Determinantal representations of the weighted core-EP, DMP, MPD, and CMP inverses of matrices with quaternion and complex elements.

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Abstract

In this paper, we extend notions of the weighted core-EP right and left inverses, the weighted DMP and MPD inverses, and the CMP inverse to matrices over the quaternion skew field \( \mathbb{H} \) that have some features in comparison to these inverses over the complex field. We give the direct methods of their computing, namely, their determinantal representations by using noncommutative column and row determinants previously introduced by the author. As the special cases, by using the usual determinant, we give their determinantal representations for matrices with complex entries as well. A numerical example to illustrate the main result is given.

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1 Introduction

In the whole article, the notations \( \mathbb{R} \) and \( \mathbb{C} \) are reserved for fields of the real and complex numbers, respectively. \( \mathbb{H}^{m \times n} \) stands for the set of all \( m \times n \) matrices over the quaternion skew field

\[ \mathbb{H}^{m \times n} = \{ h_0 + h_1i + h_2j + h_3k \mid i^2 = j^2 = k^2 = ijk = -1, h_0, h_1, h_2, h_3 \in \mathbb{R} \}. \]

\( \mathbb{H}^{m \times n} \) determines its subset of matrices with a rank \( r \). For given \( h = h_0 + h_1i + h_2j + h_3k \in \mathbb{H} \), the conjugate of \( h \) is \( \overline{h} = a_0 - h_1i - h_2j - h_3k \). For \( A \in \mathbb{H}^{m \times n} \), the symbols \( A^* \) and \( \text{rk}(A) \) specify the conjugate transpose and the rank of \( A \), respectively. A matrix \( A \in \mathbb{H}^{n \times n} \) is Hermitian if \( A^* = A \). The index of \( A \in \mathbb{H}^{n \times n} \), denoted \( \text{Ind} A = k \), is the smallest positive number such that \( \text{rk}(A^{k+1}) = \text{rk}(A^k) \).

Due to [7] the definition of the weighted Drazin inverse can be generalized over \( \mathbb{H} \) as follows.

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Definition 1.1. For \( A \in \mathbb{H}^{m \times n} \) and \( W \in \mathbb{H}^{n \times m} \), the W-weighted Drazin inverse of \( A \) with respect to \( W \), denoted by \( A_{d,W} \), is the unique solution to equations,
\[
(WA)^{k+1}XW = (AW)^k, \quad XWAX = X, \quad AWX = XWA,
\]
where \( k = \text{Ind}(AW) \).

The properties of the complex W-weighted Drazin inverse can be found in [45, 52, 53, 56]. These properties can be generalized to \( \mathbb{H} \). Among them, if \( A \in \mathbb{H}^{m \times n} \) with respect to \( W \in \mathbb{H}^{n \times m} \) and \( k = \max\{\text{Ind}(AW), \text{Ind}(WA)\} \), then
\[
A_{d,W} = A((WA)^d)^2 = ((AW)^d)^2 A.
\]

Let \( A \in \mathbb{H}^{n \times n} \) and \( W = I_n \) be the identity matrix of order \( n \). Then \( X = A^d \) is the Drazin inverse of \( A \). In particular, if \( \text{Ind} A = 1 \), then the matrix \( X \) is called the group inverse and it is denoted by \( X = A^# \).

Using the Penrose equations [41], the Moore-Penrose inverse of a quaternion matrix can be defined as well (see, e.g. [19]).

Definition 1.2. The Moore-Penrose inverse of \( A \in \mathbb{H}^{n \times m} \) is called the exclusive matrix \( X \), denoted by \( A^\dagger \), satisfying the following four equations
\[
AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.
\]

\( P_A := AA^\dagger \) and \( Q_A := A^\dagger A \) are the orthogonal projectors onto the range of \( A \) and the range of \( A^* \), respectively. For \( A \in \mathbb{C}^{n \times m} \), the symbols \( \mathcal{N}(A) \) and \( \mathcal{R}(A) \) will denote the kernel and the range space of \( A \), respectively.

The core inverse was introduced by Baksalary and Trenkler in [2]. Later, it was investigated by S. Malik in [34] and S.Z. Xu et al. in [54], among others.

Definition 1.3. [2] A matrix \( X \in \mathbb{C}^{n \times n} \) is called the core inverse of \( A \in \mathbb{C}^{n \times n} \) if it satisfies the conditions
\[
AX = P_A, \quad \text{and} \quad \mathcal{R}(X) = \mathcal{R}(A).
\]
When such matrix \( X \) exists, it is denoted \( A^\# \).

In 2014, the core inverse was extended to the core-EP inverse defined by K. Manjunatha Prasad and K.S. Mohana [32]. Determinantal formulas for the core EP generalized inverse in complex matrices has been derived in [42] based on the determinantal representation of a reflexive inverse obtained in [15].

Other generalizations of the core inverse were recently introduced for \( n \times n \) complex matrices, namely BT inverses [3], DMP inverses [33], and CMP inverses [35], etc. The characterizations, computing methods, some applications of the core inverse and its generalizations were investigated in complex matrices and rings (see, e.g. [6, 11, 13, 32, 33, 37, 38, 39, 41, 51]).

Only recently generalizations of the core inverse were extended to rectangular matrices by using the weighted Drazin inverse. Among them, the W-weighted core-EP inverse in complex matrices was introduced in [15], its representations and properties were studied in [10], and generalizations of the weighted core-EP inverse were expanded over a ring with involution [38] and Hilbert space [39].
respectively. The concepts of the complex weighted DMP and CMP inverses were introduced in [36] and [39], respectively.

The main goals of this paper are extended the notions of the weighted core-EP inverses, and the weighted DMP and CMP inverses over the quaternion skew-field \( H \), and get their determinantal representations that are the direct methods of their obtaining by using determinants.

The determinantal representation of the usual inverse is the matrix with cofactors in entries that suggests a direct method of finding the inverse of a matrix. The same is desirable for the generalized inverses. But, there are various expressions of determinantal representations of generalized inverses even for matrices with complex or real entries, (see, e.g. [4, 5, 14–16, 46, 47]). In view of the noncommutativity of quaternions, the problem of the determinantal representation of quaternion generalized inverses is evidently dependent on complexities related with definition of the determinant with noncommutative entries (it is also called a noncommutative determinant).

The majorities of the previous defined noncommutative determinants are derived by transforming the quaternion matrix to an equivalent complex or real matrix (see, e.g. [1]). However, by this way it is impossible for us to give determinantal representations of quaternionic generalized inverses. Only now it can be done thanks to the theory of column-row determinants introduced by the author in [17,18]. Currently, by using of row-column determinants, determinantal representations of various generalized inverses have been derived and applied to solutions of quaternion matrix equations by the author (see, e.g. [19,20]), (among them the core inverse and its generalizations in the quaternion [30] and complex [31] cases), and by other researchers (see, e.g. [48–50]).

The paper is organized as follows. In Section 2, we start with preliminary introduction of the theory of row-column determinants and the determinantal representations of the Moore-Penrose inverse, of the Drazin and weighted Drazin inverses, and of the core inverse and its generalizations over the quaternion skew field previously obtained by using row-column determinants. In Section 3, we introduce the concepts of the left and right weighted core-EP inverses over the quaternion skew field and give their determinantal representations. In Section 4, the quaternion weighted DMP and MPD inverses are established and their determinantal representations are obtained. Determinantal representations of the quaternion CMP inverse are get in Section 5. A numerical example to illustrate the main results is considered in Section 6. Finally, in Section 7, the conclusions are drawn.

2 Preliminaries.

2.1 Elements of the theory of row-column determinants.

Suppose \( S_n \) is the symmetric group on the set \( I_n = \{1, \ldots, n\} \).

Definition 2.1. [17] The \( i \)th row determinant of \( A = (a_{ij}) \in H^{n \times n} \) is defined for any \( i \in I_n \) by setting

\[
\text{rdet}_i A = \sum_{\sigma \in S_n} (-1)^{n-r} (a_{i_{k_1}i_{k_1+1} \ldots a_{i_{k_1+i_{k_1+1}}} \ldots a_{i_{k_r}i_{k_r+1} \ldots a_{i_{k_r+i_{k_r+1}}}}),
\]

\[
\sigma = (i \cdot i_{k_1}i_{k_1+1} \ldots i_{k_1+i_{k_1+1}}) (i_{k_2}i_{k_2+1} \ldots i_{k_2+i_{k_2+1}}) \ldots (i_{k_r}i_{k_r+1} \ldots i_{k_r+i_{k_r+1}}),
\]

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where $\sigma$ is the left-ordered permutation. It means that its first cycle from the left starts with $i$, other cycles start from the left with the minimal of all the integers which are contained in it,

$$i_{k_t} < i_{k_{t+s}} \text{ for all } t = 2, \ldots, r, \ s = 1, \ldots, l_t,$$

and the order of disjoint cycles (except for the first one) is strictly conditioned by increase from left to right of their first elements, $i_{k_2} < i_{k_3} < \cdots < i_{k_r}$.

Similarly, for a column determinant along an arbitrary column, we have the following definition.

**Definition 2.2.** [17] The $j$th column determinant of $A = (a_{ij}) \in \mathbb{H}^{n \times n}$ is defined for any $j \in I_n$ by setting

$$cdet_j A = \sum_{\tau \in S_n} (-1)^{n-\tau}(a_{j_{k_1}, j_{k_1+1}, \ldots} a_{j_{k_{s-1}}, j_{k_s}} \ldots a_{j_{k_{t-1}}, j_{k_t}} a_{j_{k_t+1}, j_{k_{t+s}}}),$$

$$\tau = (j_{k_1}, \ldots, j_{k_{c-1}}, j_{k_c}) \cdots (j_{k_{l-1}}, \ldots, j_{k_l}) (j_{k_{l+1}}, \ldots, j_{k_{l+s}}),$$

where $\tau$ is the right-ordered permutation. It means that its first cycle from the right starts with $j$, other cycles start from the right with the minimal of all the integers which are contained in it,

$$j_{k_t} < j_{k_{t+s}} \text{ for all } t = 2, \ldots, r, \ s = 1, \ldots, l_t,$$

and the order of disjoint cycles (except for the first one) is strictly conditioned by increase from right to left of their first elements, $j_{k_2} < j_{k_3} < \cdots < j_{k_r}$.

The row and column determinants have the following linear properties.

**Lemma 2.1.** [17] If the $i$th row of $A \in \mathbb{H}^{n \times n}$ is a left linear combination of some row vectors, i.e. $a_i = \alpha_1 b_1 + \cdots + \alpha_k b_k$, where $\alpha_l \in \mathbb{H}$ and $b_l \in \mathbb{H}^{1 \times n}$ for all $l = 1, \ldots, k$ and $i = 1, \ldots, n$, then

$$rdet_i A_i (\alpha_1 b_1 + \cdots + \alpha_k b_k) = \sum_l \alpha_l rdet_i A_i (b_l).$$

**Lemma 2.2.** [17] If the $j$th column of $A \in \mathbb{H}^{n \times n}$ is a right linear combination of other column vectors, i.e. $a_j = c_1 \alpha_1 + \cdots + c_k \alpha_k$, where $\alpha_l \in \mathbb{H}$ and $c_l \in \mathbb{H}^{n \times 1}$ for all $l = 1, \ldots, k$ and $j = 1, \ldots, n$, then

$$cdet_j A_j (c_1 \alpha_1 + \cdots + c_k \alpha_k) = \sum_l cdet_j A_j (c_l) \alpha_l.$$

So, an arbitrary $n \times n$ quaternion matrix inducts $n$ row determinants and $n$ column determinants that are different in general. Only for a Hermitian matrix $A$, we have [17],

$$rdet_1 A = \cdots = rdet_n A = cdet_1 A = \cdots = cdet_n A \in \mathbb{R},$$

that enables to define the determinant of a Hermitian matrix by setting $\det A := rdet_i A = cdet_i A$ for all $i = 1, \ldots, n$. Its properties have been completely studied in [18]. In particular, from them it follows the definition of the determinantal rank of a quaternion matrix $A$ as the largest possible size of nonzero principal minors of its corresponding Hermitian matrices, i.e. $rk A = rk(A^+ A) = rk(\overline{A A^*})$. 

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2.2 Determinantal representations of generalized inverses.

Let \( \alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\} \) and \( \beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\} \) be subsets with \( 1 \leq k \leq \min\{m, n\} \). By \( A^\beta_\alpha \) denote a submatrix of \( A \in \mathbb{H}^{m \times n} \) with rows and columns indexed by \( \alpha \) and \( \beta \), respectively. Then, \( A^\beta_\alpha \) is a principal submatrix of \( A \) with rows and columns indexed by \( \alpha \). Moreover, for Hermitian \( A \), \( |A|_{\alpha}^\beta \) is the corresponding principal minor of \( \det A \). Suppose that

\[
L_{k,n} := \{\alpha : \alpha = (\alpha_1, \ldots, \alpha_k), \ 1 \leq \alpha_1 < \cdots < \alpha_k \leq n\}
\]

stands for the collection of strictly increasing sequences of \( 1 \leq k \leq n \) integers chosen from \( \{1, \ldots, n\} \). For fixed \( i \in \alpha \) and \( j \in \beta \), put \( I_{r,m}\{i\} := \{\alpha : \alpha \in I_{r,m}, i \in \alpha\} \), \( J_{r,n}\{j\} := \{\beta : \beta \in I_{r,n}, j \in \beta\} \).

Denote by \( a_{ij} \) and \( a_i^\alpha \) the \( j \)-th column and the \( i \)-th row of \( A \). Similarly, \( a_{ij}^\alpha \) and \( a_j^\alpha \) stand for the \( j \)-th column and the \( i \)-th row of \( A^\dagger \). By \( A_i \), \( (b) \) and \( A_j (c) \) we denote the matrices obtained from \( A \) by replacing its \( i \)-th row with the row \( b \) and its \( j \)-th column with the column \( c \), respectively.

**Theorem 2.3.** \([20]\) If \( A \in \mathbb{H}^{m \times n}_r \), then the Moore-Penrose inverse \( A^\dagger = \left( a_{ij}^\dagger \right) \in \mathbb{H}^{n \times m} \) possess the determinantal representations

\[
a_{ij}^\dagger = \sum_{\beta \in J_{r,n}\{i\}} \frac{\text{cdet}_i \left( (A^\ast A)^i \right)_{\alpha}^\beta}{\sum_{\beta \in J_{r,n}} |A^\ast A|^\beta_{\alpha}} = \sum_{\alpha \in I_{r,m}\{j\}} \frac{\text{rdet}_j \left( (AA^\ast)^\dagger \right)_{\alpha}^\beta}{\sum_{\alpha \in I_{r,m}} |AA^\ast|^\alpha_{\alpha}}. \tag{2.1}
\]

**Remark 2.4.** For an arbitrary full-rank matrix \( A \in \mathbb{H}^{m \times n}_r \), a row-vector \( b \in \mathbb{H}^{1 \times n} \), and a column-vector \( c \in \mathbb{H}^{n \times 1} \), we put, respectively,

- when \( r = m \)

\[
\text{rdet}_i \left( (AA^\ast)^i \right)_{\alpha} = \sum_{\alpha \in I_{r,m}\{i\}} \text{det} \left( (AA^\ast)^\dagger \right)_{\alpha},
\]

\[
\text{det} (AA^\ast) = \sum_{\alpha \in I_{r,m}} |AA^\ast|^\alpha_{\alpha} \quad i = 1, \ldots, m;
\]

- when \( r = n \)

\[
\text{cdet}_j \left( (A^\ast A)^j \right)_{\alpha} = \sum_{\beta \in J_{n,m}\{j\}} \text{cdet}_j \left( (A^\ast A)^{\dagger} \right)_{\beta},
\]

\[
\text{det} (A^\ast A) = \sum_{\beta \in J_{n,m}} |A^\ast A|^\beta_{\beta} \quad j = 1, \ldots, n.
\]

**Corollary 2.1.** If \( A \in \mathbb{H}^{m \times n}_r \), then the determinantal representations of the projection matrices \( A^\dagger A =: Q_A = \left( q_{ij}^A \right)_{n \times n} \) and \( AA^\dagger =: P_A = \left( p_{ij}^A \right)_{m \times m} \) can
be expressed as follows

\[ q_{ij}^A = \frac{\sum_{\beta \in J_{r,n}(i)} \text{cldet}_{i}((A^*A)_{i} \hat{a}_j)_{\beta}}{\sum_{\beta \in J_{r,n}} |A^*A|_{\beta}^{\alpha}} = \frac{\sum_{\alpha \in I_{r,n}(j)} \text{rdet}_{j} ((A^*A)_{j} \hat{a}_i)_{\alpha}}{\sum_{\alpha \in I_{r,n}} |A^*A|_{\alpha}^{\beta}}, \quad (2.3) \]

\[ p_{ij}^A = \frac{\sum_{\alpha \in I_{r,m}(j)} \text{rdet}_{j}((AA^*)_{j} \bar{a}_i)_{\alpha}}{\sum_{\alpha \in I_{r,m}} |AA^*|_{\alpha}^{\alpha}} = \frac{\sum_{\beta \in J_{r,m}(i)} \text{cldet}_{i}((AA^*)_{i} \bar{a}_j)_{\beta}}{\sum_{\beta \in J_{r,m}} |AA^*|_{\beta}^{\beta}}, \quad (2.4) \]

where \( \hat{a}_j \) and \( \bar{a}_i \), \( \hat{a}_i \), and \( \bar{a}_j \) are the \( i \)-th rows and the \( j \)-th columns of \( A^*A \in \mathbb{H}^{n \times n} \) and \( AA^* \in \mathbb{H}^{m \times m} \), respectively.

The following corollary gives determinantal representations of the Moore-Penrose inverse and of both projectors in complex matrices.

**Corollary 2.2.** \[19\] Let \( A \in \mathbb{C}^{m \times n} \). Then the following determinantal representations are obtained

(i) for the Moore-Penrose inverse \( A^\dagger = (a^\dagger_{ij})_{n \times m} \),

\[ a^\dagger_{ij} = \frac{\sum_{\beta \in J_{r,n}(i)} |(A^*A)_{i} \hat{a}_j|_{\beta} \beta}{\sum_{\beta \in J_{r,n}} |A^*A|_{\beta}^{\alpha}} = \frac{\sum_{\alpha \in I_{r,m}(j)} |(AA^*)_{j} \bar{a}_i|_{\alpha} \alpha}{\sum_{\alpha \in I_{r,m}} |AA^*|_{\alpha}^{\alpha}}; \]

(ii) for the projector \( Q_A = (q_{ij})_{n \times n} \),

\[ q_{ij} = \frac{\sum_{\beta \in J_{r,n}(i)} |(A^*A)_{i} \hat{a}_j|_{\beta} \beta \beta}{\sum_{\beta \in J_{r,n}} |A^*A|_{\beta}^{\alpha}} \]

where \( \hat{a}_j \) is the \( j \)-th column of \( A^*A \);

(iii) for the projector \( P_A = (p_{ij})_{m \times m} \),

\[ p_{ij} = \frac{\sum_{\alpha \in I_{r,m}(j)} |(AA^*)_{j} \bar{a}_i|_{\alpha} \alpha \alpha}{\sum_{\alpha \in I_{r,m}} |AA^*|_{\alpha}^{\alpha}} \]

where \( \bar{a}_i \) is the \( i \)-th row of \( AA^* \).

There are two case for determinantal representations of the W-weighted Drazin inverse over the quaternion skew field.

**Lemma 2.3.** \[22\] Let \( A \in \mathbb{H}^{m \times n} \), \( W \in \mathbb{H}^{n \times m} \), \( k = \max \{ \text{Ind} (AW), \text{Ind} (WA) \} \). Denote \( AW = V = (v_{ij})_{n \times m} \) and \( WA = U = (u_{ij})_{n \times n} \). Then for \( A_{d,W} = (a_{d,W}^{ij})_{n \times m} \), we have
(i) if \( \text{rk} U^{k+1} = \text{rk} U^k = r \),

\[
d_{ij}^{d,W} = \frac{\sum_{s=1}^{n} \left( \sum_{\alpha \in I_{r,n} \{s\}} \text{rdet}_s \left( \left( U^{2k+1} (U^{2k+1})^* \right)_{s} (\tilde{\phi}_i) \right) \alpha \right) \left( U^{2k+1} (U^{2k+1})^* \right)^{10}_a u^{(k)}_{s_j}}{\left( \sum_{\alpha \in I_{r,n}} \left| U^{2k+1} (U^{2k+1})^* \right| \alpha \right)^2}, \tag{2.5}
\]

where \( \tilde{\phi}_i \) is the \( i \)-th row of \( \tilde{\Phi} := A \Phi U^{2k+1} \in \mathbb{H}^{m \times n} \), and \( \Phi = (\phi_{tq}) \in \mathbb{H}^{n \times n} \) such that

\[
\phi_{tq} = \sum_{\alpha \in I_{r,n} \{q\}} \text{rdet}_q \left( \left( U^{2k+1} (U^{2k+1})^* \right)_{q} (\bar{u}_t) \right) \alpha.
\]

Here \( \bar{u}_t \) is the \( t \)-th row of \( U^{k}(U^{2k+1})^* =: \bar{U} \in \mathbb{H}^{n \times n} \).

(ii) if \( \text{rk} V^{k+1} = \text{rk} V^k = r \),

\[
d_{ij}^{d,W} = \frac{\sum_{t=1}^{m} v^{(k)}_{it} \sum_{\beta \in J_{r,m} \{t\}} \text{cdet}_t \left( \left( V^{2k+1} V^{2k+1} \right)_{t} (\tilde{\psi}_j) \right) \beta}{\left( \sum_{\beta \in J_{r,m}} \left| V^{2k+1} V^{2k+1} \right| \beta \right)^2}, \tag{2.6}
\]

where \( \tilde{\psi}_j \) is the \( j \)-th column of \( \tilde{\Psi} := (V^{2k+1} V^{2k} \Psi A \in \mathbb{H}^{m \times n} \), and \( \Psi = (\psi_{st}) \in \mathbb{H}^{m \times m} \) such that

\[
\psi_{st} = \sum_{\beta \in J_{r,m} \{s\}} \text{cdet}_s \left( \left( V^{2k+1} V^{2k+1} \right)_{s} (\bar{v}_t) \right) \beta.
\]

Here \( \bar{v}_t \) is the \( t \)-th column of \( (V^{2k+1})^* V^k =: \bar{V} \in \mathbb{H}^{m \times m} \).

**Lemma 2.4.** \([24]\)** Let \( A \in \mathbb{H}^{m \times n} \), \( W \in \mathbb{H}^{n \times m} \), \( k = \text{max} \{ \text{Ind} (AW) , \text{Ind} (WA) \} \).

Then for \( A_{d,W} = \left( a_{ij}^{d,W} \right) \in \mathbb{H}^{m \times m} \), we have

(i) if the matrix \( AW \in \mathbb{H}^{m \times m} \) is Hermitian and \( \text{rk}(AW)^{k+1} = \text{rk}(AW)^k = r \), then

\[
a_{ij}^{d,W} = \frac{\sum_{\beta \in J_{r,m} \{j\}} \text{cdet}_j \left( (AW)^{k+2} (\bar{v}_j) \right) \beta}{\sum_{\beta \in J_{r,m}} \left| (AW)^{k+2} \right| \beta}, \tag{2.7}
\]

where \( \bar{v}_j \) is the \( j \)-th column of \( \bar{V} = (AW)^k A \) for all \( j = 1, \ldots, n \).
Corollary 2.3. \[ \text{Then } A \alpha, \beta \text{ for any } H \]

Definition 2.5. \[ \text{introduced by the following definition.} \]

We denote them by \[ \bar{u} \text{ scalar right-multiplying, and quaternion row-vectors form a left vector} \]

\[ H \text{spaces that} \]

\[ \text{with quaternion-scalar left-multiplying. We denote them by} \]

\[ \text{Moreover, } H \text{ is Hermitian and } k \] \[ \text{is the } i \text{-th row of } \bar{U} = A(WA)^k \text{ for all } i = 1, \ldots, m. \]

The following corollary gives determinantal representations of the W-weighted Drazin inverse in complex matrices.

Corollary 2.3. \[ \text{Let } A \in \mathbb{H}^{n \times n}, \text{ W } \in \mathbb{H}^{n \times m}, k = \max\{\text{Ind (AW)}, \text{Ind (WA)}\}. \]

Then \[ A_{d,W} = \left( a_{d,W}^{i,j} \right) \in \mathbb{H}^{m \times n} \text{ can be expressed as} \]

\[ a_{d,W} = \frac{\sum_{\alpha \in I_{r,n} \backslash \{j\}} \text{rdet}_j \left( (WA)_{j}^{k+2} (\bar{u}_i) \right)^\alpha}{\sum_{\alpha \in I_{r,n}} \left| (WA)^{k+2} \right|^\alpha}. \]

where \[ \bar{u}_i \text{ is the } i \text{-th row of } \bar{U} = A(WA)^k \text{ and } \bar{v}_j \text{ is the } j \text{-th column of } \bar{V} = (AW)^k A. \]

Quaternion column-vectors form a right vector \( \mathbb{H} \)-space with quaternion-scalar right-multiplying, and quaternion row-vectors form a left vector \( \mathbb{H} \)-space with quaternion-scalar left-multiplying. We denote them by \( H_r \) and \( H_l \), respectively. Moreover, \( H_r \) and \( H_l \) possess corresponding \( \mathbb{H} \)-valued inner products by putting

\[ \langle x, y \rangle_r = \bar{y}_1 x_1 + \cdots + \bar{y}_n x_n \text{ for } x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n \in H_r, \]

\[ \langle x, y \rangle_l = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n \text{ for } x, y \in H_l, \]

that satisfy the inner product relations, namely, conjugate symmetry, linearity, and positive-definiteness but with specialties

\[ \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \text{ when } x, y, z \in H_r, \]

\[ \langle ax + by, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \text{ when } x, y, z \in H_l, \]

for any \( \alpha, \beta \in \mathbb{H} \).

So, an arbitrary quaternion matrix induce right and left vector \( \mathbb{H} \)-spaces that introduced by the following definition.

Definition 2.5. For an arbitrary matrix over the quaternion skew field, \( A \in \mathbb{H}^{m \times n} \), we denote by

- \( C_r(A) = \{ y \in \mathbb{H}^{n \times 1} : y = Ax, x \in \mathbb{H}^{n \times 1} \} \), the right column space of \( A \),
- \( N_r(A) = \{ x \in \mathbb{H}^{n \times 1} : Ax = 0 \} \), the right null space of \( A \),
- \( R_l(A) = \{ y \in \mathbb{H}^{1 \times n} : y = xA, x \in \mathbb{H}^{1 \times m} \} \), the left row space of \( A \),
- \( N_l(A) = \{ x \in \mathbb{H}^{1 \times m} : xA = 0 \} \), the left null space of \( A \).
2.3 Determinantal representations of the core inverses and the core-EP inverses

Because of quaternion noncommutativity, Definition 1.3 can be expanded to matrices over $\mathbb{H}$ as follows.

**Definition 2.6.** A matrix $X \in \mathbb{H}^{n \times n}$ is said to be the right core inverse of $A \in \mathbb{H}^{n \times n}$ if it satisfies the conditions $AX = P_A$, and $C_r(X) = C_r(A)$.

When such matrix $X$ exists, it is denoted $A^\oplus$.

**Definition 2.7.** A matrix $X \in \mathbb{H}^{n \times n}$ is said to be the left core inverse of $A \in \mathbb{H}^{n \times n}$ if it satisfies the conditions $XA = Q_A$, and $R_l(X) = R_l(A)$.

When such matrix $X$ exists, it is denoted $A^\odot$.

Similar as in [42], we introduce two core-EP inverses over quaternions.

**Definition 2.8.** A matrix $X \in \mathbb{H}^{n \times n}$ is said to be the right core-EP inverse of $A \in \mathbb{H}^{n \times n}$ if it satisfies the conditions $XAX = A$, and $C_r(X) = C_r(X^*) = C_r(A^d)$.

It is denoted $A^{\dagger}$.

The lemma below give characterization of the right core-EP inverse. Due to [42], the right weighted core-EP inverse is characterized in terms of three equations.

**Lemma 2.5.** Let $A, X \in \mathbb{H}^{n \times n}$ be such that $\text{Ind}(A) = k$. Then $X$ is the right core-EP inverse of $A$ if and only if $X$ satisfies the conditions:

$$XA^{k+1} = A^k, \ AX^2 = X, \ (AX)^* = AX \ \text{and} \ \ C_r(X) \subseteq C_r(A^k).$$

Taking into account ([42], Theorem 2.3), the following expression can be extended to quaternion matrices.

**Lemma 2.6.** Let $A \in \mathbb{H}^{n \times n}$ and let $l$ be a non-negative integer such that $l \geq k = \text{Ind}(A)$. Then $A^\ominus = A^dA^l(A^l)^\dagger$.

**Definition 2.9.** A matrix $X \in \mathbb{H}^{n \times n}$ is said to be the left core-EP inverse of $A \in \mathbb{H}^{n \times n}$ if it satisfies the conditions

$$XAX = A, \ \text{and} \ \ R_l(X) = R_l(X^*) = R_l(A^d).$$

It is denoted $A^{\ddagger}$.

**Remark 2.10.** Since $C_r((A^*)^d) = R_l(A^d)$, then the left core inverse $A^\ddagger$ of $A \in \mathbb{C}^{n \times n}$ is similar to the *core inverse introduced in [42], and the dual core-EP inverse introduced in [55].
Similarly, we have the following characterization of the left core-EP inverse.

**Theorem 2.11.** Let $X, A \in \mathbb{H}^{n \times n}$ and let $l$ be a non-negative integer such that $l \geq k = \text{Ind}(A)$. The following statements are equivalent:

(i) $X$ is the left core-EP inverse of $A$.

(ii) $A^{k+1}X = A^k$, $X^2A = X$, $(XA)^* = XA$ and $R_l(X) \subseteq R_l(A^k)$.

(iii) $X = A_\circ \circ = (A^l)\dagger A^lA^d$.

Thanks to [9], there exists the simple relation between the left and right core-EP inverses, $(A^\circ)^* = (A^\circ)_{\circ}$. So, it is enough to investigate the left core-EP inverse, and right core-EP inverse case can be investigated analogously. But in [30], we gave separately determinantal representations of both core-EP inverses.

**Theorem 2.12.** Suppose $A \in \mathbb{H}^{n \times n}$, $\text{Ind} A = k$, $\text{rk} A^k = s$, and there exist $A^\circ$ and $A_{\circ}$. Then $A^\circ = (a_{ij}^{\circ,r})$ and $A_{\circ} = (a_{ij}^{\circ,l})$ possess the determinantal representations, respectively,

\[
\begin{align*}
  a_{ij}^{\circ,r} &= \frac{\sum_{\alpha \in I_{s,n}(j)} \text{rdet}_j \left( \left( A^{k+1}(A^{k+1})^* \right)_j \left( \hat{a}_{i \cdot} \right) \right)^\alpha}{\sum_{\alpha \in I_{s,n}} \left| A^{k+1}(A^{k+1})^* \right|_\alpha^\alpha}, \\
  a_{ij}^{\circ,l} &= \frac{\sum_{\beta \in J_{s,n}(i)} \text{cdet}_i \left( \left( (A^{k+1})^*(A^{k+1})^* \right)_i \left( \hat{a}_{\cdot j} \right) \right)^\beta}{\sum_{\beta \in J_{s,n}} \left| (A^{k+1})^*(A^{k+1})^* \right|_\beta^\beta},
\end{align*}
\]

where $\hat{a}_{i \cdot}$ is the $i$-th row of $\hat{A} = A^k(A^{k+1})^*$ and $\hat{a}_{\cdot j}$ is the $j$-th column of $\hat{A} = (A^{k+1})^*A^k$.

**Corollary 2.4.** Suppose $A \in \mathbb{H}^{n \times n}$, $\text{Ind} A = 1$, and there exist $A^\circ$ and $A_{\circ}$. Then $A^\circ = (a_{ij}^{\circ,r})$ and $A_{\circ} = (a_{ij}^{\circ,l})$ can be expressed as follows

\[
\begin{align*}
  a_{ij}^{\circ,r} &= \frac{\sum_{\alpha \in I_{1,n}(j)} \text{rdet}_j \left( \left( A^2(A^2)^* \right)_j \left( \hat{a}_{i \cdot} \right) \right)^\alpha}{\sum_{\alpha \in I_{1,n}} \left| A^2(A^2)^* \right|_\alpha^\alpha}, \\
  a_{ij}^{\circ,l} &= \frac{\sum_{\beta \in J_{1,n}(i)} \text{cdet}_i \left( \left( (A^2)^*(A^2)^* \right)_i \left( \hat{a}_{\cdot j} \right) \right)^\beta}{\sum_{\beta \in J_{1,n}} \left| (A^2)^*(A^2)^* \right|_\beta^\beta},
\end{align*}
\]

where $\hat{a}_{i \cdot}$ is the $i$-th row of $\hat{A} = A(A^2)^*$ and $\hat{a}_{\cdot j}$ is the $j$-th column of $\hat{A} = (A^2)^*A$.

The following corollary gives determinantal representations of the right and left, core and core-EP inverses for complex matrices.
Corollary 2.5. Suppose \( A \in \mathbb{C}^{n \times n} \), \( \text{Ind} A = k \), and there exist \( \textbf{A}^\oplus = (a_{ij}^{\oplus,r}) \) and \( \textbf{A}^\oplus = (a_{ij}^{\oplus,l}) \). Then they have the following determinantal representations, respectively,

\[
a_{ij}^{\oplus,r} = \sum_{\alpha \in I_{s,n} \{j\}} \left| \left( A^{k+1} (A^{k+1})^\ast \right)_j \hat{a}_i \right|^\alpha, \\
a_{ij}^{\oplus,l} = \sum_{\beta \in J_{s,n} \{i\}} \left| \left( (A^{k+1})^\ast A^{k+1} \right)_i \hat{a}_j \right|^\beta,
\]

where \( \hat{a}_i \) is the \( i \)-th row of \( \hat{A} = A(A^{k+1})^\ast \) and \( \hat{a}_j \) is the \( j \)-th column of \( \hat{A} = (A^{k+1})^\ast A^{k} \).

If \( \text{Ind} A = 1 \), then \( \textbf{A}^\oplus = (a_{ij}^{\oplus,r}) \) and \( \textbf{A}^\oplus = (a_{ij}^{\oplus,l}) \) have the following determinantal representations, respectively,

\[
a_{ij}^{\oplus,r} = \sum_{\alpha \in I_{s,n} \{j\}} \left| \left( A^2 (A^2)^\ast \right)_j \hat{a}_i \right|^\alpha, \\
a_{ij}^{\oplus,l} = \sum_{\beta \in J_{s,n} \{i\}} \left| \left( (A^2)^\ast A^2 \right)_i \hat{a}_j \right|^\beta,
\]

where \( \hat{a}_i \) is the \( i \)-th row of \( \hat{A} = A(A^2)^\ast \) and \( \hat{a}_j \) is the \( j \)-th column of \( \hat{A} = (A^2)^\ast A \).

3 Concepts of quaternion W-weighted core-EP inverses and their determinantal representations

The concept of the W-weighted core-EP inverse in complex matrices was introduced by Ferreyra et al. \[13\] that can be expended to quaternion matrices as follows.

**Definition 3.1.** Suppose \( \textbf{A} \in \mathbb{H}^{m \times n}, \textbf{W} \in \mathbb{H}^{n \times m} \), and \( k = \max \{ \text{Ind}(\textbf{WA}), \text{Ind}(\textbf{AW}) \} \). The right W-weighted core-EP inverse of \( \textbf{A} \) is the unique solution to the system

\[
\textbf{WAX} = (\textbf{WA})^k \left( (\textbf{WA})^k \right)^\dagger, \quad \text{and} \ \mathcal{C}_r(\textbf{X}) \subseteq \mathcal{C}_r \left( (\textbf{AW})^k \right).
\]

It is denoted \( \textbf{A}^{\oplus, W,r} \).

Due to \[42\], the right weighted core-EP inverse over the quaternion skew field can be determined as follows.
Theorem 3.2. Let \( A, X \in \mathbb{H}_m^{m \times n}, W \in \mathbb{H}_n^{n \times m}, \) and \( k = \max\{\text{Ind}(WA), \text{Ind}(AW)\} \). The following statements are equivalent:

(i) \( X \) is the right weighted core-EP inverse of \( A \).

(ii) \( X W (AW)^{k+1} = (AW)^k \), \( A WXW = X \), and \( (WAW)^* = WAW \).

(iii) 
\[
X = A \left[ (WA)^\ominus \right]^2.
\] (3.1)

We propose to introduce a left weighted core-EP inverse as well.

Definition 3.3. Suppose \( A \in \mathbb{H}_m^{m \times n}, W \in \mathbb{H}_n^{n \times m}, \) and \( k = \max\{\text{Ind}(WA), \text{Ind}(AW)\} \). The left \( W \)-weighted core-EP inverse of \( A \) is the unique solution to the system

\[
XW A W = \left( (AW)^k \right)^\dagger (AW)^k, \quad \text{and} \quad \mathcal{R}_l(X) \subseteq \mathcal{R}_l \left( (WA)^k \right).
\] (3.2)

It is denoted \( A^{\ominus W,l} \).

Theorem 3.4. Let \( A, X \in \mathbb{H}_m^{m \times n}, W \in \mathbb{H}_n^{n \times m}, \) and \( k = \max\{\text{Ind}(WA), \text{Ind}(AW)\} \). The following statements are equivalent:

(i) \( X = \left[ (AW)^\ominus \right]^2 A \).

(ii) \( X \) is the left weighted core-EP inverse of \( A \).

(iii) \( X \) is the unique solution to the three equations:

\[
(WA)^{k+1} WX = (WA)^k, \quad XWXWA = X, \quad \text{and} \quad (XWAW)^* = XWAW.
\] (3.3)

Proof. (i) \(\rightarrow\) (ii) We show that \( X = \left[ (AW)^\ominus \right]^2 A \) satisfies the condition (3.2).

Indeed, 
\[
XWAW = \left[ (AW)^\ominus \right]^2 AWA W = \left[ (AW)^k \right]^\dagger (AW)^k \, (AW)^d A W = \left[ (AW)^k \right]^\dagger (AW)^k, \
\]
\[
\left[ (AW)^\ominus \right]^2 A = \left[ (AW)^\ominus \right]^3 AWA = \ldots = \left[ (AW)^\ominus \right]^{k+2} A (WA)^k \quad \text{i.e.} \\
\mathcal{R}_l(\left[ (AW)^\ominus \right]^{k+2} A (WA)^k) \subseteq \mathcal{R}_l \left( (WA)^k \right).
\]

(ii) \(\rightarrow\) (iii) Now, we prove that \( X = \left[ (AW)^\ominus \right]^2 A \) satisfies the equations in (3.3).

Since \((WA)^{k+1} W = W (AW)^{k+1}\) and due to Theorem 2.12,

\[
(AW)^\ominus = \left( (AW)^k \right)^\dagger (AW)^k (AW)^d.
\]
then
\[(WA)^{k+1} W [(AW)_φ]^2 A = W [(AW)^{k+1} (AW)_φ] (WA)_φ A = W (AW)^k (AW)\dagger (AW)^d A = W (AW)^k (AW)^d A = (WA)^{k+1} (WA)^d = (WA)^k.\]

By Theorem 2.12, taking into account \([(AW)_φ]^2 AW = (AW)_φ], we have
\[[(AW)_φ]^2 AW [(AW)_φ]^2 AW = [(AW)_φ]^2 AW (AW)_φ A = [(AW)_φ]^2 A.\]

Finally,
\[\left([(AW)_φ]^2 AW\right)^* = (AW)_φ AW = (AW)_φ AW.\]

Now, we prove the uniqueness of \(X\). Let
\[(WA)^{k+1} WX = (WA)^k, \quad XWXWA = X, \quad (XWAW)^* = XWAW, \quad \text{and} \quad X = [(AW)_φ]^2 A.\]

Suppose there also exists the left weighted core-EP inverse \(Y\) such that
\[(WA)^{k+1} WY = (WA)^k, \quad YWYWYA = Y, \quad \text{and} \quad (YWAW)^* = YWAW.\]

We show that \(Y = X = [(AW)_φ]^2 A\). So,
\[Y = YWYWYA = Y (WY)^2 (WA)^2 = Y (WY)^k (WA)^k = Y (WY)^k (WA)^{k+1} WX = YW^{k+1} (AW)^{k+1} X = (YWAW)^{k+1} X = \left([(AW)_φ]^2 \right)^{k+1} X = \left([(AW)^{2k+1}]^\dagger (AW)^{2k+1} [(AW)_φ]^2 A.\right.\]

By Theorem 2.11, \((AW)_φ = [(AW)^{2k+1}]^\dagger (AW)^{2k+1} (AW)_φ\). So,
\[Y = \left[(AW)^{2k+1}\right]^\dagger (AW)^{2k+1} [(AW)^{2k+1}]^\dagger (AW)^d (AW)_φ A = \left[(AW)^{2k+1}\right]^\dagger (AW)^{2k+1} (AW)_φ A = [(AW)_φ]^2 A.\]

Finally, from the uniqueness of \(X\) follows \((iii) \rightarrow (i).\)

Now, we give determinantal representations of the quaternion \(W\)-weighted core-EP inverses.
Theorem 3.5. Suppose $A \in \mathbb{H}^{m \times n}$, $W \in \mathbb{H}^{n \times m}$, and $k = \max\{\text{Ind}(WA), \text{Ind}(AW)\}$. Then the right weighted core-EP inverse $A^\Phi_\phi W_r = \left( a^\Phi_{ij} W_r \right) \in \mathbb{H}^{m \times n}$ possess the determinantal representations

$$a^\Phi_{ij} W_r = \sum_{\alpha \in \mathcal{I}_s \setminus \{j\}} \text{rdet}_j \left( \left( U^k \left( U^{k+1} \right)^* \right)_j \left( \tilde{\phi}_{i_\alpha} \right) \right)_\alpha^\alpha,$$

(3.4)

where $\tilde{\phi}_{i_\alpha}$ is the $i$-th row of $\tilde{\Phi} = \Phi U^k \left( U^{k+1} \right)^*$.

Proof. Taking into account (3.1) for $U$, we get

$$a^\Phi_{ij} W_r = \sum_{l=1}^n \sum_{j=1}^n a_{il} u_{lj} W_r u_{lj}.$$

(3.5)

Using the determinantal representation (2.9) of $U^\Phi$ gives

$$\phi_{il} = \sum_{\alpha \in \mathcal{I}_s \setminus \{j\}} \sum_{\alpha \in \mathcal{I}_s \setminus \{f\}} \text{rdet}_f \left( \left( U^k \left( U^{k+1} \right)^* \right)_f \left( \tilde{\phi}_{i_\alpha} \right) \right)_\alpha^\alpha =$$

$$= \frac{\sum_{\alpha \in \mathcal{I}_s \setminus \{f\}} \text{rdet}_f \left( \left( U^k \left( U^{k+1} \right)^* \right)_f \left( \tilde{\phi}_{i_\alpha} \right) \right)_\alpha^\alpha}{\sum_{\alpha \in \mathcal{I}_s \setminus \{f\}} \left( U^k \left( U^{k+1} \right)^* \right)_f \left( \tilde{\phi}_{i_\alpha} \right)_\alpha^\alpha},$$

where $\tilde{\phi}_{i_\alpha}$ is the $i$-th row of $\tilde{\Phi} = \Phi U^k \left( U^{k+1} \right)^*$ and $\tilde{\phi}_{i_\alpha}$ is the $i$-th row of $\tilde{\Phi} = \Phi U^k \left( U^{k+1} \right)^*$.

Denote

$$\phi_{il} = \sum_{\alpha \in \mathcal{I}_s \setminus \{f\}} \text{rdet}_f \left( \left( U^k \left( U^{k+1} \right)^* \right)_f \left( \tilde{\phi}_{i_\alpha} \right) \right)_\alpha^\alpha.$$

Substituting $\phi_{il}$ into (3.5) gives

$$a^\Phi_{ij} W_r = \sum_{l=1}^n \phi_{il} u_{lj} W_r = \sum_{f=1}^n \phi_{lf} \sum_{\alpha \in \mathcal{I}_s \setminus \{f\}} \text{rdet}_j \left( \left( U^k \left( U^{k+1} \right)^* \right)_j \left( \tilde{\phi}_{f_\alpha} \right) \right)_\alpha^\alpha,$$

$$= \frac{\sum_{\alpha \in \mathcal{I}_s \setminus \{f\}} \text{rdet}_j \left( \left( U^k \left( U^{k+1} \right)^* \right)_j \left( \tilde{\phi}_{f_\alpha} \right) \right)_\alpha^\alpha}{\sum_{\alpha \in \mathcal{I}_s \setminus \{f\}} \left( U^k \left( U^{k+1} \right)^* \right)_j \left( \tilde{\phi}_{f_\alpha} \right)_\alpha^\alpha}.$$

Putting $\sum_{f=1}^n \phi_{lf} \tilde{\phi}_{f_\alpha} = \tilde{\phi}_{i_\alpha}$ as the $i$-th row of $\tilde{\Phi} = \Phi U^k \left( U^{k+1} \right)^*$ yields (3.4).
Similarly, it can be proved theorem on the determinantal representation of the quaternion W-weighted left core-EP inverse.

**Theorem 3.6.** Suppose \( A \in \mathbb{H}^{m \times n}, W \in \mathbb{H}^{n \times m} \), and \( k = \max\{\text{Ind}(WA), \text{Ind}(AW)\} \), \( \text{rk}(AW)^k = s \). Denote \( AW := V = (v_{ij}) \in \mathbb{H}^{m \times m} \). Then the right weighted core-EP inverse \( A^{\Phi, W, l} = (a_{ij}^{\Phi, W, l}) \in \mathbb{H}^{n \times m} \) possess the determinantal representations

\[
a^{\Phi, W, l}_{ij} = \frac{\sum_{\beta \in J_{s,m} \{i\}} \cdet_i \left( \left( (V^{k+1})^* V^{k+1} \right)_{i} (\tilde{\psi}_{j})_{\beta} \right)^{\beta}}{\left( \sum_{\beta \in J_{s,m}} |(V^{k+1})^* V^{k+1}_{\beta}|^{\beta} \right)^2},
\]

where \( \tilde{\psi}_{j} \) is the \( j \)-th column of \( \tilde{\Psi} = (V^{k+1})^* V^k \Psi \). The matrix is determined \( \Psi = (\psi_{ij}) \in \mathbb{H}^{m \times n} \) as follows

\[
\psi_{ij} = \frac{\sum_{\beta \in J_{s,m} \{i\}} \cdet_i \left( \left( (V^{k+1})^* V^{k+1} \right)_{i} (\tilde{\psi}_{j})_{\beta} \right)^{\beta}}{\left( \sum_{\beta \in J_{s,m}} |(V^{k+1})^* V^{k+1}_{\beta}|^{\beta} \right)^2},
\]

where \( \tilde{\psi}_{j} \) is the \( j \)-th column of \( \tilde{V} = (V^{k+1})^* V^k A \).

It’s evident that the following corollaries can be get in the complex case.

**Corollary 3.1.** Suppose \( A \in \mathbb{C}^{m \times n}, W \in \mathbb{C}^{n \times m} \), and \( k = \max\{\text{Ind}(WA), \text{Ind}(AW)\} \), \( \text{rk}(WA)^k = s \). Denote \( WA := U = (u_{ij}) \in \mathbb{C}^{n \times n} \). Then the right weighted core-EP inverse \( A^{\Phi, W, r} = (a_{ij}^{\Phi, W, r}) \in \mathbb{C}^{m \times n} \) possess the determinantal representations

\[
a^{\Phi, W, r}_{ij} = \frac{\sum_{\alpha \in I_{s,n} \{j\}} \left| \left( U^{k+1} (U^{k+1})^* \right)_{j} (\tilde{\phi}_{i})_{\alpha} \right|^{\alpha}}{\left( \sum_{\alpha \in I_{s,n}} |U^{k+1} (U^{k+1})^*|_{\alpha}^{\alpha} \right)^2},
\]

where \( \tilde{\phi}_{i} \) is the \( i \)-th row of \( \tilde{\Phi} = \Phi U^k (U^{k+1})^* \). The matrix is determined \( \Phi = (\phi_{ij}) \) as follows

\[
\phi_{ij} = \frac{\sum_{\alpha \in I_{s,n} \{j\}} \left| \left( U^{k+1} (U^{k+1})^* \right)_{j} (\tilde{\phi}_{i})_{\alpha} \right|^{\alpha}}{\left( \sum_{\alpha \in I_{s,n}} |U^{k+1} (U^{k+1})^*|_{\alpha}^{\alpha} \right)^2},
\]

where \( \tilde{\phi}_{i} \) is the \( i \)-th row of \( \tilde{U} = AU^k (U^{k+1})^* \).

**Corollary 3.2.** Suppose \( A \in \mathbb{C}^{m \times n}, W \in \mathbb{C}^{n \times m} \), and \( k = \max\{\text{Ind}(WA), \text{Ind}(AW)\} \), \( \text{rk}(AW)^k = s \). Denote \( AW := V = (v_{ij}) \in \mathbb{C}^{m \times m} \). Then the right weighted core-EP inverse \( A^{\Phi, W, l} = (a_{ij}^{\Phi, W, l}) \in \mathbb{H}^{n \times m} \) possess the determinantal representations

\[
a^{\Phi, W, l}_{ij} = \frac{\sum_{\beta \in J_{s,m} \{i\}} \left| \left( (V^{k+1})^* V^{k+1} \right)_{i} (\tilde{\psi}_{j})_{\beta} \right|^{\beta}}{\left( \sum_{\beta \in J_{s,m}} |(V^{k+1})^* V^{k+1}_{\beta}|^{\beta} \right)^2},
\]
where \( \tilde{\psi}_j \) is the \( j \)-th column of \( \tilde{\Psi} = (V^{k+1})^* V^k \Psi \). The matrix is determined 
\[ \Psi = (\psi_{lj}) \in \mathbb{H}^{m \times n} \]
as follows
\[ \psi_{lj} = \sum_{\beta \in J_{s,m}} \left| \left( (V^{k+1})^* V^{k+1} \right)_l (\tilde{\nu}_j) \right| \beta, \]
where \( \tilde{\nu}_j \) is the \( j \)-th column of \( \tilde{\nu} = (V^{k+1})^* V^k A. \)

4 The W-weighted DMP and MPD inverses and their determinantal representations.

The concept of the DMP inverse in complex matrices was introduced in [34] by S. Malik and N. Thome that can be expended to quaternion matrices as follows.

**Definition 4.1.** Suppose \( A \in \mathbb{H}^{n \times n} \) and \( \text{Ind} A = k. \) A matrix \( X \in \mathbb{H}^{n \times n} \) is said to be the DMP inverse of \( A \) if it satisfies the conditions

\[ XAX = X, \quad XA = A^d A, \quad \text{and} \quad A^k X = A^k A^\dagger. \] (4.1)

It is denoted \( A^{d,\dagger}. \)

Similar as for complex matrices [34], if a quaternion matrix satisfies the system of equations (4.1), then it is unique and has the representation,

\[ A^{d,\dagger} = A^d A A^\dagger. \]

In [30], we also introduce the MPD inverse.

**Definition 4.2.** Suppose \( A \in \mathbb{H}^{n \times n} \) and \( \text{Ind} A = k. \) A matrix \( X \in \mathbb{H}^{n \times n} \) is said to be the MPD inverse of \( A \) if it satisfies the conditions

\[ XAX = X, \quad AX = AA^d, \quad \text{and} \quad XX^k = A^\dagger A^k. \]

It is denoted \( A^{\dagger,d}. \)

It is not difficult to show that \( A^{\dagger,d} \) is unique and it can be represented as \( A^{\dagger,d} = A^\dagger A A^d. \)

In [30], we gave the determinantal representations of the DMP and MPD inverses over the quaternion skew field.

Recently in [36], the definition of the DMP inverse of a square matrix with complex elements was extended to rectangular matrices. We extend it over the quaternion skew field.

**Definition 4.3.** Let \( A \in \mathbb{H}^{m \times n} \) and \( W \in \mathbb{H}^{n \times m} \) be a nonzero matrix. The W-weighted DMP (WDMP) inverse of \( A \) with respect to \( W \) is defined as

\[ A^{d,\dagger,W} = WA^{d,W} WAA^\dagger. \]

Similarly to complex matrices can be proved the next lemma.

**Lemma 4.1.** Let \( A \in \mathbb{H}^{n \times n} \) and \( k = \max \{ \text{Ind}(WA), \text{Ind}(AW) \}. \) The matrix \( X = A^{d,\dagger,W} \) is the unique matrix that satisfies the following system of equations

\[ XAX = X, \quad XA = WA^{d,W} WA, \quad \text{and} \quad (WA)^{k+1} X = (WA)^{k+1} A^\dagger. \] (4.2)
We propose to introduce the weighted MPD inverse as well.

**Lemma 4.2.** Let \( A \in \mathbb{H}^{n \times n} \) and \( k = \max\{\text{Ind}(WA), \text{Ind}(AW)\} \). Then the matrix \( X = A^\dagger W A W A_d^dW W \) is the unique solution to the equations

\[
XAX = X, \quad AX = AW A_d^dW W, \quad \text{and} \quad X(AW)^{k+1} = A^\dagger (AW)^{k+1}. \quad (4.3)
\]

**Proof.** From Definitions 1.1, 1.2, and taking into account (1.1), it follows

\[
\begin{align*}
A^\dagger W A W (AA^\dagger A) W A_d^dW W &= A^\dagger W (A_d^dW W A W)^dW W = \\
&= A^\dagger W A_W^dW W, \\
AA^\dagger W A_d^dW W &= A W A_d^W W, \\
A^\dagger W A_d^dW W (AW)^{k+1} &= A^\dagger W ((AW)^d)^2 AW (AW)^{k+1} = \\
&= A^\dagger W (AW)^d AW (AW)^d (AW)^{k+1} = \\
&= A^\dagger (AW)^{k+1}.
\end{align*}
\]

It means that \( X = A^\dagger W A W A_d^dW W \) is the solution to the equations (4.3).

To prove uniqueness, suppose both \( X_1 \) and \( X_2 \) are two solutions to (4.3). Using repeated applications of the equations in (4.3) and in Definition 1.1 we obtain

\[
X_1 = X_1 A X_1 = X_1 A W A_d^dW W = X_1 (AW)^2 (A_d^dW W)^2 = \ldots = \\
= X_1 (AW)^{k+1} (A_d^dW W)^{k+1} = A^\dagger (AW)^{k+1} (A_d^dW W)^{k+1} = \\
= X_2 (AW)^{k+1} (A_d^dW W)^{k+1} = X_2 A W A_d^dW W = X_2 A X_2 = X_2.
\]

It completes the proof. \( \square \)

**Definition 4.4.** Let \( A \in \mathbb{H}^{m \times n} \) and \( W \in \mathbb{H}^{n \times m} \) be a nonzero matrix. The \( W \)-weighted MPD (WMPD) inverse of \( A \) with respect to \( W \) is defined as

\[
A_{\dagger,W} = A^\dagger W A_d^dW W.
\]

Now, we give determinantal representations of the WDMP inverse. We have two cases due to Hermicity of the matrix \( WA \).

**Theorem 4.5.** Let \( A \in \mathbb{H}^{n \times n} \) and \( W \in \mathbb{H}^{n \times m} \) be a nonzero matrix. Suppose \( k = \max\{\text{Ind}(WA), \text{Ind}(AW)\} \) and \( \text{rk}(WA)^k = \text{rk} U^k = r_1 \). Then the determinantal representations of its WDMP inverse \( A_{\dagger,W} = (a_{ij}^{d,W}) \) can be expressed as follows.

(i) If \( WA \) is an arbitrary matrix, then

\[
a_{ij}^{d,W} = \sum_{\alpha \in I_{r,m}} \sum_{j} \text{rdet}_{j}((AA^*)_{j} (\tilde{\omega}_{m}))^{\alpha}_{a} \sum_{\beta \in I_{r,m}} |AA^*|^{\alpha}_{\beta} \left( \sum_{\alpha \in I_{r,m}} |U^{2k+1} (U^{2k+1})^{*}|^{\alpha}_{a} \right)^{2},
\]

where \( \tilde{\omega}_{i} \) is the \( i \)-th row of \( \tilde{\Omega} = \Omega (WA)^{k+1} A^* \). The matrix \( \Omega = (\omega_{is}) \) is such that \( \omega_{is} \) is determined by

\[
\omega_{is} = \sum_{\alpha \in I_{r_1,n}} \text{rdet}_{s} \left( \left( U^{2k+1} (U^{2k+1})^{*} \right)_{s} (\tilde{\omega}_{m}) \right)^{\alpha}_{a},
\]

\( 17 \)
where \( \hat{\phi}_l \) is the \( i \)-th row of \( \hat{\Phi} := W A \Phi U^{2k}(U^{2k+1})^* \in \mathbb{H}^{n \times n} \), and \( \Phi = (\phi_q) \in \mathbb{H}^{m \times n} \) such that

\[
\phi_q = \sum_{\alpha \in I_{r,m}(q)} \text{rdet}_q \left( \left( U^{2k+1} (U^{2k+1})^* \right)_q (\hat{u}_l) \right)_\alpha. \tag{4.6}
\]

Here \( \hat{u}_l \) is the \( l \)-th row of \( U^k (U^{2k+1})^* =: \hat{U} \in \mathbb{H}^{n \times n} \).

(ii) If \( WA \) is Hermitian, then

\[
a_{ij}^{d,W} = \sum_{\alpha \in I_{r,m}} \frac{\text{rdet}_j ([A A^*])_j, (\tilde{\omega}_l)^\alpha}{\sum_{\alpha \in I_{r,m}} |A A^*|_\alpha^2 \sum_{\alpha \in I_{r,n}} |W A|_{2+2}^\alpha}.
\tag{4.7}
\]

where \( \tilde{\omega}_l \) is the \( i \)-th row of \( \tilde{\Omega} = \Omega W A A^* \). The matrix \( \Omega = (\omega_{is}) \) is such that

\[
\omega_{is} := \sum_{\alpha \in I_{r,n}(s)} \text{rdet}_s \left( (W A)_{k+2}^s (\tilde{u}_l)_\alpha^s \right),
\]

where \( \tilde{u}_l \) is the \( i \)-th row of \( \tilde{U} = (W A)^{k+1} \).

**Proof.** Taking into account (4.1), we have

\[
a_{ij}^{d,W} = \sum_{l=1}^m \sum_{s=1}^m \sum_{f=1}^m w_{sf} a_{ij}^{d,W} w_{fj} \mathcal{P}^W_{ij}, \tag{4.8}
\]

where \( \mathcal{A}_{d,W} = (a_{ms}^{d,W}) \in \mathbb{H}^{m \times n} \), and \( \mathcal{P}_A = (p_{ij}^A) \in \mathbb{H}^{m \times m} \).

(i) Denote \( W_1 := W A A^* = (w_{ij}^{(1)}) \). By applying (2.4) for the determinantal representation of \( \mathcal{P}_A \), respectively, we have

\[
\sum_{j=1}^m w_{sf} p_{fj}^A = \sum_{j=1}^m w_{sf} \frac{\sum_{\alpha \in I_{r,m}(j)} \text{rdet}_j ((A A^*)_j, (\tilde{\omega}_l)^\alpha)}{\sum_{\alpha \in I_{r,m}} |A A^*|_\alpha^2 \sum_{\alpha \in I_{r,n}} |W A|_{2+2}^\alpha}
\]

\[
= \frac{\sum_{\alpha \in I_{r,m}(j)} \text{rdet}_j ((A A^*)_j, (w_{ij}^{(1)})^\alpha)}{\sum_{\alpha \in I_{r,m}} |A A^*|_\alpha^2 \sum_{\alpha \in I_{r,n}} |W A|_{2+2}^\alpha}. \tag{4.9}
\]

Substituting (4.9) into (4.8) and applying (2.5) for the determinantal representation of \( \mathcal{A}_{d,W} \) give

\[
a_{ij}^{d,W} = \sum_{s=1}^m a_{ij}^{d,W} \left( (U^{2k+1} (U^{2k+1})^* s, (\phi_l^1))_\alpha^s \right) \times \frac{\sum_{\alpha \in I_{r,n}(s)} \text{rdet}_s \left( (A A^*)_j, (w_{ij}^{(2)})^\alpha \right)}{\sum_{\alpha \in I_{r,m}} |A A^*|_\alpha^2 \sum_{\alpha \in I_{r,n}} |W A|_{2+2}^\alpha}. \tag{4.10}
\]
where \( \tilde{\phi}_j \) is the \( i \)-th row of \( \tilde{\Phi} := W^\dagger \Phi U^{2k}(U^{2k+1})^* \in \mathbb{H}^{m \times n} \), and \( \tilde{w}_s^{(2)} \) is the \( s \)-th row of \( \tilde{W}_2 = U^k \tilde{W} A^* = (WA)^{k+1} A^* \). Here \( \Phi = (\phi_l)_q \in \mathbb{H}^{n \times n} \) such that

\[
\phi_{lq} = \sum_{a \in I_{r,m}} \text{rdet}_q \left( (U^{2k+1}(U^{2k+1})^*)_q (\tilde{u}_l) \right)_{a},
\]

where \( \tilde{u}_l \) is the \( l \)-th row of \( \tilde{U}^k(U^{2k+1})^* =: \tilde{U} \in \mathbb{H}^{n \times n} \).

Denote

\[
\omega_{is} := \sum_{a \in I_{r,m}} \text{rdet}_s \left( (U^{2k+1}(U^{2k+1})^*)_s (\tilde{\phi}_i) \right)_{a}
\]

and construct the matrix \( \Omega = (\omega_{is}) \). Since

\[
\sum_{s=1}^n \omega_{is} \sum_{a \in I_{r,m}} \text{rdet}_s \left( (AA^*)_s (w_s^{(2)}) \right)_{a} = \sum_{a \in I_{r,m}} \text{rdet}_s ((AA^*)_s (\tilde{w}_s^{(2)})_{a},
\]

where \( \tilde{w}_s \) is the \( i \)-th row of \( \tilde{\Omega} = \Omega W_2 = \Omega (WA)^{k+1} A^* \), then finally, from (4.10) it follows (4.11).

(ii) Applying the determinantal representation (2.8) of \( A^{d,W} \) in (4.8) gives

\[
a^{d,W}_{ij} = \sum_{s=1}^n \sum_{a \in I_{r,m}} \text{det}_s \left( (WA)_{j,i}^{k+2}(\tilde{u}_i) \right)_{a} \sum_{a \in I_{r,m}} \text{det}_s ((AA^*)_j (w_s^{(2)})_{a})_{a}
\]

(4.11)

where \( \tilde{u}_i \) is the \( i \)-th row of \( \tilde{U} := (WA)^{k+1} \in \mathbb{H}^{n \times n} \), and \( w_s^{(2)} \) is the \( s \)-th row of \( W_2 = U^k \tilde{W} A^* = (WA)^{k+1} A^* \).

Denote

\[
\omega_{is} := \sum_{a \in I_{r,m}} \text{rdet}_s \left( (WA)_{j,i}^{k+2}(\tilde{u}_i) \right)_{a}
\]

and construct the matrix \( \Omega = (\omega_{is}) \). Since

\[
\sum_{s=1}^n \omega_{is} \sum_{a \in I_{r,m}} \text{rdet}_s ((AA^*)_s (\tilde{w}_s^{(2)})_{a})_{a} = \sum_{a \in I_{r,m}} \text{rdet}_s ((AA^*)_s (\tilde{w}_s^{(2)})_{a})_{a},
\]

where \( \tilde{w}_s \) is the \( i \)-th row of \( \tilde{\Omega} = \Omega W_2 = \Omega (WA)^{k+1} A^* \), then finally, from (4.11) it follows (4.17).

The following corollary can be get in the complex case.

**Corollary 4.1.** Let \( A \in \mathbb{C}^{n \times n} \) and \( W \in \mathbb{C}^{n \times m} \) be a nonzero matrix. Suppose \( k = \max \{ \text{Ind}(WA), \text{Ind}(AW) \} \) and \( \text{rk}(WA)^k = \text{rk} U^k = r_1 \). Then the determinantal representations of its WDMP inverse \( A^{d,W} = (a^{d,W}_{ij}) \) can be expressed as

\[
a^{d,W}_{ij} = \frac{\sum_{a \in I_{r,m}} |(AA^*)_j (\tilde{w}_s^{(2)})|_{a}}{\sum_{a \in I_{r,m}} |AA^*| a \sum_{a \in I_{r,m}} |(WA)^{k+2}| a}.\]
where $\tilde{w}_i$ is the $i$-th row of $\tilde{\Omega} = \Omega W A A^*$. The matrix $\Omega = (\omega_{is})$ is such that

$$\omega_{is} := \sum_{a \in J_{1,n}(s)} |(W A)_{a,k}^2(\tilde{u}_i)|_a^\alpha,$$

where $\tilde{u}_i$ is the $i$-th row of $\tilde{U} = (W A)^{k+1}$.

For the weighted MPD inverse, we have similarly.

**Theorem 4.6.** Let $A \in \mathbb{H}^{m \times n}$ and $W \in \mathbb{H}^{n \times m}$ be a nonzero matrix. Suppose $k = \max \{ \text{Ind}(WA), \text{Ind}(AW) \}$ and $\text{rk}(AW)^k = \text{rk} V_k = r_1$. Then the determinantal representations of its WMPD inverse $A^{t,d,W} = (a^{t,d,W}_{ij})$ can be expressed as follows.

(i) If the matrix $AW$ is arbitrary, then

$$a^{t,d,W}_{ij} = \frac{\sum_{\beta \in J_{r,m}(t)} \cdet_s ((A^* A)_{i,j}(\tilde{v}_j))_\beta}{\sum_{\beta \in J_{r,m}} |A^* A|_\beta^\alpha \sum_{\beta \in J_{r,m}} |(V^{2k+1})^* V^{2k+1}|_\beta^\alpha}$$

(4.12)

where $\tilde{v}_j$ is the $j$-th column of $\tilde{\Psi} = A^* (AW)^{k+1} \Psi$. The matrix $\Psi = (\psi_{ij})$ is determined by

$$v_{ij} := \sum_{\beta \in J_{r,m}(t)} \cdet_s \left( \left( (V^{2k+1})^* V^{2k+1} \right)_{i,j} (\tilde{\psi}_j) \right)_\beta$$

where $\tilde{\psi}_j$ is the $j$-th column of $\tilde{\Psi} := (V^{2k+1})^* V^{2k} \Psi AW \in \mathbb{H}^{m \times n}$. Here the matrix $\Psi = (\psi_{st}) \in \mathbb{H}^{m \times m}$ is such that

$$\psi_{st} = \sum_{\beta \in J_{r,m}(s)} \cdet_s \left( \left( (V^{2k+1})^* V^{2k+1} \right)_{s,t} (\tilde{\psi}_j) \right)_\beta$$

where $\tilde{\psi}_j$ is the $j$-th column of $\tilde{\Psi} := (V^{2k+1})^* V^k =: \tilde{V} \in \mathbb{H}^{m \times m}$.

(ii) If the matrix $AW$ is Hermitian, then

$$a^{t,d,W}_{ij} = \frac{\sum_{\beta \in J_{r,m}(t)} \cdet_s ((A^* A)_{i,j}(\tilde{v}_j))_\beta}{\sum_{\beta \in J_{r,m}} |A^* A|_\beta^\alpha \left( \sum_{\beta \in J_{r,m}} |(AW)^{k+2}|_\beta^\alpha \right)^2}$$

(4.13)

where $\tilde{v}_j$ is the $j$-th column of $\tilde{\Psi} = A^* (AW)^{k+1} \Psi$. The matrix $\Psi = (\psi_{ij})$ is determined by

$$v_{ij} = \sum_{\beta \in J_{r,m}(s)} \cdet_s \left( (AW)^{k+2}_{i,j} (\tilde{v}_j) \right)_\beta$$

where $\tilde{v}_j$ is the $j$-th column of $\tilde{V} = (AW)^{k+1}$. 


Proof. Taking into account (4.2), we have

\[ a_{ij}^{d,W} = \sum_{t=1}^{m} \sum_{s=1}^{n} \sum_{f=1}^{n} a_{ij}^{d,W} a_{is}^{d,W} w_{sf}, \]  

(4.14)

where \( A^{d,W} = (a_{ij}^{d,W}) \in \mathbb{H}^{m \times n} \) and \( Q_A = (q_A^{i}) \in \mathbb{H}^{n \times n}. \)

(i) Denote \( W_1 := A^{*} AW = (w_i^{(1)}) \). By applying (2.3) for the determinantal representation of \( Q_A \), we have

\[
\begin{align*}
\sum_{t=1}^{n} q_{it}^{A} w_{it} &= \sum_{\beta \in J_{r,n}(i)} \text{cdet}_{i} ((A^{*} A_{i})_{i} (\hat{A}_{1}))_{\beta}^{\beta} w_{it} = \\
&= \sum_{\beta \in J_{r,n}(i)} \text{cdet}_{i} ((A^{*} A_{i})_{i} (w_{i}^{(1)}))_{\beta}^{\beta} \sum_{\beta \in J_{r,n}} |A^{*} A_{i}^{\beta}_{\beta}|. 
\end{align*}
\]

(4.15)

Substituting (4.14) into (4.13) and applying (2.3) for the determinantal representation of \( A^{d,W} \), we obtain

\[
\begin{align*}
a_{ij}^{d,W} &= \sum_{t=1}^{n} \sum_{\beta \in J_{r,n}(i)} \text{cdet}_{1} ((A^{*} A_{i})_{i} (w_{i}^{(2)}))_{\beta}^{\beta} \sum_{\beta \in J_{r,n}} |A^{*} A_{i}^{\beta}_{\beta}| \\
&\quad \sum_{\beta \in J_{r,n}(i)} \text{cdet}_{1} ( ((V^{2k+1})^{*} V^{2k+1})_{i} (\hat{v}_{j})_{\beta}^{\beta} \sum_{\beta \in J_{r,n}} |(V^{2k+1})^{*} V^{2k+1}|_{\beta}^{\beta} \\
&= \sum_{\beta \in J_{r,n}(i)} \text{cdet}_{1} ( ((V^{2k+1})^{*} V^{2k+1})_{i} (\hat{v}_{j})_{\beta}^{\beta} \sum_{\beta \in J_{r,n}} |(V^{2k+1})^{*} V^{2k+1}|_{\beta}^{\beta})^{2},
\end{align*}
\]

(4.16)

where \( \hat{v}_{j} \) is the \( j \)-th column of \( \hat{\Psi} := (V^{2k+1})^{*} V^{2k} \Psi AW \in \mathbb{H}^{m \times n} \) and \( w_{i}^{(2)} \) is the \( l \)-th column of \( W_2 = A^{*} AW v^{k} = A^{*} (AW)^{k+1} \in \mathbb{H}^{m \times n} \). Here the matrix \( \Psi = (\psi_{i}) \in \mathbb{H}^{m \times n} \) is such that

\[
\psi_{i} = \sum_{\beta \in J_{r,n}(i)} \text{cdet}_{s} ( ((V^{2k+1})^{*} V^{2k+1})_{s} (\hat{v}_{i})^{\beta}_{\beta}),
\]

here \( \hat{v}_{i} \) is the \( l \)-th column of \( (V^{2k+1})^{*} v^{k} =: \hat{V} \in \mathbb{H}^{m \times n} \).

Denote

\[
\nu_{ij} := \sum_{\beta \in J_{r,n}(i)} \text{cdet}_{s} ( ((V^{2k+1})^{*} V^{2k+1})_{s} (\hat{v}_{j})^{\beta}_{\beta})
\]

and construct the matrix \( \Upsilon = (\nu_{ij}) \). Taking into account that

\[
\sum_{t=1}^{n} \sum_{\beta \in J_{r,n}(i)} \text{cdet}_{i} ((A^{*} A_{i})_{i} (w_{i}^{(2)})_{\beta}^{\beta}) \nu_{ij} = \sum_{\beta \in J_{r,n}(i)} \text{cdet}_{i} ((A^{*} A_{i})_{i} (\hat{v}_{j})_{\beta}^{\beta}),
\]
where \( \tilde{v}_j \) is the \( j \)-th column of \( \tilde{\Upsilon} = \mathbf{W}_2 \Upsilon = \mathbf{A}^*(\mathbf{A} \mathbf{W})^{k+1} \Upsilon \), finally, from (4.16) it follows (4.12).

(ii) Applying the determinantal representation (2.7) of \( \mathbf{A}^{d,W} \) in (4.14) gives

\[
a_{i,j}^{d,W} = \sum_{\beta \in J_{r,n}} \frac{\text{cdet}_i \left((\mathbf{A}^* \mathbf{A})_i \left(\mathbf{w}_t^{(2)}\right)\right)^\beta}{\left|\mathbf{A}^* \mathbf{A}\right|_\beta} \sum_{\beta \in J_{r,n}} \frac{\text{cdet}_t \left((\mathbf{A} \mathbf{W})^{k+2}_t \left(\tilde{v}_j\right)\right)^\beta}{\left|\mathbf{A} \mathbf{W}\right|_\beta} \sum_{\beta \in J_{r,1,m}(t)} \left|\mathbf{A} \mathbf{W}\right|^{k+2}_s \left(\tilde{v}_j\right)^\beta,
\]

(4.17)

where \( \mathbf{w}_t^{(2)} \) is the \( t \)-th column of \( \mathbf{W}_2 = \mathbf{A}^* \mathbf{A} \mathbf{W} \mathbf{V}^k = \mathbf{A}^*(\mathbf{A} \mathbf{W})^{k+1} \in \mathbb{H}^{m \times m} \) and \( \tilde{v}_j \) is the \( j \)-th column of \( \tilde{\mathbf{V}} = (\mathbf{A} \mathbf{W})^{k+1} \). Now, construct the matrix \( \Upsilon = (\upsilon_{ij}) \), where

\[
\upsilon_{ij} := \sum_{\beta \in J_{r,1,m}(t)} \text{cdet}_t \left((\mathbf{A} \mathbf{W})^{k+2}_t \left(\tilde{v}_j\right)\right)^\beta.
\]

Then, from (4.17) it follows (4.13).

\[\square\]

**Corollary 4.2.** Let \( \mathbf{A} \in \mathbb{C}_r^{m \times n} \) and \( \mathbf{W} \in \mathbb{C}^{n \times m} \) be a nonzero matrix. Suppose \( k = \max\{\text{Ind}(\mathbf{W} \mathbf{A}), \text{Ind}(\mathbf{A} \mathbf{W})\} \) and \( \text{rk}(\mathbf{A} \mathbf{W})^k = \text{rk} \mathbf{V}^k = r_1 \). Then the determinantal representations of its WMPD inverse \( \mathbf{A}^{†,d,W} = \left(a_{i,j}^{d,W}\right) \) can be expressed as

\[
a_{i,j}^{d,W} = \sum_{\beta \in J_{r,n}(1)} \frac{\left|(\mathbf{A}^* \mathbf{A})_i \left(\tilde{v}_j\right)\right|_\beta}{\left|\mathbf{A}^* \mathbf{A}\right|_\beta} \sum_{\beta \in J_{r,n}} \frac{\left|(\mathbf{A} \mathbf{W})^{k+2}_s \left(\tilde{v}_j\right)\right|_\beta}{\left|\mathbf{A} \mathbf{W}\right|_\beta} \sum_{\beta \in J_{r,1,m}(s)} \left|\mathbf{A} \mathbf{W}\right|^{k+2}_s \left(\tilde{v}_j\right)^\beta,
\]

where \( \tilde{v}_j \) is the \( j \)-th column of \( \tilde{\Upsilon} = \mathbf{A}^*(\mathbf{A} \mathbf{W})^{k+1} \Upsilon \). The matrix \( \Upsilon = (\upsilon_{ij}) \) is determined by

\[
\upsilon_{ij} = \sum_{\beta \in J_{r,1,m}(s)} \left|\mathbf{A} \mathbf{W}\right|^{k+2}_s \left(\tilde{v}_j\right)^\beta,
\]

where \( \tilde{v}_j \) is the \( j \)-th column of \( \tilde{\mathbf{V}} = (\mathbf{A} \mathbf{W})^{k+1} \).

Theorems 4.5 and 4.6 give the determinantal representations of the weighted DMP and DMP inverses over the quaternion skew field. For better understanding, we present the algorithm of finding one of them, for example WDM from Theorem 4.5 in the case (i).

**Algorithm 4.7.**

1. Compute the matrix \( \mathbf{U} = \mathbf{U}^k(\mathbf{U}^{2k+1})^* \).
2. Find \( \phi_{iq} \) by (4.6) for all \( i, q = 1, \ldots, n \) and construct the matrix \( \Phi = (\phi_{iq}) \).
3. Compute the matrix \( \tilde{\Phi} := \mathbf{W} \Phi \mathbf{U}^{2k}(\mathbf{U}^{2k+1})^* \).
4. By (4.5), find $\omega_s$ for all $i, s = 1, \ldots, n$ and construct the matrix $\Omega = (\omega_s)$.

5. Compute the matrix $\tilde{\Omega} := \Omega (W \Omega)^{k+1} A^*.$

6. Finally, find $a_{ij}^{d,W}$ by (4.3) for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

5 Determinantal representations of the weighted CMP inverse

In [35] by M. Mehdipour and A. Salemi the CMP inverse was investigated that can be extended to quaternion matrices as follows.

**Definition 5.1.** Suppose $A \in \mathbb{H}^{n \times n}$ has the core-nilpotent decomposition $A = A_1 + A_2$, where $\text{Ind} A_1 = \text{Ind} A$, $A_2$ is nilpotent and $A_1 A_2 = A_2 A_1 = 0$. The CMP inverse of $A$ is called the matrix $A^{c,\dagger} := A^\dagger A_1 A^\dagger$.

Similarly to complex matrices can be proved the next lemma.

**Lemma 5.1.** Let $A \in \mathbb{H}^{n \times n}$ and $W \in \mathbb{H}^{n \times m}$ be a nonzero matrix. The matrix $X = A^{c,\dagger}$ is the unique matrix that satisfies the following system of equations:

$$XAX = X, \ AXA = A_1, \ AX = A_1 A^\dagger, \ \text{and} \ \XA = A^\dagger A_1.$$ 

Moreover, $A^{c,\dagger} = A^\dagger A W A^{d,W} W A^\dagger$.

Determinantal representations of the CMP inverse over the quaternion skew field within the framework of the theory of row-column determinants are derived in [30].

Recently, Mosić [40] introduced the weighted CMP inverse of a rectangular matrix that can be extended over the quaternion skew field without any changes.

**Lemma 5.2.** Let $A \in \mathbb{H}^{m \times n}$ and $W \in \mathbb{H}^{n \times m}$ be a nonzero matrix. The system of equations

$$XAX = X, \ AX = A W A^{d,W} W A^\dagger, \ \text{and} \ \XA = A^\dagger A W A^{d,W} W A^\dagger.$$ 

is consistent and its unique solution is $X = A^\dagger A W A^{d,W} W A^\dagger$.

**Definition 5.2.** Let $A \in \mathbb{H}^{m \times n}$ and $W \in \mathbb{H}^{n \times m}$ be a nonzero matrix. The weighted CMP (WCMP) inverse of $A$ with respect to $W$ is defined as

$$A^{c,\dagger,W} := A^{c,\dagger} W A^{d,W} W A^\dagger.$$ 

Taking into account Corollary 2.1 and Lemma 2.3, it follows the next theorem about determinantal representations of the quaternion WCMP inverse.

**Theorem 5.3.** Let $A \in \mathbb{H}^{m \times n}$ and $W \in \mathbb{H}^{n \times m}$ be a nonzero matrix. Suppose $k = \max\{\text{Ind}(WA), \text{Ind}(AW)\}$. Then the determinantal representations of its WCMP inverse $A^{c,\dagger,W} = (a_{ij}^{c,\dagger,W})$ can be expressed as

(i) if $\text{rk}(WA)^k = \text{rk} U^k = r_1$, then

$$a_{ij}^{c,\dagger,W} = \sum_{\alpha \in I_{r,m}(j)} r_{\text{det}_j((\AA^\dagger)_j(\tilde{\omega}_s)_s)^{\alpha}} \left( \sum_{\beta \in I_{r,n}} |A^\dagger A_\beta|^2 \right)^{\alpha} \left( \sum_{\gamma \in I_{r_1,n}} |U^{2k+1} (U^{2k+1})^\gamma|^2 \right)^{\alpha},$$

(5.2)
where \( w_i \) is the \( i \)-th row of \( \tilde{\Omega} = \Omega(WA)^{k+1}A^* \). The matrix \( \Omega = (\omega_{iz}) \) is such that \( \omega_{iz} \) is determined by

\[
\omega_{iz} = \sum_{\beta \in J_{r,n}(i)} c_{det} \left( (A^*A)_{i,\beta} \left( \phi_{z}^{(1)} \right) \right)_{\beta}^{\beta},
\] (5.3)

Here \( \phi_{z}^{(1)} \) is the \( z \)-th column of \( \Phi_1 = A^*AW \hat{\Phi} = (\hat{\phi}_{iz}) \) and

\[
\hat{\phi}_{iz} := \sum_{\alpha \in I_{r,n}(z)} r_{det}( \left( (U^{2k+1}(U^{2k+1})^*)_{z,\alpha} \right( \hat{\phi}_{t}) )_{\alpha}^{\alpha},
\] (5.4)

where \( \hat{\phi}_{t} \) is the \( t \)-th row of \( \Phi := A\Phi U^{2k}(U^{2k+1})^* \in \mathbb{H}^{m \times n} \), and \( \Phi = (\phi_{q}) \in \mathbb{H}^{n \times n} \) such that

\[
\phi_{q} = \sum_{\alpha \in I_{r,n}(q)} r_{det}( \left( (U^{2k+1}(U^{2k+1})^*)_{q,\alpha} \right( \hat{\phi}_{t}) )_{\alpha}^{\alpha}.
\] (5.5)

Here \( \hat{u}_{t} \) is the \( t \)-th row of \( U^{(k)}(U^{2k+1})^* =: \hat{U} \in \mathbb{H}^{n \times n} \).

(ii) if \( rk(AW)^k = rk V^k = r_1 \), then

\[
a_{ij}^{c:W} = \frac{\sum_{\beta \in J_{r,n}(i)} c_{det} \left( (AA^*)_{i,\beta} \left( \hat{\nu}_{j} \right) \right)_{\beta}^{\beta}}{\left( \sum_{\alpha \in I_{r,m}(j)} |AA^*|_{\alpha}^{\alpha} \right)^{2} \sum_{\beta \in J_{r,m}(j)} |(V^{2k+1})^* V^{2k+1}|_{\beta}^{\beta}}
\] (5.6)

where \( \hat{\nu}_{j} \) is the \( j \)-th column of \( \hat{\nu} = (A^*AW)^{k+1}Y \). The matrix \( Y = (\nu_{z}) \) is determined by

\[
\nu_{z} = \sum_{\alpha \in I_{r,m}(j)} r_{det}( \left( (AA^*)_{j,\alpha} \left( \psi_{z}^{(1)} \right) \right)_{\alpha}^{\alpha},
\]

where \( \psi_{z}^{(1)} \) is the \( z \)-th row of \( \Psi^{(1)} = \tilde{\Psi}WAA^* \). Here \( \tilde{\Psi} = (\tilde{\psi}_{zs}) \) is such that

\[
\tilde{\psi}_{zs} := \sum_{\beta \in J_{r,m}(z)} c_{det}( \left( (V^{2k+1})^* V^{2k+1} \right)_{z,\beta} \left( \tilde{\psi}_{s} \right) )_{\beta}^{\beta}
\]

where \( \tilde{\psi}_{s} \) is the \( s \)-th column of \( \tilde{\Psi} := (V^{2k+1})^* V^{2k} \Psi A \in \mathbb{H}^{m \times n} \), and \( \Psi = (\psi_{\ell}) \in \mathbb{H}^{m \times m} \) is determined by

\[
\psi_{\ell} = \sum_{\beta \in J_{r,m}(s)} r_{det}( \left( (V^{2k+1})^* V^{2k+1} \right)_{s,\beta} \left( \tilde{\nu}_{t} \right))_{\beta}^{\beta}.
\]

Here \( \tilde{\nu}_{t} \) is the \( t \)-th column of \( (V^{2k+1})^* V^k =: \tilde{V} \in \mathbb{H}^{n \times m} \).

Proof. Taking into account (5.1), we have

\[
a_{ij}^{c:W} = \sum_{l=1}^{n} \sum_{t=1}^{m} \sum_{s=1}^{n} \sum_{f=1}^{m} q_{ii}^{l} w_{lt} a_{ts}^{d:W} w_{sf} p_{lj}^{f},
\] (5.7)
where $Q_A = (q_{ij}^A) \in \mathbb{H}^{n \times n}$, $A^{d,W} = (a_{ij}^{d,W}) \in \mathbb{H}^{m \times n}$, and $P_A = (p_{ij}^A) \in \mathbb{H}^{m \times m}$.

(i) Denote $W_1 := A^*AW = (w_{ij}^{(1)})$ and $W_2 := WAA^* = (w_{ij}^{(2)})$. By applying one of the cases of (2.3) and (2.4) for the determinantal representations of $Q_A$ and $P_A$, respectively, we have

$$\sum_{l=1}^n q_{il}^A w_{lt} = \sum_{l=1}^n \sum_{\beta \in J_{r,n}(t)} \frac{\text{cdet}_t((A^*A)_{ij}(\hat{a}^t_j))_\beta^\beta}{\sum_{\beta \in J_{r,n}} |A^*A|_\beta^\beta} w_{lt} = \sum_{\beta \in J_{r,n}(t)} \frac{\text{cdet}_t((A^*A)_{ij}(w_{ij}^{(1)}))_\beta^\beta}{\sum_{\beta \in J_{r,n}} |A^*A|_\beta^\beta}.$$ (5.8)

$$\sum_{f=1}^m w_{sf}^f p_{fj}^A = \sum_{f=1}^m w_{sf}^f \sum_{\alpha \in I_{r,m}(j)} \frac{r_{d,e}((AA^*)_{ij}(\tilde{a}^j_f))_\alpha^\alpha}{\sum_{\alpha \in I_{r,m}} |AA^*|_\alpha^\alpha} \sum_{\alpha \in I_{r,m}} |AA^*|_\alpha^\alpha = \frac{\sum_{\alpha \in I_{r,m}(j)} r_{d,e}((AA^*)_{ij}(w_{ij}^{(2)}))_\alpha^\alpha}{\sum_{\alpha \in I_{r,m}} |AA^*|_\alpha^\alpha}. $$ (5.9)

Substituting (5.8) and (5.9) into (5.7), denoting $W_3 = (WA)^{k+1} A^*$, and applying (2.3) for the determinantal representation of $A^{d,W}$ give

$$a_{ij}^{d,W} = \sum_{t=1}^n \sum_{s=1}^n \frac{\text{cdet}_t((A^*A)_{ij}(w_{ij}^{(1)}))_\beta^\beta}{\sum_{\beta \in J_{r,n}} |A^*A|_\beta^\beta} \times \sum_{\alpha \in I_{r,m}(s)} \frac{r_{d,e}((U^{2k+1}(U^{2k+1})^*)(\tilde{\phi}_t_{r,n}))_\alpha^\alpha u_{ks}^{(k)} \sum_{\alpha \in I_{r,m}} |U^{2k+1}(U^{2k+1})^*|_\alpha^\alpha}{\sum_{\alpha \in I_{r,m}} |AA^*|_\alpha^\alpha} \times \frac{\sum_{\alpha \in I_{r,m}(j)} r_{d,e}((AA^*)_{ij}(u_{ij}^{(2)}))_\alpha^\alpha}{\sum_{\alpha \in I_{r,m}} |AA^*|_\alpha^\alpha} \times \sum_{\beta \in J_{r,n}} |A^*A|_\beta^\beta.$$
\[
\begin{align*}
\sum_{\alpha \in I_{r,m}(z)} \text{rdet}_z \left( \left( \left( U^{2k+1} (U^{2k+1})^* \right)_z^{j} \tilde{\phi}_l \right)_{\alpha} \right) & \\
\times 
\left( \sum_{\alpha \in I_{r,m}} \left| U^{2k+1} (U^{2k+1})^* \right|_{\alpha}^\alpha \right)^2 & \\
\sum_{\alpha \in I_{r,m}(j)} \text{rdet}_z \left( (A A^*)_{j, (w^{(3)}_z)} \right)_{\alpha} & \\
\sum_{\alpha \in I_{r,m}} |AA^*|_{\alpha}^\alpha ,
\end{align*}
\]

where \( \tilde{\phi}_l \) is the \( t \)-th row of \( \tilde{\Phi} := A \Phi U^{2k+1} (U^{2k+1})^* \in \mathbb{H}^{m \times n} \), and \( \Phi = (\tilde{\phi}_l) \in \mathbb{H}^{m \times n} \) is such that

\[
\tilde{\phi}_l = \sum_{\alpha \in I_{r,m}(\{q\})} \text{rdet}_q \left( \left( U^{2k+1} (U^{2k+1})^* \right)_q^{j} (\tilde{\mathbf{u}}_l) \right)_{\alpha}.
\]

Here \( \tilde{\mathbf{u}}_l \) is the \( t \)-th row of \( U^k (U^{2k+1})^* =: \tilde{U} \in \mathbb{H}^{m \times n} \). Denote

\[
\hat{\phi}_{t,z} := \sum_{\alpha \in I_{r,m}(\{z\})} \text{rdet}_z \left( \left( U^{2k+1} (U^{2k+1})^* \right)_z^{j} (\hat{\phi}_l) \right)_{\alpha}
\]

and construct the matrix \( \hat{\Phi} = (\hat{\phi}_{t,z}) \). Then, determine

\[
\omega_{iz} = \sum_{t=1}^n \sum_{\beta \in J_{r,n}(t)} \text{cdet}_t \left( (A^* A)_{t, (w^{(1)}_t)} \right)^\beta_{\beta} \hat{\phi}_{t,z} =
\]

\[
= \sum_{\beta \in J_{r,n}(t)} \text{cdet}_t \left( (A^* A)_{t, (\phi^{(1)}_t)} \right)^\beta_{\beta}
\]

where \( \phi^{(1)}_t \) is the \( t \)-th column of \( \Phi_1 = W_1 \tilde{\Phi} = A^* A W \tilde{\Phi} \) and construct the matrix \( \Omega = (\omega_{iz}) \). Taking into account \( \sum_{\alpha \in I_{r,m}} |AA^*|_{\alpha}^\alpha = \sum_{\beta \in J_{r,n}} |A^* A|_{\beta}^\beta \), and

\[
\sum_{z=1}^n \omega_{iz} \sum_{\alpha \in I_{r,m}(j)} \text{rdet}_z \left( (A A^*)_{j, (w^{(3)}_z)} \right)_{\alpha} = \sum_{\alpha \in I_{r,m}(j)} \text{rdet}_z ((A A^*)_{j, (\tilde{w}_i)} \tilde{\omega}_i \alpha),
\]

where \( \tilde{w}_i \) is the \( i \)-th row of \( \tilde{W} = \Omega W_3 = \Omega (WA)^{k+1} A^* \), finally from \( 5.10 \), it follows \( 5.2 \).

(ii) By applying the determinantal representations \( 2.6 \) for \( A^{d,W} \) and the same as in the above point for \( Q_A \) and \( P_A \), we get

\[
a_{ij}^{c,W} = \sum_{t=1}^m \sum_{s=1}^n \sum_{\beta \in J_{r,n}(t)} \text{cdet}_t \left( (A^* A)_{t, (w^{(1)}_t)} \right)^\beta_{\beta} \sum_{\alpha \in I_{r,m}(j)} |AA^*|_{\alpha}^\alpha \times
\]

\[
26
\]
where $\tilde{\psi}_{z} := \sum_{\beta \in J_{r_1,m} \{ z \}} \text{cdet}_{z} \left( \left( (V^{2k+1})^{*} V^{2k+1} \right)_{z} \left( \tilde{\psi}_{z} \right) \right)_{\beta}^{\beta}$ and construct the matrix $\tilde{\Psi} = (\tilde{\psi}_{z})$. Then, introduce

$$v_{zj} = \sum_{z=1}^{n} \tilde{\psi}_{zs} \sum_{\alpha \in I_{r,m}(j)} \text{rdet}_{j} \left( (A^{*}A)_{j} (w_{z}^{(2)}) \right)_{\alpha}^{\alpha} = \sum_{\alpha \in I_{r,m}(j)} \text{rdet}_{j} \left( (A^{*}A)_{j} (\psi_{z}^{(1)}) \right)_{\alpha}^{\alpha},$$

where $\tilde{\psi}_{z}^{(1)}$ is the $z$-th row of $\Psi^{(1)} = \tilde{\Psi} W_{2} = \tilde{\Psi} W A A^{*}$ and construct the matrix $\tilde{Y} = (v_{zj})$. Taking into account that

$$\sum_{z=1}^{n} \sum_{\beta \in I_{r,n}(i)} \text{cdet}_{i} \left( (A^{*}A)_{i} (w_{z}^{(2)}) \right)_{\beta}^{\beta} v_{zj} = \sum_{\beta \in I_{r,n}(i)} \text{cdet}_{i} \left( (A^{*}A)_{i} (\tilde{v}_{j}) \right)_{\beta}^{\beta},$$

where $\tilde{v}_{j}$ is the $j$-th column of $\tilde{Y} = W_{3} Y = A^{*}(AW)^{k+1} Y$, finally from (5.11), it follows (5.6).

Simpler expressions of determinantal representations of the WCMP inverse can be obtained in the cases having Hermicity.
Theorem 5.4. Let $A \in \mathbb{H}^{m \times n}$ and $W \in \mathbb{H}^{n \times m}$ be a non-zero matrix. Suppose $k = \max\{\text{Ind}(WA), \text{Ind}(AW)\}$. Then the determinantal representations of its WCMP inverse $A^{c,+W} = (a_{ij}^{c,+W})$ can be expressed as

(i) if $WA$ is Hermitian and $\text{rk}(WA)^k = r_1$, then

$$
a_{ij}^{c,+W} = \frac{\sum_{\alpha \in I_{r,m}(j)} \text{rdet}_r((A^*A^+)_{ij}, (\tilde{w}^\alpha))_\alpha^\beta}{\left(\sum_{\beta \in J_{r,m}} |A^*A|_\beta^2\right)^{r/2} \sum_{\alpha \in I_{r,n}} |(AW)^{k+2}|_\alpha^\alpha}$$

where $\tilde{w}_1$ is the $i$-th row of $\tilde{\Omega} = \Omega WAA^*$. The matrix $\Omega = (\omega_{is})$ is such that

$$
\omega_{is} = \sum_{\alpha \in I_{r,n}(s)} \text{rdet}_s((WA)^{k+2}(\phi_{s1}))_\alpha^\alpha.
$$

Here $\phi_{s1}^\alpha$ is the $i$-th row of $\Phi_1 = \Phi A (WA)^k$ and the matrix $\Phi = (\phi_{it})$ is such that

$$
\phi_{it} := \sum_{\beta \in J_{r,n}(i)} \text{cdet}_i((A^*A^+)_{ij}, (w^{(1)}_1))_\beta^\beta,
$$

where $w^{(1)}_1$ is the $t$-th column of $W_1 = A^*AW$.

(ii) if $AW$ is Hermitian and $\text{rk}(AW)^k = r_1$, then

$$
a_{ij}^{c,+W} = \frac{\sum_{\beta \in J_{r,n}(i)} \text{cdet}_i((A^*A^+)_{ij}, (\tilde{v}_j))_\beta^\beta}{\left(\sum_{\alpha \in I_{r,m}} |AA^*|_\alpha^2\right)^{r/2} \sum_{\beta \in J_{r,m}} |(AW)^{k+2}|_\beta^\beta}
$$

where $\tilde{v}_j$ is the $j$-th column of $\tilde{Y} = A^*AWY$. The matrix $Y = (v_{ij})$ is determined by

$$
v_{ij} = \sum_{\beta \in J_{r,m}(i)} \text{cdet}_r((AW)^{k+2}(\psi^{(1)}_j))_\beta^\beta,
$$

where $\psi^{(1)}_j$ is the $j$-th column of $\Psi^{(1)} = (AW)^k A^*$. Here $\Psi = (\psi_{sj})$ is such that

$$
\psi_{sj} := \sum_{\alpha \in I_{r,m}(j)} \text{rdet}_j((AA^+)_{ij}, (w^{(2)}_s))_\alpha^\alpha,
$$

where $w^{(2)}_s$ is the $s$-th row of $W_2 = WAA^*$.

Proof. (i) Taking into account (5.1), applying one of the cases of (2.3) and (2.4) for the determinantal representations of $Q_A$ and $P_A$, respectively, and (2.3) for the determinantal representation of $A^{d,W}$ give
and construct the matrix \( \Phi A \) where
\[
\phi_i(\tilde{\omega}) = \sum_{\beta \in J_r, n} c \det_i \left( (A^*A)_{ji}(w_{jt}^{(1)}) \right)_{\beta}^3
\]

and construct the matrix \( \Phi = (\phi_{it}) \). Then, determine
\[
\omega_{is} = \sum_{n=1}^{m} \phi_{it} \sum_{r,m} \det_s \left( (WA)_{r,m}^{k+2}(\bar{u}_{rt}) \right)_{\alpha}^\alpha = \sum_{r,n} \det_s \left( (WA)_{r,n}^{k+2}(\phi_{it}^{(1)}) \right)_{\alpha}^\alpha
\]

where \( \phi_{it}^{(1)} \) is the \( i \)-th row of \( \Phi_i = \Phi A (WA)^k \) and construct the matrix \( \Omega = (\omega_{is}) \). Taking into account that \( \sum_{\alpha \in I_{r, n}} |A\alpha|_\beta = \sum_{\beta \in I_{r, n}} |A^*A|_{\beta \beta} \), and
\[
\sum_{s=1}^{n} \omega_{is} \sum_{\alpha \in I_{r, m}(j)} \det_j \left( (AA^*)_{ji}(w_{jt}^{(2)}) \right)_{\alpha}^\alpha = \sum_{\alpha \in I_{r, m}(j)} \det_j \left( (AA^*)_{ji}(\bar{\omega}_{it}) \right)_{\alpha}^\alpha
\]

where \( \bar{\omega}_{it} \) is the \( i \)-th row of \( \bar{\Omega} = \mathbf{\Omega} W_2 = \mathbf{\Omega} WAA^* \), finally from (5.14), it follows (5.12).

(ii) By applying (2.3) for the determinantal representation of \( A_{d,W} \) and the same determinantal representations of \( Q_A \) and \( P_A \) as in the point (i), we get
\[
\alpha_{ij}^{d,W} = \sum_{t=1}^{\mathbf{n}} \sum_{s=1}^{m} \frac{\det_i \left( (A^*A)_{ji}(w_{jt}^{(1)}) \right)_{\beta}^\beta}{\sum_{\beta \in I_{r, n}} |A\alpha|_\beta} \times \frac{\sum_{\beta \in I_{r, n}} \det_j \left( (AA^*)_{ji}(w_{jt}^{(2)}) \right)_{\alpha}^\alpha}{\sum_{\alpha \in I_{r, m}} |AA^*|_{\alpha \alpha}},
\]

(5.15)
where \( \mathbf{w}_t^{(1)} \) is the \( t \)-th column of \( \mathbf{W}_1 := \mathbf{A}^* \mathbf{AW} \), \( \mathbf{v}_s \) is the \( s \)-th column of \( \mathbf{V} = (\mathbf{AW})^k \mathbf{A} \), and \( \mathbf{w}_s^{(2)} \) is the \( s \)-th row of \( \mathbf{W}_2 := \mathbf{WAA}^* \). Denote

\[
\psi_{sj} := \sum_{\alpha \in I_t, m(j)} \text{rdet}_t \left( (\mathbf{AA}^*)_{.,j} (\mathbf{w}_s^{(2)}) \right)^\alpha
\]

and construct the matrix \( \Psi = (\psi_{sj}) \). Then, introduce

\[
\nu_{ij} = \sum_{s=1}^{n} \sum_{\beta \in J_{r, m}(t)} \text{rdet}_t \left( (\mathbf{AW})_{.,t}^k (\tilde{\mathbf{v}}_s) \right)^\beta \psi_{sj} = \sum_{\beta \in J_{r, m}(t)} \text{rdet}_t \left( (\mathbf{AW})_{.,t}^k (\psi_{(1)j}^{(1)}) \right)^\beta
\]

where \( \psi_{(1)j}^{(1)} \) is the \( j \)-th column of \( \Psi^{(1)} = (\mathbf{AW})^k \mathbf{A} \Psi \) and construct the matrix \( \Upsilon = (\nu_{ij}) \). Taking into account that

\[
\sum_{t=1}^{n} \sum_{\beta \in J_{r, m}(t)} \text{rdet}_t \left( (\mathbf{AA}^*)_{.,i} \left( \mathbf{w}_t^{(1)} \right) \right)^\beta \nu_{ij} = \sum_{\beta \in J_{r, m}(t)} \text{rdet}_t \left( (\mathbf{AA}^*)_{.,i} (\tilde{\mathbf{v}}_j) \right)^\beta
\]

where \( \tilde{\mathbf{v}}_j \) is the \( j \)-th column of \( \tilde{\mathbf{V}} = \mathbf{A}^* \mathbf{AW} \), finally from (5.15), it follows (5.16). \( \square \)

**Corollary 5.1.** Let \( \mathbf{A} \in \mathbb{C}^{m \times n} \) and \( \mathbf{W} \in \mathbb{C}^{n \times m} \) be a nonzero matrix. Suppose \( k = \max \{ \text{Ind}(\mathbf{WA}), \text{Ind}(\mathbf{AW}) \} \). Then the determinantal representations of its WCMP inverse \( \mathbf{A}^{<t,W} = \left( a_{ij}^{<t,W} \right) \) can be expressed as

\[
(i) \text{ if } \text{rk}(\mathbf{W}\mathbf{A})^k = r_1, \text{ then }
\]

\[
a_{ij}^{<t,W} = \frac{\sum_{\alpha \in I_t, m(j)} |(\mathbf{AA}^*)_{.,j} (\mathbf{\omega}_t^{(1)})|_\alpha^\alpha}{\left( \sum_{\beta \in J_{r, m}(i)} |\mathbf{A}^* \mathbf{A}|_\beta^2 \right)^2 \sum_{\alpha \in I_{r, n}} |(\mathbf{WA})^k|_\alpha^\alpha}
\]

where \( \mathbf{\omega}_t^{(1)} \) is the \( i \)-th row of \( \mathbf{\Omega} = \mathbf{WA} \mathbf{A}^* \). The matrix \( \mathbf{\Omega} = (\omega_{is}) \) is such that

\[
\omega_{is} = \sum_{\alpha \in I_{r, n}} |(\mathbf{WA})^k + (\phi_{(1)t})|_\alpha^\alpha.
\]

Here \( \phi_{(1)t} \) is the \( i \)-th row of \( \mathbf{\Phi} = \mathbf{A} (\mathbf{WA})^k \) and the matrix \( \mathbf{\Phi} = (\phi_{it}) \) is such that

\[
(\mathbf{\Phi}_{it}) := \sum_{\beta \in J_{r, n}(i)} |(\mathbf{A}^* \mathbf{A})_{.,i} (\mathbf{w}_t^{(1)})|_\beta^\beta.
\]

where \( \mathbf{w}_t^{(1)} \) is the \( t \)-th column of \( \mathbf{W}_1 = \mathbf{A}^* \mathbf{AW} \).

\[
(ii) \text{ if } \text{rk}(\mathbf{A}\mathbf{W})^k = r_1, \text{ then }
\]

\[
a_{ij}^{<t,W} = \frac{\sum_{\beta \in J_{r, n}(i)} |(\mathbf{A}^* \mathbf{A})_{.,i} (\mathbf{\nu}_j) |_\beta^\beta}{\left( \sum_{\alpha \in I_t, m} |\mathbf{AA}^*|_\alpha^2 \right)^2 \sum_{\beta \in J_{r, m}} |(\mathbf{WA})^k + (\phi_{(1)t})|_\beta^\beta}
\]

Here \( \phi_{(1)t} \) is the \( i \)-th row of \( \mathbf{\Phi} = \mathbf{A} (\mathbf{WA})^k \) and the matrix \( \mathbf{\Phi} = (\phi_{it}) \) is such that

\[
\phi_{it} := \sum_{\beta \in J_{r, n}(i)} |(\mathbf{A}^* \mathbf{A})_{.,i} (\mathbf{w}_t^{(1)})|_\beta^\beta.
\]
where $\tilde{v}_j$ is the $j$-th column of $\tilde{\textbf{Y}} = A^* \textbf{A} \textbf{W} \textbf{Y}$. The matrix $\textbf{Y} = (v_{ij})$ is determined by

$$v_{ij} = \sum_{\beta \in J_{l_1,m}(t)} \left[ (\textbf{A} \textbf{W})_t^{k+2} \left( \psi_j^{(1)} \right) \right]_{\beta}^\beta,$$

where $\psi_j^{(1)}$ is the $j$-th column of $\Psi^{(1)} = (\textbf{A} \textbf{W})^k \textbf{A} \Psi$. Here $\Psi = (\psi_{kj})$ is such that

$$\psi_{kj} := \sum_{\alpha \in J_{l_1,m}(j)} \left[ (\textbf{A} \textbf{A}^*)_{j,i} (\textbf{w}^{(2)}_s) \right]_{\alpha}^a,$$

where $\textbf{w}^{(2)}_s$ is the $s$-th row of $\textbf{W}_2 = \textbf{W} \textbf{A} \textbf{A}^*$.

Theorems 5.3 and 5.4 give determinantal representations of the WCMP inverse over the quaternion skew field. For better understanding, we present the algorithm of its finding, for example, in Theorem 5.3 the case (i). Other algorithms can be construct similarly.

Algorithm 5.5. 1. Compute the matrix $\textbf{U} = \textbf{U}^k (\textbf{U}^{2k+1})^*$. 2. Find $\phi_lq$ by \textbf{(5.5)} for all $l = 1, \ldots, n$ and $q = 1, \ldots, n$ and construct the matrix $\Phi = (\phi_{lq})$. 3. Compute the matrix $\tilde{\Phi} := \textbf{A} \Phi \textbf{U}^{2k} (\textbf{U}^{2k+1})^*$. 4. Find $\tilde{\phi}_{tqz}$ by \textbf{(5.3)} for all $t = 1, \ldots, m$ and $z = 1, \ldots, n$ and construct the matrix $\tilde{\Phi} = (\tilde{\phi}_{tqz})$. 5. Compute the matrix $\textbf{F}_1 = \textbf{A}^* \textbf{A} \textbf{W} \tilde{\Phi}$. 6. By \textbf{(5.3)}, find $\omega_{tqz}$ for all $i = 1, \ldots, m$ and $z = 1, \ldots, n$ and construct the matrix $\Omega = (\omega_{tqz})$. 7. Finally, find $a_{ij}^{cw}$ by \textbf{(6.2)} for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

6 An example

In this section, we give an example to illustrate our results. Given the matrices

$$\textbf{A} = \begin{bmatrix} 0 & i & 0 \\ k & 1 & i \\ 1 & 0 & 0 \\ 1 & -k & -j \end{bmatrix}, \quad \textbf{W} = \begin{bmatrix} k & 0 & i \\ -j & k & 0 \\ 0 & 1 & 0 \\ -k & 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (6.1)

Since

$$\textbf{V} = \textbf{A} \textbf{W} = \begin{bmatrix} -k & -j & 0 & i \\ -1 - j & i + k & j & 1 + j \\ k & 0 & i & 0 \\ -1 + k & 1 - j & i & i - k \end{bmatrix}, \quad \textbf{U} = \textbf{W} \textbf{A} = \begin{bmatrix} i & j & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\textbf{A}^* \textbf{A} = \begin{bmatrix} 3 & -2k & -2j \\ 2k & 3 & 2i \\ 2j & -2i & 2 \end{bmatrix}, \quad \textbf{A} \textbf{A}^* = \begin{bmatrix} 1 & i & 0 & -j \\ -i & 3 & k & 3k \\ 0 & -k & 1 & 1 \\ j & -3k & 1 & 3 \end{bmatrix}.$$
and \( \mathrm{rk} \mathbf{A} = 3, \mathrm{rk} \mathbf{W} = 3, \mathrm{rk} \mathbf{V} = 3, \mathrm{rk} \mathbf{V}^3 = \mathrm{rk} \mathbf{V}^2 = 2, \mathrm{rk} \mathbf{U}^2 = \mathrm{rk} \mathbf{U} = 2 \), then \( \text{Ind} \mathbf{V} = 2, \text{Ind} \mathbf{U} = 1 \), and \( k = \max \{ \text{Ind}(\mathbf{AW}), \text{Ind}(\mathbf{WA}) \} = 2 \).

We shall find the weighted DMP inverse due to Algorithm 4.7.

1. Compute the matrix \( \mathbf{\hat{U}} = \mathbf{U}^2 (\mathbf{U}^5)^* \). Since
   \[
   \mathbf{U}^2 = \begin{bmatrix}
   -1 & i + k & 0 \\
   0 & -1 & 0 \\
   0 & 0 & 0
   \end{bmatrix}, \quad \mathbf{U}^5 = \begin{bmatrix}
   i & 2 + 3j & 0 \\
   0 & k & 0 \\
   0 & 0 & 0
   \end{bmatrix},
   \]
   then
   \[
   \mathbf{\hat{U}} = (\mathbf{U}^5)^* \mathbf{U}^2 = \begin{bmatrix}
   i & 1 + j & 0 \\
   -2 + 3j & -i + 6k & 0 \\
   0 & 0 & 0
   \end{bmatrix}
   \]
   and \( \mathrm{rk} \mathbf{U}^2 = 2 \).

2. By (4.6) find \( \phi_{iq} \) for all \( i, q = 1, 2, 3 \). So, \( \Phi = \begin{bmatrix}
   i & -2 - j & 0 \\
   0 & k & 0 \\
   0 & 0 & 0
   \end{bmatrix} \).

3. Further, the matrix \( \mathbf{\hat{\Phi}} := \mathbf{WA}\Phi\mathbf{U}^4 (\mathbf{U}^5)^* = \begin{bmatrix}
   6i - k & 1 + j & 0 \\
   -2 + 3j & k & 0 \\
   0 & 0 & 0
   \end{bmatrix} \).

4. By (4.5) find \( \omega_{is} \) for all \( i, s = 1, 2, 3 \). So, we have that \( \Omega = \Phi \).

5. Compute the matrix \( \mathbf{\tilde{\Omega}} := \Omega \mathbf{U}^3 \mathbf{A}^* = \begin{bmatrix}
   0 & -k & 1 & 1 \\
   -i & 1 & 0 & k \\
   0 & 0 & 0 & 0
   \end{bmatrix} \).

6. Finally, find \( a_{ij}^{d,W} \) by (4.4) for all \( i = 1, \ldots, 4 \) and \( j = 1, 2, 3 \). So,
   \[
   a_{11}^{d,W} = \frac{\sum_{\alpha \in I_{2,4}(1)} \mathrm{rdet}_1 (\mathbf{AA}^*) (\mathbf{A} (\omega_{11}))^\alpha}{\left( \sum_{\beta \in I_{2,4}} |\mathbf{AA}^*|^\beta \sum_{\alpha \in I_{2,3}} |\mathbf{U}^5 (\mathbf{U}^5)^*|^\alpha \right)^2} = \frac{\sum_{\alpha \in I_{2,3}} |\mathbf{AA}^*|^\beta}{\left( \sum_{\beta \in I_{2,4}} |\mathbf{AA}^*|^\beta \sum_{\alpha \in I_{2,3}} |\mathbf{U}^5 (\mathbf{U}^5)^*|^\alpha \right)^2},
   \]
   \[
   = \frac{1}{2} \left( \mathrm{rdet}_1 \begin{bmatrix}
   0 & -k & 1 \\
   -i & 3 & k \\
   0 & -k & 1
   \end{bmatrix} + \mathrm{rdet}_1 \begin{bmatrix}
   0 & 1 & 1 \\
   0 & 1 & 1 \\
   j & -3k & 3
   \end{bmatrix} \right) = 0.
   \]

Continuing similarly, we obtain
   \[
   \mathbf{A}^{d,W} = \begin{bmatrix}
   0 & 0 & 1 & 0 \\
   -i & 0 & 0 & 0 \\
   0 & 0 & 0 & 0
   \end{bmatrix}.
   \]

It is easy verify that \( \mathbf{X} = \mathbf{A}^{d,W} \) from (6.2) with the given matrices (6.1) is the solution to Eqs. (6.2).
7 Conclusions

Notions of the weighted core-EP right and left inverses, the weighted DMP and MPD inverses, and the weighted CMP inverse have been extended to quaternion matrices in this paper. Due to noncommutativity of quaternions, these generalized inverses in quaternion matrices have some features in comparison to complex matrices. We have obtained their determinantal representations within the framework of the theory of column-row determinants previously introduced by the author. As the special cases, their determinantal representations in complex matrices have been obtained as well.

References

[1] Aslaksen, H.: Quaternionic determinants. Math. Intellig. 18(3), 57-65 (1996)

[2] Baksalary, O.M., Trenkler, G.: Core inverse of matrices, Linear Multilinear Algebra 58, 681-697 (2010)

[3] Baksalary, O.M., Trenkler, G.: On a generalized core inverse, Appl. Math. Comput. 236, 450-457 (2014)

[4] Bapat, R.B., Bhaskara Rao, K.P.S., Manjunatha Prasad, K.: Generalized inverses over integral domains. Linear Algebra Appl. 140, 181-196 (1990)

[5] Bhaskara Rao, K.P.S.: Generalized inverses of matrices over integral domains, Linear Algebra Appl. 49, 179-189 (1983)

[6] Chen, J., L., Zhu, H., H., Patri´cio, P., Zhang, Y.L.: Characterizations and representations of core and dual core inverses. Canad. Math. Bull. 60, 269-282 (2017)

[7] R. E. Cline, T. N. E. Greville, A Drazin inverse for rectangular matrices, Linear Algebra Appl. 29, 53-62 (1980)

[8] Cohen, N., De Leo, S.: The quaternionic determinant. Electron. J. Linear Algebra 7, 100–111 (2000)

[9] Gao, Y.F., Chen, J.L.: Pseudo core inverses in rings with involution, Comm. Algebra 46, 38-50 (2018)

[10] Gao, Y.F., Chen, J.L, Patricio, P.: Representations and properties of the W-weighted core-EP inverse, Linear Multilinear Algebra, (2018). Doi: 10.1080/03081087.2018.1535573

[11] Guterman, A., Herrero, A., Thome, N.: New matrix partial order based on spectrally orthogonal matrix decomposition, Linear Multilinear Algebra 64(3), 362-374 (2016)

[12] Ferreyra, D.E., Levis, F.E., Thome, N.: Maximal classes of matrices determining generalized inverses, Appl. Math. Comput. 333, 42-52 (2018)

[13] Ferreyra, D.E., Levis, F.E., Thome, N.: Revisiting the core-EP inverse and its extension to rectangular matrices. Quaest. Math. 41(2), 265-281 (2018)
[14] Kyrchei, I.: Analogs of the adjoint matrix for generalized inverses and corresponding Cramer rules, Linear Multilinear Algebra 56(4), 453-469 (2008)

[15] Kyrchei, I.: Explicit formulas for determinantal representations of the Drazin inverse solutions of some matrix and differential matrix equations, Appl. Math. Comput. 219, 7632-7644 (2013)

[16] Kyrchei, I.: Cramer’s rule for generalized inverse solutions. In: I. Kyrchei (Ed.), Advances in Linear Algebra Research, pp. 79–132, Nova Sci. Publ., New York, 2015.

[17] Kyrchei, I.: Cramer’s rule for quaternionic systems of linear equations. J. Math. Sci. 155(6), 839–858 (2008)

[18] Kyrchei, I.: The theory of the column and row determinants in a quaternion linear algebra. In: Albert R. Baswell (Ed.), Advances in Mathematics Research 15, pp. 301–359. Nova Sci. Publ., New York (2012)

[19] Kyrchei, I.: Determinantal representations of the Moore-Penrose inverse over the quaternion skew field. J. Math. Sci. 180(1), 23–33 (2012)

[20] Kyrchei, I.: Determinantal representations of the Moore-Penrose inverse over the quaternion skew field and corresponding Cramer’s rules. Linear Multilinear Algebra 59(4), 413-431 (2011)

[21] Kyrchei, I.: Determinantal representations of the Drazin inverse over the quaternion skew field with applications to some matrix equations. Appl. Math. Comput. 238, 193–207 (2014)

[22] Kyrchei, I.: Determinantal representations of the W-weighted Drazin inverse over the quaternion skew field. Appl. Math. Comput. 264, 453–465 (2015)

[23] Kyrchei, I.: Explicit determinantal representation formulas of W-weighted Drazin inverse solutions of some matrix equations over the quaternion skew field. Math. Probl. Eng. 8673809, 13 p. (2016)

[24] Kyrchei, I.: Determinantal representations of the Drazin and W-weighted Drazin inverses over the quaternion skew field with applications. In: Griffin, S. (Ed.), Quaternions: Theory and Applications, pp. 201–275. Nova Sci. Publ., New York, (2017)

[25] Kyrchei, I.: Weighted singular value decomposition and determinantal representations of the quaternion weighted Moore-Penrose inverse. Appl. Math. Comput. 309, 1–16 (2017)

[26] Kyrchei, I.: Determinantal representations of the quaternion weighted Moore-Penrose inverse and its applications. In: Baswell, A.R.(Ed.) Advances in Mathematics Research 23, pp. 35-96. Nova Science Publ., New York (2017)

[27] Kyrchei, I.: Determinantal representations of solutions to systems of quaternion matrix equations. Adv. Appl. Clifford Algebras 28(1), 23 (2018)
[28] Kyrchei, I.: Cramer’s Rules for Sylvester quaternion matrix equation and its special cases. Adv. Appl. Clifford Algebras 28(5), 90 (2018)

[29] Kyrchei, I.: Determinantal representations of solutions to systems of two-sided quaternion matrix equations. Linear Multilinear Algebra (2019). Doi:10.1080/03081087.2019.1614517

[30] Kyrchei, I.: Determinantal representations of the quaternion core inverse and its generalizations. Adv. Appl. Clifford Algebras 29: 104 (2019)

[31] Kyrchei, I.: Determinantal representations of the core inverse and its generalizations with applications. Journal of Mathematics 8175935, 6 p. (2018)

[32] Liu, X., Cai, N.: High-order iterative methods for the DMP inverse. Journal of Mathematics 8175935, 6 p. (2018)

[33] Ma, H., Stanimirović, P.S.: Characterizations, approximation and perturbations of the core-EP inverse. Appl. Math. Comput. 359, 404-417 (2019)

[34] Malik, S., Thome, N.: On a new generalized inverse for matrices of an arbitrary index. Appl. Math. Comput. 226, 575-580 (2014)

[35] Mehdipour, M., Salemi, A.: On a new generalized inverse of matrices. Linear Multilinear Algebra 66(5), 1046-1053 (2018)

[36] Meng, L.: The DMP Inverse for Rectangular Matrices. Filomat 31(19), 6015-6019 (2017)

[37] Mielniczuk, J.: Note on the core matrix partial ordering. Discuss. Math. Probab. Stat. 31, 71-75 (2011)

[38] Mosić, D., Deng, C., Ma, H.: On a weighted core inverse in a ring with involution. Comm. Algebra 46(6), 2332-2345 (2018)

[39] Mosić, D.: Weighted core-EP inverse of an operator between Hilbert spaces. Linear Multilinear Algebra 67(2), 278-298, (2019)

[40] Mosić, D. The CMP inverse for rectangular matrices. Aequationes Math. 92, 649-659 (2018)

[41] R. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc. 52, 406-413 (1955)

[42] Prasad, K.M., Mohana, K.S.: Core-EP inverse. Linear Multilinear Algebra 62(3), 792-802 (2014)

[43] Prasad, K.M., Raj, M.D.: Bordering method to compute Core-EP inverse. Spec. Matrices 6, 193-200 (2018)

[44] Rakić, D.S., Č. Dinčić, N., Djordjević, D.S.: Group, Moore-Penrose, core and dual core inverse in rings with involution. Linear Algebra Appl. 463, 115-133 (2014)

[45] Stanimirović, P.S., Katsikis, V.N., Ma, H.: Representations and properties of the W-weighted Drazin inverse. Linear Multilinear Algebra 65(6), 1080-1096 (2017)
[46] Stanimirović, P.S.: General determinantal representation of pseudoinverses of matrices. Mat. Vesnik. 48, 1-9 (1996)

[47] Stanimirović, P.S., Djordjevic, D.S.: Full-rank and determinantal representation of the Drazin inverse, Linear Algebra Appl. 311, 131-151 (2000)

[48] Song, G.J.: Determinantal representations of the generalized inverses $A^{(2)}_{P,S}$ over the quaternion skew field with applications. J. Appl. Math. Comput. 39, 201-220 (2012)

[49] Song, G.J.: Bott-Duffin inverse over the quaternion skew field with applications. J. Appl. Math. Comput. 41, 377-392 (2013)

[50] Song, G.J.: Characterization of the W-weighted Drazin inverse over the quaternion skew field with applications. J. Linear Algebra 26, 1-14 (2013)

[51] Wang, H.X.: Core-EP decomposition and its applications. Linear Algebra Appl. 508, 289-300 (2016)

[52] Y. Wei, Integral representation of the W-weighted Drazin inverse, Appl. Math. Comput. 144, 3-10 (2003)

[53] Y. Wei, A characterization for the W-weighted Drazin inverse and a Cramer rule for the W-weighted Drazin inverse solution, Appl. Math. Comput. 125, 303-310 (2002)

[54] Xu, S.Z., Chen, J.L., Zhang, X.X.: New characterizations for core inverses in rings with involution. Front. Math. China 12, 231-246 (2017)

[55] Zhou, M., Chen, J., Li, T., Wang, D.: Three limit representations of the core-EP inverse. Filomat 32, 5887-5894 (2018)

[56] Al-Zhour, Z., Kilicman, A., Abu-Hassa, M.H.: New representations for weighted Drazin inverse of matrices. Int. J. Math. Anal. 1(15), 697-708 (2007)