THE GROWTH OF THE RANGE OF STABLE RANDOM WALKS

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Abstract. In this article, we establish an almost sure invariance principle for the capacity and cardinality of the range for a class of $\alpha$-stable random walks on the integer lattice $\mathbb{Z}^d$ with $d > 5\alpha/2$ and $d > 3\alpha/2$, respectively. As a direct consequence, we conclude Khintchine’s and Chung’s laws of the iterated logarithm for both processes.

1. Introduction

Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of independent and identically distributed random variables with values in $\mathbb{Z}^d$, defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider a random walk $S_n = X_1 + \cdots + X_n$, $n \geq 1$, with $S_0 = 0$. The range $\mathcal{R}_n$ of the random walk $\{S_n\}_{n \geq 0}$ is defined as the random set

$$\mathcal{R}_n = \{S_0, \ldots, S_n\}, \quad n \geq 0.$$ 

The cardinality of the range is denoted by $|\mathcal{R}_n|$, $n \geq 0$.

The random walk $\{S_n\}_{n \geq 0}$ is called transient if $\sum_{n \geq 1} \mathbb{P}(S_n = 0) < \infty$; otherwise it is called recurrent. Recall that every random walk is either transient or recurrent. The capacity of a set $A \subseteq \mathbb{Z}^d$ (with respect to any transient random walk $\{S_n\}_{n \geq 0}$) is defined as

$$\text{Cap}(A) = \sum_{x \in A} \mathbb{P}(\tau^x_A = \infty),$$

where $\tau^x_A$ is the first return time of $\{S_n + x\}_{n \geq 0}$ to the set $A$, that is,

$$\tau^x_A = \inf\{n \geq 1 : S_n + x \in A\}.$$

We use notation $\tau^y_A$ if $A = \{y\}$, and we abandon the upper index if $x = 0$.

The main aim of this article is to investigate the growth of the capacity of the range. More precisely, we consider a class of symmetric stable random walks (see assumptions (A1) and (A2) below) and obtain an almost sure invariance principle which asserts that the centered stochastic process $\{\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)]\}_{n \geq 0}$ can be approximated (up to a constant) by a path of a standard Brownian motion almost surely. As a corollary we obtain Khintchine’s and Chung’s laws of the iterated logarithm. Our approach is based upon two main ingredients. The first is a decomposition of the capacity of the range which allows us to represent it as a sum of finitely many independent random variables plus an error term. The second is the Skorohod embedding theorem which we apply to replace the sequence of independent random variables that appears in the capacity decomposition with a Brownian path sampled at certain random instances.

A slick adjustment of our method allows us to obtain analogous results for the process $\{|\mathcal{R}_n|\}_{n \geq 0}$. For this case, following LeGall [20], we utilize a decomposition of the range set into two independent parts plus an error term which can be treated as an intersection time of the corresponding random walk. This, together with the Skorohod embedding theorem, enables us to conclude the almost sure invariance principle and corresponding laws of the iterated logarithm for $\{|\mathcal{R}_n|\}_{n \geq 0}$.

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Problems related to the range of random walks constitute a rich area of research in modern probability theory. The first result in this direction is due to Dvoretzky and Erdős [15] where they obtained the strong law of large numbers for \( \{ |R_n| \}_{n \geq 0} \) of a simple random walk in \( d \geq 2 \). Later on, Spitzer [29] extended this theorem to all random walks in \( d \geq 1 \). A central limit theorem (CLT) for \( \{ |R_n| \}_{n \geq 0} \) (with a normal law in the limit) was first obtained by Jain and Orey [17] for strongly transient random walks, and Jain and Pruit [18] extended this result to all random walks in \( d \geq 3 \). Le Gall [20] proved a version of a CLT for \( \{ |R_n| \}_{n \geq 0} \) of all two-dimensional random walks with zero mean and finite second moment with a non-normal law in the limit. Le Gall and Rosen [21] established the strong law of large numbers and CLT for \( \{ |R_n| \}_{n \geq 0} \) of a class of \( \alpha \)-stable random walks in \( \mathbb{Z}^d \). The law of the limiting random variable depends on the value of the ratio \( d/\alpha \) in this case.

The capacity of the range has attracted much attention in the literature as well. To understand the motivation for the study of the capacity of the range and its links to the theory of intersection of paths of random walks we refer the reader to [1] and interesting references therein. The first result concerning the limiting behavior of the process \( \{ \text{Cap}(R_n) \}_{n \geq 0} \) is due to Jain and Orey [17] where they obtained the strong law of large numbers for all transient random walks. CLT was recently proved in [2] for a simple random walk in \( \mathbb{Z}^d \) with \( d \geq 6 \). The case of a simple random walk in dimensions \( d = 4 \) and \( d = 5 \) was studied in [3] and [26] respectively, see also [10] for \( d = 3 \). In [12] we established CLT for the capacity of the range for a class of stable random walks which possess one-step loops, see also [13] for a functional CLT for such random walks.

The theory of the growth of the process \( \{ \text{Cap}(R_n) \}_{n \geq 0} \) is still in its infancy. In [1] the authors proved downward large and moderate deviation estimates and an upward large deviation principle for a symmetric simple random walk in dimensions \( d \geq 5 \). Almost sure invariance principles and laws of the iterated logarithm for the capacity of the range have not been studied so far. In this article, we establish these results for symmetric stable random walks which admit one-step loops (see assumption (A2) below), in dimensions \( d > 5\alpha/2 \), where \( \alpha \in (0, 2] \) is the index of stability of the walk. Our results, however, are also true for a symmetric simple random walk in dimensions \( d \geq 6 \), see Remark 1.3. We remark that the limit behavior of the capacity (and cardinality) of the range of stable random walks depends on the value of the ratio \( d/\alpha \). One can observe an analogy between the limit behavior of the cardinality of the range for \( d/\alpha > 3/2 \), and the capacity of the range for \( d/\alpha > 5/2 \) (as already commented in [12]). If \( d \leq 5\alpha/2 \) then we expect that the capacity of the range behaves differently than displayed in the present article as this is the case for the cardinality of the range when \( d \leq 3\alpha/2 \). We conjecture that if \( d = 5\alpha/2 \) then the Brownian path which appears in our almost sure invariance principle has to be evaluated at certain times \( \phi(n) \) for a specific function \( \phi(t) \) which is determined by the variance of the capacity of the range, see [5]. Similarly, for \( d < 5\alpha/2 \) we suspect that the Brownian motion has to be replaced with another stochastic process, see [6] where such a process is given by renormalized intersection local times of the Brownian motion in the case of the cardinality of the range of planar random walks.

As we mentioned before, our approach, which we apply for the capacity of the range, can be slightly changed and then transferred to obtain the almost sure invariance principle for \( \{ |R_n| \}_{n \geq 0} \) in dimensions \( d > 3\alpha/2 \). The study of the growth of the process \( \{ |R_n| \}_{n \geq 0} \) was initiated with the Khintchine’s law of the iterated logarithm (LIL) obtained by Jain and Pruit [19] for aperiodic random walks satisfying \( \mathbb{P}(\tau_0 = \infty) < 1 \) and such that they are either strongly transient or lie in \( \mathbb{Z}^d \) with \( d \geq 4 \). We note that if \( \mathbb{P}(\tau_0 = \infty) = 1 \) then \( |R_n| = n + 1 \) a.s. Under similar assumptions Hamana [16] proved an almost sure invariance principle (asIP). Bass and Kumagai [5] obtained asIP, and Khintchine’s and
Chung’s LILs for a class of random walks which have finite moments of order $2+\delta$, for $\delta > 0$, in dimensions $d = 2$ and $d = 3$, see also [4] for a significant extension to random walks with finite second moments. The one-dimensional case was studied by Chen in [11]. Our contribution to this topic is twofold. Firstly, our method to handle the range is entirely different than that of aforementioned articles where authors used another decomposition formulas which were based on the definition of the range rather than on the Markov property of a random walk, see eq. (3.1). Secondly, random walks satisfying (A1) clearly do not have to possess finite second moments nor finite supports. It means that by an appropriate choice of a small $\alpha$ we can achieve a rich class of random walks for which the almost sure invariance principle holds in all dimensions. To efficiently construct examples of random walks satisfying (A1) one can employ a recently developed method of discrete subordination [8]. We refer to [7] and [22] for a condition under which a subordinate random walk belongs to the domain of attraction of a stable law. Moreover, random walks constructed according to this procedure fulfil (A2) as well.

Assumptions and Main Results. In the course of study we confine our attention to aperiodic random walks only. The random walk $\{S_n\}_{n \geq 0}$ is aperiodic if the smallest additive subgroup generated by the set $\{x \in \mathbb{Z}^d : \mathbb{P}(S_1 = x) > 0\}$ is equal to $\mathbb{Z}^d$. This assumption is not restrictive, for if $\{S_n\}_{n \geq 0}$ is not aperiodic, we could then pose the problem (and prove the same theorems) on a smaller subgroup of $\mathbb{Z}^d$, see [29, pp 20].

We obtain our results for a class of symmetric $\alpha$-stable random walks in $\mathbb{Z}^d$, that is, we assume the following condition.

(A1) $\{S_n\}_{n \geq 0}$ is symmetric and it belongs to the domain of attraction of a non-degenerate $\alpha$-stable random law with $0 < \alpha \leq 2$, meaning that there exists a regularly varying function $b(x)$ with index $1/\alpha$ such that

$$\frac{S_n}{b(n)} \xrightarrow{(d)} X_\alpha,$$

where $X_\alpha$ is an $\alpha$-stable random variable on $\mathbb{R}^d$ and $(d)$ stands for the convergence in distribution.

To perform a necessary analysis of the variance of $\text{Cap}(\mathcal{R}_n)$ we need an additional assumption on the random walk $\{S_n\}_{n \geq 0}$.

(A2) $\{S_n\}_{n \geq 0}$ admits one-step loops, that is, $\mathbb{P}(X_1 = 0) > 0$.

We now state the main results of the article.\footnote{We use the standard $O$-notation: for $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to (0, \infty)$ we write $f(n) = O(g(n))$ if, and only if, there is a constant $C > 0$ such that $|f(n)| \leq C g(n)$ for all $n \geq 1$.}

Theorem 1.1. Assume (A1), (A2) and $d > 5\alpha/2$. Then, there exists a standard Brownian motion $\{B_t\}_{t \geq 0}$ defined on the same probability space (possibly enlarged) as $\{S_n\}_{n \geq 0}$, and a constant $\sigma_d > 0$ such that for any $\varepsilon > 0$,

$$\sigma_d^{-1} \{\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)]\} - B_n = \begin{cases} O(n^{9/8-d/(4\alpha)+\varepsilon}), & d \in (5\alpha/2, 3\alpha) \\ O(n^{1/4+\varepsilon}), & d \geq 3\alpha \end{cases} \text{ a.s.}$$

Applying the corresponding results for Brownian motion (see [27, Chapter 11]), we conclude Khintchine’s and Chung’s LILs.

Corollary 1.2. Under the assumptions of Theorem 1.1, it holds $\mathbb{P}$-a.s.

$$\liminf_{n \to \infty} \frac{\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)]}{\sqrt{n \log \log n}} = -\sqrt{2} \sigma_d,$$
\[
\limsup_{n \to \infty} \frac{\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)]}{\sqrt{n \log \log n}} = \sqrt{2} \sigma_d,
\]
\[
\liminf_{n \to \infty} \frac{\max_{0 \leq m \leq n} |\text{Cap}(\mathcal{R}_m) - \mathbb{E}[\text{Cap}(\mathcal{R}_m)]|}{\sqrt{n / \log \log n}} = \frac{\pi}{8} \sigma_d.
\]

Let us remark that assumption \( d > 5\alpha/2 \) implies that the random walk \( \{S_n\}_{n \geq 0} \) is strongly transient. Recall that a transient random walk \( \{S_n\}_{n \geq 0} \) is called strongly transient if \( \sum_{n \geq 1} n \mathbb{P}(S_n = 0) < \infty \); otherwise it is called weakly transient. It is known that every transient random walk is either strongly or weakly transient (see [25]). According to [25, Theorem 3.4] and [30, Theorem 7] every random walk satisfying (A1) is transient if \( d > \alpha \) and strongly transient if \( d > 2\alpha \). We remark that (strong) transience assumption is quite natural in the present context. It ensures that the range process is quite natural in the present context. It ensures that the range process is fast enough and together with assumption (A2) it enables us to control the variance of \( \text{Cap}(\mathcal{R}_n) \).

Our strategy to prove Theorem 1.1 is based upon a capacity decomposition from [2] which allows us to represent the random variable \( \text{Cap}(\mathcal{R}_n) \) as a sum of finitely many independent random variables plus an error term which is expressed in terms of the Green function of the random walk, see eq. (2.2). To show that the error term is negligible we apply estimates for its moments from [12], see Lemma 2.1. We then employ the Skorohod embedding theorem to approximate the sum of independent random variables in eq. (2.2) by a path of Brownian motion evaluated at a random time which is given by a sum of specific stopping times. To get rid of randomness coming from this sequence we prove that it satisfies a version of the strong law of large numbers. To establish this result we study the second term in the asymptotics of \( \text{Var}(\text{Cap}(\mathcal{R}_n)) \) which was proved in [12, Lemma 5.3], see Lemma 2.2.

**Remark 1.3 (Simple random walk).** Observe that Theorem 1.1 excludes a symmetric simple random walk as it does not satisfy (A2) (it clearly fulfills (A1) with \( \alpha = 2 \)). We can, however, repeat the same analysis as in Section 2, and employ results from [2] instead of the corresponding results from [12], to obtain almost sure invariance principle eq. (1.1) for such a random walk in dimensions \( d > 5 \).

Performing an analogous approach for the cardinality of the range, we establish the following result.

**Theorem 1.4.** Assume (A1) and \( d > 3\alpha/2 \). Then, there exists a standard Brownian motion \( \{B_t\}_{t \geq 0} \) defined on the same probability space (possibly enlarged) as \( \{S_n\}_{n \geq 0} \), and a constant \( \sigma_d > 0 \) such that for any \( \varepsilon > 0 \),

\[
\sigma_d^{-1} (|\mathcal{R}_n| - \mathbb{E}[|\mathcal{R}_n|]) - B_n = \begin{cases} O(n^{7/8 - d/(4\alpha) + \varepsilon}), & d \in (3\alpha/2, 2\alpha) \\ O(n^{1/4 + \varepsilon}), & d \geq 2\alpha \end{cases} \quad \text{a.s.}
\]

Similarly as before, we conclude the corresponding LILs.

**Corollary 1.5.** Under the assumptions of Theorem 1.4 it holds \( \mathbb{P} \)-a.s.

\[
\liminf_{n \to \infty} \frac{|\mathcal{R}_n| - \mathbb{E}[|\mathcal{R}_n|]}{\sqrt{n \log \log n}} = -\sqrt{2} \sigma_d,
\]
\[
\limsup_{n \to \infty} \frac{|\mathcal{R}_n| - \mathbb{E}[|\mathcal{R}_n|]}{\sqrt{n \log \log n}} = \sqrt{2} \sigma_d,
\]
\[
\liminf_{n \to \infty} \frac{\max_{0 \leq m \leq n} |\mathcal{R}_m| - \mathbb{E}[|\mathcal{R}_m|]}{\sqrt{n \log \log n}} = \frac{\pi}{8} \sigma_d.
\]
To prove Theorem 1.4 we utilize a method of splitting the range into two independent parts which was first used by Le Gall in [20]. To deal with the error term in this case we apply estimates of moments of intersection times which we extract from [21], see Section 3. Next, to show that the sequence of independent random variables from the decomposition of the range can be approximated by a Brownian path, we again apply the Skorohod embedding theorem.

2. The Capacity Process

This section is devoted to the proof of Theorem 1.1. We start by recalling necessary notation and results. We denote by \( G(x, y) \) the Green function of the random walk \( \{S_n\}_{n \geq 0} \), that is,

\[
G(x, y) = \sum_{n=0}^{\infty} \mathbb{P}(S_n = y - x), \quad x, y \in \mathbb{Z}^d.
\]

Also, for \( A, B \subseteq \mathbb{Z}^d \) we denote

\[
G(A, B) = \sum_{x \in A} \sum_{y \in B} G(x, y).
\]

According to [2, Corollary 2.1], for any \( L, n \geq 1 \) such that \( 2^L \leq n \), we have

\[
\sum_{i=1}^{2^L} \text{Cap}(\mathcal{R}_{n/2^L}^{(i)}) - 2 \sum_{i=1}^{2^L} \sum_{l=1}^{2^{l-1}} \varepsilon_{l}^{(i)} \leq \text{Cap}(\mathcal{R}_{n}) \leq \sum_{i=1}^{2^L} \text{Cap}(\mathcal{R}_{n/2^L}^{(i)}),
\]

where \( \{\mathcal{R}_{n/2^L}^{(i)}\}_{i=1, \ldots, 2^L} \) are independent and the random variable \( \mathcal{R}_{n/2^L}^{(i)} \) has the same law as \( \mathcal{R}_{[n/2^L]} \) (or \( \mathcal{R}_{[n/2^L+1]} \)). For each \( l = 1, \ldots, L \) the random variables \( \{\varepsilon_l^{(i)}\}_{i=1, \ldots, 2^{l-1}} \) are independent and \( \varepsilon_l^{(i)} \) has the same law as \( G(\mathcal{R}_{n/2^L}, \mathcal{R}_{n/2^L}) \) with \( \{\mathcal{R}_{n/2^L}^{(i)}\}_{i=1, \ldots, 2^L} \) being an independent copy of \( \{\mathcal{R}_{n/2^L}^{(i)}\}_{i=1, \ldots, 2^L} \). In the sequel we use notation

\[
\mathcal{C}_n = \text{Cap}(\mathcal{R}_{n}), \quad \mathcal{C}_{n/2^L}^{(i)} = \text{Cap}(\mathcal{R}_{n/2^L}^{(i)}),
\]

and

\[
\tilde{\mathcal{C}}_n = \mathcal{C}_n - \mathbb{E}[\mathcal{C}_n], \quad \tilde{\mathcal{C}}_{n/2^L}^{(i)} = \mathcal{C}_{n/2^L}^{(i)} - \mathbb{E}[\mathcal{C}_{n/2^L}^{(i)}].
\]

Directly from eq. (2.1) we conclude that

\[
\tilde{\mathcal{C}}_n = \sum_{i=1}^{2^L} \tilde{\mathcal{C}}_{n/2^L}^{(i)} - \varepsilon(n),
\]

where

\[
-2 \sum_{i=1}^{2^L} \sum_{l=1}^{2^{l-1}} \mathbb{E}[\varepsilon_l^{(i)}] \leq \varepsilon(n) \leq 2 \sum_{i=1}^{2^L} \sum_{l=1}^{2^{l-1}} \varepsilon_l^{(i)},
\]

with \( \varepsilon_l^{(i)} \) having the same law as \( G(\mathcal{R}_{n/2^L}, \tilde{\mathcal{R}}_{n/2^L}) \).

Recall that the function \( b(x) \) appearing in assumption (A1) is necessarily of the form

\[
b(x) = x^{1/\alpha} \ell(x), \quad x \geq 0,
\]
where \( \ell(x) \) is a slowly varying function, see [9]. Without loss of generality the function \( b(x) \) can be chosen to be continuous and monotone increasing. In the sequel, we will also frequently use the following function
\[
h_d(n) = \begin{cases} 
1, & d > 3\alpha, \\
\sum_{k=1}^{n} k^{-1}(\ell(k))^{-d}, & d = 3\alpha, \\
n^{d}(b(n))^{-d}, & d \in (2\alpha, 3\alpha).
\end{cases}
\]

We fix the constant \( \Lambda = \frac{d}{\alpha} - \frac{5}{2} \), and observe that if \( \Lambda \geq 1/2 \) then \( h_d(n) \) is non-decreasing and slowly varying. If \( \Lambda \in (0, 1/2) \) we present \( h_d(n) \) in the form
\[
h_d(n) = n^{1/2-\Lambda} s(n),
\]
with \( s(n) = (\ell(n))^{-d} \). Clearly, in this case, \( h_d(n) \) is then non-decreasing and regularly varying of index \( 1/2 - \Lambda \), which is strictly smaller than \( 1/2 \).

We start our analysis by finding the asymptotic behavior of the error term \( \mathcal{E}(n) \).

**Lemma 2.1.** Assume (\textbf{A1}) and \( d > 5\alpha/2 \). Then
\[
\mathcal{E}(n) = \begin{cases} 
O(n^{1/2-\epsilon}), & \Lambda \in (0, 1/2) \text{ and } \epsilon \in (0, \Lambda), \quad \text{a.s.} \\
O(n^\epsilon), & \Lambda \geq 1/2 \text{ and } \epsilon > 0,
\end{cases}
\]

**Proof.** For any natural odd number \( p \geq 1 \), according to eq. (2.3), we have
\[
-2^p \sum_{l_1=1}^{L} \sum_{i_1=1}^{2^{l_1}-1} \cdots \sum_{l_p=1}^{L} \sum_{i_p=1}^{2^{l_p}-1} E[\mathcal{E}_{l_1}^{(i_1)}] \cdots E[\mathcal{E}_{l_p}^{(i_p)}] \leq (\mathcal{E}(n))^p
\]
and
\[
(\mathcal{E}(n))^p \leq 2^p \sum_{l_1=1}^{L} \sum_{i_1=1}^{2^{l_1}-1} \cdots \sum_{l_p=1}^{L} \sum_{i_p=1}^{2^{l_p}-1} \mathcal{E}_{l_1}^{(i_1)} \cdots \mathcal{E}_{l_p}^{(i_p)}.
\]
According to [12, Lemma 3.2] there is \( c_1 = c_1(p) > 0 \) such that
\[
E[\mathcal{E}_{l_1}^{(i_1)}] \cdots E[\mathcal{E}_{l_p}^{(i_p)}] \leq c_1(h_d(n/2^L))^p = c_1(h_d(n))^p, \quad l = 1, \ldots, L, \quad i = 1, \ldots, 2^{l_1}-1.
\]
Thus, by Hölder’s inequality,
\[
E[\mathcal{E}_{l_1}^{(i_1)}] \cdots E_{l_p}^{(i_p)} \leq \|\mathcal{E}_{l_1}^{(i_1)}\|_p \cdots \|\mathcal{E}_{l_p}^{(i_p)}\|_p \leq c_1(h_d(n))^p,
\]
which implies that
\[
E[\|\mathcal{E}(n)\|_p^p] \leq c_1 2^p (h_d(n))^p.
\]

We set \( L = \lfloor \log_2(n^\beta) \rfloor \) with \( \beta \in (0, 1) \) (recall that \( L, n \geq 1 \) are such that \( 2^L \leq n \)). Then, \( 2^p L \leq n^{p\beta} \), and
\[
E[\|\mathcal{E}(n)\|_p^p] \leq c_1 2^p n^{p\beta}(h_d(n))^p.
\]
**Case (i).** We assume that \( \Lambda \in (0, 1/2) \), and fix \( \epsilon \in (0, \Lambda) \). By eq. (2.6) and Markov’s inequality, we arrive at
\[
P(|\mathcal{E}(n)| > n^{1/2-\epsilon}) \leq \frac{E[\|\mathcal{E}(n)\|_p^p]}{n^{p(1/2-\epsilon)}} \leq \frac{c_1 2^p n^{p(1/2-\Lambda)}(s(n))^p}{n^{p(1/2-\epsilon-\beta)}}.
\]
Since \( s(n) \) is slowly varying, for any \( \gamma > 0 \) there is \( c_2 = c_2(\gamma) > 0 \) such that \( s(n) \leq c_2 n^\gamma \) for \( n \geq 1 \). Hence,
\[
P(|\mathcal{E}(n)| > n^{1/2-\epsilon}) \leq c_1 c_2^2 2^p n^{p(\beta+\gamma-(\Lambda-\epsilon))}.
\]
Since $\Lambda - \epsilon > 0$, we can choose $\beta$, $\gamma$ and $p$ such that $p(\beta + \gamma - (\Lambda - \epsilon)) < -1$. This yields
\[
\sum_{n=1}^{\infty} \mathbb{P}(|E(n)| > n^{1/2-\epsilon}) < \infty.
\]
In view of the Borel-Cantelli lemma, $|E(n)| > n^{1/2-\epsilon}$ only finitely often a.s. which forces the result.

Case (ii). We assume that $\Lambda \geq 1/2$, and fix $\epsilon > 0$. Then, by Markov’s inequality we have that
\[
\mathbb{P}(|E(n)| > n^\epsilon) \leq \frac{\mathbb{E}[|E(n)|^p]}{n^{p\epsilon}} \leq \frac{c_12^p(h_d(n))^p}{n^{p(\beta - \epsilon)}}.
\]
Since $h_d(n)$ is slowly varying, for any $\gamma > 0$ there is $c_3 = c_3(\gamma) > 0$ such that $h_d(n) \leq c_3n^\gamma$ for $n \geq 1$. Hence,
\[
\mathbb{P}(|E(n)| > n^\epsilon) \leq c_1c_3^p2^pn^{p(\beta + \gamma - \epsilon)}.
\]
We choose $\beta, \gamma$ and $p$ such that $p(\beta + \gamma - \epsilon) < -1$, which yields
\[
\sum_{n=1}^{\infty} \mathbb{P}(|E(n)| > n^\epsilon) < \infty.
\]
Now, the assertion follows again from the Borel-Cantelli lemma. 

In the next step we study the asymptotic behavior of $\sum_{i=1}^{2\ell} \bar{C}_{n/2^\ell}$. Recall that under assumptions of Theorem 1.1, it was proved in [12, Lemmas 4.3 and 5.3] that $\{\var{C}_n/n\}_{n \geq 1}$ converges to a strictly positive limit $\sigma_d^2$. In the following crucial lemma we investigate the second order term of this asymptotics.

Lemma 2.2. Under the assumptions of Theorem 1.1, it holds that
\[
\var{C}_n = \sigma_d^2n + O(n^{1/2}h_d(n)).
\]

Proof. Analogously as in eq. (2.1) we have
\[
C_n^{(1)} + C_m^{(2)} - 2E(n, m) \leq C_{n+m} \leq C_n^{(1)} + C_m^{(2)},
\]
where $C_n^{(1)} = \text{Cap}(R_n^{(1)})$ and $C_m^{(2)} = \text{Cap}(R_m^{(2)})$ with $R_n^{(1)}$ and $R_m^{(2)}$ being independent and having the same law as $R_n$ and $R_m$ respectively. Moreover, $E(n, m)$ has the same law as $G(R_n^{(1)}, R_m^{(2)})$. By taking expectation in eq. (2.7) and then by subtracting those two relations, we get
\[
\bar{C}_n^{(1)} + \bar{C}_m^{(2)} - 2E(n, m) \leq \bar{C}_{n+m} \leq \bar{C}_n^{(1)} + \bar{C}_m^{(2)} + 2\mathbb{E}[E(n, m)],
\]
which implies
\[
|\bar{C}_{n+m} - (\bar{C}_n^{(1)} + \bar{C}_m^{(2)})| \leq 2(E(n, m) + \mathbb{E}[E(n, m)]) \leq 2(E(n, m)) + \|E(n, m)\|_2.
\]
Thus
\[
\|\bar{C}_{n+m} - (\bar{C}_n^{(1)} + \bar{C}_m^{(2)})\|_2 \leq 4\|E(n, m)\|_2.
\]
From this we obtain
\[
\|\bar{C}_{n+m}\|_2 \leq \|\bar{C}_n^{(1)} + \bar{C}_m^{(2)}\|_2 + \|\bar{C}_{n+m} - (\bar{C}_n^{(1)} + \bar{C}_m^{(2)})\|_2
\]
\[
\leq \|\bar{C}_n^{(1)} + \bar{C}_m^{(2)}\|_2 + 4\|E(n, m)\|_2
\]
\[
= (\|\bar{C}_n^{(1)} + \bar{C}_m^{(2)}\|_2^2) + 4\|E(n, m)\|_2.
\]
By independence of $C_n^{(1)}$ and $C_m^{(2)}$, we conclude that
\[
\|\bar{C}_{n+m}\|_2 \leq (\|\bar{C}_n^{(1)}\|_2^2 + \|\bar{C}_m^{(2)}\|_2^2)^{1/2} + 4\|E(n, m)\|_2.
\]
and whence
\[ \|\bar{c}_{n+m}\|^{2}_{2} \leq \|\bar{c}_{n}^{(1)}\|^{2}_{2} + \|\bar{c}_{m}^{(2)}\|^{2}_{2} + 8(\|\bar{c}_{n}^{(1)}\|^{2}_{2} + \|\bar{c}_{m}^{(2)}\|^{2}_{2})^{1/2}\|\mathcal{E}(n, m)\|_{2} \\
+ 16\|\mathcal{E}(n, m)\|^{2}_{2}. \]

According to [12, Lemmas 3.2 and 4.3], there is \( c_{4} > 0 \) such that
\[ (2.8) \quad \|\bar{c}_{n}\|_{2} \leq c_{4}\sqrt{n} \quad \text{and} \quad \|\mathcal{E}(n, m)\|_{2} \leq c_{4}h_{d}(n + m), \quad n, m \geq 1. \]

Since the index of regular variation of \( h_{d}(n) \) is strictly smaller than \( 1/2 \), we arrive at
\[ \|\bar{c}_{n+m}\|^{2}_{2} \leq \|\bar{c}_{n}^{(1)}\|^{2}_{2} + \|\bar{c}_{m}^{(2)}\|^{2}_{2} + 8c_{4}^{2}\sqrt{n + m}h_{d}(n + m) + 16c_{4}^{2}(h_{d}(n + m))^{2} \]
\[ \leq \|\bar{c}_{n}^{(1)}\|^{2}_{2} + \|\bar{c}_{m}^{(2)}\|^{2}_{2} + c_{5}\sqrt{n + m}h_{d}(n + m), \]
for some \( c_{5} > 0 \) large enough.

Analogously as above we have
\[ (\|\bar{c}_{n}^{(1)}\|^{2}_{2} + \|\bar{c}_{m}^{(2)}\|^{2}_{2})^{1/2} = \|\bar{c}_{n}^{(1)} + \bar{c}_{m}^{(2)}\|_{2} \]
\[ \leq \|\bar{c}_{n+m}\|_{2} + \|\bar{c}_{n+m} - (\bar{c}_{n}^{(1)} + \bar{c}_{m}^{(2)})\|_{2} \]
\[ \leq \|\bar{c}_{n+m}\|_{2} + 4\|\mathcal{E}(n, m)\|_{2}, \]
which implies
\[ \|\bar{c}_{n}^{(1)}\|^{2}_{2} + \|\bar{c}_{m}^{(2)}\|^{2}_{2} \leq \|\bar{c}_{n+m}\|^{2}_{2} + 8\|\bar{c}_{n+m}\|_{2}\|\mathcal{E}(n, m)\|_{2} + 16\|\mathcal{E}(n, m)\|^{2}_{2}. \]

Using eq. \((2.8)\) (and properties of \( h_{d}(n) \)) we conclude that
\[ \|\bar{c}_{n}^{(1)}\|^{2}_{2} + \|\bar{c}_{m}^{(2)}\|^{2}_{2} \leq \|\bar{c}_{n+m}\|^{2}_{2} + 8c_{4}^{2}\sqrt{n + m}h_{d}(n + m) + 16c_{4}^{2}(h_{d}(n + m))^{2} \]
\[ \leq \|\bar{c}_{n+m}\|^{2}_{2} + c_{5}\sqrt{n + m}h_{d}(n + m). \]

We write
\[ x_{n} = \text{Var}(\mathcal{C}_{n}) = \|\bar{c}_{n}\|^{2}_{2} \quad \text{and} \quad b_{n} = c_{5}\sqrt{n}h_{d}(n), \quad n \geq 1. \]

We have shown that
\[ x_{n} + x_{m} - b_{n+m} \leq x_{n+m} \leq x_{n} + x_{m} + b_{n+m}, \quad n, m \geq 1, \]
and from [12, Lemmas 4.3 and 5.3] we know that
\[ \lim\limits_{n \to \infty} \frac{x_{n}}{n} = \sigma_{d}^{2} > 0. \]

Take \( n = m = 2^{k-l}l \) for \( k, l \geq 1 \). Then one easily checks that
\[ \left| \frac{x_{2^{k-l}l}}{2^{k-l}} - \frac{x_{2^{k-l-1}l}}{2^{k-l-1}} \right| \leq \frac{b_{2^{k-l}}}{2^{k-l}}, \quad k, l \geq 1. \]

Next, observe that
\[ \sum_{k=1}^{\infty} \left( \frac{x_{2^{k-l}l}}{2^{k-l}} - \frac{x_{2^{k-l-1}l}}{2^{k-l-1}} \right) = \lim_{N \to \infty} \sum_{k=1}^{N} \left( \frac{x_{2^{k-l}l}}{2^{k-l}} - \frac{x_{2^{k-l-1}l}}{2^{k-l-1}} \right) = \sigma_{d}^{2} - \frac{x_{l}}{l}, \quad l \geq 1 \]
and whence
\[ \left| \frac{x_{n}}{n} - \sigma_{d}^{2} \right| = \sum_{k=1}^{\infty} \left( \frac{x_{2^{k-n}2^{k-n}}}{2^{k-n}} - \frac{x_{2^{k-n-1}2^{k-n-1}}}{2^{k-n-1}} \right) \leq \sum_{k=1}^{\infty} \frac{b_{2^{k-n}}}{2^{k}} \]
\[ \leq \frac{c_{5}}{\sqrt{n}} \sum_{k=1}^{\infty} \frac{h_{d}(2^{k-n})}{2^{k/2}}, \quad n \geq 1. \]

By recalling the definition of \( \{b_{n}\}_{n \geq 1} \), we conclude that
\[ \left| \frac{x_{n}}{n} - \sigma_{d}^{2} \right| \leq \frac{c_{5}}{\sqrt{n}} \sum_{k=1}^{\infty} \frac{h_{d}(2^{k-n})}{2^{k/2}}, \quad n \geq 1. \]
Consequently and the proof is finished.

For \( \Lambda \) hold

\[
\sum_{k=1}^{\infty} \frac{(2^k n)^{1/2-\Lambda} s(2^k n)}{2^{k/2}} = c_5 n^{-\Lambda} \sum_{k=1}^{\infty} 2^{-k/2} s(2^k n).
\]

Since \( s(n) \) is slowly varying, according to [9, Theorem 1.5.6], there is a constant \( c_6 > 0 \) such that \( s(2^k n) \leq c_6 2^{k/4} s(n) \) for all \( k, n \geq 1 \). Hence

\[
\frac{x_n}{n} - \sigma_d^2 \leq c_5 c_6 n^{-\Lambda} s(n) \sum_{k=1}^{\infty} 2^{-k/2} = c_7 n^{-\Lambda} s(n), \quad n \geq 1,
\]

which yields

\[
|x_n - \sigma_d^2 n| \leq c_7 n^{1-\Lambda} s(n) = c_7 n^{1/2} h_d(n), \quad n \geq 1.
\]

Case (ii). If \( \Lambda \geq 1/2 \), then \( h_d(n) \) is slowly varying and thus there is \( c_8 > 0 \) such that \( h_d(2^k n) \leq c_8 2^{k/4} h_d(n) \) for all \( k, n \geq 1 \). This yields

\[
\frac{x_n}{n} - \sigma_d^2 \leq c_5 c_8 h_d(n) \sum_{k=1}^{\infty} 2^{-k/4} = c_9 \frac{h_d(n)}{\sqrt{n}}, \quad n \geq 1.
\]

Consequently

\[
|x_n - \sigma_d^2 n| \leq c_9 n^{1/2} h_d(n), \quad n \geq 1,
\]

and the proof is finished. \( \square \)

We finally concentrate on the asymptotic behavior of the sequence \( \sigma_d^{-1} \sum_{i=1}^{2^L} \tilde{C}^{(i)}_{n/2^L} \).

We shall apply the Skorohod embedding theorem (see [28]) which asserts that there exist a standard Brownian motion \( \{B_t\}_{t \geq 0} \) and non-negative independent stopping times \( T_1, \ldots, T_{2^L} \) such that

\[
\{B_{T_{0} + \cdots + T_i} - B_{T_{0} + \cdots + T_{i-1}}\}_{i=1\ldots, 2^L} = \{\sigma_d^{-1} \tilde{C}^{(i)}_{n/2^L}\}_{i=1\ldots, 2^L},
\]

where \( T_0 = 0 \). We conclude that \( \sigma_d^{-1} \sum_{i=1}^{2^L} \tilde{C}^{(i)}_{n/2^L} \) has the same law as \( B_{T_{0} + \cdots + T_{2^L}} \). If necessary, we can enlarge the probability space in such a way that \( \{B_t\}_{t \geq 0} \) and \( \{S_n\}_{n \geq 0} \) are defined on the same \( (\Omega, \mathcal{F}, \mathbb{P}) \), see [23]. Moreover, the following moment estimates hold

\[
\mathbb{E}[T_i] = \sigma_d^{-2} \text{Var}(\tilde{C}^{(i)}_{n/2^L}) \quad \text{and} \quad \mathbb{E}[T_i^2] \leq c_{10} \sigma_d^{-4} \mathbb{E}[(\tilde{C}^{(i)}_{n/2^L})^4],
\]

for a constant \( c_{10} > 0 \) which does not depend on \( i = 1, \ldots, 2^L \). We start by showing a law of large numbers for stopping times \( T_1, \ldots, T_{2^L} \).

**Lemma 2.3.** For \( L = \lfloor \log_2(n^{\beta}) \rfloor \) with \( \beta \in (0, 1) \) we have

\[
\sum_{i=1}^{2^L} T_i = n + \mathcal{O}(n^{(1+\beta)/2} h_d(n^{1-\beta})) \quad a.s.
\]

**Proof.** By Lemma 2.2, we have

\[
\mathbb{E}[T_i] = \sigma_d^{-2} \text{Var}(\tilde{C}^{(i)}_{n/2^L}) = \frac{n}{2^L} + \mathcal{O}((n/2^L)^{1/2} h_d(n/2^L)).
\]

Since \( L = \lfloor \log_2(n^{\beta}) \rfloor \) we have \( n^\beta / 2^L \leq n^\beta \) which implies

\[
\sum_{i=1}^{2^L} \mathbb{E}[T_i] = n + \mathcal{O}(n^{(1+\beta)/2} h_d(n^{1-\beta})).
\]
By [12, Lemma 5.4], there is a constant $c_{11} > 0$ such that
\[ \mathbb{E}[T_i^2] \leq c_{10}\sigma_i^{-4}\mathbb{E}[(\bar{X}_{n,2i}^{(i)})^4] \leq c_{11}\left(\frac{n}{2L}\right)^2 \leq 4c_{11}n^{2(1-\beta)}. \]

We have then
\[ \sum_{i=1}^{\infty} \text{Var}\left(\frac{T_i - \mathbb{E}[T_i]}{\sqrt{i\log(i+1)}}\right) \leq 4c_{11}n^{2(1-\beta)}\sum_{i=1}^{\infty} \frac{1}{i\log^2(i+1)} < \infty, \]
and according to [14, Theorem 2.5.3] it holds that
\[ \sum_{i=1}^{\infty} \frac{T_i - \mathbb{E}[T_i]}{\sqrt{i\log(i+1)}} < \infty \quad \text{a.s.} \]

Next we apply Kronecker’s lemma (see [14, Theorem 2.5.5]) to the two sequences \{(T_i - \mathbb{E}[T_i])/(\sqrt{i\log(1+i)})\}_{i \geq 1} and \{\sqrt{i\log(1+i)}\}_{i \geq 1}. We conclude that
\[ \sum_{i=1}^{2^L}(T_i - \mathbb{E}[T_i]) = \mathcal{O}(n^{3/2}\log n) \quad \text{a.s.} \]

Finally, we have
\[ \sum_{i=1}^{2^L}T_i = \sum_{i=1}^{2^L}(T_i - \mathbb{E}[T_i]) + \sum_{i=1}^{2^L}\mathbb{E}[T_i] \]
\[ = n + \mathcal{O}(n^{3/2}\log n) + n + \mathcal{O}(n^{(1+\beta)/2}h_d(n^{1-\beta})) \quad \text{a.s.} \]
which proves the assertion. \hfill \Box

The next step is the following asymptotic result.

**Lemma 2.4.** Choose $L = \lfloor \log_2(n^\beta) \rfloor$ with $\beta \in (0, 1)$. Then for any $\epsilon > 0$ we have
\[ B_{T_{0+\cdots+T_{2^L}}} - B_n = \begin{cases} \mathcal{O}(n^{(1-\Lambda + \beta\Lambda)/2+\epsilon}), & \Lambda \in (0, 1/2), \\ \mathcal{O}(n^{(1+\beta)/4+\epsilon}), & \Lambda \geq 1/2, \end{cases} \quad \text{a.s.} \]

**Proof.** Fix $\epsilon > 0$ and $\beta \in (0, 1)$. For any $\gamma > 0$ there is $c_{12} = c_{12}(\gamma) > 0$ such that
\[ s(n) \leq c_{12}n^\gamma \quad \text{when} \quad \Lambda \in (0, 1/2) \quad \text{and} \quad h_d(n) \leq c_{12}n^\gamma \quad \text{when} \quad \Lambda \geq 1/2 \quad \text{for} \quad n \geq 1. \]
Thus, according to Lemma 2.3,
\[ \sum_{i=1}^{2^L}T_i = \begin{cases} n + \mathcal{O}(n^{1-\Lambda + \beta\Lambda + \gamma(1-\beta)}), & \Lambda \in (0, 1/2), \\ n + \mathcal{O}(n^{(1+\beta)/2+\gamma(1-\beta)}), & \Lambda \geq 1/2, \end{cases} \quad \text{a.s.} \]

Observe that
\[ |B_{T_{0+\cdots+T_{2^L}}} - B_n| = \begin{cases} |B_{n+\mathcal{O}(n^{1-\Lambda + \beta\Lambda + \gamma(1-\beta)})} - B_n|, & \Lambda \in (0, 1/2), \\ |B_{n+\mathcal{O}(n^{1+\beta)/2+\gamma(1-\beta)}| - B_n|, & \Lambda \geq 1/2, \end{cases} \]
\[ \leq \begin{cases} R(n, n + c_{13}n^{1-\Lambda + \beta\Lambda + \gamma(1-\beta)}), & \Lambda \in (0, 1/2), \\ R(n, n + c_{13}n^{(1+\beta)/2+\gamma(1-\beta)}), & \Lambda \geq 1/2, \end{cases} \quad \text{a.s.,} \]

for some constant $c_{13} > 0$, where
\[ R(a, b) = \sup_{a \leq s, t \leq b} |B_s - B_t|. \]

We claim that
\[ R(a, b) \overset{(d)}{=} \sup_{0 \leq s, t \leq b-a} |B_s - B_t| \leq 2 \sup_{0 \leq s \leq b-a} |B_s|. \]
Indeed, for any \( u \geq 0 \) we have
\[
\mathbb{P}(R(a,b) \leq u) = \mathbb{E}[\mathbb{1}_{\{R(a,b) \leq u\}}] = \mathbb{E}[\mathbb{1}_{\{R(0,b-a) \leq u\}} \circ \theta_a|\mathcal{F}_a]]
\]
\[
= \int_{\mathbb{R}} \mathbb{E}[\mathbb{1}_{\{R(0,b-a) \leq u\}}] \mathbb{P}_{B_a}(dx) = \mathbb{P}(R(0,b-a) \leq u),
\]
where in the third step we used the Markov property and space homogeneity of \( \{B_t\}_{t \geq 0} \).

\( \theta_t \) denotes the shift operator, and \( \mathbb{P}_{B_a}(dx) \) stands for the distribution of the random variable \( B_a \).

We next choose arbitrary \( \delta > 0 \) and distinguish between two cases.

\textbf{Case (i).} When \( \Lambda \in (0,1/2) \), we have that
\[
\mathbb{P}(|B_{T_{1}+\cdots+T_{2L}} - B_{n}| \geq 2\sqrt{2c_{13}n^{(1-\Lambda+\beta\Lambda+\gamma(1-\beta))/2+\delta}})
\]
\[
\leq \mathbb{P}(R(n,n+c_{13}n^{(1-\Lambda+\beta\Lambda+\gamma(1-\beta))}) \geq 2\sqrt{2c_{13}n^{(1-\Lambda+\beta\Lambda+\gamma(1-\beta))/2+\delta}})
\]
\[
\leq \mathbb{P}\left(2 \sup_{0 \leq s \leq c_{13}n^{(1-\Lambda+\beta\Lambda+\gamma(1-\beta))}} |B_{s}| \geq 2\sqrt{2c_{13}n^{(1-\Lambda+\beta\Lambda+\gamma(1-\beta))/2+\delta}}\right)
\]
\[
\leq 2e^{-n^{2\delta}},
\]
where in the last step we applied [24, Exercise II.1.23]. Finally, the Borel-Cantelli lemma implies that
\[
B_{T_{1}+\cdots+T_{2L}} - B_{n} = \mathcal{O}(n^{(1-\Lambda+\beta\Lambda+\gamma(1-\beta))/2+\delta}) \quad \text{a.s.}
\]

The result follows by choosing \( \gamma \) and \( \delta \) such that \( \gamma(1-\beta)/2 + \delta < \epsilon \).

\textbf{Case (ii).} When \( \Lambda \geq 1/2 \), we again obtain
\[
\mathbb{P}(|B_{T_{1}+\cdots+T_{2L}} - B_{n}| \geq 2\sqrt{2c_{13}n^{(1-\Lambda+\beta+\gamma(1-\beta))/2+\delta}})
\]
\[
\leq \mathbb{P}(R(n,n+c_{13}n^{(1-\Lambda+\beta+\gamma(1-\beta))/2+\delta}}) \geq 2\sqrt{2c_{13}n^{(1-\Lambda+\beta+\gamma(1-\beta))/2+\delta}})
\]
\[
\leq \mathbb{P}\left(2 \sup_{0 \leq s \leq c_{13}n^{(1-\Lambda+\beta+\gamma(1-\beta))/2+\delta)))} |B_{s}| \geq 2\sqrt{2c_{13}n^{(1-\Lambda+\beta+\gamma(1-\beta))/2+\delta}}\right)
\]
\[
\leq 2e^{-n^{2\delta}},
\]
which together with the Borel-Cantelli lemma gives
\[
B_{T_{1}+\cdots+T_{2L}} - B_{n} = \mathcal{O}(n^{(1+\beta)/4+\gamma(1-\beta)/2+\delta}) \quad \text{a.s.}
\]

The assertion follows by choosing again \( \gamma \) and \( \delta \) such that \( \gamma(1-\beta)/2 + \delta < \epsilon \). \qed

We finally prove Theorem 1.1.

\textbf{Proof of Theorem 1.1.} By Lemma 2.1, we have
\[
\sigma_{d}^{-1}(\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)]) - B_{n}
\]
\[
= \sigma_{d}^{-1}\tilde{C}_{n} - B_{n}
\]
\[
= \sigma_{d}^{-1}\sum_{i=1}^{2L}\tilde{C}_{n/2^L}^{(i)} - \sigma_{d}^{-1}\mathcal{E}(n) - B_{n}
\]
\[
= \sigma_{d}^{-1}\sum_{i=1}^{2L}\tilde{C}_{n/2^L}^{(i)} - B_{n} + \begin{cases} \mathcal{O}(n^{1/2-\epsilon}), & \Lambda \in (0,1/2) \quad \text{and} \quad \epsilon \in (0,\Lambda), \\ \mathcal{O}(n^{\epsilon}), & \Lambda \geq 1/2 \quad \text{and} \quad \epsilon > 0, \end{cases}
\]
Since \( \sigma_{d}^{-1}\sum_{i=1}^{2L}\tilde{C}_{n/2^L}^{(i)} \) has the same law as \( B_{T_{1}+\cdots+T_{2L}} \) and they are defined on the same probability space, we can replace the first sequence with the second in the asymptotic formula. Hence
\[
\sigma_{d}^{-1}(\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)]) - B_{n}
\]
We fix arbitrary $\varepsilon > 0$ and consider two cases.

**Case (i).** When $\Lambda \in (0, 1/2)$ we take $\epsilon = \Lambda/4$, and choose $\beta \in (0, 1)$ in Lemma 2.4 such that $\beta \Lambda/2 < \varepsilon$. This yields

$$\sigma_a^{-1}(\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)]) - B_n = \mathcal{O}(n^{1/2-\Lambda/4+\varepsilon}) + \mathcal{O}(n^{1/2-\Lambda/4})$$

$$= \mathcal{O}(n^{9/8-d/(4\alpha)+\varepsilon}) \quad \text{a.s.}$$

**Case (ii).** When $\Lambda \geq 1/2$ we choose $\epsilon > 0$ and $\beta \in (0, 1)$ in Lemmas 2.1 and 2.4 such that $\beta/4 + \epsilon < \varepsilon$. This yields

$$\sigma_a^{-1}(\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)]) - B_n = \mathcal{O}(n^{(1+\beta)/4+\varepsilon}) + \mathcal{O}(n^\beta) = \mathcal{O}(n^{1/4+\varepsilon}) \quad \text{a.s.}$$

and the proof is finished. \hfill $\square$

### 3. The Cardinality Process

The goal of this section is to prove Theorem 1.4. Results in this section correspond to the results from Section 2 and the main arguments can be easily repeated so we only present the main steps of the proofs. We start with a decomposition of the range which goes back to Le Gall [20] and which was later applied in [2, Corollary 2.1] to handle the capacity of the range. For $a, b \in [0, \infty)$, $a \leq b$, we use notation $\mathcal{R}_a = \mathcal{R}_{[a]}$ and $\mathcal{R}[a, b] = \mathcal{R}_b \setminus \mathcal{R}_{a-1}$ with $\mathcal{R}_{a-1} = \emptyset$ if $a < 1$. It holds

$$|\mathcal{R}_n| = |\mathcal{R}_{n/2} \cup \mathcal{R}[n/2, n]| = |\mathcal{R}_{n/2} \cup \mathcal{R}[n/2, 2n] - S_{[n/2]}|$$

$$= |(\mathcal{R}_{n/2} - S_{[n/2]}) \cup (\mathcal{R}[n/2, n] - S_{[n/2]})| = |\mathcal{R}_{n/2}^{(1)} \cup \mathcal{R}_{n/2}^{(2)}|$$

$$= |\mathcal{R}_{n/2}^{(1)}| + |\mathcal{R}_{n/2}^{(2)}| - |\mathcal{R}_{n/2}^{(1)} \cap \mathcal{R}_{n/2}^{(2)}|. \tag{3.1}$$

The Markov property implies that $\mathcal{R}_{n/2}^{(1)}$ and $\mathcal{R}_{n/2}^{(2)}$ are independent, and that $\mathcal{R}_{n/2}^{(2)}$ is equal in law to $\mathcal{R}_{[n/2]}$ (or $\mathcal{R}_{[n/2+1]}$). By symmetry of $\{S_n\}_{n \geq 0}$ we have that $\mathcal{R}_{n/2}^{(1)}$ has the same law as $\mathcal{R}_{[n/2]}$. Applying the same subdivision to $\mathcal{R}_{n/2}^{(1)}$ and $\mathcal{R}_{n/2}^{(2)}$ and iterating this procedure $L$ times ($2^L \leq n$) we arrive at

$$|\mathcal{R}_n| = \sum_{i=1}^{2^L} |\mathcal{R}_{n/2^L}^{(i)}| - \sum_{i=1}^{2^{2^L-1}} \sum_{i=1}^{2^{2^L-1}} \epsilon_i. \tag{3.2}$$

Here the random variables $\epsilon_i$, $i = 1, \ldots, 2^{2^L-1}$, are independent, and $\epsilon_i$ has the same law as $|\mathcal{R}_{n/2^L} \cap \mathcal{R}_{n/2^L}|$, with $\mathcal{R}_{n/2^L}$ being an independent copy of $\mathcal{R}_{n/2^L}$. Also, random variables $\mathcal{R}_{n/2^L}^{(i)}$, $i = 1, \ldots, 2^L$, are independent, and $\mathcal{R}_{n/2^L}^{(i)}$ has the same law as $\mathcal{R}_{[n/2^L]}$ (or $\mathcal{R}_{[n/2^L+1]}$).

By $I_n$ we denote the number of intersection points of two independent copies of our random walk up to time $n$, that is, $I_n = |\mathcal{R}_n \cap \mathcal{R}_n|$ with $\mathcal{R}_n$ being independent of $\mathcal{R}_n$, and having the same law. Clearly, $\epsilon_i$ is equal in law to $\epsilon_i$. According to [21, Remark after Corollary 3.2] there is a constant $c_1 > 0$ such that

$$\mathbb{E}[I_n] \leq c_1 F_d(n), \tag{3.3}$$

where

$$F_d(n) = \begin{cases} 1, & d > 2\alpha, \\ \sum_{k=1}^n k^{-1}(f'_k)^{-d} & d = 2\alpha, \\ n^2(b(n))^{-d}, & d \in (\alpha, 2\alpha), \end{cases}$$
and \( \ell(n) \) is a slowly varying function from eq. (2.4). We fix the constant \( \Delta = d/\alpha - 3/2 \) and observe that if \( \Delta \geq 1/2 \) then \( F_d(n) \) is slowly varying, see [21, Lemma 2.2], and if \( \Delta \in (0,1/2) \) we represent the function \( F_d(n) \) in the form
\[
F_d(n) = n^{1/2-\Delta}s(n),
\]
where \( s(n) \) is a slowly varying function. In this case \( F_d(n) \) is evidently regularly varying of index smaller than 1/2.

We first study the error term in eq. (3.2). According to [21, Lemma 3.1], the following estimate is valid
\[
\mathbb{E}[(I_n)^p] \leq (p!)^2(\mathbb{E}[I_n])^p, \quad p \in \mathbb{N}.
\]
Our plan is to use eqs. (3.3) and (3.5) to bound the moment of order \( p \) of the error terms \( \mathcal{E}_i \) in eq. (3.2). We have
\[
\tilde{\mathcal{R}}_n = \sum_{i=1}^{2^L} \tilde{\mathcal{R}}_{n/2^L}^{(i)} + \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \mathbb{E}[\mathcal{E}_i] - \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \mathcal{E}_i,
\]
where \( \tilde{\mathcal{R}}_n = \mathcal{R}_n - \mathbb{E}[\mathcal{R}_n] \) (and similarly \( \tilde{\mathcal{R}}_{n/2^L}^{(i)} = \mathcal{R}_{n/2^L}^{(i)} - \mathbb{E}[\mathcal{R}_{n/2^L}^{(i)}] \)). We denote
\[
\mathcal{E}(n) = \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \mathbb{E}[\mathcal{E}_i] - \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \mathcal{E}_i.
\]

**Lemma 3.1.** Assume \((A1)\) and \( d > 3\alpha/2 \). Then,
\[
\mathcal{E}(n) = \begin{cases} \mathcal{O}(n^{1/2-\epsilon}), & \Delta \in (0,1/2) \text{ and } \epsilon \in (0,\Delta), \quad \text{a.s.} \\ \mathcal{O}(n^\epsilon), & \Delta \geq 1/2 \text{ and } \epsilon > 0, \end{cases}
\]

**Proof.** We have
\[
\mathbb{E}[|\mathcal{E}(n)|^p] \leq 2^{p-1} \left( \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \mathbb{E}[\mathcal{E}_i] \right)^p + 2^{p-1} \mathbb{E} \left[ \left( \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \mathcal{E}_i \right)^p \right],
\]
where we applied the inequality \((a+b)^p \leq 2^{p-1}(a^p + b^p)\) which holds for any \( a, b \geq 0 \) and \( p \in \mathbb{N} \). As mentioned after eq. (2.4), the function \( b(n) \) is chosen in such a way that \( F_d(n) \) is increasing. By eq. (3.3), we conclude that
\[
\mathbb{E}[\mathcal{E}_i] \leq c_1 F_d(n/2^l) \leq c_1 F_d(n),
\]
and whence
\[
\left( \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \mathbb{E}[\mathcal{E}_i] \right)^p \leq \left( \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} c_1 F_d(n) \right)^p \leq c_2^{2^L}(F_d(n))^p.
\]
For the second term in eq. (3.6) we use eq. (3.5) to get that there exists a constant \( c_2 = c_2(p) > 0 \) such that
\[
\mathbb{E}[(\mathcal{E}_i)^p] \leq c_2(F_d(n/2^l))^p \leq c_2(F_d(n))^p, \quad l = 1, \ldots, L, \quad i = 1, \ldots, 2^{l-1},
\]
which implies
\[
\|\mathcal{E}_i\|_p = (\mathbb{E}[(\mathcal{E}_i)^p])^{1/p} \leq c_2^{1/p} F_d(n).
\]
Together with Hölder’s inequality this gives
\[
\mathbb{E} \left[ \left( \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} \mathcal{E}_i \right)^p \right] \leq \sum_{l_1=1}^{L} \sum_{i_1=1}^{2^{l_1-1}} \cdots \sum_{l_p=1}^{L} \sum_{i_p=1}^{2^{l_p-1}} \|\mathcal{E}_{i_1}\|_p \cdots \|\mathcal{E}_{i_p}\|_p \leq c_2^{2^L}(F_d(n))^p.
\]
We set \( L = \lfloor \log_2(n^\beta) \rfloor \) with \( \beta \in (0, 1) \). It follows that
\[
\mathbb{E}[|\mathcal{E}(n)|^p] \leq c_32^\beta n^{\beta/2} (F_d(n))^p.
\]
The assertion follows by using the same arguments as in Lemma 2.1.

By [21, Theorem 4.4], assumption (A1) implies that if \( d > 3\alpha/2 \) then the sequence \( \{\text{Var}(\mathcal{R}_n)/n\}_{n \geq 1} \) converges to a strictly positive limit \( \sigma_d^2 \). We first obtain the second order term of this asymptotics.

**Lemma 3.2.** Assume (A1) and \( d > 3\alpha/2 \). It holds that
\[
\text{Var}(\mathcal{R}_n) = \sigma_d^2 n + O(n^{1/2} F_d(n)).
\]

**Proof.** Similarly as in eq. (3.2) we can easily show that for any \( n, m \geq 1 \),
\[
(3.7) \quad |\mathcal{R}_n^{(1)}| + |\mathcal{R}_m^{(2)}| - I_{n+m} \leq |\mathcal{R}_m^{(2)}| \leq |\mathcal{R}_n^{(1)}| + |\mathcal{R}_m^{(2)}|,
\]
where \( I_n = |\mathcal{R}_n^{(1)} \cap \mathcal{R}_m^{(2)}| \). By taking expectation in eq. (3.7) and then by subtracting those two relations,
\[
\mathcal{R}_n^{(1)} + \mathcal{R}_m^{(2)} - I_{n+m} \leq \mathcal{R}_{n+m} \leq \mathcal{R}_n^{(1)} + \mathcal{R}_m^{(2)} + \mathbb{E}[I_{n+m}],
\]
which implies
\[
|\mathcal{R}_{n+m} - (\mathcal{R}_n^{(1)} + \mathcal{R}_m^{(2)})| \leq I_{n+m} + \mathbb{E}[I_{n+m}] \leq I_{n+m} + \|I_{n+m}\|_2.
\]
There is \( c_4 > 0 \) such that
\[
(3.8) \quad \|I_n\|_2 \leq c_4 F_d(n) \quad \text{and} \quad \|\mathcal{R}_n\|_2 \leq c_4 \sqrt{n}, \quad n \geq 1,
\]
see [21, Remark after Corollary 3.2 and Theorem 4.4]. Applying these estimates we can proceed similarly as in Lemma 2.2. As \( d > 3\alpha/2 \), \( F_d(n) \leq c_5 \sqrt{n} \) for some \( c_5 > 0 \) and all \( n \geq 1 \), and we obtain that for
\[
x_n = \text{Var}(\mathcal{R}_n) = \|\mathcal{R}_n\|_2^2 \quad \text{and} \quad b_n = c_6 \sqrt{n} F_d(n), \quad n \geq 1,
\]
for some \( c_6 > 0 \), we have
\[
x_n + x_m - b_{n+m} \leq x_{n+m} \leq x_n + x_m + b_{n+m}, \quad n, m \geq 1,
\]
and from [21, Theorem 4.4] we know that
\[
\lim_{n, m \to \infty} \frac{x_n}{n} = \sigma_d^2 > 0.
\]
Setting \( n = m = 2^{k-1}l \) for \( k, l \geq 1 \), we arrive at
\[
\left| \frac{x_n}{n} - \sigma_d^2 \right| \leq \frac{c_6}{\sqrt{n}} \sum_{k=1}^{\infty} \frac{F_d(2^k n)}{2^{k/2}}, \quad n \geq 1.
\]
Again, by a similar reasoning as in Lemma 2.2, the result follows.

The next step is to investigate the asymptotic behavior of \( \sum_{i=1}^{2^L} \bar{\mathcal{R}}_{n/2^L}^{(i)} \). We again apply Skorohod embedding theorem to conclude that there exist a standard Brownian motion \( \{B_t\}_{t \geq 0} \) and stopping times \( T_1, \ldots, T_{2^L} \) such that

\[
\{B_{T_0 + \cdots + T_i} - B_{T_0 + \cdots + T_{i-1}}\}_{i=1,\ldots,2^L} \overset{(d)}{=} \{\sigma_d^{-1} \bar{\mathcal{R}}_{n/2^L}^{(i)}\}_{i=1,\ldots,2^L},
\]
where \( T_0 = 0 \). We obtain that \( \sigma_d^{-1} \sum_{i=1}^{2^L} \bar{\mathcal{R}}_{n/2^L}^{(i)} \) has the same law as \( B_{T_0 + \cdots + T_{2^L}} \). Moreover, the following moment estimates hold
\[
\mathbb{E}[T_i] = \sigma_d^{-2} \text{Var}(\bar{\mathcal{R}}_{n/2^L}^{(i)}) \quad \text{and} \quad \mathbb{E}[T_{i}^2] \leq c_7 \sigma_d^{-4} \mathbb{E}[\bar{\mathcal{R}}_{n/2^L}^{(i)}]^4,
\]
for a constant \(c_7 > 0\) which does not depend on \(i = 1, \ldots, 2^L\). Recall that, if necessary, we can enlarge the probability space so that \(\{B_t\}_{t \geq 0}\) and \(\{S_n\}_{n \geq 0}\) are defined on the same \((\Omega, \mathcal{F}, \mathbb{P})\). We now establish a law of large numbers for stopping times \(T_1, \ldots, T_{2^L}\).

**Lemma 3.3.** For \(L = \lfloor \log_2(n^\beta) \rfloor\) with \(\beta \in (0, 1)\) we have

\[
\sum_{i=1}^{2^L} T_i = n + O(n^{(1+\beta)/2} F_d(n^{1-\beta})) \quad \text{a.s.}
\]

*Proof.* The proof proceeds by using the same arguments as in Lemma 2.3. \(\Box\)

Next step is to prove the result analogous to Lemma 2.4.

**Lemma 3.4.** Choose \(L = \lfloor \log_2(n^\beta) \rfloor\) with \(\beta \in (0, 1)\). Then for any \(\epsilon > 0\) we have

\[
B_{T_0 + \cdots + T_{2^L}} - B_n = \begin{cases} 
O(n^{(1-\Delta+\beta\Delta)/2+\epsilon}), & \Delta \in (0, 1/2), \\
O(n^{(1+\beta)/4+\epsilon}), & \Delta \geq 1/2,
\end{cases} \quad \text{a.s.}
\]

*Proof.* The proof follows along the same arguments as in Lemma 2.4. \(\Box\)

*Proof of Theorem 1.4.* As in Section 2, we combine Lemmas 3.1 and 3.4 to prove the desired almost sure invariance principle for the process \(\{R_n\}_{n \geq 1}\). \(\Box\)

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