Analytic Methods to Calculate Fault Trees with Loops - Restrictions of Application and Solution Uniqueness

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Abstract. One of the important tasks of the Reliability Estimation is Analysis of the Fault Tree. A problem of Fault Trees analysis is considered one of the most complex ones, since structure of such trees is characterized by a considerable number of interconnections. Usually analytical methods are used and most applicable method is Minimal Cut Sets building and calculation. Classical Fault Tree Analysis methods are applicable only for Fault Trees without loops. Loops can appear in Fault Tree, when a TOP or some intermediate gates appear as input to another gate at a lower level of the model. An occurrence of a Loop has been a problematic issue in a Fault Tree calculation.

The article relates to the uniqueness of the solution for the Fault Trees with arbitrary Loops. There are assumed, that failures of the Basic Events are non-repairable and Fault Tree gates may be expressed by two main logic gates – AND-gates and OR-gates.

1. INTRODUCTION

One of the important tasks of the Reliability Estimation is Analysis of the Fault Tree [1]. Classical Fault Tree Analysis methods (Minimal Cut Sets calculations) are applicable only for Fault Trees without loops. A variety of methods have been developed to calculate Fault Tree with loops (see, e.g., articles [1 – 9]). Fault Tree Handbook with Aerospace Applications [1] proposes general advise – "...The loops are cut (eliminated) in the fault tree...". But it is non-correct to simply "delete loop", analyst should carefully investigate concrete features of the analyzed Fault Tree and to decide, how to build equivalent Fault tree without loop.

The conventional method, presented on [2], proposes to solve the logical loop problem by breaking the logical loops at the points where the dependencies among the support systems are relatively weak and developing new fault trees without the logical loops. But this method gets us exact solution only for simple FaultTrees with loops. Yang [3] built contra-example for this approach, which shows its mistake. Consider Fault Tree of 4 TOPs and triple linear interrelated loops:

\[
A = Aa \text{ OR } (Ab \text{ AND } B) \text{ OR } (Ac \text{ AND } C) \text{ OR } (Ad \text{ AND } D) \\
B = Bb \text{ OR } (Ba \text{ AND } A) \text{ OR } (Bc \text{ AND } C) \text{ OR } (Bd \text{ AND } D) \\
C = Cc \text{ OR } (Ca \text{ AND } A) \text{ OR } (Cb \text{ AND } B) \text{ OR } (Cd \text{ AND } D) \\
D = Dd \text{ OR } (Da \text{ AND } A) \text{ OR } (Db \text{ AND } B) \text{ OR } (Dc \text{ AND } C)
\]

Where:

- A, B, C and D are Sub-Fault Trees TOPs with Loops, non-calculated directly by means of classical non-cycled fault tree analysis
- Aa, Ab, Ac, Ad, Bb, Ba, Bc, Bd, Cc, Ca, Cb, Cd, Dd, Da, Db and Dc are Basic Events.
On the [3] it is shown, that for Ad = Dc = Cb = Db = TRUE and other Basic Event values, equalled for FALSE, the value of TOP, obtained by conventional method using, is equalled to FALSE, but it isn't satisfy for above equations. Opposite, value A = TRUE is correct.

Yang [3] presented an exact analytical method to break the logical loops by means of using of the Boolean Algebra laws to transformate Fault Trees with loops to the Fault Tree without loops. It is proposed to break the logical loops in the merged fault tree by disconnecting one of the connected gates that cause the logical loops. Some modifications of this approach are considered in different articles, denoted for analysis of the Fault Tree with loops [5 – 7].

Proposed of these articles methods have following drawbacks:

- They don't formulate restrictions for its field of application. Arbitrary Fault Trees may consist of gates of different types (AND, OR, NOT, etc.) and Basic Events of different types (non-repairable, repairable, periodically tested, etc.). Really these articles consider only Fault Trees with gates AND and OR, but assumptions according Basic Event types are absent. Otherwise, for repairable Basic Events proposed methods don't get correct results.

- It isn't proved, that proposed analytic methods get us full solution, i.e. don't exist other Boolean solutions, also satisfy for analysed Fault Tree with loops. These methods use direct Boolean transformation, so they are not applicable for situation, when simultaneously two possible TOP values (both FALSE and TRUE) are satisfy for the Boolean equations and so may be solutions.

The main aim of theour article is to remove these drawbacks of the early proposed analytic methods. Remaining part of the paper is organized as follows:

- In Chapter 2 we introduce some contra-example for analytic methods, proposed for Fault Trees with Loops
- In Chapter 3 we prove Uniqueness of the solution for Arbitrary Fault Trees with Loops and Non-Repairable Basic Events
- In Chapter 4 we discuss obtained results.

2. FIELD of APPLICATION for ANALYTIC SOLUTIONS

2.1 Types of Basic Events

Consider following Fault Tree:

\[
A = Aa \text{ OR } (Ab \text{ AND } B), \\
B = Bb \text{ OR } (Ba \text{ AND } A)
\]

Where A and B are TOPs, Aa is repairable Basic Event, Ab, Bb and Ba are repairable or non-repairable Basic Events.

According analytic methods, proposed on [3 - 9] the solution will be following:
• $A = Aa \lor (Ab \land Bb)$

So, for combination \{Aa = FALSE, Ab = TRUE, Ba = TRUE, Bb = FALSE\} we get solution $A = FALSE$.

Consider following trajectories of the Basic Event state transformations and TOP A state transformation, due to equation $A = Aa \lor (Ab \land (Bb \lor (Ba \land A)))$:

$t_0$: $Aa(t_0) = FALSE, Ab(t_0) = FALSE, Ba(t_0) = FALSE, Bb(t_0) = FALSE, A(t_0) = FALSE$

$t_1$: $Aa(t_1) = TRUE, Ab(t_1) = FALSE, Ba(t_1) = FALSE, Bb(t_1) = FALSE, A(t_1) = TRUE$

$t_2$: $Aa(t_2) = TRUE, Ab(t_2) = TRUE, Ba(t_2) = FALSE, Bb(t_2) = FALSE, A(t_2) = TRUE$

$t_3$: $Aa(t_3) = TRUE, Ab(t_3) = TRUE, Ba(t_3) = TRUE, Bb(t_3) = FALSE, A(t_3) = TRUE$

$t$: $Aa(t) = FALSE, Ab(t) = TRUE, Ba(t) = TRUE, Bb(t) = FALSE, A(t) = TRUE$

where $t_0$ is Initial Time, $t_1, t_2$ and $t_3$ are some Intermediate Times, $t$ is Final Time (i.e. the time of the Fault Tree Analysis)

Is it mean, that solution, obtained by analytic methods proposed on [3 - 9], is wrong? No, this solution also may be, but for another trajectory:

$t_0$: $Aa(t_0) = FALSE, Ab(t_0) = FALSE, Ba(t_0) = FALSE, Bb(t_0) = FALSE, A(t_0) = FALSE$

$t_1$: $Aa(t_1) = TRUE, Ab(t_1) = TRUE, Ba(t_1) = FALSE, Bb(t_1) = FALSE, A(t_1) = TRUE$

$t_2$: $Aa(t_2) = FALSE, Ab(t_2) = TRUE, Ba(t_2) = TRUE, Bb(t_2) = FALSE, A(t_2) = FALSE$

$t$: $Aa(t) = FALSE, Ab(t) = TRUE, Ba(t) = TRUE, Bb(t) = FALSE, A(t) = TRUE$

Due to possible non-monotonically character of the Basic Event Aa trajectory ($FALSE \Rightarrow TRUE \Rightarrow FALSE$), we get dual solutions of the TOP for the same initial and final values of the Basic Events. So, for Fault Trees with Loops and Repairable Basic Events the solution depends not only of values of Basic Events, but also of Basic Event state transformation, i.e. of pre-history of Basic Events States. Same conclusion is correct for some other types of Basic Events – periodically tested, constant availability, etc. But for Fault Trees with Loops, which contain only Non-Repairable Basic Events, this situation (dual solution for some combination of the Basic Event values) is impossible, we will prove this in next chapter.

Comment. For first point of view, this fact (existence of the dual solutions) is some strange. Classic Fault Tree Analysis methods (Minimal Cut Sets) don't take into account pre-history of the Basic Event states and allow us to Calculate TOP state based only of the final states of the Basic Events, both for repairable and non-repairable Basic Events. But it is correct only for classic, regular Fault Tree. For some other types of the Fault Trees it is wrong. For example, for Dynamic Fault Trees, which contain such gates as PAND (Priority AND), SEQ, SPARE, FDEP, etc., it is necessary to take into account pre-history of the single gate state transformations – final states of the Basic Events don't allow us to calculate TOP state.

2.2 Types of Gates
If Fault Tree contain some NOT type gates (NOT, NOR, NAND, XOR, etc.), we also could not provide monotonicall character of the some gate trajectory and so dual solutions may exist.

So, second (a Fault Treeer Basic Event types) restriction should be following: all gates on the analysed Fault Tree may be expressed by two main logic gates – AND-gates and OR-gates.

Certainly, Fault Tree can contain some complex gates, e.g "K-out-of-N" gate, but should be possible expressed these complex gates only from AND-gates and OR-gates. For example, "2-out-of-3" gate may be expressed as:

\{Gate1 AND Gate2 AND NOT(Gate3) \} \ OR \ \{Gate1 AND Gate3 AND NOT(Gate2) \} \ OR \ \{Gate2 AND Gate3 AND NOT(Gate1) \} \ OR \ \{Gate1 AND Gate2 AND Gate3\},

but this expression contains not only AND-gates and OR-gates, but also NOT gate and so directly not applicable for analytic methods, proposed for Fault Trees with Loops. But it is also possible to expressed "2-out-of-3" gate only from AND-gates and OR-gates, without using of gate NOT:

\{ Gate1 AND Gate2 \} \ OR \ \{ Gate1 AND Gate3 \} \ OR \ \{ Gate2 AND Gate3 \}, and this Fault Tree applicable for analytic methods, proposed for Fault Trees with Loops.

3. Uniqueness of the solution

Main Theorem.

For arbitrary Fault Tree with multiple non-linear interrelated Loops, which contains only non-repairable Basic Events and all gates may be expressed only by means of AND-gates and OR-gates, for possible dual solution of some TOP all trajectory values of this TOP.

3.1 FAULT TREES with ORDINARY LOOPS

First consider Fault Tree, for which the loops are only ordinary. Ordinary loop means, that each TOP may depend step-by-step (by circle) only of one other TOP. Full Fault Tree may consist on several sub-trees and the TOP of each sub-tree depends not only of Basic Events, but also from one other TOP:

\[ TOP[1] = F(a[1], \ldots, a[n], TOP[2]) \]
\[ TOP[2] = U(a[1], \ldots, a[n], TOP[3]) \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\[ TOP[k-1] = W(a[1], \ldots, a[n], TOP[k]) \]
\[ TOP[k] = R(a[1], \ldots, a[n], TOP[1]) \]

Where:

- \( F, U, \ldots, W, R \) – some boolean expressions.
- \( a[1], \ldots, a[n] \) – Basic Events

Illustration of dependencies between fault trees:
We can sequentially substitute expressions for TOP and to get final equation
(1) \( \text{TOP} = H(a[1], \ldots, a[n], \text{TOP}), \)
where \( H \) is some boolean expression and for \( \text{TOP}[1] \) the index is omitted (i.e. \( \text{TOP} = \text{TOP}[1] \)).

Using standard rules of the Fault Tree Minimal Cut building, following representation always may be got:
\[
\text{TOP} = \{a[i_1] \text{ AND } a[i_2] \text{ AND } \ldots\} \text{ OR } \{a[j_1] \text{ AND } a[j_2] \text{ AND } \ldots\} \text{ OR } \ldots \text{ OR } \{a[m_1] \text{ AND } a[m_2] \text{ AND } \ldots\} \text{ OR } \{\text{TOP AND } a[q_1] \text{ AND } a[q_2] \text{ AND } \ldots\} \text{ OR } \ldots \text{ OR } \{\text{TOP AND } a[f_1] \text{ AND } a[f_2] \text{ AND } \ldots\}.
\]

From this representation, short form may be get (\textit{MAIN EQUATION}):
(2) \( \text{TOP} = Q_{10}(a[1], \ldots, a[n]) \text{ OR } \{Q_{11}(a[1], \ldots, a[n]) \text{ AND } \text{TOP}\} \)

Where \( Q_{10} (a[1], \ldots, a[n]) \) and \( Q_{11}(a[1], \ldots, a[n]) \) – some boolean expressions, which don't depend of \( \text{TOP} \) and depend only of Basic Events \( a[1], \ldots, a[n] \) and Fault Tree structure.

All state alternatives are shown on the Table 1, and column "Available" corresponds for the possible correct solution (" + " corresponds for available value, i.e. if "input TOP
value" = "Top value from eq. (2), " - " corresponds for non-available value, i.e. if "input TOP value" =/= "Top value from eq. (2) ). Possible Dual Solutions are signed as *Italic&Bold*

| Values of the input variables at the final time t | Output Value of TOP at the final time t |
|-----------------------------------------------|-----------------------------------------|
| Q₁₀  | Q₁₁ | TOP value | From eq. (2) | TOP value is Available ? | By means of simulation | Correct value |
| FALSE | FALSE | FALSE | FALSE | + | FALSE |
| FALSE | TRUE | *FALSE* | FALSE | + | FALSE |
| TRUE | FALSE | TRUE | - | | TRUE |
| TRUE | TRUE | TRUE | - | | TRUE |
| TRUE | TRUE | TRUE | + | | TRUE |

Table 1. States alternatives for arbitrary Fault Tree with ordinary loop

It is seen, that from 4 alternatives for Q₁₀ and Q₁₁ values, only one combination {Q₁₀ = FALSE, Q₁₁ = TRUE} gets possible dual solution: both TOP = TRUE and TOP = FALSE may be solutions of the main equation (2). For this situation it is necessary to perform event-driven (step-by-step) simulation.

Assume, that all Basic Events a[i] have value FALSE as the initial state. If some a[i](t₀) = TRUE, due to non-repairable type of the Basic Events it will be also same value on this state at the final time t (i.e. a[i](t) = TRUE), so it is necessary to perform following:
- To remove Basic Event a[i] from the gates AND, for which a[i] is input.
- To install TRUE instead of output of the gate OR, for which Basic Event a[i] is input, and etc, follow to Down-Top approach.

As at the start (i.e. at the time t₀) all Basic Events are at the state FALSE, the Q₁₀(t₀) and Q₁₁(t₀) are also have values FALSE, because they are some combinations of the AND-gates and OR-gates. So, TOP(t₀) is also has value FALSE, to satisfy eq. (2). Consider some intermediate time moment t₁ s.t. t₀ < t₁ < t, at what some Basic Event a(i) has changed his state from FALSE to TRUE (if ALL Basic Events did not change its states up time t, TOP also did not change its state and so TOP(t) = TOP(t₀) ).

Is it possible, that value of Q₁₀(t₁) = TRUE ? No, it is impossible, because all Basic Events are non-repairable and so any Boolean Function, composed of gates AND and OR, is non-decreased and at the time moment t the Boolean Function Q₁₀(t) should have value FALSE.
Is it possible, that value of $Q_{11}(t_1) = \text{TRUE}$? Yes, it is possible, because at the time moment $t$ the Boolean Function $Q_{11}(t)$ should have value $\text{TRUE}$. Also possible, that $Q_{11}(t_1) = \text{FALSE}$.

But in any case, independently of $Q_{11}(t_1)$ value (TRUE or FALSE), the new value of the the TOP at the time $t_2 = (t_1 + \Delta t)$ will be $\text{FALSE}$, because $\text{TOP}(t_2) = Q_{10}(t_1) \text{ OR } \{Q_{11}(t_1) \text{ AND TOP}(t_1)\} = \text{FALSE OR } \{Q_{11}(t) \text{ AND FALSE}\} = \text{FALSE}$. So, after each possible state changing of some Basic Event from $\text{FALSE}$ to $\text{TRUE}$ the state of the TOP isn't changed and will have same value $\text{FALSE}$.

**Conclusion.** Analysis of the Table 1 allow us to get equation $\text{TOP}(t) = Q_{10}(t)$ and so to produce following rule (named "FALSE insertion in the initial Fault Tree with loop instead of TOP input") - to calculate Fault Tree with ordinary loops it is enough to delete loops and to insert to the right part of the initial equation (2) value $\text{FALSE}$ as Input instead of the TOP input.

### 3.2. ARBITRARY FAULT TREES

Consider now arbitrary Fault Tree, for which the loops are not only ordinary and moreover, are not only linear. For example, Fault Tree with 3 non-linear interrelated loops is following:

\[
\begin{align*}
A &= Q_{10} \text{ OR } (Q_{11} \text{ AND } A) \text{ OR } (Q_{12} \text{ AND } B) \text{ OR } (Q_{13} \text{ AND } A \text{ AND } B) \\
B &= Q_{20} \text{ OR } (Q_{21} \text{ AND } A) \text{ OR } (Q_{22} \text{ AND } B) \text{ OR } (Q_{23} \text{ AND } A \text{ AND } B)
\end{align*}
\]

where $A$, $B$, and $C$ are TOPs of the Sub-Fault Trees with loops.

By means of substitution of $C$ to the first and second equation, and Boolean algebra transformations, we get following main equations (4):
where $Q_{10}$, $Q_{11}$, etc. – some boolean expressions, dependent only from Basic Events $a[1], \ldots, a[n]$, and don't dependent from TOPs A and B.

For arbitrary Fault Trees with multiple loops, where each TOP may depend Non-Linearly of several TOPs, the system of the **main equations** for the TOP[$i$] will be following

(i = 1…k):

(5)  

$$
\text{TOP}[i] = Q_{i,0} \text{ OR } \{(Q_{i,1} \text{ AND TOP}[1]) \text{ OR } \ldots \text{ OR } \{(Q_{i,k} \text{ AND TOP}[k])\} \text{ OR } \{(Q_{i,1,2} \text{ AND TOP}[1] \text{ AND TOP}[2]) \text{ OR } \ldots \text{ OR } \{(Q_{i,1,k} \text{ AND TOP}[1] \text{ AND TOP}[k])\} \text{ OR } \ldots \text{ OR } \{(Q_{i,1,2,\ldots,k} \text{ AND TOP}[1] \text{ AND TOP}[2] \text{ AND } \ldots \text{ AND TOP}[k])\}}
$$

Where $Q_{i,j,\ldots}(a[1],\ldots, a[n])$ – some Boolean expressions (MinCut Sets), depending on Basic Events $a[1], \ldots, a[n]$, and don't dependent on TOP[1],,,TOP[k].

Fragment of the full table for arbitrary Fault Tree with 3 TOPs and triple non-linear interrelated loops, corresponded for the system of equations (4), is shown on the Table 2. Column "Available" corresponds for the possible correct solutions, possible dual solutions are signed as *Italic&Bold*.

As for Fault Trees with ordinary loops it is possible to assume, that at the moment $t_0$ the values of all Basic Events $a[1], \ldots, a[n]$ are equalled to **FALSE**. It is also clear, that values of all Boolean Expressions $Q_{i,j,k}(t_0)$ are also equalled to **FALSE** and so, to satisfy eq. (5), values of all $T(t_0)$ should be also equalled to **FALSE**.
Values of the Input Variables at the final time $t$

|   | $Q_{10}$ | $Q_{11}$ | $Q_{12}$ | $Q_{13}$ | $Q_{20}$ | $Q_{21}$ | $Q_{22}$ | $Q_{23}$ | $A$   | $B$   | $A$   | $B$  | Values are available? |
|---|---------|---------|---------|---------|---------|---------|---------|---------|-------|-------|-------|------|------------------------|
|1  | False   | False   | False   | False   | False   | False   | False   | False   | False | False | False | False | +                      |
|   | False   | True    | False   | False   | False   | False   | False   | False   | False | False | False | False | -                      |
|   | True    | False   | False   | False   | False   | False   | False   | False   | False | False | False | False | -                      |
|2  | False   | False   | False   | False   | True    | False   | False   | False   | False | False | False | True  | -                      |
|   | False   | True    | False   | True    | True    | True    | True    | True    | True  | True  | True  | True  | +                      |
|3  | False   | False   | True    | False   | False   | True    | False   | False   | False | False | False | False | -                      |
|   | True    | False   | False   | False   | False   | False   | False   | False   | False | False | False | True  | +                      |
|   | True    | True    | True    | True    | True    | True    | True    | True    | True  | True  | True  | True  | +                      |

Table 2. Fragment of the full table of the states alternatives for arbitrary Fault Tree with 3 TOPs and triple non-linear interrelated loops.

**Statement 1.** For any intermediate time moment any $TOP[i]$ could not change his state from **TRUE** to **FALSE**.

To prove this statement, consider some time moment $t_1$, at what $TOP[i] = \text{TRUE}$ and some Basic Event $a[j]$ has changed his state (if all Basic Events did not change its states, each TOP also did not change its state). Due to non-repairable type of the Basic Events, the only transformation of Basic Event state from **FALSE** to **TRUE** is possible, i.e. $a[j](t_1) = \text{TRUE}$. Consider eq. (5) – it consist only of Boolean operations **AND** and **OR**, and so for time $t_2 = t_1 + \Delta t$ we get, that $TOP[i](t_2) \text{ can't decrease its value for comparison with } TOP[i](t_1) \text{ value, so } TOP[i](t_2) = \text{TRUE}$

Consider some fixed combination of the possible values of the Basic Events at the time $t$ \{a[1](t),..., a[n](t)\} and corresponding combination of values of Boolean expressions
Consider some possible dual solution for some \( \text{TOP}[i](t) \) according eq. (5). As illustration we can consider values \( \text{FALSE} \) and \( \text{TRUE} \) for \( \text{TOP} = A \) from Fault Tree (4) – see lines 2 and 3 from Table 2. There may be two alternatives:

a) For this combination of values of the Boolean expressions \( Q_{ij..}(a[1](t), \ldots, a[n](t)) \) only one \( \text{TOP} \) has possible dual solution (e.g., line 2 at the Table 2 – only \( A \) has possible dual solution)

b) For this combination of values of the Boolean expressions \( Q_{ij..}(a[1](t), \ldots, a[n](t)) \) both several \( \text{TOPs} \) simultaneously have possible dual solutions (e.g., line 3 at the Table 2 – both \( A \) and \( B \) have possible dual solutions)

Let only \( \text{TOP}[i] \) has possible dual solution for this combination of the Boolean expressions \( Q_{ij..}(a[1](t), \ldots, a[n](t)) \). From eq. (5) we get, that \( Q_{i,0}(t) = \text{FALSE} \), because in opposite case \( \text{TOP}[i](t) = \text{TRUE} \) for any possible values of the \( Q_{ij..}(t) \) and \( \text{TOP}[j](t) \) and so solution for \( \text{TOP}[i](t) \) will not be possible dual.

Comment. It is right only for \( \text{TOPs} \) with possible dual solutions. For example, for \( B \) we see, that \( Q_{20}(t) = \text{TRUE} \) (line 2 at the Table 2)

After separation of all expressions on the right part of the equation (5) for \( \text{TOP}[i] \) between expressions, which contain \( \text{TOP}[i] \), and expressions, which don't contain \( \text{TOP}[i] \), we can re-write eq. (5) in shortest form as :

(6) \( \text{TOP}[i] = Q_{i,0} \text{ OR } G_i \text{ OR } (W_i \text{ AND } \text{TOP}[i]), \)

Where:

- \( G_i \) – some Boolean expression, that don't dependent on \( \text{TOP}[i] \).
- \( W_i \) – some Boolean expression, that dependent or don't dependent on \( \text{TOP}[i] \).

For considered combination of the Boolean expressions \( Q_{ij..}(a[1](t), \ldots, a[n](t)) \) the only \( \text{TOP}[i](t) \) has possible dual solution, so all other \( \text{TOPs} \) have some fixed (single) solutions. If for these solutions and for considered combination of the Boolean expressions \( Q_{ij..}(a[1](t), \ldots, a[n](t)) \) we have, that \( G_i(t) = \text{TRUE} \), we get that \( \text{TOP}[i](t) = \text{TRUE} \) for any values of the \( W_i(t) \) and \( \text{TOP}[i](t) \), and so solution for \( \text{TOP}[i](t) \) will not be possible dual. So, \( G_i(t) = \text{FALSE} \).

Consider some intermediate time moment \( t_1 \) s.t. \( t_0 \leq t_1 \leq t \), at what some \( \text{BE} \) has changed his state from \( \text{FALSE} \) to \( \text{TRUE} \) (if all \( \text{BEs} \) did not change its states up time \( t \), all Boolean expressions \( Q_{ij..}(a[1], \ldots, a[n]) \) also did not change its state, and so only \( \text{TOP}[i](t) = \text{FALSE} \) will be solution and we haven't dual solution – see line 1 at the Table 2).

Both all \( \text{BEs} \) and all \( \text{TOPs} \) could not change its states from \( \text{TRUE} \) to \( \text{FALSE} \), so boolean expressions \( Q_{i,0} \) and \( G_i \) also could not change its states from \( \text{TRUE} \) to \( \text{FALSE} \) (because they composed only from operations \( \text{AND} \) and \( \text{OR} \)).

So, at the time \( t_1 \) for \( Q_{i,0} \) and \( G_i \) there are proved, that \( Q_{i,0}(t_1) = \text{FALSE} \) and \( G_i(t_1) = \text{FALSE} \), because both \( Q_{i,0}(t_0) = Q_{i,0}(t) = \text{FALSE} \) and \( G_i(t_0) = G_i(t) = \text{FALSE} \).
At the time $t_1$, when some BE has changed his state from $\text{FALSE}$ to $\text{TRUE}$, the value of the TOP[$i$] is as early, i.e. TOP[$i$]($t_1$) = $\text{FALSE}$. We don't know value of the $W_i(t_1)$ - it may be both $\text{TRUE}$ and $\text{FALSE}$. But in any case at the time $t_2 = t_1 + \Delta t$ (after some of the Basic Events has changed its state from $\text{FALSE}$ to $\text{TRUE}$) the value of the TOP[$i$] at this time $t_2$ will be:

$$\text{TOP}[i](t_2) = Q_{i,0}(t_1) \text{ OR } G_i(t_1) \text{ OR } \{W_i(t_1) \text{ AND } \text{TOP}[i](t_1)\} = \text{FALSE OR FALSE OR } \{W_i(t_1) \text{ AND FALSE}\} = \text{FALSE}.$$ 

So, after each transformation of the state of any of Basic Events the value of the TOP[$i$] with dual solution isn't changed and so TOP[$i$]($t$) = $\text{FALSE}$.

b) For some combination of values of the Boolean expressions $Q_{i,j,..}(a[1](t),...,a[n](t))$ both several TOPs simultaneously have dual solutions (e.g., line 3 at the Table 2 - both A and B have dual solutions).

Let TOP[$i$] and TOP[$j$] have dual solutions for this combination of the Boolean expressions $Q_{i,j,..}(a[1](t),...,a[n](t))$. From eq. (5) we get, that $Q_{i,0}(t) = \text{FALSE}$ and $Q_{j,0}(t) = \text{FALSE}$, because in opposite case TOP[$i$]($t$) = $\text{TRUE}$ or TOP[$j$]($t$) = $\text{TRUE}$ for any possible values of the $Q_{i,j,..}(t)$ and TOP[$r$]($t$) and solutions for TOP[$i$]($t$) and TOP[$j$] ($t$) will not be dual.

After separation of all expressions on the right part of the equation (5) for TOP[$i$] between expressions, which contain TOP[$i$] or TOP[$j$], and expressions, which don't contain TOP[$i$] or TOP[$j$], we can re-write eq. (5) in shortest form as :

$$(7) \quad \text{TOP}[i] = Q_{i,0} \text{ OR } G_i \text{ OR } (W_j \text{ AND } \text{TOP}[j]) \text{ OR } (W_i \text{ AND } \text{TOP}[i])$$

Where:

- $G_i$ – some Boolean expression, that don't dependent on TOP[$i$] and TOP[$j$].
- $W_j$ – some Boolean expression, that don't dependent on TOP[$i$] and dependent or don't dependent on TOP[$j$].
- $W_i$ – some Boolean expression, that dependent or don't dependent on TOP[$j$] and TOP[$i$] .

For considered combination of the Boolean expressions $Q_{i,j,..}(a[1](t),...,a[n](t))$ the only TOP[$i$]($t$) and TOP[$j$]($t$) have dual solutions, so all other TOPs have some fixed (single) solutions. If for these solutions and for considered combination of the Boolean expressions $Q_{i,j,..}(a[1](t),...,a[n](t))$ we have, that $G_i(t) = \text{TRUE}$, we get that TOP[$i$]($t$) = $\text{TRUE}$ and solution for TOP[$i$]($t$) will not be dual. So, $G_i(t) = \text{FALSE}$.

Consider some intermediate time moment $t_1$ s.t. $t_0 <= t_1 <= t$, at what some Basic Event has changed his state from $\text{FALSE}$ to $\text{TRUE}$ (if all Basic Events did not change its states up time $t$, TOP[$i$] also did not change its state, so TOP[$i$]($t$) = $\text{FALSE}$ and we haven't dual solution – see line 1 at the Table 2).
Both all Basic Events and all TOPs could not change its states from TRUE to FALSE, so boolean expressions $Q_{i,0}$ and $G_i$ also could not change its states from TRUE to FALSE (because they composed only from operations AND and OR). So, at the time $t_1$ for $Q_{i,0}$ and $G_i$ there are proved, that $Q_{i,0}(t_1) = \text{FALSE}$ and $G_i(t_1) = \text{FALSE}$, because both $Q_{i,0}(t) = Q_{i,0}(t) = \text{FALSE}$ and $G_i(t) = G_i(t) = \text{FALSE}$.

At the time $t_1$ the value of the TOP[i] is as early, i.e. $\text{TOP}[i](t_1) = \text{FALSE}$. We don't know values of the $W_i(t_1)$ and $W_j(t_1)$ - they may be both TRUE and FALSE. But in any case at the time $t_2 = t_1 + \Delta t$ (after some of the Basic Event has changed its state from FALSE to TRUE) the value of the TOP[i] at this time will be: $\text{TOP}[i](t_2) = Q_{i,0}(t_1) \text{OR } G_i(t_1) \text{ OR } \{W_i(t_1) \text{ AND } \text{TOP}[i](t_1) \} \text{ OR } \{W_j(t_1) \text{ AND } \text{TOP}[j](t_1) \} = \text{FALSE OR FALSE OR} \{W_i(t_1) \text{ AND } \text{FALSE}) \} \text{ OR } \{W_j(t_1) \text{ AND } \text{FALSE}) \} = \text{FALSE}$.

So, after each transformation of the state of any of Basic Events, if before this time moment the states of the TOP[i] and TOP[j] with possible dual solution were FALSE, the values of the TOP[i] and TOP[j] are not changed. So $\text{TOP}[i](t) = \text{FALSE}$ and $\text{TOP}[j](t) = \text{FALSE}$.

Same consideration for situations, when 3 or more TOPs simultaneously have possible dual solutions for some combination of the Boolean expressions $Q_{i,j}(a[1](t),..., a[n](t))$.

5. CONCLUSIONS

Early proposed exact analytic methods for calculation of the Arbitrary Fault Tree with Loops are analysed. It is shown, that they don't applicable for Fault Trees with repairable Basic Events, because such Fault Trees can have dual solutions, dependent on pre-history. Otherwise, it is proved, that for Fault Tree with non-repairable Basic Events, which include only gates AND, OR and based of them composed gates (as "K out of M"), the solution may be only uniqueness and so .early proposed methods are correct.

References

[1] Fault Tree Handbook with Aerospace Applications. NASA Headquarters, Washington, DC 20546. August, 2002
[2] Coles G.A and Powers T.B. Breaking the logical loop to complete the Probabilistic risk assessment. Proceeding of PSA 89: International Topical Meeting on Probability, Reliability and Safety Assessment, pp. 1155 - 1160, 1989.
[3] Yang J. E., Han S. H., Park J. H. and Jin Y. H. Analytic Method to Break Logical Loops Automatically in PSA. Reliability Eng. & System Safety, Vol. 56, pp. 101 - 105, 1997
[4] Demichela M., Piccinini N., Ciarambino I. and Contini S. How to avoid the generation of logic loops in the construction of fault tree. Reliability Eng. & System Safety, Vol. 84, pp. 197 - 207, 2004
[5] Jung W. S. and Han S. H. Development of an Analytical Method to Break Logical Loops at the System Level. Reliability Eng. & System Safety, Vol. 90, pp. 37 - 44, 2005

[6] Lim H. G. and Jang S. C. An Analytic Solution for a Fault Tree with Circular Logics in which the Systems are Linearly Interrelated. Reliability Eng. & System Safety, Vol. 92, pp. 804 - 807, 2007

[7] Vaurio J.K. A Recursive Method for Breaking Complex Logic Loops in Boolean System Models. Reliability Eng. and System Safety, Vol. 92, pp. 1473-1475, 2007

[8] Matsuoka, T. An Exact Method for Solving Logical Loops in Reliability Analysis. Reliability Eng. & System Safety, Vol. 94, pp. 1282 - 1288, 2009

[9] Matsuoka, T. Method for Solving Logical Loops in System Reliability Analysis. Nuclear Safety and Simulation, Vol. 1, Numb. 4, pp. 328 - 339, 2010