ON THE CHOW GROUPS OF CERTAIN CUBIC FOURFOLDS

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ABSTRACT. This note is about the Chow groups of a certain family of smooth cubic fourfolds. This family is characterized by the property that each cubic fourfold $X$ in the family has an involution such that the induced involution on the Fano variety $F$ of lines in $X$ is symplectic and has a $K3$ surface $S$ in the fixed locus. The main result establishes a relation between $X$ and $S$ on the level of Chow motives. As a consequence, we can prove finite–dimensionality of the motive of certain members of the family.

1. INTRODUCTION

For a smooth projective variety $X$ over $\mathbb{C}$, let $A^i(X) := CH^i(X)_{\mathbb{Q}}$ denote the Chow groups (i.e. the groups of codimension $i$ algebraic cycles on $X$ with $\mathbb{Q}$–coefficients, modulo rational equivalence). Let $A^i_{\text{hom}}(X)$ denote the subgroup of homologically trivial cycles.

When $X \subset \mathbb{P}^5(\mathbb{C})$ is a smooth cubic fourfold, we have $A^i_{\text{hom}}(X) = 0$ for $i \neq 3$, but $A^3_{\text{hom}}(X) \neq 0$ (this is related to the fact that $H^{3,1}(X) \neq 0$). The main result of this note shows that for a certain family of cubic fourfolds, the group $A^3_{\text{hom}}(X)$ is not larger than the Chow group of 0–cycles on a $K3$ surface:

**Theorem** (=theorem [3.1]). Let $X \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold defined by an equation

$$f(x_0, x_1, x_2, x_3) + (x_4)^2\ell_1(x_0, \ldots, x_3) + (x_5)^2\ell_2(x_0, \ldots, x_3) + x_4x_5\ell_3(x_0, \ldots, x_3) = 0$$

(here $f$ has degree 3 and $\ell_1, \ell_2, \ell_3$ are linear forms). There exists a $K3$ surface $S$ and a correspondence $\Gamma \in A^3(X \times S)$ inducing a split injection

$$\Gamma_* : A^3_{\text{hom}}(X) \hookrightarrow A^2_{\text{hom}}(S).$$

In a nutshell, the argument proving theorem [3.1] is as follows: cubics $X$ as in theorem [3.1] have an involution inducing a symplectic involution $\iota_F$ of the Fano variety of lines $F = F(X)$. The fixed locus of $\iota_F$ contains a $K3$ surface $S$. The inclusion $S \subset F$ being symplectic, there is a (correspondence–induced) isomorphism

$$\Gamma_* : H^{3,1}(X) \cong H^{2,0}(S).$$

Because the cubics $X$ as in theorem [3.1] form a large family, and the correspondence $\Gamma$ exists for the whole family, one can apply Voisin’s method of “spread” [33], [34], [35], [36] to this isomorphism, and obtain a statement on the level of rational equivalence which proves theorem [3.1].

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As an application of theorem 3.1, we obtain some new examples of cubics with finite–dimensional motive (in the sense of Kimura/O’Sullivan [19], [1], [17]):

**Corollary** (=corollary 4.1). Let $X$ be as in theorem 3.1, and assume $\dim H^4(X) \cap H^{2,2}(X, \mathbb{C}) \geq 20$.

Then $X$ has finite–dimensional motive.

For $X$ as in corollary 4.1, one can also prove finiteness for the Fano varieties of lines on $X$ (remark 4.2). This gives new examples of hyperkähler fourfolds with finite–dimensional motive.

**Conventions.** In this article, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we will denote by $A^j(X)$ the Chow group of $j$–dimensional cycles on $X$ with $\mathbb{Q}$–coefficients; for $X$ smooth of dimension $n$ the notations $A^j(X)$ and $A^{n-j}(X)$ are used interchangeably.

The notations $A^j_{\text{hom}}(X)$, $\text{AJ}^j(X)$ will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. For a morphism $f: X \to Y$, we will write $\Gamma_f \in A_*(X \times Y)$ for the graph of $f$. The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [29], [26]) will be denoted $M_{\text{rat}}$.

We will write $H^j(X)$ to indicate singular cohomology $H^j(X, \mathbb{Q})$.

Given a group $G \subset \text{Aut}(X)$ of automorphisms of $X$, we will write $A^j(X)^G$ (and $H^j(X)^G$) for the subgroup of $A^j(X)$ (resp. $H^j(X)$) invariant under $G$.

2. Preliminaries

2.1. Refined Künneth decomposition.

**Definition 2.1.** Let $X$ be a smooth projective variety, and $h \in \text{Pic}(X)$ an ample class. The hard Lefschetz theorem asserts that the map

$$L^{n-i}: H^i(X) \to H^{2n-i}(X)$$

obtained by cupping with $h^{n-i}$ is an isomorphism, for any $i < n$. One of the standard conjectures, often denoted $B(X)$, asserts that the inverse isomorphism is algebraic: we say that $B(X)$ holds if for any $i < n$, there exists a correspondence $C_i \in A^i(X \times X)$ such that

$$(C_i)_*: H^{2n-i}(X) \to H^i(X)$$

is an inverse to $L^{n-i}$.

**Remark 2.2.** For more on the standard conjectures, cf. [20], [21]. In this note, we will be using the following two facts: Any smooth hypersurface $X \subset \mathbb{P}^n(\mathbb{C})$ verifies $B(X)$ [20], [21]. For any smooth cubic fourfold $X \subset \mathbb{P}^5(\mathbb{C})$, the Fano variety of lines $F := F(X)$ verifies $B(F)$ (this follows from [9] Theorem 1.1, or alternatively from [23] Corollary 6).
Remark 2.3. Let $N^*H^*$ denote the coniveau filtration on cohomology \cite{6}. Vial \cite{31} has introduced a variant filtration $\tilde{N}^*H^*$, called the niveau filtration. There is an inclusion

$$\tilde{N}^j H^i(X) \subset N^j H^i(X)$$

for any $X$ and all $i, j$. Conjecturally, this is always an equality (this would follow from the standard conjecture B). If $B(X)$ holds and $j \geq \frac{i-1}{2}$, this inclusion is an equality \cite{31].

Theorem 2.4 (Vial \cite{31}). Let $X$ be a smooth projective variety of dimension $n \leq 5$. Assume $B(X)$ holds. There exists a decomposition of the diagonal

$$\Delta_X = \sum_{i,j} \pi^X_{i,j} \text{ in } H^{2n}(X \times X),$$

where the $\pi_{i,j}$'s are mutually orthogonal idempotents. The correspondence $\pi_{i,j}$ acts on $H^*(X)$ as a projector on $Gr^j_N H^i(X)$. Moreover, $\pi_{i,j}$ can be chosen to factor over a variety of dimension $i - 2j$ (i.e., for each $\pi_{i,j}$ there exists a smooth projective variety $Z_{i,j}$ of dimension $i - 2j$, and correspondences $\Gamma_{i,j} \in A^{n-j}(Z_{i,j} \times X), \Psi_{i,j} \in A^{j}(X \times Z_{i,j})$ such that $\pi_{i,j} = \Gamma_{i,j} \circ \Psi_{i,j}$ in $H^{2n}(X \times X)$).

Proof. This is a special case of \cite{31} Theorem 1]. Indeed, as mentioned in loc. cit., varieties $X$ of dimension $\leq 5$ such that $B(X)$ holds verify condition (*) of loc. cit. \hfill \Box

Remark 2.5. If $X$ is a surface, $\pi_{2,0}^X$ is the homological realization of the projector $\pi_{2,\text{tr}}^X$ constructed on the level of Chow motives in \cite{18}.

2.2. Spread.

Lemma 2.6 (Voisin \cite{33}, \cite{34}). Let $M$ be a smooth projective variety of dimension $n + 1$, and $L$ a very ample line bundle on $M$. Let

$$\pi: \mathcal{X} \to B$$

denote a family of hypersurfaces, where $B \subset |L|$ is a Zariski open. Let

$$p: \widetilde{\mathcal{X} \times_B \mathcal{X}} \to \mathcal{X} \times_B \mathcal{X}$$

denote the blow-up of the relative diagonal. Then $\widetilde{\mathcal{X} \times_B \mathcal{X}}$ is Zariski open in $V$, where $V$ is a projective bundle over $\widetilde{M \times M}$, the blow-up of $M \times M$ along the diagonal.

Proof. This is \cite{33} Proof of Proposition 3.13 or \cite{34} Lemma 1.3]. The idea is to define $V$ as

$$V := \left\{ ((x, y, z), \sigma) \mid \sigma|_z = 0 \right\} \subset \widetilde{M \times M \times |L|}.$$ 

The very ampleness assumption ensures $V \to \widetilde{M \times M}$ is a projective bundle. \hfill \Box

This is used in the following key proposition:

Proposition 2.7 (Voisin \cite{34}). Assumptions as in lemma 2.6. Assume moreover $M$ has trivial Chow groups. Let $R \in A^n(V)$. Suppose that for all $b \in B$ one has

$$H^n(X_b)_{\text{prim}} \neq 0 \quad \text{and} \quad R|_{\widetilde{X_b \times X_b}} = 0 \in H^{2n}(\widetilde{X_b \times X_b}).$$
Then there exists $\gamma \in A^n(M \times M)$ such that

$$(p_b)_*(R|_{\tilde{X}_b \times X_b}) = \gamma|_{X_b \times X_b} \in A^n(X_b \times X_b)$$

for all $b \in B$. (Here $p_b$ denotes the restriction of $p$ to $\tilde{X}_b \times X_b$, which is the blow–up of $X_b \times X_b$ along the diagonal.)

Proof. This is [34, Proposition 1.6].

The following is an equivariant version of proposition 2.7:

Proposition 2.8 (Voisin [34]). Let $M$ and $L$ be as in proposition 2.7. Let $G \subset \text{Aut}(M)$ be a finite group. Assume the following:

(i) The linear system $|L|^G := \mathbb{P}(H^0(M, L)^G)$ has no base–points, and the locus of points in $\tilde{M} \times \tilde{M}$ parametrizing triples $(x, y, z)$ such that the length 2 subscheme $z$ imposes only one condition on $|L|^G$ is contained in the union of (proper transforms of) graphs of non–trivial elements of $G$, plus some loci of codimension $> n + 1$.

(ii) Let $B \subset |L|^G$ be the open parametrizing smooth hypersurfaces, and let $X_b \subset M$ be a hypersurface for $b \in B$ general. There is no non–trivial relation

$$\sum_{g \in G} c_g \Gamma_g + \gamma = 0 \quad \text{in } H^{2n}(X_b \times X_b),$$

where $\gamma$ is a cycle in $\text{Im}(A^n(M \times M) \to A^n(X_b \times X_b))$.

Let $R \in A^n(X \times_B X)$ be such that

$$R|_{X_b \times X_b} = 0 \quad \text{in } H^{2n}(X_b \times X_b) \quad \forall b \in B.$$

Then there exists $\gamma \in A^n(M \times M)$ such that

$$R|_{X_b \times X_b} = \gamma|_{X_b \times X_b} \in A^n(X_b \times X_b) \quad \forall b \in B.$$

Proof. This is not stated verbatim in [34], but it is contained in the proof of [34, Proposition 3.1 and Theorem 3.3]. We briefly review the argument. One considers

$$V := \left\{ (x, y, z), \sigma \left| \sigma|_z = 0 \right. \right\} \subset \tilde{M} \times \tilde{M} \times |L|^G.$$

The problem is that this is no longer a projective bundle over $\tilde{M} \times M$. However, as explained in the proof of [34, Theorem 3.3], hypothesis (i) ensures that one can obtain a projective bundle after blowing up the graphs $\Gamma_g, g \in G$ plus some loci of codimension $> n + 1$. Let $M' \to \tilde{M} \times M$ denote the result of these blow–ups, and let $V' \to M'$ denote the projective bundle obtained by base–changing.

Analyzing the situation as in [34, Proof of Theorem 3.3], one obtains

$$R|_{X_b \times X_b} = R_0|_{X_b \times X_b} + \sum_{g \in G} \lambda_g \Gamma_g \quad \text{in } A^n(X_b \times X_b),$$

where $R_0 \in A^n(M \times M)$ and $\lambda_g \in \mathbb{Q}$ (this is [34, Equation (15)]). By assumption, $R|_{X_b \times X_b}$ is homologically trivial. Using hypothesis (ii), this implies that all $\lambda_g$ have to be 0. \qed
3. Main result

Theorem 3.1. Let $X \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold defined by an equation

$$f(x_0, x_1, x_2, x_3) + (x_4)^2 \ell_1(x_0, \ldots, x_3) + (x_5)^2 \ell_2(x_0, \ldots, x_3) + x_4 x_5 \ell_3(x_0, \ldots, x_3) = 0$$

(here $f$ has degree 3 and $\ell_1, \ell_2, \ell_3$ are linear forms). There exists a $K3$ surface $S$ and a correspondence $\Gamma \in A^3(X \times S)$ inducing a split injection

$$\Gamma_* : A^3_{hom}(X) \hookrightarrow A^2_{hom}(S).$$

Proof. Let us consider the involution

$$\iota : \mathbb{P}^5 \to \mathbb{P}^5,$$

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_0 : x_1 : x_2 : x_3 : -x_4 : -x_5].$$

The family of cubic fourfolds $X$ as in theorem 3.1 is exactly the family of smooth cubic fourfolds invariant under $\iota$ (this was observed in [8, Section 7], and also in [14], where this family appears as “Family V-(1)” in the classification table of [14, Theorem 0.1]). Let us denote by

$$\iota_X : X \to X$$

the involution of $X$ induced by $\iota$.

Let $F := F(X)$ denote the Fano variety parametrizing lines contained in $X$. The variety $F$ is a hyperkähler variety [4]. The involution

$$\iota_F : F \to F,$$

induced by $\iota_X$ is symplectic [8, Section 7], [14, Theorem 0.1]. The fixed locus of $\iota_F$ consists of 28 isolated points and a $K3$ surface $S \subset F$ [8, Section 7], [14, Section 4]. The involution $\iota_F$ being symplectic, the surface $S \subset F$ is a symplectic subvariety, i.e. the inclusion $\tau : S \to F$ induces an isomorphism

$$\tau^* : H^{2,0}(F) \xrightarrow{\cong} H^{2,0}(S).$$

As is readily seen, this implies there is also an isomorphism

(1) $$\tau^* : H^2_{tr}(F) \xrightarrow{\cong} H^2_{tr}(S),$$

where $H^2_{tr}() \subset H^2()$ denotes the smallest Hodge–substructure containing $H^{2,0}()$. Let $\Gamma_{BD}$ be the correspondence inducing the Beauville–Donagi isomorphism

(2) $$(\Gamma_{BD})_* : H^4(X) \xrightarrow{\cong} H^2(F)$$

[4]. (That is, let $P \subset X \times F$ denote the incidence variety, with morphisms $p : P \to F, q : P \to X$. Then $\Gamma_{BD} := \Gamma_p \circ \iota \Gamma_q \in A^3(X \times F).$)

Let us define a correspondence

$$\Gamma := \iota \Gamma_q \circ \Gamma_{BD} \in A^3(X \times S).$$

Combining isomorphisms (1) and (2), we obtain an isomorphism

$$\Gamma_* : H^4(X)/N^2 \xrightarrow{\cong} H^2_{tr}(F) \to H^2_{tr}(S).$$
A bit more formally, this implies there is an isomorphism of homological motives

$$\Gamma: (X, \pi_4^X, 0) \xrightarrow{\cong} (S, \pi_2^{S}, 0) \text{ in } \mathcal{M}_{\text{hom}}.$$  

Here, $\pi_{4,1}^X = \pi_4^X - \pi_{1,2}^X$ is a projector on $H^4(X)/N^2$; this exists thanks to theorem 2.4. The projector $\pi_{2,2r}$ is the projector on $H^2_{tr}(S)$ constructed in [18]. Let $\Psi \in A^3(S \times X)$ be a correspondence inducing an inverse to the isomorphism (3). This means that we have

$$(\Psi \circ \Gamma)_* = \text{id}: H^4(X)/N^2 \to H^4(X)/N^2,$$

which means that there is a homological equivalence of cycles

$$\Psi \circ \Gamma \circ \pi_4^X = \pi_4^X + \gamma_1 \text{ in } H^8(X \times X),$$

where $\gamma_1 \in A^4(X \times X)$ is some cycle supported on $V \times V \subset X \times X$, where $V \subset X$ is a codimension 2 closed subvariety (this is because $\gamma_1$ is supported on the support of $\pi_{4,2}^X$, which is supported on $V \times V$ as indicated, by theorem 2.4).

As $X \subset \mathbb{P}^5$ is a hypersurface, the only interesting Künneth component is $\pi_4^X$. That is, we can write

$$\Delta_X = \pi_4^X + \gamma_2 \text{ in } H^8(X \times X),$$

where $\gamma_2$ is a “completely decomposed” cycle, i.e. a cycle with support on $\cup_i V_i \times W_i \subset X \times X$, where $\dim V_i + \dim W_i = 4$. Plugging this in equation (4), we obtain a homological equivalence of cycles

$$\Psi \circ \Gamma = \Delta_X + \gamma \text{ in } H^8(X \times X),$$

where $\gamma$ is a “completely decomposed” cycle in the above sense.

We now proceed to upgrade the homological equivalence (5) to a rational equivalence. This can be done thanks to the work of Voisin on the Bloch/Hodge equivalence [33], [34], using the technique of “spread” of algebraic cycles in good families.

Following the approach of [33], [34], we put the above construction in family. We define

$$\pi: \mathcal{X} \to B$$

to be the family of all smooth cubic fourfolds given by an equation as in theorem 3.1. (That is, we let $G \subset \text{Aut}(\mathbb{P}^5)$ be the order 2 group generated by the involution $u$, and we define

$$B \subset \left( \mathbb{P}^5 \right)^G$$

as the open subset parametrizing smooth $G$–invariant cubics.) We will write $X_b := \pi^{-1}(b)$ for the fibre over $b \in B$. We also define families

$$\mathcal{F} \to B, \quad \mathcal{S} \to B$$

of Fano varieties of lines, resp. of $K3$ surfaces. (That is, $\mathcal{S} \subset \mathcal{F}$ is the fixed locus of the involution of $\mathcal{F}$ induced by $\iota$.) We will write $\mathcal{F}_b$ and $\mathcal{S}_b$ for the fibre over $b \in B$.

The correspondence $\Gamma$ constructed above readily extends to this relative setting:
Lemma 3.2. There exists a relative correspondence $\Gamma \in A^3(\mathcal{X} \times_B \mathcal{S})$, such that for all $b \in B$, the restriction

$$\Gamma_b := \Gamma|_{X_b \times S_b} \in A^3(X_b \times S_b)$$

induces the isomorphism

$$\Gamma_b : (X_b, \pi_{X_b,1}, 0) \xrightarrow{\cong} (S_b, \pi_{S_b,2}, 0) \text{ in } \mathcal{M}_{\text{hom}}$$

as in (3).

Proof. Let $\mathcal{P} \subset \mathcal{X} \times_B \mathcal{F}$ denote the incidence variety, with projections $p : \mathcal{P} \rightarrow \mathcal{F}$, $q : \mathcal{P} \rightarrow \mathcal{X}$. Let $\tau$ denote the inclusion morphism $\mathcal{S} \rightarrow \mathcal{F}$. We define

$$\Gamma := \iota_\tau \circ \Gamma_p \circ \iota_\tau \in A^3(\mathcal{X} \times_B \mathcal{S}) .$$

(For composition of relative correspondences in the setting of smooth quasi–projective families that are smooth over a base $B$, cf. [10], [15], [27], [12], [26, 8.1.2].) \hfill \square

The correspondences $\Psi$ and $\gamma$ also extend to the relative setting:

Lemma 3.3. There exist subvarieties $\mathcal{V}_i, \mathcal{W}_i \subset \mathcal{X}$ with $\text{codim}(\mathcal{V}_i) + \text{codim}(\mathcal{W}_i) = 4$, and relative correspondences

$$\Psi \in A^3(S \times_B \mathcal{X}), \quad \gamma \in A^4(\mathcal{X} \times_B \mathcal{X}),$$

where $\gamma$ is supported on $\bigcup_i \mathcal{V}_i \times_B \mathcal{W}_i$, and such that for all $b \in B$, the restrictions

$$\Psi_b := \Psi|_{S_b \times X_b} \in A^3(S_b \times X_b), \quad \gamma_b := \gamma|_{X_b \times X_b} \in A^4(X_b \times X_b)$$

verify the equality

$$\Psi_b \circ \Gamma_b = \Delta_{X_b} + \gamma_b \text{ in } H^8(X_b \times X_b)$$

as in (5).

Proof. The statement is different, but this is really the same Hilbert schemes argument as [33, Proposition 3.7]. [35, Proposition 4.25].

Let $\Gamma \in A^3(\mathcal{X} \times_B \mathcal{S})$ be the relative correspondence of lemma 3.2, and let $\Delta_X \in A^4(\mathcal{X} \times_B \mathcal{X})$ be the relative diagonal. By what we have said above, for each $b \in B$ there exist subvarieties $V_{b,i}, W_{b,i} \subset X_b$ (with $\text{dim}(V_{b,i}) + \text{dim}(W_{b,i}) = 4$), and a cycle $\gamma_b$ supported on

$$\bigcup_i V_{b,i} \times W_{b,i} \subset X_b \times X_b ,$$

and a cycle $\Psi_b \in A^3(S_b \times X_b)$, such that there is equality

$$(6) \quad \Psi_b \circ \Gamma_b = \Delta_X|_{X_b \times X_b} + \gamma_b \text{ in } H^8(X_b \times X_b).$$

The point is that the data of all the $(b, V_{b,i}, W_{b,i}, \gamma_b, \Psi_b)$ that are solutions of the equality (6) can be encoded by a countable number of algebraic varieties $p_j : M_j \rightarrow B$, with universal objects

$$\mathcal{V}_{i,j} \rightarrow M_j , \quad \mathcal{W}_{i,j} \rightarrow M_j , \quad \gamma_j \rightarrow M_j , \quad \Psi_j \rightarrow M_j$$

(where $\mathcal{V}_{i,j}, \mathcal{W}_{i,j} \subset \mathcal{X}_{M_j}$, and $\gamma_j$ is a cycle supported on $\bigcup_i \mathcal{V}_{i,j} \times M_j \mathcal{W}_{i,j}$, and $\Psi_j \in A^3(S \times M_j \mathcal{X})$, with the property that for $m \in M_j$ and $b = p_j(m) \in B$, we have

$$\gamma_j|_{X_b \times X_b} = \gamma_b \text{ in } H^8(X_b \times X_b),$$

$$\Psi_j|_{S_b \times X_b} = \Psi_b \text{ in } H^8(S_b \times X_b).$$

\hfill \square
By what we have said above, the union of the $M_j$ dominate $B$. Since there is a countable number of $M_j$, one of the $M_j$ (say $M_0$) must dominate $B$. Taking hyperplane sections, we may assume $M_0 \to B$ is generically finite (say of degree $d$). Projecting the cycles $\gamma_0$ and $\Psi_0$ to $X \times_B X$, resp. to $S \times_B X$, and then dividing by $d$, we have obtained cycles $\gamma$ and $\Psi$ as requested.

Lemma 3.3 can be succinctly restated as follows: the relative correspondence

$$R := \Psi \circ \Gamma - \Delta_X - \gamma \in A^4(X \times_B X')$$

has the property that for all $b \in B$, the restriction is homologically trivial:

$$R|_{X_b \times X_b} \in A_{hom}^4(X_b \times X_b) \quad \forall b \in B.$$  

Applying theorem 2.8 to $R$ (this is possible in view of proposition 3.4 below), we find that

$$(R + \delta)|_{X_b \times X_b} = 0 \quad \text{in} \quad A^4(X_b \times X_b) \quad \forall b \in B,$$

where $\delta$ is some cycle

$$\delta \in \text{Im} \left( A^4(P^5 \times P^5) \to A^4(X \times_B X') \right).$$

Since $A_{hom}^4(P^5 \times P^5) = 0$, we have

$$(\delta|_{X_b \times X_b})_* A_{hom}^*(X_b) = 0.$$  

For $b \in B$ general, the fibre $X_b \times X_b$ will be in general position with respect to the $\mathcal{V}_i$ and $\mathcal{W}_i$ and so

$$\dim(\mathcal{V}_i \cap X_b) + \dim(\mathcal{W}_i \cap X_b) = 4 \quad \forall i,$$

which ensures that

$$(\gamma|_{X_b \times X_b})_* A_{hom}^*(X_b) = 0.$$  

Plugging in the definition of $R$ into the rational equivalence (7), this means that

$$(\Psi|_{X_b \times X_b})_* (\Gamma|_{X_b \times X_b})_* = \text{id}: \quad A_{hom}^*(X_b) \to A_{hom}^*(X_b) \quad \text{for} \ b \in B \ \text{general},$$

which proves theorem 3.1 for $b \in B$ general.

To prove theorem 3.1 for any given $b_0 \in B$, we note that the above construction can also be made locally around the point $b_0$: in the construction of lemma 3.3 we throw away all the data $M_j$ for which the subvarieties $\mathcal{V}_{i,j}, \mathcal{W}_{i,j}$ are not all in general position with respect to $X_{b_0} \times X_{b_0}$. The union of the remaining $M_j$ will dominate an open $B' \subset B$ containing $b_0$, and so the above proof works for the cubic $X_{b_0}$.

To end the proof, it remains to verify the hypotheses of theorem 2.8 (which we applied above) are met with. This is the content of the following:

**Proposition 3.4.** Let $X \to B$ be the family of smooth cubic fourfolds as in theorem 3.1, i.e.

$$B \subset \left( \mathbb{P} H^0(\mathbb{P}^5, O_{\mathbb{P}^5}(3)) \right)^G$$

is the open subset parametrizing smooth $G$–invariant cubics, and $G = \{id, \iota\} \subset \text{Aut}(\mathbb{P}^5)$ as above. This set–up verifies the hypotheses of proposition 2.8.
Proof. Let us first prove hypothesis (i) of proposition 2.8 is satisfied.

To this end, we consider the tower of morphisms

\[ p: \mathbb{P}^5 \xrightarrow{p_1} P' := \mathbb{P}^5/G \xrightarrow{p_2} P := \mathbb{P}(1^4, 2^2), \]

where \( \mathbb{P}(1^4, 2^2) = \mathbb{P}^5/(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \) denotes a weighted projective space. Let us write \( \iota_4, \iota_5 \) for the involutions of \( \mathbb{P}^5 \) and so (using lemma 3.5 below) there exists a smooth cubic

\[ \Gamma \in \mathbb{P}^5 \]

where \( c, d \) are independent and contains \( x \). Let us write

\[ \Gamma \]

Let us now assume \( x, y \in \mathbb{P}^5 \) are two points such that

\[ (x, y) \notin \Delta_5 \cup \Gamma_{i_4} \cup \Gamma_{i_5} \cup \Gamma_i. \]

Then

\[ p(x) \neq p(y) \quad \text{in} \quad P, \]

and so (using lemma 3.5 below) there exists \( \sigma \in \mathbb{P}H^0(P', \mathcal{O}_{P'}(3)) \) containing \( p(x) \) but not \( p(y) \). The pullback \( p^* \sigma \) contains \( x \) but not \( y \), and so these points \( (x, y) \) impose 2 independent conditions on \( \mathbb{P}H^0(P, \mathcal{O}_P(3)) \).

It only remains to check that a generic element \( (x, y) \in \Gamma_{i_4} \cup \Gamma_{i_5} \) also imposes 2 independent conditions. Let us assume \( (x, y) \) is generic on \( \Gamma_4 \) (the argument for \( \Gamma_5 \) is only notationally different). Let us write \( x = [a_0 : a_1 : \ldots : a_5] \). By genericity, we may assume all \( a_i \) are \( \neq 0 \) (intersections of \( \Gamma_4 \) with a coordinate hyperplane have codimension \( > n + 1 \) and so need not be considered for hypothesis (i) of proposition 2.8). We can thus write

\[ x = [1 : a_1 : a_2 : a_3 : a_4 : a_5], \quad y = [1 : a_1 : a_2 : a_3 : -a_4 : a_5], \quad a_i \neq 0. \]

The cubic

\[ a_5x_0(x_4)^2 - a_4x_0x_4x_5 = 0 \]

is \( G \)-invariant and contains \( x \) while avoiding \( y \). This proves hypothesis (i) is satisfied.

To establish hypothesis (ii) of proposition 2.8 we proceed by contradiction. Let us suppose hypothesis (ii) is not met with, i.e. there exists a smooth cubic \( X_b \) as in theorem 3.1 and a non–trivial relation

\[ c \Delta_{X_b} + d \Gamma_{i_{X_b}} + \delta = 0 \quad \text{in} \quad H^8(X_b \times X_b), \]

where \( c, d \in \mathbb{Q}^* \) and \( \delta \in \text{Im}(A^4(\mathbb{P}^5 \times \mathbb{P}^5) \to A^4(X_b \times X_b)) \). Looking at the action on \( H^{3,1}(X_b) \), we find that necessarily \( c = -d \) (indeed, \( \delta \) does not act on \( H^{3,1}(X_b) \), and \( \iota \) acts as the identity on \( H^{3,1}(X_b) \)). That is, we would have a relation

\[ \Delta_{X_b} - \Gamma_{i_{X_b}} + \frac{1}{c} \delta = 0 \quad \text{in} \quad H^8(X_b \times X_b). \]
Looking at the action on $H^{2,2}(X_b)$, we find that

$$(\iota_{X_b})^* = \text{id} : \text{Gr}^2_F H^4(X_b, \mathbb{C})_{\text{prim}} \to \text{Gr}^2_F H^4(X_b, \mathbb{C})_{\text{prim}}.$$ 

Since there is a codimension 2 linear subspace in $\mathbb{P}^5$ fixed by $\iota$, it follows that actually

$$(\iota_{X_b})^* = \text{id} : \text{Gr}^2_F H^4(X_b, \mathbb{C}) \to \text{Gr}^2_F H^4(X_b, \mathbb{C}).$$

Consider now the Fano variety of lines $F = F(X_b)$ with the involution $\iota_F$. Using the Beauville–Donagi isomorphism [4], one obtains that also

$$(\iota_F)^* = \text{id} : \text{Gr}^1_F H^2(F, \mathbb{C}) \to \text{Gr}^1_F H^2(F, \mathbb{C}).$$

As $\dim \text{Gr}^1_F H^2(F, \mathbb{C}) = 21$, this would imply that the trace of $(\iota_F)^*$ on $\text{Gr}^1_F H^2(F, \mathbb{C})$ is 21. However, this contradicts proposition 3.6 below, and so hypothesis (ii) must be satisfied.

**Lemma 3.5.** Let $P = \mathbb{P}(1^4, 2^2)$. Let $r, s \in P$ and $r \neq s$. Then there exists $\sigma \in \mathbb{P}H^0(P, \mathcal{O}_P(3))$ containing $r$ but avoiding $s$. 

**Proof.** It follows from Delorme’s work [11, Proposition 2.3(iii)] that the locally free sheaf $\mathcal{O}_P(2)$ is very ample. This means there exists $\sigma' \in \mathbb{P}H^0(P, \mathcal{O}_P(2))$ containing $r$ but avoiding $s$. Taking the union of $\sigma'$ with a hyperplane avoiding $s$, one obtains $\sigma$ as required. 

**Proposition 3.6** (Camere [8]). Let $X_b \subset \mathbb{P}^5$ be a cubic as in theorem 3.1, and let $\iota_{X_b}$ be the involution as above. Let $F = F(X_b)$ be the Fano variety of lines, and let $\iota_F$ be the involution of $F$ induced by $\iota_{X_b}$. The trace of $(\iota_F)^*$ on the 21–dimensional vector space $\text{Gr}^1_F H^2(F, \mathbb{C})$ is 5.

**Proof.** This follows from [8, Theorem 5].

**Remark 3.7.** Let $X$ and $S$ be as in theorem 3.1. One expects there is actually an isomorphism

$$\Gamma_* : A^3_{\text{hom}}(X) \xrightarrow{\cong} A^2_{\text{hom}}(S).$$

I am unsure whether the argument of theorem 3.1 can also be used to prove surjectivity.

**Remark 3.8.** To find the K3 surface $S$ of theorem 3.1, we have used the existence of the symplectic involution $\iota_F$ on the Fano variety $F = F(X)$ of lines on the cubic fourfold $X$, for which $S \subset F$ is in the fixed locus. One could ask if there exist cubic fourfolds $X$ other than those of theorem 3.1, such that the Fano variety $F(X)$ has a symplectic automorphism with a 2–dimensional component in the fixed locus.

However, if one restricts to polarized symplectic automorphisms of $F(X)$, there are only 2 families with a surface in the fixed locus: the family of theorem 3.1 and a family with an abelian surface in the fixed locus. This follows from the classification obtained by L. Fu in [14, Theorem 0.1] (the first family is labelled “Family V-(1)”, and the second family is labelled “Family IV-(2)” in loc. cit.).

The second family (with an abelian surface in the fixed locus) is studied from the point of view of algebraic cycles in [24].
Remark 3.9. Let $X$ and $F$ be as in theorem 3.1. We mention in passing that the automorphisms $\iota$ and $\iota_F$ of $X$ resp. of $F$ act as the identity on $A^3(X)$, resp. on $A^4(F)$ (for $X$, this follows immediately from theorem 3.1).

This is proven more generally for any polarized symplectic automorphism of the Fano variety of lines of a cubic fourfold [13, Theorems 0.5 and 0.6] (for a slightly different take on this, cf. [30, Theorem 5.3]). The argument of [13] is (just like the argument proving theorem 3.1) based on the idea of spread of algebraic cycles in a family, inspired by [33], [34].

4. Finite–dimensionality

Corollary 4.1. Let $X \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold defined by an equation

$$f(x_0, x_1, x_2, x_3) + (x_4)^2 \ell_1(x_0, \ldots, x_3) + (x_5)^2 \ell_2(x_0, \ldots, x_3) + x_4x_5\ell_3(x_0, \ldots, x_3) = 0$$

Assume

$$\dim H^4(X) \cap H^{2,2}(X, \mathbb{C}) \geq 20.$$ 

Then $X$ has finite–dimensional motive.

Proof. It follows from (the proof of) theorem 3.1 there is an inclusion as direct summand

$$h(X) \subset h(S)(1) \bigoplus_j \mathbb{L}(m_j) \quad \text{in } \mathcal{M}_{\text{rat}},$$

where $S$ is a $K3$ surface. We have also seen (in the proof of theorem 3.1) there is an isomorphism

$$\Gamma_* : H^4(X)/N^2 \xrightarrow{\cong} H^{2}_{\text{tr}}(S).$$

Since the Hodge conjecture is known for $X$ (because $X$ is Fano), there is equality

$$N^2H^4(X) = H^4(X) \cap H^{2,2}(X, \mathbb{C}).$$

Thus, the hypothesis on the dimension of the space of Hodge classes implies that

$$\dim N^2H^4(X) \geq 20,$$

and so

$$\dim H^2_{\text{tr}}(S) = \dim(H^4(X)/N^2) = 23 - \dim N^2 \leq 3.$$ 

This implies the Picard number $\rho(S)$ is at least 19, and so $S$ has finite–dimensional motive [28]. In view of inclusion (8), this concludes the proof. □

Remark 4.2. Let $X$ be a cubic as in corollary 4.1. Applying [22], it follows that the Fano variety of lines $F := F(X)$ also has finite–dimensional motive.

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