A STUDY ON $q$-APPELL POLYNOMIALS FROM DETERMINANATL POINT OF VIEW

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Abstract. This research is aimed to give a determinantal definition for the $q$-Appell polynomials and show some classical properties as well as find some interesting properties of the mentioned polynomials in the light of the new definition.

1. Introduction, preliminaries and definitions

Throughout this research we always apply the following notations: $\mathbb{N}$ indicates the set of natural numbers, $\mathbb{N}_0$ indicates the set of non-negative integers, $\mathbb{R}$ indicates set of all real numbers, and $\mathbb{C}$ denotes the set of complex numbers. We refer the readers to [1] for all the following $q$-standard notations. The $q$-shifted factorial is defined as

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - q^j a), \quad n \in \mathbb{N}, \quad (a; q)_\infty = \prod_{j=0}^{\infty} (1 - q^j a), \quad |q| < 1, a \in \mathbb{C}.$$ 

The $q$-numbers and $q$-factorial are defined by

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad [0]! = 1, \quad [n]_q! = [1]_q[2]_q \cdots [n]_q, \quad n \in \mathbb{N}, a \in \mathbb{C},$$

respectively. The $q$-polynomial coefficient is defined by

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q!(n-k)_q!}.$$

The $q$-analogue of the function $(x + y)^n$ is defined by

$$(x + y)_q^n := \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q q^{1/2(k-1)} x^{n-k} y^k, \quad n \in \mathbb{N}_0.$$ 

The $q$-binomial formula is known as

$$(1-a)_q^n = \prod_{j=0}^{n-1} (1 - q^j a) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q q^{1/2(k-1)} (-1)^k a^k.$$
In the standard approach to the $q$-calculus two exponential functions are used:

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1 - (1 - q)q^k z)}, \quad 0 < |q| < 1, \ |z| < \frac{1}{|1 - q|}.$$  

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} z^n}{[n]_q!} = \prod_{k=0}^{\infty} (1 + (1 - q)q^k z), \quad 0 < |q| < 1, \ z \in \mathbb{C}.$$  

From this form we easily see that $e_q(z) E_q(-z) = 1$. Moreover,  

$$D_q e_q(z) = e_q(z), \quad D_q E_q(z) = E_q(qz),$$  

The $q$-derivative of a function $f$ at point $0 \neq z \in \mathbb{C}$ is defined as  

$$D_q f(z) := \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1.$$  

**Proposition 1.** Consider two arbitrary functions $f(z)$ and $g(z)$. The following relations hold for the $q$-derivative:  

a) if $f$ is differentiable,  

$$\lim_{q \to 1} D_q f(z) = \frac{df(z)}{dz},$$  

where $\frac{d}{dz}$ indicates the ordinary derivative is defined in Calculus.  

b) $D_q$ is a linear operator; that is, for arbitrary constants $a$ and $b$  

$$D_q(a f(z) + b g(z)) = a D_q(f(z)) + b D_q(g(z)),$$  

c)  

$$D_q(f(z)g(z)) = f(qz)D_q g(z) + g(z)D_q f(z),$$  

d)  

$$D_q \left( \frac{f(z)}{g(z)} \right) = \frac{g(qz)D_q f(z) - f(qz)D_q g(z)}{g(z)g(qz)}.$$  

For the first time in 1909 Jackson introduced the $q$-analogue of Taylor series expansion of an arbitrary function $f(z)$ for $0 < q < 1$, as follows  

$$f(z) = \sum_{n=0}^{\infty} \frac{(1 - q)^n}{(q; q)_n} D^n f(a)(z - a)_q^n,$$  

where $D^n f(a)$ is the $n^{th}$ $q$-derivative of the function $f$ at point $a$.  

Furthermore, Jackson integral of an arbitrary function $f(x)$ is defined as,  

$$\int f(x) d_q x = (1 - q) \sum_{n=0}^{\infty} x q^n f(x q^n), \quad 0 < q < 1.$$  

Appell polynomials for the first time were defined by Appell in 1880. Inspired by the work of Throne [5], Sheffer [6], and Varma [7], Al-Salam, in 1967, introduced the family of $q$-Appell polynomials $\{A_{n,q}(x)\}_{n=0}^{\infty}$, and studied some of their properties [8]. According to his definition, the n-degree polynomials $A_{n,q}(x)$ are called $q$-Appell if they hold the following $q$-differential equation  

$$D_{q,x}(A_{n,q}(x)) = [n]_q A_{n-1,q}(x), \quad n = 0, 1, 2, ...$$
Note to the fact that $A_{0,q}(x)$ is a non zero constant let say $A_{0,q}$. To begin with the relation(5) for $n = 1$, i. e.

$$D_{q,x}(A_{1,q}(x)) = [1]_q A_{0,q}(x) = A_{0,q}.$$  

Using Jackson integral for the $q$-differential equation above, we get

$$A_{1,q}(x) = A_{0,q}x + A_{1,q},$$

where $A_{1,q}$ is an arbitrary constant. We can repeat the method above to obtain $A_{2,q}(x)$, as below by starting from the property(5) for $q$-Appell polynomials

$$D_{q,x}(A_{2,q}(x)) = [2]_q A_{1,q}x = [2]_q A_{0,q}x + [2]_q A_{1,q}.$$  

Now take Jackson integral

$$A_{2,q}(x) = A_{0,q}x^2 + [2]_q A_{1,q} + A_{2,q},$$

where $A_{2,q}$ is an arbitrary constant.

By using induction on $n$ and applying similar method to the methods used for finding $A_{1,q}(x)$, $A_{2,q}(x)$ and continuing taking Jackson integrals we have

$$A_{n-1,q}(x) = A_{n-1,q} + \left[\frac{n-1}{1}\right]_q A_{n-2,q}x + \left[\frac{n-1}{2}\right]_q A_{n-3,q}x^2 + \ldots + A_{0,q}x^{n-1}.$$  

Considering the fact that for $n = 1, 2, 3, \ldots$, every $A_{n,q}(x)$ satisfies the relation(5), we can write

$$D_{q,x}(A_{n,q}(x)) = \left[n\right]_q A_{n-1,q} + \left[n\right]_q A_{n-2,q}x + \left[n\right]_q A_{n-3,q}x^2 + \ldots + \left[n\right]_q A_{0,q}x^{n-1}.$$  

Now, taking the Jackson integral of the $q$-differential equation above can lead to

$$A_{n,q}(x) = A_{n,q} + \left[n\right]_q A_{n-1,q}x + \left[\frac{n}{2}\right]_q A_{n-2,q}x^2 + \left[\frac{n}{3}\right]_q A_{n-3,q}x^3 + \ldots + \left[\frac{n}{n}\right]_q A_{0,q}x^n,$$

where $A_{n,q}$ is an arbitrary constant. Since

$$\left[\frac{n}{i}\right]_q = \left[i\right]_q^{-1},$$

so for $n = 0, 1, 2, \ldots$, we have

(6)

$$A_{n,q}(x) = A_{n,q} + \left[n\right]_q A_{n-1,q}x + \left[\frac{n}{2}\right]_q A_{n-2,q}x^2 + \left[\frac{n}{3}\right]_q A_{n-3,q}x^3 + \ldots + A_{0,q}x^n.$$  

It is worthy of note that according to the discussion above there exists a one to one correspondence between the family of $q$-Appell polynomials $\{A_{n,q}(x)\}_{n=0}^{\infty}$ and the numerical sequence $q$-Appell polynomials $\{A_{n,q}\}_{n=0}^{\infty}$, $A_{n,q} \neq 0$. Moreover, every $A_{n,q}(x)$ can be obtained recursively from $A_{n-1,q}(x)$ for $n \geq 1$. 
Also, \(q\)-Appell polynomials can be defined by means of generating function \(A_q(t)\), as follows

\[
A_q(x, t) := A_q(t)e_q(tx) = \sum_{n=0}^{\infty} A_{n,q}(x) \frac{t^n}{[n]_q!}, \quad 0 < q < 1,
\]

where

\[
A_q(t) := \sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q!}, \quad A_q(t) \neq 0,
\]

is an analytic function at \(t = 0\). \(A_{n,q}(x) := A_{n,q}(0)\), and \(e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}\).

Based on different selections for the generating function \(A_q(t)\), different families of \(q\)-Appell polynomials can be obtained. In the following we mention some of them:

a) Taking \(A_q(t) = [1]_q = 1\) leads to obtain the family including all increasing integer powers of \(x\) starting from 0,

\[\{1, x, x^2, x^3, \ldots\}\].

b) Taking \(A_q(t) = \left(\frac{\tau_q(t) - \tau_{m-1,q}(t)}{\tau_q(t) - \tau_{m-1,q}(t)}\right)^\alpha\), leads to obtain the family of generalized \(q\)-Bernoulli polynomials \(B_{n,q}^{[m-1,\alpha]}(x, 0)\), \([16]\).

c) Taking \(A_q(t) = \left(\frac{\tau_q(t) + \tau_{m-1,q}(t)}{\tau_q(t) + \tau_{m-1,q}(t)}\right)^\alpha\), leads to obtain the family of generalized \(q\)-Euler polynomials \(E_{n,q}^{[m-1,\alpha]}(x, 0)\), \([16]\).

d) Taking \(A_q(t) = \left(\frac{\tau_q(t) + \tau_{m-1,q}(t)}{\tau_q(t) + \tau_{m-1,q}(t)}\right)^\alpha\), leads to obtain the family of generalized \(q\)-Genocchi polynomials \(G_{n,q}^{[m-1,\alpha]}(x, 0)\), \([16]\).

e) Taking \(A_q(t) = \left(\frac{\lambda e_q(t) - \tau_{m-1,q}(t)}{\lambda e_q(t) - \tau_{m-1,q}(t)}\right)^\alpha\), leads to obtain the family of generalized \(q\)-Apostol Bernoulli polynomials \(B_{n,q}^{[m-1,\alpha]}(x, 0; \lambda)\) of order \(\alpha\), \([17]\).

f) Taking \(A_q(t) = \left(\frac{\tau_q(t) + \tau_{m-1,q}(t)}{\tau_q(t) + \tau_{m-1,q}(t)}\right)^\alpha\), leads to obtain the family of \(q\)-Apostol-Euler polynomials \(E_{n,q}^{[m-1,\alpha]}(x, 0; \lambda)\) of order \(\alpha\), \([17]\).

g) Taking \(A_q(t) = \left(\frac{\tau_q(t) + \tau_{m-1,q}(t)}{\lambda e_q(t) + \tau_{m-1,q}(t)}\right)^\alpha\), leads to obtain the family of \(q\)-Apostol-Genocchi polynomials \(G_{n,q}^{[m-1,\alpha]}(x, 0; \lambda)\) of order \(\alpha\), \([17]\).

h) Taking \(A_q(t) = H_q(t) = \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)} \frac{2^n}{[2n]!!}\), leads to obtain the family of \(q\)-Hermite polynomials \(H_{n,q}(x)\).

Later, in 1982, Srivastava specified more characterizations of the family of \(q\)-Appell polynomials, \([9]\). Over the past decades, \(q\)-Appell polynomials have been studied from different aspects in \([11], [12]\), using different methods such as operator algebra their properties are found in \([10]\). Also, recently, the \(q\)-difference equations satisfied by sequence of \(q\)-Appell polynomials have been derived by Mahmudov, \([13]\). In this paper, inspired by the Costabile et al.’s algebraic approach for defining Bernoulli polynomials as well as Appell polynomials, for the first time, we introduce a determinantal definition of the well known family of \(q\)-Appell polynomials, \([14], [15]\). This new algebraic definition, not only allows us to benefit from algebraic properties of determinant to prove the existing properties of \(q\)-Appell polynomials...
more simpler, but also helps to find some new properties. Moreover, this approach unifies all different families of $q$-Appell polynomials some of which are mentioned in a)-h).

In the following sections, firstly we introduce the determinantal definition of $q$-Appell polynomials and then we show that this definition matches with the classical definitions. Next we prove some classical and new properties related to this family in the light of the new definition and by using the related algebraic approaches.

2. $q$-POLYNOMIALS FROM DETERMINANTAL POINT OF VIEW

Assume that $P_{n,q}(x)$ is an $n$-degree $q$-polynomial defined as follows

\[
\begin{align*}
P_{0,q}(x) &= \frac{1}{\beta_0}, \\
P_{n,q}(x) &= \frac{(-1)^n}{(\beta_0)^n} \\
\begin{vmatrix}
1 & x & x^2 & \ldots & x^{n-1} & x^n \\
\beta_0 & \beta_1 & \beta_2 & \ldots & \beta_{n-1} & \beta_n \\
0 & \beta_0 & \frac{2}{1} & \beta_1 & \ldots & \frac{n-1}{q} & \beta_{n-2} & \frac{n}{q} & \beta_{n-1} \\
0 & 0 & \beta_0 & \ldots & \frac{n-1}{2} & \beta_{n-3} & \frac{n}{2} & \beta_{n-2} & \ldots & \beta_0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & 0 & \beta_0 & \ldots & \frac{n}{n-1} & \beta_1
\end{vmatrix}
\end{align*}
\]

where $\beta_0, \beta_1, \ldots, \beta_n \in \mathbb{R}$, $\beta_0 \neq 0$, $n = 1, 2, 3, \ldots$

Then we can obtain the following results.

**Lemma 2.** Suppose that $A_{n\times n}(x)$ is a matrix including elements $a_{ij}(x)$ which are first order $q$-differentiable functions of variable $x$. Then the $q$-derivative of $\det(A_{n\times n}(x))$ can be calculated by the following formula.

\[
D_{q,x}(\det(A_{n\times n}(x))) = D_{q,x}(a_{ij}(x))
\]

\[
= \sum_{i=1}^{n} D_{q,x}(a_{i1}(x)) a_{i2}(qx) \ldots a_{in}(qx) + \ldots + D_{q,x}(a_{i1}(x)) a_{i2}(qx) \ldots a_{in}(qx)
\]

where $D_{q,x}(a_{ij}(x))$ is a matrix including elements $D_{q,x}(a_{ij}(x))$.

**Proof.** The proof can be done by induction on $n$. \hfill \Box

**Theorem 3.** $P_{n,q}(x)$ satisfies the following identity

\[
D_{q,x}(P_{n,q}(x)) = [n]_q P_{n-1,q}(x), \quad n = 1, 2, \ldots
\]
Proof. Taking the $q$-derivative of determinant (9) with respect to $x$ by using formula (10), given in Lemma 2, we obtain

\begin{equation}
D_{q,x}(P_{n,q}(x)) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix}
0 & 1 & [2]_q x & \cdots & [n]_q x^{n-1} \\
\beta_0 & \beta_1 & \beta_2 & \cdots & \beta_n \\
0 & \beta_0 & \frac{2}{1}_q \beta_1 & \cdots & \frac{n}{1}_q \beta_{n-1} \\
0 & 0 & \beta_0 & \cdots & \frac{n}{2}_q \beta_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \frac{n}{n-1}_q \beta_1 \\
\end{vmatrix},
\end{equation}

Expanding the determinant (11) above along with the first column, we have

\begin{equation}
D_{q,x}(P_{n,q}(x)) = \frac{(-1)^{n-1}}{(\beta_0)^n} \times \begin{vmatrix}
1 & [2]_q x & \cdots & [n-1]_q x^{n-2} & [n]_q x^{n-1} \\
\beta_0 & \frac{2}{1}_q \beta_1 & \cdots & \frac{n-1}{1}_q \beta_{n-2} & \frac{n}{1}_q \beta_{n-1} \\
0 & \beta_0 & \cdots & \frac{n-1}{2}_q \beta_{n-3} & \frac{n}{2}_q \beta_{n-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \beta_0 & \frac{n}{n-1}_q \beta_1 \\
\end{vmatrix},
\end{equation}

Now, considering the fact that

\[
\frac{[i-1]_q}{[j]_q} \begin{vmatrix}
j \\
i-1 \\
\end{vmatrix}_q = \frac{[i-1]_q [j]_q!}{[j]_q [i-1]_q!} \frac{1}{[j-1]_q!} \frac{[j-1]_q!}{[i-2]_q!} \frac{1}{[i-2]_q!} = \begin{vmatrix} j-1 \\
i-2 \\
\end{vmatrix}_q,
\]

and multiplying the $j^{th}$ column of the determinant (12) by $\frac{1}{[j]_q}$, as well as the $i^{th}$ row by $[i-1]_q$ we obtain

\[
D_{q,x}(P_{n,q}(x)) = \frac{(-1)^{n-1}}{(\beta_0)^n} \times \frac{[1]_q!}{[0]_q!} \times \frac{[2]_q}{[1]_q} \times \cdots \times \frac{[n]_q}{[n-1]_q} \times
\]
(13)\[
\begin{vmatrix}
1 & x & \ldots & x^{n-2} & x^{n-1} \\
\beta_0 & \beta_1 & \ldots & \beta_{n-2} & \beta_{n-1} \\
0 & \beta_0 & \ldots & \binom{n-2}{1}_q \beta_{n-3} & \binom{n-1}{1}_q \beta_{n-2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & \beta_0 & \binom{n-1}{n-2}_q \beta_1 
\end{vmatrix},
\]

which is exactly the desired result. \qed

**Theorem 4.** The $q$-polynomials $P_{n,q}(x)$, defined in (11), can be expressed as

(14)\[ P_{n,q}(x) = \sum_{i=0}^{n} \binom{n}{j}_q \alpha_{n-j} x^j, \]

where

(15)\[
\begin{cases}
\alpha_0 = \frac{1}{\beta_0} \\
\alpha_j = \frac{(-1)^j}{(\beta_0)^j} \\
\end{cases}
\begin{vmatrix}
\beta_0 & \beta_1 & \beta_2 & \ldots & \beta_{j-1} & \beta_j \\
0 & \beta_0 & \frac{2}{1}_q \beta_1 & \ldots & \frac{j-1}{1}_q \beta_{j-2} & \frac{j}{1}_q \beta_{j-1} \\
0 & 0 & \beta_0 & \ldots & \frac{j-1}{2}_q \beta_{j-3} & \frac{j}{2}_q \beta_{j-2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & \ldots & 0 & \beta_0 & \frac{j}{j-1}_q \beta_1 
\end{vmatrix}.
\]

**Proof.** Expanding the determinant (9) along the first row, we obtain
Clearly, according to the given definition for $\alpha$, in (15), the first determinant leads to obtain $\alpha_n$, which is the coefficient of $x^0$. Also, the last determinant, which is the determinant of an upper triangular $n \times n$ matrix, will lead to obtain the coefficient of $x^n$ as follows

\[
\alpha_0 = \frac{(-1)^{2n+2}}{(\beta_0)^{n+1}} = \frac{1}{\beta_0}.
\]

To calculate the coefficient of $x^j$ for $0 < j < n$, consider the following determinant
\[= \frac{(-1)^n}{(\beta_0)^{n+1}} (-1)^{j+2} \times\]

\[
\begin{vmatrix}
\beta_0 & \beta_1 & \cdots & \beta_{j-1} & \beta_{j+1} & \cdots & \beta_n \\
0 & \beta_0 & \cdots & \frac{1}{j-1} & \frac{j+1}{j} & \cdots & \frac{n}{q} \\
0 & 0 & \cdots & \frac{1}{j-2} & \frac{j}{j} & \cdots & \frac{n}{q} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \beta_0 & \frac{j+1}{j} & \cdots & \frac{n}{q} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & \frac{n}{n-1} \\
\end{vmatrix}
\]

\[= \frac{(-1)^{n+j}}{(\beta_0)^{n+1}} (\beta_0)^j \times\]

\[
\begin{vmatrix}
\frac{j+1}{j} & \beta_1 & \cdots & \frac{n-1}{j} & \beta_{n-1} & \beta_n \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & \beta_0 & 1 \\
\end{vmatrix}
\]

Now multiplying the first column of the last determinant by \[
\begin{bmatrix}
\frac{1}{j+1} \\
\end{bmatrix}_q,
\]
we obtain

\[
= \frac{(-1)^{n+j}}{(\beta_0)^{n-j+1}} \times
\begin{vmatrix}
\frac{1}{j+1} & \beta_0 & \cdots & \frac{j+2}{j} & \beta_1 & \cdots & \frac{n}{q} \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 \\
\end{vmatrix}
\]

Further similar calculations to get coefficients 1 for the first elements of each column in determinant above leads to
In order to create coefficient 1 for the term $\beta_0$ placed in the second row of the above determinant, multiply this row by $\left[ \begin{array}{c} j+1 \\ j \end{array} \right]_q$. As we are aware of the fact that

$$
\left[ \begin{array}{c} j+2 \\ j+1 \\ j+2 \\ j \\
\end{array} \right]_q \times \left[ \begin{array}{c} j+1 \\ j \\
\end{array} \right]_q = \left[ \begin{array}{c} 2 \\
\end{array} \right]_q,
$$

and also

$$
\left[ \begin{array}{c} n \\
\end{array} \right]_q \times \left[ \begin{array}{c} j+1 \\
\end{array} \right]_q = \left[ \begin{array}{c} n-j \\
\end{array} \right]_q.
$$

Thus we have

$$
= \frac{(-1)^{n+j}}{(\beta_0)^{n-j+1}} \times \left[ \begin{array}{c} 1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} 1 \\
\end{array} \right]_q \times \ldots \times \left[ \begin{array}{c} 1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} 1 \\
\end{array} \right]_q
$$

$$
\times \left[ \begin{array}{c} \beta_1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} \beta_2 \\
\end{array} \right]_q \times \ldots \times \left[ \begin{array}{c} \beta_{n-j-1} \\
\end{array} \right]_q \times \left[ \begin{array}{c} \beta_{n-j} \\
\end{array} \right]_q
$$

$$
\times \left[ \begin{array}{c} \beta_0 \\
\end{array} \right]_q \times \left[ \begin{array}{c} \beta_0 \\
\end{array} \right]_q \times \ldots \times \left[ \begin{array}{c} \beta_0 \\
\end{array} \right]_q \times \left[ \begin{array}{c} \beta_0 \\
\end{array} \right]_q
$$

$$
\times \left[ \begin{array}{c} \beta_0 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \ldots \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q
$$

\ldots \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q
$$

\ldots \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q
$$

\ldots \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q
$$

\ldots \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q
$$

\ldots \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q
$$

\ldots \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin{array}{c} n-1 \\
\end{array} \right]_q \times \left[ \begin\
We continue this method for each row. As the number of coefficients in
\[
\begin{pmatrix} 1 \quad 1 \quad \cdots \quad 1 \end{pmatrix}_q \times \begin{pmatrix} j+1 \quad j+2 \quad \cdots \quad j+1 \end{pmatrix}_q \times \begin{pmatrix} n-1 \quad n-1 \quad \cdots \quad n-1 \end{pmatrix}_q \times \begin{pmatrix} n \quad n \quad \cdots \quad n \end{pmatrix}_q,
\]
is \(n-j\), so it is equal to the number of rows. Moreover, in each step one of the coefficients above will be cancelled by the corresponding inverse which will be multiplied later by each row. Therefore, we are sure that at the end we obtain
\[
\begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-j-1} & \beta_{n-j} \\ \beta_0 & \frac{2}{1}_q & \beta_1 & \cdots & \frac{n-j-1}{1}_q & \beta_{n-j-2} & \frac{n-j-1}{1}_q & \beta_{n-j-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_0 & \frac{n}{n-1}_q & \beta_1 \end{pmatrix} = (\frac{-1}{(\beta_0)^{n-j+1}})^j \cdot \begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_j \\ 0 & \frac{2}{1}_q & \beta_1 & \cdots & \frac{j-1}{1}_q & \beta_{j-2} & \frac{j}{1}_q & \beta_{j-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_0 & \frac{j}{j-1}_q & \beta_{j-2} \\ 0 & \cdots & \beta_0 & \frac{1}{1}_q & \beta_{j-2} \\ 0 & \cdots & \beta_0 & \frac{1}{1}_q & \beta_{j-2} \end{pmatrix}
\]
\[= \alpha_{n-j},\]
whence the result.

\[\square\]

**Corollary 5.** The following identity holds for the \(q\) polynomials \(P_{n,q}(x)\)

\[(17)\]
\[P_{n,q}(x) = \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix}_q P_{n-j,q}(0)x^j, \quad n = 0, 1, 2, \ldots\]

**Proof.** According to the definition, for \(j = 0, 1, \ldots, n\), \(P_{j,q}(x) = \alpha_j\), since \[\square\]
Corollary 6. The following relations hold for $\alpha_j$s in relation (14)

\begin{equation}
\alpha_0 = \frac{1}{\beta_0},
\end{equation}

\begin{equation}
\alpha_j = -\frac{1}{\beta_0} \sum_{i=0}^{j-1} \left[ \begin{array}{c} j \\ i \end{array} \right] \beta_{j-i} \alpha_i, \quad j = 1, 2, ..., n.
\end{equation}

Proof. The proof is done by expanding $\alpha_j$, defined in relation (15), along with the first row and also applying a similar technique to the proof of theorem 4.

Theorem 7. Suppose that $\{A_{n,q}(x)\}$ be the sequence of $q$-Appell polynomials with generating function $A_q(t)$, defined in the relations (7) and (9). If $B_0,q, B_1,q, ..., B_n,q$ with $B_0,q \neq 0$ are the coefficients of $q$-Taylor series expansion of the function $\frac{1}{A_q(t)}$ introduced in relation (9), then for $n = 0, 1, 2, ...$ we have

\begin{equation}
A_{n,q}(x) = \frac{1}{B_0,q} (x) \sum_{k=0}^{n} \left[ \begin{array}{c} x^{n-k} \\ k \end{array} \right] B_{n-k,q}.
\end{equation}

Proof. According to the relations (7) and (9), we have

\begin{equation}
A_q(t) = \sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q!} = A_{0,q} + A_{1,q}t + A_{2,q} \frac{t^2}{[2]_q!} + ... + A_{n,q} \frac{t^n}{[n]_q!} + ...,\n\end{equation}
and also

\[ A_q(t)e_q(tx) = \sum_{n=0}^{\infty} A_{n,q}(x) \frac{t^n}{[n]_q!} = A_{0,q}(x) + A_{1,q}(x)t + A_{2,q}(x)\frac{t^2}{[2]_q!} + \ldots + A_{n,q}(x)\frac{t^n}{[n]_q!} + \ldots. \]

Let \( B_q(t) = \frac{1}{A_q(t)} \). Thus, considering the hypothesis of the theorem and also noting the definition of \( q \)-Taylor series expansion of \( B_q(t) \) at \( a = 0 \) given in relation (3) we have

\[ B_q(t) = B_{0,q} + B_{1,q}t + B_{2,q}\frac{t^2}{[2]_q!} + \ldots + B_{n,q}(x)\frac{t^n}{[n]_q!} + \ldots. \]

By using Cauchy product rule for the series production \( A_q(t)B_q(t) \), we obtain

\[
1 = A_q(t)B_q(t) = \sum_{n=0}^{\infty} A_{n,q}(x)\frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}_q A_{k,q}B_{n-k,q}\frac{t^n}{[n]_q!}.
\]

Consequently,

\[
\sum_{k=0}^{n} \binom{n}{k}_q A_{k,q}B_{n-k,q} = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n > 0. \end{cases}
\]

It means that

\[ B_{0,q} = \frac{1}{A_0}, \quad B_{n,q} = -\frac{1}{A_0}(\sum_{k=1}^{n} \binom{n}{k}_q A_{k,q}B_{n-k,q}), \quad n = 1, 2, 3, \ldots. \]

Now, multiply both sides of identity (21) by \( B_q(t) = \frac{1}{A_q(t)} \), and then replace \( e_q(tx) \) by its \( q \)-Taylor series expansion, i.e. \( \sum_{k=0}^{\infty} x^n \frac{t^n}{[n]_q!} \). Therefore we obtain

\[ \sum_{k=0}^{\infty} x^n \frac{t^n}{[n]_q!} = e_q(tx) = B_q(t) \sum_{n=0}^{\infty} A_{n,q}(x)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} A_{n,q}(x)\frac{t^n}{[n]_q!}. \]

Using Cauchy product rule in the last part of relation above leads to

\[ \sum_{k=0}^{\infty} x^n \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}_q B_{n-k,q}A_{k,q}(x)\frac{t^n}{[n]_q!}. \]

Comparing the coefficients of \( \frac{t^n}{[n]_q!} \) in both sides of equation (24), we have
Writing identity (25) for \( n = 0, 1, 2, \ldots \) leads to obtain the following infinite system in the parameter \( A_{n,q}(x) \)

\[
\begin{align*}
B_{0,q} A_{0,q}(x) &= 1, \\
B_{1,q} A_{0,q}(x) + B_{0,q} A_{0,q}(x) &= x, \\
B_{2,q} A_{0,q}(x) + B_{1,q} A_{1,q}(x) + B_{0,q} A_{2,q}(x) &= x^2, \\
&\vdots \\
B_{n,q} A_{0,q}(x) + B_{n-1,q} A_{1,q}(x) + \ldots + B_{0,q} A_{n,q}(x) &= x^n,
\end{align*}
\]

(26)

As it is clear the coefficient matrix of the infinite system (26) is lower triangular. So this property helps us to find \( A_{n,q}(x) \) by applying Cramer rule to only the first \( n + 1 \) equations of this system. Hence we can obtain

\[
A_{n,q}(x) = \begin{bmatrix}
B_{0,q} & 0 & 0 & \ldots & 0 & 1 \\
B_{1,q} & B_{0,q} & 0 & \ldots & 0 & x \\
B_{2,q} & 2 & B_{1,q} & B_{0,q} & \ldots & 0 & x^2 \\
& \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
B_{n,q} & n & B_{n-1,q} & \ldots & B_{0,q} & x^{n-1} \\
& \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
B_{0,q} & 0 & 0 & \ldots & 0 & 0 \\
B_{1,q} & B_{0,q} & 0 & \ldots & 0 & 0 \\
B_{2,q} & 2 & B_{1,q} & B_{0,q} & \ldots & 0 & 0 \\
& \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
B_{n,q} & n & B_{n-1,q} & \ldots & B_{0,q} & 0 \\
& \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]
\[
\begin{vmatrix}
B_0,q & 0 & 0 & \cdots & 0 & 1 \\
B_1,q & B_0,q & 0 & \cdots & 0 & x \\
B_2,q & 2 & B_1,q & B_0,q & \cdots & 0 & x^2 \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\
B_{n-1},q & n-1 & 1 & B_{n-2},q & \cdots & \cdots & B_0,q & x^{n-1} \\
B_n,q & n & 1 & B_{n-1},q & \cdots & \cdots & B_0,q & x^n \\
\end{vmatrix}
= \frac{1}{(B_0,q)^{n+1}}
\]

Now, take the transpose of the last determinant and then interchange \(i\)th row of the obtained determinant with \(i + 1\)th row, \(i = 1, 2, \ldots, n\). This leads to obtain the desired result that is exactly relation (19).

**Theorem 8.** The following facts are equivalent for the \(q\)-Appell polynomials:

a) \(q\)-Appell polynomials can be expressed by considering the relations (5) and (6).

b) \(q\)-Appell polynomials can be expressed by considering the relations (7) and (8).

c) \(q\)-Appell polynomials can be expressed by considering the determinantal relation (19).

**Proof.**

(a \(\Rightarrow\) b) Suppose that relations (5) and (6) hold. Construct an infinite series
\[
\sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q} c_q(tx)
\]
form all constants \(A_{n,q}\) used for defining \(A_n,q(x)\) in relation (9).

Now find the following Cauchy product

\[
\sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q} c_q(tx) = \sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q} \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n-k,q} x^k \frac{t^n}{[n]_q}.
\]

From relation (6) we know that

\[
\sum_{k=0}^{n} A_{n-k,q} x^k = A_{n,q}(x),
\]

So we find that

\[
\sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q} c_q(tx) = A_{n,q}(x),
\]

whence the result.

(b \(\Rightarrow\) c) The proof follows directly from Theorem 7.

(c \(\Rightarrow\) a) The proof follows from Theorems 3 and 7.

□
As the consequence of discussion above and particularly Theorem 8, we are allowed to introduce the determinantal definition of \( q \)-Appell polynomials as follows.

**Definition 9.** \( q \)-Appell polynomials \( \{ A_{n,q}(x) \}_{n=0}^{\infty} \) can be defined as

\[
A_{n,q}(x) = \frac{1}{B_{0,q}} \begin{pmatrix}
1 & x & x^2 & \cdots & x^{n-1} & x^n \\
B_{0,q} & B_{1,q} & B_{2,q} & \cdots & B_{n-1,q} & B_{n,q} \\
0 & B_{0,q} & \left[ \begin{array}{c}
2 \\
1
\end{array} \right] & B_{1,q} & \cdots & \left[ \begin{array}{c}
n - 1 \\
1
\end{array} \right] & B_{n-2,q} & \left[ \begin{array}{c}
n \\
1
\end{array} \right] & B_{n-1,q} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & B_{0,q} & \left[ \begin{array}{c}
n \\
-1
\end{array} \right] & B_{1,q}
\end{pmatrix},
\]

where \( B_{0,q}, B_{1,q}, B_{2,q}, \ldots, B_{n,q} \in \mathbb{R} \), \( B_{0,q} \neq 0 \) and \( n = 1, 2, 3, \ldots \).

### 3. Basic Properties of \( q \)-Appell Polynomials from Determinantal Point of View

In this section by using Definition 9, we review the basic properties of \( q \)-Appell polynomials.

**Theorem 10.** For \( q \)-Appell polynomials the following identities hold

\[
A_{n,q}(x) = \frac{(-1)^n}{B_{0,q}} (x^n - \sum_{k=0}^{n-1} \left[ \begin{array}{c}
n \\
k
\end{array} \right] q B_{n-k,q} A_{k,q}(x)), \quad n = 1, 2, 3, \ldots
\]

**Proof.** Start from expanding the determinant in the Definition 9 along with the \( n + 1 \)th row

\[
A_{n,q}(x) = \frac{(-1)^n}{(B_{0,q})^{n+1}} \begin{pmatrix}
1 & x & x^2 & \cdots & x^{n-1} \\
B_{0,q} & B_{1,q} & B_{2,q} & \cdots & B_{n-1,q} \\
0 & B_{0,q} & \left[ \begin{array}{c}
2 \\
1
\end{array} \right] & B_{1,q} & \cdots & \left[ \begin{array}{c}
n - 1 \\
1
\end{array} \right] & B_{n-2,q} \\
0 & 0 & B_{0,q} & \cdots & \left[ \begin{array}{c}
n - 1 \\
2
\end{array} \right] & B_{n-3,q} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & B_{0,q} & \left[ \begin{array}{c}
n - 1 \\
-2
\end{array} \right] & B_{1,q}
\end{pmatrix}.
\]
\[ + \frac{(-1)^{n+1}}{(B_0q)^{n+1}} B_{0,q} \times \]
\[
\begin{vmatrix}
1 & x & x^2 & \ldots & x^{n-2} & x^n \\
B_0,q & B_1,q & B_2,q & \ldots & B_{n-2,q} & B_{n,q} \\
0 & B_0,q & \frac{2}{1} & B_1,q & \ldots & \frac{n-2}{1} & B_{n-3,q} \\
0 & 0 & B_0,q & \ldots & \frac{n-2}{2} & B_{n-4,q} & \frac{n}{2} & B_{n-2,q} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & \ldots & B_0,q & \frac{n-1}{n-2} & B_{2,q} \\
\end{vmatrix}
\]
\[= \frac{-1}{B_0,q} \left[ \begin{array}{c}
\frac{n}{n-1} \end{array} \right] B_{1,q} A_{n-1,q}(x) + \frac{(-1)^{n+1}}{(B_0,q)^n} \times \]
\[
\begin{vmatrix}
1 & x & x^2 & \ldots & x^{n-2} & x^n \\
B_0,q & B_1,q & B_2,q & \ldots & B_{n-2,q} & B_{n,q} \\
0 & B_0,q & \frac{2}{1} & B_1,q & \ldots & \frac{n-2}{1} & B_{n-3,q} \\
0 & 0 & B_0,q & \ldots & \frac{n-2}{2} & B_{n-4,q} & \frac{n}{2} & B_{n-2,q} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & \ldots & B_0,q & \frac{n-1}{n-2} & B_{2,q} \\
\end{vmatrix}
\]

Now repeat the same method for the last determinant

\[= \frac{-1}{B_0,q} \left[ \begin{array}{c}
\frac{n}{n-1} \end{array} \right] B_{1,q} A_{n-1,q}(x) + \frac{(-1)^{n+1}}{(B_0,q)^n} \left[ \begin{array}{c}
\frac{n-1}{n-2} \end{array} \right] B_{2,q} \times \]
\[
\begin{vmatrix}
1 & x & x^2 & \ldots & x^{n-3} & x^{n-2} \\
B_0,q & B_1,q & B_2,q & \ldots & B_{n-3,q} & B_{n-2,q} \\
0 & B_0,q & \frac{2}{1} & B_1,q & \ldots & \frac{n-3}{1} & B_{n-4,q} \\
0 & 0 & B_0,q & \ldots & \frac{n-3}{2} & B_{n-5,q} & \frac{n-2}{2} & B_{n-4,q} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & \ldots & B_0,q & \frac{n-2}{n-3} & B_{1,q} \\
\end{vmatrix}
\]
\[+ \frac{(-1)^{n+2}}{(B_0,q)^n} B_{0,q} \times \]
\[= \frac{-1}{B_0,q} \left[ \begin{array}{c}
\frac{n}{n-1} \end{array} \right] B_{1,q} A_{n-1,q}(x) + \frac{(-1)^{n-1}}{(B_0,q)^n} \left[ \begin{array}{c}
\frac{n-1}{n-2} \end{array} \right] B_{2,q} \frac{(B_0,q)^{n-1}}{(-1)^{n-2}} A_{n-2,q}(x)\]
\[ + \frac{(-1)^{n-2}}{(B_0,q)^{n-1}} \times \]

\[
\begin{array}{cccccccc}
1 & x & x^2 & \ldots & x^{n-3} & x^n \\
B_{0,q} & B_{1,q} & B_{2,q} & \ldots & B_{n-3,q} & B_{n,q} \\
0 & B_{0,q} & \left[ \begin{array}{c} 2 \\ 1 \end{array} \right]_q B_{1,q} & \ldots & \left[ \begin{array}{c} n-3 \\ 1 \end{array} \right]_q B_{n-4,q} & \left[ \begin{array}{c} n \\ 1 \end{array} \right]_q B_{n-1,q} \\
0 & 0 & B_{0,q} & \ldots & \left[ \begin{array}{c} n-3 \\ 2 \end{array} \right]_q B_{n-5,q} & \left[ \begin{array}{c} n \\ 2 \end{array} \right]_q B_{n-2,q} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & \ldots & 0 & B_{0,q} & \left[ \begin{array}{c} n-1 \\ n-2 \end{array} \right]_q B_{2,q} \\
\end{array}
\]

\[= -\frac{1}{B_{0,q}} \left[ \begin{array}{c} n \\ n-1 \end{array} \right]_q B_{1,q} A_{n-1,q}(x) - \frac{1}{B_{0,q}} \left[ \begin{array}{c} n-1 \\ n-2 \end{array} \right]_q B_{2,q} A_{n-2,q}(x) + \frac{(-1)^{n-2}}{(B_0,q)^{n-1}} \times \]

\[
\begin{array}{cccccccc}
1 & x & x^2 & \ldots & x^{n-3} & x^n \\
B_{0,q} & B_{1,q} & B_{2,q} & \ldots & B_{n-3,q} & B_{n,q} \\
0 & B_{0,q} & \left[ \begin{array}{c} 2 \\ 1 \end{array} \right]_q B_{1,q} & \ldots & \left[ \begin{array}{c} n-3 \\ 1 \end{array} \right]_q B_{n-4,q} & \left[ \begin{array}{c} n \\ 1 \end{array} \right]_q B_{n-1,q} \\
0 & 0 & B_{0,q} & \ldots & \left[ \begin{array}{c} n-3 \\ 2 \end{array} \right]_q B_{n-5,q} & \left[ \begin{array}{c} n \\ 2 \end{array} \right]_q B_{n-2,q} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & \ldots & 0 & B_{0,q} & \left[ \begin{array}{c} n-1 \\ n-2 \end{array} \right]_q B_{2,q} \\
\end{array}
\]

Continue a similar method to arrive at

\[
= -\frac{1}{B_{0,q}} \left[ \begin{array}{c} n \\ n-1 \end{array} \right]_q B_{1,q} A_{n-1,q}(x) - \frac{1}{B_{0,q}} \left[ \begin{array}{c} n-1 \\ n-2 \end{array} \right]_q B_{2,q} A_{n-2,q}(x) \\
- \ldots - \frac{1}{(B_0,q)^2} \left[ \begin{array}{c} n \\ n-1 \end{array} \right]_q \ldots - \frac{1}{(B_0,q)^2} (B_{n-1,q}-B_{0,q} x^n) \\
= -\frac{1}{B_{0,q}} \left[ \begin{array}{c} n \\ n-1 \end{array} \right]_q B_{1,q} A_{n-1,q}(x) - \frac{1}{B_{0,q}} \left[ \begin{array}{c} n-1 \\ n-2 \end{array} \right]_q B_{2,q} A_{n-2,q}(x) \\
- \ldots - \frac{1}{B_{0,q}} \left[ \begin{array}{c} n \\ n-1 \end{array} \right]_q B_{n,q} A_{n,q}(x) - \frac{1}{B_{0,q}} \left[ \begin{array}{c} n \\ n-1 \end{array} \right]_q B_{n-1,q} A_{n-1,q}(x) \\
= \frac{1}{B_{0,q}} \left( x^n - \sum_{k=0}^{n-1} \left[ \begin{array}{c} n \\ k \end{array} \right]_q B_{n-k,q} A_{k,q}(x) \right).
\]
Corollary 11. Powers of $x$ can be expressed based on $q$-Appell polynomials as

\begin{equation}
\label{power}
x^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}_q B_{n-k,q} A_{k,q}(x), \quad n = 1, 2, 3, \ldots
\end{equation}

Proof. The proof is the direct result of relation (28) in Theorem 10. \hfill \Box

Notation 12. Suppose $P_n(x)$ and $Q_n(x)$ are two polynomials of degree $n$. Let $P_n(x)$ be defined as in relation (9). Then for $n=1, 2, 3, \ldots$ we have

\begin{equation}
\label{product}
(PQ)(x) := \frac{(-1)^n}{(\beta_0)^{n+1}} \times \begin{vmatrix}
Q_0(x) & Q_1(x) & Q_2(x) & \ldots & Q_{n-1}(x) & Q_n(x) \\
\beta_0 & \beta_1 & \beta_2 & \ldots & \beta_{n-1} & \beta_n \\
0 & \beta_0 & \begin{pmatrix} 2 \\ 1 \end{pmatrix}_q & \beta_1 & \ldots & \begin{pmatrix} n-1 \\ 1 \end{pmatrix}_q & \beta_{n-2} \\
0 & 0 & \beta_0 & \ldots & \begin{pmatrix} n-1 \\ 2 \end{pmatrix}_q & \beta_{n-3} & \beta_{n-2} \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & 0 & \beta_0 & \begin{pmatrix} n-1 \\ n-1 \end{pmatrix}_q & \beta_1 \\
\end{vmatrix}
\end{equation}

Theorem 13. Suppose that $\{A_{n,q}(x)\}_{n=0}^{\infty}$ and $\{\tilde{A}_{n,q}(x)\}_{n=0}^{\infty}$ are two families of $q$-Appell polynomials. Then

a) For every $\alpha$ and $\beta \in \mathbb{R}$, \{\(\alpha A_{n,q}(x) + \beta \tilde{A}_{n,q}(x)\)\}$_{n=0}^{\infty}$ is also a family of $q$-Appell polynomials.

b) \{(A\tilde{A})_{n,q}(x)\}$_{n=0}^{\infty}$ is also a family of $q$-Appell polynomials.

Proof. a) The proof is the direct consequence of linear properties of determinant.

b) According to the determinantal definition of $q$-Appell polynomials given in Theorem 7 relation (19) and also notation (30), we have

\begin{equation}
(A\tilde{A})_{n,q}(x) = A_{n,q}(\tilde{A}_{n,q}(x)) = \frac{(-1)^n}{(\beta_0)^{n+1}} \times 
\end{equation}
Therefore we can continue as

Using formula (10) given in Lemma 2 we have

\[ D_q((A\tilde{A})_{n,q}(x)) \]
\[ = \frac{(-1)^n}{(B_{0,q})^{n+1}} \times \]

Since \( \{\tilde{A}_{n,q}(x)\}_{n=0}^{\infty} \) is a family of q-Appell polynomials, according to relation (10) we have

\[ D_q x (\tilde{A}_{n,q}(x)) = [n]_q \tilde{A}_{n-1,q}(x), \quad n = 0, 1, 2, \ldots \]

Therefore we can continue as

\[ D_q((A\tilde{A})_{n,q}(x)) = \frac{(-1)^n}{(B_{0,q})^{n+1}} \times \]
Definition 14. We define 2D $q$-Appell polynomials \( \{A_{n,q}(x,y)\}_{n=0}^{\infty} \) by means of the generating function below

\[
A_q(x,y,t) := A_q(t)e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} A_{n,q}(x,y) \frac{t^n}{[n]_q!},
\]
Theorem 16. The following fact holds for 2D $\{A_{n,q}(x, y)\}_{n=0}^\infty$

\[
\begin{align*}
A_0,q(x, y) &= \frac{1}{B_0,q} \times \\
A_n,q(x, y) &= \frac{(B_0,q)^n}{(B_0,q)^{n+1}} \times \\
& \begin{pmatrix}
1 & x + y & (x + y)^2 & \ldots & (x + y)^{n-1} & (x + y)^n \\
B_{0,q} & B_{1,q} & B_{2,q} & \ldots & B_{n-1,q} & B_{n,q} \\
0 & B_{0,q} & \begin{pmatrix} 2 \\ 1 \end{pmatrix}_q & \begin{pmatrix} n-1 \\ 1 \end{pmatrix}_q & B_{n-2,q} & \begin{pmatrix} n \\ 1 \end{pmatrix}_q \\
0 & 0 & B_{0,q} & \begin{pmatrix} n-1 \\ 2 \end{pmatrix}_q & B_{n-3,q} & \begin{pmatrix} n \\ 2 \end{pmatrix}_q \\
& & & \vdots & \vdots & \vdots \\
0 & \ldots & \ldots & \ldots & 0 & B_{0,q} \\
& & & & & \begin{pmatrix} n \\ n-1 \end{pmatrix}_q B_{1,q}
\end{pmatrix}.
\end{align*}
\]

Remark 15. From the Definition \[14\] it is clear that

\[
A_{n,q}(x, 0) = A_{n,q}(x).
\]

Theorem 16. The following fact holds for 2D $q$-Appell polynomials $\{A_{n,q}(x, y)\}_{n=0}^\infty$

\[
A_n,q(x, y) = \sum_{k=0}^{n} \binom{n}{k}_q q^{1/2(n-k)(n-k-1)} A_{k,q}(x)y^{n-k}.
\]

Proof. Proof is simple and based on properties of determinant.

Corollary 17. The following difference identity holds for $q$-Appell polynomials $\{A_{n,q}(x)\}_{n=0}^\infty$

\[
A_{n,q}(x, 1) - A_{n,q}(x) = \sum_{k=1}^{n-1} \binom{n}{k}_q q^{1/2(n-k)(n-k-1)} A_{k,q}(x), \quad n = 0, 1, 2, \ldots
\]

Proof. Using relations (33) and also (34) for $y = 1$ and $y = 0$ and replacing the results in the left side of relation (35) leads to reach to the right side of this relation.

Theorem 18. For every $t \in \mathbb{R}$, the following facts are equivalent for $q$-Appell polynomials $\{A_{n,q}(x)\}_{n=0}^\infty$

a) $A_{n,q}(x, -y) = (-1)^n A_{n,q}(0, y)$,

b) $A_{n,q}(x) = (-1)^n A_{n,q}(0)$.

Proof. (a $\Rightarrow$ b) The proof is done using part (a) for $x = 0$. 
We apply the relation (34) for the left hand side of part (a) as follows

\[ A_{n,q}(x, -y) = \sum_{k=0}^{n} \binom{n}{k} q^{1/2(n-k)(n-k-1)} A_{k,q}(x, 0) (-y)^{n-k} \]

\[ = (-1)^n \sum_{k=0}^{n} \binom{n}{k} q^{1/2(n-k)(n-k-1)} A_{k,q}(x, 0) (-1)^k y^{n-k} \]

\[ = (-1)^n \sum_{k=0}^{n} \binom{n}{k} q^{k(k-1)/2} A_{k,q}(x, 0) (-1)^{n-k} y^k. \]

Using part (b), we have

\[ A_{n,q}(x, -y) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} q^{k(k-1)/2} A_{k,q}(x, 0) (-1)^{n-k} y^k. \]

Now, using Definition 14 leads to obtain

\[ A_{n,q}(x, -y) = (-1)^n A_{n-k,q}(0, y), \]

whence the result.

**Lemma 19.** In relation (23) for the coefficients \(A_{n,q}\) and \(B_{n,q}\) we have

**Theorem 20.**

(36) \(A_{2n+1,q} = 0 \leftrightarrow B_{2n+1,q} = 0, \quad n = 0, 1, 2, \ldots\)

**Proof.** \((\Rightarrow)\) We have already known the following fact from relation (23) for \(n = 0, 1, 2, \ldots\)

\[ \begin{align*}
B_{1,q} &= -\frac{1}{A_0} A_{1,q} B_{0,q}, \\
B_{2n+1,q} &= -\frac{1}{A_0} \binom{2n+1}{k} A_{1,q} B_{2n,q} \\
&\quad + \frac{1}{A_0} \left( \sum_{k=1}^{n} \left( \binom{2n+1}{2k} A_{2k,q} B_{2n-k+1,q} + \binom{2n+1}{2k+1} A_{2k+1,q} B_{2n-k+1,q} \right) \right).
\end{align*} \]

Since \(A_{2n+1,q} = 0\) for \(n = 0, 1, 2, \ldots\), then

\[ \begin{align*}
B_{1,q} &= 0, \\
B_{2n+1,q} &= -\frac{1}{A_0} \sum_{k=1}^{n} \binom{2n+1}{2k} A_{2k,q} B_{2n-k+1,q}, \quad n = 1, 2, 3, \ldots
\end{align*} \]

Consequently, we should have \(B_{2n+1,q} = 0\) for \(n = 0, 1, 2, \ldots\)

\((\Leftarrow)\) In a similar way to the above we can prove it.

**Theorem 21.** The following facts are equivalent for \(q\)-Appell polynomials \(\{A_{n,q}(x)\}_{n=0}^{\infty}\)

a) \(A_{n,q}(-x) = (-1)^n A_{n,q}(x),\)

b) \(B_{2n+1,q} = 0, \quad \text{for} \quad n = 0, 1, 2, \ldots\)

**Proof.** According to Theorem 18 we know that

\[ A_{n,q}(-x) = (-1)^n A_{n,q}(x) \Leftrightarrow A_{n,q}(t) = (-1)^n A_{n,q}(0) \]

So using Lemma 19 we have
\( \Leftrightarrow A_{2n+1,q}(0) = (-1)^n A_{2n+1,q}(0) \Leftrightarrow A_{2n+1,q} = 0 \Leftrightarrow B_{2n+1,q} = 0. \)

\[ \text{Theorem 22. For every } n \geq 1, \text{ q-Appell polynomials } \{A_n,q(x)\}_{n=0}^\infty \text{ satisfy the following identities} \]

\[ \int_0^x A_{n,q}(t) \, dt = \frac{1}{[n+1]_q} \left( A_{n+1,q}(x) - A_{n,q}(0) \right) \quad (37) \]

\[ \int_0^x A_{n,q}(t) \, dt = \frac{1}{[n+1]_q} \sum_{k=0}^{n+1} \left[ \begin{array}{c} n+1 \\ k \end{array} \right]_q q^{1/2k(k-1)} A_{n-k,q}(0) \quad (38) \]

\[ \text{Proof: Relation (37) is the direct result of property (5) for q-Appell polynomials } \{A_n,q(x)\}_{n=0}^\infty. \text{ To prove equality (38), we start from relation (37) for } x = 1 \text{ as follows} \]

\[ \int_0^1 A_{n,q}(t) \, dt = \frac{1}{[n+1]_q} \left( A_{n+1,q}(1) - A_{n,q}(0) \right). \]

Now, find \( A_{n+1,q}(1) \) using relation (54) by assuming \( x = 0 \) and \( y = 1 \)

\[ A_{n+1,q}(1) = \sum_{k=0}^{n+1} \left[ \begin{array}{c} n+1 \\ k \end{array} \right]_q q^{1/2k(k-1)} A_{n-k,q}(0). \]

Therefore, we obtain

\[ \int_0^1 A_{n,q}(t) \, dt = \frac{1}{[n+1]_q} \sum_{k=0}^n \left[ \begin{array}{c} n+1 \\ k \end{array} \right]_q q^{1/2k(k-1)} A_{n-k,q}(0). \]

\[ \blacksquare \]

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