SCHRÖDINGER OPERATORS WITH STRONGLY SINGULAR POTENTIALS AND DE BRANGES SPACES

JONATHAN ECKHARDT

ABSTRACT. We utilize the theory of de Branges spaces to show that certain Schrödinger operators with strongly singular potentials are uniquely determined by their associated spectral measure. The results are applied in order to obtain a uniqueness theorem for perturbed Bessel operators.

1. INTRODUCTION

We consider Schrödinger operators $H$ (with separate boundary conditions), induced by the differential expression

$$\tau = -\frac{d^2}{dx^2} + q(x)$$

on some interval $(a, b)$. Here $q$ is a real-valued, locally integrable function on $(a, b)$ referred to as the potential. In [6] (see also [4], [7]), Kostenko, Sakhnovich and Teschl developed a Weyl–Titchmarsh theory for $H$ under the sole hypothesis that there is some nontrivial real entire solution $\phi$ of

$$-\phi''(z, x) + q(x)\phi(z, x) = z\phi(z, x), \quad x \in (a, b), \quad z \in \mathbb{C},$$

which lies in $L^2((a, b))$ near $a$ and satisfies the boundary condition at $a$ if $\tau$ is in the limit-circle case at $a$. Here, by real entire solution we mean that $\phi(\cdot, c)$ and $\phi'(\cdot, c)$ are real entire functions for one (and hence for all) $c \in (a, b)$. For such a solution to exist it is necessary and sufficient that for some $c \in (a, b)$, $H_{(a, c)}$ has purely discrete spectrum (see [6, Lemma 2.2]), where $H_{(a, c)}$ is the restriction of $H$ to $L^2((a, c))$ with Dirichlet boundary conditions at $c$.

In particular they were able to prove a local Borg–Marchenko uniqueness result for their singular Weyl $m$-function under restrictions on the exponential growth of the solution $\phi$. Their proof follows the simple proof of Bennewitz [2] which covers the case of regular left endpoints. However, since the spectral measure determines the singular Weyl $m$-function only up to some real entire function, their Borg–Marchenko theorem does not immediately yield a uniqueness result for the spectral measure. In fact, all one would need is some growth restriction on the difference of two singular $m$-functions with the same spectral measure. This will be done in [3] in the case when the spectrum of the operators is assumed to be purely discrete and has finite convergence exponent.

The present paper uses a completely different approach. We utilize de Branges’ theory of Hilbert spaces of entire functions in order to obtain a uniqueness result for the spectral measure. In particular we use de Branges’ ordering theorem to
conclude that the de Branges spaces associated to Schrödinger operators with the same spectral measure are equal.

In Section 2 we start with a brief review of the theory of de Branges spaces. For a detailed discussion we refer to de Branges’ book [1]. The following section introduces the de Branges spaces associated with a self-adjoint Schrödinger operator as above. The core of this section is quite similar to [10, Section 3] (see also [11]) with the sole difference that we do not assume the left endpoint to be regular. Section 4 is devoted to our uniqueness result for the spectral measure. Finally, in the last section we apply our result to perturbed Bessel operators.

2. de Branges spaces

An analytic function $N$ in the upper complex half-plane $\mathbb{C}^+$ is said to be of bounded type if it can be written as the quotient of two bounded analytic functions. For such a function the number

$$\limsup_{y \to \infty} \frac{\ln|N(iy)|}{y} \in [-\infty, \infty),$$

is referred to as the mean type of $N$.

A de Branges function is an entire function $E$, which satisfies the estimate

$$|E(z)| > |E(z^*)|, \quad z \in \mathbb{C}^+.$$ 

The de Branges space $B$ associated with such a function consists of all entire functions $F$ such that

$$\int_{\mathbb{R}} \frac{|F(\lambda)|^2}{E(\lambda)} d\lambda < \infty,$$

and such that $F/E$ and $F^\# / E$ are of bounded type in $\mathbb{C}^+$ with nonpositive mean type. Here $F^\#$ is the entire function defined by

$$F^\#(z) = F(z^*)^*, \quad z \in \mathbb{C}.$$ 

Equipped with the inner product

$$[F, G] = \frac{1}{\pi} \int_{\mathbb{R}} \frac{F(\lambda)G(\lambda)^*}{|E(\lambda)|^2} d\lambda, \quad F, G \in B,$$

the vector space $B$ turns into a Hilbert space (see [1, Theorem 21]). For each $\zeta \in \mathbb{C}$, the point evaluation in $\zeta$ is a continuous linear functional on $B$, i.e.

$$F(\zeta) = [F, K(\zeta, \cdot)], \quad F \in B,$$

where the reproducing kernel $K$ is given by (see [1, Theorem 19])

(2.1) $$K(\zeta, z) = \frac{E(z)E^\#(\zeta^*) - E(\zeta^*)E^\#(z)}{2i(\zeta^* - z)}, \quad \zeta, z \in \mathbb{C}.$$ 

Note that though there is a multitude of de Branges functions giving rise to the same de Branges space (including norms), the reproducing kernel $K$ is independent of the actual de Branges function.

Our uniqueness result relies on the ordering theorem of de Branges [1, Theorem 35]. In order to state it let $E_1, E_2$ be two de Branges functions and $B_1, B_2$ the corresponding de Branges spaces.
Theorem 2.1. Suppose $B_1$, $B_2$ are isometrically embedded in $L^2(\mathbb{R}; \rho)$, for some Borel measure $\rho$. If $E_1/E_2$ is of bounded type in the upper complex half-plane and has no real zeros or singularities, then $B_1$ contains $B_2$ or $B_2$ contains $B_1$.

Moreover, we will need the following converse statement.

Lemma 2.2. If $B_1$ is contained in $B_2$ or $B_2$ is contained in $B_1$, then $E_1/E_2$ is of bounded type in the upper complex half-plane.

Proof. Without loss of generality assume $B_1$ is contained in $B_2$. As a consequence of [1, Theorem 25], there are entire functions $F_1$, $G_1 \in B_1 \subseteq B_2$ such that 

$$E_1(z) = F_1(z) + zG_1(z), \quad z \in \mathbb{C},$$

Now the function

$$\frac{E_1(z)}{E_2(z)} = \frac{F_1(z)}{E_2(z)} + z \frac{G_1(z)}{E_2(z)}, \quad z \in \mathbb{C}^+,$$

is of bounded type since both summands are by the definition of $B_2$. □

3. Schrödinger Operators and de Branges Spaces

In this section let $(a, b)$ be some bounded or unbounded interval, $q$ a real-valued, locally integrable function on $(a, b)$ and $\tau$ the differential expression

$$\tau = -\frac{d^2}{dx^2} + q(x),$$

on $(a, b)$. By $H$ we denote some associated self-adjoint Schrödinger operator in $L^2(a, b)$ with separate boundary conditions if $\tau$ is in the limit-circle case at both endpoints. Concerning the regularity of the potential $q$ near the endpoint $a$, we will only assume that there is some real entire solution $\phi$ of

$$-\phi''(z, x) + q(x)\phi(z, x) = z\phi(z, x), \quad x \in (a, b), z \in \mathbb{C},$$

such that for each $z \in \mathbb{C}$, $\phi(z, \cdot)$ is nontrivial, lies in $L^2(a, b)$ near $a$ and satisfies the boundary condition at $a$ if $\tau$ is in the limit-circle case at $a$. Here by real entire we mean that for some (and hence for all) $c \in (a, b)$, $\phi(\cdot, c)$ and $\phi'(\cdot, c)$ are real entire functions. For the proof of our uniqueness result we will need the following simple lemma on the asymptotics of the solution $\phi$. Note that we always use the principal square root with branch cut along the negative real axis.

Lemma 3.1. For each $x, \tilde{x} \in (a, b)$ we have the asymptotics

$$\frac{\phi(z, x)}{\phi(z, \tilde{x})} = e^{(x-\tilde{x})\sqrt{-z}}(1 + o(1)),$$

as $|z| \to \infty$ along the imaginary axis.

Proof. For each $z \in \mathbb{C}$ let $c(z, \cdot)$ and $s(z, \cdot)$ be the solutions of $(\tau - z)u = 0$ with the initial conditions

$$c(z, \tilde{x}) = s'(z, \tilde{x}) = 1 \quad \text{and} \quad c'(z, \tilde{x}) = s(z, \tilde{x}) = 0.$$

Now if $x \geq \tilde{x}$ the claim follows from

$$\phi(z, x) = \phi(z, \tilde{x}) \left( c(z, x) + \frac{\phi'(z, \tilde{x})}{\phi(z, \tilde{x})} s(z, x) \right), \quad z \in \mathbb{C}\setminus\mathbb{R},$$
and the well-known asymptotics of the quotient on the right-hand side (see \cite{13} Lemma 9.19) and the solutions \( c \) and \( s \) (see \cite{13} Lemma 9.18). The case when \( x < \tilde{x} \) follows by reversing the roles of \( x \) and \( \tilde{x} \).

For each \( c \in (a, b) \) we denote by \( L^2(a, c) \) the closed linear subspace of \( L^2(a, b) \) consisting of all functions which vanish outside of \((a, c)\). Now as in the case of regular left endpoints, one may define the transform of a function \( f \in L^2(a, c) \) as

\[
\hat{f}(z) = \int_a^b \phi(z, x) f(x) dx, \quad z \in \mathbb{C}.
\]

It is a result of \cite{6} Section 3] that there is some Borel measure \( \rho \) on \( \mathbb{R} \) such that

\[
\int_\mathbb{R} |\hat{f}(\lambda)|^2 d\rho(\lambda) = \int_a^b |f(x)|^2 dx, \quad f \in L^2(a, c),
\]

for all values \( c \in (a, b) \). Moreover, this transformation uniquely extends to a unitary map from \( L^2(a, b) \) onto \( L^2(\mathbb{R}; \rho) \) and the operator \( H \) is mapped onto multiplication with the independent variable in \( L^2(\mathbb{R}; \rho) \). Note that \( \rho \) is uniquely determined by these properties and hence referred to as the spectral measure of \( H \) associated with the solution \( \phi \).

From these results one sees that the transforms of all functions \( f \in L^2(a, c) \), equipped with the norm inherited from the space \( L^2(\mathbb{R}; \rho) \), form a Hilbert space. In order to show that they even form a de Branges space, fix some \( c \in (a, b) \) and consider the entire function

\[
E(z, c) = \phi(z, c) + i\phi'(z, c), \quad z \in \mathbb{C}.
\]

Using the Lagrange identity and the fact that the Wronskian of two solutions satisfying the same boundary condition at \( a \) (if any) vanishes in \( a \), one gets

\[
\frac{E(z, c)E^\#(\zeta^*, c) - E(\zeta^*, c)E^\#(z, c)}{2i(\zeta^* - z)} = \int_a^c \phi(\zeta, x)^* \phi(z, x) dx, \quad \zeta, \ z \in \mathbb{C}^+.
\]

In particular, taking \( \zeta = z \) this shows that \( E(\cdot, c) \) is a de Branges function. Moreover, note that \( E(\cdot, c) \) does not have any real zero \( \lambda \), since otherwise both, \( \phi(\lambda, c) \) and \( \phi'(\lambda, c) \) would vanish. By \( B(c) \) we denote the de Branges space associated with the de Branges function \( E(\cdot, c) \), endowed with the inner product

\[
[F, G]_{B(c)} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{F(\lambda)G(\lambda)^*}{|E(\lambda, c)|^2} d\lambda = \frac{1}{\pi} \int_{\mathbb{R}} \frac{F(\lambda)G(\lambda)^*}{\phi(\lambda, c)^2 + \phi'(\lambda, c)^2} d\lambda, \quad F, G \in B(c).
\]

Now using \( (3.1) \) and a similar calculation as above, one shows that the reproducing kernel \( K(\cdot, \cdot, c) \) of this space is given by

\[
K(\zeta, z, c) = \int_a^c \phi(\zeta, x)^* \phi(z, x) dx, \quad \zeta, \ z \in \mathbb{C}.
\]

**Theorem 3.2.** For each \( c \in (a, b) \) the transformation \( f \mapsto \hat{f} \) is unitary from \( L^2(a, c) \) onto \( B(c) \), in particular

\[
B(c) = \left\{ \hat{f} \mid f \in L^2(a, c) \right\}.
\]

**Proof.** Fix some \( \lambda \in \mathbb{R} \) and consider the function

\[
f_\lambda(x) = \phi(\lambda, x) \mathbb{1}_{(a, c)}(x), \quad x \in (a, b).
\]
The transform of this function is given by
\[
\hat{f}_\lambda(z) = \int_a^c \phi(\lambda, x) \phi(z, x) \, dx = K(\lambda, z, c), \quad z \in \mathbb{C}.
\]
In particular this shows that the transform lies in \( B(c) \). Moreover, we have
\[
\|f_\lambda\|^2 = \int_a^c |\phi(\lambda, x)|^2 \, dx = K(\lambda, \lambda, c) = [K(\lambda, \cdot, c), K(\lambda, \cdot, c)]_{B(c)} = \|K(\lambda, \cdot, c)\|_{B(c)}^2.
\]
Hence our transform is an isometry on the linear span \( D \) of the functions \( f_\lambda, \lambda \in \mathbb{R} \).
But this span is dense in \( L^2(a, c) \) since it contains the eigenfunctions of the operator \( H_{(a, c)} \). Moreover, the linear span of the functions \( K(\lambda, \cdot, c), \lambda \in \mathbb{R} \) is dense in \( B(c) \).
Indeed, each \( F \in B(c) \) such that
\[
0 = [F, K(\lambda, \cdot, c)]_{B(c)} = F(\lambda), \quad \lambda \in \mathbb{R},
\]
vanishes identically. Thus our transformation restricted to \( D \) extends uniquely to a unitary map \( V \) from \( L^2(a, c) \) onto \( B(c) \). In order to identify \( V \) with our transformation note that for each fixed \( z \in \mathbb{C} \), both \( f \mapsto f(z) \) and \( f \mapsto V f(z) \) are continuous on \( L^2(a, c) \).

As an immediate consequence of Theorem 3.2 and the fact that our transformation from (3.1) extends to a unitary map from \( L^2(a, c) \) onto \( L^2(\mathbb{R}; \rho) \), we get the following corollary.

**Corollary 3.3.** For each \( c \in (a, b) \) the de Branges space \( B(c) \) is isometrically embedded in \( L^2(\mathbb{R}; \rho) \), that is
\[
\int_{\mathbb{R}} |F(\lambda)|^2 \, d\rho(\lambda) = \|F\|_{B(c)}^2, \quad F \in B(c).
\]
Moreover, the linear span of the spaces \( B(c), c \in (a, b) \) is dense in \( L^2(\mathbb{R}; \rho) \), i.e.
\[
(3.5) \quad \bigcup_{c \in (a, b)} B(c) = L^2(\mathbb{R}; \rho).
\]

The following corollary shows that the de Branges spaces \( B(c), c \in (a, b) \) are totally ordered, strictly increasing and continuous in some sense.

**Corollary 3.4.** If \( c_1, c_2 \in (a, b) \) with \( c_1 < c_2 \), then \( B(c_1) \) is isometrically embedded in but not equal to \( B(c_2) \). Moreover, for each \( c \in (a, b) \) we have
\[
(3.6) \quad \bigcup_{x \in (a, c)} B(x) = B(c) = \bigcap_{x \in (c, b)} B(x).
\]

**Proof.** The embedding is clear from Theorem 3.2 and Corollary 3.3. Moreover, Theorem 3.2 shows that \( B(c_2) \oplus B(c_1) \) is unitarily equivalent to \( L^2(c_1, c_2) \), hence \( B(c_1) \) is not equal to \( B(c_2) \). The second claim follows from the similar fact that
\[
\bigcup_{x \in (a, c)} L^2(a, x) = L^2(a, c) = \bigcap_{x \in (c, b)} L^2(a, x).
\]

Note that the solution \( \phi \) is not uniquely determined. Indeed, [3, Corollary 2.3] shows that any other solution with the same properties as \( \phi \) is given by
\[
\tilde{\phi}(z, x) = e^{s(z)} \phi(z, x), \quad x \in (a, b), \quad z \in \mathbb{C},
\]
where \( g \) is some real entire function. Furthermore [6] Remark 3.8] shows that the corresponding spectral measures are related by

\[
\hat{\rho} = e^{-2g} \rho.
\]

In particular they are mutually absolutely continuous. Using Theorem 3.2 it is easily seen that for each \( c \in (a, b) \), multiplication with the entire function \( e^g \) maps \( B(c) \) isometrically onto \( \tilde{B}(c) \).

4. A uniqueness result for the spectral measure

In this section let \( q_1, q_2 \) be two real-valued, locally integrable functions on intervals \((a_1, b_1)\) respectively \((a_2, b_2)\) and \( H_1, H_2 \) two associated self-adjoint Schrödinger operators with separate boundary conditions. Suppose there are nontrivial real entire solutions \( \phi_1, \phi_2 \) which are square integrable near the left endpoint and satisfy the boundary condition there, if any. As in the previous section we denote with \( \rho_1, \rho_2 \) the corresponding spectral measures, with \( E_1, E_2 \) the corresponding de Branges functions, with \( B_1, B_2 \) the corresponding de Branges spaces and with \( K_1, K_2 \) the corresponding reproducing kernels.

We say \( H_1 \) and \( H_2 \) are equal up to some shift if there is a linear function \( \eta \) with gradient one, mapping \((a_1, b_1)\) onto \((a_2, b_2)\) such that \( q_1 = q_2 \circ \eta \) and \( H_1 = U^{-1} H_2 U \), where \( U \) is the unitary map from \( L^2(a_1, b_1) \) onto \( L^2(a_2, b_2) \) induced by \( \eta \).

**Theorem 4.1.** Suppose there is some real entire function \( g \) such that

\[
e^{g(z)} \frac{E_1(z, x_1)}{E_2(z, x_2)} \quad z \in \mathbb{C}^+,
\]

is of bounded type for some \( x_1 \in (a_1, b_1) \) and \( x_2 \in (a_2, b_2) \). If \( \rho_1 = e^{-2g} \rho_2 \) then \( H_1 \) and \( H_2 \) are equal up to some shift.

**Proof.** First of all note that without loss of generality we may assume that \( g \) vanishes identically, since otherwise we replace \( \phi_1 \) with \( e^g \phi_1 \). Moreover, because of Lemma 2.2 the function in (4.1) is of bounded type for all \( x_1 \in (a_1, b_1) \) and \( x_2 \in (a_2, b_2) \). Now fix some arbitrary \( x_1 \in (a_1, b_1) \). Since for each \( x_2 \in (a_2, b_2) \), both \( B_1(x_1) \) and \( B_2(x_2) \) are isometrically contained in \( L^2(\mathbb{R}; \rho_1) \) we infer from Theorem 2.1 (note that (4.1) has no real zeros or singularities because \( E_1(\cdot, x_1) \) and \( E_2(\cdot, x_2) \) do not have real zeros) that \( B_1(x_1) \) is contained in \( B_2(x_2) \) or \( B_2(x_2) \) is contained in \( B_1(x_1) \). We claim that the infimum \( \eta(x_1) \) of all \( x_2 \in (a_2, b_2) \) such that \( B_1(x_1) \subseteq B_2(x_2) \) lies in \( (a_2, b_2) \). Indeed otherwise we either had \( B_2(x_2) \not\subseteq B_1(x_1) \) for all \( x_2 \in (a_2, b_2) \) or \( B_1(x_1) \subseteq B_2(x_2) \) for all \( x_2 \in (a_2, b_2) \). In the first case this would mean that \( B_1(x_1) \) is dense in \( L^2(\mathbb{R}; \rho) \), which is not possible in view of Corollary 3.4. The second case would imply that for every function \( F \in B_1(x_1) \) and \( \zeta \in \mathbb{C} \) we have

\[
|F(\zeta)| \leq \|F \|_{B_2(x_2)} \|K_2(\zeta, \cdot, x_2)\|_{B_2(x_2)}
\]

\[
\leq \|F \|_{B_2(x_2)} \|K_2(\zeta, \cdot, x_2)\|_{B_2(x_2)}
\]

\[
= \|F \|_{B_1(x_1)} \|K_2(\zeta, \cdot, x_2)\|_{B_2(x_2)}
\]

for each \( x_2 \in (a_2, b_2) \). Since \( K_2(\zeta, \cdot, x_2) \to 0 \) as \( x_2 \downarrow a_2 \) by (3.1), we had \( B_1(x_1) = \{0\} \) contradicting Theorem 3.2. Now from (3.4) we infer that

\[
B_2(\eta(x_1)) = \bigcup_{x_2 < \eta(x_1)} B_2(x_2) \subseteq B_1(x_1) \subseteq \bigcap_{x_2 > \eta(x_1)} B_2(x_2) = B_2(\eta(x_1)),
\]
hence $B_1(x_1) = B_2(\eta(x_1))$, including norms.

The function $\eta : (a_1, b_1) \to (a_2, b_2)$ defined this way is strictly increasing because of Corollary 3.4 and continuous by (3.6). Moreover, since for each $\zeta \in \mathbb{C}$

$$K_2(\zeta, \zeta, \eta(x_1)) = K_1(\zeta, \zeta, x_1) \to 0,$$

as $x_1 \downarrow a$ we infer that $\eta(x_1) \downarrow a_2$ as $x_1 \downarrow a_1$. Finally (3.5) shows that $\eta$ actually has to be a bijection. Using the formula for the reproducing kernels (3.4) once more we get for each $z \in \mathbb{C}$

$$\int_{a_1}^{\infty} |\phi_1(z, x)|^2 dx = \int_{a_2}^{\infty} |\phi_2(z, x)|^2 dx, \quad x_1 \in (a_1, b_1).$$

Now by the implicit function theorem $\eta$ is differentiable (note that the integrand does not vanish if $z \in \mathbb{C}\setminus\mathbb{R}$) with

$$|\phi_1(z, x_1)|^2 = \eta'(x_1)|\phi_2(z, \eta(x_1))|^2, \quad x_1 \in (a_1, b_1).$$

Using Lemma 3.1 twice we get for all $x_1, \tilde{x}_1 \in (a_1, b_1)$ the asymptotics

$$e^{-2(x_1-\tilde{x}_1)\Re\sqrt{\eta}} (1 + o(1)) = \left| \frac{\phi_1(z, x_1)}{\phi_1(z, \tilde{x}_1)} \right|^2 = \left| \frac{\eta'(x_1)}{\eta'(\tilde{x}_1)} \frac{\phi_2(z, \eta(x_1))}{\phi_2(z, \eta(\tilde{x}_1))} \right|^2 = \frac{\eta'(x_1)}{\eta'(\tilde{x}_1)} e^{-2(\eta(x_1)-\eta(\tilde{x}_1))\Re\sqrt{\eta}} (1 + o(1)),$$

as $|z| \to \infty$ along the imaginary axis. Now this shows

$$\eta(x_1) - \eta(\tilde{x}_1) = x_1 - \tilde{x}_1, \quad x_1, \tilde{x}_1 \in (a_1, b_1),$$

i.e. $\eta$ is linear with gradient one.

Using (4.2) once more, we get for each $\zeta \in \mathbb{R}$

$$\phi_1(\lambda, x_1)^2 = \phi_2(\lambda, \eta(x_1))^2, \quad x_1 \in (a_1, b_1).$$

Taking logarithmic derivatives we obtain

$$\frac{\phi_1'(\lambda, x_1)}{\phi_1(\lambda, x_1)} = \frac{\phi_2'(\lambda, \eta(x_1))}{\phi_2(\lambda, \eta(x_1))},$$

for almost all $x_1 \in (a_1, b_1)$. Differentiating this equation once more, we get

$$\frac{\phi_1''(\lambda, x_1)}{\phi_1(\lambda, x_1)} = \frac{\phi_2''(\lambda, \eta(x_1))}{\phi_2(\lambda, \eta(x_1))},$$

for almost all $x_1 \in (a_1, b_1)$. Thus also

$$q_1(x_1) = \lambda + \frac{\phi_1''(\lambda, x_1)}{\phi_1(\lambda, x_1)} = \lambda + \frac{\phi_2''(\lambda, \eta(x_1))}{\phi_2(\lambda, \eta(x_1))} = q_2(\eta(x_1))$$

for almost all $x_1 \in (a_1, b_1)$. Finally note that (4.3) implies that $\phi_1$ and $\phi_2 \circ \eta$ are linearly dependent. In particular if $\tau_1$ (and hence also $\tau_2$) is in the l.c. case at the left endpoint this shows that the boundary condition of $H_1$ and $H_2$ there is the same. Furthermore, if $\tau_1$ (and hence also $\tau_2$) is in the l.c. case at the right endpoint, then $H_1$ and $H_2$ have some common eigenvalue $\lambda$. Now the fact that $\phi_1(\lambda, \cdot)$ and $\phi_2(\lambda, \cdot)$ satisfy the respective boundary condition at the right endpoint shows that $H_1$ is equal to $H_2$ up to some shift. \hfill $\square$
Note that even if one fixes the left endpoint, the operator is determined by the spectral measure in general only up to some shift. This is due to the fact that we allowed the left endpoint to be infinite. Indeed if one takes finite fixed left endpoints, the operators are uniquely determined by the spectral measure.

**Corollary 4.2.** Suppose that \(-\infty < a_1 = a_2\) and that there is some real entire function \(g\) such that

\[
e^{g(z)} \frac{E_1(z, x_1)}{E_2(z, x_2)} \quad z \in \mathbb{C}^+,
\]

is of bounded type for some \(x_1 \in (a_1, b_1)\) and \(x_2 \in (a_2, b_2)\). If \(\rho_1 = e^{-2g} \rho_2\) then \(b_1 = b_2, q_1 = q_2\) and \(H_1 = H_2\).

**Proof.** This follows immediately from Theorem 4.1 and \(\lim_{x_1 \downarrow a_1} \eta(x_1) = a_1\). \(\square\)

Below we will see that some kind of growth restriction on the solutions \(\phi_1\) and \(\phi_2\) suffices to guarantee that (4.1) is of bounded type. However, note that this condition in Theorem 4.1 can not be dropped and some assumption has to be imposed on the solutions \(\phi_1\) and \(\phi_2\). As an example consider the interval \((0, \pi)\), the potential \(q_1 = 0\) and let \(H_1\) be the associated Schrödinger operator with Dirichlet boundary conditions. As our real entire solution \(\phi_1\) we choose

\[
\phi_1(z, x) = \frac{\sin \sqrt{zx}}{\sqrt{z}}, \quad x \in (0, \pi), \ z \in \mathbb{C}.
\]

The associated spectral measure \(\rho_1\) is given by

\[
\rho_1 = \frac{2}{\pi} \sum_{n \in \mathbb{N}} n^2 \delta_{n^2},
\]

where \(\delta_{n^2}\) is the unit Dirac measure in the point \(n^2\). Now choose some sequence \(\kappa_n, n \in \mathbb{N}\) of positive reals such that all but finitely many of these numbers are equal to one. From the solution of the inverse spectral problem in the regular case it is known (see e.g. [8], [9]) that there is some potential \(q_2 \in L^2(0, \pi)\) and a corresponding operator \(H_2\) with Dirichlet boundary conditions such that the spectral measure \(\rho_2\) associated with the real entire solution \(\phi_2\) of

\[-\phi''_2(z, x) + q_2(x)\phi_2(z, x) = z\phi_2(z, x), \quad x \in (0, \pi), \ z \in \mathbb{C},\]

with initial conditions

\[
\phi_2(z, 0) = 0 \quad \text{and} \quad \phi'_2(z, 0) = 1,
\]

is given by

\[
\rho_2 = \frac{2}{\pi} \sum_{n \in \mathbb{N}} \kappa_n n^2 \delta_{n^2}.
\]

Now pick some real entire function \(g\) such that

\[
g(n^2) = \frac{\ln \kappa_n}{2}, \quad n \in \mathbb{N},
\]

and switch to the real entire solution

\[
\tilde{\phi}_2(z, x) = e^{g(z)} \phi_2(z, x), \quad x \in (0, \pi), \ z \in \mathbb{C}.
\]

Then the spectral measure associated to this solution is equal to \(\rho_1\), but the corresponding operators \(H_1\) and \(H_2\) are different (at least if not all \(\kappa_n\) are equal to one).
However, also note that in this case (4.1) fails to be of bounded type rather badly, since the function $e^g$ is not even of finite exponential order.

We conclude this section by showing that condition (4.1) in Theorem 4.1 holds if the solutions $\phi_1, \phi_2$ satisfy some growth condition. Therefore recall that an entire function $F$ belongs to the Cartwright class $C$ if it is of finite exponential type and the logarithmic integral

$$\int_{\mathbb{R}} \frac{\ln^+ |F(x)|}{1 + x^2} dx < \infty,$$

exists, where $\ln^+$ is the positive part of the natural logarithm $\ln$. In particular note that the class $C$ contains all entire functions of exponential order less then one.

Now a Theorem of Krein [12, Theorem 6.17], [14, Section 16.1] state s that the class $C$ consists of all entire functions which are of bounded type in the upper and in the lower complex half-plane. Now since the quotient of two functions of bounded type is of bounded type itself, this immediately yields the following uniquenes s result.

**Corollary 4.3.** Suppose that $E_1(\cdot, x_1)$ and $E_2(\cdot, x_2)$ belong to the class $C$ for some $x_1 \in (a_1, b_1)$ and $x_2 \in (a_2, b_2)$. If $\rho_1 = \rho_2$ then $H_1$ is equal to $H_2$ up to some shift.

Again as in Corollary 4.2, if one takes finite fixed left endpoints, the operator is uniquely determined by the spectral measure. In particular as a special case one recovers the classical result due to Marchenko that the spectral measure uniquely determines the operator, if the left endpoint is regular. However our result covers a larger class of potentials, as we will show in the next section. There we will apply our results in order to obtain a uniqueness theorem for perturbed Bessel operators.

**5. Application to perturbed Bessel operators**

Let $l \in [-\frac{1}{2}, \infty)$, $0 < b \leq \infty$ and consider the differential expression

$$\tau = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x),$$

on $(0, b)$, where $q$ is some real-valued, locally integrable function on $(0, b)$. We will assume that the function

$$(5.1) \quad \tilde{q}(x) = \begin{cases} x|q(x)|, & \text{if } l > -\frac{1}{2}, \\ x(1 - \ln x)|q(x)|, & \text{if } l = -\frac{1}{2}, \end{cases}$$

is integrable near zero. According to [5, Theorem 2.4], $\tau$ is in the limit-circle case at zero if and only if $l \in [-\frac{1}{2}, \frac{1}{2})$. Now let $H$ be some associated self-adjoint operator with the boundary condition

$$(5.2) \quad \lim_{x \downarrow 0} x'((l + 1)f(x) - xf'(x)) = 0,$$

at zero, if necessary. In [5, Lemma 2.2] it has been shown that there is a nontrivial real entire solution $\phi$ of exponential order $\frac{1}{2}$ which lies in $L^2(0, b)$ near zero and satisfies the boundary condition (5.2) at zero if $l \in [-\frac{1}{2}, \frac{1}{2})$. Note that this solution is unique up to scalar multiples because of the growth restriction, as it has been shown in [6, Lemma 6.4]. The associated spectral measure is denoted by $\rho$.

Now in order to state our uniqueness theorem let $l_1, l_2 \in [-\frac{1}{2}, \infty)$, $0 < b_1$, $b_2 \leq \infty$ and $q_1, q_2$ be two potentials such that the functions $\tilde{q}_1, \tilde{q}_2$ defined as in (5.1) are integrable near zero. Furthermore, let $H_1, H_2$ be two corresponding self-adjoint operators with the boundary condition (5.2) at zero, if necessary. By
\( \phi_1, \phi_2 \) we denote some real entire solutions of exponential order 1/2 which lie in \( L^2(0, b) \) near zero and satisfy the boundary condition there, if any. Finally let \( \rho_1, \rho_2 \) be the associated spectral measures. Our uniqueness results from the preceding section now yield the following uniqueness theorem.

**Theorem 5.1.** If \( \rho_1 = \rho_2 \) then \( l_1 = l_2, b_1 = b_2, q_1 = q_2 \) and \( H_1 = H_2 \).

**Proof.** Since the solutions are of exponential order 1/2, we may immediately apply Corollary 4.3 and obtain

\[
\frac{l_1(l_1+1)}{x^2} + q_1(x) = \frac{l_2(l_2+1)}{x^2} + q_2(x),
\]

for almost all \( x \in (0, b_1) \) and \( H_1 = H_2 \). Now since the functions \( \overline{\tau}_1 \) and \( \overline{\tau}_2 \) are integrable near zero we infer that \( l_1(l_1+1) = l_2(l_2+1) \) and hence also \( l_1 = l_2 \). \( \square \)

**Acknowledgments.** I thank Gerald Teschl for helpful discussions and hints with respect to the literature.

**References**

[1] L. de Branges, *Hilbert spaces of entire functions*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1968.

[2] C. Bennewitz, *A proof of the local Borg–Marchenko theorem*, Comm. Math. Phys. 218 (2001), no. 1, 131–132.

[3] J. Eckhardt and G. Teschl, *Uniqueness results for Schrödinger operators with strongly singular potentials*, (in preparation).

[4] A. Kostenko and G. Teschl, *On the singular Weyl–Titchmarsh function of perturbed spherical Schrödinger operators*, J. Diff. Eq. 250, 3701–3739 (2011).

[5] A. Kostenko, A. Sakhnovich, and G. Teschl, *Inverse eigenvalue problems for perturbed spherical Schrödinger operators*, Inverse Problems 26 (2010), no. 10, 105013, 14pp.

[6] A. Kostenko, A. Sakhnovich, and G. Teschl, *Weyl–Titchmarsh theory for Schrödinger operators with strongly singular potentials*, Int. Math. Res. Not. 2011, Art. ID rnr065, 49pp.

[7] A. Kostenko, A. Sakhnovich, and G. Teschl, *Commutation methods for Schrödinger operators with strongly singular potentials*, Math. Nachr. (to appear).

[8] B. M. Levitan, *Inverse Sturm–Liouville problems*, VNU Science Press, Utrecht, 1987.

[9] J. Pöschel and E. Trubowitz, *Inverse spectral theory*, Pure Appl. Math., 130, Academic Press, Inc., Boston, MA, 1987.

[10] C. Remling, *Schrödinger operators and de Branges spaces*, J. Funct. Anal. 196 (2002), no. 2, 323–394.

[11] C. Remling, *Inverse spectral theory for one-dimensional Schrödinger operators: the A function*, Math. Z. 245 (2003), no. 3, 597–617.

[12] M. Rosenblum and J. Rovnyak, *Topics in Hardy classes and univalent functions*, Birkhäuser Verlag, Basel, 1994.

[13] G. Teschl, *Mathematical Methods in Quantum Mechanics; With Applications to Schrödinger Operators*, Graduate Studies in Mathematics, Amer. Math. Soc., Rhode Island, 2009.

[14] B. Ya. Levin, *Lectures on Entire Functions*, Transl. Math. Monographs 150, Amer. Math. Soc., Providence, RI, 1996.

E-mail address: jonathan.eckhardt@univie.ac.at