Hamiltonian treatment of Collapsing Thin Shells in Lanczos-Lovelock’s theories

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The Hamiltonian treatment for the collapse of thin shells for a family of Lanczos-Lovelock theories is studied. This formalism allows us to carry out a concise analysis of these theories. It is found that the black holes solution can be created by collapsing a thin shell. Naked singularities cannot be formed by this mechanism. Among the different Lanczos-Lovelock’s theories, the Chern-Simons’ theory corresponds to an exceptional case, because naked singularities can emerge from the collapse of a thin shell. This kind of theory does not possess a gravitational self-interaction analogous to the Newtonian case.

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I. INTRODUCTION

In a recent work [1], black hole solutions in a particular class of Lovelock’s gravitation theories were studied. These theories were selected by requiring that they have a unique Anti-de Sitter (AdS) vacuum with a fixed cosmological constant. This strongly restricts the coefficients in the Lanczos-Lovelock (LL) action [2]. For a given dimension $d$, the Lagrangians under consideration are labelled by an integer $k = 1, \ldots, \left\lfloor \frac{d-1}{2} \right\rfloor$, where the Einstein-Hilbert (EH) Lagrangian corresponds to the case $k = 1$. For $k = \left\lfloor \frac{d-1}{2} \right\rfloor$, we must distinguish between even and odd dimensions, because the theories are different. When $d$ is odd, the corresponding Lagrangian is given by the Euler-Chern-Simons form (CS) for the AdS group, whose exterior derivative is proportional to the Euler density in $2n$ dimensions [3, 4]. For $d$ even, the Lagrangian reads as the Born-Infeld form (BI). In this case the expression for the (LL) action is proportional to the Pfaffian of the 2-form $\bar{R}^{ab}$ and, in this sense, it has a Born-Infeld-like form [5]. These two cases are exceptional, because they are the only ones which allow sectors with non-trivial torsion [6]. For $d \geq 7$ there exist other interesting possibilities, which are different from EH, BI and CS. For example, the theory with $k = 2$ has been studied by several authors in different scenarios [7, 8, 9, 10].

In this LL theories for any dimensions and $k$, there exist well-behaved black hole solutions. However, we must differentiate between cases with odd and even $k$, because in theories with even $k$ an, additional solution appears, which represents a naked singularity.

It is interesting to study the black holes formations through gravitational collapses of thin shells. In the usual thin shell treatment [11, 12, 13], the analysis of collapse is based on the discontinuities of the extrinsic curvature of the world tube of the collapsing matter. However, the implementation of the Israel formalism in this kind of theories (LL) is very difficult, because the action contains high powers in the curvature and, therefore, in the extrinsic curvature. In this formalism the complicated analysis of collapse makes the treatment quite unattractive.

Another approach in studying matter collapses is the Oppenheimer-Snyder formalism, which was applied by Ilha et al [14, 15] to the case of a homogeneous collapsing dust, where the inner metric is described by the Friedmann-Robertson-Walker line element, and the external metric corresponds to the solution of the fields equations in BI or CS theories. In Ref. [15], the authors discussed the formation of a naked singularity in the CS theory.

On the other hand, an alternative way to study gravitational collapse of thin shell is the Hamiltonian treatment. This treatment was applied in Ref. [16] to the Einstein-Hilbert gravity, where the direct integration of the canonical constraints reproduces the standard shell dynamics for a number of known cases. In particular, it was applied in detail to three dimensional spacetime and the properties of the (2+1)-dimensional charged black hole collapse was further
elucidated. The Hamiltonian treatment was also extended to deal with rotating solutions in three dimensions. The general form of the equations of the shell dynamics implies the stability of black holes. As far as, black hole in this model cannot be converted into naked singularities by any shell collapse processes.

In this work we will extend the Hamiltonian formalism to our approach to the theory of LL in high dimensions, and particularly to the theories described in Ref. [1]. This formalism permits us to analyze the black holes formations in an economical way.

The plan of the paper is as follows. Section II briefly reviews the LL action and its spherically symmetric solution. Section III analyzes the collapse of a spherically symmetric shell under the Hamiltonian formalism. Finally, section IV is devoted to conclusions.

II. LL ACTION

The LL action is a polynomial of degree \( \frac{d}{2} \) in the curvature, which can be also written in terms of the Riemann curvature \( R_{ab} \) and the vielbein \( e^a \) as

\[
I(g) = \kappa \int \sum_{p=0}^{[d/2]} \alpha_p \epsilon_{a_1 \ldots a_d} R^{a_1 a_2} \ldots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \ldots e^{a_d},
\]

(1)

where \( \alpha_p \) are arbitrary constants. In the first-order formalism, the action (1) is regarded as a functional of the vielbein and the spin connection, and the corresponding field equations obtained varying with respect to \( e^a \) and \( \omega^{ab} \) reads

\[
\sum_{p=0}^{[d/2]} (d - 2p) \alpha_p \epsilon_{ab_1 \ldots b_{d-1}} R^{b_1 b_2} \ldots R^{b_{2p-1} b_{2p}} e^{b_{2p+1}} \ldots e^{b_{d-1}} = 0,
\]

(2)

\[
\sum_{p=0}^{[d-1]/2} p (d - 2p) \alpha_p \epsilon_{abc_1 \ldots c_{d-1}} R^{c_1 c_2} \ldots R^{c_{2p-1} c_{2p}} T^{c_{2p+1}} e^{c_{2p+2}} \ldots e^{c_d} = 0.
\]

(3)

Here \( T^a = de^a + \omega^a_b e^b \) is the torsion 2-form.

Note that in even dimensions, the term \( L^{(d/2)} \) is the Euler density and, therefore, does not contribute to the field equations. However, the presence of this term in the action—with a fixed weight factor—guarantees the existence of a well-defined variational principle for asymptotically locally AdS spacetimes [17, 18].

The first two terms in the LL action (1) are the cosmological and kinetic terms of the EH action. Therefore, General Relativity is contained in the LL theory as a particular case.

The linearized approximation of the LL and EH actions around a flat, torsionless background are classically equivalent [19]. However, beyond the perturbation theory the presence of higher powers of curvature in the Lagrangian make both theories radically different. In particular, black holes and big-bang solutions of (2) have different asymptotic behaviors from their EH counterparts. Hence, a generic solution of the LL action cannot be approximated by a solution of Einstein’s theory.

A. Static and Spherically Symmetric Solutions

Consider static and spherically symmetric solutions of equations (2) and (3). In Schwarzschild-like coordinates, the metric can be written as

\[
ds^2 = -N^2(r) f^2(r) dt^2 + \frac{dr^2}{f^2(r)} + r^2 d\Omega_{d-2}^2.
\]

(4)

\[\text{\(2\)}\] Wedge product between forms is understood throughout.
Replacing this Ansatz in the field equations (2) and (3) leads to the following equations for \( N(r) \) and \( f^2(r) \):

\[
\frac{dN}{dr} = 0, \quad (5)
\]

\[
\frac{d}{dr} \left( r^{d-1} \sum_p (d-2p)\alpha_p \left( \frac{1-f^2}{r^2} \right)^p \right) = 0. \quad (6)
\]

Integrating equations (5) and (6) yields

\[
N = N_\infty, \quad (7)
\]

\[
\sum_p (d-2p)\alpha_p \left( \frac{1-f^2}{r^2} \right)^p = \frac{1}{\kappa (d-2)!\Omega_{d-2}} \frac{M+C_0}{r^{d-1}}, \quad (8)
\]

where the constant of integration \( N_\infty \) relates the coordinate time to the proper time of an observer at spatial infinity. We will assume it equal to one. The constant \( M \) stands for the mass up to an additive constant \( C_0 \), which is nonzero only in the case of CS theories.

Equations (8) corresponds to the solution of field equations, which is a polynomial in \( f^2(r) \), so many roots for \( f^2(r) \) with the same mass will exist, but, with different asymptotical behavior. This means that (2) possesses, in general, several solutions with a constant curvature in the asymptotical region, making the value of the cosmological constant ambiguous. In fact, the cosmological constant could change in different regions of the same spatial section, or it could jump arbitrarily as the system evolves in time.

These problems are overcome by demanding that the theory have a unique cosmological constant. In order to satisfy this condition, we choose the coefficients \( \alpha_p \)'s as follows:

\[
\alpha_p := c_p^k = \begin{cases} \binom{d(p-k)}{p} \left( \frac{k}{p} \right), & p \leq k, \\ 0, & p > k \end{cases}, \quad (9)
\]

and

\[
\kappa = \frac{1}{2(d-2)!\Omega_{d-2}G_k}. \quad (10)
\]

With this choice, \( f^2(r) \) adopts the following form:

\[
f^2(r) = \frac{r^2}{l^2} + 1 - \chi \left( \frac{2G_kM + \delta_{d-2k-1}}{r^{d-2k-1}} \right)^{1/k}, \quad (11)
\]

where \( \chi = (\pm 1)^{k+1} \). For even \( k \), the ambiguity of sign expressed through \( \chi \) in (11) implies that there are two possible solutions, provided \( M > 0 \). The solution with \( \chi = 1 \) describes a black hole with an events horizon surrounding the singularity at the origin. The solution with \( \chi = -1 \) has a naked singularity with positive mass. If \( k \) is odd, there is no ambiguity of sign because \( \chi = 1 \). Therefore this solution corresponds to a black hole with positive mass.

From eq. (11), it is observed that for \( k = 1 \), the AdS black hole solution for EH in \( d \)-dimensional is recovered. The black hole solutions corresponding to BI and CS theories are obtained also from expression (11), setting \( k = \left[ \frac{d-1}{2} \right] \).

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3 In the first order formalism, the field equations imply that the torsion vanishes, except for BI and CS theories, so that, it is not necessarily to set \( T^a = 0 \). However, for static and spherically symmetric configurations the equation (6) implies that the torsion must vanish in these cases as well.
III. COLLAPSE OF THIN SHELLS

Let \( \Sigma_\xi \) be a time-like hypersurface, which represents the evolution of a thin shell \cite{11,12}. This hypersurface divides the spacetime into two regions; the interior denoted by \( V^(-) \) and the exterior denoted by \( V^+ \), respectively. Each of these regions contains \( \Sigma_\xi \) as a part of its boundary. We introduce into \( \Sigma_\xi \) a set of intrinsic coordinates \( \rho^a \), where the Latin indices go from 0 to \( d - 2 \), and in the regions \( V^(-) \) and \( V^+ \), the independent coordinates \( x^a_\xi \) and \( x^\rho_\xi \) are introduced, so that the parametric equations for \( \Sigma_\xi \) in these charts are \( x^a_\xi (\rho^a) \) and \( x^\rho_\xi (\rho^a) \), respectively.

At each point on \( \Sigma_\xi \) there exists a unit space-like vector \( \xi^\mu \), normal to \( \Sigma_\xi \) and pointing from \( V^(-) \) to \( V^+ \), and \( d - 1 \) vectors \( e^a_\xi = \partial x^{a\xi} / \partial \rho^a \) tangential to \( \Sigma_\xi \) in the directions of the coordinates \( \rho^a \).

The time-like hypersurface \( \Sigma_\xi \) represents the evolution of an infinitesimal \( d - 2 \)-dimensional matter thin shell. There is no matter outside the shell. Therefore, the matter moves only on the shell, so that its \( d \)-velocity \( u^a \) is normal to \( \xi^\lambda \) and vanishes outside \( \Sigma_\xi \). Moreover, an observer on the shell can refer the movement of matter to the reference points \( (\rho^1, \ldots, \rho^{d-2}) \) and the reference time \( \rho^0 = \tau \), and thus, the velocity is described by an intrinsic vector \( u^a \). The vectors \( u^a \) and \( u^\alpha \) are joined by the relation \( u^a = e^a_\alpha u^\alpha \).

The mechanical properties of matters are described by the surface energy-momentum tensor \( T_{\mu\nu} \), which is normal to \( \xi^\lambda \) and it vanishes outside \( \Sigma_\xi \). For an observer on of \( \Sigma_\xi \), the tensor \( T_{\mu\nu} \) is described by the intrinsic coordinates. For an ideal fluid, the intrinsic energy-momentum tensor has this form:

\[
T_{ab} = \sigma u_a u_b - \sigma (h_{ab} + u_a u_b),
\]

where \( \sigma \) means the rest surface mass density of the shell and \( \sigma \) the surface tension. Since, the tensor \( T_{ab} \) is confined into the hypersurface \( \Sigma_\xi \), it satisfies the continuity equation \( T_{a/b} = 0 \). Multiplying this tensor by \( u^a \), we obtain the following relation

\[
(\sigma u^a)_{/a} - \sigma (u^a)_{/a} = 0.
\]

This equation can be seen as the equation of state of the matter on the hypersurface \( \Sigma_\xi \).

The next step is to introduce a timelike ADM foliation, \( \Sigma_t \), of the spacetime. The foliation intersects the world tube of the collapsing matter, which corresponds to the thin shell of the \( \Sigma_t \) at the time \( t \). As usual, the metric tensor is decomposed by using the basis \( N^\mu = (N^t, N^i) \), where \( N^t \) represents the lapse and \( N^i \) the shift function. In this basis the line element in the coordinates \( x^a \) of the regions \( V^(-) \) and \( V^+ \) takes the form

\[
ds^2 = -(N^t)^2 dt^2 + g_{ij}(N^i dt + dx^j)(N^j dt + dx^j).
\]

In the presence of matter, and since \( N^\mu \) are Lagrange Multipliers, the total Hamiltonian becomes

\[
\mathcal{H} = N^t \mathcal{H}_\perp + N^i \mathcal{H}_i,
\]

where \( \mathcal{H}_\perp \) and \( \mathcal{H}_i \) are defined by

\[
\mathcal{H}_\perp = \mathcal{H}_{\perp}^{(g)} + \mathcal{H}_{\perp}^{(m)},
\]

and

\[
\mathcal{H}_i = \mathcal{H}_i^{(g)} + \mathcal{H}_i^{(m)},
\]

respectively. Here \( \mathcal{H}_{\perp}^{(g)} \) and \( \mathcal{H}_i^{(g)} \) correspond to the Hamiltonian terms related to the gravitational field of the LL action \cite{21}, and are given by

\[
\mathcal{H}_{\perp}^{(g)} = -\kappa \sqrt{g} \sum_p (d - 2p)! (\omega_p \gamma_{i_1 j_1} \ldots \gamma_{i_p j_p} R_{i_1 j_1} \ldots R_{i_p j_p} R_{i_2 j_2} \ldots R_{i_{p-1} j_{p-1}} R_{i_{p+1} j_{p+1}} \ldots R_{i_{2p-1} j_{2p-1}} R_{i_{2p} j_{2p}}),
\]

\footnote{Here \( \hat{\tau} \) denotes the surface tension and \( \tau \), the proper time.}
\[ \mathcal{H}_i^{(g)} = -2\pi^j_{i/j}, \]

where \( \pi^{ij} \) are the conjugate momenta to the metric tensor of the intrinsic tensor metric \( g_{ij}; \) \( g \), its determinant; and \( R_{ijkl} \) corresponds to the tensor curvature, which can be written in terms of the geometric quantities of \( \Sigma \) as

\[ R_{ijkl} = \hat{R}_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk}, \]

(20)

with \( \hat{R}_{ijkl} \) stands for the curvature tensor of the leaf \( \Sigma_t \) of the foliation and \( K_{ij} \) is the extrinsic curvature.

The momenta are defined in terms of extrinsic curvature \( K_{ij} = \frac{1}{2} N^\perp (N^i_j + N^j_i - \dot{g}_{ij}) \) as

\[ \pi^j_i = -\kappa \sqrt{g} \sum_p \frac{p!(d - 2p)!\alpha_p}{2^{p+1}} \sum_{s=0}^{p-1} D_{s(p)} \delta^{j i_1 \ldots i_{2p-1}_{i_{2s}} \ldots i_{2p}} R_{i_1 j_1 \ldots j_{2s}} \ldots R_{i_{2s-1} j_{2s}} K_{i_{2s+1}}^j \ldots K_{i_{2p-1}}^{j_{2s-1}}, \]

(21)

where

\[ D_{s(p)} = \frac{(-4)^{p-s}}{s!!(2(p-s) - 1)!!}. \]

The matter components \( \mathcal{H}_i^{(m)} \) and \( \mathcal{H}_i^{(m)} \), are given by

\[ \mathcal{H}_i^{(m)} = \sqrt{g} I_{\perp}, \]

(22)

\[ \mathcal{H}_i^{(m)} = 2\sqrt{g} I_{\perp}, \]

(23)

where \( \perp \) corresponds to a contraction with a normal vector to the hypersurface \( \Sigma_t \), \( n_\mu = (-N, 0, \ldots, 0) \).

In what follows we restricts ourselves to the spherical coordinates. We will use the proper time \( \tau \) and spherical angles as intrinsic coordinates of the hypersurface \( \Sigma_\xi \): \( \rho^a = (\tau, \theta^1, \ldots, \theta^{d-2}) \). The motion of the shell is expressed by the equation \( r = R(\tau) \). The derivative with respect to \( \tau \) is denoted by a dot. The line element of \( \Sigma_\xi \) in this coordinates is expressed by

\[ ds^2_\perp = -d\tau^2 + R^2(\tau) d\Omega_{d-2}^2. \]

(24)

The interior and exterior line element with spherical symmetry are given by

\[ ds^2_- = -f^2_-(r) dt^2_- + f^{-2}_-(r) dr^2 + r^2 d\Omega_{d-2}^2, \quad r < R(\tau), \]

(25)

and

\[ ds^2_+ = -f^2_+(r) dt^2_+ + f^{-2}_+(r) dr^2 + r^2 d\Omega_{d-2}^2, \quad r > R(\tau). \]

(26)

Interior and exterior coordinates match continuously on the \( \Sigma_\xi \), but it is found that \( t_- \neq t_+ \). In these coordinates the vectors \( u^\alpha \) and \( \xi^\alpha \) are given by

\[ u^\alpha = \left( \frac{\gamma}{f^2}, \dot{R}, 0, \ldots, 0 \right), \]

(27)

\[ \xi^\alpha = \left( \frac{\dot{R}}{f^2}, \gamma, 0, \ldots, 0 \right), \]

(28)
Integrating in the radial direction, from $R$ to $r$, we have
\[ \gamma = \sqrt{f^2 + R^2}. \] (29)

Substituting (25) and (26) into the Hamiltonian generator $\mathcal{H}_\perp$, we obtain
\[ \mathcal{H}_\perp = -\frac{\kappa(d-2)!}{r^{d-2}} \sqrt{g} \frac{d}{dr} \left\{ r^{d-1} \sum_p (d-2p)\alpha_p \left( \frac{1 - f^2}{r^2} \right)^p \right\} + \sqrt{g} T_{\perp \perp}. \] (30)

We are interested in integrating out the constraint $\mathcal{H}_\perp = 0$ across a radial infinitesimal length centered in the shell position $r = R(\tau)$ to a constant time. In this form, it is possible to express the discontinuities of geometry in terms of the projected stress $T_{\perp \perp}$. It is easy to prove that $T_{\perp \perp}$ has the same form that the one obtained in [17], due to the symmetry of the thin shell. Finally $T_{\perp \perp}$ is given by
\[ T_{\perp \perp} = \sigma \gamma \delta (r - R(\tau)). \] (31)

Integrating in the radial direction, from $R + \epsilon$ and $R - \epsilon$, we obtain in the limit $\epsilon \to 0$
\[ \kappa(d-2)! R \sum_p (d-2p)\alpha_p \left\{ \left( \frac{1 - f^2(R)}{R^2} \right)^p - \left( \frac{1 - f^2(R)}{R^2} \right) \right\} = \frac{1}{2} \sigma (\gamma_+ + \gamma_-). \] (32)

This expression has been seen as the generalization of the equation obtained for GR.

From expression (3), it is found that
\[ m_+ - m_- = \frac{1}{2} m (\gamma_+ + \gamma_-), \] (33)

where $m = \Omega_{d-2} R^{d-2} \sigma$. Expression (33) is the same to that obtained from the GR case [11]. If $\sigma > 0$ then $M_+ > M_-$, this means that if $M_-$ is the mass of a black hole inside of the shell, the final mass of the black hole will be greater, therefore its event horizon increases.

In order to complete the present picture of a radial collapse, it is necessary to analyze the consistency of the remaining nonvanishing components of the Hamiltonian treatment related to the radial and angular components. The angular contribution of the constraint (17) are identical to zero. Because the radial component is not identically zero, it is necessary to evaluate
\[ \mathcal{H}_r = -2\pi_\perp^J + 2\sqrt{g} T_{\perp r}, \] (34)

which yields
\[ \mathcal{H}_r = \kappa \sqrt{\Gamma} (d-2)! \sum_{p=0}^{k} \frac{p! (d-2p)! \alpha_p}{2^{p+1} (d-2p-1)!} \sum_{s=0}^{p-1} 2^s D_s(p) f^{2(s-p)} \left( 1 - f^2 \right)^s \frac{d}{dr} \left( r^{d-2p-1} (\alpha \hat{r})^{2p-2s-1} \right) - \frac{R^{d-2} \sqrt{\Gamma} \hat{R}}{f^2} \sigma \delta (r - R(\tau)), \] (35)

where $\Gamma$ corresponds to the determinant of the angular metric. One can expect that $\mathcal{H}_r$ to be proportional to $\mathcal{H}_\perp$, since (16) already provides the equation of motion for $R(\tau)$. It would be interesting to explicitly see that this indeed occurs. But due to equation (35), in the general case it is complicated (it is not possible to integrate) to perform this point. However, it is straightforward to prove that the correct Einstein-Hilbert limit is obtained when $k = 1$ [17].

The acceleration of the thin shell is obtained from equation (35) by differentiating with respect to proper time $\tau$, which yields
\[ m \ddot{R} = -\frac{m^2}{2 (M_+ - M_-)} \left( \gamma_+ \frac{df^2}{dR} + \gamma_- \frac{df_+^2}{dR} \right) - (d - 2) \Omega_{d-2} R^{d-3} \hat{R} \gamma_+ \gamma_-, \] (36)

Notice that we need the explicit forms of $f^2$ and $f_+^2$. The form of $f^2$ will be
The form of $f^2$ must be split into two cases; the case $d-2k-1 \neq 0$ and the case $d-2k-1 = 0$. In the latter case there exists a gap in the mass, in which the vacuum corresponds to $M_- = -(2G_k)^{-1}$.

- If $d-2k-1 \neq 0$, with $M_- = 0$ and $M_+ = M$, then equation (33) takes the form $M = \frac{1}{2} m (\gamma_+ + \gamma_-)$, so that if $\sigma > 0$, then $M > 0$. The acceleration is given by

$$m \ddot{R} = -\frac{m}{l^2} R - (d-2) \Omega_{d-2} R^{d-3} \hat{r} \gamma_+ \gamma_- - \chi (d-2k-1) m^2 \left( \frac{2G_k}{M^{k-1} R^{d-k-1}} \right)^{1/k} \gamma_-$$

(38)

The first two terms of (38) correspond to the acceleration due to AdS, and the interaction of tangent tension on the thin shell. If $\chi = 1$, for a black hole solution, then $\ddot{R} < 0$. Therefore, in this way the thin shell always collapses to a black hole.

On the other hand, when $\chi = -1$ we might think that a naked singularity, however due to that the latter term a Eq. (33) is positive and thus a repulsive gravitational force appears. It could be shown that this force therefore dominates over the other terms when $R \to 0$. Therefore, naked singularity cannot be formed through a thin shell collapse. In order to see this point we consider

$$M = \frac{m}{2} (\gamma_+ + \gamma_-),$$

(39)

where

$$\gamma_\pm = \sqrt{\dot{R}^2 + f^2_\pm},$$

(40)

and $m = \Omega_{d-2} R^{d-2} \sigma$ with $M > 0$. In this case $f^2_-$ and $f^2_+$ are given by

$$f^2_- = 1 + \frac{R^2}{l^2},$$

(41)

and

$$f^2_+ = 1 + \frac{R^2}{l^2} + \left( \frac{2G_k M}{R^{d-2k-1}} \right)^{1/k},$$

(42)

respectively. For a naked singularity it is necessary that $f^2_+ > f^2_- > 0$.

From (39) we obtain $\ddot{R}^2$,

$$\ddot{R}^2 = \left[ \frac{M}{m} - m \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-2k-1}} \right)^{1/k} \right]^2 - \left( \frac{R^2}{l^2} + 1 \right),$$

(43)

from which we could read an effective potential.

In order to give either a quantitative and qualitative discussion of this potential, let us write Eq. (43) in the form

$$\ddot{R}^2 = \alpha_+ \alpha_-$$

(44)

where

$$\alpha_\pm = \frac{M}{m} - m \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-2k-1}} \right)^{1/k} \pm \sqrt{\frac{R^2}{l^2} + 1}.$$

(45)
FIG. 1: This plot shows behavior of our effective potential and their turning point.

The physical regions are defined by $\dot{R}^2 > 0$, which implies that both $\alpha_+$ and $\alpha_-$ have the same sign. Note that for the case under study we have $\gamma_+ > 1$, due to $f^2_+ > 1$, this means that the sum $\gamma_+ + \gamma_- > 2$. Therefore, from Eq. (39) we obtain $M > m$. In order to characterize the physical regions, we need the behaviors of the $\alpha_\pm$ parameters. We will simplify our study to the dust case which means $m = \text{Constant}$. Our results are shown in Fig. 1 from which we could see different regions,

I. $\dot{R}^2 > 0$, for $R < R_1$,
II. $\dot{R}^2 < 0$, for $R_1 < R < R_2$,
III. $\dot{R}^2 > 0$, for $R_2 < R < R_3$,
IV. $\dot{R}^2 < 0$, for $R_3 < R$.

Note that the classical allowed regions are I and III. In region I note that when $R \to 0 \dot{R}^2 \to \infty$. In this case, a naked singularity emerge from a collapse of a thin shell. However we could prove that the motion in this region is prohibited by causality. In fact, from Eq. (52) (see appendix), we obtain

$$\gamma_- = \frac{M}{m} - m \left( \frac{2^{1-2k}G_k M^{1-k}}{R^{d-2k-1}} \right)^{1/k}$$

(46)

and the condition $\gamma_- > 0$, implies

$$\frac{M}{m} > m \left( \frac{2^{1-2k}G_k M^{1-k}}{R^{d-2k-1}} \right)^{1/k}.$$  

(47)

This mean that it must exist a minimum radius, which we denote by $R = R_*$ in order not to violate the conservation law expressed by Eq. (39). It is direct to prove that $R_1 < R_* < R_2$, which implies that $\alpha_+ > 0$. 


Therefore, the thin shell "movement" is allowed only in region III, this is \( R_2 < R < R_3 \). So, naked singularity, in the case of dust, cannot be formed through a thin shell collapse.

For the general case, is not simple to study equation (43), since it requires to solve equation (57) (see appendix). Also we should know the relation between \( \sigma \) and \( \hat{\tau} \) (the state equation for the shell), in order to obtain \( m = m(R) \). However, we may argue that equation (46) is of general character, so also it applies for the general case. The a minimum radius must exist, so that the thin shell does not violate equation (46). Therefore, it must exist a turning point. So, naked singularity, in the general case, cannot be formed through a thin shell collapse.

It is easy to see that in the limit \( R \to 0 \) the first term vanishes. If we consider the particular case of dust, i.e. \( \hat{\tau} = 0 \), eq. (57) implies that \( m = \text{Const.} \), therefore the last term goes to infinity for \( R \to 0 \). Thus, for the dust case the thin shell does not collapse to \( R = 0 \), because the acceleration becomes very strong.

At this point, if we considered Einstein-Hilbert limit (\( k = 1 \)) in equation (38), the last term would be reduced to \( -(d - 3)Gm^2/R^{d-2} \), thus corresponding to the Newton gravitational interaction. Therefore, for \( k \neq 1 \) this term will be a generalization of the Newton gravitational interaction, with an effective gravitational constant given by

\[
-\frac{1}{k} \left( \frac{2Gk}{M^{k-1}} \right)^{1/k} \tag{48}
\]

- If \( d - 2k - 1 = 0 \), it corresponds to the CS theory with \( M_- = -(2Gk)^{-1} \). In this case, expression (50) takes the form \( M + (2Gk)^{-1} = \frac{1}{2} m(\gamma_+ + \gamma_-) \). If \( \sigma > 0 \), then it is implied that \( M > -(2Gk)^{-1} \); therefore, the naked singularities with negative mass can emerge from the collapse of a thin shell. Here, acceleration is given by

\[
m\ddot{R} = \frac{m}{l^2} R - (d - 2) \Omega_{d-2} R^{d-3} \hat{\tau} \gamma_+ \gamma_- , \tag{49}
\]

where \( \ddot{R} < 0 \), and thus the thin shell always collapses. It is observed from eq. (49) that a term analogue to the Newton gravitational interaction does not appear.

IV. CONCLUSION AND REMARKS

We have developed the Hamiltonian formalism for the collapse of thin shells in Lanczos-Lovelock theories, and we presented given a concise analysis of the theories described in Ref. [1]. We show in these theories that the black holes solution can be created by collapsing a thin shell and naked singularities cannot be formed by this mechanism. On the other hand, if we consider theories with \( k \neq 1 \), these exhibit a generalization of the Newton gravitational interaction, and effective gravitational constant becomes given by

\[
-\frac{1}{k} \left( \frac{2Gk}{M^{k-1}} \right)^{1/k} \tag{50}
\]

Also we have shown that when we take the Einstein-Hilbert limit (\( k = 1 \)) in equation (38), the last term is reduced to \( -(d - 3)Gm^2/R^{d-2} \), which coincides with the Newton gravitational interaction.

Nevertheless among the different Lanczos-Lovelock’s theories, the Chern-Simons theory exhibits an exceptional behavior, since naked singularities can emerge from the collapsing of a thin shell. This kind of theory does not possess a gravitational self-interaction analogous to the Newtonian case.

It is straightforward to prove that in the case of electrically charged thin shells, the general form of eq. (32), governing the radial collapse in \( d \) dimensions, remains the same because the electromagnetic stress tensor contributes with a finite jump value on the \( \mathcal{H}_\perp \) and, therefore, does not contribute to the radial integral of \( \mathcal{H}_\perp \). Moreover, when we consider the charged case in CS theories, a mechanism that prevents the naked singularities formation appears.

Finally, we conjecture, by virtue of the results from Ref. [16] that the presence of an angular moment in \( 2 + 1 \) dimensions prevents the formation of naked singularities; thus, in higher dimensions, the angular moment could prevent naked singularities formations in CS theories.
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VI. APPENDIX

From Eq. (43), i.e.

\[ \dot{R}^2 = \left( \frac{M}{m} - m \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-2k-1}} \right)^{1/k} \right)^2 - \left( \frac{R^2}{l^2} + 1 \right), \]

we will obtain the acceleration of the thin shell given by Eq. (38).

Let star from

\[ \gamma_{\pm} = \frac{M}{m} \pm \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-2k-1}} \right)^{1/k} m, \]

and

\[ (\sigma u^a)_{/a} - \hat{\tau} u^a = 0. \]

This equation can be seen as the equation of state of the matter on the hypersurface \( \Sigma_{\xi} \). Using spherical coordinates and the identity \( T^a_{/u} = \partial_a (\sqrt{-h} T^u) / \sqrt{-h} \), where \( h \) is the proper metric determinant, we obtain

\[ \partial_a (\sqrt{-h} \sigma u^a) = \hat{\tau} \partial_a (\sqrt{-h} u^a). \]

The proper metric is given by

\[ ds^2 = -d\tau^2 + R^2 (\tau) d\Omega_{d-2}^2, \]

with \( \sqrt{-h} = R^{d-2} \sqrt{\Gamma} \), where \( \Gamma \) is the angular metric determinant. Besides we consider radial collapse then \( u^a = (1, 0, ..., 0) \), moreover consider proper coordinates given by \( \rho^a = (\tau, \theta^1, ..., \theta^{d-2}) \), then we get

\[ \frac{d}{d\tau} \left( R^{d-2} \sqrt{\Gamma} \sigma \right) = (d - 2) R^{d-3} \hat{\tau} \sqrt{\Gamma} \dot{R}, \]

where \( \dot{R} = dR/d\tau \) and since \( d\sqrt{\Gamma}/d\tau = 0 \) we have

\[ \frac{d}{d\tau} (R^{d-2} \sigma) = (d - 2) R^{d-3} \hat{\tau} \dot{R}. \]

Multiplying this latter equation by the angular volume of the unit sphere, \( \Omega_{d-2} \) (for \( d = 4 \) \( \Omega_2 = 4\pi \)) and defining \( m = \Omega_{d-2} R^{d-2} \sigma \), we obtain

\[ \frac{dm}{d\tau} = (d - 2) R^{d-3} \hat{\tau} \dot{R}. \]

Due to, symmetry we have \( m = m(R) \) and \( d/dt = \dot{R} d/dR \), then

\[ \frac{dm}{dR} = (d - 2) R^{d-3} \hat{\tau} \dot{R}. \]
From which we get
\[ 2\dot{R}R = -\frac{2}{l^2}R\ddot{R} \]
\[ + 2\left[ \frac{m}{m} - m \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-2k-1}} \right)^{1/k} \right] \]
\[ - \frac{M dm}{m^2 d\tau} \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-2k-1}} \right)^{1/k} \frac{dm}{d\tau} + \frac{(d - 2k - 1)}{k} \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-k-1}} \right)^{1/k} \dot{R} \].
\[ \text{(60)} \]

For the \((-\)) sign, we obtain the following
\[ 2\dot{R}R = -\frac{2}{l^2}R\ddot{R} \]
\[ + 2\gamma_- \left[ \frac{m}{m} - \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-2k-1}} \right)^{1/k} \frac{dm}{d\tau} + \frac{(d - 2k - 1)}{k} \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-k-1}} \right)^{1/k} \dot{R} \right] \].
\[ \text{(61)} \]

we could rewritten this equation in the form
\[ \ddot{R} + \frac{\gamma_-}{m} \left[ \frac{m}{m} + \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-2k-1}} \right)^{1/k} \frac{dm}{d\tau} + \frac{(d - 2k - 1)}{k} \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-k-1}} \right)^{1/k} \dot{R} \right] = \frac{\gamma_-}{m} m\dot{R} \gamma_- \].
\[ \text{(62)} \]

For \((+\)) sign, we obtain
\[ \ddot{R}R = -\frac{1}{l^2}R\ddot{R} - \gamma_- \gamma_+ \frac{dm}{m} \frac{d\tau}{\tau} \]
\[ + \frac{(d - 2k - 1)}{k} \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-k-1}} \right)^{1/k} m\dot{R} \gamma_- \]
\[ \text{(63)} \]

but \(d\frac{dm}{d\tau} = \dot{R}d\frac{dR}{d\tau}\) and using Eq. \[\text{(59)}\]
\[ \ddot{R} = -\frac{1}{l^2}R - (d - 2)R^{d-3} \gamma_+ \gamma_- \frac{m}{m} \]
\[ + \frac{(d - 2k - 1)}{k} \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-k-1}} \right)^{1/k} m\gamma_- \]
\[ \text{(64)} \]

multiplying by \(m\), we obtain finally
\[ m\ddot{R} = -\frac{m}{l^2}R - (d - 2)R^{d-3} \gamma_+ \gamma_- \frac{m}{m} \]
\[ + \frac{(d - 2k - 1)}{k} \left( \frac{2^{1-2k} G_k M^{1-k}}{R^{d-k-1}} \right)^{1/k} m^2 \gamma_- \]
\[ \text{(65)} \]

which corresponds to Eq. \[\text{(58)}\] in the main text.

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