Large Sets of $t$-Designs over Finite Fields

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**Abstract**

A $t$-$(n, k, \lambda; q)$-design is a set of $k$-subspaces, called blocks, of an $n$-dimensional vector space $V$ over the finite field with $q$ elements such that each $t$-subspace is contained in exactly $\lambda$ blocks. A partition of the complete set of $k$-subspaces of $V$ into disjoint $t$-$(n, k, \lambda; q)$ designs is called a large set of $t$-designs over finite fields. In this paper we give the first nontrivial construction of such a large set with $t \geq 2$. 
1 Introduction

A simple \( t \)-design over a finite field or, more precisely, a \( t-(n,k,\lambda;q) \) design is a set \( \mathcal{B} \) of \( k \)-subspaces of an \( n \)-dimensional vector space \( V \) over the finite field \( \mathbb{F}_q \) such that each \( t \)-subspace of \( V \) is contained in exactly \( \lambda \) members of \( \mathcal{B} \).

The study of combinatorial \( t \)-designs and Steiner systems on (finite) sets goes back to the 19th century and has a rich literature [7]. Cameron [5, 6] and Delsarte [8] extended the notions of \( t \)-designs and Steiner systems from sets to vector spaces over finite fields in the early 1970s. Recently, designs over finite fields gained a lot of interest because of applications for error-correction in networks [12].

In 1987, Thomas [24] constructed the first nontrivial simple \( t \)-designs over finite fields for \( t = 2 \). Since then, more designs over finite fields have been constructed, see [3, 4, 9, 17, 21, 22]. Specifically, in [3] the first nontrivial \( t \)-design over finite fields with \( t = 3 \) has been found and in [4] 2-(13,3,1;2) designs have been constructed. The latter ones are the first nontrivial \( t \)-designs over finite fields with \( \lambda = 1 \) and \( t = 2 \). Designs with \( \lambda = 1 \) are called \( q \)-Steiner systems.

An \( LS_q[N](t,k,n) \) large set \( \mathcal{L} \) is a set of \( N \) disjoint \( t-(n,k,\lambda;q) \) designs such that their union forms the complete set of all \( k \)-subspaces of \( V = \mathbb{F}_q^n \). Large sets of designs over finite fields have been studied for the first time by Ray-Chaudhuri and Schram [19]. There, the authors used non-simple designs. In this paper we investigate the existence of large sets of simple \( t \)-designs over finite fields.

In the case of designs on sets, large sets are intensively studied objects [10, Section II.4.4]. A celebrated result by Teirlinck [23] is that large sets of designs on sets exist for all \( t > 0 \) and \( N > 0 \).

Large sets of certain \( t-(n,k,\lambda;q) \) designs have been intensively studied in the framework of projective geometry. In geometry, \( 1-(n,k,1;q) \) designs are known as \((k-1)\)-spreads in \( PG(n-1,q) \). A large set of \( 1-(n,k,1;q) \) designs is called \((k-1)\)-parallelism of the projective geometry \( PG(n-1,q) \). A parallelism is a 1-parallelism, i.e. \( k = 2 \).

Since \( 1-(n,k,1;q) \) designs exist if and only if \( k \) divides \( n \), a necessary condition for the existence of a parallelism in \( PG(n-1,q) \) is that \( n \) must be even. Beutelspacher [2] proved the existence of a parallelism in \( PG(2^i - 1,q) \) for all \( i \geq 2 \). Later, Baker [1] and Wettl [25] gave a construction of parallelisms in \( PG(n-1,q) \) for \( n \) even. Penttila and Williams [18] studied
PG(3, q) for $q \equiv 2 \mod 3$ and constructed parallelisms subsuming the results presented in [16].

Up to now, no large sets of $t$-designs over finite fields with $t \geq 2$ have been reported. The main result of this paper is the following one:

**Theorem 1.** Nontrivial large sets of $t$-designs over finite fields exist for $t \geq 2$.

The theorem is proved by showing the existence of a large set consisting of three disjoint $2$-(8, 3, 21; 2) designs.

## 2 The Construction of Large Sets

Let $[V_k]$ denote the set of $k$-subspaces of $V$. The expression

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{(q^n-1)(q^{n-1}-1) \cdots (q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1) \cdots (q-1)}$$

is called the $q$-binomial coefficient. The set $[V_k]$ itself is already a design, the so-called trivial design, with parameters $t$-$(n, k, \lambda_{\text{max}}; q)$, where

$$\lambda_{\text{max}} = \left[ \begin{array}{c} n-t \\ k-t \end{array} \right]_q.$$

Hence, an obvious necessary condition for the existence of a $LS_q[N](t, k, n)$ large set is the equality $\lambda \cdot N = \lambda_{\text{max}}$. Moreover, since the blocks of a $t$-design also form an $i$-design as long as $0 \leq i \leq t$, we have the necessary conditions

$$N \mid \left[ \begin{array}{c} n-i \\ k-i \end{array} \right]_q \quad \text{for} \quad 0 \leq i \leq t.$$ 

The general linear group $\text{GL}(n, q)$, whose elements are represented by $n \times n$-matrices $\alpha$, acts on $[V_k]$ by left multiplication $\alpha K := \{\alpha x \mid x \in K\}$. An element $\alpha \in \text{GL}(n, q)$ is called an automorphism of a $t$-$(n, k, \lambda; q)$ design $\mathcal{B}$ if $\mathcal{B} = \alpha \mathcal{B} := \{\alpha K \mid K \in B\}$. The set of all automorphisms of a design forms a group, called the automorphism group of the design. Every subgroup of the automorphism group of a design is denoted as a group of automorphisms of the design.
If $G$ is a subgroup of $GL(n, q)$ the $G$-orbit on a $k$-subspace $K$ is denoted by $G(K) := \{ \alpha K \mid \alpha \in G \} \subseteq \binom{V}{k}$. Now, a $t$-$(n, k, \lambda; q)$ design $\mathcal{B}$ admits a subgroup $G$ of the general linear $GL(n, q)$ as a group of automorphisms if and only if $\mathcal{B}$ consists of $G$-orbits on $\binom{V}{k}$. The $G$-incidence matrix $A_{t,k}^G$ is defined to be the matrix whose rows and columns are indexed by the $G$-orbits on the set of $t$- and $k$-subspaces of $V$, respectively. The entry indexed by the orbit $G(T)$ on $\binom{V}{t}$ and the orbit $G(K)$ on $\binom{V}{k}$ is defined by $|\{K' \in G(K) \mid T \subseteq K'\}|$.

According to Kramer and Mesner [13] a simple $t$-$(n, k, \lambda; q)$ design admitting $G$ as a group of automorphisms exists if and only if there is a 0/1-column vector $x$ satisfying $A_{t,k}^G x = \lambda \mathbf{1}$, where $\mathbf{1}$ denotes the all-one column vector. The vector $x$ represents the corresponding selection of $G$-orbits on $\binom{V}{k}$.

The following algorithm describes a basic approach to find large sets. A version of this algorithm for large sets of designs on sets can be found in [14, 15].

**Algorithm A.** The algorithm computes an $LS_q[N](t, k, n)$ large set $\mathcal{L}$ consisting of $N$ $t$-$(n, k, \lambda; q)$ designs admitting $G$ as a group of automorphisms. Either the algorithm terminates with a large set or it ends without any statement about the existence.

1. **[Initialize.]** Set $\mathcal{B}$ as the complete set of $G$-orbits on $\binom{V}{k}$ and set $\mathcal{L} := \emptyset$.

2. **[Solve.]** Find a random $t$-$(n, k, \lambda; q)$ design $\mathcal{B}$ consisting of orbits of $\mathcal{B}$. If such a $t$-design exists insert $\mathcal{B}$ into $\mathcal{L}$ and continue with A3. Otherwise terminate without a large set.

3. **[Remove.]** Remove the selected orbits in $\mathcal{B}$ from $\mathcal{B}$. If $\mathcal{B} = \emptyset$ then terminate with a large set $\mathcal{L}$. Otherwise goto A2.

Algorithm A can be implemented by a slight modification of the Kramer-Mesner approach. We just have to add a further row to the Diophantine system of equations the following way:

$$
\begin{bmatrix}
A_{t,k}^G \\
\cdots y_K \\
\end{bmatrix}
\begin{bmatrix}
x \\
\end{bmatrix}
= 
\begin{bmatrix}
\lambda \\
\vdots \\
\lambda \\
0 \\
\end{bmatrix}
$$

The vector $y = [\cdots y_K \cdots]$ is indexed by the $G$-orbits on $\binom{V}{k}$ corresponding to the columns of $A_{t,k}^G$. The entry $y_K$ indexed by the $G$-orbit containing
$K$ is defined to be one if the orbit has already been covered by a selected $t$-$(n, k, \lambda; q)$ design. Otherwise it is zero. In every iteration step the vector $y$ has to be updated.

A second simple approach which might be reasonable if the number of total solutions of the Kramer-Mesner system is small uses an exact cover solver [11].

**Algorithm B.** The algorithm computes an $LS_q[N](t, k, n)$ large set $\mathcal{L}$ consisting of $N$ $t$-$(n, k, \lambda; q)$ designs admitting $G$ as a group of automorphisms. The algorithm terminates with the existence statement true or false.

B1. [Initialize.] Find all 0/1-column vectors $x_1, \ldots, x_s$ solving $A^G_{t,k}x = \lambda 1$ and form the matrix $A = [x_1 \mid \cdots \mid x_s]$.

B2. [Exact cover.] Find a 0/1-vector $y$ solving the system $Ay = 1$. If such a solution $y$ exists return true. Otherwise return false.

### 3 The Existence of $LS_2[3](2, 3, 8)$

In this section we present the construction of the first nontrival large set of designs over finite fields, a large set $\mathcal{L}$ with parameters $LS_2[3](2, 3, 8)$. The large set consists of three 2-(8, 3, 21; 2) designs $\mathcal{L} = \{B_1, B_2, B_3\}$, each admitting $G = \langle \alpha \rangle \leq \text{GL}(8, 2)$ as a group of automorphisms. The group $G$ has order 255 and is generated by a Singer cycle $\alpha$, represented by the matrix

$$
\alpha = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

The existence of 2-(8, 3, 21; 2) designs having $G$ as a group of automorphisms has been shown previously in [3]. The large set $\mathcal{L}$ was constructed with Algorithm A. Each of the three designs of $\mathcal{L}$ consists of 127 orbits of $G$ on the set of 3-subspaces of $V = \mathbb{F}_2^8$. The orbit representatives for each of
the designs $B_1$, $B_2$, and $B_3$ are depicted in Tables 1, 2, and 3. For each representative, the three column vectors

\[
\begin{bmatrix}
  x_0 & y_0 & z_0 \\
  x_1 & y_1 & z_1 \\
  \vdots & \vdots & \vdots \\
  x_7 & y_7 & z_7 \\
\end{bmatrix}
\]

spanning a 3-subspace of $V$, are encoded as a triple of the positive integers

\[
[X, Y, Z] = \left[ \sum_{i=0}^{7} x_i 2^i, \sum_{i=0}^{7} y_i 2^i, \sum_{i=0}^{7} z_i 2^i \right].
\]

### 4 Further Results

Let $K^\perp = \{ x \in V \mid \langle x, y \rangle = 0 \text{ for all } y \in K \}$ denote the orthogonal complement of a subspace $K$ of $V$ with respect to the standard inner product $\langle -,- \rangle$. By Suzuki [20, Lemma 4.3] we know that every $t-(n, k, \lambda; q)$ design $\mathcal{B}$ defines a $t-(n, n-k, \lambda^\perp; q)$ design $\mathcal{B}^\perp := \{ K^\perp \mid K \in \mathcal{B} \}$ with

\[
\lambda^\perp = \lambda \left\lceil \frac{n-t}{k-t} \right\rceil q.
\]

This design is called the *complementary design*. If a $t$-design admits $G$ as a group of automorphisms, which group does the complementary $t$-design admit?

The orthogonal complement of a subspace corresponds to the set-wise complement of a subset in the classical situation for designs on sets. There, it is clear that the automorphism group which is a subgroup of the symmetric group remains the same, since set-wise complements commute with permutations: $\pi K = \pi \overline{K}$.

For designs over finite fields the following lemma gives the answer.

**Lemma 1.** A $t-(n, k, \lambda; q)$ design $\mathcal{B}$ admits $G \leq \text{GL}(n, q)$ as a group of automorphisms if and only if the complementary $t-(n, n-k, \lambda^\perp; q)$ design $\mathcal{B}^\perp$ admits $H = \{ \alpha^T \mid \alpha \in G \}$ as a group of automorphisms.
Table 1: Design $B_1$

| 1, 112, 128 | 14, 64, 128 | 40, 112, 128 | 69, 80, 128 | 92, 32, 128 |
| 1, 48, 128  | 15, 16, 128 | 41, 112, 128 | 70, 8, 128  | 94, 32, 128  |
| 2, 80, 128  | 15, 96, 128 | 41, 64, 128  | 70, 32, 128 | 94, 96, 128  |
| 2, 96, 128  | 16, 64, 128 | 42, 48, 128  | 70, 96, 128 | 98, 8, 128   |
| 3, 48, 128  | 17, 64, 128 | 43, 80, 128  | 71, 16, 128 | 98, 16, 128  |
| 3, 64, 128  | 17, 96, 128 | 44, 16, 128  | 71, 48, 128 | 99, 16, 128  |
| 4, 32, 128  | 18, 32, 128 | 45, 80, 128  | 72, 80, 128 | 99, 36, 128  |
| 4, 48, 128  | 19, 8, 128  | 47, 112, 128 | 73, 80, 128 | 100, 8, 128  |
| 4, 72, 128  | 19, 96, 128 | 49, 40, 128  | 74, 16, 128 | 100, 24, 128 |
| 5, 72, 128  | 20, 96, 128 | 49, 64, 128  | 75, 80, 128 | 100, 40, 128 |
| 5, 80, 128  | 21, 24, 128 | 50, 64, 128  | 77, 96, 128 | 101, 16, 128 |
| 6, 32, 128  | 21, 32, 128 | 50, 84, 128  | 78, 32, 128 | 101, 112, 128|
| 6, 72, 128  | 21, 96, 128 | 52, 40, 128  | 79, 32, 128 | 102, 80, 128 |
| 6, 96, 128  | 23, 8, 128  | 55, 56, 128  | 79, 48, 128 | 103, 40, 128 |
| 7, 32, 128  | 25, 32, 128 | 55, 64, 128  | 79, 112, 128| 103, 48, 128 |
| 7, 48, 128  | 33, 48, 128 | 56, 64, 128  | 83, 8, 128  | 103, 72, 128 |
| 7, 80, 128  | 33, 64, 128 | 57, 4, 128   | 83, 96, 128 | 105, 112, 128|
| 8, 64, 128  | 33, 80, 128 | 60, 64, 128  | 84, 8, 128  | 106, 112, 128|
| 9, 64, 128  | 34, 64, 128 | 61, 64, 128  | 84, 96, 128 | 110, 16, 128 |
| 9, 96, 128  | 35, 4, 128  | 62, 64, 128  | 85, 24, 128 | 110, 48, 128 |
| 10, 32, 128 | 35, 48, 128 | 63, 64, 128  | 85, 32, 128 | 114, 40, 128 |
| 10, 64, 128 | 36, 64, 128 | 65, 32, 128  | 85, 96, 128 | 114, 120, 128|
| 12, 96, 128 | 37, 40, 128 | 66, 80, 128  | 86, 32, 128 | 122, 4, 128  |
| 13, 32, 128 | 37, 64, 128 | 67, 48, 128  | 87, 32, 128 |                |
| 13, 64, 128 | 38, 16, 128 | 67, 96, 128  | 89, 96, 128 |                |
| 14, 32, 128 | 38, 112, 128| 69, 16, 128  | 90, 32, 128 |                |
|1, 24, 128 |20, 32, 128 |38, 64, 128 |67, 112, 128 |86, 96, 128 |
|1, 32, 128 |20, 64, 128 |38, 72, 128 |68, 16, 128 |87, 8, 128 |
|1, 64, 128 |20, 72, 128 |38, 80, 128 |68, 96, 128 |87, 24, 128 |
|1, 92, 128 |20, 120, 128 |39, 80, 128 |69, 24, 128 |87, 96, 128 |
|2, 32, 128 |22, 32, 128 |39, 120, 128 |69, 32, 128 |91, 32, 128 |
|2, 64, 128 |22, 64, 128 |40, 16, 128 |69, 40, 128 |95, 96, 128 |
|3, 16, 128 |23, 64, 128 |41, 16, 128 |69, 96, 128 |98, 48, 128 |
|3, 32, 128 |25, 64, 128 |41, 48, 128 |70, 48, 128 |98, 80, 128 |
|3, 68, 128 |26, 96, 128 |42, 64, 128 |71, 32, 128 |99, 48, 128 |
|3, 80, 128 |27, 32, 128 |43, 64, 128 |71, 72, 128 |99, 80, 128 |
|3, 96, 128 |27, 64, 128 |44, 80, 128 |71, 80, 128 |100, 16, 128 |
|5, 32, 128 |27, 96, 128 |46, 16, 128 |71, 112, 128 |102, 16, 128 |
|5, 64, 128 |28, 64, 128 |46, 64, 128 |72, 32, 128 |102, 40, 128 |
|7, 96, 128 |28, 96, 128 |47, 16, 128 |72, 96, 128 |103, 16, 128 |
|8, 48, 128 |30, 64, 128 |47, 80, 128 |73, 96, 128 |108, 48, 128 |
|9, 16, 128 |31, 32, 128 |51, 64, 128 |74, 32, 128 |109, 16, 128 |
|10, 96, 128 |31, 96, 128 |52, 64, 128 |74, 96, 128 |109, 48, 128 |
|12, 32, 128 |32, 64, 128 |52, 72, 128 |76, 32, 128 |114, 4, 128 |
|12, 64, 128 |34, 20, 128 |54, 8, 128 |76, 48, 128 |114, 36, 128 |
|14, 48, 128 |34, 80, 128 |55, 24, 128 |77, 32, 128 |115, 40, 128 |
|14, 80, 128 |35, 8, 128 |65, 24, 128 |78, 96, 128 |115, 72, 128 |
|15, 32, 128 |35, 16, 128 |65, 112, 128 |82, 32, 128 |117, 120, 128 |
|15, 48, 128 |35, 36, 128 |66, 16, 128 |82, 56, 128 |118, 120, 128 |
|15, 112, 128 |37, 8, 128 |66, 48, 128 |83, 32, 128 |
|18, 64, 128 |37, 72, 128 |66, 112, 128 |84, 32, 128 |
|19, 72, 128 |37, 112, 128 |67, 80, 128 |85, 40, 128 |
Table 3: Design $B_3$

|         | 1, 16, 128 | 14, 96, 128 | 36, 120, 128 | 66, 96, 128 | 87, 72, 128 |
|---------|------------|-------------|---------------|--------------|--------------|
| 2, 120 | 15, 64, 128 | 37, 24, 128 | 67, 16, 128   | 88, 32, 128  | 99, 32, 128  |
| 2, 36, | 15, 80, 128 | 37, 80, 128 | 67, 32, 128   | 91, 64, 128  |             |
| 2, 52, | 17, 32, 128 | 39, 8, 128  | 68, 32, 128   | 92, 64, 128  |             |
| 3, 40, | 19, 32, 128 | 39, 16, 128 | 69, 48, 128   | 95, 32, 128  |             |
| 4, 56, | 19, 64, 128 | 39, 48, 128 | 69, 120, 128  | 99, 112, 128 |             |
| 4, 64, | 19, 68, 128 | 39, 64, 128 | 70, 16, 128   | 100, 112, 128|             |
| 4, 96, | 21, 64, 128 | 40, 64, 128 | 71, 8, 128    | 100, 120, 128|             |
| 5, 16, | 23, 32, 128 | 42, 80, 128 | 71, 56, 128   | 101, 24, 128 |             |
| 5, 96, | 23, 96, 128 | 43, 16, 128 | 72, 48, 128   | 101, 8, 128  |             |
| 5, 112 | 24, 32, 128 | 44, 64, 128 | 73, 16, 128   | 101, 48, 128 |             |
| 6, 56, | 24, 64, 128 | 45, 64, 128 | 73, 32, 128   | 102, 56, 128 |             |
| 6, 64, | 26, 64, 128 | 47, 64, 128 | 76, 16, 128   | 104, 16, 128 |             |
| 7, 16, | 27, 4, 128  | 48, 64, 128 | 76, 80, 128   | 104, 112, 128|             |
| 7, 64, | 28, 32, 128 | 49, 8, 128  | 76, 96, 128   | 107, 112, 128|             |
| 7, 112 | 30, 32, 128 | 53, 64, 128 | 77, 112, 128  | 107, 48, 128 |             |
| 8, 32, | 30, 96, 128 | 54, 64, 128 | 78, 16, 128   | 110, 80, 128 |             |
| 8, 80, | 31, 64, 128 | 55, 72, 128 | 79, 16, 128   | 111, 16, 128 |             |
| 8, 96, | 33, 40, 128 | 57, 64, 128 | 79, 96, 128   | 114, 8, 128  |             |
| 9, 32, | 33, 72, 128 | 58, 64, 128 | 81, 32, 128   | 114, 52, 128 |             |
| 9, 48, | 34, 48, 128 | 59, 64, 128 | 81, 96, 128   | 114, 84, 128 |             |
| 9, 80, | 34, 72, 128 | 65, 16, 128 | 82, 8, 128    | 118, 8, 128  |             |
| 10, 16 | 34, 100, 128| 65, 48, 128 | 82, 96, 128   | 118, 24, 128 |             |
| 13, 48 | 35, 64, 128 | 65, 80, 128 | 83, 40, 128   |             |             |
| 13, 96 | 35, 72, 128 | 65, 96, 128 | 86, 24, 128   |             |             |
| 14, 16 | 35, 80, 128 | 66, 32, 128 | 87, 40, 128   |             |             |
Proof. We have

\[(\alpha K)^\perp = \{x \in V \mid \langle x, \alpha y \rangle = 0 \forall y \in K\} \]
\[= \{x \in V \mid \langle \alpha^T x, y \rangle = 0 \forall y \in K\} \]
\[= \{(\alpha^T)^{-1} x \mid x \in V : \langle x, y \rangle = 0 \text{ for all } y \in K\} \]
\[= \{(\alpha^T)^{-1} x \mid x \in K^\perp\} \]
\[= (\alpha^T)^{-1}(K^\perp). \]

Since the mapping \(\alpha \mapsto (\alpha^T)^{-1}\) defines a group isomorphism between \(G\) and \(H\) the orthogonal complement maps orbits \(G(K)\) of \(\left[ \begin{array}{c} V \\ k \end{array} \right]\) onto orbits \(H(K^\perp)\) of \(\left[ \begin{array}{c} V \\ n-k \end{array} \right]\). This completes the proof. \(\square\)

The orthogonal complement defines a bijection between the set of \(k\)- and \((n-k)\)-subspaces, and hence a partition of \(\left[ \begin{array}{c} V \\ k \end{array} \right]\) into \(t-(n, k, \lambda; q)\) designs yields a partition of \(\left[ \begin{array}{c} V \\ n-k \end{array} \right]\) into \(t-(n, n-k, \lambda^\perp; q)\) designs. Finally, the existence of an \(LS_q[N](t, k, n)\) large set implies the existence of an \(LS_q[N](t, n-k, n)\) large set.

Taking the orthogonal complements of each orbit representative of the 2-(8, 3, 21; 2) designs given in the Tables[1][2] and[3] we obtain representatives of disjoint 2-(8, 5, 21; 2) designs forming an \(LS_2[3](2, 5, 8)\) large set, where each design is admitting a Singer cyclic group as a group of automorphisms by Lemma[1].

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References

[1] R. D. Baker. Partitioning the Planes of \(AG_{2m}(2)\) into 2-Designs. \textit{Discrete Mathematics}, 15:205–211, 1976.

[2] A. Beutelspacher. On Parallelisms of Finite Projective Spaces. \textit{Geometriae Dedicata}, 3:35–40, 1974.
[3] M. Braun, A. Kerber, and R. Laue. Systematic Construction of $q$-Analogs of Designs. *Designs, Codes and Cryptography*, 34:55–70, 2005.

[4] M. Braun, T. Etzion, P. J. R. Östergård, A. Vardy, and A. Wassermann. Existence of $q$-Analogs of Steiner Systems. submitted, 2013.

[5] P. J. Cameron. Generalisation of Fisher’s Inequality to Fields with More than One Element, in T. McDonough and V. Mavron, Eds., *Combinatorics*, London Mathematical Society Lecture Note Series, 13:9–13, 1974.

[6] P. J. Cameron. Locally Symmetric Designs. *Geometriae Dedicata*, 3:65–76, 1974.

[7] C. J. Colbourn and J. H. Dinitz (eds.). *Handbook of Combinatorial Designs (2nd ed.)*. CRC Press, 2007.

[8] P. Delsarte. Association Schemes and $t$-Designs in Regular Semilattices. *Journal of Combinatorial Theory, Series A*, 20:230–243, 1976.

[9] T. Itoh. A New Family of 2-Designs over $GF(q)$ Admitting $SL_m(q^l)$. *Geometriae Dedicata*, 69:261–286, 1998.

[10] G. B. Khosrovshahi and R. Laue. $t$-designs with $t \geq 3$. in C. J. Colbourn and J. H. Dinitz (eds.), *Handbook of Combinatorial Designs (2nd ed.)*, 79–101, CRC Press, 2007

[11] D. E. Knuth. Dancing Links. in J. Davies, B. Roscoe, and J. Woodcock (eds.), *Millennial Perspectives in Computer Science*, Palgrave Macmillan, Basingstoke, 187–214, 2000.

[12] R. Koetter and F. Kschischang. Coding for Errors and Erasures in Random Network Coding. *IEEE Transactions on Information Theory*, 54:3579–3591, 2008.

[13] E. Kramer and D. Mesner. $t$-Designs on Hypergraphs. *Discrete Mathematics*, 15(3):263–296, 1976.

[14] R. Laue, S. Magliveras, and A. Wassermann. New Large Sets of $t$-Designs. *Journal of Combinatorial Designs*, 9:40–59, 2001.
[15] R. Laue, G. R. Omidi, B. Tayfeh-Rezaie, and A. Wassermann. New Large Sets of $t$-Designs with Prescribed Groups of Automorphisms. *Journal of Combinatorial Designs*, 15(3):210–220, 2007.

[16] G. Lunardon. On Regular Parallelisms in $PG(3, q)$. *Discrete Mathematics*, 51:229–235, 1984.

[17] M. Miyakawa, A. Munemasa, and S. Yoshiara. On a Class of Small 2-Designs over $GF(q)$. *Journal of Combinatorial Designs*, 3:61–77, 1995.

[18] T. Penttila and B. Williams. Regular Packings of $PG(3, q)$. *European Journal of Combinatorics*, 19:713–720, 1998.

[19] D. K. Ray-Chaudhuri and E. J. Schram. A Large Set of Designs on Vector Spaces. *Journal of Number Theory*, 47:247–272, 1994.

[20] H. Suzuki. *Five Days Introduction to the Theory of Designs*. Lecture Notes, given at Osaka City Univ. in December, 1989.

[21] H. Suzuki. 2-Designs over $GF(2^m)$. *Graphs and Combinatorics*, 6:293–296, 1990.

[22] H. Suzuki. 2-Designs over $GF(q)$. *Graphs and Combinatorics*, 8:381–389, 1992.

[23] L. Teirlinck. Locally Trivial $t$-Designs and $t$-Designs without Repeated Blocks. *Discrete Mathematics*, 77:345–356, 1989.

[24] S. Thomas. Designs over Finite Fields. *Geometriae Dedicata*, 24:237–242, 1987.

[25] F. Wettl. On Parallelisms of Odd-Dimensional Finite Projective Spaces. Proceedings of the second international mathematical miniconference, part II (Budapest, 1988), Period Polytech. Transportation Engrg, 19(1-2):111–116, 1991.