QUASI-NORMAL MODES AND EXPONENTIAL ENERGY DECAY
FOR THE KERR-DE SITTER BLACK HOLE

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ABSTRACT. We provide a rigorous definition of quasi-normal modes for a rotating black
hole. They are given by the poles of a certain meromorphic family of operators and agree
with the heuristic definition in the physics literature. If the black hole rotates slowly
enough, we show that these poles form a discrete subset of \( \mathbb{C} \). As an application we
prove that the local energy of linear waves in that background decays exponentially once
orthogonality to the zero resonance is imposed.

Quasi-normal modes are the complex frequencies appearing in expansions of waves; their
real part corresponds to the rate of oscillation and the nonpositive imaginary part, to
the rate of decay. According to the physics literature \cite{24} they are expected to appear
in gravitational waves caused by perturbations of black holes (for more recent references
and findings, see for example \cite{8,25,26}). In the mathematics literature they were studied
by Bachelot and Motet-Bachelot \cite{3,4,5} and Sá Barreto and Zworski \cite{31}, who applied
the methods of scattering theory and semiclassical analysis to the case of a spherically
symmetric black hole. Quasi-normal modes were described in \cite{31} as resonances; that is,
poles of the meromorphic continuation of a certain family of operators; it was also proved
that these poles asymptotically lie on a lattice. This was further developed by Bony and
Häfner in \cite{11}, who established an expansion of the solutions of the wave equation in terms
of resonant states. As a byproduct of this result, they obtained exponential decay of local
energy for Schwarzschild–de Sitter. Melrose, Sá Barreto, and Vasy \cite{30} have extended this
result to more general manifolds and more general initial data.

In this paper, we employ different methods to define quasi-normal modes for the Kerr–de
Sitter rotating black hole. As in \cite{31} and \cite{11}, we use the de Sitter model; physically, this
 corresponds to a positive cosmological constant; mathematically, it replaces asymptotically
Euclidean spatial infinity with an asymptotically hyperbolic one. Let \( P_g(\omega), \omega \in \mathbb{C} \), be the
stationary d’Alembert–Beltrami operator of the Kerr–de Sitter metric (see Section 1 for
details). It acts on functions on the space slice \( M = (r_-, r_+) \times S^2 \). We define quasi-normal
modes as poles of a certain (right) inverse \( R_g(\omega) \) to \( P_g(\omega) \). Because of the cylindrical
symmetry of the operator \( P_g(\omega) \), it leaves invariant the space \( \mathcal{D}_k' \) of distributions with fixed
angular momentum \( k \in \mathbb{Z} \) (with respect to the axis of rotation); the inverse \( R_g(\omega) \) on \( \mathcal{D}_k' \)
is constructed by
Theorem 1. Let $P_g(\omega, k)$ be the restriction of $P_g(\omega)$ to $\mathcal{D}_k'$. Then there exists a family of operators

$$R_g(\omega, k) : L^2_{\text{comp}}(M) \cap \mathcal{D}_k' \rightarrow H^2_{\text{loc}}(M) \cap \mathcal{D}_k'$$

meromorphic in $\omega \in \mathbb{C}$ with poles of finite rank and such that $P_g(\omega, k)R_g(\omega, k)f = f$ for each $f \in L^2_{\text{comp}}(M) \cap \mathcal{D}_k'$.

Since $R_g(\omega, k)$ is meromorphic, its poles, which we call $k$-resonances, form a discrete set. One can then say that $\omega \in \mathbb{C}$ is a resonance, or a quasi-normal mode, if $\omega$ is a $k$-resonance for some $k \in \mathbb{Z}$. However, it is desirable to know that resonances form a discrete subset of $\mathbb{C}$; that is, $k$-resonances for different $k$ do not accumulate near some point. Also, one wants to construct the inverse $R_g(\omega)$ that works for all values of $k$. For $\delta_r > 0$ put

$$K_r = (r_- + \delta_r, r_+ - \delta_r), \quad M_K = K_r \times S^2,$

and let $1_{M_K}$ be the operator of multiplication by the characteristic function of $M_K$ (which will, based on the context, act $L^2(M_K) \rightarrow L^2(M)$ or $L^2(M) \rightarrow L^2(M_K)$). Then we are able to construct $R_g(\omega)$ on $M_K$ for a slowly rotating black hole:

Theorem 2. Fix $\delta_r > 0$. Then there exists $a_0 > 0$ such that if the rotation speed of the black hole satisfies $|a| < a_0$, we have the following:

1. Every fixed compact set can only contain $k$-resonances for a finite number of values of $k$. Therefore, quasi-normal modes form a discrete subset of $\mathbb{C}$.

2. The operators $1_{M_K}R_g(\omega, k)1_{M_K}$ define a family of operators

$$R_g(\omega) : L^2(M_K) \rightarrow H^2(M_K)$$

such that $P_g(\omega)R_g(\omega)f = f$ on $M_K$ for each $f \in L^2(M_K)$ and $R_g(\omega)$ is meromorphic in $\omega \in \mathbb{C}$ with poles of finite rank.

As stated in Theorem 2, the operator $R_g(\omega)$ acts only on functions supported in a certain compact subset of the space slice $M$ depending on how small $a$ is. This is due to the fact that the operator $P_g(\omega)$ is not elliptic inside the two ergospheres located near the endpoints $r = r_{\pm}$. The result above can then be viewed as a construction of $R_g(\omega)$ away from the ergospheres. However, for fixed angular momentum we are able to obtain certain boundary conditions on the elements in the image of $R_g(\omega, k)$, as well as on resonant states:

Theorem 3. Let $\omega \in \mathbb{C}$.

1. Assume that $\omega$ is not a resonance. Take $f \in L^2_{\text{comp}}(M) \cap \mathcal{D}_k'$ for some $k \in \mathbb{Z}$ and put $u = R_g(\omega, k)f \in H^2_{\text{loc}}(M)$. Then $u$ is outgoing in the following sense: the functions

$$v_{\pm}(r, \theta, \varphi) = |r - r_{\pm}|^{a_\perp^{-1}(1 + \alpha)(r_{\perp}^2 + a^2)\omega}u(r, \theta, \varphi - A_{\pm}^{-1}(1 + \alpha)a \ln |r - r_{\pm}|).$$

\[1\]In this paper, the subscript in the constants such as $\delta_r$, $C_r$, $C_\theta$ does not mean that these constants depend on the corresponding variables, such as $r$ or $\theta$; instead, it indicates that they are related to these variables.
are smooth near the event horizons \( \{ r_\pm \} \times S^2 \).

2. Assume that \( \omega \) is a resonance. Then there exists a resonant state; i.e., a nonzero solution \( u \in C^\infty(M) \) to the equation \( P_g(\omega)u = 0 \) that is outgoing in the sense of part 1.

The outgoing condition can be reformulated as follows. Consider the function \( U = e^{-i\omega t}u \) on the spacetime \( \mathbb{R} \times M \); then \( u \) is outgoing if and only if \( U \) is smooth up to the event horizons in the extension of the metric given by the Kerr-star coordinates \((t^*, r, \theta, \varphi^*)\) discussed in Section 1. This lets us establish a relation between the wave equation on Kerr-de Sitter and the family of operators \( R_g(\omega) \) (Proposition 1.2). Note that here we do not follow earlier applications of scattering theory (including [11]), where spectral theory and in particular self-adjointness of \( P_g \) are used to define \( R_g(\omega) \) for \( \text{Im} \omega > 0 \) and relate it to solutions of the wave equation via Stone’s formula. In the situation of the present paper, due to the lack of ellipticity of \( P_g(\omega) \) inside the ergospheres, it is doubtful that \( P_g \) can be made into a self-adjoint operator; therefore, we construct \( R_g(\omega) \) directly using separation of variables, cite the theory of hyperbolic equations (see Section 1) for well-posedness of the Cauchy problem for the wave equation, and prove Proposition 1.2 without any reference to spectral theory.

We now study the distribution of resonances in the slowly rotating Kerr–de Sitter case. First, we establish absence of nonzero resonances in the closed upper half-plane:

**Theorem 4.** Fix \( \delta_r > 0 \). Then there exist constants \( a_0 \) and \( C \) such that if \( |a| < a_0 \), then:

1. There are no resonances in the upper half-plane and

\[
\| R_g(\omega) \|_{L^2(M_K) \to L^2(M_K)} \leq \frac{C}{|\text{Im} \omega|^2}, \quad \text{Im} \omega > 0.
\]

2. There are no resonances \( \omega \in \mathbb{R} \setminus 0 \) and

\[
R_g(\omega) = \frac{i(1 \otimes 1)}{4\pi(1 + \alpha)(r_+^2 + r_-^2 + 2\alpha^2)\omega} + \text{Hol}(\omega),
\]

where \( \text{Hol} \) stands for a family of operators holomorphic at zero.

Next, we use the methods of [38] and the fact that the only trapping in our situation is normally hyperbolic to get a resonance free strip:

**Theorem 5.** Fix \( \delta_r > 0 \) and \( s > 0 \). Then there exist \( a_0 > 0, \nu_0 > 0 \), and \( C \) such that for \( |a| < a_0 \),

\[
\| R_g(\omega) \|_{L^2(M_K) \to L^2(M_K)} \leq C|\omega|^s; \quad |\text{Re} \omega| \geq C; \quad |\text{Im} \omega| \leq \nu_0.
\]
Theorems 4 and 5, together with the fact that resonances form a discrete set, imply that for \( \nu_0 \) small enough, zero is the only resonance in \( \{ \text{Im } \omega \geq -\nu_0 \} \). This and the presence of the global meromorphic continuation provide exponential decay of local energy.

**Theorem 6.** Let \((r, t^*, \theta, \varphi^*)\) be the coordinates on the Kerr–de Sitter background introduced in Section 1. Fix \( \delta_r > 0 \) and \( s' > 0 \) and assume that \( a \) is small enough. Let \( u \) be a solution to the wave equation \( \Box_g u = 0 \) with initial data
\[
\begin{align*}
    u|_{t^*=0} &= f_0 \in H^{3/2+s'}(M) \cap \mathcal{E}'(M_K), \\
    \partial_{t^*} u|_{t^*=0} &= f_1 \in H^{1/2+s'}(M) \cap \mathcal{E}'(M_K).
\end{align*}
\]
Also, define the constant
\[
u_0 = 1 + \frac{\alpha}{4\pi} \int_{t^*=0} * (du).
\]
Here \( * \) denotes the Hodge star operator for the metric \( g \) (see Section 1). Then
\[
\|u(t^*, \cdot) - u_0\|_{L^2(M_K)} \leq C e^{-\nu't^*}(\|f_0\|_{H^{3/2+s'}(M_K)} + \|f_1\|_{H^{1/2+s'}(M_K)}), \quad t^* > 0,
\]
for certain constants \( C \) and \( \nu' \) independent of \( u \).

For the Kerr metric, the local energy decay is polynomial as shown by Tataru and Tohaneanu \cite{34, 33}, see also the lecture notes by Dafermos and Rodnianski \cite{13} and the references below.

**Outline of the proof.** The starting point of the construction of \( R_{g}(\omega) \) is the separation of variables introduced by Teukolsky in \cite{36}. The separation of variables techniques and the related symmetries have been used in many papers, including \cite{2, 9, 11, 15, 16, 19, 20, 31, 37}; however, these mostly consider the case of zero cosmological constant, where other difficulties occur at zero energy and a global meromorphic continuation of the type presented here is unlikely. In our case, since the metric is invariant under axial rotation, it is enough to construct the operators \( R_{g}(\omega, k) \) and study their behavior for large \( k \). The operator \( P_{g}(\omega, k) \) is next decomposed into the sum of two ordinary differential operators, \( P_{r} \) and \( P_{\theta} \) (see \((1.3))\). The separation of variables is discussed in Section 1; the same section contains the derivation of Theorem 6 from the other theorems by the complex contour deformation method.

In the Schwarzschild–de Sitter case, \( P_{g} \) is just the Laplace–Beltrami operator on the round sphere and one can use spherical harmonics to reduce the problem to studying the operator \( P_{r} + \lambda \) for large \( \lambda \). In the case \( a \neq 0 \), however, the operator \( P_{g} \) is \( \omega \)-dependent; what is more, it is no longer self-adjoint unless \( \omega \in \mathbb{R} \). This raises two problems with the standard implementation of separation of variables, namely decomposing \( L^2 \) into a direct sum of the eigenspaces of \( P_{g} \). Firstly, since \( P_{g} \) is not self-adjoint, we cannot automatically guarantee
\footnote{Recently, the author has obtained stronger exponential decay results (arXiv:1010.5201), as well as a more precise description of resonances and a resonance decomposition (arXiv:1101.1260); all of these are based on the present paper.}
existence of a complete system of eigenfunctions and the corresponding eigenspaces need not be orthogonal. Secondly, the eigenvalues of $P_\theta$ are functions of $\omega$, and meromorphy of $R_\theta(\omega)$ is nontrivial to show when two of these eigenvalues coincide. Therefore, instead of using the eigenspace decomposition, we write $R_\theta(\omega)$ as a certain contour integral (2.1) in the complex plane; the proof of meromorphy of this integral is based on Weierstrass preparation theorem. This is described in Section 2.

In Section 3, we use the separation of variables procedure to reduce Theorems 1–4 to certain facts about the radial resolvent $R_r$ (Proposition 3.2). For fixed $\omega, \lambda, k$, where $\lambda \in \mathbb{C}$ is the separation constant, $R_r$ is constructed in Section 4 using the methods of one-dimensional scattering theory. Indeed, the radial operator $P_r$, after the Regge–Wheeler change of variables (4.1), is equivalent to the Schrödinger operator $P_x = D_x^2 + V_x(x)$ for a certain potential $V_x$ (4.2). (Here $x = \pm \infty$ correspond to the event horizons.) This does not, however, provide estimates on $R_r$ that are uniform as $\omega, \lambda, k$ go to infinity.

The main difficulty then is proving a uniform resolvent estimate (see (3.8)), valid for large $\lambda$ and $\text{Re} \lambda \gg |\text{Im} \lambda| + |\omega|^2 + |ak|^2$, which in particular guarantees the convergence of the integral (2.1) and Theorem 2. A complication arises from the fact that $V_x(\pm \infty) = -\omega_{\pm}^2$, where $\omega_{\pm}$ are proportional to $(r_{\pm}^2 + a^2) \omega - ak$. No matter how large $\omega$ is, one can always choose $k$ so that one of $\omega_{\pm}$ is small, making it impossible to use standard complex scaling in the case $\omega = o(k)$, due to the lack of ellipticity of the rescaled operator at infinity. To avoid this issue, we use the analyticity of $V_x$ and semiclassical analysis to get certain control on outgoing solutions at two distant, but fixed, points (Proposition 6.1), and then an integration by parts argument to get an $L^2$ bound between these two points. This is discussed in Section 6.

Finally, Section 7 contains the proof of Theorem 3. We first use the results of Sections 2–6 to reduce the problem to scattering for the Schrödinger operator $P_x$ in the regime $\lambda = O(\omega^2), k = O(\omega)$ (Proposition 7.1). In this case, we apply complex scaling to deform $P_x$ near $x = \pm \infty$ to an elliptic operator (Proposition 7.2). We then analyse the corresponding classical flow; it is either nontrapping at zero energy, in which case the usual escape function construction (as in, for example, [27]) applies (Proposition 7.3), or has a unique maximum. In the latter case we use the methods of [38] designed to handle more general normally hyperbolic trapped sets and based on commutator estimates in a slightly exotic microlocal calculus. The argument of [38] has to be modified to use complex scaling instead of an absorbing potential near infinity (see also [38, Theorem 2]).

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3Complex scaling originated in mathematical physics with the work of Aguilar–Combes [1], Balslev–Combes, and Simon. It has become a standard tool in chemistry for computing resonances. A microlocal approach has been developed by Helffer–Sjöstrand, and a more geometric version by Sjöstrand–Zworski [32] — see that paper for pointers to the literature. Complex scaling was reborn in numerical analysis in 1994 as the method of “perfectly matched layers” (see [7]). A nice application of the method of complex scaling to the Schwarzschild–de Sitter case is provided in [14].
It should be noted that, unlike [11] or [31], the construction of $R_g(\omega)$ in the present paper does not use the theorem of Mazzeo–Melrose [28] on the meromorphic continuation of the resolvent on spaces with asymptotically constant negative curvature (see also [21]). In [11] and [31], this theorem had to be applied to prove the existence of the meromorphic continuation of the resolvent for $\omega$ in a fixed neighborhood of zero where complex scaling could not be implemented.

Remark. The results of this paper also apply if the wave equation is replaced by the Klein–Gordon equation [23]

$$(\Box_g + m^2)u = 0,$$

where $m > 0$ is a fixed constant. The corresponding stationary operator is $P_g(\omega) + m^2\rho^2$; when restricted to the space $D'_k$, it is the sum of the two operators (see Section 1)

$$P_r(\omega, k; m) = P_r(\omega, k) + m^2r^2, \quad P_\theta(\omega; m) = P_\theta(\omega) + m^2a^2\cos^2\theta.$$

The proofs in this paper all go through in this case as well. In particular, the rescaled radial operator $P_\rho$ introduced in (4.2) is a Schrödinger operator with the potential

$$V_\rho(x; \omega, \lambda, k; m) = (\lambda + m^2r^2)\Delta_r - (1 + \alpha)^2((r^2 + a^2)\omega - ak)^2.$$

Since $V_\rho(\pm\infty)$ is still equal to $-\omega^2_\pm$ with $\omega_\pm$ defined in (4.3), the radial resolvent can be defined as a meromorphic family of operators on the entire complex plane. Also, the term $m^2r^2\Delta_r$ in the operator $P_\rho$ becomes of order $O(h^2)$ under the semiclassical rescaling and thus does not affect the arguments in Sections 6 and 7.

The only difference in the Klein–Gordon case is the absence of the resonance at zero: 1 is no longer an outgoing solution to the equation $P_\rho u = 0$ for $\omega = k = \lambda = 0$. Therefore, there is no $u_0$ term in Theorem 6, and all solutions to (0.1) decay exponentially in the compact set $M_K$.

1. **Kerr–de Sitter metric**

The Kerr–de Sitter metric is given by the formulas [12]

$$g = -\rho^2\left(\frac{d\tau^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right)$$

$$-\frac{\Delta_\theta \sin^2\theta}{(1 + \alpha)^2\rho^2}(a\,dt - (r^2 + a^2)\,d\varphi)^2$$

$$+ \frac{\Delta_r}{(1 + \alpha)^2\rho^2}(dt - a\sin^2\theta\,d\varphi)^2.$$

Here $M_0$ is the mass of the black hole, $\Lambda$ is the cosmological constant (both of which we assume to be fixed throughout the paper), and $a$ is the angular momentum (which we
assume to be bounded by some constant, and which is required to be small by most of our theorems);

\[ \Delta_r = (r^2 + a^2) \left(1 - \frac{\Lambda r^2}{3}\right) - 2M_0 r, \]
\[ \Delta_\theta = 1 + \alpha \cos^2 \theta, \]
\[ \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \alpha = \frac{\Lambda a^2}{3}. \]

We also put
\[ A_\pm = \mp \partial_r \Delta_r (r_\pm) > 0. \]

The metric is defined for \( \Delta_r > 0 \); we assume that this happens on an open interval \( 0 < r_- < r < r_+ < \infty \). (For \( a = 0 \), this is true when \( 9\Lambda M_0^2 < 1 \); it remains true if we take \( a \) small enough.) The variables \( \theta \in [0, \pi] \) and \( \varphi \in \mathbb{R}/2\pi \mathbb{Z} \) are the spherical coordinates on the sphere \( S^2 \). We define the space slice \( M = (r_-, r_+) \times S^2 \); then the Kerr–de Sitter metric is defined on the spacetime \( \mathbb{R} \times M \).

The d’Alembert–Beltrami operator of \( g \) is given by

\[ \Box_g = \frac{1}{\rho^2} D_r (\Delta_r D_r) + \frac{1}{\rho^2 \sin \theta} D_\theta (\Delta_\theta \sin \theta D_\theta) \]
\[ + \frac{(1 + \alpha)^2}{\rho^2 \Delta_\theta \sin^2 \theta} \left( a \sin^2 \theta D_t + D_\varphi \right)^2 \]
\[ - \frac{(1 + \alpha)^2}{\rho^2 \Delta_r} \left( (r^2 + a^2) D_t + a D_\varphi \right)^2. \]

(Henceforth we denote \( D = \frac{i}{\iota} \partial_r \).) The volume form is

\[ dVol = \frac{\rho^2 \sin \theta}{(1 + \alpha)^2} \ dt \, dr \, d\theta \, d\varphi. \]

If we replace \( D_t \) by a number \( -\omega \in \mathbb{C} \), then the operator \( \Box_g \) becomes equal to \( P_g(\omega)/\rho^2 \), where \( P_g(\omega) \) is the following differential operator on \( M \):

\[ P_g(\omega) = D_r (\Delta_r D_r) - \frac{(1 + \alpha)^2}{\Delta_r} \left( (r^2 + a^2) \omega - a D_\varphi \right)^2 \]
\[ + \frac{1}{\sin \theta} D_\theta (\Delta_\theta \sin \theta D_\theta) + \frac{(1 + \alpha)^2}{\Delta_\theta \sin^2 \theta} (a \omega \sin^2 \theta - D_\varphi)^2. \tag{1.1} \]

We now introduce the separation of variables for the operator \( P_g(\omega) \). We start with taking Fourier series in the variable \( \varphi \). For every \( k \in \mathbb{Z} \), define the space

\[ \mathcal{D}'_k = \{ u \in \mathcal{D}' \mid (D_\varphi - k)u = 0 \}. \tag{1.2} \]

This space can be considered as a subspace of \( \mathcal{D}'(M) \) or of \( \mathcal{D}'(S^2) \) alone, and

\[ L^2(M) = \bigoplus_{k \in \mathbb{Z}} (L^2(M) \cap \mathcal{D}'_k); \]
the right-hand side is the Hilbert sum of a family of closed mutually orthogonal subspaces.

Let $P_g(\omega, k)$ be the restriction of $P_g(\omega)$ to $D'_k$. Then we can write

$$P_g(\omega, k) = P_r(\omega, k) + P_\theta(\omega)|_{D'_k},$$

where

$$P_r(\omega, k) = D_r(\Delta_r D_r) - \frac{(1 + \alpha)^2}{\Delta_r}((r^2 + a^2)\omega - ak)^2,$$

$$P_\theta(\omega) = \frac{1}{\sin \theta}D_\theta(\Delta_\theta \sin \theta D_\theta) + \frac{(1 + \alpha)^2}{\Delta_\theta \sin^2 \theta}(a\omega \sin^2 \theta - D_\omega)^2$$

are differential operators in $r$ and $(\theta, \varphi)$, respectively.

Next, we introduce a modification of the Kerr-star coordinates (see [13, Section 5.1]). Following [33], we remove the singularities at $r = r_{\pm}$ by making the change of variables $(t, r, \theta, \varphi) \rightarrow (t^*, r, \theta, \varphi^*)$, where

$$t^* = t - F_t(r), \quad \varphi^* = \varphi - F_\varphi(r).$$

Note that $\partial_{t^*} = \partial_t$ and $\partial_{\varphi^*} = \partial_\varphi$. In the new coordinates, the metric becomes

$$g = -\rho^2\left(\frac{dr^2}{\Delta_r} + d\theta^2\right)$$

$$-\frac{\Delta_\theta \sin^2 \theta}{(1 + \alpha)^2 \rho^2}[a dt^* - (r^2 + a^2) d\varphi^* + (aF_t'(r) - (r^2 + a^2)F_\varphi'(r))dr]^2$$

$$+ \frac{\Delta_r}{(1 + \alpha)^2 \rho^2}[dt^* - a \sin^2 \theta d\varphi^* + (F_t'(r) - a \sin^2 \theta F_\varphi'(r))dr]^2.$$

The functions $F_t$ and $F_\varphi$ are required to be smooth on $(r_-, r_+)$ and satisfy the following conditions:

- $F_t(r) = F_\varphi(r) = 0$ for $r \in K_r = [r_- + \delta_r, r_+ - \delta_r]$;
- $F_t'(r) = \pm(1 + \alpha)(r^2 + a^2)/\Delta_r + F_{t\pm}(r)$ and $F_\varphi'(r) = \pm(1 + \alpha)a/\Delta_r + F_{\varphi\pm}(r)$, where $F_{t\pm}$ and $F_{\varphi\pm}$ are smooth at $r = r_{\pm}$, respectively;
- for some (a-independent) constant $C$ and all $r \in (r_-, r_+)$,

$$\frac{(1 + \alpha)^2(r^2 + a^2)^2}{\Delta_r} - \Delta_r F_t'(r)^2 - (1 + \alpha)^2 a^2 \geq \frac{1}{C} > 0.$$

Under these conditions, the metric $g$ in the new coordinates is smooth up to the event horizons $r = r_{\pm}$ and the space slices

$$M_{t_0} = \{t^* = t_0 = \text{const}\} \cap (\mathbb{R} \times M), \quad t_0 \in \mathbb{R},$$

are space-like. Let $\nu_t$ be the time-like normal vector field to these surfaces, chosen so that $g(\nu_t, \nu_t) = 1$ and $\langle dt^*, \nu_t \rangle > 0$. 

We now establish a basic energy estimate for the wave equation in our setting. Let \( u \) be a real-valued function smooth in the coordinates \((t^*, r, \theta, \varphi^*)\) up to the event horizons. Define the vector field \( T(du) \) by
\[
T(du) = \partial_t u \nabla_g u - \frac{1}{2} g(du, du) \nu_t.
\]
Since \( \nu_t \) is timelike, the expression \( g(T(du), \nu_t) \) is a positive definite quadratic form in \( du \).

For \( t_0 \in \mathbb{R} \), define \( E(t_0)(du) \) as the integral of this quadratic form over the space slice \( M_{t_0} \) with the volume form induced by the metric.

**Proposition 1.1.** Take \( t_1 < t_2 \) and let \( \Omega = \{ t_1 \leq t^* \leq t_2 \} \times M \).

Assume that \( u \) is smooth in \( \Omega \) up to its boundary and solves the wave equation \( \Box_g u = 0 \) in this region. Then
\[
E(t_2)(du) \leq e^{C_e(t_2-t_1)} E(t_1)(du)
\]
for some constant \( C_e \) independent of \( t_1 \) and \( t_2 \).

**Proof.** We use the method of [35, Proposition 2.8.1]. We apply the divergence theorem to the vector field \( T(du) \) on the domain \( \Omega \). The integrals over \( M_{t_1} \) and \( M_{t_2} \) will be equal to \(-E(t_1)\) and \( E(t_2)\). The restriction of the metric to tangent spaces of the event horizons is nonpositive and the field \( \nu_t \) is pointing outside of \( \Omega \) at \( r = r_\pm \); therefore, the integrals over the event horizons will be nonnegative. Finally, since \( \Box_g u = 0 \), one can prove that \( \text{div} T(du) \) is quadratic in \( du \) and thus
\[
|\text{div} T(du)| \leq C g(T(du), \nu_t).
\]
Therefore, the divergence theorem gives
\[
E(t_2) - E(t_1) \leq C \int_{t_1}^{t_2} E(t_0) \, dt_0.
\]
It remains to use Gronwall’s inequality. \( \Box \)

The geometric configuration of \( \{ t^* = t_1 \}, \{ t^* = t_2 \}, \{ r = r_\pm \}, \) and \( \nu_t \) with respect to the Lorentzian metric \( g \) used in Proposition 1.1 combined with the theory of hyperbolic equations (see [13, Proposition 3.1.1], [22, Theorem 23.2.4], or [36, Sections 2.8 and 7.7]), makes it possible to prove that for each \( f_0 \in H^1(M), \ f_1 \in L^2(M) \), there exists a unique solution
\[
u(t^*, \cdot) \in C([0, \infty); H^1(M)) \cap C^1([0, \infty); L^2(M))\]
to the initial value problem
\[
\Box_g u = 0, \ \ \ u|_{t^*=0} = f_0, \ \ \partial_{t^*} u|_{t^*=0} = f_1. \quad (1.4)
\]
We are now ready to prove Theorem 6. Fix \( \delta_r > 0 \) and assume that \( a \) is chosen small enough so that Theorems 2–5 hold. Assume that \( s' > 0 \) and \( u \) is the solution to (1.4)
with $f_0 \in H^{3/2+s'} \cap \mathcal{E}'(M'_K)$ and $f_1 \in H^{1/2+s'} \cap \mathcal{E}'(M'_K)$, where $M'_K$ is fixed and compactly contained in $M_K$. By finite propagation speed (see [36, Theorem 2.6.1 and Section 2.8]), there exists a function $\chi(t) \in C^\infty(0, \infty)$ independent of $u$ and such that $\chi(t^*) = 1$ for $t^* > 1$, and for $t^* \in \text{supp}(1 - \chi)$, supp $u(t^*, \cdot) \subset M_K$. By Proposition 1.1 we can define the Fourier-Laplace transform

$$\hat{\chi u}(\omega) = \int e^{it^*\omega} \chi(t^*)u(t^*, \cdot) \, dt^* \in H^{3/2+s'}(M), \text{ Im } \omega > C_e.$$

Put $f = \rho^2 \Box_g(\chi u) = \rho^2[\Box_g, \chi]u$; then

$$f \in H^{1/2+s'}_{\text{comp}}(\mathbb{R}; L^2(M) \cap \mathcal{E}'(M_K)).$$

Therefore, one can define the Fourier-Laplace transform $\hat{f}(\omega) \in L^2 \cap \mathcal{E}'(M_K)$ for all $\omega \in \mathbb{C}$, and we have the estimate

$$\int |\omega|^{2s+1} ||\hat{f}(\omega)||_{L^2(M)}^2 \, d\omega \leq C(||f_0||_{H^{3/2+s'}}^2 + ||f_1||_{H^{1/2+s'}}^2).$$

where integration is performed over the line $\{\text{Im } \omega = \nu = \text{const}\}$ with $\nu$ bounded.

**Proposition 1.2.** We have for $\text{Im } \omega > C_e$,

$$\hat{\chi u}(\omega)|_{M_K} = R_g(\omega)\hat{f}(\omega).$$

**Proof.** Without loss of generality, we may assume that $u \in C^\infty \cap D'_k$ for some $k \in \mathbb{Z}$; then $R_g(\omega)\hat{f}(\omega)$ can be defined on the whole $M$ by Theorem 1. Fix $\omega$ and put

$$\Phi(\omega) = e^{i\omega F_i(r)} \hat{\chi u}(\omega) - R_g(\omega, k)\hat{f}(\omega) \in C^\infty(M).$$

Since $\rho^2 \Box_g(\chi u) = f$, we have

$$P_g(\omega)(e^{i\omega F_i(r)} \hat{\chi u}(\omega)) = \hat{f}(\omega);$$

therefore, $P_g(\omega)\Phi(\omega) = 0$. Note also that $\Phi$ is smooth inside $M$ because of ellipticity of the operator $P_g(\omega)$ on $D'_k$ (see [36, Section 7.4] and the last step of the proof of Theorem 1).

Now, if we put

$$U(t, \cdot) = e^{-it\omega}\Phi(\omega)(\cdot),$$

then $\Box_g U = 0$ inside $M_S$. However, by Theorem 3, $U$ is smooth in the $(r, t^*, \theta, \varphi^*)$ coordinates up to the event horizons and its energy grows in time faster than allowed by Proposition 1.1; therefore, $\Phi = 0$.

We now restrict our attention to the compact $M_K$, where in particular $t = t^*$ and $\varphi = \varphi^*$. By the Fourier Inversion Formula, for $t > 1$ and $\nu > C_e$,

$$u(t)|_{M_K} = (2\pi)^{-1} \int e^{-i(t\omega + iv)}R_g(\omega + iv)\hat{f}(\omega + iv) \, d\omega.$$

Fix positive $s < s'$. By Theorems 4 and 5, there exists $\nu_0 > 0$ such that zero is the only resonance with $\text{Im } \omega \geq -\nu_0$. Using the estimates in these theorems, we can deform the
contour of integration above to the one with \( \nu = -\nu_0 \). Indeed, by a density argument we may assume that \( u \in C^\infty \), and in this case, \( \hat{f}(\omega) \) is rapidly decreasing as \( \text{Re} \omega \to \infty \) for \( \text{Im} \omega \) fixed. We then get

\[
\begin{align*}
\left. u(t) \right|_{K_r} &= \frac{1 + \alpha}{4\pi(r_+^2 + r_-^2 + 2a^2)} (\hat{f}(0), 1)_{L^2(K_r)} \\
&\quad + (2\pi)^{-1} e^{-\nu_0 t} \int e^{-it\omega} R_g(\omega - i\nu_0) \hat{f}(\omega - i\nu_0) \, d\omega.
\end{align*}
\]

(1.5)

We find a representation of the first term above in terms of the initial data for \( u \) at time zero. We have

\[
(\hat{f}(0), 1)_{L^2(K_r)} = \int_{M_K \times \mathbb{R}} \Box_g (\chi u) \, d\text{Vol}.
\]

Here \( d\text{Vol} \) is the volume form induced by \( g \). Integrating by parts, we get

\[
\int_{M_K \times \mathbb{R}} \Box_g (\chi u) \, d\text{Vol} = -\int_{t \geq 0} \Box_g ((1 - \chi) u) \, d\text{Vol} = \int_{t = 0} *(du).
\]

(1.6)

Here \( * \) is the Hodge star operator induced by the metric \( g \), with the orientation on \( M \) and \( \mathbb{R} \times M \) chosen so that \( *(dt) \) is positively oriented on \( \{ t = 0 \} \).

Finally, the \( L^2 \) norm of the integral term in \( (1.5) \) can be estimated by

\[
C e^{-\nu_0 t} \int \langle \omega \rangle^{s - s' - 1/2} \| \langle \omega \rangle^{s' + 1/2} \hat{f}(\omega - i\nu_0) \|_{L^2(K_r)} \, d\omega
\]

\[
\leq C \| f_0 \|_{H^{s'+1/2}} + \| f_1 \|_{H^{s'+1/2}}.
\]

since \( \langle \omega \rangle^{s - s' - 1/2} \in L^2 \). This proves Theorem 6.

**Remark.** In the original coordinates, \((t, r, \theta, \varphi)\), the equation \( \Box_g u = 0 \) has two solutions depending only on the time variable, namely, \( u = 1 \) and \( u = t \). Even though Theorem 6 does not apply to these solutions because we only construct the family of operators \( R_g(\omega) \) acting on functions on the compact set \( M_K \), it is still interesting to see where our argument fails if \( R_g(\omega) \) were well-defined on the whole \( M \). The key fact is that our Cauchy problem is formulated in the \( t^* \) variable. Then, for \( u = t \) the function \( f_0 = u|_{t^* = 0} \) behaves like \( \log |r - r_{\pm}| \) near the event horizons and thus does not lie in the energy space \( H^1 \). As for \( u = 1 \), our theorem gives the correct form of the contribution of the zero resonance, namely, a constant; however, the value of this constant cannot be given by the integral of \( *(du) \) over \( t^* = 0 \), as \( du = 0 \). This discrepancy is explained if we look closer at the last equation in (1.6); while integrating by parts, we will get a nonzero term coming from the integral of \( *(d\chi(t^*)) \) over the event horizons.
2. Separation of variables in an abstract setting

In this section, we construct inverses for certain families of operators with separating variables. Since the method described below can potentially be applied to other situations, we develop it abstractly, without any reference to the operators of our problem. Similar constructions have been used in other settings by Ben-Artzi–Devinatz [6] and Mazzeo–Vasy [29, Section 2].

First, let us consider a differential operator

\[ P(\omega) = P_1(\omega) + P_2(\omega) \]

in the variables \((x_1, x_2)\), where \(P_1(\omega)\) is a differential operator in the variable \(x_1\) and \(P_2(\omega)\) is a differential operator in the variable \(x_2\); \(\omega\) is a complex parameter. If we take \(\mathcal{H}_1\) and \(\mathcal{H}_2\) to be certain \(L^2\) spaces in the variables \(x_1\) and \(x_2\), respectively, then the corresponding \(L^2\) space in the variables \((x_1, x_2)\) is their Hilbert tensor product \(\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2\). Recall that for any two bounded operators \(A_1\) and \(A_2\) on \(\mathcal{H}_1\) and \(\mathcal{H}_2\), respectively, their tensor product \(A_1 \otimes A_2\) is a bounded operator on \(\mathcal{H}\) and

\[ \|A_1 \otimes A_2\| = \|A_1\| \cdot \|A_2\|. \]

The operator \(P\) is now written on \(\mathcal{H}\) as

\[ P(\omega) = P_1(\omega) \otimes 1_{\mathcal{H}_2} + 1_{\mathcal{H}_1} \otimes P_2(\omega). \]

We now wish to construct an inverse to \(P(\omega)\). The method used is an infinite-dimensional generalization of the following elementary

**Proposition 2.1.** Assume that \(A\) and \(B\) are two (finite-dimensional) matrices and that the matrix \(A \otimes 1 + 1 \otimes B\) is invertible. (That is, no eigenvalue of \(A\) is the negative of an eigenvalue of \(B\).) For \(\lambda \in \mathbb{C}\), let \(R_A(\lambda) = (A + \lambda)^{-1}\) and \(R_B(\lambda) = (B - \lambda)^{-1}\). Take \(\gamma\) to be a bounded simple closed contour in the complex plane such that all poles of \(R_A\) lie outside of \(\gamma\), but all poles of \(R_B\) lie inside \(\gamma\); we assume that \(\gamma\) is oriented in the clockwise direction. Then

\[ (A \otimes 1 + 1 \otimes B)^{-1} = \frac{1}{2\pi i} \int_\gamma R_A(\lambda) \otimes R_B(\lambda) \, d\lambda. \]

The starting point of the method are the inverses\(^4\)

\[ R_1(\omega, \lambda) = (P_1(\omega) + \lambda)^{-1}, \quad R_2(\omega, \lambda) = (P_2(\omega) - \lambda)^{-1} \]

defined for \(\lambda \in \mathbb{C}\). These inverses depend on two complex variables, and we need to specify their behavior near the singular points:

\(^4\)In this section, we do not use the fact that \(R_j(\omega, \lambda) = (P_j(\omega) \pm \lambda)^{-1}\), neither do we prove that \(R(\omega) = P(\omega)^{-1}\). This step will be done in our particular case in the proof of Theorem in the next section; in fact, \(R_1\) will only be a right inverse to \(P_1 + \lambda\). Until then, we merely establish properties of \(R(\omega)\) defined by (2.1) below.
Definition 2.1. Let $X$ be any Banach space, and let $W$ be a domain in $\mathbb{C}^2$. We say that $T(\omega, \lambda)$ is an ($\omega$-nondegenerate) meromorphic map $W \to X$ if:

1. $T(\omega, \lambda)$ is a (norm) holomorphic function of two complex variables with values in $X$ for $(\omega, \lambda) \notin Z$, where $Z$ is a closed subset of $W$, called the divisor of $T$,
2. for each $(\omega_0, \lambda_0) \in Z$, we can write $T(\omega, \lambda) = S(\omega, \lambda)/X(\omega, \lambda)$ near $(\omega_0, \lambda_0)$, where $S$ is holomorphic with values in $X$ and $X$ is a holomorphic function of two variables (with values in $\mathbb{C}$) such that:
   - for each $\omega$ close to $\omega_0$, there exists $\lambda$ such that $X(\omega, \lambda) \neq 0$, and
   - the divisor of $T$ is given by $\{X = 0\}$ near $(\omega_0, \lambda_0)$.

Note that the definition above is stronger than the standard definition of meromorphy and it is not symmetric in $\omega$ and $\lambda$. Henceforth we will use this definition when talking about meromorphic families of operators of two complex variables. It is clear that any derivative (in $\omega$ and/or $\lambda$) of a meromorphic family is again meromorphic. Moreover, if $T(\omega, \lambda)$ is meromorphic and we fix $\omega$, then $T$ is a meromorphic family in $\lambda$.

If $X$ is the space of all bounded operators on some Hilbert space (equipped with the operator norm), then it makes sense to talk about having poles of finite rank:

Definition 2.2. Let $H$ be a Hilbert space and let $T(\omega, \lambda)$ be a meromorphic family of operators on $H$ in the sense of Definition 2.1. For $(\omega_0, \lambda_0)$ in the divisor of $T$, consider the decomposition

$$T(\omega_0, \lambda) = T_H(\lambda) + \sum_{j=1}^{N} \frac{T_j}{(\lambda - \lambda_0)^j}.$$  

Here $T_H$ is holomorphic near $\lambda_0$ and $T_j$ are some operators. We say that $T$ has **poles of finite rank** if every operator $T_j$ in the above decomposition of every $\omega$-derivative of $T$ near every point in the divisor is finite-dimensional.

One can construct meromorphic families of operators with poles of finite rank by using the following generalization of Analytic Fredholm Theory:

Proposition 2.2. Assume that $T(\omega, \lambda) : \mathcal{H}_1 \to \mathcal{H}_2$, $(\omega, \lambda) \in \mathbb{C}^2$, is a holomorphic family of Fredholm operators, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are some Hilbert spaces. Moreover, assume that for each $\omega$, there exists $\lambda$ such that the operator $T(\omega, \lambda)$ is invertible. Then $T(\omega, \lambda)^{-1}$ is a meromorphic family of operators $\mathcal{H}_2 \to \mathcal{H}_1$ with poles of finite rank. (The divisor is the set of all points where $T$ is not invertible.)

Proof. We can use the proof of the standard Analytic Fredholm Theory via Grushin problems, see for example [17, Theorem C.3]. \square

We now go back to constructing the inverse to $P(\omega)$. We assume that

(A) $R_j(\omega, \lambda)$, $j = 1, 2$, are two families of bounded operators on $\mathcal{H}_j$ with poles of finite rank. Here $\omega$ lies in a domain $\Omega \subset \mathbb{C}$ and $\lambda \in \mathbb{C}$.
Figure 1. An admissible contour. The poles of $R_1$ are denoted by circles and the poles of $R_2$ are denoted by asterisks.

We want to integrate the tensor product $R_1 \otimes R_2$ in $\lambda$ over a contour $\gamma$ that separates the sets of poles of $R_1(\omega, \cdot)$ and $R_2(\omega, \cdot)$. Let $Z_j$ be the divisor of $R_j$. We call a point $\omega$ \textbf{regular} if the sets $Z_1(\omega)$ and $Z_2(\omega)$ given by

$$Z_j(\omega) = \{ \lambda \in \mathbb{C} \mid (\omega, \lambda) \in Z_j \}$$

do not intersect. The behavior of the contour $\gamma$ at infinity is given by the following

\textbf{Definition 2.3.} Let $\psi \in (0, \pi)$ be a fixed angle, and let $\omega$ be a regular point. A smooth simple contour $\gamma$ on $\mathbb{C}$ is called \textbf{admissible} (at $\omega$) if:

- outside of some compact subset of $\mathbb{C}$, $\gamma$ is given by the rays $\text{arg } \lambda = \pm \psi$, and
- $\gamma$ separates $\mathbb{C}$ into two regions, $\Gamma_1$ and $\Gamma_2$, such that sufficiently large positive real numbers lie in $\Gamma_2$, and $Z_j(\omega) \subset \Gamma_j$ for $j = 1, 2$.

(Henceforth, we assume that $\text{arg } \lambda \in [-\pi, \pi]$. The contour $\gamma$ and the regions $\Gamma_j$ are allowed to have several connected components.)

Existence of admissible contours and convergence of the integral is guaranteed by the following condition:

(B) For any compact $K_\omega \subset \Omega$, there exist constants $C$ and $R$ such that for $\omega \in K_\omega$ and $|\lambda| \geq R$,

- for $|\text{arg } \lambda| \leq \psi$, we have $(\omega, \lambda) \not\in Z_1$ and $\|R_1(\omega, \lambda)\| \leq C/|\lambda|$, and
- for $|\text{arg } \lambda| \geq \psi$, we have $(\omega, \lambda) \not\in Z_2$ and $\|R_2(\omega, \lambda)\| \leq C/|\lambda|$.

It follows from (B) that there exist admissible contours at every regular point. Take a regular point $\omega$, an admissible contour $\gamma$ at $\omega$, and define

$$R(\omega) = \frac{1}{2\pi i} \int_\gamma R_1(\omega, \lambda) \otimes R_2(\omega, \lambda) \, d\lambda. \quad (2.1)$$

Here the orientation of $\gamma$ is chosen so that $\Gamma_1$ always stays on the left. The integral above converges and is independent of the choice of an admissible contour $\gamma$. Moreover, the set of regular points is open and $R$ is holomorphic on this set. (We may represent $R(\omega)$ as a
locally uniform limit of the integral over the intersection of $\gamma$ with a ball whose radius goes to infinity.)

The main result of this section is

**Proposition 2.3.** Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces, and $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is their Hilbert tensor product. Let $R_1(\omega, \lambda)$ and $R_2(\omega, \lambda)$ be two families of bounded operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, for $\omega \in \Omega \subset \mathbb{C}$ and $\lambda \in \mathbb{C}$. Assume that $R_1$ and $R_2$ satisfy assumptions (A)–(B) and the nondegeneracy assumption

(C) The set $\Omega_R$ of all regular points is nonempty.

Then the set of all non-regular points is discrete and the operator $R(\omega)$ defined by (2.1) is meromorphic in $\omega \in \Omega$ with poles of finite rank.

The rest of this section contains the proof of Proposition 2.3. First, let us establish a normal form for meromorphic decompositions of families in two variables:

**Proposition 2.4.** Let $T(\omega, \lambda)$ be meromorphic (with values in some Banach space) and assume that $(\omega_0, \lambda_0)$ lies in the divisor of $T$. Then we can write near $(\omega_0, \lambda_0)$

$$T(\omega, \lambda) = \frac{S(\omega, \lambda)}{Q(\omega, \lambda)},$$

where $S$ is holomorphic and $Q$ is a monic polynomial in $\lambda$ of degree $N$ and coefficients holomorphic in $\omega$; moreover, $Q(\omega_0, \lambda) = (\lambda - \lambda_0)^N$. The divisor of $T$ coincides with the set of zeroes of $Q$ near $(\omega_0, \lambda_0)$.

**Proof.** Follows from Definition 2.1 and Weierstrass Preparation Theorem. \qed

**Proposition 2.5.** Assume that $Q_j(\omega, \lambda)$, $j = 1, 2$, are two monic polynomials in $\lambda$ of degrees $N_j$ with coefficients holomorphic in $\omega$ near $\omega_0$. Assume also that for some $\omega$, $Q_1$ and $Q_2$ are coprime as polynomials. Then there exist unique polynomials $p_1$ and $p_2$ of degree no more than $N_2 - 1$ and $N_1 - 1$, respectively, with coefficients meromorphic in $\omega$ and such that

$$1 = p_1 Q_1 + p_2 Q_2$$

when $p_1$ and $p_2$ are well-defined.

**Proof.** The $N_1 + N_2$ coefficients of $p_1$ and $p_2$ solve a system of $N_1 + N_2$ linear equations with fixed right-hand side and the matrix $A(\omega)$ depending holomorphically on $\omega$. If $\omega$ is chosen so that $Q_1$ and $Q_2$ are coprime, then the system has a unique solution; therefore, the determinant of $A(\omega)$ is not identically zero. The proposition then follows from Cramer's Rule. \qed

We are now ready to prove that $R(\omega)$ is meromorphic. It suffices to show that for each $\omega_0 \notin \Omega_R$ lying in the closure $\overline{\Omega_R}$, $\omega_0$ is an isolated non-regular point and $R(\omega)$ has a meromorphic decomposition at $\omega_0$ with finite-dimensional principal part. Indeed, in this
case \( \overline{\Omega_R} \) is open; since it is closed and nonempty by (C), we have \( \overline{\Omega_R} = \Omega \) and the statement above applies to each \( \omega_0 \).

Let \( Z_1(\omega_0) \cap Z_2(\omega_0) = \{\lambda_1, \ldots, \lambda_m\} \). We choose a ball \( \Omega_0 \) centered at \( \omega_0 \) and disjoint balls \( U_l \) centered at \( \lambda_l \) such that:

- for \( \omega \in \Omega_0 \), the set \( Z_1(\omega) \cap Z_2(\omega) \) is covered by balls \( U_l \) and the set \( Z_1(\omega) \cup Z_2(\omega) \)
  does not intersect the circles \( \partial U_l \);
- for \( \omega \in \Omega_0 \) and \( \lambda \in U_l \), we have \( R_j = S_{jl}/Q_{jl} \), where \( S_{jl} \) are holomorphic and \( Q_{jl} \) are monic polynomials in \( \lambda \) of degree \( N_{jl} \) with coefficients holomorphic in \( \omega \), and
  \( Q_{jl}(\omega, \lambda) = (\lambda - \lambda_l)^{N_{jl}} \);
- for \( \omega \in \Omega_0 \), the set of all roots of \( Q_{jl}(\omega, \cdot) \) coincides with \( Z_j(\omega) \cap \partial U_l \);
- there exists a contour \( \gamma_0 \) that does not intersect any \( U_l \) and is admissible for any \( \omega \in \Omega_0 \) with respect to the sets \( Z_j(\omega) \setminus \cup U_l \) in place of \( Z_j(\omega) \); moreover, each \( \partial U_l \)
  lies in the region \( \Gamma_l \) with respect to \( \gamma_0 \) (see Definition 2.3).

Let us assume that \( \omega \in \Omega_0 \) is regular. (Such points exist since \( \omega_0 \) lies in the closure of \( \Omega_R \).)

For every \( l \), the polynomials \( Q_{1l}(\omega, \lambda) \) and \( Q_{2l}(\omega, \lambda) \) are coprime; we find by Proposition 2.5 two
unique polynomials \( p_{1l}(\omega, \lambda) \) and \( p_{2l}(\omega, \lambda) \) such that

\[ 1 = p_{1l}Q_{1l} + p_{2l}Q_{2l} \]

and \( \deg p_{1l} < N_{2l}, \deg p_{2l} < N_{1l} \). The converse is also true: if all coefficients of \( p_{1l} \) and \( p_{2l} \)
are holomorphic at some point \( \omega \) for all \( l \), then \( \omega \) is a regular point. It follows immediately
that \( \omega_0 \) is an isolated non-regular point.

To obtain the meromorphic expansion of \( R(\omega) \) near \( \omega_0 \), let us take a regular point \( \omega \in \Omega_0 \)
and an admissible contour \( \gamma = \gamma_0 + \cdots + \gamma_m \), where \( \gamma_0 \) is the \( \omega \)-independent contour defined above and each \( \gamma_l \) is a contour lying in \( U_l \). The integral over \( \gamma_0 \) is holomorphic near \( \omega_0 \), while

\[
\int_{\gamma_l} R_1(\omega, \lambda) \otimes R_2(\omega, \lambda) \, d\lambda = \int_{\gamma_l} S_{1l}(\omega, \lambda) \otimes S_{2l}(\omega, \lambda) \left( p_{1l}(\omega, \lambda) + p_{2l}(\omega, \lambda) \frac{Q_{1l}(\omega, \lambda)}{Q_{2l}(\omega, \lambda)} \right) \, d\lambda
\]

\[= \int_{\partial U_l} p_{1l}S_{1l} \otimes R_2 \, d\lambda = \sum_{j=0}^{N_{2l}-1} p_{1lj}(\omega) \int_{\partial U_l} (\lambda - \lambda_l)^j S_{1l} \otimes R_2 \, d\lambda.\]

Here \( p_{1lj}(\omega) \) are the coefficients of \( p_{1l} \) as a polynomial of \( \lambda - \lambda_l \); they are meromorphic in \( \omega \) and the rest is holomorphic in \( \omega \in \Omega_0 \).

It remains to prove that \( R \) has poles of finite rank. It suffices to show that every derivative
in \( \omega \) of the last integral above at \( \omega = \omega_0 \) has finite rank. Each of these, in turn, is a finite linear combination of

\[
\int_{\partial U_l} (\lambda - \lambda_l)^j \partial^a \omega S_{1l}(\omega_0, \lambda) \otimes \partial^b \omega R_2(\omega_0, \lambda) \, d\lambda.
\]
However, since \( \partial_a \omega S_1(\omega, \lambda) \) is holomorphic in \( \lambda \in U_1 \), only the principal part of the Laurent decomposition of \( \partial_a R_2(\omega, \lambda) \) at \( \lambda = \lambda_1 \) will contribute to this integral; therefore, the image of each operator in the principal part of Laurent decomposition of \( R(\omega) \) at \( \omega_0 \) lies in \( H_1 \otimes V_2 \), where \( V_2 \) is a certain finite-dimensional subspace of \( H_2 \). It remains to show that each of these images also lies in \( V_1 \otimes H_2 \), where \( V_1 \) is a certain finite-dimensional subspace of \( H_1 \). This is done by the same argument, using the fact that
\[
\int_{\partial U_l - \gamma_l} R_1(\omega, \lambda) \otimes R_2(\omega, \lambda) d\lambda
\]
can be written in terms of \( p_{2l} \) and \( R_1 \otimes S_{2l} \) and the integral over \( \partial U_l \) is holomorphic at \( \omega_0 \). The proof of Proposition 2.3 is finished.

3. Construction of \( R_g(\omega) \)

As we saw in the previous section, one can deduce the existence of an inverse to \( P_g = P_r + P_\theta \) and its properties from certain properties of the inverses to \( P_r + \lambda \) and \( P_\theta - \lambda \) for \( \lambda \in \mathbb{C} \). We start with the latter. For \( a = 0 \), \( P_\theta \) is the (negative) Laplace–Beltrami operator for the round metric on \( S^2 \); therefore, its eigenvalues are given by \( \lambda = l(l+1) \) for \( l \in \mathbb{Z} \), \( l \geq 0 \). Moreover, if \( D'_k \) is the space defined in \( (1.2) \) and there is an eigenfunction of \( P_\theta |_{D'_k} \) with eigenvalue \( l(l+1) \), then \( l \geq k \). These observations can be generalized to our case:

**Proposition 3.1.** There exists a two-sided inverse
\[
R_\theta(\omega, \lambda) = (P_\theta(\omega) - \lambda)^{-1} : L^2(S^2) \to H^2(S^2), \ (\omega, \lambda) \in \mathbb{C}^2,
\]
with the following properties:

1. \( R_\theta(\omega, \lambda) \) is meromorphic with poles of finite rank in the sense of Definition 2.2 and it has the following meromorphic decomposition at \( \omega = \lambda = 0 \):
\[
R_\theta(\omega, \lambda) = \frac{S_{0\theta}(\omega, \lambda)}{\lambda - \lambda_\theta(\omega)}
\]
where \( S_{0\theta} \) and \( \lambda_\theta \) are holomorphic in \( a \)-independent neighborhoods of zero and
\[
S_{0\theta}(0,0) = -\frac{1 \otimes 1}{4\pi}, \ \lambda_\theta(\omega) = O(|\omega|^2).
\]

2. There exists a constant \( C_\theta \) such that
\[
\|R_\theta(\omega, \lambda)\|_{L^2(S^2) \cap D'_k \to L^2(S^2)} \leq \frac{C_\theta}{|k|^2} \quad \text{for} \ |\lambda| \leq k^2/2, \ |k| \geq C_\theta |a\omega|.
\]
\[
\|R_\theta(\omega, \lambda)\|_{L^2(S^2) \cap D'_k \to L^2(S^2)} \leq \frac{2}{|\text{Im} \lambda|} \quad \text{for} \ |\text{Im} \lambda| > C_\theta |a|(|a\omega| + |k|)|\text{Im} \omega|.
\]
3. For every $\psi > 0$, there exists a constant $C_\psi$ such that

$$
\|R_\theta(\omega, \lambda)\|_{L^2(S^2) \to L^2(S^2)} \leq \frac{C_\psi}{|\lambda|} \quad \text{for } |\arg \lambda| \geq \psi, \quad |\lambda| \geq C_\psi|a\omega|^2.
$$

(3.4)

Proof. 1. Recall (1.3) that $P_\theta(\omega)$ is a holomorphic family of elliptic second order differential operators on the sphere. Therefore, for each $\lambda$, the operator $P_\theta(\omega) - \lambda : H^2(S^2) \to L^2(S^2)$ is Fredholm (see for example [36, Section 7.10]). By Proposition 2.2, $R_\vartheta(\omega, \lambda)$ is a meromorphic family of operators $L^2 \to H^2$. We now obtain a meromorphic decomposition for $R_\vartheta$ near zero using the framework of Grushin problems [17, Appendix C]. Let $i_1 : \mathbb{C} \to L^2(S^2)$ be the operator of multiplication by the constant function 1 and $\pi_1 : H^2(S^2) \to \mathbb{C}$ be the operator mapping every function to its integral over the standard measure on the round sphere. Consider the operator $A(\omega, \lambda) : H^2 \oplus \mathbb{C} \to L^2 \oplus \mathbb{C}$ given by

$$
A(\omega, \lambda) = \begin{pmatrix} P_\theta(\omega) - \lambda & i_1 \\ \pi_1 & 0 \end{pmatrix}.
$$

The kernel and cokernel of $P_\vartheta(0)$ are both one-dimensional and spanned by 1, since this is the Laplace–Beltrami operator for a certain Riemannian metric on the sphere. (Indeed, by ellipticity these spaces consist of smooth functions; by self-adjointness, the kernel and cokernel coincide; one can then apply Green’s formula [36, (2.4.8)] to an element of the kernel and itself.) Therefore [17, Theorem C.1], the operator $B(\omega, \lambda) = A(\omega, \lambda)^{-1}$ is well-defined at $(0, 0)$; then it is well-defined for $(\omega, \lambda)$ in an $a$-independent neighborhood of zero.

We write

$$
B(\omega, \lambda) = \begin{pmatrix} B_{11}(\omega, \lambda) & B_{12}(\omega, \lambda) \\ B_{21}(\omega, \lambda) & B_{22}(\omega, \lambda) \end{pmatrix}.
$$

Now, by Schur’s complement formula we have near $(0, 0)$,

$$
R_\vartheta(\omega, \lambda) = B_{11}(\omega, \lambda) - B_{12}(\omega, \lambda)B_{22}(\omega, \lambda)^{-1}B_{21}(\omega, \lambda).
$$

However, $B_{22}(\omega, \lambda)$ is a holomorphic function of two variables, and we can find

$$
B_{22}(\omega, \lambda) = \frac{\lambda}{4\pi} + O(|\omega|^2 + |\lambda|^2).
$$

(The $\omega$-derivative vanishes at zero since $\partial_\omega P_\omega(0)|_{\vartheta_0} = 0$. To compute the $\lambda$-derivative, we use that $B_{12}(0, 0) = i_1/4\pi$ and $B_{21}(0, 0) = \pi_1/4\pi$.) The decomposition (3.1) now follows by Weierstrass Preparation Theorem.

2. We have $P_\theta(\omega) = P_\theta(0) + P_\theta'(\omega)$, where

$$
P_\theta'(\omega) = \frac{(1 + \alpha)^2 a\omega}{\Delta_\theta} (-2D_\varphi + a\omega \sin^2 \theta)
$$
is a first order differential operator and
\[ P_\theta(0) = \frac{1}{\sin \theta} D_\theta (\Delta_0 \sin \theta D_\theta) + \frac{(1 + \alpha)^2}{\Delta_0 \sin^2 \theta} D_\theta^2 : H^2(S^2) \to L^2(S^2) \]
satisfies \( P_\theta(0) \geq k^2 \) on \( \mathcal{D}_k \); therefore, if \( u \in H^2(S^2) \cap \mathcal{D}_k \), then
\[ \|u\|_{L^2} \leq \| (P_\theta(0) - \lambda) u \|_{L^2} \frac{d(\lambda, k^2 + \mathbb{R}^+)}{C_1 |a\omega|(|a\omega| + |k|)}, \]
Since \( \|P_\theta(\omega)\|_{L^2(S^2) \cap \mathcal{D}_k} \leq 2(1 + \alpha)^2 |a\omega|(|a\omega| + |k|) \), we get
\[ \|u\|_{L^2} \leq \frac{\| (P_\theta(\omega) - \lambda) u \|_{L^2}}{d(\lambda, k^2 + \mathbb{R}^+) - C_1 |a\omega|(|a\omega| + |k|)}, \tag{3.5} \]
provided that the denominator is positive. Here \( C_1 \) is a global constant.

Now, if \(|\lambda| \leq k^2/2\), then \( d(\lambda, k^2 + \mathbb{R}^+) \geq k^2/2 \) and
\[ d(\lambda, k^2 + \mathbb{R}^+) - C_1 |a\omega|(|a\omega| + |k|) \geq \frac{k^2}{4} \text{ for } |k| \geq 8(1 + C_1)|a\omega|; \]

To prove \((3.3)\), introduce
\[ \text{Im} P_\theta(\omega) = \frac{1}{2} (P_\theta(\omega) - P_\theta(\omega)^*) = \frac{2(1 + \alpha)^2}{\Delta_0} a \text{ Im} \omega (a \text{ Re} \omega \sin^2 \theta - D_\theta); \]
we have
\[ \| \text{Im} P_\theta(\omega)\|_{L^2(S^2) \cap \mathcal{D}_k} \leq 2(1 + \alpha)^2 |a \text{ Im} \omega|(|a\omega| + |k|). \]
However, for \( u \in H^2(S^2) \cap \mathcal{D}_k \),
\[ \| (P_\theta(\omega) - \lambda) u \| \cdot \| u \| \geq | \text{Im}((P_\theta(\omega) - \lambda) u, u) | \geq | \text{Im} \lambda | \cdot \| u \|^2 - |(\text{Im} P_\theta(\omega) u, u) | \]
\[ \geq (| \text{Im} \lambda | - 2(1 + \alpha)^2 |a|(|a\omega| + |k|) | \text{Im} \omega |) \| u \|^2 \]
and we are done if \( C_\theta \geq 4(1 + \alpha)^2 \).

3. If \(| \arg \lambda | \geq \psi \), then \( d(\lambda, k^2 + \mathbb{R}^+) \geq (k^2 + |\lambda|)/C_2 \); here \( C_2 \) is a constant depending on \( \psi \). We have then
\[ d(\lambda, k^2 + \mathbb{R}^+) - C_1 |a\omega|(|a\omega| + |k|) \geq \frac{1}{C_2} |\lambda| - C_3 |a\omega|^2 \]
for some constant \( C_3 \), and we are done by \((3.5)\). \( \square \)

The analysis of the radial operator \( P_r \) is more complicated. In Sections 4–6, we prove
Proposition 3.2. There exists a family of operators

\[ R_r(\omega, \lambda, k) : L^2_{\text{comp}}(r_-, r_+) \to H^2_{\text{loc}}(r_-, r_+), \ (\omega, \lambda) \in \mathbb{C}^2, \]

with the following properties:

1. For each \( k \in \mathbb{Z} \), \( R_r(\omega, \lambda, k) \) is meromorphic with poles of finite rank in the sense of Definition 2.2, and \((P_r(\omega, k) + \lambda)R_r(\omega, \lambda, k)f = f\) for each \( f \in L^2_{\text{comp}}(r_-, r_+)\). Also, for \( k = 0 \), \( R_r \) admits the following meromorphic decomposition near \( \omega = \lambda = 0 \):

\[ R_r(\omega, \lambda, 0) = \frac{S_{r0}(\omega, \lambda)}{\lambda - \lambda_r(\omega)}, \] (3.6)

where \( S_{r0} \) and \( \lambda_r \) are holomorphic in a-independent neighborhoods of zero and

\[ S_{r0}(0, 0) = \frac{1}{r_+ - r_-}, \]
\[ \lambda_r(\omega) = \frac{i(1 + \alpha)(r_+^2 + r_-^2 + 2a^2)}{r_+ - r_-}\omega + O(|\omega|^2). \]

2. Take \( \delta_r > 0 \). Then there exist \( \psi > 0 \) and \( C_r \) such that for

\[ |\lambda| \geq C_r, \ |\arg \lambda| \leq \psi, \ |ak|^2 \leq |\lambda|/C_r, \ |\omega|^2 \leq |\lambda|/C_r, \] (3.7)

\((\omega, \lambda, k)\) is not a pole of \( R_r \) and we have

\[ \|1_{K_r} R_r(\omega, \lambda, k)1_{K_r}\|_{L^2 \to L^2} \leq \frac{C_r}{|\lambda|}. \] (3.8)

Also, there exists \( \delta_{r0} > 0 \) such that, if \( K_+ = [r_+ - \delta_{r0}, r_+] \) and \( K_- = [r_-, r_- + \delta_{r0}] \), then for each \( N \) there exists a constant \( C_N \) such that under the conditions \( (3.7) \), we have

\[ \|1_{K_\pm} |r - r_{\pm}|^{iA_\pm^{-1}(1+\alpha)(r_{\pm}^2 + \alpha^2)\omega - ak}R_r(\omega, \lambda, k)1_{K_r}\|_{L^2 \to C^N(K_\pm)} \leq \frac{C_N}{|\lambda|^N}. \] (3.9)

3. There exists a constant \( C_\omega \) such that \( R_r(\omega, \lambda, k) \) does not have any poles for real \( \lambda \) and real \( \omega \) with \( |\omega| > C_\omega |ak| \).

4. Assume that \( R_r \) has a pole at \((\omega, \lambda, k)\). Then there exists a nonzero solution \( u \in C^\infty(r_-, r_+) \) to the equation \((P_r(\omega, k) + \lambda)u = 0\) such that the functions

\[ |r - r_{\pm}|^{iA_\pm^{-1}(1+\alpha)(r_{\pm}^2 + \alpha^2)\omega - ak}u(r) \]

are real analytic at \( r_{\pm} \), respectively.

5. Take \( \delta_r > 0 \). Then there exists \( C_{1r} > 0 \) such that for

\[ \text{Im} \omega > 0, \ |ak| \leq |\omega|/C_{1r}, \ |\text{Im} \lambda| \leq |\omega| \cdot \text{Im} \omega/C_{1r}, \ \text{Re} \lambda \geq -|\omega|^2/C_{1r}, \] (3.10)

\((\omega, \lambda, k)\) is not a pole of \( R_r \) and we have

\[ \|1_{K_r} R_r(\omega, \lambda, k)1_{K_r}\|_{L^2 \to L^2} \leq \frac{C_{1r}}{|\omega| \cdot \text{Im} \omega}. \] (3.11)
Given these two propositions, we can now prove Theorems 1–4.

**Proof of Theorem 1.** Take \( k \in \mathbb{Z} \) and an arbitrary \( \delta_r > 0 \); put \( H_1 = L^2(K_r), H_2 = L^2(S^2) \cap D'_k, R_1(\omega, \lambda) = R_r(\omega, \lambda, k), \) and \( R_2(\omega, \lambda) = R_\theta(\omega, \lambda)|_{D'_k} \); finally, let the angle \( \psi \) of admissible contours at infinity be chosen as in Proposition 3.2. We now apply Proposition 2.3. Condition (A) follows from the first parts of Propositions 3.1 and 3.2. Condition (B) follows from (3.4) and part 2 of Proposition 3.2. Finally, condition (C) holds because every \( \omega \in \mathbb{R} \) with \( |\omega| > C_\omega |ak| \), where \( C_\omega \) is the constant from part 3 of Proposition 3.2, is regular. Indeed, \( P_\theta(\omega) \) is self-adjoint and thus has only real eigenvalues. Now, by Proposition 2.3 we can use (2.1) to define \( R_g(\omega,k) \) as a meromorphic family of operators on \( L^2(M_K) \cap D'_k \) with poles of finite rank. This can be done for any \( \delta_r > 0 \); therefore, \( R_g(\omega,k) \) is defined as an operator \( L^2_{\text{comp}}(M) \cap D'_k \to L^2_{\text{loc}}(M) \cap D'_k \).

Let us now prove that \( P_g(\omega,k)R_g(\omega,k)f = f \) in the sense of distributions for each \( f \in L^2_{\text{comp}} \). We will use the method of Proposition 2.1. Assume that \( \omega \) is a regular point, so that \( R_g(\omega,k) \) is well-defined. By analyticity, we can further assume that \( \omega \) is real, so that \( L^2(S^2) \cap D'_k \) has an orthonormal basis of eigenfunctions of \( P_\theta(\omega) \). Then it suffices to prove that

\[
I = (R_g(\omega,k)(f_r(r)f_\theta(\theta,\varphi)), P_g(\omega)(h_r(r)h_\theta(\theta,\varphi))) = (f_r, h_r) \cdot (f_\theta, h_\theta),
\]

where \( f_r, h_r \in C_0^\infty(r_-, r_+), h_\theta \in C^\infty(S^2) \cap D'_k, \) and \( f_\theta \in D'_k \) satisfies

\[
P_\theta(\omega)f_\theta = \lambda_0 f_\theta, \quad \lambda_0 \in \mathbb{R}.
\]

Take an admissible contour \( \gamma \); then

\[
I = \frac{1}{2\pi i} \int_{\gamma} (R_r(\omega, \lambda, k)f_r, P_r(\omega, k)h_r) \cdot (R_\theta(\omega, \lambda)f_\theta, h_\theta)
+ (R_r(\omega, \lambda)f_r, h_r) \cdot (R_\theta(\omega, \lambda)f_\theta, P_\theta(\omega)h_\theta) d\lambda.
\]

However,

\[
R_\theta(\omega, \lambda)f_\theta = \frac{f_\theta}{\lambda_0 - \lambda}.
\]

It then follows from condition (B) that we can replace \( \gamma \) by a closed bounded contour \( \gamma' \) which contains \( \lambda_0 \), but no poles of \( R_r \). (To obtain \( \gamma' \), we can cut off the infinite ends of \( \gamma \) sufficiently far and connect the resulting two endpoints by the arc \( -\psi \leq \arg \lambda \leq \psi \); the
integral over the arc can be made arbitrarily small.) Then
\[ I = \frac{1}{2\pi i} \int_{\gamma'} ((1 - \lambda R_r(\omega, \lambda, k)) f_r, h_r) \cdot (R_\theta(\omega, \lambda) f_\theta, h_\theta) \]
\[ + (R_r(\omega, \lambda, k) f_r, h_r) \cdot ((1 + \lambda R_\theta(\omega, \lambda)) f_\theta, h_\theta) \, d\lambda \]
\[ = \frac{1}{2\pi i} \int_{\gamma'} (f_r, h_r) \cdot (R_\theta(\omega, \lambda) f_\theta, h_\theta) + (R_r(\omega, \lambda, k) f_r, h_r) \cdot (f_\theta, h_\theta) \, d\lambda \]
\[ = \frac{1}{2\pi i} \int_{\gamma'} \frac{(f_r, h_r)(f_\theta, h_\theta)}{\lambda_0 - \lambda} \, d\lambda = (f_r, h_r)(f_\theta, h_\theta), \]
which finishes the proof.

Finally, the operator \( P_\theta(\omega, k) \) is the restriction to \( D'_K \) of the elliptic differential operator on \( M \) obtained from \( P_\theta(\omega) \) by replacing \( D_\rho \) by \( k \) in the second term of \( (1.1) \). Therefore, by elliptic regularity (see for example [36, Section 7.4]) the operator \( R_\theta(\omega, k) \) acts into \( H^2_{loc} \).

Next, Theorem 2 follows from Theorem 1 the fact that the operator \( P_\theta(\omega) \) is elliptic on \( M_K \) for small \( a \) (to get \( H^2 \) regularity instead of \( L^2 \)), and the following estimate on \( R_\theta(\omega, k) \) for large values of \( k \):

**Proposition 3.3.** Fix \( \delta_r > 0 \). Then there exists \( a_0 > 0 \) and a constant \( C_k \) such that for \( |a| < a_0 \) and \( |k| \geq C_k(1 + |\omega|) \), \( \omega \) is not a pole of \( R_\theta(\cdot, k) \) and we have
\[ \|1_{M_K} R_\theta(\omega, k)1_{M_K}\|_{L^2(H^2_k) \to L^2} \leq \frac{C_k}{|k|^2}. \] (3.12)

**Proof.** Let \( \psi, C_r \) be the constants from part 2 of Proposition 3.2 and \( C_\theta, C_\psi \) be the constants from Proposition 3.1. Put \( \lambda_0 = k^2/3 \); if \( C_k \) is large enough, then
\[ |k| > 1 + C_\theta |a\omega|, \quad \lambda_0 > C_\psi |a\omega|^2 + C_r (1 + |\omega|^2). \]

Take the contour \( \gamma \) consisting of the rays \( \{ \arg \lambda = \pm \psi, |\lambda| \geq \lambda_0 \} \) and the arc \( \{ |\lambda| = \lambda_0, |\arg \lambda| \leq \psi \} \). By (3.2) and (3.4), all poles of \( R_\theta \) lie inside \( \gamma \) (namely, in the region \( \{ |\lambda| \geq \lambda_0, |\arg \lambda| \leq \psi \} \), and
\[ \| R_\theta(\omega, \lambda) \|_{L^2(\mathbb{S}^2) \cap D'_K \to L^2(\mathbb{S}^2)} \leq \frac{C}{|\lambda|} \] (3.13)
for each \( \lambda \) on \( \gamma \). Now, suppose that \( |a| < a_0 = (3C_r)^{-1/2} \); then (3.7) is satisfied inside \( \gamma \) and (3.12) follows from (2.1), (3.8), and (3.13). \( \square \)

**Proof of Theorem 3.** 1. Fix \( \delta_r > 0 \) such that \( \text{supp } f \subset M_K \). Take an admissible contour \( \gamma \); then by (2.1) and the fact that the considered functions are in \( D'_K \),
\[ v_\pm = \frac{1}{2\pi i} \int_{\gamma} (R^\pm_r(\omega, \lambda, k) \otimes R_\theta(\omega, \lambda)) f \, d\lambda, \] (3.14)
where
\[ R^\pm_r(\omega, \lambda, k) = |r - r_\pm|^{iA_\pm^{-1}(1+\alpha)(r_\pm^2+a^2)(\omega-ak)} R_r(\omega, \lambda, k). \]

By part 2 of Proposition 3.2 we may choose compact sets \( K_\pm \) containing \( r_\pm \) such that for each \( N \), there exists a constant \( C_N \) (depending on \( \omega, \lambda, \) and \( \gamma \)) such that
\[ \|1_{K_\pm} R^\pm_r(\omega, \lambda, k)1_{K_r}\|_{L^2 \to C^N(K_\pm)} \leq \frac{C_N}{1 + |\lambda|^2}, \quad \lambda \in \gamma. \]

(The estimate is true over a compact portion of \( \gamma \) since the image of \( R^\pm_r \) consists of functions smooth at \( r = r_\pm \), by the construction in Section 4.) Now, by (3.4) we get for some constant \( C'_N \),
\[ \|R^\pm_r(\omega, \lambda, k) \otimes R_\theta(\omega, \lambda)f\|_{C^N(K_\pm; L^2(S^2))} \leq \frac{C'_N}{1 + |\lambda|^2} ; \]
by (3.14), \( v_\pm \in C^\infty(K_\pm; L^2(S^2)) \).

Now, since \((P_r + P_\theta)u = f \) and (assuming that \( K_\pm \cap K_r = \emptyset \)) \( f|_{K_\pm \times S^2} = 0 \), we have \((P_r^\pm(\omega, k) + P_\theta(\omega))v_\pm = 0 \) on \( K_\pm \times S^2 \), where
\[ P^\pm_r(\omega, k) = |r - r_\pm|^{iA_\pm^{-1}(1+\alpha)(r_\pm^2+a^2)(\omega-ak)} P_r(\omega, k)|r - r_\pm|^{-iA_\pm^{-1}(1+\alpha)(r_\pm^2+a^2)(\omega-ak)} \]
has smooth coefficients on \( K_\pm \) (see Section 4). Then for each \( N \),
\[ P^N_\theta v_\pm = (-P^\pm_r)^N v_\pm \in C^\infty(K_\pm; L^2(S^2)); \]
since \( P_\theta \) is elliptic, we get \( v_\pm \in C^\infty(K_\pm; H^2(N(S^2))) \). Therefore, \( v_\pm \in C^\infty(K_\pm \times S^2). \)

2. Let \( \omega \) be a pole of \( R_\theta(\omega, k) \). Then \( \omega \) is not a regular point; therefore, there exists \( \lambda \in \mathbb{C} \) such that \( (\omega, \lambda) \) is a pole of both \( R_r \) and \( R_\theta \). This gives us functions \( u_r(r) \) and \( u_\theta(\theta, \varphi) \in D'_k \) such that \((P_r(\omega, k) + \lambda)u_r = 0 \) and \((P_\theta(\omega) - \lambda)u_\theta = 0 \). It remains to take \( u = u_r \otimes u_\theta \) and use part 4 of Proposition 3.2 \( \square \)

The following fact will be used in the proof of Theorem 4 as well as in Section 7:

**Proposition 3.4.** Fix \( \delta_r > 0 \). Let \( \psi, C_r \) be the constants from part 2 of Proposition 3.2 \( C_\theta, C_\psi \) be the constants from Proposition 3.1 \( \) and \( C_k \) be the constant from Proposition 3.3.

Take \( \omega \in \mathbb{C} \) and put
\[ L = (C_r(1 + C_k)^2 + C_\psi)(1 + |\omega|^2). \]
Assume that \( a \) is small enough so that Proposition 3.3 applies and suppose that \( \omega \) and \( l_1, l_2 > 0 \) are chosen so that
\[ l_1 \geq C_\psi |a\omega|^2, \quad l_2 \geq C_\theta |a|(|a\omega| + C_k(1 + |\omega|)) \text{ Im } \omega|, \quad l_2 \leq L \sin \psi. \quad (3.15) \]

Also, assume that for all \( \lambda \) and \( k \) satisfying
\[ |k| \leq C_k(1 + |\omega|), \quad -l_1 \leq \text{ Re } \lambda \leq L, \quad |\text{ Im } \lambda| \leq l_2, \quad (3.16) \]
we have the estimate
\[ \|1_{K_r} R_r(\omega, \lambda, k)1_{K_r}\|_{L^2 \to L^2} \leq C_1 \quad (3.17) \]
The admissible contour $\gamma$ used in Proposition 3.4 for some constant $C_1$ independent of $\lambda$ and $k$. Then $\omega$ is not a resonance and
\[
\|R_g(\omega)\|_{L^2(M_K) \to L^2(M_K)} \leq C_2 \left( \frac{1}{1 + |\omega|^2} + \frac{1 + C_1(l_1 + 1 + |\omega|^2)}{l_2} + \frac{C_1 l_2}{l_1} \right) \tag{3.18}
\]
for a certain global constant $C_2$.

**Proof.** First of all, by Proposition 3.3, it suffices to establish the estimate (3.18) for the operator $R_g(\omega, k)$, where $|k| \leq C_k(1 + |\omega|)$. Now, by (2.1), it suffices to construct an admissible contour in the sense of Definition 2.3 and estimate the norms of $R_r$ and $R_\theta$ on this contour. We take the contour $\gamma$ composed of:

- the rays $\gamma_{1 \pm} = \{ \arg \lambda = \pm \psi, |\lambda| \geq L \}$;
- the arcs $\gamma_{2 \pm} = \{ |\arg \lambda| \leq \psi, |\lambda| = L, \pm \text{Im} \lambda \geq l_2 \}$;
- the segments $\gamma_{3 \pm}$ of the lines $\{ \text{Im} \lambda = \pm l_2 \}$ connecting $\gamma_{2 \pm}$ with $\gamma_4$;
- the segment $\gamma_4 = \{ \text{Re} \lambda = -l_1, |\text{Im} \lambda| \leq l_2 \}$.

Then $\gamma$ divides the complex plane into two domains; we refer to the domain containing positive real numbers as $\Gamma_2$ and to the other domain as $\Gamma_1$. We claim that $R_\theta(\omega, \cdot)|_{\mathcal{D}_k}$ has no poles in $\Gamma_1$, $R_r(\omega, \cdot, k)$ has no poles in $\Gamma_2$, and the $L^2 \to L^2$ operator norm estimates
\[
\|R_\theta(\omega, \lambda)\| \leq C/|\lambda|, \quad \|1_{K_r} R_r(\omega, \lambda, k) 1_{K_r}\| \leq C/|\lambda|, \quad \lambda \in \gamma_{1 \pm}; \tag{3.19}
\]
\[
\|R_\theta(\omega, \lambda)|_{\mathcal{D}_k}\| \leq C/l_2, \quad \|1_{K_r} R_r(\omega, \lambda, k) 1_{K_r}\| \leq C/(1 + |\omega|^2), \quad \lambda \in \gamma_{2 \pm}; \tag{3.20}
\]
\[
\|R_\theta(\omega, \lambda)|_{\mathcal{D}_k}\| \leq C/l_2, \quad \|1_{K_r} R_r(\omega, \lambda, k) 1_{K_r}\| \leq C_1, \quad \lambda \in \gamma_{3 \pm}; \tag{3.21}
\]
\[
\|R_\theta(\omega, \lambda)\| \leq C/l_1, \quad \|1_{K_r} R_r(\omega, \lambda, k) 1_{K_r}\| \leq C_1, \quad \lambda \in \gamma_4 \tag{3.22}
\]
hold for some global constant $C$; then (3.18) follows from these estimates and (2.1).
First, we prove that $R_{\theta}(\omega, \cdot)|_{D_k'}$ has no poles $\lambda \in \Gamma_1$. First of all, assume that $|\lambda| \geq L$. Then $|\arg \lambda| \geq \psi$ and we can apply part 3 of Proposition 3.1; we also get the first half of (3.19). Same argument works for $\Re \lambda \leq -l_1$, and we get the first half of (3.22). We may now assume that $|\lambda| \leq L$ and $\Re \lambda \geq -l_1$; it follows that $|\Im \lambda| \geq l_2$. But in that case, we can apply (3.3), and we get the first halves of (3.20) and (3.21).

Next, we prove that $R_r(\omega, \cdot, k)$ has no poles $\lambda \in \Gamma_2$. First of all, assume that $|\lambda| \geq L$ and $\Re \lambda \geq 0$. Then $|\arg \lambda| \leq \psi$ and we can apply part 2 of Proposition 3.2; we also get the second halves of (3.19) and (3.20). Now, in the opposite case, (3.16) is satisfied and we can use (3.17) to get the second halves of (3.21) and (3.22). □

**Proof of Theorem 4.** First, we take care of the resonances near zero. By Proposition 3.3, we can assume that $k$ is bounded by some constant. Next, if $\omega = 0$ and $a = 0$, then $R_g(\omega, k)$ only has a pole for $k = 0$, and in the latter case, $\lambda = 0$ is the only common pole of $R_g(0, \cdot)$ and $R_r(0, \cdot, 0)$. (In fact, the poles of $R_g(0, \cdot)|_{D_k'}$ are given by $\lambda = l(l + 1)$ for $l \geq |k|$; an integration by parts argument shows that $R_r(0, \cdot, k)$ cannot have poles with $\Re \lambda > 0$.)

The sets of poles of the resolvents $R_g(\omega, \lambda)|_{D_k'}$ and $R_r(\omega, \lambda, k)$ depend continuously on $a$ in the sense that, if there are no poles of one of these resolvents for $(\omega, \lambda)$ in a fixed compact set for $a = 0$, then this is still true for $a$ small enough. It follows from here and the first parts of Propositions 3.1 and 3.2 that there exists $\varepsilon_\omega, \varepsilon_\lambda > 0$ such that for $a$ small enough,

- $R_g(\omega, k)$ does not have poles in $\{ |\omega| \leq \varepsilon_\omega \}$ unless $k = 0$;
- if $|\omega| \leq \varepsilon_\omega$, then all common poles of $R_g(\omega, \cdot)|_{D_k'}$ and $R_r(\omega, \cdot, 0)$ lie in $\{ |\lambda| \leq \varepsilon_\lambda \}$;
- the decompositions (3.1) and (3.6) hold for $|\omega| \leq \varepsilon_\omega$, $|\lambda| \leq \varepsilon_\lambda$;
- we have $\lambda_r(\omega) \neq \lambda_\theta(\omega)$ for $0 < |\omega| \leq \varepsilon_\omega$.

It follows immediately that $\omega = 0$ is the only pole of $R_g$ in $\{ |\omega| \leq \varepsilon_\omega \}$. To get the meromorphic decomposition, we repeat the argument at the end of Section 2 in our particular case. Note that for small $\omega \neq 0$,

$$R_g(\omega, 0) = \frac{1}{2\pi i} \int_\gamma R_r(\omega, \lambda, 0) \otimes R_\theta(\omega, \lambda)|_{D_k'} d\lambda + \text{Hol}(\omega)$$

Here $\gamma$ is a small contour surrounding $\lambda_\theta(\omega)$, but not $\lambda_r(\omega)$; the integration is done in the clockwise direction; Hol denotes a family of operators holomorphic near zero. By (3.1)
and (3.6), we have

\[ R_\gamma(\omega, 0) = \text{Hol}(\omega) + \frac{1}{2\pi i} \int_\gamma \frac{S_{r_0}(\omega, \lambda) \otimes S_{\theta_0}(\omega, \lambda)}{(\lambda - \lambda_r(\omega))(\lambda - \lambda_\theta(\omega))} \, d\lambda \]

\[ = \text{Hol}(\omega) + \frac{1}{\lambda_r(\omega) - \lambda_\theta(\omega)} \frac{1}{2\pi i} \int_\gamma (S_{r_0}(\omega, \lambda) \otimes S_{\theta_0}(\omega, \lambda)) \left( \frac{1}{\lambda - \lambda_r(\omega)} - \frac{1}{\lambda - \lambda_\theta(\omega)} \right) \, d\lambda \]

\[ = \text{Hol}(\omega) + \frac{1}{\lambda_r(\omega) - \lambda_\theta(\omega)} S_{r_0}(\omega, \lambda_\theta(\omega)) \otimes S_{\theta_0}(\omega, \lambda_\theta(\omega)) \]

\[ = \text{Hol}(\omega) + \frac{i(1 \otimes 1)}{4\pi(1 + \alpha)(r_+^2 + r_-^2 + 2a^2)\omega}. \]

Now, let us consider the case \(|\omega| > \varepsilon_\omega, \text{Im}\, \omega > 0\). We will apply Proposition 3.4 with \(l_1 = |\omega|^2/C_{1r}, l_2 = |\omega| \text{Im}\, \omega/C_{1r}\). Here \(C_{1r}\) is the constant in Proposition 3.2. Then (3.15) is true for small \(a\) and (3.17) follows from (3.16) for small \(a\) by part 5 of Proposition 3.2, with \(C_1 = C_{1r}/(|\omega| \text{Im}\, \omega)\). It remains to use (3.18).

Finally, assume that \(\omega\) is a real \(k\)-resonance and \(|\omega| > \varepsilon_\omega\). Then by Proposition 3.3 and part 3 of Proposition 3.2, if \(a\) is small enough, then the operator \(R_r(\omega, \cdot, \cdot, k)\) cannot have a pole for \(\lambda \in \mathbb{R}\). However, the operator \(P_\theta(\omega)\) is self-adjoint and thus only has real eigenvalues, a contradiction. □

### 4. Construction of the radial resolvent

In this section, we prove Proposition 3.2 except for part 2, which is proved in Section 6. We start with a change of variables that maps \((r_-, r_+)\) to \((-\infty, \infty)\):

**Proposition 4.1.** Define \(x = x(r)\) by

\[ x = \int_{r_0}^r \frac{ds}{\Delta_r(s)}. \]

(Here \(r_0 \in (r_-, r_+)\) is a fixed number.) Then there exists a constant \(X_0\) such that for \(\pm x > X_0\), we have \(r = r_\pm + F_\pm(e^{\mp A_\pm x}), \) where \(F_\pm(w)\) are real analytic on \([0, e^{-A_\pm X_0}]\) and holomorphic in the discs \(|w| < e^{-A_\pm X_0}\) ⊂ \(\mathbb{C}\).

**Proof.** We concentrate on the behavior of \(x\) near \(r_+\). It is easy to see that \(-A_+ x(r) = \ln(r_+ - r) + G(r),\) where \(G\) is holomorphic near \(r = r_+\). Exponentiating, we get

\[ w = e^{-A_+ x} = (r_+ - r)e^{G(r)}. \]

It remains to apply the inverse function theorem to solve for \(r\) as a function of \(w\) near zero. □
After the change of variables $r \to x$, we get

$$P_x(\omega, k) + \lambda = \Delta_r^{-1} P_x(\omega, \lambda, k),$$

$$P_x(\omega, k) = D_x^2 + V_x(x; \omega, \lambda, k),$$

$$V_x = \lambda \Delta_r - (1 + \alpha)^2 ((r^2 + a^2) \omega - ak)^2. \quad (4.2)$$

(We treat $r$ and $\Delta_r$ as functions of $x$ now.) We put

$$\omega_{\pm} = (1 + \alpha) ((r_\pm^2 + a^2) \omega - ak), \quad (4.3)$$

so that $V_x(\pm \infty) = -\omega_{\pm}^2$. Also, by Proposition 4.1, we get

$$V_x(x) = V_{\pm}(e^{\mp A_x}), \quad \pm x > X_0, \quad (4.4)$$

where $V_{\pm}(w)$ are functions holomorphic in the discs $\{|w| < e^{-A_{\pm}X_0}\}$.

We now define outgoing functions:

**Definition 4.1.** Fix $\omega, k, \lambda$. A function $u(x)$ (and the corresponding function of $r$) is called outgoing at $\pm \infty$ iff

$$u(x) = e^{\pm i\omega_{\pm}x} v_{\pm}(e^{\mp A_x}), \quad (4.5)$$

where $v_{\pm}(w)$ are holomorphic in a neighborhood of zero. We call $u(x)$ outgoing if it is outgoing at both infinities.

Let us construct certain solutions outgoing at one of the infinities:

**Proposition 4.2.** There exist solutions $u_{\pm}(x; \omega, \lambda, k)$ to the equation $P_x u_{\pm} = 0$ of the form

$$u_{\pm}(x; \omega, \lambda, k) = e^{\pm i\omega_{\pm}x} v_{\pm}(e^{\mp A_x}; \omega, \lambda, k),$$

where $v_{\pm}(w; \omega, \lambda, k)$ is holomorphic in $\{|w| < W_{\pm}\}$ and

$$v_{\pm}(0; \omega, \lambda, k) = \frac{1}{\Gamma(1 - 2i\omega_{\pm}A_{\pm}^{-1})}. \quad (4.6)$$

These solutions are holomorphic in $(\omega, \lambda)$ and are unique unless $n = 2i\omega_{\pm}A_{\pm}^{-1}$ is a positive integer.

**Proof.** We only construct the function $u_+$. Let us write the Taylor series for $v_+$ at zero:

$$v_+(w) = \sum_{j \geq 0} v_j w^j.$$  

Put $w = e^{-A_+ x}$; then the equation $P_x u_+ = 0$ is equivalent to

$$(A_+ w D_w - \omega_+)^2 + V_x) v_+ = 0.$$  

By (4.2) and Proposition 4.1, $V_x$ is a holomorphic function of $w$ for $|w| < W_+$. If $V_x = \sum_{j \geq 0} V_j w^j$ is the corresponding Taylor series, then we get the following system of linear equations on the coefficients $v_j$:

$$j A_+ (2i\omega_+ - j A_+) v_j + \sum_{0 \leq l \leq j} V_l v_{j-l} = 0, \quad j > 0. \quad (4.7)$$
If \( \nu \) is not a positive integer, then this system has a unique solution under the condition \( v_0 = \Gamma(1 - \nu)^{-1} \). This solution can be uniquely holomorphically continued to include the cases when \( \nu \) is a positive integer. Indeed, one defines the coefficients \( v_0, \ldots, v_\nu \) by Cramer’s Rule using the first \( \nu \) equations in (4.7) (this can be done since the zeroes of the determinant of the corresponding matrix match the poles of the gamma function), and the rest are uniquely determined by the remaining equations in the system (4.7).

We now prove that the series above converges in the disc \( \{|w| < W_+\} \). We take \( \varepsilon > 0 \); then \( |V_j| \leq M(W_+ - \varepsilon)^{-j} \) for some constant \( M \). Then one can use induction and (4.7) to see that \( |v_j| \leq C(W_+ - \varepsilon)^{-j} \) for some constant \( C \). Therefore, the Taylor series for \( v \) converges in the disc \( \{|w| < W_+ - \varepsilon\} \); since \( \varepsilon \) was arbitrary, we are done. \( \Box \)

The condition (4.6) makes it possible for \( u_\pm \) to be zero for certain values of \( \omega_\pm \). However, we have the following

**Proposition 4.3.** Assume that one of the solutions \( u_\pm \) is identically zero. Then every solution \( u \) to the equation \( P_x u = 0 \) is outgoing at the corresponding infinity.

**Proof.** Assume that \( u_+(x; \omega_0, \lambda_0, k_0) \equiv 0 \). (The argument for \( u_- \) is similar.) Put \( \nu = 2i\omega_0 A_+^{-1} \); by (4.6), it has to be a positive integer. Similarly to Proposition 4.2 we can construct a nonzero solution \( u_1 \) to the equation \( P_x u_1 = 0 \) with

\[
    u_1(x) = e^{-i\omega_0 x} \tilde{v}_1(e^{-A_+ x})
\]

and \( \tilde{v}_1 \) holomorphic at zero. We can see that \( u_1(x) = e^{i\omega_0 x} v_1(e^{-A_+ x}) \), where \( v_1(w) = w^\nu \tilde{v}_1(w) \) is holomorphic; therefore, \( u_1 \) is outgoing. Note that \( u_1(x) = o(e^{i\omega_0 x}) \) as \( x \to +\infty \).

Now, since \( u_+(x; \omega_0, \lambda_0, k_0) \equiv 0 \), we can define

\[
    u_2(x) = \lim_{\omega \to \omega_0} \frac{1}{\Gamma(1 - 2i\omega A_+^{-1})} u_+(x; \omega, \lambda_0, k_0);
\]

it will be an outgoing solution to the equation \( P_x u_2 = 0 \) and have \( u_2(x) = e^{i\omega_0 x}(1 + o(1)) \) as \( x \to +\infty \). We have constructed two linearly independent outgoing solutions to the equation \( P_x u = 0 \); since this equation only has a two-dimensional space of solutions, every its solution must be outgoing. \( \Box \)

The next statement follows directly from the definition of an outgoing solution and will be used in later sections:

**Proposition 4.4.** Fix \( \delta_r > 0 \) and let \( K_r \) be the image of the set \( K_r = (r_- + \delta_r, r_+ - \delta_r) \) under the change of variables \( r \to x \). Assume that \( X_0 \) is chosen large enough so that Proposition 4.1 holds and \( K_r \subset (-X_0, X_0) \). Let \( u(x) \in H^2_{\text{loc}}(\mathbb{R}) \) be any outgoing function in the sense of Definition 4.1 and assume that \( f = P_x u \) is supported in \( K_r \). Then:

1. \( u \) can be extended holomorphically to the two half-planes \( \{ \pm \Re z > X_0 \} \) and satisfies the equation \( P_z u = 0 \) in these half-planes, where \( P_z = \bar{D}_z^2 + V_z(z) \) and \( V_z(z) \) is well-defined by (4.4).
2. If $\gamma$ is a contour in the complex plane given by $\text{Im } z = F(\text{Re } z)$, $x_- \leq \text{Re } z \leq x_+$, and $F(x) = 0$ for $|x| \leq X_0$, then we can define the restriction to $\gamma$ of the holomorphic extension of $u$ by

$$u_\gamma(x) = u(x + iF(x))$$

and $u_\gamma$ satisfies the equation $P_\gamma u_\gamma = f$, where

$$P_\gamma = \left(\frac{1}{1 + iF'(x)D_x}\right)^2 + V_x(x + iF(x)).$$

3. Assume that $\gamma$ is as above, with $x_\pm = \pm \infty$, and $F'(x) = c = \text{const}$ for large $|x|$. Then $u_\gamma(x) = O(e^{\pm i\text{Im}(1 + ic)\omega_\pm x})$ as $x \to \pm \infty$. As a consequence, if $\text{Im}(1 + ic)\omega_\pm > 0$, then $u_\gamma(x) \in H^2(\mathbb{R})$.

We are now ready to prove Proposition 3.2.

Proof of part 1. Given the functions $u_\pm$, define the operator $S_x(\omega, \lambda, k)$ on $\mathbb{R}$ by its Schwartz kernel

$$S_x(x, x'; \omega, \lambda, k) = u_+(x)u_-(x') [x > x'] + u_-(x)u_+(x') [x < x'].$$

The operator $S_x(\omega, \lambda)$ acts $L^2_{\text{comp}}(\mathbb{R}) \to H^2_{\text{loc}}(\mathbb{R})$ and $P_xS_x = W(\omega, \lambda, k)$, where the Wronskian

$$W(\omega, \lambda, k) = u_+(x; \omega, \lambda, k) \cdot \partial_x u_-(x; \omega, \lambda, k) - u_-(x; \omega, \lambda, k) \cdot \partial_x u_+(x; \omega, \lambda, k)$$

is constant in $x$. Moreover, $W(\omega, \lambda, k) = 0$ if and only if $u_+(x; \omega, \lambda, k)$ and $u_-(x; \omega, \lambda, k)$ are linearly dependent as functions of $x$. Also, the image of $S_x$ consists of outgoing functions.

Now, we define the radial resolvent $R_x(\omega, \lambda, k) = R_x(\omega, \lambda, k)\Delta_x$, where

$$R_x(\omega, \lambda, k) = \frac{S_x(\omega, \lambda, k)}{W(\omega, \lambda, k)}.$$  (4.8)

It is clear that $R_x$ is a meromorphic family of operators $L^2_{\text{comp}} \to H^2_{\text{loc}}$ and $(P_x + \lambda)R_x$ is the identity operator. We now prove that $R_x$, and thus $R_x$, has poles of finite rank. Fix $k$ and take $(\omega_0, \lambda_0) \in \{W = 0\}$; we need to prove that for every $l$, the principal part of the Laurent decomposition of $\partial^l_\omega R_x(\omega_0, \lambda, k)$ at $\lambda = \lambda_0$ consists of finite-dimensional operators. We use induction on $l$. One has $P_x(\omega, \lambda, k)R_x(\omega, \lambda, k) = 1$; differentiating this identity $l$ times in $\omega$, we get

$$P_x(\omega_0, \lambda, k)\partial^l_\omega R_x(\omega_0, \lambda, k) = \delta_{l0}1 + \sum_{m=1}^l c_m \partial^m_\omega P_x(\omega_0, \lambda, k)\partial^{l-m}_\omega R_x(\omega_0, \lambda, k).$$

(Here $c_{ml}$ are some constants.) The right-hand side has poles of finite rank by the induction hypothesis. Now, consider the Laurent decomposition

$$\partial^l_\omega R_x(\omega_0, \lambda, k) = Q(\lambda) + \sum_{j=1}^N \frac{R_j}{(\lambda - \lambda_0)^j}.$$
Here $Q$ is holomorphic at $\lambda_0$. Multiplying by $P_x$, we get
\begin{equation*}
\sum_{j=1}^{N} \frac{P_x(\omega_0, \lambda, k)R_j}{(\lambda - \lambda_0)^j} \sim \sum_{j=1}^{N} \frac{L_j}{(\lambda - \lambda_0)^j}
\end{equation*}
up to operators holomorphic at $\lambda_0$. Here $L_j$ are some finite-dimensional operators. We then have
\begin{equation*}
P_x(\omega_0, \lambda_0, k)R_N = L_N,
\end{equation*}
\begin{equation*}
P_x(\omega_0, \lambda_0, k)R_{N-1} = L_{N-1} - (\partial_{\lambda} P_x(\omega_0, \lambda_0, k))R_N, \ldots
\end{equation*}
Each of the right-hand sides has finite rank and the kernel of $P_x(\omega_0, \lambda_0, k)$ is two-dimensional; therefore, each $R_j$ is finite-dimensional as required. (We also see immediately that the image of each $R_j$ consists of smooth functions.)

Finally, we establish the decomposition at zero. As in part 1 of Proposition 3.1 it suffices to compute $S_x(0, 0, 0)$ and the first order terms in the Taylor expansion of $W$ at $(0, 0, 0)$. We have $u_\pm(x; 0, 0, 0) = 1$ for all $x$; therefore, $S_x(x, x'; 0, 0, 0) = 1$. Next, put $\omega_\pm(x) = \partial_{\omega} u_\pm(x; 0, 0, 0)$ and $u_{\lambda\pm}(x) = \partial_{\lambda} u_\pm(x; 0, 0, 0)$. By differentiating the equation
\begin{equation*}
P_x u_\pm = 0 \text{ in } \omega \text{ and } \lambda \text{ and recalling the boundary conditions at } \pm \infty,
\end{equation*}
we get
\begin{equation*}
\partial_x^2 u_{\lambda \pm}(x) = \Delta_r,
\end{equation*}
\begin{equation*}
u_{\lambda \pm}(x) = v_{\lambda \pm}(e^{A_\pm x}), \quad \pm x \gg 0;
\end{equation*}
\begin{equation*}
\partial_x^2 u_{\omega \pm}(x) = 0,
\end{equation*}
\begin{equation*}
u_{\omega \pm}(x) = \pm i(1 + \alpha)(r_\pm^2 + a^2) x + v_{\omega \pm}(e^{A_\pm x}), \quad \pm x \gg 0,
\end{equation*}
for some functions $v_{\lambda \pm}, v_{\omega \pm}$ real analytic at zero. We then find
\begin{equation*}
\partial_\lambda W(0, 0, 0) = \partial_x(u_{-\lambda} - u_{+\lambda}) = \int_{-\infty}^{\infty} \Delta_r \, dx = r_+ - r_-,
\end{equation*}
\begin{equation*}
\partial_\omega W(0, 0, 0) = \partial_x(u_{-\omega} - u_{+\omega}) = -i(1 + \alpha)(r_+^2 + r_-^2 + 2a^2). \quad \square
\end{equation*}

**Proof of part 3.** Assume that $\omega$ and $\lambda$ are both real and $R_r$ has a pole at $(\omega, \lambda, k)$. Let $u(x)$ be the corresponding resonant state; we know that it has the asymptotics
\begin{equation*}
u_\pm(x) = e^{\pm i \omega_\pm x} U_\pm(1 + O(e^{A_\pm x})), \quad x \to \pm \infty;
\end{equation*}
\begin{equation*}
\partial_x u_\pm(x) = e^{\pm i \omega_\pm x} U_\pm(\pm i \omega_\pm + O(e^{A_\pm x})), \quad x \to \pm \infty
\end{equation*}
for some nonzero constants $U_\pm$. Since $V_x(x; \omega, \lambda, k)$ is real-valued, both $u$ and $\bar{u}$ solve the equation $(D_x^2 + V_x(x)) u = 0$. Then the Wronskian $W_u(x) = u \cdot \partial_x \bar{u} - \bar{u} \cdot \partial_x u$ must be constant; however,
\begin{equation*}
W_u(x) \to \mp 2i \omega_\pm |U_\pm|^2 \text{ as } x \to \pm \infty.
\end{equation*}
Then we must have $\omega_+ \omega_- \leq 0$; it follows immediately that $|\omega| = O(|a k|)$.

\square
Proof of part 4. First, assume that neither of $u_{\pm}$ is identically zero. Then the resolvent $R_x$, and thus $R_x$, has a pole iff the functions $u_{\pm}$ are linearly dependent, or, in other words, if there exists a nonzero outgoing solution $u(x)$ to the equation $P_x u = 0$. Now, if one of $u_{\pm}$, say, $u_+$, is identically zero, then by Proposition 4.3, $u_-$ will be an outgoing solution at both infinities. □

Proof of part 5. Assume that $u(x)$ is outgoing and $P_x(\omega, \lambda, k)u = f \in L^2(K_x)$. Since $\Im \omega > 0$, we have $\Im \omega_{\pm} > 0$ and thus $u \in H^2(\mathbb{R})$.

First, assume that $|\arg \omega - \pi/2| < \varepsilon$, where $\varepsilon > 0$ is a constant to be chosen later. Then

$$\Re V_x(x) = (1 + \alpha)^2(r^2 + a^2)^2(\Im \omega)^2 + \Re \lambda \cdot \Delta_r - (1 + \alpha)^2(r^2 + a^2)\Re \omega - ak)^2;$$

using (3.10), we can choose $\varepsilon$ and $C_{1r}$ so that $\Re V_x(x) \geq |\omega|^2/C > 0$ for all $x \in \mathbb{R}$. Then

$$\|u\|_{L^2(\mathbb{R})} \cdot \|f\|_{L^2(\mathbb{R})} \geq \Re \int \bar{u}(x)(D_x^2 + V_x(x))u(x) \, dx$$

$$\geq \int \Re V_x(x)|u|^2 \, dx \geq C^{-1}|\omega|^2\|u\|_{L^2(\mathbb{R})}^2$$

and (3.11) follows.

Now, assume that $|\arg \omega - \pi/2| \geq \varepsilon$. Then

$$\Im V_x(x) = -2(1 + \alpha)^2(r^2 + a^2)\Re \omega - ak)(r^2 + a^2)\Im \omega + \Im \lambda \cdot \Delta_r;$$

it follows from (3.10) that we can choose $C_{1r}$ so that the sign of $\Im V_x(x)$ is constant in $x$ (positive if $\arg \omega > \pi/2$ and negative otherwise) and, in fact, $|\Im V_x(x)| \geq |\omega|\Im \omega/C > 0$ for all $x$. Then (assuming that $\Im V_x(x) > 0$)

$$\|u\|_{L^2(\mathbb{R})} \cdot \|f\|_{L^2(\mathbb{R})} \geq \Im \int \bar{u}(x)(D_x^2 + V_x(x))u(x) \, dx$$

$$= \int \Im V_x(x)|u|^2 \, dx \geq C^{-1}|\omega|\Im \omega\|u\|_{L^2(\mathbb{R})}^2$$

and (3.11) follows. □

5. Preliminaries from semiclassical analysis

In this section, we list certain facts from semiclassical analysis needed in the further analysis of our radial operator. For a general introduction to semiclassical analysis, the reader is referred to [17].

Let $a(x, \xi)$ belong to the symbol class

$$S^m = \{a(x, \xi) \in C^\infty(\mathbb{R}^2) \mid \sup_{x, \xi} (\xi|^{\beta} - m |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \text{ for all } \alpha, \beta\}. $$
Here \( m \in \mathbb{R} \) and \( \langle \xi \rangle = \sqrt{1 + |\xi|^2} \). Following \cite{17} Section 8.6, we define the corresponding semiclassical pseudodifferential operator \( a^w(x, hD_x) \) by the formula

\[
a^w(x, hD_x)u(x) = \frac{1}{2\pi h} \int e^{\frac{i}{h}(x-y)\eta} a\left(\frac{x+y}{2}, \eta\right) u(y) \, dyd\eta.
\]

Here \( h > 0 \) is the semiclassical parameter. We denote by \( \Psi^m \) the class of all semiclassical pseudodifferential operators with symbols in \( S^m \). Introduce the semiclassical Sobolev spaces \( H^s_h \subset \mathcal{D}'(\mathbb{R}) \) with the norm \( \|u\|_{H^s_h} = \|\langle hD_x \rangle^s u\|_{L^2} \); then for \( a \in S^m \), we have

\[
\|a^w(x, hD_x)\|_{H^s_h \to H^{s-m}_h} \leq C,
\]

where \( C \) is a constant depending on \( a \), but not on \( h \). Also, if \( a(x, \xi) \in C_0^\infty(\mathbb{R}^2) \), then

\[
\|a^w(x, hD_x)\|_{L^2(\mathbb{R}) \to L^\infty(\mathbb{R})} \leq C h^{-1/2},
\]

where \( C \) is a constant depending on \( a \), but not on \( h \). (See \cite{17} Theorem 7.10 for the proof.)

General facts on multiplication of pseudodifferential operators can be found in \cite{17} Section 8.6. We will need the following: for each \( u \in \mathcal{D}(\mathbb{R}) \),

if \( \text{supp } a \cap \text{supp } b = \emptyset \), then \( a^w(x, hD_x)b^w(x, hD_x) = \mathcal{O}_{L^2 \to H^0_h}(h^\infty) \) for all \( N \); \( a^w(x, hD_x)b^w(x, hD_x) = (ab)^w(x, hD_x) + \mathcal{O}_{\Psi^{m+n-1}}(h) \), \( [a^w(x, hD_x), b^w(x, hD_x)] = -ih\{a, b\}^w(x, hD_x) + \mathcal{O}_{\Psi^{m+n-2}}(h^2) \).

Here \( \{\cdot, \cdot\} \) is the Poisson bracket, defined by \( \{a, b\} = \partial_x a \cdot \partial_x b - \partial_x b \cdot \partial_x a \). Also, if \( A \in \Psi^m \), then the adjoint operator \( A^* \) also lies in \( \Psi^m \) and its symbol is the complex conjugate of the symbol of \( A \).

One can study pseudodifferential operators on manifolds \cite{17} Appendix E], and in particular on the circle \( \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z} \). If \( a(x, \xi) = a(\xi) \) is a symbol on \( T^*\mathbb{S}^1 \) that is independent of \( x \), then \( a^w(hD_x) \) is a Fourier series multiplier modulo \( \mathcal{O}(h^\infty) \): for each \( N \),

if \( u(x) = \sum_{j \in \mathbb{Z}} u_j e^{ijx} \), then \( a(hD_x)u(x) = \sum_{j \in \mathbb{Z}} a(hj)u_j e^{ijx} + \mathcal{O}_{H^N_h}(h^\infty) \|u\|_{L^2} \).

In the next three propositions, we assume that \( P(h) \in \Psi^m \) and \( P(h) = p^w(x, hD_x) + \mathcal{O}_{\Psi^{m-1}}(h) \), where \( p(x, \xi) \in \mathcal{S}^m \).

\textbf{Proposition 5.1. (Elliptic estimate)} Suppose that the function \( \chi \in \mathcal{S}^0 \) is chosen so that \( |p| \geq \langle \xi \rangle^m / C > 0 \) on \( \text{supp } \chi \) for some \( h \)-independent constant \( C \). Also, assume that either the set \( \text{supp } \chi \) or its complement is precompact. Then there exists a constant \( C_1 \) such that for each \( u \in H^m_h \),

\[
\|\chi^w(x, hD_x)u\|_{H^m_h} \leq C_1 \|P(h)u\|_{L^2} + \mathcal{O}(h^\infty) \|u\|_{L^2}.
\]

\textsuperscript{5}We write \( A(h) = O_X(h^k) \) for some Fréchet space \( X \), if for each seminorm \( \| \cdot \|_X \) of \( X \), there exists a constant \( C \) such that \( \|A(h)\|_X \leq Ch^k \). We write \( A(h) = O_X(h^\infty) \) if \( A(h) = O_X(h^k) \) for all \( k \).
Proof. The proof follows the standard parametrix construction. We find a sequence of symbols \( q_j(x, \xi; h) \in S^{-m-j}, j \geq 0 \), such that for

\[
Q_N(h) = \sum_{0 \leq j \leq N} h^j q_j^w(x, hD_x),
\]

we get

\[
(Q_N(h) P(h) - 1) \chi^w(x, hD_x) = O_{\Psi^{-N-1}}(h^{N+1}); \quad (5.7)
\]

applying this operator equation to \( u \), we prove the proposition.

We can take any \( q_0 \in \Psi^{-m} \) such that \( q_0 = p^{-1} \) near supp \( \chi \); such a symbol exists under our assumptions. The rest of \( q_j \) can be constructed by induction using the equation (5.7). □

Proposition 5.2. (Gårding inequalities) Suppose that \( \chi \in C_0^\infty(\mathbb{R}^2) \).

1. If \( \text{Re} p \geq 0 \) near supp \( \chi \), then there exists a constant \( C \) such that for every \( u \in L^2 \),

\[
\text{Re}(P(h)\chi^w u, \chi^w u) \geq -Ch\|\chi^w u\|_{L^2}^2 - O(h^\infty)\|u\|_{L^2}^2. \quad (5.8)
\]

2. If \( \text{Re} p \geq 2\varepsilon > 0 \) near supp \( \chi \) for some constant \( \varepsilon > 0 \), then for \( h \) small enough and every \( u \in L^2 \),

\[
\text{Re}(P(h)\chi^w u, \chi^w u) \geq \varepsilon\|\chi^w u\|_{L^2}^2 - O(h^\infty)\|u\|_{L^2}^2. \quad (5.9)
\]

Proof. 1. Take \( \chi_1 \in C_0^\infty(\mathbb{R}^2; \mathbb{R}) \) such that \( \chi_1 = 1 \) near supp \( \chi \), but \( \text{Re} p \geq 0 \) near supp \( \chi_1 \). Then, apply the standard sharp Gårding inequality [17, Theorem 4.24] to the operator \( \chi_1^w P(h) \chi_1^w \) and the function \( \chi^w u \), and use [5.2].

2. Apply part 1 of this proposition to the operator \( P(h) - 2\varepsilon \). □

Proposition 5.3. (Exponentiation of pseudodifferential operators) Assume that \( G \in C_0^\infty(\mathbb{R}^2), s \in \mathbb{R} \), and define the operator \( e^{sG^w} : L^2 \to L^2 \) as

\[
e^{sG^w} = \sum_{j \geq 0} \frac{(sG^w)^j}{j!}.
\]

Assume that \( |s| \) is bounded by an \( h \)-independent constant. Then:

1. \( e^{sG^w} \in \Psi^0 \) is a pseudodifferential operator.
2. \( e^{sG^w} P(h)e^{-sG^w} = P(h) + ish(H_pG)^w + O_{L^2 \to L^2}(h^2) \).

Proof. 1. See for example [17, Theorem 8.3] (with \( m(x, \xi) = 1 \)). The full symbol of \( e^{sG^w} \) can be recovered from the evolution equation satisfied by this family of operators; we see that it is equal to 1 outside of a compact set.

2. It suffices to differentiate both sides of the equation in \( s \), divide them by \( h \), and compare the principal symbols. □
6. Analysis near the zero energy

In this section, we prove part 2 of Proposition 3.2. Take $h > 0$ such that $\text{Re} \lambda = h^{-2}$. Put

$$\tilde{\mu} = h \text{Im} \lambda, \; \tilde{k} = hk, \; \tilde{\omega} = h\omega, \; \tilde{\omega}_\pm = h\omega_\pm;$$

then (3.7) implies that

$$|\mu| \leq \varepsilon_r, \; |a \tilde{k}| \leq \varepsilon_r, \; |\tilde{\omega}| \leq \varepsilon_r, \; |\tilde{\omega}_\pm| \leq \varepsilon_r,$$

where $\varepsilon_r > 0$ and $h$ can be made arbitrarily small by choice of $C_r$ and $\psi$. If $P_x$ is the operator in (4.2), then $P_x = h^{-2}\tilde{P}_x$, where

$$\tilde{P}_x(h; \tilde{\omega}, \tilde{\mu}, \tilde{k}) = h^2 D_x^2 + \tilde{V}_x(x; \tilde{\omega}, \tilde{\mu}, \tilde{k}),$$

$$\tilde{V}_x(x; \tilde{\omega}, \tilde{\mu}, \tilde{k}) = (1 + i\tilde{\mu})\Delta_x - (1 + \alpha)^2((r^2 + a^2)\tilde{\omega} - a\tilde{k})^2.$$ 

Now, we use Proposition 4.4. Let $u$ be an outgoing function in the sense of Definition 4.1 and assume that $f = \tilde{P}_x u$ is supported in $K_x$. Then $u$ satisfies (4.5) for $|x| > X_0$ and some functions $v_\pm$. Fix $x_+ > X_0$ and consider the function

$$v_1(y) = v_+(e^{-A_+ (x_+ + iy)}; \omega, \lambda, k), \; y \in \mathbb{R}. \quad (6.2)$$

This is a $2\pi/A_+$-periodic function; we can think of it as a function on the circle. It follows from the differential equation satisfied by $v_+$ together with Cauchy-Riemann equations that $Q(h)v_1(y) = 0$, where

$$Q(h; \tilde{\omega}, \tilde{\mu}, \tilde{k}) = (-i h D_y + \tilde{\omega}_+)^2 + \tilde{V}_x(x_+ + iy; \tilde{\omega}, \tilde{\mu}, \tilde{k}).$$

Let $q(y, \eta)$ be the semiclassical symbol of $Q$:

$$q(y, \eta) = (-i\eta + \tilde{\omega}_+)^2 + \tilde{V}_x(x_+ + iy).$$

For small $h$, the function $v_1(y)$ has to be (semiclassically) microlocalized on the set $\{q = 0\}$. Since the symbol $q$ is complex-valued, in a generic situation this set will consist of isolated points. Also, since $v_1$ is the restriction to a certain circle of the function $v_+$, which is holomorphic inside this circle, it is microlocalized in $\{\eta \leq 0\}$. Therefore, if the equation $q(y, \eta) = 0$ has only one root with $\eta \leq 0$, then the function $v_1$ has to be microlocalized at this root. If furthermore $q$ satisfies Hörmander’s hypoellipticity condition, one can obtain an asymptotic decomposition of $v_1$ in powers of $h$. We will only need a weak corollary of such decomposition; here is a self-contained proof of the required estimates:

**Proposition 6.1.** Assume that $x_+ > X_0$ is chosen so that:

- the equation $q(y, \eta) = 0$, $y \in S^1$, has exactly one root $(y_0, \eta_0)$ such that $\eta_0 < 0$;
- the equation $q(y, \eta) = 0$ has no roots with $\eta = 0$;
- the condition $i\{q, \bar{q}\} < 0$ is satisfied at $(y_0, \eta_0)$;
- $\text{Re}(\eta_0 + i\tilde{\omega}_+) < 0$. 

Proof. (6.3) Therefore, we can decompose 1 = \chi + \chi_+ + \chi_0, where \chi_+ depends only on the \eta variable, is supported in \{\eta > 0\}, and is equal to 1 for large positive \eta and near every root of the equation q(y, \eta) = 0 with \eta > 0. Since \chi_+ is holomorphic at zero, its Taylor series provides the Fourier series for \chi_+; it then follows from (6.5) that

\|
(1 - \chi^w(y, hD_y))v_1\|_{\mathcal{H}_h^N} = O(h^\infty)\|v_1\|_{L^2},
\]

(6.3)
\]

\|
(hD_y - \eta(y))v_1\|_{\mathcal{H}_h^N} = O(h)\|v_1\|_{L^2},
\]

(6.4)
\]

\|
v_1\|_{L^2} \leq Ch^{1/4}\|v_1(y_0)\|,
\]

(6.5)
\]

\|
(hD_y - \eta_0)v_1(y_0)\| \leq Ch^{1/2}\|v_1\|_{L^2},
\]

(6.6)
\]

\|
Re\left(\frac{h\partial_x u_+(x + iy_0)}{u_+(x + iy_0)}\right) \leq -\frac{1}{C} < 0.
\]

(6.7)
\]

Similar statements are true for \chi_+ replaced by \chi_-, with the opposite inequality sign in (6.7).

Proof. (6.3): We know that

\inf\{\eta \mid q(y, \eta) = 0, (y, \eta) \neq (y_0, \eta_0)\} > 0.
\]

Therefore, we can decompose 1 = \chi + \chi_+ + \chi_0, where \chi_+ depends only on the \eta variable, is supported in \{\eta > 0\}, and is equal to 1 for large positive \eta and near every root of the equation q(y, \eta) = 0 with \eta > 0. Since \chi_+ is holomorphic at zero, its Taylor series provides the Fourier series for \chi_+; it then follows from (6.5) that

\|
\chi^w_+(y, hD_y)v_1\|_{\mathcal{H}_h^N} = O(h^\infty)\|v_1\|_{L^2}.
\]

Next, the symbol q is elliptic near supp \chi_0; therefore, by Proposition 5.1 (whose proof applies without changes to our case), since Q(h)v_1 = 0, we have

\|
\chi^w_0(y, hD_y)v_1\|_{\mathcal{H}_h^N} = O(h^\infty)\|v_1\|_{L^2}.
\]

This finishes the proof.

(6.4): Take a small cutoff \chi as above, and factor q = (\eta - \eta(y))q_1, where q_1(y, \eta) is nonzero near supp \chi. We then find a compactly supported symbol r_1 with r_1q_1 = 1 near supp \chi. Now, we have

\|
\chi^w(y, hD_y)(r_1^w(y, hD_y)q_1^w(y, hD_y) - 1)(hD_y - \eta(y))v_1\|_{\mathcal{H}_h^N} = O(h)\|v_1\|_{L^2},
\]

(6.4)
\]

\|
(1 - \chi^w(y, hD_y))(r_1^w(y, hD_y)q_1^w(y, hD_y) - 1)(hD_y - \eta(y))v_1\|_{\mathcal{H}_h^N} = O(h^\infty)\|v_1\|_{L^2},
\]

(6.4)
\]

\|
r_1^w(y, hD_y)(q_1^w(y, hD_y)(hD_y - \eta(y)) - Q(h))v_1\|_{\mathcal{H}_h^N} = O(h)\|v_1\|_{L^2}.
\]

(6.5)
\]

It remains to add these up.

(6.5): We cut off v_1 to make it supported in a small \varepsilon-neighborhood of y_0. Put \phi = (h\partial_{y_y} - i\eta(y))v_1; we know that \|\phi\|_{L^2} \leq Ch\|v_1\|_{L^2}. Now, put

\phi(y) = \int_{y_0}^y \eta(y') dy'.
\]
The condition \(i\{q, \bar{q}\}_{(y_0, \eta_0)} < 0\) is equivalent to
\[
\text{Im} \partial_y \eta(y_0) > 0;
\]
it follows that
\[
\text{Im}(\Phi(y) - \Phi(y')) \geq \beta((y - y_0)^2 - (y' - y_0)^2)
\]
for some \(\beta > 0\), \(|y - y_0| < \varepsilon\), and \(y'\) between \(y\) and \(y_0\). (To see that, represent the left-hand side as an integral.) Now,
\[
v_1(y) = e^{i\Phi(y)/h}v_1(y_0) + h^{-1} \int_{y_0}^{y} e^{i(\Phi(y) - \Phi(y'))/h} f(y') \, dy'.
\]
Let \(Tf(y)\) be the second term in the sum above; it suffices to prove that
\[
\|Tf\|_{L^2(y_0 - \varepsilon, y_0 + \varepsilon)} \leq C h^{-1/2} \|f\|_{L^2(y_0 - \varepsilon, y_0 + \varepsilon)}.
\]
This can be reduced to the inequalities
\[
\sup_{0 \leq y - y_0 < \varepsilon} \int_{y_0}^{y} |e^{i(\Phi(y) - \Phi(y'))/h}| \, dy' = O(h^{1/2}),
\]
\[
\sup_{0 \leq y' - y_0 < \varepsilon} \int_{y_0 + \varepsilon}^{y_0} |e^{i(\Phi(y) - \Phi(y'))/h}| \, dy = O(h^{1/2}),
\]
and similar inequalities for the case \(y, y' < y_0\). We now use (6.8); after a change of variables, it suffices to prove that
\[
\sup_{y > 0} \int_{y}^{y_0} e^{(y')^2 - y^2} \, dy' < \infty, \sup_{y' > 0} \int_{y'}^{\infty} e^{(y')^2 - y^2} \, dy < \infty.
\]
To prove the first of these inequalities, make the change of variables \(y' = ys\); then the integral becomes
\[
\int_{0}^{1} ye^{ys^2(1 - 1)} \, ds.
\]
However, \(ye^{ys^2(1 - s^2)} \leq C(1 - s^2)^{-1/2}\), and the integral of the latter converges.

After the change of variables \(y = y' + s\), the integral of the second inequality above becomes
\[
\int_{0}^{\infty} e^{-2ys - s^2} \, ds.
\]
This can be estimated by \(\int e^{-s^2} \, ds\).

(6.6): Let \(\chi \in C_0^\infty(\mathbb{R}^2)\) have \(\chi = 1\) near \((y_0, \eta_0)\). Combining (5.1) and (6.4) with
\[
\|1 - \chi w(y, hD_y)(hD_y - \eta(y)) v_1\|_{L^\infty} = O(h^\infty) \|v_1\|_{L^2},
\]
we get \(\|hD_y - \eta(y) v_1\|_{L^\infty} = O(h^{1/2}) \|v_1\|_{L^2}\); it remains to take \(y = y_0\).

(6.7): Follows immediately from (6.5), (6.6), (6.2), Cauchy-Riemann equations, and the fact that \(\text{Re}(\eta_0 + i\tilde{\omega}+) < 0\). □
If $\tilde{P}_x$ were a semiclassical Schrödinger operator with a strictly positive potential, then a standard integration by parts argument would give us $\|u\|_{L^2} \leq C\|\tilde{P}_x u\|_{L^2}$ on any interval for each function $u$ satisfying the condition (6.7) at the right endpoint of this interval and the opposite condition at its left endpoint. We now generalize this argument to our case. Assume that we have vertical control at the points $x_\pm$, $\pm x_\pm > X_0$, and let $(y_\pm, \eta_\pm)$ be the corresponding microlocalization points. Let $\gamma$ be a contour in the $z$ plane; we say that we have horizontal control on $\gamma$ if:

- $\gamma \cap \{|\text{Re } z| \leq X_0\} \subset \mathbb{R}$;
- the endpoints of $\gamma$ are $z_\pm = x_\pm + iy_\pm$;
- $\gamma$ is given by $\text{Im } z = F(\text{Re } z)$, where $F$ is a smooth function and $F'(x_\pm) = 0$;
- $\text{Re}[(1 + iF'(x_\pm))\tilde{V}_x(x + iF(x))] \geq \frac{1}{C_1} > 0$ for all $x$.

Now, let $u$ be as in the beginning of this section and define $u_\gamma(x)$, $x_- \leq x \leq x_+$, by Proposition 4.4. Then $\tilde{P}_\gamma u_\gamma = f$, where

$$\tilde{P}_\gamma = \left( \frac{1}{1 + iF'(x)} hD_x \right)^2 + \tilde{V}_x(x + iF(x)).$$

If we have vertical control at the endpoints of $\gamma$, then by (6.7),

$$\pm \text{Re}(\bar{u}_\gamma(x_\pm)h\partial_x u_\gamma(x_\pm)) \leq -|u_\gamma(x_\pm)|^2/C < 0.$$

Now, assume that we have horizontal control on $\gamma$. Then we can integrate by parts to get

$$\int_{x_-}^{x_+} \text{Re}(\bar{u}_\gamma(1 + iF'(x))f) \, dx = \int_{x_-}^{x_+} \text{Re}(\bar{u}_\gamma \cdot (1 + iF'(x))\tilde{P}_\gamma u_\gamma) \, dx$$

$$= \int_{x_-}^{x_+} \text{Re} \left[ \frac{hD_x u_\gamma}{1 + iF'(x)} \right] \, dx + \int_{x_-}^{x_+} \text{Re}[(1 + iF'(x))\tilde{V}_x(x + iF(x))] \cdot |u_\gamma|^2 \, dx$$

$$- h^2 \text{Re}(\bar{u}_\gamma \partial_x u_\gamma)|_{x=x_-}^{x_+} \geq \frac{1}{C_1} (\|u_\gamma\|^2_{L^2} + h(\|u_\gamma(x_+)\|^2 + |u_\gamma(x_-)|^2)).$$

Therefore,

$$\|u_\gamma\|_{L^2} \leq C\|f\|_{L^2}.$$
It follows that the operator $R_x$ from (4.8) is correctly defined and
\[ \| K_x R_x K_x \|_{L^2 \to L^2} \leq Ch^2, \]
This proves the estimate (3.8) under the assumptions made above.

We now prove (3.9). We concentrate on the estimate on $K_+$; the case of $K_-$ is considered in a similar fashion. First of all, it follows from (6.9) that
\[ |u_\gamma(x_+)| \leq Ch^{-1/2}\|f\|_{L^2}. \tag{6.10} \]

Now, assume that we have vertical control at every point of the interval $I_+ = [x_+, x_+ + 1]$ and let $(y(x), \eta(x))$, $x \in I_+$, be the corresponding microlocalization points. Let $v_x(z) = e^{-i\omega_+ z}u(z)$ and put $v_2(x) = v_x(x + iy(x))$; then
\[ |v_2(x_+)| \leq C e^{\Im(\omega_+ z_+)} |u_\gamma(x_+)|. \tag{6.11} \]
Now, by Proposition 6.1, we have
\[ |h\partial_x \ln |v_2(x)| - \eta(x)| \leq Ch^{3/4}, \quad x \in I_. \tag{6.12} \]
Integrating (6.12) and combining it with (6.10) and (6.11), we see that if
\[ \Im(\tilde{\omega}_+ z_+) + \int_{x_+}^{x_+ + 1} \eta(x) \, dx < -2\delta_0 \tag{6.13} \]
for some $\delta_0 > 0$, then $|v_2(x_+ + 1)| \leq C e^{-\delta_0/h}\|f\|_{L^2}$. Next, $v_2(x_+ + 1)$ is the value of $v$ at the microlocalization point; therefore, by Proposition 6.1 and (5.1),
\[ \sup_{y \in \mathbb{R}} |v_x(x_+ + 1 + iy)| \leq Ch^{-1/4} e^{-\delta_0/h}\|f\|_{L^2}. \]
Finally, recall that $v_x(z) = v_w(e^{-A_+ z})$, where the function $v_w(w)$ is holomorphic inside the disc $B_w = \{|w| \leq e^{-A_+(x_+ + 1)}\}$. The change of variables $w \to r$ is holomorphic by Proposition 4.1; let $K^w_+$ be the image of $K_+$ under this change of variables. If $\delta r_0$ is small enough, then $K^w_+$ lies in the interior of $B_w$; then by the maximum principle and Cauchy estimates on derivatives, we can estimate $\|v_w\|_{C^N(K^w_+)}$ for each $N$ by $O(h^\infty)\|f\|_{L^2}$. This completes the proof of (3.9) if the conditions above are satisfied.

To prove part 2 of Proposition 3.2, it remains to establish both vertical and horizontal control in our situation:

**Proposition 6.2.** Assume that $\delta_\epsilon > 0$. Then there exist $\epsilon_\epsilon$ and $x_\pm$, $\pm x_\pm > X_0$, such that under the conditions (6.1),
- we have vertical control at every point of the intervals $I_+ = [x_+, x_+ + 1]$ and $I_- = [x_--1, x_-]$;
- we have horizontal control on a certain contour $\gamma$;
- the inequality (6.13) (and its analogue on $I_-$) holds.
Proposition 7.1. Fix $a, k, \tilde{\omega}, \tilde{\eta}$ such that $\tilde{\omega}_\pm = 0$ and $q(y, \eta) = -\eta^2 + \Delta_r(x_+ + iy)$. Therefore, if we choose $x_+$ large enough, there exists exactly one solution $(y_0, \eta_0)$ to the equation $q(y, \eta) = 0$ with $\eta \leq 0$, and this solution has $y_0 = 0$. It is easy to verify that in that case we have vertical control on $I_+$. Similarly one can choose the point $x_-; moreover, we can assume that $K_r \subset (x_-, x_+)$ after the change of variables $r \to x$. Next, since $\tilde{V}_x = \Delta_r$, we can take $\gamma$ to be the interval $[x_-, x_+]$ of the real line. The condition \eqref{6.13} holds because $\eta(x) < 0$ for every $x$ and $\tilde{\omega}_\pm = 0$.

Now, fix $x_\pm$ as above. The parameters of our problem are $a$, varying in a compact set, $\Lambda$ and $M$, both fixed, and $ak, \tilde{\mu}, \tilde{\omega}$. By the implicit function theorem, if the last three parameters are small enough, the (open) conditions of vertical control and the condition \eqref{6.13} are still satisfied, yielding $y_{\pm}$ close to zero. Then one can take the contour $\gamma$ defined by $\text{Im } z = F(\text{Re } z)$, where $F = 0$ near $K_r$, $F(x_{\pm}) = y_{\pm}$, and $F$ is small in $C^\infty$. For small values of $ak, \tilde{\mu}, \tilde{\omega}$, we will still have horizontal control on this $\gamma$, proving the proposition. $\square$

7. Resonance free strip

In this section, we prove Theorem 5. First of all, by Proposition 3.4, it suffices to prove

**Proposition 7.1.** Fix $\delta_r > 0, \varepsilon_\varepsilon > 0$, and a large constant $C'$. Then there exist constants $a_0 > 0$ and $C''$ such that for
\[
|\text{Re } \lambda| + k^2 \leq C'|\text{Re } \omega|^2, \quad |a| < a_0, \quad |\text{Re } \omega| \geq 1/C'',
\]
\[
|\text{Im } \omega| \leq 1/C'', \quad |\text{Im } \lambda| \leq |\text{Re } \omega|/C''
\]
we have
\[
\|1_{K_r} R_r(\omega, \lambda, k) 1_{K_r}\|_{L^2 \to L^2} \leq C'' \omega^{1 - \varepsilon_\varepsilon}.
\]

Indeed, we take $C'$ large enough so that $C''(1 + |\omega|^2 + L \leq C'|\omega|^2/2$; then, we put $l_1 = L$ and $l_2 = \|\text{Re } \omega\|/C''$.

Next, we reformulate Proposition 7.1 in semiclassical terms. Without loss of generality, we may assume that $\text{Re } \omega > 0$. Put $h = (\text{Re } \omega)^{-1}$ and consider the rescaled operator
\[
\tilde{P}_x = h^2 P_x = h^2 D_x^2 + (\tilde{\lambda} + ih\tilde{\mu})\Delta_r - (1 + \alpha)^2((r^2 + a^2)(1 + ih\nu) - a\tilde{k})^2.
\]
Here $P_x$ is the operator in \eqref{4.2} and
\[
\tilde{\lambda} = h^2 \text{Re } \lambda, \quad \tilde{k} = hk, \quad \tilde{\mu} = h \text{Im } \lambda, \quad \nu = \text{Im } \omega.
\]
Then it suffices to prove that for $h$ small enough and under the conditions
\[
|\tilde{\lambda}| \leq C', \quad |\tilde{k}| \leq C', \quad |\tilde{\mu}| \leq 1/C'', \quad |\nu| \leq 1/C'', \quad (7.1)
\]
for each $f(x) \in L^2 \cap \mathcal{E}'(K_\varepsilon)$ and solution $u(x)$ to the equation $\tilde{P}_x u = f$ which is outgoing in the sense of Definition 4.1, we have
\[
\|u\|_{L^2(K_\varepsilon)} \leq Ch^{-1-\varepsilon_\varepsilon}\|f\|_{L^2}. \quad (7.2)
\]
Figure 4. The contour used for complex scaling

(Here $K_x$ is the image of $K_r = (r_− + δ_r, r_+ − δ_r)$ under the change of variables $r → x$.) We write $\tilde{P}_x = h^2 D_x^2 + \tilde{V}_0 + ih\tilde{V}_1$, where

$$\tilde{V}_0 = \tilde{\lambda}\Delta_r - (1 + \alpha)^2(r^2 + a^2 - a\tilde{k})^2,$$

$$\tilde{V}_1 = \tilde{\mu}\Delta_r - \nu(1 + \alpha)^2(r^2 + a^2)(r^2 + a^2)(2 + ih\nu) - 2a\tilde{k})^2.$$

We note that $\tilde{V}_0$ is real-valued and $\|\tilde{V}_1\|_{L∞} ≤ C/C''$ for some global constant $C$.

We now apply the method of complex scaling. (This method was first developed by Aguilar and Combes in [1]; see [32] and the references there for more recent developments.) Consider the contour $γ$ in the complex plane given by $\text{Im } x = F(\text{Re } x)$, with $F$ defined by

$$F(x) = \begin{cases} 0, & |x| ≤ R; \\ F_0(x - R), & x ≥ R; \\ -F_0(-x - R), & x ≤ −R. \end{cases} (7.3)$$

Here $R > X_0$ is large and $F_0 ∈ C^∞_0(0, ∞)$ is a fixed function such that $F_0' ≥ 0$ and $F_0'' ≥ 0$ for all $x$ and $F_0'(x) = 1$ for $x ≥ 1$. (We could use a contour which forms an arbitrary fixed angle $\tilde{\theta} ∈ (0, π/2)$ with the horizontal axis for large $x$; we choose the angle $π/4$ to simplify the formulas.) Now, let $u$ be an outgoing solution to the equation $\tilde{P}_x u = f ∈ L^2 \cap \mathcal{E}'(K_x)$, as above. By Proposition 4.4, we can define the restriction $u_γ$ of $u$ to $γ$ and $\tilde{P}_γ u_γ = f$, where

$$\tilde{P}_γ = \left(\frac{h}{1 + IF'(x)}D_x\right)^2 + \tilde{V}_0(x + IF(x)) + ih\tilde{V}_1(x + IF(x)).$$

Also, for $a$ and $h$ small enough, $u_γ$ lies in $H^2(\mathbb{R})$. Therefore, in order to prove (7.2), it is enough to show that for each $u_γ ∈ H^2(\mathbb{R})$, we have

$$\|u_γ\|_{L^2(\mathbb{R})} ≤ Ch^{-1 - \varepsilon}\|\tilde{P}_γ u_γ\|_{L^2(\mathbb{R})}. \quad (7.4)$$
Let $p_0$ and $p_\gamma$ be the semiclassical principal symbols of $\hat{P}_x$ and $\hat{P}_\gamma$:

$$p_0(x, \xi) = \xi^2 + \tilde{V}_0(x),$$

$$p_\gamma(x, \xi) = \frac{\xi^2}{(1 + iF'(x))^2} + \tilde{V}_0(x + iF(x)).$$

The key property of the operator $\hat{P}_\gamma$, as opposed to $\hat{P}_x$, is ellipticity at infinity, which follows from the fact that $\tilde{V}_0(\pm \infty) = -\tilde{\omega}_0^2$, where

$$\tilde{\omega}_0^2 = (1 + \alpha)(r_+^2 + a^2 - a\dot{k}) \geq 1/C > 0$$

if $a$ is small enough. Certain other properties of the symbol $p_\gamma$ can be derived using only the behavior of $\tilde{V}_0$ near infinity given by (4.4); we state them for a general class of potentials:

**Proposition 7.2.** Assume that $V(x)$, $x > 0$, is a real-valued potential such that for $x > X_0$, we have $V(x) = V_+(e^{-A_+x})$ for a certain constant $A_+ > 0$ and a function $V_+(w)$ holomorphic in $\{|w| < e^{-A_+x_0}\}$; assume also that $V_+(0) < 0$. Let $F(x)$ be as in (7.3), for $R > X_0$, and put

$$p(x, \xi) = \xi^2 + V(x),$$

$$p_\gamma(x, \xi) = \frac{\xi^2}{(1 + iF'(x))^2} + V(x + iF(x)).$$

Then there exists a constant $C_c$ such that for $R$ large enough and $\delta > 0$ small enough,

$$\text{if } x \geq R + 1, \text{ then } |p_\gamma(x, \xi)| \geq 1/C_c > 0, \tag{7.5}$$

$$\text{if } |p_\gamma(x, \xi)| \leq e^{-A_+R}, \text{ then } \text{Im } p_\gamma(x, \xi) \leq 0, \tag{7.6}$$

$$\text{if } |p_\gamma(x, \xi)| \leq \delta, \text{ then } |p(x, \xi)| \leq C_c\delta, \text{ } |\nabla(\text{Re } p_\gamma - p)(x, \xi)| \leq C_c\delta. \tag{7.7}$$

Similar facts hold if $V$ is defined on $x < 0$ instead.

**Proof.** Without loss of generality, we assume that $A_+ = 1$ and $V_+(0) = -1$. First of all, if $x \geq R + 1$, then

$$p_\gamma(x, \xi) = -i\xi^2/2 + V(x + iF(x)) = -i\xi^2/2 - 1 + O(e^{-R}).$$

For $R$ large enough, we then get $|p_\gamma(x, \xi)| \geq 1/2$, thus proving (7.5).

For the rest of the proof, we may assume that $R \leq x \leq R + 1$. Then, since $F_0'$ is increasing, we get $0 \leq F(x) \leq F'(x)$. Suppose that $|p_\gamma(x, \xi)| \leq \delta$; then

$$\frac{\xi^2}{(1 + iF'(x))^2} = -V(x + iF(x)) + O(\delta) \tag{7.8}$$

$$= -V(x)(1 + O(\delta + e^{-R}F(x))) = 1 + O(\delta + e^{-R}).$$

Taking the arguments of both sides, we get

$$F'(x) \leq C(\delta + e^{-R}F(x)) \leq C\delta + Ce^{-R}F'(x).$$
Then for $R$ large enough,

$$|p_\gamma(x, \xi)| \leq \delta \to F'(x) \leq C\delta.$$  

This proves (7.7), if we note that $F''$ is bounded and

$$\text{Re} \ p_\gamma(x, \xi) - p(x, \xi) = \xi^2G_1(F'(x)^2) + G_2(F(x), x)$$

for certain smooth functions $G_1$ and $G_2$ that are equal to zero at $F' = 0$ and $F = 0$, respectively.

Now, putting $\delta = e^{-R}$ and taking the arguments and then the absolute values of both sides of (7.8), we get for $|p_\gamma| \leq \delta$,

$$F'(x) = O(e^{-R}), \quad \xi^2 = 1 + O(e^{-R}).$$

Therefore,

$$\text{Im} \ p_\gamma(x, \xi) = -2F'(x) + O(e^{-R}(F(x) + F'(x))) = F'(x)(-2 + O(e^{-R})),$$

which proves (7.6). \hfill \square

Now, we study the trapping properties of the Hamiltonian flow of $p_0$ at the zero energy:

**Proposition 7.3.** There exist constants $C_V$ and $\delta_V$ such that for a small enough and every $\tilde{\lambda}, \tilde{k}$ satisfying (7.1), at least one of the three dynamical cases below holds:

1. $\tilde{V}_0 \leq -\delta_V$ everywhere;
2. $\{|\tilde{V}_0| \leq \delta_V\} = [x_1, x_2] \sqcup [x_3, x_4]$, where $-C_V \leq x_1 < x_2 < x_3 < x_4 \leq C_V$ and $\tilde{V}_0 \geq 1/C_V$ on $[x_1, x_2]$, $\tilde{V}_0 \leq -1/C_V$ on $[x_3, x_4]$;
3. $\{\tilde{V}_0 \geq -\delta_V\} = [x_1, x_2]$ with $|x_j| \leq C_V$, $\tilde{V}_0'' \leq -1/C_V$ on $[x_1, x_2]$.

**Proof.** First of all, if $\tilde{\lambda}$ is small enough or $\tilde{\lambda} < 0$, then we have $\tilde{V}_0 < 0$ everywhere and therefore case (1) holds for $\delta_V$ small enough. Therefore, we may assume that $1/C \leq \tilde{\lambda} \leq C$ for some constant $C$. Now, we write

$$\tilde{V}_0(x) = G_V(r)(F_V(r) - \tilde{\lambda}^{-1}),$$

$$G_V(r) = \tilde{\lambda}(1 + \alpha)^2(r^2 + a^2 - a\tilde{k})^2,$$

$$F_V(r) = \frac{\Delta r}{(1 + \alpha)^2(r^2 + a^2 - a\tilde{k})^2}.$$  

Note that $1/C \leq G_V(r) \leq C$ for $a$ small enough, some constant $C$, and all $r$. As for $F_V$, there exists $\varepsilon > 0$ such that for $a$ small enough, $\partial_r F_V(r) \geq 1/C > 0$ for $r \leq 3M - \varepsilon$, $\partial_r F_V(r) \leq -1/C < 0$ for $r \geq 3M + \varepsilon$, and $\partial^2_r F_V(r) \leq -1/C < 0$ for $|r - 3M| \leq \varepsilon$. Indeed, this is true for $a = 0$ and follows for small $a$ by a perturbation argument. Let $r_0 \in [3M - \varepsilon, 3M + \varepsilon]$ be the point where $F_V$ achieves its maximal value. Take small $\delta_1 > 0$; then we have one of the following three cases, each of which in turn implies the corresponding case in the statement of this proposition:
that Proposition 7.2 holds. The first two cases are in Proposition 7.3 are nontrapping; it 
see [18, Proposition A.6]. Now, given Proposition 7.2, we can choose
where $U$
value; therefore, there still exists an escape function $G$ consisting of a single point ($\delta$
set consisting of a single point ($\delta$
Figure 5. Three cases for the potential $V_0$ and its Hamiltonian flow near 
the zero energy. The horizontal lines correspond to $V_0 = \pm \delta_V$.

(1) $F_V(r_0) - \tilde{\lambda}^{-1} \leq -\delta_1$. Then $V_0(x) < -\delta_V$ for all $x$ and $\delta_V > 0$ small enough.

(2) $F_V(r_0) - \tilde{\lambda}^{-1} \geq \delta_1$. Then for $\delta_2 < \delta_1/2$, $\{ |F_V - \tilde{\lambda}^{-1}| \leq \delta_2 \} = [x_1, x_2] \cup [x_3, x_4]$, where $x_2 < x_3$, $x_j$ are bounded by a global constant (since $\tilde{\lambda}$ is bounded from above), and $\partial_r F_V(r) > 1/C_\delta > 0$ for $x \in [x_1, x_2]$, $\partial_r F_V(r) < -1/C_\delta < 0$ for $x \in [x_3, x_4]$. Here $C_\delta$ is a constant depending on $\delta_1$, but not on $\delta_2$. It follows that for $\delta_2$ small enough depending on $\delta_1$, we have $V''_0(x) > 0$ for $x \in [x_1, x_2]$ and $V''_0(x) < 0$ for $x \in [x_3, x_4]$; also, for $\delta_V$ small enough, we have $\{|V_0| \leq \delta_V \} \subset [x_1, x_2] \cup [x_3, x_4]$.

(3) $|F_V(r_0) - \tilde{\lambda}^{-1}| < \delta_1$. Then $\{ F_V - \tilde{\lambda}^{-1} > -\delta_1 \} = [x_1, x_2]$ with $\partial^2_r F_V(r) < -1/C < 0$ 
for $x \in [x_1, x_2]$. For $\delta_1$ small enough, we then get $V''_0 < -1/C < 0$ for $x \in [x_1, x_2]$, 
and for $\delta_V$ small enough, we have $\{ V_0 \geq -\delta_V \} \subset [x_1, x_2]$. □

We are now ready to prove (7.4) and, therefore, Theorem 5. Fix $R$ large enough so that Proposition 7.2 holds. The first two cases are in Proposition 7.3 are nontrapping; it 
follows that there exists an escape function $G \in C_0^\infty(\mathbb{R}^2)$ such that $H_{p_0} G < 0$ on $\{|p_0| \leq \delta_V/2 \} \cap \{|x| \leq R + 2 \}$. In the third case, we have hyperbolic trapping with the trapped 
set consisting of a single point $(x_0, 0)$, where $x_0$ is the point where $V_0$ achieves its maximal value; therefore, there still exists an escape function $G \in C_0^\infty(\mathbb{R}^2)$ such that $H_{p_0} G \leq 0$ on $\{|p_0| \leq \delta_V/2 \} \cap \{|x| \leq R + 2 \}$ and $H_{p_0} G < 0$ on $\{|p_0| \leq \delta_V/2 \} \cap \{|x| \leq R + 2 \} \setminus U(x_0, 0)$, where $U$ is a neighborhood of $(x_0, 0)$ which can be made arbitrarily small by the choice of $G$ (see [8 Proposition A.6]). Now, given Proposition 7.2, we can choose $\delta_0 > 0$ such that

$$\text{Im } p_{r_0} \leq 0 \text{ on } \{|p_{r_0}| \leq \delta_0\} \quad (7.9)$$
and for cases (1) and (2) of Proposition 7.3 we have
\[
H_{\text{Re}p_0} G \leq -1/C < 0 \quad \text{on} \quad \{|p_{\gamma_0}| \leq \delta_0\},
\] (7.10)
and for case (3) of Proposition 7.3 we have
\[
H_{\text{Re}p_0} G \leq 0 \quad \text{on} \quad \{|p_{\gamma_0}| \leq \delta_0\},
\]
\[
H_{\text{Re}p_0} G \leq -1/C < 0 \quad \text{on} \quad \{|p_{\gamma_0}| \leq \delta_0\} \setminus U(x_0,0).
\] (7.11)

Armed with these inequalities, we can handle the nontrapping cases even without requiring that \(\mu\) and \(\nu\) be small. The statement below follows the method initially developed in [27] and is a special case of the results in [14, Chapter 6]; however, we choose to present the proof in our simple case:

**Proposition 7.4.** Assume that either case (1) or case (2) of Proposition 7.3 holds. Then for \(\lambda\) and \(k\) bounded by \(C', \bar{\mu}\) and \(\nu\) bounded by some constant, and \(h\) small enough, we have
\[
\|u_\gamma\|_{L^2} \leq C'h^{-1}\|\tilde{P}_\gamma u_\gamma\|_{L^2}
\] (7.12)
for each \(u_\gamma \in H^2(\mathbb{R})\).

**Proof.** Take \(\chi \in C_0^\infty(\mathbb{R}^2)\) such that \(\text{supp} \chi \subset \{|p_{\gamma_0}| < \delta_0\}\), but \(\chi = 1\) near \(\{p_{\gamma_0} = 0\}\). Next, take \(s > 0\), to be chosen later, and put
\[
\tilde{P}_{\gamma,s} = e^{scw} \tilde{\rho} e^{-sGw}, \quad u_{\gamma,s} = e^{sGw} \chi w u_\gamma.
\]
Take \(\chi_1 \in C_0^\infty(\mathbb{R}^2)\) supported in \(\{|p_{\gamma_0}| < \delta_0\}\), but such that \(\chi_1 = 1\) near \(\text{supp} \chi\). Then by part 1 of Proposition 5.3 and (5.2),
\[
\|(1 - \chi_1 w) u_{\gamma,s}\| = O(h^\infty)\|u_\gamma\|.
\] (7.13)
(In the proof of the current proposition, as well as the next one, we only use \(L^2\) norms.) Also, for some \(s\)-dependent constant \(C\),
\[
C^{-1}\|\chi w u_s\| \leq \|u_{\gamma,s}\| \leq C\|u_\gamma\|.
\]
Now, by part 2 of Proposition 5.3 we have
\[
\tilde{P}_{\gamma,s} = \tilde{P}_{\gamma,0} + ihV_1 + ish(H_{\text{Re}p_{\gamma,0}} G)w + O(h^2).
\]
Here \(\tilde{P}_{\gamma,0}\) is the principal part of \(\tilde{P}_{\gamma}\) (without \(V_1\)) and the constant in \(O(h^2)\) depends on \(s\). We then have
\[
\text{Im}(\tilde{P}_{\gamma,s} \chi_1 w u_{\gamma,s}, \chi_1 w u_{\gamma,s}) = \text{Im}(\tilde{P}_{\gamma,0} \chi_1 w u_{\gamma,s}, \chi_1 w u_{\gamma,s}) + h \text{Re}(V_1 \chi_1 w u_{\gamma,s}, \chi_1 w u_{\gamma,s}) + h\text{Im}(H_{\text{Re}p_{\gamma,0}} G)w \chi_1 w u_{\gamma,s}, \chi_1 w u_{\gamma,s}) + O(h^2)\|\chi_1 w u_{\gamma,s}\|^2.
\]
By (7.9) and part 1 of Proposition 5.2
\[
\text{Im}(\tilde{P}_{\gamma,0} \chi_1 w u_{\gamma,s}, \chi_1 w u_{\gamma,s}) \leq C h\|\chi_1 w u_{\gamma,s}\|^2 + O(h^\infty)\|u_\gamma\|^2.
\]
Next, by (7.10) and part 2 of Proposition 5.2
\[
((H_{\text{Re}p_{\gamma,0}} G)w \chi_1 w u_{\gamma,s}, \chi_1 w u_{\gamma,s}) \leq -C^{-1}\|\chi_1 w u_{\gamma,s}\|^2 + O(h^\infty)\|u_\gamma\|^2.
\]
Adding these up, we get
\[
\text{Im}(\tilde{P}_{\gamma,s} w u_{\gamma,s}) \leq -h(C_1^{-1} s - C_1 - O(h))\|\chi_1^w u_{\gamma,s}\|^2 + O(h^\infty)\|u_{\gamma}\|^2.
\]
Here the constants in \(O(\cdot)\) depend on \(s\), but the constant \(C_1\) does not. Therefore, if we choose \(s\) large enough and \(h\)-independent, then for small \(h\) we have the estimate
\[
\|\chi_1^w u_{\gamma,s}\|^2 \leq C h^{-1} \|\tilde{P}_{\gamma,s} \chi_1^w u_{\gamma,s}\| \cdot \|\chi_1^w u_{\gamma,s}\| + O(h^\infty)\|u_{\gamma}\|^2.
\]
Together with (7.13), this gives
\[
\|u_{\gamma}\|^2 \leq C h^{-1 - \varepsilon_c} \|\tilde{P}_{\gamma} u_{\gamma}\| \cdot \|u_{\gamma}\| + C h^{-1} \|[\tilde{P}_{\gamma}, \chi^w] u_{\gamma}\| \cdot \|u_{\gamma}\| + O(h^\infty)\|u_{\gamma}\|^2.
\]
Applying Proposition 5.1 to estimate \((1 - \chi^w) u_{\gamma}\) and the commutator term above, we get the estimate (7.12).

\[\square\]

**Remark.** The method described above can actually be used to obtain a logarithmic resonance free region; however, since we expect the resonances generated by trapping to lie asymptotically on a lattice as in [31], we only go a fixed amount deep into the complex plane.

The third case in Proposition 7.3 is where trapping occurs, and we analyse it as in [38]; (See also [10] for a different method of solving the same problem.)

**Proposition 7.5.** Assume that case (3) in Proposition 7.3 holds, and fix \(\varepsilon_c > 0\). Then for \(\tilde{\lambda}\) and \(k\) bounded by \(C'\) and for \(\tilde{\mu}, \nu, h\) small enough, we have
\[
\|u_{\gamma}\|_{L^2} \leq C h^{-1 - \varepsilon_c} \|\tilde{P}_{\gamma} u_{\gamma}\|_{L^2}
\]
for each \(u_{\gamma} \in H^2(\mathbb{R})\).

**Proof.** First, we establish [38] Lemma 4.1] in our case. Let \(x_0\) be the point where \(\tilde{V}_0\) achieves its maximum value. We may assume that \(|p_0(x_0, 0)| = |\tilde{V}_0(x_0)| < \delta_0/2\); otherwise, we are in one of the two nontrapping cases. Put
\[
\tilde{\xi}(x) = \text{sgn}(x - x_0) \sqrt{\tilde{V}_0(x_0) - \tilde{V}_0(x)};
\]
since \(\tilde{V}_0''(x_0) < 0\), it is a smooth function. Then, define the functions \(\varphi \pm(x, \xi) = \xi \mp \tilde{\xi}(x)\). We have
\[
H_{p_0} \varphi \pm(x, \xi) = \mp c(x, \xi) \varphi \pm(x, \xi),
\]
where \(c(x, \xi) = 2\tilde{\xi}(x)\) is greater than zero near the trapped point \((x_0, 0)\). Also, \(\{\varphi_+, \varphi_-\} = c(x, \xi)\). Next, take \(\tilde{h} > h\) and large \(C_0 > 0\), let \(\chi_0 \geq 0\) be supported in a small neighborhood of \((x_0, 0)\) with \(\chi_0 = 1\) near this point, and define the modified escape function [38] (4.6]
\[
G_1(x, \xi) = -\chi_0(x, \xi) \log \frac{\varphi_-^2(x, \xi) + h/\tilde{h}}{\varphi_+^2(x, \xi) + h/\tilde{h}} + C_0 \log(1/h) G(x, \xi).
\]
Here $G$ is an escape function satisfying (7.11). We can write
\[
H_{ReP_{\gamma,0}} G_1 = -2\chi_0 c \left( \frac{\varphi_-^2}{\varphi_-^2 + h/h} + \frac{\varphi_+^2}{\varphi_+^2 + h/h} \right) - (H_{p_0} \chi_0) \log \frac{\varphi_-^2 + h/h}{\varphi_+^2 + h/h} + C_0 \log (1/h) H_{ReP_{\gamma,0}} G(x, \xi).
\] (7.15)

Take $\chi_1$ supported in $\{|p_{\gamma_0}| < \delta_0\}$, but equal to 1 near $\{p_{\gamma_0} = 0\}$. Then one can use the uncertainty principle [38, Section 4.2] to show that if $\chi_2$ is supported inside $\{\chi_0 = 1\}$, but $\chi_2 = 1$ near $(x_0, 0)$, then for each $v \in L^2$,
\[
((H_{ReP_{\gamma,0}} G_1)^w \chi_1^w v, \chi_1^w v) \leq (-C^{-1}\tilde{h} + O(h^2))||\chi_1^w v||^2 + O(\log (1/h))||(1 - \chi_2^w)\chi_1^w v||^2
\]
\[
-C_0 C^{-1} \log (1/h) ||(1 - \chi_2^w)\chi_1^w v||^2 + O(C_0 h \log (1/h))||\chi_1^w v||^2 + O(h^\infty)||v||^2
\]
\[
\leq -(C^{-1}\tilde{h} - O(h^2) + C_0 h \log (1/h)))||\chi_2^w||^2
\]
\[
-(C_0 C^{-1} \log (1/h) - O(C_0 h \log (1/h) + \log (1/h)))||(1 - \chi_2^w)\chi_1^w v||^2 + O(h^\infty)||v||^2.
\]

If we fix $C_0$ large enough and $\tilde{h}$ small enough and assume that $h$ small enough, then
\[
((H_{ReP_{\gamma,0}} G_1)^w \chi_1^w v, \chi_1^w v) \leq -C^{-1} \log (1/h) ||(1 - \chi_2^w)\chi_1^w v||^2 - C^{-1}\tilde{h} ||\chi_1^w v||^2 + O(h^\infty)||v||^2.
\]

Next, we conjugate by exponential pseudodifferential weights. First of all, one can prove that
\[
||G_1^w||_{L^2 \rightarrow L^2} \leq C \log (1/h);
\]
therefore,
\[
||e^{sG_1^w}||_{L^2 \rightarrow L^2} \leq h^{-C|s|}.
\]

Let $\chi$ be supported in $\{\chi_1 = 1\}$, but $\chi = 1$ near $\{p_{\gamma_0} = 0\}$, and
\[
P_{\gamma,s} = e^{sG_1^w} P_{\gamma} e^{-sG_1^w}, \quad u_{\gamma,s} = e^{sG_1^w} \chi^w u_{\gamma};
\]
then [38, Section 4.3]
\[
P_{\gamma,s} = P_{\gamma} + ish(H_{p_{\gamma_0}} G_1)^w + O(s^2 \tilde{h} + sh^{3/2} \tilde{h}^{3/2} + h^2).
\]

Therefore, since $\text{Im} \ p_{\gamma_0} = 0$ near $\text{supp} \ \chi_2$,
\[
\text{Im}(\tilde{P}_{\gamma,s} \chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) = \text{Im}(\tilde{P}_{\gamma_0} \chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) + h \text{Re}(V_1 \chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s})
\]
\[
+ sh \text{Re}((H_{ReP_{\gamma,0}} G_1)^w \chi_1^w u_{\gamma,s}, \chi_1^w u_{\gamma,s}) + O(s^2 \tilde{h} + sh^{3/2} \tilde{h}^{3/2} + h^2)||\chi_1^w u_{\gamma,s}||^2
\]
\[
\leq O(h)||1 - \chi_2^w|| \chi_1^w u_{\gamma,s}||^2 + h ||V_1||_{L^\infty} ||\chi_1^w u_{\gamma,s}||^2 - C^{-1} s \text{sh} \log (1/h) ||(1 - \chi_2^w)\chi_1^w u_{\gamma,s}||^2
\]
\[
-C^{-1} s \text{sh} \tilde{h} ||\chi_1^w u_{\gamma,s}||^2 + O(s^2 \tilde{h} + sh^{3/2} \tilde{h}^{3/2} + h^2)||\chi_1^w u_{\gamma,s}||^2 + O(h^\infty)||u_{\gamma}||^2.
\]
Here $\tilde{P}_{\gamma}$ is the principal part of $\tilde{P}_{\gamma}$, as before. If we choose $s$ small enough independently of $h$, then for small $h$,

$$\text{Im}(\tilde{P}_{\gamma,s}w_{\gamma,s,\chi}u_{\gamma,s}) \leq -C_{1}sh\log(1/h)\|1 - \chi_{2}w_{\gamma,s}\|^{2}$$

$$-h(C^{-1}s\tilde{h} - \|V_{1}\|_{L^{\infty}})\|\chi_{1}w_{\gamma,s}\|^{2} + O(h^{\infty})\|u_{\gamma}\|^{2}.$$ 

Now, $\|V_{1}\|_{L^{\infty}}$ can be made very small by choosing $\tilde{\mu}$ and $\nu$ small enough. Then, we get

$$\|\chi_{2}w_{\gamma}\|^{2} \leq C_{1}h^{-1-C_{s}}\|u_{\gamma}\| \cdot \|P_{2}\chi_{2}w_{\gamma}\| + O(h^{\infty})\|u_{\gamma}\|^{2}.$$ 

By proceeding as in the end of Proposition 7.4, we get (7.14), provided that $s$ is small enough. □

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QUAS-NORMAL MODES FOR KERR–DE SITTER

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