ON THE STABILITY OF RADIAL SOLUTIONS TO AN ANISOTROPIC
GINZBURG-LANDAU EQUATION

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Abstract. We study the linear stability of entire radial solutions $u(re^{i\theta}) = f(r)e^{i\theta}$, with positive increasing profile $f(r)$, to the anisotropic Ginzburg-Landau equation

$$-\Delta u - \delta (\partial_x + i\partial_y)^2 u = (1 - |u|^2)u, \quad -1 < \delta < 1,$$

which arises in various liquid crystal models. In the isotropic case $\delta = 0$, Mironescu showed that such solution is nondegenerately stable. We prove stability of this radial solution in the range $\delta \in (\delta_1, 0]$ for some $-1 < \delta_1 < 0$, and instability outside this range. In strong contrast with the isotropic case, stability with respect to higher Fourier modes is not a direct consequence of stability with respect to lower Fourier modes. In particular, in the case where $\delta \approx -1$, lower modes are stable and yet higher modes are unstable.

1. Introduction

Given $\delta \in (-1, 1)$ and $u: \mathbb{R}^2 \to \mathbb{C}$, we consider the anisotropic energy

$$E[u] = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u|^2 + \delta \Re \left\{ (\partial_\eta \bar{u})^2 \right\} + \frac{1}{4} (1 - |u|^2)^2 \, dx,$$

where $\partial_\eta = \partial_x + i \partial_y$.

Minimizers and stable critical points of $E$ are relevant in describing 2D point defects (or 3D straight-line defects) in some liquid crystal configurations (e.g. smectic-$C^*$ thin films [4] and nematics close to the Fréedericksz transition [2]). This energy can also be viewed as a toy model to understand intricate phenomena triggered by elastic anisotropy in the more complex Landau-de Gennes energy [11].

Remark 1.1. The anisotropic term $\Re \left\{ (\partial_\eta \bar{u})^2 \right\}$ can be rewritten as

$$\Re \left\{ (\partial_\eta \bar{u})^2 \right\} = (\nabla \cdot u)^2 - (\nabla \times u)^2,$$

so that, in view of the identity $|\nabla u|^2 = (\nabla \cdot u)^2 + (\nabla \times u)^2 - 2 \det(\nabla u)$, energy (1) differs from

$$\tilde{E}[u] = \int_{\mathbb{R}^2} \frac{k_s}{2} (\nabla \cdot u)^2 + \frac{k_b}{2} (\nabla \times u)^2 + \frac{1}{4} (1 - |u|^2)^2, \quad k_s = 1 + \delta, \ k_b = 1 - \delta,$$

only by the integral of the null Lagrangian $\det(\nabla u)$. This is precisely the form that appears in [4] where minimizers of

$$\tilde{E}_\varepsilon[u] = \int_{\Omega} \frac{k_s}{2} (\nabla \cdot u)^2 + \frac{k_b}{2} (\nabla \times u)^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2$$

are investigated in the limit as $\varepsilon \to 0^+$ in a bounded planar domain $\Omega$.  

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Critical points of $E$ are solutions of the Euler-Lagrange equation

$$\mathcal{L}_\delta u = (|u|^2 - 1)u \quad \text{in } \mathbb{R}^2$$
$$\mathcal{L}_\delta u := \Delta u + \delta \partial_\eta \bar{u}.$$  

We are interested in symmetric solutions of the form

$$u(re^{i\theta}) = f(r)e^{i\alpha} \quad \text{for some } \alpha \in \mathbb{R},$$

with a radial profile $f(r)$ satisfying

$$f(0) = 0, \quad \lim_{r \to +\infty} f(r) = 1, \quad |f(r)| > 0 \quad \forall r \in (0, \infty).$$

Formally, one can always look for solutions of (3) in the form (4) (as a consequence of the $O(2)$-invariance of $E$), and $f$ must solve

$$Tf + \delta e^{-2i\alpha} T\bar{f} = \left(|f|^2 - 1\right) f, \quad T = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2}.$$  

At this point we see a fundamental difference with respect to the isotropic case $\delta = 0$. If $\delta = 0$, one can find solutions as above for a real-valued function $f$, which moreover does not depend on $\alpha$. In the anisotropic case $\delta \neq 0$, as remarked in [2], the function $f$ can be real-valued only if $\alpha \equiv 0$ modulo $\pi/2$. In that case, the existence and uniqueness of a solution satisfying (5) follows from the case $\delta = 0$ (see [1, 6]). Otherwise, the function $f$ must be complex valued.

**Remark 1.2.** Another difference with respect to the isotropic case is that for $\delta \neq 0$ the Ansatz $u(re^{i\theta}) = f(r)e^{i\alpha}e^{i\theta}$ cannot provide a solution when the winding number $m$ is $\neq 1$.

In [2], the core energies of the two symmetric solutions corresponding to $\alpha = 0, \pi/2$ are compared, to find that the lowest energy corresponds to $\alpha = 0$ for $\delta < 0$ and $\alpha = \pi/2$ for $\delta > 0$. In view of Remark 1.1 this is consistent with the fact that $\nabla \times e^{i\theta} = 0$, while $\nabla \cdot ie^{i\theta} = 0$; indeed, for $\delta < 0$ the energy $\tilde{E}[u]$ in Remark 1.1 penalizes more strongly the term $(\nabla \times u)^2$ than the term $(\nabla \cdot u)^2$, since in this case $k_b = 1 - \delta > k_s = 1 + \delta$. In [4, Proposition 3.1] the authors use this to show that minimizers of (2) behave like $e^{i\alpha}e^{i\theta}$ around point defects, with $\alpha \equiv 0$ (resp. $\pi/2$) modulo $\pi$ if $\delta < 0$ (resp. $\delta > 0$). These results tell us, for $\delta \neq 0$, which one is the minimizing behavior at infinity.

Here, in contrast, we fix the far-field behavior and investigate the local stability of radial solutions with respect to compactly supported perturbations. For the isotropic case $\delta = 0$, this study has been performed in [12] (see also [4]), and the radial solution is stable. In the anisotropic situation $\delta \neq 0$ we find that the corresponding symmetric solution stays stable for negative $\delta$ close to zero and it loses stability for $\delta$ either positive or close to minus one (see Theorem 1.3 for precise statements).

It can be readily seen that the case $\alpha = \pi/2$ corresponds to $\alpha = 0$, after changing the sign of $\delta$. Accordingly, we only treat the case where $\alpha = 0$. That is, we investigate the linear stability of solutions $u$ of the form

$$u_{\text{rad}}^\delta(r, \theta) = f(r)e^{i\theta}, \quad f : (0, +\infty) \to (0, +\infty) \quad \text{with } f(0) = 0, \quad \lim_{r \to +\infty} f(r) = 1.$$
Let us note that the equation satisfied by \( u_\text{rad}^\delta \), (3), reduces to the following ODE for \( f \)
\[
(1 + \delta) T f = (f^2 - 1) f, \quad T = \frac{d}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2}.
\]

As pointed out in [2], the rescaling of the variable by \((1 + \delta)\frac{1}{2}\) simplifies (7) to the standard ODE corresponding to the isotropic case \(\delta = 0\). Whence, existence and uniqueness of \( f \) follow from [1, 6]. Moreover, it is known that \( f \) takes values in \((0,1)\) and is strictly increasing.

The second variation of the energy \( \mathcal{E} \) around \( u_\text{rad}^\delta \) is the quadratic form
\[
\Omega_\text{rad}^\delta[v] = \int_{\mathbb{R}^2} |\nabla v|^2 + \delta \text{Re} \left\{ (\partial_y v)^2 \right\} - (1 - |u_\text{rad}^\delta|^2)|v|^2 + 2(\nabla u_\text{rad}^\delta \cdot v)^2 \, dx
\]
(8)  \[
= \int_{\mathbb{R}^2} |\nabla v|^2 + \delta \text{Re} \left\{ (\partial_y v)^2 \right\} - (1 - f^2)|v|^2 + 2f^2(e^{i\theta} \cdot v)^2 \, dx
\]
associated to the linear operator obtained by linearizing (3) around \( u_\text{rad}^\delta \):
\[
\mathcal{L}(u_\text{rad}^\delta)[v] = -\Omega_\text{rad}^\delta v - (1 - |u_\text{rad}^\delta|^2)v + 2(\nabla u_\text{rad}^\delta \cdot v)u_\text{rad}^\delta,
\]
where \( u \cdot v := \text{Re} \{ uv \bar{\varepsilon} \} \) denotes the standard inner product of complex-valued functions.

Taking into account the asymptotic expansion \( f(r) = 1 + O(r^{-2}) \) as \( r \to \infty \) (see [1, 6]), it follows that the energy space of \( \Omega_\text{rad}^\delta \) naturally corresponds to
\[
\mathcal{H} := \left\{ v \in H^1_{\text{loc}}(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{1}{r^2}|v|^2 + (e^{i\theta} \cdot v)^2 \, dx < +\infty \right\}.
\]

Also, the translational invariance of \( \mathcal{E} \) readily provides two elements of \( \mathcal{H} \) at which \( \Omega_\text{rad}^\delta \) vanishes, namely
\[
\partial_x u_\text{rad}^\delta = e^{i\theta} \left( f' \cos \theta - i \frac{f}{r} \sin \theta \right), \quad \partial_y u_\text{rad}^\delta = e^{i\theta} \left( f' \sin \theta + i \frac{f}{r} \cos \theta \right),
\]
and the linear space they generate is denoted by
\[
K_0 = \text{span}\{\partial_x u_\text{rad}^\delta, \partial_y u_\text{rad}^\delta\}.
\]

Our main result shows that the symmetric solution \( u_\text{rad}^\delta \) is stable when \( \delta \leq 0 \) is small, and unstable otherwise:

**Theorem 1.3.** Let \( u_\text{rad}^\delta \) denote the radial solution \( 6 \) of the anisotropic Ginzburg-Landau equation \( 3 \), and let \( \Omega_\text{rad}^\delta \) denote the quadratic form \( 8 \) associated to the energy \( \mathcal{E} \) around \( u_\text{rad}^\delta \). Then, there exists a unique number \( \delta_1 \in (-1,0) \) such that
- for every \( \delta \in (\delta_1,0] \), \( u_\text{rad}^\delta \) is nondegenerately stable: namely,
  \[
  \Omega_\text{rad}^\delta[v] > 0 \quad \text{for all } v \in H \setminus K_0,
  \]
- for every \( \delta \in (-1,\delta_1) \cup (0,1) \), \( u_\text{rad}^\delta \) is linearly unstable: namely,
  \[
  \Omega_\text{rad}^\delta[v] < 0 \quad \text{for some } v \in H.
  \]

**Remark 1.4.** The most relevant range from the stand point of physics is \( \delta \in (-1,0] \) since for \( \delta > 0 \) the far-field behavior corresponding to \( \alpha = 0 \) is non-minimizing, and this translates here into instability of the radial solution.
Remark 1.5. In the stability range $\delta \in (\delta_1, 0]$, a contradiction argument as in [5, Lemma 3.1] provides a coercivity estimate of the form

$$Q_{\delta \text{rad}}[v] \geq C(\delta) \int_{\mathbb{R}^2} |\nabla v|^2 \, dx \quad \forall v \in K_0^+ : \int_{S^1} (i e^{i\theta}) \cdot v(r e^{i\theta}) \, d\theta = 0 \quad \forall r > 0,$$

where $\perp$ denotes orthogonality in $\mathcal{H}$. Using this coercivity for $\delta = 0$, one can deduce stability for small negative $\delta$ via a relatively simple perturbation argument, combined with properties of the lower modes in §3. Instead, we will give a more quantitative proof, which provides an explicit range for stability: we deduce that $\delta_1 \leq -1/\sqrt{5}$.

Our proof of Theorem 1.3 follows the general strategy of [12]: we decompose $v$ into Fourier modes

$$v = e^{i\theta} \sum_{n \in \mathbb{Z}} w_n(r) e^{in\theta},$$

and we are led to studying the sign of $Q_{\delta \text{rad}}$, separately, for each mode

$$e^{i\theta} \left( w_n(r) e^{in\theta} + w_{-n}(r) e^{-in\theta} \right).$$

As in [12], the lower modes $n = 0$ and $n = 1$ play a special role. They can be studied via an appropriate decomposition already used in [12] (see also [5]). For any $\delta \in (-1, 0]$ we find that these lower modes are stable, while for $\delta > 0$ the mode corresponding to $n = 0$ is unstable.

A major difference of the present work compared to [12] (or similar results in [8–10]) pertains to the higher modes $n \geq 2$. In contrast with the cited works, stability for the higher modes is not an obvious consequence of stability for the lower modes. More precisely in the isotropic case we have

$$Q_{\text{rad}}^0 \left[e^{i\theta} \left( w_+(r) e^{in\theta} + w_-(r) e^{-in\theta} \right) \right] \geq Q_{\text{rad}}^0 \left[e^{i\theta} \left( w_+(r) e^{i\theta} + w_-(r) e^{-i\theta} \right) \right] \quad \forall n \geq 1,$$

but for $\delta \neq 0$ this is not valid anymore, see (16). This feature is new and specific to the anisotropic case $\delta \neq 0$. Our strategy to study the sign of these higher modes is based on the same decomposition used for $n = 1$, and a careful balance of the contributions of additional terms, which end up causing instability for $\delta \approx -1$.

The article is organized as follows. In Section 2 we recall the splitting property of the quadratic form $Q_{\delta \text{rad}}$ with respect to Fourier expansion. In Section 3 we study the stability of lower modes, and in Section 4 the instability of higher modes. In Section 5 we give the proof of Theorem 1.3. In addition, we included Appendix A to recall the details of the decomposition used to study the lower modes, adapted to our notations.

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2. Fourier splitting

Recall that \( f(r) = f_0((1 + \delta)^{-\frac{1}{2}} r) \) where \( f_0 \) is the classical Ginzburg-Landau vortex profile corresponding to the case \( \delta = 0 \). That is, the unique solution of

\[
 f''_0 + \frac{1}{r} f'_0 - \frac{1}{r^2} f_0 = -(1 - f_0^2) f_0, \quad f_0 > 0 \text{ on } (0, +\infty), \quad f_0(0) = 0, \quad \lim_{r \to +\infty} f_0(r) = 1. 
\]  

We rescale variables and consider \( Q_\delta[v] = \Omega_{\text{rad}}^\delta[\tilde{v}] \) where \( \tilde{v}(\tilde{x}) = v((1 + \delta)^{-\frac{1}{2}} \tilde{x}) \), so that

\[
 Q_\delta[v] = \int_{\mathbb{R}^2} |\nabla v|^2 + \delta \text{Re} \left\{ (\partial_\eta \tilde{v})^2 \right\} + (1 + \delta) \left\{ 2 f_0^2 (e^{i\theta} \cdot v)^2 - (1 - f_0^2) |v|^2 \right\} \, dx, 
\]  

which corresponds to the second variation of the appropriately rescaled energy around \( u_{\text{rad}}^0 \).

Following [12] we decompose \( v \) using Fourier series, as

\[
 v = e^{i\theta} w = e^{i\theta} \sum_{n \in \mathbb{Z}} w_n(r) e^{in\theta}, 
\]  

where we have conveniently shifted the index \( n - 1 \mapsto n \).

This decomposition provides a “diagonalization” of the linearized operator:

**Lemma 2.1.** The quadratic form (10) splits as

\[
 Q_\delta[v] = Q_\delta \left[ w_0(r) e^{i\theta} \right] + \sum_{n \geq 1} Q_\delta \left[ e^{i\theta} \left( w_n(r) e^{in\theta} + w_{-n}(r) e^{-in\theta} \right) \right]. 
\]

**Proof of Lemma 2.1.** Lemma 2.1 essentially asserts that the family of functions

\[
 \{ w_0(r) e^{i\theta}, \{ e^{i\theta} \left( w_n(r) e^{in\theta} + w_{-n}(r) e^{-in\theta} \right) : n \geq 1 \}, 
\]

is orthogonal for the quadratic form \( Q \). This quadratic form (10) is composed of three terms. For the first term,

\[
 \int_{\mathbb{R}^2} |\nabla v|^2 \, dx, 
\]

the orthogonality of (12) is a standard fact (recall e.g. in [12]). For the third term,

\[
 \int_{\mathbb{R}^2} \left\{ f_0^2 (e^{i\theta} \cdot v)^2 - (1 - f_0^2) |v|^2 \right\} \, dx, 
\]

the orthogonality of (12) is proved in [12]. The novelty here, with respect to [12], concerns the anisotropic term

\[
 \int_{\mathbb{R}^2} \text{Re} \left\{ (\partial_\eta \tilde{v})^2 \right\} \, dx. 
\]

The orthogonality of (12) for this anisotropic term, as a matter of fact, follows from the calculations in [3 § 3.2]. As our notations are different, we sketch a proof here for the reader’s convenience.

We compute

\[
 \partial_\eta \tilde{v} = e^{i\theta} \partial_\eta \tilde{v} + \frac{ie^{i\theta}}{r} \partial_\eta \tilde{v} = \sum_{n \in \mathbb{Z}} \left( \tilde{w}'_n + \frac{1 + n}{r} \tilde{w}_n \right) e^{-in\theta}, 
\]

where we have conveniently shifted the index \( n - 1 \mapsto n \).
and deduce, using the orthogonality of \( \{ e^{in\theta} \} \) in \( L^2(S^1) \),

\[
\int_{S^1} \Re \left\{ (\partial_{\bar{\eta}} \bar{v})^2 \right\} \, d\theta
\]
\[
= \Re \left\{ \sum_{n,m \in \mathbb{Z}} \left( \bar{w}'_n + \frac{1+n}{r} \bar{w}_n \right) \left( \bar{w}'_m + \frac{1+m}{r} \bar{w}_m \right) \int_{S^1} e^{-i(n+m)\theta} \, d\theta \right\}
\]
\[
= \Re \left\{ \sum_{n \in \mathbb{Z}} \left( \bar{w}'_n + \frac{1+n}{r} \bar{w}_n \right) \left( \bar{w}'_{-n} + \frac{1-n}{r} \bar{w}_{-n} \right) \right\}
\]

This implies the announced orthogonality and completes the proof of Lemma 2.1. \( \square \)

According to the decomposition of Lemma (2.1), we define the quadratic forms

\[
Q^\delta_0[\varphi] = \frac{1}{2\pi} Q^\delta \left[ \varphi(r)e^{i\theta} \right]
\]
\[
Q^\delta_n[\varphi, \psi] = \frac{1}{2\pi} Q^\delta \left[ e^{i\theta} \left( \varphi(r)e^{in\theta} + \psi(r)e^{-im\theta} \right) \right]
\]

for \( \varphi \in \mathcal{H}_0 \),

where \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) are the natural spaces corresponding to the conditions \( \varphi(r)e^{i\theta} \in \mathcal{H} \) and \( e^{i\theta} (\varphi(r)e^{in\theta} + \psi(r)e^{-im\theta}) \in \mathcal{H} \) for \( n \geq 1 \), respectively.

\[
\mathcal{H}_0 = \left\{ \varphi \in H^1_{loc}(0, \infty) : \int_0^{+\infty} \left( |\varphi'|^2 + \left| \frac{\varphi}{r} \right|^2 + \Re \{ \varphi \}^2 \right) r \, dr < +\infty \right\},
\]
\[
\mathcal{H}_1 = \left\{ (\varphi, \psi) \in (H^1_{loc}(0, \infty))^2 : \int_0^{+\infty} \left( |\varphi'|^2 + |\psi'|^2 + \left| \frac{\varphi}{r} \right|^2 + \left| \frac{\psi}{r} \right|^2 + \left| \varphi + \bar{\psi} \right|^2 \right) r \, dr < +\infty \right\}
\]

**Remark 2.2.** Using the density of smooth functions in \( H^1_{loc} \) and cut-off functions \( \chi_\varepsilon \) such that \( \mathbf{1}_{2\varepsilon < r < \varepsilon^{-1}} \leq \chi_\varepsilon(r) \leq \mathbf{1}_{\varepsilon < r < 2\varepsilon^{-1}} \) and \( |\chi_\varepsilon(r)| \leq C/r \), we see that smooth test functions with compact support in \( (0, \infty) \) are dense in \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \). Hence, in the sequel, we will always be able to perform calculations assuming, without loss of generality, that \( \varphi \) and \( \psi \) are such test functions.
The quadratic forms $Q_0^\delta$ and $Q_n^\delta$ are explicitly given by

\begin{align}
Q_0^\delta[\varphi] &= \int_0^\infty \left( |\varphi'|^2 + \frac{1}{r^2} |\varphi|^2 + \delta \Re \left( \left( \varphi' + \frac{1}{r} \varphi \right)^2 \right) \right. \\
&\quad \left. + (1 + \delta) \left\{ 2f_0^2 (\Re \{\varphi\})^2 - (1 - f_0^2) |\varphi|^2 \right\} \right) rdr,
\end{align}

\begin{align}
Q_n^\delta[\varphi, \psi] &= \int_0^\infty \left( |\varphi'|^2 + |\psi'|^2 + \frac{(1+n)^2}{r^2} |\varphi|^2 + \frac{(1-n)^2}{r^2} |\psi|^2 \\
&\quad + 2\delta \Re \left\{ \left( \varphi' + \frac{1+n}{r} \varphi \right) \left( \psi' + \frac{1-n}{r} \psi \right) \right\} \\
&\quad + (1 + \delta) \left\{ f_0^2 (|\varphi| + |\psi|)^2 - (1 - f_0^2) (|\varphi|^2 + |\psi|^2) \right\} \right) rdr.
\end{align}

*Remark 2.3.* For every $n \geq 1$ there is a further splitting, namely

$$Q_n^\delta[\varphi, \psi] = Q_n^\delta[\Re \{\varphi\}, \Re \{\psi\}] + Q_n^\delta[\Im \{\varphi\}, -\Im \{\psi\}].$$

Consequently, it will be sufficient to consider real-valued test functions $\varphi, \psi$.

### 3. Study of the lower modes $Q_0^\delta$ and $Q_1^\delta$

We show that $Q_0^\delta$ is positive for $\delta \leq 0$, but it can become negative for $\delta > 0$. In addition, we prove that $Q_1^\delta$ is nonnegative for all $\delta \in (-1, 0]$.

#### 3.1. Positivity of $Q_0^\delta$ for $\delta \in (-1, 0]$.

Let us recall from (13) that $Q_0^\delta$ is given by

$$Q_0^\delta[\varphi] = \int_0^\infty \left( |\varphi'|^2 + \frac{1}{r^2} |\varphi|^2 + \delta \Re \left( \left( \varphi' + \frac{1}{r} \varphi \right)^2 \right) \right. \\
&\quad \left. + (1 + \delta) \left\{ 2f_0^2 (\Re \{\varphi\})^2 - (1 - f_0^2) |\varphi|^2 \right\} \right) rdr
$$

We now introduce the quadratic form

$$A_0[\varphi] := Q_0^\delta[\varphi] \\
= \int_0^\infty \left( |\varphi'|^2 + \frac{1}{r^2} |\varphi|^2 + 2f_0^2 (\Re \{\varphi\})^2 - (1 - f_0^2) |\varphi|^2 \right) rdr.$$
It is known that $A_0[\varphi] > 0$, unless $\varphi = 0$ (see Appendix A for more details). Moreover, we have the identity

$$Q_0^\delta[\varphi] = (1 + \delta)A_0[\text{Re} \{\varphi\}] + (1 - \delta)A_0[i\text{Im} \{\varphi\}] - 2\delta \int (1 - f_0^2)(\text{Im} \{\varphi\})^2 r \, dr$$

$$+ \delta \int_0^\infty \frac{d}{dr} \left[ (\text{Re} \{\varphi\})^2 - (\text{Im} \{\varphi\})^2 \right] dr$$

$$= (1 + \delta)A_0[\text{Re} \{\varphi\}] + (1 - \delta)A_0[i\text{Im} \{\varphi\}] - 2\delta \int (1 - f_0^2)(\text{Im} \{\varphi\})^2 r \, dr,$$

which is valid for any $\varphi \in C_c^\infty(0, \infty)$, hence for $\varphi \in H_0$ thanks to Remark 2.2. Since $1 - f_0^2 \geq 0$, we deduce the positivity of $Q_0^\delta$ for every $\delta \in (-1, 0]$.

3.2. **Instability for $\delta > 0$.** Using the formula (18) obtained for $A_0$ in Appendix A, we see that for any compactly supported real-valued test function $\chi$ we have

$$Q_0^\delta[i\chi_0\chi] = (1 - \delta) \int f_0^2(\chi')^2 r \, dr - 2\delta \int (1 - f_0^2)f_0^2 \chi^2 r \, dr.$$

Applying this to $\chi_n(r) = \chi_1(r/n)$, for a fixed test function $\chi_1$, and using the asymptotic expansion [1, 6]:

$$f_0(r) = 1 - \frac{1}{2r^2} + O(r^{-4}) \quad \text{as } r \to \infty,$$

we see that

$$\lim_{n \to \infty} Q_0^\delta[i\chi_0\chi_n] = (1 - \delta) \int (\chi_1')^2 r \, dr - 2\delta \int \frac{\chi_1^2}{r^2} r \, dr.$$

When $\delta > 0$, this expression must be negative for some $\chi_1$, since Hardy’s inequality is known to fail in two dimensions. Explicitly, by choosing

$$\chi_1(r) = \sin(\sqrt{\lambda \ln r})1_{(1, e^{\pi/\sqrt{\lambda}})}(r)$$

for $\lambda = \frac{\delta}{1 - \delta} > 0$,

we have that $\chi_1 \in H^1(0, \infty)$ is compactly supported, and

$$\lim_{n \to \infty} Q_0^\delta[i\chi_0\chi_n] = -\delta \int \frac{\chi_1^2}{r^2} r \, dr < 0.$$

Whence, for $\delta > 0$, the mode of order 0 already brings instability. This comes as no surprise as this mode corresponds to infinitesimal rotations (see Appendix A), and we know that the far-field behavior $e^{i\theta}$ is unstable: rotating this far-field behavior decreases the energy.
3.3. Positivity of $Q_1^\delta$ for $\delta \leq 0$. Recall, according to (14), that $Q_1^\delta$ is given by

$$Q_1^\delta[\varphi, \psi] = \int_0^\infty \left[ |\varphi'|^2 + |\psi'|^2 + \frac{4}{r^2} |\varphi|^2 ight. $$

$$+ 2\delta \text{Re} \left\{ \left( \varphi' + \frac{2}{r} \varphi \right) \overline{\psi'} \right\} $$

$$+ (1 + \delta) \left\{ f_0^2 |\varphi + \overline{\psi}'|^2 - (1 - f_0^2) \left( |\varphi|^2 + |\psi|^2 \right) \right\} r \, dr.$$ 

We introduce the quadratic form $A_1 := Q_1^0$, namely

$$A_1[\varphi, \psi] = \int_0^\infty \left[ |\varphi'|^2 + |\psi'|^2 + \frac{4}{r^2} |\varphi|^2 ight. $$

$$+ f_0^2 |\varphi + \overline{\psi}'|^2 - (1 - f_0^2) \left( |\varphi|^2 + |\psi|^2 \right) \left. \right] r \, dr.$$

It is a known fact that $A_1$ is nonnegative on $H_1$, and vanishes exactly at pairs $(\varphi, \psi)$ corresponding to maps $v$ which are linear combinations of $\partial_x u_{rad}^0$ and $\partial_y u_{rad}^0$ (see Appendix A for more details). Moreover, we have

$$Q_1^\delta[\varphi, \psi] - (1 + \delta) A_1[\varphi, \psi] = -\delta \int_0^\infty \left[ |\varphi'|^2 + |\psi'|^2 + \frac{4}{r^2} |\varphi|^2 \right] r \, dr$$

$$+ 2\delta \int_0^\infty \text{Re} \left\{ \left( \varphi' + \frac{2}{r} \varphi \right) \overline{\psi'} \right\} r \, dr$$

$$= -\delta \int_0^\infty \left| \varphi' + \frac{2}{r} \varphi - \overline{\psi}' \right|^2 r \, dr - 2\delta \int_0^\infty \frac{d}{dr} \left[ |\varphi|^2 \right] dr$$

$$= -\delta \int_0^\infty \left| \varphi' + \frac{2}{r} \varphi - \overline{\psi}' \right|^2 r \, dr,$$

for $(\varphi, \psi) \in (C_c^\infty(0, \infty))^2$, hence for all $(\varphi, \psi) \in H_1$. From this identity we infer that $Q_1^\delta \geq 0$ for every $\delta \in (-1, 0]$, and equality can only occur when $v$ is a linear combination of $\partial_x u_{rad}^0$ and $\partial_y u_{rad}^0$.

4. Study of the higher modes $Q_n^\delta$ for $n \geq 2$

4.1. Positivity of $Q_n^\delta$ for $n \geq 2$ and $\delta \in [-1/\sqrt{5}, 0]$. Let us recall: in the isotropic case, the positivity of $Q_n^\delta$ (any $n \geq 2$) is a consequence of the fact that $Q_n^0 \geq Q_1^0$. Here, from the
definition (14) of $Q^\delta_n$, we have

\begin{align*}
Q^\delta_n[\varphi, \psi] - Q^\delta_1[\varphi, \psi] &= (n - 1) \int_0^\infty \left[ \frac{n + 3}{r^2} |\varphi|^2 + \frac{n - 1}{r^2} |\psi|^2 - 2\delta \frac{n + 1}{r^2} \text{Re}\{\bar{\varphi}\bar{\psi}\} \\
&\quad + 2\frac{\delta}{r} \text{Re}\{\bar{\varphi}'\bar{\psi}' - \varphi'\bar{\psi}\} \right] r \, dr.
\end{align*}

Unlike what happens in the isotropic case, this does not obviously have a sign (because of the last term which contains derivatives).

It seems reasonable to use a decomposition for $\varphi, \psi$ adapted to $Q^\delta_1$, as in Appendix A. Accordingly, we define for any real-valued test functions $\zeta, \eta$, the adapted quadratic form

$$B^\delta_n[\zeta, \eta] = \frac{1}{2} Q^\delta_n[f_0'\zeta - r^{-1}f_0\eta, f_0'\zeta + r^{-1}f_0\eta]$$

Decomposing

$$Q^\delta_n = (1 + \delta)A_1 + Q^\delta_1 - (1 + \delta)A_1 + Q^\delta_n - Q^\delta_1$$

and using the above expressions of $Q^\delta_n - Q^\delta_1$ (16) and $Q^\delta_1 - (1 + \delta)A_1$ (15), we have, for real-valued $(\varphi, \psi) \in H_1$:

\begin{align*}
Q^\delta_n[\varphi, \psi] &= (1 + \delta)A_1[\varphi, \psi] \\
&\quad - \delta \int_0^\infty \left( \varphi' + \frac{2}{r} \varphi - \psi' \right)^2 r \, dr \\
&\quad + (n - 1) \int_0^\infty \left[ \frac{n + 3}{r^2} \varphi^2 + \frac{n - 1}{r^2} \psi^2 - 2\delta \frac{n + 1}{r^2} \varphi \psi \right] \\
&\quad + 2\delta(n - 1) \int_0^\infty \frac{1}{r} (\varphi\psi' - \varphi'\psi) \, r \, dr.
\end{align*}

When plugging in $\varphi = f_0'\zeta - r^{-1}f_0\eta$, $\psi = f_0'\zeta + r^{-1}f_0\eta$, the first term significantly simplifies thanks to the formula (19) for $A_1$ in Appendix A. For the other terms we directly expand

\begin{align*}
\varphi' + \frac{2}{r} \varphi - \psi' &= 2f_0'\zeta - \eta - 2f_0\eta', \\
\frac{n + 3}{r^2} \varphi^2 + \frac{n - 1}{r^2} \psi^2 - 2\delta \frac{n + 1}{r^2} \varphi \psi &= 2(1 - \delta) \frac{n + 1}{r^2} (f_0'\zeta)^2 + 2(1 + \delta) \frac{n + 1}{r^2} \left( \frac{f_0}{r} \eta \right)^2 - \frac{8}{r^2} f_0' f_0 \frac{f_0}{r} \eta, \\
\varphi\psi' - \varphi'\psi &= 2 \left( \frac{f_0}{r} \eta \right)' f_0'\zeta - 2(f_0')' f_0\eta.
\end{align*}
from which it follows that

$$B^\delta_n[\zeta, \eta] = (1/2)Q^\delta_n[f'_0\zeta - r^{-1}f_0\eta, f'_0\zeta + r^{-1}f_0\eta]$$

can be rewritten as

$$B^\delta_n[\zeta, \eta] = (1 + \delta) \int_0^\infty \left[ \frac{f_0^2}{r^2}(\eta')^2 + (f'_0)^2(\zeta')^2 + \frac{2}{r^2}f_0f'_0(\eta - \zeta)^2 \right] r dr$$

$$- 2\delta \int_0^\infty \left[ \frac{f_0'}{r} \left( \eta - \zeta \right) + \frac{f_0}{r^2} \eta' \right]^2 r dr$$

$$+ (n - 1) \int_0^\infty \left[ (1 - \delta) \frac{n + 1}{r^2}(f'_0\zeta)^2 + (1 + \delta) \frac{n + 1}{r^2} \left( \frac{f_0}{r} \right)^2 - \frac{4}{r^2} \left( f'_0 \zeta \right) \left( \frac{f_0}{r} \right) \right] r dr$$

$$+ 2\delta(n - 1) \int_0^1 \frac{1}{r} \left[ \left( \frac{f_0}{r} \right)' f'_0 \zeta - (f'_0 \zeta) \frac{f_0}{r} \eta \right] r dr.$$ Integrating by parts, the last integral becomes

$$\int_0^\infty \frac{1}{r} \left[ \left( \frac{f_0}{r} \right)' f'_0 \zeta - (f'_0 \zeta) \frac{f_0}{r} \eta \right] r dr = 2 \int_0^\infty \frac{1}{r} \left( \frac{f_0}{r} \right)' f'_0 \zeta r dr$$

$$= 2 \int_0^\infty \left[ \left( f'_0 - \frac{f_0}{r} \right) f'_0 \eta \zeta + \frac{f_0}{r^2} f_0^2 \right] r dr.$$ We use the first positive term in (17) in order to absorb this latter term: thanks to the identity

$$(1 + \delta) \frac{f_0^2}{r^2}(\eta')^2 + 4\delta(n - 1) \frac{f_0}{r^2} f'_0 \zeta = (1 + \delta) \left( \frac{f_0}{r} \eta' + \frac{2\delta}{1 + \delta} (n - 1) f'_0 \zeta \right)^2$$

$$- \frac{4\delta^2}{1 + \delta} (n - 1)^2 (f'_0)^2 \left( \frac{\zeta}{r} \right)^2,$$

we rewrite (17) as

$$B^\delta_n[\zeta, \eta] = B^\delta_{n,1}[\zeta, \eta] + (n - 1)B^\delta_{n,2}[\zeta, \eta],$$

$$B^\delta_{n,1}[\zeta, \eta] = (1 + \delta) \int_0^\infty \left[ \left( \frac{f_0}{r} \right)' f'_0 \eta \eta + \frac{2\delta}{1 + \delta} (n - 1) f'_0 \zeta \right]^2 r dr$$

$$+ 2 \int_0^\infty \left\{ (1 + \delta) f'_0 f_0 \left( \eta - \zeta \right)^2 r^2 - \delta \left[ f'_0 f_0 \left( \eta - \zeta \right) + \frac{f_0}{r} \eta' \right]^2 \right\} r dr,$$

$$B^\delta_{n,2}[\zeta, \eta] = \int_0^\infty q^\delta_n(r) \left[ \frac{f'_0}{r} \frac{\zeta}{r} - \frac{f_0}{r} \eta' \right] r dr,$$

and $q^\delta_n(r)$ is the quadratic form on $\mathbb{R}^2$ given by

$$q^\delta_n[r, X, Y] = a_nX^2 + b_nY^2 + 2c(r)XY,$$

$$a_n = (1 - \delta)(n + 1) - 4\frac{\delta^2}{1 + \delta} (n - 1)$$

$$b_n = (1 + \delta)(n + 1)$$

$$c(r) = -2 - 2\delta \left( 1 - r^2 f'_0 \right) f_0$$
We readily see that $B_n^{\delta, 1}$ is nonnegative for $\delta \leq 0$. Moreover, since $1 > rf'_0/f_0 > 0$ [7 Proposition 2.2], for $\delta \leq 0$, it follows that 
\[|c(r)| \leq 2.\]

As $b_n > 0$, a sufficient condition for $q_n^{\delta}(r)$ to be positive definite for all $r > 0$ is 
\[4 < a_n b_n = (1 - \delta^2)(n + 1)^2 - 4\delta^2(n^2 - 1).\]

This amounts to the condition 
\[0 < \alpha(\delta)n^2 + \beta(\delta)n + \gamma(\delta),\]

where 
\[\alpha(\delta) = 1 - 5\delta^2,\]
\[\beta(\delta) = 2(1 - \delta^2),\]
\[\gamma(\delta) = -3(1 - \delta^2).\]

For $\delta \in [-1/\sqrt{5}, 0]$ we have $\alpha(\delta), \beta(\delta) \geq 0$ so that the above polynomial in $n$ is nondecreasing on $[0, +\infty)$. Hence, it is positive for all values of $n \geq 2$ if and only if it is positive for $n = 2$. That is, 
\[0 < 4\alpha(\delta) + 2\beta(\delta) + \gamma(\delta) = 5 - 21\delta^2.\]

We deduce that $q_n^{\delta}$ is a positive definite quadratic form for all $n \geq 2$ whenever $\delta \in [-1/\sqrt{5}, 0]$. In particular, $B_n^{\delta, 2} \geq 0$ and therefore $Q_n^{\delta} \geq 0$ for $\delta \in [-1/\sqrt{5}, 0]$, with equality only at $(0, 0)$.

4.2. Instability for $\delta \approx -1$. In this section we show that $Q_n^{\delta}$ can take negative values for $\delta \approx -1$ and $n \geq 1$ large enough. To this end, we choose $\eta = \zeta$ in (17), to obtain 
\[\hat{B}_n^{\delta}[\zeta] = B_n^{\delta}[\zeta, \zeta] = (1 - \delta) \int_0^\infty f_0^2 r^2 (\zeta')^2 r dr + (1 + \delta) \int_0^\infty (f'_0)^2(\zeta')^2 r dr + (n - 1) \int_0^\infty \frac{\zeta^2}{r^2} \alpha_n^{\delta}(r) r dr.
\]

\[\alpha_n^{\delta}(r) = (1 - \delta)(n + 1)(f'_0)^2 + (1 + \delta)(n + 1) \left(\frac{f_0}{r}\right)^2 - 2(2 + \delta)f'_0 f_0 + 2\delta(f'_0)^2 - 2\delta f_0 f''_0.\]

Using the asymptotics of $f_0$ (116) 
\[f_0(r) = 1 - \frac{1}{2}r^{-2} + O(r^{-4}), \quad f'_0(r) = r^{-3} + O(r^{-5}), \quad f''_0(r) = -3r^{-4} + O(r^{-6}),\]

we find, for $r \to +\infty$, 
\[\alpha_n^{\delta}(r) = \frac{(1 + \delta)(n + 1)}{r^2} \left(1 - \frac{1}{r^2}\right) - 4 \frac{1 - \delta}{r^4} + O(r^{-6}).\]

For $\delta = -1$ the leading order is negative. Hence, there exists $\varepsilon > 0$ and a compact interval $[r_0, r_0 + 1]$ on which $\alpha_n^{-1} \leq -2\varepsilon$. Thus, we deduce that for all $n \geq 2$ there exists $\delta_n > -1$ such that for all $\delta \in (-1, \delta_n]$ 
\[-\varepsilon \geq \alpha_n^{\delta}(r), \quad \forall r \in [r_0, r_0 + 1].\]
Choosing a nonzero test function $\zeta_0$ with support in $[r_0, r_0 + 1]$, we obtain

$$\dot{B}_n[^{\delta}[\zeta_0] \leq C_1(\zeta_0) - (n - 1)\varepsilon C_2(\zeta_0) \quad \forall \delta \in (-1, \delta_n],$$

for some $C_1(\zeta_0), C_2(\zeta_0) > 0$. If $n$ is large enough this becomes negative. Compared to the isotropic case this is a really new situation: lower modes are positive but higher modes can bring instability.

5. Proof of Theorem 1.3

In what precedes we have shown that $u_\text{rad}^{\delta}$ is nondegenerately stable for small $\delta \leq 0$, and unstable for $\delta > 0$ and $\delta$ close to $-1$. In particular, setting

$$\delta_1 = \sup\{\delta \in (-1, 0) : u_\text{rad}^{\delta} \text{ is unstable}\},$$

we know that $-1 < \delta_1 < 0$. It remains to show that $u_\text{rad}^{\delta}$ is unstable for all $\delta \in (-1, \delta_1)$, and nondegenerately stable for $\delta \in (\delta_1, 0]$.

Let $\delta' \in (-1, \delta_1)$ be such that $u_\text{rad}^{\delta'}$ is unstable, that is, $Q^{\delta'}[v] < 0$ for some choice of $v \in H$. Given that $\delta \mapsto Q^{\delta}[v]$ is an affine function which is nonnegative for $\delta = 0$ and negative for $\delta = \delta'$, we deduce that $Q^{\delta}[v] < 0$ for all $\delta \leq \delta'$. Therefore, $u_\text{rad}^{\delta}$ is unstable for all $\delta \in (-1, \delta')$. By arbitrariness of $\delta'$ we deduce that $u_\text{rad}^{\delta}$ is unstable for all $\delta \in (-1, \delta_1)$.

Let us now fix $\delta \in (\delta_1, 0]$. By definition of $\delta_1$, $u_\text{rad}^{\delta}$ is not unstable for all $\delta \in (\delta_1, 0]$. In other words, $Q^{\delta}[v]$ is nonnegative for all $v \in H$. It remains to show that, in fact, $Q^{\delta}[v] > 0$ for all $v \in H \setminus \text{span}(\partial_x u_\text{rad}^0, \partial_y u_\text{rad}^0)$. We observe that the function $\delta \mapsto Q^{\delta}[v]$ is affine for any given $v \in H \setminus \text{span}(\partial_x u_\text{rad}^0, \partial_y u_\text{rad}^0)$; it is positive for $\delta = 0$ because $u_\text{rad}^0$ is nondegenerately stable, and it is nonnegative for $\delta \in (\delta_1, 0)$. Thus, it must be strictly positive for $\delta \in (\delta_1, 0)$. This proves the desired nondegenerate stability in the announced range.

APPENDIX A. Positivity of $A_0, A_1$

We sketch here the approach in [12], adapted to our notation (see also [5]), based on Hardy-type decompositions to show positivity of the two following quadratic forms

$$A_0[\varphi] = \int_0^\infty \left[ |\varphi'|^2 + \frac{1}{r^2} |\varphi|^2 + 2 f_0^2 (\text{Re}\{\varphi\})^2 - (1 - f_0^2) |\varphi|^2 \right] rdr,$$

$$A_1[\varphi, \psi] = \int_0^\infty \left[ |\varphi'|^2 + |\psi'|^2 + 4 \frac{1}{r^2} |\varphi|^2 + f_0^2 |\varphi + \psi|^2 - (1 - f_0^2) \left( |\varphi|^2 + |\psi|^2 \right) \right] rdr.$$
By density of test functions, and since \( (18) \), so that

\[
\int_0^\infty \left[ (f_0')^2|\varphi'|^2 + 2f_0f_0'\varphi' \cdot \varphi' + \frac{f_0^2}{r^2}|\varphi|^2 - (1 - f_0^2)f_0^2|\varphi|^2 \right] rdr = 0,
\]

so that

\[
(18) \quad A_0[f_0\varphi] = \int_0^\infty \left[ f_0^2|\varphi'|^2 + 2f_0^4(\text{Re} \{\varphi'\})^2 \right] rdr.
\]

By density of test functions, and since \( f_0 > 0 \), we deduce that \( A_0[\varphi] > 0 \) for any non-zero \( \varphi \in \mathcal{H}_0 \). Moreover \( A_0[\varphi] \approx 0 \) exactly when \( \varphi \approx if_0 \). This corresponds to the fact that in the isotropic case \( \delta = 0 \),

\[
\partial_\alpha[\psi^\alpha u_{\text{rad}}^\delta]|_{\alpha = 0} = if_0 e^{i\theta}
\]
solves the linearized equation due to rotational invariance.

For \( A_1 \), it is convenient to start by splitting it as

\[
A_1[\varphi, \psi] = A_1[\text{Re} \{\varphi\}, \text{Re} \{\psi\}] + A_1[\text{Im} \{\varphi\}, -\text{Im} \{\psi\}],
\]

so we may just treat the case of real-valued test functions \( \varphi, \psi \). Guided by the fact that

\[
\partial_z u_{\text{rad}}^0 = e^{i\theta}(f_0^2 \cos \theta - i\frac{f_0}{r} \sin \theta), \quad \partial_y u_{\text{rad}}^0 = e^{i\theta}(f_0^2 \sin \theta + i\frac{f_0}{r} \cos \theta),
\]

solve the linearized equation around \( u_{\text{rad}}^0 \), one uses the ansatz

\[
\varphi = f_0^0 \zeta - \frac{f_0}{r} \eta, \quad \psi = f_0^0 \zeta + \frac{f_0}{r} \eta,
\]

for some real-valued \( \eta, \zeta \in C_c^\infty(0, \infty) \). Testing equation \((9)\), solved by \( f_0 \), against \( f_0r^{-2}\eta^2 \) we obtain

\[
\int_0^\infty \left[ \left( \left( \frac{f_0}{r} \right)' \right)^2 \eta^2 + 2 \left( \frac{f_0}{r} \right)' \frac{f_0}{r} \eta \eta' + \frac{2}{r^2} f_0^2 \eta^2 - \frac{2}{r^3} f_0 f_0' \eta^2 - (1 - f_0^2)\frac{f_0^2}{r^2} \eta^2 \right] rdr = 0,
\]

and similarly testing \((9)\) against \((f_0^0\zeta^2)'\) we find

\[
\int_0^\infty \left[ (f_0')^2\zeta^2 + 2f_0^0f_0''\zeta \zeta' + \frac{2}{r^2}(f_0')^2\zeta^2 - \frac{2}{r^3} f_0 f_0' \zeta^2 + (3f_0^2 - 1)(f_0^2)\zeta^2 \right] rdr = 0.
\]

As a consequence of these two identities, we learn

\[
(19) \quad A_1 \left[ f_0^0 \zeta - r^{-1} f_0 \eta, f_0^0 \zeta + r^{-1} f_0 \eta \right] = 2 \int_0^\infty \left[ \frac{f_0^2}{r^2} (\eta')^2 + (f_0')^2 (\zeta')^2 + \frac{2}{r^3} f_0 f_0' (\eta - \zeta)^2 \right] rdr.
\]

Since \( f_0, f_0' > 0 \) one may consider the choice

\[
\zeta = \frac{1}{2f_0^0}(\varphi + \psi), \quad \eta = \frac{r}{2f_0}(\psi - \varphi),
\]
and deduce from the above that $A_1[\varphi, \psi] > 0$ for all non-zero $(\varphi, \psi) \in \mathcal{H}_1$. Moreover $A_1[\varphi, \psi] = 0$ exactly when $(\varphi, \psi)$ is in the real linear span of
\[
\left( f'_0 - \frac{f_0}{r}, f'_0 + \frac{f_0}{r} \right), \quad \left( i \left( f'_0 - \frac{f_0}{r} \right), -i \left( f'_0 + \frac{f_0}{r} \right) \right),
\]
which corresponds to the fact that $\partial_x u^0_{\text{rad}}$ and $\partial_y u^0_{\text{rad}}$ solve the linearized equation.

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