REMARKS ON GRAVITY, ENTROPY, AND INFORMATION

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Abstract. This is a partially survey collection of material on gravity, entropy, and information with some new heuristic results related to the WDW equation.

Contents

1. INTRODUCTION 1
2. DEDONDER-WEYL THEORY 2
3. COVARIANT QFT 3
4. EXACT UNCERTAINTY AND WDW 10
   4.1. EXACT UNCERTAINTY 11
   4.2. WDW 13
   4.3. SOME FUNCTIONAL CALCULUS 15
5. REMARKS ON ENTROPY 17
   5.1. ENTROPY AND THE EINSTEIN EQUATIONS 19
6. WDW AND THE EINSTEIN EQUATIONS 22
   6.1. MULTIFINGERED TIME 27
7. TIME 30
   7.1. EXTRINSIC CURVATURE AND TIME 32
References 35

1. INTRODUCTION

We gather here some material relating covariant quantum field theory (QFT) à la deDonder-Weyl, Bohmian mechanics, the WDW equation, differential entropy, and Fisher information. Some of this is speculative and/or heuristic but the themes suggested seem worth pursuing. In particular

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one sees apparently deep connections between physics and information theory, which theme was enunciated many years ago by B. Frieden [24], J.A. Wheeler, and others.

2. DE DONDER-WEYL THEORY

We begin with a sketch of the deDonder-Weyl theory and some applications following [14, 43, 44, 55, 56, 57] (cf. also [33, 38, 39, 40, 51, 53, 68, 75]). First from [43, 44] we give some information and discussion of deDonder-Weyl (dDW) theory. Recall that the dDW formula for the classical Euler-Lagrange (EL) equations takes the form

\[ \frac{\partial p_i^a}{\partial x^i} = -\frac{\partial H}{\partial y^a} \quad \frac{\partial y^a}{\partial x^i} = \frac{\partial H}{\partial p_i^a} \quad p_i^a = \frac{\partial L}{\partial (\partial_i y^a)} \quad H = p_i^a \partial_i y^a - L \]

The (manifest) covariance simply means that space and time variables enter the theory on a completely equal footing. The HJ equation of dDW theory is

\[ \frac{\partial}{\partial t} S + H \left( y^a, \pi^a \right) = 0 \]

Then in [44] one writes for \( \Sigma \) a Cauchy surface, \( \Sigma(y = y(x), t = \text{constant}) \)

\[ S = \int_{\Sigma} (S^\mu \omega_\mu) |_{\Sigma} = \int_{\Sigma} S^i |_{\Sigma} \]

where \( \omega_\mu = \partial_\mu \left( dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^t \right) \) and \( S^i(y^a = y^a(x), x, t) \) is a restriction of the time-like component of the \( S^\mu(y^a, x^\mu) \) to \( \Sigma \).

REMARK 2.1. We recall (cf. [52]) for a vector \( X = X^j \partial_j \) one defines

\[ X | f = 0; \quad X | \omega^1 = \omega^1(X) = \langle \omega^1, X \rangle; \quad X | \omega_i dx^i = \]

\[ = (X | \omega_i) dx^i + \omega_i (X | dx^i) = \omega_i dx^i (X) = \omega_i X^i \]

Also the Lie derivative is defined via (2B) \( \mathcal{L}_X \omega = X | \omega - d(X | \omega). \)

We recall now that canonical field quantization for say \( L = (1/2) \partial_\mu y \partial^\mu y - V(y) \) involves \( \pi(x) = (\partial L / \partial (\partial_t y(x))) \) with

\[ H = \int d^3x (\partial_t y(x) \pi(x) - L) = \int d^3x \left( \frac{1}{2} \pi^2(x) + (\partial_t y(x))^2 + V(x) \right) \]

for \( \pi(x) = \partial_t y(x) \). Then one takes \( \hat{\pi}(x) = -i \delta / \delta y(x) \) and \( i \partial_t \psi = \hat{H} \psi \) where

\[ \hat{H} = \frac{1}{2} \int dx \left( -\frac{\delta^2}{\delta y^2(x)} + (\partial_t y(x))^2 + V(x) \right) \]
Now going to dDW theory look at
\[ (2.7) \quad \partial_t S^t + \partial_i S^i + \frac{1}{2} \partial_j S^\mu \partial_j S^\mu + V = 0 \]
and the standard functional HJ equation reads (cf. (2.2))
\[ (2.8) \quad \partial_t S + \frac{1}{2} \int d\mathbf{x} \left[ \left( \frac{\delta S}{\delta y(\mathbf{x})} \right)^2 + (\partial_t y(\mathbf{x}))^2 + 2V \right] = 0 \]
From (2.3) one obtains then
\[ (2.9) \quad \partial_t S = \int d\mathbf{x} \partial_t S^i|_{\Sigma}; \quad \frac{\delta S}{\delta y(\mathbf{x})} = \partial_y S^i|_{\Sigma} \]
The equation for \( S^\mu|_{\Sigma} \) can be obtained from the dDW HJ equation by
noticing that when acting on \( S^\mu|_{\Sigma} \) the spatial derivative \( \partial_i \) turns into the
total derivative \( (d/dx^i) = \partial_i + \partial_i y(\mathbf{x}) \partial_y \), the last term of which should
be compensated. Thus the equation for \( S^\mu|_{\Sigma} \) assumes the form (signature
\((- - + +))\)
\[ (2.10) \quad \partial_t S^\mu|_{\Sigma} + \frac{d}{dx^i} S^i|_{\Sigma} - \partial_i y(\mathbf{x}) \partial_y S^i|_{\Sigma} + \frac{1}{2} (\partial_y S^i|_{\Sigma})^2 - \frac{1}{2} (\partial_y S^i|_{\Sigma})^2 + V = 0 \]
Substituting \( \partial_i S^i|_{\Sigma} \) from this equation into the right side of (2.9A) and
using (2.9B) one obtains
\[ (2.11) \quad \partial_t S + \int d\mathbf{x} \left( \frac{1}{2} \left( \frac{\delta S}{\delta y(\mathbf{x})} \right)^2 + \frac{d}{dx^i} S^i|_{\Sigma} - \partial_i y(\mathbf{x}) \partial_y S^i|_{\Sigma} - \frac{1}{2} (\partial_y S^i|_{\Sigma})^2 + V \right) = 0 \]
The second term under the integral does not contribute since it is a total
divergence (this point may need further clarification in some gravitational
models). The third and forth terms together lead to \( (1/2)(\partial_i y(\mathbf{x}))^2 \) because
in dDW theory \( \partial_y S^i = p_i^\dagger \) and for a scalar field \( p_i^\dagger|_{\Sigma} = -\partial_i y(\mathbf{x}) \). We have
therefore obtained the functional HJ equation (10.8) as a consequence of
the dDW HJ equation (10.7) restricted to the Cauchy surface \( \Sigma \) and a
natural hypothesis (10.3) on relating the HJ eikonal functional \( S \) to the
dWD eikonal functions \( S^\mu|_{\Sigma} \).

3. COVARIANT QFT

One shows here following [55, 56, 57, 58] that the deterministic evolution
of quantum fields is a covariant version of the Bohmian hidden variable
interpretation of quantum field theory (QFT). The deDonder-Weyl (dDW)
covariant canonical formalism is exploited in a novel manner and a co-
variant Bohmian formulation is not postulated but derived; this suggests
that the Bohmian interpretation could be the missing link between QM
and GR. The dDW formalism treats space and time variables on an equal
footing. Thus given a Lagrangian \( L(y^a, \partial_\mu y^a, x^\nu) \) with field variables \( y^a \)
and $\mu, \nu = 1, \cdots, n$) one defines polynomials $p_\alpha^\mu = \partial L/\partial (\partial_\mu y^\alpha)$ and a dDW Hamiltonian (cf. 2.1) $H = \partial_\mu y^\alpha p_\alpha^\mu - L$ such that the Euler-Lagrange (EL) field equations take the form (cf. (2.1))

$$
\partial_\mu y^\alpha = \frac{\partial H}{\partial p_\alpha^\mu}; \quad \partial_\mu p_\alpha^\mu = -\frac{\partial H}{\partial y^\alpha}
$$

The fields are treated as a multitime dDW system evolving in space and time (not just in time) and everything is manifestly covariant. Consequently this is an ideal framework for quantum gravity. Following now [55] (cf. also [14]) one writes (using only one field $\phi$ for illustration)

$$
\mathfrak{A} = \int d^4x \mathfrak{L}; \quad \mathfrak{L} = \frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) - V(\phi); \quad \pi^\mu = \frac{\partial \mathfrak{L}}{\partial (\partial_\mu \phi)} = \partial^\mu \phi
$$

The covariant canonical equations of motion and dDW Hamiltonian (not related to the energy density) are

$$
\partial_\mu \phi = \frac{\partial S}{\partial \pi^\mu}; \quad \partial_\mu \pi^\mu = -\frac{\partial S}{\partial \phi}; \quad S(\pi^\mu, \phi) = \pi^\mu \partial_\mu \phi - \mathfrak{L} = \frac{1}{2}\pi^\mu \pi_\mu + V
$$

By introducing the local vector $S^\mu(\phi(x), x)$ the dynamics can also be described by the covariant dDW Hamilton-Jacobi equation and equation of motion

$$
S\left(\frac{\partial S^\mu}{\partial \phi}, \phi\right) + \partial_\mu S^\mu = 0; \quad \partial_\mu \phi = \pi^\mu = \frac{\partial S^\mu}{\partial \phi}
$$

Note here that $\partial_\mu$ acts only on the second argument of $S^\mu(\phi(x), x)$ and the corresponding total derivative is $d_\mu = \partial_\mu + (\partial_\mu \phi)(\partial/\partial \phi)$. To describe the relation between the covariant HJ equation and the conventional HJ equation one writes from (3.3)

$$
\frac{1}{2} \frac{\partial S_\mu}{\partial \phi} \frac{\partial S^\mu}{\partial \phi} + V + \partial_\mu S^\mu = 0; \quad \frac{1}{2} \frac{\partial S_\mu}{\partial \phi} \frac{\partial S^\mu}{\partial \phi} = \frac{1}{2} \frac{\partial S^0}{\partial \phi} \frac{\partial S^0}{\partial \phi} + \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi)
$$

where $i = 1, 2, 3$ are the space indices and one notes also that (3A) $\partial_\mu S^\mu = \partial_0 S^0 + d_i S^i - (\partial_i \phi)(\partial^i \phi)$. Now introduce the quantity $\mathcal{S} = \int d^3x S^0$ leading to

$$
\frac{\partial S^0(\phi(x), x)}{\partial \phi(x)} = \frac{\delta \mathcal{S}([\phi(x, t)], t)}{\delta \phi(x, t)}; \quad \frac{\delta}{\delta \phi(x, t)} = \frac{\delta}{\delta \phi(x)} \bigg|_{\phi(x)=\phi(x,t)}
$$

Putting the second equation of (3.5) and (3.6) into the first equation of (3.5) yields upon integration then (cf. (2.8))

$$
\int d^3x \left[ \frac{1}{2} \left( \frac{\delta \mathcal{S}}{\delta \phi(x,t)} \right)^2 + \frac{1}{2}(\nabla \phi)^2 + V(\phi) \right] + \partial_t \mathcal{S} = 0
$$

which is the standard non-covariant HJ equation (recall here $\partial^\mu \phi = \partial S^i/\partial \phi$ and see Section 2 for a more detailed derivation). The time evolution of the
field $\phi(x,t)$ is now given via (3B) $\partial_t \phi(x,t) = \delta S/\delta \phi(x,t)$ (from the time component in (3.4)) and one notes that in deriving (3.7) it was necessary to use the space part of the equations of motion in (3.4); this will be important in the quantum extension below.

We recall that QFT can be formulated in the Schrödinger picture via

(3.8) $\hat{H} \psi = i\hbar \partial_t \psi; \quad \hat{H} = \int d^3x \left[ -\frac{\hbar^2}{2} \left( \frac{\delta}{\delta \phi(x)} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right]$

Write now (3C) $\psi([\phi(x)],t) = \mathcal{R}([\phi(x)],t) \exp(iS([\phi(x)],t)/\hbar)$ and (3.8) will be equivalent to a set of two real equations

(3.9) $\int d^3x \left[ \frac{1}{2} \left( \frac{\delta S}{\delta \phi(x)} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi) + \Omega \right] + \partial_t S = 0;$

$\int d^3x \left[ \frac{\delta \mathcal{R}}{\delta \phi(x)} \frac{\delta S}{\delta \phi(x)} + J \right] + \partial_t \mathcal{R} = 0; \quad \Omega = -\frac{\hbar^2}{2} \frac{\delta^2 \mathcal{R}}{\delta \phi^2(x)}; \quad J = \frac{\mathcal{R}}{2} \frac{\delta^2 S}{\delta \phi^2(x)}$

The second equation is equivalent to

(3.10) $\partial_t \mathcal{R}^2 + \int d^3x \frac{\delta}{\delta \phi(x)} \left( \mathcal{R}^2 \frac{\delta S}{\delta \phi(x)} \right) = 0$

and this represents the unitarity of the theory since it provides a norm (3D) $\int [d\phi(x)]_\psi \bar{\psi} \psi = \int [d\phi(x)] \mathcal{R}^2$ that does not depend on time (some argument is needed here). One must also stipulate that the quantity $\exp(iS/\hbar)$ be single valued. This formulation also suggests an interesting Bohmian interpretation stating that the quantum fields have a deterministic time evolution given by the classical equation (3B) and the statistical predictions will be equivalent to those of the conventional interpretation (cf. [14, 55] for discussion). Comparing now (3.9) with (3.7) we see that the quantum field satisfies an equation similar to the classical one except for the additional nonlocal quantum potential $\Omega$. There are no contradictions here with the Bell theory (which specifies local hidden variables) and the quantum equation of motion will be

(3.11) $\partial^\mu \partial_\mu \phi + \frac{\partial V(\phi)}{\partial \phi} + \frac{\delta Q}{\delta \phi(x,t)} = 0$

where $Q = \int d^3x \Omega$.

We will now need a covariant version of the Bohm theory which goes as follows. One wants first a quantum version of the classical covariant dDW HJ equation in (3.5) and one formulates the classical version first in a somewhat different way. Thus let $A([\phi],x)$ be a functional of $\phi$ and a function of $x$; define then ($\star$) $dA/d\phi(x) = \int d^4x' (\delta A([\phi],x')/\delta \phi(x))$ where $\delta/\delta \phi(x)$ is a spacetime functional derivative. If $A([\phi],x) = A(\phi(x),x)$ (local functional)
\[
\frac{\delta A([\phi], x')}{\delta \phi(x)} = \frac{\delta A([\phi], x')}{\delta \phi(x, x')} \delta(x' - x') \frac{\partial A([\phi], x)}{\partial \phi(x)} = \int d^3x' A([\phi], x', x^0) \frac{\delta}{\delta \phi(x, x^0)} \frac{\partial A([\phi], x)}{\partial \phi(x)} \frac{\delta}{\delta \phi(x, x^0)}
\]

One can write the HJ equation in (3.5) as

\[
\frac{1}{2} \frac{dS_\mu}{d\phi} \frac{dS^\mu}{d\phi} + V + \partial_\mu S^\mu = 0
\]

which is appropriate for the quantum modification. Similarly the classical equations of motion in (3.4) can be written as (3F) \( \partial^\mu \phi = dS^\mu / d\phi \). This leads now to the quantum analogue of the classical covariant equation, namely

\[
\frac{1}{2} \frac{dS_\mu}{d\phi} \frac{dS^\mu}{d\phi} + V + \Omega + \partial_\mu S^\mu = 0
\]

(cf. [3]). Here (3.14) is manifestly covariant provided that \( \Omega \) in (3.9) can be written in a covariant form (see below for this). One can then show that (3.14) implies (3.9) provided \( S^0 \) is local in time (so that (3.12) can be used - cf. (3.6)) and \( S^i \) must be completely local so that \( dS^i / d\phi = \partial S^i / \partial \phi \) and hence \( d_i S^i = \partial_i S^i + (\partial_i \phi)(dS^i / d\phi) \) (cf. (3.4)). Thus in the covariant quantum theory based on the dDW formalism one must require the validity of (3F) and this is nothing but a covariant version of the Bohmian equations of motion written for an arbitrarily nonlocal \( S^\mu \). To produce covariant versions of the remaining terms in (3.9) introduce a vector \( R^\mu([\phi], x) \) which generates a preferred foliation of spacetime with \( R^\mu \) normal to the leaves of the foliation. Then introduce (3G) \( \mathcal{R}([\phi], \Sigma) = \int_{\Sigma} d\Sigma_\mu R^\mu \) where \( \Sigma \) is a 3-D leaf generated by \( R^\mu \). Similarly a covariant version of \( \mathcal{S} \) is (3H) \( \mathcal{S}([\phi], \Sigma) = \int_{\Sigma} d\Sigma_\mu S^\mu \) with \( \Sigma \) again generated by \( R^\mu \). The covariant version of (3C) is then (3I) \( \psi([\phi], \Sigma) = \mathcal{R}([\phi], \Sigma)e^{i\mathcal{S}([\phi], \Sigma)/\hbar} \) and for \( R^\mu \) one postulates the equation

\[
\frac{dR^\mu}{d\phi} \frac{dS^\mu}{d\phi} + 3 + \partial_\mu S^\mu = 0
\]

In this manner a preferred foliation emerges dynamically as a foliation generated by the solution \( R^\mu \) of (3.15) and (3.14). Note that \( R^\mu \) plays no classical role and the existence of a preferred foliation is a purely quantum effect. Now the relation between (3.15) and (3.9) is obtained by assuming that nature has chosen a solution of the form \( R^\mu = [R^0, 0, 0, 0] \) where \( R^0 \) is local in time and by integration of (3.15) over \( d^3x \) with \( S^0 \) local one
sees that (3.15) is truly a covariant substitute for (3.9). Finally one has covariant versions of \( \mathcal{Q} \) and \( \mathcal{J} \) in the form

\[
\mathcal{Q} = -\frac{\hbar^2}{\mathcal{R}} \frac{\delta^2 \mathcal{R}}{\delta \Sigma \phi^2(x)}; \quad \mathcal{J} = \frac{\mathcal{R}}{2} \frac{\delta^2 \mathcal{S}}{\delta \Sigma \phi^2(x)}
\]

where \( \delta/\delta \Sigma \phi(x) \) is a version of (3.6) in which \( \Sigma \) is generated by \( R^\mu \). Here \( \Sigma \) depends on \( x (x \in \Sigma) \) and \( \Sigma \) is kept fixed in the variation \( \delta \Sigma \phi(x) \). Thus (3.14)-(3.15) with (3.16) represent a covariant substitute for the functional SE (3.8) equivalent to (3.9). The covariant Bohmian equations (3F) imply a covariant version of (3.11), namely

\[
\partial^\mu \partial_\mu \phi + \frac{\partial V}{\partial \phi} + \frac{d\mathcal{Q}}{d\phi} = 0
\]

Since the last term can also be written as \( \delta (\int d^4x \mathcal{Q})/\delta \phi(x) \) the equation of motion (3.17) can be obtained by varying the quantum action (3J) \( \mathcal{A}_Q = \int d^4x \mathcal{S}_Q = \int d^4x (\mathcal{L} - \mathcal{Q}) \). To summarize one can say that the conventional SE corresponds to a special class of solutions of the covariant canonical quantization of fields given by (3.14), (3.15), and (3.16) for which \( R^i = 0, S^i \) is local, and \( R^0, S^0 \) are local in time.

Generalizations are included in [55] dealing with a larger number of fields and curved spacetimes. We indicate some of the equations and refer to [55] for discussion. Thus let \( \phi(x) = \{\phi_a(x)\} \) be a collection of fields with action

\[
\mathcal{L} = \frac{1}{2} G^{ab}(\phi, x) g^{\mu\nu}(x)(\partial_\mu \phi_a)(\partial_\nu \phi_b) + F^{a\mu}(\phi, x) \partial_\mu \phi_a - V(\phi, x)
\]

In particular \( G^{ab}, F^{a\mu}, \) and \( V \) are proportional to \( |g|^{1/2} \) for convenience in calculations etc. so \( |g|^{1/2} \) is included in the definition of \( \mathcal{L} \). One writes \( G_{ab} G^{bc} = \delta^c_a \) and since \( G^{ab} \sim |g|^{1/2} \) one notes that if \( \partial_\mu \phi_a \) is a tensor then \( \partial^\mu \phi^a \) is a tensor density. The canonical momenta are \( \pi^{a\mu} = \partial \mathcal{L}/\partial (\partial_\mu \phi_a) = \partial^\mu \phi^a + F^{a\mu} \) and the dDW Hamiltonian is

\[
\mathcal{H} = \pi^{a\mu} \partial_\mu \phi_a - \mathcal{L} = \frac{1}{2} (\partial^\mu \phi^a)(\partial_\mu \phi_a) + V =
\]

\[
= \frac{1}{2} \pi^{a\mu} \pi_{a\mu} - \pi^{a\mu} F_{a\mu} + \frac{1}{2} F^{a\mu} F_{a\mu} + V
\]

The corresponding covariant canonical equations of motion are then

\[
\partial_\mu \phi_a = \frac{\partial \mathcal{H}}{\partial \pi_{a\mu}} = \pi_{a\mu} - F_{a\mu}; \quad \partial^\mu \pi_{a\mu} = -G_{ab} \frac{\partial \mathcal{H}}{\partial \phi_b} \equiv -\partial_a \mathcal{H}
\]

Here \( \partial_a = G_{ab} \partial^b \neq \partial^b G_{ab} \) (since \( G_{ab} \) depends on \( \phi \)) and the covariant HJ equations are

\[
\pi^{a\mu} = \frac{\partial \mathcal{S}^a}{\partial \phi_a} \equiv \partial^a \mathcal{S}^a;
\]
\[
\frac{1}{2} (\partial^a S^\mu)(\partial_a S_\mu) - F_{a\mu} \partial^a S^\mu + \frac{1}{2} F^{a\mu} F_{a\mu} + V + \partial_\mu S^\mu = 0
\]

The total derivative is \( d_\mu = \partial_\mu + (\partial_\mu \phi_a) \partial^a \) and one shows explicitly that (3.21) is covariant (cf. [55]). The general covariant generalization of (⋆) depends on the tensor nature of A and here A is a vector density \( A^\mu \) so one writes

\[ (3.22) \quad dA^\mu([\phi], x) \equiv e^\mu_{\bar{\alpha}}(x) \frac{\delta A^\mu([\phi], x)}{\delta \phi(x)} \]

where \( e^\mu_{\bar{\alpha}} \) is the tetrad satisfying

\[ e^\mu_{\bar{\alpha}} e^\bar{\alpha}_\nu = g^\nu_{\mu} \] (\( \bar{\alpha} \) is an index in the \( SO(1,3) \) group). Now in (3.21) one replaces the derivative \( \partial^a \) with \( d^a = d/d\phi_a \) and adds the Q term where

\[ (3.23) \quad \Omega = -\frac{\hbar^2}{2\mathcal{R}} \frac{\delta}{\delta \Sigma \phi_a} G_{ab} \frac{\delta}{\delta \Sigma \phi_b} \mathcal{R} \]

Then (3.15) generalizes to

\[ (3.24) \quad (d^a R^\mu)(d_a S_\mu) - F_{a\mu} d^a R^\mu + 3 + \partial_\mu R^\mu = 0; \quad 3 = \frac{\mathcal{R}}{2 \delta \Sigma \phi_a} \left( G_{ab} \frac{\delta S}{\delta \Sigma \phi_b} - F_{a\mu} r_\mu \right) \]

where \( r^\mu = R^\mu/(R^\lambda R_\lambda)^{1/2} \). The orderings in (3.23) are chosen to lead to a SE with a Hermitian Hamiltonian. One uses now the manifestly covariant forms

\[ (3.25) \quad \mathcal{R}([\phi], \Sigma) = \int_\Sigma d\Sigma \mu \tilde{R}^\mu; \quad \mathcal{S}([\phi], \Sigma) = \int_\Sigma d\Sigma \mu \tilde{S}^\mu \]

where \( \tilde{S}^\mu \) and \( \tilde{R}^\mu = R^\mu/|g|^{1/2} \) are vectors. The Bohmian equations of motion

\[ (3.26) \quad \partial_\mu \phi_a = d_a S_\mu - F_{a\mu} \]

are then equivalent to the equations obtained by varying the quantum action (3J) and we refer to [55] for details and further generalization.

In [55], in addition to the calculations involving \( G_{ab} \) (cf. (3.18)-(3.26)) one discusses quantum gravity as follows. The classical gravitational action is

\[ \mathfrak{A} = \int d^4 x |g|^{1/2} R \]

where \( R \) is the scalar curvature and to write the Lagrangian in a form appropriate for a canonical treatment one sets

\[ (3.27) \quad |g|^{1/2} R = \frac{1}{2} G^{\alpha\beta\mu\nu}(\partial_\mu g_{\alpha\beta})(\partial_\nu g_{\gamma\delta}) + \text{total derivative} \]

The total derivative term is ignored and one assumes that \( G^{\alpha\beta\mu\nu} \) and its inverse depend on \( g_{\alpha\beta} \) but not on its derivatives (cf. [40] [64]). The fields \( \phi_a \) are the components \( g_{\alpha\beta} \) of the metric and all 10 components will be
quantized (in contrast to the conventional noncovariant canonical quantization where only the space components are quantized). One finds then (as before) the following quantum equations

\[ \frac{1}{2} G_{\alpha\beta\gamma\delta} \frac{dS^\mu}{dg_{\alpha\beta}} \frac{dS^\nu}{dg_{\gamma\delta}} + \Omega + \partial_\mu S^\mu = 0; \quad \Omega = -\frac{\hbar^2}{2\sqrt{\gamma}} \frac{\delta}{\delta_{g_{\alpha\beta}}} G^r_{\alpha\beta\gamma\delta} \frac{\delta}{\delta_{g_{\gamma\delta}}} \mathcal{R}; \]

\[ G_{\alpha\beta\gamma\delta} \frac{dR^\mu}{dg_{\alpha\beta}} \frac{dS^\nu}{dg_{\gamma\delta}} + \mathcal{J} = 0; \quad \mathcal{J} = \frac{R}{2} \delta_{g_{\alpha\beta}} G^r_{\alpha\beta\gamma\delta} \frac{\delta}{\delta_{g_{\gamma\delta}}} \mathcal{S} \]

where \( G^r_{\alpha\beta\gamma\delta} = G_{\alpha\beta\mu\gamma\delta} r^\mu r^\nu \) (recall \( r^\mu = R^\mu / (R^\lambda R_\lambda)^{1/2} \)). The Bohmian equations of motion \( \partial_\mu g_{\alpha\beta} = G_{\alpha\beta\mu\gamma\delta} (dS^\gamma/dg_{\gamma\delta}) \) are equivalent to the equations of motion obtained via the quantum action \( \mathcal{A}_Q = \int d^4x (|g|^{1/2} R - Q) \) and this leads to the equation of motion

\[ R^{\mu\nu} - \frac{g^{\mu\nu}}{2} R + |g|^{-1/2} \frac{d\Omega}{dg_{\mu\nu}} = 0 \]

The potential \( \Omega \) is a scalar density so one can write \( \Omega = |g|^{1/2} \tilde{\Omega} \) where \( \tilde{\Omega} \) is a scalar and (3.29) becomes

\[ R^{\mu\nu} + \frac{d\tilde{Q}}{dg_{\mu\nu}} = 0 \]

Another suggestive form is

\[ \frac{g^{\mu\nu}}{2} R - R^{\mu\nu} = 8\pi G_N T^{\mu\nu}; \quad T^{\mu\nu} = \frac{1}{16\pi G_N} \left( 2 \frac{d\tilde{\Omega}}{dg_{\mu\nu}} + g^{\mu\nu} \tilde{\Omega} \right) \]

Note that (3.31) implies that the Bohmian equations of motion are fully covariant. By contrast if the quantization of gravity is based on the conventional canonical WDW equation that does not treat space and time on an equal footing then the Bohmian interpretation leads to an equation similar to (3.31) but with a non-covariant energy-momentum tensor of the form \( T^{ij} \propto d\tilde{\Omega}/dg_{ij} \) and \( T^{0\mu} \propto \tilde{\Omega} g^{0\mu} \). One recalls also that the WDW quantization corresponds to the case in which \( \tilde{R}^i = 0 \) and \( \tilde{S}^i \) is local while \( \tilde{S}^0, \tilde{R}^0 \) are functionals local in time.

**Remark 3.1.** In [57] the problem of time in quantum gravity is addressed by weakening the Hamiltonian constraint \( \hat{H} = 0 \) to \( <\psi|\hat{H}|\psi> = 0 \) which is consistent with the classical Hamiltonian constraint. This can be written as (we shift \( g \rightarrow h \) here in thinking of applications below to the deWitt metric and \( 3h \sim \hbar \))

\[ \int \mathcal{D}h \psi^* \hat{H} \psi = 0 \]
and for $\psi = R \exp(iS/h)$, $\hat{H}\psi = i\hbar \partial_t \psi$ and a stipulation $(d/dt) \int \mathcal{D}h \psi^* \psi = 0$ one finds that (3.32) holds if $\partial_t S = 0$ (note $R \sim \mathcal{R}$ and $S \sim \mathcal{S}$ here). Hence the (weak) Hamiltonian constraint (3.32) is consistent with $\psi = R(h,t) \exp(iS(h)/\hbar)$ (implies seems too strong here). The point here is to allow $i\hbar \partial_t \psi = \hat{H}\psi$ but insist that this not contradict $\hat{H} = 0$ in the classical limit. Consider then $H = \tilde{G}_{AB}(h)\pi^A\pi^B + V(h)$ ($h = \{h_A\}$ and $\tilde{G}_{AB} = \tilde{G}_{BA}$) or explicitly

\begin{equation}
\tilde{G}_{AB}\pi^A\pi^B \equiv \kappa \int d^3x \tilde{G}_{ijkl}\pi^i\pi^j\pi^k\pi^l; \quad V = -\kappa^{-1} \int d^3x \sqrt{|h|^3} R
\end{equation}

where $\kappa = 8\pi G$ and

\begin{equation}
\tilde{G}_{ijkl} = \frac{\sqrt{|h|}}{2}(h_{ik}h_{j\ell} + h_{jk}h_{i\ell} - h_{ij}h_{k\ell})
\end{equation}

($\tilde{G}_{ijkl}$ differs from $G_{ijkl}$ by a factor of $\sqrt{h}$ and this can be absorbed in $\mathcal{D}h$ as needed yielding $\mathcal{D}h$). In the quantum case $\pi^A$ becomes $\tilde{\pi}^A = -i\hbar(\delta/\delta h^A) \equiv -i\hbar\partial^A$ and different orderings of the $\tilde{\pi}^A$ in $\hat{H}$ become important. Some argument shows that a form (3K) $\hat{H} = \tilde{\pi}^A\tilde{G}_{AB}\tilde{\pi}^B + V$ implies $|\psi\rangle$ as well as all $|\psi_1\rangle|\psi_2\rangle$ are time independent since

\begin{equation}
\frac{d}{dt} \int \mathcal{D}h \psi^*_1 \psi_2 = \hbar \int \mathcal{D}h \partial^A[\tilde{G}_{AB}(\psi^*_1 i \tilde{\pi}^B \psi_2)]
\end{equation}

which vanishes because the integral over a total derivative vanishes (thus unitary time evolution implies the sandwich ordering). Moreover for $h \to 0$ (with $c = 1$) one obtains densities

\begin{equation}
\tilde{G}_{AB}\partial^A S \partial^B S + V = 0; \quad \partial_t R^2 + \partial^A[2R^2 \tilde{G}_{AB}\partial^B S] = 0
\end{equation}

which is the classical HJ equation (via $\pi^A = \partial^A S$) and

\begin{equation}
\dot{h}_A = \partial_t h_A = \frac{\partial H}{\partial \pi^A} = 2\tilde{G}_{AB}\pi^B; \quad \partial_t \rho + \partial^A(\rho \dot{h}_A) = 0 \quad (\sim \frac{d\rho}{dt} = 0)
\end{equation}

for $\rho = R^2$. Hence in fact the conventional strong form of the Hamiltonian constraint (leading to $\partial_t \rho = 0$) does not have the correct classical limit, but the weaker form does.

4. EXACT UNCERTAINTY AND WDW

In [17] we sketched some new heuristic results concerning WDW and exact uncertainty following [14 33 34 35 36 74].
4.1. EXACT UNCERTAINTY. Basically following e.g. \[34, 36\] one defines Fisher information via (4A) \[ F_x = \int dx P(x) |\partial_x \log(P(x))|^2 \] and a Fisher length by \[ \delta x = F^{-1/2}_x \] where \( P(x) \) is a probability density for a 1-D observable \( x \). The Cramer-Rao inequality says \( \text{Var}(x) \geq F^{-1}_x \) or simply \( \Delta x \geq \delta x \). For a quantum situation with \( P(x) = |\psi(x)|^2 \) and \( \psi \) satisfying a SE one finds immediatly

\[
F_X = \int dx |\psi|^2 \left[ \frac{\psi'}{\psi} + \frac{\psi''}{\psi} \right]^2 dx = 4 \int dx \delta' \delta' + \int dx |\psi|^2 \left[ \frac{\psi'}{\psi} - \frac{\psi''}{\psi} \right]^2 = \frac{4}{\hbar^2} [ < p^2 >_\psi - < p^2 >_{cl} ]
\]

where \( p_{cl} = (\hbar/2i) (|\psi' / \psi| - (\psi'' / \psi)) \) is the classical momentum observable conjugate to \( x \) (\( \sim S_X \) for \( \psi = R\exp(iS/h) \)). Setting now \( p = p_{cl} + p_{nc} \) one obtains after some calculation (4B) \( F_X = (4/\hbar^2)(\Delta p_{nc})^2 = 1/(\delta x)^2 \Rightarrow \delta x \Delta p_{nc} = \hbar/2 \) as a relation between nonclassicality and Fisher information. Note \( < p >_{\psi} = p_{cl} >_{\psi} \), \( \partial_x |\psi|^2 + \partial_x [\psi|^2 m^{-1} p_{cl}] = 0 \) from the SE, and \( (\Delta x)(\Delta p) \geq (\delta x)(\Delta p_{nc}) \).

We recall also that from (11) \( F_x \) is proportional to the difference of a quantum and a classical kinetic energy. Thus \( (\hbar^2/4)F_x(1/2m) = (1/2m) < p^2 >_{\psi} -(1/2m) < p^2 >_{cl} >_{\psi} \) and \( E_F = (\hbar^2/8m)F_x \) is added to \( E_{cl} \) to get \( E_{quant} \). By deBroglie-Bohm (dBB) theory there is a quantum potential

\[
Q = \frac{\hbar^2}{8m} \left( \left( \frac{P'}{P} \right)^2 - 2 \frac{P''}{P} \right); \quad P = |\psi|^2
\]

and evidently (4C) \( < Q >_{\psi} = \int P Q dx = (\hbar^2/8m)F_x \) (upon neglecting the boundary integral term at \( \pm \infty \) - i.e. \( P' \to 0 \) at \( \pm \infty \)).

Now the exact uncertainty principle (cf. \[34, 36, 74\]) looks at momentum fluctuations (4D) \( P = \nabla S + f \) with \( < f > = f = 0 \) and replaces a classical ensemble energy \( < E >_{cl} \) by (4E) \( P \sim |\psi|^2 \)

\[
< E > = \int dx P \left[ (2m)^{-1} |\nabla S + f|^2 + V \right] = < E >_{cl} + \int dx P \left\{ \frac{\nabla f}{2m} \right\}
\]

Upon making an assumption of the form (4E) \( f = \alpha(x, P, S, \nabla P, \nabla S, \cdots) \) one looks at a modified Hamiltonian (4F) \( \hat{H}_q[P, S] = \hat{H}_{cl} + \int dx P(\alpha/2m) \). Then, assuming

1) Causality - i.e. \( \alpha \) depends only on \( S, P \) and their first derivatives
2) Independence for fluctuations of noninteracting uncorrelated ensembles
3) \( f \to L^T f \) for invertible linear coordinate transformations \( x \to L^{-1} x \)
(4) Exact uncertainty - i.e. $\alpha = \bar{f} \cdot \bar{f}$ is determined solely by uncertainty in position

one arrives at

$$\tilde{H}_q = \tilde{H}_c + c \int dx \frac{\nabla P \cdot \nabla P}{2mP}$$

and putting $\hbar = 2\sqrt{c}$ with $\psi = \sqrt{P}exp(iS/\hbar)$ a SE is obtained.

As pointed out in [15] in the SE situation with $Q$ as in (4.2), in 3-D one has

$$\int P Q d^3x \sim -\frac{\hbar^2}{8m} \int \left[ 2\Delta P - \frac{1}{P}(\nabla P)^2 \right] d^3x = \frac{\hbar^2}{8m} \int \frac{1}{P}(\nabla P)^2 d^3x$$

since $\int_\Omega \Delta P d^3x = \int_{\partial \Omega} \nabla P \cdot n d\Sigma$ can be assumed zero for $\nabla P = 0$ on $\partial \Omega$. Hence

We recall now (cf. [11, 14, 16]) that the relation between the SE and the quantum potential (QP) is not 1-1. The QP $Q$ depends on the wave function $\psi = R exp(iS/\hbar)$ via $Q = -(\hbar^2/2)(\Delta R/R)$ for the SE and thus the solution of a quantum HJ equation, involving $S$ and $R$(via $Q$), requires the companion “continuity” equation to determine $S$ and $R$ (and hence $\psi$). There is some lack of uniqueness since $Q$ determines $R$ only up to uniqueness for solutions of $\Delta R + (2m/\hbar^2)QR = 0$ and even then the HJ equation $S_t + \cdots = 0$ could introduce still another arbitrary function (cf. [14, 16]).

**THEOREM 4.1.** Given that any quantum potential for the SE has the form (4.2) (with $\nabla P = 0$ on $\partial \Omega$) it follows that the quantization can be identified with momentum fluctuations of the type studied in [36] and thus has information content as described by the Fisher information. Thus we see that given a SE described via a probability distribution $P (= |\psi|^2)$ one can identify this equation as a quantum model arising from a classical Hamiltonian $\tilde{H}_c$ perturbed by a Fisher information term as in (4.4). Thus the quantization involves an information content with entropy significance (cf. here [15, 60] for entropy connections). This suggests that any quantization of $\tilde{H}_c$ arises (or can arise) through momentum perturbations related to Fisher information and it also suggests that $P = |\psi|^2$ (with $\int P d^3x = 1$) should be deemed a requirement for any solution $\psi$ of the related SE (note $\int P d^3x = 1$ eliminates many putative counterexamples). Thus once $P$ is specified as a probability distribution for a wave function $\psi = \sqrt{P}exp(iS/\hbar)$ arising from a SE corresponding to a quantization of $\tilde{H}_c$, then $Q$ can be expressed via Fisher information. Similarly given $Q$ as a Fisher information perturbation of $\tilde{H}_c$, then $\tilde{H}_q$ can be expressed via Fisher information perturbation of $\tilde{H}_c$ (arising from momentum fluctuations involving $P$).
as in (4.4)) there is a unique wave function \( \psi = \sqrt{P} \exp(iS/\hbar) \) satisfying the corresponding SE.

4.2. WDW. The same sort of arguments can be applied for the WDW equation following [34, 35, 69, 74, 81] (cf. also [5, 20, 31, 47, 50, 51, 53, 72, 76, 77, 79, 82, 83, 84, 85, 86] for WDW). Thus take an ADM situation

(4.6) \( ds^2 = -(N^2 - h^{ij}N_iN_j) + 2N_i dx^i dt + h_{ij} dx^i dx^j \)

and assume dynamics generated by an action (4G) \( A = \int dt [\hat{H} + \int \mathcal{D}h P \partial_\psi S] \). One will have equations of motion (4H) \( \partial_t P = \delta \hat{H}/\delta S \) and \( \partial_t S = -\delta \hat{H}/\delta P \) (cf. [14, 35]). A suitable “classical” Hamiltonian is

(4.7) \( \hat{H}_c[P, S] = \int \mathcal{D}h PH_0 \left[ h_{ij}, \frac{\delta S}{\delta h_{ij}} \right] \); \( H_0 = \int dx \left[ N \left( \frac{1}{2} G_{ijkl} \pi^{ij} \pi^{kl} + V(h_{ij}) \right) - 2N_i \nabla_j \pi^{ij} \right] \)

where \( G_{ijkl} \) is the deWitt (super)metric (4I) \( G_{ijkl} = \left(1/\sqrt{h}\right) (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}) \) and \( V \sim \sqrt{\hbar}(2\Lambda^{-3}R) \). Then thinking of \( \pi^{ij} = \delta S/\delta h_{ij} + f^{ij} \) and e.g. \( \hat{H}_q = \hat{H}_c + (1/2) \int \mathcal{D}h P \int dx N G_{ijkl} f^{ij} f^{kl} \) one arrives via exact uncertainty at a Fisher information contribution (cf. [17, 24, 34, 35])

(4.8) \( \hat{H}_q[P, S] = \hat{H}_c + \frac{c}{2} \int \mathcal{D}h \int dx N G_{ijkl} \frac{\delta P}{\delta h_{ij}} \frac{\delta P}{\delta h_{kl}} \sim \hat{H}_c + \frac{c}{2} \int \mathcal{D}h N \mathcal{Q} \)

with \( h = 2\sqrt{c} \) and \( \psi = \sqrt{P} \exp(iS/\hbar) \) resulting in (for \( N = 1 \) and \( N_i = 0 \))

(4.9) \( \left[ -\frac{h^2}{2} \frac{\delta}{\delta h_{ij}} G_{ijkl} \frac{\delta}{\delta h_{kl}} + V \right] \psi = 0 \)

with a sandwich ordering (\( G_{ijkl} \) in the middle - cf. also Section 3 and [51]). In general there are also constraints

(4.10) \( \frac{\delta \psi}{\delta N} = \frac{\delta \psi}{\delta N_i} = \partial_t \psi = 0; \nabla_j \left( \frac{\delta \psi}{\delta h_{ij}} \right) = 0 \)

We note here (keeping \( N = 1 \) with \( N_i = 0 \))

(4.11) \( \int \frac{\delta}{\delta h_{ij}} \left( G_{ijkl} \frac{\delta}{\delta h_{kl}} \sqrt{P} e^{iS/\hbar} \right) = \left[ \frac{\delta G_{ijkl}}{\delta h_{ij}} \left( \frac{1}{2} P^{-1/2} \frac{\delta P}{\delta h_{kl}} + \frac{i P^{1/2}}{\hbar} \frac{\delta S}{\delta h_{kl}} \right) + \right. \)

\( +G_{ijkl} \left\{ \frac{1}{4} P^{-3/2} \frac{\delta P}{\delta h_{kl}} \frac{\delta P}{\delta h_{ij}} + \frac{1}{2} P^{-1/2} \frac{\delta^2 P}{\delta h_{kl} \delta h_{ij}} - \frac{P^{1/2}}{h^2} \frac{\delta S}{\delta h_{kl}} \frac{\delta S}{\delta h_{ij}} + \right. \)

\( + \frac{i}{2h} P^{-1/2} \left( \frac{\delta P}{\delta h_{kl}} \frac{\delta S}{\delta h_{ij}} + \frac{\delta S}{\delta h_{kl}} \frac{\delta P}{\delta h_{ij}} + \frac{i P^{1/2}}{h} \frac{\delta^2 S}{\delta h_{kl} \delta h_{ij}} \right) \right\} e^{iS/\hbar} \)
Therefore writing out the WDW equation gives (cf. [17])

\[
(4.12) \quad -\frac{\hbar^2}{4P} \frac{\delta}{\delta h_{ij}} \left[ G_{ijkl} \frac{\delta P}{\delta h_{kl}} \right] + \frac{\hbar^2}{8P^2} G_{ijkl} \frac{\delta P}{\delta h_{k\ell}} \frac{\delta P}{\delta h_{ij}} + G_{ijkl} \left[ \frac{\hbar^2}{8P} \frac{\delta^2 P}{\delta h_{ij} \delta h_{ij}} + \frac{1}{2} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{ij}} \right] + V = 0;
\]

\[
2P \frac{\delta G}{\delta h_{ij}} \frac{\delta S}{\delta h_{k\ell}} + G \left( \frac{\delta P}{\delta h_{k\ell}} \frac{\delta S}{\delta h_{ij}} + \frac{\delta S}{\delta h_{k\ell}} \frac{\delta P}{\delta h_{ij}} \right) + 2PG \frac{\delta^2 S}{\delta h_{k\ell} \delta h_{ij}} = 0
\]

It is useful here to compare with \(-\frac{\hbar^2}{2m} \psi'' + V\psi = 0\) which for \(\psi = R \exp(iS/h)\) yields

\[
(4.13) \quad \frac{1}{2m} S_x^2 + V + Q = 0; \quad Q = -\frac{\hbar^2}{4m} R'' = \frac{\hbar^2}{8m} \left[ \frac{2P''}{P} - \left( \frac{P'}{P} \right)^2 \right]
\]

along with \(\delta(R^2 S') = \delta(PS') = 0\) (leading to (4.5)). The analogues here are then in particular

\[
(4.14) \quad \frac{1}{2m} S_x^2 \sim 2G_{ijkl} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{k\ell}} \quad Q = \frac{\hbar^2}{8m} \left[ \frac{2P''}{P} - \left( \frac{P'}{P} \right)^2 \right] \sim \frac{\hbar^2}{4P} \frac{\delta}{\delta h_{ij}} \left[ G_{ijkl} \frac{\delta P}{\delta h_{k\ell}} \right] + G_{ijkl} \left\{ \frac{\hbar^2}{8P^2} \frac{\delta P}{\delta h_{k\ell}} \frac{\delta P}{\delta h_{ij}} + \frac{\hbar^2}{4P} \frac{\delta^2 P}{\delta h_{ij} \delta h_{k\ell}} \right\}
\]

We note that the Q term arises directly from

\[
(4.15) \quad Q = -\frac{\hbar^2}{2} P^{-1/2} \frac{\delta}{\delta h_{ij}} \left( G_{ijkl} \frac{\delta P^{1/2}}{\delta h_{k\ell}} \right)
\]

and corresponds really to a density with

\[
(4.16) \quad \int \mathcal{D}h P Q = -\frac{\hbar^2}{2} \int \mathcal{D}h P^{1/2} \frac{\delta}{\delta h_{ij}} \left( G_{ijkl} \frac{\delta P^{1/2}}{\delta h_{k\ell}} \right)
\]

But from \(\int \mathcal{D}h \delta[ ] = 0\) one has (cf. (4.3))

\[
(4.17) \quad \int \mathcal{D}h P^{1/2} \frac{\delta}{\delta h_{ij}} \left( G_{ijkl} \frac{\delta P^{1/2}}{\delta h_{k\ell}} \right) = -\int \mathcal{D}h \frac{\delta P^{1/2}}{\delta h_{ij}} G_{ijkl} \frac{\delta P^{1/2}}{\delta h_{k\ell}}
\]

**THEOREM 4.2.** Given a WDW equation of the form (4.9) with associated quantum potential given via (4.15) (or (4.16)) it follows that the quantum potential gives rise to momentum fluctuations of Fisher information type as in (4.8) (for \(N = 1\)). Thus let us assume there exists a suitable \(\mathcal{D}f\) as in Section 4.3 below which is a measure in the (super)space of fields \(h\). Then there is an integration by parts formula (4.20) which removes the need for considering surface terms in integrals \(\int d^3x\) (cf. [20] for cautionary remarks about Green’s theorem, etc.). Consequently given
a WDW equation of the form (4.9) with corresponding $Q$ as in (4.15) (and $\psi = \sqrt{P} \exp(iS/\hbar)$, one can show that the equation can be modelled on a perturbation of a classical $\tilde{H}_c$ via a Fisher information type perturbation as in (4.8) (cf. here [14, 15, 17, 24, 35]). Here $P$ represents a probability density of fields $h_{ij}$ which determine $G_{ijk\ell}$ (and $V$ incidentally) and the very existence of a quantum equation (i.e. WDW) seems to require entropy type input via Fisher information fluctuation of fields. This suggests that quantum gravity requires a statistical spacetime (an idea that has appeared before - cf. [13]).

**REMARK 4.1.** We note from [5, 23, 25, 30, 94] that the “superspace” = $Riem/Diff$ with the deWitt metric $G_{ijk\ell} = G_{k\ell ij}$ is a collection of manifolds called a stratified manifold and therefore the calculations involving $\mathcal{D}h$ here (as well as in [34, 35]) must be appropriately determined. ■

4.3. SOME FUNCTIONAL CALCULUS. We go here to [12, 14, 35, 37] and will first sketch the derivation of (3.4) following [34, 35] (cf. also [12]). The relevant functional calculus goes as follows. One defines a functional $F$ of fields $f$ and sets

$$\delta F = F[f + \delta f] - F[f] = \int dx \frac{\delta F}{\delta f} \delta f_x$$

Here e.g. $dx \sim d^4x$ and in the space of fields there is assumed to be a measure $\mathcal{D}f$ such that $\int \mathcal{D}f \equiv \int \mathcal{D}f'$ for $f' = f + h$ (cf. [12, 35]). Then evidently (4J) $\int \mathcal{D}f (\delta F/\delta f) = 0$ when $\int \mathcal{D}f F[f] < \infty$. Indeed

$$0 = \int \mathcal{D}f (F[f + \delta f] - F[f]) = \int dx \delta f_x \left( \int \mathcal{D}f \frac{\delta F}{\delta f_x} \right)$$

and this provides an integration by parts formula

$$\int \mathcal{D}f P \left( \frac{\delta F}{\delta f} \right) = - \int \mathcal{D}f \left( \frac{\delta P}{\delta f} \right) F$$

for $P[f]$ a probability density functional. Classically a probability density functional arises in discussing an ensemble of fields and conservation of probability requires

$$\partial_t P + \sum_a \int dx \frac{\delta}{\delta f^a_x} \left( P \frac{\delta H}{\delta g^a_x} \bigg|_{g = \delta S/\delta f} \right)$$

where $g^a_x$ is the momentum corresponding to $f^a_x$; thus one assumes a motion equation

$$\partial_t S + H \left( f, \frac{\delta S}{\delta f}, t \right) = 0$$
The equations of motion here are then

\[ \frac{\partial_t P}{\Delta S} = \Delta \tilde{H} \]
\[ \frac{\partial_t S}{\Delta P} = -\Delta \tilde{H} \]

where \((4K)\) \(\tilde{H}(P, S, t) = \langle H \rangle = \int \mathcal{D}fPH(f, (\delta S/\delta f), t)\). The variational theory here involves functionals \(I[F] = \int \mathcal{D}f \xi(F, \delta F/\delta f)\) and one can write

\[ \Delta I = I[F + \Delta F] - I[F] = \int \mathcal{D}f \left[ \frac{\partial \xi}{\partial F} \Delta F + \int dx \left( \frac{\partial \xi}{\partial (\delta F/\delta f_x)} \right) \frac{\delta(\Delta F)}{\delta f_x} \right] = \]

\[ = \int \mathcal{D}f \left[ \frac{\partial \xi}{\partial F} - \int dx \frac{\delta}{\delta f_x} \left( \frac{\partial \xi}{\partial (\delta F/\delta f_x)} \right) \right] \Delta F + \]
\[ + \int dx \int \mathcal{D}f f \frac{\delta}{\delta f_x} \left[ \left( \frac{\partial \xi}{\partial (\delta F/\delta f_x)} \right) \delta F \right] \]

Assuming the term \(\int \mathcal{D}f [\Delta F]\) is finite the last integral vanishes and one obtains \((4L)\) \(\Delta I = \int \mathcal{D}f (\Delta I/\Delta F) \Delta F\), thus defining a variational derivative

\[ \frac{\Delta I}{\Delta F} = \frac{\partial \xi}{\partial F} - \int dx \frac{\delta}{\delta f_x} \left( \frac{\partial \xi}{\partial (\delta F/\delta f_x)} \right) \]

In the Hamiltonian theory one can work with a generating function \(S\) such that \((4M)\) \(g = \delta S/\delta f\) and \(\partial_t f + H(f, \delta S/\delta f, t) = 0\) (HJ equation) and solving this is equivalent to \(\partial_t f = \delta H/\delta g\) and \(\partial_t g = -\delta H/\delta f\) (cf. \[35\]). Once \(S\) is specified the momentum density \(g\) is determined via \(g = \delta S/\delta f\) and an ensemble of fields is specified by a probability density functional \(P[f]\) (and not by a phase space density functional \(\rho[f, g]\). In the HJ formulation one writes \((4N)\) \(V_x[f] = \partial f_x/\partial t = (\delta H/\delta g)\big|_{g=\delta S/\delta f}\) and hence the associated continuity equation \(\partial_t \int \mathcal{D}f P\) is

\[ \partial_t P + \int dx \frac{\delta}{\delta f_x}[PV_x] = 0 \]

provided \(<V_x>\) is finite.

Now after proving (4.4) one proceeds as follows to produce a SE. The Hamiltonian formulation gives \((4O)\) \(\partial_t P = \Delta \tilde{H}/\Delta S\) and \(\partial_t S = -\Delta \tilde{H}/\Delta P\) where the ensemble Hamiltonian is

\[ \tilde{H} = \tilde{H}(P, S, t) = \langle H \rangle = \int \mathcal{D}f PH[f, \delta S/\delta f, t] \]

where \(P\) and \(S\) are conjugate variables. The equations \((4O)\) arise from \(\Delta \tilde{A} = 0\) where \(\tilde{A} = \int dt [\tilde{H} + \int \mathcal{D}f S \partial_t P\). One specializes here to quadratic
Hamiltonian functions

\[ H_c[f, g, t] = \sum_{a,b} dx K^{ab}_x[f] g^a_x g^b_x + V[f] \]

and to this is added a term as in (4.4) to get \( \tilde{H} \) (which does not depend on \( S \)). Hence from (4O) with \( \partial_t f_x = \delta H_c / \delta g_x \) one obtains following (4.26)

\[ \partial_t P + \int dx \frac{\delta}{\delta f_x} \left[ P \frac{\delta H}{\delta g_x} \right]_{g=S/\delta f} = 0 \]

(cf. 4.25)). The other term in \( \tilde{H} \) is simply

\[ \left( \frac{\hbar}{4} \right) \int \mathcal{D} f \int PK^{ab}_x(\delta P/\delta f_x^a)(\delta P/\delta f_x^b)(1/P^2) \]

and this provides a contribution to the HJ equation via \( \partial_t S = -\Delta \tilde{H} / \Delta P \) which will have the form

\[ Q = -\frac{\hbar^2}{4} P^{-1/2} \int dx \frac{\delta}{\delta f_x^2} \left( K^{ab}_x P^{1/2} \frac{\delta P}{\delta f_x^2} \right) \sim \int dx Q \]

corresponding to (4.15). We note further then from (4.17)

\[ Q \sim \frac{\hbar^2}{2} \int dx G_{ijkl} \frac{\delta P^{1/2}}{\delta h_{ij}} \frac{\delta P^{1/2}}{\delta h_{kl}} \sim \frac{\hbar^2}{8} \int dx G_{ijkl} \frac{1}{P} \frac{\delta P}{\delta h_{ij}} \frac{\delta P}{\delta h_{kl}} \]

as in (4.8). Hence Theorem 4.2 is established under the hypotheses indicated concerning \( \mathcal{D} f \) etc. Some care is needed here in distinguishing densities \( Q \) from \( Q \); in view of \( \int dx \int \mathcal{D} h = \int \mathcal{D} h \int dx \) one can move terms around rather freely.

5. REMARKS ON ENTROPY

One recalls (cf. [14, 15, 26]) that with the SE (under certain circumstances) one has a differential entropy \( \mathcal{S} = -\int dx \rho \log(\rho) \) (1-D for simplicity here) with \( \partial_t \rho = -\partial (v \rho) \) and \( v = -u = -D \partial \log(\rho) \) (diffusion current) leading to

\[ \partial_t \mathcal{S} = -\int dx \rho (\log(\rho) + 1) = \int dx (\log(\rho) + 1) \partial (v \rho) = -\int dx \rho D \partial \log(\rho) = D \int \frac{(\partial \rho)^2}{\rho} \]

Thus the Fisher information is the time derivative of the differential entropy and there should be some analogue of this for WDW. There is not a priori
a natural time evolution for WDW but Section 3 provides a way around this. In any case one might look for a formula of the form
\begin{equation}
\delta \int \mathcal{D} h F(S, P, h_{ij}) = \int \mathcal{D} h \left[ \frac{\delta F}{\delta S} \delta S + \cdots \right] = \int \mathcal{D} h \frac{\delta P^{1/2}}{\delta h_{ij}} G_{ijkl} \frac{\delta P^{1/2}}{\delta h_{kl}}
\end{equation}
where \( F \) represents some kind of entropy term. Note from \([35]\) that \( f_{ij} \sim (1/P)(\delta P/\delta h_{ij}) = \delta \log(P)/\delta h_{ij} \) is claimed to be inconsistent with \( f_{ij} = 0 \), but for \( < f_{ij} > = \tau_{ij} = \int \mathcal{D} h f_{ij} \) we get \( \int \mathcal{D} h (\delta \log(P)/\delta h_{ij}) = 0 \) automatically. Hence referring now to Section 3, in particular (3.9) - (3.10) and Remark 3.1, one thinks of \( R^2 = \rho (\equiv P) \) and looks at (3.31) (with \( Q \) and \( J \) added). The second equation is in fact fixed by the sandwich ordering as
\begin{equation}
\partial_t \rho + \partial^A [2 \rho \tilde{G}_{AB} \partial^B S] = 0
\end{equation}
where \( \partial^B S = -i \hbar (\delta/\delta h_B) S \). Now recall from \([14]\) that in a Brownian motion situation the use of a drift velocity \( u = D \nabla \log(\rho) = -v = -(1/m) \nabla S \) is natural \( (D = \hbar/2m) \). Another context involving the SE with statistical geometry and a Weyl space produces a Weyl vector \( \phi_i = -\partial_i \log(\rho) \) related to an osmotic velocity field. Thus a relation \( u = -c \phi = c \nabla \log(\rho) \) can be envisioned with \( \rho = P \sim R^2 \) so that, instead of dealing with \( \delta S/\delta h_{ij} = \pi^{ij} - (1/P)(\delta P/\delta h_{ij}) \) one is motivated to consider
\begin{equation}
\frac{\delta S}{\delta h_B} \sim -c \frac{\delta P}{P \delta h_B}
\end{equation}
provided one is only interested in metric fluctuations (there is no particle mass here to impede this). In this case on could work with (5.3) as \((-i \hbar)^2 = -\hbar^2\)
\begin{equation}
\partial_t P - \frac{\delta}{\delta h_A} \left[ 2 P \tilde{G}_{AB} \frac{\hbar^2 c}{P} \frac{\delta P}{\delta h_B} \right] = 0
\end{equation}
Then for a differential entropy defined via \((5A)\) \( S = -(1/c) \int dx \int \tilde{\mathcal{D}} h P \log(P) \) one would have
\begin{equation}
S_t \sim -\frac{\hbar^2}{c} \int dx \int \tilde{\mathcal{D}} h P_t [1 + \log(P)] = -\hbar^2 \int dx \int \tilde{\mathcal{D}} h [1 + \log(P)] \left[ \frac{\delta}{\delta h_A} \left( 2 \tilde{G}_{AB} \frac{\delta P}{\delta h_B} \right) \right] = \hbar^2 \int dx \int \tilde{\mathcal{D}} h \left( \frac{2 P}{P} \tilde{G}_{AB} \frac{\delta P}{\delta h_B} \right) \sim 16 \int \mathcal{D} h \int dx P Q \sim 16 \int \tilde{\mathcal{D}} h P Q
\end{equation}
(cf. (4.16), (4.17), and (4.32)). One arrives then at a heuristic result

**THEOREM 5.1.** Given the weak constraint situation of Remark 3.1 and assuming only metric fluctuations satisfying (5.4) one can define a differential entropy \((5A)\) and express the Fisher information (expressed via the
quantum potential $Q$) as a time derivative $\partial_t S$. A similar theorem holds using the covariant Bohmian formulation with $R^i = 0$, $S^i$ local etc.

**Remark 5.1.** In [13] one develops a theory of a time direction hidden in quantum mechanics based on $Q(t) > 0$ where $Q$ is the quantum potential. The idea is that $\int^t Q(\tau) d\tau$ is a monotone increasing function of time which can be useful to characterize the direction of time. We will not go into the idea of a knowledge functional $K$ here except to remark that $K \sim Q$ (up to a factor of $\hbar^2/2$). In any event this also seems to be compatible with entropy change as envisioned in (5.1) and Theorem 5.1.

5.1. **Entropy and the Einstein Equations.** In [61] (cf. also [15 18 62 63 64 65]) one takes an entropy functional $(u^a = \bar{x}^a - x^a$ is a perturbation)

(5.7) \[ S = \frac{1}{8\pi} \int d^4x \sqrt{|g|} \left[ M^{abcd} \nabla_a u_b \nabla_c u_d + N_{ab} u^a u^b \right] \]

Extremizing with respect to $u_b$ leads to $(N_{ab} u^a u^b = N^{ab} u_a u_b)$

(5.8) \[ \nabla_a \left( M^{abcd} \nabla_c \right) u_d = N^{bd} u_d \]

Note $\int d^4x \sqrt{-g} f \nabla_a u_b = - \int d^4x \sqrt{-g} u_b \nabla_a f$ since via [3] one can write $\delta \sqrt{-g} = -(1/2) \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$ and $\nabla_a g^{\mu\nu} = 0$. Choosing $M$ and $N$ such that (5.8) (for all $u_d$) implies the Einstein equations entails

(5.9) \[ M^{abcd} = g^{ad} g^{bc} - g^{ab} g^{cd}, \quad N_{ab} = 8\pi \left( T_{ab} - \frac{1}{2} g_{ab} T \right) \]

Consequently $S$ becomes

(5.10) \[ S = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left[ (\nabla_a u^b)(\nabla_b u^a) - (\nabla_b u^b)^2 + N_{ab} u^a u^b \right] = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left[ Tr(J^2) - (Tr(J))^2 + 8\pi \left( T_{ab} - \frac{1}{2} g_{ab} T \right) u^a u^b \right] \]

where $J^b_a = \nabla_a u^b$. Note here

(5.11) \[ \int d^4x \sqrt{-g} g^{ad} g^{bc} \nabla_a u_b \nabla_c u_d = \int d^4x \sqrt{-g} (\nabla_a u^b)(\nabla^c u^d) \]

and also

(5.12) \[ \nabla_a \left( M^{abcd} \nabla_c \right) u_d = \nabla_a \left[ g^{ad} g^{bc} - g^{ab} g^{cd} \right] \nabla_c u_d = \nabla_a g^{ad} g^{bc} \nabla_c u_d - \nabla_a g^{ab} g^{cd} \nabla_c u_d = \nabla_a \nabla^b u^d - \nabla^b \nabla_a u^d \sim (\nabla_a \nabla^b - \nabla^b \nabla_a) u^d \]

Further (as in [51 52])

(5.13) \[ M^{abcd} \nabla_a u_b \nabla_c u_d = g^{ad} g^{bc} \nabla_a u_b \nabla_c u_d - g^{ab} g^{cd} \nabla_a u_b \nabla_c u_d = \nabla^d u_b \nabla^b u_d - \nabla^b \nabla^d u_d = \nabla^d u_d - \nabla_a u^a \nabla^a u^d \]
which confirms (5.10). We record also from [50] that
\[(5.14) \quad (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \alpha(w) = R(\alpha, \partial_\mu, \partial_\nu, w)\]
which identifies \(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \) with \(R_{\mu\nu}\) and allows us to imagine (5.12)
as \(R^b_au^a\) with Einstein equations
\[(5.15) \quad R^b_au^a = N^b_ag^c_du^a_d \quad (= N^bc_dg^ca_d)\]
for example, which is of course equivalent to \(R_{ab} = N_{ab}\) (cf. also [33]). Note also \(G_{ab} = R_{ab} - (1/2)Rg_{ab} = kT_{ab}\) implies that \(R^\mu_\nu - (1/2)R\delta^\mu_\nu = kT^\mu_\nu\) which upon contraction gives \(R = -kT\) (since \(\delta^\mu_\nu = 4\)) and hence \(R_{ab} = k(T_{ab} - (1/2)Tg_{ab})\).

For completeness we sketch here a derivation of the Einstein equations from an action principle (cf. [3] [19] [54] [93]). The Einstein-Hilbert action is
\[A = \int_\Omega [\mathcal{L}_G + \mathcal{L}_M]d^4x \quad \text{where} \quad \mathcal{L}_G = (1/2\chi)\sqrt{-\hat{g}}\hat{R} \quad (\chi = 8\pi \text{ and } 4\hat{R} \text{ is the Ricci scalar})\]Following [19] we list a few useful facts first (generally we will write if necessary \(g_{ab}T^{cb} = T^c_a\) and \(g_{ab}T^{bc} = T^c_\alpha\)).

1. \(\nabla_\gamma g^{\alpha\beta} = 0\) (by definitions of covariant derivative and Christoffel symbols).
2. \(\partial \sqrt{-g} = (1/2)\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta}\) and \((\delta g_{\alpha\beta})g^{\alpha\beta} = -(\delta g^{\alpha\beta})g_{\alpha\beta}\) (see e.g. [33] for the calculation).
3. For a vector field \(v^a\) one has \(\nabla_\alpha v^a = \partial_\alpha(\sqrt{-g}v^a)/(1/\sqrt{-g})\) and \(\nabla_\beta T^{\alpha\beta} = \partial_\beta(\sqrt{-g}T^{\alpha\beta})+(1/\sqrt{-g})\Gamma^\alpha_\beta_\gamma T^{\gamma\alpha})(\text{from } \Gamma^\alpha_\beta_\gamma = (1/2)(\partial_\alpha g_{\mu\nu})g^{\mu\nu} \text{ and } \partial_\alpha(\log(\sqrt{-g}) = \Gamma^\alpha_\beta_\alpha)\).
4. For two metrics \(g, g^*\) one shows that \(\delta \Gamma^\alpha_\beta_\gamma = \Gamma^\alpha_\beta_\gamma - \Gamma^\alpha_\beta_\gamma\) is a tensor.
5. \(\delta R_{\alpha\beta} = \nabla_\sigma(\delta \Gamma^\alpha_\beta_\sigma) - \nabla_\beta(\delta \Gamma^\alpha_\sigma_\beta)\) (see [19] for the calculations).
6. Recall also Stokes theorem \(\int_\Omega \nabla_\sigma v^\sigma \sqrt{-\hat{g}}d^4x = \int_\partial\Omega \hat{\nabla}v^\sigma \sqrt{-\hat{g}}d^3\Sigma\).

Now requiring a stationary action for arbitrary \(\delta g^{ab}\) (with certain derivatives of the \(g^{ab}\) fixed on the boundary of \(\Omega\) one obtains (\(\mathcal{L}_M\) is the matter Lagrangian)
\[(5.16) \quad \delta I = \frac{1}{2\chi} \int_\Omega \left( R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R \right) \sqrt{-g}\delta g^{\alpha\beta}d^4x + \frac{1}{2\chi} \int_\Omega g^{\alpha\beta} \sqrt{-g}\delta R_{\alpha\beta}d^4x + \int_\Omega \frac{\delta \mathcal{L}_M}{\delta g^{\alpha\beta}} \delta g^{\alpha\beta}d^4x = 0\]
The second term can be written
\[(5.17) \quad \frac{1}{2\chi} \int_\Omega g^{\alpha\beta} \sqrt{-g}\delta R_{\alpha\beta}d^4x = \frac{1}{2\chi} \int_\Omega g^{\alpha\beta} \sqrt{-g}[\nabla_\sigma(\delta \Gamma^\alpha_\beta_\sigma) - \nabla_\beta(\delta \Gamma^\alpha_\sigma_\beta)]d^4x = \frac{1}{2\chi} \int_\Omega \sqrt{-g}[\nabla_\sigma(g^{\alpha\beta} \delta \Gamma^\alpha_\beta_\sigma) - \nabla_\beta(g^{\alpha\beta} \delta \Gamma^\alpha_\sigma_\beta)]d^4x = \]
Thus e.g. think of functionals $F$ and $\delta(x^\alpha) = (\sqrt{-g}g^{\alpha\beta}\delta \Gamma^\xi_{\alpha\beta}) - (\sqrt{-g}g^{\alpha\beta}\delta \Gamma^\xi_{\alpha\beta})$. This can be transformed into an integral over the boundary $\partial \Omega$ where it vanishes if certain derivatives of $g_{\alpha\beta}$ are fixed on the boundary. In fact the integral over the boundary $\partial \Omega = S$ can be written as $\sum (\epsilon/2\chi) \int S_i \gamma_{\alpha\beta} \delta N^\alpha_{\alpha\beta} d^4 x$ where $\epsilon_i = n_i \cdot n_i = \pm 1$ ($n_i$ normal to $S_i$) and $\gamma_{\alpha\beta} = g_{\alpha\beta} - \epsilon_i n_{\alpha} \cdot n_{\beta}$ is the 3-metric on the hypersurface $S_i$ (cf. [96]). Further

$$\tag{5.18} \tilde{N}^\alpha_{\alpha\beta} = \sqrt{|\gamma|}(K^\alpha_{\alpha\beta} - \dot{K}^\alpha_{\alpha\beta}) = -\frac{1}{2} g^\alpha_{\alpha\beta} \gamma^\beta\nu \mathcal{L}_n (g^{-1} \gamma_{\mu\nu})$$

where $K_{\alpha\beta} = -(1/2) \mathcal{L}_n \gamma_{\alpha\beta}$ is the extrinsic curvature of each $S_i$ and $\mathcal{L}_n$ is the Lie derivative. Consequently if the quantities $\tilde{N}^\alpha_{\alpha\beta}$ are fixed on the boundary for an arbitrary $\dot{g}_{\alpha\beta}$ one gets from the first and last equations in (5.10) the Einstein field equations

$$\tag{5.19} G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \chi T_{\alpha\beta}; \ T_{\alpha\beta} = -\frac{2}{3} \frac{\delta \mathcal{L}_M}{\delta g^{ab}} + \mathcal{L}_M g_{\alpha\beta}$$

We note here that

$$\tag{5.20} \delta \int \mathcal{L}_m \sqrt{-g} d^4 x = \int \frac{\delta \mathcal{L}_m}{\delta g^{ab}} \sqrt{-g} d^4 x + \int \mathcal{L}_m \delta \sqrt{-g} d^4 x = $$$$= \int \frac{\delta \mathcal{L}_m}{\delta g^{ab}} \sqrt{-g} d^4 x - \frac{1}{2} \int \mathcal{L}_m g^{ab} (\delta g^{ab}) \sqrt{-g} d^4 x$$

A factor of 2 then arises from the $2\chi$ in (5.16).

**REMARK 5.2.** Let us rephrase some of this following [93] for clarity. Thus e.g. think of functionals $F(\psi)$ with $\psi = \psi_\lambda$ a one parameter family and set $\delta \psi = (d\psi/\lambda)_{\lambda=0}$. For $F(\psi)$ one writes then $dF/\lambda = \int \phi \delta \psi$ and sets $\phi = (dF/\delta \psi)_{\psi_0}$. Then (assuming all functional derivatives are symmetric with no loss of generality) one has for $\mathcal{L}_G = \sqrt{-g} R$ and $S_G = \int \mathcal{L}_G d^4 x$

$$\tag{5.21} \frac{d \mathcal{L}_G}{d \lambda} = \sqrt{-g} (\delta R_{ab}) g^{ab} + \sqrt{-g} R_{ab} \delta g^{ab} + R \delta \sqrt{-g}$$

But $g^{ab} \delta R_{ab} = \nabla_a v_a$ for $v_a = \nabla^a (\delta g_{ab}) - \delta g^{cd} \nabla_d (\delta g_{cd})$. Further $\delta \sqrt{-g} = -(1/2) \sqrt{-g} g_{ab} \delta g^{ab}$ so one has

$$\tag{5.22} \frac{d S_G}{d \lambda} = \int \frac{d \mathcal{L}_G}{d \lambda} d^4 x = \int \nabla_a v_a \sqrt{-g} d^4 x + \int \left( R_{ab} - \frac{1}{2} g_{ab} R \right) (\delta g^{ab}) \sqrt{-g} d^4 x$$

Discarding the first term as a boundary integral we get the first term in (5.16).
REMARK 5.3. From [61] we see that the entropy in $S$ in (5.7) reduces to a 4-divergence when the Einstein equations are satisfied “on shell” making $S$ a surface term

\[
S = \frac{1}{8\pi} \int_V d^4x \sqrt{-g} \nabla_i (u^b \nabla_b u^i - u^i \nabla_b u^b) = \frac{1}{8\pi} \int_{\partial V} d^3x \sqrt{h_n} (v^b \nabla_b u^i - u^i \nabla_b u^b)
\]

Thus the entropy of a bulk region $V$ of spacetime resides in its boundary $\partial V$ when the Einstein equations are satisfied. In varying (5.7) to obtain (5.8) one keeps the surface contribution to be a constant. Thus in a semiclassical limit when the Einstein equations hold to the lowest order the entropy is contributed only by the boundary term and the system is holographic. ■

REMARK 5.4. Let us call attention here to [22, 41] where a very different approach is made to derive the Einstein equations from thermodynamics using entropy ideas. Using non-equilibrium thermodynamics one finds also that entropy dependence on the Ricci scalar can be accommodated. ■

6. WDW AND THE EINSTEIN EQUATIONS

We sketch here the derivation of the Einstein equations from quantum geometrodynamics following Gerlach [29]. He works with the Einstein HJ (EHJ) equation in the Perez form (cf. [70])

\[
3R + h^{-1} \left( \frac{1}{2} h_{ij} h_{k\ell} - h_{ik} h_{j\ell} \right) \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{k\ell}}
\]

where $h_{ij}$ is the metric of the spatial hypersurface $\Sigma$. One defines (6A) $\delta S = \int [\delta S/\delta h_{ij}(x)] \delta h_{ij}(x) d^3x$ with integration over $\Sigma$ and assumes that $S$ is a function of the 3-geometry only, namely (6B) $S = S[3, \Phi]$ (i.e. $S$ is coordinate independent). Assume further the principle of constructive interference (see below) and that either $\Sigma$ is finite with no boundary or that $\Sigma$ is asymptotically flat. Under these conditions one proves that there are 4 functions $N, N_i (i = 1, 2, 3)$ which together with $h_{ij}$ give a spacetime metric

\[
ds^2 = h_{ij} (N^i dx^0 + dx^i) (N^j dx^0 + dx^j) - N^2(dx^0)^2 = h_{ij} dx^i dx^j + 2N_i dx^i dx^0 + (N_j N^j - N^2)(dx^0)^2
\]

which satisfies the Einstein field equations. Further the manifestly covariant equations of geometrodynamics

\[
\frac{\delta h_{ij}}{\delta \sigma} = \frac{\delta H}{\delta \pi^{ij}(x)}, \quad \frac{\delta \pi^{ij}(x)}{\delta \sigma} = - \frac{\delta H}{\delta h_{ij}(x)}
\]
hold where $H = H[h_{ij}]$ and $\pi^{ij} = (\delta S/\delta h_{ij})$. Here $\sigma$ is the Tomonaga-Schwinger many fingered time parameter (cf. [14] [56] [80] [92]). One notes that the dynamical phase functional $S = S[h_{ij}]$ is required to be a functional of the 3-geometry alone, regardless of coordinates so one writes (6C) $S = S[3\mathfrak{G}]$ which means that (6D) $\nabla_j[\delta S/\delta h_{ij}] = 0$. To see this consider $h_{ij}(x)$ with $x^i \rightarrow x'^i = x^i + \epsilon i(x)$ while preserving the geometry where

$$h_{ij}'(x) = h_{ij}(x) + \delta h_{ij}(x); \quad \delta h_{ij}(x) = -\epsilon (\nabla_j \xi_i + \nabla_i \xi_j)$$

The ostensible change in $S$ would be

$$\delta S = \int \frac{\delta S}{\delta h_{ij}(x)} \delta h_{ij}(x)d^3x = -2\epsilon \int \frac{\delta S}{\delta h_{ij}(x)} \nabla_j \xi_i d^3x = 2\epsilon \int \nabla_j \left( \frac{\delta S}{\delta h_{ij}(x)} \right) \xi_i d^3x$$

However the $3\mathfrak{G}$ itself is not changed so $\delta S$ must vanish which means that (6D) holds.

Now the phase functional is defined on superspace (i.e. the set of equivalence classes of spacelike $h_{ij}(x)$) that can be transformed into each other by spatial coordinate transformations. One considers a solution to the EHJ equation satisfying (6D) $\nabla_j[\delta S/\delta h_{ij}] = 0$ to forego the details of some messy parametrization scheme. Now from (6E) $\alpha(u) \rightarrow \alpha(u) + \delta \alpha(u)$ and $\beta(u) \rightarrow \beta(u) + \delta \beta(u)$. Some argument (cf. [29]) yields then (6F) $(\delta S/\delta^3\mathfrak{G})(\alpha + \delta \alpha, \beta + \delta \beta)d^3\mathfrak{G} = (\delta S/\delta^3\mathfrak{G})(\alpha, \beta)d^3\mathfrak{G}$. One is concerned here not with e.g. $\delta h_{ij}/\delta \sigma$ where $\sigma$ is the Tomonaga time parameter (discussed in Section 6.1 below) but rather with a parametrization independent quantity such as (6G) $\delta^2\mathfrak{G} = \int (\delta^3\mathfrak{G}/\delta \sigma) \delta \sigma d^3x$ or equivalently with $\delta h_{ij} = \int (\delta h_{ij}/\delta \sigma) \delta \sigma d^3x$. Thus the focus of attention is $\delta h_{ij}$ rather than $\delta h_{ij}/\delta \sigma$ and this allows one to forego the details of some messy parametrization scheme. Now from (6F) a necessary condition for the “vector” $\delta^2\mathfrak{G}/\delta \sigma(x)$ to be tangent to a history through $3\mathfrak{G}$ is (6H) $\delta(\delta S/\delta^3\mathfrak{G})(\delta^3\mathfrak{G}/\delta \sigma) = 0$ where $(\delta S/\delta^3\mathfrak{G})$ denotes the change due to an arbitrary infinitesimal variation in $(\alpha(u), \beta(u))$. Then the EHJ equation together with (6H) contains all of general relativity (GR), exhibited then as (see below)

$$3R - \left( \frac{\delta S}{\delta^3\mathfrak{G}} \right) \left( \frac{\delta S}{\delta^3\mathfrak{G}} \right)^* = 0; \quad \delta \left( \frac{\delta S}{\delta^3\mathfrak{G}} \right) \frac{\delta^3\mathfrak{G}}{\delta \sigma} = 0$$

The starred vector $(\delta S/\delta^3\mathfrak{G})^*$ is the dual with respect to the deWitt metric and an easy way now of obtaining the Einstein field equations (Efe) is to use the language of tensor analysis. Then the tangent vector in (6H) becomes

$$\frac{\delta^3\mathfrak{G}}{\delta \sigma(x')} \rightarrow \frac{\delta h_{ij}(x, \sigma(x'))}{\delta \sigma(x')}; \quad \frac{\delta S}{\delta^3\mathfrak{G}} \rightarrow \frac{\delta S}{\delta h_{ij}(x)} \equiv \pi^{ij}(x)$$
where (6I) \( \nabla_j \pi^{ij} = 0 \) in order for \( S \) to depend only on \( ^3G \). With this notation (6.8) becomes

\[
(6.8) \int \delta \pi^{ij}(x) \frac{\delta h_{ij}(x)}{\delta \sigma(x')} d^3 x = 0; \quad ^3R + h^{-1} \left( \frac{1}{2} h_{ij} h_{k\ell} - h_{ik} h_{j\ell} \right) \pi^{ij} \pi^{k\ell} = 0
\]

One enunciates the principle of constructive interference stated in (6.8) now as follows: In order that a change \( \delta h_{ij} \) or equivalently \( \delta h_{ij}/\delta \sigma \) be a vector tangent to a history it is necessary that there exist a \( \pi^{ij}(x) \) with the property

\[
(6.9) \int \pi^{ij}(x) \frac{\delta h_{ij}(x)}{\delta \sigma(x')} d^3 x = extremum
\]

if one changes the integration constants \( \alpha(u), \beta(u) \) slightly. One discusses the fact that freedom in adjusting the integration constants corresponds to freedom in choosing the momentum density and relates this to the idea of having a complete solution \( S \) as a functional of the maximum number of possible independent constants (cf. [29] for details).

Now going to the Efe one replaces (6.9) with the help of (6G) by writing (6J) \( \int \pi^{ij} \delta h_{ij} d^3 x \) and this must be an extremum with respect to variations in \( \pi^{ij}(x) \) subject to the restrictions

\[
(6.10) \quad R_0 = h^{1/2} \left[ \frac{3}{2} R + h^{-1} \left( \frac{1}{2} h_{ij} h_{k\ell} - h_{ik} h_{j\ell} \right) \right] \pi^{ij} \pi^{k\ell} = 0;
\]

\[
\nabla_j \pi^{ij} = 0; \quad \pi^{ij}(x) = \frac{\delta S}{\delta h_{ij}}
\]

One can take the restrictions on \( S \) into account in the extremum principle by multiplying them by yet to be determined functions \( \delta M(x) \) and \( 2\delta M_i(x) \) and add to (6J) to get \( \int [\pi^{ij} \delta h_{ij} + \delta M R_0 + 2\delta M_i \nabla_j \pi^{ij}] d^3 x \). Now consider changes in the integral due to arbitrary variations in \( \pi^{ij} \); an integration by parts yields

\[
(6.11) \int_{\Sigma} \left[ \delta h_{ij} \delta \pi^{ij} + \delta M \left( \frac{\delta R_0}{\delta \pi^{ij}} \right) \delta \pi^{ij} - 2\delta \nabla_j M_i \delta \pi^{ij} \right] d^3 x + \int_{\partial \Sigma} \delta M_i \delta \pi^{ij} dS_j
\]

The surface term vanishes due to boundary conditions and one emphasizes that the change \( \delta h_{ij} \) has nothing to do with the variations in \( \pi^{ij} \). The arbitrary changes in \( \pi^{ij} \) fall into two classes (A) Those that satisfy the variation equations (6I) and (6.8) and (B) Those that do not. The principle of constructive interference requires that the variations of the integral (6.11) vanish for class (A) variations. Consequently the coefficients of these variations must vanish and one then adjusts the functions \( \delta M \) and \( \delta M_i \) so
that the coefficients of the class (B) variations also vanish. The result of

(6.12) \[ \delta h_{ij} = -2\delta M(h)^{-1/2} \left( \frac{1}{2} g_{ij} \pi^k_k - \pi_{ij} \right) + (\delta \nabla_j M_i + \nabla_i M_j) \]

This equation relates the change in \( h_{ij} \) between two close 3-geometries to the momentum \( \pi_{ij} \) and is discussed in [29]. In order to put this in a more familiar form one notes that

(6.13) \[ \delta M = \int \frac{\delta M}{\delta \sigma(x')} \delta \sigma(x') d^3x'; \quad \delta M_i = \int \frac{\delta M_i}{\delta \sigma(x')} \delta \sigma(x') d^3x' \]

Hence

(6.14) \[ \frac{\delta h_{ij}}{\delta \sigma(x')} = -2\frac{\delta M}{\delta \sigma(x')} g^{-1/2} \left( \frac{1}{2} g_{ij} \pi^k_k - \pi_{ij} \right) + \frac{\delta \nabla_j M_i + \delta \nabla_i M_j}{\delta \sigma(x')} \]

which amounts to

(6.15) \[ \frac{\delta M(x, \sigma)}{\delta \sigma(x')} [(1/2)g_{ij} \pi^k_k - \pi_{ij}] = \frac{1}{2} \left( \frac{\delta \nabla_j M_i}{\delta \sigma(x')} + \frac{\delta \nabla_i M_j}{\delta \sigma(x')} - \frac{\delta h_{ij}}{\delta \sigma(x')} \right) g^{1/2} \]

These equations are still manifestly covariant and by introducing

(6.16) \[ H_0(x') = -\int \frac{\delta M(x)}{\delta \sigma(x')} R_0(x) d^3x; \quad H_1(x') = -2 \int \frac{\delta M_i(x)}{\delta \sigma(x')} \nabla_j \pi^j_i(x) d^3x \]

one can rewrite (6.14) as

(6.17) \[ \frac{\partial h_{ij}(x, \sigma)}{\partial \sigma(x')} = \frac{\delta (H_0(x') + H_1(x'))}{\delta \pi^j_i(x)} \]

Note that \( h_{ij}(x, \sigma) \) is a functional of \( \sigma(x') \) and introduce a particular parameter for the hypersurface, say \( \sigma(x') = t \), in which case

(6.18) \[ \frac{\partial h_{ij}(x, t)}{\partial t} = \int \frac{\delta h_{ij}(x, \sigma)}{\delta \sigma(x')} d^3x' \]

Integrating (6.14)-(6.15) in \( x' \) gives

(6.19) \[ \frac{\partial h_{ij}}{\partial t} = -2N(h)^{-1/2} [(1/2)g_{ij} \pi^k_k - \pi_{ij}] + 2\nabla_j N_i \equiv \]

\[ \equiv (1/2)h_{ij} \pi^k_k - \pi_{ij} = h^{1/2}(\nabla_j N_i + \nabla_i N_j - \partial_t h_{ij})/2N \]

where

(6.20) \[ \nabla_j N_i = \int \frac{\delta \nabla_i M_j}{\delta \sigma(x')} d^3x'; \quad N = \int \frac{\delta M}{\delta \sigma(x')} d^3x' \]

Now two conclusions can be drawn from (6.12) and (6.14)

- The term \( \nabla_j N_i \) transforms like a 3-tensor so \( N_i \) is a covariant 3-vector
- The factor \( N \) transforms like a 3-scalar.
In addition the two equations serve two purposes

- (6.12) reveals how a tangent vector \( \delta^3 \mathcal{G} / \delta \sigma(x') \) must be related to \( \pi^{ij} = \delta S / \delta h_{ij} \) if this vector is tangent to its history
- (6.19) serves as a definition of the extrinsic curvature if one sets

\[
(6.21) \quad h^{-1/2} [(1/2) g_{ij} \pi^k_k - \pi_{ij}] = K_{ij}
\]

provided that one identifies the hypersurface parameter \( t \) with the fourth coordinate and the functions \( N \) and \( N_i \) with the lapse and shift functions (cf. (6.2))

Having determined how \( h_{ij} \) varies along a classical history (half of the dynamical equations) one does the same thing for \( \pi_{ij} \) via

\[
(6.22) \quad \frac{\delta \pi^{ij}}{\delta \sigma} = \int \frac{\delta \pi^{kl}(x)}{\delta h_{ij}(x')} \frac{\delta h_{ij}(x')}{\delta \sigma} d^3x' = \int \frac{\delta \pi^{ij}(x')}{\delta h_{kl}(x)} \frac{\delta h_{ij}(x')}{\delta \sigma} d^3x
\]

The EHJ equation (6.1) holds for all \( 3 \mathcal{G} \) and hence the functional derivatives of (6.1) and (6D) with respect to \( h_{ij}(x) \) must vanish at all functions \( h_{ij} \), so

\[
(6.23) \quad 0 = \int \frac{\delta H_0}{\delta \pi^{ij}(x')} \frac{\delta \pi^{ij}(x')}{\delta h_{kl}(x')} d^3x' + \frac{\delta H_0}{\delta h_{kl}(x)}
\]

\[
0 = \int \frac{\delta H_1}{\delta \pi^{ij}(x')} \frac{\delta \pi^{ij}(x')}{\delta h_{kl}(x')} d^3x' + \frac{\delta H_1}{\delta h_{kl}(x)}
\]

To evaluate the expression on the right in (6.22) put in (6.17) for \( \delta h_{ij} / \delta \sigma \) to get

\[
(6.24) \quad \frac{\delta \pi^{kl}(x)}{\delta \sigma} = \int \frac{\delta \pi^{ij}(x')}{\delta h_{kl}(x')} \left( \frac{\delta H_0}{\delta \pi^{ij}(x')} + \frac{\delta H_1}{\delta \pi^{ij}(x')} \right) d^3x'
\]

But via (6.22) the right side of this reduces to

\[
(6.25) \quad \frac{\delta \pi^{kl}(x)}{\delta \sigma} = - \frac{\delta (H_0 + H_1)}{\delta h_{kl}}
\]

Hence the change in \( \pi^{kl} \) for a given test function \( \delta \sigma(x') \) is

\[
(6.26) \quad \delta \pi^{kl}(x) = - \int \frac{\delta [H_0(x') + H_1(x')]}{\delta h_{kl}(x)} \delta \sigma(x') d^3x'
\]

The ensuing momentum equations (6.25–6.26) are also manifestly covariant.

One has now obtained 3 constraint equations (6D) and two sets of equations (6.17) and (6.25); it remains to show that these plus the EHJ equation are equivalent to the ten Efe. First it is shown (cf. [29]) that \( \delta \pi^{ij} + \pi^{ij} \) is
a legitimate momentum in that it satisfies (6.1) and (6D). Then write the available equations (6.8, 6.17, 6.25), and (6I) together as

\begin{equation}
3R + h^{-1}[(1/2)h_{ij}h_{k\ell} - h_{ik}h_{j\ell}]\pi^{ij}\pi^{k\ell} = 0; \nabla_j\pi^{ij} = 0;
\end{equation}

\begin{equation}
\frac{\delta h_{ij}(x)}{\delta \sigma(x')} = \frac{\delta[H_0(x') + H_1(x')]}{\delta \pi^{ij}(x')}; \frac{\delta \pi^{ij}}{\delta \sigma(x')} = -\frac{\delta[H_0(x') + H_1(x')]}{\delta h_{ij}(x)}
\end{equation}

However the first 2 equations are essentially contained in the last 2 equations (once this holds at the initial point). The last two equations are covariant and hold on every 3-D slice through spacetime. That the above four equations imply the ten Efe can be best seen by observing that these equations can be derived from a variational principle whose Lagrangian is (6.28)

\begin{equation}
\mathcal{L} = \int \left[ \frac{\delta h_{ij}}{\delta \sigma(x')} \pi^{ij} + \frac{\delta M}{\delta \sigma(x')} \times [h^{1/2}3R + h^{-1/2}((1/2)\pi^{ij}\pi^{ij} - \pi_{ij}\pi^{ij})] + 2\frac{\delta M}{\delta \sigma(x')} \nabla^i\pi^{ij} - 2(\frac{\delta \pi^{ij}}{\delta \sigma(x')} - \frac{1}{2}\pi^{ij}\frac{\delta M}{\delta \sigma(x')} + h^{1/2}\frac{\delta M}{\delta \sigma(x')})_i \right] d^3x' = \partial_t h_{ij}\pi^{ij} + N[h^{1/2}3R + h^{-1/2}((1/2)\pi^{ij}\pi^{ij})] - 2N_i\nabla_j\pi^{ij} - (2\pi^{ij}N_j - (1/2)\pi^{ij}N^i + h^{1/2}N^{ij})_i,
\end{equation}

(the notation \(f_i\) presumably means \(\partial f\)). This Lagrangian for the 3+1 formulation is equal to (6K) \(\mathcal{L} = (\pi^4)g^{1/2}R\). The necessary identifications with the 4-geometry are then

\begin{equation}
\pi^{ij} = h(4T_0^0 - h_{mn}4\Gamma^k_{m\ell}h^{k\ell})^4g^{im}g^{jn}; (Nh)^{1/2} = (-\pi^4)^{1/2}
\end{equation}

Denoting the Efe by \(G_{\mu\nu} = 0 (\mu, \nu = 0, 1, 2, 3)\) then (6.8) and (6I) are \(G_0^i = 0\) while (6.25) is a linear combination of these equations together with the remaining 6 Efe where (6.17) serves as the definition of \(\pi^{ij}(x)\).

Putting in now \(\psi = \exp(iS/h)\) in superspace, \(S\) is the solution of the EHJ equation \(\nabla^2 - (\delta S/\delta \Theta)(\delta S/\delta \Theta)^* = 0\).

### 6.1 MULTIFINGERED TIME.

The discussion of the multifingered time (MFT) of Tomonaga in [29] can be improved as in [56] (cf. also [14] [80] [92]). Let \(x = \{x^\mu\} = (x^0, x)\) be spacetime coordinates. A timelike Cauchy hypersurface \(\Sigma\) can be defined via a function \(T(x)\) via the equation ((6M) \(x^0 = T(x)\)). If \(T(x)\) is given then \(x \in \Sigma\) is correct and if \(\sigma \subset \Sigma\) then \(T_\sigma\) denotes the set of values for \(x \in \sigma\). For a scalar field \(\phi\) one describes its dynamics via

\begin{equation}
\hat{H}(x)\psi[\phi, T] = \frac{\delta \psi[\phi, T]}{\delta T(x)}
\end{equation}
A wave functional \( \psi[\phi, T] \) can be viewed as a functional of \( \phi \) and (6.30) shows how \( \psi \) changes for an infinitesimal change \( \delta T(x) \) (we will occasionally omit boldface on \( x \) now). Thus (6.30) is a generalized SE but it does not involve any preferred foliation of spacetime. Since \( \Sigma \) is determined by \( T \) one can say that \( \rho[\phi, T] = |\psi[\phi, T]|^2 \) is the probability density for the field to have the value \( \phi \) at time \( T \) but remember that \( T \) is a collection of real parameters with one real parameter for each point \( x \). Consider now a free scalar field with Hamiltonian density (6N) \( H(x) = -(1/2)(\delta^2/\delta \phi^2(x)) + (1/2)[(\nabla \phi(x))^2 + m^2 \phi^2(x)] \). Then writing \( \psi = R \exp(iS) \) one obtains

\[
\frac{1}{2} \left( \frac{\delta S}{\delta \phi(x)} \right)^2 + \frac{1}{2}[(\nabla \phi(x))^2 + m^2 \phi^2(x)] + \Omega(x, \phi, T) + \frac{\delta S}{\delta T(x)} = 0
\]

(6.31)

\[
\frac{\delta \rho}{\delta T(x)} + \frac{\delta}{\delta \phi(x)} \left( \rho \frac{\delta S}{\delta \phi(x)} \right) = 0; \quad \Omega = -\frac{1}{2} \frac{\delta^2 R}{\delta \phi^2(x)}
\]

(6.32)

The Bohmian interpretation involves a deterministic time dependent hidden variable such that the time evolution of this variable is consistent with the probabilistic interpretation of \( \rho \). This is naturally achieved by introducing a MFT field \( \Phi(x, T) \) satisfying the MFT Bohmian equation of motion

\[
\frac{\delta \Phi(x, T)}{\delta T(x')} = \frac{\delta^3(x - x')}{\delta \phi(x)} \left. \frac{\delta S}{\delta \phi(x)} \right|_{\Phi=\phi}; \quad \int_{\sigma_x} d^3 x' \frac{\delta \Phi(x, T)}{\delta T(x')} = \left. \frac{\delta S}{\delta \phi(x)} \right|_{\Phi=\phi}
\]

(6.33)

where \( \sigma_x \) is an arbitrarily small region around \( x \). The second equation in (6.33) is the MFT version of the usual single-time Bohmian equation of motion \( \partial_t \Phi(x, t) = (\delta S/\delta \phi(x))|_{\Phi=\phi} \) whereas the first equation is more fundamental since no \( \sigma_x \) is involved. For comparison purposes however integration within \( \sigma_x \) is useful; e.g. using (6.31) and (6.33) one has

\[
\left[ \left( \int_{\sigma_x} d^3 x' \frac{\delta}{\delta T(x')} \right)^2 - \nabla_x^2 + m^2 \right] \Phi(x, T) = -\int_{\sigma_x} d^3 x' \partial \Omega(x', \phi, T) \left|_{\Phi=\phi} \right.
\]

(6.34)

This can be viewed as an MFT Klein-Gordon equation with a quantum term added. Note that officially one should write \( \Phi(x, T(x)) = \phi(x, x^0) = \Phi(x) \) and we assume this is understood throughout.

Now to provide a manifestly covariant QFT one introduces \( s = (s^1, s^2, s^3) \) which serve as coordinates on a 3-D manifold; then write \( x^\mu = X^\mu(s) \) leading to one equation \( f(x^0, x^1, x^2, x^3) = 0 \) determining a 3-D hypersurface in spacetime. Assume a background metric \( g_{\mu\nu}(x) \) is given with induced metric

\[
h_{ij}(s) = g_{\mu\nu}(X(s)) \frac{\partial X^\mu(s)}{\partial s^i} \frac{\partial X^\nu(s)}{\partial s^j}
\]

(6.35)
on the hypersurface. A normal and unit normal to this surface is then
\[ \tilde{n}_\mu(s) = \epsilon_{\mu\alpha\beta\gamma} \frac{\partial X^\alpha}{\partial s^1} \frac{\partial X^\beta}{\partial s^2} \frac{\partial X^\gamma}{\partial s^3}; \quad n^\mu(s) = \frac{g^{\mu\nu} \tilde{n}_\nu}{\sqrt{|g_{\alpha\beta} \tilde{n}_\alpha \tilde{n}_\beta|}} \]

Now the equations above can be written in a covariant form via
\[ \frac{\delta}{\delta X^\mu(s)} \rightarrow \frac{\delta}{\delta X^\mu(s)} \equiv \frac{\delta}{\delta X^\mu(s)} \]

The Tomonaga-Schwinger equation (6.30) becomes then
\[ \hat{H}(s)\psi[\phi, X] = i n^\mu(s) \frac{\delta \psi[\phi, X]}{\delta X^\mu(s)} \]

and for free fields the Hamiltonian density operator in curved spacetime is
\[ \hat{H} = -\frac{1}{2|h|^{1/2}} \frac{\delta^2}{\delta \phi^2(s)} + \frac{|h|^{1/2}}{2} [-h^{ij}(\partial_i \phi)(\partial_j \phi) + m^2 \phi^2] \]

The Bohmian equations of motion (6.33) become
\[ \frac{\delta \Phi(s, X)}{\delta \tau(s')} = \frac{\delta^3(s - s')}{|h(s)|^{1/2}} \frac{\delta S}{\delta \phi(s)} \bigg|_{\phi=\Phi} \]

and (6.34) becomes
\[ \left( \int_{x'} d^3 s' \frac{\delta}{\delta \tau(s')} \right)^2 + \nabla_i \nabla_i + m^2 \right] \Phi(s, X) = - \int_{x'} d^3 s' \frac{\delta \Omega(s', \phi, X)}{\sqrt{|h|}} \frac{\delta \phi(s)}{\delta \phi(s)} \bigg|_{\phi=\Phi} \]

where \( \nabla_i \) is the covariant derivative in \( s^i \) and
\[ \Omega(s, \phi, X) = - \frac{1}{\sqrt{|h(s)|}} \frac{1}{2R} \frac{\delta^2 R}{\delta \phi^2(s)} \]

There is a sort of gauge freedom associated related to the covariance due to the freedom in choosing the \( X^\mu(s) \). For a timelike hypersurface the simplest choice of gauge is \( X^i(s) = s^i \). This choice implies \( \delta X^i(s) = 0 \) which leads to some of the previous equations prior to covariance. For example (6.40) becomes
\[ (g^{00}(x))^{1/2} \frac{\delta \Phi(x, X^0)}{\delta X^0(x')} = \frac{\delta^3(x - x')}{|h(x)|^{1/2}} \frac{\delta S}{\delta \phi(x)} \bigg|_{\phi=\Phi} \]

which is the curved spacetime version of (6.33). The covariant formulation of QFT leads to a covariant MFT Bohmian interpretation of quantum fields which also does not involve a preferred foliation of spacetime. The covariant Bohmian dynamics does not depend on the choice of coordinates but when a choice is made then the solution of the MFT Bohmian equations of motion can be written so that the MFT nature of the field is not manifest. However the Bohmian equation of motion retains its covariant form.
7. TIME

We have seen how MFT arises in QFT and we want to examine this further in connection with gravity. We begin with remarks based on [2, 3, 9, 10, 13, 21, 22, 27, 28, 31, 45, 46, 48, 49, 55, 56, 57, 67, 87, 94, 95, 96, 97]. We note first that time can arise naturally for WDW when using the dDW theory but a MFT approach seems to require the semiclassical approach and some interaction with matter (see however [29] as discussed in Section 6). Weakening the Hamiltonian constraint as in [57] (discussed in Remark 3.1) also provides a time. The semiclassical approach is illustrated in [10, 29, 48] for example and we will sketch some of this here following [31].

We forgo the sandwich ordering here for convenience - it remains our principal ordering candidate however. Thus consider \((c = 1)\)

\[
H \psi[h_{ab}, \phi] = \left( -16\pi G h^2 G_{abcd} \frac{\delta^2}{\delta h_{ab} \delta h_{cd}} - \frac{\sqrt{h}}{16\pi G} (R - 2\Lambda) + H_m \right) \psi = 0
\]

The integrated form of (7.1) is

\[
\int d^3 x N H \psi \equiv H^N \psi = (H_N^G + H_N^m) \psi = 0
\]

One uses now an Ansatz \((M = 32\pi G)^{-1}\)

\[
\psi = \exp \left[ i(M S_0 + S_1 + M^{-1} S_2 + \cdots) / \hbar \right]
\]

leading to a set of equations of consecutive orders in M. The highest order \(M^2\) shows that \(S_0\) depends only on the 3-metric \(h\) (cf. [48]) and the next order \(M\) gives the HJ equation for the gravitational field

\[
H_x = \frac{1}{2} G_{abcd} \frac{\delta S_0}{\delta h_{ab}} \frac{\delta S_0}{\delta h_{cd}} - 2\sqrt{h} (R - 2\Lambda) = 0
\]

Note that these depend on the lapse function \(N(x)\). At the next order \(M^0\) it is convenient to introduce a functional (7A) \(\psi = D(h_{ab}) \exp(i S_1 / \hbar)\) and require that \(D\) satisfies

\[
G_{abcd} \frac{\delta S_0}{\delta h_{ab}} \frac{\delta D}{\delta h_{cd}} - \frac{1}{2} G_{abcd} \frac{\delta^2 S_0}{\delta h_{ab} \delta h_{cd}} D = 0
\]

(note \(D\) corresponds to the vanVleck determinant). The important observation here is that \(\psi\) obeys the equation

\[
i \hbar G_{abcd} \frac{\delta S_0}{\delta h_{ab}} \frac{\delta \psi}{\delta h_{cd}} = H_m \psi
\]

which can be rewritten in terms of vector fields

\[
\chi(x) = G_{abcd} \frac{\delta S_0}{\delta h_{ab}(x)} \frac{\delta}{\delta h_{cd}(x)} = -2K_{cd} \frac{\delta}{\delta h_{cd}(x)}; K_{cd} = \frac{1}{2} G_{abcd} \frac{\delta S_0}{\delta h_{ab}}
\]
where $K_{cd}$ has the meaning of an extrinsic curvature. If one now writes (7B) $\chi(x) = (\delta/\delta \tau(x))$ then (7.6) would be a Tomonaga-Schwinger equation with respect to the MFT $\tau(x)$ (note $\tau$ is really a function on $\text{Riem}(\Sigma)$). However this leads to a contradiction since $[(\delta/\delta \tau(x), \delta/\delta \tau(y))] = 0$ of necessity but $[H_m(x), H_m(y)] \neq 0$. One writes then (7C) $i h \chi^N = H^{N}_m \psi$ and (with some argument) shows that in fact

$$ (7.8) \ [\chi^N, \chi^M] = -2 \int_x (N \partial_a M - M \partial_a N) \Delta_b \left( \frac{\delta}{\delta h_{ab}} \right) = \int \Delta_R h_{ab} \frac{\delta}{\delta h_{ab}} $$

where (7D) $K^a = h^{ab}(N \partial_b M - M \partial_b N)$. Hence $[\chi^N, \chi^M] \neq 0$ and time functions as above can never be introduced (because the Ricci scalar $R$ is not ultralocal in $h_{ab}$). The vector fields $\chi^N$ are generators of a hypersurface deformation normal to itself and the commutator generates stretchings of the hypersurface. A proper understanding of (7.14) and its compatibility with (7.6) is obtained however if one expands the diffeomorphism constraints in powers of G (or M) which gives (7E) $2h_{bc} \frac{\delta S_0}{\delta h_{ab}} = 0$ (cf. [47] for notation - $D_a \sim$ covariant derivative). The highest order M yields (since $S_0$ does not depend on the scalar field $\phi$) (7F) $2h_{bc} \frac{\delta S_0}{\delta h_{ab}} = 0$ (diffeomorphism invariance of $S_0$). The next order $M^0$ leads to a condition on $\psi$, namely

$$ (7.9) \ 2h_{bc} \frac{\delta \psi}{\delta h_{ab}} = \phi_x \frac{\delta \psi}{\delta \phi} $$

Since $D$ depends only on the 3-metric (cf. (7G)) it is appropriate to demand that it be diffeomorphism invariant by itself, i.e. (7G) $h_{bc} \frac{\delta D}{\delta h_{ab}} = 0$. From (7.15) one finds then (7H) $2h_{bc} \frac{\delta \psi}{\delta h_{ab}} = \phi_x \frac{\delta \psi}{\delta \phi}$ which is of the same form as the general solution (7E). Thus it expresses the invariance of the wave functional $\psi[h_{ab}, \phi]$ with respect to simultaneous diffeomorphisms of the metric and matter field. The consistency condition for (7C) is (7I) $[\chi^N, \chi^M] \psi = [H^M_m, H^N_m] \psi$. This however is nothing but the momentum constraint in this order of approximation, namely (7H), since $[\chi^N, \chi^M]$ generates a diffeomorphism of the metric, (7.9), and $[H^M_m, H^N_m]$ closes on the momentum density of matter which generates a diffeomorphism of the matter field. Thus in the full theory the momentum constraints provide the integrability conditions for the Tomonaga-Schwinger equations (7C).

In the explicit case of a scalar field one has e.g.

$$ (7.10) \ [H^M_m, H^N_m] = - \int_x (N \partial_a M - M \partial_a N) h^{ab} \phi \frac{\delta}{\delta \phi} $$

Although a family of time functions $\tau(x)$ on $\text{Riem}(\Sigma)$ does not exist one can integrate (7C) along the vector field $\chi^N$ for one particular choice of N and this defines a global time parameter $t$ with respect to which one global SE
can be written down. It is in this sense that QFT with respect to a chosen foliation emerges from full quantum gravity. If there are no such general time functions on $Riem(\Sigma)$ what about $S(\Sigma) = Riem(\Sigma)/Diff(\Sigma)$? To answer this one projects the vector fields $\chi^N$ to S which is possible since $\chi^N$ is invariant under diffeomorphisms - referring to [31] for details one arrives at

\[(7.11) \quad \pi_*[\chi^N, \chi^M] = [\pi_*\chi^N, \pi_*\chi^M] = 0\]

and there exist functions $\tilde{\tau}^N$ on S such that $(7J) \chi^N = \delta/\delta \tilde{\tau}^N$ where $\tilde{\chi} = \pi_*\chi$, etc. However the WDW operator is only defined on $Riem(\Sigma)$ and some of the intervening calculations do not make sense on S. There is further discussion of anomalies, etc. that is worth reading. This paper corrects some confusion about the existence of Tomonaga-Schwinger times on $Riem(\Sigma)$ in other papers (e.g. [10, 48]) and one should also exercise caution in this respect relative to the calculations from [29] in Section 6.

7.1. EXTRINSIC CURVATURE AND TIME. We go now to some papers [2, 3, 27, 28, 45, 71, 96, 97] where from [2] one recalls that it is not N but the slicing density $\alpha(x, t) = Nh^{-1/2}$ is the freely specifiable quantity for the lapse. One writes then

\[(7.12) \quad ds^2 = -N^2 dt^2 + h_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)\]

(in what follows $R \sim 3^R$). The momentum conjugate to the metric is a density of weight one $\pi^{ij} = h^{1/2}(Kh^{ij} - K^{ij})$ where $K^{ij}$ is the extrinsic curvature with trace K. The natural time derivative for evolution $\hat{\partial}_0$ acts in the normal future direction to the spacelike slice $\Sigma$ and is denoted by an over-dot; one has $\hat{\partial}_0 = \partial_t - \Sigma_\beta$ where $\Sigma_\beta$ is the Lie derivative along the shift $\beta$. Every foliation is described by a wave equation for N for some value of $\alpha$ thus making N a dynamical variable. The Hamiltonian constraint does not fix the time but does fix the proper time rate $d\tau/dt = ah^{1/2} = N$ along the normal $\partial_0$. Using $\alpha$ has the effect of altering the Hamiltonian density from H to

\[(7.13) \quad \tilde{H} = h^{1/2}H = \pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2 - hR\]

which is of scalar weight 2 and a rational function of the metric. $\tilde{H}$ will be referred to as the Hamiltonian density and may not vanish. This leads to a modification of the ADM action as in [4, 91], namely $(16\pi G = c = 1)$

\[(7.14) \quad S(h, \pi, \alpha, \beta) = \int d^4x(\pi^{ij}\dot{h}_{ij} - \alpha\tilde{H})\]

(one assumes $N \sim 1 + O(r^{-1})$). Explicitly the Lie derivative term in $\dot{\pi}^{ij}$ is, up to a divergence, $(7K) \quad 2\beta^i \nabla_j \pi^i_j = -\beta^i H_i$. 
Consider now a general variation of the modified Hamiltonian density

\[ \delta \tilde{H} = (2\pi_{ij} - h_{ij}\pi)\delta \pi^{ij} + (2\pi^{ik}\pi_k^j - \pi \pi_{ij}) + hR^{ij} - hh^{ij}R)\delta h_{ij} - h(\nabla^i \nabla^j \delta h_{ij} - h^{ij}\nabla^k \nabla_k \delta h_{ij}) \]

Note that this does not involve either the Hamiltonian or momentum densities; in contrast the variation of the ADM Hamiltonian density \( \delta H = \delta(h^{-1/2}\tilde{H}) \) does contain a term proportional to the Hamiltonian density. Requiring that \( S \) above be stationary under a variation with respect to \( \pi^{ij} \) gives the definition of the extrinsic curvature

\[ \dot{h}_{ij} = \alpha \frac{\delta \tilde{H}}{\delta \pi^{ij}} = \alpha(2\pi_{ij} - h_{ij}\pi) \equiv -2NK_{ij} \]

Requiring stationarity under a variation in \( h_{ij} \) gives the equation of motion

\[ \ddot{\pi}^{ij} = -\alpha \frac{\delta \tilde{H}}{\delta h_{ij}} = -\alpha h(R^{ij} - h^{ij}R) - \alpha(2\pi^{ik}\pi_k^j - \pi \pi_{ij}) - h(\nabla^i \nabla^j \alpha - h^{ij}\nabla^k \nabla_k \alpha) \]

The slicing density \( \alpha \) and the shift \( \beta^i \) are not to be varied; instead the constraints are imposed on initial data and are preserved dynamically as shown below. Thus consider the familiar 3+1 identities

\[ h_{ij} \equiv -2NK_{ij}; \quad K_{ij} \equiv N(R_{ij} - 4\pi_{ij} + K K_{ij} - K_{ik}K^k_j - N^{-1}\nabla_i \nabla_j N) \]

One recalls also that \( h^{-1}\dot{h} = h^{ij}\dot{h}_{ij} = -2NK \). Now pass to canonical variables and use (7.18) to arrive at

\[ \dot{\pi}^{ij} \equiv Nh^{1/2}(Rh^{ij} - R^{ij}) - Nh^{-1/2}(2\pi^{ik}\pi_k^j - \pi \pi_{ij}) + h^{1/2}(\nabla^i \nabla^j N - h^{ij}\nabla^k \nabla_k N) + Nh^{1/2}\mathfrak{R}^{ij}; \quad \mathfrak{R}^{ij} = 4R_{ij} - h_{ij}4R^k_k \]

One sees that the equations of motion (7.16)-(7.17) derived from the action principle are (7.18)-(7.19) when \( 4R_{ij} - h_{ij}4R^k_k = 0 \). Thus to say that (7.17) holds is to assert that \( 4R^{ij} = 0 \). In fact the equations of motion hold strongly independent of whether the constraints are satisfied or not and this is not true in the ADM formulation because of the presence of the Hamiltonian density in the equations of motion for \( \pi^{ij} \). This difference can be explained more fully as follows. Given \( G_{\mu\nu} = 4R_{\mu\nu} - (1/2)g_{\mu\nu}4R^k_k \) and the observation that \( 2G^0_0 = 4R^0_0 - 4R^k_k \) one has \( G_{ij} + h_{ij}G^0_0 \equiv 4R_{ij} - h_{ij}4R^k_k \). The vanishing of the right side does not depend on either the Hamiltonian or momentum densities and is equivalent to \( 4R_{ij} = 0 \) or \( G_{ij} = -h_{ij}G^0_0 \). Thus while \( 4R_{\mu\nu} = 0 \) and \( G_{\mu\nu} = 0 \) are equivalent \( R_{ij} = 0 \) and \( G_{ij} = 0 \) are not equivalent as equations of motion - unless the Hamiltonian density \( H = 2h^{1/2}G^0_0 \) vanishes exactly (i.e. unless the Hamiltonian constraint holds). The ADM action principle is equivalent to
\( G_{ij} = 0 \) and one recalls that the use of \( R_{ij} \) instead of \( G_{ij} \) has always been preferred by the French school. This raises the important principle that a constrained Hamiltonian theory should be well behaved even when the constraints are violated. There is much further calculation in this direction which we omit here (cf. [71, 97]).

**REMARK 7.1.** There is a great deal of material now available on general relativity in terms of Ashtekar variables (see e.g. [6, 7, 8, 32, 42, 47, 53, 76, 78, 81, 88, 89, 90] for a very incomplete list of references on loop quantum gravity, etc.). In [89] for example one recasts the WDW equation in the new variables in terms of the 3-geometry elements \( C \) and \( K \) where \( C \) is the Chern-Simons functional and \( \mathcal{R} \) is the integral of the trace of the extrinsic curvature (cf. also [88]).
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