Reidemeister torsion, hyperbolic three-manifolds, 
and character varieties

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Abstract

This is a survey on Reidemeister torsion for hyperbolic three-manifolds 
of finite volume. Torsions are viewed as topological invariants and also as functions on the variety of representations in $SL_2(\mathbb{C})$. In both cases, the torsions may also be computed after composing with finite dimensional representations of $SL_2(\mathbb{C})$. In addition the paper deals with the torsion of the adjoint representation as a function on the variety of $PSL_{n+1}(\mathbb{C})$-characters, using that the first cohomology group with coefficients twisted by the adjoint is the tangent space to the variety of characters.

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1 Introduction

The goal of this paper is to survey some results on Reidemeister torsions of orientable, hyperbolic three-manifolds of finite volume. Torsions are viewed as invariants of hyperbolic manifolds and also as functions on the variety of \( \text{SL}_2(\mathbb{C}) \)-characters.

The paper uses combinatorial torsion, though some relevant developments described here are proved using analytic torsion, which is briefly mentioned. No details on other kinds of torsion are provided, like \( L^2 \)-torsion. There are remarkable recent surveys on twisted Alexander polynomials [69, 33] and on abelian torsions [59]. There are also the classical surveys of Milnor [68] and Turaev [95], as well as the books by Turaev [96, 97] and Nicolaescu [80].

Besides Section 2 (devoted to general tools on torsions), only orientable hyperbolic three-manifolds \( M^3 \) of finite volume are considered. The hyperbolic structure is unique, by Mostow-Prasad theorem, so their holonomy is unique up to conjugacy. It lifts to a representation in \( \text{SL}_2(\mathbb{C}) \). The lift is naturally associated to a spin structure \( \sigma \) and it is acyclic, by a theorem of Raghunathan. Thus this yields a topological invariant of the oriented manifold with a spin structure (Definition 3.4 in the closed case and Definition 3.10 in the non-compact case):

\[
\tau(M^3, \sigma) \in \mathbb{C}^*. 
\]

Its main properties are described, in particular, its behavior by Dehn filling allows to construct sequences of closed manifolds whose volume stays bounded but the torsion converges to infinity (Corollary 3.28). This must be compared with Theorem 3.7 (due to Bergeron and Venkatesh) on the asymptotic behavior of torsions by coverings, that yields sequences of coverings \( M^3_n \to M^3 \) so that \( |\tau(M^3_n, \sigma)| \) grows with the exponential of the volume of \( M^3_n \). For those sequences
the injectivity radius converges to infinity. Some results suggest that short geodesics may play a role in the behavior of this torsion, for instance the surgery formula, Proposition 3.25.

Additionally, the paper deals with finite dimensional irreducible representations of $SL_2(\mathbb{C})$. Their composition with the lift of the holonomy provides a family of invariants of a closed hyperbolic manifold, oriented and with a spin structure. A remarkable theorem of Müller relates the asymptotic behavior of those invariants with the volume (Theorem 4.5).

For simplicity, finite volume hyperbolic three-manifolds are assumed to have a single cusp. I am interested in the torsion as a function on the variety of characters. The distinguished component is the component of the variety of $SL_2(\mathbb{C})$-characters that contains a lift of the holonomy of the complete structure. It is easy to see that the torsion is a rational (meromorphic) function on this curve. It is shown here that for a knot exterior (more generally, for a manifold with first Betti number 1), this torsion is a regular function (it has no poles) on the distinguished component.

The paper also discusses the functions obtained by composing the representation with the irreducible representations of $SL_2(\mathbb{C})$. In particular the torsion for the adjoint representation occurs in the volume conjecture, which is also quickly mentioned but not analyzed.

Finally I describe the torsion of the adjoint representation as a function on the variety of characters in $PSL_{n+1}(\mathbb{C})$. I considered the case for $PSL_2(\mathbb{C})$ in [82] and Kitayama and Terashima discuss the general case for $PSL_{n+1}(\mathbb{C})$ in [55]. The relevant fact for this torsion is a result of André Weil, that identifies the tangent space to the variety of characters with the first cohomology group with coefficients twisted by the adjoint representation (under generic hypothesis). In this way this torsion is related to local parameterizations of the deformation space, which amounts to choose peripheral curves. In particular, a formula for the change of curve is provided, and this allows to define a volume form under some circumstances (not at the holonomy of the complete structure, but for instance for characters in $SU(n)$), as done by Witten for surfaces and Dubois for knot exteriors and $SU(2)$. In addition Weil’s interpretation allows to compute the torsion for surface bundles from the tangent map of the monodromy on the deformation space of the fibre.

The paper is organized as follows. Section 2 is devoted to the preliminaries on combinatorial torsion, including examples as Seifert fibered manifolds, Witten’s theorem on the volume form on representations of surfaces, and Johnson’s construction of an analog to Casson’s invariant. This section concludes with a brief recall of analytic torsion and Cheeger-Müller theorem on the equivalence of both. Section 3 discusses the torsion of a hyperbolic three-manifold for the lift of the holonomy in $SL_2(\mathbb{C})$. It addresses first closed manifolds and then the cusped ones, considering both the invariant and the function on the distinguished component of the variety of characters. Section 4 deals with the analog to the previous section, but instead of the torsion of representations in $SL_2(\mathbb{C})$, it considers torsions of compositions with finite dimensional representations of $SL_2(\mathbb{C})$. In particular it mentions the torsion of the adjoint representation as part of the volume conjecture. Section 5 deals with the the torsion of the adjoint on the variety of characters in $PSL_{n+1}(\mathbb{C})$, in particular recalling the work of Kitayama and Terashima and the author.
The paper concludes with two appendices. Appendix \[A\] is devoted to the proof of a technical result: the trivial representation does not lie in the distinguished component of the variety of characters if the first Betti number is one. This is used in Section \[\ref{sec:tor}\] to prove that the torsion in $\text{SL}_2(\mathbb{C})$ is a regular function without poles. Appendix \[B\] provides results in cohomology that are required for the torsions of Sections \[\ref{sec:tor}\] and \[\ref{sec:cohom}\]. It is based mainly on a vanishing theorem of Raghunathan and it also discusses bases for the cohomology groups.

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## 2 Combinatorial and analytic torsions

This section overviews the definition and main properties of combinatorial torsion, as they are used later in some proofs. It also recalls briefly the definition of analytic torsion, as it is used in many relevant results.

### 2.1 Combinatorial torsion

For a detailed definition of combinatorial torsion see \[\cite{95, 65, 59, 80}\]. This section follows mainly \[\cite{82}\], in particular its convention with the power $(-1)^i + 1$ instead of $(-1)^i$ for the alternated product. (See Remark \[\ref{rem:convention}\] on this convention.) The definition is given for chain and for cochain complexes, in a way that both homology and cohomology give the same definition of the torsion of a manifold.

#### 2.1.1 Torsion of a chain complex

Let $F$ be a field and $C_* = (C_*, \partial)$ a chain complex of finite dimensional $F$-vector spaces:

$$C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0.$$  

The subspaces of boundaries and cycles are denoted by $B_i = \text{Im}(C_{i+1} \xrightarrow{\partial} C_i)$ and $Z_i = \ker(C_i \xrightarrow{\partial} C_{i-1})$ respectively, the homology is denoted by by $H_i = Z_i/B_i$.

Assume $c_i = \{c_{i,1}, \ldots, c_{i,i_i}\}$ is an $F$-basis for $C_i$ and $h_i = \{h_{i,1}, \ldots, h_{i,r_i}\}$, is a basis for $H_i$, if nonzero. For the definition of torsion, a basis $b_i = \{b_{i,1}, \ldots, b_{i,r_i}\}$ for $B_i$ is also required. Using the exact sequences:

$$0 \rightarrow Z_i \rightarrow C_i \xrightarrow{\partial} B_{i-1} \rightarrow 0 \quad (1)$$
$$0 \rightarrow B_i \rightarrow Z_i \rightarrow H_i \rightarrow 0, \quad (2)$$

lift $b_{i-1}$ to $\tilde{b}_{i-1} \subset C_i$ in \[1\] and $h_i$ to $\tilde{h}_i \subset Z_i \subset C_i$ in \[2\] and construct a new basis for $C_i$:

$$b_i \cup \tilde{b}_{i-1} \cup \tilde{h}_i, \quad (3)$$

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where \(\sqcup\) denotes the disjoint union. The bases are compared with the determinant of the corresponding matrix. Given two bases \(\alpha = \{\alpha_1, \ldots, \alpha_r\}\) and \(\beta = \{\beta_1, \ldots, \beta_r\}\) for \(F^r\), if \((\eta_{ij}) \in M_{n}(F)\) is the matrix that relates the bases, i.e. if \(\alpha_i = \sum_j \eta_{ij} \beta_j\), set
\[
[\alpha, \beta] = \det(\eta_{ij}) \in F^*.
\] (4)

**Definition 2.1.** The torsion of the chain complex \(C_\ast\) with bases \(\{c_i\}\) and bases \(\{h_i\}\) for \(H_i\) is:

\[
\text{tor}(C_\ast, \{c_i\}, \{h_i\}) = \prod_{i=0}^{d} [b^i \sqcup \tilde{b}_{i-1} \sqcup \tilde{h}_i, c_i]^{(-1)^{i+1}} \in F^* / \{\pm 1\}.
\] (5)

As it is an alternated product, it is easy to see that it does not depend on the choice of \(b_i\), the basis for \(B_i\), and it is also straightforward that it does not depend on the lifts \(\tilde{b}_{i-1}\) and \(\tilde{h}_i\).

**Remark 2.2.** In the alternated product defining the torsion of a complex [5], authors use \((-1)^i\) instead of \((-1)^{i+1}\). Definition 2.1 follows the convention of [86, 97, 99, 57, 54] for instance, but opposite to [65, 21] and all papers on analytic torsion [13, 71] and quantum invariants [102, 3, 24, 37, 20, 44, 76, 19, 81, 23]. I followed this convention in [82] but not in [63]. This convention is better suited for polynomials, also for functions on the deformation space; the opposite one is more useful for the interpretation of the torsion as a volume form.

We follow a different definition for the torsion of a cocomplex, which gives a nonstandard statement of Lemma 2.4 (in the standard version it is corrected by a power \((-1)^n\)).

There is a formula of change of bases. For different choices \(c'_i\) and \(h'_i\) of bases for \(C_i\) and \(H_i\) we have

\[
\frac{\text{tor}(C_\ast, \{c'_i\}, \{h'_i\})}{\text{tor}(C_\ast, \{c_i\}, \{h_i\})} = \prod_{i=0}^{d} \left[\frac{[c'_i, c_i]}{[h'_i, h_i]}\right]^{(-1)^i}.
\] (6)

The proof is straightforward, see [65, 82, 80] for details.

**Definition 2.3.** Let \(C^\ast = (C^\ast, \delta)\) be a cochain complex with bases \(\{c^i\}\), and cohomology bases \(\{h^i\}\), by constructing the \(b^i\) in a similar way, its torsion is defined as

\[
\text{tor}(C^\ast, \{c^i\}, \{h^i\}) = \prod_{i=0}^{d} [b^i \sqcup \tilde{b}_{i-1} \sqcup \tilde{h}_i, c_i]^{(-1)^i} \in F^* / \{\pm 1\}.
\] (7)

The convention of powers \((-1)^i\) and \((-1)^{i+1}\) has been changed in this definition, for the purpose of Lemma 2.4.

**Lemma 2.4.** Let \((C_i)^\ast = \text{hom}_F(C_i, F)\) be the dual cocomplex, with coboundary \(\delta : C^\ast_i \to C^\ast_{i-1}\) defined by \(\delta(\theta) = \theta \circ d\). If \((h_i)^\ast\) is dual to \(h_i\), then

\[
\text{tor}(C_\ast, \{h_i\}) = \text{tor}((C_\ast)^\ast, \{(h_i)^\ast\}).
\]

Notice that this lemma uses Definition 2.3 of a cochain complex. If instead of the cocomplex one considers the dual complex, one must re-index the dimension \(i\) by \(d-i\); then the torsion of the complex is replaced by its \((-1)^{d-i}\)-power! Then, one may see [60, 67] for a proof with this version of Lemma 2.4.
2.1.2 Twisted chain complexes

Let $K$ be a finite CW-complex. This paper is mostly interested in 3-manifolds but also in surfaces and in $S^1$, for which there is a canonical choice of PL-structure. Let 

$$\rho: \pi_1 K \to \text{SL}_n(F)$$

be a representation of its fundamental group. Consider the chain complex of vector spaces 

$$C_*(K; \rho) := F^n \otimes\rho C_*(\tilde{K}; \mathbb{Z})$$

where $C_*(\tilde{K}; \mathbb{Z})$ denotes the simplicial complex of the universal covering and $\otimes\rho$ means that one takes the quotient of $F^n \otimes\mathbb{Z} C_*(\tilde{K}; \mathbb{Z})$ by the $\mathbb{Z}$-module generated by $\rho(\gamma) v \otimes c - v \otimes \gamma \cdot c$, where $v \in F$, $\gamma \in \pi_1 K$ and $c \in C_*(\tilde{K}; \mathbb{Z})$, and $\cdot$ stands for transpose. Namely

$$v \otimes \gamma \cdot c = \rho(\gamma)^t v \otimes c \quad \forall \gamma \in \pi_1 K.$$

Instead of the transpose, one could use the inverse. For some representations the inverse and transpose are the same or conjugate, but this is not true in general, and here this is relevant for duality with cohomology.

The boundary operator is defined by linearity and

$$\partial(v \otimes c) = v \otimes \partial c,$$

for $v \in F$ and $c \in C_*(\tilde{K}; \mathbb{Z})$. The homology of this complex is denoted by $H_*(K; \rho)$.

Analogously, one considers the cocomplex of cochains

$$C^*(K; \rho) := \text{hom}_{\pi_1 K}(C_*(\tilde{K}; \mathbb{Z}), F^n)$$

that has a natural coboundary operator to define the cohomology $H^*(K; \rho)$.

Then $C_*(K; \rho)$ is a cocomplex of finite dimensional $F$-vector spaces. Choose $\{v_1, \ldots, v_n\}$ a $F$-basis for $F^n$ and let $\{e_i^1, \ldots, e_i^j\}$ denote the set of $i$-cells of $K$.

Then $c_i = \{v_r \otimes e_s^i \mid r \leq n, s \leq j_i\}$ is a $F$-basis for $C_i(K; \rho)$.

Let $h_i$ be a basis for $H_i(K; \rho)$. One can now define the torsion by means of chain complexes:

**Definition 2.5.** The torsion of $(K, \rho, \{h_i\})$ is 

$$\text{tor}(K, \rho, \{h_i\}) = \text{tor}(C_*(K; \rho), \{c_i\}, \{h_i\}) \in F^*/\{\pm 1\}.$$ 

**Remark 2.6.** (a) This torsion does not depend on the lifts of the cells $\tilde{e}_i$ nor the basis $\{v_1, \ldots, v_n\}$ of $F^n$.

(b) It does not depend either on the conjugacy class of $\rho$, taking care that the bases for the homology are in correspondence via the natural isomorphism between the homology groups of the conjugate representations.

**Remark 2.7** (Sign indeterminacy). The torsion lies in $F^*/\{\pm 1\}$, but there are ways to avoid the sign indeterminacy:
(a) When both the Euler characteristic $\chi(K)$ and $n = \dim \rho$ are even, then the sign of this torsion is also well defined, i.e. it lives in $F^*$. 

(b) When $\chi(K)$ is even but $n = \dim \rho$ is odd, the ordering of the cells is relevant: If two cells are permuted in the construction, then the sign of the torsion is changed. To overcome this and to get an invariant in $F^*$, Turaev [95] noticed that given an ordering of the basis in homology with constant real coefficients, $H_*(K; \mathbb{R})$, there is a natural way to order the cells of $K$, (so that the torsion with trivial coefficients is then positive). So if one has an orientation of the the homology $H_*(K; \mathbb{R})$, then there is a choice for the sign of the torsion.

**Remark 2.8.** It is a topological invariant but not in its relative version, cf [80, Remark 2.12]. Namely:

(a) It is an invariant of the simple homotopy type of $K$. Hence, by Chapman’s theorem [12], $\text{tor}(K, \rho, \{h_i\})$ is invariant by homeomorphisms. Working with manifolds of dimension $\leq 3$, uniqueness of the triangulation up to subdivision is an alternative to Chapman’s theorem.

(b) However there is a well defined notion of torsion of a pair of CW-complexes $(K, L)$, but then the torsion is not a topological invariant of a pair. In fact Milnor used Reidemeister torsion of a pair to distinguish two homeomorphic simplicial complexes that are not combinatorially equivalent [66].

We deal now with the construction using cohomology. Consider elements $(\tilde{e}_i^*) \otimes v : C_i(K; \mathbb{Z}) \to F^*$ as the morphism of $\pi_1K$-modules defined by

$$
((\tilde{e}_i^*) \otimes v)(\tilde{e}_j^*) = \begin{cases} 
v & \text{if } s = r, \\
0 & \text{if } s \neq r.
\end{cases}
$$

Then $\tilde{e}^j = \{(\tilde{e}_i^*) \otimes v_j\}_{1 \leq i \leq n}$ is a basis for $C^*(K; \rho)$. Using this basis, one may define the torsion of the cocomplex $C_*(K; \rho)$, $\{\tilde{e}^j\}$, and $\{h^j\}$ a basis in cohomology. Remarks 2.6, 2.7, and 2.8 also hold true for the torsion defined from cohomology. By using Lemma 2.4, we have:

**Remark 2.9.** The complexes $\tilde{C}^*(K; \rho)$ and $C_*(K; \rho)$ are dual. In addition

$$
\text{tor}(\tilde{C}^*(K; \rho), \{(\tilde{e}^j), \{(h^j)^*\}) = \text{tor}(C_*(K; \rho), \{e^j\}, \{h_i\}).
$$

Hence both homology or cohomology can be used to define the torsion.

Here Poincaré duality and duality between homology and cohomology is not discussed even if it is used, see [82] for instance.

### 2.2 Geometric properties of combinatorial torsion

This subsection recalls the basic properties of combinatorial torsion, the main one being Mayer-Vietoris, useful for cut and paste. Bundles over $S^1$ are also considered, in particular instead of just the torsion it is more convenient to consider the twisted Alexander polynomial, which is a torsion by [53]. The subsection finishes with examples.
2.2.1 Mayer-Vietoris

Let $K$ be a CW-complex, with subcomplexes $K_1 \subset K_2 \subset K$ so that $K = K_1 \cup K_2$. Let $\rho: \pi_1 K \to \text{SL}_n(F)$ be a representation. Consider the diagram of inclusions

$$
\begin{array}{ccc}
K_1 \cap K_2 & \xrightarrow{j_1} & K_1 \\
\downarrow i_2 & & \downarrow j_1 \\
K_2 & \xrightarrow{j_2} & K.
\end{array}
$$

There is a Mayer-Vietoris exact sequence in homology with twisted coefficients, if the representations on $\pi_1 K_1$, $\pi_1 K_2$ and $\pi_1 L$ are the restrictions:

$$
\cdots \to \bigoplus_L H_i(L; \rho) \xrightarrow{i_1 \oplus i_2} H_i(K_1; \rho) \oplus H_i(K_2; \rho) \xrightarrow{j_1 \oplus j_2} H_i(K; \rho) \to \bigoplus_L H_{i-1}(L; \rho) \to \cdots (8)
$$

where $L$ runs on the connected components of $K_1 \cap K_2$. Notice that different choices of base-points yield canonical isomorphisms between the homology groups, hence we can consider $K_1 \cap K_2$ not connected.

Choose a basis for each of these cohomology groups $h_*$ for $H_*(K; \rho)$, $h_{1*}$ for $H_*(K_1; \rho)$, $h_{2*}$ for $H_*(K_2; \rho)$, and $h_{L*}$ for $H_*(L; \rho)$. The long exact sequence (8) is viewed as a complex. Its torsion is denoted by $\text{tor}(H, h_*)$.

**Proposition 2.10** (Mayer-Vietoris). Let $K$ be a CW-complex, with subcomplexes $K_1 \cap K_2$ so that $K = K_1 \cup K_2$. Let $\rho: \pi_1 K \to \text{SL}_n(F)$ be a representation and choose basis in homology. Then

$$
\text{tor}(K, h_*; \rho) = \frac{\text{tor}(K_1, h_{1*}; \rho) \text{tor}(K_2, h_{2*}; \rho)}{\prod_L \text{tor}(L, h_{L*}; \rho) \text{tor}(H, h_*)}
$$

where $L$ runs on the connected components of $K_1 \cap K_2$.

This is used in surgery formulas (e.g. Proposition 3.25) or for the mapping torus (Proposition 2.14). The proof can be found in [65] or in [52] Section 0.4.

2.2.2 Polynomials

Here some properties of twisted Alexander polynomials viewed as torsions are briefly discussed, since they are convenient for describing some results of Reidemeister torsion. See the recent surveys [69, 33] for more details on twisted Alexander polynomials.

Start with a surjective morphism $\phi: \pi_1 K \to \mathbb{Z}$. Instead of a representation $\rho: \pi_1 K \to \text{SL}_n(F)$, consider the twisted representation

$$
\rho \otimes \phi: \pi_1 K \to \text{GL}_n(F(t))
$$

where $F(t)$ is the field of fractions of the polynomial ring $F[t]$.

If $H_*(K; \rho) = 0$, then $H_*(K; \rho \otimes \phi) = 0$ [68, 54] and there is a well defined torsion, or twisted polynomial:

$$
\Delta_{K, \rho, \phi}(t) = \text{tor}(K, \rho, \phi) \in F(t)[\pm t^n].
$$
As the determinant is not one, there is an indeterminacy factor \( t^k \), for some integer \( k \in \mathbb{Z} \).

**Remark 2.11.** Assume \( H_*(K; \rho) = 0 \). Then

\[
\Delta_{K, \rho, \phi}(1) = \text{tor}(K, \rho).
\]

This is proved from the map at the chain level \( C_*(K; \rho \otimes \phi) \to C_*(K; \rho) \) induced by evaluation \( t = 1 \).

If \( \Delta_{K, \rho, \phi} = \sum a_i t^i \), then its degree can be defined as:

\[
\deg(\Delta_{K, \rho, \phi}) = \max\{i - j \mid a_i, a_j \neq 0\}.
\]

**Proposition 2.12.** If \( M^3 \) is a three-manifold and \( \Sigma \subset M^3 \) is a surface dual to \( \phi \) such that \( H_0(\Sigma; \rho) = 0 \), then

\[
\deg \Delta_{M^3, \rho, \phi} \leq -\chi(\Sigma)n.
\]

This proposition can be proved using Mayer-Vietoris to a tubular neighborhood of the surface and its exterior.

For a knot \( K \subset S^3 \), there is a natural surjection \( \phi : \pi_1(S^3 - K) \to \mathbb{Z} \). Let \( A_K \) denote its Alexander polynomial. In [57] (this reference is based on a preprint from 1990) Lin defined a twisted Alexander polynomial \( A^\rho_K \) that lives in \( F(t) \), that was later modified by Wada [99].

Milnor in 1962 for the untwisted Alexander polynomials, and Kitano in 1996 for the twisted ones, proved:

**Theorem 2.13.** For a knot exterior \( S^3 - K \), we have:

(a) [67] For the trivial representation:

\[
\Delta_{S^3 - K, 1, \phi} = \text{tor}(S^3 - K, \phi) = \frac{A_K(t)}{(t-1)}.
\]

(b) [53] For \( \rho \) a non-trivial acyclic representation:

\[
\Delta_{S^3 - K, \rho, \phi} = \text{tor}(S^3 - K, \rho \otimes \phi) = A^\rho_K(t).
\]

Milnor’s formula, for the untwisted polynomial, has been generalized to links and to other three-manifolds in 1986 by Turaev [95]. See also [33, 69].

The following is straightforward from Mayer-Vietoris, Proposition 2.10

**Proposition 2.14.** Let \( K \) be a CW complex, \( f:|K| \to |K| \) a homeomorphism with mapping torus \( M_f \). Then

\[
\Delta_{M_f, \rho, \phi} = \prod_{i=0}^{\dim K} \det(f_i - t \text{Id})^{(-1)^{i+1}},
\]

where \( f_i : H_i(K; \rho) \to H_i(K; \rho) \) denotes the induced map in homology. In particular, by Remark 2.11

\[
\text{tor}(M_f, \rho) = \prod_{i=0}^{\dim K} \det(f_i - \text{Id})^{(-1)^{i+1}}.
\]
Another relevant issue for this polynomial is Turaev’s interpretation of the twisted Alexander polynomial: namely this polynomial encodes the torsion of the finite cyclic coverings ([95], see also [25, 83, 89, 85, 109]).

**Proposition 2.15.** Let \( K, \rho \pi_1 K \to \text{SL}_n(F), \) and \( \phi \pi_1 K \to \mathbb{Z} \) be as above. Let \( K_m \to K \) be the cyclic covering of order \( m \) corresponding to the kernel of \( \phi \) composed with the projection \( \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \). Then

(a) \( H_*(K_m; \rho) = 0 \) if and only if \( \Delta_{M^3, \rho, \phi}(\zeta) \neq 0 \) for every \( \zeta \in \mathbb{C} \) satisfying \( \zeta^m = 1 \).

(b) If \( H_*(K_m; \rho) = 0 \) then \( \text{tor}(K_m, \rho) = \prod_{i=0}^{m-1} \Delta_{M^3, \rho, \phi}(\zeta^i) \)

for \( \zeta \in \mathbb{C} \) a primitive \( m \)-root of the unity.

### 2.2.3 Examples

The first examples below can be found in many references, for instance [80, Chapter 2].

**Example 2.16** (The circle). Consider the circle \( S^1 \). A representation \( \rho \) of its fundamental group \( \pi_1 S^1 \cong \mathbb{Z} \) is determined by the image of its generator, that we denote by \( A \in \text{SL}_n(F) \). Notice that the homology and cohomology of \( S^1 \) twisted by \( \rho \) is determined by \( H_0(S^1; \rho) \), by duality between homology and cohomology and vanishing of the Euler characteristic. Since \( H^0(S^1; \rho) \) is isomorphic to the subspace of invariant elements \( (F^n)^{\rho(\mathbb{Z})} = \ker(A - \text{Id}) \),

\[
H_*(S^1; \rho) = 0 \text{ if and only if } \det(A - \text{Id}) \neq 0.
\]

On the other hand, \( S^1 \) is just the mapping torus of the identity map on the point \(*\). The homology of the point is \( H_0(\ast; \rho) \cong F^n \) and the action of the return map is multiplication by \( A \). Thus, by Proposition 2.14

\[
\Delta(S^1, \rho \otimes \phi) = \frac{1}{\det(A - t\text{Id})} \quad \text{and} \quad \text{tor}(S^1, \rho) = \frac{1}{\det(A - \text{Id})}.
\]

In fact using Proposition 2.14 is a fancy way of computing the torsion of the circle. It is more natural to view Proposition 2.14 as a generalization of the torsion of the circle.

**Example 2.17** (The 2-torus). The homology and the cohomology of the two-torus \( T^2 \), \( H_*(T^2; \rho) \) and \( H^*(T^2; \rho) \), are determined by \( H^0(T^2; \rho) \) which is the subspace of \( F^n \) of invariant elements. This assertion follows from the different dualities (Poincaré, and homology/cohomology) and the Euler characteristic. Thus if \( F^n \) has no nonzero invariant elements by \( \rho(\pi_1 T^2) \), then \( H_*(T^2; \rho) = 0 \). In this case

\[
\text{tor}(T^2; \rho) = 1.
\]

This can be proved viewing \( T^2 \) as the mapping torus of the identity on \( S^1 \) and Proposition 2.14 as the action on \( H^1(S^1; \rho) \) is the same as on \( H^0(S^1; \rho) \) and the corresponding terms cancel. See also [48].
Example 2.18 (Lens spaces). Let \( p, q \) be integers so that \( p \geq 2, p \geq q \geq 1 \) and \( p \) and \( q \) are coprime. View the three-sphere as the unit sphere in \( \mathbb{C}^2 \): \( S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \), and consider the action of the cyclic group of order \( p \) generated by the transformation

\[
S^3 \to S^3 \quad (z_1, z_2) \mapsto (e^{2\pi i/p}z_1, e^{2\pi i/p}z_2).
\]

The quotient by this action is the Lens space \( L(p, q) = S^3/\sim \). Consider the Heegaard decomposition into two solid torus \( S \times S^1 \) and by the previous examples:

\[
\pi_1(S^3) \cong \mathbb{Z}^3 \
\]

are the torus that lift respectively to \( \pi \times H \). By irreducibility \( \rho \) \( \rho \) maps the soul of \( L \) generated by the transformation \( \rho \). His result is reproduced here. Previously, Freed had computed a torsion for Brieskorn spheres \([31]\), and of course Franz \([29]\) and Reidemeister torsion of a Seifert fibered three-manifold with a representative \( \rho \).

Notice that the determinant of \( \rho \) is not one, but has norm one. Thus the topological invariant is obtained once taking the module \(|\mathrm{tor}(L(p,q),\rho)|\) and considering all nontrivial \( p \)-th roots of unity. This is the original example of Franz \([29]\) and Reidemeister \([30]\). See also \([35, 17, 14]\), for instance.

Example 2.19 (\( \Sigma \times S^1 \)). Consider the product of a compact oriented surface, possibly with boundary, and the circle, \( \Sigma \times S^1 \). Let \( \rho : \pi_1(\Sigma \times S^1) \to \text{SL}_n(\mathbb{C}) \) be an irreducible representation. In particular, \( \rho \) maps the generator of the factor \( \pi_1 S^1 \) to a central matrix, namely \( \rho(t) = \omega \text{Id} \) with \( \omega^n = 1 \). We view \( \Sigma \times S^1 \) as the mapping torus of the identity on \( \Sigma \) and apply Proposition 2.11.

By irreducibility \( H_0(\Sigma; \rho) \cong H_2(\Sigma; \rho) = 0 \) and \( \dim H_1(\Sigma; \rho) = -n \chi(\Sigma) \). As the action of \( t \) on \( H_1(\Sigma; \rho) = \mathbb{C}^{-n\chi(\Sigma)} \) is multiplication by \( \omega \):

(a) \( \rho \) is acyclic iff \( \omega \neq 1 \).

(b) when \( \omega \neq 1 \), then \( \mathrm{tor}(\Sigma \times S^1, \rho) = (\omega - 1)^{-n \chi(\Sigma)} \).

Example 2.20 (Seifert fibered manifolds). In \([50, 52]\) Kitano computed the Reidemeister torsion of a Seifert fibered three-manifold with a representation in \( \text{SL}_n(\mathbb{C}) \). His result is reproduced here. Previously, Freed had computed a torsion for Brieskorn spheres \([31]\), and of course Franz \([29]\) and Reidemeister \([30]\) for lens spaces.

Let \( M^3 \) be a Seifert fibered manifold, whose base is a compact surface \( \Sigma \), possibly with boundary, with \( c \) cone points corresponding to singular fibres. Let \( \rho : \pi_1 M^3 \to \text{SL}_n(\mathbb{C}) \) be an irreducible representation. In particular \( \rho \) maps the fibre to \( \omega \text{Id} \), where \( \omega^n = 1 \). For the \( i \)-th cone point, let \( (\alpha_i, \beta_i) \) denote the Seifert coefficients, and let \( c_i \) denote the loop such that \( c_i^{\alpha_i} f^{\beta_i} = 1 \). Let \( \{\lambda_{i,1}, \ldots, \lambda_{i,n}\} \) denote the eigenvalues of \( \rho(c_i) \). Choose integers \( (r_i, s_i) \) such that \( \alpha_i s_i - \beta_i r_i = 1 \).
Let $\hat{\Sigma}$ denote the surface $\Sigma$ minus the cone points, i.e. the space of regular fibres, so that $\chi(\hat{\Sigma}) = \chi(\Sigma) - c$.

Assume $M^3$ is not a solid nor a thick torus, then \[50, 52\]:

(a) $\rho$ is acyclic if and only if $\omega \neq 1$ and $\lambda_{i,j}^{r_i} \omega^{s_i} \neq 1$, for every $i = 1, \ldots, c$, $j = 1, \ldots, n$.

(b) If it $\rho$ is acyclic, then $\text{tor}(M^3, \rho) = (\omega - 1)^{-n\chi(\hat{\Sigma})} \prod_{i=1}^{c} \prod_{j=1}^{n} \frac{1}{(\lambda_{i,j}^{r_i} \omega^{s_i} - 1)}$.

Not being a solid torus $S^1 \times D^2$ nor a thick torus $S^1 \times S^1 \times [0, 1]$ implies that $\chi(\hat{\Sigma}) < 0$.

When the base $\Sigma$ is orientable, this is a straightforward consequence of Examples 2.19, 2.16, and 2.17.

When $\Sigma$ is not orientable, choose a reversing orientation curve $\sigma \subset \hat{\Sigma}$, so that a tubular neighborhood of $\Sigma$ is a Möbius band and its complement is orientable, with the same Euler characteristic. By using the fibration, this decomposes $M^3$ as a Seifert manifold $N$ with orientable base and $Q$, the orientable circle bundle over the Möbius strip, with $N \cap Q = \partial Q \cong S^1 \times S^1$. In particular $\pi_1 N$ surjects onto $\pi_1 M^3$, so the induced representation on $N$ is irreducible and we use the theorem in the case with orientable base. The curve $\sigma$ may be chosen so that $\rho|_{\pi_1 Q}$ is non-trivial, hence $Q$ and $\partial Q$ are acyclic, and the assertion about acyclicity follows from Mayer-Vietoris (Proposition 2.10) and the case with orientable base. In addition, in the acyclic case, the torsion of the 2-torus is trivial (Example 2.17), and so is the torsion of $Q$ that retracts to a Klein bottle (hence it is also a mapping torus). Thus $\text{tor}(M^3, \rho) = \text{tor}(N, \rho)$ and the proof is concluded.

**Remark 2.21.** For a Seifert fibered manifold, the torsion is constant on the components of the variety of representations.

See [105][106] for the asymptotic behavior of these torsions.

**Example 2.22 (Torus knots).** The previous example may be applied to a torus knot, with coefficients $p, q$, that are relatively prime positive integers. It is Seifert fibered, with base a disc and two singular fibres, with coefficients $(p, 1)$ and $(q, 1)$. Its components of the variety of representations in $\text{SL}_2(\mathbb{C})$ are determined by the choice of the eigenvalues of the corresponding elements, $\{e^{\pm \pi i k_1 \theta / p}\}$ and $\{e^{\pm \pi i k_2 \theta / q}\}$, with $0 < k_1 < p$ and $0 < k_2 < q$ and $k_1 \equiv k_2 \mod 2$. As the fibre is mapped to $(-1)^{k_1}$ times the identity, the representation is acyclic only for $k_1$ odd. The torsion is

$$\frac{1}{\left(1 - \cos \frac{\pi k_1}{p}\right)\left(1 - \cos \frac{\pi k_2}{q}\right)}.$$ 

The description of the components of the variety of representations in $\text{SL}_3(\mathbb{C})$ is more involved [75], as there are much more possibilities for the eigenvalues. For instance, for the trefoil knot $(p = 2, q = 3)$ there are no acyclic irreducible representations in $\text{SL}_3(\mathbb{C})$.  

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For $p = 2$, $q = 5$, there are two components of acyclic representations in $\text{SL}_3(\mathbb{C})$ \[\rho\]. For one of the components its eigenvalues are $\{e^{2\pi i/3}, e^{-2\pi i/3}, e^{2\pi i/3}\}$ and $\{e^{2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3}\}$, for the other, the complex conjugates. For both the torsion is

$$\frac{4/3}{1 + 2\cos \frac{2\pi}{3}} = 2 - \frac{2}{3}\sqrt{5}.$$ 

See \[103\] for a discussion on the torsion of torus knots.

**Example 2.23** (Volume form on representations of surfaces). Let $\Sigma$ denote a surface of genus $g > 1$ and $G$ a compact Lie group. Denote by $X(\Sigma, G)$ the variety of representations of $\pi_1\Sigma$ in $G$ up to conjugacy. Let $X^*(\Sigma, G)$ denote the subset of representations such that the infinitesimal commutator of the image is trivial. This is equivalent to $H^0(\Sigma; \text{Ad}\rho) = 0$, and for $G$ linear, this holds true when $\rho$ is irreducible.

If $\chi_\rho \in X^*(\Sigma, G)$, by definition $H^0(\Sigma; \text{Ad}\rho) = 0$. Hence $H^2(\Sigma; \text{Ad}\rho) = 0$ and $H^1(\Sigma; \text{Ad}\rho)$ is identified to the tangent space of $X^*(\Sigma, G)$ at the character of $\rho$ \[34\], see Theorem \[5.8\]. Reidemeister torsion on homology defines a volume form, by the formula of change of basis in homology \[90\]. Namely, if $2r = \dim X^*(\Sigma, G)$,

$$\text{vol}_{\text{tor}}: \wedge^{2r} H^1(\Sigma; \text{Ad}\rho) \to \mathbb{R}$$

$$h_1 \wedge \cdots \wedge h_{2r} \mapsto \left(\text{tor}(\Sigma, \rho, \{h_1, \ldots, h_{2r}\})\right)^{-1}$$

is a well defined linear isomorphism. The sign can be controlled by means of orientation.

The space $X^*(\Sigma, G)$ has a well defined symplectic structure, due to Atiyah-Bott and Goldman \[34\], that we denote by $\omega$. In particular $\frac{\omega}{2\pi}$ is a natural volume form. Witten proved in \[103\] that they are the same form:

**Theorem 2.24** (Witten). For a compact Lie group $G$

$$\frac{1}{(2\pi)^{2r}} \text{vol}_{\text{tor}} = \frac{\omega^r}{r!}.$$ 

The proof uses a symplectic structure on a chain complex, which has been further developed by Sözen, cf \[91, 90\].

**Example 2.25.** Johnson used the point of view of volume for constructing the torsion from a Heegaard splitting, in a hand written paper that unfortunately was never published. Consider a closed 3-manifold $M^3$ with a Heegaard decomposition: i.e. $M^3 = B_1 \cup \Sigma B_2$, where $B_1$ and $B_2$ are handlebodies such that $\Sigma = B_1 \cap B_2 = \partial B_1 = \partial B_2$ is a surface of genus $\geq 2$. This yields a commutative diagram of fundamental groups and their representation spaces:

$$\pi_1\Sigma \xrightarrow{i_1} \pi_1 B_1 \xrightarrow{i_{1*}} \pi_1 M$$

$$\pi_1 B_2 \xrightarrow{i_{2*}} \pi_1 B_1 \xrightarrow{j_{1*}} X^*(\Sigma, G)$$

$$X^*(B_1, G) \xrightarrow{j_{1*}} X^*(M^3, G)$$

$$X^*(B_2, G) \xrightarrow{j_{2*}} X^*(M^3, G)$$

for $G = \text{SU}(2)$. Assume we have a representation $\rho \in X^*(M^3, G)$ that is infinitesimally rigid (i.e. $H^1(M^3; \text{Ad}\rho) = 0$.) Then, by using a Mayer-Vietoris
argument, Johnson shows that \( X^*(B_1, G) \) and \( X^*(B_2, G) \) intersect transversally at the character \( \rho_0 \) in \( X^*(\Sigma, G) \). Johnson uses Reidemeister torsions to define volume forms \( \text{vol}_{B_i} \) and \( \text{vol}_{X^*(\Sigma, G)} \) (as the groups involved are free or surface groups). He defines an invariant as the ratio between \( \text{vol}_{\Sigma} \) and \( \text{vol}_{B_1} \wedge \text{vol}_{B_2} \). What he proves is:

\[
\text{tor}(M^3, \rho_0) = \frac{\text{vol}_{B_1} \wedge \text{vol}_{B_2}}{\text{vol}_{\Sigma}},
\]

cf. Proposition 2.10. Under Johnson hypothesis there are finitely many acyclic conjugacy classes of representations in \( \text{SU}(2) \) and he considers the addition of all Reidemeister torsions. Using this Heegaard splitting, this is analogous to Casson’s invariant \([36]\), by taking into account additionally this volume on the varieties of characters.

The point of view of volume à la Johnson has also been used by Dubois in \([21, 22]\), we will comment on it in Example \([5, 13]\).

### 2.3 Analytic torsion

Consider now a smooth compact manifold \( M \) and a representation

\[
\rho: \pi_1 M \to \text{SL}_n(\mathbb{R}).
\]

The manifold \( M \) has a unique \( C^1 \)-triangulation, so one can view it as a CW-complex and compute its torsion. Consider the associated flat bundle

\[
E_{\rho} = \widetilde{M} \times \mathbb{R}^n / \pi_1 M,
\]

where \( \pi_1 M \) acts on the universal covering \( \widetilde{M} \) by deck transformations and on \( \mathbb{R}^n \) via \( \rho \). The space of \( E_{\rho} \)-valuated differential forms is denoted by \( \Omega^p(E_{\rho}) \), and its de Rham cohomology by \( H^*(M; E_{\rho}) \). To simplify, we assume that \( \rho \) is acyclic, namely, by de Rham theorem we assume that

\[
H^*(M; E_{\rho}) \cong H^*(M; \rho) = 0.
\]

We choose a Riemannian metric \( g \) on \( M \) and a metric \( \mu \) on the bundle \( E_{\rho} \) (notice that since we do not assume \( \rho \) to be orthogonal, perhaps the metric \( \mu \) cannot be chosen to be flat). This yields a metric on \( \Omega^p(E_{\rho}) \). Using it, we may define the adjoint to the differential and the Laplacian \( \Delta_{\rho}(\rho) \) on \( \Omega^p(E_{\rho}) \). As it is an elliptic operator, it has a discrete spectrum \( 0 < \lambda_0 \leq \lambda_1 \leq \cdots \to \infty \). The zeta function is defined on the complex half-plane \( \text{Re}(s) \geq n/2 \):

\[
\zeta_{\rho}(s) = \sum_{\lambda_i} \lambda_i^{-s}
\]

and it extends to the a meromorphic function on the complex plane, homomorphic at \( s = 0 \).

**Definition 2.26.** The analytic torsion is defined as:

\[
\text{tor}_a(M, \rho, g, h) = \exp \left( \frac{1}{2} \sum_{p=0}^{\dim M} (-1)^p p \zeta_{\rho}'(0) \right).
\]

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In [71] Müller proves that if \( \dim M \) is odd and, as we assume \( \rho \) is acyclic, then it is independent of \( g \) and \( h \). It can also be defined using the trace of the heat operator:

\[
\text{tor}_{\text{an}}(M, \rho) = \exp \left( \frac{1}{2} \sum_{p=0}^{\dim M} (-1)^p \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^\infty \left( \text{Tr}(e^{-t\Delta_\rho}) e^{st} \right) dt \right) \right)_{s=0}. \tag{12}
\]

In the following theorem notice that we use the convention for Reidemeister torsion opposite to the usual in analytic torsion.

**Theorem 2.27** ([13, 71]). Let \( M \) be a closed hyperbolic manifold of odd dimension and \( \rho : \pi_1(M) \to \text{SL}_n(\mathbb{R}) \). Then

\[
\text{tor}_{\text{an}}(M, \rho) = \frac{1}{|\text{tor}(M, \rho)|}.
\]

This theorem was first proved by Cheeger [13] and Müller [70] independently for orthogonal representations, and later by Müller [71] for unimodular ones. In addition, acyclicity of \( \rho \) is not required by choosing orthonormal harmonic basis in cohomology.

This subsection concludes recalling a theorem of Fried. Let \( \rho : \pi_1 M \to \text{SO}(n) \) be a representation of a hyperbolic manifold. For \( s \in \mathbb{C} \) with \( \text{Re}(s) \) sufficiently large, consider

\[
R_\rho(s) = \prod_\gamma \det(\text{Id} - \rho(\gamma) e^{-s l(\gamma)})
\]

where the product runs over the prime, closed geodesics of \( M \) and \( l(\gamma) \) denotes the length of \( \gamma \). This is called the Ruelle zeta function.

**Theorem 2.28** ([32]). Let \( M \) be a closed hyperbolic manifold of odd dimension and assume that \( \rho : \pi_1 M \to \text{SO}(n) \) is acyclic. Then \( R(s) \) extends meromorphically to \( \mathbb{C} \) and

\[
R_\rho(0) = \text{tor}_{\text{an}}(M, \rho)^2.
\]

This theorem has been extended by Wotzke [104] to other representations of hyperbolic manifolds, see also [72].

### 3 Torsion of hyperbolic three-manifold with representations in \( \text{SL}_2(\mathbb{C}) \)

This section is devoted to the torsion of orientable hyperbolic 3-manifolds, using the representations in \( \text{SL}_2(\mathbb{C}) \) obtained as a lift of the holonomy representation (the choice of the lifts depends on the choice of a spin structure). It starts with closed manifolds, for which this representation is acyclic. Then it considers manifolds of finite volume with one cusp, for which it is also acyclic. Besides the invariant itself, it analyzes the function on the variety of characters defined by the torsion. This is applied for instance to study the behavior of torsion under Dehn filling.
3.1 Torsions from lifts of the holonomy representation

Let $M^3$ be a closed, orientable hyperbolic 3-manifold. Its holonomy representation is unique up to conjugation:

$$\text{hol} : \pi_1 M^3 \to \text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}_2(\mathbb{C}).$$

To get a natural representation in a linear group one can lift the holonomy to $\text{SL}_2(\mathbb{C})$. Such a lift always exists [15] and it depends naturally on a choice of a spin structure, because the group of isometries is naturally identified with the frame bundle of $\mathbb{H}^3$ and $\text{SL}_2(\mathbb{C})$ with the spin bundle, cf. [63].

3.1.1 Lifts of the holonomy

There is natural action of $H^1(M^3; \mathbb{Z}/2\mathbb{Z})$ on the set of lifts $\varphi$ to $\text{SL}_2(\mathbb{C})$ of the holonomy representation: viewing $H^1(M^3; \mathbb{Z}/2\mathbb{Z})$ as morphisms from $\pi_1 M^3$ to $\mathbb{Z}/2\mathbb{Z}$, a morphism $\varphi : \pi_1 M^3 \to \mathbb{Z}/2\mathbb{Z}$ maps a representation $\varphi$ to $(-1)^\varphi \varphi$.

Proposition 3.1. (a) There is a natural bijection between the set of lifts of the holonomy representation and the set of spin structures. This is an isomorphism of affine spaces on the vector space $H^1(M^3; \mathbb{Z}/2\mathbb{Z})$.

(b) If $M^3$ and $\overline{M}^3$ are the same manifold with opposite orientations, then there is a natural bijection between spin structures on $M^3$ and on $\overline{M}^3$ so that lifts of the holonomy correspond to complex conjugates.

Item (a) can be found essentially in [15], see also [63].

A quick way of proving Item (b) is using the bijection of Item (a), knowing that complex conjugation in $\text{PSL}_2(\mathbb{C})$ is the result of composing with an isometry that changes the orientation of $\mathbb{H}^3$. On the other hand, this bijection can be constructed explicitly from frame bundles and spin, but details are not given here.

For a spin structure $\sigma$, the corresponding lift of the holonomy, according to Proposition 3.1(a), will be denoted by

$$\varphi_\sigma : \pi_1 M^3 \to \text{SL}_2(\mathbb{C}).$$

The behavior of torsion by mutation is also interesting. Mutation is the operation that consists in cutting along a genus two surface, applying the hyperelliptic involution and gluing again. Notice that one does not require the surface to be essential, thus the genus two surface can be replaced by a (properly embedded) torus with one or two punctures or a sphere with three or four punctures (in particular a Conway sphere for a knot exterior). See [27].

Let $(M^3)^\mu$ denote the result of mutation, by [57] $(M^3)^\mu$ is hyperbolic with the same volume as $M^3$.

Remark 3.2. There is a natural correspondence between the spin structures on $M^3$ and the spin structures on $(M^3)^\mu$.

Here is an explanation of the remark, using the natural bijection between lifts of the holonomy and spin structures in Proposition 3.1. Assume that $\Sigma$ separates $M^3$ in two components $M_1$ and $M_2$. Then

$$\pi_1 M^3 \cong \pi_1 M_1 \ast_{\pi_1 \Sigma} \pi_1 M_2.$$
If $\varrho_\sigma : \pi_1(M^3) \to \text{SL}_2(\mathbb{C})$ is a lift of the holonomy, then $\varrho_\sigma(\mu) : \pi_1(M^3)^\mu \to \text{SL}_2(\mathbb{C})$ is defined so that $\varrho_\sigma(\mu)|_{\pi_1(M_1)} = \varrho_\sigma|_{\pi_1(M_1)}$ and $\varrho_\sigma(\mu)|_{\pi_1(M_2)}$ is conjugate to $\varrho_\sigma|_{\pi_1(M_2)}$ by a matrix in $\text{PSL}_2(\mathbb{C})$ that realizes the involution $\mu$ on $\Sigma$. When $\Sigma$ does not separate, then the construction is similar from the presentation of $\pi_1M^3$ as HNN-extension.

### 3.1.2 Torsions for closed 3-manifolds

The following theorem is a particular case of Raghunathan’s. With other cohomology results, it is discussed in Appendix [5.2]. In particular the following theorem is stated in Corollary [5.2].

**Theorem 3.3.** Let $M^3$ be a closed, orientable, hyperbolic 3-manifold. Then any lift of its holonomy representation is acyclic.

With Mostow rigidity, this yields immediately a topological invariant of the spin manifold.

**Definition 3.4.** Let $M^3$ be a compact, oriented, hyperbolic 3-manifold with spin structure $\sigma$. The torsion of $(M^3, \sigma)$ is defined as

$$
\tau(M^3, \sigma) := \text{tor}(M^3, \varrho_\sigma) \in \mathbb{C}^*
$$

where $\varrho_\sigma = \text{hol}$ is the lift of the holonomy hol corresponding to the spin structure $\sigma$.

**Remark 3.5.** There is no sign indeterminacy, i.e. it is a well defined complex number, because $\varrho_\sigma$ is a representation in $\mathbb{C}^2$, which is even dimensional, and $\chi(M^3) = 0$, see Remark [2.7].

Here are some of its properties.

**Proposition 3.6.** The torsion $\tau(M^3, \sigma)$ in Definition [3.4] has the following properties:

(a) It is a topological invariant of the spin manifold $(M^3, \sigma)$.

(b) There are examples of manifolds $M^3$ with two spin structures $\sigma$ and $\sigma'$ such that $\tau(M^3, \sigma) \neq \tau(M^3, \sigma')$.

(c) Let $\overline{M^3}$ denote the manifold $M^3$ with opposite orientation. If one changes the orientation and the spin structure accordingly as in Proposition [3.7] then the torsion is the complex conjugate $\tau(\overline{M^3}, \overline{\sigma}) = \overline{\tau(M^3, \sigma)}$.

(d) Let $(M^3)^\mu$ denote the result of mutation. If $\sigma^\mu$ denotes the corresponding spin structure as in Remark [3.2] then

$$
\tau((M^3)^\mu, \sigma^\mu) = \tau(M^3, \sigma).
$$

(e) Let $M^3$ be an oriented hyperbolic manifold with one cusp and with spin structure $\sigma$. The set of modules of the torsions obtained by Dehn filling on $M^3$, $|\tau(M^3, \sigma)|$ so that $\sigma$ extends to $M^3_{p/q}$, is dense in the interval

$$
\left[ \frac{1}{4} |\tau(M^3, \sigma)|, +\infty \right).
$$
Item (a) follows from uniqueness of the hyperbolic structure, by Mostow rigidity. To prove (b) it suffices to compute an example, this is done in Corollary 3.27. Item (c) is straightforward from Proposition 3.1. Item (d) is proved in [61]. Finally, (e) is proved later when discussing cusped manifolds, as this will follow immediately from a surgery formula.

Item (e) shows that this torsion is not obviously related to the hyperbolic volume. The following theorem, a particular case of [6, Thm. 4.5], finds a relation (we use the convention of torsion opposite to [6]):

**Theorem 3.7** (Bergeron and Venkatesh, [6]). Let $M_3$ be a compact oriented hyperbolic 3-manifold. Assume that $M_3^\ast \to M_3$ is a sequence of coverings such that the injectivity radius of $M_3^\ast$ converges to infinity. Then

$$\lim_{n \to \infty} \frac{\log |\tau(M_3^\ast, \sigma)|}{\log \deg(M_3^\ast \to M_3)} = \frac{11}{12\pi} \log \text{vol}(M_3).$$

Equivalently;

$$\lim_{n \to \infty} \frac{\log |\tau(M_3^\ast, \sigma)|}{\log \deg(M_3^\ast \to M_3)} = \frac{11}{12\pi} \log \text{vol}(M_3).$$

This theorem relies on analytic torsion and on $L^2$-torsion, as $\frac{11}{12\pi}$ is the $L^2$-torsion of $\mathbb{H}^3$. The proof uses the $L^2$-Laplacian of hyperbolic space, and it is based on approximations of averages of the trace of the difference of heat kernels, see Equation (12). They require the notion of strong acyclicity (the property in Theorem B.1) to avoid eigenvalues of the Laplacian approaching to zero. This has been generalized in [1], in particular without requiring that the $M_3^\ast$ are coverings. See also [73].

### 3.1.3 Cusped hyperbolic 3-manifolds

In this subsection, $M^3$ denotes a finite volume hyperbolic manifold, i.e. a manifold whose ends are cusps.

**Assumption 3.8.** Assume that $M^3$ has a single cusp.

This is done not only to simplify notation, but because with more cusps some further issues need to be discussed [63]. Again one has:

**Theorem 3.9.** Let $M^3$ be a closed, orientable, hyperbolic 3-manifold with one cusp. Then any lift to $\text{SL}_2(\mathbb{C})$ of its holonomy representation is acyclic.

This is proved for instance in [62], it is a particular case of Theorem B.7 in Appendix B. With more cusps this is may not hold true for all lifts of the holonomy representation, i.e. for all spin structures. It is true provided that for each cusp the trace of the peripheral elements is not identically $+2$ (for some elements it is $-2$). This is always the case if there is a single cusp [62, 10].

One may as well define the same torsion as in Definition 3.4:

**Definition 3.10.** Let $M^3$ be a compact, oriented, hyperbolic 3-manifold with one cusp, and let $\sigma$ denote a spin structure on $M^3$. The torsion of $(M^3, \sigma)$ is defined as

$$\tau(M^3, \sigma) := \text{tor}(M^3, \sigma) \in \mathbb{C}^\ast,$$

where $\sigma = \text{hol}$ is the lift of the holonomy corresponding to $\sigma$.

This torsion has the same properties as in the closed case, Proposition 3.6.
3.1.4 The twisted polynomial

It is relevant to mention the twisted polynomial for hyperbolic knot exteriors corresponding to a lift of the holonomy $\varrho$ constructed by Dunfield, Friedl, and Jackson in [26]. Given a hyperbolic knot $K \subset S^3$, choose the spin structure on $M^3 = S^3 - K$ such that the trace of the meridian is $+2$ (the trace of the longitude is always $-2$ by [10], see also [62, Corollary 3.10]) and consider the abelianization $\phi : \pi_1 M^3 \to \mathbb{Z}$. Dunfield, Friedl, and Jackson study the polynomial

$$\Delta_K(t) := \Delta_{M^3, \varrho \phi}(t)$$

following the notation of Subsection 2.2.2. By Proposition 2.12 its degree is $\leq 2(2g(K) + 1)$, where $g(K)$ is the genus of the knot, i.e., the minimal genus of a Seifert surface. Numerical evidence (knots up to 15 crossings) yield them to conjecture:

**Conjecture 3.11 ([26]).** For a hyperbolic knot $K$:

(a) $\deg \Delta_K(t) = 2(2g(K) + 1)$.

(b) $\Delta_K$ is monic if and only if $K$ is a fibered knot.

The equality of the degree has been proved by Morifuji and Tan for some families of two bridge knots, see [69] and references therein. Agol and Dunfield have shown:

**Theorem 3.12 ([2]).** For libroid hyperbolic knots,

$$\deg \Delta_K(t) = 2(2g(K) + 1).$$

Being libroid means the existence of a collection of disjointly embedded minimal genus Seifert surfaces in the exterior of the knot so that their open complement is a union of books of $I$-bundles, in a way that respects the structure of sutured manifold. See [2].

In the remarkable paper [26] the authors also conjecture that being monic determines whether the knot is fibered or not, and rise many interesting questions about this polynomial and its relationship with other invariants.

3.2 Torsion on the variety of characters

Let $M^3$ be a hyperbolic, oriented manifold with one cusp. A relevant difference with the closed case is the fact that the holonomy of $M^3$ can be deformed in the variety of representations (to holonomies of non-complete structures).

3.2.1 The distinguished curve of characters

The variety of $\text{SL}_2(\mathbb{C})$-representations of $M^3$ is the set

$$\text{hom}(\pi_1 M^3, \text{SL}_2(\mathbb{C})), $$

which it is an affine algebraic set: if a generating set of $\pi_1 M^3$ has $k$ elements, then $\text{hom}(\pi_1 M^3, \text{SL}_2(\mathbb{C}))$ embeds in $\text{SL}_2(\mathbb{C})^k \subset \mathbb{C}^{4k}$, by mapping a representation to the image of its generators. The algebraic equations are induced by the relations of the group.
The group \( \text{PSL}_2(\mathbb{C}) \) acts on \( \text{hom}(\pi_1 M^3, \text{SL}_2(\mathbb{C})) \) by conjugation, and the affine algebraic quotient is the variety of characters

\[
X(M^3) := X(M^3, \text{SL}_2(\mathbb{C})) = \text{hom}(\pi_1 M^3, \text{SL}_2(\mathbb{C}))/\text{PSL}_2(\mathbb{C}).
\]

This is defined in terms of the invariant functions: \( X(M^3, \text{SL}_2(\mathbb{C})) \) is the algebraic affine set whose function ring \( \mathbb{C}[X(M^3, \text{SL}_2(\mathbb{C}))] \) is the ring of invariant functions

\[
\mathbb{C}[\text{hom}(\pi_1 M^3, \text{SL}_2(\mathbb{C}))]/\text{PSL}_2(\mathbb{C}).
\]

By [93], see also [7, Appendix B] each component that contains the lift of the holonomy of \( M^3 \) is a curve.

**Definition 3.13.** An irreducible component of \( X(M^3, \text{SL}_2(\mathbb{C})) \) that contains a lift of the holonomy is called a distinguished component and it is denoted by \( X_0(M^3) \).

For many manifolds, e.g. for 2-bridge knot exteriors, there is a unique distinguished component. A priori there could be more components, but the definition makes sense because they would be isomorphic. More precisely, there are two characters of the holonomy representation in \( \text{PSL}_2(\mathbb{C}) \) that are complex conjugate from each other, that correspond to the different orientations. When lifting them to \( \text{SL}_2(\mathbb{C}) \), this gives \( 2|H^1(M^3; \mathbb{Z}/2\mathbb{Z})| \) characters, two for each spin structure. The corresponding components \( X_0(M^3) \) are isomorphic by means of the natural action of \( H^1(M^3; \mathbb{Z}/2\mathbb{Z}) \) and complex conjugation.

Recently, Casella, Luo, and Tillmann [11] have shown an example of hyperbolic manifold with one cusp \( M^3 \) such that the holonomy characters of the different orientations lie in different components of \( X(M^3, \text{PSL}_2(\mathbb{C})) \). To my knowledge, the following question is still open:

**Question 3.14.** Once \( M^3 \) is oriented, are all the lifts of the oriented holonomy contained in a single irreducible component of \( X(M^3, \text{SL}_2(\mathbb{C})) \)?

The distinguished component \( X(M^3, \text{SL}_2(\mathbb{C})) \) is a curve and it was studied by Thurston in his proof of the hyperbolic Dehn filling theorem [93]. More precisely, in a neighborhood of the holonomy of the complete structure of \( M^3 \), the representations are holonomies of incomplete structures, and in some cases the completion is a Dehn filling. This is discussed in Paragraph 3.2.3.

### 3.2.2 Torsion on the distinguished curve of characters

We say that a character \( \chi \in X(M^3) \) is trivial if it takes values in \( \{\pm 2\} \), i.e. it is a lift or the trivial character in \( \text{PSL}_2(\mathbb{C}) \).

An irreducible character is the character of a unique conjugacy class of representations. A reducible character can correspond to more conjugacy classes, but if the character is non-trivial, then either all representations with this character are acyclic, either none of them is (Lemma A.3).

**Definition 3.15.** Define the torsion function on \( X_0(M^3) \), minus the trivial character:

\[
T_M(\chi_\rho) = \begin{cases} 
\text{tor}(M^3, \rho) & \text{if } \rho \text{ is acyclic;} \\
0 & \text{if } \chi_\rho \text{ is non-trivial and } \rho \text{ non-acyclic},
\end{cases}
\]

where \( \chi_\rho \) denotes the character of \( \rho \).
Proposition 3.16. For a hyperbolic oriented manifold with one cusp, the torsion defines a rational function on $X_0(M^3)$, $\mathbb{T}_M \in \mathbb{C}(X_0(M^3))$, which is regular away from the trivial character.

In particular, if the trivial character is not contained in $X_0(M^3)$ (e.g. if $b_1(M^3) = 1$), then $\mathbb{T}_M \in \mathbb{C}[X_0(M^3)]$, i.e. it is holomorphic (with no poles)

$$\mathbb{T}_M : X_0(M^3) \to \mathbb{C}.$$

Remark 3.17. We prove that the trivial representation cannot be approached by irreducible ones when $b_1(M^3) = 1$ in Appendix A. This is always the case when $M^3$ is a knot exterior.

Proof. The dimension of each cohomology group is upper semi-continuous on the representation (see [42, Lemma 3.2], this is a particular case of the semi-continuity theorem [38, Ch. III, Theorem 12.8]). With Lemma A.3, we can conclude that acyclicity holds true in a dense open Zariski domain after removing the trivial character if required. The fact that the function $\mathbb{T}_M$ is algebraic on this domain is clear, as this is defined from polynomials on the entries of $\rho$. Invariance by conjugation is one of the properties of the torsion. Thus it remains to deal with the points where it is not acyclic. Recall that by Lemma A.3 a representation with nontrivial character is acyclic if and only all representations with the same character are.

First notice that $H^0(M^3; \rho)$ is trivial when the character $\chi_\rho$ is non-trivial, because this cohomology group is naturally isomorphic to the space of invariants $\mathbb{C}^{\rho(\pi_1, M^3)}$. More precisely, $\mathbb{C}^{\rho(\pi_1, M^3)}$ is non-trivial only when all elements in $\rho(\pi_1, M^3)$ have 1 as eigenvalue, which means that their trace is 2, i.e. $\chi_\rho$ is trivial. Thus by duality $H_0(M^3; \rho) = 0$ when $\chi_\rho$ is non-trivial.

Now fix a representation $\rho_1$ which is not acyclic and non-trivial. Non-acyclicity implies that $H_1(M^3; \rho_1) \neq 0$ and $H_2(M^3; \rho_1) \neq 0$, as $H_0(M^3; \rho_1) = 0$, the homotopical dimension of $M^3$ is 2, and $\chi(M^3) = 0$. Notice that $M^3$ has the simple homotopy type of a 2-complex (see [80, Page 54] for instance), that can be used to compute the torsion. Using the notation of Section 2.1, fix a basis $\{v_1, v_2\}$ for $\mathbb{C}^2$ and lifts of cells $\tilde{e}_j^i$ of a triangulation of $M^3$. Then define a family of basis $c_i(\rho)$ by varying the representation in $v_k \Phi_\rho \tilde{e}_j^i$. Now choose $\tilde{b}_1(\rho) = c_2(\rho)$ and $b_0(\rho)$, a linear combination of $c_1(\rho)$ with constant coefficients (though $c_1(\rho)$ changes with $\rho$), so that $\partial(b_0(\rho_1)) = c_0(\rho_1)$. The function

$$\rho \mapsto [\partial \tilde{b}_1(\rho) \cup \tilde{b}_0(\rho), c_1(\rho)]/[\partial b_0(\rho), c_0(\rho)]$$

(16)

is well defined in the set where its denominator does not vanish. This is a Zariski open set that contains $\rho_1$. On this set $\partial \tilde{b}_1(\rho) = \partial v_2(\rho)$ has maximal rank iff $\rho$ is acyclic, thus the function (16) vanishes when $\rho$ is not acyclic, and when $\rho$ is acyclic (16) is the torsion.

Example 3.18. For the figure eight knot, in [51] Kitano computes it:

$$\mathbb{T}_M(\rho) = 2 - 2 \text{tr}(\rho(m)) = 2 - 2 \chi_\rho(m),$$

where $m$ denotes the meridian of the knot. Notice that here the function only depends on $\text{tr}(\rho(m))$, namely the evaluation of the character at $m$, and one does not need to describe the variety of characters. In general, as $\text{tr}(\rho(m))$ is a non-constant function on the curve $X_0(M^3)$, $\mathbb{T}_M(\rho)$ and $\chi_\rho(m) = \text{tr}(\rho(m))$ are related by a polynomial equation; compare with [25].
3.2.3 Dehn filling space

Again let $M^3$ be an oriented, finite volume hyperbolic manifold with one cusp. Consider the peripheral torus $T^2$, which is the boundary of a compact core of $M^3$. Choose a frame on the peripheral torus $T^2$, i.e. two simple closed curves that generate $\pi_1 T^2$, denote this frame by $\{m, l\}$. The notation suggests that the canonical choice for a knot exterior is the pair meridian-longitude.

A Dehn filling on $M^3$ is the result of gluing a solid torus $D^2 \times S^1$ to a compact core $M^3$ along the boundary. Up to homeomorphism, this manifold depends only on the (unoriented) homology class in $T^2$ of the meridian, i.e. the curve $\partial D^2 \times \{\ast\}$. This curve is written as $\pm (pm + ql)$, and the Dehn filling is denoted by $M_{p/q}$.

When $|p| + |q|$ is sufficiently large, by Thurston’s theorem $M_{p/q}$ is hyperbolic. To prove it, he introduces the Thurston’s parameters of the Dehn filling space, by writing, for representations close to the holonomy of the complete hyperbolic structure,
\[
\rho(m) = \pm \begin{pmatrix}
e^{u/2} & 1 \\
0 & e^{-u/2}
\end{pmatrix} \quad \rho(l) = \pm \begin{pmatrix}e^{v/2} & f(u) \\
0 & e^{-v/2}
\end{pmatrix}
\]
with $u, v \in \mathbb{C}$ in a neighborhood of the origin. For the holonomy of the complete structure, $u = v = 0$ and write
\[
cs = cs(l, m) = f(0) \in \mathbb{C} - \mathbb{R}.
\]

**Definition 3.19.** The parameter $u \in U \subset \mathbb{C}$ as above is called the Thurston parameter and $cs = cs(l, m) \in \mathbb{C} - \mathbb{R}$ the cusp shape of the complete structure.

The Thurston parameter $u$ is in fact a parameter of a double branched covering of a neighborhood of the lift of the holonomy $g$, as the local parameter of $X_0(M^3)$ is $\text{tr}(\rho(m)) = \pm 2 \cosh \frac{u}{2}$. For the complete structure, the peripheral group acts as a lattice on a horosphere, that we view as $\mathbb{C}$. Up to affine equivalence, this lattice is given by $m \mapsto 1, l \mapsto cs$.

**Lemma 3.20** (Neumann and Zagier [73]). There exists a (standard) neighborhood of the origin $U \subset \mathbb{C}$ such that:

(a) The map $U \rightarrow X_0(M^3)$ such that $u \mapsto \chi_\rho$, where $\rho$ is as in (17) is a double branched covering of a neighborhood in $X_0(M^3)$ of the character of the holonomy of the complete structure.

(b) $\nu$ is an analytic odd function on $u$ that satisfies
\[
\nu(u) = \text{cs} u + O(u^3).
\]

(c) $f(u) = \sinh(v)/\sinh(u)$.

The generalized Dehn filling coefficients are the $(p, q) \in \mathbb{R}^2 \cup \{\infty\}$ such that
\[
pu + qv = 2\pi i
\]
when $u \neq 0$ and $\infty$ when $u = 0$.

**Theorem 3.21**. [72] The generalized Dehn filling coefficients define a homeomorphism between $U$ and a neighborhood of $\infty$ in $S^2 = \mathbb{R}^2 \cup \{\infty\}$. If a pair of coprime integers $(p, q) \in \mathbb{Z}^2$ lies in the image of this homeomorphism, then $M_{p/q}^3$ is hyperbolic, with holonomy whose restriction to $M^3$ satisfies [13].
We are interested in properties of $M_{p/q}^3$.

**Theorem 3.22** (Neumann and Zagier [79]).

$$\text{vol}(M_{p/q}^3) = \text{vol}(M^3) - \pi^2 \frac{\text{Im}(cs)}{|p + cs q|^2} + O\left(\frac{1}{|p + cs q|^4}\right).$$

We are also interested in the complex length of the soul of the solid torus added by Dehn filling, which is a short geodesic. Let $r$ and $s$ be integers satisfying $ps - qr = 1$, this complex length is $ru + sv$. A straightforward computation yields [82]:

**Remark 3.23.** The complex length of the core of the solid torus $M_{p/q}^3$ is

$$ru + sv = 2\pi \left(\frac{r}{p} + \frac{s}{q}\right) = 2\pi \left(\frac{s}{q} + \frac{u}{p}\right).$$

Given a spin structure $\sigma$ on $M^3$, it may extend or not to $M_{p,q}^3$. It is easy to give a characterization using the bijection of Proposition 3.1.

**Lemma 3.24.** [63] A spin structure $\sigma$ on $M$ extends to $M_{p,q}^3$ iff the corresponding lift of the holonomy $\gamma_\sigma$ (for the complete structure on $M^3$) satisfies

$$\text{tr}(\gamma_\sigma (pm + ql)) = -2.$$  

### 3.2.4 Torsion for Dehn fillings

Let $|p| + |q|$ be sufficiently large and let $\sigma$ be a spin structure on $M^3$ extendible to $M_{p/q}^3$. Let $\gamma_{p/q,\sigma}$ denote the lift of the holonomy of the restriction to $M^3$ of the hyperbolic structure on $M_{p/q}^3$, and $\chi_{p/q,\sigma}$ the corresponding character. Let $\lambda_{p/q}$ denote the complex length of the soul of the filling torus. There is a surgery formula:

**Proposition 3.25.** Let $M^3$ be as above. For $|p| + |q|$ sufficiently large:

$$\tau(M_{p/q}^3, \sigma) = \frac{T_M(\chi_{p/q, \sigma})}{2(1 - \cosh(\lambda_{p/q}/2))}.$$  

This proposition can be proved using Mayer-Vietoris (Proposition 2.10) to the pair $(M^3, D^2 \times S^1)$. The proof can be found for instance in [49] for the figure eight knot, and in [44] for twist knots, but it applies to every cusped manifold, and it is essentially done too in [35, Proposition 4.10]. The only computation required is the torsion of the solid torus added by Dehn filling, or its core geodesic, which is (by Example 2.10):

$$\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} e^{\lambda_{p/q}/2} & 0 \\ 0 & e^{-\lambda_{p/q}/2} \end{pmatrix} = \frac{1}{2(1 - \cosh(\lambda_{p/q}/2))}.$$  

**Remark 3.26.** The complex length $\lambda_{p/q}$ is defined up to addition of a term in $2\pi i \mathbb{Z}$. The spin structure (or equivalently a choice of lift of the holonomy) determines $\lambda_{p/q}$ up to $4\pi i \mathbb{Z}$, as the holonomy of this curve is conjugate to

$$\begin{pmatrix} e^{\lambda_{p/q}/2} & 0 \\ 0 & e^{-\lambda_{p/q}/2} \end{pmatrix}. $$
Corollary 3.27. There exist closed hyperbolic manifolds $N^3$ with two spin structures $\sigma$ and $\sigma'$ so that $\tau(N^3, \sigma) \neq \tau(N^3, \sigma')$.

Proof. The manifolds are obtained by Dehn filling on the figure eight knot. Consider the sequence $p_n/q_n = 2n$ so that both spin structures on the figure eight knot exterior $M^3$ extend to $M^3_{p_n/q_n}$, i.e. $H_1(M^3_{2n}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. The corresponding lifts differ on the sign of the trace of the meridian. As the core of the solid torus is homologous to the meridian, the sign of the traces of these cores differ. Hence for one of the lifts the complex length (modulo $4\pi i\mathbb{Z}$) is $\lambda_{2n}$, for the other one it is $\lambda_{2n} + 2\pi i$ by Remark 3.26. By Remark 3.23 one may approximate

$$\lambda_{2n} = \frac{2\pi i}{2n} + O\left(\frac{1}{n^2}\right).$$

The goal is to show that the limit

$$\lim_{n \to +\infty} \frac{\tau(M^3_{2n}, \sigma)}{\tau(M^3_{2n}, \sigma')}$$

is not 1. By Proposition 3.25 it is the product of the limits:

$$\lim_{n \to +\infty} \frac{1 - \cosh(\lambda_{2n}/2)}{1 - \cosh((\lambda_{2n} + 2\pi i)/2)} = \lim_{n \to +\infty} \frac{1 - \cos \frac{\pi}{2n}}{1 + \cos \frac{\pi}{2n}} = 0$$

(21)

(using the approximation (19)) and

$$\lim_{n \to +\infty} \frac{T^M(\chi_{2n, \sigma})}{T^M(\chi_{2n, \sigma}')} = \frac{\tau(M^3, \sigma)}{\tau(M^3, \sigma')}$$

(22)

As for one of the spin structures the trace of the meridian is 2 and for the other $-2$, by Example 3.18 the limit in (22) is $(2 - 2 \cdot (-2))/(2 - 2 \cdot 2) = -3$. Thus the limit (21) vanishes, hence it is not 1.

The previous argument applies to any knot exterior, the precise value of the torsion is not needed. Corollary 3.27 holds true also for cusped manifolds, by applying an analogous formula for partial surgery.

Corollary 3.28. The set of modules of the torsions $|\tau(M^3_{p/q}, \sigma)|$ obtained by Dehn filling on $M^3$, so that the spin structure $\sigma$ on $M^3$ extends to $M^3_{p/q}$, is dense in

$$\left[\frac{1}{4}|\tau(M^3, \sigma)|, +\infty\right).$$

Proof. By the surgery formula, Proposition 3.25 it suffices to show that

$$|1 - \cosh(\lambda_{p/q}/2)|$$

is dense in the interval $[0, 2]$. Let $r, s \in \mathbb{Z}$ be such that $ps - qr = 1$, by Remark 3.23 $\lambda_{p/q} = 2\pi i \frac{r}{p} + \frac{u}{p}$ and $u \to 0$. Then the result follows from the density of $\cos(\pi \lambda^2_p)$ in $[-1, 1]$. Notice that extendibility of $\sigma$ is just determined by a condition $ap + bq \equiv 0 \mod 2$, for $a, b \in \mathbb{Z}/2\mathbb{Z}$ depending on $\sigma$.

As the surgery formula shows, this density is a consequence of the contribution of the core geodesics. Notice that in Theorem 3.7 the hypothesis require the length of geodesics to be bounded below away from zero.
3.2.5 Branched coverings on the figure eight

We next consider another family of examples. We consider it only for the figure eight knot, but it generalizes to other manifolds.

Consider $M_n^3$ the $n$-th cyclic branched covering of $S^3$, branched along the figure eight knot. It is hyperbolic for $n \geq 4$. Its torsion may be computed by means of the twisted Alexander polynomial, by using the formula à la Fox, Proposition 2.15 due to Turaev [95], see also [25, 83, 89, 85, 109]. More precisely, this polynomial is

$$p_k(t) = t^2 - 2 \chi(m)t + 1,$$

by [60], cf. [54]. Notice that its value at $1$ agrees with Example 3.18. The holonomy of $M_n^3$ extends to a holonomy of the quotient orbifold $M_n^3/(\mathbb{Z}/n\mathbb{Z})$ in $\text{PSL}_2(\mathbb{C})$. It induces a representation of the exterior of the not $S^3 - \mathcal{K}$ that lifts to $\text{SL}_2(\mathbb{C})$, that we denote by $\rho_n$. Since a rotation of angle $2\pi/n$ has trace $\pm 2\cos(\pi/n)$, we have

$$\chi_{\rho_n}(m) = \text{tr}(\rho_n(m)) = \pm 2\cos(\pi/n).$$

Using Proposition 2.15 if $\Sigma_n \subset M_n$ is the lift of the branching locus, then

$$\text{tor}(M_n^3 - \Sigma_n, \sigma) = \prod_{k=0}^{n-1} p_{\chi_n}(e^{2\pi i k/n}) = \prod_{k=0}^{n-1} \left( e^{4\pi i k/n} \mp 4\cos(\pi/n) e^{2\pi i k/n} + 1 \right).$$

By the surgery formula (Proposition 3.25) and knowing that the complex length of the core geodesic is $\approx \sqrt{\frac{2}{\pi}} + O \left( \frac{1}{n^2} \right)$ (Remark 3.23),

$$\text{tor}(M_n^3, \sigma) = \frac{1/2}{1 - \cosh \left( \sqrt{\frac{2}{\pi}} + O \left( \frac{1}{n^2} \right) \right)} \prod_{k=0}^{n-1} \left( e^{4\pi i k/n} \mp 4\cos(\pi/n) e^{2\pi i k/n} + 1 \right). \quad (23)$$

In particular,

$$\lim_{n \to \infty} \frac{\log |\text{tor}(M_n^3, \sigma)|}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \left| e^{4\pi i k/n} \mp 4\cos(\pi/n) e^{2\pi i k/n} + 1 \right|. \quad (24)$$

Here we used that $\lim_{x \to 0} x \log x = 0$ to get rid of the first term, outside the product, in (23). As $t^2 \mp 4t + 1$ has no roots in the unit circle, the limit (24) converges to

$$\frac{1}{2\pi} \int_{|z|=1} \log |z^2 \mp 4z + 1|.$$

Thus by Jensen’s formula

$$\lim_{n \to \infty} \frac{\log |\text{tor}(M_n^3, \sigma)|}{n} = \log |2 + \sqrt{3}|.$$

This approach uses the ideas on Mahler measure of [89]. It applies to a hyperbolic knot provided that the torsion polynomial has no roots in the unit circle. It is easy to prove that the roots of the polynomial are not roots of the unity, but this does not discard roots in the unit circle. For the adjoint representation, the nonexistence of roots in the unit circle has been proved by Kapovich [46].

25
4 Torsions for higher dimensional representations of hyperbolic three-manifolds

This section is devoted to further Reidemeister torsions naturally associated to hyperbolic three-manifolds, those obtained by using the (finite dimensional and linear) representations of $\text{SL}_2(\mathbb{C})$.

4.1 Representations of $\text{SL}_2(\mathbb{C})$

Let $V_{k+1}$ be the space of degree $k$ homogeneous polynomials in two variables:

$$V_{k+1} := \{ P \in \mathbb{C}[X,Y] \mid P \text{ is homogeneous of degree } k \}.$$

The group $\text{SL}_2(\mathbb{C})$ acts naturally on $V_{k+1}$ by precomposition on polynomials:

$$\text{SL}_2(\mathbb{C}) \times V_{k+1} \rightarrow V_{k+1} \quad (A, P) \mapsto P \circ A^t.$$

We consider the transpose matrix $A^t$ so that the action is on the left. We could also have considered the inverse instead of the transpose, as there exist a matrix $B \in \text{SL}_2(\mathbb{C})$ such that $A^{-1} = BA^tB^{-1}$ for every $A \in \text{SL}_2(\mathbb{C})$. This defines a representation

$$\text{Sym}^k : \text{SL}_2(\mathbb{C}) \rightarrow \text{SL}_{k+1}(\mathbb{C}).$$

Not only $\text{Sym}^k$ has determinant 1, but preserves a non-degenerate bilinear form that is symmetric for $k$ even, and skew-symmetric for $k$ odd. So

$$\text{Sym}^k : \text{SL}_2(\mathbb{C}) \rightarrow \begin{cases} \text{SO}(2l + 1, \mathbb{C}) & \text{for } k = 2l \\ \text{Sp}(2l + 2, \mathbb{C}) & \text{for } k = 2l + 1 \end{cases}$$

For $k = 1$, $\text{Sym}$ is the standard representation and the bilinear form is the determinant of the $2 \times 2$ matrix obtained from two vectors. Then the form invariant for $\text{Sym}^k$ is the $k$-th symmetric product of this determinant.

We shall also consider the complex conjugate $\text{Sym}^k$, which is antiholomorphic.

**Proposition 4.1.** Classification of irreducible representations of $\text{SL}_2(\mathbb{C})$, cf. [8].

(a) Any irreducible holomorphic representation of $\text{SL}_2(\mathbb{C})$ is equivalent to $\text{Sym}^k$ for a unique $k \geq 0$.

(b) Any irreducible representation of $\text{SL}_2(\mathbb{C})$ is equivalent to

$$\text{Sym}^{k_1, k_2} := \text{Sym}^{k_1} \otimes \text{Sym}^{k_2}$$

for a unique pair of integers $k_1, k_2 \geq 0$.

Notice that $\text{Sym}^{k} = \text{Sym}^{k, 0}$ and that $\text{Sym}^{0} = \text{Sym}^{0, 0}$ is the trivial representation in $\mathbb{C}$. The adjoint representation $\text{Ad} : \text{SL}_2(\mathbb{C}) \rightarrow \text{SO}(\mathfrak{sl}_2(\mathbb{C}))$ is equivalent to $\text{Sym}^2$, the invariant form being the Killing form.

The representation $\text{Sym}^{k, k}$ takes values in $\text{SL}_{(k+1)^2}(\mathbb{R})$. More precisely, the image of $\text{Sym}^{k, k}$ is contained in $\text{SO}(p, q)$, with

$$p = \frac{k^2 + 3k + 2}{2} \quad \text{and} \quad q = \frac{k^2 + k}{2}.$$
For instance, \( \text{Sym}^{1,1} : \text{PSL}_2(\mathbb{C}) \to \text{SO}_0(3,1) \) is the canonical isomorphism between different presentations of the group of orientation preserving isometries of hyperbolic 3-space. This representation is used to study infinitesimal deformations of conformally flat structures on a hyperbolic 3-manifold. The representation \( \text{Sym}^{2,2} : \text{PSL}_2(\mathbb{C}) \to \text{SO}_0(6,3) \) is used in the study of infinitesimal deformations of projective structures on a hyperbolic 3-manifold.

Let \( g_\sigma : \pi_1 M^3 \to \text{SL}_2(\mathbb{C}) \) denote a lift corresponding to a spin structure \( \sigma \), we shall denote its composition with \( \text{Sym}^{k_1,k_2} \) by

\[
\hat{g}_\sigma^{k_1,k_2} := \text{Sym}^{k_1,k_2} \circ g_\sigma,
\]

and \( \hat{g}_\sigma^k = \hat{g}_\sigma^0 \).

**Theorem 4.2** (Raghunathan). Let \( M^3 \) be a compact hyperbolic orientable 3-manifold and \( \text{hol} \) a lift of its holonomy. If \( k_1 \neq k_2 \) then

\[
H^*(M^3; \hat{g}_\sigma^{k_1,k_2}) = 0.
\]

This theorem, due to Raghunathan, is commented in Appendix B with other facts about cohomology.

### 4.2 Higher torsions for closed manifolds

From Theorem 4.2 we can define torsions.

**Definition 4.3.** Let \( M^3 \) be a closed oriented hyperbolic three-manifold, with spin structure \( \sigma \), and let \( \hat{g}_\sigma^{k_1,k_2} = \text{Sym}^{k_1,k_2} \circ g_\sigma \) be as in (25).

For \( k_1 \neq k_2 \), define

\[
\tau^{k_1,k_2}(M^3, \sigma) := \text{tor}(M^3, \hat{g}_\sigma^{k_1,k_2}) \in \mathbb{C}^*/\pm 1,
\]

and for \( k \geq 1 \)

\[
\tau^{k}(M^3, \sigma) := \tau^{k,0}(M^3, \sigma) = \text{tor}(M^3, \hat{g}_\sigma^k).
\]

When \( k = 1 \) one obtains the torsion of Definition 3.4

\[
\tau^1(M^3, \sigma) = \tau^{1,0}(M^3, \sigma) = \tau(M^3, \sigma).
\]

**Remark 4.4.**

1. For \( k_1 \) or \( k_2 \) odd, there is no sign indeterminacy as it is a representation in \( \mathbb{C}^{(k_1+1)(k_2+1)} \), which is even dimensional (Remark 2.7).

2. For \( k_1 + k_2 \) even, \( \text{Sym}^{k_1} \circ \text{Sym}^{k_2} \) factors through \( \text{PSL}_2(\mathbb{C}) \), hence it does not depend on the spin structure. In addition, the sign indeterminacy can be avoided by using Turaev’s refined torsion and an orientation in homology, provided by Poincaré duality \([96, 97]\).

3. By construction, \( \tau^{k_1,k_2}(M^3, \sigma) = \tau^{k_2,k_1}(M^3, \sigma) \).

4. When \( k_1 = k_2 \), \( g^{k,k} \) may be not acyclic (for instance when it contains a totally geodesic surface \([64]\), but sometimes it can be acyclic (e.g. almost all Dehn fillings on two bridge knots for \( k = 1 \) by \([13, 88, 28]\) or on the figure eight knot for \( k = 2 \) by \([12]\)). When \( g^{k,k} \) is acyclic, then \( \tau^{k,k}(M^3, \sigma) \) is also well defined.
5. Changing the orientation. Let $M^3$ and $\overline{M}^3$ denote the same manifold with opposite orientations and let $\sigma$ and $\overline{\sigma}$ denote the corresponding spin structures as in Proposition 3.1. Then

$$\tau^{k_1,k_2}(M^3, \overline{\sigma}) = (-1)^{b_1(M^3)(k_1+k_2)} \tau^{k_1,k_2}(M^3, \sigma),$$

where $b_1(M^3)$ denotes the first Betti number. Hence if $M^3$ is amphichiral, then $\tau^{k_1,k_2}(M^3, \sigma)$ is real for $b_1(M^3)(k_1+k_2)$ even and purely imaginary for $b_1(M^3)(k_1+k_2)$ odd.

In [72] W. Müller found a beautiful theorem about the asymptotic behavior when $k \to \infty$:

**Theorem 4.5 ([72]).** Let $M$ be a compact, oriented, hyperbolic 3-manifold with a spin structure $\sigma$. Then

$$\lim_{k \to \infty} \frac{\log |\tau^k(M^3, \sigma)|}{k^2} = \frac{1}{4\pi} \text{vol}(M^3).$$

(28)

In particular, the volume can be computed from the sequence of torsions.

Again, with our convention our torsion is the inverse of the usual analytic torsion, so Equation (28) has opposite sign to [72].

As noticed in [72], this theorem implies that there are finitely many manifolds with the same sequence of torsions.

**Question 4.6.** Is there a finite collection of integers $0 < k_1 < \cdots < k_n$ such that there are finitely many manifolds with the same $k_1, \ldots, k_n$-torsions?

The answer is yes under the extra hypothesis that the volume is bounded above. More precisely, an infinite sequence of manifolds with bounded volume accumulates (for the geometric topology) to a finite set of cusped manifolds with finite volume, this is Jørgensen-Thurston properness of the volume function [92, 93]. For such a sequence of manifolds $M_n^3$, $\tau^2(M_n^3, \sigma) \to \infty$, by Corollary 4.25, see also [82, Corollaire 4.18].

A priori, for some $N \in \mathbb{N}$ there could exist an infinite sequence of hyperbolic manifolds whose volume goes to infinity and with the same $k$-torsions for $k \leq N$. By Theorem 3.7 the injectivity radius of this sequence would not converge to infinity.

Müller proves Theorem 4.5 using analytic torsion and Ruelle zeta functions. There is first a theorem of Wotzke [104] that generalizes Fried’s theorem (Theorem 2.28) and from this he obtains a functional equation that relates the volume, the Ruelle zeta functions and the torsions. Theorem 4.5 has been generalized by Müller and Pfaff [74] by working directly on the trace of the heat kernel in [12].

**Theorem 4.7 ([74]).** Let $M^3$ be a compact, oriented, hyperbolic 3-manifold with a spin structure $\sigma$. Then

$$\lim_{k_1 \to \infty} \frac{\log |\tau^{k_1,k_2}(M^3, \sigma)|}{k_1^2} = \frac{1}{4\pi} (k_2 + 1) \text{vol}(M^3).$$

(29)

Again, the sign convention here differs from [74].
4.3 Functions on the variety of characters

In this subsection $M^3$ denotes an orientable hyperbolic 3-manifold of finite volume, with a single cusp. The subsection discusses the torsion as function on the variety of characters.

4.3.1 Generic cohomology on the distinguished component

Some cohomology preliminaries are required before defining the torsion of $\text{Sym}^{k_1,k_2}$ as function on $X_0(M^3)$. To simplify, use the notation

$$\rho^{k_1,k_2} := \text{Sym}^{k_1,k_2} \circ \rho$$

for any representation $\rho : \pi_1 M^3 \to \text{SL}_2(\mathbb{C})$.

**Definition 4.8.** A character $\chi \in X_0(M^3)$ is called $(k_1,k_2)$-exceptional if there exists a representation $\rho \in \text{hom}(\pi_1 M^3, \text{SL}_2(\mathbb{C}))$ with character $\chi_{\rho} = \chi$ such that $H^0(M^3; \rho^{k_1,k_2}) \neq 0$.

The set of $(k_1,k_2)$-exceptional characters is denoted by $E^{k_1,k_2}$.

If a character is irreducible, then the set of representations with this character is precisely a conjugation orbit, but for reducible characters there may be representations with the same characters that are not conjugate.

Notice that in Proposition 3.16 we have shown that $E^{1,0}$ consist of the trivial character, when it belongs to $X_0(M^3)$ (hence $E^{1,0} = \emptyset$ when $b_1(M^3) = 1$).

**Lemma 4.9.** If $k_1 \neq k_2$, then a $(k_1,k_2)$-exceptional character is reducible. In particular the $(k_1,k_2)$-exceptional set $E^{k_1,k_2}$ is a finite subset of $X_0(M^3)$.

The proof of this lemma is provided in Appendix C (Lemma C.9).

For further results, the following definition is convenient.

**Definition 4.10.** A subset of $X_0(M^3)$ is generic if it is non empty and Zariski open. Here the ground field is assumed to be $\mathbb{C}$ when working with an holomorphic representation $\text{Sym}^k$, and $\mathbb{R}$ for $\text{Sym}^{k_1,k_2}$ when $k_2 \neq 0$.

For the generic behavior of other cohomology groups, one must distinguish the case where $k_1$ or $k_2$ is odd from the case they are both even. All proofs are postponed to Appendix C.

**Proposition 4.11.** Let $M^3$ be orientable, hyperbolic and with one cusp. Assume $k_1 \neq k_2$.

(a) If $k_1$ or $k_2$ is odd, then

$$\left\{ \chi_{\rho} \in X_0(M^3) \mid \rho^{k_1,k_2} \text{ is acyclic} \right\}$$

is generic.

(b) If both $k_1$ and $k_2$ are even, then

$$\left\{ \chi_{\rho} \in X_0(M^3) \mid \dim H_i(M^3; \rho^{k_1,k_2}) = 1 \text{ for } i = 1,2 \text{ and } 0 \text{ otherwise} \right\}$$

is generic.
When \( k_1 \) and \( k_2 \) are even, we need to discuss the choice of basis in homology and perhaps we still need to remove a Zarsiki closed subset. We shall consider, for \( k_1 \) and \( k_2 \) even,

\[
F^{k_1, k_2} = \left\{ \chi_{\rho} \in X_0(M^3) \mid \dim H^0(T^2; \rho^{k_1, k_2}) \neq 1 \right\} \cup E^{k_1, k_2},
\]

which is Zarsiki closed subset of \( X_0(M^3) \) (over \( \mathbb{C} \) for \( k_2 = 0 \), over \( \mathbb{R} \) otherwise). In particular it is finite when \( k_2 = 0 \) or \( k_1 = 0 \).

**Lemma 4.12.** Let \( T^2 \) denote the peripheral torus. If both \( k_1 \) and \( k_2 \) are even, then \( F^{k_1, k_2} \) is a Zarsiki closed subset of \( X_0(M^3) \) (over \( \mathbb{C} \) for \( k_2 = 0 \) or \( k_1 = 0 \), over \( \mathbb{R} \) otherwise). In particular it is finite when \( k_2 = 0 \) or \( k_1 = 0 \).

Consider the cap product

\[
\cap: H^0(T^2; \rho^{k_1, k_2}) \times H_1(T^2; \mathbb{C}) \to H_1(T^2; \rho^{k_1, k_2}).
\]

It can be described as follows. Choose \( a \in H^0(T^2; \rho^{k_1, k_2}) \), \( a \neq 0 \), and view it as an element of \( C^{(n_1+1)(n_2+1)} \) invariant by the action of \( \rho^{k_1, k_2}(\pi_1 T^2) \). Any element in \( H_1(T^2; \mathbb{C}) \) is represented by a simplicial cycle: \( z \in C_*(K; \mathbb{C}) \) with \( \partial z = 0 \) (here \( K \) is a triangulation of \( T^2 \)). Choose a lift of \( z \) to the universal covering, \( \tilde{z} \in C_*(\tilde{K}; \mathbb{C}) \), then \( a \otimes \tilde{z} \in C_1((K; \rho^{k_1, k_2}) \) is a cocycle and

\[
a \cap [z] = [a \otimes \tilde{z}],
\]

where the brackets denote the class in homology.

In the next proposition \([T^2] \in H_2(T^2; \mathbb{Z})\) denotes a fundamental class, \( \{\} \) the linear span, and \( i:T^2 \to M^3 \) the inclusion.

**Proposition 4.13.** Let \( k_1 \neq k_2 \) be even integers. For \( 0 \neq a \in H^0(T^2; \rho^{k_1, k_2}) \) and \( 0 \neq \gamma \in H_1(T^2; \mathbb{C}) \), the set

\[
\left\{ \chi_{\rho} \in X_0(M^3) \mid \{i_*(a \cap \gamma)\} = H_1(M^3; \rho^{k_1, k_2}) \right\}
\]

is generic.

### 4.3.2 Higher torsions on the variety of characters

Again assume that \( M^3 \) is an oriented hyperbolic 3-manifold of finite volume with a single cusp. We define the Reidemeister torsion on the distinguished curve of characters \( X_0(M^3) \).

**Definition 4.14.** For \( k_1 \) or \( k_2 \) odd, with \( k_1 \neq k_2 \), and for \( \chi_{\rho} \in X_0(M^3) - E^{k_1, k_2} \), where \( E^{k_1, k_2} \) is as in Definition 4.8, define

\[
\mathcal{T}^{k_1, k_2}_M(\chi_{\rho}) = \begin{cases} 
\text{tor}(M^3, \rho^{k_1, k_2}) & \text{if } \rho^{k_1, k_2} \text{ is acyclic;} \\
0 & \text{otherwise.}
\end{cases}
\]

For \( k_1 \) and \( k_2 \) even, with \( k_1 \neq k_2 \), \( \gamma \) a peripheral curve, and \( \chi_{\rho} \in X_0(M^3) - F^{k_1, k_2} \) (where \( F^{k_1, k_2} \) is as in (30)), define

\[
\mathcal{w}^{k_1, k_2, \gamma}_M(\chi_{\rho}) = \begin{cases} 
\text{tor}(M^3, \rho^{k_1, k_2}, \{a \cap \gamma \cap \gamma, a \cap [T^2]\}) & \text{if } \{a \cap \gamma\} = H_1(M^3; \rho^{k_1, k_2}), \\
0 & \{a \cap \gamma\} = H_1(M^3; \rho^{k_1, k_2}), \( a \cap [T^2]\) = H_2(M^3; \rho^{k_1, k_2});
\end{cases}
\]

otherwise.
We also use the notation

\[ m^k = m^k,0 \]  for \( k \) odd or even respectively.

The proof of Proposition 4.16 with minor changes yields the following:

**Proposition 4.15.** For \( k_1 \neq k_2, m^{k_1,k_2} \) and \( m^{k_1,k_2}_{M,\gamma} \) are nonzero (real) analytic functions on \( X_0(M^3) - E^{k_1,k_2} \) for \( k_1 \) or \( k_1 \) odd, or \( X_0(M^3) - F^{k_1,k_2} \) for \( k_1 \) and \( k_2 \) even, holomorphic when \( k_2 = 0 \) and antiholomorphic when \( k_1 = 0 \).

When \( H^1(M^3;\mathbb{Z}) \cong \mathbb{Z} \) there is a well defined Alexander polynomial \( \Delta(t) \), and we can give conditions for the regularity of \( T^k_M \) and \( m^k_{M,\gamma} \) on the whole curve \( X_0(M^3) \).

**Proposition 4.16.** Assume that \( H^1(M^3;\mathbb{Z}) \cong \mathbb{Z} \) and that no root of \( \Delta(t) \) is a root of unity. Then for \( k \) odd \( E^{k,0} = \emptyset \). Hence for \( k \) odd \( m^k_M \) is a holomorphic function on \( X_0(M^3) \).

**Proof.** The proof of Proposition 3.16 applies here with a minor modification. Notice that the condition on the homology implies that the trivial character does not lie in \( X_0(M^3) \). On the other hand, the reducible characters in \( X_0(M^3) \) are the characters of the composition of a surjection

\[ \pi_1 M^3 \to \mathbb{Z} \]

with a representation that maps the generator of the cyclic group to

\[ A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \]

with \( \Delta(\lambda^2) = 0 \) [9,13]. Since \( \lambda \) is not a root of unity and \( k \) is odd, \( \text{Sym}^k(A) \) has no eigenvalue equal to one, hence \( H^0(M^3;\rho^k) = 0 \) for every \( \rho \) with character in \( X_0(M^3) \).

**Proposition 4.17.** Assume that \( M^3 = S^3 - S \) is a knot exterior in the sphere. Assume also that no root of \( \Delta(t) \) is a root of unity and all roots have multiplicity one. Then for \( k \) even, \( T^k_{M,\gamma} \) can be extended to the whole curve \( X_0(M^3) \).

**Proof.** There are two issues to discuss: the dimension of \( H^0(T^2;\rho) \) and the vanishing or not of \( H^0(M^3;\rho) \). For the first one, the assumption \( M^3 = S^3 - S \) implies that \( \rho \) is never trivial on the peripheral torus \( T^2 \). It my happen that \( \rho(\pi_1 T^2) \) is non-trivial but \( \text{Sym}^k(\rho(\pi_1 T^2)) \) has an invariant subspace of dimension \( \geq 2 \). However, this issue only occurs when \( \rho(\pi_1 T^2) \) is contained in a 1-parameter subgroup \( G \) of \( \text{SL}_2(\mathbb{C}) \) conjugate to the group of diagonal matrices. Then \( \text{Sym}^k(G) \) has only one invariant subspace, that varies analytically on \( \rho \). Thus the element \( a \) in the expression of the cup products can be chosen to depend analytically on \( \rho \).

Next discuss \( H^0(M^3;\rho) \). The argument in Proposition 4.16 fails in the case \( k \) even because \( \text{Sym}^k \) has always an invariant subspace, thus for those reducible representations \( H^0(M^3;\rho^k) = 0 \) and further discussion on reducible representations is required. Namely, at a reducible representation, for the corresponding abelian representation \( \rho \) as in the proof of Proposition 4.16 the hypothesis that
λ is not a root of unity implies that \(\text{Sym}^k(A)\) has a single eigenvalue equal to one, hence \(\dim H^0(M^3;\rho^k) = 1\). The results of \([43, 41]\) and the hypothesis that the root is simple, yield that there are, up to conjugation, two representations \(\rho'\) and \(\rho''\) that are non abelian and have the same character as \(\rho\). In particular \(H^0(M^3;\rho'(k_1k_2)) \cong H^0(M^3;\rho''(k_1k_2)) = 0\). Now, using the arguments of Proposition \([43, 41]\) we may define the torsion in a Zariski dense subset of the component of \(\text{hom}(\pi_1M^3, \text{SL}_2(C))\) that corresponds to \(X_0(M^3)\). It suffices to consider an open neighborhood of \(\rho'\) for the usual topology that projects to a neighborhood of \(\chi_\rho\) \([43, 41]\), which proves that the torsion on a punctured neighborhood of \(\chi_\rho\) extends to \(\chi_\rho\).

The assertion for the case \(k = 2\) has been proved by Yamaguchi in \([107]\). More precisely he computes the precise value of \(T^2_{M,\gamma}\) at the reducible character.

**Example 4.18.** Some torsion functions are computed here for the figure eight knot exterior. Let \(\theta = \theta_m\) denote the evaluation of a character of a meridian: \(\theta_m(\chi) = \chi(m)\) (i.e. the trace of a meridian). As mentioned in Example \([82]\) in \([51]\) Kitano computes:

\[T_M = T^1_M = 2 - 2\theta.\]

Further computations with the help of symbolic software yield:

\[
\begin{align*}
T^3_M &= - (\theta^2 - 2\theta - 2)^2 \\
T^5_M &= 2(\theta - 1) (\theta^4 + 2\theta^3 - 13\theta^2 - 20\theta^5 + 49\theta^4 + 48\theta^3 - 33\theta^2 - 18\theta - 18) \\
T^7_M &= - (\theta - 1)^2 (2\theta^7 - 4\theta^6 - 21\theta^5 + 19\theta^4 + 57\theta^3 + 13\theta^2 - 18\theta - 6)^2 \\
T^9_M &= 2(\theta - 1) (\theta^{12} + 2\theta^{11} - 13\theta^{10} - 13\theta^9 + 27\theta^8 - \theta^7 + 95\theta^6 + 90\theta^5 \\
&\quad - 148\theta^4 - 74\theta^3 + 61\theta^2 + 12\theta - 6)^2 .
\end{align*}
\]

Remark that these torsions are functions on the variable \(\theta = \theta_m\), the trace of the meridian. This is not always the case, for instance for \(T^2_{M,\gamma}\), computed in \([82]\) (and in Section \([51]\) and described below. We also computed:

\[
\begin{align*}
T^2_{M,\lambda} &= 3 + 3\theta^4 - 3\theta^3 - 14\theta^2 - 3\theta^2 + 13\theta + \theta^3 - 4\theta^2 + 4\theta^2 - 2\theta^2 - 2 \\
&\quad - 16\theta^2 - \theta^4 + \theta^2 + \eta(\theta^2 - \theta^2 - \theta - 1)
\end{align*}
\]

where \(\eta\) is a variable that satisfies \(\eta^2 = (\theta^2 - 1)(\theta^2 - 5)\). This variable can be written in terms of the traces of other elements, see Equation \([49]\).

In subsection \([5.2]\) other torsions are computed. If \(m\) and \(l\) denote respectively the meridian and the longitude, by \([50]\), \([51]\), \([54]\):

\[T^2_{M,l} = 5 - 2\theta^2, \quad T^2_{M,m} = \frac{1}{2}\eta, \quad \text{and} \quad T^4_{M,l} = 8(2 - \theta^2).
\]

### 4.4 Evaluation at the holonomy

#### 4.4.1 Cohomology at the character of the holonomy

In order to evaluate the torsion functions at the lift of the holonomy, some results on the cohomology for this representation are required. Again the proofs are postponed to Appendix \([3]\).
Theorem 4.19 (Theorem [B.7]). Let \( \rho_1 M^3 \to \text{SL}_2(\mathbb{C}) \) be a lift of the holonomy.

(a) If \( k_1 + k_2 \) is odd, then \( T^{k_1,k_2} \) is acyclic.

(b) If \( k_1 + k_2 \) is even, \( k_1 \neq k_2 \), then

\[
\dim H_i(M^3; T^{k_1,k_2}) = \begin{cases} 
1 & \text{if } i = 1, 2, \\
0 & \text{otherwise}.
\end{cases}
\]

Notice that when both \( k_1 \) and \( k_2 \) are odd, the homology of \( T^{k_1,k_2} \) differs from the generic homology of \( \text{Sym}^{k_1,k_2} \) on \( \chi_0(M^3) \), there is a discontinuity in the dimension of the cohomology, which is generically trivial.

We also need to discuss the basis. Let \( i: T^2 \to M^3 \) denote the inclusion.

Proposition 4.20 (Corollary [B.6] and Proposition [B.8]). Let \( \rho_1 M^3 \to \text{SL}_2(\mathbb{C}) \) be a lift of the holonomy. Assume that \( k_1 + k_2 \) is even.

(a) \( \dim H^0(T^2; \rho) = 1 \).

(b) For \( 0 \neq a \in H^0(T^2; \rho) \), \( \langle \uparrow (a \cap [T^2]) \rangle = H_2(M^3; T^{k_1,k_2}) \).

(c) For \( 0 \neq a \in H^0(T^2; \rho) \) and \( 0 \neq \gamma \in H_1(T_2; \mathbb{Z}) \):

(i) \( \langle \uparrow (a \cap \gamma) \rangle = H_1(M^3; T^{k_1,k_2}) \) if \( k_1 = 0 \) or \( k_2 = 0 \); in addition

\[
\uparrow (a \cap \gamma) = \text{cs}(\gamma_1, \gamma_2) \uparrow (a \cap \gamma_2)
\]  \( (32) \)

where \( \text{cs}(\gamma_1, \gamma_2) \) is the cusp shape (Definition [3.19]).

(ii) \( \uparrow (a \cap \gamma) = 0 \) if \( k_1 \neq 0 \) and \( k_2 \neq 0 \).

4.4.2 Torsion at the characters of the holonomy

Proposition 4.21. Let \( \chi_\rho \) be the character of a lift \( \rho \) of the holonomy representation

(a) When \( k_1 + k_2 \) is odd, \( T^{k_1,k_2}_M(\chi_\rho) = 0 \).

In particular, when \( k \) is odd, \( T^{k_1}_M(\chi_\rho) = 0 \).

(b) When \( k_1 \neq k_2 \) are both odd, \( T^{k_1,k_2}_M(\chi_\rho) = 0 \).

(c) When \( k_1 \neq k_2 \) are both even and \( k_1, k_2 \neq 0 \), for \( 0 \neq \gamma \in H_1(T_2; \mathbb{Z}) \), we have

\[
T^{k_1,k_2}_M(\chi_\rho) = 0.
\]

(d) For \( k \) even and \( 0 \neq \gamma \in H_1(T_2; \mathbb{Z}) \), \( T^{k}_M,\gamma(\chi_\rho) = 0 \). In addition

\[
\frac{T^{k}_M,\gamma_1(\chi_\rho)}{T^{k}_M,\gamma_2(\chi_\rho)} = \text{cs}(\gamma_2, \gamma_1),
\]

where \( 0 \neq \gamma_1, \gamma_2 \in H_1(T_2; \mathbb{Z}) \).
Proof. For item (a), Theorem 4.19 guarantees that the representation is acyclic and therefore $\tau_M^{k_1,k_2}(\chi_\theta) \neq 0$. For (b), the same theorem tells that the cohomology of $g^{k_1,k_2}$ is nonzero in dimension 1 and 2. Since it is generically zero, what we have is a discontinuity in the dimension of the cohomology groups. On the other hand, the zeroth homology group of $g^{k_1,k_2}$ vanishes, thus $T_M^{k_1,k_2}(\chi_\theta) = 0$. The proof of (b) is analogous, using the vanishing of the cup product in dimension one (not two) of the cap product in Proposition 4.20. The non-vanishing of the cup product of Proposition 4.23 for $k_1$ even and $k_2 = 0$ yields the first part of (b), the second part follows from Equation (32).

Theorem 4.22 ([63]). For any peripheral curve $\gamma$,

$$\lim_{k \to \infty} \frac{\log |\tau_M^{2k}(\chi_\theta)|}{(2k)^2} = \lim_{k \to \infty} \frac{\log |\tau_M^{2k+1}(\chi_\theta)|}{(2k+1)^2} = \frac{1}{4\pi} \text{vol}(M^3).$$ (33)

Again, the sign convention is the opposite to [63]. This theorem is proved from M"{u}ller’s Theorem 4.5 in the closed case, by using the approximation of the cusped manifolds by closed manifolds obtained by Dehn filling. Since M"{u}ller’s proof uses Ruelle zeta functions, the key point is to understand geodesics (of bounded lengths) of the closed manifolds that approximate a cusped one.

4.4.3 Dehn filling

The following is a generalization to other representations of the formula for Dehn filling in Proposition 4.25. In particular the same context and notation is used.

Proposition 4.23. Let $M^3$ be as above. For $|p| + |q|$ sufficiently large:

(a) For $k_1$ or $k_2$ odd,

$$\tau^{k_1,k_2}(M_{p,q}^3, \sigma) = T_M^{k_1,k_2}(\chi_{p/q},\sigma) \prod_{i_1=0}^{k_1} \prod_{i_2=0}^{k_2} \frac{1}{1 - (\lambda_{p/q})^{2i_1-k_1}(\lambda_{p/q})^{2i_2-k_2}}.$$ (46)

(b) For $k_1$ and $k_2$ even, $k_1 \neq k_2$,

$$\tau^{k_1,k_2}(M_{p,q}^3) = T_M^{k_1,k_2}(\chi_{p,q}) \prod_{(i_1,i_2) \in \mathcal{I}} \frac{1}{1 - (\lambda_{p/q})^{2i_1-k_1}(\lambda_{p/q})^{2i_2-k_2}},$$

where $(i_1,i_2) \in \mathcal{I}$ if $0 \leq i_1 \leq k_1$, $0 \leq i_2 \leq k_2$, and $i_1 \neq \frac{k_1}{2}$ or $i_2 \neq \frac{k_2}{2}$.

Corollary 4.24.

$$\lim_{p^2 + q^2 \to \infty} |\tau^{2k}(M_{p/q}^3, \sigma)| = +\infty.$$ (47)

Proof. Use first Proposition 4.23 (a). Since the character $\chi_{p,q}$ converges to $\chi_\theta$, $T_M^{2k}(\chi_\theta) \neq 0$, and $|\text{cs}(pm + ql,m)| = |p + q \text{ cs}(l,m)| \to \infty$, one has

$$T_M^{2k}(\chi_{p,q}) \to \infty.$$ (48)

As $|\lambda_{p/q}| \to 1$, the surgery formula in Proposition 4.23 yields the result.

The same proof as Corollary 3.28 yields:
Corollary 4.25. Given a spin structure, The set of modules of the torsions \( \tau^{k_1,k_2}(M^3/p_iq_i) \) obtained by Dehn filling on \( M^3 \), so that \( \sigma \) can be extended, is dense in
\[
\left[ \frac{1}{4} \tau^{2k_1+1,k_2}(M^3,\sigma), +\infty \right).
\]

When \( k_1 + k_2 \) is even and \( k_1 k_2 \neq 0 \), then \( \tau^{k_1,k_2}(\chi_\sigma) = 0 \) (for \( k_1 \) odd) and \( \tau^{k_1,k_2}_M(\chi_\sigma) = 0 \) (for \( k_1 \) even). In this case, we cannot get conclusions from the surgery formula.

4.5 Quantum invariants

When \((k_1,k_2) = (2,0)\), \( \text{Sym}^2 \) is the adjoint representation and its torsion occurs in the volume conjecture.

The role of the torsion in the expansion of the path integral is already mentioned in the work of Witten [102] and Bar-Natan Witten [3]. Of course the work of Kashaev [47] and Murakami-Murakami [77] play a key role in the conjecture.

For a knot \( K \), let \( J_N(K;z) \) denote the colored Jones polynomial of \( K \). If the knot is hyperbolic, let \( u \) denote Thurston’s parameter of the Dehn filling space (Definition 3.19). Denote the corresponding character by \( \chi_u \) and let \( \text{CS}(K,\chi_u) \) denote the \( \mathbb{C} \)-valued Chern-Simons invariant of a representation with character \( \chi_u \), namely the real part is minus the Chern-Simons invariant and the imaginary part is the volume of the representation. The following version is taken from [76] by H. Murakami, who attributes it to [37, 20].

Conjecture 4.26. Let \( K \) be a hyperbolic knot. There exists a neighborhood \( U \subset \mathbb{C} \) of the origin such that for \( u \in U - \pi i \mathbb{Q} \), we have the following asymptotic equivalence:
\[
J_u(K;e^{2\pi i u/N}) \sim \sqrt{-\pi} \left( \tau_{K,m}^2(\chi_u) \right)^{1/2} e^{\pi u/2} \text{CS}(K,\chi_u)
\]
where \( m \) denotes the meridian of the knot.

Again this paper uses a different convention for torsion from [76] and the other references, as it is the opposite to the convention for analytic torsion. It has been checked for the figure eight knot and \( u \) real, \( 0 < u < \log\left((3 + \sqrt{5})/2\right) \) by Murakami in [76]. For torus knots, the volume is zero, but not the Chern Simons invariant nor the torsion, and the asymptotic computations of Dubois-Kashaev [24] and Hikami-Murakami [44] support the corresponding conjecture for torus knots.

Related to this conjecture, the torsion is also involved in a potential function, introduced by Yokota [110]. From a diagram of the projection of a knot \( K \), in [81] Ohtsuki and Takata define \( \omega_2(K)^{-1} \) as the modified Hessian of the potential function of the diagram. They justify (formally) that \( \sqrt{\omega_2(K)} \) is the term that appears in the asymptotic development of the Kashaev invariant and therefore they conjecture
\[
\frac{1}{\omega_2(K)} = \pm 2 i \tau_{K,m}^2(\chi_\sigma).
\]
In [81] they prove that the conjecture holds true for two bridge knots.
There is also a remarkable contribution of Dimofte and Garoufalidis [19], that define an invariant from an ideal triangulation of a knot exterior and an enhanced Neumann-Zagier datum. Enhanced Neumann-Zagier datum means that, besides the complex collection shape parameters of the ideal hyperbolic tetrahedra, they use matrices with integer coefficients that describe how to glue the tetrahedra and a collection of integers that code a combinatorial flattening (introduced in [78] by Neumann to calculate the Chern-Simons invariant combinatorially). Form these data, Dimofte and Garoufalidis construct an invariant of the hyperbolic manifold and they check numerically that it equals $1/R_{K,m}(\chi_0)$ up to some constant.

5 Representations in $\text{PSL}_{n+1}(\mathbb{C})$ and the adjoint

In [82], I considered the torsion corresponding to the adjoint representation as a function on the variety of characters in $\text{SL}_2(\mathbb{C})$, in fact $\text{PSL}_2(\mathbb{C})$. Kitayama and Terashima [55] considered the analog for $\text{PSL}_{n+1}(\mathbb{C})$, i.e. the torsion for the adjoint representation.

Along the section $M_3$ denotes an oriented finite volume hyperbolic three-manifold with one cusp, the case with more cusps would require further notation, but here it is essentially the same. Consider the composition of the holonomy with $\text{Sym}^n$:

$$\varphi^0 : \pi_1 M_3 \to \text{PSL}_{n+1}(\mathbb{C}).$$

As we deal with representations in $\text{PSL}_2(\mathbb{C})$, the holonomy does not need to be lifted to $\text{SL}_2(\mathbb{C})$.

**Theorem 5.1** ([62, 60]). The character of $\varphi^0$ is a smooth point of the variety of characters $X(M^3, \text{PSL}_{n+1}(\mathbb{C}))$, of local dimension $n$. It is locally parametrized by the symmetric functions on any peripheral curve.

**Corollary 5.2.** There exists a unique irreducible component of the character variety $X(M^3, \text{PSL}_{n+1}(\mathbb{C}))$ that contains the character of $\varphi^0$.

This component is called the distinguished component and it is denoted by

$$X_0(M^3, \text{PSL}_{n+1}(\mathbb{C})).$$

In the proof of Theorem 5.1, one shows that $\mathfrak{sl}_{n+1}(\mathbb{C})^{\text{Ad} \varphi^0(\pi_1 T^2)} \cong \mathbb{C}^n$. In addition, if $\{a_1, \ldots, a_n\}$ is a basis for $\mathfrak{sl}_{n+1}(\mathbb{C})^{\text{Ad} \varphi^0(\pi_1 T^2)}$, then

$$\{a_1 \cap [T^2], \ldots, a_n \cap [T^2]\}$$

is a basis for $H_2(M^3; \text{Ad} \varphi^0)$ and

$$\{a_1 \cap [\gamma], \ldots, a_n \cap [\gamma]\}$$

a basis for $H_1(M^3; \text{Ad} \varphi^0)$, here $\gamma$ is a peripheral curve, non-trivial in $H_1(T^2; \mathbb{Z})$. See [62, 60, 55] for details. For a generic character $\chi_\rho$ in $X_0(M^3, \text{PSL}_{n+1}(\mathbb{C}))$, $\mathfrak{sl}(n, \mathbb{C})^{\text{Ad} \rho(\pi_1 T^2)} \cong \mathbb{C}^n$ and a similar construction applies to find a basis for $H_1(M^3; \text{Ad} \rho)$. Generalizing the construction of [82], Kitayama and Terashima [55] define the torsion. Here we consider the possibility that the function vanishes, but before an exceptional set must be removed.

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Definition 5.3. A representation $\rho: \pi_1 M^3 \to \text{PSL}_{n+1}(\mathbb{C})$ is exceptional if:

(a) $\mathfrak{sl}_{n+1}(\mathbb{C}) \overset{\text{Ad} \rho(\pi_1 M^3)}{\to} \mathbb{C} \neq 0$ or

(b) $\dim \left( \mathfrak{sl}_{n+1}(\mathbb{C}) \overset{\text{Ad} \rho(\pi_1 T^2)}{\to} \mathbb{C} \right) > n$, for the peripheral torus $T^2$.

The set of characters of exceptional representations is denoted by

$$E_{n+1} \subset X_0(M^3, \text{PSL}_{n+1}(\mathbb{C}))$$

**Lemma 5.4.** The set $E_{n+1}$ is Zariski closed in $X_0(M^3, \text{PSL}_{n+1}(\mathbb{C}))$.

**Proof.** The defining properties are Zariski closed in $\hom(\pi_1 M^3, \text{PSL}_{n+1}(\mathbb{C}))$ and invariant by conjugation, so the projection to $X_0(M^3, \text{PSL}_{n+1}(\mathbb{C}))$ is a Zariski closed subset, see [98] for instance. \qed

**Definition 5.5.** Let $\chi_\rho \in X_0(M^3, \text{PSL}_{n+1}(\mathbb{C})) - E_{n+1}$, choose $\{a_1, \ldots, a_n\}$ a basis for $\mathfrak{sl}_{n+1}(\mathbb{C}) \overset{\text{Ad} \rho(\pi_1 T^2)}{\to} \mathbb{C}$ and let $\gamma$ be a peripheral curve.

(a) If $\{a_1 \cap [\gamma], \ldots, a_n \cap [\gamma]\}$ is a basis for $H_1(M^3; \text{Ad} \rho)$, define

$$\mathcal{T}_{(M, \gamma)}(\chi_\rho) = \text{TOR}(M^3, \{a_1 \cap [\gamma], \ldots, a_n \cap [\gamma]\}, \{a_1 \cap [T^2], \ldots, a_n \cap [T^2]\}); \quad (34)$$

(b) otherwise set

$$\mathcal{T}_{(M, \gamma)}(\chi_\rho) = 0. \quad (35)$$

This yields a function

$$\mathcal{T}_{M, \gamma}: X_0(M^3, \text{PSL}_{n+1}(\mathbb{C})) - E_{n+1} \to \mathbb{C}. \quad (36)$$

It can be checked that it is a well defined regular function, using the ideas of [82][55] and Proposition 5.16. For instance, when $\{a_1 \cap [\gamma], \ldots, a_n \cap [\gamma]\}$ is a basis for $H_1(M^3; \text{Ad} \rho)$, then $\{a_1 \cap [T^2], \ldots, a_n \cap [T^2]\}$ is a basis for $H_2(M^3; \text{Ad} \rho)$ (because $H_2(M^3; \text{Ad} \rho)$ is naturally isomorphic to $H_2(T^2; \text{Ad} \rho) \cong H^0(T^2; \text{Ad} \rho)$).

When there may be confusion, the index $n+1$ may be included in the notation:

$$\mathcal{T}_{M, \gamma}^{n+1}: X_0(M^3, \text{PSL}_{n+1}(\mathbb{C})) - E_{n+1} \to \mathbb{C}. \quad (37)$$

The following proposition relates $\mathcal{T}_{M, \gamma}$ with the torsion $T_{(M, \gamma)}^{2n}$ when we consider symmetric powers of representations in $\text{PSL}_2(\mathbb{C})$.

**Proposition 5.6.** For a generic character $\chi_\rho \in X_0(M^3, \text{PSL}_2(\mathbb{C}))$, including the holonomy $\chi_\rho$,

$$\mathcal{T}_{M, \gamma}^{n+1}(\chi_\rho^n) = \prod_{i=1}^n T_{M, \gamma}^{2i}(\chi_\rho).$$

**Corollary 5.7.** For a generic character $\chi_\rho \in X_0(M^3, \text{PSL}_2(\mathbb{C}))$, including the holonomy $\chi_\rho$,

$$\mathcal{T}_{M, \gamma}^{n+1}(\chi_\rho) = \mathcal{T}_{M, \gamma}^{n}(\chi_\rho) T_{M, \gamma}^{2n}(\chi_\rho).$$

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Proof of Proposition 5.6. The proof is straightforward from Klebsh-Gordan formula:

$$\text{Ad} \circ \text{Sym}^n = \bigoplus_{i=1}^{n} \text{Sym}^{2i}$$

and multiplicativity of the torsion for sums of representations.

The next theorem due to Weil [100], see also [58], gives a nice interpretation to $T_{\gamma}M$:

**Theorem 5.8 (100).** Let $\Gamma$ be a finitely generated group and $\rho$ an irreducible representation with character $\chi_{\rho} \in X(\Gamma, \text{PSL}_{n+1}(C)) - E_{n+1}$. Then there is a natural isomorphism

$$T^{\text{Zar}}_{\chi_{\rho}} X(\Gamma, \text{PSL}_{n+1}(C)) \cong H^1(\Gamma; \text{Ad}\rho)$$

where $T^{\text{Zar}}$ denotes the Zariski tangent space as a scheme.

In particular, if $\phi: \Gamma \rightarrow \Gamma'$ is a group morphism, then the induced map in cohomology corresponds to the tangent map

$$d\phi^*: T^{\text{Zar}}_{\chi_{\rho}} X(\Gamma', \text{PSL}_{n+1}(C)) \rightarrow T^{\text{Zar}}_{\phi^*\chi_{\rho}} X(\Gamma, \text{PSL}_{n+1}(C)).$$

Few comments are in order here. First at all, the condition that $\chi_{\rho} \notin E_{n+1}$ is used to say that the infinitesimal commutator of $\text{Ad}\rho$ is trivial, i.e. $H^0(\pi_1 M^3; \text{Ad}\rho) = 0$. Secondly, the variety of characters is not a variety but a scheme: the defining polynomial ideal may be not reduced, thus we must consider the Zariski tangent space of the scheme, perhaps not reduced. Finally, just mention that there are generalizations of this result when $\rho$ is reducible, see [4, 5].

This interpretation has a nice application for surface bundles over the circle, following again [82, 55]. Assume that $M^3$ is a bundle over a circle, with fibre a punctured surface $\Sigma$ and monodromy $\varphi: \Sigma \rightarrow \Sigma$, i.e.

$$M^3 = \Sigma \times [0, 1]/(x, 1) \sim (\varphi(x), 0).$$

It has a natural epimorphism $\pi_1 M^3 \rightarrow \mathbb{Z}$, corresponding to the projection of the fibration $M^3 \rightarrow S^1$. The induced map on the monodromy is denoted by

$$\varphi_*: X(\Sigma, \text{PSL}_n(C)) \rightarrow X(\Sigma, \text{PSL}_n(C))$$

and the restriction to $\pi_1 \Sigma$ of characters in $X(M^3, \text{PSL}_n(C))$ restrict to the fixed point set of $X(\Sigma, \text{PSL}_n(C))$. By Weil’s theorem, the map induced in $H^1(\Sigma; \text{Ad}\rho)$ can be interpreted as the differential $d\varphi_*$. Thus, using Proposition 2.14 the twisted polynomial is

$$\det(d\varphi_* - t \text{Id}).$$

Its evaluation at $t = 1$ vanishes because $H^* (M^3; \text{Ad}\rho) = 0$. The polynomial is divisible by $(t - 1)^n$, corresponding to the invariant curve $\gamma = \partial \Sigma$, as $\dim H^1(\gamma; \text{Ad}\rho) = n$. An argument using the exact sequences and the basis of homology yields the following result:
Proposition 5.9. If $M^3$ is a punctured surface bundle, and $\gamma$ the boundary of the fibre. Then

$$T_{M,\gamma} = \det \left. \left( \frac{d\varphi^* - t \Id}{(t-1)^n} \right) \right|_{t=1}. $$

This result is very useful for computing the torsion, it allows to obtain it from the variety of characters or moduli spaces without knowing the representation. For instance Kitayama and Terashima use cluster algebras to compute it [55]. In Subsection 5.2 we recall the method of [82] to compute it.

5.1 Local parameters and change of curve

Consider $\mathfrak{h} \subset \mathfrak{sl}_{n+1}(\mathbb{C})$ the Cartan subalgebra of diagonal matrices, in particular a diagonal matrix in $\mathrm{PSL}_{n+1}(\mathbb{C})$ lies in $\exp \mathfrak{h}$.

For a generic character $\chi \in X_0(M^3, \mathrm{PSL}_{n+1}(\mathbb{C}))$, one would like to consider in a neighborhood $U$ of $\chi$:

$$\log_\gamma : U \subset X_0(M^3, \mathrm{PSL}_{n+1}(\mathbb{C})) \to \mathfrak{h}$$

but a priori this may not be well defined. Notice that there are indeterminacies due to the action of the Weyl group (permutation of elements in the diagonal) and indeterminacies due to the complex logarithm. This motivates the following definition.

Definition 5.10. A representation $\rho \in \mathrm{hom}(\pi_1 M^3, \mathrm{PSL}_{n+1}(\mathbb{C}))$ is chamber regular if there exists a peripheral element $\gamma \in \pi_1 T^2$ such that $\rho(\gamma)$ has $n+1$ different eigenvalues.

In terms of Lie groups this is a regularity condition: $\rho(\pi_1 T^2)$ is contained in a Cartan subgroup and in the interior of the Weyl chamber of $\mathrm{PSL}_{n+1}(\mathbb{C})$.

By [60], the symmetric functions on the eigenvalues of $\rho(\gamma)$ define a local biholomorphism in a neighborhood of $\gamma^n$, hence all eigenvalues of $\rho(\gamma)$ are different in a Zariski open set. Thus:

Remark 5.11. The set of chamber regular characters is a non-empty Zariski open subset of $X_0(M^3, \mathrm{PSL}_2(\mathbb{C}))$.

For a chamber regular character and a peripheral element $\gamma \in \pi_1 T^2$, there exist a neighborhood $U \subset X_0(M^3, \mathrm{PSL}_{n+1}(\mathbb{C}))$ such that the logarithm $\log_\gamma$ as in (39) is defined in $U$. Notice that the eigenvalues of the image of $\gamma$ do not need to be different, provided that there is an element in the peripheral group with different eigenvalues.

Next consider a nonzero element $a \in \mathfrak{h}$. Using the Killing form, we define $a^* \in \mathfrak{h}$ to be the pairing with $a$.

Lemma 5.12. Let $\rho$ be a chamber regular representation, $\gamma$ a peripheral curve and $a \in \mathfrak{h}$. Viewing $H_1(M^3; \text{Ad } \rho)$ as cotangent space:

$$a \cap [\gamma] = d(a^* \log_\gamma). \quad (40)$$

This must be compared with [82 Lemma 3.20]. Before proving this lemma, let us discuss its consequences.
**Proposition 5.13.** Let $\chi_\rho \in X_0(M^3; \text{PSL}_{n+1}(\mathbb{C}))$ be a chamber regular character and $\gamma$ a peripheral curve. Then $T_{M,\gamma}(\chi_\rho) \neq 0$ if and only if $\chi_\rho$ is a scheme reduced smooth point of $X_0(M^3; \text{PSL}_{n+1}(\mathbb{C}))$ and $\log_\gamma$ is a local parameter.

The proof follows easily from Lemma 5.12 and we just sketch it. Namely, the condition of being scheme reduced and smooth is equivalent to saying that $\dim H^1(M^3; \text{Ad } \rho) = n$. By the standard arguments of the long exact sequence of the pair $(M^3, T^2)$ this implies that $\dim H^0(T^2; \text{Ad } \rho) = n$ and that $\{a_1 \cap [T^2], \ldots, a_n \cap [T^2]\}$ is a basis for $H^2(M^3; \text{Ad } \rho)$. Then Lemma 5.12 yields that $\{a_1 \cap [\gamma], \ldots, a_n \cap [\gamma]\}$ is a basis for $H^1(M^3; \text{Ad } \rho)$ iff $\log_\gamma$ is a local parameter.

For a chamber regular character, the condition of Proposition 5.13 is equivalent to the notion of $\gamma$-regularity of [55, Definition 3.2].

Notice finally that for any non-trivial peripheral curve $\gamma$, for the lift of the holonomy $\hat{\rho}$, even if it is not chamber regular, $T_{M,\gamma}(\chi_{\hat{\rho}}) \neq 0$, by [55, Theorem 3.4].

**Proposition 5.14.** Let $\gamma_1, \gamma_2$ be two peripheral elements. In a Zariski open domain in $X_0(M^3, \text{PSL}_{n+1}(\mathbb{C}))$

$$\frac{T_{M,\gamma_1}}{T_{M,\gamma_2}} = J(\log_{\gamma_1} \log^{-1}_{\gamma_2}).$$

Notice that the Jacobian $J(\log_{\gamma_1} \log^{-1}_{\gamma_2})$ is well defined generically on the distinguished component $X_0(M^3, \text{PSL}_{n+1}(\mathbb{C}))$, i.e. in a non-empty open Zariski set, as $\log_{\gamma_2}$ is a local parameter in an open set. In addition, this Jacobian is independent of the parametrization of $\hat{\rho}$, as any change of parametrization cancels in the quotient. The proof of Proposition 5.14 is straightforward from Lemma 5.12 and the formula of change of basis in homology (40).

Proposition 5.14 does not cover $\chi_{\hat{\rho}}$, the character of $\text{Sym}^n$ of the holonomy of the complete hyperbolic structure, as it is not chamber regular. However, Corollary 5.7 and Proposition 4.21 yield:

$$T_{M,\gamma_1}^{n+1}(\chi_{\hat{\rho}}) = \text{cs}(\gamma_2, \gamma_1)^n T_{M,\gamma_1}^{n+1}(\chi_{\hat{\rho}}),$$

where $\text{cs}(\gamma_2, \gamma_1)$ denotes the cusp shape (Definition 5.19).

**Proof of Lemma 5.12** Assume that $\rho$ is a generic representation so that $\rho(\gamma)$ is diagonal with different eigenvalues. Let $\mathfrak{h}$ denote the Cartan algebra of diagonal matrices, and choose a non-zero element $a \in \mathfrak{h}$. In particular $a \cap \gamma \in H_1(M^3; \text{Ad } \rho)$. Consider the dual of $a$:

$$a^*: \mathfrak{h} \to \mathbb{C}, \quad h \mapsto B(a, h)$$

where $B$ denotes the Killing form. Consider also a first order deformation $\rho_t$ of $\rho$, i.e.

$$\rho_t = (1 + t \dot{\rho} + O(t^2)) \rho$$

(41)

where $\dot{\rho} : \pi_1 M^3 \to \mathfrak{sl}_{n+1}(\mathbb{C})$ is a group cocycle, that we project to $H^1(M^3; \text{Ad } \rho)$.

Consider finally the Kronecker pairing

$$\langle \cdot \rangle : H^k(M^3; \text{Ad } \rho) \times H_k(M^3; \text{Ad } \rho) \to \mathbb{C}$$
defined as follows. Let $z \in C^k(M^3; \text{Ad}\rho)$ be a cocycle, with $z: C^k(\tilde{M}^3; \mathbb{Z}) \to \mathfrak{sl}_{n+1}(\mathbb{C})$, and let $h \otimes m \in C^k(M^3; \text{Ad}\rho)$, where $h \in \mathfrak{sl}_{n+1}(\mathbb{C})$ and $m \in C^k(\tilde{M}^3; \mathbb{Z})$.

At the (co-)chain level, the Kronecker pairing is

$$\langle z, h \otimes m \rangle = B(h, z(m))$$

(42)

where $B$ denotes again the Killing form. This induces a non-degenerate pairing between homology and cohomology [82]. After all those preliminaries, to establish the lemma one must prove the following equality

$$\langle \dot{\rho}, a \cap [\gamma] \rangle = \left. \frac{d}{dt} a^* \log \rho_t(\gamma) \right|_{t=0}$$

(43)

To prove (43), start with the group cohomology version of (42) in the current context (see [82]):

$$\langle \dot{\rho}, a \cap [\gamma] \rangle = B(a, \dot{\rho}(\gamma))$$

(44)

From (41) evaluated at $\gamma$ and taking logarithms:

$$\log \rho_t(\gamma) = \log \rho(\gamma) + t \dot{\rho}(\gamma) + O(t^2))$$

(45)

Then (43) follows from $a^*$ applied to (45), then differentiating and applying (44) to the result.

Lemma 5.12 may be related to the results of Goldman [35].

Example 5.15 (Volumes on $X(K, SU(2))$). In [21, 22] Dubois makes a relevant contribution of torsions as volume form on the variety of characters of a knot in $SU(2)$. Here the Cartan algebra has dimension one and the logarithm of a matrix in $SU(2)$ is an angle. Dubois volume form is, up to sign,

$$\text{vol}_{\text{tor}} = \pm \frac{d\varphi_{\gamma}}{T_{m, \gamma}}$$

where $\varphi_{\gamma}$ denotes the angle of the representation of the peripheral curve $\gamma$. Notice that by Proposition 5.14 this form is independent of the choice of the peripheral curve. In [21, 22], using Turaev’s refinement of torsion and a good choice of $\varphi_{\gamma}$, Dubois avoids the sign indeterminacy. He also shows that it equals to a volume form defined from a Heegaard splitting, à la Johnson (Example 2.25). This is related to the construction of an orientation on the space of representations of Heusener [40].

Proposition 5.14 suggests that a similar volume form can be constructed for representations in $SU(n+1)$, taking a convenient parametrization of the Cartan algebra (i.e. infinitesimal angles). It remains to know also whether the variety of characters of a knot in $SU(n+1)$ is non-empty and $n$-dimensional. This holds true for instance for two bridge knots. Another issue is to compute explicitly the variety of $SU(n+1)$ characters for a knot.

5.2 An example

Let me use Proposition 5.9 to compute $T_{M, \gamma}$ for the figure eight knot for $\text{PSL}_2(\mathbb{C})$ and $\text{PSL}_3(\mathbb{C})$. For $\text{PSL}_2(\mathbb{C})$ this is done in [82], but I recall it here for completeness.
Let $\Gamma = \pi_1 M$ denote the fundamental group of the figure eight knot exterior. The presentation

$$\Gamma = \langle a, b, m \mid m a m^{-1} = a b, m b m^{-1} = b a b \rangle$$

corresponds to the fact that it is fibered over the circle, with fibre a punctured torus, whose fundamental group is the free group $F_2 = \langle a, b \rangle$. The element $m$ is also a meridian curve of the knot. The monodromy $\phi: F_2 \to F_2$ satisfies

$$\phi(a) = a b \quad \text{and} \quad \phi(b) = b a b.$$

Since $F_2$ is the derived subgroup of $\Gamma$, every representation of $\Gamma$ in $\text{PSL}_{n+1}(\mathbb{C})$ restricts to a representation of $F_2$ in $\text{SL}_{n+1}(\mathbb{C})$ that is fixed by the monodromy $\phi^*$. The set of fixed characters is denoted by

$$X(F_2, \text{SL}_{n+1}(\mathbb{C}))^{\phi^*}.$$

The following lemma is proved in [82, 39] (using that $F_2$ is the commutator of $\Gamma$):

**Lemma 5.16.** For $n = 1, 2$, the restriction map induces an isomorphism

$$X_{irr}(\Gamma, \text{PSL}_{n+1}(\mathbb{C})) \cong X_{irr}(F_2, \text{SL}_{n+1}(\mathbb{C}))^{\phi^*},$$

where the subindex $_{irr}$ stands for irreducible characters.

### 5.2.1 Computations for $X(M^3, \text{PSL}_2(\mathbb{C}))$

By Fricke-Klein theorem, the variety of characters $X(F^2, \text{SL}_2(\mathbb{C}))$ is isomorphic to $\mathbb{C}^3$. More precisely, defining

$$\alpha_1(\rho) = \text{tr}(\rho(a)), \quad \alpha_2(\rho) = \text{tr}(\rho(b)), \quad \alpha_3(\rho) = \text{tr}(\rho(a b)).$$

Fricke-Klein theorem asserts that $(\alpha_1, \alpha_2, \alpha_3)$ are global coordinates of $X(F^2, \text{SL}_2(\mathbb{C}))$. Using the relations of traces, $\forall A, B \in \text{SL}_2(\mathbb{C})$:

$$\text{tr}(A^{-1}) = \text{tr}(A), \quad \text{tr}(AB) = \text{tr}(AB^{-1}) - \text{tr}(A) \text{tr}(B),$$

we may deduce:

$$\phi^*(\alpha_1) = \alpha_3, \quad \phi^*(\alpha_2) = \alpha_2 \alpha_3 - \alpha_1, \quad \phi^*(\alpha_3) = \alpha_2^2 \alpha_3 - \alpha_1 \alpha_3 - \alpha_2.$$

Thus $\phi^*(\alpha_1) = \alpha_2$ is equivalent to

$$\alpha_3 = \alpha_1, \quad \alpha_1 + \alpha_2 = \alpha_1 \alpha_2.$$

Then, the torsion polynomial is

$$\det(d\phi^* - t \cdot \text{Id}) = (t - 1)(t^2 + (1 - 2\alpha_1 \alpha_2) t + 1). \quad (46)$$

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Removing the factor \( (t - 1) \) and evaluation at \( t = 1 \) yields
\[
T_{M,l} = 3 - 2\alpha_1\alpha_2 = 3 - 2\alpha_1 - 2\alpha_2 \tag{47}
\]
Using that the trace of the longitude \( l = [a, b] \) satisfies \( \theta_l = \alpha_1^2 + \alpha_2^2 - \alpha_1 - \alpha_2 - 2 \), we get \( T_{M,l}^2 = 17 + 4\theta_l \). If \( \theta_m \) denotes the trace of the meridian, the distinguished component \( X_0(\Gamma, \text{PSL}_2(\mathbb{C})) \) is the curve
\[
\alpha_1^2 + \alpha_1 - 1 = \theta_m^2 (\alpha_1 - 1). \tag{48}
\]
(Let me emphasize that \( \theta_l \) and \( \theta_m \) denote traces and not angles, I made this choice because \( \tau \) is already used for torsion.) To get the variety \( \text{PSL}_2(\mathbb{C}) \) characters instead of \( \text{SL}_2(\mathbb{C}) \), we get rid of four variables (we are left with \( \beta_1, \beta_2, \beta_3, \) and \( \beta_4 \)), and we deduce that
\[
X_{l} = \{ \alpha_1 + \alpha_2 = \theta_m^2 - 1 \text{ and } \alpha_1 - \alpha_2 = \pm\sqrt{\theta_m^2 - 5}\theta_m - 1 \}. \tag{49}
\]
Thus
\[
T_{M,l} = T_{M,l}^2 = 5 - 2\theta_m^2. \tag{50}
\]
Proposition 5.14 can be worked out \([82]\) to yield:
\[
T_{M,m} = T_{M,m}^2 = \pm \frac{\alpha_1 - \alpha_2}{2} = \pm \frac{\alpha_1 + 1 - \frac{\theta_m^2}{2}}{2} = \pm \frac{1}{2} \sqrt{\theta_m^2 - 5}\theta_m - 1. \tag{51}
\]

5.2.2 Computations for \( X(M^3, \text{PSL}_3(\mathbb{C})) \)

By a theorem of Lawton \([56]\) and Will \([101]\), \( X(F_2, \text{PSL}_3(\mathbb{C})) \) is a double branched covering of \( \mathbb{C}^n \) with coordinates
\[
\beta_1(\rho) = \text{tr}(\rho(a)) \\
\beta_2(\rho) = \text{tr}(\rho(a^{-1})) \\
\beta_3(\rho) = \text{tr}(\rho(b)) \\
\beta_4(\rho) = \text{tr}(\rho(b^{-1})) \\
\beta_5(\rho) = \text{tr}(\rho(ab)) \\
\beta_6(\rho) = \text{tr}(\rho(b^{-1}a^{-1})) \\
\beta_7(\rho) = \text{tr}(\rho(a^{-1}b)) \\
\beta_8(\rho) = \text{tr}(\rho(ab^{-1}))
\]
and the trace of \( l = [a, b] \) and its inverse, \( \theta_{l^1} \), are a degree two extension of the coordinates \( \beta_1, \ldots, \beta_8 \), and \( \theta_l \) and \( \theta_{l^1} \) are Galois conjugate.

As \( \phi(l) = l \), we may work in \( \mathbb{C}^n \). Following \([99]\), \( \phi^* \) can be computed as
\[
\phi^*(\beta_1) = \beta_5 \\
\phi^*(\beta_2) = \beta_6 \\
\phi^*(\beta_3) = -\beta_1\beta_4 + \beta_3\beta_5 + \beta_8 \\
\phi^*(\beta_4) = -\beta_2\beta_3 + \beta_4\beta_6 + \beta_7 \\
\phi^*(\beta_5) = -\beta_1\beta_4\beta_6 + \beta_3\beta_5^2 - \beta_3\beta_6 + \beta_5\beta_8 + \beta_2 \\
\phi^*(\beta_6) = -\beta_2\beta_3\beta_6 + \beta_4\beta_6^2 - \beta_4\beta_5 + \beta_6\beta_7 + \beta_1 \\
\phi^*(\beta_7) = \beta_3 \\
\phi^*(\beta_8) = \beta_4.
\]
Now, setting \( \beta_i = \phi^*(\beta_i) \), we get rid of four variables (we are left with \( \beta_1, \beta_2, \beta_3, \) and \( \beta_4 \), and we deduce that \( X(F_2, \text{PSL}_3(\mathbb{C})) \)) has three components:
(a) $V_0$, with equations $\beta_1 = \beta_2$, $\beta_3 = \beta_4$.

(b) $V_1$, with equations $\beta_1 = \beta_2 = 1$.

(c) $V_2$, with equations $\beta_3 = \beta_4 = 1$.

The component $V_0$ is the restriction of the distinguished component $X_0(M^3, \text{PSL}_3(\mathbb{C}))$. On this component, the torsion polynomial is

$$
\det(d\phi - t \text{Id}) = (t - 1)^2 \left( t^2 + (2 - \beta_1 \beta_3) t + 1 \right) \times \left( t^4 + (-\beta_1 \beta_3 - 2 \beta_1 - 2 \beta_3) t^3 + (6 \beta_1 \beta_3 + 2) t^2 + (-\beta_1 \beta_3 - 2 \beta_1 - 2 \beta_3) t + 1 \right).
$$

(52)

After getting rid of $(t - 1)^2$ and evaluating at $t = 1$, we get

$$
T_{M, t} = (4 - \beta_1 \beta_3) (1 - \beta_1)(1 - \beta_3).
$$

(53)

Next Corollary 5.7 is used to compute $T_{M, t}^4$. The relation between traces in $\text{SL}_2(\mathbb{C})$ and their image in $\text{SL}_3(\mathbb{C})$ via $\text{Sym}$ yields

$$
\beta_1 = \beta_2 = \alpha_1^2 - 1,
$$

$$
\beta_3 = \beta_4 = \alpha_2^2 - 1.
$$

Using these identities in (53) we get

$$
T_{M, t} \circ \text{Sym} = (3 - 2\alpha_1 \alpha_2) (2 - \alpha_1^2) (2 - \alpha_2^2).
$$

Thus by applying Corollary 5.7 and (47), one gets

$$
T_{M, t}^4 = 4(2 - \alpha_1^2)(2 - \alpha_2^2) = 8(1 - \alpha_1 \alpha_2) = 8(2 - \theta_m^2).
$$

(54)

**Appendix A**  Not approximating the trivial representation

For a manifold $M^3$ there are components of $X(M^3, \text{PSL}_2(\mathbb{C}))$ that consist of characters of abelian representations. When $b_1(M^3) = 1$ those components are curves, and their union is denoted by

$$
X^{ab}(M^3, \text{PSL}_2(\mathbb{C})).
$$

The number of components of $X^{ab}(M^3, \text{PSL}_2(\mathbb{C}))$ depends on the torsion of $H_1(M^3; \mathbb{Z})$.

Being irreducible is a Zariski open property for a character \cite{16}. On the other hand, every reducible character is also the character of an abelian representation. This yields that $X^{ab}(M^3, \text{PSL}_2(\mathbb{C}))$ are precisely the components consisting only of reducible characters.

**Lemma A.1.** Assume that $b_1(M^3) = 1$. Then the trivial character belongs to a single irreducible component of $X(M^3, \text{PSL}_2(\mathbb{C}))$, which is one of the curves of $X^{ab}(M^3, \text{PSL}_2(\mathbb{C}))$.

**Corollary A.2.** If $b_1(M^3) = 1$, then the trivial character does not belong to $X_0(M^3, \text{PSL}_2(\mathbb{C}))$. 44
Proof of Lemma A.1. The proof uses the projection
\[ \pi: \text{hom}(\pi_1M^3, \text{PSL}_2(\mathbb{C})) \to X(M^3, \text{PSL}_2(\mathbb{C})) \quad \rho \mapsto \chi_{\rho} \] (55)
and the dimension of its fibre. For the trivial character \( \chi_0 \), a representation \( \rho \in \pi^{-1}(\chi_0) \) is conjugate to
\[ \rho(\gamma) = \begin{pmatrix} 1 & h(\gamma) \\ 0 & 1 \end{pmatrix}, \quad \forall \gamma \in \pi_1M^3, \] (56)
where \( h: \pi_1M^3 \to \mathbb{C} \) is a group homomorphism. (Not to be confused with the cusp shape of peripheral representations, as \( h \) is defined in the whole group \( \pi_1M^3 \).) Conjugating the representation \( \rho \) by a diagonal matrix means replacing the morphism \( h \) by a multiple. Thus, as \( b_1(M^3) = 1 \), there are only two orbits by conjugation in \( \pi^{-1}(\chi_0) \): the trivial and the non-trivial morphism \( h: \pi_1M^3 \to \mathbb{C} \). By looking at the dimension of the stabilizers, these orbits have dimension either 0 (for \( h \) trivial) or 2 (for \( h \) non-trivial). Hence the dimension of \( \pi^{-1}(\chi_0) \) is 2. On the other hand, on components \( Y \) of \( X(M^3, \text{PSL}_2(\mathbb{C})) \) that contain irreducible representations, the generic dimension of \( \pi^{-1} \) is 3, the dimension of \( \text{PSL}_2(\mathbb{C}) \). Since this dimension is upper semi-continuous, the trivial character cannot belong to an irreducible component of \( X(M^3, \text{PSL}_2(\mathbb{C})) \) that contains irreducible characters. Hence it must belong to a component whose characters are all reducible.

Lemma A.3. Let \( \rho_1, \rho_2 \in \text{hom}(\pi_1M^3, \text{SL}_2(\mathbb{C})) \) have the same character. If \( \chi_{\rho_1} = \chi_{\rho_2} \) is nontrivial, then \( \rho_1 \) is acyclic if and only if \( \rho_2 \) is acyclic.

Proof. For every character there is a unique closed orbit by conjugation, so that every other orbit accumulates to it, see [58]. So we may assume that the conjugation orbit of \( \rho_2 \) accumulates to \( \rho_1 \). By semi-continuity, \( \rho_1 \) acyclic implies that so is \( \rho_2 \). Next assume that \( \rho_1 \) is not acyclic. Up to conjugacy, there exists a group homomorphism \( \phi: \pi_1M^3 \to \mathbb{Z} \) and \( \lambda \in \mathbb{C} - \{0, \pm 1\} \) such that
\[ \rho_1(\gamma) = \begin{pmatrix} \lambda^{\phi(\gamma)} & 0 \\ 0 & \lambda^{-\phi(\gamma)} \end{pmatrix} \quad \text{and} \quad \rho_2(\gamma) = \begin{pmatrix} \lambda^{\phi(\gamma)} & f(\gamma) \\ 0 & \lambda^{-\phi(\gamma)} \end{pmatrix} \]
for every \( \gamma \in \pi_1M^3 \). Here \( f: \pi_1M^3 \to \mathbb{C} \) is a crossed morphism: \( f(\gamma_1\gamma_2) = f(\gamma_1) + \lambda^{2\phi(\gamma)}f(\gamma_2) \), for all \( \gamma_1, \gamma_2 \in \pi_1M^3 \). The homology of \( \rho_1 \) decomposes as a direct sum of \( \pi_1M^3 \)-modules: \( \mathbb{C} \oplus \{0\} \) and \( \{0\} \oplus \mathbb{C} \), and both are nonzero (one is dual from the other). The \( \rho_2 \)-module does not decompose, but there is an exact sequence of \( \pi_1M^3 \)-modules
\[ 0 \to \mathbb{C} \oplus \{0\} \to \mathbb{C}^2 \to \mathbb{C} \to 0 \]
where \( \pi_1M^3 \) acts on \( \mathbb{C}^2 \) via \( \rho_2 \), and the action on the other modules is the same as for \( \rho_1 \). From the corresponding long exact sequence in homology it follows easily that \( \rho_2 \) is not acyclic.

Appendix B  Cohomology on the variety or characters

The aim of this appendix is to provide references and proofs for the result in cohomology of Section 4.2.45
B.1 The complete structure

Let $M^3$ be a hyperbolic orientable 3-manifold and

$$\varrho = \tilde{\text{hol}}: \pi_1 M^3 \rightarrow \text{SL}_2(\mathbb{C})$$

a lift of its holonomy. As before denote by

$$\varrho^{k_1,k_2} := \text{Sym}^{k_1,k_2} \circ \varrho: \pi_1 M^3 \rightarrow \text{SL}_{(k_1+1)(k_2+1)}(\mathbb{C}).$$

(57)

Let

$$E_{k_1,k_2} = M \times_{\varrho^{k_1,k_2}} \mathbb{C}^{(k_1+1)(k_2+1)}$$

(58)

be the flat bundle twisted by $\varrho^{k_1,k_2}$ as in (9). In particular its de Rham cohomology is isomorphic to the simplicial cohomology of $\varrho^{k_1,k_2}$ by de Rham theorem (10).

Choosing a Hermitian metric on the fibre of the bundle $E_{k_1,k_2} \rightarrow M^3$ and a Riemannian metric on $M^3$, there is a product on $E_{k_1,k_2}$-valued differential forms $\Omega^*(M^3; E_{k_1,k_2})$ by integration on $M^3$, that we denote by $\langle \cdot, \cdot \rangle$. In particular it makes sense to talk about $L^2$-forms, as the forms with finite norm.

Theorem B.1 (Raghunathan [84]). For $k_1 \neq k_2$, there exists a uniform constant $c_{k_1,k_2} > 0$ with the following property. For every hyperbolic orientable 3-manifold $M^3$, and every differential form $\omega \in \Omega^*(M^3; E_{k_1,k_2})$ with compact support,

$$\langle \Delta \omega, \omega \rangle \geq c_{k_1,k_2} \langle \omega, \omega \rangle.$$

This property implies strong acyclicity, as it yields that the spectrum of $\Delta$ is bounded below by the uniform constant $c_{k_1,k_2} > 0$.

Corollary B.2 (Theorem 4.2). Let $M^3$ be a closed, orientable, hyperbolic 3-manifold, then $\varrho^{k_1,k_2}$ is acyclic for $k_1 \neq k_2$.

In particular $\varrho^k = \text{Sym}^k \circ \tilde{\text{hol}}$ is acyclic for $k \geq 1$.

Remark B.3. This corollary does not hold true when $k_1 = k_2$. Millson [64] showed that it fails when $k_1 = k_2 > 0$ and $M^3$ contains a totally geodesic embedded surface, by means of bending.

B.1.1 The finite volume case

We next discuss the consequences in the finite volume case of Theorem B.1. The following corollary does not assume finite volume.

Corollary B.4. Let $M^3$ be an orientable hyperbolic 3-manifold and $\tilde{\text{hol}}$ a lift of its holonomy. For $k_1 \neq k_2$ every closed $L^2$-form in $\Omega^*(M^3; E_{k_1,k_2})$ is exact.

In particular every element in $H^*(M^3; \varrho^{k_1,k_2})$ is represented by a form that is not $L^2$.

In order to apply this corollary, we need first to compute the homology and cohomology of the peripheral torus. All the information on the dimension is given by $H^0(T^2; \varrho^{k_1,k_2})$, which is the set of invariants of the module $\mathbb{C}^{(k_1+1)(k_2+1)}$ by the action of $\varrho^{k_1,k_2}(\pi_1 T^2)$.

The following implies Item (a) of Proposition 4.20.
Lemma B.5. Let $M^3$ be as above and let $T^2 \subset M^3$ be the peripheral torus. The invariant subspace of the peripheral group is

$$H^0(T^2; g^{k_1,k_2}) \cong \begin{cases} 0 & \text{if } k_1 + k_2 \text{ is odd,} \\
\mathbb{C} & \text{if } k_1 + k_2 \text{ is even.} \end{cases}$$

Proof. The lift of the holonomy restricted to $\pi_1 T^2 \cong \mathbb{Z}^2$ is written as

$$(n_1, n_2) \mapsto (-1)^{(n_1, n_2)} \begin{pmatrix} 1 & n_1 + n_2 \cdot \text{cs} \\
0 & 1 \end{pmatrix}$$

where cs $\in \mathbb{C} - \mathbb{R}$ is the cusp shape in Definition 3.19 and $c: \mathbb{Z}^2 \to \mathbb{Z}/2\mathbb{Z}$ is a surjection (here it is relevant that $c$ is non-trivial). From the representation, it is straightforward that if $k_1 + k_2$ is odd, then there is an element whose eigenvalues are all $(-1)$, and for $k_1 + k_2$ even, the subspace of invariants is generated by the monomial $X^{k_1} \cdot X^{k_2}$.

From Poincaré duality we get information on $H^2$, but also on $H^1$ because $\chi(T^2) = 0$. We also know the dimension of the homology groups by duality. Thus we have:

Corollary B.6. Let $M^3$ and $T^2$ as above.

(a) If $k_1 + k_2$ is odd then

$$H^*(T^2; g^{k_1,k_2}) \cong H_*(T^2; g^{k_1,k_2}) = 0.$$  

(b) If $k_1 + k_2$ is even, then

$$\dim H^i(T^2; g^{k_1,k_2}) = \dim H_i(T^2; g^{k_1,k_2}) = \begin{cases} 1 & \text{for } i = 0, 2, \\
2 & \text{for } i = 1, \\
0 & \text{otherwise.} \end{cases}$$

Theorem B.7 (Theorem 4.19). Let $M^3$ be a hyperbolic orientable 3-manifold with a single cusp, and let $T^2$ be a peripheral torus.

(a) When $k_1 + k_2$ is odd, then $H_*(M^3; g^{k_1,k_2}) = 0$.

(b) When $k_1 + k_2$ is even with $k_1 \neq k_2$, then

(i) $H_i(M^3; g^{k_1,k_2}) = 0$ for $i \neq 1, 2$,

(ii) $H_2(M^3; g^{k_1,k_2}) \cong H_2(T^2; g^{k_1,k_2}) \cong \mathbb{C}$,

(iii) $H_1(M^3; g^{k_1,k_2}) \cong \mathbb{C}$.

Proof. The group $H^0(M^3; g^{k_1,k_2})$ vanishes, as this is the subspace of fixed vectors, and both $\varrho$ and $\text{Sym}^{k_1} \otimes \text{Sym}^{k_2}$ are irreducible. By Corollary B.4, the map $H^1(M^3; T^2; g^{k_1,k_2}) \to H^1(M^3; g^{k_1,k_2})$ vanishes. Thus $H^1(M^3; g^{k_1,k_2}) \to H^1(T^2; g^{k_1,k_2})$ is injective, by the long exact sequence in cohomology of the pair. With Poincaré duality and duality between homology and cohomology, we get Item (b). Poincaré duality also yields that $H^1(T^2; g^{k_1,k_2}) \to H^2(M^3, T^2; g^{k_1,k_2})$ is surjective. Then the lemma follows from the long exact sequence in cohomology and the duality between homology and cohomology.
B.1.2 A basis in cohomology

To describe a basis for $H_i(M^3; g^{k_1,k_2})$ when $i = 1, 2$, recall the cap product defined in Equation (31).

$$\cap: H^0(T^2; g^{k_1,k_2}) \times H_i(T^2; \mathbb{C}) \rightarrow H_i(T^2; g^{k_1,k_2}).$$

Let $i: T^2 \rightarrow M^3$ denote the inclusion, and $i_*: H_j(T^2; g^{k_1,k_2}) \rightarrow H_j(M^3; g^{k_1,k_2})$, the induced map.

**Proposition B.8.** Let $M^3$ and $T^2$ be as above, $k_1 \neq k_2$, $k_1 + k_2$ even. Choose $a \in H^0(T^2; g^{k_1,k_2})$, $a \neq 0$. Then:

(a) $i_*(a \cap [T^2])$ is a basis for $H_2(M^3; g^{k_1,k_2})$, where $[T^2] \in H_2(T^2; \mathbb{Z})$ is a fundamental class.

(b) $i_*(a \cap [\gamma])$ is a basis for $H_1(M^3; g^{k_1,0})$ or $H_1(M^3; g^{0,k_2})$, for any $[\gamma] \in H_1(T^2; \mathbb{Z})$, $[\gamma] \neq 0$.

(c) If $\tilde{a}$ denotes a lift of the holonomy of the complete structure, for any pair of peripheral curves $0 \neq [\gamma_1],[\gamma_2] \in H^1(T^2; \mathbb{Z})$ and for $a \in H^0(T^2; g^{k,0})$

$$a \cup [\gamma_2] = \text{cs}(\gamma_2, \gamma_1) a \cap [\gamma_1]$$

in $H_1(T^2; \rho^{k,0})$.

(d) When $k_1 k_2 \neq 0$, the cap product

$$\cap: H^0(T^2; g^{k_1,k_2}) \times H_1(T^2; \mathbb{C}) \rightarrow H_1(T^2; g^{k_1,k_2})$$

is the trivial map (i.e. identically zero).

**Proof.** For Item (a), $a \cap [T^2]$ is a basis for $H_2(T^2; g^{k_1,k_2})$, by Poincaré duality and Lemma B.5. Item (b) is proved in [60] for $k_2 = 0$. It holds true for $k_1 = 0$ by complex conjugation.

To prove the other two items, we choose a cell decomposition of the torus from a square with opposite edges identified. Namely there is a 2-cell represented by a square, whose sides are two copies of the 1-cells: $e_1$ and $e_2$ (with respective homology classes $[e_1] = [m]$ and $[e_2] = [l]$ respectively) and the vertices are four copies of the 0-cell. Choose lifts to the universal covering so that

$$\partial \tilde{e}^2 = (1 - l)\tilde{e}^1 + (m - 1)\tilde{e}_1^2,$$

where $m$ and $l$ generate $\pi_1 T^2$. We may assume that

$$g(m) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad g(l) = \pm \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$$

where $\eta = \text{cs}(l, m) \in \mathbb{C} - \mathbb{R}$.

We prove now (c). $\text{Sym}^k$ acts on the space of degree $k$ homogeneous polynomials on $X$ and $Y$; the one dimensional space invariant by $g^k = g^{k,0}$ is generated by $a = X^k$ (Lemma B.10). By (59):

$$\partial(X^{k-1}Y \otimes \tilde{e}^2) = -\eta a \otimes \tilde{e}_1 + a \otimes \tilde{e}_1^2$$

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which in cohomology translates to \(-\eta a \cap [m] + a \cap [l] = 0\). This proves (c) for a system of generators \([\{m\}, [l]\}] \) of \(H_2(T^2; \mathbb{Z})\), and it holds in general by linearity.

We prove finally (d). \(\text{Sym}^{k_1} \otimes \text{Sym}^{k_2}\) acts on the space of degree \(k_1\) homogeneous polynomials on \(X\) and \(Y\), multiplied by degree \(k_2\) homogeneous polynomials on \(X\) and \(Y\); the one dimensional space invariant by \(\rho^{k_1,k_2}\) is generated by \(a = X^{k_1} X^{k_2}\). By (59):

\[
\partial(X^{k_1-1} Y X^{k_2} \otimes \bar{e}^2) = -\eta a \otimes \bar{e}_1^3 + a \otimes \bar{e}_1^2 \\
\partial(X^{k_1} X^{k_2-1} Y \otimes e^2) = -\eta a \otimes \tilde{e}_1^1 + a \otimes \tilde{e}_1^2
\]

Namely, in homology \(-\eta a \cap [m] + a \cap [l] = -\eta a \cap [m] + a \cap [l] = 0\). As \(\eta \notin \mathbb{R}\), the claim follows.

**B.2 Generic representations in the distinguished component.**

Next comes the proof of genericity results used in Paragraph 4.3.1. Recall the notation

\[\rho^{k_1,k_2} := \text{Sym}^{k_1,k_2} \rho.\]

Recall also that a character \(\chi \in X_0(M^3)\) is called \((k_1,k_2)\)-exceptional if there exists a representation \(\rho \in \text{hom}(\pi_1 M^3, \text{SL}_2(\mathbb{C}))\) with character \(\chi_\rho = \chi\) such that \(H^0(M^3; \rho^{k_1,k_2}) \neq 0\). The set of \((k_1,k_2)\)-exceptional characters is denoted by \(E^{k_1,k_2}\).

**Lemma B.9.** If \(k_1 \neq k_2\), then a \((k_1,k_2)\)-exceptional character is reducible. In particular the \((k_1,k_2)\)-exceptional set \(E^{k_1,k_2}\) is a finite subset of \(X_0(M^3)\).

**Proof.** In the holomorphic case \((k_2 = 0)\), assume that \(H^0(M^3; \rho^k) \neq 0\). Then there is a non-trivial subspace of \(\mathbb{C}^{k+1}\) fixed by \(\text{Sym}^k(\rho(\pi_1 M^3))\), where \(\rho(\pi_1 M^3)\) denotes the Zariski closure of \(\rho(\pi_1 M^3)\). Since \(\text{Sym}^k\) is irreducible, this means that \(\rho(\pi_1 M^3)\) is not the full group \(\text{SL}_2(\mathbb{C})\), which in the holomorphic setting means that \(\rho\) is reducible.

When \(k_2 \neq 0\), one can only work with the real Zariski closure, and the previous argument yields that either \(\rho\) is reducible or it is contained in a real subgroup conjugate to \(\text{PSL}_2(\mathbb{R})\) or \(\text{SU}(2)\). The restriction of \(\text{Sym}^{k_1} \otimes \text{Sym}^{k_2}\) to those real subgroups is equivalent to \(\text{Sym}^{k_1} \otimes \text{Sym}^{k_2}\), which by Klebsch-Gordan formula decomposes as

\[\text{Sym}^{k_1} \otimes \text{Sym}^{k_2} = \text{Sym}^{k_1+k_2} \oplus \text{Sym}^{k_1+k_2-2} \oplus \ldots \oplus \text{Sym}^{k_1-k_2}.\]  

(60)

As \(k_1 \neq k_2\), the powers of \(\text{Sym}\) in (60) are non-trivial, hence the argument in the holomorphic case applies again.

Recall that we say that a property is generic when it holds true for a non-empty Zariski open subset of \(X_0(M^3)\), and that the ground field is either \(\mathbb{C}\) or \(\mathbb{R}\), depending on whether the discussion is in the holomorphic setting, for \(\text{Sym}^k\), or not, for \(\text{Sym}^{k_1,k_2}\).

**Lemma B.10.** Let \(M^3\) be a hyperbolic manifold with one cusp and let \(k_1 \neq k_2 \in \mathbb{N}\) be such that \(k_1\) or \(k_2\) is odd. Then the set

\[\{\chi_\rho \in X_0(M^3) \mid \dim H^0(T^2; \rho^{k_1,k_2}) = 0\}\]

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is a non-empty Zariski open subset of the curve \( X_0(M^3) \).

**Proof.** By upper semi-continuity of the dimension of the cohomology (see the proof of Proposition \[3.10\]), it suffices to show that \( \dim H^0(T^2; \rho_{k_1,k_2}) = 0 \) for some representation \( \rho \) with character in \( X_0(M^3) \). When \( k_1 \) is odd and \( k_2 \) is even, or vice-versa, then it holds for the lift of the holonomy of the complete structure, by Lemma \[B.5\]. If both \( k_1 \) and \( k_2 \) are odd, consider an orbifold Dehn filling on \( M^3 \) with filling curve \( \gamma \) that consists in adding a solid torus with singular core curve with branching order \( n \). By the Dehn filling theorem, for \( n \) large enough it is hyperbolic and the restriction \( \rho \) of its holonomy has character in \( X_0(M^3) \). The holonomy of the curve \( \gamma \) is conjugate to

\[
\rho(\gamma) = \begin{pmatrix}
e e^{\frac{i\pi}{2}} & 0 \\
0 & e^{-\frac{i\pi}{2}}
\end{pmatrix}.
\]

As the core of the geodesic has non-trivial length, the complex length of its holonomy has nonzero real part. Thus we can find a peripheral curve \( \gamma' \) with holonomy

\[
\rho(\gamma') = \begin{pmatrix}
\lambda & 0 \\
0 & \frac{1}{\lambda}
\end{pmatrix}
\]

with \( \lambda \) not real nor unitary. This yields that \( \rho^{k_1,k_2}(\gamma') \) has no eigenvalue 1, hence \( \dim H^0(T^2; \rho_{k_1,k_2}) = 0 \).

Next lemma implies Lemma \[4.12\] as \( F^{k_1,k_2} = (X_0(M^3) - Z^{k_1,k_2}) \cup E^{k_1,k_2} \).

**Lemma B.11.** Let \( M^3 \) be a hyperbolic manifold with one cusp and let \( k_1 \neq k_2 \in \mathbb{N} \) be such that \( k_1 \) and \( k_2 \) are even.

(a) The set of characters

\[
Z^{k_1,k_2} = \{ \chi_\rho \in X_0(M^3) \mid \dim H_0(T^2; \rho_{k_1,k_2}) = 1 \}
\]

is a non-empty Zariski open subset of the curve \( X_0(M^3) \).

(b) For any \( [\gamma] \in H^1(T^2; \mathbb{Z}) \), \( [\gamma] \neq 0 \), the set

\[
Z^{k_1,k_2}_\gamma = \{ \chi_\rho \in Z^{k_1,k_2} \mid i_* (a \cap [\gamma]) \text{ is a basis for } H_1(M^3; \rho_{k_1,k_2}) \}
\]

is non-empty and Zariski open.

**Proof.** For (a), given any non-trivial peripheral element \( \gamma \in \pi_1 T^2 \), for a generic character \( \chi_\rho \in X_0(M^3) \), \( \rho(\gamma) \) is a diagonal matrix, with eigenvalues different from \( \pm 1 \) and so that \( \rho^{k_1,k_2}(\gamma) \) has only one eigenvalue equal to one. This property is in fact Zariski open (over \( \mathbb{C} \) when \( k_2 = 0 \), over \( \mathbb{R} \) otherwise), cf. the proof of Lemma \[B.10\].

Next comes the proof of (b). Any nonzero element in \( H^1(T^2; \mathbb{Z}) \) is represented by a simple closed curve \( \gamma \). Consider the orbifold \( \mathcal{O}_n \) obtained by Dehn filling \( M^3 \) along \( \gamma \), so that the soul of the solid torus is the branching locus, with branching index \( n \in \mathbb{N} \). For \( n \) sufficiently large, \( \mathcal{O}_n \) is hyperbolic, and its holonomy restricts to a representation with character \( \chi_\rho \in X_0(M^3) \). As \( \text{Sym}^3 \otimes \text{Sym}^2 \) of the holonomy of \( \mathcal{O}_n \) is acyclic, a Mayer-Vietoris argument
yields that $a \cap \gamma$ is a basis for $H^1(M^3; \rho^{k_1,k_2})$. Then the argument for a generic $\rho$ follows from semi-continuity.

Notice that for all but finitely many curves $\gamma$ we may take a trivial branching index $n = 1$ and work with manifolds instead of orbifolds. However, in order to consider all filling slopes, one needs to work with orbifolds.

For any character $\chi_\rho$ in $Z_{k_1,k_2}^k \gamma$, by the long exact sequence in cohomology and Poincaré duality, the inclusion induces an isomorphism, $H_2(M; \rho^{k_1,k_2}) \cong H_2(T^2; \rho^{k_1,k_2})$. In addition, Poincaré duality again gives a natural isomorphism $H_2(T^2; \rho^{k_1,k_2}) \cong H_0(T^2; \rho^{k_1,k_2})$ which yields that $a \cap [T^2]$ is a basis for $H_2(T^2; \rho^{k_1,k_2})$. Thus Lemma B.11(b) yields Proposition 4.13.

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