Perturbative connection formulas for Heun equations

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Received 31 August 2022
Accepted for publication 19 October 2022
Published 3 November 2022

Abstract
Connection formulas relating Frobenius solutions of linear ODEs at different Fuchsian singular points can be expressed in terms of the large order asymptotics of the corresponding power series. We demonstrate that for the usual, confluent and reduced confluent Heun equation, the series expansion of the relevant asymptotic amplitude in a suitable parameter can be systematically computed to arbitrary order. This allows to check a recent conjecture of Bonelli-Iossa-Panea Lichtig-Tanzini expressing the Heun connection matrix in terms of quasiclassical Virasoro conformal blocks.

Keywords: Heun equation, connection problem, conformal field theory

(Some figures may appear in colour only in the online journal)

1. Introduction

Heun’s differential equation was introduced in 1889 [Heun] as the general linear 2nd order ODE with four Fuchsian singular points. Being the simplest generalization of the Gauss hypergeometric equation, it is at the same time highly nontrivial because of a qualitatively novel feature—the presence of accessory parameters. A long list of applications of Heun’s equation and its confluent versions includes such diverse topics as e.g. the theory of conformal mappings, Painlevé functions, black hole scattering and quantum optics; see [Ron, Hor] for an extensive bibliography.

It was discovered by Zamolodchikov [Z86] that Heun accessory parameter function is directly related to the quasiclassical 4-point Virasoro conformal block. The Heun equation

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appears in this setting \cite{T10, LLNZ} as a limit of the BPZ decoupling equations \cite{BPZ} for conformal blocks with degenerate insertions. Under the AGT correspondence \cite{AGT}, the quasi-classical limit of Liouville CFT corresponds to the Nekrasov-Shatashvili limit \cite{NS} of the corresponding 4D $\mathcal{N} = 2$ supersymmetric SU(2) gauge theories. This has brought a new surge of interest in the study of equations of Heun type and their applications in the past decade, see e.g. \cite{AGH, BGG, CC21, CN, LeNo, LiNa, NC, PP14, PP17}.

A remarkable further step was recently made in \cite{BIPT21, BIPT22} where it was realized that (a class of) connection problems for Heun equations are also solvable in terms of quasi-classical conformal blocks. The main idea behind this statement is the calculation of quantum (operator-valued) monodromy of conformal blocks with respect to positions of degenerate fields using the elementary fusion and braiding transformations\footnote{The approach of loc. cit. uses in addition the DOZZ formula for Liouville structure constants. However, it is not strictly necessary: the locality of the elementary fusion/braiding operations is already sufficient to arrive to the same result, as discussed in \cite{Lis, CFMP} and section 2 of the present manuscript.}. The procedure is closely related to the study of Verlinde loop operators in \cite{AGGTV} and it has already been used in \cite{ILT} to express solutions of Fuchsian systems and associated isomonodromic tau functions via $c=1$ conformal blocks. The Heun connection problem similarly emerges in the analysis of the $c \to \infty$ limit of quantum monodromy. In practical terms, the relation to CFT allows to compute explicitly a perturbative expansion of the Heun connection matrix in a suitable parameter that can be efficiently used in applications \cite{CFMP, DGILZ}.

The aim of the present work is to show that equivalent series expansions of the Heun connection coefficients can be derived in a rigorous way, without relying on CFT intuition. First terms of these expansions and their counterparts for Mathieu equation can indeed be computed in a rigorous way, without relying on CFT intuition. First terms of these expansions and their counterparts for Mathieu equation can indeed be computed in a rigorous way, without relying on CFT intuition.

Our main results are summarized as follows.

**Theorem 1.** Let us write the normal form of the reduced confluent Heun equation as

\[
\begin{aligned}
\frac{d^2}{dz^2} + \frac{1}{2} \frac{1 - \theta_0^2}{z^2} + \frac{1}{4} \frac{1 - \theta_1^2}{(z - 1)^2} + \frac{\theta_0^2 + \theta_1^2 - \omega^2 - \frac{1}{2} - \lambda}{z(z - 1)} \psi(z) = 0,
\end{aligned}
\]

with $\theta_0, \theta_1 \not\in \mathbb{Z}/2$. Denote by $\psi_0^\pm(z), \psi_1^\pm(z)$ its Frobenius solutions at $z = 0, 1$ normalized as

\[
\psi_0^\pm(z) = z^{\frac{1}{2} \mp \theta_0} \left[ 1 + \sum_{k=1}^{\infty} \psi_{0,k}^\pm z^k \right] \quad \text{as } z \to 0^+, \quad (1.2a)
\]

\[
\psi_1^\pm(z) = (1 - z)^{\frac{1}{2} \mp \theta_1} \left[ 1 + \sum_{k=1}^{\infty} \psi_{1,k}^\pm (z - 1)^k \right] \quad \text{as } z \to 1^-.
\]

The connection between the two Frobenius bases is given by

\[
\psi_\epsilon^0(z) = \sum_{\epsilon'=\pm} \mathbb{C}(\epsilon \theta_0, \epsilon' \theta_1) \psi_{\epsilon'}^1(z), \quad \epsilon = \pm,
\]

\[
(1.3)
\]
where the function $C(\theta_0, \theta_1)$ admits the following representation in terms of continued fractions:

$$C(\theta_0, \theta_1) = \frac{\Gamma(1 - 2\theta_0) \Gamma(2\theta_1)}{\Gamma(\frac{1}{2} + \theta_1 - \theta_0 + \omega) \Gamma(\frac{1}{2} + \theta_1 - \theta_0 - \omega)} \times \exp \sum_{k=1}^{\infty} \ln \left(1 - \frac{\lambda \beta_k}{1 - \lambda \beta_k} \right), \quad (1.4)$$

with

$$\beta_k = \frac{k(k - 2\theta_0)}{(k + \frac{1}{2} - \theta_0 + \theta_1)^2 - \omega^2} \left((k - \frac{1}{2} - \theta_0 + \theta_1)^2 - \omega^2\right)^{-1}. \quad (1.5)$$

**Theorem 2.** Consider the normal form of the Heun equation,

$$\left\{ \begin{array}{l}
\frac{d^2}{dz^2} + \frac{\frac{1}{2} - \theta_0^2}{z^2} + \frac{1}{2} - \frac{\theta_1^2}{(z - 1)^2} + \frac{1}{2} - \frac{\theta_1^2}{(z - 1)^2} + \theta_0^2 + \theta_1^2 + \theta_2^2 - \frac{1}{z(z - 1)} \\
+ \frac{(r - 1) (\omega^2 + \theta_1^2 - \theta_2^2 - \frac{1}{z(z - 1)} \psi(z) = 0, \quad (1.6)
\end{array} \right.$$

and assume that $|r| > 1$ and $\theta_0, \theta_1 \notin \mathbb{Z}/2$. Let $\psi_0(z), \psi_1(z)$ denote its Frobenius solutions normalized as in (1.2). The relation between the two bases is given by (1.3), where the function $C(\theta_0, \theta_1)$ can be written as

$$C(\theta_0, \theta_1) = \frac{\Gamma(1 - 2\theta_0) \Gamma(2\theta_1) (1 - \lambda) - \frac{1}{2} - \theta_0}{\Gamma(\frac{1}{2} + \theta_1 - \theta_0 + \omega) \Gamma(\frac{1}{2} + \theta_1 - \theta_0 - \omega)} \times \exp \sum_{k=1}^{\infty} \ln \left(1 - \lambda \alpha_{k-1} - \frac{\lambda \beta_k}{1 - \lambda \alpha_k} - \frac{\lambda \beta_k}{1 - \lambda \beta_k} \right), \quad (1.7)$$

with $\lambda = \frac{1}{2}$ and

$$\alpha_k = -\frac{(k + \frac{1}{2} - \theta_0 - \theta_1)^2 - \theta_0^2 - \theta_2^2 + \omega^2}{(k + \frac{1}{2} - \theta_0 + \theta_1)^2 - \omega^2}, \quad (1.8a)$$

$$\beta_k = \frac{k(k - 2\theta_0)(k - \theta_0 + \theta_1 - \theta_2)}{(k + \frac{1}{2} - \theta_0 + \theta_1)^2 - \omega^2} \left((k - \frac{1}{2} - \theta_0 + \theta_1)^2 - \omega^2\right). \quad (1.8b)$$

Explicit perturbative solution of the connection problem for small $\lambda$ is obtained by truncating the infinite fractions in (1.4) and (1.7) at the desired order, expanding the result in powers of $\lambda$ and algorithmically computing the resulting infinite sums in terms of polygamma functions. The results match the coefficients of the perturbative series predicted by CFT considerations from [BIPT21, BIPT22].

The paper is organized as follows. In section 2, we recall the arguments leading to the conjectural solution (2.20) of the Heun connection problem in terms of quasiclassical conformal blocks. Section 3 discusses the Schäfke-Schmidt connection formula (corollary 3.2). The proofs of theorems 1 and 2 as well as the corresponding result for the confluent Heun equation are given in section 4. We conclude with a brief discussion of open questions.
2. CFT derivation of Trieste formula

2.1. Fusion transformations of conformal blocks

Let us choose \( n \geq 3 \) distinct points \( t = \{t_k\}_{k=0,\ldots,n-1} \) on the Riemann sphere \( \mathbb{C}P^1 \). We fix from the outset \( t_0 = 0, t_1 = 1, t_{n-1} = \infty \) and assume that \( |t_1| < |t_2| < \ldots < |t_{n-2}| \). Spherical Virasoro conformal block \( \mathcal{F}(t) \) is a multivariate series that can be assigned to any trivalent tree with \( n \) external vertices labeled by \( t \) whose every edge \( e \) is equipped with a weight \( \Delta_e \in \mathbb{C} \). For example, the tree

\[
\begin{array}{c}
\infty \\
\Delta_{n-1} \\
\vdots \\
\Delta_3 \\
\Delta_2 \\
\Delta_1 \\
0
\end{array}
\]

is represented by a series of the form

\[
\mathcal{F}(t) = \mathcal{F}_{3pt}(t) \mathcal{F}(t), \quad \mathcal{F}_{3pt}(t) = \prod_{\ell=1}^{n-2} \frac{\tilde{\Delta}_\ell - \tilde{\Delta}_{\ell-1} - \Delta_\ell}{t_\ell},
\]

\[
\mathcal{F}(t) = \sum_{k \in \mathbb{N}^{n-1}} \mathcal{F}_k \left( \frac{t_1}{t_2}, \frac{t_2}{t_3}, \ldots, \frac{t_{n-2}}{t_{n-1}} \right),
\]

where \( \tilde{\Delta}_0 = \Delta_0, \tilde{\Delta}_{n-2} = \Delta_{n-1} \). The coefficients \( \mathcal{F}_k \) are rational functions of the conformal weights \( \Delta, \tilde{\Delta} \) and the Virasoro central charge \( c \). They are uniquely determined by the Virasoro commutation relations and normalization \( \mathcal{F}_0 = 1 \). It will be convenient for us to parameterize the central charge and conformal weights as

\[
c = 1 + 6 \left( b + b^{-1} \right)^2, \quad \Delta = \frac{1}{2} \left( b + b^{-1} \right)^2 - p^2.
\]

Conformal block series are believed to be convergent and analytically continuable to the universal cover of the configuration space \( \text{Conf}_n(\mathbb{C}P^1) \). Different trees with the same external dimensions give rise to conformal blocks related by sequences of elementary fusion and braiding transformations.

Of special interest for us are conformal blocks involving an additional degenerate field \( \Phi_{(1,2)}(z) \). Such a field has Liouville momentum \( p_{(1,2)} = -b - b^{-1}/2 \) and will be represented by a dashed line in the diagrams below. Two important properties of degenerate fields are

- **Simple fusion rules**: the OPE of \( \Phi_{(1,2)} \) with a generic Virasoro primary with momentum \( p \) contains only two conformal families with momenta \( p_{\pm} = p \pm \frac{b}{2} \).
- **BPZ decoupling equations**: these are linear homogeneous PDEs satisfied by the relevant conformal blocks. For \( t_0, t_1, t_{n-1} \) fixed at 0, 1 and \( \infty \), they have the form

\[
\mathcal{D}_{\text{BPZ}} = \frac{1}{b^2} \frac{\partial^2}{\partial z^2} - \left( \frac{1}{z} + \frac{1}{z-1} \right) \frac{\partial}{\partial z} + \sum_{k=2}^{n-2} \frac{t_k (t_k - 1)}{z(z-1)(z-t_k)} \frac{\partial}{\partial t_k} + \sum_{k=0}^{n-2} \frac{\Delta_k}{(z-t_k)} + \frac{\Delta_{n-1} - \Delta_{(1,2)} - \sum_{k=0}^{n-2} \Delta_k}{z(z-1)}.
\]

(2.3)
In the case of three generic primaries and one degenerate field, the BPZ equation becomes an ODE equivalent to Gauss hypergeometric equation. The corresponding 3+1 point conformal blocks provide bases of its Frobenius solutions at different Fuchsian singular points 0, 1, \( \infty \), depending on the choice of the generic edge the degenerate field is attached to. In particular,

\[
\mathcal{F}_{p_0, p_1, p_\infty}(z) = \mathcal{F}_{p, p_\infty}(1 - z),
\]

where

\[
\mathcal{F}_{p_0, p_1, p_\infty}(z) = z^{\frac{1 + p_0^2}{8} + b p_0} (1 - z)^{\frac{1 + p_1^2}{8} + b p_1} \times {}_2F_1 \left[ \begin{array}{c}
\frac{1}{2} + b (p_1 + p_\infty + p_0), \\
\frac{1}{2} + b (p_1 - p_\infty + p_0)
\end{array}; 1 + 2 b p_0 \right].
\]

The classical connection formulas for \( {}_2F_1 \) hypergeometric functions can be interpreted as the fusion transformation for 3+1 point conformal blocks,

\[
\sum_{\epsilon'} F_{\epsilon \epsilon'} (p_0, p_1, p_\infty) = \sum_{\epsilon'} F_{\epsilon' \epsilon} (p_0, p_1, p_\infty), \quad \epsilon, \epsilon' = \pm.
\]

The elements of the fusion matrix \( F_{\epsilon \epsilon'} (p_0, p_1, p_\infty) = F (\epsilon p_0, \epsilon' p_1, p_\infty) \) do not depend on \( z \) and are explicitly expressed in terms of gamma functions:

\[
F (p_0, p_1, p_\infty) = \frac{\Gamma (1 - 2 b p_0) \Gamma (2 b p_1)}{\Gamma \left( \frac{1}{2} + b (p_1 + p_\infty + p_0) \right) \Gamma \left( \frac{1}{2} + b (p_1 - p_\infty + p_0) \right)}.
\]

Locality of the fusion transformations implies that

\[
\sum_{\epsilon'} F_{\epsilon' \epsilon} (p_0, p_1, p_\infty) = \sum_{\epsilon'} F_{\epsilon' \epsilon} (p_0, p_1, p_\infty), \quad \epsilon, \epsilon' = \pm.
\]

with the same fusion matrix \( F \). The box denotes any fixed admissible tree with momenta assigned to edges. This exact fusion relation for multipoint conformal blocks with degenerate fields will play a crucial role below. Let us also record the normalizations
which are straightforward to obtain from the leading order of the OPE of the degenerate field with the relevant primary. In the above, it is implicitly assumed that $z \in (0, 1)$. Fractional powers for generic complex values are defined by analytic continuation.

2.2. Quasiclassical limit

Consider the limit where all external and internal momenta and the Virasoro central charge are sent to infinity according to

$$b \to 0, \quad p_k \to 0, \quad bp_k \to \theta_k \quad \forall k.$$  \hfill(2.10)

A famous conjecture of Zamolodchikov \cite{Z86} states that the corresponding behavior of conformal blocks is given by

$$\ln F\left\{\left\{b^{-1}\theta_k\right\}; t\right\} = b^{-2} \mathcal{W}\left(\left\{\theta_k\right\}; t\right) + O(1) \quad \text{as } b \to 0. \quad (2.11)$$

This is a statement about the existence of the $b \to 0$ limit of the coefficients of the multivariate formal series $b^2 \ln F\left\{\left\{b^{-1}\theta_k\right\}; t\right\}$. The corresponding multivariate formal series $\mathcal{W}\left(\left\{\theta_k\right\}; t\right)$ is called quasiclassical conformal block. Global analytic structure of $\mathcal{W}\left(\left\{\theta_k\right\}; t\right)$ lacks even a conjectural description already in the simplest 4-point case.

The quasiclassical conformal blocks are conjecturally related to accessory parameters of ODEs of Heun type \cite{Z86, T10, LLNZ}. The link may be established by considering an additional degenerate field $\Phi(z)$. In the quasiclassical limit, its conformal dimension remains finite: $\Delta_{(1, 2)} = -\frac{1}{2}$ as $b \to 0$. Conformal block with such an extra insertion is expected to behave in the quasiclassical limit as

$$\Psi_{\pm}(z) = \Psi(z; t) \exp\{b^{-2} \mathcal{W}(t)\} \left[1 + o(1)\right] \quad \text{as } b \to 0. \quad (2.12)$$

Substituting this asymptotics into BPZ equation, the latter transforms into an ODE,

$$\left[\frac{d^2}{dz^2} + \frac{\delta_k}{(z-t_k)^2} + \frac{\delta_{n-1} - \sum_{k=0}^{n-2} \delta_k}{z(z-1)} + \frac{t_k - 1}{z(z-1)(z-t_k)}\right] \Psi_{\pm}(z) = 0, \quad (2.13)$$

with

$$\delta_k = \frac{1}{4} - \theta_k^2, \quad k = 0, \ldots, n - 1, \quad (2.14a)$$
\[ \delta_k = t_k \frac{\partial W}{\partial t_k}, \quad k = 2, \ldots, n - 2. \]  

Equation (2.13) is the normal form of the most general linear 2nd order ODE with \( n \) Fuchsian singularities. Rescaled external momenta are directly related to local monodromy exponents \( \frac{1}{2} \pm \delta_k \) at the singular points; \( n - 3 \) rescaled internal momenta such as \( \sigma = bp_\sigma \) encode exponents of composite monodromy and parameterize accessory parameters \( \delta_2, \ldots, \delta_{n-2} \).

### 2.3. Trieste connection formula

The structure of the operator product expansions encoded in the conformal block diagrams implies that the amplitudes \( \Psi_{\pm}(z) \) have \( z \to 0 \) expansions of the form

\[ \Psi_{\pm}(z) = \mathcal{N}_{\pm} z^{1 \pm \theta_0} \left[ 1 + \sum_{k=1}^{\infty} \psi_{\pm}^{(k)} z^k \right]. \]  

Therefore, these amplitudes give a basis of Frobenius solutions of (2.13). The normalization coefficients \( \mathcal{N}_{\pm} \) can be determined from the leading OPE term in (2.9a):

\[ \mathcal{N}_c = \lim_{b \to 0} \exp \left\{ b^{-2} W(t) \right\} = \mathcal{N} \exp \left\{ \frac{\epsilon}{2} \frac{\partial W}{\partial \theta_1} \right\}, \quad \epsilon = \pm. \]  

Here \( \mathcal{N} \) denotes the limit

\[ \mathcal{N} = \lim_{b \to 0} \exp \left\{ b^{-2} W(t) \right\}, \]  

which is related to the \( O(1) \) correction in (2.11). We will not attempt to derive a more explicit expression of \( \mathcal{N} \) since this prefactor is inessential for our purposes.

Indeed, we could repeat the same analysis for the quasiclassical limit of conformal blocks appearing in the right hand side of (2.8). The corresponding amplitudes give a basis of Frobenius solutions of (2.13) at the singular point \( z = 1 \). The analogs of the normalization prefactors (2.16) for these solutions are \( \mathcal{N} \exp \left\{ \frac{\epsilon}{2} \frac{\partial W}{\partial \theta_1} \right\} \) with the same \( \mathcal{N} \) defined by (2.17). Given the exact fusion relation (2.8), the only missing ingredient needed to derive the connection formula between the two bases is the quasiclassical limit of the fusion matrix (2.7). It is obviously given by

\[ F_{cl}(\theta_0, \theta_1, \theta_\infty) = \frac{\Gamma(1 - 2\theta_1) \Gamma(2\theta_1)}{\Gamma\left(\frac{1}{2} + \theta_1 - \theta_0 + \theta_\infty\right) \Gamma\left(\frac{1}{2} + \theta_1 - \theta_0 - \theta_\infty\right)}, \]  

which is related to the \( O(1) \) correction in (2.11). We will not attempt to derive a more explicit expression of \( \mathcal{N} \) since this prefactor is inessential for our purposes.
This yields the following statement. Let us denote by $\psi_{\pm}^{[0]}(z)$, $\psi_{\pm}^{[1]}(z)$ two pairs of normalized Frobenius solutions of the generalized Heun equation (2.13) at the points $z = 0$ and $z = 1$:

$$
\psi_{\pm}^{[0]}(z) = z^{1/2} \left[ 1 + \sum_{k=1}^{\infty} \psi_{\pm,k}^{[0]} z^k \right],
$$

$$
\psi_{\pm}^{[1]}(z) = (1-z)^{1/2} \left[ 1 + \sum_{k=1}^{\infty} \psi_{\pm,k}^{[1]} (z-1)^k \right].
$$

(2.19)

The connection between the two bases is given by

$$
\psi_{\pm}^{[0]}(z) = \sum_{\epsilon,\epsilon'} C(\epsilon \theta_0, \epsilon' \theta_1, \sigma) \psi_{\pm}^{[1]}(z), \quad \epsilon, \epsilon' = \pm,
$$

(2.20a)

$$
C(\theta_0, \theta_1, \sigma) = F_{cl}(\theta_0, \theta_1, \sigma) \exp \left[ \frac{1}{2} \left( \frac{\partial \mathcal{W}}{\partial \theta_1} - \frac{\partial \mathcal{W}}{\partial \theta_0} \right) \right]
$$

(2.20b)

An equivalent result was first obtained in the 4-point case (the usual Heun equation) in [BIPT22], cf equations (4.1.16) and (4.1.17) therein. We will refer to it as the Trieste formula.

The dressing of the quasiclassical fusion matrix $F_{cl}$ by conformal block derivatives $\frac{\partial \mathcal{W}}{\partial \theta_0}, \frac{\partial \mathcal{W}}{\partial \theta_1}$ is a consequence of the shifts of the external momenta by $\pm \frac{\lambda}{2}$ in the leading OPE terms in (2.9).

Connection matrices between Frobenius solutions at $z = 0$ and other singular points $t_2, \ldots, t_{n-1}$ can be computed in an analogous way using along with fusion the elementary braiding transformations. In general, the transformed conformal blocks also involve shifts of the internal momenta by half-integer multiples of $b$, which in the quasiclassical limit produce derivatives with respect to rescaled internal momenta.

The monodromy of solutions of (2.13) can be written in terms of the connection matrices. A class of monodromy invariants has a particularly simple expression which does not involve conformal block functions explicitly. For example, the spectrum of the monodromy matrix $M_0M_1$ around two singular points $z = 0, 1$ is given by $\text{Sp}M_0M_1 = \{-e^{\pm 2\pi i \sigma}\}$. In a similar fashion, the other rescaled internal momenta are related to exponents of composite monodromy along the cycles encoded in the conformal block diagram interpreted as a pants decomposition of $\mathbb{CP}^1 \setminus \mathbb{t}$.

Under minimal adjustments, the above argument can also be carried out for irregular conformal blocks leading to connection formulas for a number of confluent Heun equations [BIPT21, BIPT22]. We also note a recent work [JN] where results related to (2.20) were derived on the gauge theory side of the AGT correspondence (cf e.g. formulas of section 6.1 therein).

2.4. Practical implementation for Heun equation

An important subtlety with the application of Trieste formula is that the connection coefficients are expressed in terms of monodromy invariants (such as $\sigma$) instead of parameters $\{f_{\pm}\}$ of the generalized Heun equation. The connection between the two is given by the equation (2.14b) that has to be solved perturbatively.

For reader’s convenience, let us outline the procedure in the case of the usual (4-point) Heun equation written in the normal form
It then becomes clear that the coefficients for any $t$, $\delta \theta^2$ for $k=0,1,t,\infty$ and $1 < |t| < \infty$. The perturbative expansion of the quasiclassical conformal block $\mathcal{W}(t)$ in $1/t$ has the form
\begin{equation}
\mathcal{W}(t) = (\delta_\infty - \delta_\sigma - \delta_t) \ln t + \sum_{k=1}^\infty \mathcal{W}_k t^{-k}.
\end{equation}

Here $\delta_\sigma = \frac{1}{4} - \sigma^2$ and the first coefficients are given by
\begin{equation}
\begin{align}
\mathcal{W}_1 &= \frac{(\delta_\sigma - \delta_0 + \delta_1)(\delta_\sigma - \delta_\infty + \delta_t)}{2\delta_\sigma}, \\
\mathcal{W}_2 &= \frac{(\delta_\sigma - \delta_0 + \delta_1)^2(\delta_\sigma - \delta_\infty + \delta_t)^2}{8\delta^3_\sigma} \left( \frac{1}{\delta_\sigma - \delta_0 + \delta_1} + \frac{1}{\delta_\sigma - \delta_\infty + \delta_t} - \frac{1}{2\delta_\sigma} \right) \\
&+ \frac{(\delta_\sigma^2 + 2\delta_\sigma(\delta_0 + \delta_1) - 3(\delta_0 - \delta_1)^2)(\delta_\sigma^2 + 2\delta_\sigma(\delta_\infty + \delta_t) - 3(\delta_\infty - \delta_t)^2)}{16\delta^3_\sigma(4\delta_\sigma + 3)}.
\end{align}
\end{equation}

The calculation of subsequent coefficients can be carried out by using any suitable method (e.g. Zamolodchikov recursion [Z84] or Nekrasov functions [Nek]) to generate conformal block expansion and computing the quasiclassical limit thereof.

Parameterize the accessory parameter as
\begin{equation}
\delta' = \delta_\infty - \delta_\sigma - \delta_t = -\frac{1}{4} - \theta_\infty^2 + \omega^2 + \theta_t^2.
\end{equation}
The meaning of the new parameter $\omega$ introduced instead of $\delta'$ is the limiting value of the composite monodromy exponent $\sigma$ as $t \to \infty$. Indeed, Zamolodchikov relation $\delta' = t^2 \frac{\mathcal{W}}{\mathcal{W}}$ leads to the perturbative series
\begin{equation}
\sigma(t) = \omega + \sum_{k=1}^\infty \sigma_k t^{-k}.
\end{equation}
For any $k$, the coefficient $\sigma_k$ is determined by $\mathcal{W}_1, \ldots, \mathcal{W}_k$ and is given by a rational function of $\theta = (\theta_0, \theta_1, \theta_t, \theta_\infty)$ and $\omega$, e.g.
\begin{equation}
\sigma_1 = \frac{\left( \frac{1}{4} - \omega^2 + \theta_0^2 - \theta_1^2 \right) \left( \frac{1}{4} - \omega^2 + \theta_\infty^2 - \theta_t^2 \right)}{4\omega \left( \frac{1}{4} - \omega^2 \right)}.
\end{equation}

Expanding the first factor in the Trieste formula (2.20), one can write
\begin{equation}
\ln \frac{\mathcal{F}_1(\theta_0, \theta_1, \sigma)}{\mathcal{F}_1(\theta_0, \theta_1, \omega)} = \sum_{k=0}^\infty \left[ (-1)^k \psi^{(k)} \left( \frac{1}{4} + \theta_1 - \theta_0 - \omega \right) \\
- \psi^{(k)} \left( \frac{1}{4} + \theta_1 - \theta_0 + \omega \right) \right] \frac{(\sigma - \omega)^{k+1}}{(k+1)!}.
\end{equation}
It then becomes clear that the coefficients $f_k$ of the perturbative expansion
\begin{equation}
\ln C(\theta_0, \theta_1, \sigma) = \ln \mathcal{F}_1(\theta_0, \theta_1, \omega) + \sum_{k=1}^\infty f_k t^{-k}
\end{equation}
are given by linear combinations of polygamma functions \( \psi^{(n)} \left( \frac{1}{2} + \theta_1 - \theta_0 \pm \omega \right) \) \((n = 0, \ldots, k - 1)\) with coefficients rational in \( \theta \) and \( \omega \). In particular,

\[
f_1 = -\left( \frac{1}{4} - \omega^2 + \theta_0^2 - \theta_1^2 \right) \left( \frac{1}{2} - \omega^2 + \theta_0^2 - \theta_1^2 \right) \left[ \psi \left( \frac{1}{2} + \theta_1 - \theta_0 + \omega \right) - \psi \left( \frac{1}{2} + \theta_1 - \theta_0 - \omega \right) \right] - \frac{(\theta_0 + \theta_1) \left( \frac{1}{2} - \omega^2 + \theta_0^2 - \theta_1^2 \right)}{2 \left( \frac{1}{4} - \omega^2 \right)}.
\]

The expressions for \( \sigma_2 \) and \( f_2 \) are omitted for the sake of brevity, yet they are completely straightforward to compute from (2.22), (2.23) and (2.27).

### 3. Schäfke-Schmidt connection formula

Consider a more general linear ODE

\[
\frac{d^2}{dz^2} + \frac{\frac{1}{2} - \theta_0^2}{z^2} + \frac{\frac{1}{2} - \theta_1^2}{(z-1)^2} + \frac{U(z)}{z(z-1)} \psi(z) = 0,
\]

where \( U(z) \) is analytic inside the disk \(|z| < R\) with \( R > 1\). The generalized Heun equation (2.13) is a special case of (3.1). Throughout the rest of the paper, it is assumed that \( \theta_0, \theta_1 \notin \mathbb{Z}/2\), in which case there exist unique bases of normalized Frobenius solutions \( \psi^{(0)}_\pm(z), \psi^{(1)}_\pm(z) \) whose expansions near the Fuchsian singularities \( z = 0, 1 \) have the form (2.19).

The Wronskian \( W(\psi_a, \psi_b) = \psi_a \psi_b' - \psi_b \psi_a' \) of any pair of solutions of (3.1) is independent of \( z \). It is straightforward to check that

\[
W \left( \psi^{(0)}_+, \psi^{(0)}_- \right) = 2\theta_0, \quad W \left( \psi^{(1)}_+, \psi^{(1)}_- \right) = -2\theta_1.
\]

The connection formula

\[
\psi^{(0)}_\pm(z) = \sum_{\epsilon, \epsilon'} C_{\epsilon \epsilon'} \psi^{(1)}_{\epsilon'}(z), \quad \epsilon, \epsilon' = \pm
\]

then implies that

\[
C_{\epsilon \epsilon'} = C(\epsilon \theta_0, \epsilon' \theta_1), \quad C(\theta_0, \theta_1) = -\frac{1}{2\theta_1} W \left( \psi^{(0)}_+, \psi^{(1)}_- \right).
\]

The connection coefficients are thus expressed in terms of a single function \( C(\theta_0, \theta_1) \), just as in (2.20a) above. This is a consequence of the invariance of (3.1) with respect to the sign flips \( \theta_0 \mapsto -\theta_0, \theta_1 \mapsto -\theta_1 \) of local monodromy exponents.

From (3.2) and (3.3) it follows that \( \det C = -\frac{\theta_0}{\theta_1} \). Expressing the composite monodromy around 0 and 1 in terms of \( C_{\epsilon \epsilon'} \), we can also write

\[
\text{Tr} \begin{pmatrix} C_{++} & C_{+-} \\ C_{-+} & C_{--} \end{pmatrix} \begin{pmatrix} e^{2\pi i \theta_1} & 0 \\ 0 & e^{-2\pi i \theta_1} \end{pmatrix} \begin{pmatrix} C_{--} & C_{-+} \\ C_{++} & C_{+-} \end{pmatrix} \begin{pmatrix} e^{2\pi i \theta_0} & 0 \\ 0 & e^{-2\pi i \theta_0} \end{pmatrix} = -2 \cos 2\pi \sigma \cdot \det C.
\]

This yields the relations

\[
C_{++} C_{+-} = -\frac{\theta_0 \cos \pi (\theta_1 \mp \theta_0 + \sigma) \cos \pi (\theta_1 \mp \theta_0 - \sigma)}{\sin 2\pi \theta_0 \sin 2\pi \theta_1}.
\]
It is a simple exercise to check that these two identities are satisfied by the conjectural formula (2.20) of the previous section.

Equation (3.4) is already sufficient for numerical evaluation of the connection coefficients $C_{\epsilon \epsilon'}$. Indeed, using (3.1) one may generate the expansions (2.19) truncated at sufficiently large order, and then compute the Wronskians at an intermediate point $\tilde{z} \in (0, 1)$. Our analytic perturbative calculation for Heun equations will be based on a result of Schäfke and Schmidt [SS] which relates $C(\theta_0, \theta_1)$ to the asymptotics of expansion coefficients of suitably modified Frobenius solutions.

We start by recalling a statement which originates from a classical work of Darboux [Dar, p. 10].

**Proposition 3.1.** Let $u(z)$ be a function having exactly one branch point $z = 1$ inside a circle $|z| = R > 1$. If $u(z)$ can be represented as

$$u(z) = v(z) + (1 - z)^{-\theta} w(z),$$

with $v(z), w(z)$ analytic in a neighborhood of $z = 1$, then the coefficients of the Taylor expansion $u(z) = \sum_{k=0}^{\infty} u_k z^k$ at $z = 0$ have the asymptotics

$$u_k = w(1) \frac{k^{\theta-1}}{\Gamma(\theta)} \left[1 + o(1)\right] \quad \text{as } k \to \infty. \quad (3.8)$$

**Proof.** Consider the contour $\mathcal{C} = \mathcal{C}_r \cup \mathcal{C}_+ \cup \mathcal{C}_- \cup \mathcal{C}_{R'}$ represented in figure 1(a), where $R' < R$ and $r$ is chosen sufficiently small so that $v(z), w(z)$ are analytic inside an open disk containing $\mathcal{C}_r$. Clearly, $u_k = \frac{1}{2\pi i} \oint_{\mathcal{C}} z^{-k-1} u(z) \, dz$. The integrals along $\mathcal{C}_{R'}$ and $\mathcal{C}_\pm$ are exponentially suppressed as $O(R'^{-k})$ and $O\left((1 + r)^{-k}\right)$ as $k \to \infty$. Substituting (3.7) into the remaining integral along $\mathcal{C}_r$, we further notice that the contribution of $v(z)$ vanishes. The contribution of $w(z)$ can be estimated as
Connection matrix relating the normalized Frobenius bases of solutions of (3.1) at \( z = 0, 1 \) is given by

\[
C(\theta_0, \theta_1) = \Gamma(2\theta_1) \lim_{k \to \infty} k^{1-2\theta_1} u_k,
\]

where \( u_k \) denote the coefficients of the Taylor expansion at 0 of the function

\[
u(z) = C(\theta_0, -\theta_1) z^{-\frac{1}{2}+\theta_0} (1-z)^{-\frac{1}{2}+\theta_1} \psi_+^{[1]}(z),
\]

and

\[
w(z) = C(\theta_0, \theta_1) z^{-\frac{1}{2}+\theta_0} (1-z)^{-\frac{1}{2}+\theta_1} \psi_+^{[1]}(z).
\]

Indeed, \( \nu(z) \) and \( w(z) \) are holomorphic in a neighborhood of \( z = 1 \) and \( w(1) = C(\theta_0, \theta_1) \).

The relation (3.7) comes from (3.3) with \( \epsilon = + \).

Let us illustrate the formula (3.10) with a toy example of the Gauss hypergeometric equation. The latter corresponds to the choice \( U(z) = \theta_0^2 + \theta_1^2 - \theta_\infty^2 - \frac{1}{2} \). The ODE (3.1) for \( \psi_+^{[0]}(z) \) then transforms into the canonical form of the hypergeometric equation satisfied by \( u(z) \):

\[
\left[ z (1-z) \frac{d^2}{dz^2} + (1-2\theta_0 - 2(1-\theta_0 + \theta_1)z) \frac{d}{dz} + \theta_\infty^2 - (\frac{1}{2} - \theta_0 + \theta_1)^2 \right] u(z) = 0.
\]

This in turn yields a 2-term linear recurrence relation

\[
u_{k+1} = \left( \frac{1}{2} - \theta_0 + \theta_1 + k \right)^2 - \theta_\infty^2 \nu_k.
\]

Given the initial condition \( \nu_0 = 1 \), its solution reads

\[
u_k = \left( \frac{1}{2} - \theta_0 + \theta_1 + \theta_\infty \right)_k \left( \frac{1}{2} - \theta_0 + \theta_1 - \theta_\infty \right)_k,\]

where \( (\lambda)_k \) denotes the Pochhammer symbol. Substituting this expression into (3.10) and computing the limit, one finds that \( C(\theta_0, \theta_1) = F_{cl}(\theta_0, \theta_1, \theta_\infty) \), with \( F_{cl} \) given by (2.18). We thereby recover the standard hypergeometric connection formulas.
4. Application to Heun equations

4.1. Reduced confluent Heun equation (RCHE)

We now proceed to the connection problem for Heun equations with at least two Fuchsian singularities: reduced confluent, confluent and usual Heun equation. To explain the idea with a minimal amount of fuss, it is convenient to start with the reduced confluent case. This corresponds to setting in (3.1)

\[ U(z) = \theta_0^2 + \omega z - \frac{1}{2} \lambda z, \]

where \( \omega \) plays the role of accessory parameter and \( \lambda \) is a coupling constant. We will be mainly interested in the weak coupling regime \( |\lambda| \ll 1 \).

The function \( u(z) \) defined by (3.11) satisfies the canonical form of the RCHE:

\[
\left[ z (1-z) \frac{d^2}{dz^2} + (1-2\theta_0 - 2(1-\theta_0+\theta_1)z) \frac{d}{dz} + \omega^2 \left( \frac{1}{2} - \theta_0 + \theta_1 \right)^2 + \lambda \omega \right] u(z) = 0. \tag{4.2}
\]

Lemma 4.1. The connection matrix relating Frobenius solutions of the RCHE at \( z = 0, 1 \) is given by

\[
C(\theta_0, \theta_1) = \frac{\Gamma(1-2\theta_0)\Gamma(2\theta_1)}{\Gamma\left(\frac{1}{2} + \theta_1 - \theta_0 + \omega\right)\Gamma\left(\frac{1}{2} + \theta_1 - \theta_0 - \omega\right)} a_{\infty}, \tag{4.3}
\]

where \( a_{\infty} = \lim_{k \to \infty} a_k \) is the limit of the sequence defined by the 3-term recurrence relation

\[
a_{k+1} - a_k = -\lambda \beta_k a_{k-1}, \tag{4.4}
\]

with \( a_{-1} = 0, a_0 = 1 \) and \( \beta_k \) defined by (1.5).

Proof. Equation (4.2) implies the following 3-term recurrence for the coefficients \( u_k \):

\[
(k+1)(k+1-2\theta_0)u_{k+1} = \left(\frac{1}{2} - \theta_0 + \theta_1 + k\right)^2 - \theta_0^2 \right) u_k - \lambda u_{k-1}, \tag{4.5}
\]

which in the case \( \lambda = 0 \) reduces to (3.14). Introducing rescaled coefficients \( a_k = u_k/\lambda^{\theta_0} \), it is straightforward to show that they satisfy (4.4) and (1.5). The statement of the lemma then follows from corollary 3.2 and the hypergeometric example in the end of the previous section.

In order to find the perturbative expansion of \( a_{\infty} \) for small \( \lambda \), we first rewrite it as a determinant of an infinite tridiagonal matrix.

Lemma 4.2. The limiting value \( a_{\infty} \) can be written as

\[
a_{\infty} = \det \begin{pmatrix} 1 & -1 \\ -\lambda \beta_1 & 1 & -1 \\ -\lambda \beta_2 & 1 & -1 \\ -\lambda \beta_3 & 1 & \\ & & 1 \end{pmatrix}. \tag{4.6}
\]

Proof. Interpret the recurrence relations (4.4) with \( k = 0, \ldots, n - 1 \) in combination with the initial condition \( a_0 = 1 \) as a linear system for \( a_0, \ldots, a_n \). Solving it for \( a_n \) and sending \( n \to \infty \) gives the representation (4.6).
One could attempt to write a formal power series expansion

\[
a_\infty = 1 - \lambda \sum_{k=1}^{\infty} \beta_k + \lambda^2 \sum_{k=1}^{\infty} \sum_{k'=k+2}^{\infty} \beta_k \beta_{k'} + \ldots
\]

(4.7)

It is however more efficient to expand \( \ln a_\infty \) as this eliminates nested sums.

**Lemma 4.3.** We have

\[
\ln a_\infty = -\sum_{n=1}^{\infty} \frac{\text{Tr} A^{2n}}{2n} \lambda^n,
\]

(4.8)

where \( A \) denotes the infinite matrix

\[
A = \begin{pmatrix}
0 & 1 \\
\beta_1 & 0 & 1 \\
\beta_2 & 0 & 1 \\
& & \ddots
\end{pmatrix}.
\]

(4.9)

We have, in particular,

\[
\text{Tr} A^2 = \sum_{k=1}^{\infty} 2\beta_k, \quad \text{Tr} A^4 = \sum_{k=1}^{\infty} \left( 4\beta_k \beta_{k+1} + 2\beta_k^2 \right),
\]

(4.10a)

\[
\text{Tr} A^6 = \sum_{k=1}^{\infty} \left( 6\beta_k \beta_{k+1} \beta_{k+2} + 6\beta_k^2 \beta_{k+1} + 6\beta_k \beta_{k+1}^2 + 2\beta_k^3 \right), \ldots
\]

(4.10b)

The general expression for \( \text{Tr} A^{2n} \) in terms of \( \beta_k \)'s can be written in a combinatorial way. Consider a staircase walk \( W \) on the \( n \times n \) square grid going from \((0,0)\) to \((n,n)\) as shown in the example in figure 2(a). The vertical edges of \( W \) intersect \( \ell \leq n \) consecutive diagonals parallel to the main one and passing through the midpoints of the edges. Order these diagonals in the northwest direction and denote the corresponding intersection numbers by \( \mu_1, \ldots, \mu_\ell \). Obviously, \( \mu_1 + \ldots + \mu_\ell = n \). In this way we assign to every walk its type \( \mu = (\mu_1, \ldots, \mu_\ell) \) — a composition (ordered partition) of \( n \) into \( \ell \) parts. It will be denoted \( \mu \vdash n \) similarly to ordinary partitions. The total number of compositions of \( n \) is known to be equal to \( 2^n - 1 \). We may then write

\[
\text{Tr} A^{2n} = \sum_{k=1}^{\infty} \sum_{\mu \vdash n} N_\mu \beta_k^{\mu_1} \beta_{k+1}^{\mu_2} \ldots \beta_{k+\ell}^{\mu_\ell},
\]

(4.11)

where \( N_\mu \) denotes the number of staircase walks of type \( \mu \). For example, the term \( 6\beta_k \beta_{k+1}^2 \) in (4.10b) corresponds to six possible walks of type \((1,2)\) represented in figure 2(b). The total number of staircase walks is \( \sum_{\mu \vdash n} N_\mu = \binom{2n}{n} \).

We now finish the proof of theorem 1 from the Introduction and simultaneously obtain an explicit expression for the coefficients \( N_\mu \).

**Lemma 4.4.** The quantity \( a_\infty \) admits the following infinite fraction representation:

\[
\ln a_\infty = \sum_{k=1}^{\infty} \ln \left( 1 - \frac{\lambda \beta_k}{1 - \lambda \beta_{k+1} + \ldots} \right).
\]

(4.12)
Proof. Consider a sequence of infinite determinants
\[
D_k = \det \begin{pmatrix}
   1 & -1 & & \\
   -\lambda \beta_k & 1 & -1 & \\
   -\lambda \beta_{k+1} & 1 & & \\
   & & & \\
\end{pmatrix}, \quad k = 1, 2, \ldots.
\] (4.13)

Obviously, \(a_\infty = D_1\). Expanding \(D_k\) with respect to the first row or the first column, one finds the following 3-term linear recurrence relation
\[
D_k = D_{k+1} - \lambda \beta_k D_{k+2}.
\] (4.14)

It in turn implies that the ratios \(\eta_k = \frac{D_k}{D_{k+1}}\) satisfy a 2-term nonlinear recurrence
\[
\eta_k = 1 - \frac{\lambda \beta_k}{1 - \lambda \beta_{k+1}}.
\] (4.15)

The statement of the lemma then immediately follows from \(\ln D_1 = \sum_{k=1}^{\infty} \ln \eta_k\).

Corollary 4.5. The integers \(N_\mu\) in (4.11) are given by the following products of binomial coefficients:
\[
N_\mu = \frac{2^n}{\mu_1} \prod_{m=1}^{\ell-1} \left( \frac{\mu_m + \mu_{m+1} - 1}{\mu_{m+1}} \right).
\] (4.16)

Proof. The factor \(\beta_\mu^k\) can appear in (4.11) only from the \(\mu_1\)th term in the expansion of the logarithm in \(\ln \left( 1 - \frac{\lambda \beta_k}{1 - \lambda \beta_{k+1}} \right)\), i.e. from \(-1/\mu_1 \left( \frac{\lambda \beta_k}{1 - \lambda \beta_{k+1}} \right)^{\mu_1}\). Likewise, the next factor \(\beta_\mu^{k+1}\) can only be produced by the \(\mu_2\)th term in the binomial expansion of \(\left( 1 - \frac{\lambda \beta_{k+1}}{1 - \lambda \beta_{k+2}} \right)^{-\mu_1}\) given by \((-1)^{\mu_2} \left( \frac{\lambda \beta_{k+1}}{1 - \lambda \beta_{k+2}} \right)^{\mu_2}\). Continuing the same procedure, we see that the coefficient of \(\beta_\mu^k \beta_{k+1}^{\mu_1} \cdots \beta_{k+\ell}^{\mu_\ell}\) in the expansion of \(\ln \eta_k\) is equal to
\[
- \left( -1 \right)^{\mu_2 + \mu_3 + \cdots + \mu_\ell} \frac{1}{\mu_1} \left( \frac{-\mu_1}{\mu_2} \right) \left( \frac{-\mu_2}{\mu_3} \right) \cdots \left( \frac{-\mu_{\ell-1}}{\mu_\ell} \right).
\] (4.17)
The representation (4.16) is nothing but a rewrite of the latter expression taking into account the coefficient $\frac{1}{2}$ in front of $\text{Tr} A^{2\ell}$ in (4.8).

The formulas such as (4.8), (4.11) and (4.16) allow to compute the expansion of $\ln \alpha_\infty$ in powers of $\lambda$ to arbitrary order. In practice, it is more convenient to use (4.12), truncate the infinite fraction at appropriate order and expand the result. The coefficients $c_n$ of

$$\ln \alpha_\infty = \sum_{n=1}^{\infty} c_n \lambda^n,$$  \hspace{1cm} (4.18)

are then given by linear combinations of sums $\sum_{k=1}^{\infty} \beta_k^{\mu_1 \mu_2 \cdots \mu_\ell}$, cf (4.10). Since $\beta_k$ defined by (1.5) is a rational function of $k$, such sums can be calculated in a closed form in terms of rational and polygamma functions. Indeed, using partial fraction decomposition of $\beta_k^{\mu_1 \mu_2 \cdots \mu_\ell}$ with respect to $k$, any such sum can be written as a linear combination of sums of the form

$$\sum_{k=1}^{\infty} \frac{1}{(k-x)^{\mu+1}} = (-1)^{\mu+1} \frac{\psi^{(\mu)}(1-x)}{n!}, \hspace{1cm} n \geq 1, \hspace{1cm} (4.19a)$$

$$\sum_{k=1}^{\infty} \left( \frac{n}{k-x_m} \right) = - \sum_{m=1}^{n} y_m \psi(1-x_m), \hspace{1cm} \sum_{m=1}^{n} y_m = 0. \hspace{1cm} (4.19b)$$

The property $\sum_{m=1}^{n} y_m = 0$ is ensured automatically by the asymptotic behavior $\beta_k = O\left(\frac{1}{k^\ell}\right)$ as $k \to \infty$. For example, it follows from (4.10a) that

$$c_1 = - \frac{1}{4} \left( \frac{1}{4} - \theta_0^2 + \theta_1^2 - \omega^2 \right) \left( \psi \left( \frac{1}{4} - \theta_0 + \theta_1 + \omega \right) - \psi \left( \frac{1}{4} - \theta_0 + \theta_1 - \omega \right) \right)$$

$$+ \frac{\theta_0 + \theta_1}{2 \left( \frac{1}{4} - \omega^2 \right)}, \hspace{1cm} (4.20a)$$

$$c_2 = - \frac{3}{32 \omega^2 - (\frac{1}{4} - \omega^2)^2} \left( \psi^{(1)} \left( \frac{1}{4} - \theta_0 + \theta_1 + \omega \right) - \psi^{(1)} \left( \frac{1}{4} - \theta_0 + \theta_1 - \omega \right) \right)$$

$$+ \frac{\left( 60 \omega^4 - 35 \omega^2 + 2 \right) \left( \theta_0^2 - \theta_1^2 \right)^2}{256 \omega^3 \left( \frac{1}{4} - \omega^2 \right)^2 \left( 1 - \omega^2 \right)} - \frac{3 \left( \theta_0^2 + \theta_1^2 \right)}{32 \omega \left( 1 - \omega^2 \right) \left( \frac{1}{4} - \omega^2 \right)}$$

$$- \frac{3 \left( \theta_0^2 + \theta_1^2 \right)}{64 \omega^3 \left( \frac{1}{4} - \omega^2 \right)^2} + \frac{2 - 3 \omega^2}{64 \omega^3 \left( 1 - \omega^2 \right)}$$

$$\times \left( \psi \left( \frac{1}{4} - \theta_0 + \theta_1 + \omega \right) - \psi \left( \frac{1}{4} - \theta_0 + \theta_1 - \omega \right) \right)$$

$$+ \frac{\theta_0 + \theta_1}{4 \left( \frac{1}{4} - \omega^2 \right)^2} - \frac{3 \left( \theta_0 - \theta_1 \right)}{32 \left( \frac{1}{4} - \omega^2 \right) \left( 1 - \omega^2 \right)} - \frac{25 - 52 \omega^2}{128 \left( \frac{1}{4} - \omega^2 \right)^2 \left( 1 - \omega^2 \right)} \left( \theta_0 - \theta_1 \right) \left( \theta_0 + \theta_1 \right)^2.$$

This calculation confirms the validity of the Trieste formula for RCHE [BIPT22]. We have compared our rigorous results against CFT predictions going up to corrections of order $\lambda^3$ and found complete agreement. There are no obstacles other than computer time to check higher orders which essentially amount to comparison of huge rational expressions.
4.2. Heun equation (HE)

In the case of the usual Heun equation (2.21), it is convenient to consider a modification of the Schäfke-Schmidt formula. Instead of $u(z)$ defined by (3.11), we introduce

$$u(z) := z^{-\frac{1}{2} + \theta_0} (1-z)^{-\frac{1}{2} - \theta_1} (t-z)^{-\frac{1}{2} + \theta_0} \psi_+^{[0]}(z)$$

$$= t^{-\frac{1}{2} + \theta_0} \left[ 1 + \sum_{k=1}^{\infty} u_k z^k \right].$$  \hspace{1cm} (4.21)

The effect of the extra factor $(t-z)^{-\frac{1}{2} + \theta_0}$ in the above is a slight change of the formula for the connection coefficient as compared to (3.10):

$$C(\theta_0, \theta_1) = \Gamma(2\theta_1) \left( 1 - \frac{1}{2} \right)^{-\theta_0} \lim_{k \to \infty} k^{-2\theta_1} u_k.$$  \hspace{1cm} (4.22)

The function $u(z)$ defined by (4.21) satisfies the canonical form of HE,

$$\left[ \frac{d^2}{dz^2} + \left( \frac{1 - 2\theta_0}{z} + \frac{1 + 2\theta_1}{z - 1} + \frac{1 - 2\theta_0}{z - t} \right) \frac{d}{dz} + \frac{((\theta_0 - \theta_1 + \theta_1 - 1)^2 - \theta_0^2)}{z(z - 1)(z - t)} \right] u(z) = 0,$$  \hspace{1cm} (4.23)

where

$$q = t (\theta_0 - \theta_1 - \frac{1}{2})^2 - t\omega^2 + (\theta_0 + \theta_1 - \frac{1}{2})^2 - \theta_0^2 - \theta_1^2 + \omega^2.$$  \hspace{1cm} (4.24)

The main advantage of (4.23), obtained at the expense of losing the symmetry $\theta_1 \leftrightarrow -\theta_1$, is that it yields a 3-term recurrence relation for the coefficients $u_k$. Also note that for $t \to \infty$, equation (4.23) transforms into hypergeometric equation (3.13) (with $\theta_\infty$ replaced by $\omega$), which implies that

$$u_k^{t \to \infty} = \frac{(\frac{1}{2} - \theta_0 + \theta_1 + \omega)_{k-1}}{k!} \frac{(\frac{1}{2} - \theta_0 + \theta_1 - \omega)}{(1 - 2\theta_0)_{k}}.$$  \hspace{1cm} (4.25)

Introducing the rescaled coefficients $a_k = u_k / u_k^{t \to \infty}$, it becomes straightforward to write the Heun counterpart of lemma (4.1):

**Lemma 4.6.** The connection matrix relating normalized Frobenius solutions of the HE at $z = 0, 1$ is given by

$$C(\theta_0, \theta_1) = \frac{\Gamma(1 - 2\theta_0) \Gamma(2\theta_1)}{\Gamma\left(\frac{1}{2} + \theta_1 - \theta_0 + \omega\right) \Gamma\left(\frac{1}{2} + \theta_1 - \theta_0 - \omega\right)} \left( 1 - \lambda \right)^{-\theta_0} a_\infty,$$  \hspace{1cm} (4.26)

where $\lambda = \frac{1}{2}$ and $a_\infty = \lim_{k \to \infty} a_k$ is the limit of the sequence defined by the 3-term recurrence relation

$$a_{k+1} - a_k = -\lambda (\alpha_k a_k + \beta_k a_{k-1}),$$  \hspace{1cm} (4.27)

with $a_{-1} = 0$, $a_0 = 1$ and $\alpha_k, \beta_k$ given by (1.8).

The coefficients $\alpha_k, \beta_k$ in (1.8) no longer tend to 0 as $k \to \infty$. This makes infinite determinants generalizing (4.6) ill-defined. Nevertheless it is still possible to obtain an analog of lemma 4.4.
Lemma 4.7. The quantity $a_{\infty}$ admits the following infinite fraction representation:

$$\ln a_{\infty} = -\ln (1-\lambda) + \sum_{k=1}^{\infty} \ln \left( 1 - \lambda \alpha_{k-1} - \frac{\lambda \beta_k}{1 - \lambda \alpha_k - \frac{\lambda \beta_{k+1}}{1 - \lambda \alpha_k - \frac{\lambda \beta_{k+2}}{\ddots}}} \right).$$  \hspace{1cm} (4.28)

Proof. Let $a_k^{(N)}$ denote the solution of the recurrence relation (4.27) satisfying the initial conditions $a_{N-1}^{(N)} = 0$, $a_N^{(N)} = 1$. We are ultimately interested in the limiting value of $a_k^{(0)}$ as $k \to \infty$.

Since the sequences $a_k^{(N)}$ and $a_k^{(N+1)}$ are linearly independent, there exist $C_1$ and $C_2$ such that $a_k^{(N+2)} = C_1 a_k^{(N+1)} + C_2 a_k^{(N)}$. It can be deduced from the initial conditions on three sequences that $C_1 = t \beta_{N+1} - \alpha_N \beta_{N+1}$, $C_2 = -t \beta_{N+1}^{-1}$, so that

$$a_k^{(N)} - a_k^{(N+1)} = -\lambda \left( \alpha_N a_k^{(N+1)} + \beta_N a_k^{(N+2)} \right).$$  \hspace{1cm} (4.29)

Define $D_{N+1} := \lim_{k \to \infty} a_k^{(N)}$. The existence of the limit is ensured by the Schäfke-Schmidt formula. Note in particular that the quantity $a_{\infty}$ that we are after coincides with $D_1$. The recurrence (4.29) implies that

$$D_N - D_{N+1} = -\lambda (\alpha_N D_{N+1} + \beta_N D_{N+2}).$$  \hspace{1cm} (4.30)

This relation is the Heun counterpart of (4.14). Now we can proceed similarly to the proof of lemma 4.4. Introducing the ratios $\eta_k = \frac{D_k}{D_{k+1}}$, one obtains a 2-term Riccati recurrence $\eta_k = 1 - \lambda \alpha_k - \lambda \beta_k / \eta_{k+1}$. It is solved by the continued fraction

$$\eta_k = 1 - \lambda \alpha_k - \frac{\lambda \beta_k}{1 - \lambda \alpha_k - \frac{\lambda \beta_{k+1}}{\ddots}}.$$  \hspace{1cm} (4.31)

We may write

$$D_1 = \ln D_{N+1} + \sum_{k=1}^{N} \ln \eta_k.$$  \hspace{1cm} (4.32)

The existence of the limit $N \to \infty$ of the sum in the second term does not require that $\alpha_k, \beta_k \to 0$ as $k \to \infty$. In fact, it suffices to have $\alpha_k + \beta_k = O \left( \frac{1}{k} \right)$, which holds for $\alpha_k, \beta_k$ defined by (1.8). Therefore it remains to compute the limit

$$D_\infty = \lim_{N \to \infty} D_{N+1} = \lim_{N \to \infty} \lim_{M \to \infty} \det \left( \begin{array}{ccccccc} 1 - \lambda \alpha_N & -1 & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\lambda \beta_{N+1} & 1 - \lambda \alpha_{N+1} & -1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -\lambda \beta_{M+1} & 1 - \lambda \alpha_{M+1} & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -\lambda \beta_{M+L} & 1 - \lambda \alpha_{M+L} \end{array} \right).$$  \hspace{1cm} (4.33)

Since $\alpha_k = -1 + O \left( \frac{1}{k} \right)$, $\beta_k = 1 + O \left( \frac{1}{k} \right)$ as $k \to \infty$, it follows that $D_\infty$ coincides with the limiting value $\lim_{M \to \infty} \tilde{D}_M$ of the determinant of an $M \times M$ tridiagonal Toeplitz matrix,

$$\tilde{D}_M = \det \left( \begin{array}{ccccccc} 1 + \lambda & -1 & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\lambda & 1 + \lambda & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 + \lambda & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 + \lambda \end{array} \right).$$  \hspace{1cm} (4.34)
The latter determinant can be easily evaluated (e.g. by induction on \( M \)) to \( \tilde{D}_M = \sum_{k=0}^{M} \lambda^k \), which implies that \( D_\infty = 1 / (1 - \lambda) \). In combination with (4.32), this yields (4.28) and concludes the proof of theorem 2.

As before for RCHE, the coefficients in the expansion \( \ln a_\infty = \sum_{n=1}^{\infty} c_n \lambda^n \) are thus given by sums of explicit rational expressions and can be evaluated in a closed form. For example, \( c_1 = 1 - \sum_{k=1}^{\infty} (\alpha_k + \beta_k) = 1 - \theta_t + f_1 \),

(4.35)

where \( f_1 \) is given by (2.29). Similarly computing subsequent coefficients, we have successfully checked the Trieste formula (2.20) up to order \( \lambda^3 \).

4.3. Confluent Heun equation (CHE)

The case of the CHE

\[
\left( \frac{d^2}{dz^2} + \frac{1 - \theta_0}{z^2} + \frac{1 - \theta_1}{(z-1)^2} - \frac{\lambda^2}{4z} - \frac{\lambda \theta_1 + \theta_0 + \omega^2 - 1}{z(z-1)} \right) \psi(z) = 0,
\]

(4.36)

is intermediate between the RCHE and HE. Since it can be analyzed in a similar manner, we limit ourselves to the statement of the final result.

**Theorem 3.** The connection relation between the normalized Frobenius solutions \( \psi_{10}^\pm (z) \), \( \psi_{11}^\pm (z) \) of the CHE is given by (1.3), with \( C(\theta_0, \theta_1) \) defined by

\[
C(\theta_0, \theta_1) = \frac{\Gamma(1 - 2\theta_0) \Gamma(2\theta_1) e^{\lambda/2}}{\Gamma\left(\frac{1}{2} + \theta_1 - \theta_0 + \omega\right) \Gamma\left(\frac{1}{2} + \theta_1 - \theta_0 - \omega\right)} \times \exp \left\{ \sum_{k=1}^{\infty} \ln \left( 1 - \lambda \alpha_{k-1} - \frac{\lambda \beta_k}{1 - \lambda \alpha_k - \frac{\lambda \beta_{k+1}}{1 - \lambda \alpha_{k+1}} - \frac{1}{4}} \right) \right\},
\]

(4.37)

where

\[
\alpha_k = \frac{k + \frac{1}{2} - \theta_0 - \theta_1}{(k + \frac{1}{2} - \theta_0 + \theta_1)^2 - \omega^2},
\]

(4.38a)

\[
\beta_k = -\frac{k(2\theta_0)(k - \theta_0 + \theta_1 - \theta_1)}{\left( (k + \frac{1}{2} - \theta_0 + \theta_1)^2 - \omega^2 \right) \left( (k + \frac{1}{2} - \theta_0 + \theta_1)^2 - \omega^2 \right)}.
\]

(4.38b)

At the formal level, the formulas (4.37) and (4.38) can be obtained from their Heun analogs (1.7) and (1.8) in theorem 2 in the limit

\[
\theta_t = \frac{\Lambda + \theta_0}{2}, \quad \theta_\infty = \frac{\Lambda - \theta_0}{2}, \quad \lambda_{HE} = \frac{\lambda_{CHE}}{\Lambda}, \quad \Lambda \to \infty.
\]

(4.39)

Further limit to RCHE is more subtle and involves in addition a \( \lambda \)-dependent redefinition of the accessory parameter \( \omega \).

5. Discussion and outlook

We have developed a procedure of the perturbative solution of the connection problem for the usual, confluent and reduced confluent Heun equation between Frobenius solutions associated
to different Fuchsian singular points. It confirms the validity of the Trieste formula (2.20) and its confluent variants. Compared to CFT approach, theorems 1–3 become particularly efficient when monodromy exponents \( \theta \) and accessory parameter \( \omega \) are assigned specific numerical values, since in the former setting one needs to keep \( \theta_0, \theta_1 \) arbitrary for the computation of derivatives \( \frac{\partial W}{\partial \theta_0}, \frac{\partial W}{\partial \theta_1} \).

It would be interesting to generalize our approach to the connection problem involving irregular singularities using an appropriate modification of the Schäfke-Schmidt formula. We note that in this case there also exist perturbative expansions \([\text{BIPT22}]\) predicted by CFT. In addition to CHE and RCHE, this concerns the doubly-confluent Heun equation and its reduced and doubly-reduced version.

Another appealing direction would be to study the connection problem for different confluent Heun equations at strong coupling \( |\lambda| \gg 1 \) (instead of the weak coupling considered here). Indeed, in this regime there already exist analogs of the Zamolodchikov conjecture \([\text{LiNa}]\) for the confluent and biconfluent Heun equation. Moreover, the expansion of the relevant accessory parameter function in CHE can be obtained from 3-term recurrence relations and continued fractions \([\text{CC22}]\).

To the authors’ knowledge, there is no rigorous proof of the Zamolodchikov relation \( \varepsilon = i \frac{\partial W}{\partial \theta} \) between the Heun accessory parameter and quasiclassical conformal block available yet. Moreover, without control of analytic properties of such conformal blocks it is unclear how to interpret this statement beyond the level of formal series. The Trieste formula, on the other hand, can be formulated in purely mathematical terms. One may define the function \( W(t) \) by (2.14b) in which case (2.20) yields a highly nontrivial conjectural relation between the Heun accessory parameter of Floquet type and the connection matrix. We expect it to be related to the extended symplectic structure on the space of monodromy data of Fuchsian systems introduced in \([\text{BK}]\).

### Data availability statement

No new data were created or analysed in this study.

### Acknowledgments

The authors are grateful to G Bonelli, C Iossa, D Panea Lichtig, A Tanzini, B Carneiro da Cunha and V P Gusynin for stimulating discussions.

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