Challenging the cosmic censorship conjecture from a Gauss-Bonnet sector

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(Dated: December 11, 2020)

The Dvali-Gabadadze-Porrati (DGP) brane-world model is employed to study the gravitational collapse of dust, with a Gauss-Bonnet (GB) term present in the five-dimensional bulk. We find that, within the normal (non self-accelerating) DGP branch and due to the curvature effects from the GB component on the brane, the black hole singularity acquires modified features. More precisely, during collapse and for a finite comoving time, before a singularity would emerge at the zero physical radius, the first time derivative of the Hubble rate diverges, whereas the brane energy density and the Hubble rate remain finite. This is a peculiar behaviour which displays similar properties to the sudden singularity occurring in particular late-time cosmological frameworks. Furthermore, the question whether this altered singularity can be visible to an external observer or it will be hidden by a black hole horizon, is addressed. We establish that, depending on the given induced-gravity parameter and the GB coupling constant, there exists a threshold mass, for the collapsing dust, below which no trapped surfaces evolve as the collapse proceeds towards the singularity. In other words, a naked sudden singularity may form.

PACS numbers: 04.20.Dw, 04.50.Kd, 04.20.Jb, 04.70.Bw

I. INTRODUCTION

Recent astrophysical evidence including the first image of a black hole [1] and gravitational wave detection [2][3] constitute remarkable achievements. In particular, they have the potential to enhance immensely the study of phenomena with large masses producing events bearing startling energy ranges. A strong gravitational field, governed by the theory of general relativity (GR), plays a dominant role, in so far as thorough analysis has demonstrated. But can such observational procedures advise us into selecting paths to further probe the nature of our universe? Concretely, identifying on the one hand, domains where to defy our appraisal of gravitational collapse. And, on the other hand, suggesting therefore whether there is a broader gravitational theory beyond GR.

The process of the gravitational collapse of a sufficiently massive star leads to the formation of a spacetime singularity [4][5]. These are regions with zero physical radii where the energy density and the curvature of spacetime blow up. Under such extreme physical conditions, extending them towards and in the Planck regime, we expect that the theory of GR must be replaced by a more complete theory of gravity. This could require to find a quantum theory of gravity which modifies the spacetime geometry at Planck ranges [7][11].

In past decades there has been huge effort towards formulating a theory to be valid in very small length scales and high energy density. Accordingly, it would be of interest to examine the presence or otherwise, the absence of spacetime singularities in these theories. Moreover, it would be pertinent to investigate if and how a new theory of gravity, incorporating geometrical sectors other than GR terms, would modify the nature of any such singularities. In GR, the cosmic censorship conjecture (CCC) states that, the central curvature singularity arising in a continual gravitational contraction must be hidden inside a black hole horizon [12][13]. Thereby, it is not possible for future-directed null geodesics, originating from vicinity of the singularities, to escape to infinity leaving the center visible to outside observers (i.e. by forming a naked singularity). For particular matter distributions, and for various models of gravity, the fate of CCC and its possible violation has been extensively studied in the literature [15][19] (see also [20][21]). Hence, challenging the scope of that conjecture within a new theory of gravity is a valuable purpose.

A significant scope of approaches, within either astrophysical and cosmological settings, have contributed to the literature of modified gravity and singularities, for example [19][22][31]. Noteworthy, string and M theories are two of the main candidates to unify quantum mechanics and GR [32][34]. It is of relevance to point that the low energy limit of these higher dimensional theories exist in a four-dimensional (4D) world. Inspired by these theories, brane-world models were proposed in which the observable world is a 4D brane embedded in a 5D bulk [35][39]. These models are characterized by the feature that standard matter fields are confined to the (3 + 1)D brane whereas gravity propagates in the higher dimensional bulk. Moreover, gravity appears 4D on the brane via the warping of the bulk dimension. In other words,
from the point of view of the brane observer, gravity appears 5D at high energy whereas it is revealed 4D in the low energy regime.

An interesting brane-world scenario was subsequently suggested by Dvali, Gabadadze and Porrati (DGP) [36], which assumes an Einstein-Hilbert brane action where the brane Ricci scalar can be interpreted as arising from a quantum effect due to the interaction between the bulk gravitons and the matter on the brane [35]. In this model, the induced gravity term on the brane dominates below a certain “cross-over” scale $r_c$, so gravity becomes 4D. However, at scales greater than $r_c$, gravity “leaks off” the brane (the 5D Ricci term in the DGP action begins to dominate over the 4D Ricci term) so it appears to be 5D to observers on the brane. Thereby, the bulk is no longer required to be warped in the DGP model, so that the DGP brane is instead embedded in an infinite dimensional Minkowski bulk. Therefore, the DGP brane-world model modifies the low energy regime of a gravitational theory (i.e., IR modifications). From a geometrical point of view, adding a (UV-like) Gauss-Bonnet (GB) term in 5D is natural: the most general unique action in 5D must include a GB sector [28, 40–42]. This geometrical term thus modifies gravity in the high energy regime.

Within the context of the above paragraphs, our objective herein this paper is to apply a specific IR-UV combination to gravity to investigate the fate of gravitational collapse at high energies and to understand the consequences as far as CCC is concerned (see for example [43, 44]). We thus consider a DGP-GB brane model where a dust fluid is present in the brane, with the bulk characterized by higher curvature effects from GB terms. We will show that such setting contains interesting new type of (gravitational collapse) singularities (see e.g., [28, 45, 46]). In addition, the induced-gravity parameter and the GB coupling constant, comply to bring about a threshold mass. Depending on its value, trapped surfaces may evolve as the collapse proceeds or, instead a naked singularity may form. This can be of relevance within the scope of appraising the CCC within a given IR-UV combination towards a gravity theory.

We thus organize this paper as follows. In Sec. II, we describe a homogeneous dust fluid undergoing isotropic and homogeneous collapse on the DGP brane (with a GB term). Then, in Sec. III, we will present different classes of solutions to the modified Friedmann equation, where the physical reasonability of the solutions will be thoroughly analyzed. In Sec. IV, by selecting the relevant solutions for gravitational collapse, we will investigate the fate of the singularity emerging on the brane. We will show that a GR singularity will be replaced by a different type, which we label as sudden (naked/black hole) singularity. In Sec. V by applying appropriate junction conditions on the dust boundary, we will analyze the evolution of trapped surfaces and will discuss the possible candidates for the exterior spacetime geometry both for the formation of a black hole and of a naked singularity. Finally, in Sec. VI we will present the conclusion and discussions concerning our work.

II. GRAVITATIONAL COLLAPSE IN THE DGP-GB MODEL

The marginally bound collapse of a dust sphere is the simplest case of the gravitational collapse within the Oppenheimer-Snyder model [47] (see also [48, 50]). In this model, the end product of the collapse process is a black hole in spacetime, that is, after a concrete time elapses from the initial configuration, the collapsing star proceeds towards the horizon and it continues until a spacetime singularity is formed. Therefore, the final singularity will be hidden behind the Schwarzschild horizon.

Our purpose in this section is to construct a class of continual collapse model in a brane-world scenario, such that the interplay between the brane and bulk (which is modified due to the higher order curvature terms) could alter the nature of singularity or/and development of trapped surfaces in the spacetime. We require that the weak and null energy conditions are preserved throughout the collapse, though the emergent effective pressure may be negative in the vicinity of the singularity. We thus introduce the gravitational collapse of a homogeneous dust cloud on a DGP brane while a GB term is considered on the bulk. We first extract the dynamical equations for the collapse scenario. Then, we will discuss the physical reasonability of the model by analysing the energy conditions.

A. The model

The general gravitational action with induced gravity on the ($\mathbb{Z}_2$ symmetric) brane (DGP term) and quadratic Gauss-Bonnet term is

\[
S = \int d^5x \sqrt{-g} [M_5^4 R_{(5)}^3] + \int d^4x \sqrt{-g} M_4^2 (R_{(4)} + \mathcal{L}_{\text{matt}}) + \frac{\alpha}{2} \int d^5x \sqrt{-g} [M_5^3 R_{(5)}^2 - 4 R_{(5)} ab R^{ab} + \mathcal{R}_{(5)}^{abcd} R^{abcd}] ,
\]

(2.1)

where $\alpha(\geq 0)$ is the coupling constant associated to the GB term in the Minkowski bulk. The GB term corresponds to the leading order quantum correction to gravity in an effective action approach of string theory [51], where $\alpha$ is related to string scale and can be identified with the inverse of string tension [52]. The case $\alpha = 0$ corresponds to the standard DGP model [55, 36]. Moreover, $M_4$ and $M_5$ are the fundamental four and five dimensional Planck masses, respectively.

By considering a spherically symmetric, homogeneous dust fluid on the brane, the interior spacetime can have a
FLRW metric \[18\] [19]. Then, by choosing the marginally bound \((k = 0)\) case, the interior metric is of the form
\[
ds^2 = -dt^2 + a^2(t) \left( dr^2 + r^2 d\Omega^2 \right),
\]
where \(d\Omega^2\) denotes the line element on a two-sphere. The generalized Friedmann equation for the brane, equipped with the metric \[(2.2)\], then reads \[40\]
\[
H^2 = \frac{\kappa_{(5)}^2}{6\kappa_c} \rho \pm \frac{1}{r_c} \left( 1 + \frac{8}{3}\alpha H^2 \right) H,
\]
where \(\kappa_{(4)}^2 = M_{(4)}^2\) and \(\kappa_{(5)}^2 = M_{(5)}^{-2}\) are four and five-dimensional gravitational constants, respectively; and \(r_c\) is the induced-gravity “cross-over” length scale which marks the transition from 4D to 5D gravity:
\[
r_c = \frac{\kappa_{(5)}^2}{2\kappa_{(4)}^2}.
\]
The two signs on the right hand side of Eq. \[(2.3)\] are associated with two branches DGP(\(\pm\)), in the case \(\alpha = 0\), which correspond to different embeddings of the brane in the Minkowski bulk. Moreover, \(H\) is the Hubble parameter of the brane, and \(\rho\) represents the total energy density of the brane, i.e. the dust fluid, which is given by
\[
\rho = \rho_0 \left( \frac{a_0}{a} \right)^{-3}.
\]
Here, \(a_0\) and \(\rho_0\) are respectively, the initial values of the scale factor and the energy density of the brane at the beginning of the collapse.

To deal with the Friedmann equation it is helpful to introduce the following dimensionless variables:
\[
\bar{H} := \frac{8}{3} \frac{\alpha}{r_c} H, \quad \bar{\rho} := \frac{32}{27} \frac{\kappa_{(5)}^2}{r_c^3} \rho, \quad b := \frac{8}{3} \frac{\alpha}{r_c^2}.
\]
Then, we can express the Friedmann equation Eq. \[(2.3)\] as
\[
\bar{H}^2 = \bar{\rho} + \epsilon(b\bar{H} + \bar{H}^3),
\]
where \(\epsilon = +1, -1\) in Eq. \[(2.3)\] correspond, respectively, to the (+) sign, which is usually called (in the cosmological context) as the self-accelerating branch, and the (−) sign, which is called the normal branch. Note that, the two cubic equations can be transformed to each other by changing the sign of \(\bar{H}\), therefore, the solution to the (+) branch is simply equal to the negative of the solution to the (−) branch.

### B. Energy conditions

For a desired branch, once the mathematical solutions to the modified Friedmann equation \[(2.7)\] are known, one needs to discuss their physical implications in the collapse scenario. In particular, to assure physical reasonability of the solutions, a set of energy conditions must be satisfied by the collapsing system \[5\]. There are two important energy conditions to be satisfied by all (classical) matter fields. The first is the weak energy condition (WEC), which requires that the energy density measured by any local time-like observer must be non-negative. And the second one is the null energy condition (NEC), which ensures that the flow of the energy for any time-like observer is not space-like (i.e. the speed of the energy flow does not exceed the speed of the light).

In order to study the energy conditions in their well-known general relativistic form, it is convenient to introduce an effective energy density \(\rho_{\text{eff}}\) on the brane which obeys the equation of state of a perfect fluid, equipped with an effective parameter \(w_{\text{eff}}\) and an effective pressure \(p_{\text{eff}}\). For a given \(\epsilon\), we thus rewrite the modified Friedmann equation \[(2.3)\] based on these effective quantities as
\[
H^2 = \frac{\kappa_{(4)}^2}{3} \rho_{\text{eff}},
\]
where \(\rho_{\text{eff}}\) is defined as
\[
\rho_{\text{eff}} := \rho + \frac{6}{\kappa_{(5)}^2} \epsilon \left( 1 + \frac{8}{3}\alpha H^2 \right) H.
\]
This effective energy density and its corresponding pressure should satisfy an effective conservation equation as follow:
\[
\dot{\rho}_{\text{eff}} + 3H (\rho_{\text{eff}} + p_{\text{eff}}) = 0,
\]
where, a dot denotes a derivative with respect to the proper time. Then, \(\dot{\rho}_{\text{eff}}\) is obtained by taking a time derivative of Eq. \[(2.9)\], as
\[
\dot{\rho}_{\text{eff}} = \dot{\rho} + \frac{6}{\kappa_{(5)}^2} \epsilon \left( 1 + 8\alpha H^2 \right) \dot{H}.
\]
By setting Eq. \[(2.11)\] into the conservation equation \[(2.10)\], we find the effective pressure \(p_{\text{eff}}\) as
\[
p_{\text{eff}} = -\dot{\rho}_{\text{eff}} + \frac{2r_c H \rho}{2r_c H - \epsilon(1 + 8\alpha H^2)}.
\]
Note that, when simplifying the above equation, we have used the conservation equation for the energy-momentum tensor of the dust fluid, i.e. \(\dot{\rho} + 3H \rho = 0\).

Now that we have the effective energy density and the effective pressure, we can write the WEC and the NEC for the effective energy-momentum tensor as:
\[
\text{WEC : } \rho_{\text{eff}} = \rho + \frac{6}{\kappa_{(5)}^2} \epsilon \left( 1 + \frac{8}{3}\alpha H^2 \right) H \geq 0, \quad (2.13)
\]
\[
\text{NEC : } \rho_{\text{eff}} + p_{\text{eff}} = \frac{2r_c H \rho}{2r_c H - \epsilon(1 + 8\alpha H^2)} \geq 0. \quad (2.14)
\]
To simplify our analysis of the energy conditions, we rewrite the above equations in terms of the dimensionless quantities as
parameters:

\[
\text{WEC:} \quad \tilde{\rho} + \epsilon (b + H^2) \dot{H} \geq 0, \quad (2.15) \\
\text{NEC:} \quad \frac{2\dot{H} \tilde{\rho}}{2\dot{H} - \epsilon (b + 3H^2)} \geq 0. \quad (2.16)
\]

The WEC can be examined by comparing Eq. (2.15) with the Friedmann equation (2.7), noting that

\[
\tilde{\rho} + \epsilon (b + H^2) \dot{H} = H^2 \geq 0. \quad (2.17)
\]

Therefore, the WEC always holds. For the NEC to be satisfied, the denominator of the Eq. (2.16) should be always negative. This is equivalent to the condition below:

\[
2\dot{H} - \epsilon (b + 3H^2) < 0. \quad (2.18)
\]

In the above discussion, we have followed the fact that \(\rho > 0\) for the energy density of the dust, whereas \(\dot{H} < 0\) guarantees a collapsing process.

For the \((-)\) branch, Eq. (2.18) is satisfied if,

\[
\mathcal{H}_1^{(-)} < \dot{H} < \mathcal{H}_2^{(-)}, \quad (2.19)
\]

where

\[
\mathcal{H}_1^{(-)} = -\frac{1}{3} (1 + \sqrt{1 - 3b}), \quad (2.20) \\
\mathcal{H}_2^{(-)} = -\frac{1}{3} (1 - \sqrt{1 - 3b}). \quad (2.21)
\]

Similarly, for the \((+)\) branch, the condition (2.18) implies that

\[
\dot{H} < \mathcal{H}_2^{(+)}, \quad \text{or} \quad \dot{H} > \mathcal{H}_1^{(+)}, \quad (2.22)
\]

where

\[
\mathcal{H}_1^{(+)} = \frac{1}{3} (1 + \sqrt{1 - 3b}), \quad (2.23) \\
\mathcal{H}_2^{(+)} = \frac{1}{3} (1 - \sqrt{1 - 3b}). \quad (2.24)
\]

The first condition is satisfied because for a collapse process we always have \(\dot{H} < 0\). However, the second condition is not the case in the present context.

From the NEC (2.19) for the \((-)\) branch, it is clear that the Hubble rate is limited by the two values \(\mathcal{H}_1^{(-)}, \mathcal{H}_2^{(-)} < 0\). In contrary, in the \((+)\) branch, the NEC (2.22) does not provide a lower limit for \(\dot{H}\). This permits the divergence of \(\dot{H}\) (with negative sign) at the late time stages of the collapse, which subsequently violates the WEC (2.15). We are interested in a situation where, without considering any matter that violates the NEC on the brane, the effective energy density (2.5) will behave like a phantom component on the brane (that depends explicitly on \(\dot{H}\)). This would smoothen the divergent nature of the dust fluid at the collapse end state (cf. 20, 47, 53). We, henceforth in the rest of this paper, will only consider a \((-)\) branch.

In the next section we will analyse the solutions to the Friedmann equation (2.7) for the \((-)\) branch and study their physical implications for different collapse scenarios.

III. THE SOLUTIONS

For the \((-)\) branch, the (dimensionless) Friedmann equation (2.7) can be written as

\[
\dot{H}^3 + \dot{H}^2 + b\dot{H} - \tilde{\rho} = 0. \quad (3.1)
\]

The number of the solutions to the cubic equations (3.1) depends on the sign of the discriminant function \(N\), defined as

\[
N = Q^3 + S^2, \quad (3.2)
\]

where \(Q\) and \(S\) are,

\[
Q := \frac{1}{3} \left( b - \frac{1}{3} \right), \quad S := \frac{1}{6} b + \frac{1}{2} \tilde{\rho} - \frac{1}{27}. \quad (3.3)
\]

If \(N > 0\), then we have three roots, one of which is real and the other two are complex conjugates. For \(N = 0\), all of the solutions are real and two of them are equal. And finally if \(N < 0\), all roots are real [51].

To simplify the analysis of the sign of \(N\) we rewrite it in terms of two new variables \(\rho_+\) and \(\rho_-\) as [25]

\[
N = \frac{1}{4} (\tilde{\rho} - \rho_+) (\tilde{\rho} - \rho_-), \quad (3.4)
\]

where

\[
\rho_+ := \frac{2}{27} \left[ 1 + \sqrt{(1 - 3b)^3} \right] - \frac{b}{3}, \quad (3.5) \\
\rho_- := \frac{2}{27} \left[ 1 - \sqrt{(1 - 3b)^3} \right] - \frac{b}{3}. \quad (3.6)
\]

Now we look at the behavior of \(\rho_+\) with respect to the parameter \(b\). If \(b\) takes a value in the range \(0 < b < 1/4\), \(\rho_+\) is always positive while \(\rho_-\) is always negative. Then depending on the relative values of \(\rho_+\) and \(\rho_-\) there are three cases:

i) for \(\tilde{\rho} > \rho_+\) \(\Rightarrow N > 0\): “high energy regime”;

ii) for \(\tilde{\rho} = \rho_+\) \(\Rightarrow N = 0\): “limiting regime”;

iii) for \(\tilde{\rho} < \rho_-\) \(\Rightarrow N < 0\): “low energy regime”.

For the range of parameter \(b \geq 1/4\), \(N\) is always positive and there exists a unique real solution.

A. The case \(0 < b < \frac{1}{4}\)

As mentioned earlier, here we will have three different cases depending on the relative values of the energy density \(\tilde{\rho}\) of the brane. However, the energy density is
time-dependent (it blueshifts in time; i.e. the energy density grows as the collapse proceeds) so that depending on the initial energy density \( \bar{\rho}_0 \) of the brane, defined by

\[
\bar{\rho}_0 := \frac{32}{27} \frac{\kappa^2 \alpha^2}{r_c^3} \rho_0 = \frac{\kappa^2}{3} \rho_0 \, r_c^2 \rho_0 , \tag{3.7}
\]

\( N \) and consequently the solutions to the Hubble rate will change.

1. The initial condition \( \bar{\rho}_0 > \bar{\rho}_+ \)

This corresponds to a case where the brane starts its evolution initially in the high energy regime. Since \( \bar{\rho} \) is an incremental function of time, \( \bar{\rho} > \bar{\rho}_0 > \bar{\rho}_+ \), thus the brane remains in this regime in the rest of its progress. In this case \( N > 0 \) and we will have a unique (real) solution:

\[
H_1(\eta) = \frac{1}{3} \left[ 2\sqrt{1 - 3\bar{\rho}} \cosh \left( \frac{\eta}{3} \right) - 1 \right] , \tag{3.8}
\]

where we have defined the parameter \( \eta \) as

\[
cosh(\eta) := \frac{S}{\sqrt{-Q^2}} , \quad \sinh(\eta) := \sqrt{\frac{N}{-Q^2}} . \tag{3.9}
\]

Note that this solution is always positive so it does not represent a collapsing scenario. Therefore, this initial condition, i.e. \( \bar{\rho}_0 > \bar{\rho}_+ \), is not physically relevant for our study.

2. The initial condition \( \bar{\rho}_0 \leq \bar{\rho}_+ \)

This case represents a collapse scenario initiated in a low energy regime. Then, as the brane energy density blueshifts, the collapse can evolve through all three regimes. In the following we will analyse these stages:

(i) As long as the energy density of the brane is in the range \( \bar{\rho}_0 < \bar{\rho} < \bar{\rho}_+ \), i.e. the brane evolves in the low energy regime, there exist three different solutions for the Hubble rate:

\[
\tilde{H}_1(\theta) = -\frac{1}{3} \left[ 2\sqrt{1 - 3\bar{\rho}} \cos \left( \frac{\theta + \pi}{3} \right) + 1 \right] , \tag{3.10}
\]

\[
\tilde{H}_2(\theta) = \frac{1}{3} \left[ 2\sqrt{1 - 3\bar{\rho}} \cos \left( \frac{\theta}{3} \right) - 1 \right] , \tag{3.11}
\]

\[
\tilde{H}_3(\theta) = -\frac{1}{3} \left[ 2\sqrt{1 - 3\bar{\rho}} \cos \left( \frac{\theta - \pi}{3} \right) + 1 \right] , \tag{3.12}
\]

where, \( 0 < \theta \leq \theta_0 \) is defined as follows:

\[
\cos(\theta) := \frac{S}{\sqrt{-Q^2}} , \quad \sin(\theta) := \sqrt{\frac{N}{Q^2}} . \tag{3.13}
\]

with \( \theta_0 \) being its value at the initial configuration:

\[
\cos(\theta_0) := \frac{9b + 27\bar{\rho}_0 - 2}{2\sqrt{(1 - 3\bar{\rho})^3}} . \tag{3.14}
\]

Likewise, the energy density of the brane evolves as

\[
\bar{\rho}(\theta) = \frac{2}{27} \left[ 1 + \sqrt{(1 - 3\bar{\rho})^3 \cos(\theta)} \right] - \frac{b}{3} . \tag{3.15}
\]

From Eqs. (3.10)-(3.12) it is clear that the first and the last solutions, i.e. \( \tilde{H}_1(\theta) \) and \( \tilde{H}_3(\theta) \), are negative. However, in the range \( 0 < \theta \leq \theta_0 \), \( |\tilde{H}_1(\theta)| \) gives a descending Hubble rate from its initial condition at \( \theta_0 \) towards \( \theta = 0 \) (cf. dashed blue curve in the right panel of Fig. 1); thus it cannot be a suitable solution for the collapsing process. In contrary, the solution \( \tilde{H}_2(\theta) \) is ascending in the range \( \theta = \theta_0 \) and \( \theta = 0 \) (cf. solid blue curve in the right panel of Fig. 1), so it illustrates a collapse scenario. Accordingly, the second solution, i.e. \( \tilde{H}_2(\theta) \), is always positive performing an expanding phase which is not of our interest in this range (cf. dashed red curve in the right panel of Fig. 1).

Following the above arguments, we have a single physically reasonable solution which is given by Eq. (3.10). This solution should respect the energy condition (2.19), i.e. for all \( \theta \) this solution should satisfy \( \tilde{H}_1(\theta) < \tilde{H}_1(\theta_0) < \tilde{H}_2(-) \), thus

\[
\cos \left( \frac{\theta_0 + \pi}{3} \right) > \cos \left( \frac{\theta + \pi}{3} \right) > \cos \left( \frac{2\pi}{3} \right) . \tag{3.16}
\]

This gives a condition on the brane energy density \( \bar{\rho}_- < 0 < \bar{\rho}_0 < \bar{\rho} \) which is already satisfied. Consequently, \( \bar{\rho}_0 \) implies

\[
\cos(\theta_0) > \frac{9b - 2}{2\sqrt{(1 - 3\bar{\rho})^3}} =: \cos(\theta_{\max}) , \tag{3.17}
\]

thus, \( 0 < \theta \leq \theta_0 < \theta_{\max} \).

(ii) Once the energy density of the brane reaches the value \( \bar{\rho}_+ \), the discriminant function \( N \) vanishes and the collapse gets to the limiting regime. There exist two real solutions in this limit as

\[
\tilde{H}_1 = -\frac{1}{3} \left( \sqrt{1 - 3\bar{\rho}} + 1 \right) = \tilde{\mathcal{H}}_1(-) , \tag{3.18}
\]

\[
\tilde{H}_2 = \frac{1}{3} \left( 2\sqrt{1 - 3\bar{\rho}} - 1 \right) , \tag{3.19}
\]

which are constants. The solution \( \tilde{H}_2 \) above is negative being the limit of Eq. (3.10) when \( \theta \to 0 \). On the other hand, \( \tilde{H}_2 \) is always positive being the limit of Eq. (3.11), thus it is not associated with a collapse scenario.

(iii) If the energy density of the brane continues growing, the collapse then enters the high energy regime
In this regime $N > 0$, so that we have only one real solution for the Hubble rate. This solution is given by Eq. (3.8) for $\eta > 0$. Subsequently, since this solution is always positive, it cannot be a reasonable candidate for the Hubble rate.

In summary, in the range $0 < b < 1/4$, the physically relevant solutions for the collapse on the brane is provided by Eq. (3.10) in the low energy regime and Eq. (3.18) in the limiting regime.

**B. The case $\frac{1}{4} \leq b < \frac{1}{3}$**

In this range, since $N > 0$, there exists a unique real solution for $\dot{H}$ which is described by the Eq. (3.8). Therefore, this range of $b$ does not provide a physically suitable solution to the collapse scenario.

**C. The case $b = \frac{1}{3}$**

For this value of $b$, the densities (3.5) and (3.6) are equal and become $\dot{\rho}_+ = \dot{\rho}_- = -1/27 < 0$. Therefore, $N > 0$ and there is a unique solution which is given by

$$\dot{H}_1(t) = \frac{1}{3} \left[ \left(1 + 27\dot{\rho}\right)^{1/3} - 1 \right].$$

This solution is always positive and hence inappropriate for describing our desirable contracting phenomenon.

**D. The case $b > \frac{1}{3}$**

In this case $\rho_+$ and $\rho_-$ are complex conjugates which implies that $N > 0$. Thus, the solution reads as

$$\dot{H}_1(\vartheta) = \frac{1}{3} \left[ 2\sqrt{3b - 1} \sinh \left( \frac{\vartheta}{3} \right) - 1 \right],$$

where, $\vartheta$ is defined by

$$\sinh(\vartheta) := \frac{S}{\sqrt{Q^2}}, \quad \cosh(\vartheta) := \sqrt{\frac{N}{Q^2}}.$$  (3.22)

Since $\vartheta$ is always positive, the above solution takes always positive values. Therefore, we discard this solution.

For the sake of completeness in this section, let us analyze the dust collapse in the standard DGP model. For this case, no GB component is present on the (Minkowski) bulk, so by setting $\alpha = 0$ in the Friedmann equation (2.3), for a normal branch, we get

$$H^2 = \frac{\kappa_5^2}{6r_c} \rho - \frac{H}{r_c}.$$  (3.23)

This is a quadratic equation for $H$ which has only one negative solution:

$$H(\rho) = -\frac{1}{2r_c} \left( 1 + \sqrt{1 + \frac{4}{3} \kappa_4^2 r_c^2 \rho} \right).$$  (3.24)

This solution represents a dynamical evolution of a dust cloud, started evolving from an initial density $\rho_0 > 0$, towards $a = 0$. Thus, at the center, both the energy density and the Hubble rate blow up and the collapse ends up in a shell-focusing singularity.

In the next sections we will elaborate more on the physical implications of the solutions (3.10) and (3.18) obtained for the range $0 < b < 1/4$.

**IV. THE FATE OF THE SINGULARITY**

In the previous section we have found the physically reasonable solution (3.18) to the generalized Friedmann equation (3.1). In this section, we will look into the fate of the collapse provided by this solution and the types of the singularities it may encounter.

We start with analyzing the behaviour of the first (comoving) time derivative of the Hubble rate. By taking the time derivative of the generalized Friedmann equation (3.1) we get

$$\dot{H} = -\frac{\kappa^2_{(5)} H \rho}{2H + \frac{1}{8a} (1 + 8aH^2)},$$

in which we have substituted for $\dot{\rho}$, from the energy conservation equation $\dot{\rho} = -3H\rho$ for the dust. It turns out that some interesting features of the collapse can be achieved once the denominator of Eq. (4.1) vanishes:

$$8aH^2 + 2r_c H + 1 = 0.$$  (4.2)

This quadratic equation has two real roots

$$H_1 = -\frac{r_c}{8\alpha} \left( 1 + \sqrt{1 - 3b} \right),$$

$$H_2 = -\frac{r_c}{8\alpha} \left( 1 - \sqrt{1 - 3b} \right).$$  (4.4)

These roots are in fact equivalent respectively, to the lower and the upper bound (2.20) and (2.21) of the $\dot{H}$ provided by the NEC (2.19). The only physically relevant root of Eq. (4.2) which adjusts to the solution (3.10) is $H_1$, i.e. the limiting solution; $H_1 \equiv b r_c H_1 = \bar{H}_1^{-}$. This solution indicates that, for an initial configuration in the low energy regime, as the dust cloud collapses towards the limiting regime, through the solution (3.10), the first time derivative of the Hubble rate diverges, whereas, the energy density and the Hubble rate itself remain finite:

$$\theta \to 0: \quad \bar{\rho} \to \bar{\rho}_+, \quad \bar{H}_1(\theta) \to \bar{H}_1^{-}, \quad \dot{\bar{H}}_1 \to \infty.$$  (4.5)
Therefore, a peculiar abrupt event occurs in a finite (comoving) time $t_{\text{sing}}$ in the limiting regime, prior to the formation of the shell-focusing singularity at $R = r_a = 0$. In a (late-time) cosmological setting, such an abrupt event is called a sudden singularity [28, 45, 46]. Following such denomination, we thus call it, within the present setting, a sudden black hole/naked singularity depending on whether or not the singularity will be trapped by an apparent horizon at the final stages of the collapse.

Finally, using the relation between the Hubble parameter and the energy density, $\dot{\rho} + 3H\rho = 0$, one can write the Hubble rate as a function of comoving time. Then, by integrating this equation, the time remaining before the dust brane hits the sudden singularity is obtained as

$$t_{\text{sing}} - t_0 = -\frac{br_c}{3} \int_{\bar{\rho}_0}^{\bar{\rho}_+} \frac{d\bar{\rho}}{\bar{H}(\bar{\rho})},$$

where, $t_0$ and $t_{\text{sing}}$ denote the present time and the time at the sudden singularity, respectively. To evaluate the integral (4.6), it is convenient to rewrite the solution (3.10) explicitly as a function of $\bar{\rho}$ as

$$\bar{H}_1(\bar{\rho}) = -\frac{1}{2} (s_+ + s_-) - \frac{i\sqrt{3}}{2} (s_+ - s_-) - \frac{1}{3},$$

where,

$$s_{\pm}(\bar{\rho}) := \left[ S(\bar{\rho}) \pm \sqrt{Q^3 + S^2(\bar{\rho})} \right]^\frac{1}{2}.$$

The result is shown in Fig. 2 where $t_{\text{sing}} - t_0$ is evaluated for different values of the energy density in the range $\bar{\rho}_0 < \bar{\rho} < \bar{\rho}_+$. As a specific case, given by the solution (3.24) for the standard DGP model, the time derivative of the Hubble rate is derived from the Friedmann equation (3.23) as

$$\dot{H} = \frac{\kappa_4^2 H\rho}{2H + 1/r_c}.$$  

This is equivalent to the Eq. (4.1) for $\alpha = 0$ which implies that, for the case $H = -1/2r_c$, the time derivative of Hubble parameter diverges. This value for $H$ does not display a physically relevant situation, i.e. matching with the solution (3.24). Therefore, only a shell-focusing singularity at $R = 0$ can occur in this case.

V. TRAPPED SURFACES AND THE EXTERIOR GEOMETRY

To complete our model of gravitational collapse in DGP-GB brane scenario, we need to match the interior spacetime (2.2) to a suitable exterior geometry. In an

1 Because of the extra-dimensional effects, i.e., the confinement of the dust cloud to the brane while the gravitational field can access the extra dimension, and the non-local gravitational interaction between the brane and the bulk, the standard 4D matching conditions (cf. [55, 56]) on the brane are much more complicated.
FIG. 2. Variation of the comoving time \( t - t_0 \) in terms of the energy density \( \bar{\rho} \) of the brane. The time left before the brane hits the sudden singularity, \( t_{\text{sing}} - t_0 \), is shown at the singular energy density \( \bar{\rho}_+ \). For fixed values of \( r_v = 0.2 \) and \( b = 1/8 \), we get \( \bar{\rho}_+ \approx 0.069 \) and \( t_{\text{sing}} - t_0 \approx 0.45 \).

advanced Eddington-Finkelstein-like coordinates \((v, r_v)\), a general exterior metric is written as \([60]\):

\[
ds^2 = -f(v, r_v) \, dv^2 - 2dv \, dr_v + r_v^2 \, d\Omega^2. \tag{5.1}
\]

In this coordinates system, the trapping horizon is given simply by the relation \( f(v, r_v) = 0 \). In general, the formation of the (shell-focusing) singularity is independent of matching the interior and the exterior spacetimes. For GR settings, when the interior matter content is a pressure-less dust fluid, an exterior Schwarzschild geometry can be matched at the boundary of the two regions \([5, 47]\). However, in the herein effective theory, due to presence of an effective nonzero pressure \((2.12)\), induced by the DGP-GB geometrical terms, the exterior region cannot be a Schwarzschild spacetime. It follows that, a more realistic picture should involve a radiation zone and matter emissions in the outer region. Therefore, we match the interior, at the boundary \( \Sigma \) of the dust cloud, to an exterior geometry described by a general class of non-stationary metric, called the generalized Vaidya spacetime \([61]\). Its geometry is described by the metric \((5.1)\) where the boundary function \( f(v, r_v) \) is defined as

\[
f(v, r_v) := 1 - \frac{2GM(v, r_v)}{r_v}. \tag{5.2}
\]

Here, \( G \) is the 4D Newton’s constant given by the term \( \kappa_{(4)}^2 = 8\pi G \), and \( M \) is called the generalized Vaidya mass which depends on the retarded null coordinates \((v, r_v)\).

According to GR, the matching conditions allow us to study the formation of apparent horizons that we refer to it later. Thereby, in the herein effective model for the collapse of a fluid, having the effective density and pressure \( \bar{\rho}_{\text{eff}} \) and \( \bar{p}_{\text{eff}} \), the Israel-Darmois junction conditions \([55, 56]\) should be satisfied at the boundary \( \Sigma \) of the fluid (with the boundary shell \( r = r_b \)). This implies that, at \( \Sigma \), two fundamental forms, namely, the metric and the extrinsic curvature must match. Matching the area radius at the boundary results in \( R(t, r_b) = r_b a(t) = r_v \). Likewise, matching the first and second fundamental forms on \( \Sigma \), for the FLRW interior metric and the generalized Vaidya exterior metric, leads to \([20]\):

\[
F_{\text{eff}}(t, r_b) = 2GM(v, r_v), \tag{5.3}
\]

\[
(dv/dt)_\Sigma = (1 + r_b \ddot{a}) \left( 1 - \frac{F_{\text{eff}}}{R} \right)^{-1}, \tag{5.4}
\]

\[
GM(v, r_v, r_v) = \frac{F_{\text{eff}}}{2R} + r_v^2 \ddot{a}, \tag{5.5}
\]

where, \( F_{\text{eff}}(t, r_b) \) is interpreted as the effective Misner-Sharp mass function \([62]\), being the total effective mass within the boundary shell labelled by \( r_b \) at time \( t \), which satisfies the relation

\[
F_{\text{eff}} = \frac{\kappa_{(4)}^2}{3} \rho_{\text{eff}} R^3. \tag{5.6}
\]

Note that, in the GR context, \( \rho_{\text{eff}} \) is replaced by \( \rho \), so that the effective mass function becomes the usual gravitational mass \( F = (\kappa_{(4)}^2/3)\rho R^3 \). For the dust ball within \( r_b \), by setting \( \rho = \rho_0 (R_0/R)^3 \) (where \( R_0 \equiv r_b \rho_0 \) is the physical radius of the boundary shell), this yields

\[
F = 2Gm_0 = \text{const.}, \tag{5.7}
\]

where \( m_0 \) is the total mass of the star (i.e. the physical mass contained inside \( R_0 \)).

A. Evolution of trapped surfaces

The geometry of the trapped surfaces is a key to ascertain the formation of a black hole or a naked singularity. The ratio of the effective mass to the physical radius of the collapse \( R \) determines the properties of the trapped surface. When the mass function satisfies the inequality \( F_{\text{eff}} < R \) no trapped surfaces would form during the dynamical evolution of the collapse, whereas for the case \( F_{\text{eff}} \geq R \) trapped surfaces will emerge. We now proceed by studying the behavior of \( F_{\text{eff}} \), evaluated at the solution \([5, 47]\). By substituting the effective energy density
for the (−) branch, into Eq. (5.6) we obtain
\[ F_{\text{eff}} = F - \frac{1}{r_c} \left( 1 + \frac{8}{3} \alpha H^2 \right) H R^3 = 2GM. \]  
(5.8)

To examine the visibility of the sudden singularity, we rewrite the condition \( F_{\text{eff}} < R \) in a dimensionless form
\[ g(\bar{\rho}) := \bar{\rho} - (b + H^2) \bar{H} - \bar{H}^2 < 0, \]  
(5.9)

where
\[ \bar{H}_h(\bar{\rho}) := -\frac{br_c}{R_0} \left( \frac{\bar{\rho}}{\bar{\rho}_0} \right)^{\frac{1}{b}}. \]  
(5.10)

The Hubble rate \( \bar{H} \) in Eq. (5.9) is the solution to the Friedmann equation (3.1). So, the terms \( \bar{\rho} - (b + H^2)\bar{H} \) can be replaced by \( \bar{H}^2 \). This yields
\[ g(\bar{\rho}) := \bar{H}^2(\bar{\rho}) - \bar{H}_h^2(\bar{\rho}) < 0, \quad \text{for} \quad \bar{\rho}_0 \leq \bar{\rho} \leq \bar{\rho}_+. \]  
(5.11)

This inequality implies that, to avoid the trapped surfaces forming, for all \( \bar{\rho}_0 \leq \bar{\rho} \leq \bar{\rho}_+ \), the value \( |\bar{H}_1(\bar{\rho})| \) should be always smaller than \( |\bar{H}_h(\bar{\rho})| \), i.e. \( \bar{H}_h(\bar{\rho}) < \bar{H}_1(\bar{\rho}) < 0 \), from the initial configuration up to the formation of the sudden singularity. In the case where (5.11) does not hold, i.e. \( g(\bar{\rho}) \geq 0 \), trapped surfaces would form and the sudden singularity will be covered by the black hole horizon. The equality gives the energy density \( \bar{\rho}_h \) or the physical radius \( R_h \), at the apparent horizon:
\[ \bar{H}_1(\bar{\rho}_h) = -\frac{br_c}{R_0} \left( \frac{\bar{\rho}_+}{\bar{\rho}_0} \right)^{\frac{1}{3}} \quad \text{or} \quad \bar{H}_1(R_h) = -\frac{br_c}{R_h}. \]  
(5.12)

Once the energy density \( \bar{\rho}_h \) is found from the left equation above, the time during which the horizon forms, i.e. \( t_h - t_0 \), can be determined from the Fig. 2.

Fig. 3 depicts the condition (5.11) for the evolution of the apparent horizon obtained for different values of initial conditions \( R_0 \) and \( \bar{\rho} \). The “red curve” represents an initial condition for which an apparent horizon forms before the brane reaches to the sudden singularity. The “gray curve” shows a condition for the formation of a naked sudden singularity, and the “blue curve” represents a limiting situation, where a horizon forms at the singular epoch. It turns out that, if (5.11) holds initially, i.e., \( g(\bar{\rho}_0) < 0 \), then to ensure that trapped surfaces will never evolve during the collapse, \( g(\bar{\rho}) \) would vanish only where \( \bar{\rho} > \bar{\rho}_+ \). Let us be more precise as follows. Suppose that (5.11) holds at the initial configuration. Accordingly, this gives the condition
\[ H_0 > -\frac{1}{R_0}, \]  
(5.13)

between the initial Hubble rate \( H_0 = \bar{H}_1(\bar{\rho}_0) \) and the initial physical radius \( R_0 \) of the dust. Then, the necessary and sufficient condition for formation of trapped surfaces during the collapse is that \( g(\bar{\rho}_+) \geq 0 \). This yields
\[ \frac{br_c}{R_0} \left( \frac{\bar{\rho}_+}{\bar{\rho}_0} \right)^{\frac{1}{3}} \leq \frac{1}{3} \left( \sqrt{1 - 3b} + 1 \right). \]  
(5.14)

Now, by substituting \( \bar{\rho}_+ \) from Eq. (3.5) into the relation above, after some simplification, Eq. (5.14) becomes
\[ m_0 \geq m_\star, \]  
(5.15)
where, \( m_* \) is a constant, with the dimension of mass, defined by
\[
m_* := \frac{br_c}{2G} \frac{2 \left[ 1 + \sqrt{(1 - 3b)^3} - 9b \right]}{(\sqrt{1 - 3b} + 1)^3},
\]
(5.16)
which explicitly depends only on the DGP-GB parameters \( b \) and \( r_c \). The right plot in the Fig. 3 represents the behaviour of the \( m_* \) with respect to \( b \) in the range \( 0 < b < 1/4 \) for a fixed value of \( r_c \). Clearly, the parameter \( m_* \) represents a threshold mass for the black hole formation. In other words, if the initial mass \( m_0 \) of the dust cloud is smaller that \( m_* \), then no horizon would form during the collapse up to the sudden singularity. But, if \( m_0 \geq m_* \), then formation of trapped surfaces will be ensured and the final sudden singularity will be hidden behind a black hole horizon.

For the standard DGP model, with the solution (3.24), the effective mass function (5.8) reduces to
\[
F_{\text{eff}} = F + \frac{1}{2r_c} \left[ 1 + \sqrt{1 + \frac{4}{3} \kappa_4 r_c^2 \rho} \right] R^3.
\]
(5.17)
This expression is always positive so that \( F_{\text{eff}} > R \), which implies that the singularity will be hidden behind the event horizon of a black hole. To be more precise, let us rewrite the condition (5.11) for the solution (3.24) as
\[
m_0 < -\frac{4\pi \rho}{3H^3(\rho)}, \quad \text{for all} \quad \rho \geq \rho_0.
\]
(5.18)
If this condition holds initially, in order to avoid the horizon formation, the dust cloud should remain untrapped until when the singularity at \( R = 0 \) is reached. It follows that, in the singular limit, i.e. as \( R \to 0 \), where the dust density blows up as \( \rho \to \infty \), the right hand side of the inequality (5.18) vanishes:
\[
-\frac{4\pi \rho/3}{H^3(\rho)} \approx \frac{1}{\rho^{1/2}} \to 0 \quad \Rightarrow \quad m_0 < 0.
\]
(5.19)
This implies that, for condition (5.18) to be satisfied during the collapse, the star mass \( m_0 \) should be negative in the vicinity of the singularity \( R = 0 \), which cannot be true. Therefore, for all initial star mass \( m_0 > 0 \), in the standard DGP model, formation of the black hole is inevitable.

B. The exterior geometry

The matching conditions (5.3)-(5.5) lead to the exterior geometry (5.1) with the boundary function
\[
f(R) = 1 - \frac{F_{\text{eff}}}{R} = 1 - \frac{\kappa_4}{3} \rho_{\text{eff}} R^2
= 1 - H^2 R^2.
\]
(5.20)
It is clear that, on the onset of the collapse, the condition (5.13) for the absence of trapped surfaces (in the interior region) can be translated to the condition \( f(R_0) > 0 \) on the boundary function (in the exterior geometry). Now, for the interior solution \( \bar{H}_1(\bar{\rho}) \), we get
\[
f(R) = 1 - \frac{GM(R)}{R} = 1 - \left( \frac{\bar{H}_1 R}{br_c} \right)^2,
\]
(5.21)
where \( M(R) \) is the effective mass of the exterior geometry. The physical solution \( \bar{H}_1 \) should be written now in terms of the exterior parameters (i.e. \( m_0 \) and \( R \)). Given the solution (4.7), the expression for \( S(\bar{\rho}) \) in terms of the exterior parameters can be rewritten from Eq. (3.3) as
\[
S(R) = b - \frac{1}{27} + b^2 r_c^2 G m_0 R^3.
\]
(5.22)
Now, in terms of the parameters \( m_0, b \) and \( r_c \), we can write the solution \( f(R) \) as
\[
f(R) = 1 - \frac{R^2}{b r_c^2} \left[ 1 + i\sqrt{3} \left( \frac{b}{6} - \frac{1}{27} + \frac{b^2 r_c^2 G m_0}{R^3} \right)^2 \right]
- \sqrt{Q^3 + \left( \frac{b}{6} - \frac{1}{27} + \frac{b^2 r_c^2 G m_0}{R^3} \right)^2 - \left( \frac{1}{3} \right)^2}.
\]
(5.23)
The behaviour of \( f(R) \) is plotted in the Fig. 4. It is shown that, there exist three different cases depending on the initial mass \( m_0 \): If \( m_0 < m_* \) (gray solid curve), then \( f(R) \) is always positive which represents a spacetime geometry with no horizon. On the other hand, for \( m_0 > m_* \) (red curve), there exists a trapped region, where \( f(R) < 0 \), which covers the sudden singularity. This represents a black hole geometry. Finally, the case \( m_0 = m_* \) represents a limiting condition for the formation of the black hole. In other words, in this case the black hole horizon forms only on the sudden singularity.

Finally, by substituting the original DGP solution (3.24) into (5.20), the boundary function becomes
\[
f(R) = 1 - \frac{2Gm_0}{R} - \frac{R^2}{2r_c^2} \left( 1 + \sqrt{1 + \frac{8\pi^2 G m_0}{R^3}} \right).
\]
(5.24)
This function represents a black hole spacetime whose geometry is different from the Schwarzschild one, the difference being involved in the last term (cf. the dashed curve in Fig. 4). As it is clear from the Fig. 4, at high en-
energies in the vicinity of the general relativistic singularity (i.e., in the limit \( R \ll r_c \)), the second term in Eq. (5.24) is negligible so that the standard DGP model becomes purely 4D, while at low energies (i.e., \( R \gg r_c \)) the second term becomes dominant. This is a consequence of the fact that, the original DGP model provides only IR modifications to the brane and cannot change the nature of the UV regime.

In addition, in Fig. 4 the boundary function \( f(R) \) in the GR and the standard DGP settings were plotted for the mass \( m_0 = 0.002 < m_* \). Clearly, for this range of star mass, trapped surfaces form as the collapse proceeds. However, the GB modification to the high energy regime, prevents the formation of the horizons by setting a threshold mass \( m_* \) for the horizon to develop.

VI. CONCLUSIONS AND OUTLOOK

In this paper, we have investigated the gravitational collapse of a dust cloud in an induced gravity model, specifically where a dust fluid is present on a DGP brane, whereas a GB term is provided for the bulk. In particular, by considering a normal (non self-accelerating) DGP branch, we studied the evolution equation for the collapse and derived different classes of solutions.

Concretely, by applying suitable physical conditions on these solutions (i.e., qualifying a contracting brane and the required energy conditions), we obtained a new collapse scenario with a distinct set of characteristic features. To be more precise, this scenario represents a process in which the brane starts its evolution from an appropriate initial condition towards vanishing physical radius \( R = 0 \), where the general relativistic (shell-focusing) singularity is located. However, prior to the formation of this shell-focusing singularity, the brane reaches a regime where the Hubble rate and the energy density get finite specific values, while the (comoving) time derivative of the Hubble rate diverges. Aware of a labeling chosen for a phenomenon in late-time cosmological setting, we analogously suggest herein to call this abrupt event a sudden naked/black hole singularity, depending on the avoidance/formation of the apparent horizon.

Subsequently, we studied the evolution of the trapped surfaces on the brane. In doing so, we aimed at establishing whether the specific IR-UV combination employed within this manuscript, would allow new features regarding the CCC. In particular, whether a GB sector would introduce any challenging appraisal. We found that, the formation of trapped surfaces depends only on the initial value of the star mass \( m_0 \). Specifically, there exists a threshold mass \( m_* \), below which no horizon can form. This is indeed, a consequence of the presence of the GB term in the action, implying significant modifications at high energies. In particular, from further utilizing appropriate junction conditions on the 2-boundary surface of the dust cloud, we obtained a suitable exterior geometry which matches to the interior region at the boundary surface. It turns out that, this exterior geometry represents a black hole only if \( m_0 \) is larger or equal to \( m_* \). Otherwise, the sudden singularity will be visible to the distant observers through the exterior region. It is worthy of note that, in the absence of the GB term on the bulk (i.e. for the standard DGP model), as the collapse proceeds, the formation of the trapped surfaces is inevitable. Meanwhile, as the brane enters the high energy regime–where gravity becomes 4D on the brane–the induced gravity alone does not change the process of the singularity formation at \( R = 0 \), thus, the final shell-focusing singularity will be covered by a black hole horizon.

Gravitational waves can provide an observational source to test some of the novel predictions and modifications to GR at high energies. The herein DGP-GB brane-world scenario introduces significant changes to the dynamics of the gravitational collapse and perturbations thereon, offering interesting and potentially testable implications for high energy astrophysics. In particular, from the brane viewpoint, the bulk effects, i.e. the high-energy corrections and the corresponding Kaluza-Klein modes, act as source terms for the brane perturbations on the interior spacetime. These effects can be carried out to the exterior spacetime geometry of the emergent (non-
Schwarzschild) black hole/naked singularity through the matching conditions on the 2-boundary surface. Consequently, probing such effects through observations of gravitational waves could eventually reveal non-standard features of the spacetime in the high energy regime.

ACKNOWLEDGMENTS

PVM and YT acknowledge the FCT grants UIDB-MAT/00212/2020 and UIDP-MAT/00212/2020 at CMA-

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This article is based upon work conducted within the Action CA18108–Quantum gravity phenomenology in the multi-messenger approach–supported by the COST (European Cooperation in Science and Technology).
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