A theoretical study on the damping of collective excitations in a Bose-Einstein condensate

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Abstract

We study the damping of low-lying collective excitations of condensates in a weakly interacting Bose gas model within the framework of imaginary time path integral. A general expression of the damping rate has been obtained in the low momentum limit for both the very low temperature regime and the higher temperature regime. For the latter, the result is new and applicable to recent experiments. Theoretical predictions for the damping rate are compared with the experimental values.

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The recent realization of Bose-Einstein condensation (BEC) in dilute atomic vapors [1, 2] has opened the door to experimentally study weakly interacting dilute quantum gases for which microscopic theories have been well studied for decades. In particular, it has been shown that the frequencies of the low-lying collective excitations of the condensates [3, 4] agree excellently with theoretical predictions based on a mean-field theory [5]. The damping in such collective modes has been experimentally discovered by Jin et al [3] and Mewes et al [4]. A very recent experiment [6] has extended the study of low-lying collective excitations of condensates to include higher temperatures in which the damping of these excitations exhibits dramatic temperature dependence. So far there is no theoretical prediction for it in such a temperature regime.

This paper aims to provide a simple way of calculating the damping rate of collective excitations in the above temperature regime. The experimental setup we are considering is a dilute gas of atoms confined in a trap potential. Recently, Chou, Yang and Yu [7] argued that the local density approximation can be applied into the present problem. In other words, the trap potential may be treated as a slowly varying external potential and enters into the theory only as a modification to the chemical potential. To simplify the theory, let us assume that the trap is absent in deriving the damping rate since we are only interested in finding the leading term for it. In the following, we shall study a model of weakly interacting bosonic particles in an infinite free space using the imaginary time path integral method which allows us to take into account excitation collision processes that are believed to be the causal of the damping.

In the low momentum and low temperature limit, one usually finds an effective field
theory in which only slowly varying degrees of freedom appear explicitly with interactions that include the effects of fast varying fields that have been integrated out first. One version of such an effective field theory has been established for a non-ideal Bose gas in the low temperature regime by Popov. The theory treats the system effectively as that of quasiparticles described by two slowly varying real scalar fields $\phi(x)$ and $\sigma(x)$. Here, the $\sigma(x)$ field describes the density fluctuations, written as $\sigma(x) = n(x) - n_0$ with $n_0$ the density of the ground state. The $\phi(x)$ field is the Goldstone field in this theory appearing originally in the phase of original particle fields. Notice that we have written the four-dimensional Euclidean spacetime as $x = (x, \tau)$ with $\tau = it$ denoting the Euclidean “time”. According to Popov, the Euclidean action for such a system can be written by ($\hbar \equiv 1$ henceforth)

\[
S[\phi, \sigma] = \int_0^\beta d\tau \int_{-\infty}^\infty d^3x \left\{ i \frac{\partial^2 p}{\partial \mu \partial n} \sigma \partial_\tau \phi(x) - \frac{1}{2m} \frac{\partial p}{\partial \mu} \nabla \phi(x) \cdot \nabla \phi(x) - \frac{1}{2} \frac{\partial^2 p}{\partial \mu^2} (\partial_\tau \phi(x))^2 \\
+ \frac{1}{2} \frac{\partial^2 p}{\partial n^2} \sigma^2(x) - \frac{\nabla \sigma(x) \cdot \nabla \sigma(x)}{8mn_0} - \frac{\sigma(x)(\nabla \phi(x) \cdot \nabla \phi(x))}{2m} \right\}
\]

(1)

where $m$ is the particle mass and $p$ is the pressure of the system given as the function of the chemical potential $\mu$ and the particle density $n$. Notice that $\phi(x)$ and $\sigma(x)$ are periodic in “time” $\tau$ with period $\beta = 1/(k_B T)$. If $v = (1/m)\nabla \phi$ is identified with the phonon velocity field, one can easily verify that the action (1) corresponds to a Hamiltonian that is one form of Landau-Khalatnikov hydrodynamic Hamiltonian. In the low temperature regime such that the non-condensate fraction of particles is very small, the expression of the pressure $p(\mu, n)$ can be well approximated by that of $T = 0$. For a weakly interacting dilute gas, it is given by

\[
p = \mu n - \frac{t_0}{2} n^2
\]

where $t_0 = \frac{4\pi a}{m}$ characterizes the interaction of particles with the $s$-wave scattering length $a$. 

2
It follows that
\[
\frac{\partial^2 p}{\partial \mu \partial n} = 1, \quad \frac{\partial p}{\partial \mu} = n_0, \quad \frac{\partial^2 p}{\partial \mu^2} = 0, \quad \text{and} \quad \frac{\partial^2 p}{\partial n^2} = -t_0.
\] (2)

For simplicity, we write \(\phi\) and \(\sigma\) fields into a real scalar doublet as \(\Phi = (\phi, \sigma)\). The Green’s functions (or propagators) are defined by the matrix
\[
G(x - x') = \langle T \{\Phi(x)\Phi(x')\} \rangle = \frac{\int [\Pi_x \Phi(x)] \Phi(x') e^{S[\phi, \sigma]}}{\int [\Pi_x \Phi(x)] e^{S[\phi, \sigma]}}
\]
with \(T\) denoting a time-ordered product. The Fourier transform of \(G(x)\) is defined through
\[
G(k) = \frac{1}{\beta(2\pi)^3} \sum_{\nu} \int d^3k G(k) e^{-i(kx - \omega_{\nu} \tau)},
\] (3)
where the notation \(k \equiv (k, i\omega_{\nu})\) is understood and Matsubara frequencies \(\omega_{\nu} = 2\pi \nu / \beta\) \((\nu = 0, \pm 1, \pm 2, \cdots)\). The quadratic part of the action \(S\) can be written into the form of
\[
S_{\text{QUAD}} = \int d^4x d^4x' \Phi(x) \mathcal{D}(x, x') \Phi(x')
\]
and the free Green’s function \(G_0(k)\) is equal to the inverse of the matrix \(\mathcal{D}\). Thus, we find
\[
G_0^{-1}(k) = \begin{pmatrix}
\frac{n_0 k^2}{m} & \omega_{\nu} \\
-\omega_{\nu} & t_0 + \frac{k^2}{4mn_0}
\end{pmatrix},
\] (4)
It follows that
\[
G_0(k) = \begin{pmatrix}
t_0 + k^2 / (4mn_0) & -\omega_{\nu} \\
\omega_{\nu}^2 + c^2(k) & \omega_{\nu}^2 + c^2(k) \\
\omega_{\nu} & \omega_{\nu}^2 + c^2(k) \\
\omega_{\nu}^2 + c^2(k) & \omega_{\nu}^2 + c^2(k)
\end{pmatrix},
\] (5)
where the spectrum
\[
\epsilon(k) = \sqrt{\left(\frac{k^2}{2m}\right)^2 + c^2 k^2}
\] (6)
with \(c \equiv \sqrt{t_0 n_0 / m}\). The propagators \(G_0\) are represented by Feynman diagrams in Fig. [1].

Further, the cubic term of the action ([1]) describes the interaction of three excitations that
are known as phonons in the low momentum region, giving rise to a vertex of

$$\delta^3(k_1 + k_2 + k_3)\delta_{\nu_1+\nu_2+\nu_3,0} \frac{(k_1 \cdot k_2)}{m}$$

represented by the last diagram of Fig. 1. Obviously, it can be treated as perturbation in the low momentum limit.

The spectrum of collective excitations is given by the poles of the exact Green’s function

$$G(k) = G_0(k) + G_0(k)\Pi(k)G(k)$$ where $$\Pi(k)$$ denotes the matrix of the self-energy parts.

It follows that the spectrum is determined by

$$\det G^{-1}(k) = \det(G_0^{-1}(k) - \Pi(k)) = 0. \quad (7)$$

Now that we are working in the imaginary time formalism, let us make the analytical continuation $$i\omega_n \rightarrow \omega + i\eta$$ ($$\eta \equiv 0^+$$) after the Matsubara frequency sum and write $$\omega = \text{Re}\omega - i\gamma(k)$$ with $$\gamma$$ denoting the damping rate. Then, keeping only terms up to one-loop order \[14\], we find from Eqs. (4) and (7) that

$$\gamma(k) = \frac{1}{2\text{Re}\omega} \left[ (t_0 + \frac{k^2}{4mn_0})\text{Im}\Pi_{\phi\phi}(k, \omega + i\eta) + \frac{n_0k^2}{m}\text{Im}\Pi_{\sigma\sigma}(k, \omega + i\eta) \right] - \text{Re}\Pi_{\sigma\sigma}(k, \omega + i\eta) \quad (8)$$

with the understanding that all $$\omega$$’s are replaced with $$\epsilon(k)$$ after the analytical continuation.

Up to one-loop order, there are six diagrams (see Fig. 2) for the self-energy matrix $$\Pi(k, \omega)$$ but only the last five (b)-(f) contribute to the damping. After collecting contributions from all related diagrams, we have

$$\gamma(k) = \gamma_1(k) + \gamma_2(k), \quad (9)$$

with

$$\gamma_1(k) = \frac{1}{32\pi^2} \int d^3k' \delta(\epsilon(k) - \epsilon(k') - \epsilon(k - k'))[f(\epsilon(k')) - f(-\epsilon(k - k'))] \times$$
where \( f \) is the well-known Beliaev’s result \[15\]. The product is determined by

\[
\left\{ \frac{(k - k')^2(k \cdot k')^2\epsilon(k')\epsilon(k)}{2m_0k^2k'^2\epsilon(k - k')} + \frac{(k \cdot k')(k \cdot (k - k'))\epsilon(k)}{2m_0k^2} \\
+ \frac{k^2(k' \cdot (k - k'))^2\epsilon(k')\epsilon(k - k')}{4m_0k^2(k - k')^2\epsilon(k)} + \frac{(k' \cdot (k - k'))(k \cdot k')\epsilon(k')}{mn_0k^2} \right\}
\]

(10)

and

\[
\gamma_2(k) = \frac{1}{32\pi^2} \int d^3k' \delta(\epsilon(k) + \epsilon(k') - \epsilon(k + k')) [f(\epsilon(k')) - f(\epsilon(k + k'))] \times
\left\{ \frac{\epsilon(k)}{2m_0} \left[ \frac{k^2(k \cdot (k + k'))^2\epsilon(k + k')}{k^2(k + k')^2\epsilon(k')} + \frac{(k + k')^2(k \cdot k')^2\epsilon(k')}{k^2k'^2\epsilon(k + k')} \right] + \frac{(k \cdot k')(k \cdot (k + k'))\epsilon(k)}{mn_0k^2} + \frac{k^2(k' \cdot (k + k'))^2\epsilon(k')\epsilon(k + k')}{2m_0k^2(k + k')^2\epsilon(k)} \right\},
\]

(11)

where \( f(\epsilon) = 1/(\exp(\beta\epsilon) - 1) \).

If \( T = 0 \), one can easily verify that for small \( k (ck \ll n_0t_0) \)

\[
\gamma_{T=0}(k) = \frac{3k^5}{640\pi mn_0}
\]

(12)

which is the well-known Beliaev’s result \[15\]. The product \( n_0t_0 \) characterizes the strength of particle interactions.

For \( T \neq 0 \) and small \( k \) such that \( ck \ll k_B T \) and \( ck \ll n_0t_0 \), we find that the damping rate to the lowest order in \( k \) is determined by

\[
\gamma(k) = \frac{ckk_0^5}{16\pi mn_0k_B T} \mathcal{J} \left( \frac{n_0t_0}{k_B T} \right)
\]

(13)

where \( k_0 \equiv \sqrt{mn_0t_0} \) and the function \( \mathcal{J}(x) \) is given by

\[
\mathcal{J}(x) = \frac{\pi^2}{6x^3} + \int_0^\infty d\xi \left[ \frac{2\xi^2}{(1 + \xi^2)^{3/2}} + \frac{3}{2(1 + \xi^2)} - \frac{2}{(1 + \xi^2)^2} \right] \frac{\xi^2 e^{\xi x}}{(e^{\xi x} - 1)^2}.
\]

If the limit \( \frac{k_B T}{n_0t_0} \rightarrow 0 \) is taken, Eq. (13) reduces to the familiar form

\[
\gamma(k) = \frac{3\pi^3k(k_B T)^4}{40mn_0e^4}
\]

(14)
which was given by Hohenberg and Martin [16]. Though, we shall see in the following paragraph that Eq. (14) is invalid in the recent experimental temperature regime. In fact, both expressions (12) and (14) were already derived by Popov in Ref. [4] from the effective action (11). But, to the best of my knowledge, the general expression (13) is first obtained here. It is valid as long as the system is in the low temperature region such that the number of particles in the excited states is much less than that in the condensate.

The damping of collective excitations in BEC has been measured in the dilute atomic vapors of both 87Rb and sodium at JILA and MIT, respectively [3, 4, 6]. For the experiment of 87Rb at JILA, the condensates are produced in a trap with frequencies of \( \nu_r = 129\text{Hz} \) radially and \( \nu_z = 365\text{Hz} \) axially. For a typical condensate of \( N_{\text{BEC}} = 4500 \) atoms, the density of a condensate can be estimated as \( n_0 \approx 1.4 \times 10^{14}\text{cm}^{-3} \) [17]. Further, the scattering length for a 87Rb vapor may be taken roughly as \( a = 103\text{Bohr} \) [18]. It turns out that the typical interaction energy \( n_0 t_0 \approx 57\text{nK} \). Therefore, the damping rate is expected to be given by Eq. (13) instead of Eq. (14) in the temperature region \((30, 300)\text{nK}\) where it was measured (see Figs. 1 and 3 of Ref. [6]). We also checked that for the lowest excitation mode \((m = 2)\) both \( ck/(k_B T) \) and \( ck/(n_0 t_0) \) are less than 0.2 for any \( T \) above 50nk. Fig. 3 shows theoretical predictions for the damping rate of collective excitations based on Eq. (13) in comparison with the experimental data from Ref. [6]. The theoretical curve \((b)\) seems to fit the experiment well apart from that the theoretical values are varying with the density of condensates approximately according to \( \gamma \propto 1/\sqrt{n_0^3} \). However, a Bose condensate in a trap is essentially inhomogeneous and we will not try to determine a proper average density \( n_0 \) here. Next, let us check whether the above experiment is in the low temperature region.
For $T < 100\text{nK}$, we read off from Fig. 1 of Ref. [6] that the corresponding $T' < 0.6$ (where $T' \equiv T/T_c$) and the condensate fraction $N_{\text{BEC}}/N$ is greater than or about 80%. Hence, the higher order corrections to Eq. (13) due to finite temperature effects only can be estimated from the calculation of Feynman diagrams less than $(1 - N_{\text{BEC}}/N)^2 = 4\%$. On the other hand side, Ref. [6] shows that more than 50% of particles reside in noncondensates when $T' > 0.8$ and correspondingly $T > 175\text{nK}$ roughly. This means that the damping rate shall be no longer determined by Eq. (13) in that temperature regime. I do not know yet whether there exists any theoretical prediction for it.

In Fig. 4, we plot the damping rate of the collective excitation of frequency 30Hz for the sodium gas system of Mewes et al [4] using the $s$-wave scattering length $a = 65\text{Bohr}$ [20]. The decay time of 250(40)ms was found experimentally when a nearly pure condensate was formed at the temperature $T \approx 0.5T_c$ [21]. The Bose-Einstein transition temperature is determined theoretically according to $T_c = \frac{\hbar \bar{\omega}}{k_B}(N/1.202)^{1/3}$ [22,20] with $\bar{\omega}$ the geometric mean of the harmonic trap frequencies $\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$. For that the trap used in Ref. [4] is of frequencies of 250Hz radially and 19Hz axially and typically holds a total number $N \approx 5 \times 10^6$ of atoms when the decay time was measured, we have $T_c \simeq 800\text{nK}$. For a typical condensate density $n_0 = 3 \times 10^{14}\text{cm}^{-3}$ [20], we can read off from the curve (c) of Fig. 4 that $\gamma \simeq 4.4\text{s}^{-1}$ for temperature $T = 400\text{nK}$. This damping rate corresponds to a decay time of about 230ms that agrees very well with the experimental value. Also, since at such a temperature the condensate fraction of atoms is around 90%, higher order corrections to $\gamma$ due to the finite temperature effect alone are quite small ($< 1\%$). Also, we checked that both $ck/(kBT)$ and $ck/(n_0t_0)$ are less than one percent. That is to say, the system is well in
the low momentum region.

In conclusion, this paper analytically calculates the damping rate of collective excitations for a dilute Bose gas model in a temperature regime where theoretical predictions did not exist previously. Although the model has ignored the contribution of the trap and has required that the condensate be homogeneous, it produces results for the damping of collective excitations that are in good agreement with the experiments. Our study also reveals that the damping is due to the three-phonon collision processes at finite temperature. This paper serves a very preliminary study of this subject and fully understanding it requires a complete analysis to take into account the contribution of the trap and the inhomogeneity of the system.

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Figure 1: Diagrams for propagators and vertex.
Figure 2: One-loop diagrams for the self-energy matrix $\Pi(k)$. 
Figure 3: The damping rate of collective excitations in a $^{87}$Rb atomic gas. Three solid lines indicate theoretical predictions from Eq. (13). For those, the excitation frequency is taken as $\omega/2\pi = 1.4\nu_r = 180.6$Hz corresponding to the mode $m = 2$ and the condensate densities are: (a) $n_0 = 1.0 \cdot 10^{14}$cm$^{-3}$; (b) $n_0 = 2.0 \cdot 10^{14}$cm$^{-3}$; and (c) $n_0 = 3.0 \cdot 10^{14}$cm$^{-3}$. Two discrete curves are replotted from the data of Ref. [6]. (The experimental data is the courtesy of M. R. Matthews [13].)
Figure 4: The damping rate of collective excitations in a sodium atomic gas. All solid lines are plotted according to Eq. (13). The excitation frequency is 30Hz and the condensate densities are: (a) $n_0 = 1.0 \cdot 10^{14}\text{cm}^{-3}$; (b) $n_0 = 2.0 \cdot 10^{14}\text{cm}^{-3}$; (c) $n_0 = 3.0 \cdot 10^{14}\text{cm}^{-3}$; and (d) $n_0 = 4.0 \cdot 10^{14}\text{cm}^{-3}$.