Finite automata for Schreier graphs of virtually free groups

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(Joint work with P. Silva, X. Soler-Escrivà)
1. The bijection between subgroups of $F_A$ and Stallings automata

2. Many applications

3. Moving out of free groups

4. Stallings sections

5. Virtually free groups
Outline

1. The bijection between subgroups of $F_A$ and Stallings automata
2. Many applications
3. Moving out of free groups
4. Stallings sections
5. Virtually free groups
Notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet ($n$ letters).
- $\tilde{A} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- Usually, $A = \{a, b, c\}$.
- $\tilde{A}^*$ the free monoid on $\tilde{A}$ (words on $A^\pm 1$).
- $1$ denotes the empty word, and $|\cdot|$ the length of words.
- $\sim$ is the eq. rel. generated by $a_i a_i^{-1} \sim a_i^{-1} a_i \sim 1$.
- $R_A = \{\text{reduced words}\} \subset \tilde{A}^*$.
- $\overline{w}$ is the reduced word for $w$.
- $F_A = \tilde{A}^*/\sim$ is the free group on $A$ (words on $A^\pm 1$ modulo $\sim$).
- $\pi: \tilde{A}^* \rightarrow F_A$ the natural projection (a morphisms of monoids).
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A Stallings automata is a finite $A$-labeled oriented graph with a distinguished vertex, $(X, v)$, such that:

1. $X$ is connected,
2. no vertex of degree 1 except possibly $v$ ($X$ is a core-graph),
3. no two edges with the same label go out of (or in to) the same vertex.

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Stallings (building on previous works) gave a bijection between finitely generated subgroups of $F_A$ and Stallings automata:

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\{\text{f.g. subgroups of } F_A \} \leftrightarrow \{\text{Stallings automata}\},
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Reading the subgroup from the automata

Definition

To any given (Stallings) automaton \((X, v)\), we associate its fundamental group:

\[ \pi(X, v) = \{ \text{labels of closed paths at } v \} \leq F_A, \]

clearly, a subgroup of \(F_A\).

\[ \pi(X, \bullet) = \{1, a, a^{-1}, bab, bc^{-1}b, \]
\[ babab^{-1}cb^{-1}, \ldots\} \]

\[ \pi(X, \bullet) \not\ni bc^{-1}bcaa \]

Membership problem in \(\pi(X, \bullet)\) is solvable.
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we can fold and identify vertices $u$ and $v$ to obtain

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Lemma (Stallings)

If \((X, v) \rightsquigarrow (X', v')\) is a Stallings folding then \(\pi(X, v) = \pi(X', v')\).

Given a f.g. subgroup \(H = \langle w_1, \ldots, w_m \rangle \leq F_A\) (we assume \(w_i\) are reduced words), do the following:

1. Draw the flower automaton,
2. Perform successive foldings until obtaining a Stallings automaton, denoted \(\Gamma(H)\).

Well defined?
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The bijection

**Lemma**

The automaton $\Gamma(H)$ does not depend on the sequence of foldings.

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The automaton $\Gamma(H)$ does not depend on the generators of $H$.

**Theorem**

The following is a bijection between f.g subgroups and Stallings automata:

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Corollary (Nielsen-Schreier)

Every subgroup of $F_A$ is free.

- We have proved the finitely generated case, but everything extends easily to the general case.
- The original proof (1920’s) is combinatorial and much more technical.
- Everything now is nicely algorithmic.
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Membership & containment

(Membership)

Does \( w \) belong to \( H = \langle w_1, \ldots, w_m \rangle \) ?

- Construct \( \Gamma(H) \),
- Check whether \( w \) is readable as a closed path in \( \Gamma(H) \) (at the basepoint).

(Containment)

Given \( H = \langle w_1, \ldots, w_m \rangle \) and \( K = \langle v_1, \ldots, v_n \rangle \), is \( H \leq K \) ?

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(Computing a basis)

Given $H = \langle w_1, \ldots, w_m \rangle$, find a basis for $H$.

- Construct $\Gamma(H)$,
- Choose a maximal tree,
- Read the corresponding basis.

(Conjugacy)

Given $H = \langle w_1, \ldots, w_m \rangle$ and $K = \langle v_1, \ldots, v_n \rangle$, are they conjugate (i.e. $H^x = K$ for some $x \in F_A$)?

- Construct $\Gamma(H)$ and $\Gamma(K)$,
- Check whether they are “equal” up to the basepoint.
- Every path between the two basepoints spells a valid $x$. 
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Enric Ventura (UPC)  Finite automata for Schreier graphs of virtually free groups  May 19th, 2012  22 / 40
Given $H = \langle w_1, \ldots, w_m \rangle$, we can decide whether $H \leq_{f.i.} F_A$; and, if yes, compute a set of coset representatives.

If $H \leq_{f.i.} F_A$ is of index $[F : H]$, then $r(H) = 1 + [F : H] \cdot (r(F_A) - 1)$.

Every f.g. subgroup $H \leq_{fg} F_A$ is a free factor of a finite index one, $H \leq_{ff} H * L \leq_{f.i.} F_A$. 
Finite index subgroups

(Finite index)

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(Schreier index formula)

If \( H \leq_{f.i.} F_A \) is of index \([F : H]\), then
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r(H) = 1 + [F : H] \cdot (r(F_A) - 1).\]

Theorem (M. Hall)

Every f.g. subgroup \( H \leq_{fg} F_A \) is a free factor of a finite index one, \( H \leq_{ff} H \ast L \leq_{f.i.} F_A \).
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Theorem (M. Hall)
Theorem (Howson)

The intersection of finitely generated subgroups of $F_A$ is again finitely generated.

Theorem

We can effectively compute a basis for $H \cap K$ from a set of generators for $H$ and from $K$.

Theorem (H. Neumann)

$\tilde{r}(H \cap K) \leq 2\tilde{r}(H)\tilde{r}(K)$, where $\tilde{r}(H) = \max\{0, r(H) - 1\}$. 
Intersection of subgroups

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$\tilde{r}(H \cap K) \leq 2\tilde{r}(H)\tilde{r}(K)$, where $\tilde{r}(H) = \max\{0, r(H) - 1\}$.
Our goal

Can we extend this to other families of groups $G = \langle A \mid R \rangle$?

- f.g. subgroups $H \leq G$ are not free in general,
- there exist subgroups $H \leq F_2 \times F_2$ with unsolvable membership problem,
- ... for general $G$ this is asking too much.

(Goal 1)

Put conditions to the presentation $G = \langle A \mid R \rangle$ to recreate the bijection with f.g. subgroups and the membership problem, algorithmically.

(Goal 2)

Identify which are the groups admitting such a presentation.
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The Schreier graph

Definition

The Schreier graph \( \Gamma(G, H, A) \) of a subgroup \( H \leq G = \langle A \mid R \rangle \) w.r.t. \( A \) is:

- vertices: left cosets of \( G \) modulo \( H \), \( V = \{Hg \mid g \in G\} \),
- edges: \( Hg \xrightarrow{a} Hga \), for \( g \in G \) and \( a \in A \),
- basepoint: \( H \cdot 1 \).

Note that \( \Gamma(G, H, A) \) is finite if and only if \( H \leq_{f.i.} G \).

Definition

The core of a graph \((\Gamma, v)\) is the smallest subgraph containing \( v \) and having the same fundamental group; i.e. \( c(\Gamma) \) is \( \Gamma \) after deletion of all "pending trees".
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The *core* of a graph \( (\Gamma, v) \) is the *smallest* subgraph containing \( v \) and having the same fundamental group; i.e. \( c(\Gamma) \) is \( \Gamma \) after deletion of all "pending trees".
The key observation

Observation

$\Gamma(H)$ is the core of the Schreier graph $\Gamma(F_A, H, A)$, for $H \leq F_A$.

(Key observation)

In the free case, $\Gamma(H)$ is the "central" part of $\Gamma(F_A, H, A)$, i.e. it is a part of $\Gamma(F_A, H, A)$ such that

- it is finite,
- it is computable from a set of generators for $H$,
- it is big enough to remember $H$.

(Finite groups)

If $G = \langle A | R \rangle$ is finite and $H \leq G$, then we can take $\Gamma(H)$ to be the whole $\Gamma(G, H, A)$...
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If G = ⟨A | R⟩ is finite and H ≤ G, then we can take Γ(H) to be the whole Γ(G, H, A)...
For all the talk, \( G = \langle A \mid R \rangle \) and \( \pi: \tilde{A}^* \rightarrow G \).

**Definition**

A section of \( \pi \) is a subset \( S \subseteq \tilde{A}^* \) such that \( S\pi = G \) and \( S^{-1} = S \).

**Definition**

Given a section \( S \subseteq \tilde{A}^* \) and \( H \leq_{f.g.} G \), define \( \Gamma(G, H, A) \cap S \) to be the smallest subgraph of \( \Gamma(G, H, A) \) where you can read all \( w \in S \) as closed paths at the basepoint.

**Observation**

In the free case, \( \pi: \tilde{A}^* \rightarrow F_A \), \( S = R_A \) is a section, and \( \Gamma(F_A, H, A) \cap S = \Gamma(H) \).
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1. The bijection between subgroups of $F_A$ and Stallings automata
2. Many applications
3. Moving out of free groups
4. Stallings sections
5. Virtually free groups
Stallings sections

**Definition**

A section \( S \subseteq \tilde{A}^* \) is a Stallings section if

(S0) \( S \) is a regular language and effectively computable,

(S1) \( \forall g \in G, \quad S_g = g\pi^{-1} \cap S \) is rational and effectively computable,

(S2) \( \forall g, h \in G, \quad S_{gh} \subseteq \overline{S_gS_h} \).

**Observation**

If \( A \) is an automaton and \( L \subseteq \tilde{A}^* \) is regular and effectively computable then \( A \cap L \) is regular and effectively computable.
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If $A$ is an automaton and $L \subseteq \tilde{A}^*$ is regular and effectively computable then $A \cap L$ is regular and effectively computable.
Proposition

For the free group $F_A = \langle A \mid - \rangle$, $S = R_A$ is a Stallings section.

Proof. $R_A \pi = F_A$ and $R_A^{-1} = R_A$.

(S0) $R_A$ is rational and effectively computable by

Theorem (Benois)

$L \subseteq \tilde{A}^*$ rational $\Rightarrow \overline{L} \subseteq \tilde{A}^*$ is rational and effectively computable.

(S1) $\forall g \in F_A$, $S_g = \{\overline{g}\}$ rational and effectively computable.

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Finite groups

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Proposition

Suppose $\langle A \mid R \rangle \simeq G \simeq \langle A' \mid R' \rangle$. Then, there exists a Stallings section for $\pi : \tilde{A}^* \to G$ if and only if there exists a Stallings section for $\pi' : \tilde{A'}^* \to G$.

Proof. Take a monoid morphism $\varphi : \tilde{A}^* \to \tilde{A'}^*$ such that $\varphi \pi' = \pi$. If $S$ is a Stallings section for $\pi : \tilde{A}^* \to G$, then $S\varphi$ will be a Stallings section for $\pi' : \tilde{A'}^* \to G$, and vice versa. □

So, existence of a Stallings section is a group property, independent of the presentation.
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Proof. Take a monoid morphism $\varphi : \tilde{A}^* \to \tilde{A}'^*$ such that $\varphi \pi' = \pi$.

If $S$ is a Stallings section for $\pi : \tilde{A}^* \to G$, then $S\varphi$ will be a Stallings section for $\pi' : \tilde{A}'^* \to G$, and viceversa. □

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Suppose $\langle A \mid R \rangle \simeq G \simeq \langle A' \mid R' \rangle$. Then, there exists a Stallings section for $\pi : \tilde{A}^* \to G$ if and only if there exists a Stallings section for $\pi' : \tilde{A}'^* \to G$.

Proof. Take a monoid morphism $\varphi : \tilde{A}^* \to \tilde{A}'^*$ such that $\varphi \pi' = \pi$. If $S$ is a Stallings section for $\pi : \tilde{A}^* \to G$, then $S\varphi$ will be a Stallings section for $\pi' : \tilde{A}'^* \to G$, and viceversa. □

So, existence of a Stallings section is a group property, independent of the presentation.
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So, existence of a Stallings section is a group property, independent of the presentation.
Lemma

Let $S$ be a Stallings section for $\pi : \tilde{A}^* \to G$, let $H \leq_{f.g.} G$, and let $A$ be an inverse automata such that

- $S_H \subseteq L(A) \subseteq H\pi^{-1}$,
- there is no path $p \xrightarrow{w} q$ with $p \neq q$ and $w\pi = 1$.

Then, $\Gamma(G, H, A) \cap S = A \cap S$.

Theorem

Let $S$ be a Stallings section for $\pi : \tilde{A}^* \to G$. For every $H \leq_{f.g.} G$, $\Gamma(G, H, A) \cap S$ is effectively computable and satisfies

$S_H \subseteq L(\Gamma(G, H, A) \cap S) \subseteq H\pi^{-1}$. 
Constructing $\Gamma(G, H, A) \cap S$

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Corollary

Let $S$ be a Stallings section for $\pi : \tilde{A}^* \to G$, $H \leq_{f.g.} G$, and $g \in G$. TFAE:

(a) $g \in H$,
(b) $S_g \subseteq L(\Gamma(G, H, A) \cap S)$,
(c) $S_g \cap L(\Gamma(G, H, A) \cap S) \neq \emptyset$.

Hence, the membership problem is solvable in $G$.

Proof.

(a) $\Rightarrow$ (b). If $g \in H$ then $S_g \subseteq S_H \subseteq L(\Gamma(G, H, A) \cap S)$.
(b) $\Rightarrow$ (c). $S_g \neq \emptyset$ because $S$ is a section.
(c) $\Rightarrow$ (a). Take $g \in S_g \cap L(\Gamma(G, H, A) \cap S)$ and we have $g = s_\pi \in H$.

The decidability comes from (S1) and the Theorem (and intersection of regular languages being regular and computable). □
Corollary

Let $S$ be a Stallings section for $\pi: \tilde{A}^* \rightarrow G$, $H \leq_f G$, and $g \in G$. TFAE:

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The bijection between subgroups of $F_A$ and Stallings automata

Many applications

Moving out of free groups

Stallings sections

Virtually free groups
**Theorem**

If $G_1$ and $G_2$ are groups with Stallings sections, and $H$ is a finite subgroup of both, then the amalgamated product $G_1 \ast_H G_2$ also admits a Stallings section.

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If $G$ is a group with a Stallings section and $K$ is a finite subgroup, then the HNN extension $G \ast_K$ also admits a Stallings section.

**Corollary**

Virtually free groups admit Stallings sections.
Amalgamation and HNN

After several quite technical computations...

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A finitely generated group $G$ admits a Stallings section if and only if $G$ is virtually free.

Proof.
Playing with a Stallings section we first prove that the word problem submonoid $1_{\pi^{-1}}$ is context-free.
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THANKS