Holographic entropy bounds in the inflationary universe

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Abstract

We introduce the relation between the holographic entropy bounds and the inflationary universe. First the holographic entropy bounds for radiation-dominated universe, radiation-dominated universe with a positive cosmological constant are introduced. For an exact de Sitter phase, we use the maximal entropy bound. We classify the inflation based on the quasi-de Sitter spacetime into three steps: slow-roll period of inflation, epoch of reheating, and radiation-dominated era. Then we study how to apply three entropy bounds to the three steps of the inflation. Finally we discuss our results.

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1 Introduction

The inflation turned out to be a successful tool to resolve the problems of the hot big bang model [1]. Thanks to the recent observations of the cosmic microwave background anisotropies and large scale structure galaxy surveys, it has become widely accepted by the cosmology community [2]. The idea of inflation is based on the very early universe dominance of vacuum energy density of a hypothetical scalar field, the inflaton. This produces the quasi-de Sitter spacetime [3] and during the slow-roll period, the equation of state can be approximated by the vacuum state as \( p \approx -\rho \) [4]. After that there must exist a strong non-adiabatic and out-of-equilibrium phase called reheating to produce a large increase of the entropy. But we don’t know exactly how inflation started.

To solve this problem we have to build cosmology from the quantum gravity, but now we are far from it. Although we are lacking for a complete understanding of the quantum gravity, there exists the holographic principle. This principle is mainly based on the idea that for a given volume \( V \), the state of maximal entropy is given by the largest black hole that fits inside \( V \). ’t Hooft and Susskind [5] argued that the microscopic entropy \( S \) associated with the volume \( V \) should be less than the Bekenstein-Hawking entropy: \( S \leq A/4G \) in the units of \( c = \hbar = 1 \) [6]. Here the horizon area \( A \) of a black hole equals the surface area of the boundary of \( V \). That is, if one reconciles quantum mechanics and gravity, the observable degrees of freedom of the three-dimensional universe comes from a two-dimensional surface. Actually holographic area bounds limit the number of physical degrees of freedom in the bulk spacetime.

The implications of the holographic principle for cosmology have been investigated in the literature. Following an earlier work by Fischler and Susskind [7] and works in [8, 9], it was argued that the maximal entropy inside the universe is produced by the Hubble horizon-size black hole. Roughly the total entropy should be less or equal than the Bekenstein-Hawking entropy of the Hubble-size black hole \( (\approx H V_H/4G_{n+1}) \) times the number \( (N_H \approx V/V_H) \) of Hubble regions in the universe. Hence one obtains an upper bound on the total entropy which is proportional to \( H V/4G_{n+1} \).

Furthermore, Verlinde obtained the pre-factor as \( (n - 1) \) and proposed the new holographic bound like Eq.(2.6) in a radiation-dominated phase by introducing three entropies [10]: Bekenstein-Verlinde entropy \( (S_{BV}) \), Bekenstein-Hawking entropy \( (S_{BH}) \), and Hubble entropy \( (S_H) \). For example, such a radiation is given by a conformal field theory (CFT) with a large central charge dual to the AdS-black hole [11]. It indicated an interesting relationship between the Friedmann equation governing the cosmological evolution and the square-root form of entropy-energy relation, called Cardy-Verlinde formula [12].
Although the Friedmann equation has a geometric origin and the Cardy-Verlinde formula is designed only for the matter content, this strongly suggested that both sets may arise from a single underlying fundamental theory. In addition, this approach showed new features about the mathematical structure of the Friedmann equation [13].

In this work we will explore the implications of the holographic principle for the inflationary universe. We classify the inflation based on the quasi-de Sitter spacetime into three steps: (i) slow-roll period of inflation (SR), (ii) epoch of reheating (RH), and (iii) radiation-dominated era (RD). In order to obtain the holographic entropy bounds for these steps, we introduce three kinds of the universe with the equation of state: an exact de Sitter phase with a positive cosmological constant, a radiation-dominated universe with a positive cosmological constant, and a radiation-dominated universe. In order to get the entropy bounds for a radiation-dominated universe with a positive cosmological constant, we need to define the entropy for the cosmological constant \( S_\Lambda \) and the cosmological D-entropy \( S_D \) [14]. Furthermore, we use the entropy bounds for de Sitter space.

The organization of this paper is as follows. In section 2, we discuss the holographic entropy bounds for radiation-dominated, radiation-dominated with a positive cosmological constant. Section 3 is devoted to the description of the inflationary universe based on the quasi-de Sitter spacetime. And then we study how to apply the entropy bounds to the three steps of the inflation: SR, RH, and RD. Finally we discuss our results in section 4.

## 2 Cosmological entropy bounds

### 2.1 Three entropies

Let us start a \((n + 1)\)-dimensional Friedmann-Robertson-Walker (FRW) metric

\[
ds^2 = -d\tau^2 + R(\tau)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega_{n-1}^2 \right],
\]

where \( R \) is the scale factor of the universe and \( d\Omega_{n-1}^2 \) denotes the line element of a \((n - 1)\)-dimensional unit sphere. Here \( k = -1, \ 0, \ 1 \) represent that the universe is open, flat, closed, respectively. The evolution is determined by the two FRW equations

\[
\begin{align*}
H^2 &= \frac{16\pi G_{n+1}}{n(n-1) V} E - \frac{k}{R^2} + \frac{1}{l_{n+1}^2}, \\
\dot{H} &= -\frac{8\pi G_{n+1}}{n-1} \left( \frac{E}{V} + p \right) + \frac{k}{R^2},
\end{align*}
\]

\[
(2.2)
\]
where $H$ represents the Hubble parameter with the definition $H = \dot{R}/R$ and the overdot stands for derivative with respect to the cosmic time $\tau$, $E$ is the energy of matter filling the universe, and $p$ is its pressure. $V$ is the volume of the universe, $V = R^n \Sigma^n_k$ with $\Sigma^n_k$ being the volume of a $n$-dimensional space with $k$, and $G_{n+1}$ is the Newton constant in $(n + 1)$ dimensions. Here we assume the equation of state for matter: $p = \omega \rho$, $\rho = E/V$. For our purpose, we include the curvature scale of de Sitter space $l_{n+1}$ which relates to the cosmological constant via $1/l_{n+1}^2 = 2\Lambda_{n+1}/n(n - 1)$. Youm [15] considered the cosmological constant as the energy density $p_\Lambda = -\rho_\Lambda = -\Lambda_{n+1}$ to obtain the entropy bounds for a vacuum-like matter with $\omega = -1$. Hereafter we do not follow this direction to obtain the entropy bound for a positive cosmological constant.

Verlinde has introduced three entropies for a closed radiation-dominated universe [10]:

$$S_{BV} = \frac{2\pi}{n} ER$$

$$S_{BH} = (n - 1) \frac{V}{4G_{n+1}R}$$

$$S_H = (n - 1) \frac{HV}{4G_{n+1}}.$$  \hfill (2.3)

$S_{BV} \leq S_{BH}$ is supposed to hold for a weakly self-gravitating universe ($HR \leq 1$), while $S_{BV} \geq S_{BH}$ works when the universe is in the strongly self-gravitating phase ($HR \geq 1$). It is interesting to note that for $k = 1, HR = 1$, one finds that three entropies are identical: $S_{BV} = S_{BH} = S_H$.

### 2.2 Radiation-dominated universe

First we start with $k = 1, \Lambda_{n+1} = 0$ case because this case gives us a concrete and clear relation. We define a quantity $E_{BH}$ which corresponds to energy needed to form a universe-size black hole: $S_{BH} = (n - 1)V/4G_{n+1}R \equiv 2\pi E_{BH} R/n$. With this quantity, the Friedmann equations (2.2) can be further cast to

$$S_H = \frac{2\pi R}{n} \sqrt{E_{BH}(2E - E_{BH})},$$

$$E_{BH} = n(E + pV - T_H S_H).$$  \hfill (2.4)

where the Hubble temperature ($T_H$) is given by $T_H = -\frac{\dot{H}}{2\pi H}$. On the other hand, the entropy of radiation and its Casimir energy can be described by the Cardy-Verlinde and

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1In [16] the first one is called the Bekenstein entropy. In fact this bound is slightly different from the original Bekenstein entropy [6] by a numerical factor $1/n$. So we call this the Bekenstein-Verlinde entropy. This could be viewed as the counterpart of the Bekenstein entropy in the cosmological setting [10].
Smarr formulae

\[ S = \frac{2\pi R}{n} \sqrt{E_c(2E - E_c)}, \]

\[ E_c = n(E + pV - TS). \]  \hspace{1cm} (2.5)

Here \( T \) stands for the temperature of radiation with \( \omega = 1/3 \). These can describe the entropy \( S \) of a CFT-radiation living on a \( n \)-dimensional sphere with radius \( R \). Here \( E \) is the total energy of the CFT and \( E_c \) stands for the Casimir energy of the system, non-extensive part of the total energy. Suppose the entropy of radiation in the FRW universe can be described by the Cardy-Verlinde formula. Comparing (2.4) with (2.5), one finds that if \( E_{BH} = E_c \), then \( S_H = S \) and \( T_H = T \). At this stage we introduce the Hubble bounds for entropy, temperature and Casimir energy \[10\]

\[ S \leq S_H, \quad T \geq T_H, \quad E_c \leq E_{BH}, \quad \text{for} \quad HR \geq 1 \]  \hspace{1cm} (2.6)

which are relation between geometric and matter quantities. The Hubble entropy bound is saturated by the entropy of radiation filling the universe if the Casimir energy \( E_c \) is enough to form a universe-size black hole. At this moment, equations (2.4) and (2.5) coincide exactly. This implies that the first Friedmann equation somehow knows the entropy formula of a square-root form for radiation-matter filling the universe. For example, let us consider a moving brane universe in the background of the 5D Schwarzschild-AdS black hole. Savonije and Verlinde \[11\] found that when the brane crosses the black hole horizon, the Hubble entropy bound is saturated by the entropy of black hole(=the entropy of the CFT-radiation). Also the Hubble temperature and energy \((T_H, E_{BH})\) equals to the temperature and Casimir energy \((T, E_c)\) of the CFT-radiation dual to the AdS black hole at this moment.

Up to now we discuss \( k = 1 \) case only. For later purpose, we list the Hubble entropy bound for arbitrary \( k \) \[17\]

\[ S \leq S_H, \quad \text{for} \quad HR \geq \sqrt{2 - k}. \]  \hspace{1cm} (2.7)

On the other hand, the Bekebstein-Verlinde entropy bound for arbitrary \( k \) is given by

\[ S \leq S_{BV}, \quad \text{for} \quad HR \leq \sqrt{2 - k}. \]  \hspace{1cm} (2.8)
2.3 Radiation-dominated universe with a positive cosmological constant

For a radiation-dominated FRW universe with $k = 1, \Lambda_{n+1} \neq 0$, we have to introduce the cosmological entropy $S_{\Lambda}$, cosmological D-entropy $S_{D}$ and D-temperature $T_{D}$ as \[ S_{\Lambda} = (n - 1)\frac{V}{4G_{n+1}t_{n+1}}, \quad S_{D} = \sqrt{|S_{H}^{2} - S_{\Lambda}^{2}|}, \quad T_{D} = -\frac{\dot{H}}{2\pi \sqrt{|1/p^{2} - H^{2}|}}. \] (2.9)

We note that the cosmological D-entropy $S_{D}$ is constructed by analogy with the difference (D) between the entropy of exact de Sitter space and that of asymptotically de Sitter space. Further we assume that three entropies in Eq. (2.3) are still useful for describing the radiation-dominated universe with $\Lambda_{n+1} \neq 0$. In the case of $\Lambda_{n+1} \rightarrow 0$, one recovers the radiation-dominated universe without a cosmological constant.

\[ S_{\Lambda} \rightarrow 0, \quad S_{D} \rightarrow S_{H}, \quad T_{D} \rightarrow T_{H}. \] (2.10)

For $S_{H} \geq S_{\Lambda}$, the Friedmann equations in Eq. (2.1) can be rewritten as

\[ S_{D} = \frac{2\pi R}{n}\sqrt{E_{BH}(2E - E_{BH})}, \]
\[ E_{BH} = n(E + pV - T_{D}S_{D}), \] (2.11)

while the entropy and Casimir energy of radiation can be expressed as

\[ S = \frac{2\pi R}{n}\sqrt{E_{c}(2E - E_{c})}, \]
\[ E_{c} = n(E + pV - TS). \] (2.12)

On the other hand, for $S_{H} \leq S_{\Lambda}$, the equations can be rewritten as

\[ S_{D} = \frac{2\pi R}{n}\sqrt{E_{BH}(E_{BH} - 2E)}, \]
\[ E_{BH} = n(E + pV - T_{D}S_{D}) \] (2.13)

and the entropy and Casimir energy are

\[ S = \frac{2\pi R}{n}\sqrt{E_{c}(E_{c} - 2E)}, \]
\[ E_{c} = n(E + pV - TS). \] (2.14)

\footnote{Although Bousso argued that a cosmological constant did not carry entropy \[18\], there is no contradiction to introducing the corresponding entropy. Actually $S_{\Lambda}$ is closely related to the de Sitter entropy of $S_{dS}$. This is given by the Bekenstein-Hawking entropy of the de Sitter horizon $((n - 1)V_{dS}/4G_{n+1}t_{n+1} \approx S_{dS})$ times the number ($N_{dS} = V/V_{dS}$) of de Sitter regions in the universe.}
As is shown in Eq. (2.10), the cosmological D-entropy plays the same role as the Hubble entropy does in the case without cosmological constant.

Now we are in a position to discuss how the entropy bounds are changed. The first Friedmann equation can be rewritten as

\[(HR)^2 - \frac{R^2}{l_{n+1}^2} = 2 \frac{S_{BV}}{S_{BH}} - k.\] (2.15)

Using this relation, in case of \(k = 1, \Lambda_{n+1} = 0\), one finds \(HR \geq 1 \rightarrow S_{BV} \geq S_{BH}\), while \(HR \leq 1 \rightarrow S_{BV} \leq S_{BH}\). Hence this leads to the Hubble entropy bound of \(S \leq S_H\) for \(HR \geq 1\), whereas the Bekenstein-Verlinde entropy bound of \(S \leq S_{BV}\) for \(HR \leq 1\). Other cases of \(k = -1, 0\) are shown explicitly in Eqs. (2.7) and (2.8).

For \(k = -1, 0, 1\) and \(\Lambda_{n+1} \neq 0\), \((HR)^2 - \frac{R^2}{l_{n+1}^2} \geq 2 - k \rightarrow S_{BV} \geq S_{BH}\), while \((HR)^2 - \frac{R^2}{l_{n+1}^2} \leq 2 - k \rightarrow S_{BV} \leq S_{BH}\). Thus this leads to the D-entropy bound for the strongly self-gravitating universe:

\[S \leq S_D, \text{ for } HR \geq \sqrt{2 - k + \frac{R^2}{l_{n+1}^2}},\] (2.16)

whereas the Bekenstein-Verlinde entropy bound is found for the weakly self-gravitating system:

\[S \leq S_{BV}, \text{ for } HR \leq \sqrt{2 - k + \frac{R^2}{l_{n+1}^2}}.\] (2.17)

When the D-entropy bound is saturated by the entropy \(S\) of radiation, both sets of equations (2.11) [or (2.13)] and (2.12) [or (2.14)] coincide with each other, just like the case without the cosmological constant. We note from Eq. (2.15) that one cannot find the relation of \(S_D = S_{BV} = S_{BH}\) for \(HR = 1\), unless \(\Lambda_{n+1} = 0, k = 1\). Here we obtain an important relation for \(k = 0\) case from Eq. (2.15) as

\[Hl_{n+1} \geq 1.\] (2.18)

This relation comes out because three entropies all in Eq. (2.3) should be positive definitely. Applying this to the entropy, we have

\[S_H \geq S_\Lambda.\] (2.19)

Hence the other case of \(S_H \leq S_\Lambda\) in Eqs. (2.13) together with (2.14) is not allowed for flat \((k = 0)\) cosmological evolution. In other words, this case is forbidden due to the evolution equation.
2.4 de Sitter Universe with a positive cosmological constant

Now we wish to discuss the entropy bounds for the exact de Sitter evolution with $\Lambda_{n+1}$. This corresponds to the perfect fluid $p = -\rho$ with $\rho = \Lambda_{n+1}$. We need this case because the equation of state for slow-roll period of inflation is not defined clearly but it is approximately given by $p \approx -\rho$. In other words, the first Friedmann equation does not play an important role in determining the corresponding entropy bounds. What we can obtain at most here is that the cosmological entropy $S_\Lambda$ is equal to the Hubble entropy $S_H$. In other words, Eq.(2.15) with $k = 0$ shows that in a universe dominated by the cosmological constant, the solution is an exponential expansion of rate of $R(\tau) \propto e^{H\tau} = e^{\tau/l_{n+1}}$.

3 Inflationary Universe and Entropy Bounds

3.1 Inflationary Universe on the quasi-de Sitter space

We adopt an idealized model of the inflationary model based on the quasi-de Sitter spacetime. In what follows we work with the (3+1)-dimensional flat FRW slicing of de Sitter spacetime, because this maps directly onto the FRW spacetime of the post inflationary universe. The line element which covers half of the full de Sitter solution is given by

$$ds^2_{FRW-\text{as}} = -d\tau^2 + \exp[2H\tau](dr^2 + r^2d\Omega_2^2).$$

(3.1)

Another slicing of an exact de Sitter spacetime is given by the static coordinates

$$ds^2_{S-deS} = -(1 - H^2\tilde{r}^2)dt^2 + (1 - H^2\tilde{r}^2)^{-1}d\tilde{r}^2 + \tilde{r}^2d\Omega_2^2,$$

(3.2)

where $H^{-1} = l_4$ is the size of the event (cosmological) horizon. The Gibbons-Hawking temperature is $T_{GH} = H/2\pi$ and the area of the event horizon is $A = 4\pi/H^2$. These coordinates cover the entire causal diamond accessible to any actual observation by an observer at the origin of $r = 0$. Hence this finite spatial region is subjected to the holographic entropy bound of $S \leq S_\text{as} = A/4G = \pi/GH^2 = \pi l_4^2/G$. We call this a “hot tin

However, assuming the adiabatic expansion of the FRW universe, one has the lower entropy bound for the de Sitter expansion for $k = 1$ $S \geq S_0\left(\frac{4}{H^2}\right)^n$ for $HR \geq 1$, while one finds the constant lower bound $S \geq \hat{S}_0\left(\frac{1}{2\pi}\right)^n$ for $HR \leq 1$. Here we do not follow this approach.

The coordinates of $\tau, r$ and $t, \tilde{r}$ are related by the transformations $r = e^{Ht}\sqrt{1 - H^2\tilde{r}^2}, \tau = t + \frac{1}{2\pi}\ln[1 - H^2\tilde{r}^2]$. 

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can” in the sense that an observer in the causal patch (the interior of the can) is surrounded by a hot horizon (the walls) [3]. The physical degrees of freedom, accessible to the observer, are in thermal equilibrium with the Gibbons-Hawking temperature. Apparently an inflating region resembles this exact de Sitter space, where the apparent horizon and the event horizon coincide and thus there is no exterior of the can. However an actual situation is slightly different. Unlike the exact de Sitter space, during inflation many modes are expelled from the apparent horizon. In this case, the space is not an exact de Sitter one (hat tin can) but a quasi-de Sitter space (hot porous tin can). Authors in ref. [20] accounted this leaking entropy $S_L$ from the apparent horizon to obtain a holographic limitation of the effective field theory for inflation.

In order to show the route of information flow from inflation to observable anisotropy, let us see the Penrose diagram in Fig.1.

Figure 1: Penrose diagram of an inflationary cosmology based on quasi-de Sitter space and Friedmann-Robertson-Walker space

As usual, points denote two-spheres($S^2$), the left-hand edge represents the world line of an observer at the origin. Others are boundaries at infinity. The lower half stands for a quasi-de Sitter space (QdS) and the upper half for a FRW spacetime. The join between
them is the epoch of reheating (RH) and shaded regions of each show the regions within
the apparent horizon of an observer at the origin: one is an apparent event horizon
for QdS and the other is an apparent particle horizon for FRW. REC (BBN) represent
spacelike hypersurfaces for the recombination epoch and big bang nucleosynthesis epoch.
Two regions in QdS are necessary, one appropriate for matching onto FRW and the other
for holographic analysis. Actually perturbations are imprinted by fluctuating quantum
field (QF) on the scale of the apparent event horizon during the slow-roll period of inflation
(SR). The apparent horizon grows slightly during SR, as is shown by two closely parallel
null lines. This happens because the spacetime becomes asymptotically de Sitter space
due to the increase of entropy during SR. The intersection of our past light cone (null line)
with REC is the two-sphere of the last scattering surface (LS) for the cosmic background
radiation. A particular timelike trajectory of a comoving sphere (CMS) is shown. The
radiation-dominated era (RD) is from the end of RH to the time of REC and the matter-
dominated era (MD) is extended from REC to the present: US, NOW. Finally a high
frequency gravitational wave background (GWR) can reach US via direct null trajectories.

3.2 Entropy bounds in the inflationary universe

We begin with a scalar field (φ ≡ φ(τ) : inflaton). This gives us the energy density and
pressure

\[ \rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \quad p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi). \]  

(3.3)

Note that although the scalar field acts as a perfect fluid, it does not possess any equation
of state like \( p_\phi = \omega_\phi \rho_\phi \) exactly. To get a good approximation during inflation, we consider
only an inflaton coupled to gravity minimally as

\[ 3H^2 \frac{\dot{\phi}^2}{2M_p^2} + \frac{V(\phi)}{M_p^2}, \quad \ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = 0, \]  

(3.4)

where the overdot is the time derivative and the Planck mass is given by \( M_p = 1/\sqrt{8\pi G_N} \)
in the units of \( c = \hbar = 1 \). The first equation is obtained from Eqs.(2.2) and (3.3), whereas
the second from the conservation law of \( \dot{\rho} + 3H(\rho + p) = 0 \). Inflation occurs when the
potential energy of the scalar is dominant in Eq.(3.3). Then this situation is approximated
by the slow-roll period of inflation (SR). This is formally defined by \( |\epsilon| \) and \( |\delta| \ll 1 \), where
\( \epsilon = \frac{3\dot{\phi}^2}{2V}, \delta = \frac{\ddot{\phi}}{H\dot{\phi}}. \) Actually the slow-roll approximation corresponds to dropping the terms
of order \( O(\epsilon, \delta) \). Then the equation (3.3) leads to

\[ 3H^2 \approx \frac{V(\phi)}{M_p^2}, \quad 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} \approx 0, \]  

(3.5)
Now let us discuss the evolution of physical scales as the universe evolves. The relevant
picture is introduced in Fig. 2 which may be acceptable in the inflationary cosmology [21].

An important question concerning a given scale is whether it is larger or smaller than
the Hubble horizon $H^{-1}$. A physical scale $R$ starts well inside the horizon and then crosses
the horizon $R = H^{-1}$ at time ($\bullet$) during SR. It stays outside the horizon during time :
after the inflation (SR), RH, and RD. It crosses the horizon $R = H^{-1}$ again at later time
($\star$) and reenters the horizon in the period of RD. We wish to describe the inflationary
universe mainly in terms of three periods: SR, RH, and RD.

(i) SR: In the slow-roll approximation the potential and thus the Hubble radius of
$H^{-1}(\approx M_p/\sqrt{3V})$ can be taken to be constant over each Hubble time during inflation.
Hence this can be approximated by de Sitter universe with a positive cosmological constant.
Hence the holographic entropy bound is that the observable entropy of the universe
cannot exceed the entropy of exact de Sitter space $S_{\text{dS}}$, $S \leq S_{\text{dS}} = A/4G = \pi l_4^2/G \approx \pi M_p^2/3GV$. This is an absolute maximal entropy because of the nature of de Sitter space.

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5Strictly speaking, we should refer to “Hubble distance” or “Hubble length” for $H^{-1}$.
That is, the bound includes all degrees of freedom of matter fields as well as all quantum degrees of freedom of the spacetime itself. This is the hat tin can-picture of de Sitter space. There exist another picture of the hot porous tin can where the total entropy in SR is divided into the dominant equilibrium contribution approximately by the \( S_{\text{dS}} \) and a small non-equilibrium contribution by the leaking entropy from the apparent horizon \( S_{L} \).

(ii) RH : The slow-roll period will cease as the inflaton moves to the minimum of the potential, where it oscillates and decays into radiation, reheating the inflated universe to give a large increase of the entropy. Although this is a non-adiabatic and out-of-equilibrium phase, we may approximate this phase by the radiation-dominated universe with a positive cosmological constant. Here we still need a positive cosmological term because during RH the universe exhausts the vacuum energy. The promising entropy bound is \( S \leq S_{D} \) in Eq.(2.16) together with \( S_{H} \geq S_{\Lambda} \) in Eq.(2.19).

(iii) RD : After RH, the evolution of the universe is described by the \( k = 0 \) radiation-dominated one. Then the desired entropy bound is given by \( S \leq S_{H} \) in Eq.(2.7) for the left hand side of the reference point (\( \ast \)) and \( S \leq S_{BV} \) in Eq.(2.8) for the right hand side.

4 Discussion

We discuss the relationship between three periods of the quasi-de Sitter spacetime and three kinds of holographic entropy bounds. Although this relation is not confirmed completely, our approach may be useful for estimating the physical degrees of freedom by the holographic principle. Actually the \( k = 0 \) case is more suitable for describing the inflationary universe than \( k = 1 \) because the \( k \)-curvature term becomes less important compared with the vacuum energy density term. However, applying for the holographic entropy bounds to the inflation, \( k = 1 \) case is more intuitive than \( k = 0 \).

This is so because \( k = 1 \) case gives us the direct relations for the radiation-dominated universe: the Hubble bound of \( S \leq S_{H} \) is valid for \( R \geq H^{-1} \), while the Bekenstein-Verlinde bound of \( S \leq S_{BV} \) is valid for \( R \leq H^{-1} \). We note that the Hubble bound is based on how many Hubble-size black holes exist within the radius \( R \) of the universe. When the radius \( R \) of the universe is smaller than the Hubble radius, one should reconsider the validity of the Hubble bound. In this case, the appropriate entropy bound is the Bekenstein-Verlinde bound. At the reference point (\( \ast \)) of \( R = H^{-1} \), one finds that \( S_{H} = S_{BH} \), where \( S_{BH} \) is the Bekenstein-Hawking entropy of the universe-size black hole. Actually this is not to serve as a bound on the total entropy but rather on the sub-extensive component of the entropy for a finite system. For \( k = 0 \) case, the Hubble bound of \( S \leq S_{H} \) is valid for \( R \geq \sqrt{2}H^{-1} \), while the Bekenstein-Verlinde bound of \( S \leq S_{BV} \) is valid for \( R \leq \sqrt{2}H^{-1} \).
The reference scale is slightly shifted from $H^{-1}$ to $\sqrt{2}H^{-1}$ compared to $k = 1$ case. However the connection between entropy bounds and scales is still transparent.

For the reheating after inflation, one finds from Fig.2 that $R > H^{-1}$. In the case of $k = 0$, the cosmological D-bound of $S \leq S_D$ is valid for $R \geq H^{-1}\sqrt{2 + R^2/l_4^2}$, while the Bekenstein-Verlinde bound of $S \leq S_{BV}$ is valid for $R \leq H^{-1}\sqrt{2 + R^2/l_4^2}$. Here the cosmological D-entropy $S_D$ is given by $\sqrt{S_H^2 - S_\Lambda^2}$ with $S_H \geq S_\Lambda$ from Eq. (2.19). $S_H$ is based on the Bekenstein-Hawking entropy of Hubble-size black hole $(2V_H/4G_4H^{-1} = 2\pi H^{-2}/3G_4 \equiv 2S_{HS}/3)$ times the number ($N_H = V/V_H$) of Hubble regions in the universe, while $S_\Lambda$ is based on the Bekenstein-Hawking entropy for de Sitter-event horizon $(2V_{dS}/4G_4l_4 = 2S_{dS}/3)$ times the number ($N_{dS} = V/V_{dS}$) of de Sitter region in the universe. From Eq. (2.18) and Fig.2 we have $l_4 \geq H^{-1}$. Then one has $S_{dS} \geq S_{HS}$, $V_{dS} \geq V_H$, $N_H \gg N_{dS}$. Hence one achieves $S_H \geq S_\Lambda$. Choosing $R \geq l_4$ during RH, then one finds the relation $R \geq l_4 \geq H^{-1}$. In this case the cosmological D-entropy bound provides an upper bound for RH, as the Hubble bound did in RD. From Fig.2, the other case of $R \leq H^{-1}\sqrt{2 + R^2/l_4^2}$ seems to not occur in the reheating phase.

Finally for the slow-roll period we approximate the entropy bound of the universe by the single de Sitter entropy : $S \leq S_{dS}$.

In conclusion, to estimate the entropy of the inflation we apply three holographic entropy bounds to the three steps of SR, RH, and RD. The relevant entropy bounds are the de Sitter entropy bound for SR: $S \leq S_{dS}$, the D-entropy bound for RH : $S \leq S_D$, and the Hubble entropy bound for RD : $S \leq S_H$. We usually assume the large increase of entropy via the reheating process in the inflationary scenario. In this work we just employ the entropy bounds for a radiation-dominated universe with a positive cosmological constant for RH. We do not yet have a detail mechanism to produce the large entropy in a holographic way.

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