On flux integrals for generalized Melvin solution related to simple finite-dimensional Lie algebra

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Abstract A generalized Melvin solution for an arbitrary simple finite-dimensional Lie algebra \( \mathcal{G} \) is considered. The solution contains a metric, \( n \) Abelian 2-forms and \( n \) scalar fields, where \( n \) is the rank of \( \mathcal{G} \). It is governed by a set of \( n \) moduli functions \( H_s(z) \) obeying \( n \) ordinary differential equations with certain boundary conditions imposed. It was conjectured earlier that these functions should be polynomials—the so-called fluxbrane polynomials. These polynomials depend upon integration constants \( q_s \), \( s = 1, \ldots, n \). In the case when the conjecture on the polynomial structure for the Lie algebra \( \mathcal{G} \) is satisfied, it is proved that 2-form flux integrals \( \Phi^s \) over a proper 2d submanifold are finite and obey the relations

\[
q_s \Phi^s = 4\pi n_s h_s,
\]

where \( h_s > 0 \) are certain constants (related to dilatonic coupling vectors) and the \( n_s \) are powers of the polynomials, which are components of a twice dual Weyl vector in the basis of simple (co-)roots, \( s = 1, \ldots, n \). The main relations of the paper are valid for a solution corresponding to a finite-dimensional semi-simple Lie algebra \( \mathcal{G} \). Examples of polynomials and fluxes for the Lie algebras \( A_1, A_2, A_3, C_2, G_2 \) and \( A_1 + A_1 \) are presented.

1 Introduction

In this paper we start with a generalization of a Melvin solution [1], which was presented earlier in Ref. [2]. It appears in the model which contains a metric, \( n \) Abelian 2-forms and \( l \geq n \) scalar fields. This solution is governed by a certain non-degenerate (quasi-Cartan) matrix \( (A_{ss'}) \), \( s, s' = 1, \ldots, n \). It is a special case of the so-called generalized fluxbrane solutions from Ref. [3]. For fluxbrane solutions see Refs. [4–28] and the references therein. The appearance of fluxbrane solutions was motivated by superstring/M theory.

The generalized fluxbrane solutions from Ref. [3] are governed by moduli functions, \( H_s(z) > 0 \), defined on the interval \((0, +\infty)\), where \( z = \rho^2 \) and \( \rho \) is a radial variable. These functions obey a set of \( n \) non-linear differential master equations governed by the matrix \( (A_{ss'}) \), equivalent to Toda-like equations, with the following boundary conditions imposed:

\[
H_s(+0) = 1, s = 1, \ldots, n.
\]

In this paper we assume that \( (A_{ss'}) \) is a Cartan matrix for some simple finite-dimensional Lie algebra \( \mathcal{G} \) of rank \( n \) \( (A_{ss} = 2 \) for all \( s \) \). According to a conjecture suggested in Ref. [3], the solutions to the master equations with the boundary conditions imposed are polynomials:

\[
H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k,
\]

where the \( P_s^{(k)} \) are constants. Here \( P_s^{(n_s)} \neq 0 \) and

\[
n_s = 2 \sum_{s'=1}^{n} A^{ss'},
\]

where we denote \( (A^{ss'})^{-1} \)(. The integers \( n_s \) are components of a twice dual Weyl vector in the basis of simple (co-)roots [29].

The set of fluxbrane polynomials \( H_s(z) \) defines a special solution to open Toda chain equations [30,31] corresponding to a simple finite-dimensional Lie algebra \( \mathcal{G} \) [32]. In Refs. [2,33] a program (in Maple) for the calculation of these polynomials for the classical series of Lie algebras \( (A-, B-, C- \) and \( D-\)series) was suggested. It was pointed out in Ref. [3] that the conjecture on the polynomial structure of \( H_s(z) \) is valid for Lie algebras of the \( A- \) and \( C-\)series. In Ref. [34] the conjecture from Ref. [3] was verified for the Lie algebra \( E_6 \) and certain duality relations for six \( E_6 \)-polynomials were proved. In Sect. 2 we present the generalized Melvin solution from
Ref. [2]. In Sect. 3 we deal with the generalized Melvin solution for an arbitrary simple finite-dimensional Lie algebra $\mathcal{G}$. Here we calculate 2-form flux integrals $\Phi^s = \int_M \Phi^s$, where $F^s$ are 2-forms and $M_s$ is a certain 2d submanifold. These integrals (fluxes) are finite when moduli functions are polynomials. In Sect. 3 we consider examples of fluxbrane polynomials and fluxes for the Lie algebras: $A_1$, $A_2$, $A_3$, $C_2$, $G_2$ and $A_1 + A_1$.

2 The solutions

We consider a model governed by the action

$$ S = \int d^Dx \sqrt{|g|} \left\{ R[g] - h_{a\beta}g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \frac{1}{2} \sum_{s=1}^n \exp[2\lambda_s(\varphi)](F^s)^2 \right\}$$

(2.1)

where $g = g_{MN}(x)dx^M \otimes dx^N$ is a metric, $\varphi = (\varphi^a) \in \mathbb{R}^l$ is a set of scalar fields, $(h_{a\beta})$ is a constant symmetric non-degenerate $l \times l$ matrix ($l \in \mathbb{N}$), $F^s = dA^s = \frac{1}{2} F^s_{MN} dx^M \wedge dx^N$ is a 2-form, $\lambda_s$ is a 1-form on $\mathbb{R}^l$: $\lambda_s(\varphi) = \lambda_{s\alpha} \varphi^\alpha$, $s = 1, \ldots, n$. Here $(\lambda_{s\alpha})$, $s = 1, \ldots, n$, are dilatonic coupling vectors. In (2.1) we denote $|g| = |\det(g_{MN})|$, $(F^s)^2 = F^s_{MN} F^s_{NP} g^{MN} g^{NP}$, $s = 1, \ldots, n$.

Here we start with a family of exact solutions to field equations corresponding to the action (2.1) and depending on one variable $\rho$. The solutions are defined on the manifold

$$ M = (0, +\infty) \times M_1 \times M_2,$$

(2.2)

where $M_1$ is a one-dimensional manifold (say $S^1$ or $\mathbb{R}$) and $M_2$ is a (D-2)-dimensional Ricci-flat manifold. The solution reads [2]

$$ g = \left( \prod_{s=1}^n H_s^{2h_s/(D-2)} \right) \left\{ w d\rho \otimes d\rho + \left( \prod_{s=1}^n H_s^{-2h_s} \right) \rho^2 d\varphi \otimes d\varphi + g^2 \right\},$$

(2.3)

$$ \exp(\varphi^a) = \prod_{s=1}^n H_s^{h_s \lambda_s^a},$$

(2.4)

$$ F^s = q_s \left( \prod_{s' \neq s}^n H_{s'}^{-A_{s's}} \right) \rho d\rho \wedge d\varphi,$$

(2.5)

$s = 1, \ldots, n$; $\alpha = 1, \ldots, l$, where $w = \pm 1$, $g^1 = d\rho \otimes d\rho$ is a metric on $M_1$ and $g^2$ is a Ricci-flat metric on $M_2$. Here $q_s \neq 0$ are integration constants, $q_s = -Q_s$ in the notations of Ref. [2], $s = 1, \ldots, n$.

The functions $H_s(z) > 0$, $z = \rho^2$, obey the master equations

$$ \frac{d}{dz} \left( \frac{z}{H_s} \frac{d}{dz} H_s \right) = P_s \prod_{s' = 1}^n H_{s'}^{-A_{s's}^s},$$

(2.6)

with the following boundary conditions:

$$ H_s(+0) = 1,$$

(2.7)

where

$$ P_s = \frac{1}{4} K_s q_s^2,$$

(2.8)

$s = 1, \ldots, n$. The boundary condition (2.7) guarantees the absence of a conic singularity [in the metric (2.3)] for $\rho = +0$.

The parameters $h_s$ satisfy the relations

$$ h_s = K_s^{-1}, \quad K_s = B_{ss} > 0,$$

(2.9)

where

$$ B_{ss'} \equiv 1 + \frac{1}{2 - D} + \lambda_{s\alpha} \lambda_{s'\beta} h^{\alpha\beta},$$

(2.10)

$s, s' = 1, \ldots, n$, with $(h^{\alpha\beta}) = (h_{a\beta})^{-1}$. In the relations above we denote $\lambda_s^a = h^{\alpha\beta} \lambda_{s\beta}$ and

$$(A_{s's'}) = (2B_{ss'}/B_{s's'}).$$

(2.11)

The latter is the so-called quasi-Cartan matrix.

We note that the constants $B_{ss'}$ and $K_s = B_{ss}$ have a certain mathematical sense. They are related to scalar products of certain vectors $U^s$ (brane vectors, or $U$-vectors), which belong to a certain linear space ("truncated target space", for our problem it has dimension $l + 2$), i.e. $B_{ss'} = (U^s, U^{s'})$ and $K_s = (U^s, U^s)$ [35–37]. The scalar products of such a type are of physical significance, since they appear for various solutions with branes, e.g. black branes, $S$-branes, fluxbranes etc. Several physical parameters in multidimensional models with branes, e.g. the Hawking-like temperatures and the entropies of black holes and branes, PPN parameters, Hubble-like parameters, fluxes etc., contain such scalar products; see [36,37] and Sect. 3 of this paper. The relation (2.11) defines generalized intersection rules for branes which were suggested in [35]. The constants $K_s$ are invariants of dimensional reduction. It is well known, see [37] and the references therein, that $K_s = 2$ for branes in numerous supergravity models, e.g. in dimensions $D = 10, 11$.

It may be shown that if the matrix $(h_{a\beta})$ has an Euclidean signature and $l \geq n$, and $(A_{s's'})$ is a Cartan matrix for a simple Lie algebra $\mathcal{G}$ of rank $n$, there exists a set of co-vectors $\lambda_1, \ldots, \lambda_n$ obeying (2.11) (for $l = n$ see Remark 1 in the next section). Thus the solution is valid at least when $l \geq n$ and the matrix $(h_{a\beta})$ is positive-definite.
The solution under consideration is a special case of the fluxbrane (for \( w = +1 \), \( M_1 = S^1 \)) and S-brane \((w = -1)\) solutions from [3] and [25], respectively.

If \( w = +1 \) and the (Ricci-flat) metric \( g^2 \) has a pseudo-Euclidean signature, we get a multidimensional generalization of Melvin’s solution [1].

In our notations Melvin’s solution (without scalar field) corresponds to \( D = 4, n = 1, l = 0, M_1 = S^1 (0 < \phi < 2\pi), M_2 = \mathbb{R}^2, g^2 = -dt \otimes dt + dx \otimes dx \) and \( G = A_1 \).

For \( w = -1 \) and \( g^2 \) of Euclidean signature we obtain a cosmological solution with a horizon (as \( \rho = +0 \)) if \( M_1 = \mathbb{R} \) (\( -\infty < \phi < +\infty \)).

### 3 Flux integrals for a simple finite-dimensional Lie algebra

Here we deal with the solution which corresponds to a simple finite-dimensional Lie algebra \( G \), i.e. the matrix \( A = (A_{ij}) \) is coinciding with the Cartan matrix of this Lie algebra. We put also \( n = l, w = +1 \) and \( M_1 = S^1, h_{a\beta} = \delta_{a\beta} \) and denote \( (\lambda_{a\mu}) = (\lambda^a) = \lambda_s, s = 1, \ldots, n \).

Due to (2.9)–(2.11) we get

\[
K_s = \frac{D - 3}{D - 2} + \lambda^2_s, \quad (3.1)
\]

\[
h_s = K^{-1}_s, \quad \lambda_s, \lambda_l = \frac{1}{2} K_i A_{il} - \frac{D - 3}{D - 2} \equiv \Gamma_{si}. \quad (3.2)
\]

\( s, l = 1, \ldots, n \). [Equation (3.1) is a special case of (3.2)].

It follows from (2.9)–(2.11) that

\[
\frac{h_i}{h_j} = \frac{K_j}{K_i} = B_{jj}/B_{ii} = B_{ji}/B_{ij} = A_{ji}/A_{ij} \quad (3.3)
\]

for any \( i \neq j \) obeying \( A_{ij}, A_{ji} \neq 0; i, j = 1, \ldots, n \). It may be readily shown from (3.3) that the ratios \( h_i/h_j = K_i/K_j \) are fixed numbers for any given Cartan matrix \( (A_{ij}) \) of a simple (finite-dimensional) Lie algebra \( G \). (This follows from (3.3) and the connectedness of the Dynkin diagram of a simple Lie algebra.) The ratios (3.3) may be written as follows:

\[
\frac{h_i}{h_j} = \frac{K_j}{K_i} = \frac{r_j}{r_i} \quad (3.4)
\]

\( i \neq j \), where \( r_i = (\alpha_i, \alpha_i) \) is the length squared of a simple root \( \alpha_i \) corresponding to the Lie algebra \( G \). Here we use the notations \( A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j) \); \( i, j = 1, \ldots, n \). Equation (3.4) implies

\[
K_i = \frac{1}{2} K r_i, \quad (3.5)
\]

\( i = 1, \ldots, n \), where \( K > 0 \). (For simply laced \((A, D, E)\) Lie algebras all \( r_i \) are equal.)

**Remark 1** For large enough \( K \) in (3.5) there exist vectors \( \lambda_s \) obeying (3.2) [and hence (3.1)]. Indeed, the matrix \((\Gamma_{si})\) is positive-definite if \( K > K_s \), where \( K_s \) is some positive number. Hence there exists a matrix \( \Lambda \), such that \( \Lambda^T \Lambda = \Gamma \). We put \((\Lambda_{as}) = (\lambda^a_s)\) and get the set of vectors obeying (3.2).

Now let us consider the oriented 2-dimensional manifold \( M_s = (0, +\infty) \times S^1 \). The flux integrals

\[
\Phi^s = \int_{M_s} F^s = \int_0^{+\infty} \int_0^{2\pi} \frac{d\rho}{\rho} \rho B^s(\rho^2) \quad (3.6)
\]

where

\[
B^s(\rho^2) = q_s \prod_{l=1}^n \left( H_l(\rho^2)^{-A_{sl}} \right), \quad (3.7)
\]

are convergent for all \( s \), if the conjecture for the Lie algebra \( G \) (on polynomial structure of moduli functions \( H_s \)) is obeyed for the Lie algebra \( G \) under consideration.

Indeed, due to the polynomial assumption (1.1) we have

\[
H_s(\rho^2) \sim C_s \rho^{2n_s}, \quad C_s = p_s^{(n_s)}, \quad (3.8)
\]

as \( \rho \to +\infty; s = 1, \ldots, n \). From (3.7), (3.8) and the equality \( \sum_l A_{is} m_l = 2 \), following from (1.2), we get

\[
B^s(\rho^2) \sim q_s C^s \rho^{-4}, \quad C^s = \prod_{l=1}^n C_i^{-A_{sl}}, \quad (3.9)
\]

and hence the integral (3.6) is convergent for any \( s = 1, \ldots, n \).

By using the master equations (2.6) we obtain

\[
\int_0^{+\infty} d\rho \rho B^s(\rho^2) = q_s P_s^{-1} \frac{1}{2} \int_0^{+\infty} dz \left( \frac{z}{H_s} \frac{d}{dz} H_s \right)
\]

\[
= \frac{1}{2} q_s P_s^{-1} \lim_{z \to +\infty} \left( \frac{z}{H_s} \frac{d}{dz} H_s \right)
\]

\[
= \frac{1}{2} n_s q_s P_s^{-1}, \quad (3.10)
\]

which implies [see (2.8)]

\[
\Phi^s = 4\pi n_s q_s^{-1} h_s, \quad (3.11)
\]

\( s = 1, \ldots, n \).

Thus, any flux \( \Phi^s \) depends upon one integration constant \( q_s \neq 0 \), while the integrand form \( F^s \) depends upon all constants: \( q_1, \ldots, q_n \).
We note that for $D = 4$ and $g^2 = -dt \otimes dt + dx \otimes dx$, $q_i$ is coinciding with the value of the $x$-component of the $i$th magnetic field on the axis of symmetry.

In the case of the Gibbons–Maeda dilatonic generalization of the Melvin solution, corresponding to $D = 4, n = 1$ and $\mathcal{G} = A_1$ [5], the flux from (3.11) $(s = 1)$ is in agreement with that considered in Ref. [26]. For Melvin’s case and some higher dimensional extensions (with $\mathcal{G} = A_1$) see also Ref. [14].

Due to (3.4) the ratios
$$q_i \Phi^i = \frac{n_j h_i}{n_j h_j} = \frac{n_j r_j}{n_j r_i},$$
are fixed numbers depending upon the Cartan matrix $(A_{ij})$ of a simple finite-dimensional Lie algebra $\mathcal{G}$.

**Remark 2** The relation for flux integrals (3.11) is also valid when the matrix $\left(\lambda_{a'}^{\alpha}\right)$ is a Cartan matrix of a finite-dimensional semi-simple Lie algebra $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_k$, where $\mathcal{G}_1, \ldots, \mathcal{G}_k$ are simple Lie (sub)algebras. In this case the Cartan matrix $(A_{ij})$ has a block-diagonal form, i.e. $(A_{ij}) = \text{diag} \left( \left( A_{i_1 j_1}^{(1)} \right), \ldots, \left( A_{i_k j_k}^{(k)} \right) \right)$, where $\left( A_{i_k j_k}^{(a)} \right)$ is the Cartan matrix of the Lie algebra $\mathcal{G}_a, a = 1, \ldots, k$. The set of polynomials in this case splits in a direct union of sets of polynomials corresponding to the Lie algebras $\mathcal{G}_1, \ldots, \mathcal{G}_k$. Equations (3.4) and (3.12) are valid, when the indices $i, j$ correspond to one $a$th block, $a = 1, \ldots, k$. The quantities $q_i \Phi^i$ and $q_j \Phi^j$ corresponding to different blocks are independent. Equation (3.5) should be replaced by
$$K_{a} = \frac{1}{2} K^{(a)} r_{a}, \quad K^{(a)} > 0,$$
for any index $a$, corresponding to the $a$th block; $a = 1, \ldots, k$. The existence of dilatonic coupling vectors $\lambda_1$ obeying (3.2) [(and (3.1)] just follows from the arguments of Remark 1, if we put all $K^{(a)} = K > 0$.

The manifold $M_s = (0, +\infty) \times S^1$ is isomorphic to the manifold $\mathbb{R}^2 \setminus \{0\}$. The solution (2.3)–(2.5) may be understood (or rewritten by pull-backs) as defined on the manifold $\mathbb{R}^2 \times M_2$, where the coordinates $\rho, \phi$ are understood as coordinates on $\mathbb{R}^2$. They are not globally defined. One should consider two charts with coordinates $\rho, \phi = \phi_1$ and $\rho, \phi = \phi_2$, where $\rho > 0, 0 < \phi_1 < 2\pi$ and $-\pi < \phi_2 < \pi$. Here $\text{exp}(i \phi_1) = \text{exp}(i \phi_2)$. In both cases we have $x = \rho \cos \phi$ and $y = \rho \sin \phi$, where $x, y$ are standard coordinates on $\mathbb{R}^2$. Using the identity $\rho d\rho \wedge d\phi = dx \wedge dy$ we get
$$F_s = q_s \prod_{i' = 1}^n \left( H_{i'}(x^2 + y^2) \right)^{-A_{i'i}} dx \wedge dy,$$
and
$$s = 1, \ldots, n. \text{ The 2-forms (3.14) are well defined on } \mathbb{R}^2.$$
Indeed, due to the conjecture from Ref. [3] any polynomial $H_1(z)$ is a smooth function on $\mathbb{R} = (-\infty, +\infty)$ which obeys $H_1(z) > 0$ for $z \in (-\varepsilon_s, +\infty)$, where $\varepsilon_s > 0$. This is valid due to the conjecture from Ref. [3] $H_2(z) > 0$ for $z > 0$ and $H_2(+0) = 1$. Thus, $\left( \prod_{i'=1}^n (H_{i'}(x^2 + y^2))^{-A_{i'i'}} \right)$ is a smooth function since it is a composition of two well-defined smooth functions $\left( \prod_{i'=1}^n (H_{i'}(z))^{-A_{i'i'}} \right)$ and $z = x^2 + y^2$.

Now we show that there exist 1-forms $A^s$ obeying $F_s = dA_s$ which are globally defined on $\mathbb{R}^2$. We start with the open submanifold $\mathbb{R}_e^2$. The 1-forms
$$A_s = \left( \int_0^\rho d\bar{\rho} B^s (\bar{\rho}^2) \right) d\phi = \frac{1}{2} \left( \int_0^{\rho^2} d\bar{\rho} B^s (\bar{\rho}) \right) d\phi$$
are well defined on $\mathbb{R}_e^2$ (here $d\phi = (x^2 + y^2)^{-1}(-y dx + x dy)$) and obey $F_s = dA_s, s = 1, \ldots, n$. Using the master equation (2.6) we obtain
$$A_s = \frac{q_s}{2 P_s} \left( \int_{\rho^2} \frac{d\bar{\rho}}{\bar{\rho}} \left( \frac{\bar{\rho}}{H_s (\bar{\rho})} \right) \right) d\phi = \frac{2h_s}{q_s} H_s (\rho^2) \bar{\rho}^2 d\phi,$$  
$s = 1, \ldots, n. \text{ Here } H_s = \frac{d}{d\rho} H_s. \text{ Due to the relation } \rho^2 d\phi = -(y dx + x dy), \text{ we obtain}
$$A_s = \frac{2h_s}{q_s} H_s (x^2 + y^2)^{-1}(-y dx + x dy),$$
$s = 1, \ldots, n. \text{ The 1-forms (3.17) are well-defined smooth 1-forms on } \mathbb{R}^2.$

We note that in the case of the Gibbons–Maeda solution [5] corresponding to $D = 4, n = 1$ and $\mathcal{G} = A_1$ the gauge potential from (3.16) coincides (up to notations) with that considered in Ref. [7].

Now we verify our result (3.11) for flux integrals by using the relations for the 1-forms $A^s$. Let us consider a 2d oriented manifold (disk) $D_R = \{(x, y) : x^2 + y^2 \leq R^2\}$ with the boundary $\partial D_R = C_R = \{(x, y) : x^2 + y^2 = R^2\}$. $C_R$ is a circle of radius $R$. It is an 1d oriented manifold with the orientation (inherited from that of $D_R$) obeying the relation $\int_{C_R} d\phi = 2\pi$. Using the Stokes–Cartan theorem we get
$$\int_{D_R} F_s = \int_{D_R} dA_s = \int_{C_R} A_s = 4\pi h_s H_s (R^2) \frac{H_s (R^2)}{q_s}, \quad (3.18)$$
$s = 1, \ldots, n. \text{ By using the asymptotic relation (3.8) we find}
$$\lim_{R \to +\infty} \int_{D_R} F_s = \frac{4\pi h_s n_s}{q_s}, \quad (3.19)$$
s = 1, \ldots, n, \text{ in agreement with (3.11).}
Remark 3. We note (for completeness) that the metric and scalar fields for our solution with \( w = +1 \) and \( l = n \) can be extended to the manifold \( \mathbb{R}^2 \times M_2 \). Indeed, in the coordinates \( x, y \) the metric \((2.3)\) and scalar fields \((2.4)\) read as follows:

\[
g = \left( \prod_{s=1}^{n} H_s^{2h_s/(D-2)} \right) \{ dx \otimes dx + dy \otimes dy + f(-ydx + xdy)^2 + g^2 \},
\]

\[
\varphi^a = \sum_{s=1}^{n} h_s \lambda_s^a \ln H_s,
\]

\[
a = 1, \ldots, l. \quad \text{Here } H_s = H_s(x^2 + y^2), \quad s = 1, \ldots, n, \quad \text{and } f = f(x^2 + y^2), \quad \text{where}
\]

\[
f(z) = \left( \prod_{s=1}^{n} (H_s(z))^{-2h_s} - 1 \right) z^{-1},
\]

for \( z \neq 0 \) and \( f(0) = \lim_{z \to 0} f(z) \) (the limit does exist). The function \( f(z) \) is smooth in the interval \((-\varepsilon, +\infty)\) for some \( \varepsilon > 0 \). Indeed, it is smooth in the interval \((0, +\infty)\) and holomorphic in the domain \([z|0 < |z| < \varepsilon]\) for a small enough \( \varepsilon > 0 \). Since the limit \( \lim_{z \to 0} f(z) \) does exist the function \( f(z) \) is holomorphic in the disc \([z|0 < |z| < \varepsilon]\) and hence it is smooth in the interval \((-\varepsilon, +\infty)\). This implies that the metric is smooth on the manifold \( \mathbb{R}^2 \times M_2 \). (See the text after Eq. (3.14).) The scalar fields are also smooth on \( \mathbb{R}^2 \times M_2 \).

4 Examples

Here we present fluxbrane polynomials corresponding to the Lie algebras \( A_1, A_2, A_3, C_2, G_2, A_1 + A_1 \) and related fluxes. Here as in [32] we use other parameters \( p_s \) instead of \( P_s \):

\[
p_s = P_s/n_s,
\]

\[
s = 1, \ldots, n.
\]

**A1-case.** The simplest example occurs in the case of the Lie algebra \( A_1 = sl(2) \). Here \( n_1 = 1 \). We get [3]

\[
H_1 = 1 + p_1 z
\]

and

\[
\Phi^1 = 4\pi q_1^{-1} h_1,
\]

which is also valid for Melvin’s solution with \( D = 4 \) and \( h_1 = 2 \).

**A2-case.** For the Lie algebra \( A_2 = sl(3) \) with the Cartan matrix

\[
(A_{s,s'}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}
\]

we have [3, 25, 32] \( n_1 = n_2 = 2 \) and

\[
H_1 = 1 + 2p_1 z + p_1 p_2 z^2,
\]

\[
H_2 = 1 + 2p_2 z + p_1 p_2 z^2.
\]

We get in this case

\[
(\Phi^1, \Phi^2) = 8\pi h(q_1^{-1}, q_2^{-1}),
\]

where \( h_1 = h_2 = h \).

**A3-case.** The polynomials for the \( A_3 \)-case read as follows [32, 33]:

\[
H_1 = 1 + 3p_1 z + 3p_1 p_2 z^2 + p_1 p_2 p_3 z^3,
\]

\[
H_2 = 1 + 4p_2 z + 3\left(p_1 p_2 + p_2 p_3\right) z^2
\]

\[
+ 4p_1 p_2 p_3 z^3 + p_1 p_2^2 p_3 z^4,
\]

\[
H_3 = 1 + 3p_3 z + 3p_2 p_3 z^2 + p_1 p_2 p_3 z^3.
\]

Here we have \((n_1, n_2, n_3) = (3, 4, 3)\) and

\[
(\Phi^1, \Phi^2, \Phi^3) = 4\pi h(3q_1^{-1}, 4q_2^{-1}, 3q_3^{-1})
\]

with \( h_1 = h_2 = h_3 = h \).

**C2-case.** For the Lie algebra \( C_2 = so(5) \) with the Cartan matrix

\[
(A_{s,s'}) = \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}
\]

we get \( n_1 = 3 \) and \( n_2 = 4 \). For \( C_2 \)-polynomials we obtain [25, 32]

\[
H_1 = 1 + 3p_1 z + 3p_1 p_2 z^2 + p_1^2 p_2 z^3,
\]

\[
H_2 = 1 + 4p_2 z + 6p_1 p_2 z^2 + 4p_1^2 p_2 z^3 + p_1^3 p_2 z^4.
\]

In this case we find

\[
(\Phi^1, \Phi^2) = 4\pi (3h_1 q_1^{-1}, 4h_2 q_2^{-1})
\]

where \( h_1 = 2h_2 \).

**G2-case.** For the Lie algebra \( G_2 \) with the Cartan matrix

\[
(A_{s,s'}) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}
\]

we have [25, 32] \( n_1 = n_2 = 2 \) and

\[
H_1 = 1 + 2p_1 z + p_1 p_2 z^2,
\]

\[
H_2 = 1 + 2p_2 z + p_1 p_2 z^2.
\]
we get \( n_1 = 6 \) and \( n_2 = 10 \). In this case the fluxbrane polynomials read [25, 32]

\[
H_1 = 1 + 6p_1z + 15p_1p_2z^2 + 20p_1^2p_2z^3 + 15p_1^3p_2z^4 + 6p_1^3p_2^2z^5 + p_1^4p_2^2z^6,
\]

(4.17)

\[
H_2 = 1 + 10p_2z + 45p_1p_2z^2 + 120p_1^2p_2z^3 + p_1^2p_2(135p_1 + 75p_2)z^4 + 252p_1^3p_2^2z^5 + p_1^2p_2^2(75p_1 + 135p_2)z^6 + 120p_1^4p_2^3z^7 + 45p_1^5p_2^3z^8 + 10p_1^6p_2^3z^9 + p_1^6p_2^4z^{10}.
\]

(4.18)

We are led to the relations

\[
(\Phi^1, \Phi^2) = 4\pi (6h_1q_1^{-1}, 10h_2q_2^{-1})
\]

(4.19)

where \( h_1 = 3h_2 \).

**Case.** For the semi-simple Lie algebra \( A_1 + A_1 \) we obtain \( n_1 = n_2 = 1 \),

\[
H_1 = 1 + p_1z, \quad H_2 = 1 + p_2z,
\]

(4.20)

and

\[
(\Phi^1, \Phi^2) = 4\pi (q_1^{-1}h_1, q_2^{-1}h_2),
\]

(4.21)

where \( h_1 \) and \( h_2 \) are independent, as well as the quantities \( q_1\Phi^1 \) and \( q_2\Phi^2 \).

5 Conclusions

Here we have considered a multidimensional generalization of Melvin’s solution corresponding to a finite-dimensional Lie algebra \( \mathcal{G} \). We have assumed that the solution is governed by a set of \( n \) fluxbrane polynomials \( H_1(z) \), \( s = 1, \ldots, n \). These polynomials define special solutions to open Toda chain equations corresponding to the Lie algebra \( \mathcal{G} \).

The polynomials \( H_s(z) \) depend also upon parameters \( q_s \), which are coinciding for \( D = 4 \) (up to a sign) with the values of colored magnetic fields on the axis of symmetry.

We have calculated 2d flux integrals \( \Phi^s = \int F^s, s = 1, \ldots, n \). Any flux \( \Phi^s \) depends only upon one parameter \( q_s \), while the integrand \( F^s \) depends upon all parameters \( q_1, \ldots, q_n \). The relation for flux integrals (3.11) is also valid when the matrix \( (A_{\alpha\beta}) \) is a Cartan matrix of a finite-dimensional semi-simple Lie algebra \( \mathcal{G} \).

Here we have considered examples of polynomials and fluxes for the Lie algebras \( A_1, A_2, A_3, C_2, G_2 \) and \( A_1 + A_1 \). The approach of this paper will be used for a calculation of certain flux integrals for forms \( F^s \) of arbitrary ranks corresponding to certain fluxbrane solutions (of electric type by \( p \)-brane notation or magnetic type by fluxbrane classification) governed by fluxbrane polynomials [38].

An open problem is to find the fluxes for the solutions which are related to infinite-dimensional Lorentzian Kac–Moody algebras, e.g. hyperbolic ones [39, 40]. In this case one should deal with phantom scalar fields in the model (2.1) and non-polynomial solutions to Eqs. (2.6). Another possibility is to study the convergence of flux integrals for non-polynomial solutions for moduli functions corresponding to non-Cartan matrices \( (A_{\alpha\beta}) \) (e.g. for the model with two 2-forms from Ref. [41]).

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We remind the reader that an electric (magnetic) \( p \)-brane corresponds to a magnetic (electric) \( F(D - 3 - p) \) fluxbrane; see [3] and the references therein.
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