UNIFIED PRODUCTS FOR LEIBNIZ ALGEBRAS. APPLICATIONS

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Abstract. Let \( g \) be a Leibniz algebra and \( E \) a vector space containing \( g \) as a subspace. All Leibniz algebra structures on \( E \) containing \( g \) as a subalgebra are explicitly described and classified by two non-abelian cohomological type objects: \( \mathcal{H}L_2^2(V, g) \) provides the classification up to an isomorphism that stabilizes \( g \) and \( \mathcal{H}L^2(V, g) \) will classify all such structures from the viewpoint of the extension problem - here \( V \) is a complement of \( g \) in \( E \). A general product, called the unified product, is introduced as a tool for our approach. The crossed (resp. bicrossed) products between two Leibniz algebras are introduced as special cases of the unified product: the first one is responsible for the extension problem while the bicrossed product is responsible for the factorization problem. The description and the classification of all complements of a given extension \( g \subseteq C \) of Leibniz algebras are given as a converse of the factorization problem. They are classified by another cohomological object denoted by \( \mathcal{H}A^2(h, g | (\triangleright, \triangleleft, \leftarrow, \rightarrow)) \), where \( (\triangleright, \triangleleft, \leftarrow, \rightarrow) \) is the canonical matched pair associated to a given complement \( h \). Several examples are worked out in details.

Introduction

Leibniz algebras were introduced by Bloh [8] under the name of D-algebras and rediscovered later on by Loday [20] as non-commutative generalizations of Lie algebras. A systematic study of Leibniz algebras was initiated in [21], [14]. Since then, Leibniz algebras generated a lot of interest and became a field of study in its own right. Several classical theorems known in the context of Lie algebras were extended to Leibniz algebras, there exists a (co)homology theory for them, the classification of certain types of Leibniz algebras of a given (small) dimension was recently performed, their interaction with vertex operator algebras, the Godbillon-Vey invariants for foliations or differential geometry was highlighted. For more details and motivations we refer to [5], [6], [7], [9], [10], [11], [12], [13], [15], [16], [17], [18], [20], [21], [25], [26], [27] and the references therein. The starting point of this paper is the following question:

Extending structures problem. Let \( g \) be a Leibniz algebra and \( E \) a vector space containing \( g \) as a subspace. Describe and classify the set of all Leibniz algebra structures \( [-, -] \) that can be defined on \( E \) such that \( g \) becomes a Leibniz subalgebra of \( (E, [-, -]) \).

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In this paper, we will provide an answer to the above problem as follows: first we will describe explicitly all Leibniz algebra structures on $E$ which contain $g$ as a subalgebra; then we will classify them up to a Leibniz algebra isomorphism $\varphi : E \to E$ that stabilizes $g$, that is $\varphi$ acts as the identity on $g$. The extending structures (ES) problem was formulated at the level of groups in [3] and for arbitrary categories in [1] where it was studied for quantum groups; recently we approached the problem for Lie algebras [4].

The ES problem is a very difficult one. If $g = \{0\}$, then the ES problem asks for the classification of all Leibniz algebra structures on an arbitrary vector space $E$, which is of course a hopeless problem for vector spaces of large dimension: the classification of all 3-dimensional (resp. 4-dimensional) Leibniz algebras was finished only recently in [13] (resp. [11]). For this reason, from now on we will assume that $g \neq \{0\}$. Even though the ES problem is a difficult one, we can still provide detailed answers to it in certain special cases which depend on the choice of the Leibniz algebra $g$ and mainly on the codimension of $g$ in $E$. It generalizes and unifies the extension problem and the factorization problem.

The extension problem asks for the classification of all extensions of $h$ by $g$ and it was first studied in [21] for $g$ abelian; in this case all such extensions are classified by the second cohomology group $\text{HL}^2(h, g)$ [21, Proposition 1.9]. The fact that $g$ is abelian is essential in proving this classification result. However, a classification result can still be proved in the non-abelian case and the classification object denoted by $\mathbb{H}L^2(h, g)$ will generalize the second cohomology group $\text{HL}^2(h, g)$ and it will be explicitly constructed as a special case of the ES problem. The main drawback of this construction is the fact that $\mathbb{H}L^2(h, g)$ does not arise as a cohomology group of a certain complex, it will be constructed using the theory of crossed products for Leibniz algebras. To conclude, the extension problem appears as a special case of the ES problem as follows: if in the ES problem we replace the condition "$g$ is a Leibniz subalgebra of $(E, [-, -])$" by a more restrictive one, namely "$g$ is a two sided ideal of $E$ and the quotient $E/g$ is isomorphic to a given Leibniz algebra $h"", then what we obtain is in fact the extension problem.

On the other hand, if in the ES problem we add the additional hypothesis "the complement of $g$ in $E$ is isomorphic to a given Leibniz algebra $h"" we obtain the factorization problem for Leibniz algebras which can be explicitly formulated as follows: describe and classify all Leibniz algebras $\Xi$ that factorize through two given Leibniz algebras $g$ and $h$, i.e. $\Xi$ contains $g$ and $h$ as Leibniz subalgebras such that $\Xi = g + h$ and $g \cap h = \{0\}$. Exactly as in the case of Lie algebras [23, Theorem 4.1], [22, Theorem 3.9] we will introduce the concept of a matched pair of Leibniz algebras and we will associate to it a bicrossed product which will be responsible for the factorization problem. However, in this case the definition of the concept of a matched pair for Leibniz algebras (Definition 4.5) is a lot more elaborated and difficult then the one for Lie algebras. The last section is devoted to the converse of the factorization problem. It consists of the following question introduced in [2] in the context of Hopf algebras and Lie algebras:

**Classifying complements problem.** Let $g \subseteq \mathfrak{E}$ be an extension of Leibniz algebras. If a complement of $g$ in $\mathfrak{E}$ exists, describe explicitly and classify all complements of $g$ in $\mathfrak{E}$, i.e. Leibniz subalgebras $h$ of $\mathfrak{E}$ such that $\mathfrak{E} = g + h$ and $g \cap h = \{0\}$.

The paper is organized as follows: in Section 2 we introduce the abstract construction of the unified product $g \times V$ for Leibniz algebras: it is associated to a Leibniz algebra
\( \mathfrak{g} \), a vector space \( V \) and a system of data \( \Omega(\mathfrak{g}, V) = (\prec, \succ, \leftarrow, \rightarrow, f, \{-,-\}) \) called an extending datum of \( \mathfrak{g} \) through \( V \). Theorem 2.3 establishes the set of axioms that has to be satisfied by \( \Omega(\mathfrak{g}, V) \) such that \( \mathfrak{g} \ltimes V \) with a given bracket becomes a Leibniz algebra, i.e. a unified product. In this case, \( \Omega(\mathfrak{g}, V) = (\prec, \succ, \leftarrow, \rightarrow, f, \{-,-\}) \) will be called a Leibniz extending structure of \( \mathfrak{g} \) through \( V \). Now let \( \mathfrak{g} \) be a Leibniz algebra, \( E \) a vector space containing \( \mathfrak{g} \) as a subspace and \( V \) a given complement of \( \mathfrak{g} \) in \( E \). Theorem 2.5 provides the answer to the description part of the ES problem: there exists a Leibniz algebra structure \([\cdot,\cdot]\) on \( E \) such that \( \mathfrak{g} \) is a subalgebra of \((E, [\cdot,\cdot])\) if and only if there exists an isomorphism of Leibniz algebras \((E, [\cdot,\cdot]) \cong \mathfrak{g} \ltimes V\), for some Leibniz extending structure \( \Omega(\mathfrak{g}, V) = (\prec, \succ, \leftarrow, \rightarrow, f, \{-,-\}) \) of \( \mathfrak{g} \) through \( V \). The theoretical answer to the classification part of the ES problem is given in Theorem 2.9: we will construct explicitly a relative cohomology 'group', denoted by \( \mathcal{H}L^2_\partial(V, \mathfrak{g}) \), which will be the classifying object of all extending structures of the Leibniz algebra \( \mathfrak{g} \) to \( E \); the classification is given up to an isomorphism of Leibniz algebras which stabilizes \( \mathfrak{g} \). The construction of the second classifying object, denoted by \( \mathcal{H}L^2(V, \mathfrak{g}) \) is also given and it parameterizes all extending structures of \( \mathfrak{g} \) to \( E \) up to an isomorphism which simultaneously stabilizes \( \mathfrak{g} \) and co-stabilizes \( V \) - i.e. this classification is given from the point of view of the extension problem. In Section 3 we give some explicit examples of computing \( \mathcal{H}L^2_\partial(V, \mathfrak{g}) \) and \( \mathcal{H}L^2(V, \mathfrak{g}) \) in the case of flag extending structures as defined in Definition 3.1. The main result of this section is Theorem 3.6: several special cases of it are discussed and explicit examples are given. Section 4 deals with two main special cases of the unified product. The crossed product of two Leibniz algebras is introduced as a special case of the unified product. Corollary 4.1 shows that any extension of a given Leibniz algebra \( \mathfrak{h} \) by a Leibniz algebra \( \mathfrak{g} \) is equivalent to a crossed product extension and the classifying object \( \mathcal{HL}^2(\mathfrak{h}, \mathfrak{g}) \) of all extensions of \( \mathfrak{h} \) by \( \mathfrak{g} \) is constructed in Corollary 4.2 as a generalization of the second Loday-Pirashvili cohomology group \( \mathcal{HL}^2(\mathfrak{h}, \mathfrak{g}) \) [21]. The concept of matched pair of Leibniz algebras is introduced in Definition 4.5 generalizing the one for Lie algebras [22, Theorem 3.9], [23, Theorem 4.1]. To any matched pair of Leibniz algebras the bicrossed product is constructed as the tool responsible for the factorization problem (Corollary 4.6). Finally, Section 5 gives the full answer to the classifying complements problem for Leibniz algebras. The main result of the section is Theorem 5.5: if \( \mathfrak{g} \) is a Leibniz subalgebra of \( \mathfrak{E} \) and \( \mathfrak{h} \) is a fixed complement of \( \mathfrak{g} \) in \( \mathfrak{E} \), then the isomorphism classes of all complements of \( \mathfrak{g} \) in \( \mathfrak{E} \) are parameterized by a certain cohomological object denoted by \( \mathcal{HA}^2(\mathfrak{h}, \mathfrak{g} | (\cdot, \cdot, \leftarrow, \rightarrow)) \) which is explicitly constructed, where \((\cdot, \cdot, \leftarrow, \rightarrow)\) is the canonical matched pair associated to the factorization \( \mathfrak{E} = \mathfrak{g} \# \mathfrak{h} \). The key points in proving this result are Theorem 5.2 and Theorem 5.3.

1. Preliminaries

All vector spaces, linear or bilinear maps are over an arbitrary field \( k \). A map \( f : V \to W \) between two vector spaces is called the trivial map if \( f(v) = 0 \), for all \( v \in V \). Let \( g \leq E \) be a subspace in a vector space \( E \); a subspace \( V \) of \( E \) such that \( E = g + V \) and \( V \cap g = 0 \) is called a complement of \( g \) in \( E \). Such a complement is unique up to an isomorphism and its dimension is called the codimension of \( g \) in \( E \). A Leibniz algebra is a vector space
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\[ [g, h, l] = [[g, h], l] - [[g, l], h] \]  
\[ \text{for all } g, h, l \in g. \]

for all \( g, h, l \in g \). Any Lie algebra is a Leibniz algebra, and a Leibniz algebra satisfying \( [g, g] = 0 \), for all \( g \in g \) is a Lie algebra. The typical example of a Leibniz algebra is the following [20]: let \( g \) be a Lie algebra, \((M, \langle \rangle)\) a right \( g \)-module and \( \mu : M \to g \) a \( g \)-equivariant map, i.e. \( \mu(m \cdot g) = [\mu(m), g] \), for all \( m \in M \) and \( g \in g \). Then \( M \) is a Leibniz algebra with the bracket \([m, n]_{(\langle, \mu)} := m \cdot \mu(n)\), for all \( m, n \in M \). Another important example was constructed in [19]: if \( g \) is a Lie algebra, then \( g \otimes g \) is a Leibniz algebra with the bracket given by \([x \otimes y, a \otimes b] := [x, [a, b]] \otimes y + x \otimes [y, [a, b]]\), for all \( x, y, a, b \in g \). For other interesting examples of Leibniz algebras we refer to [21].

Let \( g \) be a Leibniz algebra. A subspace \( I \leq g \) is called a two-sided ideal of \( g \) if \([x, g] \in I \) and \([g, x] \in I \), for all \( x \in I \) and \( g \in g \). \( g \) is called perfect if \([g, g] = g \) and abelian if \([g, g] = 0 \). By \( Z(g) \) we shall denote the center of \( g \), that is the two-sided ideal consisting of all \( g \in g \) such that \([g, x] = 0 \), for all \( x \in g \). \( \text{Der}(g) \) stands for the space of all derivations of \( g \), that is, all linear maps \( \Delta : g \to g, \) such that for any \( g, h \in g \) we have

\[ \Delta([g, h]) = [\Delta(g), h] + [g, \Delta(h)] \]

A key role in the construction of the different non-abelian cohomological objects arising from the classification part of the ES problem will be played both by the classical space of derivations and by the space of anti-derivations as defined below.

**Definition 1.1.** An *anti-derivation* of a Leibniz algebra \( g \) is a linear map \( D : g \to g \) such that

\[ D([g, h]) = [D(g), h] - [D(h), g] \]

for all \( g, h \in g \). We denote by \( \text{ADer}(g) \) the space of all anti-derivations of \( g \).

**Example 1.2.** For a Lie algebra \( g \) we have that \( \text{ADer}(g) = \text{Der}(g) \) but in the case of Leibniz algebras the two spaces are, in general, not equal. The next example illustrates this. Let \( g \) be the 3-dimensional Leibniz algebra with the basis \( \{e_1, e_2, e_3\} \) and the bracket defined by: \([e_1, e_3] = e_2, [e_3, e_3] = e_1\). A straightforward computation shows that the set \( \text{Der}(g) \) (resp. \( \text{ADer}(g) \)) coincides with the set of all arrays \( \Delta \) (resp. \( D \)) of the form:

\[ \Delta = \begin{pmatrix} 2b_1 & 0 & b_2 \\ b_2 & 3b_1 & b_3 \\ 0 & 0 & b_1 \end{pmatrix} \quad \text{(resp. } D = \begin{pmatrix} 0 & 0 & d_1 \\ 0 & 0 & d_2 \\ 0 & 0 & d_3 \end{pmatrix} \text{)} \]

for all \( b_1, b_2, b_3, d_1, d_2, d_3 \in k \).

The following new concept will play an important role in the construction of the two non-abelian cohomological objects introduced in Section 3 and Section 4.

**Definition 1.3.** Let \( g \) be a Leibniz algebra. A *pointed double derivation* of \( g \) is a triple \((g_0, D, \Delta)\), where \( g_0 \in g \) and \( D, \Delta : g \to g \) are linear maps satisfying the following
The compatibility conditions (\(D\)) denote by \(X\) and \(E\) extension problem. Any two cohomologous brackets on \(X\) and \(E\) are of the set of all isomorphism classes of Leibniz algebra structures on \(g\).

\[
D(g_0) = [g, g_0] = [g, D(h) + \Delta(h)] = D^2(g) + D(\Delta(g)) = 0 \quad (4)
\]
\[
D^2(g) + \Delta(D(g)) = [g_0, g] \quad (5)
\]
\[
\Delta([g, h]) = [\Delta(g), h] + [g, \Delta(h)] \quad (6)
\]
\[
D([g, h]) = [D(g), h] - [D(h), g] \quad (7)
\]

We denote by \(D(g)\) the space of all pointed double derivations.

The compatibility conditions (6)-(7) show that \(\Delta\) (resp. \(D\)) is a derivation (resp. an antiderivation) of \(g\), hence \(D(g) \subseteq g \times ADer(g) \times Der(g)\). If \(g\) is a Lie algebra then we can easily see that the space \(D(g)\) coincides with the set of all pairs \((g_0, D) \in Z(g') \times Der(g)\) such that \(D(g_0) = 0\). An example of computing the space \(D(g)\) is given in Example 4.4.

Let \(g\) be a Leibniz algebra, \(E\) a vector space such that \(g\) is a subspace of \(E\) and \(V\) a complement of \(g\) in \(E\), i.e. \(V\) is a subspace of \(E\) such that \(E = g + V\) and \(V \cap g = 0\). For a linear map \(\varphi : E \to E\) we consider the diagram:

\[
\begin{array}{ccc}
\mathcal{G} & \overset{i}{\longrightarrow} & E \\
\downarrow{\text{Id}} & & \downarrow{\varphi} \\
\mathcal{G} & \underset{i}{\longrightarrow} & E \\
\end{array}
\]

\[
\begin{array}{ccc}
E & \overset{\pi}{\longrightarrow} & V \\
\downarrow{\text{Id}} & & \downarrow{\text{Id}} \\
E & \underset{\pi}{\longrightarrow} & V \\
\end{array}
\]

where \(\pi : E \to V\) is the canonical projection of \(E = g + V\) on \(V\) and \(i : \mathcal{G} \to E\) is the inclusion map. We say that \(\varphi : E \to E\) stabilizes \(\mathcal{G}\) (resp. co-stabilizes \(V\)) if the left square (resp. the right square) of the diagram (8) is commutative.

Two Leibniz algebra structures \(\{-, -\}\) and \(\{-, -\}'\) on \(E\) containing \(\mathcal{G}\) as a Leibniz subalgebra are called equivalent and we denote this by \((E, \{-, -\}) \equiv (E, \{-, -\}')\), if there exists a Leibniz algebra isomorphism \(\varphi : (E, \{-, -\}) \to (E, \{-, -\}')\) which stabilizes \(\mathcal{G}\). \(\{-, -\}\) and \(\{-, -\}'\) are called cohomologous, and we denote this by \((E, \{-, -\}) \approx (E, \{-, -\}')\), if there exists a Leibniz algebra isomorphism \(\varphi : (E, \{-, -\}) \to (E, \{-, -\}')\) which stabilizes \(\mathcal{G}\) and co-stabilizes \(V\), i.e. the diagram (8) is commutative.

\(\equiv\) and \(\approx\) are both equivalence relations on the set of all Leibniz algebras structures on \(E\) containing \(\mathcal{G}\) as subalgebra and we denote by \(\text{ExtdL}(E, g)\) (resp. \(\text{ExtdL}'(E, g)\)) the set of all equivalence classes via \(\equiv\) (resp. \(\approx\)). \(\text{ExtdL}(E, g)\) is the classifying object of the ES problem: by explicitly computing \(\text{ExtdL}(E, g)\) we obtain a parametrization of the set of all isomorphism classes of Leibniz algebra structures on \(E\) that stabilize \(\mathcal{G}\). \(\text{ExtdL}'(E, g)\) gives a classification of the ES problem from the point of view of the extension problem. Any two cohomologous brackets on \(E\) are of course equivalent, hence there exists a canonical projection:

\[
\text{ExtdL}'(E, g) \to \text{ExtdL}(E, g)
\]

For two sets \(X\) and \(Y\) we shall denote by \(X \sqcup Y\) the coproduct in the category of sets of \(X\) and \(Y\), i.e. \(X \sqcup Y\) is the disjoint union of \(X\) and \(Y\).
2. Unified products for Leibniz algebras

In this section we shall give the theoretical answer to the ES-problem by constructing two cohomological type objects which will parameterize $\text{ExtdL}(E, g)$ and $\text{ExtdL}'(E, g)$. First we introduce the following:

Definition 2.1. Let $g$ be a Leibniz algebra and $V$ a vector space. An extending datum of $g$ through $V$ is a system $\Omega(g, V) = (\triangleleft, \triangleright, \triangleleft, \triangleright, f, \{−, −\})$ consisting of six bilinear maps

\[ \triangleleft : V \times g \rightarrow V, \quad \triangleright : V \times g \rightarrow g, \quad \triangleleft : g \times V \rightarrow g, \quad \triangleright : g \times V \rightarrow V \]

\[ f : V \times V \rightarrow g, \quad \{−, −\} : V \times V \rightarrow V \]

Let $\Omega(g, V) = (\triangleleft, \triangleright, \triangleleft, \triangleright, f, \{−, −\})$ be an extending datum. We denote by $g \ltimes_{\Omega(g, V)} V = g \ltimes V$ the vector space $g \ltimes V$ together with the bilinear map $[-, -] : (g \ltimes V) \times (g \ltimes V) \rightarrow g \ltimes V$ defined by:

\[ [(g, x), \ (h, y)] := ([g, h] + x \triangleright h + g \triangleleft y + f(x, y), \ \{x, y\} + x \triangleleft h + g \triangleright y) \quad (9) \]

for all $g, h \in g$ and $x, y \in V$. The object $g \ltimes V$ is called the unified product of $g$ and $\Omega(g, V)$ if it is a Leibniz algebra with the bracket given by (9). In this case the extending datum $\Omega(g, V) = (\triangleleft, \triangleright, \triangleleft, \triangleright, f, \{−, −\})$ is called a Leibniz extending structure of $g$ through $V$. The maps $\triangleleft, \triangleright, \triangleleft$ and $\triangleright$ are called the actions of $\Omega(g, V)$ and $f$ is called the cocycle of $\Omega(g, V)$.

Example 2.2. The unified product is a very general construction: in particular, the hemisemidirect product introduced in differential geometry [18, Example 2.2] is a special case of it. Let $g$ be a Lie algebra and $(V, \triangleleft)$ be a right $g$-module. Then $g \ltimes V$ is a Leibniz algebra with the bracket $[(g, x), \ (h, y)] := ([g, h], x \triangleleft h)$, for all $g, h \in g, x, y \in V$ called the hemisemidirect product of $g$ and $V$. This Leibniz algebra is not a Lie algebra if $g$ acts nontrivially on $V$. The hemisemidirect product is a special case of the unified product if we let $\triangleright, \triangleleft, \triangleright, f$ and $\{−, −\}$ to be the trivial maps and $g$ to be a Lie algebra.

Let $\Omega(g, V)$ be an extending datum of $g$ through $V$. The bracket defined by (9) has a rather complicated formula; however, for some specific elements we obtain easier forms which will be very useful for future computations. More precisely, the following relations hold in $g \ltimes V$ for any $g, h \in g, x, y \in V$:

\[ [(g, 0), \ (h, 0)] = ([g, h], 0), \quad [(g, 0), \ (0, y)] = (g \triangleleft y, g \triangleright y) \quad (10) \]

\[ [(0, x), \ (h, 0)] = (x \triangleright h, x \triangleleft h), \quad [(0, x), \ (0, y)] = (f(x, y), \{x, y\}) \quad (11) \]

The next theorem provides the set of axioms that need to be fulfilled by an extending datum $\Omega(g, V)$ such that $g \ltimes V$ is a unified product.

Theorem 2.3. Let $g$ be a Leibniz algebra, $V$ a vector space and $\Omega(g, V)$ an extending datum of $g$ by $V$. Then $g \ltimes V$ is a unified product if and only if the following compatibility conditions hold for any $g, h \in g, x, y, z \in V$:

(L1) $(V, \triangleleft)$ is a right $g$-module, i.e. $x \triangleleft [g, h] = (x \triangleleft g) \triangleleft h - (x \triangleleft h) \triangleleft g$

(L2) $x \triangleright [g, h] = [x \triangleright g, h] - [x \triangleright h, g] + (x \triangleleft g) \triangleright h - (x \triangleleft h) \triangleright g$

(L3) $[g, h] \triangleright x = g \triangleright (h \triangleright x) + (g \triangleright x) \triangleleft h$
Proof. The proof relies on a detailed analysis of the Leibniz identity for the bracket given by (9), similar to the one provided in the proof of [4, Theorem 2.2] corresponding to the Lie algebra case. There are, however, two significant differences which require some extra care, namely: the bracket on the Leibniz algebra $g$ is not anti-symmetric and the Leibniz identity is not invariant under circular permutations. As the computations are rather long but straightforward we will only indicate the essential steps of the proof, the details being left to the reader. To start with, we note that $g \ltimes V$ is a Leibniz algebra if and only if Leibniz’s identity holds, i.e.:

$$[[g, x], [(h, y), (l, z)]] = [[(g, x), (h, y)], (l, z)] - [[(g, x), (l, z)], (h, y)]$$

(12)

for all $g, h, l \in g$ and $x, y, z \in V$. Since in $g \ltimes V$ we have $(g, x) = (g, 0) + (0, x)$ it follows that (12) holds if and only if it holds for all generators of $g \ltimes V$, i.e. the set

$$\{(g, 0) \mid g \in g\} \cup \{(0, x) \mid x \in V\}.$$  

Hence we are left to deal with eight cases which are necessary and sufficient for testing the compatibility condition (12). First, we should notice that (12) holds for the triple $(g, 0), (h, 0), (l, 0)$, since in $g \ltimes V$ we have that $[(g, 0), (h, 0)] = [(g, h), 0]$. Now, taking into account (10), we obtain that (12) holds for $(g, 0), (h, 0), (0, x)$ if and only if

$$[[g, x], [h, y], (l, z)] = [[(g, x), (h, y)], (l, z)] - [[(g, x), (l, z)], (h, y)]$$

(12)

i.e. if and only if (L3) and (L4) hold. A similar computation proves that the compatibility condition (12) is fulfilled for the triple $(g, 0), (0, x), (h, 0)$ if and only if:

$$[[g, h], x] = [[g, x], [h, 0], (l, z)] - [[g, x], [h, 0], (l, z)] - [[g, x], (l, z)], (h, 0)]$$

We are left with three more cases to study. First, observe that (12) holds
for \((0, x), (g, 0), (h, 0)\) if and only if
\[
(x \triangleright [g, h], x \triangleleft [g, h]) = ([x \triangleright g, h] - [x \triangleright h, g] + (x \triangleleft g) \triangleright h - (x \triangleleft h) \triangleright g, (x \triangleleft g) \lhd h - (x \triangleright h) \lhd g)
\]
which is equivalent to the fact that axioms \((L1)\) and \((L2)\) hold. Analogously, we can show that \((L2)\) holds for \((0, x), (0, y), (g, 0)\) if and only if \((L7)\) and \((L8)\) hold. Finally, it is straightforward to see that \((L2)\) holds for \((0, x), (g, 0), (0, y)\) if and only if the following two compatibilities are fulfilled:
\[
\{x, y\} \triangleright g = (x \triangleright g) \leftarrow y + f(x \triangleleft g, y) - [f(x, y), g] - x \triangleright (g \leftarrow y) - f(x, g \rightarrow y)
\]
\[
\{x, y\} \triangleleft g = \{x \triangleleft g, y\} + (x \triangleright g) \rightarrow y - x \triangleleft (g \leftarrow y) - \{x, g \rightarrow y\}
\]
Since we are looking for a minimal set of axioms we obtain, assuming that \((L7)\) and \((L8)\) hold, that \((L2)\) is fulfilled for the triple \((0, x), (g, 0), (0, y)\) if and only if \((L12)\) and \((L13)\) hold.

From now on, a Leibniz extending structure of \(g\) through \(V\) will be viewed as a system \(\Omega(g, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \{\cdot, \cdot\})\) satisfying the compatibility conditions \((L1)-(L14)\). We denote by \(\mathcal{LZ}(g, V)\) the set of all Leibniz extending structures of \(g\) through \(V\).

**Example 2.4.** We provide the first example of a Leibniz extending structure and the corresponding unified product. More examples will be given in Section 3 and Section 4.

Let \(\Omega(g, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \{\cdot, \cdot\})\) be an extending datum of a Leibniz algebra \(g\) through a vector space \(V\) such that its actions are all the trivial maps, i.e., \(x \triangleleft g = x \triangleright g = g \rightarrow x = g \leftarrow x = 0\), for all \(x \in V\) and \(g \in g\). Then, \(\Omega(g, V) = (f, \{\cdot, \cdot\})\) is a Leibniz extending structure of \(g\) through \(V\) if and only if \((V, \{\cdot, \cdot\})\) is a Leibniz algebra and \(f : V \times V \rightarrow g\) is an abelian 2-cocycle, that is
\[
[g, f(x, y)] = [f(x, y), g] = 0, \quad f(x, \{y, z\}) - f(\{x, y\}, z) + f(\{x, z\}, y) = 0 \quad (13)
\]
for all \(g \in g, x, y, z \in V\). The first part of \((13)\) shows that the image of \(f\) is contained in the center of \(g\), while the second part is a 2-cocycle condition (see [21, Section 1.7]) which follows from the axiom \((L5)\). In this case, the associated unified product \(g \ltimes \Omega(g, V)\) \(V\) will be called the twisted product of the Leibniz algebras \(g\) and \(V\).

Let \(\Omega(g, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \cdot)\) \(\in \mathcal{LZ}(g, V)\) be a Leibniz extending structure and \(g \ltimes V\) the associated unified product. Then the canonical inclusion
\[
i_g : g \rightarrow g \ltimes V, \quad i_g(g) = (g, 0)
\]
is an injective Leibniz algebra map. Therefore, we can see \(g\) as a subalgebra of \(g \ltimes V\) through the identification \(g \cong i_g(g) \cong g \times \{0\}\). Conversely, we have the following result which provides an answer to the description part of the ES problem:

**Theorem 2.5.** Let \(g\) be a Leibniz algebra, \(E\) a vector space containing \(g\) as a subspace and \([-\cdot, \cdot]\) a Leibniz algebra structure on \(E\) such that \(g\) is a Leibniz subalgebra in \((E, [-\cdot, \cdot])\). Then there exists a Leibniz extending structure \(\Omega(g, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, [-\cdot, \cdot])\) of \(g\) through a subspace \(V\) of \(E\) and an isomorphism of Leibniz algebras \((E, [-\cdot, \cdot]) \cong g \ltimes V\) that stabilizes \(g\) and co-stabilizes \(V\).
Proof. Since \( k \) is a field, there exists a linear map \( p : E \to g \) such that \( p(g) = g \), for all \( g \in g \). Then \( V := \ker(p) \) is a complement of \( g \) in \( E \). We define the extending datum \( \Omega(g, V) = (\triangleleft, \triangleright) \), associated to all Leibniz extending structures \( \Omega(g, V) \), for a given complement \( V \) of \( g \). Now, the map \( \varphi : g \times V \to E \), \( \varphi(g, x) := g + x \), is a linear isomorphism between the direct product of vector spaces \( g \times V \) and the Leibniz algebra \( (E, [\cdot, \cdot]) \) with the inverse given by \( \varphi^{-1}(y) := (p(y), y - p(y)) \), for all \( y \in E \). Hence, there exists a unique Leibniz algebra structure on \( g \times V \) such that \( \varphi \) is an isomorphism of Leibniz algebras and this unique bracket is given by:

\[
[(g, x), (h, y)] := \varphi^{-1}([\varphi(g, x), \varphi(h, y)])
\]

for all \( g, h \in g \) and \( x, y \in V \). Using Theorem 2.3, the proof will be finished if we prove that this bracket is the one defined by (9) associated to the system \((\triangleleft, \triangleright), \triangleleft_p, \triangleright_p, \rightarrow \rightarrow_p, f = f_p, \{-, -\}, p \) constructed above. Indeed, for any \( g, h \in g \), \( x, y \in V \) we have:

\[
[(g, x), (h, y)] = \varphi^{-1}([\varphi(g, x), \varphi(h, y)]) = \varphi^{-1}([g + x, h + y]) = \varphi^{-1}([g, h] + [g, y] + [x, h] + [x, y]) = ([g, h] + p([g, y]) + p([x, h]) + p([x, y]), [g, y] + [x, h] + [x, y] - p([g, y]) - p([x, h]) - p([x, y]) = ([g, h] + x \triangleright g + g \triangleleft x + f(x, y), \{x, y\} + x \triangleleft h + g \rightarrow y)
\]

as needed. Thus, \( \varphi : g \times V \to E \) is an isomorphism of Leibniz algebras and the following diagram is commutative

\[
\begin{array}{ccc}
g & \xrightarrow{i} & g \times V & \xrightarrow{q} & V \\
\downarrow{Id} & & \downarrow{p} & & \downarrow{Id} \\
g & \xrightarrow{i} & E & \xrightarrow{\pi} & V
\end{array}
\]

where \( \pi : E \to V \) is the projection of \( E = A + V \) on the vector space \( V \) and \( q : A \ltimes V \to V \), \( q(g, x) := x \) is the canonical projection. \( \square \)

Based on Theorem 2.5, the classification of all Leibniz algebra structures on \( E \) that contain \( g \) as a Leibniz subalgebra reduces to the classification of all unified products \( g \ltimes V \), associated to all Leibniz extending structures \( \Omega(g, V) \), for a given complement \( V \) of \( g \) in \( E \). In order to construct the cohomological objects \( \mathcal{H}L^2_E(g, V) \) and \( \mathcal{H}L^2(V, g) \) which will parameterize the classifying sets \( \text{ExtGL}(E, g) \) and respectively \( \text{ExtGL'}(E, g) \) we need the following technical result:
Lemma 2.6. Let $\Omega(g, V) = \langle \triangleleft, \triangleright, \leftarrow, \rightarrow, f; \{-, -\}\rangle$ and $\Omega'(g, V) = \langle \triangleleft', \triangleright', \leftarrow', \rightarrow', f'; \{-, -\}\rangle$ be two Leibniz extending structures of $g$ through $V$ and $g \times V, g \times' V$ the associated unified products. Then there exists a bijection between the set of all morphisms of Leibniz algebras $\psi : g \times V \rightarrow g \times' V$ which stabilizes $g$ and the set of pairs $(r, v)$, where $r : V \rightarrow g, v : V \rightarrow V$ are two linear maps satisfying the following compatibility conditions for any $g \in g, x, y \in V$:

(ML1) $v(g \rightarrow x) = g \rightarrow' v(x)$;
(ML2) $v(x \triangleleft g) = v(x) \triangleleft' g$;
(ML3) $x \triangleright g + r(x \triangleleft g) = [r(x), g] + v(x) \triangleright' g$;
(ML4) $g \leftarrow x + r(g \rightarrow x) = [g, r(x)] + g \leftarrow' v(x)$;
(ML5) $v\{x, y\} = r(x) \rightarrow' v(y) + v(x) \leftarrow' v(y) + v(x) \triangleright' r(y) + f'(v(x), v(y))$;
(ML6) $f(x, y) + r\{x, y\} = [r(x), r(y)] + r(x) \leftarrow' v(y) + v(x) \triangleright' r(y) + f'(v(x), v(y))$

Under the above bijection the morphism of Leibniz algebras $\psi = \psi_{(r, v)} : g \times V \rightarrow g \times' V$ corresponding to $(r, v)$ is given for any $g \in g$ and $x \in V$ by:

$$\psi(g, x) = (g + r(x), v(x))$$

Moreover, $\psi = \psi_{(r, v)}$ is an isomorphism if and only if $v : V \rightarrow V$ is an isomorphism and $\psi = \psi_{(r, v)}$ co-stabilizes $V$ if and only if $v = \text{Id}_V$.

Proof. A linear map $\psi : g \times V \rightarrow g \times' V$ which stabilizes $g$ is uniquely determined by two linear maps $r : V \rightarrow g, v : V \rightarrow V$ such that $\psi(g, x) = (g + r(x), v(x))$, for all $g \in g$, and $x \in V$. Indeed, by denoting $\psi(0, x) = (r(x), v(x)) \in g \times V$ for all $x \in V$, we obtain:

$$\psi(g, x) = \psi((g, 0) + \psi(0, x)) = \psi(g, 0) + \psi(0, x) = (g + r(x), v(x))$$

Let $\psi = \psi_{(r, v)}$ be such a linear map, i.e. $\psi(g, x) = (g + r(x), v(x))$, for some linear maps $r : V \rightarrow g, v : V \rightarrow V$. We will prove that $\psi$ is a morphism of Leibniz algebras if and only if the compatibility conditions (ML1)–(ML6) hold. It is enough to prove that the compatibility

$$\psi([[(g, x), (h, y)]] = [[\psi(g, x), \psi(h, y)]]$$

holds for all generators of $g \times V$. First of all, it is easy to see that (14) holds for the pair $(g, 0), (h, 0)$, for all $g, h \in g$. Now we prove that (14) holds for the pair $(g, 0), (0, x)$ if and only if (ML1) and (ML4) hold. Indeed, $\psi([[(g, 0), (0, x)]] = [\psi(g, 0), \psi(0, x)]$ is equivalent to $\psi(g \leftarrow x, g \rightarrow x) = [(g, 0), (r(x), v(x))]$ and hence to $(g \leftarrow x + r(g \rightarrow x), v(g \rightarrow x)) = [(g, r(x)] + g \leftarrow' v(x), g \rightarrow' v(x)]$, i.e. to the fact that (ML1) and (ML4) hold.

Next we prove that (14) holds for the pair $(0, x), (g, 0)$ if and only if (ML2) and (ML3) hold. Indeed, $\psi([[(0, x), (g, 0)]] = [\psi(0, x), \psi(g, 0)]$ is equivalent to $\psi(x \triangleright g, x \triangleleft g) = [(r(x), v(x)), (g, 0)]$ and therefore to $(x \triangleright g + r(x \triangleleft g), v(x \triangleleft g)) = [(r(x), g] + v(x) \triangleright' g, v(x) \triangleleft' g)$, i.e. to the fact that (ML2) and (ML3) hold.

To this end, we prove that (14) holds for the pair $(0, x), (0, y)$ if and only if (ML5) and (ML6) hold. Indeed, $\psi([[(0, x), (0, y)]] = [\psi(0, x), \psi(0, y)]$ is equivalent to $\psi(f(x, y), \{x, y\}) = [(r(x), v(x)), (r(y), v(y))]$; thus it is equivalent to: $(f(x, y) + r\{x, y\}, v\{x, y\})$.
As a conclusion of this section, the theoretical answer to the ES-problem follows:

Definition 2.8. Two Leibniz extending structures $\Omega(g, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \{\cdot, \cdot\})$ and $\Omega'(g, V) = (\triangleleft', \triangleright', \leftarrow', \rightarrow', f', \{\cdot, \cdot\}')$ are called equivalent and we denote this by $\Omega(g, V) \cong \Omega'(g, V)$ if there exists a pair $(r, v)$ of linear maps, where $r : V \to g$ and $v \in \text{Aut}_h(V)$ such that $(\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \{\cdot, \cdot\})$ is implemented from $(\triangleleft', \triangleright', \leftarrow', \rightarrow', f', \{\cdot, \cdot\}')$ using $(r, v)$ via:

\[
\begin{align*}
x \triangleleft g &= v^{-1}(v(x) \triangleright g) \\
g \rightarrow x &= v^{-1}(g \leftarrow v(x)) \\
x \triangleright g &= [r(x), g] + v(x) \triangleright g - r \circ v^{-1}(v(x) \triangleright' g) \\
g \leftarrow x &= [g, r(x)] + g \leftarrow' v(x) - r \circ v^{-1}(g \rightarrow' v(x)) \\
\{x, y\} &= v^{-1}(r(x) \rightarrow' v(y) + v(x) \triangleright' r(y) + \{v(x), v(y)\}') \\
f(x, y) &= [r(x), r(y)] + r(x) \leftarrow' v(y) + v(x) \triangleright' r(y) + f'(v(x), v(y)) - r \circ v^{-1}(r(x) \rightarrow' v(y) + v(x) \triangleright' r(y) + \{v(x), v(y)\}')
\end{align*}
\]

for all $g \in g$, $x, y \in V$.

Using Lemma 2.6, we obtain that $\Omega(g, V) \cong \Omega'(g, V)$ if and only if there exists $\psi : g \times V \to g \times' V$ an isomorphism of Leibniz algebras that stabilizes $g$, where $g \times V$ and $g \times' V$ are the corresponding unified products. On the other hand, the isomorphisms between two unified products that stabilize $g$ and co-stabilize $V$ are decoded by the following:

Definition 2.8. Two Leibniz extending structures $\Omega(g, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \{\cdot, \cdot\})$ and $\Omega'(g, V) = (\triangleleft', \triangleright', \leftarrow', \rightarrow', f', \{\cdot, \cdot\}')$ are called cohomologous and we denote this by $\Omega(g, V) \approx \Omega'(g, V)$ if and only if $\triangleleft = \triangleleft'$, $\rightarrow = \rightarrow'$ and there exists a linear map $r : V \to g$ such that for any $g \in g$, $x, y \in V$:

\[
\begin{align*}
x \triangleright g &= [r(x), g] + x \triangleright' g - r(x \triangleright g) \\
g \leftarrow x &= [g, r(x)] + g \leftarrow' x - r(g \rightarrow' x) \\
\{x, y\} &= r(x) \rightarrow' y + x \triangleright' r(y) + \{x, y\}' \\
f(x, y) &= [r(x), r(y)] + r(x) \leftarrow' y + x \triangleright' r(y) + f'(v(x), v(y)) - r \circ v^{-1}(r(x) \rightarrow' y + x \triangleright' r(y) + \{x, y\}')
\end{align*}
\]

As a conclusion of this section, the theoretical answer to the ES-problem follows:
Theorem 2.9. Let \( g \) be a Leibniz algebra, \( E \) a vector space that contains \( g \) as a subspace and \( V \) a complement of \( g \) in \( E \). Then:

1. \( \equiv \) is an equivalence relation on the set \( \mathcal{LZ}(g, V) \) of all Leibniz extending structures of \( g \) through \( V \). If we denote \( \mathcal{HL}^2_0(V, g) := \mathcal{LZ}(g, V) / \equiv \), then the map

\[
\mathcal{HL}^2_0(V, g) \rightarrow \text{ExtdL}(E, g), \quad (\langle, \rangle, \leftarrow, \rightarrow, f, \{\cdot, \cdot\}) \mapsto (g \ltimes V, [\cdot, \cdot])
\]

is bijective, where \( (\langle, \rangle, \leftarrow, \rightarrow, f, \{\cdot, \cdot\}) \) is the equivalence class of \( (\langle, \rangle, \leftarrow, \rightarrow, f, \{\cdot, \cdot\}) \) via \( \equiv \).

2. \( \approx \) is an equivalence relation on the set \( \mathcal{LZ}(g, V) \) of all Leibniz extending structures of \( g \) through \( V \). If we denote \( \mathcal{HL}^2(V, g) := \mathcal{LZ}(g, V) / \approx \), then the map

\[
\mathcal{HL}^2(V, g) \rightarrow \text{ExtdL}'(E, g), \quad (\langle, \rangle, \leftarrow, \rightarrow, f, \{\cdot, \cdot\}) \mapsto (g \ltimes V, [\cdot, \cdot])
\]

is bijective, where \( (\langle, \rangle, \leftarrow, \rightarrow, f, \{\cdot, \cdot\}) \) is the equivalence class of \( (\langle, \rangle, \leftarrow, \rightarrow, f, \{\cdot, \cdot\}) \) via \( \approx \).

3. Flag extending structures of Leibniz algebras. Examples

After we have provided a theoretical answer to the ES problem in Theorem 2.9, we are left to compute the classifying object \( \mathcal{HL}^2_0(V, g) \) for a given Leibniz algebra \( g \) that is a subspace in a vector space \( E \) with a complement \( V \) and then to describe all Leibniz algebra structures on \( E \) which extend the one of \( g \). In this section we propose an algorithm to tackle the problem for a large class of such structures.

Definition 3.1. Let \( g \) be a Leibniz algebra and \( E \) a vector space containing \( g \) as a subspace. A Leibniz algebra structure on \( E \) is called a flag extending structure of \( g \) if there exists a finite chain of Leibniz subalgebras of \( E \)

\[
g = E_0 \subset E_1 \subset \cdots \subset E_m = E
\]

such that \( E_i \) has codimension 1 in \( E_{i+1} \), for all \( i = 0, \ldots, m - 1 \).

As an easy consequence of Definition 3.1 we have that \( \dim_k(V) = m \), where \( V \) is the complement of \( g \) in \( E \). In what follows we will provide a way of describing all flag extending structures of \( g \) to \( E \) in a recursive manner which relies on the first step, namely \( m = 1 \). Therefore, we start by describing and classifying all unified products \( g \ltimes V_1 \), for a 1-dimensional vector space \( V_1 \). This procedure can be iterated by replacing the initial Leibniz algebra \( g \) with a unified product \( g \ltimes V_1 \) obtained in the previous step. After \( m \) steps we arrive at the description of all flag extending structures of \( g \) to \( E \). We start by introducing the following two concepts:

Definition 3.2. Let \( g \) be a Leibniz algebra. A flag datum of the first kind of \( g \) is a 5-tuple \((g_0, \alpha, \lambda, D, \Delta)\), where \( g_0 \in g, \alpha \in k, \lambda : g \rightarrow k, D \) and \( \Delta : g \rightarrow g \) are linear maps satisfying the following compatibilities for any \( g, h \in g \):

1. \( \lambda([g, h]) = 0, \lambda(D(g)) + \alpha \lambda(g) = 0, \lambda(\Delta(g)) = 0; \)
2. \( D(g_0) = -\alpha g_0, \lambda(g_0) = -\alpha^2, \alpha \Delta(g) = -[g, g_0]; \)
3. \( [g, \Delta(h)] + [g, D(h)] = -\lambda(h) \Delta(g); \)
Let \( g \) be a Leibniz algebra. A flag datum of the second kind of \( g \) is a quadruple \((g_0, \nu, D, \Delta)\), where \( g_0 \in g \), \( \nu : g \to k \), \( \nu \neq 0 \) is a non-trivial map, \( D \) and \( \Delta : g \to g \) are linear maps satisfying the following compatibilities for any \( g, h \in g \):

\[
\begin{align*}
(G1) \quad & \nu([g, h]) = 0, \quad [g, g_0] = 0, \quad D(g_0) = 0, \quad \nu(g_0) = 0; \\
(G2) \quad & D^2(g) + D(\Delta(g)) = 0, \quad D(g) + \nu(D(g)) + \nu(\Delta(g)) = 0; \\
(G3) \quad & \Delta([g, h]) = \Delta(g, h) + [g, \Delta(h)]; \\
(G4) \quad & D([g, h]) = [D(g), h] + [g, D(h)] + \nu(h)\Delta(g) + \nu(g)\Delta(h); \\
(G5) \quad & D([g, h]) = [D(g), h] - [D(h), g] - \nu(g)D(h) + \nu(h)D(g);
\end{align*}
\]

We denote by \( F_1(g) \) the set of all flag datums of the first kind of \( g \).

**Definition 3.3.** Let \( g \) be a Leibniz algebra. A flag datum of the second kind of \( g \) is a quadruple \((g_0, \nu, D, \Delta)\), where \( g_0 \in g \), \( \nu : g \to k \), \( \nu \neq 0 \) is a non-trivial map, \( D \) and \( \Delta : g \to g \) are linear maps satisfying the following compatibilities for any \( g, h \in g \):

\[
\begin{align*}
\text{(G1)} & \quad \nu([g, h]) = 0, \quad [g, g_0] = 0, \quad D(g_0) = 0, \quad \nu(g_0) = 0; \\
\text{(G2)} & \quad D^2(g) + D(\Delta(g)) = 0, \quad D(g) + \nu(D(g)) + \nu(\Delta(g)) = 0; \\
\text{(G3)} & \quad \Delta([g, h]) = \Delta(g, h) + [g, \Delta(h)]; \\
\text{(G4)} & \quad D([g, h]) = [D(g), h] + [g, D(h)] + \nu(h)\Delta(g) + \nu(g)\Delta(h); \\
\text{(G5)} & \quad D([g, h]) = [D(g), h] - [D(h), g] - \nu(g)D(h) + \nu(h)D(g);
\end{align*}
\]

We denote by \( F_2(g) \) the set of all flag datums of the second kind of \( g \) and by \( F(g) := F_1(g) \sqcup F_2(g) \), the disjoint union of the two sets. The elements of \( F(g) \) will be called flag datums of \( g \). \( F(g) \) contains the space of pointed double derivations \( D(g) \) of \( g \) via the canonical embedding: \( D(g) \to F_1(g), \ (g_0, D, \Delta) \to (g_0, 0, 0, D, \Delta) \). The next proposition shows that the space \( F(g) \) is the counterpart for Leibniz algebras of what we have called in [4, Definition 4.2] the space of twisted derivations of a Lie algebra:

**Proposition 3.4.** Let \( g \) be a Leibniz algebra and \( V \) a vector space of dimension 1 with a basis \( \{x\} \). Then there exists a bijection between the set \( \mathcal{LZ}(g, V) \) of all Leibniz extending structures of \( g \) through \( V \) and \( F(g) = F_1(g) \sqcup F_2(g) \).

Under the above bijective correspondence the Leibniz extending structure \( \Omega(g, V) = (\prec, \triangleright, \precarrow, \trianglerightarrow, \{\precarrow, \trianglerightarrow\}) \) corresponding to \((g_0, \alpha, \lambda, D, \Delta) \in F_1(g) \) is given by:

\[
\begin{align*}
\prec g &= \lambda(g)x, \quad \triangleright g = D(g), \quad f(x, x) = g_0 \\
\precarrow g &= \Delta(g), \quad g \precarrow x = 0, \quad \{x, x\} = \alpha x
\end{align*}
\]

while the Leibniz extending structure \( \Omega(g, V) = (\prec, \triangleright, \precarrow, \trianglerightarrow, \{\precarrow, \trianglerightarrow\}) \) corresponding to \((g_0, \nu, D, \Delta) \in F_2(g) \) is given by:

\[
\begin{align*}
\prec g &= -\nu(g)x, \quad \triangleright g = D(g), \quad f(x, x) = g_0 \\
\precarrow g &= \Delta(g), \quad g \precarrow x = \nu(g)x, \quad \{x, x\} = 0
\end{align*}
\]

for all \( g \in g \).

**Proof.** We have to compute the set of all bilinear maps \( \prec : V \times g \to V \), \( \triangleright : V \times g \to g \), \( \precarrow : g \times V \to g \), \( \trianglerightarrow : g \times V \to V \), \( f : V \times V \to g \) and \( \{\precarrow, \trianglerightarrow\} : V \times V \to V \) satisfying the compatibility conditions \((L1) - (L14)\) of Theorem 2.3. Since \( V \) has dimension 1 there exists a bijection between the set of all bilinear maps \( \prec : V \times g \to V \) and the set of all linear maps \( \lambda : g \to k \) and the bijection is given such that the action \( \prec : V \times g \to g \) associated to \( \lambda \) is given by the formula: \( x \prec g := \lambda(g)x \), for all \( g \in g \). In the same manner, the action \( \precarrow : g \times V \to V \) is uniquely determined by a linear map \( \nu : g \to k \) such that
$g \rightarrow x = \nu(g)x$, for all $g \in g$. Similarly, the bilinear maps $\triangleright: V \times g \rightarrow g$, $\langle: g \times V \rightarrow g$ are uniquely implemented by linear maps $D = D_\triangleright: g \rightarrow g$ respectively $\Delta = \Delta_\langle: g \rightarrow g$ via the formulas: $x \triangleright g := D(g)$ and $g \langle x := \Delta(g)$, for all $g \in g$. Finally, any bilinear map $f: V \times V \rightarrow g$ is uniquely implemented by an element $g_0 \in g$ such that $f(x, x) = g_0$ and any bracket $\{-, -\}: V \times V \rightarrow V$ is uniquely determined by a scalar $\alpha \in k$ such that $\{x, x\} = \alpha x$.

Now, the compatibility condition (L9) is equivalent to $\alpha \nu(g) = 0$, for all $g \in g$, while (L14) gives $\nu(g)(\nu(h) + \lambda(h)) = 0$, for all $g, h \in g$. If $\nu = 0$, the trivial map on $g$, then (L9) and (L14) are trivially fulfilled and the rest of the compatibility conditions of Theorem 2.3 came down to (F1)-(F7) from the definition of $F_1(g)$. This can be proved by a routine computation: for instance, axiom (L1) is equivalent to $\lambda([g, h]) = 0$ while axiom (L2) is equivalent to (F7). Otherwise, if $\nu \neq 0$ implies that $\alpha = 0$ and $\lambda = -\nu$. Based on this, it is straightforward to see that the compatibility conditions (L1) - (L14) of Theorem 2.3 are equivalent to (G1) - (G5) from the definition of $F_2(g)$.

Let $(g_0, \alpha, \lambda, D, \Delta) \in F_1(g)$. The unified product $g \ltimes (g_0, \alpha, \lambda, D, \Delta) V$ associated to the Leibniz extending structure given by (16) - (17) will be denoted by $g_1(x \mid (g_0, \alpha, \lambda, D, \Delta))$ and has the bracket defined by:

$$ [(g, 0), (h, 0)] = ([g, h], 0), \quad [(g, 0), (0, x)] = (\Delta(g), 0) \quad (20) $$

$$ [(0, x), (0, x)] = (g_0, \alpha x), \quad [(0, x), (g, 0)] = (D(g), \lambda(g)x) \quad (21) $$

for all $g, h \in g$. On the other hand, for $(g_0, \nu, D, \Delta) \in F_2(g)$, the unified product $g \ltimes (g_0, \nu, D, \Delta) V$ associated to the Leibniz extending structure given by (18)-(19) will be denoted by $g_2(x \mid (g_0, \nu, D, \Delta))$ and has the bracket defined by:

$$ [(g, 0), (h, 0)] = ([g, h], 0), \quad [(g, 0), (0, x)] = (\Delta(g), \nu(g)x) \quad (22) $$

$$ [(0, x), (0, x)] = (g_0, 0)), \quad [(0, x), (g, 0)] = (D(g), -\nu(g)x) \quad (23) $$

for all $g, h \in g$. Thus, we have obtained the following:

**Corollary 3.5.** Any Leibniz algebra that contains a given Leibniz algebra $g$ as a subalgebra of codimension 1 is isomorphic to a Leibniz algebra of type $g_1(x \mid (g_0, \alpha, \lambda, D, \Delta))$, for some $(g_0, \alpha, \lambda, D, \Delta) \in F_1(g)$ or to a Leibniz algebra of type $g_2(x \mid (g_0, \nu, D, \Delta))$, for some $(g_0, \nu, D, \Delta) \in F_2(g)$.

Now, we shall classify these Leibniz algebras up to an isomorphism that stabilizes $g$, i.e. we give the first explicit classification result for the ES-problem. This is the key step in the classification of all flag extending structures of $g$.

**Theorem 3.6.** Let $g$ be a Leibniz algebra of codimension 1 in the vector space $E$. Then:

$$ \text{ExtDl}(E, g) \cong \mathcal{H}\mathcal{L}_g^2(k, g) \cong (F_1(g)/ \equiv_1) \sqcup (F_2(g)/ \equiv_2), $$

where:

$\equiv_1$ is the equivalence relation on the set $F_1(g)$ defined as follows: $(g_0, \alpha, \lambda, D, \Delta) \equiv_1 (g_0', \alpha', \lambda', D', \Delta')$ if and only if $\lambda = \lambda'$ and there exists a pair $(q, G) \in k^* \times g$ such that
for any \( g \in \mathfrak{g} \):

\[
\begin{align*}
g_0 & = q^2 g_0' + [G,G] + q D'(G) + q E'(G) - q \alpha' G - \lambda'(G) G \\
\alpha & = q \alpha' + \lambda'(G) \quad (24) \\
D(g) & = q D'(g) + [G,g] - \lambda'(g) G \\
\Delta(g) & = q \Delta'(g) + [g,G] \quad (25)
\end{align*}
\]

\( \equiv_2 \) is the equivalence relation on \( \mathcal{F}_2(\mathfrak{g}) \) given by: \( (g_0, \nu, D, \Delta) \equiv_2 (g_0', \nu', D', \Delta') \) if and only if \( \nu = \nu' \) and there exists a pair \( (q, G) \in k^* \times \mathfrak{g} \) such that for any \( g \in \mathfrak{g} \):

\[
\begin{align*}
g_0 & = q^2 g_0' + [G,G] + q D'(G) + q E'(G) \\
D(g) & = q D'(g) + [G,g] + \nu'(g) G \\
\Delta(g) & = q \Delta'(g) + [g,G] - \nu'(g) G \quad (26)
\end{align*}
\]

The bijection between \( (\mathcal{F}_1(\mathfrak{g})/ \equiv_1) \sqcup (\mathcal{F}_2(\mathfrak{g})/ \equiv_2) \) and \( \text{Ext}^\mathbb{L}_E(\mathcal{F}, \mathfrak{g}) \) is given by:

\[
\begin{align*}
(g_0, \alpha, \lambda, D, \Delta)^{-1}_{1} \mapsto g_1(x | (g_0, \alpha, \lambda, D, \Delta)) \quad \text{and} \quad (g_0, \nu, D, \Delta)^{-1}_{2} \mapsto g_2(x | (g_0, \nu, \lambda, D, \Delta))
\end{align*}
\]

Proof. The proof relies on Proposition 3.4 and Theorem 2.9. Let \( V \) be a complement of \( \mathfrak{g} \) in \( E \) having \( \{x\} \) as a basis. Since \( \dim_k(V) = 1 \), any linear map \( r : V \rightarrow \mathfrak{g} \) is uniquely determined by an element \( G \in \mathfrak{g} \) such that \( r(x) = G \), where \( \{x\} \) is a basis in \( V \). On the other hand, any automorphism \( \nu \) of \( V \) is uniquely determined by a non-zero scalar \( q \in k^* \) such that \( \nu(x) = qx \). Based on these facts, a little computation shows that the compatibility conditions from Definition 2.7, imposed for the Leibniz extending structures (16)-(17) and respectively (18)-(19), take precisely the form given in the statement of the theorem. We should mention here that a Leibniz extending structure given by (16) - (17) is never equivalent in the sense of Definition 2.7 to a Leibniz extending structure given by (18)-(19), thanks to the compatibility condition (ML1) of Lemma 2.6. Therefore, we obtain the disjoint union from the statement and the proof is finished.

Remark 3.7. In the context of Theorem 3.6 we also have that

\[
\text{Ext}^\mathbb{L}_E(\mathcal{F}, \mathfrak{g}) \cong \mathcal{H}^2(\mathcal{L}, \mathfrak{g}) \cong (\mathcal{F}_1(\mathfrak{g})/ \approx_1) \sqcup (\mathcal{F}_2(\mathfrak{g})/ \approx_2),
\]

where:

\( \approx_i \) is the following relation on \( \mathcal{F}_i(\mathfrak{g}) \): \( (g_0, \alpha, \lambda, D, \Delta) \approx_1 (g_0', \alpha', \lambda', D', \Delta') \) if and only if \( \lambda = \lambda' \) and there exists \( G \in \mathfrak{g} \) such that relations (24)-(27) hold for \( q = 1 \) and respectively \( (g_0, \nu, D, \Delta) \approx_2 (g_0', \nu', D', \Delta') \) if and only if \( \nu = \nu' \) and there exists \( G \in \mathfrak{g} \) such that (28)-(30) hold for \( q = 1 \).

Theorem 3.6 takes a simplified form for perfect Leibniz algebras. Indeed, let \( \mathfrak{g} \) be a perfect Leibniz algebra, i.e. \( \mathfrak{g} \) is generated as a vector space by all brackets \( \{x, y\} \). Then (G1) shows that \( \mathcal{F}_2(\mathfrak{g}) \) is the empty set since, by definition, an element \( \nu \) of a quadruple \( (g_0, \nu, D, \Delta) \in \mathcal{F}_2(\mathfrak{g}) \) is a non-trivial map. Thus, we have that \( \mathcal{F}(\mathfrak{g}) = \mathcal{F}_1(\mathfrak{g}) \). Let now \( (g_0, \alpha, \lambda, D, \Delta) \in \mathcal{F}_1(\mathfrak{g}) \); it follows from (F1) and (F2) that \( \lambda = 0 \), the trivial map, and \( \alpha = 0 \). Furthermore, we can easily see that for a perfect Leibniz algebra \( \mathfrak{g} \), \( \mathcal{F}(\mathfrak{g}) \) identifies with the set of all triples \( (g_0, D, \Delta) \), where \( g_0 \in \mathfrak{g}, D, \Delta : \mathfrak{g} \rightarrow \mathfrak{g} \) are linear
maps satisfying the compatibilities (4)-(7), that is $\mathcal{F}(g) \cong \mathcal{D}(g)$, where $\mathcal{D}(g)$ is the space of all pointed double derivations of $g$ as defined in Definition 1.3.

Two pointed double derivations $(g_0, D, \Delta)$ and $(g_0', D', \Delta')$ are equivalent and we write $(g_0, D, \Delta) \equiv (g_0', D', \Delta')$ if and only if there exists a pair $(q, G) \in k^* \times g$ such that:

\[
g_0 = q^2 g_0' + [G, G] + q D'(G) + q \Delta'(G)
\]
\[
D - q D' = [G, -], \quad \Delta - q \Delta' = [-, G]
\]

On the other hand, two pointed double derivations $(g_0, D, \Delta)$ and $(g_0', D', \Delta')$ are cohomologous and we write $(g_0, D, \Delta) \approx (g_0', D', \Delta')$ if and only if there exists $G \in g$ such that:

\[
g_0 = g_0' + [G, G] + D'(G) + \Delta'(G)
\]
\[
D - D' = [G, -], \quad \Delta - \Delta' = [-, G]
\]

Taking into account the unified product defined by (20)-(21) we obtain:

**Corollary 3.8.** Let $g$ be a perfect Leibniz algebra having $\{e_i \mid i \in I\}$ as a basis. Then any Leibniz algebra $E$ containing $g$ as a subalgebra of codimension 1 has the bracket $[-, -]_E$ defined on the basis $\{x, e_i \mid i \in I\}$ by:

\[
[e_i, e_j]_E := [e_i, e_j], \quad [x, x]_E := g_0, \quad [e_i, x]_E := \Delta(e_i), \quad [x, e_i]_E := D(e_i)
\]

for all $(g_0, D, \Delta) \in \mathcal{D}(g)$. Furthermore, $\mathcal{HL}_g^2(k, g) \cong \mathcal{D}(g)/\equiv$ and $\mathcal{HL}_g^2(k, g) \cong \mathcal{D}(g)/\approx$, where $\equiv$ (resp. $\approx$) is the relation defined by (31)-(32) (resp. (33)-(34)).

On the other hand, we have the following result for abelian Leibniz algebras:

**Example 3.9.** Let $g$ be a vector space with $\{e_i \mid i \in I\}$ as a basis viewed as an abelian Leibniz algebra. Then, there exist three families of Leibniz algebras that contain $g$ as a subalgebra of codimension 1. They have $\{x, e_i \mid i \in I\}$ as a basis and the bracket given for any $i \in I$ as follows:

$g_{11}^{(g_0, D, \Delta)}$:

$[e_i, e_j] = 0$, $[e_i, x] = \Delta(e_i)$, $[x, x] = g_0$, $[x, e_i] = D(e_i)$

for all triples $(g_0, D, \Delta) \in g \times \text{Hom}_k(g, g)^2$ such that $g_0 \in \text{Ker}(D)$ and $D \circ \Delta = \Delta \circ D = -D^2$. The Leibniz algebra $g_{11}^{(g_0, D, \Delta)}$ is the unified product associated to the flag datum of the first kind $(g_0, \alpha, \lambda, D, \Delta)$ for which $\alpha := 0$ and $\lambda := 0$.

The second family of Leibniz algebras has the bracket given as follows:

$g_{12}^{(u, h_0, \lambda)}$:

$[e_i, e_j] = 0$, $[e_i, x] = 0$, $[x, e_i] = u \lambda(e_i) h_0 + \lambda(e_i) x$

$[x, x] = -u^2 \lambda(h_0) h_0 - u \lambda(h_0) x$

for all triples $(u, h_0, \lambda) \in k^* \times g \times \text{Hom}_k(g, k)$ such that $\lambda \neq 0$. The Leibniz algebra $g_{12}^{(u, h_0, \lambda)}$ is the unified product associated to the flag datum of the first kind $(g_0, \alpha, \lambda, D, \Delta)$ for which $g_0 := u^2 \lambda(h_0) h_0$, $\alpha := -u \lambda(h_0)$, $\Delta := 0$ and $D(g) := u \lambda(g) h_0$, for all $g \in g$.

Finally, for the last family of Leibniz algebras the bracket is given as follows:

$g_2^{(u, g_0, h_0, \nu)}$:

$[e_i, e_j] = 0$, $[e_i, x] = -[x, e_i] = -u \nu(e_i) h_0 + \nu(e_i) x$, $[x, x] = g_0$
for all \((u, g_0, h_0, \nu) \in k^* \times g^2 \times \text{Hom}_k(g, k)\) such that: \(2g_0 = 0, \nu(g_0) = 0\) and \(\nu \neq 0\).

The Leibniz algebra \(g_2^{(u, g_0, h_0, \nu)}\) is the unified product associated to the flag datum of the second kind \((g_0, \nu, D, \Delta)\) for which \(D(g) := u \nu(g)h_0\) and \(\Delta(g) := -u \nu(g)h_0\), for all \(g \in g\).

The above results are obtained by a straightforward computation which relies on the explicit description of the set \(F(g)\) for an abelian Leibniz algebra. For instance, axiom (F3) from the flag datum of first kind, for an abelian Leibniz algebra, takes the form \(\lambda(h)\Delta(g) = 0\), for all \(g, h \in g\). Therefore, we have to consider two cases in order to describe the set \(F_1(g)\): the first one corresponds to \(\lambda = 0\), while for the second one we have \(\lambda \neq 0\). The corresponding unified products are the ones given by the first two families of Leibniz algebras. In order to describe the set \(F_2(g)\) we mention that the condition \(2g_0 = 0\) is derived from axiom (G2). In this case the corresponding unified product is the one given by the last family of Leibniz algebras.

Next we provide some explicit examples. First, we prove that any Leibniz algebra which contains a semisimple Lie algebra \(g\) as a subalgebra of codimension 1 is in fact a Lie algebra and the classifying objects \(H^2_0(k, g)\) and \(H^2(k, g)\) are both singletons.

**Example 3.10.** Let \(g\) be a semisimple Lie algebra of codimension 1 in the vector space \(E\) and \(\{e_i \mid i = 1, \cdots, n\}\) a basis of \(g\). Then any Leibniz algebra structure on \(E\) that contains \(g\) as a subalgebra is isomorphic to the Lie algebra having \(\{x, e_i \mid i = 1, \cdots, n\}\) as a basis and the bracket \([-,-]_E\) defined by for any \(i = 1, \cdots, n\) by:

\[
[e_i, e_j]_E := [e_i, e_j], \quad [x, e_i]_E = -[e_i, x]_E := [h_0, e_i], \quad [x, x] = 0
\]

for some \(h_0 \in g\). Furthermore, \(\mathcal{H}L^2_0(k, g) = \mathcal{H}L^2(k, g) = 0\).

Indeed, we apply Corollary 3.8 taking into account that any semisimple Lie algebra \(g\) is perfect, \(\text{Im}(g) = \text{Der}(g)\) and \(Z(g) = 0\). Let \((g_0, D, \Delta) \in F(g)\) be a flag datum of \(g\); then, since \(g\) has a trivial center we obtain from \(4\) that \(g_0 = 0\). Moreover, as \(g\) is perfect it follows again from \(4\) that \(\Delta = -D\). Thus, \(F(g) = \text{Der}(g)\). Since \(g\) is semisimple any derivation \(D \in \text{Der}(g)\) is inner, i.e. there exists \(h_0 \in g\) such that \(D = [h_0, -]\). Thus, the Leibniz algebra \(g_1(x \mid (g_0, \alpha, \lambda, D, \Delta)) = g_1(x \mid D)\) defined by \(20\) and \(21\) takes the form given in the statement. Moreover, two derivations \(D = [h_0, -]\) and \(D' = [h'_0, -]\) are equivalent in the sense of \(34\) if and only if there exists \(G \in g\) such that \(h_0 = h'_0 + G\), i.e. any two derivations are cohomologous. This shows that \(\mathcal{H}L^2(k, g)\) is a singleton having only 0 as an element and so is \(\mathcal{H}L^2_0(k, g)\) being a quotient of it.

Now we will provide an explicit example which highlights the efficiency of Theorem 3.6. More precisely, we will describe all 4-dimensional Leibniz algebras that contain a given non-perfect 3-dimensional Leibniz algebra \(g\) as a subalgebra. Then we will be able to compute the classifying object \(\text{Ext}dL_k(k^4, g) \cong \mathcal{H}L^2_0(k, g)\). The detailed computations are rather long but straightforward and can be provided upon request.

**Example 3.11.** Let \(g\) be the 3-dimensional Leibniz algebra with the basis \(\{e_1, e_2, e_3\}\) and the bracket defined by: \([e_1, e_3] = e_2, [e_3, e_3] = e_1, [e_1, e_2] = e_3\).

Then, there exist four families of 4-dimensional Leibniz algebras which contain \(g\) as a subalgebra: they have \(\{e_1, e_2, e_3, x\}\) as a basis and the bracket is given as follows (the
first three families of Leibniz algebras can be defined over any field \( k \) while in case of the fourth family we need to distinguish between fields of characteristic 2 and those of characteristic different than 2):

(1) If \( \text{char}(k) \neq 2 \) then the four families of Leibniz algebras that contain \( g \) as a subalgebra are the following:

\[
\mathfrak{g}_{11}^{(b_1, b_2, c, d_1, d_2)} : \quad [e_1, e_3] = e_2, \quad [e_3, e_3] = e_1, \\
[e_1, x] = b_1 e_2, \quad [e_3, x] = b_1 e_1 + b_2 e_2, \\
[x, x] = b_1 d_1 e_1 + c e_2, \quad [x, e_3] = d_1 e_1 + d_2 e_2
\]

for all \( b_1, b_2, c, d_1, d_2 \in k \). The Leibniz algebra \( \mathfrak{g}_{11}^{(b_1, b_2, c, d_1, d_2)} \) is the unified product associated to the flag datum of the first kind \((g_0, \alpha, \lambda, D, \Delta)\) defined as follows: \( \alpha := 0, \lambda := 0, g_0 := b_1 d_1 e_1 + c e_2 \) and \( D, \Delta \) are given by

\[
D := \begin{pmatrix} 0 & 0 & d_1 \\ 0 & 0 & d_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Delta := \begin{pmatrix} 0 & 0 & b_1 \\ b_1 & 0 & b_2 \\ 0 & 0 & 0 \end{pmatrix}
\]

The second family of Leibniz algebras has the bracket given by:

\[
\mathfrak{g}_{12}^{(b_1, b_2, b_3, c, d)} : \quad [e_1, e_3] = e_2, \quad [e_3, e_3] = e_1, \quad [e_1, x] = 2b_1 e_1 + b_2 e_2, \\
[e_2, x] = 3b_1 e_2, \quad [e_3, x] = b_2 e_1 + b_3 e_2 + b_1 e_3, \\
[x, x] = (2b_1 d + b_2^2 - b_1 b_3) e_1 + c e_2, \quad [x, e_3] = b_2 e_1 + d e_2 - b_1 e_3
\]

for all \( b_1 \in k^* \) and \( b_2, b_3, c, d \in k \). The Leibniz algebra \( \mathfrak{g}_{12}^{(b_1, b_2, b_3, c, d)} \) is the unified product associated to the flag datum of the first kind \((g_0, \alpha, \lambda, D, \Delta)\) defined as follows: \( \alpha := 0, \lambda := 0, g_0 := (2b_1 d + b_2^2 - b_1 b_3) e_1 + c e_2 \) and \( D, \Delta \) are given by

\[
D := \begin{pmatrix} 0 & 0 & b_2 \\ 0 & 0 & d \\ 0 & 0 & -b_1 \end{pmatrix}, \quad \Delta := \begin{pmatrix} 2b_1 & 0 & b_2 \\ b_2 & 3b_1 & b_3 \\ 0 & 0 & b_1 \end{pmatrix}
\]

The third family of Leibniz algebras has the bracket given by:

\[
\mathfrak{g}_{13}^{(\alpha, \lambda_0, d_1, d_2)} : \quad [e_1, e_3] = e_2, \quad [e_3, e_3] = e_1, \quad [e_1, x] = \alpha \lambda_0^{-1} e_2, \\
[e_3, x] = \alpha \lambda_0^{-1} e_1, \quad [x, x] = \alpha \lambda_0^{-1} (d_1 e_1 + d_2 e_2 - \alpha e_3) + \alpha x, \\
[x, e_3] = d_1 e_1 + d_2 e_2 - \alpha e_3 + \lambda_0 x
\]

for all \( \lambda_0 \in k^* \) and \( \alpha, d_1, d_2 \in k \). The Leibniz algebra \( \mathfrak{g}_{13}^{(\alpha, \lambda_0, d_1, d_2)} \) is the unified product associated to the flag datum of the first kind \((g_0, \alpha, \lambda, D, \Delta)\) defined as follows: \( \lambda(e_1) = \lambda(e_2) := 0, \lambda(e_3) := \lambda_0 \neq 0, g_0 := \alpha \lambda_0^{-1} d_1 e_1 + \alpha \lambda_0^{-1} d_2 e_2 - \alpha^2 \lambda_0^{-1} e_3 \) and \( D, \Delta \) are given by

\[
D := \begin{pmatrix} 0 & 0 & d_1 \\ 0 & 0 & d_2 \\ 0 & 0 & -\alpha \end{pmatrix}, \quad \Delta := \begin{pmatrix} 0 & 0 & \alpha \lambda_0^{-1} \\ \alpha \lambda_0^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
Finally, the last family of Leibniz algebras has the bracket defined as follows:

\[ [e_1, e_3] = e_2, \quad [e_3, e_3] = e_1, \quad [e_1, x] = \nu_0^{-1} d_3 e_2, \]
\[ [e_3, x] = (-d_1 + 2\nu_0^{-1} d_3) e_1 - (d_2 - \nu_0^{-1} d_1 + \nu_0^{-2} d_3) e_2 - d_3 e_3 + \nu_0 x, \]
\[ [x, x] = \nu_0^{-2} d_3^2 e_1 + (\nu_0^{-2} d_1 d_3 - \nu_0^{-3} d_3^2) e_2, \]
\[ [x, e_3] = d_1 e_1 + d_2 e_2 + d_3 e_3 - \nu_0 x \]

for all \( \nu_0 \in k^* \) and \( d_1, d_2, d_3 \in k \). The Leibniz algebra \( \mathfrak{g}_{21}^{(\nu_0, d_1, d_2, d_3)} \) is the unified product associated to the flag datum of the second kind \((g_0, \nu, D, \Delta)\) defined as follows: \( \nu(e_1) = \nu(e_2) := 0, \quad \nu(e_3) := \nu_0 \neq 0, \quad g_0 := \nu_0^{-2} d_3^2 e_1 + (\nu_0^{-2} d_1 d_3 - \nu_0^{-3} d_3^2) e_2 \) and \( D, \Delta \) are given by

\[
D := \begin{pmatrix} 0 & 0 & d_1 \\ 0 & 0 & d_2 \\ 0 & 0 & d_3 \end{pmatrix} \quad \Delta := \begin{pmatrix} 0 & 0 & -d_1 + 2\nu_0^{-1} d_3 \\ \nu_0^{-1} d_3 & 0 & -d_2 + \nu_0^{-1} d_1 - \nu_0^{-2} d_3 \\ 0 & 0 & -d_3 \end{pmatrix}
\]

(35)

(2) If char(k) = 2, then the four families of Leibniz algebras that contain \( \mathfrak{g} \) as a subalgebra are the following: \( \mathfrak{g}_{11}^{(b_1, b_2, c, d_1, d_2)}, \mathfrak{g}_{12}^{(b_1, b_2, b_3, c, d)}, \mathfrak{g}_{13}^{(c, \lambda_0, d_1, d_2)} \) defined above together with the family of Leibniz algebras defines as follows:

\[ [e_1, e_3] = e_2, \quad [e_3, e_3] = e_1, \quad [e_1, x] = \nu_0^{-1} d_3 e_2, \]
\[ [e_3, x] = -d_1 e_1 - (d_2 - \nu_0^{-1} d_1 + \nu_0^{-2} d_3) e_2 - d_3 e_3 + \nu_0 x, \]
\[ [x, x] = \nu_0^{-2} d_3^2 e_1 + c e_2, \quad [x, e_3] = d_1 e_1 + d_2 e_2 + d_3 e_3 - \nu_0 x \]

for all \( \nu_0 \in k^* \) and \( c, d_1, d_2, d_3 \in k \). The Leibniz algebra \( \mathfrak{g}_{22}^{(c, \nu_0, d_1, d_2, d_3)} \) is the unified product associated to the flag datum of the second kind \((g_0, \nu, D, \Delta)\) defined as follows: \( \nu(e_1) = \nu(e_2) := 0, \quad \nu(e_3) := \nu_0 \neq 0, \quad g_0 := \nu_0^{-2} d_3^2 e_1 + c e_2 \) and \( D, \Delta \) are given by \((35)\).

The proof is a purely computational one and we will only indicate the main steps. We start by computing \( F_1(\mathfrak{g}) \). First, notice that a linear map \( \lambda : \mathfrak{g} \rightarrow k \) satisfies the first compatibility of (F1), i.e. \( \lambda([g, h]) = 0 \) if and only if \( \lambda \) is given by \( \lambda(e_1) = \lambda(e_2) = 0 \) and \( \lambda(e_3) = \lambda_0 \), for some \( \lambda_0 \in k \). For such a \( \lambda \) we can easily show that a pair \((D, \Delta)\) satisfies the compatibilities (F6) and (F7) if and only if we have:

\[
D = \begin{pmatrix} 0 & 0 & d_1 \\ 0 & 0 & d_2 \\ 0 & 0 & d_3 \end{pmatrix} \quad \Delta = \begin{pmatrix} 2b_1 & 0 & b_2 \\ b_2 & 3b_1 & b_3 \\ 0 & 0 & b_1 \end{pmatrix}
\]

for some \( d_1, d_2, d_3, b_1, b_2, b_3 \in k \). Let now \( \alpha \in k \) and consider \( g_0 = c_1 e_1 + c_2 e_2 + c_3 e_3 \), for some \( c_1, c_2, c_3 \in k \). We can easily see that the 5-tuple \((g_0, \alpha, \lambda, D, \Delta)\) satisfies the compatibilities (F1) - (F7), i.e. it is a flag datum of the first kind if and only if coincides with one of the three flag datums described in (1). For instance, the compatibility (F1) is fulfilled if and only if \( \lambda_0(\alpha + d_3) = \lambda_0 b_1 = 0 \). This last equality leads us to consider two cases, namely \( \lambda_0 = 0 \) or \( \lambda_0 \neq 0 \). It is now straightforward to describe \( F_1(\mathfrak{g}) \) (without depending on the characteristic of \( k \)).

Analogously, we can describe \( F_2(\mathfrak{g}) \). A non-trivial map \( \nu : \mathfrak{g} \rightarrow k \) satisfies the first compatibility of (G1) if and only if \( \nu \) is given by \( \nu(e_1) = \nu(e_2) = 0 \) and \( \nu(e_3) = \nu_0 \),
for some $\nu_0 \in k^*$. For such a map $\nu$, we can easily show that $(D, \Delta)$ satisfies
the compatibilities (G4) and (G5) if and only if $D$ and $\Delta$ are given by (35). By considering
again $g_0 = c_1 e_1 + c_2 e_2 + c_3 e_3$, for some $c_1, c_2, c_3 \in k$ we see that the compatibility (G1)
is fulfilled if and only if $c_3 = 0$. The last compatibility of (G3) is equivalent to:
$$2c_1 = 2\nu_0^{-2} d_1^2, \quad c_1 + 2\nu_0 e_2 = 2\nu_0^{-1} d_1 d_3 - \nu_0^{-1} d_3^2$$
The above two compatibilities are the ones that lead us to the description of $\mathcal{F}_2(g)$
depending on the characteristic of $k$. Moreover, these computations provide also the
description of the classifying object $\mathcal{H}L_2^2(k, g)$. If $\text{char}(k) \neq 2$ then
$$\mathcal{H}L_2^2(k, g) \cong (k^5/\equiv_{11}) \sqcup ((k^* \times k^4)/\equiv_{12}) \sqcup ((k^* \times k^3)/\equiv_{13}) \sqcup ((k^* \times k^3)/\equiv_2) \quad (36)$$
where $\equiv_{1i}$ are the equivalence relations (24)-(27) while $\equiv_2$ is the equivalence relation
(28)-(30). In the case when $\text{char}(k) = 2$, then the last term of (36) is replaced by
$(k^* \times k^3)/\equiv_2$.

4. Special cases of unified products

In this section we deal with two special cases of the unified product namely the crossed
(resp. bicrossed) product of two Leibniz algebras. We emphasize the problem for which
each of these products is responsible. We use the following convention: if one of the maps
$\triangleleft, \triangleright, \leftarrow, \rightarrow, f$ or $\{-,-\}$ of an extending datum $\Omega(g, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \{-,-\})$ is
trivial then we will omit it from the 6-tuple $(\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \{-,-\})$.

Crossed products and the extension problem for Leibniz algebras. We shall highlight a first special case of the unified product, namely the crossed product of Leibniz
algebras that is the key player in the study of the extension problem in its full generality.
Let $\Omega(g, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \{-,-\})$ be an extending datum of a Leibniz algebra $g$
through $V$ such that $\triangleleft$ and $\rightarrow$ are both trivial, i.e. $x \triangleleft g = g \rightarrow x = 0$, for all $x \in V$ and
$g \in g$. Then, it follows from Theorem 2.3 that $\Omega(g, V) = (\triangleright, \leftarrow, f, \{-,-\})$ is a Leibniz
extension structure of $g$ through $V$ if and only if $(V, \{-,-\})$ is a Leibniz algebra and
$(g, V, \triangleright, \leftarrow, f)$ is a crossed system of Leibniz algebras, i.e. the following compatibilities
hold for any $g, h \in g$ and $x, y, z \in V$:

(CS1) $[g, h] \leftarrow x = [g, h \leftarrow x] + [g \leftarrow x, h]$;
(CS2) $g \leftarrow \{x, y\} = (g \leftarrow x) \leftarrow y - (g \leftarrow y) \leftarrow x - [g, f(x, y)]$;
(CS3) $x \triangleright f(y, z) = f(x, y) \leftarrow z - f(x, z) \leftarrow y + f\{x, y, z\} - f\{x, y\} - f(x, y, z)$;
(CS4) $x \triangleright [g, h] = [x \triangleright g, h] - [x \triangleright h, g]$;
(CS5) $\{x, y\} \triangleright g = x \triangleright (y \triangleright g) + (x \triangleright g) \leftarrow y - [f(x, y), g]$;
(CS6) $[g, h \leftarrow x] + [g, x \triangleright h] = 0$;
(CS7) $x \triangleright (y \triangleright g) + x \triangleright (g \leftarrow y) = 0$.

In this case, the associated unified product $g \times_{\Omega(g, V)} V$ will be denoted by $g^{\#f}_{\triangleright, \leftarrow} V$ and
we shall call it the crossed product of the Leibniz algebras $g$ and $V$. Hence, the crossed
product associated to the crossed system $(g, V, \triangleright, \leftarrow, f)$ is the Leibniz algebra defined as
follows: $g^{\#f}_{\triangleright, \leftarrow} V = g \times V$ with the bracket given for any $g, h \in g$ and $x, y \in V$ by:
The crossed product of Leibniz algebras is the object responsible for answering the following special case of the ES problem, which is a generalization of the extension problem: Let $g$ be a Leibniz algebra, $E$ a vector space containing $g$ as a subspace. Describe and classify all Leibniz algebra structures on $E$ such that $g$ is a two-sided ideal of $E$. The classical extension problem initiated in [21] is a special case of this question if we require the additional assumption on the quotient $E/g$ to be isomorphic to a given Leibniz algebra $h$.

Indeed, let $(g, V, \triangleright, \triangleleft, f)$ be a crossed system of two Leibniz algebras. Then, $g \cong g \times \{0\}$ is a two-sided ideal in the crossed product $g\#_{\triangleright,\triangleleft} V$ since $[(g, 0), (h, y)] := ([g, h] + g \triangleleft y, 0)$ and $[(g, x), (h, 0)] := ([g, h] + x \triangleright h, 0)$. Conversely, we have:

**Corollary 4.1.** Let $g$ be a Leibniz algebra, $E$ a vector space containing $g$ as a subspace. Then any Leibniz algebra structure on $E$ that contains $g$ as a two-sided ideal is isomorphic to a crossed product of Leibniz algebras $g\#_{\triangleright,\triangleleft} V$ and the isomorphism can be chosen to stabilize $g$ and co-stabilize $V$.

**Proof.** Let $[-,-]$ be a Leibniz algebra structure on $E$ such that $g$ is a two-sided ideal in $E$. In particular, $g$ is a subalgebra of $E$ and hence we can apply Theorem 2.5. In this case the actions $\triangleleft = \triangleleft_p$ and $\triangleright = \triangleright_p$ of the Leibniz extending structure $\Omega(g, V) = (\triangleleft_p, \triangleright_p, f_p, \{-,-\}_p)$ constructed in the proof of Theorem 2.5 are both trivial since for any $x \in V$ and $g \in g$ we have that $[x, g], [g, x] \in g$ and hence $p([x, g]) = [x, g]$ and $p([g, x]) = [g, x]$. Thus, $x \triangleleft_p g = g \triangleright_p x = 0$ and hence the Leibniz extending structure $\Omega(g, V) = (\triangleleft, \triangleright, \triangleleft, \triangleright, f, \{-,-\})$ constructed in the proof of Theorem 2.5 is precisely a crossed system of Leibniz algebras and the unified product $g \ltimes_{\Omega(g, V)} V = g\#_{\triangleright,\triangleleft} V$ is the crossed product of $g$ and $V = \text{Ker}(p)$. □

Let $g$ and $h$ be two given Leibniz algebras. The extension problem asks for the classification of all extensions of $h$ by $g$, i.e., of all Leibniz algebras $E$ that fit into an exact sequence

$$0 \longrightarrow g \overset{i}{\longrightarrow} E \overset{\pi}{\longrightarrow} h \longrightarrow 0$$

(38)

The classification is up to an isomorphism of Leibniz algebras that stabilizes $g$ and co-stabilizes $h$ and we denote by $\mathcal{E}P(h, g)$ the isomorphism classes of all extensions of $h$ by $g$ up to this equivalence relation. If $g$ is abelian, then $\mathcal{E}P(h, g) \cong \text{HL}^2(h, g)$, where $\text{HL}^2(h, g)$ is the second cohomology group [21, Proposition 1.9]. The crossed product is the tool to approach the extension problem in its full generality, leaving aside the abelian case. Let us explain this briefly. Consider $g$ and $h$ be two Leibniz algebras and we denote by $\mathcal{CS}(h, g)$ the set of all triples $(\triangleright, \triangleleft, f)$ such that $(g, h, \triangleright, \triangleleft, f)$ is a crossed system of Leibniz algebras. First we remark that, if $(g, h, \triangleright, \triangleleft, f)$ is a crossed system, then the crossed product $g\#_{\triangleright,\triangleleft} h$ is an extension of $h$ by $g$ via

$$0 \longrightarrow g \overset{\iota}{\longrightarrow} g\#_{\triangleright,\triangleleft} h \overset{\pi_n}{\longrightarrow} h \longrightarrow 0$$

(39)
where \( i_{\mathfrak{g}}(g) = (g, 0) \) and \( \pi_{\mathfrak{h}}(g, h) = h \) are the canonical maps. Conversely, Corollary 4.1 shows that any extension \( \mathcal{E} \) of \( \mathfrak{h} \) by \( \mathfrak{g} \) is equivalent to a crossed product extension of the form \((39)\). Thus, the classification of all extensions of \( \mathfrak{h} \) by \( \mathfrak{g} \) reduces to the classification of all crossed products \( \mathfrak{g} \mathcal{L} \mathfrak{h} \) associated to all crossed systems of Leibniz algebras \((\mathfrak{g}, \mathfrak{h}, \triangleright, \leftarrow, f)\). Definition 2.8, in the special case of crossed systems, takes the following simplified form: two triples \((\triangleright, \leftarrow, f)\) and \((\triangleright', \leftarrow', f')\) of \( \mathcal{CS}(\mathfrak{h}, \mathfrak{g}) \) are cohomologous and we denote this by \((\triangleright, \leftarrow, f) \approx (\triangleright', \leftarrow', f')\) if there exists a linear map \( r : \mathfrak{h} \to \mathfrak{g} \) such that:

\[
\begin{align*}
x \triangleright g &= x \triangleright' g + [r(x), g] \\
g \leftarrow x &= g \leftarrow' x + [g, r(x)] \\
f(x, y) &= f'(x, y) + [r(x), r(y)] - r(\{x, y\}) + r(x) \leftarrow' y + x \triangleright' r(y)
\end{align*}
\]

for all \( g \in \mathfrak{g}, x, y \in \mathfrak{h} \). Then, as we mentioned before Definition 2.8, \((\triangleright, \leftarrow, f) \approx (\triangleright', \leftarrow', f')\) if and only if there exists \( \psi : \mathfrak{g} \mathcal{L} \mathfrak{h} \to \mathfrak{g} \mathcal{L} \mathfrak{h}' \) an isomorphism of Leibniz algebras that stabilizes \( \mathfrak{g} \) and co-stabilizes \( \mathfrak{h} \). As a special case of Theorem 2.9, we obtain the theoretical answer to the extension problem in the general (non-abelian) case:

**Corollary 4.2.** Let \( \mathfrak{g} \) and \( \mathfrak{h} \) be two arbitrary Leibniz algebras. Then \( \approx \) is an equivalence relation on the set \( \mathcal{CS}(\mathfrak{h}, \mathfrak{g}) \) of all crossed systems and the map

\[
\mathbb{H}^2(\mathfrak{h}, \mathfrak{g}) := \mathcal{CS}(\mathfrak{h}, \mathfrak{g})/\approx \to \mathcal{EP}(\mathfrak{h}, \mathfrak{g}) , \quad \overline{(\triangleright, \leftarrow, f)} \mapsto \mathfrak{g} \mathcal{L} \mathfrak{h}
\]

is a bijection between sets, where \( \overline{(\triangleright, \leftarrow, f)} \) is the equivalence class of \((\triangleright, \leftarrow, f)\) via \( \approx \).

If \( \mathfrak{g} \) is an abelian Leibniz algebra, then \( \mathbb{H}^2(\mathfrak{h}, \mathfrak{g}) \) coincides with the second cohomology group \( \mathbb{H}^2(\mathfrak{h}, \mathfrak{g}) \) constructed in [21]. The explicit answer to the extension problem for two given Leibniz algebras \( \mathfrak{g} \) and \( \mathfrak{h} \) will be given once we compute the non-abelian cohomological object \( \mathbb{H}^2(\mathfrak{h}, \mathfrak{g}) \) which in general is a highly non-trivial problem. A detailed study of this object for various Leibniz algebras will be given elsewhere; here we give only one example that corresponds to the case when \( \mathfrak{h} := k \), the abelian Leibniz algebra of dimension 1, as this is a special case of Theorem 3.6 and Remark 3.7.

**Corollary 4.3.** Let \( \mathfrak{g} \) be a Leibniz algebra with \( \{e_i \mid i \in I\} \) as a basis. Then

\[
\mathbb{H}^2(k, \mathfrak{g}) \cong \mathcal{D}(g)/\approx
\]

where \( \mathcal{D}(g) \) is the space of all pointed double derivations of \( \mathfrak{g} \) and \( \approx \) is the equivalence relation defined by \((33)-(34)\). In particular, any extension of \( k \) by \( \mathfrak{g} \) is isomorphic to the Leibniz algebra having \( \{x, e_i \mid i \in I\} \) as a basis and the bracket \([-, -]_{(g_0, D, \Delta)}\) defined for any \( i \in I \) by:

\[
\begin{align*}
[e_i, e_j]_{(g_0, D, \Delta)} := [e_i, e_j], & \quad [x, e_i]_{(g_0, D, \Delta)} := g_0 \\
[e_i, x]_{(g_0, D, \Delta)} := \Delta(e_i), & \quad [x, e_i]_{(g_0, D, \Delta)} := D(e_i)
\end{align*}
\]

for some \((g_0, D, \Delta) \in \mathcal{D}(g)\).

**Proof.** Follows from Theorem 3.6 since the set of crossed systems \( \mathcal{CS}(k, \mathfrak{g}) \) is precisely the set \( \Omega(\mathfrak{g}, k) = (\triangleright, \leftarrow, \to, f, \{-, -\}) \) of all Leibniz extending structures of \( \mathfrak{g} \) through \( k \) having the actions \( \triangleright \) and \( \leftarrow \) both trivial. Moreover, any extension \( E \) of \( k \) by \( \mathfrak{g} \) is a
Leibniz algebra containing \( g \) as a subalgebra of codimension 1. In this context, the compatibility conditions (F1)-(F7) that define a flag datum of the first kind collapses to (4)-(7). The fact that \( < \) is the trivial action implies that \( \lambda = 0 \). The Leibniz algebra from the statement is the unified (crossed) product defined by (20)-(21).

In the next example we compute explicitly the object \( \text{HL}^2(k, g) \) for a certain Leibniz algebra \( g \).

**Example 4.4.** Let \( g \) be the 3-dimensional Leibniz algebra with the basis \( \{e_1, e_2, e_3\} \) and the bracket defined by: \( [e_1, e_3] = e_2, [e_3, e_3] = e_1 \). A little computation, similar to one performed in Example 3.11, shows that the set \( \mathcal{D}(g) \) identifies with the set of all 6-tuples \((c, b_1, b_2, b_3, d_1, d_2) \in k^6 \) which satisfy:

\[
b_1 (d_1 - b_2) = 0
\]

The bijection is defined such that \((g_0, D, \Delta) \in \mathcal{D}(g) \) corresponding to \((c, b_1, b_2, b_3, d_1, d_2) \) is given by

\[
g_0 := (2b_1d_2 + b_2d_1 - b_1b_3) e_1 + c e_2, \quad D := \begin{pmatrix}
0 & 0 & d_1 \\
0 & 0 & d_2 \\
0 & -b_1 & 0
\end{pmatrix}, \quad \Delta := \begin{pmatrix}
2b_1 & 0 & b_2 \\
b_2 & 3b_1 & b_3 \\
0 & 0 & b_1
\end{pmatrix}
\]

The compatibility condition \( b_1 (d_1 - b_2) = 0 \) imposes a discussion on whether \( b_1 = 0 \) or \( b_1 \neq 0 \). This leads to the description of \( \text{HL}^2(k, g) \) as the following coproduct of sets:

\[
\text{HL}^2(k, g) \cong (k^5/\approx_1) \sqcup (k^* \times k^4/\approx_2),
\]

where:

\( \approx_1 \) is the following relation on \( k^5 \):

\[
(c, b_2, b_3, d_1, d_2) \approx_1 (c', b'_2, b'_3, d'_1, d'_2) \text{ if and only if there exist } u, v \in k \text{ such that}
\]

\[
c = c' + uv, \quad b_2 = b'_2 + v, \quad b_3 = b'_3 + v, \quad d_1 = d'_1 + v, \quad d_2 = d'_2 + u
\]

and \( \approx_2 \) is the relation on \( k^* \times k^4 \) defined by:

\[
(b_1, c, b_3, d_1, d_2) \approx_2 (b'_1, c', b'_3, d'_1, d'_2) \text{ if and only if } b_1 = b'_1, b_3 = b'_3, d_2 = d'_2 \text{ and there exist } v, w \in k \text{ such that}
\]

\[
c = c' + vd'_1 + 3wb'_1 + (d_1 - d'_1)(v + d'_2 + b'_3)
\]

**Bicrossed products and the factorization problem for Leibniz algebras.** The concept of a matched pair of Lie algebras was introduced in [23, Theorem 4.1] and independently in [22, Theorem 3.9]. For any such matched pair of Lie algebras a new Lie algebra, called the **bicrossed product** is constructed under the name of bicrossproduct in [23, Theorem 4.1], double cross sum in [24, Proposition 8.3.2], double Lie algebra [22, Definition 3.3]. Now we shall introduce the concept of a matched pair of Leibniz algebras. As we will see, in this case the definition is a lot more laborious.

**Definition 4.5.** A **matched pair** of Leibniz algebras is a system \((g, h, <, \triangleright, \Leftarrow, \triangleright)\) consisting of two Leibniz algebras \((g, \{, , \}), (h, \{, , \})\) and four bilinear maps \( < : g \times g \rightarrow h, \triangleright : h \times g \rightarrow g, \Leftarrow : g \times h \rightarrow g, \triangleright : g \times h \rightarrow h \) satisfying the following compatibilities for any \( g, h \in g, x, y \in h \):

- \((MP1)\) \((h, <)\) is a right \( g \)-module, i.e. \( x < [g, h] = (x < g) < h - (x < h) < g \);
- \((MP2)\) \((g, \Leftarrow)\) is a right \( h \)-module, i.e. \( g \Leftarrow \{x, y\} = (g \Leftarrow x) \Leftarrow y - (g \Leftarrow y) \Leftarrow x \);
- \((MP3)\) \( x \triangleright [g, h] = [x \triangleright g, h] - [x \triangleright h, g] + (x \Leftarrow g) \triangleright h - (x \Leftarrow h) \triangleright g \);
To start with, notice that any bicrossed product

\[ \{ x, y \} \triangleleft g = x \triangleleft (y \triangleright g) + (x \triangleright g) \rightharpoonup y + \{ x, y \triangleleft g \} + \{ x \triangleleft g, y \}; \]

\[ \{ x, y \} \triangleright g = x \triangleright (y \triangleright g) + (x \triangleright g) \leftarrow y; \]

\[ [g, h] \leftarrow x = [g, h \leftarrow x] + [g \leftarrow x, h] + g \leftarrow (h \rightarrow x) + (g \rightarrow x) \triangleright h; \]

\[ \{ x, y \} \leftarrow g = g \leftarrow (h \rightarrow x) + (g \rightarrow x) \triangleright h; \]

\[ g \rightarrow \{ x, y \} = (g \rightarrow x) \rightarrow y - (g \rightarrow y) \rightarrow x + \{ g \rightarrow x, y \} - \{ g \rightarrow y, x \}; \]

\[ [g, h] \rightarrow x = [g \rightarrow (h \rightarrow x) + (g \rightarrow x) \triangleright h; \]

\[ g \rightarrow \{ x, y \} = (g \rightarrow x) + (g \rightarrow y) + \{ x, y \triangleleft g \} + \{ x, g \rightarrow y \} = 0; \]

\[ g \rightarrow (h \rightarrow x) + g \rightarrow (x \triangleleft h) = 0. \]

Let \((g, h, \triangleleft, \triangleright, \leftarrow, \rightarrow)\) be a matched pair of Leibniz algebras. Then \(g \ast h := g \times h\), as a vector space, with the bracket defined for any \(g, h \in g\) and \(x, y \in h\) by

\[ [(g, x), (h, y)] := ([g, h] + x \triangleright h + g \leftarrow y, \{ x, y \} + x \triangleleft h + g \rightarrow y) \]

is a Leibniz algebra called the bicrossed product associated to the matched pair of Leibniz algebras \((g, h, \triangleleft, \triangleright, \leftarrow, \rightarrow)\). This fact can be proved directly, but it can also be derived as a special case of Theorem 2.3. Indeed, let \(g\) be a Leibniz algebra and \(\Omega(g, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \{ - \}, \{ - \})\) an extending datum of \(g\) through \(V\) such that \(f\) is the trivial map, i.e. \(f(x, y) = 0\), for all \(x, y \in V\). Then, we can easily see that \(\Omega(g, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, \{ - \}, \{ - \})\) is a Leibniz extending structure of \(g\) through \(V\) if and only if \((V, \{ - \}, \{ - \})\) is a Leibniz algebra and \((V, \triangleleft, \triangleright, \leftarrow, \rightarrow)\) is a matched pair of Leibniz algebras in the sense of Definition 4.5. In this case, the associated unified product \(g \times_{\Omega(g, V)} V = g \ast V\) is the bicrossed product of the matched pair \((g, V, \triangleleft, \triangleright, \leftarrow, \rightarrow)\) as defined by (40).

The bicrossed product of two Leibniz algebras is the construction responsible for the factorization problem, which is a special case of the ES problem and can be stated as follows: Let \(g\) and \(h\) be two given Leibniz algebras. Describe and classify all Leibniz algebras \(\Xi\) that factorize through \(g\) and \(h\), i.e. \(\Xi\) contains \(g\) and \(h\) as Leibniz subalgebras such that \(\Xi = g \ast h\) and \(g \cap h = \{ 0 \}\). Indeed, using Theorem 2.5 we can prove the following:

**Corollary 4.6.** A Leibniz algebra \(\Xi\) factorizes through \(g\) and \(h\) if and only if there exists a matched pair of Leibniz algebras \((g, h, \triangleleft, \triangleright, \leftarrow, \rightarrow)\) such that \(\Xi \cong g \ast h\).

**Proof.** To start with, notice that any bicrossed product \(g \ast h\) factorizes through \(g \cong g \times \{ 0 \}\) and \(h \cong \{ 0 \} \times h\). Conversely, assume that \(\Xi\) factorizes through \(g\) and \(h\). Let \(p : \Xi \rightarrow g\) be the \(k\)-linear projection of \(\Xi\) on \(g\), i.e. \(p(g + x) := g\), for all \(g \in g\) and \(x \in h\). Now, we apply Theorem 2.5 for \(V := \ker(p) = h\). Since \(V\) is a Leibniz subalgebra of \(E := \Xi\), the map \(f = f_p\) constructed in the proof of Theorem 2.5 is the trivial map as \([x, y] \in V = \ker(p).\) Thus, the Leibniz extending structure \(\Omega(g, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \{ - \}, \{ - \})\) constructed in the proof of Theorem 2.5 is precisely a matched pair of Leibniz algebra and the unified product \(g \times_{\Omega(g, V)} V = g \ast V\) is the bicrossed product of the matched pair \((g, V, \triangleleft, \triangleright, \leftarrow, \rightarrow)\). Explicitly, the matched pair \((g, h, \triangleleft, \triangleright, \leftarrow, \rightarrow)\) is given by:

\[ x \triangleright g := p([x, g]), \quad x \triangleleft g := [x, g] - p([x, g]) \]

\[ g \leftarrow x := p([g, x]), \quad g \rightarrow x := [g, x] - p([g, x]) \]
for all $x \in \mathfrak{h}$ and $g \in \mathfrak{g}$.

From now on the matched pair constructed in (41) and (42) will be called the canonical matched pair associated to the factorization $\Xi = \mathfrak{g} + \mathfrak{h}$ of $\Xi$ through $\mathfrak{g}$ and $\mathfrak{h}$. Based on Corollary 4.6 the factorization problem can be restated in a computational manner as follows: Let $\mathfrak{g}$ and $\mathfrak{h}$ be two given Leibniz algebras. Describe the set of all matched pairs $(\mathfrak{g}, \mathfrak{h}, \triangleleft, \triangleright, \leftarrow, \rightarrow)$ and classify up to an isomorphism all bicrossed products $\mathfrak{g} \bowtie \mathfrak{h}$. A detailed study of this problem will be given somewhere else.

**Example 4.7.** Let $\mathfrak{g}$ be the 3-dimensional Leibniz algebra considered in Example 3.11 and $\mathfrak{k}$ be the 1-dimensional (abelian) Leibniz algebra. Then all bicrossed products $\mathfrak{g} \bowtie \mathfrak{k}$ can be explicitly described as a special case of Example 3.11. To this end we need to consider $g_0 := 0$ and $\alpha := 0$ in the unified products associated to all flag datums of the first kind provided in Example 3.11 and $g_0 := 0$ in the unified products associated to all flag datums of the second kind. For instance, by taking $d_3 = 0$ in the Leibniz algebra $\mathfrak{g}^{(0, d_1, d_2, d_3)}_{21}$ of Example 3.11 we obtain the bicrossed product $\mathfrak{g} \bowtie \mathfrak{k}$ which is a 4-dimensional Leibniz algebra with the basis $\{e_1, e_2, e_3, x\}$ and the bracket given by:

\[
\begin{align*}
[e_1, e_3] &= e_2, & [e_3, e_3] &= e_1, & [e_3, x] &= -d_1 e_1 + (d_2 - \nu_0^{-1} d_1) e_2 + \nu_0 x, \\
[x, e_3] &= d_1 e_1 + d_2 e_2 - \nu_0 x
\end{align*}
\]

for all $\nu_0 \in k^*$ and $d_1, d_2 \in k$. This Leibniz algebra is the bicrossed product associated to the following matched pair $(\mathfrak{g}, \mathfrak{k}, \triangleleft, \triangleright, \leftarrow, \rightarrow)$:

\[
\begin{align*}
x \triangleleft e_3 &= -\nu_0 x, & x \triangleright e_3 &= d_1 e_1 + d_2 e_2, \\
 e_3 \rightarrow x &= \nu_0 x, & e_3 \leftarrow x &= -d_1 e_1 + (d_2 + \nu_0^{-1} d_1) e_2
\end{align*}
\]

where the undefined actions are zero and $x$ is a basis of $\mathfrak{k}$. Another example of a matched pair of Leibniz algebras and the corresponding bicrossed product will be given in Example 5.6.

5. **Classifying complements for extensions of Leibniz algebras**

This section is devoted to the classifying complements (CC) problem whose statement was given in the Introduction. Let $\mathfrak{g} \subseteq \Xi$ be a Leibniz subalgebra of $\Xi$. A Leibniz subalgebra $\mathfrak{h}$ of $\Xi$ is called a complement of $\mathfrak{g}$ in $\Xi$ (or a $\mathfrak{g}$-complement of $\Xi$) if $\Xi = \mathfrak{g} + \mathfrak{h}$ and $\mathfrak{g} \cap \mathfrak{h} = \{0\}$. If $\mathfrak{h}$ is a complement of $\mathfrak{g}$ in $\Xi$, Corollary 4.6 shows that $\Xi \cong \mathfrak{g} \bowtie \mathfrak{h}$, where $\mathfrak{g} \bowtie \mathfrak{h}$ is the bicrossed product associated to the canonical matched pair of the factorization $\Xi = \mathfrak{g} + \mathfrak{h}$ as constructed in (41) and (42).

We denote by $\mathcal{F}(\mathfrak{g}, \Xi)$ the (possibly empty) isomorphism classes of all $\mathfrak{g}$-complements of $\Xi$. The factorization index of $\mathfrak{g}$ in $\Xi$ is defined by $|\Xi : \mathfrak{g}|^F := |\mathcal{F}(\mathfrak{g}, \Xi)|$ as a numerical measure of the (CC) problem.

**Definition 5.1.** Let $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft, \leftarrow, \rightarrow)$ be a matched pair of Leibniz algebras. A linear map $r : \mathfrak{h} \to \mathfrak{g}$ is called a deformation map of the matched pair $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft, \leftarrow, \rightarrow)$ if the following compatibility holds for any $x, y \in \mathfrak{h}$:

\[
r([x, y]) - [r(x), r(y)] = x \triangleright r(y) + r(x) \leftarrow y - r(x \triangleleft r(y) + r(x) \rightarrow y)
\]

(43)
We denote by \(\mathcal{DM}(\mathfrak{h}, \mathfrak{g} | \langle, \langle, \rightarrow \rangle)\) the set of all deformation maps of the matched pair \((\mathfrak{g}, \mathfrak{h}, \triangleright, \leftarrow, \rightarrow)\). The trivial map \(r(x) = 0\), for all \(x \in \mathfrak{h}\), is of course a deformation map. The right hand side of (43) measures how far \(r : \mathfrak{h} \rightarrow \mathfrak{g}\) is from being a Leibniz algebra map. Using this concept which will play a key role in solving the (CC) problem, we introduce the following deformation of a Leibniz algebra:

**Theorem 5.2.** Let \(\mathfrak{g}\) be a Leibniz subalgebra of \(\Xi\), \(\mathfrak{h}\) a given \(\mathfrak{g}\)-complement of \(\Xi\) and \(r : \mathfrak{h} \rightarrow \mathfrak{g}\) a deformation map of the associated canonical matched pair \((\mathfrak{g}, \mathfrak{h}, \triangleright, \leftarrow, \rightarrow)\).

1. Let \(f_r : \mathfrak{h} \rightarrow \Xi = \mathfrak{g} \bowtie \mathfrak{h}\) be the \(k\)-linear map defined for any \(x \in \mathfrak{h}\) by:
   \[
   f_r(x) = (r(x), x)
   \]
   Then \(\bar{\mathfrak{h}} := \text{Im}(f_r)\) is a \(\mathfrak{g}\)-complement of \(\Xi\).
2. \(\mathfrak{h}_r := \mathfrak{h}\), as a vector space, with the new bracket defined for any \(x, y \in \mathfrak{h}\) by:
   \[
   [x, y]_r := [x, y] + x \triangleright r(y) + r(x) \leftarrow y
   \]
   is a Leibniz algebra called the \(r\)-deformation of \(\mathfrak{h}\). Furthermore, \(\mathfrak{h}_r \cong \bar{\mathfrak{h}}\), as Leibniz algebras.

**Proof.** (1) To start with, we will prove that \(\bar{\mathfrak{h}} = \{(r(x), x) \mid x \in \mathfrak{h}\}\) is a Leibniz subalgebra of \(\mathfrak{g} \bowtie \mathfrak{h} = \Xi\). Indeed, for all \(x, y \in \mathfrak{h}\) we have:

   \[
   [(r(x), x), (r(y), y)] = (r(x), r(y)] + x \triangleright r(y) + r(x) \leftarrow y, [x, y] + x \triangleright r(y) + r(x) \rightarrow y
   \]

   \[
   \overset{(40)}{=} (r([x, y] + x \triangleright r(y) + r(x) \rightarrow y), [x, y] + x \triangleright r(y) + r(x) \rightarrow y)
   \]

   i.e. \([r(x), x), (r(y), y)] \in \bar{\mathfrak{h}}\). Moreover, it is straightforward to see that \(\mathfrak{g} \cap \bar{\mathfrak{h}} = \{0\}\) and \((g, x) = (g - r(x), 0)] + (r(x), x) \in \mathfrak{g} + \bar{\mathfrak{h}}\) for all \(g \in \mathfrak{g}, x \in \mathfrak{h}\). Here, we view \(\mathfrak{g} \cong \mathfrak{g} \times \{0\}\) as a subalgebra of \(\mathfrak{g} \bowtie \mathfrak{h}\). Therefore, \(\mathfrak{h}\) is a \(\mathfrak{g}\)-complement of \(\Xi = \mathfrak{g} \bowtie \mathfrak{h}\).

(2) We denote by \(\tilde{f}_r : \mathfrak{h} \rightarrow \bar{\mathfrak{h}}\) the linear isomorphism induced by \(f_r\). We will prove that \(\tilde{f}_r\) is also a Leibniz algebra map if we consider on \(\mathfrak{h}\) the bracket given by (44). Indeed, for any \(x, y \in \mathfrak{h}\) we have:

   \[
   \tilde{f}_r([x, y]_r) \overset{(44)}{=} \tilde{f}_r([x, y] + x \triangleright r(y) + r(x) \rightarrow y)
   \]

   \[
   = (r([x, y] + x \triangleright r(y) + r(x) \rightarrow y), [x, y] + x \triangleright r(y) + r(x) \rightarrow y)
   \]

   \[
   \overset{(43)}{=} ([r(x), r(y)] + x \triangleright r(y) + r(x) \leftarrow y, [x, y] + x \triangleright r(y) + r(x) \rightarrow y)
   \]

   \[
   \overset{(40)}{=} [(r(x), x), (r(y), y)] = [\tilde{f}_r(x), \tilde{f}_r(y)]
   \]

Therefore, \(\mathfrak{h}_r\) is a Leibniz algebra and the proof is now finished. \(\square\)

The following is the converse of Theorem 5.2: it proves that all \(\mathfrak{g}\)-complements of \(\Xi\) are \(r\)-deformations of a given complement.
Theorem 5.3. Let \( g \) be a Leibniz subalgebra of \( \Xi \), \( h \) a given \( g \)-complement of \( \Xi \) with the associated canonical matched pair of Leibniz algebras \((g, h, \triangleright, \triangleleft, \mapsto)\). Then \( h \) is a \( g \)-complement of \( \Xi \) if and only if there exists an isomorphism of Leibniz algebras \( \Xi \cong h_r \), for some deformation map \( r : h \rightarrow g \) of the matched pair \((g, h, \triangleright, \triangleleft, \mapsto)\).

Proof. Let \( h \) be an arbitrary \( g \)-complement of \( \Xi \). Since \( \Xi = g \oplus h = g \oplus h \) we can find four \( k \)-linear maps:

\[
u : h \rightarrow g, \quad v : h \rightarrow h, \quad t : h \rightarrow g, \quad w : h \rightarrow h
\]

such that for all \( x \in h \) and \( y \in h \) we have:

\[
x = u(x) \oplus v(x), \quad y = t(y) \oplus w(y)
\]

(45)

By an easy computation it follows that \( v : h \rightarrow h \) is a linear isomorphism of vector spaces. We denote by \( \tilde{v} : h \rightarrow g \triangleright h \) the composition:

\[
\tilde{v} : h \xrightarrow{v} h \xrightarrow{i} \Xi = g \triangleright h
\]

Therefore, we have \( \tilde{v}(x) \stackrel{(45)}{=} (-u(x), x) \), for all \( x \in h \). Then we shall prove that \( r := -u \) is a deformation map and \( h \cong h_r \). Indeed, \( h = \text{Im}(v) = \text{Im}(\tilde{v}) \) is a Leibniz subalgebra of \( \Xi = g \triangleright h \) and we have:

\[
[(r(x), x), (r(y), y)] \stackrel{(40)}{=} [(r(x), r(y)) + x \triangleright r(y) + r(x) \triangleleft y, [x, y] + x \triangleleft r(y) + r(x) \mapsto y] = (r(z), z)
\]

for some \( z \in h \). Thus, we obtain:

\[
r(z) = [r(x), r(y)] + x \triangleright r(y) + r(x) \triangleleft y, \quad z = [x, y] + x \triangleleft r(y) + r(x) \mapsto y
\]

(46)

By applying \( r \) to the second part of (46) it follows that \( r \) is a deformation map of the matched pair \((g, h, \triangleright, \triangleleft, \mapsto)\). Furthermore, (46) and (44) show that \( v : h \rightarrow h \) is also a Leibniz algebra map which finishes the proof. \( \square \)

In order to provide the classification of all complements we introduce the following:

Definition 5.4. Let \((g, h, \triangleright, \triangleleft, \mapsto, \mapsto)\) be a matched pair of Leibniz algebras. Two deformation maps \( r, R : h \rightarrow g \) are called equivalent and we denote this by \( r \sim R \) if there exists \( \sigma : h \rightarrow h \) a \( k \)-linear automorphism of \( h \) such that for any \( x, y \in h \):

\[
\sigma([x, y]) - [\sigma(x), \sigma(y)] = \sigma(x) \triangleleft R(\sigma(y)) + R(\sigma(x)) \mapsto \sigma(y) - \sigma(x \triangleleft r(y)) - \sigma(r(x) \mapsto y)
\]

To conclude this section, the following result provides the answer to the \((\text{CC})\) problem for Leibniz algebras:

Theorem 5.5. Let \( g \) be a Leibniz subalgebra of \( \Xi \), \( h \) a \( g \)-complement of \( \Xi \) and \((g, h, \triangleright, \triangleleft, \mapsto, \mapsto)\) the associated canonical matched pair. Then \( \sim \) is an equivalence relation on the set \( \mathcal{DM}(h, g | (\triangleright, \triangleleft, \mapsto, \mapsto)) \) and the map

\[
\mathcal{H}A^2(h, g | (\triangleright, \triangleleft, \mapsto, \mapsto)) := \mathcal{DM}(h, g | (\triangleright, \triangleleft, \mapsto, \mapsto))/ \sim \rightarrow F(g, \Xi), \quad \tau \mapsto h_r.
\]
is a bijection between $\mathcal{HA}^2(\frak{h}, \frak{g}|(>, <, \leftarrow, \rightarrow))$ and the isomorphism classes of all $\frak{g}$-complements of $\Xi$. In particular, the factorization index of $\frak{g}$ in $\Xi$ is computed by the formula:

$$[\Xi : \frak{g}]^f = |\mathcal{HA}^2(\frak{h}, \frak{g}|(>, <, \leftarrow, \rightarrow))|$$

**Proof.** Follows from Theorem 5.3 taking into account the fact that two deformation maps $r$ and $R$ are equivalent in the sense of Definition 5.4 if and only if the corresponding Leibniz algebras $\frak{h}_r$ and $\frak{h}_R$ are isomorphic.

**Example 5.6.** Let $\frak{h}$ be the abelian Lie algebra of dimension 2 with basis $\{f_1, f_2\}$ and $\frak{g}$ the Lie algebra with basis $\{e_1, e_2\}$ and the bracket: $[e_2, e_1] = -[e_1, e_2] = e_2$. Then there exists a matched pair of Leibniz algebras $(\frak{g}, \frak{h}, <, >, \leftarrow, \rightarrow)$, where the non-zero values of the actions are given as follows:

$$f_1 < e_1 := f_1, \quad f_2 < e_1 := f_2, \quad f_1 > e_1 := e_2, \quad e_1 \leftarrow f_1 := -e_2, \quad e_1 \rightarrow f_1 := -f_1$$

The bicrossed product $\Xi = \frak{g} \bowtie \frak{h}$ associated to this matched pair is the following 4-dimensional Leibniz algebra having $\{e_1, e_2, f_1, f_2\}$ as a basis and the bracket given by:

$$[e_2, e_1] = -[e_1, e_2] = e_2, \quad [f_1, e_1] = f_1 + e_2, \quad [e_1, f_1] = -f_1 - e_2, \quad [f_2, e_1] = f_2$$

Furthermore, the deformation maps associated with the above matched pair of Leibniz algebras are given as follows:

$$\tau(\gamma, \delta) : \frak{h} \rightarrow \frak{g}, \quad \tau(f_1) = \gamma e_2, \quad \tau(f_2) = \delta e_2$$

$$r_{\alpha, \beta} : \frak{h} \rightarrow \frak{g}, \quad r(f_1) = \alpha e_2 + \beta e_1, \quad r(f_2) = 0$$

for some scalars $\alpha, \beta, \gamma, \delta \in k$. One can easily see that $\frak{h}_{r_{\gamma, \delta}}$ coincides with the Lie algebra $\frak{h}$ for all $\gamma, \delta \in k$ while $\frak{h}_{r_{\alpha, \beta}}$ has the bracket given by:

$$[f_1, f_2]_{r_{\alpha, \beta}} = [f_1, f_1]_{r_{\alpha, \beta}} = [f_2, f_2]_{r_{\alpha, \beta}} = 0, \quad [f_2, f_1]_{r_{\alpha, \beta}} = \beta f_2$$

Therefore, if $\beta = 0$ then $\frak{h}_{r_{\alpha, \beta}}$ again coincides with $\frak{h}$. If $\beta \neq 0$, then for any $\alpha \in k$ and $\beta \in k^*$, the Leibniz algebra $\frak{h}_{r_{\alpha, \beta}}$ is isomorphic to the Leibniz algebra $\frak{k}$ with basis $\{F_1, F_2\}$ and the bracket given by: $[F_1, F_1] = F_2, \quad [F_2, F_1] = F_2$. The isomorphism $\psi : \frak{k} \rightarrow \frak{h}_{r_{\alpha, \beta}}$ is given by: $\psi(F_1) := f_1 + f_2, \quad \psi(F_2) := \beta f_2$. Since obviously $\frak{k}$ is not isomorphic to the abelian Lie algebra $\frak{h}$ we obtain that the extension $\frak{g} \subseteq \Xi$ has factorization index $[\Xi : \frak{g}]^f = 2$.

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