MAHLER FORMULA FOR SELF MAPS ON THE N-DIMENSIONAL PROJECTIVE SPACE

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Abstract. The Mahler formula gives an expression for the height of an algebraic number, as the integral of the log of the minimal equation with respect to the Haar measure on the circle. In the present work we prove that a similar result holds for nice self maps on the n-dimensional projective space. The height of the number is replaced by the canonical height of a hypersurface, and the Haar measure is replaced by the canonical invariant measure.

1. Introduction

A classical formula of Mahler [Mah60] states that, if $P = (\lambda : 1) \neq \infty$ is a point in $\mathbb{P}^1_{\mathbb{Q}}$, the naive height $h_{nv}(P)$ can be related to the integral of the log of the minimal equation $F$ of $\lambda$, with respect to the Haar measure $d\theta$ on $S^1$. The identity we find in this case is:

$$h_{nv}(P) = \frac{1}{\deg(F)} \int_{S^1} \log |F(z)| d\theta.$$ 

Now, the naive height is the canonical height (c.f. later) associated to the morphisms $\phi_n : t \to t^n$ on $\mathbb{P}^1$. This means that $h_{nv}(\phi_n(t)) = nh_{nv}(t)$ and $h_{nv}(t) \geq 0$. On the other hand the Haar measure $d\theta$ on $S^1$ is invariant under the action of this endomorphisms, in the sense that $\phi^*_n d\theta = nd\theta$ and $\phi_n d\theta = d\theta$. It was already established in [PST05] that a more general equation is true for arbitrary dynamical systems on $\mathbb{P}^1$. The terms “bad reduction” and “integral at a finite place” appear for the first time while dealing with dynamical systems in dimension one. We prove in the following work that the situation is similar for dynamical systems on $\mathbb{P}^n$, provided that the morphism admits a “good model”. Suppose that we are working with a number field $K$. The general Mahler formula we prove states that, if $\varphi : \mathbb{P}^n_K \to \mathbb{P}^n_K$ admits a
model \( \varphi = (p_0 : ... : p_n) : \mathbb{P}^n_K \to \mathbb{P}^n_K \), such that \((p_0, ..., p_n)\) represents a regular sequence inside \( \mathcal{O}_K[T_0, ..., T_n] \) then:

\[
h_{\varphi}(D^+) - h_{\varphi}(D^-) = \sum_{v/\infty} \int_{\mathbb{P}^n_L} \log |F|_v d\mu_{\varphi,v} + E(F, v \text{ finite})
\]

where we have the following:

(i) \( F = F^+/F^- \) is a rational function on \( \mathbb{P}^n_K \).
(ii) \( h_{\varphi}(D^+) \) and \( h_{\varphi}(D^-) \) represent the canonical heights of the cycles \( D^+ \) and \( D^- \) respectively, where \( \text{div}(F) = D^+ - D^- \).
(iii) For every place \( v \) of \( K \) at infinity, \( d\mu_{\varphi,v} \) represents an invariant measure relative to \( \varphi \) on \( \mathbb{P}^n_C \).
(iv) Even though the sequence \((p_0, ..., p_n)\) is regular in \( \mathcal{O}_K[T_0, ..., T_n] \), the map \( \varphi = (p_0 : ... : p_n) : \mathbb{P}^n_K \to \mathbb{P}^n_K \) may not be a well defined map on \( \mathbb{P}^n_K \). For example \( \varphi : \mathbb{P}^2_\mathbb{Q} \to \mathbb{P}^2_\mathbb{Q} \) defined over \( \mathbb{Q} \) as \( \varphi(x : y : z) = (y^2 - 3z^2 : x^2 - 3yz^2 : zy) \) does not extend to a map on \( \mathbb{P}^2_{\overline{\mathbb{Q}}} \). The term \( E(F, v \text{ finite}) \) in the formula, is arising from the blow-up we may need to do in order to extend the map \( \varphi \) to an integral model. It depends in fact only on a finite number of finite places, which we call places of bad reduction.

When the map \( \varphi : \mathbb{P}^n_K \to \mathbb{P}^n_K \) defines a map (which we are calling here with the same name) \( \varphi : \mathbb{P}^n_{\mathcal{O}_K} \to \mathbb{P}^n_{\mathcal{O}_K} \), the term \( E(F, v \text{ finite}) = 0 \).

We will explain several particular cases and consequences of our formula; the classical formula of Mahler will be between them. Let us assume for the moment that our map admits a model such that \( h_{\varphi}(D^-) = 0 \). This condition will prove to be natural for polynomial functions in dimension one. If we pick the equation \( F \) such that \( v(F) = 0 \) for every finite place \( v \) (the valuation \( v \) naturally extends to rational functions on \( \mathbb{P}^n_K \)), and \( E(F, v \text{ finite}) < 0 \), we get the inequality:

\[
h_{\varphi}(D^+) \leq \sum_{v/\infty} \int_{\mathbb{P}^n_L} \log |F|_v d\mu_{\varphi,v}.
\]

When the map \( \varphi \) has good reduction everywhere \((E(F, v \text{ finite}) = 0)\) and \( v(F) = 0 \) for every finite \( v \), the above inequality becomes an equality. Particular cases of this formula can be found in [Mai00]. Let’s see now the dimension one case, which was treated in [PST05]. Suppose that \( \varphi = (p_0 : p_1) : \mathbb{P}^1 \to \mathbb{P}^1 \) is a map on the Riemann sphere and \( F \) is a polynomial equation. We can always change coordinates to get \( T_1/p_1 \), which will make \( h_{\varphi}(D^-) = h_{\varphi}(\infty) = 0 \). Also by base change we can assume that \((p_0, p_1)\) is a regular sequence. As a consequence of Proposition 5.3 and definition 5.6 we will be able to consider the
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The term $E(F, v_{\text{finite}})$ as sum of “integrals” over the finite places of $K$, i.e. $E(F, v_{\text{finite}}) = \sum_{v_{\text{finite}}} \int_{\mathbb{P}^1_{\mathbb{C}, v}} \log |F| d\mu_{\varphi, v}$. The measure $d\mu_{\varphi}$ at each place over infinity is nothing but the Brolin measure [Bro65], further studied by other authors such as, Lyubich [Lyu83] and Freire, Lopez, and Mañé [Man83]. Taking a point $P = (\lambda : 1) \neq \infty$ in $\mathbb{P}^1_K$ and $F$ the minimal equation of $\lambda$ over $\mathcal{O}_K$ (which we assume with no common factors, i.e. $v(F) = 0$ at every finite valuation), the formula we found takes a more symmetric shape,

$$h_{\varphi}(D^+) = \sum_v \int_{\mathbb{P}^1_{\mathbb{C}}} \log |F|_v d\mu_{\varphi, v}.$$

When the map $\varphi = (p_0 : p_1)$ is defined over $\mathbb{Q}$ and has good reduction everywhere (which is the case of the morphisms $\phi_n : t \to t^n$), we can take $F$ to be the minimal equation of $\lambda \in \mathbb{Q}$ over $\mathbb{Z}$ and get:

$$h_{\varphi}(P) = \frac{1}{\deg(F)} \int_{\mathbb{P}^1_{\mathbb{C}}} \log |F| d\mu_{\varphi}.$$

This last expression is the Mahler formula for rational morphisms on $\mathbb{P}^1$ in absence of bad reduction.

Going back to the general case, in this paper we work with a map $\varphi = (p_0 : ... : p_n) : \mathbb{P}^n_K \to \mathbb{P}^n_K$. It may not be possible to extend this map to a well defined map on $\mathbb{P}^n_{\mathcal{O}_K}$, exactly because the $p_i$ may have common zeroes along a subscheme $Y_1$ of $\mathbb{P}^n_K$. The technique we will use is arithmetic intersection theory. We will work with arithmetic varieties $X_k$ and rational maps $\sigma_k : X_k \to \mathbb{P}^n_K$, determined by blowing-up subschemes $Y_k$ in $\mathbb{P}^n_K$. We will establish the equality of cycles:

$$\text{div}(F_k) = D_k - \deg(F)\infty_k + \sum_{v,i} x_{v,i,k}C_{v,i,k}$$

$$- \deg(F) \sum_{v,i} y_{v,i,k}C_{v,i,k} + \sum_{\text{finite } v} v(F)X_{v,k},$$

where

(i) $F$ is a polynomial in the $n$ variables variables $T_0/T_n, ..., T_{n-1}/T_n$ and $F_k = \sigma_k^* F$;
(ii) $\text{div}(F) = D - \deg(F)\infty + \sum_{\text{finite } v} v(F)X_v$ and $D_k$ is the proper transform of $D$ by $\sigma_k$;
(iii) the divisor $\infty_k$ is determined by the equation $\sigma_k^* T_n = 0$ in $X_k$ and;
(iv) the $C_{v,i,k}$ are the different components of the exceptional divisor of the blow up.
Then we will intersect both sides of the above expression with a particular class of curves on $X_k$ and finally we will get the Mahler formula from a limit argument on $k$.

1.1. **Notation and conventions.** Unless otherwise stated $K$ will denote a number field with ring of integers $\mathcal{O}_K$. For a place $v$ of $K$, $\mathbb{C}_v = \bar{K}_v$ will denote the completion of the localization $K_v$ at $v$. $\mathbb{P}_K^n$ will denote the $n$-dimensional projective space over $K$ and similar for $\mathbb{P}_{\mathcal{O}_K}^n$. The symbol $\mathcal{L}$ will be used to denote line bundles on different kind of varieties. $V$ will denote an algebraic variety (must of the time projective) of dimension $n$. $M$ will denote a complex projective variety of dimension $n$ and, if $\mathcal{L}$ is a line bundle on $M$, the term $c_1(\mathcal{L}, ||.) = c_1(\bar{\mathcal{L}})$ will denote a $(1, 1)$ current, similar to the first Chern form of $\mathcal{L}$. In the presentation of the arithmetic as well as the geometric interception theory, $X$ will denote a Macaulay arithmetic variety of absolute dimension $n + 1$ over $\text{Spec}(\mathcal{O}_K)$. This means that all local rings are Macaulay and there exist a flat, proper and finite type map $f: X \to \text{Spec}(\mathcal{O}_K)$ whose fibres $X_v$ over the places $v$ of $K$ are projective varieties of dimension $n$. For an arithmetic variety $X$ and a line bundle $\mathcal{L}$ on $X$, we denote by $\mathcal{L}_v = \mathcal{L} \otimes_{\mathcal{O}_X} \text{Spec}(K_v)$ the restriction of $\mathcal{L}$ to the fibre $X_v$. The line bundle $\mathcal{L}$ will comes sometimes equipped with hermitian metrics $||.||_{P,v}$ on the fibres $\mathcal{L}_{P,v}$ over each point $P \in X_v$. For a set $\mathcal{L}_1, ..., \mathcal{L}_i$ of metrized line bundles on an arithmetic variety $X$, the expression $\hat{\text{deg}}(\hat{c}_1(\mathcal{L}_1) ... \hat{c}_1(\mathcal{L}_i)|Z)$ will represent the arithmetic intersection degree of the line bundles $\mathcal{L}_1, ..., \mathcal{L}_i$ over a cycle $Z \subset X$ of dimension $i$. We will pay special attention, in sections 4, 5 and 6 of this paper, to the arithmetic variety $X = \mathbb{P}_{\mathcal{O}_K}^n$ as well as arithmetic varieties $X_k$ that arise from blowing-up subschemes $Y_k$ of $X$. By a model for a map $\varphi: \mathbb{P}_K^n \to \mathbb{P}_K^n$ we mean a system $(p_0 : ... : p_n)$ of polynomials in $K[T_0, ..., T_n]$ representing the map in the coordinates $(T_0 : ... : T_n)$. The ideal generated by a system of polynomials $p_0, ..., p_m$ will be denoted by $\langle p_0, ..., p_m \rangle$ and the symbol rad$(I)$ will be used to denote the radical of any ideal $I$.

2. **Self maps on algebraic varieties**

In this section we will give some examples of self maps on algebraic varieties. Let $\varphi: V \to V$ be a map of the algebraic variety $V$ to itself. Under certain conditions on the variety $V$ (existence of a line bundle $\mathcal{L}$ with good properties), we will associate to $V$ and $\mathcal{L}$ a canonical height function and a canonical measure. The canonical height will be a generalization of both, the Neron-Tate on abelian varieties and the naive height on $\mathbb{P}^1$. The canonical measure will be a generalization of
Brolin’s measure for maps on \( \mathbb{P}^1 \). Let’s start by giving some examples of self maps on algebraic varieties.

**Example 2.1.** Suppose that \( K \) is a field. A map \( \varphi : \mathbb{P}^n_K \to \mathbb{P}^n_K \) of algebraic degree \( d \) is given by a set of degree \( d \) homogenous polynomials \( p_0(T_0, ..., T_n), ..., p_n(T_0, ..., T_n) \in K[T_0, ..., T_n] \), such that \( \text{rad}(\langle p_0, ..., p_n \rangle) = \langle T_0, ..., T_n \rangle \).

**Example 2.2.** Let \( A \) be an abelian variety. The multiplication by \( n \) morphism \([n] : A \to A\) represents a self map on \( A \).

**Example 2.3.** Consider the smooth toric variety \( \mathbb{P}(\Delta) \) defined over \( \bar{\mathbb{Q}} \). For each \( p \geq 2 \) there is a morphism \([p] : \mathbb{P}(\Delta) \to \mathbb{P}(\Delta)\) associated with the multiplication by \( p \) in \( \Delta \).

### 2.1. Canonical height.

Suppose that \( V \) is a projective variety defined over a number field \( K \), and \( \varphi : V \to V \) a morphism, with the property that there exist a line bundle \( \mathcal{L} \) on \( V \) and a real number \( \alpha > 1 \) such that \( \varphi^* \mathcal{L} \sim \mathcal{L}^\alpha \). Suppose also that \( h_\mathcal{L} \) represents a height function associated to \( \mathcal{L} \) (for a detailed discussion see B-3 in [HiS00]). Then we can find (see for example B-4 in [HiS00]) a positive height function \( h_\varphi \) on \( V(\bar{K}) \), defined as the limit \( h_\varphi(P) = \lim_{k \to \infty} \frac{h_\mathcal{L}(\varphi^k(P))}{\alpha^k} \) with the properties:

1. \( h_\varphi \) satisfies Northcott’s theorem: points with coordinates in \( \bar{K} \) with bounded degree and bounded height are finite in number.
2. \( h_\varphi(\varphi(P)) = \alpha h_\varphi(P) \).
3. \( h_\varphi \) is a non-negative function.
4. \( h_\varphi(P) = 0 \) if and only if \( P \) has a finite forward orbit under iteration of the map \( \varphi \).
5. \(|h_\varphi(P) - h_\mathcal{L}(P)|\) is bounded on \( V(\bar{\mathbb{Q}}) \).

Condition (iv) above expresses the general ideal that canonical heights should reflect the complexity of algebraic cycles under iteration of maps. With the use of the arithmetic intersection theory [Zhan95] we can extend the positive function \( h_\varphi \) to algebraic cycles in \( V \). The value \( h_\varphi(Y) \) will denote the canonical height of a \( p \)-cycle \( Y \) inside \( V \) and we will have some similar properties (Check [Zhan95] theorem 2.4):

1. \( h_\varphi(Y) \geq 0 \).
2. \( h_\varphi(Y) \) satisfy the functional equation \( h_\varphi(\varphi_*Y) = \alpha h_\varphi(Y) \).
3. If the orbit \( \{Y, f(Y), ...\} \) is finite, then \( h_\varphi(Y) = 0 \).

In section 3 of this paper, we will develop the necessary arithmetic intersection theory to define the height \( h_{\mathcal{L}_1, ..., \mathcal{L}_n} \) attached to a collection of metrized line bundles \((\mathcal{L}_i, \| \cdot \|_i)\). In [Zhan95] this results are generalized to define heights associated to limits of metrics, the so-called...
Let $M$ be a number field and $v$ a place of $K$. Consider the projective variety $V$ defined over $\bar{K}_v$ and $L$ a line bundle on $V$ such that $\phi : L^\alpha \rightarrow \varphi^* L$ for some $\alpha > 1$. Assume that we have chosen a continuous and bounded metric $\|\cdot\|_v$ on each fibre of $L_v$.

The following theorem is due to Shouwu Zhang [Zhan95]:

**Theorem 2.4.** The sequence defined recurrently by $\|\cdot\|_{v,1} = \|\cdot\|_v$ and $\|\cdot\|_{v,n} = (\varphi^* \varphi^* \|\cdot\|_{v,n-1})^{1/\alpha}$ for $n > 1$, converge uniformly on $V(\bar{K}_v)$ to a metric $\|\cdot\|_{\varphi,v}$ (independent of the choice of $\|\cdot\|_{v,1}$) on $L_v$ which satisfies the equation $\|\cdot\|_{\varphi,v} = (\varphi^* \bar{\varphi}^* \|\cdot\|_{\varphi,v})^{1/\alpha}$.

**Proof.** See theorem (2.2) in S. Zhang [Zhan95]. Denote by $h$ the continuous function $\log \|\cdot\|_1$ on $V(\bar{K}_v)$. Then

$$\log \|\cdot\|_n = \log \|\cdot\|_1 + \sum_{k=0}^{n-2} \frac{1}{\alpha} (\varphi^* \bar{\varphi}^*)^k h.$$ 

Since $\|(\frac{1}{\alpha} \varphi^* \bar{\varphi}^*)^k h\|_{\sup} \leq (\frac{1}{\alpha})^k \|h\|_{\sup}$, it follows that the series given by the expression $\sum_{k=0}^{\infty} (\frac{1}{\alpha} \varphi^* \bar{\varphi}^*)^k h$ converges absolutely to a bounded and continuous function $h^\varphi$ on $V(\bar{K}_v)$. Let $\|\cdot\|_{\varphi,v} = \|\cdot\|_1 \exp(h^\varphi)$, then $\|\cdot\|_n$ converges uniformly to $\|\cdot\|_{\varphi,v}$ and it is not hard to check that $\|\cdot\|_{\varphi,v}$ satisfies

$$\|\cdot\|_{\varphi,v} = (\varphi^* \bar{\varphi}^* \|\cdot\|_{\varphi,v})^{1/\alpha},$$

which was the result we wanted to prove. \qed

**Definition 2.5.** The metric $\|\cdot\|_{\varphi,v}$ is called the canonical metric on $L_v$.

**Example 2.6.** Consider the line bundle $L = \mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n_{\mathbb{Q}}$ and the map $\phi_2(T_0 : \ldots : T_n) = (T_0^2 : \ldots : T_n^2)$. If we choose the Fubini-Study metric $\|(\lambda_0 T_0 + \ldots + \lambda_n T_n)(a_0 : \ldots : a_n)\|_F = \frac{\sum \lambda_i a_i}{\sqrt{\sum a_i^2}}$ as our smooth metric $\|\cdot\|_1$ on $L_{\mathbb{C}}$, the limit metric we get at infinity is:

$$\|(\lambda_0 T_0 + \ldots + \lambda_n T_n)(a_0 : \ldots : a_n)\|_{nv} = \frac{\sum \lambda_i a_i}{\sup_i(\|a_i\|)}.$$

2.3. **Canonical measure and integral at infinite places.** In this part we set up what will be called the integral at infinite places attached to a map $\varphi : V \rightarrow V$ and a place $v$ of $K$ over infinity. We need to develop some analytic theory related to the complex points of $V$.

Let $M$ be a $n$-dimensional complex projective variety, $\varphi : M \rightarrow M$ a map on $M$ and $L$ an ample line bundle on $M$. Suppose also that

adelic metrized line bundles. The canonical height will naturally arise associated to a special kind of adelic metric on our line bundle $L$, which will be called the canonical metric.
for some $\alpha > 1$ we have $\phi: \mathcal{L}^\alpha \overset{\sim}{\rightarrow} \varphi^* \mathcal{L}$, and that the line bundle $\mathcal{L}$ is equipped with the canonical metric $\|\cdot\|_\varphi$ on the fibres. Let $U \subset M$ be an open set. The function $x \mapsto -\log \|s(P)\|_\varphi$ for a non-zero holomorphic section $s$ on $U$, is not necessarily smooth, and due to this fact, the first Chern “form” $c_1(\mathcal{L}, ||\cdot||) = \frac{1}{(2\pi i)} \partial \bar{\partial} \log \|s_1(P)\|_\varphi$ may be no more than a distribution. We would like to define the product $c_1(\mathcal{L}, ||\cdot||) \wedge \ldots \wedge c_1(\mathcal{L}, ||\cdot||)$. Unfortunately for general currents we do not have a product as we do for smooth currents. Results of Bedford, Taylor and Demailly \cite{BeT82, Dem92, Dem93} and \cite{Dem97}, allow us to consider a product of currents with good properties. We follow the presentation in \cite{Mai00}.

**Definition 2.7.** (Lelong). Let $U$ be an open set of complex manifold $M$ of dimension $n$. A current $T \in D^{p,p}(U)$ is said to be positive ($T \geq 0$) if for every choice of $C^\infty (1,0)$-forms $\alpha_1, \ldots, \alpha_{n-p}$ with compact support on $U$, the distribution $T \wedge (i\alpha_1 \wedge \bar{\alpha}_1) \ldots \wedge (i\alpha_{n-p} \wedge \bar{\alpha}_{n-p})$ is a positive measure on $U$.

**Example 2.8.** A locally integrable function $u$ on $M$ is says to be plurisubharmonic if the hessian $i\mathcal{D} \mathcal{D} u = i \sum \partial^2 u / \partial z_j \partial \bar{z}_m \wedge \bar{z}_m \geq 0$ on $M$. For basic properties of plusisubharmonic functions see for example \cite{Dem92} or \cite{Dem93}.

**Example 2.9.** Let $Y$ be an algebraic $p$–cycle on $M$. The $(p,p)$–current $\delta_Y$ of integration on $Y$, is a positive current on $M$.

**Definition 2.10.** (Bedford-Taylor). Let $T$ be a positive closed current of type $(p,p)$ and $u$ a plurisubharmonic function locally bounded on $U$. We define the product $(dd^c u) \wedge T = dd^c(uT)$.

**Remark 2.11.** The product $Tu$ is well defined and in general we can define $(dd^c u_1) \wedge (dd^c u_2) \ldots (dd^c u_q) \wedge T = dd^c(u_1 \wedge (dd^c u_2) \ldots (dd^c u_q) \wedge T)$. By prop 1.2 in \cite{Dem97}, the current $(dd^c u_1) \wedge (dd^c u_2) \ldots (dd^c u_q) \wedge T$ is a positive closed current of bidegree $(p+q, p+q)$.

**Lemma 2.12.** Let $\mathcal{L}$ be an ample line bundle on $M$ and let $\varphi: M \rightarrow M$ be a map with the property that for some $\alpha > 1$, there is an isomorphism $\phi: \mathcal{L}^\alpha \overset{\sim}{\rightarrow} \varphi^* \mathcal{L}$. Assume that $\|\cdot\|_\varphi$ is the canonical metric on $\mathcal{L}$. The function $P \mapsto -\log \|s(P)\|_\varphi$ is plurisubharmonic on the open set $U = M - \text{div}(s)$ and therefore the current $i\partial \bar{\partial}(-\log \|s(P)\|_\varphi)$ is a positive current on $U$.

**Proof.** The proof is basically taken from \cite{Kaw00}. Consider the continuous and positive function $H(P) = \frac{\|s(P)\|_\varphi}{\|s(P)\|_\varphi^2}$ on $M$. Define $c = \min_{P \in M} H(P)$. By changing $\varphi$ by $c\varphi$ if necessary, we can have $H > 1$.
(this will not affect the result by the second part of theorem 2.2 in [Zhan95]). In general we have
\[ \frac{\|s\|_n}{\|s\|_{n-1}} = \frac{\|s \circ \varphi\|_{n-1}}{\|s \circ \varphi\|_{n-2}}, \]
and then \(-\log \|s(P)\|_n \leq -\log \|s(P)\|_{n-1}\) for every \(n > 2\) and the sequence \(\{-\log \|\cdot\|_n\}_{n=1}^\infty\) is a non increasing sequence of plurisubharmonic function converging to \(-\log \|s(P)\|_\varphi.\)

\[\square\]

**Proposition 2.13.** Let \(s_i (i=1..q)\) be sections of the line bundles \(L_i\) respectively, such that the divisors \(\text{div}(s_i)\) meet properly on \(M\). Let us denote \(c_1(L_i) = c_1(L_i, \|\cdot\|_\varphi)\) the Chern “form” associated to the canonical metric studied in proposition 2.4, then the current
\[-\log \|s_i\|_\varphi c_1(L_1) \ldots c_1(L_{i-1}) \cdot \delta_{\text{div}(s_{i+1})} \ldots \delta_{\text{div}(s_q)}\]
is a well defined current and
\[\int_M (-\log \|s_i\|_n) c_1(L_1, \|\cdot\|_n) \ldots c_1(L_{i-1}, \|\cdot\|_n) \cdot \delta_{\text{div}(s_{i+1})} \ldots \delta_{\text{div}(s_q)}\]
tends to
\[\int_M (-\log \|s_i\|_\varphi) c_1(L_1) \ldots c_1(L_{i-1}) \cdot \delta_{\text{div}(s_{i+1})} \ldots \delta_{\text{div}(s_q)}.\]

**Proof.** We have that \(c_1(L_i, \|\cdot\|_\varphi)\) is a positive current that can be written locally in the form \(dd^c u\) where \(u = -\log \|\cdot\|_\varphi\) is a plurisubharmonic function on \(M\). On the other hand \(\delta_{\text{div}(s_i)}\) are closed and positive (Check theorem 3.5 in [Dem92]). The sequence of currents
\[\{-\log \|s_i\|_n c_1(L_1, \|\cdot\|_n) \ldots c_1(L_{i-1}, \|\cdot\|_n) \cdot \delta_{\text{div}(s_{i+1})} \ldots \delta_{\text{div}(s_q)}\}_{n=1}^\infty\]
converge weakly to \(-\log \|s_i\|_\varphi c_1(L_1) \ldots c_1(L_{i-1}) \cdot \delta_{\text{div}(s_{i+1})} \ldots \delta_{\text{div}(s_q)}\) after the following general proposition proved in [Dem97]. \[\square\]

**Proposition 2.14.** (Demailly) Denote by \(L(u)\) the set of point where the plurisubharmonic function \(u\) is not locally bounded. Let \(U\) be an open set of \(M\) and \(T \in D_{+}^{p,p}(U)\) a positive closed current of type \((p,p)\). Let also \(u_1, \ldots, u_q\) be plurisubharmonic functions on \(U\), such that for every choice of indices \(j_1 < j_2 \ldots < j_m\) inside \(\{1, 2, \ldots, q\}\) the intersection \(L(u_{j_1}) \cap \ldots \cap L(u_{j_m}) \cap \text{Supp}(T)\) is contained in an analytic set of complex dimension \(\leq n - p + m\). One can then construct the currents \(u_1(\ddc u_2) \wedge \ldots \wedge (\ddc u_q) \wedge T\) and \((\ddc u_1) \wedge \ldots \wedge (\ddc u_q) \wedge T\) of mass locally finite over \(U\) and uniquely characterized by the fact: for every non-increasing sequences \((u_1^k), \ldots, (u_q^k)\) of plurisubharmonic functions converging punctually to \(u_1, \ldots, u_q\) respectively, we have that \(u_1^k(\ddc u_2^k) \wedge \ldots \wedge (\ddc u_q^k) \wedge T\) and \((\ddc u_1^k) \wedge \ldots \wedge (\ddc u_q^k) \wedge T\) converge
Proposition 2.17. Suppose that \( \int \) and \( T \) with 

\[ \text{Proof.} \]

Let \( T \) with \( \sum \) or [Dem93]. Thm. 2.5 and Pro. 2.9. \( \square \)

\[ \square \]

which is the result we wanted to prove.

The canonical current associated to \( \varphi \) is defined as \( T_\varphi = c_1(\mathcal{L}, \|\|_\varphi) \). The canonical distribution associated to \( \varphi \) is

\[ d\mu_\varphi = c_1^n(\mathcal{L}_\varphi, \|\|_\varphi). \]

Proposition 2.16. The canonical distribution is in fact a measure, which we call the canonical measure.

\[ \text{Proof.} \] The (1, 1)–current \( T_\varphi \) can be identified with an expression \( T_\varphi = \sum_{i,j} T_{i,j}dz_i \wedge dz_j \) where the coefficients \( T_{i,j} \) are distributions. Consider the \( (n-1,0)–\)form \( \alpha = \sum_{|I|=n-1} \alpha_I dz_I \). The fact that \( T_\varphi \geq 0 \) forces \( \sum T_{i,j} \alpha_{n-i} \bar{\alpha}_{n-j} \) to be a positive measure for every \( \alpha \). As a consequence the \( T_{i,j} \) are complex measures with \( T_{i,j} = \bar{T}_{i,j} \). In the same way the \( p–\)current \( \bigwedge_{i=1}^p T_\varphi \) has measures as coefficients for each \( p \). \( \square \)

Proposition 2.17. Suppose that \( A \) is a subset of \( M \) such that \( \mu(A) = \int_A d\mu_\varphi < \infty \), then

(i) \( \mu_\varphi(\varphi(A)) = (\deg \varphi)^n \mu_\varphi(A) < \infty \) whenever \( \varphi|A \) is injective,

(ii) \( \mu_\varphi(\varphi^{-1}(A)) = \mu_\varphi(A) < \infty \).

\[ \text{Proof.} \]

Let \( \deg(\varphi) = d \) be the algebraic degree of \( \varphi \). Take an open set \( W \) with \( \mu(W) < \infty \). Assume that we have \( \varphi^{-1}(W) = \bigcup_{i=1}^m U_i \) where \( \varphi : U_i \to W \) is injective for each \( i \) and let \( U \) denotes any of the \( U_i \). Consider \( n \) local sections \( s_i \neq 0 \) of \( \mathcal{O}(1) \) holomorphic on \( W \), in this case \( d\mu_\varphi = \frac{1}{(\pi i)^n} \partial \overline{\partial} (g_1) \cdot \ldots \cdot (\pi i)^n \partial \overline{\partial} (g_n) \), where \( g_i = \log(\|s_i(P)\|_\varphi) \) and we have

\[ \mu_\varphi(W) = \int_W \frac{1}{(\pi i)^n} \partial \overline{\partial} \log(\|s_1(P)\|_\varphi) \ldots \frac{1}{(\pi i)^n} \partial \overline{\partial} \log(\|s_n(P)\|_\varphi) \]

\[ = \int_U \frac{1}{(\pi i)^n} \partial \overline{\partial} \log(\|s_1(\varphi(P))\|_\varphi) \ldots \frac{1}{(\pi i)^n} \partial \overline{\partial} \log(\|s_n(\varphi(P))\|_\varphi) \]

\[ = d^n \int_U \frac{1}{(\pi i)^n} \partial \overline{\partial} \log(\|\varphi^*s_1(P)\|_\varphi) \ldots \frac{1}{(\pi i)^n} \partial \overline{\partial} \log(\|\varphi^*s_n(P)\|_\varphi) \]

\[ = d^n \mu_\varphi(U), \]

and

\[ \mu_\varphi(\varphi^{-1}(W)) = d^n \mu_\varphi(U) = \mu_\varphi(W), \]

which is the result we wanted to prove. \( \square \)
Definition 2.18. Let $V$ be a projective variety defined over a number $K$, $\varphi : V \to V$ a map on $V$ and $(\mathcal{L}, \|\cdot\|_v)$ a metrized ample line bundle on $V$ with the property that there exist an isomorphism $\psi : \mathcal{L}^a \xrightarrow{\sim} \varphi^* \mathcal{L}$. Let $v : K \hookrightarrow \mathbb{C}$ be a place of $K$ over infinity. The canonical measure $d\mu_{\varphi,v}$ is the canonical measure on $V \otimes_v \mathbb{C}$ associated to $\varphi_v = \varphi \otimes_v \mathbb{C}$ and $\mathcal{L}_v = \mathcal{L} \otimes_v \mathbb{C}$.

2.4. Examples. Here we revisit some examples of self maps on algebraic varieties with extra information about canonical heights and measure.

Example 2.19. Suppose that we are working with a number field $K$ and $\varphi = (p_0 : \ldots : p_n) : \mathbb{P}^n_K \to \mathbb{P}^n_K$ is a map on the $n$-dimensional projective space over $K$. In general it is hard to get a closed form for the iterate of such a map. In case that for some natural $k$ we have $p_i(T_0, \ldots, T_n) = T_i^k$ for each $0 \leq i \leq n$, we obtain the so-called naive height on $\mathbb{P}^n_K$:

$$h_{nv}(t_0 : \ldots : t_n) = \frac{1}{[K : \mathbb{Q}]} \log \prod_{\text{places } v \text{ of } K} \sup(t_0|_v, \ldots, t_n|_v)^{N_v},$$

where $N_v = [K_v : \mathbb{Q}_w]$ and $w$ is the place of $\mathbb{Q}$ such that $v \mid w$. The associated measure $d\mu_{\varphi}$ is the normalized Haar measure on the Torus $S^1 \times \ldots \times S^1$. If $T_0, T_1, \ldots, T_n$ represent projective coordinates in $\mathbb{P}^n$, the canonical metric at infinity whose curvature gives the canonical measure is

$$\|(\lambda_0 T_0 + \ldots + \lambda_n T_n)(a_0 : \ldots : a_n)\|_{nv} = \frac{|\lambda_0 a_0 + \ldots + \lambda_n a_n|}{\sup(|a_0|, \ldots, |a_n|)}.$$

Example 2.20. Let $A$ be an abelian variety and $\mathcal{L}$ a symmetric line bundle on $A$. The multiplication by $n$, $[n] : A \to A$, satisfies $[n]^* \mathcal{L} \cong \mathcal{L}^{n^2}$, the canonical height is the Neron-Tate height $\hat{h}_{NT}$ and the canonical measure, after results in [Mor85], is the Haar measure on $A$.

Example 2.21. For the smooth toric variety $\mathbb{P}(\Delta)$ and the morphism $[p] : \mathbb{P}(\Delta) \to \mathbb{P}(\Delta)$ we have $[p]^*(\mathcal{O}_{\mathbb{P}(\Delta)}) \cong \mathcal{O}_{\mathbb{P}(\Delta)}^p$ and we can build a canonical height $\hat{h}_{\mathcal{O}_{\mathbb{P}(\Delta)}}$. In [Mai00], V. Maillot presents explicit formulae for the canonical measure as well as several properties of the canonical height. The Mahler formula is established in this case as a consequence of the vanishing of the canonical multithread of the whole variety $\mathbb{P}(\Delta)$. 
3. Arithmetic intersection theory

In this section we develop the arithmetic intersection theory for arithmetic varieties with Cohen-Macaulay local rings. The \( n+1 \)-dimensional variety \( X \) will be Macaulay and equipped with a finite type, flat and proper map \( f : X \to \text{Spec}(\mathcal{O}_K) \), where \( K \) is a number field. First we introduce the geometric intersection:

**Definition 3.1.** We say that a \( q \)-cycle \( D \) in \( X \) is locally regular complete intersection (l.r.c.i.) if it is locally given by the intersection of a regular sequence of length \( n+1-q \).

**Remark 3.2.** A l.r.c.i. \( n \)-cycle \( D \) is just a Cartier divisor.

**Definition 3.3.** Suppose that the \( q_1 \)-cycle \( D_1 \) is l.r.c.i. and \( D_2 \) is any \( q_2 \)-cycle. Assume that they have no common components. Then the intersection is given by

\[
(D_1.D_2) = \sum_{i=0}^{q_1} (-1)^i \text{Tor}_i(\mathcal{O}(D_1), \mathcal{O}(D_2)).
\]

In case \( q_1 + q_2 \leq n+1 \) we define the degree of the intersection as

\[
\text{deg}(D_1.D_2) = \sum_{i=0}^{q_1} (-1)^i \text{length}(	ext{Tor}_i(\mathcal{O}(D_1), \mathcal{O}(D_2))).
\]

**Definition 3.4.** If a cycle \( D_1 \) is such that for some natural \( m \), \( mD \) is l.r.c.i., we can extend the intersection as \( (D_1.D_2) = 1/m(D_1.D_2) \). An \( n \)-cycle with this property is called a \( \mathbb{Q} \)-Cartier divisor.

The intersection just defined satisfy many of the desirable properties for intersection numbers. Symmetry is clear from the definition and associativity is a consequence of the convergence of the Tor spectral sequence [Ser75]. Bilinearity is a consequence of the lemma:

**Lemma 3.5.** Let \( A \) be a commutative ring, \( I \) an ideal and \( f \) a non-zero divisor in \( A \). Then we have the exact sequence:

\[
0 \to A/I \to A/fI \to A/fA \to 0.
\]

**Proof.** The canonical map from \( A/fI \to A/fA \) has kernel \( fA/fI \). On the other hand the map \( \gamma : A \to fA/fI \), given by \( \gamma(1) = [f] \) has kernel \( I \), because if two elements \( a \in A \) and \( i \in I \) satisfy \( af = if \), then \( a = i \) because \( f \) is not a zero divisor. \( \square \)

**Proposition 3.6.** Let \( X \) be a projective arithmetic variety, \( C \) a Cohen-Macaulay projective curve in \( X \) and \( D \) a Cartier divisor, then

\[
\text{deg}(D.C) = \text{deg}_{C} \mathcal{O}_X(D)|_C.
\]
Proof. If $C$ is a projective curve and $L$ is a line bundle on $X$ a projective variety, one can then speak of $(L.C)$ for $L$ is the difference in $\text{Pic}(X)$ between two very ample line bundles each of them having sections with no common components with $C$. The result follows because a line bundle on a Cohen Macaulay curve has a well defined degree. \qed

**Definition 3.7.** If $\mathcal{L}$ is a line bundle on $X$ we will denote by $c_1(\mathcal{L})$ the class of divisors determine by $\mathcal{L}$ and by $c_1(\mathcal{L})^i$ the intersection of any element in this class with itself $i$ times.

**Proposition 3.8.** If $D_2$ has codimension $n+1$ and $D_1$ is locally given by one equation, then $\deg(D_1.D_2) = 0$.

Proof. Suppose that $f$ is the equation defining $D_1$ and $I$ is the ideal of $D_2$ in the local ring $\mathcal{O}_x = A$. The assumption on the dimension of $D_2$ implies that the modules $A/I$ and $\text{Tor}_i(A/I,A/f)$ are of finite length for all $i$. Now the result follows because the length is an additive function and we have the exact sequence:

$$0 \to \text{Tor}_1(A/I,A/f) \to A/I \to A/I \to A/(I+(f)) \to 0.$$ \qed

**Proposition 3.9.** Suppose that $\sigma: X_1 \to X$ is a map of projective Arithmetic varieties over $\text{Spec}(\mathcal{O}_K)$. Let $C$ be a closed $1-$cycle of $X_1$ and $\mathcal{L}$ a line bundle on $X$. If $C$ is contracted by $\sigma$ to a subscheme of $X$ of codimension $n+1$ the intersection number $\deg(\sigma^*(L).C)$ is zero.

Proof. $L$ can be realized as the line bundle associated to the difference of two very ample divisors on $X$ each of them having no intersection with $\sigma_*(C)$. The reciprocal images of these divisors in $Y$ do not meet $C$ and the result follows. \qed

The geometric intersection, however, does not take into account the places of $K$ over infinity. An original idea of Arakelov [Ara74], later developed by Szpiro [Sz95], Bost [BG91], Gillet [BG91], Soulé [BG91], Zhang [Zha92, Zha95], Faltings and many others, allow us to consider an intersection theory that equally value all places of $K$.

**Definition 3.10.** Let $X$ be a Cohen-Macaulay arithmetic variety of dimension $n+1$, defined over a number field $K$ and $\mathcal{L} = (\mathcal{L}, ||.||)$ a hermitian line bundle on $X$. Suppose that $v_1, ..., v_s$ are the different places of $K$ over infinity. We will denote by $c_1(\mathcal{L})$ the vector $(c_1(\mathcal{L}), c_1(\mathcal{L})_{v_1}, ..., c_1(\mathcal{L})_{v_s})$, where $c_1(\mathcal{L})$ is the class of equivalent divisors determined by $\mathcal{L}$ and $c_1(\mathcal{L})_{v_i}$ is the $(1,1)-$current associated to the metric $||.||_{v_i}$ on $\mathcal{L}_{v_i}$. 
Proposition 3.11. Let $X$ be a Cohen-Macaulay arithmetic variety of dimension $n+1$, defined over a number field $K$. Let $Z \in Z_k(X)$ be a cycle on $X$ and $\mathcal{L}_1, ..., \mathcal{L}_k$ a set of hermitian line bundles on $X$. Then the number $\deg_Z(\mathcal{O}_1(\mathcal{L}_1) ... \mathcal{O}_k(\mathcal{L}_k)) \in \mathbb{R}$, is completely determined by the properties:

(i) is $k$–linear.

(ii) is symmetric.

(iii) for $k = 0$ and $Z = \sum_i n_i P_i \ (P_i \in X_v)$, we have then $\deg_Z = \sum_i n_i N_P \log N(v)$ where $N_P = [K(P) : K].$

(iv) for $k \geq 1$ and $s_k \neq 0$, a section of $\mathcal{L}_k$ which meets $Z$ properly, we have

$$\hat{\deg}_Z(\mathcal{O}_1(\mathcal{L}_1) ... \mathcal{O}_k(\mathcal{L}_k)|Z) = \hat{\deg}_Z(\mathcal{O}_1(\mathcal{L}_1) ... \mathcal{O}_{k-1}(\mathcal{L}_{k-1})|Z, \text{div}(s_k))$$

$$- \sum_{v/\infty} \int_{X(C)} \delta_{Z(C)} \log \|s_k\|_{k,v} c_1(\mathcal{L}_1)_v ... c_1(\mathcal{L}_{k-1})_v,$$

where $\sum_{v/\infty}$ is taken over the places of $K$ at infinity.

Proof. Conditions (i), (iii) and (iv) are sufficient to determine recursively the number $\hat{\deg}_Z(\mathcal{O}_1(\mathcal{L}_1) ... \mathcal{O}_k(\mathcal{L}_k))$. Suppose that we consider sections $s_i$ of the line bundles $\mathcal{L}_i$ respectively, such that the divisors $\text{div}(s_i)$ meet properly in $X$. Let $v$ be a place of $K$ over infinity. Introducing the star product $g_{1,v} * g_{2,v} * ... * g_{k,v}$ of the currents $g_i = - \log \|s_i\|_v$ we can state a non-recursive formula of the arithmetic degree

$$\hat{\deg}_Z(\mathcal{O}_1(\mathcal{L}_1) ... \mathcal{O}_k(\mathcal{L}_k)) = (\text{div}(s_1) \text{div}(s_k))_{\text{finite}} + \sum_{v/\infty} \int_{Z(C)} g_{1,v} * ... * g_{k,v},$$

where the first term on the right is representing the weighted sum

$$(\text{div}(s_1) \text{div}(s_k))_{\text{finite}} = \sum_{e \text{ finite}} \deg(\text{div}(s_1)_e ... \text{div}(s_k)_e) \log N(v).$$

The condition (ii) will be a consequence of the following lemma. □

Lemma 3.12. The arithmetic degree $\hat{\deg}_Z(\mathcal{O}_1(\mathcal{L}_1) ... \mathcal{O}_k(\mathcal{L}_k))$ is a symmetric function of the $\mathcal{L}_i$.

Proof. The geometric intersection is symmetric on the divisors $\text{div}(s_i)$. Let’s concentrate then in the term involving Green functions over a fixed place $v$. Suppose that $g_i$ (for $i = 1, 2$) are Green currents of “log” type along the cycles $Z_1$ and $Z_2$ and relative to $v$ (for the existence see lemma 1.2.2 of [BGS04]). We have $g_1 * g_2 = g_2 * g_1 + \delta T_1 + \delta T_2$ for some currents $T_1 \in D^{p-1,p}$ and $T_2 \in D^{p,p-1}$, then $\int_Z g_1 * g_2 = \int_Z g_2 * g_1 + \int_Z \delta T_1 + \int_Z \delta T_2$ and by Stokes theorem we obtain the symmetry for the arithmetic degree. □
Proposition 3.13. Suppose that \( \mathcal{L} \) is a hermitian line bundle on \( X \) and \( f \in K(X) \) is a rational function on \( X \). Then:
\[
\deg(\hat{c}_1(\mathcal{O}(f)).\hat{c}_1^n(\mathcal{L})) = 0.
\]

Proof. The curvature of the trivial bundle \( c_1(\mathcal{O}(f)) = 0 \). Using this result and the symmetry of the arithmetic degree we can reduce to the case of dimension 1, which is nothing else but the product formula (see for example the treatment in [Szp]).

Proposition 3.14. Suppose that \( \mathcal{L} \) is a hermitian line bundle on \( X \) and \( f \in K(X) \) is a rational function on \( X \) then,
\[
\deg(\hat{c}_1^n(\mathcal{L})|\text{div}(f)) = \sum_{v/\infty} \int_{X_v} \log |f|_v d\mu_v.
\]

Proof. We have that \( \deg(\hat{c}_1(\mathcal{O}(f)).\hat{c}_1^n(\mathcal{L})) = 0 \) and also that
\[
\deg(\hat{c}_1(\mathcal{O}(f)).\hat{c}_1^n(\mathcal{L})) = \deg(\hat{c}_1(\mathcal{L})^n|\text{div}(f)) - \sum_{v/\infty} \int_{X_v(\mathcal{C})} \log |f|_v(c_1(\mathcal{L}))^n.
\]
which gives the formula we wanted.

Remark 3.15. In the notation of [PST05] we can write
\[
(\text{div}(f).\mathcal{L})_{\text{Ar}} = \deg(\hat{c}_1(\mathcal{L})|\text{div}(f)).
\]

Definition 3.16. Let \( Y \in Z_q(X) \) be a \( q \)-cycle inside the arithmetic variety \( X \) and \( \mathcal{L}_1, ..., \mathcal{L}_q \) ample line bundle on \( X \). The real number
\[
h_{\mathcal{L}_1, ..., \mathcal{L}_q}(Y) = \deg_Y(\hat{c}_1(\mathcal{L}_1)...\hat{c}_1(\mathcal{L}_q)|Y)
\]
is called the multi-height of \( Y \) relative to \( \mathcal{L}_1, ..., \mathcal{L}_q \).

Let’s denote by \( X_K \) the generic fibre of \( X \). Suppose that we have a map \( \varphi : X_K \to X_K \) and an ample line bundle \( \mathcal{L} \) on \( X_K \) such that for some number \( \alpha \), we have the isomorphism of line bundles \( \psi : \mathcal{L}^\alpha \cong \varphi^* \mathcal{L} \). Then we can build a sequence of arithmetic varieties \( X_k, k = 1, 2, ..., \) models of \( X_K \) over \( \text{Spec}(\mathcal{O}_K) \), and metrized line bundles \( (\mathcal{L}_k, ||.||_k) \) on \( X_k \), such that \( ||.||_{k+1,v} = (\psi^*\varphi^* ||.||_{k,v})^{1/\alpha} \) for each place \( v \). The detailed discussion of the construction of the \( \mathcal{L}_k \) can be found in page 10 and 11 of [Zhan95], it constitutes an example of a sequence of adelic metrized line bundles, that is, a sequence of line bundles with good metrics at every place. The numbers \( \deg_Y(\hat{c}_1((\mathcal{L}_k, ||.||_k))^{\dim(Y)+1}|Y) \) converge for every \( p \)-cycle \( Y \subset X_K \) (see theorem 1.4 of [Zhan95]) and the limit will be called \( \deg(c_1(\mathcal{L}, ||.||_v)^{\dim Y+1}|Y) \). This number is now not depending on the \( X_k \).
\textbf{Definition 3.17.} Under the conditions just discussed we define 
\[ h_\varphi(Y) = \frac{\deg(c_1(L, ||\varphi||^{\dim+1}|Y))}{c_1(L)^{\dim+1}}. \]

\textbf{Proposition 3.18.} Consider a rational function \( F \) on \( X_K \) and \( L \) a line bundle on \( X_K \) with conditions as before. Then, If the map \( \varphi : X_K \rightarrow X_K \) extends to a map \( \varphi : X \rightarrow X \), we have 
\[ h_\varphi(\text{div}(F)) = \sum_{v/\infty} \int_{X_v} \log |F|^v \frac{d\mu_{\varphi,v}}{c_1(L)^n}. \]

\textit{Proof.} This is a consequence of proposition 3.14 and definition 3.17. □

4. The Blow Up

Let \( \varphi : \mathbb{P}^n_K \rightarrow \mathbb{P}^n_K \) be a map with \( \varphi^*\mathcal{O}(1) \cong \mathcal{O}(d) \) and defined over a number field \( K \). A model for \( \varphi \) over \( \mathcal{O}_K \) is a vector \((p_0, q_1, ..., q_n)\) of elements of \( \mathcal{O}_K[T_0, T_1, ..., T_n]_d \), such that our map is expressed as 
\[ \varphi = (p_0 : q_1 : ... : q_n) : \mathbb{P}^n_K \rightarrow \mathbb{P}^n_K. \] We will be interested in models of \( \varphi \) such that \((p_0, q_1, ..., q_n)\) form a regular sequence.

\textbf{Lemma 4.1.} Let \( A = \mathcal{O}_K[T_0, T_1, ..., T_n] \) and \( p_0, p_1, ..., p_n \in A \). Then, the following two statements are equivalent:

(i) the sequence \( p_0, p_1, ..., p_n \) is a regular sequence.
(ii) \( \text{dim}(V((p_0, p_1, ..., p_n))) = 0. \)

\textit{Proof.} The ring \( A \) is Cohen-Macaulay of dimension \( n+1 \), therefore the sequence \( p_0, ..., p_n \) is regular if and only if it is a maximal system of parameters. □

\textbf{Corollary 4.2.} Let \( A = \mathcal{O}_K[T_0, T_1, ..., T_n] \) and \( p_0, p_1, ..., p_n \) a regular sequence in \( A \). Then the sequence \( p_k = \{ (p_{k0} : p_{k1} : ... : p_{kn}) \} \), defined recursively by 
\[ p_0 = (p_0 : p_1 : ... : p_n) \quad p_{ki} = p_{k-1,i}(p_0 : p_1 : ... : p_n) \quad 0 \leq i \leq n \quad k > 0 \]
is also a regular sequence for all \( k \).

\textit{Proof.} If \( p_{k-1} \) is a regular sequence, \( \text{dim}(V((p_{k-1,0}, p_{k-1,1}, ..., p_{k-1,n}))) = 0 \), because \( \varphi \) is a finite map, \( \text{dim}(V((p_{k,0}, p_{k,1}, ..., p_{k,n}))) = 0 \) and \( p_k \) is also a regular sequence. □

\textbf{Definition 4.3.} Denote \( \mathbb{P}^n_{\mathcal{O}_K} \) by \( X \) and by \( Y_k \), the closed subscheme of \( X \) defined by the ideal \( I_k = \langle p_{k,0}, p_{k,1}, ..., p_{k,n} \rangle \). The model \( X_k \) is defined by the property that \( \sigma^k : X_k \rightarrow X = \mathbb{P}^n_{\mathcal{O}_K} \) is the blowing up of \( Y_k \). The exceptional divisor will be denoted by \( E_k \) and its irreducible components by \( C_{v,i,k} \), in such a way that we have a finite sum 
\[ E_k = \sum_{v,i>0} r_{v,i,k} C_{v,i,k}. \]
In the rest of this subsection we will work with $X = \mathbb{P}^n_{\mathcal{O}_K}$ and a map 
$\varphi : \mathbb{P}^n_K \to \mathbb{P}^n_K$ represented by a regular sequence $(p_0, ..., p_n)$ in $\mathcal{O}_K$. In
the way we have defined the map $\sigma^k$, we have $\sigma^k I_k = \mathcal{O}_{X_k}(-E_k)$, by the
universal property of the blow-up. By the same property, the surjection
$\mathcal{O}^2_{X_k} \to \sigma^k (\mathcal{O}_X(d)) \otimes \mathcal{O}_{X_k}(-E_k)$, gives rise to a map $\varphi_k : X_k \to X$. By
definition of the map $\varphi_k$ we have $\varphi_k^* \mathcal{O}_X(1) = \sigma^k (\mathcal{O}_X(d)) \otimes \mathcal{O}_{X_k}(-E_k)$. We
will denote by $\mathcal{L}_0$ the line bundle $\mathcal{O}(1)$ on $X = \mathbb{P}^n_{\mathcal{O}_K}$ and $\mathcal{L}_k = \varphi^k \mathcal{L}_0$
on the model $X_k$.

**Proposition 4.4.** The scheme $X_k$ is Macaulay and $Y_k$ is a subscheme of codimension $n + 1$ in $\mathcal{O}_K$ and does not meet the generic fiber $X_K$. Each component $C_{v,i,k}$ is isomorphic to the projective space of dimension $n$ over the residual field $K_{v,i,k}$ of the close point image of $C_{v,i,k}$. If $X_{k,v}$ does not meet $Y_k$, then $X_{k,v}$ is isomorphic to $\mathbb{P}^n_{K_v}$.

**Proof.** Since $I_k$ is finitely generated the scheme $X_k = \text{proj}(\bigoplus_{n \geq 0} I^n_k)$ is locally complete intersection in $\mathbb{P}^n_{\mathcal{O}_K}$ and therefore Macaulay. The sequence $(p_{k,0}, p_{k,1}, ..., p_{k,n})$ being regular in $\mathcal{O}_K[T_0, T_1, ..., T_n]$ forces $Y_k$ to be of codimension $n + 1$. By definition of the blow-up we get that the components $C_{v,i,k}$ are isomorphic respectively to $\mathbb{P}^n_{K_{v,i,k}}$. The total fiber $F_i$ and $E_k$ are Cartier divisors, therefore the components $C_{v,i,k}$ are $Q$-Cartier divisors, because they don’t meet each other. The intersection of $\mathcal{L}_k$ with itself $n$ times $c_1(\mathcal{L}_k)^n$ represent a Cohen Macaulay curve on $X$. Using propositions 4.3 and 4.6 we have

$$\hat{\deg}(c_1(\mathcal{L}_k)^n|_{C_{v,k}}) = \deg(c_1(\mathcal{O}_{X_k}(-E_k))^n|_{C_{v,k}})$$
$$= \deg(\mathcal{O}_{\mathbb{P}^n_{K_{v,i,k}}}(1)) = [K_{v,i,k} : K_v] \log |N(v)|.$$

Also for places $v$ of good reduction,

$$\deg(c_1(\mathcal{L}_k)^n|_{X_{k,v}}) = \deg(\mathcal{O}_{\mathbb{P}^n_{K_v}}(1)) = \log |N(v)|.$$

In this way the proposition gives us a way to compute the arithmetic intersection of $\mathcal{L}_k$ with the different vertical components of $X_k$. \qed

**Definition 4.5.** Recall that $f : X \to \text{Spec}(\mathcal{O}_K)$ is an Arithmetic variety. The projection $f(Y_1) \subset \text{Spec}(\mathcal{O}_K)$ will be called the places of bad reduction of $\varphi$.

**Remark 4.6.** The only places $v$ appearing in the exceptional divisors $E_k = \sum_{v,i>0} r_{v,i,k} C_{v,i,k}$ are the places of bad reduction.

Let $F_0 \in \mathcal{O}_K[T_0, ..., T_n]$. We can assume that $v(F_0) = 0$ for every place $v$ of bad reduction because there is only finitely many places of bad reduction and any Dedekind domain with finitely many primes (like $\cap_v \mathcal{O}_v(z)$) is unique factorization domain. Now consider the rational
function $F = F_0/T_n^{\deg(F_0)}$ on $\mathbb{P}^n_{\mathcal{O}_K}$ and $F_k = \sigma_k^*F$. The symbol $\infty_k$ will be denoting the divisor of $X_k$ defined by the equation $\sigma_k^*T_n = 0$ and in particular $\infty = \text{div}(T_n)$. If we define the irreducible horizontal divisor $D$ in $X$ by the equation $\text{div}(F) = D - \deg(F)\infty + \sum_{v \text{ finite}} v(F)X_v$, we can establish the following lemma.

**Lemma 4.7.** There exist non-negative integers $x_{v,i,k}$ and $y_{v,i,k}$ depending only on $D$, such that $\text{div}(F_k)$ can be written as

$$\text{div}(F_k) = D_k - \deg(F)\infty_k + \sum_{v,i} x_{v,i,k}C_{v,i,k}$$

where $D_k$ is the proper transform of $D$ by $\sigma_k$.

**Proof.** We have the formula for the divisor $\text{div}(F_k)$:

$$\text{div} F_k = \text{div} \sigma_k^*F = \sigma_k^* \text{div} F = \sigma_k^*(D) - n\sigma_k^*(\infty) + \sum_{v \text{ finite}} v(F)\sigma_k^*(X_{v,k}),$$

but now, for certain non-negative integers $x_{v,i,k}$ the reciprocal image of the effective divisor $D$ is $\sigma_k^*(D) = D_k + \sum_{v,i} x_{v,i,k}C_{v,i,k}$. Now also for certain non-negative integers $y_{v,i,k}$ we have $\sigma_k^*(\infty) = \infty_k + \sum_{v,i} y_{v,i,k}C_{v,i,k}$ and the proof is finished.

**Corollary 4.8.** With the notation as before we have $\sigma_k^*(\infty) = \infty_k + \sum_{v,i} y_{v,i,k}C_{v,i,k}$ and $\sigma_k^*(D) = D_k + \sum_{v,i} x_{v,i,k}C_{v,i,k}$.

### 4.1. Negativity conditions.

It is interesting to look at a particular type of models.

**Definition 4.9.** We say that a model $(p_0, \ldots, p_n)$ satisfies negativity conditions if we have $\text{rad}(\langle p_{k,0}, p_{k,1}, \ldots, p_{k,n}, T_n \rangle) = \langle T_0, T_1, \ldots, T_n \rangle$ in $\mathcal{O}_K[T_0, \ldots, T_n]$, for every $k$.

**Example 4.10.** This condition is satisfied for example if we are considering a map $\varphi = (p_0 : p_1) : \mathbb{P}^1 \to \mathbb{P}^1$, on the Riemann sphere and $p_0$ is a monic polynomial in the variable $T_0$.

**Lemma 4.11.** If the model have negativity conditions the proper transform $\infty_k$ of $\infty$ in $X_k$ is equal to the reciprocal image $\sigma_k^*(\infty)$.

**Proof.** It is enough to show that $\infty$ does not meet $Y_k$ and this is a consequence of the fact that the ideal $\text{rad}(\langle p_{k,0}, p_{k,1}, \ldots, p_{k,n}, T_n \rangle) = \langle T_0, T_1, \ldots, T_n \rangle$. 

\qed
Lemma 4.12. If the model has negativity conditions, there exist non-negative integers \( x_{v,i,k} \) depending only on \( D \), such that \( \text{div}(F_k) \) can be written as

\[
\text{div}(F_k) = D_k - \deg(F)\infty_k + \sum_{v,i,k} x_{v,i,k} C_{v,i,k} + \sum_{\text{finite } v} v(F)X_{v,k}
\]

where \( D_k \) is the proper transform of \( D \) by \( \sigma_k \).

Proof. The results follows by Lemma 4.1 and Lemma 4.7. \( \square \)

5. Finite places

Let \( v \) denote a valuation on \( K, \mathcal{O}_v \) the set of elements \( z \in K \) such that \( v(z) \geq 0 \), \( x \) will denote a vector \( x = (x_0, \ldots, x_n) \in \tilde{K}^n \) and \( P \in \mathbb{P}^n(\tilde{K}) \) a point in the \( n \)-projective space over \( \tilde{K} \). The valuation \( v \) is assumed to be extended to an algebraic closure \( \tilde{K} \) of \( K \). For a vector \( x = (x_0, x_2, \ldots, x_n) \) we define \( v(x) = \min_i \{v(x_i)\} \). For a polynomial \( p \in \tilde{K}[T_0, \ldots, T_n] \) we take \( v(p) \) as the valuation of the vector formed by its coefficients. For a sequence of polynomials \( (p_0, \ldots, p_n) \) we put \( v(p_0, \ldots, p_n) = \min_i \{v(p_i)\} \). Suppose that \( \varphi : \mathbb{P}_K^n \to \mathbb{P}_K^n \) is a rational map of algebraic degree \( d \) given by homogeneous polynomials \( (p_0 : \ldots : p_n) \) over \( \mathcal{O}_v \), then we can define a map

\[
S_v : \tilde{K}^{n+1} - (0, \ldots, 0) \times \mathcal{O}_v[T_0, \ldots, T_n]_d^{n+1} \to \mathbb{R}_{\geq 0}
\]

\[
S_v(x, (p_0, \ldots, p_n)) = v(p_0(x), \ldots, p_n(x)) - v(x^0_0, \ldots, x^d_n)
\]

The map \( S_v \) is in fact a well defined map \( S_v : \mathbb{P}_K^n \times \mathcal{O}_v[T_0, \ldots, T_n]_d^{n+1} \to \mathbb{R}_{\geq 0} \), which we still denote by \( S_v(P,(p_0,\ldots,p_n)) \). To see this, take any two sets of homogenous coordinates for \( P \), say \( (x_0, \ldots, x_n) \) and \( (y_0, \ldots, y_n) = \lambda(x_0, \ldots, x_n) \), then the valuation \( v(p_0(\lambda x), \ldots, p_n(\lambda x)) = dv(\lambda) + v(p_0(x), \ldots, p_n(x)) \) and \( v(\lambda^d x^d) = dv(\lambda) + v(x^d) \), and the result follows.

Definition 5.1. Suppose that the polynomial \( F \) has divisor \( \text{div}(F) = D - \deg(F)\infty - \sum_{\text{finite } v} v(F)X_v \), then we define:

\[
E(F, v \text{ finite}) = -\lim_k \sup \sum_v \log |N(v)| \left( \frac{\sum_{P \in D} S_v(P, p_{k,0}, \ldots, p_{k,n})}{d_{nk}} - \deg(F) \sum_{P \in \infty} \frac{S_v(P, p_{k,0}, \ldots, p_{k,n})}{d_{nk}} - v(F) \right).
\]

Remark 5.2. We have \( S_v(P, p_{k,0}, \ldots, p_{k,n}) > 0 \) only for finitely many \( P \), because the sequence \( (p_{k,0}, \ldots, p_{k,n}) \) is regular. In dimension one we can actually change the lim sup of the formula into a lim.
5.1. **Convergence of each v-adic integral in dimension one.**

This part basically follows section 5.1 of [PST05]. Suppose that we are working in dimension one, i.e. with a map $\varphi = (p_0 : p_1) : \mathbb{P}^1 \to \mathbb{P}^1$ on the Riemann Sphere. Let’s keep the notation of the previous subsection, that is $S_v(x, (p_0, p_1)) = v(p_0(x), p_1(x)) - v(x_0^d, x_1^d)$.

**Proposition 5.3.** The sequence

$$h_k(P) = \left\{ \frac{S_v(P, (p_{k,0}, p_{k,1}))}{d^k} \right\}_k$$

(i) is bounded and increasing, and therefore convergent to a function, which we denote $h_{p_0,p_1,v}(P)$.

(ii) $h_{p_0,p_1,v}(\varphi(P)) = d h_{p_0,p_1,v}(P)$.

The proof will be the result of the application of two lemmas:

**Lemma 5.4.** Suppose that we denote $P_k = (p_{k,0}(P) : p_{k,1}(P))$, then we have the equality

$$S_v(P, (p_{k+1,0}, p_{k+1,1})) = dS_v(P, (p_{k,0}, p_{k,1})) + S_v(P, (p_{0}, p_{1})).$$

**Proof.** Assume that $P = (x_0 : x_1)$ and $v(x) = v((x_0, x_1)) = 0$. If we set $x_k = (p_{k,0}(x), p_{k,1}(x))$ we have the equalities

$$S_v(P, (p_{k+1,0}, p_{k+1,1})) = v(p_{k+1,0}(x), p_{k+1,1}(x))$$

$$= v(p_0(x_k), p_1(x_k)) - v(x_k^d) + v(x_k^d)$$

$$= dS_v(P, (p_{k,0}, p_{k,1})) + S_v((p_k, (p_{0}, p_{1}))),$$

which gives the result we were trying to prove. $\square$

**Lemma 5.5.** The function $S_v(P, (p_0, p_1))$ is bounded on $\mathbb{P}^1(K)$, so we can define

$$R_v(p_0, p_1) = \sup_{P \in \mathbb{P}^1} \{ S_v(P, (p_0, p_1)) \}.$$

**Proof.** There exist elements $0 \neq b_i \in \mathcal{O}_v$, where $0 \leq i \leq 1$, such that $b_i x_i^d \equiv 0((p_0, p_1))$. If $P = (x_0 : x_1) \in \mathbb{P}^1$, with $x_1 \neq 0$, then $S_v(P, (p_0, p_1)) \leq v(b_i)$. So in general $\sup_P \{ S_v(P, (p_0, p_1)) \} \leq \sup_i \{ v(b_i) \}$. $\square$

Now we can proceed to prove proposition 5.3

**Proof.** From lemma 5.3 we get $0 \leq h_{k+1}(P) - h_k(P) \leq R_v(p_0, p_1)/d^{k+1}$, so $\{ h_k(P) \}$ is bounded by $R_v(p_0, p_1)/(d - 1)$ and therefore converges.
On the other hand
\[ h_k(\varphi(P)) - dh_k(P) = \frac{S(P_1, (p_{k,0}, p_{k,1}))}{d^k} - \frac{dS(P, (p_{k,0}, p_{k,1}))}{d^k} \]
\[ = S(P, (p_{k+1,0}, p_{k+1,1})) - \nu(x_1) - \frac{dS(P, (p_{k,0}, p_{k,1}))}{d^k} \]
\[ = S(P_k, (p_0, p_1)) - \nu(x_1) \leq R_v(p_0, p_1) - \nu(x_1). \]

By passing to the limit we get \( h_{v, p_0, p_1}(\varphi(P)) = dh_{v, p_0, p_1}(P). \)

**Definition 5.6.** Suppose that \( \varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K \). We will define the “integral” of \( \log |F|_v \) over the finite place \( v \) of a polynomial \( F = K[z_1, ..., z_n] \) as
\[
\int_{\mathbb{P}^1_K \setminus \mathbb{P}^1_K} \log |F|_v d\mu_{v, \varphi} = \log |N(v)| \left( \sum_{P \in D} h_{v, p_0, p_1}(P) \right.
\]
\[ - \deg(F) \sum_{P \in \infty} h_{v, p_0, p_1}(P) - \nu(F) \bigg), \]

and then
\[
E(F, v \text{ finite}) = \sum_v \int_{\mathbb{P}^1_K \setminus \mathbb{P}^1_K} \log |F|_v d\mu_{v, \varphi}. \]

5.2. **Geometry of** \( E(F, v \text{ finite}) \). We want to relate \( E(F, v \text{ finite}) \) with the geometry of the blow-up. Suppose that \( \sigma_k : (\mathbb{P}^n)_k \to \mathbb{P}^n \) is the blow-up associated with the model \((p_{k,0} : ... : p_{k,n})\). Writing
\[
(2) \quad \sigma_k^*D = D_k + \sum_{v,i} x_{v,i,k} C_{v,i,k} \quad \sigma_k^*(\infty) = \infty_k + \sum_{v,i} y_{v,i,k} C_{v,i,k}, \]

where \( C_{v,i,k} \) are the different components of the exceptional fibre above \( v \), and \( K_{v,i,k} \) denotes the field of definition of the close point corresponding to \( C_{v,i,k} \), we can state:

**Proposition 5.7.** For every \( v \) finite place of \( K \), we have:
\[
\sum_i x_{i,v,k}[K_{v,i,k} : K_v] = \sum_{P \in D} S_v(P, p_{k,0}, ..., p_{k,n}),
\]
\[
\sum_i y_{i,v,k}[K_{v,i,k} : K_v] = \sum_{P \in \infty} S_v(P, p_{k,0}, ..., p_{k,n}).
\]

**Proof.** Let \( \sigma_{\mathcal{O}_{X_v}} : (X_k)_v \to X_{\mathcal{O}_v} \) be the blow-up \( \sigma_k : X_k \to X \), composed with the base extension to \( \text{Spec}(\mathcal{O}_v) \). Let \( D_v \) be the localization of \( D \) at \( v \). We have the equation \( \sigma_{\mathcal{O}_{\mathcal{O}_{X_v}}}^*(D_v) = D_{v,k} + f_{v,k} C_{v,k} \) for some horizontal divisor \( D_{v,k} \) in \( X_k \) and non-negative integers \( f_{v,k} \). Suppose that \( J_v \) is the ideal sheaf of \( D_v \) in \( X_{\mathcal{O}_v} \), then \( \sigma_{\mathcal{O}_{\mathcal{O}_{X_v}}}(J_v) \) will correspond
to the ideal sheaf of \( D_{v,k} + f_{v,k}C_{v,k} \). We are going to assume that there is only one \( P = (a_0 : \ldots : a_n) \in X_v \cap D \), because otherwise we will blow up one point at a time. The subscheme of \((X_k)_v\) determined by \( \sigma_{O_v,k}(J_v) \) is isomorphic to

\[
\text{Proj}(O_v[T_0, \ldots, T_n]/(p_{k,i}(a_0, \ldots, a_n)T_j - p_{k,j}(a_0, \ldots, a_n)T_i)) \cong \text{Proj}(O_v[T_0, \ldots, T_n]/\pi^{r,v}(\bar{p}_{k,i}(a_0, \ldots, a_n)T_j - \bar{p}_{k,j}(a_0, \ldots, a_n)T_i)) \\
\cong \text{Proj}(⟨O_v/\pi^{r,v}O_v⟩|T_0, \ldots, T_n⟩ \cup \text{Spec}(O_v)) \quad = r_{k,v,P}^{\mathbb{P}^{n}_{K_v}} \cup \text{Spec}(O_v).
\]

where \( \bar{p}_{k,i}(a_0, \ldots, a_n) = p_{k,i}(a_0, \ldots, a_n)/\pi^{r,v} \) for all \( 0 \leq i \leq n \) and the valuation \( r_{k,v,P} = S_v(P, p_{k,0}, \ldots, p_{k,n}) \).

On the other hand \( \sigma_{\mathbb{P}^{n}_{K_v}} x_{v,i,k} = x_{v,i,k}^{\mathbb{P}^{n}_{K_{v,i,k}}} = x_{v,i,k}^{K_{v,i,k} : K_v} \mathbb{P}^{n}_{K_v} \). As a consequence of this two facts \( x_{v,i,k}^{K_{v,i,k} : K_v} = S_v(P, p_{k,0}, \ldots, p_{k,n}) \) and

\[
\sum_i x_{v,i,k}^{K_{v,i,k} : K_v} = \sum_{P \in D} S_v(P, p_{k,0}, \ldots, p_{k,n}).
\]

The second part is analogous using \( \deg(F)\infty \) instead of \( D \). \( \Box \)

**Remark 5.8.** The expression \( E(F, v \text{ finite}) \) previously defined, takes the form

\[
E(F, v \text{ finite}) = -\limsup_k \sum_v \log |N(v)| \left( \frac{\sum_i x_{v,i,k}^{K_{v,i,k} : K_v}}{d^{nk}} - \deg(F) \frac{y_{v,i,k}^{K_{v,i,k} : K_v}}{d^{nk}} - v(F) \right).
\]

**Remark 5.9.** In case we are working in dimension one, that is with a map \( \varphi = (p_0 : p_1) : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \), we can do a more precise computation of the contributions of \( y_{i,v,k} \) (See lemma 5.9 in [PST05]). In this case, assuming that \( p_0 = A_dT_d^d + \ldots \), the integral at a finite place \( v \) takes the particular form

\[
\int_{C_v} \log |F|_v d\mu_{v,\varphi} = -\lim_k \log |N(v)| \left( \lim_k \frac{x_{v,i,k}^{K_{v,i,k} : K_v}}{d^k} - \deg(F) \frac{A_d[K_{v,i,k} : K_v]}{d - 1} - v(F) \right)
\]

and again

\[
E(F, v \text{ finite}) = \sum_v \int_{C_v} \log |F| d\mu_{v,\varphi}.
\]

**Remark 5.10.** The geometry of \( E(F, v \text{ finite}) \) allows to express in a better way the negativity conditions for a model, indeed if our model...
(p_0 : ... : p_n) have negativity conditions and we are able to pick the equation F such that v(F) = 0 for every v, then E(F, v finite) ≤ 0.

6. Mahler formula

In this section we present the main result of this paper. Suppose that ϕ : \( \mathbb{P}^n_K \rightarrow \mathbb{P}^n_K \) is a map on the n-dimensional projective space, given by a model \((p_0 : ... : p_n) : \mathbb{P}^n_K \rightarrow \mathbb{P}^n_K\), with the property that \( (p_0, ..., p_n) \) is a regular sequence in \( \mathcal{O}_K \). Recall the non-standard models \( \sigma_k : X_k \rightarrow \mathbb{P}^n \) that we introduced before as the blows up of \( \mathbb{P}^n \) at the subschemes \( Y_k \). Denote by \( K_{v,i,k} \) the field of definition of \( C_{v,i,k} \) and by \( K_v \) the local field of \( K \) at \( v \). Recall that \( F_k = \sigma_k^* F \) for a polynomial \( F = F(T_0/T_n, ..., T_{n-1}/T_n) \) and \( L_k = \varphi_k^* L_0 \), for \( L_0 = \mathcal{O}(1) \). The divisors \( \text{div}(F) = D - \deg(F) \infty + \sum_{v \text{ finite}} v(F)X_v \) and \( \text{div}(T_n) = \infty \) represent divisors in the arithmetic variety \( \mathbb{P}^n_{\mathcal{O}_K} \).

**Theorem 6.1.** With the conditions and notations of the above paragraph, we have the equality:

\[
h_{\varphi}(D) = \sum_{v/\infty} \int_{\mathbb{P}^1_C} \log |F|_v d\mu_{\varphi,v} + E(F, v \text{ finite}) + \deg(F)h_{\varphi}(\infty).
\]

**Proof.** We are going to make use of the arithmetic intersection theory on \( X_k \). Let’s compute \( \text{deg}(\hat{c}_1(L_k)^n| \text{div}(F_k)) \). We have the following:

(i) \( \text{deg}(\hat{c}_1(L_k)^n| \text{div}(F_k)) = d^{nk} \sum_{v | \infty} \int_{\mathbb{P}^1_C} \log |F|_v c_1(v, ||.||)^n \) by proposition 3.14.

(ii) \( \text{deg}(\hat{c}_1(L_k)^n| C_k) = [K_{v,i,k} : K_v] \log |N(v)| \) by proposition 4.4.

(iii) \( \text{deg}(\hat{c}_1(L_k)^n| X_{k,v}) = \log |N(v)| \) also by proposition 4.4.

Let’s recall the formula [1]

\[
\text{div}(F_k) = D_k - \deg(F) \infty_k + \sum_{v,i} x_{v,i,k} C_{v,i,k} - \deg(F) \sum_{v,i} y_{v,i,k} C_{v,i,k} + \sum_{v \text{ finite}} v(F)X_{v,k}.
\]
Now we are going to let \( \hat{\deg}(\hat{c}_1(L_k)^n|.) \) act on each side,
\[
\hat{\deg}(\hat{c}_1(L_k)^n| \text{div}(F_k)) = h_{L_k^n}(D_k) - (\deg(F))h_{L_k^n}(\infty_k)
+ \sum_{i, v} x_{v, i, k} \log |N(v)|[K_{v, i, k} : K_v]
- \deg(F) \sum_{i, v} y_{v, i, k} \log |N(v)|[K_{v, i, k} : K_v]
+ d^n k \sum_v v(F) \log |N(v)|,
\]

dividing by \( d^n k \) and taking limits gives us that the limit
\[
\lim_k \sum_{i, v} (x_{v, i, k} \log |N(v)|[K_{v, i, k} : K_v] - \deg(F) y_{v, i, k} \log |N(v)|[K_{v, i, k} : K_v])
\]
exists and
\[
\sum_v \int_{\mathbb{P}^n(C)} \log |F|_v d\mu_{\varphi, v} = -E(F, v \text{ finite}) + h_{\varphi}(D_Q) - \deg(F) h_{\varphi}(\infty),
\]
which was the result we wanted to prove.

**Corollary 6.2.** Let \( F \) be a rational function on \( \mathbb{P}_K^n \) and \( \text{div}(F) = D^+ - D^- \) then
\[
h_{\varphi}(D^+) - h_{\varphi}(D^-) = \sum_{v} \int_{\mathbb{P}^n(C)} \log |F|_v d\mu_{\varphi, v} + E(F, v \text{ finite})
\]

**Proof.** The rational function \( F \) can be written as the quotient \( F^+/F^- \) of two homogeneous polynomial equations \( F^+ \) and \( F^- \) of the same degree. Then we apply the previous result to \( F^+ \) and \( F^- \).

**Corollary 6.3.** If \( \varphi \) has a model such that the divisor \( \infty \) has a finite forward orbit \( \{\infty, \varphi(\infty), \ldots\} \) (which forces \( h_{\varphi}(\infty) = 0 \)), then
\[
h_{\varphi}(D_Q) = \sum_{v} \int_{\mathbb{P}^n(C)} \log |F|_v d\mu_{\varphi, v} + E(F, v \text{ finite}).
\]

**Corollary 6.4.** Suppose that \( \varphi = (p_0 : p_1) : \mathbb{P}^1 \to \mathbb{P}^1 \) and we have chosen coordinates such that \( T_n/p_1 \). The integral of the log the minimal equation \( F \) of a point \( P = (\lambda : 1) \in \mathbb{P}^1 \) is related to the height of \( P \) by the formula
\[
h_{\varphi}(P) = \frac{1}{\deg(F)} \sum_{v} \int_{\mathbb{P}^1_{K_v}} \log |F|_v d\mu_{\varphi, v}.
\]

**Proof.** This is a combination of the previous corollary, proposition 5.3 and definition 5.6.
Corollary 6.5. Assume $h_\varphi(\infty) = 0$. Let $(p_0 : \ldots : p_n)$ be a model for $\varphi : \mathbb{P}^n \to \mathbb{P}^n$, satisfying negativity conditions and with $v(F) = 0$ for every finite place $v \in O_k$. Then

$$h_\varphi(D_\mathbb{Q}) \leq \sum_{v/\infty} \int_{\mathbb{P}_v^2} \log |F|_v d\mu_{\varphi,v}.$$ 

Example 6.6. The rational map $\varphi : \mathbb{P}^2_\mathbb{Q} \to \mathbb{P}^2_\mathbb{Q}$ given by the model $\varphi(x : y : z) = (x^3 + 3x^2y : y^3 - 3y^2z : xz)$, has good reduction everywhere, that is, the expression $E(F,v_{\text{finite}}) = 0$.

Example 6.7. The rational map $\varphi : \mathbb{P}^2_\mathbb{Q} \to \mathbb{P}^2_\mathbb{Q}$ given by the model $\varphi(x : y : z) = (x^3 - 3y^2z : x^2 - 3y^2z : y)$ has bad reduction at 3. The reduced map is not defined at the point $(0 : 0 : 1) \in \mathbb{P}^2_\mathbb{Q}$. The model satisfy negativity conditions and $h_\varphi(\infty) = 0$. A polynomial equation $F = c_mz^m + \ldots$ has $c_m \equiv 0(\text{mod } 3)$ if and only if $E(F,v_{\text{finite}}) \neq 0$.

Example 6.8. The rational map $\varphi : \mathbb{P}^2_\mathbb{Q} \to \mathbb{P}^2_\mathbb{Q}$ given by the model $\varphi(x : y : z) = (3y^2 - 5z^2 : 3x^2 - 5y^2 : z)$. The reduced maps is not well defined at the points $(1 : 0 : 0) \in \mathbb{P}^2_\mathbb{Q}_3$ and $(0 : 0 : 1) \in \mathbb{P}^2_\mathbb{Q}_5$.

Example 6.9. Let $p$ be a prime number. The map $\varphi : \mathbb{P}^1_\mathbb{Q} \to \mathbb{P}^1_\mathbb{Q}$ given by the model $\varphi(x : y) = (px^2 + y^2 : py^2)$ has bad reduction at $p$. Following lemma 5.9 in [PST05], if $F$ is a polynomial equation, we have $E(F,v_{\text{finite}}) = \log(p)$.

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