THE BEST CONSTANT IN THE KHINTCHINE INEQUALITY
OF THE ORLICZ SPACE $L_{\psi_2}$ FOR EQUIDISTRIBUTED
RANDOM VARIABLES ON SPHERES

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Abstract. We compute the best constant in the Khintchine inequality for
equidistributed random variables on the $N$-sphere in the Orlicz space $L_{\psi_2}$.

1. Introduction

The classical Khintchine inequality compares the $L_p$-norm of a sum of Rade-
macher variables with the $\ell_2$-norm of the coefficients of the sum. The computation
of the best possible constants has attracted a lot of interest. For the classical case,
Haagerup found the best constants for general $p \in (1, \infty)$ in [1]. Also Khintchine
inequalities for different kinds of random variables were investigated, for example rotationally invariant random vectors in [3]. A second variation of the problem changes the underlying space. The Khintchine inequality in Orlicz spaces has been
considered in various cases, the first example is a paper by Rodin and Semyonov [7].

Let $q > 0$ and $\psi_q(x) := \exp(x^q) - 1$ for $x \in \mathbb{R}$. By $||| \cdot |||_{\psi_q}$ we denote the norm of
the Orlicz space $L_{\psi_q}(\Omega, \Sigma, \mu)$. This is given by

$$|||X|||_{\psi_q} := \inf\{c > 0 \mid E\left[\psi_q \left(\frac{\|X\|}{c}\right)\right] \leq 1\},$$

for $X \in L_{\psi_q}$. By $\|\cdot\|$ we denote the Euclidean norm. For $q \leq 2$ one can still compare
the $L_{\psi_q}$-norm and the $\ell_2$-norm, see [4]. For $q > 2$, Pisier proved that the Lorentz
sequence spaces $\ell_{q',\infty}$ ($1/q + 1/q' = 1$), instead of $\ell_2$ come into play, see [6]. This
fact was already mentioned by Rodin and Semyonov [7].

Here we compute the best constant for the Orlicz space $L_{\psi_2}$ and equidistributed
variables on $N$-dimensional spheres. We apply the technique from [5]. Peskir
reduces the case of the Orlicz space to the classical Khintchine inequality in $L_q$.
The optimality of the constants from $L_q$ carries over to $L_{\psi_2}$. The same reduction
technique can be used for variables on spheres. König and Kwapien computed the
optimal constants in [3]. Again the optimality carries over. In this paper we prove
the following result.

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on the sphere.
Theorem 1.1. Let $X_j$, $j = 1, \ldots, n$ be an i.i.d. sequence of equidistributed random variables on the $N$-sphere $S^{N-1}$. For all $a = a_1, \ldots, a_n \in \mathbb{R}$ we have

$$\left\| \sum_{j=1}^{n} a_j X_j \right\|_{\psi_2} \leq b(N) \left( \sum_{j=1}^{n} a_j^2 \right)^{\frac{1}{2}},$$

where the constant $b(N) := \sqrt{\frac{2}{N}} \sqrt{\frac{1}{1 - (\frac{2}{N})^N}}$ is optimal.

Note that $b(N)$ decreases to $\frac{1}{\sqrt{\ln 2}}$ for $N \to \infty$. In Section 2 we prove that the inequality is true. Therefore we consider the series expansion of the exponential function. Then we apply the Khintchine inequality from [3]. In Section 3 we show that the constant $b(N)$ can not be smaller. We show that with $Y_n := \sum_{j=1}^{n} \frac{1}{\sqrt{n}} X_j$ we get asymptotic equality in Theorem 1.1 for $n \to \infty$.

2. Proof of the inequality

Let $C > 0$. Applying Beppo-Levi we may interchange the limit and the expected value.

$$E \left[ \exp \left( \frac{\left\| \sum_{j=1}^{n} a_j X_j \right\|^2}{C^2 \sum_{j=1}^{n} ||a_j||^2} \right) \right]$$

$$= E \left[ \sum_{k=0}^{\infty} \frac{1}{k!} C^{2k} \left( \sum_{j=1}^{n} ||a_j||^2 \right)^k \left\| \sum_{j=1}^{n} a_j X_j \right\|^{2k} \right]$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} C^{2k} \left( \sum_{j=1}^{n} ||a_j||^2 \right)^k E \left[ \left\| \sum_{j=1}^{n} a_j X_j \right\|^{2k} \right]$$

(2.1)

Now we apply König’s and Kwapién’s Khintchine inequality for variables on the sphere and use the constants for $p = 2k$, which gives $(\tilde{b}(2k))^{2k} = \left( \frac{2k}{N} \right)^k \left( \frac{\Gamma(k + \frac{N}{2})}{\Gamma(\frac{N}{2})} \right)$, see [3, Theorem 3]. We obtain

$$E \left[ \left\| \sum_{j=1}^{n} a_j X_j \right\|^{2k} \right] \leq \left( \tilde{b}(2k) \left( \sum_{j=1}^{n} ||a_j||^2 \right)^{\frac{k}{2}} \right)^{2k} = \tilde{b}(2k) \left( \sum_{j=1}^{n} ||a_j||^2 \right)^k.$$

This holds for all $k \in \mathbb{N}$ and therefore for every summand in (2.1). Note that $\tilde{b}(2k)$ does not depend on $n$.

Therefore we get

$$E \left[ \exp \left( \frac{\left\| \sum_{j=1}^{n} a_j X_j \right\|^2}{C^2 \sum_{j=1}^{n} ||a_j||^2} \right) \right] \leq \sum_{k=0}^{\infty} \frac{1}{k!} C^{2k} \left( \frac{2k}{N} \right)^k \left( \frac{\Gamma(k + \frac{N}{2})}{\Gamma(\frac{N}{2})} \right).$$

(2.2)

Note that $\Gamma(k + \frac{N}{2}) = \Gamma \left( \frac{N}{2} \right) \prod_{l=1}^{k} (l - \frac{N}{2}).$

Consider the function $f(x) := (1 - \frac{x}{N})^{-\frac{N}{2}}$. The right-hand side of inequality (2.2) is the Taylor expansion of the function $f$ at the point $x = \frac{1}{C^2}$. 


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So we get

$$E \left[ \exp \left( \frac{\left\| \sum_{j=1}^{n} a_j X_j \right\|^2}{C^2 \sum_{j=1}^{n} \|a_j\|^2} \right) \right] \leq f \left( \frac{1}{C^2} \right).$$

Now let $C := b(N) = \sqrt{\frac{2}{N}} \sqrt{\frac{1}{1-(\frac{1}{2})^N}}$. Then $f \left( \frac{1}{C^2} \right) = 2$ and this proves that the inequality from Theorem 1.1 holds true.

3. PROOF OF THE OPTIMALITY

In this section let $X_j, j \in \mathbb{N}$ be an i.i.d. family of equidistributed random variables on the sphere $S^{N-1}$. Denote $Y_n := \sum_{j=1}^{n} \frac{1}{\sqrt{n}} X_j$.

Lemma 3.1. Let $C \geq \sqrt{\frac{2}{N}} \sqrt{\frac{1}{1-(\frac{1}{2})^N}}$. Then the family of random variables

$$\left( \exp \left( \frac{\left\| \sum_{j=1}^{n} \frac{1}{\sqrt{n}} X_j \right\|}{C} \right) \right), n \in \mathbb{N}$$

is uniformly integrable.

Proof. According to [2, Theorem 6.19] it suffices to prove that for some $p > 1$,

$$I(p) := \sup_{n \in \mathbb{N}} E \left[ \left( \exp \left( \frac{\|Y_n\|}{C} \right) \right)^p \right] < \infty.$$

First note that for a $N$-dimensional Gaussian variable $Z$ we have

$$E \left[ \|X_j\|^{2k} \right] = 1 \leq E \left[ \|Z\|^{2k} \right].$$

Using a theorem of Zolotarev [8, Theorem 3] this implies

$$\mathbb{P} (\|Y_n\| > t) \leq \exp(-Nq(t)),$$

where $q(t) = \frac{1}{2}(t^2 - \ln t - 1)$. For large $t$ we have $t^2 - \ln t - 1 > \gamma t^2$ for some $\gamma$ close to 1, say $\gamma \in (\frac{1}{2}, 1)$.

Therefore we find

$$I(p) = \sup_{n \in \mathbb{N}} \int_{0}^{\infty} \mathbb{P} \left( \exp \left( \frac{\|Y_n\|^2}{C^2} \right) > t \right) \, dt$$

$$= 1 + \sup_{n \in \mathbb{N}} \int_{1}^{\infty} \mathbb{P} \left( \|Y_n\| > \frac{C}{\sqrt{p}} \sqrt{\ln(t)} \right) \, dt$$

$$\leq 1 + \int_{1}^{\infty} t^{-\frac{N}{2} C^2 \gamma} \, dt.$$

So we can choose $p \in (1, \frac{N}{2} C^2 \gamma)$ such that the latter integral is finite. $\square$

Lemma 3.2. Let $Z$ be a $N$-dimensional Gaussian variable. Then we have

$$\|Z\|_{\psi^2} = \frac{\sqrt{2}}{\sqrt{1-(\frac{1}{2})^N}}.$$
Proof. Let $C > \sqrt{2}$. We compute

$$
\mathbb{E} \left[ \exp \left( \frac{\|Z\|^2}{C^2} \right) \right] = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp \left( \frac{\|x\|^2}{C^2} \right) \exp \left( -\frac{\|x\|^2}{2} \right) \, dx
$$

$$
= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp \left( -\frac{1}{2} t^2 \left( \frac{C^2 - 2}{C^2} \right) \right) \, dt
$$

$$
= \left( \frac{C^2}{C^2 - 2} \right)^{\frac{N}{2}}.
$$

Now we have $\left( \frac{C^2}{C^2 - 2} \right)^{\frac{N}{2}} \leq 2$ if and only if $C \geq \sqrt{\frac{2}{1 - (\frac{1}{2})^N}}$. This proves the lemma.

\[\square\]

Lemma 3.3. Let $Z$ be a $N$-dimensional Gaussian variable. Then we have

$$
\lim_{n \to \infty} \left\| \sum_{j=1}^{n} \frac{1}{\sqrt{n}} X_j \right\|_{\psi_2} = \|Z\|_{\psi_2}.
$$

Proof. Assume $\limsup_{n \to \infty} \|Y_n\|_{\psi_2} > \|Z\|_{\psi_2}$. Then there exists a subsequence $n_k, k \in \mathbb{N}$ and some $\epsilon > 0$ such that

$$
\|Y_{n_k}\|_{\psi_2} > \|Z\|_{\psi_2} + \epsilon.
$$

According to Lemma 3.1 the family $\left( \exp \left( \frac{\|Y_n\|}{\|Z\|_{\psi_2} + \epsilon} \right)^2 \right), n \in \mathbb{N}$ is uniformly integrable. Also

$$
G_n := \exp \left( \frac{\|Y_n\|}{\|Z\|_{\psi_2} + \epsilon} \right)^2 - \exp \left( \frac{\|Z\|}{\|Z\|_{\psi_2} + \epsilon} \right)^2, n \in \mathbb{N}
$$

is uniformly integrable. For $M > 0$ we have

$$
\int G_n \, d\mathbb{P} \leq \int_{\{G_n \leq M\}} G_n \, d\mathbb{P} + \sup_{n \in \mathbb{N}} \int_{\{G_n > M\}} G_n \, d\mathbb{P}.
$$

For every fixed $M > 0$, the first integral tends to 0 for $n \to \infty$ by the central limit theorem. The second integral tends to 0 for $M \to \infty$ due to the uniform integrability. Therefore

$$
\lim_{n \to \infty} \int \exp \left( \frac{\|Y_n\|}{\|Z\|_{\psi_2} + \epsilon} \right)^2 \, d\mathbb{P} = \int \exp \left( \frac{\|Z\|}{\|Z\|_{\psi_2} + \epsilon} \right)^2 \, d\mathbb{P}.
$$
This implies
\[ 2 \geq \int \exp \left( \frac{\|Z\|}{\|Z\|_{\psi^2}} \right) \, dP \]
\[ > \int \exp \left( \frac{\|Z\|}{\|Z\|_{\psi^2} + \epsilon} \right) \, dP \]
\[ = \lim_{n \to \infty} \int \exp \left( \frac{\|Y_n\|}{\|Z\|_{\psi^2} + \epsilon} \right) \, dP \]
\[ = \lim_{k \to \infty} \int \exp \left( \frac{\|Y_{nk}\|}{\|Z\|_{\psi^2} + \epsilon} \right) \, dP \]
\[ \geq 2, \]
which is a contradiction. Therefore \( \limsup_{n \to \infty} \|Y_n\|_{\psi^2} \leq \|Z\|_{\psi^2} \). In the same way we show \( \liminf_{n \to \infty} \|Y_n\|_{\psi^2} \geq \|Z\|_{\psi^2} \). □

This finishes the proof of our Theorem.

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