Phase diagram and correlation functions of the anisotropic imperfect Bose gas in $d$ dimensions

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Abstract. We study an anisotropic variant of the $d$-dimensional imperfect Bose gas, where the asymptotic behaviour of the dispersion $\epsilon_k$ at vanishing momentum $k$ may differ from the standard quadratic form. The analysis reveals the key role of the shift exponent $\psi$ governing the asymptotic behaviour of the critical temperature $T_c(\mu)$ as a function of the chemical potential $\mu$ at $T_c \rightarrow 0$. We argue that the universality classes of Bose–Einstein condensation admitted by the model may be classified according to the allowed values of $\psi$ so that spatial dimensionality has only an indirect impact on the transition properties. We analyse the correlation function of the model and discuss its asymptotics depending on the direction. Both for the perfect and imperfect anisotropic Bose gases, the correlation function $\chi(x)$ at $T > T_c$ turns out to show either exponential decay or exponentially damped oscillatory behaviour depending on the orientation of $x$ with respect to the dispersion anisotropies.

Keywords: Bose–Einstein condensation, cold atoms, correlation functions, quantum criticality
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**Contents**

1. Introduction ................................................................. 2
2. Model and the saddle-point method .................................... 3
   2.1. Genesis of the dispersion asymptotics .......................... 5
   2.2. The ‘cone’ dispersion .................................................. 6
3. Phase diagram ............................................................... 6
   3.1. Lower critical dimension .............................................. 6
   3.2. Critical line ............................................................... 8
   3.3. Critical behaviour ....................................................... 8
4. Comparison to the ideal Bose gas ........................................ 10
5. Correlation function ....................................................... 12
   5.1. Correlation function of the perfect anisotropic Bose gas:
        asymptotic behaviour ................................................... 12
      5.1.1. Case $\epsilon_k = \frac{1}{2} \sum_{j=1}^{d-1} k_j^2 + \frac{1}{2} k_d^2$ and $x = x e_1$ ........................................ 13
      5.1.2. Case $\epsilon_k = \frac{1}{2} \sum_{j=1}^{d-1} k_j^2 + \frac{1}{2} k_d^2$ and $x = x e_d$ ........................................ 13
      5.1.3. Case $\epsilon_k = \frac{1}{2} \sum_{j=1}^{d-1} k_j^2 + \frac{1}{2} k_d^{2m}$ $(m \in \mathbb{N})$ ........................................ 14
   5.2. Correlation function of the perfect anisotropic Bose gas:
        general expressions ...................................................... 14
   5.3. Correlation function of the imperfect anisotropic Bose gas ..................................................... 16
6. Remarks on the relation to the spherical model ....................... 16
7. Summary ........................................................................ 17
   Acknowledgments ..................................................................... 18
   References ........................................................................ 18

### 1. Introduction

Bose gases are among the few systems that have remained within central focus of condensed matter physics for many decades. Certainly the detailed context of these studies evolved over time. In particular, the last years witnessed remarkable progress in experimental control over ultra-cold atomic gases (bosons in particular) in optical lattices [1, 2]. An accurate theoretical [3] description of such systems necessarily involves a lattice and inter-particle interactions. For the majority of theoretical models, such as the Bose–Hubbard model [4], these may be treated only approximately, and, in most cases, involve costly numerical techniques. Recognised theoretical approaches to interacting Bose systems on a lattice include, among others, strong-coupling expansions [5, 6], reflection positivity [7], renormalisation group [8–12] and bosonic dynamical mean field theory [13, 14].
Bearing the above in mind it seems worthwhile investing some effort in development of simplified theoretical models which are susceptible to an exact analytical treatment. In the present paper we adapt a model known as the imperfect Bose gas (IBG) [15–19] to the anisotropic (d-dimensional) case. This may result from an underlying lattice. We show how tuning the lattice parameters may lead to varying the asymptotic behaviour of the dispersion $\epsilon_k$, which in turn alters the universality class of the transition as well as the lower and upper critical dimensions as compared to the standard isotropic gas in the continuum. The present model is exactly solvable and allows for tracking the emergence of universality without a recourse to renormalisation group theory. The anisotropies of the dispersion are also a source of interesting behaviour of the correlation function $\chi(x)$ which, as we demonstrate, may (in the normal phase) show either monotous exponential decay, or exponentially damped oscillations, depending on the direction of $x$. This property occurs both for the perfect (noninteracting) and imperfect Bose gases.

The outline of the paper is as follows: in section 2 we introduce the model together with the saddle-point technique leading to its exact solution. Of high relevance is the asymptotic form of the dispersion $\epsilon_k$, which we relate to specific lattice models. Section 3 contains a presentation of our results concerning the phase diagram and properties of the Bose–Einstein condensation. These are compared to the case of the noninteracting Bose gas [20, 21] in the subsequent section 4. We analyse the correlation function of both the perfect and imperfect Bose gases in section 5 with emphasis on the asymptotic features and properties related to the anisotropy. We discuss relations between the imperfect Bose gas and the spherical model in section 6. We summarise the paper in section 7.

2. Model and the saddle-point method

We consider a system of spinless bosons at a fixed temperature $T$, chemical potential $\mu$ and volume $V = L^d$. The system is governed by the Hamiltonian

$$\hat{H} = \sum_k \epsilon_k \hat{n}_k + \frac{a}{2V} \hat{N}^2.$$  

We impose periodic boundary conditions and the dispersion relation $\epsilon_k$ is, for now, unspecified. The $k$ summation runs over the first Brillouin zone. We will in particular discuss the hypercubic lattice in section 2.1. The quantity $\hat{N} = \sum_k \hat{n}_k$ is the total particle number operator. The repulsive mean-field interaction term $H_{mf} = \frac{a}{2V} \hat{N}^2$ $(a > 0)$ may be derived from the long-range repulsive part $v(r)$ of a 2-particle interaction potential upon performing the Kac scaling limit $\lim_{\gamma \to 0} \gamma^d v(\gamma r)$, i.e. for vanishing interaction strength and diverging range. Therefore, the interaction employed in equation (1) corresponds to a limiting situation where the force between two particles is taken sufficiently long ranged and weak so that the actual distance between them becomes less and less relevant. The presence of the $1/V$ factor assures extensivity of the system. The continuum version of the model with $\epsilon_k \sim k^2$ was studied in [15–19, 22–24]. The presence of the mean-field interaction $H_{mf}$ has a profound impact on the properties of
Bose–Einstein condensation and makes it substantially different as compared to the ideal Bose gas [21].

Following the line of [19] the grand canonical partition function \( \Xi(T, V, \mu) \) can be represented via the contour integral

\[
\Xi(T, V, \mu) = -ie^{\frac{\mu^2}{2a}}V \left( \frac{V}{2\pi a \beta} \right)^{1/2} \int_{\theta \beta - i\infty}^{\theta \beta + i\infty} ds e^{-s V \phi(s)},
\]

where the parameter \( \theta < 0 \) is arbitrary and

\[
\phi(s) = \frac{s^2}{2a \beta} + \frac{\mu s}{a} - \frac{1}{\sqrt{V}} \sum_{k \neq 0} \ln \frac{1}{1 - e^{-s \epsilon_k}} - \frac{1}{\sqrt{V}} \ln \frac{1}{1 - e^s}
\]

with \( \beta = (k_B T)^{-1} \). We isolated the last term and assumed that there is a single minimum of the dispersion \( \epsilon_k \) in the Brillouin zone, which occurs at \( k = 0 \) with \( \epsilon_{k=0} = 0 \).

The integration contour in equation (2) may be deformed to pass through a saddle point \( \bar{s} \), located in the negative part of the real axis in the corresponding complex plane (see below). The factor \( V \) in the exponential in equation (2) then assures that the saddle-point approximation to the integral in equation (2) becomes exact for \( V \to \infty \), i.e.

\[
\lim_{V \to \infty} \frac{1}{V} \log \Xi(T, V, \mu) = \frac{\beta \mu^2}{2a} - \phi(\bar{s}).
\]

This reduces the problem of evaluating the integral in equation (2) to solving the saddle-point equation

\[
\phi'(\bar{s}) = 0
\]

and subsequently computing \( \phi(\bar{s}) \). Explicitly equation (5) reads

\[
-\frac{\bar{s}}{a \beta} + \frac{\mu}{a} = \frac{1}{\sqrt{V}} \sum_{k \neq 0} \frac{1}{e^{\beta \epsilon_k} - 1} + \frac{1}{\sqrt{V}} \frac{1}{e^{-\bar{s}} - 1}.
\]

Thus far we adapted the reasoning of [19] to the present, more general case where the dispersion \( \epsilon_k \) remains unspecified. For the continuum case, where \( \epsilon_k \sim k^2 \) the \( k \)-summation in equation (6) may be performed for \( V \to \infty \) yielding a Bose function. This is not the case here, where even convergence properties determining the lower critical dimension for condensation do depend on \( \epsilon_k \).

We now consider the thermodynamic limit \( V \to \infty \) where \( \sum_{k \neq 0} \to \frac{V}{(2\pi)^d} \int d^d k \). The term involving the \( k \) summation occurring in equation (6) is then replaced by the integral

\[
I_d(\beta, \bar{s}) = \frac{1}{(2\pi)^d} \int_{BZ} d^d k \frac{1}{e^{\beta \epsilon_k} - 1}.
\]

In the thermodynamic limit and for \( T \leq T_c \) or \( T \to T_c^+ \) we have \( \bar{s} \to 0^- \) (see section 3) and therefore the integral is dominated by the vicinity of \( k = 0 \).

In what follows we will specify to the cases where the asymptotic behaviour of the dispersion \( \epsilon_k \) around \( k = 0 \) may be cast in the form

\[
\text{https://doi.org/10.1088/1742-5468/aabc7c}
\]
Phase diagram and correlation functions of the anisotropic imperfect Bose gas in $d$ dimensions

\[ \epsilon_k \approx \sum_{i=1}^{d} t_i |k_i|^{\alpha_i}, \]  

(8)

where $\alpha_i$ and $t_i$ are positive real numbers for $i \in \{1 \ldots d\}$. We discuss this restriction and the specific relevant cases below in section 2.1.

2.1. Genesis of the dispersion asymptotics

Consider now a system of noninteracting bosons on a hypercubic lattice, subject to periodic boundary conditions. The corresponding Hamiltonian reads

\[ H_{\text{free}} = -\sum_{(xy)} t_{xy} c_x^\dagger c_y = \sum_k \epsilon_k c_k^\dagger c_k, \]  

(9)

where $x$ and $y$ label the lattice points, $(xy)$ denotes a pair of points, the hoppings $t_{xy}$ are assumed to be real, and the second equality follows from diagonalising the Hamiltonian with the Fourier transform. The dispersion is expressed as

\[ \epsilon_k = \sum_x 2t_x (1 - \cos (k \cdot x)) + \text{const} \]  

(10)

with $t_x = t_{0x}$. One can add a constant to the original Hamiltonian to assure that the constant appearing in equation (10) vanishes. For positive hopping coefficients the dispersion is always quadratic around $k = 0$. This does not have to be the case in situations where the hoppings carry different signs. For example, consider the case

\[ \epsilon_k = 2t \sum_{i=1}^{d} (1 - \cos k_i) + 2t_2 (1 - \cos 2k_1). \]  

(11)

If the $t, t_2$ couplings fulfil: $t_2 = -t/4$, the coefficient of $k_1^2$ vanishes, and the asymptotics takes the form

\[ \epsilon_k \sim t \sum_{i=2}^{d} k_i^2 + \tau_1 k_1^4, \]  

(12)

which is anomalously flat in the direction 1.

Now generalise the above case by taking $t_{e_i} = t > 0$ for $i = 1, \ldots, d$; $t_{2e_1} = t_2$, $t_{3e_1} = t_3, \ldots, t_{me_1} = t_m$. The analysis then shows that by a suitable choice of the hopping constants $t_2, \ldots, t_m$ one can achieve vanishing of the derivatives of $\epsilon_k$ in direction 1 up to the order $2m$. The asymptotic form of the dispersion then reads

\[ \epsilon_k \sim t \sum_{i=2}^{d} k_i^2 + \tau_m k_1^{2m}, \quad \tau_m > 0. \]  

(13)

Such a construction can be carried out independently in each of the spatial directions, leading to the asymptotics of the form of equation (8). The resulting exponents $2m_i$ are even natural numbers, which may by made arbitrarily large by judicious choices of the hoppings. For the sake of generality we allow the exponents appearing in equation (8) to be arbitrary positive real numbers. Observe however that such a generalization can be realised only by involving infinite-range hoppings.

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The asymptotics given by equation (8) displays a minimum at \( k = 0 \) being an *isolated point*. By modifications of certain assumptions of the model, one can obtain a *non isolated* minimum. For instance, if we relax the demand of strict translational invariance to translational invariance with some period, we can achieve a completely flat dispersion in one or more directions. Examples of lattices with such dispersions appear in the Hubbard models with ‘flat-band ferromagnetism’ [25] (the ‘sawtooth’ one-dimensional lattice, or two-dimensional kagomé lattice). The analysis of systems with non-isolated critical points (‘completely flat dispersion’) will not be considered here.

2.2. The ‘cone’ dispersion

It is clear that the asymptotic formula assumed in equation (8) also applies to the ultrarelativistic Bose gas in continuum, where the relativistic dispersion is dominated by the kinetic contribution, i.e.

\[
\sqrt{m^2c^4 + c^2p^2} \approx c|p|.
\]

(14)

Of course, we consider only the kinetic energy term and neglect QFT effects such as creation of particle-antiparticle pairs etc.

The model with relativistic dispersion can be viewed as of academic interest only, however it is also possible to achieve the ‘cone’ asymptotic form by a suitable tuning of the hopping constants, potentially involving long-ranged hoppings.

3. Phase diagram

We now analyse the properties of the model defined by equation (1) and assuming the asymptotic behaviour of \( \epsilon_k \) given by equation (8).

3.1. Lower critical dimension

The last term in equation (6) has the interpretation of the condensate density \( n_0 \) [19]. The system finds itself in the low-temperature phase (i.e. the phase hosting the Bose–Einstein condensate) if \( n_0 \) remains finite in the thermodynamic limit. If the integral \( I_d(\beta, \bar{s}) \) (see equation (7)) diverges for \( \bar{s} = 0 \), then equation (6) admits a solution for \( \bar{s} < 0 \) at any \( V \), and \( \lim_{V \rightarrow \infty} \bar{s} < 0 \) at any \( T > 0 \). Indeed, the left-hand side (LHS) of equation (6), viewed as a function of \( \bar{s} \), may be represented as a straight line of negative slope. On the other hand, the first term on the right-hand side (RHS) represents (for \( \bar{s} < 0 \) a continuous convex function unbounded from below. If this function diverges for \( \bar{s} \rightarrow 0^- \), these two curves must intersect at some \( \bar{s} < 0 \). This implies that the last term of equation (6) vanishes in the thermodynamic limit, therefore condensation does not occur at any \( T > 0 \), and, in consequence, the system finds itself below the lower critical dimension. Conversely, if the integral in equation (6) is finite at \( \bar{s} = 0 \), than by lowering \( T \) one may shift the straight line corresponding to the LHS of equation (6) such that the two curves do not intersect. The difference is then compensated by the last term of equation (6). For a graphical illustration we refer to figure 1 of [19].
We therefore seek to derive a condition for the convergence/divergence of the $d$-dimensional integral of equation (7). The integral is dominated by the vicinity of $k = 0$ and the replacement of $\epsilon_k$ by its asymptotics at small $k$ has no impact on convergence. We use the asymptotics given in equation (8) and perform the variable transformation

$$x_i^2 = \beta t_i |k_i|^\alpha_i \quad \text{for} \quad i = 1 \ldots d.$$  \hspace{1cm} (15)

This brings the integral to the form

$$I = \pi^{-d} \int dx_1 \ldots dx_d \prod_{i=1}^{d} \frac{x_i^{1+\frac{1}{\alpha_i}}}{\alpha_i} \left( \frac{1}{\beta t_i} \right)^{\frac{1}{\alpha_i}} \frac{1}{e^{x_i^2} - 1},$$  \hspace{1cm} (16)

where

$$r_d^2 = x_1^2 + \ldots x_d^2.$$  \hspace{1cm} (17)

The expression for $I$ differs from the integral in equation (7) only by a finite expression.

We subsequently introduce the spherical coordinates

$$x_i = \begin{cases} r_d \cos \theta_i & i = 1 \\ r_d \left( \prod_{j=1}^{i-1} \sin \theta_j \right) \cos \theta_i & i \in \{2, \ldots, d - 1\} \\ r_d \left( \prod_{j=1}^{d-1} \sin \theta_j \right) & i = d \end{cases}.$$  \hspace{1cm} (18)

The angular integrations then give a constant and the convergence of the integral becomes equivalent to finiteness of the expression

$$I = \text{const} \int_0^\Lambda dr \frac{r^{-1+\frac{1}{\psi}}}{e^{x^2} - 1},$$  \hspace{1cm} (19)

where we replaced $r_d$ with $r$ and introduced

$$\frac{1}{\psi} = \sum_{i=1}^{d} \frac{1}{\alpha_i}.$$  \hspace{1cm} (20)

The cutoff $\Lambda > 0$ may be taken arbitrarily small without influencing convergence. The integral of equation (19) converges for $\psi < 1$ and diverges otherwise. The condition for the lower critical dimension $d_L$ therefore reads

$$\frac{1}{\psi} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \ldots + \frac{1}{\alpha_{d_L}} = 1.$$  \hspace{1cm} (21)

We conclude this subsection by stating that condensation occurs if and only if

$$\psi < 1.$$  \hspace{1cm} (22)

In the most standard case where $\alpha_i = 2$ for each $i$, we recover $d_L = 2$. Obviously, higher values of the exponents $\{\alpha_i\}$ tend to rise $d_L$. Using the observations from section 2.1 we note that $d_L$ may be pushed to arbitrarily high values. In the next subsection we demonstrate that $\psi$ may be given the interpretation in terms of the shift exponent, which governs the low temperature asymptotics of the phase boundary $T_c(\mu)$. 

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3.2. Critical line

If $d > d_L$ and the thermodynamic state is adjusted to be exactly at the phase transition, we have $\bar{s} \to 0^-$, and, in addition, $n_0 \to 0$ for $V \to \infty$. Equation (6) then yields the condition for the condensation temperature $T_c$ as a function of the chemical potential (or vice versa).

$$\frac{\mu}{a} = \frac{1}{(2\pi)^d} \int_{BZ} d^d k \frac{1}{e^{\epsilon_k/(k_B T_c)} - 1}.$$  \hspace{1cm} (23)

We note that the expression on the RHS of the above equation (with $T_c$ replaced by $T$) coincides with the expression for the critical density $n_c^p$ of a perfect Bose gas at a given temperature $T$ [21]. We therefore recover the relation

$$\mu_c(T) = a n_c^p(T).$$ \hspace{1cm} (24)

The dependence of $T_c$ on $\mu$ as given implicitly by equation (23) is nonuniversal, i.e. depends on the specific form of $\epsilon_k$. We note however, that in the limit of $T_c \to 0$ the integral becomes dominated by $k \approx 0$. This allows us to replace $\epsilon_k$ with its asymptotic form given by equation (8) and investigate the universal asymptotic behaviour of $T_c(\mu)$ at low temperatures. By performing the transformation of equation (15), passing to the spherical coordinates equation (18), we extract the dependence $\mu(T_c)$ in the form

$$\mu \simeq \text{const} \times T_c^{\frac{\psi}{d}}$$ \hspace{1cm} (25)

with $\psi$ given by equation (20). The non-universal prefactor involves the product

$$\prod_{i=1}^d \left[ \frac{2}{\alpha_i} \left( \frac{k_B T_c}{t_i} \right)^{\frac{1}{\alpha_i}} \right]$$

and the integral from equation (16). The exponent $\psi$ governing the asymptotic dependence $T_c(\mu)$ is however universal and given by equation (20). For $\epsilon_k \sim k^2$ it agrees with the known renormalisation-group result for $\phi^4$-type effective field theory, which reads [28–30]

$$\psi = \frac{z}{d + z - 2}.$$ \hspace{1cm} (26)

For the presently relevant case of interacting Bose particles the dynamical exponent $z = 2$. In consequence the above formula agrees with equation (20) as long as $\alpha_i = 2$ for each $i \in \{1 \ldots d\}$. Interestingly, neither the above standard formula, nor our equation (22) allows for the possibility of realising a phase diagram with $\psi > 1$.

3.3. Critical behaviour

We now analyse the saddle-point equation (6) asymptotically close to the phase transition, on the high-temperature side, where $\bar{s}$ can be used as a small parameter. We split the integration domain (Brillouin zone) into a union of an arbitrary open neighbourhood $K_\Lambda^d$ of $k = 0$, which is of a characteristic (arbitrarily small) size $\Lambda$, and the remainder $\mathcal{R}_\Lambda$, so that in equation (7)

$$\int_{BZ} = \int_{K_\Lambda^d} + \int_{\mathcal{R}_\Lambda}.$$ \hspace{1cm} (27)
Phase diagram and correlation functions of the anisotropic imperfect Bose gas in $d$ dimensions

The contribution to the integral $I_d(\beta, \bar{s})$ in equation (7) from the remainder region is then regular and one may simply expand the integrand in $\bar{s}$ around zero $e^{-\bar{s}} = 1 - \bar{s} + \ldots$. This gives rise to terms linear in $\bar{s}$ in equation (6). In the region $K^A_d$ it is legitimate to use the asymptotic form of $\epsilon_k$ given by equation (8). We then proceed along a line similar to section 3.1. We perform the rescaling given by equation (15) and pass to the spherical coordinates equation (18). It is then advantageous to specify $K^A_d$ in such a way that the transformation maps it onto a $d$-dimensional ball of a radius $\Lambda'$. One then performs the angular integrations and the asymptotic form of equation (6) yields

$$-\frac{\bar{s}}{a\beta} + \frac{\mu}{a} = A_d \beta^{-1/\psi} \left[ G_d \int_0^{\Lambda'} dr \frac{r^{2/\psi - 1}}{e^{-\bar{s}r^2} - 1} + c_0 + c_1 \bar{s} + \ldots \right],$$

(28)

where $A_d = \frac{1}{\pi^d} \left[ \prod_{i=1}^d \left( \frac{1}{\alpha_i} \right)^{-1/\alpha_i} \right]$, $G_d$ is a numerical constant arising from the angular integration, and, finally $c_0$ and $c_1$ are non-universal constants contributed by the integration over the $R^A$ region. The neglected terms are of order $\bar{s}^2$. We now observe that by extending the integration domain in equation (28) from $\Lambda'$ to infinity, we produce another contribution analytical in $\bar{s}$, i.e.

$$\int_0^{\Lambda'} dr \frac{r^{2/\psi - 1}}{e^{-\bar{s}r^2} - 1} = \int_0^{\infty} dr \frac{r^{2/\psi - 1}}{e^{-\bar{s}r^2} - 1} + d_0 + d_1 \bar{s} + \ldots.$$  

(29)

The power series $d_0 + d_1 \bar{s} + \ldots$ may be combined with $c_0 + c_1 \bar{s} + \ldots$ in equation (28). Subsequently we write

$$\int_0^{\infty} dr \frac{r^{2/\psi - 1}}{e^{-\bar{s}r^2} - 1} = \frac{1}{2} \Gamma\left(\frac{1}{\psi}\right) g_{1/\psi}(e^{\bar{s}}),$$

(30)

where we used the integral representation of the Bose functions

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^{\infty} dx \frac{x^{n-1}}{e^x - 1}.$$  

(31)

For clarity we now rewrite equation (28) in the form

$$-\frac{\bar{s}}{a\beta} + \frac{\mu}{a} = B_d \beta^{-1/\psi} g_{1/\psi}(e^{\bar{s}}) + C_d \beta^{-1/\psi} + E_d \beta^{-1/\psi} \bar{s} + \ldots,$$  

(32)

where we incorporated all the non-universal ($T$-independent) constants into $B_d$, $C_d$ and $E_d$. The asymptotic form of the Bose functions is given by [21]

$$g_n(e^{\bar{s}}) - \zeta(n) \approx \begin{cases} 
\Gamma(1 - n) |\bar{s}|^{n-1} & 1 < n < 2 \\
|\bar{s}| \log(|\bar{s}|) & n = 2 \\
-\zeta(n-1) |\bar{s}| & n > 2.
\end{cases}$$

(33)

Equation (32) is structurally similar to the one studied for the continuum gas [22, 23], the major difference being that the role of the dimensionality $d$ has now been taken over by the quantity $2/\psi$. The leading $\bar{s}$-dependent term in equation (32) is linear for $\frac{1}{\psi} > 2$, and in consequence the dependence of $\bar{s}$ on $d$ drops out. The condition for the upper critical dimension $d_U$ therefore reads

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\[
\frac{1}{\psi} = 2,
\] (34)

which generalises the result \( d_U = 4 \) [22] to systems characterised by an arbitrary dispersion admitted by equation (8). On the other hand, for \( \frac{1}{\psi} < 2 \) the linear terms in equation (32) can be dropped. By plugging the asymptotic formula of equation (33) into equation (32) we find the asymptotic solution for \( \bar{s} \) in the vicinity of the phase transition, i.e. for \( \epsilon = (\mu - \mu_c)/\mu_c \ll 1 \) in the form

\[
\bar{s} \sim \begin{cases} 
-\epsilon^{\psi/(1-\psi)} & 1 < 1/\psi < 2 \\
\epsilon/\log(\epsilon) & 1/\psi = 2 \\
-\epsilon & 1/\psi > 2 
\end{cases}.
\] (35)

By computing the grand canonical free energy \( \omega \) from equation (4) it is possible to extract the exponent \( \alpha \) defined by \( \omega^{\text{sing}} \sim |\epsilon|^{2-\alpha} \). One obtains

\[
\alpha = \begin{cases} 
\frac{1-2\psi}{1-\psi} & 1 < 1/\psi < 2 \\
0 & 1/\psi > 2 
\end{cases}.
\] (36)

For the case \( \alpha_i = 2 \) for \( i \in \{1 \ldots d\} \) one recovers the exponents specific to the Berlin-Kac universality class [26].

4. Comparison to the ideal Bose gas

In this section, we analyse the ideal Bose gas with the dispersion asymptotics given by equation (8). This cannot be done by taking \( a \to 0 \) since this operation does not commute with the thermodynamic limit. The calculation is standard, so we quote only the key points.

The general expression for the grand partition function \( \Xi \) of the Bose gas with the one-particle dispersion \( \epsilon_k \) is given by [20, 21]

\[
\ln \Xi = -\sum_{k \neq 0} \ln(1 - ze^{-\beta \epsilon_k}) - \ln(1 - z),
\]

where \( z = e^{\beta \mu} \) is determined by

\[
N = \sum_{k \neq 0} \frac{ze^{-\beta \epsilon_k}}{1 - ze^{-\beta \epsilon_k}} + \frac{z}{1 - z}.
\]

In the thermodynamic limit we have

\[
\begin{align*}
\lim_{\infty} \frac{1}{V} \ln \Xi &= -B_d(\beta, z), \\
\lim_{\infty} \frac{1}{V} \frac{z}{1 - z} &= B_d(\beta, z) + \lim_{\infty} \frac{1}{V} \frac{z}{1 - z}.
\end{align*}
\] (37)

where:
The phase diagram of the ideal Bose gas has little in common with that of the IBG, since it is not defined for $\mu > 0$, and condensation may occur only for $\mu = 0$. The present analysis proceeds analogously to the case of the standard continuum model [20, 21] and we here list only the aspects of present relevance.

- The Bose–Einstein condensation is present if the integral (39) is convergent for $z = 1$, otherwise there is no phase transition. Observe that the convergence condition for $b_d(\beta,1)$ is identical as for the integral of equation (7) for $\bar{s} = 0$. In consequence the system displays Bose–Einstein condensation exclusively for $\psi < 1$ which is the same as for the IBG. Note however, that the quantity $\psi$ cannot be given the interpretation in terms of the shift exponent (which now has no sense at all), since condensation is restricted to $\mu = 0$.

- The critical exponents of these two models do not coincide. Let us exemplify this with the behaviour of the specific heat $C$. For the noninteracting Bose gas it turns out to behave as in the continuum model in $d$ dimensions [21, 27], where however the dimension $d$ of the continuum model is replaced by $d_{ef} = \frac{2}{\psi}$. We obtain the following behaviour near the critical temperature ($n$ is fixed and we consider the temperature dependence):

(i) The specific heat is a continuous (although non-smooth) function of temperature for $d_{ef} < 4$. The left- and right-side derivatives at the critical point are finite.

(ii) For $d_{ef} = 4$, the specific heat is also continuous at $T_c$, but the right-hand derivative is infinite.

(iii) For $d_{ef} > 4$ there is a discontinuity in the specific heat itself. Note that such a feature is characteristic to Landau theory.

The asymptotic behaviour of $b(\beta, z)$ for $z \approx 1$ does not enter into the behaviour of the specific heat—only the convergence properties of $b(\beta, z)$ are important.

- The asymptotic behaviour of $b(\beta, z)$ for $z \approx 1$ however influences the behaviour of specific heat near $T = 0$. It turns out that we have the following asymptotics:

$$C \sim T^{\frac{1}{\psi}}.$$  \hspace{1cm} (40)

In $d = 3$, for the standard quadratic asymptotics of $\epsilon_k$ we recover the behaviour characteristic to the continuum model: $C \sim T^{3/2}$, and for $\alpha_1 = \alpha_2 = \alpha_3 = 1$, we have: $C \sim T^3$, i.e. the same behaviour as for photons in a continuum model.
5. Correlation function

In this section we analyse the correlation function of the anisotropic imperfect Bose gas defined in section 2. As was shown in [31] for the isotropic case, this can be obtained from the correlation function of the noninteracting Bose gas upon replacing the chemical potential $\mu$ with $\tilde{\mu} = \mu - an(T, \mu)$. The derivation of this result is not sensitive to the particular form of the dispersion and straightforwardly carries over to the present, more general case. The parameter $\tilde{\mu}(T, \mu)$ is obtained from

$$n(T, \mu) = -\frac{\partial \omega}{\partial \mu} = \frac{\mu}{a} - \beta^{-1} \frac{\partial}{\partial \mu} \phi(s(T, \mu), T, \mu),$$

which leads to

$$n(T, \mu) = \frac{\mu}{a} - \frac{\tilde{s}}{a\beta^i},$$

and, in consequence

$$\tilde{\mu} = \beta^{-1} \tilde{s}.$$ (43)

This relation may be viewed as an interpretation of the parameter $\tilde{s}$ in terms of a renormalised chemical potential. With the above knowledge, the problem now becomes reduced to an analysis of the correlation function of the anisotropic perfect Bose gas, followed by the replacement $\mu \rightarrow \tilde{\mu} = \beta^{-1} \tilde{s}$ and an extraction of the physically meaningful quantities with particular focus on the transition at $\tilde{s} \rightarrow 0^-$.

Clearly, the exponent governing the singularity of the correlation length at the phase transition depends on the direction. What is less obvious, also the presence (or absence) of oscillations of the correlation function $\chi(x)$ as function of the distance $|x|$ (at $T > T_c$) does depend on the direction of $x$ relative to the anisotropies of the dispersion.

5.1. Correlation function of the perfect anisotropic Bose gas: asymptotic behaviour

The density–density correlation function is defined by

$$\chi(x_1, x_2) = n_2(x_1, x_2) - n^2,$$ (44)

where the two-particle density $n_2(x_1, x_2)$ is related to the 2-particle (grand canonical) density matrix $\rho_2(x_1, x_2, x'_1, x'_2)$ by

$$n_2(x_1, x_2) = \rho_2(x_1, x_2, x'_1, x'_2),$$ (45)

and $n$ denotes the density. For noninteracting bosons one finds (see e.g. [21])

$$\rho_2(x_1, x_2, x'_1, x'_2) = \rho_1(x_1, x'_1)\rho_1(x_2, x'_2) + \rho_1(x_2, x'_1)\rho_1(x_1, x'_2),$$ (46)

where the one-particle density matrix $\rho_1(x_1, x_2)$ is in the thermodynamic limit given by

$$\rho_1(x_1, x'_1) = \int \frac{d^d k}{(2\pi)^d} \rho k e^{i k (x_1 - x'_1)},$$ (47)

and therefore depends on $x = x_1 - x'_1$. Here
\[ \rho_k = \left( z^{-1} e^{\beta k} - 1 \right)^{-1} = \sum_{j=1}^{\infty} z^j e^{-\beta jk} \]  

is the Bose–Einstein distribution. Using the above, we evaluate \( n_2(x_1, x_2) = \rho_2(x_1, x_2, x_1, x_2) \) and find

\[ n_2(x_1, x_2) = \rho_1^2(x_1, x_2) + n^2. \]  

In consequence:

\[ \chi(x_1, x_2) = \chi(x_1 - x_2) = \rho_1^2(x_1 - x_2) = \rho_1^2(x). \]  

In what follows, we denote \( \rho_1(x) \) as \( \rho(x) \), so that \( \chi(x) = \rho(x)^2 \).

For \( |x| \to \infty \) the summation over \( j \) may be replaced by an integration over a continuous variable \( \sum_j \to \int dt \). The subsequent change of variables \( t = \frac{k}{x} \) brings \( \rho(x) \) (given by equation (47)) to the following form

\[ \rho(x) \simeq \int \frac{d^d k}{(2\pi)^d} \int_0^\infty dt x e^{\Phi(k,t)}, \]  

where

\[ \Phi(k, t) = -\beta t e_k + \frac{kx}{x} + \beta \mu t. \]  

The asymptotic behaviour of \( \rho(x) \) for \( x \to \infty \) can now be extracted with the steepest-descent method. Evaluating the derivatives of \( \Phi(k, t) \) yields the saddle-point condition

\[ \begin{cases} e_k = 0 \\ \beta t e_k |_{t_0} - i \frac{2\mu}{x} = 0 \end{cases} \]  

for \( t_0, e_k \). Below we solve equation (53) for specific cases and evaluate the asymptotic form of the correlation function.

5.1.1. Case \( e_k = \frac{1}{2} \sum_{j=1}^{d-1} k_j^2 + \frac{1}{4} k_d^4 \) and \( x = xe_1 \). In this case \( x \) is taken perpendicular to the anisotropy. Equation (53) then yields the solution

\[ \begin{align*} e_k &= 0 \\ t_0 &= \beta^{-1}(-2\mu)^{-\frac{1}{2}} \end{align*} \]  

with \( \Re(t_0) > 0 \). From the Laplace formula one now obtains

\[ \rho(x) \sim e^{\Phi(k_0, t_0)} = e^{-x/2\xi_\perp}, \]  

which allows us to identify the correlation length \( \xi_\perp = \left[ \sqrt{2\mu}/2 \right]^{-1} \) together with the critical exponent \( \nu_\perp = \frac{1}{2} \), which is precisely the same as for the standard isotropic case.

5.1.2. Case \( e_k = \frac{1}{2} \sum_{j=1}^{d-1} k_j^2 + \frac{1}{4} k_d^4 \) and \( x = xe_d \). Taking \( x \) along the anisotropy, we obtain the following two solutions to equation (53) with \( \Re(t_0) > 0 \):

\[ \begin{align*} k_0^{(1)} &= \left( \frac{4\mu}{\tau} \right)^{1/4} e^{\frac{1}{\tau} \mu} e_d, \\ t_0^{(1)} &= \frac{1}{\beta \tau} \left( \frac{4\mu}{\tau} \right)^{-3/4} e^{\frac{1}{\tau} \mu}, \end{align*} \]  

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Phase diagram and correlation functions of the anisotropic imperfect Bose gas in $d$ dimensions

$$k_0^{(2)} = \left( \frac{-4\mu}{\tau} \right)^{1/4} e^{\pi i d}, \quad t_0^{(2)} = \frac{1}{\beta \tau} \left( \frac{-4\mu}{\tau} \right)^{-3/4} e^{-\frac{\mu}{\tau}}.$$  \hspace{1cm} (57)

In consequence,

$$x \Phi(k_0^{(l)}, t_0^{(l)}) = \frac{1}{\sqrt{2}} (-1 \pm i) \left( \frac{-4\mu}{\tau} \right)^{1/4} x \quad \text{for} \quad l \in \{1, 2\}.$$  \hspace{1cm} (58)

The moduli of both contributions are equal and therefore both points contribute to the density matrix. We find

$$\rho(x) \sim e^{x \Phi(k_0^{(1)}, t_0)} + e^{x \Phi(k_0^{(2)}, t_0)} = 2 \cos \left[ \left( \frac{-\mu}{\tau} \right)^{1/4} x \right] e^{-\left( \frac{-\mu}{\tau} \right)^{1/4} x}.  \hspace{1cm} (59)$$

This allows us to read off the exponent $\nu_0 = \frac{1}{4}$. We observe the oscillatory behaviour of $\rho(x)$, which may be contrasted with the case $x = xe_1$ equation (55), where we found monotonous decay.

5.1.3. Case $\epsilon_k = \frac{1}{2} \sum_{j=1}^{d-1} k_j^2 + \frac{7}{4} k_d^2$ ($m \in \mathbb{N}$). The analysis of the previous section may be generalised to a situation where $\epsilon_k = \frac{1}{2} \sum_{j=1}^{d-1} k_j^2 + \frac{7}{4} k_d^2$ and $m > 1$ is a natural number. For the ‘radial’ case ($x = xe_1$) we obtain a monotonous behaviour of $\rho(x)$ with the exponent $\nu_\perp = \frac{1}{2}$. For the ‘axial’ case ($x = xe_d$) one finds oscillatory behaviour of $\rho(x)$ and $\nu_\parallel = \frac{1}{2m}$.

5.2. Correlation function of the perfect anisotropic Bose gas: general expressions

Above we extracted the asymptotic behaviour of $\rho(x)$ at large $|x|$ in the ‘radial’ and ‘axial’ directions. For $\epsilon_k = \frac{1}{2} \sum_{j=1}^{d-1} k_j^2 + \frac{7}{4} k_d^2$ one may however also obtain an exact expression for $\rho(x)$ valid for any $x$. This is achieved by factorising the integral in equation (47) and directly evaluating the expression. One finds

$$\rho(x) = \sum_{j=1}^{\infty} z^j \prod_{a=1}^{d-1} F_2(x_\alpha, \beta, j) F_4(x, \beta, j),  \hspace{1cm} (60)$$

where

$$F_2(x_\alpha, \beta, j) = \sqrt{\frac{2\pi}{\beta j}} \exp \left( -\frac{x_\alpha^2}{2 \beta j} \right);  \hspace{1cm} (61)$$

and

$$F_4(x, \beta, j) = \frac{\pi}{\Gamma(\frac{3}{4})} \frac{\sqrt{\beta j} \, _0F_2\left(1, \frac{3}{4}; \frac{x^4}{4 \beta j}\right)}{\Gamma\left(\frac{3}{4}\right)} - \frac{x^2 \, _0F_2\left(\frac{5}{4}, \frac{3}{4}; \frac{x^4}{4 \beta j}\right)}{\Gamma(\frac{3}{4})}.  \hspace{1cm} (62)$$

Above we put $\tau = 1$. The symbol $\, _0F_2\left(1, \frac{3}{4}; \frac{x^4}{4 \beta j}\right)$ denotes one of the generalised hypergeometric functions. The generalised hypergeometric function is a function depending on $p + q$ parameters $a_1, \ldots, a_p, b_1, \ldots, b_q$ and the variable $z$. Its definition reads \cite{32, 33}
Phase diagram and correlation functions of the anisotropic imperfect Bose gas in $d$ dimensions

\[
\rho(x_1,0,0) = \sum_{k=0}^{\infty} \left( a_1 \right)_k \cdots \left( a_p \right)_k \left( b_1 \right)_k \cdots \left( b_q \right)_k \frac{1}{k!} z^k.
\]

In the above formula, $(a)_k$ denotes the Pochhammer symbol.

Despite being exact, equation (60) is inconvenient for extracting the asymptotic expressions derived in the previous section. It may however be easily plotted for arbitrary choices of $x$ without recourse to asymptotic expansions. This is done for $d = 3$ in figures 1–3. The resulting plots confirm the monotonous character of $\rho(x)$ in the radial direction.

**Figure 1.** The density matrix $\rho(x)$ in the radial direction (for $x = xe_1$) for a sequence of $z$ approaching Bose–Einstein condensation. The plotted curves correspond to $\beta = 1$ and $z = 0.1$ (the lowest curve), $z = 0.5$, $z = 0.8$, $z = 1$ (the highest curve). The quantity $\rho(x)$ decreases monotonously in agreement with the results of section 5.1.1.

**Figure 2.** The density matrix $\rho(x)$ in the axial direction (for $x = xe_3$) for a sequence of $z$ approaching Bose–Einstein condensation. The plotted curves correspond to $\beta = 1$ and $z = 0.1$ (the lowest curve), $z = 0.5$, $z = 0.8$, $z = 1$ (the highest curve). The quantity $\rho(x)$ shows oscillatory behaviour in agreement with the results of section 5.1.2.
direction ($\mathbf{x} = x\mathbf{e}_1$) and the occurrence of damped oscillations in the axial direction ($\mathbf{x} = x\mathbf{e}_3$). More generally, oscillations occur for $\mathbf{x}$ such that $x\mathbf{e}_3 \neq 0$, as can be seen in figure 3.

5.3. Correlation function of the imperfect anisotropic Bose gas

As explained at the beginning of this section, the correlation function of the imperfect Bose gas can now be obtained by performing the replacement $\mu \rightarrow \tilde{\mu} = \beta^{-1} \bar{s}$ in the results obtained above in the absence of any interactions. We find

$$\nu^{(IBG)}_{\perp/\parallel} = \begin{cases} \frac{\psi}{1-\psi} \nu_{\perp/\parallel} & 1 < 1/\psi < 2 \\ \nu_{\perp/\parallel} & 1/\psi > 2 \end{cases} \quad (63)$$

as the exponents controlling the divergence of the correlation lengths at Bose–Einstein condensation. The exponents $\nu_{\perp}$ and $\nu_{\parallel}$ were obtained explicitly in the section 5.1.3 for the case of $\epsilon_{k} = \frac{1}{2} \sum_{j=1}^{d-1} k_{j}^2 + \frac{\tau}{4} k_{d}^{2m}$, with integer $m$. Note that equation (63) holds in the high-temperature phase.

For the special case of $\alpha_i = 2$ for $i \in \{1 \ldots d\}$ one recovers $\nu = \frac{1}{d-2}$ below the upper critical dimension, which then equals 4.

6. Remarks on the relation to the spherical model

The classical spherical model [26, 34] is among the most recognised statistical physics models that display a phase transition, and, at the same time, admit an analytical solution. As already remarked, the IBG and the spherical model fall into the same bulk universality class at $T > 0$, which is also specific to the $\mathcal{N} \rightarrow \infty$ limit of $O(\mathcal{N})$-symmetric models [35, 36]. In addition to the most studied cases (such as only nearest-neighbour interactions) leading to the standard Berlin-Kac exponents, one may manipulate the
model couplings $J_{ij}$ (competing ferro- and antiferromagnetic interactions) and obtain a picture similar to the one described in section 3 for the IBG [37, 38].

Despite this above affinity between the two models we would like to point out a difference. While the IBG Hamiltonian contains a kinetic energy contribution, the spherical model does not. As a result, the IBG is naturally equipped with two parameters ($T$ and $\mu$) to tune the transition and is well defined for $T \to 0$. In consequence, its phase diagram realises the standard scenario of quantum criticality for $T$ small [23, 39]. This is not quite the case for the Berlin-Kac model, where a realisation of quantum criticality requires defining its quantum extension, which in practice usually amounts to adding an extra kinetic term to the Hamiltonian. This procedure is however not unique and leads to several different universality classes for $T$ approaching zero, which, for example, may be characterised by distinct $\psi$ exponents. Different variants of the quantum spherical model have been addressed in [40–47]. For some of them, the non-standard critical behaviour was observed for fine-tuned coupling constants [48, 49].

Another interesting connection between the IBG and the spherical model occurs via the Casimir forces. Reference [22] found the Casimir scaling functions different by a factor of 2 from those of the spherical model. An explanation for this fact was recently given in [24]. The anisotropies of the correlation functions should find a reflection in the form of the scaling functions for the Casimir energy, which may be an interesting direction for future studies.

7. Summary

Realising the imperfect Bose gas on a lattice allows for engineering the kinetic couplings in such a way that the asymptotic behaviour of the dispersion is altered [50]. This influences the universality class of Bose–Einstein condensation. By means of an exact analysis of the considered model, we have argued that the universality class is only indirectly related to the system dimensionality. Instead, it is fully specified by the shift exponent $\psi$. This in turn is given by equation (20), and, for the present system, is not determined exclusively by the system dimensionality $d$ and the dynamical exponent $z$. The derived condition for the occurrence of the Bose–Einstein condensate (at arbitrarily low but finite $T$) reads $1/\psi > 1$. Non-classical critical behaviour occurs for $1/\psi \in ]1, 2]$. The universality class is then the same as that of the classical spherical model, albeit in dimensionality $d_{\text{eff}} = \frac{2}{\psi}$. For $1/\psi > 2$ Bose–Einstein condensation is characterised by classical (Landau) critical exponents. Our analysis indicates (both for the perfect and imperfect Bose gases) a qualitative change in the behaviour of the correlation function introduced by the anisotropies. The character of the divergence of the correlation length acquires a dependence on direction (the exponents $\nu_\perp$ and $\nu_\parallel$ are of different values). The relation between these exponents in the cases of perfect and imperfect Bose gases is linear with a coefficient specified by the shift exponent $\psi$ - see equation (63). The correlation function may display either a monotonous decay, or exponentially damped oscillations. The latter is found more generic in the simplest case of $\sim k^2$ dispersion in all directions but one. This is in contrast to the standard isotropic case with $\epsilon_k \sim k^2$ where the oscillations are absent.
Our result for the correlation function poses a natural question whether similar effects (presence or absence of oscillations of $\rho(x)$ in anisotropic Bose gases) occur also for interacting systems. It would also be interesting to investigate the finite-size effects in systems with such anisotropies, in particular see if and how the direction dependence of the correlation function and its oscillatory behaviour transfers to the scaling functions for the Casimir energy.

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18
Phase diagram and correlation functions of the anisotropic imperfect Bose gas in \( d \) dimensions

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