Singularities in \( d = 4, N = 1 \) Heterotic String Vacua

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Singularities in the Yukawa and gauge couplings of \( N = 1 \) compactifications of the \( SO(32) \) heterotic string are discussed.

Perturbative \( d = 4, N = 1 \) heterotic vacua are characterized by a \( c = 9 \) conformal field theory (CFT) with \( (0,2) \) worldsheet supersymmetry and the choice of a \( c = 22 \) vector bundle. The space-time spectrum features the gravitational multiplet, non-Abelian vector multiplets, charged chiral matter multiplets \( Q^I, \tilde{Q}^I \) and gauge neutral chiral moduli multiplets \( M^i \). The couplings of these multiplets are described by an effective Lagrangian which is constrained by \( N = 1 \) supersymmetry to only depend on three arbitrary functions: the real Kähler potential \( K \), the holomorphic superpotential \( W \) and the holomorphic gauge kinetic function \( f \). Due to their holomorphicity the latter two obey a non-renormalization theorem\footnote{For a recent discussion see ref.\cite{2}.}. The superpotential \( W \) receives no perturbative corrections and one only has

\[
W = W^{(0)} + W^{(NP)},
\]

where \( W^{(0)} \) denotes the tree level contribution while \( W^{(NP)} \) summarizes the non-perturbative corrections. \( W^{(0)} \) contains mass terms and Yukawa couplings both of which are generically moduli dependent. \( \tilde{Q}^I \) are the massive charged multiplets of the string vacuum while \( Q^I \) denotes the massless multiplets. The massive modes \( \tilde{Q}^I \) are commonly integrated out of the effective Lagrangian since their typical mass is of order of the Planck scale \( M_{Pl} \). However, since their masses can be moduli dependent they might become light in special regions of the moduli space. Therefore we choose to keep such massive multiplets in the effective theory. Thus a generic tree level superpotential is given by

\[
W^{(0)} = m_I(M^i) \tilde{Q}^I \tilde{Q}^I + Y_{IJK}(M^i) Q^I Q^J Q^K + \ldots,
\]

(2)

where all gauge quantum numbers of \( Q^I \) and \( \tilde{Q}^I \) are suppressed.

The real part of the gauge kinetic function \( f \) determines the inverse gauge coupling according to \( g^{-2} = \text{Re}f \). The holomorphic \( f \) receives no perturbative corrections beyond one-loop and one has

\[
f = f^{(0)} + f^{(1)} + f^{(NP)},
\]

(3)

where \( f^{(1)} \) is the one-loop correction.

In most (if not all) heterotic string vacua \( W \) and \( f \) are singular functions on the moduli space\footnote{For a recent discussion see ref.\cite{2}.}. For example, in a heterotic vacuum obtained as a Calabi–Yau compactification on a quintic hypersurface in \( \mathbf{CP}^4 \) with defining polynomial \( p = \sum_{\alpha=1}^5 X_\alpha^5 - 5\psi X_1 X_2 X_3 X_4 X_5 \) one finds\footnote{For a recent discussion see ref.\cite{2}.}

\[
Y \sim M^{-1}, \quad f \sim \log M,
\]

(4)

where \( M \equiv 1 - \psi^5 \). It is of interest to understand the physical origin of such singularities. In this talk we expand on a suggestion by Kachru, Seiberg and Silverstein\footnote{For a recent discussion see ref.\cite{2}.} that non-perturbative

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gauge dynamics is the cause of singularities in $N = 1$ heterotic vacua.

For type II vacua compactified on Calabi–Yau manifolds (or equivalently on $(c = \bar{c} = 9)$ (2, 2) CFTs) Strominger \[5\] gave a consistent picture of the physical mechanism responsible for the singularities. For such vacua the space-time effective theory is $N = 2$ supersymmetric and, as a consequence, there is only a logarithmic singularity in the gauge couplings $g^{-2}$ but no power like behaviour in any of the other couplings. Specifically, for the quintic threefold one finds

$$g^{-2} \sim \log M \bar{M} + \ldots . \quad (5)$$

Such a correction to the gauge couplings is induced in quantum field theories at one-loop by charged matter multiplets of mass $M$. Strominger suggested that there are indeed non-perturbative charged states in the type II string spectrum which cause the singularity in the gauge couplings. In perturbative string theory all such states are integrated out and one only sees their effects in the moduli dependence of the gauge couplings. However, as $M \to 0$ some of the non-perturbative states become massless and it is no longer legitimate to integrate them out. Thus the singularity signals light states and the break down of an effective low energy theory which does not properly include all the light degrees of freedom.

The relevant non-perturbative states arise in $N = 2$ hypermultiplets and they carry $U(1)$ charge of Ramond-Ramond vector bosons. As a consequence, they do not have the canonical couplings to the dilaton but their mass only depends on the scalar fields of the $N = 2$ vector multiplets in the string spectrum. Therefore, the corrections to the gauge couplings even though being non-perturbative appear without any dilaton dependence and thus ‘compete’ with tree level effects.

In string vacua with $N = 1$ supersymmetry the situation is more complicated. The power-like singularity of the tree level Yukawa couplings cannot be explained by states becoming massless. Instead it was suggested \[3\] that at least some of the singularities are caused by strong coupling dynamics of an asymptotically free non-perturbative gauge group. Such non-perturbative gauge groups are known to arise at special points in the moduli space of heterotic string vacua in six space-time dimensions and 8 supercharges \[9\]. Such vacua can be constructed either as an abstract (0, 4) CFT or as geometrical $K3$ compactification of the $d = 10$ heterotic string. Consistency requires that the number of instantons of the gauge group ($E_8 \times E_8$ or $SO(32)$) is non-vanishing and equal to 24. Associated with these instantons is a (quaternionic) moduli space which parameterizes their size, location and embedding into the gauge group. This moduli space has singularities at the points where instantons shrink to zero size. The physical origin of these singularities is either a set of gauge bosons becoming massless or an entire string becoming tensionless \[4\] \[13\]. Both of these effects are invisible in string perturbation theory; they are non-perturbative in that they occur in regions of the moduli space where string perturbation theory breaks down.

The moduli space of $k$ $SO(32)$ instantons is isomorphic to the Higgs branch of an $Sp(2k)$\[5\] gauge theory with 32 half-hypermultiplets in the fundamental (2, $k$) 2-dimensional representation, a traceless antisymmetric tensor in the $k(2k-1)-1$-dimensional representation and a singlet.\[6\] The singularities of this moduli space precisely occur where some (or all) of the $Sp(2k)$ gauge bosons become massless.

The situation is more involved when an $SO(32)$ instanton shrinks on a singularity of the underlying CFT or K3 manifold. Here we only discuss the simplest case of an $A_1$ singularity. One has to distinguish between instantons ‘with vector structure’ and ‘without vector structure’ \[12\]. For instantons without vector structure the moduli space is conjectured to be isomorphic to the Higgs branch of a $U(2k)$ gauge theory with 16 hypermultiplets in the fundamental (2, $k$) 2-dimensional representation and two antisymmet-

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3As a consistency check it was shown that the non-perturbative states also render the coupling of $R^2$ logarithmically singular at $M \to 0$ where the coefficient of the singularity counts the number of massless states \[4\].

4By $Sp(2k)$ we mean the rank $k$ symplectic group whose fundamental representation has dimension $2k$.

5This terminology refers to properties of the $SO(32)$ connection at infinity. See refs. \[14\] \[15\] for details.
ric tensors \( \mathbf{4} \) in the \( k(2k - 1) \)-dimensional representation \( \mathbf{16} \times \mathbf{16} \).

Small instantons with vector structure on an \( A_1 \) singularity show a more complicated behaviour \( \mathbf{4} \times \mathbf{8} \). The case where less than four instantons coalesce on the singularity is not fully understood yet while the moduli space of four (and more) instantons on the singularity has a Higgs branch and a Coulomb branch. On the Coulomb branch the dimension of the moduli space has been reduced by 29 but an additional tensor multiplet is present.

In this talk we focus on the \( SO(32) \) heterotic string and only consider the non-perturbative effects associated with non-perturbative gauge bosons.\(^6\) In this case the gauge group \( G \) of the string vacuum is a product of the perturbative gauge group \( G_{\text{pert}} \) and the non-perturbative gauge group \( G_{\text{NP}} \)

\[
G = G_{\text{pert}} \times G_{\text{NP}}
\]

where \( G_{\text{NP}} \) is a subgroup of either \( Sp(2k) \) or \( U(2k) \).\(^7\)

Upon toroidally compactifying the above theories one obtains \( N = 2 \) string vacua in \( d = 4 \). String vacua with \( N = 1 \) supersymmetry arise when one compactifies not on \( K^3 \times T^2 \) but on a Calabi–Yau threefold. However, there is a particular class of threefolds – \( K^3 \) fibrations – which is closely related to the six-dimensional heterotic vaca. \( K^3 \) fibrations are 3-dimensional Calabi–Yau manifolds where a \( K^3 \) is fibred over a \( \mathbb{P}^1 \) base. If the base is large the adiabatic argument applies \( \mathbf{20} \) and the singularities of the \( K^3 \) fibres are inherited from the corresponding six-dimensional vacuum \( \mathbf{4} \).

A specific class of \( K^3 \) fibrations are the quintic hypersurfaces defined in weighted projective space \( \mathbf{20} \). For compactifications of the \( SO(32) \) heterotic string on \( (0,2) \)

\(^6\) We thank P. Aspinwall and K. Intriligator for a useful correspondence on this point.

\(^7\) Singularities in the \( E_8 \times E_8 \) heterotic string and chirality changing phase transitions have recently been discussed in ref. \( \mathbf{21} \).

\(^8\) By giving an appropriate vacuum expectation value to one of the two antisymmetric tensors of \( U(2k) \) one arrives at the \( Sp(2k) \) gauge theory with the precise spectrum given above.

deformations of such Calabi–Yau spaces the non-perturbative spectrum is computed in ref. \( \mathbf{22} \) for the case of a single small instanton at a smooth point in the \( K^3 \) fibre.\(^8\) It is found that the non-perturbative gauge group in the four-dimensional vacuum is given by \( G_{\text{NP}} = Sp(2) \cong SU(2) \) and out of the 32 half-hypermultiplets in \( d = 6 \) only four doublets in chiral \( N = 1 \) supermultiplets (which we denote by \( q_i, i = 1, \ldots, 4 \)) survive in \( d = 4 \). This resulting gauge theory is asymptotically free \( (b_{SU(2)} > 0) \) and thus becomes strongly coupled below its characteristic scale \( \Lambda_{SU(2)} \). It has the additional property that

\[
c := T(ad) - \sum_r n_r T(r) = 0,
\]

where \( T(r) \) is the index in the representation \( r, T(ad) \) is the index in the adjoint representation and \( n_r \) counts the number of chiral multiplets in representation \( r \). (In this notation the one-loop coefficient of the \( \beta \)-function is given by \( b = 3T(ad) - \sum_r n_r T(r) \).) The coefficient \( c \) also appears in the anomaly equation of the R-symmetry. An anomaly free R-symmetry implies \( \sum_r n_r T(r) R_c = -c \), where \( R_c \) is the \( R \)-charge of the superfield. As a consequence, whenever \( c = 0 \) one can choose \( R = 0 \) for all superfields, and hence no non-perturbative superpotential \( W^{(NP)} \) can be generated by the strong coupling dynamics \( \mathbf{23} \). This conclusion is believed to hold irrespective of the precise form of tree level superpotential.

Below \( \Lambda_{SU(2)} \) the theory confines and the surviving degrees of freedom are the gauge singlets

\[
M_{ij} := q_i \cdot q_j.
\]

\( M_{ij} \) is antisymmetric and obeys the constraint \( PfM := \frac{1}{4} \epsilon^{ijkl} M_{ij} M_{kl} = 0 \); thus there are five physical degrees of freedom in the effective theory. Quantum mechanically the constraint is modified and reads \( \mathbf{23} \)

\[
PfM = \Lambda^4_{SU(2)} \cdot
\]

\(^9\) It is important to consider a \( (0,2) \) deformation since on the \( (2,2) \) locus the spin connection is embedded in the gauge connection and a small instanton necessarily has to shrink on a \( K^3 \) singularity.
So altogether the superpotential is given by
\begin{equation}
W = Y_{IJK}(M) Q^I Q^J Q^K + m_I(M) \hat{Q}^I \hat{Q}^J + \lambda (\text{Pf} M - \Lambda^4) + \text{non-renormalizable terms} + \text{stringy non-perturbative terms},
\end{equation}

where \( \lambda \) is a Lagrange multiplier. Note that even though no non-perturbative superpotential is generated by the strongly coupled gauge theory there are ‘stringy’ non-perturbative corrections of order \( \mathcal{O}(e^{-g_{\text{string}}^2}) \) where \( g_{\text{string}} \) is the string coupling constant or equivalently the vacuum expectation value of the dilaton multiplet.

The non-renormalizable interactions typically include mass terms for the \( M_{ij} \). One way to see this is to note that a corresponding \( N = 2 \) vacuum includes a chiral multiplet \( \phi \) in the adjoint representation with couplings (in \( N = 1 \) notation)
\begin{equation}
W = \kappa^{ij} q_i \cdot \phi \cdot q_j + \mu \text{Tr} \phi^2. \tag{11}
\end{equation}

\( N = 2 \) enforces \( \mu = 0 \) and \( \kappa^{ij} = \sqrt{2} \delta^{ij} \) while for \( N = 1 \) \( \phi \) becomes heavy and \( \kappa \) arbitrary. Integrating out \( \phi \) and diagonalizing the resulting mass matrix leads to
\begin{equation}
W = \frac{1}{2} \sum_{i,j} m_{ij} M_{ij}^2. \tag{12}
\end{equation}

(We choose \( M_{21} = 1 \) throughout this talk.) The number of moduli (or equivalently the dimension of the moduli space) is given by 5 minus the number of non-vanishing mass terms \( m_{ij} \).

Ref. [6] considered the case of a one-dimensional moduli space, that is a superpotential with four mass terms
\begin{equation}
W = Y_{IJK}(M) Q^I Q^J Q^K + m_I(M) \hat{Q}^I \hat{Q}^J + \lambda (\text{Pf} M - \Lambda^4) + m_{13}M_{13}^2 + m_{14}M_{14}^2 + m_{23}M_{23}^2 + m_{24}M_{24}^2 + \ldots. \tag{13}
\end{equation}

Integrating out the massive modes via \( \frac{\delta W}{\delta M} = 0 \) results in
\begin{align*}
M_{23} = M_{24} = M_{13} = M_{14} = 0, \\
M_{12}M_{34} = \Lambda^4. \tag{14}
\end{align*}

(Note that in string theory \( \langle Q \rangle = 0 \) holds by construction.)

In string perturbation theory both the Yukawa couplings \( Y_{IJK}(M) \) and the masses \( m_I(M) \) are given as power series expansion in the moduli. However, the strong coupling effects which are responsible for generating the non-perturbative constraint \( \text{Pf} M = \Lambda_{SU(2)}^4 \) remove the origin of the moduli space and render the perturbative expansion of the Yukawa couplings singular. For example
\begin{equation}
Y_{IJK}(M) \sim M_{12} + M_{34} + \ldots \tag{15}
\end{equation}
produces a singularity
\begin{equation}
Y_{IJK}(M) \sim \frac{\Lambda^4}{M_{34}} + \ldots \text{ as } M_{34} \to 0. \tag{16}
\end{equation}

Furthermore, the \( SU(2) \) scale \( \Lambda_{SU(2)} \) depends on the gauge coupling \( g_{SU(2)} \) of the non-perturbative \( SU(2) \) gauge bosons via \( \Lambda_{SU(2)} \sim \exp(-\frac{8\pi^2}{g_{SU(2)}^2}) \). It was shown in refs. [26,27] that the gauge couplings of the non-perturbative gauge bosons do not depend on \( g_{\text{string}} \) and hence the Yukawa couplings in eq. (16) do not depend on \( g_{\text{string}} \) either. Thus, exactly as in the \( N = 2 \) case, the non-perturbative effect which generates the singularity in eq. (16) does not have the standard dilaton or \( g_{\text{string}} \) dependence but instead competes with tree level couplings. (Of course, this is a requirement for a consistent explanation of the singularities in the Yukawa couplings.)

The mass terms \( m_I \) either become large or small at the singular points in the moduli space. We already learned from the \( N = 2 \) analysis that light states \( (m_I(M) \sim M_{34}) \) which are charged under \( G_{\text{pert}} \) produce a singularity in the associated gauge coupling at one loop; they contribute a correction
\begin{equation}
g^{-2} = \sum_j \frac{b_j}{16\pi^2} \log|m_j|^2 + \ldots. \tag{17}
\end{equation}

The coefficient of the singularity is set by the multiplicity of the light modes and their gauge quantum numbers. Unfortunately, in string theory this coefficient is so far only known for \( (2,2) \) vacua [28] and hence a more detailed comparison with the \( (0,2) \) models considered in ref. [6] cannot be presented. Conversely, it is not known at present how to repeat the analysis of ref. [6] for
(2, 2) vacua since the non-perturbative physics in the corresponding \( d = 6 \) vacua is not fully understood.

Nevertheless, it is interesting to display the coefficients of the singularity in (2, 2) vacua. One finds for all (2, 2) vacua of the \( SO(32) \) heterotic string, where \( G_{\text{pert}} = SO(26) \times U(1) \), the constraint
\[
16\pi^2 (g_{SO(26)}^{-2} - \frac{1}{6} g_{U(1)}^{-2}) = -16F_1 ,
\]
while for (2, 2) vacua of the \( E_8 \times E_8 \) heterotic string \( (G_{\text{pert}} = E_8 \times E_8) \) one has
\[
16\pi^2 (g_{E_8}^{-2} - g_{E_8}^{-2}) = -12F_1 .
\]
\( F_1 \) is the topological index defined in ref. 2\(^1\) which for the quintic hypersurface in \( \mathbb{CP}^4 \) is given by
\[
F_1 = -\frac{1}{12} \log \bar{M} M + \ldots .
\]
One can easily check that eqs. (17)–(20) are not easy to satisfy. One finds only two sensible solutions for \( SO(26) \times U(1) \)\(^2\): either one has one pair of \( SO(26) \) singlets with \( U(1) \) charges \( \pm 2 \) or four pairs of \( SO(26) \) singlets with \( U(1) \) charges \( \pm 1 \). (Here \( m = M \) was assumed, \( m \) being the mass of the light states; if \( m = M^2 \) \( (m = M^2) \) one can also have two (one) pairs with \( U(1) \) charges \( \pm 1 \).) In particular no states carrying \( SO(26) \) charge can become light at the singularity. Furthermore, there is no sensible solution for \( E_8 \times E_8 \) at all. This might indicate that in (2, 2) vacua a different mechanism is responsible for the singularities in the gauge couplings as well as the Yukawa couplings\(^2\).

In most Calabi–Yau vacua the singularity of the Yukawa couplings has a more complicated structure than just a simple pole. In particular one observes generically a singular locus with more than one component where the different components can intersect in various ways. Such a behaviour is reproduced by \( k \) small instantons located at the same point in moduli space or in other words by \( G_{\text{NP}} = Sp(2k) \). For this case the analysis of ref. 3\(^1\) can be repeated without major modifications. One finds that out of the 32 half-hypermultiplets in the fundamental \(^3\) representation of the six-dimensional vacua again four chiral \(^4\) multiplets remain in \( d = 4 \). In addition, the antisymmetric tensor \(^5\) survives as a chiral multiplet. As before, the resulting spectrum in \( d = 4 \) is an asymptotically free gauge theory with the additional property that \( c_{Sp(2k)} = 0 \) for all values of \( k \). Thus, no non-perturbative superpotential can be generated by the strongly coupled \( Sp(2k) \) gauge theory. Fortunately, this gauge theory has been analysed in some detail in ref. 2\(^6\) (see also 3\(^7\)). The physical degrees of freedom below \( \Lambda_{Sp(2k)} \) are found to be
\[
M_{ij}^l := q_i \cdot A^l \cdot q_j, \quad i,j = 1, \ldots , 4, \quad l = 0, \ldots , k - 1 ,
\]
\[
T_r := \frac{1}{4r} \text{Tr} A^r, \quad r = 2, \ldots , k ,
\]
where \( A \) denotes the antisymmetric tensor. As for the \( SU(2) \) gauge theory the \( M_{ij}^l \) and \( T_r \) are not all independent but related by constraint equations. These constraints are rather involved and for simplicity we focus on \( Sp(4) \) henceforth. For this case one has
\[
T_2 \text{Pf} M^0 + \frac{1}{2} \text{Pf} M^1 = 2\Lambda^6 ,
\]
\[
c_{ijkl} M_{ij}^0 M_{kl}^1 = 0 ,
\]
where classically the left hand side of the first equation vanishes. Hence, there are \( 2 \cdot 6 + 1 - 2 = 11 \) physical degrees of freedom in the effective theory. Exactly as in the previous \( SU(2) \) example these states can be massive and the superpotential has the generic form
\[
W = Y_{ijk} (M, T) Q^i Q^j Q^k + m_i (M, T) \hat{Q}^i \hat{Q}^j + \lambda \ (T_2 \text{Pf} M^0 + \frac{1}{2} \text{Pf} M^1 - 2\Lambda^6 )
\]
\[
+ \mu \ c_{ijkl} M_{ij}^0 M_{kl}^1 + \frac{1}{2} \sum_{i,j} (m_{ij}^0 (M_{ij}^0)^2 + (m_{ij}^1 (M_{ij}^1)^2) + m_{ij}^2 T_{ij}^2
\]
\[
+ \ldots .
\]

The dimension of the moduli space depends on the number of non-vanishing mass terms in eq. (23). Consider, for example, the case \( m_{12}^1 =
$m_{34}^0 = m = 0$ which results in a two-dimensional moduli space. The equations of motion are solved by

\[
\begin{align*}
\lambda &= \mu = M_{ij}^1 = 0 \quad \forall i, j \\
M_{13}^0 &= M_{14}^0 = M_{23}^0 = M_{24}^0 = 0 \\
T_2 M_{12}^0 M_{34}^0 &= 2\Lambda^6. 
\end{align*}
\]

A Yukawa coupling of the form

\[ Y_{IJK} \sim T_2 + M_{12}^0 + M_{34}^0 + \ldots \] (25)

now produces a singularity

\[ Y_{IJK}(M) \sim \frac{2\Lambda^6}{M_{12}^0 M_{34}^0} + \ldots \] (26)

as $M_{34}^0 \to 0$ or $M_{12}^0 \to 0$. This is an example for a Yukawa coupling which depends on two intersecting singular lines.

Different combinations of mass terms can lead to different properties of the singular locus. For example, consider the case $m_{12}^0 = m_{34}^0 = m_{12}^1 = m_{34}^1 = m = 0$ for which the equations of motion lead to

\[
\begin{align*}
\lambda &= \mu = 0 \\
M_{13}^0 &= M_{14}^0 = M_{23}^0 = M_{24}^0 = 0 \\
M_{12}^1 M_{34}^1 + M_{12}^2 M_{34}^2 &= 0 \\
T_2 M_{12}^0 M_{34}^0 + \frac{1}{2} M_{12}^1 M_{34}^1 &= 2\Lambda^6. 
\end{align*}
\]

If we solve the last two equations for $T_2$ and $M_{34}^0$, say, we find for a Yukawa coupling of the form

\[ Y(M) \sim T_2 + M_{12}^0 + M_{34}^0 + M_{12}^1 + M_{34}^1 + \ldots \] (28)

the singularities

\[ Y(M) \sim \frac{(M_{12}^0)^2}{2(M_{12}^0)^2} - \frac{2M_{12}^1\Lambda^6}{(M_{12}^0)^2 M_{12}^4} - \frac{M_{12}^0 M_{12}^4}{M_{12}^4}. \] (29)

as $M_{12}^0$, $M_{12}^1$, or $M_{34}^0 \to 0$. There are three singular components but at most two of them appear in the same term of the Yukawa couplings, i.e., only two of them are intersecting. If the dimension of the moduli space equals the number of intersecting singular components then there cannot be more than two singular lines in the $Sp(4)$ model.

Singular Yukawa couplings with more than two intersecting components can arise in an $Sp(2k)$ gauge theory once we have $k > 2$. More precisely, in order to generate $l$ intersecting singular components we need to look at a quantum constraint of the form

\[ T_2 T_3 \ldots T_l \text{Pf} M^0 + \ldots = \Lambda^{2k+2} \] (30)

to find the minimal $k$ required. This yields

\[ 4k \geq l(l + 1) + 2 \] (31)
as a necessary condition. A proof of this relation together with an extended analysis of the singularities in the Yukawa couplings will be presented elsewhere.

Finally let us consider the case where $G_{NP} = SU(2k)$. This gauge group arises when $k$ instantons without vector structure shrinks on an $A_1$ singularity of the $K3$ fibre. Again the computation of the $d = 4$ spectrum can be performed following the methods of ref. The one finds two flavours of fundamentals, i.e. $2[\square + \bigotimes]$, as well as one flavour of antisymmetric tensors $[\bigotimes + \[\square ]]$. As before, this an asymptotically free gauge theory with $c_{SU(2k)} = 0$ for any $k$. Consequently this theory confines below $\Lambda_{SU(2k)}$ but no non-perturbative superpotential is generated by the strong coupling dynamics.

For simplicity we focus on $G_{NP} = SU(4)$ where $\bigotimes \cong \[\square ]$ holds. Therefore the two antisymmetric tensors $A_r$, $r = 1, 2$, transform as a doublet under an additional $SU(2)$ flavour symmetry. The low energy degrees of freedom are found to be

\[
\begin{align*}
M_{ij}^0 &= q_i q_j, \quad i, j = 1, 2, \\
M_{ij}^1 &= q_i A^2 q_j, \\
H_r &= q A_r q, \quad r = 1, 2, \\
\bar{H}_r &= \bar{q} A_r \bar{q}, \\
T_{rs} &= A_r A_s .
\end{align*}
\] (32)

These singlet fields satisfy the additional constraints

\[ \det T \det M^0 - \epsilon^{rs} \epsilon^{tu} T_r H_s \bar{H}_u - \det M^1 = \Lambda^8, \]

\[ \epsilon^{ij} \epsilon^{kl} M_{ik}^0 M_{jl}^1 + \epsilon^{rs} H_r \bar{H}_s = 0 . \] (33)

The structure of the singularities again depends on the mass terms for the low energy degrees of freedom.
freedom. For example, mass terms for the fields $M^1, H, \bar{H}, M_0^1, M_0^2$ and $T_{12}$ lead to the equations of motion $M^1 = H = \bar{H} = M_0^1 = M_0^2 = T_{12} = 0$ and result in a three-dimensional moduli space satisfying the constraint

$$T_{11} T_{22} M_0^{11} M_0^{22} = \Lambda^8.$$  

In this case the singular locus of the Yukawa couplings has three intersecting components. A more detailed analysis of the $SU(2k)$ gauge theories will also be presented in ref. [31].

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