Weyl Time-Evolution-Operator in anisotropic Bianchi-type-I universes

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Abstract

Dirac’s equation in anisotropic Bianchi-type-I background space-times is treated utilizing orthonormal frames. Specializing to the massless case and power law scale factors \( \alpha_j(t) = t^{q_j} \) \((q_1 = q_2 \neq q_3)\) of the metric, a general expression of the Weyl time-evolution-operator is derived. It is shown that this outcome agrees with results of exactly soluble models. It also agrees with approximate solutions of models where no exact results are available, e.g. if the background is given by the anisotropic planar Kasner spacetime.

The time-evolution-operator approach goes beyond a standard asymptotic calculation, since the initial value problem can be fully taken into account. Within the framework of this formalism it is also possible to study small deviations from flat backgrounds. In particular, it is shown that the limiting case of vanishing anisotropy renders the correct result.

I. INTRODUCTION

The behavior of particles in curved backgrounds obeying the Dirac equation has long been of considerable interest in cosmology and astrophysics.

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An early discussion of the massless case has been given by Brill and Wheeler [1]. An investigation of quantized spin $\frac{1}{2}$ fields in conformally flat Friedmann-Lemaître-Robertson-Walker (fFLRW) universes has been performed by Parker. He showed that no production of massless particles can take place in those spacetimes [2]. Further questions in connection with the quantization of fermionic fields in fFLRW and FLRW backgrounds have also been addressed in the literature [3].

At least in some cases it is relatively simple to find exact (massive) classical solutions of Dirac’s equation in isotropic backgrounds, which is no longer true for anisotropic backgrounds. For example, Barut and Duru investigated massless and massive fermions in two fFLRW spacetimes with power law expansion, and in the steady-state part of de Sitter spacetime [4]. The latter model had also been studied by Cotaescu, who in addition performed the quantization of the spin $\frac{1}{2}$ field [5], and by Candelas and Raine using a path integral approach for the quantization of massive scalar and Dirac fields [6]. Massless and massive fermion propagators in fFLRW backgrounds with constant deceleration have been treated by Koksma and Prokopec [7].

If one is occupied with questions concerning the early universe, then one inevitably encounters the problem of anisotropy. Although the present day universe seems to be isotropic to a very high degree, this needed not necessarily be the case in an early phase of development of the universe. If one starts e.g. with an anisotropically expanding Kasner universe and takes into account quantum effects in the vicinity of the initial singularity, then, according to Zel’dovich [8], this could lead to an isotropization of the universe at the Planck time-scale due to particle creation processes. A semiclassical calculation by Hu and Parker [9] lent further credit to this point of view. They investigated a quantized massless conformal scalar field in a Bianchi-type-I (BI) universe with axial symmetry. A further motivation for the interest in Kasner universes as background spacetimes is given by the work of Belinskii, Khalatnikov, and Lifshitz [10] and Misner [11], who realized that Bianchi-type-IX universes can be described by sequences of Kasner spacetimes, if one moves backward in time toward the initial singularity. More recently, the issue of anisotropy and in particular of BI and Kasner spacetimes has been discussed in the context of preinflationary scenarios of the universe [12], [13], and also in vector inflation [14] and gauge inflation models [15].

Classical solutions of Dirac’s equation in BI spacetimes have been investigated e.g. by Henneaux who considered gravitational and spinor fields being both invariant under a special group of transformations [16], or by Saha and
Boyadjiev who studied interacting but space-independent spinor and scalar fields [17]. To obtain analytical results when quantizing fermionic fields propagating in anisotropic spacetimes one usually has to resort to a perturbative treatment of the background. One considers e.g. in a method utilized by Zel'dovich and Starobinsky small anisotropic perturbations about a fFLRW spacetime [18]. For a model with a special form of weak anisotropy, Birrell and Davies performed the quantization of the massive scalar field [19], and Lotze treated the quantization of the corresponding massive spin \( \frac{1}{2} \) field [20].

It remains a fact, however, that mostly no exact classical (even massless) spinor solutions in anisotropic background spacetimes are at disposal. One can still try to calculate solutions valid at late and early times, resp., but such a calculation suffers from various shortcomings. Above all it is not clear how (and whether at all) the small-time and asymptotic solutions found in this way match in the sense that to an asymptotic solution a corresponding early time solution could be uniquely related (and vice versa). This suggests to tackle the problem by trying to determine approximate solutions of the Weyl-Dirac equation in BI backgrounds. These approximate solutions are characterized by the property of matching the exact solutions at early and late times as well. The approximate Weyl time-evolution-operator (TEO) derived in this work provides by construction these solutions.

This paper is organized as follows: In sect. II. we set up Dirac's equation in BI backgrounds. In sect. III. we specialize to the case of massless fermions in axially symmetric or planar BI (pBI) spacetimes with power law scale factors. For those backgrounds an exact expression of the Weyl TEO is found. This result can be used to derive a simple parameter transformation (PT) which generates all solutions of a given equivalence class of exact massless spinor mode solutions with pBI backgrounds, provided one knows a single arbitrary solution of this class [21]. It is shown that this PT is equivalent to a special conformal map, which in practice is often more convenient to use. In this section we also determine an analytic expression for the approximate Weyl TEO, which is nonperturbative w.r.t. the underlying spacetime. This outcome is compared in sect.s IV. A, B and C with the solutions of exactly soluble models. In sect. IV. D the background is described by Kasner's anisotropic planar vacuum solution, for which no exact spinor solutions are known. In sect. V. we investigate the Weyl TEO in anisotropically perturbed conformally flat backgrounds, and in sect. VI. some general properties of the TEO are discussed.
II. ORTHONORMAL FRAMES AND DIRAC EQUATION

In a coordinate frame with cosmic time $t$ the line element of a BI universe is defined by

$$ds^2 = dt^2 - \sum_{i=1}^{3} \alpha_i^2(t)(dx^i)^2$$  \hspace{1cm} (1)

with metric tensor $g = g_{\mu\nu}dx^\mu \otimes dx^\nu$, where $g_{\mu\nu} = \text{diag}(1, -\alpha_1^2, -\alpha_2^2, -\alpha_3^2)$. A natural choice of an orthonormal frame at $p \in M$ is given by the covectorfields $\Theta^0 = dt$, $\Theta^j = \alpha_j(t)dx^j$ (no sum), which constitute bases of the fibers $T_p^*(M)$ of the cotangent bundlespace $\mathcal{B}(M, T_p^*(M)) \equiv T^*(M) = \bigcup_{p \in M} T_p^*(M)$. The base space $M$ denotes a differentiable pseudo-Riemannian manifold endowed with metric $g$. Likewise, $e^0 = \partial^0 \equiv \partial_0$, $e^j = \alpha^{-1}_j(t)\partial_j$ (no sum) are the corresponding basis vectorfields of the fibers of the tangent bundlespace $\mathcal{B}(M, T_p(M)) \equiv T(M) = \bigcup_{p \in M} T_p(M)$. The line element (1) is then given by $ds^2 = (\Theta^0)^2 - (\Theta^1)^2 - (\Theta^2)^2 - (\Theta^3)^2$, and the metric tensor assumes the form $g = \eta_{\mu\nu}\Theta^\mu \otimes \Theta^\nu$ with $\eta_{\mu\nu} = g(e_\mu, e_\nu) \equiv \text{diag}(+1, -1, -1, -1)$, since $\Theta^\mu(e_\nu) = \delta^\mu_\nu$.

The exterior derivative of the covector $\Theta^j$ is

$$d\Theta^j = -C^j_{0j} \Theta^0 \wedge \Theta^j,$$  \hspace{1cm} (2)

and the 3 nonvanishing commutation coefficients $C^\alpha_{\mu\nu}$ are

$$C^j_{0j} = -\dot{\alpha}_j/\alpha_j$$  \hspace{1cm} (3)

($j = 1, 2, 3$, no summation). In general holds: $d\Theta^\mu = -C^\mu_{\alpha\beta} \Theta^\alpha \wedge \Theta^\beta$ and correspondingly: $[e_\mu, e_\nu] = C^\alpha_{\mu\nu} e_\alpha$ with $C^\alpha_{\mu\nu} = -C^\alpha_{\nu\mu}$.

Only in bundlespaces where the fibers are given by the tangent spaces $T_p(M)$ of the underlying base manifold $M$, one can define a soldering form $\Xi = e_\nu \otimes \Theta^\nu$ [22], [23]. This soldering form can be viewed as a vector-valued 1-form, and applying the exterior covariant derivative to $\Xi$ results in $D \wedge \Xi = e_\nu \otimes T^\nu$, where

$$T^\nu = d\Theta^\nu + \omega^\nu_\mu \wedge \Theta^\mu$$  \hspace{1cm} (4)

denotes the torsion 2-form. The exterior covariant derivative of the soldering form must be distinguished from its covariant derivative, which is
trivially zero here: $D \Xi = 0$. To obtain Dirac’s equation in a curved spacetime one starts with the following ansatz of the connection 1-form:

$$\omega^\nu_\mu = L^\nu_\mu \Theta^\alpha. \quad (5)$$

Because we are dealing with a (pseudo-) Riemannian manifold, the two conditions metricity

$$dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu} \quad (6)$$

and vanishing torsion

$$T^\nu = 0, \quad (7)$$

can be utilized to determine $\omega_{\mu\nu}$. Vanishing torsion implies $C^\mu_{\alpha\beta} = L^\mu_\beta - L^\mu_\alpha$, and from metricity follows $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and $L_{\alpha\beta\gamma} = -L_{\beta\alpha\gamma}$, since $g_{\mu\nu} = \eta_{\mu\nu}$ in an orthonormal frame. Thus, $C^j_{0j} = -L^j_{0j} = L_{j0j}$ (no sum), and one finds for the nonzero entries of the connection 1-form:

$$\omega^j_0 = -C^j_{0j} \Theta^j \equiv \omega^0_j. \quad (8)$$

The covariant differentiation of a Dirac bispinor is given by the 1-form

$$(D\psi)^I = d\psi^I + \frac{1}{8} \omega^\mu_\nu [\gamma_\mu, \gamma^\nu]_L \psi^I \quad (9)$$

(with spinor indices $I, L = 1, 2$), where the $\gamma^\mu$ are flat space Dirac matrices in standard representation: $\gamma^0 = i\sigma^3 \otimes \sigma^0$, $\gamma^j = i\sigma^2 \otimes \sigma^j$. From (9) follows $[D, \gamma_5] = 0$ (with $\gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$). Hence, in the massless case the chirality operator $\gamma_5$ generates a global symmetry of the free Lagrangian. In spacetime (1) one gets owing to (8), (9)

$$\gamma^\mu (D\psi)(e_\mu) \equiv \gamma^\mu D e_\mu \psi = \left\{ \gamma^\mu e_\mu - \frac{1}{4} \sum_{j=1}^{3} C^j_{0j} \gamma_j [\gamma_0, \gamma_j] \right\} \psi \quad (10)$$

so that Dirac’s equation takes the form

$$\left\{ e_0 - \gamma_0 \sum_{j=1}^{3} \gamma_j e_j + C + i\gamma_0 m \right\} \psi = 0, \quad (11)$$

\footnote{For the analogous treatment of Maxwell’s equations see ref. [24].}
where \( C := -\frac{1}{2} \sum_{j=1}^{3} C_{0j} = \partial_t \ln(|g|^{1/4}) \) and \( g = \det g_{\mu\nu} \). Spatial translation invariance of the spacetime (11) motivates the ansatz

\[
\psi_k(x, t) = c_k e^{ikx} \begin{pmatrix} \varphi(k, t) \\ \chi(k, t) \end{pmatrix}
\]

with normalization constant \( c_k \), and \( \varphi, \chi \) denote Weyl-spinors. Eq. (11) can then be rearranged as a coupled system for those spinors:

\[
e_0 \left( e^{+imt}|g|^{1/4} \varphi(k, t) \right) - ip\sigma e^{+imt}|g|^{1/4} \chi(k, t) = 0
\]

\[
e_0 \left( e^{-imt}|g|^{1/4} \chi(k, t) \right) - ip\sigma e^{-imt}|g|^{1/4} \varphi(k, t) = 0
\]

where the physical 3-momentum \( p \) is defined to be

\[
p_j(t) = k_j/\alpha_j(t).
\]

In the massless case the problem simplifies as the system (13) decouples by setting \( \chi = \mp \varphi \). The negative sign defines the eigenspinor of \( \gamma_5 \) with eigenvalue -1, while the positive sign defines that one with eigenvalue +1. From (12) follows for the two chirality eigenstates: \( \psi^{(\mp)}_k = c_k^{(\mp)} e^{ikx} (\varphi^{(\mp)}, \mp \varphi^{(\mp)}) \). We focus now on the massless case and introduce the Weyl-spinors

\[
\phi^{(\mp)}(k, t) = \begin{pmatrix} \phi_1^{(\mp)}(k, t) \\ \phi_2^{(\mp)}(k, t) \end{pmatrix}
\]

\[
\phi_L^{(\mp)}(k, t) = |g(t)|^{1/4} \exp \left[ \pm (-1)^L i \int_{t_0}^{t} p_3(x) dx \right] \varphi_L^{(\mp)}(k, t).
\]

From (13) one gets

\[
\partial_t \phi^{(j, \mp)}(k, t) - \Omega^{(\mp)}(k, t) \phi^{(j, \mp)}(k, t) = 0,
\]

where the two linearly independent spinor solutions for each chirality state are labeled by \( j \), and where the matrix \( \Omega^{(\mp)} \) is defined by

\[
\Omega^{(\mp)}(k, t) = \begin{pmatrix} 0 & \mathcal{P}^{(\mp)}(k, t) \\ -[\mathcal{P}^{(\mp)}(k, t)]^* & 0 \end{pmatrix}
\]

with...
\[ P^{(\mp)}(k, t) = \pm (ip_1 + p_2) \exp \left[ \mp 2i \int_{t_{\tilde{A}}}^t p_3(x) dx \right], \quad (18) \]

\( t_{\tilde{A}} \geq 0 \). The system \((16), (17)\) has the following useful property: Given any solution \( \Phi \equiv (\Phi_1, \Phi_2)^T \), then \( \Psi \equiv (\Psi_1, \Psi_2)^T \) with \( \Psi_1 = \Phi_2^*, \Psi_2 = -\Phi_1^* \) is a second solution orthogonal to \( \Phi \) w.r.t. the hermitean scalar product

\[ (\Phi(k, t), \Psi(k, t)) = \sum_{L=1}^{2} \Phi_L^*(k, t) \Psi_L(k, t). \quad (19) \]

Solving \((16)\) one gets with \((12), (15)\) the pertaining four bispinor solutions:

\[ \psi^{(j, \mp)}_k(x, t) = c^{(j, \mp)}_k e^{ikx} \left( \begin{array}{c} \varphi^{(j, \mp)}(k, t) \\ \mp \varphi^{(j, \mp)}(k, t) \end{array} \right), \quad (20) \]

\( (j = 1, 2) \). Note that \((18)\) implies \( P^{(-)}(k, t) \rightarrow P^{(+))(k, t)} \) when \( k \rightarrow -k \). Hence, to obtain all solutions of \((16)\), one must only find the two solutions of, say, the negative chirality case. The positive chirality solutions are then:

\[ \phi^{(j, +)}(k, t) = \phi^{(j, -)}(-k, t). \quad (21) \]

An appropriate scalar product for Dirac-spinors is defined by

\[ \langle u, v \rangle = \int_{\Sigma} * F_{u,v} \]

with \( \Sigma \) a spacelike Cauchy hypersurface and * the duality operator. Furthermore, \( F_{u,v} \) is given by

\[ F_{u,v}(x) \equiv [F_{u,v}(x)]_\mu \Theta^\mu := \overline{u(x)} \gamma_\mu v(x) \Theta^\mu, \quad (23) \]

where \( u \) and \( v \) are solutions of \((11)\), and \( \overline{\Phi} \) denotes the Dirac adjoint. The integral on the r.h.s. of equation \((22)\) is independent of the choice of the hypersurface \( \Sigma \), which can be seen as follows: The 4-form

\[ * \delta F_{u,v} = -\Theta^0 \wedge \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \{ \epsilon_\mu ([F_{u,v}]^\mu) + C^\mu_{\mu\nu} [F_{u,v}]^\nu \} \]

(with \( \delta \) the codifferential or adjoint operator) vanishes, since
\[ e_\mu([F_{u,v}]^\mu) + C_{\mu\nu}^{\mu}[F_{u,v}]^\nu = 0. \tag{24} \]

One obtains now the desired result as a consequence of Gauss' theorem: \( \int_{\partial V} * F_{u,v} = \int_V * \delta F_{u,v} = 0. \) Relation (24) appears e.g. also in Maxwell theory as Lorenz gauge condition, if instead of (23) the 1-form \( A = A_\mu(x) \Theta^\mu \) is used \[24\]. The inner product (22) is sesquilinear, satisfies \( \langle u, v \rangle = \langle v, u \rangle^* \) and is positive definite. Hence, eqs. (22) and (23) define a hermitean scalar product, which will be used to define the normalization condition for Dirac-spinors:

\[ <\psi_{(jc)}^{(k)} , \psi_{(j'c')}^{(k')} > = \delta_{jj'} \delta_{cc'} \delta(k - k') \tag{25} \]

with \( j, j' = 1, 2; c, c' = -, + \). The covectors \( \Theta^\mu \) are related to the covectors of the coordinate frame via \( \Theta^\mu = D^\mu_l(x) dx^l \) (greek indices refer to the orthonormal, latin ones to the coordinate frame), where the \( D^\mu_l(x) \) denote the vierbein fields. Correspondingly, one has for the vectors \( e_\mu, \partial_l \) the relation \( e_\mu = (D^{-1}(x))^l_\mu \partial_l \). In a coordinate frame, eq. (23) reads:

\[ F_{u,v}(x) = (\tilde{F}_{u,v}(x))_l dx^l \equiv u(x) \tilde{\gamma}_l(x)v(x) dx^l \quad \text{with} \quad \tilde{\gamma}_l(x) = D^\mu_l(x) \gamma_\mu, \]

and (24) takes then the form \( \partial_l(\sqrt{|g|}(\tilde{F}_{u,v})_l^l) = 0 \).

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III. WEYL THEORY IN PLANAR BI UNIVERSES

For simplicity we specialize now to pBI spacetimes. Moreover, since we are also interested in the evolution of a BI background at very early times, we assume that the scale factors are of power law type. This is motivated by the fact that close to the singularity, BI spacetimes can be well approximated by suitable vacuum Kasner geometries \[25\]. Hence, in the following we consider background geometries described by eq. (1) with \( \alpha_1 = \alpha_2 = t^\nu, \alpha_3 = t^{1-\nu} \):

\[ ds^2 = dt^2 - t^{2\nu}(dx^1)^2 - t^{2\nu}(dx^2)^2 - t^{2\nu}(dx^3)^2, \tag{26} \]

\( (\mu > 0) \). The components of the 3-momentum \( p \) are \( p_1 = k_1 t^{-\nu}, p_2 = k_2 t^{-\nu} \) and \( p_3 = k_3 t^{1-\nu} \). In the following we put \( k_3 \neq 0 \). Defining

\[ k_\pm = (k_2 \pm i k_1) e^{\pm 2ik_3 t^{\nu}/\mu} \tag{27} \]

2The case \( k_3 = 0 \) is exactly solvable, see sect. IV.A.
one has with (18)

\[ P^{(-)}(k, t) = k^\nu e^{-2ik_3 t^\mu / \mu}. \]  

(28)

In what follows we seek negative chirality solutions of eq. (16). To this end we define the (negative chirality) operator \( \hat{\Omega}_k \) acting on the function \( \xi \):

\[ \hat{\Omega}_k[\xi] = \xi_A + \int_{t_A}^{t} \Omega^{(-)}(k, y) \xi(k, y) dy \]  

(29)

with \( \Omega^{(-)} \) given in (17), \( \xi_A := \xi(k, t_A) \) and \( t_A \geq t_A \geq 0 \). With the help of this operator the problem of solving the system of differential equations (16) can be cast into the equivalent fixed point problem \( \hat{\Omega}_k[\phi^{l, -}] = \phi^{l, -} \). It can be shown that such a fixed point must exist, and that it is the only one. Moreover, for any arbitrary continuous function \( \xi \) holds [26]: \( \hat{\Omega}_k^n[\xi] \to \phi \), i.e.:

\[ \phi^{l, -}(k, t) = \lim_{n \to \infty} \left( \hat{\Omega}_k^n[\xi] \right) (t) = \left[ 1 + \sum_{n=1}^{\infty} \int_{t_A}^{t} dt_1 \int_{t_A}^{t_1} dt_2 ... \int_{t_A}^{t_{n-1}} dt_n \Omega^{(-)}(k, t) \right] \xi^{(l)}(k, t_A) \]  

(30)

with \( 1_n \) the \( n \times n \) unit matrix. If one adopts the initial condition \( \xi^{(l)}(k, t_A) = \phi^{l, -}(k, t_A) \) as a special choice, then eq. (30) can be written in terms of a time ordered exponential:

\[ \phi^{l, -}(k, t) = T \exp \left( \int_{t_A}^{t} \Omega^{(-)}(k, y) dy \right) \phi^{l, -}(k, t_A). \]

This expression satisfies by definition eq. (16). It is reminiscent of a result by Tsamis and Woodard [27] who determined massless scalar field solutions in fFLRW backgrounds with arbitrary expansion factor.

For convenience we introduce the new time variable

\[ s(t) = t^\mu \]  

(31)

with \( s_A := s(t_A) \), and define the parameter

\[ \delta = (1 - \nu) / \mu. \]  

(32)
This parameter can vanish or even take on negative values provided \( t_A > 0 \). Eq. (30) can now be written in compact form:

\[
\phi^{(l,-)}(k, t) = K_k^{(-)}(t|t_A) \phi^{(l,-)}(k, t_A),
\]

where the Weyl TEO \( K_k^{(-)} \) for the negative chirality spinors reads:

\[
K_k^{(-)}(t|t_A) = \sum_{n=0}^{\infty} \begin{pmatrix} I_n(s) & 0 \\ 0 & I_n^*(s) \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{k_\lambda}{\kappa} \end{pmatrix}^n
\]

with \( I_0 := 1, I_n(s) := I_n(s; \sigma_A, 1) \) for \( n \geq 1 \), where

\[
I_n(s; a, b) = \left( \frac{k_3 s}{\mu} \right)^n \int_a^b \sigma_1 \int_{\sigma_A}^{\sigma_1} \cdots \int_{\sigma_A}^{\sigma_n-1} \prod_{l=1}^{n} \sigma_{l}^{i} e^{i(-1)^{l} \sigma_{l} \tau} \]

\( (\sigma_A \leq a \leq b \leq 1) \), and where

\[
\tau(s) = 2k_3 s/\mu \equiv 2k_3 t^\mu/\mu,
\]

\[
\sigma_A(s) = s_A/s \equiv \tau_A/\tau \geq 0,
\]

\[
\kappa = \sqrt{k_1^2 + k_2^2} \equiv \sqrt{k_+ k_-}, \quad \eta_\delta = \frac{\kappa^2}{2|k_3|^2}.
\]

It will turn out that, provided \( \mu, \delta > 0 \), the quantity \( \eta_\delta \) will play the role of a natural expansion parameter at late times \( \tau \), i.e. it can be taken to be small when \(|\tau|\) becomes large. Note that \(|\tau| \gg 1\) does not imply: \( s \gg 1 \). It can also be satisfied for any \( s = O(\mu) \) by \(|k_3| \gg 1\). As a consequence, \(|k_3|\) can be regarded as a large and \( \eta_\delta \) as a small quantity, if \(|\tau| \gg 1\).

Equations (33) - (35) together with (15), (20) represent the two linearly independent exact negative chirality Weyl spinor solutions of a massless spin \( \frac{1}{2} \) field in spacetimes (26). Recently, motivated by this result, a special parameter transformation (PT) has been introduced, which transforms an exact (and approximate, resp.) massless spinor solution \( \psi \) propagating in a background \( ds^2 \), into a new exact (and approximate, resp.) solution \( \psi' \) with different background \( ds'^2 \) [21]. The new line element \( ds'^2 \) and the new solution \( \psi' \) belong to the same equivalence class of line elements as \( ds^2 \), and to
the same equivalence class of exact (and approximate, resp.) solutions as \( \psi \).

The explicit transformation of (26) can be performed according to the rule:

\[
\nu \rightarrow \nu' = 1 - a(1 - \nu), \quad \mu \rightarrow \mu' = a\mu, \tag{39}
\]

\((a \neq 0 \text{ real})\). This transformation can be viewed as a map \((M, g) \rightarrow (M', g')\), where w.r.t. the coordinate frame

\[
g_{\alpha\beta}(x) = \text{diag} (1, -x_0^{2\nu}, -x_0^{2\nu}, -x_0^{2-2\mu}), \tag{40}
\]

and

\[
g'_{\alpha\beta}(x) = \text{diag} (1, -x_0'^{2\nu'}, -x_0'^{2\nu'}, -x_0'^{2-2\mu'}). \tag{41}
\]

\(M, M'\) are differentiable manifolds endowed with metrices \(g\) and \(g'\), and \(\dim M' \equiv \dim M = n\) (with \(M = M'\) and \(n = 4\) here).

As an alternative one could employ the (nonlinear) diffeomorphism \(f^{(a)} : (M, g) \rightarrow (\tilde{M}, \tilde{g})\), defined by:

\[
x_0 \rightarrow \tilde{x}_0 \equiv x_0^a, \quad x_j \rightarrow \tilde{x}_j \equiv ax_j, \tag{42}
\]

\((j = 1, 2, 3; x_0 \equiv t > 0, a \neq 0)\) together with the additional restriction:

\[
\tilde{k} \tilde{x} = k x. \tag{43}
\]

\(\tilde{M}\) denotes a third differentiable manifold with metric \(\tilde{g}\), and \(\dim \tilde{M} \equiv n\).

In practice it is often more convenient to utilize \(f^{(a)}\) instead of (39). Applying this diffeomorphism to (20) and (26), one obtains the new line element \(d\tilde{s}^2\) and spinor \(\tilde{\psi}\), which are Weyl-related to their PT-transformed counterparts, i.e.: \(d\tilde{s}^2 = \Lambda^2 ds^2\) and \(\tilde{\psi} = \Lambda^{-3/2}\psi'\), with \(\Lambda(x) := ax_0^{a-1}\) [21].

In more mathematical terms, the relationship between \(f^{(a)}\) and PT (39) can be described as follows: Consider a tangent vectorfield \(X_p \in T_p(M)\) to \(M\) at \(p\) and define the vector space homomorphism \(f^{(a)}_\ast : T_p(M) \rightarrow T_q(\tilde{M})\) by \(f^{(a)}_\ast(X_p)(h) := X_p(h \circ f^{(a)})\). The points \(p \in M\) and \(q = f^{(a)}(p) \in \tilde{M}\) are in local coordinates given by \((x_0, ..., x_{n-1})\) and \((\tilde{x}_0, ..., \tilde{x}_{n-1})\), and \(h \in C^\infty(\tilde{M})\). \(f^{(a)}_\ast\) is the differential or Jacobian of \(f^{(a)}\). The dual map \(f^{(a)}_* : T^*_q(\tilde{M}) \rightarrow T^*_p(M)\) is defined by \((f^{(a)}_* \omega_q)(X_p) := \omega_q(f^{(a)}_\ast(X_p))\), with \(\omega_q \in T^*_q(M)\). Now, consider \(f^{(a)}\) as conformal map satisfying:

\[
f^{(a)}_* \tilde{g}|q = \Lambda^2 g|_p, \tag{44}
\]
where \( \tilde{g} \in B(\hat{M}, T^*_q(\hat{M}) \otimes T^*_q(\hat{M})) \equiv T^*(\hat{M}) \otimes T^*(\hat{M}) \) and likewise \( g \in T^*(M) \otimes T^*(M) \) are locally represented by \( \tilde{g}|_q = \tilde{g}_{\mu \nu}(x) \, d\bar{x}^\mu \otimes d\bar{x}^\nu \) and \( g|_p = g_{\mu \nu}(x) \, dx^\mu \otimes dx^\nu \). Since with basis vectors \( \{\partial_\alpha\}_\alpha \in T_p(M) \) holds:

\[
(f_q^{(a)\ast} \tilde{g}|_q)(\partial_\alpha, \partial_\beta) = \tilde{g}|_q(f_p^{(a)}(\partial_\alpha), f_p^{(a)}(\partial_\beta)),
\]
eq (44) assumes in a coordinate neighborhood the form:

\[
\tilde{g}_{\alpha \beta}(\bar{x}) \frac{\partial(\bar{x}^\alpha \circ f^{(a)})}{\partial x^\nu} \frac{\partial(\bar{x}^\beta \circ f^{(a)})}{\partial x^\mu} = \Omega^2(x) g_{\mu \nu}(x).
\]

(45)

\( \bar{x}^\alpha \) denotes the \( \alpha \)-th coordinate function \( \bar{U}_q \to \mathbb{R} \), with \( \bar{x}^\alpha \circ f^{(a)}(p) \equiv \bar{x}^\alpha(q) := \bar{x}_\alpha (\bar{U}_q \subset \hat{M}) \). Then one gets with (40), (42), (45):

\[
\tilde{g}_{\alpha \beta}(\bar{x}) = \text{diag} (1, -\bar{x}_0^{2\tilde{\nu}}, -\bar{x}_0^{2\tilde{\mu}}, -\bar{x}_0^{2-2\tilde{\mu}}),
\]

where

\[
\tilde{\nu} = 1 - \bar{a}(1 - \nu), \quad \tilde{\mu} = \bar{a} \mu, \quad (\bar{a} \equiv 1/a).
\]

(46)

Eq. (46) defines the to (39) inverse PT, a consequence of \( f^{(a)\ast} \) having the "opposite direction" compared to \( f^{(a)} \). As result we get: \( \tilde{g} \equiv g' \). Thus, (12) together with constraint (13) is equivalent to (39) together with normalization condition (27) for massless Dirac-spinors [3]

As has already been mentioned, the PT (and the diffeomorphism \( f^{(a)} \), resp.) generates all solutions of a given equivalence class of exact spinor solutions, with the only prerequisite being the knowledge of a single arbitrary exact solution of this class. It is now a general feature of every such equivalence class that in the limiting case \( a \to 0 \) its solutions must always approach those solutions of the Weyl-Dirac equation with background spacetime given by that special fFLRW line element (26) with \( \nu = 1, \mu = 0 \) [21]. It is shown in App. C that the solutions (33) - (35) indeed satisfy this criterion.

The above presented results are exact. Since an analytical computation of eq.s (34), (35) can only be carried out for a few special cases such as \( \delta = 1 \) or \( k_3 = 0 \) (see sect.s IV, A, B), we will focus attention on the determination of an analytical expression for the approximate Weyl TEO. Before starting we note that according to (35) holds:

\[
I_n(s) \equiv I_n^{(1)}(\bar{\tau})I_n^{(2)}(\tau) \quad \text{with} \quad \bar{\tau}(s) := \kappa s^6/\mu \propto \sqrt{\mu_6} |\tau|.
\]

In all asymptotic calculations that follow it is always understood

\[3\text{ Eq. (27)} \text{ enforces the correct transformation of the normalization constants } c_k^{(j, \mp)} \text{ of the spinors (29), when the PT is applied, which is automatically taken care of by } f^{(a)} \text{ together with (13). Of course, (39) and (12) are not equivalent, neither are (27) and (13).}\]
that $|\tau| \gg 1$ ($s \gg \mu/2|k_3|$) applies, so that the phase factors in (35) rapidly oscillate. Clearly, $|\tau| \gg 1$ does not necessarily imply $|\tilde{\tau}| \gg 1$, and vice versa. Furthermore, in all that follows we choose $s_A$ so, that it is certainly within the small-time regime. Hence, $s_A \lesssim \mu/2|k_3|$. We put now $\epsilon_l = 1 - \sigma_l$, i.e. for $\sigma_A > 0$: $0 \leq \epsilon_1 \leq \ldots \leq \epsilon_{n-1} \leq \epsilon_n < 1$, and write for $n \neq l$: $\sigma_l^{\delta-1} \approx e^{(1-\delta)(1-\sigma_l)}$, i.e.

$$\prod_{l=1}^{n} \sigma_l^{\delta-1} \approx \sigma_n^{\delta-1} e^{(1-\delta)(n-1)} \exp \left[ (\delta - 1) \sum_{l=1}^{n-1} \sigma_l \right].$$

(47)

Insertion of (47) into (35) gives:

$$\widetilde{I}_n(s; a, b) = e^{\delta-1} \left( \frac{k s^\delta}{e^{\delta-1} \mu} \right)^n \int_a^b d\sigma_1 \ldots \int_{\sigma_A}^{\sigma_n} d\sigma_n \sigma_n^{\delta-1} e^{i(1-\sigma_n)\tau} \prod_{l=1}^{n-1} e^{(\delta-1)\sigma_l + i(1-\sigma_l)\tau}$$

where analogously to the above definition holds: $\tilde{I}_0(s) := 1$, $\tilde{I}_n(s) := \tilde{I}_n(s; \sigma_A, 1)$. Substituting $\tilde{I}_n(s)$ for $I_n(s)$ in (34) leads to the approximate negative chirality Weyl TEO $\tilde{K}_k^{(-)}(t|t_A)$. Note that we have fully retained in (47) the n-th factor $\sigma_n^{\delta-1}$. It ensures that $\tilde{K}_k^{(-)}$ represents at early and late times a sensible approximation of the exact Weyl TEO (34), (35). This can be easily verified for small times, because $I_n(s) \equiv \tilde{I}_n(s)$ for $n = 0, 1$. These are already the most dominant terms when $s(t) - s_A \ll 1$. Moreover, in this case holds: $\sigma_A \approx 1$, so that $\sigma_l \approx 1$ and hence $\epsilon_l \ll 1$ for all $l$. As a consequence, $I_n(s) \approx \tilde{I}_n(s)$ for $n > 1$ and thus $\tilde{K}_k^{(-)} \approx K_k^{(-)}$. The more complicated asymptotic case is dicussed in App. A.

In the following only spacetimes satisfying $0 < \delta \leq 1$, $\mu > 0$ will be considered. Leaving details of the calculation to App. A, one finally gets (tildes will be dropped, since from now on, except in sect. IV, B, we are only concerned with the approximate Weyl TEO):

$$K_k^{(-)}(t|t_A) = \begin{pmatrix} (K_k^{(-)}(t|t_A))_{11} & (K_k^{(-)}(t|t_A))_{12} \\ (K_k^{(-)}(t|t_A))^*_{12} & (K_k^{(-)}(t|t_A))^*_{11} \end{pmatrix}$$

(49)

with
\[
\left( K_{k}^{(-)}(t|t_{A}) \right)_{11} = 1 - \frac{\kappa s_{\delta}}{\mu} \int_{0}^{1} dz \frac{e^{\lambda z} z^{1-\delta}}{1-z} \frac{R(z; x)}{Z(0; x)},
\]
\[
\left( K_{k}^{(-)}(t|t_{A}) \right)_{12} = \frac{k_{\perp} s_{\delta}}{\mu} e^{-i\tau} \int_{0}^{1-\delta} dz \frac{e^{\lambda z} z^{1-\delta}}{1-z} \frac{Z(z; x)}{Z(0; x)},
\]

where (with Bessel functions \( J_{\pm \lambda} \))

\[
R(z; x) = J_{-\lambda}(D x) J_{\lambda}(D x e^{(1-\delta)z}) - J_{\lambda}(D x) J_{-\lambda}(D x e^{(1-\delta)z}),
\]
\[
Z(z; x) = J_{-\lambda}(D x) J_{-\lambda^{*}}(D x e^{(1-\delta)z}) + J_{\lambda}(D x) J_{\lambda^{*}}(D x e^{(1-\delta)z}).
\]

and \( F_{\delta}(x) := F_{1}(\delta; \delta+1; x) \) denotes a confluent hypergeometric function.

\[
x(s) = \frac{\kappa s_{\delta}}{\mu(1-\delta)}, \quad \lambda(s) = \frac{1}{2} + \frac{i \tau(s)}{2(1-\delta)},
\]
\[
D(\delta) = \frac{1}{\delta} \frac{(1-\delta)F_{\delta}(\delta-1)}{\sinh(1-\delta)}. \]

The constant \( D \) results from a second approximation (s. App. A). Its purpose is to improve in the limit \( \delta \to 0 \) eq.s (50), (51), especially if \( s_{A} = 0, \) or \( \sigma_{A} \to 0. \) We can undo this approximation by simply setting \( D \equiv 1. \)

The asymptotic expansions of eq.s (50) for \( 0 < \delta \leq 1/2 \) can be found in App. B, and it is shown in App. C that the approximate solutions (49) - (51) tend (with \( D \equiv 1 \)) under PT (39) and (42), (43), resp., for \( a \to 0 \) to the correct fFLRW result.

**IV. EXACT AND APPROXIMATE RESULTS**

We consider first three exactly solvable models serving as testing ground for eq.s (33), (49) - (51): massless fermions moving in hypersurfaces \( x^{3} = \text{const.} \) (\( \delta \) arbitrary), in a fFLRW \( (\delta = 1) \) and in an anisotropic stiff-fluid background \( (\delta = 1/2). \) In the fourth example, where the background is given by the planar anisotropic Kasner universe \( (\delta = 1/4), \) no exact solutions are known.
A. Propagation in hypersurfaces

In contrast to the case $|k_3| \gg 1$, which, as we have seen in sect. III, can be related to the asymptotic regime, the case $|k_3| \to 0$ has nothing to do with the limiting case $t \to 0$. Rather, if $k_3 = 0$, one gets spinors propagating in timelike hypersurfaces $x^3 = \text{const}$. From (16), (17) and (28) follows then:

$$\partial_t \phi_1^{(j,-)} = k_+ t^{-\nu} \phi_2^{(j,-)}, \quad \partial_t \phi_2^{(j,-)} = -k_- t^{-\nu} \phi_1^{(j,-)}.$$  

The exact solutions are given by

$$\phi^{(1,-)}(k, t) = A_1 \begin{pmatrix} \cos y \\ -\frac{\kappa}{k_+} \sin y \end{pmatrix}, \quad \phi^{(2,-)}(k, t) = A_2 \begin{pmatrix} \sin y \\ \frac{\kappa}{k_+} \cos y \end{pmatrix}$$  

with $k \equiv (k_1, k_2, 0)^T$, and

$$y = \kappa \frac{t^{1-\nu} - t_A^{1-\nu}}{1 - \nu}$$  

which can be either directly calculated or by virtue of (33) - (35) with $I_n(s) \equiv \left[\frac{\kappa(s^k - s^d)}{\mu^d}\right]^n/n!$. These solutions are oscillating with time-dependent frequency $\omega(t) = \kappa t^{-1} \ln(t/t_A)$, if $\nu = 1$, and $\omega(t) = \kappa t^{-\nu} / (1 - \nu)$ otherwise.

Approximation (47) cannot be expected to yield sensible results if $k_3 = 0$, but ignoring this for the time being one obtains (with $D = 1$) from (A21):

$$V_k(z; s) \xrightarrow{k_3 \to 0} i \sin[x(s)(e^{(1-\delta)z} - 1)], \quad U_k(z; s) \xrightarrow{k_3 \to 0} \cos[x(s)(e^{(1-\delta)z} - 1)],$$  

and executing the integrals in (50) leads with

$$\xi(s) = x(s)(e^{(1-\delta)(1-\sigma_A(s))} - 1)$$  

to:

$$K_k^{(-)}(t|t_A) \approx \left( \begin{array}{c} \frac{\cos[\xi(s)]}{\kappa} - \frac{k_3}{\kappa} \sin[\xi(s)] \\ -\frac{k_3}{\kappa} \sin[\xi(s)] \cos[\xi(s)] \end{array} \right).$$  

The closer $\delta$ or $\sigma_A$ is to unity, the better this result will be. In fact, eq. (57) would represent the exact TEO following directly from eq.s (33), (35), if one replaces $\xi$ with $\xi_{\text{exact}} \equiv y = x(1 - \delta)(1 - \sigma_A^d) / \delta$. Using
\[ \phi^{(1,-)}(\mathbf{k}, t_A) = A_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi^{(2,-)}(\mathbf{k}, t_A) = \frac{\kappa A_2}{k_+} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

one arrives by use of (33), (57) again at (54) with, however, \( y = \xi_{exact} \). For \( \delta \to 0 \) (\( \nu \to 1 \)) TEO (57) does not reproduce the correct logarithmic behavior. Nevertheless, the approximate expressions (50) produce reasonable results for \( \delta \) not too small.

If one applies in addition the second approximation (53), one gets:

\[ K_k^{(-)}(t|t_A) \approx \begin{pmatrix} 1 - \frac{1}{D_k} + \frac{\cos[D_k]}{D_k} & \frac{\kappa k_+}{\kappa} \sin[D_k] \\ \frac{\kappa k_+}{\kappa} \cos[D_k] & 1 - \frac{1}{D_k} + \frac{\kappa}{\kappa} \cos[D_k] \end{pmatrix} \]

This operator is unitary only for \( D = 1 \) and \( D \to \infty \). Hence, (53) is not applicable when \( k_3 = 0 \).

**B. Flat FLRW background**

We now study the behavior of massless spin \( \frac{1}{2} \) particles in a radiation-dominated universe with line element (26) where \( \mu = \nu = 1/2 \) (\( \delta = 1 \)). In this case the TEO given by eq.s (49), (50) is exact and reads:

\[
\begin{align*}
\left( K_k^{(-)}(t|t_A) \right)_{11} & = e^{-2ik_3(\sqrt{t} - \sqrt{t_A})} \left\{ \cos[2k(\sqrt{t} - \sqrt{t_A})] + i \frac{k_3}{k} \sin[2k(\sqrt{t} - \sqrt{t_A})] \right\} \\
\left( K_k^{(-)}(t|t_A) \right)_{12} & = \frac{k_+}{k} e^{-2ik_3(\sqrt{t} + \sqrt{t_A})} \sin[2k(\sqrt{t} - \sqrt{t_A})].
\end{align*}
\]  

(58)

In order to compare with previous results [4], we set \( t_A = t = 0 \) and use

\[ \phi^{(j,-)}(\mathbf{k}, 0) = \begin{pmatrix} 1 \\ \frac{(\xi_3 - k_3)}{k_1 - i k_2} \end{pmatrix} \]

as initial condition. The negative chirality bispinor solutions are given by

\[
\begin{align*}
\psi_k^{(j,-)}(\mathbf{x}, t) & = c_k^{(j,-)} e^{i k \mathbf{x}} \begin{pmatrix} \varphi^{(j,-)}(\mathbf{k}, t) \\ -\varphi^{(j,-)}(\mathbf{k}, t) \end{pmatrix}, \\
\varphi^{(j,-)}(\mathbf{k}, t) & = |g(t)|^{-1/4} e^{2i(-1)^j \kappa \text{sign} k_3 \sqrt{t}} \begin{pmatrix} 1 \\ \frac{(\xi_3 - k_3)}{k_1 - i k_2} \end{pmatrix}
\end{align*}
\]

(59)
(\|g\| \equiv t^3). Analogously, the positive chirality solutions are owing to \(21\):

\[
\psi^{(j,+)}_k(x, t) = c^{(j,+)}_k e^{ikx} \left( \begin{array}{c}
\varphi^{(j,+)}(k, t) \\
\varphi^{(j,+)}(k, t)
\end{array} \right)
\]

\[
\varphi^{(j,+)}(k, t) = |g(t)|^{-1/4} e^{-2i(-1)^j \kappa \text{sgn}k_3 \sqrt{\tau}} \left( \begin{array}{c}
\frac{1}{k_1 - ik_2} \\
\frac{-1}{k_1 - ik_2}
\end{array} \right),
\]

where \(\text{sgn}k_3 = \text{sgn}(k - |k_3|)/2k\). The four solutions \((59), (60)\) establish an orthogonal system of bispinors w.r.t. \(22\). Suitable linear combinations of these chirality eigenspinors yield the outcome of ref. [4], for example:

\[
\sin \zeta \, \psi^{(1,+)}_k(x, t) + \cos \zeta \, \psi^{(2,-)}_k(x, t) =
\]

\[
\frac{1}{\sqrt{2}(2\pi)^{3/2}} e^{ikx} e^{-2i\kappa \text{sgn}k_3 \sqrt{\tau}} \left( \begin{array}{c}
\text{sgn}k_3 \\
0 \\
-k_3 \\
0 + ik_3
\end{array} \right).
\]

C. Anisotropic stiff-fluid model

Next we investigate exact solutions of Einstein’s field equations when the material content is described by a ”perfect fluid” [31]. A special case is given by line element \(26\) with \(\mu = 1, \nu = 1/2 \) \((\delta = 1/2)\). Starting with \((16), (17)\) and \((28)\) one obtains for the negative chirality case \((j = 1, 2)\):

\[
\partial_t \phi^{(j,-)}_1 = \frac{k_+}{\sqrt{\tau}} e^{-i\tau} \phi^{(j,-)}_2, \quad \partial_t \phi^{(j,-)}_2 = -\frac{k_-}{\sqrt{\tau}} e^{-i\tau} \phi^{(j,-)}_1,
\]

where \(\tau \equiv 2k_3 t\). Exact solutions of this system are given by \(30\)

\[
\phi^{(1,-)}_1(k, t) = (-i\tau)^{-\frac{3}{4}} e^{-i\tau} W_{-\frac{1}{4} - i\eta \frac{1}{4}}(i\tau)
\]

\[
\phi^{(1,-)}_2(k, t) = \sqrt{-2i k_3 k_+} (-i\tau)^{-\frac{3}{4}} e^{i\tau} [i\eta W_{-\frac{1}{4} - i\eta \frac{1}{4}}(i\tau) - W_{\frac{3}{4} - i\eta \frac{1}{4}}(i\tau)]
\]
We compare this outcome with TEO result $\text{(50)}$. For $|\tau| \gg 1$, one has:

\begin{align*}
\phi^{(1,-)}_k(k, t) &\sim e^{i\frac{\tau}{4}\text{sign} k_3} (-i\tau)^{-\eta}\left(1 + \frac{1}{1-i\eta-2\eta^2} + O\left(\frac{1}{\tau}\right)\right) \\
\phi^{(2,-)}_k(k, t) &\sim (-i\tau)^{\eta}\left(1 + \frac{1}{1-i\eta-2\eta^2} + O\left(\frac{1}{\tau}\right)\right)
\end{align*}

(67)
\[
\begin{align*}
\left( K_k^{(-)}(t|0) \right)_{11} & \sim [1 + O(\eta)] \\
\left( K_k^{(-)}(t|0) \right)_{12} & \sim \frac{k_2 + i k_1}{\sqrt{-2ik_3}} \left[ -i \sqrt{\pi} \text{sign} k_3 + \frac{e^{-i\tau}}{\sqrt{-i\tau}} [1 + O(\eta)] \right],
\end{align*}
\]

(68)

following from (316). Choosing as initial conditions

\[
\phi^{(j,-)}(k,0) = \begin{pmatrix} \phi_1^{(j,-)}(k,0) \\ \phi_2^{(j,-)}(k,0) \end{pmatrix}
\]

with at least one component satisfying \( \phi_L^{(j,-)}(k,0) \neq 0 \), one gets for \( |\tau| \to \infty \) the two asymptotic Weyl spinor solutions via (33). Conversely, owing to (33) one can also deduce the initial condition from the asymptotic expressions (68). To see this explicitly, use the asymptotic conditions

\[
\phi_1^{(1,-)}(k,t) \xrightarrow{|\tau| \to \infty} 0, \quad \phi_2^{(1,-)}(k,t) \xrightarrow{|\tau| \to \infty} \text{const.} \neq 0,
\]

(69)

which are compatible with the asymptotic expansion (67) of the exact result. As a consequence, the spinor components \( \phi_L^{(1,-)}(k,t) \) \( (L = 1, 2) \) must obey a second condition at \( t = 0 \) following from (68), (69):

\[
\phi_2^{(1,-)}(k,0) = -\frac{\sqrt{2i\kappa_3}}{k_2 + ik_1} \frac{1}{\Gamma(\frac{1}{2})} \left[ 1 + O(\eta) \right] \phi_1^{(1,-)}(k,0).
\]

(70)

This condition represents according to (66) to lowest order in \( \eta \) the correct initial value at \( t = 0 \) of the exact solution \( \phi^{(1,-)}(k,t) \).

The asymptotic expansion of \( \phi^{(1,-)}(k,t) \) reads with (68), (70):

\[
\frac{\phi^{(1,-)}(k,t)}{\phi_1^{(1,-)}(k,0)} = \frac{1}{\sqrt{\pi}} \left( \frac{e^{-i\tau}}{\sqrt{-i\tau}} \left[ 1 + O(\eta) \right] + O \left( \eta \sqrt{-i\tau} \right) \right).
\]

(71)

The determination of the asymptotic expansion of \( \phi^{(2,-)}(k,t) \) by use of (68) proceeds along the same lines as before: We demand instead of (69) as asymptotic condition (guaranteeing orthogonality of \( \phi^{(1,-)} \) and \( \phi^{(2,-)} \)):

\[
\phi_1^{(2,-)}(k,t) \xrightarrow{|\tau| \to \infty} \text{const.} \neq 0, \quad \phi_2^{(2,-)}(k,t) \xrightarrow{|\tau| \to \infty} 0,
\]

(72)

which implies analogous to (70) the second condition:
\[
\phi^{(2,-)}_2(k,0) = \frac{i(k_2 - i k_1) \text{sign} k_3}{\sqrt{2i k_3}} \Gamma \left( \frac{1}{2} \right) [1 + O(\eta)] \phi^{(2,-)}_1(k,0). \quad (73)
\]

Again, this result is to lowest order in \(\eta\) the correct initial condition for the exact solution \(\phi^{(2,-)}(k,t)\). For \(|\tau| \gg |1|\) one obtains

\[
\frac{\phi^{(2,-)}(k,t)}{\phi^{(2,-)}_1(k,0)} = \left( \frac{1 + O(\eta) + O \left( \frac{\eta}{\sqrt{\tau}} \right)}{i \frac{(k_2 - i k_1) \text{sign} k_3}{\sqrt{2i k_3}} e^{\frac{\tau}{2}} [1 + O(\eta)]} \right). \quad (74)
\]

Comparison of (71), (74) with the asymptotic expansions (67) of the exact results shows agreement apart from the prefactors \((\pm i\tau)^{\frac{\pm}{2}}\eta\). These are essentially given by a phase factor with vanishing frequency at large times (s. App. A) and can be disregarded in the asymptotic limit.

For \(t \to 0\) one gets with eq.s (49) - (51):

\[
K^{(-)}_k(t|0) = \left( \frac{1}{\frac{\mu}{\nu} \delta} s^\delta \left[ 1 + O(s^{2\delta}) \right] \right), \quad (75)
\]

and inserting the exact initial values \(\phi^{(j,-)}(k,0)\) into (33) yields the correct behavior at early times, eq. (66).

D. Anisotropic axisymmetric Kasner model

In this model the background is described by line element (26), where \(\mu = 4/3, \nu = 2/3\) (\(\delta = 1/4\)). This vacuum solution evolves for \(t \to 0\) as the preinflationary limiting case of a pBI background geometry [13].

The system of differential equations reads (\(\tau \equiv 3k^3 t^{4/3}/2\)):

\[
\partial_t \phi^{(j,-)}_1 = \frac{k_+}{t^{2/3}} e^{\frac{3\Phi}{2} k_3 t^{4/3}} \phi^{(j,-)}_2, \quad \partial_t \phi^{(j,-)}_2 = -\frac{k_-}{t^{2/3}} e^{\frac{3\Phi}{2} k_3 t^{4/3}} \phi^{(j,-)}_1. \quad (76)
\]

From (33), (49), (B11) and (B14) one gets (for simplicity: \(t_A = 0\)):
\[ \phi^{(1,-)}_1(k,t) \sim \left( 1 + 2i(1 - \delta) D G(\tau) \left[ \frac{\{1 - iD^2 G(\tau)\}}{\delta} \right] F_\delta(\delta - 1) + \frac{e^{3(1-\delta)}}{\delta} iD^2 G(\tau) F_\delta(3(\delta - 1)) \right. \]

\[ + \left. \frac{e^{-i\tau [1 + D^2 E_1(\tau)]}}{|\tau|^{\delta}} \frac{\Gamma(\delta)}{(-i \text{sign} k_3)^{\delta}} \phi^{(1,-)}_1(k,0) \right) + i \frac{k_+ \sqrt{2\eta \delta}}{\kappa} \left( \frac{\mu}{2} \right)^{\delta} \left( e^{iD^2 \tau E_1(\tau)} \frac{\Gamma(\delta)}{(i \text{sign} k_3)^{\delta}} \phi^{(1,-)}_2(k,0) \right) + \text{...} \]

(77)

where \( E_j, G \) and \( F_\alpha(x) \) have been defined in (B12), (B15). Neglecting terms of order \( \eta \delta \) and imposing again (69) one obtains from (77) the condition:

\[ \phi^{(1,-)}_1(k,0) = -\frac{3}{4} \frac{1}{\Gamma(1/4)} k_+ \left( \frac{3i}{2} k_3 \right)^{-1/4} \phi^{(1,-)}_2(k,0). \] (78)

In the same way one finds with the asymptotic condition (72):

\[ \phi^{(2,-)}_1(k,0) = \frac{4}{3} \frac{1}{\Gamma(1/4)} \frac{1}{k_-} \left[ \left( \frac{3i}{2} k_3 \right)^{1/4} \phi^{(2,-)}_2(k,0) \right] \] (79)

To determine for \(|\tau| \gg 1\) the spinors \( \phi^{(1,-)} \) and \( \phi^{(2,-)} \), one substitutes eq. (78) into (77) and gets \((\delta \equiv 1/4)\):

\[ \phi^{(1,-)}_1(k,t) \sim A^{(1,-)}_1(k) \frac{e^{-i\tau}}{|\tau|^{1-\delta}} [1 + O(\eta \delta \tau^{2\delta-1})] [1 + O(\eta \delta)] \phi^{(1,-)}_2(k,0) \] (80)

with \( A^{(1,-)}_1(k) := \frac{i}{2} \text{sign} k_3 \left( \frac{\mu}{2} \right)^{\delta-1} k_+ / |k_3|^\delta \), and use of (78) and

\[ \phi^{(1,-)}_2(k,t) = \left( -K^{(-)}_k(t|0) \right)_{12} \phi^{(1,-)}_1(k,0) + \left( K^{(-)}_k(t|0) \right)_{11} \phi^{(1,-)}_2(k,0) \] together with (B11), (B14) yields:

21
\begin{equation}
\phi_2^{(1,-)}(k, t) \sim [1 - i \text{sign} k_3 (\mu/2)^{2\delta - 2} (D/2) \{ _1F_1(1; 1 + \delta; 1 - \delta)/\delta + O(\eta_\delta) \} \\
\times \eta_\delta |\tau|^{2\delta - 1} + O(|\tau|^{\delta - 1})] [1 + O(\eta_\delta)] \phi_2^{(1,-)}(k, 0)
\end{equation}

(81)

where the spinor \( \phi_2^{(1,-)}(k, 0) \) on the r.h.s of eq.s (80), (81) is the lowest order term of the \( \eta_\delta \)-expansion of the exact solution \( \phi_2^{(1,-)}(k, t) \) at \( t = 0 \). In the same manner one obtains:

\begin{equation}
\phi_1^{(2,-)}(k, t) \sim [1 + i \text{sign} k_3 (\mu/2)^{2\delta - 2} (D/2) \{ _1F_1(1; 1 + \delta; 1 - \delta)/\delta + O(\eta_\delta) \} \\
\times \eta_\delta |\tau|^{2\delta - 1} + O(|\tau|^{\delta - 1})] [1 + O(\eta_\delta)] \phi_1^{(2,-)}(k, 0)
\end{equation}

(82)

\begin{equation}
\phi_2^{(2,-)}(k, t) \sim A_2^{(2,-)}(k) e^{i\tau |\tau|^{\delta - 1} [1 + O(\eta_\delta \tau^{2\delta - 1})] [1 + O(\eta_\delta)] \phi_2^{(2,-)}(k, 0)
\end{equation}

(83)

with \( A_2^{(2,-)} = -(A_1^{(1,-)})^* \). As above \( \phi_2^{(2,-)}(k, 0) \) denotes the lowest order term of the \( \eta_\delta \)-expansion of the exact solution \( \phi_2^{(2,-)}(k, t) \) at \( t = 0 \). The bispinor solutions are again given by (65) (with \(|g(t)| \equiv t^2\)).

The asymptotic and early-time solutions to (76) have recently been determined by use of a standard approach and PT (39) [21]. It has been found agreement in the small- and large-time regimes with the above solutions. The proper matching of these two regimes, however, can only be performed by use of the TEO results.

V. EXPANSION ABOUT CONFORMALLY FLAT SPACETIMES

The class of conformally flat spacetimes is given by (26) with \( \delta = 1 \). In the following we will study spacetimes with small deviations off conformal flatness defined by \( \delta := 1 - \epsilon \) (\( \epsilon \) a small positive parameter). This does not necessarily imply that we are dealing with weakly anisotropic spacetimes only. If one uses as measure of anisotropy the quantity \( \Delta H/H \), with \( H := \sum_j \partial_t \ln \alpha_j/3 \) the average Hubble parameter, and \( \Delta H := \partial_t \ln \alpha_1 - \partial_t \ln \alpha_3 \) the difference of the directional Hubble parameters, it follows that the parameter \( \epsilon \) is, apart from \( \mu \) being close to unity, related to weak anisotropy via

\begin{equation}
\Delta H/H \approx \mu \epsilon/(1 - \mu)
\end{equation}

(83)
But if $\mu \approx 1$, the anisotropy will indeed be large and even tends to infinity when $\mu \to \mu_0 = (1 - 2\epsilon/3)^{-1}$, while $\epsilon \ll 1$ is still maintained.

Before starting with the computation it should be mentioned that for arbitrarily small $\epsilon > 0$ not any two such anisotropic spacetimes of the equivalence class of spacetimes satisfying $\delta = 1 - \epsilon$ are conformally equivalent. Furthermore, for any nonzero $\epsilon$ this class of spacetimes comprises for $\delta < 1$ already all possible types of anisotropic universes with axial symmetry: expansion along all spatial axes, expansion along the $x^1-$ and $x^2-$ axes and contraction along the $x^3-$ axis with increasing, constant and decreasing 3-volume, resp., and contraction along all spatial axes.

To derive suitable expressions for $R, Z$ defined in (51), we utilize the product formula for Bessel functions [29]:

$$J_{\pm \lambda}(x) J_{\pm \lambda}(x e^{\epsilon z}) = e^{\pm \lambda \epsilon z} \sum_{n=0}^{\infty} \left[ \frac{x}{2} \left(1 - e^{2\epsilon z} \right) \right]^n \frac{n!}{n!} J_{\pm \lambda+n}(x) J_{\pm \lambda}(x)$$

$$J_{\pm \lambda}(x) J_{\pm \lambda}(x e^{\epsilon z}) = e^{\pm \lambda \epsilon z} \sum_{n=0}^{\infty} \left[ \frac{x}{2} \left(1 - e^{2\epsilon z} \right) \right]^n \frac{n!}{n!} J_{\pm \lambda+n}(x) J_{\pm \lambda}(x),$$

(84)

with $2\lambda \equiv 1 + i\tau/\epsilon$ and $x \equiv \kappa s^{1-\epsilon}/\mu \epsilon$. Owing to (B4) holds:

$$J_{\pm \lambda+n}(x) J_{\mp \lambda}(x) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dv \, e^{\pm 2\lambda v} e^{in \epsilon \tau v} J_n(2x \cos v),$$

$$J_{\pm \lambda+n}(x) J_{\pm \lambda}(x) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dv \, e^{\mp \tau v/\epsilon} e^{-in \epsilon \tau v} J_{n+1}(2x \cos v).$$

(85)

By splitting the domain and shifting $v \to \frac{\pi}{\epsilon} \left( \frac{n}{2} - v \right)$, one gets:

$$R(z; s) = \frac{i \epsilon}{\pi \tau} e^{\frac{\pi |s|}{2}} \sum_{n=0}^{\infty} \frac{[i \epsilon \kappa_3 \Delta x^{1-e^{2\epsilon z}}]^n}{n!}$$

$$\times \left[ (-1)^{n+1} e^{\lambda \epsilon z} \int_0^\infty dv \, e^{-v} e^{i(n+1)\epsilon v/\tau} J_n[2\Delta x \sin(\epsilon v/|\tau|)] \right]$$

$$+ e^{-\lambda \epsilon z} \int_0^\infty dv \, e^{-v} e^{-i(n-1)\epsilon v/\tau} J_n[2\Delta x \sin(\epsilon v/|\tau|)] \right],$$

(86)
using (51), (84), and neglecting terms $\sim e^{-\text{const} \cdot \tau/\epsilon}$. Provided $|\tau| \gtrsim 1$, the sines in (86) can be linearized, and the integrals can be replaced by [33]:

$$w_n^\pm = \int_0^\infty dv e^{-\rho_n^\pm v} J_n[D_A v] = \frac{(\frac{D_A}{\sqrt{(\rho_n^\pm)^2 + D^2 A_k^2 + \rho_n^\pm}}) \frac{v}{n}}{(\sqrt{D_A^2 + D^2 A_k^2})}, \quad (87)$$

with

$$A_k(s) = \frac{2e \epsilon}{|\tau|} \equiv \frac{\kappa}{|\kappa|} s^{-\epsilon}, \quad \rho_n^\pm(\tau) = 1 - i(1 \pm n) \frac{\epsilon}{\tau}. \quad (88)$$

Insertion of this result into (86) gives

$$R(z; s) \approx -\frac{i}{\pi} e^{\frac{z|z|}{2}} (e^{\lambda x + i \lambda_k} - e^{-\lambda x - i \lambda_k}) \quad (89)$$

with

$$\lambda_k(z; s) = \frac{e^{2z} - 1}{2\epsilon} y_k(s), \quad y_k(s) = \frac{\tau(s)}{2} \frac{D^2 A_k^2(s)}{\sqrt{1 + D^2 A_k^2(s)}}. \quad (90)$$

Here, use has been made of $\rho_n^\pm \approx 1$, which is permissible for large $|\tau|$: substituting (87) for the integrals in (86) yields a series with summands $(n e^{-\rho_n^\pm v} J_n[D_A v])/n!$ where $n_\pm := \pm i(e^{2z} - 1)/4\epsilon$. We split now the summation into $\sum_{n=0}^\infty + \sum_{n=N}^\infty$, with $N(\tau; \epsilon) := \lfloor \frac{\epsilon}{|\tau|} \rfloor \gg 1$ the greatest integer function, and $\epsilon$ a small positive constant. Then one can replace $w_n^\pm$ with $w_n := w_n^\pm(\rho_n^\pm \equiv 1)$ in the first sum, and adding up the first sum to infinity while simultaneously subtracting exactly these additional terms in the second sum gives the desired result plus an additional series with $n \in [N; \infty)$. But the second series is absolutely convergent, hence bounded, and can therefore be made arbitrarily small for sufficiently large $|\tau|$.

Eqs. (89), (A22) put together give

$$\frac{R(z; s)}{Z(0; s)} = -\frac{i D_A k(s) e^{-\epsilon \lambda^* z}}{2 \sqrt{1 + D^2 A_k^2(s)}} \left[ e^{\epsilon z + i \lambda_k} - e^{-i \epsilon z - i \lambda_k} \right] \left[ 1 + O(e^{-\frac{|z|}{\epsilon}}) \right]. \quad (91)$$

Introducing now

$$Q_\pm(k; s) = \frac{\sqrt{1 + D^2 A_k^2(s)} \pm 1}{\sqrt{1 + D^2 A_k^2(s)}}, \quad (92)$$
one gets in the same manner as before with (84), (51):

\[
Z(z; s) = e^{\frac{\pi |v|}{2D\pi x(s)}} \left[ Q_+ e^{-\lambda^* z + i\Omega_k} + Q_- e^{\lambda^* z - i\Omega_k} \right],
\]

(93)

and finally with (50):

\[
\left( K^{(-)}(t | t_A) \right)_{11} = 1 + \frac{i y_k(s)}{2D} Q_+ \left[ \mathcal{I}_1(k; s) - \mathcal{I}_2(k; s) \right],
\]

\[
\left( K^{(-)}(t | t_A) \right)_{12} = \frac{k_+}{2\mu} s^{1-\epsilon} e^{-i\tau} \left[ Q_- \mathcal{I}_1^*(k; s) + Q_+ \mathcal{I}_2^*(k; s) \right],
\]

(94)

where

\[
\mathcal{I}_1(k; s) = \int_0^{1-\sigma_A} dz \frac{e^{\epsilon z + i\Omega_k(z;s)}}{(1-z)^\epsilon}, \quad \mathcal{I}_2(k; s) = \int_0^{1-\sigma_A} dz \frac{e^{-i\epsilon z - i\Omega_k(z;s)}}{(1-z)^\epsilon},
\]

(95)

with \(0 \leq \sigma_A(s) \equiv s_A/s \equiv \tau_A/\tau \leq 1\). For \(\sigma_A > 0\), one obtains:

\[
\mathcal{I}_1 \equiv \frac{1}{2\epsilon} \int_0^\epsilon dv e^{iy_k(s)\frac{v}{2\epsilon}} \left[ 1 - \epsilon \left\{ \ln(1 + v) - \ln \left( 1 - \frac{\ln(1 + v)}{2\epsilon} \right) \right\} + O(\epsilon^2) \right],
\]

(96)

where we defined

\[
\mathcal{E}(s) = e^{2\epsilon[1-\sigma_A(s)]} - 1.
\]

(97)

Because \(v \leq \mathcal{E} = O(\epsilon) \ll 1\), one has \(\ln(1 + v) \approx v\), where convergence of the r.h.s. of (96) requires \(v/2\epsilon \leq \mathcal{E}/2\epsilon < 1\), and gets eventually [33]:

\[
\mathcal{I}_1(k; s) = \frac{e^{i[1-\mathcal{S}]y_k(s)} - 1}{iy_k(s)} + \epsilon \frac{e^{iy_k}}{iy_k} \left[ - \{1 - \mathcal{S} + \ln(\mathcal{S})\} e^{-i\mathcal{S}y_k} + \ln(\mathcal{S}) + \frac{e^{-i\mathcal{S}y_k} - e^{-iy_k}}{iy_k} + \tilde{\text{Ci}}(\mathcal{S}y_k) - i \text{Si}(\mathcal{S}y_k) - \left\{ \text{Ci}(y_k) - i \text{Si}(y_k) \right\} \right] + O(\epsilon^2)
\]

(98)

with constraint \(\mathcal{S} > 0\), where \(\mathcal{S}(s) = 1 - \mathcal{E}(s)/2\epsilon\), \(\tilde{\text{Ci}}(x) = \text{Ci}(x) - \log x\).
Ci(x), Si(x) are the Cosine and Sine integrals, resp. Note that the r.h.s. of (98) is only defined for $S > 0!$ Since $S \to -\epsilon$ when $s \to \infty$, we need the analytic continuation of $I_1$:

\[
I_1^{(e)}(k; z) = \frac{e^{i[1-S(z)]y_k(z)} - 1}{iy_k(z)} + \epsilon \frac{e^{iy_k}}{iy_k} \left[ - \left\{ 1 - S + \log(S) \right\} e^{-iS y_k} + \log(S) \right.
\]

\[
+ \frac{e^{-iS y_k} - e^{-iy_k}}{iy_k} + \tilde{\text{Ci}}(S y_k) - i \text{Si}(S y_k) - \left\{ \tilde{\text{Ci}}(y_k) - i \text{Si}(y_k) \right\} \right] + O(\epsilon^2)
\]

where $z$ is now a complex variable, and $\ln[S(s)]$ has to be replaced with the principal branch of the complex logarithm, $\log[S(z)]$, with branch cut

\[
\{ S(z) = re^{i\varphi} | r \geq 0, \varphi = \alpha - \pi (0 < \alpha \ll 1) \}.
\]

More precisely, $I_1^{(e)}(k; z)$ represents the analytic continuation of $I_1(k; s)$ along a contour which starts on the real axis at the point $z = s_A > 0$ ($S(s_A) = 1$), encircles the branch-point $z_0 = \frac{s_A}{\epsilon} [1 + O(\epsilon)]$ ($S(z_0) = 0$) counter-clockwise along a semi-circle with radius $\rho < \epsilon$, and after returning back to the real axis at $z_0 + \rho$, proceeds to $+\infty$ (corresponding to $0 > S(z) \geq -\epsilon + O(\epsilon^2)$).

The asymptotic series of $I_1^{(c)}$ valid for $s \geq \frac{s_A}{\epsilon} + \rho \Rightarrow S < 0$ reads [23]:

\[
I_1^{(c)}(k; s) \sim e^{i[1-S(s)]y_k(s)} - 1 + \epsilon \frac{e^{iy_k}}{iy_k} \left[ - e^{-iS y_k} (1 - S + \log S) \right.
\]

\[
+ \frac{e^{-iS y_k} - e^{-iy_k}}{iy_k} + i e^{-iS y_k} \sum_{n=0}^{\infty} (-iS y_k)^{-n} n! + O(\epsilon^2),
\]

where $y_k(s) \sim s^{1-2\epsilon}$ when $s \to \infty$. The second integral $I_2$ is given by:

\[
I_2(k; s) = \frac{e^{-iA(k); s}}{1 - \epsilon} \left\{ F_{1-\epsilon} [iA(k; s)] - \left( \frac{T_A}{\tau} \right)^{1-\epsilon} F_{1-\epsilon} \left( i \frac{T_A}{\tau} A(k; s) \right) \right\}
\]

\[
(101)
\]

$S(z)$ is no entire function, in contrast to $\text{Si}(z)$ and $\tilde{\text{Ci}}(z)$.
with \( F_\alpha(x) \equiv {}_1F_1(\alpha; \alpha + 1; x) \) and

\[
\mathcal{A}(k; \epsilon) = \tau(s) + y_k(s) [1 + O(\epsilon)]. \tag{102}
\]

Insertion of \( (100) \) and the asymptotic expansion of \( (101) \) into \( (94) \) yields for fixed \( \epsilon > 0 \) the asymptotic behavior of the Weyl TEO in weakly perturbed conformally flat spacetimes:

\[
\left( K_\kappa^{(-)}(t|t_A) \right)_{11} \sim e^{i(1-S(s))y_k(s)} \left[ 1 - \frac{\epsilon}{2} \left( 3 - e^{-i(1-S(s))y_k(s)} + O(s_A/s) + 2\log(-\epsilon + s_A/s + ...) \right) + O(\epsilon^2) + O(\tau^{-\epsilon}) \right]
\]

(\( \mathcal{D} = 1 + \epsilon/2 + O(\epsilon^2) \)), and a similar calculation gives:

\[
\left( K_\kappa^{(-)}(t|t_A) \right)_{12} \sim -i \frac{k_+}{2k_3} e^{i\tau(s)A^2_k(s)} [1 + i(1 - e^{\tau_A})] \left[ 1 + O(\epsilon) + O(\tau^{-\epsilon}) \right]. \tag{104}
\]

Some time ago Chimento and Mollerach attempted to obtain massive spinor solutions in BI backgrounds \[34\]. They found only two independent solutions, and they also argued that a smooth transition from spinor solutions in a BI background to solutions in a fFLRW background is not possible. In a subsequent work by Castagnino et al. \[35\] it had been pointed out that the ansatz used in \[34\] is not the most general one, and provided one can use a separation ansatz for the spinor field it is always possible to obtain four independent solutions.

Furthermore, at least in the case of massless spinor solutions in anisotropic pBI backgrounds, it can now explicitly be shown with the above outcome that for vanishing anisotropy the correct solutions in fFLRW backgrounds can be obtained without encountering any discontinuity: Since for fixed \( s > 0 \) holds:

\[
A_k(s) \xrightarrow{\epsilon \to 0} \frac{k}{|k_3|}, \quad y_k(s) \xrightarrow{\epsilon \to 0} \frac{\text{sign}k_3}{\mu} (k - |k_3|)s, \quad \mathcal{A}(k; s) \xrightarrow{\epsilon \to 0} \frac{\text{sign}k_3}{\mu} (k + |k_3|)s,
\]

and

\[
F_{1-\epsilon}[i\mathcal{A}(k; s)] \xrightarrow{\epsilon \to 0} e^{i\frac{\text{sign}k_3}{\mu} (k + |k_3|)s} - \frac{1}{i \frac{\text{sign}k_3}{\mu} (k + |k_3|)s},
\]

27
eq.s (94) with (99), (101) (these are no asymptotic expressions!) reduce to the matrix elements of the exact Weyl TEO with fFLRW backgrounds:

\[
\begin{align*}
\left(K_k^{(-)}(t|t_A)\right)_{11} & \xrightarrow{\epsilon \to 0} e^{-\frac{2i k_3}{\mu} \Delta s} \left[\cos(k \Delta s/\mu) + i \left(\frac{k_3}{k}\right) \sin(k \Delta s/\mu)\right], \\
\left(K_k^{(-)}(t|t_A)\right)_{12} & \xrightarrow{\epsilon \to 0} \left(\frac{k_+}{k}\right) e^{-\frac{i 2i k_3}{\mu} (\Delta s + 2 s_A)} \sin(k \Delta s/\mu),
\end{align*}
\]

(\Delta s := s - s_A, k_+ defined in (27)). For \(\mu = 1/2\) one recovers eq.s (58). We stress that once the asymptotic expansion of the TEO has been used, \(\epsilon\) must stay finite!

VI. THE WEYL AND DIRAC TEO

We study now the TEO for Weyl spinors and also the corresponding expression for massless Dirac spinors in a little more detail. Starting with the negative chirality Weyl TEO acting on \(\phi^{(j,-)}(\mathbf{k}, t)\) at time \(t > t_A\):

\[
\phi^{(j,-)}(\mathbf{k}, t) = K_k^{(-)}(t|t_A) \phi^{(j,-)}(\mathbf{k}, t_A)
\]

where \(K_k^{(-)}\) has been given in (49), (50), one has for small \(\Delta t := t - t_A\)

\[
K_k^{(-)}(t|t_A) = \begin{pmatrix} 1 & \frac{k_+}{1} e^{-2i k_3 t'/\mu + \frac{\mu \delta - 1}{\mu} \Delta t} \\ -k_- e^{2i k_3 t'/\mu + \frac{\mu \delta - 1}{\mu} \Delta t} & 1 \end{pmatrix} + O[(\Delta t)^2].
\]

(107)

For \(t_A = 0\) this Weyl TEO is given by equation (75). Owing to (17), (18), eq. (107) can be rewritten as

\[
K_k^{(-)}(t|t_A) = 1 + \Omega^{(-)}(\mathbf{k}, t) \Delta t + O[(\Delta t)^2]
\]

so that (106) and (107) represent the integrated system (16) up to order \(\Delta t\). With (108) one shows that \(K_k^{(-)}(t|t_A)\) infinitesimally satisfies

\[
K_k^{(-)}(t_C|t_B) K_k^{(-)}(t_B|t_A) = K_k^{(-)}(t_C|t_A).
\]

(109)

We turn next to the TEO for bispinors \(\psi^{(j,-)}\). Eq. (15) can be cast into

\[
\varphi^{(j,-)}(\mathbf{k}, t) = |g(t)|^{-1/4} Q^{-1}(k_3; t) \phi^{(j,-)}(\mathbf{k}, t)
\]
with
\[
Q(k_3, t) = \begin{pmatrix} Q_{11}(k_3, t) & 0 \\ 0 & Q_{11}(k_3, t)^* \end{pmatrix}, \quad Q_{11}(k_3, t) = \exp \left[ -i \int_{\tilde{t}_A}^t p_3(x) dx \right]
\]
(110)

\[
(0 \leq \tilde{t}_A \leq t_A). \text{ Hence, } \varphi^{(-)}(k, t) = K^{(-)}_k(t|t_A) \varphi^{(-)}(k, t_A) \text{ is equivalent to (106), and the modified Weyl TEO reads:}
\]
\[
K^{(-)}_k(t|t_A) = \frac{|g(t_A)/g(t)|^{1/4} Q^{-1}(k_3, t) K^{(-)}_k(t|t_A) Q(k_3, t_A)}. \quad (111)
\]

The Dirac TEO for the negative chirality bispinor (20) is given by
\[
K^{(-)}_k(t|t_A) = 1_2 \otimes K^{(-)}_k(t|t_A)
\]
(112)

The expression for the positive chirality TEO can be obtained with (21) and (106). One gets
\[
K^{(+)}_k(t|t_A) = K^{(-)}_k(t|t_A), \quad \text{and the corresponding Weyl TEO for the spinors } \varphi^{(+)} \text{ is given by}
\]
\[
K^{(+)}_k(t|t_A) = \frac{|g(t_A)/g(t)|^{1/4} Q(k_3, t) K^{(+)}_k(t|t_A) Q^{-1}(k_3, t_A)}. \quad (113)
\]

Thus, the positive chirality Dirac TEO assumes the form
\[
K^{(+)}_k(t|t_A) = 1_2 \otimes K^{(+)}_k(t|t_A),
\]
(114)

and owing to \(K^{(+)}_k(t|t_A) = K^{(-)}_k(t|t_A)\) one has
\[
K^{(+)}_k(t|t_A) = K^{(-)}_k(t|t_A).
\]
(115)

It can be easily verified that the operators \(K^{(+)}_k(t|t_A)\) satisfy a relation analogous to (109), and that holds:
\[
\left( K^{(\pm)}_k(t|t_A) \right) ^\dagger K^{(\pm)}_k(t|t_A) = |g(t_A)/g(t)|^{1/2} \left[ \left| \left( K^{(-)}_k(t|t_A) \right)_{11} \right|^2 + \left| \left( K^{(-)}_k(t|t_A) \right)_{12} \right|^2 \right]^{1/4}
\]
(116)

where with (B14) follows at large times:
\[
\left| \left( K_k^{(-)}(t|t_A) \right) \right|_{11}^2 = 1 + O(\mathcal{D} \eta_\delta |\tau|^{\delta - 1}) \tag{117}
\]

and likewise, utilizing \([B11]\):

\[
\left| \left( K_k^{(-)}(t|t_A) \right) \right|_{12}^2 = \Gamma^2(\delta) \left( \mu/2 \right)^{2\delta - 2} \eta_\delta \left[ 1/2 + O(\eta_\delta |\tau|^{2\delta - 1}) \right]. \tag{118}
\]

With

\[
* F_{k_k}^{(\pm)}(j^\pm_1, j^\pm_2, \psi_k^{(i, \pm)}(x, t_A)) = \frac{1}{3!} \left( \psi_k^{(j, \pm)}(x, t_A) \right)^\dagger \left( K_k^{(\pm)}(t|t_A) \right)^\dagger \gamma^0 \times \gamma^\nu K_k^{(\pm)}(t|t_A) \psi_k^{(i, \pm)}(x, t_A) \epsilon_{\alpha\beta\gamma\nu} \Theta^\alpha \land \Theta^\beta \land \Theta^\gamma
\]

and \([22]\) one obtains

\[
\left\langle \psi_k^{(j, \pm)}(t) , \psi_k^{(i, \pm)}(t) \right\rangle \equiv \left\langle K_k^{(-)}(t|t_A) \psi_k^{(j, \pm)}(t_A), K_k^{(-)}(t|t_A) \psi_k^{(i, \pm)}(t_A) \right\rangle
\]

\[
= \int_{\Sigma_{t_A}} \Theta^1 \land \Theta^2 \land \Theta^3 \left( \psi_k^{(j, \pm)}(x, t_A) \right)^\dagger \psi_k^{(i, \pm)}(x, t_A)
\]

\[
\times [1 + O(\eta_\delta) + O(\eta_\delta |\tau|^{2\delta - 1})]
\]

\[
= \left\langle \psi_k^{(j, \pm)}(t_A), \psi_k^{(i, \pm)}(t_A) \right\rangle \left[ 1 + O(\eta_\delta) + O(\eta_\delta |\tau|^{2\delta - 1}) \right]
\]

\[
(119)
\]

The choice of the hypersurface has been such that \(\Theta^0 = 0\) within \(\Sigma_{t_A}\), and use has been made of

\[
\epsilon_{\alpha\beta\gamma\nu} \Theta^\alpha \land \Theta^\beta \land \Theta^\gamma(t) = 3! \sqrt{|g(t)/g(t_A)|} \Theta^1 \land \Theta^2 \land \Theta^3(t_A).
\]

For \(\delta = 1/2\), eq. \([119]\) can be confirmed in an analogous way. When \(\delta = 1 - \epsilon\) (\(\epsilon \ll 1\)), one has with eqs \([103], \[104]\):

\[
\left| \left( K_k^{(-)}(t|t_A) \right) \right|_{11}^2 + \left| \left( K_k^{(-)}(t|t_A) \right) \right|_{12}^2 = 1 + \eta_1 [1 - e^{\tau_A} + e^{2\tau_A}/2 + O(\epsilon)]
\]

\[
- \epsilon \{3 + 2 \ln |1 - \epsilon + \tau_A/\tau + ...| + \cos[(1 - S)x_k] + O(\epsilon) + O(\tau^{-\epsilon})\} + O(\eta_1 - \epsilon \tau^{-\epsilon})
\]

so that

\[
\left\langle \psi_k^{(j, \pm)}(t) , \psi_k^{(i, \pm)}(t) \right\rangle = \left\langle \psi_k^{(j, \pm)}(t_A), \psi_k^{(i, \pm)}(t_A) \right\rangle \left[ 1 + O(\eta_1)
\right.
\]

\[
+ O \left( \epsilon [\ln \epsilon + O(1) + ...] + O(\eta_1 - \epsilon \tau^{-\epsilon}) \right].
\]

\[
(120)
\]
As a result, for $0 < \delta \leq 1/2$, and $\delta = 1 - \epsilon$ the Dirac TEOs $K^\pm_k$ are at large times, apart from terms of order $\epsilon \ln \epsilon$ and $\eta_\delta$, unitary operators.

At early times one gets either from (75):

$$\left| \left( K^\pm_k(t|0) \right) \right|_1^2 + \left| \left( K^\pm_k(t|0) \right) \right|_2^2 = 1 + O(t^{2\mu \delta}),$$

(121)

$t_A = t_\tilde{A} = 0$, or from (107):

$$\left| \left( K^\pm_k(t|t_A) \right) \right|_1^2 + \left| \left( K^\pm_k(t|t_A) \right) \right|_2^2 = 1 + O[(\Delta t)^2].$$

(122)

$t_A \geq t_\tilde{A} \geq 0$). For $t \ll 1$ and $\Delta t = t - t_A \ll 1$, resp., one finds

$$\left\langle \psi^{(j,\pm)}_k(t), \psi^{(l,\pm)}_k(t) \right\rangle = \left\langle \psi^{(j,\pm)}_k(t_A), \psi^{(l,\pm)}_k(t_A) \right\rangle.$$  

(123)

Thus the $K^\pm_k(t|t_A)$ are unitary when $t$ and $\Delta t$, resp., tend to zero.

VII. CONCLUSION

Starting with the formulation of Dirac’s equation in anisotropic Bianchi-type-I (BI) spacetimes w.r.t. anholonomic orthonormal frames and specializing to the massless case in planar BI background spacetimes, we derived the exact formal Weyl time-evolution-operator (TEO) of this problem, eq.s (33) - (35). Based on this solution one can introduce a simple parameter transformation (PT) capable of generating exact massless spinor solutions. It has been shown that this PT is equivalent to a constrained conformal map, which is often more convenient to use than the PT. Furthermore, an analytical outcome of the approximate Weyl TEO has been determined. The basic outcome given by eq.s (49) - (51) is valid for all spacetimes (26) satisfying $0 < \delta \equiv (1 - \nu)/\mu \leq 1$.

Comparing this approximate TEO result with solutions of exactly soluble models, we found that it reproduces for $k_3 \neq 0$ the exact early- and late-time behavior of the anisotropic stiff-fluid spinor solutions ($\delta = 1/2$), apart from a phase with vanishing frequency. Moreover, for flat FLRW models ($\delta = 1$) the approximate TEO is exact. Remarkably, the approximate TEO yields also useful results even when $k_3 = 0$, although it can not be properly defined any more in this case.
Especially in cases where no exact solutions are available, the application of the TEO technique allows at least the determination of approximate solutions, which behave at early and late times like the exact solutions. This has been demonstrated for massless fermions in the anisotropic axisymmetric Kasner background ($\delta = 1/4$). Further, approximate Weyl TEOs valid for $0 < \delta \leq 1/2$ and $\delta = 1 - \epsilon$ ($0 < \epsilon \ll 1$) have explicitly been derived. Using these results, it has been shown that for all models with planar BI backgrounds near conformal flatness the smooth transition to conformal flatness is possible, and that the approximate TEO becomes exact in this case.

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APPENDIX A: CALCULATION OF THE APPROXIMATE WEYL TEO

The approximation we are discussing in this appendix consists in replacing in eq. (35) for $l \neq n$ the function ($0 < \delta < 1$)

$$f_1(\sigma_l) = \sigma_l^{\delta-1}$$

(A1)

with

$$f_2(\sigma_l) = e^{(\delta-1)(\sigma_l-1)},$$

(A2)

where both functions are monotonically decreasing, $f_1 > f_2$ for $0 \leq \sigma_l < 1$, and $f_1 \approx f_2$ when $\sigma_l \approx 1$. In other words, we are concerned with $\tilde{I}_n(s)$ given by (48) instead of $I_n(s)$ defined in (35), where as a consequence of keeping in $\tilde{I}_n(s)$ the exact n-th integral holds: $\tilde{I}_n(s) = I_n(s)$ for $n = 0, 1$. Before we carry out the calculation of $\tilde{I}_n$, it is worth discussing the applicability of this approximation in the asymptotic regime. We introduce the quantity

$$\Sigma_{l;n} = \sum_{m=l}^{n} (-1)^{m+l} \sigma_m.$$
satisfying \( \sigma_l \geq \Sigma_{l;n} \geq \sigma_l - \sigma_{l+1} \). This is a consequence of the domain of integration in eqs. (35) and (48), resp., which is the n-volume \( V_n(\sigma_1, \ldots, \sigma_n) \) defined by \( 1 \geq \sigma_1 \geq \sigma_2 \ldots \geq \sigma_n \geq \sigma_A \). Due to the oscillating term \( \exp[-i \Sigma_{l;n} \tau] \) in the integrands in (35), (48), only the integration over those subsets of \( V_n \) is of importance at large \( \tau(s) \), for which the phase slowly varies, that is \( \Delta \Sigma_{l;n} \ll 1 \). For example, the subsets \( V_n|\sigma_n \geq 1 - \varepsilon \) or \( V_n|b(s) \geq \sigma_1 \) with

\[
\mathbf{b}(s) := \sigma_A(s) + \varepsilon, \quad \varepsilon \ll 1
\]  

(A4)

fulfill this requirement, because in either case holds: \( \Delta \Sigma_{l;n} \ll \varepsilon \). This inequality is valid for any \( n \). Clearly, the above approximation is sensible when integrating over \( V_n|\sigma_n \geq 1 - \varepsilon \), since then \( f_1 \approx f_2 \). The integration over \( V_n|b \geq \sigma_1 \) is more complicated. In fact, the latter case corresponds to the most critical part of integration as regards the above described approximation, because \( f_1(\sigma_i) \) tends to infinity when \( \sigma_i \to 0 \), while \( f_2(\sigma_i) \) stays finite. But the dominant contributions to the exact and approximate TEO (34) are in this case given by \( I_0, I_1 \) and \( \tilde{I}_0, \tilde{I}_1 \), resp. (with \( \tilde{I}_n \equiv I_n \) for \( n = 0, 1 \)), and we get only small corrections from higher terms \( I_n, \tilde{I}_n \) \((n \geq 2)\) to \( I_0, I_1 \) and \( \tilde{I}_0, \tilde{I}_1 \), resp. These are in the exact case of order \( \eta_\delta \varepsilon^{2\delta} \) and in the approximate case at least of order \( \varepsilon^{2\delta} \). The basic difference between the exact and approximate case consists in that there are no \( O(\eta_\delta) \)-corrections for \( \tilde{I}_0 = 1 \) if \( \delta < 1/2 \). This follows from the absence of terms \( \sim \tau^{-\delta} \) in the approximate case, in contrast to the exact case (A7). For example, \( I_2 \) contains a constant term \( \sim \eta_\delta \), and this term is not present in \( \tilde{I}_2 \), see also the discussion below.

In general, the integration over \( V_n \) in the expressions for \( \tilde{I}_n \) and \( I_n \) can be rewritten as \((n > 1)\):

\[
\int_{V_n} [d\sigma]^n = \int_{V_n|\sigma_n \geq 1 - \varepsilon} [d\sigma]^n + \int_{V_n|\sigma_n \geq \sigma_1} [d\sigma]^n \equiv \int_{V_n|b \geq \sigma_1} [d\sigma]^n
\]

\[
+ \int_1^b d\sigma_1 \left\{ \int_{V_{n-1}|b \geq \sigma_2} [d\sigma]^{n-1} + \int_{b}^{\sigma_1} [d\sigma] + \int_{V_{n-2}|b \geq \sigma_3} [d\sigma]^{n-2} \right\}
\]  

(A5)

where the integration over \( V_{n-1}|b \geq \sigma_2 \) is the second most critical one. However, with the same argumentation as above, the contribution to TEO (34) from this integration can be disregarded, and due to an accordingly repeated
argumentation, it is sufficient to consider the multiple integral

\[ \int_b^{\sigma_n-2} d\sigma_1 f_r(\sigma_1)e^{-i\sigma_1\tau} \int_b^{\sigma_n-1} d\sigma_2 f_r(\sigma_2)e^{-i\sigma_2\tau} \int_{\sigma_A}^{\sigma_n-1} d\sigma_3 f_r(\sigma_3)e^{-i\sigma_3\tau} \]

\[ (r = 1, 2). \] The asymptotic expansion of the exact n-th integral reads:

\[ \int_{\sigma_A}^{\sigma_n-1} d\sigma_n f_1(\sigma_n)e^{-i\sigma_n\tau} = \left[ \Gamma(\delta) e^{[\text{sign}(-1)^n\tau]\pi\delta} - \frac{\tau^\delta}{\delta} F_\delta(i(-1)^n\tau_A) \right] + \frac{f_1(\sigma_{n-1}) e^{-i\sigma_{n-1}\tau}}{i(-1)^n\tau} + O\left(\frac{1}{\tau^2}\right), \]

\[ (\tau_A \equiv 2k_3s_A/\mu). \] Note that the first two terms are \( \sigma_{n-1}\)-independent and \(~ \sim \tau^{-\delta}. \) These terms can not be obtained within the above introduced approximation, i.e. if one had used \( f_2 \) instead of \( f_1 \) in (A7). This is another reason why in the approximate expressions \( \tilde{I}_n \) defined in (48) the n-th integral is the exact one (s. also (A6)). On the other hand, for any \( \varepsilon > 0 \) holds:

\[ \int_b^{\sigma_n-1} d\sigma_L f_L(\sigma_L)e^{-i\sigma_L\tau} = \frac{f_r(\sigma_{L-1})e^{-i\sigma_{L-1}\tau} - f_r(b)e^{-i\sigma_L\tau}}{i(-1)^T} \left[ 1 + O\left(\frac{1}{\tau}\right) \right], \]

Thus there are, in contrast to (A7), in (A8) in the case \( r = 1 \) no more terms of order \( \tau^{-\delta} \) here. Hence, for nonzero \( \varepsilon \) the exact \( (r = 1) \) and the approximate integral \( (r = 2) \) exhibit the very same asymptotic behavior. Since arbitrary powers of \( f_r \) are again described by these functions (with suitable rescaling of \( \delta \)), and since the same holds also true - up to constant factors - for integrals of arbitrary powers of \( f_r \), it follows with (A7) and (A8) that expression (A6) shows in both cases \( (r = 1, 2) \) to leading order the same asymptotic behavior. These considerations imply that the approximate Weyl TEO (34), (48) agrees asymptotically with the exact Weyl TEO (34), (35). This is confirmed by the results of sect. IV, and by comparison of the expressions \( I_2, \tilde{I}_2 \) given below. Recalling that for the lowest order diagonal
and off-diagonal entries of the TEO holds: \( I_0 \equiv \tilde{I}_0, I_1 \equiv \tilde{I}_1 \), one obtains immediately with \((35)\)

\[
I_0(s) = 1, \quad I_1(s) = \frac{\kappa s^\delta}{\mu} \left[ F_\delta(-i\tau) - \sigma_A^\delta F_\delta(-i\tau_A) \right]. \tag{A9}
\]

As before, \( F_\delta(x) \equiv 1 F_1(\delta; \delta + 1; x) \). For \( \tilde{I}_2 \) one gets \([33]\):

\[
\tilde{I}_2(s) = \left( \frac{\kappa s^\delta}{\mu} \right)^2 \frac{\delta}{T} \left[ \frac{F_\delta(\delta - 1) - \sigma_A^\delta F_\delta([\delta - 1] \sigma_A)}{e^{\delta - 1}} - \frac{F_\delta(i\tau) - \sigma_A^\delta F_\delta(i\tau_A)}{e^{i\tau}} \right] \tag{A10}
\]

\((T := i\tau + 1 - \delta)\), while \( I_2 \) is given by:

\[
I_2(s) = \left( \frac{\kappa s^\delta}{\mu} \right)^2 \left[ \mathcal{F}_\delta(-i\tau) - \sigma_A^{2\delta} \mathcal{F}_\delta(-i\tau_A) - \sigma_A^{\delta} F_\delta(-i\tau) F_\delta(i\tau_A) \right. \\
+ \sigma_A^{2\delta} |F_\delta(i\tau_A)|^2, \tag{A11}
\]

where \( \mathcal{F}_\delta(x) := 2 F_2(1, 2\delta; 2\delta + 1, \delta + 1; x)/2 \) denotes a generalized hypergeometric function. In order to derive \((A11)\) we used \([36]\):

\[
\int_0^y dx \: x^{2\delta - 1} 1 F_1(1; \delta + 1; -i\tau x) = \frac{y^{2\delta}}{\delta} \mathcal{F}_\delta(-i\tau y).
\]

Note that \( \tilde{I}_2(s) \to I_2(s) \) for \( \delta \to 1 \). For simplicity we put now \( s_A = 0 \) (i.e. \( \sigma_A = 0 \)) and obtain for \( 0 < \delta < 1 \) the asymptotic expressions \([29], [37]\):

\[
I_1(s) \sim \mathcal{R}_k(\delta) \left\{ e^{-i\pi\delta \text{sign} k_3} \Gamma(\delta) + \frac{e^{-i\tau} \{1 + O([\delta - 1])\}}{(-i\tau)^{1-\delta}} \right\},
\]

\[
I_2(s) \sim \mathcal{R}_k^2(\delta) \left\{ \frac{1}{1 - 2\delta} \left[ C_\delta + \frac{1}{(-i\tau)^{1-2\delta}} \right] + \Gamma(\delta) \frac{e^{-i\tau} \{1 + O([\delta - 1])\}}{(-i\tau)^{1-\delta}} \right\},
\]

\[
\tilde{I}_2(s) \sim \mathcal{R}_k^2(\delta) \left\{ -\frac{1}{1 - 2\delta} \frac{F_1(1; 1 + \delta; 1 - \delta)/\delta}{(-i\tau)^{1-2\delta}} + \Gamma(\delta) \frac{e^{-i\tau} \{1 + O([\delta - 1])\}}{(-i\tau)^{1-\delta}} \right\}, \tag{A12}
\]

with \( C_\delta := [\Gamma(\delta)e^{-i\pi\delta \text{sign} k_3}]^2 \left( \frac{1}{2} - \delta \right)/\sin(\pi[\frac{1}{2} - \delta]) \), and \( \mathcal{R}_k := \kappa(i\mu/2k_3)^\delta \) a constant of order \( \sqrt{\eta_\alpha} \). The time-dependent terms of \( I_2, \tilde{I}_2 \) agree, apart from
the constant factors in front of the $\tau^{2\delta-1}$-terms, whereas the constant term only exists in the asymptotic expansion of $I_2$. It adds for $\delta \neq 1/2$ a constant $O(\eta_\delta)$-term to the lowest order diagonal entry of the TEO, $I_0$. Note that the term $\sim (-i\tau)^{\delta-1}e^{-i\tau}$ is also present in the asymptotic expressions for $I_1 \equiv \tilde{I}_1$, but differs by a factor of order $\sqrt{\eta_\delta}$ from the corresponding term in $I_2$, $\tilde{I}_2$.

The term proportional to $[C_\delta + (-i\tau)^{2\delta-1}]/(1 - 2\delta)$ is critical since it diverges for $\delta \geq 1/2$ in the asymptotic limit. In particular, it can not be canceled by higher order terms $I_n = O(\eta^{n/2})$ ($n > 2$). This seemingly violates unitarity of the exact Weyl TEO for $\delta \geq 1/2$. The way out of this problem is given by the requirement that such a term must for $n > 2$ appear in every $I_n$, and all those terms must add up to a phase factor $e^{\omega(t)}t$ with $\eta_\delta$-dependent frequency $\omega$. While this can not be proven for the exact TEO, the feasible summation of the terms $\tilde{I}_n$ yields in fact phase factors of precisely this form in the asymptotic limit.

It is instructive to look at the special and exactly solvable case $\delta = 1/2$ a little more closely. One obtains (with $\mu = 1$, $\tau = 2k_3s \equiv 2k_3t$):

$$I_2(s) \sim -i\eta \left[ \log(i\tau) - \psi(1/2) - \Gamma(1/2) \frac{e^{-i\tau}}{\sqrt{-i\tau}} + O(1/i\tau) \right], \quad (A13)$$

where $R_2(1/2) = i\eta$, $\eta \equiv \eta_{1/2} \text{sign}k_3 = \kappa^2/2k_3$. On the other hand, one gets immediately from $\tilde{A12}$:

$$\tilde{I}_2(s) \sim -i\eta \left\{ 2_1F_1(1; 3/2; 1/2) - \Gamma(1/2) \frac{e^{-i\tau}}{\sqrt{-i\tau}} + O(1/i\tau) \right\}. \quad (A14)$$

The essential difference to $\tilde{A13}$ consists in the absence of the logarithmic term. This term is just the correct $O(\eta)$-term of the $\eta$-expansion of the prefactor $(i\tau)^{-\eta}$ appearing in the exact solutions (67). The higher $O(\eta)$-terms of the diagonal elements of the exact Weyl TEO must be contained in $I_{2l} = O(\eta^l)$, and of the off-diagonal elements in $I_{2l-1} = O(\eta^{(2l-1)/2})$ ($l > 1$). The phase factor $(i\tau)^{-\eta}$ is, up to a constant, given by $e^{i\omega(t)t}$ with asymptotically vanishing frequency $\omega(t) = |\eta| \ln|2k_3t|/t$. Hence, it follows for $\delta = 1/2$ that the approximate Weyl TEO result can not reproduce this factor.

---

5 See eqs (B11), (B14); note that $\tau E_j(\tau) \propto \eta^{\tau 2\delta-1} \propto \omega(\tau t) t$ with $\omega \propto t^{1-2\nu-\mu} \rightarrow 0$ when $t \rightarrow \infty$.

6 Note that $2_1F_1(1; 3/2; 1/2) \approx -\psi(1/2)$
We now turn to the explicit computation of the approximate Weyl TEO in background spacetimes (26) with $0 < \delta \leq 1$. Eq. (48) can be cast into

$$\tilde{I}_n(s) = s^\delta e^{-i\frac{\pi s}{2}} \int_0^{1-\sigma_A} dz \ h_n(1-\sigma_A-z) \ (g_{0,n-1} \ast g_{1,n-1} \ast \ldots \ast g_{n-1,n-1})(z; s)$$

(A15)

with

$$g_{l,j}(z; s) = \exp \left\{ \left[ (1-\delta)(j-l) + i(-1)^l \tau(s) / 2 \right] z \right\},$$

$$h_{ij}(z; s) = (z + \sigma_A)^{\delta-1} \exp \left\{ i(-1)^j \tau(s) (z + \sigma_A) / 2 \right\},$$

(A16)

where the asterisk denotes the Laplace convolution product. Utilizing the inverse Laplace transformation one gets

$$(g_{0,n-1} \ast g_{1,n-1} \ast \ldots \ast g_{n-1,n-1})(z; s) = \sum_{l=1}^{n} \frac{e^{G_{l,n}(s)z}}{\prod_{m=1, m \neq l}^{n} \Delta G_{l,m}(s)}$$

(A17)

with $\Delta G_{l,m} := G_{l,j} - G_{m,j}$ and $G_{l,j}(s) := -i(-1)^l \tau(s) / 2 + (1-\delta)(j-l)$. Insertion of (A15), (A17) into (34) yields for the TEO a matrix expression, where in the entries terms of the form

$$\sum_{n=1}^{\infty} \left( \frac{i KS^\delta}{\mu} \right)^{2n-1} \sum_{l=1}^{2n} \frac{e^{G_{l,2n}(s)z}}{\prod_{m=1, m \neq l}^{2n} \Delta G_{l,m}(s)}$$

appear. Exploiting properties of $G_{l,j}$ and $\Delta G_{l,m}$ such as $G_{k,j} = G_{k,l} + (1-\delta)(j-l)$ and $\Delta G_{l,2k,m+2k} = \Delta G_{l,m}$ ($l < m$), and suitably rearranging the summands, one gets after some algebra:

$$\sum_{n=1}^{\infty} \left( \frac{i KS^\delta}{\mu} \right)^{2n-1} \sum_{l=1}^{2n} \frac{e^{G_{l,2n}(s)z}}{\prod_{m=1, m \neq l}^{2n} \Delta G_{l,m}(s)} = -e^{G_{1,1}(s)z} S_1 S_4^* + e^{G_{2,2}(s)z} S_2 S_3^*$$

$$= -e^{-i\tau z/2} V_k^*(z; s),$$

37
\[
\sum_{n=1}^{\infty} \left(\frac{iK_\delta}{\mu}\right)^{2(n-1)2n-1} e^{G_{i,2n-1}(s)z} \sum_{i=1}^{2n-1} \prod_{m=1, m \neq l}^{\Delta G_{i,m}(s)} = e^{G_{1,1}(s)z} S_1 S_3 + e^{G_{2,2}(s)z} S_2 S_4 =: e^{i\tau z/2} U_k^* (z; s), \tag{A18}
\]

where

\[
S_1 = \Gamma(\lambda) \left(\frac{x}{2}\right)_{\lambda}^\lambda J_{-\lambda}(x), \quad S_3 = \Gamma(\lambda^*) \left(\frac{x e^{(1-\delta)z}}{2}\right)_{\lambda}^\lambda J_{-\lambda}(x e^{(1-\delta)z}),
\]

\[
S_2 = -i \Gamma(\lambda^*) \left(\frac{x}{2}\right)_{\lambda}^\lambda J_{\lambda}(x), \quad S_4 = i \Gamma(\lambda) \left(\frac{x e^{(1-\delta)z}}{2}\right)_{\lambda}^\lambda J_{\lambda}(x e^{(1-\delta)z}), \tag{A19}
\]

with \(x, \lambda\) defined in (52), and \(J_{\lambda}\) denotes Bessel’s function of the first kind, \(\Gamma\) the Gamma function \[28\]. With (A18) substituted into (A17) one eventually obtains from (34), (A15) the approximate Weyl TEO (49) with

\[
\left( K_{k}^{(-)} (t|t_A) \right)_{11} = 1 + \frac{iK_\delta}{\mu} \int_0^1 dz (1-z)^{\delta-1} V_k(z; x),
\]

\[
\left( K_{k}^{(-)} (t|t_A) \right)_{12} = \frac{k_+ s^\delta}{\mu} e^{-i\tau} \int_0^1 dz (1-z)^{\delta-1} U_k(z; x), \tag{A20}
\]

where

\[
V_k(z; x) = i e^{(1-\delta)\lambda^*(s)z} R(z; x)/Z(0; x),
\]

\[
U_k(z; x) = e^{(1-\delta)\lambda(s)z} Z(z; x)/Z(0; x). \tag{A21}
\]

and

\[
R(z; x) = J_{-\lambda}(x)J_{\lambda}(x e^{(1-\delta)z}) - J_{-\lambda}(x)J_{-\lambda}(x e^{(1-\delta)z}),
\]

\[
Z(z; x) = J_{-\lambda}(x)J_{-\lambda}(x e^{(1-\delta)z}) + J_{\lambda}(x)J_{\lambda}(x e^{(1-\delta)z}),
\]

\[
\frac{1}{Z(0; x)} = \frac{\pi x}{2 \cosh(\frac{\pi x}{2(1-\delta)})} = \frac{x}{2} \Gamma(\lambda^*) \Gamma(\lambda). \tag{A22}
\]

When \(\sigma_A \to 0\) and \(k_3 \neq 0\), this approximation can be improved for \(0 < \delta \leq 1\) by replacing \(f_2\) with \(F_2 := D_1 f_2\). The constant \(D_1\) is determined via the normalization

\[
38
\]
\[
\int_0^1 d\sigma_1 F_2(\sigma_1) / \int_0^1 d\sigma_1 f_1(\sigma_1) = 1,
\]
and one gets:
\[
\mathcal{D}_1(\delta) = \frac{1}{\delta} \frac{1 - \delta}{e^{1-\delta} - 1} \xrightarrow{\delta \to 1} 1. \tag{A23}
\]

This normalization ensures for all higher order terms \( \tilde{I}_n (n \geq 2) \) the correct (for \( \sigma_A \to 0 \): divergent) behavior when \( \delta \) approaches zero.

The asymptotic expansions of \( K_{11}^{(-)} \) and \( K_{12}^{(-)} \) for \( 0 < \delta \leq 1/2 \) have been derived in App. B. These calculations suggest to introduce a second constant
\[
\mathcal{D}_2(\delta) = \frac{2}{e^{1-\delta} + 1} \xrightarrow{\delta \to 1} 1, \tag{A24}
\]
satisfying \( \mathcal{D}_2 \approx 1 \) if \( 0 < \delta \leq 1 \). \( \mathcal{D}_2 \) combines with \( \mathcal{D}_1 \) to
\[
\mathcal{D}(\delta) = \mathcal{D}_1(\delta) \mathcal{D}_2(\delta) \equiv \frac{1}{\delta} \frac{(1 - \delta) F_1(\delta; \delta + 1; \delta - 1)}{\sinh(1 - \delta)}. \tag{A25}
\]
To implement this modification, simply substitute \( \mathcal{D} x \) for \( x \) in (A22).

**APPENDIX B: ASYMPTOTIC EXPANSION OF THE APPROXIMATE WEYL TEO**

It is possible to derive for \( 0 < \delta \leq 1/2 \ (k_3 \neq 0) \) an analytic asymptotic expression of the approximate Weyl TEO. To this end one must determine the asymptotic behavior of \( R \) and \( Z \) given by eq.s (51) and (A22), resp. This means that an asymptotic calculation of \( J_\nu(x) \) must be carried out where both quantities, order \( \nu(s) \) and variable \( x(s) \), increase indefinitely if \( s \to \infty \).

We rewrite \( Z \) as:
\[
Z(z; s) = \frac{2\lambda(s)}{\mathcal{D} X(z; s)} Z_+(z; s) - \hat{Z}_+(z; s) + \hat{Z}_-(z; s) \tag{B1}
\]
with
\[
Z_{\pm}(z; s) := J_{\pm\lambda}(\mathcal{D} X) J_{\mp\lambda}(\mathcal{D} x), \quad \hat{Z}_{\pm}(z; s) := J_{1\pm\lambda}(\mathcal{D} X) J_{1\mp\lambda}(\mathcal{D} x). \tag{B2}
\]
and

\[ X(z; s) := x(s)e^{(1-\delta)z}. \] (B3)

Here, use has been made of the identity \(J_{\alpha-1}(y) = 2\alpha J_{\alpha}(y)/y - J_{\alpha+1}(y),\)
\(x, \lambda\) have been given in (52), and \(D\) in (53). 
A helpful relation is [29]:

\[ J_\mu(\alpha y)J_\nu(\beta y) = \frac{(2\alpha)^{\mu}(2\beta)^{\nu}}{\pi} \int_{-\pi/2}^{\pi/2} d\theta e^{i\theta(\mu-\nu)} \left( \frac{\cos \theta}{\Lambda(\theta)} \right)^{\mu+\nu} J_{\mu+\nu} \Lambda(\theta) y \] (B4)

with \(\Lambda(\theta) := \sqrt{2\cos \theta (\alpha^2 e^{i\theta} + \beta^2 e^{-i\theta})},\) valid for \(\text{Re}(\mu + \nu) > -1\).

With \(\alpha = e^{z(1-\delta)}, \beta = 1\) follows, that

\[ \frac{2\lambda}{\partial X} Z_+(z; s) = \frac{1}{\pi Dx} e^{-(1-\delta)\lambda^* z} e^{\pi/2[a]} \]
\[ \times \int_0^{\pi/2[a]} d\tau e^{-(1-i\alpha)\tau} J_0 \left[ \Lambda \left( \frac{\pi}{2} + \alpha v; z \right) |Dx| \right] (1 + O(e^{-\pi/2[a]})) \]
\[ \tilde{Z}_+(z; s) = -\frac{2\alpha Dx}{\pi} e^{(1-\delta)(1+\lambda)z} e^{\pi/2[a]} \]
\[ \times \int_0^{\pi/2[a]} d\tau e^{-(1-2i\alpha)\tau} \sin(\alpha \tau) J_0 \left[ \Lambda \left( \frac{\pi}{2} + \alpha v; z \right) |Dx| \right] (1 + O(e^{-\pi/2[a]})) \] (B5)

and

\[ Z_-(z; s) = \frac{-i\alpha(s)}{\pi} e^{-(1-\delta)\lambda z} e^{\pi/2[a]} \]
\[ \times \int_0^{\pi/2[a]} d\tau e^{-(1-i\alpha)\tau} J_0 \left[ \Lambda \left( \frac{\pi}{2} - \alpha v; z \right) |Dx| \right] (1 + O(e^{-\pi/2[a]})) \]
\[ \tilde{Z}_-(z; s) = \frac{2\alpha Dx}{\pi} e^{(1-\delta)\lambda^* z} e^{\pi/2[a]} \]
\[ \times \int_0^{\pi/2[a]} d\tau e^{-\tau} \sin(\alpha \tau) J_1 \left[ \Lambda \left( \frac{\pi}{2} - \alpha v; z \right) |Dx| \right] (1 + O(e^{-\pi/2[a]})) . \] (B6)

\[ ^\dagger\text{This condition is not satisfied in the original expression for } Z, s. \text{ eq. (A22).} \]
where

\[ a(s) = (1 - \delta)/\tau(s), \]  

and \( \tau \) as in eq. (36). In the asymptotic case \(|\tau| \gg 1\), the upper limit of the integrals in (B5), (B6) can be replaced by infinity. By virtue of

\[ \Lambda(\pi/2 \mp av; z) |Dx| = 2B_{\pm}(z; s) \sqrt{v} \left\{ 1 \mp i \coth[(1 - \delta)z] \frac{av}{2} + O[(av)^2] \right\} \]

with

\[ B_{\pm}(z; s) = \sqrt{\pm ia(s)} \sinh[(1 - \delta)z] e^{(1-\delta)z/2} |D(\delta)x(s)| \]

and \( J_\alpha(\beta y) = \beta^\alpha \sum_{n=0}^\infty [y(1 - \beta)^2/2]^n J_{\alpha+n}(y)/n! \) [29] one gets as asymptotic expansions of the above integrals in terms of Kummer's function \(_1F_1^*\):

\[ \int_0^\infty dv e^{-(1-ia)v} J_0[\Lambda(\pi/2 + av; z)] |Dx| \sim _1F_1^* \left(1; 1; -\frac{B_{\pm}^2(z; s)}{1 - ia}\right) (1 - ia)^{-1} \]

\[ - 2ia B_{\pm}^2(z; s) \coth[(1 - \delta)z] _1F_1^* \left(3; 2; -\frac{B_{\pm}^2(z; s)}{1 - ia}\right) (1 - ia)^{-3} + ... \]

\[ \int_0^\infty dv e^{-(1-2ia)v} \sin(av) \frac{J_1[\Lambda(\pi/2 + av; z)] |Dx|}{\Lambda(\pi/2 + av; z)|Dx|} \sim \]

\[ - \frac{i}{4} \left[ _1F_1^* \left(1; 2; -\frac{B_{\pm}^2(z; s)}{1 - 3ia}\right) \right] - \frac{1}{1 - 3ia} \]

\[ \int_0^\infty dv e^{-v} \sin(av) \frac{J_1[\Lambda(\pi/2 - av; z)] |Dx|}{\Lambda(\pi/2 - av; z)|Dx|} \sim \]

\[ - \frac{i}{4} \left[ _1F_1^* \left(1; 2; -\frac{B_{\pm}^2(z; s)}{1 - ia}\right) \right] - \frac{1}{1 + ia} \]

Substituting these expansions for the integrals in (B5), (B6) and using \(_1F_1(a + 1; a; x) = (1 + x/a)e^x\), \(_1F_1(1; 2; x) = (e^x - 1)/x\), one arrives at
\[ Z(z; s)/Z(0; s) \sim \mathcal{L}_1(z; s) + \mathcal{L}_2(z; s) + \mathcal{L}_3(z; s)[1 + O(e^{-\pi/|a|})] \quad (B8) \]

with

\[ \mathcal{L}_1(z; s) := e^{-\lambda^* (1-\delta)z} e_{-\delta} \{ 1 - L_1(z; s) - L_2(z; s) - \ldots \} \]
\[ \mathcal{L}_2(z; s) := \frac{e^{\lambda(s)(1-\delta)z}}{2 \sinh((1-\delta)z)} \left[ e_{-\delta} \{ 1 - L_2(z; s) - \ldots \} - e_{1-\delta} \{ 1 - L_3(z; s) - \ldots \} \right] \quad (B9) \]
\[ \mathcal{L}_3(z; s) := \frac{e^{-\lambda(s)(1-\delta)z}}{2 \sinh((1-\delta)z)} \left[ e_{-\delta} \{ 1 + L_2(z; s) + \ldots \} - e_{1-\delta} \{ 1 - L^*_2(z; s) + \ldots \} \right] \]

and

\[ L_1(z; s) = \frac{2ia(s) \coth((1-\delta)z) B^2(z; s)}{[1-i\alpha(s)]^2} \]
\[ L_2(z; s) = \frac{-ia(s) \coth((1-\delta)z) B^4_1(z; s)}{[1-i\alpha(s)]^3} \]
\[ L_3(z; s) = \left( \frac{1 - i \alpha(s)}{1 - 3i \alpha(s)} \right)^3 L_2(z; s). \]

This outcome is valid for \( 0 < \delta < 1/2 \). Eq.s \( (B8), (B9) \) into \( (A21) \) leads to the following asymptotic expansion (discarding terms of order \( e^{-\pi/|a(s)|} \))

\[ \int_0^{1-\sigma \Lambda(s)} dz (1-z)^{\delta-1} U_k(z; s) \sim \sum_{j=1}^3 \int_0^{1-\sigma \Lambda(s)} dz (1-z)^{\delta-1} e^{(1-\delta)\lambda z} \mathcal{L}_j(z; s) \quad (B10) \]

where the l.h.s. is up to a prefactor the off-diagonal element \( K_{12}(-) \) of the TEO. It is now a straightforward but tedious calculation to evaluate the r.h.s. of \( (B10) \). The final result reads eventually:
\( (K_{k}^{(-)}(t|t_A))_{12} \sim \frac{k_{+} \sqrt{2\eta_0}}{\kappa} \left( \frac{\mu}{2} \right)^\delta |\tau|^{\delta} e^{-i\tau} \left[ e^{i\tau (1+D^2E_1(\tau))} \frac{1}{\delta} \left\{ 1 + O(\eta_0 D^2|\tau|^{2\delta-2}) \right\} \right. \)

\times \left\{ F_\delta(-i\tau [1 + D^2E_2(\tau)]) - \left( \frac{\tau_A}{\tau} \right)^\delta F_\delta(-i\tau_A [1 + D^2E_2(\tau)]) \right\} 

\left. + \frac{e^{1-\delta} \eta_0 (\mu/2)^{2\delta-2}}{2\delta} D^2 |\tau|^{2\delta-2} \left\{ 1 + O(\eta_0 D^2 |\tau|^{2\delta-1}) \right\} \right. 

\times \left\{ F_\delta(\delta - 1) - \left( \frac{\tau_A}{\tau} \right)^\delta F_\delta([\delta - 1][\tau_A/\tau]) \right\} 

\left. + \frac{e^{2-2\delta} \eta_0 (\mu/2)^{2\delta-2}}{2\delta} D^2 |\tau|^{2\delta-2} e^{i\tau} \left\{ 1 + O(\eta_0 D^2 |\tau|^{2\delta-1}) \right\} \right. 

\times \left\{ F_\delta(2[\delta - 1] - i\tau) - \left( \frac{\tau_A}{\tau} \right)^\delta \right. F_\delta \left( 2[\delta - 1][\tau_A/\tau] - i\tau_A \right) \right\} \right] \]

(B11)

with \( F_\alpha(x) := \frac{\Gamma(\alpha)}{\Gamma(\alpha + 1)} \), and

\[
E_1(\tau) = \frac{e^{2(1-\delta)} - 1}{4(1-\delta)} \eta_0 \left( \frac{\mu}{2} \right)^{2\delta-2} \left| \frac{\tau}{\tau - 1} \right|^{2\delta-2}, \quad E_2(\tau) = \frac{(1-\delta)e^{1-\delta}}{\sinh(1-\delta)} E_1(\tau). \]

(B12)

The determination of the asymptotic limit of \( K_{11}^{(-)} \) proceeds analogously.

With \( R := Z_+ - Z_- \), the same manipulations as above lead to

\[
R(z; s)/Z(0; s) \sim -i\alpha D x(s)/(1 - i\alpha) \]

\[
\times \left[ e^{\lambda(s)(1-\delta)z} e^{-B_1^2(z,s)} \left\{ 1 - L_1(z; s) \left[ 1 - B_1^2(z; s)/(2 - 2i\alpha) \right] + \ldots \right\} \right. 

\left. - e^{-\lambda(s)(1-\delta)z} e^{B_1^2(z,s)} \left\{ 1 - L_1(z; s) \left[ 1 + B_1^2(z; s)/(2 - 2i\alpha) \right] - \ldots \right\} \right]. \]

(B13)

For \( \delta < 1/2 \) one obtains with (A20), (A21) for large times:
\[
\left( K_k^{(-)}(t|t_A) \right)_{11} \sim 1 + 2i \left( 1 - \delta \right) D G(\tau) \left[ \frac{e^{1-\delta}}{\delta} \{1 - iD^2G(\tau)\} \right. \\
\times \left\{ F_\delta(\delta - 1) - \left( \frac{T_A}{\tau} \right) \delta F_\delta \left[ (\delta - 1) \frac{T_A}{\tau} \right] \right\} \\
+ i \left( \frac{e^{3(1-\delta)}}{\delta} D^2G(\tau) \right) \left\{ F_\delta(3[\delta - 1]) - \left( \frac{T_A}{\tau} \right) \delta F_\delta \left( 3[\delta - 1] \frac{T_A}{\tau} \right) \right\} \\
+ O(D^4G^2(\tau)) - \frac{e^{-i\tau(1+D^2E_1(\tau))}}{\delta} \{ 1 + O(\eta D^2|\tau|^{2\delta-2}) \} \\
\times \left\{ F_\delta(i\tau[1 + D^2E_2(\tau)]) - \left( \frac{T_A}{\tau} \right) \delta F_\delta(i\tau[1 + D^2E_2(\tau)]) \right\} \right] \\
\] 
with \( E_1, E_2, F_\alpha \) defined above and 
\[
G(\tau) = \frac{e^{\delta-1}}{2 \sinh(1 - \delta)} \tau E_1(\tau). 
\] 

If \( \delta = 1/2 \), one obtains with (13) and the asymptotic expansion of:

\[
\left( K_k^{(-)}(t|t_A) \right)_{11} \sim 1 + i D \left( \frac{1}{2} \right) \frac{\eta}{\mu} \left[ 2 e^{1/2} \{ 1 + O(\eta) \} \right. \\
\times \left\{ F_{1/2}(-1/2) + O(\sqrt{T_A/\tau}) \right\} + O(e^{-i\tau/\sqrt{|\tau|}}) \\
\left( K_k^{(-)}(t|t_A) \right)_{12} \sim \frac{k_\perp}{\kappa} \left| \frac{\eta}{\mu} \right| \sqrt{|\tau|} \left[ 2 \{ 1 + O(\eta) + O(\eta/\tau) \} \\
\times \left\{ F_\delta(-i\tau[1 + O(\eta/\tau)]) - O(\sqrt{T_A/\tau}) \right\} + O(\eta/\tau) \right]. 
\]

**APPENDIX C: PT LIMITING CASE**

We derived in sect. III. the exact expression for the Weyl TEO, eq.s (34), (35) and also the related expression for the approximate Weyl TEO given
by (49) - (51). For any \( \delta \leq 1 \), the exact and the approximate Weyl TEO solutions as well correspond to the in [21] introduced equivalence classes of exact spinor solutions \([ \bar{\delta} ]\) and their approximate counterparts. It has been explicitly demonstrated there, that by use of (39) (or (42) and (43)) one gets for the classes \([ \bar{\delta} ]\) and \([ \bar{\delta}/2 ]\) the correct spinor solutions with fFLRW background \( ds^2 = dt^2 - t^2(dx^2 + dy^2 + dz^2) \), when \( a \to 0 \). This line element is special in that it corresponds to the only fixed point of PT (39).

We are now going to show that this result is true for all equivalence classes \([ \bar{\delta} ]\) and also for all classes of approximate spinor solutions \((\delta \leq 1)\). In the exact case we start with (34), (35) and apply (42), (43), that is we consider \( I_n(s^a; \sigma_A^a, 1) \) with \( k' := k/a \) and \( \tau' := 2k_3s^a/a \mu (a, \sigma_A > 0) \):

\[
I_n(s^a; \sigma_A^a, 1) = \left( \frac{k\sigma_A^a}{a\mu} \right)^n \int_0^1 ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n \prod_{l=1}^n \sigma_l^{1-1/s_l} e^{i(1-s_l)} (C1)
\]

For \( a \ll 1 \) holds: \( \sigma_j = 1 - \epsilon_j (\epsilon_j \ll 1, j = 1, \ldots, n) \), which means that for \( a \to 0 \) approximation (47) becomes exact. As a result, exact (eq.s (34), (35)) and approximate expressions (eq.s (34), (48)) must approach the same result when \( a \to 0 \). It is therefore sufficient to consider the approximate TEO given by eq.s (50), (51), but here necessarily with \( D = 1 \). This is so because (53), with \( D \neq 1 \), spoils the PT limiting case, since \( D(\delta) \) is invariant under PT (39) (and obviously under (42), (43), resp).

We start with rewriting eq.s (50), (51) appropriately, using (42) and (43):

\[
\left( K_{\kappa/a}(\mu/a, \mu_A) \right)_{11} = 1 - \frac{k\sigma_A^a}{a\mu} \int_0^{1-\sigma_A^a} dz \left( \frac{e^{i\lambda'z}}{1-z} \right)^{1-\delta} \frac{R(z; x')}{Z(0; x)},
\]

\[
\left( K_{\kappa/a}(\mu/a, \mu_A) \right)_{12} = \frac{k_3s^a}{a\mu} e^{-i\tau'} \int_0^{1-\sigma_A^a} dz \left( \frac{e^{i\lambda'z}}{1-z} \right)^{1-\delta} \frac{Z(z; x')}{Z(0; x')},
\]

(\text{where} \( x \to x' := \kappa s^a/a (1-\delta) \), \( \lambda \to \lambda' := 1/2 + i\tau'/2(1-\delta) \), and also eq.s (55) (substitute \( 1-\delta \) for \( \epsilon \))):

\[
J_{+\lambda' + n}(x') J_{-\lambda'}(x') = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dv e^{\frac{\pi}{4}v} e^{-i\nu} J_{n+1}(2x' \cos v). \quad (C3)
\]
If \( a \ll 1 \), eq. (C3) assumes the form:

\[
J_{\pm \lambda^* n}(x') J_{\pm \lambda}(x') = (\pm i \text{sign})^n \frac{1 - \delta}{\pi \tau} e^{-\pi i |\tau'|} \left[ 1 + O(\text{e}^{-\text{const.}\pi |\tau'|}) \right]
\]

\[
\times \int_0^{\infty} dv e^{-[1+iO(a)]v} J_{n \pm 1} \left( \frac{\kappa (1 + O(a))}{|k_3|} v \right)
\]

\[
a \to 0 \quad \frac{1 - \delta}{\pi \tau} e^{\pi i |\tau'|} \left| k_3 \right| k (\pm i)^n \left( \frac{k - |k_3|}{\kappa} \right)^{n \pm 1}.
\]

From inspection of (C2) one infers \( z = O(a) \). For convenience we introduce the new integration variable: \( \zeta := z/(1 - \sigma a) \) with \( \zeta \in [0; 1] \). Then:

\[
\frac{\pi \tau'}{1 - \delta} e^{-\pi i |\tau'|} Z(s^a; z) \xrightarrow{a \to 0} \frac{\kappa \left| k_3 \right|}{k} \left\{ \frac{e^{i \zeta k \text{sign} k_3 \ln(t/t_A)}}{k \text{sign} k_3 - k_3} + \frac{e^{-i \zeta k \text{sign} k_3 \ln(t/t_A)}}{k \text{sign} k_3 + k_3} \right\},
\]

where (51), (84), (85), (C4) have been utilized. Analogously one gets:

\[
\frac{\pi \tau'}{1 - \delta} e^{-\pi i |\tau'|} R(s^a; z) \xrightarrow{a \to 0} \frac{2\left| k_3 \right|}{k} \sin \left( \zeta k \text{sign} k_3 \ln \frac{t}{t_A} + O(a) \right),
\]

and insertion of eq.s (C5), (C6) into (C2) (with \( Z(0; x') \) as in (A22)) gives

\[
\left( K_k^{(-)}(t|t_A) \right)_{11} = e^{-ik_3 \ln(t/t_A)} \left[ \cos \left( k \ln \frac{t}{t_A} \right) + i \frac{k_3}{k} \sin \left( k \ln \frac{t}{t_A} \right) \right],
\]

\[
\left( K_k^{(-)}(t|t_A) \right)_{12} = \frac{k_+}{k} e^{-ik_3 \ln(t/t_A)} \sin \left( k \ln \frac{t}{t_A} \right),
\]

with \( k_+ := (k_2 + ik_1) e^{2ik_3 \ln(t_A/t)} \). This is the exact Weyl TEO in background spacetime \( ds^2 = dt^2 - t^2(dx^2 + dy^2 + dz^2) \), which is a special result of (105) for \( \delta = 1, \mu \to 0, \nu \equiv 1 - \mu \to 1 \) (recall that \( s \equiv t^\mu \)). While the limiting case of conformal flatness leads to the entire class \( \{ \bar{1} \} \) of exact spinor solutions, the PT limiting case yields only (as it should) that solution, which corresponds to the PT fixed point \( \mu = 0, \nu = 1 \) [21].
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