Finite-time Landauer principle at strong coupling

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Landauer’s principle gives a fundamental limit to the thermodynamic cost of erasing information. Its saturation requires a reversible isothermal process, and hence infinite time. We develop a finite-time version of Landauer’s principle for a bit encoded in the occupation of a single fermionic mode, which can be strongly coupled to a reservoir. By solving the exact non-equilibrium dynamics, we optimize erasure processes (taking both the fermion’s energy and system-bath coupling as control parameters) in the slow driving regime through a geometric approach to thermodynamics. We find analytic expressions for the thermodynamic metric and geodesic equations, which can be solved numerically. Their solution yields optimal processes that allow us to characterize a finite-time correction to Landauer’s bound, fully taking into account non-markovian and strong coupling effects.

Any logical operation that is irreversible will incur a thermodynamic cost in the form of heat dissipated into the environment. Landauer’s principle quantifies this relation between information processing and thermodynamics with the bound $Q \geq k_B T \ln 2$ for the erasure of a single bit of information \cite{landauer56}. Here $Q$ is the dissipated heat, $k_B$ is the Boltzmann constant and $T$ is the absolute temperature at which the process is taking place. In recent years, this principle has been intensively studied within the fields of stochastic and quantum thermodynamics \cite{esposito08,mazza11,fazio12,gorini14,gallavotti15,allesandrini16,allen16,leigh17,moreno17,allahverdyan17,allahverdyan17b,allahverdyan17c}, and has been approached in several experimental platforms \cite{milburn00,duan01,peterson16,peterson16b,ivanov17,peterson17a,peterson17b,peterson17c}.

Landauer’s bound cannot be saturated with finite resources, namely time and energy - an impossibility that is intimately related to the second and third law of thermodynamics. In finite time, using tools from optimal transport theory \cite{marzocchi18,marzocchi18b,peterson19} and thermodynamic geometry \cite{alleiner08,alleiner09,alleiner11,alleiner15}, optimal erasure protocols have been derived both for classical systems described by overdamped Langevin dynamics \cite{alleiner12,alleiner13} and open quantum systems described by Lindblad master equations \cite{allen15,peterson16a,peterson16b,peterson17a,peterson17c,peterson18}. Such optimal protocols naturally lead to a finite-time correction to Landauer’s bound in different physical set-ups, which has given rise to the term \textit{finite-time Landauer principle} \cite{alleiner11,alleiner15,alleiner17}. For a slowly driven (quantum) two-level system weakly coupled to a thermal bath, the finite-time bound takes the simple form \cite{alleiner17}

\begin{equation}
Q \geq k_B T \left( \ln 2 + \frac{\pi^2}{4 \Gamma \tau} \right) + O \left( \frac{1}{\Gamma^2 \tau^2} \right),
\end{equation}

where $\tau$ is the total time of the process and $\Gamma$ is the thermalisation rate. As expected from the second law of thermodynamics, the finite-time correction is positive and when $\Gamma \tau \to \infty$ we recover the standard bound. We also note that the optimal protocol saturating the finite bound eq. (1) has been recently implemented in a semiconductor quantum dot \cite{alleiner17}.

Despite this remarkable progress, previous works on the \textit{finite-time Landauer principle} have focused in Markovian systems which, for quantum systems, can be guaranteed by a sufficiently weak interaction between system and bath. In the presence of strong coupling \cite{modi07,modi08,modi10,modi11,bennett11,banuls11,banuls12}, we expect both new opportunities arising due to faster relaxation rates and non-Markovian dynamics \cite{allen96,allen99,allen00,allen01,allen02,allen03,allen04,allen05,allen06,allen07,allen08,allen09,allen10,allen11,allen12,allen13,allen14}, as well as challenges due to the presence of new sources of irreversibility \cite{alicki90,alicki91,alicki92,alicki93,alicki94}. The goal of this work is to take a first step into this exciting regime by deriving a finite-time correction to Landauer’s principle for a single fermion that can interact strongly with a reservoir, as described by the resonant-level model \cite{timm97,timm98,timm99}. In order to generalise eq. (1) to the strong coupling regime, we exploit the framework of thermodynamic geometry \cite{alleiner08,alleiner09,alleiner11,alleiner15,alleiner17}, which has proven extremely successful to devise minimally dissipative processes both in classical \cite{alleiner12,alleiner13} and quantum systems \cite{alleiner14,alleiner15,alleiner16,alleiner17,alleiner18,alleiner19,alleiner20,alleiner21,alleiner22}.

Framework. We consider a driven system $S$ that can be put in contact with a thermal bath $B$, so that the total time-dependent Hamiltonian reads:

\begin{equation}
H(t) = H_S(t) + H_B.
\end{equation}

Here, $H_S(t)$, $H_B$ are the externally controllable Hamiltonian of $S$ and $B$, respectively, whereas $H_B$ is the Hamiltonian of $B$. The state $\rho(t)$ of SB evolves as $\rho(t) = U(t) \rho(0) U^\dagger(t)$ with $U(t) = T \exp \left( -\frac{i}{\hbar} \int_0^t ds H(s) \right)$. The work cost induced by driving $H(t)$, with $t \in [0, \tau]$, reads:

\begin{equation}
W = \int_0^\tau ds \text{Tr} \left[ \rho(s) \dot{H}(s) \right] = \text{Tr} \left[ H(\tau) \rho(\tau) - H(0) \rho(0) \right].
\end{equation}

Focusing on protocols where $H_B(0) = H_B(\tau) = 0$, we can naturally identify from the first law of thermodynamics $W = Q + \Delta E_S$ with $\Delta E_S = \text{Tr} \left[ H_S(\tau) \rho(\tau) - H_S(0) \rho(0) \right]$, the dissipated heat

\begin{equation}
Q = \text{Tr} \left[ H_B(\rho(\tau) - \rho(0)) \right]
\end{equation}

as the total energy absorbed by the bath \cite{binder01}.

Assuming that the initial state of SB is a thermal state:

$\rho(0) = e^{-\beta H(0)}/Z(0)$ with $Z(t) \equiv \text{Tr} \left[ e^{-\beta H(t)} \right]$, eq. (3) can be re-expressed as \cite{alleiner17}:

\begin{equation}
W = \Delta F + k_B T \Sigma,
\end{equation}

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where $\Delta F = k_B T \ln[Z(0)/Z(\tau)]$ is the change of equilibrium free energy of SB, and the entropy production $\Sigma$ can be expressed as: $\Sigma = S\left(\rho(\tau)|\rho(0)\right)$. The entropy production $\Sigma \geq 0$ accounts for the irreversible energetic contribution in finite-time processes, and depends on the production $\Sigma\geq 0$. Let us expand the Hamiltonian in what follows through the framework of quantum thermodynamic geometry \cite{39}. Thermodynamic geometry in strongly coupled systems. Let us expand the Hamiltonian $H(t)$ in eq. (2) as $H(t) = \sum_{ij} \lambda_{ij}(t)X_j$ where $\{\lambda_{ij}(t)\}$ are the externally controllable parameters and $\{X_j\}$ are the corresponding observables. In order to apply the geometric approach, we need to impose more structure on the possible evolutions $U(t)$ generated by eq. (2). We require two basic ingredients:

**Requirement 1: Thermalization.** In absence of driving, the conjugated observables $X_j$ thermalise. More precisely, for a frozen Hamiltonian $H(t)$, we have

$$\lim_{s \to \infty} \text{Tr}[\hat{U}_i(s)\rho(0)\hat{U}_j(s)X_j] = (X_j(t))_{eq},$$

where $\hat{U}_i(s) = e^{-iH(t)s}$, $(X_j(t))_{eq} = \text{Tr}[\omega_j(t)X_j]$, and $\beta$ is implicitly defined by the initial energy of the total system. In the context considered here, namely purely unitary dynamics of SB, this condition is satisfied both by non-integrable systems satisfying the ETH hypothesis \cite{94, 95} and also for integrable systems typically appearing in open quantum systems \cite{96–99}.

**Requirement 2: Slow driving,** so that the system remains close to the instantaneous equilibrium state while being driven. This enables us to keep only leading terms when making a linear-response expansion in the driving speed, which can be expressed as:

$$(X_j(t)) = (X_j(t))_{eq} + \sum_i m_{ij}\dot{\lambda}_i(t) + \ldots$$

The coefficients $m_{ij}$, which depend on the point $\{\lambda_i(t)\}$, can in principle be derived from the exact equations of motion.

Combining the expansion of eq. (7) with eq. (3) and eq. (5), we obtain:

$$k_B T \Sigma = \sum_{ij} \int_0^\tau dt \dot{\lambda}_i(t)m_{ij}(t)\dot{\lambda}_j(t)$$

at leading order in the inverse of the driving speed. Because of the second law of thermodynamics, it follows that $m_{ij}$ can be expressed as a metric, i.e., a symmetric, positive-definite $m \geq 0$ operator that depends smoothly on the point $\{\lambda_i(t)\}$. We can associate a length to a protocol by defining $L = \int_0^\tau dt \sum_{ij} \lambda_i(t)m_{ij}(t)\dot{\lambda}_j(t)$. It is related to the entropy production via a Cauchy-Schwarz inequality:

$$k_B T \Sigma \geq \frac{1}{\tau} L^2,$$

where equality is satisfied by protocols with constant entropy production rate. Furthermore, to minimize the entropy production of any (slow) protocol we have to find the shortest path between the desired initial and final value of the Hamiltonian’s parameters. This corresponds to a geodesic path, with length $\mathcal{L}$, which naturally defines a minimal entropy production

$$k_B T \Sigma_{\text{min}} = \frac{1}{\mathcal{L}} \mathcal{L}^2.$$

We can find $\Sigma_{\text{min}}$ by solving the geodesic equation that is derived from the metric and computing its length \cite{34, 39}.

**Thermodynamic geometry of the resonant-level model.** Having explained the general ideas behind our work, we now focus on finite-time driving processes of a single fermionic mode coupled to a fermionic bath, which can e.g. describe a single-electron quantum dot. The total Hamiltonian reads:

$$H(t) = \epsilon(t)\hat{a}^\dagger \hat{a} + \sum_{k=1}^n \omega_k \hat{b}_k^\dagger \hat{b}_k + g(t) \sum_{k=1}^n \lambda_k \hat{a}^\dagger \hat{b}_k + \lambda_k^* \hat{b}_k^\dagger \hat{a}.$$
where \( f_{\beta}(\omega) = (1+e^{\beta \omega})^{-1} \) is the Fermi-Dirac distribution and we defined the propagator

\[
G(t,s) = \exp \left[ -\frac{1}{\hbar} \int_s^t \, d\tau \, \mu(\tau) + i\varepsilon(\tau) \right],
\]

with \( \mu(t) := \frac{1}{2} g(t)^2 \). From these expressions we can exactly compute the thermodynamic work eq. (3), which reads:

\[
W = \int_0^T \, dt \, \dot{\varepsilon}(t)p(t) + \dot{\mu}(t)v(t)/g(t).
\]

From the exact solutions for \( p(t) \) and \( v(t) \), in App. A we show that Requirement 1 is satisfied, and hence \( W = \Delta F \) in the quasistatic limit. For slow but finite-time processes, we perform a slow driving expansion of eq. (13) and eq. (14) (details in App. B) using that the thermalization rate of the system is \( \Gamma := \frac{2}{\hbar} \int_0^T \, dt \, \mu(t), \) so that the expansion can be performed in orders of \( 1/(\tau \Gamma) \). We then obtain an expansion for \( W \) analogous to eq. (5) where the entropy production \( \Sigma \) is described by eq. (8) with \( \lambda(t) = (\varepsilon(t), \mu(t)) \) and the thermodynamic metric

\[
m(t) = \frac{\hbar}{\pi} \int_{-\infty}^{\infty} d\omega \, f_{\beta}^{(\varepsilon)}(\omega)m_{\omega}(\varepsilon(t) - \omega, \mu(t)) ,
\]

where

\[
m_{\omega}(\varepsilon, \mu) = \frac{1}{(\mu^2 + \varepsilon^2)^2} \begin{pmatrix} 4\varepsilon \mu^2 & -\mu(\mu^2 - 3\varepsilon^2) \\ -\mu(\mu^2 - 3\varepsilon^2) & 2\varepsilon(\varepsilon^2 - \mu^2) \end{pmatrix}.
\]

This metric gives a geometrical description of slow thermodynamic protocols performed on the system. By solving the geodesic equations [100], we can find the geodesic length \( L \) and hence the minimal entropy production eq. (10). Before attempting to solve them, we now study the high and low temperature limits to gain further analytical insights.

High temperature limit \((\beta \varepsilon, \beta \mu \ll 1)\). Since the terms of eq. (17) quickly decay at high frequencies we can perform the high temperature expansion \( f_{\beta}^{(\varepsilon)}(\omega) = 1 - \frac{1}{2} \beta \omega + O(\beta^3 \omega^3) \) directly in the metric. At leading order, we find:

\[
m_{HT} = \frac{\hbar \beta}{8\mu} ,
\]

which coincides with the metric of an angle distance in the \((\varepsilon, \mu)\) space - hence the metric is singular. If we reparameterize \((\varepsilon, \mu)\) as \((r \cos \phi, r \sin \phi)\) we find \( k_B T \Sigma_{min} = \frac{\hbar}{2\pi} (\Delta \phi)^2 \). Therefore any protocol that keeps \( \phi(t) \) constant is a geodesic, leading to the minimal entropy production:

\[
k_B T \Sigma_{min} \bigg|_{T=0} = \frac{\hbar}{\pi \tau} (\Delta \phi)^2 ,
\]

with \( \phi = \arctan(\mu/\varepsilon) \). Note that there are multiple (infinitely many) geodesics for any pair of boundary points. This fact prevents us from continuing the expansion to further orders in temperature. Nevertheless, this limit provides analytical insights on optimal protocols with \( \beta \varepsilon \) or \( \beta \mu \gg 1 \). In particular, we note that there is no need for a diverging coupling even when \( \varepsilon(\tau) \rightarrow \infty \) as, once \( \mu \) has become large, eq. (24) shows that it is optimal to reduce the coupling while increasing the energy.

Landauer erasure. In what follows, we focus on minimizing \( \Sigma \) in an erasure process, which imposes specific boundary conditions to the geodesic equations. We assume that we have no prior knowledge of the system, therefore its initial state is \( \rho_S(0) = |1\rangle/2 \). This translates in taking \( \varepsilon(0) = 0 \) so that it coincides with the thermal state of \( H_S \). For the qubit to be erased we want its final state to be \( \rho_S(\tau) \approx |0\rangle/|0\rangle \) (i.e. \( p(\tau) \approx 0 \)). Since the driving is done slowly, \( p(t) \) is always close to its thermal expectation value. Therefore by choosing \( \beta \varepsilon(\tau) \rightarrow \infty \) we ensure \( p(\tau) \approx 0 \). For the coupling, the boundary conditions are \( \mu(0) = \mu(\tau) = 0 \), because we want to think of this as an “eraser machine” that the qubit is “brought to” at the start and “retrieved from” at the end. Given this family of protocols, we recognise from eq. (5) that \( W = k_B T \ln(2 + \Sigma) \), and similarly \( Q = k_B T \ln(2 + \Sigma) \), thus justifying the minimisation of \( \Sigma \) as given in eq. (8). After the qubit has been decoupled (i.e. \( t > \tau \)), we

\[1\] Strictly speaking, in order to ensure consistency with the slow driving limit, \( \beta \varepsilon(\tau) \) has to remain finite (so that the speed \( \lambda \) remains finite). However, the final population \( p(\tau) \) is exponentially small with \( \beta \varepsilon(\tau) \), leading to exponentially small corrections. Our results are valid up to such corrections, and for sufficiently large \( \tau \) to ensure the validity of the approximation.
The entropy production of the optimal erasure protocol as a function of the final energy, compared to the high temperature regime cost eq. (22). (right) Comparison of the entropy production for a geodesic protocol in which one parameter is varied at a time (with \( \mu \) being increased until \( \mu^* \)) and the weak coupling approximation eq. (1); the minimal possible entropy production, \( \tau \Sigma_{\text{min}}/\beta = 2.57946 \pm 1 \cdot 10^{-5} [\hbar] \), obtained when both parameters are changed simultaneously is also shown.

Interestingly, we now argue that the form of the correction eq. (25) is in fact general of any Landauer erasure protocol with control on S and the SB coupling. Indeed, first note that the geodesic length \( L \) is dimensionless and can only depend on \( \beta \) and the boundary conditions as we optimize over \( \mu \) and \( \varepsilon \). In an erasure process, the boundary conditions read: \( \varepsilon(0) = 0, \varepsilon(\tau) \to \infty \), and \( \mu_i(0) = \mu_i(\tau) = 0 \) where \( i \) runs over all the possible control parameters on SB. But this implies that \( L \) is independent of them and hence of \( \beta \). Therefore \( k_B T \Sigma_{\text{min}} \) will take the form of a constant, independent of any parameter of the system and bath, divided by \( \tau \). This is a crucial difference from eq. (1).

This simple argument based on dimensional analysis thus shows that eq. (25) is rather general, with the value of \( a \) depending on the specific implementation (e.g. the ohmicity of the bath). It is important to highlight that the bound eq. (25) implies that, even when having access to arbitrary strong SB interactions (naively taking \( \Gamma \to \infty \) in eq. (1)), infinite time is still required for perfect erasure due to the quantum-mechanical correction derived here.

Finally, we analyze a scenario where the coupling is kept constant while \( \varepsilon(t) \) is driven, which is motivated both by experimental set-ups and for a comparison with the weakly interacting case. Therefore, we restrict to one-parameter protocols consisting of the three following steps: 1. while keeping \( \varepsilon \) at 0 we turn on the coupling to some value \( \mu_i \); 2. while keeping the coupling fixed we bring \( \varepsilon \) from 0 to some value \( \varepsilon_* \gg k_B T \); 3. while keeping \( \varepsilon \) constant we turn off the coupling. Each step contributes positively to the entropy production, and its minimisation is discussed in App. E. In Fig. 1, we show \( \Sigma_{\text{min}} \) for different values of \( \mu_* \), ranging from the weak to the super-strong coupling regime. It can be appreciated how eq. (1) breaks down, and also how such one-parameter protocols become close to the fundamental limit eq. (25) for \( \beta \mu_* > 1 \).

Conclusions and outlook. In this work, we have developed optimal finite-time protocols for Landauer erasure of a single bit encoded in the occupation of a single
fermionic mode, which can be strongly coupled to a reservoir. This has been achieved by extending a geometric approach to quantum thermodynamics to strongly coupled open systems - to be contrasted with recent claims of failure of this approach in quantum systems [102]. Two main insights arise from our work: First, optimal erasure processes require the presence of strong coupling, thus motivating the study of quantum thermodynamics beyond the standard Markovian approximation. Second, despite strong coupling enabling faster erasure, there is a fundamental quantum correction that prevails for arbitrary strong coupling, see eq. (25) - to be compared with eq. (1) derived in the weak coupling regime. This correction is independent of the properties of the bath (temperature, relaxation rates) and arises due to the quantum-mechanical evolution underlying the dissipative process. This work opens exciting directions for the future. On the one hand, the level of experimental control required to implement such protocols is in principle possible in quantum dots [103–106], where the energy-level ε(t) and coupling g(t) can be independently controlled, even by several orders of magnitude [107]. On the other hand, a simple dimensional argument shows that the form eq. (25) is general of erasure processes with control on both S and the SB coupling. It would definitely be interesting to characterise the dependence of a in the nature of the bath and the SB coupling (e.g. its spectral density), and more generally to derive similar quantum-mechanical finite-time corrections that are independent of the specific implementation.

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Supplemental Material for “Finite-time Landauer principle at strong coupling”

Appendix A: Solving the exact dynamics

1. Solving the Heisenberg equations

We consider a single fermionic mode coupled to a fermionic bath. Without loss of generality we can set the chemical potential to 0 and the ground state of the two-level system to 0. The Hamiltonians of the system and bath are

\[ \hat{H}_{S}(t) = \varepsilon(t)\hat{a}^{\dagger}\hat{a}, \]
\[ \hat{H}_{B} = \sum_{k=1}^{n} \omega_{k}\hat{b}_{k}^{\dagger}\hat{b}_{k}, \]

where \( \hat{a}^{\dagger} \) is the creation operator of the two-level system and \( \hat{b}_{k}^{\dagger} \) is the creation operator of a bath mode with frequency \( \omega_{k} \). These ladder operators follow the canonical anticommutation relations. For the interaction between system and bath, the Hamiltonian is

\[ \hat{H}_{\text{int}}(t) = g(t)\hat{V} = g(t)\sum_{k=1}^{n} \lambda_{k}\hat{a}^{\dagger}\hat{b}_{k} + \lambda_{k}^{*}\hat{a}\hat{b}_{k}^{\dagger}, \]

where the \( \lambda_{k} \) are the interaction weights.

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\[ \hat{H}_{S}(t) = \varepsilon(t)\hat{a}^{\dagger}\hat{a}, \]
\[ \hat{H}_{B} = \sum_{k=1}^{n} \omega_{k}\hat{b}_{k}^{\dagger}\hat{b}_{k}, \]

where \( \hat{a}^{\dagger} \) is the creation operator of the two-level system and \( \hat{b}_{k}^{\dagger} \) is the creation operator of a bath mode with frequency \( \omega_{k} \). These ladder operators follow the canonical anticommutation relations. For the interaction between system and bath, the Hamiltonian is

\[ \hat{H}_{\text{int}}(t) = g(t)\hat{V} = g(t)\sum_{k=1}^{n} \lambda_{k}\hat{a}^{\dagger}\hat{b}_{k} + \lambda_{k}^{*}\hat{a}\hat{b}_{k}^{\dagger}, \]

where the \( \lambda_{k} \) are the interaction weights.

Will will consider that \( \varepsilon(t) \) and \( g(t) \) are the control parameters to then perform the erasure of information in the single mode of the system. We now proceed to solve the dynamics of the system and bath in the Heisenberg picture. For an operator \( \hat{A} \) in the Schrödinger picture, we denote by \( \hat{A}_{H}(t) \) the corresponding operator in the Heisenberg picture. The evolution of \( \hat{A}_{H}(t) \) is defined by the Heisenberg equation of motion:

\[ \frac{d}{dt}\hat{A}_{H}(t) = i\left[\hat{H}(t),\hat{A}_{H}(t)\right], \]

where \( \hat{H} = \hat{H}_{S}(t) + \hat{H}_{\text{int}}(t) + \hat{H}_{B} \) and \( \hbar = 1 \). Applying this equation to the ladder operators \( \hat{a}_{H}(t) \) and \( \hat{b}_{k,H}(t) \) we find the following system of \( n + 1 \) equations:

\[ \frac{d}{dt}\hat{a}_{H}(t) = -i\varepsilon(t)\hat{a}_{H}(t) - ig(t)\sum_{k} \lambda_{k}\hat{b}_{k,H}(t), \]
\[ \frac{d}{dt}\hat{b}_{k,H}(t) = -i\omega_{k}\hat{b}_{k,H}(t) - ig(t)\lambda_{k}^{*}\hat{a}_{H}(t). \]

By defining \( \tilde{u}_{k}(t) = e^{i\omega_{k}t}\hat{b}_{k,H}(t) \) we can see that eq. (A6) becomes

\[ e^{-i\omega_{k}t}\frac{d}{dt}\tilde{u}_{k}(t) = -ig(t)\lambda_{k}^{*}\hat{a}_{H}(t), \]

which is solved by \( \tilde{u}_{k}(t) = \tilde{u}_{k}(0) - i\lambda_{k}^{*}\int_{0}^{t} ds e^{i\omega_{k}s}g(s)\hat{a}_{H}(s) \). We therefore find

\[ \hat{b}_{k,H}(t) = e^{-i\omega_{k}t}\hat{b}_{k} - i\lambda_{k}^{*}\int_{0}^{t} ds g(s)\hat{a}_{H}(s)e^{i\omega_{k}(s-t)}, \]

where we used that \( \tilde{u}_{k}(0) = \hat{b}_{k,H}(0) = \hat{b}_{k} \). Therefore

\[ \sum_{k} \lambda_{k}\hat{b}_{k,H}(t) = \hat{\xi}(t) - i\int_{0}^{t} ds \chi(s-t)g(s)\hat{a}_{H}(s), \]

where we defined the noise operator \( \hat{\xi}(t) = \sum_{k} e^{-i\omega_{k}t}\lambda_{k}\hat{b}_{k} \) and \( \chi(t) = \sum_{k} e^{i\omega_{k}t}|\lambda_{k}|^{2} \). We can notice that \( \chi(t) \) is the (symmetrized) noise correlation function:

\[ \left\langle \left\{ \hat{\xi}^{\dagger}(t),\hat{\xi}(0) \right\} \right\rangle = \sum_{j,k} e^{i\omega_{j}t}\lambda_{j}\lambda_{j}^{*}\left\langle \left\{ \hat{b}_{j}^{\dagger},\hat{b}_{j} \right\} \right\rangle = \sum_{k} e^{i\omega_{k}t}|\lambda_{k}|^{2} = \chi(t), \]
and its Fourier transform is the (unit-less) spectral density of the bath \( \mathcal{J}(\omega) = 2\pi \sum_k |\lambda_k|^2 \delta(\omega - \omega_k) \) By inserting eq. (A8) into equation eq. (A5) we find an equation of motion for \( \hat{a}_H(t) \):

\[
\frac{d}{dt} \hat{a}_H(t) = -i\varepsilon(t)\hat{a}_H(t) - ig(t)\hat{\xi}(t) - g(t) \int_0^t ds \chi(s-t)g(s)\hat{a}_H(s) .
\] (A9)

In order to solve eq. (A9) we will need to explicitly take the continuum limit so that our bath indeed becomes a bath. We can take its spectral density to be either a Lorentzian \( \mathcal{J}(\omega) = \frac{\Lambda^2}{\omega^2 + \frac{\Lambda^2}{4}} \) or a pass-band \( \mathcal{J}(\omega) = \Theta(\Lambda - |\omega|) \) (\( \Theta \) is the Heaviside step function). We will need to assume that we are working in the wide-band approximation \( \Lambda \to \infty \). More practically, we are assuming that the bath interaction is the same over the energies we are spanning with the system. This limit allows us to say that the noise correlation function is negligible for time differences larger than zero:

\[
\lim_{\Lambda \to \infty} \lim_{n \to \infty} \chi(t) = \delta(t) .
\]

In this limit eq. (A9) becomes considerably simpler:

\[
\frac{d}{dt} \hat{a}_H(t) = - \left( i\varepsilon(t) + \frac{1}{2} g(t)^2 \right) \hat{a}_H(t) - ig(t)\hat{\xi}(t) .
\] (A10)

Similarly to how we solved eq. (A6), we define \( \hat{u}(t) = \exp \left[ \int_0^t z(s)ds \right] \hat{a}_H(t) \) for \( z(t) := i\varepsilon(t) + \frac{1}{2} g(t)^2 \). Now we have

\[
e^{-\int_0^t z(s)ds} \frac{d}{dt} \hat{u}(t) = -ig(t)\hat{\xi}(t) ,
\]

which is solved by \( \hat{u}(t) = \hat{u}(0) - i \int_0^t ds g(s) \exp \left[ \int_0^s z(r)dr \right] \hat{\xi}(s) \). We therefore find the solution of the evolution of the ladder operator of the distinguished mode:

\[
\hat{a}_H(t) = G(t,0)\hat{a} - i \int_0^t ds g(s)G(t,s)\hat{\xi}(s)
\] (A11)

where we defined the propagator \( G(t,s) = \exp \left[ -\int_s^t z(r)dr \right] \). And from eq. (A7) we find the solution for the bath modes:

\[
\hat{b}_{k,H}(t) = e^{-i\omega_k t}\hat{b}_k - i\lambda_k^* \int_0^t ds g(s)G(s,0)\hat{a}e^{i\omega_k(s-t)} - \lambda_k^* \int_0^t ds \int_0^s dr g(s)g(r)G(s,r)\hat{\xi}(r)e^{i\omega_k(s-t)} .
\] (A12)

2. Relevant observables

Since we are performing an erasure, we will assume the system starts in a factorized state and that the bath starts in a thermal state at inverse temperature \( \beta \) state with respect to its Hamiltonian:

\[
\hat{\rho}(0) = \hat{\rho}_S(0) \otimes \frac{e^{-\beta H_B}}{Z_B} , \quad Z_B = \text{Tr}[e^{-\beta H_B}] .
\]

We are interested in computing the occupation probability of the excited level \( p(t) = \langle \hat{a}^\dagger \hat{a} \rangle \) and the system-bath interaction potential \( v(t) = \langle \hat{V} \rangle \). From eq. (A11) we have

\[
p(t) = \text{Tr} \left[ \hat{\rho}(0)\hat{a}_H^\dagger(t)\hat{a}_H(t) \right] = |G(t,0)|^2 p(0) + \int_0^t ds d\rho g(s)g^*(t,s)G(t,s)G(t,s)\text{Tr} \left[ \frac{e^{-\beta H_B}}{Z_B} \hat{\xi}^\dagger(s)\hat{\xi}(r) \right] ,
\] (A13)

where we used the CAR to get \( \text{Tr}[\hat{a}^\dagger \hat{\xi}(s)] = 0 \) and drop the cross terms. Further using the CAR we simplify the remaining trace in the integral:

\[
\text{Tr} \left[ \frac{e^{-\beta H_B}}{Z_B} \hat{\xi}^\dagger(s)\hat{\xi}(r) \right] = \sum_k e^{i\omega_k(s-r)}|\lambda_k|^2 \text{Tr} \left[ \frac{e^{-\beta H_B}}{Z_B} \hat{b}_k^\dagger \hat{b}_k \right] = \sum_k e^{i\omega_k(s-r)}|\lambda_k|^2 f_{\beta}(\omega_k) ,
\] (A14)
where \( f_\beta(\omega) = (1 + e^{\beta \omega})^{-1} \) is the Fermi-Dirac distribution. We can apply the continuum limit to eq. (A14) by using the equality \( 2\pi \sum_k |\lambda_k|^2 h(\omega_k) = \int d\omega \mathfrak{J}(\omega) h(\omega) \), which holds for any function \( h \) by definition of \( \mathfrak{J} \). We can then apply the wideband limit by using \( \lim_{\Lambda \to \infty} \lim_{n \to \infty} \mathfrak{J}(\omega) = 1 \). We find

\[
\text{Tr} \left[ \frac{e^{-\beta H_B}}{Z_B} \hat{\xi}(s) \hat{\xi}(r) \right] = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(s-r)} \mathfrak{J}(\omega) f_\beta(\omega) \xrightarrow{\Lambda \to \infty} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(s-r)} f_\beta(\omega). \tag{A15}
\]

Applying eq. (A15) to eq. (A13) we get

\[
p(t) = |G(t,0)|^2 p(0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ f_\beta(\omega) \int_0^t ds \int_0^s dr \ g(s) g(r) G^*(t,s) G(t,r) e^{i\omega(s-r)}. \tag{A16}
\]

Using eq. (A8), the CAR and eq. (A11), we have

\[
v(t) = \text{Tr} \left[ \hat{\rho}(0) \hat{a}_H(t) \left( \sum_k \lambda_k \hat{b}_H(t) \right) \right] + \text{h.c.} \\
= \text{Tr} \left[ \hat{\rho}(0) \hat{a}_H(t) \hat{\xi}(t) \right] - i \int_0^t ds \ c(s-t) g(s) \text{Tr} \left[ \hat{\rho}(0) \hat{a}_H(t) \hat{a}_H(s) \right] + \text{h.c.} \xrightarrow{\Lambda \to \infty} \text{Tr} \left[ \hat{\rho}(0) \hat{a}_H(t) \hat{\xi}(t) \right] - i \int_0^t ds \ c(s-t) g(s) \text{Tr} \left[ \hat{\rho}(0) \hat{a}_H(t) \hat{a}_H(s) \right] + \text{h.c.},
\]

\[
= \text{Tr} \left[ \hat{\rho}(0) \hat{a}_H(t) \hat{\xi}(t) \right] + \text{h.c.},
\]

\[
= i \int_0^t ds \ g(s) G^*(t,s) \text{Tr} \left[ e^{-\beta H_B} Z_B \hat{\xi}(s) \hat{\xi}(t) \right] + \text{h.c.},
\]

\[
= \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \ f_\beta(\omega) \int_0^t ds \ g(s) G^*(t,s) e^{i\omega(s-t)} + \text{h.c.}.
\]

So we have

\[
v(t) = \frac{1}{\pi} \imath \int_{-\infty}^{\infty} d\omega \ f_\beta(\omega) \int_0^t ds \ g(s) G(t,s) e^{i\omega(t-s)} . \tag{A17}
\]

### 3. Proof of requirement 1

We will now proceed to prove that, in absence of driving, \( p(t) \) and \( v(t) \) thermalize. We do so in two steps, we first simplify the expressions of eq. (A16) and eq. (A17) for \( \varepsilon(t) = \varepsilon \) and \( g(t) = g \) and compute the infinite time limit. Then we compute the thermal expectation value of the corresponding observables and prove that the obtained expressions are the same.

#### a. Infinite time limit in absence of driving

By assuming that the driving parameters are kept constant the propagator becomes

\[
G(t,s) = e^{-(t-s)}(\frac{i}{2}g^2 + i\varepsilon) . \tag{A18}
\]

This allows us to compute the time integrals in eq. (A16) and eq. (A17):

\[
p(t) = p(0) e^{-g^2 t} + \frac{g^2}{2\pi} \int_{-\infty}^{\infty} d\omega \ f_\beta(\omega) \frac{1 - 2e^{-g^2 t/2} \cos((\omega - \varepsilon)t) + e^{-g^2 t}}{g^4/4 + (\omega - \varepsilon)^2} , \tag{A19}
\]

\[
v(t) = \frac{g}{\pi} \int_{-\infty}^{\infty} d\omega \ f_\beta(\omega) \frac{(\omega - \varepsilon) \left[ 1 - e^{-g^2 t/2} \cos((\omega - \varepsilon)t) \right] - \frac{1}{2} g^2 e^{-g^2 t/2} \sin((\omega - \varepsilon)t)}{g^4/4 + (\omega - \varepsilon)^2} . \tag{A20}
\]
As a side-note, it is interesting to note that the frequency integral of eq. (A19) can be solved to give the following expression for the occupation probability

\[
p(t) = 1 + p(0)e^{-g^2t} + \frac{\sinh(\beta \varepsilon)}{\cosh(\beta \varepsilon) + \cos(\frac{\beta g^2}{2})} + \left(1 + e^{-g^2t}\right) \left[\frac{1}{2} - \frac{1}{\pi} \Im \psi^{(0)} \left(\frac{1}{2} - \frac{\beta}{2\pi} \left(\frac{g^2}{2} + i\varepsilon\right)\right)\right] + \frac{e^{-g^2t}}{\pi} 3B \left(\frac{e^{2\pi i/\beta}; 1 + \frac{\beta}{2\pi} \left(\frac{g^2}{2} - i\varepsilon\right); 0\right) ,
\]

(A21)

where \(\psi^{(0)}(z) = \frac{d}{dz} \ln \Gamma(z)\) is the digamma function (defined as the logarithmic derivative of the Gamma function) and \(B(x; a, b) = \int_0^x ds \ s^{a-1}(1 - s)^{b-1}\) is the incomplete beta function. This expression is useful for numerical implementations as it is faster to compute than the integral of eq. (A19).

By taking the limit \(t \to \infty\) in eq. (A19) and eq. (A20) we find

\[
\lim_{t \to \infty} p(t) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} f_\beta(\omega) \frac{g^2/2}{g^4/4 + (\omega - \varepsilon)^2} ,
\]

(A22)

\[
\lim_{t \to \infty} v(t) = g \int_{-\infty}^{\infty} \frac{d\omega}{\pi} f_\beta(\omega) \frac{\omega - \varepsilon}{g^4/4 + (\omega - \varepsilon)^2} .
\]

(A23)

Here we can notice that if we take the Laplace transform of the propagator we obtain

\[
\tilde{G}(z) := \int_0^{\infty} dt \ G(t, 0)e^{-zt} = \frac{1}{z + i\varepsilon + g^2/2} ,
\]

(A24)

which allows us to rewrite eq. (A22) and eq. (A23) as

\[
\lim_{t \to \infty} p(t) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} f_\beta(\omega) \Re \left[\tilde{G}(-i\omega)\right] ,
\]

(A25)

\[
\lim_{t \to \infty} v(t) = g \int_{-\infty}^{\infty} \frac{d\omega}{\pi} f_\beta(\omega) \Im \left[\tilde{G}(-i\omega)\right] .
\]

(A26)

b. Thermal expectation value

We now compute the expectation value of \(\hat{a}^\dagger \hat{a}\) and \(\hat{V}\) when the state is a Gibbs state

\[
\hat{\omega}_\beta := \frac{e^{-\beta \hat{H}}}{Z} = \frac{\exp\left[\beta \varepsilon \hat{a}^\dagger \hat{a} - \beta g \hat{V} - \beta \hat{H}_B\right]}{Z} , \quad Z = \text{Tr}[e^{-\beta \hat{H}}].
\]

Therefore we want to find \(p_{th} := \text{Tr}[\hat{\omega}_\beta \hat{a}^\dagger \hat{a}]\) and \(v_{th} := \text{Tr}[\hat{\omega}_\beta \hat{V}]\). Using the fact that the total Hamiltonian is quadratic, we can diagonalize it to rewrite it in the following way

\[
\hat{H} = \sum_k \varepsilon_k \hat{c}_k^\dagger \hat{c}_k , \quad (A27)
\]

where \(\varepsilon_k\) are eigen-energies and \(\hat{c}_k\) are fermionic ladder operators that follow the CAR: \(\{\hat{c}_j^\dagger, \hat{c}_k\} = \delta_{jk} 1\), \(\{\hat{c}_j, \hat{c}_k\} = 0\). They are related to the original ones in the following way

\[
\hat{a} = \sum_k \langle 0|\hat{a}|k\rangle \hat{c}_k , \quad (A28)
\]

\[
\hat{b}_j = \sum_k \langle 0|\hat{b}_j|k\rangle \hat{c}_k , \quad (A29)
\]

where \(|k\rangle = \hat{c}_k^\dagger |0\rangle\) are 1-particle eigenstates of the Hamiltonian with eigenvalue \(\varepsilon_k\). Inserting this in the expression for the thermal expectation of the probability of occupation we find

\[
p_{th} = \frac{1}{Z} \sum_{jk} \langle j|\hat{a}^\dagger|0\rangle \langle 0|\hat{a}|k\rangle \text{Tr}[e^{-\beta \hat{H}} \hat{c}_j^\dagger \hat{c}_k] = \sum_k |\langle k|\hat{a}^\dagger|0\rangle|^2 \text{Tr}[e^{-\beta \varepsilon_k \hat{c}_k^\dagger \hat{c}_k^\dagger \hat{c}_k}] = \sum_k |\langle k|\hat{a}^\dagger|0\rangle|^2 f_\beta(\varepsilon_k) .
\]

(A30)
From eq. (A11) it is easy to see that we can write the propagator in the following way
\[ G(t, 0) = \langle 0|\hat{a}_H(t)\hat{a}^\dagger|0\rangle = \langle 0|\hat{U}(t)\hat{a}\hat{U}^\dagger(t)|\hat{a}^\dagger|0\rangle = \sum_k e^{-i\varepsilon_k t} \langle 0|\hat{U}(t)\hat{a}|k\rangle\langle k|\hat{a}^\dagger|0\rangle = \sum_k e^{-i\varepsilon_k t} |\langle k|\hat{a}^\dagger|0\rangle|^2, \] (A31)
where we used the fact that the vacuum state does not evolve \( \hat{U}(t)|0\rangle = |0\rangle \) and that since we are performing no driving we have \( \hat{U}(t) = e^{-it\sum_k \varepsilon_k \hat{c}_k^\dagger \hat{c}_k} \). Note that the sum needs only to be over 1-particle states as there is a scalar product with the 1-particle state \( \hat{a}^\dagger|0\rangle \). By now defining \( \varphi(\omega) := \sum_k |\langle k|\hat{a}^\dagger|0\rangle|^2 \delta(\omega - \varepsilon_k) \) we can identify
\[ G(t, 0) = \int_{-\infty}^{\infty} d\omega \ \varphi(\omega)e^{-i\omega t}, \] (A32)
\[ p_{th} = \int_{-\infty}^{\infty} d\omega \ f_\beta(\omega)\varphi(\omega). \] (A33)
Considering eq. (A25) it is clear that if \( \varphi(\omega) = \frac{1}{\pi} \Re \left[ \hat{G}(-i\omega) \right] \) then we have proven \( p_{th} = \lim_{t \to \infty} p(t) \). Therefore we compute the Laplace transform of \( G(t, 0) \) using eq. (A32)
\[ \hat{G}(-i\omega) = \int_0^\infty dt \ G(t, 0)e^{i\omega t}, \]
\[ = \int_0^\infty dt \int_{-\infty}^{\infty} d\omega' \ e^{i(\omega-\omega')t}\varphi(\omega'), \]
\[ = \int_{-\infty}^{\infty} d\omega' \varphi(\omega') \int_0^\infty dt \ e^{i(\omega-\omega')t}\Theta(t), \]
\[ = \int_{-\infty}^{\infty} d\omega' \varphi(\omega') \int_{-\infty}^{\infty} dt \ \Theta(t)e^{i(\omega-\omega')t}, \]
\[ = \pi\varphi(\omega) + iP. \int_{-\infty}^{\infty} d\omega' \frac{\varphi(\omega')}{\omega-\omega'}, \]
where \( P. \) denotes the Cauchy principal value, \( \Theta(t) \) is the Heaviside step function and we used that its Fourier transform is (in a distributional sense) \( \int dt \ e^{\omega t}\Theta(t) = \pi\delta(s) + P.\frac{s}{\omega} \). Since \( \varphi(\omega) \) is by definition a real function we can see that \( P. \int_{-\infty}^{\infty} d\omega' \frac{\varphi(\omega')}{\omega-\omega} \) is a real number. Therefore we can conclude \( \varphi(\omega) = \frac{1}{\pi} \Re \left[ \hat{G}(-i\omega) \right] \). Which concludes the proof of the thermalization of \( p(t) \).
To prove the thermalization of \( v(t) \) we proceed in a similar fashion. We start by computing \( v_{th} \)
\[ v_{th} = \frac{1}{Z} \sum_{jk} \lambda_j \langle k|\hat{a}^\dagger|0\rangle \langle 0|\hat{b}_j|k\rangle \text{Tr} \left[ e^{-\beta H}\hat{c}_k^\dagger \hat{c}_k \right] + h.c. = \sum_{jk} f_\beta(\varepsilon_k) \left( \lambda_j \langle k|\hat{a}^\dagger|0\rangle \langle 0|\hat{b}_j|k\rangle + \lambda_j^* \langle k|\hat{b}_j^\dagger|0\rangle \langle 0|\hat{a}|k\rangle \right). \] (A34)
To proceed we have to define the following cross-propagators
\[ \lambda_j K_j(t) := \langle 0|\hat{a}_H(t)\hat{b}_j^\dagger|0\rangle = \sum_k e^{-i\varepsilon_k t} \langle k|\hat{b}_j^\dagger|0\rangle \langle 0|\hat{a}|k\rangle = \int_{-\infty}^{\infty} d\omega \ \psi_j(\omega)e^{-i\omega t}, \] (A35)
\[ \lambda_j^* H_j(t) := \langle 0|\hat{b}_j \hat{H}(t)|\hat{a}^\dagger|0\rangle = \sum_k e^{-i\varepsilon_k t} \langle k|\hat{a}^\dagger|0\rangle \langle 0|\hat{b}_j|k\rangle = \int_{-\infty}^{\infty} d\omega \ \psi_j^*(\omega)e^{-i\omega t}, \] (A36)
where we defined \( \psi_j(\omega) = \sum_k \langle k|\hat{b}_j^\dagger|0\rangle \langle 0|\hat{a}|k\rangle \delta(\omega - \varepsilon_k) \). By further defining \( \psi_0(\omega) := \sum_k \lambda_k^* \psi_k(\omega) \) and \( \psi(\omega) = \psi_0(\omega) + \psi_0^*(\omega) \) we can see that
\[ K(t) := \sum_j |\lambda_j|^2 (K_j(t) + H_j(t)) = \int_{-\infty}^{\infty} d\omega \ \psi(\omega)e^{-i\omega t}, \] (A37)
\[ v_{th} = \int_{-\infty}^{\infty} d\omega \ f_\beta(\omega)\psi(\omega). \] (A38)
Therefore, similarly to the case of \( G(t, 0) \), we have
\[ \tilde{K}(-i\omega) = \pi\psi(\omega) + iP. \int_{-\infty}^{\infty} d\omega' \frac{\psi(\omega')}{\omega-\omega'}, \]
and in particular $\pi \psi(\omega) = \Re \left[ \hat{K}(-i\omega) \right]$ (since $\psi(\omega)$ is real by definition). Hence, by eq. (A26) and eq. (A38), the last step to prove that $v(t)$ thermalizes is to check that $\Re \left[ \hat{K}(-i\omega) \right] = g^3 \left[ \tilde{G}(-i\omega) \right]$. To do so we start by computing the components of $K(t)$: from eq. (A11) we can see that

$$K_j(t) = -ig \int_0^t ds \ G(t,s)e^{-i\omega_j s} ,$$

$$= -ig \int_0^t ds \ e^{-(t-s)(\frac{1}{2}g^2+i\varepsilon)} e^{-i\omega_j s} ,$$

$$= -ig e^{-(t\frac{1}{2}g^2+i\varepsilon)} \int_0^t ds \ e^{s(\frac{1}{2}g^2+i(\varepsilon-\omega_j))} ,$$

$$= -ig \frac{e^{-i\omega_j t} - e^{-(\frac{1}{2}g^2+i\varepsilon)t}}{\frac{1}{2}g^2 + i(\varepsilon-\omega_j)} = -ig \frac{e^{-i\omega_j t} - G(t,0)}{\frac{1}{2}g^2 + i(\varepsilon-\omega_j)} ;$$

and from eq. (A12)

$$H_j(t) = -ig \int_0^t ds \ G(s,0)e^{i\omega_j (s-t)} = -ige^{-i\omega_j t} \int_0^t ds \ e^{-(\frac{1}{2}g^2+i(\varepsilon-\omega_j)s)} = -ige^{-i\omega_j t} e^{-(\frac{1}{2}g^2+i\varepsilon)t} = K_j(t) .$$

Since the time dependence is contained in the exponentials, it is straightforward to compute the Laplace transform

$$\tilde{K}_j(z) = \tilde{H}_j(z) = \frac{-ig}{\frac{1}{2}g^2 + i(\varepsilon-\omega_j)} \int_0^\infty dt \left[ e^{-i\omega_j t} - G(t,0) \right] e^{-zt}$$

$$= -ig \frac{1}{z + i\omega_j} \left[ \frac{1}{z + i\varepsilon + g^2/2} \right] = -ig \frac{1}{z + i\omega_j} \tilde{G}(z) .$$

Therefore we find

$$\tilde{K}(-i\omega) = \frac{g}{\pi} \tilde{G}(-i\omega) P. \int_{-\infty}^\infty d\omega' \frac{1}{\omega - \omega'} = -ig \tilde{G}(-i\omega) ,$$

which allows us to conclude $\psi(\omega) = g^3 \left[ \tilde{G}(-i\omega) \right]$. This concludes the proof of the thermalization of $v(t)$.

**Appendix B: Slow driving expansion**

1. Deriving the thermodynamic metric

We are interested in performing an erasure protocol and minimizing the work cost of performing it. The erasure protocol is one where $\varepsilon(0) = 0$, $\varepsilon(\tau) \gg 1$ and $g(0) = g(\tau) = 0$, for $\tau$ the total time of the protocol. The work cost of a protocol where we control $\varepsilon$ and $g$ is

$$W = \int_0^\tau dt \ Tr \left[ \hat{p}(t) \frac{d}{dt} \hat{H}(t) \right] = \int_0^\tau dt \ \dot{\varepsilon}(t)p(t) + \dot{g}(t)v(t) .$$

To get a correction to Landauer’s bound for finite time protocols, and to work with more tractable expressions, we expand eq. (B1) in the long times limit up to first order. To do that we first need to make some notation changes. First we make the time parameter in $\varepsilon$ and $g$ dimensionless, so that the protocol starts at “time” input parameter 0 and ends at “time” input parameter 1. So we have the following mappings: $t \rightarrow \tau t$, $\int dt \rightarrow \tau \int dt$ and $\frac{d}{dt} \rightarrow \tau^{-1} \frac{d}{\tau t}$. Second we need to “extract” the evolution timescale of the system in order to make the slow driving expansion. From eq. (A16), eq. (A17) and the definition of the propagator (or more clearly form eq. (A19) and eq. (A20)) it is quite clear that the relaxation timescale of the system, at any point of the evolution, is of the order $(g(t)^2)^{-1}$. Hence we are going to take the average of the square of the coupling as normalizing factor, we therefore define (in normalized time) $\Gamma := \int_0^1 dt \ g(t)^2$. We now define a normalized version of our control parameters:

$$\varepsilon(t) := \frac{1}{\Gamma} \varepsilon(t) \text{,} \quad \gamma(t) := \frac{1}{2\Gamma} g(t)^2 .$$

(B2)
We can therefore write the expression for work cost in this new convention
\[
W = \int_0^1 dt \, \dot{e}(t) p(t) + \dot{e}(t) \sqrt{\frac{\Gamma}{2\gamma(t)}} \, v(t) .
\] (B3)

We can also rewrite the propagator
\[
G(t, s) = \exp \left[ -\tau \Gamma \int_s^t dr \, \gamma(r) + i\epsilon(r) \right] ,
\] (B4)

and the expectation values of the observables
\[
p(t) = |G(t, 0)|^2 p_0 + \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \, f_\beta(\omega \Gamma) \left| \tau \Gamma \int_0^t ds \, \gamma(s) \frac{1}{2} G(t, s) e^{i\tau \omega(t-s)} \right|^2 ,
\] (B5)
\[
v(t) = \frac{\tau \Gamma \sqrt{2\Gamma}}{\pi} \Im \int_{-\infty}^{\infty} d\omega \, f_\beta(\omega \Gamma) \int_0^t ds \, \gamma(s) \frac{1}{2} G(t, s) e^{i\tau \omega(t-s)} ,
\] (B6)

where we were able to insert a phase in the time integral of eq. (B5) because of the absolute value and rescaled \( \omega \) by \( \Gamma \). We can see that we have to expand in \( 1/\tau \Gamma \) the same integral for both eq. (B5) and eq. (B6). To do that we do partial integration. First, we can notice that
\[
G_\omega(t, s) := G(t, s) e^{i\tau \omega(t-s)} = \exp \left[ -\tau \Gamma \int_s^t dr \, \gamma(r) + i(\epsilon(r) - \omega) \right] .
\]

Furthermore we have
\[
\frac{d}{ds} G_\omega(t, s) = \tau \Gamma [\gamma(s) + i(\epsilon(s) - \omega)] G_\omega(t, s) .
\]

Therefore we can write
\[
\tau \Gamma \int_0^t ds \, \gamma(s) \frac{1}{2} G_\omega(t, s) = \int_0^t ds \, \frac{\gamma(s) \frac{1}{2}}{\gamma(s) + i(\epsilon(s) - \omega)} \frac{d}{ds} G_\omega(t, s) ,
\]
\[
= \frac{\gamma(s) \frac{1}{2}}{\gamma(s) + i(\epsilon(s) - \omega)} G_\omega(t, s) \bigg|_s^t - \int_0^t ds \, G_\omega(t, s) \frac{d}{ds} \frac{\gamma(s) \frac{1}{2}}{\gamma(s) + i(\epsilon(s) - \omega)} ,
\] (B7)

where we can evaluate the first part as \( \frac{\gamma(s) \frac{1}{2}}{\gamma(s) + i(\epsilon(s) - \omega)} G_\omega(t, s) \bigg|_s^t = \frac{\gamma(t) \frac{1}{2}}{\gamma(t) + i(\epsilon(t) - \omega)} + O(e^{-\tau \Gamma}) \). We absorbed the \( G_\omega(t, 0) \) term in \( O(e^{-\tau \Gamma}) \). For the second term we evaluate the derivative and continue integrating by parts
\[
\int_0^t ds \, \frac{\gamma(s) \frac{1}{2}}{\gamma(s) + i(\epsilon(s) - \omega)} \left( \gamma(s) + i(\epsilon(s) - \omega) \right)^2 G_\omega(t, s)
\]
\[
= \frac{1}{\tau \Gamma} \frac{\gamma(s) \frac{1}{2}}{\gamma(s) + i(\epsilon(s) - \omega)} \left( \gamma(s) + i(\epsilon(s) - \omega) \right)^3 G_\omega(t, s) \bigg|_s^t
\]
\[
- \frac{1}{\tau \Gamma} \int_0^t ds \, G_\omega(t, s) \frac{d}{ds} \frac{\gamma(s) \frac{1}{2}}{\gamma(s) + i(\epsilon(s) - \omega)} \left( \gamma(s) + i(\epsilon(s) - \omega) \right)^3 .
\] (B8)

Similarly as in eq. (B7), we keep only the evaluation at \( s = t \) for the first term because the evaluation at \( s = 0 \) is of order \( O(e^{-\tau \Gamma}) \). Whereas the remaining integral will also have to be evaluated by parts, and in doing so we will obtain another power of \( 1/\tau \Gamma \). But since we are only interested in the first order correction and the integral will only yield terms of order \( O(1/\tau^2 \Gamma^2) \) we don’t need to compute it. By combining eq. (B7) and eq. (B8) we finally find
\[
\tau \int_0^t ds \, \gamma(s) \frac{1}{2} G_\omega(t, s) = \frac{\gamma(t) \frac{1}{2}}{\gamma(t) + i(\epsilon(t) - \omega)} - \frac{1}{\tau \Gamma} \frac{\gamma(t) \frac{1}{2}}{\gamma(t) + i(\epsilon(t) - \omega)} \left( \gamma(t) + i(\epsilon(t) - \omega) \right)^3 + O(\frac{1}{\tau^2 \Gamma^2}) .
\] (B9)
The absolute value squared of eq. (B9) is
\[
\left| \tau T \int_0^t ds \gamma(s)^\frac{1}{2} G_\omega(t, s) \right|^2 = \frac{\gamma(t)}{\gamma(t)^2 + (\epsilon(t) - \omega)^2} + \frac{1}{\tau T} \frac{4\dot{\gamma}(t)\gamma(t)^2(\epsilon(t) - \omega) + \dot{\gamma}(t)\gamma(t) \left( \gamma(t)^2 - 3(\epsilon(t) - \omega)^2 \right)}{(\gamma(t)^2 + (\epsilon(t) - \omega)^2)^3} + \mathcal{O}(\frac{1}{\tau^2 T^2}) . \quad (B10)
\]

Combining this with eq. (B5) we find the expansion of \( p(t) \) in the slow driving regime
\[
p(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega f_\beta(\omega \Gamma) \frac{\gamma(t)}{\gamma(t)^2 + (\epsilon(t) - \omega)^2} + \frac{1}{\tau T} \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega f_\beta(\omega \Gamma) \frac{4\dot{\epsilon}(t)\gamma(t)^2(\epsilon(t) - \omega) + \dot{\gamma}(t)\gamma(t) \left( \gamma(t)^2 - 3(\epsilon(t) - \omega)^2 \right)}{(\gamma(t)^2 + (\epsilon(t) - \omega)^2)^3} + \mathcal{O}(\frac{1}{\tau^2 T^2}) . \quad (B11)
\]

Whereas the imaginary part of eq. (B9) is
\[
\frac{\tau T}{\gamma(t)^\frac{1}{2}} \Im \int_0^t ds \gamma(s)^\frac{1}{2} G_\omega(t, s) = -\frac{\epsilon(t) - \omega}{\gamma(t)^2 + (\epsilon(t) - \omega)^2} - \frac{1}{\tau T} 2\dot{\gamma}(t)(\epsilon(t) - \omega) \left( \gamma(t)^2 - (\epsilon(t) - \omega)^2 \right) - \dot{\gamma}(t)\gamma(t) \left( \gamma(t)^2 - 3(\epsilon(t) - \omega)^2 \right) \frac{(\gamma(t)^2 + (\epsilon(t) - \omega)^2)^3}{(\gamma(t)^2 + (\epsilon(t) - \omega)^2)^3} + \mathcal{O}(\frac{1}{\tau^2 T^2}) . \quad (B12)
\]

Therefore the slow driving expansion of \( v(t) \) is
\[
\frac{1}{\sqrt{2\pi\gamma(t)}} v(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega f_\beta(\omega \Gamma) \frac{\epsilon(t) - \omega}{\gamma(t)^2 + (\epsilon(t) - \omega)^2} - \frac{1}{\tau T} \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega f_\beta(\omega \Gamma) 2\dot{\gamma}(t)(\epsilon(t) - \omega) \left( \gamma(t)^2 - (\epsilon(t) - \omega)^2 \right) - \dot{\gamma}(t)\gamma(t) \left( \gamma(t)^2 - 3(\epsilon(t) - \omega)^2 \right) \frac{(\gamma(t)^2 + (\epsilon(t) - \omega)^2)^3}{(\gamma(t)^2 + (\epsilon(t) - \omega)^2)^3} + \mathcal{O}(\frac{1}{\tau^2 T^2}) . \quad (B13)
\]

Therefore we can see that we can rewrite the work cost of the protocol as
\[
W = W^{(0)} + \frac{1}{\tau T} W^{(1)} + \mathcal{O}(\frac{1}{\tau^2 T^2}) . \quad (B14)
\]

The leading order term is
\[
W^{(0)} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega f_\beta(\omega) \int_0^1 dt \frac{\dot{\gamma}(t)(\epsilon(t) - \omega) - \dot{\gamma}(t)(\epsilon(t) - \omega/\Gamma)}{\gamma(t)^2 + (\epsilon(t) - \omega/\Gamma)^2} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega f_\beta(\omega) \left( \arctan(\frac{\gamma(0)}{\epsilon(0) - \omega/\Gamma}) - \arctan(\frac{\gamma(1)}{\epsilon(1) - \omega/\Gamma}) \right) , \quad (B15)
\]

where we re-scaled \( \omega \) by \( \Gamma \). Here we were able to perform the integral independently of the function describing the control parameters. Therefore \( W^{(0)} \) only depends on their initial and final value. More importantly, we can identify the instantaneous thermal expectation values of \( p(t) \) and \( v(t) \) (from eq. (A22) and eq. (A23)) in the time integral of \( W^{(0)} \). This implies very directly that \( W^{(0)} = \Delta F \).

We can notice that we can write \( W^{(1)} \) as
\[
W^{(1)} = \int_0^1 dt \bar{\lambda}_t^T m(\bar{\lambda}_t) \bar{\lambda}_t , \quad (B16)
\]

with \( \bar{\lambda}_t = (\epsilon(t), \gamma(t))^T \) and the metric
\[
m(\bar{\lambda}) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega f_\beta(\omega) m_w(\epsilon - \omega/\Gamma, \gamma) , \quad (B17)
\]
for
\[ m_\omega(\epsilon, \gamma) = \frac{1}{(\gamma^2 + \epsilon^2)^3} \left( \frac{4e\gamma^2}{\gamma(\gamma^2 - 3\epsilon^2)} \frac{\gamma(\gamma^2 - 3\epsilon^2)}{2\epsilon(\epsilon^2 - \gamma^2)} \right) . \] (B18)

Since the leading order is independent of the path taken in parameter space, minimizing the work cost of erasure only implies minimizing \( W^{(1)} \), i.e., the entropy production \( k\beta T\Sigma \). As we see from eq. (B16) it is equivalent to finding the shortest path in a metric space described by the metric \( m(\lambda) \). The length of this shortest path is known as thermodynamic length. In the main text eq. (17) and eq. (18) represent the metric when the problem is rewritten in terms of the unit-full parameters.

2. Weak coupling limit

Previous works on optimization of finite-time Landauer erasure have focused on the Markovian regime [41–51], corresponding to the weak coupling limit. We analyze this regime in this section. First we assume that the coupling remains unchanged during the protocol, which means \( \Gamma = g^2 \) and \( \gamma = 1 \). We start by rewriting \( p(t) \) from eq. (B11) in a more convenient manner under this first assumption:

\[ p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ f_\beta(\omega) \frac{\Gamma}{\Gamma^2/4 + (\varepsilon(t) - \omega)^2} + \frac{1}{\Gamma^2} \int_{-\infty}^{\infty} d\omega \ f_\beta(\omega) \frac{\dot{\varepsilon}(t)(\varepsilon(t) - \omega)\Gamma^3}{(\Gamma^2/4 + (\varepsilon(t) - \omega)^2)^3} + \mathcal{O}(\frac{1}{\Gamma^2}) , \] (B19)

where we used eq. (B2) to go back to unit-full parameters and re-scaled \( \omega \) by \( \Gamma \). Integrating by parts the second integral we get

\[ p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ f_\beta(\omega) \frac{\Gamma}{\Gamma^2/4 + (\varepsilon(t) - \omega)^2} + \frac{1}{\Gamma} \frac{\beta\varepsilon(t)}{8\pi} \int_{-\infty}^{\infty} d\omega \ f_\beta(\omega)(1 - f_\beta(\omega)) \frac{\Gamma^3}{(\Gamma^2/4 + (\varepsilon(t) - \omega)^2)^2} , \] (B20)

where we used that \( \frac{d}{d\omega} f_\beta(\omega) = -\beta f_\beta(\omega)(1 - f_\beta(\omega)) \), \( \frac{1}{\pi} \frac{\Gamma}{\Gamma^2/4 + (\varepsilon(t) - \omega)^2} = \frac{4\varepsilon(t)}{(\Gamma^2/4 + (\varepsilon(t) - \omega)^2)^2} \) and dropped the \( \mathcal{O}(1/\Gamma^2) \) to make the notation lighter. We now take the weak coupling limit, but we do so while keeping the slow driving assumption: \( \tau\Gamma \gg 1 \). Using the results of [108] we get the following

\[ \lim_{\Gamma \to 0} \frac{1}{\pi} \frac{\Gamma/2}{(\Gamma^2/4 + (\varepsilon(t) - \omega)^2)^2} = \delta(\varepsilon(t) - \omega) , \]

\[ \lim_{\Gamma \to 0} \frac{1}{\pi} \frac{\Gamma^3/8}{(\Gamma^2/4 + (\varepsilon(t) - \omega)^2)^3} = \frac{1}{2} \delta(\varepsilon(t) - \omega) , \]

where the equalities are meant in a distributional sense. Therefore we find that the occupation probability in the weak coupling limit is

\[ p(t) = f_\beta(\varepsilon(t)) + \frac{1}{\tau\Gamma} \beta\varepsilon(t) f_\beta(\varepsilon(t))[1 - f_\beta(\varepsilon(t))] . \] (B21)

This result coincides with applying a slow driving expansion to a simple exponential relaxation model with characteristic time \( \Gamma (\dot{p} = -\tau\Gamma[p-f_\beta(\varepsilon(t))] \)). Computing the work cost yields

\[ W = \Delta F + \frac{1}{\tau\Gamma\beta} \int_0^t dt \ \dot{\varepsilon}(t)^2 f_\beta(\varepsilon(t))[1 - f_\beta(\varepsilon(t))] , \] (B22)

with \( \Delta F = \beta^{-1} \ln \frac{1 + e^{-\beta\varepsilon(0)}}{1 + e^{-\beta\varepsilon(t)}} \). We will now minimize the work cost of the erasure protocol, similar optimizations have been done before in [47, 109, 110]. From variational calculus we know that the extremal function of the integral in eq. (B22) will keep the integrand constant. So we can solve the variational problem as follows:

\[ \dot{\varepsilon}(t) \sqrt{f_\beta(\varepsilon(t))[1 - f_\beta(\varepsilon(t))]} = K_w , \]

\[ \int_{\varepsilon(0)}^{\varepsilon(t)} e^{-\beta\varepsilon/2} d\varepsilon = K_w \int_0^t dt' , \]

\[ 2 \arctan(e^{\beta\varepsilon(t)/2}) - \frac{\pi}{2} = \beta K_w t , \]

\[ \varepsilon(t) = 2\beta^{-1} \ln \tan \left( \beta K_w t/2 + \frac{\pi}{4} \right) , \] (B23)
with $K_\omega = 2\beta^{-1}(\arctan(e^{\beta \varepsilon(1)/2}) - \frac{\pi}{4})^{\beta \varepsilon(1) \rightarrow \infty}$, $\frac{\pi}{2} \beta^{-1}$. We therefore find

$$\varepsilon_{\text{weak}}(t) = 2\beta^{-1} \ln \tan \left( \frac{\pi}{4} (t + 1) \right),$$  \hspace{1cm} (B24)

and recover the result of eq. (1):

$$W = k_B T \left( \ln 2 + \frac{\pi^2}{4 \tau \Gamma} \right).$$  \hspace{1cm} (B25)

### 3. Solving the integral of the thermodynamic metric and finding the symmetry

We will now try to find a more tractable version of the metric in eq. (B17). First we notice that

$$m_\omega(\epsilon, \gamma) = -\frac{d}{d\epsilon} \frac{1}{(\gamma^2 + \epsilon^2)^{\frac{1}{2}}} \left( \frac{\gamma^2}{-\epsilon} - \frac{-\epsilon^2}{\epsilon} \right) = -\frac{d}{d\epsilon} m_0(\epsilon, \gamma).$$  \hspace{1cm} (B26)

We can remark that $m_0$ coincides with a metric of an angle distance in the $(\epsilon, \gamma)$ space. To solve the integral of eq. (B17) we will go in Fourier space. For a function $h(\epsilon)$ its Fourier transform $\tilde{h}(\xi)$ has the defining property

$$h(\epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \, \tilde{h}(\xi) e^{i\xi \epsilon}.$$  \hspace{1cm} (B27)

Even though $f_\beta(\omega)$ is not an integrable function we can find its Fourier transform in a distributional sense

$$\tilde{f}_\beta(\xi) = \pi \delta(\xi) + \frac{\pi i}{\beta \sinh (\pi \xi / \beta)}.$$  \hspace{1cm} (B28)

For $m_0$ we find

$$\tilde{m}_0(\xi, \gamma) = \frac{\pi}{2} \Theta(\xi) e^{-\xi \gamma} \left( \gamma^{-1} + \frac{i\xi}{\gamma} - \xi \right) + \frac{\pi}{2} \Theta(-\xi) e^{\xi \gamma} \left( \gamma^{-1} - \frac{i\xi}{\gamma} - \xi \right),$$  \hspace{1cm} (B29)

with $\tilde{m}_0(0, \gamma) = \frac{\pi}{2\gamma} \delta$. Therefore we can rewrite eq. (B17) as

$$m_{\tilde{\lambda}} = -\frac{1}{4\pi^3} \frac{d}{d\epsilon} \int_{-\infty}^{\infty} d\omega d\xi d\xi' \, \tilde{f}_\beta(\xi') \tilde{m}_0(\xi, \gamma) e^{i\omega \xi'} e^{i(\epsilon - \omega / \Gamma) \xi},$$

$$= -i \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\omega d\xi d\xi' \, \xi \tilde{f}_\beta(\xi') \tilde{m}_0(\xi, \gamma) e^{i\epsilon \xi} e^{i\omega (\xi' - \xi / \Gamma)},$$

$$= -i \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\xi d\xi' \, \xi \tilde{f}_\beta(\xi, \gamma) \tilde{m}_0(\xi, \gamma) \delta(\xi' - \xi / \Gamma) e^{i\epsilon \xi},$$

$$= -i \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\xi \, \xi \tilde{f}_\beta(\xi / \Gamma) \tilde{m}_0(\xi, \gamma) e^{i\epsilon \xi},$$

$$= \frac{1}{2\beta \pi} \int_{-\infty}^{\infty} d\xi \, \xi \frac{\tilde{m}_0(\xi, \gamma)}{\sinh (\frac{\pi \xi}{\beta \Gamma})} e^{i\epsilon \xi},$$  \hspace{1cm} (B30)
where we used eq. (B27), eq. (B28) and the fact that $\int_{-\infty}^{\infty} d\omega \, e^{i\omega x} = 2\pi \delta(x)$. If we now insert eq. (B29) and flip the sign in the second integral we find

$$m(\tilde{\lambda}) = \frac{1}{4\beta} \int_{0}^{\infty} d\xi \frac{\xi \cdot e^{-\xi(z-i\epsilon)}}{\sinh(\frac{\pi \xi}{2\beta})} \left( \frac{1}{i\xi} \left( \xi^{-1} + \xi^{-1} - \xi \right) + \frac{1}{4\beta} \int_{-\infty}^{0} d\xi \frac{\xi \cdot e^{i\xi(z+i\epsilon)}}{\sinh(\frac{\pi \xi}{2\beta})} \left( \xi^{-1} - \xi^{-1} - \xi \right) \right),$$

$$= \frac{1}{4\beta} \int_{0}^{\infty} d\xi \frac{\xi \cdot e^{-\xi z}}{\sinh(\frac{\pi \xi}{2\beta})} \left( \frac{1}{i\xi} \left( \xi^{-1} + \xi^{-1} - \xi \right) + \frac{1}{4\beta} \int_{0}^{\infty} d\xi \frac{\xi \cdot e^{i\xi(z+i\epsilon)}}{\sinh(\frac{\pi \xi}{2\beta})} \left( \xi^{-1} - \xi^{-1} - \xi \right) \right),$$

where

$$= \frac{1}{4\beta} \int_{0}^{\infty} d\xi \frac{\xi \cdot e^{-\xi z}}{\sinh(\frac{\pi \xi}{2\beta})} \left( \frac{1}{i\xi} \left( \xi^{-1} + \xi^{-1} - \xi \right) + \frac{1}{4\beta} \int_{0}^{\infty} d\xi \frac{\xi \cdot e^{i\xi(z+i\epsilon)}}{\sinh(\frac{\pi \xi}{2\beta})} \left( \xi^{-1} - \xi^{-1} - \xi \right) \right).$$

We will now be able to compute these integrals in terms of poly-gamma functions. The poly-gamma function of order $m \geq 0$ is defined as $\psi^{(m)}(z) := \frac{d^{m+1}}{dz^{m+1}} \ln(\Gamma(z))$. For $m > 0$ and $\Re[z] > 0$ they have an integral representation:

$$\psi^{(m)}(z) = (-1)^{m+1} \int_{0}^{\infty} d\xi \frac{\xi^{m} \cdot e^{-\xi z}}{1 - e^{-\xi}}.$$  \hspace{1cm} (B32)

Using the fact that $\psi^{(m)}(z) = \frac{2e^{-z}}{z^{m+1}}$, a change of variable and eq. (B32) we find

$$m(\tilde{\lambda}) = \frac{1}{\beta \gamma} \int_{0}^{\infty} d\xi \frac{\xi \cdot e^{-\xi z}}{1 - e^{-\xi}} + \frac{1}{\beta} \int_{0}^{\infty} d\xi \frac{\xi \cdot e^{i\xi(z+i\epsilon)}}{1 - e^{-\xi}},$$

$$\text{or}

$$m(\tilde{\lambda}) = \frac{\beta \Gamma^{2}}{4\pi^{2} \gamma} \int_{0}^{\infty} d\xi \frac{\xi \cdot e^{-\xi z}}{1 - e^{-\xi}} + \frac{\beta \Gamma^{2}}{8\pi^{3}} \int_{0}^{\infty} d\xi \frac{\xi \cdot e^{i\xi(z+i\epsilon)}}{1 - e^{-\xi}},$$

$$\text{or}

$$m(\tilde{\lambda}) = \frac{\beta \Gamma^{2}}{4\pi^{2} \gamma} \Re[\psi^{(1)}(\frac{1}{2} + \frac{\beta \Gamma}{2\pi}(\gamma + i\epsilon))]$$

We can notice that the metric explicitly depends on $\Gamma$ in such a way that it seems that the solution for the geodesic should depend on this scale factor. Though this dependence disappears if we re-parameterize the problem in terms of its original unit-full parameters. We start by rewriting the work as

$$W = \Delta F + W^{(1)} + \mathcal{O}(\frac{1}{\tau^{2} \Gamma^{2}}),$$

where we redefined $W^{(1)}$ with the unit-full parameters $\tilde{\lambda} = (\varepsilon(t), \mu(t))^{T}$ (with $\mu(t) := \frac{1}{2} g(t)^{2} = \Gamma(t)$):

$$W^{(1)} = \frac{1}{\tau} \int_{0}^{\tau} dt \tilde{\lambda}^{T} m(\tilde{\lambda}) \tilde{\lambda},$$

$$m(\tilde{\lambda}) = \frac{\beta}{4\pi^{2} \mu} \Re[\psi^{(1)}(\frac{1}{2} + \frac{\beta \Gamma}{2\pi}(\gamma + i\epsilon))].$$

We remind the reader that $z = \mu + i\varepsilon$. This metric is the same as the one presented in the main text. Despite not looking very approachable, eq. (B36) is a much more tractable version of eq. (17) when it comes to numerical implementations (as the polygamma functions are computed much faster than integrals) and analytical studies of the geometric properties of thermodynamic protocols.

We can notice from eq. (B35) and eq. (B36) is that there is a symmetry in the corrective term. If we perform the following transformation:

$$\varepsilon(t) \rightarrow \lambda \varepsilon(t),$$

$$\mu(t) \rightarrow \lambda \mu(t),$$

$$\beta \rightarrow \lambda^{-1} \beta,$$

$$\text{we get}$$

(B37)
for \( \lambda > 0 \); then \( W^{(1)} \) remains unchanged. This symmetry allows us to conclude that the minimal value of \( W^{(1)} \) to perform Landauer erasure will be of the form \( c/\tau \) where \( c \) is a constant that does not depend on any physical quantity.

**Appendix C: High temperature limit**

In order to find the minimal dissipation in the multi-variable case we need to numerically solve the equations of motion given by the exact metric, which (unsurprisingly) are very untractable analytically. But instead of solving an initial value problem (which we can always solve by numerical integration, in principle) we are trying to solve a boundary value problem. Generically, to solve a BVP numerically, the solver will try many IVPs until the wanted BVP is reached. But here we can notice that we can turn the BVP into an IVP by taking an analytical approximation of the problem around the point \((\varepsilon, \mu) = (0, 0)\).

As we have seen in eq. (B37) there is an underlying symmetry in this problem, so a limit where \( \varepsilon \) and \( \mu \) are infinitesimal is the same as a limit where \( \beta \) is infinitesimal but \( \varepsilon \) and \( \mu \) finite. Formally we are requiring \( \beta|\mu + i \varepsilon| \ll 1 \), which is a high-temperature limit. It is important to note that despite the fact that this approximation will yield some analytical results on how to optimize a protocol in the high temperature regime it will not give us a result that is relevant for Landauer erasure because to perform erasure we are assuming that we reach \( \beta \varepsilon \approx 1 \), which is a low temperature limit.

One way to obtain an analytical result in this framework is by going back to eq. (B17) and apply the high-temperature expansion of the Fermi-Dirac distribution: \( f_\beta(\omega) = \frac{1}{2} - \frac{1}{2} \beta \omega + \mathcal{O}(\beta^3 \omega^3) \). Since \( m_0(\pm \infty, \gamma) = 0 \) the first term of the metric in this expansion is 0. But from the next order we find (in unit-full parameters)

\[
\lambda^i + \Gamma^i_{jk} \dot{\lambda}^j \dot{\lambda}^k = 0 ,
\]

where we assumed the Einstein tensorial notation and \( \Gamma^i_{jk} \) are the Christoffel symbols

\[
\Gamma^i_{jk} := \frac{1}{2} m^{id} (\partial_j m_{kl} + \partial_k m_{jl} - \partial_l m_{jk}) .
\]

Here we have \( m^{id} = \frac{8 \mu}{\beta^2} \delta^{id} \) and \( \partial_\mu m_{bc} = -\frac{\beta}{8 \mu^2} \delta_{ab} \delta_{bc} \). We therefore find

\[
\Gamma^i_{jk} = -\frac{1}{2 \mu}(\delta_{jm} \delta^i_k + \delta_{km} \delta^i_j - \delta_{ij} \delta_{jk}) ,
\]

which can be rewritten as

\[
\Gamma^\varepsilon = -\frac{1}{2 \mu} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \Gamma^\mu = \frac{1}{2 \mu} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .
\]

We get the following differential equations for \( \mu \) and \( \varepsilon \):

\[
\dot{\varepsilon} \mu = \varepsilon \dot{\mu} , \quad 2 \mu \ddot{\mu} = \dot{\mu}^2 - \dot{\varepsilon}^2 .
\]

From the first equation we can see that \( \int d\varepsilon / \dot{\varepsilon} = \int d\mu / \mu \), therefore \( \dot{\varepsilon} = C \mu \) for some constant \( C \) (we can already see as a sanity check that \( \dot{\varepsilon} \) never changes sign in an optimal protocol). The equation for \( \mu \) becomes

\[
2 \mu \ddot{\mu} = \dot{\mu}^2 - C^2 \mu^2 ,
\]

when we consider that \( \mu = g^2 / 2 \) and \( \mu = \dot{g}^2 + g \ddot{g} \) we can see that

\[
\ddot{g} = -C^2 g / 4 .
\]
Taking into account that the boundary conditions for \( g \) are \( g(0) = g(1) = 0 \) we find that \( g(t) = A \sin(k \pi t) \) for some constant \( A, k \in \mathbb{N}^* \) and \( C = 2k \pi \). By choosing \( \varepsilon(0) = 0 \) and \( \varepsilon(1) = \varepsilon_* \) we have \( \varepsilon(t) = k \pi A^2 \int_0^t ds \sin(k \pi s)^2 \), therefore \( A^2 = \frac{2}{k \pi} \). Thus the optimal protocol, portrayed in Fig. 2, is

\[
\varepsilon(t) = \varepsilon_* \left( t - \frac{\sin(2k \pi t)}{2k \pi} \right), \quad \mu(t) = \frac{\varepsilon_*}{k \pi} \sin(k \pi t)^2 .
\]

We therefore find the dissipated work for the high temperature limit by inserting this in eq. (B35) and taking \( k = 1 \):

\[
W^{(1)}_{HT} = \frac{\pi \beta \varepsilon_*}{2 \tau} .
\]

We can see that in this scenario the corrective term grows extensively with the final energy \( \varepsilon_* \), combining this with the fact that the exact metric goes to 0 faster than \( O(|z^{-1}|) \) we can presume that most of the dissipation in the exact protocol is caused by the part of the protocol that matches with the high temperature regime.

**Appendix D: Low temperature limit**

We will now study the limit of \( T \to 0 \) we have \( f_\beta(\omega) \to f_\infty(\omega) = \Theta(-\omega) \), where \( \Theta \) is the Heaviside step function. Therefore the integral of eq. (B17) becomes (in unit-full parameters)

\[
m_{T=0}(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \Theta(-\omega)m_\omega(\varepsilon - \omega, \mu) ,
\]

\[
= \frac{1}{\pi} \int_{\varepsilon}^{\infty} d\omega \left( \frac{\mu^2}{\mu^2 + \varepsilon^2} \right) m_\omega(\varepsilon, \mu) ,
\]

\[
= \frac{1}{\pi} \int_{\varepsilon}^{\infty} d\omega \frac{d}{d\omega} m_\omega(\varepsilon, \mu) ,
\]

\[
= \frac{1}{\pi} m_0(\varepsilon, \mu) .
\]

where we used eq. (B26) and the fact that \( m_0(+\infty, \mu) = 0 \). Thus we have

\[
m_{T=0}(\lambda) = \frac{1}{\pi} \frac{1}{(\mu^2 + \varepsilon^2)^2} \left( \frac{\mu^2}{\varepsilon^2} \right) .
\]
We can compute the integrand of $W^{(1)}$ to find

$$\dot{\lambda}^T m_{T=0}(\dot{\lambda}) \dot{\lambda} = \frac{1}{\pi} \frac{\varepsilon^2 \mu^2 - 2 \dot{\varepsilon} \dot{\mu} \varepsilon + \dot{\mu}^2 \varepsilon^2}{(\mu^2 + \varepsilon^2)^2} = \frac{1}{\pi} \left( \frac{\varepsilon \dot{\mu} - \dot{\varepsilon} \mu}{\mu^2 + \varepsilon^2} \right)^2 \tag{D3}$$

By defining the coordinates $\vec{\lambda} = (r, \phi)^T$, such that $\varepsilon = r \cos \phi$ and $\mu = r \sin \phi$, we have $\phi = \arctan \frac{\mu}{\varepsilon}$. Therefore

$$\dot{\lambda}^T m_{T=0}(\dot{\lambda}) \dot{\lambda} = \frac{\dot{\phi}^2}{\pi} \tag{D4}$$

From which we can deduce the metric in these new coordinates

$$m_{T=0}^{(r,\phi)}(\vec{\lambda}_{r,\phi}) = \frac{1}{\pi} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{D5}$$

Crucially, we can notice that this metric is singular: here changes in the coordinate $r$ do not cause an increase in the work cost. Therefore we can parameterize geodesics at zero temperature as follows

$$\varepsilon(t) = r(t) \cos(\phi(0)(1 - t) + \phi(1)t) \quad \mu(t) = r(t) \sin(\phi(0)(1 - t) + \phi(1)t) \tag{D6}$$

where $r(t)$ is any function that satisfies the boundary conditions. By using eq. (B35) we can find the dissipated work

$$W^{(1)}_{T=0} = \frac{(\Delta \phi)^2}{\pi T} \tag{D7}$$

for $\Delta \phi = \phi(1) - \phi(0)$.

### Appendix E: One-parameter case

Because of the metric we obtain in eq. (B36) it is quite clear that, in the two-parameter case, we will not be able to solve analytically the geodesics for the full problem. This even prevents us form finding an analytical expression for the distance between two points in the parameter space, as it is the length of the shortest path (for which we have no expression). But by fixing one parameter to an arbitrary value and solving for the other we can use the fact that geodesics always have a conserved quantity along their path (the integrand: $\dot{\lambda}^T m(\dot{\lambda}) \dot{\lambda}$) to avoid solving the geodesic equation and finding an explicit formula for the length and geodesic. We point out the fact that the symmetry mentioned in App. B does not lead to a conserved quantity because $\beta$ is a constant of the system instead of a function of time for which we are solving.

Here we will take the erasure protocol to be made of three parts, which will be optimized separately: 1. we turn on the coupling to some value $\mu_*$, while keeping the energy at zero; 2. while keeping the coupling at $\mu_*$ we increase the energy from zero to infinity; 3. we turn the coupling off. Incidentally, this type of protocols are more realistic for an experimental realization as often setups are not able to control optimally energy and coupling at the same time. And even if the control over the coupling is only to turn it on to some value and turn it off, step 2 will remain valid. Furthermore, previous studies done at weak coupling essentially are described by this type of protocol; but they are in a regime where step 1 and 3 can be neglected. Therefore we can compare the results of this section to those of the weak coupling limit.

We start by looking at step 3, in the limit of $\beta \varepsilon \to \infty$ we actually reach a scenario described by the $T = 0$ limit. Therefore the length of this step is described by eq. (D5), for any finite value of $\mu_*$ the angle span of this step is trivially 0. Therefore, up to first order, this step will not cause any extra dissipation, no matter how it is realized.

We notice that we can write the length of the first step as follows

$$L_1 = \int_0^1 dt |\dot{\mu}(t)| m_{\mu}(0, \mu(t))^{1/2} = \int_0^{\mu(1)} d\mu \ m_{\mu}(0, \mu)^{1/2} \tag{E1}$$

where we used the fact that, since the metric is not explicitly time-dependent, the sign of $\dot{\mu}$ has to be always positive for this step. With eq. (B36) we find the following expression for the length

$$L_1 = \frac{1}{\sqrt{2\pi}} \int_0^{\beta \mu_*/2\pi} \sqrt{\Re \left[ \frac{1}{x} \psi^{(1)} \left( \frac{1}{2} + x \right) + \psi^{(2)} \left( \frac{1}{2} + x \right) \right]} dx \tag{E2}$$
Now that we have an expression for $L_1$ we can recover an equation for $\mu(t)$. We can use the fact that the integrand of the time integral in eq. (E1) is constant to obtain

\[ t L_1 = \frac{1}{\sqrt{2\pi}} \int_0^{\beta \mu(t)/2\pi} \sqrt{\mathbb{R} \left[ \frac{1}{x} \psi^{(1)} \left( \frac{1}{2} + x \right) + \psi^{(2)} \left( \frac{1}{2} + x \right) \right]} dx , \]  

(E3)

which gives an implicit definition of $\mu(t)$, or rather an explicit definition of its inverse $t(\mu)$.

By following the same procedure as in step 1 we can recover the length of step 2

\[ L_2 = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{\mathbb{R} \left[ \frac{2\pi \beta \mu}{2\pi} \psi^{(1)} \left( \frac{1}{2} + \frac{\mu}{2\pi} + iy \right) - \psi^{(2)} \left( \frac{1}{2} + \frac{\mu}{2\pi} + iy \right) \right]} dy , \]  

(E4)

and the implicit definition of $\varepsilon(t)$

\[ L_2 t = \frac{1}{\sqrt{2\pi}} \int_0^{\beta \varepsilon(t)/2\pi} \sqrt{\mathbb{R} \left[ \frac{2\pi \beta \mu}{2\pi} \psi^{(1)} \left( \frac{1}{2} + \frac{\mu}{2\pi} + iy \right) - \psi^{(2)} \left( \frac{1}{2} + \frac{\mu}{2\pi} + iy \right) \right]} dy . \]  

(E5)

These implicit definitions of $\varepsilon(t)$ and $\mu(t)$ can be solved numerically, the results are shown in Fig. 3.

![Fig. 3. Parametrization of $\mu(t)$ and $\varepsilon(t)$ described by eq. (E3) and eq. (E5) for multiple values of $\beta \mu_\ast$.](image)

The question remains about how to subdivide optimally the protocol times of step 1 ($\tau_1$) and step 2 ($\tau_2 = \tau - \tau_1$). The total excess work is given by $W^{(1)} = L_2^2/\tau_1 + L_2^2/\tau_2$, by taking the derivative and imposing it to be zero we find

\[ \tau_1 = \frac{L_1}{L_2 + L_1} \tau . \]  

(E6)

And therefore we find

\[ W^{(1)} = \frac{1}{\tau} \left( L_1 + L_2 \right)^2 , \]  

(E7)

which is indeed what was to be expected, as $L_1 + L_2$ is the total length of the protocol.

We now discuss how we can obtain an exact version of eq. (1), so that it applies also in the strong coupling regime. First we can notice that since $L_1 > 0$ and $L_2 > 0$ we have $W^{(1)} \geq L_2^2/\tau$. Then by considering that $m_{\varepsilon \varepsilon}(\varepsilon, \mu_\ast)$ is a one-dimensional metric it has to be positive by definition. Therefore the integrand of eq. (E4) is always positive. Next we can consider the fact that

\[ \lim_{\mu_\ast \to \infty} \frac{2\pi \beta \mu}{2\pi} \psi^{(1)} \left( \frac{1}{2} + \frac{\mu}{2\pi} + iy \right) - \psi^{(2)} \left( \frac{1}{2} + \frac{\mu}{2\pi} + iy \right) = 0 . \]  

(E8)

Therefore, for all $\mu_\ast$ and all $\varepsilon$

\[ m_{\varepsilon \varepsilon}(\varepsilon, \mu_\ast) \geq \lim_{\mu_\ast \to \infty} m_{\varepsilon \varepsilon}(\varepsilon, \mu_\ast) , \]  

(E9)
which allows us to conclude $L_2 \geq \lim_{\mu_* \to \infty} L_2$. For any finite $\mu_*$ large enough $m_{\mu\mu}(\varepsilon, \mu_*)$ can be approximated by $[m_T=0]_{\varepsilon}(\varepsilon, \mu_*)$. The angle spanned by the integral of $L_2$ is $\pi/2$. Therefore for all $\mu_*$

$$\int_0^\infty d\varepsilon [m_T=0]_{\varepsilon}(\varepsilon, \mu_*)^{1/2} = \frac{\sqrt{\pi}}{2},$$  \hfill (E10)

and since the approximation becomes exact in the limit $\mu_* \to \infty$ we have $\lim_{\mu_* \to \infty} L_2 = \sqrt{\pi}/2$. Therefore,

$$W^{(1)} \geq \frac{\pi}{4\tau}.$$ \hfill (E11)

In Fig. 4 we show the minimal value of $W^{(1)}$ for step 2 as a function of $\beta\mu_*$, and we compare it to eq. (1) and to eq. (E11). We can see how, when the coupling becomes small, the exact curve agrees with eq. (1).

**FIG. 4.** Comparison of the excess work $W^{(1)} = k_B T \Sigma$ for a slow erasure protocol at constant coupling in the exact description (eq. (E4)) with the weak coupling approximation (eq. (1)) and the lower bound of eq. (E11).

Appendix F: Numerical solution to the general case

We now discuss how the numerical problem of finding the optimal erasure protocol was approached. Having found the metric eq. (B36), all we have to do to find the optimal erasure protocol is to solve the geodesic equations

$$0 = \ddot{\varepsilon} + \Gamma_{\varepsilon\varepsilon}^\mu \dot{\varepsilon}^2 + 2\Gamma_{\varepsilon\mu}^\mu \dot{\varepsilon} \dot{\mu} + \Gamma_{\mu\mu}^\mu \dot{\mu}^2,$$

$$0 = \ddot{\mu} + \Gamma_{\mu\mu}^\mu \dot{\mu}^2 + 2\Gamma_{\mu\varepsilon}^\mu \dot{\varepsilon} \dot{\mu} + \Gamma_{\mu\mu}^\mu \dot{\mu}^2;$$  \hfill (F1)

with the Christoffel symbols defined as in eq. (C3). Though the differential equations we get are quite untractable and cannot be solved analytically, we won’t even write them here as they are very long and will not bring any

**FIG. 5.** A series of optimal protocols depicted for multiple values of $\beta \varepsilon(1)$. They all start with zero energy and coupling and end with finite energy and zero coupling. In the limit of large $\beta \varepsilon(1)$ they can be considered as erasure protocols. Shown in the parameter space (left) and as a function of time (centre and right).
insight. Therefore we will solve them numerically, and indeed eq. (F1) is quite practical for numerical integration since the second derivative of the parameters can be easily isolated. The boundary conditions we impose for the erasure protocol are \( \varepsilon(0) = \mu(0) = \mu(1) = 0, \varepsilon(1) \gg k_B T \). Indeed we cannot impose \( \beta \varepsilon(1) = \infty \) because we are performing numerics, therefore we set it to be an arbitrarily large value.

But at this point we can notice that close to \( t = 0 \), by continuity, we are satisfying the conditions for the high temperature approximation. And at \( t = 0 \) the approximation becomes exact. Therefore the initial conditions of an optimal erasure protocol in the general case must match with the initial conditions of the protocols that we previously studied in the high-temperature regime. From a numerical perspective it is much more preferable to solve an initial value problem instead of a boundary value problem. Therefore we used the numerical solver DOP853, implemented in the scipy library in python, to solve eq. (F1) with the initial conditions given by eq. (D6).

To be precise, we cannot start the integration from \( t = 0 \) as the metric is formally divergent at \((\varepsilon, \mu) = (0, 0)\), therefore we evaluate eq. (D6) at an infinitesimal time and integrate from there. The specific value we choose for \( \varepsilon_0 \) sets the value of \( \varepsilon(1) \) that is reached in a monotonous way. When \( \varepsilon_0 \) is chosen small the protocol closes matches with those of eq. (D6) (as long as \( \varepsilon(1) \) is also small). Then for larger values of \( \varepsilon_0 \) we get more interesting behavior, as is shown in Fig. 5.

When one changes the value of \( k \) in eq. (D6) the value of \( \varepsilon(1) \) that is reached is different. But, as is shown in Fig. 6, by numerically searching values of \( \varepsilon_0 \) such that the same \( \varepsilon(1) \) is reached for different values of \( k \) we find that the protocols end up being the same.

Finally we thought it might be interesting to compare the best one-parameter protocol to the geodesic erasure protocol we find numerically. And we can see from Fig. 7 that, despite seeming very different in the path taken in the parameter space, when we look at the functions of time they are actually quite similar. The apparent difference happens because the part of the protocol for \( \beta \varepsilon \gg 1 \) is done very quickly since the metric is vanishing in that region.

![FIG. 6. Comparing two optimal erasure protocol with \( \beta \varepsilon(1) \approx 21 \) for two different values of \( k \). The same is found for other values of \( k \).](image1)

![FIG. 7. Comparing an optimal erasure protocol \((\beta \varepsilon(1) \approx 50)\) to an optimized \((\beta \mu_\ast \approx 1.863)\) erasure protocol where we change only one parameter at a time.](image2)