Averaging Inhomogeneous Cosmologies
– a Dialogue

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Abstract.
The averaging problem for inhomogeneous cosmologies is discussed in the form of a disputation between two cosmologists, one of them (RED) advocating the standard model, the other (GREEN) advancing some arguments against it. Technical explanations of these arguments as well as the conclusions of this debate are given by BLUE.

1. The conjecture about the average flow

The standard model which, on some large scale, is defined as a homogeneous and isotropic solution of Einstein’s equations for gravitationally interacting matter, has proved to be remarkably robust against various observational challenges especially of the recent past. It is this robustness together with a list of theoretical and observational arguments which makes it hard to see any need for an alternative\(^2\) to the standard model. Nevertheless, there exist some simple arguments which let the standard model appear dogmatic and a replacement overdue, while most scientific activity in the field is directed towards a consolidation of the standard model. It is fair to say that most of the work, which is directed towards consolidation, is already based implicitly on the assumption that the standard model gives the correct picture.

I here sketch a possible dialogue which we can watch without risk of being biassed by some prejudice: we have two people who try to defend

\(^1\)Proc. 2nd SFB Workshop on Astro–particle physics, Ringberg 1996, Proceedings Series SFB 375/P002 (1997), R. Bender, T. Buchert, P. Schneider and F.v. Feilitzsch (eds.).

\(^2\)‘alternative’ is not meant in the sense of invoking physical laws other than general relativity and making generalizations which depart from the standard kinematical properties of an, on average, homogeneous–isotropic universe. Rather we think at improving on the standard model in its presently employed form. Compare also the discussion in (Ellis et al. 1997).
their points of view and both of them might be biassed, but both advance arguments which can be proved or disproved. This dialogue is mirrored in an ongoing debate in the field of astrostatistics about the existence of evidence for a scale of homogeneity (e.g., Davis 1996 and Pietronero 1996), a subject which is also dealt with in several contributions to this volume (see: Kerscher et al., Martínez, Sylos Labini et al.) and was the subject of a panel discussion held during the meeting.

Let us start with the advocate of the standard model RED: “The (large–scale) standard model is a solution of Friedmann’s equations:

$$3 \left(\frac{\dot{a}}{a}\right)^2 - 8\pi G \frac{M}{a^3} + \frac{k}{2a^2} - \Lambda = 0 ; \quad \rho_H a^3 =: M = \text{const.},$$

where $\Lambda$ is the cosmological constant, $k$ is related to the constant curvature of the model at time $t_0$, $\rho_H(t)$ the value of the homogeneous density at time $t$, and $a(t)$ is the scale function of the isotropically expanding (or contracting) universe.” Also: “This model is unstable to perturbations in the density field and/or the velocity field, respectively”, which is the well–known content of gravitational instability; we may call this property local gravitational instability. In spite of this instability, RED supports the following conjecture about the global properties of the Universe (a statement which will come again in various refined versions later on):

**Conjecture (Version 1):** “The Universe can be approximated by the standard model, if averaged on some large scale $L$, e.g. for the inhomogeneous density field we would have: $\langle \rho(\mathbf{x}, t) \rangle_L(t) = \rho_H(t)$ for all times.”

Here, we may think for simplicity that the brackets $\langle \mathcal{A} \rangle$ denote Euclidean spatial averages of some tensor field as a function of Eulerian coordinates and time, $\mathcal{A}(\mathbf{x}, t)$. The Newtonian case serves as a good illustration. Below, we shall explain that the arguments carry over to Riemannian spaces and general relativity.

GREEN replies: “I don’t expect that spatial averaging and time evolution commute as a result of the nonlinearity of the basic system of equations.”

BLUE explains: “If we average the cosmological fields of density and velocity at some initial time $t_0$ (at recombination\(^4\)) and use these average values (which are remarkably isotropic according to the microwave background measurements) as initial data of a homogeneous–isotropic solution, then, e.g., the value of $\rho_H(t)$ at time $t$ is expected to differ from the average field $\langle \rho \rangle_L(t)$ of the inhomogeneous initial data evolved to the time $t$.

\(^3\)For convenience we restrict the discussion to the matter dominated era.

\(^4\)This does, however, not imply that the ‘backreaction’ discussed hereafter will be unimportant at earlier times; it may even be relevant in the early universe as pointed out by Mukhanov et al. (1997).
This has been particularly emphasized by Ellis (1984).” Indeed, GREEN is right: the explanation of non–commutativity is given by BLUE in terms of the 
\textit{commutation rule} for the expansion scalar defined as the divergence of the velocity field \( \theta = \nabla \cdot \vec{v} \) (Buchert, Ehlers 1997):

\[
\frac{d}{dt} \langle \theta \rangle_D - \langle \frac{d}{dt} \theta \rangle_D = \langle \theta^2 \rangle_D - \langle \theta \rangle_D^2 .
\] (2)

“Equation (2) shows that, on any spatial domain \( D \), the evolution of the average quantity differs from the averaged evolved one, the difference being given by a fluctuation term. For details and discussions of what follows see (Buchert 1996 and Buchert & Ehlers 1997); for the relation to dynamical models see (Ehlers & Buchert 1997).”

Also: “Equation (2) only assumes mass conservation, i.e., we follow a tube of trajectories so that the mass in the spatial averaging domain \( D \) is conserved in time. This is a sensible assumption, since we want to extend the spatial domain to the whole universe later on.”

2. A generalized expansion law

BLUE goes further by specifying the local dynamical law for the expansion scalar. This is furnished by Raychaudhuri’s equation: “Introducing a scale factor via the volume \( V = |D| \), \( a_D(t) := V^{1/3} \) (so, we do not care about the shape of the spatial domain \( D \); it may expand anisotropically), Raychaudhuri’s equation for \( \dot{\theta} \) may be inserted into the commutation rule (2) resulting in the \textit{generalized Friedmann equations}:

\[
3 \frac{\ddot{a}_D}{a_D} + 4 \pi G \langle \rho \rangle_D - \Lambda = 0 , \quad \frac{d}{dt} \langle \rho \rangle_D + 3 \frac{\dot{a}_D}{a_D} \langle \rho \rangle_D = 0 ,
\] (3a, b)

with the \textit{effective} source term which involves averages over fluctuation terms of the expansion, the shear scalar \( \sigma \) and the vorticity scalar \( \omega \):

\[
4 \pi G \rho_{\text{eff}} := 4 \pi G \langle \rho \rangle_D - \langle Q \rangle_D ,
\] (3c)

\[
\langle Q \rangle_D = \frac{2}{3} \langle (\theta - \langle \theta \rangle_D)^2 \rangle_D - 2 \langle \sigma^2 - \omega^2 \rangle_D .
\] (3d)

Thus, as soon as inhomogeneities are present, they are sources of the equation governing the average expansion. They may be negative or positive giving rise to an additional effective (dynamical) density which we may measure by the dimensionless ratio

\[
B_D := \frac{\langle Q \rangle_D}{4 \pi G \langle \rho \rangle_D} .
\] (4)

For \( B_D = 1 \), the source due to ‘backreaction’ is, on the averaging domain, equal to that of the averaged matter density. The effective density does, in
general, not obey a continuity equation like the matter density; an effective mass is either produced or destroyed in the course of structure formation.”

RED: “In principle you are right, but I doubt that the effect is quantitatively significant.”

GREEN: “Irrespective of the global relevance of this term, it will play an interesting role on scales where its value is non-negligible; for dominating shear fluctuations the ‘backreaction’ could fake a ‘dark matter’ component, since the mass in the standard Friedmann equation will be overestimated in this case.”

3. Inhomogeneous Newtonian cosmologies which are Friedmannian on average

RED is going to advance a strong argument in favour of the standard model: “In Newtonian theory the ‘backreaction’ term (4) vanishes by averaging over the whole Universe, if the latter is topologically closed, i.e., compact and without boundary.”

Indeed, he succeeds in writing the local term $Q$ as a divergence of some vector field $\vec{\Psi}$, $Q = \nabla \cdot \vec{\Psi}$, if he assumes space to be Euclidean. “Hence, using Gauß’s theorem, we may transform the volume integral in the average $\langle Q \rangle_D$ into a surface integral over the boundary $\partial D$ of the averaging domain. For compact universes without boundary (e.g., a 3-torus $T$) this surface integral is zero; we obtain $\langle Q \rangle_T = 0$ on the torus.”

BLUE: “As a side-result RED proved that the currently employed models for large-scale structure, analytical or N-body simulations, are constructed correctly: the assumption of periodic boundary conditions is equivalent to using as the 3-space a hypertorus, not $\mathbb{R}^3$. These models so far have been assumed to be Friedmannian on average by construction rather than derivation.”

4. Generalized expansion law in general relativity

GREEN adds a disclaimer: “The above argument depends on the flatness of space.”

RED: “But, as was shown in (Buchert & Ehlers 1997) the generalized Friedmann equations (3) also hold for irrotational flows in general relativity.”

GREEN: “Yes, but not the fact that the local term $Q$ can be written as a divergence of some vector field, which makes, besides spatial curvature, a crucial difference.”

BLUE illustrates this last statement by GREEN by some technical explanations: “If one introduces normal (Gaussian) coordinates $X^i$ and thus foliates spacetime into flow-orthogonal hypersurfaces of constant time (this is only possible for irrotational flows), then Eqs. (3) also hold. Since Eqs. (3) are equations for scalar quantities, they are manifestly covariant and
will hold in any coordinate system. The crucial difference to the Newtonian treatment, however, is the non–integrability of inhomogeneous deformations (as defined below): Write the metric of the spatial hypersurfaces as a quadratic form \( g_{ij} = \eta^a_i \eta^a_j \) involving the one–forms \( \eta^a = \eta^a_i dX^i \), then it is necessary and sufficient for the metric to be flat that the one–forms are exact, i.e., \( \eta^a = df^a \), and the coefficients \( \eta^a_i \) reduce to a deformation gradient with respect to Lagrangian coordinates, \( df^a_i \equiv \partial f^a / \partial X^i \); in other words, the coefficient matrix \( \eta^a_i \) which measures the deformations is integrable. The non–integrability of \( \eta^a_i \) implies the non–existence of a vector field \( \vec{\Psi} \). Therefore, we cannot shift a volume averaged quantity to a contribution on the boundary of the averaging domain and the conclusion on the vanishing of ‘backreaction’ for models with non–Euclidean space sections cannot be drawn using RED’s argument.”

GREEN: “This remark by BLUE is bad news in so far as we expect the valid theory to be general relativity on the large scales under consideration.”

BLUE: “Again, we can formulate a side–result concerning current models of large–scale structure: If we model structure in, e.g., an undercritical–density universe (the total density parameter being \( \Omega < 1 \)), then the model has to be interpreted as a Newtonian model. It makes sense to speak about hypersurfaces of constant negative curvature for the average model, but in that case there currently exists no proof that the average model is Friedmannian for closed, curved spaceforms. Moreover, it is then not even possible to introduce a simple hypertorus topology, since this is incompatible with a hyperbolic geometry (compare, e.g., Lachièze–Rey & Luminet 1995).”

5. Enforcing closure for spatial hypersurfaces

RED summarizes the preceeding findings and reformulates his statement:

Conjecture (Version 2): “On some large scale \( L \) we still may approximate the metric of spatial hypersurfaces by the flat Euclidean metric. Then, we have two options (which both are connected with the requirement of periodic boundary conditions):

Either,

• We live in a “small universe” (Ellis 1971, Ellis & Schreiber 1986), i.e., space is genuinely compact without boundary and has a finite size \( L^3 \). This we may call topological closure condition (Option A).

Or,

• The value of \( \langle Q \rangle_L \) is numerically negligible on the scale \( L \). This would support the generally held view and we may call this technical closure condition (Option B), since then Option A is a justified approximation.”
GREEN: “I agree that a compact universe is appealing, since only then we have some hope to explore a substantial fraction of its volume; however, it may then not have flat space sections.”

6. Is there a compensation of fluctuation terms?

GREEN accepts the flatness of space as a working hypothesis in order not to complicate the discussion: “If we don’t have Option A (in which case RED’s argument is exact), we have to examine the ‘backreaction’ term $\langle Q \rangle_D$ in more detail quantitatively.”

GREEN develops his argument: “The terms in $\langle Q \rangle_D$ have positive contributions (vorticity and expansion fluctuations), $\langle Q^+ \rangle_D$, and negative ones (shear fluctuations), $\langle Q^- \rangle_D$”, and “Each individual fluctuation term has fixed sign and, thus, does not vanish on any scale”.

BLUE: “An immediate consequence of the positivity of these terms is that their values may decay with scale until we reach a representative volume of the Universe, but as soon as we have reached this, these terms approach a finite positive value, even on a scale on which we may assume periodic boundary conditions.”

GREEN concludes that “The requirement of vanishing or smallness of the sum $\langle Q^+ \rangle_D + \langle Q^- \rangle_D$ implies a conspiracy between vorticity, shear and expansion fluctuations, which is not to be expected a priori.”

The final refinement of RED’s statement therefore assumes the form:

**Conjecture (Version 3):** “On some large scale $L$ we still may approximate the metric of spatial hypersurfaces by the flat Euclidean metric. In general we may not expect that the Universe is genuinely periodic on the scale $L$. However, on that scale, the term $\langle Q \rangle_L$ has a negligible value:

Either, because:
- Each of the terms $\langle Q^+ \rangle_L$ and $\langle Q^- \rangle_L$ is numerically small, so that the conspiracy assumption does not matter.

Or, because (if the terms are not numerically small):
- The inhomogeneities evolve such that $\langle Q^+ \rangle_D + \langle Q^- \rangle_D \to 0$ for scales approaching $L$ and for all times.”

BLUE: “Both options imply assumptions on the initial fluctuation spectrum, and both formulate properties of gravitational dynamics which is, in principle, testable.”

RED: “I agree that we want, under all circumstances, avoid “fine-tuning”; the standard model should be generic for a wide range of dynamical models for the evolution of inhomogeneities.”
7. A physical model

GREEN now argues on the grounds of gravitational dynamics: “Up until now there is no dynamical model which includes the full ‘backreaction’ (apart from perturbative studies which may capture some of the effect – see Futamase 1989, 1996, Bildhauer 1990 and Bildhauer & Futamase 1991a,b, as well as Russ et al. 1997); the main problem to construct such a model is the following: not only the inhomogeneities affect the global expansion, but also any model for the evolution of inhomogeneities will depend on how the average evolves in time.”

BLUE details: “The latter is principally known from the linear theory of gravitational instability: if the universe expands faster, then the inhomogeneities have a harder time to form. Here, we are faced with a nonlinear self–interaction problem:

We may start with a flat Friedmann model as background (the average of the Ricci scalar is zero), and some model for the inhomogeneities relative to this background. From the inhomogeneous model we calculate the ‘backreaction’ (this was recently attempted by Russ et al. 1997). However, if there is a nonvanishing ‘backreaction’ on the global scale $L$, then this procedure gives us just the first step in the sense of an iteration; in the second step we would have a curved background (the average of the Ricci scalar is nonzero), and we would have to construct an inhomogeneous model for a curved background including the ‘backreaction’ from the first iteration. In turn the second iteration would yield the ‘backreaction’ for this model, and the full ‘backreaction’ could be calculated after $N$ steps of this procedure provided there is convergence to a solution.”

GREEN: “It is clear that we are far from being able to investigate such a model. For example, in a curved background we can neither use simple periodic boundary conditions, nor can we work with the standard Fourier transformation; we would have to work with eigenfunctions on curved spaces and would have to respect the compatibility with some, in general, nontrivial topologies.” One “way out” is to cheat: GREEN bases his further argumentation on the standard Newtonian model: “We may use the standard model which is mathematically well–defined as the average over a general inhomogeneous but periodic Newtonian model, and let the box of the simulation extend to very large scales. Then, the ‘backreaction’ can be calculated for subensembles of the simulation box on scales which we consider representative for the Universe.” This possible study will at least give us some quantitative clues of the effect; it is the subject of an ongoing work (Buchert et al. 1997) which BLUE is going to sketch in the next section.
8. A dynamical approach to cosmic variance

“Let us assume, in agreement with Conjecture 3 by RED, that the space sections are Euclidean and that the inhomogeneities can be subjected to periodic boundary conditions on some large scale. Usually, this scale is set to be around 300Mpc/h, mainly because of limits on CPU power using N–body simulations. However, a recent analysis of the IRAS 1.2 Jy catalogue (see Kerscher et al., this volume) has demonstrated that fluctuations in the matter distribution do not vanish on that scale. A mock catalogue of the IRAS sample produced by a simulation with a box size of 250Mpc/h enforces the fluctuations to vanish on the periodicity scale, and the corresponding analysis of the mock sample shows disagreement in all moments (except the first) of the matter distribution with the observed data (see Kerscher et al., this volume and Kerscher et al. 1997). This example shows that not only fluctuations in the average density are an indication for inhomogeneities, but averages over higher–order moments of the density field (e.g. reflected by averaged shear fluctuations\(^5\)) create huge (phase–correlated), possibly low–amplitude structures.

Thus, even if we do not argue globally about the ‘backreaction problem’ (the ‘backreaction’ is zero by construction due to the assumed Newtonian description and the periodicity), this effect has to be seriously considered on scales of current all–sky surveys. This study entails, from a dynamical point of view, a quantification of cosmic variance within the standard model.

We therefore have to run simulations of a considerably larger spatial extent; we may use for simplicity “truncated Lagrangian schemes”. These schemes have been shown to agree with N–body results down to scales around the correlation length (Melott et al. 1994, Weiß et al. 1996) and, thus, are suitable tools to realize boxes of Gigaparsec extent. For this purpose it may be considered sufficient to use the “truncated first–order scheme” (known as TZA; “Truncated Zel’dovich Approximation”, Coles et al. 1993).

In a work in progress Buchert et al. (1997) consider two COBE normalized cosmogonies, Standard–CDM and a CDM model with cosmological constant. Both cosmogonies are realized for a box of 1.8 Gpc/h with an effective resolution of 3000\(^3\) Lagrangian fluid elements. The simulation box is then subdivided into smaller boxes and the ensemble average is taken over values of the dimensionless relative ‘backreaction’ (4). Other quantities like the expected Hubble constant or the expected density parameter including ‘backreaction’ are also studied both numerically and analytically.

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\(^5\)Shear fluctuations are accessible through observations: the Mark III catalogue of peculiar–velocities offers this access, which is an especially interesting study, since reconstruction of the density field is performed just on the scales where a large effect is to be expected.
The expected ‘backreaction’, i.e., the quantity $|Q_D|$ averaged over all subensembles on the respective scale and over 8 random phase realizations, normalized by the background density $4\pi G\rho_H$. Two box sizes (600Mpc/h and 1.8Gpc/h) of a Standard–CDM model are shown together demonstrating that the results of the two runs match. The ‘backreaction’ is represented as a function of the scale $L \equiv a_D = |D|^{1/3}$ (measured in Mpc/h) in linear/linear (left panel) and log/log (right panel) format. This dimensionless quantity is still of the same order as the actual matter density on scales around 100Mpc/h. Normalized in addition by the r.m.s. density fluctuations, calculated with top–hat smoothing, we get a value which is almost constant with scale of about 70.

A plot of the absolute value of $Q_D$, normalized by the global mean density, for the initial conditions SCDM against scale is shown in Figure 1, which already gives a clear representation of the scale dependence of the effect under study. The absolute values are shown here, because $Q_D$ might be negative or positive in subsamples, and the overall sum is, by assumption, zero. The absolute value is then an estimate of the expected ‘backreaction’ on some scale. However, in some subsamples the effect may be smaller or larger.

Three preliminary conclusions may be drawn from the first results of this study:

• The magnitude of the ‘backreaction’ source term is of the same order as the mean density and higher on scales $\lambda < 100$Mpc/h for SCDM. It quickly drops to a 10% effect on scales of $\lambda \approx 200$Mpc/h.

• The magnitude of the ‘backreaction’ source term is proportional to the r.m.s. density fluctuations almost independent of scale. Compared to the density fluctuations it amounts to a factor of about 70 for SCDM.
• Using the “Zel’dovich approximation” we can calculate analytically the ‘backreaction’ $B_D$. This calculation shows that $B_D$ is a growing function of time in an expanding universe.

9. BLUE’s Conclusions

“This debate brought up one fundamental result which supports RED: any inhomogeneous Newtonian cosmology, whose flat space sections are confined to a length–scale $L$ on which the matter variables are periodic, averages out to the standard model. On this global scale there is no ‘backreaction’ and the cosmological parameters of the homogeneous–isotropic solutions of Friedmann’s equations are well–defined also for the average cosmology on that scale. Setting up simulations of large–scale structure in this way is correct, and a global comoving coordinate system can be introduced to scale the whole cosmology. This validates the common way of constructing inhomogeneous models of the Universe.

On the other hand, GREEN’s arguments initiate two well–justified ways of saying that this architecture is “forced” due to the settings of a) excluding curvature of the space sections, and b) requiring periodic boundary conditions:

One way is to analyze the effect of ‘backreaction’ on scales smaller than the periodicity scale, and base this analysis upon the standard model, however, by extending the spatial size of periodicity to very large scales. The results obtained in the framework of a well–tested approximation scheme are three–fold: first, they show the importance of the influence of the inhomogeneities on average properties of a chosen spatial domain, although the “forcing conditions” bring the effect down to zero on the scale $L$. Second, the ‘backreaction’ value is numerically small on large scales, but it is always larger than the r.m.s. density fluctuations; in other words: taking the amplitudes of density fluctuations serious (e.g. by normalizing the cosmogony on some large scale) always implies the presence of, e.g., shear fluctuations which are neglected on that scale in the standard model. Third, they show that the effect is a growing function of time. From the latter result we may establish the notion of global gravitational instability of the standard model as opposed to the well–known local instability: it states that, as soon as the ‘backreaction’ has a non–zero value at some time, this value will be increasing; the average model drifts away from the standard model.

The second “forcing” is due to the Newtonian treatment. Being justified on smaller scales, a Newtonian model is expected to fail just when we approach the large scales of periodicity which we have to consider to justify neglection of the ‘backreaction’ effect. Setting up a general–relativistic model unavoidably implies the presence of local curvature and will, in general, yield a nonvanishing average curvature for inhomogeneous models. Neither simple periodic boundary conditions can be employed, nor can be
proved that the ‘backreaction’ effect should vanish globally, at least for compact space sections without boundary.”

10. Summary

Both, RED and GREEN, are right, but if RED’s assumptions are weakened, the resulting cosmology has much richer properties and cannot be confined to a simple box: it will take its additional degrees of freedom to evolve away from the standard model. GREEN’s more general view suffers from the fact that an alternative model is yet not formulated, but it is definitely within reach.

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