Konishi Anomalies and $N = 1$ Curves

K. Landsteiner *

Instituto de Física Teórica, Universidad Autónoma de Madrid
28049 Madrid, Spain

Abstract: We present a brief summary of exact results on the non-perturbative effective superpotential of $N = 1$ supersymmetric gauge theories based on generalized Konishi anomaly equations. In particular we consider theories with classical gauge groups and chiral matter in two-index tensor representations. All these theories can be embedded into theories with unitary gauge group and adjoint matter. This embedding can be used to derive expressions for the exact non-perturbative superpotential in terms of the $1/N$ expansion of the free energy of the related matrix models.

1 Introduction

Dijkgraaf and Vafa conjectured in [1] that the exact non-perturbative superpotential of a $N = 1$ supersymmetric gauge theory can be computed in terms of the free energy of a related matrix model. The original example was based on a theory with $SU(N)$ gauge group and matter in the adjoint representation. In [2] it was pointed out that the loop equations in the large $N$ limit of the matrix model are equivalent to a set of generalized Konishi anomaly relations truncated to the chiral ring of the theory in question (see also [3]). This was extended to theories with classical gauge groups and chiral matter multiplets in more general representations. A particular interesting example proved to be the theory based on symplectic gauge groups $Sp(N)$ and matter transforming in the antisymmetric representation [4]. These theories presented a puzzle as pointed out in [5]. The matrix model free energy did not seem to reproduce the effective superpotential in the cases that were already known from field theory arguments based on holomorphy. This puzzle could be resolved by Cachazo in [6] using an embedding of the theory with symplectic gauge group and antisymmetric matter into a theory with unitary gauge group and adjoint matter. This embedding is most naturally understood in the string realization of the theories and the earlier noted discrepancy could be traced back to nontrivial contributions of orientifolds [7, 8]. The string theory realization also allowed a better understanding of the embedding and an extension to all theories with orthogonal or symplectic gauge group and two-index matter.

What was missing so far in the literature was the case of unitary gauge groups with symmetric or antisymmetric matter. It has been treated now in [9, 10, 11]. Although the model has a somewhat different flavour from the previously described ones (since it

*corresponding author E-mail: Karl.Landsteiner@uam.es
needs two chiral multiplets in conjugate representation) it can be treated in a completely analogous way by using a composite adjoint field.

2 Konishi Anomalies

We will now briefly summarize the main features of the generalized Konishi anomaly equations of the theories with classical gauge groups and a single matter multiplet in a two-index tensor representation.

The theories under consideration have a superpotential of the form

\[ W = \sum_i \frac{g_k}{k+1} \mathrm{tr} (\Phi^{k+1}) \]  

where \( \Phi \) is a chiral superfield transforming as \( \Phi \rightarrow g \Phi g^{-1} \) under the a gauge transformation \( g \). For unitary gauge groups that means it transforms under the adjoint and for symplectic and orthogonal it should be a symmetric or antisymmetric two-index tensor representation.

Two generating functions of correlators of chiral operators are of particular interest. They are defined as

\[ T(z) = \left\langle \mathrm{tr} \left( \frac{1}{z - \Phi} \right) \rightangle, \quad R(z) = -\frac{1}{32\pi^2} \left\langle \mathrm{tr} \left( \frac{W^\alpha W_\alpha}{z - \Phi} \right) \rightangle. \]

where \( W^\alpha \) is the gaugino. Note that the first \( d + 2 \) coefficients of the Laurent expansion of \( T(z) \) are the operators defining the superpotential. The strategy is therefore to determine the generating functions \( T(z) \) and \( R(z) \). Then we can integrate the differential equations \( \frac{\partial W_{\text{eff}}}{\partial g_k} = \mathrm{tr} (\Phi^{k+1}) \) and determine the effective superpotential up to a term that is independent of the couplings \( g_k \).

Since we need to know only correlators of chiral operators we can restrict ourselves to the chiral ring. The anomaly equation which allows the determination of \( T(z) \) and \( R(z) \) gives rise to a relation in the chiral ring

\[ \left\langle \delta \Phi_I \frac{\partial W}{\partial \Phi_I} \rightangle = -\frac{1}{32\pi^2} \left\langle W^\alpha I W_{\alpha,J} K \frac{\partial (\delta \Phi_K)}{\partial \Phi_I} \right\rangle. \]

The left hand side is the response of the classical theory to a arbitrary holomorphic field transformation \( \delta \Phi \) whereas the right hand side is a (anomalous) quantum effect. The capital indices denote a basis in the representation of the field \( \Phi \). The anomaly is a generalization \[2], \[3\] of the well-known Konishi anomaly \[12\]. The anomaly equation \[3\] has been shown to be exact non-perturbatively in \[13\].

Suitably chosen variations \( \delta \Phi \) lead to two relations

\[ \frac{1}{2} R^2(z) = W'(z) R(z) + f(z) \]  
\[ T(z) R(z) = W'(z) T(z) + X(R) + c(z) \]

\[ R(z) \] takes values on a hyperelliptic Riemann surface \( \Sigma \) and \( T(z) \) is a meromorphic differential on \( \Sigma \), \( f(z) \) and \( c(z) \) are polynomials

\[ f(z) = -\frac{1}{32\pi^2} \frac{(W'(z) - W'(\Phi)) W^2}{z - \Phi}, \quad c(z) = \frac{(W'(z) - W'(\Phi))}{z - \Phi}. \]
There is a one-to-one correspondence between the coefficients of these polynomials and
the period integrals
\[ S_i = \oint_{C_i} R(z)dz \leftrightarrow f(z) \text{, } N_i = \oint_{C_i} T(z)dz \leftrightarrow c(z). \tag{8} \]

The physical interpretation is that the cuts on the Riemann surface represent a vacuum
in which the gauge group \( G \) is broken to a subgroup \( \prod_i G_i \). The rank of each factor group
\( G_i \) is determined by \( N_i \) and the gaugino condensate in a factor group is given by \( S_i \). \(^{[2]}\)

The form of the anomaly relations depend on the gauge group and matter content only
through the term labelled as \( X(R) \), specifically it is given by
\[
\begin{align*}
&\bullet U(N) \text{ and } \phi \text{ adjoint: } X(R) = 0 \\
&\bullet SO(N)/SP(N) \text{ and } \phi \text{ adjoint: } X(R) = -2\epsilon z R(z), \epsilon = \pm 1 \\
&\bullet SO(N)/SP(N) \text{ and } \phi^T = \epsilon \phi: X(R) = 2\epsilon R'(z), \epsilon = \pm 1
\end{align*}
\]
where in the last two lines \( \epsilon = +1 \) for \( SO(N) \) and \( \epsilon = -1 \) for \( SP(N) \). \( X(R) \) determines
subleading \( 1/N \) corrections in the effective superpotential.

Let us now have a look to the new model with symmetric and antisymmetric tensors
of \( U(N) \). We take two matrix valued superfields
\[ X^T = \epsilon X \text{, } Y^T = \epsilon Y \tag{9} \]
and transforming as \( X \to UXU^T \text{ and } y \to U^*YU \) under \( U(N) \) gauge transformation. We
can form the composite adjoint \( \Phi = XY \) and define a superpotential and the generating
functions of chiral correlators just as in \(^{[1]} \), \(^{[2]} \). To discuss the classical theory it is
useful to compute the F-flatness conditions as \( X \frac{\partial}{\partial X} W(\Phi) = \Phi W'(\Phi) \). This shows that
the eigenvalues of the composite adjoint have either to vanish or be one of the \( d \) roots of
\( W'(z) = 0 \). The classical gauge symmetry breaking pattern is
\[ U(N) \to \begin{cases} U(N_0) \bigotimes_{i=1}^d SO(N_i), \ & \epsilon = +1 \\
U(N_0) \bigotimes_{i=1}^d SP(N_i), \ & \epsilon = -1 \end{cases} \tag{10} \]
The unitary factor group corresponds to the \( N_0 \) fold degeneracy of the zero-eigenvalue
of \( \Phi \). Since the non-zero eigenvalues ultimately come from symmetric or antisymmetric
matrices the gauge group is broken to factors of either orthogonal or symplectic groups.

The variations
\[ \delta X = \frac{1}{z - XY} X \text{, } \delta X = -\frac{1}{32\pi^2} \frac{W^2}{z - XY} X \tag{11} \]
lead to the anomaly relations
\[
\begin{align*}
\frac{1}{2} z R^2(z) & = z W'(z) R(z) + f(z) \tag{12} \\
z T(z) R(z) & = z W'(z) T(z) + 2\epsilon z R'(z) + c(z) \tag{13}
\end{align*}
\]
where the polynomials \( f(z) \) and \( c(z) \) are defined by
\[ f(z) = -\frac{1}{32\pi^2} (z W'(z) - XY W'(XY)) \frac{W^2}{z - XY}, \quad c(z) = \frac{(z W'(z) - XY W'(XY))}{z - XY} \tag{14} \]
We also note that the Riemann surface \( \Sigma \) defined by \(^{[12]} \) has a fixed branch point at
\( z = 0 \).

\(^{[1]}\)It is \( N_0 \)if the factor group is unitary and \( N_0/2 \) if it is symplectic or orthogonal.
3 The Matrix Model

The partition function of the related matrix model is given by

$$Z = \frac{1}{|G|} \int_{\Gamma} d\hat{X} d\hat{Y} e^{-\frac{1}{\kappa} \text{tr} [W(\hat{X}\hat{Y})]} ,$$

(15)

where $\hat{X}$ and $\hat{Y}$ are the matrices corresponding to the chiral multiplets $X, Y$ of the gauge theory. They are thus complex $\hat{N} \times \hat{N}$ matrices obeying $(\hat{X}^T, \hat{Y}^T) = \epsilon(\hat{X}, \hat{Y})$. $|G|$ is a normalization factor including the volume of the gauge group and $\Gamma$ is a suitably chosen path in the configuration space $\mathcal{M}$ of the matrices $\hat{X}, \hat{Y}$ with $\dim_{\mathbb{R}}(\Gamma) = \dim_{\mathbb{C}}(\mathcal{M})$.

We define the matrix model resolvent $\omega(z) = \kappa \text{tr} \left( \frac{1}{z-\Phi} \right)$ and the polynomial $\hat{f}(z) = -\kappa \text{tr} \left( \frac{zW'(z)}{z-\Phi} - \Phi W'(\Phi) \right)$. Standard arguments [10] lead then to the exact loop equation

$$\left\langle \frac{1}{2} z \omega^2(z) - zW'(z)\omega(z) - \frac{\epsilon}{2} z \omega'(z) - \hat{f}(z) \right\rangle = 0$$

(16)

The large $\hat{N}$ (or equivalently $\kappa$) expansion $\langle \omega(z) \rangle = \sum_{k=0}^{\infty} \kappa^k \omega_k(z)$ leads to an identification of the matrix model quantities with the field theory quantities according to

$$\omega_0(z) = R(z), \quad \delta \omega_0(z) + 4 \omega_1(z) = T(z)$$

(17)

$$\hat{f}(z) = f(z), \quad \delta \hat{f}(z) = c(z)$$

(18)

filling fraction $\kappa \hat{N}_i = $ gaugino condensate $S_i$

(19)

where we used the differential operator $\delta = \sum_{i=0}^{d} N_i \frac{\partial}{\partial S_i}$. Finally the superpotential of the field theory can be computed from the matrix model free energy as

$$W_{\text{eff}} = \sum_{i=0}^{d} N_i \frac{\partial F_0}{\partial S_i} + 4F_1 + \tau S$$

(20)

where $F_{0,1}$ are the leading and subleading contributions to the matrix model free energy in the $1/\hat{N}$ expansion and $\tau$ is the tree level gauge coupling [1].

4 Superpotential

In order to give a prescription of how to compute the superpotential we will follow [6], [7], [8] and embed the theory into one with unitary gauge group and elementary adjoint matter. It is useful now to go to a double cover of the $z$-plane and introduce

$$\zeta^2 = z, \quad R(\zeta) = \zeta R(\zeta^2), \quad T(\zeta) = \zeta T(\zeta^2), \quad W(\zeta) = \frac{1}{2} W(\zeta^2)$$

(21)

Except for the cut touching the origin $z = 0$ this doubles the number of cuts as indicated in the following figure.

The period integrals are

$$\oint_{C_0} \bar{R}(\zeta) d\zeta = S_0, \quad \oint_{C_0} \bar{T}(\zeta) d\zeta = N_0$$

(22)

$$\oint_{C_i} \bar{R}(\zeta) d\zeta = S_i, \quad \oint_{C_i} \bar{T}(\zeta) d\zeta = N_i/2$$

(23)
We introduce now $\tilde{y}^2 = (\tilde{W'})^2 + \tilde{f}$, $\tilde{T} = \tilde{T} + \Psi$ and $\tilde{c} = \tilde{c} + \bar{W}'' - \frac{\bar{W}'}{\zeta}$. where

$$\tilde{T} = \frac{\tilde{c}}{\tilde{y}}, \quad \Psi = \epsilon\left(\frac{\tilde{y'}}{\tilde{y}} - \frac{1}{\zeta}\right)$$  \hspace{1cm} (24)

The reason for introducing this new variables is that the pair $(\tilde{y}, \tilde{T})$ fulfill now the same relations as the pair $(y = R + W', T)$ of a theory with unitary gauge group and elementary adjoint matter! We have therefore a rather specific embedding of the original theory into one with unitary gauge group and elementary, adjoint matter. Notice that the period integrals of $\tilde{T}$ and $\bar{T}$ are related by

$$\oint \bar{T} = \oint \tilde{T} + \oint \Psi$$  \hspace{1cm} (25)

and the ranks of the product gauge groups are then determined by

$$\left\{ \frac{N_0}{N_1} \right\} = \tilde{N}_i + \epsilon(1 - \delta_{i,0})$$

The superpotential of embedding theory can be computed in the well-known way [1] whereas the original theory has an additional contribution from $1/N$ corrections due to non-orientable graphs. The relevant terms are

$$W_{eff}^F = \sum_{i=0} \tilde{N}_i \frac{\partial F_0}{\partial S_i} = N_0 \frac{\partial F_0}{\partial S_0} + \sum_{i>0} 2\left(\frac{N_i}{2} - \epsilon\right) \frac{\partial F_0}{\partial S_i}$$  \hspace{1cm} (26)

$$W_{eff}^F = \sum_{i=0} N_i \frac{\partial F_0}{\partial S_i} + 4F_1.$$  \hspace{1cm} (27)

The first line gives the contribution to the superpotential using the unitary embedding theory with adjoint matter whereas the second line is the same contribution expressed in the original theory with (anti)symmetric matter. Comparing the two we can extract the contribution $F_1$. We summarize the subleading $1/N$ contributions in the different cases in the table

| Theory                        | $F_1$ |
|-------------------------------|-------|
| $U(N)$ + (anti)symmetric      | $F_1 = -\frac{\epsilon}{2} \sum_{i=1}^{d} \frac{\partial F'}{\partial S_i}$ |
| $SO(N)/SP(N) +$ adjoint       | $F_1 = -\frac{\epsilon}{2} \sum_{i=1}^{d} \frac{\partial F}{\partial S_i}$ |
| $SO(N)/SP(N) +$ (anti)symmetric | $F_1 = -\frac{\epsilon}{2} \sum_{i=1}^{d} \frac{\partial F}{\partial S_i}$ |

This extends the results for $F_1$ already obtained in [6, 7, 8] to the case of unitary gauge group and two-index tensor representations. In all the theories under consideration the subleading contributions can be computed from the the leading $F_0$ as a variation with respect to gaugino bilinear. Only the gaugino condensates in $SO/SP$ factor groups contribute! In particular this means that for $SO/SP$ with adjoint only the cut at the origin gives rise to $F_1$, for $SO/SP$ with (anti)symmetric matter all the cuts contribute and for $U(N)$ with antisymmetric matter only the cut at the origin does not contribute to $F_1$. 

![Figure 1: Going to the double cover](source)
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