On a certain analogy between hydrodynamic flow in porous media and heat conductance in solids

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We consider a porous medium being saturated with a pore fluid (Biot’s theory). The fluid is assumed as incompressible. It is shown that the general integral of the elastic and pressure equations can be written in form of a time dependent vectorpotential \( F \) being a solution of a homogeneous, fourth order differential equation. The obtained equation for \( F \) is of a more general form than the corresponding thermo-elastic vectorpotential, being a solution of a time dependent and inhomogeneous vector bi-Laplacian. Both vectorpotentials do, however, agree for stationary problems in general and for certain particular boundary conditions (irrotational deformations). An example of an irrotational deformation is studied in detail, exhibiting known properties of classical vector diffusion.

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I. INTRODUCTION

Imposing an inhomogeneous pressure/temperature field on a (porous) material causes in general mechanical deformations and stresses. These ‘induced’ elastic deformations in turn can and do influence the further mass/heat transport. The problem is described as a coupled field problem consisting of (at least) two equations (a) an elastic equation describing mechanical displacements due to a source field (pressure/temperature gradient field) and (b) at least one transport equation for mass/heat with a source term originating from the mechanical deformation (divergence of the displacement field).

The entire description bases on continuity equations and constitutive laws which are discussed in textbooks in great detail [1]. Perhaps less known is Biot’s theory of porous media for the case of symmetric elasticity, i.e. \( \sigma_{ij} = \sigma_{ji} \). Deformations are characterized by the strains \( 2u_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i) \). Employing Hooke’s linear relation between stresses and strains for isotropic materials one obtains Lamé’s equation for the displacements,

\[
(1 - 2\nu)\Delta \mathbf{u} + \text{grad} \text{ div } \mathbf{u} + \frac{2}{E}(1 + \nu)(1 - 2\nu) \mathbf{f} = \mathbf{0},
\]

with Poisson contraction number \( \nu \) and Young modulus \( E \). There are a few quite general results about this equation, which will be useful in the following. Taking the divergence of Eq.(2.3) one has,

\[
\Delta \text{ div } \mathbf{u} = -\frac{(1 + \nu)(1 - 2\nu)}{(1 - \nu)E} \text{ div } \mathbf{f},
\]

i.e., \( \text{div } \mathbf{u} \) is the solution of a Poisson equation and for vanishing body forces \( \mathbf{f} \) it is even a potential function. The physical significance of \( \text{div } \mathbf{u} \) is that it measures the local relative volume change due to the deformation which is an invariant. Pure shear deformations are unreflected in Eq.(2.3) because they preserve the volume. Of course all information on vorticity has been lost by passing from Eq.(2.2) to Eq.(2.3). On the other hand taking the Laplacian on Eq.(2.2) and inserting Eq.(2.3) into yields,

\[
\Delta \Delta \mathbf{u} = \frac{2(1 + \nu)}{E} \left( \frac{1}{2(1 - \nu)} \text{grad} \text{ div } \mathbf{f} - \Delta \mathbf{f} \right).
\]
This equation is of fourth order while Eq. (2.2) is of second order; hence their solutions will be in general not identical. In his seminal work Galerkin resolved this problem. He introduced another vector function $\mathbf{F}$ satisfying the (inhomogeneous) biharmonic equation,

$$\Delta \Delta \mathbf{F} = -\frac{2}{E} (1 + \nu) \mathbf{f}, \quad (2.5)$$

with $\mathbf{f}$ being the same body force as in Eq. (2.2). He showed that the integral of Eq. (2.2) can be expressed in form of derivatives of $\mathbf{F}$,

$$\mathbf{u} = \Delta \mathbf{F} - \frac{1}{2(1 - \nu)} \text{grad div } \mathbf{F}. \quad (2.6)$$

We would like to note that the foregoing considerations made no particular assumptions on the body force field $\mathbf{f}$ in particular they apply to conservative fields. Typically $\mathbf{f}$ is explicitly known like for gravitational forces, or it is itself a solution of another independent equation. The latter is the case for thermoelastic problems, in which

$$\mathbf{f} = -\alpha \, K \, \text{grad } T \quad (2.7)$$

where $\alpha$ is the heat expansion coefficient, the $K = \frac{E}{3(1 - 2\nu)}$ accounts for the material’s compressibility, and $T$ is the temperature field. Typically Eqs. (2.2) and (2.7) are supplemented by a heat conductance equation,

$$\chi \Delta T = \partial_t T, \quad (2.8)$$

with $\chi$ being the coefficient of heat conductance divided by the specific heat capacity. Equation (2.5) then becomes a biharmonic equation with time dependent – but known inhomogeneities – which do not dependent on the deformation themselves. As will be shown further below does flow in porous materials imply in general ‘body forces’ being deformation dependent themselves. The deformation acts back on the flow. We will come back to this point in more detail further below.

2. Flow equation

Let's consider fluid flow in a porous continuum. The porosity (local volumetric fluid fraction) can be described by a scalar field $\phi (\mathbf{r}, t)$. It is common to assume that the relative fluid-solid velocity $\phi (\mathbf{v} - \mathbf{u})$ can be expressed by Darcy’s relation,

$$\mathbf{f} = -\text{grad } p = \phi \frac{\kappa}{\mu} (\mathbf{v} - \mathbf{u}), \quad (2.9)$$

where $\mathbf{v}$ and $\mathbf{u}$ are the fluid and solid velocities, $\kappa$ the mechanical permeability, $\mu$ the fluid viscosity, and $p$ the hydrostatic pressure field.

The continuity equations for solid and fluid mass are,

$$\frac{\partial}{\partial t} \left( (1 - \phi) \rho_s \right) + \text{div} \left( (1 - \phi) \rho_s \mathbf{u} \right) = 0, \quad (2.10)$$

and

$$\frac{\partial}{\partial t} \left( \phi \rho_f \right) + \text{div} \left( \phi \rho_f \mathbf{v} \right) = 0, \quad (2.11)$$

respectively, where $\rho_s$ and $\rho_f$ denote the solid and fluid mass densities. In the following we make the two important assumptions that the mass densities are constants ($\rho_s = \text{const.}$ and $\rho_f = \text{const.}$).

From Eqs. (2.9), (2.10) and (2.11) and the assumption of constant mass densities follows a) that the local volume is conserved because the inflow of fluid volume into a volume element equals the solid volume outflow, and b) that the flow equation takes the following simple form ($\kappa/\mu = \text{const.}$),

$$\frac{\kappa}{\mu} \Delta p = \text{div } \mathbf{u}, \quad (2.12)$$

i.e. a Poisson equation for the pressure with the rate of the relative elastic volume change as source term. The porosity $\phi$ does not enter explicitly into the flow equation because of constant mass densities. As soon as $\mathbf{u}$ has been determined from the coupled Eqs. (2.2) and (2.12) $\phi$ can be computed from Eq. (2.10).

B. The coupled system

1. The Galerkin-Biot potential

After the above preparations we will focus on the coupled Eqs. (2.2) and (2.12) describing elastic–flow interactions for incompressible pore fluids. Note that

$$\mathbf{f} = -\text{grad } p, \quad (2.13)$$

in Eq. (2.2), comp. Eq. (2.2). In general Lamb’s vector equation (2.2) is much harder to solve than the scalar Poisson Equation (2.12). However, for the very particular case of irreotational displacements Lamb’s equation can be integrated easily, see Sec. IIIA. In general (and also for most boundary conditions of practical interest) displacements and velocities will have a non-zero rotational part. Therefore $\text{div } \mathbf{u}$ or its time derivative will not contain the full information about displacements and strains, i.e. all elastic volume preserving deformations (shears) are unreflected in it.

It has been convincingly demonstrated that for heat-conductance in solids the relative volume changes, $\text{div } \mathbf{u}$, actually do also appear in Eq. (2.8). However, the corresponding coefficient is for most solids so small that this term can be safely disregarded simplifying the theoretical treatment considerable. An analogy to heat-conductance in solids one can think of a strongly compressible fluid, i.e. a gas in a porous medium. One can neglect the back-influence of the pressure induced elastic distortion on the pore pressure field and is basically left with the equations describing a corresponding thermo-elastic problem. On the other hand does flow of an nearly incompressible fluid in a porous medium represent the ‘opposite’ case: in order flow to happen elastic deformation must take place. Without elastic deformation there is no flow and vice versa.
We reconsider the flow problem Eq. (2.5) and make some small refinements. Of course the pore pressure field is unknown in Eq. (2.3) since we have not incorporated any information from Eq. (2.13). This needs to be done in order to obtain a fully decoupled equation for displacements. Starting from Eq. (2.12) we have,

\[ f = - \text{grad } p = \frac{\mu}{\kappa} \text{grad} \int_V \frac{\text{div } \tilde{u}(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} dV'. \quad (2.14) \]

This equation still needs to be complemented by boundary conditions for the pressure field. Inserting Eq. (2.4) into Eq. (2.14) and integrating employing Helmholtz's theorem one obtains a relation between \( f \) and \( \mathbf{F} \) which is in turn inserted in Eq. (2.7),

\[ \Delta \Delta \mathbf{F} = \frac{1}{D} \text{grad div } \tilde{\mathbf{F}}, \quad (2.15) \]

with the abbreviation

\[ D = \frac{\kappa E(1 - \nu)}{\mu (1 + \nu)(1 - 2\nu)}. \quad (2.16) \]

Equation (2.13) is the main general result in this section. It needs to be complemented by boundary and initial conditions for displacements and pressure. It is not difficult to see from Eqs. (2.6) and (2.15) that if curl \( u = 0 \) the displacements are characterized by a vector-diffusion equation. However, the general case with non-vanishing curl is much more complicated (it is described by an integro-differential equation in \( u \)).

2. Dispersion relation

While theoretical simplifications within models are usually appreciated a remaining question is whether the simplified model assumptions do lead to a consistent physical picture concerning certain criteria. One such criterion is stability. For example, is it obvious that the model equations are stable if one disregards inertial forces for the elastic continuum phase? This problem is most convenient considered investigating the dispersion relation \( \omega(\mathbf{k}) \) obtained from an expansion after eigenfunctions. For brevity we only present the basic result in the following.

Assuming certain mathematical integrability conditions on displacements \( \tilde{u}(\mathbf{r}, t) \) and pressure \( p(\mathbf{r}, t) \) both fields can be represented as Fourier-Integrals,

\[ \tilde{u}(\mathbf{r}, t) = \int d\mathbf{k} \tilde{u}(\mathbf{k}) e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}, \quad (2.17) \]

and

\[ p(\mathbf{r}, t) = \int d\mathbf{k} \tilde{p}(\mathbf{k}) e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}. \quad (2.18) \]

Inserting this two representations into the basic Eqs. (2.3) and (2.12) one obtains the dispersion relation,

\[ \omega(\mathbf{k}) = i D \mathbf{k}^2, \quad (2.19) \]

with \( D \) being already defined in Eq. (2.16) The dispersion relation has the form of a diffusion only problem, i.e., all solutions are damped in time and therefore the equations are stable. The damping is controlled by a constant of diffusion \( D \) in a rough estimate of order \( 10^{-1} \text{m}^2 \text{s}^{-1} \) which is orders of magnitude larger than for molecular diffusion (we estimated \( D \) from \( \kappa \approx 10^{-15} \text{m}^2, E \approx 10^{11} \text{Pa}, \mu \approx 10^{-3} \text{Pa s} \) and \( \nu = 0.25 \)).

III. THE POINT SOURCE

HYDRAULIC LOADING)

In the following we consider a particular simple case in three dimension: at time \( t_0 \) the material is loaded by a pressure peak \( \delta(t)\delta(\mathbf{r}) \). Dimensionless quantities for the length scale, stress and time scale will be useful (denoted by a star):

\[ r^* = \frac{r}{\kappa^{1/2}}, \quad p^* = \frac{p}{G}, \quad t^* = \frac{t G}{\mu}. \quad (3.1) \]

with \( \kappa, 2G = E/(1 + \nu) \) and \( \mu \) being the permeability, elastic modulus and fluid viscosity respectively.

A. Pressures and Displacements

Because the chosen symmetry of boundary conditions guarantees the existence of an irrotational solution Lamé’s equation Eq. (2.3) simplifies considerably,

\[ \text{div } \mathbf{u}^* = a^*2(p^* - p_0^*), \quad (3.2) \]

where \( p_0^* \) denotes the equilibrium pressure and \( 2a^*2 = (1 - 2\nu)/(1 - \nu) \) a material constant. Note that the solution for \( \mathbf{u}^* \) can be obtained by direct integration of Eq. (3.2),

\[ \mathbf{u}^*(\mathbf{r}^*) = -\frac{a^*2}{4\pi} \text{grad}^* \int_{V^*} dV^* \frac{p^*(\mathbf{r}^*) - p_0^*}{|\mathbf{r}^* - \mathbf{r}^'|}. \quad (3.3) \]

Equation (3.3) can be inserted into the pressure equation (2.12) giving an (inhomogeneous) heat conductance equation,

\[ \Delta^* p^* - a^*2 \frac{\partial p^*}{\partial t^*} = -\delta(t^*)\delta(\mathbf{r}^*), \quad (3.4) \]

with three dimensional solution (Green’s function),

\[ p^*(\mathbf{r}^*, t^*) - p_0^* = \frac{1}{a^*2(A/\pi)^{3/2}} e^{-A r^*2}, \quad (3.5) \]

where,

\[ A = \frac{a^*2}{4t^*} = \frac{(1 - 2\nu)\mu}{8(1 - \nu)Gt}. \quad (3.6) \]

Thus incompressible flow in a porous medium whose displacement field is irrotational corresponds to a heat conductance problem in a fiktive non-porous medium. One can insert the pressure field Eq. (3.5) into Eq. (3.3) and evaluate the integral,

\[ \mathbf{u}^*(\mathbf{r}^*) = \frac{1}{2\pi^{3/2}} \frac{1}{r^*2} \int_0^{\sqrt{A}r^*} e^{-z^2} dz - \frac{\sqrt{A}}{|r^*|} e^{-A r^*2} \mathbf{e}_r. \quad (3.7) \]
In Fig. 1 we show the dimensionless radial displacements, $u_{r^*}$, for two different times.

![Graph of radial displacement $u_{r^*}$ vs. radius $r$]

FIG. 1. Radial displacement according to Eq. (3.7) for two different times $A = 1.0$ and $A = 0.5$.

The displacements being zero at the symmetry center raise to a maximum value and drop for larger radii towards zero again. As can be seen from Eq. (3.7) do the displacements follow a scaling form. The scaling variable $\delta = \sqrt{Ar^*}$ may be anticipated for a diffusion-related problem.

### B. Deformations

It is obvious that the deformations can be written in scaling form,

$$u_{r^*} = \left(\frac{A}{\pi}\right)^{3/2} H(\delta), \quad (3.8)$$

with $H(\delta)$ being a scaling function of the form,

$$H(\delta) = e^{-\delta^2} + \frac{e^{-\delta^2}}{\delta^2} - \frac{1}{\delta^3} \int_{\delta}^{\infty} dz e^{-z^2}, \quad (3.9)$$

being plotted in Fig. 3.

![Graph of scaling function $H(\delta)$ vs. reduced radius $\delta$]

FIG. 2. Scaling function $H(\delta)$ as function of the reduced radius $\delta = \sqrt{Ar^*}$, comp. Eqs. (3.3), (3.6) and (3.9). The function, being essentially a rescaled radial deformation, shows four regions of interest: (a) for small radii the positive large but finite deformation, (b) for $\delta_0 \approx 1$ a root, (c) a negative, minimum deformation at $\delta_c \approx 1.6$ and (d) a decay towards zero for large radii.

Let $\delta_0$ denote the root of the foregoing equation. We find,

$$\delta_0 = \sqrt{Ar^*_0} \approx 1, \quad (3.10)$$

within three percent error. The volumes exhibiting positive and negative stresses are therefore separated by a surface of radius $r^*_0$ at time $A$ around the center, or equivalently $r_0 = 2\sqrt{Dt}$ with diffusivity $D$ defined in Eq. (2.16), eg. an order of magnitude estimate gives $r_0 \approx 20$ cm for $t = 0, 1$ sec for the neutral zone. Another interesting property is the asymptotic behaviour of the deformation for $\delta \to \infty$. The asymptotics is given as,

$$\lim_{\delta \to \infty} \frac{\partial u^*}{\partial r^*} = -\frac{1}{2\pi r^*^3} = -\frac{1}{2\pi} \left(\frac{\kappa^{1/2} \nu}{r}\right)^3. \quad (3.11)$$

It is interesting to note that neither elastic constants nor fluid properties enter this relation. Only the permeability enters this compressional far-field decay. This relation should be compared to the simple case of a cavity of radius $R$ bearing pressure $-p$ within an infinite non-Biot medium with vanishing pressure at infinity,

$$u_{rr} = -\frac{p(1 + \nu)}{E} \left(\frac{R}{r}\right)^3 = -\frac{p^*}{2} \left(\frac{R}{r}\right)^3, \quad (3.12)$$

where one measures the bearing pressure in units of the shear modul $G = \frac{E}{2(1 + \nu)}$. Both relations are essentially the same, where the typical length-scale $\kappa^{1/2}$ in Eq. (3.11) given by the solids permeability acts similar to the cavity-size in non-Biot elasticity. In principle this should allow to measure the permeability experimentally for $\delta \to \infty$, which means in practice for distances $r >> 2\sqrt{Dt}$. This observation leads us to the conclusion that the more complicated 'Biot elasticity' reduces for length scales much larger than the scale of diffusion towards conventional linear elasticity.

The maxima of the tensile deformations, $\delta \to 0$, are readily calculated using $\lim_{\delta \to 0} H(\delta) = 1/3$.

$$\lim_{\delta \to 0} \frac{\partial u^*}{\partial r^*} = \frac{A^{3/2}}{3\pi^{3/2}}. \quad (3.13)$$

This value does not depend on the material’s permeability.

Before turning towards the mechanical stresses we quickly summarize the results for the polar deformations $u_{\theta^*\delta^*}$. Fig. 3 shows the polar deformations in a rescaled form.
Reduced Radius δ

0.0 0.2 0.4 0.6 0.8

Scaling Function Exp(-δ^2) - Η(δ)

FIG. 3. Polar deformation \( u_ΦΦ = u_Φ^*/r^* \). Because the materials moves outwards from the injection center all deformations are positive. Note that the polar deformations are monotonically decreasing for increasing δ while the radial deformations exhibit a negative minimum.

One has,

\[
\begin{align*}
    u_ΦΦ^* &= \frac{1}{2} \left( \frac{A}{\pi} \right)^{3/2} (e^{-δ^2} - H(δ)), \\
    \text{with } H(δ) \text{ and } δ \text{ as defined in Eq. (3.9).}
\end{align*}
\]

The limiting cases are,

\[
\lim_{δ \to 0} u_ΦΦ^* = \lim_{δ \to 0} u_ΦΦ^*,
\]

and

\[
\lim_{δ \to ∞} u_ΦΦ^* = -\frac{1}{2} \lim_{δ \to ∞} u_ΦΦ^*.
\]

C. Stresses

We now proceed to calculate the radial stresses \( σ_rr^* \) which we measure in units of the shear modul \( G \),

\[
\begin{align*}
    σ_rr^* &= \frac{2}{(1-2ν)} \left( (1-ν)u_ΦΦ^* + 2νu_ΦΦ^* \right),
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
    \frac{1-2ν}{2} \left( \frac{A}{π} \right)^{-3/2} σ_rr^* &= (1-ν)H(δ) + ν(δ^2 - H(δ)).
\end{align*}
\]

We have plotted the rescaled stresses Eq. (3.18) in Fig. 4 for Poisson numbers \( ν = 0.2 \) and \( ν = 0.4 \).

FIG. 4. Rescaled radial stress \( Σ_σ^*_rr = \frac{1-2ν}{2} \left( \frac{A}{π} \right)^{-3/2} σ_σ^*_rr \) as a function of the rescaled radius δ for two values of Poisson numbers. For \( ν \to 1/2 \) the scaling function decays like \( Σ_σ^*_rr = e^{-δ^2} \), which is positive only.

C. Stresses

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\end{align*}
\]

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The characteristic points of the rescaled stress \( Σ_σ^*_rr \) do now depend on \( ν \) albeit not very strongly, i.e., for lower \( ν \) one finds lower maximum tensile stresses, lower roots \( δ_0 \) and higher compressive stresses. Generally spoken do the polar deformations \( u_ΦΦ^* \) strengthen the tensile stresses and weaken the compressive region. Higher contraction numbers \( ν \) do stronger weight polar deformations on radial stresses. The two limiting cases for the radial stresses are,

\[
\lim_{δ \to 0} σ_rr^* = \frac{2}{3} \left( \frac{A}{π} \right)^{3/2}, \quad \text{and}
\]

\[
\lim_{δ \to ∞} σ_rr^* = \frac{1}{π(1-2ν)r^*}. 
\]

IV. CONCLUSION

We have considered a simplistic model for the fluid flow within a porous elastic solid. The presented description is given by linear equations. Essential simplifications are: Vanishing inertial forces of the solid phase and vanishing fluid compressibility. The first assumption relieves us of the problem of wave propagation in Biot media. The second assumption allows us to construct a linear description. We tried to obtain some general results for the linear problem. It appears that the general integrals for Biot’s and the thermo-elastic problem are different even within a linear description. While in both cases the body force field is governed by scalar potentials, e.g. pressure and temperature respectively, Biot’s problem is a fully cross-coupled field problem. This changes the structure of the governing Galerkin-Biot vector-potential, Eq. (2.15), substantially in such a way that mixed space-time derivatives do appear. However, we were able to show that an irrotational displacement field
garantuees an equivalence of both problems. Though irrotational displacement fields are rare in geometries of practical interest, we studied the point symmetric case in three dimensions in order to exemplify the fluids action on the elastic strain/stress distribution. We quantitatively discussed the situation that arises when a pressure pulse within a 'borhole' drives fluid through the solids pore-space. We found that radial tensile stresses arise in a region where non-permeable solids exhibit only compressive stresses. This surely is of interest to selected problem in soil mechanics and fracture mechanics, i.e. hydraulic fracturing.

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[5] We have calculated \( \lim_{\delta \to 0} H(\delta) \) employing an infinite recursion of L'Hospital's rule, leading to a geometric series.