Exact solution of cluster model with next-nearest-neighbor interaction

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A one-dimensional cluster model with next-nearest-neighbor interactions and two additional composite interactions is solved; the free energy is obtained and a correlation function is derived exactly. The model is diagonalized by a transformation obtained automatically from its interactions, which is an algebraic generalization of the Jordan-Wigner transformation. The gapless condition is expressed as a condition on the roots of a cubic equation, and the phase diagram is obtained exactly. We find that the distribution of roots for this algebraic equation determines the existence of long-range order, and we again obtain the ground-state phase diagram. We also derive the central charges of the corresponding CFT. Finally, we note that our results are universally valid for an infinite number of solvable spin chains whose interactions obey the same algebraic relations.

1. Introduction

Quantum spin models have been widely investigated as a basic theme in statistical physics, mathematical physics, and condensed matter physics. In one-dimension, there exist many solvable systems, and there also exist general relationships with two-dimensional classical systems. Specifically, a one-dimensional XY-model was introduced in [1]-[4] and was later found to be equivalent to the two-dimensional rectangular Ising model[5].

In the 2000s, the cluster models attracted wide attention. The ground state of this model is called the cluster state, which is a candidate of a resource for measurement-based quantum computation(MBQC)[6][7][8]. The one-dimensional cluster model was first introduced and solved by Suzuki[5] and has been investigated by numerous researchers[9]-[22].

A new method of fermionization has recently been introduced[23], and an infinite number of new solvable models have been reported in [23]-[25]. In this method, the transformation that diagonalizes the system is obtained automatically from the interactions of the model. The Hamiltonian is diagonalized using only algebraic relations of the interactions, and therefore, the equivalences of the models are immediately understood. In the case of the XY model, the transformation results in the Jordan-Wigner transformation[26][2]; thus, this new transformation can be regarded as the algebraic generalization of the Jordan-Wigner transformation.
This algebraic method has been developed into a graph theoretical method of fermionization[27], in which transformations of operators are expressed by deformations of graphs, and the kernels of their adjacency matrices provide conserved quantities of the systems.

In this paper, we investigate the cluster model with next-nearest-neighbor interactions and two additional composite interactions. The Hamiltonian is

\[ -\beta \mathcal{H} = K_0 \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x + K_1 \sum_{j=1}^{N} \sigma_j^x \sigma_{j+2}^x + K_2 \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x \sigma_{j+2}^x + K_{-1} \sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^y \sigma_{j+2}^y + K_{-2} \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^y \sigma_{j+2}^y, \]

where \( 1_j \) is the identity operator at site \( j \). When \( K_0 = K_1 = 0 \), this model becomes the cluster model with next-nearest-neighbor interactions and found to be basically equivalent [24] to the XY chain. However, the model (1) cannot be diagonalized by the Jordan-Wigner transformation. Our formula can diagonalize this Hamiltonian. The formula in this paper is summarized as follows. Let the number of sites be \( N = 2M \), where \( M \) is even, and let us assume a cyclic boundary condition \( \sigma_{N+i}^x = \sigma_i^x \) \( (k = x, y, z) \). Let us consider two series of operators \( \{ \eta_j \} \) and \( \{ \zeta_j \} \), which are defined as

\[
\eta_{2j-1} = \sigma_{2j-1}^x \sigma_{2j}^x \sigma_{2j+1}^x, \quad \eta_{2j} = \sigma_{2j}^x \sigma_{2j+1}^x \sigma_{2j+2}^x, \quad (1 \leq j \leq \frac{N}{2} - 1) \]

\[
\eta_{N-1} = \sigma_1^x \sigma_0^x \quad \eta_N = \sigma_1^x \sigma_2^x, \quad (2)
\]

and

\[
\zeta_{2j-1} = \sigma_{2j-1}^x \sigma_{2j}^x \sigma_{2j+1}^x, \quad \zeta_{2j} = \sigma_{2j}^x \sigma_{2j+1}^x \sigma_{2j+2}^x, \quad (1 \leq j \leq \frac{N}{2} - 1) \]

\[
\zeta_{N-1} = \sigma_N^x \sigma_{N-1}^x \quad \zeta_N = \sigma_1^x \sigma_2^x. \quad (3)
\]

Note that the operators \( \eta_j \) and \( \zeta_k \) are the interactions found in the Hamiltonian (1), and the operators \( \eta_j \) and \( \zeta_k \) commute with each other for all \( j \) and \( k \). Next, following equation (2.9) in [23], we introduce transformations \( \varphi_1(j), \varphi_2(j), \varphi_3(j), \varphi_4(j) \) \( (1 \leq j \leq M) \) as

\[
\varphi_1(j) = \frac{1}{\sqrt{2}} e^{i \frac{\pi}{2}(k-1)} \eta_0 \eta_1 \cdots \eta_k \quad (k = 2j - 2),
\]

\[
\varphi_2(j) = \frac{1}{\sqrt{2}} e^{i \frac{\pi}{2}(k-1)} \eta_0 \eta_1 \cdots \eta_k \quad (k = 2j - 1),
\]

\[
\varphi_3(j) = \frac{1}{\sqrt{2}} e^{i \frac{\pi}{2}(k-1)} \zeta_0 \zeta_1 \cdots \zeta_k \quad (k = 2j - 2),
\]

\[
\varphi_4(j) = \frac{1}{\sqrt{2}} e^{i \frac{\pi}{2}(k-1)} \zeta_0 \zeta_1 \cdots \zeta_k \quad (k = 2j - 1), \quad (4)
\]

where

\[
\eta_0 = i \sigma_1^x \sigma_2^x \quad \text{and} \quad \zeta_0 = i \sigma_2^x \sigma_3^x \quad \text{(5)}
\]

are the initial operators. Then, the operators \( \varphi_l(j) \) satisfy

\[
\{ \varphi_l(j), \varphi_m(k) \} = \delta_{lm} \delta_{jk} \quad (l, m = 1, 2),
\]

\[
\{ \varphi_l(j), \varphi_m(k) \} = \delta_{lm} \delta_{jk} \quad (l, m = 3, 4),
\]

\[
\varphi_l(j) \varphi_m(k) = \varphi_m(k) \varphi_l(j) \quad (l = 1, 2, m = 3, 4), \quad (6)
\]
for all $j$ and $k$. Therefore, the operators $\phi_l(j)$ form two series of Majorana fermions. The Hamiltonian (1) is expressed as the sum of two-body products of $\phi_l(j)$. Hence, the Hamiltonian (1) can be diagonalized. The transformations from $\eta_j$ to $\phi_1$ and $\phi_2$ ($\zeta_j$ to $\phi_3$ and $\phi_4$) are clearly different from the Jordan-Wigner transformation as shown in (12)-(15). From the transformations (4), we can derive the free energy and a correlation function exactly.

Next, let us consider the algebraic equation
\[
\alpha_2 z^3 + \alpha_1 z^2 - z + \alpha_{-1} = 0,
\]
where $\alpha_{-1} = K_{-1}/K_0$, $\alpha_1 = K_1/K_0$, $\alpha_2 = K_2/K_0$, and also consider three subsets of the complex plane
\[
C_u = \{ z \in \mathbb{C} \mid |z| = 1 \}, \quad D_O = \{ z \in \mathbb{C} \mid |z| > 1 \}, \quad D_I = \{ z \in \mathbb{C} \mid |z| < 1 \}.
\]
We find that there is no energy gap above the ground state if and only if at least one root of equation (7) belongs to the unit circle $C_u$. We also find that the existences of long-range orders are classified by locations of the roots of the cubic equation (7). The boundary of the phase in terms of the long-range order satisfies the gapless condition; therefore, we obtain a consistent phase diagram from two different procedures. We also determine the universality classes of these phase transitions by deriving the central charges of the corresponding CFT.

We also find that the phase diagram has a symmetry under a shift of indices from $j$ to $j + 1$ for $K_j$. These results are obtained using only the algebraic relations of the interactions. Hence, the results we obtained are universally valid for models consisting of the interactions that obey the same algebraic relations.

In section 2, we diagonalize the Hamiltonian (1) exactly by applying the transformation (4) and obtain the free energy. In section 3, we consider the gapless condition and obtain a corresponding phase diagram. In section 4, first a correlation function is introduced, and the asymptotic limit is obtained exactly. Next, we classify the existences of long-range orders in terms of locations of the roots of equation (7), and again, we obtain the same phase diagram. In section 5, we derive the central charges from the finite size behavior of the energy spectrum. In section 6, we consider the symmetry of the phase diagram, and illustrate that the results obtained in this paper are valid for an infinite number of Hamiltonians that satisfy our condition.

2. Diagonalization and the free energy

In this section, we diagonalize the Hamiltonian (1) and derive the free energy. The operators $\eta_j$ and $\eta_k$ ($\zeta_j$ and $\zeta_k$) are called adjacent if $(j,k) = (j,j+1)$ ($1 \leq j \leq N - 1$) or $(j,k) = (N,1)$. Then, the operators $\{\eta_j\}$ in (2) satisfy the relations
\[
\eta_j \eta_k = \begin{cases} 
-\eta_k \eta_j & \text{if } j \text{ and } k \text{ are adjacent} \\
\eta_k \eta_j & \text{if } j \text{ and } k \text{ are not adjacent} \\
1 & j = k,
\end{cases}
\]
and $\{\zeta_j\}$ satisfy the same relations replacing $\eta_j$ by $\zeta_j$ in (9). Note that
\[
(\eta_1 \cdots \eta_k)^2 = (\zeta_1 \cdots \zeta_k)^2 = (-1)^{k-1} \quad (k < N).
\]
Other interactions in (1) are obtained from \( \eta_j \) and \( \zeta_k \) as

\[
\begin{align*}
\eta_{2j-1} \eta_{2j} \eta_{2j+1} &= \sigma_{2j-1}^x \sigma_{2j}^y \sigma_{2j+1}^x \sigma_{2j+2} \sigma_{2j+3}, \\
\eta_{2j} \eta_{2j+1} \eta_{2j+2} &= -\sigma_{2j}^x \sigma_{2j+1}^x \sigma_{2j+2}^x \sigma_{2j+3} \sigma_{2j+4}, \\
\zeta_{2j-1} \zeta_{2j} \zeta_{2j+1} &= \sigma_{2j}^x \sigma_{2j+1}^y \sigma_{2j+2}^x \sigma_{2j+3} \sigma_{2j+4}, \\
\zeta_{2j} \zeta_{2j+1} \zeta_{2j+2} &= -\sigma_{2j+1}^x \sigma_{2j+2}^x \sigma_{2j+3} \sigma_{2j+4} \sigma_{2j+5}.
\end{align*}
\]

Then the Hamiltonian (1) is written in terms of \( \eta_j \) and \( \zeta_k \) as

\[
-\beta \mathcal{H} = K_0 \sum_{j=1}^{M} (\eta_{2j-1} + \zeta_{2j-1}) + K_1 \sum_{j=1}^{M} (\eta_{2j} + \zeta_{2j}) - K_2 \sum_{j=1}^{M} (\eta_{2j} \eta_{2j+1} \eta_{2j+2} + \zeta_{2j} \zeta_{2j+1} \zeta_{2j+2}) + K_{-1} \sum_{j=1}^{M} (\eta_{2j-1} \eta_{2j} \eta_{2j+1} + \zeta_{2j-1} \zeta_{2j} \zeta_{2j+1}).
\]

Next, let us consider the transformations \( \varphi_k(j) \) \( (k = 1, 2, 3, 4, \, 1 \leq j \leq M) \) introduced in (4). They are written in terms of the Pauli operators as

\[
\begin{align*}
\varphi_1(j) &= \frac{1}{\sqrt{2}} \left( \prod_{\nu=1}^{j-1} 1_{2\nu-1} \sigma_{2\nu}^x \right) \sigma_{2j-1}^x \sigma_{2j}^x, \\
\varphi_2(j) &= \frac{1}{\sqrt{2}} \left( \prod_{\nu=1}^{j-1} 1_{2\nu-1} \sigma_{2\nu}^y \right) 1_{2j-1} \sigma_{2j}^y \sigma_{2j+1}^x, \\
\varphi_3(j) &= \frac{1}{\sqrt{2}} \left( \prod_{\nu=1}^{j-1} 1_{2\nu} \sigma_{2\nu}^x \right) \sigma_{2j}^y \sigma_{2j+1}^x, \\
\varphi_4(j) &= \frac{1}{\sqrt{2}} \left( \prod_{\nu=1}^{j-1} 1_{2\nu} \sigma_{2\nu}^y \right) 1_{2j} \sigma_{2j+1}^y \sigma_{2j+2}^x \ (j = 1, 2, 3 \cdots).
\end{align*}
\]

They are clearly different from the Jordan-Wigner transformation. The initial operators \( \eta_0 = i \sigma_1^x \sigma_2^x \) and \( \zeta_0 = i \sigma_2^y \sigma_3^x \) are not necessary to diagonalize the Hamiltonian, but introduced to avoid boundary terms in (13) - (16). The initial operators satisfy the relations

\[
\begin{align*}
\eta_0^2 &= -1, \quad \zeta_0^2 = -1, \quad \eta_0 \zeta_0 = \zeta_0 \eta_0, \\
\eta_0 \eta_j &= -\eta_j \eta_0, \quad \eta_0 \eta_j = \eta_j \eta_0 \quad (2 \leq j \leq N), \\
\zeta_0 \zeta_j &= -\zeta_j \zeta_0, \quad \zeta_0 \zeta_j = \zeta_j \zeta_0 \quad (2 \leq j \leq N).
\end{align*}
\]

From (10) and (17), we find that

\[
\{ \varphi_l(j), \varphi_m(k) \} = \delta_{lm} \delta_{jk} \quad (l, m = 1, 2 \text{ or } l, m = 3, 4, \quad 1 \leq j, k \leq M),
\]
and $\varphi_l(j)$ ($l=1,2$) and $\varphi_m(k)$ ($m=3,4$) commute with each other. Moreover, we find
\begin{align}
(+2i)\varphi_2(j)\varphi_1(j+2) &= \eta_{2j}\eta_{2j+1}\eta_{2j+2} = -\sigma^x_{2j}\sigma^x_{2j+1}\sigma^x_{2j+2}\sigma^x_{2j+3}\sigma^x_{2j+4}, \\
(-2i)\varphi_2(j)\varphi_1(j+1) &= \eta_{2j} = \sigma^x_{2j}\sigma^x_{2j+1}, \quad \text{(19)} \\
(+2i)\varphi_2(j)\varphi_1(j) &= \eta_{2j-1} = \sigma^x_{2j-1}\sigma^x_{2j+1}, \\
(-2i)\varphi_2(j)\varphi_1(j-1) &= \eta_{2j-3}\eta_{2j-2}\eta_{2j-1} = \sigma^x_{2j-3}\sigma^y_{2j-2}\sigma^x_{2j-1}\sigma^x_{2j+1}, \\
\end{align}
and
\begin{align}
(+2i)\varphi_4(j)\varphi_3(j+2) &= \zeta_{2j}\zeta_{2j+1}\zeta_{2j+2} = -\sigma^x_{2j+1}\sigma^x_{2j+2}\sigma^x_{2j+3}\sigma^x_{2j+4}\sigma^x_{2j+5}, \\
(-2i)\varphi_4(j)\varphi_3(j+1) &= \zeta_{2j} = \sigma^x_{2j+1}\sigma^x_{2j+2}\sigma^x_{2j+3}, \quad \text{(20)} \\
(+2i)\varphi_4(j)\varphi_3(j) &= \zeta_{2j-1} = \sigma^x_{2j-1}\sigma^x_{2j+1}, \\
(-2i)\varphi_4(j)\varphi_3(j-1) &= \zeta_{2j-3}\zeta_{2j-2}\zeta_{2j-1} = \sigma^x_{2j-3}\sigma^y_{2j-2}\sigma^y_{2j-1}\sigma^x_{2j+1}\sigma^x_{2j+2}. \\
\end{align}

Then the Hamiltonian is written as
\begin{align}
-\beta H &= (-2i)K_{-1}\sum_{j=1}^{M} (\varphi_2(j)\varphi_1(j-1) + \varphi_4(j)\varphi_3(j-1)) \\
&\quad + (+2i)K_0\sum_{j=1}^{M} (\varphi_2(j)\varphi_1(j) + \varphi_4(j)\varphi_3(j)) \\
&\quad + (-2i)K_1\sum_{j=1}^{M} (\varphi_2(j)\varphi_1(j+1) + \varphi_4(j)\varphi_3(j+1)) \\
&\quad + (-2i)K_2\sum_{j=1}^{M} (\varphi_2(j)\varphi_1(j+2) + \varphi_4(j)\varphi_3(j+2)), \quad \text{(21)}
\end{align}
which is the sum of two-body products of the Majorana fermion operators $\varphi_l(j)$.

For the purpose to check the boundary condition, let us consider the boundary terms
\begin{align}
\varphi_2(M)\varphi_1(M+l) &= \frac{i}{\sqrt{2}}(-\sigma^x_2\sigma^x_4\cdots\sigma^x_N)\sigma^x_N\sigma^x_1 \cdot \varphi_1(M+l), \quad \text{(22)} \\
\varphi_4(M)\varphi_3(M+l) &= \frac{i}{\sqrt{2}}(-\sigma^x_1\sigma^x_3\cdots\sigma^x_{N-1})\sigma^x_1\sigma^x_2 \cdot \varphi_3(M+l). \quad \text{(23)}
\end{align}

Then the cyclic boundary condition $\sigma^k_{N+l} = \sigma^k_l$ yields, for $l=1$ and 2, that
\begin{align}
\varphi_1(M+l) &= \begin{cases} 
\varphi_1(l) & \sigma^x_2\sigma^x_4\cdots\sigma^x_N = -1 \\
-\varphi_1(l) & \sigma^x_2\sigma^x_4\cdots\sigma^x_N = 1
\end{cases} \\
\varphi_3(M+l) &= \begin{cases} 
\varphi_3(l) & \sigma^x_1\sigma^x_3\cdots\sigma^x_{N-1} = -1 \\
-\varphi_3(l) & \sigma^x_1\sigma^x_3\cdots\sigma^x_{N-1} = 1
\end{cases} \quad \text{(24)}
\end{align}
and
\begin{align}
\varphi_3(M+l) &= \begin{cases} 
\varphi_3(l) & \sigma^x_1\sigma^x_3\cdots\sigma^x_{N-1} = -1 \\
-\varphi_3(l) & \sigma^x_1\sigma^x_3\cdots\sigma^x_{N-1} = 1
\end{cases} \quad \text{(25)}
\end{align}

Next let us introduce the Fourier transformation
\begin{align}
\varphi_l(j) &= \frac{1}{\sqrt{M}} \sum_{0 \leq q < \pi} (e^{iqj}C_l(q) + e^{-iqj}C_l^+(q)), \quad \text{(26)}
\end{align}
where

\[ \{ C_l^\dagger (p), C_m(q) \} = \delta_{lm} \delta_{pq}, \]
\[ \{ C_l^\dagger (p), C_m^\dagger (q) \} = \{ C_l(p), C_m(q) \} = 0 \quad (l, m = 1, 2 \text{ or } l, m = 3, 4), \quad (27) \]

and \( C_l^\dagger (p), C_l(p) \) \((l = 1, 2)\) and \( C_m^\dagger (q), C_m(q) \) \((m = 3, 4)\) commute with each other. From (24) and (25), we find

\[ q = \frac{2k}{M} \pi \quad (k = 0, 1, \ldots, \frac{M}{2} - 1) \quad (28) \]

when \( \sigma_2^\dagger \sigma_4^\dagger \cdots \sigma_N^\dagger = -1 \) and when \( \sigma_1^\dagger \sigma_3^\dagger \cdots \sigma_{N-1}^\dagger = -1 \). We also find

\[ q = \frac{2k - 1}{M} \pi \quad (k = 1, 2, \ldots, \frac{M}{2}) \quad (29) \]

when \( \sigma_2^\dagger \sigma_4^\dagger \cdots \sigma_N^\dagger = 1 \) and when \( \sigma_1^\dagger \sigma_3^\dagger \cdots \sigma_{N-1}^\dagger = 1 \). Without loss of generality, we can assume (29). The Hamiltonian is then expressed as

\[ - \sum_{0 < q < \pi} (W_{12}(q) + W_{34}(q)), \quad (30) \]

where

\[ W_{12}(q) = 2iL(q)C_2^\dagger (q)C_1(q) + 2iL(q)^*C_2(q)C_1^\dagger (q), \]
\[ W_{34}(q) = 2iL(q)C_4^\dagger (q)C_3(q) + 2iL(q)^*C_4(q)C_3^\dagger (q), \quad (31) \]

and

\[ L(q) = K_{-1}e^{-iq} - K_0 + K_1e^{iq} + K_2e^{2iq}. \quad (32) \]

Here \( L(q)^* \) is the complex conjugate of \( L(q) \). For each \( q > 0 \), the Hamiltonian is therefore the sum of two commutative operators. We obtain

\[ A(q) = L(q)L(q)^* = K_{-1}^2 + K_0^2 + K_1^2 + K_2^2 - 2(K_{-1}K_0 + K_0K_1 - K_1K_2) \cos q \]
\[ + 2(K_{-1}K_1 - K_0K_2) \cos 2q + 2K_{-1}K_2 \cos 3q, \quad (33) \]

which is real and non-negative.

With respect to the basis \(|0\rangle, C_1^\dagger (q)|0\rangle, C_2^\dagger (q)|0\rangle, C_3^\dagger (q)C_1^\dagger (q)|0\rangle, W_{12}(q)\) is expressed as

\[ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2iL(q)^* & 0 \\ 0 & -2iL(q) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (34) \]

and its eigenvalues are found to be 0, 0, ±2√\( A(q) \). In the same way, the eigenvalues of \( W_{34}(q) \) are found to be 0, 0, ±2√\( A(q) \).
The partition function is obtained as
\[
Z = \prod_{0<q<\pi} (e^0 + e^0 + e^{\Lambda(q)} + e^{-\Lambda(q)})^2 \\
= \prod_{0<q<\pi} (e^{\frac{3}{2}\Lambda(q)} + e^{-\frac{3}{2}\Lambda(q)})^4,
\]
where \(\Lambda(q) = 2\sqrt{A(q)}\). The free energy is
\[
-\beta f = \lim_{N \to \infty} \frac{\log Z}{N} \\
= \frac{1}{\pi} \int_{0}^{\pi} \log(e^{\frac{3}{2}\Lambda(q)} + e^{-\frac{3}{2}\Lambda(q)})dq,
\]
where we have used that \(\Delta q = \frac{2}{N/2\pi}\).

When \(K_2 = K_{-1} = 0\), the free energy (36) becomes identical to that of the XY chain. In fact, if we introduce
\[
\eta_{2j-1}^{(1)} = \sigma_{2j-1}^x \sigma_{2j}^x, \quad \eta_{2j}^{(1)} = \sigma_{2j}^y \sigma_{2j+1}^y, \quad \zeta_{2j-1}^{(1)} = \sigma_{2j-1}^x \sigma_{2j}^x, \quad \zeta_{2j}^{(1)} = \sigma_{2j}^y \sigma_{2j+1}^y,
\]
then the operators \(\eta_{j}^{(1)}\) and \(\zeta_{j}^{(1)}\) satisfy the same relations as \(\eta_{j}\) and \(\zeta_{j}\), and \(K_0 \sum_{j} (\eta_{2j-1}^{(1)} + \zeta_{2j-1}^{(1)}) + K_1 \sum_{j} (\eta_{2j}^{(1)} + \zeta_{2j}^{(1)})\) are nothing but the interactions of the XY chain. Thus the model (1) with \(K_2 = K_{-1} = 0\) and the XY chain obey the same algebraic relation, and therefore result in the same free energy.

3. Gapless condition and the phase diagram
Let us consider the gapless condition that \(A(q) = |L(q)|^2 = 0\) with some \(q\). The condition \(\alpha_2 z^3 + \alpha_1 z^2 - z + \alpha_{-1} = 0\),
\[
(38)
\]
where \(z = e^{iq}\), \(\alpha_{-1} = K_{-1}/K_0\), \(\alpha_1 = K_1/K_0\), \(\alpha_2 = K_2/K_0\). Thus the model is gapless if and only if the algebraic equation (38) has a root that belongs to \(C_u = \{z \in \mathbb{C} | |z| = 1\}\).

We find that the condition is equivalent to (X1) or (X2) or (X3), where
\[
(X1) \quad \alpha_1 = -\alpha_{-1} + (\alpha_2 - 1) \quad \text{or} \quad \alpha_1 = -\alpha_{-1} - (\alpha_2 - 1),
\]
\[
(X2) \quad \alpha_{-1} \neq 0 \quad \text{and} \quad \alpha_1 = \alpha_{-1} - \frac{\alpha_2(1+\alpha_2)}{\alpha_{-1}} \quad \text{and} \quad (1+\alpha_2)^2 - 4\alpha_2^2 < 0,
\]
\[
(X3) \quad \alpha_{-1} = 0 \quad \text{and} \quad -2 < \alpha_1 < 2 \quad \text{and} \quad \alpha_2 = -1.
\]
A derivation is given in Appendix A. These (X1) - (X3) form the boundary of each phase in the diagram shown in Figure 1 and 2.

4. Correlation function and the ground state phase transition
Next let us investigate a ground-state correlation function and its asymptotic behavior. The correlation function we consider is
\[
I_1(2n) = \langle \sigma_{2j}^x \sigma_{2j+2n}^x \rangle_0,
\]
(40)
Fig. 1  The phase diagram on $\alpha_{-1} - \alpha_1$ plane for each $\alpha_2 \geq 0$. Two lines denote $\alpha_1 = -\alpha_{-1} \pm (\alpha_2 - 1)$, and $C(\alpha_2)$ denotes the curve $\alpha_1 = \alpha_{-1} - \frac{\alpha_2(1 + \alpha_2)}{\alpha_{-1}}$, $P\left(\frac{1 + \alpha_2}{2}, \frac{1 - 3\alpha_2}{2}\right)$, $Q(-\alpha_2, 1)$, $P'(-\frac{1 + \alpha_2}{2}, -\frac{1 - 3\alpha_2}{2})$, and $Q'(\alpha_2, -1)$ are the multicritical points. Regions characterized by $I_1(\infty) > 0$ and $I_2(\infty) > 0$ are colored by blue and green, respectively. The central charge on each critical line is derived in section 5.
Fig. 2  The phase diagram for each $\alpha_2 < 0$. The red segment corresponds to $(X3)$. In the case $\alpha_2 = -1$, the point $O(0,0)$ does not satisfy $(X2)$ but satisfy $(X3)$. 
where \( \langle \cdot \rangle_0 \) is the ground-state expectation. In the thermodynamic limit, we obtain the exact expression

\[
I_1(2n) = \left( \frac{2}{i} \right)^n \left| \begin{array}{cccc}
D(1) & D(2) & \cdots & D(n) \\
D(0) & D(1) & \cdots & D(n-1) \\
\vdots & \vdots & \ddots & \vdots \\
D(2-n) & D(3-n) & \cdots & D(1)
\end{array} \right| ,
\]

(41)

where

\[
D(r) = \frac{i}{4\pi} \int_{-\pi}^{\pi} dq \left( e^{-iqr} \sqrt{A(q)L(q)^*} \right).
\]

(42)

A derivation of (41) and (42) is given in Appendix B.

The determinant (41) is a Toeplitz determinant, i.e. \((j, k)\) elements depend only on \(k - j = r\). We can therefore apply the Szegő’s theorem to obtain the asymptotic limit \(n \to \infty\) of the determinant. Let \(C(r) = \frac{2}{i} D(r+1)\), then (41) is expressed as

\[
\left| \begin{array}{cccc}
C(0) & C(1) & \cdots & C(n-1) \\
C(-1) & C(0) & \cdots & C(n-2) \\
\vdots & \vdots & \ddots & \vdots \\
C(1-n) & C(2-n) & \cdots & C(0)
\end{array} \right| .
\]

(43)

Let us introduce \(f(p)\) by the relation

\[
C(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ipr} f(p) dp.
\]

(44)

Then the Szegő’s theorem says that the asymptotic form of (41) is

\[
\lim_{n \to \infty} \frac{\det(\{C(r)\})}{\lambda^n} = \exp \left( \sum_{n=1}^{\infty} ng_n g_{-n} \right),
\]

(45)

where

\[
g_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ipn} \log f(p) dp
\]

(46)

and \(\lambda = \exp g_0\). Note that \(g_{-n} = -g_n\). We obtain, from (42) and (44) that

\[
f(p) = \frac{\sqrt{L(p)L(p)^*}}{L(p)^*} e^{-ip} = \frac{\sqrt{e^{-ip} L(p)}}{e^{ip} L(p)^*},
\]

(47)

thus

\[
\log f(p) = \frac{1}{2} \log(\alpha_{-1} e^{-2ip} - e^{-ip} + \alpha_1 + \alpha_2 e^{ip}) - \frac{1}{2} \log(\alpha_{-1} e^{2ip} - e^{ip} + \alpha_1 + \alpha_2 e^{-ip}).
\]

With the use of the equality

\[
\alpha_{-1} e^{-2ip} - e^{-ip} + \alpha_1 + \alpha_2 e^{ip} = A(1 + a_1 e^{ip})(1 + a_2 e^{-ip})(1 + a_3 e^{-ip}),
\]

(48)

\[
A = (-t)\alpha_2, \quad a_1 = \frac{-1}{t}, \quad a_2 + a_3 = \frac{\alpha_1}{\alpha_2} + t, \quad a_2 a_3 = \frac{-1}{t} \frac{\alpha_{-1}}{\alpha_2},
\]

(49)
where \( t \) is a real solution of the equation \( \alpha_1 - t + \alpha_1 t^2 + \alpha_2 t^3 = 0 \), and the factor \( A \) is independent of \( p \), we obtain
\[
\log f(p) = \frac{1}{2} \left( \log(1 + a_1 e^{ip}) + \log(1 + a_2 e^{-ip}) + \log(1 + a_3 e^{-ip}) \right) - \frac{1}{2} \left( \log(1 + a_1 e^{ip}) + \log(1 + a_2 e^{ip}) + \log(1 + a_3 e^{ip}) \right).
\]

(50)

Then we consider the following four cases:

(i) \(|a_1| < 1 \) and \(|a_2| < 1 \) and \(|a_3| < 1 \),

(ii) \(|a_1| > 1 \) and \(|a_2| > 1 \) and \(|a_3| < 1 \),

(iii) \(|a_1| > 1 \) and \(|a_2| < 1 \) and \(|a_3| > 1 \),

(iv) the other cases.

**case(i)** Let \(|b| < 1 \), then
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inp} \log(1 + be^{ip}) dp = \begin{cases} 
\frac{(-1)^{n-1} b^n}{n} & (n > 0) \\
0 & otherwise
\end{cases}
\]

(51)

and
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inp} \log(1 + be^{-ip}) dp = \begin{cases} 
\frac{(-1)^{n-1} b^{-n}}{-n} & (n < 0) \\
0 & otherwise
\end{cases}.
\]

(52)

From (46), (50), (51) and (52), we obtain
\[
g_0 = 0, \quad gn = \frac{1}{2} \left( \frac{(-1)^{n-1}}{n} (a_1^n - a_2^n - a_3^n) \right) (n > 0).
\]

(53)

Then from (45) we obtain
\[
\lim_{n \to \infty} I_1(2n) = \left[ (1 - a_1^2)(1 - a_2^2)(1 - a_3^2)(1 - a_1 a_2)^{-2}(1 - a_1 a_3)^{-2}(1 - a_2 a_3)^2 \right]^{\frac{1}{4}}.
\]

(54)

**case(ii)** Let \(|b| > 1 \), then
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inp} \log(1 + be^{ip}) dp = \frac{(-1)^n}{n} + \begin{cases} 
\frac{(-1)^{n-1}}{n} \left( \frac{1}{b} \right)^n & (n > 0) \\
0 & otherwise
\end{cases}
\]

(55)

and
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inp} \log(1 + be^{-ip}) dp = \frac{(-1)^n}{-n} + \begin{cases} 
\frac{(-1)^{n-1}}{n} \left( \frac{1}{b} \right)^n & (n < 0) \\
0 & otherwise
\end{cases}.
\]

(56)

From (46), (50), (55) and (56), we obtain
\[
\lim_{n \to \infty} I_1(2n) = \left[ \left( 1 - \frac{1}{a_1^2} \right) \left( 1 - \frac{1}{a_2^2} \right) \left( 1 - a_3^2 \right) \left( 1 - \frac{1}{a_1 a_2} \right)^{-2} \left( 1 - a_3 a_1 \right)^2 \left( 1 - a_3 a_2 \right)^{-2} \right]^{\frac{1}{4}}.
\]

(57)

The limit \( \lim_{n \to \infty} I_1(2n) \) in the case (iii) is obtained from (57) with \( a_2 \) and \( a_3 \) replaced by \( a_3 \) and \( a_2 \), respectively. In the case (iv), it is easy to derive that \( \sum_{n=1}^{\infty} ng_n g_{-n} = -\infty \) and \( \lim_{n \to \infty} I_1(2n) = 0 \). From (54) and (57), we find that \( I_1(\infty) = \lim_{n \to \infty} I_1(2n) > 0 \) if and only if \( a_1 \), \( a_2 \) and \( a_3 \) satisfy (i) or (ii) or (iii).
Next let us consider the ground state phase diagram. From (48), we obtain
\[ \alpha_2(e^{ip})^3 + \alpha_1(e^{ip})^2 - e^{ip} + \alpha_{-1} = Aa_1(e^{ip} + \frac{1}{a_1})(e^{ip} + a_2)(e^{ip} + a_3). \] (58)
Therefore, \(-1/a_1\), \(-a_2\) and \(-a_3\) are the roots of the cubic equation
\[ \alpha_2z^3 + \alpha_1z^2 - z + \alpha_{-1} = 0. \] (59)
Hence (i) - (iii) is satisfied if and only if

one root of (59) belongs to \(D_O\) and the other two roots belong to \(D_I\),

where \(D_O = \{ z \in \mathbb{C} \mid |z| > 1 \}\) and \(D_I = \{ z \in \mathbb{C} \mid |z| < 1 \}\). The equation (59) is nothing but the equation (38) (the same equation appears after (49)). Therefore the boundary of the region \(I_1(\infty) > 0\) satisfies the gapless condition, i.e. the curves corresponding to the gapless condition form the boundary of the ground state phase transition in terms of the correlation function (40). About the other regions bounded by the gapless curves, we consider correlation functions

\[ I_l(2n) = \left( \frac{2}{r} \right)^n \left\langle \prod_{k=1}^{n} \varphi_2(j + k - l)\varphi_1(j + k) \right\rangle_0, \quad (l = -1, 0, 2), \] (61)
namely
\[ I_{-1}(2n) = \left\langle \prod_{k=1}^{n} \sigma_2^x(j+k+1)-3\sigma_2^y(j+k+1)-2\sigma_2^y(j+k+1)\sigma_2^x(j+k+1+1) \right\rangle_0 \]
\[ = \left\langle \sigma_2^x_{2j+1}\sigma_2^y_{2j+2}\sigma_2^z_{2j+3} \cdot 12j+4 \cdot \cdots \cdot 12(j+n) \right\rangle_0, \] (62)
\[ I_0(2n) = \left\langle \prod_{k=1}^{n} (-\sigma_2^x_{2j+k-1}\sigma_2^z_{2j+k}) \sigma_2^x_{2j+k+1} \right\rangle_0 \]
\[ = (-1)^n \left\langle \sigma_2^x_{2j+1} \left( \sigma_2^x_{2j+1} \sigma_2^y_{2j+1} \sigma_2^z_{2j+1} \cdot \cdots \cdot \sigma_2^z_{2j+n} \sigma_2^y_{2j+n} \sigma_2^x_{2j+n+1} \right) \right\rangle_0 \] (63)
\[ I_2(2n) = \left\langle \prod_{k=1}^{n} (-\sigma_2^x_{2j+k-2}\sigma_2^y_{2j+k-2}+1\sigma_2^z_{2j+k-2}) \sigma_2^x_{2j+k-2} \sigma_2^y_{2j+k} \sigma_2^z_{2j+k} \right\rangle_0 \]
\[ = \left\langle \sigma_2^x_{2j-2}\sigma_2^y_{2j-1}\sigma_2^z_{2j} \right. \]
\[ \cdot \left( 12j+1 \sigma_2^z_{2j+1} \sigma_2^z_{2j+2} \sigma_2^z_{2j+3} \cdot \cdots \cdot 12(j+n-5)\sigma_2^z_{2j+n-5} \sigma_2^z_{2j+n-4} \sigma_2^y_{2j+n-4} \right) \]
\[ \cdot \left. \sigma_2^x_{2j+n-2} \sigma_2^y_{2j+n-1} \sigma_2^z_{2j+n} \right\rangle_0. \] (64)

Similarly we find that
\(a\) \(I_{-1}(\infty) > 0\) if and only if all the roots of (59) belong to \(D_O\),

\(b\) \(I_0(\infty) > 0\) if and only if

one root of (59) belongs to \(D_I\) and the other two roots belong to \(D_O\),

\(c\) \(I_2(\infty) > 0\) if and only if all the roots of (59) belong to \(D_I\).

The results are shown in Figure 1 and 2.
The term proportional to 1
From (72), we find that the conformal invariant normalization is
The term proportional to 1

5. Central charge

Let us consider the central charges of the theories on the critical lines (X1) - (X3). The points P, Q, P’ and Q’ in Figure 1 and 2 are the multicritical points and will be considered at the end of this section. Except for these points, first we consider (X1) in particular the case \( \alpha_1 = -\alpha_1 - (\alpha_2 - 1) \). Then

\[ L(q) = K_0 e^{-iq}(e^{iq} - 1)g(e^{iq}), \]
where \( g(z) = \alpha_2 z^2 + (1 - \alpha_1)z - \alpha_1, \) and \( g(e^{iq}) \neq 0 \).

Then, we obtain

\[ \Lambda(q) = 2|L(q)| = 4|K_0||\sin \frac{q}{2}||g(e^{iq})|. \]  

(68)

When \( \alpha_2 \neq 1 \), we obtain the dispersion relation \( \Lambda(q) \cong 4|K_0||p|C \), where \( C = |g(1)| \) and

\[ p = \frac{2k - 1}{N} - \frac{1}{2} \frac{2k - 1}{M} = \frac{q}{2}. \]

Thus the conformal invariant normalization of the Hamiltonian is obtained[28] from the condition

\[ 4|K_0|C = 1. \]  

(69)

The term proportional to \( 1/N \) in the \( 1/N \) expansion of the ground-state energy \( E_0 = -\sum_{0 < q < \pi} \Lambda(q) \) is obtained from

\[-4|K_0|C \sum_{0 < q < \pi} |\sin \frac{q}{2}| = -4|K_0|C \frac{1}{2\sin \frac{\pi}{N}} \]

\[ = -2|K_0|C \frac{N}{\pi} - 2|K_0|C \frac{\pi}{12N} + O(\frac{1}{N^3}), \]  

(70)

where we have used the relation

\[ \sum_{k=1}^{M/2} \left| \sin \frac{12k - 1}{M} \pi \right| = \frac{1}{2\sin \frac{\pi}{M}}. \]  

(71)

The second term \(-2|K_0|C\pi/6N\) is, from (69), equal to \(-\pi/12N\). From the condition[29][30] that this term should be equal to \(-c\pi/6N\), we obtain the central charge \( c = 1/2 \). On the other critical line \( \alpha_1 = -\alpha_1 + (\alpha_2 - 1) \), we similarly obtain \( c = 1/2 \).

When \( \alpha_2 = 1 \), we obtain

\[ L(q) = K_0 e^{-iq}(e^{2iq} - 1)(e^{iq} - \alpha_1). \]

We find two elementary excitations \( v_1|q| \) and \( v_2|q - \pi| \). If \( \alpha_1 \neq 0 \), then \( v_1 \neq v_2 \). If \( \alpha_1 = 0 \), then \( v_1 = v_2 \), and considering again the term proportional to \( 1/N \), we obtain \( c = 1 \).

Next, let us consider (X2) and (X3). The equation (59) has a real root \( B \) and two imaginary roots \( e^{iq_0} \) and \( e^{-iq_0} \), where \( |B| \neq 1 \) and \( 0 < q_0 < \pi \). Then we obtain

\[ L(q) = K_0 e^{-iq}(e^{iq} - \alpha_1)(e^{iq} - e^{iq_0})(e^{iq} - e^{-iq_0}) \]

\[ \Lambda(q) = 4|K_0\alpha_2|\sqrt{1 + B^2 - 2B\cos q} \cos q_0 - \cos q |. \]  

(72)

From (72), we find that the conformal invariant normalization is

\[ 8|K_0\alpha_2|\sqrt{1 + B^2 - 2B\cos q_0} \sin q_0 = 1. \]  

(73)

The term proportional to \( 1/N \) is obtained from

\[ -4|K_0\alpha_2|\sqrt{1 + B^2 - 2B\cos q_0} \frac{\sin q_0}{\sin \frac{\pi}{M}}. \]  

(74)
From (73) and (74), we find that $-\pi/6N$ should be equal to $-e\pi/6N$, and obtain the central charge $c = 1$.

At the point Q (at the point Q'), two linear elementary excitations $v_1|q|$ and $v_2|q - q_0|$ where $q_0 \neq 0, 2\pi/3$ ( $v_1|q - \pi|$ and $v_2|q - q_0|$ where $q_0 \neq \pi, \pi/3$ ) appear, but $v_1 \neq v_2$.

At the point P (at the point P'), then $q_0 = 0$ ($q_0 = \pi$), and we find excitations proportional to $|q|^2$ (proportional to $|q - \pi|^2$) when $\alpha_2 \neq 1, -1/3$, and find excitations proportional to $|q|^3$ and $|q - \pi|$ (proportional to $|q - \pi|^2$ and $|q|$) when $\alpha_2 = 1$, and an excitation proportional to $|q|^3$ (proportional to $|q - \pi|^3$) when $\alpha_2 = -1/3$.

### 6. Symmetry and Generalizations

Let us first consider the symmetry of the phase diagram. When $K_0 \neq 0$ and $\alpha_2 = 0$, the gapless condition is simplified and expressed as

\[
\begin{align*}
(X1) \quad &(-\alpha_1) = -(-\alpha_{-1}) \pm 1, \\
(X2) \quad &(-\alpha_{-1}) \neq 0, \quad (-\alpha_1) = (-\alpha_{-1}), \quad 1 - 4(-\alpha_{-1})^2 < 0, \\
(X3) \quad &\text{cannot be satisfied.} \quad (75)
\end{align*}
\]

When $K_1 \neq 0$ and $\alpha_{-1} = 0$, let us consider the variables $\beta_{-1} = K_1/K_0$, $\beta_0 = K_0/K_1$, and $\beta_2 = K_2/K_1$, i.e. the normalization by $K_1$ instead of $K_0$. In this case the condition is expressed as

\[
\begin{align*}
(X1) \quad &\beta_2 = -(-\beta_0) \pm 1, \\
(X2) \quad &\text{cannot be satisfied.} \\
(X3) \quad &\beta_{-1} = 0, \quad \beta_2 = (-\beta_0), \quad (-\beta_0) < -1/2 \text{ or } 1/2 < (-\beta_0). \quad (76)
\end{align*}
\]

From (21), we find the natural coupling constants of this model are $K_{-1}$, $-K_0$, $K_1$, $K_2$. Thus when we consider the shift of indices of $K_j$ from $j$ to $j + 1$, the correspondences of $\alpha_j$ and $\beta_j$ are

\[
(-\alpha_{-1}) = K_{-1}/K_0 \mapsto -K_0/K_1 = (-\beta_0), \quad (-\alpha_1) = K_1/K_0 \mapsto K_2/K_1 = \beta_2. \quad (77)
\]

When we replace $\alpha_j$ with corresponding $\beta_j$ in (75), we find that the condition (75) becomes (76). In both cases, the phase diagrams are also identical to that of the XY chain with an external field.

Next we will consider a generalization of the results obtained in Section 2-5. We solve the model (1) using only the algebraic relations of the operators (9). Hence the model (12) generated from the operators which satisfy (9) can be simultaneously diagonalized, and its string order parameter (40) yields the same phase diagram as shown in Figure 1 and 2. For example, from the operators (37), the model

\[
K_0 \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x + K_1 \sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^y + K_2 \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z \sigma_{j+2}^z \sigma_{j+3}^z + K_{-1} \sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^y \sigma_{j+2}^y \sigma_{j+3}^y \quad (78)
\]

is generated from (12), its string order parameter (40) in this case is

\[
I_1(n) = \langle \prod_{k=1}^{n} (\sigma_{2(j+k)}^y - 1\sigma_{2(j+k)}^y) \rangle_0 = \langle \prod_{k=1}^{2n} \sigma_{2j+k}^y \rangle_0, \quad (79)
\]

and this model results in the free energy (36) and the phase diagram shown in Figure 1 and 2.
We introduce, in this paper, two series of operators \( \{ \eta_j \} \) and \( \{ \zeta_j \} \), which commute with each other. As a result, our model factorizes into two commutative spin chains as discussed in [24] and [31]. One series of operators, however, can yield the free energy (36) and the phase diagram shown in Figure 1 and 2. For example, the series of operators,

\[
\eta_{2j-1}^{(3)} = \sigma_j^z, \quad \eta_{2j}^{(3)} = \sigma_j^x \sigma_{j+1}^x
\]

yields the Hamiltonian

\[
-\beta \mathcal{H} = K_0 \sum_{j=1}^{N} \sigma_j^z + K_1 \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x + K_2 \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z \sigma_{j+2}^z + K_3 \sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^y.
\]

This Hamiltonian is composed of the XY interactions, an external field and the cluster interactions. The string order parameter (40) in this case is

\[
I_1(n) = \langle \prod_{k=1}^{n} (\sigma_{j+k-1}^x \sigma_{j+k}^x) \rangle_0 = \langle \sigma_j^x \sigma_{j+n}^x \rangle_0,
\]

and this model also results in (36) and the diagram in Figure 1 and 2. In case of both (37) and (80), the transformation (4) results in the Jordan-Wigner transformation, and in case of (2) and (3), the transformation becomes (13) - (16).

Generally, an infinite number of operators that satisfy the condition (9), including (37) and (80), are given in Table 1 and 2 in [24]. Our present results are universally valid for these infinite number of solvable spin chains.

7. Conclusion

In this paper, we obtain the exact solution of the model (1), which is a cluster model that considers next-nearest-neighbor interactions and two additional composite interactions. We introduce the series of operators (2) and (3), which satisfy the algebraic relation (9). Then, we introduce the transformations (4), and the model (1) is diagonalized. We derive the free energy (36), consider a correlation function (40), and derive its exact expressions (41) and (42). We obtained the ground-state phase diagram, as shown in Figure 1 and 2, from the gapless condition, and from the asymptotic behavior of the correlation function, which is classified by the location of the roots of an algebraic equation (59). We also derive the central charges of the corresponding CFT. The exact solution is obtained using only the algebraic relation (9) of the interactions. Finally, we note that an infinite number of solvable models, generated from operators that satisfy (9), yield exactly the same results obtained in this paper. Our transformation can be regarded as an algebraic generalization of the Jordan-Wigner transformation. This paper provides a nontrivial example that cannot be solved by the Jordan-Wigner transformation but can be solved by our method.

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A. The gapless condition

Here we show the derivation of the condition that the equation (38) has a root which belongs to \( C_u \) if and only if (X1) or (X2) or (X3) is satisfied.
First, if the equation (38) has a root 1 or -1, then (X1) is satisfied. Next, we assume that the equation (38) has imaginary roots which belong to \( \mathbb{C}_u \). When \( \alpha_2 \neq 0 \) and \( \alpha_{-1} \neq 0 \), let \( c, \bar{c} \) and \( B \) be the roots of the equation (38), where \( c, \bar{c} \) are imaginary and \( B \) is real. Then from the cubic equation (38), we obtain
\[
c + \bar{c} + B = -\frac{\alpha_1}{\alpha_2}, \quad c\bar{c}B = -\frac{\alpha_{-1}}{\alpha_2}.
\] (A1)
Thus \( c \) and \( \bar{c} \) are the imaginary roots of the quadratic equation
\[
z^2 + \left( \frac{\alpha_1}{\alpha_2} + B \right)z - \frac{\alpha_{-1}}{B\alpha_2} = 0.
\] (A2)
From (A2), we obtain
\[
(\frac{\alpha_1}{\alpha_2} + B)^2 + \frac{4\alpha_{-1}}{B\alpha_2} < 0 \quad \text{(A3)}
\]
and
\[
|c| = |\bar{c}| = \sqrt{-\frac{\alpha_{-1}}{B\alpha_2}}.
\] (A4)
From \(|c| = |\bar{c}| = 1\) and (A4), we obtain
\[
B = -\frac{\alpha_{-1}}{\alpha_2}. \quad \text{(A5)}
\]
By inserting (A5) to the equation (38), we obtain
\[
\alpha_1 = \alpha_{-1} - \frac{\alpha_2(1 + \alpha_2)}{\alpha_{-1}}. \quad \text{(A6)}
\]
On the other hand, From (A3), (A5) and (A6), we find
\[
(1 + \alpha_2)^2 - 4\alpha_{-1}^2 < 0. \quad \text{(A7)}
\]
From (A6) and (A7) we obtain (X2).

When \( \alpha_2 \neq 0 \) and \( \alpha_{-1} = 0 \), the equation (38) becomes \( z(\alpha_2z^2 + \alpha_1z - 1) = 0 \). Thus the imaginary roots of (38) are
\[
z = -\frac{\alpha_1 \pm i\sqrt{-4\alpha_2 - \alpha_1^2}}{2\alpha_2}, \quad \text{(A8)}
\]
where \(-4\alpha_2 - \alpha_1^2 > 0\) is necessary. From (A8), we find \(|z| = \sqrt{-\frac{\alpha_1^2}{\alpha_2}}\), and the condition (X3) is obtained.

When \( \alpha_2 = 0 \), the equation (38) becomes a quadratic equation \( \alpha_1z^2 - z + \alpha_{-1} = 0 \). From this equation, we obtain the condition that \( \alpha_1 = \alpha_{-1} \) and \( 1 - 4\alpha_{-1}^2 < 0 \), which is contained in (X2). Consequently, (X1) or (X2) or (X3) is a necessary condition.

In contradiction, when (X1) or (X3) is satisfied, it is easy to show that the equation (38) has a root which belongs to \( \mathbb{C}_u \). When (X2) is satisfied, the equation (38) becomes \((\alpha_2z + \alpha_{-1})(\alpha_{-1}z^2 - (\alpha_2 + 1)z + \alpha_{-1}) = 0\). Thus (38) has imaginary roots
\[
z = \frac{(1 + \alpha_2) \pm i\sqrt{4\alpha_{-1}^2 - (\alpha_2 + 1)^2}}{2\alpha_{-1}}, \quad \text{(A9)}
\]
from which we find \(|z| = 1\). Therefore, (X1) or (X2) or (X3) is sufficient.
B. Correlation functions

Here we derive the expression (41) and (42) of the ground-state correlation function defined in (40). The Hamiltonian is diagonalized by the canonical transformation

$$\xi_k(q) = \frac{1}{\sqrt{2A(q)}} (-iL(q)C_k(q) + \sqrt{A(q)}C_{k+1}(q)) \quad (k = 1, 3),$$

$$\xi_k(q) = \frac{1}{\sqrt{2A(q)}} (-iL(q)C_{k-1}(q) - \sqrt{A(q)}C_k(q)) \quad (k = 2, 4)$$

as

$$-\beta \mathcal{H} = \sum_{0 < q < \pi} A(q) \{ \xi_1(q)\xi_1(q) - \xi_2(q)\xi_2(q) + \xi_3(q)\xi_3(q) - \xi_4(q)\xi_4(q) \}. \quad (B2)$$

From (B1), We obtain

$$C^j_2(q)C_1(q) = \frac{i}{2L(q)} \frac{\sqrt{A(q)}}{\sqrt{L(q)}} (\xi^j_1(q)\xi_1(q) + \xi^j_2(q)\xi_2(q) - \xi^j_3(q)\xi_3(q) - \xi^j_4(q)\xi_4(q)). \quad (B3)$$

On the other hand,

$$\sigma^x_{2j}\sigma^x_{2j+2n} = \prod_{k=1}^n \sigma^x_{2(j+k)-2}\sigma^x_{2(j+k)} - 1 \sigma^x_{2(j+k)}$$

and from (19) $\sigma^x_{2j}\sigma^x_{2j+2} = \frac{1}{2}\varphi_2(j)\varphi_1(j + 1)$, therefore

$$I_1(2n) = \left(\frac{2}{i}\right)^n \left\langle \prod_{k=1}^n \varphi_2(j+k-1)\varphi_1(j+k) \right\rangle_0.$$ \quad (B5)

Using the Wick’s theorem, $I(2n)$ is expressed as

$$I_1(2n) = \left(\frac{2}{i}\right)^n \sum_P \text{sgn}(P) \prod_{k=1}^n \langle \varphi_2(j+k-1)\varphi_1(P(j+k+1)) \rangle_0, \quad (B6)$$

where $P$ are the permutations of the indices. The sum (B6) is the determinant

$$\left(\frac{2}{i}\right)^n \begin{vmatrix} G_{j+1,j+1} & G_{j+2,j+1} & \cdots & G_{j,n+1} \\ G_{j+1,j+2} & G_{j+1,j+2} & \cdots & G_{j+1,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{j+1,n-1,n+1} & G_{j+1,n-1,n+2} & \cdots & G_{j+1,n-1,n+1} \end{vmatrix}, \quad (B7)$$

where $G_{l,m} = \langle \varphi_2(l)\varphi_1(m) \rangle_0$. Because of the translational invariance, (B7) is expressed as (41). From (B3) and the fact that the ground-state correlation functions satisfy $\langle \xi_1^j(q)\xi_1(q) \rangle_0 = 1$, $\langle \xi_2^j(q)\xi_2(q) \rangle_0 = \langle \xi_3^j(q)\xi_3(q) \rangle_0 = \langle \xi_4^j(q)\xi_4(q) \rangle_0 = 0$, we obtain

$$G_{j,j+r} = \frac{1}{M} \sum_{0 < q < \pi} [e^{iqr} G^j_2(q)G_1(q)0 + e^{-iqr} G_2(q)G_1^j(q)0]$$

$$= \frac{1}{M} \frac{i}{2} \sum_{0 < q < \pi} \left( e^{iqr} \frac{\sqrt{A(q)}}{L(q)} + e^{-iqr} \frac{\sqrt{A(q)}}{L(q)^*} \right). \quad (B8)$$

In the thermodynamic limit, (B8) results in (42).

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