A Note on Rough Parametric Marcinkiewicz Functions

Laith Hawawsheh¹, Ahmad Al-Salman²,∗ and Shaher Momani³,4

¹ School of Basic Sciences and Humanities, German Jordanian University, Amman, Jordan
² Department of Mathematics, Sultan Qaboos University, P.O. Box 36, Al-Khod 123 Muscat, Sultanate of Oman
³ Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan
⁴ Department of Mathematics and Sciences, College of Humanities and Sciences, Ajman University, Ajman, UAE

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Abstract. In this note, we obtain sharp $L^p$ estimates of parametric Marcinkiewicz integral operators. Our result resolves a long standing open problem. Also, we present a class of parametric Marcinkiewicz integral operators that are bounded provided that their kernels belong to the sole space $L^1(S^{n-1})$.

Key Words: Marcinkiewicz integrals, parametric Marcinkiewicz functions, rough kernels, Fourier transform, Marcinkiewicz interpolation theorem.

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1 Introduction

Let $n \geq 2$ and $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. Suppose that $\Omega$ is a homogeneous function of degree zero on $\mathbb{R}^n$ that satisfies $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0.$$  \hspace{1cm} (1.1)

In 1960, Hörmander (see [6]) introduced the following parametric Marcinkiewicz function $\mu^\rho_\Omega$ of higher dimension by

$$\mu^\rho_\Omega f(x) = \left( \int_{-\infty}^{\infty} \left( \int_{|y| \leq 2^t} f(x-y)|y|^{-\rho+\rho} \Omega(y)dy \right)^2 dt \right)^{\frac{1}{2}},$$  \hspace{1cm} (1.2)

∗Corresponding author. Email addresses: Laith.hawawsheh@gju.edu.jo (L. Hawawsheh), alsalman@squ.edu.om (A. Al-Salman), s.momani@ju.edu.jo (S. Momani)

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where $\rho > 0$. When $\rho = 1$, the corresponding operator $\mu_{\Omega} = \mu_{\Omega}^1$ is the classical Marcinkiewicz integral operator introduced by Stein (see [7]). When $\Omega \in Lip_\alpha(S^{n-1})$, $(0 < \alpha \leq 1)$, Stein proved that $\mu_{\Omega}$ is bounded on $L^p$ for all $1 < p \leq 2$. Subsequently, Benedek-Calderón-Panzone proved the $L^p$ boundedness of $\mu_{\Omega}$ for all $1 < p < \infty$ under the condition $\Omega \in C^1(S^{n-1})$ (see [4]). Since then, the $L^p$ boundedness of $\mu_{\Omega}$ has been investigated by several authors. For background information, we advise readers to consult [1–3, 7], among others.

Concerning the problem whether there are some $L^p$ results on $\mu_{\Omega}^{\rho}$ similar to those on $\mu_{\Omega}$ when $\Omega$ satisfies only some size conditions, Ding, Lu, and Yabuta (see [5]) studied the general operator

$$h^{\mu}_{\Omega,h}f(x) = \left( \int_{-\infty}^{\infty} \left| 2^{-n} \int_{|y| \leq 2^t} f(x-y) \frac{|y|^{-n+p}}{|y|} h(|y|) \Omega(y) \, dy \right|^q \, dt \right)^{\frac{1}{q}}, \quad (1.3)$$

where $h$ is a radial function on $\mathbb{R}^n$ satisfying $h(|x|) \in L^q(L^2)(\mathbb{R}^+)$, $1 \leq q \leq \infty$, where the class $L^q(L^2)(\mathbb{R}^+)$ is defined by

$$L^q(L^2)(\mathbb{R}^+) = \left\{ h : |h|_{L^q(L^2)(\mathbb{R}^+)} = \sup_{j \in \mathbb{Z}} \left( \int_{2^j}^{2^{j+1}} |h(r)|^q \frac{dr}{r} \right)^{\frac{1}{q}} < \infty \right\}.$$ 

For $q = \infty$, we set $L^\infty(L^\infty)(\mathbb{R}^+) = L^\infty(\mathbb{R}^+)$. It is clear that

$$L^\infty(L^\infty)(\mathbb{R}^+) \subset L^\infty(L^q)(\mathbb{R}^+) \subset L^\infty(L^2)(\mathbb{R}^+) \subset L^\infty(L^1)(\mathbb{R}^+),$$

$1 < q < r < \infty$. Ding, Lu, and Yabuta (see [5]) proved the following result:

**Theorem 1.1** ([5]). Suppose that $\Omega \in L(\log^+ L)(S^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying (1.1) and $h(|x|) \in L^q(L^2)(\mathbb{R}^+)$ for some $1 < q \leq \infty$. If $Re(\rho) = \alpha > 0$, then $|h^{\mu}_{\Omega,h}f|_2 \leq C \alpha^{-1} |f|_2$, where $C$ is independent of $\rho$ and $f$.

In [1], Al-Salman and Al-Qassem considered the $L^p$ boundedness of $\mu_{\Omega,h}^{\rho}$ for $p \neq 2$. which was left open in [5]. They proved the following result:

**Theorem 1.2** ([1]). Suppose that $\Omega \in L(\log^+ L)(S^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying (1.1). If $h(|x|) \in L^q(L^2)(\mathbb{R}^+)$, $1 < q \leq \infty$, and $\alpha = Re(\rho) > 0$, then $\left| \mu_{\Omega,h}^{\rho}f \right|_p \leq C \alpha^{-1} |f|_p$, for all $1 < p < \infty$, where $C$ is independent of $\rho$ and $f$.

In light of Theorem 1.1, it is clear that the dependence of the $L^p$ bounds on $\alpha$ in Theorem 1.2 is not sharp. More precisely, we have the following long standing natural open problem:

**Problem:**

(a) Is the power $(-1/2)$ of $\alpha$ in Theorem 1.1 sharp?
(b) Does the result in Theorem 1.2 hold with power of \( \alpha \) greater than \((-1)\)?

It is our aim in this note to consider this problem. In fact, we shall prove the following result which completely resolves the above problem:

**Theorem 1.3.** Suppose that \( \Omega \in L(\log^+ L)(S^{n-1}) \) is a homogeneous function of degree zero on \( \mathbb{R}^n \) satisfying (1.1). If \( h(|x|) \in l^\infty (L^q)(\mathbb{R}^+) \), \( 1 < q \leq \infty \), and \( \alpha = \text{Re}(\rho) > 0 \), then

\[
\left| \mu_{\Omega,h}^\rho f \right|_p \leq C_{\alpha} \alpha^{-\frac{1}{p}} |f|_p \quad \text{for all } 1 < p < \infty,
\]

where \( C \) is independent of \( \rho \) and \( f \). Moreover, the power \((-1/p)\) is sharp in the sense that it cannot be replaced by larger power.

It is clear that Theorem 1.3 substantially improves Theorem 1.2 as far as the power of \( \alpha \) is concerned. Concerning the function \( \Omega \), we present in Section 3 of this note a subclass of the class \( l^\infty (L^q)(\mathbb{R}^+) \) where the corresponding operator \( \mu_{\Omega,h}^\rho \) is bounded on \( L^2 \) under the sole integrability condition \( \Omega \in L^1(S^{n-1}) \).

Throughout the rest of the paper the letter \( C \) will stand for a constant but not necessarily the same one in each occurrence.

## 2 Proof of main result

This section is devoted to present a proof of Theorem 1.3. We start by recalling the following well known interpolation theorem:

**Theorem 2.1 ([8]).** Let \( T \) be a sublinear operator satisfying

\[
|T(f)|_{L^{p_1}(\mathbb{R}^n)} \leq C_{p_1} |f|_{L^{p_1}(\mathbb{R}^n)}
\]

and

\[
|T(f)|_{L^{p_2}(\mathbb{R}^n)} \leq C_{p_2} |f|_{L^{p_2}(\mathbb{R}^n)}
\]

for some \( 1 \leq p_1, p_2 \leq \infty \) and \( C_{p_1}, C_{p_2} > 0 \). Then for all \( \theta \in [0, 1] \), we have

\[
|T(f)|_{L^{p_\theta}(\mathbb{R}^n)} \leq C_{p_\theta} |f|_{L^{p_\theta}(\mathbb{R}^n)},
\]

where \( p_\theta \) satisfies \( \frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \) and \( C_{p_\theta} = C_{p_1} C_{p_2}^{1-\theta} \).

**Proof of Theorem 1.3.** The proof is based on an interpolation argument. By Theorem 1.1 and Theorem 1.2, we have

\[
\left| \mu_{\Omega,h}^\rho f \right|_2 \leq \frac{C}{\sqrt{\alpha}} |f|_2
\]

(2.1)
and
\[
\left| \mu_{\Omega,h}^\epsilon f \right|_{1+\epsilon} \leq \frac{C}{\epsilon} |f|_{1+\epsilon},
\]
(2.2)
for any \( \epsilon > 0 \). Thus, (2.1) and (2.2) show that the operator \( \mu_{\Omega,h}^\epsilon \) is a bounded operator from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \) and from \( L^{1+\epsilon}(\mathbb{R}^n) \) to \( L^{1+\epsilon}(\mathbb{R}^n) \), respectively. Thus, by Theorem 2.1, (2.1) and (2.2), we have
\[
\left| \mu_{\Omega,h}^\epsilon f \right|_p \leq C_{\alpha}^\epsilon \left( \frac{1}{p^\epsilon} \right) |f|_p
\]
(2.3)
for all \( 1 + \epsilon < p < 2 \). Letting \( \epsilon \to 0^+ \), we would get
\[
\left| \mu_{\Omega,h}^\epsilon f \right|_p \leq \frac{C_{\alpha}}{\alpha^p} |f|_p
\]
for \( 1 < p < 2 \). Similarly, for \( M > 2 \), we have by Theorem 1.2 that
\[
\left| \mu_{\Omega,h}^\epsilon f \right|_M \leq \frac{C_{\alpha}}{\alpha^M} |f|_M.
\]
(2.4)
Interpolating between (2.1) and (2.4) yields
\[
\left| \mu_{\Omega,h}^\epsilon f \right|_p \leq C_{\alpha}^\epsilon \left( \frac{1}{p^\epsilon} \right) |f|_p
\]
(2.5)
for all \( 2 < p < M \). Letting \( M \to \infty \) gives
\[
\left| \mu_{\Omega,h}^\epsilon f \right|_p \leq \frac{C_{\alpha}}{\alpha^p} |f|_p
\]
for \( 2 < p < \infty \).

Now, we show that the power \( (1/p) \) is sharp. We shall work out the case \( p = 2 \) and \( \rho = \alpha \) is a positive real number. We shall also assume \( 0 < \alpha < 1 \). Set
\[
\Omega(x) = (x_1)' = \frac{x_1}{|x|}.
\]
Then \( \Omega \) satisfies (1.1) and \( \Omega \in L^2(S^{n-1}) \). On the other hand, let \( f(x) = x_1 \) if \( |x| < 1 \) and \( f(x) = 0 \) if \( |x| \geq 1 \). Then \( f \in L^2(\mathbb{R}^n) \). In fact,
\[
|f|_2 = \frac{1}{\sqrt{n+2}} |\Omega|_2.
\]
Now,
\[
\left| \mu_{\Omega,h}^\epsilon f \right|_2^2 \geq \int_{|x|<1} \int_3^\infty \int_{S^{n-1}} \int_0^t \Omega(y') f(x - ry') \frac{dr}{t^{1-a}} d\sigma(y') \left| t^\frac{\epsilon}{12} dx \right|^2 dt \int_{t^{1+2\alpha}} dx.
\]
By noticing that $f(x - ry') = 0$ whenever $|x| < 1$ and $r > 2$, it follows from the last integral that

\[
\begin{align*}
\mu_{\Omega,h}^\rho f_2^2 &\geq \int_{|x|<1} \int_0^\infty \left| \int_{S^{n-1}} \Omega(y')(x_1 - ry_1') \frac{dr}{r^{1-\alpha}} d\sigma(y') \right|^2 \frac{dt}{t^{1+2\alpha}} dx \\
&= \int_{|x|<1} \int_0^\infty \left( \Omega(y') \right)^2 r^\alpha d\sigma(y') \left| \int_0^2 \frac{dr}{r^{1-\alpha}} \right|^2 \frac{dt}{t^{1+2\alpha}} dx \\
&= |\Omega|^2 \left( \frac{2^\alpha + 1}{1 + \alpha} \right) \left( \frac{1}{32\alpha} \right) \frac{1}{2\alpha} |B(0,1)| \\
&\geq C \sqrt{\alpha} |f_2|, \quad (2.6)
\end{align*}
\]

where $|B(0,1)|$ is the volume of the ball $B(0,1) = \{ x \in \mathbb{R}^n : |x| < 1 \}$ and $C$ is a constant independent of $\alpha$. Here, (2.6) follows by (1.1). This completes the proof. $\square$

3 Further study

As pointed out in the introduction section, in this section we present a subclass of the class $L^\infty(L^q)(\mathbb{R}^+) \ni \mu_{\Omega,h}^\rho$ is bounded on $L^2$ under the condition $\Omega \in L^1(\mathbb{S}^{n-1})$. If $q = \infty$, $L^\infty(L^\infty)(\mathbb{R}^+) = L^\infty(\mathbb{R}^+)$. For $1 \leq q < \infty$, let $\mathcal{D}_q$ be the space of all measurable radial functions $h$ on $\mathbb{R}^n$ which satisfy

\[
\frac{h(r)}{r^{1/q}} \in L^\infty(L^q)(\mathbb{R}^+), \quad (3.1a)
\]

\[
\sum_{j=1}^\infty \left( \int_{2^j}^{2^{j+1}} |h(r)|^q \frac{dr}{r} \right)^{1/q} < \infty. \quad (3.1b)
\]

It is obvious that $\mathcal{D}_q \subset L^\infty(L^q)(\mathbb{R}^+)$ and this inclusion is proper for $1 \leq q < \infty$. In fact, for $j \in \mathbb{Z}^-$, we have

\[
\left( \int_{2^j}^{2^{j+1}} |h(r)|^q \frac{dr}{r} \right)^{1/q} = \left( \int_{2^j}^{2^{j+1}} \frac{h(r)}{r^{1/q}} \frac{dr}{r} \right)^{1/q} \leq C \left| h/r^{1/q} \right|_{L^\infty(L^q)(\mathbb{R}^+)}.
\]

On the other hand, for $j \in \mathbb{Z}^+$, by (3.1b) we have

\[
\left( \int_{2^j}^{2^{j+1}} |h(r)|^q \frac{dr}{r} \right)^{1/q} \leq \sum_{j=1}^\infty \left( \int_{2^j}^{2^{j+1}} |h(r)|^q \frac{dr}{r} \right)^{1/q} < \infty.
\]

Notice further that the constant functions are contained in $L^\infty(L^q)(\mathbb{R}^+)$ but not in $\mathcal{D}_q$.

On the other hand,

\[
\mathcal{D}_q \not\subset L^\infty(\mathbb{R}^+). \quad (3.2)
\]
To see (3.2), we construct a function \( h \in \mathcal{D}_q \setminus L^\infty(\mathbb{R}^+) \). For convenience, we consider the case \( q = 2 \). Define \( h \) on \( \mathbb{R}^+ \) by \( h(r) = \sqrt[n]{r} \), if \( r \in [1 + \frac{1}{n+1}, 1 + \frac{1}{n}] \), \( n \in \mathbb{N} \) and \( h(r) = 0 \) otherwise. It is clear that \( h \) is not bounded. To see that \( h \in \mathcal{D}_q \), we first observe that since \( h(r) = 0 \) for all \( r \geq 2 \), it follows that \( h \) satisfies (3.1b). To see that \( h \) satisfies (3.1a), notice

\[
\left( \int_1^\infty \frac{h(r)}{r^{1/2}} \frac{dr}{r} \right)^{1/2} = \left( \sum_{n=1}^\infty \int_{1+\frac{1}{n}}^{1+\frac{1}{n+1}} \frac{h(r)}{r^{1/2}} \frac{dr}{r} \right)^{1/2} = \left( \sum_{n=1}^\infty \frac{\sqrt[n]{r}}{n(n+1)} \right)^{1/2} < \infty.
\]

Now, we have the following result:

**Theorem 3.1.** If \( h \in \mathcal{D}_q \) for some \( 1 \leq q < \infty \) and \( \Omega \in L^1(S^{n-1}) \) is a homogeneous function of degree zero on \( \mathbb{R}^n \) satisfying (1.1), then \( \mu_{\Omega,h} \) is bounded on \( L^2(\mathbb{R}^n) \).

**Proof.** By simple change of variables and Plancherel’s theorem, we have

\[
\left\| \mu_{\Omega,h} \right\|_{L^2}^2 \leq \int_{\mathbb{R}^n} \hat{f}(\xi) \left( \int_0^\infty t^{-\rho} \int_{|y| \leq t} e^{-2\pi i y \cdot \xi} |y|^{-n+\rho} h(|y|) \Omega(y)dy \right)^2 \frac{dt}{t} d\xi. \tag{3.3}
\]

On the other hand, by Minkowski’s integral inequality, we have

\[
\left( \int_0^\infty t^{-\rho} \int_{|y| \leq t} e^{-2\pi i y \cdot \xi} |y|^{-n+\rho} h(|y|) \Omega(y)dy \right)^2 \frac{dt}{t} \leq \int_0^\infty \left( \int_{S^{n-1}} e^{-2\pi i \rho y \cdot \xi} h(r) \Omega(y') \chi_{[0,t]}(r) r^{n-1} d\sigma(y') \right)^2 \frac{dr}{r^{1+2\rho}} \int_0^\infty \frac{dt}{t^{1+2\rho}} \int_{S^{n-1}} e^{-2\pi i \rho y \cdot \xi} h(r) \Omega(y') \chi_{[0,t]}(r) r^{n-1} d\sigma(y') \frac{dr}{r^{1+2\rho}} \frac{dt}{t^{1+2\rho}} \]

\[
= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_{S^{n-1}} e^{-2\pi i \rho y \cdot \xi} h(r) \Omega(y') d\sigma(y') \frac{dr}{r^{1+a}} \int_0^\infty \frac{dt}{t^{1+2\rho}} \]

\[
= \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_{S^{n-1}} e^{-2\pi i \rho y \cdot \xi} h(r) \Omega(y') d\sigma(y') \frac{dr}{r^{1+a}} \int_0^\infty \frac{dt}{t^{1+2\rho}}. \tag{3.4}
\]

In view of (3.4), we need only to show that

\[
\sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \int_0^\infty \int_{S^{n-1}} e^{-2\pi i \rho y \cdot \xi} h(r) \Omega(y') d\sigma(y') \frac{dr}{r} < \infty. \tag{3.5}
\]

We consider two cases:
Case 1. If $|\xi| > 2$, then

$$
\int_0^\infty \left| \int_{S^{n-1}} e^{-2\pi i \xi \cdot \Omega} h(r) \Omega(y') d\sigma(y') \right| \frac{dr}{r} 
$$

$$
= \int_0^{2/|\xi|} \left| \int_{S^{n-1}} e^{-2\pi i \xi \cdot \Omega} h(r) \Omega(y') d\sigma(y') \right| \frac{dr}{r} 
$$

$$
+ \int_{2/|\xi|}^1 \left| \int_{S^{n-1}} e^{-2\pi i \xi \cdot \Omega} h(r) \Omega(y') d\sigma(y') \right| \frac{dr}{r} 
$$

$$
+ \int_1^\infty \left| \int_{S^{n-1}} e^{-2\pi i \xi \cdot \Omega} h(r) \Omega(y') d\sigma(y') \right| \frac{dr}{r} 
$$

$$
=: I + II + III. 
$$

(3.6)

By the cancellation property (1.1), we get

$$
I = \int_0^{2/|\xi|} \left| \int_{S^{n-1}} e^{-2\pi i \xi \cdot \Omega} h(r) \Omega(y') d\sigma(y') \right| \frac{dr}{r} 
$$

$$
= \int_0^{2/|\xi|} \left| \int_{S^{n-1}} (e^{-2\pi i \xi \cdot \Omega} - 1) h(r) \Omega(y') d\sigma(y') \right| \frac{dr}{r} 
$$

$$
= \sum_{l=0}^{\infty} \int_{2^{-1}/|\xi|}^{2/|\xi|} \left| \int_{S^{n-1}} (e^{-2\pi i \xi \cdot \Omega} - 1) h(r) \Omega(y') d\sigma(y') \right| \frac{dr}{r} 
$$

$$
\leq C \| \Omega \|_{L^1(S^{n-1})} \| h \|_{L^\infty(L^q)(\mathbb{R}^n)^n}, 
$$

(3.7)

where the last inequality was obtained using (3.1b). Next, choose $j_0 \in \mathbb{Z}$ such that $2^{j_0} \leq 2/|\xi|$. Then

$$
II = \int_{2^{-1}/|\xi|}^{1} \left| \int_{S^{n-1}} e^{-2\pi i \xi \cdot \Omega} h(r) \Omega(y') d\sigma(y') \right| \frac{dr}{r} 
$$

$$
\leq \int_{2^{j_0}}^1 \left| \int_{S^{n-1}} e^{-2\pi i \xi \cdot \Omega} h(r) \Omega(y') d\sigma(y') \right| \frac{dr}{r} 
$$

$$
\leq \| \Omega \|_{L^1(S^{n-1})} \| h / r^{1/q} \|_{L^\infty(L^q)(\mathbb{R}^n)^n} \sum_{j=j_0+1}^{\infty} (2^{j-1})^{1/q} 
$$

$$
\leq C \| \Omega \|_{L^1(S^{n-1})} \| h / r^{1/q} \|_{L^\infty(L^q)(\mathbb{R}^n)^n}, 
$$

(3.8)

where $C$ does not depend on the choice of $j_0$. Finally, notice that

$$
III = \int_1^\infty \left| \int_{S^{n-1}} e^{-2\pi i \xi \cdot \Omega} h(r) \Omega(y') d\sigma(y') \right| \frac{dr}{r} 
$$

$$
\leq C \| \Omega \|_{L^1(S^{n-1})}, 
$$

(3.9)
where the last inequality was obtained using (3.1b). This proves (3.5) for all $\xi \in \mathbb{R}^n$ with $|\xi| > 2$.

Case 2. If $|\xi| \leq 2$, then

$$
\int_0^\infty \left| \int_{S^{n-1}} e^{-2\pi i r y' \cdot \xi} h(r) \Omega(y') \, d\sigma(y') \right| \frac{dr}{r} = \int_0^2 \left| \int_{S^{n-1}} e^{-2\pi i r y' \cdot \xi} h(r) \Omega(y') \, d\sigma(y') \right| \frac{dr}{r} + \int_2^\infty \left| \int_{S^{n-1}} e^{-2\pi i r y' \cdot \xi} h(r) \Omega(y') \, d\sigma(y') \right| \frac{dr}{r}. \quad (3.10)
$$

To estimate (3.10), we follow similar argument as in Case 1. This completes the proof.

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