THE CATLIN MULTITYPE AND BIHOLOMORPHIC
EQUIVALENCE OF MODELS

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Abstract. We consider an alternative approach to a fundamental CR invariant – the Catlin multitype. It is applied to a general smooth hypersurface in $\mathbb{C}^{n+1}$, not necessarily pseudoconvex. Using this approach, we prove biholomorphic equivalence of models, and give an explicit description of biholomorphisms between different models. A constructive finite algorithm for computing the multitype is described. The results can be viewed as providing a necessary step in understanding local biholomorphic equivalence of Levi degenerate hypersurfaces of finite Catlin multitype.

1. Introduction

The subject of this paper is local biholomorphic geometry of Levi-degenerate hypersurfaces in $\mathbb{C}^{n+1}$, and a fundamental CR invariant – the Catlin multitype. We consider a constructive approach, which allows to understand the local equivalence problem on the level of weighted homogeneous models.

The problem of local biholomorphic equivalence for real hypersurfaces in complex space has a long history (we refer to the survey articles [1], [17] for a historical account). In recent years, the problem has been intensively studied on Levi degenerate manifolds, mostly using the extrinsic approach of Poincaré and Moser. In fact, a result of [11] indicates that the intrinsic approach of Cartan, Chern and Tanaka is in general not available in the degenerate setting.

We start by reviewing some motivating facts from complex dimension two. The lowest order CR invariant of a smooth hypersurface $M \subseteq \mathbb{C}^2$ at a point $p \in M$ is the type of the point, introduced by J. J. Kohn in his pioneering work [12]. The type is an integer measuring the maximal order of contact between $M$ and complex curves passing through $p$. In terms of coordinates, the point is of finite type $k$ if and only if there exist local holomorphic coordinates $(z, w)$ such that the defining equation for $M$ takes form

$$\text{Im } w = P(z, \bar{z}) + o(\text{Re } w, |z|^k),$$

where $P$ is a polynomial.
where $P$ is a nonzero homogeneous polynomial of degree $k$ without harmonic terms. The manifold $\text{Im } w = P(z, \bar{z})$ is the model hypersurface at $p$. Here $P$ is determined uniquely up to linear transformations in the complex tangential variable $z$, and one immediately obtains important invariants from the coefficients of $P$ (see e.g. [13], [14]).

To study higher order invariants, consider a biholomorphic transformation

$$w^* = g(z, w), \quad z^* = f(z, w),$$

which preserves the local description (1). The main tool for analyzing the action of (2) on the defining equation of $M$ is the generalized Chern-Moser operator

$$(3) \quad L(f, g) = \text{Re} \left\{ ig(z, \text{Re } w + iP(z, \bar{z})) + 2 \frac{\partial P}{\partial z} f(z, \text{Re } w + iP(z, \bar{z})) \right\},$$

whose existence is a fundamental consequence of the finite type condition. Examining the kernel and image of $L$ one can construct a complete set of local invariants ([14]).

In higher dimensions, local geometry of Levi degenerate hypersurfaces is far more complicated, even on the initial level. Invariants relevant for analysis of the inhomogeneous Cauchy-Riemann equations are now obtained by considering orders of contact with singular complex varieties. If $d_k$ denotes the maximal order of contact of $M$ with complex varieties of dimension $k$ at $p$, the n-tuple $(d_n, \ldots, d_1)$ is called the D’Angelo multitype of $M$ at $p$ ([7]).

For pseudoconvex hypersurfaces, D. Catlin ([4]) introduced a different notion of multitype, using a more algebraic approach. The entries of the Catlin multitype take rational values, but need not be integers, anymore. This approach provides a defining equation analogous to (1), and a well defined weighted-homogeneous model, an essential tool for local analysis (see e.g. [9], [10]).

There is a class of hypersurfaces on which the two multitypes coincide (termed semiregular [8], or h-extendible [18]), but in the most interesting instances, the two multitypes are not equal.

In this paper we use Catlin’s definition of multitype for a general smooth hypersurface in $\mathbb{C}^{n+1}$. The definition itself is nonconstructive, and the corresponding models are not uniquely defined. In order to study higher order CR invariants it becomes essential to understand the non-uniqueness in the definition of models. In particular, it is not a priori clear if all models have to be biholomorphically equivalent (for
pseudoconvex h-extendible hypersurfaces this problem was considered in [16].

Hypersurfaces of finite Catlin multitype provide the natural class of manifolds for which a generalization of the Chern-Moser operator is well defined.

We denote again by $P$ the leading weighted homogeneous polynomial determined by Catlin’s construction, and consider a biholomorphic transformation

$$w^* = w + g(z, w), \quad z_i^* = z_i + f_i(z, w).$$

The operator now takes form

$$L(f, g) = \text{Re} \left \{ i g(z, \text{Re} w + i P(z, \bar{z})) + 2 \sum_{j=1}^{n} \frac{\partial P}{\partial z_j} f_j(z, \text{Re} w + i P(z, \bar{z})) \right \}.$$

The first necessary step in understanding this operator is to consider the strictly subhomogeneous level, in the sense of Definition 2.3 below. Our results imply, in particular, that the kernel of $L$ is always trivial on this level. Analysis of the kernel and image of $L$, and applications to the local equivalence problem is the subject of a forthcoming article.

The paper is organized as follows. In Section 2 we define the Catlin multitype of a general smooth hypersurface in $\mathbb{C}^{n+1}$. This leads to distinguished weighted coordinate systems. Then we consider the associated weighted homogeneous transformations, and define their subhomogeneous and superhomogeneous analogs. In Section 3 we analyze model hypersurfaces, and define a normalization, which is used in an essential way in the following section.

Section 4 considers the biholomorphic equivalence problem for models. We prove that all models at a given point are biholomorphically equivalent, by explicitly described polynomial transformations. Using this result we give in Section 5 a constructive finite algorithm for computing the multitype.

2. Hypersurfaces of Finite Multitype

Let $M \subseteq \mathbb{C}^{n+1}$ be a smooth hypersurface (not necessarily pseudoconvex), and $p$ be a Levi degenerate point on $M$. We will assume that $p$ is a point of finite type in the sense of Bloom and Graham. Throughout the paper, the standard multiindex notation will be used.

Let $(z, w)$ be local holomorphic coordinates centered at $p$, where $z = (z_1, z_2, ..., z_n)$ and $z_j = x_j + iy_j$, $w = u + iv$. The hyperplane
\{ v = 0 \} is assumed to be tangent to \( M \) at \( p \). We describe \( M \) near \( p \) as the graph of a uniquely determined real valued function

\[
v = \Psi(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n, u).
\]

The definition of multitype is based on weighted coordinate systems. Roughly speaking, the weights measure the order of vanishing of a suitably chosen defining function in each of the variables. As the first step, the weights of the complex nontangential variables \( w, u \) and \( v \) are set equal to one. Then we consider the complex tangential variables.

**Definition 2.1.** A weight is an \( n \)-tuple of nonnegative rational numbers \( \Lambda = (\lambda_1, \ldots, \lambda_n) \), where \( 0 \leq \lambda_j \leq \frac{1}{2} \), and \( \lambda_j \geq \lambda_{j+1} \), such that for each \( k \) either \( \lambda_k = 0 \), or there exist nonnegative integers \( a_1, \ldots, a_k \) satisfying

\[
\sum_{j=1}^k a_j \lambda_j = 1.
\]

If \( \Lambda \) is a weight, the weighted degree of a monomial \( c_{\alpha\beta} z^\alpha \bar{z}^\beta u^l \) is defined to be

\[
l + \sum_{i=1}^n (\alpha_i + \beta_i) \lambda_i.
\]

A polynomial \( P(z, \bar{z}, u) \) is \( \Lambda \)-homogeneous of weighted degree \( \kappa \) if it is a sum of monomials of weighted degree \( \kappa \).

The weighted length of a multiindex \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is defined by

\[
|\alpha|_\Lambda = \lambda_1 \alpha_1 + \cdots + \lambda_n \alpha_n.
\]

Similarly, if \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_n) \) are two multiindices, the weighted length of the pair \((\alpha, \hat{\alpha})\) is

\[
|(\alpha, \hat{\alpha})|_\Lambda = \lambda_1 (\alpha_1 + \hat{\alpha}_1) \cdots + \lambda_n (\alpha_n + \hat{\alpha}_n).
\]

The weighted order of a differential operator \( \frac{\partial^{|\alpha+\hat{\alpha}|+l}}{\partial z^\alpha \partial \bar{z}^\beta \bar{u}^l} \) is equal to

\[
l + |(\alpha, \hat{\alpha})|_\Lambda.
\]

A weight \( \Lambda \) will be called distinguished if there exist local holomorphic coordinates \((z, w)\) in which the defining equation of \( M \) takes form

\[
v = P(z, \bar{z}) + o_\Lambda(1),
\]

where \( P(z, \bar{z}) \) is a nonzero \( \Lambda \)-homogeneous polynomial of weighted degree one without pluriharmonic terms, and \( o_\Lambda(1) \) denotes a smooth
function whose derivatives of weighted order less than or equal to one vanish.

The fact that distinguished weights do exist follows from the assumption of Bloom-Graham finite type \((2)\).

**Definition 2.2.** Let \(\Lambda_M = (\mu_1, \ldots, \mu_n)\) be the infimum of distinguished weights with respect to the lexicographic ordering. The multitype of \(M\) at \(p\) is defined to be the \(n\)-tuple \((m_1, m_2, \ldots, m_n)\), where \(m_j = \frac{1}{\mu_j}\) if \(\mu_j \neq 0\) and \(m_j = \infty\) if \(\mu_j = 0\). If none of the \(m_j\) is infinity, we say that \(M\) is of finite multitype at \(p\).

Note that since the definition of multitype considers all distinguished weights, the infimum is a biholomorphic invariant, and we may speak of the multitype.

Coordinates corresponding to a distinguished weight \(\Lambda\), in which the local description of \(M\) has form \((7)\), with \(P\) being \(\Lambda\)-homogeneous, will be called \(\Lambda\)-adapted.

\(\Lambda_M\) will be called the multitype weight, and \(\Lambda_M\)-adapted coordinates will be also referred to as multitype coordinates.

It is easy to verify that for any \(\delta > 0\) there are only finitely many possible rational values for any weight entry, satisfying \(\lambda_i > \delta\). It follows that if \(M\) is of finite multitype at \(p\), \(\Lambda_M\)-adapted coordinates do exist.

From now on we assume that \(p \in M\) is a point of finite multitype.

Let \(t\) denote the number of different entries appearing in the multitype weight, and \(\nu_j, j = 1, \ldots, t\), be the length of the \(j\)-th constant piece of the multitype weight. Hence, denoting \(k_j = \sum_{i=1}^{j} \nu_i\), we have \(\mu_1 = \cdots = \mu_{k_1} > \mu_{k_1+1} = \cdots = \mu_{k_2} > \cdots = \mu_{k_{t-1}} > \mu_{k_{t-1}+1} = \cdots = \mu_n\).

We define a 'generating' sequence of weights \(\Lambda_1, \ldots, \Lambda_t\) as follows. \(\Lambda_1\) is the constant \(n\)-tuple \((\mu_1, \ldots, \mu_1)\) and \(\Lambda_t = \Lambda_M\) is the multitype weight. For \(1 < j < t\), the weight \(\Lambda_j = (\lambda_1^j, \ldots, \lambda_n^j)\) is defined by \(\lambda_i^j = \mu_i\) for \(i \leq k_{j-1}\), and \(\lambda_i^j = \mu_{k_{j-1}+1}\) for \(i > k_{j-1}\).

If \((7)\) is the defining equation in some multitype coordinates, we define a model hypersurface associated to \(M\) at \(p\) as

\[
M_H = \{(z, w) \in \mathbb{C}^{n+1} \mid v = P(z, \bar{z})\}.
\]

In order to analyse biholomorphisms between models, we will use the following terminology.

**Definition 2.3.** Let \(\Lambda = (\lambda_1, \ldots, \lambda_n)\) be a weight. A transformation \(w^* = w + g(z_1, \ldots, z_n, w), \quad z_i^* = z_i + f_i(z_1, \ldots, z_n, w)\)
preserving form (6) is called
- \( \Lambda \)-homogeneous if \( f_i \) is a \( \Lambda \)-homogeneous polynomial of weighted degree \( \lambda_i \) and \( g \) is a \( \Lambda \)-homogeneous polynomial of weighted degree one,
- \( \Lambda \)-subhomogeneous if \( f_i \) is a polynomial consisting of monomials of weighted degree less or equal to \( \lambda_i \) and \( g \) consists of monomials of weighted degree less or equal to one,
- \( \Lambda \)-superhomogeneous if the Taylor expansion of \( f_i \) consists of terms of weighted degree greater or equal to \( \lambda_i \) and \( g \) consists of terms of weighted degree greater or equal to one.

Note that we only consider nonsingular transformations (with non-vanishing Jacobian at the origin).

We now fix \( \Lambda_M \)-adapted coordinates, and write the corresponding leading polynomial \( P \) as

\[
P(z, \bar{z}) = \sum_{|(|\alpha|, |\hat{\alpha}|)|\Lambda_M = \mu} A_{\alpha, \hat{\alpha}} z^\alpha \bar{z}^{\hat{\alpha}}.
\]

Let \( P^k \) denote the restriction of \( P \) to the first \( k \) coordinate axes,

\[
P^k(z_1, \ldots, z_k, \bar{z}_1, \ldots, \bar{z}_k) = P(z_1, \ldots, z_k, 0, \ldots, 0, \bar{z}_1, \ldots, \bar{z}_k, 0, \ldots, 0).
\]

It follows from the definition that \( \Lambda_M \)-homogeneous transformations are of the form

\[
z_i^* = z_i + \sum_{|\alpha|\Lambda_M = \mu_i} C_{\alpha} z^\alpha, \quad w^* = cw + \sum_{|\alpha|\Lambda_M = \mu} D_{\alpha} z^\alpha
\]

where \( c \in \mathbb{R}^* \).

The set of such transformations forms a group, which will be denoted by \( \mathcal{H} \). The subgroup of \( \mathcal{H} \) consisting of transformations for which \( g = 0 \) (preserving the \( w \) variable) will be denoted by \( \mathcal{H}^Z \). Finally, let \( \mathcal{L} \) denote the subgroup of \( \mathcal{H}^Z \), consisting of all linear transformations in \( \mathcal{H}^Z \).

3. A NORMALIZATION OF THE MODEL

We will use the truncated leading polynomial \( P^k, k = 1, \ldots, n \), to define a normalization condition corresponding to \( \Lambda_M \)-homogeneous changes in the \( z_k \) variable.

Definition 3.1. Multitype coordinates \((z, w)\) are called regular, if for
each \( k = 1, \ldots, n \),

\[
\frac{\partial P^k}{\partial z_k}(z_1, \ldots, z_k, \bar{z}_1, \ldots, \bar{z}_k)
\]

is not identically zero.

The following lemma shows that regular coordinates do exist, and are in fact generic among multitype coordinates.

**Lemma 3.1.** Let \((z,w)\) be multitype coordinates. Then there exists a transformation \( H \in \mathcal{H}^Z \), such that the new coordinates are regular.

**Proof:** The proof is by induction. We will assume that \( \frac{\partial P^j}{\partial z_j} \) is not identically zero for all \( j < k \), and find transformations which preserve this and attain the condition for \( P^k \).

Let \( \kappa' \) be the largest index such that \( \mu_k = \mu_{\kappa'} \). Clearly, \( P^{\kappa'} \) has to depend on \( z_k \), otherwise we could lower the weight of \( z_k \) and obtain a lexicographically smaller distinguished weight, contradicting the definition of \( \Lambda_{M} \). Pick any monomial in \( P^k \) containing \( z_k \), say

\[
A_{\beta,\hat{\beta}} z^\beta \bar{z}^{\hat{\beta}},
\]

where \( A_{\beta,\hat{\beta}} \neq 0 \), and \((\beta, \hat{\beta})\) satisfies \( \beta_j = \hat{\beta}_j = 0 \) for \( j > k' \), and \( \beta_k + \hat{\beta}_k \neq 0 \). Consider all terms in \( P^k \) with the same initial part in the variables \( z_1, \ldots, z_{k-1} \),

\[
\left( \prod_{j<k} z_j^{\beta_j} \bar{z}_j^{\hat{\beta}_j} \right) Q(z_k, \ldots, z_{k'}, \bar{z}_k, \ldots, \bar{z}_{k'}).\]

Clearly, for a generic linear transformation of the variables \( z_k, \ldots, z_{k'} \), in the new coordinates the corresponding homogeneous polynomial \( Q^* \) does not vanish on the \( z_k \) axis. It follows that the restriction of \( P^* \) to \( z_{k+1} = \cdots = z_n = 0 \) depends on \( z_k \). This finishes the proof.

The following definition singles out a leading term in \( P \) for each of the variables.

**Definition 3.2.** Let \((z,w)\) be regular coordinates. The leading term in the variable \( z_k \) is given by the lexicographically smallest multiindex pair \( \Gamma^k = (\gamma^k, \hat{\gamma}^k) \), such that

\[
\gamma^k_j = \hat{\gamma}^k_j = 0 \quad \text{for} \quad j = k + 1, \ldots, n,
\]

\[
\gamma^k_k + \hat{\gamma}^k_k \neq 0 \quad \text{and} \quad A_{\gamma^k, \hat{\gamma}^k} \neq 0.
\]
The leading terms are used to define a normalization of $P$.

**Definition 3.3.** Let $(z, w)$ be regular coordinates. We say that the leading polynomial $P$, given by (9), is normalized if for every $k$

\[ (i) \quad A_{\gamma_k, \hat{\gamma}_k} = 1 \]

and

\[ (ii) \quad A_{\alpha, \hat{\alpha}} = 0 \]

for any multiindex pair $(\alpha, \hat{\alpha})$ such that $\alpha = \gamma_k$, $\hat{\alpha}_j = \hat{\gamma}_j$ for $j < k$, $\hat{\alpha}_k = \gamma_k^k - 1$ and $|\hat{\alpha}|_{\Lambda_M} = |\hat{\gamma}_k|_{\Lambda_M}$.

We will denote by $\epsilon^k$ the multiindex of length $n$ whose $k$-th component is equal to one and other components are zero.

It is straightforward to show that the normalization of $P$ can indeed be attained by a $\Lambda_M$-homogeneous transformation.

**Lemma 3.2.** There exist regular coordinates in which the leading polynomial $P(z, \bar{z})$ is normalized.

**Proof:** By induction. Let us assume that we have found regular coordinates such that (i) and (ii) are satisfied for all $\Gamma^j$ with $j < k$ (note that $\Gamma^j$ are determined by the coordinates). We will change the variable $z_k$ in such a way that (i) and (ii) is satisfied also for $\Gamma^k$. The transformation will be of the form

\[ z_k = \sum_{|\alpha|_{\Lambda_M} = \mu_k} C_\alpha (z^*)^\alpha, \quad z_j = z_j^* \text{ for } j \neq k, \]

where $C_\alpha \neq 0$ implies $\alpha_j = 0$ for $j < k$. Substituting into $v = P(z, \bar{z})$, we determine the coefficients which attain the normalization condition for $k$. This gives

\[ C_\alpha = -A_{\gamma_k, \hat{\gamma}_k - \epsilon^k + \alpha} + \ldots, \]

for $\alpha \neq \epsilon^k$. The condition $A_{\gamma_k, \hat{\gamma}_k} = 1$ is attained by taking $C_{\epsilon^k}$ as a solution to

\[ A_{\gamma_k, \hat{\gamma}_k}(C_{\epsilon^k})^{\gamma_k} (\tilde{C}_{\epsilon^k})^{\hat{\gamma}_k} = 1. \]

That finishes the proof.

4. **Biholomorphic equivalence of models**

In this section we consider the local equivalence problem for models. We start by showing that a transformation preserves form (7) if and only if it is superhomogeneous.
Theorem 4.1. A biholomorphic transformation takes $\Lambda_M$-adapted coordinates into $\Lambda_M$-adapted coordinates if and only if it is $\Lambda_M$-superhomogeneous.

Proof. We first prove the only if part of the statement. Consider a transformation
\begin{equation}
\begin{align*}
z^* &= z + f(z, w) \\
w^* &= w + g(z, w),
\end{align*}
\end{equation}
which takes $\Lambda_M$-adapted coordinates $(z, w)$ into $\Lambda_M$-adapted coordinates $(z^*, w^*)$.

Let $v^* = F^*(z^*, \bar{z}^*, u^*)$ be the defining equation of $M$ in the new coordinates. Substituting (15) into $v^* = F^*(z^*, \bar{z}^*, u^*)$ we obtain the transformation formula
\begin{equation}
\begin{align*}
F^*(z + f(z, u + iF(z, \bar{z}, u)), z + f(z, u + iF(z, \bar{z}, u)), u + \operatorname{Re} g(z, u + iF(z, \bar{z}, u)) &= F(z, \bar{z}, u) + \operatorname{Im} g(z, u + iF(z, \bar{z}, u)).
\end{align*}
\end{equation}

Without any loss of generality, we may assume that $P$ is normalized (applying a $\Lambda_M$-homogeneous transformation in the source space, if necessary). On the other hand, we do not assume that $P^*$ is normalized. Instead, in the target space we use an element of $\mathcal{L}$ to normalize the linear part of the transformation and assume that the Jacobi matrix of the transformation at the origin is the unit matrix.

By induction we will show that the transformation has to be superhomogeneous with respect to all weights in the generating sequence $\Lambda_1, \Lambda_2, \ldots, \Lambda_t$.

For $l = 1$, we have $\Lambda_1 = (m_1, \ldots, m_1)$. Hence $\Lambda_1$-homogeneous transformations in $\mathcal{H}^\mathbb{Z}$ are linear, and the claim is obvious.

Let $l > 1$ and assume the transformation is $\Lambda_j$-superhomogeneous for all $j$ with $j < l$. We will prove that it is also $\Lambda_l$-superhomogeneous. Note that $\lambda_{j}^l < \lambda_{j-1}^l$ if and only if $j > k_{l-1}$.

We separate the strictly subhomogeneous part (with respect to $\Lambda_l$) of the inverse transformation, and write
\begin{equation}
\begin{align*}
z_i &= z_i^* + \sum_{|\alpha| \lambda_i < \lambda_i^l} C_{\alpha}^i(z^*)^\alpha + O_{\Lambda_i}(\lambda_i^l).
\end{align*}
\end{equation}

Note that $w = w^* + o_{\Lambda_i}(1)$, since $P$ and $P^*$ contain no pluriharmonic terms. In this notation, let
\[\Theta = \{(i, \alpha) \in \mathbb{Z}^{n+1}; C_{\alpha}^i \neq 0\} \cup \{(i, \alpha) \in \mathbb{Z}^{n+1}; C_{\alpha}^i = 0\} \quad \text{for } i = 1, \ldots, t, \alpha \in \mathbb{Z}^n, \lambda_i^l \geq \lambda_i^j.

The elements of $\Theta$ have the following immediate properties. If $(i, \alpha) \in \Theta$, then $\alpha_j = 0$ for $j \leq i$, since $\lambda_i^l \geq \lambda_i^j$. Further, by $\Lambda_{l-1}$-superhomogeneity,
each of the terms appearing in (17) must contain at least one of the
variables \(z_{k_l-1+1}, \ldots, z_n\). We denote
\[
S(\alpha) = \alpha_{k_l-1+1} + \cdots + \alpha_n.
\]
Hence \(S(\alpha) > 0\) for all \((i, \alpha) \in \Theta\).

Analogous notation will be also used for multiindex pairs:
\[
S(\alpha, \hat{\alpha}) = \alpha_{k_l-1+1} + \cdots + \alpha_n + \hat{\alpha}_{k_l-1+1} + \cdots + \hat{\alpha}_n.
\]

Let \(m_S\) be the minimal value of \(S(\alpha)\) as \((i, \alpha)\) ranges over \(\Theta\). For
\((i, \alpha) \in \Theta\) consider the ”gap”
\[
G(i, \alpha) = \lambda^l_i - \sum_{j=1}^n \alpha_j \lambda^j_i.
\]

Among all pairs \((i, \alpha)\) in \(\Theta\) for which \(S(\alpha) = m_S\), let \(\Xi\) denote the
set of those for which \(G(i, \alpha)\) is maximal. Next, let \(m\) be the smallest
integer such that \((m, \alpha) \in \Xi\) for some \(\alpha\). Now we fix one such pair,
\((m, \delta) \in \Xi\) and consider the corresponding monomial in (17)
\[
C^m_{(0, \ldots, 0, \delta_{m+1}, \ldots, \delta_n)} \prod_{j>m}(z_j^*)^{\delta_j},
\]
where \(\delta = (0, \ldots, 0, \delta_{m+1}, \ldots, \delta_n)\). Note that \(m \leq k_{l-1}\).

Substituting (17) into
\[
v = \sum_{|(\alpha, \hat{\alpha})|_{\Lambda_l}=1} A_{\alpha, \hat{\alpha}} z^\alpha \bar{z}^{\hat{\alpha}} + o_{\Lambda_l}(1),
\]
we compute the coefficient of
\[
(z^*)^{\gamma^m} (z^*)^{\hat{\gamma}^m - \epsilon^m + \delta}.
\]
Since \(F\) starts with weight one, it is enough to consider the strictly
subhomogeneous part of the transformation. Hence we need to consider
the expansion of
\[
F(z_1^* + \sum_{|\alpha|_{\Lambda_l} < \lambda^l_1} C^1_\alpha (z^*)^\alpha, \ldots, z_n^* + \sum_{|\alpha|_{\Lambda_l} < \lambda^l_n} C^n_\alpha (z^*)^\alpha, \bar{z}_1^* + \cdots, 0)\]

First, consider terms coming from the leading polynomial. If for some
multiindex pair \((\beta, \hat{\beta})\) the coefficient \(A_{\beta, \hat{\beta}}\) enters the equation for (18),
then by the choice of \((m, \delta)\) there exists a multiindex \(\alpha\) and \(j \in \{1, \ldots, n\}\)
such that
\[
\hat{\gamma}^m - \epsilon^m + \delta = \hat{\beta} - \epsilon^j + \alpha,
\]
and $\beta = \gamma^m$. But, again by the choice of $(m, \delta)$, the gaps satisfy
\[ |\delta - \epsilon^m|_{\Lambda_i} = |\alpha - \epsilon^j|_{\Lambda_i}, \]
so
\[ |\hat{\gamma}^m|_{\Lambda_i} = |\hat{\beta}|_{\Lambda_i}. \]
Moreover, $\hat{\gamma}^m_j = \hat{\beta}_j$ for all $j < m$, which gives contradiction with the normalization of $P$. Note that
\[ \gamma^m - \epsilon_m = \beta - \epsilon_j \]
is impossible, since it forces $\lambda_j = \lambda_m$, which contradicts the linear part of the transformation being the identity.

It remains to prove that terms of weight greater than one in $F$ do not enter the equation for $(18)$. Let $F_{\alpha, \hat{\alpha}, z^\alpha \bar{z}^\hat{\alpha} u^l}$ be such a term, where $|\alpha, \hat{\alpha}|_{\Lambda_M} > 1$. By the choice of $(m, \delta)$, in order to influence a term of weight $1 - G(m, \delta)$ in $F^*$ we have to substitute at least twice a term with the lowest value of $S(\alpha)$, or a term with a higher value of $S(\alpha)$. In both cases the value of $S(\alpha, \hat{\alpha})$ for the resulting term is bigger than $m_S = S(\gamma^m - \epsilon^m + \delta, \hat{\gamma}^m)$.

Thus we have proved that a transformation which takes $\Lambda_M$-adapted coordinates into $\Lambda_M$-adapted coordinates is $\Lambda_M$-superhomogeneous. The converse follows immediately from $(16)$.

Now we can describe explicitly biholomorphisms between different models.

**Theorem 4.2.** Let $M_H$ and $\tilde{M}_H$ be two models for $M$ at $p$. Then there is a $\Lambda_M$-homogeneous transformation which maps $M_H$ to $\tilde{M}_H$. In particular, all models are biholomorphically equivalent by a polynomial transformation. Proof: By the previous proposition, the coordinates in which $M_H$ is the model are related to those in which $\tilde{M}_H$ is the model by a $\Lambda_M$-superhomogeneous transformation. But terms of weight greater than $\lambda_i$ in $f_i$ influence only terms of weight greater then one in $F^*$. Hence $\tilde{M}_H$ is obtained by the homogeneous part of this transformation.

## 5. Computing the multitype

Using Theorem 4.1, the process of computing multitype can be described as follows.

In the first step, we consider local holomorphic coordinates in which the leading polynomial in the variables $z, \bar{z}$ contains no pluriharmonic term. The first multitype component $m_1$ is then equal to the degree of
this polynomial. Hence \( m_1 = \frac{1}{\mu_1} \) is equal to the Bloom-Graham type of \( M \) at \( p \), and we set \( \Lambda_1 = (\mu_1, \ldots, \mu_1) \).

In the second step, consider all \( \Lambda_1 \)-homogeneous transformations and choose one which makes the leading polynomial \( P_1 \) independent of the largest number of variables. Let \( d_1 \) denote this number. Permuting variables, if necessary, we can assume that in such coordinates,

\[
v = P_1(z_1, \ldots, z_{n-d_1}, \tilde{z}_1, \ldots, \tilde{z}_{n-d_1}) + Q_1(z, \tilde{z}) + o(u),
\]

where \( P_1 \) is \( \Lambda_1 \)-homogeneous of weighted degree one, and \( Q_1 \) is \( o_{\Lambda_1}(1) \). Since \( \Lambda_1 \)-homogeneous transformations are linear, and using the fact that for any weight \( \Lambda \) which is lexicographically smaller than \( \Lambda_1 \), \( \Lambda \)-adapted coordinates are also \( \Lambda_1 \)-adapted, it follows that \( \mu_1 = \mu_2 = \cdots = \mu_{n-d_1} \) and \( \mu_{n-d_1+1} < \mu_1 \). Let

\[
Q_1(z, \tilde{z}) = \sum_{|\alpha, \hat{\alpha}|_{\Lambda_1} > 1} C^1_{\alpha, \hat{\alpha}} z^\alpha \tilde{z}^{\hat{\alpha}},
\]

and denote

\[
\Theta_1 = \{ (\alpha, \hat{\alpha}) \mid C^1_{\alpha, \hat{\alpha}} \neq 0 \text{ and } \sum_{i=1}^{n-d_1} \alpha_i + \hat{\alpha}_i < m_1 \}.
\]

For each \((\beta, \hat{\beta}) \in \Theta_1\) consider the number

\[
W_1(\beta, \hat{\beta}) = \frac{1 - \sum_{i=1}^{n-d_1} (\beta_i + \hat{\beta}_i) \mu_1}{\sum_{i=n-d_1+1}^{n} \beta_i + \hat{\beta}_i}.
\]

The weight \( \Lambda_2 \) is defined by \( \lambda^2_j = \mu_1 \) for \( j \leq n - d_1 \), and

\[
\lambda^2_j = \max_{(\alpha, \hat{\alpha}) \in \Theta_1} W_1(\alpha, \hat{\alpha})
\]

for \( j > n - d_1 \).

By the definition of \( \Lambda_2 \), the leading polynomial with respect to \( \Lambda_2 \) in the above coordinates depends on at least \( n - d_1 + 1 \) variables. This ends the second step.

Now we continue the process. In the \( j \)-th step, \( j > 2 \), we use the coordinates obtained in the previous step, and consider all \( \Lambda_{j-1} \)-homogeneous transformations. We denote by \( d_{j-1} \) the largest number of variables which do not appear in the leading polynomial after the transformation, and fix such a coordinate system. It is easy to show, using the same arguments as in Theorem 4.1., that any transformation which takes \( \Lambda_{j-1} \)-adapted coordinates into \( \Lambda_{j-1} \)-adapted coordinates has to be \( \Lambda_{j-1} \)-superhomogeneous. If \( d_{j-1} < d_{j-2} \), using this and the fact that for any weight \( \Lambda \) which is lexicographically smaller than \( \Lambda_{j-1} \),
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Λ-adapted coordinates are also Λ\(_{j-1}\)-adapted, it follows that we have determined the \((d_{j-2} - d_{j-1})\) multitype entries

\[ \mu_{n-d_{j-2}+1} = \cdots = \mu_{n-d_{j-1}} = \lambda_{n-d_{j-2}+1}^{j-1}, \]

and set \(\lambda^j_i = \mu_i\) for \(i \leq n - d_{j-1}\). To define the remaining entries of \(\Lambda_j\), we write

\[ v = P_{j-1}(z_1, \ldots, z_{n-d_{j-1}}, \bar{z}_1, \ldots, \bar{z}_{n-d_{j-1}}) + Q_{j-1}(z, \bar{z}) + o(u), \]

where \(P_{j-1}\) is \(\Lambda_{j-1}\)-homogeneous of weighted degree one, and \(Q_{j-1}\) is \(o(\Lambda_{j-1})\),

\[ Q_{j-1}(z, \bar{z}) = \sum_{|\alpha, \hat{\alpha}|, |\lambda_{j-1}| > 1} C_{\alpha, \hat{\alpha}}^{j-1} z^\alpha \bar{z}^{\hat{\alpha}}. \]

Let

\[ \Theta_{j-1} = \{(\alpha, \hat{\alpha}) \mid C_{\alpha, \hat{\alpha}}^{j-1} \neq 0 \text{ and } \sum_{i=1}^{n-d_{j-1}} (\alpha_i + \hat{\alpha}_i)\mu_i < 1\}. \]

As before, denote

\[ W_{j-1}(\beta, \hat{\beta}) = \frac{1 - \sum_{i=1}^{n-d_{j-1}} (\beta_i + \hat{\beta}_i)\mu_i}{\sum_{i=n-d_{j-1}+1}^{n} \beta_i + \hat{\beta}_i}. \]

The remaining entries of \(\Lambda_j\) are defined by

\[ \lambda^j_i = \max_{\alpha \in \Theta} W_{j-1}(\alpha, \hat{\alpha}) \]

for \(j > n - d_{j-1}\).

If \(d_{j-1} = d_{j-2}\), we only use (20) to determine \(\lambda^j_{n-d_{j-1}+1}, \ldots, \lambda^j_{n}\). No multitype component is determined at this step.

It is immediate to verify that the process terminates after finitely many steps, and determines all components of the multitype weight.

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