GRAPHS WITH THE SAME TRUNCATED CYCLE MATROIDS

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Abstract

The classical Whitney’s 2-Isomorphism Theorem describes the families of graphs having the same cycle matroid. In this paper we describe the families of graphs having the same truncated cycle matroid and prove, in particular, that every 3-connected graph, except for $K_4$, is uniquely defined by its truncated cycle matroid.

Key words: graph, matroid, cycle matroid, truncated matroid.

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1 Introduction

In 1933 H. Whitney [3] described the families of graphs having the same cycle matroid. He also proved, in particular, that every 3-connected graph is uniquely defined by its cycle matroid [4]. In this paper we describe the families of graphs having the same truncated cycle matroid and prove, in particular, that every 3-connected graph, except for $K_4$, is uniquely defined by its cycle matroid. The dual version of our paper is described by R. Chen and Z. Gao [1].

2 Main notions, notation, and simple observations

Given two finite sets $X$ and $Y$, let $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$.

Given a finite set $E$, let $2^E$ denote the set of all subsets of $E$ and $\binom{E}{k}$ denote the set of all $k$-element subsets of $E$.

Given $S \subseteq 2^E$, let $\Delta(S) = \Delta\{S : S \in S\}$.

2.1 On graphs

A graph $G$ is a triple $(V, E, \phi)$ such that $V$ and $E$ are disjoint finite sets, $V \cap V = \emptyset$, $V \neq \emptyset$, and $\phi : E \rightarrow \binom{V}{2}$. We will also put $V = V(G)$ and $E = E(G)$.

The elements of $V = V(G)$ and $E = E(G)$ are called vertices and edges of graph $G$, respectively. If $\phi(e) = \{u, v\}$, we say that vertices $u$ and $v$ are incident to edge $e$ and are the end-vertices of $e$ in $G$.

We say that graph $G' = (V', E', \phi')$ is a subgraph of graph $G = (V, E, \phi)$ and write $G' \leq G$ if $V' \subseteq V$, $E' \subseteq E$, and function $\phi'$ is a restriction of function $\phi$ on $V'$.
The **degree** \(d(v,G)\) of vertex \(v\) in \(G\) is the number of edges incident to \(v\) in \(G\).

A **cycle** in graph \(G\) is an \(\leq\)-minimal subgraph of \(G\) with every vertex of degree two. An \((x, y)\)-**path** in graph \(G\) is an \(\leq\)-minimal subgraph of \(G\) with exactly two vertices \(x\) and \(y\) of degree one.

A cycle \(C\) in graph \(G\) is called **Hamiltonian**, if \(V(C) = V(G)\). A cycle \(Q\) in graph \(G\) is called **quasi-Hamiltonian**, if \(V(Q) = V(G) \setminus q\) for some \(q \in V(G)\).

A forest in graph \(G\) is a subgraph of \(G\) with no cycles. A forest \(F\) in graph \(G\) is called **maximal** if \(F\) is not a proper subgraph of another forest in \(G\).

A graph \(G\) with at least two vertices is connected if \(G\) has an \((x, y)\)-path for every two vertices \(x\) and \(y\) in \(G\). Obviously, a maximal forest in a connected graph \(G\) is a tree \(T\) with \(V(T) = V(G)\) (called a **spanning tree** of \(G\)).

Let \(G = (V, E, \phi)\) and \(X \subseteq E\). Let \(G[X]\) be the graph such that \(E(G[X]) = X\) and \(V(G[X])\) is the set of vertices of \(G\) incident to at least one edge in \(X\). We say that \(G[X]\) is the **subgraph** of \(G\) induced by the edge subset \(X\).

A graph \(G\) is called **even** if every vertex of \(G\) is incident to an even number of edges in \(G\) (i.e. the degree of every vertex in \(G\) is even). For example, a cycle is an even graph.

Graphs \(G = (V, E, \phi)\) and \(G' = (V', E', \phi')\) are **equal** if \(V = V', E = E'\), and \(\phi = \phi'\).

An **isomorphism** from \(G = (V, E, \phi)\) to \(G' = (V', E', \phi')\) is a pair \((\nu, \varepsilon)\), where \(\nu : V \to V'\) and \(\varepsilon : E \to E'\) are bijections such that \(\phi(e) = \{x, y\} \iff \phi'(\varepsilon(e)) = \{\nu(x), \nu(y)\}\). Graphs \(G\) and \(G'\) are isomorphic (denoted by \(G \sim G'\)) if there exists an isomorphism from \(G\) to \(G'\) (or, equivalently, an isomorphism from \(G'\) to \(G\)).

A **vertex star** in graph \(G\) is the set of edges in \(G\) incident to the same vertex. Graphs \(G\) and \(G'\) are strongly isomorphic (denoted by \(G \approx G'\)) if they have the same family of vertex stars.

### 2.2 On matroids

Let \(M\) be a matroid on the ground set \(E\) with the set of bases \(\mathcal{B} = \mathcal{B}(M)\) and the set of circuits \(\mathcal{C}(M)\). A circuit \(H\) in \(M\) is called **Hamiltonian** if the number of elements in \(H\) is equal to the number of elements in a base of \(M\) plus one.

We will need the following (known and easy to prove) fact.

**Claim 2.1** Let \(M\) be a matroid on the ground set \(E\) and

\[ B_t(M) = \{ B - x : B \in \mathcal{B}(M), x \in B \}. \]

Then

(c1) \(B_t\) is the set of bases of a matroid on the ground set \(E\) (denoted by \(M_t\)) and

(c2) \(\mathcal{C}(M_t) = \mathcal{C}(M) \cup \mathcal{B}(M) \setminus \mathcal{H}(M)\), where \(\mathcal{H}(M)\) is the family of Hamiltonian circuits of \(M\).

Matroid \(M_t\) is called the the **truncation of matroid** \(M\) or, simply, a **truncated matroid**.
2.3 On matroids of a graph

Let $G = (V, E, \phi)$ be a graph with the vertex set $V$, the non-empty edge set $E$, and the incident function $\phi$. Let $\mathcal{B}(G)$ be the family of the edge sets of maximal forests and $\mathcal{C}(G)$ be the family of the edge sets of cycles in graph $G$.

It is known and easy to prove that $\mathcal{B}(G)$ is the set of bases and $\mathcal{C}(G)$ is the set of circuits of a matroid $M$ on the ground set $E$ called the cycle matroid of graph $G$ and denoted by $M(G)$.

Obviously, if graph $G$ is connected, then $\mathcal{B}(G)$ is the family of the edge sets of spanning trees in $G$ and $\mathcal{B}_t(G)$ is the family of the edge-sets of maximal two-component forests in $G$.

3 Main Result

We need the following (easy to prove) claim.

Claim 3.1 Suppose that

(a1) $G$ is a graph and $Y, Z \subseteq E(G)$ and
(a2) $G[Y]$ is an even subgraph of $G$ and $G[Z]$ is a cycle of $G$.

Then

(c1) $G[Y \Delta Z]$ is also an even subgraph of $G$, and therefore
(c2) if $\mathcal{F}$ is a family of the edge sets of cycles in $G$, then $G[\Delta \mathcal{F}]$ is an even subgraph of $G$.

Put $\mathcal{C}(M_t(G)) = \mathcal{C}_t(G)$. From the definition of $M_t(G)$ we have:

Lemma 3.2 Let $G$ be a connected graph with no loops and no parallel edges.

Then $\mathcal{C}_t(G) = \mathcal{T}(G) \cup \mathcal{Q}(G) \cup \mathcal{S}(G)$, where

$\mathcal{T}(G)$ is the family of the edge-sets of spanning trees of $G$,
$\mathcal{Q}(G)$ is the family of the edge-sets of quasi-Hamiltonian cycles of $G$, and
$\mathcal{S}(G)$ is the family of the edge-sets of small cycles of $G$.

We say that a pair of graphs $\{G, F\}$ satisfies condition $\mathcal{K}$ if the following conditions hold:

(b0) $G$ and $F$ are connected graphs,
(b1) $G$ and $F$ are graphs with no loops and no parallel edges,
(b2) $E(G) = E(F) = E$,
(b3) $M_t(G) = M_t(F)$, and
(b4) $M(G) \neq M(F)$.

Lemma 3.3 Suppose that a pair of graphs $\{G, F\}$ satisfies condition $\mathcal{K}$.

Then there exists $X \subseteq E$ such that exactly one of the following holds:

(c1) $G[X]$ is a quasi-Hamiltonian cycle in $G$ and $F[X]$ is a spanning tree in $F$ or
(c2) $F[X]$ is a quasi-Hamiltonian cycle in $F$ and $G[X]$ is a spanning tree in $G$.
Proof. Put $\mathcal{C}(M_t(G)) = \mathcal{C}_t(G)$. By Lemma 3.2, $\mathcal{C}_t(G) = T(G) \cup Q(G) \cup S(G)$. Obviously,

$$M_t(G) = M_t(F) \Leftrightarrow \mathcal{C}_t(G) = \mathcal{C}_t(F).$$

It is easy to see that $S(G) = S(F)$.

Therefore,

$$M_t(G) = M_t(F) \Leftrightarrow T(G) \cup Q(G) = T(F) \cup Q(F).$$

Now

$$M(G) \neq M(F) \Rightarrow T(G) \neq T(F) \Leftrightarrow T(G) \Delta T(F) \neq \emptyset.$$ 

Thus, there exists $X \subseteq E$ such that at least one of the following holds:

- $X \in Q(G) \setminus Q(F)$ and $X \in T(F) \setminus T(G)$ or
- $X \in Q(F) \setminus Q(G)$ and $X \in T(G) \setminus T(F)$.

□

Claim 3.4 The edge set $X$ in Lemma 3.3 is not the symmetric difference of the edge sets of small cycles in $G$ (i.e. of members of set $S(G)$).

Proof. Suppose, on the contrary, that $X = \Delta F$ for some family of edge sets of cycles in $G$. Then by Claim 3.1, $F[\Delta F]$ is an even graph and therefore is not a spanning tree in graph $F$. This contradicts Lemma 3.3. □

Without loss of generality we can assume by Lemma 3.3 that $G[X]$ is a quasi-Hamiltonian cycle in $G$ and $F[X]$ is a spanning tree in $F$.

Let $G[X] = R$ and $F[X] = T$. Then $V(G) \setminus V(R)$ consists of one vertex that we denote by $c$. Since graph $G$ is connected, there exists an edge $s$ incident to $c$ and to a vertex (say, $r$) in $R$. Thus, $R' = R \cup (csr)$ is a connected spanning subgraph of $G$, and so if $e$ is an edge in $G$, then one of its end-vertices is in $R$ and the other one is either equal to $c$ or is also in $R$.

We call cycle $R$ the rim of $G$, an edge of $G$ incident to $c$ a spoke of $G$, and an edge $h$ in $E(G) \setminus E(R)$ with both end-vertices in $R$ a chord of $G$.

Claim 3.5 Graph $G$ has no chords, and therefore $G$ has at least one spoke.

Proof. Suppose, on the contrary, that $G$ has a chord $h$ with the end-vertices $u$ and $v$. Then $h$ belongs to two cycles $C_1$ and $C_2$ in $G$ such that

1. $E(C_1), E(C_2) \in S(G) = S(F)$ and
2. $E(C_1) \Delta E(C_2) = E(R)$.

This contradicts Claim 3.1. □

Claim 3.6 Suppose that $s$ and $s'$ are some different spokes in $G$ with the end-vertices $r$ and $r'$ in $R$, respectively. Let $P = P(r, r')$ be a shortest path in $R$ with the end-vertices $r$ and $r'$. Then path $P$ has at most two edges.

Proof. Let $H = rscs'r'$ and let $P' = P'(r, r')$ be the path in $R$ such that $P \cup P' = R$. Let $C = P \cup H$ and $C' = P' \cup H$. Then $E(R) = E(C) \Delta E(C')$. Suppose, on the contrary, that
Proof By Claim 3.5, \( R(E(G)) \geq 3 \). Then also \( e(P') \geq 3 \), and so \( e(R) \geq 2p \geq 6 \). Also \( e(C) = e(P) + 2 \) and \( e(C') = e(P') + 2 \). Therefore \( e(C) < e(R) \) and \( e(C') < e(R) \). Then, \( E(C), E(C') \in S(G) \) and \( E(C) \Delta E(C') = E(R) \). Now by Lemma 3.3 \( F[E(C) \Delta E(C')] \) is an even graph and therefore is not a spanning tree in graph \( F \). This contradicts Lemma 3.3.

Claim 3.7 Graph \( G \) has at least one and at most three spokes. Moreover,

(c1) if \( G \) has two spokes \( s_1 \) and \( s_2 \), then their end-vertices \( r_1 \) and \( r_2 \) in \( R \) are on distance one or two in \( R \) (see Fig. 3.7[c1,1] and Fig.3.7[c1,2]) and

(c2) if \( G \) has three spokes \( s_1, s_2, \) and \( s_3 \), then their end-vertices \( r_1, r_2, \) and \( r_3 \) belong to a 3-vertex path in \( R \) (see Fig.3.7[c2,p1]).

Proof By Claim 3.5 \( G \) has at least one spoke. If \( G \) has at most two spokes, then our claim (c1) follows from Claim 3.6.

Now we prove (c2). Suppose that \( G \) has exactly three spokes \( s_1, s_2, \) and \( s_3 \) with the end-vertices \( r_1, r_2, \) and \( r \) in \( R \), respectively.

(p1) Suppose first that some two vertices from \( \{r_1,r_2,r\} \), say, \( r_1 \) and \( r_2 \), are adjacent in \( R \). If \( r \) is adjacent to \( r_1 \) or \( r_2 \), then \( r_1, r_2, \) and \( r \) belong to a 3-vertex path in \( R \), and we are done (see Fig.3.7[c2,p1]). Therefore \( r \) is on distance at least 2 from \( r_1 \) and \( r_2 \) in \( R \). Then \( E(R) = X \) is the symmetric difference of the edge sets of three small cycles. This contradicts Claim 3.4.

(p2) Now suppose that no two vertices from \( \{r_1,r_2,r\} \) are adjacent in \( R \). Then by Claim 3.6 every two vertices from \( r_1, r_2, r \) are on distance 2 in \( R \). Therefore \( R \) is a 6-edge cycle and \( E(R) = X \) is the symmetric difference of the edge sets of three squares \( S_i, i \in \{1,2,3\} \) in \( G \). This contradicts Claim 3.4.

Put \( M_t = M_t(G) = M_t(F) \).

Claim 3.8 Suppose that

(a1) a pair of graphs \( \{G,F\} \) satisfies condition \( \mathcal{K} \) and

(a2) \( G \) has exactly one spoke \( s \) with its end-vertex \( r \) in \( R = G[X] \).

Then

(c1) either \( F \) is isomorphic to \( G \) or \( F \) is a cycle and

(c2) \( M_t \) is isomorphic to a uniform matroid \( U_{m-2,m} = (E, \mathcal{B}) \), where \( m \) is the number of elements in \( E \) and \( \mathcal{B} = \binom{E}{m-2} \) is the set of bases of matroid \( U_{m-2,m} \).

Proof As we have assumed above, \( R = G[X] \) is a quasi-hamiltonian cycle in \( G \) and \( T = F(X) \) is a spanning tree in \( F \). Then \( T \cup s = F[X \cup s] \) has a cycle, say \( D \). If \( D \) is a small cycle, then \( E(D) \) induces a small cycle in \( G \), a contradiction. Therefore one of the following holds:

(h1) \( D \) is a quasi-hamiltonian cycle in \( F \), and so \( F \) is isomorphic to \( G \) or

(h2) \( D \) is a hamiltonian cycle in \( F \), and so \( D = F \) is a cycle.

Therefore (c1) holds. Moreover, in both cases \( M_t \) satisfies (c2).
Claim 3.9 Suppose that
(a1) a pair of graphs \{G, F\} satisfies condition \(K\) and
(a2) \(G\) has exactly two spokes \(s_1\) and \(s_2\) and their end-vertices \(r_1\) and \(r_2\) are incident to an
edge, say \(x\), in \(R = G[X]\).

Then
(c1) graphs \(G\) and \(F\) are isomorphic,
(c2) graph \(F\) can be obtained from graph \(G\) by a series of some Whitney 2-vertex cut switches
and, possibly, also by exchanging two edges \(e\) and \(s_i\) for some \(i \in \{1, 2\}\) (which is not a
Whitney graph operation), and
(c3) graphs \(G\) and \(F\) are 2-connected.

Proof By Claim 3.3 \(R = G[X]\) is a quasi-hamiltonian cycle in \(G\) and \(F[X]\) is a spanning
tree \(T\) in \(F\). Since \(G\) has exactly two spokes \(s_1\) and \(s_2\) and their end-vertices \(r_1\) and \(r_2\) in
\(R = G[X]\) are on distance one in \(R\), there is an edge \(x\) in \(R\) such that \(\Delta = c_{s_1}r_1xr_2s_2c\) is a
3-cycle in \(G\). Clearly, \(e \in E(R) \cap E(T)\). Then \(E(\Delta)\) is the edge-set of a 3-cycle \(Z\) in \(F\).

Let \(V(Z) = \{z, z_1, z_2\}\), where vertices \(z_1, z_2\) are incident to edge \(x\), and so vertex \(z\) is not
incident to edge \(x\). Let \(zPz'\) be a shortest path in tree \(T\) from \(z\) to \(\{z_1, z_2\}\). Let \(z' = z_1,\)
and so \(s_1\) is the edge in \(Z\) incident to vertices \(z\) and \(z_1\) in \(T\). Obviously, \(x \notin E(P)\). If
\(e(P) < e(T - x)\), then \(P \cup s_1\) is a small cycle in \(F\), and therefore \(E(P) \cup s_1\) is the edge
set of a small cycle in \(G\) distinct from \(\Delta\), a contradiction. Then \(e(P) = e(T - x)\), and so
\(E(P) = E(T - x)\), a tree \(T\) is the path \(zPz_1z_2\), and \(F = T \cup s_2\). Thus, \(F\) is isomorphic to
\(G\), and therefore (c1) holds. It also follows that (c2) and (c3) holds.

Claim 3.10 Suppose that
(a1) a pair of graphs \{G, F\} satisfy condition \(K\) and
(a2) \(G\) has exactly two spokes \(s_1\) and \(s_2\) and their end-vertices \(r_1\) and \(r_2\) in \(R\) are the end-
vertices of a two-edge path \(r_1x_1rx_2r_2\) in \(R\).

Then
(c1) graphs \(G\) and \(F\) are isomorphic,
(c2) graph \(F\) can be obtained from graph \(G\) by a series of some Whitney 2-vertex cut switches
and, possibly, also by exchanging two edges \(e_i\) and \(s_j\) for some \(i, j \in \{1, 2\}\) (which is not a
Whitney graph operation), and
(c3) graphs \(G\) and \(F\) are 2-connected.

Proof By (a2), \(\bowtie = c_{s_1}r_1x_1rx_2r_2s_2c\) is an induced 4-cycle in \(G\). Then \(E(\bowtie)\) is also the edge-
set of an induced 4-cycle in \(F\). Let \(R^*\) be the cycle in \(G\) with \(E(R^*) = (E(R) \setminus \{x_1, x_2\}) \cup
\{s_1, s_2\}\).

Consider the path \(Z = R - r\) with the end-vertices \(r_1\) and \(r_2\). Then \(Z = R^* - c\). Obviously,
\(e(Z) \geq 1\). Also \(\{s_1, s_2, x_1, x_2\}\) is the edge set of a 4-cycle in \(F\).

Suppose first that \(Z\) has one edge, say \(z\). Then \(Q_c = (R \cup z) \setminus r\) and \(Q_r = R^* \cup z\) \setminus c\)
are triangles and the only quasi-hamiltonian cycles in \(G\). By Lemma 3.3, we can assume
that $E(Q_c)$ induces a spanning tree in $F$. Then $v(F) = v(G) = 4$, $E(F) = E(G)$, and $e(F) = e(G) = 5$. Therefore graphs $G$ and $F$ are isomorphic.

Now suppose that $e(Z) \geq 2$. Let $z_1, z_2 \in E(Z)$, where $z_1 \neq z_2$. Let $Z_i = Z - z_i$ and $T_i = Z_i \cup (r_1s_1e_2r_2) \cup \{x_i, r\}$, where $i \in \{1, 2\}$. Then each $T_i$ is a spanning tree of $G$ and $G = T_1 \cup T_2$. Let $E_i = E(T_i)$. Obviously, $F = F(E_1) \cup F(E_2)$.

We claim that each $F(E_i)$ is also a spanning tree of $F$. Suppose, on the contrary, that, say, $F(E_1)$ is not a spanning tree of $F$. Then $D = F(E_1)$ is a quasi-hamiltonian cycle in $F$ containing a path with three edges $x_1, s_1, s_2$. Hence $x_2$ is a cord of cycle $D$. Then the edge-set $(E(D) \cup x_2) \setminus \{s_1, s_2, x_1\}$ induces a small cycle in $F$ but not in $G$, a contradiction. It follows that graphs $G$ and $F$ are isomorphic. Therefore (c1) holds. It follows that (c2) and (c3) also hold. 

\begin{claim}
Suppose that
\begin{enumerate}[(a1)]
\item a pair of graphs $\{G, F\}$ satisfies condition $\mathcal{K}$ and
\item $G$ has exactly three spokes $s_1, s_2, s_3$ such that end-vertices $r_1$, $r_2$, and $r$ belong to a 3-vertex path $r_1x_1r_2x_2$ in $R$ and spoke $s$ is incident to $r$.
\end{enumerate}

Then
\begin{enumerate}[(c1)]
\item graphs $G$ and $F$ are isomorphic,
\item graph $F$ can be obtained from graph $G$ by a series of some Whitney 2-vertex cut switches and, possibly, also by exchanging two edges $e_i$ and $s_i$ for some $i \in \{1, 2\}$ (which is not a Whitney graph operation), and
\item graphs $G$ and $F$ are 2-connected.
\end{enumerate}
\end{claim}

\begin{proof}
From (a2) it follows (as above) that in graph $G$ there is a vertex $r$ and edges $x_1$ and $x_2$ in $R$ such that $\diamondsuit = c_5r_1x_1r_2x_2$ is a 4-cycle in $G$. Let $G' = G \setminus s$ and $F' = F \setminus s$. Then by Claim 3.10, $G'$ and $F'$ are isomorphic. If $F = F' \cup s$ is not isomorphic to $G$, then $s$ belongs to a small cycle $D$ in $F$, which is also a small cycle in $G$, a contradiction. Therefore (c1) holds. It also follows that (c2) holds.

It is easy to prove the following

\begin{claim}
Let $H$ be a non-connected graph with no loops, no parallel edges, no isolated vertices, and with the set of components $\{C_i : i = 1, ..., k\}$. Let $V = \{v_i \in V(C_i) : i = 1, ..., k\}$ and let $H^*$ be the graph obtained from $H$ by identifying the vertices in $V$ with a new vertex $v$.

Then
\begin{enumerate}[(c1)]
\item $v(H^*) \geq 3$ and $H^*$ is of connectivity one (i.e. is connected, but not 2-connected),
\item $M(H) = M(H^*)$,
\item $M_i(H) = M_i(H^*)$, and
\item $H^*$ satisfies condition $\mathcal{K}$.
\end{enumerate}
\end{claim}

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Claim 3.13 Suppose that a pair of graphs \( \{G, F\} \) satisfies all condition in \( K \), except for condition \((b0)\), namely, at least one of \( G, F \), say \( G \), is not a connected graph. Then \( G \) has two components, namely, a one-edge component and a cycle.

Proof By Claim 3.12, \( G^* \) satisfies condition \( K \).

By Claims 3.8, 3.9, 3.10 and 3.11, the graphs satisfying the conditions of these Claims are 2-connected. Since graph \( G^* \) is not 2-connected, \( G^* \) does not satisfy the assumptions of these Claims. Thus, \( G^* \) satisfies the assumptions of Claim 3.8. Therefore \( G \) has exactly two components, namely, a one-edge component and a cycle.

From the above we have in particular:

Claim 3.14 Every 3-connected graph, except for \( K_4 \), is uniquely defined by its truncated cycle matroid.

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