A new model for quantum games based on the Marinatto–Weber approach

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Abstract
The Marinatto–Weber approach to quantum games is a straightforward way to apply the power of quantum mechanics to classical game theory. In the simplest case, the quantum scheme is that players manipulate their own qubits of a two-qubit state either with the identity 1 or the Pauli operator σx. However, such a simplification of the scheme raises doubt as to whether it could really reflect a quantum game. In this paper we put forward examples which may constitute arguments against the present form of the Marinatto–Weber scheme. Next, we modify the scheme to eliminate the undesirable properties of the protocol by extending the players’ strategy sets.

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1. Introduction
The Marinatto–Weber (MW) scheme [1] has become one of the most frequently used schemes for quantum games. Though it was created for research on Nash equilibria in quantum 2 × 2 games, it has also found an application in studying some of the refinements of a Nash equilibrium, such as evolutionarily stable strategies [2, 3]. Moreover, the MW scheme turns out to be applicable to finite extensive games [4]. Among other applications, the problem of duopoly is worthy noting. It was shown in [5] that the MW scheme can be used in Stackelberg’s model of duopoly. The paper initiated further studies on the quantum Stackelberg duopoly [6, 7] and the quantum approach to Bertrand duopoly [8]. These recent papers show uninterrupted interest in research on quantum games played according to the MW idea and they provide sufficient motivation to study this protocol.

2. Preliminary to the MW approach
The MW scheme was originally designed for a 2 × 2 game

\[
\begin{pmatrix}
 l & r \\
 b & \end{pmatrix}
\begin{pmatrix}
 (a_{00}, b_{00}) & (a_{01}, b_{01}) \\
 (a_{10}, b_{10}) & (a_{11}, b_{11}) \\
\end{pmatrix}, \quad \text{where} \quad a_{ij} \in \mathbb{R}.
\]

(1)
Each of the two players acts with the identity \( \sigma_i \) on his own qubit of some fixed two-qubit state \( \rho_{in} = |\psi_{in}\rangle \langle \psi_{in}| \), which is called the initial state. The players’ payoffs are determined by measurement of the resulting final state \( \rho_{fn} \). Formally, the final state takes the following form:

\[
\rho_{fn} = pq\mathbb{1} \otimes \rho_{in}\mathbb{1} \otimes \mathbb{1} + (1 - p)q\sigma_x \otimes \rho_{in}\mathbb{1} \otimes \mathbb{1} + (1 - p)(1 - q)\mathbb{1} \otimes \sigma_x \rho_{in}\sigma_x \otimes \mathbb{1},
\]

(2)

where \( p \) and \( q \) are the probabilities of choosing the identity \( \mathbb{1} \) by player 1 and player 2, respectively. Then the pair of players’ payoffs \( (\pi_1, \pi_2) \) depends on \( p \) and \( q \), and through the measurement operator

\[
X = (a_{00}, b_{00})|00\rangle \langle 00| + (a_{01}, b_{01})|01\rangle \langle 01| + (a_{10}, b_{10})|10\rangle \langle 10| + (a_{11}, b_{11})|11\rangle \langle 11|,
\]

(3)

is given by the formula

\[
(\pi_1, \pi_2)(p, q) = \text{tr}(X \rho_{fn}).
\]

(4)

The MW protocol can also be expressed by a useful diagram

\[
\begin{array}{l}
|\psi_{in}\rangle = \sum_{ij} \lambda_{ij}|ij\rangle \quad \xrightarrow{\sigma_x} \quad \frac{1}{\sigma_x} \\
(a_{00}, b_{00}) (a_{01}, b_{01}) (a_{10}, b_{10}) (a_{11}, b_{11}) \quad \xrightarrow{\text{tr}} \quad (a_{00}, \beta_{00}) (a_{01}, \beta_{01}) (a_{10}, \beta_{10}) (a_{11}, \beta_{11}).
\end{array}
\]

(5)

where \( |\psi_{in}\rangle = \sum_{ij} \lambda_{ij}|ij\rangle \) is a two-qubit state and

\[
(a_{00}, \beta_{00}) = \sum_{ij} |\lambda_{ij}|^2 (a_{ij}, b_{ij}), \quad (a_{01}, \beta_{01}) = \sum_{ij} |\lambda_{ij}|^2 (a_{ij}, b_{ij});
\]

\[
(a_{10}, \beta_{10}) = \sum_{ij} |\lambda_{ij}|^2 (a_{ij}, b_{ij}), \quad (a_{11}, \beta_{11}) = \sum_{ij} |\lambda_{ij}|^2 (a_{ij}, b_{ij}).
\]

(6)

The bimatrix on the left of diagram (5) represents the classical (input) \( 2 \times 2 \) game while the bimatrix of the right is the output game obtained by applying the MW protocol.

2.1. Advantages of the MW approach

The simplicity of the calculations and the possibility to easily extend the scheme to consider more complex games than \( 2 \times 2 \) can be counted among the advantages of the MW scheme. The MW approach to \( n \times m \) bimatrix games is defined by the initial state represented by a vector from \( \mathbb{C}^n \otimes \mathbb{C}^m \) and appropriate \( n \) and \( m \) unitary operators for player 1 and 2, respectively (see [9]). In any such case the dimension of the output game is always equal to the input game, which does not make the output game more difficult to deal with. In particular, determining Nash equilibria or evolutionarily stable strategies have the same level of difficulty in both the classical and quantum games.

The MW approach has also achieved popularity because it has found application in infinite games; that is in games where the players’ strategy sets are infinite. For example, in [5] the authors showed a way to apply the MW approach to Stackelberg’s model of duopoly—a game in which the players’ strategy sets are identified with interval \([0, \infty)\) (to learn the model of duopoly, see the original paper [10] or textbook [11]).

3. Undesirable features of the MW approach

In spite of the simplicity and large number of applications, the MW scheme has been driven by more complex schemes: the Eisert–Wilkens–Lewenstein approach [12] in the case of quantum \( 2 \times 2 \) games and the Li et al approach [13] in the case of various types of duopolies. Below we point out the main flaws in the MW approach.
3.1. Problem of non-classical games induced by separable initial states

A common criticism of the MW scheme is that the initial state has an excessive impact on the output game. In other words, the initial state plays the main role in producing the non-classical output game and the output games may vary significantly from each other depending on the initial state. However, in our view, this is not a disadvantage of the scheme. Quantum information techniques like superdense coding, the Greenberger–Horne–Zeilinger (GHZ) example, the violation of the Clauser–Horne–Shimony–Holt (CHSH) inequality and many others are also based on preparing some special quantum state and (or) operators acting on it so that non-classical results could be obtained. But in every such protocol the quantum state has to be an entangled one. Unfortunately, the MW scheme outputs a non-classical game even if the initial state is separable. Let us take the following example:

\[
\begin{pmatrix}
5 & 3 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
3 & 5
\end{pmatrix}
\langle \psi_{\text{in}} \rangle = \frac{1}{\sqrt{2}} (|00\rangle + |01\rangle)
\xrightarrow{\begin{pmatrix}
\mathbb{I} & \sigma_x \\
\sigma_x & \mathbb{I}
\end{pmatrix}}
\begin{pmatrix}
3 & 2 \\
2 & 3
\end{pmatrix}
\begin{pmatrix}
2 & 3 \\
3 & 2
\end{pmatrix}.
\]

(7)

Though the outcome corresponding to any strategy profile of the output game can be obtained through a suitable strategy profile in the input game, the output game is not equivalent to the classical one. The outcomes greater than 3 are not available in the output game and the only reasonable result of the game is (3, 2)—the result that is never chosen by rational players in the input game.

3.2. Insufficient range of players’ actions

The two element set \{\mathbb{I}, \sigma_x\} of players’ strategies seems to be appropriate in the MW scheme. The identity \mathbb{I} and the Pauli matrix \sigma_x are supposed to represent classical moves, while the initial quantum state \langle \psi_{\text{in}} \rangle plays the role of a joint strategy. However, such a setting can lead to a situation in which a player has no influence on the outcome of the output game. This can be easily observed with example (7). The second qubit is prepared in the superposition \langle |0\rangle + |1\rangle \rangle/\sqrt{2}. Then, no matter what operation (any probability distribution on \mathbb{I} and \sigma_x) the second player performs on his own qubit, he cannot change the state and therefore he cannot influence the outcome of the game. In this case the second player would need operators

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\quad \text{and} \quad
\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}
\]

(8)
in order to obtain the states |0\rangle and |1\rangle, respectively. Only then would the outcomes of the input game in (7) be achievable.

In fact, any separable state other than \langle i/j \rangle can lead through the MW approach to output games which are not equivalent to classical ones. The way to remove this property (keeping the general MW approach unchanged) is to broaden the range of unitary operations for players. Since a qubit in a separable state takes the effective form

\[
\cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle,
\]

two-parameter unitary operators are required to obtain the states |0\rangle and |1\rangle from state (9). For this reason, let us consider unitary operators

\[
U(\theta, \varphi) = \begin{pmatrix}
\cos \frac{\theta}{2} & e^{i\varphi} \sin \frac{\theta}{2} \\
e^{i\varphi} \sin \frac{\theta}{2} & -\cos \frac{\theta}{2}
\end{pmatrix}
\]

(10)
and assume that each player $i$'s strategies are triples

for player 1: \( (p, \theta_1, \varphi_1) : 0 \leq p \leq 1, 0 \leq \theta_1 \leq \pi, 0 \leq \varphi_1 \leq 2\pi \). \hspace{1cm} (11)

for player 2: \( (q, \theta_2, \varphi_2) : 0 \leq q \leq 1, 0 \leq \theta_2 \leq \pi, 0 \leq \varphi_2 \leq 2\pi \). \hspace{1cm} (12)

Next, we define the final state

\[
\rho_{\text{fin}} = pqU_1 \otimes U_2 \rho_\text{in} U_1^\dagger \otimes U_2^\dagger + p(1-q)U_1 \otimes V_2 \rho_\text{in} U_1^\dagger \otimes V_2^\dagger + (1-p)qU_1 \otimes U_2 \rho_\text{in} U_1^\dagger \otimes U_2^\dagger + (1-p)(1-q)V_1 \otimes V_2 \rho_\text{in} V_1^\dagger \otimes V_2^\dagger,
\]

where

\[
U_i := U(\theta_i, \varphi), \quad V_i = U(\pi - \theta_i, \varphi_i - \pi).
\]

(14)

Then for any separable pure state

\[
|\psi_\text{in}\rangle = e^{i\delta} \left( \cos \frac{\theta_1}{2} |0\rangle + e^{i\varphi_1} \sin \frac{\theta_1}{2} |1\rangle \right) \otimes \left( \cos \frac{\theta_2}{2} |0\rangle + e^{i\varphi_2} \sin \frac{\theta_2}{2} |1\rangle \right),
\]

(15)

each player $i$ can select appropriate $\theta_i$ and $\varphi_i$ such that the operators (14) create the final state

\[
\rho_{\text{fin}} = pq|00\rangle\langle 00| + p(1-q)|01\rangle\langle 01| + (1-p)q|10\rangle\langle 10| + (1-p)(1-q)|11\rangle\langle 11|,
\]

(16)

which implies the input (classical) game in the MW scheme.

The extension (10)–(14) removes the players’ powerlessness against maximal superpositions. Moreover, it guarantees that the classical game can be reconstructed in the case of separable states. The solution, however, seems a bit artificial. One may question why the players are not allowed to use the full range of unitary operators or why a probability distribution over the two-parameter operators is not given by a probability density function. Moreover, the refinement (10)–(14) does not solve the following problem.

3.3. Choice of the initial state

The choice of the initial state has a significant impact on the output game. However, the MW protocol does not allow the players to take a part in choosing the initial state. It causes many undesirable features of the scheme. In particular, there is a possibility that the output game induced by some fixed initial state could favour one of the players and be unjust to the other one, even if the strategic positions in the input game are completely identical. In other words, it would be the case that a player is not satisfied by the initial state and she would be willing to change the initial state if she were allowed to. Since the initial state is supposed to be the players’ joint strategy, the players should have some influence in choosing the initial state, otherwise it stands in contradiction with the game-theoretical sense of strategy being an element chosen by a player.

Furthermore, the fact that the initial state is treated as the joint strategy ought to be taken into consideration in game-theoretical solution concepts. For example, in a Nash equilibrium no player gains by deviating from the equilibrium strategy. Therefore, each player should have the possibility of rejecting the initial state $|\psi_\text{in}\rangle$ if the ‘classical’ state $|00\rangle$ would increase his payoff.

4. A new model based on the MW scheme

Let us modify the MW scheme to obtain a protocol which is free from the flaws listed in the previous section. In order to do that let us redefine the final state (2) keeping the operator (3) and payoff functional (4) unchanged. We assume now that the players now are additionally
allowed to choose whether they play the classical or quantum game. That is, each player decides to choose either 1 or $\sigma_i$ and at the same time whether to perform the chosen local action on $|00\rangle$ or $|\psi_m\rangle$. Since the quantum state $|\psi_m\rangle$ is considered the joint strategy, we assume that if both players pick the action ‘quantum’, the chosen actions from the set $\{1, \sigma_1\}$ are performed on state $|\psi_m\rangle$, otherwise the players apply the local actions to state $|00\rangle$.

Denote by $C$ and $Q$ the actions ‘classical’ and ‘quantum’, respectively. In such a scheme each player has four pure strategies which we denote by $C \times 1, C \times \sigma_i, Q \times 1$ and $Q \times \sigma_i$. Thus, the extended MW (eMW) scheme is formally defined as follows.

**Definition 4.1.** If $|\psi_m\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ represents a pure quantum state, $\tau_1 = (p_1, p_2, p_3, p_4)$ and $\tau_2 = (q_1, q_2, q_3, q_4)$ are mixed strategies over $C \times 1, C \times \sigma_i, Q \times 1$ and $Q \times \sigma_i$ of player 1 and 2, respectively, i.e. $p_i, q_i \geq 0$, $\sum_i p_i = 1$ and $\sum_i q_i = 1$ and $X$ is as in (3), then the extended final state $\rho_{ext}$ is given by

$$\rho_{ext} = I \otimes \rho \left| (p_1 q_1 + p_1 q_3 + p_3 q_1) |00\rangle \langle 00| + p_3 q_1 |\psi_m\rangle \langle \psi_m| I \otimes I + \sigma_i \right| (p_1 q_2 + p_1 q_4 + p_3 q_2) |00\rangle \langle 00| + p_3 q_4 |\psi_m\rangle \langle \psi_m| \sigma_i \otimes I + \sigma_i \right| (p_2 q_2 + p_2 q_4 + p_4 q_2) |00\rangle \langle 00| + p_4 q_4 |\psi_m\rangle \langle \psi_m| \sigma_i \otimes \sigma_i$$

and the payoff pair is $\pi(\tau_1, \tau_2) = tr(X \rho_{ext})$.

Let us see how formula (17) works. If, for example, player 1 chooses operation 1 and the actions ‘classical’ and ‘quantum’ with equal probability (which means choosing $C \times 1$ and $Q \times I$ with equal probability) and player 2 picks $Q \times \sigma_i$, then $p_1 = p_3 = 1/2$ and $q_4 = 1$. It implies that formula (17) takes on $\rho_{ext} = (|01\rangle \langle 01| + I \otimes \sigma_i |\psi_m\rangle \langle \psi_m| \sigma_i) / 2$.

Like in the MW scheme there exists a convenient way to express the eMW scheme using the bimatrix form, in this case by means of a $4 \times 4$ bimatrix game. The individual entries of the bimatrix can be obtained by determining the payoffs $\pi(\tau_1, \tau_2)$ for each of the 16 possible pure profiles

$$(\tau_1, \tau_2) \in \{|C, Q| \times \{|1, \sigma_i\} \} \times \{|C, Q| \times \{|1, \sigma_i\} \}.$$

(18)

Another way to obtain the bimatrix form is to use diagram (5). Indeed, formula (17) says that players play the game on the right of diagram (5) if they both have chosen $Q$. In this case $p_1 = p_2 = q_1 = q_2 = 0$ and then formula (17) coincides with the final state (2). In the other cases (i.e. given that $p_2 q_3 = p_3 q_2 = p_4 q_1 = p_4 q_4 = 0$) the players play the classical game (on the left of diagram (5)) because they perform $I$ and $\sigma_i$ on state $|00\rangle \langle 00|$ then. We thus obtain the following bimatrix counterparts:

$C \times \{1, \sigma_i\} \rightarrow (a_{00}, b_{00}) \rightarrow (a_{01}, b_{01})$

$C \times \{1, \sigma_i\} \rightarrow (a_{10}, b_{10}) \rightarrow (a_{11}, b_{11})$

$Q \times \{1, \sigma_i\} \rightarrow (a_{00}, b_{00}) \rightarrow (a_{01}, b_{01})$

$Q \times \{1, \sigma_i\} \rightarrow (a_{10}, b_{10}) \rightarrow (a_{11}, b_{11})$

(19)

where $(a_{ij}, b_{ij})$ are as in (6). The eMW scheme generalizes the classical way of playing games. If $|\psi_m\rangle = |00\rangle$, the strategies from the set $\{C, Q\} \times 1$ and $\{C, Q\} \times \sigma_i$ are equivalent to each other, i.e. they induce the same payoff pairs for any pure strategy played by the opponent. Therefore, the quotient game obtained by removing equivalent strategies (leaving the one representative) coincides with the original one in respect of the game outcomes and players’ strategic positions.

In general, the eMW scheme provides output games quite different from ones induced by the MW scheme. Since a unilateral deviation from the classical action $C$ does not make
the players play on $|\psi_m\rangle$, a Nash equilibrium in an input game (1) remains the equilibrium in the quantum counterpart (19). Moreover, in a case that the players do not find the quantum strategy mutually advantageous, the classical equilibria are the only ones in game (19). This property is well illustrated by the following example.

**Example 4.2.** Let us consider a $2 \times 2$ game given by bimatrix

$$
\begin{pmatrix}
    l & r \\
    t & (\alpha, \beta) & (\gamma, \gamma) \\
    b & (\gamma, \gamma) & (\beta, \alpha)
\end{pmatrix},
$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha > \beta > \gamma$.

This is the general form of a game called ‘battle of the sexes’ (the game on the left of diagram (7) is the particular case). The example was used in [1] to show that the players playing the quantum strategy $|\psi_m\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ have access to Nash equilibria with better results than in the case of playing only classical strategies (see also [14] and [15]). The eMW scheme shows, however, that state $(|00\rangle + |11\rangle)/\sqrt{2}$ would never be chosen by the players. Indeed, if the players are allowed to decide whether to choose the entangled state or not, they face the following game:

$$
\begin{align*}
    C \times 1 & \quad C \times \sigma_x & \quad Q \times 1 & \quad Q \times \sigma_x \\
    C \times 1 & \quad (\alpha, \beta) & \quad (\gamma, \gamma) & \quad (\alpha, \beta) & \quad (\gamma, \gamma) \\
    C \times \sigma_x & \quad (\gamma, \gamma) & \quad (\beta, \alpha) & \quad (\gamma, \gamma) & \quad (\beta, \alpha) \\
    Q \times 1 & \quad (\alpha, \beta) & \quad (\gamma, \gamma) & \quad \left(\frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2}\right) & \quad (\gamma, \gamma) \\
    Q \times \sigma_x & \quad (\gamma, \gamma) & \quad (\beta, \alpha) & \quad (\gamma, \gamma) & \quad \left(\frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2}\right)
\end{align*}
$$

(21)

Now, if the players were restricted to using only $Q \times \{1, \sigma_x\}$, game (21) would come down to one associated with the MW scheme and strategy profiles: $(Q \times 1, Q \times 1)$, $(Q \times \sigma_x, Q \times \sigma_x)$ and one in which each player plays $Q \times 1$ and $Q \times \sigma_x$, with equal probability would constitute Nash equilibria. However, these profiles are no longer equilibria in game (21). The first player’s best response to $Q \times 1$ of player 2 is $C \times 1$ instead of $Q \times 1$. The second pure profile is not an equilibrium for similar reasons. Next, for the probability distribution $(1/2, 1/2)$ over $Q \times \{1, \sigma_x\}$ played by one of the players, the best response of the other player is $C \times 1$. As a result, since a Nash equilibrium is considered a necessary condition for a strategy profile to be a reasonable one, the quantum strategy $(|00\rangle + |11\rangle)/\sqrt{2}$ would not be chosen by the players. It does not have to indicate, however, that the players cannot benefit from quantum strategies. Let us consider a similar (in terms of the MW scheme) state $(|01\rangle + |10\rangle)/\sqrt{2}$. The MW approach through this state also implies two pure Nash equilibria with outcome $(\alpha + \beta)/2$ for each player and the Nash equilibrium in mixed strategies. However, only $(Q \times 1, Q \times \sigma_x)$ remains an equilibrium if the players are able to decide whether to use the quantum state or not since they play the game given by bimatrix

$$
\begin{align*}
    C \times 1 & \quad C \times \sigma_x & \quad Q \times 1 & \quad Q \times \sigma_x \\
    C \times 1 & \quad (\alpha, \beta) & \quad (\gamma, \gamma) & \quad (\alpha, \beta) & \quad (\gamma, \gamma) \\
    C \times \sigma_x & \quad (\gamma, \gamma) & \quad (\beta, \alpha) & \quad (\gamma, \gamma) & \quad (\beta, \alpha) \\
    Q \times 1 & \quad (\alpha, \beta) & \quad (\gamma, \gamma) & \quad \left(\frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2}\right) & \quad (\gamma, \gamma) \\
    Q \times \sigma_x & \quad (\gamma, \gamma) & \quad (\beta, \alpha) & \quad (\gamma, \gamma) & \quad \left(\frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2}\right)
\end{align*}
$$

(22)

The opposite players’ preferences in the classical battle of the sexes game (player 1 prefers $(C \times \sigma_x, C \times \sigma_x)$, whereas player 2 prefers $(C \times \sigma_x, C \times \sigma_x)$ if the game is played classically) make $(Q \times 1, Q \times \sigma_x)$ the most reasonable profile.
5. Disadvantages of the MW approach versus the eMW scheme

Since a quantum state $|\psi\rangle$ is considered to be a players’ joint strategy, the two additional actions: $C$ (classical) and $Q$ (quantum) appear to be a natural extension of the MW scheme. It turns out that this extension is sufficient to remove the disadvantages we listed in section 3. The problem of the powerlessness of players’ actions (when the qubits are maximal superpositions) is not a concern in the extended scheme. The action $C$ played by just one of the players turns the game into a classical one. For the same reason, each outcome of the classical game is still available in the quantum counterpart (19). The extended scheme also removes, in some sense, the problem concerning the choice of the initial state since the players can decide whether to play the quantum state or not. The fact that the initial state has to be chosen by an arbiter beforehand just means that there are infinitely many possible quantum extensions of some classical game. Every such extension is associated with some initial quantum state then.

5.1. Division of games into classical and quantum ones

It has been proved that classical correlations can always be associated with separable states [16] and any pure entangled state violates some Bell inequality [17, 18]. So, an interesting problem is to examine if the eMW scheme could give us a similar hierarchical structure in quantum game theory.

First, let us assign to the mixed state (17) a set of pure quantum states by means of the following simple fact.

Fact 5.1. Let $X$ be as in (3). For any density operator $\rho$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ there exists a pure state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ and a mixed state $\sum_{ij=0,1} \lambda_{ij} |ij\rangle \langle ij|$ such that

$$\text{tr}(X \rho) = \text{tr}(X |\psi\rangle \langle \psi|) = \text{tr} \left( X \sum_{ij=0,1} \lambda_{ij} |ij\rangle \langle ij| \right).$$

(23)

Proof 5.1. Let $\rho = \sum_{i,j,k,l=0,1} O_{ijkl} |ij\rangle \langle kl|$ be a density operator. Putting state $|\psi\rangle = \sum_{i,j=0,1} \sqrt{O_{ij}} |ij\rangle$ and numbers $\lambda_{ij} = O_{ij}$ we obtain equality (23). □

Thus, we can always relate the final state $\rho_{\text{ext}}$ to an equivalent pure state (with respect to the measurement $X$). Such a assignment is necessary because specification of the MW scheme allows players to obtain strictly non-classical results with separable mixed states. For instance, the separable state $\rho_{\text{ext}} = (|00\rangle \langle 00| + |11\rangle \langle 11|)/2$ given by taking $p_3 = 1$, $q_1 = q_3 = 1/2$ and $|\psi_{\text{in}}\rangle = |11\rangle$ in (17) induces $\text{tr}(X \rho_{\text{ext}}) = ((a_{00}, b_{00}) + (a_{11}, b_{11}))/2$—the outcome that is not available with the use of classical mixed strategies. However, if we restrict ourselves to pure states, only the entangled pure state $(|00\rangle + e^{i\phi} |11\rangle)/\sqrt{2}$ (up to the phase factor) generates that outcome. Note that the state $|\psi_{\text{in}}\rangle$ is just a part of $\rho_{\text{ext}}$. Thus, it does not have to be entangled so that the final state generates a non-classical game.

Example 5.1. Let us consider the following bimatrix game:

$$\begin{pmatrix}
 l & r \\
 t & (5, 5) & (0, 4) \\
 b & (4, 0) & (2, 2)
\end{pmatrix}.$$  

(24)

The game exhibits a conflict between two reasonable types of equilibria. The profile $(b, r)$ is a risk dominant equilibrium [19] (see also [15]). If one of the players is uncertain about the move of the other one, a risk dominant strategy turns out to be a reasonable choice. On the other hand, the profile $(t, l)$ is a payoff dominant equilibrium for which both players receive
much more. The uncertainty about the result of game (24) disappears in the quantum domain. The final state $\rho_{\text{ext}}$ for $|\psi_{\text{in}}\rangle = |11\rangle$ implies bimatrix

$$
\begin{pmatrix}
C \times \mathbb{1} & C \times \sigma_s & Q \times \mathbb{1} & Q \times \sigma_s \\
(5, 5) & (0, 4) & (5, 5) & (0, 4) \\
(4, 0) & (2, 2) & (4, 0) & (2, 2) \\
(5, 5) & (0, 4) & (2, 2) & (4, 0) \\
(4, 0) & (2, 2) & (0, 4) & (5, 5)
\end{pmatrix}
$$

(25)

The profile $(Q \times 1, Q \times \mathbb{1})$ corresponding to the risk-dominant outcome is no longer an equilibrium in (25). If the players decide to play quantum strategies, the equilibrium $(Q \times \sigma_s, Q \times \sigma_s)$ that provides the players with the payoff dominant outcome is the unique reasonable profile.

We hypothesize that the association with a pure separable state is a sufficient condition on $\rho_{\text{ext}}$ for a game to be classical. It is not work in the case of the original MW scheme. In fact, if $|\psi_{\text{in}}\rangle$ is separable, we can always find a separable pure state that is equivalent to (2) (with regard to measurement (3)) but, as diagram (7) shows, the output game does not coincide with the classical one. In the eMW scheme the players have a greater impact on the final state $\rho_{\text{ext}}$ despite the same local operators $\mathbb{1}$ and $\sigma_s$. It follows from the fact that the players do not perform $\mathbb{1} \otimes \mathbb{1}$, $\mathbb{1} \otimes \sigma_s$, $\sigma_s \otimes \mathbb{1}$ and $\sigma_s \otimes \sigma_s$ on the whole quantum state but on the appropriate parts of it. This distinction is relevant as it allows us to formulate the following proposition.

**Proposition 5.1.** A quotient game of a game specified by definition 4.1 coincides with the input game (1) if and only if for any players’ strategies $\tau_1$ and $\tau_2$, the final state (17) satisfies equation (23) for some separable pure state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$.

**Proof 5.2.** Let us first assume that the quotient game is equal to the input game (1). In this case the pairs of strategies $C \times \mathbb{1}$, $Q \times \mathbb{1}$ and $C \times \sigma_s$, $Q \times \sigma_s$ have to be equivalent to each other which imposes $|\psi_{\text{in}}\rangle = |00\rangle$ (up to the phase factor) on the final state (17) and therefore

$$
|\psi\rangle = (\sqrt{p_1 + p_3}|0\rangle + \sqrt{p_2 + p_4}|1\rangle) \otimes (\sqrt{q_1 + q_3}|0\rangle + \sqrt{q_2 + q_4}|1\rangle).
$$

(26)

Then for any $p_i$, $q_i$, a separable state $|\psi\rangle = (\sqrt{p_1 + p_3}|0\rangle + \sqrt{p_2 + p_4}|1\rangle) \otimes (\sqrt{q_1 + q_3}|0\rangle + \sqrt{q_2 + q_4}|1\rangle)$ (27) satisfies the equation $\text{tr}(X|\psi\rangle\langle\psi|) = \text{tr}(X|\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|)$.

On the other hand, let us determine the general form of the final state $\rho_{\text{ext}}$. The equation (23) shows that we can replace $|\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|$ with some mixed state $\sum_{i,j=0,1} \eta_{ij} |ij\rangle\langle ij|$ without loss of generality. Thus the general final state can be written as follows:

$$
\rho_{\text{ext}} = (p_1 q_1 + p_1 q_3 + p_3 q_1 + p_3 q_3) |00\rangle\langle 00| + (p_1 q_2 + p_2 q_1 + p_2 q_3 + p_3 q_2) |01\rangle\langle 01| + (p_1 q_3 + p_3 q_1 + p_3 q_3 + p_3 q_1) |10\rangle\langle 10| + (p_2 q_2 + p_3 q_2 + p_3 q_3 + p_3 q_1) |11\rangle\langle 11|.
$$

(28)

Now, note that if $|\psi\rangle$ is separable, the third term in equation (23) satisfies a separability condition $\lambda_{00}\lambda_{11} = \lambda_{01}\lambda_{10}$. Since for any $p_i$, $q_i$, state (28) is associated with some separable state, the separability condition gives equations

$$
\begin{align*}
(2 + \eta_{10})\eta_{01} = (1 + \eta_{00})\eta_{11} & \quad \text{for } p_1 = p_4 = q_1 = q_3 = \frac{1}{2}, \\
(2 + \eta_{01})\eta_{10} = (1 + \eta_{00})\eta_{11} & \quad \text{for } p_1 = p_3 = q_1 = q_4 = \frac{1}{2}, \\
(3 + \eta_{00})\eta_{11} = \eta_{01}\eta_{10} & \quad \text{for } p_1 = p_3 = q_1 = q_3 = \frac{1}{2}.
\end{align*}
$$

(29)
Together with the normalization condition $\sum_{i,j} \eta_{i,j} = 1$, equations (29) imply $\eta_{00} = 1$. As a result, state (28) determines a game which corresponds to the input one (up to a quotient game).

Proposition 5.1 tells us that $|\psi_m\rangle \neq |00\rangle$ in (17) provides the players with non-classical strategies and, as examples 4.2 and 5.1 show, may significantly influence the game result. The non-classical result can be obtained only if the final state relates to some entangled pure state. Thus, like in the case of theory of quantum correlation, there exists a hierarchical structure for quantum games in regard to separable and entangled states.

6. Further generalization of the scheme

6.1. More quantum strategies

The eMW approach can be generalized to allow for more than one joint strategy $|\psi_m\rangle$. Denote by $|\psi_1\rangle, \ldots, |\psi_n\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ the $n$ available joint quantum strategies and by $Q_1, \ldots, Q_n$, the corresponding ‘quantum’ actions. Then, together with the local actions $\mathbb{I}$ and $\sigma_i$, each player has the $2(n + 1)$-element set of strategies $\{C, Q_1, \ldots, Q_n\} \times \{\mathbb{I}, \sigma_1, \ldots, \sigma_n\}$. Similarly to (19), we assume that the quantum strategy $|\psi_i\rangle$ is played only if both players choose $Q_i$. Otherwise, the players perform the local operations on $|00\rangle$. As a result, the output game can be characterized by the matrix

\[
\begin{pmatrix}
C \times \{\mathbb{I}, \sigma_1\} & Q_1 \times \{\mathbb{I}, \sigma_2\} & \cdots & Q_n \times \{\mathbb{I}, \sigma_n\}
\end{pmatrix}
\]

Formally, the construction of the appropriate final state is as follows. Denote by $(p_1, p_2, \ldots, p_{2(n+1)})$ and $(q_1, q_2, \ldots, q_{2(n+1)})$ the mixed strategies of player 1 and 2, respectively. Probabilities $p_1$ and $p_2$ are associated with actions $C \times \mathbb{I}$ and $C \times \sigma_i$, the pair $p_{2i+1}, p_{2i+2}$ (the pair $q_{2i+1}, q_{2i+2}$) concerns $Q_i \times \mathbb{I}$ and $Q_i \times \sigma_i$ for $i = 1, 2, \ldots, n$. For each $i, j \in \{0, 1, \ldots, n\}$, we define a density operator $\rho_{ij}$,

\[
\rho_{ij} = p_{2i+1}q_{2j+1}|\phi_{ij}\rangle \langle \phi_{ij}| + p_{2i+2}q_{2j+2}|\phi_{ij}\rangle \langle \phi_{ij}|
\]

where

| $\phi_{ij}$ if $i = j \neq 0$; $\langle \phi_{ij}| \sigma_i \otimes 1 + p_{2i+2}q_{2j+2}\sigma_i |\phi_{ij}\rangle \sigma_i \otimes \sigma_j$, $|00\rangle$ otherwise.

Then the general final state corresponding to (30) is $\rho_{\text{ext}} = \sum_{i,j=0}^n \rho_{ij}$.

6.2. More local strategies

We showed in [9] how to construct the scheme for $n \times m$ bimatrix games according to the MW approach. In this case, player 1 (player 2) has $n$ operators $U_i$ ($m$ operators $V_i$) defined on $\mathbb{C}^m$ that act on basis states $\{|0\rangle, |1\rangle, \ldots, |n-1\rangle\}$ ($\{|0\rangle, |1\rangle, \ldots, |m-1\rangle\}$) as follows:

\[
\begin{align*}
U_0|i\rangle &= |i\rangle, & V_0|i\rangle &= |i\rangle, \\
U_1|i\rangle &= |i + 1 \text{ mod } n\rangle, & V_1|i\rangle &= |i + 1 \text{ mod } m\rangle, \\
& \vdots &\vdots \\
U_{n-1}|i\rangle &= |i + (n-1) \text{ mod } n\rangle & V_{m-1}|i\rangle &= |i + (l-1) \text{ mod } m\rangle.
\end{align*}
\]
The construction of the final state $\rho_{ext}$ is analogous to definition 4.1 but the set of pure strategies for player 1 and 2 is now $[C, Q] \times \{U_0, \ldots, U_{n-1}\}$ and $[C, Q] \times \{V_0, \ldots, V_{m-1}\}$, respectively.

7. Conclusions

During the last 14 years of research into quantum games, the MW [1] and Eisert–Wilkens–Lewenstein (EWL) [12] concepts have become the most commonly used quantum schemes for $2 \times 2$ games. However, the undesirable features of the MW scheme question if it may reflect a real quantum game. Our protocol is closer to the EWL concept. In both the EWL and eMW schemes there is the need for entangled states to generate non-classical results. Next, turning the quantum game into a classical one can be obtained just by the restriction of players strategies. It is also possible to study the case where only one of the players has access to the quantum strategy (in the EWL it is performed by the appropriate restriction of the player’s strategy set). It could be done by allowing only one of the players to decide whether to play the state $|00\rangle$ of $|\psi\rangle$, i.e., only one of the players has access to the actions $C$ and $Q$.

At the same time, the eMW scheme is much simpler that the EWL scheme for studying high-dimensional bimatrix games. Thus, it may constitute an alternative to the EWL scheme.

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