Improved Bounds for Rectangular Monotone Min-Plus Product and Applications

Anita Dürr∗

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Abstract

In a recent breakthrough paper, Chi et al. (STOC’22) introduce an \(\tilde{O}(n^{3+\omega/2})\) time algorithm to compute Monotone Min-Plus Product between two square matrices of dimensions \(n \times n\) and entries bounded by \(O(n)\). This greatly improves upon the previous \(\tilde{O}(n^{12+\omega/5})\) time algorithm and as a consequence improves bounds for its applications. Several other applications involve Monotone Min-Plus Product between rectangular matrices, and even if Chi et al.’s algorithm seems applicable for the rectangular case, the generalization is not straightforward. In this paper we present a generalization of the algorithm of Chi et al. to solve Monotone Min-Plus Product for rectangular matrices with polynomial bounded values. We next use this faster algorithm to improve running times for the following applications of Rectangular Monotone Min-Plus Product: \(M\)-bounded Single Source Replacement Path, Batch Range Mode, \(k\)-Dyck Edit Distance and 2-approximation of All Pairs Shortest Path. We also improve the running time for Unweighted Tree Edit Distance using the algorithm by Chi et al.

Keywords. Monotone Min-Plus Product; Bounded-Difference Min-Plus Product; Fine-Grained Complexity

1 Introduction

The Min-Plus Product \(C\) between two \(n \times n\) matrices \(A\) and \(B\) is defined as the \(n \times n\) matrix \(C = A \star B\) with entries \(C_{i,j} = \min_{k \in [n]} \{A_{i,k} + B_{k,j}\}\) for all \(i,j \in [n]\). The naive algorithm computes \(C\) in \(O(n^3)\) time while the currently fastest algorithm by Williams \cite{21,22} runs in time \(n^3/2^{O(\sqrt{\log n})}\). However, since the All Pairs Shortest Path (APSP) problem is equivalent to the Min-Plus Product, under the APSP Hypothesis there is no algorithm solving Min-Plus Product in truly subcubic running time of \(O(n^{3-\varepsilon})\) for some \(\varepsilon > 0\). Nevertheless, it is sometimes possible to break that barrier if we assume some structure on the matrices. In their recent breakthrough paper, Chi et al. \cite{7} show that if \(B\) has monotone rows then one can solve the Min-Plus Product in time \(\tilde{O}(n^{3+\omega/2})\), where \(\omega\) is the exponent of fast matrix multiplication. This improves upon the previously best known algorithm by Gu et al. \cite{13} for Square Monotone Min-Plus Product which had a running time of \(\tilde{O}(n^{2+\omega})\).

Fortunately, Min-Plus Product, as a fundamental problem, can be used to solve many other applications in which it is easy to assume that one of the matrices is monotone. Therefore the result of Chi et al. yields improvements for several applications. A direct consequence is for instance the improvement of algorithms solving the Language Edit Distance, RNA-folding and Optimum Stack Generation problems, which where reduced to Square Monotone Min-Plus Product by Bringmann et al. \cite{2}. However there are several other applications of Monotone Min-Plus Product in which the matrices are rectangular: the Single Source Replacement Path (SSRP) with bounded edge-weights (Gu et al. \cite{13}), Batch Range Mode (Williams and Xu \cite{23}), \(k\)-Dyck Edit Distance (Fried et al. \cite{10}), as well as a 2-approximation of APSP (Deng et al. \cite{8}). Even though the techniques used in \cite{7} to solve Square Monotone Min-Plus Product seem to be applicable for rectangular matrices, a generalization of the algorithm is not straightforward and needs to be done carefully. In this paper we show the generalization for the rectangular case and then improve the running time for the above mentioned problems. We also improve the running time for Unweighted Tree Edit Distance, which was reduced to Square Monotone Min-Plus Product by Mao \cite{17}, using the algorithm by Chi et al. \cite{7}.

∗anita.durr@epfl.ch (École Polytechnique Fédérale de Lausanne, Switzerland)

1 We use the notation \(\tilde{O}\) to hide a polylogarithmic factor.
Definitions We restrict ourselves to integer matrices only. A matrix $X$ is said to be monotone if each row is a non-decreasing sequence and every entry is non-negative and bounded by $O(n^\mu)$ for some $\mu \geq 0$. When the monotonicity holds for the columns we call the matrix column-monotone. We also consider Min-Plus Product between rectangular matrices $A$ and $B$ of dimensions $n \times n^\beta$ and $n^\beta \times n$ for some $\beta \geq 0$. Define $\omega(\beta)$ to be the exponent of fast rectangular matrix multiplication between two matrices of dimensions $n \times n^\beta$ and $n^\beta \times n$. So $\omega = \omega(1)$. We focus on the Rectangular Monotone Min-Plus Product parameterized by $(\beta, \mu)$, which is the Min-Plus Product between matrices $A$ and $B$ of dimensions $n \times n^\beta$ and $n^\beta \times n$ where $B$ is monotone with non-negative entries bounded by $O(n^\mu)$.

Results Our contribution is two fold. First we provide an algorithm computing the Rectangular Monotone Min-Plus Product in time $\tilde{O}(n^{\frac{1+\beta+\mu+\omega(\beta)}{2}})$, as stated in Theorem 1. We remark that the algorithm directly generalizes the algorithm provided by Chi et al. [7] for the special setting $\beta = \mu = 1$ and that in that case one falls back on their results. Next, we detail how the result of Theorem 1 is used to improve running times of $M$-bounded SSRP, Batch Range Mode, $k$-Dyck Edit Distance, 2-approximation APSP. Indeed in those applications the dependency on the Monotone Min-Plus Product is not direct but depends on some parameters that need to be optimized according to the new running time given by Theorem 1. We also simplify and improve the running time for Unweighted Tree Edit Distance. The improved bounds are summarized in Table 1. Additionally, we discuss one consequence of Chi et al.’s algorithm for the lower bound of SSRP algorithms.

**Theorem 1.** Let $\beta, \mu$ be non-negative real numbers. Let $A$ be an $n \times n^\beta$ integer matrix and $B$ an $n^\beta \times n$ integer monotone matrix with non-negative entries bounded by $O(n^\mu)$. Then the Min-Plus Product $A \ast B$ can be computed in $\tilde{O}(n^{\frac{1+\beta+\mu+\omega(\beta)}{2}})$ time in expectation.

A similar result for Theorem 1 and the improved running time for the $k$-Dyck Edit Distance problem was obtained independently and in parallel by Fried et al. [9].

Monotone and Bounded-Difference Min-Plus Product Strictly speaking, except for SSRP and Batch Range Mode, the above mentioned applications don’t use Monotone Min-Plus Product but Bounded-Difference Min-Plus Product. A matrix $X$ is $\delta$-bounded-difference (or simply bounded-difference if $\delta$ is a constant) if $|X_{i,j} - X_{i,j+1}| \leq \delta$ and $|X_{i,j} - X_{i+1,j}| \leq \delta$ for all $i,j$. One can reduce a $\delta$-bounded difference matrix $X$ to a monotone matrix $X'$ in quadratic time by setting $X'_{i,j} = X_{i,j} + \delta j - X_{i,1}$ for all $i,j$. In that case elements of $X'$ are non-negative and bounded by $O(n\delta)$. In most of the above mentioned applications, a Min-Plus Product is performed between two bounded-difference matrices. This means that a stronger structure is assumed on the matrices than just the monotonicity of one matrix and this could allow for faster algorithms. In fact, before the recent result of Chi et al. [7] (and its generalization via Theorem 1), the best known algorithm by Chi et al. [6] for Square Bounded-Difference Min-Plus Product had a running time of $\tilde{O}(n^{2.779})$, while the best known algorithm by Gu et al. [13] for square Monotone Min-Plus Product (when $\mu = 1$) had a running time of $\tilde{O}(n^{2.8653})$. However Theorem 1 improves on both of those bounds and thus improves the running time for both Monotone and Bounded-Difference Min-Plus Product.

Related work It is worth mentioning that other problems such as Dynamic Range Mode (Gu et al. [13]), 3SUM+ and 3SUM+ in preprocessed universe (Chan and Lewenstein [5]) were solved alongside Min-Plus Product.
Chan and Lewenstein reduce Min-Plus Convolution to 3SUM+ and a better algorithm for Min-Plus Convolution thus doesn’t entail a better algorithm for 3SUM+. However the tools and techniques used in Chan and Lewenstein [5] for Min-Plus Convolution are adapted to 3SUM+ and 3SUM+ in a preprocessed universe to obtain a faster algorithm. Similarly, in Gu et al. [13] the Min-Plus Product algorithm is adapted for Dynamic Range Mode. In both those work Min-Plus Product is not used as a black-box, hence our result does not directly imply an improvement.

Finally, we would like to mention that Chi et al. also solve Monotone Min-Plus Convolution in \( \tilde{O}(n^{1.5}) \) time with the same technique as in [7] for Monotone Min-Plus Product. This is a major improvement upon the previously \( \tilde{O}(n^{1.869}) \) time algorithm by Chan and Lewenstein [5]. Since Binary Jumbled Indexing (or Histogram Indexing) can be reduced to Monotone Min-Plus Convolution in quadratic time (see Chan and Lewenstein [5]), this also directly yields an \( \tilde{O}(n^{1.5}) \) time algorithm for it. Additionally, earlier this year Bringmann and Cassis [1] used Chi et al.’s result to get new running times for Knapsack.

## 2 Rectangular Monotone Min Plus Product

In this section we reformulate and generalize the algorithm presented by Chi et al. [7] to compute the Monotone Min-Plus Product for arbitrary \( \beta, \mu \geq 0 \), thus proving Theorem 1. We recall that all the ideas of the presented algorithm are from Chi et al. [7] and that the only novelty is to extend the setting of the matrices. We also like to note that the following algorithm can be adapted to solve Min-Plus Product when \( \beta \) is column-monotone instead of row-monotone by using the same method as explained by Chi et al. in [7, Appendix A]. The running time is slightly different and we sketch a proof in Appendix B. Also, since \((A \ast B)^T = B^T \ast A^T\), this algorithm and the one described in Appendix B can be used if \( A \) is column- or row-monotone respectively.

**Overview**  
Let \( A \) and \( B \) be two matrices of dimensions \( n \times n^\beta \) and \( n^\beta \times n \) such that \( B \) is monotone and with entries bounded by \( O(n^h) \). The idea of the algorithm is to divide the matrix \( C = A \ast B \) into two matrices \( C^* \) and \( \tilde{C} \) such that \( C^* \) and \( \tilde{C} \) be such that for any \( i \in [n], j \in [n^\beta], \) \( k \in [n^\beta] \), and \( u \) updates \( s_i \leftarrow \min\{s_i, u\} \) for each \( i \in [n] \). A query operation returns the current value of an arbitrary element \( s_i \). We will also multiply matrices of polynomials: if \( A \) and \( B \) are matrices of polynomials of degree at most \( d \) and of dimensions \( n \times n^\beta \) and \( n^\beta \times n \), then we can compute the product \( A \cdot B \) in \( \tilde{O}(dn^{\omega(\beta)}) \) time using Fast Fourier Transform and fast matrix multiplication.

The algorithm uses the **segment tree** data structure which allows for a sequence of integers \( S = \{s_1, \ldots, s_n\} \) to perform the following update and query operations in \( O(\log n) \) deterministic worst-case time. An update on \( S \) is defined by an interval \([i, j] \subseteq [n]\) and an integer \( u \) and updates \( s_i \leftarrow \min\{s_i, u\} \) for each \( i \in [i, j] \). A query operation returns the current value of an arbitrary element \( s_i \).

The computation of the remainder matrix \( \tilde{C} = (C \mod p) \) is more complex. We will proceed iteratively, each step computing one bit of \( \tilde{C} \). For that, we define \( h \) to be such that \( 2^h \leq \beta < 2^{h+1} \) and for each \( i = h, h - 1, \ldots, 0 \) let \( A^{(i)} = \left[ \frac{A \mod p}{2^i} \right] \) and \( B^{(i)} = \left[ \frac{B \mod p}{2^i} \right] \). We then construct a matrix \( C^{(i)} \) approximating \( \left[ \frac{C \mod p}{2^i} \right] \) such that in the end \( C^{(0)} = \tilde{C} \). Note that \( C^{(i)} \) will not be the Min-Plus Product \( A^{(i)} \ast B^{(i)} \), as this would be too costly, but is only an approximation of it. \( C^{(i)} \) is computed using \( C^{(i+1)}, A^{(i+1)}, B^{(i+1)}, A^{(i)} \) and \( B^{(i)} \). Lemma 7 ensures that the correct \( C_{i,j}^{(i)} \) is equal to the sum \( A_{i,k}^{(i)} + B_{k,j}^{(i)} \) for some \( k \) satisfying both conditions:

- \( A_{i,k}^{(i+1)} + B_{k,j}^{(i+1)} = C_{i,j}^{(i+1)} + b \) for some \( b \in \{-10, \ldots, 10\} \)
- \( A_{i,k}^{(i)} + B_{k,j}^{(i)} = C_{i,j}^{(i)} \)

The algorithm first focuses on the first condition, and then removes all candidates that don’t satisfy the second condition. In fact, terms \( A_{i,k}^{(i)} + B_{k,j}^{(i)} \) satisfying the first condition can be filtered out by using a common polynomial multiplication trick: construct two matrices of polynomials \( A^p, B^p \) on two variables \( x \) and \( y \) such that for any \( i \in [n], k \in [n^\beta], j \in [n] \)

\[
A_{i,k}^p = x^{A_{i,k}^{(i)}} \cdot y^{A_{i,k}^{(i+1)}},
\]

and

\[
B_{k,j}^p = x^{B_{k,j}^{(i)}} \cdot y^{B_{k,j}^{(i+1)}}.
\]

When computing the (standard) product \( C^p = A^p \cdot B^p \), we will get for any \( i, j \in [n] \)

\[
C_{i,j}^p = \sum_{k \in [n^\beta]} x^{A_{i,k}^{(i)} + B_{k,j}^{(i)}} \cdot y^{A_{i,k}^{(i+1)} + B_{k,j}^{(i+1)}}
\]
So to filter out terms \( A_{i,k}^{(l)} + B_{k,j}^{(l)} \) satisfying the first condition of Lemma 7, we can compare the \( y \)-degrees of \( C_{i,j}^p \) to \( C_{i,j}^{(l+1)} + b \) for every offset \( b \in \{-10, \ldots, 10\} \) and keep the corresponding \( x \)-degrees. Now it might be the case that those terms don’t satisfy the second condition, i.e. are such that \( A_{i,k}^{*} + B_{k,j}^{*} \neq C_{i,j}^{*} \). This is why we construct for all \( b \in \{-10, \ldots, 10\} \) and \( l \in \{h, \ldots, 0\} \) the set \( T_b^{(l)} \) of all terms such that the first condition of Lemma 7 holds, but not the second. We then subtract the erroneous terms selected by \( T_b^{(l)} \) to keep only the terms of Lemma 7. In order to efficiently compute the \( T_b^{(l)} \) we will use previously computed \( T_b^{(l+1)} \). To bound the size of the set \( T_b^{(l)} \) we will use the monotonicity of \( B^{(l)} \), \( C^{(l)} \) but we will also need the integer \( p \) to be a prime number (see Lemma 3).

Finally, a last detail to address is the running time for computing \( C^p \). The \( x \) and \( y \)-degrees are bounded by \( O(n^\omega) \) and play a multiplicative factor in the computation of \( C^p \). We can improve this by replacing the \( x \)-degrees by \( A_{i,k} = 2A_{i,k}^{(l+1)} \) and \( B_{k,j} = 2B_{k,j}^{(l+1)} \) respectively. Then the \( x \)-degree becomes 0 or 1, and the computation of \( C^p \) takes \( \tilde{O}(n^\omega \cdot n^{\omega(\beta)}) \) time.

2.1 Algorithm

We now describe the full algorithm in detail. Let \( A \) and \( B \) be matrices as in Theorem 1 of dimensions \( n \times n^\beta \) and \( n^\beta \times n \) and such that \( B \) is monotone with \( O(n^\omega) \) bounded entries for some reals \( \beta, \mu \geq 0 \). Following the same reasoning as in [7], we can assume that all entries of \( A \) are non-negative and at most \( O(n^\mu) \). As a consequence, the matrix \( C = A \ast B \) is row-monotone with non-negative entries at most \( O(n^\omega) \).

Without loss of generality, we can assume that \( n \) is a power of \( 2 \). Define a constant parameter \( \alpha \in (0, \mu) \) to be optimized later. Pick uniformly at random a prime number \( p \) in the range \([40n^\omega, 80n^\omega]\) and define the integer \( h \) such that \( 2^{h-1} \leq p < 2^h \). Similarly as in [7], we can make the following assumption (see Lemma 2).

**Assumption 1.** For every \( i, j \) either \( (A_{i,j} \mod p) < p/3 \) or \( A_{i,j} = +\infty \). For every \( i, j \) \( (B_{i,j} \mod p) < p/3 \) and each row of \( B \) is non-decreasing.

The idea of the algorithm is to compute two matrices \( C^* \) and \( \tilde{C} \) such that \( C^*_{i,j} = |C_{i,j}/p| \) and \( \tilde{C}_{i,j} = C_{i,j} \mod p \) for each \( i, j \). Then, to recover the original value \( C_{i,j} \) simply compute \( p \cdot C^*_{i,j} + \tilde{C}_{i,j} \). We compute \( C^* \) using a combinatorial approach, while we compute \( \tilde{C} \) iteratively, computing each bit at a time.

**Compute the quotient matrix \([C/p]\)** Define \( A^* \) and \( B^* \) as \( A^*_{i,j} = [A_{i,j}/p] \) if \( A_{i,j} \) is finite, otherwise \( A^*_{i,j} = +\infty \), and \( B^*_{i,j} = [B_{i,j}/p] \). Then, by Assumption 1, the product \( C^*_{i,j} = A^*_{i,j} \ast B^*_{i,j} \) is such that if \( C_{i,j} \) is finite then \( C^*_{i,j} = \lfloor C_{i,j}/p \rfloor \).

Note that \( B^* \) is row-monotone with entries bounded by \( O(n^{\omega-\alpha}) \). Thus in each row \( k \) of \( B^* \) we can define at most \( O(n^{\omega-\alpha}) \) intervals \([j_0, j_1] \subset [n] \) such that \( B_{k,j_0-1}^* \neq B_{k,j_1}^* = B_{k,j_1+1}^* = \cdots = B_{k,j_l}^* \neq B_{k,j_{l+1}}^* \). To compute \( C^* \) we first initialize each entry to \(+\infty\). We then loop over each \( i \in [n] \), each \( k \in [n] \) and each interval \([j_0, j_1] \subset [n] \) as defined above for the \( k \)-th row of \( B^* \) and update \( C^*_{ij} \leftarrow \min\{C^*_{ij}, A_{i,k}^* + B_{k,j_l}^*\} \) for each \( j \in [j_0, j_1] \). This operation is correct since \( B_{k,j_l}^* = B_{k,j_{l+1}}^* \) for every \( j \in [j_0, j_{l+1}] \). Updating all elements \( C^*_{ij} \) for \( j \in [j_0, j_{l+1}] \) can be done in logarithmic time using a segment tree. Hence the total running time to compute \( C^* \) is the range of \( i \) times the range of \( k \) times the number of intervals in each row of \( B^* \), i.e. \( \tilde{O}(n^{1+\beta+\mu-\alpha}) \).

**Compute the remainder matrix \((C \mod p)\)** For every \( 0 \leq l \leq h \), construct an \( n \times n^\beta \) and \( n^\beta \times n \) matrices \( A^{(l)} \) and \( B^{(l)} \) as \( A^{(l)}_{i,j} = \left\lfloor \frac{A_{i,j} \mod p}{2^l} \right\rfloor \) if \( A_{i,j} \) is finite, \( A^{(l)}_{i,j} = +\infty \) otherwise and \( B^{(l)}_{i,j} = \left\lfloor \frac{B_{i,j} \mod p}{2^l} \right\rfloor \). We will iteratively construct an \( n \times n \) matrix \( C^{(l)} \) approximating \( \left\lfloor \frac{C_{i,j} \mod p}{2^l} \right\rfloor \) if \( C_{i,j} \) is finite. \( C^{(l)} \) isn’t defined directly but constructed such that it satisfies the following properties if \( C_{i,j} \) is finite:

\[
\begin{align*}
1) \quad & \left\lfloor \frac{C_{i,j} \mod p}{2^l} - 2(2^l-1) \right\rfloor \leq C^{(l)}_{i,j} \leq \left\lfloor \frac{C_{i,j} \mod p + 2(2^l-1)}{2^l} \right\rfloor \\
2) \quad & \text{for } j_0 < j_1, \text{ if } C^{(l)}_{i,j_0} = C^{(l)}_{i,j_1} \text{ then } C^{(l)}_{i,j_0}, \ldots, C^{(l)}_{i,j_1} \text{ are monotonically non-decreasing}
\end{align*}
\]

Note that \( C^{(l)} \) is not necessary equal to \( A^{(l)} \ast B^{(l)} \). We iteratively calculate matrix \( C^{(l)} \) for \( l = h, h-1, \ldots, 0 \) satisfying the above properties. Since \( \left\lfloor \frac{C_{i,j} \mod p}{2^l} \right\rfloor \) can be seen as the approximation of \( C_{i,j} \mod p \) up to the \( l \)-th bit, at each iteration we refine our approximation and get in the end \( \tilde{C} = C^{(0)} \). To compute \( C^{(l)} \) from \( C^{(l+1)} \) we look for the minimum value of \( A_{i,k}^{(l)} + B_{k,j}^{(l)} \) such that \( k \) satisfies both conditions:
• \( A_{i,k}^{(l+1)} + B_{k,j}^{(l+1)} = C_{i,j}^{(l+1)} + b \) for some \( b \in \{-10, \ldots, 10\} \)

• \( A_{i,k}^* + B_{k,j}^* = C_{i,j}^* \)

By Lemma 7 this ensures that \( C^{(l)} \) satisfies both properties (1) and (2).

We will filter out terms according to the first condition and then remove the ones that don’t satisfy the second condition. We explain later how the filtering according to the first condition is done and for now define the set of erroneous terms, which are the terms satisfying the first condition but not the second. For that, let us first make a series of observations. All elements in \( A^{(l)}, B^{(l)}, C^{(l)} \) are infinite or non-negative integers at most \( O(n^\alpha/2^l) \). Furthermore, by property (2) of \( C^{(l)} \), and since \( B \) is row-monotone and elements of \( B \) and \( C \) are bounded by \( O(n^\mu) \), every row of \( B^{(l)} \) and \( C^{(l)} \) is composed of \( O(n^\mu/2^l) \) intervals of equal elements. Also, each pair of rows of \( B^{(l)}, C^{(l)} \) can be divided into \( O(n^\mu/2^l) \) segments, where we define a segment as follow:

**Definition 1.** For \( i, j_0, j_1 \in [n], k \in [n^\beta] \) and \( j_0 \leq j_1 \) define a segment w.r.t \( B^{(l)} \) and \( C^{(l)} \) to be the tuple \( (i, k, [j_0, j_1]) \) such that for all \( j \in [j_0, j_1] \) \( B_{k,j}^{(l)} = B_{k,j_0}^{(l)}, B_{k,j}^{*} = B_{k,j_0}^{*}, \) and \( C_{i,j}^{(l)} = C_{i,j_0}^{(l)}, C_{i,j}^{*} = C_{i,j_0}^{*}. \)

We can then define the set of erroneous terms \( T^{(l)}_b \) for each offset \( b \in \{-10, \ldots, 10\} \) and every \( l \in \{0, \ldots, h\} \) as the set of segments \( (i, k, [j_0, j_1]) \) w.r.t \( B^{(l)}, C^{(l)} \) such that \( A_{i,k} < +\infty \) and \( A_{i,k}^* + B_{k,j}^* = C_{i,j_0}^{(l)} + b \) and \( A_{i,k}+ B_{k,j}^* \neq C_{i,j_0}^{*}. \)

We will show in Lemma 3 that the size of \( T^{(l)}_b \) can be bounded by \( \tilde{O}(n^{1+\beta+\mu-\alpha}). \)

**First iteration** The algorithm starts with \( l = h \), so \( A^{(l)}, B^{(l)}, C^{(l)} \) are zero matrices in the first iteration. \( T^{(h)} \) is an empty set for \( b \neq 0 \) and \( T^{(h)}_0 \) is the set of segments \( (i, k, [j_0, j_1]) \) w.r.t \( B^{(h)}, C^{(h)} \) such that \( A_{i,k} < +\infty \) and \( A_{i,k}^* + B_{k,j}^* \neq C_{i,j_0}^{*}. \)

Since the number of segments in each pair of rows of \( B^{(h)}, C^{(h)} \) is at most \( O(n^\mu/p) = O(n^{\mu-\alpha}) \) and since the number of pairs of rows of \( B^{(h)}, C^{(h)} \) is \( O(n^{1+\beta}) \), we can bound the size of \( T^{(h)}_b \) for every \( b \in \{-10, \ldots, 10\} \) by \( O(n^{1+\beta+\mu-\alpha}) \). Hence the first iteration does not take more than \( O(n^{1+\beta+\mu-\alpha}) \) time.

Each following iteration of \( l = h-1, \ldots, 0 \) consists of three phases: a polynomial matrix multiplication using \( A^{(l)}, A^{(l+1)}, B^{(l)} \) and \( B^{(l+1)} \) that allows to filter out candidates for \( C^{(l)} \); a subtraction of the erroneous terms \( T^{(l+1)} \); and finally the computation of \( T^{(l)}_b \) using \( T^{(l+1)}_b \). We describe the three phases for a fixed \( l \in \{0, \ldots, h-1\} \).

**Multiply matrices of polynomials** Construct two polynomial matrices \( A^p \) and \( B^p \) on variables \( x, y \) as \( A_{i,k}^p = 0 \) if \( A_{i,k} \) is infinite, otherwise:

\[
A_{i,k}^p = x^{A_{i,k}^{(l)}-2A_{i,k}^{(l+1)}} \cdot y^{A_{i,k}^{(l+1)}}
\]

and

\[
B_{k,j}^p = x^{B_{k,j}^{(l)}-2B_{k,j}^{(l+1)}} \cdot y^{B_{k,j}^{(l+1)}}
\]

Compute the standard matrix multiplication \( C^p = A^p \cdot B^p \). Since \( A_{i,k}^{(l)} - 2A_{i,k}^{(l+1)} \) and \( B_{k,j}^{(l)} - 2B_{k,j}^{(l+1)} \) are 0 or 1, the \( x \)-degree is bounded by 1. The \( y \)-degree is bounded by \( O(p) = O(n^\alpha) \). Hence computing \( C^p \) takes \( \tilde{O}(n^{2(\beta+\alpha)}) \) time.

**Filter candidates and subtract erroneous terms** The previous computation of \( C^p \) allows us to filter out candidate terms \( A_{i,k}^{(l)} \) and \( B_{k,j}^{(l+1)} \) that satisfy the first condition for \( C_{i,j}^{(l)} \), i.e. \( A_{i,k}^{(l+1)} + B_{k,j}^{(l+1)} = C_{i,j}^{(l+1)} + b \) for some \( b \in \{-10, \ldots, 10\} \). Indeed we have that for every \( i, j \in [n]^2 \):

\[
C_{i,j}^p = \sum_{k \in [n^\beta]} \sum_{A_{i,k} < +\infty} x^{A_{i,k}+B_{k,j}^{(l+1)}-2A_{i,k}^{(l+1)}-2B_{k,j}^{(l+1)}} \cdot y^{A_{i,k}^{(l+1)}+B_{k,j}^{(l+1)}}
\]

If \( C_{i,j}^p = 0 \), set \( C_{i,j}^{(l)} = +\infty \). Otherwise, build the following polynomial for each \( b \in \{-10, \ldots, 10\} \):

\[
C_{i,j,b}^p(x) = \sum_{d=C_{i,j}^{(l)}+b} \lambda x^d \quad \text{where } \lambda^e \cdot y^d \text{ is a term in } C_{i,j}^p
\]
This construction takes $\tilde{O}(n^{1+\beta+\alpha})$ time for all $b \in \{-10, \ldots, 10\}$ and $i, j \in [n]^2$. We then compute the polynomial corresponding to the erroneous terms:

$$R_{i,j,b}(x) = \sum_{j \in [j_0, j_1]} x^{A_{i,k}^{(l)} + B_{k,j}^{(l)} - 2A_{i,k}^{(l+1)} - 2B_{k,j}^{(l+1)}}.$$

Since $B_{k,j}^{(l)} - 2B_{k,j}^{(l+1)}$ is 0 or 1, there can be at most two different segments $(i, k, [j_0, j_1]) \in T_b^{(l+1)}$ corresponding to $B_{k,j}^{(l)}$. This means that for a fixed $b \in \{-10, \ldots, 10\}$, we can construct the polynomials $R_{i,j,b}(x)$ for all $i, j \in [n]^2$ by considering each segment $(i, k, [j_0, j_1])$ of $T_b^{(l+1)}$ and adding the corresponding term to at most two different $R_{i,j,b}(x)$ and $R_{i',j',b}(x)$. This takes $\tilde{O}(T_b^{(l+1)})$ time for each offset $b$.

Finally, set $C_{i,j}^{(l)} = \min_{b \in \{-10, \ldots, 10\}} \{s_{i,j,b} + 2(C_{i,j}^{(l+1)} + b)\}$, where $s_{i,j,b}$ denotes the smallest degree of $x$ monomials in the difference $C_{i,j,b}^{p}(x) - R_{i,j,b}(x)$. This phase takes $\tilde{O}(n^{1+\beta+\alpha} + |T_b^{(l+1)}|)$ time in total, which by Lemma 3 is $\tilde{O}(n^{1+\beta+\mu-\alpha})$.

**Computing $T_b^{(l)}$ from $T_b^{(l+1)}$.** By Lemma 4, both $B_{k,j}^{(l)} - 2B_{k,j}^{(l+1)}$ and $C_{i,j}^{(l)} - 2C_{i,j}^{(l+1)}$ are between 0 and $O(1)$. So each segment w.r.t. $B^{(l+1)}$, $C^{(l+1)}$ contains at most $O(1)$ segments w.r.t. $B^{(l)}$, $C^{(l)}$. Furthermore, by Lemma 5, we have $\bigcup_{b=-10}^{10} T_b^{(l)} \subset \bigcup_{b=-10}^{10} T_b^{(l+1)}$. So to construct $T_b^{(l)}$, we need to look at each sub-segment of each segment in $\bigcup_{b=-10}^{10} T_b^{(l+1)}$ and put it in the set $T_b^{(l)}$ it belongs to. Each segment in $T_b^{(l+1)}$ can be split in at most $O(1)$ sub-segments and we can use binary search to find the splitting points. Hence the construction of $T_b^{(l)}$ for all $b \in \{-10, \ldots, 10\}$, takes $\tilde{O}(\bigcup_{b=-10}^{10} T_b^{(l+1)})$ time, which by Lemma 3 is $\tilde{O}(n^{1+\beta+\mu-\alpha})$.

**Total running time** The expected running time of the algorithm is bounded by $\tilde{O}(n^{1+\beta+\mu-\alpha + \omega(\beta)+\alpha})$. Setting $\alpha = \frac{1+\beta+\mu-\omega(\beta)}{2}$ we equalize both terms and get $\tilde{O}(n^{1+\beta+\mu-\omega(\beta)})$.

### 2.2 Proof of correctness

**Lemma 2.** The computation of $A \star B$ where $A$ and $B$ are from Theorem 1 can be done by computing a constant number of Min-Plus Products $A^{i} \star B^{i}$ of matrices $A^{i}, B^{i}$ satisfying Assumption 1.

The proof of analogous Lemma 3.4 in [7] applies verbatim. The idea is to construct three copies of $A$ and three copies of $B$ containing the values respectively in $[0, p/3]$, $[p/3, 2p/3]$ and $[2p/3, p]$ when taking the modulo $p$. We can then remove an appropriate offset to get values modulo $p$ at most $p/3$, while maintaining monotonicity of rows in the copies of $B$.

**Lemma 3.** In expectation, for every $b \in \{-10, \ldots, 10\}$ and every $l \in \{0, \ldots, h\}$, $T_b^{(l)}$ contains $\tilde{O}(n^{1+\beta+\mu-\alpha})$ segments.

**Proof.** Assume that $2^l < p/100$ and consider a segment $(i, k, [j_0, j_1])$ w.r.t $B^{(l)}, C^{(l)}$. Take $j \in [j_0, j_1]$ where $A_{i,k}$ is finite and $A_{i,k}^{*} + B_{k,j}^{*} \neq C_{i,j}^{*}$. By Assumption 1 we have that $(C_{i,j} \mod p) < 2p/3$, hence $|A_{i,k} + B_{k,j} - C_{i,j}| \geq p/3$. Indeed, if $A_{i,k}^{*} + B_{k,j}^{*} \geq C_{i,j}^{*} + 1$ then

$$\frac{A_{i,k}}{p} + \frac{B_{k,j}}{p} \geq \left[\frac{A_{i,k}}{p}\right] + \left[\frac{B_{k,j}}{p}\right] \geq \left[\frac{C_{i,j}}{p}\right] + 1 \geq \frac{C_{i,j}}{p} - \frac{2}{3} + 1 = \frac{C_{i,j}}{p} + \frac{1}{3},$$

so $A_{i,k} + B_{k,j} \geq C_{i,j} + p/3$. Similarly if $A_{i,k}^{*} + B_{k,j}^{*} \leq C_{i,j}^{*} - 1$ then

$$\frac{A_{i,k} - 1}{3} + \frac{B_{k,j} - 1}{3} \leq \left[\frac{A_{i,k}}{p}\right] + \left[\frac{B_{k,j}}{p}\right] \leq \left[\frac{C_{i,j}}{p}\right] - 1 \leq \frac{C_{i,j}}{p} - 1.$$

So we have $|A_{i,k} + B_{k,j} - C_{i,j}| \geq p/3$ in both cases.
We want to bound the probability that \((i, k, [j_0, j_1])\) appears in \(T_b^{(l)}\), i.e. the probability that \(\left\lfloor \frac{A_{i,k} \mod p}{2^l} \right\rfloor + \left\lfloor \frac{B_{k,j} \mod p}{2^l} \right\rfloor = C_{i,j}^{(l)} + b\) holds, which is
\[
-4 \leq A_{i,k} \mod p \mod \frac{2^l}{2^l} + B_{k,j} \mod p \mod \frac{2^l}{2^l} - C_{i,j} \mod p \mod \frac{2^l}{2^l} - b \leq 4
\]
. Let \(C_{i,j} = A_{i,q} + B_{q,j}\). So if \((i, k, [j_0, j_1]) \in T_b^{(l)}\) then \((A_{i,k} + B_{k,j} - A_{i,q} - B_{q,j}) \mod p \in [2^l(b - 4), 2^l(b + 4)]\). This means that \(A_{i,k} + B_{k,j} - A_{i,q} - B_{q,j}\) is congruent modulo \(p\) to one of the \(O(2^l)\) remainders. It holds that for all remainder \(r \in [2^l(1 - 1, 2^l(4 + 4))], (|b| \leq 10),\)
\[
|r| \leq 14 \cdot 2^l < p/6 \leq \frac{1}{2} |A_{i,k} + B_{k,j} - A_{i,q} - B_{q,j}|
\]
. If \(A_{i,k}, A_{i,q}\) are finite and \(B_{k,j}, B_{q,j}\) are from the original \(B\) (see Lemma 2), \([A_{i,k} + B_{k,j} - A_{i,q} - B_{q,j} - r]\) is a positive number bounded by \(O(n^\alpha)\). The probability \(P\) that \(p \in [40n^\alpha, 80n^\alpha]\) divides \(A_{i,k} + B_{k,j} - A_{i,q} - B_{q,j} - r\) is the quotient of the number of divisors of \(A_{i,k} + B_{k,j} - A_{i,q} - B_{q,j} - r\) over the number of primes in \([40n^\alpha, 80n^\alpha]\). There are at most \(O(\log n^\alpha)\) divisors of a \(O(n^\alpha)\) number. By the prime number Theorem (Jameson [14]), there is at least \(\Omega(n^\alpha/\log n^\alpha)\) prime numbers in \([40n^\alpha, 80n^\alpha]\). Hence \(P = O(\log n^\alpha/\log n^\alpha) = O(n^{-\alpha})\).

In Lemma 2, if \(B_{k,j}, B_{q,j}\) are not from the original \(B\) then they are set artificially to numbers congruent modulo \(p\) to 0, \([p/3]\) or \([2p/3]\). In that case, it still holds that \(p\) divides \(A_{i,k} + B_{k,j} - A_{i,q} - B_{q,j} - r\) with probability \(O(n^{-\alpha})\). To see this we can condition on different cases. For instance, if \(B_{k,j}\) is congruent modulo \(p\) to \([p/3]\) and \(B_{q,j}\) is from the original \(B\), then we want that \(p\) divides \(A_{i,k} + [p/3] - A_{i,q} - B_{q,j} - r\). Since 3 does not divide \(p\), \(3[p/3]\) is \(p + 1\) or \(p + 2\). So \(p\) divides \(3(A_{i,k} - A_{i,q} - B_{q,j} - r) + 1\) or \(3(A_{i,k} - A_{i,q} - B_{q,j} - r) + 2\). The probability that this is the case is \(O(n^{-\alpha})\) in both cases. Other cases of \(B_{k,j}, B_{q,j}\) are similar. Summing conditional probabilities for all cases, we obtain that \(p\) divides \(A_{i,k} + B_{k,j} - A_{i,q} - B_{q,j} - r\) with probability \(O(n^{-\alpha})\).

Finally, there are \(O(2^l)\) such possible remainders \(r\), so the probability that \((i, k, [j_0, j_1])\) appears in \(T_b^{(l)}\) is \(O(2^l n^{-\alpha})\). By linearity of expectation, since there are \(O(\frac{n^{\alpha + \beta + \mu}}{2^l})\) segments \((i, k, [j_0, j_1])\) w.r.t. \(B^{(l)}, C^{(l)}\), the expected number of segments in \(T_b^{(l)}\) is \(O(n^{\alpha + \beta + \mu - \alpha})\). 

**Lemma 4.** In each iteration \(l = h, \ldots, 0\) and for every \(i, j \in [n]^2\), we have that \(C_{i,j}^{(l)} - 2C_{i,j}^{(l+1)} \in [-7, 8]\.

**Proof.** The proof of analogous Lemma 3.5 in [7] applies verbatim.

**Lemma 5.** For every \(l = h - 1, \ldots, 0\), we have that \(\bigcup_{b=-10}^{10} T_b^{(l)} \subseteq \bigcup_{b=-10}^{10} T_b^{(l+1)}\).

**Proof.** The proof of analogous Lemma 3.6 in [7] applies verbatim.

**Lemma 6.** For every \(l = h, \ldots, 0\), if \(A_{i,k} + B_{k,j} = C_{i,j}\), then there is some \(b \in \{-10, \ldots, 10\}\) such that \(A_{i,k}^{(l)} + B_{k,j}^{(l)} = C_{i,j}^{(l)} + b\).

**Proof.** The proof of analogous Lemma 3.8 in [7] applies verbatim.

**Lemma 7.** For every \(l = h - 1, \ldots, 0\) and \(i, j \in [n]^2\), if we set \(C_{i,j}^{(l)}\) to the minimum value of \(A_{i,k}^{(l)} + B_{k,j}^{(l)}\) such that \(k\) satisfies that:

- \(A_{i,k}^{(l+1)} + B_{k,j}^{(l+1)} = C_{i,j}^{(l+1)} + b\) for some \(b \in \{-10, \ldots, 10\}\)
- \(A_{i,k} + B_{k,j} = C_{i,j}\)

then \(C_{i,j}^{(l)}\) satisfies both properties (1) and (2).

**Proof.** The proof of correctness of Section 3.2.1 in [7] applies verbatim.

## 3 Applications

In this section we list problems for which we can improve the bounds on the running time using Theorem 1. We also improve running time for Unweighted Tree Edit Distance using the algorithm by Chi et al. [7] for Square Monotone Min-Plus Product. A summary of the improved bounds is given in Table 1 but we detail the computations here.
3.1 Single Source Replacement Paths

In the SSRP problem, one is given an edge-weighted graph $G = (V, E, w)$ and a source $s \in V$ and has to find for all pairs $(v, e) \in V \times E$ the shortest path from $s$ to $v$ without using the edge $e$. In [13], Gu et al. use a Monotone Min-Plus Product algorithm to solve the $M$-bounded SSRP problem, which is the SSRP setting with edge-weights in $\{−M, \ldots, M\}$.

Their algorithm has a running time of (see [13, Proof of Lemma 27])

$$\hat{O}(n^{\mu+\omega} + n^{\mu+\zeta+\omega(1-\zeta)} + n^{3-2\zeta} + n^{v(1-\zeta)+1+\mu}),$$

where $M = n^\mu$ is the bound on the edge-weights, $\zeta \in [0, 1]$ is a parameter to optimize and $\varphi(\beta, \mu)$ is the function returning the smallest number such that the Monotone Min-Plus Product between matrices $A$ and $B$ of dimensions $n \times n^\beta$ and $n^\beta \times n$ with values of $B$ non-negative and at most $O(n^\mu)$ can be computed in $\hat{O}(n^{\varphi(\beta, \mu)})$ time. We recall that the interesting regime is when $0 \leq \mu \leq 3 - \omega$. Using the bound on $\varphi(\beta, \mu)$ given in [13], the running time is expressed as $\hat{O}(M^{\frac{5}{2}} + n^{\frac{36-7\omega}{36}})$ using $\omega$ or as $\hat{O}(M^{0.8043}n^{2.4957})$ using known bounds for fast rectangular matrix multiplication.

Theorem 1 yields the tighter bound $\varphi(1-\zeta, 1+\mu) \leq 3 + \mu - \zeta + \omega(1 - \zeta)/2$. We can thus upper bound the running time of the algorithm by $\hat{O}(n^{\mu+\omega} + n^{\mu+\zeta+\omega(1-\zeta)} + n^{3-2\zeta} + n^{3 + \mu - \zeta + \omega(1 - \zeta)/2})$. Let’s call $A := \mu + \omega$, $B := \mu + \zeta + \omega(1 - \zeta)$, $C := 3 - 2\zeta$ and $D := (3 + \mu - \zeta + \omega(1 - \zeta)/2$ the exponents of the different terms. We optimize $\zeta$ such that the exponent of $n$ equals $\min_{\zeta \in [0, 1]} \max\{A, B, C, D\}$ and will obtain running times of $\hat{O}(M^{\frac{5}{2}} + n^{\frac{36-7\omega}{36}})$ using $\omega$ and $\hat{O}(M^{0.8825}n^{2.4466})$ using fast rectangular matrix multiplication. Remark that since $\mu \leq 3 - \omega$ the term $\hat{O}(n^3) = \hat{O}(Mn^\omega)$ is not dominant and thus can be omitted.

To express the running time as a function $\omega$, we bound $\omega(1-\zeta) \leq (1-\zeta) \cdot \omega + 2\zeta$ by implementing rectangular matrix multiplication by cutting rectangular matrices into square matrices and get:

$$B \geq \mu + \zeta + (1 - \zeta) \cdot \omega + 2\zeta$$

$$= \mu + \omega + \zeta \cdot (3 - \omega)$$

$$D \leq (3 + \mu - \zeta + (1 - \zeta) \cdot \omega + 2\zeta)/2$$

$$= (3 + \mu + \omega + \zeta \cdot (1 - \omega))/2$$

Now equalizing $B', C$ and $D'$ we find $\zeta = \frac{3 - \mu - \omega}{5 - \omega}$ which is indeed in $[0, 1]$ for $0 \leq \mu \leq 3 - \omega$. Hence the dominant term is $B' = \frac{2\mu + 2\zeta - \omega}{5 - \omega}$. This yields an expected running time of $\hat{O}(M^{\frac{5}{2}} + n^{\frac{36-7\omega}{36}})$.

To improve this bound we can use fast rectangular matrix multiplication and known bounds on $\omega(\beta)$. Indeed, since the function $\omega(\beta)$ is convex in $\beta$, we can compute an upper bound on $\omega(\beta_0)$ and $\omega(\beta_1)$ for $\beta_0 < \beta_1$ and interpolate an upper bound on $\omega(\beta) \leq \frac{\omega(\beta_0)(\beta_1 - \beta) + \omega(\beta_1)(\beta - \beta_0)}{\beta_1 - \beta_0}$ for values $\beta \in [\beta_0, \beta_1]$. Let’s set $\zeta$ as a linear function of $\mu$, i.e. $\zeta = a + b \cdot \mu$ for some real numbers $a, b \neq 0$. Then by optimizing $\max\{B, C, D\}$ with plugged in $\mu = 0$ we obtain the optimal value for $\zeta_0 = a$. When plugging in $\mu = 3 - \omega$ we get the optimal value for $\zeta_1$ and compute $b = \frac{3 - \mu - \omega}{5 - \omega}$. Using the current best bound on fast matrix multiplication [16] we get $a = 0.2767$ and $b = -0.4412$. Set $\beta = 1 - \zeta$ and choose the bounds $\beta_0 = 1 - \zeta_0 = 1 - a = 0.7233$ and $\beta_1 = 1$. We get the following upper bound for $\zeta \in [0, 0.2767]$.

$$\omega(1 - \zeta) \leq \frac{\omega(\beta_0)(\beta_1 - (1 - \zeta)) + \omega(\beta_1)(1 - \zeta - \beta_0)}{\beta_1 - \beta_0}$$

$$= \frac{\omega(\beta_0)(a + b \cdot \mu) + \omega(\beta_1)(-b \cdot \mu)}{a}$$

$$= \omega(\beta_0) + \frac{b}{a}(\omega(\beta_0) - \omega(\beta_1))\mu$$

Using the values of $a$ and $b$ previously computed and the bounds on fast matrix multiplication [15] and fast rectangular matrix multiplication [16], we obtain $\omega(1 - \zeta) \leq 2.1698 + 0.3237\mu$. Finally, if we use this bound to optimize $\max\{B, C, D\}$, we get a running time of $\hat{O}(M^{0.8825}n^{2.4466})$.

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2As for Min-Plus Product, SSRP was shown in [19, 20] to be sub-cubically equivalent to APSP. So under the APSP Hypothesis there is no truly sub-cubic time algorithm to solve SSRP in the general case. Gu et al. [13] consider bounded weights to achieve a truly sub-cubic algorithm.
we use the algorithm for Bounded-Difference Min-Plus Product in Bringmann et al. show a faster algorithm running in time \( \tilde{O}(n^{3.5}) \). Gu et al. show that SSRP with weights in \( \{1, 1\} \) seems harder than SSRP with weights in \( \{0, 1\} \). They prove that if there exists a \( T(n) \) time algorithm for SSRP in an \( n \)-vertex graphs with edge-weights in \( \{-1, 0, 1\} \), then there exists an \( O(T(n)\sqrt{n}) \) time algorithm for the Bounded-Difference Min-Plus Product of \( n \times n \) matrices. So if SSRP with \( \{-1, 0, 1\} \) weights can be solved in \( \tilde{O}(n^2) \) time (if \( \omega = 2 \)), then the Bounded-Difference Min-Plus Product could be solved in \( \tilde{O}(n^{2.5}) \). The existence of the second algorithm seemed unlikely at the time of publication of [13]. However the new algorithm by Chi et al. [7] yields an \( \tilde{O}(n^{2.5}) \) time algorithm if \( \omega = 2 \). Hence the above reasoning to lower bound the running time of SSRP with negative weights by \( \tilde{O}(n^2) \) (if \( \omega = 2 \)) does not hold anymore.

### 3.2 Batch Range Mode

Let \( a \) be an array of elements. A range mode query asks for the most frequent element in a given contiguous interval of \( a \). In the Batch Range Mode problem, the array \( a \) and all queries are given in advance, and the task is to compute all query answers.

The Batch Range Mode problem is solved by Williams and Xu [23] via a black box computation of a Monotone Min-Plus Product. Their deterministic algorithm runs in \( \tilde{O}(n^{1+\tau} + n^{(1-\tau)\varphi(1, \frac{1}{1-\tau})}) \) time (see [13, Proof of Theorem 4]), where \( \tau \) is a parameter in \([0, 1]\) and \( \varphi(\beta, \mu) \) is the function returning the smallest number such that the Monotone Min-Plus Product between matrices \( A \) and \( B \) of dimensions \( n \times n^3 \) and \( n^3 \times n \) with values of \( B \) non-negative and at most \( O(n^\mu) \) can be computed in \( \tilde{O}(n^{\varphi(\beta, \mu)}) \) time.

Gu et al. [13] bound this running time by plugging in their algorithm for Monotone Min-Plus Product and obtained a running time \( \tilde{O}(n^{\frac{1}{1-\tau}}) \). Gao and He [11] showed a faster algorithm running in time \( \tilde{O}(n^{\frac{1}{1-\tau}}) \). By using the algorithm of Williams and Xu [23] with the Monotone Min-Plus Product algorithm of Theorem 1 we improve the bound even further. Indeed,

\[
(1 - \tau) \cdot \varphi(1, \frac{\tau}{1-\tau}) \leq (1 - \tau) \cdot \frac{2 + \varphi(1, \frac{\tau}{1-\tau})}{2} = 1 + \frac{\omega}{2} - \frac{\tau}{2} (1 + \omega)
\]

and equalizing this term to \( 1 + \tau \) we get \( \tau = \frac{\omega}{2(1+\omega)} \) and the running time becomes \( \tilde{O}(n^{1+\psi(k)}) = \tilde{O}(n^{\frac{3+2\omega}{2(1+\omega)}}) \). With current best bound on fast matrix multiplication [15] this is \( \tilde{O}(n^{1.4416}) \).

### 3.3 \( k \)-Dyck Edit Distance

The Dyck Edit Distance of a sequence \( S \) of parenthesis (of various types) is the smallest number of edit operations (insertions, deletions, and substitutions) needed to transform \( S \) into a sequence of balanced opening and closing parenthesis. Fried et al. [10] study the \( k \)-Dyck Edit Distance problem in which the input is an \( n \)-length sequence of parenthesis \( S \) and a positive integer \( k \) and the task is to compute the Dyck Edit Distance of \( S \) if it is at most \( k \), otherwise to return \( k + 1 \).

Fried et al. show how to solve it in time \( \tilde{O}(n + k^2 \cdot \psi(k, \sqrt{k})) \) (see [10, Theorem 1.1]), where \( \psi(a, b) \) is the running time to compute the Min-Plus Product between two bounded-difference matrices of dimensions \( a \times b \) and \( b \times a \). To bound \( \psi(a, b) \), Fried et al. use the algorithm for Bounded-Difference Min-Plus Product in Bringmann et al. [2] and get a running time of \( \tilde{O}(n + k^{4.7829}) \).

Our new result of Theorem 1 yields a tighter bound of \( \psi(n, n^3) \leq \tilde{O}(n^{\frac{2+\beta+\omega(\beta)}{2}}) \). Hence the running time improves to \( \tilde{O}(n + k^{4.4412}) \), which is \( \tilde{O}(n + \sqrt{\log n}) \) with the current best bound on fast rectangular matrix multiplication [16]. Actually if \( k \geq \sqrt{n} \), Fried et al. show a faster algorithm running in time \( \tilde{O}(\psi(n, k)) \) (see [10, Section 4]), which we can directly bound to \( \tilde{O}(n^{\frac{2+\beta+\omega(\beta)}{2}}) = \tilde{O}(n^{\sqrt{\log n}}) \).

The same improved bound was obtained independently and in parallel by Fried et al. [9].

### 3.4 2-approximation APSP

The All Pairs Shortest Path (APSP) problem asks for a given graph \( G = (V, E) \) the shortest path between all pairs of vertices. Deng et al. [8] consider a 2-approximation to APSP for undirected unweighted graphs, which is a
mapping $\tilde{d} : V \times V \to \mathbb{N}$ such that $d_{\text{OPT}}(u, v) \leq \tilde{d}(u, v) + 2$ for every vertices $u$ and $v$, where $d_{\text{OPT}}(u, v)$ is the shortest path from $u$ to $v$.

The 2-approximation to APSP is computed by Deng et al. by using Bounded-Difference Min-Plus Product as a black-box. When $n = |V|$, the running time of their algorithm is $\tilde{O}(n^{2t/2} + \psi(n, n/t))$ (see [8, Theorem 11]), where $t \geq 1$ is a parameter and $\psi(a, b)$ is the running time to compute the Min-Plus Product between two bounded-difference matrices of dimensions $a \times b$ and $b \times a$. Deng et al. use the algorithm for Bounded-Difference Min-Plus Product in Bringmann et al. [2] and get a running time of $\tilde{O}(n^{2.2867})$. With Theorem 1 we have an improved upper bound on $\psi$ and can thus compute a 2-approximation of APSP in $\tilde{O}(n^{2t/2} + \frac{2^{3.5+\omega} \log_2(n/t)}{t^2})$ time. By setting $t = n^{0.5185}$, the running time becomes $\tilde{O}(n^{2.293})$ using current best bounds on fast rectangular matrix multiplication [16].

### 3.5 Unweighted Tree Edit Distance

Given an alphabet $\Sigma$ and a rooted ordered tree with node labels in $\Sigma$, we consider two types of operations: relabeling a node to another symbol of $\Sigma$ and deleting a node so that they become identical.

In [17], Mao shows the first truly sub-cubic algorithm to solve the Unweighted Tree Edit Distance. For two trees of size $n$ and $m$, the simplified expression of the running time is $\tilde{O}(n \cdot m^{\frac{3n-1}{n+m}})$ (see A), where $\alpha$ is the exponent of the Bounded-Difference Min-Plus Product between square matrices. Using the algorithm for Bounded-Difference Min-Plus Product of Bringmann et al. [2], the running time can be bounded by $\tilde{O}(n \cdot m^{1.9514})$ time, which is subcubic if $m = O(n)$. We remark here that the running time $\tilde{O}(n \cdot m^{1.9546})$ announced in [17] is slightly worse due to a numerical error.

Since a $n \times n$ bounded-difference matrix can be transformed into a row-monotone matrix with non-negative entries bounded by $O(n)$, the work of Chi et al. [7] allows us to get a tighter bound on $\alpha \leq \frac{3n-1}{n+m}$. We can thus bound the running time by $\tilde{O}(n \cdot m^{\frac{3n-1}{3n+3}})$. With the current best bound on fast matrix multiplication [15], this is bounded by $\tilde{O}(n \cdot m^{1.9149})$.

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\[\text{In the weighted case, the cost of deletions and relabeling is a function of the symbol on the nodes. Bringmann et al. [3, 4] show that the Weighted Tree Edit Distance does not admit any truly subcubic algorithm under the APSP Hypothesis, if } T_1 \text{ and } T_2 \text{ have both } n \text{ nodes and } |\Sigma| = \Theta(n).\]
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A Simplified Running Time for Unweighted Tree Edit Distance

We explain here how the running time of the algorithm for Unweighted Tree Edit Distance of Mao [17] can be expressed as a function of the running time for Bounded-Difference Min-Plus Product.

We use the notations of the article [18] without redefining them. In [18, Section 4.2.3], the running time of [18, Algorithm 2] is shown to be $O\left(\sum \text{MUL}(\Delta, m) + n \cdot m \cdot \Delta^2\right)$, with $\Delta$ a parameter to optimize, $n = |T_1|$, $m = |T_2|$ and MUL($m, n$) is the function defined in [18, Theorem 4.5]. This theorem gives a (numerical) bound on MUL($m, n$) which is proven in [18, Section 4.4, Algorithm 6]. The running time of [18, Algorithm 6] is $O(n^2 \cdot m^2 / \delta)$ for the recursive calls and $O(n^2 \cdot \delta^{\alpha - 2})$ for the final part, where $\alpha$ is the exponent for Bounded-Difference Min-Plus Product and $\delta$ is a parameter to optimize. Here, we do not replace $\alpha$ by the bound given by Bringmann et al. [2] as done in [18], but directly bound MUL($m, n$) by $O(n^2 \cdot m^2 / \delta + n^2 \delta^{\alpha - 2})$. To equalize both terms we set $\delta = m^{2\alpha - 1}$ and get the bound MUL($m, n$) = $O(n^2 \cdot m^{2\alpha - 1})$. We can now replace MUL($\Delta, m$) in the total running time of [18, Algorithm 2] and obtain $O(n \cdot m^2 \cdot \Delta^{\frac{2\alpha - 2}{2 \alpha - 1}} + n \cdot m \cdot \Delta^2)$. We equalize both terms by setting $\Delta = m^{2\alpha - 1}$ and get the following expression of the running time for Unweighted Tree Edit Distance: $O(n \cdot m^{\frac{3\alpha - 1}{2 \alpha - 1}})$.

One can verify that we obtain the same result as [18, Theorem 4.5] and [18, Theorem 1.1] if we replace $\alpha$ in the final expression by the numerical bound given by Bringmann et al. [2]: $\alpha \leq 2.8244$ in the randomized algorithm and $\alpha \leq 2.8603$ for the deterministic case.

B Min-Plus Product when $B$ is column-monotone

We want to compute the Min-Plus Product $C = A \star B$ of the $n^a \times n$ matrix $A$ and the $n \times n^b$ matrix $B$ where the columns of $B$ are monotonously non-decreasing and entries of $B$ are positive and bounded by $O(n^\omega)$. Without detailing every computation again, we explain why $C$ can be computed in $O(n \cdot \omega(n, \omega, b, c))$ time, where $\omega(a, b, c)$ is the lowest exponent of the multiplication of matrices of dimensions $n^a \times n^b$ and $n^b \times n^c$. If matrices $A$ and $B$ have dimension $n \times n^\beta$ and $n^\beta \times n^\gamma$ respectively, the running time becomes $O(n^{\frac{\omega(\beta, \beta + 3\gamma + \gamma^2)}{\beta}})$. Note that this is different from the $O(n^{\omega(\beta, \beta + 3\gamma + \gamma^2)})$ time algorithm for the Min-Plus Product when $B$ is row-monotone. Intuitively, the algorithm iterates over the rows of $A$, over the rows of $B$ (which corresponds to the columns of $A$) and the columns of $B$.  

\[4\] In [18, Section 4.4] the parameter is called $\Delta$, but we rename it to $\delta$ to avoid confusion with the $\Delta$ parameter of the running time of [18, Algorithm 2].

\[5\] This is up to a small numerical error done by [18]. In fact when applying [18, Theorem 4.5] in [18, Section 4.2.3], the exponent of $\Delta$ should be 0.0962 and not 0.0952.
When $B$ is row-monotone, iterating over the columns of $B$ can be reduced to a number of steps depending on $\mu$ and iterating over the rows of $A$ and the rows of $B$ takes $O(n \cdot n^2)$ time. However when $B$ is column-monotone, the acceleration happens when iterating over the rows of $B$ and hence iterating over the two other ranges takes $O(n \cdot n)$ time.

**Quotient matrix** The algorithm starts by sampling a prime number $p$ from the range $[40n^3, 80n^3]$ uniformly at random, where $\beta \in (0, \mu]$ is some parameter we tune later. Then we can still assume Assumption 1. We first compute the quotient matrix $\tilde{C} = \tilde{A} \ast \tilde{B}$ where $\tilde{A}_{ij} = \left\lfloor \frac{A_{ij}}{p} \right\rfloor$ if $A_{ij}$ is finite, otherwise $\tilde{A}_{ij} = +\infty$, and similarly for $\tilde{B}$. The entries of $\tilde{B}$ are bounded by $O(n^\mu - \beta)$ so since the number of intervals of $+\infty$ values in each column of $\tilde{B}$ is bounded by $O(n^\mu - \beta)$, there are $O(n^\mu - \beta)$ intervals of the same value in each column of $\tilde{B}$. We can thus compute $\tilde{C}$ using a segment tree structure by iterating over all $i \in [n^\alpha]$, over all $j \in [n^\gamma]$ and over all intervals $[k_0, k_1]$ of the $j$th column of $\tilde{B}$. This computation takes $O(n^{\alpha + \gamma + \mu - \beta})$ time. By Assumption 1, $\tilde{C}_{ij} = \left\lceil \frac{C_{ij}}{p} \right\rceil$ if $C_{ij}$ is finite, otherwise $\tilde{C}_{ij} = +\infty$.

**Remainder matrix** We then compute the remainder matrix $(C \mod p)$ recursively. Let $h$ be the integer such that $2^{h-1} \leq p < 2^h$ and define for $\ell = 0, 1, \ldots, h$ matrices $A^{(\ell)}$ and $B^{(\ell)}$ where $A^{(\ell)}_{ij} = \left\lceil \frac{A_{ij}}{p^{2^\ell}} \right\rceil$ if $A_{ij}$ is finite, otherwise $A^{(\ell)}_{ij} = +\infty$, and similarly for $B^{(\ell)}$. Observe that the values of $A^{(\ell)}$ and $B^{(\ell)}$ are bounded by $O(n^\beta)$ and that each column of $B^{(\ell)}$ contains at most $O(n^\mu / 2^\ell)$ intervals of the same value. We define a segment w.r.t. $\ell$ to be a tuple $(i, [k_0, k_1], j)$ such that for any $k \in [k_0, k_1]$ $A^{(\ell)}_{ik} = A^{(\ell)}_{ik} \tilde{A}_{ik} = \tilde{A}_{ik0}$, $B^{(\ell)}_{kj} = B^{(\ell)}_{kj} = \tilde{B}_{kj}$ and $\tilde{B}_{kj} = \tilde{B}_{kj}$. For a fixed $i$ and fixed $j$, there are $O(n^{\mu / 2^\ell})$ segments. Next define the set $T^{(\ell)}_b$ to be the set of segments w.r.t. $\ell$ such that $A^{(\ell)}_{ik0} + B^{(\ell)}_{kj0} = C^{(\ell)}_{ij} + b$ but $\tilde{A}_{ik0} + \tilde{B}_{kj0} \neq \tilde{C}_{ij}$ for some offset $b \in [-10, 10]$. We recursively compute the matrix $C^{(\ell)}$ where $C^{(\ell)}_{ij} = \left\lfloor \frac{(C_{ij} \mod p) + 2(2^\ell - 1)}{p} \right\rfloor$ for decreasing values of $\ell = h, h - 1, \ldots, 0$.

**Base case** In the base case $\ell = h$, since $p < 2^h$ all matrices $A^{(h)}$, $B^{(h)}$ and $C^{(h)}$ have zero entries. $T^{(h)}$ is thus the set of all segments w.r.t. $h (i, [k_0, k_1], j)$ such that $\tilde{A}_{ik0} + \tilde{B}_{kj0} \neq \tilde{C}_{ij}$ and $T^{(h)}$ is the empty set for $b \neq 0$. $T^{(h)}$ can be computed in $O(n^{\alpha + \gamma + \mu - \beta})$ time by iterating over all the $i \in [n^\alpha]$, all the $j \in [n^\gamma]$ and all the intervals of the $j$th column of $\tilde{B}$.

**Recursive case** Each recursive step consists of computing $C^{(\ell)}$ from $T^{(\ell + 1)}_b$ and then computing $T^{(\ell)}_b$ from $T^{(\ell + 1)}_b$. To compute $C^{(\ell)}$ from $T^{(\ell + 1)}_b$ we first use fast matrix multiplication on matrices of polynomials, then we subtract the erroneous terms and finally look for the lowest value. The matrices we multiply are $A^p$ and $B^n$ where $A^p_{ik} = x^{A^{(\ell)}_{ik} - 2A^{(\ell)}_{ik0}} y^{A^{(\ell)}_{ik0}}$ if $A_{ik}$ is finite and 0 otherwise, and similarly for $B^p$. The $x$-degree is 0 or 1 and the $y$-degree is bounded by $O(n^\beta)$. Computing $C^\ell = A^p \times B^p$ thus takes $O(n^{\omega(\alpha, 1, \gamma) + \beta})$ time. We then compute a polynomial of candidate terms $C^{(\ell)}_{ij} = \sum_{x = C^{(\ell + 1)}_{ij} - b} \lambda x \in C^\ell$ for every $i \in [n^\alpha]$, every $j \in [n^\gamma]$ and every $b \in [-10, 10]$.

Since $C^{(\ell + 1)}_{ij}$ can take up to $O(n^\beta)$ values, this takes $O(n^{\alpha + \beta + \gamma})$ time. Finally we use $T^{(\ell + 1)}_b$ to compute the set of erroneous terms $R^p_{ij} = \sum_{(i, [k_0, k_1], j) \in T^{(\ell + 1)}_b} x^{A^{(\ell + 1)}_{ik} + B^{(\ell + 1)}_{kj} - 2(A^{(\ell + 1)}_{ik0} + B^{(\ell + 1)}_{kj0})}$. This takes time $O(|T^{(\ell + 1)}_b|)$. It still holds that $\bigcup_{b = -10}^{10} T^{(\ell)}_b = \bigcup_{b = -10}^{10} T^{(\ell + 1)}_b$ and since each segment in $T^{(\ell + 1)}_b$ breaks into a constant number of segments of $T^{(\ell)}_b$ (for $b' \in [-10, 10]$), the algorithm simply needs to sort the sub-segments in the correct set, which can be done using binary search. Therefore computing all $T^{(\ell)}_b$ from $T^{(\ell + 1)}_b$ can be done in $O(|T^{(\ell + 1)}_b|)$.

**Total runtime** Finally, we need to bound the size of $T^{(\ell)}_b$. We can use the same argument as previously using a prime number theorem. A fixed segment such that $\tilde{A}_{ik0} + \tilde{B}_{kj0} \neq \tilde{C}_{ij}$ is in $T^{(\ell)}_b$ with probability $O(2^\ell / n^\beta)$ and there are $O(n^{\alpha + \gamma + \mu / 2^\ell})$ such segments so we can bound the expected size of $T^{(\ell)}_b$ by $\mathbb{E}_p(|T^{(\ell)}_b|) = O(n^{\alpha + \gamma + \mu - \beta})$. In the end the total running time is $O(n^{\omega(\alpha, 1, \gamma) + \beta + n^{\alpha + \gamma + \mu - \beta} + n^{\alpha + \gamma + \beta}})$. Optimizing over $\beta$ this yields $O(n^{\omega(\alpha, 1, \gamma) + \alpha + \gamma + \mu + \beta})$. 

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