Port-Hamiltonian Realizations of Nonminimal Linear Time Invariant Systems

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Abstract

The question when a general linear time invariant control system is equivalent to a port-Hamiltonian system is answered. Several equivalent characterizations are derived which extend the characterizations of [62] to the general non-minimal case and to the case where the feedthrough term does not have an invertible symmetric part. An explicit construction of the transformation matrices is presented in terms of the computation of invariant subspaces of even matrix pencils. This construction includes the potential of incorporating a perturbation to a nearby port-Hamiltonian system, when such a transformation does not otherwise exist. Results are illustrated via numerical examples.

Keywords: port-Hamiltonian system, passivity, stability, system transformation, Kalman-Yacubovich-Popov inequality, Lyapunov inequality, even pencil, quadratic eigenvalue problem.

AMS subject classification. 93A30, 93B17, 93B11.

1 Introduction

The synthesis of system models that describe complex physical phenomena often involves the coupling of independently developed subsystems originating within different disciplines. Systematic approaches to coupling such diversely generated subsystems prudently follows a system-theoretic network paradigm that focuses on the transfer of energy, mass, and other conserved quantities among the subsystems. When the subsystem models themselves arise from variational principles, then the aggregate system typically has structural features that reflects underlying conservation laws and very often it may be characterized as a port-Hamiltonian (PH) system, see [5, 6, 23, 41, 46, 48, 51, 52, 55, 56, 57, 58, 59, 60] for some major references and [47] for a detailed survey covering also the case of descriptor systems. Although

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PH systems may be formulated in a more general framework, we will restrict ourselves to *input-state-output PH systems*, which have the form

\[
\dot{x} = (J - R) \nabla_x \mathcal{H}(x) + (F - P)u(t),
\]

\[
y(t) = (F + P)^T \nabla_x \mathcal{H}(x) + (S + N)u(t),
\]

where \(x : [0, \infty) \to \mathbb{R}^n\) is the \(n\)-dimensional state vector; \(\mathcal{H} : \mathbb{R}^n \to [0, \infty)\) is the Hamiltonian, a continuously differentiable scalar-valued vector function, describing the distribution of internal energy among the energy storage elements of the system; \(J = -J^T \in \mathbb{R}^{n \times n}\) is the structure matrix describing the energy flux among energy storage elements within the system; \(R = R^T \in \mathbb{R}^{n \times n}\) is the dissipation matrix describing energy dissipation/loss in the system; \(F \pm P \in \mathbb{R}^{n \times m}\) are the port matrices, describing the manner in which energy enters and exits the system, and \(S + N\), with \(S = S^T \in \mathbb{R}^{m \times m}\) and \(N = -N^T \in \mathbb{R}^{m \times m}\), describing the direct feed-through of input to output. The matrices, \(R, P,\) and \(S\) must satisfy

\[
K = \begin{bmatrix}
R & P \\
P^T & S
\end{bmatrix} \geq 0; \quad (2)
\]

that is, \(K\) is symmetric positive-semidefinite. This implies, in particular, that \(R\) and \(S\) are also positive semidefinite, \(R \geq 0\) and \(S \geq 0\).

Port-Hamiltonian systems generalize the classical notion of Hamiltonian systems expressed in our notation as \(\dot{x} = J V \nabla \mathcal{H}(x)\). The analog of the conservation of energy for Hamiltonian systems is for PH systems [1], the dissipation inequality:

\[
\mathcal{H}(x(t_1)) - \mathcal{H}(x(t_0)) \leq \int_{t_0}^{t_1} y(t)^T u(t) \, dt, \quad (3)
\]

which has a natural interpretation as asserting that the increase in internal energy of the system, as measured by \(\mathcal{H}\), cannot exceed the total work done on the system. \(\mathcal{H}(x)\) is a storage function associated with the supply rate, \(y(t)^T u(t)\). In the language of system theory, \([3]\) constitutes the property that \([1]\) is a passive system \([18]\).

One may verify with elementary manipulations that the inequality in \([3]\) is an immediate consequence of the inequality in \([2]\), and holds even when the coefficient matrices \(J, R, F, P, S,\) and \(N\) depend on \(x\) or explicitly on time \(t\) (see, \([41]\)) or, indeed (with care taken to define suitable operator domains), when they represent linear operators acting on infinite dimensional spaces \([32, 58]\). Notice that with a null input, \(u(t) = 0\), the dissipation inequality asserts that \(\mathcal{H}(x)\) is non-increasing along any unforced system trajectory. Thus, \(\mathcal{H}(x)\) defines a Lyapunov function for the unforced system, so PH systems are implicitly Lyapunov stable \([30]\). Similarly, \(\mathcal{H}(x)\) is non-increasing along any system trajectory that produces a null output, \(y(t) = 0\), so PH systems also have Lyapunov stable zero dynamics \([17]\).

PH systems constitute a class of systems that is closed under power-conserving interconnection. This means that port-connected PH systems produce an aggregate system that must also be port-Hamiltonian. This aggregate system will then be guaranteed to be both stable and passive. Modeling with PH systems, thus, represents physical properties in such a way as to facilitate automated modeling \([34]\) while encoding physical properties explicitly into the structure of the equations. This framework also provides a compelling motivation to identify and preserve PH structure whenever it is present in order to produce high quality reduced order surrogate models, see \([3, 27, 53]\).
Remark 1 In [6, 46, 17] the class of PH systems has been extended to include input-state-output PH descriptor (PHDAE) systems, which have the form

\[ \begin{align*}
    &\dot{E}x = (J - R)e + (F - P)u(t), \\
    &y(t) = (F + P)^T e + (S + N)u(t).
\end{align*} \tag{4} \]

Here we have taken \( \nabla_x \mathcal{H}(x) = ETe(x) \), with an auxiliary vector function \( e : \mathbb{R}^n \to \mathbb{R}^n \) and where \( E = ET \in \mathbb{R}^{n \times n} \) may be singular (allowing then for the incorporation of algebraic constraints). For the moment, we focus on the case that \( E \) is nonsingular and without loss of generality can be taken to be the identity matrix.

Consider a general linear time-invariant (LTI) system:

\[ \begin{align*}
    \dot{x} &= Ax + Bu, \\
    y &= Cx + Du, 
\end{align*} \tag{5} \]

with \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}, \) and \( D \in \mathbb{R}^{m \times m} \). Following the previous discussion leading to (3), the system (5) is passive if there exists a continuously differentiable storage function \( \mathcal{H} : \mathbb{R}^n \to [0, \infty) \) such that (3) holds for all admissible inputs \( u \) [63].

Two natural questions arise: When is (5) equivalent to a port-Hamiltonian (descriptor) system? What is the set of equivalence transformations?

Here specifically, we consider the potential for equivalence to LTI port-Hamiltonian systems taking the form

\[ \begin{align*}
    \dot{\Xi} &= (J - R)Q \Xi + (F - P) \varphi, \\
    \eta &= (F + P)^T Q \Xi + (S + N) \varphi, 
\end{align*} \tag{6} \]

with \( J = -J^T, R = R^T \geq 0, Q = QT > 0, S = ST \geq 0, N = -NT, \) where \( J, R, Q \in \mathbb{R}^{n \times n}, F, P \in \mathbb{R}^{n \times m}, S, N \in \mathbb{R}^{m \times m}, \) and \( K \) as defined in [2] is positive semidefinite.

Remark 2 Note that by introducing \( \zeta = Q \Xi \) and \( E = Q^{-1} \), we could alternatively discuss an equivalent descriptor formulation

\[ \begin{align*}
    \dot{\zeta} &= (J - R) \zeta + (F - P) \varphi, \\
    \eta &= (F + P)^T \zeta + (S + N) \varphi. 
\end{align*} \tag{7} \]

The notion of system equivalence that we consider here engages three invertible transformations connecting (3) and (6), one on each of the input, output, and state space:

\[ \begin{align*}
    u(t) &= \tilde{V} \varphi(t), \quad \eta(t) = V^T y(t), \quad \text{and} \quad x(t) = T^{-1} \Xi(t) \quad (\text{with } \tilde{V}, V, T \text{ invertible}).
\end{align*} \]

Within this context, the supply rate associated with (5) is transformed as

\[ y(t)^T u(t) = \eta(t)^T V^{-1} \tilde{V} \varphi(t). \]

We wish to constrain the permissible transformations characterizing system equivalence so as to be power conserving; that is, so that supply rates remain invariant, i.e. \( y(t)^T u(t) = \eta(t)^T \varphi(t) \). Thus, we assume that \( \tilde{V} = V \) and we say that (5) is equivalent to a system of the form (6) if there exist invertible matrices, \( V \) and \( T \), such that

\[ \begin{align*}
    u(t) &= V \varphi(t), \quad \eta(t) = V^T y(t), \quad \text{and} \quad x(t) = T^{-1} \Xi(t). 
\end{align*} \tag{8} \]
Since PH systems are structurally passive, our starting point is the following characterization of passivity introduced in [63] for minimal linear time invariant systems. The system (5) is minimal if it is both controllable and observable. The system (5) (and more specifically, the pair of matrices \((A, B)\) with \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}\)) is controllable if rank \(\begin{bmatrix} sI - A & B \end{bmatrix} = n\) for all \(s \in \mathbb{C}\). Similarly, the system (5) (and the pair \((A, C)\) with \(A \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{m \times n}\)) is observable if rank \(\begin{bmatrix} sI - A & C \end{bmatrix} = n\) for all \(s \in \mathbb{C}\).

**Theorem 1** ([63]) Assume that the LTI system (5) is minimal. The matrix inequality

\[
\begin{bmatrix}
A^T Q + QA & QB - C^T \\
B^T Q - C & -(D + D^T)
\end{bmatrix} \leq 0
\]  

has a solution \(Q = Q^T > 0\) if and only if (5) is a passive system, in which case:

i) \(H(x) = \frac{1}{2}x^T Q x\) defines a storage function for (5) associated with the supply rate \(y^T u\), satisfying (3).

ii) There exist maximum and minimum symmetric solutions to (9):

\[0 < Q_- \leq Q_+\] such that for all symmetric solutions \(Q\) to (9),

\[Q_- \leq Q \leq Q_+\].

This result has an immediate consequence for PH realizations.

**Corollary 2** Assume that the LTI system (5) is minimal. Then (5) has a PH realization if and only if it is passive. Moreover, if (5) is passive then every equivalent system to (5) (as generated by transformations as in (8)) may also be directly expressed as a PH system.

**Proof.** If a minimal system of the form (5) has a PH realization then a fortiori it is passive. Conversely, if (5) is passive then (9) has a positive definite solution \(\hat{Q} = \hat{Q}^T = T^T T\), written in terms of a Cholesky or “square root factor” \(T\). Then we can define directly

\[
\begin{align*}
Q &= I, \quad J = \frac{1}{2}(TAT^{-1} - (TAT^{-1})^T), \quad R = -\frac{1}{2}(TAT^{-1} + (TAT^{-1})^T) \\
F &= \frac{1}{2}(TB + (CT^{-1})^T), \quad P = \frac{1}{2}(-TB + (CT^{-1})^T), \\
S &= \frac{1}{2}(D + D^T), \quad N = \frac{1}{2}(D - D^T)
\end{align*}
\]

and (9) can be written in terms of these defined quantities as

\[
-2 \begin{bmatrix}
T^T & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
R & P \\
P^T & S
\end{bmatrix}
\begin{bmatrix}
T & 0 \\
0 & I
\end{bmatrix} \leq 0,
\]

which verifies (2). Thus, \(J, R, Q, F, P, S, N\) as defined in (10) will indeed determine a PH system.

**Remark 3** Note that instead of the transformation in (10) we may instead use the descriptor formulation (7) with

\[
\begin{align*}
E &= \hat{Q}, \quad J = \frac{1}{2}(\hat{Q}A - (\hat{Q}A)^T), \quad R = -\frac{1}{2}(\hat{Q}A + (\hat{Q}A)^T), \\
F &= \frac{1}{2}(\hat{Q}B + C^T), \quad P = \frac{1}{2}(-\hat{Q}B + C^T), \\
S &= \frac{1}{2}(D + D^T), \quad N = \frac{1}{2}(D - D^T)
\end{align*}
\]

that avoids the factorization of \(\hat{Q}\) and the similarity transformation with \(T\).
Note that, so far, we have discussed the existence of a positive definite solution \( Q \) of (9) under the assumption of minimality of the system. However, such solutions may also exist in the case that the system is not minimal. A detailed analysis of the intricate relationship between passive systems, port-Hamiltonian descriptor systems and the solvability of the KYP inequality has recently been presented in [19]. This analysis is particularly important when the system matrices arise from an interpolatory realization or a model reduction process where the resulting systems may be non-minimal or otherwise very close approximations of non-minimal systems. In these circumstances, the computation of a PH representation may be very sensitive to small perturbations arising from measurement or round-off errors.

Consider the following example from [5].

**Example 1** The system \( \dot{x} = -x, y = u \) is both stable and passive but not minimal. In this case, the inequality (9) is satisfied with any (scalar) \( X > 0 \), the Hamiltonian may be defined as \( H(x) = \frac{X}{2} x(t)^2 \), and the dissipation inequality evidently holds since, for \( t_1 \geq t_0 \),

\[
H(x(t_1)) - H(x(t_0)) = \frac{X}{2} (x(t_0) e^{-(t_1-t_0)})^2 - \frac{X}{2} (x(t_0))^2 \\
= \frac{X}{2} (x(t_0))^2 (e^{-2(t_1-t_0)} - 1) \leq 0 \leq \int_{t_0}^{t_1} y(t) u(t) \, dt.
\]

In Section 3.4, we will analyze how the conditions for the existence of solutions can be verified using numerical methods and how to calculate solutions to (9).

Note that when a system of the form (5) is generated by an interpolatory realization or other model reduction strategies, it may only be a close approximation to a passive system even when the original system is passive. In this case the inequality (9) might not be solvable, but a solution may exist for an adjacent system that arises from a small perturbation of the coefficients \( A, B, C, D \). How to obtain such a perturbation is an important research topic, see [1, 13, 26] and the references therein. We will construct such a perturbation in our procedure and thereby, (referencing the pH representation) make a nearly passive system passive.

**Remark 4** Since the matrix inequality (9) typically has an infinite number of solutions, an important question is how best to use the freedom in the choice of the solution of (9). A natural goal might be to minimize distances to instability or non-passivity or other robustness measures such as the so-called analytic center of the solution set, see [3, 45]. It is currently a mostly open problem to characterize the solution set of (9) and to analyze different robustness measures, see [50] for partial results.

The matrix inequality (9) implies the **Lyapunov inequality** in the upper left block,

\[
A^T Q + QA \leq 0, \quad (12)
\]

and via Lyapunov’s theorem [40], this guarantees that the unforced system \( \dot{x} = Ax \) is **stable**, i.e., \( A \) has all eigenvalues in the closed left half plane and those on the imaginary axis are semisimple; if the inequality (12) is strict, then the system is **asymptotically stable**, i.e., all eigenvalues of \( A \) are in the open left half plane. Passivity is encoded in the solvability of the full matrix inequality (9); **strict passivity** occurs when the inequality is strict, i.e., if the dissipation inequality (3) is strict, see [35] for a detailed analysis.
Remark 5 In order to characterize the boundary of the solution sets of the LMI s (9) and (12) one needs to study the case when either the inequalities in (9) and (12) are not strict or when the resulting solutions are only semidefinite (or both). Extreme points of the solution set of (9) have recently been characterized in [50].

This paper is organized as follows: We discuss the solution of the Lyapunov and Riccati matrix inequalities in the general situation of non-minimal systems in Section 2 and for the case that the symmetric part of $D$ is not invertible in Section 3. We present procedures to construct explicit transformations mapping a general linear time-invariant system to a port-Hamiltonian form (6) in Sections 3. Computational methods to determine a PH representation from a standard state space system rely heavily on the solution of Lyapunov and Riccati inequalities, see Section 4.

We begin our analysis in the next section by recalling some well-known results for Lyapunov and Riccati inequalities.

2 Lyapunov and Riccati inequalities

The solutions of Lyapunov and Riccati inequalities as they arise in the characterizations (9) and (12) are typically addressed through semidefinite programming, see [11]. Instead, we discuss an explicit characterization associated with invariant subspaces.

2.1 Solution of Lyapunov inequalities.

The stability of $A$ is a necessary condition for (12) and (9) which require that $T^T A^T T \preceq 0$, or equivalently, that the Lyapunov inequality (12) has a positive definite solution $Q = T^T T$. It is well known that the equality case in (12) always has a positive definite solution if $A$ is stable, see [40]. In the following we recall, see e.g. [11], a characterization of the complete set of solutions of the inequality case.

If $A$ is stable, but not asymptotically stable, then due to the fact that the eigenvalues on the imaginary axis are all semi-simple, using the real Jordan form of $A$, see e.g. [31], there exist a nonsingular matrix

$$M \in \mathbb{R}^{n \times n}$$

such that

$$M A M^{-1} = \text{diag} (A_1, \alpha_2 J_2, \ldots, \alpha_r J_r),$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$ is asymptotically stable, $\alpha_2, \ldots, \alpha_r \geq 0$ are real and distinct, and $J_j = \begin{bmatrix} 0 & I_{n_j} \\ -I_{n_j} & 0 \end{bmatrix}$, $j = 2, \ldots, r$. To characterize the solution set of (12), we make the ansatz that

$$Q = M^T \text{diag} (Q_1, \hat{Q}_2, \ldots, \hat{Q}_r) M,$$

and separately consider the determination of the block $Q_1$ and the other blocks. Let $\mathcal{W}(n_1)$ be the set of symmetric positive semidefinite matrices $\Theta_1 \in \mathbb{R}^{n_1 \times n_1}$ with the property that $\Theta_1 x \neq 0$ for any eigenvector $x$ of $A_1$. Then for any $\Theta_1 \in \mathcal{W}(n_1)$ we define $Q_1$ to be the unique symmetric positive definite solution of the Lyapunov equation $A_1^T Q_1 + Q_1 A_1 = -\Theta_1$, see [40]. The other matrices $\hat{Q}_j$, $j = 2, \ldots, r$ are chosen of the form

$$\hat{Q}_j = \begin{bmatrix} Y_j & Z_j \\ -Z_j & Y_j \end{bmatrix} > 0,$$
with $Z_j = -Z_j^T$, when $\alpha_j > 0$ or an arbitrary $\hat{Q}_j > 0$ when $\alpha_j = 0$.

We have the following characterization of the solution set of (12).

**Lemma 3** Let $A \in \mathbb{R}^{n \times n}$. Then the Lyapunov inequality (12) has a symmetric positive definite solution $Q \in \mathbb{R}^{n \times n}$ if and only if $A$ is stable.

If $A$ is asymptotically stable, then the solution set is given by the set of all symmetric positive definite solutions of the Lyapunov equation $A^TQ + QA = -\Theta$, where $\Theta$ is any symmetric positive semidefinite matrix with the property that $\Theta x \neq 0$ for any eigenvector $x$ of $A$, or in other words $(A, \Theta)$ is observable.

If $A$ is stable, but not asymptotically stable, then with the transformation (13), any solution of (12) has the form (13) solving the Lyapunov equation

$$A^TQ + QA = -M^T \text{diag}(\Theta_1, 0, \ldots, 0)M$$

with $\Theta_1 \in \mathcal{W}(n_1)$.

**Proof.** Consider a transformation to the form (13) and set, for a symmetric matrix $Q$, $M^{-T}QM^{-1} = [Q_{ij}]$ partitioned accordingly. Form

$$M^{-T}(A^TQ + QA)M^{-1} = \begin{bmatrix}
A^T_{11}Q_{11} + Q_{11}A_{11} & A^T_{12}Q_{12} + \alpha_2Q_{12}J_{22} & \cdots & A^T_{1r}Q_{1r} + \alpha_rQ_{1r}J_{rr} \\
\alpha_2J^T_2Q_{12} + Q_{12}^TJ_{12} & A^T_2Q_{22} + \alpha_2J^T_2Q_{22} + \alpha_2J^T_2Q_{22} & \cdots & \alpha_2J^T_2Q_{2r} + \alpha_2Q_{2r}J_{rr} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_rJ^T_rQ_{1r} + Q_{1r}^TJ_{1r} & \alpha_rJ^T_rQ_{2r} + \alpha_2Q_{2r}J_{rr} & \cdots & \alpha_rJ^T_rQ_{rr} + \alpha_rQ_{rr}J_{rr}
\end{bmatrix},$$

again partitioned accordingly. For any $j = 2, \ldots, r$ with $\alpha_r \neq 0$ partition

$$Q_{jj} = \begin{bmatrix}
Q_{j,11}^T & Q_{j,12}^T \\
Q_{j,12}^T & Q_{j,22}^T
\end{bmatrix}$$

according to the block structure in $J_j$, so that

$$J_{j}^TQ_{jj} + Q_{jj}J_{j} = \begin{bmatrix}
-Q_{j,12}^T - Q_{j,12} & Q_{j,11} - Q_{j,22} \\
Q_{j,11} - Q_{j,22} & Q_{j,12} + Q_{j,12}
\end{bmatrix}. $$

Then, $\alpha_j(J_j^TQ_{jj} + Q_{jj}J_{j}) \leq 0$ if and only if $Q_{j,12}^T + Q_{j,12}^T = 0$ and it follows that $Q_{j,11} - Q_{j,22} = 0$. Thus any matrix of the form $Q_{jj} = \begin{bmatrix} Y_j & Z_j \\
-Z_j & Y_j\end{bmatrix} > 0$ with $Z_j = -Z_j^T$, is a positive definite solution. If $\alpha_j = 0$, then clearly any $Q_{jj} > 0$ is a solution.

In both cases the resulting diagonal blocks $Q_{jj} > 0$ satisfy

$$\alpha_j(J_j^TQ_{jj} + Q_{jj}J_{j}) = 0.$$

Thus, for $i \neq j$, the blocks $Q_{ij}$ have to satisfy one of the two matrix equations

$$A^T_iQ_{1j} + \alpha_jQ_{1j}J_{j} = 0, \quad \alpha_iJ^T_iQ_{ij} + \alpha_jQ_{ij}J_{j} = 0,$$

which implies that $Q_{ij} = 0$ for all $i \neq j$, since the spectra of the respective pairs $A_1, \alpha_1J_{j}$ and $\alpha_iJ_{j}, \alpha_jJ_{j}$ are different and thus the corresponding Sylvester equations only have 0 as a solution (10).
For any positive semidefinite matrix $\Theta_1$ satisfying $\Theta_1 x \neq 0$ for any eigenvector $x$ of $A_1$, by Lyapunov’s Theorem [10], there exists a symmetric positive definite matrix $Q_1$ satisfying $A_1^T Q_1 + Q_1 A_1 = -\Theta_1$. Thus, the Lyapunov inequality [12] has a solution $Q > 0$ and it has the form (14) satisfying (15). The results for an asymptotically stable $A$ can be easily derived as a special case with all $\hat{Q}_2, \ldots, \hat{Q}_r$ void in (14).

Conversely, if the Lyapunov inequality has a solution $Q > 0$, and if $x \neq 0$ is an eigenvector of $A$, i.e., $Ax = \lambda x$, then
\[(\bar{\lambda} + \lambda) x^T Q x \leq 0.\]

Since $x^T Q x > 0$, it follows that $\bar{\lambda} + \lambda \leq 0$, i.e., the real part of $\lambda$ is non-positive. Suppose that $A$ has a purely imaginary eigenvalue $i\alpha$. Since a similarity transformation of $A$ does not change the Lyapunov inequality (12), we may assume that $A$ is in (complex) Jordan canonical form, and we may consider each Jordan block separately. If $A_k$ is a single real Jordan block of size $k \times k$, $k > 1$ with eigenvalue $i\alpha$, then for any Hermitian matrix $\hat{Q}_k = [q_{ij}]$ we have that
\[
A_k^T Q_k + Q_k A_k = \begin{bmatrix}
0 & q_{11} & \cdots & q_{1,k-1} \\
q_{11} & q_{12} + \bar{q}_{12} & \cdots & q_{1,k+q_{2,k-1}} \\
\vdots & \vdots & \ddots & \vdots \\
q_{1,k-1} & q_{2,k-1} + \bar{q}_{1,k} & \cdots & q_{k-1,k+q_{k-1,k}}
\end{bmatrix}.
\]

This matrix cannot be negative semidefinite, since otherwise $q_{11} = 0$, which would contradict that $Q_k$ is positive definite.

Therefore, $A$ has to be stable, i.e., if it has purely imaginary eigenvalues, these eigenvalues must be semi-simple.

**Remark 6** The construction in Lemma 3 relies on the computation of the real Jordan form of $A$, which usually cannot be computed in finite precision arithmetic. For the numerical computation of the solution it is better to use the real Schur form, see [24].

### 2.2 Solution of Riccati inequalities in the case $D + D^T > 0$.

By Corollary 2 we can characterize (at least in the minimal case) the existence of a transformation to PH form via the existence of a symmetric positive definite matrix $Q$ solving the Kalman-Yakubovich-Popov (KYP) linear matrix inequality
\[
\begin{bmatrix}
A^T Q + Q A & Q B - C^T \\
B^T Q - C & -(D + D^T)
\end{bmatrix} \leq 0.
\]

(16)

It is clear that for (16) to hold, $\hat{S} := \frac{1}{2}(D + D^T)$ has to be positive semidefinite but let us first consider the case that $\hat{S} > 0$. The case that $\hat{S}$ is singular will be discussed in Section 3.

Under this condition we now discuss the solvability of (16) in the general case that the system may be either non-controllable or non-observable. For this we will have to identify the controllable and observable subsystems.

Since $\hat{S} > 0$, by using Schur complements, we have that (16) is equivalent to the Riccati inequality
\[
(A - B \hat{S}^{-1} C)^T Q + Q (A - B \hat{S}^{-1} C) + Q B \hat{S}^{-1} B^T Q + C^T \hat{S}^{-1} C \leq 0.
\]

(17)
To study the solvability of (17), we first investigate the influence of the purely imaginary eigenvalues of $A$ on the solvability of (16) and (17). Clearly, a necessary condition for (16) to be solvable is that $Q$ satisfies $A^T Q + QA \preceq 0$. Following Lemma 3, if $A$ has the form (13), then $Q$ must have the form (14), and $A^T Q + QA$ has the form (15). Written in compact form, we get
\[
MAM^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad M^{-T}QM^{-1} = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix},
\]
where
\[
A_2 = \text{diag}(\alpha_2 J_2, \ldots, \alpha_r J_r) = -A_2^T, \quad Q_2 = \text{diag}(Q_2, \ldots, Q_r),
\]
satisfies $A_2^T Q_2 + Q_2 A_2 = 0$, and
\[
M^{-T}(A^T Q + QA)M^{-1} = \begin{bmatrix} A_1^T Q_1 + Q_1 A_1 & 0 \\ 0 & 0 \end{bmatrix}.
\]
Setting
\[
MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CM^{-1} = \begin{bmatrix} C_1 & C_2 \end{bmatrix},
\]
and premultiplying $M^{-T}$ and post-multiplying $M^{-1}$ to the first block row and column of (16), respectively, one has that
\[
\begin{bmatrix} A_1^T Q_1 + Q_1 A_1 & 0 & Q_1 B_1 - C_1^T \\ 0 & 0 & Q_2 B_2 - C_2^T \\ B_1^T Q_1 - C_1 & B_2^T Q_2 - C_2 & -\hat{S} \end{bmatrix} \preceq 0.
\]
Therefore, to have a positive definite solution of (17), $Q_2$ must be positive definite satisfying
\[
B_1^T Q_2 = C_2, \quad A_2^T Q_2 + Q_2 A_2 = 0,
\]
and $Q_1$ must be a positive definite solution of the further reduced matrix inequality
\[
\begin{bmatrix} A_1^T Q_1 + Q_1 A_1 & Q_1 B_1 - C_1^T \\ B_1^T Q_1 - C_1 & -\hat{S} \end{bmatrix} \preceq 0
\]
or equivalently $Q_1$ has to satisfy the *reduced Riccati inequality*
\[
\Psi(Q_1) := (A_1 - B_1 \hat{S}^{-1} C_1)^T Q_1 + Q_1 (A_1 - B_1 \hat{S}^{-1} C_1) + Q_1 B_1 \hat{S}^{-1} B_1^T Q_1 + C_1^T \hat{S}^{-1} C_1 \preceq 0.
\]
To study the solvability of (20), we recall that by construction, $A_1$ is asymptotically stable. We claim that $A_1 - B_1 \hat{S}^{-1} C_1$ is necessarily asymptotically stable as well. To show this, suppose that the inequality (20) has a solution $Q_1 > 0$. Since $Q_1 B_1 \hat{S}^{-1} B_1^T Q_1 + C_1^T \hat{S}^{-1} C_1 \geq 0$, it follows from Lemma 3 that $A_1 - B_1 \hat{S}^{-1} C_1$ is stable, and there exists an invertible matrix $M_1$ such that $M_1 (A_1 - B_1 \hat{S}^{-1} C_1) M_1^{-1} = \text{diag}(\hat{A}_1, \alpha_2 J_2, \ldots, \alpha_r J_r)$ is in real Jordan form as in (13), where $\hat{A}_1$ is asymptotically stable, $M_1^{-T} Q_1 M_1^{-1}$ has the form (14) and following (15), we have
\[
M_1^{-T} (Q_1 B_1 \hat{S}^{-1} B_1^T Q_1 + C_1^T \hat{S}^{-1} C_1) M_1^{-1} = \text{diag}(\hat{\Theta}, 0, \ldots, 0) \geq 0.
\]
Then, due to the positive definiteness of \( \hat{S} \) it follows that \( C_1 M_1^{-1} = \begin{bmatrix} C_{11} & 0 & \ldots & 0 \end{bmatrix} \), and by making use of the block diagonal structure of \( M_1^{-T} Q_1 M_1^{-1} \), we also have \( M_1 B_1 = \begin{bmatrix} B_{11}^T & 0 & \ldots & 0 \end{bmatrix}^T \). Thus it follows that

\[
M_1 A_1 M_1^{-1} = \text{diag}(\hat{A}_{11} + B_1 \hat{S}^{-1} C_{11}, \alpha_2 J_2, \ldots, \alpha_r J_r).
\]

Since \( A_1 \) is asymptotically stable, all \( \alpha_j J_j \) must be void, which implies that \( A_1 - B_1 \hat{S}^{-1} C_1 \) must be asymptotically stable as well.

In order to characterize the solution of the reduced Riccati inequality \( (20) \), we first have to identify what happens if the system is not minimal. To numerically check minimality, we can use the orthogonal version of the Kalman decomposition, \cite{33, 61}, see also \cite{49}.

**Lemma 4** Consider a general system of the form \( (5) \). Then there must exist a real orthogonal matrix \( U \) such that

\[
U^T A U = \begin{bmatrix}
\hat{A}_{11} & 0 & \hat{A}_{13} \\
\hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} \\
0 & 0 & \hat{A}_{33}
\end{bmatrix} =: \begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} \\
\hat{A}_{13} & \hat{A}_{22} \\
0 & \hat{A}_{33}
\end{bmatrix}
\]

\[
(21)
\]

\[
U^T B = \begin{bmatrix}
\hat{B}_1 \\
\hat{B}_2 \\
0
\end{bmatrix} =: \begin{bmatrix}
\hat{B}_1 \\
\hat{B}_2 \\
0
\end{bmatrix}, \quad CU = \begin{bmatrix}
\hat{C}_1 & 0 & \hat{C}_3
\end{bmatrix} =: \begin{bmatrix}
\hat{C}_1 & \hat{C}_2
\end{bmatrix},
\]

where the pairs \((\hat{A}_{11}, \hat{B}_1)\) and \((\hat{A}_{11}, \hat{B}_1)\) are controllable and the pair \((\hat{A}_{11}, \hat{C}_1)\) is observable.

**Proof.** Applying first the controllability and then the observability staircase forms of \cite{61} to the system, there exists a real orthogonal matrix \( U \) such that \( U^T A U \) is in the form \( (21) \) with \((\hat{A}_{11}, \hat{B}_1)\) controllable and \((\hat{A}_{11}, \hat{C}_1)\) observable. Using the block structure in Lemma 4 we have that \((\hat{A}_{11}, \hat{B}_1)\) is controllable as well. \( \square \)

The next lemma considers the Riccati inequality \( (20) \), where now the coefficients are transformed into the form found in \( (21) \).

**Lemma 5** For the reduced Riccati inequality \( (20) \), suppose that both \( A_1 \) and \( A_1 - B_1 \hat{S}^{-1} C_1 \) are asymptotically stable, and that \( \hat{S} > 0 \). Then there must exist an invertible transformation to the condensed form \( (21) \) of \( A_1, B_1, C_1 \), such that

\[
U^T(A_1 - B_1 \hat{S}^{-1} C_1) U = \begin{bmatrix}
\hat{A}_{11} - B_1 \hat{S}^{-1} \hat{C}_1 & \hat{A}_{12} - B_1 \hat{S}^{-1} \hat{C}_2 \\
0 & \hat{A}_{22}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A_{11} - B_1 S^{-1} C_1 & 0 & A_{13} - B_1 S^{-1} C_3 \\
A_{21} - B_2 S^{-1} C_1 & A_{22} & A_{23} - B_2 S^{-1} C_3 \\
0 & 0 & A_{33}
\end{bmatrix}, \quad (22)
\]

where \( A_{11} - B_1 \hat{S}^{-1} C_1, A_{22}, \) and \( A_{33} \) are all asymptotically stable. In addition, \( A_{11} \) also is asymptotically stable, \((\hat{A}_{11}, \hat{B}_1)\) and \((A_{11}, B_1)\) are controllable, and \((A_{11}, C_1)\) is observable.
Proof. The proof is straightforward and omitted. □

By the previous lemma we can check controllability and observability but these properties can also be read off from Lagrangian invariant subspaces (if they exist) of certain Hamiltonian matrices associated with the equality case in (17), respectively (20). In particular, we have

**Lemma 6** Suppose that $\hat{S} > 0$ and that the Hamiltonian matrix

$$H := \begin{bmatrix} A - BS^{-1}C & BS^{-1}B^T \\ -C^T S^{-1}C & -(A - BS^{-1}C)^T \end{bmatrix}$$ (23)

has a Lagrangian invariant subspace, i.e., there exist square matrices $W_1, W_2, Z$ such that $W_1^T W_2 = W_2^T W_1$, $[W_1^T W_2^T]^T$ has full column rank, and

$$H \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} Z.$$ (24)

If the pair $(A, B)$ is controllable then $W_1$ is invertible, and if $(A, C)$ is observable then $W_2$ is invertible.

Proof. Suppose that there exist $W_1$ and $W_2$ satisfy (24). Without loss of generality we may assume that $[W_1^T W_2^T]^T$ has orthogonal columns. Then it is well known, [14, 15, 53] that there exists a CS decomposition, i.e., there exist orthogonal matrices $U, V$ and diagonal matrices $\Delta, \Gamma$ such that

$$U^T W_1 V = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \quad U^T W_2 V = \begin{bmatrix} \Gamma & 0 \\ 0 & I \end{bmatrix},$$

where $\Delta$ is invertible, and $\Delta^2 + \Gamma^2 = I$. (Note that the zero block in $U^T W_1 V$ may be void.)

Partition

$$U^T(A - BS^{-1}C)U = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad U^TB S^{-1}B^T U = \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^T & G_{22} \end{bmatrix},$$

$$U^TC^TS^{-1}CU = \begin{bmatrix} E_{11} & E_{12} \\ E_{12}^T & E_{22} \end{bmatrix}, \quad V^TZV = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix}.$$ (25)

Then (24) takes the form

$$\begin{bmatrix} A_{11} & A_{12} & G_{11} & G_{12} \\ A_{21} & A_{22} & G_{12}^T & G_{22} \\ -E_{11} & -E_{12} & -A_{11}^T & -A_{12}^T \\ -E_{12}^T & -E_{22} & -A_{12}^T & -A_{22} \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}. \quad (25)$$

By comparing the (2,2) block on both sides one has $G_{22} = 0$. Due to the positive definiteness of $\hat{S}$ and thus the positive semi-definiteness of $U^TB S^{-1}B^T U$, then also $G_{12} = 0$. By comparing the (1,2) blocks on both sides of (25) and using the nonsingularity of $\Delta$, it follows that $Z_{12} = 0$. Then by comparing the (3,2) block on both sides of (25) one has $A_{21} = 0$. Hence,

$$U^T(A - BS^{-1}C)U = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad U^TB S^{-1}B^T U = \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$
implying that \((A - BS^{-1}C, B)\) is not controllable. On the other hand \((A - BS^{-1}C, B)\) is controllable if and only if \((A, B)\) is controllable due to the relation

\[
\begin{bmatrix}
  sI - (A - BS^{-1}C) & B \\
  sI - A & I - S^{-1}C
\end{bmatrix} = \left[
\begin{array}{cc}
  I & 0 \\
  S^{-1}C & I
\end{array}
\right].
\]

With an analogous proof, \(W_2\) is invertible when \((A, C)\) is observable. 

Note that \(W_1\) may be still invertible even if \((A, B)\) is not controllable, for instance, when \(B = 0\) and \(C = 0\). The same applies to \(W_2\).

The following lemma is also well-known, see e.g. [39, 43], but we present a proof in our notation.

**Lemma 7** Suppose that \((A, B)\) is controllable, \((A, C)\) is observable, \(A\) is asymptotically stable, and \(S > 0\). Then the Riccati equation

\[
(A - BS^{-1}C)^TQ + Q(A - S^{-1}C) + QBS^{-1}B^TQ + C^T S^{-1}C = 0
\]

has a solution \(Q > 0\) if and only if the Hamiltonian matrix in (23) has a Lagrangian invariant subspace satisfying (24). When such a Lagrangian invariant subspace exists, \(Q = W_2 W_1^{-1} > 0\) solves (26).

**Proof.** Suppose that \(Q > 0\) solves (26). It is straightforward to prove that \([W_1^T \ W_2^T]^T = \begin{bmatrix} I & Q \end{bmatrix}\) satisfies the required conditions and (24) with \(Z = A + BS^{-1}(B^TQ - C)\).

Suppose there are \(W_1, W_2,\) and \(Z\) satisfying (24). by Lemma 7 both \(W_1\) and \(W_2\) are invertible. Define \(Q = W_2 W_1^{-1}\), which is invertible and also symmetric because of the relation \(W_1^T W_2 = W_2^T W_1\). From (24), we then obtain

\[
H \begin{bmatrix} I \\ Q \end{bmatrix} = \begin{bmatrix} I \\ Q \end{bmatrix} (W_1 Z W_1^{-1}),
\]

which implies that \(Q\) solves (26). The positive definiteness of \(Q\) follows from the fact that \(Q\) is invertible, \(A - BS^{-1}C\) is asymptotically stable, and \(QBS^{-1}B^TQ + C^T S^{-1}C \geq 0\). 

**Remark 7** By Lemma 7 we have seen that if the pair \((A, C)\) is not observable then the matrix \(W_2\) in (24) may or may not be invertible. Hence, if \(W_1\) is invertible and \(W_2\) is not, then a symmetric solution to (26) still exits but is only positive semidefinite. This gives a characterization of the boundary of the solution set of the matrix inequality (16), see also [19] for a detailed discussion of the existence of solution to the matrix inequality (16) if the controllability and observability conditions are violated.

Lemma 7 shows that under the conditions of controllability and observability the Riccati equation (26) has a solution \(Q > 0\) whenever the Hamiltonian matrix \(H\) has a Lagrangian invariant subspace. The existence of such an invariant subspace depends only on the purely imaginary eigenvalues of \(H\), e.g. [22]. When such an invariant subspace exists, then there are many such invariant subspaces. Note that the eigenvalues of \(H\) are symmetric with respect to the imaginary axis in the complex plane. If (24) holds, then the union of the eigenvalues of \(Z\) and \(-Z^T\) form the spectrum of \(H\). One particular choice is that the spectrum of \(Z\) is in the closed left half complex plane, another choice is that it is in the closed right half complex plane. The two corresponding solutions of the Riccati equation (26) are the minimal solution \(Q_-\) and the maximal solution \(Q_+\) and all other solutions of the Riccati equation lie (in the Loewner ordering of symmetric matrices) between these extremal solutions.
Example 2 Consider the example $B = \hat{S} = 1$, $C = -1$, $A = -1 - \alpha$, where $\alpha > 0$. So $A$ and $A - BS^{-1}C = -\alpha$ are both asymptotically stable. The Riccati equation (26) is $q^2 - 2aq + 1 = 0$, which does not have a real positive semidefinite solution when $\alpha \in (0, 1)$. If $\alpha = 1$, then it has a unique solution $q = 1$ associated with $Z = 0$. If $\alpha > 1$, then it has two solutions $\alpha \pm \sqrt{\alpha^2 - 1} > 0$ with $Z = \pm \sqrt{\alpha^2 - 1}$.

Suppose that $A_1, B_1, C_1$ from (20) have been transformed via an orthogonal matrix $U$ into the form (21). Partition

$$U^TQ_1U = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{12}^T & \tilde{Q}_{22} \end{bmatrix}.$$ 

The reduced Riccati inequality (20) then is equivalent to

$$U^T\Psi(Q_1)U = \begin{bmatrix} \Psi_{11}(Q_1) & \Psi_{12}(Q_1) \\ \Psi_{12}(Q_1)^T & \Psi_{22}(Q_1) \end{bmatrix} \leq 0,$$  

(27)

where (suppressing arguments for compactness)

$$\Psi_{11} = (\hat{A}_{11} - \hat{B}_1\hat{S}^{-1}\hat{C}_1)^T\tilde{Q}_{11} + \tilde{Q}_{11}(\hat{A}_{11} - \hat{B}_1\hat{S}^{-1}\hat{C}_1) + \tilde{Q}_{11}\hat{B}_1\hat{S}^{-1}\hat{B}_1^T\tilde{Q}_{11} + \hat{C}_1^T\hat{S}^{-1}\hat{C}_1, \\
\Psi_{12} = (\hat{A}_{11} - \hat{B}_1\hat{S}^{-1}\hat{C}_1 - \hat{B}_1^T\hat{Q}_{11})^T\tilde{Q}_{12} + \tilde{Q}_{12}\hat{A}_{12} + \tilde{Q}_{11}(\hat{A}_{12} - \hat{B}_1\hat{S}^{-1}\hat{C}_2) + \hat{C}_1^T\hat{S}^{-1}\hat{C}_2, \\
\Psi_{22} = \hat{A}_{12}^T\hat{Q}_{22} + \hat{Q}_{22}\hat{A}_{12} + (\hat{A}_{12} - \hat{B}_1\hat{S}^{-1}\hat{C}_2)^T\tilde{Q}_{12} + \tilde{Q}_{12}(\hat{A}_{12} - \hat{B}_1\hat{S}^{-1}\hat{C}_2) + \hat{C}_2^T\hat{S}^{-1}\hat{C}_2.$$ 

For (20) to have a positive definite solution $Q_1 > 0$, it is necessary that

$$\Psi_{11}(Q_1) = \tilde{\Psi}_{11}(\tilde{Q}_{11}) \leq 0$$

has a positive definite solution $\tilde{Q}_{11}$ or equivalently, that the dual Riccati inequality

$$\Phi(\tilde{Y}) := (\hat{A}_{11} - \hat{B}_1\hat{S}^{-1}\hat{C}_1)\tilde{Y} + \tilde{Y}(\hat{A}_{11} - \hat{B}_1\hat{S}^{-1}\hat{C}_1)^T + \tilde{Y}\hat{C}_1^T\hat{S}^{-1}\hat{C}_1\tilde{Y} + \hat{B}_1\hat{S}^{-1}\hat{B}_1^T \leq 0$$

has a solution $\tilde{Y} = \tilde{Q}_{11}^{-1} > 0$. Using the partitioning in (21) for $\tilde{Y} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}$, then one can write this as

$$\Phi(\tilde{Y}) = \begin{bmatrix} \Phi_{11}(\tilde{Y}) & \Phi_{12}(\tilde{Y}) \\ \Phi_{12}(\tilde{Y})^T & \Phi_{22}(\tilde{Y}) \end{bmatrix} \leq 0,$$

where

$$\Phi_{11}(\tilde{Y}) = \Phi_{11}(Y_{11}) = (A_{11} - B_1S^{-1}C_1)Y_{11} + Y_{11}(A_{11} - B_1S^{-1}C_1)^T + Y_{11}C_1^T\hat{S}^{-1}C_1Y_{11} + B_1S^{-1}B_1^T.$$ 

It is necessary that $\Phi_{11}(Y_{11}) \leq 0$ has a solution $Y_{11} > 0$, or equivalently that the dual inequality

$$(A_{11} - B_1S^{-1}C_1)^TQ_{11} + Q_{11}(A_{11} - B_1S^{-1}C_1) + Q_{11}B_1S^{-1}B_1^TQ_{11} + C_1^T\hat{S}^{-1}C_1 \leq 0,$$  

(28)

has a solution $Q_{11} > 0$. This is equivalent to the fact that the equality case in (28) has a positive definite solution, see [11, 39]. Since the corresponding Hamiltonian matrix is

$$H_{11} = \begin{bmatrix} A_{11} - B_1S^{-1}C_1 & B_1S^{-1}B_1^T \\ -C_1^T\hat{S}^{-1}C_1 & -(A_{11} - B_1S^{-1}C_1)^T \end{bmatrix},$$ 

(29)
and since \((A_{11}, B_1, C_1)\) is minimal by Lemma \([7, 28]\) has a solution \(Q_{11} > 0\) if and only if \(H_{11}\) has a Lagrangian invariant subspace. This and the condition that \(A_{11} - B_1S^{-1}C_1\) is asymptotically stable are necessary conditions for the solvability of (19) or (20).

We now show through an explicit construction that these two conditions are also sufficient for the existence of a positive definite solution of (19) or (20). Together with (18), they constitute necessary and sufficient conditions for the solvability of (16) or (17).

The Hamiltonian matrix corresponding to \(\tilde{\Psi}_{11}(Q_{11})\) is

\[
\tilde{H}_{11} = \begin{bmatrix}
\tilde{A}_{11} - \tilde{B}_1S^{-1}\tilde{C}_1 & B_1S^{-1}B_1^T \\
-C_1^T S^{-1}\tilde{C}_1 & -\left(A_{11} - \tilde{B}_1S^{-1}\tilde{C}_1\right)^T
\end{bmatrix}
= \begin{bmatrix}
A_{11} - B_1S^{-1}C_1 & 0 \\
A_{21} - B_2S^{-1}C_1 & A_{22}
\end{bmatrix}
\begin{bmatrix}
B_1S^{-1}B_1^T & B_1S^{-1}B_1^T \\
B_2S^{-1}B_2^T & B_2S^{-1}B_2^T
\end{bmatrix}
- \begin{bmatrix}
-C_1^T S^{-1}C_1 & 0 \\
0 & 0
\end{bmatrix}
- \begin{bmatrix}
-(A_{11} - B_1S^{-1}C_1)^T & -(A_{21} - B_2S^{-1}C_1)^T \\
0 & 0
\end{bmatrix}.
\]

The spectrum of \(\tilde{H}_{11}\) is the union of the spectra of the submatrices \(H_{11}, A_{22}\), and \(-A_{22}\). Since \(A_{22}\) is asymptotically stable, if \(H_{11}\) does not have purely imaginary eigenvalues, so is \(\tilde{H}_{11}\). Hence, \(\tilde{H}_{11}\) has a Lagrangian invariant subspace. Consider the Riccati equation

\[
\tilde{\Psi}_{11} + \tilde{\Xi}_{11} = 0,
\]

with \(\tilde{\Xi}_{11} \geq 0\) being chosen such that \((\tilde{A}_{11} - \tilde{B}_1S^{-1}\tilde{C}_1, \tilde{C}_1^T S^{-1} \tilde{C}_1 + \tilde{\Xi}_{11})\) is observable; it is clear that such a \(\tilde{\Xi}_{11}\) always exists. Recall that \((\tilde{A}_{11} - \tilde{B}_1S^{-1}\tilde{C}_1, \tilde{B}_1)\) is controllable. The Riccati equation (30) corresponds to the Hamiltonian matrix

\[
\tilde{H}_{11}(\tilde{\Xi}_{11}) := \tilde{H}_{11} - \begin{bmatrix}
0 & 0 \\
0 & \tilde{\Xi}_{11}
\end{bmatrix}.
\]

For a sufficiently small (in norm) \(\tilde{\Xi}_{11}\), by continuity, \(\tilde{H}_{11}(\tilde{\Xi}_{11})\) has a Lagrangian invariant subspace (e.g., when \(\tilde{\Xi}_{11}\) is chosen small enough so that no eigenvalues of \(\tilde{H}_{11}(\tilde{\Xi}_{11})\) are on the imaginary axis), and by Lemma \([7, 28]\) \(\tilde{\Psi} = -\tilde{\Xi}_{11} \leq 0\) has a positive definite solution \(Q_{11}\), where \(Q_{11}\) can be chosen so that all the eigenvalues of \(A_{11} - \tilde{B}_1S^{-1}(\tilde{C}_1 - \tilde{B}_1^T \tilde{Q}_{11})\) are in the closed left half complex plane.

If \(H_{11}\) as given in (29) has purely imaginary eigenvalues (potentially including 0) and we assume that it has a Lagrangian invariant subspace, then (28), written with equality, still has a solution \(Q_{11}^0 > 0\) such that all eigenvalues of \(A_{11} - B_1S^{-1}(C_1 - B_1^T Q_{11}^0)\) are in the closed left half complex plane. Let \(\tilde{Q}_{11}^0 = \begin{bmatrix} Q_{11}^0 & 0 \\ 0 & 0 \end{bmatrix} \geq 0\). Then \(\tilde{\Psi}_{11}(\tilde{Q}_{11}^0) = 0\). Subtracting this from (30) yields the Riccati equation

\[
(\tilde{A}_{11}^0)^T \tilde{Y} + \tilde{Y} \tilde{A}_{11}^0 + \tilde{Y} \tilde{B}_1 S^{-1} \tilde{B}_1^T \tilde{Y} + \tilde{\Xi}_{11} = 0,
\]

where \(\tilde{Y} = \tilde{Q}_{11} - \tilde{Q}_{11}^0\) and

\[
\tilde{A}_{11}^0 = \tilde{A}_{11} - \tilde{B}_1S^{-1}(\tilde{C}_1 - \tilde{B}_1^T \tilde{Q}_{11}^0) = \begin{bmatrix} A_{11} - B_1S^{-1}(C_1 - B_1^T Q_{11}^0) & 0 \\ A_{21} - B_2S^{-1}(C_1 - B_1^T Q_{11}^0) & A_{22} \end{bmatrix}.
\]
Since the eigenvalues of \( A_{11} - B_1 \hat{S}^{-1}(C_1 - B_1^T Q_{11}^0) \) are in the closed half complex plane, there is an invertible block lower triangular matrix

\[
\tilde{L} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & I \end{bmatrix}
\]
such that

\[
\tilde{L} \tilde{A}^0_{11} \tilde{L}^{-1} = \tilde{L} \begin{bmatrix} A_{11} - B_1 \hat{S}^{-1}(C_1 - B_1^T Q_{11}^0) & 0 \\ A_{21} - B_2 \hat{S}^{-1}(C_1 - B_1^T Q_{11}^0) & A_{22} \end{bmatrix} \tilde{L}^{-1} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},
\]

where \( \Sigma_1 \) has only purely imaginary eigenvalues and \( \Sigma_2^0 \) is asymptotically stable. The block \( L_{11} \) is used for the similarity transformation for disconnecting the diagonal blocks \( \Sigma_1 \) and \( \Sigma_2^0 \). Repartition

\[
\tilde{L} B_1 = \begin{bmatrix} B_1^0 \\ B_2^0 \end{bmatrix}
\]

according to \( \tilde{L} \tilde{A}^0_{11} \tilde{L}^{-1} \). We look for a solution of (31) of the form

\[
\tilde{L}^{-T} \tilde{Y} \tilde{L}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & Y_2 \end{bmatrix}
\]

By taking a congruence transformation on both sides of (31) with \( \tilde{L}^{-T} \) on the left and \( \tilde{L}^{-1} \) on the right, the resulting equation reduces to

\[
(\Sigma_2^0)^T Y_2 + Y_2 \Sigma_2^0 + Y_2 B_2^0 \hat{S}^{-1}(B_2^0)^T Y_2 + \Xi_{22} = 0
\]

for a suitably chosen \( \Xi_{22} \geq 0 \), and \( \Xi_{11} = \tilde{L}^T \begin{bmatrix} 0 & 0 \\ 0 & \Xi_{22} \end{bmatrix} \tilde{L} \geq 0 \). Since \( (\hat{A}_{11}^0, \hat{B}_1) \) is controllable, so is \( (\Sigma_2^0, B_2^0) \). Recall also that \( \Sigma_2^0 \) is asymptotically stable. Analogous to the previous case one can choose (a sufficiently small) \( \Xi_{22} \geq 0 \) with \( (\Sigma_2^0, \Xi_{22}) \) observable and then the reduced Riccati equation has a positive definite solution \( Y_2 \) with the eigenvalues of \( \Sigma_2^0 + B_2^0 \hat{S}^{-1}(B_2^0)^T Y_2 \) in the closed left half complex plane. Then

\[
\tilde{Q}_{11} = \tilde{Q}_{11}^0 + \tilde{L}^T \begin{bmatrix} 0 & 0 \\ 0 & Y_2 \end{bmatrix} \tilde{L}
\]
solves (30). Since

\[
\tilde{L}(\hat{A} - \hat{B}_1 \hat{S}^{-1}(\hat{C}_1 - \hat{B}_1^T \hat{Q}_{11})) \tilde{L}^{-1} = \tilde{L} \hat{A}^0_{11} \tilde{L} + \tilde{L} B_1 \hat{S}^{-1}(\hat{L} \tilde{B}_1)^T \begin{bmatrix} 0 & 0 \\ 0 & Y_2 \end{bmatrix}
\]

\[
= \begin{bmatrix} \Sigma_1 & B_1^0 \hat{S}^{-1}(B_2^0)^T Y_2 \\ 0 & \Sigma_2^0 + B_2^0 \hat{S}^{-1}(B_2^0)^T Y_2 \end{bmatrix},
\]

15
the eigenvalues of $\tilde{A} - \tilde{B}_1 \tilde{S}^{-1}(\tilde{C}_1 - \tilde{B}_1^T \tilde{Q}_{11})$ are in the closed left half complex plane. Because $Q_{11} > 0$, $\mathbf{Y}_2 > 0$ and

$$
\mathbf{L}^{-T} Q_{11} \mathbf{L}^{-1} = \mathbf{L}^{-T} Q_{11}^0 \mathbf{L}^{-1} + \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{Y}_2 \end{bmatrix} = \begin{bmatrix} L_{11}^T Q_{11}^0 L_{11}^{-1} & 0 \\ 0 & \mathbf{Y}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{Y}_2 \end{bmatrix},
$$

one has $\tilde{Q}_{11} > 0$. Hence, in either case the equation (30) has a positive definite solution $\tilde{Q}_{11}$ with all the eigenvalues of $\tilde{A} - \tilde{B}_1 \tilde{S}^{-1}(\tilde{C}_1 - \tilde{B}_1^T \tilde{Q}_{11})$ in the closed half complex plane for some $\mathbf{Ξ}_{11} > 0$.

Once we have such a solution $\tilde{Q}_{11}$, we can solve $\tilde{\Psi}_{12} = 0$ for $\tilde{Q}_{12}$. This Sylvester equation has a unique solution because $\tilde{A}_{22} = 33$ is asymptotically stable and the eigenvalues of $A_{11} - B_1 S^{-1}(C_1 - B_1^T Q_{11})$ are in the closed left half complex plane.

Having (leaving off arguments) solved the equality $\tilde{\Psi}_{12} = 0$, we finally need to solve the inequality $\tilde{\Psi}_{22} \leq 0$. We may consider the Lyapunov equation

$$
\tilde{\Psi}_{22} = -\mathbf{Ξ}_{22} \leq 0.
$$

Since $\tilde{A}_{22} = 33$ is asymptotically stable, for any $\mathbf{Ξ}_{22} > 0$ it always has a solution $\tilde{Q}_{22}$. Then $Q_1 = U \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{12}^T & \tilde{Q}_{22} \end{bmatrix} U^T$ solves the Riccati equation

$$
\Psi(Q_1) = -U \begin{bmatrix} \mathbf{Ξ}_{11} & 0 \\ 0 & \mathbf{Ξ}_{22} \end{bmatrix} U^T \leq 0.
$$

With the assumption that $A_1 - B_1 S^{-1} C_1$ is asymptotically stable, this $Q_1$ must be positive semidefinite. Suppose that $Q_1 U x = 0$ for some $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0$. The vector $x_2 \neq 0$, since otherwise $\tilde{Q}_{11} x_1 = 0$ contradicting the positive definiteness of $\tilde{Q}_{11}$. From $\Psi(Q_1) = -U \begin{bmatrix} \mathbf{Ξ}_{11} & 0 \\ 0 & \mathbf{Ξ}_{22} \end{bmatrix} U^T$, one has

$$
x^T \begin{bmatrix} \mathbf{Ξ}_{11} & 0 \\ 0 & \mathbf{Ξ}_{22} \end{bmatrix} x = x_1^T \tilde{\mathbf{Ξ}}_{11} x_1 + x_2^T \tilde{\mathbf{Ξ}}_{22} x_2 = 0.
$$

But if we choose $\tilde{\mathbf{Ξ}}_{22} > 0$, this is not possible and thus $Q_1$ must be positive definite. Therefore, choosing a positive definite $\tilde{\mathbf{Ξ}}_{22}$ guarantees the corresponding solution $Q_1$ to be positive definite.

**Remark 8** In the construction of positive definite solutions to (9) that we have described, we have chosen $\tilde{\mathbf{Ξ}}_{11}$ ($\tilde{\mathbf{Ξ}}_{22}$) and $\tilde{\mathbf{Ξ}}_{22}$ in order to guarantee the existence of positive definite solutions of the individual occurring Riccati and Lyapunov equations.

In a more general framework we can choose such a perturbation to turn the inequality in (9) into an equality by adding a positive semidefinite matrix to the left hand side to guarantee
the existence of positive definite solutions. The set of all positive semidefinite perturbations of this kind will then help to characterize the solution set of (9). To do this in detail is beyond the scope of this paper and has recently been investigated in [50].

We summarize the conditions for the existence of a positive definite solution of the matrix inequality (16) in the following theorem.

**Theorem 8** Consider a general system of the form (5) with \( A \) stable and \( \hat{S} = D + D^T > 0 \). Let \( M \) be invertible such that

\[
MAM^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CM^{-1} = \begin{bmatrix} C_1 & C_2 \end{bmatrix},
\]

where \( A_2 \) is diagonalizable and contains all the purely imaginary eigenvalues of \( A \). Let \( A_1, B_1, C_1 \) have the condensed form (22) with an orthogonal matrix \( U \). Then the matrix inequality (16) has a positive definite solution \( Q \) if and only if the following conditions hold.

(a) There exists a positive definite matrix \( Q_2 \) satisfying

\[
B_2^TQ_2 = C_2, \quad A_2Q_2 = Q_2A_2.
\]

(b) The block \( A_{11} - B_1\hat{S}^{-1}C_1 \) is asymptotically stable.

(c) The Hamiltonian matrix \( H_{11} \) defined in (29) has a Lagrangian invariant subspace.

If these conditions hold, then the linear matrix inequality (16) has a positive definite solution of the form

\[
Q = M^T \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} M > 0, \quad \text{where } Q_1 > 0 \text{ solves (20), } Q_2 \text{ is determined from condition (a), and}
\]

\[
(A - B\hat{S}^{-1}C)^TQ + Q(A - B\hat{S}^{-1}C) + QB\hat{S}^{-1}B^TQ + C^T\hat{S}^{-1}C
\]

\[
= -M^T \begin{bmatrix} \Xi_{11} & 0 \\ 0 & \Xi_{22} \end{bmatrix} M \leq 0,
\]

where \( \Xi_{11}, \Xi_{22} \) are chosen as in the above explicit construction.

**Proof.** The proof follows from the explicit construction. \( \Box \)

**Remark 9** Relation (24) is equivalent to the property that the even matrix pencil

\[
\mathcal{N} - \mathcal{M} := \lambda \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ A^T & 0 & C^T \\ B^T & C & \hat{S} \end{bmatrix}
\]

is regular and has index at most one, i.e., the eigenvalues at \( \infty \) are all semi-simple, (this follows because \( \hat{S} > 0 \)) and that it has a deflating subspace spanned by the columns of \( \begin{bmatrix} Q & -I \\ -I & Y^T \end{bmatrix} \), with \( Y = \hat{S}^{-1}(C - B^TQ) \), i.e.,

\[
\begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q \\ -I \\ Y \end{bmatrix} (A - BY) = \begin{bmatrix} 0 & A & B \\ A^T & 0 & C^T \\ B^T & C & \hat{S} \end{bmatrix} \begin{bmatrix} Q \\ -I \\ Y \end{bmatrix}.
\]

\( \Box \)
A pencil $\lambda \mathcal{N} - \mathcal{M}$ is called even if $\mathcal{N} = -\mathcal{N}^T$ and $\mathcal{M} = \mathcal{M}^T$. Since even pencils have the Hamiltonian spectral symmetry in the finite eigenvalues, see [16], this means that there are equally many eigenvalues in the open left and in the open right half plane.

Numerically, to compute $Q$ it is preferable to work with the even pencil (33) rather than with the Hamiltonian matrix (23), since explicit inversion of $\hat{S}$ is avoided. Numerically stable structure preserving methods for this task are available, see [8, 9, 12].

**Remark 10** To show that the system (5) is passive, it is sufficient that the linear matrix inequality (16) has a positive semidefinite solution $Q$. In this case the conditions for the existence of solutions to (9) can be relaxed. First of all, the condition $B_2^T Q_2 = C_2$ can be relaxed to $\text{Ker} B_2 \subseteq \text{Ker} C_2^T$, rank $C_2 B_2 = \text{rank} C_2$, and $C_2 B_2 \geq 0$. Also, $A_2$ may have purely imaginary eigenvalues with Jordan blocks. For example in the extreme case when $C_2 = 0$, $Q_2$ always satisfies the conditions for any $A_2$.

Secondly, for (19) or (20), we still require that $A_{11} - B_1 \hat{S}^{-1} C_1$ is asymptotically stable and that the Hamiltonian matrix $H_{11}$ in (29) has a Lagrangian invariant subspace. Since this only requires that $Q_1 \geq 0$, a solution can be determined in a simpler way. We may simply set $Q_{11} = Q_{11}^0$ and $\hat{\Psi}_{12} = 0$ has a form $\hat{\Psi}_{12} = \begin{bmatrix} Q_{12} \\ 0 \end{bmatrix}$. To solve $\hat{\Psi}_{22} = 0$ for $Q_{22}$, one can show $Q_1 = U \begin{bmatrix} Q_{11} \\ Q_{12} \\ \hat{Q}_{22} \end{bmatrix} U^T \geq 0$ and solves the Riccati equation $\Psi(Q_1) = 0$. Also, in some circumstances the condition that $\hat{A}_{22} = A_{33}$ is asymptotically stable can be relaxed. In these relaxed cases however, it is necessary to transform the system to a descriptor formulation, see [19].

### 3 Construction of port-Hamiltonian realizations

In the last section we have seen that the existence of a port-Hamiltonian realization for (5) reduces to the existence of a nonsingular matrix $T \in \mathbb{R}^{n \times n}$ or a positive definite matrix $Q = T^T T$ such that the matrix inequality (9) holds. Note that since $Q = T^T T$, (9) is equivalent to

$$W + W^T \geq 0,$$

where

$$W = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -A & -B \\ C & D \end{bmatrix} \begin{bmatrix} T \\ 0 \end{bmatrix}^{-1},$$

for an invertible matrix $T$.

We develop here a constructive procedure to check these conditions. In our previous considerations the matrix $V$ that is used for a basis change in the input space need only be invertible, but to implement the transformations in a numerical stable manner, we will require in the following that $V$ be real orthogonal.

#### 3.1 The case that $\hat{S} = D + D^T \geq 0$ is singular.

Suppose first that the matrix $\hat{S} = D + D^T \geq 0$ is singular. Consider an orthogonal matrix $V_0 = [V_{0,1}, V_{0,2}]$, where $V_{0,1}$ is chosen so that its columns form an orthonormal basis of the kernel of $\hat{S}$. To construct such a $V_0$ we can use a singular value or rank-revealing $QR$
decomposition. Then we have
\[ \hat{\mathbf{S}} = \mathbf{V}_0^T \mathbf{D} \mathbf{V}_0 + \mathbf{V}_0^T \mathbf{D}^T \mathbf{V}_0 = \mathbf{V}_0^T (\mathbf{D} + \mathbf{D}^T) \mathbf{V}_0 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{S}_2 \end{bmatrix}, \]
where \( \mathbf{S}_2 = \mathbf{S}_2^T > 0 \). Set
\[ \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{C}_1^T & \mathbf{C}_2^T \end{bmatrix} = \mathbf{B} \mathbf{V}_0, \]
\[ \begin{bmatrix} \mathbf{C}_1^T & \mathbf{C}_2^T \end{bmatrix} := \mathbf{C}^T \mathbf{V}_0, \]
each partitioned compatibly with \( \mathbf{V}_0 \) as in (35).

**Remark 11** If \( \hat{\mathbf{S}} \) is invertible but nearly singular, then the coefficients in (17) may suffer a large relative perturbation. In this case, it is appropriate to regularize the problem by perturbing \( \hat{\mathbf{S}} \) to a nearby positive semidefinite (but singular) problem where the matrix \( \mathbf{S}_2 \) is well conditioned with respect to inversion. In our procedure we do this by setting small positive eigenvalues of \( \hat{\mathbf{S}} \) to zero.

Scaling the second block row and column of the matrix inequality (34) with \( \mathbf{V}_0^T \) and \( \mathbf{V}_0 \) respectively, we obtain the matrix inequality
\[
\begin{bmatrix}
-(\mathbf{T} \mathbf{A} \mathbf{T}^{-1})^T - \mathbf{T} \mathbf{A} \mathbf{T}^{-1} & -\mathbf{B}_1^T + (\mathbf{C}_1^T \mathbf{T}^{-1})^T & -\mathbf{B}_2^T + (\mathbf{C}_2^T \mathbf{T}^{-1})^T \\
-(\mathbf{B}_1)^T + \mathbf{C}_1 \mathbf{T}^{-1} & 0 & 0 \\
-(\mathbf{B}_2)^T + \mathbf{C}_2 \mathbf{T}^{-1} & 0 & \mathbf{S}_2
\end{bmatrix} \geq 0
\]
which has an invertible solution \( \mathbf{T} \) if and only if the matrix inequality
\[
\begin{bmatrix}
-(\mathbf{T} \mathbf{A} \mathbf{T}^{-1})^T - \mathbf{T} \mathbf{A} \mathbf{T}^{-1} & -\mathbf{B}_2^T + (\mathbf{C}_2 \mathbf{T}^{-1})^T \\
-(\mathbf{B}_2)^T + \mathbf{C}_2 \mathbf{T}^{-1} & \mathbf{S}_2
\end{bmatrix} \geq 0
\]
has an invertible solution \( \mathbf{T} \) satisfying the constraint \((\mathbf{B}_1)^T \mathbf{C}_1 \mathbf{T}^{-1} = 0\). We characterize conditions when this constraint is satisfied in the following subsections. But notice that we are, in effect, restricting the input and output space to the invertible part of \( \mathbf{D} + \mathbf{D}^T \). Once these restricted transformation matrices have been constructed, full transformations satisfying the given constraint can be obtained by extending to the full space.

### 3.2 Construction in the case \( \mathbf{D} = -\mathbf{D}^T \)

To explicitly construct the transformation to port-Hamiltonian form let us first discuss the extreme case that \( \hat{\mathbf{S}} = 0 \), i.e., that \( \mathbf{D} = -\mathbf{D}^T \). Considering the matrix \( \mathbf{K} \) in (2), to satisfy \( \mathbf{K} \geq 0 \), we must have \( \mathbf{P} = 0 \), and the block \( \mathbf{V}_{0,2} \) in the transformation of the feedthrough term is void, while \( \mathbf{V}_0 = \mathbf{V}_{0,1} \) is any orthogonal matrix.

**Corollary 9** For a state-space system of the form (5) with \( \mathbf{D} + \mathbf{D}^T = 0 \) the following two statements are equivalent:

1. There exists a change of basis \( \mathbf{x} = \mathbf{T}^{-1} \mathbf{z} \) with an invertible matrix \( \mathbf{T} \in \mathbb{R}^{n \times n} \) such that the resulting realization in the new basis has port-Hamiltonian structure as in (6).
2. There exists an invertible matrix \( \mathbf{T} \) such that
   a) \( (\mathbf{B} \mathbf{T})^T = \mathbf{C} \mathbf{T}^{-1} \) and
   b) \( (\mathbf{T} \mathbf{A} \mathbf{T}^{-1})^T + \mathbf{T} \mathbf{A} \mathbf{T}^{-1} \leq 0 \). (39)
Note that without the constraint (39a), if \( A \) is stable, then by Lemma 3 the second condition (39b) can always be satisfied. Adding the constraint (39a), however, makes the question nontrivial.

We have the following characterization of the transformation matrices \( T \) that satisfy (39a).

**Lemma 10** Consider \( B, C^T \in \mathbb{R}^{n \times m} \), and assume that \( \text{rank} \, B = r \).

a) There exists an invertible transformation \( T \) satisfying condition (39a) if and only if \( \text{Ker} \, C^T = \text{Ker} \, B \), \( \text{rank} \, CB = r \) and \( CB \geq 0 \), or equivalently, there exists an invertible (orthogonal) matrix \( W \) such that

\[
BW = \begin{bmatrix} B_1 & 0 \end{bmatrix}, \quad C^T W = \begin{bmatrix} C_1^T & 0 \end{bmatrix}, \quad C_1 B_1 = YY^T > 0,
\]

where \( B_1, C_1^T \in \mathbb{R}^{n \times r} \) have full column rank and \( Y \in \mathbb{R}^{r \times r} \) is invertible.

b) Let \( N_B \in \mathbb{R}^{n \times (n-r)} \) have columns that form a basis of \( \text{Ker} \, B^T \). If Condition a) is satisfied, then any \( T \) satisfying condition (39a) has the form \( T = UT_Z T_0 \) with

\[
T_0 = \begin{bmatrix} N_B^T \\ Y^{-1} C_1 \end{bmatrix}, \quad T_Z = \begin{bmatrix} Z & 0 \\ 0 & I \end{bmatrix},
\]

where \( U \in \mathbb{R}^{n \times n} \) is an arbitrary orthogonal matrix and \( Z \in \mathbb{R}^{(n-r) \times (n-r)} \) is an arbitrary nonsingular matrix.

**Proof.** Condition (39a) is equivalent to \( C = B^T T^T T \) and the following conditions will be necessary for the existence of an invertible \( T \) with this property:

\[
\text{Ker} \, C^T = \text{Ker} \, T^T TB = \text{Ker} \, B, \quad 0 \leq CB = B^T T^T TB, \quad \text{rank} \, CB = \text{rank} \, B^T T^T TB = \text{rank} \, B = r.
\]

Conversely, observe that \( \text{Ker} \, C^T = \text{Ker} \, B \) is equivalent to the existence of an orthogonal matrix \( W \in \mathbb{R}^{m \times m} \) such that

\[
BW = \begin{bmatrix} B_1 & 0 \end{bmatrix}, \quad C^T W = \begin{bmatrix} C_1^T & 0 \end{bmatrix},
\]

with \( B_1, C_1^T \in \mathbb{R}^{n \times r} \) having full column rank. The conditions \( \text{rank} \, CB = r \) and \( CB \geq 0 \) together are equivalent to \( \tilde{C}_1 \tilde{B}_1 > 0 \). Thus, there must exist an invertible matrix \( Y \) (e.g., a Cholesky factor or a positive-definite square root) such that \( C_1 B_1 = YY^T \).

The matrix \( T_0 \) as in (40) is then well defined. Furthermore, \( T_0 \) is invertible, since if \( T_0 y = 0 \) for some vector \( y \), then \( C_1 y = 0 \) and \( N_B^T y = 0 \). The latter statement implies that \( y \in \text{Ran}(B) \), so \( y = B_1 z \) for some \( z \) and, furthermore, \( C_1 B_1 z = 0 \). This in turn implies that \( z = 0 \) and \( y = 0 \); so \( T_0 \) is injective, and hence invertible.

The invertibility of \( T_0 \) implies that

\[
B^T T_0^T T_0 = W^{-T} \begin{bmatrix} B_1^T \\ 0 \end{bmatrix} \begin{bmatrix} N_B & C_1^T Y^{-T} \end{bmatrix} \begin{bmatrix} N_B^T \\ Y^{-1} C_1 \end{bmatrix} = W^{-T} \begin{bmatrix} (C_1 B_1) (C_1 B_1)^{-1} C_1 \\ 0 \end{bmatrix} = W^{-T} \begin{bmatrix} C_1 \\ 0 \end{bmatrix} = C.
\]
Hence, \([39]a\) holds with \(T = T_0\).

Now suppose that \(T\) is any invertible transformation satisfying \([39]a\). Then,

\[
B^T T^T (TT_0^{-1}) = CT_0^{-1} = B^T T_0^T,
\]

which is equivalent to

\[
B^T T_0^T ((TT_0^{-1})^T (TT_0^{-1}) - I) = [0 \ Y] ((TT_0^{-1})^T (TT_0^{-1}) - I) = 0.
\]

From this, it follows that

\[
(TT_0^{-1})^T (TT_0^{-1}) = \begin{bmatrix} Z^T Z & 0 \\ 0 & I \end{bmatrix}
\]

for some invertible matrix \(Z\), and that \(TT_0^{-1} \begin{bmatrix} Z^{-1} & 0 \\ 0 & I \end{bmatrix} = U\) must be real orthogonal.

In order to explicitly construct a transformation matrix \(T_0\) as in \([40]\), it will be useful to construct bi-orthogonal bases for the two subspaces \(\text{Ker} B^T\) and \(\text{Ker} C\). Toward this end, let \(N_C\) contain columns that form a basis of \(\text{Ker} C\), so that \(\text{Ran} N_C = \text{Ker} C\). Such a matrix is easily constructed in a numerically stable way via the singular value decomposition or a rank-revealing \(QR\) decomposition of \(C\), see \([24]\). Since \(B\) and \(C^T\) are assumed to satisfy \(\text{Ker} B = \text{Ker} C^T\), we have singular value decompositions

\[
B = \begin{bmatrix} U_B,1 & U_B,2 \end{bmatrix} \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} V_B^T, \quad C = U_C \begin{bmatrix} \Sigma_C & 0 \\ 0 & 0 \end{bmatrix} V_C^{T,1} V_C^{T,2}
\]

(43)

with \(\Sigma_B, \Sigma_C \in \mathbb{R}^{r \times r}\) both invertible. We then obtain

\[
N_C = V_C^{T,2}, \quad N_B = U_B,2.
\]

(44)

Observe that \(N_B^T N_C\) is nonsingular, since if \(N_B^T N_C Y = 0\) and \(z = N_C Y\), then \(N_B^T z = 0\) implies that \(z \in \text{Ran} B\). But then, \(z = B_1 w = N_C Y\) implies \(C_1 B_1 w = 0\), where \(B_1\) and \(C_1\) are defined in \([42]\), and so, \(w = 0\) and hence \(z = 0\). Then, since \(N_C\) has full column rank, we have \(Y = 0\). Thus, \(N_B^T N_C\) is injective, hence invertible.

Performing another singular value decomposition, \(N_B^T N_C = \tilde{U} \Delta \tilde{V}^T\), with \(\Delta\) positive diagonal, and \(\tilde{U}, \tilde{V}\) real orthogonal, we can perform a change of basis \(\tilde{N}_B = N_B \tilde{U} \Delta^{-1/2}\) and \(\tilde{N}_C = N_C \tilde{V} \Delta^{-1/2}\) and obtain that the columns of \(\tilde{N}_B\) form a basis for \(\text{Ker} B^T\), the columns of \(\tilde{N}_C\) form a basis for \(\text{Ker} C\) and these two bases are bi-orthogonal, i.e., \(\tilde{N}_B^T \tilde{N}_C = I\), and we have

\[
T_0 = \begin{bmatrix} \tilde{N}_B^T \\ Y^{-1} C_1 \end{bmatrix}, \quad T_0^{-1} = \begin{bmatrix} \tilde{N}_C & B_1 Y^{-T} \end{bmatrix}.
\]

(45)

Note that \(W, B_1, C_1\) in Lemma \([10]\) can be determined by the SVDs in \([43]\).

Using the formula \([40]\), we can express the conditions for a transformation to PH form that we have obtained so far in a more concrete way.

**Corollary 11** Consider system \([5]\) with \(D = -D^T\) and \(\text{rank} B = r\). Let the columns of \(\tilde{N}_B\) and \(\tilde{N}_C\) span the kernels of \(B^T\) and \(C\) and satisfy \(\tilde{N}_B^T \tilde{N}_C = I\). Then system \([5]\) is equivalent to a PH system if and only if

1. \(\text{Ker} C^T = \text{Ker} B\), \(\text{rank} CB = r\), \(CB \geq 0\), and
2. there exists an invertible matrix $Z$ such that
\[
\begin{bmatrix}
Z & 0 \\
0 & I
\end{bmatrix}
T_0 A T_0^{-1}
\begin{bmatrix}
Z^{-1} & 0 \\
0 & I
\end{bmatrix}
+ 
\begin{bmatrix}
Z & 0 \\
0 & I
\end{bmatrix}
T_0 A T_0^{-1}
\begin{bmatrix}
Z^{-1} & 0 \\
0 & I
\end{bmatrix}
\] \leq 0, \tag{46}
\]
and $T_0$, $T_0^{-1}$ are defined in (45).

Proof. The condition follows from Corollary 9 and the representation (40) by setting $U = I$ and $T_0$ as in (45). \qed

3.3 Construction in the case of general $D$

For the case that $D$ is general we will present a recursive procedure which is analogous to the index reduction procedure for differential-algebraic equations in [37]. The first step is to perform the transformations (35), (36), and to obtain the following characterization when a transformation to port-Hamiltonian form (6) exists.

Lemma 12 Consider system (5) transformed as in (35) and (36). Then the system is equivalent to a port-Hamiltonian system of the form (6) if and only if

1. $\ker C_T^1 = \ker B_1$, rank $C_1 B_1 = \text{rank } B_1$, and $C_1 B_1 \geq 0$, or equivalently, there exists an invertible (orthogonal) matrix $W$ such that
\[
B_1 W = \begin{bmatrix}
\hat{B}_1 & 0
\end{bmatrix}, \quad C_T^1 W = \begin{bmatrix}
\hat{C}_1^T & 0
\end{bmatrix}, \quad \hat{C}_1 \hat{B}_1 = YY^T > 0, \tag{47}
\]
where $\hat{B}_1, \hat{C}_1^T \in \mathbb{R}^{n \times r}$ have full column rank and $Y \in \mathbb{R}^{r \times r}$ is invertible, and

2. there exists an invertible matrix $Z$ such that
\[
\hat{Y} + \hat{Y}^T \geq 0, \tag{48}
\]
with
\[
\hat{Y} := \begin{bmatrix}
Z & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
-T_0 A T_0^{-1} & -T_0 B_2 \\
C_2 T_0^{-1} & V_{\hat{B}_2}^T D V_{\hat{B}_2}
\end{bmatrix}
\begin{bmatrix}
Z & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}^{-1} \tag{49}
\]
and
\[
T_0 = \begin{bmatrix}
\hat{N}_{B_1}^T \\
\hat{Y}^{-1} \hat{C}_1
\end{bmatrix}, \quad T_0^{-1} = \begin{bmatrix}
\hat{N}_{C_1} & \hat{B}_1 \hat{Y}^{-1}
\end{bmatrix}, \tag{50}
\]
and the columns of full rank matrices $\hat{N}_{B_1}$ and $\hat{N}_{C_1}$ form the kernels of $B_1^T$, $C_1$ respectively, and satisfy $\hat{N}_{B_1}^T \hat{N}_{B_1} = I$, $\hat{N}_{C_1} \hat{N}_{C_1} = I$.

Proof. Condition (39a) in this case has the form $(TB_1)^T = C_1 T^{-1}$. If this condition holds, then applying the result of Lemma 10 to $B_1$ and $C_1$ one has $T$ as the one in Lemma 10 b) with $T_0$ of the form as (40). The result is proved by applying this formula to (34). \qed
We can repartition the middle factor of $\tilde{Y}$ in (49) as

$$
\begin{bmatrix}
-T_0AT_0^{-1} & -T_0B_2 \\
C_2T_0^{-1} & V_{0,2}^TDV_{0,2}
\end{bmatrix} =
\begin{bmatrix}
-N_{B_1}^TA\tilde{N}C_1 & -N_{B_1}^TA\tilde{B}_1Y^{-T} & -N_{B_1}^TB_2 \\
-Y^{-1}C_1A\tilde{N}C_1 & -Y^{-1}C_1A\tilde{B}_1Y^{-T} & -Y^{-1}C_1B_2 \\
C_2\tilde{N}C_1 & C_2\tilde{B}_1Y^{-T} & V_{0,2}^TDV_{0,2}
\end{bmatrix}
$$

Thus, we have that

$$
\tilde{Y} = \begin{bmatrix} Z & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -\tilde{A}_1 & -\tilde{B}_1 \\ \tilde{C}_1 & D_1 \end{bmatrix} \begin{bmatrix} Z & 0 \\ 0 & I \end{bmatrix}^{-1},
$$

and condition (48) is exactly the same as the condition (34), just replacing $T$, $A$, $B$, $C$, $D$ with $Z$, $\tilde{A}_1$, $\tilde{B}_1$, $\tilde{C}_1$, $D_1$. Hence, the existence of $Z$ can be checked again by using Lemma 12.

This implies that the procedure of checking the existence of a transformation from (35) to a PH system can be performed in a recursive way. One first performs the transformation (35) and checks whether a condition as in Part 1 of Lemma 12 does not hold, in which case a transformation does not exist. Otherwise, one checks whether $D_1 + D_1^T$ in (51) is invertible. If it is and if the associated matrix inequality does not have a positive definite solution, then a transformation does not exist. Otherwise, a transformation exists and a corresponding transformation matrix $T$ can be constructed by computing $Z$ satisfying $\tilde{Y}(Z) + \tilde{Y}(Z)^T \geq 0$ and the matrix $T_0$ is formed accordingly. In the remaining case, i.e., a condition as in Part 1 of Lemma 12 holds and $D_1 + D_1^T$ is singular, the process is repeated.

To formalize the recursive procedure, let

$$G_0 = \begin{bmatrix} -A & -B \\ C & D \end{bmatrix},$$

and suppose that (39a) is satisfied. Then form

$$G_1 := \tilde{T}_0 \tilde{V}_0^TG_0 \tilde{V}_0 \tilde{T}_0^{-1},$$

where

$$\tilde{T}_0 = \begin{bmatrix} T_0 & 0 \\ 0 & I \end{bmatrix}, \quad \tilde{V}_0 = \begin{bmatrix} I & 0 \\ 0 & V_0 \end{bmatrix},$$

$V_0 = [V_{0,2} \ V_{0,1}]$ is the matrix in the decomposition (35) times a permutation that interchanges the last block columns and $T_0$ is obtained from (60). In this way we obtain

$$G_1 = \begin{bmatrix}
-N_{B_1}^TA\tilde{N}C_1 & -N_{B_1}^TA\tilde{B}_1Y^{-T} & -N_{B_1}^TB_2 \\
-Y^{-1}C_1A\tilde{N}C_1 & -Y^{-1}C_1A\tilde{B}_1Y^{-T} & -Y^{-1}C_1B_2 \\
C_2\tilde{N}C_1 & C_2\tilde{B}_1Y^{-T} & V_{0,2}^TDV_{0,2}
\end{bmatrix} \begin{bmatrix} 0 \\ \Gamma \\ V_{0,1}^TDV_{0,1} \end{bmatrix},$$

where, by using (47),

$$C_1T_0^{-1} = \begin{bmatrix} 0 & W^{-T} \end{bmatrix} \tilde{C}_1 \begin{bmatrix} 0 & \tilde{B}_1Y^{-T} \end{bmatrix} = \begin{bmatrix} 0 & W^{-T} \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & \Gamma \end{bmatrix} = \begin{bmatrix} 0 & \Gamma \end{bmatrix},$$

$$T_0B_1 = \begin{bmatrix} 0 & W^{-1} \end{bmatrix} \begin{bmatrix} Y^{-1}\tilde{C}_1 & \tilde{B}_1 \\ 0 & W^{-1} \end{bmatrix} = \begin{bmatrix} [Y & 0] & W^{-1} \end{bmatrix} = \begin{bmatrix} 0 & \Gamma \end{bmatrix}.$$
By (35), we have
\[ V_{0,1}^T D V_{0,2} = -(V_{0,2}^T D V_{0,1})^T, \quad V_{0,1}^T D V_{0,1} = -(V_{0,1}^T D V_{0,1})^T. \]

So we can express \( G_1 \) as
\[
G_1 = \begin{bmatrix}
-\tilde{A}_1 & -\tilde{B}_1 \\
\tilde{C}_1 & \tilde{D}_1 \\
0 & \Gamma_1 \\
\Phi_1
\end{bmatrix}, \quad \Phi_1 = -\Phi_1^T.
\]

If \[
\begin{bmatrix}
-\tilde{A}_1 & -\tilde{B}_1 \\
\tilde{C}_1 & \tilde{D}_1
\end{bmatrix}
\]
satisfies condition (39a), then in an analogous way we construct
\[
\tilde{T}_1 = \text{diag}(T_1, I, I), \quad \tilde{V}_1 = \text{diag}(I, V_1, I)
\]
such that
\[
G_2 = \tilde{T}_1 \tilde{V}_1^T G_1 \tilde{V}_1 \tilde{T}_1^{-1} = \begin{bmatrix}
-\tilde{A}_2 & -\tilde{B}_2 \\
\tilde{C}_2 & \tilde{D}_2 \\
0 & \Gamma_2 \\
\Phi_2
\end{bmatrix}
\]
and we continue. Since the size of the matrices \( T_j \) is monotonically decreasing, this procedure will terminate after a finite number of \( k \) steps with
\[
G_k = \tilde{T}_{k-1} \tilde{V}_{k-1}^T \ldots \tilde{T}_0 \tilde{V}_0^T G_0 \tilde{V}_0 \tilde{T}_0^{-1} \ldots \tilde{T}_{k-1} \tilde{T}_{k-1}^{-1} = \begin{bmatrix}
-\tilde{A}_k & -\tilde{B}_k \\
\tilde{C}_k & \tilde{D}_k \\
0 & \Gamma_k \\
\Phi_k
\end{bmatrix}
\]
where \( \tilde{D}_k + \tilde{D}_k^T \) is positive definite, \( \Phi_k = -\tilde{\Phi}_k^T \),
\[
\tilde{T}_j = \text{diag}(T_j, I, I), \quad \tilde{V}_j = \text{diag}(I, V_j, I),
\]
and \( \ell_j \) is the size of \( T_j \) for \( j = 0, \ldots, k-1 \).

If there exists an invertible matrix \( T_k \) such that
\[
\begin{bmatrix}
T_k & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
-\tilde{A}_k & -\tilde{B}_k \\
\tilde{C}_k & \tilde{D}_k
\end{bmatrix}
\begin{bmatrix}
T_k^{-1} & 0 \\
0 & I
\end{bmatrix} + \begin{bmatrix}
T_k & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
-\tilde{A}_k & -\tilde{B}_k \\
\tilde{C}_k & \tilde{D}_k
\end{bmatrix}
\begin{bmatrix}
T_k^{-1} & 0 \\
0 & I
\end{bmatrix}^T \geq 0,
\]
see the solvability conditions in the previous section, then with \( \tilde{T}_k = \text{diag}(T_k, I, I) \), we have
\[
\tilde{T}_k G_k \tilde{T}_k^{-1} + (\tilde{T}_k G_k \tilde{T}_k^{-1})^T \geq 0.
\]
(52)

Observe that for each \( i, j \) with \( j \geq i \) the matrices \( \tilde{V}_i \) and \( \tilde{V}_i^T \) commute with \( \tilde{T}_j \) and \( \tilde{T}_j^{-1} \), and thus setting
\[
\tilde{T} = \tilde{T}_k \ldots \tilde{T}_0, \quad \tilde{V} = \tilde{V}_0 \ldots \tilde{V}_{k-1},
\]
then
\[
\tilde{T}_k G_k \tilde{T}_k^{-1} = \tilde{V}^T \tilde{T} G_0 \tilde{T}^{-1} \tilde{V},
\]
and inequality (52) implies that
\[
\tilde{T} G_0 \tilde{T}^{-1} + (\tilde{T} G_0 \tilde{T}^{-1})^T \geq 0.
\]
Then the desired transformation matrix \( T \) is positioned in the top diagonal block of \( \tilde{T} \), and the matrix \( V \) is positioned in the bottom diagonal block of \( \tilde{V} \).
Remark 12 The recursive procedure described above requires at each step the computation of three singular value decompositions in order to check the ranks of the matrices $\tilde{B}_j$ and $\tilde{C}_j$ and in order to construct bi-orthogonal bases so that (50) holds. While each step of this procedure can be implemented in a numerically stable way, the consecutive rank decisions make the aggregate procedure difficult to analyze, similar to the case of staircase algorithms [16, 20, 21]. In general the strategy should be adapted toward the goal of obtaining a realization in port-Hamiltonian form that is robust to small perturbations, see [3, 45] for some ways to do this.

3.4 Explicit solution of linear matrix inequalities via even pencils

We have seen that to check the existence of the transformation to port-Hamiltonian form and to explicitly construct the transformation matrices $T$, $V$ is equivalent to consider the solution of the linear matrix inequality (9). As we have discussed before, the best way to do this is via the transformation of the even pencil (32). In this subsection we combine the recursive procedure with the construction of a staircase like form for this even pencil.

For a given real symmetric matrix $Q$ denote by

$$
\Psi_0(Q) := \begin{bmatrix} -A^TQ - QA & C^T - QB \\ C - B^TQ & D + D^T \end{bmatrix}
$$

the corresponding block matrix, which is supposed to be positive semidefinite.

Let $B_1, C_1$ be defined as in (35), (36). If $B_1, C_1$ satisfy Part 1. of Lemma 12 then, since $\text{Ker} \ C_1^T = \text{Ker} B_1$, there exist real orthogonal matrices $U_1, V_1$ (which can be obtained by performing a permuted singular value decomposition of $B_1$) such that

$$
\tilde{U}_1^T B_1 \tilde{V}_1 = \begin{bmatrix} 0 & 0 \\ \Sigma_B & 0 \end{bmatrix}, \quad \tilde{V}_1^T C_1 \tilde{U}_1 = \begin{bmatrix} C_{11} & C_{12} \\ 0 & 0 \end{bmatrix}
$$

(53)

where $\Sigma_B$ is invertible and $C_{12} \Sigma_B$ is real symmetric and positive definite. Transforming the desired $Q$ correspondingly as

$$
\tilde{Q} := \tilde{U}_1^T Q \tilde{U}_1 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix},
$$

then, since the linear matrix inequality (9) implies $\tilde{Q}(\tilde{U}_1^T B_1) = \tilde{U}_1^T C_1^T$, it follows in the transformed variables that

$$
Q_{22} := C_{12} \Sigma_B^{-1} > 0, \quad Q_{12} := C_{11} \Sigma_B^{-1}.
$$

(54)

It remains to determine $Q_{11}$ so that $\tilde{Q} > 0$. To achieve this, we set

$$
Q_0 := \begin{bmatrix} Q_{12} Q_{22}^{-1} Q_{12}^T \\ Q_{12}^T Q_{22} \end{bmatrix} = T_0^{-T} \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} \end{bmatrix} T_0^{-1}, \quad T_0 = \begin{bmatrix} I & 0 \\ -Q_{22}^{-1} Q_{12}^T & I \end{bmatrix}
$$

(55)

and we clearly have that $Q_0 \geq 0$. Then we can rewrite $\tilde{Q}$ as

$$
\tilde{Q} = \begin{bmatrix} Q_{11} & 0 \\ 0 & 0 \end{bmatrix} + Q_0 = T_0^{-T} \left( \begin{bmatrix} Q_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) T_0^{-1} = T_0^{-T} \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} T_0^{-1},
$$

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partitioned analogously and we obtain that \( \tilde{Q} > 0 \) if and only if \( Q_1 > 0 \). Let \( V_0 \) be the orthogonal matrix given in (35). By performing a congruence transformation on \( \Psi_0(Q) \) with

\[
Z_0 = \begin{bmatrix} T_0 & 0 \\ 0 & V_0 \end{bmatrix}, \quad T_0 := \tilde{U}^1T_0, \quad V_0 := V_0 \begin{bmatrix} \tilde{V}_1 & 0 \\ 0 & I \end{bmatrix}
\]

and using the fact that \( \tilde{Q}(\tilde{U}_T^1B_1\tilde{V}_1) = (\tilde{V}_T^1C_1\tilde{U}_1)^T \) for any real symmetric \( Q_1 \), it follows that

\[
Z_0^T\Psi_0(Q)Z_0 = \begin{bmatrix}
-(T_0^{-1}AT_0)^T T_0^TQT_0 - T_0^TQT_0(T_0^{-1}AT_0) & T_0^TC_2^T - T_0^TQT_0(T_0^{-1}B_2) \\
0 & 0
\end{bmatrix}
\]

Partitioning

\[
T_0^{-1}AT_0 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad T_0^{-1}B_2 = \begin{bmatrix} B_{13} \\ B_{23} \end{bmatrix}, \quad C_2T_0 = \begin{bmatrix} C_{31} & C_{32} \end{bmatrix}
\]

and using the fact that

\[
T_0^TQT_0 = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_{22} \end{bmatrix}
\]

we obtain that

\[
Z_0^T\Psi_0(Q)Z_0 = \begin{bmatrix}
-A_{11}^TQ_1 - Q_1A_{11} & -Q_1A_{12} - A_{21}^TQ_{22} & 0 & C_{31}^T - Q_1B_{13} \\
-A_{12}^TQ_1 - Q_{22}A_{21} & -A_{22}^TQ_{22} - Q_{22}A_{22} & 0 & C_{32}^T - Q_{22}B_{23} \\
0 & 0 & 0 & 0 \\
C_{31} - B_{13}^TQ_1 & C_{32} - B_{23}^TQ_{22} & 0 & S_2
\end{bmatrix}
\]

In this way, we have that (9) holds for some \( Q > 0 \) if and only if

\[
\Psi_1(Q_1) := \begin{bmatrix}
-A_{11}^TQ_1 - Q_1A_{11} & -Q_1A_{12} - A_{21}^TQ_{22} & 0 & C_{31}^T - Q_1B_{13} \\
-A_{12}^TQ_1 - Q_{22}A_{21} & -A_{22}^TQ_{22} - Q_{22}A_{22} & 0 & C_{32}^T - Q_{22}B_{23} \\
C_{31} - B_{13}^TQ_1 & C_{32} - B_{23}^TQ_{22} & 0 & S_1
\end{bmatrix}
\]

holds for some real symmetric positive definite \( Q_1 \), where

\[
A_1 = A_{11}, \quad C_1 = \begin{bmatrix} -Q_{22}A_{21} \\ C_{31} \end{bmatrix}, \quad B_1 = \begin{bmatrix} A_{12} & B_{13} \end{bmatrix},
\]

and

\[
D_1 + D_1^T = \begin{bmatrix}
-A_{22}^TQ_{22} - Q_{22}A_{22} & C_{32}^T - Q_{22}B_{23} \\
C_{32} - B_{23}^TQ_{22} & S_2
\end{bmatrix}
\]

This construction has reduced the solution of the linear matrix inequality (9) to the solution of a smaller linear matrix inequality of the same form. Thus, we can again proceed in a recursive manner with the same reduction process until either the condition in Part 1. of Lemma 10 no longer holds (in which case no solution exists) or \( D_k + D_k^T \) is positive definite for some \( k \).
This reduction process can be considered as the construction of a structured staircase form for the even pencil \([32]\). By applying a congruence transformation to the pencil \([32]\) with the matrix
\[
Y_0 = \begin{bmatrix}
T_0^{-T} & 0 & 0 \\
0 & T_0 & 0 \\
0 & 0 & V_0
\end{bmatrix},
\]
it follows that
\[
\lambda \begin{bmatrix}
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\text{I} & 0 & 0 & 0 & 0 & 0 \\
0 & -\text{I} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} - \begin{bmatrix}
A_{11} & A_{12} & 0 & 0 & B_{13} \\
0 & 0 & A_{21} & A_{22} & \Sigma_B & 0 & B_{23} \\
A^T_{11} & A^T_{12} & 0 & 0 & 0 & 0 & C^T_{31} \\
A^T_{12} & A^T_{22} & 0 & 0 & C^T_{31} & 0 & C^T_{32} \\
0 & \Sigma_B & 0 & C_{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B^T_{13} & B^T_{23} & C_{31} & C_{32} & 0 & 0 & S_2
\end{bmatrix}.
\]

By performing another congruence transformation with the matrix
\[
\tilde{Y}_0 = \begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & -Q_{22} & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & I
\end{bmatrix},
\]
the pencil becomes
\[
\lambda \begin{bmatrix}
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\text{I} & 0 & 0 & 0 & 0 & 0 \\
0 & -\text{I} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} - \begin{bmatrix}
0 & 0 & A_{11} & A_{12} & A_{21} & A_{22} & \Sigma_B & 0 & B_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{23} \\
\text{A}^T_{11} & \text{A}^T_{12} & 0 & -A^T_{21}Q_{22} & 0 & 0 & C^T_{31} \\
\text{A}^T_{12} & \text{A}^T_{22} & -Q_{22}A_{21} & -A^T_{22}Q_{22} - Q_{22}A_{22} & 0 & 0 & C^T_{32} - Q_{22}B_{23} \\
0 & \Sigma_B & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B^T_{13} & B^T_{23} & C_{31} & C_{32} - B^T_{23}Q_{22} & 0 & 0 & S_2
\end{bmatrix}.
\]

By further moving the last block row and column to the fifth position and then the 2nd block
row and column to the fifth position, i.e., by performing a congruence permutation with
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

one obtains
\[
\lambda \begin{bmatrix}
0 & I & 0 & 0 & 0 \\
-I & 0 & 0 & 0 & 0 \\
0 & 0 & -\Gamma_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} - \begin{bmatrix}
0 & A_1 & B_1 \\
A_1^T & 0 & C_1 \\
B_1^T & C_1 & D_1 + D_1^T \\
0 & A_{21} & \Delta_1 T \\
0 & 0 & \Sigma_1 T
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \hat{\Sigma}_1 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

where \( \Gamma_1 = \begin{bmatrix} I & 0 \end{bmatrix} \), \( \Delta_1 = \begin{bmatrix} A_{22} & B_{23} \end{bmatrix} \), \( \Sigma_1 := \Sigma_B \)

and \( A_1, B_1, C_1, D_1 + D_1^T \) are as defined before. In this way, we may repeat the reduction process on the (1,1) block, which corresponds to \( \Psi_1 \). In order to exploit the block structures of the pencil we use a slightly different compression technique for \( D_1 + D_1^T \). Note that we may write
\[
D_1 + D_1^T = \begin{bmatrix} D_{11} & D_{12} \\
D_{12}^T & S_2 \end{bmatrix},
\]

with \( S_2 \) symmetric positive definite. Then we have
\[
D_1 + D_1^T = \begin{bmatrix} I & 0 \\
S_2^{-1}D_{12}^T & I \end{bmatrix}^T \begin{bmatrix} D_{11} - S_2^{-1}D_{12}D_{12}^T & 0 \\
0 & S_2 \end{bmatrix} \begin{bmatrix} I & 0 \\
S_2^{-1}D_{12} & I \end{bmatrix}.
\]

Let
\[
D_{11} - D_{12}S_2^{-1}D_{12}^T = Z_1 \begin{bmatrix} 0 & 0 \\
0 & \tilde{S}_2 \end{bmatrix} Z_1^T,
\]

where \( \tilde{S}_2 \) is invertible and \( Z_1 \) is orthogonal. Then
\[
\begin{bmatrix} Z_1 & \tilde{S}_2^{-1}D_{12}Z_1 \end{bmatrix}^T (D_1 + D_1^T) \begin{bmatrix} Z_1 & \tilde{S}_2^{-1}D_{12}Z_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\
0 & \tilde{S}_2 & 0 \\
0 & 0 & \tilde{S}_2 \end{bmatrix} = : \begin{bmatrix} 0 & 0 \\
0 & \tilde{S}_2 \end{bmatrix}.
\]

A necessary condition for the existence of a transformation to PH form is that \( \tilde{S}_2 \) > 0 or equivalently \( \tilde{S}_2 \) > 0. If this holds, then using the fact that
\[
\begin{bmatrix} Z_1 & \tilde{S}_2^{-1}D_{12}Z_1 \end{bmatrix}^T \Gamma_1 = \begin{bmatrix} Z_1^T \\
0 \end{bmatrix},
\]

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by performing a congruence transformation on the 3rd block rows and columns with

\[
\begin{bmatrix}
Z_1 \\
-S_2^{-1}D_{12}^T Z_1
\end{bmatrix}
\]

and another congruence transformation on the fourth block row and column with \( Z_1 \) we obtain the pencil

\[
\lambda \begin{bmatrix}
0 & \mathbf{I} & 0 & 0 \\
-\mathbf{I} & 0 & 0 & 0 \\
0 & 0 & -\Gamma_{11} & 0 \\
0 & 0 & 0 & -\Gamma_{21}
\end{bmatrix}
- \begin{bmatrix}
0 & \mathbf{A}_1 & \mathbf{B}_{11} & \mathbf{B}_{12} \\
\mathbf{A}_1^T & 0 & \mathbf{C}_{11}^T & \mathbf{C}_{21}^T \\
\mathbf{B}_{11}^T & \mathbf{C}_{11} & 0 & 0 \\
\mathbf{B}_{12}^T & \mathbf{C}_{21} & 0 & \hat{\mathbf{S}}_2
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \Delta_{11} & \Delta_{21} & \Delta_{31} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

where

\[
\begin{bmatrix}
\Gamma_{11} \\
\Gamma_{12}
\end{bmatrix} = \Gamma = \begin{bmatrix}
\mathbf{I} & 0 \\
0 & \mathbf{I}
\end{bmatrix}.
\]

In order to proceed, \( \mathbf{B}_{11}, \mathbf{C}_{11}^T \) must satisfy the same conditions as \( \mathbf{B}_1, \mathbf{C}_1^T \). If these conditions hold, then we can perform a second set of congruence transformation and transform the pencil to

\[
\lambda \begin{bmatrix}
0 & \mathbf{I}_\ell & 0 & 0 \\
-\mathbf{I}_\ell & 0 & 0 & 0 \\
0 & 0 & -\Gamma_2 & 0 \\
0 & 0 & 0 & -\tilde{\Gamma}_{11}
\end{bmatrix}
- \begin{bmatrix}
0 & \mathbf{A}_2 & \mathbf{B}_2 \\
\mathbf{A}_2^T & \mathbf{C}_2^T & \Theta_2^T & 0 \\
\mathbf{B}_2^T & \mathbf{C}_2 & \mathbf{D}_2 + \mathbf{D}_4^T \\
0 & \Theta_2 & \Delta_2 & 0 \\
0 & 0 & 0 & \Sigma_2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \tilde{\Delta}_{11} & 0 & 0 \\
0 & 0 & \tilde{\Delta}_{21} & 0 \\
0 & 0 & 0 & \tilde{\Delta}_{31}
\end{bmatrix},
\]

where

\[
\tilde{\Gamma}_{11} = \begin{bmatrix}
0 \\
\Gamma_{21}
\end{bmatrix}, \quad \tilde{\Gamma}_{11} = \Gamma_{31}, \quad \Gamma_2 = \begin{bmatrix}
\mathbf{I} \\
0
\end{bmatrix},
\]

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and with the partitioning $\Delta_{11} = \begin{bmatrix} \Delta_{11,1} \\ \Delta_{11,2} \end{bmatrix}$,

$$\Delta_{11} = \Delta_{11,1}, \quad \Delta_{21} = \begin{bmatrix} \Delta_{11,2} \\ \Delta_{31} \end{bmatrix}, \quad \begin{bmatrix} \Delta_{31} \\ \Delta_{41} \end{bmatrix} = \Delta_{21}.$$  

This reduction process continues as long as all the required conditions hold, until for some $k$, $D_k + D_k^T > 0$. If this is the case, then the pencil (32) is reduced to an even pencil that has the eigenvalue $\infty$ with equal algebraic and geometric multiplicity (it is of index one)

$$\lambda \begin{bmatrix} 0 & I_\ell & 0 \\ -I_\ell & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A_k & B_k \\ A_k^T & 0 & C_k^T \\ B_k & C_k & D_k + D_k^T \end{bmatrix},$$  

and if it has a deflating subspace associated with a set of $\ell$ finite eigenvalues chosen such that the deflating subspace is as in (33), and from this we can compute a Hermitian positive definite matrix $Q$ associated with (56).

Note the above process is actually a special staircase form reduction process that deflates the singular part and higher index of the eigenvalue infinity of the even pencil (32), [12, 16].

**Remark 13** Note that to check the passivity of (5) it is only necessary to have a positive semidefinite solution to (9). Thus, if one only wants to check passivity, then Part 1. in Lemma 10 can be relaxed to $\ker B_1 \subseteq \ker C_1^T$. In this case the transformation to the form (53) can still be made, but $Q_{22}$ in (54) is only positive semidefinite, and $\ker Q_{22} \subseteq \ker Q_{12}$. Then $Q_0$ can still be defined but instead of $Q_{11}^{-1}$ one needs to use the Moore-Penrose pseudoinverse, see [24], of $Q_{11}$. However, in this case $Q_0$ and the resulting solution $Q$ cannot be positive definite. Thus, in this situation, (5) may be a passive system that cannot be transformed to a standard port-Hamiltonian system of the form (6) system.

A simple example is the scalar system

$$\dot{z} = -z + 2u, \quad y = 0z + 0u.$$  

This system is passive (but not strictly passive) and the matrix inequality (9) has the unique singular solution $Q = 0$. So this system cannot be transformed to a port-Hamiltonian system of the form (6). In this case then one has to use a descriptor formulation (7).

To illustrate the analysis procedures, consider the following example.

**Example 3** In the finite element analysis of disc brake squeal [25], the model is a very large-scale second-order system of differential equations with approximately a million degrees of freedom, that furthermore also depends on parameters, e.g., the disc speed $\omega$. If no further constraints are incorporated, then in the stationary case the system takes the form

$$M\ddot{q} + \left( C_1 + \frac{\omega}{\omega_r} C_R + \frac{\omega}{\omega_r} C_G \right) \dot{q} + \left( K_1 + K_R + \left( \frac{\omega}{\omega_r} \right)^2 K_G \right) q = B u, \quad y = B^T q,$$

where $M = M^T > 0$ is the mass matrix, $C_1 = C_1^T \geq 0$ models material damping, $C_G = -C_G^T$ models gyroscopic effects, $C_R = C_R^T \geq 0$ models friction induced damping, $K_1 = K_1^T > 0$ is the stiffness matrix, $K_R = K_2 + N$ with $K_2 = K_2^T$ and $N = -N^T$, is a nonsymmetric
matrix modeling circulatory effects, $K_G = K_G^T \geq 0$ is the geometric stiffness matrix, and $\omega$ is the rotational speed of the disc with reference velocity $\omega_r$. In industrial brake models, the matrices $D := C_1 + \frac{\omega}{\omega_r} C_R$, and $N$ are sparse and have very low rank (approx. 2000) corresponding to finite element nodes associated with the brake pad. Setting $G := \frac{\omega}{\omega_r} C_G$, $K = K_1 + K_2 + (\frac{\omega}{\omega_r})^2 K_G$, we may assume that $K > 0$. Here in the practical design a shim is attached to the brake pad which may be interpreted as choosing the input as output feedback $u = D_S B^T q$ in order to stabilize the system in a given range of disk speeds.

Then, introducing $p = Mq$, we can write the system in first order form

$$
\begin{bmatrix}
\frac{\dot{p}}{\dot{q}}
\end{bmatrix} = (J - R) Q \begin{bmatrix}
p \\
q
\end{bmatrix},
$$

where

$$
J := \begin{bmatrix}
-G \\
(I + \frac{1}{2}(N - BD_S B^T) K^{-1})^T \\
0
\end{bmatrix},
$$

$$
R := \begin{bmatrix}
D \\
(\frac{1}{2}(N - BD_S B^T) K^{-1})^T \\
0
\end{bmatrix},
$$

$$
Q = \begin{bmatrix}
M^{-1} & 0 \\
0 & K
\end{bmatrix}.
$$

Since regardless of the choice of $D_S$, the matrix $R$ is indefinite as long as $N \neq 0$ (then $N - BD_S B^T \neq 0$) it is clear that for this system we cannot read off its stability and it is definitely unstable if $x^T R x < 0$ for some eigenvector $x$ of $J$.

**Remark 14** Many of the results described in the paper can be extended to the case of general descriptor systems having the form (4) or (7) but with singular $E$. This is an open problem and currently a topic of active research.

**Remark 15** In this paper we have always looked for positive definite solutions $Q$ of the KYP matrix inequality (9). However, we could relax this requirement to the case that $Q$ is positive semidefinite. In this case several further subtle problems arise, and one definitely would need to consider descriptor systems. See [19, 42, 48] for a detailed discussion.

**Remark 16** In view of the fact that the KYP matrix inequality (9) may have many solutions, one may choose the solution so that the resulting pH representation is robust to data or numerical errors. To do this one could choose a robustness measure like to the distance to instability or non-passivity or to consider a representation that is far away from the boundary of the solution set of (9). Partial results in this direction have been obtained in [3, 5, 44].

### 4 Numerical Considerations

In this section we discuss some numerical issues that arise when implementing the procedure as discussed in the last section. An associated MATLAB script is available for download on the MathWorks FileExchange (under nearby-pH-realization) and at GitHub.

Most steps of our procedure are implemented in a straightforward way using standard techniques from numerical linear algebra that ensure backward stability. There are a few places where forward stability may be lost and large relative errors could occur. In particular,

\[ \text{https://github.com/christopherbeattie/nearby-pH-realization} \]
the similarity transformations with respect to $T_i$ in (52) as well as the linear solves implicit in the inversions of $Q_{22}$ in (55) used in forming the matrices $T_i$. If these are ill-conditioned then large relative errors may arise.

Another difficulty is the non-uniqueness of solutions to the linear matrix inequality (16). To make the solution unique one can optimize a quality measure like the distance to instability or non-passivity, or try to find the analytic center of the solution, see [3, 45]. All these are difficult and expensive optimization problems on top of all the computational work that has to be carried out. How to do this efficiently is an open question, even for the case that the system is minimal and $\hat{S}$ is well-conditioned with respect to inversion. The difficulty arises, in particular, since the solution sets of (16) and (17) are rather difficult to characterize, see [50] for a detailed analysis based on eigenvalue perturbation theory.

Remark 17 Note that if the skew-symmetric/symmetric pencil (56) has purely imaginary eigenvalues, then the solution of the Riccati equation associated with (17) can only be computed with the Newton method of [7] which has been implemented e.g. in [10]. In this case none of the usual approaches utilizing invariant subspaces will be fully satisfactory and even the Newton method might not achieve a quadratic rate of convergence, displaying only linear convergence [28].

We have tested our procedure for a large number of examples with randomly generated stable and passive systems which were produced from pH systems by multiplying out the factors. In each example the procedure yielded the same pH representation.

Remark 18 As we have discussed in Section 3, we reduce the problem to a subproblem, where $\hat{C}$ is invertible and thus where (17) can be formed and is solvable. For this we have to make several rank decisions or regularization steps which result in small perturbations. This is often justified, since the coefficient matrices $A, B, C, D$ are typically not exact because they arise often from a data based realization, interpolation or model reduction process. So we can make small perturbations to these data to regularize the problem if this does not change the resulting (hopefully) robust pH representation.

This is common practice when solving (10), see e.g. [2, 29], where often $\hat{S}$ is perturbed to be invertible, so that the pencil (32) is regular and of index at most one, see Remark 9. However, as discussed in Section 3, see Remark 11 since we are able to deal with a singular $\hat{S}$ we perform the regularization in a different way. For this we have to perform the rank decisions in (35), (47), (53), which are critical in the process. As in most staircase algorithms [61], it is recommended to make conservative decisions, i.e. to assume smaller rank if the decision is difficult using the usual rank decision procedures [20, 21].

Remark 19 Note that if there is no solution to the linear matrix inequality (9) our procedure will produce a positive definite solution $Q$ to a slightly perturbed linear matrix inequality for which a solution can be assured, see Example 4 below.

Remark 20 Since the solution to (9) is not unique, even if we start with a pH realization of a system, i.e., when the system is in the form (11) with quadratic Hamiltonian $H = \frac{1}{2}x^T Q x$, our procedure in general, may not return the same representation, see e.g. Example 4 below.
Example 4 Consider a system (5) with $A = (J - R)Q$, $B = F - P$, $C = (F + P)^T$, $D = S + N$, where $J = 0$, $N = 0$, $Q = I_4$,

$$
R = \begin{bmatrix}
1 & 2 \\
2 & 3 \\
3 & 4
\end{bmatrix},
B = \begin{bmatrix}
1 & 2 \\
1 & -1 \\
1 & 3
\end{bmatrix},
P = \begin{bmatrix}
\epsilon & -1 \\
\epsilon & 1 \\
\epsilon & -1
\end{bmatrix},
S = \begin{bmatrix}
\epsilon & 0 \\
0 & 1
\end{bmatrix}.
$$

We ran our script with different values of $\epsilon = 0, \pm 2 \cdot 10^{-16}, \ldots, \pm 2 \cdot 10^{-9}$. When $\epsilon$ is very close to 0 but negative, the system is slightly non-passive, since the matrix $W$ has a negative eigenvalue. Our script always returns a positive definite $Q$ that changes only slightly as $\epsilon$ varies (regardless of sign), while the inequality (16) remains semidefinite but numerically singular with four very small nonzero eigenvalues. For $\epsilon = 0, \pm 10^{-16}$ the pencil (32) has index $\geq 2$, otherwise the index is 1. Nonetheless, our procedure works in all cases, see Figure 1. Running our script for a system (5) with the same coefficients and $\epsilon$ values as in Example 4, but $Q$ random positive definite, we obtain the results in Figure 2.

Remark 21 In Section 3 we have developed a method that allows to identify the solvability of the matrix inequality (9) via the computation of a staircase like form and deflating subspaces of the even matrix pencil (32). In numerical practice, to avoid the use of consecutive rank decisions, one may apply a so called derivative array approach, see [37], where the parts associated with the finite eigenvalues, the part associated with the infinite eigenvalues and the singular part are separated by one sided transformations of (32) from the left generated based on an extended pencil associated with a derivative array of the DAE associated with the even pencil. For even pencils this follows directly from a procedure developed in [38] for time varying DAEs and gives a reduced system that is associated with an even pencil and an unstructured part associated with the eigenvalue $\infty$. We do not present this approach here, see [38, 36] for details.
5 Conclusions

In this paper we have extended the characterization when a system is equivalent to a port-Hamiltonian system to the case of general non-minimal systems and to the case that the symmetric part of the feedthrough matrix is singular. We have presented an explicit procedure for the construction of the transformation matrices, which has been implemented as a numerical method. The method works in all tested and synthetically constructed problems. Using small perturbations to the system coefficients the method can be also be used when the system is not stable or not passive. Our procedure generates a nearby system that has a pH representation.

Open problems include the question how to parameterize the positive definite solutions of (9) in terms of eigenvalues or pseudospectra of the system matrix and the choice of adequate robustness measures to select an optimal solution of (9). The extension of the approach to differential-algebraic equations is another important research topic.

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Figure 2: Deviation of $Q$ and residual for (9) from solution of $\epsilon = 0$ for varying $\epsilon$ and random positive definite $Q$. 

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