Contraction Bidimensionality of Geometric Intersection Graphs

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Received: 6 June 2021 / Accepted: 14 November 2021 / Published online: 24 January 2022
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Abstract
Given a graph $G$, we define $\text{bcg}(G)$ as the minimum $k$ for which $G$ can be contracted to the uniformly triangulated grid $\Gamma_k$. A graph class $\mathcal{G}$ has the SQGC property if every graph $G \in \mathcal{G}$ has treewidth $O(\text{bcg}(G)^c)$ for some $1 < c < 2$. The SQGC property is important for algorithm design as it defines the applicability horizon of a series of meta-algorithmic results, in the framework of bidimensionality theory, related to fast parameterized algorithms, kernelization, and approximation schemes. These results apply to a wide family of problems, namely problems that are contraction-bidimensional. Our main combinatorial result reveals a wide family of graph classes that satisfy the SQGC property. This family includes, in particular, bounded-degree string graphs. This considerably extends the applicability of bidimensionality theory for contraction bidimensional problems.

Keywords Treewidth · Bidimensionality · Parameterized algorithms

1 Introduction
Treewidth is one of most well-studied parameters in graph algorithms. It serves as a measure of how close a graph is to the topological structure of a tree (see Sect. 2 for the formal definition). Gavril is the first to introduce the concept in [30] but it obtained its name in the second paper of the Graph Minors series of Robertson and Seymour.

An extended abstract of this article appeared in the Proceedings of the 12th International Symposium on Parameterized and Exact Computation, IPEC 2017, September 6–8, 2017, Vienna, Austria [2].

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in [38]. Treewidth has extensively used in graph algorithm design due to the fact that a wide class of intractable problems in graphs becomes tractable when restricted on graphs of bounded treewidth [1,5,6]. Before we present some key combinatorial properties of treewidth, we need some definitions.

**Graph contractions and minors.** Our first aim is to define two parameterized versions of the contraction relation on graphs.

**Definition 1** (Contractions) Given a non-negative integer c, two graphs H and G, and a surjection σ : V(G) → V(H) we write $H \leq^c G$ if

- for every $x \in V(H)$, the graph $G[\sigma^{-1}(x)]$ is a non-empty graph (i.e., a graph with at least one vertex) of diameter at most c and
- for every $x, y \in V(H)$, $\{x, y\} \in E(H) \iff G[\sigma^{-1}(x) \cup \sigma^{-1}(y)]$ is connected.

We say that $H$ is a c-diameter contraction of $G$ if there exists a surjection $\sigma : V(G) \rightarrow V(H)$ such that $H \leq^c G$ and we write this $H \leq_c G$. Moreover, if $\sigma$ is such that for every $x \in V(H), |\sigma^{-1}(x)| \leq c'$, then we say that $H$ is a c’-size contraction of $G$, and we write $H \leq^{(c')} G$. Given two graphs $G$ and $H$, if there exists an integer $c$ such that $H \leq^c G$, then we say that $H$ is a contraction of $G$, and we write $H \leq G$. Moreover, if there exists a subgraph $G'$ of $G$ such that $H \leq G'$, we say that $H$ is a minor of $G$ and we write this $H \leq G$. Given a graph $H$, we denote by excl($H$) the class of graphs that exclude $H$ as a minor.

**1.1 Combinatorics of Treewidth**

One of the most celebrated structural results on treewidth is the following:

**Proposition 2** There is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that every graph excluding a $(k \times k)$-grid as a minor has treewidth at most $f(k)$.

A proof of Proposition 2 appeared for the first time by Robertson and Seymour in [39]. Other proofs, with better bounds to the function $f$, appeared in [40] and later in [17] (see also [33,35]). Currently, the best bound for $f$ is due to Chuzhoy, who recently proved in [4] that $f(k) = k^9 \cdot \log^{O(1)} k$. On the other hand, it is possible to show that Proposition 2 is not correct when $f(k) = \mathcal{O}(k^2 \cdot \log k)$ (see [43]).

The potential of Proposition 2 on graph algorithms has been capitalized by the theory of bidimensionality that was introduced in [10] and has been further developed in [9,13,14,16,21–23,26,29,32]. This theory offered general techniques for designing efficient fixed-parameter algorithms and approximation schemes for NP-hard graph problems in broad classes of graphs (see [8,11,12,15,20]). In order to present the result of this paper we first give a brief presentation of this theory and of its applicability.

**Optimization parameters and bidimensionality.** A graph parameter is a function $p$ mapping graphs to non-negative integers. We say that $p$ is a minimization graph parameter if $p(G) = \min\{k \mid \exists S \subseteq V(G) : |S| \leq k \text{ and } \phi(G, S) = \top\} \cup \{\bot\}$, where $\phi$ is a some predicate on $G$ and $S$. Similarly, we say that $p$ is a maximization graph parameter if in the above definition we replace min and $\leq$ by max and $\geq$ respectively. Minimization or maximization parameters are briefly called optimization parameters.
Definition 3 (Bidimensionality) Given two real functions $f$ and $g$, we use the term $f \gtrapprox g$ to denote that $f(x) \geq g(x) - o(g(x))$. A graph parameter $p$ is minor-closed (resp. contraction-closed) when $H \preceq G \Rightarrow p(H) \leq p(G)$ (resp. $H \preceq G \Rightarrow p(H) \leq p(G)$). We can now give the two following definitions:

- $p$ is minor-bidimensional if $p$ is minor-closed,
- $p$ is contraction-bidimensional if $p$ is contraction-closed,

for some $\delta > 0$. In the above definitions, we use $\boxtimes_k$ for the $(k \times k)$-grid and $\Gamma_k$ for the uniformly triangulated $(k \times k)$-grid (see Fig. 1). If $p$ is a minimization (resp. maximization) graph parameter, we denote by $\Pi_p$ the problem that, given a graph $G$ and a non-negative integer $k$, asks whether $p(G) \leq k$ (resp. $p(G) \geq k$). We say that a problem is minor/contraction-bidimensional if it is $\Pi_p$ for some bidimensional optimization parameter $p$.

A (non exhaustive) list of minor-bidimensional problems is: VERTEX COVER, FEEDBACK VERTEX SET, LONGEST CYCLE, LONGEST PATH, CYCLE PACKING, PATH PACKING, DIAMOND HITTING SET, MINIMUM MAXIMAL MATCHING, FACE COVER, and MAX BOUNDED DEGREE CONNECTED SUBGRAPH. Some problems that are contraction-bidimensional (but not minor-bidimensional) are CONNECTED VERTEX COVER, DOMINATING SET, CONNECTED DOMINATING SET, CONNECTED FEEDBACK VERTEX SET, INDUCED MATCHING, INDUCED CYCLE PACKING, CYCLE DOMINATION, CONNECTED CYCLE DOMINATION, $d$-SCATTERED SET, INDUCED PATH PACKING, $r$-CENTER, CONNECTED $r$-CENTER, CONNECTED DIAMOND HITTING SET, UNWEIGHTED TSP TOUR (see [14,20,44]).

Subquadratic grid minor/contraction property. In order to present the meta-algorithmic potential of bidimensionality theory we need to define some property on graph classes that defines the horizon of its applicability.

Definition 4 (SQGC and SQGC) Let $G$ be a graph class. We say that $G$ has the subquadratic grid minor property (SQGM property for short) if there exist a constant $1 \leq c < 2$ such that every graph $G \in G$ which excludes $\boxtimes_t$ as a minor, for some
Fig. 2 The applicability of bidimensionality theory. The green lines correspond the consequences [32] while the red lines correspond to the result of this paper (Color figure online)

integer $t$, has treewidth $O(t^c)$. In other words, this property holds for $G$ if Proposition 2 can be proven for a sub-quadratic $f$ on the graphs of $G$.

Similarly, we say that $G$ has the subquadratic grid contraction property (SQGC property for short) if there exist a constant $1 \leq c < 2$ such that every graph $G \in \Gamma$ which excludes $\Gamma_t$ as a contraction, for some integer $t$, has treewidth $O(t^c)$. For brevity we say that $\mathcal{G} \in \text{SQGM}(c)$ (resp. $\mathcal{G} \in \text{SQGC}(c)$) if $\mathcal{G}$ has the SQGM (resp SQGC) property for $c$. Notice that $\text{SQGC}(c) \subseteq \text{SQGM}(c)$ for every $1 \leq c < 2$.

1.2 Algorithmic Implications

The meta-algorithmic consequences of bidimensionality theory are summarised as follows. Let $G \in \text{SQGM}(c)$, for $1 \leq c < 2$, and let $p$ be a minor-bidimensional-optimization parameter.

[A] As it was observed in [10], the problem $\Pi_p$ can be solved in $2^{o(k)} \cdot n^{O(1)}$ steps on $G$, given that the computation of $p$ can be done in $2^{O(tw(G))} \cdot n^{O(1)}$ steps (here $tw(G)$ is the treewidth of the input graph $G$). This last condition can be implied by a purely meta-algorithmic condition that is based on some variant of Modal Logic [37]. There is a wealth of results that yield the last condition for various optimization problems either in classes satisfying the SQGM property [18,18,19,41,42] or to general graphs [3,7,24].

[B] As it was shown in [26] (see also [27]), when the predicate $\phi$ can be expressed in Counting Monadic Second Order Logic (CMSOL) and $p$ satisfies some additional combinatorial property called separability, then the problem $\Pi_p$ admits a linear kernel, that is a polynomial-time algorithm that transforms $(G, k)$ to an equivalent instance $(G', k')$ of $\Pi_p$ where $G'$ has size $O(k)$ and $k' \leq k$.

[C] It was proved in [22] (see also [25] and [28]), that the problem of computing $p(G)$ for $G \in \mathcal{G}$ admits a Efficient Polynomial Approximation Scheme (EPTAS) — that is an $\epsilon$-approximation algorithm running in $f(\frac{1}{\epsilon}) \cdot n^{O(1)}$ steps — given that $\mathcal{G}$ is hereditary and $p$ satisfies the separability property and some reducibility property (related to CMSOL expresibility).

All above results have their counterparts for contraction-bidimensional problems with the difference that one should instead demand that $\mathcal{G} \in \text{SQGC}(c)$. Clearly, the applicability of all above results is delimited by the SQGM/SQGC property. This
is schematically depicted in Fig. 2, where the green triangles triangles indicate the applicability of minor-bidimensionality and the red triangle indicate the applicability of contraction-bidimensionality. The aforementioned $\Omega(k^2 \cdot \log k)$ lower bound to the function $f$ of Proposition 2, indicates that $\text{SQGM}(c)$ does not contain all graphs (given that $c < 2$).

As an example we mention the well known $d$-Domination Set problem (for some $d \geq 1$), asking whether a graph $G$ has a set $S$ of at most $k$ vertices such that every vertex in $G$ is within distance at most $d$ from some vertex of $S$. $d$-Domination Set is contraction bidimensional problem that satisfies the additional meta-algorithmic conditions in [A], [B], and [C]. This implies that it can be solved in $2^{O(\sqrt{k})} \cdot n$ time, it admits a linear kernel, and its optimization version admits an EPTAS on every graph class that has the SQGC property.

The emerging direction of research is to detect the most general classes in $\text{SQGM}(c)$ and $\text{SQGC}(c)$. Concerning the $\text{SQGC}$ property, the following result was proven in [15].

**Proposition 5** For every graph $H$, $\text{excl}(H) \in \text{SQGM}(1)$.

A graph $H$ is an apex graph if it contains a vertex whose removal from $H$ results to a planar graph. For the $\text{SQGC}$ property, the following counterpart of Proposition 5 was proven in [21].

**Proposition 6** For every apex graph $H$, $\text{excl}(H) \in \text{SQGC}(1)$.

Notice that both above results concern graph classes that are defined by excluding some graph as a minor. For such graphs, Proposition 6 is indeed optimal. To see this, consider $K_h$-minor free graphs where $h \geq 6$ (these graphs are not apex graphs). Such classes do not satisfy the $\text{SQGC}$ property: take $\Gamma_k$, add a new vertex, and make it adjacent, with all its vertices. The resulting graph excludes $\Gamma_k$ as a contraction and has treewidth $> k$.

### 1.3 String Graphs

An important step extending the applicability of bidimensionality theory further than $H$-minor free graphs, was done in [23] (see also [25]).

**Definition 7** (String graphs, map graphs, and unit disk graphs) *Unit disk* graphs are intersections graphs of unit disks in the plane and *map* graphs are intersection graphs of face boundaries of planar graph embeddings. We denote by $\mathcal{U}_d$ the set of unit disk graphs (resp. of $\mathcal{M}_d$ map graphs) of maximum degree $d$.

The following was proved in [23,25].

**Proposition 8** For every positive integer $d$, $\mathcal{U}_d \in \text{SQGM}(1)$ and $\mathcal{M}_d \in \text{SQGM}(1)$.

Proposition 8 was further extended for intersection graphs of more general geometric objects (in 2 dimensions) in [32]. To explain the results of [32] we need to define a more general model of intersection graphs.
Definition 9 (String graphs) Let $\mathcal{L} = \{L_1, \ldots, L_k\}$ be a collection of lines in the plane. We say that $\mathcal{L}$ is normal if there is no point belonging to more than two lines. The intersection graph $G_\mathcal{L}$ of $\mathcal{L}$, is the graph whose vertex set is $\mathcal{L}$ and where, for each $i, j$ where $1 \leq i < j \leq k$, the edge $\{L_i, L_j\}$ has multiplicity $|L_1 \cap L_2|$. We denote by $S_d$ the set containing every graph $G_\mathcal{L}$ where $\mathcal{L}$ is a normal collection of lines in the plane and where each vertex of $G_\mathcal{L}$ has edge-degree at most $d$. i.e., is incident to at most $d$ edges. We call $S_d$ string graphs with edge-degree bounded by $d$.

It is easy to observe that $\mathcal{U}_d \cup \mathcal{M}_d \subseteq S_{f(d)}$ for some quadratic function $f$. Indeed, given a graph $G$ in $\mathcal{U}_d$, for each unit disk of its representation in the plane, we can create a string that corresponds to the perimeter of the disk. As all the disks are of the same size, the intersection graph of the strings is homeomorphic to $G$. The same applies for map graphs by considering the boundaries of the faces and creating a string for each of them. Moreover, apart from the classes considered in [23], $S_d$ includes a much wider variety of classes of intersection graphs [32]. As an example, consider $C_{d, \alpha}$ as the class of all graphs that are intersection graphs of $\alpha$-convex bodies $^1$ in the plane and have edge-degree at most $d$. In [32], it was proven that $C_{d, \alpha} \subseteq S_c$ where $c$ depends (polynomially) on $d$ and $\alpha$. Another interesting class from [32] is $\mathcal{F}_{H, \alpha}$ containing all $H$-subgraph free intersection graphs of $\alpha$-fat$^2$ families of convex bodies. Notice $\mathcal{U}_d$ can be seen as a special case of both $C_{d, \alpha}$ and $\mathcal{F}_{H, \alpha}$. (See [36] for other examples of classes included in $S_d$.)

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1 We call a set of points in the plane a body if it is homeomorphic to the closed disk $\{(x, y) | x^2 + y^2 \leq 1\}$. A 2-dimensional body $B$ is a $\alpha$-convex if every two of its points can be the extremes of a line $L$ consisting of $\alpha$ straight lines and where $L \subseteq B$. Convex bodies are exactly the 1-convex bodies.

2 A collection of convex bodies in the plane is $\alpha$-fat if the ratio between the maximum and the minimum radius of a circle where all bodies of the collection can be circumscribed and inscribed respectively, is upper bounded by $a$. 

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Fig. 4 An example of the proof of Proposition 12. In the leftmost figure we see a collection of lines in the plane $L = \{L_1, \ldots, L_k\}$ whose intersection graph $G_L$ is depicted in the rightmost figure and has maximum edge degree 9 because of line $c$ that meets other lines in 9 points, therefore $G_L \in \mathcal{S}_9$. To see why $G_L \in \mathcal{P}(1, 9)$, one may see the leftmost figure are a planar graph $P \in \mathcal{P}$ where vertices of degree 1 are discarded. The vertices of this planar graph can be seen as the result of the contraction of the red edges (seen as subgraphs of diameter 1) in the graph $J$ in the middle, i.e., $P \leq^{(1)} J$. Finally, the intersection graph $G_L$ can be seen as a result of the contraction in $J$ of each one of the paths, on at most 9 vertices, to a single vertex. Therefore $G_L \leq^{9} J$, hence $G_L$ belongs in the $(1, 9)$-extension of planar graphs.

1.4 Our Contribution

Graph class extensions.

**Definition 10** $(c_1, c_2)$-extension) Given a class of graph $G$ and two integers $c_1$ and $c_2$, we define the $(c_1, c_2)$-extension of $G$, denoted by $G^{(c_1, c_2)}$, as the set containing every graph $H$ such that there exist a graph $G \in G$ and a graph $J$ that satisfy $G \leq^{(c_1)} J$ and $H \leq^{c_2} J$ (see Fig. 3 for a visualization of this construction). Keep in mind that $G^{(c_1, c_2)}$ and $G^{(c_2, c_1)}$ are two different graph classes. We also denote by $\mathcal{P}$ the class of all planar graphs.

Using the above notation, the two combinatorial results in [32] can be rewritten as follows:

**Proposition 11** Let $c_1 \geq 1$ and $c_2 \geq 0$ be two integers. If $G \in SQGC(c)$ for some $1 \leq c < 2$, then $G^{(c_1, c_2)} \in SQGM(c)$.

**Proposition 12** For every $d \in \mathbb{N}$, $S_d \subseteq \mathcal{P}(1, d)$.

We visualise the idea of the proof of Proposition 12 by some example, depicted in Fig. 4. In Lemma 25 we use the same idea for a more general result. Fig. 4 motivates the definition of the $(c_1, c_2)$-extension of a graph class. Intuitively, the fact that $H \in G^{(c_1, c_2)}$ expresses the fact that $H$ can be seen as a “bounded” distortion of a graph in $G$ (after a fixed number of “de-contractions” and contractions).

Proposition 6, combined with Proposition 11, provided the wider, so far, framework on the applicability of minor-bidimensionality: $SQGM(1)$ contains $excl(H)^{(c_1, c_2)}$ for every apex graph $H$ and positive integers $c_1$, $c_2$. As, by Proposition 6, $\mathcal{P} \in SQGC(1)$, Proposition 11 and Proposition 12 directly classifies in $SQGM(1)$ the graph class $S_d$, and therefore a large family of bounded degree intersection graphs (including $U_d$ and $M_d$). As a result of this, the applicability of bidimensionality theory for minor-bidimensional problems has been extended to much wider families (not necessarily minor-closed) of graph classes of geometric nature [32].
Our Main Result.

Definition 13 (Intersection graphs) Given a graph $G$ and a set $S \subseteq V(G)$ we say that $S$ is a connected set of $G$ if $G[S]$ is a connected graph. We also define by $C(G)$ the set of all connected subsets of $V(G)$. Given a $C \subseteq C(G)$, we define the intersection graph of $C$ in $G$, denoted by $IG(C)$, as the graph whose vertex set is $C$, where two vertices $C_1$ and $C_2$ of $IG(C)$ are connected by an edge if $C_1 \cap C_2 \neq \emptyset$, and, moreover, the multiplicity of the edge $\{C_1, C_2\}$ is equal to $|V(C_1 \cap C_2)|$. Given a graph class $\mathcal{G}$ we define the following class of graphs $\inter(\mathcal{G}) = \bigcup_{G \in \mathcal{G}} \{IG(C) | C \subseteq C(G)\}$.

In other words, $\inter(\mathcal{G})$ contains all the intersection graphs of the connected vertex subsets of each of the graphs in $\mathcal{G}$. Given a $d \in \mathbb{N}$, we define $\inter_d(\mathcal{G})$ as the set of graphs in $\inter(\mathcal{G})$ that have edge-degree at most $d$.

However, also the degree bound is maintained, as indicated by the following easy lemma.

Lemma 14 For every $d \in \mathbb{N}$, $S_d \subseteq \inter_d(\mathcal{P}) \subseteq S_d'$, for some $d' = O(d^2)$.

Proof (Proof (sketch).) We deal with the less trivial statement that $\inter_d(\mathcal{P}) \subseteq S_{O(d^2)}$. For this, let $H \in \inter_d(\mathcal{P})$ such that $H = IG(C)$ for some collection $C$ of connected subsets of $V(G)$, for some $G \in \mathcal{P}$. We choose the planar graph $G$ so that $|V(G)| + |E(G)|$ is minimized. This means that for every $C \in \mathcal{C}$, $G[C]$ is a tree on at most $2d$ vertices. If we now replace each tree $G[C]$ by a string “surrounding” it is easy to observe that two such string cannot have more than $O(d^2)$ points in common.

Observe that Proposition 11 exhibits some apparent “lack of symmetry” as the assumption is “qualitatively stronger” than the conclusion. This does not permit the application of bidimensionality for contraction-bidimensional parameters on classes further than those of apex-minor free graphs. In other words, the results in [32] covered, for the case of $S_d$, the green triangles in Fig. 2 but left the red triangles open. The main result of this paper is to fill this gap by proving the following extension of Proposition 11. The main result of this paper is the following.

Theorem 15 Let $c_1$ and $c_2$ be two positive integers. If $G \in \sqgc(c)$ for some $1 \leq c < 2$, then $G^{(c_1,c_2)} \in \sqgc(c)$.

Consequences. We call a graph class monotone if it is closed under taking of subgraphs, i.e., every subgraph of a graph in $\mathcal{G}$ is also a graph in $\mathcal{G}$. A powerful consequence of Theorem 15 is the following (the proof is postponed in Sect. 3).

Theorem 16 If $\mathcal{G}$ is a monotone graph class, where $G \in \sqgc(c)$ for some $1 \leq c < 2$, and $d \in \mathbb{N}$, then $\inter_d(\mathcal{G}) \in \sqgc(c)$.

Combining Proposition 6 and Theorem 16 we obtain that $\sqgc(1)$ contains $\inter_d(excl(H))$ for every apex graph $H$. This extends the applicability horizon of
2 Definitions and Preliminaries

We denote by $\mathbb{N}$ the set of all non-negative integers. Given $r, q \in \mathbb{N}$, we define $[r, q] = \{r, \ldots, q\}$ and $[r] = [1, r]$.

All graphs in this paper are undirected, loop-less, and may have multiple edges. If a graph has no multiple edges, we call it simple. Given a graph $G$, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. Let $x$ be a vertex or an edge of a graph $G$ and likewise for $y$; their distance in $G$, denoted by $\text{dist}_G(x, y)$, is the smallest number of vertices of a path in $G$ that contains them both. Moreover if $G$ is a graph and $x, y \in V(G)$, we denote by $N_G^c(x)$, for each $c \in \mathbb{N}$, the set $\{y \mid y \in V(G), \text{dist}_G(x, y) \leq c + 1\}$. For any set of vertices $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of $G$ induced by the vertices from $S$. If $G[S]$ is connected, then we say that $S$ is a connected vertex set of $G$. We define the diameter of a connected subset $S$ as the maximum pairwise distance between any two vertices of $S$. The edge-degree of a vertex $v \in V(G)$ is the number of edges that are incident to it (multi-edges contribute with their multiplicity to this number).

**Definition 17** (Grids) The $(k \times k)$-grid, denoted by $\boxplus_k$, is the graph whose vertex set is $[0, k - 1]^2$ and two vertices $(i, j)$ and $(i', j')$ are adjacent if $|i - i'| + |j - j'| = 1$. For $k \geq 3$, the graph $\Gamma_k$ (resp. $\hat{\Gamma}_k$), is defined if we add in $\boxplus_k$ all the edges in $\{(i + 1, j), (i, j + 1) \mid (i, j) \in [0, k - 2]^2\}$ as well as all the edges between $(k - 1, k - 1)$ (resp. a new vertex $a$) and the vertices in $\{(i, j) \mid [0, k - 1]^2 \setminus [1, k - 2]^2\}$ that have not been added already. For an example of $\Gamma_k$ (resp. $\hat{\Gamma}_k$), see Fig. 1 (resp. Fig. 6). Notice that $\Gamma_k$ is a triangulation of $\boxplus_k$. In each of these graphs we denote the vertices of the underlying grid by their coordinates $(i, j) \in [0, k - 1]^2$ agreeing that...
the upper-left corner (i.e., the unique vertex of degree 3) is the vertex \((0, 0)\). \(\hat{\Gamma}_k\) has two vertices of degree 3, the top left and the bottom right of the grid part. We call \(\Gamma_k\) the \textit{uniformly triangulated grid} and \(\hat{\Gamma}_k\) the \textit{extended uniformly triangulated grid}.

**Definition 18** (Treewidth) A \textit{tree-decomposition} of a graph \(G\), is a pair \((T, X)\), where \(T\) is a tree and \(X = \{X_t : t \in V(T)\}\) is a family of subsets of \(V(G)\), called \textit{bags}, such that the following three properties are satisfied:

- \(\bigcup_{t \in V(T)} X_t = V(G)\),
- for every edge \(e \in E(G)\) there exists \(t \in V(T)\) such that \(e \subseteq X_t\), and
- \(\forall v \in V(G)\), the set \(T_v = \{t \in V(T) \mid v \in X_t\}\) is a connected vertex set of \(T\).

The \textit{width} of a tree-decomposition is the cardinality of the maximum size bag minus 1 and the \textit{treewidth} of a graph \(G\) is the minimum width over all the tree-decompositions of \(G\). We denote the treewidth of \(G\) by \(\text{tw}(G)\).

**Lemma 19** Let \(G\) be a graph and let \(H\) be a \(c\)-size contraction of \(G\). Then \(\text{tw}(G) \leq (c + 1) \cdot (\text{tw}(H) + 1) - 1\).

**Proof** By definition, since \(H\) is a \(c\)-size contraction of \(G\), there is a mapping between each vertex of \(H\) and a connected set of at most \(c\) edges in \(G\), so that by contracting these edge sets we obtain \(H\) from \(G\). The endpoints of these edges form disjoint connected sets in \(G\), implying a partition of the vertices of \(G\) into connected sets \(\{V_x \mid x \in V(H)\}\), where \(|V_x| \leq c + 1\) for any vertex \(x \in V(H)\).

Consider now a tree decomposition \((T, \mathcal{X}')\) of \(H\). We claim that the pair \((T, \mathcal{X}')\), where \(X_t' := \bigcup_{x \in X_t} V_x\) for \(t \in T\) is a tree decomposition of \(G\). Clearly all vertices of \(G\) are included in some bag, since all vertices of \(H\) did. Every edge of \(G\) with both endpoints in the same part of the partition is in a bag, as each of these vertex sets is placed as a whole in the same bag. If \(e\) is an edge of \(G\) with endpoints in different parts of the partition, say \(V_x\) and \(V_y\), then this implies that \(\{x, y\} \in E(H)\). Thus, there is a node \(t\) of \(T\) for which \(x, y \in X_t\) and therefore \(e \subseteq X_t'\). Moreover, the continuity property remains unaffected, since for any vertex \(x \in V(H)\) each vertex in \(V_x\) induces the same subtree in \(T\) that \(x\) did. \(\square\)

In Table 1 we present all the notation that we use in this paper.
3 Proof of Theorem 16

We start with the following useful property of the contraction relation. We use $\delta(G)$ for the minimum number of edges that are incident to a vertex of the graph $G$. Given a vertex $v$ in $G$ incident to exactly two edges $e_1 = \{v, x\}$ and $e_2 = \{v, y\}$, the dissolution of $v$ in $G$ is the operation of removing $e_1$ and $e_2$ from $G$ and then we add the edge $\{x, y\}$. If the graph $H$ occurs from $G$ after applying some (possibly empty) sequence of vertex dissolutions, then we say that $H$ is a dissolution of $G$. We also say that $H$ is a topological minor of $G$ if $H$ the dissolution of some subgraph of $G$.

**Lemma 20** Let $Q$ be a graph where $\delta(Q) \geq 3$ and let $H, G$ be graphs where $H$ is a dissolution of $G$. If $Q \leq G$, then $Q \leq H$.

**Proof** As $Q \leq G$ then there exist $\sigma : V(G) \rightarrow V(Q)$ such that for all $x \in V(Q)$, $G[\sigma^{-1}(x)]$ is a non-empty graph and for every $x, y \in V(Q)$, $\{x, y\} \in E(Q) \iff G[\sigma^{-1}(x) \cup \sigma^{-1}(y)]$ is connected.

Let $v$ be a vertex in $G$ incident to exactly two edges $e_1 = \{v, v'\}$ and $e_2 = \{v, v''\}$, and let $G'$ be the graph obtained from $G$ after the dissolution of $v$. Let $\sigma' : V(G') \rightarrow V(Q)$ such that for all $z \in V(G')$, $\sigma'(z) = \sigma(z)$. As the dissolution maintains connectivity, we have that for every $x, y \in V(Q)$, $\{x, y\} \in E(Q) \iff G[\sigma'^{-1}(x) \cup \sigma'^{-1}(y)]$ is connected. Moreover, as $\delta(Q) \geq 3$, we know that for each $x \in V(Q)$, there exists $z \in \sigma^{-1}(x)$ such that $z$ has edge degree at least 3. In particular we know that $z$ is different from $v$. So we have that $G[\sigma'^{-1}(x)]$ is a non-empty graph. Thus $Q \leq G'$. The lemma follows by iterating this argument. \qed

**Definition 21** (The function $bcg$) Given a graph $G$, we define $bcg(G)$ as the maximum $k$ for which $G$ can be contracted to the uniformly triangulated grid $\Gamma_k$. 

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Notice that \( \text{bcg} \) is a contraction-closed parameter, i.e., if \( H \leq G \), then \( \text{bcg}(H) \leq \text{bcg}(G) \).

**Lemma 22** Let \( H \) and \( G \) be two graphs. If \( H \) is a dissolution of \( G \), then \( \text{bcg}(H) = \text{bcg}(G) \).

**Proof** The fact that \( \text{bcg}(H) \leq \text{bcg}(G) \) follows from the fact that \( H \) is also a contraction of \( G \) and taking into account the contraction-closedness of \( \text{bcg} \). The fact that \( \text{bcg}(G) \leq \text{bcg}(H) \) follows by taking into account that \( \delta(\Gamma_k) \geq 3 \) and applying inductively Lemma 20 to the vertices of degree 2 in \( G \) that need to be dissolved in order to transform \( G \) to \( H \).

**Definition 23** Given a graph class \( \mathcal{G} \), we define the dissolution closure of \( \mathcal{G} \) as the graph class \( \text{diss}(\mathcal{G}) \) containing all the dissolutions of the graphs in \( \mathcal{G} \).

We observe the following.

**Lemma 24** If \( \mathcal{G} \in \mathcal{SQGC}(c) \) for some \( 1 \leq c < 2 \), then \( \text{diss}(\mathcal{G}) \in \mathcal{SQGC}(c) \).

**Proof** Suppose that \( \mathcal{G} \in \mathcal{SQGC}(c) \) for some \( 1 \leq c < 2 \), which implies that

\[
\forall G \in \mathcal{G} \quad \text{tw}(G) \leq \lambda \cdot (\text{bcg}(G))^c. \tag{1}
\]

Let \( H \in \text{diss}(\mathcal{G}) \) and let \( G \in \mathcal{G} \) such that \( H \) is a dissolution of \( G \). By Lemma 22, \( \text{bcg}(H) = \text{bcg}(G) \) and from (1), \( \text{tw}(G) \leq \lambda (\text{bcg}(H))^c \). As \( H \) is a minor of \( G \), we have that \( \text{tw}(H) \leq \lambda (\text{bcg}(H))^c \) and the lemma follows.

The next lemma uses as a departure point the same idea as the one of proof of Proposition 12, visualized by the example of Fig. 4.

**Lemma 25** If \( \mathcal{G} \) is a graph class that is topological minor closed, then \( \text{inter}_d(\mathcal{G}) \subseteq \mathcal{G}^{(d+1,d-1)} \).

**Proof** Let \( H \) be a graph on \( h \) vertices in \( \text{inter}_d(\mathcal{G}) \), for some \( d \in \mathbb{N} \). This means that there is a graph \( G \) in \( \mathcal{G} \) such that we can see the vertices of \( H \) as a set \( \mathcal{C} = \{C_1, \ldots, C_h\} \) of connected subsets of \( G \) and each multi-edge \( e = \{C_i, C_j\} \) of \( H \) corresponds to two mutually intersecting subsets of \( \mathcal{C} \) and the multiplicity of \( e \) is \( |C_i \cap C_j| \). For every \( \{i, j\} \in \binom{[h]}{2} \), we set \( V_{i,j} = C_i \cap C_j \), \( m_{i,j} = |C_i \cap C_j| \), and we assume that \( e_{i,j} = \{C_i, C_j\} \) is a multi-edge of \( H \) of multiplicity \( m_{i,j} \) (if this edge does not exist in \( H \), then the multiplicity of \( e_{i,j} \) is 0).

We define \( V_i = \bigcup_{j \in [h]} V_{i,j} \), for every \( i \in [h] \) and also set \( V = \bigcup_{i \in [h]} V_i \). Notice that, for each \( i \in [h] \), \( |V_i| \) is upper bounded by the edge-degree, in \( H \), of the vertex \( C_i \), therefore, \( |V_i| \leq d \) for each \( i \in [h] \). Also, a vertex in \( V \) cannot belong in more than \( d+1 \) distinct \( C_i \)’s as, otherwise \( H \) would contain a clique with at least \( d + 2 \) vertices. As \( H \in \text{inter}_d(\mathcal{G}) \), this is not possible.

Recall that, for each \( i \in [h] \), \( V_i \) is a subset of \( C_i \) and let \( T_i \) be a minimum-size tree of \( G[C_i] \) containing the vertices of \( V_i \). We partition the set of vertices of \( T_i \) into three sets \( V_i, \overline{V_i}, D_i \) where among the vertices in \( V(T_i) \setminus V_i, D_i \) are the vertices of degree
2 and \( \overline{V}_i \) are the rest. By minimality, the leaves of \( T_i \) belong in \( V_i \). Moreover, there is no vertex in \( \overline{V}_i \) that belongs to some other \( \overline{V}_{i'} \), \( i \in [h] \setminus \{i\} \). We denote by \( \hat{T}_i \) the tree obtained from \( T_i \) if we dissolve in \( T_i \) all vertices of \( D_i \). That way we can still partition the vertices of each \( \hat{T}_i \), \( i \in [h] \), into \( V_i \) and \( \overline{V}_i \). Also, it is easy to see that \( \hat{T}_i \) has diameter at most \(|V_i| - 1 \leq d - 1\).

We define the graph \( G' := \bigcup_{i \in [h]} \hat{T}_i \). Notice that \( G' \) is obtained from \( \bigcup_{i \in [h]} T_i \) (that is a subgraph of \( G \)) after we dissolve all vertices in \( \bigcup_{i \in [h]} D_i \). Therefore \( G' \) is a topological minor of \( G \), thus \( G' \in \mathcal{G} \). We consider the collection \( \mathcal{T} = \{ \hat{T}_1, \ldots, \hat{T}_h \} \) of connected subgraphs of \( G' \).

We define the graph \( J \) to be the disjoint union of the \( h \) trees in \( \mathcal{T} \) in which, for each \( x \in V \), we add a clique between all the copies of \( x \). Notice that each added clique has size at least 2 and at most \( d + 1 \).

Observe now that \( G' \leq (d+1) J \), as \( G' \) is obtained after contracting in \( J \) the aforementioned pairwise disjoint cliques. Moreover, \( H \leq_d J \) as \( H \) is obtained after we contract in \( J \) each \( \hat{T}_i \) (of diameter \( \leq d - 1 \)) to a single vertex. As \( G' \in \mathcal{G} \), we conclude that \( H \in \mathcal{G}^{d+1,d-1} \) as required. \( \square \)

We are now ready to prove Theorem 16.

**Proof of Theorem 16** Let \( \mathcal{G} \) be a monotone graph class in \( SQGC(c) \) for some \( 1 \leq c < 2 \) and let \( \mathcal{D} = dis\mathcal{s}(\mathcal{G}) \). From Lemma 24, \( \mathcal{D} \in SQGC(c) \) and by the monotonicity of \( \mathcal{G} \), we have that \( dis\mathcal{s}(\mathcal{G}) \) is closed under taking of topological minors. Therefore, from Lemma 25, \( inter_d(\mathcal{D}) \subseteq \mathcal{D}^{d+1,d-1} \) and from Theorem 15, \( inter_d(\mathcal{D}) \in SQGC(c) \). The result follows because \( \mathcal{G} \subseteq \mathcal{D} \), as this implies that \( inter_d(\mathcal{G}) \subseteq inter_d(\mathcal{D}) \). \( \square \)

### 4 Proof of Theorem 15

Let \( H \) and \( G \) be graphs and \( c \) be a non-negative integer. If \( H \leq^c_G G \), then we say that \( H \) is a \( \sigma \)-contraction of \( G \), and denote this by \( H \leq^c_G G \).

Before we proceed the the proof of Theorem 15 we make first the following three observations. (In all statements, we assume that \( G \) and \( H \) are two graphs and \( \sigma : V(G) \to V(H) \) such that \( H \) is a \( \sigma \)-contraction of \( G \)).

**Observation 26** Let \( S \) be a connected subset of \( V(H) \). Then the set \( \bigcup_{x \in S} \sigma^{-1}(x) \) is connected in \( G \).

**Observation 27** Let \( S_1 \subseteq S_2 \subseteq V(H) \). Then \( \sigma^{-1}(S_1) \subseteq \sigma^{-1}(S_2) \subseteq V(G) \).

**Observation 28** Let \( S \) be a connected subset of \( V(G) \). Then the diameter of \( \sigma(S) \) in \( H \) is at most the diameter of \( S \) in \( G \).

Given a graph \( G \) and \( S_1, S_2 \subseteq V(G) \) we say that \( S_1 \) and \( S_2 \) **touch** if either \( S_1 \cap S_2 \neq \emptyset \) or there is an edge of \( G \) with one endpoint in \( S_1 \) and the other in \( S_2 \).

We say that a collection \( \mathcal{R} \) of paths of a graph is **internally disjoint** if none of the internal vertices, i.e., none of the vertex of degree 2, of some path in \( \mathcal{R} \) is a vertex of some other path in \( \mathcal{R} \). Let \( \mathcal{A} \) be a collection of subsets of \( V(G) \). We say that \( \mathcal{A} \) is a **connected packing** of \( G \) if its elements are connected and pairwise disjoint. If
additionally $A$ is a partition of $V(G)$, then we say that $A$ is a connected partition of $G$ and if, additionally, all its elements have diameter bounded by some integer $c$, then we say that $A$ is a $c$-diameter partition of $G$.

### 4.1 $\Lambda$-State Configurations

**Definition 29** ($\Lambda$-state configurations) Let $G$ be a graph. Let $\Lambda = (\mathcal{W}, \mathcal{E})$ be a graph whose vertex set is a connected packing of $G$, i.e., its vertices are connected subsets of $V(G)$. A $\Lambda$-state configuration of a graph $G$ is a quadruple $S = (\mathcal{X}, \alpha, \mathcal{R}, \beta)$ where

1. $\mathcal{X}$ is a connected packing of $G$,
2. $\alpha$ is a bijection from $\mathcal{W}$ to $\mathcal{X}$ such that for every $W \in \mathcal{W}$, $W \subseteq \alpha(W)$,
3. $\mathcal{R}$ is a collection of internally disjoint paths of $G$, and
4. $\beta$ is a bijection from $\mathcal{E}$ to $\mathcal{R}$ such that if $\{W_1, W_2\} \in \mathcal{E}$ then the endpoints of $\beta(\{W_1, W_2\})$ are in $W_1$ and $W_2$ and $\beta(\{W_1, W_2\}) \subseteq \alpha(W_1) \cup \alpha(W_2)$.

**Definition 30** (States, freeways, clouds, and coverage) A $\Lambda$-state configuration $S = (\mathcal{X}, \alpha, \mathcal{R}, \beta)$ of $G$ is complete if $\mathcal{X}$ is a partition of $V(G)$. We refer to the elements of $\mathcal{X}$ as the states of $S$ and to the elements of $\mathcal{R}$ as the freeways of $S$. We define $\text{indep}(S) = V(G) \setminus \bigcup_{X \in \mathcal{X}} X$. Note that if $S$ is a $\Lambda$-state configuration of $G$, $S$ is complete if and only if $\text{indep}(S) = \emptyset$.

Let $A$ be a $c$-diameter partition of $G$. We refer to the sets of $A$ as the $A$-clouds of $G$. We define $\text{front}_A(S)$ as the set of all $A$-clouds of $G$ that are not subsets of some $X \in \mathcal{X}$. Given a $A$-cloud $C$ and a state $X$ of $S$ we say that $C$ shadows $X$ if $C \cap X \neq \emptyset$. The coverage $\text{cov}_S(C)$ of an $A$-cloud $C$ of $G$ is the number of states of $S$ that are shadowed by $C$. A $\Lambda$-state configuration $S = (\mathcal{X}, \alpha, \mathcal{R}, \beta)$ of $G$ is $A$-normal if its satisfies the following conditions:

(A) If a $A$-cloud $C$ intersects some $W \in \mathcal{W}$, then $C \subseteq \alpha(W)$.

(B) If a $A$-cloud over $S$ intersects the vertex set of at least two freeways of $S$, then it shadows at most one state of $S$.

We define $\text{cost}_A(S) = \sum_{C \in \text{front}_A(S)} \text{cov}_S(C)$. Given $S_1 \subseteq S_2 \subseteq V(G)$ where $S_1$ is connected, we define $cc_G(S_2, S_1)$ as the (unique) connected component of $G[S_2]$ that contains $S_1$.

### 4.2 Triangulated Grids Inside Triangulated Grids

The next lemma is the main combinatorial engine of our results. We assume that $H \leq^c G$ and $\Gamma_k \leq G$. Here $H$ should be seen as the result of a “shrink” of $G$ in the sense that $G$ can be contracted to $H$ so that each vertex of $H$ is created after a bounded number of contractions. The lemma states that if $G$ can be contracted to a uniformly triangulated grid, then $H$, as a “shrunk version” of $G$, can also be contracted to a uniformly triangulated grid that is no less than linearly smaller.

The proof strategy views the graph $G$ as being contracted into a uniformly triangulated grid $\Gamma_k$ (see Fig. 7), we choose a scattered set of “capitals” in it (the black vertices in Fig. 7). Then we set up a “conquest” procedure where each capital is trying...
to expand to a country. This procedure has three phases. The first phase is the expansion face where each country tries to incorporate unconquered territories around it (the limits of this expansion is depicted by the red cycles in Fig. 7). The second phase is the clash face, where different countries are fighting for disputed territories. Finally, the third phase is the annex phase, where each country naturally incorporates remaining enclaves. The end of this war creates a set of countries occupying the whole G that, when contracted, give rise to a uniformly triangulated grid \( \Gamma_{{k'}} \), where \( k' = \Omega(k) \).

**Lemma 31** Let \( G \) and \( H \) be graphs and \( c, k \) be non-negative integers such that \( H \leq_c G \) and \( \Gamma_k \leq G \). Then \( \Gamma_{{k'}} \leq H \) where \( k' = \lfloor \frac{k-1}{2c+1} \rfloor - 1 \).

**Proof** Let \( k^* = 1 + (2c+1) \cdot (k' + 1) \) and observe that \( k^* \leq k \), therefore \( \Gamma_{{k^*}} \leq \Gamma_k \leq G \). For simplicity we use \( \Gamma = \Gamma_{{k^*}} \). Let \( \phi : V(G) \rightarrow V(H) \) such that \( H \leq \phi \leq G \) and let \( \sigma : V(G) \rightarrow V(\Gamma) \) such that \( \Gamma \leq \sigma \leq G \). We define \( \mathcal{A} = \{ \phi^{-1}(a) \mid a \in V(H) \} \). Notice that \( \mathcal{A} \) is a \( c \)-diameter partition of \( G \).

For each \((i, j) \in [(0, k'+1)^2] \), we define \( b_{i,j} \) to be the vertex of \( \Gamma \) with coordinate \((i(2c+1), j(2c+1)) \). We set \( Q_{\text{in}} = \{ b_{i,j} \mid (i, j) \in [(1, k')^2] \} \) and \( Q_{\text{out}} = \{ b_{i,j} \mid (i, j) \in [(0, k'+1)^2] \} \setminus Q_{\text{in}} \). Let also \( Q = Q_{\text{in}} \cup \{ b_{i,j} \} \) be a new element that does not belong in \( Q_{\text{in}} \). Here \( b_{i,j} \) can be seen as a vertex that “represents” all vertices in \( Q_{\text{out}} \).

Let \( q, p \) be two different elements of \( Q \). We say that \( q \) and \( p \) are linked if they both belong in \( Q_{\text{in}} \) and their distance in \( \Gamma \) is \( 2c+1 \) or one of them is \( b_{i,j} \) and the other is \( b_{i,j} \) where \( i \in [1, k') \) or \( j \in [1, k') \). \( \square \)

For each \( q \in Q_{\text{in}} \), we define \( W_q = \sigma^{-1}(q) \). \( W_q \) is connected by the definition of \( \sigma \). In case \( q = b_{i,j} \) we define \( W_q = \bigcup_{q' \in Q_{\text{out}}} \sigma^{-1}(q') \). Note that as \( Q_{\text{out}} \) is a connected set \( \Gamma \), then, by Observation 26, \( W_{b_{i,j}} \) is connected in \( G \). We also define \( \mathcal{W} = \{ W_q \mid q \in Q \} \). Given some \( q \in Q \), we call \( W_q \) the \( q \)-capital of \( G \) and a subset \( S \) of \( V(G) \) is a capital of \( G \) if it is the \( q \)-capital for some \( q \in Q \). Notice that \( \mathcal{W} \) is a connected packing of \( V(G) \).

Let \( q \in Q \). If \( q \in Q_{\text{in}} \) then we set \( N_q = N^{c}_{\Gamma}(q) \). If \( q = b_{i,j} \), then we set \( N_q = \bigcup_{q' \in Q_{\text{out}}} N^{c}_{\Gamma}(q') \). Note that for every \( q \in Q \), \( N_q \subseteq V(\Gamma) \). For every \( q \in Q \), we define \( X_q = \sigma^{-1}(N_q) \). Note that \( X_q \subseteq V(G) \). We also set \( \mathcal{X} = \{ X_q \mid q \in Q \} \). Let \( q \) and \( p \) be two linked elements of \( Q \). If both \( q \) and \( p \) belong to \( Q_{\text{in}} \) and therefore are vertices of \( \Gamma \), then we define \( Z_{p,q} \) as the unique shortest path between them in \( \Gamma \). If \( p = b_{i,j} \) and \( q \in Q_{\text{in}} \), then we know that \( q = b_{i,j} \) where \( i \in [1, k') \) or \( j \in [1, k') \). In this case we define \( Z_{p,q} \) as any shortest path in \( \Gamma \) between \( b_{i,j} \) and the vertices in \( Q_{\text{out}} \). In both cases, we define \( P_{p,q} \) by picking some path between \( W_p \) and \( W_q \) in \( G[\sigma^{-1}(V(Z_{p,q}))] \) such that \( |V(P_{p,q}) \cap W_q| = 1 \) and \( |V(P_{p,q}) \cap W_p| = 1 \).

Let \( \mathcal{E} = \{ (W_p, W_q) \mid \text{p and q are linked} \} \) and let \( \Lambda = (\mathcal{W}, \mathcal{E}) \). Notice that \( \Lambda \) is isomorphic to \( \Gamma_{{k'}} \) and consider the isomorphism that correspond each vertex \( q = b_{i,j} \), \( i, j \in [(1, k')^2] \) to the vertex with coordinates \((i, j)\). Moreover \( b_{i,j} \) corresponds to the apex vertex of \( \Gamma_{{k'}} \).

Let \( \alpha : \mathcal{W} \rightarrow \mathcal{X} \) such that for every \( q \in Q \), \( \alpha(W_q) = X_q \). Let also \( \mathcal{R} = \{ P_{p,q} \mid p, q \in Q, \text{p and q are linked} \} \). We define \( \beta : \mathcal{E} \rightarrow \mathcal{R} \) such that if \( q \) and \( p \) are linked, then \( \beta(W_q, W_p) = P_{p,q} \). We use notation \( S = (\mathcal{X}, \alpha, \mathcal{R}, \beta) \).
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Fig. 7 A visualization of the proof of Lemma 31. In this whole graph $\Gamma_k$, we initialize our research on $\hat{\Gamma}_k$, such that every internal red hexagon will become a vertex of $\hat{\Gamma}_k$ and correspond to a state and the border, also circle by a red line will become the vertex $b_{\text{out}}$. The blue edges correspond to the freeways. Red cycles correspond to the boundaries of the starting countries. Blue paths between big-black vertices are the freeways. Big-black vertices are the capitals (Color figure online)

Claim 32 $S$ is an $A$-normal $\Lambda$-state configuration of $G$.

Proof of Claim 32. We first see that $S$ is a $\Lambda$-state configuration of $G$. Condition 1 follows by the definition of $X_q$ and Observation 26. Condition 2 follows directly by the definitions of $W_q$ and $X_q$. For Condition 3, we first observe that, by the construction of $\Gamma$ and the definition of $Z_{p,q}$, for any two pairs $p, q$ and $p', q'$ of pairwise linked elements of $Q$, the paths $Z_{p,q}$ and $Z_{p',q'}$ are internally vertex disjoined paths of $\Gamma$. It implies that $P_{p,q}$ and $P_{p',q'}$ can intersect each other only on the vertices of $W_p \cup W_q \cup W_{p'} \cup W_{q'}$. But $P_{p,q}$ (resp. $P_{p',q'}$), by construction contains only two vertices of $W_p \cup W_q \cup W_{p'} \cup W_{q'}$ that are the extremities of $P_{p,q}$ (resp. $P_{p',q'}$). So $P_{p,q}$ and $P_{p',q'}$ are internally vertex disjoined, as required. For Condition 4, assume that $\{W_p, W_q\} \in \mathcal{E}$. The fact that the endpoints of $\beta(\{W_p, W_q\}) = P_{p,q}$. It remains to prove that $V(\beta(\{W_p, W_q\})) \subseteq \alpha(W_p) \cup \alpha(W_q)$ or equivalently, that $V(P_{p,q}) \subseteq X_p \cup X_q$. Observe that, if both $p, q \in Q_{\text{in}}$, then every vertex in the shortest path $Z_{p,q}$ should be within distance $c$ from either $p$ or $q$. Similarly, if $p \in Q_{\text{in}}$ and $q = b_{\text{out}}$, then every vertex in the shortest path $Z_{p,q}$ should be within distance $c$ from either $p$ or some vertex in $Q_{\text{out}}$. So for every $p, q \in Q$, with $p \neq q$, $Z_{p,q} \subseteq N_p \cup N_q$. By Observation 27, every vertex in $\sigma^{-1}(V(Z_{p,q}))$ belongs to $X_p \cup X_q$ and the required follows as $V(P_{p,q}) \subseteq \sigma^{-1}(V(Z_{p,q}))$. This completes the proof that $S$ is a $\Lambda$-state configuration of $G$. 
We now prove that $S$ is $\mathcal{A}$-normal. Recall that $\mathcal{A}$ be a $c$-diameter partition of $G$. Let $C$ be a $\mathcal{A}$-cloud and let $C' = \sigma(C)$ be a subset of $V(\Gamma)$. As $C$ is of diameter at most $c$, then, from Observation 28, $C'$ is also of diameter at most $c$. Notice that if $C$ intersects some member $W$ of $\mathcal{W}$, then $C' = \sigma(C)$ also intersects $\sigma(W)$, therefore $C'$ intersects some element of $Q_{in} \cup Q_{out}$. Assume $C'$ contains $p \in Q_{in} \cup Q_{out}$, then $C' \subseteq N_p$. From Observation 26, $C \subseteq X_p = \alpha(W_p)$, therefore $C$ satisfies Condition (A).

By construction, the distance in $\Gamma$ between two elements of $Q_{in}$ is either $2c + 1$ or at least $4c + 2$. The distance in $\Gamma$ between on elements of $Q_{in}$ and any element of $Q_{out}$ is a multiple of $2c + 1$. This implies that if $p, q \in Q$, $p \neq q$, $N_p \cap C' \neq \emptyset$, and $N_q \cap C' \neq \emptyset$, then $p$ and $q$ are linked.

By construction, if $p$ and $q$ are linked, then for every $r \in Q$ and every $u \in Z_{p,q}$, $\text{dist}_{\Gamma}(r, u) \geq \min(\text{dist}_{\Gamma}(r, p), \text{dist}_{G}(r, q))$, where for every $x \in Q_{in}$, the quantity $\text{dist}_{\Gamma}(x, b_{out})$ is interpreted as $\min(\text{dist}_{\Gamma}(x, q') \mid q' \in Q_{out})$. This implies that if $C'$ intersects $Z_{p,q}$ for some $p, q \in Q$, then for every $r \in Q \setminus \{p, q\}$, then $C'$ does not intersect $N_r$. We will use this fact in the next paragraph towards completing the proof of Condition (B).

We now claim that if $C'$ intersects two distinct paths in $\{Z_{p,q} \mid (p, q) \in Q^2, p \neq q\}$, then $C'$ intersects at most one of the sets in $\{N_{q'} \mid q' \in Q\}$. Let $Z_{p,q}$ and $Z_{p',q'}$ be two distinct paths intersected by $C'$. We argue first that $p, q, p', q'$ cannot be all different. Indeed, if this is the case, as $C'$ intersects $Z_{p,q}$ then $C'$ cannot intersect $N_{q'}$ or $N_{q''}$ as $p', q' \neq \{p, q\}$. As $Z_{p',q''} \subseteq N_{q'} \cup N_{p'}$, we have a contradiction. Assume now that $p = p'$ and $q \neq q'$. As $C'$ intersects $Z_{p,q}$, then it does not intersect $N_r$ for any $r \in Q \setminus \{p, q\}$, and as it intersects $Z_{p,q'}$, then it does not intersect $N_r$ for any $r \in Q \setminus \{p, q\}$. We obtain that $C'$ intersects at most one of the sets in $\{N_r \mid r \in Q\}$ that is $N_p$. By definition of the states, we obtain that $C$ shadows at most one state that is $X_p$. That completes the proof of condition (B). \qed

We define bellow three ways to transform a $\mathcal{A}$-state configuration of $G$. In each of them, $S = (\mathcal{X}, \alpha, \mathcal{R}, \beta)$ is an $\mathcal{A}$-normal $\mathcal{A}$-state configuration of $G$ and $C$ is an $\mathcal{A}$-cloud in $\text{front}_A(S)$.

1. The expansion procedure applies when $C$ intersects at least two freeways of $S$.

   Let $X$ be the state of $S$ shadowed by $C$ (this state is unique because of property (B) of $\mathcal{A}$-normality). We define $\langle \mathcal{X}', \alpha', \mathcal{R}', \beta' \rangle = \text{expand}(S, C)$ such that
   \begin{itemize}
   \item $\mathcal{X}' = \mathcal{X} \setminus \{X\} \cup \{X \cup C\}$,
   \item for each $W \in \mathcal{W}$, $\alpha'(W) = X'$ where $X'$ is the unique set of $\mathcal{X}'$ such that $W \subseteq X'$,
   \item $\mathcal{R}' = \mathcal{R}$, and $\beta' = \beta$.
   \end{itemize}

2. The clash procedure applies when $C$ intersects exactly one freeway $P$ of $S$.

   Let $X_1, X_2$ be the two states of $S$ that intersect this freeway. Notice that $P = \beta(\alpha^{-1}(X_1), \alpha^{-1}(X_2))$, as it is the only freeway with vertices in $X_1$ and $X_2$. Assume that $(C \cap V(P)) \cap X_1 \neq \emptyset$ (if, not, then swap the roles of $X_1$ and $X_2$). We define $\langle \mathcal{X}', \alpha', \mathcal{R}', \beta' \rangle = \text{clash}(S, C)$ as follows:
   \begin{itemize}
   \item $\mathcal{X}' = \{X_1 \cup C\} \cup \bigcup_{X \in \mathcal{X} \setminus \{X_1\}} \{G(X \setminus C, \alpha^{-1}(X))\}$ (notice that $\alpha^{-1}(X) \subseteq X \setminus C$, for every $X \in \mathcal{X}$, because of property (A) of $\mathcal{A}$-normality),
   \end{itemize}
for each $W \in \mathcal{W}$, $\alpha'(W) = X'$ where $X'$ is the unique set of $\mathcal{X}'$ such that $W \subseteq X'$,
- $\mathcal{R}' = \mathcal{R} \setminus \{P\} \cup \{P'\}$, where $P' = P_1 \cup P^* \cup P_2$ is defined as follows: let $s_i$ be the first vertex of $C$ that we meet while traversing $P$ when starting from its endpoint that belongs in $W_i$ and let $P_i$ the subpath of $P$ that we traversed that way, for $i \in \{1, 2\}$. We define $P^*$ by taking any path between $s_1$ and $s_2$ inside $G[C]$, and
- $\beta' = \beta \setminus \{(W_1, W_2), (P)\} \cup \{(W_1, W_2), (P')\}$.

3: The annex procedure applies when $C$ intersects no freeway of $S$ and touches some country $X \in \mathcal{X}$. We define $(\mathcal{X}', \alpha', \mathcal{R}', \beta') = \text{annex}(S, C)$ such that
- $\mathcal{X}' = \{X_1 \cup C\} \cup \bigcup_{X \in \mathcal{X} \setminus \{X_1\}} \{cc_G(X \setminus C, \alpha^{-1}(X))\}$ (notice that $\alpha^{-1}(X) \subseteq X \setminus C$, for every $X \in \mathcal{X}'$, because of property (A) of $A$-normality),
- for each $W \in \mathcal{W}$, $\alpha'(W) = X'$ where $X'$ is the unique set of $\mathcal{X}'$ such that $W \subseteq X'$,
- $\mathcal{R}' = \mathcal{R}$, and $\beta' = \beta$.

Claim 33 Let $S = (\mathcal{X}, \alpha, \mathcal{R}, \beta)$ be an $A$-normal $\Lambda$-state configuration of $G$, and $C \in \text{front}_A(S)$. Let $S' = \text{action}(S, C)$ where $\text{action} \in \{\text{expand}, \text{clash}, \text{annex}\}$. Then $S'$ is an $A$-normal $\Lambda$-state configuration of $G$ where $\text{cost}(S', A) \leq \text{cost}(S, A)$. Moreover, if $\text{cov}_S(C) \geq 1$, then $\text{cost}(S', A) < \text{cost}(S, A)$ and if $\text{cov}_S(C) = 0$ (which may be the case only when $\text{action} = \text{annex}$), then $|\text{indep}(S')| < |\text{indep}(S)|$.

Proof of Claim 33. We first show that $S'$ is an $A$-normal $\Lambda$-state configuration of $G$. In each case, the construction of $S'$ makes sure that $\mathcal{X}'$ is a connected packing of $G$ and that the countries are updated in a way that their capitals remain inside them. Moreover, the highways are updated so to remain internally disjoint and inside the corresponding updated countries. We next prove that $S'$ is $A$-normal. Condition (A) is invariant as the cloud we take into consideration cannot intersect any $W \in \mathcal{W}$ and a cloud intersecting some capital $W \in \mathcal{W}$ cannot be disconnected from $W$. It now remains to prove condition (B). Because of Condition 4 of the definition of a $\Lambda$-state configuration, if a cloud $C$ intersects a freeway, then it shadows at least one state. Now assume that a cloud $C$ intersects two freeways in $S'$, then by construction of $S'$, it also intersects at least the two same freeways in $S$. This along with the fact that $S$ satisfies Condition (B), implies that $S'$ satisfies condition (B) as well, as required.

Notice that, for any cloud $C^* \in A \setminus \{C\}$, if $C^*$ does not intersect a state $X$ in $S$, then the corresponding state $X'$ in $S'$, i.e., the state $X' = \alpha'(\alpha^{-1}(X))$, also does not intersect $C^*$. This means that $\text{cost}(S', A) \leq \text{cost}(S, A)$.

Notice now that by the construction of $S'$, $C$ is not in $\text{front}_A(S')$. In the case where $\text{cov}_S(C) \geq 1$ we have that $\text{cost}(S', A) < \text{cost}(S, A)$.

Notice that the case where $\text{cov}_S(C) = 0$ happens only when $\text{action} = \text{annex}$ and there is an edge with one endpoint in $C$ and one in some country $X^*$ of $S$ that does not intersect $C$. Moreover $cc_G(X \setminus C, \alpha^{-1}(X)) = X$, for every state $X$ of $S$. This implies that $\text{indep}(S') \subseteq \text{indep}(S)$. As $C \subseteq \text{indep}(S)$ and $C \cap \text{indep}(S') = \emptyset$, we conclude that $|\text{indep}(S')| < |\text{indep}(S)|$ as required. \hfill \Box

To continue with the proof of Lemma 31 we explain how to transform the $A$-normal $\Lambda$-state configuration $S$ of $G$ to a complete one. This is done in two phases. First, as long as there is an $A$-cloud $C \in \text{front}(S)$ where $\text{cov}_S(C) \geq 1$, we apply one of
the above three procedures depending on the number of freeways intersected by $C$. We again use $\mathcal{S}$ to denote the $\mathcal{A}$-normal $\Lambda$-state configuration of $G$ that is created in the end of this first phase. Notice that, as there is no $\mathcal{A}$-cloud with $cov_S(C) \geq 1$, then $\text{cost}_A(\mathcal{S}) = 0$. The second phase is the application of anex$(\mathcal{S}, C)$, as long as some $C \in \text{front}_A(\mathcal{S})$ is touching some of the countries of $\mathcal{S}$. We claim that this procedure will be applied as long as there are vertices in independ$(\mathcal{S})$. Indeed, if this is the case, the set $\text{front}_A(\mathcal{S})$ is non-empty and by the connectivity of $G$, there is always a $C \in \text{front}_A(\mathcal{S})$ that is touching some country of $\mathcal{S}$. Therefore, as $\text{cost}_A(\mathcal{S}) = 0$ (by Claim 33), procedure anex$(\mathcal{S}, C)$ will be applied again.

By Claim 33, $|\text{indep}(\mathcal{S})|$ is strictly decreasing during the second phase. We again use $\mathcal{S}$ for the final outcome of this second phase. We have that $\text{indep}(\mathcal{S}) = \emptyset$ and we conclude that $\mathcal{S}$ is a complete $\mathcal{A}$-normal $\Lambda$-state configuration of $G$ such that $|\text{front}_A(\mathcal{S})| = 0$.

We are now going to create a graph isomorphic to $\Lambda$ only by doing contractions in $G$. For this we use $\mathcal{S}$, a complete $\mathcal{A}$-normal $\Lambda$-state configuration of $G$ such that $|\text{front}_A(\mathcal{S})| = 0$, obtained as describe before. We contract in $G$ every country of $\mathcal{S}$ into a unique vertex. This can be done because the countries of $\mathcal{S}$ are touching some country of $\mathcal{S}$. This implies that there is a graph isomorphic to $\Lambda$ that is a subgraph of $G'$. So $\Gamma_k'$ is a subgraph of $G'$ with the same number of vertices. Let see $\hat{\Gamma}_k'$ as a subgraph of $G'$ and let $e$ be an edge of $G'$ that is not an edge of $\hat{\Gamma}_k'$. As $e$ is an edge of $G'$, this implies that in $G$, there is a cloud that touches some country of $\mathcal{S}$ and there is no freeway between them but still an edge. This is not possible by construction of $\mathcal{S}$. We deduce that $G'$ is isomorphic to $\hat{\Gamma}_k'$. Moreover, as $|\text{front}_A(\mathcal{S})| = 0$, then every cloud is a subset of a country. This implies that $G'$ is also a contraction of $H$. By contracting in $G'$ the edge corresponding to $\{a, (k' - 1, k' - 1)\}$ in $\hat{\Gamma}_k'$, we obtain that $\Gamma_{k'}$ is a contraction of $H$. Lemma 31 follows.

**Proof of Theorem 15** Let $\lambda$, $c$, $c_1$, and $c_2$ be integers. It is enough to prove that there exists an integer $\lambda' = O(\lambda \cdot c_1 \cdot (c_2)^c)$ such that for every graph class $\mathcal{G} \in S\mathcal{G}(c)$,

$$\forall G \in \mathcal{G} \quad \text{tw}(G) \leq \lambda \cdot (\text{beg}(G))^c \Rightarrow \quad \forall F \in \mathcal{G}^{(c_1,c_2)} \quad \text{tw}(F) \leq \lambda' \cdot (\text{beg}(F))^c.$$ 

Let $\mathcal{G} \in S\mathcal{G}(c)$ be a class of graph such that $\forall G \in \mathcal{G} \quad \text{tw}(G) \leq \lambda \cdot (\text{beg}(G))^c$. Let $H \in \mathcal{G}^{(c_1,c_2)}$ and let $G$ and $J$ be two graphs such that $G \in \mathcal{G}$, $G \leq^{(c_1)} J$, and $H \leq^{c_2} J$. $G$ and $J$ exist by definition of $\mathcal{G}^{(c_1,c_2)}$.

- By definition of $H$ and $J$, $\text{tw}(H) \leq \text{tw}(J)$.
- By Lemma 19, $\text{tw}(J) \leq (c_1 + 1)(\text{tw}(G) + 1) - 1$.
- By definition of $\mathcal{G}$, $\text{tw}(G) \leq \lambda \cdot \text{beg}(G)^c$.
- By Lemma 31, $\text{beg}(G) \leq (2c_2 + 1)(\text{beg}(H) + 2) + 1$.

If we combine these four statements, we obtain that

$$\text{tw}(H) \leq (c_1 + 1)(\lambda \cdot [(2c_2 + 1)(\text{beg}(H) + 2) + 1]^c + 1) - 1.$$
As the formula is independent of the graph class, the Theorem 15 follows. □

5 Conclusions, Extensions, and Open Problems

The main combinatorial result of this paper is that, for every $d$ and every apex-minor-free graph class $\mathcal{G}$, the intersection class $\text{inter}_d(\mathcal{G})$ has the SQGC property for $c = 1$. Certainly, the main general question is to detect even wider graph classes with the SQGM/SQGC property. In this direction, some insisting open issues are the following:

- Is the bound on the (multi-)degree necessary? Are there classes of intersection graphs with unbounded or “almost bounded” maximum degree that have the SQGM/SQGC property?
- All so far known results classify graph classes in $\text{SQGM}(1)$ or $\text{SQGC}(1)$. Are there (interesting) graph classes in $\text{SQGM}(c)$ or $\text{SQGC}(c)$ for some $1 < c < 2$ that do not belong in $\text{SQGM}(1)$ or $\text{SQGC}(1)$ respectively? An easy (but trivial) example of such a class is the class $Q_d$ of the $q$-dimensional grids, i.e., the cartesian products of $q \geq 2$ equal length paths. It is easy to see that the maximum $k$ for which an $n$-vertex graph $G \in Q_q$ contains a $(k \times k)$-grid as a minor is $k = \Theta(n^{\frac{1}{q}-\frac{1}{q}})$. On the other size, it can also be proven that $\text{tw}(G) = \Theta(n^{\frac{1}{q}})$. These two facts together imply that $Q_q \in \text{SQGM}(2 - \frac{2}{q})$ while $Q_q \notin \text{SQGM}(2 - \frac{2}{q} - \epsilon)$ for every $\epsilon > 0$.
- Usually the graph classes in $\text{SQGC}(1)$ are characterised by some “flatness” property. For instance, see the results in [31,34,34] for $H$-minor free graphs, where $H$ is an apex graph. Can $\text{SQGC}(1)$ be useful as an intuitive definition of the “flatness” concept? Does this have some geometric interpretation?

Funding. The first author was supported by ANR projects DEMOGRAPH (ANR-16-CE40-0028). The second and the third author were supported by the ANR projects DEMOGRAPH (ANR-16-CE40-0028), ESIGMA (ANR-17-CE23-0010), and the French-German Collaboration ANR/DFG Project UTMA (ANR-20-CE92-0027).

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