Massive Spin-5/2 Fields Coupled to Gravity: Tree-Level Unitarity vs. the Equivalence Principle

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ABSTRACT

I show that the gravitational scattering amplitudes of a spin-5/2 field with mass \( m \ll M_{Pl} \) violate tree-level unitarity at energies \( \sqrt{s} \approx \sqrt{m M_{Pl}} \) if the coupling to gravity is minimal. Unitarity up to energies \( \sqrt{s} \approx M_{Pl} \) is restored by adding a suitable non-minimal term, which gives rise to interactions violating the (strong) equivalence principle. These interactions are only relevant at distances \( d \lesssim 1/m \).

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Torsion-free minimal coupling to gravity is defined by the following substitutions, performed on a free lagrangian for an arbitrary field $\phi$:

$$\partial_\mu \rightarrow D_\mu, \quad \eta^{\mu\nu} \rightarrow g^{\mu\nu} = e^\mu_a e^\nu_a,$$

and by the equation $D_\mu e^\mu_a = 0$. The flat metric $\eta^{\mu\nu}$ is replaced by the curved one, and ordinary derivatives are replaced by covariant derivatives, containing the usual symmetric Christoffel symbol, and the spin-connection. The tetrads $e^\mu_a$, which transform curved indices $\mu$ into flat tangent-space indices $a$, are needed in order to describe the coupling of fermions to gravity.

The equivalence principle, in its weakest form, states that gravity couples to the stress-energy tensor of matter $T_{\mu\nu}$. To linear order in the fluctuations of the gravitational field about the flat-space background, and in the gauge $e_\mu^a - e^\mu_a = 0$,

$$S[\phi, e^\mu_a] = S[\phi]_{\text{free}} + T^{\mu\nu}[\phi]h_{\mu\nu}, \quad h_{\mu\nu} = e_{\mu\nu} - \eta_{\mu\nu}. \quad (2)$$

Here $S[\phi, e^\mu_a]$ is the action of the field $\phi$ coupled to gravity. Notice that at linear order we may identify curved- and flat-space indices. In this formulation the equivalence principle is simply a definition of the stress-energy tensor, through eq. (2).

The minimal coupling prescription is singled out by imposing that $T_{\mu\nu}$ be the (symmetrized) Noether stress-energy tensor. Needless to say, the minimal-coupling prescription is neither unambiguous nor unique: covariant derivatives do not commute, therefore, integrating by parts in a free bosonic lagrangian, and using prescription (1) one finds a different result than by performing substitution (1) directly. Moreover, general relativity allows for matter-gravity couplings proportional to the curvature tensors, and derivatives thereof. In addition to those problems, massless fields coupled to gravity may turn out to give inconsistent theories [4]. In the following, I shall deal with massive fields only, where this last problem does not arise.

The former two problems of minimal coupling are of little concern in the case of massive particles of spin $s \leq 2$ interacting with gravity. The reason behind this fact is that the minimal coupling of particles with $s \leq 2$, besides being “simple,” enjoys another remarkable property: scattering amplitudes, involving gravitational interactions only, are small for any center-of-mass energy $\sqrt{s}$ lower than the Planck scale $M_{Pl}$.

In order to convince ourselves that this property is not obvious, let us consider, for instance, the conversion of two massive, spin-$s$ particles into two gravitons. The graphs contributing to this process are depicted in fig. 1. The corresponding scattering amplitude involves the propagator $\Pi$ of the massive field $\phi$. 


Figure 1: solid lines denote massive, spin-s particles, curly lines denote gravitons.

It is well known that for \( s \geq 1 \) this propagator contains terms proportional to \( 1/m^2 \), due to the existence of (restricted) gauge invariances in the \( m \to 0 \) limit. These mass singularities could, in principle, give rise to a scattering amplitude containing terms \( \mathcal{O}(s^2/m^2M_{Pl}^2) \). Such a scattering amplitude would become large, and eventually exceed the unitarity bounds, at \( \sqrt{s} \approx \sqrt{mM_{Pl}} \). This is an energy scale much below the Planck one, when \( m \ll M_{Pl} \).

The reason why \( \mathcal{O}(s^2/m^2M_{Pl}^2) \) terms are absent for \( s=1, 3/2, 2 \) is the following. The diagrams in fig. 1 giving rise to the dangerous \( \mathcal{O}(s^2/m^2M_{Pl}^2) \) terms have the form \( J \Pi J \). The tensor current \( J \) is obtained by varying the action \( S[\phi,e]_{\mu} \) with respect to the field \( \phi \), and keeping only terms linear in the fluctuation of the metric about the flat-space background

\[
\frac{\delta S[\phi,e]_{\mu}}{\delta \phi} = J + \mathcal{O}(h^2). \tag{3}
\]

As noticed above, terms proportional to \( 1/m^2 \) in the propagator \( \Pi \) are related to gauge invariances in the massless limit. More precisely \( \Pi \phi = m^{-2} \phi \) iff \( \phi \) is a pure gauge. The standard form of \( \phi \) for \( s=1, 3/2, 2 \) reads\(^3\)

\[
s = 1 : \phi_{\mu} = \partial_{\mu} \epsilon, \quad s = 3/2 : \phi_{\mu} = \partial_{\mu} \epsilon, \quad s = 2 : \phi_{\mu\nu} = \partial_{\mu} \epsilon_{\nu} + \partial_{\nu} \epsilon_{\mu}. \tag{4}
\]

The gauge parameter \( \epsilon \) is a real scalar for \( s=1 \), a Majorana spinor for \( s=3/2 \), and a real vector for \( s=2 \). If the projection \( J \cdot \phi \) of the current \( J \) on the vectors \( \phi \) has the form \( mX \), with \( X \) any operator possessing a smooth \( m \to 0 \) limit, then, by dimensional reasons, no \( \mathcal{O}(s^2/m^2M_{Pl}^2) \) terms will arise in the scattering amplitude of fig. 1. The key observation now is that, up to \( \mathcal{O}(h^2) \) terms, \( J \cdot \phi \) equals \( \phi \cdot \delta S[\phi,e]_{\mu}/\delta \phi \), due to eq. (3): we find the projection \( J \cdot \phi \) by varying the action \( S[\phi,e]_{\mu} \) under a gauge transformation,\(^4\) and linearizing in the gravitational field \( h_{\mu\nu} \).

For generic spin this variation contains terms of the form \( mX \), hereafter called “soft,” as well as hard terms. The latter ones do not vanish in the \( m \to 0 \) limit. For \( s=1, 3/2, 2 \) though, the

\(^2\)The ‘seagull’ diagram in fig. 1 does not contribute to the leading zero-mass singularity.

\(^3\)More complicated forms of \( \phi \) can be reduced to eq. (4) by field redefinitions.

\(^4\)To be exact, under a gauge transformation of the massless, free lagrangian \( S_0[\phi] = \lim_{m \to 0} S[\phi] \).
hard terms are proportional either to the (linearized) scalar-curvature tensor $R$ or to the Ricci tensor $R_{\mu\nu}$ \[2, 3\]. These hard contributions vanish when we impose the free-graviton equations of motion (that is the linearized Einstein equations in the vacuum). We are allowed to use Einstein’s equations because the graviton lines in fig. 1 are external. Moreover, ambiguities in the ordering of covariant derivative, in the action of spin-1 and spin-2 fields, yield only harmless terms, vanishing on shell.

Up to spin 2 minimal coupling seems therefore the only natural choice for describing gravitational interactions, but the situation changes drastically for $s=5/2$.

Lagrangians for massive particles of spin 5/2 have been given by several authors \[1, 4, 5\], we adopt here the lagrangian given in refs. \[4, 6\]. The set of fields used there is the minimal one needed to describe a spin-5/2 particle \[4, 5\], namely, a Majorana tensor-spinor $\psi_{\mu\nu}$, symmetric in the $\mu, \nu$ indices, and an auxiliary Majorana spinor $\chi$. The action reads

$$S = \int d^4x e^{-\frac{1}{2} \bar{\psi}_{ab} D \psi_{ab} - \bar{\psi}_{ab} \gamma_\rho D \gamma_\rho \psi_{ca} + 2 \bar{\psi}_{ab} \gamma_\rho D e \psi_{ca} + \frac{1}{4} \bar{\psi}_{aa} D \psi_{bb} - \bar{\psi}_{aa} D b \gamma_c \psi_{bc}} + \frac{m}{2} (\bar{\psi}_{ab} \psi_{ab} - \bar{\psi}_{ab} \gamma_\rho \psi_{ca} - \frac{7}{4} \bar{\psi}_{aa} \psi_{bb} - \frac{16}{3} \bar{\psi}_{aa} \chi + \frac{32}{9} \bar{\chi} \chi).$$

Our conventions on the metric and gamma matrices follow ref. \[6\]. The covariant derivative of the field $\psi_{\mu\nu}$ is

$$D_\mu \psi_{\nu\rho} = \partial_\mu \psi_{\nu\rho} + \frac{1}{2} \sigma_{ab} \omega_{\mu}^{ab} (e) \psi_{\nu\rho} + \Gamma_\mu^\lambda \psi_\lambda \psi_\rho + \Gamma_\mu^\rho \psi_\nu \lambda.$$

The Christoffel’s symbols are the standard ones (torsion-free). The commutator of two covariant derivatives is

$$[D_\mu, D_\nu] \psi_{\rho\sigma} = \frac{1}{2} R_{\mu\nu\rho\sigma} + R_{\mu\nu\rho\lambda} \psi_{\lambda}^\sigma + R_{\mu\nu\sigma\lambda} \psi_{\rho}^\lambda.$$  

The free lagrangian possesses a restricted gauge invariance at $m = 0$ \[4, 5\]. The gauge parameter is a gamma-traceless Majorana vector spinor, and the gauge transformation reads

$$\delta \psi_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu, \quad \gamma^\mu \epsilon_\mu = 0.$$  

The free equations of motion of $\psi_{\mu\nu}$ and $\chi$ are

$$\partial^\mu \psi_{\mu\nu} = m \psi_{\mu\nu}, \quad \gamma^\mu \psi_\mu = 0, \quad \chi = 0.$$  

In order to see whether $O(s^2 / m^2 M_P^2)$ terms exist in the scattering diagrams of fig. 1 we must perform a variation of action \[8\] under the transformation \(8\), linearize in the gravitational field, and put $\psi_{\mu\nu}, \chi$ and $h_{\mu\nu}$ on shell. A short calculation gives

$$\delta S = -4 \bar{\epsilon}_\nu \gamma_\rho \psi_\lambda \sigma R^{\mu\lambda\rho\sigma} + \text{soft terms} + O(h^2).$$

The hard term in this equation is proportional to the Riemann tensor, thus, it does not vanish on shell, and the scattering amplitude of fig. 1 does contain $O(s^2 / m^2 M_P^2)$ terms.

This result means that a minimally coupled light $(m \ll M_P)$ spin-5/2 field interacts strongly with gravity even at relatively low energies $(\sqrt{s} \approx \sqrt{mM_P} \ll M_P)^5$. This scenario seems

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\[5\] A massless spin-5/2 field coupled to gravity is downright inconsistent \[1\].
bizarre: it seems natural to assume that gravitational interactions be weak up to energies \( \sqrt{s} \approx M_{Pl} \), irrespective of any particle’s mass. If we do impose this requirement, or, in other words, if we impose that gravitational tree-level amplitudes respect unitarity up to the Planck scale, we must find a way of cancelling the hard term in eq. (10), even at the price of giving up the minimal-coupling prescription. Minimal coupling has nothing really fundamental about it, whereas tree-level unitarity is a sensible requirement which, in the case of electromagnetic interactions, has been already proven fruitful \[8, 9, 10\].

Indeed, we can cancel the hard term of eq. (10) by adding to the spin-5/2 action a non-minimal coupling proportional to the Riemann tensor. Notice that a similar situation happens when higher-spin massive particles are coupled to electromagnetism \[10\]. In that case, appropriate non-minimal terms cancel tree-level unitarity violating terms in the scattering amplitudes \[10\]. In ref. \[10\] it was also shown that the cancellation between minimal and non-minimal terms does occur for charged open-string states in a constant electromagnetic background.

To prove that the cancellation takes place also for a spin-5/2 field coupled to gravity, let me add a non-minimal term to action (5)

\[
\int d^4x e^{\bar{\alpha}}(R^{\mu\nu\rho\sigma} + \frac{1}{2} \gamma^{5} \epsilon^{\nu\sigma\alpha\beta} R^{\mu\rho}_{\alpha\beta}) \psi_{\rho\sigma}.
\]

The variation of this term under transformation (8) yields

\[
-2\alpha \bar{\psi}_{\mu} R^{\mu\nu\rho\sigma} (\partial_{\nu} \psi_{\sigma\rho} - \partial_{\sigma} \psi_{\nu\rho} + \gamma^{5} \epsilon^{\nu\sigma\alpha\beta} \partial_{\alpha} \psi_{\beta\rho}) = 2\alpha m \bar{e}_{\mu} R^{\mu\nu\rho\sigma} \gamma^{\lambda} \epsilon^{\sigma\nu\rho\lambda} \psi_{\lambda\rho} + O(h^2).
\]

To get eq. (12) one must recall that, on shell

\[
D_{\mu} R^{\mu\nu\rho\sigma} = 0, \quad \phi \psi_{\mu} = m \psi_{\mu},
\]

use the Bianchi identities

\[
R^{\mu}_{[\nu\rho\sigma]} = 0, \quad \partial_{\nu} R_{\mu\rho\sigma} = 0,
\]

and the identity \( R^{\alpha\beta\gamma\delta} \epsilon^{\mu\nu\alpha\beta} = 0 \). Recalling that on shell \( \gamma^{\mu} \psi_{\mu} = 0 \), eq. (12) is transformed into

\[
-4\alpha m \bar{e}_{\nu} R^{\nu\lambda\rho\sigma} \epsilon_{\mu\rho\lambda\sigma}.
\]

By choosing \( \alpha = -1/m \) the term (15) cancels the hard term in eq. (10), and, by consequence, the terms proportional to \( s^2/m^2 M_{Pl}^2 \) in the scattering amplitude of fig. 1.

Notice that our cancellation becomes exact on any background satisfying Einstein’s vacuum equations, if we covariantize the derivatives in eq. (8). The coupling (11), needless to say, is defined only up to terms proportional to \( R \) and \( R_{\mu\nu} \).

The non-minimal term in eq. (11) violates the strong equivalence principle, and it introduce a coupling proportional to \( 1/m \). The new stress-energy tensor associated with the lagrangian containing (11) is no longer the Noether one; for instance, it contains terms with two derivatives of the field. There exist a field redefinition transforming this new stress-energy tensor into the standard one, but it is probably non local, being defined as a power series in \( 1/m \). This new
coupling is obviously compatible with all principles of general relativity. Moreover, it is negligible at large distance, i.e. in soft-graviton scattering with transferred momenta $q \lesssim m$, since it gives rise to additional interactions of strength proportional to $q^2/m$. This is to be compared with the minimal interaction, whose strength is proportional to the energy $E$ of the spin-5/2 particle.

Tree-level unitarity up to $\sqrt{s} = M_{Pl}$ seems a physically meaningful and natural requirement, unlike the minimal-coupling prescription. This requirement entails that any theory containing light spin-5/2 particles should give rise to an effective lagrangian containing the term with $\alpha = -1/m$. String theory is an example of such a theory, when the string tension $\alpha' \ll M_{Pl}^2$: it would be interesting to check whether its low-energy effective lagrangian actually contains term $\alpha$.  

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