SHARP NULL FORM ESTIMATES ON ENDLINE GEOMETRIC CONDITIONS OF THE CONE

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Abstract. We prove \( H^{\alpha_1} \times H^{\alpha_2} \to L^q_t L^r_x \) null form estimates for solutions to homogeneous wave equations with \((q, r)\) on the endline of the condition concerning geometry of the cone, except the critical index. This extends the previous endpoint result of Tao, Math. Z. 238, no. 2, 215–268, (2001) in symmetric-norms to mixed-norms and improves the local-in-time result of Tataru, MR1979953, to be global in the setting of constant variable coefficient equations, as well as the sharp off-endline estimates established by Lee and Vargas, Amer. J. Math 130 (2008), no. 5, 1279–1326, to the borderline with respect to the cone condition. Our proof is based on the multiplier theory in mixed-norms, which ultimately reduces the question to a uniform endline bilinear restriction estimates including high-low frequency interactions for a family of conic type surfaces depending on a parameter \( \sigma \), which converges to the oblique cone as \( \sigma \to 0 \). We prove this uniform estimate by using the enhanced version of the induction-on-scale method.

1. Introduction

1.1. Motivation and background. Let \( n \geq 2 \) be a fixed integer and \( \phi, \psi \) be solutions of the homogeneous wave equation in \( \mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n \)

\( \Box \phi = 0, \quad \Box \psi = 0, \quad (\Box = -\partial^2_t + \Delta_x, \ t \in \mathbb{R}, \ x \in \mathbb{R}^n) \) (1.1)

subject to the initial conditions at \( t = 0 \)

\( \phi[0] := (\phi(0, \cdot), \partial_t \phi(0, \cdot)) = (\phi_0, \phi_1), \quad \psi[0] := (\psi(0, \cdot), \partial_t \psi(0, \cdot)) = (\psi_0, \psi_1). \)

For \( \alpha_1, \alpha_2 \in \mathbb{R} \), define

\[ \|\phi[0]\|_{\dot{H}^{\alpha_1}} = \left( \|\phi_0\|_{\dot{H}^{\alpha_1}}^2 + \|\phi_1\|_{\dot{H}^{\alpha_1-1}}^2 \right)^{\frac{1}{2}}, \quad \|\psi[0]\|_{\dot{H}^{\alpha_2}} = \left( \|\psi_0\|_{\dot{H}^{\alpha_2}}^2 + \|\psi_1\|_{\dot{H}^{\alpha_2-1}}^2 \right)^{\frac{1}{2}}, \]

where \( \dot{H}^{\alpha} = \dot{H}^{\alpha}(\mathbb{R}^n) \) is the homogeneous \( L^2 \)-based Sobolev space of order \( \alpha \).

Let \( D_0, D_+, D_- \) be the Fourier multipliers defined by

\[ \widehat{D_0 f}(\tau, \xi) = |\xi| \hat{f}(\tau, \xi), \]
\[ \widehat{D_+ f}(\tau, \xi) = (|\tau| + |\xi|) \hat{f}(\tau, \xi), \]
\[ \widehat{D_- f}(\tau, \xi) = ||\tau| - |\xi|| \hat{f}(\tau, \xi), \]

where \( \hat{\cdot} \) denotes the space-time Fourier transform in sense of distributions on \( \mathbb{R}^{1+n} \) and \( (\tau, \xi) \) are the Fourier variables corresponding to \( (t, x) \).

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We are interested in the validity of the null form estimates for $\phi$ and $\psi$
\[ \|D_0^{\beta_0}D_+^{\beta_+}D_-^{\beta_-}(\phi\psi)\|_{L_t^q(\mathbb{R},L_x^r(\mathbb{R}^n))} \leq C \|\phi[0]\|_{\mathcal{H}^{\alpha_1}} \|\psi[0]\|_{\mathcal{H}^{\alpha_2}}, \tag{1.2} \]
for some finite constant $C$, with $1 \leq q, r \leq \infty$ such that the following conditions on $(q, r, \alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-)$ are imposed:

- Scaling invariance
  \[ \beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 + \frac{1}{q} - n\left(1 - \frac{1}{r}\right), \tag{1.3} \]

- Geometry of the cone
  \[ \frac{1}{q} \leq \frac{n+1}{2} \left(1 - \frac{1}{r}\right), \quad \frac{1}{q} \leq \frac{n+1}{4}, \tag{1.4} \]

- Concentration along null directions
  \[ \beta_- \geq \frac{1}{q} - \frac{n-1}{2} \left(1 - \frac{1}{r}\right), \tag{1.5} \]

- Low frequencies in $(++)$ interaction
  \[ \beta_0 \geq \frac{1}{q} - n\left(1 - \frac{1}{r}\right), \tag{1.6} \]
  \[ \beta_0 \geq \frac{2}{q} - (n+1)\left(1 - \frac{1}{r}\right), \tag{1.7} \]
  \[ \beta_0 \geq \frac{2}{q} - n\left(1 - \frac{1}{r}\right) - \frac{1}{2}, \tag{1.8} \]

- Low frequencies in $(+-)$ interaction
  \[ \alpha_1 + \alpha_2 \geq \frac{1}{q}, \tag{1.9} \]
  \[ \alpha_1 + \alpha_2 \geq \frac{3}{q} - n\left(1 - \frac{1}{r}\right), \tag{1.10} \]

- Interaction between high and low frequencies
  \[ \alpha_1, \alpha_2 \leq \beta_- + \frac{n}{2}, \tag{1.11} \]
  \[ \alpha_1, \alpha_2 \leq \beta_- + \frac{n}{2} - \frac{1}{q} + \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right), \tag{1.12} \]
  \[ \alpha_1, \alpha_2 \leq \beta_- + \frac{n}{2} - \frac{1}{q} + n\left(\frac{1}{2} - \frac{1}{r}\right), \tag{1.13} \]
  \[ \alpha_1, \alpha_2 \leq \beta_- + \frac{n}{2} - \frac{1}{q} + n\left(\frac{1}{2} - \frac{1}{r}\right) + \left(\frac{1}{2} - \frac{1}{q}\right), \tag{1.14} \]
  \[ \alpha_1, \alpha_2 \leq \beta_- + \frac{n+1}{2} - \frac{1}{q}, \tag{1.15} \]
  \[ \alpha_1, \alpha_2 \leq \beta_- + \frac{n}{2} - \frac{1}{q} + n\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{1}{r} - \frac{1}{q}, \tag{1.16} \]

- Extra conditions for low frequencies of $(+-)$ interactions in low dimensions
  \[ \alpha_1 + \alpha_2 \geq \frac{3}{q} + \frac{1}{r} - 2, \quad n = 3, \tag{1.17} \]
  \[ \alpha_1 + \alpha_2 \geq \frac{3}{q} + \frac{1}{2r} - \frac{5}{4}, \quad n = 2. \tag{1.18} \]
That the conditions (1.3)-(1.7) and (1.9)-(1.14) are necessary for (1.2) to hold was described first by Foschi and Klainerman [6], and it is conjectured by the authors [6] that for \( n \geq 2 \) and \( 1 \leq q, r \leq \infty \), these conditions should also be sufficient. The condition (1.5) is related to Lorentz invariance of the cone [6, 19]. In [13], Lee and Vargas observed that in order for (1.2) to be true, (1.15)-(1.16) and (1.8) are needed as well. Moreover, by testing a more refined example in the low frequency interactions for the (+−) case, Lee, Rogers and Vargas [12] proved that when \( n = 3, 2 \), (1.2) requires an additional conditions (1.17) and (1.18) respectively.

It is convenient to classify the conditions (1.3)-(1.18) into three categories. We reserve the same terminology for (1.3) and (1.4), to which we shall refer as the first and second categories. We shall call (1.4) the **cone condition** below for brevity. The conditions (1.5)-(1.18) are packed up as the third category, to which we refer as the **multiplier conditions**.

In the case when \( q = r = 2 \), Foschi and Klainerman [6, Theorem 1.1] has determined the exact conditions on \((\alpha_1, \alpha_2, \beta_0, \beta_+ ,\beta_-)\) for estimates (1.2) to be true in the \( L^2_{t,x}(\mathbb{R}^{1+n}) \) norm for all \( n \geq 2 \).

For cases beyond bilinear-\( L^2_{t,x} \) setting, the null form estimate encircles in some sense the bilinear Fourier restriction estimates for surfaces of disjoint conic type. For frequency localized waves without Fourier multipliers, bilinear estimates with \( q = r < 2 \) in the two dimensional case was first established by Bourgain [3] under separateness of the frequency variables. This is a genuine bilinear restriction estimate on the cone, a remarkable progress towards the conjecture of Klainerman and Machedon [6, 22]. The result was improved later by Tao, Vargas and Vega [17] and Tao-Vargas [18, 19], which shapes nowadays the standard bilinear method to the restriction and Kakeya conjectures, see [5] for more investigations. The first sharp result is obtained by Wolff [22] on the cone, except the endpoint case, which is settled by Tao [15], by building up the wave-table theory and exploiting energy concentrations in the physical space along null directions of the cone. Extensions of this result to the mixed-norms on the endline \( q = \frac{n+1}{n} \) in (1.4) with the exception of the critical index \((q_c, r_c)\), which by definition reads

\[
q_c = \max \left( 1, \frac{4}{n+1} \right), \quad r_c = \min \left( 2, \frac{n+1}{n-1} \right),
\]

(1.19)
is obtained by Temur [21], improving the previous off-endline result due to Lee and Vargas [13] to the borderline case. Note that the critical index is out of reach by the current method. Indeed, it shares a level of the same difficulty with the endpoint multilinear restriction conjecture of Bonnett-Carbery-Tao [2], a very difficult open question. Even as a weaker result, the endpoint multilinear Kakeya inequality can only be established through the intricate algebraic topological method by Guth [8]. A much simpler version of the proof is given by Carbery and Valdimarsson [4]. We find the endpoint Fourier restriction theorems become more interesting recently. We refer to [7] for its connexion with the unrectifiability of measures and [10, 14, 9] for their role in the Calderón problem.

The null form estimates (1.2) in \( q = r \geq 2 \) with nontrivial multipliers is first established by Klainerman and Tataru [11], while the \( q = r < 2 \) case first by Tao

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1 Conjecture 14.16 in [6].
2 We conjecture the two dimensional case should be available.
More refined $L^p$-null form results in symmetric norms with optimal conditions were established by Tao [15], based on the endpoint bilinear restriction theorem on the light cone including the interaction between high-low frequencies. In [13], Lee and Vargas proved the sharp null form estimates under the scaling condition (1.3) and that (1.4)-(1.16) are all valid with strict inequalities for all $n \geq 4$, and for $n = 2, 3$ with a gap $\frac{1}{q+1} \leq q \leq \frac{1}{4}$ when $2 < r \leq \infty$. This gap is supplied by Lee, Rogers and Vargas [12] when $n = 3$ incorporated with the new condition (1.17), with strict inequality, whereas only a partial result was obtained in [12] for the two dimensional case, under an apparently stronger condition $\alpha_1 + \alpha_2 > \frac{q}{q} - 1$ when $r > 2$. This is related to an open problem concerning the endpoint bilinear Strichartz estimate for one-dimensional Schrödinger equations.

Note that all of the above results concerning (1.2) are established in the spacetime norms $L^q_t L^r_x$ with $(q, r)$ being off the endline of the cone condition (1.4) and that all the multiplier conditions are assumed to hold with strict inequalities, only except that in [15, Theorem 17.3], an endpoint result was obtained for symmetric norms $q = r \geq \frac{2n}{n+1}$, and in this case, the low frequency $(++)$ interaction condition is allowed to be taken as an equality. This result was extended to mixed-norms on the endline by Tataru [20] for second order hyperbolic operators with rough coefficients, where the estimates are obtained locally in time. One of the difficulties for the global result in mixed-norms on the endline is the obstacle of using Lorentz invariance.

The purpose of this paper is to prove the null form estimates (1.2) for all $(q, r)$ on the endline of the cone condition (1.4) except the critical index $(q_c, r_c)$, with the low frequency $(++)$ interaction condition allowed to take equality, and all the other relevant sharp multiplier conditions hold with strict inequalities.

**Theorem 1.1.** Let $n \geq 2$ and $1 \leq q, r \leq \infty$. Assume the following conditions hold

$$
\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 - q^{-1}\left(\frac{n-1}{n+1}\right),
$$

$$
\frac{1}{q} = \frac{n+1}{2}\left(1 - \frac{1}{r}\right), \quad \frac{1}{q} < \min\left(1, \frac{n+1}{4}\right),
$$

$$
\beta_0 \geq 0,
$$

$$
\beta_- > \frac{2}{q(n+1)},
$$

$$
\alpha_1 + \alpha_2 > q^{-1}\left(\frac{n+3}{n+1}\right),
$$

$$
\alpha_1, \alpha_2 < \beta_- + q^{-1}\left(\frac{n-1}{n+1}\right),
$$

$$
\alpha_1, \alpha_2 < \beta_- + \frac{1}{2} - \frac{2}{q(n+1)}.
$$

Then, there is a constant $C$ depending possibly on $(q, r, \alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-)$ such that (1.2) holds for all $(\phi[0], \psi[0]) \in H^{\alpha_1}(\mathbb{R}^n) \times H^{\alpha_2}(\mathbb{R}^n)$.

On the endline of (1.4), only a partial components of the necessary conditions in (1.3)-(1.18) that matter. We selected them for sake of reading. The strict inequalities (1.23)-(1.26) are responsible for the summability on various of dyadic geometric series. In certain off-endline cases such as $(q, r) = (2, 2)$ and $(\infty, 2)$, the condition associated to interactions of low frequencies in the $(++)$ case being strict
inequality is also necessary \[6\]. Moreover, in one of these two cases, certain special choices of \((\alpha_1, \alpha_2, \beta_-)\) on the equalities of some of the multiplier conditions are known to be inadmissible.

The strict inequalities in (1.23) and (1.24) are used in the proof to tackle the fake singularity by angular compression, which is related to the cone property (c.f. Example 14.3 and Example 14.9 of [6]). It is employed as a stepstone for the bilinear Whitney type dyadic decomposition along the angular direction incorporated with a rescaling argument. This strategy is of course rather crude. To push these conditions to equalities, one may have to prove certain bilinear square-function estimates with respect to angular decompositions.

Theorem 1.1 extends the null form estimates at the endpoint in the symmetric norm of Tao \[15\] to the whole endline apart from the critical index \((q_c, r_c)\), with the low-frequency interaction \((++\)) condition (1.22) being able to attain the equality. It also extends the local-in-time estimates of Tataru \[20\] to be global-in-time for the constant coefficient wave equations. In particular, it can be added to the picture of sharp null form estimates off the endline, attributed to Lee-Vargas \[13\] and Lee-Rogers-Vargas \[12\], where the condition on \(\beta_0\) was required to hold with strict inequality.

1.2. **Endline bilinear estimates for a family of conic type surfaces.** To prove Theorem 1.1 we adopt a hybrid of the dyadic decomposition in the frequency space introduced in \[19\] and \[15, 13\], which allows to reduce (1.2) to a uniform endline bilinear restriction estimates in Theorem 1.2 below for a family of conic type surfaces depending on a small parameter \(\sigma > 0\).

This uniform bilinear estimate is interesting in its own right. As a by-product, we investigate by the end of this paper how one can combine its proof with the method of descent to get the endpoint bilinear restriction estimates on the unit sphere, an open question raised by Foschi and Klainerman \[6, Conjecture 17.2\], where the sharp non-endpoint case can be included into the elliptic type surfaces and has already been settled by Tao \[16\]. Unlike the question on paraboloids in \[23\], one has to control the effects of certain perturbations on the phase functions.

To state our uniform bilinear estimates, we first introduce some notations. For each small number \(\sigma > 0\), let \(E_{\sigma}\) be the class of \(\mathbb{R}\)-valued functions \(\Phi_{\sigma}\) defined on \(\mathbb{R}^n\) satisfying the following conditions:

- \(\Phi_{\sigma} \in C^\infty(\mathbb{R}^n \setminus \{0\})\) is homogeneous of order one,
- \(\Phi_{\sigma}(0, 1) = 0, \nabla_{\xi'} \Phi_{\sigma}(0, 1) = 0 \in \mathbb{R}^{n-1}, \nabla_{\xi'}^2 \Phi_{\sigma}(0, 1) = \text{Id}_{\mathbb{R}^{n-1}},\)
- \(\text{rank } \nabla_{\xi'} \Phi_{\sigma}(\xi', \xi_n) = n - 1,\) for all \(\xi \in \mathbb{R}^n \setminus \{0\}\)

with \(\xi := (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R},\) such that on \(\mathcal{N} = \{\xi \in \mathbb{R}^n : |\xi'| \leq 3|\xi_n|\}\) and for any multi-indices \(\alpha = (\alpha_1, \ldots, \alpha_n)\) with \(\alpha_j \in \mathbb{Z} \cap [0, \infty)\) \(\forall j,\) there exists a finite constant \(C = C_\alpha > 0\) depending only on \(n\) and \(\alpha\) for which we have

\[
|\partial^\alpha_{\xi'} (\Phi_{\sigma}(\xi) - \frac{|\xi'|^2}{2\xi_n})| \leq C \sigma^2 |\xi|^{1-|\alpha|}, \quad \forall \xi \in \mathcal{N} \cap (\mathbb{R}^n \setminus \{0\}),
\]

where \(|\alpha| := \alpha_1 + \cdots + \alpha_n.\)

Let \(\Omega \subset \mathcal{N}\) be a compact set away from the origin and define

\[
S_{\Omega}^{\Phi_{\sigma}} : f \mapsto \int_{\Omega} e^{i(x \cdot \xi + t \Phi_{\sigma}(\xi))} \hat{f}(\xi) \, d\xi,
\]
initially on Schwartz functions $S(\mathbb{R}^n)$. Then, $S_{\mathbb{R}^n}^{\Phi_{\sigma}} = e^{it\Phi_{\sigma}(D)}$ with $D = -i\nabla_x$. This corresponds to the adjoint Fourier restriction operator on the conic surface

$$C_{\Omega}^{\Phi_{\sigma}} := \{(\Phi_{\sigma}(\xi), \xi) : \xi \in \Omega\} \subset \mathbb{R}^{1+n},$$

a Monge patch given by the graph of the function $\Phi_{\sigma}$ over $\Omega$. For brevity, we shall refer to the given $\Phi_{\sigma}$ as a Monge function.

**Theorem 1.2.** Let $n \geq 2$ and $\Sigma_1, \Sigma_2$ be defined as

$$\Sigma_j = \{\xi \in \mathbb{R}^n : 1 \leq (-1)^j \xi_{n-j}, \xi_n \leq 2, |\xi''| \leq 1\}, \quad j = 1, 2,$$

where $\xi = (\xi''', \xi_{n-1}, \xi_n) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}$. Then, there exists $\sigma_0 > 0$, such that for any $\varepsilon > 0$, and $(q, r)$ satisfying

$$\frac{1}{q} < \min\left(1, \frac{n+1}{4}\right), \quad \frac{1}{q} = \frac{n+1}{2}\left(1 - \frac{1}{r}\right),$$

with $1 \leq q, r \leq \infty$, there exists $C = C_{\Sigma_1, \Sigma_2, q, r, \sigma_0, \varepsilon}$ such that

$$\left\| (S_{\Sigma_1}^{\Phi_{\sigma}} f)(S_{\Sigma_2}^{\Phi_{\sigma}} g) \right\|_{L^q_x(\mathbb{R}; L^r(\mathbb{R}^n))} \leq C \mu_{\max}\left(\frac{1}{q} - \frac{1}{r}\right)^{\varepsilon} \left\| f \right\|_{L^q} \left\| g \right\|_{L^r},$$

(1.28)

holds for all $\mu \geq 1$, $f, g \in S(\mathbb{R}^n)$ and all $\Phi_{\sigma} \in \mathcal{E}_{\sigma}$, $\forall \sigma \in (0, \sigma_0]$.

Theorem 1.2 is a stability result of the endline bilinear estimate for the oblique cones $C_{\Sigma_1} := \{(\frac{\xi'''}{\xi_{n-1}}, \xi', \xi_n) : 1 \leq \xi_{n-1} \leq 2, |\xi'| \leq 3\}$ under small perturbations as the conic type surfaces given by the graph of Monge functions in $\bigcup_{\sigma \in (0, \sigma_0]} \mathcal{E}_{\sigma}$ provided $\sigma_0$ is small enough.

The $\mu$-loss arises from the iterative argument for the essential concentration of the waves on conic regions. It is responsible for the requirement on the conditions (1.25) (1.26) with strict inequalities in Theorem 1.1. To push them to equalities, one has to remove the $\mu$-loss first. Once this having been done, it is also necessary to establish certain vector-valued bilinear adjoint restriction estimates, as pointed out in [13][15], to replace the crude argument in Section 2.

The $\mu$-loss is irrelevant to the condition $\beta_0 \geq 0$, since (1.22) will only be used in the low-frequency $(++)$ case where $\mu \sim 1$. This inequality is required to be strict in the off-endline case [13] and it is allowed to be taken as equality in the endpoint result [15]. To get the endline estimates (1.2) with all $\beta_0 \geq 0$ for the $r > 1$ case, we shall combine an argument going back to Klainerman-Tataru [11], also used in [13][13], with a Mihlin-Hörmander multiplier estimate to remove the strict inequality postulated in (1.7). The $L^q_x L^r_{\sigma} -$case will be treated separately by a classical $L^2 \times L^2 \rightarrow L^1$ paraproduct estimate, easily deduced from the Coifman-Meyer argument. Note that this strict inequality condition on $\beta_0$ does not cause any wastage for the sharp results of [13] in the off-endline case.

Theorem 1.2 is a uniform endline bilinear restriction estimates for a family of surfaces of disjoint conic type satisfying the axioms of Tao and Vargas in [18][19], including high-low frequency interactions as [15]. We imposed more restrictive conditions here in order to simplify the argument. It is plausible that one may relax the $\mathcal{E}_{\sigma}$ conditions. Notice that even for the limiting case when $\sigma = 0$, (1.28)

\[3\]The Stein operator.
is highly nontrivial. Indeed, (1.28) in symmetric norms with \( q = r = \frac{n+3}{n+1} \) on \( \mathcal{C}_{\text{obl}} \) can be easily deduced from Theorem 1.1 of [15] via changing variables

\[
(x', x_n, t) \to \left( x', x_n + \frac{t}{\sqrt{2}}, x_n - \frac{t}{\sqrt{2}} \right),
\]

with \( x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \) and \( (\xi', \xi_n) \to (\xi', \sqrt{2} \psi(\xi', \xi_n)) \) with

\[
\psi(\xi', \xi_n) = \frac{|\xi'|^2}{2\xi_n^2} - \xi_n, \quad \xi_n \neq 0.
\]

This argument clearly fails when \( q \neq r \), and we have to give a direct proof of (1.28) by adapting the induction method in [15].

The reason of resorting to the uniform bilinear estimates of Theorem 1.2 is to get over the obstruction in using the Minkowski-conformal invariance of the cone under \( \text{Conf}^+(\mathbb{R}^{n+1}) \) in mixed-norms.

This invariance employed in the symmetric-norms does not work in the mixed-norms when \( q \neq r \), since transferring between Euclidean and the null coordinates is impossible, due to intertwining of the temporal and spatial variables simultaneously via linear transformations. To overcome this, Lee and Vargas [13, §4.2, P.1313] adopted a different way of rescaling, leading naturally to a family of operators, smoothly depending on the level of angular separateness, and proved a uniform bilinear estimate for these operators for all \((q, r)\) satisfying the cone condition (1.4) with strict inequalities.

Our proof is a spontaneity of this approach and Theorem 1.2 slightly generalizes the result for specific family of Fourier extension operators in spirit of Lee-Vargas to Monge functions in the class \( E_\sigma \), not only with an enlarged class of surfaces but also including the endline case except the critical index \((q_c, r_c)\).

1.3. Outline of the proof. We briefly explain our proof of Theorem 1.1. By symmetry, we first reduce the null form estimates for solutions of wave equations to the \((++)\) and \((+\pm)\) cases associated to one-sided half waves in the canonical way as [6, 19, 15, 13, 12].

For the \( L^\infty_t L^1_x \) case, following [6], we write the products of the two waves into a bilinear operator and use the translation-modulation invariance along with the Plancherel theorem to reduce the estimate to a bilinear \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n) \) multiplier estimate which is known from the Coifman-Meyer theory.

For each \((q, r)\) on the endline with \( 1 < r < r_c \), we shall adopt the standard Littlewood-Paley decomposition, reducing to the following three cases:

\[
(a) \text{ low-low } (++), \quad (b) \text{ low-low } (+\pm), \quad (c) \text{ high-low } (+\pm),
\]

where the ‘low-low’ is short for the low-low frequency interactions and the ‘high-low’ stands for the interactions of high-low frequencies.

All of these cases can be handled under the conditions of Theorem 1.1 by using the Main Proposition in Section 2 below. To handle Case (a), we consider the interactions of the two waves creating relatively low and very low frequencies. For the first one, we use the bilinear Whitney decomposition along the angular directions to deal with the angular compressions where the transversality condition depends on the level of colinearity, to which we use a rescaling to the Main Proposition. For the second, we use an argument of Klainerman-Tataru [11] to find enough amount
of transversality. In both cases, the condition $\beta_0 \geq 0$ allows us to use a Mihlin-Hörmander type multiplier estimate in the mixed norms $[1]$. To handle Case (b), besides of the bilinear Whitney in the angular directions, we shall also make use of the decomposition along null directions, due to Lee and Vargas $[13]$. The case (c) includes the high-low frequency interactions, which can be handled in a unified way. We shall use the condition (1.25) for $q \geq 2$ and (1.26) for $q < 2$. To complete the proof, we reduce the Main Proposition to Theorem 1.2 by using the non-isotropic rescaling of Lee-Vargas $[13]$.

The major part of the paper is devoted to Theorem 1.2, for which we shall use the enhanced induction-on-scale method of Tao $[15]$. Instead of dealing with a fixed lightcone there, we get a uniform estimate for a family of surfaces, fairly well-behaved thanks to the $E_\sigma$ conditions for all $\sigma \in (0, \sigma_0]$, provided $\sigma_0$ fixed small.

Recall that in $[15]$, there are two key ingredients in proving the endpoint theorems for the lightcone.

The first one is the propagation of waves along null directions, or more precisely Huygens’ principle for wave equations, which allows to explore spatial localizations by means of an appropriate truncation operator which preserves the wave structure on the frequency space. As in $[15]$, we shall use a microlocal version of this property. Consequences of this property are two aspects. One is the energy estimate on the conic regions of opposite colours, where the first use of the transversality condition takes place. A crucial fact is the Kakeya null property, that the normal vector at any point of the cone are always in the lightray directions. In particular, the normal field is a submanifold of the unit sphere $S^m$ of codimension one. This is used to tackle the case when the energy of the waves simultaneously concentrate in a small disk. The other one is the essential concentration of waves in the physical spacetime along conic sets which allows to wrest a universal constant strictly smaller than one.

The second ingredient is the wave table theory based on a careful wave-packet decomposition and is more technically involved. The use of wave tables draws on three novelties compared with $[16, 22]$. The first one is equipping an auxiliary small constant to the previous versions of $[16]$ by an averaging argument, such that the small constant can be flexibly chosen to deal with the energy-concentrated case by using iteration. The second novelty is that in the construction of wave tables, one takes in more geometric information of interactions between two waves of opposite colours in the physical spacetime, especially that the wave table for a wave is constructed with respect to the other wave of the opposite colour, carefully modifying the initial data by using a linear transformation, depending on the opposite colour wave, of the plain wave-packets in the tube summations. This along with separated supports of quilts allows to get a pigeonhole-free version of the bilinear $L^2$–Kakeya type estimate, removing the otherwise logarithmic loss, which is acceptable in the non-endpoint case incorporated with the $\varepsilon$–removal lemma, however would block the approach to the endpoint estimates. The third one is concerning the products of the two waves, of which one is localized to high frequency and the wave table for the low frequency wave can be localized in the physical spacetime to quilts at a depth reciprocal to the high frequency, consistent with the uncertainty principle.

Based mainly on the above two innovations, the induction on scale method of Wolff $[22]$ is enhanced by Tao $[15]$ to get the endpoint bilinear restriction estimate and our task is to verify this strategy for a family of conic type surfaces. To this end, we rely on the exploitation of the gain in non-concentration of energy
analogous to the analysis for Proposition 3.6 in [15] in order to bring in a universal small constant strictly less than one. To get the uniform estimate for the family of conic surfaces in question, we find that in this step, which only plays an auxiliary role, it suffices to apply the argument as in [23] for any fixed $\Phi_\sigma$.

The paper is organized as follows. In Section 2, we prove Theorem 1.1 by reducing it to Theorem 1.2 via Littlewood-Paley theory involving spacetime multipliers in mixed-norms. The proof of Theorem 1.2 will be carried out through Section 3 to Section 5. In Section 6, we invest a possible way of combining the method of descent with the argument for Theorem 1.2 to get the endpoint case of the Foschi-Klainerman conjecture on the bilinear Fourier restriction estimate on the unit sphere.

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2. Proof of Theorem 1.1 via the Main Proposition

In this section, we first reduce Theorem 1.1 to the main proposition below by using a hybrid of the dyadic decomposition in [19] and [15], which allows us to take the equality in (1.22). The main proposition is then proved by using a non-isotropic rescaling argument and Theorem 1.2.

Given $\Omega \subset \mathbb{R}^n$, for each $n \geq 2$, define the forward/backward cone as $C^\pm = \{(\pm |\xi|, \xi) : \xi \in \mathbb{R}^n\}$. Let $u : \mathbb{R}^{1+n} \to \mathbb{C}$ be a $(\pm)$ wave (resp. a $(\mp)$ wave) if $u$ is an $L^2$-measure supported on $C^+$ (resp. on $C^-$).

Let $u$ be a $(\pm)$ wave and $v$ be a $(\mp)$ wave. The energy of $u$ and $v$ are defined as

$$E(u) = ||u(t, \cdot)||_2^2, \quad E(v) = ||v(t, \cdot)||_2^2.$$ 

By Plancherel’s theorem, we see that $E(u)$, $E(v)$ are independent of $t$.

Fix $\sigma \in (0, 1)$ and write $\xi = (\xi', \xi_n)$ with $\xi'' = (\xi_1, \ldots, \xi_{n-2})$. Define

$$\Gamma^\pm_\sigma = \{ (\tau, \xi'' \xi_{n-1}, \xi_n) : \tau = \pm |\xi|, 1 \leq \pm \xi_n \leq 2, \sigma \leq \xi_{n-1} < 2\sigma, |\xi''| \leq \sigma \}.$$ 

In this subsection, we prove Theorem 1.1 assuming the following result.

Main Proposition. Let $n \geq 2$. For any $\varepsilon > 0$ and $1 \leq q, r \leq \infty$ such that

$$\frac{1}{q} < \min \left(1, \frac{n+1}{4}\right), \quad \frac{1}{q} = \frac{n+1}{2} \left(1 - \frac{1}{r}\right),$$

there exists a finite constant $C = C_{\varepsilon, q, r, n}$ such that we have

$$\|uv\|_{L^q_t(L^r_x(\mathbb{R}^n))} \leq C \mu^{\max \left(\frac{n-1}{2} - \frac{r}{2}, 0\right) + \varepsilon} |\sigma|^{-\frac{n+1}{4} + \varepsilon} E(u)^{1/2} E(v)^{1/2},$$

for all $u$ and $v$ being either $(\pm)$ and $(\mp)$ waves such that

$$\text{supp } \tilde{u} \subset \Gamma^+_\sigma, \quad \text{supp } \tilde{v} \subset \mu \Gamma^-_\sigma, \quad (2.2)$$

or being both $(\pm)$ waves such that

$$\text{supp } \tilde{u} \subset \Gamma^+_\sigma, \quad \text{supp } \tilde{v} \subset -\mu \Gamma^-_\sigma, \quad (2.3)$$

for all dyadic number $\mu \geq 1$ and all $\sigma \in (0, 1)$. 
Remark 2.1. When \( q = \infty \) and \( r = 1 \), we may take \( \varepsilon = 0 \) in (2.1) by using the Cauchy-Schwarz inequality and energy estimates. In fact, the \( \varepsilon \)-loss might be removed for \( q \geq 2 \) by using the same method for \( (q, r) = (2, \frac{n+1}{n}) \) and interpolate with \( (q, r) = (\infty, 1) \). We will not elaborate this.

Remark 2.2. The method of the proof for the case (2.3) can be modified easily to yield the same bilinear estimate (2.1) for \( u, v \) being both \((+)\) waves with the frequency supports condition that \( \supp \tilde{u} \subset \Gamma^+ \) and \( \supp \tilde{v} \) is contained in

\[
\mu \{ (\tau, \xi', \xi_{n-1}, \xi_n) : \tau = |\xi|, 1 \leq -\xi_n \leq 2, \sigma \leq \xi_{n-1} < 2\sigma, |\xi'| \leq \sigma \}.
\]

In this case, the factor \( \sigma^{-1/4} - \) on the right side of (2.1) may be discarded.

Assuming the main proposition, we prove Theorem 1.1. Let \( \phi \) and \( \psi \) be the solutions to (2.1) with initial data \( \phi[0] \) and \( \psi[0] \) respectively. Denote

\[
\phi^\pm(t) = e^{\pm itD_0} \phi^\pm_0, \quad \psi^\pm(t) = e^{\pm itD_0} \psi^\pm_0,
\]

as the one-sided solutions (half-waves) with

\[
\phi^+_0 = \frac{1}{2} \left( \phi_0 \pm (iD_0)^{-1} \phi_1 \right), \quad \psi^+_0 = \frac{1}{2} \left( \psi_0 \pm (iD_0)^{-1} \psi_1 \right).
\]

Then, we have \( \phi = \phi^+ + \phi^- \), \( \psi = \psi^+ + \psi^- \), and

\[
\| \phi[0] \|_{\mathcal{H}^\alpha_1} = \sqrt{2} \left( \| D_0^{\alpha_1} \phi^+_0 \|_2^2 + \| D_0^{\alpha_1} \phi^-_0 \|_2^2 \right)^{1/2},
\]

\[
\| \psi[0] \|_{\mathcal{H}^\alpha_2} = \sqrt{2} \left( \| D_0^{\alpha_2} \psi^+_0 \|_2^2 + \| D_0^{\alpha_2} \psi^-_0 \|_2^2 \right)^{1/2}.
\]

Writing

\[
\phi \psi = \phi^+ \psi^+ + \phi^+ \psi^- + \phi^- \psi^+ + \phi^- \psi^-\n\]

and using the triangle inequality, it is enough, after taking conjugation, to prove the estimate (1.2) with \( \phi \psi \) being equal to \( \phi^+ \psi^+ \) and \( \phi^+ \psi^- \), to which we refer respectively as the \((++)\) and \((+−)\) case. We are thus reduced to

\[
\| D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-} (\phi^+ \psi^+) \|_{L^1_t(\mathbb{R}; L^q_x(\mathbb{R}^n))} \leq C \| D_0^{\alpha_1} \phi^+_0 \|_{L^2_t(\mathbb{R}; L^q_x(\mathbb{R}^n))} \| D_0^{\alpha_2} \psi^+_0 \|_{L^2_t(\mathbb{R}; L^q_x(\mathbb{R}^n))},
\]

(2.5)

for some constant \( C = C_\Theta \), which may depend on \( \Theta := (q, r, \beta_0, \beta_+, \beta_-, \alpha_1, \alpha_2) \), satisfying the conditions in Theorem 1.1.

As [19], it is convenient to recast (2.5) into \( C_{\alpha_1, \alpha_2}^\pm (Q^1, Q^2, D) \) below.

Definition 2.3. Let \( \alpha_1, \alpha_2 \in \mathbb{R} \), \( Q^1, Q^2 \subset \mathbb{R} \) and \( D \) be the Fourier multiplier on \( \mathbb{R}^{1+n} \) associated to a symbol \( m(\tau, \xi) \). For \( (q, r) \in [1, \infty]^2 \) satisfying (1.21), define \( C_{\alpha_1, \alpha_2}^\pm (Q^1, Q^2, D) \) to be the best constant \( C \) such that

\[
\| D(\phi^+ \psi^+) \|_{L^1_t(\mathbb{R}; L^q_x(\mathbb{R}^n))} \leq C \| \phi^+_0 \|_{\mathcal{H}^{\alpha_1}(\mathbb{R}^n)} \| \psi^+_0 \|_{\mathcal{H}^{\alpha_2}(\mathbb{R}^n)}
\]

(2.6)

holds for all \( \phi^+_0, \psi^+_0 \) whose Fourier transforms are supported in \( Q^1, Q^2 \) respectively, where \( \phi^+, \psi^\pm \) are given by (2.4).

The optimal constant \( C_{\alpha_1, \alpha_2}^\pm \) depends on the given exponents \( (q, r) \). We suppress the expression for this dependence in the notation for brevity.

With this terminology, (2.5) can be written as

\[
C_{\alpha_1, \alpha_2}^\pm (\mathbb{R}^n, \mathbb{R}^n, D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}) \lesssim \Theta 1.
\]

(2.7)
Remark 2.4. As in [19], the bilinear restriction theorem on the cone \([22, 15, 21]\) can be reformulated as \(C_{\alpha_1, \alpha_2}^{\pm} (Q^1, Q^2, 1) \lesssim 1\) for compact subsets \(Q^1, Q^2 \subset \mathbb{R}^n\) being appropriately separated. Thus, it is reasonable to integrate the bilinear restriction theorem into the more general null form theory.

Remark 2.5. If we let \(B_{\alpha_1, \alpha_2}^{\pm, r} (\mathbb{R}^n, \mathbb{R}^n, D_0^\alpha D_+^\beta + D_-^\delta)\) have the same meaning in Definition 2.3 as the optimal constant of the estimate (2.6) for \(q = r = \frac{n+2}{n-2}\), which satisfies (1.4) in the one-dimensionally reduced endline condition, then the Machedon-Klainerman conjecture for the Schrödinger equation in the \((n-1)\) dimension is implied by the following null form estimate

\[
B_{\alpha_1, \alpha_2}^{\pm, r} (\mathbb{R}^n, \mathbb{R}^n, D_0^\alpha D_+^\beta + D_-^\delta) \lesssim 1,
\]

for some suitable \(\beta_0, \beta_+, \beta_-\) satisfying (1.8). This is called the method of descent, introduced in [15 Proposition 17.5].

Remark 2.6. The estimate (2.8) remains open, while for the \(\alpha_1 + \alpha_2 > \frac{n}{n+2}\) case, it is included in a general result proved in [13]. Although a proof for (2.8) is not known, an adaptation of the method in [15] yields an intermediate result weaker than (2.8) but strong enough to imply the endline estimates on paraboloids [23]. For the endpoint estimate on the unit sphere, we need deal with perturbations on the phase functions as in Theorem 1.2.

To show (2.7), we consider the case \(q = \infty\) and \(r = 1\) first. Following [6], let

\[
W_+ (\eta, \zeta) = \frac{|\eta + \zeta|^\alpha (|\eta| + |\zeta|)^\beta - (|\eta| + |\zeta| - |\eta + \zeta|)^\beta}{|\eta|^{\alpha_1}|\zeta|^{\alpha_2}},
\]

\[
W_- (\eta, \zeta) = \frac{|\eta + \zeta|^\alpha + |\eta| + |\zeta| - |\eta| - |\zeta|)^\beta}{|\eta|^{\alpha_1}|\zeta|^{\alpha_2}}.
\]

Define the bilinear multiplier operators \(B_{(++)}\) and \(B_{(+-)}\) as

\[
B_{(\pm \pm)} (f, g) (x) = \iint_{\mathbb{R}^{2n}} e^{i x (\eta + \zeta)} W_\pm (\eta, \zeta) f(\eta) g(\zeta) d\eta d\zeta.
\]

By translation invariance and Plancherel’s theorem, (2.7) follows from

\[
\|B_{(\pm \pm)}\|_{L^2 (\mathbb{R}^n) \times L^2 (\mathbb{R}^n) \to L^1 (\mathbb{R}^n)} \lesssim 1.
\]

Noting that \(W_\pm\) is smooth on \(\mathbb{R}^{2n} \setminus \{0\}\) and homogeneous of order zero by (1.20), the classical Coifman-Meyer method yields (2.9). We omit the details.

Next, we consider the \(r > 1\) case. We shall write \(\phi \psi\) standing for \(\phi^+ \psi^+\) or \(\phi^+ \psi^-\), depending on being either in the \((++\)) or the \((+-\)) case.

For each \(\lambda \in 2^\mathbb{Z}\), let \(A_\lambda := \{ \xi \in \mathbb{R}^n : |\xi| \sim \lambda \}\). Using Littlewood-Paley, we write

\[
\phi (t) = \sum_{\nu \in 2^\mathbb{Z}} \phi_\nu (t), \quad \psi (t) = \sum_{\mu \in 2^\mathbb{Z}} \psi_\mu (t),
\]

where the partial Fourier transforms \(\hat{\phi}_\nu (t), \hat{\psi}_\mu (t)\) of \(\phi_\nu (t), \psi_\mu (t)\) with respect to the spatial variables are supported respectively in \(A_\nu, A_\mu\) for every \(t\). For (2.7), by using the Minkowski inequality, Schur’s test and symmetry, it suffices to show

\[
C_{\alpha_1, \alpha_2}^{\pm} (A_\nu, A_\mu, D_0^\alpha D_+^\beta + D_-^\delta) \lesssim \theta_t \varepsilon \left( \frac{\mu}{\nu} \right)^{-\varepsilon}, \quad \forall \mu \geq \nu,
\]

(2.10)
for some small \( \varepsilon > 0 \), with \( \Theta \) in (2.5). Scaling by (1.20), we reduce (2.10) to

\[
C_{\alpha_1, \alpha_2}^\pm (A_1, A_\mu, D_\mu^\beta D_\mu^\gamma) \lesssim \mu^{-\varepsilon}, \quad \forall \mu \geq 1.
\]  

(2.11)

Let \( \mathbf{D} := D_0^\beta D_+^\gamma D_-^\gamma \) and \( \mathbf{m} = \mathbf{m}(\tau, \xi) \) be the symbol of the multiplier \( \mathbf{D} \),

\[
\mathbf{m}(\tau, \xi) = |\xi|^\beta_0 (|\tau| + |\xi|)^\beta_+ ||\tau| - |\xi||^\beta_-. 
\]

Then \( \mathbf{D}(\phi_1 \psi_\mu)(\tau, \xi) = \mathbf{m}(\tau, \xi) (\tilde{\phi}_1 * \tilde{\psi}_\mu)(\tau, \xi) \) vanishes outside the set

\[
\text{supp}(\mathbf{m}) \cap \{(|\xi| \pm |\eta|, \zeta + \eta) : \zeta \in A_1, \eta \in A_\mu \},
\]

depending on being in the (++) or the (+) case. In the (++) case, we may confine \( \mathbf{m} \) inside the region \( |\xi| \leq \tau \) by triangle inequality, whereas in the (+) case, we may impose the condition \( |\tau| \leq |\xi| \).

Now, we introduce the \( P_\pm \) operator as in [19] corresponding to the (++) and (+) cases respectively.

In the (++) case, let \( P_+ \) be a smooth multiplier with symbol supported on \( |\xi| \ll \tau \) and write

\[
\mathbf{D}(\phi_1 \psi_\mu) = P_+ \mathbf{D}(\phi_1^+ \psi_\mu^+) + (I - P_+) \mathbf{D}(\phi_1^+ \psi_\mu^-),
\]

where the first term on the right side vanishes unless \( \mu \sim 1 \). It is easy to see that as long as the second term does not vanishes identically, \( (I - P_+) \) is frequency-localized to the region \( |\xi| \sim \tau \).

In the (+) case, we have \( |\tau| + |\xi| \sim |\xi| \lesssim \mu \). Letting \( P_- \) be given by a smooth multiplier with frequency localized on \( |\tau| + |\xi| \ll \mu \), we may write

\[
\mathbf{D}(\phi_1 \psi_\mu) = P_- \mathbf{D}(\phi_1^+ \psi_\mu^+) + (I - P_-) \mathbf{D}(\phi_1^+ \psi_\mu^-),
\]

where the first term on the right side vanishes unless \( \mu \sim 1 \), whereas the second one is truncated in the frequency space to the region \( |\tau| + |\xi| \sim \mu \).

By triangle inequalities, (2.11) is reduced to

\[
C_{\alpha_1, \alpha_2}^\pm (A_1, A_\mu, P_+ \mathbf{D}) \lesssim \mu^{-\varepsilon},
\]

(2.13)

\[
C_{\alpha_1, \alpha_2}^\pm (A_1, A_\mu, P_- \mathbf{D}) \lesssim \mu^{-\varepsilon},
\]

(2.14)

\[
C_{\alpha_1, \alpha_2}^\pm (A_1, A_\mu, (I - P_\pm) \mathbf{D}) \lesssim \mu^{-\varepsilon},
\]

(2.15)

uniformly for all \( \mu \geq 1 \). The \( (I - P_\pm) \) part corresponds to the high-low frequency interactions, which can be handled in a unified way.

We prove (2.13) (2.14) and (2.15) using the main proposition.

**Proof of (2.13):** We have \( \mu \sim 1 \) otherwise \( P_\pm \mathbf{D} = 0 \). This corresponds to the low frequency interactions in the (++) case.

Using \( 1 < q, r < \infty \) and the Mihlin-Hörmander theorem for mixed-norms, the operator \( D_0^\beta D_+^\gamma D_-^\gamma \) is bounded on \( L_q^r(\mathbb{R}^n) \), see for example [1] Theorem 7.

Here, we have used \( \beta_0 \geq 0 \) and that the symbol of the multiplier \( D_0^\beta D_+^\gamma \) is smooth on \( \mathbb{R}^{1+n} \setminus \{0\} \) and is homogeneous of order zero. Therefore, it suffices to show (2.13) with \( \beta_0 = 0 \).

Without loss of generality, we may take \( \mu = 1 \) after suitably modifying the structure constant if necessary as in [19].
Fix a small constant $\gamma_0 \in 2^\mathbb{Z}$ with $0 < \gamma_0 \ll 1$ and decompose
\[
\{(\tau, \xi) : |\xi| \leq 1\} = \bigcup_{\gamma_0 \leq \gamma \leq 1} \Omega_{\gamma},
\]
where $\Omega_{\gamma} := \{(\tau, \xi) : |\xi| \sim \gamma\}$ for each $\gamma > \gamma_0$ and $\Omega_{\gamma_0} = \{(\tau, \xi) : |\xi| \leq \gamma_0\}$.

Accordingly, decompose $P_+$ by the inhomogeneous Littlewood-Paley partition of the identity operator
\[
P_+ = \sum_{\gamma_0 \leq \gamma \leq 1} \Delta_{\gamma},
\]
where $\Delta_{\gamma} = P_+ \Delta_{\gamma}$ is the multiplier adapted to $\Omega_{\gamma}$ and $\Delta_{\gamma} \sim 1$ if $\gamma > \gamma_0$. Using Minkowski’s inequality, we may reduce (2.13) (with $\mu = 1$) to showing that
\[
C^+_{\alpha_1, \alpha_2}(A_1, A_1, \Delta, D^0_+ D^0_-) \lesssim 1, \forall \gamma \in [\gamma_0, 1].
\]
(2.16)

For each $\gamma > \gamma_0$ and each dyadic $\sigma$ with $0 < \sigma \leq 1$, we decompose $A_1$ into finitely overlapping projective sectors $\Gamma$ with angle $\sigma$. By using the bilinear Whitney type decomposition (c.f. [17, 18, 13]) and Schur’s test, we may reduce (2.16) to showing that there exist some small $\varepsilon > 0$ such that
\[
C^+_{\alpha_1, \alpha_2}(\Gamma, \Gamma', \Delta, D^0_+ D^0_-) \lesssim \sigma^\varepsilon,
\]
holds uniformly for all $\Gamma, \Gamma'$ with angle $\sigma$ such that $\angle(\Gamma, \Gamma') \sim \sigma$. Here, $\angle(\Gamma, \Gamma')$ denotes the angular distance between $\Gamma$ and $\Gamma'$.

Write $\Gamma = -\Gamma \cup \Gamma$ with $\Gamma, -\Gamma$ being the antipode components of $\Gamma$ and likewise for $\Gamma' = -\Gamma' \cup \Gamma'$, where the angle between $\Gamma, \Gamma'$ belongs to $(0, \pi/2)$.

By triangle’s inequality and symmetry, (2.17) follows from
\[
C^+_{\alpha_1, \alpha_2}(-\Gamma, \pm\Gamma', \Delta, D^0_+ D^0_-) \lesssim \sigma^\varepsilon,
\]
(2.18)

To prove (2.18), we first consider the cases $\gamma > \gamma_0$. Discarding the harmless operators $\Delta, D^0_+ D^0_- \sim 1$, we are reduced to
\[
C^+_{\alpha_1, \alpha_2}(\Gamma, \pm\Gamma', |\square|) \lesssim \sigma^\varepsilon
\]
(2.19)

where $|\square| := D_+ D_-$. To proceed, we recall the following technical lemma on the multiplier estimates for $|\square|$ taken from [13].

**Lemma 2.7.** Let $\beta_- \in \mathbb{C}$. Suppose either that $u$ and $v$ are $(+)$ and $(−)$ waves with (2.2) or that both $u$ and $v$ are $(+)$ waves such that (2.3) holds. Then, for all $1 \leq q, r \leq \infty$, $\mu \geq 1$ and $\sigma \in (0, 1)$, we have for any $N$, there is a finite constant $C_N$ independent of $\beta_-$ such that
\[
\|\|\beta_- (uv)\|_{L^q_x L^r_t} \leq C_N (1 + |\beta_-|)^N (\mu \sigma^2)^{\text{Re}(\beta_-)} \times \sum_{(k_1, k_2) \in \mathbb{Z}^n \times \mathbb{Z}^n} (1 + |k_1| + |k_2|)^{-N} \|u_{k_1} v_{k_2}\|_{L^q_x L^r_t},
\]
where for each $(k_1, k_2) \in \mathbb{Z}^n \times \mathbb{Z}^n$, $u_{k_1}$ and $v_{k_2}$ are $(+)$ and $(+)/(−)$ waves, having the same support property in the frequency space with $u, v$ and satisfying that
\[
E(u_{k_1}) = E(u), \quad E(v_{k_2}) = E(v).
\]
By using Lemma 2.7 and (1.23), we may reduce (2.19) with $\varepsilon$ small enough to
\[ C^+_{\alpha_1, \alpha_2}(\Gamma, \pm \Gamma', 1) \lesssim \sigma^{-\frac{4}{(\alpha+1)}}, \] (2.20)
which, after a suitable spatial rotation, follows from (2.11) of the Main Proposition and Remark 2.2.

It remains to consider the $\Delta_{\gamma_0}$ term. In this case, instead of using Lemma 2.7 and the bilinear Whitney, we shall use an argument as in [11, 19]. Divide $A_1$ into essentially disjoint cubes $Q$ of sidelength $\gamma_0$. Let $m_{\gamma_0}$ be the symbol of $\Delta_{\gamma_0}$ and denote $B(0, R)$ as the ball of radius $R > 0$ centered at the origin of $\mathbb{R}^n$. We write $Q \sim Q'$ if there exists $\zeta \in Q$, $\eta \in Q'$ such that $m_{\gamma_0}(|\zeta + |\eta|, \zeta + \eta) \neq 0$. Then, there is a universal constant $C$ such that $Q \sim Q'$ only when $Q + Q' \subset B(0, C\gamma_0)$. By taking $\gamma_0$ smaller if necessary, $Q \sim Q'$ only when $Q$ and $Q'$ are almost opposite. Consequently, we have
\[ \sup_Q \text{Card}\{Q' : Q \sim Q'\} + \sup_Q \text{Card}\{Q : Q \sim Q'\} \lesssim 1. \]

By using Schur’s test, we may reduce (2.16) to
\[ C^+_{\alpha_1, \alpha_2}(Q, Q', \Delta_{\gamma_0}D^{\beta,+}_+ D^{\beta,-}_-) \lesssim 1, \] (2.21)
uniformly for $\gamma_0$-cubes $Q \sim Q'$.

Noting that $D_+ \sim D_- \sim 1$, we have by Bernstein’s inequality
\[ C^+_{\alpha_1, \alpha_2}(Q, Q', \Delta_{\gamma_0}D^{\beta,+}_+ D^{\beta,-}_-) \lesssim C^+_{\alpha_1, \alpha_2}(Q, Q', 1). \]
Thus, it suffices to show
\[ C^+_{\alpha_1, \alpha_2}(Q, Q', 1) \lesssim 1, \] (2.22)
uniformly for $\gamma_0$-cubes $Q \sim Q'$.

By using rotation in the spatial variable, we may place the sets $Q$ and $Q'$ as in Remark 2.2 so that (2.22) follows. The proof of (2.13) is thus complete.

**Proof of (2.14):** We are in the low-low frequency interaction case with $\mu \sim 1$. Moreover, we have $D_0 \sim D_+$. Hence by the standard multiplier estimate (see [1]), we may reduce (2.14) to
\[ C^+_{\alpha_1, \alpha_2}(A_1, A_1, P_-D^{\beta_0+\beta_+, \beta_\mu}_+ D^{\beta,-}_-) \lesssim 1. \] (2.23)

Decompose \{(\tau, \xi) : |\tau| + |\xi| \lesssim 1\} = $\bigcup_{\gamma > 0} \Omega^+_\gamma$, where for each $\gamma \in (0, 1) \cap 2\mathbb{Z}$, \[ \Omega^+_\gamma := \{ (\tau, \xi) : |\tau| + |\xi| \sim \gamma \}. \]

For each $\gamma$, partition $\Omega^+_\gamma = \bigcup_{0 \leq \lambda \leq \gamma} \Omega_{\lambda, \gamma}$ with $\lambda$ being dyadic, and \[ \Omega^-_{\lambda, \gamma} = \{ (\tau, \xi) \in \Omega^+_\gamma : ||\tau| - |\xi|| \sim \lambda \}. \]

Let $\Delta^+_\gamma$ and $\Delta^-_{\lambda, \gamma}$ be the smooth multipliers adapted to $\Omega^+_\gamma$ and $\Omega^-_{\lambda, \gamma}$ respectively such that we may write
\[ P_+D^{\beta_0+\beta_+, \beta_\mu}_+ D^{\beta,-}_- = \sum_{0 < \gamma \leq 1} \sum_{0 < \lambda \leq \gamma} \gamma^\beta_0 + \beta_\mu + \beta_- \Delta^+_\gamma \Delta^-_{\lambda, \gamma}. \]

By Minkowski’s inequality, (2.23) is reduced to showing for some $\varepsilon > 0$
\[ C^+_{\alpha_1, \alpha_2}(A_1, A_1, \Delta^+_\gamma \Delta^-_{\lambda, \gamma}) \lesssim \gamma^{-(\beta_0 + \beta_\mu + \beta_-) + \varepsilon} \left( \frac{\lambda}{\gamma} \right)^{-\beta_- + \varepsilon}, \] (2.24)
holds uniformly for all $\lambda \leq \gamma \leq 1$. 
Let \( m_{\Lambda,\gamma}^{+,-}(\tau,\xi) \) be the symbol of \( \Delta_+^{\Lambda,\gamma} \Delta_{\Lambda,\gamma}^- \) such that \( m_{\Lambda,\gamma}^{+,-} \) is a smooth bump function adapted to \( \{ (\tau, \xi) : |\xi|+|\tau| \sim \gamma, ||\tau||-||\xi|| \sim \lambda \} \). Partition \( A_j \) into essentially disjoint sectors \( \Gamma \) of length \( \sim 1 \) in the radial direction and of angular width \( \sim \sqrt{\gamma \lambda} \).

Write \( \Gamma' \sim \Gamma'' \) if there exist \( \zeta \in \Gamma', \eta \in \Gamma'' \) such that \( m_{\Lambda,\gamma}^{+,-}(|\xi|-|\eta|, \zeta + \eta) \neq 0 \).

Since the \( \gamma \sim 1 \) case is easier and can be handled by slightly adjusting the argument, we only deal with the \( \gamma \ll 1 \) case.

Using the elementary relation

\[
|\zeta + \eta| - ||\xi|| - |\eta|| \sim \frac{|\xi|}{|\zeta + \eta|} \theta(\zeta, -\eta)^2,
\]

where \( \theta(\zeta, -\eta) \) denotes the angle formed between \( \zeta \) and \( -\eta \) (c.f. [19]), we have \( \Gamma' \sim \Gamma'' \) only when \( \zeta \sim \sqrt{\gamma \lambda} \). By Schur’s test, (2.24) reduces to

\[
\tilde{C}_{\alpha_1,\alpha_2}(\Gamma', \Gamma'', \Delta_+^{\gamma} \Delta_{\Lambda,\gamma}^-) \lesssim \gamma^{-\beta_0+\beta_+ + \beta_- + \varepsilon} \frac{\Lambda}{\gamma} - \beta_- + \varepsilon,
\]

(2.25)

uniformly for all \( \Gamma' \sim \Gamma'' \).

For each dyadic \( \sigma \) with \( 0 < \sigma \lesssim \sqrt{\gamma \lambda} \), decompose \( A_1 \) as before into finitely overlapping projective sectors \( \{ \Gamma_j \} \) with angle \( \sigma \). Using the bilinear Whitney type decomposition and Schur’s test, it suffices to show

\[
\tilde{C}_{\alpha_1,\alpha_2}(\Gamma' \cap \Gamma'', \Gamma' \cap (-\Gamma''), \Delta_+^{\gamma} \Delta_{\Lambda,\gamma}^-) \lesssim \sigma^{-\beta_0+\beta_+ + \beta_- + \varepsilon} \frac{\Lambda}{\gamma} - \beta_- + \varepsilon,
\]

(2.26)

for \( \zeta \sim \sqrt{\gamma \lambda} \). To this end, we decompose \( \Gamma' \times (-\Gamma'') \) as follows.

For each \( \xi \in \{ \log_2 \sigma, \ldots, \log_2 \sigma^{-1} \} \) with \( \sigma \in (0, 1) \cap 2\mathbb{Z} \), we write \( \Gamma' = \bigcup \Lambda_j' \Lambda_j' \), where each \( \Lambda_j' \) is a segment on \( \Gamma' \) of dimensions \( \sigma \times \cdots \times \sigma \times 2^{-1} \sigma \). For each \( \Lambda_j' \), set

\[
\Lambda_j'' = \{ \xi \in (-\Gamma'') : \text{dist}(\xi, -\Lambda_j') \in [2^{j-1} \sigma, 2^j \sigma) \}, \quad \forall \, j \geq 1,
\]

and so that we have

\[
\Gamma' \times (-\Gamma'') = \bigcup_{j=0}^{\log_2 \frac{\lambda}{\sigma}} \bigcup_{\Lambda_j', \Lambda_j''} \Lambda_j' \times \Lambda_j''.
\]

By Schur’s test, (2.26) reduces to

\[
\tilde{C}_{\alpha_1,\alpha_2}(\Lambda_j', \Lambda_j'', \Delta_+^{\gamma} \Delta_{\Lambda,\gamma}^-) \lesssim \sigma^{-\beta_0+\beta_+ + \beta_- + \varepsilon} \frac{\Lambda}{\gamma} - \beta_- + \varepsilon, \quad \forall \, j \in \mathcal{J}_\sigma,
\]

(2.27)

where the logarithmic loss \( \frac{\log 2}{\sigma} \) is safely absorbed to \( \sigma^{-\varepsilon} \) by slightly adjusting \( \varepsilon \).

By using \( \Delta_+^+ \sim \gamma^{-\beta_0+\beta_+} D_+^{\beta_0+\beta_+} \) and \( \Delta_{\Lambda,\gamma}^- \sim \lambda^{-\beta_-} D_-^{\beta_-} \), (2.27) reduces to

\[
\tilde{C}_{\alpha_1,\alpha_2}(\Lambda_j', \Lambda_j'', D_+^{\beta_0+\beta_+} D_-^{\beta_-}) \lesssim \sigma^{3\varepsilon}, \quad \forall \, j \in \mathcal{J}_\sigma,
\]

(2.28)

where we have used \( \sigma \lesssim \sqrt{\gamma \lambda} \) and \( \gamma \ll 1 \).

In view of the definition of \( \Lambda_j', \Lambda_j'' \) for \( j \geq 1 \) and the angular separation for the \( j = 0 \) case, it is readily deduced that \( D_+ \sim 2^j \sigma \) on \( C_+^{\Lambda_j'} + C_-^{\Lambda_j''} \). Therefore, by Bernstein’s estimates, Lemma 2.7 and (2.1), we have

\[
\tilde{C}_{\alpha_1,\alpha_2}(\Lambda_j', \Lambda_j'', D_+^{\beta_0+\beta_+} D_-^{\beta_-}) \lesssim 2^{(\beta_0+\beta_+) \sigma^{\beta_0+\beta_+ + \beta_- - \frac{4}{\gamma(n+1)}}}.
\]

(2.29)
For \( \beta_0 + \beta_+ - \beta_- \geq 0 \), by using \( 2^j \sigma \lesssim 1 \), the right side of (2.29) is bounded by
\[
(2^j \sigma)^{(\beta_0 + \beta_+ - \beta_-)} \sigma^{\beta_- - \frac{1}{2}} \lesssim \sigma^{3 \varepsilon}
\]
for \( \varepsilon \) sufficiently small, where we have used (1.22) with strict inequality.

If \( \beta_0 + \beta_+ - \beta_- < 0 \), discarding \( 2^j (\beta_0 + \beta_+ - \beta_-) \lesssim 1 \), and using (1.22) (1.24) with strict inequality, the right side of (2.29) is bounded by
\[
\sigma^{\alpha_1 + \alpha_2 - \frac{1}{2} \left( \frac{a - 1}{a + 1} + \frac{1}{a} \right)} \lesssim \sigma^{\alpha_1 + \alpha_2 - \frac{1}{2} \left( \frac{a + 3}{a + 1} \right)} \lesssim \sigma^{3 \varepsilon}.
\]
Combining these two cases, we obtain (2.28) and complete the proof of (2.14).

\* Proof of (2.13): \* We shall use (1.22) when \( q \geq 2 \) and use (1.24) when \( q < 2 \).

We may assume \( \mu \gg 1 \), whilst the \( \mu \sim 1 \) case can be handled using the same method as for (2.13) and (2.14).

Using \( \mu \gg 1 \) and \( (I - P_{\pm}) \), we have \(|\xi| \sim |\tau| + |\xi| \sim \mu \) for both the \((++)\) and the \((+-)\) cases. In fact, in the \((++)\) case, this follows from \(||\xi| \sim |\tau| \sim \mu \), while in the \((+-)\) case, it follows from \(||\xi| \leq |\tau| \) and \(|\tau| + |\xi| \sim \mu \).

By using the Bernstein estimate, we may reduce (2.15) to
\[
C_{\alpha_1, \alpha_2}^\pm (A_1, A_\mu, (I - P_{\pm}) D_{\beta_-}^-) \lesssim \mu^{- (\beta_0 + \beta_+ - \varepsilon)}.
\]
(2.30)
Noting that
\[
\left| \left| \left| (|\zeta| \pm |\eta|) - |\zeta + \eta| \right| \right| \right| \lesssim \frac{|\zeta| \cdot |\eta|}{|\zeta| + |\eta|} \lesssim 1,
\]
for all \((\zeta, \eta) \in A_1 \times A_\mu\), we may decompose as before
\[
D_{\beta_-}^- = \sum_{\gamma \text{ dyadic}} \gamma^{\beta_-} \Delta^-_{\eta}.
\]
By using the standard multiplier estimate, we are reduced to
\[
C_{\alpha_1, \alpha_2}^\pm (A_1, A_\mu, \Delta^-_{\eta}) \lesssim \gamma^{- \beta_- + \varepsilon} \mu^{- (\beta_0 + \beta_+ - \varepsilon)},
\]
(2.31)
for some small \( \varepsilon > 0 \).

Partition \( A_1 = \cup \Gamma \), \( A_\mu = \cup \Gamma_\mu \) where \( \{\Gamma\} \), \( \{\Gamma_\mu\} \) are collections of disjoint sectors of angular width \( \sim \gamma^2 \) and of radial width \( \sim 1 \) and \( \sim \mu \) respectively. Let \( m^-_{\gamma} \)
be the symbol of \( \Delta^-_{\eta} \). We write \( \Gamma \sim \Gamma_\mu \) if there exists \( \zeta \in \Gamma \) and \( \eta \in \Gamma_\mu \) such that \( m^-_{\gamma} ((|\zeta| \pm |\eta|, \zeta + \eta) \neq 0 \). Then, it is clear that \( \Gamma \sim \Gamma_\mu \) only when \( \zeta \in \Gamma \times \Gamma_{\mu} \).

By using Schur’s test once more, we are reduced to
\[
C_{\alpha_1, \alpha_2}^\pm (\Gamma, \Gamma_\mu, \Delta^-_{\eta}) \lesssim \gamma^{- \beta_- + \varepsilon} \mu^{- (\beta_0 + \beta_+ - \varepsilon)},
\]
(2.32)
uniformly for all \( \Gamma \sim \Gamma_{\mu} \).

Given \( \Gamma, \Gamma_{\mu} \) such that \( \Gamma \sim \Gamma_{\mu} \), we may assume that the operator \( \Delta^-_{\eta} \) in (2.32) is localized on a neighborhood of the set \( \{(|\zeta| \pm |\eta|, \zeta + \eta) : (\zeta, \eta) \in \Gamma \times \Gamma_{\mu}\} \).

We shall sketch the proof below. Let \( \tilde{\phi} \) and \( \tilde{\psi}_{\mu} \) be such that the space-time Fourier transforms \( \tilde{\phi} \) and \( \tilde{\psi}_{\mu} \) are supported on \( C^+(\Gamma) \) and \( C^\pm(\Gamma_{\mu}) \) respectively.

For each \( \mu \geq 1 \), applying the standard bilinear Whitney type decomposition in the circular directions for \( \phi \psi_{\mu} \) as above (see also 18, 15, 13), we may write
\[
\phi \psi_{\mu} = \sum_{\sigma \text{ dyadic}} \sum_{0 < \sigma \lesssim \gamma^{1/2}} \phi_{\Gamma', \Gamma_{\mu}} \psi_{\mu, \Gamma'}
\]
for each \( \mu \geq 1 \).
where $\Gamma, \Gamma'$ are sectors of the forward cone or the forward and backward cones, depending on being either in the $(++)$ or $(+-)$ case, with $\text{Ang}(\Gamma), \text{Ang}(\Gamma') \sim \sigma$, where $\text{Ang}$ stands for the aperture angle of $\Gamma, \Gamma'$. We slightly abused notations here by letting $\Gamma$ be on the cone rather than that in the proof for the other two cases (2.13), where instead of $\Gamma \subset A_1$, its lifts $C^+(\Gamma)$ are subsets of the cone.

By using Minkowski’s inequality, Cauchy-Schwarz and Plancherel’s theorem, we may reduce (2.32) further to

$$
\|D_0^\alpha D_+^{\beta_+} D_-^{\beta_-} (\phi_T \psi_{\mu, \Gamma'})\|_{L^q_t L^r_x} \leq C \left(\frac{\sigma}{\mu}\right)^{2\epsilon} \mu^{\alpha_2} E(\phi_T)^{\frac{1}{2}} E(\psi_{\mu, \Gamma'})^{\frac{1}{2}},
$$

(2.33)

where we have used $D_0 \sim D_+ \sim \mu$ and $D_- \sim \gamma^{\beta_-} D_-^{\beta_-}$.

Exploring the spatial rotation for $\phi_T$ and $\psi_{\mu, \Gamma'}$, we may assume that

$$(++) \ (\Gamma, \Gamma') = (\Gamma_0^+ + \mu \Gamma_0^-, \mu \Gamma^-_0),$$

$$(+-) \ (\Gamma, \Gamma') = (\Gamma_0^+ + \mu \Gamma_0^-, \mu \Gamma^-_0).$$

Note that $\tilde{\phi_T} \ast \tilde{\psi_{\mu, \Gamma'}}$ is supported in the region where we have $D_0 \sim D_+ \sim \mu$. To show (2.33), it suffices to show

$$
\|\Box^{\beta} (\phi_T \psi_{\mu, \Gamma'})\|_{L^q_t L^r_x} \leq C \left(\frac{\sigma}{\mu}\right)^{2\epsilon} \mu^{\alpha + \beta} \mu^{\alpha - \beta} + \alpha_2 E(\phi_T)^{\frac{1}{2}} E(\psi_{\mu, \Gamma'})^{\frac{1}{2}},
$$

(2.34)

with $(\Gamma, \Gamma')$ being equal either to $(\Gamma_0^+ + -\mu \Gamma_0^-, \mu \Gamma^-_0)$ or to $(\Gamma_0^+ + \mu \Gamma_0^-, \mu \Gamma^-_0)$.

In view of (1.20) and Lemma 2.21 we may reduce (2.34) to

$$
\|\phi_T \psi_{\mu, \Gamma'}\|_{L^q_t L^r_x} \leq C \sigma^{2\beta_- + 2\epsilon} \mu^{\beta_- - \alpha_1 + \frac{n-1}{q(n+1)} - 2\epsilon} E(\phi_T)^{\frac{1}{2}} E(\psi_{\mu, \Gamma'})^{\frac{1}{2}}.
$$

(2.35)

By using the condition (1.23) with strict inequality and taking $\epsilon$ small enough depending on $\beta_-$ and $q$, we may reduce (2.34) to

$$
\|\phi_T \psi_{\mu, \Gamma'}\|_{L^q_t L^r_x} \leq C \sigma^{\frac{1}{q(n+1)}} \mu^{\beta_- - \alpha_1 + \frac{n-1}{q(n+1)} - 2\epsilon} E(\phi_T)^{\frac{1}{2}} E(\psi_{\mu, \Gamma'})^{\frac{1}{2}},
$$

(2.36)

with $(\Gamma, \Gamma')$ being equal either to $(\Gamma_0^+ + -\mu \Gamma_0^-, \mu \Gamma^-_0)$ or $(\Gamma_0^+ + \mu \Gamma_0^-, \mu \Gamma^-_0)$.

To show (2.33), we consider the following two cases:

$-$ Case A. $2 \leq q < \infty$;

$-$ Case B. $q \epsilon < q < 2$.

In Case A, we have by using (2.21)

$$
\|\phi_T \psi_{\mu, \Gamma'}\|_{L^q_t L^r_x} \leq C \sigma^{-\frac{1}{q(n+1)}} \mu^{\beta_- - \alpha_1 + \frac{n-1}{q(n+1)} - 2\epsilon} E(\phi_T)^{\frac{1}{2}} E(\psi_{\mu, \Gamma'})^{\frac{1}{2}}.
$$

Using (1.23) with strict inequality, we have

$$
\beta_- - \alpha_1 + \frac{n-1}{q(n+1)} > 3\epsilon
$$

by taking $\epsilon$ small enough. This yields (2.36).

In Case B, we have by using the Main Proposition

$$
\|\phi_T \psi_{\mu, \Gamma'}\|_{L^q_t L^r_x} \leq C \sigma^{-\frac{1}{q(n+1)}} \mu^{\beta_- + \epsilon} E(\phi_T)^{\frac{1}{2}} E(\psi_{\mu, \Gamma'})^{\frac{1}{2}}.
$$

Using (1.20) with strict inequality, we have

$$
\beta_- - \alpha_1 + \frac{n-1}{q(n+1)} - 2\epsilon > \frac{1}{q} - \frac{1}{2} + \epsilon.
$$

by taking $\epsilon$ small enough. Thus, we have (2.36) and hence (2.14).
Collecting (2.13) (2.14) and (2.15), we have proved Theorem 1.1. It remains to prove the Main Proposition.

2.2. Reduction to Theorem 1.2. It suffices to prove (2.1) for \((q, r)\) on the endline \(\frac{2}{q} = (n+1)\left(1 - \frac{1}{r}\right)\) apart from the critical index \((q_c, r_c)\) defined in (1.19).

To handle the large angle case \(\sigma \sim 1\), we shall use the following result, which is simply a restatement of the main proposition. We take \(\sigma = \frac{1}{8}\) to illustrate the idea without loss of generality.

**Proposition 2.8.** Let \(n \geq 2\). For any \(\varepsilon > 0\) and \(1 \leq q, r \leq \infty\) such that
\[
\frac{1}{q} \leq \min\left(1, \frac{n+1}{4}\right), \quad \frac{1}{q} = \frac{n+1}{2}\left(1 - \frac{1}{r}\right),
\]

there exists a finite constant \(C = C_{\varepsilon, q, r, n}\) such that we have
\[
\|uv\|_{L^q_x L^r_t} \leq \mu^{\max\left(\frac{1}{4} - \frac{1}{q}, \varepsilon\right)} E(u)^{\frac{1}{q}} E(v)^{\frac{1}{r}}
\]
for all \(\mu \geq 1\) and all \(u, v\) being either (+) and (-) waves with
\[
\text{supp } \bar{u} \subset \Gamma_+^1, \quad \text{supp } \bar{v} \subset \mu \Gamma_+^1,
\]
or being (+) waves satisfying the conditions
\[
\text{supp } \bar{u} \subset \Gamma_+^1, \quad \text{supp } \bar{v} \subset -\mu \Gamma_+^1.
\]

This is simply an extension of Theorem 1.1 in [15] to the mixed-norms with \(\mu = 2^k\) for all \(k \geq 1\) there. We remark that if \(k \sim 1\), this result with \(q \leq 2\) has been established in [21], where the argument is technically simpler since one needs not deal with the high-low frequency interactions for \(k \gg 1\).

The essential part of the proof for the main proposition is to handle the small angular case \(\sigma \ll 1\). We shall reduce the question to Theorem 1.2 by using the angular rescaling as in [13]. Taking conjugate if necessary \(|uv| = |\bar{uv}|\), it suffices to consider the case when \(u\) and \(v\) are both (+) waves satisfying (2.3).

Let \(\sigma_0 > 0\) be given by Theorem 1.2 and
\[
\theta(\xi') = \sqrt{1 + |\xi'|^2} - 1, \quad \xi' \in \mathbb{R}^{n-1}.
\]
For each \(\sigma \in (0, \sigma_0]\), define
\[
\Phi_\sigma(\xi) = \xi_n \sigma^{-2} \theta(\sigma \xi'/\xi_n).
\]
It is easy to see that \(\Phi_\sigma\) satisfies the conditions in \(\mathcal{E}_\sigma\).

Denote by \(f(x) = u(0, x)\) and \(g(x) = v(0, x)\) and let
\[
f_\sigma(x) = \sigma^{-\frac{n-1}{2}} f(\sigma^{-1} x', x_n), \quad g_\sigma(x) = \sigma^{-\frac{n-1}{2}} g(\sigma^{-1} x', x_n),
\]
where \(x = (x', x_n)\) with \(x' \in \mathbb{R}^{n-1}\) and \(x_n \in \mathbb{R}\), so that
\[
\|f_\sigma\|_{L^2(\mathbb{R}^n)} = E(u)^{1/2}, \quad \|g_\sigma\|_{L^2(\mathbb{R}^n)} = E(v)^{1/2}.
\]
Changing variables
\[
x_n \rightarrow x_n - t, \quad (\xi', \xi_n) \rightarrow (\sigma \xi', \xi_n), \quad (x', x_n, t) \rightarrow (\sigma^{-1} x', x_n, \sigma^{-2} t),
\]
we have
\[
\|uv\|_{L^q_x L^r_t} \lesssim \sigma^{-\frac{q}{q} + (n-1)\left(1 - \frac{1}{r}\right)} \|(S_{\Omega_1}^{\Phi_\sigma} f_\sigma)(S_{\mu_{\Sigma_2}}^{\Phi_\sigma} g_\sigma)\|_{L^q_x L^r_t},
\]
with $\Sigma_1$ and $\Sigma_2$ fulfill the condition in Theorem 1.2. Applying the uniform bilinear estimate in Theorem 1.2 for all $\sigma \leq \sigma_0 \ll 1$, we obtain
\[
\|uv\|_{L^6_t L^3_x} \leq C_{\varepsilon} \mu^{\max\left(\frac{1}{q} - \frac{1}{2}, 0\right)} + \varepsilon \left(\frac{1}{\sigma}\right)^{\frac{1}{2}} \frac{n}{n-1} \mathbf{E}(u)^{\frac{1}{2}} \mathbf{E}(v)^{\frac{1}{2}},
\]
with $C_{\varepsilon}$ independent of $\sigma$. Therefore, to complete the proof of Theorem 1.1 it suffices to show Theorem 1.2 and Proposition 2.8.

In the rest part of the paper, we will only prove Theorem 1.2 with full details, and make remarks on the necessary modifications to get Proposition 2.8. We shall assume $\mu = 2^k$ for some integer $k \geq 1$.

3. Fundamental properties of waves

For any $\sigma \in (0, \sigma_0]$ with $\sigma_0$ small to be fixed. Take $\Phi_\sigma \in \mathcal{E}_\sigma$ and let
\[
F_1^\sigma(x, t) = S_{\omega_1, \Sigma_1}^\Phi(t) f_1(x), \quad F_2^\sigma(x, t) = S_{\omega_2, \Sigma_2}^\Phi(t) f_2(x),
\]
where $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$.

We shall call $F_1^\sigma$ and $F_2^\sigma$ the red and blue $\Phi_\sigma$-waves of frequency $2^m$ and $2^k$ respectively. Let $C_{\Phi_\sigma} = \{ (\Phi_\sigma(\xi), (\xi) : \xi \in \mathbb{R}^n \}$ and denote
\[
C_{\Phi_\sigma, r} = \{ (\Phi_\sigma(\xi), (\xi) : \xi \in 2^m \Sigma_1 \}, \quad C_{\Phi_\sigma, b} = \{ (\Phi_\sigma(\xi), (\xi) : \xi \in 2^k \Sigma_2 \},
\]
where $\Sigma_1$ and $\Sigma_2$ are defined in Theorem 1.2. The spacetime Fourier transforms of $F_1^\sigma$ and $F_2^\sigma$ are supported respectively on $C_{\Phi_\sigma, r}$ and $C_{\Phi_\sigma, b}$.

When there is no need to distinguish the color, we shall also call $F^\sigma$ a $\Phi_\sigma$-wave or simply a wave. The energy of a wave $F^\sigma(x, t)$ is defined as before by letting
\[
\mathbf{E}(F^\sigma) := \| F^\sigma(\cdot, t) \|_{L^2(\mathbb{R}^n)}^{1/2}.
\]

We shall prove, through Section 3 to Section 5, that there exists a $\sigma_0 > 0$ such that $\forall \varepsilon > 0$ and $(q, r)$ satisfying the conditions in Theorem 1.2 the bilinear estimate
\[
\|F_1^\sigma F_2^\sigma\|_{L^6_t L^3_x} \lesssim 2^{\gamma(q, r) \varepsilon} \mathbf{E}(F_1^\sigma)^{1/2} \mathbf{E}(F_2^\sigma)^{1/2},
\]
with $\gamma(q, r) := \max\left(\frac{1}{q} - \frac{1}{2}, 0\right) + \varepsilon$, holds for all the red and blue $\Phi_\sigma$-waves $F_1^\sigma, F_2^\sigma$ of frequency $1$ and $2^k$, uniformly with respect to $\Phi_\sigma \in \mathcal{E}_\sigma$ and $\sigma \in (0, \sigma_0]$. The implicit constant in (3.2) depends at most on $\Sigma_1, \Sigma_2, \sigma_0, q, r, \varepsilon$.

3.1. The law of propagation. Let $\Xi_j^\sigma = \{ -\nabla \Phi(\xi) : \xi \in \Sigma_j \}$ for $j = 1, 2$. Then $\Xi_1^\sigma, \Xi_2^\sigma$ are disjoint subsets of an $(n-1)$ dimensional hypersurface in $\mathbb{R}^n$.

Let $\Xi_{j, r}^\sigma, \Xi_{j, b}^\sigma$ be a small neighborhood of $\Xi_j^\sigma$ with the enlargement constant being independent of $\sigma_0$ such that if we let
\[
\Lambda_j^\sigma(z_0) := z_0 + \{ t(\omega, 1) : t \in \mathbb{R}, \omega \in \Xi_j^\sigma \}
\]
with $z_0 = (x_0, t_0)$, then $\Lambda_1^\sigma(z_0)$ and $\Lambda_2^\sigma(z_0)$ are conic hypersurfaces in $\mathbb{R}^{n+1}$ meeting transversely for all $\Phi_\sigma \in \mathcal{E}_\sigma$ and $\sigma \leq \sigma_0$ by taking $\sigma_0$ small enough.

**Proposition 3.1.** For each $j = 1, 2$, let $\Sigma_j^\sigma$ be a small neighborhood of $\Sigma_j$ such that there exists a universal constant $c_0 > 0$ such that if we let
\[
(\Xi_j^\sigma)^C := \{ -\nabla \Phi(\xi) : \xi \in \mathbb{R}^n, |\xi| = 1 \} \setminus \Xi_j^\sigma,
\]
and \( a_j \in C_0^\infty(\Sigma_j^*) \) with \( a_j(\xi) = 1 \) for all \( \xi \in \Sigma_j \), then
\[
\min_{\iota = \pm} \angle(\iota(\omega, 1), (-\nabla \Phi_\sigma(\xi), 1)) \geq c_0
\]
for all \( \omega \in (\Xi_j^{\Phi_\sigma} )^c \) and \( \xi \in \text{supp} \ a_j \). Let
\[
K_{\Phi_\sigma}^j(x, t) = \int e^{i(x \xi + i\Phi_\sigma(\xi))} a_j(\xi) \, d\xi.
\]
Then,
\[
S_{\Sigma_j}^{\Phi_\sigma}(t)f(x) = [K_{\Phi_\sigma}^j(\cdot, t) * f](x),
\]
with
\[
|K_{\Phi_\sigma}^j(x, t)| \lesssim_M (1 + \text{dist}(x, t, \Lambda_{\Phi_\sigma}) )^{-M}
\]
for all \( (x, t) \in \mathbb{R}^{n+1} \) and all integers \( M \geq 1 \). Here \( \Lambda_{\Phi_\sigma} := \Lambda_{\Phi_\sigma}(z)_{i=0}^n \) and the implicit constant in (3.3) depends on \( c_0 \), but is uniform in \( \sigma \in (0, \sigma_0] \).

**Proof.** By definition, we have (3.3). If \( (x, t) \) is in a \( O(1) \)–neighbourhood of \( \Lambda_{\Phi_\sigma} \), then (3.4) follows from the Hausdorff-Young inequality. Let \( \text{dist}(x, t, \Lambda_{\Phi_\sigma}) \geq C \) with large \( C \). Define \( L = (x + i\nabla \Phi_\sigma(\xi)) \cdot \nabla \xi \). Let \( \left| L^M e^{i(x \xi + i\Phi_\sigma(\xi))} \right| = e^{i(x \xi + i\Phi_\sigma(\xi))}, \forall M \geq 1 \).

Integrating by parts, we get (3.4). The proof is complete. \( \square \)

### 3.2. Energy estimates on conic regions of opposite colour.

**Lemma 3.2.** Let \( F_1^\sigma, F_2^\sigma \) be red and blue waves of frequency \( 2^m \) with \( m \geq 0 \) and let \( \Lambda_{\Phi_\sigma}(z_0, r) \) be an \( O(r) \)–neighbourhood of \( \Lambda_{\Phi_\sigma}(z_0) \) with \( r > 0 \) and \( z_0 = (x_0, t_0) \). Then, we have
\[
\| F_j^\sigma \|_{L^2(\Lambda_{\Phi_\sigma}^\sigma, (z_0, r))} \lesssim r^{1/2} \mathbb{E}(F_j^\sigma)^{1/2} \quad \forall j, k \in \{1, 2\}, j \neq k,
\]
for all \( z_0 \in \mathbb{R}^{n+1} \) and \( r \gtrsim 2^{-m} \). The implicit constant in (3.5) might depend on \( \sigma_0 \) but is uniform with respect to \( \sigma \) such that \( 0 < \sigma \leq \sigma_0 \).

**Proof.** By translation invariance, we may take \( z_0 \) as the origin. We only show (3.5) for \( (j, k) = (1, 2) \) and by symmetry the other case follows.

Consider first \( m = 0 \). Let
\[
\mathcal{D}_{2, t}^\sigma = \{ x \in \mathbb{R}^n ; \text{dist}(x, t, \Lambda_{\Phi_\sigma}^\sigma) \lesssim r \}.
\]
Let \( S_{1}^{\Phi_\sigma} \) be the adjoint of \( S_{1}^{\Phi_\sigma} \). By \( TT^* \), it suffices to show
\[
\left\| \int S_{1}^{\Phi_\sigma}(t) \mathbb{L}_{\mathcal{D}_{2, t}^\sigma} H(\cdot, t) \right\|_{L^2(\mathbb{R}^n)} \lesssim r^{1/2} \| H \|_{L^2(\mathbb{R}^{n+1})}
\]
for all \( H \in L^2(\mathbb{R}^{n+1}) \). Taking squares and multiplying out, we have
\[
\left\| \int S_{1}^{\Phi_\sigma}(t) \mathbb{L}_{\mathcal{D}_{2, t}^\sigma} H(\cdot, t) \right\|_{L^2(\mathbb{R}^n)}^2 \lesssim \int \mathbb{L}_{\mathcal{D}_{2, t}^\sigma}(\tilde{x}) K_{\Phi_\sigma}^1(\tilde{x} - x, t - t) \mathbb{L}_{\mathcal{D}_{2, t}^\sigma}(x) H(x, t) \overline{H(\tilde{x}, t)} \, dx \, d\tilde{x} \, dt \, d\tilde{t},
\]
where \( K_{\Phi_\sigma}^1 \) is given by Proposition 3.1 with \( a_1 \) replaced by \( |a_1|^2 \) there. Split the integral (3.6) over time variables to the \( |t - \tilde{t}| \lesssim r \) and \( |t - \tilde{t}| \gg r \) part.
By Cauchy-Schwarz and the $L^2$-boundedness of
\[ \sup_{t,\tilde{t}} \| I_{\mathcal{D}^t_x} S^{\Phi_\sigma}_1 (\tilde{t}) \circ S^{\Phi_{\sigma+\epsilon}}_1 (t) I_{\mathcal{D}^t_x} \|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} = O(1), \]
the $|t - \tilde{t}| \leq r$ part of the integral \((3.6)\) is bounded by $r \| H \|_{L^2}^2$.

Next, using \((3.4)\)
\[ |C^{\Phi_\sigma}_1 (\tilde{x} - x, \tilde{t} - t)| \lesssim_M \left( 1 + \text{dist}((\tilde{x} - x, \tilde{t} - t)), \Lambda^{\Phi_\sigma}_1 \right)^{-M} \]
and the constraints for all $x \in \mathcal{D}^r_x$, $\tilde{x} \in \mathcal{D}^r_{\tilde{x}}$, one finds that the $|t - \tilde{t}| \gg r$ part of \((3.6)\) can be bounded by
\[ \int \int (1 + |t - \tilde{t}|/r)^{-M} \| H(\cdot, t) \|_2 \| H(\cdot, \tilde{t}) \|_2 dt d\tilde{t}, \]
where, we have taken $\sigma_0$ small so that the amount of the transversality between the conic surfaces $\Lambda^{\Phi_\sigma}_1$ and $\Lambda^{\Phi_\sigma}_2$ depends only on $\Sigma_1, \Sigma_2, \sigma_0$ but independent of $\sigma \in (0, \sigma_0]$ and $\Phi_\sigma \in \mathcal{E}_\sigma$. The result follows by Schur’s test.

For the general case $m \geq 1$, we may apply the result to $F^{\Phi_\sigma}_1 (2^{-m} x, 2^{-m} t)$ and use rescaling. The proof is complete. \qed

**Corollary 3.3.** There is a constant $C > 0$ depending only on $\Sigma_1, \Sigma_2$ and $\sigma_0$ such that if $\sigma_0$ is sufficiently small, then the following property holds: Let $F^{\Phi_\sigma}_1$ be a red $\Phi_\sigma$-wave of frequency 1 and $F^{\Phi_\sigma}_2$ be a blue $\Phi_\sigma$-wave of frequency $2^k$ associated to some $\Phi_\sigma \in \mathcal{E}_\sigma$ with $\sigma \in (0, \sigma_0]$. Let $R \geq r \gg 1$, $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and set
\[ \Lambda_{\text{purple}}(z_0, r) = \Lambda^{\Phi_\sigma}_1(z_0, r) \cup \Lambda^{\Phi_\sigma}_2(z_0, r). \]
Then, we have
\[ \| F^{\Phi_\sigma}_1 F^{\Phi_\sigma}_2 \|_{L^1(Q_R \cap \Lambda_{\text{purple}}(z_0, r))} \leq C \sqrt{r R} E(F^{\Phi_\sigma}_1)^{1/2} E(F^{\Phi_\sigma}_2)^{1/2}, \]
for all $Q_R$ spacetime cube of length $R$.

**Proof.** The result follows from using Lemma 3.3 and the triangle inequality. \qed

**3.3. The careful wavepacket decomposition.** Let $N > 1$ be a large integer depending only on the dimension $n$ and $C_0 = 2^{(N/\varepsilon_0)}$ be a large integer where $\varepsilon_0 > 0$ is made small when necessary but never tending zero.

Let $0 < \varepsilon \leq 2^{-C_0}$ and $\varrho = R^{1/2}$. Define
\[ \mathcal{L} = \varepsilon^{-2} \varrho \mathbb{Z}^n, \quad \Gamma = \varrho^{-1} \mathbb{Z}^{n-1}. \]
Fix $\Phi_\sigma \in \mathcal{E}_\sigma$. For each $(v, \mu) \in \mathcal{L} \times \Gamma$, define a $(\Phi_\sigma, \varrho)$-tube $T = T^{\Phi_\sigma}_v (v, \mu)$ as
\[ T = \{ (x, t) \in \mathbb{R}^{n+1} : |x - v + t \nabla \psi_\sigma (\mu, 1)| \leq \varrho \}. \quad (3.7) \]
Denote by $\mathbf{T}_\sigma = \mathbf{T}_{\Phi_\sigma}$ the collection of such tubes and write $(v_T, \mu_T)$ as the coordinate parametrizing a given tube $T \in \mathbf{T}_\sigma$ in the sense of \((3.7)\). For any $T$, let
\[ \psi_T(x, t) = \min \{ 1, \text{dist}((x, t), T)^{-100N} \}. \]
be the bump function adapted to $T$.

A disk is a subset $D$ resident in $\mathbb{R}^{n+1}_{x, t}$ of the form
\[ D = D(x_D, t_D; r_D) = \{ (x, t) : |x - x_D| \leq r_D \}, \]
for some \((x_D, t_D) \in \mathbb{R}^{n+1}\), which is called the center of \(D\) and \(r_D > 0\), which is called the radius of \(D\). We call \(t_D\) the time coordinate of \(D\). The indicator function of \(D\) is defined as

\[
\mathbb{1}_D(x) = \begin{cases} 
1, & (x, t) \in D, \\
0, & (x, t) \notin D.
\end{cases}
\]

The bump function adapted to \(D\) is defined as

\[
w_D(x) := \left(1 + \frac{|x - x_D|}{r_D}\right)^{-N^{100}}.
\]

For any \(F \in C^\infty(\mathbb{R}^{n+1})\) and disk \(D\), we write

\[
\|F\|_{L^2(D)} := \left(\int_{|x - x_D| \leq r_D} |F(x, t_D)|^2 \, dx\right)^{\frac{1}{2}}.
\]

We use \(Q = Q(x_Q, t_Q; r_Q)\) to denote a spacetime cube \(Q\) of side-length \(r_Q\) centered at \(z_Q := (x_Q, t_Q) \in \mathbb{R}^{n+1}\). For any \(C > 0\), we write \(CQ = Q(x_Q, t_Q; Cr_Q)\), and we call \(\text{LS}(Q) := [t_Q - \frac{r_Q}{2}, t_Q + \frac{r_Q}{2}]\) the lifespan of \(Q = Q(x_Q, t_Q; r_Q)\).

Let \(Q = Q_R \subset \mathbb{R}^{n+1}\) be a spacetime cube of length \(R \geq C_0 2^k\). Let \(j \geq 0\) be an integer and partition \(Q\) into \(2^{(n+1)j}\) many subcubes of side-length \(2^{-j}R\). Denote \(Q_j(Q)\) as the collection of these cubes. Let \(J \approx \log R\) such that each \(q\) in \(Q_j(Q)\) is a cube of side-length \(\varrho\).

For any \(\Phi_\sigma\)-wave \(F^\sigma\) of frequency one whose spacetime Fourier transform \(\widehat{F^\sigma}\) is supported on the surface \(C^\Phi_\sigma_{\Sigma^*} = \{(\Phi_\sigma(\xi), \xi) : \xi \in \Sigma^*\}\), we define the margin of \(F^\sigma\)

\[
\text{marg}(F^\sigma) := \text{dist}(\text{supp}(\widehat{F^\sigma}), C^\Phi_\sigma_{\Sigma^*}),
\]

where \(\Sigma^*\) is the enlargement of \(\Sigma\) in Proposition 3.1 with \(\Sigma\) being either \(\Sigma_1\) or \(\Sigma_2\).

If \(F^\sigma\) is a \(\Phi_\sigma\)-wave of frequency two \(2^n\), we define \(\mathbb{D}_{2^m}\) by letting

\[
\mathbb{D}_{2^m} : F^\sigma(x, t) \mapsto F^\sigma(2^{-m}x, 2^{-m}t),
\]

so that \(\mathbb{D}_{2^m} F^\sigma(x, t) = S^\Phi_\sigma(t) [\mathbb{D}_{2^m} F^\sigma(\cdot, 0)](x)\) with \(\Sigma\) being equal to \(\Sigma_1\) or \(\Sigma_2\), is a wave of frequency one. Define the margin of \(F^\sigma\) as \(\text{marg}(F^\sigma) = \text{marg}(\mathbb{D}_{2^m} F^\sigma)\). Thus, by homogeneity,

\[
\text{marg}(F^\sigma) = 2^{-m} \text{dist}(\widehat{F^\sigma}, 2^m(C^\Phi_\sigma_{\Sigma^*})).
\]

Let \(Q = [-1/2, 1/2] \times \cdots \times [-1/2, 1/2]\) and \(\mathbb{1}_Q\) be the characteristic function of the unit box \(Q\). For each \(\mu \in \Gamma\) and \(\zeta \in \mathbb{R}^{n-1}\), define

\[
\mathbb{P}_{\zeta, \phi}^{\mu} : f \mapsto \int_{\Sigma^*} e^{i(x, \xi)} \mathbb{1}_{Q + \zeta}(\varphi(\xi_n^{-1} \xi' - \mu)) \widehat{f}(\xi) \, d\xi' d\xi_n,
\]

where \(Q + \zeta := \{\eta + \zeta : \eta \in Q\}\).

**Lemma 3.4.** There exists \(C > 0\) depending only on \(n\), such that the following statement holds:

For any spacetime cube \(Q = Q_R\) of side-length \(R\), for any \(\varphi \in \{0, 2^{-C_0}\}\), there is a linear map \(F^\sigma \mapsto \{F^\sigma_T\}_{T \in T_n}\) such that for any \(\Phi_\sigma\)-wave \(F^\sigma\) of frequency one, each \(F^\sigma_T\) is also a \(\Phi_\sigma\)-wave of the same colour and frequency to \(F^\sigma\) with the relaxed margin property

\[
\text{marg}(F^\sigma_T) \geq \text{marg}(F^\sigma) - C_0^{-1},
\]

(3.9)
and
\[ F_\sigma(x,t) = \sum_{T \in \mathcal{T}_\sigma} F_\sigma^T(x,t), \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}. \quad (3.10) \]

Moreover, we have:

- For any \( t \in \mathbb{L} S(C_0 Q) \) and every \( T \in \mathcal{T}_\sigma \)
  \[ \mathcal{E}(F_\sigma^T) \leq C \varpi^{-C} \int_Q \| \psi_T(\cdot, t) F_\sigma^T(\cdot, t) \|_{L^2(\mathbb{R}^n)}^2 d\xi, \quad (3.11) \]

- Concentration of \( F_\sigma^T \) on \( T \)
  \[ \| F_\sigma^T \|_{L^\infty(Q)} \leq C \left( 1 + \operatorname{dist}(T, Q) \right)^{-N} \mathcal{E}(F_\sigma)^{1/2}, \quad (3.12) \]

- No local accumulating of tube multiplicities
  \[ \sum_{T \in \mathcal{T}_\sigma, q \in Q_j(Q)} \sup_{T} \| \psi_T^{-50} F_\sigma^T \|_{L^2(C_q)} \leq C \varpi^{-C} \mathcal{E}(F_\sigma), \quad (3.13) \]

- The Bessel type inequality
  \[ \left( \sum_{\Delta} \mathcal{E}\left( \sum_{T \in \mathcal{T}_\sigma} m^{\Delta, T} F_\sigma^T \right) \right)^{1/2} \leq (1 + C \varpi) \mathcal{E}(F_\sigma)^{1/2}, \quad (3.14) \]

holds for all \( m^{\Delta, T} \geq 0 \) such that

\[ \sup_{T \in \mathcal{T}_\sigma} \sum_{\Delta} m^{\Delta, T} \leq 1 \]

where \( \sum_{\Delta} \) is summing over a finite number of \( \Delta \)'s.

**Proof.** By translation in the physical spacetime and the modulation in the frequency space, we may take \( z_Q \) to be the origin of \( \mathbb{R}^{n+1} \).

- **Step 1. The decomposition map.** Let \( \Upsilon_0 \in \mathcal{S}(\mathbb{R}^n) \) be a non-negative Schwartz function such that \( \Upsilon_0 \) is supported in \( B(0, 1/10) := \{ \xi \in \mathbb{R}^n; |\xi| \leq 1/10 \} \) and \( \Upsilon_0 \) equals to one on \( B(0, 1/20) \). Put \( \Upsilon_\nu(x) = \Upsilon_0(\varpi^2 \varrho^{-1}(x - \nu)) \), \( \nu \in \mathcal{L} \). By the Poisson summation formula, we have \( \sum_{\nu \in \mathcal{L}} \Upsilon_\nu(x) = 1 \) for all \( x \in \mathbb{R}^n \).

For each \( (\nu, \mu) \in \mathcal{L} \times \Gamma \), let for every \( x \) and \( \xi = (\xi', \xi_n) \in \Sigma^* \)
\[ a_{\nu, \mu}(x, \xi) = \Upsilon_\nu(x) (\| Q_0 * \mathcal{F} Q_0 (\varrho (\xi_n^{-1} \xi' - \mu))) \].

For any \( f \in \mathcal{S}(\mathbb{R}^n) \) such that \( \supp \hat{f} \subset \Sigma \), define
\[ f_{\nu, \mu}(x) = \int_{\Sigma^*} e^{i(x, \xi) \cdot a_{\nu, \mu}(x, \xi)} \hat{f}(\xi) d\xi. \]

Then, by Fubini’s theorem, we have
\[ f(x) = \sum_{(\nu, \mu) \in \mathcal{L} \times \Gamma} f_{\nu, \mu}(x), \quad \forall x \in \mathbb{R}^n. \]

For each \( \Phi_\sigma \)-wave \( F_\sigma \), apply this decomposition with \( f(x) = F_\sigma(x, 0) \). By using the linearity of the operator \( S_{\Sigma^*}^{\Phi_\sigma} (t) \), we have
\[ S_{\Sigma^*}^{\Phi_\sigma} (t) f(x) = \sum_{(\nu, \mu) \in \mathcal{L} \times \Gamma} S_{\Sigma^*}^{\Phi_\sigma} (t) f_{\nu, \mu}(x). \quad (3.15) \]

Letting \( F_\sigma^T(x, t) = S_{\Sigma^*}^{\Phi_\sigma} (t) f_{\nu, \mu}(x) \) with \( (\nu, \mu) = (\nu_T, \mu_T) \), we get (3.10). The relaxed margin property (3.9) holds with a fixed \( C \) independent of \( \Phi_\sigma \).
- Step 2. Proof of (3.11) (3.12) (3.13). Let \( \mathcal{B} := \{ \xi \in \mathbb{R}^n; |\xi| \leq 50n \} \) and let \( \alpha \in C_c^\infty(\mathbb{R}^{n-1}) \) be equal to one on \( \mathcal{B} \), vanishing outside \( O(1) \)-neighborhood of \( \mathcal{B} \). Let \( \beta \in C_c^\infty([\frac{1}{100}, 100]) \) be equal to one on \([\frac{1}{50}, 50] \). Let \( p(\xi', \xi_n) = \alpha(\xi')\beta(\xi_n) \) and

\[
K^\sigma_\mu(t, x) := \int_{\mathbb{R}^n} e^{i(\xi' \cdot x + \xi_n t + \nabla_x \Phi_\sigma(\xi))} p(\rho(\xi_n^{-1} \xi' - \mu), \xi_n) \, d\xi' d\xi_n.
\]

We have \( F^\sigma_T(t, \cdot) = K^\sigma_\mu(t-t_0, \cdot) \ast F^\sigma_T(t_0, \cdot) \) for every \( T \in \mathcal{T}_\sigma \) and all \( t, t_0 \in \mathbb{R} \). Changing variables, we get

\[
K^\sigma_\mu(t, x) = \varrho^{-(n-1)} \int_{\mathbb{R}^n} e^{i(\varrho^{-1} \xi' \cdot x + \xi_n t + \nabla_x \Phi_\sigma(\varrho^{-1} \xi' + \mu, 1))} p\left( \frac{\xi'}{\xi_n}, \xi_n \right) \, d\xi' d\xi_n.
\]

Taylor expanding using homogeneity and the \( \mathcal{E}_\nu \) conditions along with \( \xi_n \sim 1 \)

\[
\Phi_\sigma(\varrho^{-1} \xi' + \mu \xi_n, \xi_n) = \xi_n \Phi_\sigma(\varrho^{-1} \xi_n^{-1} \xi' + \mu, 1)
\]

\[
= \xi_n \Phi_\sigma(\mu, 1) + \varrho^{-1} \nabla_{\xi'} \Phi_\sigma(\mu, 1) \cdot \xi' + R^{-1} \Psi_{\sigma, \mu}^R(\xi),
\]

where \( \Psi_{\sigma, \mu}^R \) is homogeneous of order one and depends on the Hessian of \( \Phi_\sigma \)

\[
\sup_{0 < \sigma \leq \sigma_0 \in \mathcal{E}_\nu, |\mu| \leq 1} \xi \in \mathbb{R}| \frac{\partial^2 \Psi_{\sigma, \mu}^R(\xi)}{\partial \xi' \partial \xi_n} | \leq C_\gamma
\]

for any multi-indices \( \gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}^n \geq 0 \), we have

\[
K^\sigma_\mu(t, x) = \varrho^{-(n-1)} \int_{\mathbb{R}^n} e^{i\phi_\sigma^{\sigma, \mu}(x, t; \xi')} p\left( \frac{\xi'}{\xi_n}, \xi_n \right) \, d\xi' d\xi_n,
\]

where

\[
\phi_\sigma^{\sigma, \mu}(x, t; \xi) := \varrho^{-1}(x' + t \nabla_{\xi'} \Phi_\sigma(\mu, 1)) \cdot \xi' + (x_n + x' \cdot \mu + t \Phi_\sigma(\mu, 1)) \xi_n + \frac{t}{R} \Psi_{\sigma, \mu}^R(\xi).
\]

Letting

\[
L_1 = \frac{1 + i^{-1}(\varrho^{-1}(x' + t \nabla_{\xi'} \Phi_\sigma(\mu, 1)) + \frac{t}{R} \nabla_{\xi'} \Psi_{\sigma, \mu}^R(\xi)) \cdot \nabla_{\xi'}}{1 + |\varrho^{-1}(x' + t \nabla_{\xi'} \Phi_\sigma(\mu, 1)) + \frac{t}{R} \nabla_{\xi'} \Psi_{\sigma, \mu}^R(\xi)|^2},
\]

\[
L_2 = \frac{1 + i^{-1}(x_n + x' \cdot \mu + t \Phi_\sigma(\mu, 1) + \frac{t}{R} \partial_{\xi_n} \Psi_{\sigma, \mu}^R(\xi)) \partial_{\xi_n}}{1 + |x_n + x' \cdot \mu + t \Phi_\sigma(\mu, 1) + \frac{t}{R} \partial_{\xi_n} \Psi_{\sigma, \mu}^R(\xi)|^2},
\]

such that for any integer \( M \geq 1 \), we have

\[
L_1^M e^{i\phi_\sigma^{\sigma, \mu}(x, t; \xi)} = L_2^M e^{i\phi_\sigma^{\sigma, \mu}(x, t; \xi)} = e^{i\phi_\sigma^{\sigma, \mu}(x, t; \xi)}.
\]

Noting that

\[
1 + |\varrho^{-1}(x' + t \nabla_{\xi'} \Phi_\sigma(\mu, 1)) + \frac{t}{R} \nabla_{\xi'} \Psi_{\sigma, \mu}^R(\xi)| + |x_n + x' \cdot \mu + t \Phi_\sigma(\mu, 1) + \frac{t}{R} \partial_{\xi_n} \Psi_{\sigma, \mu}^R(\xi)|
\]

\[
\geq \frac{1}{C_0^M} \left| \varrho^{-1}(x' + t \nabla_{\xi'} \Phi_\sigma(\mu, 1)) \right| + \frac{1}{C_0} \left| x_n + x' \cdot \mu + t \Phi_\sigma(\mu, 1) \right|
\]

holds for all \( \xi \in \text{supp} \rho \) and all \( |t| \leq C_0 R \), we have by \( M \)-fold integration by parts

\[
|K^\sigma_\mu(t, x)| \lesssim_M C_0^M \varrho^{-(n-1)} \left( 1 + \varrho^{-1} |x + t \nabla \Phi_\sigma(\mu, 1)| \right)^{-M},
\]

where we have used the fact that \( \Phi_\sigma(\mu, 1) = \mu \cdot \nabla_{\xi'} \Phi_\sigma(\mu, 1) + \partial_{\xi_n} \Phi_\sigma(\mu, 1) \) and for each \( t \), the \( x' \) variable is essentially contained in the ball centered at \(-t \nabla_{\xi'} \Phi_\sigma(\mu, 1)\) of radius \( \varrho \). With (3.17) and using the same argument of dyadic decomposition and
summing up convergent geometric series exploring the decay properties of bump functions, c.f. [15] (103) P.262, we get
\[ \|F_\sigma\|_{L^2(D(x,t),\mu)} \leq C \|wD(x,t)\|_{\sigma} F_\sigma(\cdot, t)\|_{L^2(\mathbb{R}^n)} \] (3.18)
for all disk \(D\) and \(\Phi_\sigma\)-wave \(F_\sigma\) such that the spatial Fourier transform is supported on \(\xi \in \Sigma\) with \(|\xi^{-1}\xi' - \mu| \leq 50n\sigma^{-1}\). Using (3.18),
\[ F_\sigma^e(x,0) = \mathcal{F}_{\nu_T}(x) \int_{\mathcal{Q}} \mathcal{P}_{\zeta}^\mu e^{-i\zeta \cdot \mu} F_\sigma(\cdot, 0) d\zeta, \]
and that \(\mathcal{P}_{\zeta}^\mu F_\sigma(x, t)\) is a \(\Phi_\sigma\)-wave with \(\mu = \mu_T\), satisfying the conditions for the \(F^n\) in (3.18), we obtain (3.11) by Cauchy-Schwarz.

With (3.17) (3.18), the estimates (3.12) (3.13) can be easily verified by using the same proof in [15].

- Step 3, Proof of the Bessel type inequality. The argument is same to [15, 23] and we only sketch it. By Plancherel’s theorem and Minkowski’s inequality, the left side of (3.19) is less or equal to
\[ \int_Q \left( \sum_{\Delta} \left\| \sum_{T \in T_\sigma} m^{\Delta,T} \mathcal{F}_{\nu_T}(\cdot) \mathcal{P}_{\zeta}^\mu e^{-i\zeta \cdot \mu} \right\|_2^2 \right)^{\frac{1}{2}} d\zeta, \] (3.19)
with \(f(x) = F_\sigma(x,0)\). For each \(\mu \in I\), define \(\mathcal{B}_{\theta,\mu} = \mu + \frac{1}{\theta} Q\) and let
\[ O = \bigcup_{\mu \in I} \left\{ \xi \in \mathcal{B}_{\theta,\mu} ; \ \text{dist}(\xi, \mathbb{R}^{n-1} \setminus \mathcal{B}_{\theta,\mu}) \geq \omega_\theta \right\}. \]

For any \(\eta \in \theta^{-1}Q\), define
\[ P_{O+\eta} : f(x) \mapsto \int e^{i(x \cdot \xi)} \mathbf{1}_{\{ \xi ; \ \xi' \in O+\eta \}}(\xi) \tilde{f}(\xi) d\xi. \]

Splitting \(f = (P_{O+\eta} f) + (\text{Id} - P_{O+\eta}) f\) and using the triangle inequality, we have that the right side of (3.19) \(\leq I + II\), where
\[ I = \theta^{n-1} \int_{\theta^{-1}Q} \left( \sum_{\Delta} \left\| \sum_{T \in T_\sigma} m^{\Delta,T} \mathcal{F}_{\nu_T}(\cdot) \mathcal{P}_{\zeta}^\mu e^{-i\zeta \cdot \mu} \right\|_2^2 \right)^{\frac{1}{2}} d\eta, \] (3.20)
\[ II = \theta^{n-1} \int_{\theta^{-1}Q} \left( \sum_{\Delta} \left\| \sum_{T \in T_\sigma} m^{\Delta,T} \mathcal{F}_{\nu_T}(\cdot) \mathcal{P}_{\zeta}^\mu e^{-i\zeta \cdot \mu} \right\|_2^2 \right)^{\frac{1}{2}} d\eta. \] (3.21)

For \(I\), we use the Plancherel theorem and the strict orthogonality from the pairwise \(\omega_\theta \theta^{-1}\)-separateness between the connected components of \(O\), which allows a petite amplification in the frequency space due to convolution with \(\mathcal{F}_{\nu_T}\). For \(II\), by using the Plancherel theorem and the almost orthogonality followed with Cauchy-Schwarz and Fubini, we have
\[ \sup_{\xi \in \Sigma^*} \theta^{n-1} \int_{\theta^{-1}Q} \left( 1 - P_{O+\eta}(\xi'/\xi_n) \right) d\eta \lesssim_n \omega^2. \]

We refer to [15, 23] for more details. The proof is complete. \(\square\)

Remark 3.5. For waves of high frequency, say \(2^m\) with \(m \geq 1\), we may use the \(D_{2^m}\) normalizing the frequency to be one and apply Lemma 3.4 then scaling back.
Remark 3.6. In the above, we have always assumed the initial data of waves are Schwartz functions. This assumption can be removed by standard density argument.

3.4. Spatial localizations using the Huygens principle. We introduce the spatial localization operators as in [13] and summarize the properties useful in this paper to capture the energy concentration of waves. Since for any fixed $\Phi_\sigma$, the proof are same to [13], we shall omit most of technical parts.

For any $c > 0$, let $c D := D(x_D, t_D; c r_D)$. Define the disk exterior of $D$ as

$$D^\text{ext} = D^\text{ext}(x_D, t_D; r_D) = \{(x, t_D) : |x - x_D| > r_D\}.$$  (3.22)

For any $F \in \mathcal{C}_\text{loc}^\infty(\mathbb{R}^{n+1})$ and disk $D$, we write

$$\|F\|_{L^2(D^\text{ext})} := \left( \int_{|x - x_D| > r_D} |F(x, t_D)|^2 \, dx \right)^{1/2}.$$

3.4.1. The localization operator $P_D$. We introduce the localization operator $P_D$ as in [13]. Let $\Upsilon_0(x) \geq 0$ be the Schwartz function in the proof of Lemma 3.4. For every $r > 0$, set $\Upsilon_r(x) = r^{-n} \Upsilon_0(r^{-1} x)$.

Definition 3.7. Let $F^\sigma(x, t)$ be a $\Phi_\sigma$—wave of frequency 1 for some $\Phi_\sigma \in \mathcal{E}_\sigma$. For any disk $D = D(x_D, t_D; r_D)$, we define $P_D^\Phi_\sigma F^\sigma$ at time $t_D$ as

$$P_D^\Phi_\sigma F^\sigma(t_D) = \left( \mathbb{I}_D \ast \Upsilon_{r_D^{-1}} \right) F^\sigma(t_D),$$

and $\forall t \in \mathbb{R}$

$$(P_D^\Phi_\sigma F^\sigma)(t) = S_D^\Phi_\sigma(t - t_D) \left[ P_D^\Phi_\sigma F^\sigma(t_D) \right].$$

For any $\Phi_\sigma$—wave $F^\sigma$ of frequency $2^m$, we define $P_D^{\Phi_\sigma, 2^m} F^\sigma = \mathbb{D}_{2^m}^{-1} \circ P_D^{\Phi_\sigma} \circ \mathbb{D}_{2^m} F^\sigma$.

We shall abuse notations below, writing $P_D^{\Phi_\sigma}$ instead of $P_D^{\Phi_\sigma, 2^m}$ for short and the meaning is clear from context.

We show next that the operator $P_D^{\Phi_\sigma}$ localizes a $\Phi_\sigma$—wave to $D_+$ while $1 - P_D^{\Phi_\sigma}$ localizes it to the exterior of $D_-$.

Lemma 3.8. Let $m \geq 0$ and $D$ be a disk with radius $r_D = r \geq C_0 2^{-m}$. Then, for any $\Phi_\sigma$—waves $F^\sigma$ with frequency $2^m$, $P_D^{\Phi_\sigma} F^\sigma$ is a $\Phi_\sigma$—wave of frequency $2^m$ and the same colour to $F^\sigma$, satisfies the margin estimate

$$\text{marg}(P_D^{\Phi_\sigma} F^\sigma) \geq \text{marg}(F^\sigma) - C_0 (2^m r)^{-1 + \frac{2}{n}},$$

and the following local energy estimates

$$\|P_D^{\Phi_\sigma} F^\sigma\|_{L^2(D^\text{ext})} \lesssim (2^m r)^{-N} E(F^\sigma)^{1/2},$$

$$\|(1 - P_D^{\Phi_\sigma}) F^\sigma\|_{L^2(D^\text{ext})} \lesssim (2^m r)^{-N} E(F^\sigma)^{1/2},$$

$$\sup_t \|P_D^{\Phi_\sigma} F^\sigma(t)\|_{L^2(\mathbb{R}^n)}^2 \leq \|F^\sigma\|_{L^2(\mathbb{R}^n)}^2 + O((2^m r)^{-N} E(F^\sigma)),$$  (3.25)

$$\sup_t \|(1 - P_D^{\Phi_\sigma}) F^\sigma(t)\|_{L^2(\mathbb{R}^n)}^2 \leq \|F^\sigma\|_{L^2(\mathbb{R}^n)}^2 + O((2^m r)^{-N} E(F^\sigma)),$$  (3.26)

$$\sup_t \|(1 - P_D^{\Phi_\sigma}) F^\sigma(t)\|_{L^2(\mathbb{R}^n)} \leq E(F^\sigma)^{1/2}, \sup_t \|P_D^{\Phi_\sigma} F^\sigma(t)\|_{L^2(\mathbb{R}^n)} \leq E(F^\sigma)^{1/2},$$  (3.27)

where $D^\text{ext}_\pm$ is the exterior of $D_\pm := D(x_D, t_D; r(1 \pm (2^m r)^{-1/P}))$. The implicit constants are all independent of $\sigma$.

Proof. The argument is exactly same as [13] and we omit it. See also [23]. \qed
For the general case, we use the dilation operator $\Upsilon$ of 
and combine it with the constraints outside the conic set in (3.28) to use the rapid decay dividing the integration in question into the local and global parts, where for the $R, r$ the above case with $\sigma$.

Notion of the wave tables on a spacetime cube.

4.1. Notion of the wave tables on a spacetime cube. We recall first the red and blue wave tables in [15].

A wave table operation $\tilde{\mathcal{S}}$ on $Q$ with depth $j$ is a linear map such that for any $\Phi_\sigma$-wave $F^\sigma$ for some $\sigma$, we have $\tilde{\mathcal{S}}_j : F^\sigma \mapsto \mathcal{F}^\sigma_j$, with $\mathcal{F}^\sigma_j$ being of the matrix form

\[ \mathcal{F}^\sigma_j = (F^\sigma(q))_{q \in Q_j(Q)}, \]

which is called the wave table of $F^\sigma$ on $Q$, where each $F^\sigma(q)$ is a $\Phi_\sigma$-wave (of the same colour to $F^\sigma$) as well. We define $\tilde{\mathcal{S}}_0$ to be the identity map sending $F^\sigma$ to itself, which is linked in some sense to the cube $Q$. For a wave table of $F^\sigma$ with given depth $j$, we call each wave $F^\sigma(q)$ the entry of $\mathcal{F}^\sigma_j$. The Fourier support of $\mathcal{F}^\sigma_j$ is defined as

\[ \text{supp}\, \tilde{\mathcal{F}}^\sigma_j = \bigcup_{q \in Q_j(Q)} \text{supp}\, \tilde{F}^\sigma(q), \]

and the margin of $\mathcal{F}^\sigma_j$ is defined as in (3.3) with $F^\sigma$ replaced by $\mathcal{F}^\sigma_j$.

For any integer $m \geq 0$, we may apply the map $\tilde{\mathcal{S}}_m$ to each above $F^\sigma(q)$ to get

\[ \tilde{\mathcal{S}}_m(F^\sigma(q)) = (F^\sigma(q'))_{q' \in Q_m(q)}. \]

The composition $\tilde{\mathcal{S}}_m \circ \tilde{\mathcal{S}}_j$ defined this way by applying $\tilde{\mathcal{S}}_m$ to each entry of $\tilde{\mathcal{S}}_j(F^\sigma)$ gives a new wave table operation $\tilde{\mathcal{S}}_{m+j} = \tilde{\mathcal{S}}_m \circ \tilde{\mathcal{S}}_j$ with depth $m + j$. By doing so, it is possible to obtain a chain $\{\tilde{\mathcal{S}}_j\}_{j \geq 0}$ of wave table operations. This general formulation can be extended to any given $Q$. Moreover, there is in no way a prescribed paradigm of constructing wave tables. One special construction that matters for us in this paper is the one in [15] by iterating $\tilde{\mathcal{S}}_{j+C_0} := \tilde{\mathcal{S}}_{C_0} \circ \tilde{\mathcal{S}}_j.$
for each \( j \in C_0 \mathbb{Z} \). To this end, we need to give an explicit construction of \( \mathcal{F}_{C_0} \) first, to which we now turn immediately.

For any \( R \geq C_02^{2\max(k,m)} \) we define \( R_{\sigma}^{g_{-2}m \sigma} \) as the set of red waves \( F^\sigma \) with \( \tilde{F}^\sigma \) being an \( L^2 \) measure on \( C_1^{\Phi_{-2}m \sigma} \) satisfying the margin condition

\[
\text{marg}(F^\sigma) \geq (100)^{-1} - (2^{-m}R)^{-\frac{1}{2}}.
\]

Likewise, define \( B_{\sigma}^{g_{-2}m \sigma} \) as the set of blue waves \( G^\sigma \) such that \( \text{supp} \tilde{G}^\sigma \subset C_2^{g_{-2}m \sigma} \) satisfying the margin condition

\[
\text{marg}(G^\sigma) \geq (100)^{-1} - (2^{-k}R)^{-\frac{1}{2}}.
\]

We construct the wave tables for waves of frequency one first and then generalize the construction to at least one of the waves with high frequency.

Given two waves of opposite colours, we shall construct \( \mathcal{F}_{C_0} \) for one wave on any given cube with respect to the other. Let \( Q = Q_R \) be a spacetime cube of length \( R \) and \( F^\sigma \in R_{\sigma}^{g_{-1}m} \) and \( G^\sigma \in B_{\sigma}^{g_{-1}m} \) be red and blue waves. We construct the wave table of \( F^\sigma \) of depth \( C_0 \) on \( Q \) with respect to \( G^\sigma \) based on Lemma 3.4.

Denote

\[
F^\sigma = \sum_{T_1 \in T_{1}^{a}} F^\sigma_{T_1}, \quad G^\sigma = \sum_{T_2 \in T_{2}^{a}} G^\sigma_{T_2},
\]

as the decomposition into wave-packets for the red and blue waves given by (3.10). For any \( \Delta \in Q_{C_0}(Q) \), let

\[
K_Q(\Delta) = \{ q \subset \Delta; \ q \in Q_J(Q) \},
\]

with \( J \approx \log R \) such that each \( q \) in \( Q_J(Q) \) is a cube of side-length \( \sqrt{R} \). Let \( \chi \in S(\mathbb{R}^{n+1}) \) be such that the spacetime Fourier transform of \( \chi \) is compactly supported in a small neighbourhood of the origin and \( \chi \geq 1 \) on double of the unit ball, with the structure constants independent of \( \sigma \). Let \( A_q \) be the affine transform sending the John ellipsoid inside \( q \) to the unit ball such that if we let \( \chi_q = \chi \circ A_q \), then we have \( \chi_q \geq \mathbb{1}_q \), where \( \mathbb{1}_q \) is the characteristic function of \( q \).

The red wave table \( F^\sigma \) of \( F^\sigma \) with respect to the blue wave \( G^\sigma \) on \( Q \), with depth \( C_0 \) depending on \( \varpi \) and \( R \) is given by

**Definition 4.1.** For each \( \Delta \in Q_{C_0}(Q) \) and \( T_1 \in T_{1}^{a} \), define

\[
m_{T_1}^{G^\sigma} \cdot \Delta = \sum_{q \in K_Q(\Delta)} \sum_{T_2 \in T_{2}^{a}} \left\| \chi_q \psi_{T_1} \psi^{-50}_{T_2} \right\|_{L^2(\mathbb{R}^{n+1})}^2,
\]

and set

\[
m_{T_1}^{G^\sigma} = \sum_{\Delta \in Q_{C_0}(Q)} m_{T_1}^{G^\sigma} \cdot \Delta.
\]

The \((\varpi, R)\)-red wave table \( F^\sigma \) is defined by \( \mathcal{F}_{C_0}^{\varpi}(F^\sigma, G^\sigma; Q) \) at depth \( C_0 \) for \( F^\sigma \) with respect to \( G^\sigma \) over \( Q \) is defined as

\[
F^\sigma = \left( F^\sigma, \varpi(\Delta) \right)_{\Delta \in Q_{C_0}(Q)},
\]

with

\[
F_{\varpi, \Delta}(x, t) := \sum_{T_1 \in T_{1}^{a}} m_{T_1}^{G^\sigma} \cdot \Delta \frac{m_{T_1}^{G^\sigma}}{m_{T_1}^{F^\sigma}} F_{T_1}^\sigma(x, t),
\]

where \( F^\sigma \) depends on \( \varpi \) through (3.10).
By the linearity of $S^{(\Phi,\sigma)}(t)$, each $F^{(\Phi,\sigma)}(\Delta)$ is a red wave and

$$F^{\sigma} = \sum_{\Delta \in Q_{0}(Q)} F^{(\Phi,\sigma)}(\Delta).$$

Define the energy of $F^{\sigma}$ as

$$E(F^{\sigma}) := \sum_{\Delta \in Q_{0}(Q)} E(F^{(\Phi,\sigma)}(\Delta)).$$

From the Bessel type inequality \[3.14\], we have

**Lemma 4.2.** There is a constant $C_{n}$ depending only on $n$ such that we have

$$E(F^{\sigma})^{1/2} \leq (1 + C_{n} \varpi)E(F^{\sigma})^{1/2},$$

for any $(\varpi, R)$-wave tables $F^{\sigma}$ with depth $C_{0}$ over a spacetime cube $Q$.

By symmetry, we may also define the blue wave table $G^{\sigma} = \mathcal{E}^{\Phi,\sigma}(G^{\sigma}, F^{\sigma}; Q)$ of $G^{\sigma}$ with respect to any red wave $F^{\sigma}$ on the cube $Q$ as above.

The above constructions naturally extends to the case when one of the waves is of high frequency. For waves both with high frequencies, we may normalize the red wave to be frequency one by using $\mathcal{D}_{2m}$ and apply the same construction as above, then undo the scaling.

**4.2. No waste bilinear $L^{2}$-Kakeya type estimate.** Let $\varpi \in [0, 2^{-C_{0}}]$ and $j \geq 0$ be an integer. For any spacetime cube $Q$, define the $(\varpi, j)$—interior of $Q$ as

$$\mathcal{I}^{\varpi,j}(Q) := \bigcup_{q \in \mathcal{G}(Q)} (1 - \varpi)q.$$

**Proposition 4.3.** Let $F^{\sigma} \in \mathcal{A}^{\Phi,\sigma,2m}_{R}$ and $G^{\sigma} \in \mathcal{B}^{\Phi,\sigma,2k}_{R}$. For any $\varpi \in (0, 2^{-C_{0}}]$, let $Q = Q_{R}$ with $R \geq C_{0} \max\{2^{-m}, 2^{m-2k}\}$, there exists a constant $C$, depending only on $\Sigma_{1}, \Sigma_{2}, \sigma_{0}$ and independent of $C_{0}$, such that if $F^{\sigma}$ and $G^{\sigma}$ are red and blue waves satisfying the margin condition

$$\text{marg}(F^{\sigma}) \geq (2^{m}R)^{-1/2}, \text{ marg}(G^{\sigma}) \geq (2^{k}R)^{-1/2}$$

with $E(F^{\sigma}) = E(G^{\sigma}) = 1$. Let $F^{\sigma}, G^{\sigma}$ be the $(\varpi, R)$—wave tables for $F^{\sigma}$ and $G^{\sigma}$ with depth $C_{0}$ over $Q^{*} := CQ$ respectively given by Definition 4.4. Then, we have

$$\max_{\Delta, \Delta' \in Q_{0}(Q^{*})} \|F^{(\Phi,\sigma)}(\Delta) G^{(\Phi,\sigma)}(\Delta')\|_{L^{2}_{x,t}(\mathcal{I}^{\varpi,j}(Q^{*} \setminus \Delta'))} \leq C \varpi^{-\frac{m}{2}} \left(\frac{2^{m}}{R}\right)^{\frac{\sigma_{0}}{m}}. \quad (4.3)$$

**Proof.** The argument is exactly same with (54) of [15]. First, by scaling invariance, one may take $m = 0$ so that (4.3) is essentially (73) of [15]. Then, by using the explicit construction formula in Definition 4.1 with $R$ replaced by $2^{m}R$ there, and \[3.13\] of Lemma 3.4 to conclude the proof. There are two places where the transversality condition is used for the family of conic surfaces $C_{1}^{\Phi,\sigma}$ and $C_{2}^{\Phi,\sigma}$. One is the bilinear $L^{2}$—estimate in low dispersions and the other one is the energy estimate on conic regions of opposite colour of Lemma 4.2. For the first one, we refer to Lemma 14.2 and Lemma 14.3 in [15] for more details. We apply our analogue counterpart for $\Phi_{\sigma}$ surfaces with the high frequency wave at frequency $2^{k-j}$ of angular dispersion $O((2^{j}R)^{-1/2})$. It is easy to see that these two properties can be made uniform for all $\Phi_{\sigma} \in \mathcal{L}_{\sigma}$ and all $\sigma \in (0, \sigma_{0}]$ by taking $\sigma_{0}$ small enough. \(\square\)
4.3. Persistence on the non-concentration of energy.

**Definition 4.4.** For any \( r > 0 \), a spacetime cube \( Q \), a red wave \( F^\sigma \) and a blue wave \( G^\sigma \), we define the \( r \)-energy-concentration of \( F^\sigma, G^\sigma \) over \( Q \) to be the quantity

\[
E_{r,Q}(F^\sigma, G^\sigma) = \max \left\{ \frac{1}{2} E(F^\sigma)^{1/2} E(G^\sigma)^{1/2}, \sup_D \| F^\sigma \|_{L^2(D)} \| G^\sigma \|_{L^2(D)} \right\},
\]

where \( D \) ranges over all disks with \( r_D = r \) and \( t_D \in LS(Q) \).

This quantity is very robust in taking advantage of the Huygens’ principle of wave equations, although this property does not hold anymore for Schrödinger equations. However, one can recover this property by using the method of descent introduced in [15]. From this perspective, there seems a principle of adding dimensions to treat equations, although this property fails and then use method of descent to get the expected result as a limiting case.

**Proposition 4.5.** Let \( R \geq C_0 \max\{2^{-m}, 2^{m-2k}\} \) with \( m, k \geq 0 \) and \( Q = Q_R \) be a spacetime cube of side-length \( R \). For each \( r \geq R^{\frac{1}{2}+\frac{1}{\Phi}} 2^{-m(\frac{1}{2} - \frac{1}{k})} \), we define

\[
r^\# := r \left( 1 - C_0 (2^m r)^{-\frac{1}{\Phi}} \right).
\]

There exists a constant \( C > 0 \), such that if \( F^\sigma \in \mathcal{R}_R^{\Phi}, G^\sigma \in \mathcal{B}_R^{\Phi}, \) with \( E(F^\sigma) = E(G^\sigma) = 1 \) and \( F^\sigma \) is a \((\varpi, R^{1/2})\)-wave tables for \( F^\sigma \) with respect to \( G^\sigma \) over \( Q \) with depth \( C \), then

\[
\sup_{\Delta \in Q_{c_0}(Q)} E_{r^\#, Q}(F^\sigma, G^\sigma, \varpi) \leq (1 + C \varpi) E_{r^\#, Q}(F^\sigma, G^\sigma) + \mathcal{O}(\varpi^{-C} R^{-N/2})
\]

holds for all \( \varpi \in (0, 2^{-C_0}] \) and all \( Q \).

**Proof.** The argument is the same to [15] and we only sketch it. By scaling, it suffices to consider the \( m = 0 \) case. Let

\[
D = D(z_0, r^\#), \quad D' = D(z_0, r \left( 1 - C_0 \frac{1}{2} r^{-\frac{1}{\Phi}} \right)), \quad D'' = D(z_0, r)
\]

with \( z_0 = (x_0, t_0), \) \( t_0 \in LS(5Q), \) \( D \subsetneq D'_- \subsetneq D' \subsetneq D'_+ \subsetneq D'' \).

Write \( F^\sigma = P_{D^\sigma}^\frac{\Phi}{\sigma} F^\sigma + (I - P_{D^\sigma}^\frac{\Phi}{\sigma}) F^\sigma : F_1 + F_2 \) and by the linearity of \( F^\sigma \mapsto F^\sigma \), write the corresponding decomposition for each \( F^\sigma \) short for \( F^\sigma, \varpi, (\Delta) \) as \( F^\sigma_1 (\Delta) + F^\sigma_2 (\Delta) \). The \( F^\sigma_2 (\Delta) \) part follows from Lemma 12 and 3.26. The contribution of the \( F^\sigma_2 (\Delta) \) part is bounded by the error term. Indeed, the estimate follows from using the condition \( r \geq R^{\frac{1}{2}+\frac{1}{\Phi}} \) by distinguishing the following two cases for the wave packets \( F^\sigma_T \) for \( T \in T_\sigma \) in the definition of \( F^\sigma (\Delta) \):

- Case A. \( \text{dist}(T,D) \geq R^{\frac{1}{2}+\frac{1}{\Phi}} \cdot x \)
- Case B. \( \text{dist}(T,D) \leq R^{\frac{1}{2}+\frac{1}{\Phi}} \cdot x \)

For Case A, we use (3.12), (3.13) to see the contributions of \( q \) that do intersect with \( D \) is bounded by the error term. For Case B, we use (3.24) to get the result. We refer to [22] P.30 for more details.

\[ \square \]
4.4. **Wrap-up.** To state and demonstrate the main result of this section, let us introduce more notations.

Given a wave $F^\sigma$ and a cube $Q = Q_R$ of side-length $R \geq C_0 2^k$, assume that $k \in C_0 \mathbb{Z}$ large and that we have defined $\mathfrak{g}_j$ for each $j \in C_0 \mathbb{Z} \cap [0, k]$, inductively by $\mathfrak{g}_{j+C_0} = \mathfrak{g}_{C_0} \circ \mathfrak{g}_j$ so that $\mathcal{F}_{j}^\sigma = \mathfrak{g}_j(F^\sigma)$ is a wave table of $F^\sigma$ over $Q$ with depth $j$. For each $j$, we have obtained inductively a chain of wave tables

$$\mathcal{F}_{0}^\sigma \prec \mathcal{F}_{C_0}^\sigma \prec \cdots \prec \mathcal{F}_{j}^\sigma \prec \cdots \prec \mathcal{F}_{j}^\sigma.$$

For any $j' < j$, we say $\mathcal{F}_{j}^\sigma$ is a relative wave table of $\mathcal{F}_{j'}^\sigma$ with depth $j - j'$. For each entry $\mathcal{F}_{j}^\sigma(q)$ of $\mathcal{F}_{j'}^\sigma$ with $q \in Q_j(Q)$, there exists a unique $q' \in Q_{j'}(Q)$ such that $q \in Q_{j-j'}(q')$ and $\mathcal{F}_{j}^\sigma(q) = \mathcal{F}_{j}^\sigma(q')$, obtained as an entry of $\mathfrak{g}_{j-j'}(F_{\sigma}(q'))$.

For each $j$ and $0 \leq j' < j$, define the $j'-quilt$ of $F^\sigma := F_{j}^\sigma$ on $Q$ as

$$[\mathcal{F}_{j}^\sigma]_{j'} = \sum_{q' \in Q_{j'}(Q)} \mathbb{I}_{q'} |\mathcal{F}_{j}^\sigma(q')|,$$

where $\mathbb{I}_{q'}$ is the indicator function of $q'$ and $\mathcal{F}_{j}^\sigma(q')$ is the entry of $\mathcal{F}_{j'}^\sigma$. Then

$$\max_{q \in Q_j(Q)} |\mathcal{F}_{j}^\sigma(q)| \leq [\mathcal{F}_{j}^\sigma]_{j} \leq \cdots \leq [\mathcal{F}_{j}^\sigma]_{0} = [\mathcal{F}_{j}^\sigma]_0 = |F^\sigma| \mathbb{I}_Q. \quad (4.4)$$

Let

$$X_{\varpi, k}(Q) = \bigcap_{j = C_0}^{k} \varpi 2^{-(k-j)/N} \mathbb{J}_j(Q).$$

Recall that for any $\Phi_\sigma \in \Sigma_\sigma$ and $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$,

$$\Lambda^\Phi_{\text{purple}}(z_0, r) = \Lambda^\Phi_{1}(z_0, r) \cup \Lambda^\Phi_{2}(z_0, r),$$

with $\Lambda^\Phi_{1}(z_0, r)$ given in Lemma 3.2 for $i = 1, 2$. Define

$$\mathcal{X}^\varpi_{\Phi_\sigma, z_0}(Q) = X_{\varpi, k}(Q) \cap \Lambda^\Phi_{\text{purple}}(z_0, r).$$

For any $u \in C^\infty(\mathbb{R}_t^2 \times \mathbb{R}_x)$ and any measurable subset $\Omega \subset \mathbb{R}^{n+1}$, such that $\Omega = \bigcup_{l \in \mathbb{Z}} (\Pi_l \times \{t\})$ for some $\mathbb{I} \subset \mathbb{R}$, we let

$$\|u\|_{Z(\Omega)} := \left( \int_{\mathbb{I}} \left( \int_{\Pi_l} |u(x, t)|^q dx \right)^{\frac{q}{s}} dt \right)^{\frac{1}{q}} ,$$

with $(q, s) = (q^+_\gamma, r^-_\gamma) \in \Gamma$, where for any $\gamma \in \mathbb{R}$, we denote $\gamma^+$ (resp. $\gamma^-$) as a real number greater (resp. smaller) than but sufficiently close to $\gamma$. Thus, the sense of $Z(X_{\varpi, k}(Q))$ and $Z(\mathcal{X}^\varpi_{\Phi_\sigma, z_0}(Q))$ is clearly. The assumption on $k \in C_0 \mathbb{Z}$ is only for a technical reason and can be removed by scaling.

Based on the above preparations, we arrive at

**Proposition 4.6.** There exists a constant $C > 0$ depending only on $n$ such that the following statement holds.

Let $R \geq C_0 2^k$ and $\varpi \in (0, 2^{-C_0}]$. Let $F^\sigma$ and $G^\sigma$ be red and blue waves associated to $\Phi_\sigma \in \Sigma_\sigma$ for some $\sigma \in (0, \sigma_0]$ with frequency 1 and $2^k$ respectively, which obey the energy normalization

$$E(F^\sigma) = E(G^\sigma) = 1.$$
and the relaxed margin condition
\[
\min \left\{ \text{margin}(F^\sigma), \text{margin}(G^\sigma) \right\} \geq 100^{-1} - 2(2^k R^{-1})^{\frac{1}{2}}. \tag{4.5}
\]

For any spacetime cube $Q$ of sidelength $CR$, there exists a red wave table $F^\sigma$ on $Q$ with depth $k$ and frequency one, and a blue wave table $G^\sigma$ on $Q$ with depth $C_0$ and frequency $2^k$, both depending on $\varpi$ such that

- The margin condition holds
\[
\min \left\{ \text{margin}(F^\sigma), \text{margin}(G^\sigma) \right\} \geq 100^{-1} - (2^{k+C_0}/R)^{-\frac{1}{2}}. \tag{4.6}
\]
- The energy estimate holds
\[
\max \left\{ \text{E}(F^\sigma), \text{E}(G^\sigma) \right\} \leq 1 + C \varpi, \tag{4.7}
\]
- Effective approximation via $(k, C_0)$-quilts product
\[
\|F^\sigma G^\sigma\|_{Z(X = k(Q))} \leq \left\| \left[ F^\sigma \right]_k [G^\sigma]_{C_0} \right\|_{Z(X = k(Q))} + \varpi^{-C} 2^{k\gamma(q)}, \tag{4.8}
\]
- Refined approximation on conic regions
\[
\|F^\sigma G^\sigma\|_{Z(X = k, r(Q))} \leq \left\| \left[ F^\sigma \right]_k [G^\sigma]_{C_0} \right\|_{Z(X = k, r(Q))} + \varpi^{-C} 2^{k\gamma(q)} \left( 1 + \frac{R}{2^{kr}} \right)^{-\frac{1}{2}}, \tag{4.9}
\]

for all $z_0 \in \mathbb{R}^{n+1}$ and $r > 0$.

- Persistence of the non-concentration of energy
\[
\mathbf{E}_{r, \#, 5Q}(F^\sigma, G^\sigma) \leq \mathbf{E}_{r, 5Q}(F^\sigma, G^\sigma) + C \varpi + \varpi^{-C} R^{-\frac{1}{2}}, \tag{4.10}
\]
for all $r \geq C R^\frac{1}{2} \varpi$, with $r^\# := r(1 - C_0 r^{-\frac{1}{2}})$, where $\gamma(q) = \max \left( \frac{1}{q}, \frac{1}{2}, 0 \right)$ and the energy concentration for $F^\sigma, G^\sigma$ is defined as
\[
\mathbf{E}_{r, Q}(F^\sigma, G^\sigma) := \max_{\Delta, \Delta' \in \mathcal{Q}_{C_0}(Q)} \mathbf{E}_{r, Q}(F(\Delta), G(\Delta')).
\]

**Proof.** The argument is the same to [15] and we only sketch it below. We first construct the red wave table with respect to the blue wave by letting $\mathfrak{F}_0(F^\sigma)$ be the restriction of $F^\sigma$ on $Q$, and inductively $\mathfrak{F}_j + C_0 := \mathfrak{F}_{j+2^{(k-j)/2}} \circ \mathfrak{F}_j$ for every integer $j \in [0, k]$ being a multiple of $C_0$. Reverse the colour of the waves, we construct the blue wave table for $G^\sigma$ with respect to the red wave table. This is done by letting $G^\sigma$ be the table of entries given by $\mathcal{G}_{C_0}^\sigma(G^\sigma, F^\sigma(q), Q)$ for each $q \in \mathcal{Q}_k(Q)$. Telescoping the no waste bilinear $L^2$ estimate and using the same interpolation argument with the energy estimate and also Lemma 3.2 as in [23] to get the approximation as well as its refinement via $(k, C_0)$-quilts in the $Z$-norms. Finally, the persistence of the non-concentration of energy is readily deduced from Proposition 4.5. \qed

**Remark 4.7.** For Proposition 2.8, where the surface is the lightcone without using the Lee-Vargas rescaling in [13], the above construction of wave tables is readily obtained by modifying Proposition 4.1 of [15] in the step of using interpolations.
5. Proof of Theorem \ref{thm:main}

For any \( \sigma \in (0, \sigma_0] \) and \( R \geq C_0 2^k \), we denote
\[
\mathcal{R}_R^k \times \mathcal{B}_R^{\sigma,k} := \bigcup_{\Phi \in \mathcal{B}_\sigma} \left( \mathcal{R}_{\Phi}^{\sigma+1} \times \mathcal{B}_R^{\sigma,2^k} \right).
\]

**Definition 5.1.** Fix \( \sigma \in (0, \sigma_0] \). For any \( R \geq C_0 2^k \), fix \( Q_R \subset \mathbb{R}^{n+1} \) a spacetime cube of side-length \( R \). Let \( A^\sigma (R) \) be the optimal constant \( C \) such that
\[
\| F^\sigma G^\sigma \|_{Z(Q_R)} \leq C \mathcal{E}(F^\sigma)^{1/2} \mathcal{E}(G^\sigma)^{1/2},
\] (5.1)
holds for all \( (F^\sigma, G^\sigma) \in \mathcal{R}_R^\sigma \times \mathcal{B}_R^{\sigma,2^k} \) and all \( Q_R \).

By the invariance of translation in the physical spacetime and modulating the frequency variables, \( A^\sigma (R) \) is independent of the center of \( Q_R \).

Since we have
\[
\sup_{0 < \sigma \leq \sigma_0} A^\sigma (C_0 2^k) \lesssim 2^k \left( \frac{1}{4} \right),
\]
by using the same argument in \cite{13}, We shall only consider \( R \geq C_0 2^k \).

To show Theorem \ref{thm:main}, we shall prove that there is a fixed constant \( C_* \) depending only on \( n, \varepsilon, \sigma_0, \Sigma_1, \Sigma_2 \) and the \((q, s)\) exponent in \( Z\)-norm taken sufficiently close to the critical index \((q_c, r_c)\), such that
\[
\sup_{0 < \sigma \leq \sigma_0} A^\sigma (R) \leq C_* 2^{k\gamma(q, \varepsilon)},
\]
holds with \( \gamma(q, \varepsilon) = \frac{1}{q} - \frac{1}{2} + \varepsilon \) for all \( R \geq C_0 2^k \). We set
\[
\overline{A}^\sigma (R) = \sup_{\sigma \leq \sigma' \leq \sigma_0} \sup_{C_0 2^k \leq R' \leq R} A^\sigma (R'),
\]
for dealing with the margin condition. We may assume \( \overline{A}^\sigma (R) \geq 2^{k\gamma(k, \varepsilon)} \).

We consider the small scale case \( R \leq 2 C_0 2^k \) first. In this case, by using the non-endpoint argument leading to (41) in \cite{13} P. 1307, we have
\[
\overline{A}^\sigma (R) \lesssim 2^{k\gamma(q, \varepsilon)}, \quad \forall \varepsilon > 0.
\]

Thus, it suffices to consider the large scale case \( R \geq 2 C_0 2^k \). To this end, we need an auxiliary quantity linked to the energy concentration as in \cite{15}. Here, slightly different from \cite{15} where the proof deals with a fixed lightcone, we are working with a family of conic surfaces. However, we find that it suffices to study the auxiliary quantity for every fixed conic surface.

**Definition 5.2.** For any \( \sigma \in (0, \sigma_0] \), let \( \Phi_\sigma \in \mathcal{B}_\sigma \), \( R \geq 2^k C_0 \) and \( r, r' > 0 \). We define \( A^{\Phi_\sigma} (R, r, r') \) to be the optimal constant \( C \) such that
\[
\| F^\sigma G^\sigma \|_{Z(Q_R \cap A^{\Phi_\sigma}(z_0, r'))} \leq C \left( \mathcal{E}(F^\sigma)^{1/2} \mathcal{E}(G^\sigma)^{1/2} \right)^{1/2} \mathcal{E}_{r, C_0 Q_R} (F^\sigma, G^\sigma)^{1/2},
\]
holds for all \( (F^\sigma, G^\sigma) \in \mathcal{R}_R^{\Phi_\sigma+1} \times \mathcal{B}_R^{\Phi_\sigma,2^k} \) and all \( Q_R \) and all \( z_0 = (x_0, t_0) \in \mathbb{R}^{n+1} \).
5.1. Control of $\mathcal{A}_{\Phi^*}$ by $\overline{\mathcal{A}'}$ in the large scale case $R \geq 2^{C_0k}$.

**Proposition 5.3.** There is a constant $C > 0$ depending only on $n$ and $q, s$ in the $Z$-norm, but not explicitly on $C_0$, such that for any $\sigma \in (0, \sigma_0]$ and any $\Phi_{\sigma} \in \mathcal{E}_{\sigma}$, we have

$$\mathcal{A}_{\Phi^*}(R, r, C_0(r+1)) \leq (1 + C2^{-C_0})\overline{\mathcal{A}'}(R) + 2^{C_0}2^{k\gamma(q)}, \quad (5.2)$$

for all $R \geq 2^{C_0k}$ and all $r \geq R^{q+k}$. The proof is achieved in three steps.

**5.1.1. Step 1.** The non-concentrated case $r \geq C_0R$. Recall an orthogonality lemma:

**Lemma 5.4.** Let $\{F_j\}_j \subset L^q_tL^r_x(\mathbb{R}^{n+1})$ such that $\{F_j\}_j$ be red and blue waves with normalized energy. For any $Q \in \mathcal{E}_{\sigma}$ and any $\Phi_{\sigma} \in \mathcal{E}_{\sigma}$ with $\sigma \in (0, \sigma_0]$. Let $D = D(z_Q, r/2)$ and write

$$F^\sigma = P^\Phi_{D}F^\sigma + (1 - P^\Phi_{D})F^\sigma, \quad G^\sigma = P^\Phi_{D}G^\sigma + (1 - P^\Phi_{D})G^\sigma.$$ 

Using Lemma [5.4], we have

$$\max \left\{ \|((1 - P^\Phi_{D})F^\sigma)\|_{Z(Q_R)}, \|((P^\Phi_{D}F^\sigma)(1 - P^\Phi_{D})G^\sigma)\|_{Z(Q_R)} \right\} \leq C2^{k\gamma(q)}. $$

We are reduced to

$$\|(P^\Phi_{D}F^\sigma)(P^\Phi_{D}G^\sigma)\|_{Z(Q)} \leq (1 + C\|\overline{\mathcal{A}'}(R)\|_{E_{r,C_0}(Q)}(F^\sigma, G^\sigma)^{1/q} + C2^{k\gamma(q)}. \quad (5.3)$$

To see this is the case, let $F^\sigma_D$ and $G^\sigma_D$ be the wave tables for the red and blue waves $P^\Phi_{D}F^\sigma$ and $P^\Phi_{D}G^\sigma$ on an appropriately enlarged cube $Q^*$ containing $Q$. Applying Proposition [5.3], we have

$$\|(P^\Phi_{D}F^\sigma)(P^\Phi_{D}G^\sigma)\|_{Z(Q_R)} \leq (1 + C\|\overline{\mathcal{A}'}(R)\|_{E_{r,C_0}(Q)}(F^\sigma, G^\sigma)^{1/q} + C2^{k\gamma(q)}. $$

Applying Lemma [5.4] and the definition of $\overline{\mathcal{A}'}(R)$, we get

$$\left\| \sum_{\Delta \in \mathcal{Q}_{c_0}(Q^*)} \mathcal{J}^{\Delta}(F^\sigma_D, G^\sigma_D) \right\|_{Z(\Delta)} \leq \overline{\mathcal{A}'}(2^{-C_0}R) \left( \sum_{\Delta \in \mathcal{Q}_{c_0}(Q^*)} \mathcal{J}^{\Delta}(F^\sigma_D, G^\sigma_D)^{q/2} \right)^{1/q}, \quad (5.4)$$

The enlargement is for the use of the averaging Lemma 4.2 in [13], extension of Lemma 6.1 in [15] to the mixed-norms.
where we have used $\mathcal{F}_D^{\sigma, \Delta} \in \mathfrak{B}_R^{\sigma, \Delta} \subset \mathfrak{B}_R^{\sigma, k}$, $\mathcal{G}_D^{\sigma, \Delta} \in \mathfrak{B}_R^{\sigma, k}$. Using Cauchy-Schwarz, $E(\mathcal{F}_D^\sigma), E(\mathcal{G}_D^\sigma) \leq 1 + C\varpi$ by Lemma 4.2 and (1.23) of Lemma 3.8 we get (5.3). □

5.1.2. Step 2. The energy concentrated case $R^{1/2+\varpi} \leq r \leq C_0 R$.

**Proposition 5.6.** There is $C > 0$ and $\theta > 0$ such that for any $R \geq 2C_0 k$, we have for all $r, r' > 0$ with $R^{1/2+\varpi} \leq r \leq C_0 R$.

$$\mathcal{A}^{\Phi_\sigma}(R, r, r') \leq (1 + C\varpi)\mathcal{A}^{\Phi_\sigma}(R/C_0, r^{\#}, r') + \varpi^{-C_3} \left(1 + \frac{R}{2k r'}\right)^{-\theta} 2^{k\gamma(q)}$$

with $r^{\#} = r(1 - C_0 r^{-\varpi} \varpi^{-C_3})$ holds for all $0 < \varpi \leq 2^{-C_0}$ and $\Phi_\sigma \in \mathcal{E}_\sigma$ with $\sigma \in (0, \sigma_0]$. Proof. Let $F^\sigma \in \mathfrak{B}_R^{\Phi_\sigma, 1}, G^\sigma \in \mathfrak{B}_R^{\Phi_\sigma, 2k}$ be red and blue $\Phi_\sigma$–waves with normalized energy $E(F^\sigma) = E(G^\sigma) = 1$. For any $Q = Q_R$ by using Proposition 4.3 we have for all $z_0$

$$\left\| F^\sigma G^{\sigma} \right\|_{Z\left(\chi_{\Phi_\sigma}^{r, k, r}(Q)\right)} \leq (1 + C\varpi)\left\| F^\sigma \right\|_{Z\left(\chi_{\Phi_\sigma}^{r, k, r}(Q)\right)} + \varpi^{-C} \left(1 + \frac{R}{2k r'}\right)^{-\theta} 2^{k\gamma(q)}.$$  

We are reduced to showing

$$\left\| F^\sigma \right\|_{Z\left(\chi_{\Phi_\sigma}^{r, k, r}(Q)\right)} \leq (1 + C\varpi)\mathcal{A}^{\Phi_\sigma}(R/C_0, r^{\#}, r') E_{r, C_0} Q(F^\sigma, G^\sigma)^{1/q} + \varpi^{-C} R^{-N/2} 2^{k\gamma(q)}. \quad (5.5)$$

Using the definition of $\mathcal{A}^{\Phi_\sigma}(R, r, r')$, we have for all $\Delta$

$$\left\| F^\sigma G^{\sigma, \Delta} \right\|_{Z\left(\chi_{\Phi_\sigma}^{r, k, r}(\Delta)\right)} \leq \mathcal{A}^{\Phi_\sigma}(R/C_0, r^{\#}, r') E_{r, C_0} \left(\mathcal{A}^{\Phi_\sigma}(R, r, r')} E_{r, C_0} Q(F^\sigma, G^\sigma)^{1/q} + \varpi^{-C} R^{-N/2} 2^{k\gamma(q)}\right)^{1/(2q)}. \quad (5.5)$$

By Lemma 5.3 Proposition 15.5 with $2^{-C_0}CR \ll C_0^{-1} R$ so that $C_0 \Delta \subset 5Q$, we obtain by Cauchy-Schwarz’s inequality

$$\left\| F^\sigma \right\|_{Z\left(\chi_{\Phi_\sigma}^{r, k, r}(Q)\right)} \leq \sum_{\Delta \in Q_0} \left\| F^\sigma G^{\sigma, \Delta} \right\|_{Z\left(\chi_{\Phi_\sigma}^{r, k, r}(Q)\right)} \leq (1 + C\varpi)\mathcal{A}^{\Phi_\sigma}(R/C_0, r^{\#}, r') E_{r, C_0} Q(F^\sigma, G^\sigma)^{q/q'} + \varpi^{-C} R^{-qN/2} 2^{k\gamma(q)}, \quad (5.6)$$

and (5.5) follows by adjusting the constant $C$.

□

5.1.3. Step 3. Proof of Proposition 5.5. With Proposition 5.3 and 5.6 we may complete the proof of Proposition 5.5 by using the standard iteration argument. Please see [15] P.225 for details.

☐

5.2. Essential concentration along conic regions. Fix a pair of red and blue waves $(F^\sigma, G^\sigma) \in \mathfrak{B}_R^\sigma \times \mathfrak{B}_R^{\Phi_\sigma, 2k}$ with $E(F^\sigma) = E(G^\sigma) = 1$. The complete proof relies crucially on the Kakeya compression property below.

Before stating this result, let us start with a non-endpoint bilinear estimate. For $0 < r_1 < r_2 < +\infty$, we define the cubical annulus as

$$Q^{\text{ann}}(x_Q, t_Q; r_1, r_2) = Q(x_Q, t_Q; r_2) \setminus Q(x_Q, t_Q; r_1).$$
The following non-endpoint bilinear estimate holds for localized blue or red waves, on a dyadic annulus corresponds to Lemma 11.1 of Tao [15], and we omit the proof since it is rather standard using the wave-packets and a multiplicity estimate on the overlappedness of tubes. See also [23] for more details.

**Lemma 5.7.** Let $R \geq 2^{C_0k}$, $C_0k \leq r \leq R^{1/2 + \delta}$, and $D = D(z_D, C_0^{1/2}r)$ with $z_D = (x_D, t_D)$. Then, there exists $b > 0$, depending only on $\sigma_0$, $Z$ and $n$, such that for any red and blue $\Phi_{\sigma}^-$ waves $F^\sigma, G^\sigma$ of frequency 1 and $2^k$ respectively, with $E(F^\sigma) = E(G^\sigma) = 1$, we have
\[
\| (F^\sigma D^\Phi F^\sigma) G^\sigma \|_{L^\infty(Q^m(z_D, 2R))}, \| F^\sigma (D^\Phi F^\sigma) G^\sigma \|_{L^\infty(Q^m(z_D, 2R))} \lesssim R^{-b},
\]
where the implicit constants are uniform with respect to $\sigma \in (0, \sigma_0)$.

The main result of this subsection reads

**Proposition 5.8.** Let $\Phi_{\sigma}^\pm \in \mathcal{E}_\sigma$ and $(F^\sigma, G^\sigma) \in \mathcal{R}^{R^{-1}} \times \mathcal{R}^{R^{-1}}$ be the pair of red and blue waves fixed at the beginning of this section with $E(F^\sigma) = E(G^\sigma) = 1$. There exists a constant $C > 0$ depending only on $n, \varepsilon, \sigma_0$ and $q, s$, in the $Z$–norm such that for any $R \geq 2^{C_0k}$ and $\delta \in (0, 1/2)$, if $Q_R$ satisfies
\[
\| F^\sigma G^\sigma \|_{Z(Q_R)} \geq \frac{1}{2} T^\sigma(R),
\]
and we let $r_\delta$ be the supremum of all radii $r \geq C_02^k$ such that
\[
E_{r, C_0Q_R}(F^\sigma, G^\sigma) \leq 1 - \delta
\]
holds and let $r_\delta = C_02^k$ if no such radius exists, then there exists a cube $Q_{r_\delta}$ of size $\overline{r_\delta} \in \mathcal{R}^{R}$ and $z_\delta \in \mathcal{R}^{n+1}$ such that $\overline{r_\delta} \leq r_\delta$ when $r_\delta \geq 2C_0^k$, for which we have
\[
\| F^\sigma G^\sigma \|_{Z(Q_R)} \leq (1 - C(\delta + C_0^{-c})^q)^{-2/q} \| F^\sigma G^\sigma \|_{Z(Q_0^{\delta_\sigma})} + 2C_02^{k\gamma(p, c)},
\]
where $\Omega_\delta^{\gamma_\sigma} := \overline{Q_{r_\delta}} \cap \Lambda_\delta^{\gamma_\sigma}$ with $\Lambda_\delta^{\gamma_\sigma} = \Lambda_\delta^{\gamma_\sigma}(z_\delta, C_0(r_\delta + 1))$.

5.2.1. The medium or low concentration case: $r_\delta \geq R^{1/2 + 1/N}$ In this case, we show there is a constant $C$ such that for some $z_\delta \in \mathcal{R}^{n+1}$, we have
\[
\| F^\sigma G^\sigma \|_{Z(Q_R)} \leq (1 - C(\delta + C_0^{-c})^q)^{-1/q} \| F^\sigma G^\sigma \|_{Z(Q_{r_\delta} \cap \Lambda_\delta^{\gamma_\sigma})}
\]
holds with $C$ independent of $Q_R$ and $z_\delta$. In this case, $\overline{R_\delta} = R$ and $\overline{Q_{r_\delta}} = Q_R$.

By definition, there is $D_\delta = D(z_\delta, r_\delta)$ with $z_\delta = (x_0, t_0)$ and $t_0 \in LS(C_0Q_R)$, such that we have
\[
\min \left( \| F^\sigma \|_{L^2(D_\delta)}, \| G^\sigma \|_{L^2(D_\delta)} \right) \geq 1 - 2\delta.
\]
Let $D_\delta = C_0^{1/2}D_\delta = D(z_\delta, C_0^{1/2}r_\delta)$ and write
\[
F^\sigma = P_{D_\delta}^\Phi F^\sigma + (1 - P_{D_\delta}^\Phi) F^\sigma, \quad G^\sigma = P_{D_\delta}^\Phi G^\sigma + (1 - P_{D_\delta}^\Phi) G^\sigma.
\]
By Lemma 5.3 and the condition $2^{k\gamma(p, c)} \leq \overline{A}(R) \leq 2 \| F^\sigma G^\sigma \|_{Z(Q_R)}$, it suffices to show for some universal constant $C > 0$, we have
\[
\| (P_{D_\delta}^\Phi F^\sigma) G^\sigma \|_{Z(Q_R \cap \Lambda_\delta^{\gamma_\sigma})} \lesssim C_0^{-C},
\]
\[
\| (1 - P_{D_\delta}^\Phi) F^\sigma P_{D_\delta}^\Phi G^\sigma \|_{Z(Q_R \cap \Lambda_\delta^{\gamma_\sigma})} \lesssim C_0^{-C},
\]
and
\[ \|(1 - P_{D_3}^{\phi_3}) f^\sigma (1 - P_{D_3}^{\phi_3}) g^\sigma \|_{L^\infty(Q_R)} \lesssim (\delta + C_0^C) \overline{A}(R). \] (5.15)

To see (5.13) and (5.14), by using energy estimates, we are reduced to
\[ \|P_{D_3}^{\phi_3} f^\sigma\|_{L^\infty(Q_R \setminus A_3^{\phi_3})} \lesssim N \, R^{-N/2}, \quad \|P_{D_3}^{\phi_3} g^\sigma\|_{L^\infty(Q_R \setminus A_3^{\phi_3})} \lesssim N \, R^{-N/2}, \] (5.16)
which is obvious in view of Lemma 3.9 and \(r_3 \geq R^{1/2+4/N} \).

To show (5.15), by the induction argument and (5.9), (3.26), (3.27) as well as the assumption on \(r_3\), we have
\[ E((1 - P_{D_3}^{\phi_3}) f^\sigma) \lesssim \delta + R^{-N/2}, \quad E((1 - P_{D_3}^{\phi_3}) g^\sigma) \lesssim \delta + R^{-N/2}. \]

It is clear that
\[ (1 - P_{D_3}^{\phi_3}) f^\sigma \in \mathcal{Q}_R^{\phi_3 - 1}, \quad (1 - P_{D_3}^{\phi_3}) g^\sigma \in \mathcal{Q}_R^{\phi_3 - 2}, \]
with \(R' = \frac{R}{(1 + o(1))}\). This yields (5.15) by finitely partitioning \(Q_R\) and using the definition of \(A^\sigma(R')\) and the monotonicity of \(\overline{A}(R)\). The proof is complete.

5.2.2. The high concentration case: \(r_3 \leq R^{1/2+4/N}\). We turn to the case where the blue and red waves are highly concentrated. Define
\[ \overline{R}_3 = \max \left( 2^{C_0}, r_3^{1/(1/2+4/N)} \right). \]

Consider the case \(\overline{R}_3 > 2^{C_0}\). In this case, we necessarily have \(r_3 > 2^{C_0}/2\) and there is \(z_3\) such that we have (5.12). Let \(Q = Q_{\overline{R}_3}^{\overline{R}_3}\) be the cube of size \(\overline{R}_3\) centered at \(z_3\). By splitting \(Q_R = (Q_R \cap \overline{Q}) \cup (Q_R \setminus \overline{Q})\) and using Lemma 5.4, we have
\[ \|f^\sigma g^\sigma\|_{L^\infty(Q_R)} \lesssim \|f^\sigma g^\sigma\|_{L^\infty(Q)} + \|f^\sigma g^\sigma\|_{Z(Q_R \setminus \overline{Q})}. \]

For the first term on \(\overline{Q}\), the argument as in the medium or low concentration case leads to an estimate of the form (5.11) with \(Q_R\) there replaced by \(\overline{Q}\).

For the second term, write
\[ f^\sigma g^\sigma = (P_{D_3}^{\phi_3} f^\sigma g^\sigma) + ((1 - P_{D_3}^{\phi_3}) f^\sigma P_{D_3}^{\phi_3} g^\sigma) + ((1 - P_{D_3}^{\phi_3}) f^\sigma (1 - P_{D_3}^{\phi_3}) g^\sigma) \]
\[ \approx_{II} \approx_{II} \approx_{III} \]

For \(I\) and \(II\), dyadic decomposing \(Q_R \setminus \overline{Q}\) into annuli around \(z_3\) of the form \(Q_{\text{ann}}(z_3; 2^j, 2^{j+1})\) with \(2^j \geq \overline{R}_3\). Taking \(C_0\) large and applying Lemma 5.7 then summing over dyadic \(2^{-j}\), we are done.

It remains to handle the \(III\)–term. Denote
\[ f^\sigma = (1 - P_{D_3}^{\phi_3}) f^\sigma, \quad \hat{g}^\sigma = (1 - P_{D_3}^{\phi_3}) g^\sigma. \]
Note that \(f^\sigma, \hat{g}^\sigma\) are red and blue waves without the relaxed margin conditions required in \(\mathcal{B}_R^{\phi_3}\) and \(\mathcal{B}_R^{\phi_3}\). Thus, we can not apply the inductive argument as in the case when \(r_3 \geq R^{1/2+4/N}\). To over come this obstacle, we may use the same method in [23] Appendix B], to which we refer for details.

Switching back and affording a fixed universal constant, we have
\[ \|III\|_{Z(Q_R)} \lesssim \overline{A^\sigma}(R) + 2^{O_\epsilon(C_0)} 2^{k\gamma(q,\epsilon)} \] (5.17)
Thus, by using (5.5)
\[ \|III\|_{Z(Q_R)} \lesssim \delta \|f^\sigma g^\sigma\|_{Z(Q_R)} + 2^{O_\epsilon(C_0)} 2^{k\gamma(q,\epsilon)}, \]
Plugging this back we are done.
It remains to consider the case $\overline{R}_d = 2^{C_{\alpha}k}$. In this case, the energy is concentrated in a scale $\leq 2^{C_{\alpha}k}$, we use the small scale estimate at the very beginning of the section to see that $\|F^\sigma G^\sigma\|_{Z(Q)} \leq 2^{k\gamma(q, \varepsilon)}$ with $\varepsilon$ small.

Collecting all these estimates, we obtain (5.10) and the proof of Proposition 5.8 is complete.

5.3. End of the proof. We are ready to show Theorem 1.2. Let $C_1$ and $C_2$ be the structural constants given by Proposition 5.8 and Proposition 5.3 respectively. We may take $C_1$ large and take $\delta = C_0^{-C_{10}/100}$.

Let $(F^\sigma, G^\sigma) \in \mathcal{H}_R^\sigma \times \mathcal{B}_R^{\sigma, 2^k}$ such that there exists $\Phi_\sigma \in \mathcal{E}_\sigma$ for which we have $F^\sigma \in \mathcal{H}_R^{\Phi_\sigma}$ and $G^\sigma \in \mathcal{B}_R^{\Phi_\sigma, 2^k}$. For any spacetime cube $Q_R$, if it satisfies the condition (5.8), we let $z_\delta, r_\delta, D_\delta$ be given by Proposition 5.8. If $\overline{R}_d > 2^{C_{\alpha}k}$, then by definition of $\mathcal{A}^{\Phi_\sigma}(\overline{R}_d, r_\delta, C_0 r_\delta)$, (5.10), Proposition 5.3, (5.9), we get

$$\|F^\sigma G^\sigma\|_{Z(Q_R)} \leq (1 - C_1 (\delta + C_0^{-C_{10}/100}))^{-2/q} (1 - \delta)^{1/q} \left( (1 - C_2 2^{-C_0}) \overline{A}^\sigma (R) + 2^{C_2 C_0 2^{k\gamma(q, \varepsilon)}} \right) \leq 2^{Q(C_0) 2^{k\gamma(q, \varepsilon)}}.$$ 

Using $q > 1$, and taking $C_0$ large if necessary (depending only on $q, C_1$), one has $\exists \delta_0 \in (0, 1/10)$ and $0 < C_0 < \infty$, depending only on $C_0, C_1, C_2$ and $q$, such that

$$\|F^\sigma G^\sigma\|_{Z(Q_R)} \leq (1 - \delta_0) \overline{A}^\sigma (R) + C_0 2^{k\gamma(k, \varepsilon)}.$$ 

If $\overline{R}_d = 2^{C_{\alpha}k}$, then we have the trivial estimate $\|F^\sigma G^\sigma\|_{Z(Q_R)} \leq C_0 2^{k\gamma(q, \varepsilon)}$ by the small scale estimate, in view of the proof in the last subsection.

Thus, we have

$$\max_{Q_R} \max_{Q_R} \|F^\sigma G^\sigma\|_{Z(Q_R)} \leq \max_{Q_R} \max_{Q_R} \|F^\sigma G^\sigma\|_{Z(Q_R)} \leq C_0 2^{k\gamma(q, \varepsilon)} + \max_{Q_R} \max_{Q_R} \|F^\sigma G^\sigma\|_{Z(Q_R)}$$

$$= \max_{Q_R} \max_{Q_R} \left\{ (1 - \delta_0) \overline{A}^\sigma (R) + C_0 2^{k\gamma(k, \varepsilon)} \right\} \leq (1 - \delta_0) \overline{A}^\sigma (R) + 2C_0 2^{k\gamma(q, \varepsilon)}.$$ 

Noting the right side is independent of $(F^\sigma, G^\sigma) \in \mathcal{H}_R^\sigma \times \mathcal{B}_R^{\sigma, 2^k}$, we get

$$A^\sigma (R) \leq (1 - \delta_0) \overline{A}^\sigma (R) + 2C_0 2^{k\gamma(q, \varepsilon)}.$$ 

Note that $\overline{A}^\sigma (R)$ is finite for each $\sigma, R$. Taking suprema, we obtain

$$\overline{A}^\sigma (R) \leq \frac{2C_0}{\delta_0} 2^{k\gamma(q, \varepsilon)}.$$ 

Since the right side is uniform with respect to $\sigma \leq \sigma_0$, the proof is thus complete by taking suprema over $\sigma \in (0, \sigma_0]$. \hfill \Box

**Remark 5.9.** To get Proposition 2.8, the above same proof works equally well based on the the wave tables in [15], with modifications in the interpolation argument to meet the mixed-norms.
6. Endpoint bilinear restriction estimates on the sphere concerning a conjecture by Foschi and Klainerman

Let $n \geq 2$ and $\mathbb{S} = \mathbb{S}^{n-1} = \{ \xi \in \mathbb{R}^n : |\xi| = 1 \}$ be the unit sphere in $\mathbb{R}^n$. Define the Stein operator to be the adjoint of the Fourier restriction operator

$$Sf(x) = \int_{\mathbb{S}} e^{ix \cdot \xi} f(\xi) \, d\sigma(\xi),$$

with $d\sigma$ being the surface measure. It is conjectured in [6] that if we let $\Omega_1, \Omega_2$ be two disjoint compact subsets of $\mathbb{S}$ such that

$$\text{dist}(\Omega_1, \Omega_2) > 0, \text{dist}(\Omega_1, -\Omega_2) > 0,$$

then for every $p \geq \frac{n+2}{2}$, there is a finite constant $C = C_{\Omega_1, \Omega_2, p} > 0$ such that

$$\|Sf \cdot Sg\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^2(\Omega_1)} \|g\|_{L^2(\Omega_2)}$$

(6.1)

holds for all $f$ supported in $\Omega_1$, and $g$ supported in $\Omega_2$.

Nowadays, this question is usually referred as the bilinear restriction theorems on elliptic type surfaces. In [16], the sharp result was established by Tao for all $p > 1 + \frac{2}{n}$, except the endpoint. The purpose of this section is to invest a possible way of combining the method of descent with the argument for Theorem 1.2 to get the endpoint case. To illustrate the idea, we assume $\Omega_1, \Omega_2$ are contained in a small neighbourhood of the north pole $e_n = (0, \ldots, 0, 1)$. By modulating the integrand in the $L^p-$norm on the left side of (6.1), it is natural to consider the Monge function

$$\Psi(\xi') = \sqrt{1 - |\xi'|^2} - 1$$

and introducing the $\lambda-$dependent operator

$$S^\lambda \Omega f(x, t) = \int_{-1}^1 \int_{\Omega} e^{i(x' \cdot \xi' + ts + x_n \frac{\xi'(s)}{\lambda + s})} f(\xi', s) \, d\xi' \, ds,$$

where $t, s$ are auxiliary variables.

Taylor expand $\Psi(\xi') = \frac{|\xi'|^2}{2} + \mathcal{E}(\xi')$ for $|\xi'| \ll 1$ and look at the surface

$$\Sigma^\lambda := \{ (\xi', s) : |\xi'| \ll 1, |s| \leq 1 \}$$

Ignoring the error term $\mathcal{E}(\xi')$, the main term is a paraboloid as in [23], so that normal directions at each point of $\Sigma^\lambda$ should be almost contained in a conic surface due to the main term. To justify this rigorously, it remains to control the perturbation term. To this end, one may introduce a similar class of functions including $\frac{\Psi(\xi')}{\lambda + s}$ as a special case such that the stability result holds as $\lambda \to +\infty$.

Let $I = [-1, 1]$ and $C^\lambda(R)$ be the optimal constant $C$ for which

$$\|S^\lambda_{\Omega_1} f \cdot S^\lambda_{\Omega_2} g\|_{L^p(Q^\lambda_R)} \leq C \lambda^{1/p} \|f\|_{L^2(\Omega_1 \times I)} \|g\|_{L^2(\Omega_2 \times I)}$$

for all $f, g$ and all $\lambda-$stretched $R$-cube $Q^\lambda_R \subset \mathbb{R}^{n+1}_{x_n}$. Here, by stretch we mean multiplying $\lambda$ to the side along $x_n$-direction. Construct the wave tables and use bootstrap under the condition $R \leq \lambda$ ultimately to get $C^\lambda(R) \lesssim 1$ for all $R \leq \lambda$.

Let $\lambda = R$. Changing variables, and letting $R \to \infty$ using Lebesgue’s dominated convergence theorem and then Fatou’s lemma, we obtain after integrating $t$ out

**Theorem 6.1.** Under the above conditions on $\Omega_1, \Omega_2$, (6.1) holds for all $p \geq \frac{n+2}{n}$.

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5 Conjecture 17.2
References

[1] Antonić, N. and Ivec, I., On the Hörmander-Mihlin theorem for mixed-norm Lebesgue spaces. J. Math. Anal. Appl. 433 (2016), no. 1, 176–199. MR3388786.

[2] Bennett, J.; Carbery, A., and Tao, T., On the multilinear restriction and Kakeya conjectures. Acta. Math. 196 (2006), no. 2, 261–302. MR2275834.

[3] Bourgain, J. Estimates for cone multipliers. Geometric aspects of functional analysis (Israel, 1992-1994), 41–60, Oper. Theory Adv. Appl., 77, Birkhäuser, Basel, 1995. MR1353448.

[4] Carbery, A., and Valdimarsson, S. The endpoint multilinear Kakeya theorem via the Borsuk-Ulam theorem. J. Funct. Anal. 264 (2013), no. 7, 1643-1663. MR3019726.

[5] Demeter, C. Fourier restriction, decoupling, and applications. Cambridge Studies in Advanced Mathematics, 184. Cambridge University Press, Cambridge 2020. xvi+331 pp. ISBN:978-1-108-49970-5. MR3971577.

[6] Foschi, D. and Klainerman, S. Bilinear space-time estimates for homogeneous wave equations. Ann. Sci. École Norm. Sup. (4) 33 (2000), no. 2, 211–274. MR1755116.

[7] Del Nin, Giacomo, and Merlo, Andrea, Endpoint Fourier restriction and unrectifiability. Proc. Amer. Math. Soc. 150 (2022), no. 5, 2137-2144. MR4392348.

[8] Guth, L. The endpoint case of the Bennett-Carbery-Tao multilinear Kakeya conjecture. Acta Math. 295(2010), no. 2, 263-286. MR2746348.

[9] Haberman, B. Uniqueness in Calderón’s problem for conductivities with unbounded gradient. Comm. Math. Phys. 340 (2015), no. 2, 639-659. MR3397029.

[10] Ham, S.; Kown, Y., and Lee, S. Uniqueness in the Calderón problem and bilinear restriction estimates. J. Funct. Anal. 281 (2021), no. 8, Paper No.109119, 58 pp. MR4273826.

[11] Klainerman, S., and Tataru, D. On the optimal local regularity for Yang-Mills equations in $\mathbb{R}^{4+1}$. J. Amer. Math. Soc. 12 (1999), no. 1, 93–116. MR1626261.

[12] Lee, S., Rogers, K. M., and Vargas, A. Sharp null form estimates for the wave equation in $\mathbb{R}^{3+1}$. Int. Math. Res. Not., IMRN (2008), Art. ID rnn 096, 18 pp. MR2439536.

[13] Lee, S. and Vargas, A. Sharp null form estimates for the wave equation. Amer. J. Math. 130 (2008), no. 5, 1279-1326. MR2450209.

[14] Ponce-Vanegas, F. A bilinear strategy for Calderón’s problem. Rev. Mat. Iberoam. 37 (2021), no. 6, 2119-2160. MR4310288.

[15] Tao, T. Endpoint bilinear restriction theorems for the cone, and some sharp null form estimates. Math. Z. 238 (2001), no. 2, 215–268. MR1865417.

[16] Tao, T. A sharp bilinear restrictions estimate for paraboloids. Geom. Funct. Anal. 13 (2003) no. 6, 1359-1384. MR2033842.

[17] Tao, T.; Vargas, A. and Vega, L. A bilinear approach to the restriction and Kakeya conjectures. J. Amer. Math. Soc. 11 (1998), no. 4, 967–1000. MR1625056.

[18] Tao, T. and Vargas, A. A bilinear approach to cone multipliers I: Restriction estimates. Geom. Funct. Anal. 10 (2000), no. 1, 185–215. MR1748920.

[19] Tao, T. and Vargas, A. A bilinear approach to cone multipliers II: Applications. Geom. Funct. Anal. 10 (2000), no. 1, 216–258. MR1748921.

[20] Tataru, D., Null form estimates for second order hyperbolic operators with rough coefficients. Harmonic analysis at Mount Holyoke, (South Hadley, MA, 2001), 383-409, Contemp. Math., 320, Amer. Math. Soc., Providence, RI, 2003. MR1979953.

[21] Temur, F. An endline bilinear cone restriction estimate for mixed norms. Math. Z. 273 (2013), no. 3-4, 1197–1214. MR3030696.

[22] Wolff, T. A sharp bilinear cone restriction estimate. Ann. of Math. (2) 153 (2001) no. 3, 661-698. MR1836285.

[23] Yang, J. An endline bilinear restriction estimate for paraboloids. arXiv:math/2202.13905v2.

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