Abstract. In this paper, we prove a $C^{1,1}$ estimate for solutions of complex Monge-Ampère equations on compact almost Hermitian manifolds. Using this $C^{1,1}$ estimate, we show existence of $C^{1,1}$ solutions to the degenerate Monge-Ampère equations, the corresponding Dirichlet problems and the singular Monge-Ampère equations. We also study the singularities of the pluricomplex Green’s function. In addition, the proof of the above $C^{1,1}$ estimate is valid for a kind of complex Monge-Ampère type equations. As a geometric application, we prove the $C^{1,1}$ regularity of geodesics in the space of Sasakian metrics.

1. Introduction

Let $(M, \omega, J)$ be a compact almost Hermitian manifold of real dimension $2n$. We use $g$ and $\nabla$ to denote the corresponding Riemannian metric and Levi-Civita connection. In this paper, we consider the following complex Monge-Ampère equation

$$
(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = f \omega^n,
$$

$$
\omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0, \quad \sup_M \varphi = 0,
$$

(1.1)

where $f$ is a positive smooth function on $M$. Here we use $\sqrt{-1} \partial \bar{\partial} \varphi$ to denote $\frac{1}{n} (dJd\varphi)^{(1,1)}$, which agrees with the standard notation when $J$ is integrable (see Section 2 for more explanations).

The complex Monge-Ampère equation plays a significant role in complex geometry. When $(M, \omega, J)$ is Kähler, Yau [46] solved Calabi’s conjecture (see [8]) by proving the existence of solutions to (1.1). This is known as the Calabi-Yau theorem, which states that one can prescribe the volume form of a Kähler metric within a given Kähler class. There are many corollaries and applications of this result.

It is very interesting to extend the Calabi-Yau theorem to non-Kähler settings. When $(M, \omega, J)$ is Hermitian, the complex Monge-Ampère equation has been studied under some assumptions on $\omega$ (see [11, 31, 27, 43, 47]). In [44], Tosatti-Weinkove solved (1.1) for any Hermitian metric $\omega$. 

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Recently, Chu-Tosatti-Weinkove [14] solved (1.1) on compact almost Hermitian manifolds. Unlike Kähler and Hermitian cases, almost Hermitian case is much more complicated. It is hard to obtain the complex Hessian estimate by the analogous computation. Instead, they considered a quantity involving the largest eigenvalue \( \lambda_1 \) of the real Hessian \( \nabla^2 \phi \). Combining the maximum principle and a series of delicate calculations, the real Hessian estimate was obtained. Following the approach of [14], Chu-Tosatti-Weinkove [15] established the existence of \( C^{1,1} \) solutions to the homogeneous complex Monge-Ampère equation and solved the open problem of \( C^{1,1} \) regularity of geodesics in the space of Kähler metrics (see [9]). Further applications of these ideas can be found in [42, 17, 16, 13].

However, the \( C^{1,1} \) estimate in [14] depends on \( \sup_M f, \sup_M |\partial (\log f)|_g \) and lower bound of \( \nabla^2 (\log f) \). Hence, except the homogeneous complex Monge-Ampère equation, it is impossible to apply this \( C^{1,1} \) estimate in the study of degenerate complex Monge-Ampère equation. Motivated by this, we prove the following estimate, which improve the above \( C^{1,1} \) estimate.

**Theorem 1.1.** Let \( \varphi \) be a smooth solution of (1.1). Then there exists a constant \( C \) depending only on \((M,\omega,J)\), \( \sup_M f, \sup_M |\partial (f^{\frac{1}{n}})|_g \) and lower bound of \( \nabla^2 (f^{\frac{1}{n}}) \) such that

\[
\sup_M |\varphi| + \sup_M |\partial \varphi|_g + \sup_M |\nabla^2 \varphi|_g \leq C,
\]

where \( \nabla \) is the Levi-Civita connection of \( g \).

In [3], Blocki proved the similar estimates when \((M,\omega,J)\) is a compact Kähler manifold with nonnegative bisectional curvature. We point out that the above \( C^{1,1} \) estimate does not depend on the upper bound of \( \nabla^2 (f^{\frac{1}{n}}) \), which is very important for Theorem 1.3 and 1.4. Actually, in the proofs of Theorem 1.3 and 1.4 we use \( (|s|_h^2 + i^{-1})^{\frac{1}{2}} \) to approximate \( |s|_h \). However, there is no uniform upper bound of \( \nabla^2 (|s|_h^2 + i^{-1})^{\frac{1}{2}} \) for \( i \geq 1 \).

In addition, if we replace \( f^{\frac{1}{n}} \) by \( f^{\frac{1}{m}} \) for \( m > n \), then Theorem 1.4 can be proved by the similar argument of [14]. But for \( f^{\frac{1}{n}} \), it is impossible to adapt the approach of [14]. Some new techniques and auxiliary functions are needed. Later, we will discuss the proof of Theorem 1.1 in details.

For the degenerate complex Monge-Ampère equation, it is well known that the solution may be only of class \( C^{1,1} \) and not higher. As an application of Theorem 1.1 we prove the existence of \( C^{1,1} \) solutions.

**Theorem 1.2.** Let \((M,\omega,J)\) be a compact almost Hermitian manifold of real dimension \( 2n \). Suppose that \( f \) is a nonnegative function on \( M \) such that

\[
\sup_M f \leq C, \sup_M |\partial (f^{\frac{1}{n}})|_g \leq C, \nabla^2 (f^{\frac{1}{n}}) \geq -Cg,
\]
for a constant $C$. If $f \not\equiv 0$, then there exists a pair $(\varphi, b)$ where $\varphi \in C^{1,1}(M)$ and $b \in \mathbb{R}$, such that

\begin{equation}
(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = f e^b \omega^n, \\
\omega + \sqrt{-1} \partial \bar{\partial} \varphi \geq 0, \quad \sup_M \varphi = 0.
\end{equation}

In Theorem 1.2, the function $f$ may not be $C^{1,1}$. The conclusion still holds when $f$ can be approximated by a sequence of smooth positive functions $f_i$ in the sense of $C^0$, such that

$$
\sup_M f_i \leq C, \quad \sup_M |\partial(f_i^n)|_g \leq C, \quad \nabla^2(f_i^n) \geq -Cg,
$$

for a constant $C$ which is independent of $i$.

As an application of Theorem 1.2, we show existence of $C^{1,1}$ solutions to the singular Monge-Ampère equations.

**Theorem 1.3.** Let $(M, \omega)$ be a compact $n$-dimensional Kähler manifold and $L$ be a line bundle with Hermitian metric $h$. Given a section $s$ of $L$, a function $F \in C^2(M)$ and $N \geq n$ such that

\begin{equation}
\int_M |s|^N_h e^F \omega^n = \int_M \omega^n,
\end{equation}

then there exists a solution $\varphi \in C^{1,1}(M) \cap PSH(M, \omega)$ of

\begin{equation}
(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = |s|^N_h e^F \omega^n, \quad \sup_M \varphi = 0.
\end{equation}

Using blow-up construction, Theorem 1.3 can be applied in the study of the singularities of the pluricomplex Green’s function. More precisely, we prove

**Theorem 1.4.** Let $(M, \omega)$ be a compact $n$-dimensional Kähler manifold with $\text{Vol}(M, \omega) = \int_M \omega^n = 1$. Assume that $F$ is a smooth function on $M$ such that $\int_M e^F \omega^n = 1$. Let $\delta_p$ be the Dirac measure concentrated at $p$. Then for $\varepsilon_0$ sufficiently small, there exists $\varphi \in PSH(M, \omega) \cap C^{1,1}(M \setminus \{p\})$ such that $\varphi = \varepsilon_0 \log |z|^2 + C^{1,1}$ in local coordinates centered at $p$ and

\begin{equation}
(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = (1 - \varepsilon_0) e^F + \varepsilon_0 \delta_p.
\end{equation}

In [18], Coman-Guedj showed that there are examples of Kähler manifolds for which $\varepsilon_0$ cannot be taken equal to 1. Using the estimates of [46] and blow up argument, Phong-Sturm [36] proved that $\varphi = \varepsilon_0 \log |z|^2 + C^{1,1}$ near $z = p$ for any $\alpha \in (0, 1)$. Our result, which makes use of Theorem 1.2, improves this regularity to $C^{1,1}$.

For background material and further references on singular Monge-Ampère equation, and relation to singular Kähler-Einstein metrics, we refer the reader to [45, 33, 41, 30, 23, 5, 24, 40, 1]. For further information, we refer to the survey [38] and the references therein.
In the above theorems, we assume that $\partial M = \emptyset$. When $\partial M \neq \emptyset$, the Dirichlet problem has been studied extensively. Caffarelli-Kohn-Nirenberg-Spruck [7] established the classical solvability for strongly pseudoconvex domains in $\mathbb{C}^n$. Guan [26] generalized this result to general domains under the assumption of existence of a subsolution. For further references, we refer the reader to [9, 3, 37, 32].

Actually, Theorem 1.1 can be applied in the Dirichlet problem for the degenerate Monge-Ampère equation. When $\partial M \neq \emptyset$, Theorem 1.1 can be regarded as the interior estimate. In this case, the $C^{1,1}$ estimate depends on not only $(M, \omega, J)$, $\sup_M f$, $\sup_M |\partial (f^\frac{1}{n})|_g$ and lower bound of $\nabla^2 (f^\frac{1}{n})$, but also $\sup_{\partial M} |\varphi|$, $\sup_{\partial M} |\partial \varphi|_g$ and $\sup_{\partial M} |\nabla^2 \varphi|_g$. Combining this $C^{1,1}$ estimate with the boundary estimates (see [3, Theorem 3.2'] and [4, Lemma 7.16]), we obtain

**Theorem 1.5.** Let $(M, \omega, J)$ be a compact $n$-dimensional Kähler manifold with nonempty smooth boundary, which we assume is weakly pseudoconcave (or Levi-flat). Suppose that $f$ is a nonnegative function on $M$ such that

$$\sup_M f \leq C, \quad \sup_M |\partial (f^\frac{1}{n})|_g \leq C, \quad \nabla^2 (f^\frac{1}{n}) \geq -Cg,$$

for a constant $C$. We consider the Dirichlet problem

$$\omega + \sqrt{-1} \partial \bar{\partial} \varphi^n = f \omega^n, \quad \varphi = \varphi_0, \quad \text{on } \partial M,$$

where $\varphi_0$ is a smooth function on $\partial M$. If this problem admits a smooth subsolution, then there exists a solution $\varphi \in C^{1,1}(M) \cap PSH(M, \omega)$ of (1.4).

Theorem 1.5 generalizes Corollary 1.3 in [15], where Chu-Tosatti-Weinkove showed existence of $C^{1,1}$ solution to the homogeneous complex Monge-Ampère equation (i.e., $f \equiv 0$).

The complex Monge-Ampère equation also plays an important role in Sasakian geometry. For the reader’s convenience, let us recall the definition and some basic properties of Sasakian manifold. For general properties of Sasakian manifold, we refer the reader to the book [6]. A Sasakian manifold $(N, g_N)$ is a $(2m + 1)$-dimensional Riemannian manifold such that the cone manifold

$$(C(N), g_c) := (N \times \mathbb{R}^+, r^2 g_N + dr^2)$$

is Kähler. There exists a Killing field $\xi$ of unit length on $N$, which is called Reeb vector field. We define tensor field $\Phi$ and contact 1 form $\eta$ by

$$\Phi(X) = \nabla X \xi, \quad \eta(X) = g_N(X, \xi) \quad \text{for any } X \in TN,$$

where $\nabla$ is the Levi-Civita connection of $g_N$. We write $\mathcal{D} = \text{Ker}\{\eta\}$. Then $\Phi|_{\mathcal{D}}$ is a complex structure on $\mathcal{D}$, and $(\mathcal{D}, \Phi|_{\mathcal{D}}, d\eta)$ gives a transverse Kähler structure with Kähler form $\frac{1}{2} d\eta$ and Riemannian metric $g_N^T$ defined by

$$g_N^T(X, Y) = \frac{1}{2} d\eta(X, \Phi|_{\mathcal{D}}(Y)) \quad \text{for any } X, Y \in \mathcal{D}.$$
Let $D^C$ be the complexification of $D$. We have the following decomposition
\begin{equation}
D = D^{(1,0)} \oplus D^{(0,1)},
\end{equation}
where $D^{(1,0)}$ and $D^{(0,1)}$ are the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of $\Phi|_D$
A $p$ form $\theta$ on $(N, g_N)$ is called basic if
\[
 i_\xi \theta = 0 \quad \text{and} \quad L_\xi \theta = 0,
\]
where $i_\xi$ is the contraction with $\xi$ and $L_\xi$ is the Lie derivative with respect to $\xi$. In particular, when $p = 0$, we use $C^\infty_B(N)$ to denote the set of all smooth basic functions on $N$, i.e.,
\[
 C^\infty_B(N) = \{ \varphi \in C^\infty(N) \mid \xi(\varphi) = 0 \}.
\]
Let $\wedge^p_B(N)$ be the bundle of basic $p$ form. By (1.5), there is a natural decomposition of its complexification
\[
\wedge^p_B(N) \otimes \mathbb{C} = \bigoplus_{i+j=p} \wedge^{i,j}_B(N),
\]
where $\wedge^{i,j}_B(N)$ denotes the bundle of basic $(i,j)$ form. Accordingly, we define the corresponding operators $\partial_B$ and $\overline{\partial}_B$ by
\[
 \partial_B : \wedge^{i,j}_B(N) \to \wedge^{i,j+1}_B(N), \quad \overline{\partial}_B : \wedge^{i,j}_B(N) \to \wedge^{i,j+1}_B(N),
\]
and set
\[
 d_B = d|_{\wedge^p_B}, \quad d_B^c = \frac{1}{2} \sqrt{-1}(\overline{\partial}_B - \partial_B).
\]
It is clear that
\[
 d_B = \partial_B + \overline{\partial}_B, \quad d_B d_B^c = \sqrt{-1} \partial_B \overline{\partial}_B, \quad d_B^2 = 0, \quad (d_B^c)^2 = 0.
\]
Let $(N, g_N)$ be a compact $(2m+1)$-dimensional Sasakian manifold. We write $\mathcal{H}$ for the space of Sasakian metrics, which can be parameterized by the space (see [28])
\[
 \{ \varphi \in C^\infty_B(N) \mid \eta_\varphi \wedge (d\eta_\varphi)^n \neq 0 \},
\]
where
\[
 \eta_\varphi = \eta + d_B^c \varphi, \quad d\eta_\varphi = d\eta + \sqrt{-1} \partial_B \overline{\partial}_B \varphi.
\]
In [28], Guan-Zhang introduced a geodesic equation in $\mathcal{H}$. For each Sasakian potential $\varphi \in \mathcal{H}$, the tangent space $T_\varphi \mathcal{H}$ is $C^\infty_B(N)$ and $d\mu_\varphi = \eta_\varphi \wedge (d\eta_\varphi)^n$ defines a measure on $N$. On this infinite dimensional manifold $\mathcal{H}$, the Riemannian metric is defined by
\[
 (\psi_1, \psi_2)_\varphi = \int_N \psi_1 \psi_2 d\mu_\varphi \quad \text{for any} \quad \psi_1, \psi_2 \in T_\varphi \mathcal{H}.
\]
For any $\varphi_0, \varphi_1 \in \mathcal{H}$, let $\varphi : [0, 1] \to \mathcal{H}$ be a path connecting them. The corresponding geodesic equation is
\begin{equation}
 \left\{ \begin{array}{l}
 \frac{\partial^2 \varphi}{\partial t^2} - \frac{1}{4} \left| d_B \frac{\partial \varphi}{\partial t} \right|_{g_\varphi}^2 = 0, \\
 \varphi(\cdot, 0) = \varphi_0, \quad \varphi(\cdot, 1) = \varphi_1,
 \end{array} \right.
\end{equation}
where $g_\varphi$ is the Sasakian metric determined by $\varphi$, i.e.,
\[
g_\varphi = \frac{1}{2} d\eta_\varphi \circ (\text{Id} \otimes \Phi_\varphi) + \eta_\varphi \otimes \eta_\varphi, \quad \Phi_\varphi = \Phi - \xi \otimes (d^B_\varphi \circ \Phi).
\]

In [29], Guan-Zhang reduced (1.6) to the Dirichlet problem of complex Monge-Ampère type equation on the Kähler cone $N \times [1, \frac{3}{2}] \subset C(N)$. More precisely, they define a function $\psi$ and a $(1,1)$ form $\Omega_\psi$ on $N \times [1, \frac{3}{2}]$ by
\[
\psi(\cdot, r) = \varphi(\cdot, 2r - 2) + 4 \log r
\]
and
\[
(1.7) \quad \Omega_\psi = \omega_c + \frac{r^2}{2} \sqrt{-1} \left( \frac{\partial}{\partial r} \varphi - \frac{\partial \psi}{\partial r} \right).
\]
where $\omega_c$ is the Kähler form of $(C(N), g_c)$. As noted in [29], the path $\varphi$ is a geodesic connecting $\varphi_0$ and $\varphi_1$ if and only if $\psi$ solves the following Dirichlet problem on $N \times [1, \frac{3}{2}]$
\[
(1.8) \quad \begin{cases}
(\Omega_\psi)^{m+1} = 0, \\
\Omega_\psi \geq 0, \\
\psi(\cdot, 1) = \psi_1, \quad \psi(\cdot, \frac{3}{2}) = \psi_{\frac{3}{2}}.
\end{cases}
\]
In order to solve (1.8), for any $\varepsilon \in (0, 1)$, Guan-Zhang [29] considered the perturbation geodesic equation
\[
(1.9) \quad \begin{cases}
(\Omega_\psi)^{m+1} = \varepsilon f \omega_c^{m+1}, \\
\Omega_\psi > 0, \\
\psi(\cdot, 1) = \psi_1, \quad \psi(\cdot, \frac{3}{2}) = \psi_{\frac{3}{2}},
\end{cases}
\]
where $f$ is a positive basic function. They proved that there exists a smooth solution $\psi_\varepsilon$ of (1.9), and established the $C^2_w$ estimate (see [29, Theorem 1])
\[
\|\psi_\varepsilon\|_{C^2_w(N \times [1, \frac{3}{2}], g_c)} := \|\psi_\varepsilon\|_{C^1(N \times [1, \frac{3}{2}], g_c)} + \sup_{N \times [1, \frac{3}{2}]} |\Delta_c \psi_\varepsilon| \leq C,
\]
where $\Delta_c$ is the Laplace-Beltrami operator of $g_c$ and $C$ is a constant depending only on $(N, g_N)$, $\|f\|_{C^2(N \times [1, \frac{3}{2}], g_c)}$, $\|\psi_1\|_{C^{2,1}(N \times [1, \frac{3}{2}], g_N)}$ and $\|\psi_{\frac{3}{2}}\|_{C^{2,1}(N \times [1, \frac{3}{2}], g_N)}$. Letting $\varepsilon \to 0$, Guan-Zhang showed existence of $C^2_w$ solution of (1.8). This implies that any two Sasakian potentials $\varphi_0, \varphi_1$ can be joined by a $C^2_w$ geodesic. Clearly, this geodesic is $C^{1,\alpha}$ for any $\alpha \in (0, 1)$.

When $m = 1$, (1.8) is equivalent to the geodesic equation in the space of volume forms on Riemannian manifold with fixed volume (see [22]). In this setting, Chen-He [10] proved the geodesic is $C^2_w$, and Chu [12] improved this regularity to $C^{1,1}$.

Actually, Sasakian geometry can be considered as odd dimensional counterpart of Kähler geometry. The space of Kähler metrics can be endowed with a natural Riemannian structure (see [35, 39, 21]). Chen [9] showed any
two Kähler potentials can be connected by a $C^2_w$ geodesic. As mentioned before, Chu-Tosatti-Weinkove [15] improved this regularity to $C^{1,1}$.

In two settings mentioned above, the $C^{1,1}$ regularity is optimal (see [34, 20, 19]). It was expected that analogous result can be proved in the Sasakian case. In this paper, we prove the $C^{1,1}$ regularity of geodesics in the space of Sasakian metrics.

**Theorem 1.6.** Let $(N, g_N)$ be a compact $(2m + 1)$-dimensional Sasakian manifold. For any two Sasakian potentials $\varphi_0, \varphi_1 \in \mathcal{H}$, the geodesic connecting them is $C^{1,1}$.

To prove Theorem 1.6, it suffices to establish the $C^{1,1}$ estimate for the perturbation geodesic equation (1.9). Since $\Omega_\psi$ involves the first order term (see (1.7)), (1.9) is much more complicated than the standard complex Monge-Ampère equation. Fortunately, the proof of Proposition 4.1 is still valid for (1.9). In Section 6, we will introduce a kind of complex Monge-Ampère type equation (6.1), and (1.9) can be regarded as a special case of (6.1). By the same proof of Proposition 4.1, we derive the $C^{1,1}$ interior estimate for (6.1) (see Proposition 6.1), which gives an extension of Proposition 4.1. Then Theorem 1.6 follows from Proposition 6.1 and [29, Theorem 1, Proposition 3] ($C^2_w$ estimate and $C^{1,1}$ boundary estimate).

We now discuss the proof of Theorem 1.1. Zero order estimate was proved in [14]. For the first order estimate, we adapt an approach of Błocki [2, Theorem 1] in the Kähler case. However, there are more troublesome terms arising from the non-integrability. We show that these terms can be controlled in Section 3.

The heart of this paper is Section 4, where we prove the second order estimate. Compared to the second order estimate of [14], our method is quite different. The main reason is that the concavity of $(\det \tilde{g})^{\frac{1}{2}}$ is weaker than that of $\log \det \tilde{g}$. Then there are less "good" third order terms when we differentiate the equation twice. Hence, it is impossible to control "bad" third order terms by the similar argument in [14].

In order to overcome this difficulty, we apply the maximum principle to a new quantity. Compared to the quantity of [14], we add a new term involving $|\tilde{\omega}|^2_{\tilde{g}}$. Crucially, this gives more "good" third order terms, which can be used to control "bad" third order terms. On the other hand, we use covariant derivatives with respect to the Levi-Civita connection $\nabla$. Then there is no third order term when we commute derivatives (see (2.2) $(k = 2)$). And this is the main reason why we do not use the Chern connection. For general almost Hermitian manifold, $(\nabla^2 \varphi)^{(1,1)}$ is different from $\partial \overline{\partial} \varphi$ (they coincide in the Kähler case). We introduce a new tensor field $S$ (see (2.3)) to describe this difference. Because of this, more "bad" third order terms appear when we differentiate the equation twice. Fortunately, these terms can be controlled by using the maximum principle (see (4.24), (4.25)). After
a series of delicate calculations and estimates, we prove the second order estimate.

We expect that the method we introduced in this paper will adapt to other nonlinear PDEs on compact almost Hermitian manifolds.

2. Basic results and notation

Let $M$ be a compact manifold of real dimension $2n$. Recall that an almost complex structure $J$ on $M$ is a bundle automorphism of the tangent bundle $TM$ satisfying $J^2 = -\text{Id}$. Let $T^CM$ be the complexified tangent space. Then we have the natural decomposition

$$T^CM = T_{C}^{(1,0)}M \oplus T_{C}^{(0,1)}M,$$

where $T_{C}^{(1,0)}M$ and $T_{C}^{(0,1)}M$ are the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of $J$. For any 1 form $\alpha$ on $M$, we define

$$J_\alpha(V) = -\alpha(JV) \quad \text{for any } V \in TM.$$

Then we have the similar decomposition of $T^CM^*$ into the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces, spanned by the $(0,1)$ and $(1,0)$ forms respectively. And every $k$ form can be expressed uniquely as a linear combination of $(p,q)$ forms.

Let $g$ be a Riemannian metric on $M$. $(M, g, J)$ is called an almost Hermitian manifold if

$$g(V_1, V_2) = g(JV_1, JV_2) \quad \text{for any } V_1, V_2 \in TM.$$

We define $(1,1)$ form $\omega$ by

$$\omega(V_1, V_2) = g(JV_1, V_2) \quad \text{for any } V_1, V_2 \in TM.$$

It then follows that

$$g(V_1, V_2) = \omega(V_1, JV_2).$$

Hence, we often use $(M, \omega, J)$ to denote $(M, g, J)$ for convenience.

For any $(p,q)$ form $\beta$, we define

$$\partial\beta = (d\beta)^{p+1,q} \quad \text{and} \quad \overline{\partial}\beta = (d\beta)^{p,q+1}.$$

By direct calculation, for any $f \in C^2(M)$, we have

$$\sqrt{-1}\partial\overline{\partial}f = \frac{1}{2}(dJdf)^{(1,1)}.$$

For any two $(1,0)$ vector fields $X, Y$, we also have the following formula (see e.g. [32] (2.5))

$$(\partial\overline{\partial}\varphi)(X, Y) = XY(\varphi) - [X, Y]^{(0,1)}(\varphi).$$

Let $\{e_i\}_{i=1}^n$ be a local frame for $T_{C}^{(1,0)}M$. Throughout this paper, we use covariant derivatives with respect to the Levi-Civita connection $\nabla$. And the subscripts of a function $f$ always denote the covariant derivatives of $f$ with respect to $\nabla$, e.g.,

$$f_i = \nabla_{e_i}f, \quad f_{ij} = \nabla_{e_j} \nabla_{e_i}f, \quad f_{ijk} = \nabla_{e_k} \nabla_{e_j} \nabla_{e_i}f.$$
Recalling the commutation formula for covariant derivatives (Ricci identity), for any two vector fields $V_1, V_2$, we have
\[(2.2) \quad \nabla_{V_1} \nabla_{V_2} (\nabla^k f) - \nabla_{V_2} \nabla_{V_1} (\nabla^k f) = (\nabla^k f) R_m,\]
where $R_m$ is the curvature tensor of $g$ and $*$ denotes a contraction.

Next we define a tensor field $S$ by
\[(2.3) \quad \nabla e_i e_j - [e_i, e_j]^{(0,1)} = S_{ij}^p e_p + S_{ij}^{\overline{p}} e^{\overline{p}}.\]
By direct calculation, it is clear that
\[(2.4) \quad S_{ij}^p = S_{ji}^p \quad \text{and} \quad S_{ij}^{\overline{p}} = S_{ji}^{\overline{p}}.\]

For convenience, we write $	ilde{\omega} = \omega + \sqrt{-1} \partial \overline{\partial} \varphi > 0$ and let $	ilde{g}$ be the corresponding Riemannian metric. Combining (2.1) and (2.3), we see that
\[(2.5) \quad \tilde{g}_{ij} = g_{ij} + (\partial \overline{\partial} \varphi)(e_i, e_j) = g_{ij} + \varphi_{ij} + S_{ij}^p e_p + S_{ij}^{\overline{p}} e^{\overline{p}}.\]

For later use, let us recall the $L^1$ estimate and zero order estimate.

**Proposition 2.1** (Proposition 2.3 of [14]). For any $\varphi \in C^\infty(M)$ satisfying $\omega + \sqrt{-1} \partial \overline{\partial} \varphi > 0$ and $\sup_M \varphi = 0$. Then there exists a constant $C$ depending only on $(M, \omega, J)$ such that
\[\int_M |\varphi| \omega^n = \int_M (-\varphi) \omega^n \leq C.\]

**Proposition 2.2** (Proposition 3.1 of [14]). Let $\varphi$ be a solution of (1.1). Then there exists a constant $C$ depending only on $(M, \omega, J)$, $\sup_M f$ such that
\[\sup_M |\varphi| \leq C.\]

Throughout this paper, we say a constant is uniform if it depends only on $(M, \omega, J)$, $\sup_M f$, $\sup_M \partial(f^\frac{1}{n})|_g$ and lower bound of $\nabla^2 (f^\frac{1}{n})$. We also use Einstein notation convention. Sometimes, we will include the summation for clarity.

### 3. First order estimate

In this section, we prove the first order estimate. The proof is similar to [2] Theorem 1 in the Kähler case.

**Proposition 3.1.** Let $\varphi$ be a smooth solution of (1.1). Then there exists a constant $C$ depending only on $(M, \omega, J)$, $\sup_M f$, $\sup_M \partial(f^\frac{1}{n})|_g$ such that
\[\sup_M |\partial \varphi|_g \leq C.\]
Proof. We consider the following quantity

\[ Q = \log |\partial \varphi|^2 g + e^{-A\varphi}, \]

where \( A \) is a constant to be determined later. Let \( x_0 \) be the maximum point of \( Q \) and \( \{e_i\}_{i=1}^n \) be a local \( g \)-unitary frame for \( T^{(1,0)}_g M \) in a neighbourhood of \( x_0 \) such that

\[ \tilde{g}_{ij}(x_0) = \delta_{ij} \tilde{g}_{ii}(x_0). \]

To prove Proposition 3.1, it suffices to prove that

\[ |\partial \varphi|^2 g(x_0) \leq C. \]

Without loss of generality, we assume that \( |\partial \varphi|^2 g(x_0) > 1 \). By the maximum principle, at \( x_0 \), we have

\[ 0 \geq \tilde{g}^\iota (|\partial \varphi|^2 g) \]

\[ = \frac{\tilde{g}^\iota (|\partial \varphi|^2) \tilde{g}}{|\partial \varphi|^2 g} - \frac{\tilde{g}^\iota (|\partial \varphi|^2) \tilde{g}}{|\partial \varphi|^2 g} + \tilde{g}^\iota (e^{-A\varphi}) \tilde{g}. \]

Now we estimate each term in (3.2). For the first term of (3.2), using (2.2) and (2.5), we compute

\[ \tilde{g}^\iota (|\partial \varphi|^2) \tilde{g} = \sum_k \tilde{g}^\iota (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) + \sum_k \tilde{g}^\iota (\varphi_{ik} \varphi_k + \varphi_{ik} \varphi_k) \]

\[ \geq \sum_k \tilde{g}^\iota (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) + 2\text{Re} \left( \sum_k \tilde{g}^\iota (\varphi_{ik} \varphi_k) \right) - C|\partial \varphi|^2 g \sum_i \tilde{g}^\iota \]

\[ = \sum_k \tilde{g}^\iota (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) + 2\text{Re} \left( \sum_k \tilde{g}^\iota (\tilde{g}_{ik}) \varphi_k \right) \]

\[ - 2\text{Re} \left( \sum_k \tilde{g}^\iota (S_{ik} \varphi_p + S_{ik} \varphi_p) \varphi_k \right) - C|\partial \varphi|^2 g \sum_i \tilde{g}^\iota \]

\[ \geq \sum_k \tilde{g}^\iota (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) + 2\text{Re} \left( \sum_k \tilde{g}^\iota (\tilde{g}_{ik}) \varphi_k \right) \]

\[ - 2\text{Re} \left( \sum_k \tilde{g}^\iota (S_{ik} \varphi_p + S_{ik} \varphi_p) \varphi_k \right) - C|\partial \varphi|^2 g \sum_i \tilde{g}^\iota. \]

To deal with the third order term in (3.3), we differentiate (covariantly) the logarithm of (1.1)

\[ \log \frac{\text{det } \tilde{g}}{\text{det } g} = n \log (f^{-1}_{\frac{1}{n}}), \]

and we obtain

\[ \tilde{g}^\iota (\tilde{g}_{ik}) \varphi_k = \frac{n(f^{-1}_{\frac{1}{n}})}{f^{-1}_{\frac{1}{n}}}. \]
By the arithmetic-geometric mean inequality, it is clear that

\[ \frac{1}{f_n^\frac{1}{n}} = \left( \prod_i \tilde{g}_i \right)^\frac{1}{n} \leq \frac{1}{n} \sum_i \tilde{g}_i. \]

Combining (3.4) and (3.5), we have

\[ 2 \text{Re} \left( \sum_k \tilde{g}_i (\tilde{g}_i) \varphi_k \right) = 2 \text{Re} \left( \sum_k \frac{n(f_n^\frac{1}{n})_k \varphi_k}{f_n^\frac{1}{n}} \right) \geq -C |\partial \varphi|_g \sum_i \tilde{g}_i. \]

Substituting (3.6) into (3.3), we see that

\[ \tilde{g}_i (|\partial \varphi|_g^2) \geq \sum_i \tilde{g}_i (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) - C |\partial \varphi|_g \sum_i \tilde{g}_i. \]

(3.7)

where we used \(|\partial \varphi|_g^2(x_0) > 1\).

For the second term of (3.3), using \(Q_i(x_0) = 0\), it is clear that

\[ \frac{\tilde{g}_i (|\partial \varphi|_g^2)_i}{|\partial \varphi|_g^2} = -A^2 e^{-2A\varphi} \tilde{g}_i |\varphi_i|^2. \]

(3.8)

For the third term of (3.3), by (2.5), we see that

\[ \tilde{g}_i \varphi_i = \tilde{g}_i (\tilde{g}_i - g_i - S^p_i \varphi_p - S^p_i \varphi_p) \]

\[ = n - \sum_i \tilde{g}_i - \tilde{g}_i (S^p_i \varphi_p + S^p_i \varphi_p), \]

which implies

\[ \tilde{g}_i (e^{-A\varphi})_i = Ae^{-A\varphi} \sum_i \tilde{g}_i + A^2 e^{-2A\varphi} \tilde{g}_i |\varphi_i|^2 \]

\[ + Ae^{-A\varphi} \tilde{g}_i (S^p_i \varphi_p + S^p_i \varphi_p) - Ane^{-A\varphi}. \]

(3.10)

Substituting (3.7), (3.8) and (3.10) into (3.3), we obtain

\[ 0 \geq \sum_k \tilde{g}_i (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) \frac{1}{|\partial \varphi|_g^2} + (A^2 e^{-A\varphi} - A^2 e^{-2A\varphi}) \tilde{g}_i |\varphi_i|^2 \]

\[ - 2 \text{Re} \left( \frac{\sum_k \tilde{g}_i (S^p_i \varphi_p + S^p_i \varphi_p) \varphi_k}{|\partial \varphi|_g^2} \right) + Ae^{-A\varphi} \tilde{g}_i (S^p_i \varphi_p + S^p_i \varphi_p) \]

\[ + (Ae^{-A\varphi} - C) \sum_i \tilde{g}_i - Ane^{-A\varphi}. \]

(3.11)
Using $Q_p(x_0) = 0$ and (2.4), it is clear that

\begin{equation}
0 = -2 \text{Re} \left( \bar{g}^r S_{n}^p Q_p \right)
\end{equation}

\begin{align*}
&= -2 \text{Re} \left( \sum_k \bar{g}^r \left( \partial_{\varphi}^2 \varphi_{ik} \right) + A e^{-A\varphi} \bar{g}^r S_{n}^p \varphi_k \right) \\
&= -2 \text{Re} \left( \sum_k \bar{g}^r \left( \partial_{\varphi}^2 \varphi_{ik} + S_{n}^p \varphi_{ik} \right) + A e^{-A\varphi} \bar{g}^r \left( S_{n}^p \varphi_p + S_{n}^p \varphi_p \right) \right) \\
&= -2 \text{Re} \left( \sum_k \bar{g}^r \left( \partial_{\varphi}^2 \varphi_{ik} + S_{n}^p \varphi_{ik} \right) + A e^{-A\varphi} \bar{g}^r \left( S_{n}^p \varphi_p + S_{n}^p \varphi_p \right) \right),
\end{align*}

where we used $\varphi_{ik} = \varphi_{ik}$ and $\varphi_k = \varphi_k$ (Levi-Civita connection) in the last equality. Substituting (3.12) into (3.11), we obtain

\begin{equation}
0 \leq \sum_k \bar{g}^r \left( |\varphi_{ik}|^2 + |\varphi_{ik}|^2 \right) + \left( A^2 e^{-A\varphi} - A^2 e^{-2A\varphi} \right) \bar{g}^r |\varphi_i|^2 + (A e^{-A\varphi} - C) \sum_i \bar{g}^r - A e^{-A\varphi}.
\end{equation}

By the Cauchy-Schwarz inequality, we get

\begin{equation}
\sum_k \bar{g}^r |\varphi_{ik}|^2 \geq \sum_i \bar{g}^r \sum_k \varphi_{ki} \varphi_{ki} \bar{g}^r |\varphi_i|^2.
\end{equation}

Using $Q_i(x_0) = 0$, it follows that

\[ \sum_k (\varphi_{ki} \varphi_{ik} + \varphi_{ik} \varphi_k) = A e^{-A\varphi} |\partial_{\varphi}^2 \varphi_i| \varphi_i. \]

Combining this with (2.5) and (3.1), for any $\varepsilon \in (0, 1)$, we have

\[ \left( \sum_k \varphi_{ki} \varphi_k \right)^2 = |A e^{-A\varphi} |\partial_{\varphi}^2 \varphi_i - \sum_k (\bar{g}_{ik} - g_{ik} - S_{n}^p \varphi_p + S_{n}^p \varphi_p) \varphi_k|^2 \]

\[ = \left( A e^{-A\varphi} |\partial_{\varphi}^2 \varphi_i - \bar{g}_{ik} + 1) \varphi_i + \sum_k (S_{n}^p \varphi_p + S_{n}^p \varphi_p) \varphi_k | \right)^2 \]

\[ \geq (1 - \varepsilon) \left( A e^{-A\varphi} |\partial_{\varphi}^2 \varphi_i - \bar{g}_{ik} + 1) ^2 |\varphi_i|^2 - C \varepsilon |\partial_{\varphi}^4 \varphi_i | \right)^2 \]

\[ \geq (1 - \varepsilon) \left( A^2 e^{-2A\varphi} |\partial_{\varphi}^4 \varphi_i | ^2 - 2 A e^{-A\varphi} |\partial_{\varphi}^2 \varphi_i - \bar{g}_{ik} + 1) |\varphi_i|^2 - C \varepsilon |\partial_{\varphi}^4 \varphi_i | \right)^2. \]
Substituting this into (3.14), we see that

\[
\sum_k \tilde{g}^{ii} |\varphi_{ik}|^2 \geq (1 - \varepsilon) A^2 e^{-2A\varphi} \tilde{g}^{ii} |\varphi_i|^2 - 2A e^{-A\varphi} - \frac{2}{\varepsilon} \sum_i \tilde{g}^{ii} - \frac{C}{\varepsilon} \sum_i \tilde{g}^{ii} - 2A e^{-A\varphi} - 2,
\]

where we used \(|\partial\varphi|^2|_g(x_0) > 1\) in the second inequality. Substituting (3.15) into (3.13), we obtain

\[
0 \geq (A^2 e^{-A\varphi} - \varepsilon A^2 e^{-2A\varphi}) \tilde{g}^{ii} |\varphi_i|^2 + \left( A e^{-A\varphi} - \frac{C_0}{\varepsilon} \right) \sum_i \tilde{g}^{ii} - A(n + 2) e^{-A\varphi} - 2,
\]

where \(C_0\) is a constant depending only on \((M, \omega, J), \sup_M f\) and \(\sup_M |\partial(f^{1/n})|_g\).

Now we choose

\[
A = 2C_0 + 1 \quad \text{and} \quad \varepsilon = \frac{e^{A\varphi(x_0)}}{2}.
\]

Recalling \(\sup_M \varphi = 0\), we see that

\[
A^2 e^{-A\varphi} - \varepsilon A^2 e^{-2A\varphi} \geq \frac{1}{2} \quad \text{and} \quad A e^{-A\varphi} - \frac{C_0}{\varepsilon} \geq 1.
\]

It then follows that

\[
\frac{1}{2} \tilde{g}^{ii} |\varphi_i|^2 + \sum_i \tilde{g}^{ii} \leq C.
\]

From \(\sum_i \tilde{g}^{ii} \leq C\) and \(\frac{\det \tilde{g}}{\det g} \leq C\), we have \(\tilde{g}^{ii} \leq C\) for each \(i\). Combining this with (3.16), we obtain \(|\partial \varphi|^2|_g(x_0) \leq C\), as desired. \(\square\)

For later use, we state the following lemma, which follows from Proposition 3.1, (3.7) and (3.9).

**Lemma 3.2.** There exists a uniform constant \(C\) such that

\[
\tilde{g}^{ii} \varphi_i = n - \sum_i \tilde{g}^{ii} - \tilde{g}^{ii} (S_{\alpha\alpha}^p \varphi_p + S_{\alpha\alpha}^p \varphi_p)
\]

and

\[
\tilde{g}^{ii} (|\partial \varphi|^2|_g)_{\alpha\alpha} \geq \sum_k \tilde{g}^{ii} (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) - C \sum_i \tilde{g}^{ii} - 2Re \left( \sum_k \tilde{g}^{ii} (S_{\alpha\alpha}^p \varphi_p + S_{\alpha\alpha}^p \varphi_p) \varphi_k \right).
\]
4. Second order estimate

In this section, we prove the following second order estimate.

**Proposition 4.1.** Let \( \varphi \) be a smooth solution of (1.1). Then there exists a constant \( C \) depending only on \((M, \omega, J)\), \( \sup_M f \), \( \sup_M |\partial(f^\frac{1}{2})|_g \) and lower bound of \( \nabla^2(f^\frac{1}{2}) \) such that

\[
\sup_M |\nabla^2 \varphi|_g \leq C,
\]

where \( \nabla \) is the Levi-Civita connection of \( g \).

4.1. Auxiliary function. Let \( \lambda_1(\nabla^2 \varphi) \geq \lambda_2(\nabla^2 \varphi) \geq \cdots \geq \lambda_{2n}(\nabla^2 \varphi) \) be the eigenvalues of \( \nabla^2 \varphi \). Combining \( \omega + \sqrt{1-\partial \bar{\partial} \varphi} > 0 \), Proposition 3.1 and [14, (2.4),(2.5)], we see that

\[
2n \sum_{\alpha=1}^{2n} \lambda_{\alpha}(\nabla^2 \varphi) = \Delta \varphi \geq -C,
\]

where \( \Delta \) is the Laplace-Beltrami operator of \( g \). It then follows that

\[
|\nabla^2 \varphi|_g \leq C \max(\lambda_1(\nabla^2 \varphi), 0) + C.
\]

To prove Proposition 4.1, it suffice to prove \( \lambda_1(\nabla^2 \varphi) \) is uniformly bounded from above. Without loss of generality, we assume that \( D = \{ x \in M \mid \lambda_1(\nabla^2 \varphi)(x) > 0 \} \) is not empty. On this set, we define the following quantity

\[
Q = \log \lambda_1(\nabla^2 \varphi) + h_1(|\tilde{\omega}|^2_g) + h_2(|\partial \varphi|^2_g) + e^{-A \varphi},
\]

where

\[
h_1(s) = -\frac{1}{3} \log(10M^2_R - s), \quad h_2(s) = -\frac{1}{3} \log(1 + \sup_M |\partial \varphi|^2_g - s),
\]

\( M_R = \sup_M |\nabla^2 \varphi|_g + 1 \) and \( A > 1 \) is a constant to be determined later. We need to verify the function \( h_1(|\tilde{\omega}|^2_g) \) is well defined. Without loss of generality, we assume that

\[
M_R \gg 1.
\]

It then follows that

\[
|\tilde{\omega}|^2_g \leq 2n + 2|\partial \bar{\partial} \varphi|^2_g \leq 2n + 4|\nabla^2 \varphi|^2_g + C|\partial \varphi|^2_g \leq 5M^2_R,
\]

which implies that \( h_1(|\tilde{\omega}|^2_g) \) is well defined. By direct calculation, we have

\[
h_1'' = 3(h_1')^2, \quad h_2'' = 3(h_2')^2,
\]

and

\[
\frac{1}{30M^2_R} \leq h_1' \leq \frac{1}{15M^2_R}, \quad \frac{1}{C} \leq h_2' \leq C.
\]

Clearly, the function \( Q \) is continuous on its domain \( D \) and equal to \(-\infty\) on \( \partial D \). Let \( x_0 \) be the maximum point of \( Q \). Then we have \( \lambda_1(\nabla^2 \varphi)(x_0) > 0 \).
Let \( \{e_i\}_{i=1}^n \) be a local \( g \)-unitary frame for \( T^{(1,0)}_c M \) in a neighbourhood of \( x_0 \) such that
\[
(4.4) \quad \bar{g}_\alpha^\gamma(x_0) = \delta_{ij} \bar{g}_i^\alpha(x_0) \text{ and } \bar{g}_i^\gamma(x_0) \geq \bar{g}_{2i}^\gamma(x_0) \geq \cdots \geq \bar{g}_{2n}^\gamma(x_0).
\]

Since \((M, \omega, J)\) is almost Hermitian, there exists a coordinate system \((U; \{x^\alpha\}_{\alpha=1}^{2n})\) centered at \( x_0 \) such that it holds at \( x_0 \),
\[
(4.5) \quad g_{\alpha\beta} = \delta_{\alpha\beta}, \quad \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = 0 \quad \text{for } \alpha, \beta, \gamma = 1, 2, \cdots, 2n
\]
and
\[
(4.6) \quad J \partial_{2i-1} = \partial_{2i}, \quad e_i = \frac{1}{\sqrt{2}} (\partial_{2i-1} - \sqrt{-1} \partial_{2i}) \quad \text{for } i = 1, 2, \cdots, n.
\]

We want to apply the maximum principle to the quantity \( Q \) at \( x_0 \). However, \( Q \) may be not smooth at \( x_0 \) when the eigenspace of \( \lambda_1(\nabla^2 \varphi) \) has dimension great than 1. To deal with this case, we apply a perturbation argument, as in [14]. For \( \beta = 1, 2, \cdots, 2n \), we write \( V_\beta \) for the \( g \)-unit eigenvector of \( \lambda_\beta(\nabla^2 \varphi)(x_0) \) and denote the components of \( V_\beta \) by \((V_\beta^1, V_\beta^2, \cdots, V_\beta^{2n})\). Next we extend \( V_\beta \) to be vector fields near \( x_0 \) by taking the components to be constant and define a local endomorphism \( \Phi_\beta^\alpha \) by
\[
\Phi_\beta^\alpha = g^{\alpha\gamma} \varphi_{\gamma\beta} - g^{\alpha\gamma} B_{\gamma\beta}, \quad B_{\alpha\beta} = \delta_{\alpha\beta} - V_1^\alpha V_1^\beta.
\]
Let \( \lambda_1(\Phi) \geq \lambda_2(\Phi) \geq \cdots \geq \lambda_{2n}(\Phi) \) be the eigenvalues of \( \Phi \). It follows that the vector \( V_\beta(x_0) \) is still the eigenvector of \( \lambda_\beta(\Phi)(x_0) \). By the definition of \( \Phi \), at \( x_0 \), we have \( \lambda_1(\Phi) > \lambda_2(\Phi) \), which implies the eigenspace of \( \Phi \) corresponding to \( \lambda_1(\Phi) \) has dimension 1. Then \( \lambda_1(\Phi) \) is smooth near \( x_0 \). In a neighborhood of \( x_0 \), we consider the perturbed quantity \( \hat{Q} \) defined by
\[
\hat{Q} = \log \lambda_1(\Phi) + h_1(|\tilde{\omega}|^2) + h_2(|\partial \varphi|^2) + e^{-A\varphi}.
\]
Since \( \lambda_1(\Phi)(x_0) = \lambda_1(\nabla^2 \varphi)(x_0) \) and \( \lambda_1(\Phi) \leq \lambda_1(\nabla^2 \varphi) \) near \( x_0 \), \( \hat{Q} \) still attains a maximum at \( x_0 \). For convenience, we use \( \lambda_\beta \) to denote \( \lambda_\beta(\Phi) \) in the following argument.

On the other hand, by (4.4) and the definitions of \( Q, \hat{Q} \) and \( x_0 \), it is clear that
\[
(4.7) \quad \lambda_1(x_0) \leq M_R \leq C_A \lambda_1(x_0),
\]
where \( M_R = \sup_M |\nabla^2 \varphi|_g + 1 \) and \( C_A \) denotes a uniform constant depending on \( A \). Without loss of generality, we assume that \( \lambda_1(x_0) \gg 1 \) in the following argument.

4.2. Lower bound of \( \bar{g}_\mu^\nu \bar{Q}_\nu \). In this subsection, our aim is to obtain a lower bound of \( \bar{g}_\mu^\nu \bar{Q}_\nu \) at \( x_0 \). First, we compute \( \bar{g}_\mu^\nu(\lambda_1)_\mu \) and \( \bar{g}_\mu^\nu((\tilde{\omega})^2)_\mu \). Here we note that all the subscripts of a function denote the covariant derivatives with respect to the Levi-Civita connection \( \nabla \).
Lemma 4.2. At \( x_0 \), we have
\[
\tilde{g}^{\alpha}(\lambda_1)_{\eta} \geq 2 \sum_{\alpha > 1} \tilde{g}^{\alpha} |\varphi_{V_{0}V_{1}}|^{2} \lambda_{1} - \lambda_{\alpha} + \sum_{\eta \neq \eta_{0}} \tilde{g}^{\alpha} g^{\eta} |(\tilde{g}_{\eta\eta})_{V_{1}}|^{2} - C \lambda_{1} \sum_{i} \tilde{g}^{\alpha} - \tilde{g}^{\alpha} \left( S^{\rho}_{\alpha\eta} \varphi_{V_{1}V_{1}} + S^{\rho}_{\eta\eta} \varphi_{V_{1}V_{1}} \right)
\]
(4.8)
and
\[
\tilde{g}^{\alpha}(\tilde{\varphi})^{2} \geq 2 \sum_{k,l} \tilde{g}^{\alpha} (\tilde{g}_{k\eta})^{2} \geq C M_{R}^{2} \sum_{i} \tilde{g}^{\alpha} - 2 \sum_{k} \tilde{g}_{k\eta} \tilde{g}^{\alpha} \left( S^{p}_{\eta\eta} (\tilde{g}_{k\eta})_{p} + S_{\eta\eta}^{\rho} (\tilde{g}_{k\eta})_{p} \right),
\]
(4.9)
where \( M_{R} = \sup_{M} |\nabla^{2} \varphi|_{g} + 1 \).

Proof. First, let us recall the elementary formulas (see [14, Lemma 5.7]), holding at \( x_0 \),
\[
\lambda_{1}^{\alpha\beta} := \frac{\partial \lambda_{1}}{\partial \Phi_{\alpha\beta}^{\gamma}} = V_{1}^{\alpha} V_{1}^{\beta},
\]
(4.10)
\[
\lambda_{1}^{\alpha\beta,\gamma\delta} := \frac{\partial^{2} \lambda_{1}}{\partial \Phi_{\alpha\beta}^{\gamma} \partial \Phi_{\delta}^{\mu}} = \sum_{\mu > 1} \frac{V_{1}^{\alpha} V_{1}^{\beta} V_{1}^{\delta} V_{1}^{\gamma} + V_{1}^{\alpha} V_{1}^{\beta} V_{1}^{\gamma} V_{1}^{\delta}}{\lambda_{1} - \lambda_{\mu}}.
\]
For (4.8), using (4.10) and (4.5), we compute
\[
\tilde{g}^{\alpha}(\lambda_{1})_{\eta} = \tilde{g}^{\alpha} \lambda_{1}^{\alpha\beta,\gamma\delta}(\Phi_{\eta}^{\gamma})_{i}(\Phi_{\eta}^{\delta})_{k} + \tilde{g}^{\alpha} \lambda_{1}^{\alpha\beta} (\Phi_{\eta}^{\beta})_{k} - \tilde{g}^{\alpha} \lambda_{1}^{\beta\gamma\delta} \varphi_{\alpha\beta} + \tilde{g}^{\alpha} \lambda_{1}^{\alpha\beta} \varphi_{\alpha\beta} - \tilde{g}^{\alpha} \lambda_{1}^{\beta\gamma\delta} (B_{\alpha\beta})_{k}
\]
(4.11)
\[
\geq 2 \sum_{\alpha > 1} \tilde{g}^{\alpha} |\varphi_{V_{0}V_{1}}^{i}|^{2} \lambda_{1} - \lambda_{\alpha} + \tilde{g}^{\alpha} \varphi_{V_{1}V_{1}}^{i} - C \sum_{i} \tilde{g}^{\alpha}
\]
\[
\geq 2 \sum_{\alpha > 1} \tilde{g}^{\alpha} |\varphi_{V_{0}V_{1}}^{i}|^{2} \lambda_{1} - \lambda_{\alpha} + \tilde{g}^{\alpha} \varphi_{V_{1}V_{1}}^{i} - C \lambda_{1} \sum_{i} \tilde{g}^{\alpha},
\]
where we used (2.2) and (1.1) in the last inequality. Recalling (2.5) and using (2.2) again, we see that
\[
\tilde{g}^{\alpha} \varphi_{V_{1}V_{1}}^{i} = \tilde{g}^{\alpha} \left( \tilde{g}_{i}^{\alpha} - g_{i}^{\alpha} - S^{\rho}_{\eta\rho} \varphi_{\rho} - S^{\rho}_{\eta\rho} \varphi_{\rho} \right)_{V_{1}V_{1}}
\]
(4.12)
\[
\geq \tilde{g}^{\alpha} (\tilde{g}_{i})_{V_{1}V_{1}} - \tilde{g}^{\alpha} \left( S^{\rho}_{\eta\rho} \varphi_{V_{1}V_{1}}^{i} + S^{\rho}_{\eta\rho} \varphi_{V_{1}V_{1}}^{i} \right) - C \lambda_{1} \sum_{i} \tilde{g}^{\alpha}
\]
\[
\geq \tilde{g}^{\alpha} (\tilde{g}_{i})_{V_{1}V_{1}} - \tilde{g}^{\alpha} \left( S^{\rho}_{\eta\rho} \varphi_{V_{1}V_{1}}^{i} + S^{\rho}_{\eta\rho} \varphi_{V_{1}V_{1}}^{i} \right) - C \lambda_{1} \sum_{i} \tilde{g}^{\alpha}.
\]
Applying \( \nabla_{V_{1}} \) to the logarithm of (1.1), it follows that
\[
\tilde{g}^{\alpha}(\tilde{g}_{i})_{V_{1}} = \frac{n(f_{V_{1}}^{i})}{f_{V_{1}}^{i}}.
\]
(4.13)
Applying $\nabla V_1$ again, at $x_0$, we have

\[ g^\ast (\bar g, \bar \tau) V_1 = \bar g^\tau \bar \tau |(\bar g, \bar \tau)| V_1 \leq \frac{n(f_{\frac{1}{\mu}}) V_1}{f^\frac{1}{\mu}} - \frac{n|(|f_{\frac{1}{\mu}})| V_1|^2}{f^\frac{1}{\mu}}. \]

Substituting (4.13) into (4.14) and using the Cauchy-Schwarz inequality, we compute

\[ g^\ast (\bar g, \bar \tau) V_1 = \sum_{p \neq q} g^\tau \bar \tau |(\bar g, \bar \tau)| V_1 + \sum_{p} g^\tau |(\bar g, \bar \tau)| V_1 \]
\[ = \frac{n(f_{\frac{1}{\mu}}) V_1}{f^\frac{1}{\mu}} - \frac{1}{n} \left| \sum_{p} g^\tau |(\bar g, \bar \tau)| V_1 \right|^2 \]
\[ \geq \sum_{p \neq q} g^\tau \bar \tau |(\bar g, \bar \tau)| V_1 + \frac{n(f_{\frac{1}{\mu}}) V_1}{f^\frac{1}{\mu}}. \]

Combining this with (3.5), it is clear that

\[ g^\ast (\bar g, \bar \tau) V_1 \geq \sum_{p \neq q} g^\tau |(\bar g, \bar \tau)| V_1 - C \sum_{i} g^\tau. \]

Then the inequality (4.8) follows from (4.11), (4.12) and (4.15).

For (4.9), a direct calculation shows that

\[ g^\ast (\bar \omega^2) = 2 \sum_{k \neq l} g^\tau |(\bar g_k, \bar g_l)|^2 + 2 \sum_{k} g_{k\bar k} g^\tau |(\bar g_k)|^2. \]

By (2.5) and (2.2), for each $k = 1, 2, \ldots, n$, we have

\[ g^\ast (\bar g_k) = g^\tau (g_k + \bar \tau) \geq g^\tau \bar \tau |(\bar g_k)|^2 \]
\[ \geq g^\tau \bar \tau \varphi_{kk} + g^\tau \left( S_{kk} + S_{k\bar k} \varphi_{\bar k_i} \right) \]
\[ \geq g^\tau \bar \tau \varphi_{kk} + g^\tau \left( S_{kk} + S_{k\bar k} \varphi_{\bar k_i} \right) - C \lambda_1 \sum_{i} g^\tau \]
\[ \geq g^\tau \bar \tau \varphi_{kk} + g^\tau (\bar g_k) S_{kk} + g^\tau (\bar g_k) S_{k\bar k} - C \lambda_1 \sum_{i} g^\tau. \]

By the similar calculations of (4.13) and (4.15), it follows from (3.5) that

\[ |g^\ast (\bar g, \bar \tau)| = \frac{|n(f_{\frac{1}{\mu}})|}{f^\frac{1}{\mu}} \geq -C \sum_{i} g^\tau \]

and

\[ g^\ast (\bar g_k) = \sum_{p \neq q} g^\tau \bar \tau |(\bar g, \bar \tau)|^2 - C \sum_{i} g^\tau. \]
For the first term of (4.17), using (2.5), (2.2) and (4.19), we compute

\begin{align*}
\bar{g}^{\tilde{\tau}} \tilde{\varphi} \tilde{\tau}_{kk} & = \bar{g}^{\tilde{\tau}} \left( \bar{g}_{\tilde{\tau}} - g_{\tau} - S_{it}^{p} \tilde{\varphi}_{p} - S_{it}^{p} \tilde{\varphi}_{p} \right)_{kk} \\
& \geq \bar{g}^{\tilde{\tau}} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} - \bar{g}^{\tilde{\tau}} \left( S_{it}^{p} \tilde{\varphi}_{p} + S_{it}^{p} \tilde{\varphi}_{p} \right) - C \lambda_{1} \sum_{i} \bar{g}^{\tilde{\tau}} \\
& \geq \bar{g}^{\tilde{\tau}} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} - \bar{g}^{\tilde{\tau}} \left( S_{it}^{p} \tilde{\varphi}_{p} + S_{it}^{p} \tilde{\varphi}_{p} \right) - C \lambda_{1} \sum_{i} \bar{g}^{\tilde{\tau}} \\
& \geq \sum_{p \neq q} \bar{g}^{\tilde{\tau}} \bar{g}^{\tilde{\tau}} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} - \bar{g}^{\tilde{\tau}} S_{it}^{p} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} - \bar{g}^{\tilde{\tau}} S_{it}^{p} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} - C \lambda_{1} \sum_{i} \bar{g}^{\tilde{\tau}}.
\end{align*}

For the second and third terms of (4.17), by (4.18), we get

\begin{align*}
\bar{g}^{\tilde{\tau}} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} + \bar{g}^{\tilde{\tau}} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} & \geq -C \sum_{i} \bar{g}^{\tilde{\tau}}.
\end{align*}

Substituting (4.20) and (4.21) into (4.17), it is clear that

\begin{align*}
\bar{g}^{\tilde{\tau}} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} & \geq \sum_{p \neq q} \bar{g}^{\tilde{\tau}} \bar{g}^{\tilde{\tau}} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} - \bar{g}^{\tilde{\tau}} S_{it}^{p} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} - C \lambda_{1} \sum_{i} \bar{g}^{\tilde{\tau}} \\
& \geq - \bar{g}^{\tilde{\tau}} S_{it}^{p} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} - \bar{g}^{\tilde{\tau}} S_{it}^{p} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} - C \lambda_{1} \sum_{i} \bar{g}^{\tilde{\tau}}.
\end{align*}

By (4.1), we have $0 < \bar{g}_{\tilde{\tau}} \leq C \lambda_{1}$. Combining this with (4.16) and (4.22), we see that

\begin{align*}
\bar{g}^{\tilde{\tau}} \left( |\omega|_{g}^{2} \right) & \geq 2 \sum_{k,l} \bar{g}^{\tilde{\tau}} \left( |\omega|_{g}^{2} \right)_{k,l} - C \lambda_{1}^{2} \sum_{i} \bar{g}^{\tilde{\tau}} \\
& - 2 \sum_{k} \bar{g}_{kk} \bar{g}^{\tilde{\tau}} \left( S_{it}^{p} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} + S_{it}^{p} \left( \bar{g}_{\tilde{\tau}} \right)_{kk} \right).
\end{align*}

Using $\lambda_{1} \leq M_{R}$, we obtain the inequality (4.9). \hfill \Box

**Lemma 4.3.** At $x_{0}$, we have

\begin{align*}
0 & \geq \bar{g}^{\tilde{\tau}} Q_{\gamma}^{\tilde{\tau}} \\
& \geq 2 \sum_{\alpha > 1} \frac{\bar{g}^{\tilde{\tau}} |\varphi_{V_{\alpha} V_{\alpha}}|^{2}}{\lambda_{1} (\lambda_{1} - \lambda_{\alpha})} + \sum_{p \neq q} \frac{\bar{g}^{\tilde{\tau}} \bar{g}^{\tilde{\tau}} |\bar{g}_{\tilde{\tau}}|^{2} |V_{\alpha} V_{\alpha}|^{2}}{\lambda_{1} \lambda_{1}^{2}} \\
& + 2 h_{1}^{\prime} \sum_{k,l} \bar{g}^{\tilde{\tau}} \left( |\omega|_{g}^{2} \right)_{k,l} + h_{2}^{\prime} \sum_{k} \bar{g}^{\tilde{\tau}} \left( |\varphi_{kk}|^{2} + |\varphi_{kk}|^{2} \right) \\
& + h_{1}^{\prime \prime} \bar{g}^{\tilde{\tau}} \left( |\omega|_{g}^{2} \right)_{l}^{2} + h_{2}^{\prime \prime} \bar{g}^{\tilde{\tau}} \left( |\varphi_{kk}|^{2} \right)_{l}^{2} + A^{2} e^{-A_{\varphi}} \bar{g}^{\tilde{\tau}} |\varphi_{k}|^{2} \\
& + (A e^{-A_{\varphi}} - C) \sum_{i} \bar{g}^{\tilde{\tau}} - A e^{-A_{\varphi}}.
\end{align*}

(4.23)
Proposition 4.1.

Proof of Proposition 4.1.

Using \( \hat{g}^\pi \hat{Q}_{\pi} \geq J \) at \( x_0 \). Combining Lemma 3.2 \( 4.2 \) and \( h'_1 \leq \frac{1}{15M^p} \) (see (4.3)), we obtain

\[
\hat{g}^\pi \hat{Q}_{\pi} \geq J + \tilde{J},
\]

where

(4.24)

\[
\tilde{J} = -\frac{\hat{g}^\pi (S^p_{n\pi} \varphi_{V_{n\pi}} + S^\varphi_{n\pi} \chi_{V_{n\pi}})}{\lambda_1} - 2h'_1 \sum_k \tilde{g}_{k\overline{k}} \tilde{g}^\pi \left( S^p_{n\pi} (\tilde{g}_{k\overline{k}}) + S^\varphi_{n\pi} (\tilde{g}_{k\overline{k}}) \right) - 2h'_2 \Re\left( \sum_k \tilde{g}^\pi (S^p_{n\pi} \varphi_{mk} + S^\varphi_{n\pi} \chi_{mk}) \varphi_k \right) + A e^{-A\varphi} \tilde{g}^\pi (S^p_{n\pi} \varphi + S^\varphi_{n\pi} \varphi).
\]

Using \( \hat{Q}_p(x_0) = 0 \) and the similar calculation of (3.12), it then follows that

(4.25)

\[
\tilde{J} = -2\Re\left( \hat{g}^\pi S^p_{n\pi} \hat{Q}_p \right) = 0,
\]

as required. \qed

4.3. Proof of Proposition 4.1. In this subsection, we give the proof of Proposition 4.1.

• Partial second order estimate. We define

\[
I := \{ i \in \{1, \ldots, n\} \mid \tilde{g}_{1n} \geq A^3 e^{-2A\varphi} \tilde{g}_{0n} \text{ at } x_0 \}.
\]

Since \( A > 1 \) and \( \sup_M \varphi = 0 \), we have \( n \notin I \). The following lemma can be regarded as partial second order estimate.

Lemma 4.4. At \( x_0 \), we have

\[
\sum_k \sum_{i \notin I} (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) \leq C_A,
\]

where \( C_A \) is a uniform constant depending on \( A \).

Proof. Using \( \hat{Q}_i(x_0) = 0 \) and the Cauchy-Schwarz inequality, for each \( i = 1, 2, \ldots, n \), it is clear that

\[
\frac{\hat{g}^\pi |\varphi_{V_{i1}}|^2}{\lambda_1^2} = \hat{g}^\pi \left( h'_{1i}(|\varphi|^2_{g})_i + h'_{2i}(|\partial \varphi|^2_{g})_i - A e^{-A\varphi} \varphi_i \right) \leq 3(h'_{1i})^2 \hat{g}^\pi (|\varphi|^2_{g})_i + 3(h'_{2i})^2 \hat{g}^\pi (|\partial \varphi|^2_{g})_i + 3A^2 e^{-2A\varphi} \hat{g}^\pi |\varphi|^2.
\]

Combining this with Lemma 4.3 and discarding some positive terms, we obtain

\[
0 \geq h'_2 \sum_k \hat{g}^\pi (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) + (h'_1 - 3(h'_1)^2) \sum_i \hat{g}^\pi (|\varphi|^2_{g})_i^2
+ (h'_2 - 3(h'_2)^2) \sum_i \hat{g}^\pi (|\partial \varphi|^2_{g})_i^2 - C A^2 e^{-2A\varphi} \sum_i \hat{g}^\pi,
\]
where we used $\sum_i \tilde{g}_i^\tau \geq C^{-1}$ (see (3.3)). Using (4.2), (4.3), (4.4) and the definition of $I$, it is clear that
\[
0 \geq C^{-1} \sum_k \tilde{g}_k^\tau (|\varphi_{ik}|^2 + |\varphi_{\bar{k}}|^2) - C A^2 e^{-2A\varphi} \sum_i \tilde{g}_i^\tau
\]
\[
\geq C^{-1} \sum_k \sum_{i \notin I} \tilde{g}_k^\tau (|\varphi_{ik}|^2 + |\varphi_{\bar{k}}|^2) - C A^2 e^{-2A\varphi \tilde{g}^{\eta}}
\]
\[
\geq C^{-1} A^{-3} e^{2A\varphi} \tilde{g}^{\eta} \sum_k \sum_{i \notin I} (|\varphi_{ik}|^2 + |\varphi_{\bar{k}}|^2) - C A^2 e^{-2A\varphi \tilde{g}^{\eta}},
\]
as desired. \qed

Clearly, if $I = \emptyset$, then Proposition 4.1 follows from Lemma 4.4. Hence, we assume $I \neq \emptyset$ in the following argument.

- **Third order terms.** The key point is to deal with the "bad" third order term
\[
(4.27) \quad K := \frac{\tilde{g}_i^\tau |\varphi_{V_i V_i}|^2}{\lambda_1^2}.
\]
For any $\varepsilon \in (0, \frac{1}{3})$, we decompose the term $K$ into three parts as follows:
\[
K = \sum_{i \in I} \frac{\tilde{g}_i^\tau |\varphi_{V_i V_i}|^2}{\lambda_1^2} + 2\varepsilon \sum_{i \notin I} \frac{\tilde{g}_i^\tau |\varphi_{V_i V_i}|^2}{\lambda_1^2} + (1 - 2\varepsilon) \sum_{i \notin I} \frac{\tilde{g}_i^\tau |\varphi_{V_i V_i}|^2}{\lambda_1^2}
\]
\[
=: K_1 + K_2 + K_3.
\]

**Lemma 4.5.** At $x_0$, we have
\[
K_1 + K_2 \leq 3(h_1')^2 \tilde{g}^\tau (|\tilde{\omega}|^2)_{i}^2 + 3(h_2')^2 \tilde{g}^\tau (|\partial \varphi_{g}|^2)_{i}^2
\]
\[
+ 6\varepsilon A^2 e^{-2A\varphi \tilde{g}^\eta |\varphi_i|^2} + C \sum_i \tilde{g}_i^\tau.
\]

**Proof.** Using (4.26) and the definition of $I$, we obtain
\[
\sum_{i \in I} \frac{\tilde{g}_i^\tau |\varphi_{V_i V_i}|^2}{\lambda_1^2} \leq 3(h_1')^2 \sum_{i \in I} \tilde{g}_i^\tau (|\tilde{\omega}|^2)_{i}^2 + 3(h_2')^2 \sum_{i \in I} \tilde{g}_i^\tau (|\partial \varphi_{g}|^2)_{i}^2
\]
\[
+ 3 A^2 e^{-2A\varphi} \sum_{i \in I} \tilde{g}_i^\tau |\varphi_i|^2
\]
\[
\leq 3(h_1')^2 \sum_{i \in I} \tilde{g}_i^\tau (|\tilde{\omega}|^2)_{i}^2 + 3(h_2')^2 \sum_{i \in I} \tilde{g}_i^\tau (|\partial \varphi_{g}|^2)_{i}^2
\]
\[
+ \frac{3n \sup_{M} |\partial \varphi_{g}|^2}{A} \tilde{g}^{\eta}.
\]
By the similar calculation, it is clear that

\begin{equation}
2\varepsilon \sum_{i \notin I} g_i^\gamma |\varphi V_1 \bar{V}_1|^2 \leq 6\varepsilon (h_1')^2 \sum_{i \notin I} g_i^\gamma |\tilde{\omega}_g|^2 + 6\varepsilon (h_2')^2 \sum_{i \notin I} g_i^\gamma |(\partial \varphi_g)|^2 + 6A^2 e^{-2A\varphi} \sum_{i \notin I} g_i^\gamma |\varphi_i|^2.
\end{equation}

Combining (4.29), (4.30), \( \varepsilon \in (0, \frac{1}{4}) \) and \( A > 1 \), we obtain (4.28). \( \square \)

In order to deal with the term \( K_3 \), we define a local \((1,0)\) vector field by

\begin{equation}
\tilde{e}_1 = \frac{1}{\sqrt{2}} (V_1 - \sqrt{-1} JV_1).
\end{equation}

At \( x_0 \), since \( |\tilde{e}_1|^g = |JV_1|^g = 1 \), we write

\begin{equation}
\tilde{e}_1 = \sum_q \nu_q e_q, \quad \sum_q |\nu_q|^2 = 1
\end{equation}

and

\begin{equation}
JV_1 = \sum_{\alpha > 1} \mu_\alpha V_\alpha, \quad \sum_{\alpha > 1} \mu_\alpha^2 = 1,
\end{equation}

where we used the vector \( JV_1 \) is \( g \)-orthogonal to \( V_1 \).

**Lemma 4.6.** At \( x_0 \), if \( \lambda_1 \geq \frac{CA}{\varepsilon} \) for a uniform constant \( C_A \) depending on \( A \), then we have

\begin{equation}
K_3 \leq \sum_{\alpha > 1} \tilde{g}_i^\gamma |\varphi V_{\alpha} V_1|^2 + \sum_{p \neq q} \tilde{g}_p^\gamma \tilde{g}_q^\gamma |(g_{p\bar{q}}) V_1|^2 + 2h_1' \sum_{k,l} \tilde{g}_i^\gamma (\tilde{g}_{k\bar{l}}) V_1^2 + \frac{C}{\varepsilon} \sum_i \tilde{g}_i^\gamma.
\end{equation}

**Proof.** By (4.31), (2.2) and (2.5), we compute

\begin{equation}
\varphi V_1 \bar{V}_1 = \sqrt{2} \varphi V_{1\bar{1}} - \sqrt{-1} \varphi V_1 JV_1
\end{equation}

\begin{align*}
&= \sqrt{2} \sum_q \nu_q \varphi V_{1\bar{1}} - \sqrt{-1} \sum_{\alpha > 1} \mu_\alpha \varphi V_{\alpha} V_1 \\
&= \sqrt{2} \sum_q \nu_q \varphi V_1 - \sqrt{-1} \sum_{\alpha > 1} \mu_\alpha \varphi V_{\alpha} V_1 + E \\
&= \sqrt{2} \sum_q \nu_q (g_{1\bar{q}}) V_1 - \sqrt{-1} \sum_{\alpha > 1} \mu_\alpha \varphi V_{\alpha} V_1 + E \\
&= \sqrt{2} \sum_{q \notin I} \nu_q (g_{q\bar{q}}) V_1 + \sqrt{2} \sum_{q \notin I} \nu_q (g_{q\bar{q}}) V_1 - \sqrt{-1} \sum_{\alpha > 1} \mu_\alpha \varphi V_{\alpha} V_1 + E,
\end{align*}
where $E$ denotes a term satisfying $|E| \leq C\lambda_1$. Combining (4.34) with the Cauchy-Schwarz inequality, we compute

$$K_3 \leq \frac{C}{\varepsilon} \sum_{i \notin I} \frac{\tilde{g}_i^T}{\lambda_i^2} \left| \sum_{q \notin I} \nabla_q (\tilde{g}_q) V_i \right|^2 + \frac{C}{\varepsilon} \sum_i \tilde{g}_i^T + \left(1 - \varepsilon\right) \sum_{i \notin I} \frac{\tilde{g}_i^T \lambda_i^2}{\lambda_i^2} \sum_{q \notin I} \frac{\nabla_q (\tilde{g}_q) V_i - \sqrt{-1} \sum_{\alpha > 1} \mu_\alpha \varphi_{V_i V_i}}{\lambda_i^2}.$$

(4.35)

For convenience, we write $I = \{1, 2, \cdots, j\}$. Combining (4.6) and Lemma 4.4, it is clear that

$$\sum_{\alpha = 2j + 1}^{2n} \sum_{\beta = 1}^{2n} |\varphi_{\alpha\beta}| \leq C_A.$$

Since $V_1$ is the eigenvector of $\nabla^2 \varphi$ corresponding to $\lambda_1$, we have

$$|V_1^\alpha| = \left| \frac{1}{\lambda_1} \sum_{\beta = 1}^{2n} \varphi_{\alpha\beta} V_1^\beta \right| \leq \frac{C_A}{\lambda_1} \quad \text{for } \alpha = 2j + 1, \cdots, 2n.$$

Recalling the definitions of $\nu_q$ (see (4.32)) and $e_i$ (see (4.6)), we obtain

$$|\nu_q| \leq |V_1^{2q-1}| + |V_1^{2q}| \leq \frac{C_A}{\lambda_1} \quad \text{for } q \notin I.$$

(4.36)

For the first term of (4.35), by (4.36), we compute

$$\frac{C}{\varepsilon} \sum_{i \notin I} \frac{\tilde{g}_i^T}{\lambda_i^2} \left| \sum_{q \notin I} \nabla_q (\tilde{g}_q) V_i \right|^2 \leq \frac{C_A}{\varepsilon} \sum_{i \notin I} \frac{\tilde{g}_i^T (\tilde{g}_q) V_i}{\lambda_i^2} \leq \frac{C_A}{\varepsilon} \sum_{i \notin I} \frac{\tilde{g}_i^T (\tilde{g}_q) V_i}{\lambda_i^2}.$$

(4.37)

Using (2.5), we obtain $\tilde{g}_{qq} \leq C\lambda_1$ for any $q$. Hence, if $\lambda_1 \geq \frac{C_A}{\varepsilon}$, then we have

$$\frac{C_A}{\varepsilon \lambda_1^2} \leq \tilde{g}_{qq},$$

which implies

$$\frac{C_A}{\varepsilon} \sum_{i \notin I} \frac{\tilde{g}_i^T (\tilde{g}_q) V_i}{\lambda_i^4} \leq \sum_{i \notin I} \frac{\tilde{g}_i^T \tilde{g}_{qq} (\tilde{g}_q) V_i}{\lambda_1}.$$

(4.38)
By (4.25) and (4.24), we see that
\begin{equation}
(\tilde{g}_{\alpha})_k = \varphi_{\tilde{\alpha}k} + (S^p_{\tilde{\alpha}} \varphi)_{\tilde{\alpha}k} + (\bar{S}^p_{\tilde{\alpha}} \varphi)_{\tilde{\alpha}k}
\end{equation}
(4.39)
where \(E\) denotes a term satisfying \(|E| \leq C \lambda_1\). Using (4.31) and (4.32), it is clear that
\begin{equation}
V_1 = \frac{1}{\sqrt{2}}(\bar{e}_1 + \bar{e}_1) = \frac{1}{\sqrt{2}} \sum_k (\nu_k \epsilon_k + \nu_k \epsilon_k).
\end{equation}
(4.40)
Combining \(\lambda_1 \geq C_A\), (4.3) and (4.7), we have
\begin{equation}
\frac{CA}{\varepsilon \lambda_1^2} \leq \frac{1}{15M_F^2} \leq 2h' \quad \text{and} \quad \frac{CA}{\varepsilon \lambda_1^2} \leq 1.
\end{equation}
(4.41)
From (4.39), (4.40) and (4.41), it follows that
\begin{equation}
\frac{CA}{\varepsilon \lambda_1^2} \sum_{i \in I} \left| \frac{\bar{g}^\sigma_i (\tilde{g}_{\alpha})_i V_1}{\lambda_1^2} \right|^2 = \frac{CA}{\varepsilon \lambda_1^2} \sum_{i \in I} \left| \sum_k \frac{\nu_k (\tilde{g}_{\alpha}) k + \nu_k (\tilde{g}_{\alpha})_k}{\sqrt{2}} \right|^2
\end{equation}
(4.42)
\begin{equation}
\leq \frac{CA}{\varepsilon \lambda_1^2} \sum_{i,k} \bar{g}^\sigma_i |(\tilde{g}_{\alpha})_i|^2
\end{equation}
\begin{equation}
\leq \frac{CA}{\varepsilon \lambda_1^2} \sum_{i} \bar{g}^\sigma_i |(\tilde{g}_{\alpha})_i|^2 + \frac{CA}{\varepsilon \lambda_1^2} \sum_{i} \bar{g}^\sigma_i
\end{equation}
\begin{equation}
\leq 2h' \sum_{i,k,l} \bar{g}^\sigma_i |(\tilde{g}_{\alpha})_i|^2 + \sum_{i} \bar{g}^\sigma_i.
\end{equation}
Substituting (4.38) and (4.42) into (4.37), we obtain
\begin{equation}
\frac{C}{\varepsilon} \sum_{i \in I} \left| \frac{\bar{g}^\sigma_i}{\lambda_1^2} \sum_{q \in I} \nu_q (\tilde{g}_{\alpha}) V_1 \right|^2
\end{equation}
(4.43)
\begin{equation}
\leq \sum_{i \in I} \sum_{q \in I, q \neq i} \frac{\bar{g}^\sigma_i \bar{g}^\sigma_q |(\tilde{g}_{\alpha})_i V_1|^2}{\lambda_1^2} + 2h' \sum_{k,l} \bar{g}^\sigma_i |(\tilde{g}_{\alpha})_i|^2 + \sum_{i} \bar{g}^\sigma_i.
\end{equation}
Next, we deal with the third term of (4.35). For any \(\gamma > 0\), we have
\begin{equation}
(1 - \varepsilon) \sum_{i \in I} \frac{\bar{g}^\sigma_i}{\lambda_1^2} \sum_{q \in I} \nu_q (\tilde{g}_{\alpha}) V_1 - \sqrt{1 - \gamma} \sum_{a \geq 1} \mu_{\alpha} \varphi V_a V_i
\end{equation}
(4.44)
\begin{equation}
\leq (1 - \varepsilon)(1 + \gamma) \sum_{i \in I} \frac{2\bar{g}^\sigma_i}{\lambda_1^2} \sum_{q \in I} \nu_q (\tilde{g}_{\alpha}) V_1 \left| \sum_{\alpha > 1} \mu_{\alpha} \varphi V_a V_i \right|^2
\end{equation}
\begin{equation}
+ (1 - \varepsilon) \left(1 + \frac{1}{\gamma} \right) \sum_{i \in I} \frac{\bar{g}^\sigma_i}{\lambda_1^2} \left| \sum_{\alpha > 1} \mu_{\alpha} \varphi V_a V_i \right|^2.
\end{equation}
Using (4.31), (4.32) and the Cauchy-Schwarz inequality, we have

\[
\sum_{i \not\in I} \frac{2\tilde{g}^\ast i}{\lambda^2_1} \left| \sum_{q \in I} \nu_q(\tilde{g}_q) \nu_i \right|^2 \leq \sum_{i \not\in I} \frac{2\tilde{g}^\ast i}{\lambda^2_1} \left( \sum_{q} |\nu_q|^2 \tilde{g}_q \right) \left( \sum_{q \in I} |\tilde{g}^\ast_q(\tilde{g}_q) \nu_i|^2 \right) \\
= \tilde{g}(\tilde{e}_1, \tilde{e}_1) \sum_{i \not\in I} \sum_{q \in I} \frac{2\tilde{g}^\ast i \tilde{g}^\ast q}{\lambda^2_1} |(\tilde{g}_q) \nu_i|^2
\]

and

\[
\sum_{i \not\in I} \frac{\tilde{g}^\ast i}{\lambda^2_1} \left| \sum_{\alpha > 1} \mu_\alpha \varphi_{\alpha} \nu_i \right|^2 \leq \sum_{i \not\in I} \frac{\tilde{g}^\ast i}{\lambda^2_1} \left( \sum_{\alpha > 1} (\lambda_1 - \lambda_\alpha) \mu_\alpha^2 \right) \left( \sum_{\alpha > 1} |\varphi_{\alpha} \nu_i|^2 \right) \\
\leq \left( \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) \sum_{i \not\in I} \sum_{\alpha > 1} \frac{\tilde{g}^\ast i |\varphi_{\alpha} \nu_i|^2}{\lambda^2_1 (\lambda_1 - \lambda_\alpha)},
\]

where we used \(\sum_{\alpha > 1} \mu_\alpha^2 = 1\) (see (4.33)) in the last inequality. For convenience, we denote \(\tilde{g}(\tilde{e}_1, \tilde{e}_1)\) by \(\tilde{g}^-\). Substituting (4.45) and (4.46) into (4.44), we have

\[
(1 - \varepsilon) \sum_{i \not\in I} \frac{\tilde{g}^\ast i \sqrt{2} \sum_{q \in I} \nu_q(\tilde{g}_q) \nu_i - \sqrt{-1} \sum_{\alpha > 1} \mu_\alpha \varphi_{\alpha} \nu_i}{\lambda^2_1} \\
\leq (1 - \varepsilon)(1 + \gamma) \tilde{g}^- \sum_{i \not\in I} \sum_{q \in I} \frac{2\tilde{g}^\ast i \tilde{g}^\ast q |(\tilde{g}_q) \nu_i|^2}{\lambda^2_1} \\
+ (1 - \varepsilon) \left( 1 + \frac{1}{\gamma} \right) \left( \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) \sum_{i \not\in I} \sum_{\alpha > 1} \frac{\tilde{g}^\ast i |\varphi_{\alpha} \nu_i|^2}{\lambda^2_1 (\lambda_1 - \lambda_\alpha)}.
\]

Substituting (4.43) and (4.47) into (4.45), it is clear that

\[
K_3 \leq (1 - \varepsilon) \left( 1 + \frac{1}{\gamma} \right) \left( \lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) \sum_{i \not\in I} \sum_{\alpha > 1} \frac{\tilde{g}^\ast i |\varphi_{\alpha} \nu_i|^2}{\lambda^2_1 (\lambda_1 - \lambda_\alpha)} \\
+ (1 - \varepsilon)(1 + \gamma) \tilde{g}^- \sum_{i \not\in I} \sum_{q \in I} \frac{2\tilde{g}^\ast i \tilde{g}^\ast q |(\tilde{g}_q) \nu_i|^2}{\lambda^2_1} + \frac{C}{\varepsilon} \sum_i \tilde{g}^\ast i \\
+ \sum_{i \not\in I} \sum_{q \not\in I, q \neq i} \frac{\tilde{g}^\ast i \tilde{g}^\ast q |(\tilde{g}_q) \nu_i|^2}{\lambda_1} + 2h_i \sum_{k, l} \tilde{g}^\ast i |(\tilde{g}_{kl})|^2.
\]

Next, we give the proof of Lemma 4.6. We split up into two cases. The constant \(\gamma > 0\) will be different in each case.
Case 1. At \( x_0 \), we assume that

\[
\frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) \geq (1 - \varepsilon) \bar{g}_{1\bar{1}} > 0.
\]

Since \( \sum_{\alpha > 1} \mu_\alpha^2 = 1 \) (see (4.33)), it is clear that

\[
\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 = \sum_{\alpha > 1} (\lambda_1 - \lambda_\alpha) \mu_\alpha^2 > 0.
\]

Combining this with (4.49), we have

\[
\gamma := \frac{\lambda_1 - \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2}{\lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2} > 0.
\]

Substituting (4.49) and (4.50) into (4.48), we compute

\[
K_3 \leq 2(1 - \varepsilon) \sum_{i \not\in I} \sum_{\alpha > 1} \tilde{g}^i \left| \varphi_{V_\alpha V_i} \right|^2 / \lambda_1 (\lambda_1 - \lambda_\alpha) + 2 \sum_{i \not\in I} \sum_{q \in I} \tilde{g}^i \tilde{g}^{q\bar{I}} (\tilde{g}_{q\bar{I}}) V_i / \lambda_1
\]

\[
+ \sum_{i \not\in I} \sum_{q \not\in I, q \neq i} \tilde{g}^i \tilde{g}^{q\bar{I}} (\tilde{g}_{q\bar{I}}) V_i / \lambda_1 + 2 h'_i \sum_{k,l} \tilde{g}^k \tilde{g}^l (\tilde{g}_{k\bar{l}}) V_i / \lambda_1 + \frac{C}{\varepsilon} \sum_i \tilde{g}^i,
\]

which completes Case 1.

Case 2. At \( x_0 \), we assume that

\[
\frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) < (1 - \varepsilon) \bar{g}_{1\bar{1}}.
\]

Using (2.5), (4.31) and (4.33), we compute

\[
0 < \bar{g}_{1\bar{1}} = \tilde{g}(\bar{e}_1, \bar{e}_1)
\]

\[
= 1 + \partial \bar{\partial} \varphi(\bar{e}_1, \bar{e}_1)
\]

\[
\leq 1 + (\nabla^2 \varphi)(\bar{e}_1, \bar{e}_1) + C
\]

\[
\leq 1 + \frac{1}{2} (\varphi_{V_1 V_1} + \varphi_{J V_1}) + C
\]

\[
= \frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) + C.
\]

Combining (4.51) and (4.52), it is clear that

\[
\tilde{g}_{1\bar{1}} \leq \frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \lambda_\alpha \mu_\alpha^2 \right) + C \leq (1 - \varepsilon) \bar{g}_{1\bar{1}} + C,
\]
which implies

\[(4.53) \quad \tilde{g}_{11} \leq \frac{C}{\varepsilon}.\]

Using \[(4.52)\] again, we have

\[(4.54) \quad \lambda_1 - \sum_{\alpha>1} \lambda_\alpha \mu_\alpha^2 \leq 2\lambda_1 + C \leq 2(1 + \varepsilon^2)\lambda_1,\]

as long as \(\lambda_1 \geq \frac{C}{\varepsilon}\).

Now, we choose

\(\gamma := \frac{1}{\varepsilon^2}.\)

Combining \[(4.53), (4.54)\] and \(\varepsilon \in (0, \frac{1}{3})\), we have

\[(4.55) \quad (1 - \varepsilon) \left(1 + \frac{1}{\gamma}\right) \left(\lambda_1 - \sum_{\alpha>1} \lambda_\alpha \mu_\alpha^2\right) \sum_{i \notin I} \sum_{\alpha>1} \frac{\tilde{g}^|\varphi_{V_0V_i}|^2}{\lambda_1^2(\lambda_1 - \lambda_\alpha)} \leq (1 - \varepsilon)(1 + \varepsilon^2)(2 + 2\varepsilon^2)\lambda_1 \sum_{i \notin I} \sum_{\alpha>1} \frac{\tilde{g}^|\varphi_{V_0V_i}|^2}{\lambda_1^2(\lambda_1 - \lambda_\alpha)} \leq 2 \sum_{i \notin I} \sum_{\alpha>1} \frac{\tilde{g}^|\varphi_{V_0V_i}|^2}{\lambda_1(\lambda_1 - \lambda_\alpha)},\]

From \[(4.51)\] and \[(4.53)\], it follows that

\[(4.56) \quad (1 - \varepsilon)(1 + \gamma)\tilde{g}_{11} \sum_{i \notin I} \sum_{q \in I} \frac{2\tilde{g}^|\varphi_{V_0V_i}|^2}{\lambda_1^2} \leq (1 - \varepsilon) \left(1 + \frac{1}{\varepsilon^2}\right) \frac{C}{\varepsilon} \sum_{i \notin I} \sum_{q \in I} \frac{2\tilde{g}^|\varphi_{V_0V_i}|^2}{\lambda_1^2} \leq \frac{C}{\varepsilon^3} \sum_{i \notin I} \sum_{q \in I} \frac{\tilde{g}^|\varphi_{V_0V_i}|^2}{\lambda_1^2} \leq 2 \sum_{i \notin I} \sum_{q \in I} \frac{\tilde{g}^|\varphi_{V_0V_i}|^2}{\lambda_1},\]

as long as \(\lambda_1 \geq \frac{C}{\varepsilon}\). Substituting \[(4.55), (4.56)\] into \[(4.48)\], we get

\[K_3 \leq 2 \sum_{i \notin I} \sum_{\alpha>1} \frac{\tilde{g}^|\varphi_{V_0V_i}|^2}{\lambda_1(\lambda_1 - \lambda_\alpha)} + 2 \sum_{i \notin I} \sum_{q \neq i} \frac{\tilde{g}^|\varphi_{V_0V_i}|^2}{\lambda_1} + \sum_{i \notin I} \sum_{q \neq i} \frac{\tilde{g}^|\varphi_{V_0V_i}|^2}{\lambda_1} + 2h'_1 \sum_{k,l} |\tilde{g}(\bar{g}_{kl})|^2 + \frac{C}{\varepsilon} \sum_{i} \tilde{g}^|\varphi_{V_0V_i}|^2 \leq \sum_{\alpha>1} \frac{\tilde{g}^|\varphi_{V_0V_i}|^2}{\lambda_1(\lambda_1 - \lambda_\alpha)} + \sum_{p \neq q} \frac{\tilde{g}^|\varphi_{V_0V_i}|^2}{\lambda_1} + 2h'_1 \sum_{k,l} |\tilde{g}(\bar{g}_{kl})|^2 + \frac{C}{\varepsilon} \sum_{i} \tilde{g}^|\varphi_{V_0V_i}|^2,\]
which completes Case 2.

Combining Lemma 4.5 and 4.6 we obtain an upper bound of the "bad" third order term $K$:

$$K = K_1 + K_2 + K_3$$

\begin{align*}
&\leq 2 \sum_{\alpha>1} \frac{\tilde{g}^{\alpha\overline{\alpha}}\tilde{\varphi}_{\alpha\overline{\alpha}V_\alpha}}{\lambda_1(\lambda_1 - \lambda_\alpha)} + \sum_p \frac{\tilde{g}^{p\overline{p}}\tilde{g}^q(\tilde{g}_{q\overline{p}})_{V_\alpha}}{\lambda_1} + 2h'_1 \sum_{k,l} \tilde{g}^{\alpha\overline{\alpha}}(\tilde{g}_{k\overline{l}})_{V_\alpha} \\
&+ 3(h'_1)^2 \tilde{g}^{\alpha\overline{\alpha}}(|\tilde{\omega}|^2)_{V_\alpha} + 3(h'_2)^2 \tilde{g}^{\alpha\overline{\alpha}}(|\partial\tilde{\varphi}|^2)_{V_\alpha} \\
&+ 6\varepsilon A^2 e^{-2A\varphi} \tilde{g}^{\alpha\overline{\alpha}} |\tilde{\varphi}|^2 + \frac{C}{\varepsilon} \sum \tilde{g}^{\alpha\overline{\alpha}}.
\end{align*}

(4.57)

Now we are in a position to prove Proposition 4.1.

Proof of Proposition 4.1. Combining Lemma 4.3, (4.27), (4.57) and (4.2), it is clear that

$$0 \geq h'_1 \sum_k \tilde{g}^{\alpha\overline{\alpha}} (|\varphi|^{\alpha \overline{\alpha}} + |\tilde{\varphi}|^{\alpha \overline{\alpha}}) + (A^2 e^{-A\varphi} - 6\varepsilon A^2 e^{-2A\varphi}) \tilde{g}^{\alpha\overline{\alpha}} |\tilde{\varphi}|^2$$

$$+ \left( A^2 e^{-A\varphi} - \frac{C_0}{\varepsilon} \right) \sum \tilde{g}^{\alpha\overline{\alpha}} - A^2 e^{-A\varphi},$$

where $C_0$ is a uniform constant. We choose

$$A = 6C_0 + 1$$

and $\varepsilon = \frac{e^{A\varphi(x_0)}}{6}$. Recalling $\sup_M \varphi = 0$, we see that

$$A^2 e^{-A\varphi} - \varepsilon A^2 e^{-2A\varphi} \geq \frac{1}{6}$$

and $A^2 e^{-A\varphi} - \frac{C_0}{\varepsilon} \geq 1$.

It then follows that

$$h'_1 \sum_k \tilde{g}^{\alpha\overline{\alpha}} (|\varphi|^{\alpha \overline{\alpha}} + |\tilde{\varphi}|^{\alpha \overline{\alpha}}) + \sum \tilde{g}^{\alpha\overline{\alpha}} \leq C.$$

(4.58)

From $\sum \tilde{g}^{\alpha\overline{\alpha}} \leq C$ and $\frac{\det \tilde{g}}{\det g} \leq C$, we have $\tilde{g}^{\alpha\overline{\alpha}} \leq C$ for each $i$. Combining this with (4.3) and (4.58), we obtain $\lambda_1(x_0) \leq C$. By (4.7), it is clear that

$$\sup_M |\nabla^2 \varphi|_g + 1 = M_R \leq C,$$

as desired.

5. PROOFS OF THEOREM 1.1, 1.2, 1.3, 1.4 AND 1.5

In this section, we prove Theorem 1.1, 1.2, 1.3, 1.4 and 1.5.

Proof of Theorem 1.1. Theorem 1.1 is an immediate consequence of Proposition 2.2 and 3.4.

Proof of Theorem 1.2. By the assumptions of Theorem 1.2, there exists a sequence of positive smooth function $f_i$ on $M$ such that
(1) \( \lim_{i \to \infty} \| f_i - f \|_{C^0} = 0 \);

(2) for a uniform constant \( C \),

\[
\sup_M f_i \leq C, \quad \sup_M |\partial (f_i^{rac{1}{n}})|_g \leq C, \quad \nabla^2 (f_i^{rac{1}{n}}) \succeq -Cg.
\]

Using [14, Theorem 1.1], there exists a pair \((\varphi_i, b_i)\) where \( \varphi_i \in C^\infty(M) \) and \( b_i \in \mathbb{R} \), such that

\[
(\omega + \sqrt{-1} \partial \bar{\partial} \varphi_i)^n = f_i e^{b_i} \omega^n, \quad \omega + \sqrt{-1} \partial \bar{\partial} \varphi_i > 0, \quad \sup_M \varphi_i = 0.
\]

(5.1)

We need to prove \( |b_i| \leq C \). For the upper bound of \( b_i \), by the arithmetic-geometric mean inequality, we obtain

\[
(\det \tilde{g}/\det g)^{\frac{1}{n}} \leq \frac{1}{n} \left( n + n \sqrt{-1} \partial \bar{\partial} \varphi_i \wedge \omega^{n-1} / \omega^n \right) \leq 1 + \frac{(dJ d\varphi_i) \wedge \omega^{n-1}}{2 \omega^n}.
\]

(5.2)

Combining (5.1), (5.2) and the Stokes' formula, we compute

\[
\int_M f_i e^{\frac{b_i}{n}} \omega^n \leq \int_M \left( \frac{\det \tilde{g}}{\det g} \right)^{\frac{1}{n}} \omega^n \leq \text{Vol}(M, \omega) + \frac{1}{2} \int_M (dJ d\varphi_i) \wedge \omega^{n-1} \leq \text{Vol}(M, \omega) + C \int_M |\varphi_i| \omega^n.
\]

(5.3)

Since \( \lim_{i \to \infty} \| f_i - f \|_{C^0} = 0 \) and \( f \neq 0 \), we have

\[
\frac{1}{2} \int_M f_i^\frac{1}{n} \omega^n \leq \int_M f_i^\frac{1}{n} \omega^n \quad \text{for sufficiently large } i.
\]

(5.4)

Using (5.3), (5.4) and Proposition 2.1, it is clear that

\[
e^{b_i} \leq \left( \frac{C}{\int_M f_i^\frac{1}{n} \omega^n} \right)^n.
\]

(5.5)

Next, we will prove \( b_i \geq -C \). Let \( x_0 \) be the minimum point of \( \varphi_i \). By the maximum principle, we have

\[
e^{b_i} f_i(x_0) = \frac{(\omega + \sqrt{-1} \partial \bar{\partial} \varphi_i)^n}{\omega^n}(x_0) \geq 1,
\]

which implies

\[
e^{b_i} \geq \frac{1}{\sup_M f_i} \geq \frac{1}{C}.
\]

(5.6)

Combining (5.5), (5.6) and Theorem 1.1 we obtain

\[
|b_i| + \sup_M |\varphi_i| + \sup_M |\partial \varphi_i|_g + \sup_M |\nabla^2 \varphi_i|_g \leq C.
\]
After passing to a subsequence, we show the existence of \( C^{1,1} \) solution to (1.2). □

**Proof of Theorem 1.3.**

By (1.3), it is clear that \( s \not\equiv 0 \). For any \( i \geq 1 \), we define

\[
    f_i = (|s_i|^2 + i^{-1})^{\frac{1}{2}}.
\]

Thanks to Theorem 1.2, it suffices to verify that

\[
    \sup_M |\partial f_i|g \leq C \quad \text{and} \quad \nabla^2 f_i \geq - Cg,
\]

for a constant \( C \) which is independent of \( i \). For any point \( x_0 \in M \), there exists a local section \( s_0 \) in a neighbourhood of \( x_0 \) such that \( |s_0|^2_h \equiv 1 \). We write

\[
    s = (s_R + \sqrt{-1}s_I)s_0,
\]

where \( s_R \) and \( s_I \) are local functions near \( x_0 \). It then follows that

\[
    |s|^2_h = s_R^2 + s_I^2 \quad \text{and} \quad f_i = (s_R^2 + s_I^2 + i^{-1})^{\frac{1}{2}}.
\]

For any \( g \)-unit vector field \( V \) near \( x_0 \), we compute

\[
    V(f_i) = \frac{s_R V(s_R) + s_I V(s_I)}{(s_R^2 + s_I^2 + i^{-1})^{\frac{1}{2}}},
\]

which implies

\[
    |V(f_i)| \leq C.
\]

Applying \( V \) to (5.8), we obtain

\[
    VV(f_i) = \frac{s_R VV(s_R) + s_I VV(s_I) + (V(s_R))^2 + (V(s_I))^2}{(s_R^2 + s_I^2 + i^{-1})^{\frac{1}{2}}}
    - \frac{(s_R V(s_R) + s_I V(s_I))^2}{(s_R^2 + s_I^2 + i^{-1})^{\frac{3}{2}}}
    \geq \frac{s_R VV(s_R) + s_I VV(s_I)}{(s_R^2 + s_I^2 + i^{-1})^{\frac{1}{2}}}
    + \frac{(s_R V(s_I) + s_I V(s_R))^2}{(s_R^2 + s_I^2 + i^{-1})^{\frac{3}{2}}}
    \geq - C.
\]

Since \( x_0 \) and \( V \) are arbitrary, (5.7) follows from (5.9) and (5.10). □

We will prove Theorem 1.4 by means of blow-up construction. For the reader’s convenience, let us recall its definition first. Let \( \tilde{M} \) be the blow-up of \( M \) at \( p \) and \( \pi : \tilde{M} \to M \) be the projection map. We denote the exceptional divisor by \( E \) (i.e., \( E = \pi^{-1}(p) \)). We fix a coordinate chart \( (U; \{z^i\}_{i=1}^n) \) centered at \( p \), which we identify via \( \{z^i\}_{i=1}^n \) with the unit ball \( B_1 \subset \mathbb{C}^n \). By the exposition in [25], we identify \( \pi^{-1}(B_1) \) with \( \tilde{U} \) given by

\[
    \tilde{U} = \{(z,l) \in B_1 \times \mathbb{C}^{n-1} | z^i l^j = z^j l^i\},
\]
where \( l = [l^1, \ldots , l^n] \in \mathbb{CP}^{n-1} \). We set
\[
\tilde{U}_i = \{(z, l) \in B_1 \times \mathbb{CP}^{n-1} \mid l^i \neq 0\}.
\]
In \( \tilde{U}_i \), we have local coordinates \( \{w^i_j\}_{j=1}^n \):
\[
w^i_i = z^i \quad \text{and} \quad w^i_j = \frac{l^j}{l^i} \quad \text{for} \quad j \neq i.
\]
Hence, \( \{ (\tilde{U}_i, \{w^i_j\}_{j=1}^n) \} \) is a family of coordinate charts satisfying
\[
\tilde{U} = \bigcup_{i=1}^n \tilde{U}_i.
\]
The projection map \( \pi : \tilde{M} \to M \) is given in \( \tilde{U}_i \) by
\[
(5.11) \quad (w^1_i, \ldots , w^n_i) \to (w^i_1w^1_i, \ldots , w^i_i, \ldots , w^i_nw^n_i)
\]
and \( E \cap \tilde{U}_i \) is given by
\[
E \cap \tilde{U}_i = \{(z, l) \in B_1 \times \mathbb{CP}^{n-1} \mid w^i_i = z^i = 0\}.
\]
The line bundle \([E]\) over \( \tilde{U} \) has transition functions
\[
t_{ij} = \frac{z^i}{z^j} \quad \text{on} \quad \tilde{U}_i \cap \tilde{U}_j.
\]
Let \( s \) be the global section of \([E]\) over \( \tilde{M} \) by setting
\[
s = \begin{cases} 
    z^i & \text{on} \quad \tilde{U}_i, \\
    1 & \text{on} \quad \tilde{M} \setminus \pi^{-1}(B_{\frac{1}{2}}).
\end{cases}
\]
It follows that \( \{s = 0\} = E \). We construct a Hermitian metric \( h \) on \([E]\) as follows. Let \( h_1 \) be the Hermitian metric over \( \tilde{U} \) defined by
\[
h_1 = \frac{\sum_{j=1}^n |l^j|^2}{|l^i|^2} \quad \text{on} \quad \tilde{U}_i,
\]
and let \( h_2 \) be the Hermitian metric over \( \tilde{M} \setminus E \) such that \( |s|^2_h = 1 \). Then we define
\[
h = \rho_1 h_1 + \rho_2 h_2,
\]
where \( \{\rho_1, \rho_2\} \) is a partition of unity for the cover \( \{\pi^{-1}(B_1), \tilde{M} \setminus \pi^{-1}(B_{\frac{1}{2}})\} \) of \( \tilde{M} \). It follows that
\[
h = h_1 \quad \text{on} \quad \pi^{-1}(B_{\frac{1}{2}}).
\]
If \( \pi(\tilde{x}) = (z^1, \ldots , z^n) \in B_{\frac{1}{2}} \), then we have
\[
|s|^2_h(\tilde{x}) = \sum_{i=1}^n |z^i|^2 = |z|^2.
\]
On $\pi^{-1}(B_1^2 \setminus \{0\})$, the curvature $R(h)$ of the Hermitian metric $h$ is given by

$$R(h) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \left( \sum_{i=1}^{n} |z_i|^2 \right) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log |z|^2.$$ 

When $\varepsilon$ is sufficient small,

(5.13) \[ \tilde{\omega} = \pi^* \omega - \varepsilon R(h) \]

is a Kähler form on $\tilde{M}$ (see [23, p.178]). Moreover, the function $(\pi^* \omega)^n / \omega^n$ has analytic zeros of the form $|s|^2n - 2$. More precisely, we have the following lemma.

**Lemma 5.1.** There exists a smooth function $\tilde{F}$ on $\tilde{M}$ such that

(5.14) \[ (\pi^* \omega)^n = |s|^2n - 2 e^{\tilde{F}} \omega^n. \]

**Proof.** By the definition of blow-up construction, it is clear that $(\pi^* \omega)^n / \omega^n \neq 0$, $|s|_h^2 \neq 0$ on $\tilde{M} \setminus E$.

To prove Lemma 5.1 it suffices to prove (5.14) near $E$. By the definition of $\tilde{U}_i$, we have

$$E \subset \bigcup_{i=1}^{n} \left( \tilde{U}_i \cap \pi^{-1}(B_1^2) \right).$$

Hence, our aim is to verify

(5.15) \[ (\pi^* \omega)^n = |s|^2n - 2 e^{\tilde{F}} \omega^n \] on $\tilde{U}_i \cap \pi^{-1}(B_1^2)$, for each $i = 1, 2, \cdots, n$.

Without loss of generality, we only prove (5.15) when $i = 1$. We use $\omega_{\text{Eucl}}$ and $\tilde{\omega}_{\text{Eucl}}$ to denote the Euclidean metrics on $B_1$ and $\tilde{U}_1$, i.e.,

$$\omega_{\text{Eucl}} = \sqrt{-1} \sum_{j=1}^{n} dz^j \wedge d\bar{z}^j \quad \text{and} \quad \tilde{\omega}_{\text{Eucl}} = \sqrt{-1} \sum_{j=1}^{n} dw_1^j \wedge d\bar{w}_1^j.$$ 

Using (5.11), we compute

$$\pi^* \omega_{\text{Eucl}} = \pi^* \left( \sqrt{-1} \sum_{j=1}^{n} dz^j \wedge d\bar{z}^j \right)$$

$$= \sqrt{-1} dw_1^1 \wedge d\bar{w}_1^1 + \sqrt{-1} \sum_{j=2}^{n} d(w_1^j \bar{w}_1^j) \wedge d\overline{(w_1^j \bar{w}_1^j)}$$

$$= \sqrt{-1} \left( 1 + \sum_{j=2}^{n} |w_1^j|^2 \right) dw_1^1 \wedge d\bar{w}_1^1 + \sqrt{-1} \sum_{j=2}^{n} w_1^j \bar{w}_1^j dw_1^1 \wedge d\bar{w}_1^1$$

$$+ \sqrt{-1} \sum_{j=2}^{n} w_1^j \bar{w}_1^j dw_1^1 \wedge d\bar{w}_1^1 + \sqrt{-1} \sum_{j=2}^{n} |w_1^j|^2 dw_1^1 \wedge d\bar{w}_1^1.$$
By direct calculation, we obtain
\[(\pi^*\omega_{\text{Eucl}})^n = |w_1|^{2n-2} \left( 1 + \sum_{k=2}^{n} |w_k|^2 \right) \left( \sqrt{-1} \sum_{j=1}^{n} dw_1^j \wedge \bar{dw}_1^j \right)^n \]
\[= |w_1|^{2n-2} \omega^n_{\text{Eucl}}. \tag{5.16} \]

By (5.11) and (5.12), it is clear that
\[(5.17) \quad |s|^2_\rho(w_1, \cdots, w^n) = |w_1|^2 \left( 1 + \sum_{j=2}^{n} |w_j|^2 \right). \]

Combining (5.16) and (5.17), we have
\[(\pi^*\omega_{\text{Eucl}})^n = |s|^{2n-2}_\rho \left( 1 + \sum_{j=2}^{n} |w_j|^2 \right)^{-n+1} \tilde{\omega}_{\text{Eucl}}. \]

It follows that
\[(\pi^*\omega)^n = \pi^* \left( \frac{\omega^n}{\omega^n_{\text{Eucl}}} \right) (\pi^*\omega_{\text{Eucl}})^n \]
\[= \pi^* \left( \frac{\omega^n}{\omega^n_{\text{Eucl}}} \right) |s|^{2n-2}_\rho \left( 1 + \sum_{j=2}^{n} |w_j|^2 \right)^{-n+1} \tilde{\omega}_{\text{Eucl}} \]
\[= |s|^{2n-2}_\rho \pi^* \left( \frac{\omega^n}{\omega^n_{\text{Eucl}}} \right) \left( 1 + \sum_{j=2}^{n} |w_j|^2 \right)^{-n+1} \left( \tilde{\omega}_{\text{Eucl}} \right)^n \]
which implies (5.15), as desired. \qed

Now we are in a position to prove Theorem 1.4.

Proof of Theorem 1.4. For convenience, we use the same notations as above. To prove Theorem 1.4, we follow the approach of [36]. By (5.13), when \(\varepsilon_0\) is sufficiently small,
\[(5.18) \quad \tilde{\omega} = \pi^* \omega + \varepsilon_0 \sqrt{-1} \partial \bar{\partial} \left( \rho_1 \log |z|^2 + \rho_2 \right) \]
can be extended to a smooth Kähler form on \(\tilde{M}\). We still denote it by \(\tilde{\omega}\). By Lemma 5.1 there exists a smooth function \(\tilde{F}\) on \(\tilde{M}\) such that
\[(5.19) \quad (\pi^*\omega)^n = |s|^{2n-2}_\rho e^{\tilde{F}} \tilde{\omega}. \]

Thanks to Theorem 1.3 there exists a pair \((\tilde{\varphi}, b)\) where \(\tilde{\varphi} \in C^{1,1}(\tilde{M})\) and \(b \in \mathbb{R}\), such that
\[(\tilde{\omega} + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi})^n = |s|^{2n-2}_\rho e^{F+b} \tilde{\omega}. \]
Combining this with (5.19), we see that
\[(\tilde{\omega} + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi})^n = e^{F+b}(\pi^* \omega)^n.\]
Restricting this to \(\tilde{M} \setminus E\) and using (5.18), it is clear that
\[(\pi^* \omega + \sqrt{-1} \partial \bar{\partial} (\varepsilon_0 (\rho_1 \log |z|^2 + \rho_2) + \tilde{\varphi}))^n = e^{F+b}(\pi^* \omega)^n\]
on \(M \setminus \{p\}\).
Defining \(\varphi = \varepsilon_0 (\rho_1 \log |z|^2 + \rho_2) + \tilde{\varphi}\), we obtain
\[(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = (\varepsilon_0 \delta_p + e^{F+b})\omega^n.\]
Since \(\int_M e^F \omega^n = \int_M \omega^n = 1\), we have \(e^b = 1 - \varepsilon_0\). Then \(\varphi\) is the desired solution. □

Proof of Theorem 1.5. It suffices to establish the boundary estimate. The zero and first order estimate were proved in [3, Section 3]. When \(\partial M\) is pseudoconcave, combining [4, Lemma 7.16] and the argument of [3, Theorem 3.2'], we obtain the second order estimate. When \(\partial M\) is Levi-flat, the second order estimate was proved in [3, Theorem 3.2'].

6. Proof of Theorem 1.6

In this section, we give the proof of Theorem 1.6. First, we generalize Proposition 4.1 in a slightly more setting. Let \((M, \omega, J)\) be a compact almost Hermitian manifold of real dimension \(2n\), with nonempty smooth boundary. Let \(\{e_i\}_{i=1}^n\) be a local frame for \(T^{(1,0)}_CM\). By (2.5), the complex Monge-Ampère equation (1.1) can be expressed as
\[
\begin{align*}
\det(g_{ij} + \phi_{ij} + S_p_{ij} \phi_p + S_p_{ij}^T \phi_p) &= f \det(g_{ij}), \\
g_{ij} + \phi_{ij} + S_p_{ij} \phi_p + S_p_{ij}^T \phi_p &> 0,
\end{align*}
\]
where \(\phi_{ij} = (\nabla^2 \varphi)(e_i, e_j)\), \(\nabla\) is the Levi-Civita connection of \(g\) and \(S\) is a tensor field defined by (2.3).
Motivated by this, we introduce the following complex Monge-Ampère type equation
\[
\begin{align*}
\det(g_{ij} + \phi_{ij} + T_{ij}^p \phi_p + T_{ij}^p \phi_p^T) &= f \det(g_{ij}), \\
g_{ij} + \phi_{ij} + T_{ij}^p \phi_p + T_{ij}^p \phi_p^T &> 0,
\end{align*}
\]
where \(T\) is a tensor field satisfying
\[
T_{ij}^p = T_{ji}^p^T.
\]
The condition \(\overline{T_{ij}^p} = T_{ij}^p\) and \(\overline{T_{ij}^p} = T_{ij}^p^T\) can be regarded as an analog of (2.3).

Proposition 6.1. Let \(\varphi\) be a smooth solution of (6.1). Then there exists a constant \(C\) depending only on \(\sup_M |\varphi|\), \(\sup_M |\partial \varphi|_g\), \(\sup_{\partial M} |\nabla^2 \varphi|_g\), \((M, \omega, J)\), \(\sup_M f\), \(\sup_M |\partial (\overline{f^{1/2}})|_g\) and lower bound \(\overline{\nabla^2 (f^{1/2})}\) such that
\[\sup_M |\nabla^2 \varphi|_g \leq C.\]
Proof. We omit the proof since it is almost identical to that of Proposition 4.1 (just replacing \( S \) by \( T \)). \( \square \)

Now we are in a position to prove Theorem 1.6.

Proof of Theorem 1.6. To prove Theorem 1.6, it suffices to prove the existence of \( C^{1,1} \) solution to the geodesic equation (1.8). Our goal is to derive the \( C^{1,1} \) estimate for the perturbation geodesic equation (1.9). First, we define two Hermitian metrics on \( N \times [1, \frac{3}{2}] \) by

\[
\omega = 2r^{-2}\omega_c, \quad \tilde{\omega} = 2r^{-2}\Omega, \quad \tilde{\omega} = 2r^{-2}\Omega,
\]

where \( \omega_c \) is the Kähler form of \( g_c \). Clearly, \( (N \times [1, \frac{3}{2}], \omega) \) is a compact \((m + 1)\)-dimensional Hermitian manifold with nonempty smooth boundary. For convenience, let \( g, \tilde{g} \) be the corresponding Riemannian metrics of \( \omega, \tilde{\omega} \). We use \( \nabla \) to denote the Levi-Civita connection of \( g \). Let \( \{e_i\}_{i=1}^{m+1} \) be a local frame for \( T_C^{(1,0)} M \). Recalling (2.3), we have

\[
(\partial \overline{\partial} \psi)(e_i, \overline{e}_j) = \psi_{ij} + S^p_{ij} \psi_p + S^p_{ij} \overline{\psi}_p, \quad (\partial \overline{\partial} r)(e_i, \overline{e}_j) = r_{ij} + S^p_{ij} r_p + S^p_{ij} \overline{r}_p,
\]

where \( \psi_{ij} = (\nabla^2 \psi)(e_i, \overline{e}_j) \) and \( r_{ij} = (\nabla^2 r)(e_i, \overline{e}_j) \). Since \( \frac{\partial}{\partial r} \) is a vector field on \( C(N) \), we write

\[
\frac{\partial}{\partial r} = V^p e_p + \overline{V}^p \overline{e}_p.
\]

Combining (1.7) and (6.3), we compute

\[
\tilde{g}_{ij} = g_{ij} + (\partial \overline{\partial} \psi)(e_i, \overline{e}_j) - \frac{\partial \psi}{\partial r}(\partial \overline{\partial} r)(e_i, \overline{e}_j)
\]

\[
= g_{ij} + \psi_{ij} + S^p_{ij} \psi_p + S^p_{ij} \overline{\psi}_p
\]

\[
- (V^p \psi_p + \overline{V}^p \overline{\psi}_p) \left( r_{ij} + S^p_{ij} r_p + S^p_{ij} \overline{r}_p \right)
\]

\[
= g_{ij} + \psi_{ij} + T^p_{ij} \psi_p + \overline{T}^p_{ij} \overline{\psi}_p,
\]

where

\[
T^p_{ij} = S^p_{ij} - \left( r_{ij} + S^p_{ij} r_p + S^p_{ij} \overline{r}_p \right) V^p,
\]

\[
\overline{T}^p_{ij} = S^p_{ij} - \left( r_{ij} + S^p_{ij} r_p + S^p_{ij} \overline{r}_p \right) \overline{V}^p.
\]

It follows from (2.4) that the tensor field \( T \) satisfies (6.2). By (6.4), the perturbation geodesic equation (1.9) is equivalent to

\[
\det(g_{ij} + \psi_{ij} + T^p_{ij} \psi_p + \overline{T}^p_{ij} \overline{\psi}_p) = \varepsilon f \det(g_{ij}),
\]

\[
g_{ij} + \psi_{ij} + T^p_{ij} \psi_p + \overline{T}^p_{ij} \overline{\psi}_p > 0.
\]

By [29, Theorem 1], there exists a smooth solution of (1.9), and we denote it by \( \psi_\varepsilon \). Combining [29, Theorem 1, Proposition 3] with (6.3) and \( r \in [1, \frac{3}{2}] \),...
we obtain

\begin{equation}
\|\psi_\varepsilon\|_{C^2(N \times [1, \frac{5}{2}], g)} + \sup_{\partial(N \times [1, \frac{5}{2}])} |\nabla^2 \psi_\varepsilon|_g \leq C,
\end{equation}

where $C$ does not depend on $\varepsilon$. By the equivalence of (1.9) and (6.5), $\psi_\varepsilon$ is also a smooth solution of (6.5). Thanks to Proposition 6.1 and (6.6), we obtain the $C^{1,1}$ estimate

\[ \sup_{N \times [1, \frac{5}{2}]} |\nabla^2 \psi_\varepsilon|_g \leq C, \]

where $C$ does not depend on $\varepsilon$. Letting $\varepsilon \to 0$, we showed existence of $C^{1,1}$ solution to the geodesic equation (1.8), as required. \square

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