Burgulence and Alfvén waves heating mechanism of solar corona

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(Dated: September 26, 2018)

Heating of magnetized turbulent plasma is calculated in the framework of Burgers turbulence [A.M. Polyakov, Phys. Rev. E. 52, 6183 (1995)]. Explicit formula for the energy flux of Alfvén waves along the magnetic field is presented. The Alfvén waves are considered as intermediary between the turbulent energy and the heat. The derived results are related to a wave channel of heating of the solar corona. If we incorporate amplification of Alfvén waves by shear flow the suggested model of heating can be applied to analysis of the missing viscosity of accretion discs and to reveal why the quasars are the most powerful sources of light in the universe. We suppose that the Langevin-Burgers approach to turbulence we have applied in the current work can be also helpful for other systems where we have intensive interaction between a stochastic turbulent system and waves and can be used in many multidisciplinary researches in hydrodynamics and MHD.

PACS numbers:

I. INTRODUCTION

For more than 60 years we are still facing the perplexing astrophysical problem why the temperature of the solar corona is two orders of magnitude higher than the temperature of the photosphere; for review and references see for example Ref. [3, sec. 5.1]. The purpose of the current work is to investigate the wave mechanism of heating of the solar corona, according to which the energy is being transported by magneto-hydrodynamical waves, generated in the stochastic convective region. In the framework of this scenario the famous correlation between the solar activity and the coronal emission in the X-ray range can be easily explained. The novelty in the model is the incorporation of the Burgers approach to the turbulence in the problem for the calculation of the spectral density of the Alfvén waves. In the framework of this model we derived an explicit formulas for: 1) the spectral density of the MHD waves, 2) the total energy density and 3) the dissipated power per unit volume. In perspective the results derived here can be generalized for a realistic model of the turbulence and the distribution of density, temperature and magnetic field. In the current work the model task for calculation of the wave power, emitted by a turbulent half-space and transmitted in a non-moving magnetized plasma, is considered as an illustrative example. The homogeneous magnetic field is oriented perpendicularly to the interface of both half-spaces.

II. BURGULENCE

The Burgers approach to the turbulence gives an opportunity for approximate treatment of plenty of problems; as a tutorial lecture on Burgulence and a source of main references we recommend Ref. [5]. This approach to the turbulence has been introduced in the astrophysics by Zeldovich, but didn’t become very popular. Yet in the last decade the incorporation of the quantum field theories gave incitement to the theory of turbulence. Kolmogorov power laws have been derived and gradually the Burgers approach converted from a sophisticated high energy physics problem to a standard tool for investigation of the turbulence over different physical phenomena. The concept of random driver in the special case of formation of spicules and the problem of heating of space plasma in general was considered in the recent works Ref. [15]; here we would like to emphasize on the similarity between propagation of Alfvén waves in an infinite magnetized plasma and kink waves in solar spicules; for other papers devoted on the subject see Ref. [14]. There exist also different ways of generation of waves by turbulence where the influence of the random driver is negligible.

In the next section we will consider the application of the Burgers approach to the theory of creation of Alfvén waves from the turbulence and the stochastic granulation. In the $\beta \sim 1$ region of the solar atmosphere Alfvén waves can also be generated by resonant conversion of acoustic oscillations.

III. WAVE AMPLITUDES AND SPECTRAL DENSITIES

According to the Burgers idea the influence of the turbulent vortices on the fluid can be modeled by introduction of a random volume density of an external force $F(t, r)$ on the right hand side of the Navier-Stokes equation

$$\rho(\partial_t + \mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla p + \eta \Delta \mathbf{V} + \mathbf{j} \times \mathbf{B} + \mathbf{F}. \quad (1)$$

Analyzing low frequency and long wavelength phenomena one can approximately suppose the random force to
be a white noise with a δ-function correlation

\begin{equation}
(F(t_1, r_1)F(t_2, r_2)) = \hat{\Gamma} \rho^2 \delta(t_1 - t_2) \delta(r_1 - r_2) \mathbb{I},
\end{equation}

where \( \hat{\Gamma} \) is the Burgers parameter. For the other notions we use standard notations: \( p \) is the pressure, \( \eta \) is the viscosity, \( \mathbf{B} \) is the magnetic field.

The current \( \mathbf{j} \) is given by Ohm’s law

\begin{equation}
\mathbf{j} = \sigma \mathbf{E}', \quad \mathbf{E}' = \mathbf{E} + \mathbf{V} \times \mathbf{B},
\end{equation}

where \( \sigma \) is the electrical conductivity and \( \mathbf{E}' \) expresses the effective electric field acting on the fluid.

In quasi-stationary approximation, when the \( \partial_t \mathbf{E} \) term is negligible, the evolution of the magnetic field in a plasma with constant resistivity is

\begin{equation}
\partial_t \mathbf{B} = \text{rot} (\mathbf{V} \times \mathbf{B}) - \nu_m \text{rot} \text{rot} \mathbf{B}.
\end{equation}

We consider an incompressible fluid

\begin{equation}
\text{div} \mathbf{V} = 0
\end{equation}

with wave amplitudes for the velocity

\( \mathbf{V} = (V_x, V_y, V_z), \quad V \ll V_A \)

and the magnetic field

\( \mathbf{B} = \mathbf{B}_0 + \mathbf{B}', \quad \mathbf{B}'(t, r) = (B'_x, B'_y, B'_z), \quad B' \ll B_0 \)

being small compared to the Alfvén speed and the external magnetic field. The \( z \)-axes is chosen in a vertical direction along the constant magnetic field

\( \mathbf{B}_0 = (0, 0, B_0) = \mathbf{B}_0 \mathbf{e}_z. \)

Analogously for the pressure we also suppose small deviations from a constant value \( p_0 \)

\( p = p_0 + p' \).

The linearized system of magnetohydrodynamic (MHD) equations then reads

\begin{align*}
\partial_t \mathbf{V} &= -\frac{\nabla p'}{\rho} + \frac{\mathbf{F}}{\rho} + \frac{B_0}{\rho_0} \left( \partial_x B'_y - \partial_y B'_x \right) + \nu_k \Delta \mathbf{V}, \\
\partial_t \mathbf{B}' &= \mathbf{B}_0 \partial_z \mathbf{V} + \nu_m \Delta \mathbf{B}', \\
\text{div} \mathbf{B}' &= 0, \quad \text{div} \mathbf{V} = 0,
\end{align*}

where

\begin{equation}
\nu_m \equiv \frac{\varepsilon_0 c^2}{\sigma}, \quad \nu_k \equiv \frac{\eta}{\rho}
\end{equation}

are respectively the magnetic and kinematic viscosities. Here we would like to emphasize that the incompressibility condition \( \text{div} \mathbf{V} = 0 \) is applicable in the convective zone of the sun, where the sound speed significantly exceeds the Alfvén speed \( c_s \gg V_A \). In this region the convective circulation is treated as the influence of the turbulence in the Burgers approach. The excited Alfvén waves then propagate along the magnetic filed lines and reach the corona where the direction of the inequality could be opposite.

Let us consider the evolution of the amplitudes of standing plane waves

\begin{align*}
\mathbf{V}(t, \mathbf{r}) &= V_A \mathbf{v}_k(t) \sin(k \cdot \mathbf{r}), \\
\mathbf{B}'(t, \mathbf{r}) &= B_0 \mathbf{b}_k(t) \cos(k \cdot \mathbf{r}), \\
F(t, \mathbf{r}) &= \rho V_A \mathbf{f}_k(t) \sin(k \cdot \mathbf{r}), \\
p' &= \rho V_A \mathbf{p}_k \cos(k \cdot \mathbf{r}), \\
(f'_p(t_1)f_k(t_2)) &= \Gamma \delta(t_1 - t_2) \delta_{\mathbf{p}, \mathbf{k}} \mathbb{I}.
\end{align*}

This is a technical tool for evaluation of the average energy density used in many textbooks on statistical physics. Due to the negligible dependance on the boundary conditions usage of running waves would lead to the same result.

Substitution of this ansatz in Eq. (10) gives a separation of the variables and for the wave amplitudes we have a system of ordinary differential equations

\begin{align*}
d_t \mathbf{v} &= p \mathbf{k} + f - V_A \left( \begin{array}{c} k_x b_y - k_y b_x \\ k_y b_x - k_x b_y \\ 0 \end{array} \right) - \nu_k k^2 \mathbf{v}, \\
d_t \mathbf{b} &= V_A k_z \mathbf{v} - \nu_m k^2 \mathbf{b},
\end{align*}

where for the sake of brevity the wave-vector indices are omitted. Time differentiation of the incompressibility equation Eq. (19) and substitution in Eq. (10) gives the explicit form of the pressure

\begin{equation}
p = -\frac{\mathbf{k} \cdot \mathbf{f}}{k^2} - V_A b_z
\end{equation}

Further the substitution of the pressure from Eq. (20) in Eq. (10) gives

\begin{align*}
\dot{\mathbf{v}} &= -V_A k_z \mathbf{b} - \nu_k k^2 \mathbf{v} + \mathbf{f}_\perp, \\
\dot{\mathbf{b}} &= V_A k_z \mathbf{v} - \nu_m k^2 \mathbf{b},
\end{align*}

where

\begin{equation}
\mathbf{f}_\perp = \hat{\Pi}_\perp \cdot \mathbf{f}
\end{equation}

is the transverse projection of the external force with respect to the wave-vector \( \mathbf{k} \) by the polarization operator \( \hat{\Pi}_\perp \)

\begin{equation}
\hat{\Pi}_\perp = \mathbb{I} - \frac{\mathbf{k} \otimes \mathbf{k}}{k^2} = \left( \begin{array}{ccc}
k_x^2 + k_y^2 - k_x k_y & -k_x k_z & -k_y k_z \\
-k_x k_y & k^2 - k_z^2 & -k_x k_z \\
-k_y k_z & -k_x k_z & k^2 - k_x^2 \end{array} \right)
\end{equation}

This method for elimination of the pressure is used in Ref. [11] dedicated on amplification of Alfvén waves in
the presence of shear flow. A continuation of this research in the axial symmetric case is presented in Ref. [13].

Let us investigate the eigen-modes $\propto \exp(-i\omega t)$ discarding for a moment the influence of the external force $f$. For the case of small dissipation the secular equations for all the components are the same

$$
\begin{bmatrix}
-i\omega + \nu k^2 & -V_A k_z \\
V_A k_z & -i\omega + \nu_m k^2
\end{bmatrix} = 0,
$$

(24)

with eigen-values

$$
\omega \approx \omega_A - \frac{i\gamma}{2},
$$

(25)

where $\omega_A = V_A k_z$ is the frequency of the Alfvén waves and

$$
\gamma = \nu k^2, \quad \nu = \nu_m + \nu_k
$$

(26)

is the attenuation coefficient.

In regime of negligible magnetic viscosity the MHD system Eq. (21) turns into the second order ordinary differential equation

$$
\ddot{\vec{v}} = \nu k^2 \vec{v} - \omega_A^2 \vec{v} + \vec{f}_\perp.
$$

(27)

If we introduce the $x$-component of the displacement

$$
x(t) = \int_0^t v_x(t') dt'
$$

(28)

we obtain an effective oscillator equation under an external force

$$
\ddot{x} = \nu_k k^2 \dot{x} - \omega_A^2 x + f_{\perp},
$$

(29)

which in case of negligible kinematic viscosity has an effective energy

$$
\mathcal{E} = \frac{1}{2} (\dot{x}^2 + \omega_A^2 x^2).
$$

(30)

This is the non-perturbed by the friction and the weak random noise energy density. A harmonic oscillator under an external force is a well-known mechanical problem, see Ref. [1], Chap. V, Eq. (22,11). When the external force is a white noise a simple averaging gives that the energy is a linear function of time; in other words the random noise gives a constant power to the oscillator

$$
d_t \langle \mathcal{E} \rangle = d_t \left( \frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega_A^2 x^2 \right) = d_t \langle v_x^2 \rangle = \frac{\Gamma}{2},
$$

(31)

where

$$
\Gamma \equiv \frac{f_{\perp}}{V_A^2}
$$

(32)

is the Burgers parameter in the correlator for the plane-wave amplitudes of the transverse projection of the random force in Eq. (14)

$$
(f_{p,\perp}(t_1) f_{k,\perp}(t_2)) = \Gamma \delta(t_1 - t_2) \delta_{p,k} \delta.
$$

(33)

the technical details of this simple derivation will be omitted. Here we suppose that the wave energy does not increase significantly over one period

$$
\Gamma \ll \omega_A
$$

(34)

and the brackets stand for a period and noise averaging. In static regime the condition for dynamic equilibrium requires that the power received from the noise Eq. (31) has to be equal to the dissipated power

$$
d_t \langle \mathcal{E} \rangle = -\gamma \langle \mathcal{E} \rangle
$$

(35)

and this gives the dimensionless static averaged energy of the oscillator

$$
\mathcal{E}_{st} = \frac{\Gamma}{2\gamma} = \langle v_x^2 \rangle_{st}.
$$

(36)

Now we are going to use this model example for derivation of the spectral density of Alfvén waves. For the volume density of the average wave energy we have

$$
\langle \mathcal{E} \rangle = \frac{\langle B_z^2 \rangle}{2\mu_0} + \frac{\langle V^2 \rangle}{2\rho}.
$$

(37)

This result may be rewritten as

$$
\langle \mathcal{E} \rangle = \frac{B_0^2}{4\mu_0} \langle b^2 \rangle + \frac{V^2}{4\rho} \langle \vec{v}^2 \rangle = \frac{1}{2} \langle \vec{v}^2 \rangle p_B,
$$

(38)

where we have preformed a volume averaging

$$
\langle \sin^2(k \cdot r) \rangle = \frac{1}{2},
$$

(39)

and used the definitions for the magnetic pressure

$$
p_B \equiv \frac{B_0^2}{2\mu_0}
$$

(40)

and Alfvén speed

$$
\frac{V_A^2}{2\rho} = \frac{B_0^2}{2\mu_0}.
$$

(41)

In Eq. (38) we can substitute the static squared amplitude $\langle v_x^2 \rangle_{st}$ from Eq. (36) and the attenuation coefficient from Eq. (26). Taking into account the $y$ and $z$-components of the Alfvén wave velocities results in an additional triplication of the energy. Then for the static value of the density of the wave energy we finally derive

$$
E_k \equiv \langle \mathcal{E} \rangle_{st} = \frac{3}{4} \frac{\Gamma p_B}{\nu k^2}.
$$

(42)

This spectral density is the main detail for the statistical analysis made in the next section.
IV. ENERGY DENSITY, HEATING RATE AND ENERGY FLUX

In order to calculate the total static energy density we have to perform a summation of the spectral density Eq. \((12)\) over all wave-vectors

\[ E_{\text{tot}} = \sum_k E_k = \mathcal{V} \int \frac{d^3k}{(2\pi)^3} E_k. \quad (43) \]

The wave-vectors’ cut-off is determined by the condition for the existence of Alfvén waves, i.e. by the region where the Alfvén frequency \(\omega_A\) equals the attenuation coefficient \(\gamma\)

\[ V_A \omega_A = \nu k^2. \quad (44) \]

This equation sets the natural cut-off \(k\) for the vertical component of the wave-vector \(k_z\), parallel to the constant magnetic field \(B_0\)

\[ k = \frac{V_A}{\nu}. \quad (45) \]

Taking into consideration that the Alfvén waves are spreading axially symmetric as they follow the magnetic field lines we may rewrite the wave-vector as

\[ k^2 = k_z^2 + k_{\rho}^2, \quad (46) \]

which fixes the maximal value of the plane components of the wave-vector \(k_{\rho}\) as a function of the vertical component cut-off \(k_{z_{(\text{max})}}\)

\[ k_{(\text{max})}^2 = \sqrt{k_z(k_z - k_{z_{(\text{max})}})}. \quad (47) \]

These cut-offs will be used as boundary conditions in the calculations of the total wave energy, integrated over all possible wave-vectors in Eq. \((43)\), and the corresponding wave power. In this way considering the axial symmetry we obtain

\[ E_{\text{tot}} = \frac{3\Gamma \rho_B}{(2\pi)^3} \int_{k_z=0}^{k_z_{(\text{max})}} \int_{k_{\rho}=0}^{k_{(\text{max})}} \frac{1}{k_z^2 + k_{\rho}^2} d(\pi k_{\rho}^2) dk_z. \quad (48) \]

where we have taken into account that due to infrared divergencies the vertical component of the wave-vector should also have a minimal value, determined by the typical size \(R\) of the given magnetic field loop. Then, having in mind Eq. \((23)\), the final expression for the total energy becomes

\[ E_{\text{tot}} = \frac{3\Gamma \rho_B}{8\pi^2\nu^3} \ln \left( \frac{V_A R}{\nu} \right) - 1. \quad (49) \]

Hereby, for the Alfvén waves energy flux along the magnetic field we find

\[ S = \frac{1}{2} E_{\text{tot}} V_A = \frac{3\Gamma \rho_B}{16\pi^2\nu^3} \ln \left( \frac{V_A R}{\nu} \right) - 1. \quad (50) \]

Multiplication of the spectral density with decay \(\gamma\) gives the heating rate and summation in the wave-vectors’ space gives the dissipated by the Alfvén waves per unit volume power

\[ Q = \mathcal{V} \int E_k \gamma k \frac{d^3k}{(2\pi)^3} = \frac{\Gamma V_A}{8\pi^2\nu^3} \rho_B. \quad (51) \]

Perhaps magnetic field dependance of the heating rate can explain the correlation between the magnetic flux and X-ray luminosity\(^{[2]}\). In order to apply this result we need to know the correlation between the magnetic field and the radius of the active region.

The parameter with dimension length \(R\) which we introduced due to the infrared divergencies participate in the final formulas for the total energy density Eq. \((49)\) and energy flux Eq. \((50)\). It seems very plausible if this detail could be helpful to reveal the scaling correlations of the length of coronal loops and other observable quantities. It is premature, however, to start a detailed discussion based on a model example. The purpose of the present work is to demonstrate that the Burgers approach should be used for realistic computer simulations in the future.

V. DISCUSSION AND CONCLUSIONS

Up to now, the coronal heating problem has not a satisfactory explanation not because of a lack of interest, but rather due to difficulties concerning the simultaneous obtaining of reliable observational data for all processes running on the solar surface. For determination of the parameters of each model as a rule we are forced to use indirect means. The model, which we now present on the arena suffers the same disadvantage. For its confirmation it is necessary to know: 1) the distribution of magnetic field near the sunspots, 2) velocity-velocity correlator in horizontal direction or, which is equivalent, the correlator of the small stochastic perpendicular components of the magnetic field, etc. In short, apart from a realistic model for the turbulence we also need a realistic model for all other variables. This naturally supposes a broad interdisciplinary collaboration, which involves the whole spectrum of the solar physics, from the statistical MHD to the X-ray luminosity of the solar corona. The confirmation of the suggested model requires realistic Monte Carlo simulations, for which the analytical calculations done in the current work Eq. \((50)\), Eq. \((49)\), Eq. \((51)\) are only test examples indispensable for a more realistic treatment of the problem. We propose an oversimplified, and therefore solvable, model, which realistic generalization will hopefully lead to an adequate theory for the solar corona heating.

A natural continuation of the current investigation is to take into account the influence of the shear flow over the magnetized turbulent plasma. Preliminary numerical calculations on the behavior of MHD waves in a shear
flow show a significant amplification of the slow magnetosonic mode. The waves energy amplification is actually a transformation of the shear flow energy into wave energy which at the end dissipates into heat. This dissipation originates an effective viscosity in the shear flow and the current scenario for heating of the solar corona may also happen to be the theory for the missing viscosity in accretion disks. Understanding the origin of an additional viscosity in a magnetized turbulent plasma is rather important for the proper explanation of the shining mechanism of quasars, as well as for the angular momentum transport during formation of compact astrophysical objects.

The theory for the coronal heating takes into consideration the influence of the stochastic random noise of the turbulence and the granulation over the evolution of Alfvén waves. A similar mathematical scheme is used in the condensed matter physics for calculation of the fluctuational superconductivity. In the physics of superconductors the wave field is the effective Ginzburg-Landau wave function and the noise is produced by Langevin thermal fluctuations, which come from the inhomogeneous term in the TDGL equation. Naturally, the solid state scheme is meticulously developed and the final analytical results can be used for fitting of repeatable experimental data from current-voltage characteristics. The present work represents a realization of a similar idea in the new astrophysical plasma area.

Acknowledgments

Discussions and support by D. Damianov, R. Erdélyi, I. Rousev and I. Zhelyazkov are highly appreciated.