0, 2|x|_\infty)) \sim |x|_\infty^\theta \) and \( |R| \gtrsim 2^{-jn} \) for every \( R \in \mathcal{E}_j \), we have

\[
\left( \sum_{R \in \mathcal{E}_j} |a_R|^s \right)^{1/s} \lesssim 2^{j\theta} |x|_\infty^\frac{\theta}{s} \int_{Q_x} \left( \sum_{R \in \mathcal{E}_j} |a_R|1_R(y) \right)^s dy^{1/s}.
\]

Inserting this into the previous calculation and observing that \( \sigma > \frac{n + \theta}{s} \) we have

\[
\tilde{a}_j(x) \lesssim 2^{-j(2\sigma - \frac{2n + \theta}{s} - 2\nu)} \inf_{y \in \bar{Q}} \langle \mathcal{M}_s^\theta \rangle \left( \sum_{R \in \mathcal{E}_j} |a_R|1_R \right)(y).
\]
We arrive at (3.9) on recognizing that \( \sigma > \frac{n}{2} + \frac{\theta}{2\nu} \) if \( s \leq 1 \) and \( \sigma > n + \frac{\theta}{2\nu} \) if \( s \geq 1 \), both of which hold because of our assumption on \( \sigma \).

**Case 2:** \( |x|_\infty \leq 2(c_3 + c_4)2^j \).

Let \( \mathcal{E}_j \) be the collection of cubes defined in (2.19). Then for each \( Q \in \mathcal{E}_j \), we set \( a_Q := a_R \) whenever \( Q \subset R \). Then we have

\[
\tilde{a}_j(x) \leq \sum_{Q \in \mathcal{E}_j} \frac{|a_Q|}{(1 + 2^j d(x, Q))^{\sigma}} \quad \text{and} \quad \sum_{R \in \mathcal{E}_j} |a_R|1_R = \sum_{Q \in \mathcal{E}_j} |a_Q|1_Q.
\]

We shall discretize \( \mathbb{R}^n \) into the following ‘square annuli’ and cube centred at \( x \). For each \( m \geq 1 \), we set

\[
\mathcal{A}_0 = \mathcal{A}_0(x, j) = \{Q \in \mathcal{E}_j : |x - x_Q|_\infty \leq c_4 2^{-j}\},
\]

and \( \mathcal{A}_m = \mathcal{A}_m(x, j) = \{Q \in \mathcal{E}_j : c_4 2^{m-j} < |x - x_Q|_\infty \leq c_4 2^{m-j}\} \).

We shall also need the following cubes centred at \( x \). For each \( m \geq 0 \), we set

\[
\mathcal{B}_m = \mathcal{B}_m(x, j) = Q(x, c_4 2^{m-j+1}).
\]

These sets satisfy the following properties

\[
\#\mathcal{A}_m \lesssim 2^{mn}, \quad \mathcal{E}_j = \bigcup_{m \geq 0} \mathcal{A}_m, \quad \bigcup_{Q \in \mathcal{A}_m} Q \subset \mathcal{B}_m.
\]

By (3.10) and Hölder’s inequality if \( s > 1 \) and the \( s \)-triangle inequality otherwise,

\[
\tilde{a}_j(x) \leq \sum_{m \geq 0} \sum_{Q \in \mathcal{A}_m} \frac{|a_Q|}{(1 + 2^j d(x, Q))^{\sigma}} \lesssim \sum_{m \geq 0} 2^{-m(\sigma - \nu)(s)} \left( \sum_{Q \in \mathcal{A}_m} |a_Q|^s \right)^{1/s}.
\]

Using the last property in (3.11), along with the \( 1/s \)-triangle inequality if \( s \geq 1 \) and Hölder’s inequality otherwise, we have

\[
\left( \sum_{Q \in \mathcal{A}_m} |a_Q|^s \right)^{1/s} \leq \left( \int_{\mathcal{B}_m} \sum_{Q \in \mathcal{A}_m} |a_Q|^s |Q|^{-1} 1_Q(y) dy \right)^{1/s}.
\]

Now using the estimates \( |\mathcal{B}_m| \sim 2^{n(m-j)}, |Q| \sim 2^{-jn}, \Psi_\theta(\mathcal{B}_m) \lesssim 2^{(m-j)\theta} |x|_\infty^\theta \) and \( |x|_\infty \lesssim 2^j \) with the fact that \( \mathcal{B}_m \) contains \( x \) we have

\[
\int_{\mathcal{B}_m} \sum_{Q \in \mathcal{A}_m} |a_Q|^s |Q|^{-1} 1_Q(y) dy \lesssim 2^{n(m-j)} \frac{1}{\Psi_\theta(\mathcal{B}_m)} \int_{\mathcal{B}_m} \left( \sum_{Q \in \mathcal{A}_m} |a_Q|1_Q(y) \right)^s dy.
\]

Combining the previous three calculations gives

\[
\tilde{a}_j(x) \lesssim \inf_{y \in Q} M_s^\theta \left( \sum_{R \in \mathcal{E}_j} |a_R|1_R \right)(y) \times \sum_{m \geq 0} 2^{-m(\sigma - \nu - \frac{\theta}{2} - \frac{\theta}{2})}
\]

Our assumption on \( \sigma \) ensures the convergence of the sum, giving (3.9) and completing the proof of Lemma (3.9). \( \Box \)

**Lemma 3.6 ([30] Lemma 5).** Fix \( j \geq 0 \) and let \( g \in V_{4j} \). For each tile \( R \in \mathcal{E}_j \) we define

\[
a_R := \sup_{x \in R} |g(x)| \quad \text{and} \quad b_R := \inf_{x \in R} |g(x)|
\]
Then there exists $k \geq 1$ depending only on $n, \delta_*, \sigma$ such that for every $R \in E_j$ we have

$$a_R^* \leq c \, b_p^*, \quad \forall P \in E_{j+k}, \quad \text{with } P \cap R \neq \emptyset$$

and

$$a_R^* 1_R(x) \leq c \sum_{P \in E_{j+k}, \; P \cap R \neq \emptyset} b_p^* 1_P(x), \quad \forall x \in \mathbb{R}^n. \tag{3.13}$$

**Lemma 3.7.** Fix $0 < p \leq \infty$ and $w \in A_\infty$. Then for any $j \geq 0$ and $g \in V_A$ we have

$$\left( \sum_{R \in E_j} w(R) \max_{x \in R} |g(x)|^p \right)^{1/p} \lesssim \|g\|_{L_w^p}.$$

**Proof of Lemma 3.7.** For any tile $R \in E_j$ let $a_R$ and $b_R$ denote the collection of numbers defined in (3.12). Let $k$ be the integer from Lemma 3.6. Fix a $s \in (0, \min\{p/r_w, 1\})$ and let $\theta$ be the number from Lemma 2.5.

From the fact that the tiles in $E_j$ are all disjoint, then the inequality $a_R \leq a_R^*$ and (3.13) give

$$\left( \sum_{R \in E_j} w(R) \max_{x \in R} |g(x)|^p \right)^{1/p} \lesssim \left\| \sum_{R \in E_j} a_R^* 1_R \right\|_{L_w^p} \lesssim \left\| \sum_{R \in E_j} \sum_{P \in E_{j+k}, \; P \cap R \neq \emptyset} b_p^* 1_P \right\|_{L_w^p}.$$

Now note that $\bigcup_{R \in E_j} R \subset \bigcup_{P \in E_{j+k}} P$. Applying (3.8) with $\sigma = \theta/s + \max\{n, \frac{4}{s}\}$ we have

$$\left\| \sum_{R \in E_j} \sum_{P \in E_{j+k}, \; P \cap R \neq \emptyset} b_p^* 1_P \right\|_{L_w^p} \lesssim \left\| \sum_{P \in E_{j+k}} b_p^* 1_P \right\|_{L_w^p} \lesssim \mathcal{M}^\sigma_s \left( \sum_{P \in E_{j+k}} b_p^* 1_P \right)_{L_w^p}.$$

Finally we invoke Lemma 2.5 (b) with $q = 1$ to obtain

$$\left( \sum_{R \in E_j} w(R) \max_{x \in R} |g(x)|^p \right)^{1/p} \lesssim \left\| \sum_{P \in E_{j+k}} b_p^* 1_P \right\|_{L_w^p} \lesssim \|g\|_{L_w^p},$$

which concludes the proof.  \[ \square \]

### 3.2. Proof of Theorem 3.1

In this section we prove the frame characterizations for our weighted Besov and Triebel–Lizorkin spaces. We start by observing that (3.3) follows readily from Proposition 3.2 (see Remark 3.3). Furthermore, the independence of $\varphi$ in the definitions of the spaces $A^{\alpha,q}_{\alpha,\psi}(L)$ can be seen by following a similar argument to the unweighted case (see [30, Theorems 3.5]). Thus we only need to prove (a) and (b). We shall tackle the Triebel–Lizorkin and Besov cases separately.

**The Triebel–Lizorkin case.**

Part (a). Let $\{\psi_j\}_{j \geq 0}$ be another admissible system. We shall prove the boundedness of $T_{\psi}$. Let $0 < r < \min\{p/r_w, q\}$ and $\theta$ be the number from Lemma 2.5. Fix $\sigma = \theta/r + \max\{n, n/r\}$. By Lemma 3.4 and 3.7 with $a_R = |R|^{-1/2} |s_R|$ we have

$$|\varphi_j(\sqrt{L}) T_{\psi} s_j(x)| \lesssim \sum_{k=j-1}^{j+1} \frac{|R|^{-1/2} |s_R|}{(1 + 2^k |x - x_R|)^\sigma} \lesssim \sum_{k=j-1}^{j+1} \mathcal{M}^\sigma_s \left( \sum_{R \in E_k} |R|^{-1/2} |s_R| 1_R \right)(x).$$
Applying this estimate along with Lemma 2.5 (b) we have
\[
\|T_{\theta}^s\|_{F^{p,q}_{\alpha,w}} \lesssim \left\| \left( \sum_{j \geq 0} \left( 2^{ja} \sum_{R \in \mathcal{E}_j} M_\theta^R \left( \sum_{R \in \mathcal{E}_k} |R|^{-1/2} |s_R| 1_R \right) \right)^q \right)^{1/q} \right\|_{L_w^p}
\]
\[
\lesssim \left\| \left( \sum_{j \geq 0} M_\theta^R \left( 2^{ja} \sum_{R \in \mathcal{E}_j} |R|^{-1/2} |s_R| 1_R \right) \right)^q \right\|_{L_w^p}
\]
\[
\lesssim \left\| \left( \sum_{j \geq 0} \left( 2^{ja} \sum_{R \in \mathcal{E}_j} |R|^{-1/2} |s_R| 1_R \right)^q \right)^{1/q} \right\|_{L_w^p} = \|s\|_{F^{p,q}_{\alpha,w}}.
\]

Part (b). Let \( f \in F^{p,q}_{\alpha,w} \). Note that for each \( j \geq 0 \) we have \( \varphi_j(\sqrt{L})f \in V_j \) and hence Lemma 3.6 can be applied to \( g = \varphi_j(\sqrt{L})f \). Define the sequences
\[
\langle f^{\varphi,R} \rangle := \sup_{x \in R} |\varphi_j(\sqrt{L})f(x)|, \quad R \in \mathcal{E}_j
\]
\[
\langle f_{\varphi,P,k} \rangle := \inf_{x \in P} |\varphi_j(\sqrt{L})f(x)|, \quad P \in \mathcal{E}_{j+k}
\]
Then we firstly have
\[
\langle f^{\varphi,R} \rangle \lesssim \sum_{P \in \mathcal{E}_{j+k}, R \cap P \neq \emptyset} \langle f_{\varphi,P,k} \rangle 1_P(x), \quad x \in \mathbb{R}^n.
\]

Then by (3.14), (3.15) and (3.8) with \( 0 < r < \min\{p/r_w, q\} \) and \( \theta \) from Lemma 2.5 we have
\[
\|S_{\varphi}f\|_{F^{p,q}_{\alpha,w}} \lesssim \left\| \left( \sum_{j \geq 0} 2^{jaq} \sum_{R \in \mathcal{E}_j} \langle f^{\varphi,R} \rangle^q \right)^{1/q} \right\|_{L_w^p}
\]
\[
\lesssim \left\| \left( \sum_{j \geq 0} 2^{jaq} \left( \sum_{P \in \mathcal{E}_{j+k}} \langle f_{\varphi,P,k} \rangle^q \right) \right)^{1/q} \right\|_{L_w^p}
\]
\[
\lesssim \left\| \left( \sum_{j \geq 0} M_\theta^R \left( 2^{ja} \sum_{P \in \mathcal{E}_{j+k}} \langle f_{\varphi,P,k} \rangle 1_P \right)^q \right)^{1/q} \right\|_{L_w^p}.
\]
Now we invoke Lemma 2.5 (b) and arrive at
\[
\|S_{\varphi}f\|_{F^{p,q}_{\alpha,w}} \lesssim \left\| \left( \sum_{j \geq 0} \left( 2^{ja} \sum_{P \in \mathcal{E}_{j+k}} \langle f_{\varphi,P,k} \rangle 1_P \right)^q \right)^{1/q} \right\|_{L_w^p} \leq \|f\|_{F^{p,q}_{\alpha,w}},
\]
which completes the proof of part (b).

**The Besov case.** As in the Triebel–Lizorkin case, part (c) follows from Proposition 3.2 (see Remark 3.3).

Part (a). Let \( \{\tilde{\varphi}_j\}_{j \geq 0} \) be another admissible system. We shall prove the boundedness of \( T_{\tilde{\varphi}} \). Let \( 0 < r < \min\{p/r_w, 1\} \) and \( \theta \) be the number from Lemma 2.5 Then by Lemma 3.4
with $\sigma > \theta/r + \max\{n, n/r\}$, (3.7) with $a_R = \|R|^{-1/2}|s_R|$, and Lemma 2.5 (b) with $q = 1$ we have
\[
\|\varphi_j(\sqrt{L})(T_{\bar{V}}s)\|_{L_w^p} \lesssim \sum_{k=j-1}^{j+1} \left\| \mathcal{M}_p^p \left( \sum_{R \in \mathcal{E}_k} |R|^{-1/2}|s_R| \right) \right\|_{L_w^p} \\
\lesssim \sum_{k=j-1}^{j+1} \left\| \sum_{R \in \mathcal{E}_k} |R|^{-1/2}|s_R| \right\|_{L_w^p} \\
\lesssim \sum_{k=j-1}^{j+1} \left( \sum_{R \in \mathcal{E}_k} (w(R)^{1/p}|R|^{-1/2}|s_R|)^p \right)^{1/p}.
\]

From the preceding estimate it follows readily that
\[
\|T_{\bar{V}}s\|_{B_{\alpha,w}^{p, q}} \lesssim \left\{ \sum_{j \geq 0} 2^{j\alpha q} \left( \sum_{k=j-1}^{j+1} \left( \sum_{R \in \mathcal{E}_k} (w(R)^{1/p}|R|^{-1/2}|s_R|)^p \right)^{1/p} \right)^q \right\}^{1/q} \lesssim \|s\|_{b_{\alpha,w}^{p, q}}.
\]

Part (b). Since $\varphi_j(\sqrt{L})f \in V_{\gamma}$, we may apply Lemma 3.7 to obtain
\[
\|S_{\varphi}f\|_{b_{\alpha,w}^{p, q}} \lesssim \left\{ \sum_{j \geq 0} 2^{j\alpha q} \left( \sum_{R \in \mathcal{E}_j} w(R) |\varphi_j(\sqrt{L})f(x_R)|^p \right)^{q/p} \right\}^{1/q} \lesssim \|f\|_{B_{q,w}^{p, q}}.
\]

This concludes the Besov case and also completes the proof of Theorem 3.1.

4. EMBEDDINGS FOR WEIGHTED HERMITE SPACES

In this section we characterize continuous Sobolev-type embeddings for weighted Besov and Triebel–Lizorkin spaces associated the Hermite operator. The main result in this section is the following.

Theorem 4.1 (Embedding for sequence spaces). Let $\gamma > 0$ and $w$ be any weight. Then the following holds.
(a) For any $\alpha_1 \leq \alpha_2$ and $0 < p_1, p_2, q_1, q_2 \leq \infty$ with $0 < q_2 \leq q_1 \leq \infty$ and $\alpha_1 - \frac{\gamma}{p_1} = \alpha_2 - \frac{\gamma}{p_2}$, we have
\[
b_{\alpha_2,w}^{p_2, q_2} \hookrightarrow b_{\alpha_1,w}^{p_1, q_1}
\]
if and only if $w$ satisfies the lower bound property (1.6) of order $\gamma$.
(b) If in addition $w \in A_\infty$ then for any $\alpha_1 \leq \alpha_2$, $0 < p_1, p_2 < \infty$ and $0 < q_1, q_2 \leq \infty$ with $\alpha_1 - \frac{\gamma}{p_1} = \alpha_2 - \frac{\gamma}{p_2}$, we have
\[
f_{\alpha_2,w}^{p_2, q_2} \hookrightarrow f_{\alpha_1,w}^{p_1, q_1}
\]
if and only if $w$ satisfies the lower bound property (1.6) of order $\gamma$.

Theorem 4.1 will be used to prove Theorem 1.1 in Section 4.2. The proof of Theorem 4.1 will be given in Section 4.3. Before turning to its proof we state a crucial fact concerning the the lower bound property (1.6). Recall that the function $q(\cdot)$ was defined in (2.11).

Proposition 4.2 (Lower bound for tiles and balls). Suppose that $\gamma > 0$ and let $w$ be any weight. The following are equivalent.
(a) There exists $C > 0$ such that for every $j \geq 0$ and $R \in \mathcal{E}_j$ we have
\[
w(R) \geq C 2^{-j\gamma}.
\]
There exists \( \tilde{C} > 0 \) such that for every ball \( B \) with \( 0 < r_B \leq \varrho(x_B) \) we have
\[
w(B) \geq \tilde{C} r_B^\gamma.
\]

**Proof of Proposition 4.2.** Proof of (a) \( \Rightarrow \) (b).

Fix a ball \( B \) with \( 0 < r_B \leq \varrho(x_B) \). We observe that there exists \( j \geq 0 \) with
\[
2^j \leq |x_B|_\infty < 2^{j+1}
\]
and thus it follows that \( r_B \leq 2^{-j} \). Furthermore, there exists \( k \geq 0 \) with
\[
2^{-(j+k+1)} < r_B \leq 2^{-(j+k)}.
\]

Next choose an integer \( \ell \) such that
\[
2^\ell \geq \max \left\{ 4c_0 \sqrt{n}, \left( \frac{c_2}{4\delta_*} \right)^3 \right\}
\]
Now consider tiles from \( \mathcal{E}_{j+k+\ell} \). Firstly there is a tile \( R \) from the collection \( \mathcal{E}_{j+k+\ell} \) that contains the centre of \( B \). Indeed from (4.1) and our assumption on \( \ell \) we have
\[
|x_B|_\infty \leq 2^{j+k+\ell+1}
\]
and so by (2.16)
\[
R = Q(0, c_0 2^{-(j+k)}).
\]
Thirdly \( R \) is contained in \( B \). Indeed since \( x_B \in R \) and by (4.2) we have
\[
diam(R) = 2c_0 \sqrt{n} 2^{-(j+k+\ell+1)} \leq 2^{-(j+k+1)} < r_B.
\]
Hence \( R \subset B \). Combining the above three facts on \( B \) and \( R \) we conclude, by (4.2),
\[
w(B) \geq w(R) \geq C 2^{-(j+k+\ell)\gamma} \geq \tilde{C} r_B^\gamma
\]
with \( \tilde{C} = C 2^{-\ell \gamma} \).

We turn to the proof of (b) \( \Rightarrow \) (a). Fix \( j \geq 0 \) and let \( R \) be a tile from \( \mathcal{E}_j \). Then consider the ball \( B(x_R, r) \) where
\[
r := \min \left\{ c_1, \frac{1}{1+c_3} \right\} 2^{-j}.
\]
Then firstly we have \( B \subset R \) by (2.17). Secondly we know that \( |x_R| \leq c_3 2^j \) by (2.18). Thus \( r \leq \varrho(x_R) \) and so \( B(x_R, r) \) meets the hypothesis of Proposition 4.2 (b). Thus we have
\[
w(R) \geq w(B) \geq \tilde{C} r^\gamma = C 2^{-j \gamma}
\]
where \( C = \tilde{C} \min\{c_1, (1+c_3)^{-1}\}^\gamma \).

We are now ready to give the proofs of our embedding theorems.
4.1. Proof of Theorem\textsuperscript{4.1}. Part (a). We first prove necessity of the lower bound property. Suppose that for some $C > 0$ we have

\begin{equation}
\|s\|_{b_{q_1}^{p_1}, w} \leq C \|s\|_{b_{q_2}^{p_2}, w}
\end{equation}

for all sequences $s \in b_{q_2}^{p_2}, w$. Fix $j \geq 0$ and a tile $R \in E_j$ and consider the sequence with $s_R = 1$ and $s_{\bar{R}} = 0$ for any tiles $\bar{R} \neq R$. Then for any $p, q$ and $\alpha$ we have

$$
\|s\|_{b_{q}^{p}, w} = \left\{ 2^{j\alpha q} (w(R)|R|^{-p/2})^{q/p} \right\}^{1/q} = 2^{j\alpha} w(R)^{1/p} |R|^{-1/2}.
$$

Then (4.5) implies that

$$
2^{j\alpha} w(R)^{1/p_1} |R|^{-1/2} \leq C 2^{j\alpha_2} w(R)^{1/p_2} |R|^{-1/2},
$$

which in turn gives

$$
w(R)^{1/p_2 - 1/p_1} \geq C^{-1} 2^{-j\gamma (1/p_2 - 1/p_1)}.
$$

Now noting that $p_2 \leq p_1$ we obtain $w'(R) \geq C' 2^{-j\gamma}$, where $C' = C^{-p_1/p_2} (p_1 - p_2)$. Since this holds for any tile $R$ then Proposition\textsuperscript{4.2} yields the lower bound property for $w$ with order $\gamma$.

We now prove sufficiency for part (a). Since $q_1 \geq q_2$ and $p_1 \geq p_2$ we may apply the $q_2/q_1$-triangle inequality followed by the $p_2/p_1$-triangle inequality to obtain

$$
\|s\|_{b_{q_1}^{p_1}, w} \leq \left\{ \sum_{j \geq 0} 2^{j\alpha_1 q_2} \left( \sum_{R \in E_j} (w(R)^{1/p_1} |R|^{-1/2} s_R)^{p_2} \right)^{q_2/p_2} \right\}^{1/q_2}
$$

Now the lower bound property of $w$ and Proposition\textsuperscript{4.2} allow us to control the last expression by a constant multiple of

$$
\left\{ \sum_{j \geq 0} 2^{j\alpha_1 q_2} \left( \sum_{R \in E_j} (2^{-j\gamma (1/p_1 - 1/p_2)} w(R)^{1/p_2} |R|^{-1/2} s_R)^{p_2} \right)^{q_2/p_2} \right\}^{1/q_2},
$$

which equals $\|s\|_{b_{q_2}^{p_2}, w}$.

Part (b). Necessity can be done in the same way as part (a), after observing that for the same sequence $s$ defined in (a) we have

$$
\|s\|_{f_{\alpha}^{p_1}, w} = \left( \int_R 2^{j\alpha p} |R|^{-p/2} w(x) \, dx \right)^{1/p} = 2^{j\alpha} w(R)^{1/p} |R|^{-1/2}
$$

for any $p, q, \alpha$ and $w$.

We turn to sufficiency of the lower bound property for part (b). We shall prove that there exists $C > 0$ with

\begin{equation}
\|s\|_{f_{\alpha}^{p_1}, q_1} \leq C \|s\|_{f_{\alpha}^{p_2}, q_2}
\end{equation}

for all sequences $s \in f_{\alpha}^{p_2}, q_2$. Without loss of generality we may assume that $s = \{s_R\}_R$ is a sequence with $\|s\|_{f_{\alpha}^{p_2}, q_2} = 1$. To simplify notation we denote

$$
F_j(\alpha, q, x) := \sum_{R \in E_j} 2^{j\alpha q} (|R|^{-1/2} s_R 1_R(x))^q.
$$
We claim that (4.6) will follow from the following two estimates: there exists $\tilde{\beta} > 0$ such that for each $x \in \mathbb{R}^n$ and any $N \geq 0$,

$$\left\{ \sum_{j=0}^{N} F_j(\alpha_1, q_1, x) \right\}^{1/q_1} \leq \tilde{\beta} 2^{N\gamma/p_1};$$

(4.7)

and for any $N \geq -1$,

$$\left\{ \sum_{j=N+1}^{\infty} F_j(\alpha_1, q_1, x) \right\}^{1/q_1} \leq 2^{-N\gamma(p_2^{-1} - 1/p_1)} \left\{ \sum_{j=N+1}^{\infty} F_j(\alpha_2, q_2, x) \right\}^{1/q_2}.$$  

(4.8)

Let us show how (4.7) and (4.8) lead to (4.6). We first discretize the left hand side of (4.6) using the well known representation of the $L^p$ norm as follows.

$$\|s\|^p_{p_1; q_1; w} = \int_0^{\infty} t^{p_1-1} \left\{ \int \left\{ \sum_{j=0}^{\infty} F_j(\alpha_1, q_1, x) \right\}^{1/q_1} > t \right\} dt =: I + II,$$

where

$$I = p_1 \int_0^{\beta} t^{p_1-1} \left\{ \int \left\{ \sum_{j=0}^{\infty} F_j(\alpha_1, q_1, x) \right\}^{1/q_1} > t \right\} dt,$$

$$II = p_1 \sum_{N=0}^{\infty} \int_{\beta 2^{N+1}\gamma/p_1}^{\beta 2^{N}\gamma/p_1} t^{p_1-1} \left\{ \int \left\{ \sum_{j=0}^{\infty} F_j(\alpha_1, q_1, x) \right\}^{1/q_1} > t \right\} dt,$$

and $\beta = 2^{\tilde{\beta} 1/q_1}$ with $\tilde{\beta}$ being the constant from (4.7).

We estimate $I$ by applying (4.8) with $N = -1$ along with the substitution $t = 2^{\gamma(1/p_2 - 1/p_1)} u$.

Thus,

$$I \leq p_1 \int_0^{\beta} t^{p_1-1} \left\{ \int \left\{ \sum_{j=0}^{\infty} F_j(\alpha_2, q_2, x) \right\}^{1/q_2} > t \right\} dt$$

$$= p_1 \int_0^{\beta} t^{p_1-1} \left\{ \int \left\{ \sum_{j=0}^{\infty} F_j(\alpha_2, q_2, x) \right\}^{1/q_2} > 2^{-\gamma(p_2^{-1} - 1/p_1)} t \right\} dt$$

$$= p_1 2^{\gamma(p_1 p_2^{-1} - 1)} \int_0^{\beta 2^{\gamma(p_2^{-1} - 1)}} u^{p_1-1} \left\{ \int \left\{ \sum_{j=0}^{\infty} F_j(\alpha_2, q_2, x) \right\}^{1/q_2} > u \right\} du.$$

Now from inequality $u^{p_1} \leq u^{p_2} (2^{\gamma(1/p_2 - 1/p_1)})^{p_1-p_2}$, we have

$$I \lesssim p_2 \int_0^{\beta 2^{\gamma(p_2^{-1} - 1)}} u^{p_2-1} \left\{ \int \left\{ \sum_{j=0}^{\infty} F_j(\alpha_2, q_2, x) \right\}^{1/q_2} > u \right\} du \lesssim \|s\|^p_{p_2; q_2; w} = 1.$$

For term $II$ we first employ (4.7) and (4.8) respectively to obtain

$$II \leq p_1 \sum_{N=0}^{\infty} \int_{\beta 2^{N}\gamma/p_1}^{\beta 2^{N+1}\gamma/p_1} t^{p_1-1} \left\{ \int \left\{ \sum_{j=0}^{\infty} F_j(\alpha_1, q_1, x) \right\}^{1/q_1} > 2^{-1/q_1} t \right\} dt$$

$$\leq p_1 \sum_{N=0}^{\infty} \int_{\beta 2^{N}\gamma/p_1}^{\beta 2^{N+1}\gamma/p_1} t^{p_1-1} \left\{ \int \left\{ \sum_{j=N+1}^{\infty} F_j(\alpha_1, q_1, x) \right\}^{1/q_1} > 2^{-1/q_1} t \right\} dt$$
orthogonality to estimate (3.7) to (4.9) with \( \sigma > \theta/r \). This estimate, along with the substitution \( u = \beta t^{p_1/p_2} \), gives

\[
II \leq p_1 \sum_{N = 0}^{\infty} \int_{\beta_2 N} \frac{1}{t} \, dt \left( \left\{ x \in \mathbb{R}^n : \left( \sum_{j = N+1}^{\infty} F_j(\alpha_2, q_2, x) \right)^{1/q_2} > \beta t^{p_1/p_2} \right\} \right) dt
\]

which equals \( \| s \|_{L_p^{p_2/q_2}} \).

It remains to estimate (4.7) and (4.8). We begin with (4.7). The idea is to apply almost orthogonality to estimate \( |s_R| \). Let \( \{ \varphi_j \}_{j \geq 0} \) and \( \{ \psi_j \}_{j \geq 0} \) be admissible systems so that (3.3) holds. Then we have

\[
s_R = \langle T_{\psi}s, \varphi_R \rangle = \frac{1}{|R|^{1/2}} \varphi_j(\sqrt{L})(T_{\psi}s)(x_R),
\]

and so by Lemma 3.4 we have for any \( \sigma \geq 1 \),

\[
|s_R| \leq \left| \tau_{R}^{1/2} \right| \left| \varphi_j(\sqrt{L})(T_{\psi}s)(x_R) \right| \lesssim |R|^{1/2} \sum_{k = j+1}^{j+1} \sum_{P \in \mathcal{E}_k} \frac{|P|^{-1/2}|s_P|}{(1 + 2k|x_R - \xi_P|)^\sigma}.
\]

Now fix an \( r \in (0, \min\{p_2/r, q_2\}) \) and let \( \theta \) be the number from Lemma 2.5. Then applying (3.7) to (4.9) with \( \sigma > \theta/r + \max\{n, n/r\} \) and \( a_P = |P|^{-1/2}|s_P| \), we have

\[
|R|^{-1/2}|s_R| \lesssim \sum_{k = j+1}^{j+1} \sum_{P \in \mathcal{E}_k} \frac{|P|^{-1/2}|s_P|}{(1 + 2k|x_R - \xi_P|)^\sigma} \lesssim \sum_{k = j+1}^{j+1} \mathcal{M}_r^\theta \left( \sum_{P \in \mathcal{E}_k} |P|^{-1/2}|s_P|1_P \right)(x').
\]

for any \( x' \in Q(x, 2c_4 2^{-k}) \) and \( x \in R \).

Write \( \tilde{Q} := Q(x, 2c_4 2^{-k}) \). Then for each \( k \in \{j-1, j, j+1\} \) we have, by Lemma 2.4 (b),

\[
\inf_{x' \in \tilde{Q}} \mathcal{M}_r^\theta(F_k(\alpha_2, 1, \cdot))(x') \leq \left\{ \inf_{x' \in \tilde{Q}} \left[ \mathcal{M}_r^\theta(F_k(\alpha_2, 1, \cdot))(x') \right]^{q_2} \right\}^{1/p_2} \lesssim \left\{ \frac{1}{w(\tilde{Q})} \int_{\tilde{Q}} \left( \sum_{k \geq 0} \mathcal{M}_r^\theta(F_k(\alpha_2, 1, \cdot))(y)^{q_2} \right)^{\frac{p_2}{q_2}} w(y) \, dy \right\}^{\frac{1}{p_2}} \lesssim w(\tilde{Q})^{-1/p_2} \left\| \left( \sum_{k \geq 0} \mathcal{M}_r^\theta(F_k(\alpha_2, 1, \cdot)) \right)^{q_2} \right\|_{L^{p_2}}^{\frac{1}{q_2}} \lesssim w(\tilde{Q})^{-1/p_2} \left\| \sum_{k \geq 0} F_k(\alpha_2, 1, \cdot) \right\|_{L^{p_2}}^{q_2} = 1,
\]

Thus noting that

\[
\left\| \left( \sum_{k \geq 0} F_k(\alpha_2, 1, \cdot) \right)^{q_2} \right\|_{L^{p_2}}^{\frac{1}{q_2}} = \left\| \left( \sum_{k \geq 0} F_k(\alpha_2, q_2, \cdot) \right)^{q_2} \right\|_{L^{p_2}}^{q_2} = \| s \|_{L^{p_2/q_2}} = 1,
\]

where

\[
\mathcal{M}_r^\theta(F_k(\alpha_2, 1, \cdot)) \leq \sum_{k \geq 0} \mathcal{M}_r^\theta(F_k(\alpha_2, 1, \cdot)).
\]
we arrive at the estimate
\begin{equation}
(4.11)
\inf_{x' \in \tilde{Q}} \mathcal{M}^\theta_r \left( \sum_{P \in \mathcal{E}_k} |P|^{-1/2} |S_P|_1 \right) (x') = 2^{-k_0} \inf_{x' \in \tilde{Q}} \mathcal{M}^\theta_r \left( F_k(\alpha_2, 1, \cdot) \right) (x') \lesssim 2^{-k_0} w(\tilde{Q})^{-1/p_2}.
\end{equation}

Therefore by combining (4.10) and (4.11), and observing that because \( \sum_{R \in \mathcal{E}_j} 1_R = 1_{Q_j} \leq 1 \) and \( 2^{-k_0} \sim 2^{-j_0} \), we have for each \( x \in \mathbb{R}^n \),
\begin{equation}
(4.12)
\left\{ \sum_{j=0}^N F_j(\alpha_1, q_1, x) \right\}^{1/q_1} \lesssim \left\{ \sum_{j=0}^N \sum_{R \in \mathcal{E}_j} 2^{j \alpha q_1} 1_R(x) \left( \sum_{k=j-1}^{j+1} \inf_{x' \in \tilde{Q}} \mathcal{M}^\theta_r \left( \sum_{P \in \mathcal{E}_k} |P|^{-1/2} |S_P|_1 \right) (x') \right) \right\}^{1/q_1} \lesssim \left\{ \sum_{j=0}^N \sum_{R \in \mathcal{E}_j} 2^{j \alpha q_1} 1_R(x) \left( \sum_{k=j-1}^{j+1} 2^{-k_0} w(\tilde{Q})^{-1/p_2} \right) \right\}^{1/q_1} \lesssim \left\{ \sum_{j=0}^N 2^{j q_1(\alpha_1 - \alpha_2)} \left( \sum_{k=j-1}^{j+1} w(\tilde{Q})^{-1/p_2} \right) \right\}^{1/q_1}.
\end{equation}

To complete the estimate we apply the lower bound condition to handle \( w(\tilde{Q})^{-1/p_2} \). First note that if \( x \not\in Q(0, c_3 2^j) \) then \( 1_R(x) = 0 \) for every \( R \in \mathcal{E}_j \) and so estimate (4.11) holds trivially. Thus we may assume \( x \in Q(0, c_3 2^j) \). Let us take the ball \( B \) with \( x_B = x \) and
\[ r_B = \min \left\{ \frac{1}{1 + c_3}, c_4 \right\} 2^{-j}. \]

Then we have \( B \subset \tilde{Q} \) because \( k \in \{ j - 1, j, j + 1 \} \) implies \( r_B \leq c_4 2^{-k} \). We also have \( |x|_{\infty} \leq c_3 2^j \) because \( x \in Q(0, c_3 2^j) \), and so \( r_B \leq \varrho(x_B) \). Thus we may apply the lower bound condition to \( B \) and obtain
\[ w(\tilde{Q}) \geq w(B) \gtrsim r_B^\gamma \sim 2^{-j \gamma}. \]

Inserting this into the final line of (4.12), we have for each \( x \in Q(0, c_3 2^j) \),
\[ \left\{ \sum_{j=0}^N F_j(\alpha_1, q_1, x) \right\}^{1/q_1} \lesssim \left\{ \sum_{j=0}^N 2^{j q_1(\alpha_1 - \alpha_2)} r_B^{-\gamma q_1/p_1} \right\}^{1/q_1} = \left\{ \sum_{j=0}^N 2^{j (\gamma q_1/p_1)} \right\}^{1/q_1} \lesssim 2^{N \gamma / p_1}. \]

This proves (4.11), after taking into account that the estimate holds trivially if \( x \not\in Q(0, c_3 2^j) \).

We turn to the proof of (4.8). Now since all the tiles in \( \mathcal{E}_j \) are disjoint we have
\begin{equation}
(4.13)
\left\{ \sum_{j=N+1}^\infty F_j(\alpha_1, q_1, x) \right\}^{1/q_1} = \left\{ \sum_{j=N+1}^\infty 2^{jq_1(\alpha_1 - \alpha_2)} F_j(\alpha_2, 1, x) \right\}^{1/q_1}. \end{equation}

If \( q_1 \geq q_2 \) we apply the \( q_2/q_1 \)-triangle inequality, but if \( q_1 \leq q_2 \) then we apply Hölder’s inequality with exponent \( q_2/q_1 \) to the the terms \( 2^{jq_2(\alpha_1 - \alpha_2)/q_1} F_j(\alpha_2, 1, x) \) and \( 2^{jq_2(\alpha_1 - \alpha_2)/q_1} \). In
either case \([4.13]\) becomes

\[
\left\{ \sum_{j=N+1}^{\infty} F_j(\alpha_1, q_1, x) \right\}^{1/q_1} \leq \left\{ \sum_{j=N+1}^{\infty} 2^{j^2(q_2-\alpha_2)} F_j(\alpha_2, q_2, x) \right\}^{1/q_2} \\
\lesssim 2^{N(\alpha_1-\alpha_2)} \left\{ \sum_{j=N+1}^{\infty} F_j(\alpha_2, q_2, x) \right\}^{1/q_2},
\]

which yields \((4.8)\).

This completes the proof of \((4.7)\) and \((4.8)\) and hence of Theorem 4.1.

### 4.2. Proof of Theorem 1.1

We can now bring together the material developed throughout the paper to prove our main result stated in the Introduction. The theorem follows by combining our frame decompositions with our embedding result for the sequence spaces. In fact by invoking in turn Theorem 3.1 (a), Theorem 4.1 and Theorem 3.1 (b) we have

\[
\|f\|_{A^{p_2,q_2}_{\alpha_2,w}(L)} \lesssim \|\{\langle f, \varphi_R \rangle\}_R\|_{A_{\alpha_1,q_1}^{p_1,q_1}(\mathcal{L})} \lesssim \|\{\langle f, \varphi_R \rangle\}_R\|_{A_{\alpha_1,q_1}^{p_1,q_1}(\mathcal{L})},
\]

concluding the proof of Theorem 1.1.

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