BOUNDARY VALUES AND BOUNDARY UNIQUENESS OF J-HOLOMORPHIC MAPPINGS

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Abstract. We establish a Fatou-type Theorem for J-holomorphic mappings that are bounded in an appropriate sense and we prove the Blaschke condition for their zero sets. We also prove a Privalov-type uniqueness Theorem for pairs of J-holomorphic mappings.

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1. Introduction

Bounded holomorphic functions, defined in the unit disc $\Delta \subset \mathbb{C}$, have special properties. Namely, if $f$ is such a function then the radial limit $f^*(e^{i\theta}) := \lim_{r \to 1} f(re^{i\theta})$ exists almost everywhere and, if $f \not\equiv 0$, then the set of $\theta \in [0, 2\pi]$ such that $f^*(e^{i\theta}) = 0$ has zero length. Further, the zero set $\{\zeta_k\} \subset \Delta$ of a bounded holomorphic function satisfies the Blaschke condition.

Our goal in this paper is to generalize these results to the case of pseudoholomorphic mappings with values in almost complex manifolds. The first problem that one meets here is to introduce a correct notion of boundedness. We propose the following one:

- A $J$-holomorphic mapping $u : \Delta \to X$ into an almost complex manifold $(X, J)$ is called bounded if there exists a relatively compact domain $\Omega \supset u(\Delta)$ which admits a strictly $J$-plurisubharmonic function.

Note that this function can be always supposed to be bounded from above. In the classical function theory $|\zeta^2|$ plays the role of this function.

Likewise, a relatively compact domain $\Omega \subset X$ is called bounded if it admits a strictly $J$-plurisubharmonic function bounded from above. The definition of boundedness of holomorphic maps or domains, which we propose, requires the existence of a bounded strictly $J$-plurisubharmonic function. Neither asking relative compactness nor the existence of a
non-constant bounded $J$-plurisubharmonic function would suffice. In both cases, there could be $J$-complex lines, i.e. non-constant images of $\mathbb{C}$ under a $J$-holomorphic mapping. But that should clearly be forbidden.

We shall see that bounded $J$-holomorphic mappings have non-tangential limits for almost all $e^{i\theta} \in \partial \Delta$, i.e., that a Fatou-type theorem is valid for them.

**Theorem 1.1.** Let $u$ be a bounded $J$-holomorphic map from $\Delta$ into an almost complex manifold $(X,J)$, $J$ continuous. Then for almost every $e^{i\theta} \in \partial \Delta$ the limit $u'(e^{i\theta})$ of $u(\zeta)$ exists as $\zeta$ approaches $e^{i\theta}$ non-tangentially.

Our second result is the following.

**Theorem 1.2.** Let $u : \Delta \to X$ be a non-constant, bounded $J$-holomorphic mapping, $J \in C^{\text{Lip}}$. Then for every point $p \in X$ the set $\{\zeta_k \in \Delta : u(\zeta_k) = p\}$ satisfies the Blaschke condition

$$\sum_k (1 - |\zeta_k|) < +\infty. \quad (1.1)$$

Here we denote by $C^{\text{Lip}}$ the class of Lipschitz-continuous functions, and by $C^{1,\text{Lip}}$ the class of functions whose first derivatives are Lipschitz-continuous.

We prove in this paper the following direct analogue of the Privalov boundary uniqueness theorem.

**Theorem 1.3.** Suppose $J \in C^{1,\text{Lip}}$ and let $u_1, u_2 : \Delta \to X$ be $J$-holomorphic mappings. Assume that there is a set $E$ of positive measure on the unit circle, such that at every $e^{i\theta} \in E$ $u_1$ and $u_2$ have non-tangential boundary values and that these boundary values are equal, i.e., $u_1'(e^{i\theta}) = u_2'(e^{i\theta})$ for $e^{i\theta} \in E$. Then $u_1 \equiv u_2$.

The result is still valid with a weaker notion of non-tangential boundary value (called restricted in §3), see Theorem 3.1 below.

**Notes.**

1) We would like to draw the attention of the reader to the fact that we do not require the continuity of the strictly $J$-plurisubharmonic function in the definition of boundedness of functions and domains. Assuming continuity would allow a simplification in the proof of Theorem 1.1. However, it is not at all clear that the existence of non-continuous strictly $J$-plurisubharmonic function insures the existence of continuous ones, nor is it clear that the existence of continuous ones insures the existence of smooth ones. These are challenging problems.

2) Let us emphasize that the statement of Theorem 1.2 is not as satisfactory as the statement of Theorem 1.3. One should try to get the Blaschke condition (1.1) for the set of coincidence $\{\zeta_k : u_1(\zeta_k) = u_2(\zeta_k)\}$ of two distinct bounded $J$-holomorphic mappings. However, for the reasons explained in Section 4, this general statement is out of the range of the methods deployed in this paper which exploits plurisubharmonic functions and the notion of pluripolarity. That is the main tool that allows us to go beyond the preliminary uniqueness results established in [ISk].

3) Boundary uniqueness problems for solutions of certain PDE-s (mostly close to the Laplace equation) are commonly studied with an ad hoc assumption of some (mostly $C^1$) regularity up to the boundary, see [AB] for example. It is important to point out that we do not assume any boundary regularity of our mappings and in general there is no regularity.
2. Plurisubharmonic Functions and non-Tangential Limits

2.1 Plurisubharmonic functions

An upper semi-continuous function $\rho$ defined on an almost complex manifold $(X,J)$, with $J$ continuous, is said to be $J$-plurisubharmonic ($J$-psh for short) if its restriction to any $J$-holomorphic disc is subharmonic; i.e. if $u : \Delta \to (X,J)$ is $J$-holomorphic, then $\rho \circ u$ is subharmonic. Of course this definition is meaningful in case there is abundance of $J$-holomorphic discs (e.g. the maximum principle certainly holds), and this happen as soon as $J$ is continuous, see Appendix.

We say that $\rho$ is strictly $J$-psh if and only if, locally, $\rho + \eta$ is still $J$-psh for all functions $\eta$ of small enough $C^2$ norm. If the almost complex structure $J$ is at least of class $C^1$, there is for $C^2$ functions $\rho$ a characterization of $J$-plurisubharmonicity in terms of the Hessian (Levi form). The condition is:

$$dd^c\rho_p(v, J(p)v) \geq 0$$

for any tangent vector $v$ at any point $p \in X$. Here, for a function $\phi$, $d^c\phi(v) = -d\phi(Jv)$ for $v \in TX$. This is rather straightforward if $J$ is of class at least $C^{1,\alpha}$ for some $0 < \alpha < 1$. The $J$-holomorphic discs are then of class $C^2$ (and even $C^{2,\alpha}$). More care is needed in case of $J$ merely $C^1$ since the $J$-holomorphic discs may fail to be of class $C^2$. See Corollary 1.1 in $[P]$. For strict $J$-plurisubharmonicity the condition is that $dd^c\rho(v, Jv) > 0$ if $v \neq 0$. Equivalently the conditions can be given in terms of $\partial J\overline{\partial}J$ as in $[P]$.

2.2 Non-tangential limits

We start with the study of non-tangential boundary values of a bounded $J$-holomorphic map from the unit disc $\Delta$ in $\mathbb{C}$ with values in an almost complex manifold $(X,J)$, with $J$ merely continuous. We shall consider domains in the unit disc obtained by taking conic neighborhoods of radii. Adopting Rudin’s notation from $[Ru]$ (11.18), for $0 < \alpha < 1$ and $\theta \in [0,2\pi]$, we consider the set

$$e^{i\theta}\Omega_{\alpha} = \{\zeta \in \Delta : |\zeta - e^{i\theta}|\zeta| < \alpha(1 - |\zeta|)\}. \quad (2.1)$$

This set is a conic neighborhood of the radius $[0,e^{i\theta}]$ with vertex at $e^{i\theta}$. Let $u : \Delta \to X$ be a $J$-holomorphic map.

**Definition 2.1.** We say that $u$ has a non-tangential limit at $e^{i\theta}$ if for every $\alpha \in (0,1)$ $u(\zeta)$ has a limit as $\zeta \to e^{i\theta}$, $\zeta \in e^{i\theta}\Omega_{\alpha}$. This limit is then denoted as $u^*(e^{i\theta})$.

**Proof of the Theorem** $[LJ]$. We start the proof with two preliminaries, one on some kind of Schwarz Lemma, and one on radial limits a.e. for subharmonic functions. Let $\Omega$ be a relatively compact domain in $X$ containing $\overline{u(\Delta)}$ and $\rho$ a strictly $J$-psh function in $\Omega$. Fix some Riemannian metric $h$ on $X$. We claim that for any compact set $K \subseteq \Omega$ there exists a constant $C = C(K,h,\rho)$ such that for any $J$-holomorphic $v : \Delta \to K$ one has

$$\|dv(0)\|_h \leq C. \quad (2.2)$$

Otherwise, one gets a sequence of $J$-holomorphic $v_n : \Delta \to K$ with $\|v_n(0)\|_h \to \infty$. Then, the Brody re-parameterization Lemma (in which neither the integrability of $J$ nor its smoothness play any role) $[B]$ applies and gives a non-constant $J$-holomorphic map $v : \mathbb{C} \to K$. The function $\rho \circ v$ is then a bounded subharmonic function, hence it is constant. Since $\rho$ is strictly plurisubharmonic this is impossible.
Remark that by simple rescaling, \((2.2)\) gives \(\|dv(\zeta)\|_h \leq \frac{C}{1-|\zeta|},\) for \(v\) as above, and it follows that
\[
\text{dist}_h(v(\zeta), v(\zeta')) \leq C\delta(\zeta, \zeta'),
\]
where \(\delta\) denotes the Poincaré metric on the unit disc.

The second essential tool in the proof of Theorem \([\text{I}]\) is a Theorem of Littlewood \([\text{L}]\) that asserts that any subharmonic function on the unit disc that is bounded form above, has radial boundary values a.e., see also \([\text{II}], \text{IV.10}\). It is extremely easy to see that in general there is no non-tangential limits a.e. Under hypotheses of normality non-tangential boundary values however exist, see \([\text{M}]\). If we assumed continuity of \(\rho\), \([\text{M}]\) would apply due to \((2.3)\). We will simply have to use the argument of normality, one step later.

First, we state a generalization of Littlewood’s Theorem (where instead of taking limits along radii one takes limits along rays). A very convenient reference (where much more is done) is \([\text{D}]\). Let \(\lambda\) be a subharmonic function defined on \(\Delta\) that is bounded from above. Let \(\nu \in (-\frac{\pi}{2}, \frac{\pi}{2})\), we shall say that \(\lambda\) has boundary at \(e^{i\theta}\) in the \(\nu\)-direction if \(\lambda(e^{i\theta} - t e^{i(\theta - \nu)})\) has a limit as \(t \searrow 0\). Denote this limit by \(\lambda^{*}_{\nu}(e^{i\theta})\). Of course \(\nu = 0\) corresponds to the radial limit. The result of \([\text{D}]\) is that for any fixed \(\nu\) the limit \(\lambda^{*}_{\nu}(e^{i\theta})\) exists a.e. Moreover if \(\nu' \neq \nu\), \(\lambda^{*}_{\nu} = \lambda^{*}_{\nu'}\) a.e. The last fact is because the heart of the matter is to show that a Green potential has boundary value 0 a.e. (in the direction of \(\nu\)). See Corollary 1 on p. 520 in \([\text{D}]\).

After these preliminaries we are ready for the proof. Let \(\rho\) be a strictly \(J\)-psh function defined on a neighborhood of \(\bar{u}(\Delta)\). Taking \(\lambda = \rho \circ u\) above, we see that if \(\nu \in (-\frac{\pi}{2}, \frac{\pi}{2})\), \(\rho \circ u\) has boundary value in the \(\nu\)-direction a.e.

Let \(\varphi = (\varphi_1, \ldots, \varphi_N)\) be a \(C^1\) injective map from a neighborhood of \(\bar{\Omega}\) into \(\mathbb{R}^N\) (with \(N\) large enough). We can apply the same reasoning to the functions \(\lambda_{\varphi} := \rho \circ u + \epsilon \varphi_j \circ u\), \(j = 1, \ldots, N\) with \(\epsilon > 0\) small enough. It follows that \(u\) has boundary value \(u_{\nu}^{*}\) a.e in the \(\nu\)-direction, and as discussed above the boundary values for any two values of \(\nu\) will agree a.e.

In order to get non-tangential boundary values for \(u\) we have to use a normality argument now. Let \(F\) be a dense countable subset of \((-\frac{\pi}{2}, \frac{\pi}{2})\). There exists a set \(\Theta\) of full measure in \([0, 2\pi)\) such that for all \(\nu \in F\) and all \(\theta \in \Theta\), \(u_{\nu}^{*}(e^{i\theta})\) exists and does not depend on \(\nu\), we thus simply write \(u^{*}(e^{i\theta})\). At any such point \(e^{i\theta}\), \(u\) has a non-tangential boundary value. Indeed, if \(0 < \alpha < 1\) and \(\varepsilon > 0\) are fixed, there are \(\nu_1, \ldots, \nu_N \in F\) such that the the union of the rays from \(e^{i\theta}\) in the \(\nu_j\)-directions \((j = \{1, \ldots, N\})\) is \(\varepsilon\) dense in the Poincaré metric in \(e^{i\theta} \Omega\). There exists \(\eta > 0\) such that for \(\zeta\) on any of these \(N\) rays in the \(\nu_j\)-directions \(\text{dist}(u(\zeta), u^{*}(e^{i\theta})) \leq \varepsilon\), if \(1 - \eta < |\zeta| < 1\). Then \((2.3)\) shows that for \(\zeta \in e^{i\theta} \Omega\), \(\text{dist}(u(\zeta), u^{*}(e^{i\theta})) \leq (C+1)\varepsilon\), if \(1 - \eta < |\zeta|\).

\[\square\]

2.3. Pluripolarity

Pluripolarity refers to the \(-\infty\) sets of \(J\)-psh functions. More precisely, a subset \(V\) of an almost complex manifold \((X,J)\), \(J\) continuous, is called \textit{locally complete pluripolar} if for any point \(p \in X\) there exists a neighborhood \(U \ni p\) and a \(J\)-psh function \(\rho\) in \(U\), not identically equal \(-\infty\), such that \(M \cap U = \{x \in U : \rho(x) = -\infty\}\). If as \(U\) can be taken a neighborhood of \(V\) then \(V\) is called \textit{complete pluripolar}, or globally complete pluripolar.

The following examples of pluripolarity require some smoothness of \(J\).
- Chirka’s function: If $J$ is a $C^1$-regular almost complex structure defined near $0$ in $\mathbb{R}^{2n} = \mathbb{C}^n$, and if $J(0) = J_{st}$ (the standard complex structure), then for $A > 0$, large enough, the function $\log |z| + A|z|$, is $J$-psh near $0$ ([1R], Lemma 1.4). That implies that a point is a complete pluripolar set. The result extends to almost complex structures that are Lipschitz-continuous, see §5.2 below.

- Elkhadhra’s theorem: Let $J$ be a $C^{1,\text{Lip}}$-regular almost complex structure defined near $0$ in $\mathbb{R}^{2n}$ and let $V$ be a germ at zero of a $J$-complex submanifold of class $C^{2,\text{Lip}}$. Then there exist a neighborhood of $U$ of $0$ and a $J$-psh function $\rho$ defined in $U$ such that $\rho$ is of class $C^{2,\text{Lip}}$ on $U \setminus V$ and $V \cap U = \rho^{-1}(-\infty)$, see [E]. The function $\rho$ is of the form $-\log(-\log|f|^2) + A|z|^2$, where $z = (z_1, ..., z_n)$ are local complex coordinates centered at $0$ such that $J(0) = J_{st}$ in these coordinates and $|f|^2 = \sum_{j=1}^{p} |f_j|^2$, for appropriate functions $f_j$’s vanishing on $V$. I.e., $V$ is a $\log \log$-polar set of a $J$-psh function, unlike the case of a point, where the Chirka function has a $\log \log$-pole. Elkhadhra’s Theorem generalizes the result of [Ro1] for $J$-holomorphic curves in which already a “log log” function was used. The difference between log singularities and lesser singularities (that is essential in classical complex pluripotential theory) plays an important role in this paper, see Section 4.

**Note**. The proof of Elkhadhra’s theorem requires $C^{1,\text{Lip}}$ smoothness of $J$. In [Ro2] (where a correction for the smoothness hypothesis in [Ro1] is made) a possibly more transparent proof of pluripolarity of curves is given. That proof can be adapted to give a different proof of Elkhadhra’s Theorem, but it also very clearly requires $C^{1,\text{Lip}}$ regularity of $J$. It is clear that $J$-holomorphic curves can fail to be locally complete pluripolar if the almost complex structure is only of class $C^\alpha$ (for some $0 < \alpha < 1$). Indeed two $J$-holomorphic maps $u_1$ and $u_2$, both imbeddings, can be such that they do not coincide on any neighborhood of $0$, but they coincide on an open set whose closure contains $0$. Any $J$-psh function $\rho$ with value $-\infty$ on one of the discs near $u_1(0) = u_2(0)$ will also be $-\infty$ on the other disc. So the case left unclear is the case of structures $J \in C^{\text{Lip}}$ (or, even of smoothness $C^1$ or $C^{1,\alpha}$, $0 < \alpha < 1$). Are $J$-holomorphic curves still locally complete pluripolar and more generally does Elkhadhra’s theorem extend when $J$ is only Lipschitz-continuous?

### 3. Geometric Inclusion and Privalov-type Theorem

#### 3.1. Stratified pluripolar sets

A Privalov-type uniqueness theorem will be obtained in this paper as an immediate corollary of a more general and more ”geometric” statement, we call it the ”geometric inclusion”. Let us start with the following

**Definition 3.1.** Let $(X, J)$ be an almost complex manifold, $J$ continuous. Let $V \subset X$ be a closed subset. We say that $V$ admits a **locally complete pluripolar stratification** if $V$ can be represented as the union $V = V_k \cup V_{k-1} \cup ... \cup V_0$ where $V_0$ is closed, locally complete pluripolar in $(X, J)$ and $V_j$ is closed, locally complete pluripolar in $X \setminus \bigcup_{i=0}^{j-1} V_i$ for $j = 1, ..., k$.

In order to justify this definition let us give two examples.

1. Any $C^{2,\text{Lip}}$, $J$-complex submanifold in $(X, J)$ is locally complete pluripolar, provided that $J \in C^{1,\text{Lip}}$, see above. Especially we shall use the fact that the diagonal $\mathbb{D}$ in the Cartesian square $(X \times X, J \oplus J)$ of an almost complex manifold is locally complete pluripolar, if $J$ is of class $C^{1,\text{Lip}}$. 
2. If $J$ is an almost complex structure of class $C^{2,\alpha}$ and $V$ is a closed $J$-complex curve in $X$ then the singular part $V_0$ of $V$ is discrete and therefore locally complete pluripolar. $V_1 := V \setminus V_0$ is of class $C^{3,\alpha}$ (we only need $C^{2,Lip}$) and thus is locally the $-\infty$ set of a $J$-psh function. Therefore a $J$-complex curve admits a locally complete pluripolar stratification.

3. These two examples could be generalized as follows. Here for simplicity we shall just assume the data to be smooth. Let us define a $J$-analytic subset of an almost complex manifold $(X, J)$ as a stratified set $A = A_k \cup A_{k-1} \cup \ldots \cup A_0$, where $A_0$ is a $J$-complex submanifold in $X$ and $A_j$ is a $J$-complex submanifold of $X \setminus \bigcup_{l=0}^{j-1} A_l$. A motivation for such a definition obviously comes from the structure theorem for the usual analytic sets. From Elkhadra’s Theorem it follows that a $J$-analytic set is locally complete pluripolar stratified.

### 3.2. Geometric inclusion

In §2.1 we have given boundedness hypotheses that guarantee the existence of non-tangential limits. In this section, we proceed differently. We shall no longer require that our mappings are bounded. Moreover, unlike in Definition [2.1] we shall need only the existence of $\lim u(\zeta)$, as $\zeta$ approaches to $e^{i\theta}$, $\zeta \in e^{i\theta} \Omega_\alpha$, for some $\alpha$ possibly depending on $\theta$. In that case we say that $u$ has a restricted non-tangential limit at $e^{i\theta}$.

**Theorem 3.1.** Let $V$ be a locally complete pluripolar stratified subset of an almost complex manifold $(X, J)$, with $J$ continuous, and let $u : \Delta \to X$ be a $J$-holomorphic map. Assume that for $e^{i\theta}$ in a set $E$ of positive measure in $\partial \Delta$ the mapping $u$ has a restricted non-tangential limit $u^*(e^{i\theta})$ and that $u^*(e^{i\theta}) \in V$. Then $u(\Delta) \subset V$.

**Proof.** For a set $E \subset \partial \Delta$ and $0 < r < 1$, let

$$\Gamma_r(E, \alpha) = \bigcup_{e^{i\theta} \in E} e^{i\theta} \Omega_\alpha \cap \{|\zeta| > r\}. \quad (3.1)$$

This set is therefore made of truncated cones of fixed aperture with vertices at the points $e^{i\theta} \in E$. In the sequel we fix the minimal $j$ such that the set of $e^{i\theta}$ with $u^*(e^{i\theta}) \in V_j \setminus \bigcup_{l=0}^{j-1} V_l$ is of positive length. We denote this set still by $E$. If $j = 0$ we denote by $E$ the set of positive length such that $u^*(e^{i\theta}) \in V_0$ for $e^{i\theta} \in E$.

**Step 1.** Let $E$ be as just defined. Then there exist $p \in V_j \setminus \bigcup_{l=0}^{j-1} V_l$, a neighborhood $W$ of $p$ in $X \setminus \bigcup_{l=0}^{j-1} V_l$ and a $J$-psh function $\rho$ defined on $W$, bounded from above, with $V_j \cap W = \rho^{-1}(-\infty)$, and there exist a closed set $E_0 \subset \partial \Delta$ of positive length, $r \in (0,1)$, and $\alpha \in (0,1)$ such that $u(\Gamma_r(E_0, \alpha))$ is a relatively compact subset of $W$, and for all $e^{i\theta} \in E_0$ $u^*(e^{i\theta}) \in V_j \cap W$.

The mapping $u^* : E \to X$ is measurable as a pointwise limit of measurable mappings. By Lusin’s theorem, we can assume that $E$ is a closed set and that $u^*$ is continuous on $E$. Fix $e^{i\theta_0} \in E$ such that $E$ has positive length in any neighborhood of $e^{i\theta_0}$, set $p = u^*(e^{i\theta_0})$. Let $W_0$ and $W$ be neighborhoods of $p$, $W_0 \subset W$, $W \cap \bigcup_{l=0}^{j-1} V_l = \emptyset$ such that there exists a $J$-psh function $\rho$ defined on $W$ such that $V_j \cap W$ coincides with $\rho^{-1}(-\infty)$ in $W$. Restricting $E$ to a small neighborhood of $e^{i\theta_0}$ we can assume that for $e^{i\theta} \in E$ one has $u^*(e^{i\theta}) \in W_0$. For each $e^{i\theta} \in E$ there exists $\alpha > 0$ and $r < 1$ such that $u(e^{i\theta} \Omega_\alpha \cap \{|\zeta| > r\}) \subset W_0 \subset W_0$, where $\alpha$ and $r$ depend on $\theta$. To finish the proof of the claim we simply need to have $\alpha$
and \( r \) no longer depending on \( \theta \). For any integer \( k > 0 \), set
\[
E_k = \{ e^{i\theta} \in E : u(e^{i\theta} \Omega_k \cap \{|\zeta| > 1 - \frac{1}{k}\}) \subset \overline{W_0}\}.
\]
Each \( E_k \) is a closed hence measurable set and \( E = \bigcup_{k \geq 1} E_k \). By countable additivity there is a set \( E_k \) that has positive measure. Set \( E_0 = E_k, \, \alpha = \frac{1}{k}, \, r = 1 - \frac{1}{k} \). Then \( u(\Gamma_r(E_0, \alpha)) \subset \overline{W_0}\), and that establishes the Step 1.

**Step 2.** If for \( e^{i\theta} \) in a set \( E_0 \) of positive length in \( \partial \Delta \) one has that \( u^*(e^{i\theta}) \in V_j \cap W \) for some relatively compact \( W \subset X \setminus \bigcup_{i=0}^{j-1} V_i \) then \( u(U) \subset V_j \) for some non-empty open subset \( U \subset \Delta \).

One can replace \( E_0 \) by a subset small enough if needed, so that \( \Gamma_r(E_0, \alpha) \) is a simply connected domain with rectifiable boundary \( \gamma \). This is well explained in §2 of Chapter X in [10]. Let \( \chi \) be a conformal mapping from the unit disc \( \Delta \) onto \( \Gamma_r(E_0, \alpha) \). This map extends continuously to the closed unit disc. Set \( F = \{ e^{i\nu} \in \partial \Delta : \chi(e^{i\nu}) \in E_0 \} \). Then \( F \) is a set of positive length in \( \partial \Delta \) by F. and M. Riesz’s theorem on conformal mappings, see Theorem VIII.26 p. 318 in [1].

For \( \epsilon > 0 \) small enough, define \( \rho_\epsilon \) on \( \Delta \) by: \( \rho_\epsilon(\zeta) = \rho \circ u((1-\epsilon)\chi(\zeta)) \). Note that \( (1-\epsilon)\Gamma_r(E_0, \alpha) \subset \Gamma_{(1-\epsilon)r}(E_0, \alpha) \). Then, the functions \( \rho_\epsilon \) are subharmonic functions on \( \Delta \) that are uniformly bounded from above, and for any \( e^{i\nu} \in F \), by upper semi-continuity of \( \rho, \rho_\epsilon(e^{i\nu}) \) tends to \(-\infty \) as \( \epsilon \to 0 \). Indeed \( (1-\epsilon)\chi(e^{i\nu}) \) approaches \( \chi(e^{i\nu}) \in E_0 \) radially. So if \( e^{i\nu} \in F \), \( \chi(e^{i\nu}) = e^{i\theta} \in E_0 \), then \( u((1-\epsilon)\chi(e^{i\nu})) \to u^*(e^{i\theta}) \) as \( \epsilon \to 0 \). By the mean value property, \( \rho_\epsilon \) tends to \(-\infty \) uniformly on compact subsets of \( \Delta \). Since \( \rho \circ u(\zeta) = \rho_\epsilon(\chi^{-1}(\frac{1}{1-\epsilon}\zeta)) \) (for \( \epsilon < 1 - |\zeta| \)), \( \rho \circ u \equiv -\infty \) on \( \Gamma_r(E_0, \alpha) \), so \( u(\Gamma_r(E_0, \alpha)) \subset V_j \cap W \).

**Step 3.** In the conditions of Step 2 there exists a connected dense open subset \( A \subset \Delta \) such that \( u(A) \subset V_j \).

Let \( A \) be the set of points \( \zeta \in \Delta \) such that \( u \) maps a neighborhood of \( \zeta \) into \( V_j \), let \( \partial A \) be its boundary. Clearly \( A \) is an open subset of \( \Delta \). For each \( l \leq j \), let \( S_l = \{ \zeta \in \partial A : u(\zeta) \in V_l \} \). If \( \zeta \in S_l \), there is a \( J \)-psh function \( \rho \) defined near \( u(\zeta) \), that near this point is \(-\infty \) exactly on \( V_l \). Then \( \rho \circ u = -\infty \) near \( \zeta \) on \( S_l \), but is not identically \(-\infty \) since \( \zeta \in \partial A \) (more precisely: if \( l < j \) because \( \zeta \) is in the closure of \( A \), if \( l = j \) because \( \zeta \notin A \)). Hence each \( S_l \) is a polar subset of \( \Delta \) and so is their union that is \( \partial A \). Since \( \partial A \) is a closed polar subset of \( \Delta \), \( A \) is a connected dense open subset of \( \Delta \), as claimed.

By continuity, \( u(\Delta) \subset \bigcup_{i=0}^{j} V_i \subset V \). The proof of Theorem 3.1 is therefore completed.

\( \square \)

### 3.3. The diagonal and proof of Privalov-type Theorem

Let us derive from Theorem 3.1 the Privalov-type Theorem 1.3 from the Introduction. Let \( (\Omega_1, \rho_1) \) and \( (\Omega_2, \rho_2) \) be relatively compact domains containing \( u_1(\Delta) \) and \( u_2(\Delta) \) together with strictly \( J \)-psh functions, which clearly could be supposed to be bounded from above. Then \( \Omega := (\Omega_1 \times \Omega_2, J := J \oplus J, \lambda_1 := \rho_1 + \rho_2) \) will be the same type data for \( u := (u_1, u_2) : \Delta \to (X \times X, \tilde{J}), \) i.e., \( u \) is also bounded. Remark further that \( u := (u_1, u_2) : \Delta \to \Omega \subset X \times X \) is \( \tilde{J} \)-holomorphic.

With \( \Omega \) and \( \rho \) just defined we consider the intersection \( V := \mathbb{D} \cap \Omega \) of the diagonal \( \mathbb{D} \) of \( X \times X \) with \( \Omega \). This is a \( J \)-complex submanifold of \( \Omega \). Note that if for at least one \( e^{i\theta} \) the boundary values \( u_1^*(e^{i\theta}) \) and \( u_2^*(e^{i\theta}) \) do coincide then \( V \) is nonempty.
By the theorem of Elkhadhra $V$ is locally complete polar and therefore our problem is reduced to the study of boundary values of a $J$-holomorphic map lying on a locally complete pluripolar subset of an almost complex manifold. Theorem 3.1 is now applicable to $u$ and gives us that $u(\Delta) \subset V$. Theorem 3.3 follows.

4. Blaschke Condition for a Single Pseudoholomorphic Mapping

4.1. The Blaschke Condition: proof of Theorem 1.2

Let $\lambda_0$ be a strictly $J$-psh function defined on a neighborhood of $u(\Delta)$. Using Chirka’s function, we claim that there exists a $J$-psh function $\lambda$ defined on $\Omega$, bounded from above and such that near $p$, using local coordinates: $\lambda(z) \leq \log |z - p|$. Indeed, using local coordinates near $p$ such that $J(p) = J_{st}$, set

$$\lambda(z) = \chi(z)(\log |z - p| + A|z - p|) + B\lambda_0 - C,$$

where $\chi$ is a cut off function equal to 1 on a neighborhood of $p$. $A$ is first taken large enough in order that $\log |z - p| + A|z - p|$ is $J$-psh near $p$, then $B$, then $C$ have just to be taken large enough. We set $\rho = \lambda \circ u$. The Blaschke condition follows from the $\log$ singularity due to the following two facts:

(a) If $\rho$ is a subharmonic function defined near $\zeta_0$ in $\mathbb{C}$ and if for $\zeta$ close to $\zeta_0$ one has $\rho(\zeta) \leq \log |\zeta - \zeta_0| - C$ for some constant $C$, then $\Delta(\rho)(\{\zeta_0\}) \geq 2\pi$, where $\Delta(\rho)$ denotes the positive measure that is the distributional Laplacian of $\rho$. An immediate proof for this classical fact consists in using that, near $p$, $\rho$ is the decreasing limit as $n \to \infty$ of the functions $\rho_n = \max \{\rho, (1 - \frac{1}{n})\log |\zeta - \zeta_0| - n\}$, for which obviously $\Delta(\rho_n)(\{\zeta_0\}) = 2\pi(1 - \frac{1}{n})$.

(b) If $\rho$ is a subharmonic function on $\Delta$ that is bounded from above, then

$$\int_{\Delta} (1 - |\zeta|)\Delta \rho(\zeta) d\xi d\eta < +\infty . \tag{4.1}$$

This also is classical and it follows from the representation formula

$$\rho(0) = \frac{1}{2\pi} \int_0^{2\pi} \rho(e^{i\theta})d\theta - \frac{1}{2\pi} \int_{\Delta} \Delta(\rho)(\zeta) \log \frac{1}{|\zeta|} d\xi d\eta, \tag{4.2}$$

(see e.g. [Ga] XV.2, page 294), and from the fact that $\log \frac{1}{|\zeta|} \sim 1 - |\zeta|$ near the boundary of $\Delta$.

We can now conclude the proof. We apply (a) and (b) with $\rho = \lambda \circ u$. For any $\zeta_0$ in $\Delta$ such that $u(\zeta_0) = p$, we have for $\zeta$ close to $\zeta_0$ that $|u(\zeta) - p| \leq C|\zeta - \zeta_0|$. Hence, $\rho \leq \log |\zeta - \zeta_0| + C$, therefore, by (a), $\Delta(\rho)(\{\zeta_0\}) \geq 2\pi$. Then (4.1) gives us the desired Blaschke condition.

4.2. An example of a “bad diagonal”

In §4.1, it has been essential that the Chirka function has a logarithmic singularity, not only a pole. In [Ro2], it has been shown that $J$-holomorphic curves are not in general $-\infty$ of $J$-psh functions with logarithmic singularity. In §3.3 we applied Elkhadhra’s Theorem to a very special case, the diagonal in the product of the space with itself. Unfortunately, even in that special case, there may not exist plurisubharmonic functions with logarithmic singularity. We are thus unable to extend Theorem 1.2 to establish that the set of points where two $J$-holomorphic maps coincide, is a Blaschke sequence, under appropriate boundedness assumption.
We use the example given in [Ro2]. In that example a smooth almost complex structure $J$ on $\mathbb{C}^2 \times \mathbb{C}^2$ is constructed such that:

(i) $\{z_2 = 0\}$ is $J$-holomorphic, but is not the $-\infty$ of a $J$-psh function with logarithmic singularity, not even locally.

(ii) For every $J$-holomorphic disc $\zeta \mapsto (u_1(\zeta), u_2(\zeta))$, $u_1$ is holomorphic.

(iii) $\zeta \mapsto (u_1(\zeta), 0)$ is $J$-holomorphic if $u_1$ is holomorphic. (But $\zeta \mapsto (u_1(\zeta), a)$, for fixed $a \neq 0$ is not).

We now consider the product structure $J \oplus J$ on $\mathbb{C}^2 \times \mathbb{C}^2$, and we let $\mathbb{D}$ be the diagonal.

**Proposition 4.1.** There is no $J \oplus J$-plurisubharmonic function with logarithmic singularity along $\mathbb{D}$.

Proof. The imbedding from $(\mathbb{C}^2, J)$ into $(\mathbb{C}^2 \times \mathbb{C}^2, J \oplus J)$ defined by

$$\varphi : (z_1, z_2) \mapsto (z_1, z_2, z_1, 0)$$

is $(J, J \oplus J)$-holomorphic. Indeed, for any $J$-holomorphic map $u, \varphi \circ u$ is $J \oplus J$-holomorphic, as it follows immediately from (ii) and (iii). If $\rho$ is a $J \oplus J$-plurisubharmonic function on $(\mathbb{C}^2 \times \mathbb{C}^2, J \oplus J)$, that we shall suppose to be defined near a point $(z_1, 0, z_1, 0)$, then $\rho \circ \varphi$ is $J$-psh. If $\rho$ had a logarithmic singularity along $\mathbb{D}$ then $\rho \circ \varphi$ would have a logarithmic singularity along $z_2 = 0$, which is not possible.

5. Appendix

5.1. Existence of $J$-Complex Curves in Continuous Structures

We consider a continuous operator valued function $J$ in the unit ball $B$ of $\mathbb{R}^{2n}$. I.e., $J : B \to \text{Mat}(2n \times 2n, \mathbb{R})$. $J$ is an almost complex structure if $J^2(x) \equiv -\text{Id}$. We will use the notation $W^{1,p}$ for the Sobolev spaces of $L^p$ functions with first derivatives in $L^p$.

**Definition 5.1.** A $W^{1,1}_{\text{loc}} \cap C^0$-map $u : \Delta \to B$ is said to be $J$-holomorphic if it a.e. satisfies

$$\frac{\partial u}{\partial x} + J(u(z))\frac{\partial u}{\partial y} = 0. \quad (5.1)$$

The image $C = u(\Delta)$ is called a $J$-complex disk. $J_u(z) := J(u(z))$ can be considered as a matrix valued function on the unit disk and therefore can be viewed as a complex linear structure on the trivial bundle $E := \Delta \times \mathbb{R}^{2n}$. I.e., $J_u(z) \in C^0(\Delta, \text{End}(\mathbb{R}^{2n}))$ satisfying $J(z)^2 \equiv -\text{Id}$. The mapping $u$ can be viewed as a section of this bundle. It is known that if $J \in C^0$ then $J$-holomorphic maps belong to $W^{1,p}_{\text{loc}}$ for all $p < \infty$ (see [ISh1] Lemma 1.2.2) and therefore to $C^\alpha$ for all $\alpha < 1$ by Sobolev imbedding theorem. $J$-complex curves in continuous structures have some nice properties. For example, a Gromov compactness theorem is valid for them, see [ISh2]. However the question of existence of complex curves in continuous structures to our knowledge was never discussed in the literature, therefore we do this here.

Equation (5.1) can be rewritten as

$$\frac{\partial u}{\partial \bar{z}} + Q_u(z)\frac{\partial u}{\partial z} = 0, \quad (5.2)$$

where $\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial u}{\partial x} + J_u\frac{\partial u}{\partial y})$, $\frac{\partial u}{\partial z} = \frac{1}{2}(\frac{\partial u}{\partial x} - J_u\frac{\partial u}{\partial y})$ and

$$Q_u(z) = [J(z) + J_{\text{st}}]^{-1}[J(z) - J_{\text{st}}]. \quad (5.3)$$
Remark that $\bar{Q}$ anti commutes with $J_{st}$ and therefore is a $\mathbb{C}$-antilinear operator. Thus (5.2) can be understood as an equation for $\mathbb{C}^n$-valued map (or section) $u$. Usually it is better to consider the conjugate operator $Q$ and write (5.2) in the matrix form

$$\frac{\partial u}{\partial \bar{z}} + Q J(u(z)) \overline{\frac{\partial u}{\partial z}} = 0.$$  (5.4)

We shall look for the solutions of (5.4) in the form

$$u(z) = -T_{CG} Q(J_u(z)) \overline{\frac{\partial u}{\partial z}} + H =: \Phi(u) + H,$$  (5.5)

where $T_{CG}$ is the Cauchy-Green operator (convolution with $\frac{1}{\pi z}$), and $H$ is a holomorphic function. We shall scale the norm on $W^{1,p}(\Delta, \mathbb{C}^n)$, so that for any $f \in W^{1,p}(\Delta, \mathbb{C}^n)$

$$\|f\|_{L^\infty(\Delta)} \leq \|f\|_{W^{1,p}(\Delta)}.$$  (5.6)

**Lemma 5.1.** Suppose that $J(0) = J_{st}$, so $Q(J(0)) = 0$. Let $2 < p < \infty$, there exists $q > 0$ such that if $\|Q(J)\|_{L^\infty(B)} = q << 1$, then for $a, b \in \mathbb{C}^n$ small enough, the operator $\Phi_{a,b}$ defined by

$$\Phi_{a,b}(u) = \Phi(u) - \Phi(u)[0] - 2z(\Phi(u)[1/2] - \Phi(u)[0]) + a + 2z(b-a)$$  (5.7)

is an operator from the closed unit ball $B \subset W^{1,p}(\Delta, \mathbb{C}^n)$ into itself, that has a fixed point.

**Proof.** Unfortunately the lemma is not obtained directly by using the contraction principle, instead one uses an argument already in [NW]. The proof consists in two steps:

(i) Let $B_\infty$ be the closed unit ball in $L^\infty(\Delta, \mathbb{C}^n)$. Note that $B \subset B_\infty$ due to (5.6). For any $v \in B_\infty$, one defines the operator

$$\Phi^v_{a,b}(u) = \Phi^v(u) - \Phi^v(u)[0] - 2z(\Phi^v(u)[1/2] - \Phi^v(u)[0]) + a + 2z(b-a),$$  (5.8)

where

$$\Phi^v(u) := -T_{CG} Q(J_v(z)) \overline{\frac{\partial u}{\partial z}},$$  (5.9)

thus taking $J(v(z))$ instead of $J(u(z))$. By the contraction principle, one proves that this operator has a unique fixed point $u = T(v)$ in $B$.

(ii) Then one proves that $T$ restricted to $B$, itself has a fixed point $u$, by applying Schauder fixed point theorem. Then $\Phi^u_{a,b}(u) = u$, i.e. $\Phi_{a,b}(u) = u$, and that establishes the Lemma.

(i) Let $C_p$ be the norm of the Cauchy-Green operator as a linear map from $L^p(\Delta) \to W^{1,p}(\Delta)$. If $q < \frac{1}{8C_p}$ then for $u_1, u_2 \in B$

$$\left\|\Phi^v_{a,b}(u_1) - \Phi^v_{a,b}(u_2)\right\|_{W^{1,p}} \leq 4 \left\|T_{CG} Q(J_v) \left[\frac{\partial u_1}{\partial \bar{z}} - \frac{\partial u_2}{\partial \bar{z}}\right]\right\|_{W^{1,p}} \leq$$

$$\leq 4qC_p \|u_1 - u_2\|_{W^{1,p}} \leq \frac{1}{2} \|u_1 - u_2\|_{W^{1,p}}.$$

Since $\Phi^v_{a,b}(0) = a + 2z(b-a)$, for $a$ and $b$ small enough $\Phi^v_{a,b}$ is a contracting map from $B$ into itself. Hence it has a unique fixed point $u = T(v)$.

(ii) Let us prove now that $v \mapsto u = T(v)$ defines a continuous (in $L^\infty$-topology) map from $B_\infty$ to $B \subset B_\infty$. Indeed, we have

$$u = \Phi^v_{a,b}(u) = \Phi(u) - \Phi(u)[0] - 2z(\Phi(u)[1/2] - \Phi(u)[0]) + a + 2z(b-a)$$
and thus
\[ \|u\|_{W^{1,p}} = \|\Phi_{a,b}^v u\|_{W^{1,p}} \leq 4qC_p \|u\|_{W^{1,p}} + \|a + 2z(b-a)\|_{W^{1,p}}. \]

We get therefore
\[ \|u\|_{W^{1,p}} \leq \frac{1}{1 - 4qC_p} \|a + 2z(b-a)\|_{W^{1,p}}, \quad (5.10) \]
which means that for \(a\) and \(b\) small enough \(T\) maps \(\mathcal{B}_\infty\) to \(\mathcal{B}\).

Let \((v_n)\) be a converging sequence in \(\mathcal{B}_\infty\) with limit \(v\), and \(u_n = T v_n\). (5.10) tells us that \(u_n\) are bounded in \(W^{1,p}(\Delta)\). Therefore the sequence \(\frac{\partial u_n}{\partial z}\) is a bounded in \(L^p(\Delta)\). Let \(u = T v\), i.e., \(u\) is the unique fixed point of \(\Phi_{a,b}^v\). Write:
\[ \|u - u_n\|_{W^{1,p}} = \|\Phi_{a,b}^v(u) - \Phi_{a,b}^v(u_n)\|_{W^{1,p}} \leq 4\|\Phi^v(u) - \Phi^v(u_n)\|_{W^{1,p}} = 4\left| TCGQ(J_v) \left[ \frac{\partial u}{\partial z} - \frac{\partial u_n}{\partial z} \right] \right|_{W^{1,p}} + 4\left| TCGQ(J_v) - Q(J_{v_n}) \right|_{W^{1,p}} \leq 4qC_p \|u - u_n\|_{W^{1,p}} + 4\left| TCGQ(J_v) - Q(J_{v_n}) \right|_{W^{1,p}}. \]

Therefore
\[ \|u - u_n\|_{W^{1,p}} \leq \frac{4}{1 - 4qC_p} \left| TCGQ(J_v) - Q(J_{v_n}) \right|_{W^{1,p}} \rightarrow 0, \quad (5.11) \]
because \(Q(J_{v_n})\) converges uniformly to \(Q(J_v)\). The continuity of \(T\) is proved.

Take a closure \(\bar{\mathcal{B}}\) of \(\mathcal{B}\) in \(\mathcal{B}_\infty\). This is a convex compact subset of \(\mathcal{B}_\infty\) in \(L^\infty\)-topology. By the Schauder fixed point Theorem applied to \(T\), \(T\) has a fixed point \(u \in \bar{\mathcal{B}}\) and this \(u\), as we had proved, belongs to \(\mathcal{B}\) in fact.

\[ \square \]

**Corollary 5.1.** Let \(J\) be a continuous almost complex structure in \(\mathbb{R}^{2n}\). Then for any sufficiently close points \(a\) and \(b\) there exists a \(J\)-holomorphic map \(u : \Delta \rightarrow \mathbb{R}^{2n}\) passing through them.

The result follows from Theorem 5.1. Indeed, for the fixed point \(u\) of \(\Phi_{a,b}\) we get
\[ \frac{\partial u}{\partial z} + Q_J(u)\frac{\partial u}{\partial z} = 0, \quad u(0) = a \text{ and } u(\frac{1}{2}) = b. \]

The needed smallness of \(Q\) (equivalently smallness of \(J - J_{st}\), given \(J(0) = J_{st}\), is obtained by simple rescaling.

\[ \square \]

**Remark 1.** The standard maximum principle for \(J\)-psh functions in case of continuous \(J\)’s follows immediately.

### 5.2. Plurisubharmonicity and Pluripolarity in Lipschitz structures.

If an almost complex structure \(J\) is not differentiable, the operator \(dd_J^c\) (whose definition requires one differentiation of \(J\)) does not have an immediate meaning. For studying Lipschitz structures, instead of trying to define \(dd_J^c\) we shall proceed by approximation.

**Lemma 5.2.** Let \(J\) be a Lipschitz-continuous almost complex structure defined on a (smooth) manifold \(X\). Let \(J_k\) be a sequence of \(C^1\) almost complex structures on \(X\), with \(C^1\)-norms uniformly bounded, and converging uniformly to \(J\) as \(k \rightarrow \infty\). Let \(\rho\) be a \(C^2\) function defined on \(X\). If \(\rho\) is \(J_k\)-psh, for \(k:\) large enough, then \(\rho\) is \(J\)-psh.
Proof. The problem is local, we can always restrict our attention to a relatively compact domain in $X$, and we equip $X$ with some Riemannian metric. We shall use the following characterization of $C^1$ (but not $C^2$) subharmonic functions $\mu$, on the unit disc $\Delta$:

$$-\int_{\Delta} d\phi \wedge d^c \mu \geq 0,$$  

(5.12)

for all $\phi \in C^1_0(\Delta)$, with $\phi \geq 0$. (Here $d^c = d^c_J\mu$). The above condition gives $dd^c \mu \geq 0$, in the sense of distributions.

Let $u : \Delta \rightarrow (\mathbb{R}^{2n}, J)$ be a $J$-holomorphic map from $\Delta$ into $B$, so $u$ is of class $C^{1,\beta}$ for all $\beta < 1$. We can assume $u$ to be $C^{1,\beta}$ up to the boundary. We wish to show that $\rho \circ u$ is subharmonic. Since $u$ is $J$-holomorphic $d^c \rho \circ u = u^* d^c_J \rho$ (by the simple rules of differentiation and commutation of $d$ with the action of the almost complex structures).

So, we need to show that for any non-negative $C^1$ function $\phi$ with compact support in $\Delta$,

$$-\int_{\Delta} d\phi \wedge u^* d^c_J \rho \geq 0.$$  

(5.13)

By hypothesis we have $dd^c_J \rho(v,J_k v) \geq 0$, for all tangent vector $v$ at any point $p$. Let $\epsilon_k = \sup_p |J_k(p) - J(p)|$, so $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Since $\rho$ is of class $C^2$ and since the $C^1$ norms of $J_k$ are uniformly bounded, for some constant $K$ we have

$$dd^c_J \rho(v,Jv) \geq -K \epsilon_k \|v\|^2.$$  

(5.14)

The inequality (5.14) has a clear geometric meaning: there exists a constant $K_1 > 0$ such that for any germ of embedded $J$-holomorphic curve $u$ in $X$, oriented by $J$, $dd^c_J \rho$ induces on that curve a measure $\lambda_k$ that satisfies

$$\lambda_k \geq -K_1 \epsilon_k dm,$$  

(5.15)

where $dm$ denotes Euclidean area (Hausdorff) measure on the curve.

We now prove (5.13). By uniform convergence of $J_k$,

$$-\int_{\Delta} d\phi \wedge u^* d^c_J \rho = \lim_{k \rightarrow \infty} -\int_{\Delta} d\phi \wedge u^* d^c_J \rho .$$

Set $I_k = -\int_{\Delta} d\phi \wedge u^* d^c_J \rho$. Because $u$ is not $C^2$, the differential form $u^* d^c_J \rho$ may fail to be $C^1$, since pull back of a differential form by a map uses a derivative of the map. However Stokes formula can still be applied and it gives us $I_k = \int_{\Delta} \phi u^* dd^c_J \rho$, where the right hand side clearly makes sense. See the preliminary remark in the proof of Lemma 1.2 in [IR]. Then (5.15) yields $I_k \geq -(K_1 \sup \phi) M \epsilon_k$, where $M$ is the area of $u(\Delta)$. Therefore (5.13) follows and the Lemma is proved.

We can now generalize the plurisubharmonicity result for the Chirka function to the case of Lipschitz structures.

Lemma 5.3. Let $J$ be a Lipschitz continuous almost complex structure defined near 0 in $\mathbb{C}^n$. If $J(0) = J_{st}$, then for $A > 0$ large enough, the function $\rho$ defined by $\rho(z) = \log |z| + A |z|$ is $J$-psh.
Proof. We can approximate $J$ uniformly by a sequence $(J_k)$ of almost complex structures with $J_k(0) = J_{st}$, and with uniformly bounded $C^1$ norms. By Lemma 1.4 in [IR], for $A$ large enough, for $k$ large enough $\rho$ is $J_k$-psh (that $A$ can be chosen independently of $k$ (large) is clear from the last lines of the proof of the lemma in [IR]). Since plurisubharmonicity needs to be proven only off the $-\infty$ set, and since $\rho$ is smooth except at 0, we can apply Lemma 5.2. That establishes Lemma 5.3.

□

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