Achieving the quantum ground state of a mechanical oscillator using a Bose–Einstein condensate with back-action and cold damping feedback schemes

Sonam Mahajan¹, Neha Aggarwal¹,², Aranya B Bhattachjee² and ManMohan¹

¹ Department of Physics and Astrophysics, University of Delhi, Delhi-110007, India
² Department of Physics, ARSD College, University of Delhi (South Campus), New Delhi-110021, India

E-mail: bhattach@arsd.du.ac.in

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Abstract
We present a detailed study to show the possibility of approaching the quantum ground state of a hybrid optomechanical quantum device formed by a Bose–Einstein condensate (BEC) confined inside a high-finesse optical cavity with an oscillatory end mirror. Cooling is achieved using two experimentally realizable schemes: back-action cooling and cold damping quantum feedback cooling. In both the schemes, we found that increasing the two-body atom–atom interaction brings the mechanical oscillator to its quantum ground state. It has been observed that back-action cooling is more effective in the good cavity limit, while the cold damping cooling scheme is more relevant in the bad cavity limit. It is also shown that in the cold damping scheme, the device is more efficient in the presence of a BEC than in the absence of a BEC.

(Some figures may appear in colour only in the online journal)

1. Introduction

Optomechanical cooling of micro- and nano-mechanical resonators to their quantum ground state is used in a wide variety of sensitive measurements such as detection of weak forces [1–3], small masses [4] and small displacements [5]. Such mechanical resonators can also be used as tools for quantum metrology [6, 7] or as a medium to couple hybrid quantum systems [8, 9]. In recent years, different types of optomechanical systems have established considerable cooling of the vibrational modes of mechanical resonators interacting with an optical cavity [10, 11]. In such systems, the cavity exerts radiation pressure on the mirror. The dynamics of the mirror is coherently controlled by an external pump laser. Therefore, by modifying the pump power, a strong coupling regime can be easily achieved [12]. In such a strong coupling regime, the mechanical oscillator can be cooled to its quantum ground state using the dynamics of back-action [13, 14]. Experimentally, significant cooling of the mechanical mode of the resonator coupled to an optical cavity has been attained by using two different ways of radiation pressure interaction between the intra-cavity field and a vibrational mode: back-action cooling [10, 15, 16] and cold damping feedback [11, 17, 18]. The self-cooling of the mechanical resonator via dynamical back-action arises due to its interaction with the optical cavity through radiation pressure [10, 15, 16]. Depending upon the laser detuning, the correlations, induced by the cavity delay, between the Brownian motion of the resonator and the radiation pressure force lead to either cooling or amplification. Experimentally, a single mechanical mode has been cooled using these effects [19–21]. With the back-action cooling or self-cooling, there is a self-modification in
the dynamics of the mechanical system as the off-resonant operation of the cavity results in the retarded dynamical back-action on the system [10, 19–22]. The cold damping quantum feedback technique is used to cool the oscillating mirror by applying a viscous force on the mirror via radiation pressure generated by another intensity-modulated laser beam on the back of the mirror [23–26]. This technique increases the damping of the system without any additional noise [27, 28]. The other feedback scheme requires a very large mechanical quality factor [29] for the ground state cooling of the oscillator, whereas cooling can be much more conveniently achieved in the cold damping scheme. The optomechanical systems comprised of ultracold atoms enclosed within a cavity have grabbed much attention [30–42]. The resonance frequency of the cavity is altered due to the strong interaction of the condensed atoms with the cavity mode [43]. An all-optical transistor based on a coupled Bose–Einstein condensate (BEC) cavity system [32, 44] has been proposed [45]. A feedback-control scheme based on the phase-contrast imaging of a trapped BEC has been analysed [46, 47]. It has also been shown that in certain regimes, full quantum-field simulations are required in order to successfully cool the BEC close to the ground state [48]. The robustness of a control scheme based on a continuous centre-of-mass position measurement of a BEC is investigated when the measurement signal is corrupted by classical noise, detector inefficiencies, parameter mismatches and a time delay [49]. An optomechanical system consisting of a BEC in an optical cavity has been investigated recently [50]. By virtue of the interaction of BEC with the mechanical oscillator, such a hybrid system was helpful in achieving state engineering of the mechanical mode of an oscillator, which can be easily controlled and highly insensitive to noise effects. Experimentally, the interaction between the ultracold atoms and vibrating membrane has been studied [51]. In our previous work, we have investigated how the stochastic cooling feedback technique together with a gas of condensate confined in an optical cavity can be used to detect weak forces and coherently controls the sensitivity of this hybrid optomechanical system [52].

Motivated by these interesting features in the field of cavity, optomechanics and BEC, we compare the two schemes, i.e. back-action cooling and cold damping quantum feedback, for cooling an optomechanical system in which a BEC is coupled to an optical cavity with a movable mirror. By comparing these two schemes for such a system, we found that the cold damping quantum feedback scheme cools the mechanical oscillator to its ground state in the bad cavity limit ($\kappa \gg \omega_m$), i.e. when cavity bandwidth is greater than the mechanical frequency, while the back-action cooling is efficient in the opposite limit of the good cavity ($\kappa \ll \omega_m$), i.e. when the cavity bandwidth is smaller than the mechanical frequency. We also show that for both the schemes, in the presence of atom–atom coupling, the interaction of the vibrational mode of the mirror, single mode of the intra-cavity field and the condensate fluctuations gives rise to normal-mode splitting (NMS).

2. Model Hamiltonian

In this section, we introduce our model and describe the Hamiltonian for our system. Our model consists of an elongated cigar-shaped gas of $^{87}$Rb in the $|F = 1\rangle$ state with mass $m$ and transition frequency $\omega_p$ of the $|F = 1\rangle \rightarrow |F' = 2\rangle$ transition of the $D_2$ line. The BEC atoms strongly interact with a single one-dimensional quantized cavity mode of frequency $\omega_c$. The cavity has one mirror fixed and another mirror movable of mass $M_m$, which acts as a mechanical oscillator that is free to oscillate at mechanical frequency $\omega_m$. The coherent laser field with frequency $\omega_p$ drives the system through the fixed cavity mirror with amplitude $\eta'$. The cavity and the laser field are treated quantum mechanically as the standing wave and the pump laser field are weak (small photon number). The single-particle Hamiltonian in the rotating wave and dipole approximation for the system under consideration reads [30]

$$H_0 = H_a + H_c + H_m + H_{ca} + H_{im},$$

where $H_a = \frac{\eta^2}{2}$ represents the kinetic energy of the condensate. $H_c$ is given by $-\hbar \omega_c b^\dagger b - \hbar \eta (b - b^\dagger)$, in which the first term gives the energy of the light mode, with lowering (raising) operator $b (b^\dagger) (\{b, b^\dagger\} = 1)$ and cavity-pump detuning $\Delta_c (= \omega_p - \omega_c)$. The second term in $H_c$ is the energy due to the external pump laser. A band-pass filter is used in the detection scheme [53] such that only a single vibrational mode can be considered by neglecting the several other mechanical degrees of freedom arising from the radiation pressure. $H_m = \hbar \omega_m c^\dagger c$ gives the energy of the single vibrational mode of the movable mirror with an annihilation (creation) operator $c (c^\dagger) (\{c, c^\dagger\} = 1)$. As a result of the pressure exerted by the intra-cavity photons on the mirrors, there is an optomechanical coupling between the vibrating mirror and the cavity field. Hence, a force is exerted on the movable mirror by the electromagnetic field which depends on the number of photons in the cavity. This field is phase shifted by $2\kappa l_m$, where $k$ is the wave vector of the light field and $l_m$ is the displacement of the mirror from its equilibrium position. The scattering of the photons into other modes of the electromagnetic field can be ignored, hence validating the single-mode approximation of the cavity field. Also, a reservoir having finite temperature $T$ is attached to the movable mirror. In the absence of the radiation pressure exerted by the optical cavity field, the movable mirror can undergo Brownian motion due to its contact with thermal environment.

$$H_{ca} = \cos^2(k'x)(\hbar \omega_0 b^\dagger b + V_{cl})$$

describes the interaction of the light field with the BEC which involves the classical potential $V_{cl}$, where $\omega_0$ is the optical lattice barrier height per photon given by $\omega_0 = \frac{g_0}{\Delta_c}$, with $g_0$ and $\Delta_c (= \omega_p - \omega_c)$ representing the mirror–atom coupling and atom–pump detuning, respectively. We take $\omega_0 > 0$ as in this case the condensate atom moves towards the nodes of the optical field due to which the lowest bound state is localized at these positions. Therefore, compared to the case $\omega_0 < 0$, the interaction between the optical cavity field and the condensate atom is decreased for $\omega_0 > 0$. $H_{im} = -\hbar \kappa \omega_m b^\dagger b (c + c^\dagger)$ represents the energy due to the nonlinear dispersive coupling between the position quadrature.
of the movable mirror and the intensity of the light field where $\epsilon$ is the mirror–photon coupling. The Hamiltonian in the second quantized form with the two-body interaction is given as

$$H = \int d^3r \Psi^\dagger(\vec{r}) H_0 \Psi(\vec{r}) \frac{4\pi \hbar^2}{2m} + \int d^3r \Psi^\dagger(\vec{r}) [\hat{H} + \frac{\hbar^2}{2m}\nabla^2 - \hbar \xi_{\vec{r}}] \Psi(\vec{r}),$$

where $\Psi(\vec{r})$ is the atom field operator and $a_\imath$ is the s-wave scattering length. Using $\Psi(\vec{r}) = \sum a_\imath w(\vec{r} - \vec{r}_\imath)$, we derive the Bose–Hubbard Hamiltonian. Here, $w(\vec{r} - \vec{r}_\imath)$ is the Wannier function and $a_\imath$ is the corresponding annihilation operator for the $\imath$th bosonic atom. We consider a deep lattice so that overlapping between the Wannier functions of atoms in neighbouring wells is negligible. Extremely high tunability of the optical lattice is one of its well-known advantages. Hence, tunnelling can be made very slow by tuning the lattice potential [54, 55]. Experimentally, the optical lattice depth is tuned such that timescales over which the experiment is performed are smaller than the timescales over which tunnelling takes place; hence the tunnelling of the atoms into neighbouring wells can be neglected in deriving the Hamiltonian. Therefore, the Hamiltonian reads

$$H = K_0 \sum a_\imath^\dagger a_\imath - h\Delta, b^\dagger b + \hbar \omega_m c^\dagger c + P_0(h\nu_0 b^\dagger b + V_\xi),$$

$$\times \sum j a_j^\dagger a_j - \hbar \omega_0 b^\dagger b(c + c^\dagger) - \frac{\hbar}{2} \sum j a_j^\dagger a_j a_j a_j,$$

where $K_0 = \int d^3r w(\vec{r} - \vec{r}_\imath)\left\{ -\frac{\hbar^2 \nabla^2}{2m} \right\} w(\vec{r} - \vec{r}_j)$ is the on-site kinetic energy of the atoms. $P_0 = \int d^3r w(\vec{r} - \vec{r}_\imath) \cos^2(k' x) w(\vec{r} - \vec{r}_j)$ is the effective on-site potential energy of the atoms. The fast term in the above Hamiltonian represents the two-body atom–atom coupling, where $\nu = \frac{\hbar^2 \omega_0}{2m} \int d^3r |w(\vec{r})|^2$ represents the effective on-site atom–atom interaction energy.

3. Linearization of quantum Langevin equations

Now, we study the quantum Langevin equations (QLEs) of the system. The QLEs of motion for the boson field operator $a_\imath$, cavity photons $b$ and movable mirror (mode) operator $c$ are given as

$$\dot{a}_\imath(t) = -\frac{K_0}{h} a_\imath(t) - \frac{P_0}{h} (h\nu_0 b^\dagger(t) b(t) + V_\xi) a_\imath(t) - \frac{\hbar}{2} a_j^\dagger a_j a_\imath a_\imath(t),$$

$$\dot{b}(t) = -iP_0 \nu_0 b(t) \sum a_j^\dagger(t) a_j(t) + i\Delta, b(t) + i\epsilon \omega_m b(t) \times [c(t) + c^\dagger(t)] + \eta \frac{\kappa}{2} b(t) + \sqrt{\kappa b_m(t)},$$

$$\dot{c}(t) = -i\omega_m c(t) + i\epsilon \omega_m b^\dagger(t) b(t) - \Gamma_m c(t) + \sqrt{\Gamma_m} \xi_m(t).$$

Due to the leakage of photons from the mirror, the cavity field is damped. Therefore, we have introduced $\kappa$ as the cavity field damping rate. Also, the interaction of the movable mirror with environment damps the mechanical mode with the damping rate $\Gamma_m$. During the experiment, the condensate atoms are robust and therefore the depletion of atoms is not significant. Since the condensate temperature is much smaller than $\hbar \omega / k_B$, so there is negligible coherent amplification of the condensate motion. The mechanical mode is also affected by a random Brownian force which has $\xi$ as a zero mean value. The vacuum radiation input noise is represented by $b_n(t)$. The correlation functions for the input noise operators are given in appendix A. The QLEs are linearized around the steady state as $\dot{b}(t) \rightarrow \beta + \delta b(t), c(t) \rightarrow \gamma + \delta c(t)$ and $a_\imath(t) \rightarrow \sqrt{\Delta \pm \beta 2} \Gamma / \sqrt{\kappa}$, where $\beta(t) = -\frac{\hbar \Delta + \frac{\Delta}{2} + i\nu_0 V_\xi}{\sqrt{\kappa}}, \gamma(t) = \frac{\epsilon \omega_m}{\sqrt{\kappa}}$ and $\frac{\Delta \pm \beta 2} \Gamma / \sqrt{\kappa}$ are the steady state values of the cavity mode, mechanical mode and condensate density, respectively. $N(\sum a_\imath^\dagger a_\imath(t))$ atoms occupy $M$ number of lattice sites. It is assumed that all the sites of the optical lattice are the same; therefore, $a_j(t)$ is replaced by $a(t)$. By introducing the amplitude and phase quadratures as $\delta q_\imath(t) = [a(t) + a^\dagger(t)], \delta p_\imath(t) = i[b(t) - b^\dagger(t)],$ the QLEs of motion for the boson field operator and $c(t)$ for the cavity field are given as follows:

$$\dot{\delta q_\imath}(t) = -\kappa \frac{\delta q_\imath}(t) + \sqrt{\kappa} \delta p_\imath(t) - \Delta_\imath \delta p_\imath(t),$$

$$\dot{\delta p_\imath}(t) = -\kappa \frac{\delta p_\imath}(t) - \sqrt{\kappa} \delta q_\imath(t) + 2\Delta_\imath \delta q_\imath(t) + 2 \Gamma \delta q_\imath(t),$$

$$\dot{x}\delta\xi(t) = \omega_m \delta\xi(t),$$

$$\dot{\delta y}(t) = \omega_m \delta \eta(t) + 2 \Gamma \delta q_\imath(t) - \Gamma_m \delta p_\imath(t) + W(t),$$

where $\Delta_\imath = -P_0 \nu_0 \Delta \imath + 2 \gamma \nu_0 \gamma, \gamma = P_0 \nu_0 \sqrt{N}, G = \epsilon \omega_m, \nu = K_0 / h + P_0 \nu_0 \beta^2 + P_0 \gamma / \hbar, U_{\text{eff}} = \frac{\epsilon \omega_m}{\hbar}, \beta_1 = v + U_{\text{eff}}, \beta_2 = v + 3U_{\text{eff}}$ and $W(t) = i \sqrt{\Gamma_m} [\xi_m(t) - \xi_m(t)]$, which satisfies the correlation given in appendix A.

Let us now understand how the radiation pressure cools the mechanical oscillator. The radiation pressure exerted on the movable mirror by the optical cavity mode forms a system which acts as another reservoir connected to the mechanical oscillator when the cavity is properly detuned. As a result, the effective temperature of the vibrational mode is the temperature between the two reservoirs, i.e. between the initial bath temperature and the temperature of this effective optical reservoir. This effective temperature is zero in practice. Hence the quantum ground state is achieved when the coupling to the initial reservoir, given by the damping rate of the movable mirror $\Gamma_m$, is much smaller than the coupling to the effective optical reservoir. This explains to us that the radiation pressure coupling should be strong for significant cooling of the mechanical oscillator. Now, we shall see in the following sections how the different cooling techniques help in cooling the mechanical oscillator to its quantum ground state.
4. Back-action cooling

Back-action dynamics have been realized in a diverse variety of physical systems, including those of ultracold atoms [32, 33]. The randomness present in the unavoidable stochastic back-action forces is due to the photon shot noise. These forces arise from the radiation pressure.

In this section, we evaluate effective frequency, effective damping rate and displacement spectrum for the mechanical oscillator (mirror) in the back-action cooling scheme (shown in figure 1). Also we show how the ground state cooling of the mirror is approached in this scheme.

The displacement spectrum in Fourier space is evaluated from

\[ S_q(\omega) = \frac{1}{4\pi} \int dq' \, e^{-i(\omega+q')t} \langle \delta q(\omega) \delta q(q') + \delta q(q') \delta q(q) \rangle, \]

(13)

using the correlations in Fourier space given in appendix A. Therefore, the displacement spectrum in the Fourier space for the movable mirror is given as

\[ S_q(\omega) = |\chi_{eff}(\omega)|^2[S_{th}(\omega) + S_p(\omega, \Delta_d)], \]

(14)

where \( S_{th}(\omega) \) is the thermal noise spectrum arising from the Brownian motion of the mirror and \( S_p(\omega, \Delta_d) \) is the radiation pressure spectrum including the quantum fluctuations of the condensate. \( \chi_{eff}(\omega) \) is the effective susceptibility of the oscillator.

Here,

\[ S_{th}(\omega) = \frac{\Gamma_m}{\omega_m} \coth \left( \frac{\hbar \omega}{2k_B T} \right), \]

(15)

\[ S_p(\omega, \Delta_d) = \frac{4G^2\beta^2_k(\omega^2 - \beta_1\beta_2)^2(\Delta_d^2 + \omega^2 + \frac{\omega^2}{4})}{X(\omega)}, \]

(16)

where

\[ X(\omega) = 16g^4_c\Delta_d^4, 8g^2_c\Delta_d\beta_1(\omega^2 - \beta_1\beta_2), 8g^2_c\Delta_d\beta_2(\omega^2 - \beta_1\beta_2), \]

\[ + (\omega^2 - \beta_1\beta_2)^2 \left[ \omega^2 \kappa^2 + \left( \Delta_d^2 + \frac{\omega^2}{4} \right)^2 \right]. \]

(17)

\[ \chi_{eff}(\omega) = \frac{\omega_m}{(\omega_m^2 - \omega^2 + i\omega \Gamma_m) + \chi(\omega)}. \]

(18)

where

\[ \chi_1(\omega) = \frac{4G^2\beta^2(\omega^2 - \beta_1\beta_2)}{(\omega^2 - \beta_1\beta_2)(\Delta_d^2 + \frac{\omega^2}{4} - \omega^2 + i\omega \kappa) - 4G^2\Delta_d \beta_1}. \]

(19)

\( \chi_{eff}(\omega) \) is the effective susceptibility of the resonator altered by the radiation pressure and condensate fluctuations with

\[ |\chi_{eff}(\omega)|^2 = \frac{\omega_m^2}{(\omega_m^2 - \omega^2)^2 + \omega^2 \Gamma_{eff}(\omega)^2}. \]

(20)

The effective mechanical susceptibility of the oscillator gives us the effective resonance frequency and effective damping rate as

\[ \omega_{m,eff}(\omega) = \left( \omega_m^2 + \omega_0^2 \right)^{1/2}, \]

(21)

where

\[ \omega_m^{op} = \frac{4G^2\beta^2(\omega^2 - \beta_1\beta_2)(\omega^2 - \beta_1\beta_2)}{X(\omega)} - 4G^2\Delta_d \beta_1 \]

(22)

\[ \Gamma_{eff}(\omega) = \Gamma_m = \frac{4G^2\beta^2 \Delta_d \omega_m k(\omega^2 - \beta_1\beta_2)^2}{X(\omega)}. \]

(23)

Equations (21) and (22) show that the mechanical frequency of the mirror gets modified by the quantum fluctuations of the mirror due to the radiation pressure and condensate fluctuations. This is the so-called optical spring effect. It can be observed that the frequency due to the optical spring term does not get altered significantly for high resonance frequencies, such as those of [19–21] where \( \omega_m \gtrsim 1 \) MHz.

Experimentally, the mirror may have mechanical frequency varying from \( 2\pi \times 100 \) Hz [56], to \( 2\pi \times 10 \) kHz [36] or \( 2\pi \times 73.5 \) MHz [57] with the corresponding damping rate from \( 2\pi \times 10^{-3} \) Hz [56], to \( 2\pi \times 3.22 \) Hz [36] or \( 2\pi \times 1.3 \) kHz [57]. A high-finesse Fabry–Perot optical cavity having the decay rate of \( \kappa = 2\pi \times 8.75 \) kHz [58] \( (2\pi \times 0.66 \) MHz [33]) consists of a cloud of BEC with an order of \( 10^6 \) \( ^{87} \) Rb atoms [51] interacting with the cavity field that may have a coherent coupling strength of \( g_0 = 2\pi \times 5.86 \) kHz [58] \( (2\pi \times 14.4 \) MHz [33]). The loss of photons through the cavity mirrors decreases the energy of the cavity mode which further minimizes the interaction of the atom light field. This loss of photons can be reduced in high-finesse optical cavities.

In figure 2(a), the normalized effective mechanical frequency (\( \omega_{m,eff}/\omega_m \)) of the oscillating mirror in the absence of a BEC (dashed line) is compared with the case in the presence of a BEC (solid line) as a function of the normalized frequency (\( \omega/\omega_m \)). This figure shows an extra resonance dip in the presence of a BEC. Significant deviation from the bare frequency \( \omega_m \) is observed around \( \omega \equiv \pm \omega_m \). This deviation is enhanced in the presence of a BEC. Figure 2(b) illustrates the variation of the normalized effective mechanical frequency of the mirror as a function of \( \omega/\omega_m \) for different two-body atom–atom interactions, \( U_{eff} = 0.2\omega_m \) (dashed line), \( U_{eff} = 0.4\omega_m \) (dot–dashed line) and \( U_{eff} = 0.8\omega_m \) (solid line). It can be seen from the figure that the deviation of the mechanical frequency of the mirror from its resonance frequency \( \omega_m \) decreases with the increase in the condensate two-body interaction since
Figure 2. Plot of the normalized effective mechanical frequency \( \omega_{\text{eff}} / \omega_m \) as a function of the dimensionless frequency in the back-action cooling scheme. Panel (a) shows the variation of the dimensionless effective mechanical frequency in the absence of a BEC (solid line) and in the presence of a BEC (dashed line) with \( U_{\text{eff}} = 0.8 \omega_m \). Panel (b) represents the dimensionless effective mechanical frequency for three different values of the atomic two-body interaction with \( U_{\text{eff}} = 0.2 \omega_m \) (dashed line), \( U_{\text{eff}} = 0.4 \omega_m \) (dotted line) and \( U_{\text{eff}} = 0.8 \omega_m \) (solid line). Other parameters used are \( \Gamma_m = 10^{-5} \omega_m \), \( \Delta_m = -\omega_m \), \( \kappa = 0.3 \omega_m \), \( G = 4 \omega_m \), \( \beta = 0.05 \), \( n = 0.01 \omega_m \), \( \gamma_c = 0.3 \omega_m \) and \( k_B T / \hbar \omega_m = 10^{-3} \).

The condensate becomes more robust with a higher two-body interaction. Within a set of experimentally achievable parameters, the atomic two-body interaction can be modified using the condensate cloud having dimensions 3.3 \( \mu \text{m} \) [32] (290 nm [36]) with length 20 \( \mu \text{m} \) [32] (615.5 nm [36]). It may also vary using the scattering length ranging from 10\( a_0 \) to 190\( a_0 \) (\( a_0 \) = Bohr radius) [59].

Figure 3(a) shows the normalized effective mechanical damping \( \Gamma_{\text{eff}} / \omega_m \) of the mirror as a function of the normalized frequency \( \omega / \omega_m \) in the absence of a BEC (dashed line) and the presence of a BEC (solid line). As seen from this figure, below resonance \( \omega < \omega_m \), the effective damping of the oscillator is more in the presence of a BEC, whereas above resonance \( \omega > \omega_m \), it is less in the presence of a BEC. Also, the plot of the normalized effective mechanical damping of the mirror with normalized frequency is illustrated in figure 3(b) for the three different values of condensate two-body interactions, \( U_{\text{eff}} = 0.2 \omega_m \) (dashed line), \( U_{\text{eff}} = 0.4 \omega_m \) (dot-dashed line) and \( U_{\text{eff}} = 0.8 \omega_m \) (solid line). A higher condensate two-body interaction enhances the effective damping of the mirror below resonance, whereas it decreases the effective damping of the mirror above resonance. It shows an exception for the case \( U_{\text{eff}} = 0.4 \omega_m \) as explained later. Figure 4(a) represents the displacement spectrum \( S_q(\omega) \) as a function of the dimensionless frequency \( \omega / \omega_m \) in the presence of a BEC (solid line) and the absence of a BEC (dashed line). We observe the NMS into two modes in the absence of a BEC and we find that the normal mode splits up into three modes in the presence of a BEC. This extra mode is the result of additional quantum fluctuations of the condensate (Bogoliubov mode). Also, below resonance, we observe that the amplitude of peak (1) of the displacement spectrum in the absence of a BEC is greater than that in the presence of a BEC. This is due to the higher damping rate in the presence of a BEC as illustrated in figure 3(a). This conclusion is exactly opposite to that for the case above resonance. Figure 4(a) depicts that the amplitude of peak (1) in the absence of a BEC is more than that in the presence of a BEC, whereas the amplitude of peak (2) in the absence of a BEC is less than that in the presence of a BEC. This represents the energy exchange between different modes of the system. Also, figure 4(b) illustrates the displacement spectrum...
represents the displacement spectrum for three different values of the atomic two-body interaction with the condensate two-body interaction, varying with normalized frequency for three different values of the dimensionless frequency in the back-action cooling scheme. Figure 4 shows a variation in the displacement spectrum for different conditions. The spectrum shifts towards the right with the increase in the effective phonon number of the mirror’s motion can be calculated from the mean field results in NMS. For the observation of NMS, it is significant to note that the timescale for the exchange of energy between the mechanical mode, photon mode and the Bogoliubov mode should be faster than the decoherence of each mode. The beat of laser pump photons with the photons scattered from the condensate atoms is the source of cavity field fluctuations. Since the frequency of the Bogoliubov mode of the condensate is directly proportional to $\sqrt{U_{\text{eff}}}$, figure 4(b) shows a variation in the displacement spectrum for different $U_{\text{eff}}$. The spectrum shifts towards the right with the increase in the condensate two-body interaction. Recently, an experiment revealed that the Bogoliubov mode of the cloud of ultracold atoms interacting with an optical resonator has momentum $\pm 2k_c$ ($k_c$ is the cavity wave number) [32].

The condition $\Gamma_n \ll \omega_m \ll k_BT/h$ is always considered in standard optomechanical experiments [23, 60–62]. In this limiting case, we consider the approximation that $\coth(h\omega_{0}/2k_BT) \simeq 2k_BT/h\omega_{0}$. Now, in order to achieve the ground state cooling of the mechanical resonator, we measure the average energy of the resonator in the steady state, which is given by [63]

$$U = \frac{h\omega_m}{2}[\langle \delta q^2 \rangle + \langle \delta p^2 \rangle],$$

(24)

The system has to be stable in order to be in the steady state. Hence, the stability conditions for the back-action cooling scheme given in appendix B must always be satisfied in this regime such that all the poles of $\chi_{\text{eff}}(\omega)$ lie in the lower complex half-plane. Here, equation (24) implies that the effective phonon number of the mirror’s motion can be written as

$$n_{\text{eff}} = \frac{1}{2}[\langle \delta q^2 \rangle + \langle \delta p^2 \rangle - 1],$$

(25)

where $\langle \delta q^2 \rangle$ and $\langle \delta p^2 \rangle$ represent the displacement and momentum variances of the oscillator, respectively, which are defined as [13]

$$\langle \delta q^2 \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_q(\omega),$$

(26)

$$\langle \delta p^2 \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega^2 S_q(\omega).$$

(27)

We have solved these oscillators variances numerically using MATHEMATICA 8.0. Generally, there is no equipartition of energy, $\langle \delta q^2 \rangle \neq \langle \delta p^2 \rangle$. In order to approach the ground state cooling, the effective phonon number should be less than 1, i.e. $n_{\text{eff}} < 1$, which can only be attained if the initial average-thermal excitation number $\bar{n} = \left(\exp\left[\frac{\hbar\omega_m}{k_B T}\right] - 1\right)^{-1}$ is not excessively large. This can be possible even at cryogenic temperatures if $\omega_m$ is sufficiently large.

In order to get an intuitive picture, we have plotted both the displacement and momentum variances of the mirror as a function of the dimensionless detuning $\Delta_{d}/\omega_{0m}$ in the absence and presence of a BEC as illustrated in figure 5(a). Figure 5(a) shows that, for the large values of detuning, the variances have lesser values in the absence of a BEC than in the presence of a BEC. It is evident from the figure that one can access the ground state of the system by increasing the BEC two-body interaction. This implies that $U_{\text{eff}}$ may alter the cooling process significantly. So, in order to achieve the best condition for the ground state cooling of the system, $U_{\text{eff}}$ can be used as an additional control parameter. Figure 5(b) depicts the effective phonon number $n_{\text{eff}}$ as a function of the dimensionless detuning $\Delta_{d}/\omega_{0m}$ for the absence of a BEC (thick solid line) and the three different values of $U_{\text{eff}}$, $U_{\text{eff}} = 0.4\omega_{0m}$ (dashed line), $U_{\text{eff}} = 0.8\omega_{0m}$ (thin solid line) and $U_{\text{eff}} = 1\omega_{0m}$ (dot-dashed line). Clearly, with the increase in $U_{\text{eff}}$, the minimum value of $n_{\text{eff}}$ is shifting towards zero such that, for $U_{\text{eff}} = 1\omega_{0m}$, we have $n_{\text{eff}} \simeq 0.52$ at $\Delta_{d} = -1.13\omega_{0m}$. However, the least value of $n_{\text{eff}}$ is reached in the absence of a BEC, which is nearly 0.51 at $\Delta_{d} = -1\omega_{0m}$. Hence, in the back-action cooling scheme, we have noted that the minimum value of the effective phonon number is found in the absence of a BEC. Moreover,
we have observed that for small values of $\Delta_d$, the presence of a BEC gives better results for higher $U_{\text{eff}} = 0.8 \omega_m$ and $\omega_m$ than the absence of a BEC. One can also observe from figure 5(b) that the effective temperature of the oscillator does not vary much in the presence of a BEC ($U_{\text{eff}} = 0.8 \omega_m$ and $\omega_m$) as compared to that in the absence of a BEC. This reveals that for a wide range of detuning ($\Delta_d$), the effective temperature of the mirror is not changing significantly for higher $U_{\text{eff}}$. Therefore, one can use the condensate two-body interaction as a new handle to achieve and sustain a low temperature of the mirror with BEC over a wide range of $\Delta_d$. The atomic two-body interaction is proportional to the number of atoms ($N$). In the recent past, it has been shown experimentally in [51] that increasing the number of atoms enhances the damping of the oscillating membrane which is coupled to BEC through the cavity field. This validates our work that the energy of the mirror decreases by increasing $U_{\text{eff}}$. Furthermore, it is noted that both the variances tend to $\langle \delta q^2 \rangle \approx \langle \delta p^2 \rangle \approx 1/2$ for $G = 0.5 \omega_m$ (keeping other parameters same), i.e. energy equipartition is satisfied in this chosen parameter regime but it results in the approximate same ground state oscillator energy as we have achieved in the above-mentioned general case of no energy equipartition ($\langle \delta q^2 \rangle \neq \langle \delta p^2 \rangle$). We find that the best cooling regime is achieved under the good cavity limit condition ($\kappa \ll \omega_m$) by taking $\kappa = 0.1 \omega_m$ as mentioned in [13].

The variation of the effective phonon number with $U_{\text{eff}}/\omega_m$ at $\Delta_d = -\omega_m$ is also shown in figure 6. It depicts a rapid increase in $n_{\text{eff}}$ for a range of $U_{\text{eff}}$. This corresponds to the sudden decrease in the effective damping rate of the mirror as illustrated in figure 7. It is by virtue of the fact that, in the range $0 \leq U_{\text{eff}} \leq 0.57 \omega_m$, the second term in equation (23) increases as ($\omega^2 - \beta_1 \beta_2$) decreases. On the other hand, for $U_{\text{eff}} > 0.57 \omega_m$, this term decreases since ($\omega^2 - \beta_1 \beta_2$) becomes negative. It explains the exception observed in figures 3(b) and 4(b) for $U_{\text{eff}} = 0.4 \omega_m$. In the following section, we study the system using the cold damping feedback technique.

Figure 5. Back-action cooling scheme. (a) Plot of the mirror’s position variances $\langle \delta q^2 \rangle$ (solid line) and the mirror’s momentum variances $\langle \delta p^2 \rangle$ (dot–dashed line) versus the normalized effective detuning ($\Delta_d/\omega_m$) in the absence of a BEC and for three different values of the atomic two-body interaction with $U_{\text{eff}} = 0.4 \omega_m, U_{\text{eff}} = 0.8 \omega_m$, and $U_{\text{eff}} = \omega_m$. Other parameters chosen are $\Gamma_{m} = 10^{-7} \omega_m, \kappa = 0.1 \omega_m, G = \omega_n, \beta = 0.05, \nu = 0.01 \omega_m, \Gamma_c = 0.5 \omega_m$ and $\hbar/k_B T/\hbar \omega_m = 10^{-3}$. (b) Effective phonon number with dimensionless effective detuning in the absence of a BEC (thick solid line) and for three different values of atomic two-body interaction with $U_{\text{eff}} = 0.4 \omega_m$ (dashed line), $U_{\text{eff}} = 0.8 \omega_m$ (thin solid line) and $U_{\text{eff}} = \omega_m$ (dot–dashed line). Other parameters chosen are the same as in (a).

Figure 6. The effective phonon number versus the normalized atomic two-body interaction in the back-action cooling scheme at $\Delta_d = -\omega_m$. The rest of the parameters are the same as in figure 5.

Figure 7. The normalized effective mechanical damping rate versus the normalized atomic two-body interaction at the resonance frequency ($\omega = \omega_m$) in the back-action cooling scheme. Other parameters chosen are the same as in figure 2.

5. Cold damping feedback scheme

The cold damping feedback technique (shown in figure 8) is an alternative method to improve the cooling of a mechanical
oscillator by overdamping it at the quantum level without increasing the thermal noise of the system as proposed in [17, 27, 28]. This technique has been realized experimentally [23, 24, 60]. It involves a negative derivative feedback technique. The displacement of the oscillator is measured through phase-sensitive homodyne detection of the cavity output which is fed back to the resonator using a force proportional to the oscillator velocity [11, 13].

Therefore, the QLEs involving a feedback force for this scheme are given by

$$\delta q_a(t) = \beta_1 \delta p_a(t),$$  
(28)

$$\delta p_a(t) = -\beta_2 \delta q_a(t) - 2g_\delta \delta q_b(t),$$  
(29)

$$\delta q_b(t) = -\frac{\kappa}{2} \delta q_a(t) + \sqrt{\kappa} q_a(t) - \Delta_\omega \delta p_b(t),$$  
(30)

$$\delta p_b(t) = -\frac{\kappa}{2} \delta p_b(t) - 2g_\delta \delta q_a(t) + 2G_\beta \delta q(t) + \sqrt{\kappa} p_m(t) + \Delta_\delta \delta q_b(t),$$  
(31)

$$\dot{\delta q}(t) = \omega_m \delta p(t),$$  
(32)

$$\dot{\delta p}(t) = -\omega_m \delta q(t) + 2G_\beta \delta q_b(t) - \Gamma_m \delta p(t) + W(t) - \int_0^t ds g(t - s) \delta p_{\text{ext}}(s),$$  
(33)

where $\beta_1 = v + U_{\text{eff}}$, $\beta_2 = v + 3U_{\text{eff}}$ and the filter function $g(t)$ is the causal kernel chosen as [13]

$$g(t) = g_{\text{cd}} \frac{d}{dt} [V(t) \omega_{\text{th}} e^{-\omega_{\text{th}} t}],$$  
(34)

such that

$$g(\omega) = \frac{-i \omega g_{\text{cd}} \sqrt{\kappa}}{1 - i \omega / \omega_{\text{th}}},$$  
(35)

where $g(\omega)$ is the Fourier transform of $g(t)$. Here, $g_{\text{cd}}$ is the positive feedback gain and $\lambda$ quantifies the homodyne detection efficiency of the photodetector. Generally, the detector efficiency is $\lambda < 1$ for the inefficient homodyne detection if the effect of additional noise is considered, but for ideal homodyne detection, $\lambda = 1$ [64]. $\omega_{\text{th}}^{-1}$ signifies the feedback loop delay time. Moreover, $\delta p_{\text{ext}}(s)$ represents the estimated intra-cavity phase quadrature which is evaluated by using the generalized input–output phase relation given by [64–66]

$$p_{\text{out}}(t) = \sqrt{\kappa} \sqrt{\lambda} \delta p_b(t) - \sqrt{\kappa} p_m(t) - \sqrt{1 - \lambda} \delta p(t).$$  
(36)

Here, $p_{\text{out}}(t)$ and $p_m(t)$ represent the output homodyne phase quadrature and input noise respectively. Also $p_b(t)$ is the associated vacuum field quadrature which describes the additional noise for $\lambda < 1$ such that

$$\delta p_{\text{cd}}(t) = \frac{p_{\text{out}}(t)}{\sqrt{\kappa}} = \frac{\delta \delta p_b(t)}{\sqrt{\kappa}} - \frac{\lambda}{\sqrt{\kappa}} \frac{\delta \delta p(t)}{\sqrt{\kappa}},$$  
(37)

where $p_b(t)$ and $p_m(t)$ are uncorrelated. Now the QLEs supplemented with the feedback term are solved in the frequency domain such that the displacement and momentum variances of the oscillator are given by equations (26) and (27), respectively. It gives a distinct displacement spectrum using the correlations given in appendix A. Explicitly, it can be written as

$$S_{\text{in}}(\omega) = \left[ \chi_{\text{eff}}^{\text{cd}}(\omega) \right]^2 [S_{\text{ph}}(\omega) + S_{\text{th}}(\omega)].$$  
(38)

Here, $S_{\text{in}}(\omega)$ and $S_{\text{ph}}(\omega, \Delta_\omega)$ are given by equations (15) and (16), respectively. In this scheme, the position spectrum consists of an additional feedback-induced term expressed as

$$S_{\text{in}}(\omega) = \frac{2G_\beta \sqrt{\pi} \omega (\omega^2 - \beta_1 \beta_2) [\Delta_{\omega} (\omega^2 - \beta_1 \beta_2) \omega^2 + \Delta_\delta \omega^2 + 4g_\delta^2 \left( \Delta_\delta^2 + \frac{\omega^2}{\omega_{\text{th}}^2} \right)] [g(\omega) + g(-\omega)]}{\omega_{\text{th}}^2 \omega^2 [\omega_{\text{th}}^2 - \omega^2 + i \Gamma_m] + \chi_{\text{eff}}^{\text{cd}}(\omega)},$$  
(39)

which arises since the cold damping loop feeds back the measurement noise into the dynamics of the movable mirror. $\chi_{\text{eff}}^{\text{cd}}(\omega)$ is the effective mechanical susceptibility of the movable mirror, modified by the filter function, given by

$$\chi_{\text{eff}}^{\text{cd}}(\omega) = \left( \frac{g_{\text{cd}}}{\omega_{\text{th}}^2 - \omega^2 + i \omega_{\text{th}} \Gamma_m} \right) + \chi_{\text{eff}}^{\text{cd}}(\omega),$$  
(40)

where

$$\chi_{1}^{\text{eff}}(\omega) = \frac{2G_\beta \omega_{\text{th}} \omega (\omega^2 - \beta_1 \beta_2) \left( \omega^2 - \beta_1 \beta_2 \right) \left( \omega^2 - \beta_1 \beta_2 \right)}{\left( \omega^2 - \beta_1 \beta_2 \right) \left( \Delta_\delta^2 + \frac{\omega^2}{\omega_{\text{th}}^2} \right) - 4g_\delta^2 \Delta_\delta \beta_1},$$  
(41)

It gives us the effective resonance frequency and damping rate using equation (35):

$$\omega_{\text{m}}^{\text{eff}}(\omega) = \left[ \omega_{\text{m}}^2 + \omega_{\text{th}}^2 \right]^{1/2},$$

where

$$\omega_{\text{m}}^{\text{op}}(\omega) = X_1(\omega) \left[ \delta_{\omega}^2 + \beta_1 \beta_2 \Delta_{\omega}^2 + \frac{\kappa^2}{4} - \omega^2 - 4g_\delta^2 \Delta_\delta \beta_1 \right] \times \left( 4G_\beta \Delta_\delta + \frac{2\omega^2 g_{\text{cd}} \omega_{\text{th}} \lambda}{(\omega^2 + \omega_{\text{th}}^2)} \kappa - \omega_{\text{th}} \right)$$

$$+ X_1(\omega) \left[ \omega^2 - \beta_1 \beta_2 \right] \left[ \frac{2\omega^2 g_{\text{cd}} \omega_{\text{th}} \lambda}{(\omega^2 + \omega_{\text{th}}^2)} \kappa - \omega_{\text{th}} \right],$$  
(42)
Figure 9. Plot of the normalized effective mechanical frequency ($\omega_{m}^{\text{eff}}/\omega_m$) as a function of the dimensionless frequency in the cold damping feedback scheme. Panel (a) shows the variation of the dimensionless effective mechanical frequency in the absence of a BEC (solid line) and in the presence of a BEC (dashed line) with $U_{\text{eff}} = 0.8\omega_m$. Panel (b) represents the dimensionless effective mechanical frequency for three different values of atomic two-body interaction with $U_{\text{eff}} = 0.2\omega_m$ (dashed line), $U_{\text{eff}} = 0.4\omega_m$ (dot–dashed line) and $U_{\text{eff}} = 0.8\omega_m$ (solid line). Other parameters used are $\Gamma_m = 10^{-5}\omega_m$, $\Delta = -\omega_m$, $\kappa = 1.2\omega_m$, $G = 6\omega_m$, $\beta = 0.05$, $v = 0.015\omega_m$, $\xi = 0.3\omega_m$, $\omega_0 = 4\omega_m$, $\kappa_0 = 0.8$, $\lambda = 0.8$ and $k_B T/\hbar\omega_m = 10^3$.

Figure 10. Plot of the normalized effective damping rate ($\Gamma_m^{\text{eff}}/\omega_m$) with dimensionless frequency in the cold damping feedback scheme. Panel (a) represents the variation of the dimensionless effective mechanical frequency in the absence of a BEC (dashed line) and in the presence of a BEC (solid line) with $U_{\text{eff}} = 0.8\omega_m$. Panel (b) gives the deviation of the dimensionless effective mechanical frequency for three values of the atomic two-body interaction with $U_{\text{eff}} = 0.2\omega_m$ (dashed line), $U_{\text{eff}} = 0.4\omega_m$ (dot–dashed line) and $U_{\text{eff}} = 0.8\omega_m$ (solid line). Other parameters chosen are the same as in figure 9.

\[
\Gamma_m^{\text{eff,cd}}(\omega) = \Gamma_m + X_1(\omega) \left[ \frac{\omega^2 - \beta_1 \beta_2}{\omega^2 + \omega_0^2} \right] \omega_0^2 \Delta_d \beta_1 \left( \frac{2\omega_0^2 \omega_0^2}{\omega^2 + \omega_0^2} + \frac{\omega_0^2 - \omega^2}{2} \right) - X_1(\omega) \omega_0^2 \beta_1 \beta_2 \times \left( 4G^2 \Delta_d + \frac{2\omega_0^2 \omega_0^2 \omega_0^2}{\omega^2 + \omega_0^2} \right),
\]

(43)

where

\[
X_1(\omega) = \frac{G\beta \omega_0}{\omega^2 + \omega_0^2}.
\]

(44)

In order to over damp the mechanical mirror oscillations, an additional viscous force known as the feedback force is applied in this technique which is possible only when the estimated intra-cavity phase quadrature $\delta q_{\text{cd}}$ is proportional to the oscillator position $\delta q(t)$. It can be achieved in the bad cavity limit where $\kappa > \omega_m$. Hence, for cold damping feedback, we limit our discussion to the bad cavity limit. We have plotted the effective resonance frequency ($\omega_m^{\text{eff,cd}}$) and damping rate ($\Gamma_m^{\text{eff,cd}}$) as a function of the dimensionless frequency ($\omega/\omega_m$) in figures 9 and 10 respectively in order to compare them with the back-action cooling plots. It can be seen again from figure 9 that there is no significant shift in the frequency in both the absence and presence of a BEC in the chosen parameter regime. Hence, there is negligible optical spring effect as $\omega_m^{\text{eff,cd}}$ dominates $\omega_m^{\text{cd}}$ for higher resonance frequencies. However, the mechanical damping rate shows significant variation with change in frequency. Below resonance, effective damping increases by adding a BEC to the system (see figure 10(a)) and by increasing the atom–atom interaction (see figure 10(b)) with an exception for $U_{\text{eff}} = 0.4\omega_m$. This exception in the result can be explained from figure 11, which clearly shows the variation in the mechanical damping rate with increasing $U_{\text{eff}}$. The reason for this sudden decrease in the damping rate in a particular region of $U_{\text{eff}}$ for $\omega = \omega_m$ is similar to what we have described in the case of back-action cooling. However, in the zero detuning case, one can clearly observe from
expressions (42) and (43) that \( \omega_m^{\text{eff,cd}} \) and \( \Gamma_m^{\text{eff,cd}} \) are independent of BEC parameters and behave in the same manner as in the case for the absence of a BEC. In the adiabatic limit for zero detuning, i.e., \( \kappa, \omega_b \gg \omega \), we obtain \( \omega_m^{\text{eff,cd}} \approx \omega_m \) and \( \Gamma_m^{\text{eff,cd}} = \Gamma_m + \frac{g_{\text{cd}}}{\omega_b} \langle \delta q^2 \rangle \). This shows that the effective damping rate increases in this scheme without involving any significant change in the resonance frequency of the oscillator. We have also shown the plot of the displacement spectrum \( S^d(\omega) \) as a function of the dimensionless frequency \( (\omega/\omega_m) \) in figure 12 to compare it with the corresponding curve for the back-action cooling. This scheme involves an additional feedback-induced term denoted by \( S_{\text{fb}}(\omega) \) which gives a more distinct displacement spectrum than the back-action cooling scheme. This feedback-induced term \( S_{\text{fb}}(\omega) \) can be used as an additional handle in the cold damping feedback scheme. The displacement spectrum in this scheme can be manipulated by a coherent control over the feedback parameters \( g_{\text{cd}} \) and \( \omega_b \). The variation in the amplitude of the peaks in the absence and presence of a BEC of the displacement spectrum, shown in figure 12(a), represents the energy exchange between the different modes of the system. Only two-mode splitting is observed in the absence of a BEC (see figure 12(a)), while the normal mode splits up into three modes due to the atom–atom interaction (see figure 12(b)).

Now, we characterize the steady state energy of the mirror in the cold damping feedback scheme. The optimal cooling conditions can be obtained using the stability conditions which are modified in this scheme given in appendix B. The oscillator variances represented by equations (26) and (27) and the mean energy of the oscillator given by equation (24), using the corresponding position spectrum for the cold damping case, are calculated numerically with the help of MATHEMATICA 8.0. This shows that, also using cold damping, there is no equipartition of energy, i.e., \( \langle \delta q^2 \rangle \neq \langle \delta p^2 \rangle \) in general. Figure 13(a) illustrates both the variances \( \langle \delta q^2 \rangle \) and \( \langle \delta p^2 \rangle \) as a function of the dimensionless detuning \( (\Delta_d/\omega_m) \) for the absence of a BEC and three different values of the condensate two-body interaction \( (U_{\text{eff}}), U_{\text{eff}} = 0.4\omega_m, U_{\text{eff}} = 0.8\omega_m \) and \( U_{\text{eff}} = 1\omega_m \). As can be seen from the figure, the displacement and momentum variances decrease significantly on adding BEC to the system. Moreover, both variances decrease with the increase in the condensate two-body interaction. However, as \( \Delta_d \) approaches zero, oscillator variances increase drastically in the presence of a BEC. Therefore, one can infer that the presence of a BEC helps in cooling the mirror for smaller detunings. So, in order to approach the optimal cooling conditions in the cold damping scheme, the sum of these variances given by equations (26) and (27) is to be minimized which can be done by increasing the condensate two-body interaction. The effective phonon number given by equation (25) is shown in figure 13(b) as a function of the dimensionless detuning \( (\Delta_d/\omega_m) \) for the absence of a BEC (thick solid line) and the three different values of the atom–atom interaction, \( U_{\text{eff}} = 0.4\omega_m \) (dashed line), \( U_{\text{eff}} = 0.8\omega_m \) (dot–dashed line) and \( U_{\text{eff}} = 1\omega_m \) (thin solid line). From the figure, we observe that the minimum value of the effective mean excitation number is reached for \( U_{\text{eff}} = 0.8\omega_m \) which is 0.658 with \( \Delta_d = -1.5\omega_m \). Although, as \( \Delta_d \rightarrow 0 \), \( n_{\text{eff}} \) increases considerably with increasing \( U_{\text{eff}} \). So, the ground state cooling can be approached by adding BEC to the system involving the cold damping feedback scheme for...
lower detunings. Hence, by varying the atom–atom interaction, one can optimize the cooling process to achieve the ground state. Using the cavity self-cooling scheme, the least value of $n_{\text{eff}}$ is achieved in the absence of a BEC, while using the cold damping feedback technique, the minimum value is obtained by adding BEC to the system. Moreover, with BEC, we are getting the least value of the effective phonon number using the back-action cooling scheme. We have also shown the variation of the effective phonon number $n_{\text{eff}}$ as a function of $U_{\text{eff}}/\omega_m$ in figure 14. It clearly illustrates the fact that, in the presence of a BEC, the ground state cooling of the mechanical oscillator can be achieved for the higher condensate two-body interaction only. It is observed that the second term in equation (43) becomes excessively small and even negative for very small $U_{\text{eff}}$. This results in the sudden increase in the value of $n_{\text{eff}}$ for very small atom–atom interaction as the effective mechanical damping rate of the mirror (given by equation (43)) decreases in this region. We have also examined that by taking $\omega_{\text{ph}} = 0.5 \omega_m$, keeping other parameters the same as before, the uncertainty principle is stratified for both displacement and momentum variances. This implies that $\langle \delta q^2 \rangle \approx \langle \delta p^2 \rangle \approx 1/2$; therefore, energy equipartition is satisfied. However, the ground state could not be approached in this regime, i.e. we are getting $n_{\text{eff}} > 1$.

It can be seen that in the presence of a BEC, the cold damping feedback scheme is more advantageous over the back-action cooling scheme in approaching the quantum ground state for a wide range of effective detuning. It is the well-known fact that self-cooling of the mechanical oscillator arises via the dynamics of the radiation pressure. However, the cold damping feedback technique involves an additional viscous force on the mirror which helps in further cooling of the oscillator. From our previous work [52], it can be noted that the stochastic cooling scheme gives comparatively higher oscillator energy as compared to the cold damping feedback scheme. Additionally, our system involves BEC which further enhances the cooling of the mechanical oscillator. Earlier work [50] has demonstrated that BEC can absorb excitations from the mirror through the optical field and thus helps in further cooling of the mirror. A recent experiment has also shown that the damping of the oscillating membrane, coupled to BEC through the cavity field, can be enhanced by increasing the number of atoms [51]. This provides the experimental validation of our work which gives an advantage of adding a BEC to the optomechanical system.

6. Conclusion

In this paper, we have studied how the back-action cooling and cold damping feedback schemes help in cooling the mirror to its quantum ground state by using a BEC confined in an optical cavity. The atom–atom two-body interaction can be used as a new handle to cool the quantum device in both the schemes. It provides a systematic control of the system which can be altered either by using the number of atoms or the s-wave scattering length. Both techniques...
show distinct displacement spectra involving energy exchange between the different modes of the system. A coherent control over the feedback parameters, $g_{cd}$ and $\omega_m$, can manipulate the displacement spectrum in the cold damping feedback scheme. In back-action cooling, the least value of the effective phonon number is obtained without BEC, whereas in cold damping, it is obtained with BEC. The effective temperature of the oscillator does not vary much for the higher condensate damping, it is obtained without BEC, whereas in cold scheme, the least value of the effective phonon number is obtained using cavity self-cooling with BEC. We have analysed that both the techniques help in approaching the quantum ground state of the oscillator. Discrimination in the ideal cooling conditions for these schemes exhibits that back-action dynamics is more convenient in the good cavity limit ($\kappa \ll \omega_m$), while cold damping is preferable in the bad cavity limit ($\kappa \gg \omega_m$).

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Appendix A

The following correlations are satisfied by the input noise operators [11, 17, 64, 67]:

$$\langle b_m(t)b_m(t') \rangle = \langle b_m(t)b_m(t') \rangle = 0,$$

(A.1)

$$\langle b_m(t)b_m(t') \rangle = \delta(t - t').$$

(A.2)

The noise operator due to the Brownian motion of the mirror is given as $W(t) = i \sqrt{\frac{m}{\hbar}} [\xi_m(t) - \xi_m(t)]$, which satisfies the following correlation [64]:

$$\langle W(t)W(t') \rangle = \frac{\Gamma_m}{\hbar \omega_m} \int \frac{d\omega}{2\pi} e^{-\omega(t-t')} \omega \left[ 1 + \coth \left( \frac{\hbar \omega}{2k_B T} \right) \right].$$

(A.3)

where $T$ is the finite temperature of the bath connected to the movable mirror and $k_B$ is the Boltzmann constant. Brownian noise is the random thermal noise which arises from the stochastic motion of the mechanical oscillator (mirror). It should be noted that Brownian noise is non-Markovian in nature. The quantum Brownian motion of the movable mirror gives rise to the thermal noise term in the measured phase noise spectrum of the optical field reflected from the cavity [11, 17].

The amplitude and phase quadratures of the input noise operator satisfy the following correlations in Fourier space [67]:

$$\langle \eta_m(\omega)\eta_m(\omega') \rangle = 2\pi \delta(\omega + \omega'),$$

$$\langle p_m(\omega)p_m(\omega') \rangle = 2\pi \delta(\omega + \omega'),$$

$$\langle \eta_m(\omega)p_m(\omega') \rangle = 2\pi \eta_m(\omega + \omega'),$$

$$\langle p_m(\omega)\eta_m(\omega') \rangle = -2\pi \eta_m(\omega + \omega').$$

Also in Fourier space, the correlation function for the Brownian noise operator is given as [67]

$$\langle W(\omega)W(\omega') \rangle = 2\pi \left[ \frac{m}{\hbar} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \right] \delta(\omega + \omega').$$

The correlation of vacuum field quadrature in the Fourier space is $\langle p_\xi(\omega)p_\xi(\omega') \rangle = 2\pi \delta(\omega + \omega').$

Appendix B

In the back-action cooling scheme, two non-trivial stability conditions for the system in the absence of a BEC, obtained by applying the Routh–Hurwitz criterion, are given as:

$$S_1 = a_0 > 0,$$

(B.1)

$$S_2 = (a_3a_2a_1 - a_4a_1^2 - a_6a_2^2) > 0,$$

(B.2)

where

$$a_0 = \omega_m^2 \left( \frac{k^2}{4} + \Delta_d^2 \right) + \omega_m \beta^2 G^2 \Delta_d,$$

(B.3)

$$a_1 = \Gamma_m \left( \Delta_d^2 + \frac{k^2}{4} \right) + \omega_m \kappa,$$

(B.4)

$$a_2 = \frac{k^2}{4} + \Delta_d^2 + \Gamma_m \kappa + \omega_m^2,$$

(B.5)

$$a_3 = \Gamma_m + \kappa,$$

(B.6)

$$a_4 = 1,$$

(B.7)

while the stability conditions, evaluated by applying the Routh–Hurwitz criterion, for the system in the presence of a BEC are as follows:

$$S_3 = b_0 > 0,$$

(B.8)

$$S_4 = (b_3b_2b_1 + b_6b_1b_5 - b_6b_2^2 - b_2b_3^2) > 0,$$

(B.9)

where

$$b_0 = \frac{\beta_1^2 \beta_2^2 \omega_m^2}{4} + \omega_m^2 \beta_1 \beta_2 \Delta_d^2 + 4G^2 \beta^2 \omega_m \Delta_d \beta_1 \beta_2$$

$$+ 4G^2 \Delta_d \beta_1 \omega_m^2,$$

(B.10)

$$b_1 = \beta_1 \beta_2 \kappa \omega_m^2 + \frac{\beta_1 \beta_2^2 \Gamma_m^2}{4} + \beta_1 \beta_2 \Gamma_m \Delta_d^2 + 4G^2 \Delta_d \beta_1 \Gamma_m,$$

(B.11)

$$b_2 = \frac{k^2 \omega_m^2}{4} + \omega_m^2 \Delta_d^2 + 4G^2 \beta^2 \omega_m \Delta_d + \beta_1 \beta_2 \Gamma_m \kappa$$

$$+ \frac{\beta_1 \beta_2^2 k^2}{4} + \beta_1 \beta_2 \Delta_d^2 + 4G^2 \Delta_d \beta_1 + \beta_1 \beta_2 \omega_m^2,$$

(B.12)

$$b_3 = \beta_1 \beta_2 \left( \Gamma_m + \kappa \right) + \Gamma_m \left( \Delta_d^2 + \frac{k^2}{4} \right) + \kappa \omega_m^2,$$

(B.13)

$$b_4 = \omega_m^2 + \Gamma_m \kappa + \frac{k^2}{4} + \Delta_d^2 + \beta_1 \beta_2,$$

(B.14)

$$b_5 = \Gamma_m + \kappa,$$

(B.15)

$$b_6 = 1.$$  

(B.16)

Moreover, the Routh–Hurwitz criterion is equivalent to the condition by imposing all the poles of mechanical susceptibility ($\chi_m(\omega)$) in the lower complex half-plane. Hence, one obtains a non-trivial modified stability condition.
for the cold damping case in the presence of a BEC as follows:

\[ S_5 = (c_4c_3^2 + c_2c_5^2 - c_3c_4c_3 - c_6c_3c_1) > 0, \quad (B.17) \]

where

\[ c_1 = -i \left( \Delta_2^2 + \frac{k^2}{4} \right) \left( \omega_{\text{th}}^2 + \omega_{\text{th}}\beta_1\beta_2 + \Gamma_m\beta_1\beta_2 \right) 
- i\beta_1\beta_2 \left( \omega_{\text{th}}^2 + \Gamma_m\omega_{\text{th}} + \omega^2 \kappa \right) 
- i4\Delta_1^2\Delta_2\beta_1 (\omega_{\text{th}} + \Gamma_m) - i2G\beta_{\omega_{\text{m}}G_{\text{cd}}\beta_1\beta_2\lambda_{\omega_{\text{th}}} 
- i4G^2\beta^2\Delta_1\omega_{\text{m}}, \quad (B.18) \]

\[ c_2 = \left( \Delta_2^2 + \frac{k^2}{4} \right) \left( \omega_{\text{m}}^2 + \Gamma_m\omega_{\text{th}} + \beta_1\beta_2 \right) 
+ \beta_1\beta_2 (\omega_{\text{m}}^2 + \Gamma_m\omega_{\text{th}} + \omega_{\text{th}}\kappa + \Gamma_m\kappa) 
+ \omega_{\text{th}}^2 \kappa + \omega_{\text{th}}^2 - 4G^2\Delta_1\beta_1 + G\beta_{\omega_{\text{m}}G_{\text{cd}}\lambda_{\omega_{\text{th}}} 
+ 4G^2\beta^2\omega_{\text{m}}\Delta_1, \quad (B.19) \]

\[ c_3 = i \left[ \left( \Delta_2^2 + \frac{k^2}{4} \right) (\omega_{\text{m}}^2 + \Gamma_m) + \omega_{\text{m}}^2 (\omega_{\text{m}}^2 + \Gamma_m, + \beta_1\beta_2) 
+ \kappa\omega_{\text{th}}^2 + \beta_1\beta_2 (\kappa + \Gamma_m) + 2G\beta_{\omega_{\text{m}}G_{\text{cd}}\lambda_{\omega_{\text{th}}} \right), \quad (B.20) \]

\[ c_4 = - \left[ \left( \Delta_2^2 + \frac{k^2}{4} \right)^2 + \omega_{\text{th}}^2 + \omega_{\text{th}}(\kappa + \Gamma_m) + \beta_1\beta_2 + \kappa\Gamma_m \right], \quad (B.21) \]

\[ c_5 = -i [\omega_{\text{th}} + \kappa + \Gamma_m], \quad (B.22) \]

\[ c_6 = 1. \quad (B.23) \]

In the absence of a BEC, the stability condition for the system involving cold damping feedback can be obtained by making all BEC parameters zero in equations (B.17)–(B.23).

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