FLOER FIELD THEORY FOR TANGLES

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ABSTRACT. We use quilted Floer theory to construct category-valued field theories for tangles and links using Lagrangian correspondences arising from moduli spaces of flat connections on punctured surfaces. The formal properties of the field theories are similar to those of Khovanov-Rozansky homology \cite{Khovanov} and the gauge-theoretic invariants developed by Kronheimer-Mrowka \cite{Kronheimer-Mrowka}. As an application, we show the non-triviality of certain elements in the symplectic mapping class groups of moduli spaces of flat connections on punctured spheres.

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1. INTRODUCTION

In this paper we apply the theory of pseudoholomorphic quilts and a symplectic 2-category developed in \cite{Wehrheim-Woodward, AIDS, Bar-Nat, Shklyarov} to construct category-valued functors on a category of tangles via moduli spaces of flat connections with traceless holonomies. Originally the idea (suggested to us by Seidel) was to see whether there was a relationship with the theory of Khovanov \cite{Khovanov} developing functor-valued invariants of tangles. In particular, Khovanov’s construction gives a categorical version of the Jones polynomial, in which the invariant associated to a knot is a group and the invariant associated to a tangle is a functor. Given the gauge-theoretic interpretation of Jones polynomial provided by Witten \cite{Witten} in terms of the moduli spaces of flat bundles on surfaces, several mathematicians and physicists (in particular Kronheimer-Mrowka \cite{Kronheimer-Mrowka}) have investigated whether Khovanov’s invariant has any relation to gauge theory.

The basic idea is to decompose tangles into simple tangles, associate to these Lagrangian correspondences that arise from flat connections on the tangle complement with traceless holonomy, and then apply quilted Floer homology \cite{Wehrheim-Woodward} to obtain a group. The special feature of this approach is that independence from the choice of

Partially supported by NSF grants CAREER 0844188 and DMS 0904358.
decomposition can be understood functorially: Different decompositions of a tangle are related via gluing of simple tangles (i.e. composition in the topological category), which corresponds to embedded geometric composition of Lagrangian correspondences (i.e. composition in the symplectic category), and quilted Floer homology is invariant under the latter by \[42\]. More functorially, sequences of Lagrangian correspondences related by embedded geometric composition are isomorphic in the symplectic 2-category by \[44\]. A precise version of our following main theorem is stated in Theorem 5.1.

**Theorem 1.1.** (Floer field theory for tangles) Let \(X\) be a compact oriented surface. There exists a functor from the category of (finite subsets of \(X\), tangles \(K\) in \([-1,1] \times X\) with admissible labels as in Proposition 3.8 (d)) to the category of (categories, isomorphism classes of functors) which assigns to any finite subset \(x \subset X\) with labels \(\mu\) the Donaldson-Fukaya category of the moduli space of flat bundles on \(X - x\) with holonomies given by \(\mu\).

Gauge-theoretic invariants of knots were constructed using instantons by Collin-Steer \[7\] and Kronheimer-Mrowka \[21\], see also Jacobsson-Rubinsztein \[14\] for the similarities with Khovanov homology; the latter two papers appeared at roughly the same time as the first draft of this paper was circulated. One expects the functors defined in this paper to be related to the instanton knot invariants by a version of the Atiyah-Floer conjecture. Since the Kronheimer-Mrowka invariants are not exactly the same as Khovanov’s, one hence expects our functors to not reproduce Khovanov’s either. While we do not prove invariance under isotopies of the knots resp. tangles, we do however expect our functors to be topological invariants. In the instanton setting one moreover has maps associated to four-dimensional bordisms (in particular \[21\] obtains several attractive results by relating different knot bordisms), while the salient feature of the symplectic setting developed here is that one has functors associated to tangles. However, there are natural approaches for extending our theory to isotopies and, more generally, to four dimensions by associating natural transformations to four-dimensional bordisms of bordisms. Also we should note that the algebro-geometric realization of Khovanov homology in Cautis-Kamnitzer \[5\] is also gauge-theoretic, but algebro-geometric in nature. More recently, Witten \[49\] has conjectured a gauge-theoretic interpretation of Khovanov homology.

Technically, the starting point of our constructions is the observation that given a three-dimensional bordism containing a tangle whose components are labelled by conjugacy classes of a special unitary group, the set of flat bundles\(^1\) that extend over the bordism defines a *formal* Lagrangian correspondence between the moduli spaces of flat bundles associated to the incoming and outgoing marked, labelled boundary components. In good situations, our previous work \[44\] associates to such a correspondence a functor between the Donaldson-Fukaya categories of the symplectic moduli spaces associated to the boundary components. In particular, one could fix Lagrangian submanifolds of these symplectic moduli spaces to obtain a quilted Floer homology group which serves as a putative link invariant.

\(^1\) Note that we use the language of “sets of flat bundles” loosely – see Section 3 for precise notions involving connections.
The first problem with this naive construction is that the moduli spaces of flat bundles over surfaces are in general not even smooth, let alone monotone as required for quilt invariants without Novikov coefficients or figure eight correction terms. We resolve this by making admissibility assumptions on the number of labels and conjugacy classes, which by Proposition 3.8 and Theorem 3.13 guarantee smooth monotone symplectic manifolds. The second problem with the construction is that the Lagrangian correspondence is in general a singular subset of the product. Here the key realization (and birthplace of quilted Floer theory) was that one can decompose the bordism into *elementary bordisms-with-tangles* (pieces admitting Morse functions with at most one critical point either on the bordism or the tangle) for which the associated Lagrangian correspondences are smooth and even monotone. (In fact, it suffices to break into pieces whose critical points are of the same index, which aligns with the point of view in the Atiyah-Floer conjecture and Heegaard Floer homology.) A decomposition into elementary bordisms-with-tangles is obtained by choosing a Morse function on the bordism such that the maxima resp. minima are the outgoing resp. incoming surfaces, and all critical points have different values. Then decomposition at level sets between the critical values yields a sequence of Lagrangian correspondences.

In Section 2 we phrase this strategy as Cerf decomposition in a topological tangle category, identifying Cerf moves between different decompositions in Theorem 2.4, and functor to the symplectic category for which the blueprint is given in Theorem 2.5. In Section 3 we then show that the moduli spaces of flat connections with admissible holonomy labels fit into this blueprint. In particular, the sequence of Lagrangian correspondences obtained as sketched above is independent of the choice of decomposition up to an equivalence relation generated by embedded composition of Lagrangian correspondences.

In Section 4 we introduce a suitable notion of Donaldson-Fukaya category adapted to the moduli spaces of flat bundles under consideration. Recall here that in general the objects in a Donaldson-Fukaya category are Lagrangians submanifolds equipped with some additional data, and that the morphism spaces are Lagrangian Floer cohomology groups. The slight twist here is that the minimal Maslov numbers of the Lagrangians are two, so that in general the Floer differential does not square to zero. However, in this monotone case one has an invariant in the derived category of matrix factorizations; in fact this category already appears in Khovanov-Rozansky [18]. In particular, one has a good notion of Donaldson-Fukaya category in which the morphism spaces are not groups but rather matrix factorizations.

In Section 5 we combine the constructions of Sections 2,3,4 to obtain a category-valued field theory, or rather, a functor from our tangle categories to (categories, isomorphism classes of functors). The main point is that the equivalence of generalized Lagrangian correspondences proved in Section 3 combines with the results of [44] to show that the resulting functor is independent up to isomorphism of the decomposition into elementary pieces. This section also contains an extension to graphs, needed for a surgery exact triangle, and an application of the field theories to the symplectic mapping class group of the moduli spaces of flat bundles:
Theorem 1.2. (Non-triviality of twists on moduli spaces of bundles on punctured spheres) Let $X$ be a two-sphere, $x \subset X$ an odd number of at least five marked points, $M(X,x)$ the moduli space of flat $SU(2)$-bundles with traceless holonomies on $X - x$, and $\varphi : M(X,x) \to M(X,x)$ the symplectomorphism induced by a full twist around two markings. Then $\varphi$ is not Hamiltonian isotopic to the identity.

We thank P. Seidel for encouragement and for sharing his ideas. We also thank R. Rezazadegan for helpful comments.

The purpose of this paper is to provide more details on a draft that we have circulated since 2007. While the first author wishes to note that readers should not expect her specific level of rigour in the present version, she has agreed to publication of this version in the hope that its availability will be useful to the field.

2. Field theory for tangles

In this section we introduce various notions and constructions of (topological) field theories for tangles. Roughly speaking a field theory is a functor from a cobordism category to some other category. In Section 2.1 we use embedded bordisms in cylinders to construct a category of tangles. Section 2.2 discusses Cerf decompositions in this category and shows how to use them in the construction of general field theories. Section 2.3 then specializes this construction to a symplectic target category.

2.1. The tangle category. Our language for topological field theories for tangles adapts that in Lurie [24], rephrasing the earlier definition of Atiyah.

Definition 2.1. (a) (Marked surfaces) A marking of a surface $X$ is a collection $x = \{x_1, \ldots, x_n\} \subset X$ of distinct, oriented points for some non-negative integer $n$. By comparing with the canonical orientation of a point one obtains an orientation function $\epsilon : x \to \{\pm 1\}$. A marked surface is a tuple $(X, x)$ of a compact, oriented surface $X$ equipped with a marking $x$.

(b) (Tangles) A tangle from $(X_-, x_-)$ to $(X_+, x_+)$ is a tuple $(Y, K, \phi)$ consisting of

(i) a compact oriented 3-manifold-with-boundary $Y$;
(ii) a diffeomorphism $\phi : \partial Y \to \overline{X}_- \cup X_+$ where $\overline{X}_-$ denotes the manifold $X_-$ with reversed orientation;
(iii) a compact oriented 1-dimensional submanifold $K \subset Y$ meeting the boundary transversally in $\partial K = K \cap \partial Y$, so that $\phi$ restricts to an orientation preserving identification $\partial K \cong \overline{x}_- \cup x_+$ where $\overline{x}_-$ denotes the marking $x_-$ with reversed orientation.

(c) An equivalence between two tangles $(Y_0, K_0, \phi_0) \simeq (Y_1, K_1, \phi_1)$ from $(X_-, x_-)$ to $(X_+, x_+)$ is an orientation-preserving diffeomorphism $\psi : Y_0 \to Y_1$ with $\psi(K_0) = K_1$ and $\phi_1 \circ \psi|_{\partial Y} = \phi_0$.

(d) (Tangle category) The tangle category $\text{Tan}$ is the category whose

(i) objects are marked surfaces;
(ii) morphisms are equivalence classes of tangles $[Y, K, \phi]$;
(iii) composition is defined by gluing: if \((Y_0, K_0, \phi_{01})\) is a tangle from \((X_0, z_0)\) to \((X_1, z_1)\) and \((Y_1, K_1, \phi_{12})\) is a tangle from \((X_1, z_1)\) to \((X_2, z_2)\) then we choose collar neighborhoods \((X_1 \times (-\epsilon, 0), (z_1 \times (-\epsilon, 0))) \rightarrow (Y_0, K_0)\) resp. \((X_1 \times (0, \epsilon), (z_1 \times (0, \epsilon))) \rightarrow (Y_1, K_1)\) and define the composition \((Y_0, K_0, \phi_{01}) \circ (Y_1, K_1, \phi_{12})\) to be the union
\[
(Y_0, K_0) \cup (X_1, z_1) \times (-\epsilon, \epsilon) \cup (Y_1, K_1)
\]
equipped with the diffeomorphism of the boundary to \((X_0, z_0) \sqcup (X_2, z_2)\) induced by \(\phi_{01}\) and \(\phi_{12}\);
(iv) the identity for \((X, x)\) is the equivalence class of the cylindrical bordism \([-1, 1] \times X, [-1, 1] \times x\] equipped with the obvious identification of the boundary two copies of \((X, x)\). Composition is independent, up to equivalence, of the choice of collar neighborhood and representatives, since any two collar neighborhoods are isotopic and the equivalence class of a composition of representatives is denoted
\[
[(Y_0, K_0, \phi_{01})] \circ [(Y_1, K_1, \phi_{12})] = [(Y_0, K_0, \phi_{01}) \circ (Y_1, K_1, \phi_{12})].
\]
(e) (Labelled tangles) Let \(B\) be a set, which we call a set of labels. A decorated surface resp. tangle is a marked surface \((X, z)\) resp. tangle \((Y, K, \phi)\) equipped with a labelling of the components \(z \rightarrow B\) resp. \(\pi_0(K) \rightarrow B\).
(f) (Cylindrical tangles) Let \(X\) be a fixed compact, oriented 3-manifold. A \(X\)-cylindrical tangle is a tangle in a cylindrical bordism \([-1, 1] \times X\). Cylindrical tangles with fixed \(X\) form a category \(\text{Tan}(X)\) by using the composition law described above and identifying the gluing of the two intervals \([-1, 1]\) with \([-1, 1]\).

Our field theories fit into the language of topological field theories, which roughly speaking are functors from cobordism categories equipped with additional data.

**Definition 2.2.** (Field theories) Let \(X\) be a compact oriented surface and let \(C\) be a category. A \(C\)-valued field theory for cylindrical tangles in \(X\) is a functor \(\Phi : \text{Tan}(X) \rightarrow C\).

2.2. Cerf theory for tangles. Field theories for tangles can be constructed by decomposition into elementary tangles as follows. 

**Definition 2.3.** (Morse datum) A Morse datum for a tangle \((Y, K, \phi)\) from \((X_-, z_-)\) to \((X_+, z_+)\) consists of a pair \((f, b)\) of
(i) a Morse function \(f : Y \rightarrow \mathbb{R}\) that restricts to a Morse function \(f|_K : K \rightarrow \mathbb{R}\), and
(ii) an ordered tuple \(b = (b_0 < b_1 < \ldots < b_m) \in \mathbb{R}^{m+1}\)
such that
(i) \(X_- \cong f^{-1}(b_0)\) and \(X_+ \cong f^{-1}(b_m)\) are the sets of minima resp. maxima of \(f\),
(ii) each level set \(f^{-1}(b)\) for \(b \in \mathbb{R}\) is connected, that is, \(f\) has no critical points of index 0 or 3,
(iii) \( f \) has distinct values at the critical points of \( f \) and \( f|_K \), i.e. it induces a bijection \( \text{Crit}(f) \cup \text{Crit}(f|_K) \to f(\text{Crit}(f) \cup \text{Crit}(f|_K)) \) between critical points and critical values, and

(iv) \( b_0, \ldots, b_m \in \mathbb{R} \setminus f(\text{Crit}(f) \cup \text{Crit}(f|_K)) \) are regular values of \( f \) and \( f|_K \) such that each interval \( (b_{i-1}, b_i) \) contains at most one critical value of either \( f \) or \( f|_K \).

In the special case \( Y = [b_-, b_+] \times X \), we say that \((f, b)\) is a cylindrical Morse datum for a tangle \((Y, K, \phi)\) if \( \partial_t f(x, t) > 0 \) for all \((x, t) \in Y \), and hence each level set is diffeomorphic to \( X \).

(b) (Cerf decomposition) The Cerf decomposition of a tangle \((Y, K, \phi)\) induced by a Morse datum \((f, b)\) is the sequence

\[
(Y_i := f^{-1}([b_{i-1}, b_i]), \quad K_i := Y_i \cap K, \phi_i)_{i=1, \ldots, m}
\]

of elementary tangles between the connected level sets

\[
X_i := Y_i \cap Y_{i+1} = f^{-1}(b_i), \quad X_i = K_i \cap K_{i+1} = f^{-1}(b_i) \cap K
\]

and obvious identifications of the boundary \( \phi_i \). Here we have \( X_0 \cong X_- \) and \( X_m \cong X_+ \) via the restriction of \( \phi \), \( \partial Y_i = X_{i-1} \cup X_i \). The sequence \((Y_i, K_i, \phi_i)_{i=1, \ldots, m}\) corresponds to the decomposition

\[
Y = Y_1 \cup X_1 \cup X_2 \cup X_3 \cup \cdots \cup X_{m-1} Y_m, \quad K = K_1 \cup X_1 \cup K_2 \cup X_2 \cup \cdots \cup X_{m-1} K_m.
\]

In the special case \( Y = [b_-, b_+] \times X \), a cylindrical Cerf decomposition of the tangle \( K \) is a Cerf decomposition induced by a cylindrical Morse datum.

(c) (Elementary tangles) A tangle \((Y, K, \phi)\) is a

(i) elementary tangle if \((Y, K, \phi)\) admits a Cerf decomposition with a single piece, and

(ii) an elementary cylindrical tangle if \((Y, K, \phi)\) admits a Cerf decomposition with a single piece and no critical points on \( Y \). That is, \( Y \) is a cylindrical bordism and \( f : Y \to \mathbb{R} \) is a Morse function without critical points and the restriction \( f|_K \) has at most one critical point on \( K \).

Thus a cylindrical Cerf decomposition is a decomposition of the trivial bordism \( Y = [b_-, b_+] \times X \) into cylindrical bordisms \( Y_1 \cup X_1 \cup X_2 \cup X_3 \cup \cdots \cup X_{m-1} Y_m \) which decomposes the tangle \( K = K_1 \cup X_1 \cup K_2 \cup X_2 \cup \cdots \cup X_{m-1} K_m \) into elementary cylindrical tangles \((Y_i, K_i)\).

(d) (Diffeomorphism equivalences) A diffeomorphism equivalence between a Cerf decomposition of \((Y, K, \phi)\) from \((X_-, \overline{X}_-)\) to \((X_+, \overline{X}_+)\) given by a sequence \((Y_1, K_1, \phi_1), \ldots, (Y_m, K_m, \phi_m)\) and a Cerf decomposition of \((Y', K', \phi')\) from \((X_-, \overline{X}_-)\) to \((X_+, \overline{X}_+)\) given by \((Y'_1, K'_1, \phi'_1), \ldots, (Y'_m, K'_m, \phi'_m)\) of the same length \( m \) is a collection of orientation-preserving diffeomorphisms \((\psi_1 : (Y_1, K_1) \to (Y'_1, K'_1))_{i=1, \ldots, m}\) such that \( \psi_1 \) resp. \( \psi_m \) induces the identity on \( X_- \) resp. \( X_+ \).

The following is a special case of Cerf theory, for the special case of cylindrical Cerf decompositions.

**Theorem 2.4.** (Cerf theory for tangles) Let \((Y = [-1, 1] \times X, K, \phi)\) be a cylindrical tangle. Then any two cylindrical Cerf decompositions of \((Y, K, \phi)\) are related by a finite sequence of the following moves and diffeomorphism equivalences:
(a) (Critical point cancellation) Two elementary tangles \((Y_i, K_i, \phi_i)\) and \((Y_{i+1}, K_{i+1}, \phi_{i+1})\), which carry a local minimum resp. local maximum, both of which lie on the same strand of \(K \cap (Y_i \cup Y_{i+1})\), are replaced by the elementary tangle \((Y_i, K_i, \phi_i) \circ (Y_{i+1}, K_{i+1}, \phi_{i+1})\) which admits a Morse function with no critical point;

(b) (Critical point reversal) Two elementary tangles \((Y_i, K_i, \phi_i), (Y_{i+1}, K_{i+1}, \phi_{i+1})\) which carry critical points of index \(k\) and \(l\) on strands whose intersection with \((X_i, x_i)\) is disjoint, are replaced by two elementary tangles of index \(l\) and \(k\) such that \((Y_i, K_i, \phi_i) \circ (Y_{i+1}, K_{i+1}, \phi_{i+1})\) is diffeomorphism equivalent to \((Y'_i, K'_i, \phi'_i) \circ (Y'_{i+1}, K'_{i+1}, \phi'_{i+1})\);

(c) (Cylinder gluing) Two elementary tangles \((Y_i, K_i, \phi_i), (Y_{i+1}, K_{i+1}, \phi_{i+1})\), one of which is cylindrical, are replaced by the composition \((Y_i, K_i, \phi_i) \circ (Y_{i+1}, K_{i+1}, \phi_{i+1})\).

See Figures 2 and 3 for depictions of the first two moves.

**Figure 2. Critical point cancellation**

**Sketch of proof.** The proof follows from an examination of a generic homotopy between cylindrical Morse functions defining the two Cerf decompositions. Let \((f_j, b_j), j = 0, 1\) be cylindrical Morse data for a cylindrical tangle \((Y, K, \phi)\). Let \(f_s = (1 - s)f_0 + sf_1, s \in [0, 1]\) be the linear interpolation between \(f_0\) and \(f_1\). Then \(\partial_t f_s > 0\) for all \(s \in [0, 1]\). Consider the family \(f_s|_K\). Since \(K \subset [b_-, b_+] \times X\) is a submanifold with boundary, \(f_s|_K\) has positive resp. negative normal derivative at \(\bar{x}_-\) resp. \(\bar{x}_+\). Hence \((f_s|_K)\) has singularities or critical points only on a compact set in the interior of \(K\).
Next we apply Cerf theory to the homotopy restriction to the tangle. By Cerf theory, see for example [6] and [9], after replacing \( f_{s|K} \) with a perturbation we may assume that \( f_{s|K} \) is a Morse function injective on its critical set for all but finitely many values of \( s \in [0, 1] \), and at those values at most one of the following happen: the values of \( f_{s|K} \) at two Morse singularities coincide or a birth/death singularity occurs. Since \( K \) is a smooth family of functions on \( Y \), any such perturbation has an extension to a smooth family of functions on \( Y \) with the property that \( \partial_{t} f_{s} > 0 \) for all \((x, t) \in Y \) and \( s \in [0, 1] \). So this homotopy has only finite many values \( c_1 < \ldots < c_n \) for which \( f_{c_j} \) does not satisfy (a-c) in Definition 2.4. Each of these has either a birth/death singularity (corresponding to critical point cancellation or creation) or two critical values coinciding (corresponding to reversing order of critical points) on \( K \).

Figure 3. Critical point reversal

By Theorem 2.4, in order to construct field theories for cylindrical tangles it suffices to construct the theory on elementary tangles and check that the Cerf moves are satisfied.

**Theorem 2.5.** (Field theories for tangles via elementary tangles) Let \( C \) be a category and \( X \) a compact oriented surface. Given a partially defined functor \( \Phi \) from \( \text{Tan}(X) \) to \( C \) which associates

(a) to each marking \( \underline{x} \) of \( X \), an object \( \Phi(\underline{x}) \) of \( C \);
(b) to each equivalence class of elementary cylindrical tangles \( (Y, K, \phi) \) from \( (X, \underline{x}_{-}) \) to \( (X, \underline{x}_{+}) \), a morphism \( \Phi([[Y, K, \phi]]) \) from \( \Phi(\underline{x}_{-}) \) to \( \Phi(\underline{x}_{+}) \);

satisfying the Cerf relations

(a) If \((Y, K, \phi) = ([−1, 1] \times X, [−1, 1] \times \underline{x}, \phi) \) is a trivial tangle, then \( \Phi([[Y, K, \phi]]) \) is the identity.
(b) If \((Y_1, K_1, \phi_1) \) from \( \underline{x}_0 \) to \( \underline{x}_1 \) and \((Y_2, K_2, \phi_2) \) from \( \underline{x}_1 \) to \( \underline{x}_2 \) are elementary cylindrical tangles such that \(([Y_1, K_1, \phi_1]) \circ ([Y_2, K_2, \phi_2]) \) is equivalent to a cylindrical tangle via critical point cancellation, then

\[
\Phi([[Y_1, K_1, \phi_1]]) \circ \Phi([[Y_2, K_2, \phi_2]]) = \Phi(([Y_1, K_1, \phi_1]) \circ ([Y_2, K_2, \phi_2]));
\]

(c) If \((Y_1, K_1, \phi_1), (Y_2, K_2, \phi_2) \) and \((Y'_1, K'_1, \phi'_1), (Y'_2, K'_2, \phi'_2) \) are elementary cylindrical tangles related by critical point reversal, then

\[
\Phi([[Y_1, K_1, \phi_1]]) \circ \Phi([[Y_2, K_2, \phi_2]]) = \Phi([[Y'_1, K'_1, \phi'_1]]) \circ \Phi([[Y'_2, K'_2, \phi'_2]]);
\]

(d) If \((Y_1, K_1, \phi_1), (Y_2, K_2, \phi_2) \) are composable elementary tangles, one of which is cylindrical, then

\[
\Phi([[Y_1, K_1, \phi_1]]) \circ \Phi([[Y_2, K_2, \phi_2]]) = \Phi([[Y_1, K_1, \phi_1]) \circ ([Y_2, K_2, \phi_2])]
\]
then there is a unique $C$-valued field theory extending $\Phi$.

In other words, to define a field theory for tangles it suffices to define the morphisms for elementary bordisms and prove the Cerf relations.

2.3. Symplectic-valued field theories. In this section we specialize to field theories with values in the symplectic category. A symplectic-valued field theory for tangles in particular assigns to any tangle a sequence of Lagrangian correspondences, up to equivalence, as in [44].

**Definition 2.6.** (Geometric composition of Lagrangian correspondences) Let $M_j$ be symplectic manifolds with symplectic forms $\omega_{M_j}$ for $j = 0, 1, 2$.

(a) A **Lagrangian correspondence** from $M_1$ to $M_2$ is a Lagrangian submanifold $L \subset M_1^- \times M_2$ with respect to the symplectic structure $-\omega_{M_1} \oplus \omega_{M_2}$.

(b) The **geometric composition** of Lagrangian correspondences $L_01 \subset M_0^- \times M_1$ and $L_{12} \subset M_1^- \times M_2$ is the point set

$$L_{01} \circ L_{12} := \pi_{M_0 \times M_2}((L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2)) \subset M_0 \times M_2.$$ 

(c) A geometric composition is called **transverse** if the intersection is transverse (and hence smooth) and **embedded** if, in addition, the restriction of the projection $\pi_{M_0 \times M_2}$ is an injection of the smooth intersection, hence an embedding. In that case the image is a smooth Lagrangian correspondence $L_{01} \circ L_{12} \subset M_0^- \times M_2$.

**Definition 2.7.**

(a) **(Generalized correspondences)** Let $M_-, M_+$ be symplectic manifolds. A **generalized Lagrangian correspondence** $L$ from $M_-$ to $M_+$ consists of

(i) a sequence $N_0, \ldots, N_r$ of any length $r \geq 0$ of symplectic manifolds with $N_0 = M_-$ and $N_r = M_+$,

(ii) a sequence $L_{01}, \ldots, L_{(r-1)r}$ of compact Lagrangian correspondences with $L_{(j-1)j} \subset N_{j-1}^- \times N_j$ for $j = 1, \ldots, r$.

(b) **(Algebraic composition)** Let $M, M', M''$ be symplectic manifolds. The **algebraic composition** of generalized Lagrangian correspondences $L$ from $M$ to $M'$ and $L'$ from $M'$ to $M''$ is given by concatenation

$$L \# L' := (L_1, \ldots, L_m, L'_1, \ldots, L'_{m'})$$

(c) **(Symplectic category)** The **symplectic category** $\text{Symp}^\#$ is the category whose

(i) objects are smooth symplectic manifolds

(ii) morphisms from an object $M_-$ to an object $M_+$ are generalized Lagrangian correspondences from $M_-$ to $M_+$ modulo the composition equivalence relation $\sim$ generated by

$$(\ldots, L_{(j-1)j}, L_{j(j+1)}, \ldots) \sim (\ldots, L_{(j-1)j} \circ L_{j(j+1)}, \ldots)$$

for all sequences and $j$ such that $L_{(j-1)j} \circ L_{j(j+1)}$ is transverse and embedded.
(iii) The composition of morphisms \( L \in \text{Hom}(M, M') \) and \( L' \in \text{Hom}(M', M'') \) for symplectic manifolds \( M, M', M'' \) is defined by
\[
L \circ L' := [L \# L'] \in \text{Hom}(M, M'').
\]

(iv) The identity \( 1_M \in \text{Hom}(M, M) \) is the equivalence class of the empty sequence of length zero, or equivalently, the equivalence class \( 1_M := [\Delta_M] \) of the diagonal \( \Delta_M \subset M ^- \times M \).

**Definition 2.8.**

(a) (Symplectic-valued field theories) Let \( \text{Symp} \) denote the category of (symplectic manifolds, symplectomorphisms). A symplectic-valued field theory for cylindrical tangles for a compact oriented surface \( X \) is a functor \( \Phi : \text{Tan}(X) \rightarrow \text{Symp} \).

(b) (Monotone symplectic manifolds and correspondences) A symplectic manifold \( (M, \omega) \) is monotone with monotonicity constant \( \tau > 0 \) if \( \tau c_1(M) = [\omega] \) in \( H^2(M) \). A Lagrangian submanifold \( L \subset M \) is monotone if \( 2 \int u^* \omega = \tau I(u) \) for all \( [u] \in \pi_2(M, L) \) where \( I(u) \) is the Maslov index.

(c) (Monotone symplectic category) Denote by \( \text{Symp}_\tau^\# \) the category whose objects are monotone symplectic manifolds with monotonicity constant \( \tau \) and whose morphisms are equivalence classes of compact simply-connected\(^2\) oriented monotone Lagrangian correspondences.

(d) (Monotone symplectic-valued field theories) A \( \tau \)-monotone field theory is a functor \( \Phi \) taking values in \( \text{Symp}_\tau^\# \).

3. Flat bundles on complements of tangles

In this section we construct a symplectic-valued field theory for a particular class of labelled tangle categories. For suitable choices of the labels, this field theory will be monotone. The basic construction is well-known: associated to any tangle there is a moduli space of flat bundles with fixed holonomies around the components, which if smooth defines a Lagrangian correspondence in the moduli spaces of flat bundles with fixed holonomies on the boundary. For elementary tangles, the correspondences are smooth, and we check that the Cerf relations hold.

### 3.1. Conjugacy classes.

In this section we review the parametrization of conjugacy classes in a connected, simply-connected Lie group. In essence, this is the classification of bundles on closed one-manifolds, since such bundles are classified by the conjugacy class of their holonomies. A possible reference for the following material is Pressley-Segal [34].

**Definition 3.1.** (Conjugacy classes) Let \( G \) be a compact, connected, simple, simply-connected Lie group with Lie algebra \( \mathfrak{g} \).

(a) (Weyl group) Let \( T \subset G \) be a maximal torus, its Cartan subalgebra by \( t \subset \mathfrak{g} \), and the rank by \( r = \text{rank}(G) := \text{dim} \, t \). Let \( N(T) \) denote the normalizer of \( T \), and \( W := N(T)/T \) is the Weyl group of \( G \). The action of \( W \) on \( T \) by conjugation induces an action on \( t^\vee \) denoted \( \mu \mapsto w\mu \) for \( w \in W, \mu \in t^\vee \).

\(^2\text{For convenience; alternatively one can impose further monotonicity conditions.}\)
(b) (Root space decomposition) The Lie algebra admits a direct sum decomposition $g = t \oplus \bigoplus_{i=1}^{l} g_{\alpha_i}$ where $g_{\alpha_i}$ are the two-dimensional root spaces corresponding to the roots $\pm \alpha_1, \ldots, \pm \alpha_l \in t^\vee$ defined up to sign, permuted by the action of $W$ on $t^\vee$.

(c) (Positive chamber) Let $t_+$ be a choice of positive chamber, that is, component of $t \backslash \cup \alpha \ker(\alpha)$; this fixes a choice of positive roots $\alpha_1, \ldots, \alpha_l$ by requiring $\alpha_j(t_+) \subset \mathbb{R}_{\geq 0}$. Let $\alpha_1, \ldots, \alpha_r$ be the simple roots, i.e. the minimal set such that $t_+ = \{ \xi | \alpha_j(\xi) \geq 0, j = 1, \ldots, r \}$.

(d) (Weyl alcove) Let $\alpha_0 \in \{ \alpha_1, \ldots, \alpha_l \}$ denote the highest root, defined as the unique positive root such that $\alpha_0(\xi) \geq \alpha_j(\xi)$ for all $\xi \in t_+$ and $j = 1, \ldots, l$.

(e) (Conjugacy classes) Conjugacy classes in $G$ are parametrized by the Weyl alcove $A = \{ \xi \in t_+ | \alpha_0(\xi) \leq 1 \}$.

We write the group as the disjoint union of conjugacy classes

$$G = \bigcup_{\mu \in A} C_{\mu}, \quad C_{\mu} = \{ g \exp(\mu) g^{-1} | g \in G \}.$$

A point $\mu \in A$ will be called a label.

(f) (Involution) Denote by $\ast$ the (possibly trivial) involution of the alcove defined by $\ast : A \rightarrow A$, $C_{\ast \mu} = C_{\mu}^{-1}$.

(g) (Vertices) For each $i = 1, \ldots, r$, let $\beta_i$ denote the unique vertex of $A$ such that $\alpha_j(\beta_i) = 0$ for all $i \in \{1, \ldots, r\} \setminus \{j\}$.

(h) (Fundamental coweights) Let

$$\Lambda = \exp^{-1}(1) \subset t$$

denote the coweight lattice. For each $i = 1, \ldots, r$ let $\omega_i^\vee \in \Lambda$ be the $i$-th fundamental coweight, that is, $\omega_i^\vee$ is the positive multiple of $\beta_i$ that generates $\Lambda \cap \mathbb{R}_{\geq 0} \beta_i$.

(i) (Coxeter class) Denote the half-sum of positive roots and the dual Coxeter number of $G$ [15, 6.1] by

$$\rho = \frac{1}{2} \sum_{i=1}^{l} \alpha_i \in t^\vee, \quad c = \langle \rho, \alpha_0 \rangle + 1 \in \mathbb{Z}_+$$

using the basic inner product [34, p. 49] for which $\langle \alpha_0, \alpha_0 \rangle = 2$. We identify $t$ with $t^\vee$ using the basic inner product. The image of $\rho/c$ in $t$ lies in the interior of $A$, is independent of the inner product used, and invariant under the involution in Definition 3.1; the corresponding conjugacy class is the Coxeter class.

Example 3.2. (Conjugacy classes for special unitary groups) If $G = SU(r + 1)$ then $c = r + 1$. The standard identification of the Cartan subalgebra is

$$t \cong \{ (\xi_1, \ldots, \xi_{r+1}) \in \mathbb{R}_+^{r+1} | \xi_1 + \ldots + \xi_{r+1} = 0 \}.$$
With the standard unit vectors \( e_i \in \mathbb{R}^{r+1} \) we have simple roots \( \alpha_i = e_{i+1} - e_i \) for \( i = 1, \ldots, r \), highest root \( \alpha_0 = e_1 - e_{r+1} \), and half-sum of positive roots
\[
\rho = (-r, -r - 2, \ldots, r - 2, r)/2.
\]
The Weyl alcove is
\[
\mathfrak{A} \cong \{ (\lambda_1, \ldots, \lambda_{r+1}) \in \mathfrak{t} \mid \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{r+1} \geq \lambda_1 - 1 \}.
\]
The vertices of \( \mathfrak{A} \) are
\[
\beta_i = \frac{i}{r+1}(e_1 + \ldots + e_i) + \frac{i-r-1}{r+1}(e_{i+1} + \ldots + e_{r+1}), \quad i = 1, \ldots, r
\]
and the fundamental coweights are \( \omega_i^\vee = (r+1)\beta_i \). We will use this enumeration of simple roots, vertices, and fundamental coweights whenever working with \( SU(r+1) \).

**Remark 3.3.** (Simply-connectedness of conjugacy classes) Recall that if \( G \) is a compact 1-connected Lie group, then the conjugacy classes of \( G \) are simply-connected. To see this, let \( \mathcal{A}(S^1) = \Omega^1(S^1, g) \) denote the space of connections on the trivial bundle on the circle. Let \( \theta \) be the coordinate on \( S^1 \) so that \( d\theta \in \Omega^1(S^1) \). The holonomy fibration \( \Omega G \to \Omega^1(S^1, g) \to G \) restricts to a fibration \( \Omega G \to LG \cdot \mu \theta \to C_\mu \) where \( LG \cdot \mu \) is the orbit under the loop group \( LG = \text{Map}(S^1, G) \) acting affinely on \( \mu \theta \in \Omega^1(S^1, g) \). Since \( \Omega G \) is connected and \( LG \cdot \mu \) is simply-connected [34, Section 8.6], it follows that \( C_\mu \) is simply-connected as well.

### 3.2. Moduli spaces via holonomy

We choose to describe the moduli spaces via representations of the fundamental group, rather than gauge theory as in [46].

**Definition 3.4.**

(a) (Loops around strands) Let \( X \) be a compact, connected, oriented manifold, possibly with boundary, and let \( K \subset X \) be an oriented, embedded submanifold of codimension 2. Let \( K_1, \ldots, K_n \) denote the connected components of \( K \). For each \( j = 1, \ldots, n \) let \( \gamma_j : S^1 \to X \setminus K \) be a small loop around \( K_j \), so that the induced orientation on the normal bundle of \( K_j \) agrees with that induced by the orientations of \( K_j \) and \( X \). Each \( \gamma_j \) defines a conjugacy class \( [\gamma_j] \subset \pi_1(X \setminus K) \). Here we always implicitly fix a base point in the definition of the fundamental group \( \pi_1(X \setminus K) \).

(b) (Moduli of flat bundles with fixed holonomies) For labels \( \mu = (\mu_1, \ldots, \mu_n) \in \mathfrak{A}^n \) let \( M(X, K) \) denote the moduli space of flat \( G \)-bundles on \( X \setminus K \) whose holonomy around \( \gamma_j \) lies in the conjugacy class \( C_{\mu_j} \). We call the label of the component \( K_j \) the label of the component \( K_j \). The moduli space \( M(X, K) \) of connections with fixed holonomy has a description in terms of representations of the fundamental group, which we take as a definition:
\[
M(X, K) := \{ \varphi \in \text{Hom}(\pi_1(X \setminus K), G) \mid \varphi([\gamma_j]) \subset C_{\mu_j}, \forall j \}/G.
\]

Here \( G \) acts by conjugation. In case \( X \) is not connected, this definition is replaced by the product of moduli spaces for the connected components of \( X \).

**Remark 3.5.** (Effect of orientation change of the tangle) Changing the orientation of a component \( K_j \) (i.e. of \( \gamma_j \)) corresponds to changing the label \( \mu_j \) by the involution
* of the alcove $\mathfrak{A}$ in Definition 3.1. That is, if $\tilde{K}$ denotes the tangle obtained by changing the orientation on $K_j$ and $\tilde{\mu}$ is the set of labels obtained by replacing $\mu_j$ with $\star \mu_j$ then there is a canonical homeomorphism $M(X, K) \to M(X, \tilde{K})$.

**Remark 3.6.** (Alternative description in the finite-order case) If the conjugacy classes $C_{\tilde{\mu}} = C_{\mu_1} \times \ldots \times C_{\mu_n}$ each have finite order (as in all our examples), then one can identify the moduli space $M(X, K)$ with the moduli space of flat bundles on an orbibundle over $X$, see [27, 20, 28] for two- and four-dimensional cases. However, we will avoid using the equivariant description and instead check explicitly, in specific presentations, the smoothness of those moduli spaces that enter our constructions.

3.3. **Moduli spaces of bundles for surfaces.** The key feature of moduli spaces of bundles on compact, oriented surfaces is their symplectic nature. Below we review the description of the symplectic structure in the holonomy description.

**Remark 3.7.** Let $X$ be a compact, connected, oriented surface of genus $g$, and let $\underline{\pi} = \{x_1, \ldots, x_n\}$ be a marking.

(a) (Presentation of the fundamental group) Define $\epsilon_j = \pm 1$ depending on whether the orientation of $x_j$ agrees with the standard orientation of a point. The fundamental group $\pi_1(X \setminus \underline{\pi})$ has standard presentation

$$\pi_1(X \setminus \underline{\pi}) \cong \langle \alpha_1, \ldots, \alpha_{2g}, \gamma_1, \ldots, \gamma_n \mid \prod_{j=1}^{g} [\alpha_{2j}, \alpha_{2j+1}] \prod_{j=1}^{n} \gamma_j^{\epsilon_j} = 1 \rangle,$$

where $\gamma_j$ is a loop around $x_j$, oriented corresponding to $\epsilon_j$.

(b) (Presentation of the moduli space of flat bundles) Let $\underline{\mu} \in \mathfrak{A}^n$ be a set of labels for $\underline{\pi}$. The moduli space of flat $G$-bundles with fixed holonomy can be described in terms of a standard presentation of $\pi_1(X \setminus \underline{\pi})$ by

$$M(X, \underline{\pi}) = \{ \varphi \in \text{Hom}(\pi_1(X \setminus \underline{\pi}), G) \mid \varphi(\gamma_j) \subset C_{\mu_j} \forall j \} / G$$

(c) (Symplectic form on the moduli space) For any $X, \underline{\pi}, \underline{\mu}$ the space $M(X, \underline{\pi})$ can be realized as the moduli space of flat connections on the trivial $G$-bundle over $X \setminus \underline{\pi}$ with fixed holonomies around $\underline{\pi}$ (see [27, 28]) and as such has a symplectic form. The form can be described explicitly in the holonomy description [1] as follows. First suppose that $X$ is a surface of genus $g$, without markings. Let $\theta, \bar{\theta} \in \Omega^1(G, g)^G$ be the left and right-invariant Maurer-Cartan forms. Define a form $\omega_1$ on $G^2$ by

$$\omega_1 \in \Omega^2(G^2), \quad \omega_1 = (l^* \theta \wedge r^* \bar{\theta}) / 2 + (l^* \bar{\theta} \wedge r^* \theta) / 2.$$
where \( l, r : G^2 \to G \) are the projections on the first and second factor. For \( g \geq 1 \) define two-forms \( \omega_g \in \Omega^2(G^{2g}) \) inductively by

\[
\omega_g = \omega_{g_1} + \omega_{g_2} + \langle \Phi^*_g \theta \wedge \Phi^*_g \bar{\theta} \rangle / 2
\]

where \( g = g_1 + g_2 \) is any splitting with \( g_1, g_2 \geq 1 \), \( \Phi_{g_i} : (G^2)^{g_i} \to G \) the product of commutators, and we omit pull-backs to the factors of \( G^{2g} \cong \prod G^{2g_1} \times G^{2g_2} \) from the notation. A theorem of Alekseev-Malkin-Meinrenken [1, Theorem 9.3], extending earlier work of Weinstein, Jeffrey, and Karshon, states that the restriction of \( \omega_g \) to the identity level set of \( \Phi_g \) descends to the symplectic form on the locus of irreducible connections in \( M(X, \underline{x}) \). More generally, suppose that \( (X, \underline{x}) \) is a labelled surface. A symplectic structure on \( M(X, \underline{x}) \) can be defined as follows. For any label \( \mu \in \mathfrak{A} \), define a 2-form \( \omega_\mu \) on the conjugacy class \( C_\mu \)

\[
\omega_\mu(v_\xi(v_\eta(g))), g \in C_\mu, \quad \xi, \eta \in g
\]

where \( v_\xi, v_\eta \in \text{Vect}(C_\mu) \) are the generating vector fields for \( \xi, \eta \). Define two-forms \( \omega_{g, \mu} \in \Omega^2((G^{2g} \times C_\mu)) \) inductively by \( \omega_{g, \theta} = \omega_g, \omega_{0, \{\mu\}} = \omega_\mu \), and for any splitting \( g = g_1 + g_2, \mu = \mu_1 \cup \mu_2 \), where for each \( j = 1, 2 \) either \( g_j > 0 \) or \( \mu_j \) is non-empty, setting

\[
\omega_{g, \mu} = \omega_{g_1, \mu_1} + \omega_{g_2, \mu_2} + \langle \Phi^*_g \theta \wedge \Phi^*_g \bar{\theta} \rangle / 2
\]

where

\[
\Phi_{g_1, \mu_1} : G^{2g_1} \times C_{\mu_1} \to G, \quad (h, (c_\mu)_{\mu \in \mu_1}) \mapsto \Phi_{g_1, \mu_1}(h) \prod_{\mu \in \mu_1} c_\mu.
\]

By [1, page 27], the restriction of \( \omega_{g, \mu} \) to the identity level set of \( \Phi_{g, \mu} \) descends to the symplectic form on the locus of irreducible connections in \( \tilde{M}(X, \underline{x}) \).

The rest of this section is a rather encyclopedic description of sufficient conditions for the moduli spaces to be smooth, monotone, spin etc.

**Proposition 3.8.** (Sufficient conditions for smoothness) Let \( (X, \underline{x}) \) be a marked surface of genus \( g \) with \( n \) markings \( \underline{x} \), and let \( C_\mu \) denote the corresponding collection of conjugacy classes. Each of the following is a sufficient condition for \( M(X, \underline{x}) \) to be a smooth orbifold. Conditions (c-d) in fact imply that \( M(X, \underline{x}) \) is a manifold.

(a) (Finite automorphisms) Every flat \( G \)-connection on \( X \setminus \underline{x} \) in \( C_\mu \) has finite automorphism group.

(b) (Finite stabilizer) Every solution \( (a, b) \in G^{2g} \times C_\mu \) to \( \Phi(a, b) = 1 \) has finite stabilizer under the conjugacy action of \( G \).

(c) (Parity condition) \( G = SU(2) \), all labels are \( \mu_i = 1/4 \), and the number \( n \) of markings is odd.

(d) (Coprime condition) \( G = SU(r) \), all labels \( \mu_i \) are equal to the projection of \( \rho/c \) onto an edge of \( \mathfrak{A} \) containing \( \theta \), that is, \( \mu_i = \frac{1}{r} \beta_j \) for some \( j \in \{1, \ldots, r\} \), and \( 2 \sum_{i=1}^n \mu_i = \beta_d \mod \Lambda \) for some \( d \) coprime to \( r \). In this case we call the labels admissible.
(e) (Not-in-a-wall condition) For each tuple $w_1, \ldots, w_n \in W$ we have
\begin{equation}
\langle \sum_{i=1}^{n} w_i \mu_i, \omega_j \rangle \notin \langle \Lambda, \omega_j \rangle \quad \forall j = 1, \ldots, r
\end{equation}
where $\Lambda$ is the coweight lattice of (2).

Proof. Recall that a symplectic quotient is a smooth orbifold if the isotropy groups on the level set are finite. By [1], the space $M(X, \bar{x})$ can be realized as a symplectic quotient of the Hamiltonian $G$-manifold $\tilde{M}(X, \bar{x}) := G^{2g} \times C^\infty_\mu$ with group-valued moment map $\Phi : \tilde{M}(X, \bar{x}) \to G$ given by the product (6). (a) and (b) are immediate consequences, since the identity level set of the moment map is cut out transversally if and only if all stabilizers are discrete, by [1, Definition 2.2, Condition B3]. Item (e) describes the wall structure of the moment image of $M(X, \bar{x})$. For each tuple $w = (w_1, \ldots, w_n) \in W^n$ and subset $I \subset \{1, \ldots, l\}$ such that the span of $(\alpha_i)_{i \in I}$ is not all of $t^\vee \cong t$, define the wall corresponding to $w$ by
$$\Theta_{w, I} := \exp \left( \sum_{j=1}^{n} w_j \mu_j + \text{span}(\alpha_i)_{i \in I} \right).$$

Let $T^{X,\text{sing}}$ denote the singular values of $\Phi$ contained in $T$. We claim that $T^{X,\text{sing}} \subseteq \cup_{w,I} \Theta_{w, I}$. From Remark 3.3 follows that any orbit stratum in $C_{\mu_j}$ with infinite stabilizer group contains a $T$-fixed point in its closure, since the same is true for coadjoint orbits by equivariant formality of Hamiltonian actions [11, Appendix C]. Similarly, the closure of any orbit stratum in $G^{2g}$ with infinite stabilizer is equal to $H^{2g}$, for some subgroup $H$ containing $T^{2g}$. Now $T^{2g}$ maps to the identity under the product of commutators $\Phi$. Putting everything together, any orbit-type stratum $Y$ in $\tilde{M}(X, \bar{x})$ contains a $T$-fixed point $y$ in its closure, and such a fixed point has image contained in $\exp(\sum_{j=1}^{n} w_j \mu_j)$ for some $w \in W^n$. The tangent space $T_y Y$ is a sum of root spaces, and the assumption that the stabilizer of $Y$ is infinite implies that the span of the roots appearing in the sum is not all of $t^\vee$. The claim follows.

Part (c) is a special case of (d). Part (d) is a special case of (e) in the case of the special unitary group. Indeed, suppose that each $\mu_i$ is equal to $\frac{1}{2} \beta_j$ for some $j \in \{1, \ldots, r\}$, and $2 \sum_{i=1}^{n} \mu_i = \beta_d \mod \Lambda$ for some $d$ coprime to $r$. We have $\sum w_i \mu_i - \sum \mu_i \in \Lambda/2$. The pairing $(\beta_d/2, \omega_j)$ is never an integer for $r$ coprime to $d$ and $j = 1, \ldots, r - 1$. Hence $\langle \sum_{i=1}^{n} w_i \mu_i, \omega_j \rangle$ does not lie in $Z = \langle \Lambda, \omega_j \rangle \quad \forall j = 1, \ldots, r$ as claimed. \qed

Proposition 3.9. (Admissibility on one end implies admissibility on the other)
Suppose that $K \subset Y = [-1, 1] \times X$ is a tangle labelled with elements of $B = \{\beta_i/2, i = 1, \ldots, r\}$ from $X_-$ and $X_+ = \{+1\} \times X$ and the labels for $X_- \cap K$ are admissible in the sense of (d). Then the same hold for the induced labels of $X_+ \cap K$.

Proof. The labels on $X_\pm$ are the same except for labels which have disappeared due to critical point cancellation; by assumption the labels of two strands meeting in this way are opposite up to conjugacy and so do not contribute to the sum in (d). \qed

Remark 3.10. (Smoothness for generic labels) The moduli spaces of flat bundles for types other than $A$ have orbifold singularities for generic choices of holonomies $\mu$ and at least three markings. Indeed, for types other than $A$ the group $G$ has proper
semisimple subgroups \( H \) classified by the Borel-de Siebenthal algorithm \([4]\), which appear as centralizers of non-central elements of \( G \). The moduli space of flat \( G \)-bundles will contain \( H \)-bundles even generically, and the centers of \( H \) (which will be larger than the center of \( G \), since they are centralizers) are finite, disconnected subgroups creating orbifold singularities in the moduli space of \( G \)-bundles.

Sufficient conditions for monotonicity of the moduli space \( M(X, \underline{x}) \) are provided by \([29]\):

**Definition 3.11.** (Monotone labels) A label \( \mu \in \mathfrak{A} \) is monotone if it is a projection of the Coxeter element onto a face of the Weyl alcove, that is, \( \mu = \text{proj}_\sigma(c^{-1}\rho) \) for some face \( \sigma \) of the alcove \( \mathfrak{A} \), where \( \text{proj}_\sigma \) denotes orthogonal projection onto \( \sigma \).

**Example 3.12.** (Examples of monotone labels)

(a) (Rank two case) If \( G = SU(2) \), then \( c = 2 \) and we will identify \( \mathfrak{A} \cong [0, 1/2] \) by the map \((\lambda, -\lambda) \mapsto \lambda \) so that \( C_\mu \) consists of matrices with eigenvalues \( \exp(\pm 2\pi i \mu) \). We have \( \alpha_0 = 1 \) and \( \rho = 1/2 \). If \( \mu = 1/4 \), \( C_\mu \) consists of all traceless matrices. The monotone elements are 0, 1/2, 1/4. The unique standard element is 1/4.

(b) (Rank three case) If \( G = SU(3) \), we have \( \rho = \alpha_0 = (1, 0, -1) \). See Figure 4 for the monotone labels.

![Figure 4. Monotone conjugacy classes for SU(3)](image)

**Theorem 3.13.** (Sufficient conditions for monotonicity, \([29, \text{Theorem 4.2}]\)) Let \((X, \underline{x})\) be a marked surface with labels \( \underline{\mu} \). If each label \( \mu_j \) is monotone and \( M(X, \underline{x}) \) is smooth then \( M(X, \underline{x}) \) is monotone with monotonicity constant \( \tau^{-1} = 2c \), where \( c \) is the dual Coxeter number of \((3)\).

Finally we note that the natural action of the diffeomorphism group acts by symplectomorphisms:

**Definition 3.14.** (Marking-preserving mapping class group) Let \( \text{Diff}_+(X, \underline{x}) \) be the subgroup of orientation-preserving diffeomorphisms \( \varphi \in \text{Diff}_+(X) \) that preserve the marked points, orientations, and labels,

\[
\text{Diff}_+(X, \underline{x}) = \{ \varphi \in \text{Diff}_+(X) \mid \varphi(\underline{x}) = \underline{x}, \varphi^*\underline{\epsilon} = \underline{\epsilon}, \varphi^*\underline{\mu} = \underline{\mu} \}.
\]
Here we denote by \( \varepsilon \) and \( \mu \) the maps \( x \rightarrow \{\pm 1\}, x_i \mapsto \varepsilon_i \) and \( x \rightarrow \mathfrak{F}, x_i \mapsto \mu_i \). So for \( \varphi \in \text{Diff}^+(X) \) with \( \varphi(x_i) = x_j \) the conditions are \( \varepsilon_i = \varepsilon_j \) and \( \mu_i = \mu_j \). Let \( \text{Map}_+(X, x) \) be the quotient of \( \text{Diff}^+(X, x) \) by isotopy.

**Remark 3.15.** (Spherical braid group action) The action of \( \text{Map}_+(X, x) \) on \( \pi_1(X \setminus x) \) induces an action on \( M(X, x) \) by symplectomorphisms on the smooth stratum. See for example [1, Section 9.4] for a proof from the holonomy point of view. In particular, if \( X \) is a sphere and all labels are equal, \( \mu = (\mu, \ldots, \mu) \), then the spherical braid group \( \text{Map}_+(S^2, x) \) acts on \( M(X, x) \). Explicitly if \( \sigma_i \in \text{Map}_+(S^2, x) \) is the half-twist of \( x_i \) and \( x_{i+1} \) then for suitable choice of presentation of \( \pi_1(X \setminus x) \) we have

\[
\sigma_i[b_1, \ldots, b_n] = [b_1, \ldots, b_{i-1}, b_{i+1}, b_i b_{i+1} b_i b_{i+2}, \ldots, b_n].
\]

### 3.4. Moduli spaces for tangles

**Definition 3.16.** Let \( (Y, K, \phi) \) be a labelled tangle from \( (X_-, x_-) \) to \( (X_+, x_+) \).

(a) (Restriction to the boundary) Let \( K_1, \ldots, K_p \) be the connected components of \( K \) and fix labels \( \nu = (\nu_1, \ldots, \nu_p) \in \mathfrak{F}^p \) for \( K \). In Section 3.2 we defined the moduli space \( M(Y, K) \) of flat \( G \)-bundles on \( Y \setminus K \) with holonomy around \( K_j \) in \( C_{\nu_j} \). On the boundary the labels \( \nu \) induce labels \( \nu_\pm \in \mathfrak{F}^{n_\pm} \), given by \( \nu_j \) for \( \partial K_j \). Restriction to the boundary and pull-back under \( \phi \) define a map

\[
M(Y, K) \rightarrow M(X_-, x_-)^- \times M(X_+, x_+).
\]

More precisely, the inclusion of the boundary and a choice of paths between base points induces a map of fundamental groups that is well-defined up to conjugacy. This induces a dual map from the representation variety of the bordism to the product of representation varieties of its boundary components, which is independent of the choice of path.

(b) (Correspondences for tangles) For any labelled tangle \( (Y, K, \phi) \) we denote the image of (11) by

\[
L(Y, K, \phi) \subset M(X_-, x_-)^- \times M(X_+, x_+).
\]

**Lemma 3.17.** (Correspondences for compositions) Let \( (X_i, x_i) \) be marked surfaces for \( i = 0, 1, 2, \) and let \( (Y_{01}, K_{01}, \phi_{01}) \) resp. \( (Y_{12}, K_{12}, \phi_{12}) \) be tangle from \( (X_0, x_0) \) to \( (X_1, x_1) \) resp. from \( (X_1, x_1) \) to \( (X_2, x_2) \). Let \( \nu_{01} \) and \( \nu_{12} \) be labels for the bordisms with tangles such that they induce the same label \( \mu_i \) for \( (X_1, x_1) \). Then gluing provides a bordism with tangle \( (Y_{01} \circ Y_{12}, K_{01} \circ K_{12}) \) from \( (X_0, x_0) \) to \( (X_2, x_2) \) with labels \( \nu_{01} \circ \nu_{12} \). The induced labels \( \mu_{01} \) for \( x_0 \) and \( \mu_{12} \) for \( x_2 \) are the same as the ones induced from \( \nu_{01} \) and \( \nu_{12} \), and we have the equality of subsets of \( \text{Map}_+(X_0, x_0, \mu_{01}) \times \text{Map}_+(X_2, x_2, \mu_{12}) \)

\[
L((Y_{01}, K_{01}, \phi_{01}) \circ (Y_{12}, K_{12}, \phi_{12})) = L(Y_{01}, K_{01}, \phi_{01}) \circ L(Y_{12}, K_{12}, \phi_{12}).
\]

**Proof.** By the Seifert-van Kampen theorem we have an isomorphism (using a base point on \( X_1 \))

\[
\pi_1(Y_{01} \setminus K_{01} \circ K_{12}) \cong \pi_1(Y_{01} \setminus K_{01}) *_{\pi_1(X_1 \setminus x_1)} \pi_1(Y_{12} \setminus K_{12}).
\]
In particular, any representation on \( Y_{01} \setminus Y_{12} \setminus K_{01} \setminus K_{12} \) induces representations on both sides, whose restriction to \( X_1 \setminus x_1 \) agree. Conversely, any pair of representations on the two sides, whose restrictions to \( X_1 \setminus x_1 \) are conjugate, we can conjugate one of the sides so that the restrictions agree. This induces a representation on the glued space. □

Lemma 3.18. (Correspondences for elementary tangles)

(a) (Cylindrical bordisms) Suppose that \( (Y, K, \phi) \) admits a Morse function \( f : Y \to [-1, 1] \) with no critical points on \( Y \) or \( K \). Then \( f : (Y, K) \to [-1, 1] \) is a “marked” fiber bundle which admits a trivialization

\[ T : ([−1, 1] \times X_−, [−1, 1] \times \underline{x}_−) \to (Y, K), \]

where \( T|_{f^{-1}(\text{min} f)} = \text{Id}_{\underline{x}_-} \) and \( \phi : T|_{f^{-1}(\text{max} f)} : (X_-, \underline{x}_-) \to (X_+, \underline{x}_+) \) is an isomorphism of marked surfaces. The correspondence associated to \( (Y, K, \phi) \) is then the graph:

\[ L(Y, K, \phi) = \text{graph}((\phi^{-1})^*) \subset M(X_-, \underline{x}_-) \times M(X_+, \underline{x}_+). \]

(b) (Elementary tangles) Let \( X \) be a compact oriented surface, \( Y = [-1, 1] \times X \) and suppose that \( K \) contains a single critical point that is a maximum, and so consists of \( n - 2 \) strands meeting both the incoming and outgoing boundary, and one strand that connects two incoming markings \( x_i, x_j \), as in Figure 5. The map from \( L(Y, K, \phi) \) to \( M(X_+, \underline{x}_+) \) induced by pullback is a coisotropic embedding, and the map \( \pi_- \) from \( L(Y, K, \phi) \) to \( M(X_-, \underline{x}_-) \) is a fiber bundle with fiber \( C_{\mu \mu_i} \cong C_{\mu \mu_j} \).

(c) (Elementary bordisms) Suppose that \( Y \) contains a single critical point of \( f \) of index 1. The map \( \pi_+ \) from \( L(Y, K, \phi) \) to \( M(X_+, \underline{x}_+) \) is a coisotropic embedding, and the map \( \pi_- \) from \( L(Y, K, \phi) \) to \( M(X_-, \underline{x}_-) \) is a fiber bundle with fiber \( G \).

Furthermore, in each case \( L(Y, K, \phi) \) is a smooth Lagrangian correspondence in \( M(X_-, \underline{x}_-) \times M(X_+, \underline{x}_+) \).

\[ \begin{array}{c}
\text{Figure 5. A cup}
\end{array} \]

Proof. (a) Suppose that \( (Y, K, \phi) \) admits a Morse function \( f : Y \to [-1, 1] \) with no critical points on \( Y \) or \( K \). Fix a metric on \( Y \) with the property that the gradient flow \( \phi_t \) of \( f \) preserves \( K \), and scale it so that \( \frac{df}{dt} \phi_t = 1 \). Then \( T(x_-, \min f + t) = \phi_t(x_-) \) gives the claimed trivialization. Note that any element of \( L(Y, K, \phi) \) arises from a representation of \( Y \setminus K \), which is the pullback \((T^{-1})^* \rho \) of a representation \( \rho \) of the trivial cylinder over \( X_\setminus \underline{x}_- \). Since the restrictions of \( \rho \) to the two boundary
components are clearly conjugate, the boundary restriction of \((T^{-1})^*\rho\) will be the graph of pullback under \(\phi^{-1} = T^{-1}|_{X_+} : X_+ \to X_-\). Moreover, \(\phi^{-1}\) preserves the orientations and labels and hence induces a symplectomorphism of moduli spaces \(M(X_-,\underline{w}_-) \to M(X_+,\underline{w}_+).\) So \(L(Y,K,\phi)\) is the graph of a symplectomorphism and hence a smooth Lagrangian correspondence.

In case (b), to prove that \(L(Y,K,\phi)\) is a smooth, Lagrangian fiber bundle we choose a system of generators of the fundamental groups of the punctured surfaces such that in the holonomy description

\[
L(Y,K,\phi) = \left\{ \left( [a_1, \ldots, a_{2g}, c_1, \ldots, c_n + 2], \left( a_1, \ldots, a_{2g}, c_1, \ldots, \hat{c}_i, \ldots, \hat{c}_j, \ldots, c_n + 2 \right) \right) | c_i c_j = 1 \right\}.
\]

Such a system of generators can be found as follows. Let \(k_0 \in K\) denote the unique critical point, by assumption a maximum. Choose

\[
b_\pm := f(X_+) > c_+ > f(k_0) > c_- > f(X_-) =: b_-
\]

such that we obtain the following normal form on \(f^{-1}(\langle c_-, c_+ \rangle)\). Then the bordisms \(f^{-1}(\langle b_-, c_- \rangle)\) and \(f^{-1}(\langle b_+, c_+ \rangle)\) are cylindrical, and any choice of generators for the fundamental groups of \(f^{-1}(c_\pm)\) induces generators for the fundamental groups of \(f^{-1}(b_\pm)\). Consider the bordism \(f^{-1}(\langle c_-, c_+ \rangle)\). By taking \(c_\pm\) sufficiently close to \(f(k_0)\), one sees that \(f^{-1}(c_+)\) is obtained from \(f^{-1}(c_-)\) by replacing a twice punctured disk \(D_-\) by a disk \(D_+\) without punctures, and the two punctures labelled \(i, j\) in \(D_-\) are connected in \(f^{-1}(\langle c_-, c_+ \rangle)\) by a small cap. Choose a system of generators for \(f^{-1}(c_-)\) that except for the generators around the \(i\)-th and \(j\)-th markings do not meet \(D_-\), and the corresponding system of generators for \(f^{-1}(c_+)\). Since the \(i\)-th and \(j\)-st strands are connected by a cap, the holonomies around the punctures \(x_i, x_j \in X_+\) are inverse, up to conjugacy, and hence \(\mu_i = \mu_j\). In these coordinates \(\pi_- : L(Y,K,\phi) \to M(X_-,\underline{w}_-)\) clearly is a smooth fibration. We can identify the fiber with the anti-diagonal

\[
\Delta_i := \{ (c_i, c_i^{-1}) | c_i \in C_{\mu_i} \} \subseteq C_{\mu_i} \times C_{\mu_i} = C_{\mu_i} \times C_{\mu_j}
\]

with either of the conjugacy classes \(C_{\mu_i}\) or \(C_{\mu_j}\) by projection. The symplectic form on \(M(X_+,\underline{w}_+)\) is given by reduction from the 2-form \((9)\) on \(G^{2g} \times C_{\mu}, \cong G^{2g} \times C_{\mu_i} \times C_{\mu_i} \times C_{\mu_j}.\) In the latter splitting the 2-form is

\[
\omega_{g,\underline{w}} + \omega_{0,\{\mu_i, \mu_j\}} + \langle \Phi_{g,\underline{w}} \theta \wedge \Phi_{0,\{\mu_i, \mu_j\}} \rangle / 2,
\]

and one can check that the second and third term vanish on \(G^{2g} \times C_{\mu_i} \times \Delta_i.\) The same holds after taking quotients, hence \(L(Y,K,\phi)\) is isotropic and half-dimensional, thus Lagrangian. This also implies that \(\pi_+ : L(Y,K,\phi) \to M(X_+,\underline{w}_+)\) is a coisotropic embedding. Case (c) is similar.

**Proposition 3.19.** (Lagrangian correspondences for elementary tangles) If \((Y,K,\phi)\) is a elementary tangle as in Definition 2.3 from \((X_-,\underline{w}_-)\) to \((X_+,\underline{w}_+)\) and the labels \(\underline{w}\) for the components of \(K\) are such that the moduli spaces \(M(X_\pm,\underline{w}_\pm)\) are smooth manifolds (see Proposition 3.8) then the moduli space \(L(Y,K,\phi)\) is a smooth Lagrangian correspondence from \(M(X_-,\underline{w}_-)\) to \(M(X_+,\underline{w}_+).\)
Proof. There are three cases to consider, depending on whether a critical point occurs in the tangle, in the ambient bordism, or not at all. In the first case, the critical point must be a maximum or minimum. Up to symmetry, this is exactly the setting of Lemma 3.18 (b), so the claim follows. For a critical point in the ambient bordism the index of the critical point must be either one or two; up to symmetry this is Lemma 3.18 (c). The third case is left to the reader. \(\square\)

Remark 3.20. Relative spin structures will be needed later to provide orientations on moduli spaces of holomorphic quilts as in \([47]\). Recall from \([8, 47]\) that

(a) (Relative spin structures) A relative spin structure for a Lagrangian submanifold \(L \subset M\) is a lift of the structure group of \(TL\) from \(SO\) to \(Spin\) in the relative Čech homology of \(L \to M\). A relative spin structure is equivalent to a trivialization of the second Stiefel-Whitney class \(w_2(TL)\) in the space of relative chains for \(L \to M\).

(b) (Background classes) Any relative spin structure has a background class \(b \in H_2(M, \mathbb{Z}_2)\), given by the class of the image of the trivialization under \(C_1(M, L, \mathbb{Z}_2) \to C_2(M, \mathbb{Z}_2)\).

(c) (Equivalence) Two relative spin structures are equivalent mod \(w_2(TM)\) if their background classes \(b, b' = b + w_2(TM)\) are related by adding Stiefel-Whitney classes \(w_2(TM)\), and the trivialization corresponds to the canonical trivialization of \(w_2(TM)\) along a Lagrangian arising from the isomorphism \(TM|_L \cong TL \oplus TL^\vee\).

Remark 3.21. (Background classes for moduli of bundles) Suppose that \(x_\pm\) consists of \(n_\pm\) markings with positive resp. negative orientation, and \(\mu_\pm\) are the labels of the points with positive resp. negative orientation. We take as background classes for \(M(X_i, x_\pm)\) the Stiefel-Whitney classes for conjugacy classes associated to the positively or negatively oriented markings

\[
b_\pm(X_i, x_\pm) := w_2(T(\Pi_{j=1}^{n_\pm} C_{\mu_\pm})/G) \in H^2(M(X_i, x_\pm), \mathbb{Z}_2).
\]

These are equivalent modulo \(w_2(TM(X_i, x_\pm))\), since \(G\) is equivariantly spin for \(G\) simply-connected.

Lemma 3.22. (Relative spin structures) Let \((Y, K, \phi)\) be an oriented elementary tangle from \((X_-, x_-)\) to \((X_+, x_+)\), so that \(X_+ \cong X_-\) and \(x_+\) has at least as many elements as \(x_-\). Let \(\nu\) be a labelling of \(K\) such that the moduli spaces \(M(X_i, x_\pm)\) are smooth manifolds. Then \(L(Y, K, \phi)\) is simply-connected and canonically oriented, and has a unique relative spin structure with background class \((b_-(X_-, x_-), b_+(X_+, x_+))\) of \((13)\), compatible under composition modulo the classes \(w_2(TM(X_i, x_\pm))\).

Proof. Suppose first that \(K\) contains a critical point of index 1, with a strand connecting the incoming strands marked \(i\) and \(j\). Suppose that the orientation of \(x_j\) resp. \(x_i\) is the same resp. opposite of the standard orientation of a point. We show that \(L(Y, K, \phi)\) is simply-connected. By Lemma 3.18, \(L(Y, K, \phi)\) is diffeomorphic to a \(C_{\mu_\pm}\)-bundle over \(M(X_-, x_-)\). The base \(M(X_-, x_-)\) is simply-connected by the Atiyah-Bott method for bundles with fixed holonomy \([30]\). The conjugacy classes of \(G\) are simply-connected, by Remark 3.3. Hence so is \(L(Y, K, \phi)\).
Next we show that \( L(Y, K, \phi) \) is canonically oriented. Since the base is simply-connected and the structure group of the bundle is connected, an orientation \( L(Y, K, \phi) \) is induced by the symplectic orientation on the base \( M(X_-, \underline{\underline{x}}_-) \) and the orientation on the fiber \( C_{\mu_j} \) given by the volume form constructed in [2, Section 3.5].

Finally, by the assumption on the size of \( \underline{\underline{x}}_+ \), the map \( L(Y, K, \phi) \rightarrow M(X_+, \underline{\underline{x}}_+) \) is an embedding. In the case of no critical point this map is a diffeomorphism and so \( w_2(L(Y, K, \phi)) \) is the pull-back of \( w_2(M(X_+, \underline{\underline{x}}_+)) \). This implies that \( L(Y, K, \phi) \) has a relative spin structure. In the case of a single critical point of index 0, \( L(Y, K, \phi) \) is a symplectic quotient of the diagonal \( C_{\nu} = C_{\mu}^{j} \) for some \( i, j \) with the same labels \( \nu := \mu_{+, i} = \mu_{+, j} \) but with opposite orientations. The relative spin structure for the diagonal embedding \( C_{\nu} \rightarrow C_{\mu}^{j} \) with respect to either the positive or negative factor induces a relative spin structure for the embedding \( L(Y, K, \phi) \rightarrow M(X_+, \underline{\underline{x}}_+) \). The background class of this spin structure is given by \( w_2 \) of the bundle descended from \( TC_{\mu_+} \). Since \( L(Y, K, \phi) \) is simply connected, this relative spin structure is unique for this background class. The other cases (critical point of index 0, or no critical point) are similar. Compatibility under composition follows from uniqueness. \( \square \)

3.5. Symplectic-valued field theory. Putting everything together we construct a functor from the tangle category to the category of (symplectic manifolds, equivalence classes of generalized Lagrangian correspondences.)

**Definition 3.23.** (Admissible tangle category) Fix coprime integers \( r, d > 0 \). Let \( G = SU(r) \) and \( B \) be the set of labels as in Definition 3.8 (d). Let \( X \) be a compact oriented surface. We denote by \( \text{Tan}(X, r, d) \) the category of cylindrical \( B \)-labelled tangles in \( X \) resp. markings and labellings on \( X \) such that the labels are admissible in the sense of 3.8 (d).

**Remark 3.24.** (a) (The rank two case) In the simplest case \( r = 2, d = 1 \) the category \( \text{Tan}(X, r, d) \) is the category of tangles in \( X \) such that the number of components labelled with the label corresponding to the conjugacy class of \( SU(2) \) of trace zero is odd.

(b) (Orbifold singularities?) Possibly one could be slightly more general here and work with various other assumptions which guarantee at most orbifold singularities in the moduli spaces of flat bundles on marked, labelled surfaces. This would require Lagrangian Floer theory for orbifolds.

**Theorem 3.25.** (Symplectic-valued field theory for admissible tangles) For \( r, d > 0 \) coprime, partially define a functor \( \Phi : \text{Tan}(X, r, d) \rightarrow \text{Symp}_{1/2r}^{\#} \) by mapping an elementary tangle \((Y, K, \phi)\) to the Lagrangian correspondence \( L(Y, K, \phi) \). Then \( \Phi \) extends to a \( \text{Symp}_{1/2r}^{\#} \)-valued field theory.

**Proof.** It suffices to check the conditions in Theorem 2.5. Any diffeomorphism equivalence of Cerf decompositions induces a symplectomorphism equivalence of generalized Lagrangian correspondences, by pull-back. The Cerf relations in Theorem 2.5 follow from suitable equivariant versions of the results of [46]. However, we prefer to give an explicit computation.
Consider first the case of critical point cancellation. We may suppose that $K_1$ is a
cup connecting the strands $j, j-1$, and $K_{i+1}$ is a cup connecting the strands $j, j+1$
In terms of the holonomies around the strands $a_1, \ldots, a_{2g-2}$ for $x_i$, $b_1, \ldots, b_{2g}$ for
$c_1, \ldots, c_{2g-2}$ for $x_{i+1}$ we have
\[
L(Y_i, K_i, \phi_i) = \begin{cases}
    b_{j-1} = b_j^{-1} & 	ext{for } k < j - 1, \\
    b_k = a_k & 	ext{for } k > j \\
    b_k = a_{k-2} & 	ext{for } k > j + 1
\end{cases} \subseteq G^{n_i}/G,
\]
\[
L(Y_{i+1}, K_{i+1}, \phi_{i+1}) = \begin{cases}
    b_j = b_{j+1}^{-1} & 	ext{for } k < j \\
    b_k = c_k & 	ext{for } k > j \\
    b_k = c_{k-2} & 	ext{for } k > j + 1
\end{cases} \subseteq G^{n_{i+1}}/G.
\]
Their composition $L(Y_i, K_i, \phi_i) \circ L(Y_{i+1}, K_{i+1}, \phi_{i+1})$ is set theoretically the diagonal
in $M(X_{i-1}, x_i) \times M(X_{i+1}, x_{i+1})$. To check transversality, let
\begin{align*}
T_{[a,b]}(M(X_{i-1}, x_i) \times M(X_i, x_i)) &= \{(\xi_{i-1}, \xi_i) \in T_a G^{n_{i-1}} \times T_b G^{n_i}\} \\
T_{[b,c]}(M(X_i, x_i) \times M(X_{i+1}, x_{i+1})) &= \{(\xi_i, \xi_{i+1}) \in T_b G^{n_i} \times T_c G^{n_{i+1}}\}.
\end{align*}
Then the tangent space to the product of correspondences is
\[
T_{[a,b,c]}(L(Y_i, K_i, \phi_i) \times L(Y_{i+1}, K_{i+1}, \phi_{i+1})) = \{b_{j-1}\xi_{j-1}b_j = -\xi_j, \ -\xi_j' = -b_{j+1}\xi_{j+1}'b_j\}
\]
where $b_jx_{j-1}b_j$ denotes push-forward of the tangent vector $\xi_{j-1}$ under left multiplication
by the group element $b_{j-1}$ and right multiplication by the group element $b_j$. This intersects $T_{[a,b,c]}(M(X_{i-1}, x_i) \times \Delta M(X_i, x_i) \times M(X_{i+1}, x_{i+1}))$ transversally.

Hence the composition $L(Y_i, K_i, \phi_i) \circ L(Y_{i+1}, K_{i+1}, \phi_{i+1})$ is smooth and embedded,
equal to the diagonal. This shows invariance of the partially defined functor $\Phi$
under the Cerf move of critical point cancellation. Invariance under critical point
switches is similar. Relative spin structures were constructed in Lemma 3.22. □

Remark 3.26. (Tangles in non-trivial bordisms?) It seems likely that, by a more
detailed examination of Cerf theory, one can allow simultaneously tangles and non-
trivial bordisms, but we have not checked the details.

4. Matrix factorizations via Floer theory

In this section we describe a framework for Floer theory for monotone Lagrangians
with minimal Maslov number two, which applies in particular to Lagrangian Floer
theory in moduli spaces of parabolic bundles. More generally Fukaya-Oh-Ohta-Ono
[8] associate to any symplectic manifold a homotopy type of a (possibly curved) $A_\infty$
category. For monotone pairs of Lagrangians the Floer differential $\partial$ for a cyclic
generalized Lagrangian correspondence $L$ does not necessarily square to zero, but
the isomorphism class of the object $CF(L) := (CF(L), \partial)$ in the derived category of
matrix factorizations is independent of all choices. In our application to $SU(r)$ knot
Floer cohomology, the functor associated to a trivalent graph will be an object in
such a derived category, and the language is chosen to make it match up with that
in Khovanov-Rozansky [18]. Even in the case that the differentials have vanishing
square, working in the derived category has certain advantages. For example, it
makes duals and tensor products work the way they should. The derived category
construction discussed here is separate from the derived category construction applied by Kontsevich to Fukaya’s $A_\infty$ category. (Here the morphisms in the category are already objects in some derived category.)

4.1. **Matrix factorizations.** Categories of matrix factorizations are defined as follows, see e.g. [32, p.17].

**Definition 4.1.** (Category of matrix factorizations) For any $w \in \mathbb{Z}$, let $\operatorname{Fact}(w)$ denote the category of factorizations of $w \operatorname{Id}$.

(a) The objects of $\operatorname{Fact}(w)$ consist of pairs $(C, \partial)$, where

(i) $C$ is a $\mathbb{Z}_2$-graded free abelian group $C = C^0 \oplus C^1$;

(ii) $\partial$ is a group homomorphism $\partial : C^* \to C^{*+1}$, satisfying $\partial^2 = w \operatorname{Id}$.

(b) For objects $(C, \partial)$ and $(C', \partial')$, the space of morphisms $\operatorname{Hom}_{\operatorname{Fact}}((C, \partial), (C', \partial'))$ is the space of grading preserving maps $f : C^* \to (C')^*$ such that $f \partial = \partial' f$.

**Remark 4.2.** (a) (Duals) Given an object $(C, \partial) \in \operatorname{Obj}(\operatorname{Fact}(w))$, there exists a dual object $(C, \partial)^\vee = (C^\vee, \partial^\vee)$, where $C^\vee = \operatorname{Hom}(C^0, \mathbb{Z}) \oplus \operatorname{Hom}(C^1, \mathbb{Z})$ and $\partial^\vee$ is the dual of $\partial$. Similarly for a morphism $f : (C, \partial) \to (C', \partial')$ we obtain a dual morphism $f^\vee : (C', \partial')^\vee \to (C, \partial)^\vee$. Thus we obtain a contravariant dualization functor

$$\operatorname{Fact}(w) \to \operatorname{Fact}(w), \ (C, \partial) \mapsto (C, \partial)^\vee.$$

(b) (Tensor products) Similarly, there is a covariant tensor product functor

$$\text{(15) } \operatorname{Fact}(w_1) \times \operatorname{Fact}(w_2) \to \operatorname{Fact}(w_1 + w_2),$$

$$((C_1, \partial_1), (C_2, \partial_1)) \mapsto (C_1 \otimes C_2, \partial_1 \otimes \operatorname{Id} + \operatorname{Id} \otimes \partial_2).$$

Here $\otimes$ is the graded tensor product, so that

$$(\operatorname{Id} \otimes \partial_2)(x_1 \otimes x_2) = (-1)^{|x_1|} x_1 \otimes \partial_2 x_2$$

for homogeneous $x_1 \in C^{|x_1|}_1$.

(c) (Preservation of potential) The element $w$ is often called a potential, although here it is just a number. Any morphism between matrix factorizations $f : (C, \partial) \to (C', \partial')$ with potentials $w, w'$ satisfies

$$w' \circ f = \partial'^2 \circ f = f \circ \partial^2 = f \circ w.$$ 

In particular, if $f$ is bijective then $w = w'$.

(d) (Cohomology) For any matrix factorization $(C, \partial)$ let $H((C, \partial) \otimes_{\mathbb{Z}} \mathbb{Z}_w)$ denote the cohomology of the differential $\partial \otimes \operatorname{Id} : C \otimes_{\mathbb{Z}} \mathbb{Z}_w \to C \otimes_{\mathbb{Z}} \mathbb{Z}_w$ obtained from $\partial$ by tensoring with $\mathbb{Z}_w$. Any morphism in $\operatorname{Fact}(w)$ defines a homomorphism of the corresponding cohomology groups, and so we have a cohomology with coefficients functor to the category $\operatorname{Ab}$ of $\mathbb{Z}_2$-graded abelian groups,

$$\text{(16) } \operatorname{Fact}(w) \to \operatorname{Ab}, \ (C, \partial) \mapsto H((C, \partial) \otimes_{\mathbb{Z}} \mathbb{Z}_w).$$

**Definition 4.3.** (Derived category of matrix factorizations)
(a) Let \((C^0_\ast, \partial_0), (C^1_\ast, \partial_1)\) be matrix factorizations. A morphism \(f : (C^0_\ast, \partial_0) \to (C^1_\ast, \partial_1)\) is called \emph{null-homotopic} if there exists a map \(h : C^0_\ast \to C^{\ast-1}_1\) such that \(f = h\partial_0 + \partial_1 h\).

(b) The \emph{derived category of matrix factorizations} \(D\text{Fact}(w)\) is the category with the same objects as \(\text{Fact}(w)\), and morphisms given by the quotient of morphisms in the category \(\text{Fact}(w)\) by null-homotopic morphisms.

(c) The \emph{trivial object} in \(D\text{Fact}(w)\) is the trivial complex \(C^0 = C^1 = \{0\}\) equipped with the trivial differential \(\partial\). (Note that \(\partial^2 = w\text{Id}\), for any \(w\).)

\textbf{Remark 4.4. (Exact triangles)}

(a) \(D\text{Fact}(w)\) is naturally a triangulated category, with distinguished “exact” triangles given by the \emph{mapping cone construction}: Given a morphism of matrix factorizations \(f : (C^1_\ast, \partial_1) \to (C^2_\ast, \partial_2)\), its mapping cone is the factorization

\[
\text{Cone}(f) := (C^1_1 \oplus C^1_2, \begin{pmatrix} -\partial_1 & 0 \\ f & \partial_2 \end{pmatrix}).
\]

The exact triangles in \(D\text{Fact}(w)\) are by definition those isomorphic to triangles

\[
\ldots \to C_1 \to C_2 \to \text{Cone}(f) \to C^1_1 \to \ldots.
\]

In particular, if \(C^1_1 \xrightarrow{f} C^2_2 \to C^3_3 \to C^1_1\) is an exact triangle then \(C^3_3\) is (non-canonically) isomorphic to the mapping cone on \(f\). The proofs are the same as for the case \(w = 0\) of complexes, see e.g. \([10]\).

(b) The cohomology with coefficients functor \((16)\) factors through the derived category to give a functor \(D\text{Fact}(w) \to \text{Ab}\). Any exact triangle in \(D\text{Fact}(w)\) gives rise to a long exact sequence of cohomology groups with coefficients in \(\mathbb{Z}_w\).

4.2. Derived Floer theory for a pair of monotone Lagrangians. In this section we extend results of Oh \([31]\) on Floer theory in the presence of Maslov index two disks to cyclic generalized Lagrangian correspondences.

\textbf{Definition 4.5. (Moduli of Maslov-index-two pseudoholomorphic disks)} Let \(D \subset \mathbb{C}\) be the unit disk and fix the base point \(1 \in \partial D\). Let \((M, \omega)\) be a compact monotone symplectic manifold and \(L \subset M\) an oriented monotone Lagrangian submanifold. For any \(J \in \mathcal{J}(M, \omega)\) and submanifold \(X \subset L\), let \(\mathcal{M}_2^2(L, J, X)\) denote the moduli space of \(J\)-holomorphic disks \(u : (D, \partial D) \to (M, L)\) with Maslov number 2 and one marked point satisfying \(u(1) \in X\), modulo the action of the group \(\text{Aut}(D, \partial D, 1)\) of automorphisms of the disk fixing \(1 \in \partial D\).

Oh \([31]\) proves that the moduli space of disks above gives rise to a well-defined number:

\textbf{Proposition 4.6. (Disk invariant of a Lagrangian)} For any \(\ell \in L\) there exists a comeager subset \(\mathcal{J}^{\text{reg}}(\ell) \subset \mathcal{J}(M, \omega)\) such that \(\mathcal{M}_2^2(L, J, \{\ell\})\) is a finite set. Any relative spin structure on \(L\) induces an orientation on \(\mathcal{M}_2^2(L, J, \{\ell\})\). Letting \(\epsilon :
\( \mathcal{M}_I^2(L, J, \{\ell\}) \to \{\pm 1\} \) denote the map comparing the given orientation to the canonical orientation of a point, the disk number of \( L \),
\[ w(L) := \sum_{[u] \in \mathcal{M}_I^2(L, J, \{\ell\})} \epsilon([u]), \]
is independent of \( J \in J_{\text{reg}}(\ell) \) and \( \ell \in L \).

We will now extend the definition of quilted Floer cohomology, using the setup of [44] but dropping the assumption on minimal Maslov number at least three.

**Definition 4.7.**

(a) (Symplectic backgrounds) Fix a monotonicity constant \( \tau \geq 0 \) and an even integer \( N > 0 \). A symplectic background is a tuple \((M, \omega, b, \text{Lag}_N(M))\) as follows.

(i) (Bounded geometry) \( M \) is a smooth compact manifold;

(ii) (Monotonicity) \( \omega \) is a symplectic form on \( M \) which is monotone, i.e. \( [\omega] = \tau c_1(TM) \);

(iii) (Background class) \( b \in H^2(M, \mathbb{Z}_2) \) is a background class, which will be used for the construction of orientations; and

(iv) (Maslov cover) \( \text{Lag}_N(M) \to \text{Lag}(M) \) is an \( N \)-fold Maslov cover such that the induced 2-fold Maslov covering \( \text{Lag}_2(M) \) is the oriented double cover.

We often refer to a symplectic background \((M, \omega, b, \text{Lag}_N(M))\) as \( M \).

(b) (Lagrangian branes) brane structure on a compact Lagrangian \( L \) consists of an orientation, relative spin structure, and grading. An admissible Lagrangian brane is a compact oriented Lagrangian with brane structure with torsion fundamental group. (One can also assume other conditions which give monotonicity for pseudoholomorphic curves with boundary in these Lagrangians, or work with Novikov rings etc.)

**Theorem 4.8.** (Matrix factorizations via Floer theory) Let \( L = (L_j(j+1))_{j=0,...,r} \) be a cyclic generalized Lagrangian correspondence between symplectic backgrounds \( M_j, j = 0, \ldots, r \) with the same monotonicity constant \( \tau \geq 0 \), the Lagrangian correspondences \( L_j(j+1) \) are equipped with admissible brane structures. Then, for any \( H \in \text{Ham}(L) \), widths \( \delta = (\delta_j > 0)_{j=0,...,r} \), and for \( J \) in a comeager subset \( J_{\text{reg}}^\text{hyp}(L, H) \subset J_t(L) \), the Floer differential \( \partial : CF(L) \to CF(L) \) satisfies
\[ \partial^2 = w(L) \text{Id}, \quad w(L) = \sum_{j=0}^{r} w(L_{j(j+1)}). \]

The image \( CF(L) \) of \((CF(L), \partial)\) in \( D\text{Fact}(w(L)) \) is independent of the choice of \( H \) and \( J \), up to isomorphism.

**Proof.** We sketch the proof, following Oh [31] in the case of \( \mathbb{Z}_2 \) coefficients. For any \( x_{\pm} \in \mathcal{I}(L) \), the zero dimensional component \( \mathcal{M}(x_{-}, x_{+})_0 \) of Floer trajectories is a finite set. As in [31, Proposition 4.3] the one-dimensional component \( \mathcal{M}(x_{-}, x_{+})_1 \) is smooth, but the “compactness modulo breaking” in part (c) does not hold in general: Apart from the breaking of trajectories, a sequence of Floer trajectories of Maslov index 2 could in the Gromov compactification converge to a constant
trajectory and either a sphere bubble of Chern number one or a disk bubble of Maslov number two. All other bubbling effects are excluded by monotonicity. Thus failure of “compactness modulo breaking” can occur only when $\underline{x} = \underline{x}^+$. The proof follows from the claim that each one-dimensional moduli space $\mathcal{M}(x, x)$ of self-connecting trajectories has a compactification as a one-dimensional manifold with boundary

$$\partial \mathcal{M}(x, x) \cong \bigcup_{y \in \mathcal{I}(L)} (\mathcal{M}(x, y) \times \mathcal{M}(y, x)) \cup \bigcup_{j=0, \ldots, r} \mathcal{M}^2_j(L_{j(j+1)}, J_{j(j+1)}, \{(x_j, x_{j+1})\})^-$$

and that furthermore the orientations on these moduli spaces induced by the relative spin structures are compatible with the inclusion of the boundary. Here $\mathcal{M}^2_j(\ldots)$ denotes the moduli space $\mathcal{M}_j(\ldots)$ with orientation reversed. The subset $J_{\mathrm{reg}}^+\{L; H\}$ consists of collections of time-dependent almost complex structures $J_j : [0, \delta_j] \to \mathcal{J}(M, \omega_j)$ for which all $\mathcal{M}(\underline{x}, \underline{x})$ are smooth and the universal moduli spaces of spheres $\mathcal{M}^1_j(M_j, \{J_j(t)\}_{t \in [0, \delta_j]}, \{x_j\})$ are empty for all $\underline{x} = (x_j)_{j=0, \ldots, r} \in \mathcal{I}(L)$. This excludes the Gromov convergence to a constant trajectory and a sphere bubble. We can now restrict to those $L \in J_{\mathrm{reg}}^+(L; H)$ such that

$$J_{j(j+1)} := (-J_j(\delta_j)) \oplus J_{j+1}(0) \in J_{\mathrm{reg}}^+(L_{j(j+1)}, \{(x_j, x_{j+1})\})$$

for all $\underline{x} \in \mathcal{I}(L)$ and $j = 0, \ldots, r$. This still defines a comeager subset in $J(L)$.

To finish the proof of the claim we use a gluing theorem of non-transverse type for pseudoholomorphic maps with Lagrangian boundary conditions, which can be adapted from [26, Chapter 10] as follows: We replace $L$ with its translates under the Hamiltonian flows of $H$, so that the Floer trajectories are unperturbed $J_j$-holomorphic strips (where the $J_j$ have suffered some Hamiltonian transformation, too). Pick $[v_{j(j+1)}] \in \mathcal{M}^2_j(L_{j(j+1)}, J_{j(j+1)}, \{(x_j, x_{j+1})\})$ and a representative $v_{j(j+1)}$.

The gluing construction gives a map

$$\mathcal{M}(x, x) \to \mathcal{M}(x, x)$$

(17)

$$\mathcal{M}(x, x) \to \mathcal{M}(x, x)$$

to the moduli space of parametrized Floer trajectories of index 2, where $T > 0$ is a real parameter. This construction first identifies $v_{j(j+1)}$ with a map from the half space $\mathbb{H} \cong D \setminus \{1\}$ to $M_j^+ \times M_{j+1}$. For the pregluing choose a gluing parameter $\tau \in (T, \infty)$. Outside of a half disk of radius $\frac{1}{2}\tau^{1/2}$ around 0, interpolate the map to the constant solution $(x_j, x_{j+1})$ outside of the half disk of radius $\tau^{1/2}$ using a slowly varying cutoff function in submanifold coordinates of $L_{j(j+1)} \subset M_j^+ \times M_{j+1}$ near $(x_j, x_{j+1})$. Then rescale this map by $\tau$ to a half-disk of radius $\tau^{-1/2}$ centered around 0 in the strip $\mathbb{R} \times [0, \tau^{-1/2}]$, again extended constantly. The components give an approximately $J_{j+1}$-holomorphic map $u_{j+1} : \mathbb{R} \times [0, \tau^{-1/2}] \to M_{j+1}$ and, after reflection, an approximately $J_j$-holomorphic map $u_j : \mathbb{R} \times (\delta_j - \tau^{-1/2}, \delta_j] \to M_j$. For $T \geq \max\{\delta_j^{-2}, \delta_{j+1}^{-2}\}$ these strips can be extended to width $\delta_j$ resp. $\delta_{j+1}$. Together with the constant solutions $u_\ell \equiv x_\ell$ for $\ell \notin \{j, j+1\}$ we obtain a tuple

$$u = (u_\ell : \mathbb{R} \times [0, \delta_\ell] \to M_\ell)_{\ell=0, \ldots, r}$$
that is an approximate Floer trajectory. An application of the implicit function theorem gives an exact solution for $T$ sufficiently large. The uniqueness part of the implicit function theorem gives that $[v_{j(j+1)}]$ is an isolated limit point of $\mathcal{M}(x, x)_1$, so that $\overline{\mathcal{M}(x, x)}_1$ is a one-dimensional manifold with boundary in a neighborhood of the nodal trajectory with disk bubble $[v_{j(j+1)}]$.

It remains to examine the effect of the gluing on orientations for which we need to recall the construction of orientations in [47]. Choose a parametrization $[T, \infty] \to \overline{\mathcal{M}(x, x)}_1$, $\infty \mapsto [v_{j(j+1)}]$ homotopic to the gluing map. Now the action on orientations is given by the action on local homology groups, and homotopic maps induce the same action. So by replacing the gluing map with this parametrization we may assume that the gluing map is an embedding. The moduli space $\mathcal{M}^2_1(\mathcal{L}_{j(j+1)}, J_{j(j+1)}, \{(x_j, x_{j+1})\})$ has orientation at $[u]$ induced by determinant line $\det(D_u)$ and the determinant of the infinitesimal automorphism group $\text{aut}(\mathbb{R} \times [0, 1]) \cong \mathbb{R}$. On the other hand, the orientation on $\mathcal{M}^2_1(\mathcal{L}_{j(j+1)}, J_{j(j+1)}, \{(x_j, x_{j+1})\})$ is induced by the orientation of the determinant line $\det(D_{v_{j(j+1)}})$ and the determinant line of the automorphism group $\text{Aut}(D, \partial D, 1) \cong (0, \infty) \times \mathbb{R}$. The gluing map induces an orientation-preserving isomorphism of determinant lines $\det(D_u) \rightarrow \det(D_{v_{j(j+1)}})$ by [47]. Under gluing the second factor in $\text{Aut}(D, \partial D, 1)$ agrees with the translation action on $\mathbb{R} \times [0, 1]$. On the other hand, the first factor agrees approximately with the gluing parameter, in the sense that gluing in a dilation of $v_{j(j+1)}$ by a small constant $\rho > 0$ using gluing parameter $\tau$ is approximately the same as gluing in $v_{j(j+1)}$ with gluing parameter $\tau \rho$. Thus $v_{j(j+1)}$ represents a boundary point of $\overline{\mathcal{M}(x, x)}_1$ with the opposite of the induced orientation from the interior. Summing over the boundary of the one-dimensional manifold $\partial \overline{\mathcal{M}(x, x)}_1$ proves that $\partial^2 - \sum_{j=0}^r w(L_{j(j+1)}) \text{Id} = 0$.

The proof that the image of $(\text{CF}(\mathcal{L}), \partial)$ in the derived category of matrix factorizations is independent of all choices up to isomorphism is essentially the same as that in [44], which produces a pair of chain maps whose compositions are null homotopic. \qed

**Remark 4.9.** (Matrix factorizations for Floer theory of a pair of Lagrangians) In the special case $\mathcal{L} = (L_0, L_1)$ of a cyclic correspondence consisting of two Lagrangian submanifolds $L_0, L_1 \subset M$ we have $w(\mathcal{L}) = w(L_0) - w(L_1)$. Indeed the $-J_1$-holomorphic discs with boundary on $L_1 \subset M^{-} \times \{\text{pt}\}$ are identified with $J_1$-holomorphic discs with boundary on $L_1 \subset M$ via a reflection of the domain, which is orientation reversing for the moduli spaces.

**Definition 4.10.** (Derived matrix factorizations for generalized correspondences) Let $\mathcal{L}$ be a cyclic generalized Lagrangian correspondence as in Theorem 4.8.

(a) The derived Floer factorization $\text{CF}(\mathcal{L})$ is the image of the Floer matrix factorization $(\text{CF}(\mathcal{L}), \partial)$ in $\text{Obj}(\text{DFact}(w(\mathcal{L})))$.

(b) The Floer cohomology with coefficients in $\mathbb{Z}_w$, $w := w(\mathcal{L})$ is the image

$$HF(L; \mathbb{Z}_w) := H((\text{CF}(\mathcal{L}), \partial) \otimes_{\mathbb{Z}} \mathbb{Z}_w),$$

of $(\text{CF}(\mathcal{L}), \partial)$ under the cohomology with coefficients functor (16).
Remark 4.11. (a) (Comparison with the usual definition) In the case \( w(L) = 0 \) this definition coincides with the usual definition of Floer cohomology with \( \mathbb{Z} \) coefficients in the sense that the functor taking cohomology with \( \mathbb{Z} \) coefficients from \( D \text{Fact}(0) \) to the category of finitely generated \( \mathbb{Z}_{2} \)-graded abelian groups induces a bijection between isomorphism classes of objects.

(b) (Independence of cohomology from all choices) Theorem 4.8 and Remark 4.4 show that the Floer cohomology \( HF(L; \mathbb{Z}_{w}) \) is independent of all choices up to isomorphism in the category of abelian groups.

(c) (Case of two equal Lagrangians) The differential for a monotone pair \( L = (L, \psi(L)) \) with any symplectomorphism \( \psi \in \text{Symp}(M) \) always squares to zero, since \( w(L) = w(\psi(L)) \).

Remark 4.12. (Behavior of Floer theory under duals and products) Duals and products behave as expected for the derived Floer invariants. (Note that even in the case of Maslov number at least three, the derived invariants have better properties than the Floer cohomology in this respect.)

(a) Suppose that \( (L_{0}, L_{1}) \) is pair of admissible Lagrangian branes. Then \( CF(L_{0}, L_{1}) \) is canonically isomorphic to \( CF(L_{1}, L_{0})^{\vee} \). To see this we use the canonical identification of the generators of \( CF(L_{0}, L_{1}) \) and \( CF(L_{1}, L_{0}) \). Then \( \partial_{L_{0}, L_{1}} \) is given by the transpose \( \partial_{L_{1}, L_{0}}^{T} \) of \( \partial_{L_{1}, L_{0}} \). The bijection between trajectories is given by rotating the strip with boundary \( L_{0}, L_{1} \) by 180 degrees.

(b) Suppose that \( L_{0}^{0}, L_{1}^{0} \subset M_{0} \) and \( L_{0}^{1}, L_{1}^{1} \subset M_{1} \) are admissible Lagrangian branes. Then \( CF(L_{0}^{0} \times L_{0}^{1}, L_{1}^{0} \times L_{1}^{1}) \) is canonically isomorphic to \( CF(L_{0}^{0}, L_{0}^{1}) \odot CF(L_{1}^{0}, L_{1}^{1}) \). The right hand side is the matrix factorization whose differential is the graded tensor product \( \partial_{L_{0}^{0}, L_{0}^{1}} \odot \text{Id} + \text{Id} \odot \partial_{L_{1}^{0}, L_{1}^{1}} \).

In our main Theorem in [44] of the minimal Maslov number at least three was needed only for the definition of the Floer cohomologies. We thus obtain the following generalization.

Theorem 4.13. (Behavior of Floer theory under geometric composition) Let \( L = (L_{01}, \ldots, L_{r(r+1)}) \) be a cyclic generalized Lagrangian correspondence with admissible brane structure. Suppose that for some \( 1 \leq j \leq r \) the composition \( L_{(j-1)j} \circ L_{jj(j+1)} \) is embedded and the modified sequence \( L' := (L_{01}, \ldots, L_{(j-1)j} \circ L_{jj(j+1)}, \ldots, L_{r(r+1)}) \) is monotone. Then, with respect to the induced brane structure, we have \( w(L) = w(L') = w \) and there exists a canonical isomorphism in \( D \text{Fact}(w) \) between \( CF(L) \) and \( CF(L') \), induced by the canonical identification of intersection points.

Proof. The bijection between the trajectory spaces for small widths and for the composed Lagrangian correspondence in [44] only requires that the minimal Maslov number of the Lagrangians is at least two (which is automatic in the monotone orientable case). The comparison of orientations in [47] is also independent of Maslov indices, hence the morphism \( f : CF(L) \to CF(L') \) given by the canonical identification of intersection points satisfies \( f \circ \partial = \partial' \circ f \), where \( \partial \) and \( \partial' \) are the Floer differentials on \( CF(L) \) resp. \( CF(L') \). Similarly, the inverse \( f^{-1} : CF(L') \to CF(L) \) satisfies \( \partial' f^{-1} = f^{-1} \partial \) and hence is another morphism in \( \text{Fact}(w) \), inverse to \( f \). So
f defines an bijection between the derived Floer factorizations \( CF(L) \) and \( CF(L') \).

By Remark 4.2 (c), \( w(L) = w(L') =: w \). \( \square \)

4.3. Derived relative invariants.

**Definition 4.14.** (Relative invariants as morphisms of matrix factorizations) Given a quilted surface \( S \), a collection \( M \) of symplectic backgrounds with the same monotonicity constant \( \tau \geq 0 \) and a collection \( L \) of compact Lagrangians with admissible brane structures let

\[
C \Phi_S : \bigotimes_{e \in E_+(S)} CF(L_e) \to \bigotimes_{e \in E_-(S)} CF(L_e)
\]

denote the map obtained by counting quilts as in [44], where the derived objects are the images of the corresponding chain groups in the derived category of matrix factorizations. The proof that one obtains a morphism of matrix factorizations is exactly the same as in the Floer cohomology case, since in fact we used the assumption on the minimal Maslov number only to make the Floer cohomology groups well-defined. Note in particular that the tensor products of derived matrix factorizations are elements of \( D \text{Fact}(w) \) with the same \( w \) for incoming and outgoing ends. This is since each noncompact seam \( \sigma \) of \( S \) connects either an incoming and an outgoing end, contributing \( w(L_\sigma) \) to both sides, or it connects two ends of the same kind, contributing \( w(L_\sigma) + w(L'_\sigma) = 0 \) to one side.

**Example 4.15.** (a) (Identity morphisms) Given an admissible Lagrangian brane \( L \subset M \) one obtains an identity morphism \( I_L : \mathbb{Z} \to CF(L, L) \), where \( \mathbb{Z} \) is the trivial complex in degree 0. The differential for \( CF(L, L) \) automatically squares to zero by Remark 4.11, so after passing to cohomology the identity morphism \( I_L \) induces the identity object \( 1_L \in HF(L, L) \).

(b) (Derived composition) Given a monotone triple \( L_0, L_1, L_2 \subset M \) of admissible Lagrangian branes one obtains a derived composition morphism

\[
[C \mu_2 : CF(L_0, L_1) \otimes CF(L_1, L_2) \to CF(L_0, L_2)]
\]

in \( D \text{Fact}(w) \) for \( w = w(L_0) - w(L_1) + w(L_1) - w(L_2) = w(L_0) - w(L_2) \). The derived composition morphism is also associative.

4.4. Donaldson-Fukaya category of Lagrangians.

**Definition 4.16.** (Donaldson-Fukaya category via matrix factorizations) Let \( (M, \omega) \) be a symplectic background. Let Don\((M)\) be the Donaldson-Fukaya category whose

(a) (Objects) objects are the set of admissible Lagrangian branes as in [?], but without the assumption on the minimal Maslov number; and

(b) (Morphisms) for any pair of objects \( (L_0, L_1) \), the “space” of morphisms \( \text{Hom}(L_0, L_1) := CF(L_0, L_1) \) is an object in the disjoint union category

\[
D \text{Fact} = \coprod_{w \in \mathbb{Z}} D \text{Fact}(w).
\]
That is, Obj(\(D\)Fact) is the disjoint union of the objects of \(D\)Fact(\(w\)), and there are no morphisms between objects with different values of \(w\). Composition is given by \(D\mu_2\). The generalized Donaldson-Fukaya category Don\(\#\)(\(M\)) is defined similarly, by allowing generalized branes.

The results of [42] hold with appropriate modifications of categories to categories enriched in matrix factorizations. In particular,

**Theorem 4.17.** (Functor for a geometric composition of Lagrangian correspondences) Let \(M_0, M_1, M_2\) be symplectic backgrounds with the same monotonicity constant and \(L_{01} \subset M_0^- \times M_1, L_{12} \subset M_1^- \times M_2\) compact, oriented, simply-connected Lagrangian correspondences equipped with admissible brane structures. If \(L_{01} \circ L_{12}\) is transverse and embedded into \(M_0^- \times M_2\) then \(\Phi(L_{01} \circ L_{12})\) are isomorphic in the category of functors from Don\(\#\)(\(M_0\)) to Don\(\#\)(\(M_2\)), after a shift in background classes by \(w_2(M_2)\).

**Corollary 4.18.** (Categorification functor) For any \(\tau > 0\), the assignments \(M \mapsto\) Don\(\#\)(\(M\)) for symplectic backgrounds \(M\) with monotonicity constant \(\tau\) and \(L \mapsto\) \(\Phi(L)\) for generalized Lagrangian correspondences \(L\) with admissible brane structures define a categorification functor from Symp\(\#\) to the category of (categories, isomorphism classes of functors).

**Remark 4.19.** (a) (Categorification two-functor) Consider the enriched 2-category whose objects are symplectic backgrounds, morphisms are Lagrangian correspondences, and for each pair of 1-morphisms we have as 2-morphisms an object of \(D\)Fact. Categorification extends to a 2-functor to the 2-category whose objects are small categories enriched in \(D\)Fact, 1-morphisms are functors, and 2-morphisms are natural transformations. One expects natural transformations to 2-tangles that fit into this framework.

(b) (Comparison with curved \(A_\infty\) categories) The matrix factorizations here appear in [8] in the following guise. A curved \(A_\infty\) category consists of a collection of objects and composition maps \(\mu_d, d \geq 0\), satisfying the \(A_\infty\) associativity relations modified to include for any object \(L\) an element \(\mu_0(1) \in \text{Hom}(L, L)\). The first \(A_\infty\) relation for \(\mu_1 : \text{Hom}(L_0, L_1) \to \text{Hom}(L_0, L_1)\) is then \((\mu_1)^2 = \mu_2(\mu_0^{L_0}(1) \otimes 1 - 1 \otimes \mu_0^{L_1}(1))\).

5. FLOER FIELD THEORY FOR TANGLES AND GRAPHS

5.1. FLOER FIELD THEORY FOR TANGLES. In this section we combine the results of the previous two sections to obtain a field theories with values enriched in matrix factorizations. Combining Theorem 3.25 and the functor of Corollary 4.18 we obtain the following more precise version of Theorem 1.1:

**Theorem 5.1.** (Category-valued field theory for tangles) For any coprime positive integers \(r, d\), the partially defined functor \(\Phi\) from \(\text{Tan}(X, r, d)\) to (categories, isomorphism classes of functors) which assigns

(a) to any labelled surface \((X, \varphi)\) the Donaldson-Fukaya category Don\(\#\)(\(M(X, \varphi)\)) and
(b) to any equivalence class of elementary tangles \([Y,K,\phi]\) the corresponding quilt functor \(\Phi(L(Y,K,\phi))\) extend to a category-valued field theory for tangles.

**Proof.** We already checked in 3.25 that the correspondences are simply-connected and compact, and their equivalence class is independent of the choice of Morse datum. For the relative spin structures compatible under composition see Remark 3.21. \(\square\)

In order to define group-valued invariants of manifolds containing links, we apply the following device suggested to us by Seidel.

**Definition 5.2.**
(a) (Trivial tangle) Let \(|r+1| \subset [-1,1] \times S^2\) be the tangle with \(r + 1\) trivial strands labelled by \(\beta_1/2\).

(b) (Sphere-summed tangle) Suppose \(K\) is a link in \(S^3\) with components labelled by standard conjugacy classes in \(SU(r)\). Let \((S^3,K)\#([-1,1] \times S^2,|r+1|)\) denote the connect sum of \((S^3,K)\) with \(([-1,1] \times S^2,|r+1|)\), as in Figure 6 for the case \(r = 2\).

(c) (Sphere-summed Floer theory) For any link \(K \subset S^3\), define the sphere-summed Floer matrix factorization

\[CFS(K) := CF(\Phi((S^3,K)\#([-1,1] \times S^2,|r+1|))\text{pt},\text{pt}).\]

---

Figure 6. Adding three trivial strands

The objects associated to links so defined are matrix factorizations with differential square equal to zero and hence give rise to honest link homology groups:

**Proposition 5.3.** (Matrix factorization for a link as the Lagrangian Floer theory for a pair) Suppose \(K\) is the braid closure of a braid \(\beta\) in the spherical braid group \(B_n\). Let \(\otimes\) denote the tangle from \(2n\) markings to \(0\) markings which matches the \(n-i\)-th marking with the \(n+i-1\)-th marking, for \(i = 1, \ldots, n\), so that

\[L(\otimes) = \{[g_1, \ldots, g_{2n+r+1}] \in M(S^2, \mathbb{Z}), \quad g_j g_{2n+1-j} = 1, \quad j = 1, \ldots, n\},\]

and let \(L(\oplus)\) be its transpose. There exists an isomorphism of matrix factorizations

\[CFS(K) \rightarrow CF(L(\otimes), (\beta \times 1_n)L(\oplus)).\]

---

3We do not prove here that these invariants are isotopy invariants; rather they are invariants in the sense that they only depend on the diffeomorphism type of the pair of the manifold and the link.
Definition 5.5. \((\text{Sphere-summed Floer homology})\) For any link \(\nu\) in \(S^3\), \(\text{Sphere-summed Floer homology}\) of \(\nu\) is the symplectic topology of represen-
tation varieties. Recall from (\ref{definition_sphere_summed}) the Floer cohomology is isomorphic to the Morse cohomology by Theorem 4.17. Applying the same result to the index one critical points completes the proof. \(\square\)

Corollary 5.4. \((\text{Floer theory for a link is a chain complex})\) For any link \(K \subset S^3\), the coboundary \(\partial_{K}\) for \(CFS(K)\) has \(\partial_{K}^2 = 0\).

Statement. \((\text{Sphere-summed Floer homology})\) For any link \(K \subset S^3\), the \(SU(r)\) sphere-summed link Floer homology of \(K\) is the homology \(HFS(K)\) of \(CFS(K)\), independent up to isomorphism of the choice of decomposition.

Example 5.6. \((\text{Sphere-summed Floer homology of the unknot})\) Take the cylindrical Cerf decomposition of the unknot \(\circ\) consisting of a cup \(\cup\) and cap \(\cap\), so that

\[
HFS(\circ) = HF(L(\cup), L(\cap))
\]

where

\[
L(\cup) = \{[g_1, \ldots, g_{r+3}] \in M(S^2, \{x_1, \ldots, x_{r+3}\}), g_1g_2 = 1\}.
\]

and \(L(\cap) = L(\cup)^T\). The map from \(M(S^2, \{x_1, \ldots, x_{r+3}\})\) to \(M(S^2, \{x_1, \ldots, x_{r+1}\})\) forgetting \(g_1, g_2\) is a fiber bundle with fiber the conjugacy class \(C\) labelling the 1-st and 2-nd strands, which is isomorphic to a partial flag variety. Now \(\mathcal{C}\) admits a Morse function with only even indices, given for example by a component of a moment map. By Pozniak \[33\] the Floer cohomology is isomorphic to the Morse cohomology:

\[
HFS(\circ) = HF(L, L) = H(\mathcal{C}, \mathbb{Z}).
\]

For example, if the label is \(\nu = \exp(\omega^\nu/2)\) then \(\mathcal{C} \cong \mathbb{C}P^{r-1}\) and \(HFS(\circ) = H(\mathbb{C}P^{r-1}, \mathbb{Z})\).

Remark 5.7. \((\text{Torus-summed matrix factorizations})\) Kronheimer-Mrowka \[21\] investigate the similarity with Khovanov-Rozansky homology \[18\] in greater detail. Their device to define link invariants is slightly different than that suggested to us by Seidel: they form the connect sum with \([-1,1] \times T^2\), equipped with the (unique up to isomorphism) bundle of degree coprime to the rank.

5.2. Application to symplectic mapping classes. In this section we give an application of the functors described above to the symplectic topology of representation varieties. Recall from (\ref{torus_summed}) that orientation-preserving diffeomorphisms of a compact, oriented surface induce symplectomorphisms of the moduli spaces of flat bundles. In this section we study the case of marked spheres and show that certain of these symplectomorphisms are non-trivial in the symplectic mapping class group.

We introduce the following notation. For \(\mu \in \mathfrak{A}\) let \(M_n(\mu)\) be the moduli space of flat bundles on the sphere \(X\) with a set of markings \(g\) of order \(n\), \(G = SU(2)\), and all labels equal to the label \(\mu\). Recall that any smooth projective complex-algebraic Fano surface is isomorphic to one of the del Pezzo surfaces \(D_b\), obtained by blowing up \(\mathbb{P}^2\) at \(b < 9\) points.
Proposition 5.8. (Identification of the first non-trivial moduli space with monotone labels as a del Pezzo) For \( \mu \in \mathbb{A} \simeq [0,1/2] \) the moduli space \( M_5(\mu) \) is diffeomorphic to the smooth manifold underlying

(a) (First Chamber) the del Pezzo \( D_4 \) for \( \mu \in (0,1/5) \);
(b) (Second Chamber) the del Pezzo \( D_5 \), for \( \mu \in (1/5,1/2) \); and
(c) (Third Chamber) the empty manifold, for \( \mu \in (1/2,1/2] \).

For \( \mu = 0,1/5,2/5 \) the moduli space contains reducibles.

Proof. To determine the chamber structure, it suffices by Proposition 3.8 to find the moduli spaces containing reducibles. These are \( M_5(\mu) \) with \( \mu = 0,1/5,2/5 \) corresponding to commuting elements \( g_1,\ldots,g_5 \) such that \( g_1\ldots g_5 = 1 \).

The moduli space \( M_5(\mu) \) is Fano in the first chamber \( \mu < 1/5 \), since it is a quotient of \( (S^2)^5 \) (see [29]) and in the second by Theorem 3.13 since for \( \mu = 1/4 \) it is monotone. To identify it as a del Pezzo it suffices to determine the second Betti number. The Betti numbers of \( M_5(\mu) \) in the first chamber \( \mu < 1/5 \) can be computed by the method of Kirwan [19], since in this case the moduli space is the geometric invariant theory quotient of \( (\mathbb{P}^1)^n \) by the diagonal action of \( SL(2,\mathbb{C}) \). For \( n \geq 1 \) let

\[
P_n(\mu,t) = \sum_{j=0}^{\infty} \text{rank}(H^j(M_n(\mu)))t^j
\]
denote the Poincaré polynomial of \( M_n(\mu), \mu < 1/n \). By [19, p.193]

\[
P_n(\mu,t) = (1 + t^2)^n(1 - t^4)^{-1} - \sum_{\frac{4}{5} < r \leq n} \left( \begin{array}{c} n \\ r \end{array} \right) t^{2(r-1)}(1 - t^2)^{-1}.
\]

In particular,

\[
P_5(\mu,t) = 1 + 5t^2 + t^4 \text{ if } \mu < 1/5.
\]

The Poincaré polynomial for the second chamber can be computed by two techniques: the original approach of Atiyah-Bott [3], extended to the parabolic case by Nitsure [30], and the recursive approach of Thaddeus [40]. In the special case \( \mu = 1/4 \), the Atiyah-Bott approach gives

\[
P_5(\mu,t) = (1 + t^2)^n(1 - t^2)^{-1}(1 - t^4)^{-1} - 2^{n-1}t^n(1 - t^2)^{-2}
\]

where the first term is the contribution from the equivariant cohomology of the affine space of connections and the second is the contribution from the unstable strata corresponding to abelian orbifold connections, c.f. Street [38, Theorem 3.8]. Hence

\[
P_5(\mu,t) = 1 + 6t^2 + t^4 \text{ if } 1/5 < \mu < 2/5.
\]

For the third chamber, one observes that the moduli space \( M_5(\mu) \) fibers over solutions to \( g_1g_2g_3g_4g_5 = 1 \) with each \( g_i \) having eigenvalues \( \exp(\pm 2\pi i \mu) \) gives rise to a geodesic pentagon in \( SU(2) \cong S^3 \) with edge lengths \( 2\pi \mu \). Replacing each \( g_i \) with \( -g_i \) gives rise to a non-closed geodesic 5-gon with edge lengths \( 2\pi \mu - \pi \) connecting antipodes in \( S^3 \). For \( \mu > 2/5 \), these edge lengths are less than \( \pi/5 \) and so cannot connect antipodal points, which have distance \( \pi \), by the triangle inequality for the spherical metric. \( \square \)
Theorem 5.9. (Graph of the square of a Dehn twist is not the diagonal up to isomorphism) Let $\sigma_{12}^{(5)}$ be the half twist around the first two markings in the spherical braid group $B_5$, and $(\sigma_{12}^{(5)})^2$ its square. Let
\[ \Gamma((\sigma_{12}^{(5)})^2) \subset M_5(1/4)^- \times M_5(1/4) \]
be the graph of its action on the moduli space of bundles $M_5(1/4)$. Let $\Delta_5 \subset M_5(1/4)^- \times M_5(1/4)$ be the diagonal. Then $\Delta_5$ is not isomorphic to $\Gamma((\sigma_{12}^{(5)})^2)$ in Don $\#(M_5(1/4), M_5(1/4))$.

Proof. This is essentially a result of Seidel [35, Example 1.13]. Let $\Gamma(\sigma_{12}^{(5)})$ denote the Lagrangian associated to the half-twist, and $\Gamma((\sigma_{12}^{(5)})^{-1})$ the Lagrangian associated to the half-twist inverse. If $f \in \text{Hom}(\Delta_5, \Gamma((\sigma_{12}^{(5)})^2))$ were an isomorphism, the composition with $\Gamma((\sigma_{12}^{(5)})^{-1})$ would induce an isomorphism
\[ \Gamma((\sigma_{12}^{(5)})^{-1}) = \Delta_5 \circ \Gamma((\sigma_{12}^{(5)})^{-1}) \rightarrow \Gamma((\sigma_{12}^{(5)})^2) \circ \Gamma((\sigma_{12}^{(5)})^{-1}) = \Gamma(\sigma_{12}^{(5)}). \]
Any such isomorphism is automatically compatible with the module structure over $\text{Hom}(\Delta_5, \Delta_5) = \text{QH}(M_5(1/4))$. The rest of Seidel’s argument is the same. \qed

Theorem 5.10. (Non-triviality of squares of Dehn twists) For $n \geq 1$ let
\[ (\sigma_{12}^{(2n+3)})^2 \in \text{Diff}(M_{2n+3}(1/4)) \]
denote the symplectomorphism associated to the square of the half-twist of strands 1, 2,
\[ \Gamma((\sigma_{12}^{(2n+3)})^2) \subset M_{2n+3}(1/4)^- \times M_{2n+3}(1/4) \]
the corresponding Lagrangian, and $\Delta_{2n+3} \subset M_{2n+3}(1/4)^- \times M_{2n+3}(1/4)$ the diagonal. $\Gamma((\sigma_{12}^{(2n+3)})^2)$ is not isomorphic to $\Delta_{2n+3}$ in Don $\#(M_{2n+3}(1/4), M_{2n+3}(1/4))$.

Proof. The argument is an induction on the positive integer $n$. The case $n = 1$ is Seidel’s Theorem 5.9. Suppose that the statement in the Theorem holds for integers less than $n$ and suppose that there exists an isomorphism
\[ f \in \text{Hom}(\Delta_{2n+3}, \Gamma((\sigma_{12}^{(2n+3)})^2)). \]
Let $K_{\cup}, K_{\cap}$ denote a cup, cap at the 3, 4 strands resp. 4, 5 strands. Then, thinking of a braid as a special case of equivalence class of tangles, we have a Cerf decomposition expressing the square $(\sigma_{12}^{(2n+1)})^2$ in terms of $(\sigma_{12}^{(2n+3)})^2$

\[ (\sigma_{12}^{(2n+1)})^2 = K_{\cup} \cup (\sigma_{12}^{(2n+3)})^2 \cup K_{\cap} \]
as in Figure 7. Any isomorphism in $\text{Hom}(\Delta_{2n+3}, \Gamma((\sigma_{12}^{(2n+3)})^2))$ would therefore induce an isomorphism in the Donaldson-Fukaya category of correspondences

\[ L(K_{\cup}) \circ \Gamma((\sigma_{12}^{(2n+3)})^2) \circ L(K_{\cap}) \rightarrow L(K_{\cup}) \circ \Delta_{2n+3} \circ L(K_{\cap}) \]
by [44, Theorem 8.6] and therefore an isomorphism $\Gamma((\sigma_{12}^{(2n+1)})^2) \rightarrow \Delta_{2n+1}$ which is impossible by the inductive hypothesis. \qed
Theorem 1.2 from the Introduction follows immediately, since Hamiltonian isotopy implies isomorphism in the Donaldson-Fukaya category. In particular, the homomorphism from the braid group to the symplectic mapping class group of the moduli space of bundles does not factor through the symmetric group.

Remark 5.11. It would be interesting to know for which labels the braid group action on $M_n(\mu)$ factors through the symmetric group and to identify the kernel and image of the map $B_n \to \text{Map}(M_n(\mu), \omega)$. In the case without labels, M. Callahan (unpublished) announced a similar result for a separating Dehn twist of a genus two surface, in the moduli space of fixed-determinant bundles of rank two and degree one. Callahan’s result together with the results of this paper would imply that a separating Dehn twist is not symplectically isotopic to the identity in any genus. Analogous results for surfaces without markings are proved in I. Smith [37]. See Keating [16] for related results.

Remark 5.12. (a) (Surgery exact triangle) The surgery exact triangles for the theory (in the rank two are a consequence of a generalization of a triangle for Dehn twists by Seidel, proved in [45].

(b) (Relationship to Seidel-Smith-Manolescu theory?) The nilpotent slices used in the definition of the Seidel-Smith [36] and Manolescu [25] invariants embed in the moduli spaces of parabolic bundles, for sufficiently small weights; the map is given by assigning to a given matrix in the nilpotent slice the quasi-parabolic structure given by the eigenspaces, which sit inside a canonical rank 2 sub-bundle of the trivial bundle of rank 2n. One expects a spectral sequence relating the instanton knot homologies with the Khovanov-Rozansky invariants [18], similar to that developed by Kronheimer-Mrowka [21].

5.3. Field theory for graphs. In this section we sketch an extension to functors for graphs in trivial bordisms. Graphs naturally arise in the surgery exact triangle for higher-rank tangle functors, see [45].

Definition 5.13. (Graphs) Let $(X_-,\mathcal{I}_-),(X_+\mathcal{I}_+)$ be compact, connected, oriented marked surfaces. A graph from $(X_-,\mathcal{I}_-)$ to $(X_+\mathcal{I}_+)$ consists of an oriented compact
connected manifold $Y$, an embedding $|\Gamma| \hookrightarrow Y$ of the underlying topological space $|\Gamma|$ that maps
(a) the valence one vertices of $\Gamma$ to the boundary of $Y$,
(b) the valence greater-than-one vertices of $\Gamma$ to the interior of $Y$, and
(c) the interior of each edge to the interior of $Y$;
and an identification $\phi : \partial(Y, \Gamma) \to (X_-, \underline{x}_-) \cup (X_+, \underline{x}_+)$, with orientation on the first factor reversed. For simplicity we omit the embedding from the notation and denote by $\Gamma \subset Y$ the image of the given embedding. An equivalence of graphs is a diffeomorphism inducing the identity on the incoming and outgoing boundary components $(X_\pm, \underline{x}_\pm)$.

**Definition 5.14.** (Cerf decompositions for graphs) Let $(Y, \Gamma, \phi)$ consist of a trivial bordism $Y = [b_-, b_+] \times X$ of a closed, connected, oriented surface $X$ and an oriented graph $\Gamma$ in $Y$.

(a) A cylindrical Morse datum for $(Y, \Gamma, \phi)$ consists of a pair $(f, b)$ consisting of a smooth function $f : Y \to \mathbb{R}$ and a collection $b = (b_0 < \ldots < b_m = b_+)$ of real numbers such that

(i) each $f^{-1}(b_j)$ contains no critical points of $f|_\Gamma$ or interior vertices
(ii) each $f^{-1}(b_{k-1}, b_k)$ contains at most one critical point of $f|_\Gamma$ or vertex of $\Gamma$.
(iii) $\{b_+\} \times X$ resp. $\{b_-\} \times X$ is the set of maxima resp. minima of $f$;
(iv) $\partial_t f(t, x) > 0$ for all $(t, x) \in Y$;
(v) $f$ restricts to a Morse function on each edge of $\Gamma$;
(vi) the restriction of $f$ to any edge has critical points only on the interior of the edge; and
(vii) $f|_\Gamma$ is injective on the union of the critical set of $f|_\Gamma$ and the set of valence-greater-than-one vertices of $\Gamma$.

(b) Any cylindrical Morse datum $(f, b)$ of $(Y, \Gamma, \phi)$ gives rise to a cylindrical Cerf decomposition of $(Y, \Gamma, \phi)$ into elementary bordisms-with-graphs

$$(Y_j := f^{-1}([b_{j-1}, b_j]), \Gamma_j := Y_j \cap \Gamma, \phi_j), \quad j = 1, \ldots, m.$$ That is, each $Y_j$ is cylindrical and $\Gamma_j$ has at most one critical point or vertex.

**Theorem 5.15.** (Cerf theory for graphs) Any two cylindrical Cerf decompositions of a graph in a cylindrical bordism are related by a finite sequence of critical point cancellations or creations, critical point/vertex order reversals, and vertex/critical point cancellations, and gluing elementary graphs with no critical points or vertices to adjacent elementary graphs. See Figures 9, 10, 11.

**Sketch of proof.** The proof is similar to that of Theorem 2.4. The new feature is that the edges have boundary points, a critical point may run into a vertex; the move depicted in Figure 11 results. Let $(f_j, b_j)$ be two Morse data for $(Y, \Gamma, \phi)$. A generic homotopy $\tilde{f} : Y \times [0, 1] \to \mathbb{R}$ has the properties that (i) $\partial_t \tilde{f}(y, s) > 0$ for all $(y, s) \in Y \times [0, 1]$ and (ii) the restriction $\tilde{f}|_e$ of $\tilde{f}$ to each edge $e$ gives a good homotopy in the sense that $\tilde{f}|_e$ is a Morse function injective on its critical set except
for a finite number of times $s_1, \ldots, s_n \in [0, 1]$ where at most one of the following occur:

(a) critical point cancellation occurs in the interior
(b) a critical point occurs at an endpoint or
(c) two critical points or endpoints have the same value.

Indeed, for no critical point cancellation to occur at the endpoints it suffices that for each endpoint \( p \) and time \( s \), either \( df_{s|c}(p) \) or \( d^2f_{s|c}(p) \) is non-zero, which is always the case for generic homotopies.

**Definition 5.16.** (a) (Labels meeting a vertex) Let \((Y, \Gamma, \phi)\) be an elementary graph with a single vertex \( v \). Denote the boundary of \( Y \) by \( \partial Y = X_- \cup X_+ \). Let \( B(v) \subset Y \) be a small open ball containing \( v \), and

\[
S(v) = \partial B(v), \quad \underline{\mathcal{V}}(v) := S(v) \cap \Gamma
\]

denote the sphere around the vertex and the intersections with the graph. The complement \( Y \setminus B(v) \) of \( B(v) \) can be viewed as a three-dimensional bordism from \( X_- \cup S(v) \) to \( X_+ \), containing a tangle \( \Gamma \setminus (B(v) \cap \Gamma) \). Let \( \underline{\mu}_\pm \) denote the labels for \( \underline{\mathcal{V}}_\pm = \Gamma \setminus X_\pm \), and \( \underline{\mu}(v) \) denote the set of labels for \( \underline{\mathcal{V}}(v) \), given by the labels of the edges incoming to a vertex and the images of the labels under the involution \(*\) for the outgoing edges.

(b) (Admissible labellings) A set of labels \( \underline{\mu}(v) \) at a vertex \( v \) is vertex-admissible if the moduli space of flat bundles on the punctured sphere \( M(S(v), \underline{\mathcal{V}}(v), \underline{\mu}(v)) \) is either empty or a point. An vertex-admissible labelling of \( \Gamma \) is a labelling of the edges of \( \Gamma \) by admissible labels, such that at each vertex the collection of labels is vertex-admissible. An vertex-admissible graph is a graph equipped with a vertex-admissible labelling.

(c) (Standard labellings) Let \( \beta_j \) denote the \( j \)-th fundamental coweight of \( SU(r) \). Denote by \( j = \beta_j/2 \). Suppose that \( G = SU(r+1) \). A standard labelling of \( \Gamma \) is a labelling of each edge by 1 or 2, so that each vertex is trivalent with labels 1, 1, 2. The triple 1, 1, 2 is analogous to Khovanov-Rozansky’s 1, 2 (or thin, thick) labels \([18]\).

**Lemma 5.17.** The moduli space \( M(S^2, 1, 1, *2) \) is a point, hence any standard labelling of a bordism-with-graph is admissible.

**Proof.** The moduli space \( M(S^2, 1, 1, *2) \) is the space of equivalence classes of pairs \((g_1, g_2) \in \mathcal{C}_2^2 \) with \( g_1g_2 \in \mathcal{C}_2 \). After conjugation we may assume

\[
g_1 = \text{diag}(-\exp(\pi i/r), \exp(\pi i/r), \ldots, \exp(\pi i/r)).
\]

The centralizer of \( g_1 \) is therefore

\[
Z = S(U(1) \times U(r - 1)) \cong SU(r - 1).
\]

Let \( O \subset G \) denote the one-parameter subgroup generated by rotation in the first two coordinates in \( \mathbb{C}^r \). Since \( g_1 \) is the product of \( \text{diag}(-1, 1, \ldots, 1) \) with a central element in \( U(r) \), the adjoint action of \( g_1 \) on \( O \) is \( g_1og_1^{-1} = o^{-1} \). This implies that

\[
o \in \text{Ad}(o^{1/2})g_1 \in \mathcal{C}_1, \forall o \in O.
\]

Now \( \mathcal{C}_1 \) is a symmetric space of rank one, and in particular \( Z \) acts transitively on the unit sphere in \( T_g \mathcal{C}_1 \) which implies that the map \( O g_1 \rightarrow \mathcal{C}_1/Z \) is surjective.
Therefore after conjugation by an element of $Z$ we may assume that
\[ g_2 = og_1 = g_1 o^{-1} \]
for some $o \in O$. Also note that since $O$ is conjugate to the one-parameter subgroup generated by the first coroot $\alpha_1^\vee$ the square of $C_1$ in $G$ is
\[ C_1^2 = \text{Ad}(G)\{g_2^2 o, o \in O\} = \bigcup_{\epsilon \in [0,-1/2]} C_{\omega_1 + \epsilon \alpha_1^\vee} \]
the union of conjugacy classes of $\exp(\omega_1 + \epsilon \alpha_1)$ where $\epsilon \in [0,-1/2]$. In particular, since
\[ \omega_2/2 = \omega_1 - \alpha_1 \]
the conjugacy class $C_2$ of $\exp(\omega_2/2)$ appears in $C_1^2$. Hence the moduli space is nonempty, and a dimension count shows that it is dimension zero, hence a point. \( \square \)

**Lemma 5.18.** (Correspondence for vertex-admissible graphs is simply-connected and relatively spin) Let $\Gamma$ be a elementary graph containing a single vertex with incoming labels $1, 1$ and outgoing labels $2$. Then $L(Y, \Gamma, \phi)$ embeds in $M(X_-, \underline{x}_-)$ and is an $S^2$-bundle over $M(X_+, \underline{x}_+)$, and in particular admits a relative spin structure with background class $(w_2(M(X_-, \underline{x}_-)), 0)$.

**Proof.** Let $Y, \Gamma$ be as in the statement of the Lemma. By Lemma 5.17 the correspondence $L(Y, \Gamma, \phi)$ may be identified with the set of points in the moduli space for the incoming surface
\[ M(X_-, \underline{x}_-) = (G^{2g} \times C_{\underline{\mu}_-} \setminus \{1,1\} \times C_1 \times C_1) // G \]
such that the product of the last two factors lies in $C_2$. Thus the map to $M(X_-, \underline{x}_-)$ is an embedding and the map to the moduli space for the outgoing surface
\[ M(X_+, \underline{x}_+) = (G^{2g} \times C_{\underline{\mu}_-} \setminus \{1,1\} \times C_2) // G \]
has fiber equal to the quotient of stabilizers
\[ S(U(2) \times U(r-2)) / (S(U(1) \times U(r-1)) \cap \text{Ad}(\sigma_{12})S(U(1) \times U(r-1))) \cong S^2 \]
where $\sigma_{12}$ is the $(12)$ permutation matrix. The Lemma follows. \( \square \)

**Definition 5.19.** (Correspondence for vertex-admissible labellings) Suppose $(Y, \Gamma)$ is a graph with labelling $\nu$ that is vertex-admissible for each vertex. Let $M(Y, \Gamma)$ denote the moduli space of flat bundles on the complement of $\Gamma$ in $Y$ with holonomies around the edges of $\Gamma$ given by $\nu$. Restriction to the boundary and pullback under the boundary identification define a map
\[ M(Y, \Gamma) \to M(X_-, \underline{x}_-) \times M(X_+, \underline{x}_+). \]  
Denote the image of (19) by $L(Y, \Gamma, \phi)$.

**Lemma 5.20.** Let $(Y, \Gamma, \phi)$ be an elementary graph containing a single vertex and $\nu$ an admissible labelling of the edges of $\Gamma$. Then $L(Y, \Gamma, \phi)$ is a smooth Lagrangian correspondence from $M(X_-, \underline{x}_-)$ to $M(X_+, \underline{x}_+)$. 
Proof. We write $\mu_+ - \mu(v)$ resp. $\mu_+ \cap \mu(v)$ for the labels of those markings in $x_+$ that are not resp. are connected to $v$ by an edge. By (9), the symplectic forms on the two ends are those obtained by reduction from

$$(20) \quad \omega_{|g,\mu_+ - \mu(v)} + \omega_{|0,\mu_+ \cap \mu(v)} + (1/2)\langle \Phi^* g,\mu_+ - \mu(v) \rangle \wedge \Phi^* 0,\mu_+ \cap \mu(v) \rangle.$$ 

Let $d$ be the value of $f$ at the vertex and $\epsilon$ a small number. The level sets $f^{-1}(d - \epsilon)$ are isomorphic, by the flow used in the proof of Lemma 3.18. Choose a presentation for the fundamental group of $f^{-1}(d - \epsilon)$; then a presentation for the fundamental group of $f^{-1}(d - \epsilon)$ is obtained by replacing the generators for the strands incoming to the vertex with those outgoing. With respect to this set of generators, the correspondence defined by the bordism is

$$(21) \quad \prod_{\mu \in \mu_+ \cap \mu(v)} c_{c} = \prod_{\mu \in \mu_+ \cap \mu(v)} c_{c}$$

and descending to the quotient. This equation defines an isotropic submanifold of $C_{\mu_+ \cap \mu(v)} \times C_{\mu_+ \cap \mu(v)}$ since the moduli space for the sphere around the vertex is a point by Lemma 5.17. It follows from (20) that the (21) defines an isotropic, hence Lagrangian submanifold of the product $M(X_-,x_-) \times M(X_+,x_+)$.

The following associates a generalized Lagrangian correspondence to any graph with admissible labelling:

**Definition 5.21.** (Generalized Lagrangian correspondence for a decorated graph) Let $(f,b)$ be a cylindrical Cerf decomposition of $\Gamma$ equipped with vertex-admissible, monotone, spin labels $\mu$. Let $L(Y_j,\Gamma_j,\phi_j) \subset M(X_{j-1},x_{j-1}) \times M(X_j,x_j)$ denote the Lagrangian submanifold of the product $M(X_-,x_-) \times M(X_+,x_+)$. Define

$L(Y,\Gamma,\phi) := (L(Y_1,\Gamma_1,\phi_1),\ldots,L(Y_m,\Gamma_m,\phi_m)).$

**Proposition 5.22.** (Independence of the generalized Lagrangians from all choices up to equivalence) Let $(Y,\Gamma,\phi)$ be an admissible decorated graph from $(X_-,x_-)$ to $(X_+,x_+)$. Then the generalized Lagrangian correspondence $L(Y,\Gamma,\phi)$ is independent, up to equivalence, of the choice of Cerf decomposition.

**Proof.** By Theorem 5.15 it suffices to check that the generalized Lagrangian correspondences are invariant up to composition equivalence under the Cerf moves. The proof is similar to that of Theorem 3.25 and left to the reader.

**Definition 5.23.** (Decorated Graphs) For coprime integers $r,d > 0$ and a compact oriented surface $X$ let $\text{Graph}(X,r,d)$ denote the category of graphs whose

(a) objects are collections of distinct oriented points of $X$ with admissible labels;
(b) morphisms are equivalence classes of cylindrical graphs $([-1,1] \times X,\Gamma,\phi)$ with admissible labellings as in Definition 5.16;
(c) composition is given by gluing, and
(d) identity is the equivalence class of the trivial graph.

The following extends Theorem 3.25 to graphs.
Theorem 5.24. (Symplectic-valued field theory for graphs) For coprime integers $r, d > 0$, the partially defined functor $\Phi : \text{Graph}(X, r, d) \to \text{Symp}_{1/2r}$ for elementary graphs extend to a field theory for graphs in $X$.

Proof. By Proposition 5.22, it suffices to show that the correspondences are equipped with relative spin structures; these are provided by Lemma 3.22 for correspondences involving critical points, and Lemma 5.18 for correspondences involving vertices. □

Using Corollary 4.18 we obtain a category-valued field theory for graphs in particular assigning to any graph with admissible labels a functor between Donaldson-Fukaya categories.

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