Graphs with $C_3$-free vertices are not universal fixers

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Abstract

A non-isolated vertex $x \in V(G)$ is called $C_3$-free if $x$ belongs to no triangle of $G$. In [1] Burger, Mynhardt and Weakley introduced the idea of universal fixers. Let $G = (V, E)$ be a graph with $n$ vertices and $G'$ a copy of $G$. For a bijective function $\pi : V(G) \mapsto V(G')$, we define the prism $\pi G$ of $G$ as follows: $V(\pi G) = V(G) \cup V(G')$ and $E(\pi G) = E(G) \cup E(G') \cup M_\pi$, where $M_\pi = \{u \pi(u) : u \in V(G)\}$. Let $\gamma(G)$ be the domination number of $G$. If $\gamma(\pi G) = \gamma(G)$ for any bijective function $\pi$, then $G$ is called a universal fixer. In [1] it is conjectured that the only universal fixer is the edgeless graph $K_n$.

In this note, we prove that any graph $G$ with $C_3$-free vertices is not a universal fixer.

Keywords: dominating sets, universal fixers.

Subject Classification: 05C69

1 Introduction

Let $G = (V, E)$ be an undirected graph. The neighborhood of a vertex $v \in V(G)$ in $G$ is the set $N_G(v)$ of all vertices adjacent to $v$ in $G$. For a set $X \subseteq V(G)$, the open neighborhood $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$, the closed neighborhood, $N_G[X] = N_G(X) \cup X$ and we denote by $[X]$ the subgraph of $G$ induced by the set of vertices $X$. For two sets of vertices $X, Y \subseteq V(G)$, we denote by $E(X, Y)$ the set of edges $xy \in E(G)$ such that $x \in X$ and $y \in Y$.

A set $D \subseteq V(G)$ is a dominating set of $G$ if $N_G[D] = V$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in $G$. A $\gamma$-set of $G$ is a dominating set of $G$ of cardinality $\gamma(G)$. If a set of vertices $A \subseteq V(G)$ is dominated by a set of vertices $D \subseteq V(G)$, that is, if for every vertex $v \in A$ there is $u \in D$ such that $vu \in E(G)$, we write $D \triangleright A$. A set $S \subset V(G)$ is a 2-packing of $G$ if $N_G[u] \cap N_G[v] = \emptyset$ for every distinct $u, v \in S$.

Definition 1 Let $G = (V, E)$ be a graph and $G'$ a copy of $G$. For a bijective function $\pi : V(G) \mapsto V(G')$, we define the prism $\pi G$ of $G$ as follows:

$$V(\pi G) = V(G) \cup V(G') \text{ and } E(\pi G) = E(G) \cup E(G') \cup M_\pi,$$

where $M_\pi = \{u \pi(u) : u \in V(G)\}$.

It is clear that every permutation $\pi$ of $V(G)$ defines a bijective function from $V(G)$ to $V(G')$, so we indistinctly use the permutation $\pi$ of $V(G)$ or the associated bijection $\pi : V(G) \mapsto V(G')$.
It is known [3] that for any permutation \( \pi \) and any graph \( G \), the domination numbers \( \gamma(G) \) and \( \gamma(\pi G) \) have the following relation: \( \gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G) \). The graph \( G \) is called a universal fixer if \( \gamma(\pi G) = \gamma(G) \) for every permutation \( \pi \) of \( V(G) \).

Universal fixers were studied in [3] for several classes of graphs and it was conjectured that the edgeless graph \( K_n \) is the only universal fixer. In [2], [4] and [5] it is shown that regular graphs, claw-free graphs and bipartite graphs are not universal fixers.

2 Useful lemmas

In what follows, we suppose that the graph \( G = (V,E) \) has \( n \) vertices. For \( x \in V(G) \), the copy of \( x \) in \( V(G') \) is denoted by \( x' \). Similarly, if \( A \subset V(G) \), we denote \( A' = \{u' \in V(G') : u \in A\} \).

**Definition 2** A \( \gamma \)-set \( A \) of \( G \) is called a separable \( \gamma \)-set (or an \( A_1-\gamma \)-set) if it can be partitioned into two nonempty subsets \( A_1 \) and \( A_2 \), such that \( A_1 \geq V - A \).

In [3], the following lemma is proved:

**Lemma 1** If \( A = A_1 \cup A_2 \) is an \( A_1-\gamma \)-set of \( G \), then:

1. \( A_2 \) is an independent set;
2. \( E(A_1, A_2) = \emptyset \);
3. \( A_2 \) is a 2-packing of \( G \).

For an \( A_1-\gamma \)-set \( A \) in \( G \) and a permutation \( \pi \), we denote \( B'_1 = \pi(A_1), B'_2 = \pi(A_2) \) and \( B' = \pi(A) \). It is clear that \( B' = B'_1 \cup B'_2 \). If the permutation \( \pi \) is the identity, then \( B'_1 = A'_1 \) and \( B'_2 = A'_2 \).

**Definition 3** We say that an \( A_1-\gamma \)-set \( A = A_1 \cup A_2 \) is effective under some permutation \( \pi \) if \( B' = \pi(A) \) is a \( B'_2-\gamma \)-set in \( G' \), where \( B'_2 = \pi(A_2) \).

**Observation 1** By Lemma 1, if an \( A_1-\gamma \)-set \( A \) is effective under some permutation \( \pi \), then \( B'_1 = \pi(A_1) \) is an independent set, \( E(B'_1, B'_2) = \emptyset \) and \( B'_1 \) is a 2-packing in \( G' \).

The next Theorem is proved in [3].

**Theorem 2** A graph \( G \neq K_n \) is a universal fixer if for every permutation \( \pi \) of \( V(G) \) there exists a separable \( \gamma \)-set which is effective under \( \pi \).

3 Main result

**Definition 4** Let \( G \) be a graph. We say that a non-isolated vertex \( x \) is \( C_3 \)-free if \( x \) belongs to no triangle of \( G \).

Observe that \( x \in V(G) \) is \( C_3 \)-free if and only if \( |V[G[x]| \cong K_{1,m} \) with \( m \geq 1 \).

Our main result is the following theorem.

**Theorem 3** If \( G \) has a \( C_3 \)-free vertex \( x \), then \( G \) is not a universal fixer.

**Proof.** Let \( G \) be a graph and \( x \) a \( C_3 \)-free vertex of \( G \). Denote by \( X \) the subgraph \( \langle V[G[x]| \cong K_{1,m} \) of \( G \) for \( m \geq 1 \). We define \( \pi : V(G) \to V(G') \) such that:
1. \( \pi(v) = v' \) for every \( v \in V(G) - X \),
2. \( \pi(v) \neq v' \) for every \( v \in X \) and
3. if \( m \geq 2 \) and \( \pi(v) = u' \), then \( \pi(u) \neq v' \) for every \( u, v \in X \).

Let \( A = A_1 \cup A_2 \) be an arbitrary separable \( \gamma \)-set of \( G \). We prove that \( A \) is not effective under \( \pi \).

Since \( X = \lbrack N_G \{x\} \rbrack \), we get that \( |A \cap X| \geq 1 \) for every dominating set \( A \). We consider the following three cases.

**Case 1.** \( A \cap X = \{v\} \).

(1) If \( v \in A_1 \), then \( \pi(A_2) = A_2' = B_2' \) and \( v' \in V(G') - B' \). By Lemma 1, we have that \( E(A_1, A_2) = \emptyset \). Thus \( E(\{v'\}, A_2') = \emptyset \) and it is a contradiction to the fact that \( B_2' = A_2' \not> V(G') - B' \).

(2) If \( v \in A_2 \) and \( \pi(v) = w' \), then the definition of \( A_1 \) implies that there exists \( u \in A_1 \) such that \( uw \in E(G) \). Hence \( u'w' \in E(G') \) and consequently, \( E(\pi(A_1), \pi(A_2)) \neq \emptyset \), which is a contradiction to Observation 1.

**Case 2.** \( A \cap X = \{u, v\} \). Recall that for every \( u, v \in X \) we have that either \( uv \in E(G) \), or \( uv \notin E(G) \) and \( N(u) \cap N(v) \neq \emptyset \).

(1) If \( u, v \in A_1 \), then \( |B_1' \cap X'| = 2 \). Thus \( B_1' \) is neither independent nor a 2-packing, a contradiction to Observation 1.

(2) If \( u, v \in A_2 \), then \( A_2 \) is neither independent nor a 2-packing, a contradiction to Lemma 1.

From (1) and (2), we conclude that

\[
|A_1 \cap X| \leq 1 \text{ and } |A_2 \cap X| \leq 1
\]  

(1)

for every \( A_1 \)-\( \gamma \)-set and therefore we only have to consider the case when \( u \in A_1 \) and \( v \in A_2 \). Since \( E(A_1, A_2) = \emptyset \), vertices \( u, v \) and \( x \) are all distinct. Analogously, since \( E(B_1', B_2') = \emptyset \), vertices \( \pi(u), \pi(v) \) and \( x' \) are all distinct too. By the definition of \( \pi \), if \( \pi(y) = z' \), then \( \pi(z) \neq y' \) for every \( y \in X \). Thus there exists \( z \in X \cap (V(G) - A) \) such that either \( \pi(u) = z' \neq v' \) or \( \pi(v) = z' \neq u' \).

1. Suppose \( \pi(u) = z' \neq u' \). Since \( A_1 \not> V - A \), there exists \( w \in A_1 \) such that \( zw \in E(G) \) and \( \pi(w) = w' \in B_1' \). Then \( z'w' \in E(B_1', B_1') \), which is a contradiction with Observation 1.

2. Suppose that \( \pi(v) = z' \neq u' \). Since \( A_1 \not> V - A \), there exists \( w \in A_1 \) such that \( zw \in E(G) \) and \( \pi(w) = w' \in B_1' \). Then \( z'w' \in E(B_1', B_2') \), which is a contradiction with Observation 1.

**Case 3.** \( |A \cap X| \geq 3 \). In this case, \( |A_1 \cap X| \geq 2 \) or \( |A_2 \cap X| \geq 2 \), which is impossible by (1) of Case 2.

From these cases, we conclude that every separable \( \gamma \)-set is not effective under \( \pi \). By Theorem 2, the graph \( G \) is not a universal fixer.

Recall that the girth of a graph \( G \) is the length of a shortest cycle contained in the graph. Notice that in a graph \( G \) with girth four or more, every vertex is \( C_3 \)-free. In particular, the bipartite graphs satisfy this condition.

**Acknowledgements**

We thank Bernardo Llano for useful comments. The authors thank the financial support received from Grant UNAM-PAPIIT IN-117812 and SEP-CONACyT.
References

[1] A. P. Burger, C.M. Mynhardt, W.D. Weakley, On the domination number of prisms of graphs, Discussiones Mathematicae Graph Theory 24, (2004), no. 2, 303-318.

[2] A.P. Burger, C.M. Mynhardt, Regular graphs are not universal fixers, Discrete Mathematics 310, (2010), no. 2, 364-368.

[3] C.M. Mynhardt, Z. Xu, Domination in Prisms of Graphs: Universal Fixers, Utilitas Mathematica 78, (2009), 185-201.

[4] E.J. Cockayne, R.G. Gibson, C.M. Mynhardt Claw-free graphs are not universal fixers, Discrete Mathematics 309, (2009), no. 1, 128-133.

[5] R.G. Gibson Bipartite graphs are not universal fixers, Discrete Mathematics 308, (2008), no. 24, 5937-5943.