On two cohomological Hall algebras

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We compare two cohomological Hall algebras (CoHA). The first one is the preprojective CoHA introduced in [19] associated with each quiver $Q$ and each algebraic oriented cohomology theory $A$. It is defined as the $A$-homology of the moduli of representations of the preprojective algebra of $Q$, generalizing the $K$-theoretic Hall algebra of commuting varieties of Schiffmann-Vasserot [15]. The other one is the critical CoHA defined by Kontsevich-Soibelman associated with each quiver with potentials. It is defined using the equivariant cohomology with compact support with coefficients in the sheaf of vanishing cycles. In the present paper, we show that the critical CoHA, for the quiver with potential of Ginzburg, is isomorphic to the preprojective CoHA as algebras. As applications, we obtain an algebra homomorphism from the positive part of the Yangian to the critical CoHA.

Keywords: Quiver variety; Hall algebra; Yangian; quiver with potentials

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1. Introduction

In this paper, we study the relation between two cohomological Hall algebras (CoHA). One arises from the study of the symmetry of the cohomology of Nakajima quiver varieties; the other from Donaldson-Thomas theory of 3-Calabi-Yau categories.

The former CoHA, called the preprojective CoHA, was introduced in [19] for each quiver $Q$ and each algebraic oriented cohomology theory $A$ in the sense of Levine-Morel [11]. This is a CoHA associated with the 2-Calabi-Yau category of representations of the preprojective algebra of the quiver $Q = (I, H)$, where $I$ is the set of vertices and $H$ is the set of arrows. The preprojective CoHA, denoted by $P(A, Q) = \bigoplus_{\nu \in \mathbb{N}^I} P(A, Q)_\nu$, was used to construct an affine quantum group associated with the Kac-Moody Lie algebra $\mathfrak{g}_Q$ of $Q$ and $A$, which acts on the $A$-homology of quiver varieties. Nakajima-type operators are lifted as elements in the

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preprojective CoHA. In the case when \( A \) is the Chow group, we recovered the action of the Yangian on quiver varieties constructed in [18]. Another special case when \( A \) is the \( K \)-theory and \( Q \) is the Jordan quiver, the preprojective CoHA is the \( K \)-theoretic Hall algebra of commuting varieties studied by Schiffmann-Vasserot, which has been proven to be isomorphic to the elliptic Hall algebra [15]. The preprojective CoHA was used in [20] to construct affine quantum groups coming from arbitrary formal group laws, which includes the quantizations of Manin triples in [5, §4] as special cases, and in [21] to construct a Drinfeld realization of the elliptic quantum group and its action on the equivariant elliptic cohomology of quiver varieties which was previously unknown. The definition of this algebra is briefly recalled in §4.

Let \( M(v,w) \) be the Nakajima quiver variety with dimension vector \( v,w \in \mathbb{N} \) (see §4.2). By [19, theorem B], there is a homomorphism of \( \mathbb{N}^I \)-graded \( \mathbb{Q}[[t_1,t_2]] \)-algebras

\[
a^p : \mathcal{P}(A,Q) \to \text{End} \left( \bigoplus_{v \in \mathbb{N}^I} A_{G_w \times T}(\mathcal{M}(v,w)) \right).
\]

Here \( T = \mathbb{G}_{m,2} \), whose action is specified in assumption 4.1.

The latter CoHA was defined by Kontsevich-Soibelman in the cornerstone work [10], associated with the category of representations of a quiver with potential, called the critical CoHA. Recall that associated with any quiver with potential, there is a Jacobian algebra \( J_W \), which is defined as the quotient of the path algebra by the ideal generated by the derivatives of the potential ([3], see also §2.2). The category of modules over \( J_W \) is an abelian heart of a standard \( t \)-structure in a 3-Calabi-Yau category [9]. Kontsevich-Soibelman constructed a critical CoHA, whose underlying vector space is the critical cohomology of the representation space of \( J_W \) [10]. Here critical cohomology means cohomology valued in a vanishing cycle.

In the present paper, we focus on the case when \( A \) is the Borel-Moore homology \( H^{BM} \) for the proprojective CoHA. For the critical CoHA, for the same quiver \( Q = (I,H) \), we consider the extended quiver with potential \( (\hat{Q},W) \) studied by Ginzburg in [7]. Here the set of vertices of \( \hat{Q} \) is \( I \), and its set of arrows is \( H \sqcup H^{op} \sqcup C \) with \( H^{op} \) in bijection with \( H \), and for each \( a \in H \), the corresponding arrow in \( H^{op} \), denoted by \( a^* \), is \( a \) with orientation reversed. The set \( C \) is \( \{l_i \mid i \in I\} \), with \( l_i \) an edge loop at the vertex \( i \in I \). The potential of \( \hat{Q} \) is

\[
W := \sum_{i \in I} l_i \cdot \sum_{a \in H} [a,a^*].
\]

We study the critical CoHA \( \mathcal{H}^c(\hat{Q},W) := \bigoplus_{v \in \mathbb{N}^I} \mathcal{H}^c(\hat{Q},W)_v \) associated with \( (\hat{Q},W) \), defined in §2 in the same way as Kontsevich–Soibelman [10], taking into account a \( T \)-action on the representation space of \( \hat{Q} \).

We construct an isomorphism of the corresponding preprojective CoHA and the critical CoHA. Thus, the present paper can be considered as a step towards establishing a relation between instanton counting and Donaldson–Thomas invariants of Calabi–Yau 3-folds. Such a relation has been predicted in special cases on the level of generating functions in the study of geometric engineering (see [17]). As an application of our comparison, we obtain an algebra homomorphism of the positive part of Yangian into the critical CoHA, which is an embedding for ADE type quivers.
Theorem 1.1 (theorem 6.1). With the $T$-action on the representation space of $\hat{Q}$ described in assumption 4.1, the following holds.

1. For any $w \in \mathbb{N}^I$, there is a homomorphism of $\mathbb{N}^I$-graded $\mathbb{Q}[t_1, t_2]$-algebras
   
   \[ a^c : \mathcal{H}^c(\hat{Q}, W) \to \text{End} \left( \bigoplus_{v \in \mathbb{N}^I} H_{G_w \times T}^{BM}(\mathfrak{M}(v, w)) \right) . \]

2. For any $v \in \mathbb{N}^I$, let $(-1)^{(\frac{v}{2})} := \prod_{i \in I} (-1)^{(\frac{v_i}{2})}$. There is an isomorphism of $\mathbb{N}^I$-graded associative algebras
   
   \[ \Xi : \mathcal{P}(BM, Q) \to \mathcal{H}^c(\hat{Q}, W) \]
   
   whose restriction to the degree-$v$ piece is
   
   \[ \Xi_v : \mathcal{P}(BM, Q)_v \to \mathcal{H}^c(\hat{Q}, W)_v, \quad f \mapsto (-1)^{(\frac{v}{2})} f. \]

3. In the setup of (2.2), we have
   
   \[ a^c(\Xi_{v_1}(x)) \left( (-1)^{(\frac{v}{2})} m \right) = a^p(x)(m) \cdot (-1)^{(\frac{v_1 + v_2}{2})}, \]
   
   for any $w, v_1, v_2 \in \mathbb{N}^I$, $x \in \mathcal{P}_{v_1}$, and $m \in H_{G_w \times T}^{BM}(\mathfrak{M}(v_2, w))$.

The algebra $\mathcal{P}(BM, Q)$ has a presentation as a shuffle algebra (see [19, §3.2]), hence consequently so is $\mathcal{H}^c(\hat{Q}, W)$. Moreover, for special type of quivers, $\mathcal{P}(BM, Q)$ is known to be related to various versions of Yangians, which are affine type quantum groups. Therefore, the above theorem also provides a relation between $\mathcal{H}^c(\hat{Q}, W)$ with these Yangians.

The main technical ingredient in passing from the critical cohomology to the Borel-Moore homology is the dimension reduction [2, theorem A1], which is a cohomological version of the dimension reduction [1]. In appendix A, we prove that the dimension reduction is compatible with certain pull-backs and push-forwards. This allows us to give a description of the critical CoHA in terms of Borel-Moore homology, and hence prove theorem 1.1.

2. The critical cohomological Hall algebra

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver, where $\Gamma_0$ is the set of vertices, and $\Gamma_1$ the set of arrows. We denote the path algebra of $\Gamma$ by $\mathbb{C}\Gamma$. Let $W$ be a potential of $\Gamma$, that is, $W = \sum_u c_u u$ with $c_u \in \mathbb{C}$, and $u$’s are cycles in $\Gamma$.

Given a cycle $u = a_1 \cdots a_n$ and an arrow $a \in \Gamma_1$. The cyclic derivative is defined to be

\[ \frac{\partial u}{\partial a} = \sum_{i, a_i = a} a_{i+1} \cdots a_n a_1 \cdots a_{i-1} \in \mathbb{C}\Gamma. \]

as an element of $\mathbb{C}\Gamma$. We extend the cyclic derivative to the potential by linearity.

For any dimension vector $v = (v^i)_{i \in \Gamma_0} \in \mathbb{N}^{\Gamma_0}$, the representation space of $\Gamma$ with dimension vector $v$ is denoted by $\mathfrak{M}_{\Gamma, v}$. That is, let $V = \{V^i\}_{i \in \Gamma_0}$ be a $|\Gamma_0|$-tuple
of vector spaces so that \( \dim(V^i) = v^i \). Then,

\[
M_{\Gamma,v} := \bigoplus_{h \in \Gamma} \text{Hom}(V^{\text{out}(h)}, V^{\text{in}(h)}).
\]

The group \( G_v := \prod_{i \in I} \text{GL}_{v^i} \) acts on the representation space \( M_{\Gamma,v} \) via conjugation.

We recall the critical CoHA defined by Kontsevich-Soibelman, and give an equivalent description for a special type of potential interesting to us.

### 2.1. Critical CoHA via critical cohomology

For a quiver with potential \((\Gamma, W)\) and dimension vector \( v \in \mathbb{N}^I \), denote by \( \text{tr}(W)_v \) the trace function on \( M_{\Gamma,v} \). Let \( \text{Crit}(\text{tr} W_v) \) be the critical locus of \( \text{tr} W_v \).

We define \( \text{tr}(W) \) on \( M_{\Gamma,v} \) as follows. Let \( \varphi_{\text{tr} W_v} \) be the vanishing cycle complex on \( M_{\Gamma,v} \) with support on \( \text{Crit}(\text{tr} W_v) \).

We have an isomorphism

\[
H^*_{c,G_v}(M_{\Gamma,v}, \varphi_{\text{tr} W_v}) \cong H^*_{c,G_v}(\text{Crit}(\text{tr} W_v), \varphi_{\text{tr} W_v}).
\]

For \( v_1, v_2 \in \mathbb{N}^I \) such that \( v = v_1 + v_2 \), let \( V_1 \subset V \) be a \(|\Gamma_0|\)-tuple of subspaces of \( V \) with dimension vector \( v_1 \).

Define \( M_{\Gamma,v_1,v_2} := \{ x \in M_{\Gamma,v} \mid x(V_1) \subset V_1 \} \). We write \( G := G_v \) for short. Let \( P \subset G_v \) be the parabolic subgroup preserving the subspace \( V_1 \) and \( L := G_{v_1} \times G_{v_2} \) be the Levi subgroup of \( P \). We have the following correspondence of \( L \)-varieties.

\[
M_{\Gamma,v_1} \times M_{\Gamma,v_2} \xleftarrow{p} M_{\Gamma,v_1,v_2} \xrightarrow{\eta} M_{\Gamma,v_1+v_2}, \tag{2.1}
\]

where \( p \) is the natural projection and \( \eta \) is the embedding. The trace functions \( \text{tr}(W)_{v_1} \) on \( M_{\Gamma,v_1} \) induce a function \( \text{tr} W_{v_1} \oplus \text{tr} W_{v_2} \) on the product \( M_{\Gamma,v_1} \times M_{\Gamma,v_2} \). We define \( \text{tr}(W)_{v_1,v_2} \) on \( M_{\Gamma,v_1,v_2} \) to be

\[
\text{tr}(W)_{v_1,v_2} := p^*(\text{tr} W_{v_1} \oplus \text{tr} W_{v_2}) = \eta^*(\text{tr} W_{v_1+v_2}).
\]

Note that we have \( p^{-1}(\text{Crit}(\text{tr} W_{v_1}) \times \text{Crit}(\text{tr} W_{v_2})) \supseteq \eta^{-1}(\text{Crit}(\text{tr} W_{v_1+v_2})) \).

For simplicity, we assume that \( \text{Crit}(\text{tr} W_v) \subseteq (\text{tr} W_v)^{-1}(0) \) so that we have the Thom-Sebastiani isomorphism as below [12].

The Hall multiplication \( m^c \) of the critical CoHA is the composition of the following [10, 16].

1. The Thom–Sebastiani isomorphism

\[
H^*_{c,G_{v_1}}(M_{\Gamma,v_1}, \varphi_{\text{tr} W_{v_1}})^\vee \otimes H^*_{c,G_{v_2}}(M_{\Gamma,v_2}, \varphi_{\text{tr} W_{v_2}})^\vee \cong H^*_{c,L}(M_{\Gamma,v_1,v_2}, \varphi_{\text{tr} W_{v_1+v_2}})^\vee.
\]

2. Using the fact that \( M_{\Gamma,v_1,v_2} \) is an affine bundle over \( M_{\Gamma,v_1} \times M_{\Gamma,v_2} \), and \( \text{tr} W_{v_1,v_2} \) is the pullback of \( \text{tr} W_{v_1} \oplus \text{tr} W_{v_2} \), we have

\[
p^*: H^*_{c,L}(M_{\Gamma,v_1} \times M_{\Gamma,v_2}, \varphi_{\text{tr} W_{v_1+v_2}})^\vee \cong H^*_{c,L}(M_{\Gamma,v_1,v_2}, \varphi_{\text{tr} W_{v_1,v_2}})^\vee
\]
(3) Using the fact \( \text{tr} W_{v_1,v_2} \) is the restriction of \( \text{tr} W_v \) to \( M_{\Gamma,v_1,v_2} \). We have

\[
\eta_* : H^*_{c,L}(M_{\Gamma,v_1,v_2}, \varphi_{\text{tr} W_{v_1,v_2}})^\vee \to H^*_{c,L}(M_{\Gamma,v}, \varphi_{\text{tr} W_v})^\vee.
\]

(4) Pushforward along \( G \times P M_{\Gamma,v} \to M_{\Gamma,v}, (g, m) \mapsto gm^{-1} \), we get

\[
H^*_{c,L}(M_{\Gamma,v}, \varphi_{\text{tr} W_v})^\vee \cong \gamma H^*_{c,P}(M_{\Gamma,v}, \varphi_{\text{tr} W_v})^\vee
\cong H^*_{c,G}(G \times P M_{\Gamma,v}, \varphi_{\text{tr} W_v})^\vee \to H^*_{c,G}(M_{\Gamma,v}, \varphi_{\text{tr} W_v})^\vee.
\]

**Definition 2.1.** The critical CoHA is \( \mathcal{H}^c(\Gamma, W) := \bigoplus_{v \in \mathbb{N}^0} H^*_{c,G,v}(M_{\Gamma,v}, \varphi_{\text{tr} W_v})^\vee \), endowed with the Hall multiplication \( m^c \) described above.

**Remark 2.2.**

(1) Let \( J_{\Gamma,W} \) be the Jacobian algebra of the quiver with potential \( (\Gamma, W) \). (See [3] for details.) Representations of the Jacobian algebra \( \text{Rep}(J_{\Gamma,W}, v) \) as a Zariski closed subvariety of \( M_{\Gamma,v} \) is the same as \( \text{Crit}(\text{tr} W_v) \). Thus, we have

\[
H^*_{c,G,v}(M_{\Gamma,v}, \varphi_{\text{tr} W_v}) \cong H^*_{c,G,v}(\text{Rep}(J_{\Gamma,W}, v), \varphi_{\text{tr} W_v}).
\]

(2) The definition of critical CoHA in [10] is more general. The critical cohomology of more general subvarieties \( M^\text{op}_v \subset \text{Crit}(\text{tr} W_v) \) was considered. In our setup, we only take the maximal choice \( M^\text{op}_v = \text{Crit}(\text{tr} W_v) \), and we assume the quiver with potential admits a cut satisfying assumption 2.3.

(3) In the definition given in [10], cohomological degree was taken into consideration.

**2.2. Quiver with potential and cut**

A cut \( C \) of \( (\Gamma, W) \) is a subset \( C \subset \Gamma_1 \) such that \( W \) is homogeneous of degree 1 with respect to the grading defined on arrows by

\[
\deg a = \begin{cases} 
1 & : a \in C, \\
0 & : a \notin C.
\end{cases}
\]

In this section, we assume the quiver with potential \( (\Gamma, W) \) admits a cut \( C \). Furthermore, we assume the following.

**Assumption 2.3.** The cut \( C \) consists of only edge-loops.

In particular, under assumption 2.3, each term in \( W \) has at least one edge-loop. Moreover, we have the isomorphism \( M_{C,v} \cong \bigoplus_{i \in \Gamma_0} (gl_v)^{n_i} \), where \( n_i \) is the number of loops in \( C \) at vertex \( i \in \Gamma_0 \). As before, we have \( M_{C,v_1,v_2} = \{ x \in M_{C,v_1+v_2} \mid x(V_1) \subset V_1 \} \). Note that if \( C \) consists of exactly one edge-loop for each vertex, we identify \( M_{C,v} \) with \( gl_v = \bigoplus_{i \in \Gamma_0} gl_v^i \), the Lie algebra of \( G_v \), and \( M_{C,v_1,v_2} \) with the parabolic subalgebra \( p_{v_1,v_2} \), which is the Lie algebra of \( P_{v_1,v_2} \).

Let \( \Gamma\backslash C \) be the new quiver obtained from \( \Gamma \) by removing the cut \( C \). We have the decomposition \( M_{\Gamma,v} \cong M_{\Gamma\backslash C,v} \oplus M_{C,v} \) for \( v \in \mathbb{N}^{\Gamma_0} \). Write \( v = v_1 + v_2 \). By (2.1), there are the two correspondences \( M_{C,v_1} \times M_{C,v_2} \to M_{C,v_1,v_2}, M_{C,v_1,v_2} \to M_{C,v}, \).
and $M_{\Gamma \setminus C,v_1} \times M_{\Gamma \setminus C,v_2} \hookrightarrow M_{\Gamma \setminus C,v_1,v_2} \twoheadrightarrow M_{\Gamma \setminus C,v}$. It induces the following diagram with the square being Cartesian

\[
\begin{array}{ccc}
M_{\Gamma \setminus C,v_1} \times M_{\Gamma \setminus C,v_2} & \xrightarrow{p_1} & (M_{\Gamma \setminus C,v_1} \times M_{\Gamma \setminus C,v_2}) \oplus M_{C,v_1,v_2} \\
& \xrightarrow{id} & (M_{\Gamma \setminus C,v_1} \times M_{\Gamma \setminus C,v_2}) \oplus M_{C,v} \\
& \xrightarrow{p_2} & \tilde{\nu} \circ j_2 \times \id_{M_{C,v}} \\
M_{\Gamma \setminus C,v_1,v_2} \oplus M_{C,v_1,v_2} & \xrightarrow{i_3} & M_{\Gamma \setminus C,v_1,v_2} \oplus M_{C,v} \\
& \xrightarrow{id} & M_{\Gamma \setminus C,v} \oplus M_{C,v}.
\end{array}
\]

The embeddings $j_2 \times \id_{M_{C,v}}$ and $j_3 := j_2 \times \id_{M_{C,v_1,v_2}}$ come from the inclusion $j_2 : M_{\Gamma \setminus C,v_1} \times M_{\Gamma \setminus C,v_2} \subset M_{\Gamma \setminus C,v_1,v_2}$.

For each $a \in C$, the derivative $\partial W/\partial a$ is a linear combination of cycles of $\Gamma \setminus C$, as $W$ is homogenous of degree 1. For any $x \in M_{\Gamma \setminus C,v}$, denote by $\partial W/\partial a(x)$ the composition of the linear maps $x = (x_h)_{h \in \Gamma_1 \setminus C}$ along $\partial W/\partial a$. Then, $\partial W/\partial a(x))_{a \in C}$ is an element in $\bigoplus_{i \in \Gamma_0} (\mathfrak{g}t_{\mathfrak{g}})^{i} \cong M_{C,v}$.

Consider the quotient of the path algebra $\mathbb{C}(\Gamma \setminus C)$ by the relations $\{\partial W/\partial a \mid a \in C\}$. The representation variety of this quotient algebra is denoted by

$$J_{\Gamma \setminus C,v} := \{x \in M_{\Gamma \setminus C,v} \mid \partial W/\partial a(x) = 0, \forall a \in C\}.$$ 

We have the inclusion $J_{\Gamma \setminus C,v} \oplus M_{C,v} \subset M_{\Gamma \setminus C,v} \oplus M_{C,v} = M_{\Gamma,v}$. The two maps $p_1, i_1$ in diagram (2.2) induce the following two maps

$$J_{\Gamma \setminus C,v_1} \times J_{\Gamma \setminus C,v_2} \times M_{C,v_1} \times M_{C,v_2} \leftarrow \mathfrak{pt}$$

$$\xrightarrow{i_1} J_{\Gamma \setminus C,v_1} \times J_{\Gamma \setminus C,v_2} \times M_{C,v}$$

Define

$$J_{\Gamma \setminus C,v_1,v_2} := M_{\Gamma \setminus C,v_1,v_2} \cap J_{\Gamma \setminus C,v_1+v_2} = \{x \in M_{\Gamma \setminus C,v_1,v_2} \mid \partial W/\partial a(x) = 0, \forall a \in C\}.$$ 

In diagram (2.2), replacing $M_{\Gamma \setminus C,v}$ by $J_{\Gamma \setminus C,v}$ and similarly for the other dimension vectors, we get a similar diagram with the maps denoted by the original maps with an overline. But $J_{\Gamma \setminus C,v_1,v_2} = \eta_1^{-1} \mathcal{J}_{\Gamma \setminus C}(J_{\Gamma \setminus C,v_1+v_2}) \subseteq \mathfrak{p}_1^{-1} \mathcal{J}_{\Gamma \setminus C}(J_{\Gamma \setminus C,v_1} \times J_{\Gamma \setminus C,v_2})$.

Let $\text{pr} : M_{C,v_1,v_2} \to M_{C,v_1} \times M_{C,v_2}$ be the natural projection. Consider the following subvariety $Y \subset M_{C,v_1,v_2} \times M_{\Gamma \setminus C,v_1} \times M_{\Gamma \setminus C,v_2}$.

$$Y := \{(l, x) \mid l \in M_{C,v_1,v_2}, x \in M_{\Gamma \setminus C,v_1} \times M_{\Gamma \setminus C,v_2}, (\partial W/\partial a)_{a \in C}(x) = \text{pr}(l)\}.$$
We have the following diagram.

\[
\begin{array}{ccc}
M_{\Gamma \setminus C,v_1} \times M_{\Gamma \setminus C,v_2} \times M_{C,v} & \xrightarrow{\iota \times \text{id} M_{C,v}} & Y \times M_{C,v} \\
J_{\Gamma \setminus C,v_1} \times J_{\Gamma \setminus C,v_2} \times M_{C,v} & \xleftarrow{\iota \times \text{id} M_{C,v}} & \rightarrow \end{array}
\]

The maps in diagram (2.3) are as follows. Any map in (2.3) restricting on \( M_{C,v} \) is the identity map. The map \( \omega \) is induced from the natural map \( M_{\Gamma \setminus C,v_1,v_2} \rightarrow Y, x \mapsto ((\partial W/\partial a)_{a \in C}(x), \Pr(x)) \), where \( \Pr : M_{\Gamma \setminus C,v_1,v_2} \rightarrow M_{\Gamma \setminus C,v_1} \times M_{\Gamma \setminus C,v_2} \) is the natural projection. The map \( \iota \) is the restriction of \( \omega \). The map \( \iota : M_{\Gamma \setminus C,v_1} \times M_{\Gamma \setminus C,v_2} \rightarrow Y \) is given by \( x \mapsto ((\partial W/\partial a)_{a \in C}(x), x) \).

**Lemma 2.4.** There is a natural isomorphism

\[
(J_{\Gamma \setminus C,v_1} \times J_{\Gamma \setminus C,v_2}) \times_Y M_{\Gamma \setminus C,v_1,v_2} \cong J_{\Gamma \setminus C,v_1,v_2}.
\]

**Proof.** Indeed,

\[
\begin{align*}
(J_{\Gamma \setminus C,v_1} \times J_{\Gamma \setminus C,v_2}) \times_Y M_{\Gamma \setminus C,v_1,v_2} & = \{(x_1,x_2) \in J_{\Gamma \setminus C,v_1} \times J_{\Gamma \setminus C,v_2}, x \in M_{\Gamma \setminus C,v_1,v_2} \mid \Pr(x) = (x_1, x_2), (\partial W/\partial a)_{a \in C}(x) = 0 \} \\
& = \{x \in M_{\Gamma \setminus C,v_1,v_2} \mid (\partial W/\partial a)_{a \in C}(x) = 0 \} = J_{\Gamma \setminus C,v_1,v_2}.
\end{align*}
\]

This completes the proof. \( \square \)

2.3. Another description of the critical CoHA

We follow the convention in [2], let \( D^b(X) \) be the derived category of constructible sheaves of \( \mathbb{Q} \)-vector spaces on a variety \( X \), and \( \mathbb{D} \) be the Verdier duality functor for \( D^b(X) \). Denote by \( H^*_c(X)^\vee \) the Verdier dual of the compactly supported cohomology of \( X \). Write the structure map \( X \rightarrow \text{pt} \) as \( p_X \). The Borel-Moore homology of \( X \) is

\[
H^\text{BM}_*(X) \cong H^*_c(X)^\vee \cong \mathbb{D}_{\text{pt}}(p_X)_!(\mathbb{Q}_X) = p_X_!\mathbb{D}\mathbb{Q}_X.
\]

If \( X \) carries a \( G \)-action, we denote by \( H^*_c,G(X)^\vee \) the corresponding equivariant cohomology of \( X \).

The following equivalent description of the critical CoHA using Borel-Moore homology will be proved in §3. Let \( e(\iota) \) be the equivariant Euler class of the embedding \( \iota \) in (2.3).

**Theorem 2.5.** Assume the quiver with potential \((\Gamma, W)\) admits a cut \( C \) satisfying assumption 2.3. Then there is an isomorphism of graded vector spaces \( \mathcal{H}^c(\Gamma, W) \cong \bigoplus_{v \in EN^0} H^\text{BM}_v(J_{\Gamma \setminus C,v} \times M_{C,v,\mathbb{Q}}), \) under which the multiplication \( m^c \) on \( \mathcal{H}^c(\Gamma, W) \) is equal to \( \frac{i}{i} \circ e(\iota) \cdot \frac{i}{i} \circ \iota \).
The maps in the composition are the following.

1. The Künneth morphism $H_{G^1}^{BM}(J_{Γ\setminus C,v_1} \times MC_{v_1}) \otimes H_{G^2}^{BM}(J_{Γ\setminus C,v_2} \times MC_{v_2}) \to H_{G}^{BM}(J_{Γ\setminus C,v_1} \times J_{Γ\setminus C,v_2} \times MC_{v_1} \times MC_{v_2})$.

2. $i_{1*} ◦ P_{I*} : H_1^{BM}(J_{Γ\setminus C,v_1} \times J_{Γ\setminus C,v_2} \times MC_{v_1} \times MC_{v_2}) \to H_1^{BM}(J_{Γ\setminus C,v_1} \times J_{Γ\setminus C,v_2} \times MC_{v_2})$.

3. Composing the refined Gysin pullback$^1$ of $ω \times id_{MC_{v_1}}$ along $ω \times id_{MC_{v_2}}$, which for simplicity is denoted by $ω_ω^\sharp$, with $1/e(ι)$, we have the following map

$$\frac{1}{e(ι)}ω_ω^\sharp : H_1^{BM}(J_{Γ\setminus C,v_1} \times J_{Γ\setminus C,v_2} \times MC_{v}) \to H_1^{BM}(J_{Γ\setminus C,v_1} \times J_{Γ\setminus C,v_2} \times MC_{v}),$$

4. The pushforward $i_{2*} : H_1^{BM}(J_{Γ\setminus C,v_1} \times J_{Γ\setminus C,v_2} \times MC_{v}) \to H_1^{BM}(J_{Γ\setminus C,v} \times MC_{v})$.

5. Pushforward along $G \times P (J_{Γ\setminus C,v} \times MC_{v}) \to J_{Γ\setminus C,v} \times MC_{v}, (g,m) \mapsto gmg^{-1}$, we get $H_1^{BM}(J_{Γ\setminus C,v} \times MC_{v}) \cong H_1^{BM}(G \times P (J_{Γ\setminus C,v} \times MC_{v})) \to H_1^{BM}(J_{Γ\setminus C,v} \times MC_{v})$.

This composition $i_{2*} \circ 1/e(ι)ω_ω^\sharp \circ i_{1*} \circ P_{I*}$ a priori is only defined after inverting $e(ι)$. However, it follows from theorem 2.5 that it is well-defined before localization.

**Remark 2.6.** The multiplication $i_{2*} \circ 1/e(ι)ω_ω^\sharp \circ i_{1*} \circ P_{I*}$ does not depend on the choices of $Y$ in correspondence (2.3), as long as lemma 2.4 holds. Indeed, assume we have the following diagram

$\begin{array}{c}
Y' & \leftarrow & M_{Γ\setminus C,v_1,v_2} \\
\downarrow & & \downarrow \\
Y & \leftarrow & M_{Γ\setminus C,v_1,v_2} \\
\downarrow & & \downarrow \\
J_{Γ\setminus C,v_1} \times J_{Γ\setminus C,v_2} & \leftarrow & J_{Γ\setminus C,v_1,v_2}
\end{array}$

with all squares Cartesian and $Y \to Y'$ an closed embedding of manifolds. Then, by the excess intersection formula (see [11, theorem 6.6.9]), we have $ω_ω^\sharp \cong ω_ω^\sharp e(E)$, where $E$ is the excess normal bundle of $Y \to Y'$. This implies that

$$\frac{1}{e(ι)}ω_ω^\sharp = \frac{1}{e(ι')}ω_ω^\sharp.$$

$^1$We follow the convention of [11, §6.6.2] for refined Gysin pullback and use subindex to remember the domain and target. However, we adapt $^\sharp$ instead of $^!$ to avoid confusion with $^!$-pullback.
2.4. An auxiliary CoHA

By theorem 2.5, we can alternatively define the critical CoHA as the vector space \( \mathcal{H}^c(\Gamma, W, C) = \bigoplus_{v \in \mathbb{N}_{\Gamma_0}} H_{BM}^{G_v}(J_{\Gamma \backslash C, v} \times M_{C, v}, \mathbb{Q}) \) endowed with the multiplication \( \overline{t}_2 \circ 1/e(\omega)\overline{t}_1 \circ \overline{p}_1 \). This description has the advantage of being defined for any algebraic oriented cohomology theory in the sense of Levine-Morel [11], without resorting to a theory of ‘exponential motives’. That is, for any oriented cohomology theory \( A \), we can define \( \mathcal{H}^c(A, \Gamma, W, C) \) to be the graded vector space \( \bigoplus_{v \in \mathbb{N}_{\Gamma_0}} A_{G_v}(J_{\Gamma \backslash C, v} \times M_{C, v}, \mathbb{Q}) \) endowed with the multiplication \( m^c = \overline{t}_2 \circ 1/e(\omega)\overline{t}_1 \circ \overline{p}_1 \).

For example, \( A \) can be taken to be the intersection theory CH (that is, the Chow group, see [6]). There is a cycle map \( \text{cl} : CH_*(X) \to H_{BM}^{2*}(X) \). It induces an algebra homomorphism \( \mathcal{H}(CH, \Gamma, W, C) \to \mathcal{H}(BM, \Gamma, W, C) \).

As an intermediate object in comparing the preprojective CoHA in §4 with the critical CoHA of Kontsevich-Soibelman, we further introduce an auxiliary cohomological Hall algebra associated with the data \( (\Gamma, W, C) \) and \( A \), as

\[
\mathcal{H}^{\text{aux}}(\Gamma, W, C) := \bigoplus_{v \in \mathbb{N}_{\Gamma_0}} \mathcal{H}^{\text{aux}}(\Gamma, W, C)_v = \bigoplus_{v \in \mathbb{N}_{\Gamma_0}} A_{G_v \times T}(J_{\Gamma \backslash C, v} \times M_{C, v}, \mathbb{Q}),
\]

with multiplication defined by the composition

\[
m^\text{aux}_{v_1, v_2} := \overline{t}_2 \circ \omega^j \circ \overline{t}_1 \circ \overline{p}_1.
\]

Unlike \( m^c \), \( m^\text{aux}_{v_1, v_2} \) does depend on the specific choices of \( Y \) in the correspondence (2.3).

**Proposition 2.7.** The maps \( \{m^\text{aux}_{v_1, v_2}\}_{v_1, v_2 \in \mathbb{N}} \) define an associative \( \mathbb{N}^l \)-graded algebra structure on \( \mathcal{H}^{\text{aux}}(\Gamma, W, C) \).

**Proof.** This follows from a similar proof as that of [19, theorem 4.1]. \( \square \)

3. Proof of theorem 2.5

In this section, we prove theorem 2.5 and hence work under the assumption therein.

3.1. Proof of theorem 2.5

The proof of theorem 2.5 amounts to showing that the dimension reduction in theorem A.1 is compatible with some pullbacks and pushforwards. We present some general facts in appendix A for the convenience of the readers.

In the setup of theorem A.1, we take \( X = M_{\Gamma \backslash C, v}, Y = M_{\Gamma, v}, A^n = M_{C, v} \), and \( f = \text{tr} W_v \). Then, we have the subvariety \( Z = \{ x \in M_{\Gamma \backslash C, v} \mid \text{tr}(W_v)(x, l) = 0, \forall l \in M_{C, v} \} \).

**Lemma 3.1.** The subvariety \( Z \) of \( M_{\Gamma \backslash C, v} \) can be identified with \( J_{\Gamma \backslash C, v} \).
Proof. By definition, the potential $W$ is homgeneous of degree 1. This implies
\[ \text{tr } W = \text{tr } \sum_{a \in C} (\partial W/\partial a) a. \tag{3.1} \]

For any $x \in J_{\Gamma \backslash C, v}$, we have $\partial W/\partial a(x) = 0$, for all $a \in C$. Clearly, the equality (3.1) implies $\text{tr}(W_v)(x, l) = \text{tr} \sum_{a \in C} (\partial W/\partial a(x) \cdot l_a) = 0$, for all $l = (l_a)_{a \in C} \in M_{C, v}$. This shows $x \in Z$.

On the other hand, for any $x \in Z$, we have $\text{tr}(W_v)(x, l) = 0$, for all $l = (l_a)_{a \in C} \in M_{C, v}$. Then, by the equality (3.1) we have $\text{tr}(\partial W/\partial a(x) \cdot l_a) = 0$, for any matrix $l_a$. This shows the vanishing $(\partial W/\partial a)(x) = 0$, for all $a \in C$. Therefore, $x \in J_{\Gamma \backslash C, v}$. \hfill $\square$

Lemma 3.1, together with theorem A.1 and proposition A.2, yields the following.

**Corollary 3.2.** There is a canonical isomorphism as vector spaces,
\[ H^*_c(G_v)(M_{\Gamma \backslash C, v}, \varphi_{tr} W_v)^\vee \cong H^*_c(G_v)(J_{\Gamma \backslash C, v} \times M_{C, v}, \mathbb{Q})^\vee, \quad \text{for } v \in \mathbb{N}^{\Gamma_0}, \]
which is compatible with the Thom-Sebastiani isomorphism and the Künneth isomorphism.

The following lemma will be proved in §3.2.

**Lemma 3.3.** The isomorphism in theorem A.1 intertwines
\[ e_1(j_2 \times \text{id}_{M_{C, v}})_*: H^*_c(L)(M_{\Gamma \backslash C, v_1} \times M_{\Gamma \backslash C, v_2} \times M_{C, v}, \varphi_{tr} W_{v_1 + v_2})^\vee \]
\[ \to H^*_c(L)(M_{\Gamma \backslash C, v_1} \times M_{C, v}, \varphi_{tr} W_{v_1 + v_2})^\vee \]
and
\[ H^*_L(J_{\Gamma \backslash C, v_1} \times J_{\Gamma \backslash C, v_2} \times M_{C, v}) \cong \to H^*_L(J_{\Gamma \backslash C, v_1} \times J_{\Gamma \backslash C, v_2} \times M_{C, v}) \]
\[ \cong \downarrow \cong \]
\[ H^*_L(J_{\Gamma \backslash C, v_1} \times J_{\Gamma \backslash C, v_2} [\text{dim } M_{C, v}] - \omega_2^\vee e(j_2) \to H^*_L(J_{\Gamma \backslash C, v_1} \times J_{\Gamma \backslash C, v_2} [\text{dim } M_{C, v}]). \]

Proof of theorem 2.5. With notations in (2.2) and (3.2), the composition of §2.1(2) and (2.3) in the multiplication on $H^c(\Gamma, W)$ of Kontsevich-Soibelman is equivalent to
\[ i_{2*} \circ i_{3*} \circ p^*_3 \circ p^*_1 : H^*_c(L)(M_{\Gamma \backslash v_1} \times M_{\Gamma \backslash v_2}, \varphi_{tr} W_{v_1} \oplus W_{v_2})^\vee \to H^*_c(L)(M_{\Gamma \backslash v_1 + v_2}, \varphi_{tr} W_{v_1 + v_2})^\vee. \]

In the following 4 steps, we show that $i_{2*} \circ i_{3*} \circ p^*_3 \circ p^*_1$ is the same as $\overline{t}_{2*} \circ 1/e(i)\omega_2^\vee \circ \overline{t}_{1*} \circ \overline{p}_1^\vee$ under the isomorphism of corollary 3.2.

**Step 1.** We have the following commutative diagram
\[ H^*_c(L)(M_{\Gamma \backslash v_1} \times M_{\Gamma \backslash v_2}, \varphi_{tr} W_{v_1} \oplus W_{v_2})^\vee \]
\[ \downarrow \cong \]
\[ H^*_c(L)(J_{\Gamma \backslash C, v_1} \times J_{\Gamma \backslash C, v_2} \times M_{C, v_1} \times M_{C, v_2})^\vee \]
\[ \leftarrow p^*_1 \]
\[ \leftarrow \]
\[ H^*_c(L)(J_{\Gamma \backslash C, v_1} \times J_{\Gamma \backslash C, v_2} \times M_{C, v_1} \times M_{C, v_2})^\vee \]
\[ \leftarrow \overline{p}_1^\vee \]
\[ \leftarrow \]
\[ H^*_c(L)(J_{\Gamma \backslash C, v_1} \times J_{\Gamma \backslash C, v_2} \times M_{C, v_1} \times M_{C, v_2})^\vee. \]
Indeed, in lemma A.5, we take \( X' = X = \mathbb{M}_{\Gamma \setminus C, v_1} \times \mathbb{M}_{\Gamma \setminus C, v_2} \), \( g \) to be the identity map on \( X \), and \( h : \mathbb{M}_{C, v_1, v_2} \to \mathbb{M}_{C, v_1} \times \mathbb{M}_{C, v_2} \) the natural projection. Note that the map \( p_1 = g \times h \) is an affine bundle. Then lemma A.5 implies that the vanishing cycle pullback \( p_1^* \) is the same as \( \overline{p_1}^* \).

**Step 2.** The map \( p_3 : \mathbb{M}_{\Gamma, v_1, v_2} \to \mathbb{M}_{\Gamma \setminus C, v_1} \times \mathbb{M}_{\Gamma \setminus C, v_2} \times \mathbb{M}_{C, v_1, v_2} \) is also an affine bundle, with tr\( W_{v_1, v_2} \) on the total space \( \mathbb{M}_{\Gamma, v_1, v_2} \) obtained by pulling back of tr\( W_{v_1} \oplus \text{tr} W_{v_2} \) from \( \mathbb{M}_{\Gamma, v_1} \times \mathbb{M}_{\Gamma, v_2} \). Hence, \( j \beta^* : H^*_{c, L}(\mathbb{M}_{\Gamma, v_1} \times \mathbb{M}_{\Gamma, v_2} \times \mathbb{M}_{C, v_1, v_2}, \varphi_{\text{tr} W_{v_1} \oplus W_{v_2}})^\vee \to H^*_{c, L}(\mathbb{M}_{\Gamma \setminus C, v_1} \times \mathbb{M}_{\Gamma \setminus C, v_2} \times \mathbb{M}_{C, v_1, v_2}, \varphi_{\text{tr} W_{v_1, v_2}})^\vee \) is \( p_3^* \) followed by the Euler class \( j_3 \), that is, \( p_3^* = 1/e(j_3)j_3^* \). Plugging this into the equality \( i_3j_3^* = \left((j_2 \times \text{id}_{\mathbb{M}_{C, v}})i_1\right)_* \), we get that \( i_3 \circ p_3^* \) is the same as \( 1/e(j_3) \circ \left((j_2 \times \text{id}_{\mathbb{M}_{C, v}})i_1\right)_* \).

**Step 3.** The map \( i_1 \) is a section of an affine bundle. The function \( \text{tr} W_{v_1, v_2} \) on \( \mathbb{M}_{\Gamma \setminus C, v_1} \times \mathbb{M}_{\Gamma \setminus C, v_2} \times \mathbb{M}_{C, v} \) is obtained by pulling back \( \text{tr} W_{v} \) from \( \mathbb{M}_{\Gamma, v_1} \times \mathbb{M}_{\Gamma, v_2} \times \mathbb{M}_{C, v_1, v_2} \). Hence, we have the commutative diagram

\[
\begin{array}{c}
H^*_{c, L}(\mathbb{M}_{\Gamma \setminus C, v_1} \times \mathbb{M}_{\Gamma \setminus C, v_2} \times \mathbb{M}_{C, v_1, v_2}, \varphi_{\text{tr} W_{v_1} \oplus W_{v_2}})^\vee \\
\downarrow i_{1*} \\
H^*_{c, L}(\mathbb{M}_{\Gamma \setminus C, v_1} \times \mathbb{M}_{\Gamma \setminus C, v_2} \times \mathbb{M}_{C, v_1, v_2}, \varphi_{\text{tr} W_{v_1 \oplus v_2}})^\vee
\end{array}
\]

since both \( i_{1*} \) and \( \overline{i}_{1*} \) are multiplication by the Euler class.

Precomposing \( i_{1*} = \overline{i}_{1*} \) with equality from lemma 3.3, we have \( e(i)(j_2 \times \text{id}_{\mathbb{M}_{C, v}})i_{1*} = \omega^x_{\text{tr}}(j_2)\overline{i}_{1*} \). This implies the equality \( 1/e(i)\omega^x_{\text{tr}} \circ \overline{i}_{1*} = 1/e(j_3) \circ \left((j_2 \times \text{id}_{\mathbb{M}_{C, v}})i_1\right)_* \), with the right hand side equal to \( i_3^*p_3^* \) by Step 2.

**Step 4.** In lemma A.4, we take \( X \) to be \( \mathbb{M}_{\Gamma \setminus C, v_1, v_2} \), and \( X' \) to be \( \mathbb{M}_{\Gamma \setminus C, v_1, v_2} \), \( h : \mathbb{M}_{C, v} \to \mathbb{M}_{C, v} \) the identity map. Then, \( i_2 = g \times h \). By lemma A.4, the Borel-Moore homology pushforward

\[
\overline{i}_{2*} : H_{c, L}(\mathbb{J}_{\Gamma \setminus C, v_1, v_2} \times \mathbb{M}_{C, v})^\vee \to H_{c, L}(\mathbb{J}_{\Gamma \setminus C, v_1, v_2} \times \mathbb{M}_{C, v})^\vee
\]

coincides with the vanishing cycle pushforward

\[
i_{2*} : H^*_{c, L}(\mathbb{M}_{\Gamma \setminus C, v_1, v_2} \times \mathbb{M}_{C, v}, \varphi_{\text{tr} W_{v_1 \oplus v_2}})^\vee \to H^*_{c, L}(\mathbb{M}_{\Gamma \setminus C, v_1, v_2} \times \mathbb{M}_{C, v}, \varphi_{\text{tr} W_{v_1 \oplus v_2}})^\vee.
\]

**3.2. Proof of lemma 3.3**

Recall in (2.3), we have the map \( \omega : \mathbb{M}_{\Gamma \setminus C, v_1, v_2} \to Y \), given by \( x \mapsto ((\partial W/\partial a)_{a \in C}(x), \text{pr}(x)) \), with pr \( : \mathbb{M}_{\Gamma \setminus C, v_1, v_2} \to \mathbb{M}_{\Gamma \setminus C, v_1} \times \mathbb{M}_{\Gamma \setminus C, v_2} \) being the natural projection. We introduce a variety

\[
X := \{(l, x) \in \mathbb{M}_{C, v_1, v_2} \times \mathbb{M}_{\Gamma \setminus C, v_1, v_2} \mid (\partial W/\partial a)_{a \in C}(\text{pr}(x)) = \text{pr}(l)\},
\]

where \( \text{pr}(l) \) is the image of \( l \) under the projection \( \text{pr} : \mathbb{M}_{C, v_1, v_2} \to \mathbb{M}_{C, v_1} \times \mathbb{M}_{C, v_2} \). Now \( \omega \) is equal to the composition of the following two maps

\[a : \mathbb{M}_{\Gamma \setminus C, v_1, v_2} \to X, x \mapsto ((\partial W/\partial a)_{a \in C}(x), x), \quad b : X \to Y, (l, x) \mapsto (l, \text{pr}(x)),\]

and

\[b : X \to Y, (l, x) \mapsto (l, \text{pr}(x)).\]
with \(a\) being a closed embedding and \(b\) being an affine bundle. Using these notations, we have the following diagram with \(\omega, \varpi,\) and \(i\) the same as in diagram (2.3).

Here \(l : J_{\Gamma \setminus C, v_1} \times J_{\Gamma \setminus C, v_2} \to Y\) is the embedding, and \(\tilde{j}_2\) is the zero-section of the affine bundle \(b\).

**Lemma 3.4.** Notations as above, under the isomorphism in theorem A.1, the morphism

\[
(j_2 \times \text{id}_{M_{C,v}})_* : H^*_c,L(M_{\Gamma \setminus C, v_1} \times M_{\Gamma \setminus C, v_2} \times M_{C,v}, \varphi_{\text{tr}}w_{v_1+v_2})^\vee \\
\to H^*_c,L(M_{\Gamma \setminus C, v_1, v_2} \times M_{C,v}, \varphi_{\text{tr}}w_{v_1+v_2})^\vee
\]

is given by \(\mathcal{D} \circ (p_{J_{v_1} \times J_{v_2}})_!l^*b_*a_!(\mathbb{Q}_{M_{\Gamma \setminus C, v_1, v_2}} \to j_2)_*\mathbb{Q}_{M_{\Gamma \setminus C, v_1} \times M_{\Gamma \setminus C, v_2}}[-\dim M_{C,v}]\).

---

\(^2\)Here we extend the constructions equivariantly in the same way as in [2]. That is, assume \(G\) embeds into \(GL_C(n)\), for some \(n \in \mathbb{N}\). Let \(EG_N\) be the space of \(n\)-tuples of linearly independent vectors in \(\mathbb{C}^N\), which is denoted by \(fr(n,N)\) in [2]. Thus, \(EG_N / GL_C(n)\) is the Grassmannian of linear subspaces of dimension \(n\) in \(\mathbb{C}^N\). Let \(X_N := EG_N \times_G X\) with natural maps \(\{h_N : X_N \to X_{N+1}\}\). In what follows, \(Q_X\) should be understood as \(\{Q_{X_N}\}\) together with the natural maps \(\{(h_N)_*Q_{X_N} \to Q_{X_{N+1}}\}\). For an equivariant map \(f : X \to Y\) between smooth varieties, we have the system of maps \(\{(f_N)_*Q_{X_N} \to Q_{Y_N}[-\dim f]\}\) and \(\{Q_{Y_N} \to (f_N)_*Q_{X_N}\}\), together with commutative squares

\[
(h_N)_!Q_{Y_N} \to (f_{N+1})_!Q_{X_N},
\]

and similarly for the \(!\)-version. We omit the system from the notations. For example, in the proof of lemma 3.4, the morphism in each line should be understood as a system of morphisms. The equalities in the proof come from natural isomorphism of functors, hence give rise to equalities of commutative squares.
Proof. Let \( \pi : M_{\Gamma \setminus C:v_1,v_2} \times M_{C,v} \to M_{\Gamma \setminus C:v_1,v_2} \) be the projection. By theorem A.1, \((j_2 \times \text{id}_{M_{C,v}})_* \) is

\[
\mathbb{D} \circ (p_{M_{\Gamma \setminus C:v_1,v_2}}) ! \pi_! \pi^* i_2^* (Q_{M_{\Gamma \setminus C:v_1,v_2}} \to j_2_! Q_{M_{\Gamma \setminus C:v_1,v_2} \times M_{\Gamma \setminus C,v_2}}) = \mathbb{D} \circ (p_{M_{\Gamma \setminus C:v_1,v_2}}) ! \pi_! \pi^* i_2^* (Q_{M_{\Gamma \setminus C:v_1,v_2}} \to j_2_! Q_{M_{\Gamma \setminus C,v_1} \times M_{\Gamma \setminus C,v_2}})[−\dim M_{C,v}]
\]

Here the first isomorphism follows from projection formula (see for example [4, theorem 2.3.29]) and the fact that \( \pi_! Q = Q[−\dim M_{C,v}] \). \(\square \)

Lemma 3.5. The Gysin pullback \(\omega^d_{BM} : H^BM(J_{\Gamma \setminus C,v_1} \times J_{\Gamma \setminus C,v_2}) \to H^BM(J_{\Gamma \setminus C:v_1,v_2})\) is given by \(\mathbb{D} \circ (p_{J_{\Gamma \setminus C:v_1}} \times J_{\Gamma \setminus C,v_2}) ! \circ l^*\) applied to the following composition of morphisms

\[ b \circ \pi_{Q_{M_{\Gamma \setminus C:v_1,v_2}}} \to b_! Q_X[−\dim a] \cong Q_Y[−\dim a − \dim b]. \]

Proof. Let \( X' \subset X \) be the inverse image of \( J_{\Gamma \setminus C:v_1} \times J_{\Gamma \setminus C,v_2} \) under the map \( b \). Then, we have the following diagram with \( l' \) induced by \( l \).

\[
\begin{align*}
Y & \xleftarrow{b} X \xleftarrow{a} M_{\Gamma \setminus C:v_1,v_2} \\
\uparrow l & \quad \uparrow a \quad \uparrow l' \\
J_{\Gamma \setminus C:v_1} \times J_{\Gamma \setminus C,v_2} & \xleftarrow{b'} X' \xleftarrow{a} J_{\Gamma \setminus C:v_1,v_2}
\end{align*}
\]

It is well-known (see, e.g., [8, lemma 2.1.2]) that the map \( a^* \) is given by \((p_{X'}^*)_! l'^* (Q_X \to a_* Q_{M_{\Gamma \setminus C:v_1,v_2}})\). We have the following equalities

\[
(p_{X'}^*)_! l'^* (Q_X \to a_* Q_{M_{\Gamma \setminus C:v_1,v_2}})
\]

\[
= (p_{J_{\Gamma \setminus C:v_1}} \times J_{\Gamma \setminus C,v_2}) ! l'_! b'^* (Q_X \to a_* Q_{M_{\Gamma \setminus C,v_1,v_2}})
\]

\[
= (p_{J_{\Gamma \setminus C:v_1}} \times J_{\Gamma \setminus C,v_2}) !_ l'_! b_! (Q_X \to a_* Q_{M_{\Gamma \setminus C,v_1,v_2}})
\]

\[
= \mathbb{D} \circ (p_{J_{\Gamma \setminus C:v_1}} \times J_{\Gamma \setminus C,v_2}) !_ l'_! b_! (a_! Q_{M_{\Gamma \setminus C:v_1,v_2}} \to Q_X[−\dim a])
\]

The map \( b \) is an affine bundle. Hence, \( b^* \) is induced by \( b_! Q_X \cong Q_Y[−\dim b] \). The assertion now follows from \( \omega_{BM}^d = a^* \circ b^* \). \(\square \)

Notations as in (2.2) and (3.2), \( e(a) \) and \( e(j_2) \) are the Euler classes of the two embeddings \( a \) and \( j_2 \). Note that \( e(j_2) = e(j_3) \) and \( e(a) = e(i) \). Combining lemmas 3.4 and 3.5, we get lemma 3.3.
Proof of lemma 3.3. By lemma 3.4, \( e(a)(j_2 \times \text{id}_{\mathbb{M}_{C,v}})_* \) is the same as applying \( \mathbb{D} \circ (p_{J_{v_1} \times J_{v_2}})^{l^*} \) to the following composition

\[
\begin{align*}
& b_1 \left( a_1 \mathbb{Q}_{M_{\Gamma \setminus C,v_1,v_2}} \to \mathbb{Q}_{X}[-\dim a] \to a_4 \mathbb{Q}_{M_{\Gamma \setminus C,v_1,v_2}}[-\dim a] \right) \\
& = a_1 \mathbb{Q}_{M_{\Gamma \setminus C,v_1,v_2}}[-\dim a] \to a_1 j_2 \mathbb{Q}_{M_{\Gamma \setminus C,v_1} \times M_{\Gamma \setminus C,v_2}}[-\dim a].
\end{align*}
\]

By lemma 3.5, \( \omega_{2}^x e(j_2) \) is \( \mathbb{D} \circ (p_{J_{v_1} \times J_{v_2}})^{l^*} \) applied to the composition

\[
\begin{align*}
& b a_1 \mathbb{Q}_{M_{\Gamma \setminus C,v_1,v_2}} \to \mathbb{Q}_{Y}[-\dim a - \dim b] \\
& = b j_2 \mathbb{Q}_{Y}[-\dim a - \dim b] \to b_1 \mathbb{Q}_{X}[-\dim a] \to b j_2 \mathbb{Q}_{Y}[-\dim a].
\end{align*}
\]

We have the commutative diagram

\[
\begin{array}{ccc}
\mathbb{Q}_{M_{\Gamma \setminus C,v_1,v_2}}[-\dim a] & \xrightarrow{\text{id}} & \mathbb{Q}_{X} \xrightarrow{a_4} \mathbb{Q}_{M_{\Gamma \setminus C,v_1,v_2}} \xrightarrow{a_1 j_2} \mathbb{Q}_{M_{\Gamma \setminus C,v_1} \times M_{\Gamma \setminus C,v_2}} \\
\downarrow & & \downarrow \\
\mathbb{Q}_{M_{\Gamma \setminus C,v_1,v_2}}[-\dim a] & \xrightarrow{\tilde{j}_2} & \mathbb{Q}_{Y}[-\dim b] \to \mathbb{Q}_{X} \to \tilde{j}_2 \mathbb{Q}_{Y}
\end{array}
\]

Applying the functor \( (p_{J_{v_1} \times J_{v_2}})^{l^*} b_1 [-\dim a] \) to the diagram above gives the desired equality. \( \square \)

4. The preprojective cohomological Hall algebra

This section is a brief review of the main results about the preprojective CoHA from [19]. All the details can be found in loc. cit.

4.1. The preprojective CoHA

Let \( Q = (I, H) \) be a quiver, where \( I \) is the set of vertices, and \( H \) the set of arrows. Let \( \mathcal{Q} \) be the double quiver, and \( \Pi_{Q} \) be the preprojective algebra of \( Q \). By definition, the vertices of \( \mathcal{Q} \) are the same as the vertices of \( Q \), and the arrows of \( \mathcal{Q} \) are the arrows of \( Q \) together with arrows \( h^*: j \to i \) for each arrow \( h: i \to j \) of \( Q \). The preprojective algebra \( \Pi_{Q} \) is defined as the quotient of the path algebra of \( \mathcal{Q} \) by the relation \( \sum_{h \in H} [h, h^*] \).

We have the following moment map on \( T^* \mathbb{M}_{Q,v} = \mathbb{M}_{Q,v}^v \).

\[
\mu_v : T^* \mathbb{M}_{Q,v} \to \mathfrak{g}_v^*, \quad (x, x^*) \mapsto [x, x^*].
\]

Denote by \( \Lambda_v \) the preimage of 0 under the map \( \mu_v \). The subvariety \( \Lambda_v \subset \mathbb{M}_{Q,v}^v \) parametrizes the representations of \( \Pi_{Q} \) of dimension vector \( v \).

We fix a weight function \( m : H \coprod H^* \to \mathbb{Z} \), and define the torus \( T = \mathbb{G}_m \) action on \( T^* \mathbb{M}_{Q,v} \) as follows (see also [13, (2.7.1) and (2.7.2)]). For \( h \in H \) from \( i \) to \( j \), the element \( (t_1, t_2) \in T \) acts on the factor \( \text{Hom}(V^i, V^j) \) by \( t_1^{m_h} \), and on the factor \( \text{Hom}(V^j, V^i) \) by \( t_2^{-m_h} \). We let \( T \) act on the Lie algebra \( \mathfrak{g}_v \) of \( G_v \) by weight \( t_1 t_2 \). It induces a \( T \)-action on \( \Lambda_v \) if the moment map is \( T \)-equivariant, or equivalently, if we have the following ([19, assumption 3.1])).
Assumption 4.1. We assume the weight functions $m$ and a subtorus $D \subseteq T$ are compatible, in the sense that $t_1^{mh} t_2^{nh} = t_1 t_2$ on $D$, for all $h$.

One example of $T$ action satisfying assumption 4.1 is the following.

Example 4.2. Let $D = G_m$ with coordinate $h$, sitting in $T$ via $t_1 = t_2 = h/2$. For any pair of vertices $i$ and $j$, label the arrows from $i$ to $j$ by $h_1, \ldots, h_a$. The pairs of integers are $m_{h_p} = a + 2 - 2p$ and $m_{h_q} = -a + 2p$.

Let $A$ be an oriented cohomology theory with $A(pt) = R$. We consider the $\mathbb{N}^I$-graded $R[t_1, t_2]$-module

$$\mathcal{P}(A, Q) := \bigoplus_v \mathcal{P}_v(A, Q) = \bigoplus_v A_{G_v \times T}(A_v).$$

For each pair $v_1, v_2 \in \mathbb{N}^I$, the multiplication map $m^p_{v_1, v_2} : \mathcal{P}_{v_1} \otimes \mathcal{P}_{v_2} \to \mathcal{P}_{v_1 + v_2}$ is as follows.

Recall that $M_{Q, v_1 + v_2} = \{ x \in M_{Q, v_1 + v_2} \mid x(V_i) \subset V_i \}$. We have the following correspondence of $G \times T$-varieties

$$G \times P (M_{Q,v_1} \times M_{Q,v_2}) \leftarrow G \times P M_{Q,v_1,v_2} \rightarrow M_{Q,v}.$$  \hfill (4.1)

Let $\Lambda_{v_1,v_2} := \Lambda_v \cap M_{Q,v_1,v_2} = \{ (x,x^*) \in M_{Q,v_1,v_2} \mid [x,x^*] = 0 \}$, and $X = G \times P (M_{Q,v_1} \times M_{Q,v_2})$. By [19, lemma 5.1], we have

$$T^* X = G \times P \{(c,x,x^*) \in \mathfrak{p}_{v_1,v_2} \times (M_{Q,v_1} \times M_{Q,v_2}) \mid [x,x^*] = \text{pr}(c)\},$$

where $\text{pr} : \mathfrak{p}_{v_1,v_2} \to \mathfrak{g}_{v_1} \times \mathfrak{g}_{v_2}$ is the natural projection.

By the standard formalism of Lagrangian correspondences (see, e.g., [15, §7] and [19, §1.4]), the correspondence (4.1) induces the following commutative diagram of $G \times T$-varieties.

$$
\begin{array}{c}
G \times P (\Lambda_{v_1} \times \Lambda_{v_2}) & \xleftarrow{\overline{\phi}} & G \times P \Lambda_{v_1,v_2} & \xrightarrow{\overline{\psi}} & \Lambda_v \\
\downarrow & & \downarrow & & \downarrow \\
G \times P (M_{Q,v_1} \times M_{Q,v_2}) & \xleftarrow{\overline{T}} & T^* X & \xrightarrow{\overline{\phi}} & G \times P M_{Q,v_1,v_2} \\
& & & \downarrow & & \downarrow \\
& & & \Lambda_v & & \Lambda_v \\
\end{array}
$$

(4.2)

Note that the left square of (4.2) is a Cartesian square.

The map $m^p_{v_1, v_2}$ is defined to be the composition of the following morphisms.

1. The Künneth morphism $\mathcal{P}_{v_1} \otimes \mathcal{P}_{v_2} \to A_{L \times T}(\Lambda_{v_1} \times \Lambda_{v_2})$.

2. The natural isomorphism $A_{L \times T}(\Lambda_{v_1} \times \Lambda_{v_2}) \cong A_{G \times T}(G \times P (\Lambda_{v_1} \times \Lambda_{v_2}))$.

3. The following composition from diagram (4.2), with $\phi^d_\overline{\phi}$ being the Gysin pullback of $\phi$

$$
A_{G \times T}(G \times P (\Lambda_{v_1} \times \Lambda_{v_2})) \xrightarrow{\phi^d_\overline{\phi}} A_{G \times T}(G \times P \Lambda_{v_1,v_2}) \xrightarrow{\overline{\psi}} A_{G \times T}(\Lambda_v) \cong \mathcal{P}_0.
$$
4.2. Representations from quiver varieties

Let $Q^\triangledown$ be the framed quiver. Recall that the set of vertices of $Q^\triangledown$ is $I \sqcup I'$, with a bijection $I \rightarrow I'$ sending $i$ to $i'$. The set of edges of $Q^\triangledown$ is, by definition, a disjoint union of $H$ and a set of additional edges $j_i : i \rightarrow i'$, one for each vertex $i \in I, i' \in I'$. Let $\overline{Q^\triangledown}$ be the double of $Q^\triangledown$.

Let $w \in \mathbb{N}^I$ be the framing of $Q^\triangledown$. Let $V, W$ be two $I$-tuples of vector spaces with dimension vectors $v, w \in \mathbb{N}^I$. Let $M_{\overline{Q^\triangledown},(v,w)}$ be the representation space of the double framed quiver $\overline{Q^\triangledown}$. We have the isomorphism

$$M_{\overline{Q^\triangledown},v,w} \cong M_{Q,v} \oplus \text{Hom}_Q(V, W) \oplus \text{Hom}_Q(W, V).$$

For a fixed $I$-tuple of subvector spaces $V_1 \subset V$, such that dim($V_1$) = $v_1$, recall that we have $M_{Q,(v_1,v_2)} = \{b \in M_{Q,v_1+v_2} \mid b(V_1) \subset V_1\}$. Define

$$M_{\overline{Q^\triangledown},(v_1,v_2,v,w)} := \{(b, i, j) \mid b \in M_{Q,(v_1,v_2)}, i \in \text{Hom}(W,V), j \in \text{Hom}(V,W), \text{Im}(i) \subset V_1\}.$$

It gives the following correspondence of $G_v \times G_w \times T$-varieties

$$G_v \times p (M_{\overline{Q^\triangledown},(v_1,w)} \times M_{\overline{Q^\triangledown},v_2}) \leftarrow G_v \times p M_{\overline{Q^\triangledown},(v_1,v_2,w)} \rightarrow M_{\overline{Q^\triangledown},(v,w)}. \quad (4.3)$$

Let $\mu_{v,w} : M_{\overline{Q^\triangledown},(v,w)} \rightarrow g^*_v$ be the moment map for the $G_v$-action. Denote by $A_{v,w}$ the preimage of 0 under $\mu_{v,w}$, and let $A_{v_1,v_2,w} := A_{v_1+v_2,w} \cap M_{\overline{Q^\triangledown},(v_1,v_2,w)}$ be the intersection.

Consider the stability condition $\theta^+ = (1, 1, \cdots, 1)$ induced by the character $G_v \rightarrow G_m : (g_i)_{i \in I} \mapsto \prod_{i \in I} \text{det}(g_i) - 1$. Let $M_{\overline{Q^\triangledown},(v,w)}^{ss}$ be the set of semistable points of $M_{\overline{Q^\triangledown},(v,w)}$ under the $G_v$-action. The point $(x, x^*, i, j) \in A_{v,w}$ is $\theta^+$-semistable ([7, corollary 5.1.9]), if and only if, the following holds: For any collection of vector subspaces $S = (S_i)_{i \in k} \subset V = (V_i)_{i \in k}$, which is stable under the maps $x$ and $x^*$, if $S_k \subset \ker(j_k)$ for any $k \in I$, then $S = 0$. Note that this choice of stability condition is essential in constructing a right action of $P$ (see [19, remark 5.5] for details).

Consider the variety $X = G \times p (M_{Q^\triangledown,(v_1,w)} \times M_{Q^\triangledown,v_2})$. There is a bundle projection $T^*X \rightarrow G \times p (M_{Q^\triangledown,(v_1,w)} \times M_{Q^\triangledown,v_2})$. We define $T^*X^*$ to be the preimage of $G \times p (M_{Q^\triangledown,(v_1,w)} \times M_{Q^\triangledown,v_2})$ under this bundle projection. By [19, lemma 5.1], we have

$$T^*X^* = G \times p \{(c, (x, x^*, i, j), (y, y^*)) \in p_{v_1,v_2} \times M_{Q^\triangledown,v_1,w}^{ss} \times M_{Q^\triangledown,v_2} \mid [x, x^*] + i \circ j = pr_1(c), [y, y^*] = pr_2(c)\},$$
where \( pr_i : p_{v_1,v_2} \to g_{v_i} \) is the natural projection, \( i = 1, 2 \). The correspondence (4.3) induces the following commutative diagram of \( G_v \times T \times G_w \)-varieties.

\[
\begin{array}{ccc}
G \times P (\Lambda^{ss}_{v_1,w} \times \Lambda_{v_2}) & \xrightarrow{\psi} & G \times P \Lambda^{ss}_{v_1,v_2,w} \\
\downarrow & & \downarrow \\
G \times P (M^{ss}_{Q,v_1,v_2,w}) & \xrightarrow{\phi} & M^{ss}_{Q,v_1,v_2,w}
\end{array}
\]

Here the left square of diagram (4.4) is a pullback diagram.

For any dimension vectors \( v, w \in \mathbb{N}^I \), we have the isomorphism

\[
\mathcal{M}(v, w) := A_{T \times G_w}(\mathcal{M}(v, w)) \cong A_{G_v \times T \times G_w}(\Lambda^{ss}_{v_1,w}).
\]

The action of \( \mathcal{P}(A, Q) \) on \( \mathcal{M}(w) := \bigoplus_v \mathcal{M}(v, w) \), as a map

\[
a_{v_1,v_2} : \mathcal{M}(v_1, w) \otimes \mathcal{P}_{v_2} \to \mathcal{M}(v_1 + v_2, w)
\]

for each pair \( v_1, v_2 \in \mathbb{N}^I \), is defined to be the composition of the Künneth morphism

\[
A_{G_{v_1} \times G_w \times T} (\Lambda^{ss}_{v_1,w} \otimes A_{G_{v_2} \times T} (\Lambda_{v_2})) \to A_{L \times T \times G_w} (\Lambda^{ss}_{v_1,v_2})
\]

with the morphism

\[
\psi_* \circ \phi_{\phi}^*: A_{G_v \times T \times G_w} (G \times P (\Lambda^{ss}_{v_1,w} \times \Lambda_{v_2})) \to A_{G_v \times T \times G_w} (\Lambda^{ss}_{v,v}) = \mathcal{M}(v, w).
\]

Here \( \phi_{\phi}^* \) is the refined Gysin pullback of \( \phi \) in diagram (4.4).

5. Preprojective CoHA vs critical CoHA

We compare the preprojective CoHA of quiver \( Q \) with the auxiliary CoHA introduced in § 2.3 applied to a special quiver \( \hat{Q} \) with potential \( W \) as below.

5.1. A special case of potential and cut

**Example 5.1.** Let \( Q = (I, H) \) be any quiver. Let \( \Gamma \) be the extended quiver \( \hat{Q} \) introduced by Ginzburg in [7]. More precisely, \( \hat{Q} \) has the same set of vertices as \( Q = (I, H) \), and the following set of arrows:

1. an arrow \( a : i \to j \) for any arrow \( a : i \to j \) in \( Q \),
2. an arrow \( a^* : j \to i \) for any arrow \( a : i \to j \) in \( Q \),
3. a loop \( l_i : i \to i \) for any vertex \( i \) in \( Q \).

Define a potential \( W \) on \( \hat{Q} \) by the formula

\[
W = \sum_{(a : i \to j) \in H} (l_j a a^* - l_i a^* a) = \sum_{i \in I} l_i \cdot \sum_{a \in H} [a, a^*].
\]

Let \( C = \{ l_i \mid i \in I \} \) be the cut of the pair \( (\hat{Q}, W) \). In this case, the space \( \mathcal{I}_{\Gamma \setminus C, W} \) is the representation space of the preprojective algebra \( \Pi_Q := \mathbb{C}\hat{Q}/(\sum_{a \in H} [a, a^*]). \) And
for any \( v \in \mathbb{N}^I \) we have the equality
\[
J_{\Gamma \setminus C,v} \times M_{C,v} = \Lambda_v \times g_v.
\]

**Example 5.2.** Another example of the quiver with potential is \((\hat{Q}, W, C)\). Define a new quiver \( \hat{Q} \) to have the same set of vertices as \( Q^\forall \) and the following arrows:

1. an arrow \( a : i \to j \) for any arrow \( a : i \to j \) in \( Q^\forall \)
2. an arrow \( a^* : j \to i \) for any arrow \( a : i \to j \) in \( Q^\forall \)
3. a loop \( l_i : i \to i \) for any vertex \( i \) in \( Q \).

Let \( C = \{ l_i \mid i \in I \} \) be the cut and \( W = \sum_{k \in I} l_k \cdot (\sum_{a \in H}[a,a^*] + ij) \) be the potential on \( \hat{Q} \). In this case, the space \( J_{\Gamma \setminus C,v} \) is the representation space of \( \mathbb{C}Q^\forall / (\sum_{a \in H}[a,a^*] + ij) \). Hence, we have the equality
\[
J_{\Gamma \setminus C,v} \times M_{C,v} = \Lambda_{v,w} \times g_v.
\]

The natural projection \( \pi : \Lambda_{v,w} \times M_{C,v} \to \Lambda_{v,w} \) is a \( G_v \times G_w \)-equivariant vector bundle. We consider \( \pi^{-1}(\Lambda_{v,w}^{ss,v}) = \Lambda_{v,w}^{ss} \times M_{C,v} \). Then \( (\Lambda_{v,w}^{ss} \times M_{C,v})/G_v \) is an equivariant vector bundle on the quiver variety \( \mathbb{M}(v, w) = \Lambda_{v,w}^{ss}/G_v \).

### 5.2. The auxiliary CoHA in the case when \( \Gamma = \hat{Q} \)

Recall in § 2.4, the auxiliary cohomological Hall algebra associated with the data \((\hat{Q}, W, C)\) and arbitrary cohomology theory \( A \) is
\[
\mathcal{H}^{aux}(\hat{Q}, W, C) = \bigoplus_{v \in \mathbb{N}^\Gamma} A_{v,v} \times T(\Lambda_v \times g_v, \hat{Q}).
\]

Similar to (2.3) and (4.2), we have the following correspondence.

\[
\begin{array}{c}
G \times P(M_{\hat{Q},v_1} \times M_{\hat{Q},v_2}) \xrightarrow{i_1} G \times P(Y \times g_{v_1} \times g_{v_2}) \xrightarrow{q} G \times P(M_{\hat{Q},v_1,v_2}) \xrightarrow{\hat{q}} M_{\hat{Q},v_1,v_2} \\
G \times P(\Lambda_{v_1} \times \Lambda_{v_2} \times g_{v_2}) \xrightarrow{\tau} G \times P(\Lambda_{v_1,v_2} \times g_{v_1,v_2}) \xrightarrow{i_2} M_{\hat{Q},v_1,v_2} \xrightarrow{i_3} \Lambda_v \times g_v.
\end{array}
\]

(5.1)

Here the map \( \hat{q} \) is defined by \((g, x) \mapsto gxg^{-1}\). The map \( q \) is given by the product of \( \omega : M_{\hat{Q},v_1,v_2} \to Y \) (in diagram (2.3)) and the natural projection \( p_{v_1,v_2} \to g_{v_1} \times g_{v_2} \). Recall that \( T \) acts on the Lie algebra \( g_v^* \) by weight \( t_1, t_2 \). Consequently, the action of \( T \) on \( g_v \) is such that both \( \mathbb{G}_m \)-factors of \( T \) have weight -1. We assume the extra \( T \)-action on \( \Lambda_v \times g_v \) restricted to \( \Lambda_v \) satisfies assumption 4.1, so that the correspondence is equivariant.

The Hall multiplication \( m^{aux}_{v_1,v_2} \) of \( \mathcal{H}^{aux}(\hat{Q}, W, C) \) defined in § 2.4 is the same as the composition of the following morphisms.

1. The Künneth morphism
\[
A_{G_{v_1} \times T}(\Lambda_{v_1} \times g_{v_1}) \otimes A_{G_{v_2} \times T}(\Lambda_{v_2} \times g_{v_2}) \to A_{L \times T}(\Lambda_{v_1} \times g_{v_1} \times \Lambda_{v_2} \times g_{v_2}).
\]
(2) The isomorphisms:
\[ A_{L \times T} (\Lambda_{v_1} \times g_{v_1} \times \Lambda_{v_2} \times g_{v_2}) \cong A_{G \times T} (G \times p (\Lambda_{v_1} \times g_{v_1} \times \Lambda_{v_2} \times g_{v_2})). \]

(3) The refined Gysin pullback along \( q \) in (5.1):
\[ q^*_q : A_{G \times T} (G \times p (\Lambda_{v_1} \times g_{v_1} \times \Lambda_{v_2} \times g_{v_2})) \rightarrow A_{G \times T} (G \times p (\Lambda_{v_1,v_2} \times p_{v_1,v_2})). \]

(4) The pushforward \( \overline{\eta}_s \) in (5.1):
\[ \overline{\eta}_s : A_{G \times T} (G \times p (\Lambda_{v_1,v_2} \times p_{v_1,v_2})) \rightarrow A_{G \times T} (\Lambda_v \times g_v). \]

5.3. Action on the cohomology of quiver varieties

Similar to \( \S \) 4.2, we show \( H^{aux}(\hat{Q}, W, C) \) acts on the equivariant \( A \)-homology of the Nakajima quiver varieties. As in \( \S \) 5.2, we will also take the \( T \)-action into consideration.

Notations as in \( \S \) 4.2. For any \( v \in \mathbb{N}^I \), we have the isomorphism
\[ \mathcal{M}(v, w) := A_{G_w \times T} (\mathfrak{g}(v, w)) \cong A_{G_w \times G_v \times T} (\Lambda^s_{w,v} \times \mathfrak{g}_v). \]

For any \( v_1, v_2, w \in \mathbb{N}^I \), \( v = v_1 + v_2 \), we define a map
\[ c_{v_1,v_2}^{aux} : \mathcal{M}(v_1, w) \otimes H^{aux}(\hat{Q}, W, C) \rightarrow \mathcal{M}(v, w). \]

Let \( V, W \) be two \( I \)-tuple of vector spaces with dimension vectors \( v, w \in \mathbb{N}^I \). Recall we have the isomorphism
\[ M_{\hat{Q}^{ss},v,w} \cong M_{\hat{Q}^{ss},(v,w)} \times g_v \cong M_{\hat{Q}^{ss},v} \times \text{Hom}_Q(V, W) \times \text{Hom}_Q(W, V) \times g_v. \]

We start with the correspondence
\[ M_{\hat{Q}^{ss},(v_1,w)} \times M_{\hat{Q}^{ss},(v_2,w)} \xrightarrow{p} M_{\hat{Q}^{ss},(v_1,v_2,w)} \rightarrow M_{\hat{Q}^{ss},(v,w)}, \]
where \( M_{\hat{Q}^{ss},(v_1,v_2,w)} := \{ (b, i, j) | b \in M_{\hat{Q}^{ss},(v_1,v_2)}, i \in \text{Hom}(W, V), j \in \text{Hom}(V, W), \text{Im}(i) \subset V_1 \} \). For \( (b, i, j) \in M_{\hat{Q}^{ss},(v_1,v_2,w)} \), denote by \( (pr_1(b), pr_2(b)) \) the projection of \( b \) to \( M_{\hat{Q}^{ss},v_1} \times M_{\hat{Q}^{ss},v_2} \). Let \( i_{V_1} : W \rightarrow V_1 \) be the co-restriction of \( i \) on \( V_1 \), and \( j_{V_1} \) the restriction of \( j : V \rightarrow W \) on \( V_1 \). The map \( p \) is defined to be \( p : (b, i, j) \mapsto (pr_1(b), pr_2(b), j_{V_1}, i_{V_1}) \).

Let \( \pi_i : p_{v_1,v_2} \rightarrow g_v \) be the natural projection, \( i = 1, 2 \). Consider the varieties
\[ Y^s := \{ (l, (x, x^*, i, j), (y, y^*)) \in p_{v_1,v_2} \times M_{\hat{Q}^{ss},(v_1,w)} \}
\[ \times M_{\hat{Q}^{ss},v_2} | [x, x^*] + i \circ j = \pi_1(l), [y, y^*] = \pi_2(l) \}, \]
\[ M_{\hat{Q}^{ss},(v_1,v_2,w)} = M_{\hat{Q}^{ss},(v_1,v_2,w)} \cap (M_{\hat{Q}^{ss},(v_1,w)} \times g_v) \]
\[ = \{ (l, x, x^*, i, j) \in M_{\hat{Q}^{ss},(v_1,v_2,w)} | l \in p_{v_1,v_2}, (x, x^*) \in M_{\hat{Q}^{ss},(v,w)} \}. \]

Define a map from \( q : M_{\hat{Q}^{ss},(v_1,v_2,w)} \rightarrow g_v \times g_v \times Y^s \) by \( (l, x, x^*, i, j) \mapsto (\pi_1(l), pr_2(l), (x, x^*) + i \circ j, pr_1(x), pr_2(x), j_{V_1}, i_{V_1}) \). We have the embedding \( \Lambda_{v,w} \times g_v \subset M_{\hat{Q}^{ss},(v,w)} \). Let \( \Lambda_{v_1,v_2,w} \) be the intersection of \( \Lambda_{v_1,w} \times g_v \) with \( M_{\hat{Q}^{ss},(v_1,v_2,w)} \).
We have the following correspondence similar to (4.4), with the left square being Cartesian.

\[
\begin{array}{c}
G \times \mathcal{P} \left( \Lambda_{v_1,w} \times g_{v_1} \times \Lambda_{v_2} \right) \xrightarrow{\eta} G \times \mathcal{P} \left( \Lambda_{v_1,v_2,w} \times p_{v_1,v_2} \right) \xrightarrow{\hat{\eta}} \Lambda_{v,w} \times g_v \\
G \times \mathcal{P} (Y^s \times g_{v_1} \times g_{v_2}) \downarrow \downarrow q G \times \mathcal{P} (\Lambda_{v_1,v_2,w}) \downarrow \downarrow \mathcal{M}^{ss}_{G^*}(v_1,v_2,w) \xrightarrow{\hat{\eta}} \mathcal{M}^{ss}_{G^*}(v_1,v_2,w) \times g_v
\end{array}
\]

The action map \( a^\text{aux}_{v_1,v_2} \) on \( \mathcal{M}(w) = \bigoplus_{v \in \mathbb{N}^I} \mathcal{M}(v,w) \) is defined to be the composition of the following morphisms.

1. The Künneth morphism

\[ A_{G_{v_1} \times G_{v_2} \times T} (\Lambda_{v_1,w} \times g_{v_1} \times \Lambda_{v_2} \times g_{v_2}) \rightarrow A_{\mathcal{L} \times G_{v_2} \times T} (\Lambda_{v_1,w} \times g_{v_1} \times \Lambda_{v_2} \times g_{v_2}) \]

\[ \cong A_{G \times G_{v_2} \times T} (G \times p (\Lambda_{v_1,w} \times g_{v_1} \times \Lambda_{v_2} \times g_{v_2})). \]

2. The refined Gysin pullback \( q_\sharp^\ast \):

\[ A_{G \times G_{v_2} \times T} (G \times p (\Lambda_{v_1,w} \times g_{v_1} \times \Lambda_{v_2} \times g_{v_2})) \rightarrow A_{G \times G_{v_2} \times T} (G \times p (\Lambda_{v_1,v_2,w} \times p_{v_1,v_2})). \]

3. The pushforward \( \hat{\eta}_\ast \) in the correspondence (5.2)

\[ \hat{\eta}_\ast : A_{G \times G_{v_2} \times T} (G \times p (\Lambda_{v_1,v_2,w} \times p_{v_1,v_2})) \rightarrow A_{G \times G_{v_2} \times T} (\Lambda_{v_1+v_2,w} \times g_v). \]

We have the following

**Theorem 5.3.** For any \( w \in \mathbb{N}^I \), the maps \( a^\text{aux}_{v_1,v_2} \) define an algebra homomorphism \( \mathcal{H}^\text{aux}(\hat{Q}, W, C) \rightarrow \text{End}(\mathcal{M}(w)). \)

The proof of theorem 5.3 goes the same way as that of [19, theorem 5.4], taking into account the Lie algebra factors in the spaces which do not show up in loc. cit.

### 5.4. Multiplications of \( \mathcal{H}^\text{aux}(\hat{Q}, W, C) \) and \( \mathcal{P}(A, Q) \)

By the fact that \( A_{G \times T} (\Lambda_v \times g_v) \cong A_{G \times T} (\Lambda_v) \), there is an isomorphism of vector spaces \( \mathcal{P}_v(Q) \cong \mathcal{H}^\text{aux}_v(\hat{Q}, W, C) \) for any \( v \in \mathbb{N}^I \). However, this isomorphism is not compatible with the multiplications.
PROPOSITION 5.4. Let \( m^{\text{aux}} \) be the multiplication of \( \mathcal{H}^{\text{aux}}(\hat{Q}, W, C) \). Then, for \( x \in \mathcal{H}^{\text{aux}}_{v_1} \) and \( y \in \mathcal{H}^{\text{aux}}_{v_2} \), we have

\[
m^{\text{aux}}(x \otimes y) = \overline{\psi}_* \left( e(\gamma) \cdot \phi^\sharp \left( x \otimes y \right) \right),
\]

where \( \phi \) is as in §4.1, (4.2) and \( e(\gamma) \) is the equivariant Euler class of the closed embedding

\[
\gamma : G \times P \left( \Lambda_{v_1, v_2} \times p_{v_1, v_2} \right) \hookrightarrow (G \times P \Lambda_{v_1, v_2}) \times g_{v_1 + v_2},
\]

\[
(g, a, b) \mapsto ((g, a), gb^{-1}), \quad \text{for } a \in \Lambda_{v_1 + v_2}, b \in p_{v_1, v_2}.
\]

Proof. Recall the following diagram of correspondences:

\[
\begin{array}{ccc}
G \times P \left( \Lambda_{v_1} \times \Lambda_{v_2} \right) & \xrightarrow{\overline{\varphi}} & G \times P \Lambda_{v_1, v_2} & \xrightarrow{\overline{\psi}} & \Lambda_{v_1 + v_2} \\
\pi_{v_1} \times \pi_{v_2} & & \pi & & \pi_{v_1 + v_2} \\
G \times P \left( \Lambda_{v_1} \times g_{v_1} \times \Lambda_{v_2} \times g_{v_2} \right) & \xleftarrow{\overline{\eta}} & G \times P \left( \Lambda_{v_1, v_2} \times p_{v_1, v_2} \right) & \xleftarrow{\overline{\eta}} & \Lambda_{v_1 + v_2} \times g_v
\end{array}
\]

where the map \( \pi \) is a vector bundle with fibre \( p_{v_1, v_2} \). Under the natural isomorphisms induced by affine bundles, we have \( q^\sharp \eta = \phi^\sharp \overline{\psi} \). Note that the right square is not a Cartesian diagram. We factor the map \( \overline{\eta} \) as \( (\overline{\psi} \times \text{id}) \circ \gamma \), where \( \gamma \) is a closed embedding.

\[
\begin{array}{ccc}
G \times P \Lambda_{v_1, v_2} & \xrightarrow{\gamma} & (G \times P \Lambda_{v_1, v_2}) \times g_{v_1 + v_2} & \xrightarrow{\overline{\psi} \times \text{id}} & \Lambda_{v_1 + v_2} \times g_{v_1 + v_2} \\
\pi \times \text{pr}_1 & & \pi \times \text{pr}_1 & & \pi \times \text{pr}_1 \\
G \times P \left( \Lambda_{v_1, v_2} \times p_{v_1, v_2} \right) & \xleftarrow{\overline{\varphi}} & G \times P \left( \Lambda_{v_1, v_2} \right) & \xleftarrow{\overline{\psi}} & \Lambda_{v_1 + v_2}
\end{array}
\]

(5.3)

Clearly, the square in above diagram is a pullback diagram and satisfies the conditions in [19, lemma 1.16]. The pushforward \( \gamma_* : A_{G \times T}(G \times P \Lambda_{v_1, v_2}) \rightarrow A_{G \times T}(G \times P \Lambda_{v_1, v_2}) \) is given by \( x \mapsto x \cdot e(\gamma) \). The proposition now follows from the definition of Hall multiplication \( m^{\text{aux}} \) of \( \mathcal{H}^{\text{aux}}(\hat{Q}, W, C) \). \( \square \)

Recall that \( \gamma \) is the map

\[
\gamma : G \times P \left( p_{v_1, v_2} \times \Lambda_{v_1, v_2} \right) \hookrightarrow G \times P \Lambda_{v_1, v_2} \times g_{v_1 + v_2}, \quad (g, a, b) \mapsto ((g, a), gb^{-1}).
\]

In particular, the normal bundle to \( \gamma \) can be identified with the normal bundle of \( p_{v_1, v_2} \) in \( g_{v_1 + v_2} \).

Let \( a^p \) be the (right) action of \( \mathcal{P}(A, Q) \) on \( \mathcal{M}(w) \). By construction, \( a^p(m \otimes x) = \overline{\psi}_* \phi^\sharp \left( m \otimes x \right) \), where \( \phi \) and \( \overline{\psi} \) are as in §4.2, (4.4).
Proposition 5.5. Let $a^{\text{aux}}$ be the (right) action of $\mathcal{H}^{\text{aux}}(\hat{Q}, W, C)$ on $\mathcal{M}(w)$. We then have:

$$a^{\text{aux}}(m \otimes x) = \overline{\psi}_*(e(\gamma) \cdot \phi^x_\phi(m \otimes x)).$$

Proof. The proof is similar as the proof of proposition 5.4. □

Recall that $m^p(x \otimes y) = \overline{\psi}_* \phi^x_\phi(x \otimes y)$, $x \in \mathcal{P}_v$ and $y \in \mathcal{P}_v$. Proposition 5.4 compares the multiplications of $\mathcal{P}(A, Q)$ and $\mathcal{H}^{\text{aux}}(\hat{Q}, W, C)$, but it does not give an algebra homomorphism between the two algebras.

6. Conclusions

In this section, we compare the preprojective CoHA of $Q$ and the critical CoHA defined by Kontsevich-Soibelman associated with the quiver with potential $(\hat{Q}, W)$, hence prove theorem 1.1. By theorem 2.5, it suffices to compare the preprojective CoHA with $\mathcal{H}_{\text{c}}(\hat{Q}, W, C) = \mathcal{H}^{\text{aux}}(\hat{Q}, W, C)$, with the multiplication $m^c = \overline{i_2}_* \circ \frac{1}{e(\gamma)} \omega_\gamma^+ \circ \overline{i_1}_* \circ \overline{\rho}^+$.

In this section we take $A$ to be the Borel-Moore homology. In remark 6.2 we comment on the generality in which the main theorem of this section holds.

6.1. The main theorem

Recall that in (2.3), we have the following map

$$\iota: G \times_P (M_{Q,v_1} \times M_{Q,v_2}) \hookrightarrow G \times_P Y.$$  

Let $e(\iota)$ be the equivariant Euler class of the normal bundle of $\iota$. By [19, §3.2], the normal bundle of $\iota$ is isomorphic to $T^*G/P$ as a bundle over the Grassmannian $G/P$, which equivariantly is $G \times_P (g/v, p)^*$. Here the $T$-action on $M_{C,v} \equiv g_v$ is specified in §4. Recall that the normal bundle of $\gamma$ is the normal bundle of $p_{v_1, v_2}$ in $g_{v_1 + v_2}$, with the $T$-action induced from that on $g$. Therefore, we have

$$e(\gamma) = \prod_{i \in I} (-1)^{v_i v_2} e(\iota) = (-1)^{v_1 v_2} e(\iota),$$

where $(-1)^{v_1 v_2} := \prod_{i \in I} (-1)^{v_i v_2}$ for any $v_1, v_2 \in \mathbb{N}^I$.

We have the main theorem of this section.

Theorem 6.1.

(1) Under assumption 4.1, there is an isomorphism of $\mathbb{N}^I$-graded associative algebras $\Xi: \mathcal{P}(BM, Q) \to \mathcal{H}(\hat{Q}, W)$ whose restriction to the degree-$v$ piece is

$$\Xi_v: \mathcal{P}(BM, Q)_v \to \mathcal{H}^c(\hat{Q}, W)_v, \quad f \mapsto f \cdot (-1)^{v_2},$$

where $(-1)^{v_2} := \prod_{i \in I} (-1)^{v_i v_2}$. Here we define $(-1)^{v_2} = 0$ if $n = 0, 1$. 


\( a^\varepsilon(\Xi_v(x))(m \cdot (-1)^{\binom{v}{2}}) = (a^p(x)(m)) \cdot (-1)^{\binom{v_1+v_2}{2}} \)

for any \( w, v_1, v_2 \in N^t, x \in P_v, \) and \( m \in H^B_{G \times \mathbb{Y}}(\mathcal{M}(v_2, w)) \).

**Proof.** We only prove (2.1). Part (2.2) follows from a similar argument. By [2, corollary A.9] (recalled below as theorem A.1), \( P(BM, Q) \) is isomorphic to \( \mathcal{H}^C(\hat{Q}, W) \) as abelian groups. It suffices to show the map \( \Xi_v \) respects the multiplication structure.

For \( x \in P_v \) and \( y \in P_v \), by definition, \( m^P(x \otimes y) = \overline{\psi}_s \phi_\varepsilon(x \otimes y) \), where \( \phi \) and \( \overline{\psi} \) are as in §4.1 (4.2). On \( \mathcal{H}^{\text{aux}}(\hat{Q}, W, C) = P(BM, Q) \), by proposition 5.4 we have

\[ m^{\text{aux}}(x \otimes y) = \overline{\psi}_s \left( e(\gamma) \cdot \phi_\varepsilon(x \otimes y) \right) = (-1)^{v_1v_2} \overline{\psi}_s \left( e(1) \cdot \phi_\varepsilon(x \otimes y) \right). \]

On the other hand, on \( \mathcal{H}^C(\hat{Q}, W, C) = \mathcal{H}^{\text{aux}}(\hat{Q}, W, C) \), by theorem 2.5 we have \( m^C = \overline{r}_{2s} \circ 1/e(1) \omega_{2s} \circ \overline{p}_{1s} \circ \overline{p}_1^* \), while \( m^{\text{aux}} = \overline{r}_{2s} \circ \omega_{2s} \circ \overline{p}_{1s} \circ \overline{p}_1^* \). Therefore, \( m^P(x \otimes y) = (-1)^{v_1v_2} m^C(x \otimes y) \).

Now we verify that the isomorphism of vector spaces \( \Xi : (P(BM, Q), m^P) \rightarrow (\mathcal{H}^C(\hat{Q}, W, C), m^C) \) is an algebra homomorphism. Indeed, on the one hand, we have

\[ \Xi(m^P(x \otimes y)) = \Xi((-1)^{v_1v_2} m^C(x \otimes y)) = (-1)^{\binom{v_1+v_2}{2}} m^C(x \otimes y). \]

On the other hand, we have

\[ m^C(\Xi(x) \otimes \Xi(y)) = m^C((-1)^{\binom{v}{2}}(x) \otimes (-1)^{\binom{v}{2}}(y)) = (-1)^{\binom{v}{2}}(-1)^{\binom{v}{2}} m^C(x \otimes y). \]

The equality \( \binom{v_1}{2} + \binom{v_2}{2} + v_1v_2 = \binom{v_1+v_2}{2} \), for any \( i \in I \), shows \( \Xi(m^P(x \otimes y)) = m^C(\Xi(x) \otimes \Xi(y)) \). Thus, \( \Xi \) is an algebra homomorphism. \( \square \)

**Remark 6.2.** Let \( A \) be an arbitrary oriented cohomology theory. Let \( (R, F) \) be the formal group law associated with \( A \). Then, theorem 6.1 holds as long as \( F \) is an odd function in the sense that \( x - F y = -(y - F x) \).

**Appendix A. Borel-Moore homology and critical cohomology**

In this section, we show the compatibility of push-forwards and pull-backs in the Borel-Moore homology and the critical cohomology.

**A.1. From the critical cohomology to ordinary cohomology**

We compare the critical cohomology with the ordinary cohomology in this section, following the appendix of [2]. Let \( \pi : Y = X \times \mathbb{A}^n \rightarrow X \) be the trivial vector bundle, carrying a scaling \( \mathbb{G}_m \) action on the fibre \( \mathbb{A}^n \). Let \( f : Y = X \times \mathbb{A}^n \rightarrow \mathbb{A}^1 \) be a \( \mathbb{G}_m \)–equivariant function with respect to the natural scaling \( \mathbb{G}_m \) action on the target. For simplicity we assume \( f^{-1}(0) \supset \text{Crit}(f) \). Define \( Z \subset X \) to be the reduced scheme...
consisting of points $z \in X$, such that $\pi^{-1}(z) \subset f^{-1}(0)$. To summarize the notations, we have the diagram:

$$
\begin{array}{c}
Z \times \mathbb{A}^n \xrightarrow{i \times \text{id}} X \times \mathbb{A}^n \\
\downarrow \pi_Z \downarrow i \\
Z \xrightarrow{i} X \xrightarrow{p} \text{pt.}
\end{array}
$$

Let $\varphi_f$ be the vanishing cycle functor for $f$. Following the convention of [2], we consider $\varphi_f$ as a functor $D^b(Y) \to D^b(Y)$ between the derived categories of $Y = X \times \mathbb{A}^n$. By an abuse of notation, we will abbreviate the vanishing cycle complex $\varphi_f^! Y[-1]$ to $\varphi_f$. The support of $\varphi_f$ is on the critical locus of $f$. For a $G$-variety $X$, let $H_{c,G}^*(X)$ be the equivariant cohomology with compact support. Let $H_{c,G}^*(X)$ be its Verdier dual.

**Theorem A.1 ([2], theorem A.1 and corollary A.9).** There is a natural isomorphism of functors $D^b(X) \to D^b(Y)$, $\pi_1 \varphi_f \pi^*[-1] \cong \pi_1 i_* i^*$. In particular, we have $H_{c,G}^*(Y, \varphi_f) \cong H_{c,G}^*(Z \times \mathbb{A}^n, \mathbb{Q})$.

Indeed, by definition, we have

$$H_{c}^*(Y, \varphi_f) = p_Y ! \varphi_f[-1] (Q_Y) = p_1 \pi_1 \varphi_f[-1] (\pi^*(Q_X)).$$

On the other hand, we have the isomorphism

$$H_{c}^*(Z \times \mathbb{A}^n, \mathbb{Q}) = p_1 \pi_1 (i \times \text{id}_! )_! Q_Z \times \mathbb{A}^n = p_1 \pi_1 (i \times \text{id}_! )_! \pi_2^* Q_Z = p_1 \pi_1 \pi_2^* i_* i^* Q_X.$$

Thus, the isomorphism $H_{c,G}^*(Y, \varphi_f) \cong H_{c,G}^*(Z \times \mathbb{A}^n, \mathbb{Q})$ follows from the isomorphism of the two functors in theorem A.1, which is shown in [2, theorem A.1].

**Proposition A.2 ([2, proposition A.8]).** The following diagram of isomorphisms commutes.

$$
\begin{array}{ccc}
H_{c}^*(f_1^{-1}(0), \varphi_{f_1}) \otimes H_{c}^*(f_2^{-1}(0), \varphi_{f_2}) & \xrightarrow{\text{TS}} & H_{c}^*(f_1^{-1}(0) \times f_2^{-1}(0), \varphi_{f_1 \oplus f_2}) \\
\downarrow \cong & & \downarrow \cong \\
H_{c}^*(Z_1 \times \mathbb{A}^{n_1}, \mathbb{Q}) \otimes H_{c}^*(Z_2 \times \mathbb{A}^{n_2}, \mathbb{Q}) & \xrightarrow{\text{Ku}} & H_{c}^*(Z_1 \times Z_2 \times \mathbb{A}^{n_1+n_2}, \mathbb{Q}),
\end{array}
$$

where TS is the Thom-Sebastiani isomorphism, Ku is the Künneth isomorphism, and the vertical isomorphisms are as in theorem A.1 of [2].

**A.2. Compatibility of push-forwards and pullbacks**

In this section, we show the isomorphism in theorem A.1 is compatible with pullbacks and proper pushforwards.

Let $g : X \to X'$ be a morphism and $g \times h : Y = X \times \mathbb{A}^n \to Y' = X' \times \mathbb{A}^m$ be the morphism of the trivial bundles, where $h : \mathbb{A}^n \to \mathbb{A}^m$ is a linear morphism. Let $f' : X' \times \mathbb{A}^m \to \mathbb{A}^1$ be a function, and $f := f' \circ (g \times h)$. 
Lemma A.3. Assume $h$ is a surjective linear map, then there is a map $g_Z : Z \rightarrow Z'$ induced by $g : X \rightarrow X'$ so that $Z$ is the fibre product of $X$ and $Z'$, that is, the bottom square in the following diagram is Cartesian. In particular, in this case the top square in the following diagram is also Cartesian.

Proof. This follows from the definitions of $Z$ and $Z'$, and the assumption that $h$ is surjective. □

For the purpose of the present paper we only consider cases in which $h$ is surjective.

Lemma A.4. With notations as above, assume $g$ is a proper morphism, $m = n$ and $h$ is the identity map. Then, the following diagram commutes.

\[
\begin{array}{ccc}
H^*_c(X \times \mathbb{A}^n, \varphi_f) & \rightarrow & H^*_c(X' \times \mathbb{A}^m, \varphi_{f'}) \\
\cong & & \cong \\
H^*_c(Z \times \mathbb{A}^n, \mathbb{Q}) & \rightarrow & H^*_c(Z' \times \mathbb{A}^m, \mathbb{Q}).
\end{array}
\]

In the diagram, the vertical isomorphisms are given in theorem A.1.

Proof. By the commutativity of vanishing cycle functors with proper pushforwards, the commutativity of the diagram in the lemma is equivalent to the commutativity of the following diagram:

\[
p_{X' \times \mathbb{A}^m}(\varphi_f)[-1] \left( Q_{X' \times \mathbb{A}^m} \rightarrow (g \times h)_* Q_{X \times \mathbb{A}^n} \right) \cong \left( Q_{X' \times \mathbb{A}^m} \rightarrow (g \times h)_* Q_{X \times \mathbb{A}^n} \right).
\]

Applying the two functors

\[
F = p_{X'}(\pi_{X'})! \cdot \varphi_f(\pi_{X'})^*[−1], \quad G = p_X(\pi_{X'})!(π_{X'})^* i'_* i'^* 
\]

to the morphism $(Q_{X'} \rightarrow g_* Q_X)$ gives the desired commutativity. □
Now we consider pullbacks.

**Lemma A.5.** With notations as in diagram (A.1), then, the following diagram, in which the vertical isomorphisms are given in theorem A.1, commutes in the following two cases

1. $m = n$, and $h$ is the identity map;
2. $h$ is a surjective linear map, $X' = X$ and $g : X \to X'$ is the identity map.

\[
\begin{array}{ccc}
H^*_c(X' \times \mathbb{A}^m, \varphi_f')^\vee & \xrightarrow{(g \times h)^*} & H^*_c(X \times \mathbb{A}^n, \varphi_f)^\vee \\
\cong & \downarrow & \cong \\
H^*_c(Z' \times \mathbb{A}^m, \mathbb{Q})^\vee & \xrightarrow{(g \times h)^*} & H^*_c(Z \times \mathbb{A}^n, \mathbb{Q})^\vee.
\end{array}
\]

**Proof.** In the case $m = n$, and $h$ is the identity map, the commutativity is equivalent to the commutativity of the following diagram:

\[
p(X' \times \mathbb{A}^m | \varphi_f'[-1])(g \times h) : \mathbb{Q}_{X \times \mathbb{A}^n} \to \mathbb{Q}_{X' \times \mathbb{A}^m}[\dim(g \times h)]
\]

Applying the two functors

\[
F = pX' \times \mathbb{A}^n | \varphi_f' \pi_{X'}[-1], \quad G = pX' \times \mathbb{A}^m | \pi_{X'} \iota^!\iota^*
\]

to the morphism $(g|_{\mathbb{Q}_X} \to \mathbb{Q}_{X'}[\dim g])$ gives the desired commutativity.

In the case when $g$ is the identity map, by lemma A.3, the induced map $g_{\mathbb{Z}}$ is an isomorphism. Both the top and the bottom of the diagram become the natural isomorphism induced by an affine bundle.

\[\square\]

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Preprojective CoHA and critical CoHA

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