ON PARALLELISABLE NS-NS BACKGROUNDS

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Abstract. We classify non-dilatonic NS-NS type II supergravity backgrounds admitting a consistent absolute parallelism. They are all given by parallelised Lie groups admitting scalar flat bi-invariant lorentzian metrics. There are seven different classes, some of them containing moduli. For each class we determine the amount of supersymmetry which is preserved: there are examples with 16, 18, 20, 22, 24, 28 and 32 supersymmetries.

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1. INTRODUCTION

The purpose of this note is to present a classification of non-dilatonic parallelisable NS-NS backgrounds of ten-dimensional type II supergravity. In this note, which has been prompted in part by the preprints [35] and [33], I attempt to place these backgrounds in an appropriate mathematical context. The main result is that they are given by ten-dimensional Lie groups admitting a bi-invariant scalar flat metric. We then classify these groups up to local isometry and explore the amount of supersymmetry that these backgrounds preserve. Let us start by introducing the context.

We will be dealing with the common sector of ten-dimensional type II supergravity, the so-called NS-NS backgrounds where none of the RR fluxes are turned on. The data for such backgrounds consists of

- a ten-dimensional lorentzian spin manifold $(M, g),$
- a metric connection $D$ with closed torsion three-form $H,$ and
- a dilaton $\phi$

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subject to the equations of motion obtained by varying the action functional, which takes the form
\[
\int_M e^{-2\phi} \left( R + 4\|d\phi\|^2 - \frac{1}{2}\|H\|^2 \right) \text{dvol}_g .
\]  
(1)

For the purposes of this note we will say that \((M, g)\) admits a consistent absolute parallelism (or is parallelisable, for short) if \(D\) is flat. In this note we will classify the parallelisable ten-dimensional spacetimes up to local isometry and determine which of them are non-dilatonic supergravity backgrounds.

One can of course analyse a more general problem: namely classifying those parallelised supergravity backgrounds for which the dilaton is not constant. Besides the dilaton equation of motion, there is another condition on a nonconstant dilaton. Because the torsion three-form \(H\) is closed, \((M, g)\) is parallelisable if and only if it is locally isometric to a Lie group admitting a bi-invariant metric. In particular this means that \(H\) is parallel, hence co-closed. This imposes constraints on the derivative of the dilaton. From the Maxwell equation for \(H\), we see that
\[
d\phi \wedge \star H = 0 ,
\]
which is equivalent to
\[
\iota_{\text{grad } \phi} H = 0 ,
\]
so that relative to a basis of left-invariant vector fields, the gradient of \(\phi\) only has components along the centre of the Lie algebra. In other words, if \(\theta\) denotes the left-invariant Maurer–Cartan one-form then \(\theta(\text{grad } \phi)\) is a central element of the Lie algebra. We will assume from now on that the dilaton is constant, although this is not a consequence of parallelisability. In fact, since we will start by classifying the parallelisable geometries, it is then a simple matter (which we will nevertheless not address in this note) to determine which dilatons are consistent with the parallelisable spaces. This may in turn constraint the geometry further and in any case will alter the analysis of the supersymmetry of the backgrounds, to which we know turn.

An important invariant of a supergravity background is the amount of supersymmetry that it preserves, measured by the dimension of the (linear) space of Killing spinors. In principle this could be any integer from 0 to 32, but it will be severely constrained in the backgrounds under consideration. In type II supergravity, there are two types of Killing spinor equation, resulting from the supersymmetry variations of the gravitino and of the dilatino. The gravitino variation gives a differential equation which says that a Killing spinor is covariantly constant with respect to the spin connection associated with \(D\). Since \(D\) is flat for a parallelisable manifold, this condition does not reduce the amount of supersymmetry of the background, a fact also observed in [35]. In contrast, the dilatino variation gives an algebraic condition which says that a Killing spinor is annihilated under Clifford multiplication by the torsion three-form \(H\) (for constant dilaton). For a Lie group admitting a bi-invariant lorentzian metric, \(H\) is given essentially by the structure constants relative to a frame consisting of left-invariant vector fields.

The rest of the paper is organised as follows. In Section 2 we will recall the known results about parallelisable manifolds. In Section 3 we will prove that manifolds admitting flat metric connections with closed torsion three-form are locally isometric to Lie groups admitting a bi-invariant metric. In Section 4 we will determine all the ten-dimensional Lie groups with bi-invariant lorentzian metrics, up to local isometry. We will also determine which of them is a non-dilatonic background. In Section 5 we will determine the amount of supersymmetry preserved by each of these backgrounds. Finally in Section 6 we summarise our results.
2. Manifolds admitting absolute parallelisms

In this section we recall the known results about parallelisable manifolds.

A differentiable manifold $M$ is said to admit an absolute parallelism if it admits a smooth trivialisation of the frame bundle $B \to M$. Such a trivialisation consists of a smooth global frame and hence also trivialises the tangent bundle; whence manifolds admitting absolute parallelisms are parallelisable in the topological sense. The reduction theorem for connections on principal bundles (see, for example, [27, Section II.7]) allows us to think of absolute parallelisms in terms of holonomy groups of connections. Indeed, an absolute parallelism is equivalent to a smooth connection on the frame bundle with trivial holonomy. This implies, in particular, that the connection is flat and if the manifold is simply-connected then flatness is also sufficient.

So far these notions are purely (differential) topological and make no mention of metrics or any other structure on the manifold. The question arises whether there is a metric on $M$ which is consistent with a given absolute parallelism, so that parallel transport is an isometry; or turning the question around, whether a given pseudo-riemannian manifold $(M, g)$ admits a consistent absolute parallelism. In terms of connections, a consistent absolute parallelism is equivalent to a metric connection with torsion with trivial holonomy; or, locally, to a flat metric connection with torsion.

Elie Cartan and Schouten [10, 9] essentially solved the riemannian case by generalising Clifford’s parallelism on the 3-sphere in two different ways. The three-sphere can be understood both as the unit-norm quaternions and also as the Lie group $SU(2) = \text{Sp}(1)$. The latter characterisation generalises to other (semi)simple Lie groups, whereas the former gives rise to the parallelism of the 7-sphere thought of as the unit-norm octonions. It follows from the results of Cartan and Schouten that a simply-connected irreducible riemannian manifold admitting a consistent absolute parallelism (equivalently a flat metric connection) is isometric to one of the following: the real line, a simple Lie group with the bi-invariant metric induced from a multiple of the Killing form, or the round 7-sphere.

Their proofs might have had gaps which were addressed by Wolf [39, 40], who also generalised these results to arbitrary signature, subject to an algebraic curvature condition saying that the pseudo-riemannian manifold $(M, g)$ is of “reductive type,” a condition which is automatically satisfied in the riemannian case. (See Wolf’s paper for the precise condition.) In the case of lorentzian signature, Cahen and Parker [7] showed that one can relax the “reductive type” condition; completing the classification of absolute parallelisms consistent with a lorentzian metric.

Wolf also showed that if one also assumes that the torsion is parallel, then, in any signature, $(M, g)$ is locally isometric to a Lie group with a bi-invariant metric. In fact, it is possible to show (see below) that one obtains the same result starting with the weaker hypothesis that the torsion three-form is closed, which is the case needed in supergravity.

In particular, since the 7-sphere is not a Lie group, it follows that its torsion three-form is not closed; although it is co-closed. This follows from the fact that the round 7-sphere $S^7$ admits geometric Killing spinors. Recall that a 7-manifold admits a $G_2$ structure if and only if it is spin [28]. It follows moreover that associated with the $G_2$ structure there is a canonical non-vanishing spinor field $\psi$. If $\nabla \psi = 0$ then the $G_2$ structure is said to be parallel and the manifold has $G_2$ holonomy. If $\nabla_X \psi = \lambda X \cdot \psi$ ($\lambda \neq 0$), so that the spinor field is Killing (in the geometric sense), then the $G_2$ structure is said to be nearly parallel. The different types of $G_2$ structures in 7-manifolds have been classified by Fernández and Gray [17] by studying the algebraic type of $\nabla H$, or equivalently $\nabla \psi$. For the nearly parallel $G_2$
structures, the torsion three-form $H$ satisfies $dH = \lambda \star H$, for some nonzero real number $\lambda$. (If $\lambda = 0$ then the manifold has $G_2$ holonomy and the $G_2$ structure is parallel.) Thus, $H$ is co-closed, but not closed. Since $H^3(S^7) = 0$, the three-form $H$ cannot be both closed and co-closed. This means that $\text{AdS}_3 \times S^7$, although parallelisable, is not a parallelisable supergravity background. This does not mean that metrically $\text{AdS}_3 \times S^7$ cannot be a supergravity background provided we turn on the dilaton and other fluxes, and indeed such backgrounds are known [14].

The results of Cahen and Parker [7] actually show that in lorentzian signature one gets for free that the torsion is parallel. Therefore it follows that an indecomposable lorentzian manifold $(M, g)$ admits a consistent absolute parallelism if and only if it is locally isometric to a lorentzian Lie group with bi-invariant metric. In particular, $\text{AdS}_7$ is not parallelisable, even if we do not impose any conditions on the torsion three-form, such as that it be closed. Therefore $\text{AdS}_7 \times S^3$ cannot be a parallelisable supergravity background.

It may seem surprising that the naive continuation to lorentzian signature does not work. One can understand this more conceptually by realising that the consistent absolute parallelism of $S^7$ arises from the identification of $S^7$ as the sphere of unit octonions. There is no real division algebra, however, whose unit “sphere” has lorentzian signature. There is however a split version of the octonions which does give rise to a consistent absolute parallelism in a space form of signature $(3, 4)$. There is also a consistent absolute parallelism in a complexified version of the seven-sphere $\text{SO}(8, \mathbb{C})/\text{SO}(7, \mathbb{C})$. Both these results are contained in [39, 40].

3. Flat metric connections with closed torsion

We will now show that a pseudo-riemannian manifold $(M, g)$ with a flat metric connection with closed torsion three-form is locally isometric to a Lie group admitting a bi-invariant metric. This section is based on work with Ali Chamseddine and Wafic Sabra [11].

Let $(M, g)$ be a pseudo-riemannian manifold and let $D$ be a metric connection with torsion $T$. In other words, $Dg = 0$ and for all vector fields $X, Y$ on $M$, $T : \Lambda^2 TM \to TM$ is defined by $T(X, Y) = D_X Y - D_Y X - [X, Y]$. In terms of the torsion-free Levi-Civita connection $\nabla$, we have $D_X Y = \nabla_X Y + \frac{1}{2} T(X, Y)$. Since both $Dg = 0$ and $\nabla g = 0$, $T$ is skew-symmetric:

$$g(T(X, Y), Z) = -g(T(X, Z), Y), \quad (2)$$

for all vector fields $X, Y, Z$ and gives rise to a torsion three-form $H \in \Omega^3(M)$, defined by $H(X, Y, Z) = g(T(X, Y), Z)$. We will assume that $H$ is closed and in this section we will characterise those manifolds for which $D$ is flat.

Let $R^D$ denote the curvature tensor of $D$, defined by $R^D(X, Y) Z = D_{[X,Y]} Z - D_X D_Y Z + D_Y D_X Z$. Our strategy will be to consider the equation $R^D = 0$, decompose it into types and solve the corresponding equations. We will find that $T$ is parallel with respect to both $\nabla$ and $D$, and this will imply that $(M, g)$ is locally a Lie group with a bi-invariant metric and $D$ the parallelising connection of Cartan and Schouten [10].
The curvature \( R^D \) is given by
\[
R^D(X, Y)Z = R(X, Y)Z - \frac{1}{3}((\nabla_X T)(Y, Z) + \frac{1}{2}((\nabla_Y T)(X, Z)) - \frac{1}{4}T(X, T(Y, Z)) + \frac{1}{4}T(Y, T(X, Z)),
\]
where \( R = R^\nabla \) is the curvature of the Levi-Civita connection. The tensor
\[
R^D(X, Y, Z, W) := g(R^D(X, Y)Z, W)
\]
takes the following form
\[
R^D(X, Y, Z, W) = R(X, Y, Z, W)
- \frac{1}{2}g((\nabla_X T)(Y, Z), W) + \frac{1}{2}g((\nabla_Y T)(X, Z), W)
+ \frac{1}{4}g(T(X, T(Y, Z)), W) - \frac{1}{4}g(T(Y, T(X, Z)), W),
\]
where we have defined the Riemann tensor as usual:
\[
R(X, Y, Z, W) := g(R(X, Y)Z, W).
\]
Using equation (2) we can rewrite \( R^D \) as
\[
R^D(X, Y, Z, W) = R(X, Y, Z, W)
- \frac{1}{2}g((\nabla_X T)(Y, Z), W) + \frac{1}{2}g((\nabla_Y T)(X, Z), W)
+ \frac{1}{4}g(T(X, T(W, X)), T(Y, Z)) - \frac{1}{4}g(T(Y, T(X, Z)), W),
\]
which is manifestly skew-symmetric in \( X, Y \) and in \( Z, W \). Observe that unlike \( R \), the torsion terms in \( R^D \) do not satisfy the first Bianchi identity. Therefore breaking \( R^D \) into algebraic types will give rise to more equations and will eventually allow us to characterise the data \((M, g, T)\) for which \( R^D = 0 \).

Indeed, let \( R^D = 0 \) and consider the identity
\[
\mathfrak{S}_{XYZ} R^D(X, Y, Z, W) = 0,
\]
where \( \mathfrak{S} \) denotes signed permutations. Since \( R \) does obey the Bianchi identity
\[
\mathfrak{S}_{XYZ} R(X, Y, Z, W) = 0,
\]
we obtain the following identity
\[
\mathfrak{S}_{XYZ} g((\nabla_X T)(Y, Z), W) = -\frac{1}{2} \mathfrak{S}_{XYZ} g(T(W, X), T(Y, Z)). \tag{3}
\]
Now we use the fact that the torsion three-form \( H \) is closed, which can be written as
\[
g((\nabla_X T)(Y, Z), W) - g((\nabla_Y T)(X, Z), W)
+ g((\nabla_Z T)(X, Y), W) - g((\nabla_W T)(X, Y), Z) = 0,
\]
or equivalently,
\[
g((\nabla_W T)(X, Y), Z) = \frac{1}{2} \mathfrak{S}_{XYZ} g((\nabla_X T)(Y, Z), W).
\]
This turns equation (3) into
\[
g((\nabla_W T)(X, Y), Z) = -\frac{1}{4} \mathfrak{S}_{XYZ} g(T(W, X), T(Y, Z)). \tag{4}
\]
From this equation it follows that
\[
g((\nabla_W T)(X, Y), Z) = -g((\nabla_X T)(W, Y), Z),
\]
so that \( g((\nabla_W T)(X, Y), Z) \) is totally skew-symmetric. This means that \( \nabla H = dH = 0, \) whence \( H \) and hence \( T \) are parallel. Therefore equation (4) simplifies to
\[
\mathfrak{S}_{XYZ} g(T(W, X), T(Y, Z)) = 0. \tag{5}
\]
Let us remark that $\nabla H = 0$ and equation 6 implies that $DH = 0$ as well. Indeed,

$$(\mathcal{L}_H)X(Y, Z) = WH(X, Y, Z) - H(D_W X, Y, Z) - H(X, D_W Y, Z) - H(X, Y, D_W Z)$$

$$= WH(X, Y, Z) - H(\nabla X, Y, Z) - H(X, \nabla W Y, Z) - H(X, Y, \nabla W Z)$$

$$- \frac{1}{2}H(T(W, X), Y, Z) - \frac{1}{2}H(X, T(W, Y), Z) - \frac{1}{2}H(X, Y, T(W, Z))$$

$$= (\nabla_W H)(X, Y, Z) - \frac{1}{2} \xi \kappa X Y Z g(T(W, X), T(Y, Z)),$$

whence if $\nabla H = 0$ and 6 holds, then $DH = 0$ as well.

Equation 6 is precisely the statement that the skew-endomorphism $\eta_T \in \mathfrak{so}(TM)$ defined by $\eta_T T(X) = T(W, X)$ leaves the torsion three-form $H$ invariant. Indeed, the action of $\eta_T$ on $H$ is given by

$$(\eta_T H)(X, Y, Z) = -H(\eta_T X, Y, Z) - H(X, \eta_T Y, Z) - H(X, Y, \eta_T Z)$$

$$= -H(T(W, X), Y, Z) - H(X, T(W, Y), Z) - H(X, Y, T(W, Z))$$

$$= -H(Y, Z, T(W, X)) + H(X, Z, T(W, Y)) - H(X, Y, T(W, Z))$$

$$= -g(T(Y, Z), T(W, X)) + g(T(X, Z), T(W, Y)) - g(T(X, Y), T(W, Z))$$

$$= -g(T(W, X), T(Y, Z)) - g(T(W, Y), T(Z, X)) - g(T(W, Z), T(X, Y))$$

$$= -\xi \kappa X Y Z g(T(W, X), T(Y, Z)).$$

We pause to remark parenthetically that this shows that equation 6 is an instance of the Plücker relations in [13]. More familiar, perhaps, is the fact that equation 6 is the Jacobi identity for $T$. Indeed, notice that

$$g(T(W, X), T(Y, Z)) = H(W, X, T(Y, Z))$$

$$= H(X, T(Y, Z), W) = g(T(X, T(Y, Z)), W),$$

whence equation 6 is satisfied if and only if

$$\xi \kappa X Y Z g(T(X, T(Y, Z))) = 0. \tag{6}$$

This means that the tangent space $T_pM$ of $M$ at every point $p$ becomes a Lie algebra where the Lie bracket is given by the restriction of $T$ to $T_pM$. More is true and the restriction to $T_pM$ of the metric $g$ gives rise to an (ad-)invariant scalar product:

$$g(T(X, Y), Z) = g(X, T(Y, Z)).$$

By a theorem of Wolf [29] [41] (based on earlier work of Élie Cartan and Schouten [10] [31]) if $(M, g)$ is complete then it is a discrete quotient of a Lie group with a bi-invariant metric. In general, we can say that $(M, g)$ is locally isometric to a Lie group with a bi-invariant metric.

Indeed, since $D$ is flat, there exists locally a parallel frame $\{\xi_i\}$ for $TM$. Since $\xi_i$ is parallel, from the definition of the torsion,

$$T(\xi_i, \xi_j) = -[\xi_i, \xi_j].$$

Moreover, since $T$ is parallel relative to $D$, we see that $[\xi_i, \xi_j]$ is also parallel with respect to $D$, whence it can be written as a linear combination of the $\xi_i$ with constant coefficients. In other words, they span a real Lie algebra $\mathfrak{g}$. The homomorphism $\mathfrak{g} \to C^\infty(M, TM)$ whose image is the subalgebra spanned by the $\{\xi_i\}$ integrates, once we choose a point in $M$, to a local diffeomorphism $G \to M$. This is also an isometry if we use on $G$ the metric induced from the one on the Lie algebra, whence we conclude that $(M, g)$ is locally isometric to a Lie group with a bi-invariant metric.

To conclude let us make the observation that the condition $R^\nabla = 0$ allows us to express the Riemann curvature in terms of $T$ as follows:

$$R(X, Y)Z = \frac{1}{4} T(T(X, Y), Z), \tag{7}$$

which agrees with the standard expression for the Riemann curvature of a bi-invariant metric on a Lie group (see, e.g., 27 Ch. X, Prop. 2.12) if we identify
–T with the Lie bracket, as was done above. Contracting the above expression we obtain an expression for the Ricci curvature which agrees with the equation of motion for the metric coming from type II supergravity. Furthermore, it is clear from equation (1) that a constant dilaton is consistent with its equation of motion if and only the lagrangian vanishes. Computing the scalar curvature from equation (1) that a constant dilaton is consistent with its equation of motion if and only if the scalar curvature vanishes, or equivalently ∥H∥² = 0.

4. PARALLELISABLE SUPERGRAVITY BACKGROUNDS

Summarising the above discussion, the parallelisable supergravity backgrounds are locally isometric to Lie groups admitting a bi-invariant lorentzian metric with vanishing scalar curvature. Equivalently, they are in one-to-one correspondence with ten-dimensional Lie algebras admitting a lorentzian ad-invariant metric and such that the structure constants satisfy

\[ f_{abc}f^{abc} = 0. \]

For a recent treatment of lorentzian Lie algebras (albeit in six dimensions) the reader is referred to the forthcoming work [11]. Here we simply summarise the result.

It follows from the structure theorem of Medina and Revoy [20] (see also [19] for a refinement) that an indecomposable lorentzian Lie algebra is either isomorphic to \( \mathfrak{so}(1,2) \) with (a multiple of) the Killing form, or else is solvable and can be described as a double extension \( \mathfrak{d}_{2n+2} := \mathfrak{d}(E^{2n}, R) \) of the abelian Lie algebra \( E^{2n} \) with the (trivially invariant) euclidean metric by a one-dimensional Lie algebra acting on \( E^{2n} \) via a non-degenerate skew-symmetric linear map \( J : E^{2n} \to E^{2n} \).

More concretely, the double extension \( \mathfrak{d}_{2n+2} \) has underlying vector space \( \mathcal{V} = E^{2(d-1)} \oplus R \oplus R \), and if \((v,v^-,v^+),(w,w^-,w^+) \in \mathcal{V}\), then their Lie bracket is given by

\[
[(v,v^-,v^+),(w,w^-,w^+)] = (v^-J(w) - w^-J(v), 0, v \cdot J(w))
\]

and their inner product follows by polarisation from

\[
\| (v,v^-,v^+) \|^2 = v \cdot v + 2v^+v^-.
\]

The unique simply-connected Lie group with Lie algebra \( \mathfrak{d}_{2n+2} \) is a solvable \((2n+2)\)-dimensional Lie group admitting a bi-invariant metric

\[
ds^2 = 2dx^+dx^- - (Jx,Jx)(dx^-)^2 + (dx,dx), \tag{8}
\]

relative to natural coordinates \((x,x^-,x^+)\).

Because \( J \) is non-degenerate and skew-symmetric, it can always be skew-diagonalised via an orthogonal transformation. The skew-eigenvalues \( \lambda_1, \ldots, \lambda_n \), which are different from zero, can be arranged so that they obey: 0 < \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \). Finally a positive rescaling of \( J \) can be absorbed into reciprocal rescalings of \( x^\pm \), so that we can set \( \lambda_n \), say, equal to 1 without loss of generality. Therefore we see that the moduli space of metrics (8) is given by an \((n-1)\)-tuple \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \) where 0 < \( \lambda_1 \leq \cdots \leq \lambda_{n-1} \leq 1 \). We will call such a metric \( \text{CW}_{2n+2}(\lambda) \), as they are particular cases of Cahen–Wallach spacetimes [8].

Before determining the ten-dimensional lorentzian Lie algebras, let us observe that the above remarks allow us to compare with the results of [35] Section 4] on parallelisable plane waves. First of all it can be shown [39, 40] that parallelisable manifolds are necessarily locally symmetric. Hence parallelisable plane waves have to be locally isometric to Cahen–Wallach spacetimes. From the results of Cahen and Parker in the general case or from what was proven above for the case of closed torsion three-form, we know that they also have to be locally isometric to Lie groups with a bi-invariant metric. The problem of determining which Cahen–Wallach spacetimes are locally isometric to Lie groups with a bi-invariant metric was solved in [35] Section 2.3] and the answer consists precisely of the metrics given above in [8], in agreement with the results of [35] Section 4].
From the above remarks one can determine the ten-dimensional lorentzian Lie algebras: they are either \(\mathfrak{so}(1,2) \oplus \mathfrak{g}_7\) or \(\mathfrak{d}_{2n+2} \oplus \mathfrak{g}_{8-2n}\), where \(\mathfrak{g}_d\) is a \(d\)-dimensional reductive Lie algebra with a positive-definite metric. It is easy to come up with Table 1 where we have written the Lie algebras and the corresponding spacetimes (up to local isometry).

| Lie algebra | Spacetime |
|-------------|-----------|
| \(\mathfrak{so}(1,2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}\) | \(\text{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}\) |
| \(\mathfrak{so}(1,2) \oplus \mathfrak{su}(2) \oplus \mathbb{E}^4\) | \(\text{AdS}_3 \times S^3 \times \mathbb{R}^4\) |
| \(\mathfrak{d}_{10}\) | \(\text{CW}_{10}(\lambda)\) |
| \(\mathfrak{d}_8 \oplus \mathbb{E}^2\) | \(\text{CW}_8(\lambda) \times \mathbb{R}^2\) |
| \(\mathfrak{d}_6 \oplus \mathfrak{su}(2) \oplus \mathbb{R}\) | \(\text{CW}_6(\lambda) \times S^3 \times \mathbb{R}\) |
| \(\mathfrak{d}_6 \oplus \mathbb{E}^4\) | \(\text{CW}_6(\lambda) \times \mathbb{R}^4\) |
| \(\mathfrak{d}_4 \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)\) | \(\text{CW}_4(\lambda) \times S^3 \times S^3\) |
| \(\mathfrak{d}_4 \oplus \mathfrak{su}(2) \oplus \mathbb{E}^3\) | \(\text{CW}_4(\lambda) \times S^3 \times \mathbb{R}^3\) |
| \(\mathfrak{d}_4 \oplus \mathbb{E}^6\) | \(\text{CW}_4(\lambda) \times \mathbb{R}^6\) |
| \(\mathbb{E}^{1,1} \oplus \mathfrak{su}(3)\) | \(\mathbb{R}^{1,1} \times \text{SU}(3)\) |
| \(\mathbb{E}^{1,3} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)\) | \(\mathbb{R}^{1,3} \times S^3 \times S^3\) |
| \(\mathbb{E}^{1,6} \oplus \mathfrak{su}(2)\) | \(\mathbb{R}^{1,6} \times S^3\) |
| \(\mathbb{E}^{1,9}\) | \(\mathbb{R}^{1,9}\) |

Table 1. Ten-dimensional parallelisable spacetimes.

Finally we impose the condition that the scalar curvature should vanish, which is the consistency condition for a constant dilaton. Since \(\text{CW}(\lambda)\) is scalar flat, any spacetime of the form \(\text{CW}_{2n}(\lambda) \times \mathbb{E}^{10-2n}\) is scalar flat. Any background containing an \(S^3\) (or \(\text{SU}(3)\)) factor can only be scalar flat if there is a factor with negative scalar curvature to balance it; namely \(\text{AdS}_3\). Therefore apart from the \(\text{CW}_{2n}(\lambda) \times \mathbb{E}^{10-2n}\) backgrounds, only \(\text{AdS}_3 \times S^3 \times \mathbb{R}\) and \(\text{AdS}_3 \times S^3 \times \mathbb{R}^3\) can possibly be consistent non-dilatonic backgrounds. For a space form, such as \(\text{AdS}_3\) and \(S^3\), the scalar curvature is inversely proportional to the square of the radius of curvature and the proportionality constant only depends on the dimension, here 3. Therefore the scalar curvature of the product \(\text{AdS}_3 \times S^3 \times \mathbb{R}\) is proportional to \(-1/R_1^2 + 1/R_2^2 + 1/R_3^2\) where \(R_1\) is the radius of curvature of \(\text{AdS}_3\) and \(R_2\) and \(R_3\) are the radii of curvature of the spheres. Therefore \(\text{AdS}_3 \times S^3 \times \mathbb{R}\) is a consistent non-dilatonic background if and only if \(1/R_1^2 = 1/R_2^2 + 1/R_3^2\). Similarly, \(\text{AdS}_3 \times S^3 \times \mathbb{R}^4\) is a consistent background if and only if the radii of curvature of the two non-flat factors agree.

In summary, the non-dilatonic parallelisable NS-NS supergravity backgrounds are listed in Table 2 where we have also listed the amount of supersymmetry that is preserved. This depends on the moduli. The details of how the third column was arrived at appear in the next section: in all cases except for one, there is no distinction between the type IIA and type IIB theories.
Table 2. Ten-dimensional non-dilatonic parallelisable NS-NS backgrounds. We have adorned with (A) or (B) cases which only occur for type IIA or IIB, respectively.

| Lie algebra | Spacetime | Supersymmetry |
|-------------|-----------|---------------|
| \(\mathfrak{so}(1,2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}\) | \(\text{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}\) | 16 |
| \(\mathfrak{so}(1,2) \oplus \mathfrak{su}(2) \oplus \mathbb{E}^4\) | \(\text{AdS}_3 \times S^3 \times \mathbb{R}^4\) | 16 |
| \(\mathfrak{d}_{10}\) | \(\text{CW}_{10}(\lambda)\) | 16, 18(A), 20, 22(A), 24(B), 28(B) |
| \(\mathfrak{d}_8 \oplus \mathbb{E}^2\) | \(\text{CW}_8(\lambda) \times \mathbb{R}^2\) | 16, 20 |
| \(\mathfrak{d}_6 \oplus \mathbb{E}^4\) | \(\text{CW}_6(\lambda) \times \mathbb{R}^4\) | 16, 24 |
| \(\mathfrak{d}_4 \oplus \mathbb{E}^6\) | \(\text{CW}_4(\lambda) \times \mathbb{R}^6\) | 16 |
| \(\mathbb{E}^{1,9}\) | \(\mathbb{R}^{1,9}\) | 32 |

5. Supersymmetry

As mentioned in the introduction, the amount of supersymmetry preserved by a non-dilatonic parallelisable supergravity background is determined by the dilatino variation and is measured by the dimension of the kernel of the operation of Clifford multiplication by the torsion three-form. Because of the bi-invariance of the metric, this is a condition which can be analysed at the identity, whence at the level of the Lie algebra.

Let \(\mathfrak{g}\) be one of the above ten-dimensional lorentzian Lie algebras. We will let \([-,-]\) and \(\langle-,-\rangle\) denote the Lie bracket and the invariant metric. Let \(e_a\) be a pseudo-orthonormal frame and define \(H_{abc} := \langle[e_a,e_b],e_c\rangle\). Let \(\Gamma_a\) be the corresponding basis for the Clifford algebra \(\text{Cl}(1,9)\). As a real associative algebra, \(\text{Cl}(1,9) \cong \text{Mat}(32,\mathbb{R})\) whence there is a unique irreducible Clifford module \(S\): real and of dimension 32. Under the spin group \(\text{Spin}(1,9)\), \(S\) breaks up into \(S_+ \oplus S_-\) according to chirality. The relevant spinor representation in type IIA is the module \(S\) itself, whereas in type IIB it is the complexification \(S_+ \otimes \mathbb{C}\).

We are interested in the kernel of the Clifford endomorphism \(H = \frac{1}{6}H_{abc}\Gamma^{abc}\) acting on the relevant module. Notice that \(H\) exchanges chirality, whence \(H : S_+ \rightarrow S_+\). Let \(H_\pm\) denote the restriction of \(H\) to \(S_\pm\). We will denote by \(H_A\) and \(H_B\) the relevant Clifford endomorphism in type IIA and type IIB, respectively. Then notice that \(\dim \ker H_A = \dim \ker H_+ + \dim \ker H_-\), whereas \(\dim \ker H_B = 2(\dim \ker H_+ + \dim \ker H_-)\), whence for type IIB, there is an even number of supersymmetries preserved. It is now a simple matter to scan the Lie algebras in Table 2 and examine the endomorphism \(H\) for each one.

Before doing so, it is convenient to write each of the different types of Lie algebras which appear below relative to a pseudo-orthonormal frame and compute the Clifford endomorphism \(H\) explicitly. In some cases this is easily done in an explicit representation.

For example, for \(\mathfrak{so}(1,2)\) we can choose as basis \(X_0 = i\sigma_3, X_1 = \sigma_1\) and \(X_2 = \sigma_2\), where the \(\sigma_i\) are the (hermitian) Pauli matrices. The Lie bracket is

\[
[X_0, X_1] = -2X_2 \quad [X_1, X_2] = 2X_0 \quad [X_0, X_2] = 2X_1
\]

and hence \(H_{012} = \langle [X_0, X_1], X_2 \rangle = -2\), whence the Clifford endomorphism is

\[
H = -2\Gamma^{012}.
\]
Now this corresponds to AdS$_3$ with “unit” radius of curvature. If we want to consider AdS$_3$ with radius of curvature $\tilde{R}$ then we have to rescale the metric $\langle -, - \rangle$ by $\tilde{R}^2$, and the $\Gamma^a$ matrices by $\tilde{R}^{-1}$, whence the Clifford endomorphism becomes

$$H = -2\tilde{R}^{-1}\Gamma^{012}.$$  

Similarly, for $\mathfrak{su}(2)$, we can take as basis $X_1 = i\sigma_1$, $X_2 = i\sigma_2$ and $X_3 = i\sigma_3$. The Lie bracket is

$$[X_1, X_2] = -2X_3 \quad [X_2, X_1] = -2X_1 \quad [X_3, X_1] = 2X_2$$

and hence $H_{123} = \langle [X_1, X_2], X_3 \rangle = -2$, whence the Clifford endomorphism, once we introduce the $S^3$ radius $R$, is

$$H = -2R^{-1}\Gamma^{123}.$$  

Finally, we discuss the double extensions associated to CW$_{2n+2}(\lambda)$. Relative to a lightcone basis $X_1, X_+, X_-$, with $\langle X_i, X_j \rangle = \delta_{ij}$ and $\langle X_+, X_- \rangle = 1$, the Lie bracket is

$$[X_-, X_i] = J_{ij}X_j \quad [X_1, X_3] = J_{ij}X_+,$$

whence the only nonzero component of $H_{abc}$ is $H_{ij-} = \langle [X_i, X_j], X_- \rangle = J_{ij}$, whence the Clifford endomorphism is given by

$$H = J_{ij}\Gamma^i\Gamma^j.$$  

We now have all the necessary ingredients to compute the dimension of the kernel of the Clifford endomorphism $H$ in each of the parallelisable supergravity backgrounds classified in the previous section and hence determine the amount of supersymmetry which they preserve.

5.1. AdS$_3 \times S^3 \times S^3 \times \mathbb{R}$. Introducing radii $R_1$, $R_2$ and $R_3$ for the AdS$_3$, and the two $S^3$-spheres respectively, the Clifford endomorphism is (up to an overall scale) given by

$$H = R_1^{-1}\Gamma^{012} + R_2^{-1}\Gamma^{345} + R_3^{-1}\Gamma^{678}.$$  

This endomorphism is invertible unless $R_1^{-2} = R_2^{-2} + R_3^{-2}$, which is precisely the consistency condition for a constant dilaton. In this case the kernel is sixteen-dimensional. The generator $\Gamma_9$ anticommutes with $H$ and hence preserves the kernel of $H$. Since $\Gamma_9$ is invertible and exchanges chirality, we see that $\dim \ker H_+ = \dim \ker H_- = 8$. Therefore $\dim \ker H_A = \dim \ker H_B = 16$ and this is a half-BPS background for both type IIA and type IIB supergravity. This is in agreement with the supergravity results of [12, 6, 24] and from conformal field theory in [15].

5.2. AdS$_3 \times S^3 \times \mathbb{R}^4$. This is the limit $R_3 \to \infty$ of the previous example. Therefore $R_1 = R_2$ and the background preserves 16 supersymmetries both for type IIA and IIB [29].

5.3. CW$_{10}(\lambda)$. The Clifford endomorphism in this case takes the form

$$H = (\lambda_1\Gamma^{12} + \lambda_2\Gamma^{34} + \lambda_3\Gamma^{56} + \lambda_4\Gamma^{78})\Gamma_+,$$

where we have reintroduced the scale $\lambda_4$ for convenience. Because of the $\Gamma_+$, such endomorphism has kernel and in fact, the dimension of the kernel is at least 16, both for type IIA and type IIB. The kernel may be larger, however, depending on whether the endomorphism

$$J = \lambda_1\Gamma^{12} + \lambda_2\Gamma^{34} + \lambda_3\Gamma^{56} + \lambda_4\Gamma^{78}$$

has any kernel in the subspace $\ker \Gamma_-$ of the relevant Clifford module.

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1 This is not a coincidence. If the Clifford endomorphism $H$ has kernel, so does its square. Using the Jacobi identity, $H^2 = ||H||^2$, whence supersymmetry implies that $||H||^2 = 0$ or, equivalently, that the scalar curvature vanishes.
This question can be analysed group-theoretically (see, for example, \cite{11} Section 2.2 for a more detailed analysis in a related problem) by interpreting $J$ as an element of the Cartan subalgebra of $\mathfrak{so}(8) \subset \mathfrak{so}(1, 9)$ acting on the 16-dimensional representation $\ker \Gamma_-$ (for type IIA) or on the complexification of the 8-dimensional representation $\ker \Gamma_- \cap S_+$ (for type IIB).

In terms of $\mathfrak{so}(8)$ irreducible representations, $\Delta := \ker \Gamma_- = \Delta_+ \oplus \Delta_-$, where $\Delta_\pm$ are the half-spin representations: $\mathbf{8}$ and $\mathbf{\bar{8}}$, respectively. The weights of $\Delta$ with respect to the above basis for the Cartan subalgebra of $\mathfrak{so}(8)$ are $(\pm 1, \pm 1, \pm 1, \pm 1)$ where the signs are uncorrelated, for a total of $2^4 = 16$ weights. The weights for which the products of the signs is $\pm 1$ correspond to $\Delta_\pm$.

Let us first of all consider the case of type IIA. The action of $J$ on the representation $\Delta$ is diagonal with eigenvalues $\pm \lambda_1 \pm \lambda_2 \pm \lambda_3 \pm \lambda_4$. Setting each of these expressions to zero gives rise to 8 hyperplanes in the four-dimensional parameter space of the $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. For $\lambda$ away from such hyperplanes, the solution preserves no extra supersymmetry. However for $\lambda$ in the union of the hyperplanes, the solution preserves extra supersymmetry—how much depending on how many of these hyperplanes $\lambda$ belongs.

If $\lambda$ belongs to one and only one hyperplane, then there are two extra supersymmetries, for a total of 18. If $\lambda$ belongs to the intersection of two (but not more) hyperplanes and that, as should be the case for CW$_{10}$, no $\lambda_i$ should vanish, then there are two more for a total of 20. If $\lambda$ belongs to the intersection of three (but not more) hyperplanes and again with no $\lambda_i$ vanishing, there are two more zero eigenvalues for a total of 22. There are no $\lambda$ in the intersection of four hyperplanes with all $\lambda_i$ nonzero.

Let us now consider type IIB. We must restrict ourselves to the four hyperplanes for which the products of the signs is positive and count them with multiplicity four. If $\lambda$ lies in precisely one of the hyperplanes, then there are an additional 4 supersymmetries for a total of 20. If $\lambda$ lies in the intersection of precisely two hyperplanes, but again with no $\lambda_i$ vanishing, then there are an additional 4 for a total of 24. If $\lambda$ lies in the intersection of precisely three hyperplanes (and again no $\lambda_i$ vanishing) then there are an additional 4 for a total of 28 supersymmetries. There are no nonzero $\lambda$ in the intersection of all four hyperplanes.

5.4. CW$_8(\lambda) \times \mathbb{R}^2$. This corresponds to $\lambda_4 = 0$ in the previous case, say. There is no distinction here between type IIA and type IIB, because $\Gamma^8$, say, exchanges chirality and anticommutes with $H$.

Generically there will not be further supersymmetries; but if $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ lies in the union of the four hyperplanes $\lambda_1 \pm \lambda_2 \pm \lambda_3 = 0$ (uncorrelated signs) then there will be supersymmetry enhancement. If $\lambda$ lies in one and only one hyperplane there will be an extra 4 supersymmetries, for a total of 20. Any $\lambda$ in the intersection of any two of these hyperplanes automatically has some $\lambda_i = 0$, whence it does not correspond to this background.

5.5. CW$_6(\lambda) \times \mathbb{R}^4$. This case corresponds to putting $\lambda_3 = 0$ in the above case. Now for generic $\lambda = (\lambda_1, \lambda_2)$ there are no extra supersymmetries, but for $\lambda$ in the union of the two hyperplanes $\lambda_1 \pm \lambda_2 = 0$ there is supersymmetry enhancement. The only allowed possibility (since $\lambda \neq 0$) is that it lies in precisely one of hyperplanes. In this case there are 8 extra supersymmetries for a total of 24.

5.6. CW$_4(\lambda) \times \mathbb{R}^6$. The Clifford endomorphism is

$$H = \lambda_1 \Gamma^{12} \Gamma_+,$$

which clearly has no extra supersymmetries than those in the kernel of $\Gamma_+$. Hence this backgrounds preserves 16 supersymmetries.
This is of course the flat vacuum solution which preserves all supersymmetries.

6. Summary

In summary, we have classified (up to local isometry) all the non-dilatonic parallelisable NS-NS backgrounds of ten-dimensional type II supergravity. Parallelisability implies that the geometry is that of one of the parallelised lorentzian Lie groups listed in Table 1 and demanding that a constant dilaton obeys its equation of motion reduces the possibilities further to those in Table 2. We have moreover shown that they preserve 16, 18, 20, 22, 24 or 32 supersymmetries for type IIA, and 16, 20, 24, 28 or 32 for type IIB. The cases with geometry \( CW_{2n} \times E^{10-2n} \) are symmetric plane waves and our results agree with those in [35]. Since symmetric plane waves are in particular homogeneous, all the plane waves in the tables are contained in the classification of homogeneous plane waves of Blau and O’Loughlin [5]. Some of the plane waves in the tables have also appeared in [31, 13, 23, 3].

Being lorentzian Lie groups admitting a bi-invariant metric and having no other fields turned on but the NS-NS three-form, string propagation on these parallelisable backgrounds is described by a WZW model and hence amenable to standard techniques in conformal field theory. In particular it is possible to determine the symmetric D-branes for all the backgrounds in Table 2 using the techniques of [1, 16, 37] and indeed for many of these backgrounds this has already been done [36, 22, 21, 2, 38].

The spacetimes in Table 2 are related by three types of limits: large radius limits which in essence flatten different factors in the metric, degenerations of the Cahen–Wallach metrics by taking some of the eigenvalues \( \lambda_i \) to zero, and Penrose–Güven limits [34, 26].

As was explained in [38] for \( \text{AdS}_3 \times S^3 \rightarrow CW_4(\lambda) \), but the idea clearly generalises, the Penrose limits can be understood as group contractions. Indeed, suppose that \( \gamma \subset G \) is a null geodesic. It is determined uniquely by its initial point \( \gamma(0) \) and its initial direction \( \dot{\gamma}(0) \) is the celestial sphere at \( T_{\gamma(0)}G \). From the covariance property of [4, Section 2.4] we can apply an isometry to \( \gamma \) without changing the Penrose limit (up to isometry). Using left-translations, say, we can take \( \gamma(0) \) to be the identity. Then \( \dot{\gamma}(0) \) is a null vector in the Lie algebra, which generates a one-parameter subgroup \( H \subset G \). Since the metric on \( G \) is bi-invariant, one-parameter subgroups are geodesics, whence \( \gamma = H \). Then as shown for a particular example in [35] (see also [32]) the Penrose limit along \( H \) is the Inönü–Wigner contraction of \( G \) along \( H \).

It is also possible to argue in what superficially appears to be more generality, that parallelisability is a hereditary property of the Penrose limit. This is because the condition of parallelisability can be phrased in terms of the existence of parallel sections in a bundle with connection. As discussed in [4], extending the results in [24], parallel sections are preserved in the Penrose limit, hence if there exists a parallel frame before the limit there continues to be one afterwards. There is no need for such a general argument, though, as the results in this paper and in [7] allows us to limit ourselves to Lie groups with bi-invariant metrics, a class of spaces preserved by group contractions.

Finally let me remark that to complete the classification of parallelisable NS-NS backgrounds there remains to study the possibility of turning on the dilaton. As explained briefly in the introduction, there are two conditions to impose: first the equation of motion of the dilaton itself and secondly the gradient is constrained to lie along central directions of the Lie algebra. Given the explicit structure of the Lie algebra it is only a matter of patience to determine the possibilities.
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