Polynomial interpolation and Gaussian quadrature for matrix valued functions

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Abstract

The techniques for polynomial interpolation and Gaussian quadrature are generalized to matrix-valued functions. It is shown how the zeros and rootvectors of matrix orthonormal polynomials can be used to get a quadrature formula with the highest degree of precision.

1 Introduction

The aim of this paper is to construct quadrature formulas, using orthogonal matrix polynomials, to approximate matrix integrals. We will give an expression for the quadrature coefficients and show that the constructed formula has the highest possible degree of accuracy and converges to the exact value of the matrix integral. All these ideas are generalizations of the classical Gaussian quadrature rules for the scalar case.

In Section 2 we will give a survey of definitions and properties of matrix polynomials. These can be found in the book on matrix polynomials by I. Gohberg, P. Lancaster and L. Rodman [10] and in the survey on orthogonal matrix polynomials by L. Rodman [12]. In Section 3 we introduce orthogonal matrix polynomials on the real line and discuss some properties which we will need in the next sections. These orthogonal matrix polynomials have been considered earlier by Delsarte, Genin and Kamp [4] and Geronimo [7]. As in the scalar case, the theory of approximate integration uses results from the theory of interpolation. In Section 4 we will discuss polynomial interpolation and particularly the interpolation problem of Lagrange. We will give an expression for the Lagrange interpolation polynomial in the general case and then apply this result to the case of a Jordan pair \((X, J)\) of the orthonormal matrix polynomial \(P_n(x)\). These results are also generalizations of the known results for the scalar case. The interpolation problem has also been treated in [3], [4], [5] and [6], but their approach is an algebraic one. We have restricted our attention to the interpolation formula we needed for the construction of Gaussian quadrature rules. The Gaussian quadrature formula is then constructed in

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Section 5, where we give a formula for the quadrature coefficients and show that the quadrature rule converges under appropriate conditions.

2 Matrix polynomials

If \( A_0, A_1, \ldots, A_n \) are elements of \( \mathbb{R}^{p \times p} \) and \( A_n \neq 0 \), then we call
\[
P(x) = A_n x^n + A_{n-1} x^{n-1} + \ldots + A_1 x + A_0,
\]
a matrix polynomial of degree \( n \). This matrix polynomial is monic when \( A_n = I \), the identity matrix. A point \( x_0 \) is a zero of \( P(x) \), if \( \det P(x_0) = 0 \). Note that if the leading coefficient of \( P(x) \) is non-singular, \( \det P(x_0) \) is a polynomial of degree \( np \). Another important notion associated with matrix polynomials are Jordan chains. A sequence of \( p \)-dimensional column vectors \( v_0, v_1, \ldots, v_k \) is called a right Jordan chain of length \( k + 1 \) of a monic matrix polynomial \( \hat{P}(x) \) corresponding to \( x_0 \), if \( v_0 \neq 0 \) and
\[
\sum_{i=0}^{l} \frac{1}{i!} \hat{P}^{(i)}(x_0)v_{l-i} = 0, \quad l = 0, 1, \ldots, k.
\]
The initial vector \( v_0 \neq 0 \) is called a rootvector of \( \hat{P}(x) \) corresponding to \( x_0 \). Note that in [10] and [12] the zeros are called eigenvalues and the rootvectors are called eigenvectors. \( \hat{P}^{(i)}(x) \) is the \( i \)th derivative of \( \hat{P}(x) \) with respect to \( x \) and this means that we take the \( i \)th derivative of every element of \( \hat{P}(x) \) with respect to \( x \).

In analogy with the definition of a right Jordan chain, we call a sequence of \( p \)-dimensional row vectors \( w_0, w_1, \ldots, w_k \) a left Jordan chain of length \( k + 1 \) of a monic matrix polynomial \( \hat{P}(x) \) corresponding to \( x_0 \), if \( v_0 \neq 0 \) and
\[
\sum_{i=0}^{l} \frac{1}{i!} w_{l-i} \hat{P}^{(i)}(x_0) = 0, \quad l = 0, 1, \ldots, k.
\]

Jordan chains are not unique: a matrix polynomial can have different Jordan chains of various lengths and different rootvectors, corresponding to the same zero. In what follows we formulate the definitions and properties for right Jordan chains, unless explicitly mentioned.

A convenient way of writing a Jordan chain is given by the following property.

**Proposition 2.1** ([10], p. 27) The vectors \( v_0, v_1, \ldots, v_k \) form a right Jordan chain of the monic matrix polynomial \( P(x) = I x^n + A_{n-1} x^{n-1} + \ldots + A_1 x + A_0 \) corresponding to \( x_0 \) if and only if \( v_0 \neq 0 \) and
\[
X_0 J_0^n + A_{n-1} X_0 J_0^{n-1} + \ldots + A_1 X_0 J_0 + A_0 X_0 = 0,
\]
where \( X_0 = \left( \begin{array}{ccc} v_0 & \ldots & v_k \end{array} \right) \) is a \( p \times (k + 1) \) matrix and \( J_0 \) is a Jordan block of size \( (k + 1) \times (k + 1) \) with \( x_0 \) on the main diagonal.

Observe that the equations
\[
\sum_{i=0}^{l} \frac{1}{i!} L^{(i)}(x_0)v_{l-i} = 0, \quad l = 0, 1, \ldots, k,
\]
where \( L(x) = L_n x^n + L_{n-1} x^{n-1} + \ldots + L_1 x + L_0 \), with \( L_i \in \mathbb{R}^{p \times q} \) (i=1,2,..,n), can always be written as
\[
L_n X_0 J_0^n + L_{n-1} X_0 J_0^{n-1} + \ldots + L_1 X_0 J_0 + L_0 X_0 = 0,
\]
where \( X_0 = \begin{pmatrix} v_0 & \ldots & v_k \end{pmatrix} \) is a \( p \times (k + 1) \) matrix and \( J_0 \) is a Jordan block of size \((k + 1) \times (k + 1)\) with \( x_0 \) on the main diagonal.

In the following definitions and properties we restrict ourselves to monic \( p \times p \) matrix polynomials, but most of the theory can also be given in the context of regular matrix polynomials. These are matrix polynomials which satisfy \( \det P(x) \neq 0 \).

Now we introduce the notion of a canonical set of Jordan chains. Let
\[
v^{(i)}_{j,0}, v^{(i)}_{j,1}, \ldots, v^{(i)}_{j,\mu^{(i)}_j-1}, \quad j = 1, 2, \ldots, s_i,
\]
be a set of Jordan chains of a \( p \times p \) monic matrix polynomial \( \hat{P}(x) \) corresponding to the zero \( x_i \). Then we call the set canonical if the rootvectors \( v^{(i)}_{1,0}, v^{(i)}_{2,0}, \ldots, v^{(i)}_{s_i,0} \) are linearly independent and \( \sum_{j=1}^{s_i} \mu^{(i)}_j = m_i \), where \( m_i \) is the multiplicity of \( x_i \) as zero of \( \hat{P}(x) \). Such a canonical set of Jordan chains is not unique, but the number of chains and their length depend only upon \( \hat{P}(x) \) and \( x_i \) and do not depend on the choice of canonical set. A canonical set of Jordan chains can be associated with a pair of matrices \((X_i, J_i)\), which is called the Jordan pair of \( \hat{P}(x) \) corresponding to \( x_i \) and defined as follows:
\[
X_i = (v^{(i)}_{1,0} \ldots v^{(i)}_{1,\mu^{(i)}_1-1} \ldots v^{(i)}_{s_i,0} \ldots v^{(i)}_{s_i,\mu^{(i)}_{s_i}-1}) \quad \text{a } p \times m_i \text{ dimensional matrix}
\]
and
\[
J_i = \text{diag}(J_{i,1}, J_{i,2}, \ldots, J_{i,s_i}) \quad \text{a } m_i \times m_i \text{ dimensional matrix with }\]
\[
J_{i,j} = \begin{pmatrix}
    x_i & 1 & \cdots & \cdots \\
    \cdots & \cdots & \cdots & \cdots \\
    \cdots & \cdots & \cdots & \cdots \\
    x_i & 1 & \cdots & \cdots \\
\end{pmatrix} \quad \text{a } \mu^{(i)}_j \times \mu^{(i)}_j \text{ dimensional matrix.}
\]

A pair of matrices \((X, J)\) where \( X \) is a \( p \times np \) dimensional matrix and \( J \) a \( np \times np \) dimensional Jordan matrix is called a Jordan pair for the monic matrix polynomial \( \hat{P}(x) \) if
\[
X = (X_1 \ X_2 \ \ldots \ X_k) \quad \text{and} \quad J = \text{diag}(J_1 \ J_2 \ \ldots \ J_k),
\]
where \((X_i, J_i)\) is a Jordan pair of \( \hat{P}(x) \) corresponding to \( x_i \) and \( k \) is the number of different zeros of \( \hat{P}(x) \). Jordan pairs have the following important property:

**Proposition 2.2** ([10], p. 45) *Let \((X, J)\) be a pair of matrices where \( X \) is of size \( p \times np \) and \( J \) is a Jordan matrix of size \( np \times np \). Then \((X, J)\) is a Jordan pair of the monic matrix polynomial \( \hat{P}(x) = Ix^n + A_{n-1} x^{n-1} + \ldots + A_1 x + A_0 \) if and only if
\[
(1)
\]
\[
col(XJ)^{n-1}_{t=0} = \begin{pmatrix} X \\ XJ \\ XJ^2 \\ \cdots \\ XJ^{n-1} \end{pmatrix}
\]
is a non-singular \( np \times np \) matrix.*
(2) $XJ^n + A_{n-1}XJ^{n-1} + \ldots + A_1XJ + A_0X = 0$.

The requirement that $J$ is Jordan is not essential. We call a pair of matrices $(X, T)$ where $X$ is of size $p \times np$ and $T$ is a $np \times np$ dimensional matrix, a standard pair for the monic matrix polynomial $\hat{P}(x) = Ix^n + A_{n-1}x^{n-1} + \ldots + A_1x + A_0$ if

1. $\text{col}(XT^l)_{l=0}^{n-1}$ is a non-singular $np \times np$ matrix,

2. $XT^n + A_{n-1}XT^{n-1} + \ldots + A_1XT + A_0X = 0$.

This means that every Jordan pair is a standard pair and every standard pair $(X, T)$ for which $T$ is a Jordan matrix is a Jordan pair.

With every standard pair $(X, T)$ of a monic matrix polynomial $\hat{P}(x)$ we can associate a third matrix $Y$ of size $np \times p$, with

$$Y = \begin{pmatrix} X & 0 \\ XJ & 0 \\ XJ^2 & \vdots \\ \vdots & \vdots \\ XJ^{n-1} & 0 \\ \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \end{pmatrix}.$$ 

The triple $(X, T, Y)$ is called a standard triple for $\hat{P}(x)$ and if $T = J$, a Jordan matrix, then $(X, J, Y)$ is called a Jordan triple. One can proof that $(X', C_1, Y')$, where

$$X' = \begin{pmatrix} I & 0 & \ldots & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \ldots & I \\ -A_0 & -A_1 & -A_2 & \ldots & -A_{n-1} \end{pmatrix} \quad \text{and} \quad Y' = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I \end{pmatrix},$$

is a standard triple of $\hat{P}(x) = Ix^n + A_{n-1}x^{n-1} + \ldots + A_1x + A_0$. Two standard triples $(X, T, Y)$ and $(X', T', Y')$ are similar if there exists an invertible $np \times np$ matrix $S$ such that

$$X' = XS \quad ; \quad T' = S^{-1}TS \quad \text{and} \quad Y' = S^{-1}Y.$$ 

This matrix $S$ is uniquely defined by,

$$S = (\text{col}(XT^l)_{l=0}^{n-1})^{-1}(\text{col}(X'T'^l)_{l=0}^{n-1}).$$

Consider a Jordan triple $(X, J, Y)$, then we already know that the columns of $X$, when decomposed into blocks consistently with the decomposition of $J$ into blocks, form right Jordan chains for $\hat{P}(x)$. In analogy with this, we can give a similar meaning to the rows of $Y$. Indeed, the rows of $Y$, partitioned into blocks consistently with the decomposition of $J$ into blocks and taken in reverse order, form left Jordan chains for $\hat{P}(x)$. The notion of standard triples is important for the following representation theorem:

**Theorem 2.3** ([4], p. 58) *Let $\hat{P}(x) = Ix^n + A_{n-1}x^{n-1} + \ldots + A_1x + A_0$ be a monic matrix polynomial of degree $n$ with standard triple $(X, T, Y)$. Then $\hat{P}(x)$ admits the following representations :*
(1) $\hat{P}(x) = x^n - XT^n(V_1 + V_2x + \ldots + V_nx^{n-1})$, where $V_i$ are $np \times p$ matrices such that 
$(V_1 \ V_2 \ldots \ V_n) = (col(\ X^{t,0}t_0))^{-1}$.

(2) $\hat{P}(x) = x^n - (W_1 + W_2x + \ldots + W_nx^{n-1})T^nY$, where $W_i$ are $p \times np$ matrices such 
that $col(W_i)^n_{t=1} = (Y \ TY \ T^2Y \ldots \ T^n Y)^{-1}$.

Note that those forms are independent of the choice of the standard triple.

Finally we have some properties about the divisibility of matrix polynomials. We say 
that the matrix polynomials $Q(x)$ and $R(x)$ are the right quotient and the right remainder, 
respectively, of $P(x)$ on division by $D(x)$ if 

$$P(x) = Q(x)D(x) + R(x)$$

and if the degree of $R(x)$ is less than that of $D(x)$. The right division of matrix polynomials 
of the same order is always possible and unique, provided the divisor is a polynomial with 
non-singular leading coefficient (see [3, p. 78]). In a similar manner we can define the left 
division: $P(x) = D(x) \hat{Q}(x) + \hat{R}(x)$.

**Proposition 2.4** Let $P(x) = A_m x^m + A_{m-1} x^{m-1} + \ldots + A_1 x + A_0$ be a matrix polynomial 
of degree $m$ and let $D(x) = B_n x^n + B_{n-1} x^{n-1} + \ldots + B_1 x + B_0$ be a matrix polynomial 
of degree $n$, with a non-singular leading coefficient $B_n$ and with Jordan pair $(X, J)$. Then 
$D(x)$ is a right divisor of $P(x)$ if and only if

$$A_m X J^m + A_{m-1} X J^{m-1} + \ldots + A_1 X J + A_0 X = 0.$$ 

**Proof**

First of all we denote that every Jordan chain $v_0, v_1, \ldots, v_k$ of $D(x)$ and corresponding to 
x0 is also a Jordan chain of $Q(x)D(x)$, corresponding to the same zero:

$$\sum_{i=0}^{l} \frac{1}{i!}[Q(x)D(x)]^{(i)}(x_0)v_{l-i} = \sum_{i=0}^{l} \sum_{t=0}^{i} \frac{1}{t!} \binom{i}{t} Q^{(t)}(x_0)D^{(i-t)}(x_0)v_{l-i}$$

$$= \sum_{t=0}^{l} \frac{1}{t!} Q^{(t)}(x_0) \sum_{i=t}^{l} \frac{1}{(i-t)!} D^{(i-t)}(x_0)v_{l-i}$$

$$= \sum_{t=0}^{l} \frac{1}{t!} Q^{(t)}(x_0) \sum_{s=0}^{l-t} \frac{1}{s!} D^{(s)}(x_0)v_{l-t-s}$$

$$= 0 \quad \text{for} \quad l = 0, 1, \ldots, k.$$

Suppose $D(x)$ is a right divisor of $P(x)$, this means $P(x) = Q(x)D(x)$. But then every 
Jordan chain of $D(x)$ is also a Jordan chain of $P(x)$. In particular we have

$$A_m X J^m + A_{m-1} X J^{m-1} + \ldots + A_1 X J + A_0 X = 0.$$ 

Suppose now we know that the above mentioned equation holds. This means that every 
Jordan chain of the Jordan pair of $D(x)$ is also a Jordan chain of $P(x)$. Moreover, since 
the leading coefficient of $D(x)$ is non-singular, the right division is possible and unique, 
$P(x) = Q(x)D(x) + R(x)$. So, every Jordan chain of the Jordan pair of $D(x)$ is also a
Jordan chain of $R(x)$. But this implies that the matrix polynomial $R(x)$ of degree $\leq n - 1$ satisfies

$$
\begin{pmatrix}
R_0 & R_1 & \ldots & R_{n-1}
\end{pmatrix}
\begin{pmatrix}
X \\
XJ \\
\vdots \\
XJ^{n-1}
\end{pmatrix} = 0.
$$

Since $(X, J)$ is a Jordan pair, the $np \times np$ dimensional matrix $\text{col}(XJ)^{n-1}_{i=0}$ is non-singular and thus $R(x) = 0$. ■

Note that this proposition was also proved in [3, Thm. 2.1 on p. 333].

3 Orthogonal matrix polynomials on the real line

A symmetric $p \times p$ matrix-valued function $W(x)$, integrable over $[a, b]$ is called a weight matrix function if $W(x) \geq 0$ and $\det W(x) \neq 0$ holds almost everywhere (see [3]). The notation $X \leq Y$ for symmetric matrices means that $Y - X$ is positive semidefinite. The matrix integral

$$
\int_a^b F(x) W(x) G(x)^T \, dx
$$

where $F(x)$ and $G(x)$ are continuous matrix-valued functions, is defined in a natural way. The $(i, j)$th element is given by a sum of integrals:

$$
\sum_{s=1}^p \sum_{t=1}^p \int_a^b F(x)_{i,s} W(x)_{s,t} G(x)^T_{t,j} \, dx.
$$

Let $\mathbb{R}^{p \times p}[x]$ be the set of polynomials in a real variable $x$ and whose coefficients are $p \times p$ matrices with real entries. If $P(x)$ and $Q(x)$ are elements of this set, then we define a matricial inner product on $\mathbb{R}^{p \times p}[x]$ as follows:

$$
\langle P(x), Q(x) \rangle_L = \int_a^b P(x) W(x) Q(x)^T \, dx.
$$

This matricial inner product has some properties which we will recognize as generalizations of the properties of the scalar inner product.

**Proposition 3.1**

1. $\langle P, Q \rangle_L = \langle Q, P \rangle_L^T$ where $P, Q \in \mathbb{R}^{p \times p}[x]$.
2. $\langle C_1 P_1 + C_2 P_2, Q \rangle_L = C_1 \langle P_1, Q \rangle_L + C_2 \langle P_2, Q \rangle_L$ where $C_1, C_2 \in \mathbb{R}^{p \times p}$ and $P_1, P_2, Q \in \mathbb{R}^{p \times p}[x]$.
3. $\langle xP, Q \rangle_L = \langle P, xQ \rangle_L$ where $P, Q \in \mathbb{R}^{p \times p}[x]$.
4. Let $P(x) \in \mathbb{R}^{p \times p}[x]$, then $\langle P, P \rangle_L$ is positive semidefinite and even positive definite if $\det P(x) \neq 0$.
5. Let $P \in \mathbb{R}^{p \times p}[x]$, then $\langle P, P \rangle_L = 0$ if and only if $P = 0$.

These properties are easily proved by means of straightforward computation.
A generalization of the Gram-Schmidt orthonormalisation procedure for the set 
\( \{I, xI, x^2I, \ldots\} \) with respect to the matricial inner product \( \langle ., . \rangle \)
will give a set of orthonormal matrix polynomials \( \{P_n(x)\}_{n=0}^{\infty} \) which satisfy

\[
\int_a^b P_n(x) W(x) P_m(x)^T \, dx = \delta_{n,m} I.
\]

Moreover, \( P_n(x) \) is a matrix polynomial of degree \( n \), with a non-singular leading coefficient and is defined upon a multiplication on the left by an orthogonal matrix.

As in the scalar case, these orthonormal matrix polynomials are orthogonal to every
matrix polynomial of lower degree and they satisfy a three-term recurrence relation.

\[
x P_n(x) = D_{n+1} P_{n+1}(x) + E_n P_n(x) + D_n^T P_{n-1}(x), \quad n \geq 0,
\]

where \( D_n \) is a positive definite matrix and \( E_n \) is a symmetric matrix. The orthonormal
polynomials are defined only up to a left orthogonal factor and it is convenient to choose
this factor in such a way that the recurrence coefficients \( D_n \) are symmetric. We assumed,
without loss of generality that \( \int_a^b W(x) \, dx = I \). Furthermore we have the Christoffel-
Darboux formula :

\[
P_n(y)^T D_{n+1} P_{n+1}(x) - P_{n+1}(y)^T D_{n+1} P_n(x) = (x - y) \sum_{i=0}^{n} P_i(y)^T P_i(x).
\]

(see [7]). If we take \( x = y \), we get

\[
P_n(x)^T D_{n+1} P_{n+1}(x) = P_{n+1}(x)^T D_{n+1} P_n(x)
\]

so that \( P_n(x)^T D_{n+1} P_{n+1}(x) \) is a symmetric matrix. By means of straightforward computation we get the following equation :

\[
\sum_{i=0}^{n} P_i(x)^T P_i(x) = P_{n+1}(x)^T D_{n+1} P_n(x) - P_n(x)^T D_{n+1} P_{n+1}(x).
\]

The matrix

\[
K_n(x,y) = \sum_{i=0}^{n} P_i(y)^T P_i(x)
\]

is a positive definite matrix and we call it the reproducing kernel because of the following property.

**Proposition 3.2** Let \( \Pi_m(x) \) be a matrix polynomial of degree \( m \leq n \), then

\[
\langle \Pi_m(x), K_n(x,y) \rangle_L = \Pi_m(y)
\]

**Proof**

If we write \( \Pi_m(x) \) in terms of the orthonormal matrix polynomials \( P_0(x), \ldots, P_m(x) \) :

\[
\Pi_m(x) = \sum_{i=0}^{m} A_i P_i(x),
\]
we have for $m \leq n$

$$\langle \Pi_m(x), K_n(x, y) \rangle_L = \sum_{i=0}^{m} \sum_{j=0}^{n} A_i \langle P_i(x), P_j(x) \rangle_L P_j(y)$$

$$= \sum_{i=0}^{m} A_i P_i(y)$$

$$= \Pi_m(y). \blacksquare$$

In the scalar case all the zeros of an orthonormal polynomial are simple. This is not the case for orthonormal matrix polynomials, but nevertheless we can proof a similar property.

**Proposition 3.3** The zeros of the orthonormal matrix polynomial $P_n(x)$ have a multiplicity $\leq p$, where $p$ is the size of the matrices.

**Proof**

Let $x_0$ be a zero of $P_n(x)$ with multiplicity $m > p$. Consider a canonical set of right Jordan chains corresponding to $x_0$

$$v_{i,0}, v_{i,1}, \ldots, v_{i,\mu_i-1}, \quad i = 1, 2, \ldots, s.$$  

This means that the $p$-dimensional column vectors $v_{1,0}, v_{2,0}, \ldots, v_{s,0}$ are linearly independent and $\sum_{i=1}^{s} \mu_i = m$ (Section 1). Since $m > p$, there has to be a Jordan chain of length $> 1$. Suppose $v_0$ and $v_1$ are the two leading vectors of this chain, then they satisfy

$$P_n(x_0)v_0 = 0 \quad \text{and} \quad v_0 \neq 0,$$

$$P'_n(x_0)v_0 + P_n(x_0)v_1 = 0.$$

Using these equations, we get

$$v_0^T K_{n-1}(x_0, x_0)v_0 = v_0^T P_n(x_0)' D_n P_{n-1}(x_0)v_0 - v_0^T P_{n-1}(x_0)' D_n P_n(x_0)v_0$$

$$= -v_1^T P_n(x_0) D_n P_{n-1}(x_0)v_0$$

$$= -v_1^T P_{n-1}(x_0) D_n P_n(x_0)v_0$$

$$= 0.$$

But $K_{n-1}(x_0, x_0)$ is a symmetric and positive definite matrix and $v_0 \neq 0$. So the multiplicity of $x_0$ as zero of $P_n(x)$ has to be $\leq p$. \blacksquare

**Corollary.**

In this proof we showed that the length of a Jordan chain of $P_n(x)$ cannot be greater than 1. Thus a canonical set of right Jordan chains of $P_n(x)$ corresponding with a zero $x_0$ consist of $m$ linearly independent, non-zero $p$-dimensional column vectors, where $m$ is the multiplicity of $x_0$ as zero of $P_n(x)$.
4 Polynomial interpolation

4.1 Polynomial interpolation in general

Consider a \( p \times p \) matrix-valued function \( F(x) \) and \( k \) different points \( x_1, x_2, \ldots, x_k \) with multiplicity resp. \( m_1, m_2, \ldots, m_k \) where \( \sum_{i=1}^{k} m_i = np \). With every point \( x_i, i = 1, 2, \ldots, k \), we associate a set of \( p \)-dimensional column vectors

\[
\begin{aligned}
&v_{1,0}^{(i)}, v_{1,1}^{(i)}, \ldots, v_{1,\mu_1^{(i)}-1}^{(i)}, v_{2,0}^{(i)}, v_{2,1}^{(i)}, \ldots, v_{2,\mu_2^{(i)}-1}^{(i)}, \ldots, v_{s_i,0}^{(i)}, v_{s_i,1}^{(i)}, \ldots, v_{s_i,\mu_{s_i}^{(i)}-1}^{(i)}
\end{aligned}
\]

where \( \sum_{j=1}^{s_i} \mu_j^{(i)} = m_i \) and \( v_{1,0}^{(i)}, v_{2,0}^{(i)}, \ldots, v_{s_i,0}^{(i)} \) are non-zero, linearly independent vectors.

When we put these vectors in a \( p \times np \) dimensional matrix, we get

\[
X = (X_1 \ X_2 \ \ldots \ X_k) \quad \text{where} \quad X_i = (v_{1,0}^{(i)} \ \ldots \ v_{1,\mu_1^{(i)}-1}^{(i)} \ \ldots \ v_{s_i,0}^{(i)} \ \ldots \ v_{s_i,\mu_{s_i}^{(i)}-1}^{(i)}).
\]

The square \( np \times np \) matrix \( J \) is given by

\[
J = \text{diag}(J_1, J_2, \ldots, J_k) \quad \text{where} \quad J_i = \text{diag}(J_{i,1}, J_{i,2}, \ldots, J_{i,s_i})
\]

and

\[
J_{i,j} = \begin{pmatrix}
x_i & 1 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
x_i & 1 & \cdots & \cdots \\
\end{pmatrix}.
\]

Here \( J_i \) is a \( m_i \times m_i \) matrix and \( J_{i,j} \) a \( \mu_j^{(i)} \times \mu_j^{(i)} \) matrix. We require that the matrix \( \text{col}(X J^t)_{t=0}^{n-1} \) is non-singular. Note that this is a necessary and sufficient condition in order that \( (X, J) \) is a Jordan pair of a monic matrix polynomial of degree \( n \) (Section 1).

We will construct a matrix polynomial \( P(x) = A_{n-1}x^{n-1} + \ldots + A_1x + A_0 \), which satisfies

\[
\sum_{l=0}^{q} \frac{1}{l!} P^{(l)}(x_i) v_{l,q-l}^{(i)} = \sum_{l=0}^{q} \frac{1}{l!} F^{(l)}(x_i) v_{l,q-l}^{(i)}
\]

and this for

\[
q = 0, 1, \ldots, \mu_t^{(i)} - 1, \quad t = 1, 2, \ldots, s_i \quad \text{and} \quad i = 1, 2, \ldots, k.
\]

The coefficients of \( P(x) \) can be determined by solving a linear system of \( np^2 \) unknowns and \( np^2 \) equations, because \( \sum_{i=1}^{k} \sum_{t=1}^{s_i} \mu_t^{(i)} = np \).

**Theorem 4.1** The general polynomial interpolation problem has a unique solution.

**Proof**

Let \( P_1(x) \) and \( P_2(x) \) be two solutions of the polynomial interpolation problem. Then \( P_1(x) - P_2(x) \) is a matrix polynomial of degree \( \leq n - 1 \) which satisfies

\[
\sum_{l=0}^{q} \frac{1}{l!} [P_1^{(l)}(x_i) - P_2^{(l)}(x_i)] v_{l,q-l}^{(i)} = 0,
\]

9
where
\[ q = 0, 1, \ldots, \mu_i^{(i)} - 1, \quad t = 1, 2, \ldots, s_i \quad \text{and} \quad i = 1, 2, \ldots, k. \]

From this we get that the vectors
\[ v_{t,0}^{(i)}, v_{t,1}^{(i)}, \ldots, v_{t,\mu_i^{(i)} - 1}^{(i)}, \quad t = 1, 2, \ldots, s_i, \quad i = 1, 2, \ldots, k \]
form a Jordan chain for the matrix polynomial \( P_1(x) - P_2(x) = B_{n-1}x^{n-1} + \ldots + B_1x + B_0, \) corresponding to \( x_i. \) But this implies that
\[ B_{n-1}X_{i,t}J_{i,t}^{n-1} + B_{n-2}X_{i,t}J_{i,t}^{n-2} + \ldots + B_1X_{i,t}J_{i,t} + B_0X_{i,t} = 0, \]
where \( t = 1, 2, \ldots, s_i \) and \( i = 1, 2, \ldots, k. \) The \( p \times \mu_i^{(i)} \) dimensional matrix \( X_{i,t} \) and the \( \mu_i^{(i)} \times \mu_i^{(i)} \) dimensional matrix \( J_{i,t} \) are given by
\[
X_{i,t} = \begin{pmatrix}
  v_{t,0}^{(i)} & v_{t,1}^{(i)} & \cdots & v_{t,\mu_i^{(i)} - 1}^{(i)}
\end{pmatrix}
\]
and
\[
J_{i,t} = \begin{pmatrix}
  x_i & 1 & & \\
  \vdots & \ddots & \ddots & \\
  & & x_i & 1 \\
  & & & x_i
\end{pmatrix}.
\]

Hence we get
\[
\begin{pmatrix}
  B_0 & B_1 & \ldots & B_{n-1}
\end{pmatrix}
\begin{pmatrix}
  X_{i,t} \\
  X_{i,t}J_{i,t} \\
  \vdots \\
  X_{i,t}J_{i,t}^{n-1}
\end{pmatrix}
= 0, \quad t = 1, 2, \ldots, s_i, \quad i = 1, 2, \ldots, k,
\]
or
\[
\begin{pmatrix}
  B_0 & B_1 & \ldots & B_{n-1}
\end{pmatrix}
\begin{pmatrix}
  X \\
  XJ \\
  \vdots \\
  XJ^{n-1}
\end{pmatrix}
= 0.
\]

Since \( col(XJ_i^{t})_{i=0}^{n-1} \) is a non-singular matrix, we have
\[ B_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, n-1. \]

And this implies that \( P_1(x) = P_2(x). \) □

**Corollary**

If \( F(x) \) is a matrix polynomial of degree \( \leq n - 1, \) then \( P(x) = F(x). \)

The interpolation problem can be formulated in terms of vector polynomials instead of the sets of \( p \)-dimensional vectors. Define the \( p \)-dimensional vectors
\[
\sum_{l=0}^{q} \frac{1}{l!} F^{(l)}(x_i) v_{t,q-l}^{(i)} = z_{t,q}^{(i)}
\]
and the \( p \)-dimensional vector polynomials
\[
v_t^{(i)}(x) = v_{t,0}^{(i)} + v_{t,1}^{(i)}(x - x_i) + v_{t,2}^{(i)}(x - x_i)^2 + \ldots + v_{t,\mu_i^{(i)} - 1}^{(i)}(x - x_i)^{\mu_i^{(i)} - 1}
\]
\[ z_t^{(i)}(x) = z_{t,0}^{(i)} + z_{t,1}(x-x_i) + z_{t,2}(x-x_i)^2 + \ldots + z_{t,\mu_t^{(i)}-1}(x-x_i)^{\mu_t^{(i)}-1} \]

for \( q = 0, 1, \ldots, \mu_t^{(i)} - 1, t = 1, 2, \ldots, s_i \) and \( i = 1, 2, \ldots, k \). In this case we are looking for a matrix polynomial \( P(x) \) which satisfies

\[
\left. \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}} P(x)v_t^{(i)}(x) \right|_{x=x_i} = \left. \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}} z_t^{(i)}(x) \right|_{x=x_i}
\]

for \( \alpha = 1, 2, \ldots, \mu_t^{(i)}, t = 1, 2, \ldots, s_i \) and \( i = 1, 2, \ldots, k \). In addition we know \( v_t^{(i)}(x_i) \) are non-zero and linearly independent vectors.

If we set \( s_i = 1, \) \( i = 1, 2, \ldots, k \), we have \( k \) different points \( x_1, x_2, \ldots, x_k \), \( k \) vector polynomials \( v_1^{(i)}(x), v_2^{(i)}(x), \ldots, v_k^{(i)}(x) \) for which \( v_i^{(i)}(x_i) \neq 0, i = 1, 2, \ldots, k \) and \( k \) vector polynomials \( z_1^{(i)}(x), z_2^{(i)}(x), \ldots, z_k^{(i)}(x) \) and we want to find a matrix polynomial \( P(x) \) which satisfies

\[
\left. \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}} P(x)v_1^{(i)}(x) \right|_{x=x_i} = \left. \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}} z_1^{(i)}(x) \right|_{x=x_i}, \quad \alpha = 1, 2, \ldots, m_i, \quad i = 1, 2, \ldots, k.
\]

This is the interpolation problem treated in [1].

We will put \( \mu_t^{(i)} = 1 \) instead of \( s_i = 1 \) in the interpolation problem that we will discuss and which we need for the construction of the Gaussian quadrature rule.

### 4.2 The interpolation problem of Lagrange

If all \( \mu_j^{(i)} = 1 \), we get the interpolation problem of Lagrange. This means that we have \( k \) different points \( x_1, x_2, \ldots, x_k \) with multiplicity resp. \( m_1, m_2, \ldots, m_k \) and \( \sum_{i=1}^{k} m_i = np \). Every point \( x_i, i = 1, 2, \ldots, k \), is associated with a set of non-zero, linearly independent \( p \)-dimensional column vectors \( v_{i,1}, v_{i,2}, \ldots, v_{i,m_i} \). Let \( X \) and \( J \) be defined as follows

\[
X = (v_{1,1} \ldots v_{1,m_1} \ldots v_{k,1} \ldots v_{k,m_k}) \quad \text{and} \quad J = \text{diag}(x_1, x_2, \ldots, x_2, \ldots, x_k),
\]

then we require that the matrix \( \text{col}(XJ)^{n-1} \) is non-singular.

We will construct the interpolation matrix polynomial \( P(x) \) of degree \( n-1 \) such that

\[
P(x_i) v_{i,j} = F(x_i) v_{i,j}, \quad j = 1, \ldots, m_i, \quad i = 1, \ldots, k.
\]

**Special case.**

If \( p = 1 \), then all the multiplicities have to be equal to 1. This means we have \( n \) different points \( x_1, x_2, \ldots, x_n \) and every point \( x_i \) is associated with a non-zero number \( v_i, i = 1, 2, \ldots, k \). The matrix

\[
\begin{pmatrix}
X \\
XJ \\
XJ^2 \\
\vdots \\
XJ^{n-1}
\end{pmatrix} =
\begin{pmatrix}
v_1 & v_2 & \ldots & v_n \\
v_1 x_1 & v_2 x_2 & \ldots & v_n x_n \\
v_1 x_1^2 & v_2 x_2^2 & \ldots & v_n x_n^2 \\
\vdots & \vdots & \vdots & \vdots \\
v_1 x_1^{n-1} & v_2 x_2^{n-1} & \ldots & v_n x_n^{n-1}
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & \ldots & 1 \\
x_1 & x_2 & \ldots & x_n \\
x_1^2 & x_2^2 & \ldots & x_n^2 \\
\vdots & \vdots & \vdots & \vdots \\
x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1}
\end{pmatrix} \begin{pmatrix}
v_1 & 0 & 0 & \ldots & 0 \\
v_2 & 0 & 0 & \ldots & 0 \\
0 & v_3 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & v_n
\end{pmatrix}
\]
is non-singular because of our choice of the points \( x_1, x_2, \ldots, x_n \) and the non zero-numbers \( v_1, v_2, \ldots, v_n \).
The interpolation polynomial satisfies
\[
p(x_i)v_i = f(x_i)v_i, \quad i = 1, 2, \ldots, n.
\]
But this is equivalent with
\[
p(x_i) = f(x_i), \quad i = 1, 2, \ldots, n,
\]
because \( v_i \neq 0 \), for \( i = 1, 2, \ldots, n \). So we find the polynomial interpolation problem of Lagrange, for real functions \( f : \mathbb{R} \to \mathbb{R} \).

If \( m_i = p \cdot d_i, i = 1, 2, \ldots, k \) and \( \text{col}(X_i J_i^1)_{l=0}^{d_i-1} \) are non-singular matrices, then \( (X_i, J_i) \) is the Jordan pair of a monic matrix polynomial \( L_i(x) \) of degree \( d_i \). Because \( d_i \geq 1 \), \( R_i(x) = F(x_i) \) will be a matrix polynomial of degree \( < d_i \) and this for \( i = 1, 2, \ldots, k \).
Now we get the following interpolation problem: given \( L_1(x), L_2(x), \ldots, L_k(x) \), monic matrix polynomials of degree \( d_1, d_2, \ldots, d_k \) respectively and \( R_1(x), R_2(x), \ldots, R_k(x) \) matrix polynomials of degree \( < d_i \) respectively. Find a matrix polynomial \( P(x) \) such that
\[
P(x) = S_i(x)L_i(x) + R_i(x), \quad i = 1, 2, \ldots, k
\]
for some matrix polynomials \( S_1(x), S_2(x), \ldots, S_k(x) \). This interpolation problem is treated in \[8\], \[9\] and \[4\].

**Theorem 4.2** The interpolation matrix polynomial of the Lagrange interpolation problem is given by
\[
P(x) = \sum_{i=1}^{k} F(x_i)(0 \ldots 0 v_{i,1} \ldots v_{i,m_i} 0 \ldots 0) (V_1 + V_2 x + \ldots + V_n x^{n-1}),
\]
where
\[
(V_1 V_2 \ldots V_n) = (\text{col}(X J^1_{l=0}^{n-1})_{l=0}^{n-1})^{-1},
\]
with \( V_i \) a \( np \times p \) matrix.

**Proof**
The interpolation matrix polynomial \( P(x) = A_{n-1}x^{n-1} + \ldots + A_1 x + A_0 \) is determined by a linear system of \( np^2 \) equations and \( np^2 \) unknowns,
\[
P(x_i)v_{i,j} = F(x_i)v_{i,j}, \quad j = 1, \ldots, m_i, \quad i = 1, \ldots, k.
\]
In block form we get
\[
\begin{pmatrix}
X \\
XJ \\
XJ^2 \\
\vdots \\
XJ^{n-1}
\end{pmatrix}
\begin{pmatrix}
(F(x_1)v_{1,1} F(x_1)v_{1,2} \ldots F(x_1)v_{1,m_1} F(x_2)v_{2,1} \ldots F(x_k)v_{k,m_k})
\end{pmatrix}
\]
or

\[ A_j = (F(x_1)v_{1,1}, F(x_1)v_{1,2}, \ldots, F(x_1)v_{1,m_1}, F(x_2)v_{2,1}, \ldots, F(x_k)v_{k,m_k}) V_{j+1}, \]

for \( j = 0, 1, \ldots, n-1 \). But then we have

\[ P(x) = (F(x_1)v_{1,1}, \ldots, F(x_1)v_{1,m_1}, F(x_2)v_{2,1}, \ldots, F(x_k)v_{k,m_k}) (V_1 + V_2 x + \ldots + V_n x^{n-1}) \]

or

\[ P(x) = \sum_{i=1}^{k} F(x_i) (0 \ldots 0 v_{i,1} \ldots v_{i,m_i} 0 \ldots 0) (V_1 + V_2 x + \ldots + V_n x^{n-1}), \]

with \((m_1 + \ldots + m_{i-1})\) zeros before \( v_{i,1} \) and \((m_{i+1} + \ldots + m_k)\) zeros after \( v_{i,m_i} \).

\[ \textbf{Theorem 4.3} \text{ The interpolation matrix polynomial of the Lagrange interpolation problem is given by } P(x) = \sum_{i=1}^{k} F(x_i) W_i(x), \]

where

\[ W_i(x) = \frac{1}{x - x_i} \begin{pmatrix} v_{i,1} & \ldots & v_{i,m_i} \end{pmatrix} \begin{pmatrix} w_{i,1}^T \\ \vdots \\ w_{i,m_i}^T \end{pmatrix} \hat{Q}_n(x). \]

The vector \( w_{i,j}^T \) is the \((m_1 + \ldots + m_{i-1} + j)\)th row from \( V_n \) and \( \hat{Q}_n(x) \) is the monic matrix polynomial of degree \( n \) with Jordan pair \((X, J)\).

Note that the vectors \( w_{i,j} \) are left rootvectors of the matrix polynomial \( \hat{Q}_n(x) \) (Section 1).

\[ \textbf{Proof} \]

The proof of this theorem consists of three parts:

(a) \( JV_i - V_n X J^n V_1 = 0 \) and \( JV_i - V_n X J^n V_i = V_{i-1} \), where \( i = 2, \ldots, n \).

To show these relations we consider the similar standard pair \((X', C_1)\) with

\[ X' = (I \ 0 \ \ldots \ 0) \]

and

\[ C_1 = \begin{pmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I \\ -B_0 & -B_1 & -B_2 & \ldots & -B_{n-1} \end{pmatrix}, \]
where \( \hat{Q}_n(x) = Ix^n + B_{n-1}x^{n-1} + \ldots + B_1x + B_0 \), the monic matrix polynomial with Jordanpair \((X, J)\). This means that \( X = X'S \) and \( J = S^{-1}C_1S \) with \( S = col(XJ^n)_{i=0} \) (see Section 1). The matrix \( col(X'C_1^n)_{i=0} \) satisfies
\[
\begin{pmatrix}
X' \\
X'C_1 \\
X'C_1^2 \\
\vdots \\
X'C_1^{n-1}
\end{pmatrix} =
\begin{pmatrix}
I & 0 & \ldots & 0 \\
0 & I & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & I
\end{pmatrix},
\]

because
\[
X'C_1^k
= (0 \ldots I \ldots 0) \quad \text{with } I \text{ on the } (k+1)\text{th place and } k = 0, 1, \ldots, n-1,
= (-B_0 - B_1 \ldots - B_{n-1}) \quad \text{for } k = n.
\]
So the matrices \( V'_i \) are given by
\[
V'_i = \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} \quad \text{with } I \text{ on the } i\text{th place.}
\]

But then we have
\[
C_1 \begin{pmatrix}
I \\
\vdots \\
0
\end{pmatrix} - \begin{pmatrix}
0 \\
\vdots \\
I
\end{pmatrix} (-B_0 - B_1 \ldots - B_{n-1}) \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} = \begin{pmatrix}
I \\
\vdots \\
0
\end{pmatrix} - \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]
\[
or
C_1V'_1 - V'_nX'C_1^nV'_1 = 0.
\]

Consequently,
\[
S^{-1}C_1SS^{-1}V'_1 - S^{-1}V'_nX'SS^{-1}C_1^nSS^{-1}V'_1 = 0
\]
or
\[
JV_1 - V_nXJ^nV_1 = 0.
\]

For \( i = 2, 3, \ldots, n \) we have
\[
C_1 \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} - \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} (-B_0 - B_1 \ldots - B_{n-1}) \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]
\[
or
C_1 - \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} (-B_0 - B_1 \ldots - B_{n-1}) \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}.
\]
From the property proved in (b) we get
\[ j \text{row, where } I \text{ if we call} \]

or
\[ \text{of the equation on the left with this matrix, we get} \]

so that
\[ \text{The monic matrix polynomial } \hat{Q}_n(x) \text{ with Jordan pair } (X, J) \text{ is given by} \]

(see Section 1). But this implies we immediately get the required expression from the relations proved in (a).

(c) \[ W_i(x) = \frac{1}{x-x_i} \begin{pmatrix} v_{i,1} & \ldots & v_{i,m_i} \end{pmatrix} \begin{pmatrix} w_{i,1}^T \\ \vdots \\ w_{i,m_i}^T \end{pmatrix} \hat{Q}_n(x). \]

From the property proved in (b) we get
\[ (xI - J)^{-1}V_n\hat{Q}_n(x) = (V_1 + V_2x + \ldots + V_nx^{n-1}). \]

If we call \( I_{m_i} \) the \( np \times np \) diagonal matrix with 1 on the \((m_1 + \ldots + m_{i-1} + j)\)th row, where \( j = 1, 2, \ldots, m_i \) and 0 on all the other places and we multiply both sides of the equation on the left with this matrix, we get
\[ \frac{1}{x-x_i} \begin{pmatrix} 0 \\ \vdots \\ w_{i,1}^T \\ \vdots \\ w_{i,m_i}^T \end{pmatrix} \hat{Q}_n(x) = I_{m_i} (V_1 + V_2x + \ldots + V_nx^{n-1}). \]

Multiplication on the left of this equation with the matrix \( X \), gives
\[ \frac{1}{x-x_i} \begin{pmatrix} v_{i,1} & v_{i,2} & \ldots & v_{i,m_i} \end{pmatrix} \begin{pmatrix} w_{i,1}^T \\ w_{i,2}^T \\ \vdots \\ w_{i,m_i}^T \end{pmatrix} \hat{Q}_n(x) =
\]

\[ (0 0 \ldots 0 v_{i,1} \ldots v_{i,m_i} 0 \ldots 0) (V_1 + V_2x + \ldots + V_nx^{n-1}), \]
such that the previous theorem gives the required expression

\[ P(x) = \sum_{i=1}^{k} F(x_i) \frac{1}{x - x_i} \begin{pmatrix} v_{i,1} & \cdots & v_{i,m_i} \end{pmatrix} \begin{pmatrix} u_{i,1}^T \\ \vdots \\ u_{i,m_i}^T \end{pmatrix} \hat{Q}_n(x). \]

We get another representation of the interpolation matrix polynomial if we choose the points \( x_i \) and the corresponding vectors in a special way. This representation shall be very useful for the construction of quadrature formulas.

**Theorem 4.4** Let \((X, J)\) be a Jordan pair of the orthonormal matrix polynomial \( P_n(x) \), then the interpolation matrix polynomial of the Lagrange interpolation problem is given by

\[ P(x) = \sum_{i=1}^{k} F(x_i) W_i(x), \]

with \( k \) the number of different zeros \( x_i \), \( m_i \) the multiplicity of \( x_i \), \( v_{i,j} \) the vectors associated with \( x_i \) and

\[ W_i(x) = \begin{pmatrix} v_{i,1} & \cdots & v_{i,m_i} \end{pmatrix} K_i^{-1} \begin{pmatrix} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{pmatrix} \begin{pmatrix} P_{n+1}(x_i)^T D_{n+1} P_n(x) \\ x - x_i \end{pmatrix}. \]

The \( m_i \times m_i \) dimensional matrix \( K_i \) satisfies

\[ K_i = - \begin{pmatrix} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{pmatrix} K_{n-1}(x_i, x_i) \begin{pmatrix} v_{i,1} & \cdots & v_{i,m_i} \end{pmatrix}, \]

with

\[ K_{n-1}(x, y) = \sum_{j=0}^{n-1} P_j(y)^T P_j(x). \]

Note that the Jordan chains of the orthonormal matrix polynomials \( P_n(x) \) have length 1. So, if \( x_1, x_2, \ldots, x_k \) are the zeros of the orthonormal matrix polynomials \( P_n(x) \) with multiplicity resp. \( m_1, m_2, \ldots, m_k \), then every \( x_i \) is associated with non-zero and linearly independent vectors \( v_{i,1}, \ldots, v_{i,m_i} \). Furthermore \( \sum_{i=1}^{k} m_i = np \) and the matrix \( col(XJ)^{n-1} \) is non-singular. In other words, these are exactly the requirements which the points and the vectors for the Lagrange interpolation problem, have to fulfil.

**Proof**

From the identity of Christoffel-Darboux and from the fact that \( v_{i,j} \) is a rootvector of \( P_n(x) \) associated with the zero \( x_i \), we get

\[ v_{i,j}^T \sum_{l=0}^{n-1} P_l(x_i)^T P_l(x) = -v_{i,j}^T \frac{P_{n+1}(x_i)^T D_{n+1} P_n(x)}{x - x_i}. \]  \( (1) \)

This implies that the left side of the equation, as well the right side is a vector polynomial of degree \( n - 1 \). In other words the \( m_i \times p \) dimensional matrix

\[ \begin{pmatrix} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{pmatrix} P_{n+1}(x_i)^T D_{n+1} P_n(x) \]
is divisible by \( x - x_i \).

Set

\[
W_i(x) = A \begin{pmatrix} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{pmatrix} \frac{P_{n+1}(x_i)^TD_{n+1}P_n(x)}{x-x_i},
\]

then this matrix polynomial of degree \( n - 1 \) satisfies

\[
W_i(x_i) v_{l,j} = 0, \quad j = 1, 2, \ldots, m_i, \quad l = 1, \ldots, i - 1, i + 1, \ldots k,
\]

because there is no singularity for \( x_i \neq x \) and \( P_n(x_i)v_{l,j} = 0 \). Now we determine the \( p \times m_i \) dimensional matrix \( A \) such that

\[
W_i(x_i) v_{i,j} = v_{i,j}, \quad j = 1, 2, \ldots, m_i.
\]

From (I) we get the following equation:

\[
v_{i,j}^T \sum_{l=0}^{n-1} P_l(x_i)^TP_l(x) v_{i,s} = \left( v_{i,j}^T P_{n+1}(x_i)^TD_{n+1}P_n(x) \right) v_{i,s} - \left( v_{i,j}^T P_{n+1}(x_i)^TD_{n+1}P_n(x) \right) v_{i,s}
\]

Let \( x \) approach \( x_i \), then this becomes

\[
v_{i,j}^T \sum_{l=0}^{n-1} P_l(x_i)^TP_l(x_i) v_{i,s} = -v_{i,j}^TP_{n+1}(x_i)^TD_{n+1}P_n'(x_i)v_{i,s} = -(K_{i,j,s}).
\]

So, the matrix \( A \) satisfies

\[
A \begin{pmatrix} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{pmatrix} P_{n+1}(x_i)^TD_{n+1}P_n'(x_i)v_{i,s} = v_{i,s}, \quad s = 1, 2, \ldots, m_i,
\]

or

\[
A \begin{pmatrix} (K_{i,1,s}) \\ (K_{i,2,s}) \\ \vdots \\ (K_{i,m_i,s}) \end{pmatrix} = v_{i,s}, \quad s = 1, 2, \ldots, m_i.
\]

Consequently, \( A \) satisfies

\[
A \begin{pmatrix} (K_{i,1,1}) & (K_{i,1,2}) & \cdots & (K_{i,1,m_i}) \\ (K_{i,2,1}) & (K_{i,2,2}) & \cdots & (K_{i,2,m_i}) \\ \vdots & \vdots & \ddots & \vdots \\ (K_{i,m_i,1}) & (K_{i,m_i,2}) & \cdots & (K_{i,m_i,m_i}) \end{pmatrix} = \begin{pmatrix} v_{i,1} & v_{i,2} & \cdots & v_{i,m_i} \end{pmatrix}
\]

or

\[
A = \begin{pmatrix} v_{i,1} & v_{i,2} & \cdots & v_{i,m_i} \end{pmatrix} K_i^{-1}.
\]
Bringing everything together, we get
\[ W_i(x) = \left( v_{i,1} \ldots v_{i,m_i} \right) K_i^{-1} \left( \begin{array}{c} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{array} \right) \frac{P_{n+1}(x_i)^T D_{n+1} P_n(x)}{x-x_i}, \]

where
\[ K_i = - \left( \begin{array}{c} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{array} \right) K_{n-1}(x_i, x_i) \left( \begin{array}{c} v_{i,1} \ldots v_{i,m_i} \end{array} \right). \]

Finally we have to show that the matrix \( K_i \) is non-singular. Therefore we show that \(-K_i\) is a symmetric and positive definite matrix. From the fact that \( K_{n-1}(x_i, x_i) \) is a symmetric and positive definite matrix, we get
\[ -K_i^T = \left( \begin{array}{c} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{array} \right) K_{n-1}(x_i, x_i) \left( \begin{array}{c} v_{i,1} \ldots v_{i,m_i} \end{array} \right) = -K_i. \]

Let \( w \) be a \( m_i \)-dimensional vector, then we have
\[
\left( \begin{array}{c} w_1 \\ \vdots \\ w_{m_i} \end{array} \right) (-K_i) \left( \begin{array}{c} w_1 \\ \vdots \\ w_{m_i} \end{array} \right) = (w_1 v_{i,1}^T + \ldots + w_{m_i} v_{i,m_i}^T) K_{n-1}(x_i, x_i) (w_1 v_{i,1} + \ldots + w_{m_i} v_{i,m_i}).
\]

This expression is always \( \geq 0 \) and only equal to 0 if the vector
\[ w_1 v_{i,1} + \ldots + w_{m_i} v_{i,m_i} \]
is equal to 0. But the vectors \( v_{i,1}, \ldots, v_{i,m_i} \) are linearly independent, so that \( w^T (-K_i) w = 0 \) if and only if \( w = 0 \). So \(-K_i\) is a symmetric and positive definite matrix and taking the inverse of it will not be a problem. ■

Special case.
If \( p = 1 \) we consider \( n \) different points \( x_1, \ldots, x_n \) and associate a non zero number \( v_i \) with every point \( x_i \). The interpolation polynomial for a function \( f(x) \) is given by
\[ p(x) = \sum_{i=1}^{n} f(x_i) w_i(x), \]

with
\[ w_i(x) = -v_i \frac{-1}{v_i K_{n-1}(x_i, x_i) v_i} p_{n+1}(x_i) d_{n+1} p_n(x) \]
\[ = \frac{-p_{n+1}(x_i) k_{n,n} p_n(x)}{k_{n+1,n+1} K_{n-1}(x_i, x_i) x-x_i} \]
\[ = \frac{p_n(x)}{(x-x_i) p'_n(x_i)}. \]
Example.
Consider the orthonormal matrix polynomials on the interval \([-1, 1]\) with respect to the weight matrix function

\[
W(x) = \begin{pmatrix}
\frac{1}{\pi}(1-x^2)^{-1/2} & 0 \\
0 & \frac{2}{\pi}(1-x^2)^{1/2}
\end{pmatrix}.
\]

The recursion coefficients are given by:

\[
E_n = 0 \quad \text{for} \quad n \geq 0,
\]

\[
D_1 = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{2}
\end{pmatrix} \quad \text{and} \quad D_n = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix} \quad \text{for} \quad n \geq 1.
\]

In this case the orthonormal matrix polynomials are

\[
P_0(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_1(x) = \begin{pmatrix} \sqrt{2}x & 0 \\ 0 & 2x \end{pmatrix},
\]

\[
P_2(x) = \begin{pmatrix} 2\sqrt{2}x^2 - \sqrt{2} & 0 \\ 0 & 4x^2 - 1 \end{pmatrix}, \quad P_3(x) = \begin{pmatrix} 4\sqrt{2}x^3 - 3\sqrt{2}x & 0 \\ 0 & 8x^3 - 4x \end{pmatrix}.
\]

In general, we have

\[
P_n(x) = \begin{pmatrix} \sqrt{2}T_n(x) & 0 \\ 0 & U_n(x) \end{pmatrix}.
\]

Consider \(P_2(x)\), then the zeros and the corresponding rootvectors are given by

\[
x_1 = \frac{1}{\sqrt{2}}, \quad x_2 = -\frac{1}{\sqrt{2}}, \quad x_3 = \frac{1}{2}, \quad x_4 = -\frac{1}{2},
\]

\[
v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.
\]

This leads to

\[
K_1 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+1 & 0 \\ 0 & 1+2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -2
\]

and

\[
W_1(x) = \begin{pmatrix}
1 & -\frac{1}{2} \\
0 & x - \frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
2\sqrt{2}x^2 - \sqrt{2} & 0 \\
0 & 4x^2 - 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{\sqrt{2}}(x + \frac{1}{\sqrt{2}}) & 0 \\
0 & 0
\end{pmatrix}.
\]

In a similar manner we find

\[
K_2 = -32, \quad W_2(x) = \begin{pmatrix}
\frac{1}{\sqrt{2}}(x - \frac{1}{\sqrt{2}}) & 0 \\
0 & 0
\end{pmatrix},
\]

\[
K_3 = -18, \quad W_3(x) = \begin{pmatrix}
0 & 0 \\
0 & (x + \frac{1}{2})
\end{pmatrix},
\]

\[
K_4 = -4, \quad W_4(x) = \begin{pmatrix}
0 & 0 \\
0 & x + \frac{1}{2}
\end{pmatrix}.
\]
\[ K_4 = -8, \quad W_4(x) = \begin{pmatrix} 0 & 0 \\ 0 & -(x - \frac{1}{2}) \end{pmatrix}. \]

Let \( F(x) \) be given by
\[
F(x) = \begin{pmatrix} 2x + 5 & 6x \\ 7 & 4x - 3 \end{pmatrix},
\]
then the interpolating matrix polynomial is given by
\[
P(x) = \begin{pmatrix} 5 + \sqrt{2} & 3\sqrt{2} \\ 7 & 2\sqrt{2} - 3 \end{pmatrix} W_1(x) + \begin{pmatrix} 5 - \sqrt{2} & -3\sqrt{2} \\ 7 & -2\sqrt{2} - 3 \end{pmatrix} W_2(x) \\
+ \begin{pmatrix} 6 & 3 \\ 7 & -1 \end{pmatrix} W_3(x) + \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix} W_4(x) \\
= \begin{pmatrix} 2x + 5 & 6x \\ 7 & 4x - 3 \end{pmatrix} = F(x).
\]

Let \( F(x) \) be given by
\[
F(x) = \begin{pmatrix} x^2 + 1 & 6x \\ 7x + 1 & 5x^2 - 1 \end{pmatrix},
\]
then the interpolating matrix polynomial is given by
\[
P(x) = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{3\sqrt{2}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{2}} \end{pmatrix} W_1(x) + \begin{pmatrix} \frac{3}{2} & -3\sqrt{2} \\ \frac{7\sqrt{2}}{\sqrt{2}} + 1 & \frac{3}{2} \end{pmatrix} W_2(x) \\
+ \begin{pmatrix} \frac{5}{2} & 3 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} W_3(x) + \begin{pmatrix} \frac{5}{2} & -3 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} W_4(x) \\
= \begin{pmatrix} \frac{3}{2} & 6x \\ 7x + 1 & \frac{1}{4} \end{pmatrix}.
\]

5 Gaussian quadrature

Let \( W(x) \) be a matrix weight function defined on the interval \([a, b]\). Then we are going to approximate the integral of matrix functions by means of a sum of the form
\[
\int_a^b F(x) W(x) G(x)^T dx \simeq \sum_{i=1}^k F(x_i) \Lambda_i G(x_i)^T,
\]
where \( \Lambda_i \in \mathbb{R}^{p \times p} \).

5.1 Quadrature formulas

The numerical integration problem exists in finding points \( x_i \) and matrices \( \Lambda_i \), where \( i = 1, \ldots, k \), so that the formula will have the highest possible degree of precision. This degree of precision of a quadrature formula is equal to \( m (\in \mathbb{N}) \) if
\[
\int_a^b x^l W(x) dx - \sum_{i=1}^k x_i^l \Lambda_i = 0 \quad \text{for} \quad l = 0, 1, \ldots, m, \\
\neq 0 \quad \text{for} \quad l = m + 1.
\]
Note that this agrees with the accuracy of the quadrature formula
\[ \int_a^b F(x) W(x) G(x)^T \, dx \simeq \sum_{i=1}^k F(x_i) \Lambda_i G(x_i)^T, \]
for matrix polynomials which satisfy \( \deg F(x) + \deg G(x) \leq m. \)

It will be convenient to choose
\[ \Lambda_i = \begin{pmatrix} v_{i,1} & \ldots & v_{i,m_i} \end{pmatrix} A_i \begin{pmatrix} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{pmatrix}, \]
with \( v_{i,1}, \ldots, v_{i,m_i} \) linearly independent, non-zero vectors and \( \sum_{i=1}^k m_i = np. \) The numerical integration problem exists than in determining \( x_i \), vectors \( v_{i,j} \) and matrices \( A_i \), where \( j = 1, \ldots, m_i \) and \( i = 1, \ldots, k \), so that the formula will have the highest possible degree of precision. We will show now that \( 2n - 1 \) is the highest degree of precision for a quadrature formula of the postulated form.

**Theorem 5.1** A quadrature formula of the form
\[ \int_a^b F(x) W(x) G(x)^T \, dx \simeq \sum_{i=1}^k F(x_i) \Lambda_i G(x_i)^T \]
cannot be exact for all matrix polynomials \( F(x) \) and \( G(x) \) satisfying \( \deg F(x) + \deg G(x) \leq 2n. \)

**Proof**
Let \( F(x) = G(x) = \hat{Q}_n(x) \), where \( \hat{Q}_n(x) \) is the monic matrix polynomial of degree \( n \) which satisfies \( \hat{Q}_n(x_i)v_{i,j} = 0 \) for \( j = 1, 2, \ldots, m_i \) and \( i = 1, 2, \ldots, k \). In this case we have
\[ \int_a^b F(x) W(x) G(x)^T \, dx = \int_a^b \hat{Q}_n(x) W(x) \hat{Q}_n(x)^T \, dx > 0. \]
On the other hand we have
\[ \sum_{i=1}^k F(x_i) \Lambda_i G(x_i)^T = \sum_{i=1}^k \hat{Q}_n(x_i) \begin{pmatrix} v_{i,1} & \ldots & v_{i,m_i} \end{pmatrix} A_i \begin{pmatrix} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{pmatrix} \hat{Q}_n(x_i)^T = 0, \]
so that the quadrature formula cannot be exact in this case. \( \blacksquare \)

In the following properties we will show that it is possible to construct a quadrature formula with degree of precision \( 2n - 1. \)

**Theorem 5.2** Let \( (X, J) \) be a Jordan pair of the orthonormal matrix polynomial \( P_n(x) \) on the interval \( [a, b] \) and with respect to the weight matrix function \( W(x) \). Then we have
\[ \int_a^b F(x) W(x) G(x)^T \, dx \simeq \sum_{i=1}^k F(x_i) \Lambda_i G(x_i)^T, \quad (2) \]
where \( k \) is the number of different zeros \( x_i \), \( m_i \) is the multiplicity of \( x_i \), \( v_{i,j} \) are the vectors associated with \( x_i \),

\[
\Lambda_i = \left( \begin{array}{c} v_{i,1} \\ \vdots \\ v_{i,m_i} \end{array} \right) L_i^{-1} \left( \begin{array}{c} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{array} \right)
\]

and

\[
L_i = \left( \begin{array}{c} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{array} \right) K_{n-1}(x_i, x_i) \left( \begin{array}{c} v_{i,1} \\ \vdots \\ v_{i,m_i} \end{array} \right).
\]

This quadrature formula is exact for matrix polynomials \( F(x) \) and \( G(x) \) which satisfy

\[
\deg F(x) + \deg G(x) \leq 2n - 1.
\]

**Proof**

If we replace \( F(x) \) and \( G(x) \) by their interpolating matrix polynomial from the Lagrange interpolation problem (see Theorem 4.4), then we get

\[
\int_a^b F(x) W(x) G(x)^T \, dx \simeq \sum_{i=1}^k \sum_{j=1}^k F(x_i) \Lambda_{i,j} G(x_j)^T,
\]

with

\[
\Lambda_{i,j} = \left( \begin{array}{c} v_{i,1} \\ \vdots \\ v_{i,m_i} \end{array} \right) K_i^{-1} \left( \begin{array}{c} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{array} \right)
\]

\[
\times P_{n+1}(x_i)^T D_{n+1} \int_a^b \frac{P_n(x)}{x-x_i} W(x) \frac{P_n(x)^T}{x-x_j} \, dx D_{n+1} P_{n+1}(x_j)
\]

\[
\times \left( \begin{array}{c} v_{j,1} \\ \vdots \\ v_{j,m_j} \end{array} \right) K_j^{-1} \left( \begin{array}{c} v_{j,1}^T \\ \vdots \\ v_{j,m_j}^T \end{array} \right).
\]

From the identity of Christoffel-Darboux we get

\[
\left( \begin{array}{c} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{array} \right) K_{n-1}(x, x_i) = - \left( \begin{array}{c} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{array} \right) \frac{P_{n+1}(x_i)^T D_{n+1} P_n(x)}{x-x_i},
\]

so that

\[
\Lambda_{i,j} = \left( \begin{array}{c} v_{i,1} \\ \vdots \\ v_{i,m_i} \end{array} \right) K_i^{-1} \left( \begin{array}{c} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{array} \right) \int_a^b K_{n-1}(x, x_i) W(x) K_{n-1}(x, x_j)^T \, dx
\]

\[
\times \left( \begin{array}{c} v_{j,1} \\ \vdots \\ v_{j,m_j} \end{array} \right) K_j^{-1} \left( \begin{array}{c} v_{j,1}^T \\ \vdots \\ v_{j,m_j}^T \end{array} \right)
\]

\[
= \left( \begin{array}{c} v_{i,1} \\ \vdots \\ v_{i,m_i} \end{array} \right) K_i^{-1} \left( \begin{array}{c} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{array} \right) K_{n-1}(x_i, x_i) \left( \begin{array}{c} v_{j,1} \\ \vdots \\ v_{j,m_j} \end{array} \right) K_j^{-1} \left( \begin{array}{c} v_{j,1}^T \\ \vdots \\ v_{j,m_j}^T \end{array} \right).
\]
But for \( i \neq j \) we have
\[
v_{i,s}^T K_{n-1}(x_j, x_i) v_{j,t} = v_{i,s}^T K_n(x_j, x_i) v_{j,t} = v_{i,s}^T P_n(x_i)^T D_{n+1} P_{n+1}(x_j) - P_{n+1}(x_i)^T D_{n+1} P_n(x_j) v_{j,t} = 0,
\]
so that \( \Lambda_{i,j} = 0 \) for \( i \neq j \). For \( i = j \) we get
\[
\Lambda_i = \Lambda_{i,i} = - \begin{pmatrix} v_{i,1} & \ldots & v_{i,m_i} \end{pmatrix} K_i^{-1} K_i^{-1} \begin{pmatrix} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{pmatrix} = \begin{pmatrix} v_{i,1} & \ldots & v_{i,m_i} \end{pmatrix} (-K_i)^{-1} \begin{pmatrix} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{pmatrix},
\]
where
\[
-K_i = \begin{pmatrix} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{pmatrix} K_{n-1}(x_i, x_i) \begin{pmatrix} v_{i,1} & \ldots & v_{i,m_i} \end{pmatrix},
\]
which was to be proved.
Now we show that this quadrature formula is exact for matrix polynomials \( F(x) \) and \( G(x) \) satisfying
\[
\deg F(x) + \deg G(x) \leq 2n - 1.
\]
If \( F(x) \) and \( G(x) \) are matrix polynomials with degree \( \leq n-1 \), then the quadrature formula will be exact because the interpolation matrix polynomials are exactly \( F(x) \) and \( G(x) \). Let \( F(x) \) be a matrix polynomial of degree \( n + l \) and \( G(x) \) a matrix polynomial of degree \( n - l - 1 \), where \( l = 0, 1, \ldots, n - 1 \). Then \( G(x) \) satisfies the equation
\[
G(x) = \sum_{i=1}^{k} G(x_i) W_i(x),
\]
because \( \deg G(x) \leq n - 1 \). On the other hand \( F(x) \) is approximated by a matrix polynomial of degree \( n - 1 \):
\[
F(x) \simeq \sum_{i=1}^{k} F(x_i) W_i(x) = P(x).
\]
This implies that \( F(x) - P(x) \) is a matrix polynomial of degree \( n + l \), with zeros \( x_1, x_2, \ldots, x_k \) and eigenvectors \( v_{1,1}, \ldots, v_{1,m_1}, \ldots, v_{k,1}, \ldots, v_{k,m_k} \). But then we have
\[
F(x) - P(x) = R(x) P_n(x),
\]
with \( R(x) \) a matrix polynomial of degree \( l \), (see Section 1). So we get
\[
\int_a^b F(x) W(x) G(x)^T dx = \int_a^b P(x) W(x) G(x)^T dx + \int_a^b R(x) P_n(x) W(x) G(x)^T dx
\]
\[
= \sum_{i=1}^{k} P(x_i) \Lambda_i G(x_i)^T + \int_a^b \left( \sum_{i=0}^{l} R_i x^i \right) P_n(x) W(x) G(x)^T dx
\]

\[
\sum_{i=1}^{k} P(x_i) \Lambda_i G(x_i)^T + \sum_{i=0}^{l} R_i \int_{a}^{b} x^j P_n(x) W(x) G(x)^T \, dx \\
= \sum_{i=1}^{k} P(x_i) \Lambda_i G(x_i)^T + \sum_{i=0}^{l} R_i \int_{a}^{b} P_n(x) x^i G(x)^T \, dx
\]

The last integral will vanish because \( \deg x^i G(x) \leq l + n - l - 1 = n - 1 \) and \( P_n(x) \) is orthogonal to every matrix polynomial of lower degree. Further we have

\[
P(x_i) \Lambda_i = P(x_i) \begin{pmatrix} v_{i,1} & \cdots & v_{i,m_i} \end{pmatrix} L_i^{-1} \begin{pmatrix} v_{i,1}^T \\ \vdots \\ v_{i,m_i}^T \end{pmatrix}
\]

so that

\[
\int_{a}^{b} F(x) W(x) G(x)^T \, dx = \sum_{i=1}^{k} F(x_i) \Lambda_i G(x_i)^T.
\]

The case where \( \deg G(x) \geq n \) is treated in a similar way, but we can also take the transpose of the above mentioned equation.

We showed that using the zeros and the rootvectors of the orthonormal matrix polynomials on the interval \([a, b]\) with respect to the weight matrix function \( W(x) \) leads to a quadrature formula with degree of precision \( 2n - 1 \).

Note that \( L_i \) is a symmetric and positive definite matrix (see proof of Theorem 4.4) and so \( \Lambda_i \) is a symmetric and positive semidefinite matrix.

**Example.**

Consider the orthonormal matrix polynomials on the interval \([-1, 1]\) with respect to the weight matrix function

\[
W(x) = \begin{pmatrix} \frac{1}{\pi} (1-x^2)^{-1/2} & 0 \\ 0 & \frac{2}{\pi} (1-x^2)^{1/2} \end{pmatrix}.
\]

We already mentioned that

\[
P_2(x) = \begin{pmatrix} 2\sqrt{2}x^2 - \sqrt{2} & 0 \\ 0 & 4x^2 - 1 \end{pmatrix},
\]

with zeros

\[
x_1 = \frac{1}{\sqrt{2}}, \quad x_2 = -\frac{1}{\sqrt{2}}, \quad x_3 = \frac{1}{2}, \quad x_4 = -\frac{1}{2}
\]

and rootvectors

\[
v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.
\]
The quadrature coefficients can be computed as follows:

\[
\Lambda_1 = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\left[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1+1 & 0 \\
0 & 1+2
\end{pmatrix}
\begin{pmatrix}
1 & 0
\end{pmatrix}
\right]^{-1}
\begin{pmatrix}
1 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 1/2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1/2 & 0 \\
0 & 0
\end{pmatrix}
\].

In a similar way we find

\[
\Lambda_2 = \begin{pmatrix}
1/2 & 0 \\
0 & 0
\end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix}
0 & 0 \\
0 & 1/2
\end{pmatrix}, \quad \Lambda_4 = \begin{pmatrix}
0 & 0 \\
0 & 1/2
\end{pmatrix}.
\]

Suppose we have the following matrix polynomials

\[
F(x) = \begin{pmatrix}
x^2 + 1 & 6x \\
7x + 1 & 5x^2 - 1
\end{pmatrix} \quad \text{and} \quad G(x) = \begin{pmatrix}
2x + 5 & 6x \\
7 & 4x - 3
\end{pmatrix},
\]

then

\[
\int_a^b F(x) W(x) G(x)^T \, dx = \left( \begin{array}{cc}
\frac{3}{\sqrt{2}} & 3\sqrt{2} \\
7 + \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}}
\end{array} \right) \Lambda_1 \left( \begin{array}{cc}
5 + \sqrt{2} & 7 \\
3\sqrt{2} & 2\sqrt{2} - 3
\end{array} \right) + \left( \begin{array}{cc}
-3\sqrt{2} & 7 \\
\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}}
\end{array} \right) \Lambda_2 \left( \begin{array}{cc}
5 - \sqrt{2} & 7 \\
-3\sqrt{2} & -2\sqrt{2} - 3
\end{array} \right) + \left( \begin{array}{cc}
6 & -1 \\
\frac{5}{4} & \frac{1}{4}
\end{array} \right) \Lambda_3 \left( \begin{array}{cc}
4 & 7 \\
3 & -1
\end{array} \right) + \left( \begin{array}{cc}
\frac{5}{4} & -3 \\
\frac{1}{4} & \frac{1}{4}
\end{array} \right) \Lambda_4 \left( \begin{array}{cc}
4 & 7 \\
-3 & -5
\end{array} \right) = \begin{pmatrix}
33/2 & 33/2 \\
12 & 25/4
\end{pmatrix}.
\]

5.2 Convergence

The Gaussian quadrature formula converges to the exact value of the matrix integral of the functions \(F(x)\) and \(G(x)\), even without imposing severe conditions on those functions.

**Theorem 5.3** Let \(F(x)\) and \(G(x)\) be continuous matrix functions on the finite interval \([a, b]\), then we have

\[
\int_a^b F(x) W(x) G(x)^T \, dx = \lim_{n \to \infty} \sum_{i=1}^{k} F(x_i^{(n)}) \Lambda_i^{(n)} G(x_i^{(n)})^T,
\]

with \(x_i^{(n)}\) the zeros of the orthonormal matrix polynomial \(P_n(x)\) on the interval \([a, b]\) with respect to the weight matrix function \(W(x)\) and with \(\Lambda_i^{(n)}\) the corresponding quadrature coefficients.

In the proof of this theorem we will use the following proposition of matrix norms:
Proposition 5.4 If $A_1, A_2, \ldots, A_k$ are symmetric positive semidefinite $p \times p$-matrices, then they satisfy

$$
\sum_{i=1}^{k} \|A_i\|_2 \leq p \left\| \sum_{i=1}^{k} A_i \right\|_2,
$$

with

$$
\|A\|_2 = \sqrt{\rho(A^*A)} = \rho(A)
$$

and $\rho(A)$ is the spectral radius of $A$.

Proof
Suppose $\alpha_1^{(i)} \geq \ldots \geq \alpha_p^{(i)} \geq 0$ are the eigenvalues of $A_i$ and $\gamma_1 \geq \ldots \geq \gamma_p \geq 0$ are those of $\sum_{i=1}^{k} A_i$. Then we have

$$
\sum_{i=1}^{k} \|A_i\|_2 = \sum_{i=1}^{k} \alpha_1^{(i)} \leq \sum_{i=1}^{k} \sum_{j=1}^{p} \alpha_j^{(i)} = \sum_{i=1}^{k} \text{tr}(A_i) = \text{tr}(\sum_{i=1}^{k} A_i)
$$

$$
= \sum_{j=1}^{p} \gamma_j \leq p \gamma_1 = p \left\| \sum_{i=1}^{k} A_i \right\|_2.
$$

If the interval $[a, b]$ is finite we have another useful inequality, namely

$$
\| \int_{a}^{b} F(x) \, d\delta(x) \| \leq \int_{a}^{b} \| F(x) \| \, d\delta(x),
$$

(see [13, p. 78]).

Proof of Theorem 5.3
We will distinguish 3 cases:

1. $G(x) = I$.

Since the matrix function $F(x)$ is continuous, every $F_{i,j}(x)$ will be continuous. This means that every element can be approximated arbitrarily close by a polynomial and thus

$$
\| F(x) - Q_m(x) \|_2 \leq \varepsilon.
$$

So we have

$$
\| \int_{a}^{b} F(x) \, W(x) \, dx - \sum_{i=1}^{k} F(x_i^{(n)}) \Lambda_i^{(n)} \|_2
$$

$$
\leq \| \int_{a}^{b} F(x) \, W(x) \, dx - \int_{a}^{b} Q_m(x) \, W(x) \, dx \|_2
$$

$$
+ \| \int_{a}^{b} Q_m(x) \, W(x) \, dx - \sum_{i=1}^{k} Q_m(x_i^{(n)}) \Lambda_i^{(n)} \|_2
$$

$$
+ \| \sum_{i=1}^{k} Q_m(x_i^{(n)}) \Lambda_i^{(n)} - \sum_{i=1}^{k} F(x_i^{(n)}) \Lambda_i^{(n)} \|_2.
$$

But the first term of this sum satisfies

$$
\| \int_{a}^{b} F(x) \, W(x) \, dx - \int_{a}^{b} Q_m(x) \, W(x) \, dx \|_2 \leq \int_{a}^{b} \| F(x) - Q_m(x) \|_2 \| W(x) \|_2 \, dx
$$

$$
\leq \varepsilon \int_{a}^{b} \| W(x) \|_2 \, dx.
$$
The second term
\[ \left\| \int_a^b Q_m(x) W(x) \, dx - \sum_{i=1}^k Q_m(x_i^{(n)}) \Lambda_i^{(n)} \right\|_2 = 0, \]
for \( m \leq 2n - 1 \) and the third term can be bounded as follows:
\[ \left\| \sum_{i=1}^k Q_m(x_i^{(n)}) \Lambda_i^{(n)} - \sum_{i=1}^k F(x_i^{(n)}) \Lambda_i^{(n)} \right\|_2 \leq \sum_{i=1}^k \left\| Q_m(x_i^{(n)}) - F(x_i^{(n)}) \right\|_2 \left\| \Lambda_i^{(n)} \right\|_2 \]
\[ \leq \epsilon \sum_{i=1}^k \left\| \Lambda_i^{(n)} \right\|_2. \]

So we have
\[ \left\| \int_a^b F(x) W(x) \, dx - \sum_{i=1}^k F(x_i^{(n)}) \Lambda_i^{(n)} \right\|_2 \leq \epsilon \left( \int_a^b \left\| W(x) \right\|_2 \, dx + \sum_{i=1}^k \left\| \Lambda_i^{(n)} \right\|_2 \right). \]

But from the previous proposition we get
\[ \sum_{i=1}^k \left\| \Lambda_i^{(n)} \right\|_2 \leq p \sum_{i=1}^k \left\| \Lambda_i^{(n)} \right\|_2 = p \left\| \int_a^b W(x) \, dx \right\|_2 \leq p \int_a^b \left\| W(x) \right\|_2 \, dx \]
and thus we have
\[ \left\| \int_a^b F(x) W(x) \, dx - \sum_{i=1}^k F(x_i^{(n)}) \Lambda_i^{(n)} \right\|_2 \leq \epsilon (1 + p) \int_a^b \left\| W(x) \right\|_2 \, dx. \]

But this means that
\[ \int_a^b F(x) W(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^k F(x_i^{(n)}) \Lambda_i^{(n)}. \]

(2) \( G(x) \) is a matrix polynomial.

Suppose \( G(x) = G_1 x^l + G_{1-1} x^{l-1} + \ldots + G_1 x + G_0 \), then we have
\[
\int_a^b F(x) W(x) G(x)^T \, dx = \sum_{j=0}^l \left( \int_a^b F(x) W(x) x^j \, dx \right) G_j^T \\
= \sum_{j=0}^l \left( \int_a^b x^j F(x) W(x) \, dx \right) G_j^T \\
= \sum_{j=0}^l \lim_{n \to \infty} \sum_{i=1}^k (x_i^{(n)})^j F(x_i^{(n)}) \Lambda_i^{(n)} G_j^T \\
= \lim_{n \to \infty} \sum_{i=1}^k F(x_i^{(n)}) \Lambda_i^{(n)} \sum_{j=0}^l (x_i^{(n)})^j G_j^T \\
= \lim_{n \to \infty} \sum_{i=1}^k F(x_i^{(n)}) \Lambda_i^{(n)} G(x_i^{(n)})^T. 
\]

Which implies that the theorem holds in the case that \( G(x) \) is a matrix polynomial.
(3) $G(x)$ is a continuous matrix function on the interval $[a, b]$. In this case, both functions $F(x)$ and $G(x)$ can be approximated arbitrarily closely by a matrix polynomial:

$$\|F(x) - Q_m(x)\|_2 \leq \varepsilon \quad \text{and} \quad \|G(x) - P_l(x)\|_2 \leq \varepsilon.$$ 

This gives

$$\| \int_a^b F(x) W(x) G(x)^T dx - \sum_{i=1}^k F(x_i^{(n)}) \Lambda_i^{(n)} G(x_i^{(n)})^T \|_2$$

$$\leq \| \int_a^b F(x) W(x) G(x)^T dx - \int_a^b Q_m(x) W(x) G(x)^T dx$$

$$- \int_a^b F(x) W(x) P_l(x)^T dx + \int_a^b Q_m(x) W(x) P_l(x)^T dx \|_2$$

$$+ \| \int_a^b Q_m(x) W(x) G(x)^T dx - \sum_{i=1}^k Q_m(x_i^{(n)}) \Lambda_i^{(n)} G(x_i^{(n)})^T \|_2$$

$$+ \| \int_a^b F(x) W(x) P_l(x)^T dx - \sum_{i=1}^k F(x_i^{(n)}) \Lambda_i^{(n)} P_l(x_i^{(n)})^T \|_2$$

$$+ \| \sum_{i=1}^k Q_m(x_i^{(n)}) \Lambda_i^{(n)} G(x_i^{(n)})^T + \sum_{i=1}^k F(x_i^{(n)}) \Lambda_i^{(n)} P_l(x_i^{(n)})^T$$

$$- \sum_{i=1}^k Q_m(x_i^{(n)}) \Lambda_i^{(n)} P_l(x_i^{(n)})^T - \sum_{i=1}^k F(x_i^{(n)}) \Lambda_i^{(n)} G(x_i^{(n)})^T \|_2.$$ 

From the previous case we get that the second and the third term are arbitrarily small ($< \varepsilon$) for $n$ sufficiently large. The fourth term is equal to 0 if $m + l \leq 2n - 1$ and the other two terms can be bounded as follows,

$$\| \int_a^b [F(x) - Q_m(x)] W(x) [G(x) - P_l(x)]^T dx \|_2$$

$$\leq \int_a^b \| F(x) - Q_m(x) \|_2 \| W(x) \|_2 \| G(x) - P_l(x) \|_2 dx$$

$$\leq \varepsilon^2 \int_a^b \| W(x) \|_2 dx$$

and

$$\| \sum_{i=1}^k [F(x_i^{(n)}) - Q_m(x_i^{(n)})] \Lambda_i^{(n)} [G(x_i^{(n)}) - P_l(x_i^{(n)})]^T \|_2$$

$$\leq \sum_{i=1}^k \| F(x_i^{(n)}) - Q_m(x_i^{(n)}) \|_2 \| \Lambda_i^{(n)} \|_2 \| G(x_i^{(n)}) - P_l(x_i^{(n)}) \|_2$$

$$\leq \varepsilon^2 \sum_{i=1}^k \| \Lambda_i^{(n)} \|_2.$$
Hence for \( n \) large enough

\[
\| \int_a^b F(x) W(x) G(x)^T dx - \sum_{i=1}^k F(x_{i}^{(n)}) \Lambda_{i}^{(n)} G(x_{i}^{(n)})^T \|_2 \\
\leq 2\varepsilon + \varepsilon^2 \left( \int_a^b \|W(x)\|_2 dx + \sum_{i=1}^k \|\Lambda_{i}^{(n)}\|_2 \right) \\
\leq 2\varepsilon + \varepsilon^2 (1 + p) \int_a^b \|W(x)\|_2 dx,
\]

which proves the convergence of the Gaussian quadrature formula. 

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