EXISTENCE OF STRONG SOLUTION FOR THE CAUCHY PROBLEM OF FULLY COMPRESSIBLE NAVIER-STOKES EQUATIONS IN TWO DIMENSIONS

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Abstract. We study the Cauchy problem for the equations describing a viscous compressible and heat-conductive fluid in two dimensions. By imposing a weight function to initial density to deal with Sobolev embedding in critical space, and constructing an ad-hoc truncation to control the quadratic nonlinearity appeared in energy equation, we establish the local in time existence of unique strong solution with large initial data. The vacuum state at infinity or the compactly supported density is permitted. Moreover, we provide a different approach and slightly improve the weighted $L^p$ estimates in [19, Theorem B.1].

1. Introduction. The motion of a compressible viscous, heat-conducting, isotropic Newtonian fluid is governed by the following equations (cf. [8, 19])

$$
\begin{cases}
\frac{\rho_t + \text{div} (\rho u)}{\rho} = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla (\rho \theta) = \mu \Delta u + (\mu + \lambda) \text{div} u, \\
c_v [((\rho \theta)_t + \text{div}(\rho u \theta))] + \rho \theta \text{div} u = \kappa \Delta \theta + \frac{\mu}{2} |\nabla u + (\nabla u)^T|^2 + \lambda (\text{div} u)^2.
\end{cases}
$$

(1.1)

Here $x \in \mathbb{R}^2, t > 0$, and the unknown functions $\rho(x,t), u(x,t), \theta(x,t)$ denote the density, the velocity field, the absolutely temperature, respectively. The viscosity coefficients $\mu$ and $\lambda$ are constant and satisfy physical restriction $\mu > 0, \mu + \lambda \geq 0$. The constant $\kappa > 0$ is the heat-conduction coefficient, and $c_v > 0$ is the heat capacity of the gas at constant volume.

As one of the most important systems in continuum mechanics, there is a vast literature studying the existence of solutions for equations (1.1) under different boundary information. The study on the existence and long-time behavior of solutions to one dimensional problem is satisfactory, see [11, 13, 23, 24] and the references therein. In multi-dimensional case, the local existence of classical solutions are surveyed by Nash [22] and Serrin [25]. Matsumura-Nishida [20] obtained the global solution and its asymptotic so long as the initial data has a small disturbance around a non-vacuum equilibrium. If the vacuum state appears initially, Lions [19] and...
Feireisl-Novotny-Petzeltová [9] developed the existence theories of global weak solutions for isentropic flow if the adiabatic exponent is large. For the heat-conductive flows, we refer to the works by Bresch-Desjardins [2] and Feireisl [8] for the discussion on the existence of weak solutions. When it comes to the strong/classical solutions with large initial data and vacuum, Kim-Cho-Choe [5, 6, 7] obtained the local in time existence for isentropic and non-isentropic flows, where the authors imposed some initial compatibility conditions to remedy the time degeneracy in momentum/energy equations. We remark that, in three dimensions, $L^6$ norm of $u_t$ (or $\theta_t$) could be controlled by its gradient $L^2$ norm. However, the argument in [5, 6, 7] cannot directly be applied to unbounded domains in two dimensions. Recently, Li-Liang [15] considered the two dimensional Cauchy problem for the barotropic compressible flows, and obtained the existence of unique strong and classical solutions. The main idea in [15] is using the spatial weighted estimate on density, along with the Hardy type and Poincaré type inequalities, to control the velocity as it is always accompanied with density in momentum equations. See also [17] for the incompressible flows, and [18] for the full Navier-Stokes with zero heat-conduction.

Motivated by the papers [5, 6, 7] by Kim-Cho-Choe, as well as our previous results [15, 17, 18], we aim to establish an existence result of strong solution to equations (1.1) with the constructive conditions

$$ (u, \theta)(x,t) \to (0,0) \text{ as } |x| \to \infty \quad (1.2) $$

and

$$ (\rho, u, \theta)(x,0) = (\rho_0 \geq 0, u_0, \theta_0 \geq 0), \quad x \in \mathbb{R}^2. \quad (1.3) $$

The theorem below states our main results.

**Theorem 1.1.** For given constants $a \in (1,2)$, $q \in (2,\infty)$, $\eta_0 > 0$, and function

$$ \bar{x} = \sqrt{(e + |x|^2)\ln^{1+\eta_0}(e + |x|^2)}, \quad (1.4) $$

assume that the initial functions $(\rho_0, u_0, \theta_0)$ satisfy

$$ 0 \leq \bar{x}^a \rho_0 \in L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), $$

$$ \rho_0(|u_0|^2 + \theta_0^2) \in L^1(\mathbb{R}^2), \quad \nabla u_0 \in H^1(\mathbb{R}^2), \quad \nabla \theta_0 \in H^1(\mathbb{R}^2), \quad (1.5) $$

assume in addition that the following compatibility conditions

$$ -\mu \Delta u_0 - (\mu + \lambda)\nabla \div u_0 + \nabla(\rho_0 \theta_0) = \sqrt{\rho_0}g_1 \quad (1.6) $$

and

$$ -\kappa \Delta \theta_0 - \frac{\mu}{2} |\nabla u_0 + (\nabla u_0)^{tr}|^2 - \lambda(\div u_0)^2 = \sqrt{\rho_0}g_2 \quad (1.7) $$

hold true for $g_1, g_2 \in L^2(\mathbb{R}^2)$.

Then there is a small $T_* > 0$, such that the Cauchy problem (1.1)-(1.3) has a unique strong solution $(\rho, u, \theta)$ over $[0, T_*) \times \mathbb{R}^2$, with properties

$$ \begin{cases} 
\bar{x}^a \rho \in L^\infty(0, T_*; L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2)), \\
\sqrt{\rho}u, \sqrt{\theta} \in L^\infty(0, T_*; L^2(\mathbb{R}^2)), \\
\nabla u, \nabla \theta \in L^\infty(0, T_*; H^1(\mathbb{R}^2)), \quad \nabla^2 u, \nabla^2 \theta \in L^2(0, T_*; L^q(\mathbb{R}^2)), \\
\sqrt{\rho}u_t, \sqrt{\theta} \in L^\infty(0, T_*; L^2(\mathbb{R}^2)), \quad \nabla u_t, \nabla \theta_t \in L^2(0, T_*; L^2(\mathbb{R}^2)). \quad (1.8) 
\end{cases} $$

**Remark 1.** By (1.8), we easily check that the solution $(\rho, u, \theta)$ in Theorem 1.1 satisfies equations (1.1) almost everywhere in $\mathbb{R}^2 \times (0, T_*)$.

**Remark 2.** Although Theorem 1.1 is only concerned with a strong solution, it is hard to remove the initial compatibility conditions (1.6)-(1.7).
Methodology. The main difficulty is the appearance of vacuum and the critical space dimension. The failure of application of Sobolev embedding in critical space was offset by introducing a weight function to density for isentropic flows; see [15]. Here, new difficulties arise due to the appearance of energy equation, as well as the coupling of velocity field with temperature.

- The quadratic nonlinearity

\[ Q(\nabla u) = \frac{\mu}{2} |\nabla u + (\nabla u)^t|^2 + \lambda (\text{div} u)^2. \]  

(1.9)

When we multiply the energy equation by \( \theta \) and integrate it, we have trouble in controlling \( \int \theta Q(\nabla u) \), because no \( \| \theta \|_{L^p} \) is available, (no weighted estimate on the gradient of the velocity is known). To proceed, we estimate

\[
\int \theta Q(\nabla u) \leq C \| \theta \bar{x}^{-\frac{2}{n}} \|_{L^{\frac{n}{2}}} \| \sqrt{Q(\nabla u)} \|_{L^{\frac{8}{n+4}}} \| \bar{x}^\frac{8}{n+4} \sqrt{Q(\nabla u)} \|_{L^2},
\]

and, for the last term \( \| \bar{x}^\frac{8}{n+4} \sqrt{Q(\nabla u)} \|_{L^2} \), we control it by the left-hand side of energy equation (1.1), however, doing so leaves the \( \int \bar{x}^B \Delta \theta \) in trouble. Fortunately, the convexity of \( \bar{x}^B \), non-negativity of \( \theta \), and formula of partial integration imply

\[
\int \bar{x}^B \Delta \theta \sim \int \Delta \bar{x}^B \theta \geq 0. \tag{1.10}
\]

We will discuss (1.10) in detail in Lemma 3.1 below.

- The strong coupling of velocity field with temperature makes the \textit{a priori} estimates on approximating solutions even more delicate. For example, if we multiply (1.1)3 by \( \theta_t \), we obtain (see (3.59))

\[
\frac{d}{dt} \| \nabla \theta_t \|_{L^2}^2 + \| \sqrt{\rho \theta_t} \|_{L^2}^2 \leq \| \nabla \theta_t \|_{L^2}^2 + \ldots
\]

In order to control \( \| \nabla \theta_t \|_{L^2}^2 \), we differentiate (1.1)3 and multiply it by \( \theta_t \), to find

\[
\frac{d}{dt} \| \sqrt{\rho \theta_t} \|_{L^2}^2 + \| \nabla \theta_t \|_{L^2}^2 \leq C \int \theta_t Q(\nabla u) \theta_t + \ldots
\]

\[
\leq C \| \theta_t \bar{x}^{-\frac{2}{n}} \|_{L^{\frac{n}{2}}} \| \sqrt{Q(\nabla u) \bar{x}^\frac{1}{2}} \|_{L^2} \| \bar{x}^\frac{1}{2} \|_{L^{\frac{8}{n+4}}} \| \nabla u \|_{L^2}^2 + \ldots
\]

\[
\leq \frac{1}{2} \| \nabla \theta_t \|_{L^2}^2 + C(1 + \| \sqrt{\rho \theta_t} \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2) \| \nabla u_t \|_{L^2}^2 + \ldots
\]

where the last inequality owes to the dependence of \( A \) on \( \| \sqrt{\rho \theta_t} \|_{L^2} \) and \( \| \nabla \theta \|_{L^2} \) (see (3.34)-(3.35), (3.39)). We observe that the interdependence of each other is on different level. See the detailed deduction in Section 3.

With the above ideas, we prove Theorem 1.1 by means of \textit{domain expansion} technique. In Section 2 we collect some useful known lemmas, In Section 3 we consider the approximating solutions in bounded balls \( B_R \), and derive the \textit{a priori} estimates which are independent of the size of \( B_R \), the final Section 4 is devoted to proving Theorem 1.1 via a standard limit procedure.

2. Some auxiliary lemmas. The first lemma is for the existence results of (1.1) in a bounded ball \( B_R = \{ x \in \mathbb{R}^2 : |x| < R \} \) with strictly positive initial density, whose proof is similar to that in [7] by Cho-Kim.
Lemma 2.1. Assume that
\[ \rho_0^R \in H^2(B_R), \quad \inf_{x \in \partial B_R} \rho_0^R(x) > 0, \quad u_0^R, \theta_0^R \in H^1_0 \cap H^2(B_R). \quad (2.11) \]
Then the equations (1.1) with the boundary conditions
\[ u^R(x,t) = 0, \quad \theta^R(x,t) = 0, \quad x \in \partial B_R, \quad t > 0 \quad (2.12) \]
has a unique strong solution \((\rho^R, u^R, \theta^R)\) over \(B_R \times [0,T_R]\) for a \(T_R > 0\), which satisfies
\[ \begin{cases} \rho^R \in C \left([0,T_R]; H^2\right), & \rho(x,t) > 0 \text{ on } \overline{B_R} \times [0,T_R], \\ u^R, \theta^R \in C \left([0,T_R]; H^1_0 \cap H^2\right) \cap L^2 \left(0,T_R; H^3\right), \\ u_i^R, \theta_i^R \in L^\infty \left(0,T_R; L^2\right) \cap L^2 \left(0,T_R; H^1_0\right) \end{cases} \quad (2.13) \]

Lemma 2.2 (Gagliardo-Nirenberg). \([10, 14]\) Let \(\Omega \subseteq \mathbb{R}^N\) be a bounded or unbounded domain with piecewise smooth boundaries. It holds that for any \(v \in W^{1,q}(\Omega) \cap L^r(\Omega)\)
\[ \|v\|_{L^p(\Omega)} \leq C_1 \|v\|_{L^r(\Omega)} + C_2 \|\nabla v\|_{L^q(\Omega)} \|v\|_{L^{1-\gamma}(\Omega)}, \quad (2.14) \]
where the constant \(C_i (i = 1, 2)\) depends only on \(p, q, r, \gamma\), the exponents \(0 \leq \gamma \leq 1, 1 \leq q, r \leq \infty\) satisfy \(\frac{1}{p} = \gamma \left(\frac{1}{q} - \frac{1}{N} + (1 - \gamma)\right)\) and
\[ \begin{align*} &\min\{r, Nq - q\} \leq p \leq \max\{r, \frac{Nq}{N-q}\}, \quad \text{if } q < N; \\ &r \leq p < \infty, \quad \text{if } q = N; \\ &r \leq p \leq \infty, \quad \text{if } q > N. \end{align*} \]
Moreover, \(C_1 = 0\) in case of \(v|\partial \Omega = 0\) or \(\int_\Omega v \, dx = 0\).
As an application of (2.14) for either \(\Omega = \mathbb{R}^2\) or \(\Omega = B_R\), we have
\[ \|v\|_{L^p(\Omega)} \leq C \|v\|_{L^2(\Omega)} \|\nabla v\|_{H^1(\Omega)}^{\frac{q}{2}} \quad \forall \ p \in [2, \infty), \quad (2.15) \]
and
\[ \|v\|_{L^q(\Omega)} \leq C \|v\|_{L^q(\Omega)} + C \|\nabla v\|_{L^q(\Omega)}^{\frac{q}{2}} \|\nabla v\|_{L^q(\Omega)}^{\frac{q}{2}}, \quad \forall \ q \in (2, \infty), \quad (2.16) \]
provided the quantities of right-hand side are finite.

The following result was proved in [19, Theorem B.1] by P. Lions.

Lemma 2.3. Let \(\tilde{\theta} > 2, \ \Omega = \mathbb{R}^2\) or \(\Omega = B_R (R \geq 1)\), and let \(\tilde{D}^{1,2}(\Omega) = \{v \in H^1_{\text{loc}}(\Omega) : \nabla v \in L^2(\Omega)\}\). Then there is some \(C\) independent of \(\Omega\), such that for all \(v \in \tilde{D}^{1,2}(\Omega)\) and \(m \in [2, \infty)\), the following inequality holds true
\[ \left(\int_{\Omega} \frac{|v|^m}{(e + |x|^2) \ln^2(e + |x|^2)} \, dx\right)^{\frac{1}{m}} \leq C \left(\|v\|_{L^2(B_1)} + \|\nabla v\|_{L^2(\Omega)}\right). \quad (2.17) \]

Proof. The original proof requires \(\tilde{\theta} > 1 + \frac{m}{2}\). Here we relax it to the case \(\tilde{\theta} > 2\). The details is available in Appendix (\(\Omega\)).

Remark 3. In (2.17), we have \(\tilde{\theta} = 2\) in case of \(m = 2\). To prove this, we define the smooth cut-off \(\phi^R\) such that
\[ 0 \leq \phi^R \leq 1, \quad \phi^R = 1 \text{ in } B_{\frac{3}{2}}, \quad \phi^R = 0 \text{ in } \mathbb{R}^2 \setminus B_R, \quad |\nabla^k \phi^R| \leq CR^{-k}, \quad k = 1, 2. \quad (2.18) \]
Simple calculation gives
\[ 2 \int_{\mathbb{R}^2} \frac{(1 - \phi^4)v^2}{|x|^2 \ln |x|^2} = 2 \int_{\mathbb{R}^2} \frac{(1 - \phi^4)(x \cdot \nabla)v \cdot v}{|x|^2 \ln |x|^2} - \int_{\mathbb{R}^2} \frac{x \cdot \nabla \phi^4 v^2}{|x|^2 \ln |x|^2} \leq \int_{\mathbb{R}^2} \frac{(1 - \phi^4)v^2}{|x|^2 \ln |x|^2} + C\|\nabla v\|^2_{L^2(\mathbb{R}^2)} + C \int_{\{2 \leq |x| \leq 4\}} v^2. \]

The following weighted $L^p$ bounds will play a critical role in our proof.

**Lemma 2.4.** Let $\eta_0, \bar{x}$ be as in (1.4), and let $\Omega = \mathbb{R}^2$ or $\Omega = B_R$ $(R \geq 1)$. Suppose the non-negative function $\rho \in L^1(\Omega) \cap L^\infty(\Omega)$ satisfies for some $\eta_1 \in [1, R)$$$
\int_{B_{\eta_1}} \rho dx \geq M > 0. \tag{2.19}$$

Then it satisfies, for any $\eta \in (0, 1]$ and $\varepsilon > 0$,
\[ \|v \varepsilon^{-1}\|_{L^2(\Omega)} + \|v \varepsilon^{-\eta}\|_{L^{2+\varepsilon}(\Omega)} \leq C \left( \|\sqrt{\rho}v\|_{L^2(\Omega)} + (1 + \|\rho\|_{L^\infty(\Omega)}) \|\nabla v\|_{L^2(\Omega)} \right), \tag{2.20} \]
where the constant $C$ may depend on $\varepsilon, \eta$ and $\eta_1$, but not on $\Omega$.

**Proof.** Set $(v^2)_{B_{\eta_1}} = \frac{1}{|B_{\eta_1}|} \|v\|^2_{L^2(B_{\eta_1})}$. Utilizing (2.19), Poincaré and Hölder inequalities, we have
\[ \frac{M}{|B_{\eta_1}|} \|v\|^2_{L^2(B_{\eta_1})} \leq \int_{B_{\eta_1}} \rho \left( (v^2)_{B_{\eta_1}} - v^2 \right) + \int_{B_{\eta_1}} \rho v^2 \leq C\|\rho\|_{L^\infty} \|v\|_{L^1(B_{\eta_1})} + \int_{B_{\eta_1}} \rho v^2 \leq \frac{M}{2|B_{\eta_1}|} \|v\|^2_{L^2(B_{\eta_1})} + C\|\rho\|^2_{L^\infty} \|\nabla v\|^2_{L^2(\Omega)} + \int_{B_{\eta_1}} \rho v^2, \]
that is,
\[ \|v\|^2_{L^2(\Omega)} \leq C \left( \|\sqrt{\rho}v\|^2_{L^2(\Omega)} + \|\rho\|^2_{L^\infty(\Omega)} \|\nabla v\|^2_{L^2(\Omega)} \right). \]
This and (2.17) gives the desired (2.20). \hfill \Box

**Lemma 2.5.** [1, 5] For every $p \in (1, \infty)$ and given functions $F, G \in L^p(B_R)$, let $v, w \in H^1(B_R) \cap W^{1,p}(B_R)$ solve respectively
\[ \mu \Delta v + (\mu + \lambda)\nabla \text{div} v = F \text{ in } B_R, \quad v = 0 \text{ on } \partial B_R, \]
and
\[ \kappa \Delta w = G \text{ in } B_R, \quad w = 0 \text{ on } \partial B_R. \]
Then there is a constant $C$ which is independent of $R$, such that
\[ \|\nabla^2 v\|_{L^p(B_R)} \leq C\|F\|_{L^p(B_R)} \quad \text{and} \quad \|\nabla^2 w\|_{L^p(B_R)} \leq C\|G\|_{L^p(B_R)}. \]

3. **A priori estimates.** The main task of this section is to derive the uniform a priori estimates on the solutions $(\rho^R, u^R, \theta^R)$ of the problem (1.1) and (2.11)-(2.12) obtained in Lemma 2.1. In the following we will drop the superscript and denote simply by $(\rho, u, \theta)$ for the sake of simplicity. In addition, we also use the following simplified conventions
\[ \int f = \int_{B_R} f dx, \quad L^p = L^p(B_R), \quad W^{k,p} = W^{k,p}(B_R), \quad H^k = W^{k,2}(B_R), \]
for every $p \in [1, \infty]$ and $k = 1, 2$.

The first lemma is for a weighted integral of the nonlinearity $Q(\nabla u)$ in energy equation.

**Lemma 3.1.** Let $(\rho, u, \theta)$ be the solution to the problem (1.1) and (2.11)-(2.12). Then,

$$
\int_{B_R} Q(\nabla u) |x|^{\tilde{b}} \leq \int_{B_R} |c_v(\rho \theta_t + \rho u \nabla \theta) + \rho \theta \text{div} u| |x|^{\tilde{b}},
$$

(3.21)

where the constant $\tilde{b} > 0$, and $Q(\nabla u)$ is defined in (1.9).

**Proof.** Since $\theta_0 \geq 0$, apply standard maximum principle to (1.1) (cf. [8]) to receive

$$
\inf_{x \in \mathbb{R}^2, t \geq 0} \theta(x, t) \geq 0.
$$

(3.22)

Next, for fixed $R$ and constant $d > 0$, introduce

$$
\varphi(r) = \begin{cases} 
\frac{R^{\tilde{b}+1}}{d(R-d)^\tilde{b}} \ln \frac{R}{R-d}, & 0 \leq r \leq R - d; \\
\frac{R^{\tilde{b}+1}}{d} \frac{1}{r^\tilde{b}} \ln \frac{R}{r}, & R - d \leq r \leq R.
\end{cases}
$$

(3.23)

Direct calculation shows $\varphi(r) \geq 0$ and $\varphi(R) = 0$. Furthermore,

$$
\varphi'' + \frac{1 + 2\tilde{b}}{r} \varphi' + \tilde{b}^2 \frac{\varphi}{r^2} \geq 0 \quad \text{if} \quad r < R - d
$$

(3.24)

and

$$
\varphi'' + \frac{1 + 2\tilde{b}}{r} \varphi' + \tilde{b}^2 \frac{\varphi}{r^2} = 0 \quad \text{if} \quad R - d < r.
$$

(3.25)

Let $\varphi^\epsilon(|x|)$ be the mollification of $\varphi(|x|)$ over $B_R$. Multiply (1.1)$_3$ by $\varphi^\epsilon(|x|)|x|^{\tilde{b}}$ to find

$$
\int_{B_R} Q(\nabla u) \varphi^\epsilon(|x|)|x|^{\tilde{b}} + \kappa \int_{B_R} \Delta \theta \varphi^\epsilon(|x|)|x|^{\tilde{b}}
$$

$$
= \int_{B_R} (c_v(\rho \theta_t + \rho u \cdot \nabla \theta) + \rho \theta \text{div} u) \varphi^\epsilon(|x|)|x|^{\tilde{b}}.
$$
Therefore, we obtain (3.21) by sending $d$ to zero in (3.28). 

By the fact $\Delta |x|^b \geq 0$, (3.22)-(3.23), the first two terms on the right side of (3.27) guarantees that

$$\int_{B_R} \Delta \varphi^c |x|^b \ = \ \int_{\partial B_R} \frac{\partial \theta}{\partial n} \varphi^c |x|^b + \int_{B_R} \theta (\Delta \varphi^c |x|^b + 2 |\nabla \varphi^c| |x|^b + |\varphi^c| \Delta |x|^b)$$

$$= \int_{\partial B_R} \frac{\partial \theta}{\partial n} \varphi^c |x|^b + \int_{|x| < R-d} \theta \varphi^c \Delta |x|^b$$

$$+ \int_{|x| > R-d} \theta (\Delta \varphi^c |x|^b + 2 |\nabla \varphi^c| |x|^b + |\varphi^c| \Delta |x|^b).$$

By the fact $\Delta |x|^b \geq 0$, (3.22)-(3.23), the first two terms on the right side of (3.27) satisfies

$$\lim_{\epsilon \to 0} \int_{\partial B_R} \frac{\partial \theta}{\partial n} \varphi^c |x|^b = \int_{\partial B_R} \frac{\partial \theta}{\partial n} \varphi^c |x|^b = 0 \quad \text{and} \quad \int_{B_{R-d}} \theta \varphi^c \Delta |x|^b \geq 0.$$

As to the last term in (3.27), the continuity of $\varphi^c$ and $\varphi''$ in $\{x : R - d < |x|\}$, (3.24)-(3.25) guarantee that

$$\lim_{\epsilon \to 0} \int_{|x| < R-d} \theta (\Delta \varphi^c |x|^b + 2 |\nabla \varphi^c| |x|^b + |\varphi^c| \Delta |x|^b)$$

$$= \lim_{\epsilon \to 0} \int_{|x| < R-d} \theta |x|^b (\varphi'' + \frac{1 + 2 \check{b}}{|x|} (\varphi')' + \check{b}^2 |x|^2 \varphi)$$

$$= \int_{|x| < R-d} \theta |x|^b (\varphi'' + \frac{1 + 2 \check{b}}{|x|} \varphi' + \check{b}^2 |x|^2 \varphi)$$

$$= 0.$$

This proves (3.26), and thus,

$$\int_{B_R} Q (\nabla u) \varphi |x|^b \leq \int_{B_R} |c_v (\rho \theta_t + \rho u \nabla \theta) + \rho \theta v u| |x|^b.$$

Observe that, for fixed $R$,

$$\lim_{d \to 0} \varphi(|x|) = 1.$$

Therefore, we obtain (3.21) by sending $d$ to zero in (3.28). \hfill \Box

For convenience, we define

$$\psi(T) = 1 + \sup_{0 \leq t \leq T} \left( \| \bar{x}^a \rho \|_{L^1 \cap H^1 \cap W^{1,q}} + \| \sqrt{\rho} \|_{L^2} + \| \nabla u \|_{L^2} + \| \nabla \theta \|_{L^2} + \| \sqrt{\rho} (|u| + |\theta|) \|_{L^2} \right)$$

(3.29)

and

$$C_0 \geq 1 + \| \bar{x}^a \rho \|_{L^1 \cap H^1 \cap W^{1,q}} + \| \sqrt{\rho} (|u_0| + \theta_0) \|_{L^2}^2$$

$$+ \| \nabla u_0 \|_{H^1} + \| \nabla \theta_0 \|_{H^1} + \| g_1 \|_{L^2} + \| g_2 \|_{L^2},$$

where $a$, $q$, and $g_1$, $g_2$ are as defined in Theorem 1.1.
Introduce smooth cut-off $\zeta_n$ satisfying

$$0 \leq \zeta_n \leq 1, \quad \zeta_n = 1 \text{ in } B_{R - \frac{1}{n}}, \quad \zeta_n \to 1 \text{ in } B_R, \quad |\nabla \zeta_n| \leq 2n.$$  

By (2.13), it satisfies, for any fixed $R$,

$$\int_0^t \int_{B_R} \Delta \theta \zeta_n = -\int_0^t \int_{B_R} \nabla \theta \nabla \zeta_n \leq C \int_0^t \|\nabla \theta\|_{L^\infty(\text{dist}(x, \partial B_R) < \frac{1}{n})} \to 0 \quad (n \to \infty).$$

If we multiply the momentum equations (1.1)$_2$ by $u$, the energy equation (1.1)$_3$ by $\zeta_n$, we easily deduce

$$\int \left(\frac{1}{2} |u|^2 + c_v \rho \theta\right) (x, t) = \int \left(\frac{1}{2} \rho_0 |u_0|^2 + c_v \rho_0 \theta_0\right) \leq C_0, \quad t \geq 0. \quad (3.30)$$

By virtue of (2.18), (3.30), we multiply the mass equation (1.1)$_1$ by $\theta R$ to receive

$$\frac{d}{dt} \int \rho \theta R \geq -C \|\rho\|_{L^2} \|\sqrt{\rho} u\|_{L^2} \geq -C_1. \quad (3.31)$$

Assume that, in addition to (2.11), there is some large $R_0 \in (1, \frac{4}{\pi})$ to satisfy

$$\|\rho_0\|_{L^1(B_{R_0})} \geq \frac{1}{2}. \quad (3.32)$$

With (3.32), integrating (3.31) yields for $T_1 \leq \min\{1, (4C_1)^{-1}\}$,

$$\inf_{t \in [0, T_1]} \int_{B_{2R_0}} \rho \geq \inf_{t \in [0, T_1]} \int_{B_{2R_0}} \rho R^2 \geq \int_{B_{2R_0}} \rho_0 R^2 - C_1 T_1 \geq \frac{1}{4}. \quad (3.33)$$

On account of (2.20), (3.29)-(3.30), (3.33), we have for all $t \in [0, T_1]$

$$\|u \tilde{x}^{-1}\|_{L^2} + \|\theta \tilde{x}^{-1}\|_{L^2} + \|u \tilde{x}^{-\eta}\|_{L^{\frac{2+\eta}{\eta}}} + \|\theta \tilde{x}^{-\eta}\|_{L^{\frac{2+\eta}{\eta}}} \leq C_{\eta, \epsilon} \psi^2(t), \quad (3.34)$$

where $\eta \in (0, 1]$ and $\epsilon > 0$.

Next, thanks to (2.15), (3.29), (3.34), (1.1)$_2$, we deduce from Lemma 2.5 that

$$\|\nabla^2 u\|_{L^2} \leq C \left(\|\rho u\|_{L^2} + \|\rho \nabla \theta\|_{L^2} + \|\theta \nabla \rho\|_{L^2}\right) + C \|\rho u \cdot \nabla u\|_{L^2}
\leq C \|\rho\|_{L^\infty} \|\sqrt{\rho} u\|_{L^2} + \|\rho \tilde{x}^{-\eta}\|_{W^{1, q}} \left(\|\nabla \theta\|_{L^2} + \|\tilde{x}^{-\eta} \theta\|_{L^{\frac{2+\eta}{\eta}}}\right)
\quad + C \|\rho \tilde{x}^{-\eta}\|_{L^\infty} \|\tilde{x}^{-\eta}\|_{L^4} \|\nabla u\|_{L^{\frac{2}{n}}} \|\nabla u\|_{H^1}
\leq C \psi^7 + \frac{1}{2} \|\nabla^2 u\|_{L^2}, \quad (3.35)$$

and hence

$$\|\nabla^2 u\|_{L^2} \leq C \psi^7, \quad \text{and} \quad \|\nabla u\|_{L^r} \leq C \psi^{\frac{2r+12}{r}}, \quad \forall \ r \in (2, \infty). \quad (3.36)$$

Again using Lemma 2.5 to equations (1.1)$_3$, we estimate

$$\|\nabla^2 \theta\|_{L^2} \leq C \left(\|\rho \theta\|_{L^2} + \|\rho \theta \nabla u\|_{L^2} + \|\nabla u\|_{L^4}^2\right) + C \|\rho u \nabla \theta\|_{L^2}
\leq C \psi^8 + \frac{1}{2} \|\nabla^2 \theta\|_{L^2}. \quad (3.37)$$

This and inequalities (3.29), (3.34), (3.36) imply

$$\|\nabla^2 \theta\|_{L^2} \leq C \psi^8, \quad \text{and} \quad \|\nabla \theta\|_{L^r} \leq C \psi^{\frac{2r+14}{r}}, \quad \forall \ r \in (2, \infty). \quad (3.37)$$
Proposition 1. Let $\psi$ be as defined in (3.29), and let $(\rho, u, \theta)$ be a solution to the IBV problem (1.1) and (2.11)-(2.12). Then, it satisfies for all $T \in [0, T_1]$}

$$
\psi(T) + \int_{0}^{T} \left( \|\nabla u(t)\|_{L^2}^2 + \|\nabla \theta(t)\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2 \right) dt \leq C,
$$

(3.38)

where, and in what follows, $C$ and $C_i$ denote the generic constants which may depend on $\mu, \lambda, \kappa, c_v, a, \beta, q, R_0$, and the initial data $C_0$, but not on the size of $B_R$; additionally, $C_\alpha$ is used to emphasize that $C$ relies upon $\alpha$.

Proof. We subdivide process into different steps.

**Step 1.** estimate on $\|\sqrt{\rho \theta}\|_{L^2}^2$.

It is clear to see that, for $0 < \beta < \bar{b} \leq \frac{9}{2}$, 

$$
\bar{x}^\beta \leq C(1 + |x|^\bar{b}) < C\bar{x}^{\frac{9}{2}}.
$$

Thus, from (3.22) and Lemma 3.1 we deduce 

$$
\int_{\Omega} Q(\nabla u)\bar{x}^\beta 
\leq C\|\nabla u\|_{L^2}^2 + C \int (\rho|\theta| + \rho|u|\|\nabla \theta\| + \rho \theta|\text{div} u|) |x|^{\bar{b}}
\leq C\|\nabla u\|_{L^2}^2 + C\|\sqrt{\rho \bar{x}^{\frac{9}{2}}}\|_{L^2} (\|\sqrt{\rho \theta}\|_{L^2} + \|\sqrt{\rho u}\|_{L^2} \|\nabla \theta\|_{L^2} + \|\nabla u\|_{L^2} \|\sqrt{\rho \theta}\|_{L^2})
\leq C_0 + C \psi^3.
$$

(3.39)

After multiplied by $\theta$, it satisfies from (1.1) that 

$$
c_v \frac{d}{dt} \|\sqrt{\rho \theta}\|_{L^2}^2 + \kappa \|\nabla \theta\|_{L^2}^2 \leq C \int \rho \theta^2|\text{div} u| + \int \theta Q(\nabla u).
$$

(3.40)

By virtue of (3.29), (3.34), (3.36), (3.39), we estimate 

$$
\int \rho \theta^2|\text{div} u| \leq \|\rho \bar{x}^\alpha\|_{L^\infty} \|\theta \bar{x}^{-\frac{\alpha}{2}}\|_{L^4}^2 \|\nabla u\|_{L^2} \leq C \psi^6
$$

and 

$$
\int \theta Q(\nabla u) \leq \|\theta \bar{x}^{-\frac{\alpha}{2}}\|_{L^\infty} \|\bar{x}^{\frac{\alpha}{2}} \sqrt{Q(\nabla u)}\|_{L^2} \|\nabla u\|_{L^\infty}^2 \leq C_0 + C \psi^9.
$$

(3.41)

With the last two inequalities in hand, we integrate (3.40) and obtain 

$$
\|\sqrt{\rho \theta}\|_{L^2}^2 + \int_{0}^{T} \|\nabla \theta\|_{L^2}^2 \leq 2C_0 + CT \psi^9, \quad \forall \ T \in [0, T_1].
$$

(3.42)

**Step 2.** estimate on $\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho u}_t\|_{L^2}^2 + \int_{0}^{T} \|\nabla u_t\|_{L^2}^2$.

Multiply (1.1)$_2$ by $u_t$ gives that 

$$
\frac{d}{dt} \left( \mu \int |\nabla u|^2 + \frac{\mu + \lambda}{2} \int |\text{div} u|^2 \right) + \|\sqrt{\rho u}_t\|_{L^2}^2 = - \int \rho u \cdot \nabla u \cdot u_t - \int \nabla P u_t.
$$

(3.43)

Thanks to (3.29), (3.34), (3.36), 

$$
- \int \rho u \cdot \nabla u \cdot u_t \leq \|\sqrt{\rho u_t}\|_{L^2} \|\nabla u\|_{L^4} \|\sqrt{\rho \bar{x}^{\frac{9}{2}}}\|_{L^\infty} \|u \bar{x}^{-\frac{9}{2}}\|_{L^4} \leq C \psi^8.
$$
The Cauchy inequality shows

\[- \int \nabla P u_t = R \int \rho \theta \nabla u_t \leq C_{\delta_1} \psi^3 + \delta_1 \| \nabla u_t \|_{L^2}^2,\]

where the small constant \( \delta_1 > 0 \) is to be determined. Hence, it satisfies for all \( T \in [0, T_1] \)

\[
\| \nabla u \|_{L^2}^2 + \int_0^T \| \sqrt{\rho} u_t \|_{L^2}^2 \leq C_0 + C_{\delta_1} T \psi^8 + \delta_1 \int_0^T \| \nabla u_t \|_{L^2}^2.
\]

(3.44)

Next, differentiate (1.1) in time and multiply it by \( u_t \), to find

\[
\frac{d}{dt} \| \sqrt{\rho} u_t \|_{L^2}^2 + \mu \int |\nabla u_t|^2 + (\mu + \lambda) \int (\nabla u_t)^2 = - \int \rho \theta |u_t|^2 - \int (\rho u_t) \nabla u u_t - \int u_t \cdot \nabla P_t.
\]

(3.45)

Defining

\[
\tilde{g}_t = \rho_0^{-\frac{1}{2}} (\mu \triangle u_0 + (\mu + \lambda) \nabla \text{div} u_0 - \nabla (\rho_0 \theta_0)),
\]

it has

\[
\int \rho |u_t|^2 (x, 0) \leq \lim_{t \to 0} \sup_{t \leq 0} \int \rho^{-1} |\mu \triangle u + (\mu + \lambda) \nabla \text{div} u - \nabla (\rho \theta) - \rho u \cdot \nabla u|^2
\]

\[
\leq \| \tilde{g}_t \|_{L^2}^2 + C_0^5 \leq C C_0^5.
\]

(3.46)

Observe from (2.16), (3.29), (3.34), (3.36) that

\[
\| u \tilde{x}^{-\frac{3}{2}} \|_{L^\infty} \leq C \left( \| u \tilde{x}^{-\frac{3}{2}} \|_{L^\infty} + \| \nabla (u \tilde{x}^{-\frac{3}{2}}) \|_{L^3} \right)
\]

\[
\leq C_a \left( \| u \tilde{x}^{-\frac{3}{2}} \|_{L^\infty} + \| \nabla u \|_{L^3} + \| \tilde{x}^{-\frac{3}{2} + \epsilon} \|_{L^3} \right)
\]

\[
\leq C_a \psi^3,
\]

(3.47)

and thus

\[
\| \sqrt{\rho} u \|_{L^\infty} \leq \| \rho \tilde{x}^\frac{1}{2} \|_{L^\infty} \| u \tilde{x}^{-\frac{3}{2}} \|_{L^\infty} \leq C \psi^4.
\]

(3.48)

Similarly

\[
\| \sqrt{\rho} \theta \|_{L^\infty} \leq C \psi^4.
\]

(3.49)

Using (3.34), (3.36), (3.48)-(3.49), one deduces

\[- \int \rho_1 |u_t|^2 = -2 \int \rho u u_t \nabla u_t
\]

\[
\leq \frac{\mu}{6} \| \nabla u_t \|_{L^2}^2 + C \| \sqrt{\rho} u \|_{L^\infty} \| \sqrt{\rho} u_t \|_{L^2} \leq \frac{\mu}{6} \| \nabla u_t \|_{L^2}^2 + C \psi^{10},
\]

\[- \int (\rho u_t) \nabla u u_t = \int (u \nabla \rho + \rho \nabla \text{div} u) u \nabla u u_t = \int \rho u_t \nabla u u_t
\]

\[
\leq \| \rho \tilde{x}^\frac{1}{2} \|_{W^{1, \infty}} \| \nabla u \|_{L^2} \| \tilde{x}^{-\frac{3}{2}} \|_{L^\infty} \| \tilde{x}^{-\frac{3}{2}} \|_{L^\infty} \leq \frac{\mu}{6} \| \nabla u_t \|_{L^2}^2 + C \psi^{14},
\]

\[- \int \rho \psi |u_t|^2 \leq \frac{\mu}{6} \| \nabla u_t \|_{L^2}^2 + C \psi^{10}.
\]
and
\[-\int \nabla P_t u_t = -R \int (u \nabla \rho + \rho \nabla u) \theta \nabla u_t + R \int \rho \theta \nabla u_t \]
\[\leq \|\tilde{x}^a\|_{W^{1,4}} \|u \tilde{x}^a - \frac{t}{2}\|_{L^4} \|\theta \tilde{x}^a - \frac{t}{2}\|_{L^4} \leq C \|\nabla u_t\|_{L^2}
\]
\[+ (\|\rho\|_{L^\infty} \|\nabla u\|_{L^2} + \|\rho \theta\|_{L^2}) \|\nabla u_t\|_{L^2} \leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C\psi^{11}.
\]
Substituting the above inequalities into (3.45), integrating it, and using (3.46), we obtain
\[\|\sqrt{\rho} u_t\|_{L^2}^2 + \int_0^T \|\nabla u_t\|_{L^2}^2 \leq \|\sqrt{\rho} u_t(x, 0)\|_{L^2}^2 + C T \psi^{14}.
\] (3.50)
Therefore, combining (3.44) with (3.50), choosing \(\delta_1\) small, we get for all \(T \in [0, T_1]\)
\[\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho} u_t\|_{L^2}^2 + \int_0^T \left(\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2\right) dt \leq C \|\psi\|_{C_0^5} + C T \psi^{14}.
\] (3.51)

**Step 3.** estimate on \(\|\tilde{x}^a\|_{L^1 \cap H^1 \cap W^{1,4}}\).
It follows from (2.16), (3.29), (3.34), (3.36)-(3.37) that
\[\|\rho u_t\|_{L^4} = \|\rho \tilde{x}^a u_t \tilde{x}^{-a}\|_{L^4} \leq C \psi^2 \left(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u_t\|_{L^2}\right),
\]
\[\|\rho u \nabla u\|_{L^2} \leq C \|\rho u\|_{L^\infty} \|\nabla u\|_{L^2} \leq C \psi^{12},
\]
and
\[\|\nabla (\rho \theta)\|_{L^4} \leq C \|\nabla \rho \tilde{x}^a\|_{L^4} \|\theta \tilde{x}^{-a}\|_{L^\infty} + \|\rho\|_{L^\infty} \|\nabla \theta\|_{L^4} \leq C \psi^5 + C \psi \|\nabla \theta\|_{L^2} \leq C \psi^9.
\]
Therefore, by (3.51), Lemma 2.5, and (1.1)_2, we deduce for all \(T \in [0, T_1]\)
\[\int_0^T \|\nabla u\|_{L^2}^2 \leq C \int_0^T \left(\|\rho u_t\|_{L^4}^2 + \|\rho u \nabla u\|_{L^4}^2 + \|\nabla (\rho \theta)\|_{L^4}^2\right)
\]
\[\leq C T \psi^{24} + C \psi^4 \int_0^T \|\nabla u_t\|_{L^2}^2 \leq C \|\psi\|_{C_0^5} + C T \psi^{24}.
\] (3.52)

Next, one derives from (1.1)_1 that
\[(\rho \tilde{x}^a)_t + u \cdot \nabla (\rho \tilde{x}^a) + \rho \tilde{x}^a \nabla u - \rho \tilde{x}^a \cdot \nabla \Phi = 0,
\] (3.53)
which implies after integration
\[\frac{d}{dt} \int \rho \tilde{x}^a \leq C \int \rho |u| \tilde{x}^{a-1} \ln^{1+n_0}(e + |x|^2)
\]
\[\leq C \left(\int \rho |u|^2\right)^\frac{1}{2} \left(\int \rho \tilde{x}^{2a-2} \ln^{2(1+n_0)}(e + |x|^2)\right)^\frac{1}{2}
\]
\[\leq C \|\sqrt{\rho} u\|_{L^2} \|\tilde{x}^{a/2} \ln^{1+n_0}(e + |x|^2)\|_{L^\infty} \left(\int \rho \tilde{x}^a\right)^\frac{1}{2}.
\]
By (3.30) and the fact \(a \in (1, 2)\), it has
\[\sup_{t \in [0, T_1]} \|\tilde{x}^a\|_{L^1} \leq C \|\psi\|_{C_0}.
\] (3.54)
Step 4. estimate on \( \| \nabla \theta \|_{L^2}^2 + \| \sqrt{\rho \theta} \|_{L^2}^2 + \int_0^T \| \nabla \theta \|_{L^2}^2 \).

Multiply (1.1) by \( \theta_t \) gives
\[
\frac{k}{2} \frac{d}{dt} \| \nabla \theta \|_{L^2}^2 + c_v \| \sqrt{\rho \theta} \|_{L^2}^2 = -c_v \int \rho u \cdot \nabla \theta_t - R \int \rho \theta_t \text{div} u + \int Q(\nabla u) \theta_t.
\] (3.58)

By virtue of (3.29), (3.34), (4.9), one has
\[
\int \rho u \theta_t \nabla \theta \leq \| \rho \theta \|_{L^2} \| u \|_{L^4} \| \xi \|_{L^6} \| \theta_t \|_{L^{4 \cdot 4}} \leq \| \theta_t \|_{L^{4 \cdot 4}} \| \nabla \theta \|_{L^2} \leq C_{\delta_2} \psi^{10} + \frac{\delta_2}{2} \| \nabla \theta \|_{L^2}^2,
\]
and
\[
\int \rho \theta_t \text{div} u \leq C \| \sqrt{\rho \theta} \|_{L^\infty} \| \sqrt{\rho \theta} \|_{L^2} \| u \|_{L^2} \leq C \psi^6,
\]
where the constant \( \delta_2 > 0 \) is small and to be determined. The same argument as (3.41) runs that
\[
\int \theta_t Q(\nabla u) \leq \| \theta_t \|_{L^\frac{8}{3}} \| \nabla \xi \|_{L^2} \| \sqrt{Q(\nabla u)} \|_{L^2} \| u \|_{L^\frac{8}{3}} \leq \frac{\delta_2}{2} \| \nabla \theta_t \|_{L^2}^2 + C C_0 \psi^{10}.
\]

Substituting the last three inequalities into (3.58) and integrating the resulting expression give rise to
\[
\| \nabla \theta \|_{L^2}^2 + \int_0^T \| \sqrt{\rho \theta} \|_{L^2}^2 \leq C_0 + \delta_2 \int_0^T \| \nabla \theta_t \|_{L^2}^2 + C_{\delta_2} C_0 T \psi^{10}, \quad \forall \ T \in [0, T_1].
\] (3.59)

By defining
\[
\tilde{g}_2 = \rho_0^{-\frac{1}{2}} \left( \kappa \nabla \theta_t + \frac{\mu}{2} \left| \nabla u_0 + (\nabla u_0)^{tr} \right|^2 + \lambda (\text{div} u_0)^2 \right),
\]
we have
\[ \int \rho |\theta_t|^2 (x, 0) \leq \| \tilde{g}_2 \|_{L^2}^2 + CC_5 \leq CC_5. \] (3.60)

Differentiating (1.1) in time, multiplying it by \( \theta_t \), we have
\[
\frac{c_v}{2} \frac{d}{dt} \| \sqrt{\rho} \theta_t \|_{L^2}^2 + \kappa \| \nabla \theta_t \|_{L^2}^2
\]
\[ = -c_v \int \rho |\theta_t|^2 - c_v \int (\rho u_t) \nabla \theta_t - \int P_t \text{div} u \theta_t - \int P \text{div} u_t \theta_t
\]
\[ + \mu \int (\nabla u + \nabla u^{tr}) : (\nabla u + \nabla u^{tr}) \theta_t + 2 \lambda \int \text{div} \text{div} u \theta_t \theta_t
\]
\[ = \sum_{i=1}^{6} I_i. \]

It satisfies from (1.1), (3.29), (3.48) that
\[ I_1 = -c_v \int \rho |\theta_t|^2 = -2c_v \int \rho u \nabla \theta_t \theta_t
\]
\[ \leq \frac{\kappa}{8} \| \nabla \theta_t \|_{L^2}^2 + C \| \sqrt{\rho} u \|_{L^\infty} \| \sqrt{\rho} \theta_t \|_{L^2}^2 \leq \frac{\kappa}{8} \| \nabla \theta_t \|_{L^2}^2 + C \psi^{10}. \]

In view of (3.29), (3.34), (3.36)-(3.37), (3.48)-(3.49), we deduce
\[ I_2 = - \int (\rho u_t) \nabla \theta_t
\]
\[ = \int (u \nabla \rho + \rho \text{div} u) u \theta_t \nabla \theta - \int \rho u_t \theta_t \nabla \theta
\]
\[ \leq \| \rho \tilde{x}^a \|_{H^{1, \infty}} \| u \tilde{x}^{-\frac{3}{4}} \|_{L^\infty} \| \theta_t \tilde{x}^{-\frac{3}{4}} \|_{L^2} \| \nabla \theta \|_{L^2}
\]
\[ + \| \sqrt{\rho} u \|_{L^\infty} \| \nabla u \|_{L^4} \| \nabla \theta \|_{L^4} \| \sqrt{\rho} \theta_t \|_{L^2} + \| \rho \tilde{x}^a \|_{L^\infty} \| \tilde{x}^{-\frac{3}{4}} \theta_t \|_{L^4} \| \sqrt{\rho} u_t \|_{L^2}
\]
\[ \leq \frac{\kappa}{8} \| \nabla \theta_t \|_{L^2}^2 + C \psi^{14}, \]

and
\[ I_3 = \int (\rho \theta_t) \text{div} u \theta_t
\]
\[ = - \int (u \nabla \rho + \rho \text{div} u) \theta_t \text{div} u + \int \rho |\theta_t|^2 \text{div} u
\]
\[ \leq \frac{\kappa}{8} \| \nabla \theta_t \|_{L^2}^2 + C \psi^{14}. \]

The Cauchy inequality together with (3.29) and (3.49) show
\[ I_4 = \int P \text{div} u \theta_t
\]
\[ \leq \| \nabla u_t \|_{L^2}^2 + C \| \sqrt{\rho} \theta \|_{L^\infty} \| \sqrt{\rho} \theta_t \|_{L^2}^2
\]
\[ \leq \| \nabla u_t \|_{L^2}^2 + C \psi^{10}. \]

In order to deal with \( I_5 \) and \( I_6 \), we need the following inequalities (The proof is available in Appendix (I))
\[ |\nabla u + \nabla u^{tr}|^2 \leq CQ(\nabla u) \quad \text{and} \quad (\text{div} u)^2 \leq CQ(\nabla u). \] (3.62)
It gives from (2.20), (3.33), (3.57) that
\[
\| \vartheta \bar{x}^{- \frac{1}{2}} \|_{L^2_t \mathcal{W}^1_x} \leq C(\| \sqrt{\vartheta} \partial_t \|_{L^2} + (1 + \| \vartheta \|_{L^\infty}) \| \nabla \theta \|_{L^2}) \\
\leq C \| \sqrt{\vartheta} \partial_t \|_{L^2} + C \exp \{ C \vartheta \psi^{13} + C C_0^8 \} \| \nabla \theta \|_{L^2}
\]
and from (3.30), (3.39), (5.1), (3.57) that
\[
\| \sqrt{Q(\nabla u)} \bar{x}^{\frac{3}{2}} \|_{L^2_t}^2 \\
\leq C \| \nabla u \|_{L^2}^2 + C \| \sqrt{\vartheta} \bar{x}^{\frac{3}{2}} \|_{L^2_t} \| \nabla \theta \|_{L^2} + \| \nabla u \|_{L^2} \| \sqrt{\vartheta} \|_{L^2} \\
\leq C C_0^5 + C \psi^{14} + C \exp \{ C \vartheta \psi^{13} + C C_0^8 \} (\| \nabla \theta \|_{L^2} + C_0 \| \nabla \theta \|_{L^2} + C_0^5 + t \psi^{14}) \\
\leq \exp \{ C \vartheta \psi^{21} + C C_0^8 \} (1 + \| \nabla \theta \|_{L^2}).
\]
Next, due to (2.20), (3.35), (3.42), (3.51),
\[
\| \nabla u \|_{L^{\infty} \mathcal{W}^1_x} \leq C \| \nabla u \|_{L^2} + C \| \nabla^2 u \|_{L^2} \\
\leq C (1 + \| \vartheta \bar{x}^{- a} \|_{L^\infty} \| \nabla u \|_{L^2}) \| \nabla u \|_{L^2} \\
+ C \| \vartheta \bar{x}^{- a} \|_{W^{1,4}} \| \nabla u \|_{L^2} + C \| \vartheta \bar{x}^{- a} \|_{W^{1,4}} \left( \| \nabla \theta \|_{L^2} + \| \bar{x}^{- a} \|_{L^{\infty}} \right) \\
\leq \exp \{ C \sqrt{\vartheta} \psi^{21} + C C_0^8 \} (1 + \| \nabla \theta \|_{L^2}).
\]
The above inequalities and (3.62) ensure that
\[
I_5 + I_6 \leq C \| \vartheta \bar{x}^{- \frac{1}{2}} \|_{L^2_t \mathcal{W}^1_x} \| \sqrt{Q(\nabla u)} \bar{x}^{\frac{3}{2}} \|_{L^2_t} \| \nabla u \|_{L^2} \| \nabla u \|_{L^{\infty} \mathcal{W}^1_x} \\
\leq \frac{K}{8} \| \nabla \theta \|_{L^2} + C \psi^2 \\
+ \exp \{ C \sqrt{\vartheta} \psi^{21} + C C_0^8 \} (1 + \| \sqrt{\vartheta} \|_{L^2} + \| \nabla \theta \|_{L^2}) \| \nabla u \|_{L^2}.
\]
Therefore, utilizing (3.59)-(3.60), the estimates on $I_1 \sim I_6$, choosing $\delta_2 > 0$ small, we deduce from (3.61) for all $T \in [0, T_1]$
\[
\sup_{t \in [0, T]} (\| \nabla \theta \|_{L^2} + \| \sqrt{\vartheta} \partial_t \|_{L^2}) + \int_0^T \left( \| \sqrt{\vartheta} \partial_t \|_{L^2} + \| \nabla \theta \|_{L^2} \right) \\
\leq C C_0^5 + C T \psi^{20} + \exp \{ C \sqrt{\vartheta} \psi^{21} + C C_0^8 \} \int_0^T (1 + \| \sqrt{\vartheta} \partial_t \|_{L^2} + \| \nabla \theta \|_{L^2}) \| \nabla u \|_{L^2}.
\]
By the Young’s inequality and (3.51), it has
\[
\exp \{ C \sqrt{\vartheta} \psi^{21} + C C_0^8 \} \int_0^T (1 + \| \sqrt{\vartheta} \partial_t \|_{L^2} + \| \nabla \theta \|_{L^2}) \| \nabla u \|_{L^2} \\
\leq \frac{1}{2} \sup_{t \in [0, T]} (\| \nabla \theta \|_{L^2} + \| \sqrt{\vartheta} \partial_t \|_{L^2}) + \exp \{ C \sqrt{\vartheta} \psi^{21} + C C_0^8 \} \left( \int_0^T \| \nabla u \|_{L^2} \right)^2 \\
\leq \frac{1}{2} \sup_{t \in [0, T]} (\| \nabla \theta \|_{L^2} + \| \sqrt{\vartheta} \partial_t \|_{L^2}) + \exp \{ C \sqrt{\vartheta} \psi^{56} + C C_0^{20} \}.
\]
Thus, (3.64) satisfies for all $T \in [0, T_1]$
\[
\sup_{t \in [0, T]} (\| \nabla \theta \|_{L^2} + \| \sqrt{\vartheta} \partial_t \|_{L^2}) + \int_0^T (\| \sqrt{\vartheta} \partial_t \|_{L^2} + \| \nabla \theta \|_{L^2}) \leq \exp \{ C \sqrt{\vartheta} \psi^{56} + C C_0^{20} \}.
\]
Proof of Proposition 1. By the definition of $\psi$, and the estimates established in Steps 1-4, we obtain
\[
\psi(t) \leq \exp\{C\sqrt{t}\psi^{56} + CC_0^{20}\} \leq C_2 \exp\{C_3\sqrt{t}\psi^{56}\}, \quad \forall \ t \in [0,T_1].
\]
Choosing
\[
T_* = \min\left\{T_1, \left(\frac{\ln 2}{(2C_2)^{56}C_3}\right)^2\right\},
\]
we have
\[
\psi(T_*) \leq 2C_2. \quad (3.66)
\]
On account of (3.66), as well as (3.36)-(3.37), (3.48)-(3.49), (3.57), (3.65), one deduces for $t \in [0,T_*]$
\[
\|\rho\theta\|_{L^q} = \|\rho \overline{x} \theta \overline{x}^{-a}\|_{L^q} \leq \|\rho \overline{x}^{-a}\|^2 \left(\|\sqrt{\rho} \theta\|_{L^2} + \|\nabla \theta\|_{L^2}\right) \leq C(1 + \|\nabla \theta\|_{L^2}),
\]
and
\[
\|\rho u \nabla \theta\|_{L^q} \leq \|\sqrt{\rho} \overline{x}^{-a}\|_{L^q} \leq \|\rho u\|_{L^q} \leq C, \quad (3.67)
\]
Thus, the above inequalities, (3.65), (1.1), and Lemma 2.5 guarantee that
\[
\int_0^{T_*} \|\nabla^2 \theta\|_{L^q}^2 \leq C. \quad (3.67)
\]
In conclusion, the desired (3.38) follows from (3.51)-(3.52), (3.57), (3.65)-(3.67). The proof of Proposition 1 is completed. \hfill \Box

4. Proof of Theorem 1.1. Now we are in a position to prove Theorem 1.1.

4.1. Existence of solutions
Let the initial functions $(\rho_0, u_0, \theta_0)$ satisfy the hypotheses (1.5)-(1.7). Without loss of generality we assume $\|\rho_0\|_{L^1(\mathbb{R}^2)} = 1$, and for some $R_0 \in [1, \frac{R}{2})$,
\[
\|\rho_0\|_{L^1(B_{R_0})} \geq \frac{3}{4}.
\]
The approximate sequence $\{\hat{\rho}_0^R \geq 0\}$ such that
\[
\overline{x}^a \hat{\rho}_0^R \to \overline{x}^a \rho_0 \quad \text{in} \quad L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2) \quad \text{and} \quad \|\hat{\rho}_0^R\|_{L^1(B_{R_0})} \geq \frac{1}{2}. \quad (4.68)
\]
Consider the following elliptic equations in terms of $u_0^R$
\[
\begin{cases}
-\mu \Delta u_0^R - (\mu + \lambda) \nabla \text{div} u_0^R + \rho_0^R u_0^R = -\nabla (\rho_0 \theta_0) * j_{R^{-1}} + \sqrt{\rho_0^R} h_1^R, & \text{in} \ B_R, \\
u_0^R = 0, & \text{on} \ \partial B_R,
\end{cases}
\]
where $\rho_0^R = \hat{\rho}_0^R + R^{-1} e^{-|x|^2} \geq 0$ and $h_1^R = (\sqrt{\rho_0 u_0} + g_1) * j_{R^{-1}}$, with $j_{R^{-1}}$ being the mollifier of width $R^{-1}$. Following the same steps as that in [15], we have
\[
\lim_{R \to \infty} \left(\|\nabla (u_0^R - u_0)\|_{H^1(\mathbb{R}^2)} + \left\|\sqrt{\rho_0^R} u_0^R - \sqrt{\rho_0} u_0\right\|_{L^2(\mathbb{R}^2)}\right) = 0. \quad (4.70)
\]
Next, consider the elliptic equations of $\theta^R_0$

$$
\begin{cases}
-\kappa \Delta \theta^R_0 + \rho^R_0 \theta^R_0 = \frac{\mu}{2} \left| \nabla u^R \right|^2 + \lambda (\text{div} u^R)^2 + \sqrt{\rho^R_0} h^R_2, & \text{in } B_R, \\
\theta^R_0 = 0, & \text{on } \partial B_R,
\end{cases}
$$

(4.71)

where $h^R_2 = (\sqrt{\rho^R_0} + g_2) \ast j_{R^{-1}}$. We will prove that

$$
\lim_{R \to \infty} \left( \left\| \nabla (\theta^R_0 - \theta_0) \right\|_{H^1(\mathbb{R}^2)} + \left\| \sqrt{\rho^R_0} \theta^R_0 - \sqrt{\rho_0} \theta_0 \right\|_{L^2(\mathbb{R}^2)} \right) = 0.
$$

(4.72)

To this end, using (4.68), (4.70)-(4.71), the same method as in (3.39) runs that

$$
\int_{B_R} Q(\nabla u^R) x^b \leq C + C \int_{B_R} (\rho_0 \theta^R_0 + \sqrt{\rho^R_0} h^R_2) x^b \leq C + C \| \sqrt{\rho^R_0} \theta^R_0 \|_{L^2},
$$

which together with (4.70) and (3.41) implies

$$
\int_{\mathbb{R}^2} Q(\nabla u^R) \theta^R_0 \leq C \| \sqrt{Q(\nabla u^R) x^2} \|_{L^2(\mathbb{R}^2)} \| \nabla u^R \|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \| x^{-\frac{2}{3}} \theta^R_0 \|_{L^2(\mathbb{R}^2)}
\leq \frac{1}{4} \left( \left\| \nabla \theta^R_0 \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \sqrt{\rho^R_0} \theta^R_0 \right\|_{L^2(\mathbb{R}^2)}^2 \right) + C.
$$

(4.73)

Extending $\theta^R_0$ by zero to $\mathbb{R}^2$, multiplying (4.71) by $\theta^R_0$, and utilizing (4.73), we receive

$$
\left\| \nabla \theta^R_0 \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \sqrt{\rho^R_0} \theta^R_0 \right\|_{L^2(\mathbb{R}^2)}^2 \leq \int_{\mathbb{R}^2} Q(\nabla u^R) \theta^R_0 + C \left\| \sqrt{\rho^R_0} \theta^R_0 \right\|_{L^2(\mathbb{R}^2)} \left\| h^R_2 \right\|_{L^2(\mathbb{R}^2)}
\leq \frac{1}{2} \left( \left\| \nabla \theta^R_0 \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \sqrt{\rho^R_0} \theta^R_0 \right\|_{L^2(\mathbb{R}^2)}^2 \right) + \left\| h^R_2 \right\|_{L^2(\mathbb{R}^2)}^2 + C.
$$

This, along with (3.34) and regularity of elliptic equation, provides

$$
\left\| x^{-\gamma} \theta^R_0 \right\|_{L^{\frac{2+\gamma}{\gamma}}(\mathbb{R}^2)} \leq C \quad \text{and} \quad \left\| \nabla^2 \theta^R_0 \right\|_{L^2(\mathbb{R}^2)} \leq C.
$$

Therefore, as $R \to \infty$,

$$
\sqrt{\rho^R_0} \theta^R_0 \to \sqrt{\rho_0} \theta_0 \quad \text{in } L^2(\mathbb{R}^2), \quad \bar{x}^{-\gamma} \theta^R_0 \to \bar{x}^{-\gamma} \theta_0 \quad \text{in } L^{\frac{2+\gamma}{\gamma}}(\mathbb{R}^2).
$$

(4.74)

$$
\nabla \theta^R_0 \to \nabla \theta_0 \quad \text{in } H^1(\mathbb{R}^2)
$$

Furthermore, $\theta_0$ solves (4.71) in distribution sense. Recalling (1.7), one has

$$
\theta_0 = \theta_0.
$$

(4.75)

Next, thanks to (1.7) (4.70), (4.74)-(4.75), we multiply (4.71) by $\theta^R_0$ to receive

$$
\limsup_{R \to \infty} \left( \kappa \int_{\mathbb{R}^2} \left| \nabla \theta^R_0 \right|^2 + \int_{\mathbb{R}^2} \rho^R_0 |\theta^R_0|^2 \right) \leq \int_{\mathbb{R}^2} \kappa |\nabla \theta_0|^2 + \int_{\mathbb{R}^2} \rho_0 |\theta_0|^2.
$$

This together with weakly lower semi-continuity and arbitrariness of the sub-sequence yields

$$
\lim_{R \to \infty} \int_{\mathbb{R}^2} |\nabla \theta^R_0|^2 = \int_{\mathbb{R}^2} |\nabla \theta_0|^2, \quad \lim_{R \to \infty} \int_{\mathbb{R}^2} \rho^R_0 |\theta^R_0|^2 = \int_{\mathbb{R}^2} \rho_0 |\theta_0|^2.
$$

(4.76)

Identities (4.76) and (4.74) guarantee that the convergence of $\nabla \theta^R_0$ and $\sqrt{\rho^R_0} \theta^R_0$ is strong. Moreover, if we differentiate (4.71), multiply it by $\nabla \theta^R_0$, utilize (1.7), (4.70),
(4.74)-(4.76), we easily infer the strong convergence of \(\nabla^2 \theta_0^R\), and hence prove the validity of (4.72).

At this point, we have constructed the approximations \((\rho_0^R, u_0^R, \theta_0^R)\) to original initial data \((\rho_0, u_0, \theta_0)\). The corresponding solution \((\rho^R, u^R, \theta^R)\) to (1.1) with (2.12) and initial \((\rho_0^R, u_0^R, \theta_0^R)\) satisfies all the estimates established in previous Section 3.

Extending the \((\rho_0^R, u_0^R, \theta_0^R)\) by zero outside the domain \(B_R\), and denote by \(\bar{\rho}^R = \rho^R \rho^R\), \(\bar{u}^R = \rho^R u^R\), \(\bar{\theta}^R = \rho^R \theta^R\),

with \(\phi^R\) being defined in (2.18). First, inequalities (3.38) and (3.30) show

\[
\sup_{t \in [0,T_\ast]} \left( \|\sqrt{\bar{\rho}^R} \bar{u}^R\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\bar{\rho}^R} \bar{\theta}^R\|_{L^2(\mathbb{R}^2)} + \|\bar{\rho}^R \bar{\theta}^R\|_{L^1 \cap H^1 \cap W^{1,q}(\mathbb{R}^2)} \right)
\leq C \sup_{t \in [0,T_\ast]} \left( \|\sqrt{\bar{\rho}^R} \bar{u}^R\|_{L^2} + \|\sqrt{\bar{\rho}^R} \bar{\theta}^R\|_{L^2} + \|\bar{\rho}^R \bar{\theta}^R\|_{L^1 \cap H^1 \cap W^{1,q}} \right) \tag{4.77}
\leq C.
\]

By Poincaré inequalities, it takes from (3.36)-(3.38), (2.18) that

\[
\sup_{t \in [0,T_\ast]} \left( \|\nabla \bar{u}^R\|_{H^1(\mathbb{R}^2)} + \|\nabla \bar{\theta}^R\|_{H^1(\mathbb{R}^2)} \right)
\leq C \sup_{t \in [0,T_\ast]} \left( \|\nabla \bar{u}^R\|_{H^1} + \|\nabla \bar{\theta}^R\|_{H^1} + \|\bar{u}^R \nabla \phi^R\|_{H^1} + \|\bar{\theta}^R \nabla \phi^R\|_{H^1} \right) \tag{4.78}
\leq C,
\]

and

\[
\int_0^{T_\ast} \left( \|\nabla^2 \bar{u}^R\|_{L^1(\mathbb{R}^2)}^2 + \|\nabla^2 \bar{\theta}^R\|_{L^1(\mathbb{R}^2)}^2 \right)
\leq C \int_0^{T_\ast} \left( \|\nabla^2 \bar{u}^R\|_{L^2}^2 + \|\nabla^2 \bar{\theta}^R\|_{L^2}^2 \right)
+ C \sup_{0 \leq t \leq T_\ast} \left( \|\nabla \bar{u}^R\|_{L^2}^2 + \|\nabla \bar{\theta}^R\|_{L^2}^2 + \|\bar{u}^R \nabla^2 \phi^R\|_{L^2}^2 + \|\bar{\theta}^R \nabla^2 \phi^R\|_{L^2}^2 \right) \tag{4.79}
\leq C + C \sup_{t \in [0,T_\ast]} \|\nabla \bar{u}^R\|_{H^1}^2 \leq C.
\]

Next, by (3.38), (3.47), and (3.52), one has

\[
\int_0^{T_\ast} \|\bar{x}^2 \bar{\partial}_t \bar{\rho}^R\|_{L^\infty(\mathbb{R}^2)} \leq C \int_0^{T_\ast} \left( \|\bar{x}^2 \bar{u}^R \nabla \bar{\rho}^R\|_{L^\infty}^2 + \|\bar{x}^2 \bar{\rho}^R \Delta \bar{u}^R\|_{L^\infty}^2 \right)
\leq C \int_0^{T_\ast} \left( \|\bar{u}^R \bar{x}^2 \bar{\partial}_t\|_{L^\infty}^2 + \|\bar{u}^R \Delta\bar{u}^R\|_{L^2}^2 \right) \|\bar{x}^2 \bar{\rho}^R\|_{W^{1,q}}^2 \tag{4.80}
\leq C.
\]

It yields from (3.38) that

\[
\sup_{t \in [0,T_\ast]} \left( \|\sqrt{\bar{\rho}^R} \bar{u}^R\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\bar{\rho}^R} \bar{\theta}^R\|_{L^2(\mathbb{R}^2)} \right)
\leq C \sup_{t \in [0,T_\ast]} \left( \|\sqrt{\bar{\rho}^R} \bar{u}^R\|_{L^2} + \|\sqrt{\bar{\rho}^R} \bar{\theta}^R\|_{L^2} \right) \tag{4.81}
\leq C.
\]
In terms of Poincaré inequality, (3.37), we obtain
\[
\int_0^{T_\ast} \left( \| \nabla \tilde{u}_t \|_{L^2(\mathbb{R}^2)}^2 + \| \tilde{\theta}_t \|_{L^2(\mathbb{R}^2)}^2 \right) \leq C \int_0^{T_\ast} \left( \| \nabla u_t \|_{L^2(\mathbb{R}^2)}^2 + \| \theta_t \|_{L^2(\mathbb{R}^2)}^2 + \| u_t \|_{L^2(\mathbb{R}^2)}^2 + \| \phi_t \|_{L^2(\mathbb{R}^2)}^2 \right) \leq C.
\]

(4.82)

In summary, inequalities (4.77)-(4.82) ensure that the sequence \((\tilde{\rho}, \tilde{u}, \tilde{\theta})\) converges, up to some subsequences, to limit functions \((\rho, u, \theta)\), that is,
\[
\begin{aligned}
\tilde{x}^\alpha \tilde{\rho}^R &\to \tilde{x}^\alpha \rho \quad \text{in } C_c^\infty(\mathbb{R}^2 \times [0, T_*]) \quad \text{for any } N < \infty; \\
\tilde{x}^{-1} \tilde{u}^R &\to \tilde{x}^{-1} u, \quad \tilde{x}^{-1} \tilde{\theta}^R \to \tilde{x}^{-1} \theta \quad \text{weakly * in } L^\infty(0, T_*; L^2(\mathbb{R}^2)); \\
\sqrt{\tilde{\rho}_t} \tilde{u}_t &\to \sqrt{\rho} u_t, \quad \sqrt{\tilde{\theta}_t} \tilde{\rho}_t \to \sqrt{\theta} \tilde{\rho}_t \quad \text{weakly * in } L^\infty(0, T_*; L^2(\mathbb{R}^2)); \\
\nabla \tilde{u}_t &\to \nabla u_t, \quad \nabla \tilde{\theta}_t \to \nabla \theta_t \quad \text{weakly in } L^2(0, T_*; L^2(\mathbb{R}^2)); \\
\nabla \tilde{u}_t &\to \nabla u_t, \quad \nabla \tilde{\theta}_t \to \nabla \theta_t \quad \text{strongly in } L^2(0, T_*; L^2(\mathbb{R}^2)).
\end{aligned}
\]

Therefore, let \(\phi \in C_c^\infty(\mathbb{R}^2 \times [0, T_*])\), and choose \(\phi(\tilde{x}^R)^4\) as a test function for (1.1) with (2.12) and \((\rho_0^R, u_0^R, \theta_0^R)\), we conclude that \((\rho, u, \theta)\) solve the Cauchy problem (1.1)-(1.3) and satisfy (1.8) by sending \(R \to \infty\).

4.2 Uniqueness of the solutions

It suffices to prove that, in Theorem 1.1, two solutions \((\rho, u, \theta)\) and \((\tilde{\rho}, \tilde{u}, \tilde{\theta})\) stemming from the same initial data must be identical. Set
\[
\Phi = \rho - \tilde{\rho}, \quad \Psi = u - \tilde{u}, \quad \Theta = \theta - \tilde{\theta}.
\]

Subtracting (1.1) satisfied by \((\rho, u, \theta)\) and \((\tilde{\rho}, \tilde{u}, \tilde{\theta})\) yields
\[
\Phi_t + \tilde{u} \cdot \nabla \Phi + \Phi \text{div} \tilde{u} + \rho \text{div} \Psi + \Psi \cdot \nabla \rho = 0.
\]

Choose \(\beta \in (1, a)\) such that \(2 < \frac{2q}{q-(q-2)(a-\beta)} < q\) and multiply it by \(2\Phi \tilde{x}^\beta\), to deduce
\[
\frac{d}{dt}\| \Phi \tilde{x}^\beta \|_{L^2}^2 \leq C \left( \| \tilde{\nabla} \tilde{u} \|_{L^\infty} + \| \tilde{u} \tilde{x}^{-\frac{\alpha}{2}} \|_{L^\infty} \right) \| \Phi \tilde{x}^\beta \|_{L^2}^2 + C \| \tilde{\rho} \tilde{x}^\beta \|_{L^\infty} \| \nabla \Psi \|_{L^2} \| \Phi \tilde{x}^\beta \|_{L^2} \\
+ C \| \Phi \tilde{x}^\beta \|_{L^2} \| \Psi \tilde{x}^{-(a-\beta)} \|_{L^{\frac{2q}{q-(q-2)(a-\beta)}}} \| \tilde{x}^\alpha \nabla \rho \|_{L^\frac{2q}{q-(q-2)(a-\beta)}} \leq C \left( 1 + \| \nabla \tilde{u} \|_{L^\infty} \right) \left( \| \Phi \tilde{x}^\beta \|_{L^2}^2 + \| \sqrt{\tilde{\rho}} \Psi \|_{L^2}^2 \right) + \frac{L_t}{4} \| \nabla \Psi \|_{L^2}^2,
\]

where the second inequality follows from (1.8) and (3.34).

The momentum equations provide us
\[
\rho \Psi_t + \rho u \cdot \nabla \Psi - \mu \Delta \Psi - (\mu + \lambda) \nabla (\text{div} \Psi) = -\rho \Psi \cdot \nabla \tilde{u} - \Phi (\tilde{u}_t + \tilde{u} \cdot \nabla \tilde{u}) - \nabla (\rho \Theta + \Phi \tilde{\theta}).
\]

(4.84)

Thanks to (1.8), (3.34), (3.47), it satisfies that
\[
- \int_{\mathbb{R}^2} \nabla (\rho \Theta + \Phi \tilde{\theta}) \Psi \leq C \| \nabla \Psi \|_{L^2} \left( \| \rho \Theta \|_{L^2} + \| \tilde{\theta} \tilde{x}^{-\beta} \|_{L^\infty} \| \tilde{x}^\beta \Phi \|_{L^2} \right) \leq C \| \nabla \Psi \|_{L^2} \left( \| \sqrt{\rho} \Theta \|_{L^2} + \| \tilde{x}^\beta \Phi \|_{L^2} \right)
\]
and
\[- \int \Phi (|\bar{u}| + |\bar{u}| \|\nabla \bar{u}|) \Psi \leq C \|\Phi \bar{x}^\beta\|_{L^2} \|\Psi \bar{x}^{- \frac{\beta}{2}} \|_{L^4} \left(\|\bar{u}\bar{x}^{- \frac{\beta}{2}} \|_{L^4} + \|\nabla \bar{u}\|_{L^4} \right) + \|\nabla \bar{u}\|_{L^4} \right) \] (4.85)
\[ \leq C \|\Phi \bar{x}^\beta\|_{L^2} \left( \|\sqrt{\Psi}\|_{L^2} + \|\nabla \Psi\|_{L^2} \right) \left(1 + \|\nabla \bar{u}\|_{L^2} \right). \]

With the above inequality and (4.83), we multiply (4.84) by \(\Psi\) and deduce
\[ \frac{d}{dt} \|\sqrt{\bar{\rho}}\|_{L^2}^2 + \|\nabla \Theta\|_{L^2}^2 \leq C \left(1 + \|\nabla \bar{u}_t\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^\infty} \right) \left( \|\Phi \bar{x}^\beta\|_{L^2}^2 + \|\sqrt{\Psi}\|_{L^2}^2 + \|\sqrt{\Theta}\|_{L^2}^2 \right). \] (4.86)

Finally, from (1.1)_3 we deduce
\[ \rho \Theta_t + \rho \cdot \nabla \Theta - \kappa \Delta \Theta \]
\[ = -\rho \Psi \cdot \nabla \bar{\theta} - \Phi (\bar{\theta}_t + \bar{u} \cdot \nabla \bar{\theta}) - \rho \theta \text{div} \Psi - (\rho \Theta + \Phi \bar{\theta}) \text{div} \bar{u}
\[ + \frac{\mu}{2} \left( \nabla \Psi + (\nabla \Psi)^{tr} \right) \cdot (\nabla u + (\nabla u)^{tr} + \nabla \bar{u} + (\nabla \bar{u})^{tr} \right) + \lambda \text{div} \Psi \text{div} (u + \bar{u}). \]

After multiplied by \(\Theta\), it takes
\[ \frac{d}{dt} \|\sqrt{\bar{\rho}}\|_{L^2}^2 + \|\nabla \Theta\|_{L^2}^2 \leq C \|\Phi \bar{x}^\beta\|_{L^2} \left( \|\sqrt{\bar{\rho}}\|_{L^2} + \|\nabla \Theta\|_{L^2} \right) \left(1 + \|\nabla \bar{u}_t\|_{L^2} \right). \] (4.87)

We need to control the right-hand side terms in (4.87). Similar to (4.85), it has
\[ \|\Phi (\bar{\theta}_t + \bar{u} \cdot \nabla \bar{\theta})\Theta\|_{L^1} \leq C \|\Phi \bar{x}^\beta\|_{L^2} \left( \|\sqrt{\bar{\rho}}\|_{L^2} + \|\nabla \Theta\|_{L^2} \right) \left(1 + \|\nabla \bar{u}_t\|_{L^2} \right). \]

Next, by (1.8) and (3.49), it yields
\[ C \|\rho \cdot \nabla \Theta\|_{L^1} \leq C \|\nabla \Psi\|_{L^2} \|\sqrt{\bar{\rho}}\|_{L^2} \|\nabla \Theta\|_{L^2} \leq \|\nabla \Psi\|_{L^2}^2 + C \|\sqrt{\bar{\rho}}\|_{L^2}^2, \]

and
\[ \| (\rho \Theta + \Phi \bar{\theta}) \text{div} \bar{u} \Theta\|_{L^1} \]
\[ \leq \|\nabla \bar{u}\|_{L^\infty} \|\sqrt{\bar{\rho}}\|_{L^2}^2 + C \|\Phi \bar{x}^\beta\|_{L^2} \|\bar{x}^{- \frac{\beta}{2}}\|_{L^4} \|\nabla \bar{u}\|_{L^4} \|\bar{x}^{- \frac{\beta}{2}}\|_{L^\infty} \]
\[ \leq \|\nabla \bar{u}\|_{L^\infty} \|\sqrt{\bar{\rho}}\|_{L^2}^2 + C \|\Phi \bar{x}^\beta\|_{L^2} \left( \|\sqrt{\bar{\rho}}\|_{L^2} + \|\nabla \Theta\|_{L^2} \right). \]

Remember that Lemma 3.1, (3.62), (1.8), the similar method as (3.63) runs that
\[ \| (\nabla \Psi + (\nabla \Psi)^{tr}) \cdot (\nabla u + (\nabla u)^{tr} + \nabla \bar{u} + (\nabla \bar{u})^{tr} \right) \Theta\|_{L^1} + C \|\text{div} \Psi \text{div} (u + \bar{u})\Theta\|_{L^1} \]
\[ \leq C \|\nabla \Psi\|_{L^2} \|\bar{x}^{- \frac{\beta}{2}}\|_{L^\infty} \|\bar{x}^{- \frac{\beta}{2}}\|_{L^4} \|\nabla \bar{u}\|_{L^4} \|\bar{x}^{- \frac{\beta}{2}}\|_{L^\infty} \]
\[ \leq C \|\nabla \Psi\|_{L^2} \|\bar{x}^{- \frac{\beta}{2}}\|_{L^\infty} \|\bar{x}^{- \frac{\beta}{2}}\|_{L^4} \|\nabla \bar{u}\|_{L^4} \|\bar{x}^{- \frac{\beta}{2}}\|_{L^\infty} \]
\[ \leq C \|\nabla \Psi\|_{L^2} \|\sqrt{\bar{\rho}}\|_{L^2} + \|\nabla \Theta\|_{L^2} \].
Taking the above inequalities into account, we deduce from (4.87) that
\[
\frac{d}{dt} \parallel \sqrt{\rho} \Theta \parallel_{L^2}^2 + \parallel \nabla \Theta \parallel_{L^2}^2 \\
\leq C \left(1 + \parallel \nabla \bar{\theta} \parallel_{L^\infty} + \parallel \nabla \bar{u} \parallel_{L^\infty}\right) \left(\parallel \Phi \bar{x}^\beta \parallel_{L^2}^2 + \parallel \sqrt{\rho} \Psi \parallel_{L^2}^2 + \parallel \sqrt{\rho} \Theta \parallel_{L^2}^2\right) \\
+ C \parallel \nabla \Psi \parallel_{L^2}^2.
\] (4.88)

Set
\[G := \parallel \sqrt{\rho} \Theta \parallel_{L^2}^2 + \parallel \sqrt{\rho} \Psi \parallel_{L^2}^2 + \parallel \Phi \bar{x}^\beta \parallel_{L^2}^2.\]

Multiplying (4.84) by a large constant, adding it up to (4.88), we conclude
\[
G'(t) + \parallel \nabla \Psi \parallel_{L^2}^2 + \parallel \nabla \Theta \parallel_{L^2}^2 \\
\leq C \left(1 + \parallel \nabla \bar{u} \parallel_{L^2}^2 + \parallel \nabla \bar{\theta} \parallel_{L^\infty} + \parallel \nabla \bar{u} \parallel_{L^\infty}\right) G.
\] (4.89)

Since \(G(0) = 0\), using (1.8) and integrating (4.89) give birth to \(G(t) = 0\). Hence,
\[
\rho = \bar{\rho}, \ u = \bar{u}, \ \theta = \bar{\theta}, \ a.e. \ (x,t) \in \mathbb{R}^2 \times (0, T^*).
\]
So far we complete the proof of Theorem 1.1.

**Appendix (I): Proof of (3.62).**

**Proof.** We only discuss the case of \(\lambda < 0\). The Cauchy inequality shows
\[
Q(\nabla u) = \frac{\mu}{2} |\nabla u + \nabla u^{tr}|^2 + \lambda (\text{div} u)^2 \\
= 2\mu \left(\frac{\partial u_1}{\partial x_1}\right)^2 + 2\mu \left(\frac{\partial u_2}{\partial x_2}\right)^2 + \mu \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)^2 \\
+ \lambda \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \lambda \left(\frac{\partial u_2}{\partial x_2}\right)^2 + 2\lambda \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} \\
\geq 2\mu \left(\frac{\partial u_1}{\partial x_1}\right)^2 + 2\mu \left(\frac{\partial u_2}{\partial x_2}\right)^2 + \mu \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)^2 + 2\lambda \left(\frac{\partial u_1}{\partial x_1}\right)^2 + 2\lambda \left(\frac{\partial u_2}{\partial x_2}\right)^2 \\
\geq 2(\mu + \lambda) \left(\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2\right) + (\mu + \lambda) \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)^2 \\
\geq \frac{\mu + \lambda}{2} |\nabla u + \nabla u^{tr}|^2.
\]

Similarly,
\[
Q(\nabla u) = 2\mu \left(\frac{\partial u_1}{\partial x_1}\right)^2 + 2\mu \left(\frac{\partial u_2}{\partial x_2}\right)^2 + \mu \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)^2 + \lambda (\text{div} u)^2 \\
\geq \mu \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)^2 + \lambda (\text{div} u)^2 \\
\geq (\mu + \lambda) (\text{div} u)^2.
\]

\(\square\)
Appendix (II): Proof of (2.17).

Proof. Express the left-hand side of (2.17) as

\[ I_1 + I_2 := \int_{\mathbb{R}^2} \frac{|v_{B_1}|^m}{(e + |x|^2) \ln^\theta (e + |x|^2)} + \int_{\mathbb{R}^2} \frac{|v - v_{B_1}|^m}{(e + |x|^2) \ln^\theta (e + |x|^2)}. \]

We easy check that for \( \tilde{\theta} > 2 \)

\[ I_1 \leq |v_{B_1}|^m \int_{\mathbb{R}^2} \frac{1}{(e + |x|^2) \ln^\theta (e + |x|^2)} \leq C\|v\|_{L^1(B_1)}^m. \tag{4.90} \]

Put \( T_k = B_{2^k} \setminus B_{2^{k-1}} \) and decompose

\[ I_2 = \left( \int_{B_1} + \sum_{k=1}^{\infty} \int_{T_k} \right) \frac{|v - v_{B_1}|^m}{(e + |x|^2) \ln^\theta (e + |x|^2)}. \tag{4.91} \]

The definition of \( BMO_m \) space (cf. [21, Chapter IV]) implies for all \( m \in [1, \infty) \)

\[ \int_{B_1} \frac{|v - v_{B_1}|^m}{(e + |x|^2) \ln^\theta (e + |x|^2)} \leq C \int_{B_1} |v - v_{B_1}|^m \leq C \|v\|_{BMO_m(\mathbb{R}^2)}^m. \tag{4.92} \]

Let us control other terms in (4.91). Since \( (e + |x|^2) \ln^\theta (e + |x|^2) \geq 4^{(k-1)} \ln^\theta (e + 4^{k-1}) \) for all \( x \in T_k \), then

\[ \int_{T_k} \frac{|v - v_{B_1}|^m}{(e + |x|^2) \ln^\theta (e + |x|^2)} \leq \frac{1}{4^{(k-1)} \ln^\theta (e + 4^{k-1})} \int_{B_{2^k}} |v - v_{B_1}|^m. \tag{4.93} \]

Notice that

\[ \int_{B_{2^k}} |v - v_{B_1}|^m \leq \int_{B_{2^k}} |v - v_{B_{2^k}}|^m + \int_{B_{2^k}} |v_{B_{2^k}} - v_{B_1}|^m \]

\[ \leq |B_{2^k}| \left( \|v\|_{BMO_m(\mathbb{R}^2)}^m + \sum_{j=1}^{k} |v_{B_{2^j}} - v_{B_{2^{j-1}}}|^m \right) \tag{4.94} \]

where the last inequality owes to

\[ |v_{B_{2^k}} - v_{B_{2^{k-1}}}|^m = \frac{1}{|B_{2^{k-1}}|} \int_{B_{2^{k-1}}} (v_{B_{2^k}} - v)^m \]

\[ \leq \frac{1}{|B_{2^{k-1}}|} \int_{B_{2^{k-1}}} |v_{B_{2^k}} - v_{B_{2^{k-1}}}|^m \quad \text{(Hölder inequality)} \]

\[ \leq \frac{4}{|B_{2^k}|} \int_{B_{2^k}} |v_{B_{2^k}} - v_{B_{2^{k-1}}}|^m \leq 4 \|v\|_{BMO_m(\mathbb{R}^2)}^m. \]

Combining (4.93)-(4.94), we get

\[ \int_{T_k} \frac{|v - v_{B_1}|^m}{(e + |x|^2) \ln^\theta (e + |x|^2)} \leq C_k \|v\|_{BMO_m(\mathbb{R}^2)}^m, \]

where \( c_k = \frac{|B_{2^k}|(1+4k)}{4^{k-1} \ln^\theta (e + 4^{k-1})} \) satisfying \( c_1 = 20 \) and \( c_k = \frac{64\pi}{(k-1)^{\theta-1}} \) if \( k \geq 2 \).
Since \( \sum_{k=1}^{\infty} c_k < \infty \) if \( \bar{\theta} > 2 \), we infer
\[
\sum_{k=1}^{\infty} \int_{T_k} \frac{|v - v_{B_1}|^m}{(e + |x|^2)^{\bar{\theta}}} \leq \|v\|_{BMO_m(\mathbb{R}^2)}^m \sum_{k=1}^{\infty} c_k \leq C \|v\|_{BMO_m(\mathbb{R}^2)}^m.
\]
This and (4.90)-(4.92) ensure that
\[
\int_{\mathbb{R}^2} \frac{|v|^m}{(e + |x|^2)^{\bar{\theta}}(e + |x|^2)} dx \leq C \|v\|_{BMO_m(\mathbb{R}^2)}^m + C \|v\|_{L^2(\mathbb{R}^2)}^m.
\]
However, the John-Nirenberg Lemma ([21, Page 246]) and Poincaré guarantee
\[
\|v\|_{BMO_m(\mathbb{R}^2)} \leq C \|v\|_{BMO(\mathbb{R}^2)} \leq C \|\nabla v\|_{L^2(\mathbb{R}^2)}.
\]
This combines with (4.95) conclude (2.17). \( \square \)

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