QUANTIZED FLAG MANIFOLDS AND IRREDUCIBLE ∗-REPRESENTATIONS

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Abstract. We study irreducible ∗-representations of a certain quantization of the algebra of polynomial functions on a generalized flag manifold regarded as a real manifold. All irreducible ∗-representations are classified for a subclass of flag manifolds containing in particular the irreducible compact Hermitian symmetric spaces. For this subclass it is shown that the irreducible ∗-representations are parametrized by the symplectic leaves of the underlying Poisson bracket. We also discuss the relation between the quantized flag manifolds studied in this paper and the quantum flag manifolds studied by Soibel’man, Lakshimibai & Reshetikhin, Jurčo & Štovíček and Korogodsky.

1. Introduction

The irreducible ∗-representations of the “standard” quantization $\mathbb{C}_q[U]$ of the algebra of functions on a compact connected simple Lie group $U$ were classified by Soibel’man [40]. He showed that there is a 1–1 correspondence between the equivalence classes of irreducible ∗-representations of $\mathbb{C}_q[U]$ and the symplectic leaves of the underlying Poisson bracket on $U$ (cf. [39], [40]).

This Poisson bracket is sometimes called Bruhat-Poisson, because its symplectic foliation is a refinement of the Bruhat decomposition of $U$ (cf. Soibel’man [39], [40]). The symplectic leaves are naturally parametrized by $W \times T$, where $T \subset U$ is a maximal torus and $W$ is the Weyl group associated with $(U, T)$.

The 1–1 correspondence between equivalence classes of irreducible ∗-representations of $\mathbb{C}_q[U]$ and symplectic leaves of $U$ can be formally explained by the observation that in the semi-classical limit the kernel of an irreducible ∗-representation should tend to a maximal Poisson ideal. The quotient of the Poisson algebra of polynomial functions on $U$ by this ideal is isomorphic to the Poisson algebra of functions on the symplectic leaf.

In recent years many people have studied quantum homogeneous spaces (see for example [14], [31], [3], [13], [30], [35], [1]). The results referred to above raise the obvious question whether the irreducible ∗-representations of quantized function algebras on $U$-homogeneous spaces can be classified and related to the symplectic foliation of the underlying Poisson bracket. This question was already raised in a paper by Lu & Weinstein [25, Question 4.8], where they studied certain Poisson brackets on $U$-homogeneous spaces that arise as a quotient of the Bruhat-Poisson bracket on $U$.

To our knowledge, affirmative answers to the above mentioned question have been given so far for only three different types of $U$-homogeneous spaces, namely...
Podleś’s family of quantum 2-spheres [36] (the relation with the symplectic foliation of certain covariant Poisson brackets on the 2-sphere seems to have been observed for the first time by Lu & Weinstein [26]), odd-dimensional complex quantum spheres $SU(n + 1)/SU(n)$ (cf. Vaksman & Soibel’man [14]), and Stiefel manifolds $U(n)/U(n - l)$ (cf. Podkolzin & Vainerman [35]).

In this paper we study the irreducible $*$-representations of a certain quantized $*$-algebra of functions on a generalized flag manifold. To be more specific, let $G$ denote the complexification of $U$, and let $P \subset G$ be a parabolic subgroup containing the standard Borel subgroup $B_+$ with respect to a fixed choice of Cartan subalgebra and system of positive roots (compatible with the choice of Bruhat-Poisson bracket on $U$, see [25]). The generalized flag manifold $U/K$ naturally becomes a Poisson $U$-homogeneous space (cf. Lu & Weinstein [25]). The quotient Poisson bracket on $U/K$ is also called Bruhat-Poisson in [25], since its symplectic leaves coincide with the Schubert cells of the flag manifold $G/P \simeq U/K$.

It is straightforward to realize a quantum analogue $\mathbb{C}_q[K]$ of the algebra of polynomial functions on $K$ as a quantum subgroup of $\mathbb{C}_q[U]$. The corresponding $*$-subalgebra $\mathbb{C}_q[U/K]$ of $\mathbb{C}_q[K]$-invariant functions in $\mathbb{C}_q[U]$ may be regarded as a quantization of the Poisson algebra of functions on $U/K$ endowed with the Bruhat-Poisson bracket. The main result in this paper is a classification of all the irreducible $*$-representations of $\mathbb{C}_q[U/K]$ for an important subclass of flag manifolds containing in particular the irreducible Hermitian symmetric spaces of compact type. For this subclass we show that the equivalence classes of irreducible $*$-representations are parametrized by the Schubert cells of $U/K$. Let us emphasize that we regard here the flag manifold $U/K$ as a real manifold. This means that the algebra of functions on $U/K$ has a natural $*$-structure, which survives quantization and allows us to study $*$-representations in a way analogous to Soibel’man’s approach [40].

For an arbitrary generalized flag manifold $U/K$ we describe in detail how irreducible $*$-representations of $\mathbb{C}_q[U]$ decompose under restriction to $\mathbb{C}_q[U/K]$. This decomposition corresponds precisely to the way symplectic leaves in $U$ project to Schubert cells in the flag manifold $U/K$. It leads immediately to a classification of the irreducible $*$-representations of the $C^*$-algebra $\mathbb{C}_q(U/K)$, where $\mathbb{C}_q(U/K)$ is obtained by taking the closure of $\mathbb{C}_q[U/K]$ with respect to the universal $C^*$-norm on $\mathbb{C}_q[U]$. The equivalence classes of irreducible $*$-representations of $\mathbb{C}_q(U/K)$ are naturally parametrized by the symplectic leaves of $U/K$ endowed with the Bruhat-Poisson bracket.

For the classification of the irreducible $*$-representations of the quantized function algebra $\mathbb{C}_q[U/K]$ itself it is important to have a kind of Poincaré-Birkhoff-Witt (PBW) factorization of $\mathbb{C}_q[U/K]$ (which in turn is closely related to the irreducible decomposition of tensor products of certain finite-dimensional irreducible $U$-modules). Such a factorization is needed in order to develop a kind of highest weight representation theory for $\mathbb{C}_q[U/K]$. In Soibel’man’s paper [14], a crucial role is played by a similar factorization of $\mathbb{C}_q[U]$. From Soibel’man’s results one easily derives a factorization of the algebra $\mathbb{C}_q[U/T]$ (corresponding to $P$ minimal parabolic in $G$).

In this paper we derive a PBW type factorization for a different subclass of flag manifolds using the so-called Parthasarathy-Ranga Rao-Varadarajan (PRV) conjecture. This conjecture was formulated as a follow-up to certain results in the paper [34] and was independently proved by Kumar [18] and Mathieu [24] (see also Littelmann [22]). The subclass of flag manifolds $U/K$ we consider here can
be characterized by the two conditions that \((U, K)\) is a Gel’fand pair and that the Dynkin diagram of \(K\) can be obtained from the Dynkin diagram of \(U\) by deleting one node (cf. Koornwinder \cite{Koornwinder}). Note that the corresponding \(P \subset G\) is always maximal parabolic. These two conditions are satisfied for the irreducible compact Hermitian symmetric pairs \((U, K)\).

Roughly speaking, the PBW factorization in the above mentioned cases states that the quantized function algebra \(C_q[U/K]\) coincides with the quantized algebra of zero-weighted complex valued polynomials on \(U/K\). The quantized algebra of zero-weighted complex valued polynomials can be naturally defined for arbitrary generalized flag manifold \(U/K\). It is always a \(*\)-subalgebra of \(C_q[U/K]\) and invariant under the \(C_q[U]\)-coaction (we shall call it the factorized \(*\)-algebra associated with \(U/K\)). The factorized \(*\)-algebra is closely related to the quantized algebra of holomorphic polynomials on generalized flag manifolds studied by Soibel’man \cite{Soibelman}, Lakshmibai & Reshetikhin \cite{LakshmibaiReshetikhin}, \cite{LakshmibaiReshetikhin2}, and Jurčo & Štovíček \cite{JurcoStovicek} (for the classical groups) as well as to the function spaces considered recently by Korogodsky \cite{Korogodsky}.

In this paper we classify the irreducible \(*\)-representations of the factorized \(*\)-algebra associated with an arbitrary flag manifold \(U/K\) and we show that the equivalence classes of irreducible \(*\)-representations are naturally parametrized by the symplectic leaves of \(U/K\) endowed with the Bruhat-Poisson bracket. In particular, we obtain a complete classification of the irreducible \(*\)-representations of \(C_q[U/K]\) whenever a PBW type factorization holds for \(C_q[U/K]\) (i.e., \(C_q[U/K]\) is equal to its factorized \(*\)-algebra).

The paper is organized as follows. In section 2 we review the results by Lu & Weinstein \cite{LuWeinstein} and Soibel’man \cite{Soibelman} concerning the Bruhat-Poisson bracket on \(U\) and the quotient Poisson bracket on a flag manifold. In section 3 we recall some well-known results on the “standard” quantization of the universal enveloping algebra of a simple complex Lie algebra and its finite-dimensional representations. We also recall the construction of the corresponding quantized function algebra \(C_q[U]\) and give some commutation relations between certain matrix coefficients of irreducible corepresentations of \(C_q[U]\). They will play a crucial role in the classification of the irreducible \(*\)-representations of the factorized \(*\)-algebra.

In section 4 we define the quantized algebra \(C_q[U/K]\) of functions on a flag manifold \(U/K\) and its associated factorized \(*\)-subalgebra. We prove that the factorized \(*\)-algebra is equal to \(C_q[U/K]\) for the subclass of flag manifolds referred to above.

In section 5 we study the restriction of an arbitrary irreducible \(*\)-representation of \(C_q[U]\) to \(C_q[U/K]\). We use here Soibel’man’s explicit realization of the irreducible \(*\)-representations of \(C_q[U]\) as tensor products of irreducible \(*\)-representations of \(C_q[SU(2)]\) (cf. \cite{Soibelman}, see also \cite{Soibelman2}, \cite{Soibelman3} for \(SU(n)\)). As a corollary we obtain a complete classification of the irreducible \(*\)-representations of the \(C^*\)-algebra \(C_q(U/K)\).

Section 6 is devoted to the classification of the irreducible \(*\)-representations of the factorized \(*\)-algebra associated with an arbitrary flag manifold. The techniques in section 6 are similar to those used by Soibel’man \cite{Soibelman} for the classification of the irreducible \(*\)-representations of \(C_q[U]\), and to those used by Joseph \cite{Joseph} to handle the more general problem of determining the primitive ideals of \(C_q[U]\).

2. Bruhat-Poisson brackets on flag manifolds

In this section we review some results by Soibel’man \cite{Soibelman} and Lu & Weinstein \cite{LuWeinstein} concerning the Bruhat-Poisson bracket on a compact connected simple Lie group.
Let $g$ be a complex simple Lie algebra with a fixed Cartan subalgebra $h \subset g$. Let $G$ be the connected simply connected Lie group with Lie algebra $g$ (regarded here as a real analytic Lie group).

Let $R \subset h^*$ be the root system associated with $(g,h)$ and write $g_\alpha$ for the root space associated with $\alpha \in R$. Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be a basis of simple roots for $R$, and let $R^+$ (resp. $R^-$) be the set of positive (resp. negative) roots relative to $\Delta$. We identify $h$ with its dual by the Killing form $\kappa$. The non-degenerate symmetric bilinear form on $h^*$ induced by $\kappa$ is denoted by $(\cdot, \cdot)$. Let $W \subset \text{GL}(h^*)$ be the Weyl group of the root system $R$ and write $s_i = s_{\alpha_i}$ for the simple reflection associated with $\alpha_i \in \Delta$.

For $\alpha \in R$ write $d_\alpha := (\alpha, \alpha)/2$. Let $H_\alpha \in h$ be the element associated with the coroot $\alpha^\vee := d_\alpha^{-1} \alpha \in h^*$ under the identification $h \simeq h^*$. Let us choose nonzero $X_\alpha \in g_\alpha$ ($\alpha \in R$) such that for all $\alpha, \beta \in R$ one has $[X_\alpha, X_\beta] = H_\alpha, \kappa(X_\alpha, X_\beta) = d_\alpha^{-1}$ and $[X_\alpha, X_\beta] = c_{\alpha, \beta} X_{\alpha + \beta}$ with $c_{\alpha, \beta} = -c_{-\alpha, -\beta} \in \mathbb{R}$ whenever $\alpha + \beta \in R$. Let $h_0$ be the real form of $h$ defined as the real span of the $H_\alpha$'s ($\alpha \in R$). Then
\begin{equation}
(2.1) \quad u := \sum_{\alpha \in R^+} \mathbb{R}(X_\alpha - X_{-\alpha}) \oplus \sum_{\alpha \in R^+} \mathbb{R}i(X_\alpha + X_{-\alpha}) \oplus i h_0
\end{equation}
is a compact real form of $g$.

Set $b := h_0 \oplus n_+$ with $n_+ := \sum_{\alpha \in R^+} g_\alpha$. Then, by the Iwasawa decomposition for $g$, the triple $(g, u, b)$ is a Manin triple with respect to the imaginary part of the Killing form $\kappa$ (cf. [25, §4]). Hence $u, b$ and $g$ naturally become Lie bialgebras. The dual Lie algebra $u^*$ is isomorphic to $b$, and $g$ may be identified with the classical double of $u$. The cocommutator $\delta : g \to g \wedge g$ of the Lie bialgebra $g$ is coboundary, i.e.,
\[ \delta(X) = (\text{ad}_X \otimes 1 + 1 \otimes \text{ad}_X) r, \]
with the classical $r$-matrix $r \in g \wedge g$ given by the following well-known skew solution of the Modified Classical Yang-Baxter Equation,
\begin{equation}
(2.2) \quad r = i \sum_{\alpha \in R^+} d_\alpha (X_{-\alpha} \otimes X_\alpha - X_\alpha \otimes X_{-\alpha}) \in u \wedge u.
\end{equation}
The cocommutator on $u$ coincides with the restriction of $\delta$ to $u$.

The corresponding Sklyanin bracket on the connected subgroup $U \subset G$ with Lie algebra $u$ has
\begin{equation}
(2.3) \quad \Omega_g = l_{g}^{\otimes 2} r - r_{g}^{\otimes 2} r
\end{equation}
as its associated Poisson tensor. Here $l_g$ resp. $r_g$ denote infinitesimal left resp. right translation. This particular Sklyanin bracket is often called Bruhat-Poisson, since its symplectic foliation is closely related to the Bruhat decomposition of $G$. Let us explain this in more detail.

Let $B$ be the connected subgroup of $G$ with Lie algebra $b$, let $T \subset U$ be the maximal torus in $U$ with Lie algebra $h_0$, and set $B_+ := TB$. The analytic Weyl group $N_U(T)/T$, where $N_U(T)$ is the normalizer of $T$ in $U$, is isomorphic to $W$. More explicitly, the isomorphism sends the simple reflection $s_i$ to exp($\pi(X_{\alpha_i}, - X_{-\alpha_i})$)/$T$. The double $B_+\text{-cosets}$ in $G$ are parametrized by the elements of $W$. 

$U$ and its flag manifolds. For unexplained terminology in this section we refer the reader to [3] and [2].

We identify $X$ here as a real analytic Lie group.)
Hence one has the Bruhat decomposition
\[ G = \coprod_{w \in W} B_+ w B_+. \]

By [45, Prop. 1.2.3.6] the Bruhat decomposition has the following refinement:
\[ G = \coprod_{m \in N_U(T)} B m B. \]  

(2.4)

For \( m \in N_U(T) \) we set \( \Sigma_m := U \cap B m B \). Then \( \Sigma_m \neq \emptyset \) for all \( m \in N_U(T) \), and we have the disjoint union
\[ U = \coprod_{m \in N_U(T)} \Sigma_m. \]  

(2.5)

Now recall that multiplication \( U \times B \to G \) is a global diffeomorphism by the Iwasawa decomposition of \( G \). So for any \( b \in B \) and \( u \in U \) there exists a unique \( u^b \in U \) such that \( bu \in u^b B \). As is easily verified, the map
\[ (u, b) \mapsto u^b \]  

(2.6)

is a right action of \( B \) on \( U \), and the corresponding decomposition of \( U \) into \( B \)-orbits coincides with the decomposition (2.5). On the other hand, if we regard \( B \) as the Poisson-Lie group dual to \( U \), the action (2.6) becomes the right dressing action of the dual group on \( U \) (cf. [25, Thm. 3.14]). Since the orbits in \( U \) under the right dressing action are exactly the symplectic leaves of the Poisson bracket on \( U \) (cf. [38, Thm. 13], [25, Thm. 3.15]), it follows that (2.5) coincides with the decomposition of \( U \) into symplectic leaves (cf. [40, Theorem 2.2]).

Next, we recall some results by Lu & Weinstein [25] concerning certain quotient Poisson brackets on generalized flag manifolds. Let \( S \subset \Delta \) be a set of simple roots, and let \( P_S \) be the corresponding standard parabolic subgroup of \( G \). The Lie algebra \( p_S \) of \( P_S \) is given by
\[ p_S := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma_S} \mathfrak{g}_\alpha \]  

(2.7)

with \( \Gamma_S := R^+ \cup \{ \alpha \in R | \alpha \in \text{span}(S) \} \). Let \( I_S \) be the Levi factor of \( p_S \),
\[ I_S := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma_S \cap (-\Gamma_S)} \mathfrak{g}_\alpha. \]  

(2.8)

and set \( \mathfrak{k}_S := p_S \cap \mathfrak{u} = I_S \cap \mathfrak{u} \). Then \( \mathfrak{k}_S \) is a compact real form of \( I_S \). Set \( K_S := U \cap P_S \subset U \), then \( K_S \subset U \) is a Poisson-Lie subgroup of \( U \) with Lie algebra \( \mathfrak{k}_S \) (cf. [23, Thm. 4.7]). Hence there is a unique Poisson bracket on \( U/K_S \) such that the natural projection \( \pi : U \to U/K_S \) is a Poisson map. This bracket is also called Bruhat-Poisson. It is covariant in the sense that the natural left action \( U \times U/K_S \to U/K_S \) is a Poisson map.

Let \( W_S \) be the subgroup of \( W \) generated by the simple reflections in \( S \). One has \( P_S = B_+ W_S B_+ \) (cf. [15, Thm. 1.2.1.1]). From this one easily deduces that the double cosets \( B_+ x P_S \) (\( x \in G \)) are parametrized by the elements of \( W/W_S \). Hence one has the Schubert cell decomposition of \( U/K_S \cong G/P_S \):
\[ U/K_S = \coprod_{\overline{w} \in W/W_S} X_{\overline{w}}, \quad X_{\overline{w}} := (U \cap B_+ w P_S)/K_S \cong B_+ w/P_S, \]  

(2.9)

where \( \overline{w} \in W/W_S \) is the right \( W_S \)-coset in \( W \) which contains \( w \).
Now, by [23, Prop. 4.5], the subgroup $K_S$ is invariant under the action of $B$, which implies that the $B$-action descends to $U/K_S$. The orbits in $U/K_S$ coincide exactly with the Schubert cells. By [23, Thm. 4.6] the symplectic leaves of the Poisson manifold $U/K_S$ are exactly the orbits under the $B$-action. We conclude (cf. [23, Thm. 4.7]):

**Theorem 2.1.** The decomposition into symplectic leaves of the flag manifold $U/K_S$ endowed with the Bruhat-Poisson bracket coincides with its decomposition into Schubert cells.

Consider now the set of minimal coset representatives

$$W^S := \{ w \in W \mid l(ws_{\alpha}) > l(w) \quad \forall \alpha \in S \}. \tag{2.10}$$

$W^S$ is a complete set of coset representatives for $W/W_S$. Any element $w \in W$ can be uniquely written as a product $w = w_1w_2$ with $w_1 \in W^S$, $w_2 \in W_S$. The elements of $W^S$ are minimal in the sense that

$$l(w_1w_2) = l(w_1) + l(w_2), \quad (w_1 \in W^S, w_2 \in W_S), \tag{2.11}$$

where $l(w) := \#(R^+ \cap wR^-)$ is the length function on $W$.

Observe that $\pi$ maps the symplectic leaf $\Sigma_m \subset U$ onto the symplectic leaf $X_{w(m)} \subset U/K_S$, where $w(m) := m/T \in W$. We write $\pi_m : \Sigma_m \to X_{w(m)}$ for the surjective Poisson map obtained by restricting $\pi$ to the symplectic leaf $\Sigma_m$. The minimality condition (2.10) translates to the following property of the map $\pi_m$.

**Proposition 2.2.** Let $m \in N_U(T)$. Then $\pi_m : \Sigma_m \to X_{w(m)}$ is a symplectic automorphism if and only if $w(m) \in W^S$.

**Proof.** For $w \in W$ set

$$n_w := \bigoplus_{\alpha \in R^+ \cap wR^-} g_{\alpha}, \quad N_w := \exp(n_w).$$

Observe that the complex dimension of $N_w$ is equal to $l(w)$. Write $\text{pr}_U : G \simeq U \times B \to U$ for the canonical projection. It is well known that for $m \in N_U(T)$ and for $w \in W^S$ with representative $m_w \in N_U(T)$, the maps

$$\phi_m : N_{w(m)} \to \Sigma_m, \quad n \mapsto \text{pr}_U(nm),$$

$$\psi_w : N_w \to X_{\pi_w}, \quad n \mapsto \pi(\text{pr}_U(nm))$$

are surjective diffeomorphisms (see for example [1, Proposition 1.1 & 5.1]). The map $\psi_w$ is independent of the choice of representative $m_w$ for $w$. It follows now from (2.11) by a dimension count that $\pi_m$ can only be a diffeomorphism if $w(m) \in W^S$. On the other hand, if $m \in N_U(T)$ such that $w(m) \in W^S$, then $\pi_m = \psi_{w(m)} \circ \phi_{m}^{-1}$ and hence $\pi_m$ is a diffeomorphism.

Soibel’man [10] gave a description of the symplectic leaves $\Sigma_m$ ($m \in N_U(T)$) as a product of two-dimensional leaves which turns out to have a nice generalization to the quantized setting (cf. section 5). For $i \in [1, r]$, let $\gamma_i : SU(2) \hookrightarrow U$ be the embedding corresponding to the $i$th node of the Dynkin diagram of $U$. After a possible renormalization of the Bruhat-Poisson structure on $SU(2)$, $\gamma_i$ becomes an embedding of Poisson-Lie groups. Recall that the two-dimensional leaves of $SU(2)$ are given by

$$S_t := \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in SU(2) \mid \arg(\beta) = \arg(t) \right\} \quad (t \in \mathbb{T})$$
where $T \subset \mathbb{C}$ is the unit circle in the complex plane. The restriction of the embedding $\gamma_i$ to $S_1 \subset SU(2)$ is a symplectic automorphism from $S_1$ onto the symplectic leaf $\Sigma_{m_i} \subset U$, where $m_i = \exp \left( \mathbf{i} (X_{\alpha_i} - X_{-\alpha_i}) \right)$. Recall that $m_i \in N_U(T)$ is a representative of the simple reflection $s_i \in W$.

For arbitrary $m \in N_U(T)$ let $w(m) = s_{i_1} s_{i_2} \cdots s_{i_l}$ be a reduced expression for $w(m) := m/T \in W$, and let $t_m \in T$ be the unique element such that $m = m_{i_1} m_{i_2} \cdots m_{i_l} t_m$. Note that $t_m$ depends on the choice of reduced expression for $w(m)$. The map

$$(g_1, \ldots, g_l) \mapsto \gamma_{i_1} (g_1) \gamma_{i_2} (g_2) \cdots \gamma_{i_l} (g_l) t_m$$

defines a symplectic automorphism from $S_1^{\times l}$ onto the symplectic leaf $\Sigma_m \subset U$ (cf. [10] \S 2, [12]). Note that the image of the map is independent of the choice of reduced expression for $w(m)$, although the map itself is not.

Combined with Proposition 2.2 we now obtain the following description of the symplectic leaves of the generalized flag manifold $U/K_S$.

**Proposition 2.3.** Let $m \in N_U(T)$ and set $w := m/T \in W$. Let $w_1 \in W^S$, $w_2 \in W^S$ be such that $w = w_1 w_2$ and choose reduced expressions $w_1 = s_{i_1} \cdots s_{i_p}$ and $w_2 = s_{i_{p+1}} \cdots s_{i_l}$. Then the map

$$(g_1, g_2, \ldots, g_l) \mapsto \gamma_{i_1} (g_1) \gamma_{i_2} (g_2) \cdots \gamma_{i_l} (g_l) / K_S$$

is a surjective Poisson map from $S_1^{\times l}$ onto the Schubert cell $X_{\mathbf{m}}$. It factorizes through the projection $pr : S_1^{\times l} = S_1^{\times p} \times S_1^{\times (l-p)} \to S_1^{\times p}$. The quotient map from $S_1^{\times p}$ onto $X_{\mathbf{m}}$ is a symplectic automorphism. In particular, we have

$$X_{\mathbf{m}} = (\Sigma_{m_{i_1}} \Sigma_{m_{i_2}} \cdots \Sigma_{m_{i_l}})/K_S.$$ 

See [24] for more details in the case of the full flag manifold $(K_S = T)$.

### 3. Preliminaries on the Quantized Function Algebra $\mathbb{C}_q[U]$}

In this section we introduce some notations which we will need throughout the remainder of this paper. First, we recall the definition of the quantized universal enveloping algebra associated with the simple complex Lie algebra $\mathfrak{g}$. We use the notations introduced in the previous section.

Set $d_i := d_{\alpha_i}$ and $H_i := H_{\alpha_i}$ for $i \in [1, r]$. Let $A = (a_{ij})$ be the Cartan matrix, i.e. $a_{ij} := d_{\alpha_i}^{-1} (\alpha_i, \alpha_j)$. Note that $H_i \in \mathfrak{h}$ is the unique element such that $\alpha_j (H_i) = a_{ij}$ for all $j$. The weight lattice is given by

$$(3.1) \quad P = \{ \lambda \in \mathfrak{h}^* \mid \lambda (H_i) = (\lambda, \alpha_i^\vee) \in \mathbb{Z} \quad \forall i \}.$$ 

The fundamental weights $\varpi_{\alpha_i} = \varpi_i$ $(i \in [1, r])$ are characterized by $\varpi_i (H_j) = (\varpi_i, \alpha_j^\vee) = \delta_{ij}$ for all $j$. The set of dominant weights $P_+$ resp. regular dominant weights $P_+$ is equal to $\mathbb{K}$-span$\{ \varpi_{\alpha_i} \}_{i \in \Delta}$ with $\mathbb{K} = \mathbb{Z}_+$ resp. $\mathbb{N}$.

We fix $q \in (0, 1)$. The quantized universal enveloping algebra $U_q(\mathfrak{g})$ associated with the simple Lie algebra $\mathfrak{g}$ is the unital associative algebra over $\mathbb{C}$ with generators
In particular, $U_Q$ write $\alpha$ rem for $U$ positive roots. We have the direct sum decomposition $U$ in (2.1).

where

\[ K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1 \]

\[ K_i X_j^\pm K_i^{-1} = q_i^{\pm \alpha(H_j)} X_j^\pm \]

\[ X_i^+ X_j^- - X_j^- X_i^+ = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \]

\[ \sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s} (X_i^\pm)^{1-a_{ij}-s} X_j^\pm (X_i^\pm)^s = 0 \quad (i \neq j) \]

where $q_i := q^{d_i}$.

\[ [a]_q := \frac{q^a - q^{-a}}{q - q^{-1}} \quad (a \in \mathbb{N}), \quad [0]_q := 1, \]

\[ [a]_q! := [a]_q[a - 1]_q \ldots [1]_q, \]

and

\[ \binom{a}{n}_q := \frac{[a]_q!}{[a - n]_q! [n]_q!}. \]

A Hopf algebra structure on $U_q(\mathfrak{g})$ is uniquely determined by the formulas

\[ \Delta(X_i^+) = X_i^+ \otimes 1 + K_i \otimes X_i^+, \quad \Delta(X_i^-) = X_i^- \otimes K_i^{-1} + 1 \otimes X_i^-, \]

\[ \Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \]

\[ S(K_i^{\pm 1}) = K_i^{\mp 1}, \quad S(X_i^+) = -K_i^{-1} X_i^+, \quad S(X_i^-) = -X_i^- K_i, \]

\[ \varepsilon(K_i^{\pm 1}) = 1, \quad \varepsilon(X_i^\pm) = 0. \]

In fact, $U_q(\mathfrak{g})$ may be regarded as a quantization of the co-Poisson-Hopf algebra structure (cf. \cite[Ch. 6]{Bou96}) on $U(\mathfrak{g})$ induced by the Lie bialgebra $(\mathfrak{g}, -i\delta)$, $\delta$ being the cocommutator of $\mathfrak{g}$ associated with the $r$-matrix \cite{Drinfeld81}. $U_q(\mathfrak{g})$ becomes a Hopf $*$-algebra with $*$-structure on the generators given by

\[ (K_i^{\pm 1})^* = K_i^{1 \mp 1}, \quad (X_i^+)^* = q_i^{-1} X_i^- K_i, \quad (X_i^-)^* = q_i K_i^{-1} X_i^+. \]

In the classical limit $q \to 1$, the $*$-structure becomes an involutive, conjugate-linear anti-automorphism of $\mathfrak{g}$ with $-1$ eigenspace equal to the compact real form $\mathfrak{u}$ defined in \cite{Kac83}.

Let $U^\pm = U_q(\mathfrak{n}_\pm)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $X_i^\pm$ ($i = [1, r]$) and write $U^0 := U_q(\mathfrak{h})$ for the commutative subalgebra generated by $K_i^{\pm 1}$ ($i = [1, r]$). Let us write $Q$ (resp. $Q^+$) for the integral (resp. positive integral) span of the positive roots. We have the direct sum decomposition

\[ U^\pm = \bigoplus_{\alpha \in Q^+} U_{\pm \alpha}, \]

where $U_{\pm \alpha} := \{ \phi \in U^\pm | K_i \phi K_i^{-1} = q_i^{\pm \alpha(H_i)} \phi \}$. The Poincaré-Birkhoff-Witt Theorem for $U_q(\mathfrak{g})$ states that multiplication defines an isomorphism of vector spaces

\[ U^- \otimes U^0 \otimes U^+ \to U_q(\mathfrak{g}). \]

In particular, $U_q(\mathfrak{g})$ is spanned by elements of the form $b_{\eta \zeta} K^\alpha a_{\zeta}$ where $b_{\eta \zeta} \in U_{\eta \zeta}$, $a_{\zeta} \in U^+_\zeta$ ($\eta, \zeta \in Q_+$) and $\alpha \in Q$. Here we used the notation $K^\alpha = K_{\alpha_1} \cdots K_{\alpha_r}$ if $\alpha = \sum k_i \alpha_i$. 
For a left \( U_q(\mathfrak{g}) \)-module \( V \), we say that \( 0 \neq v \in V \) has weight \( \mu \in \mathfrak{h}^* \) if \( K_i \cdot v = q^{\mu(\alpha_i)}v \) for all \( i \). We write \( V_\mu \) for the corresponding weight space. Recall that a \( P \)-weighted finite-dimensional irreducible representation of \( U_q(\mathfrak{g}) \) is a highest weight module \( V = V(\lambda) \) with highest weight \( \lambda \in P_+ \). If \( v_\lambda \in V(\lambda) \) is a highest weight vector, we have \( V(\lambda) = \sum_{\alpha \in Q^+} U^-\alpha v_\lambda \) by the PBW Theorem, hence the set of weights \( P(\lambda) \) of \( V(\lambda) \) is a subset of the weight lattice \( P \) satisfying \( \mu \leq \lambda \) for all \( \mu \in P(\lambda) \). Here \( \leq \) is the dominance order on \( P \) (i.e. \( \mu \leq \nu \) if \( \nu - \mu \in Q^+ \) and \( \mu < \nu \) if \( \mu \leq \nu \) and \( \mu \neq \nu \).

We define irreducible finite dimensional \( P \)-weighted right \( U_q(\mathfrak{g}) \)-modules with respect to the opposite Borel subgroup. So the irreducible finite dimensional right \( U_q(\mathfrak{g}) \)-module \( V(\lambda) \) with highest weight \( \lambda \in P_+ \) has the weight space decomposition \( V(\lambda) = \sum_{\alpha \in Q^+} v_\lambda U^+\alpha \), where \( v_\lambda \in V(\lambda) \) is the highest weight vector of \( V(\lambda) \). The weights of the right \( U_q(\mathfrak{g}) \)-module \( V(\lambda) \) coincide with the weights of the left \( U_q(\mathfrak{g}) \)-module \( V(\lambda) \) and the dimensions of the corresponding weight spaces are the same.

The quantized algebra \( \mathbb{C}_q[G] \) of functions on the connected simply connected complex Lie group \( G \) with Lie algebra \( \mathfrak{g} \) is the subspace in the linear dual \( U_q(\mathfrak{g})^* \) spanned by the matrix coefficients of the finite-dimensional irreducible representations \( V(\lambda) \) (\( \lambda \in P_+ \)). The Hopf \( \ast \)-algebra structure on \( U_q(\mathfrak{g}) \) induces a Hopf \( \ast \)-algebra structure on \( \mathbb{C}_q[G] \subset U_q(\mathfrak{g})^* \) by the formulas

\[
(\phi \psi)(X) = (\phi \otimes \psi)\Delta(X), \quad 1(X) = \varepsilon(X)
\]

\[
\Delta(\phi)(X \otimes Y) = \phi(XY), \quad \varepsilon(\phi) = \phi(1)
\]

\[
S(\phi)(X) = \phi(S(X)), \quad (\phi^\ast)(X) = \phi(S(X)^\ast)
\]

where \( \phi, \psi \in \mathbb{C}_q[G] \subset U_q(\mathfrak{g})^* \) and \( X, Y \in U_q(\mathfrak{g}) \). The algebra \( \mathbb{C}_q[G] \) can be regarded as a quantization of the Poisson algebra of polynomial functions on the algebraic Poisson-Lie group \( G \), where the Poisson structure on \( G \) is given by the Sklyanin bracket associated with the classical r-matrix \(-ir\) (cf. (2.2)). Since the \( \ast \)-structure on \( \mathbb{C}_q[G] \) is associated with the compact real form \( U \) of \( G \) in the classical limit, we will write \( \mathbb{C}_q[U] \) for \( \mathbb{C}_q[G] \) with this particular choice of \( \ast \)-structure. Note that \( \mathbb{C}_q[U] \) is a \( U_q(\mathfrak{g}) \)-bimodule with the left respectively right action given by

\[
(X, \phi)(Y) := \phi(YX), \quad (\phi, X)(Y) := \phi(XY)
\]

where \( \phi \in \mathbb{C}_q[U] \) and \( X, Y \in U_q(\mathfrak{g}) \). The finite-dimensional irreducible \( U_q(\mathfrak{g}) \)-module \( V(\lambda) \) of highest weight \( \lambda \in P_+ \) is known to be unitarizable (say with inner product \( (\cdot, \cdot) \)). So we can choose an orthonormal basis consisting of weight vectors

\[
\{ v^{(i)}_\mu | \mu \in P(\lambda), i = [1, \dim(V(\lambda)_\mu)] \},
\]

where \( v^{(i)}_\mu \in V(\lambda)_\mu \) (we omit the index \( i \) if \( \dim(V(\lambda)_\mu) = 1 \)). Set

\[
C^\lambda_{\mu,i;\nu,j}(X) := (X, v^{(i)}_\mu, v^{(j)}_\nu), \quad X \in U_q(\mathfrak{g})
\]

for \( \mu, \nu \in P(\lambda) \) and \( 1 \leq i \leq \dim(V(\lambda)_\mu), 1 \leq j \leq \dim(V(\lambda)_\nu) \). If \( \dim(V(\lambda)_\mu) = 1 \) respectively \( \dim(V(\lambda)_\nu) = 1 \) we omit the dependence on \( i \) respectively \( j \) in \( (3.8) \).

It is sometimes also convenient to use the notation

\[
C^\lambda_{\nu,w}(X) := (X, v, w), \quad v, w \in V(\lambda), X \in U_q(\mathfrak{g})
\]

Note that when \( \lambda \) runs through \( P_+ \) and \( \mu, i, \nu \) and \( j \) run through the above-mentioned sets the matrix elements \( (3.8) \) form a linear basis of \( \mathbb{C}_q[G] \). Furthermore,
we have the formulas
\begin{equation}
\Delta (C_{\mu,i;\nu,j}^\lambda) = \sum_{\sigma,s} C_{\mu,i;\sigma,s}^\lambda \otimes C_{\sigma,s;\nu,j}^\lambda, \tag{3.9}
\end{equation}
\begin{equation}
\varepsilon (C_{\mu,i;\nu,j}^\lambda) = \delta_{\mu,\nu} \delta_{i,j}, \quad (C_{\mu,i;\nu,j}^\lambda)^* = S (C_{\nu,j;\mu,i}^\lambda). \tag{3.10}
\end{equation}

(Sums for which the summation sets are not specified are taken over the “obvious” choice of summation sets). Using the relations (3.9) and the Hopf algebra axiom for the antipode we obtain
\begin{equation}
\sum_{\sigma,s} (C_{\sigma,s;\mu,i}^\lambda)^* C_{\sigma,s;\nu,j}^\lambda = \delta_{\mu,\nu} \delta_{i,j}.
\end{equation}

The elements \((C_{\mu,i;\nu,j}^\lambda)^*)\) are matrix coefficients of the contragredient representation \(V(\lambda)^* \cong V(-\sigma_0 \lambda)\) (here \(\sigma_0\) is the longest element in \(W\)). To be precise, let \(\pi : U_q(\mathfrak{g}) \to \text{End}(V(\lambda))\) be the representation of highest weight \(\lambda\), and let \((\cdot, \cdot)\) be an inner product with respect to which \(\pi\) is unitarizable. Fix an orthonormal basis of weight vectors \(\{v_u\}^r_i\). Let \((\pi^*, V(\lambda)^*)\) be the contragredient representation, i.e. \(\pi^*(X)\phi = \phi \circ \pi(S(X))\) for \(X \in U_q(\mathfrak{g})\) and \(\phi \in V(\lambda)^*\). For \(u \in V(\lambda)\) set \(u^* := (\cdot, u) \in V(\lambda)^*\). We define an inner product on \(V(\lambda)^*\) by
\begin{equation}
(u^*, v^*) := (\pi(K^{-2\rho}) v^* , u^*), \quad u, v \in V(\lambda),
\end{equation}
where \(\rho = 1/2 \sum_{\alpha \in R^+} \alpha \in \mathfrak{h}^*\). By using the fact that \(S^2(u) = K^{-2\rho} u K^{2\rho}\) \((u \in U_q(\mathfrak{g}))\) one easily deduces that \(\pi^*\) is unitarizable with respect to the inner product \((\cdot, \cdot)\) on \(V(\lambda)^*\) and that \(\{\phi_{-\mu}^{(i)} := q^{(i)} \rho (v_{-\mu}^{(i)})^*\}\) is an orthonormal basis of \(V(\lambda)^*\) consisting of weight vectors (here \(\phi_{-\mu}^{(i)}\) has weight \(-\mu\)). Defining the matrix coefficients \(C_{-\mu,i;\nu,j}^\sigma\) of \((\pi^*, V(\lambda)^*)\) with respect to the orthonormal basis \(\{\phi_{-\mu}^{(i)}\}\), we then have
\begin{equation}
(C_{\mu,i;\nu,j}^\lambda)^* = q^{(i\cdot \nu,\mu)} C_{-\mu,i;\nu,j}^\sigma, \tag{3.11}
\end{equation}
(cf. [10] Prop. 3.3). A fundamental role in Soibelman’s theory of irreducible \(*\)-representations of \(C_q[U]\) is played by a Poincaré-Birkhoff-Witt (PBW) type factorization of \(C_q[U]\). For \(\lambda \in P_+\), set
\begin{equation}
B_\lambda := \text{span}\{C_{\nu;j}^\lambda | v \in V(\lambda)\}. \tag{3.12}
\end{equation}
Note that \(B_\lambda\) is a right \(U_q(\mathfrak{g})\)-submodule of \(C_q[U]\) isomorphic to \(V(\lambda)\). Set
\begin{equation}
A^+ := \bigoplus_{\lambda \in P_+} B_\lambda, \quad A^{++} := \bigoplus_{\lambda \in P_{++}} B_\lambda. \tag{3.13}
\end{equation}
The subalgebra and right \(U_q(\mathfrak{g})\)-module \(A^+\) is equal to the subalgebra of left \(U^+\)-invariant elements in \(C_q[U]\) (cf. [8]). The existence of a PBW type factorization of \(C_q[U]\) now amounts to the following statement.

**Theorem 3.1.** [10] The multiplication map \(m : (A^{++})^* \otimes A^{++} \to C_q[U]\) is surjective.

A detailed proof can be found in [8] Prop. 9.2.2. The proof is based on certain results concerning decompositions of tensor products of irreducible finite-dimensional \(U_q(\mathfrak{g})\)-modules which can be traced back to Kostant in the classical case [10, Theorem 5.1]. The close connection between Theorem 3.1 and the decomposition of
tensor products of irreducible $U_q(g)$-modules becomes clear by observing that
\begin{equation}
(B_\lambda)^* B_\mu \simeq V(\lambda)^* \otimes V(\mu)
\end{equation}
as right $U_q(g)$-modules.

Important for the study of $*$-representations of $C_q[U]$ is some detailed information about the commutation relations between matrix elements in $C_q[U]$. In view of Theorem 3.1, we are especially interested in commutation relations between the $C^\lambda_{\mu,\nu};\Lambda$ and $C^\lambda_{\nu,\lambda;\Lambda}$, resp. between the $C^\lambda_{\mu,\nu};\Lambda$ and $(C^\lambda_{\nu,\lambda;\Lambda})^*$, where $\lambda, \Lambda \in P_+$. To state these commutation relations we need to introduce certain vector subspaces of $C_q[U]$. Let $\lambda, \Lambda \in P_+$ and $\mu \in P(\lambda)$, $\nu \in P(\Lambda)$, then we set
\begin{equation}
N(\mu, \lambda; \nu, \Lambda) := \text{span}\{C^\lambda_{\nu,\mu};\Lambda C^\lambda_{\mu,\nu};\Lambda | (v, w) \in sN\},
\end{equation}
where $sN := sN(\mu, \lambda; \nu, \Lambda)$ is the set of pairs $(v, w) \in V(\lambda)_\mu \times V(\Lambda)_\nu$ with $\mu' > \mu$, $\nu' < \nu$ and $\mu' + \nu' = \mu + \nu$. Furthermore set
\begin{equation}
O(\mu, \lambda; \nu, \Lambda) := \text{span}\{((C^\lambda_{\nu,\mu};\Lambda)^* C^\lambda_{\mu,\nu};\Lambda | (v, w) \in sO\},
\end{equation}
where $sO := sO(\mu, \lambda; \nu, \Lambda)$ is the set of pairs $(v, w) \in V(\lambda)_\mu \times V(\Lambda)_\nu$ with $\mu' < \mu$, $\nu' < \nu$ and $\mu - \mu' = \nu - \nu'$. If $sN$ (resp. $sO$) is empty, then let $N = N^{opp} = \{0\}$ (resp. $O = O^{opp} = \{0\}$). We now have the following proposition.

**Proposition 3.2.** Let $\lambda, \Lambda \in P_+$ and $\nu \in V(\lambda)_\mu$, $\nu \in V(\Lambda)_\nu$.

(i) The matrix elements $C^\lambda_{\nu,\mu};\Lambda$ and $C^\lambda_{\mu,\nu};\Lambda$ satisfy the commutation relation
\begin{equation}
C^\lambda_{\nu,\mu};\Lambda C^\lambda_{\mu,\nu};\Lambda = q^{(\lambda,\Lambda)-(\mu,\nu)} C^\lambda_{\mu,\nu};\Lambda C^\lambda_{\nu,\mu};\Lambda \mod N(\mu, \lambda; \nu, \Lambda).
\end{equation}
Moreover, we have $N = N^{opp}$.

(ii) The matrix elements $(C^\lambda_{\nu,\mu};\Lambda)^*$ and $C^\lambda_{\mu,\nu};\Lambda$ satisfy the commutation relation
\begin{equation}
(C^\lambda_{\nu,\mu};\Lambda)^* C^\lambda_{\mu,\nu};\Lambda = q^{(\mu,\nu)-(\lambda,\Lambda)} C^\lambda_{\mu,\nu};\Lambda (C^\lambda_{\nu,\mu};\Lambda)^* \mod O(\mu, \lambda; \nu, \Lambda).
\end{equation}
Moreover, we have $O = O^{opp}$.

Soibelman [10] derived commutation relations using the universal $R$-matrix whereas Joseph [8, Section 9.1] used the Poincaré-Birkhoff-Witt Theorem for $U_q(g)$ and the left respectively right action $R^{(\omega)}$ of $U_q(g)$ on $C_q[U]$. Although the commutation relations formulated here are slightly sharper, the proof can be derived in a similar manner and will therefore be omitted.

As a corollary of Proposition 3.2(i) we have

**Corollary 3.3.** Let $\lambda, \Lambda \in P_+$ and $\nu \in V(\lambda)_\mu$, $\nu \in V(\Lambda)_\nu$. Then
\begin{equation}
C^\lambda_{\nu,\mu};\Lambda C^\lambda_{\mu,\nu};\Lambda = q^{(\mu,\nu)-(\lambda,\Lambda)} C^\lambda_{\mu,\nu};\Lambda C^\lambda_{\nu,\mu};\Lambda \mod N(\nu, \Lambda; \mu, \lambda).
\end{equation}

Note that Proposition 3.2(i) and Corollary 3.3 give two different ways to rewrite $C^\lambda_{\nu,\mu};\Lambda C^\lambda_{\mu,\nu};\Lambda$ as elements of the vector space
\begin{equation}
W_{\lambda,\Lambda} := \text{span}\{C^\lambda_{\nu,\mu};\Lambda C^\lambda_{\mu,\nu};\Lambda | \nu' \in V(\lambda), \ w' \in V(\Lambda)\}.
\end{equation}
We will need both “inequivalent” commutation relations (Proposition 3.2(i) and Corollary 3.3) in later sections. It follows in particular that, when $\nu' \in V(\lambda)$ and $w' \in V(\Lambda)$ run through a basis, the elements $C^\lambda_{\nu,\mu};\Lambda C^\lambda_{\mu,\nu};\Lambda$ are (in general) linearly dependent. This also follows from the following two observations. On the one hand,
Let $U$ be an algebra subalgebra, given explicitly by (2.7). We define the quantized universal enveloping algebra generated by $K$ associated with the Levi factor $l$. Any finite-dimensional irreducible components too.

By contrast, the commutation relation given in Proposition (3.2) is unique in the sense that, when $v \in V(\lambda)$ and $w \in V(\Lambda)$ run through a basis, the $C^\Lambda_{W,\alpha}(C^\Lambda_{W,\alpha})^*$ are linearly independent (cf. (3.14)).

We end this section by recalling the special case $g = sl(2, \mathbb{C})$. Set

$$t_{11} := C_{w_1; w_1}, \quad t_{12} := C_{\varpi_i; -\varpi_i},$$
$$t_{21} := C_{\varpi_i; \varpi_i}, \quad t_{22} := C_{-\varpi_i; -\varpi_i},$$

(3.18) Then it is well known that the $t_{ij}$’s generate the algebra $\mathbb{C}[SU(2)]$. The commutation relations

$$t_{k1}t_{k2} = qt_{k2}t_{k1}, \quad t_{1k}t_{2k} = qt_{2k}t_{1k} \quad (k = 1, 2),$$
$$t_{12}t_{21} = t_{21}t_{12}, \quad t_{11}t_{22} - t_{22}t_{11} = (q - q^{-1})t_{12}t_{21},$$
$$t_{11}t_{22} - qt_{12}t_{21} = 1$$

(3.19) characterize the algebra structure of $\mathbb{C}[SU(2)]$ in terms of the generators $t_{ij}$. The $*$-structure is uniquely determined by the formulas $t_{11}^* = t_{22}, \quad t_{12}^* = -qt_{21}$.

4. Quantized universal enveloping algebra on generalized flag manifolds

Let $S$ be any subset of the simple roots $\Delta$. We will sometimes identify $S$ with the index set $\{i \mid \alpha_i \in S\}$. Let $p_S \subset g$ be the corresponding standard parabolic subalgebra, given explicitly by (2.7). We define the quantized universal enveloping algebra $U_q(p_S)$ associated with the Levi factor $l_S$ of $p_S$ as the subalgebra of $U_q(g)$ generated by $K_i^{\pm 1}$ ($i \in [1, r]$) and $X_i^{\pm}$ ($i \in S$). Note that $U_q(l_S)$ is a Hopf $*$-subalgebra of $U_q(g)$.

For later use in this section we briefly discuss the finite-dimensional representation theory of $U_q(l_S)$. Recall that $l_S$ is a reductive Lie algebra with centre

$$Z(l_S) = \bigcap_{i \in S} \text{Ker}(\alpha_i) \subset \mathfrak{h}.$$ (4.1)

Moreover, we have direct sum decompositions

$$\mathfrak{h} = Z(l_S) \oplus \mathfrak{h}_S, \quad l_S = Z(l_S) \oplus \mathfrak{l}_S^0,$$ (4.2)

where $\mathfrak{h}_S = \text{span}\{H_i\}_{i \in S}$ and $\mathfrak{l}_S^0$ is the semisimple part of $l_S$. The semisimple part $\mathfrak{l}_S^0$ is explicitly given by

$$\mathfrak{l}_S^0 := \mathfrak{h}_S \oplus \bigoplus_{\alpha \in \Gamma_S \cap (-\Gamma_S)} \mathfrak{g}_\alpha.$$ (4.3)

We define the quantized universal enveloping algebra $U_q(l_S)$ associated with the semisimple part $\mathfrak{l}_S^0$ of $l_S$ as the subalgebra of $U_q(g)$ generated by $K_i^{\pm 1}$ and $X_i^{\pm}$ for all $i \in S$. Observe that $U_q(\mathfrak{l}_S)$ is a Hopf $*$-subalgebra of $U_q(g)$.

**Proposition 4.1.** Any finite-dimensional $U_q(l_S)$-module $V$ which is completely reducible as $U_q(\mathfrak{h})$-module, is completely reducible as $U_q(l_S)$-module.

**Proof.** Let $V$ be a finite-dimensional left $U_q(l_S)$-module which is completely reducible as $U_q(\mathfrak{h})$-module. Then the linear subspace

$$V^+ := \{v \in V \mid X_i^+ v = 0 \quad \forall i \in S\}$$
is $U_q(\mathfrak{b})$-stable and splits as a direct sum of weight spaces. Let $\{v_i\}$ be a linear basis of $V^+$ consisting of weight vectors, and set $V_i := U_q(\mathfrak{f}_0^S)v_i$. Since $U_q(\mathfrak{f}_0^S)$ is the quantized universal enveloping algebra associated with a semisimple Lie algebra, it follows that $V = \sum_i V_i$ is a decomposition of $V$ into irreducible $U_q(\mathfrak{f}_0^S)$-modules.

On the other hand, the $V_i$ are $U_q(\mathfrak{g})$-stable since the vectors $v_i$ are weight vectors. Hence $V = \sum_i V_i$ is a decomposition of $V$ into irreducible $U_q(\mathfrak{g})$-modules.

There are obvious notions of weight vectors and weights for $U_q(\mathfrak{g})$-modules. With a suitably extended interpretation of the notion of highest weight, the irreducible finite-dimensional $U_q(\mathfrak{g})$-modules may be characterized in terms of highest weights. We shall only be interested in irreducible $U_q(\mathfrak{g})$-modules with weights in the lattice $P$. For instance, the restriction of an irreducible $P$-weighted $U_q(\mathfrak{g})$-module to $U_q(\mathfrak{g})$ decomposes into such irreducible $U_q(\mathfrak{g})$-modules.

Branching rules for the restriction of finite-dimensional representations of $U_q(\mathfrak{g})$ to $U_q(\mathfrak{g})$ are determined by the behaviour of the corresponding characters. Since the characters for $P$-weighted irreducible finite-dimensional representations of $U_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$ are the same as for the corresponding representations of $\mathfrak{g}$ and $\mathfrak{g}$, one easily derives the following proposition.

**Proposition 4.2.** Let $\lambda \in P_*$. The multiplicity of any $P$-weighted irreducible $U_q(\mathfrak{g})$-module in the irreducible decomposition of the restriction of the $U_q(\mathfrak{g})$-module $V(\lambda)$ to $U_q(\mathfrak{g})$ is the same as in the classical case.

Next, we define the quantized algebra of functions on $U/K$. The mapping $\iota_S^* : U_q(\mathfrak{g})^* \to U_q(\mathfrak{g})^*$ dual to the Hopf $*$-embedding $\iota_S : U_q(\mathfrak{g}) \hookrightarrow U_q(\mathfrak{g})$ is surjective, and we set

$$C_q[L_S] := \iota_S^*(C_q[G]) = \{ \phi \in \iota_S(C_q[G]) \}.$$  

The formulas \([3,5]\) uniquely determine a Hopf $*$-algebra structure on $C_q[L_S]$, and $\iota_S^*$ then becomes a Hopf $*$-algebra morphism. We write $C_q[K_S]$ for $C_q[L_S]$ with this particular choice of $*$-structure. Assume now that $S \neq \Delta$. Define a $*$-subalgebra $C_q[U/K_S] \subset C_q[U]$ by

$$C_q[U/K_S] := \{ \phi \in C_q[U] \mid (\text{id} \otimes \iota_S^*)\Delta(\phi) = \phi \otimes 1 \}$$

$$= \{ \phi \in C_q[U] \mid X.\phi = \varepsilon(X)\phi, \quad \forall X \in U_q[\mathfrak{g}] \}$$

The algebra $C_q[U/K_S]$ is a left $C_q[U]$-subcomodule of $C_q[U]$. We call it the quantized algebra of functions on the generalized flag manifold $U/K$.

In a similar way, one can define the quantized function algebra $C_q[K_S^0]$ corresponding to the semisimple part $K_S^0$ of $K_S$ as the image of the dual of the natural embedding $U_q(\mathfrak{f}_0^S) \hookrightarrow U_q(\mathfrak{g})$. Its Hopf $*$-algebra structure is again given by the formulas \([3,5]\). The subalgebra $C_q[U/K_S^0]$ then consists by definition of all right $C_q[K_S^0]$-invariant elements in $C_q[U]$. Note that $C_q[U/K_S^0] \subset C_q[U]$ is a left $U_q(\mathfrak{h})$-submodule and that $C_q[U/K_S^0]$ coincides with the subalgebra of $U_q(\mathfrak{h})$-invariant elements in $C_q[U/K_S^0]$.

We now turn to PBW type factorizations of the algebra $C_q[U/K_S]$. Write $P(S)$, $P_+ (S)$, resp. $P_+ (S)$ for $K$-span $\{ \psi_{\alpha} \}_{\alpha \in S}$ with $K = \mathbb{Z}$, $\mathbb{Z}_+$ resp. $\mathbb{N}$. Set $S^c := \Delta \setminus S$. The quantized algebra $A^\text{hol}_S$ of holomorphic polynomials on $U/K_S$ is defined by

$$A^\text{hol}_S := \bigoplus_{\lambda \in P_+(S^c)} B_\lambda \subset C_q[U],$$
where $B_\lambda$ is given by (3.12) (cf. [19], [20], [41], 1 and [13]). Note that $A^\text{hol}_S$ is a right $U_q(\mathfrak{g})$-comodule subalgebra of $\mathbb{C}_q[U]$, (3.3) being the (multiplicity free) decomposition of $A^\text{hol}_S$ into irreducible $U_q(\mathfrak{g})$-modules. The right $U_q(\mathfrak{g})$-module algebra $(A^\text{hol}_S)^* \subset \mathbb{C}_q[U]$ is called the quantized algebra of antiholomorphic polynomials on $U/K_S$.

**Lemma 4.3.** The linear subspace

$$A^0_S := m((A^\text{hol}_S)^* \otimes A^\text{hol}_S) \subset \mathbb{C}_q[U],$$

where $m$ is the multiplication map of $\mathbb{C}_q[U]$, is a right $U_q(\mathfrak{g})$-submodule $*$-subalgebra of $\mathbb{C}_q[U]$.

**Proof.** Proposition 3.2(ii) implies that $A^0_S$ is a subalgebra of $\mathbb{C}_q[U]$. The other assertions are immediate. □

The subalgebra $A^0_S$ may be considered as a quantum analogue of the algebra of complex-valued polynomial functions on the real manifold $U/K^0_S$.

**Remark 4.4.** In the classical setting ($q = 1$), the algebra $A^0_S$ ($\#S^c = 1$) can be interpreted as algebra of functions on the product of an affine spherical $G$-variety with its dual. The $G$-module structure on $A^0_S$ is then related to the doubled $G$-action (see [32], [33] for the terminology). These (and related) $G$-varieties have been studied in several papers, see for example [33], [32] and [23].

The algebra $A^0_S \subset \mathbb{C}_q[U]$ is stable under the left $U_q(\mathfrak{h})$-action, so we can speak of $U_q(\mathfrak{h})$-weighted elements in $A^0_S$. Let $A_S$ be the left $U_q(\mathfrak{h})$-invariant elements of $A^0_S$. Then $A_S \subset \mathbb{C}_q[U]$ is a right $U_q(\mathfrak{g})$-module $*$-subalgebra of $\mathbb{C}_q[U]$. We now have the following lemma.

**Lemma 4.5.** We have $A^0_S \subset \mathbb{C}_q[U/K^0_S]$, so in particular $A_S \subset \mathbb{C}_q[U/K_S]$. Furthermore,

$$(4.6) \quad A_S = \text{span}\{(C^\lambda_{wv})^* C^\lambda_{vw} \mid \lambda \in P_+(S^c), \; v, w \in V(\lambda)\}. $$

**Proof.** Choose $\lambda \in P_+(S^c)$ and $i \in S$. Then we have $X^+_i \cdot v_\lambda = 0$ and $K_i \cdot v_\lambda = v_\lambda$. It follows that $\mathbb{C} v_\lambda \subset V(\lambda)$ is a one-dimensional $U_q(\mathfrak{sl}(2; \mathbb{C}))$-submodule, where we consider the $U_q(\mathfrak{sl}(2; \mathbb{C}))$ action on $V(\lambda)$ via the embedding $\phi_i: U_q(\mathfrak{sl}(2; \mathbb{C})) \hookrightarrow U_q(\mathfrak{g})$. It follows that $X^-_i \cdot v_\lambda = 0$. This readily implies that $A^0_S \subset \mathbb{C}_q[U/K^0_S]$.

The remaining assertions are immediate. □

**Definition 4.6.** We call $A_S \subset \mathbb{C}_q[U/K_S]$ the factorized $*$-subalgebra associated with $U/K_S$.

In view of Theorem 4.1 there is reason to expect that the factorized algebra $A_S$ is equal to $\mathbb{C}_q[U/K_S]$ for any generalized flag manifold $U/K_S$. Although we cannot prove this in general, we do have a proof (cf. Theorem 4.1) for a certain subclass of generalized flag manifolds that we shall define and classify in the following proposition. For the proof in these cases we use the so-called Parthasarathy-Ranga Rao-Varadarajan (PRV) conjecture, which was proved independently by Kumar [18] and Mathieu [20]. The PRV conjecture gives information about which irreducible constituents occur in tensor products of irreducible finite-dimensional $\mathfrak{g}$-modules. It seems likely that a proof for arbitrary generalized flag manifold $U/K_S$ would require further detailed information about irreducible decompositions of tensor products of finite-dimensional representations of $\mathfrak{g}$.
Recall the notations introduced in section 2. The following proposition was observed by Koornwinder [14].

**Proposition 4.7.** (14) Let $U$ be a connected, simply connected compact Lie group with Lie algebra $u$, and let $p \subset g$ be a standard maximal parabolic subalgebra. Let $K \subset U$ be the connected subgroup with Lie algebra $k := p \cap u$. Then $(U, K)$ is a Gelfand pair if and only if one of the following three conditions is satisfied:

(i) $(U, K)$ is an irreducible compact Hermitian symmetric pair;

(ii) $(U, K) \simeq (SO(2l + 1), U(l))$, \quad $(l \geq 2)$;

(iii) $(U, K) \simeq (Sp(l), U(1) \times Sp(l - 1))$, \quad $(l \geq 2)$.

**Proof.** For a list of the irreducible compact Hermitian symmetric pairs see [14, Ch. X, Table V]. The proposition follows from this and the classification of compact Gelfand pairs $(U, K)$ with $U$ simple (cf. [17, Tabelle 1]).

Let $(U, K)$ be a pair from the list (i)–(iii) in Proposition 4.7, and let $(u, t)$ be the associated pair of Lie algebras. Then $k = ts$ for some subset $S \subset \Delta$ with $\# S^c = 1$. We call the simple root $\alpha \in S^c$ the Gel’fand node associated with $(U, K)$.

**Proposition 4.8.** Let $(U, K)$ be a pair from the list (i)–(iii) in Proposition 4.7, and let $\omega := \omega_\alpha$. Let $\{\mu_1, \ldots, \mu_\ell\}$ be the fundamental spherical weights of $(U, K)$. Then every fundamental spherical representation $V(\mu_i)$ occurs in the decomposition of $V(\omega)^* \otimes V(\omega)$.

**Proof.** For the proof we use the PRV conjecture, which states the following. Let $\lambda, \mu \in P_+$ and $w \in W$. Let $[\lambda + w\mu]$ be the unique element in $P_+$ which lies in the $W$-orbit of $\lambda + w\mu$. Then $V([\lambda + w\mu])$ occurs with multiplicity at least one in $V(\lambda) \otimes V(\mu)$. The procedure is now as follows. For a pair $(U, K)$ from the list (i)–(iii) of Proposition 4.7 we write down the fundamental spherical weights $\{\mu_i\}_{i=1}^\ell$ in terms of the fundamental weights $\{\omega_i\}_{i=1}^\ell$ (cf. [17, Tabelle 1], or in the case of Hermitian symmetric spaces one can also write them down from the corresponding Satake diagrams [13]). Then we look for Weyl group elements $w_i \in W$ such that

$$[\omega - w_i \omega] = \mu_i, \quad (i = [1, \ell])$$

(here we used that $V(\omega)^* \simeq V(-\sigma_0 \omega)$).

As an example, let us follow the procedure for the compact Hermitian symmetric pair $(U, K) = (SO(2l), U(l))$ $(l \geq 2)$. We use the standard realization of the root system $R$ of type $D_l$ in the $l$-dimensional vector space $V = \sum_{i=1}^l \mathbb{R} \varepsilon_i$, with basis given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i = [1, l - 1])$ and $\alpha_l = \varepsilon_{l-1} + \varepsilon_l$. The fundamental weights are given by

$$\omega_1 = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_i, \quad (i < l - 1),$$

$$\omega_{l-1} = (\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{l-1} - \varepsilon_l)/2,$$

$$\omega_l = (\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{l-1} + \varepsilon_l)/2.$$

We set $\omega = \omega_1$ (i.e. $S^c = \{\alpha_l\}$). Let $\sigma_i$ be the linear map defined by $\varepsilon_j \mapsto -\varepsilon_j$ $(j = i, i + 1)$ and $\varepsilon_j \mapsto \varepsilon_j$ otherwise. Then $\sigma_i \in W$ $(i = [1, l - 1])$. If $l = 2l' + 1$, then

$$\omega - \sigma_1 \sigma_3 \ldots \sigma_{2l-3-} \omega = \omega_{2l-1}, \quad (i = [1, l' - 1]),$$

$$\omega - \sigma_1 \sigma_3 \ldots \sigma_{2l-1-} \omega = \omega_{l-1} + \omega_l.$$
If \( l = 2l' \) then we have
\[
\varpi - \sigma_1 \sigma_3 \ldots \sigma_{2l'-1} \varpi = \varpi_{2l'}, \quad (i = [1, l' - 1]),
\]
(4.8)
\[
\varpi - \sigma_1 \sigma_3 \ldots \sigma_{2l'-1} \varpi = 2 \varpi_{l'}.
\]
By comparison with [17, Tabelle 1] we see from (4.7) (resp. (4.8)) that all the fundamental spherical weights of the pair \((U, K) = (SO(2l), U(l))\) have been obtained. The other cases are checked in a similar manner.

The question naturally arises whether the fundamental spherical representations occur with multiplicity one in \( V(\varpi) \otimes V(\varpi) \) and whether they exhaust (together with the trivial representation) the irreducible components of \( V(\varpi) \otimes V(\varpi) \). For the complex Grassmannians \((U, K) = (SU(p + l), S(U(p) \times U(l)))\) this is indeed the case (this can be easily proved using the Pieri formula for Schur functions [28, Chapter I, (5.17)], see [4] for more details). For the general case we do not know the answer, but it is true that the irreducible decomposition of \( V(\varpi) \otimes V(\varpi) \) is multiplicity free and that all irreducible components are spherical representations. This is an easy consequence of the proof of the following main theorem of this section.

**Theorem 4.9.** The factorized \( * \)-subalgebra \( A_S \) is equal to \( \mathbb{C}_q[U/K_S] \) if
(i) \( S = \emptyset \), i.e. \( U/K_S = U/T \) is the full flag manifold;
(ii) \#\( S^c = 1 \) and the simple root \( \alpha \in S^c \) is a Gel’fand node.

**Proof.** To prove (i) we look at the simultaneous eigenspace decomposition of \( \mathbb{C}_q[U] \) with respect to the left \( U_q(\mathfrak{h}) \)-action on \( \mathbb{C}_q[U] \). The simultaneous eigenspace corresponding to the character \( \varepsilon \) on \( U_q(\mathfrak{h}) \) is exactly \( \mathbb{C}_q[U/T] \). Using Soibel’man’s factorization of \( \mathbb{C}_q[U] \) (cf. Theorem 3.1 and Lemma 4.5), it is then easily checked that \( \mathbb{C}_q[U/T] = A_0 \). To prove (ii) we note that
\[
\bigoplus_{i=1}^{l} V(\mu_i) \hookrightarrow V(\varpi)^{\ast} \otimes V(\varpi) \simeq (B_\varpi)^{\ast} B_\varpi \subset A_S
\]
as right \( U_q(\mathfrak{g}) \)-modules by Proposition 4.8 (here we use the notations as introduced in Proposition 4.8). Now \( \mathbb{C}_q[U] \) is an integral domain (cf. [3] Lemma 9.1.9 (i)), hence \( v_\lambda v_\mu \in A_S \) is a highest weight vector of highest weight \( \lambda + \mu \) if \( v_\lambda, v_\mu \in A_S \) are highest weight vectors of highest weight \( \lambda \) respectively \( \mu \). It follows that
\[
\bigoplus_{n_i \in \mathbb{Z}_+} V(n_1 \mu_1 + \ldots n_l \mu_l) \hookrightarrow A_S
\]
as right \( U_q(\mathfrak{g}) \)-modules. On the other hand we have the decomposition
\[
\mathbb{C}_q[U/K_S] \simeq \bigoplus_{n_i \in \mathbb{Z}_+} V(n_1 \mu_1 + \ldots n_l \mu_l)
\]
of \( \mathbb{C}_q[U/K_S] \) into irreducible right \( U_q(\mathfrak{g}) \)-modules by Proposition 4.2. This implies \( A_S = \mathbb{C}_q[U/K_S] \). 

In the remainder of the paper we study the irreducible \( * \)-representations of the \( * \)-algebras \( A_S \) and \( \mathbb{C}_q[U/K_S] \). In the next section we first consider the restriction of the irreducible \( * \)-representations of \( \mathbb{C}_q[U] \) to the \( * \)-algebras \( A_S \) and \( \mathbb{C}_q[U/K_S] \).
5. RESTRICTION OF IRREDUCIBLE ∗-REPRESENTATIONS TO \( C_q[U/K] \)

Let us first recall some results from Soibel’man \[40] concerning the irreducible ∗-representations of \( C_q[U] \). Let \( \{e_i\}_{i \in \mathbb{Z}_+} \) be the standard orthonormal basis of \( I_2(\mathbb{Z}_+) \). Write \( B(I_2(\mathbb{Z}_+)) \) for the algebra of bounded linear operators on \( I_2(\mathbb{Z}_+) \). Then the formulas

\[
\begin{align*}
\pi_q(t_{11})e_j &= \sqrt{(1-q^{2j})}e_{j-1}, && \pi_q(t_{12})e_j = -q^{j+1}e_j, \\
\pi_q(t_{21})e_j &= q^je_j, && \pi_q(t_{22})e_j = \sqrt{(1-q^{2j+1})}e_{j+1}
\end{align*}
\]

(here \( \pi_q(t_{11})e_0 = 0 \)) uniquely determine an irreducible ∗-representation

\[
\pi_q : C_q[SU(2)] \to B(I_2(\mathbb{Z}_+)).
\]

Now the dual of the injective Hopf ∗-algebra morphism \( \phi_i : U_q(\mathfrak{su}(2; \mathbb{C})) \to U_q(\mathfrak{g}) \) corresponding to the \( i \)-th node of the Dynkin diagram \( (i \in [1, r]) \) is a surjective Hopf ∗-algebra morphism \( \phi_i^* : C_q[U] \to C_q[SU(2)] \). Hence we obtain irreducible ∗-representations \( \pi_i := \pi_q \circ \phi_i^* : C_q[U] \to B(I_2(\mathbb{Z}_+)). \)

On the other hand, there is a family of one-dimensional ∗-representations \( \tau_i \) of \( C_q[U] \) parametrized by the maximal torus \( t \in T \cong T^r \) \( (T \subset \mathbb{C} \text{ denoting the unit circle in the complex plane}) \). More explicitly, let \( \iota_T : U_q(\mathfrak{h}) \to U_q(\mathfrak{g}) \) be the natural Hopf ∗-algebra embedding, and set \( C_q[T] := \text{span}\{\phi_i\}_{\mu \in P} \subset U_q(\mathfrak{h})^* \), where \( \phi_\mu(K^r) := q^{\mu, \sigma} \) for \( \sigma \in Q \). As in (5.3) we get a Hopf ∗-algebra structure on \( C_q[T] \).

Then \( \iota_T^* : C_q[U] \to C_q[T], \iota_T^*(\phi) := \phi \circ \iota_T \) is a surjective Hopf ∗-algebra morphism. Any irreducible ∗-representation of \( C_q[T] \) is one-dimensional and can be written as \( \tau_t(\phi_\mu) := t^\mu \) for a unique \( t \in T \cong T^r \). Here \( t^\mu := t_1^{m_1} \cdots t_r^{m_r} \) for \( \mu = \sum_{i=1}^r m_i \omega_i \). We obtain a one-dimensional ∗-representation \( \tau_i := \tau_t \circ \iota_T^* \) of \( C_q[U] \), which is given explicitly on matrix elements \( C_{\mu;i,v;\nu,j}^\lambda \) by the formula

\[
\tau_t(C_{\mu;i,v;\nu,j}^\lambda) = \delta_{\mu,r} \delta_{i,j} t^{\mu}.
\]

The following theorem completely describes the irreducible ∗-representations of \( C_q[U] \).

**Theorem 5.1 (Soibel’man \[40\]).** Let \( \sigma \in W \), and fix a reduced expression \( \sigma = s_{i_1} s_{i_2} \cdots s_{i_t} \). The ∗-representation

\[
\pi_\sigma := \pi_{i_1} \otimes \pi_{i_2} \otimes \cdots \otimes \pi_{i_t}
\]

does not depend on the choice of reduced expression (up to equivalence). The set

\[
\{\pi_\sigma \otimes \tau_t | t \in T, \sigma \in W\}
\]

is a complete set of mutually inequivalent irreducible ∗-representations of \( C_q[U] \).

Here tensor products of ∗-representations are defined in the usual way by means of the coalgebra structure on \( C_q[U] \). The irreducible representation \( \pi_\sigma \) with respect to the unit element \( e \in W \) is the one-dimensional ∗-representation associated with the counit \( e \) on \( C_q[U] \). In Soibel’man’s terminology, the representations \( \pi_\sigma \otimes \tau_t \) are said to be associated with the Schubert cell \( X_\sigma \) of \( U/T \) (cf. section 2).

We also mention here an important property of the kernel of \( \pi_\sigma \), which we will repeatedly need later on. Let \( U_q(\mathfrak{b}^+ \mathfrak{h}) \) be the subalgebra of \( U_q(\mathfrak{g}) \) generated by the \( K_i^{\pm1} \) and the \( X_i^+ \) \( (i \in [1, r]) \). For any \( \lambda \in P_+ \), the ∗-representation \( \pi_\sigma \) satisfies

\[
\pi_\sigma(C_{\nu;\lambda}^{\lambda}) = 0 \quad (\nu \not\in U_q(\mathfrak{b}^+)\nu_\sigma\lambda), \quad \pi_\sigma(C_{\nu;\lambda;\nu_\sigma\lambda}^{\lambda}) \neq 0
\]
An alternative characterization of \( \tau \) (cf. [10, Theorem 5.7]). Formula (5.4) combined with \([1\text{, Lemma 2.12}] \) shows that the classical limit of the kernel of \( \tau \) formally tends to the ideal of functions vanishing on \( X_\sigma \).

Fix now a subset \( S \subseteq \Delta \). We freely use the notations introduced earlier. Our next goal is to describe how the \( * \)-representations \( \tau \) decompose under restriction to the subalgebra \( C_q[U/K_S] \). Consider the selfadjoint operators

\[
\mathbb{L}_{\sigma,\lambda,\lambda} := \pi_{\sigma}((C_{\sigma,\lambda,\lambda}^*C_{\sigma,\lambda,\lambda})^*)
\]

for \( \lambda \in P_+ (S^\circ) \). Let \( \sigma = s_{i_1} \cdots s_{i_l} \) be a reduced expression for \( \sigma \), and set \( \pi_{\sigma} = \pi_{i_1} \otimes \pi_{i_2} \otimes \cdots \otimes \pi_{i_l} \). Then it follows from \([10, \text{Proof of Prop. 5.2}] \) (see also \([10, \text{Proof of Prop. 5.8}] \)) that

\[
\pi_{\sigma}(C_{\sigma,\lambda,\lambda}^*) = c_{\pi_{i_1}}(t_{21}(\lambda,\gamma^\vee) \otimes \pi_{i_1}(t_{21}(\lambda,\gamma^\vee)) \otimes \cdots \otimes \pi_{i_l}(t_{21}(\lambda,\gamma^\vee))
\]

where the scalar \( c \in \mathbb{T} \) depends on the particular choices of bases for the irreducible representations \( V_\mu \) (\( \mu \in P_+ \)), and with

\[
\gamma_k := s_{i_k} s_{i_{k-1}} \cdots s_{i_{k+1}}(\alpha_{i_k}) \quad (1 \leq k \leq l - 1), \quad \gamma_l := \alpha_{i_l}.
\]

The proof of \((5.3) \), which was given in \([10] \) under the assumption that \( \lambda \in P_+ \), is in fact valid for all dominant weights \( \lambda \in P_+ \). It follows from \((5.1) \), \((5.5) \) and \((5.6) \) that \( l_2(\mathbb{Z}^+_+) \otimes t(\sigma) \) decomposes as an orthogonal direct sum of eigenspaces for \( L_{\sigma,\lambda,\lambda} \),

\[
l_2(\mathbb{Z}^+_+) \otimes t(\sigma) = \bigoplus_{\gamma \in I(\lambda)} H_\gamma(\lambda),
\]

where \( I(\lambda) \subset (0, 1] \) denotes the set of eigenvalues of \( L_{\sigma,\lambda,\lambda} \) and \( H_\gamma(\lambda) \) denotes the eigenspace of \( L_{\sigma,\lambda,\lambda} \) corresponding to the eigenvalue \( \gamma \in I(\lambda) \) (we suppress the dependence on \( \sigma \) if there is no confusion possible). Observe that \( 1 \in I(\lambda) \) and that \( L_{\sigma,\lambda,\lambda} \) is injective.

Recall the definition of the set \( W^S \) of minimal coset representatives (cf. \((2.17) \)). An alternative characterization of \( W^S \) is given by

\[
W^S = \{ \sigma \in W \mid \sigma(R_+^\circ) \subset R^+ \},
\]

where \( R_+^\circ := R^+ \cap \text{span}\{S\} \) (cf. \([1, \text{Prop. 5.1 (iii)}]\)). Using this alternative description of \( W^S \) we obtain the following properties of \( L_{\sigma,\lambda,\lambda} \) for \( \lambda \in P_+ (S^\circ) \).

**Proposition 5.2.** Suppose that \( \sigma \in W^S \) and \( \lambda \in P_+ (S^\circ) \). Then

(i) \( L_{\sigma,\lambda,\lambda} \) is a compact operator;

(ii) The eigenspace \( H_1(\lambda) \) of \( L_{\sigma,\lambda,\lambda} \) corresponding to the eigenvalue 1 is spanned by the vector \( e_0 \otimes t(\sigma) \).

**Proof.** Fix a \( \lambda \in P_+ (S^\circ) \), and let \( \sigma = s_{i_1} s_{i_2} \cdots s_{i_l} \) be a reduced expression of a minimal coset representative \( \sigma \in W^S \). It is well known that

\[
R^+ \cap \sigma^{-1}(R^-) = \{ \gamma_k \}_{k=1}^l,
\]

where the \( \gamma_k \) are defined by \((5.5) \). We have \( \gamma_k \in R^+ \setminus R_+^\circ \) by \((5.5) \). It follows that \( (\lambda, \gamma_k^\vee) > 0 \) for all \( k \), since \( \lambda \in P_+ (S^\circ) \). By \((5.1) \) and \((5.4) \) it follows that \( H_1(\lambda) = \text{span}(e_0 \otimes t(\sigma)) \) and that \( H_\gamma(\lambda) \) is finite-dimensional for all \( \gamma \in I(\lambda) \). Since the spectrum of \( L_{\sigma,\lambda,\lambda} \) (which is equal to \( I(\lambda) \cup \{0\} \)) does not have a limit point except 0, we conclude that \( L_{\sigma,\lambda,\lambda} \) is a compact operator (cf. \([37, \text{Theorem 12.30}] \)).
Let us recall the following well known inequalities for weights of finite-dimensional irreducible representations of $\mathfrak{g}$ (or, equivalently, $U_q(\mathfrak{g})$).

**Proposition 5.3.** Let $\lambda \in P_+$ and $\mu, \nu \in P(\lambda)$. Then $(\lambda, \lambda) \geq (\mu, \nu)$, and equality holds if and only if $\mu = \nu \in W\lambda$.

For a proof of the proposition, see for instance [10, Prop. 11.4]. The proof is based on the following lemma, which we will also need later on. The lemma is a slightly weaker version of [10, Lemma 11.2].

**Lemma 5.4.** Let $\lambda \in P_+$ and $\mu \in P(\lambda) \setminus \{\lambda\}$, and let $m_i \in \mathbb{Z}_+$ ($i \in [1, r]$ be the expansion coefficients defined by $\lambda - \mu = \sum_i m_i \alpha_i$. Then there is an $1 \leq i \leq r$ with $m_i > 0$ and $H(H_i) \neq 0$.

We now have the following proposition, which can be regarded as a quantum analogue of the “if” part of Proposition 2.2.

**Proposition 5.5.** Let $\sigma \in W^{\mathbb{S}}$. Then $\pi_\sigma$ restricts to an irreducible $*$-representation of the factorized $*$-algebra $A_S$. In particular, $\pi_\sigma$ restricts to an irreducible $*$-representation of $C_q[U/K_S]$.

**Proof.** Let $\lambda \in P_+(S^c)$ and $\sigma \in W^{\mathbb{S}}$. Suppose $H \subset l_2(\mathbb{Z}_+) \otimes \mathbb{C}^n$ is a non-zero closed subspace invariant under $\pi_\sigma|_{A_S}$. Set $\gamma := \|L_{\sigma_\lambda} \lambda | \mu \|$. Then $\gamma > 0$, since $L_{\sigma_\lambda}$ is injective and $\gamma$ is an eigenvalue of $L_{\sigma_\lambda} \lambda | H$ by Proposition 3.2(i). Let $H_\gamma$ be the corresponding eigenspace. We claim that

$$
(5.11) \quad \pi_\sigma(\langle C_{\mu,\lambda}^\lambda * C_{\mu,\lambda}^\lambda \rangle) H_\gamma = 0, \quad \mu \neq \sigma \lambda.
$$

Suppose for the moment that the claim is correct. Then (5.10) and (5.11) imply $\gamma = 1$, hence $H_\gamma = \text{span} \{e_0^\otimes (\sigma)\}$ by Proposition 5.2(ii). So every non-zero closed invariant subspace contains the vector $e_0^\otimes (\sigma)$. Since $H^\perp$ is also a closed invariant subspace, we must have $H^\perp = \{0\}$, i.e. $H = l_2(\mathbb{Z}_+) \otimes \mathbb{C}^n$. Remains therefore to prove the claim (5.11). By (5.4) we have $\pi_\sigma(\langle C_{\mu,\lambda}^\lambda \rangle) = 0$ if $\mu < \sigma \lambda$. Hence

$$
L_{\sigma_\lambda} \lambda \pi_\sigma(\langle C_{\mu,\lambda}^\lambda * C_{\mu,\lambda}^\lambda \rangle) = q^{2(\lambda, \lambda) - 2(\mu, \sigma \lambda)} \pi_\sigma(\langle C_{\mu,\lambda}^\lambda * C_{\mu,\lambda}^\lambda C_{\sigma_\lambda,\lambda}^\lambda \rangle) = q^{2(\lambda, \lambda) - 2(\mu, \sigma \lambda)} \pi_\sigma(\langle C_{\mu,\lambda}^\lambda \rangle) \langle C_{\sigma_\lambda,\lambda}^\lambda \rangle \langle C_{\mu,\lambda}^\lambda \rangle = q^{2(\lambda, \lambda) - 2(\mu, \sigma \lambda)} \pi_\sigma(\langle C_{\mu,\lambda}^\lambda \rangle) \langle C_{\sigma_\lambda,\lambda}^\lambda \rangle \langle C_{\mu,\lambda}^\lambda \rangle = q^{2(\lambda, \lambda) - 2(\mu, \sigma \lambda)} \pi_\sigma(\langle C_{\mu,\lambda}^\lambda \rangle) \langle C_{\sigma_\lambda,\lambda}^\lambda \rangle \langle C_{\mu,\lambda}^\lambda \rangle,
$$

where we used Proposition 3.2(i) in the second equality and Proposition 3.2(ii) in the first and third equality. So (5.11) will then follow from

$$
(5.12) \quad \pi_\sigma(\langle C_{\sigma_\lambda,\lambda}^\lambda \rangle) H_\gamma = 0, \quad \mu \neq \sigma \lambda,
$$

in view of the injectivity of $L_{\sigma_\lambda} \lambda$. Fix $h \in H_\gamma$ and $\mu \in P(\lambda)$ with $\mu \neq \sigma \lambda$. By Lemma 4.5 we have $(C_{\sigma_\lambda,\lambda}^\lambda)^* C_{\mu,\lambda}^\lambda \in A_S \subset C_q[U/K_S]$, hence the vector

$$
(5.13) \quad \tilde{h} := \pi_\sigma(\langle C_{\sigma_\lambda,\lambda}^\lambda \rangle^* C_{\mu,\lambda}^\lambda) h
$$

lies in the invariant subspace $H$. Again using the commutation relations given in Proposition 5.2 and Corollary 3.3, we see that $\tilde{h}$ is an eigenvector of $L_{\sigma_\lambda} \lambda$ with eigenvalue $\tilde{\gamma} := q^{2(\lambda, \sigma^{-1} (\mu - \lambda)) \gamma}$. We have $\tilde{\gamma} > \gamma$ by Proposition 5.3. By the maximality of $\gamma$, we conclude that $\tilde{h} = 0$. This proves (5.12), hence also the claim (5.11). □
Definition 5.6. We say that the irreducible \( \pi_\sigma \) (\( \sigma \in W^S \)) of \( \mathbb{C}_q[U/K_S] \) is associated with the Schubert cell \( \lambda \sqsubset \Delta \). The following proposition can be regarded as a quantum analogue of Proposition 2.2 as well as of the “only if” part of Proposition 2.2.

Proposition 5.7. Let \( \sigma \in W \), and let \( \sigma = uv \) be the unique decomposition of \( \sigma \) with \( u \in W^S \) and \( v \in W_S \). For \( \pi_\sigma = \pi_u \otimes \pi_v \) (cf. [11, 2.11]) and \( t \in T \), we have
\[
(\pi_\sigma \otimes \tau_t)(a) = \pi_u(a) \otimes \text{id}^{\otimes (tv)}, \quad a \in \mathbb{C}_q[U/K_S].
\]

Proof. Recall that the one-dimensional \( * \)-representation \( \tau_t \) factorizes through \( \iota_T^* : C_q[U] \to C_q[T] \) and that \( \pi_v \) factorizes through \( \phi_i^* : \mathbb{C}_q[U] \to \mathbb{C}_q[SU(2)] \). The maps \( \iota_T^* \) and \( \phi_i^* (i \in S) \) factorize through \( \iota_T^* \) since the ranges of \( \iota_T \) and \( \phi_i (i \in S) \) lie in the Hopf-subalgebra \( U_q(ks) \). Hence \( \pi_v \otimes \tau_t = \pi_v \circ \iota_T^* \). Then we have for \( a \in \mathbb{C}_q[U/K_S] \),
\[
(\pi_\sigma \otimes \tau_t)(a) = (\pi_u \otimes \pi_v \otimes \tau_t) \circ \Delta (a)
= (\pi_u \otimes \pi_v, t) \circ (\text{id} \otimes \iota_T^*) \Delta (a)
= \pi_u(a) \otimes \pi_v, t(1) = \pi_u(a) \otimes \text{id}^{\otimes (tv)},
\]
which completes the proof of the proposition.

Lemma 5.8. The \( * \)-representations \( \{\pi_\sigma\}_{\sigma \in W^S} \), considered as \( * \)-representations of \( A_S \) respectively \( \mathbb{C}_q[U/K_S] \), are mutually inequivalent.

Proof. Let \( \sigma, \sigma' \in W^S \) with \( \sigma \neq \sigma' \) and \( \lambda \in P_+(S^c) \). Then \( \sigma \lambda \neq \sigma' \lambda \), since the isotropy subgroup \( \{ \sigma \in W \mid \sigma \lambda = \lambda \} \) is equal to \( W_S \) by Chevalley’s Lemma (cf. [11, Prop. 2.72]). Without loss of generality we may assume that \( \sigma \lambda \neq \sigma' \lambda \). Then we have \( \pi_\sigma((C^\lambda q_{\sigma \lambda})^* C^{\lambda q}_{\sigma' \lambda}) = 0 \) by [5, 3.4]. On the other hand, \( L_{\sigma \lambda} \) is injective. It follows that \( \pi_{\sigma} \neq \pi_{\sigma'} \) as \( * \)-representations of \( A_S \).

Let now \( \| . \|_u \) be the universal \( C^* \)-norm on \( \mathbb{C}_q[U] \) (cf. [3, §4]), so
\[
\| a \|_u := \sup_{\sigma \in W, t \in T} \| (\pi_\sigma \otimes \tau_t)(a) \|, \quad a \in \mathbb{C}_q[U].
\]

Let \( \mathbb{C}_q(U) \) (resp. \( \mathbb{C}_q(U/K_S) \)) be the completion of \( \mathbb{C}_q[U] \) (resp. \( \mathbb{C}_q[U/K_S] \)) with respect to \( \| . \|_u \). All \( * \)-representations \( \pi_\sigma \otimes \tau_t \) of \( \mathbb{C}_q[U] \) extend to \( * \)-representations of the \( C^* \)-algebra \( \mathbb{C}_q(U) \) by continuity. The results of this section can now be summarized as follows.

Theorem 5.9. Let \( S \sqsubset \Delta \). Then \( \{\pi_\sigma\}_{\sigma \in W^S} \) is a complete set of mutually inequivalent irreducible \( * \)-representations of \( \mathbb{C}_q(U/K_S) \).

Proof. This follows from the previous results, since every irreducible \( * \)-representation of \( \mathbb{C}_q(U/K_S) \) appears as an irreducible component of \( \sigma_{\mathbb{C}_q(U/K_S)} \) for some irreducible \( * \)-representation \( \sigma \) of \( \mathbb{C}_q(U) \) (cf. [3, Prop. 2.10.2]).

Theorem 5.9 does not imply that \( \{\pi_\sigma\}_{\sigma \in W^S} \) is a complete set of irreducible \( * \)-representations of the \( * \)-algebra \( \mathbb{C}_q[U/K_S] \) itself. Indeed, it is not clear that any irreducible \( * \)-representation of \( \mathbb{C}_q[U/K_S] \) can be continuously extended to a \( * \)-representation of \( \mathbb{C}_q(U/K_S) \). In the remainder of this paper we will deal with the classification of the irreducible \( * \)-representations of \( A_S \). In particular, this will yield a complete classification of the irreducible \( * \)-representations of \( \mathbb{C}_q[U/K_S] \) for the generalized flag manifolds \( U/K_S \) for which the PBW factorization is valid (cf. Theorem 4.9).
6. Irreducible \( \ast \)-representations of \( A_S \)

Let \( S \subseteq \Delta \) be any subset. In this section we show that \( \{ \pi_{\sigma} \}_{\sigma \in W^S} \) exhausts the set of irreducible \( \ast \)-representations of \( A_S \) (up to equivalence). We fix therefore an arbitrary irreducible \( \ast \)-representation \( \tau : A_S \to B(H) \) and we will show that \( \tau \simeq \pi_{\sigma} \) for a (unique) \( \sigma \in W^S \). In order to associate the proper minimal coset representative \( \sigma \in W^S \) with \( \tau \), we need to study the range \( \tau(A_S) \subseteq B(H) \) of \( \tau \) in more detail. For \( \lambda \in P_+(S^c) \) and \( \mu, \nu \in P(\lambda) \), let \( \tau^\lambda(\mu; \nu), \tau^\lambda(\nu) \subseteq B(H) \) be the linear subspaces

\[
\tau^\lambda(\mu; \nu) := \{ \tau((C^\lambda_{\nu,\nu})^*(C^\lambda_{\mu,\mu})^*) | v \in V(\lambda)_\mu, w \in V(\lambda)_\nu \},
\]

\[
\tau^\lambda(\nu) := \{ \tau((C^\lambda_{\nu,\nu})^*) | v \in V(\lambda), w \in V(\lambda)_\nu \}.
\]

For \( \lambda \in P_+(S^c) \) set

\[
D(\lambda) := \{ \nu \in P(\lambda) | \tau^\lambda(\nu) \neq \{0\} \}
\]

and let \( D_m(\lambda) \) be the set of weights \( \nu \in D(\lambda) \) such that \( \nu' \notin D(\lambda) \) for all \( \nu' < \nu \). By Lemma 3.1, we have \( D(\lambda) \neq \emptyset \), hence also \( D_m(\lambda) \neq \emptyset \). We start with a lemma which is useful for the computation of commutation relations in \( \tau(A_S) \subseteq B(H) \).

Lemma 6.1. Let \( \lambda, \Lambda \in P_+(S^c) \) and \( \nu \in D_m(\lambda) \). Let \( v \in V(\lambda), \nu' \in V(\lambda)_\nu \) with \( \nu' < \nu \) and \( w, w' \in V(\lambda) \). Then the product of the four matrix elements \( (C^\lambda_{\nu,\nu})^*, (C^\lambda_{\nu',\nu'})^*, (C^\Lambda_{\nu,\nu})^*, \) and \( (C^\Lambda_{\nu',\nu'})^* \), taken in an arbitrary order, is contained in \( \operatorname{Ker}(\tau) \).

Proof. Since \( \operatorname{Ker}(\tau) \) is a two-sided \( \ast \)-ideal in \( A_S \), it follows from the definitions that

\[
(C^\lambda_{\nu,\nu})^* (C^\lambda_{\nu',\nu'})^* (C^\Lambda_{\nu,\nu})^* C^\lambda_{\nu',\nu} \in \operatorname{Ker}(\tau).
\]

If the product of the four matrix coefficients is taken in a different order, then we can rewrite it by Proposition 3.2 and by Corollary 3.3 as a linear combination of products of matrix elements

\[
(C^\lambda_{\nu,\nu})^* (C^\lambda_{\nu',\nu'})^* (C^\Lambda_{\nu,\nu})^* C^\lambda_{\nu',\nu} = (C^\Lambda_{\nu,\nu})^* C^\lambda_{\nu',\nu}.
\]

with \( x' \in V(\lambda)_{\nu'} \) and \( \nu'' \leq \nu' < \nu \). These are all contained in \( \operatorname{Ker}(\tau) \), since \( \nu \in D_m(\lambda) \).

\[
\]

Lemma 6.2. Let \( \lambda \in P_+(S^c) \) and \( \nu \in D_m(\lambda) \). Then

(i) \( \tau^\lambda(\nu; \nu) \neq \{0\} \);

(ii) \( \nu = \sigma \lambda \) for some \( \sigma \in W^S \).

Proof. Let \( \lambda \in P_+(S^c) \) and \( \nu \in D_m(\lambda) \). Fix weight vectors \( v \in V(\lambda)_\mu, w \in V(\lambda)_\nu \) such that \( T_{v,w} := \tau((C^\lambda_{\nu,\nu})^* (C^\lambda_{\nu,\nu})^*) \neq 0 \). By Lemma 6.1, we compute

\[
(T_{v,w})^* T_{v,w} = q^{(\mu,\nu) - (\lambda,\lambda)} \tau((C^\lambda_{\nu,\nu})^* (C^\lambda_{\nu,\nu})^* C^\lambda_{\nu,\nu}).
\]

where we used Proposition 3.2(ii) in the first equality and Proposition 3.2(i) in the second equality. On the other hand, \( (T_{v,w})^* T_{v,w} \neq 0 \) since \( B(H) \) is a \( C^* \)-algebra, so we conclude that \( T_{w,w} \neq 0 \). In particular, \( \tau^\lambda(\nu; \nu) \neq \{0\} \). Formula (6.3) for \( v = w \) gives

\[
0 \neq (T_{w,w})^* T_{w,w} = \tau((C^\lambda_{\nu,\nu})^* (C^\lambda_{\nu,\nu})^*) T_{w,w} = q^{(\lambda,\lambda)} T_{w,w} T_{w,w}^*.
\]
where we have used Proposition 3.2(ii) in the last equality. It follows that \((\lambda, \lambda) = (\nu, \nu)\), since \(T_{w;w}\) is selfadjoint. By Proposition 5.3 we obtain \(\nu = \sigma \lambda\) for some \(\sigma \in W^S\).

For \(\lambda \in P_+(S^c)\) and \(\nu \in D_m(\lambda)\) we set

\[
L_{\nu;\lambda} := \tau((C^\lambda_{\nu;\lambda})^* C^\lambda_{\nu;\lambda}).
\]

This definition makes sense since \(\dim(V(\lambda)_{\nu}) = 1\) by Lemma 6.2(ii). Furthermore, \(L_{\nu;\lambda}\) is a non-zero selfadjoint operator which commutes with the elements of \(\tau(A_S)\) in the following way.

**Lemma 6.3.** Let \(\lambda, \Lambda \in P_+(S^c)\) and \(\nu \in D_m(\lambda)\). For \(v \in V(\lambda)_{\mu}, w \in V(\Lambda)_{\mu'}\) we have

\[
L_{\nu;\lambda} \tau((C^\lambda_{\nu;\lambda})^* C^\lambda_{\nu;\lambda}) = q^{2(\nu;\mu'-\mu)} \tau((C^\lambda_{\nu;\lambda})^* C^\lambda_{\nu;\lambda}) L_{\nu;\lambda}.
\]

**Proof.** By Lemma 6.4 and the commutation relations of section 3 we compute

\[
L_{\nu;\lambda} \tau((C^\lambda_{\nu;\lambda})^* C^\lambda_{\nu;\lambda}) = q^{(\lambda;\Lambda) - (\nu;\mu)} \tau((C^\lambda_{\nu;\lambda} C^\lambda_{\nu;\lambda})^* C^\lambda_{\nu;\lambda} C^\lambda_{\nu;\lambda})
\]

\[
= q^{2(\lambda;\Lambda) - 2(\nu;\mu)} \tau((C^\lambda_{\nu;\lambda})^* (C^\lambda_{\nu;\lambda})^* C^\lambda_{\nu;\lambda} C^\lambda_{\nu;\lambda})
\]

\[
= q^{(\nu;\mu') + (\lambda;\Lambda) - 2(\nu;\mu)} \tau((C^\lambda_{\nu;\lambda})^* (C^\lambda_{\nu;\lambda})^* C^\lambda_{\nu;\lambda} C^\lambda_{\nu;\lambda})
\]

\[
= q^{2(\nu;\mu'-\mu)} \tau((C^\lambda_{\nu;\lambda})^* C^\lambda_{\nu;\lambda}) L_{\nu;\lambda},
\]

where we used Proposition 3.2(ii) for the first and fourth equality, Proposition 3.2(i) for the second equality, and Corollary 5.3 for the third equality.

It follows from Lemma 6.3 that \(\text{Ker}(L_{\nu;\lambda}) \subseteq H\) is a closed invariant subspace. By the irreducibility of \(\tau\), we thus obtain the following corollary.

**Corollary 6.4.** Let \(\lambda \in P_+(S^c)\) and \(\nu \in D_m(\lambda)\). Then \(L_{\nu;\lambda}\) is injective.

The minimal coset representative \(\sigma\) of Lemma 5.2(ii) is unique and independent of \(\lambda \in P_+(S^c)\) in the following sense.

**Lemma 6.5.** There exists a unique \(\sigma \in W^S\) such that \(D_m(\lambda) = \{\sigma \lambda\}\) for all \(\lambda \in P_+(S^c)\).

**Proof.** Let \(\Lambda \in P_+(S^c)\) and \(\nu \in D_m(\Lambda)\). Then there exists a unique \(\sigma \in W^S\) such that \(\nu = \sigma \Lambda\) by Lemma 5.2(ii) and by Chevalley’s Lemma (cf. [11, Prop. 2.27]). Fix furthermore arbitrary \(\lambda \in P_+(S^c)\) and \(\nu' \in D_m(\lambda)\). Choose a \(\sigma' \in W\) such that \(\nu' = \sigma' \lambda\). By Lemma 6.4 and the commutation relations of section 3, we compute

\[
L_{\nu;\lambda} L_{\nu';\lambda} = q^{(\lambda;\Lambda) - (\nu;\nu')} \tau((C^\lambda_{\nu;\lambda} C^\lambda_{\nu;\lambda})^* C^\lambda_{\nu;\lambda} C^\lambda_{\nu;\lambda})
\]

\[
= q^{3(\lambda;\Lambda) - 3(\nu;\nu')} \tau((C^\lambda_{\nu;\lambda})^* (C^\lambda_{\nu;\lambda})^* C^\lambda_{\nu;\lambda} C^\lambda_{\nu;\lambda})
\]

\[
= q^{2(\lambda;\Lambda) - 2(\nu;\nu')} L_{\nu';\lambda} L_{\nu;\lambda},
\]

where we used Proposition 3.2(ii) in the first and third equality and Proposition 3.2(i) twice in the second equality. If we repeat the same computation, but now using Corollary 5.3 twice in the second equality, then we obtain

\[
L_{\nu;\lambda} L_{\nu';\lambda} = q^{2(\nu;\nu') - 2(\lambda;\lambda)} L_{\nu';\lambda} L_{\nu;\lambda},
\]

hence

\[
(q^{2(\lambda;\Lambda) - 2(\nu;\nu')} - q^{2(\lambda;\Lambda) - 2(\nu;\nu')}) L_{\nu';\lambda} L_{\nu;\lambda} = 0.
\]
By Corollary 6.6 we have \( L_{\nu':\lambda} L_{\nu:\Lambda} \neq 0 \), so we conclude that
\[
(\lambda, \lambda) - (\nu, \nu') = (\lambda, \lambda - \sigma^{-1} \sigma' \lambda) = 0.
\]
Since \( \Lambda \in P_+^{(S^c)} \) and \( \lambda \in P_+^{(S^c)} \), it follows from Lemma 5.4 that \( \lambda = \sigma^{-1} \sigma' \lambda \), i.e. \( \nu' = \sigma \lambda \). Hence, \( D_n(\lambda) = \{ \sigma \lambda \} \) for all \( \lambda \in P_+^{(S^c)} \).

In the remainder of this section we write \( \sigma \) for the unique minimal coset representative such that \( D_n(\lambda) = \{ \sigma \lambda \} \) for all \( \lambda \in P_+^{(S^c)} \). We are going to prove that \( \tau \simeq \pi_{\sigma} \). First we look for the analogue of the distinguished vector \( e_{0^{(\sigma)}} \) (cf. Proposition 5.2(ii)) in the representation space \( H \) of \( \tau \).

The spectrum \( \hat{I}(\lambda) \) of \( L_{\sigma \lambda: \lambda} \) is contained in \([0, \infty)\), since \( L_{\sigma \lambda: \lambda} \) is a positive operator. By considering the spectral decomposition of \( L_{\sigma \lambda: \lambda} \), one obtains the following corollary of Lemma 6.5 and [12, Lemma 4.3].

**Corollary 6.6.** Let \( \lambda \in P_+^{(S^c)} \). Then \( \hat{I}(\lambda) \subset [0, \infty) \) is a countable set with no limit points, except possibly 0.

The proof of Corollary 6.6 is similar to the proof of [41, Prop. 3.9] and of [12, Prop. 4.2].

By Corollary 6.6 we have an orthogonal direct sum decomposition
\[
H = \bigoplus_{\gamma \in \hat{I}(\lambda) \cap R > 0} H_{\gamma}(\lambda)
\]
into eigenspaces of \( L_{\sigma \lambda: \lambda} \), where \( H_{\gamma}(\lambda) \) is the eigenspace of \( L_{\sigma \lambda: \lambda} \) corresponding to the eigenvalue \( \gamma \). Let \( \gamma_0(\lambda) > 0 \) be the largest eigenvalue of \( L_{\sigma \lambda: \lambda} \).

**Lemma 6.7.** Let \( \lambda \in P_+^{(S^c)} \), \( v \in V(\lambda) \), \( w \in V(\lambda)_\nu \) and assume that \( \nu \neq \sigma \lambda \). Then \( \tau (\langle C_{v_\lambda: v}^\lambda \rangle^* C_{w_\lambda: v}^\lambda)(H_{\gamma_0(\lambda)}(\lambda)) = \{0\} \).

**Proof.** Let \( \lambda \in P_+^{(S^c)} \), \( v \in V(\lambda)_\mu \) and \( w \in V(\lambda)_\nu \). By Lemma 6.1 and the commutation relations in section 3, we compute
\[
L_{\sigma \lambda: \lambda} \tau (\langle C_{v_\lambda: v}^\lambda \rangle^* C_{w_\lambda: v}^\lambda) = \tau (C_{v_\lambda: v}^\lambda (C_{w_\lambda: v}^\lambda C_{v_\lambda: v}) C_{w_\lambda: v}) = q^{2(\lambda, \lambda) - 2(\mu, \sigma \lambda) - 2(\mu, \sigma \lambda) + 2(\lambda, \lambda) - 2(\mu, \sigma \lambda)} \tau (\langle C_{v_\lambda: v}^\lambda \rangle^* C_{w_\lambda: v}^\lambda) C_{w_\lambda: v},
\]
where we used Proposition 3.2(ii) in the second and third equality and Proposition 3.2(ii) in the first and third equality. This computation, together with the injectivity of \( L_{\sigma \lambda: \lambda} \), shows that it suffices to give a proof of the lemma for the special case that \( v = v_{\sigma \lambda} \). So we fix \( h \in H_{\gamma_0(\lambda)}(\lambda) \) and \( w \in V(\lambda)_\nu \) with \( \nu \in P(\lambda) \) and \( \nu \neq \sigma \lambda \). It follows from Lemma 6.3 that \( \hat{h} := \tau (\langle C_{v_{\sigma \lambda}: v}^\lambda \rangle^* C_{v_{\sigma \lambda}: v}) h \) is an eigenvector of \( L_{\sigma \lambda: \lambda} \) with eigenvalue \( \gamma_0(\lambda) = q^{2(\lambda, \sigma^{-1}(\nu) - \lambda)} \gamma_0(\lambda) \). By Proposition 5.3 we have \( \gamma_0(\lambda) > \gamma(\lambda) \), hence \( \hat{h} = 0 \) by the maximality of the eigenvalue \( \gamma_0(\lambda) \).

**Corollary 6.8.** \( \gamma_0(\lambda) = 1 \) for all \( \lambda \in P_+^{(S^c)} \).

**Proof.** Follows from (3.10) and Lemma 6.7.

The linear subspace of \( C[\lambda[U] \] spanned by the matrix elements \( \{ C_{\mu: \nu}^{\lambda: \lambda} \}_{\mu \in P_+} \) is a subalgebra of \( C[\lambda[U] \) with algebraic generators \( \sum_{i \in [1, r]} C_{\mu: \nu}^{\lambda: \lambda} (i \in [1, r]) \), since \( C_{\mu: \nu} \sum_{i \in [1, r]} C_{\mu: \nu} = \lambda_{\mu: \nu} C_{\sigma(\mu: \nu)}^{\mu+\nu} \), where the scalar \( \lambda_{\mu: \nu} \in \mathbb{T} \) depends on the particular choices of orthonormal bases for the finite-dimensional irreducible representations \( V(\mu) \) and
\begin{proof} By Lemma 6.7 and Lemma 6.10 we obtain for any \(0 < h \in H\),
\begin{equation}
\| L_{\sigma \mu} \| = \| h \|.
\end{equation}
This follows from the eigenspace decomposition (6.5) for \(L_{\sigma \mu}\) and the fact that 1 is the largest eigenvalue of \(L_{\sigma \mu}\). Let \(\lambda \in \mathcal{P}_+ (S^c)\) and choose arbitrary \(i \in S^c\). Then \(\lambda = \mu + \omega_i\) for certain \(\mu \in \mathcal{P}_+ (S^c)\). By (6.6), we obtain for \(h \in H_1 (\lambda)\),
\begin{align*}
\| h \| &= \| L_{\sigma \mu} h \| = \| L_{\sigma \mu} L_{\sigma \mu} h \| \\
&\leq \| L_{\sigma \mu} \| \| h \| \leq \| h \|,
\end{align*}
\end{proof}

Then \(H_1 \subset H_1 (\lambda)\) for all \(\lambda \in \mathcal{P}_+ (S^c)\). In particular, \(H_1 \neq \{0\}\).

**Lemma 6.9.** \(H_1 = H_1 (\lambda)\) for all \(\lambda \in \mathcal{P}_+ (S^c)\). In particular, \(H_1 \neq \{0\}\).

**Proof.** For \(\mu \in \mathcal{P}_+ (S^c)\) we have \(\| L_{\sigma \mu} \| = 1\). Moreover, for any \(h \in H\),
\begin{equation}
(6.6) \quad h \in H_1 (\mu) \quad \iff \quad \| L_{\sigma \mu} h \| = \| h \|.
\end{equation}

This follows from the eigenspace decomposition (6.3) for \(L_{\sigma \mu}\) and the fact that 1 is the largest eigenvalue of \(L_{\sigma \mu}\). Let \(\lambda \in \mathcal{P}_+ (S^c)\) and choose arbitrary \(i \in S^c\). Then \(\lambda = \mu + \omega_i\) for certain \(\mu \in \mathcal{P}_+ (S^c)\). By (6.6), we obtain for \(h \in H_1 (\lambda)\),
\begin{align*}
\| h \| &= \| L_{\sigma \mu} h \| = \| L_{\sigma \mu} L_{\sigma \mu} h \| \\
&\leq \| L_{\sigma \mu} \| \| h \| \leq \| h \|,
\end{align*}
\end{proof}

hence we have equality everywhere. By (6.8), it follows that \(H_1 \subset H_1 (\lambda)\). Since \(i \in S^c\) was arbitrary, we conclude that \(h \in H_1\).

**Lemma 6.10.** Let \(\lambda \in \mathcal{P}_+ (S^c)\). For all \(v \in V(\lambda)\mu\) with \(\mu \neq \sigma \lambda\) we have
\begin{equation}
\tau((C^\lambda_{\nu \lambda})^* C^\lambda_{\nu \lambda}) (H_1) \subset H_1^\perp.
\end{equation}

**Proof.** Let \(\Lambda \in \mathcal{P}_+ (S^c)\), \(\lambda \in \mathcal{P}_+ (S^c)\), and \(v \in V(\lambda)\mu\) with \(\mu \neq \sigma \lambda\) and \(\mu \in \mathcal{P}(\lambda)\). Then
\begin{equation}
(6.9) \quad L_{\sigma \lambda} \tau(((C^\lambda_{\nu \lambda})^* C^\lambda_{\nu \lambda}) (H_1) = q^{2(\Lambda, \lambda - \sigma^{-1}(\mu))} \tau(((C^\lambda_{\nu \lambda})^* C^\lambda_{\nu \lambda}) L_{\sigma \lambda} \Lambda)
\end{equation}
by Lemma 6.3. By Lemma 6.4 we have \((\Lambda, \lambda - \sigma^{-1}(\mu)) > 0\). Hence,
\begin{equation}
\tau(((C^\lambda_{\nu \lambda})^* C^\lambda_{\nu \lambda}) (H_1) = \tau(((C^\lambda_{\nu \lambda})^* C^\lambda_{\nu \lambda}) (H_1 (\lambda)) \\
\subset \bigoplus_{\gamma < 1} H_1 (\lambda) = H_1 (\lambda)^\perp = H_1^\perp,
\end{equation}
which completes the proof of the lemma.

**Corollary 6.11.** \(\dim(H_1) = 1\).

**Proof.** By Lemma 6.3 and Lemma 6.10 we obtain for any \(0 \neq h \in H_1\),
\begin{equation}
\tau(\mathcal{A} \mathcal{S}) h \subset \text{span} \{ h \} \oplus H_1^\perp,
\end{equation}
where the overbar means closure. By the irreducibility of \(\tau\), we conclude that \(\text{span} \{ h \} = H_1\).

Any vector \(h \in H_1\) with \(\| h \| = 1\) can serve now as the analogue in the representation space \(H\) of the distinguished vector \(e^0_0^0 \tau(\sigma)\) in the representation space of \(\pi_\sigma\). By comparing the Gel’fand-Naimark-Segal states of \(\tau\) and \(\pi_\sigma\) taken with respect to the cyclic vector \(h \in H_1\) \((\| h \| = 1)\) resp. \(e^0_0^0 \tau(\sigma)\), we obtain the following lemma.
Lemma 6.12. We have $\tau \simeq \pi_\sigma$ as irreducible $*$-representations of $A_S$.

Proof. Fix an $h \in H_1$ with $\|h\| = 1$, and define the Gel’fand-Naimark-Segal states $\phi_\tau, \phi_{\pi_\sigma} : A_S \to \mathbb{C}$ by

$$\phi_\tau(a) := (\tau(a)h, h), \quad \phi_{\pi_\sigma}(a) := (\pi_\sigma(a)e_0^{\otimes l(\sigma)}, e_0^{\otimes l(\sigma)}).$$

Then we have for $\phi = \phi_\tau$ (resp. $\phi = \phi_{\pi_\sigma}$),

$$\phi((C_{\mu,i}^\lambda)^* C_{\nu,j}^\lambda) = \delta_{\mu,\sigma\lambda} \delta_{\nu,\sigma\lambda}$$

for $\lambda \in P_+(S^c)$, $\mu, \nu \in P(\lambda)$, $i \in [1, \text{dim}(V(\lambda)_\mu)]$, and $j \in [1, \text{dim}(V(\lambda)_\nu)]$. Indeed, (6.11) for $\phi = \phi_\tau$ follows from Lemma 5.7 and Lemma 6.10. For $\phi = \phi_{\pi_\sigma}$, recall that $\pi_\sigma$ is an irreducible $*$-representation of $A_S$ (Proposition 5.3). We have seen in the previous section that $L_{\sigma\lambda\lambda} = \pi_\sigma((C_{\sigma\lambda\lambda}^\lambda)^* C_{\sigma\lambda\lambda}^\lambda)$ is injective for all $\lambda \in P_+(S^c)$, hence $\sigma\lambda \in D(\lambda)$ (cf. (6.2)) for all $\lambda \in P_+(S^c)$. By (5.2), we actually have $\sigma\lambda \in D_m(\lambda)$ for all $\lambda \in P_+(S^c)$. Hence the labeling $\sigma \in W^c$ of $\pi_\sigma$ coincides with its (unique) minimal coset representative defined in Lemma 5.5. Furthermore, the one-dimensional subspace $H_1$ for $\pi_\sigma$ is equal to $\text{span}\{e_0^{\otimes l(\sigma)}\}$ (cf. Proposition 5.3(ii), Lemma 6.11). So (6.11) for $\phi = \phi_{\pi_\sigma}$ follows again from Lemma 6.7 and Lemma 6.10.

By linearity it follows from (6.11) that $\phi_\tau = \phi_{\pi_\sigma}$, hence $\tau$ and $\pi_\sigma$ are unitarily equivalent $*$-representations (cf. [7, Prop. 2.4.1]).

We may summarize the results of this section as follows.

Theorem 6.13. For all $S \subseteq \Delta$, $\{\pi_\sigma\}_{\sigma \in W^c}$ is a complete set of mutually inequivalent, irreducible $*$-representations of the factorized $*$-subalgebra $A_S$.

Combining Proposition 5.7, Theorem 4.9 and Theorem 6.13 we obtain the following theorem.

Theorem 6.14. $\{\pi_\sigma\}_{\sigma \in W^c}$ is a complete set of mutually inequivalent, irreducible $*$-representations of $\mathcal{C}_q[U/K_S]$ in the the following cases:

(i) $S = \emptyset$, i.e. $U/K_S = U/T$ is the full flag manifold;

(ii) $\#S^c = 1$ and the simple root $\alpha \in S^c$ is a Gel’fand node.

For these cases the restriction to $\mathcal{C}_q[U/K_S]$ of the universal $C^*$-norm on $\mathcal{C}_q[U]$ coincides with the universal $C^*$-norm on $\mathcal{C}_q[U/K_S]$.

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