Some results on circulant and skew circulant type matrices with k-Fibonacci sequences

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Abstract. In this paper, we consider the circulant matrices, skew circulant matrices and skew left circulant matrices with k-Fibonacci sequences. We compute their eigenvalues by diagonalizing these matrices. Furthermore, the maximum column sum matrix norms, the maximum row sum matrix norms, the Frobenius norms, the spectral norms, and the bounds for the spread of these matrices are given with the properties of k-Fibonacci sequences.

1. Introduction

Circulant matrices and their generalization have a wide range of applications in signal coding theory, image processing, digital image disposal, self-regress design and so on [1][2][3][4][5]. Specifically, the computing of the eigenvalues, norms and bounds of spread of the nonsingular circulant matrices plays a basic role in the above applications. Though various computation algorithms have been reported [6], the computation complexity of these algorithms becomes rather high with the increasing of the order of matrices.

Jaiswal evaluated some determinants of circulant matrices whose elements are the generalized Fibonacci numbers [7]. Subsequently, Lind computed the determinants of circulant and skew-circulant matrices involving Fibonacci numbers [8]. These works aroused promptly wide public concern, and people began researching into circulant type matrices with special sequences [9][10][11]. Recently, Falcón proposed k-Fibonacci sequence and obtained Binet's formulas [12][13]. For any positive real number \( k \) and \( n \geq 1 \), the k-Fibonacci sequences \( \{ F_{k,n} \}_{n \in \mathbb{N}} \) are defined recurrently by

\[
F_{k,n+1} = k F_{k,n} + F_{k,n-1},
\]

where \( F_{k,0} = 0 \) and \( F_{k,1} = 1 \). In particular, \( \{ F_{1,n} \}_{n \in \mathbb{N}} \) is the classical Fibonacci sequence, \( \{ F_{2,n} \}_{n \in \mathbb{N}} \) is the classical Pell sequence, and \( \{ F_{3,n} \}_{n \in \mathbb{N}} \) is the product of Fibonacci and Lucas numbers. Thus, it is very interesting to study the explicit formulae for the eigenvalues of the circulant and skew circulant type matrices with k-Fibonacci sequences, which include many existing sequences as special cases.

The main aim of this paper is to establish some useful formulae for the eigenvalues of circulant, skew circulant and skew left circulant matrices whose elements are the k-Fibonacci sequences. Using the properties of k-Fibonacci sequences [14][15], the maximum column sum matrix norms, the maximum row sum matrix norms, the Frobenius norms, the spectral norms and bounds for the spread of these matrices will also be given. In particular, we will make the following contributions:

- We obtain an explicit formula for spectral norms of circulant matrices with k-Fibonacci sequence. We mention a related work in [16], where the upper and lower bounds for the spectral norms of circulant matrices with \((k,h)\)-Fibonacci numbers are discussed.
2. Preliminaries

Lemma 2.1. (Binet’s formula [12]) The $n$th $k$-Fibonacci number is given by

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2},$$ (1)

where $r_1, r_2$ are the roots of the characteristic equation $r^2 - kr - 1 = 0$, and $r_1 > r_2$.

Lemma 2.2. (see [13]) Let $S_{k,n}$ be the sum of the first $n+1$ terms of the $k$-Fibonacci sequence, that is

$$S_{k,n} = \sum_{j=0}^{n} F_{k,j}.$$ (ii)

(i) $S_{k,n} = \frac{1}{k}(F_{k,n+1} + F_{k,n}) - \frac{1}{k}$; (ii) $\sum_{j=1}^{n} F_{k,2j} = \frac{1}{k}(F_{k,2n+1} - 1)$; (iii) $\sum_{j=0}^{n} F_{k,2j+1} = \frac{1}{k} F_{k,2n+2}$. (2)

Lemma 2.3. (see [15]) Let $(F_{k,n})$ be $k$-Fibonacci sequence, then

$$\sum_{j=0}^{n} F_{k,j}^2 = \frac{1}{k} F_{k,n} F_{k,n+1}$$ (3)

Lemma 2.4. Let $(F_{k,n})$ be $k$-Fibonacci sequence, then

$$\sum_{j=0}^{n} j F_{k,j} = \frac{2}{k^2} + \frac{kn + k}{k^2} F_{k,n} + \frac{kn - 2}{k^2} F_{k,n+1}.$$ (4)

Proof.

$$\sum_{j=0}^{n} j F_{k,j} = \sum_{j=0}^{n} j \cdot \frac{r_1^j - r_2^j}{r_1 - r_2} = \frac{1}{k} \left[ \frac{r_1^2 (1 - r_1^{n-1}) + 1 - nr_1^{n+1}}{1 - r_1} - \frac{r_2^2 (1 - r_2^{n-1}) + 1 - nr_2^{n+1}}{1 - r_2} \right]$$

$$= \frac{1}{k^2} \left[ 2 - F_{k,n+1} + (kn - 2) F_{k,n} + (kn - 1) F_{k,n+1} \right] = \frac{2}{k^2} + \frac{kn + k}{k^2} F_{k,n} + \frac{kn - 2}{k^2} F_{k,n+1}.$$ (5)

3. Circulant matrix with the $k$-Fibonacci sequence

The $n \times n$ circulant matrix $C_n := \text{Circ}(c_1, \ldots, c_n)$ is defined as

$$C_n := \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & \cdots & c_1 \end{pmatrix}.$$ (6)

Let $F_{k,n} := \text{Circ}(F_{k,1}, F_{k,2}, \ldots, F_{k,n})$ be the circulant matrices associated with the $k$-Fibonacci sequences. In this section, we obtain an expression for eigenvalues of $F_{k,n}$ and compute four kinds of norms and bounds for the spread of $F_{k,n}$.

Theorem 3.1. Let $F_{k,n} := \text{Circ}(F_{k,1}, F_{k,2}, \ldots, F_{k,n})$ be a circulant matrix, then the eigenvalues of $F_{k,n}$ are

$$1 - F_{k,n+1} - F_{k,n} e^{\frac{2\pi (j-1)}{n}} e^{\frac{2\pi (j-1)}{4n}}, (j = 1, \ldots, n).$$ (7)
Proof. By Theorem 7 in [17], let \( \chi(x) = \sum_{j=1}^{n} F_{k,j} x^{j-1} \) and \( \theta_j = e^{i \frac{2\pi(j-1)}{n}} (j=1,\cdots,n) \), we know the eigenvalues of \( F_{k,n} \) are
\[
\chi(\theta_j) = \sum_{i=1}^{n} F_{k,i} e^{i \frac{2\pi(j-1)}{n}} = \frac{1}{r_1 - r_2} \sum_{i=1}^{n} (r_1^i - r_2^i) e^{i \frac{2\pi(j-1)}{n}} = \frac{1}{r_1 - r_2} \left[ r_1 (1 - r_1^n) - r_2 (1 - r_2^n) \right] = \frac{1}{1 - r_2 e^{i \frac{2\pi(j-1)}{n}}} \left[ r_1 - r_2 - (r_1^{n+1} - r_2^{n+1}) - (r_1^n - r_2^n) e^{i \frac{2\pi(j-1)}{n}} \right] = \frac{1}{1 - r_1 e^{i \frac{2\pi(j-1)}{n}}} \frac{2\pi(j-1)}{n} \right] = \frac{1}{1 - (r_1 + r_2) e^{i \frac{2\pi(j-1)}{n}}} + \frac{r_1 r_2 e^{i \frac{2\pi(j-1)}{n}}}{1 - (r_1 + r_2) e^{i \frac{2\pi(j-1)}{n}}} \right) (8)
\]

Theorem 3.2. Let \( F_{k,n} := \text{Circ}(F_{k,1}, F_{k,2}, \cdots, F_{k,n}) \) be a circulant matrix, then the maximum column sum matrix norm, the maximum row sum matrix norm, the Frobenius norm, and the spectral norm of \( F_{k,n} \) are respectively given as follows
\[
\| F_{k,n} \|_1 = \frac{1}{k} (F_{k,n+1} + F_{k,n}) - \frac{1}{k},
\]
\[
\| F_{k,n} \|_\infty = \frac{n}{k} F_{k,n} F_{k,n+1},
\]
\[
\| F_{k,n} \|_2 = \frac{1}{k} (F_{k,n+1} + F_{k,n}) - \frac{1}{k}.
\]
Proof. By (2), we have
\[
\| F_{k,n} \|_1 = \frac{1}{k} (F_{k,n+1} + F_{k,n}) - \frac{1}{k}.
\]
Furthermore, by (3) we know
\[
(\| F_{k,n} \|_\infty)^2 = n \sum_{j=1}^{n} F_{k,j}^2 = \frac{n}{k} F_{k,n} F_{k,n+1} + F_{k,n} F_{k,n+1} - \frac{1}{k}.
\]
According to Theorem 1 in [18], since \( F_{k,n} \) is normal, irreducible and entrywise nonnegative, then
\[
\| F_{k,n} \|_2 = \frac{1}{k} (F_{k,n+1} + F_{k,n}) - \frac{1}{k}.
\]

Theorem 3.3. Let \( F_{k,n} := \text{Circ}(F_{k,1}, F_{k,2}, \cdots, F_{k,n}) \) be a circulant matrix, and \( s(F_{k,n}) \) denote the spread of \( F_{k,n} \). Then we have
\[
s(F_{k,n}) \leq \sqrt{2n(\frac{1}{k} F_{k,n} F_{k,n+1} - 1)},
\]
\[
s(F_{k,n}) \geq \frac{n}{n-1} \left[ \frac{1}{k} (F_{k,n+1} + F_{k,n}) - \frac{1}{k} \right].
\]
Proof. Let \( tr(F_{k,n}) \) denote the trace of \( F_{k,n} \), then \( tr(F_{k,n}) = nF_{k,1} \). By (10), we know
\[
s(F_{k,n}) \leq \sqrt{2 \| F_{k,n} \|_\infty^2 - \frac{2}{n} \left( tr(F_{k,n}) \right)^2} \leq \sqrt{2n(\frac{1}{k} F_{k,n} F_{k,n+1} - 1)}.
\]
Since
\[
\sum_{s=t}a_{st} = n\left[\frac{1}{k}(F_{k,n+1} + F_{k,n}) - \frac{1}{k} - 1\right].
\]  
(17)

then, we have
\[
s(F_{k,n}) \geq \frac{1}{n-1}\left| \sum_{s=t}a_{st} \right| = \frac{n}{n-1}\left[\frac{1}{k}(F_{k,n+1} + F_{k,n}) - \frac{1}{k} - 1\right].
\]

4. Skew circulant matrix with the k-Fibonacci sequence

The \(n \times n\) skew circulant matrix \(SC_n := SCirc(c_1, \ldots, c_n)\) is define as
\[
SC_n := \begin{bmatrix}
c_1 & c_2 & \cdots & c_n \\
-c_n & c_1 & \cdots & c_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
-c_{n-1} & c_n & \cdots & c_2 \\
-c_2 & c_3 & \cdots & c_1
\end{bmatrix}.
\]  
(19)

Let \(SF_{k,n} := SCirc(F_{k,1}, F_{k,2}, \ldots, F_{k,n})\) be the skew circulant matrices associated with the \(k\)-Fibonacci sequences. In this section, we obtain the expression for eigenvalues of \(SF_{k,n}\), and compute four kinds of norms and bounds for the spread of \(SF_{k,n}\).

Theorem 4.1. Let \(SF_{k,n} := SCirc(F_{k,1}, F_{k,2}, \ldots, F_{k,n})\) be a skew circulant matrix, then the eigenvalues of \(SF_{k,n}\) are
\[
\begin{align*}
\lambda_j &= e^{\frac{i\pi x(j-1)}{n}} (j = 1, \ldots, n), \\
\lambda_j &= e^{\frac{i\pi x(j-1)}{n}} (j = 1, \ldots, n),
\end{align*}
\]  
(20)

Proof. Let \(\chi(x) = \sum_{l=1}^{n} F_{k,l} x^{l-1}\) and
\[
\theta_j = e^{\frac{i\pi x(j-1)}{n}} (j = 1, \ldots, n).
\]  
(21)

Then, by Theorem 7 in [17], we can compute the eigenvalues of \(SF_{k,n}\) as follows
\[
\begin{align*}
\chi(\theta_j) &= \sum_{l=1}^{n} F_{k,l} e^{\frac{i\pi x(j-1)}{n}} \\
&= \frac{1}{r_1 - r_2} \sum_{l=1}^{n} (r_1^{l-1} - r_2^{l}) e^{\frac{i\pi x(j-1)}{n}} = \frac{1}{r_1 - r_2} \left[ \frac{r_1 (1 + r_1^n)}{n} - \frac{r_2 (1 + r_2^n)}{n} \right] \\
&= \frac{1}{r_1 - r_2} \left[ \frac{r_1 - r_2 + (r_1^{n+1} - r_2^{n+1}) + (r_1^n - r_2^n) e^{\frac{i\pi x(j-1)}{n}}}{n} \right] = \frac{1 + F_{k,n+1} + F_{k,n} e^{\frac{i\pi x(j-1)}{n}}}{n}.
\end{align*}
\]  
(22)

For convenience of presentation, we introduce some notations. We use \(A^T\), \(A^*\), \(A^H\) to denote the transpose, the conjugate, and the conjugate transpose of \(A\), respectively. We also use \(\lambda_{\text{max}}(A)\) to represent the maximum eigenvalue of \(A\).

Theorem 4.2. Let \(SF_{k,n} := SCirc(F_{k,1}, F_{k,2}, \ldots, F_{k,n})\) be a skew circulant matrix, then the maximum column sum matrix norm, the maximum row sum matrix norm, the Frobenius norm, and the spectral norm of \(SF_{k,n}\) are respectively given as follows
\[
\|SF_{k,n}\|_\infty = \|SF_{k,n}\|_F = \frac{1}{k} (F_{k,n+1} + F_{k,n}) - \frac{1}{k}.
\]  
(23)
is defined by (22). Therefore, there exists unitary matrix. Noticing (32) then we have the following bounds for the spectral of SF, which shows SC is normal matrix. Thus SF, is normal matrix. Therefore, there exists unitary matrix Q such that

\[ Q^H \cdot SF, \cdot Q = \text{diag}(\chi(\theta_1), \ldots, \chi(\theta_n)). \]

Multiplying the above two equations, we have

\[ Q^H \cdot SF,^T \cdot SF, = \text{diag}(\chi^*(\theta_1), \ldots, \chi^*(\theta_n)). \]

Using (30), we obtain that

\[ \| SF, \|_2 = \sqrt{\max_{1 \leq j \leq n} \chi(\theta_j) \chi^*(\theta_j)}. \]

Theorem 4.3. Let SF, := SCirc(F1,1, F1,2, \ldots, Fk,n) be a skew circulant matrix, then we have the following bounds for the spectral of SF,:

\[ s(SF,) \leq \sqrt{2n \left( \frac{1}{k} F_{k,n} F_{k,n+1} - 1 \right)}, \]

\[ s(SF,) \geq \frac{1}{n-1} \left[ \left( \frac{4}{k^2} - \frac{n-2}{k} \right) F_{k,n+1} + \left( \frac{4}{k^2} - \frac{n}{k} \right) F_{k,n} - \left( \frac{4}{k^2} + \frac{n+2}{k} + n \right) \right]. \]

Proof. It is obvious that

\[ s(SF,) \leq \sqrt{2n \left( \frac{1}{k} F_{k,n} F_{k,n+1} - 1 \right)}. \]

By (4), we have

\[ \sum_{s=1}^{n} a_s = \sum_{l=2}^{n} (n - (l-1)) F_{k,l} - \sum_{l=2}^{n} (l-1) F_{k,l} = (n+2) \sum_{l=2}^{n} F_{k,l} - 2 \sum_{l=2}^{n} l F_{k,l} = (n+2) \left( \frac{1}{k} F_{k,n+1} + F_{k,n} \right) - \frac{1}{k} - 2 \left( \frac{2}{k^2} + \frac{kn+k-2}{k^2} F_{k,n} + \frac{kn-2}{k^2} F_{k,n+1} - 1 \right) = \left( \frac{4}{k^2} - \frac{n-2}{k} \right) F_{k,n+1} + \left( \frac{4}{k^2} - \frac{n}{k} \right) F_{k,n} + \left( \frac{4}{k^2} + \frac{n+2}{k} + n \right). \]
Hence, we get\[ s(SF_{k,n}) \geq \frac{1}{n-1} \left[ \left( \frac{4}{k^2} - \frac{n-2}{k} \right) F_{k,n+1} + \left( \frac{4}{k^2} - \frac{n}{k} \right) F_{k,n} - \left( \frac{4}{k^2} + \frac{n+2}{k} + n \right) \right].\]

5. Skew left circulant matrix with the k-Fibonacci sequence

The \( n \times n \) skew left circulant matrix \( SLC_n := SL_{circ}(c_1, \ldots, c_n) \) is define as
\[
SLC_n := \begin{pmatrix}
c_1 & c_2 & \cdots & c_n \\
c_2 & c_3 & \cdots & -c_1 \\
c_3 & c_4 & \cdots & -c_2 \\
\vdots & \vdots & \ddots & \vdots \\
c_n & -c_1 & \cdots & -c_{n-1}
\end{pmatrix},
\]
(37)

Let \( SLF_{k,n} := SL_{circ}(F_{k,1}, \ldots, F_{k,n}) \) be the skew left circulant matrices associated with \( k \)-Fibonacci sequences. In this section, we obtain the expression for eigenvalues of \( SLF_{k,n} \), and compute four kinds of norms and bounds for the spread of \( SLF_{k,n} \).

Theorem 5.1. Let \( SLF_{k,n} := SL_{circ}(F_{k,1}, F_{k,2}, \ldots, F_{k,n}) \) be a skew left circulant matrix, then the eigenvalues of \( SLF_{k,n} \) are
\[
\pm \sqrt{\varphi(\vartheta_j)\psi(\vartheta_j)} (t = 1, \ldots, \frac{n}{2}), \quad \text{if } n \text{ is even}
\]
\[
\pm \sqrt{\varphi(\vartheta_j)\psi(\vartheta_j)} (t = 1, \ldots, \frac{n-1}{2}) \text{ and } \frac{1}{k} (F_{k,n+1} - F_{k,n} + 1) \quad \text{if } n \text{ is odd}.
\]
(38)

Here \( \vartheta_j(t = 1, \ldots, \frac{n}{2}) \), \( \varphi(x) \) and \( \psi(x) \) are defined by (21), (43) and (45).

Proof. Let \( J_n = (e_n, e_{n-1}, \ldots, e_1) \) stand for the \( n \times n \) reverse unit matrix ( \( e_i \) denotes the \( i \)th column of \( n \times n \) unit matrix). Note that,
\[
J_n \cdot SL_{circ}(F_{k,1}, \ldots, F_{k,n}) = SC_{circ}(F_{k,n}, -F_{k,1}, \ldots, -F_{k,n-1})
\]
(39)

and
\[
SL_{circ}(F_{k,1}, \ldots, F_{k,n}) \cdot J_n = SC_{circ}(F_{k,n}, F_{k,n-1}, \ldots, F_{k,1}).
\]
(40)

By Theorem 7 in [17], there exists
\[
V = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
\vartheta_1 & \vartheta_2 & \vartheta_3 & \cdots & \vartheta_n \\
\vartheta_1^2 & \vartheta_2^2 & \vartheta_3^2 & \cdots & \vartheta_n^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vartheta_1^{n-1} & \vartheta_2^{n-1} & \vartheta_3^{n-1} & \cdots & \vartheta_n^{n-1}
\end{pmatrix}
\]
(41)
such that
\[
SC_{circ}(F_{k,n}, -F_{k,1}, \ldots, -F_{k,n-1}) = V \text{diag}(\varphi(\vartheta_1), \ldots, \varphi(\vartheta_n)) V^{-1}.
\]
(42)

where
\[
\varphi(x) = F_{k,n} - F_{k,1}x - F_{k,2}x^2 - \cdots - F_{k,n-1}x^{n-1},
\]
(43)
and
\[
SC_{circ}(F_{k,n}, F_{k,n-1}, \ldots, F_{k,1}) = V \text{diag}(\psi(\vartheta_1), \ldots, \psi(\vartheta_n)) V^{-1}.
\]
(44)

Here
\[
\psi(x) = F_{k,n} + F_{k,n-1}x + F_{k,n-2}x^2 + \cdots + F_{k,1}x^{n-1}.
\]
(45)

Using (39), (40), (42) and (44), we have that
\[
J_n \cdot SL_{circ}(F_{k,1}, \ldots, F_{k,n})^2 J_n = V \text{diag}(\varphi(\vartheta_1)\psi(\vartheta_1), \ldots, \varphi(\vartheta_n)\psi(\vartheta_n)) V^{-1}.
\]
(46)
For \( t = 1, \ldots, \lfloor \frac{n}{2} \rfloor \), where \( \lfloor x \rfloor \) denotes the maximum integer is less than \( x \), there hold

\[
\varphi(\partial_t) = F_{k,n} - \sum_{u=1}^{n-1} F_{k,u} \partial_t^u,
\]

and

\[
\varphi(\partial_{n+1-t}) = F_{k,n} + \sum_{u=1}^{n-1} F_{k,u} \partial_t^{n-u}.
\]

Considering that

\[
\partial_t \partial_{n+1-t} = 1 \quad \text{and} \quad \partial_t^n = \partial_{n+1-t} = -1.
\]

We have

\[
\partial_t^{n-u} = -(\partial_t^{n+1-t})^{-u} = -\partial_t^u.
\]

Combining (47), (48) and (50), we have

\[
\varphi(\partial_t) = \varphi(\partial_{n+1-t}).
\]

In a similar manner, we have \( \varphi(\partial_{n+1-t}) = \varphi(\partial_t) \). Thus,

\[
\varphi(\partial_t) \varphi(\partial_t) = \varphi(\partial_{n+1-t}) \varphi(\partial_{n+1-t}).
\]

This means that if \( n \) is even, the eigenvalues of \( \text{SLF}_{k,n} \) are

\[
\pm \sqrt{\varphi(\partial_t) \varphi(\partial_t)}, \quad (t = 1, \ldots, n).
\]

If \( n \) is odd, the eigenvalues of \( \text{SLF}_{k,n} \) are still

\[
\pm \sqrt{\varphi(\partial_t) \varphi(\partial_t)}, \quad (t = 1, \ldots, n - \frac{1}{2}).
\]

and the eigenvalues left is equal to the trace of \( \text{SLF}_{k,n} \), namely \( \frac{1}{k}(F_{k,n+1} - F_{k,n} + 1) \). Now we obtain (38).

Theorem 5.2. Let \( \text{SLF}_{k,n} = SL\text{Circ}(F_{k,1}, F_{k,2}, \ldots, F_{k,n}) \) be a skew left circulant matrix, then the maximum column sum matrix norm, the maximum row sum matrix norm, the Frobenius norm, and the spectral norm of \( \text{SLF}_{k,n} \) are given by

\[
\| \text{SLF}_{k,n} \|_1 = \| \text{SLF}_{k,n} \|_\infty = \frac{n}{k} (F_{k,n+1} + F_{k,n}) - \frac{1}{k},
\]

\[
\| \text{SLF}_{k,n} \|_F = \frac{n}{k} F_{k,n} F_{k,n+1},
\]

\[
\| \text{SLF}_{k,n} \|_2 = \begin{cases} 
\max_{1 \leq u \leq \frac{n}{2}} \sqrt{\varphi(\partial_t) \varphi(\partial_t)}, & \text{if } n \text{ is even,} \\
\max_{1 \leq u \leq \frac{n-1}{2}} \sqrt{\varphi(\partial_t) \varphi(\partial_t)}, -\left(\frac{F_{k,n+1} - F_{k,n} + 1}{k}\right), & \text{if } n \text{ is odd,}
\end{cases}
\]

Here \( \partial_t (t = 1, \ldots, \lfloor \frac{n}{2} \rfloor) \), \( \varphi(x) \) and \( \psi(x) \) are defined by (21), (43) and (45).

Proof. The procedure for deducing (53) and (54) is essentially the same as in Theorem 3.3, and the proof is omitted here. It is obvious that \( \text{SLF}_{k,n} \) is a symmetry matrix, hence \( \text{SLF}_{k,n} \) is a normal matrix.

Then

\[
\| \text{SLF}_{k,n} \|_2 = \sqrt{\lambda_{\max} (\text{SLF}_{k,n}^T \cdot \text{SLF}_{k,n})} = \sqrt{\lambda_{\max} (\text{SLF}_{k,n}^2)}.
\]

By (38), if \( n \) is even, we have

\[
\| \text{SLF}_{k,n} \|_2 \leq \max_{1 \leq u \leq \frac{n}{2}} \sqrt{\lambda_{\max}(\varphi(\partial_t) \varphi(\partial_t))}.
\]

If \( n \) is odd, we have

\[
\| \text{SLF}_{k,n} \|_2 \leq \max_{1 \leq u \leq \frac{n-1}{2}} \sqrt{\lambda_{\max}(\varphi(\partial_t) \varphi(\partial_t))}.
\]
\[ \| SLF_{k,n} \|_2 = \max \left\{ \max_{1 \leq n \leq 2^k} \sqrt{\phi(\theta_j)\mu(\theta_j)} \cdot \frac{1}{k} (F_{k,n+1} - F_{k,n} + 1) \right\}. \]

Theorem 5.3. Let \( SLF_{k,n} = SLCirc(F_{k,1}, F_{k,2}, \ldots, F_{k,n}) \) be a skew left circulant matrix, then the bounds for the spread of \( SLF_{k,n} \) are

\[
\begin{align*}
2F_{k,n} & \leq s(SLF_{k,n}) \leq \frac{2n}{k} F_{k,n} F_{k,n+1}, & \text{if } n \text{ is even;} \\
2F_{k,n} & \leq s(SLF_{k,n}) \leq \frac{2n}{k} F_{k,n} F_{k,n+1} - \frac{2}{nk^2} (F_{k,n+1} - F_{k,n} + 1)^2, & \text{if } n \text{ is odd.}
\end{align*}
\]

Proof. If \( n \) is even, \( tr(SLF_{k,n}) = 0 \), then we have

\[ s(SLF_{k,n}) \leq \frac{2n}{k} F_{k,n} F_{k,n+1}. \]  

(58)

If \( n \) is odd, by (2) we have

\[ tr(SLF_{k,n}) = \sum_{j=0}^{n-1} F_{k,2j+1} - \sum_{j=1}^{n-1} F_{k,2j} = \frac{1}{k} (F_{k,n+1} - F_{k,n} + 1). \]

Thus

\[ s(SLF_{k,n}) \leq \frac{2n}{k} F_{k,n} F_{k,n+1} - \frac{2}{nk^2} (F_{k,n+1} - F_{k,n} + 1)^2. \]

In addition, since \( SLF_{k,n} \) is a symmetric matrix, then we have

\[ s(SLF_{k,n}) \geq 2 \max_{s \neq t} |a_{st}| = 2F_{k,n}. \]

6. Conclusion

Circulant, skew circulant and skew left circulant matrices play an important role in engineering application. In this paper, circulant, skew circulant and skew left circulant matrices with \( k \)-Fibonacci sequences are considered. Firstly, eigenvalues of these matrix are discussed and the explicit eigenvalue formulae by diagonalizing these matrices are presented. Furthermore, four kinds of norms including the maximum column sum matrix norms, the maximum row sum matrix norms, the Frobenius norms, the spectral norms, and the bounds for the spread of these matrices are also studied. Our work will be useful complement to the existing researches on circulant and skew circulant type matrices.

7. References

[1] Zhao W 2009 The inverse problem of anti-circulant matrices in signal processing Proc. Paci.-Asia Conf. on Know. Engi. and Soft. Engi. (Shen-zhen/ China) pp 47-50
[2] Zhao G 2009 The improved nonsingularity on the r-circulant matrices in signal processing Proc. Int. Conf. on Comp. Tech. and Deve. (Kota Kinabalu/Malaysia) pp 564-567
[3] Liu V C and Vaidyanathan P P 1988 Circulant and skew-circulant matrices as new normal-form realization of IIR digital filters IEEE Transactions on Circuits and Systems 35(6) pp 625-635
[4] Mayer A, Castiaux A and Vigneron J P 1998 Electronic Green scattering with n-fold symmetry axis from block circulant matrices Comp. Phys. Comm. 109(1) pp 81-89
[5] Ng M K 2003 Circulant and skew-circulant splitting methods for Toeplitz systems Jour. of Comp. and Appl. Math. 159(1) pp 101-108
[6] Davis P J 1979 Circulant matrices (John Wiley Sons/ New York)
[7] Jaiswal D V 1969 On determinants involving generalized Fibonacci numbers Fibo. Quart 7 pp 319-330
[8] Lind D A 1970 A Fibonacci circulant Fibo. Quart 8 pp 449-455
[9] Yazlik Y and Taskara N 2013 On the inverse of circulant matrix via generalized k-Horadam numbers \textit{Appl. Math. and Comp.} 223 pp 191-196
[10] Bozkurt D and Tam T Y 2012 Determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas Numbers \textit{Appl. Math. and Comp.} 219(2) pp 544-551
[11] Bueno A C F 2013 Left circulant matrices with arithmetic sequence \textit{South Asian Jour. of Math.} 3(5) pp 322-325
[12] Sergio F and A Plaza 2007 The k-Fibonacci sequence and the Pascal 2-triangle \textit{Chao. Soli. and Frac.} 33(1) 38-49
[13] Sergio F and Plaza A 2007 On the Fibonacci k-numbers \textit{Chao. Soli. and Frac.} 32(5) pp 1615-1624
[14] Bolat C 2010 On the properties of k-Fibonacci numbers \textit{Int. J. Cont. Math. Scie.} 5(22) pp 1097-1105
[15] Kilic E 2008 Sums of the squares of terms of sequence \{un\} \textit{Indi. Acad. Sci. (Math. Sci.)} 118(1) pp 27-41
[16] Shen S 2010 On the spectral norms of r-circulant matrices with the k-Fibonacci and k-Lucas Numbers \textit{Int. J. Cont. Math. Scie.} 5(12) pp 569-578
[17] Zhang Y, Zhang H S and Chen G Y 2013 A note on the square roots of a class of circulant matrices \textit{Jour. of Appl. Math.} 2013(601243) pp 1-6
[18] Ipek A 2011 On the spectral norms of circulant matrices with classical Fibonacci and Lucas numbers entries \textit{Appl. Math. and Comp.} 217(12) pp 6011-6012

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