Characterisation for Exponential Stability of port-Hamiltonian Systems

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Given an energy-dissipating port-Hamiltonian system, we characterise the exponential decay of the energy via the model ingredients under mild conditions on the Hamiltonian density $H$. In passing, we obtain generalisations for sufficient criteria in the literature by making regularity requirements for the Hamiltonian density largely obsolete. The key assumption for the characterisation (and thus the sufficient criteria) to work is a uniform bound for a family of fundamental solutions for some non-autonomous, finite-dimensional ODEs. Regularity conditions on $H$ for previously known criteria such as bounded variation are shown to imply the key assumption. Exponentially stable port-Hamiltonian systems with densities in $L_\infty$ only are also provided.

Keywords port-Hamiltonian systems, Exponential stability, $C_0$-semi-groups, Infinite-dimensional systems theory

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1 Introduction

In [20] van der Schaft described the framework of port-Hamiltonian systems. It has since then triggered manifold research and ideas. For this we refer to [10, 21, 2, 19, 23] and the references therein, see also [9] for a survey. The basic idea is to describe a physical – mostly energy conserving or at least energy dissipating – phenomenon in terms of a partial differential equation in the underlying physical domain together with suitable boundary conditions. These boundary conditions now are what is thought of as so-called ports. At these ports one can steer and measure data. Thus, the basic system and particularly the one-plus-one-dimensional port-Hamiltonian system serves as a prototype boundary control system. The emphasis is on hyperbolic type partial differential equations. Quite naturally, it is of interest to understand those boundary conditions leading to an evolution of the state variable to have exponentially decaying energy. More precisely, we consider the operator

$$A := P_1 \partial_x \mathcal{H} + P_0 \mathcal{H}$$

as an operator in $H := L^2(a, b)^d$ (as sets) endowed with the scalar-product $(u, v) \mapsto \langle H(u, v) \rangle_{L^2}$. Here, $(a, b) \subseteq \mathbb{R}$ is a bounded interval, $P_1 = P_1^* \in \mathbb{R}^{d \times d}$ is an invertible $d \times d$-matrix, $P_0 = -P_0^* \in \mathbb{R}^{d \times d}$ and $\partial_x$ is the usual weak derivative operator. The mapping $\mathcal{H} : (a, b) \rightarrow \mathbb{R}^{d \times d}$ is measurable, attains values in the non-negative self-adjoint matrices and is strictly bounded away from 0 and bounded above. The port-Hamiltonian operator $A$ is now accompanied with suitable boundary conditions encoded in a full rank matrix $W \in \mathbb{R}^{d \times 2d}$ satisfying

$$W \left( \begin{array}{cc} P_1 & -P_1 \\ 1_d & 1_d \end{array} \right)^{-1} \left( \begin{array}{cc} 0 & 1_d \\ 1_d & 0 \end{array} \right) \left( W \left( \begin{array}{cc} P_1 & -P_1 \\ 1_d & 1_d \end{array} \right)^{-1} \right)^* \geq 0$$

(WB)

in the way that

$$\text{dom}(A) = \left\{ u \in H ; \mathcal{H} u \in H^1(a, b)^d, W \left( \begin{array}{c} (\mathcal{H} u)(b) \\ (\mathcal{H} u)(a) \end{array} \right) = 0 \right\}.$$  

(A2)

The above conditions render $A$ to be the generator of a $C_0$-semi-group of contractions on $H$ (see e.g. [10] Theorem 7.2.4] or [8] Theorem 1.1], note also [8] Theorem 1.5] for conditions for $A$ being the generator of a general $C_0$-semi-group). For the theory of $C_0$-semi-groups we refer to the monographs [7, 12]. It is the aim of this article to characterise all such settings for which this semi-group admits exponential decay. The details of the definitions are given in Section 2. For the time being we comment on approaches available in the literature and difficulties as well as elements of our strategy to obtain our characterisation result. We remark that stability and stabilisation of port-Hamiltonian systems is an important topic in control theory, see e.g. [22, 16, 18, 8, 17].

By the celebrated Lumer–Phillips theorem (see, e.g., [7] 3.15 Theorem]) for $A$ to generate a semi-group of contractions it is equivalent that $A$ is $m$-dissipative. This property is independent of the Hamiltonian density $\mathcal{H}$ encoding the material coefficients in actual physical systems. Hence, well-posedness and energy dissipation is not hinging on the actual measurements of the material parameters. Thus, one might think that, too, exponential stability is independent of the Hamiltonian. This is,
however, not the case as the example in [6, Section 5] demonstrates. This is even more so surprising as the up to now only\footnote{If $H$ is smooth, a strict inequality in \eqref{WB} leads to exponential stability for the port-Hamiltonian semi-group as well. However, by [10, Lemma 9.1.4] and its proof, a strict inequality in \eqref{WB} implies the validity of the condition in Theorem \ref{thm:main}.} available condition yielding an exponentially stable semi-group is (almost) independent of the actual Hamiltonian.

**Theorem 1.1** ([17, Theorem 3.5]). Let $H$ be of bounded variation and $A$ given by $[A1]$ and $[A2]$ is $m$-dissipative. If there exist $c > 0$ and $\eta \in \{a, b\}$ such that for all $u \in \text{dom}(A)$

$$\langle u, Au \rangle_H \leq -c \|Hu(\eta)\|^2,$$

then $A$ generates an exponentially stable $C_0$-semi-group.

Note that the example in [6] assuring the dependence of $H$ whether or not the $C_0$-semi-group is exponentially stable, uses constant $H$. Hence, even in the class of constant Hamiltonians, the above condition is not a characterisation. Also, as a second drawback of the results available in the literature, the Hamiltonian always needs to satisfy certain regularity requirements. Apart from the more recent advancement in Theorem 1.1 the results in [10, Theorem 9.1.3] require continuous differentiability or Lipschitz continuity, see [22, Theorem III.2]. We refer to the results in [2] for non-autonomous set ups.

The present article aims at replacing the regularity conditions altogether. For this define $\Phi_t$ to be the fundamental solution associated with

$$u'(x) = itP_1^{-1}(H(x)^{-1} - P_0)u(x) \quad (x \in (a, b))$$

with $\Phi_t(a) = 1_d := \text{diag}(1, \ldots, 1) \in \mathbb{R}^{d \times d}$. The key assumption we shall impose here is

$$\sup_{t \in \mathbb{R}} \|\Phi_t\|_\infty = \sup_{t \in \mathbb{R}} \sup_{x \in (a, b)} \|\Phi_t(x)\| < \infty. \tag{B}$$

We shall see below that (B) is satisfied in many relevant cases, e.g., if $H$ is scalar or of bounded variation. The main theorem of the present article is a characterisation result of exponential stability in case (B) is satisfied.

**Theorem 1.2.** Assume condition (B) and that $A$ given by $[A1]$ and $[A2]$ is $m$-dissipative; i.e. $A$ generates a contraction semi-group. Then the following conditions are equivalent:

(i) $A$ generates an exponentially stable $C_0$-semi-group.

(ii) For all $t \in \mathbb{R}$

$$T_t := W\left(\begin{array}{c} \Phi_t(b) \\ 1_d \end{array}\right)$$

is invertible with $\sup_{t \in \mathbb{R}} \|T_t^{-1}\| < \infty$. 


To the best of our knowledge this is the first characterisation result for exponential stability of port-Hamiltonian systems. Meanwhile, however, that asymptotic stability has been characterised, though, see [24]. Together with a set of examples warranting condition (B), Theorem 1.2 contains all the available sufficient criteria for exponential stability of port-Hamiltonian systems as respective special cases as we will demonstrate below. In particular, we shall provide a condition on the connection of $\mathcal{H}$ and $P_1$ guaranteeing the satisfaction of (B) — independently of any regularity requirements for $\mathcal{H}$. A special case for this setting is that of scalar-valued Hamiltonian densities $H$, that is, when we find a bounded scalar function $h: (a, b) \to \mathbb{R}$ such that $H(x) = h(x)1_d$ for almost every $x$. The corresponding theorem characterising exponential stability is then, in fact, fairly independent of $h$ in the following sense:

**Theorem 1.3.** Assume $\mathcal{H}$ to be scalar-valued and that $A$ given by (A1) with $P_0 = 0$ and (A2) is $m$-dissipative. Then the following conditions are equivalent

(i) $A$ generates an exponentially stable $C_0$-semi-group.

(ii) For all $t \in \mathbb{R}$

$$\tau_t := W\left( e^{itP_1^{-1}} \right)$$

is invertible with $\sup_{t \in \mathbb{R}} \|\tau_t^{-1}\| < \infty$.

The sufficient criterion Theorem 1.1 for scalar-valued $\mathcal{H}$ reads as follows.

**Theorem 1.4.** Let $\mathcal{H}$ be scalar-valued and $A$ given by (A1) and (A2) is $m$-dissipative. If there exist $c > 0$ and $\eta \in \{a, b\}$ such that for all $u \in \text{dom}(A)$

$$\langle u, Au \rangle_H \leq -c\|H(u)(\eta)\|^2,$$

then $A$ generates an exponentially stable $C_0$-semi-group.

We will demonstrate below that the condition in [17, Theorem 3.5] implies (ii) from Theorem 1.2, thus providing an independent proof of [17, Theorem 3.5] in a more general situation. By means of counterexamples, we will show that the invertibility of $T_t$ is not sufficient for the uniform boundedness of $T_t^{-1}$. It will rely on future research to assess whether condition (B) is needed at all in the present analysis. However, as we will illustrate in Section 6, it is satisfied under very mild conditions on the operators involved, which covers most (if not all) systems considered in the literature so far. Nevertheless, from a mathematical point of view, it is still interesting to study the necessity of condition (B). This leads us to two open problems:

**Problem 1.5.** Characterise all $P_1$ and $\mathcal{H}$ such that (B) holds.

It may well be that (B) is not needed altogether:

**Problem 1.6.** Is (B) necessary for exponential stability of the port-Hamiltonian system at hand?
We will provide a short outline of the manuscript next. We recall some of the basic results related to port-Hamiltonian systems relevant to this article in Section 2. More so, we shall revisit the generation theorem for port-Hamiltonian systems and rephrase the same in order to have a better fit to the rationale to follow. The main result together with its proof is then presented in Section 3. In Section 4, we specialise the result to constant energy densities $H$ and provide some examples and non-examples. Particularly, we apply our characterisation to [6, Section 5]. Section 5 is devoted to frame the already known positive definiteness type condition at the boundary into the present setting. More precisely, we shall show that the criterion from Theorem 1.1 can be derived from our characterisation in Section 5. Section 6 is devoted to a discussion of (B). We sum up our findings in the concluding Section 7.

2 Generation Theorem Revisited

In this section, we recall the functional analytic setting of port-Hamiltonian systems and detail some results from the literature characterising the generation property of $A$. Moreover, we slightly reformulate said generation theorem into a form more suitable for our purposes. To start out with we fix $d \in \mathbb{N}$ and let $P_1, P_0 \in \mathbb{R}^{d \times d}$ be matrices such that $P_1 = P_1^*$ is invertible and $P_0 = -P_0^*$. Moreover, let $\mathcal{H} : (a, b) \to \mathbb{R}^{d \times d}$ be a measurable function such that

$$\exists m, M > 0 \forall x \in (a, b) : m \leq \mathcal{H}(x) = \mathcal{H}(x)^* \leq M,$$

where the inequalities are meant in the sense of positive definiteness. We use $\mathcal{H}$ to define a new inner product on $L_2(a, b)^d$ by setting

$$\langle u, v \rangle_H := \langle \mathcal{H}u, v \rangle_{L_2(a, b)^d} \quad (u, v \in L_2(a, b)^d)$$

and denote the Hilbert space $L_2(a, b)^d$ equipped with this new inner product by $H$. It follows that $H = L_2([a, b]; \mathbb{R}^d)$ is equipped with the norm

$$\|u\|_H := \|\mathcal{H}^{\frac{1}{2}} u\|_{L_2(a, b; \mathbb{R}^d)}.$$

Finally, we define the port-Hamiltonian (operator)

$$A : \text{dom}(A) \subseteq H \to H,$$

$$u \mapsto P_1 (\mathcal{H}u)' + P_0 \mathcal{H}u,$$

with a suitable domain $\text{dom}(A)$ satisfying

$$\text{dom}(\partial_0 \mathcal{H}) \subseteq \text{dom}(A) \subseteq \text{dom}(\partial \mathcal{H}),$$

where $\partial_0$ denotes the distributional derivative on $L_2(a, b)^d$ with domain $H^1_0(a, b)^d$. We recall the following characterisation for $m$-dissipativity of port-Hamiltonians:

**Theorem 2.1** ([10, Theorem 7.2.4]). Let $A$ be as in (1). Then $A$ is $m$-dissipative (and hence, $A$ generates a contraction semi-group) if and only if there exists a matrix $W \in \mathbb{R}^{d \times 2d}$ with $\text{rk} W = d$ and

$$W \left( \begin{array}{cc} P_1 & -P_1 \\ 1_d & 1_d \end{array} \right) \left( \begin{array}{cc} 0 & 1_d \\ 1_d & 0 \end{array} \right) \left( W \left( \begin{array}{cc} P_1 & -P_1 \\ 1_d & 1_d \end{array} \right) \left( \begin{array}{cc} 0 & 1_d \\ 1_d & 0 \end{array} \right) \right)^* \geq 0 \quad (2)$$
such that
\[
\text{dom}(A) = \left\{ u \in H : \mathcal{H}u \in H^1(a,b)^d, \ W \left( \left( \mathcal{H}u \right)(b) \right) = 0 \right\}. \tag{3}
\]

The desired reformulation of Theorem 2.1 requires some prerequisites. For this, note since \( P_1 \) is self-adjoint and invertible, we can decompose the space \( \mathbb{R}^d \) into \( \mathbb{R}^d = E_+ \oplus E_- \) with
\[
E_+ := \text{lin}\{ v \in \mathbb{R}^d ; \exists \lambda > 0 : P_1 v = \lambda v \}, \quad E_- := \text{lin}\{ w \in \mathbb{R}^d ; \exists \lambda < 0 : P_1 w = \lambda w \}.
\]

We denote by \( \iota_{\pm} : E_{\pm} \to \mathbb{R}^d \) the canonical embeddings. Note that \( P_1^\pm := \iota_{\pm}^* P_1 \iota_{\pm} : \mathbb{R}^d \to \mathbb{R}^d \) is then the orthogonal projection on \( E_{\pm} \) and \( \iota_{\pm}^* \iota_{\pm} : E_{\pm} \to E_{\pm} \) is just the identity. Moreover, we set
\[
P_1^+ := \iota_+^* P_1 \iota_+ : E_+ \to E_+ \quad \quad P_1^- := \iota_-^* (-P_1) \iota_- : E_- \to E_-.
\]

Note that \( P_1^+ \) and \( P_1^- \) are both strictly positive self-adjoint operators. Moreover, we set
\[
Q_+ := \iota_+ \left( P_1^+ \right)^{1/2} \iota_+^* : \mathbb{R}^d \to \mathbb{R}^d, \quad Q_- := \iota_- \left( P_1^- \right)^{1/2} \iota_-^* : \mathbb{R}^d \to \mathbb{R}^d.
\]

We equip the spaces \( E_+ \) and \( E_- \) with the norms
\[
\| x \|_{E_+} := \| \iota_+ \left( P_1^+ \right)^{-1/2} x \|_{\mathbb{R}^d}, \quad \| y \|_{E_-} := \| \iota_- \left( P_1^- \right)^{-1/2} y \|_{\mathbb{R}^d}.
\]

The next lemma is a standard fact from linear algebra.

**Lemma 2.2.** Let \( W, \tilde{W} \in \mathbb{R}^{d \times 2d} \) such that \( \text{rk} W = d \) and \( \ker W = \ker \tilde{W} \). Then there exists \( K \in \mathbb{R}^{d \times d} \) invertible with
\[
W = K \tilde{W}.
\]

**Proof.** Since \( \ker W = \ker \tilde{W} \) and \( \dim \ker W = d \), we infer that both \( W \) and \( \tilde{W} \) are onto. Hence, the mappings
\[
W_1 : \ker(W)^\perp \to \mathbb{R}^d \quad \text{and} \quad \tilde{W}_1 : \ker(W)^\perp \to \mathbb{R}^d
\]
are bijections. We set \( K := W_1 \tilde{W}_1^{-1} \) and obtain
\[
W u = W_1 P_{(\ker W)^\perp} u = K \tilde{W}_1 P_{(\ker W)^\perp} u = K \tilde{W} u
\]
for each \( u \in \mathbb{R}^d \). \qed

**Lemma 2.3.** Let \( W \in \mathbb{R}^{d \times 2d} \). Then the following statements are equivalent

(i) \( W \) has rank \( d \) and satisfies
\[
W \left( \begin{array}{cc} P_1 & -P_1 \\ 1 & 1 \end{array} \right)^{-1} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( W \left( \begin{array}{cc} P_1 & -P_1 \\ 1 & 1 \end{array} \right)^{-1} \right)^* \geq 0. \tag{4}
\]
(ii) There exists a matrix $M \in \mathbb{R}^{d \times d}$ with $\|M\| \leq 1$ and an invertible matrix $K \in \mathbb{R}^{d \times d}$ such that

$$W = K \left( Q_+ - MQ_- \quad Q_- - MQ_+ \right).$$

Moreover, $\|M\| < 1$ is equivalent to strict positive definiteness in $\mathcal{H}$.

Proof. (i) $\Rightarrow$ (ii): We write $W = \left( W_1 \quad W_2 \right)$ with $W_1, W_2 \in \mathbb{R}^{d \times d}$. An easy calculation reveals that

$$W_2P_1^{-1}W_2^* \leq W_1P_1^{-1}W_1^*.$$  

We consider now the adjoint mapping $W^* : \mathbb{R}^d \to \mathbb{R}^{2d}$. For $z = (x, y) \in \text{ran} W^*$, we find $u \in \mathbb{R}^d$ such that $x = W_1^* u$ and $y = W_2^* u$. The latter gives

$$\|\iota_+^* y\|_{E_+}^2 - \|\iota_-^* y\|_{E_-}^2 = \langle \iota_+ \left( P_1^+ \right)^{-1} \iota_+^* y, y \rangle_{\mathbb{R}^d} - \langle \iota_- \left( P_1^- \right)^{-1} \iota_-^* y, y \rangle_{\mathbb{R}^d}$$

$$= \langle P_1^{-1} y, y \rangle_{\mathbb{R}^d}$$

$$= \langle P_1^{-1} W_2^* u, W_2^* u \rangle_{\mathbb{R}^d}$$

$$\leq \langle W_1 P_1^{-1} W_1^* u, u \rangle_{\mathbb{R}^d}$$

$$= \langle P_1^{-1} x, x \rangle_{\mathbb{R}^d}$$

and thus,

$$\|\iota_+^* y\|_{E_+}^2 - \|\iota_-^* y\|_{E_-}^2 \leq \|\iota_+^* x\|_{E_+}^2 + \|\iota_-^* y\|_{E_-}^2.$$  

(5)

Next, consider the mapping

$$S := \left( P_+ \quad P_- \right) W^* : \mathbb{R}^d \to \mathbb{R}^d, \quad u \mapsto P_+ W_1^* u + P_- W_2^* u.$$  

This mapping is linear and one-to-one and hence, a bijection. Indeed, if $Su = P_+ W_1^* u + P_- W_2^* u = 0$, then $P_+ W_1^* u = 0 = P_- W_2^* u$. By (ii) it follows that $P_+ W_1^* u = 0 = P_- W_2^* u$ and consequently, $W^* u = 0$. Since $W^*$ has rank $d$, it is one-to-one and thus, $u = 0$. We now define the mapping

$$C : \mathbb{R}^d \to \mathbb{R}^d, \quad v \mapsto \left( P_- \quad P_+ \right) W^* S^{-1} v$$

and set

$$M := - (Q_+ + Q_-) C^* \left( \iota_+ \left( P_1^+ \right)^{-\frac{1}{2}} \iota_+^* + \iota_- \left( P_1^- \right)^{-\frac{1}{2}} \iota_-^* \right).$$

We now prove that the matrices $W$ and $\tilde{W} := \left( Q_+ - MQ_- \quad Q_- - MQ_+ \right)$ have the same kernels. Indeed,

$$(x, y) \in \ker W \iff (x, y) \in (\text{ran } W^*)^\perp,$$

$$\Leftrightarrow \forall u \in \mathbb{R}^d : \langle x, W_1^* u \rangle + \langle y, W_2^* u \rangle = 0,$$

$$\Leftrightarrow \forall u \in \mathbb{R}^d : \langle P_+ x, P_+ W_1^* u \rangle + \langle P_- y, P_- W_2^* u \rangle + \langle P_+ x, P_+ W_1^* u \rangle + \langle P_+ x, P_- W_2^* u \rangle = 0,$$

$$\Leftrightarrow \forall u \in \mathbb{R}^d : \langle P_+ x + P_- y, C S u \rangle + \langle P_+ x + P_- y, S u \rangle = 0,$$

$$\Leftrightarrow \forall u \in \mathbb{R}^d : \langle P_+ x + P_- y + C^* (P_+ x + P_- y), S u \rangle = 0,$$
\[
\begin{align*}
\Leftrightarrow & -C^*(P_- x + P_+ y) = P_+ x + P_- y \\
\Leftrightarrow & -(Q_+ + Q_-)C^*(P_- x + P_+ y) = Q_+ x + Q_- y \\
\Rightarrow & M (Q_+ y + Q_- x) = Q_+ x + Q_- y,
\end{align*}
\]

implying

\[(x, y) \in \ker W \Leftrightarrow (Q_+ - MQ_-) x + (Q_- - MQ_+) y = 0;
\]
i.e. \( \ker W = \ker \widetilde{W} \). Employing Lemma 2.2 we find \( K \in \mathbb{R}^{d \times d} \) invertible such that

\[W = K \widetilde{W} = K \left( Q_+ - MQ_- \quad Q_- - MQ_+ \right).\]

It remains to show that \( \|M\| \leq 1 \). For this, let \( v \in \mathbb{R}^d \). Since \( S \) is onto, we find \( u \in \mathbb{R}^d \) such that

\[Q_+ v + Q_- v = Su = P_+ W_1^+ u + P_- W_2^+ u = P_+ x + P_- y,
\]

where we set \((x, y) := W^* u \in \text{ran} W^*\); thus, \( Q_- v = P_- y \) and \( Q_+ v = P_+ x \). We compute, using (5) for \((x, y)\)

\[
\|M^* v\|^2 = \|_{\left( \iota_+ \left( P_1^+ \right)^{-\frac{1}{2}} \iota_+^* + \iota_- \left( P_1^- \right)^{-\frac{1}{2}} \iota_-^* \right)} C \left(Q_+ v + Q_- v \right)\|^2 \\
= \|_{\left( \iota_+ \left( P_1^+ \right)^{-\frac{1}{2}} \iota_+^* + \iota_- \left( P_1^- \right)^{-\frac{1}{2}} \iota_-^* \right)} CSu\|^2 \\
= \|_{\left( \iota_+ \left( P_1^+ \right)^{-\frac{1}{2}} \iota_+^* + \iota_- \left( P_1^- \right)^{-\frac{1}{2}} \iota_-^* \right)} \left(P_+ x + P_- y\right)\|^2 \\
\leq \|_{E_+} \iota_+^* y \|^2 + \|_{E_-} \iota_-^* x \|^2 \\
= \|_{E_+} \iota_+^* x \|^2 + \|_{E_-} \iota_-^* y \|^2 \\
= \|_{E_+} \iota_+ \left( P_1^+ \right)^{-\frac{1}{2}} \iota_+^* x + \iota_- \left( P_1^- \right)^{-\frac{1}{2}} \iota_-^* y \|^2 \\
= \|_{E_-} \iota_- \left( P_1^- \right)^{-\frac{1}{2}} \iota_-^* Q_+ v + \iota_- \left( P_1^- \right)^{-\frac{1}{2}} \iota_-^* Q_- v \|^2 \\
= \|_{E_-} \iota_- \left( P_1^- \right)^{-\frac{1}{2}} \iota_-^* Q_- v \|^2 \\
\]

which yields \( \|M\| = \|M^*\| \leq 1 \).

(ii) \Rightarrow (i): Assume

\[
W = K \left( Q_+ - MQ_- \quad Q_- - MQ_+ \right)
\]

for \( K, M \in \mathbb{R}^{d \times d} \) with \( \|M\| \leq 1 \) and \( K \) invertible. We show that \( W \) has rank \( d \). Since \( K \) is invertible, it suffices to show that \( \tilde{W} := \left( Q_+ - MQ_- \quad Q_- - MQ_+ \right) \) has rank \( d \), which in turn is equivalent to \( \ker \left( \widetilde{W} \right)^* = \{0\} \). So, let \( u \in \ker(W)^* \); that is,

\[
Q_+ u = Q_- M^* u, \quad Q_- u = Q_+ M^* u.
\]

Since \( Q_- \) and \( Q_+ \) attain values in \( E_- \) and \( E_+ \), respectively, we infer \( Q_- u = Q_+ u = 0 \) and hence \( P_- u = P_+ u = 0 \), which imply \( u = 0 \). It remains to show (4). As shown above, this is equivalent to (5). So let \((x, y) = W^* u \) for some \( u \in \mathbb{R}^d \); that is

\[
x = (Q_+ - Q_- M^*) K^* u, \quad y = (Q_- - Q_+ M^*) K^* u.
\]
Then we compute
\[\|\iota_+^* y\|_{E_+}^2 + \|\iota_-^* x\|_{E_-}^2 = \|\iota_+ (P_1^+) - \frac{1}{2} \iota_+^* y\|_{\mathbb{R}^d}^2 + \|\iota_- (P_1^-) - \frac{1}{2} \iota_-^* x\|_{\mathbb{R}^d}^2 = \|P_+ M^* K^* u\|_{\mathbb{R}^d}^2 + \|P_- M^* K^* u\|_{\mathbb{R}^d}^2 = \|M^* K^* u\|_{\mathbb{R}^d}^2 \leq \|K^* u\|_{\mathbb{R}^d}^2 = \|P_+ K^* u\|_{\mathbb{R}^d}^2 + \|P_- K^* u\|_{\mathbb{R}^d}^2 = \|\iota_+ (P_1^+) - \frac{1}{2} \iota_+^* x\|_{\mathbb{R}^d}^2 + \|\iota_- (P_1^-) - \frac{1}{2} \iota_-^* y\|_{\mathbb{R}^d}^2 = \|\iota_+^* x\|_{E_+}^2 + \|\iota_-^* y\|_{E_-}^2.\]

For the final claim, we note that the strict positive definiteness in (i) is equivalent to
\[\|\iota_+^* y\|_{E_+}^2 + \|\iota_-^* x\|_{E_-}^2 < \|\iota_+^* x\|_{E_+}^2 + \|\iota_-^* y\|_{E_-}^2 \quad ((x, y) \in \text{ran } W^* \setminus \{0\}),\]
which by the computations above is equivalent to \(\|M\| = \|M^*\| < 1.\)

Using the latter lemma, we obtain the following characterisation result for \(A\) generating a contraction semi-group.

**Theorem 2.4.** The following statements are equivalent:

(i) \(A\) is \(m\)-dissipative,

(ii) \(A\) is dissipative and there exists \(W \in \mathbb{R}^{d \times 2d}\) such that
\[\text{dom}(A) = \left\{ u \in H; \mathcal{H} u \in H^1([a, b]; \mathbb{R}^d), W \left( \begin{array}{c} (\mathcal{H} u)(b) \\ (\mathcal{H} u)(a) \end{array} \right) = 0 \right\},\]

(iii) there exists \(W \in \mathbb{R}^{d \times 2d}\) with maximal rank, such that
\[W \left( \begin{array}{cc} P_1 & -P_1 \\ 1 & 1 \end{array} \right)^{-1} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) W \left( \begin{array}{cc} P_1 & -P_1 \\ 1 & 1 \end{array} \right)^{-1} \geq 0 \quad (6)\]
\[\text{and}\]
\[\text{dom}(A) = \left\{ u \in H; \mathcal{H} u \in H^1([a, b]; \mathbb{R}^d), W \left( \begin{array}{c} (\mathcal{H} u)(b) \\ (\mathcal{H} u)(a) \end{array} \right) = 0 \right\},\]

(iv) There exists a matrix \(M \in \mathbb{R}^{d \times d}\) with \(\|M\| \leq 1\) such that
\[\text{dom}(A) = \left\{ u \in H; \mathcal{H} u \in H^1([a, b]; \mathbb{R}^d), Q_-(\mathcal{H} u)(a) + Q_+(\mathcal{H} u)(b) = M (Q_-(\mathcal{H} u)(b) + Q_+(\mathcal{H} u)(a)) \right\}.

Moreover, in case (iv) we have
\[\langle Au, u \rangle_H = \frac{1}{2} \left( \|M (Q_-(\mathcal{H} u)(b) + Q_+(\mathcal{H} u)(a))\|^2 - \|Q_-(\mathcal{H} u)(b) + Q_+(\mathcal{H} u)(a)\|^2 \right).\]
for each \( u \in \text{dom}(\mathcal{A}) \). Furthermore, the matrix \( W \) in (ii) or (iii) can be expressed in terms of the matrix \( M \) in (iv) by

\[
W = K \left( Q_+ - MQ_- \quad Q_- - MQ_+ \right)
\]

for some invertible matrix \( K \in \mathbb{R}^{d \times d} \).

**Proof.** The equivalence of statements (i)-(iii) is well-known, cf. \[8, \text{Theorem 1}\] and \[10, \text{Theorem 7.2.4}\]. Moreover, the equivalence of (iii) and (iv) and the relation of the matrices \( W \) and \( M \) follows from Lemma 2.3. The only thing to be shown is the formula for \( \langle Au, u \rangle_M \). So, let \( u \in \text{dom}(\mathcal{A}) \). Then

\[
2\langle Au, u \rangle_M = 2\langle P_1 \partial_1 \mathcal{H} u, \mathcal{H} u \rangle + 2\langle P_0 \mathcal{H} u, \mathcal{H} u \rangle
\]

\[
= 2\langle P_1 \mathcal{H} u, \mathcal{H} u \rangle - \langle P_1 \mathcal{H} (u) (a), (\mathcal{H} u) (a) \rangle
\]

\[
= \langle Q_1^2 (\mathcal{H} u) (b), (\mathcal{H} u) (b) \rangle - \langle Q_2^2 (\mathcal{H} u) (b), (\mathcal{H} u) (b) \rangle
\]

\[
- \langle Q_2^2 (\mathcal{H} u) (a), (\mathcal{H} u) (a) \rangle + \langle Q_2^2 (\mathcal{H} u) (a), (\mathcal{H} u) (a) \rangle
\]

\[
= \| Q_+ (\mathcal{H} u) (b) + Q_- (\mathcal{H} u) (a) \|^2 - \| Q_+ (\mathcal{H} u) (a) + Q_- (\mathcal{H} u) (b) \|^2
\]

\[
= \| M (Q_+ (\mathcal{H} u) (a) + Q_- (\mathcal{H} u) (b)) \|^2 - \| Q_+ (\mathcal{H} u) (a) + Q_- (\mathcal{H} u) (b) \|^2
\]

which shows the assertion. \( \square \)

**Remark 2.5.** The anonymous referee kindly provided us with an alternative proof for the equivalence of (i) and (iv) in Theorem 2.1 using theory of port-Hamiltonian systems instead of Lemma 2.3. We sketch this proof as follows: By Theorem 2.1 item (i) in Theorem 2.4 is equivalent to \( \mathcal{H} \) with domain as in \[43\] being a generator of a contraction semigroup. By unitary equivalence one can assume that \( \mathcal{H} = 1_d \) (see \[10, \text{Lemma 7.2.3}\] or Lemma 3.1 below). Again by unitary equivalence it suffices to treat the case \( P_1 = \begin{pmatrix} \Lambda & 0 \\ 0 & \Theta \end{pmatrix} \begin{pmatrix} P_1^+ & 0 \\ 0 & -P_1^- \end{pmatrix} \). In this setup one can apply \[10, \text{Theorem 13.3.1}\]. Indeed, \( \tilde{A}u = P_1 u' \) generates a \( C_0 \)-semigroup on \( L_2(a, b)^d \) if and only if its domain is given by

\[
\text{dom}(\tilde{A}) = \left\{ u \in H^1(a, b)^d ; \quad KP_1 \begin{pmatrix} u^+(b) \\ u^-(a) \end{pmatrix} + LP_1 \begin{pmatrix} u^+(a) \\ u^-(b) \end{pmatrix} = 0 \right\},
\]

for some matrices \( K, L \), where \( K \) is invertible. Hence, (i) is equivalent to \( \tilde{A} \) with domain as above being dissipative (note that an operator is m-dissipative if and only if it is dissipative and generates a \( C_0 \)-semigroup). By invertibility of \( K \) the boundary condition for \( \tilde{A} \) can be rewritten as

\[
(Q_+ + Q_-) \begin{pmatrix} u^+(b) \\ u^-(a) \end{pmatrix} = \sqrt{P_1} \begin{pmatrix} u^+(b) \\ u^-(a) \end{pmatrix} = M \sqrt{P_1} \begin{pmatrix} u^+(a) \\ u^-(b) \end{pmatrix} = M (Q_+ + Q_-) \begin{pmatrix} u^+(a) \\ u^-(b) \end{pmatrix},
\]

for some matrix \( M \). Then (standard) integration by parts reveals that \( \tilde{A} \) is dissipative if and only if \( \|M\| \leq 1 \) (for the only if part use Lemma 2.3).
3 Characterisation of Exponential Stability

In this section we prove the main theorem of the present manuscript. We start off with an elementary observation, see also [13].

Lemma 3.1. Let $a, b \in \mathbb{R}$ with $a < b$ and $d \in \mathbb{N}$. Moreover, let $\mathcal{H} \in L_\infty(a, b; \mathbb{R}^{d \times d})$ with $0 < m \leq \mathcal{H}(x) = \mathcal{H}(x)^\top$ for almost every $x \in (a, b)$ and let $H$ be the Hilbert space $L_2(a, b; \mathbb{R}^d)$ endowed with the inner product $\langle u, v \rangle_H := \langle H u, v \rangle_{L_2}$.

Define

$$ S : L_2(a, b)^d \rightarrow H, \quad u \mapsto H^{-1}u. $$

Then $S$ is a Banach space isomorphism with $S^*v = v$ for each $v \in L_2(a, b)^d$.

Proof. $S$ clearly is bounded, one-to-one and onto; i.e., a Banach space isomorphism. Moreover, for $u \in L_2(a, b)^d$, $v \in H$ we compute

$$ \langle Su, v \rangle_H = \langle H H^{-1}u, v \rangle_{L_2(a, b)^d} = \langle u, v \rangle_{L_2(a, b)^d}, $$

which shows $S^*v = v$. \hfill \Box

For $t \in \mathbb{R}$ we consider the following ordinary differential equation

$$ v'(x) = P_1^{-1}(it\mathcal{H}(x)^{-1} - P_0)v(x) $$

and denote by $\Phi_t : [a, b] \rightarrow \mathbb{R}^{d \times d}$ the fundamental matrix satisfying $\Phi_t(a) = 1_d$.

Lemma 3.2. For $t \in \mathbb{R}$ and $x \in [a, b]$ we have $\Phi_t(x)^{-1} = P_1^{-1}\Phi_t(x)^*P_1$. In particular $\|\Phi_t(x)^{-1}\| \leq \|P_1^{-1}\||P_1||\Phi_t(x)\|$.

Proof. We compute the derivative of $\Psi : x \mapsto \Phi_t(x)^{-1}$. We have

$$ \Psi'(x) = -\Psi(x)\Phi_t'(x)\Psi(x) $$

$$ = -\Psi(x)P_1^{-1}(it\mathcal{H}(x)^{-1} - P_0)\Phi_t(x)\Psi(x) $$

$$ = -\Psi(x)P_1^{-1}(it\mathcal{H}(x)^{-1} - P_0). $$

Hence,

$$ (\Psi^*)(x) = \Psi'(x)^* = (it\mathcal{H}(x)^{-1} - P_0)P_1^{-1}\Psi(x)^* $$

and

$$ (P_1^{-1}\Psi^*)'(x) = P_1^{-1}(it\mathcal{H}(x)^{-1} - P_0)P_1^{-1}\Psi(x)^*. $$

Thus, $P_1^{-1}\Psi^*$ solves the same ODE as $\Phi_t$ does; taking into account the initial values it follows that

$$ P_1^{-1}\Psi^*(x) = \Phi_t(x)P_1^{-1} $$

and hence,

$$ \Phi_t(x)^{-1} = P_1^{-1}\Phi_t(x)^*P_1. $$

\hfill \Box
The theorem underlying the proof of our characterisation of exponential stability is the celebrated result by Gearhart–Prüß.

**Theorem 3.3** (Gearhart–Prüß, see [14]). A generates an exponentially stable $C_0$-semi-group on a Hilbert space $H$, if, and only if,

$$\sup_{z \in \mathbb{C}_{\text{Re}>0}} \|(z - A)^{-1}\|_{L(H)} < \infty. \quad (7)$$

The decisive step to reach our goal now becomes the following result.

**Theorem 3.4.** Let $A$ be as in Theorem 2.4 (iii) and assume that $\sup_{t \in \mathbb{R}} \|\Phi_t\|_{\infty} < \infty$. Then the following statements are equivalent:

(i) $i\mathbb{R} \subseteq \rho(A)$ and $\sup_{t \in \mathbb{R}} \|(it - A)^{-1}\| < \infty$,

(ii) for all $t \in \mathbb{R}$ the matrix

$$T_t := W \left( \begin{array}{c} \Phi_t(b) \\ 1 \end{array} \right) \in \mathbb{R}^{d \times d}$$

is invertible and $\sup_{t \in \mathbb{R}} \|T_t^{-1}\| < \infty$.

**Proof.** We use the operator $S$ as given in Lemma 3.1. First note that for $f \in H$ we find $u \in \text{dom}(A)$ with $(it - A)u = f$ if and only if

$$S^*(it - A)SS^{-1}u = S^*f$$

and

$$S^*(it - A)S = (itS^*S - S^*AS) = (it\mathcal{H}^{-1} - A),$$

where

$$A: \text{dom}(A) \subseteq L_2([a, b]; \mathbb{R}^d) \rightarrow L_2([a, b]; \mathbb{R}^d), \quad v \mapsto P_1v' + P_0v$$

and

$$\text{dom}(A) = \left\{ v \in H^1([a, b]; \mathbb{R}^d) : W \left( \begin{array}{c} v(b) \\ v(a) \end{array} \right) = 0 \right\}.$$ 

Now $(it - A)u = f$ if and only if $v := S^{-1}u \in \text{dom}(A)$ satisfies

$$it\mathcal{H}^{-1}v - P_1v' - P_0v = S^*f = f,$$

which in turn is equivalent to

$$v' = P_1^{-1}(it\mathcal{H}^{-1} - P_0)v - P_1^{-1}f.$$ 

Note that then

$$v(x) = \Phi_t(x)v_0 - \Phi_t(x) \int_a^x \Phi_t(s)^{-1}P_1^{-1}f(s) \, ds$$

for some $v_0 = v(a) \in \mathbb{R}^d$. Then $v \in \text{dom}(A)$ is equivalent to

$$0 = W \left( \begin{array}{c} v(b) \\ v(a) \end{array} \right)$$

for some $v_0 = v(a) \in \mathbb{R}^d$.
\[
W \left( \Phi_t(b) \right) v_0 - W \left( \Phi_t(b) \int_a^b \Phi_t(s)^{-1} P_1^{-1} f(s) \, ds \right) = T_t v_0 - W \left( \Phi_t(b) \int_a^b \Phi_t(s)^{-1} P_1^{-1} f(s) \, ds \right).
\]

(i) \Rightarrow (ii): We first show that \( T_t \) is invertible. For this, let \( v_0 \in \mathbb{R}^d \) with \( T_t v_0 = 0 \). Set \( v(x) := \Phi(x) v_0 \). Then by what we have shown above \( u := Sv \in \text{dom}(A) \) and satisfies \((i - A) u = 0\) and thus, \( u = 0 \). The latter gives \( v = 0 \) and thus, \( v(a) = v_0 = 0 \). So \( T_t \) is invertible. Assume now that
\[
\sup_{t \in \mathbb{R}} \| T_t^{-1} \| = \infty.
\]
Since \( t \mapsto T_t^{-1} \) is bounded on compact sets, we find a sequence \((t_n)_n\) with \( t_n \to \infty \) such that \( \| T_{t_n}^{-1} \| \to \infty \). By the uniform boundedness principle we find \( z \in \mathbb{R}^d \) such that
\[
\| T_{t_n}^{-1} z \| \to \infty \quad (n \to \infty).
\]
Since \( W \) has full rank, we find \( y, \tilde{y} \in \mathbb{R}^d \) such that \( z = W \left( \begin{array}{c} y \\ \tilde{y} \end{array} \right) \). Set now \( y_n := -y + \Phi_{t_n}(b) \tilde{y} \) and note that \((y_n)_n\) is bounded, since \((\Phi_{t_n}(b))_n\) is bounded. Further, set
\[
f_n(x) := - \frac{1}{b - a} P_1 \Phi_{t_n}(x) \Phi_{t_n}(b)^{-1} y_n \quad (x \in [a, b]).
\]
Then \( f_n \in L_\infty([a, b]; \mathbb{R}^d) \subseteq H \) and \((f_n)_n\) is uniformly bounded in \( L_\infty([a, b]; \mathbb{R}^d) \), where we use Lemma 3.2. Set now \( u_n := (i t_n - A)^{-1} f_n \) and note that \((u_n)_n\) is uniformly bounded in \( H \) by assumption. Hence, so is \( v_n := S^{-1} u_n \) and moreover,
\[
v_n(x) = \Phi_{t_n}(x) v_{0,n} - \Phi_{t_n}(x) \int_a^x \Phi_{t_n}(s)^{-1} P_1^{-1} f_n(s) \, ds
\]
with
\[
T_{t_n} v_{0,n} = W \left( \int_a^b \Phi_t(s)^{-1} P_1^{-1} f_n(s) \, ds \right) = -W \left( \begin{array}{c} y_n \\ 0 \end{array} \right)
\]
\[
= W \left( \begin{array}{c} y - \Phi_{t_n}(b) \tilde{y} \\ 0 \end{array} \right) = W \left( \begin{array}{c} y \\ \tilde{y} \end{array} \right) - T_{t_n} \tilde{y} = z - T_{t_n} \tilde{y}.
\]
The latter gives
\[
T_{t_n}^{-1} z = v_{0,n} + \tilde{y}.
\]
Hence \( \| v_{0,n} \| \to \infty \), but on the other hand
\[
\| v_{0,n} \| = \frac{1}{\sqrt{b - a}} \| v_{0,n} \|_{L_2([a,b]; \mathbb{R}^d)} \leq C \left( \| v_n \|_{L_2([a,b]; \mathbb{R}^d)} + \| f_n \|_{L_2([a,b]; \mathbb{R}^d)} \right),
\]
for some constant \( C > 0 \), where we again invoke Lemma 3.2. This yields a contradiction.

(ii) \Rightarrow (i): By our considerations at the beginning of the proof, we obtain
\[
(i - A) u = f
\]
if and only if \( u = Sv \) for
\[
v(x) = \Phi_t(x)v_0 - \Phi_t(x)\int_a^x \Phi_t(s)^{-1}P_1^{-1}f(s)\,ds
\]
with
\[
T_tv_0 = W \left( \Phi_t(b)\int_a^b \Phi_t(s)^{-1}P_1^{-1}f(s)\,ds \right).
\]

Since the last equality can be solved uniquely, since \( T_t \) is invertible, we infer that \( i \in \rho(A) \) and by Lemma 3.2 we have
\[
\| (i - A)^{-1}f \|_H \leq \| S \|\| v \|_{L^2} \leq C (\| v_0 \| + \| f \|_{L^2}) \leq \tilde{C}\| f \|_H,
\]
where we have used that \( \| \cdot \|_H \) and \( \| \cdot \|_{L^2} \) are equivalent and \( T_t^{-1} \) is uniformly bounded.

Our main result may now be stated as follows.

**Theorem 3.5.** Let \( P_1, P_0 \in \mathbb{R}^{d \times d} \) such that \( P_1 = P_1^* \) is invertible, \( P_0 = -P_0 \), and \( \mathcal{H} : [a, b] \to \mathbb{R}^{d \times d} \) be a measurable function such that \( \exists m, M > 0 \forall x \in [a, b] : m \leq \mathcal{H}(x) = \mathcal{H}(x)^* \leq M \).

We consider \( A \subseteq P_1\partial_x \mathcal{H} + P_0 \mathcal{H} \) on the Hilbert space \( H := (L^2(a, b)^d, (\cdot, \cdot)_{L^2}) \) with domain
\[
\text{dom}(A) = \left\{ u \in H : \mathcal{H}u \in H^1([a, b])^d, W \left( \begin{array}{c} (\mathcal{H}u)(b) \\ (\mathcal{H}u)(a) \end{array} \right) = 0 \right\}
\]
for a matrix \( W \in \mathbb{R}^{d \times 2d} \) satisfying \( \text{rk} W = d \) and
\[
W \left( \begin{array}{cc} P_1 & -P_1 \\ 1_d & 1_d \end{array} \right)^{-1} \left( \begin{array}{cc} 0 & 1_d \\ 1_d & 0 \end{array} \right) W \left( \begin{array}{cc} P_1 & -P_1 \\ 1_d & 1_d \end{array} \right)^{-1} \geq 0.
\]

Moreover, assume that the fundamental matrix \( \Phi_t \) associated to the ODE-system
\[
u'(x) = P_1^{-1}(it\mathcal{H}(x)^{-1} - P_0)u(x) \quad (x \in [a, b])
\]
with \( \Phi_t(a) = 1_d \) for each \( t \in \mathbb{R} \) satisfies \( \sup_{t \in \mathbb{R}} \| \Phi_t \| < \infty \). Then the following statements are equivalent:

(i) the (contraction-)semi-group \( (e^{tA})_{t \geq 0} \) is exponentially stable,

(ii) for each \( t \in \mathbb{R} \) the matrix
\[
T_t := W \left( \begin{array}{c} \Phi_t(b) \\ I_d \end{array} \right)
\]

is invertible with \( \sup_{t \in \mathbb{R}} \| T_t^{-1} \| < \infty \).

**Proof.** The assumptions guarantee that \( A \) is \( m \)-dissipative by Theorem 2.4. Thus, using Theorem 3.3 (ii) holds, if and only if, condition (i) in Theorem 3.4 is satisfied. Thus, the claim follows from Theorem 3.4. \( \square \)
4 A First Application

In this section, we specialise to the case $P_0 = 0$. Thus, throughout this section, let

\[ A : \text{dom}(A) \subseteq H \rightarrow H \\
\quad u \mapsto P_1(Hu)', \]

be a port-Hamiltonian operator as in \[ \text{(1)} \] with $P_0 = 0$. Furthermore, we assume $A$ to generate a $C_0$-semi-group of contractions. Hence, by Theorem 2.1 we find a full rank matrix $W \in \mathbb{R}^{d \times 2d}$ satisfying

\[ \text{(2)} \]

such that

\[ \text{dom}(A) = \left\{ u \in H ; Hu \in H^1(a,b)^d, W \begin{pmatrix} (Hu)(b) \\ (Hu)(a) \end{pmatrix} = 0 \right\}. \]

We recall $\Phi_t$, the fundamental solution associated with $u'(x) = itP^{-1}_1H^{-1}_0(x)u(x)$ \hspace{1cm} ($x \in (a,b)$)

such that $\Phi_t(a) = 1_d$ and condition [B] stating $\sup_{t \in \mathbb{R}} \| \Phi_t \|_\infty < \infty$.

We specialise this result to the case when the Hamiltonian density $H$ is constant. For this we state a special case of Theorem 6.7, which we prove in Subsection 6.2.

**Proposition 4.1.** Let $H_0 = H_0^* \in \mathbb{R}^{d \times d}$ be non-negative and invertible. Then for all $Q_1 = Q_1^* \in \mathbb{R}^{d \times d}$ invertible

\[ \sup_{t \in \mathbb{R}} \| e^{itQ_1H_0} \| < \infty. \]

**Corollary 4.2.** Assume, additionally to the assumptions in this section, $H(x) = H_0$ for some $H_0 \in \mathbb{R}^{d \times d}$ and all $x \in (a,b)$. Then the following conditions are equivalent:

(i) $A$ generates an exponentially stable $C_0$-semi-group.

(ii) For all $t \in \mathbb{R}$

\[ T_t := W \begin{pmatrix} e^{itP^{-1}_1H_0^{-1}} \\ 1_d \end{pmatrix} \]

is invertible with $\sup_{t \in \mathbb{R}} \| T_t^{-1} \| < \infty$.

**Proof.** Since $H$ is constant equal $H_0$, $\Phi_t(x) = e^{it(x-a)P^{-1}_1H_0^{-1}}$. Thus, by Proposition 4.1 [B] holds, which in turn leads to the applicability of Theorem 3.5. To obtain the claim it thus suffices to note that $t$ runs through all reals, making the prefactor $(b-a)$ superfluous, and to read off the equivalence stated in Theorem 3.5. \( \square \)

Before we put this characterisation result into perspective of the results available in the literature, we provide an example that confirms that invertibility for $T_t$ alone is not sufficient to deduce the uniform bound of the inverses.
Example 4.3. Consider the matrices $M := -\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (note that $\|M\| = 1$) and $\mathcal{H}(x) := \mathcal{H}_0 := \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}^{-1}$ for all $x \in (a, b) := (0, 1)$ as well as $P_1 = -1_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. By Theorem 2.4 (iv)

\[
W = \begin{pmatrix} -M \\ 1_2 \end{pmatrix}
\]

leads to $\mathcal{A}$ generate a contraction semi-group. We consider

\[
T_t = W \begin{pmatrix} e^{itP_1^{-1}\mathcal{H}_0^{-1}} \\ 1_2 \end{pmatrix} = \begin{pmatrix} -M \\ 1_2 \end{pmatrix} \begin{pmatrix} e^{-it\mathcal{H}_0^{-1}} \\ 1_2 \end{pmatrix} = -Me^{-it\mathcal{H}_0^{-1}} + 1_2
\]

and analyse the respective inverses for all $t \in \mathbb{R}$. We first show that $T_t$ is invertible. For this, we compute

\[
\det(T_t) = \det \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{-i\sqrt{2}t} \end{pmatrix} \right) = \det \left( 1 + \frac{1}{2}e^{-it} + \frac{1}{2}e^{-i\sqrt{2}t} - \frac{1}{2}e^{-i(1+\sqrt{2})t} \right) = 1 + \frac{1}{2} \left( e^{-it} + e^{-i\sqrt{2}t} \right).
\]

Since $|e^{-it}| = |e^{-i\sqrt{2}t}| = 1$, the determinant vanishes if and only if $e^{-it} = e^{-i\sqrt{2}t} = -1$. However, since $\sqrt{2}$ is irrational, this cannot happen for any $t \in \mathbb{R}$. Hence, $T_t$ is invertible for all $t \in \mathbb{R}$. Moreover, choosing $t_k := -3\pi k$ for $k \in \mathbb{Z}$, we obtain

\[
e^{-it_k} = -1, \quad e^{-i\sqrt{2}t_k} = e^{i\sqrt{2}\pi} e^{2\pi \sqrt{2}k}
\]

and since $\{e^{i2\pi \sqrt{2}k} : k \in \mathbb{Z}\}$ lies dense in the unit sphere $S_1$ (see, e.g., [5, Theorem 3.13]) the net $(\det(T_{t_k}))_{k \in \mathbb{Z}}$ accumulates at 0. Since $T_t$ itself is bounded in $t$, it follows that $(T_t^{-1})_{t \in \mathbb{R}}$ is unbounded.

Next, one could ask whether or not the exponential stability of the semi-group generated by $\mathcal{A}$ depends on the Hamiltonian density $\mathcal{H}$. [6, Section 5] provided an example confirming dependence. We reprove this result with the characterisation from above.

Example 4.4. Let $\theta \in \mathbb{R}$, $\theta > -1$. Consider the matrices $M := \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ (note that $\|M\| = 1$) and $\mathcal{H}(x) := \mathcal{H}_\theta := \begin{pmatrix} 1 + \theta & 0 \\ 0 & 1 \end{pmatrix}$ for all $x \in (a, b) := (0, 1)$ as well as $P_1 = 1_2$. By Theorem 2.4 (iv)

\[
W = \begin{pmatrix} 1_2 \\ -M \end{pmatrix}
\]

leads to $\mathcal{A}_\theta := P_1 \partial_x \mathcal{H}_\theta = \partial_x \mathcal{H}_\theta$ generate a contraction semi-group. In order to assess for which $\theta$ this semi-group is exponentially stable, using Corollary 4.2 we consider

\[
T_{t_\theta} := W \begin{pmatrix} e^{it(\theta-a)P_1^{-1}\mathcal{H}_\theta^{-1}} \\ 1_2 \end{pmatrix} = \begin{pmatrix} 1_2 \\ -M \end{pmatrix} \begin{pmatrix} e^{it\mathcal{H}_\theta^{-1}} \\ 1_2 \end{pmatrix}
\]

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\[ e^{it\mathcal{H}_{\theta}} - M = \left( \begin{array}{cc} e^{it\mathcal{H}_{\theta}} & 0 \\ 0 & e^{it} \end{array} \right) - \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array} \right) \]

and compute its determinant by

\[ \det T_{\theta}^t = \frac{1}{2} e^{it\mathcal{H}_{\theta}} - \frac{1}{2} e^{it} + e^{it\frac{2\pi i}{3}}. \]

Since \( T_{\theta}^t \) is uniformly bounded in \( t \), the uniform boundedness of \((T_{\theta}^t)^{-1}\) is equivalent to the uniform boundedness of \( \frac{1}{\det T_{\theta}^t} \). For \( \theta = 0 \), we obtain that \( T_{\theta}^t \) is invertible with bound for the inverse uniform in \( t \), leading to \( A_0 \) generating an exponentially stable semi-group. If \( t = 3\pi \) and \( \theta = 1/2 \), then \( T_{\theta}^t \) is not invertible and, hence, \( A_{1/2} \) does not generate an exponentially stable semi-group. A closer look at the proof of Theorem 3.4 reveals that \( 3\pi i \in \sigma_p(A_{1/2}) \) and hence, the generated semi-group is not even asymptotically stable.

5 A Sufficient Criterion for Exponential Stability

It is the aim of this section to put the characterisation into perspective of the literature. For this, throughout this section, we let

\[ A : \text{dom}(A) \subseteq H \to H \]
\[ u \mapsto P_1(\mathcal{H}u)' + P_0 \mathcal{H}u, \]

be a port-Hamiltonian operator as in (1). We recall \( \Phi_t \), the fundamental solution associated with

\[ u'(x) = P_1^{-1}(it\mathcal{H}(x)^{-1} - P_0)u(x) \quad (x \in (a,b)) \]

such that \( \Phi_t(a) = 1_d \) and condition \( \{B\} \) stating \( \sup_{t \in \mathbb{R}} \|\Phi_t\|_\infty < \infty \).

**Theorem 5.1.** Assume \( \{B\} \) and that \( A \) generates a contraction semi-group. If there exist \( c > 0 \) and \( \eta \in \{a,b\} \) such that for all \( u \in \text{dom}(A) \)

\[ \langle u, Au \rangle_H \leq -c\|\mathcal{H}u(\eta)\|^2, \quad (8) \]

then \( A \) generates an exponentially stable \( C_0 \)-semi-group.

**Remark 5.2.** In Subsection 6.2 we confirm that Theorem 5.1 implies Theorem 1.1. In fact, we shall provide criteria warranting \( \{B\} \) to be satisfied. One of them being \( \mathcal{H} \) to be of bounded variation, see also Section 6. Another criterion requires a structural hypothesis on the interplay of \( \mathcal{H} \) and \( P_1 \), which is independent of regularity, thus, showing that the statements in Theorem 5.1 and Theorem 1.1 are not equivalent, eventually proving that Theorem 5.1 is a proper generalisation of Theorem 1.1.

**Lemma 5.3.** Let \( A \) be as in Theorem 2.4. Then the mapping

\[ \text{tr} : \text{dom}(A) \to \mathbb{R}^d \quad u \mapsto Q_-(\mathcal{H}u)(b) + Q_+(\mathcal{H}u)(a) \]

is onto.
Proof. Let \( y \in \mathbb{R}^d \) and \( M \) as in Theorem 2.4 (iv). We define a function \( v_+: [a, b] \to E_+ \) by
\[
v_+(t) := \frac{t-a}{b-a} \left( P_1^+ \right)^{-\frac{1}{2}} \iota^*_+ My + \frac{t-b}{a-b} \left( P_1^- \right)^{-\frac{1}{2}} \iota^*_+ y
\]
and similarly \( v_-: [a, b] \to E_- \) by
\[
v_-(t) := \frac{t-a}{b-a} \left( P_1^- \right)^{-\frac{1}{2}} \iota^*_- My + \frac{t-b}{a-b} \left( P_1^- \right)^{-\frac{1}{2}} \iota^*_- y.
\]
Then clearly, \( v = \iota_+ v_+ + \iota_- v_- \in H^1([a, b]; \mathbb{R}^d) \) and it satisfies
\[
Q_- v(b) + Q_+ v(a) = \iota_- \iota^*_+ y + \iota_+ \iota^*_+ y = y,
\]
\[
Q_- v(a) + Q_+ v(b) = \iota_- \iota^*_+ My + \iota_+ \iota^*_+ My = My,
\]
which shows on the one hand that \( u := H^{-1}v \in \text{dom}(A) \) and on the other hand that \( \text{tr} \ u = y \).

It is possible to recast the condition in Theorem 5.1 as an inequality merely containing finite-dimensional spaces. This is a simple albeit decisive observation for the proof of Theorem 5.1

**Lemma 5.4.** Let \( A \) be as in Theorem 2.4 with \( M \) as in Theorem 2.4 (iv). Moreover let \( c > 0 \) and \( \eta \in \{a, b\} \). Then the following statements are equivalent:

(a) For all \( u \in \text{dom}(A) \)
\[
\langle Au, u \rangle_H \leq -c \| (Hu) (\eta) \|^2.
\]

(b) For all \( y \in \mathbb{R}^d \) either
\[
\|y\|^2 - \|My\|^2 \geq 2c \left( \|\iota^*_- My\|_{E_-}^2 + \|\iota^*_+ y\|_{E_+}^2 \right)
\]
if \( \eta = a \) or
\[
\|y\|^2 - \|My\|^2 \geq 2c \left( \|\iota^*_- y\|_{E_-}^2 + \|\iota^*_+ My\|_{E_+}^2 \right)
\]
if \( \eta = b \).

Proof. By Theorem 2.4 we have
\[
\langle Au, u \rangle_H = \frac{1}{2} \left( \| M (Q_- (Hu) (b) + Q_+ (Hu) (a)) \|^2 - \| Q_- (Hu) (b) + Q_+ (Hu) (a) \|^2 \right)
\]
\[
= \frac{1}{2} \left( \| \text{tr} \ u \|^2 - \| \text{tr} \ u \|^2 \right)
\]
for all \( u \in \text{dom}(A) \). Moreover,
\[
(Hu) (a) = \iota_+ (P_1^+)^{-\frac{1}{2}} \iota^*_+ Q_+ (Hu) (a) + \iota_- (P_1^-)^{-\frac{1}{2}} \iota^*_- Q_- (Hu) (a)
\]
\[
= \iota_+ (P_1^+)^{-\frac{1}{2}} \iota^*_+ \text{tr} \ u + \iota_- (P_1^-)^{-\frac{1}{2}} \iota^*_- M \text{tr} \ u,
\]
\[
(Hu) (b) = \iota_+ (P_1^+)^{-\frac{1}{2}} \iota^*_+ Q_+ (Hu) (b) + \iota_- (P_1^-)^{-\frac{1}{2}} \iota^*_- Q_- (Hu) (b)
\]
\[
= \iota_+ (P_1^+)^{-\frac{1}{2}} \iota^*_+ M \text{tr} \ u + \iota_- (P_1^-)^{-\frac{1}{2}} \iota^*_- \text{tr} \ u.
\]
Now the assertion follows from Lemma 5.3
Proposition 5.5. Let \( O \in L^\infty([a, b]; \mathbb{R}^{d \times d}) \) such that \( O(x) \) is self-adjoint for a.e. \( x \in [a, b] \). Moreover, let \( \Pi : [a, b] \to \mathbb{R}^{d \times d} \) be the fundamental solution to the differential equation

\[
v'(x) = P_1^{-1}(iO(x) - P_0)v(x)
\]

with \( \Pi(a) = 1_d \). Then the following statements hold:

(a) The matrix

\[
V := Q_+ + Q_+ \Pi(b)
\]

is invertible with \( \|V^{-1}\| \leq C_1 \|\Pi(b)\| + C_2 \), where \( C_1, C_2 \) are just depending on the values of \( P_1 \). Moreover, the matrix

\[
U := (Q_+ \Pi(b) + Q_+) V^{-1}
\]

is unitary.

(b) For \( y \in \mathbb{R}^d \) we have the following estimates

\[
\|y\|^2 \leq \|V\|^2 \left( \|\iota_+^* U y\|^2_{E_+} + \|\iota_-^* y\|^2_{E_-} \right)
\]

and

\[
\|y\|^2 \leq \|\Pi(b)^{-1}\|^2 \|V\|^2 \left( \|\iota_+^* y\|^2_{E_+} + \|\iota_-^* U y\|^2_{E_-} \right).
\]

Proof. (a) We recall from Lemma 3.2 that \( \Pi(b)^{-1} = P_1^{-1} \Pi(b)^* P_1 \) or equivalently \( P_1 = \Pi(b)^* P_1 \Pi(b) \). We set \( W := Q_+ \Pi(b) + Q_+ \) and compute

\[
V^* V = (Q_+ + \Pi(b)^* Q_+)(Q_+ + Q_+ \Pi(b))
\]

\[
= Q_+^2 + \Pi(b)^* Q_+^2 \Pi(b)
\]

\[
= Q_+^2 + P_1 + \Pi(b)^* (P_1 + Q_+^2) \Pi(b)
\]

\[
= Q_+^2 + \Pi(b)^* Q_+^2 \Pi(b)
\]

\[
= W^* W.
\]

In particular, we have \( \|V x\| = \|W x\| \) for each \( x \in \mathbb{R}^d \). Thus, if \( V x = 0 \), then \( W x = 0 \) and hence, \( Q_+ x = Q_+ V x = 0 \) as well as \( Q_+ x = Q_+ W x = 0 \). The latter gives \( P_1 x = (Q_+^2 - Q_+^2) x = 0 \) and thus, \( x = 0 \) showing the invertibility of \( V \). Moreover, for \( x \in \mathbb{R}^d \) we compute

\[
\|U x\|^2 = \langle WV^{-1} x, WV^{-1} x \rangle
\]

\[
= \langle V^{-1} x, W^* WV^{-1} x \rangle
\]

\[
= \langle V^{-1} x, V^* x \rangle = \|x\|^2,
\]

showing that \( U \) is unitary. It remains to prove the estimate for the norm of the inverse of \( V \). We set \( D := \iota_+^* \Pi(b) \iota_+ \) and \( C := \iota_-^* \Pi(b) \iota_+ \) compute for \( x \in E_+ \)

\[
\| (P_1^+) \frac{1}{2} x \|^2 = \langle P_1^+ x, x \rangle = \langle \iota_+^* P_1 \iota_+ x, x \rangle
\]

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\[
\langle \iota_+^* \Pi(b)^* P_1 \Pi(b) \rangle \iota_+ x, x \rangle \\
= \langle P_1 \Pi(b) \iota_+ x, \Pi(b) \iota_+ x \rangle \\
= \langle P_1 (\iota_+ \iota_+^* + \iota_- \iota_-^*) \Pi(b) \iota_+ x, \Pi(b) \iota_+ x \rangle \\
= \langle P_1^+ Dx, Dx \rangle - \langle P_1^- Cx, Cx \rangle \\
\leq \| (P_1^+)^{\frac{1}{2}} Dx \|^2 \leq \| (P_1^+)^{\frac{1}{2}} \|^2 \| Dx \|^2.
\]

Hence, \( D \) is invertible with \( \| D^{-1} \| \leq \frac{\| (P_1^+)^{-\frac{1}{2}} \|}{\| (P_1^+)^{\frac{1}{2}} \|}. \) Since \( V \) is unitarily equivalent (via the decomposition \( \mathbb{R}^d = E_+ \oplus E_- \)) to the matrix

\[
\begin{pmatrix}
(P_1^+)^{\frac{1}{2}} D & (P_1^+)^{\frac{1}{2}} B \\
0 & (P_1^-)^{-\frac{1}{2}}
\end{pmatrix}
\]

with \( B : = \iota_+^* \Pi(b) \iota_- \), its inverse is unitarily equivalent to

\[
\begin{pmatrix}
D^{-1} (P_1^+)^{-\frac{1}{2}} & -D^{-1} B (P_1^-)^{-\frac{1}{2}} \\
0 & (P_1^-)^{-\frac{1}{2}}
\end{pmatrix}
\]

and thus, the desired estimate for \( \| V^{-1} \| \) follows.

(b) Let \( y \in \mathbb{R}^d \). We compute

\[
\| \iota_+^* U y \|^2_{E_+} = \| \iota_+ (P_1^+)^{-\frac{1}{2}} \iota_+^* U y \|^2 \\
= \| \iota_+ \iota_+^* V^{-1} y \|^2
\]

and

\[
\| \iota_-^* y \|^2_{E_-} = \| \iota_- (P_1^-)^{-\frac{1}{2}} \iota_-^* y \|^2 \\
= \| \iota_- (P_1^-)^{-\frac{1}{2}} \iota_-^* V V^{-1} y \|^2 \\
= \| \iota_- \iota_-^* V^{-1} y \|.
\]

Consequently, we obtain

\[
\| \iota_+^* U y \|^2_{E_+} + \| \iota_-^* y \|^2_{E_-} = \| V^{-1} y \|^2 \geq \frac{1}{\| V \|^2} \| y \|^2,
\]

which proves the first estimate. Similarly, we compute

\[
\| \iota_-^* U y \|^2_{E_-} = \| \iota_- (P_1^-)^{-\frac{1}{2}} \iota_-^* U y \|^2 \\
= \| \iota_- \iota_-^* \Pi(b) V^{-1} y \|^2
\]

as well as

\[
\| \iota_+^* y \|^2_{E_+} = \| \iota_+ (P_1^+)^{-\frac{1}{2}} \iota_+^* y \|^2
\]

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\[ = \|t_+(P^+_1)^{-1} t_+^* V V^{-1} y\|^2 \]
\[ = \|t_+^* \Pi(b) V V^{-1} y\|^2. \]

Hence, we obtain
\[ \|t_+^* U y\|_{E_-}^2 + \|t_+^* y\|_{E_+}^2 = \|\Pi(b) V^{-1} y\|^2 \geq \frac{1}{\|\Pi(b)^{-1}\|^2 \|V\|^2 \|y\|^2}. \]

Hence, we obtain
\[ \|t_+^* U y\|_{E_-}^2 + \|t_+^* y\|_{E_+}^2 = \|\Pi(b) V^{-1} y\|^2 \geq \frac{1}{\|\Pi(b)^{-1}\|^2 \|V\|^2 \|y\|^2}. \]

**Proof of Theorem 6.1** By Theorem 3.3 we need to prove that
\[ T_t := W \left( \begin{array}{c} \Phi_t(b) \\ 1 \end{array} \right) \]
is invertible for each \( t \in \mathbb{R} \) with \( \sup_{t \in \mathbb{R}} \| T_t^{-1} \| < \infty \). By Theorem 2.4 the matrix \( W \) can be expressed by
\[ W = K \left( \begin{array}{cc} Q_+ - MQ_- & Q_- - MQ_+ \end{array} \right) \]
with \( \|M\| \leq 1 \) and \( K \) invertible given as in Theorem 2.4 (iv). Hence, \( T_t \) has the form
\[ T_t = K ((Q_+ - MQ_-) \Phi_t(b) + (Q_- - MQ_+)) = K (Q_- + Q_+ \Phi_t(b) - M(Q_- \Phi_t(b) + Q_+)) . \]

As in Proposition 5.5 (a) we set
\[ V_t := Q_- + Q_+ \Phi_t(b) \]
which is an invertible matrix by Proposition 5.5 (a) and
\[ U_t := (Q_- \Phi_t(b) + Q_+) V_t^{-1} \]
is unitary by Proposition 5.5 (a). Let \( y \in \mathbb{R}^d \). We prove the assertion by showing that there exists some \( \kappa > 0 \) independently of \( y \) with
\[ \|y\| \leq \kappa \|T_t y\| \quad (t \in \mathbb{R}). \]

Using the representation above, we have
\[ T_t y = K (V_t y - M(Q_+ \Phi_t(b) + Q_-) y) = K (1 - MU_t) V_t y. \]

Since \( \|y\| = \|V_t^{-1} V_t y\| \leq \|V_t^{-1}\| \|V_t y\| \) and \( \sup_{t \in \mathbb{R}} \|V_t^{-1}\| < \infty \) by Proposition 5.5 (a), it suffices to prove
\[ \|V_t y\| \leq \kappa \|\tilde{T}_t y\| \quad (t \in \mathbb{R}), \]
where \( \tilde{T}_t = K^{-1} T_t \). We employ Lemma 5.4 to obtain
\[ \|y\|^2 - \|My\|^2 \geq 2c \left( \|\Pi(b)^{-1}\|^2 \|V\|^2 \|y\|^2 + \|\Pi(b)^{-1}\|^2 \|V\|^2 \|y\|^2 \right) \]
if \( \eta = a \) or
\[
\|y\|^2 - \|My\|^2 \geq 2c \left( \|\iota_* y\|^2_{E_+} + \|\iota^*_+ My\|^2_{E_+} \right)
\]
if \( \eta = b \). Let us start with the case \( \eta = a \). Applying this inequality to \( U_i V_i y \) and using that \( V_i y = \tilde{T}_i y + MU_i V_i y \) as well as that \( U_i \) is isometric, we get
\[
\|V_i y\|^2 = \|\tilde{T}_i y\|^2 + 2(du_i V_i y, \tilde{T}_i y) + \|du_i V_i y\|^2 \\
\leq \|\tilde{T}_i y\|^2 + 2(du_i V_i y, \tilde{T}_i y) + \|U_i V_i y\|^2 - 2c \left( \|\iota_* du_i V_i y\|^2_{E_+} + \|\iota^*_+ U_i V_i y\|^2_{E_+} \right) \\
= \|\tilde{T}_i y\|^2 + 2(V_i y - \tilde{T}_i y, \tilde{T}_i y) + \|V_i y\|^2 - 2c \left( \|\iota_* V_i y - \iota^*_+ \tilde{T}_i y\|^2_{E_+} + \|\iota^*_+ U_i V_i y\|^2_{E_+} \right) \\
= -\|\tilde{T}_i y\|^2 + 2(V_i y, \tilde{T}_i y) + \|V_i y\|^2 - 2c \left( \|\iota_* V_i y - \iota^*_+ \tilde{T}_i y\|^2_{E_+} + \|\iota^*_+ U_i V_i y\|^2_{E_+} \right).
\]
Thus, we have
\[
0 \leq -\|\tilde{T}_i y\|^2 + 2(V_i y, \tilde{T}_i y) - 2c \left( \|\iota_* V_i y - \iota^*_+ \tilde{T}_i y\|^2_{E_+} + \|\iota^*_+ U_i V_i y\|^2_{E_+} \right) \\
= -\|\tilde{T}_i y\|^2 + 2(V_i y, \tilde{T}_i y) - 2c \left( \|\iota_* V_i y\|^2_{E_+} + \|\iota^*_+ \tilde{T}_i y\|^2_{E_+} - 2(\iota^*_+ (P_i^-)^{-1} \iota^*_+ \tilde{T}_i y, V_i y) + \|\iota^*_+ U_i V_i y\|^2_{E_+} \right) \\
\leq 2(V_i y, \tilde{T}_i y) - 2c \left( \|\iota_* V_i y\|^2_{E_+} - 2(\iota^*_+ (P_i^-)^{-1} \iota^*_+ \tilde{T}_i y, V_i y) + \|\iota^*_+ U_i V_i y\|^2_{E_+} \right),
\]
which yields
\[
\|\iota_* V_i y\|^2_{E_+} + \|\iota^*_+ U_i V_i y\|^2_{E_+} \leq \frac{1}{2c} \left( V_i y, \tilde{T}_i y + 2c(\iota^*_+ (P_i^-)^{-1} \iota^*_+ \tilde{T}_i y) \right) \\
\leq \frac{1}{2c} \left( \varepsilon \|V_i y\|^2 + \frac{1}{\varepsilon} \|\tilde{T}_i y + 2c(\iota^*_+ (P_i^-)^{-1} \iota^*_+ \tilde{T}_i y) \|^2 \right)
\]
for each \( \varepsilon > 0 \). Invoking Proposition 5.5 (b), we have
\[
\|V_i y\|^2 \leq \|V_i\|^2 \left( \|\iota_* V_i y\|^2_{E_+} + \|\iota^*_+ U_i V_i y\|^2_{E_+} \right) \leq \frac{\|V_i\|^2}{2c} \left( \varepsilon \|V_i y\|^2 + \frac{1}{\varepsilon} \|\tilde{T}_i y + 2c(\iota^*_+ (P_i^-)^{-1} \iota^*_+ \tilde{T}_i y) \|^2 \right).
\]
Hence, choosing \( \varepsilon := \frac{c}{\|V_i\|^2} \), we derive
\[
\|V_i y\|^2 \leq \left( \frac{\|V_i\|^2}{c} \right)^2 \|\tilde{T}_i y + 2c(\iota^*_+ (P_i^-)^{-1} \iota^*_+ \tilde{T}_i y) \|^2 \leq \kappa \|\tilde{T}_i y\|^2,
\]
for some \( \kappa > 0 \) independent of \( t \) (note that \( \text{sup} \|V_i\| < \infty \)). This proves the assertion for the case \( \eta = a \). If \( \eta = b \), an analogous computation gives
\[
\|V_i y\|^2 \leq -\|\tilde{T}_i y\|^2 + 2(V_i y, \tilde{T}_i y) + \|V_i y\|^2 - 2c \left( \|\iota_* V_i y - \iota^*_+ \tilde{T}_i y\|^2_{E_+} + \|\iota^*_+ U_i V_i y\|^2_{E_+} \right)
\]
and hence,
\[
0 \leq 2(V_i y, \tilde{T}_i y) - 2c \left( \|\iota_* V_i y\|^2_{E_+} - 2(\iota^*_+ (P_i^+)^{-1} \iota^*_+ \tilde{T}_i y, V_i y) + \|\iota^*_+ U_i V_i y\|^2_{E_+} \right).
\]
Thus, we infer
\[ \| \nu_+^* V_t y \|_{E^+}^2 + \| \nu_-^* U_t V_t y \|_{E^-}^2 \leq \frac{1}{c} (V_t y, \bar{T}_t y + 2c \lambda_+ (P_1^+)^{-1} \nu_+^* \bar{T}_t y) \]
\[ \leq \frac{1}{2c} \left( \frac{1}{c} \| V_t y \|^2 + \frac{1}{c} \| \bar{T}_t y + 2c \lambda_+ (P_1^+)^{-1} \nu_+^* \bar{T}_t y \|^2 \right) . \]

Hence, involving the second estimate in Proposition 5.5 (b), we infer
\[ \| V_t y \|^2 \leq \frac{\| \Phi_t(b)^{-1} \|^2 \| V_t \|^2}{2c} \left( \frac{1}{c} \| V_t y \|^2 + \frac{1}{c} \| \bar{T}_t y + 2c \lambda_+ (P_1^+)^{-1} \nu_+^* \bar{T}_t y \|^2 \right) \]
and choosing \( \epsilon := \frac{\| \Phi_t(b)^{-1} \|^2 \| V_t \|^2}{2c} \), we end up with
\[ \| V_t y \|^2 \leq \left( \frac{\| \Phi_t(b)^{-1} \|^2 \| V_t \|^2}{2c} \right) \frac{1}{c} \| \bar{T}_t y + 2c \lambda_+ (P_1^+)^{-1} \nu_+^* \bar{T}_t y \|^2 \leq \tilde{\kappa} \| \bar{T}_t y \|^2 , \]
for some \( \tilde{\kappa} > 0 \) independent of \( t \) (note that sup\(_t \| \Phi_t(b)^{-1} \| < \infty \) by Lemma 3.2).

We obtain another sufficient condition for exponential stability.

**Theorem 5.6.** Let \( \mathcal{A} \) be as in Theorem 2.4 and assume \((B)\). Moreover assume that \( W \) satisfies
\[ W \begin{pmatrix} P_1 & -P_1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W \begin{pmatrix} P_1 & -P_1 \\ 1 & 1 \end{pmatrix}^{-1} > 0 . \]

Then \( \mathcal{A} \) generates an exponentially stable \( C_0 \)-semi-group on \( H \).

**Proof.** Again, we need to prove that \( T_t := W \left( \Phi_t(b) \right) \) is invertible with sup\(_t \| T_t^{-1} \| < \infty \). By Lemma 2.3 we find a matrix \( M \) with \( \| M \| < 1 \) and an invertible matrix \( K \) such that
\[ W = K \begin{pmatrix} Q_+ - M Q_- & Q_- - M Q_+ \end{pmatrix} \]
and thus, \( T_t \) can be expressed as
\[ T_t = K (Q_+ \Phi_t(b) - M Q_- \Phi_t(b) + Q_- - M Q_+) = K (Q_- + Q_+ \Phi_t(b) - M (Q_- \Phi_t(b) + Q_+)) . \]

Using the matrices \( V_t := Q_- + Q_+ \Phi_t(b) \) and \( U_t := (Q_- \Phi_t(b) + Q_+) V_t^{-1} \), we infer that \( U_t \) is unitary and
\[ T_t = K (1 - MU_t) V_t . \]

Since \( K \) and \( V_t \) are both invertible and sup\(_t \| V_t^{-1} \| < \infty \) by Proposition 5.5 (a), it suffices to show that \( 1 - MU_t \) is invertible and its inverse is uniformly bounded in \( t \). This, however, follows from the Neumann series, since \( \| M \| < 1 \) and \( \| U_t \| = 1 \). Hence \( (1 - MU_t)^{-1} = \sum_{k=0}^{\infty} (MU_t)^k \) and
\[ \| (1 - MU_t)^{-1} \| \leq \sum_{k=0}^{\infty} \| M \|^k = \frac{1}{1 - \| M \|} . \]
6 The Condition (B)

This section is devoted to a discussion of condition (B). For this, throughout this section, we let \( a, b \in \mathbb{R} \), \( a < b \) and

\[
\mathcal{H} : [a, b] \to \mathbb{R}^{d \times d} \in L_\infty(a, b; \mathbb{R}^{d \times d})
\]

satisfying \( \mathcal{H}(x) = \mathcal{H}(x)^* \geq m \) for some \( m > 0 \). Furthermore, let \( P_1 = P_1^* \in \mathbb{R}^{d \times d} \) invertible. For \( t \in \mathbb{R} \) we define the fundamental matrix \( \Phi_t \in C([-\frac{1}{2}, \frac{1}{2}]; \mathbb{C}^{d \times d}) \) associated to

\[
\Phi_t(x) = P_1^{-1}(it\mathcal{H}(x)^{-1} - P_0)u(x) \in \mathbb{C}^d \quad (x \in (a, b))
\]

subject to \( \Phi_t(a) = 1_d \). In this section, we focus on the condition

\[
\sup_{t \in \mathbb{R}} \| \Phi_t \|_\infty < \infty.
\]

Whilst we do not yet know of any counterexamples, we managed to provide sufficient conditions on \( P_1 \) and \( \mathcal{H} \) warranting (B). These conditions either require some compatibility properties for \( P_1 \) and \( \mathcal{H} \) or regularity properties for \( \mathcal{H} \). In any case, these conditions are somewhat independent of \( P_0 \) as the next result confirms. For this, we use the short-hand \( \Phi_t, P_0 \) to denote the above fundamental solution for some fixed \( P_0 \).

**Proposition 6.1.** In the setting of this section we have

\[
\sup_{t \in \mathbb{R}} \| \Phi_{t, P_0} \|_\infty < \infty \iff \sup_{t \in \mathbb{R}} \| \Phi_{t, 0} \|_\infty < \infty.
\]

We recall an estimate of general nature.

**Lemma 6.2.** Let \( \mathcal{O} \in L_\infty(a, b; \mathbb{R}^{d \times d}) \) and \( \Psi \in C([a, b]; \mathbb{R}^{d \times d}) \) be the fundamental solution of

\[
u'(x) = \mathcal{O}(x)u(x)
\]

with \( \Psi(a) = 1_d \). If \( f \in L_1(a, b; \mathbb{R}^d) \), then any continuous solution, \( u \), of

\[
u'(x) = \mathcal{O}(x)u(x) + f(x)
\]

satisfies

\[
\| u(x) \| \leq \| \Psi \|_\infty \| u(a) \| + \| \Psi \|_\infty \| \Psi(\cdot)^{-1} \|_\infty \int_a^x |f(s)| \, ds.
\]

**Proof.** We employ the variations of constants formula

\[
u(x) = \Psi(x)u(a) + \int_a^x \Psi(x)\Psi(s)^{-1}f(s) \, ds,
\]

which can be readily verified. We, thus, estimate

\[
\| u(x) \| \leq \| \Psi \|_\infty \| u(a) \| + \| \Psi \|_\infty \| \Psi(\cdot)^{-1} \|_\infty \int_a^x |f(s)| \, ds.
\]
Next we address the fact that $P_0$ can, in fact, be assumed to be 0.

Proof of Proposition 6.1. Let $P_0 = -P_0^*$ and assume that $\sup_{t \in \mathbb{R}} \| \Phi_{t,P_0} \|_\infty < \infty$. Let $u_0 \in \mathbb{R}^d$ be a unit vector and consider the differential equation
\[ u'(x) = P_1^{-1} it \mathcal{H}(x)^{-1} u(x), \quad u(a) = u_0. \]  
Denote by $u_t$ its solution. Next, let $v_t$ be the unique solution of
\[ u'(x) = P_1^{-1}(it \mathcal{H}(x)^{-1} - P_0)u(x), \quad u(a) = u_0. \]  
Then
\[ u'(x) - v'(x) = P_1^{-1}(it \mathcal{H}(x)^{-1} - P_0)(u_t(x) - v_t(x)) + P_1^{-1}P_0u_t(x). \]  
By Lemma 6.2 and Lemma 3.2 we infer
\[ \|u_t(x) - v_t(x)\| \leq \|\Phi_{t,P_0}\|_\infty^2 \|P_1\| \|P_1^{-1}\|^2 \|P_0\| \int_a^x \|u_t(s)\| \, ds. \]  
Hence, using the assumption, we obtain
\[ \|u_t(x)\| \leq \|v_t(x)\| + \|u_t(x) - v_t(x)\| \leq \|\Phi_{t,P_0}\|_\infty + \|\Phi_{t,P_0}\|_\infty^2 \|P_1\| \|P_1^{-1}\|^2 \|P_0\| \int_a^x \|u_t(s)\| \, ds. \]  
Gronwall’s lemma thus confirms that
\[ \|u_t(x)\| \leq \|\Phi_{t,P_0}\|_\infty \exp \left( (b - a)\|\Phi_{t,P_0}\|_\infty^2 \|P_1\| \|P_1^{-1}\|^2 \|P_0\| \right). \]  
Computing the supremum over $t \in \mathbb{R}$ yields the assertion.

Next, let us assume that $\sup_{t \in \mathbb{R}} \| \Phi_{t,0} \|_\infty < \infty$. Similarly, as before, let $u_0 \in \mathbb{R}^d$ be a unit vector and let $u_t$ and $v_t$ be the respective solutions of (9) and (10). Considering
\[ u'(x) - v'(x) = P_1^{-1} it \mathcal{H}(x)^{-1}(u_t(x) - v_t(x)) + P_1^{-1}P_0v_t(x) \]  
and estimating as before, we eventually get the assertion as above. \qed

We may now turn to the structural assumption connecting the positive and negative spectral subspaces of $P_1$ and the mapping properties of $\mathcal{H}$.

6.1 A compatibility condition of $\mathcal{H}$ and $P_1$

We start off with a condition irrespective of any regularity of $\mathcal{H}$. We recall from Section 2
\[ E_+ = \text{lin}\{x \in \mathbb{R}^d : \exists \lambda > 0 : P_1 x = \lambda x\}, \]
\[ E_- = \text{lin}\{x \in \mathbb{R}^d : \exists \lambda < 0 : P_1 x = \lambda x\}. \]  
Then $E_+ \oplus E_- = \mathbb{R}^d$ in the sense of an orthogonal sum, since $P_1$ is self-adjoint and invertible.

The desired result reads as follows
Theorem 6.3. Assume that, for almost every \( x \in (a,b) \),
\[
\mathcal{H}(x)[E_+] \subseteq E_+.
\]
Then \( (B) \) holds.

Proof. By Proposition 6.1 without restriction, we may assume \( P_0 = 0 \). We consider the case \( E_+ = \mathbb{R}^d \) first. Let \( u_0 \in \mathbb{R}^d \) and let \( u_t \) be the solution of
\[
u'(x) = P_1^{-1}i\mathcal{H}(x)^{-1}u(x), \quad u(a) = u_0.
\]
Multiplying the equation by \( P_1^{1/2} \) we obtain
\[
\left(P_1^{1/2}u\right)'(x) = itP_1^{-1/2}\mathcal{H}(x)^{-1}P_1^{-1/2}\left(P_1^{1/2}u\right)(x).
\]
Hence, the equation satisfied by \( u_t \) is equivalent to \( w = P_1^{1/2}u_t \) solving
\[
w'(x) = itP_1^{-1/2}\mathcal{H}(x)^{-1}P_1^{-1/2}w(x) \quad w(a) = P_1^{1/2}u_0.
\]
By self-adjointness of \( \mathcal{H}(x) \) and \( P_1 \) it follows that \( tP_1^{-1/2}\mathcal{H}(x)^{-1}P_1^{-1/2} \) is self-adjoint. Thus, we deduce
\[
\frac{1}{2}\frac{d}{dx}\|w(x)\|^2 = \text{Re}\langle w(x), itP_1^{-1/2}\mathcal{H}(x)^{-1}P_1^{-1/2}w(x)\rangle = 0.
\]
Thus, \( \|w(x)\| \leq \|P_1^{1/2}u_0\| \), which proves the assertion for \( E_+ = \mathbb{R}^d \).

For the general case, it follows that the assumption guarantees that \( \mathcal{H}(x) \) reduces \( E_+ \) and, hence, also \( E_- \). The same properties follow for \( \mathcal{H}(x)^{-1} \). Hence, the system is actually block-diagonal, with each block similar to the type considered in the special case (for the \( E_- \)-block use the previous rationale multiplying by \( (P_1^-)^{1/2} \)). This shows the assertion. \( \square \)

Example 6.4. Let \( \mathcal{H} \) be scalar-valued; i.e., there exists bounded scalar function \( h : [a,b] \to \mathbb{R} \) such that \( \inf_{x \in [a,b]} h(x) > 0 \) with \( \mathcal{H}(x) = h(x)1_d \) for almost every \( x \in [a,b] \). Then the hypothesis in Theorem 5.3 is satisfied and hence, \( (B) \) holds for the corresponding \( (\Phi_t)^{1} \).

Proof of Theorem 1.3 By Theorem 5.3 we need to look at \( \Phi_t \) for scalar-valued \( \mathcal{H} \). Thus, let \( \mathcal{H} = h1_d \) for some scalar function \( h \). Then differentiation shows that
\[
\Phi_t(x) = e^{it\int_a^x h(\sigma)^{-1}d\sigma P_1^{-1}}.
\]
As \( h \) is scalar, by Example 6.4 \( t \mapsto \|\Phi_t\|_{\infty} \) is bounded. Since \( \Phi_t(b) = e^{it\int_a^b h(\sigma)^{-1}d\sigma P_1^{-1}} \) and \( \int_a^b h(\sigma)^{-1}d\sigma \neq 0 \), the second condition Theorem 1.3 is equivalent to the second one in Theorem 3.5. This shows the assertion. \( \square \)

With the results of this section, we can also prove another theorem from the introduction.

Proof of Theorem 1.4 The claim follows using Example 6.4 and Theorem 5.1. \( \square \)
The next example provides a set-up for which the Hamiltonian density can be as rough as \( L_\infty \), but the corresponding port-Hamiltonian operator still generates an exponentially stable semi-group.

Example 6.5. Let

\[
P_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} -1 & -\frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix}, \quad P_0 = 0
\]

and \( \mathcal{H} \) be scalar-valued. Then the corresponding port-Hamiltonian \( A \) generates a contraction semi-group. Furthermore, using the formula in (ii) in Theorem 1.3, we get

\[
\tau_t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} = \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix}
\]

Next,

\[
\det \tau_t = (\frac{1}{2}e^{-it} - 1)(\frac{1}{2} - e^{it}) + \frac{1}{4}e^{-it} = \frac{1}{2}e^{-it} - 1 + e^{it}.
\]

This expression is \( 2\pi \)-periodic. It thus, suffices to consider \( t \in [0, 2\pi] \). Since \( \text{Im} \det \tau_t = \frac{1}{2} \sin t = 0 \) if and only if \( t \in \{0, \pi\} \). For these values, however, we have \( \det \tau_0 = \frac{1}{2} \) and \( \det \tau_\pi = -\frac{3}{2} \). By continuity, \( \min_{t \in \mathbb{R}} |\det \tau_t| = \min_{t \in [0,2\pi]} |\det \tau_t| > 0 \). Thus, Cramer’s rule implies that \( \tau_t \) is invertible with uniform bound for the inverse. Hence, the corresponding port-Hamiltonian semi-group is exponentially stable.

6.2 A regularity condition on \( \mathcal{H} \)

The aim of this subsection is to show that Theorem 5.1 contains all the cases already contained in Theorem 1.1. For this recall the setting from the beginning of Section 6; also we briefly define what it means for a function to be of bounded variation.

Definition 6.6. Let \( I \subseteq \mathbb{R} \) be an open interval, \( \alpha \in L_1(I) \). Then \( \alpha \) is of bounded variation, if

\[
\sup \{ |\int_I \alpha(x)\phi'(x) \, dx|; \phi \in C^1_c(I), ||\phi||_\infty \leq 1 \} < \infty.
\]

By the Riesz–Markov representation theorem, there exists a unique signed Radon measure, \( D_x \alpha \), on the Borel sets of \( I \) such that

\[
-\int_I \alpha \phi' = \int_I \phi \, dD_x \alpha =: \langle D_x \alpha, \phi \rangle,
\]

for each \( \phi \in C^1_c(I) \). Moreover

\[
||D_x \alpha|| := |D_x \alpha|(I) = \sup \{ |\int_I \alpha(x)\phi'(x) \, dx|; \phi \in C^1_c(I), ||\phi||_\infty \leq 1 \},
\]

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where $|D_x \alpha|$ denotes the total variation of the measure $D_x \alpha$. The space 

$$BV(I) := \{ \alpha \in L_1(I); \alpha \text{ of bounded variation} \}$$

becomes a Banach space, if endowed with the norm given by 

$$\|\alpha\|_{BV} := \|\alpha\|_{L_1} + \|D_x \alpha\|.$$ 

A matrix-valued function $G: (a, b) \to \mathbb{R}^{d \times d}$ is called of bounded variation, if all its components are of bounded variation. In this case we set 

$$D_x G := (D_x G_{ij})_{i,j \in \{1, \ldots, n\}}$$

as a matrix-valued measure.

**Theorem 6.7.** Assume that $\mathcal{H}$ is of bounded variation. Then $[B]$ is satisfied.

This results immediately yields a proof for Proposition 4.1: 

**Proof of Proposition 4.1.** Any constant is of bounded variation. Thus, the result follows from Theorem 6.7. \qed

For the proof of Theorem 6.7, some preliminaries are in order. The material is widely known. We shall, however, summarise and prove some particular findings needed in the present situation. Note that the author of [11] focuses on right-continuous instead of left-continuous functions. The arguments, however, are similar in either cases so we still quote the results without proof.

**Theorem 6.8 ([11, Theorem 7.2 and Theorem 5.13]).** Let $I = (a, b) \subseteq \mathbb{R}$ be an interval and $\alpha \in L_1(I)$. Then the following conditions are equivalent:

(i) $\alpha \in BV(I)$,

(ii) there exists a left-continuous representative, $\alpha_{lc}$, of $\alpha$ such that 

$$\text{var}(\alpha_{lc}) := \sup_{a < t_0 < t_1 < \cdots < t_n < b} \sum_{j=1}^{n} |\alpha(t_j) - \alpha(t_{j-1})| < \infty.$$ 

In either case, $\text{var}(\alpha_{lc}) = \|D_x \alpha\|$ and $\alpha_{lc}$ may be chosen according to 

$$\alpha_{lc}(x) := \begin{cases} D_x \alpha([t_0, x)), & x > t_0, \\ -D_x \alpha([x, t_0)), & x \leq t_0, \end{cases}$$

for a Lebesgue point $t_0 \in (a, b)$ of $\alpha$.

An immediate consequence of the previous theorem is that if the Hamiltonian energy density $\mathcal{H}$ is of bounded variation, the same is true for $\mathcal{H}^{-1}$.

**Proposition 6.9.** Let $\mathcal{O}: (a, b) \to \mathbb{R}^{d \times d} \in L_\infty(a, b; \mathbb{R}^{d \times d})$ be of bounded variation and assume that $\text{Re}\, \mathcal{O}(x) \geq m$ for some $m > 0$ and a.e. $x \in (a, b)$. Then $x \mapsto \mathcal{O}(x)^{-1}$ is of bounded variation.
We estimate Hence, by Theorem 6.8, the assertion follows.

where \( \alpha \in \text{Theorem 6.10} \).

Let \( \phi \in C_c^1(I) \). Assume that \( \alpha \) is continuously differentiable. Then we compute

\[
- \int_I \alpha u \phi' \, d\lambda = - \int_I \alpha ((u\phi)' - u'\phi) \, d\lambda \\
= \langle D_x\alpha, u\phi \rangle + \int_I u'\alpha \phi \, d\lambda \\
= \langle uD_x\alpha + u'\alpha \, d\lambda, \phi \rangle
\]

We estimate

\[
| \int_I \alpha u \phi' \, d\lambda | \leq \|D_x\alpha\| \|u\phi\|_\infty + \|\alpha\|_\infty \|u'\phi\|_{L^1}
\]
\[
\leq \|D_x\alpha\|\|u\|_\infty \|\phi\|_\infty + \|\alpha\|_\infty \|u'\|_{L_2} \|\phi\|_{L_2}
\]
\[
\leq \|D_x\alpha\|\|u\|_{H^1} \|\phi\|_\infty + \|\alpha\|_\infty \|u\|_{H^1} \sqrt{b-a} \|\phi\|_\infty
\]
\[
= (\|D_x\alpha\|\|u\|_{H^1} + \|\alpha\|_\infty \|u\|_{H^1} \sqrt{b-a}) \|\phi\|_\infty.
\]

Hence, \( \alpha u \in BV(I) \) and
\[
\|D_x(\alpha u)\| \leq \|D_x\alpha\|\|u\|_{H^1} + \|\alpha\|_\infty \|u\|_{H^1} \sqrt{b-a}.
\]
Moreover, we estimate
\[
\|\alpha u\|_{L^1} \leq \|u\|_\infty \|\alpha\|_{L^1} \leq c\|u\|_{H^1} \|\alpha\|_{L^1},
\]
by the Sobolev embedding theorem. In particular, let \((u_n)_n\) be a sequence of continuously differentiable functions converging to \(u\) in \(H^1\). Then, by the estimates above, \(\alpha u \in BV(I)\). Moreover, using the product formula from the beginning of the proof for \(u_n\) and letting \(n \to \infty\), we infer
\[
D_x(\alpha u) = uD_x\alpha + u'\alpha \, d\lambda.
\]

Next, we recall Gronwall’s inequality for locally finite measures.

**Definition 6.11.** Let \(I \subseteq \mathbb{R}\) be an interval. We call a Borel measure \(\mu\) on \(I\) locally finite, if for all compact \(K \subseteq I\), \(\mu(K) < \infty\).

**Theorem 6.12** ([18, Lemma A.1]). Let \(I \subseteq \mathbb{R}\) be an interval. Let \(u: I \to \mathbb{R}\) and \(\alpha: I \to [0, \infty)\) measurable. Assume that for a locally finite Borel measure \(\mu\) on \(I\), we have \(u\) is locally integrable and there exists \(a \in I\) such that for all \(t > a\)
\[
u(t) \leq \alpha(t) + \int_{(a,t)} |u(s)| \, d\mu(s).
\]
Then
\[
u(t) \leq \alpha(t) + \int_{(a,t)} \alpha(s) \exp(\mu((s,t))) \, d\mu(s) \quad (t > a).
\]

Now, we are in the position to show the main result of this subsection.

**Proof of Theorem 6.7** Let \(v \in H^1(a, b)^d\) and \(t \in \mathbb{R}\) satisfy
\[
v'(x) = itP_1^{-1}H(x)^{-1}v(x), \quad (x \in (a, b)).
\]
By Proposition 6.9 \(O: x \mapsto H(x)^{-1}\) is of bounded variation. Referring to Theorem 6.8 without loss of generality, we may assume that \(O\) is left-continuous. Next, note that
\[
\langle v(x), O(x)v(x) \rangle_{\mathbb{C}^d} = \sum_{j=1}^d \sum_{k=1}^d v_j(x)^* O_{jk}(x)v_k(x)
\]
\[
= \langle O(x), v_j(x)^* v(x)^T \rangle_{\mathbb{C}^{d \times d}}
\]

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for all \( x \in (a, b) \). Thus, by Theorem 6.10 we obtain
\[
D_x \langle v, \mathcal{O}v \rangle_{C^d} = \langle \mathcal{O}, \partial_x ((v_j^*)^+ v^T) \rangle d\lambda + \langle D_x \mathcal{O}, vv^T \rangle \\
= (\langle v', \mathcal{O}v \rangle + \langle v, \mathcal{O}v' \rangle) d\lambda + \langle D_x \mathcal{O}, vv^T \rangle \\
= \left( \langle (itP_1^{-1} \mathcal{O}v, \mathcal{O}v) + (\mathcal{O}v, itP_1^{-1} \mathcal{O}v) \rangle d\lambda + \langle D_x \mathcal{O}, vv^T \rangle \right) \\
= \langle D_x \mathcal{O}, vv^T \rangle.
\]

Since \( v \) is continuous by the Sobolev embedding theorem, \( \langle v, \mathcal{O}v \rangle_{C^d} \) is also left-continuous. For \( s, t \in (a, b) \) with \( s < t \) we therefore obtain by Theorem 6.8
\[
\langle v(t), \mathcal{O}(t)v(t) \rangle_{C^d} = \langle v(s), \mathcal{O}(s)v(s) \rangle + D_x \langle v, \mathcal{O}v \rangle_{C^d([s, t])} \\
= \langle v(s), \mathcal{O}(s)v(s) \rangle + \sum_{j=1}^{d} \sum_{k=1}^{d} \int_{[s,t]} v_j^*(\sigma) v_k(\sigma) dD_x \mathcal{O}_{jk}(\sigma).
\]

Hence, using our general assumption on boundedness of \( \mathcal{H} \) and \( \mathcal{H}(x) \geq m \), for some \( c > 0 \) we estimate
\[
c \|v(t)\|^2 \leq \langle v(t), \mathcal{O}(t)v(t) \rangle \\
\leq \|v(s)\|^2 \left( \frac{1}{m} + \frac{1}{2} \int_{[s,t]} \|v(\sigma)\|^2 d\mu(\sigma) \right),
\]
where \( \mu := \sum_{j,k=1}^{d} |D_x \mathcal{O}_{jk}| \). Note that \( \mu \) is a finite measure on \((a, b)\). Using Theorem 6.12 we get
\[
\|v(t)\|^2 \leq \frac{1}{cm} \|v(s)\|^2 \left( 1 + \frac{1}{2c} \int_{[s,t]} e^{\frac{1}{2c}\mu((\sigma,t))} d\mu(\sigma) \right) \\
\leq \frac{1}{cm} \|v(s)\|^2 \left( 1 + \frac{1}{2c} e^{\frac{1}{2c}\mu((a,b))} \mu((a,b)) \right),
\]

Letting now \( s \to a \) in the last inequality, we derive
\[
\|v(t)\| \leq C \|v(a)\|^2 \quad (t \in [a, b])
\]
for some constant \( C > 0 \), proving the desired result. \( \square \)

We finally obtain a proof of Theorem 1.1.

**Proof of Theorem 1.1.** The statement is an immediate consequence of Theorem 5.1 and Theorem 6.7. \( \square \)

### 7 Conclusion

We presented a new characterisation of exponential stability for port-Hamiltonian systems. The characterisation works only if a certain family of fundamental solutions of a non-autonomous ODE-system is uniformly bounded. Whether or not this boundedness is needed lies beyond the scope of this article and can be considered an open problem for the time-being. We emphasise that as the strategy above uses existence and boundedness of the resolvent of the generator on the imaginary axis it might be possible, invoking results such as [1, 4], to show explicit algebraic decay for certain set-ups. Whether at all these set-ups exists and how they maybe characterised will be addressed in future work.
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