\[ L^2 \times L^2 \times L^2 \rightarrow L^{2/3} \] boundedness for trilinear multiplier operator

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Abstract

This paper discusses the boundedness of the trilinear multiplier operator, as well as that of the general multilinear multiplier operator, when the multiplier satisfies a certain degree of smoothness but with no decaying condition and is \( L^q \)-integrable, with an admissible range of \( q \). The arguments present here follow those of [7] closely.

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1 Introduction

Let \( d \geq 1 \). Let \( m(\xi, \eta, \delta) \) be a function on \( \mathbb{R}^{3d} \). Define an operator \( T_m \) as follows.

\[
T_m(f, g, h)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m(\xi, \eta, \delta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\delta) e^{2\pi i x (\xi + \eta + \delta)} d\xi d\eta d\delta.
\]

This paper aims to give an explicit result on the \( L^2 \times L^2 \times L^2 \rightarrow L^{2/3} \) boundedness of \( T_m \) given smoothness and integrability of \( m \). The author anticipates the use of such result in an upcoming project. The crucial point, that will be needed for later use, is the dependence of the operator norm \( \|T_m\| \) on \( \|m\|_{L^r} \) (see 3.6). At the time of starting this paper, the author was not aware of any such explicit result on trilinear multiplier operator on the market. Unfortunately, due to the lack of duality theory for \( L^s, 0 < s < 1 \), one can’t extend easily this result to other exponents in the Banach range, \( L^{p_1} \times L^{p_2} \times L^{p_3} \rightarrow L^r \), with \( 1/p_1 + 1/p_2 + 1/p_3 = 1/r \). Such obstacle is not met in the case of bilinear setting; see [7].

There is a body of literature regarding the boundedness of multiplier operator in the linear and bilinear settings, with various conditions on \( m \), ranging from decay to smoothness to integrability.
In the bilinear setting, one classical condition on \( m \) to guarantee the boundedness of \( T_m \) on the Banach range is the Coifman-Meyer condition [2]:

\[
|\partial^{\alpha} m(\xi, \eta)| \leq C_\alpha |(\xi, \eta)|^{-|\alpha|}
\]

for sufficiently many \( \alpha \). Moreover, in the bilinear setting, if one is to merely impose uniform bounds on derivatives of \( m \): \( \|\partial^{\alpha} m\|_{L^\infty} \leq C_\alpha < \infty \), then one needs to make other compromises. It was shown in [1] that such uniform boundedness on \( m \) alone is not sufficient to make \( T_m \) a bounded operator on \( L^2 \times L^2 \to L^1 \). It was shown in [6] that if one further imposes \( L^2 \)-integrability of \( m \), one can get back a bounded operator. Moreover, the same authors in [6] showed that there is a short range of integrability that one can impose in order to secure boundedness of \( T_m \). This paper follows the approaches taken in [7] closely. That means, the multipliers considered here, \( m \) as functions on \( R^3 \), only have uniform derivative bounds plus some integrability. This paper also discusses the optimal range of integrability of \( m \) on \( R^3 \), as in [7], as well as similar boundedness of \( T_m \) when \( m \) is a function on \( R^{d \otimes n} \), \( n \geq 1 \), \( d \geq 1 \). See Section 6.

There are other venues in the multilinear settings, where positive boundedness results have been established when the multiplier smoothness is much compromised. See [8] for the case of multipliers from Sobolev spaces, and [5] for the case of \( L^r \)-based Sobolev spaces.

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2 Notation explanation

\( N_0 \): the set of nonnegative integers

\( \ominus \): orthogonal complement

\( \{F, M\}^{d \bullet} \): \( d \)-tuples whose elements are either \( F \) or \( M \) and at least one of those must be \( M \)

\( \mathcal{C}_J \): the space of all continuous functions whose derivatives up to, and including, order \( J \)th are continuous and which have compact support

\( \kappa Q \): a cube (interval) that has the same center as the cube (interval) \( Q \) and \( \kappa \) times the side-length of \( Q \)

\( |\cdot| \): either means an absolute value or the Euclidean norm of a vector or the Lebesgue measure of a full-dimensional set

\( \mathcal{F}^{-1} \): the Fourier inverse transform

\( \lfloor \cdot \rfloor \): the integer part of a (positive) number

3 Some multiresolution analysis background

The approach followed in this discussion requires an understanding of multiresolution analysis and wavelets. In this section, necessary background is introduced. The following facts can be found in [10].

First, one starts with an orthonormal basis of \( L^2(\mathbb{R}) \).

Definition 1. An (inhomogeneous) multiresolution analysis is a sequence \( \{V_j : j \in \mathbb{N}_0\} \) of subspaces of \( L^2(\mathbb{R}) \) such that

a) \( V_0 \subset V_1 \subset \cdots \subset V_j \subset V_{j+1} \subset \cdots \) spans \( \bigcup_{j \geq 0} V_j = L^2(\mathbb{R}) \).

b) \( f \in V_0 \) iff \( f(x-n) \in V_0 \) for any \( n \in \mathbb{Z} \).

c) \( f \in V_j \) iff \( f(2^{-j} \cdot) \in V_0 \) for \( j \in \mathbb{N} \).
d) There exists $\phi_F \in V_0$ such that $\{\phi(-n)\}_{n \in \mathbb{Z}}$ is an orthonormal basis in $V_0$.

Because of the properties (c), (d), $\phi_F$ is called the scaling function or the father wavelet of such system.

For $j \in \mathbb{N}_0$, let $W_j = V_{j+1} \ominus V_j$. Let $\phi_M \in W_0$ be such that $\{\phi_M(-n)\}_{n \in \mathbb{Z}}$ forms an orthonormal basis in $W_0$. Such existence is guaranteed by wavelet theory [10]. This function $\phi_M$ is called the mother wavelet associated with $\phi_F$.

One now needs an orthonormal system of $L^2(\mathbb{R}^d)$. Such system can be generated from one of one dimension. For $n = (n_r)_{1 \leq r \leq d} \in \mathbb{Z}^d$, let

$$\Phi_n(x) = \prod_{r=1}^{d} \phi_F(x_r - n_r),$$

with $x \in \mathbb{R}^d$.

Denote $G = (G_1, \ldots, G_d) \in \{F, M\}^d$ - in other words, a $d$-tuple of types, $F$ or $M$ (father wavelet or mother wavelet, respectively). Let

$$\Phi_n^G(x) = \prod_{r=1}^{d} \phi_{G_r}(x_r - n_r).$$

Finally, let $G^0 = \{\{F, \cdots, F\}\}$ and $G^j = \{F, M\}^d$ for all $j \in \mathbb{N}$. Assume that $\|\phi_F\|_{L^2} = \|\phi_M\|_{L^2} = 1$, then one has the following promise.

**Proposition 2.** The system below forms an orthonormal basis in $L^2(\mathbb{R}^d)$,

$$\Phi_n^{j,G}(x) = \begin{cases} \Phi_n(x) & \text{for } j = 0, G \in G^0, n \in \mathbb{Z}^d \\ 2^{d/2} \Phi_n(2^j x) & \text{for } j \in \mathbb{N}, G \in G^j, n \in \mathbb{Z}^d \end{cases} \quad (3.1)$$

**Remark 3:** One can also require that $\phi_F, \phi_M \in C^j_{\beta}$ and that $\int x^\alpha \phi_M(x) \, dx = 0$ for all $|\alpha| \leq K$; here $J, K$ are sufficiently large positive integers for the following calculations to hold. For the guarantee of smoothness and moment cancellations, see Theorem 1.61 and remark 1.62 in [9].

In what follows one is concerned with functions on $\mathbb{R}^{3d}$ is in the place of $d$ - hence for example, $d$ is replaced by $3d$ in 3.1.

**Remark 4:** By definition of $\Phi_n^{j,G}$ in 3.1, each $\Phi_n^{j,G}$ with $j \in \mathbb{N}_0, n \in \mathbb{Z}^{3d}$ can be written as $\Phi_n^{j,G}(k,l,u)$ with $k, l, u \in \mathbb{Z}^d$, and moreover, $\Phi_n^{j,G}(k,l,u) = \omega_{1,k} \omega_{2,l} \omega_{3,u}$ with $\omega_{1,k}, \omega_{2,l}, \omega_{3,u}$ being functions of only variables $x_1, \cdots, x_d; x_{d+1}, \cdots, x_{2d}; x_{2d+1}, \cdots, x_{3d}$, respectively. By the assumption made in Remark 3, $\omega_{1,k}, \omega_{2,l}, \omega_{3,u} \in C^j_{\beta}$, for some sufficiently large $J$. The supports of $\omega_{1,k}$ have finite overlaps in $k$, and similarly for $\omega_{2,l}, \omega_{3,u}$. Moreover

$$\|\omega_{1,k}\|_{L^\infty}, \|\omega_{2,l}\|_{L^\infty}, \|\omega_{3,u}\|_{L^\infty} \leq 2^{d/2} \quad (3.2)$$

Let $m(\xi, \eta, \delta)$ be a function on $\mathbb{R}^{3d}$ where $\xi, \eta, \delta \in \mathbb{R}^d$. The following lemma is essentially given in [6].

**Lemma 5.** Let $K$ be a positive integer. Assume that $m \in C^{K+1}$ is a function on $\mathbb{R}^{3d}$ such that

$$\sup_{|\alpha| \leq K+1} |\partial^\alpha m|_{L^\infty} \leq C_0 < \infty. \quad (3.3)$$

Then one has,

$$|\langle \Phi_n^{j,G}, m \rangle| \leq C_0 \cdot 2^{-(K+1+d)j} \quad (3.4)$$

provided that $\phi_M$ has $K$ vanishing moments.

**Remark 5:** See the appendix for a brief discussion about the proof of this lemma.
3.1 Main theorem

Utilizing Lemma 5, a result that one wants to arrive at, is,

**Theorem 6.** Let $1 \leq q < 3$ and set $M_q = \left[\frac{6d}{3q - q}\right] + 1$. Let $m(\xi, \eta, \delta)$ be a function in $L^q(\mathbb{R}^d) \cap C^M(\mathbb{R}^d)$ satisfying

$$\|\hat{\partial}^\alpha m\|_{L^q} \leq C_0$$

for all $|\alpha| \leq M_q$. Then the trilinear operator $T_m$ with the multiplier $m$ satisfies

$$\|T_m\|_{L^2 \times L^2 \times L^2 \rightarrow L^{2/3}} \leq C_{\alpha, d, q} \|m\|_{L^2}^{q/3}.$$ 

Conversely, there is a function $m \in \bigcap_{q > 3} L^q(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d)$ such that the associate operator $T_m$ does not map $L^2 \times L^2 \times L^2 \rightarrow L^{2/3}$.

4 Sufficiency

Let $j, G$ be as in 3.1 and $n = (k, l, u) \in \mathbb{Z}^d$. Set

$$b^{j, G}_n = \langle \Phi^{j, G}_n, m \rangle.$$ 

Then (see the appendix)

$$\|m\|_{L^q} \approx_d \left( \sum_{(j, G) \in \mathbb{Z}^d} \|b^{j, G}_n \chi_{Q_{jn}}\|_{L^q}^2 \right)^{1/2},$$

with $Q_{jn}$ being a cube centered at $2^{-j}n$ with side-length $2^{1-j}$. Let $\tilde{Q}_{jn} = (1/2)Q_{jn}$. Now fix $j, G$ and refer to $b^{j, G}_n$ as simply $b_n$. Then due to the pairwise disjoint of the cubes $\tilde{Q}_{jn}$ in $n$ (for fixed $j, G$), one also has from 4.1 that

$$\|m\|_{L^q} \geq 2^{3jd/2} \left( \sum_{n \in \mathbb{Z}^d} \|b_n \chi_{\tilde{Q}_{jn}}\|_{L^q}^2 \right)^{1/2} \geq 2^{3jd/2} \left( \sum_{n \in \mathbb{Z}^d} \|b_n \chi_{\tilde{Q}_{jn}}\|_{L^q}^2 \right)^{1/2} = 2^{3jd/2} \left( \sum_{n \in \mathbb{Z}^d} \|b_n \|_{L^q}^2 \right)^{1/2}.$$ 

The disjointness of the cubes $\tilde{Q}_{jn}$ is used in the last two equalities above. Let $b = (b_n)_{n \in \mathbb{Z}^d}$. Then the calculation above says,

$$\|b\|_{L^q} \leq 2^{3jd(1/4 - 1/2)} \|m\|_{L^q}.$$ 

(4.2)

Let $r \in \mathbb{N}_0$. Define the following decomposition sets

$$U^r = \{(k, l, u) \in \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d : 2^{-r-1}|b|_{l^r} < |b_{(k, l, u)}| \leq 2^{-r}|b|_{l^r}\},$$

$$U_{1,r} = \{(k, l, u) \in U_r : \text{card}\{(l, u) : (k, l, u) \in U_r\} \geq 2^{2q/3} \|b\|_{l^q}^{2q/3} \|b\|_{l^{-2q/3}}^{2q/3}\},$$

$$U_{2,r} = \{(k, l, u) \in U_r : \text{card}\{(l, u) : (k, l, u) \in U_r\} < 2^{2q/3} \|b\|_{l^q}^{2q/3} \|b\|_{l^{-2q/3}}^{2q/3}\}.$$ 

Let $E = \{k : (k, l, u) \in U_{1,r} \text{ for some } u\}$. Then from the definitions of $U_r$ and $U_{1,r}$,

$$\text{card } E \cdot 2^{-rq/3-3} \|b\|_{l^q}^{2q/3} \|b\|_{l^{-2q/3}}^{2q/3} \leq \sum_{(k, l, u) \in U_{1,r}} |b_{(k, l, m)}|^q \leq \|b\|_{l^q}^q,$$

which, together with 4.2 leads to,

$$\text{card } E \leq 2^d 2^{qr/3} \|b\|_{l^q}^{q/3} \|b\|_{l^{-2q/3}}^{q/3}.$$ 

(4.3)
As discussed in Remark 4, let $m^{1,r} := \sum_{(k,l,u) \in U_{1,r}} b_{(k,l,u)} \omega_1 \omega_2 \omega_3$ and $m^{1,r}$ denote the operator associated with the multiplier $m^{1,r}$. Likewise, let $m^{2,r} := \sum_{(k,l,u) \in U_{2,r}} b_{(k,l,u)} \omega_1 \omega_2 \omega_3$, and define $T_{m^{2,r}}$ similarly as with $T_{m^{1,r}}$.

Let $f, g, h \in L^2(\mathbb{R}^d)$. Then one has,

$$
\|T_{m^{2,r}}(f, g, h)\|_{L^2}^{2/3} = \sum_{(k,l,u) \in U_{2,r}} b_{(k,l,u)} F^{-1}(\omega_1 \hat{f}) F^{-1}(\omega_2 \hat{g}) F^{-1}(\omega_3 \hat{h}) \|_{L^2}^{2/3} \\
\leq \sum_{(l,u) : (k,l,u) \in U_{2,r}} \|F^{-1}(\omega_2 \hat{g}) F^{-1}(\omega_3 \hat{h})\|_{L^2} \sum_{k : (k,l,u) \in U_{2,r}} b_{(k,l,u)} F^{-1}(\omega_1 \hat{f}) \|_{L^2}^{2/3} \\
\leq \sum_{(l,u) : (k,l,u) \in U_{2,r}} \|\omega_2 \hat{g}\|_{L^2} \|\omega_3 \hat{h}\|_{L^2} \sum_{k : (k,l,u) \in U_{2,r}} b_{(k,l,u)} \omega_1 \hat{f} \|_{L^2}^{1/3} \\
\leq (\sum_{l \in \mathbb{Z}^d} \|\omega_2 \hat{g}\|_{L^2}^2 \|\omega_3 \hat{h}\|_{L^2}^2)^{1/3} \left( \sum_{(l,u) : (k,l,u) \in U_{2,r}} \|\omega_1 \hat{f}\|_{L^2} \sum_{k : (k,l,u) \in U_{2,r}} b_{(k,l,u)} \omega_1 \hat{f} \|_{L^2} \right)^{1/3} \\
\leq (2^{3d/2})^{2/3} (2^{-r} \|g\|_{L^2})^{2/3} \|f\|_{L^2}^{2/3} \|\omega_3 \hat{h}\|_{L^2}^{2/3} \|h\|_{L^2}^{2/3} (\sum_{l \in \mathbb{Z}^d} \|\omega_2 \hat{g}\|_{L^2}^2 \|\omega_3 \hat{h}\|_{L^2}^2)^{1/3}. \quad (4.4)
$$

The tools used in the above calculation are: Hölder’s inequality; Plancherel’s theorem; Cauchy-Schwarz inequality; Remark 4 and its 3.2; definitions of $U_{1,r}, U_{2,r}$. Taking the 3/2th power of 4.4 allows one to obtain:

$$
\|T_{m^{2,r}}(f, g, h)\|_{L^2}^{2/3} \leq d (2^{3d/2})^{2/3} (2^{-r} \|g\|_{L^2})^{2/3} \|f\|_{L^2}^{2/3} \|\omega_3 \hat{h}\|_{L^2}^{2/3} \|h\|_{L^2}^{2/3} (\sum_{l \in \mathbb{Z}^d} \|\omega_2 \hat{g}\|_{L^2}^2 \|\omega_3 \hat{h}\|_{L^2}^2)^{1/3}. \quad (4.5)
$$

Similarly, for the operator $T_{m^{1,r}}$:

$$
\|T_{m^{1,r}}(f, g, h)\|_{L^2}^{2/3} = \sum_{(k,l,u) \in U_{1,r}} b_{(k,l,u)} F^{-1}(\omega_1 \hat{f}) F^{-1}(\omega_2 \hat{g}) F^{-1}(\omega_3 \hat{h}) \|_{L^2}^{2/3} \\
\leq \sum_{(l,u) : (k,l,u) \in U_{1,r}} \|F^{-1}(\omega_2 \hat{g}) F^{-1}(\omega_3 \hat{h})\|_{L^2} \sum_{k : (k,l,u) \in U_{1,r}} b_{(k,l,u)} F^{-1}(\omega_1 \hat{f}) \|_{L^2}^{2/3} \\
\leq \sum_{(l,u) : (k,l,u) \in U_{1,r}} \|\omega_2 \hat{g}\|_{L^2} \|\omega_3 \hat{h}\|_{L^2} \sum_{k : (k,l,u) \in U_{1,r}} b_{(k,l,u)} \omega_1 \hat{f} \|_{L^2}^{1/3} \\
\leq (\sum_{k \in \mathbb{Z}^d} \|\omega_2 \hat{g}\|_{L^2}^2 \|\omega_3 \hat{h}\|_{L^2}^2)^{1/3} \left( \sum_{(l,u) : (k,l,u) \in U_{1,r}} \|\omega_1 \hat{f}\|_{L^2} \sum_{k : (k,l,u) \in U_{1,r}} b_{(k,l,u)} \omega_1 \hat{f} \|_{L^2} \right)^{1/3} \\
\leq d (2^{3d/2})^{2/3} \|f\|_{L^2}^{2/3} (\sum_{k \in \mathbb{Z}^d} \|b_{(k,l,u)}\| \sum_{l \in \mathbb{Z}^d} \omega_2 \hat{g} \|_{L^2} \sum_{l \in \mathbb{Z}^d} \omega_3 \hat{h} \|_{L^2})^{1/3} \\
\leq d (2^{3d/2})^{2/3} \|f\|_{L^2}^{2/3} \|g\|_{L^2}^{2/3} \|h\|_{L^2}^{2/3} (\sum_{k \in \mathbb{Z}^d} \|b_{(k,l,u)}\| \sum_{l \in \mathbb{Z}^d} \omega_2 \hat{g} \|_{L^2} \sum_{l \in \mathbb{Z}^d} \omega_3 \hat{h} \|_{L^2})^{1/3} \\
\leq d (2^{3d/2})^{2/3} \|f\|_{L^2}^{2/3} \|g\|_{L^2}^{2/3} \|h\|_{L^2}^{2/3} (\sum_{k \in \mathbb{Z}^d} \|b_{(k,l,u)}\| \sum_{l \in \mathbb{Z}^d} \omega_2 \hat{g} \|_{L^2} \sum_{l \in \mathbb{Z}^d} \omega_3 \hat{h} \|_{L^2})^{1/3}. \quad (4.6)
$$

The tools used here are: Plancherel’s theorem; Hölder’s inequality; Cauchy-Schwarz inequality; Remark 4 and its 3.2 and 4.3. Taking the 3/2th power of 4.6 allows one to obtain:

$$
\|T_{m^{1,r}}(f, g, h)\|_{L^2}^{2/3} \leq d (2^{3d/2})^{2/3} \|f\|_{L^2}^{2/3} \|g\|_{L^2}^{2/3} \|h\|_{L^2}^{2/3} (\sum_{k \in \mathbb{Z}^d} \|b_{(k,l,u)}\| \sum_{l \in \mathbb{Z}^d} \omega_2 \hat{g} \|_{L^2} \sum_{l \in \mathbb{Z}^d} \omega_3 \hat{h} \|_{L^2})^{1/3}. \quad (4.7)
$$

Let $m^r := m^{1,r} + m^{2,r}$. Put 3.4 - which says that $\|b\|_{L^r} \leq C_0 2^{-(K+1+d)}$ - 4.2, 4.6, 4.7 altogether, one has:

$$
\|T_{m^r}\|_{L^2 \times L^2 \times L^2} \leq d (2^{3d/2})^{2/3} \|f\|_{L^2}^{2/3} \|g\|_{L^2}^{2/3} \|h\|_{L^2}^{2/3} (\sum_{k \in \mathbb{Z}^d} \|b_{(k,l,u)}\| \sum_{l \in \mathbb{Z}^d} \omega_2 \hat{g} \|_{L^2} \sum_{l \in \mathbb{Z}^d} \omega_3 \hat{h} \|_{L^2})^{1/3}. \quad (4.8)
$$
The above bound depends on \( r, j, G \). Hence in order for \( \| T_m \|_{L^2 \times L^2 \times L^2 \to L^{2/3}} < \infty \), one needs the corresponding powers in 4.8 to be summable. In other words, one should have

\[
q/3 - 1 < 0,
\]

and

\[
K + d + 1 > (5d/2 - qd/2)/(1 - q/3).
\]

Hence if one chooses \( 1 < q < 3 \) and \( K = \lfloor \frac{4d}{3} \rfloor + 1 =: M_q \), one can sum 4.8 in \( r \) then in \( j, G \) to obtain 3.6.

### 4.1 Remark

The trilinear model is a good example on how to extend to higher \( n \)-multilinear case. In the calculations of 4.4 above, when the count is along the "small slabs", the \( n - 1 \)-dimensional nature of \( U_{2,r} \) allows one to take out \( n - 1 \) factors. Conversely, in 4.6, when the 1-dimensional count for the large "slabs" are small, one can take out one factor first, leaving \( n - 1 \) factors in the \( L^{2/(n-1)} \) norm. Applying Hölder's inequality for these \( n - 1 \) factors yields each factor an \( L^2 \) norm. The remaining details are similar. See also Section 6.

The idea behind the composition of \( U_1, U_2, U_r \) is as follows. The "volume" (cardinality) of the set \( U_r \) is at most around \( 2^{q/3} \| |b||_{l^q} \| b \|_r \)^3. If one envisions this as a \( n \)-dimensional volume, then one can observe \( n - 1 \)-dimensional "slabs" that are particularly small or large \( (U_{2,r} \text{ or } U_1, \text{ respectively}) \). Intuitively, a \( n - 1 \)-dimensional slab is large if its cardinality is a bit more than \( \| U_r \|^{(n-1)/n} \). The count for large \( n - 1 \)-dimensional slabs can’t be so large. But this intuition also works for 1-dimensional columns that are particularly short or long; in which case, one can define a long column is a column whose size is a bit more than \( |U_r|^{1/n} \). It might be more intuitive to consider division of an \( n \)-dim object along \( n - 1 \)-dim sub-objects instead, even though the sets considered here are all discrete. Suppose:

\[
U_r = \{(k, l, u) \in 2^d \times 2^d \times 2^d : 2^{-r-1} |b|_l < |b|_{k,l,u} \} \leq 2^{-r} |b|_l \},
\]

\[
U_{1, r} = \{(k, l, u) \in U_r : \text{card} \{u : (k, l, u) \in U_r\} \geq 2^{q/3} \| |b|_{l^q} \| b \|_r^{-q/3} \},
\]

\[
U_{2, r} = \{(k, l, u) \in U_r : \text{card} \{u : (k, l, u) \in U_r\} < 2^{q/3} \| |b|_{l^q} \| b \|_r^{-q/3} \}.
\]

Let \( E = \{(k, l) : (k, l, u) \in U_{1, r} \text{ for some } u\} \). Then from the definitions of \( U_r \) and \( U_{1, r} \),

\[
\text{card} E \leq 2^{q/3} \| |b|_{l^q} \| b \|_r^{-q/3}.
\]

One has,

\[
|T_{m^r}(f, g, h)|_{L^{2/3}} \leq \| \sum_{(k, l, u) \in U_{2, r}} b_{(k, l, u)} F^{-1}(\omega, \hat{f}) F^{-1}(\omega, \hat{g}) F^{-1}(\omega, \hat{h}) \|_{L^{2/3}}^{2/3}
\]

\[
\leq \| \sum_{u \in 2^d} |F^{-1}(\omega, \hat{u})| \cdot \sum_{(k, l, u) \in U_{2, r}} b_{(k, l, u)} F^{-1}(\omega, \hat{f}) F^{-1}(\omega, \hat{g}) F^{-1}(\omega, \hat{h}) \|_{L^{2/3}}^{2/3}
\]

\[
\leq d \| \sum_{u \in 2^d} \omega \cdot |\hat{u}|_{L^2}^{2/3} \cdot \left( \sum_{(k, l, u) \in U_{2, r}} b_{(k, l, u)} F^{-1}(\omega, \hat{f}) F^{-1}(\omega, \hat{g}) F^{-1}(\omega, \hat{h}) \|_{L^1} \right)^{1/3}
\]

\[
\leq d \left( 2^{d/2} \right)^{2/3} \| \hat{f} \|_{L^2}^{2/3} \cdot \| \hat{g} \|_{L^2}^{2/3} \cdot \| \hat{h} \|_{L^2}^{2/3} \cdot \left( \sum_{u \in 2^d} |b_{(k, l, u)}| \right)^{2/3}
\]

\[
\leq \left( 2^{d/2} \right)^{2/3} \left( 2^{-r} \| b \|_r \right)^{2/3} \cdot \left( 2^{q/3} \| |b|_{l^q} \| b \|_r^{-q/3} \right) \cdot \left( \sum_{u \in 2^d} |b_{(k, l, u)}| \right)^{2/3}
\]

which yields

\[
|T_{m^r}(f, g, h)|_{L^{2/3}} \leq d \left( 2^{3d/2} \right)^{2/3} \left( 2^{-r} \| b \|_r \right)^{2/3} \cdot \left( 2^{q/3} \| |b|_{l^q} \| b \|_r^{-q/3} \right)^{2/3} \cdot \| f \|_{L^2} \cdot \| g \|_{L^2} \cdot \| h \|_{L^2}.
\]
For the operator $T_{m^1, r}$,

\[
\|T_{m^1, r}(f, g, h)\|_{L^{2/3}}^{2/3} = \left\| \sum_{(k, l, u) \in U_{1, r}} b_{(k, l, u)} \mathcal{F}^{-1}(\omega_1 k \hat{f}) \mathcal{F}^{-1}(\omega_2 l \hat{g}) \mathcal{F}^{-1}(\omega_3 u \hat{h}) \right\|_{L^{2/3}}^{2/3}
\]

\[
\leq \left\| \sum_{(k, l) \in E} |\mathcal{F}^{-1}(\omega_1 k \hat{f})||\mathcal{F}^{-1}(\omega_2 l \hat{g})| \right\| \sum_{u: (k, l, u) \in U_{1, r}} b_{(k, l, u)} \mathcal{F}^{-1}(\omega_3 u \hat{h}) \right\|_{L^{2/3}}^{2/3}
\]

\[
\leq \left( \sum_{(k, l) \in E} \|\omega_1 k \hat{f}\|_{L^2} \|\omega_2 l \hat{g}\|_{L^2} \right)^{1/3} \left( \sum_{u: (k, l, u) \in U_{1, r}} \|b_{(k, l, u)}\|_{L^2} \right)^{2/3}
\]

\[
\leq_d (2^{2j/2}2^3(2^{-r}\|b\|_v)^2/3 |f|_{L^2}^{2/3} |g|_{L^2}^{2/3} |h|_{L^2}^{2/3} (\sum_{(k, l) \in E})^{1/3}
\]

\[
\leq q_d (2^{2j/2}2^3(2^{-r}\|b\|_v)^2/3 |f|_{L^2}^{2/3} |g|_{L^2}^{2/3} |h|_{L^2}^{2/3} (2^{2q/3} |b|_{v}^{2q/3} |b|_{v}^{-2q/3})^{1/3},
\]

which yields

\[
\|T_{m^1, r}(f, g, h)\|_{L^{2/3}} \leq q_d 2^{9j/2}2^3 |b|_{v}^{2q/3} |b|_{v}^{-2q/3} |f|_{L^2}^{2} |g|_{L^2}^{2} |h|_{L^2}^{2}.
\]

5 Necessity

Let $\phi$ be a Schwartz function on $\mathbb{R}$ whose Fourier transform has support in a symmetric interval $I$ and let $\{a_j\}_{j \geq 1}, \{b_j\}_{j \geq 1}, \{c_j\}_{j \geq 1}$ be two sequences of nonnegative numbers with only finitely many nonzero terms. This function $\phi$ is not related to the wavelet functions in the previous sections. Define $f, g, h$ by

\[
\hat{f}(\xi) = \sum_{j \geq 1} a_j \phi(\xi_1 - j) \prod_{r \geq 2} \phi(\xi_r - 1), \quad (5.1)
\]

\[
\hat{g}(\eta) = \sum_{j \geq 1} b_j \phi(\eta_1 - j) \prod_{r \geq 2} \phi(\eta_r - 1), \quad (5.2)
\]

\[
\hat{h}(\delta) = \sum_{j \geq 1} c_j \phi(\delta_1 - j) \prod_{r \geq 2} \phi(\delta_r - 1). \quad (5.3)
\]

Then $f, g, h$ are Schwartz functions whose $L^2$ norms are bounded by a constant multiple of $\left(\sum_{j \geq 1} a_j^2\right)^{1/2}, \left(\sum_{j \geq 1} b_j^2\right)^{1/2}, \left(\sum_{j \geq 1} c_j^2\right)^{1/2}$, respectively.

Let $\{s_j(t)\}_{j \geq 1}$ denote the sequence of Rademacher functions [3]. Let $\{v_j\}_{j \geq 1}$ be a bounded sequence of nonnegative numbers. For $t \in [0, 1]$, consider $m_t$ by

\[
m_t(\xi, \eta, \delta) = \sum_{j \geq 1} \sum_{k \geq 1} \sum_{l \geq 1} v_{j+k+l}s_j s_k s_l \psi(\xi_1 - j) \psi(\eta_1 - k) \psi(\delta_1 - l) \prod_{r \geq 2} \psi(\xi_r - 1) \psi(\eta_r - 1) \psi(\delta_r - 1). \quad (5.4)
\]

Here $\psi$ is a smooth function on $\mathbb{R}$ supported in the interval $J = 10I$ and assuming value $1$ in $cJ$, with $c$ small enough so that $I \subset cJ$. Then from the definitions $5.1, 5.2, 5.3$,

\[
T_{m_t}(f, g, h)(x) = \sum_{j \geq 1} \sum_{k \geq 1} \sum_{l \geq 1} a_j b_k c_l v_{j+k+l} s_j s_k s_l \phi(x_1)^3 e^{2\pi i x_1 (j+k+l)} \prod_{r \geq 2} e^{6\pi i x_r} \phi(x_r)^3
\]

\[
= \sum_{l \geq 3} v_{Sl}(t) \phi(x_1)^3 e^{2\pi i x_1 l} \sum_{j=1}^{l-1} \sum_{k=1}^{l-j-1} a_j b_k c_{l-j-k} \prod_{r \geq 2} e^{6\pi i x_r} \phi(x_r)^3. \quad (5.5)
\]
Utilizing 5.5, Khinchin’s inequality [3] and Fubini’s theorem, one then has

\[
\int_0^1 \| T_{m}(f, g, h) \|_{L_{2/3}^2}^{2/3} \, dt \\
= \left( \int_R |\phi(y)|^2 \, dy \right) \left( \int_0^1 \left| \sum_{j \geq 3} v_j \phi(x_j) \sum_{k=1}^{l-1} a_j b_k c_{l-j-k} \right|^{2/3} \, dx_1 \right)^{1/3} \\
\approx c_{L_2} \int_R \left( \sum_{j \geq 3} \left| \sum_{k=1}^{l-1} a_j b_k c_{l-j-k} \right|^2 \right) \, dx_1 \approx c_{L_2} \left( \sum_{j \geq 3} \left( \sum_{k=1}^{l-1} a_j b_k c_{l-j-k} \right)^2 \right)^{1/3}.
\]

Fix a positive integer \( N \geq 2 \) and set \( a_j^N = b_j^N = c_j^N = 2^{-N/2} \) if \( j = 2^N, \ldots, 2^{N+1} - 1 \) and zero otherwise. Observe that with this agreement, \( a_j b_k c_{l-j-k} \) is only nonzero for a finite number of terms for each \( l \). Moreover, \( \sum_{j \geq 3} \sum_{k=1}^{l-1} a_j b_k c_{l-j-k} = 0 \) if \( l > 2^{N+3} \) or if \( l < 2^N \). In effect,

\[
\sum_{j \geq 3} \sum_{k=1}^{l-1} a_j b_k c_{l-j-k} \geq c_{L_2} \sum_{j \geq 3} 2^{-3N/2} \geq c2^{N+2} > 0.
\]

In 5.7 above, the constants \( C, c \) in the first instance can be found as follows. Independently of \( N \), \( a_j, b_k, c_{l-j-k} \) are "activated" (nonzero) for ranges of \( j, k, l \) that have the same length. Hence within those ranges of \( j, k \), one can fit in a "box" of sizes \([2^N, C2^N] \times [c2^N, C2^N] \), for some appropriate \( c, C \). The constant \( c \) in the second instance of 5.7 is unrelated to the previous one.

Define \( v_l = (l - 1)^{-1} (\log(l - 1))^{1/2} \) and define \( f^N, g^N, h^N \) similarly to \( f, g, h \), respectively, only with \( a_j, b_j, c_j \) being replaced by \( a_j^N, b_j^N, c_j^N \), respectively. Then from 5.6, 5.7 and simple calculus,

\[
\int_0^1 \| T_{m}(f, g, h) \|_{L_{2/3}^2}^{2/3} \, dt \geq c_{L_2} 2^{-N/3} \left( \sum_{2^N \leq l \leq C2^N} (l - 1)^{-2} \log(l - 1) \right)^{1/3} \\
\geq c_{L_2} 2^{-N/3} (\log(C2^N))^{1/3} \left( \sum_{2^N \leq l \leq C2^N} (l - 1)^{-2} \right)^{1/3} \\
\geq 2^{N/3} c N^{1/3} \left( \int_{2^N}^{C2^N} y^{-2} \, dy \right)^{1/3} \geq c N^{1/3}.
\]

Above, \( c, C \) denote positive numbers that possibly change from one instance to the next, with \( C > c \). This means that for every \( N \geq 2 \) one can find \( t_N \in [0, 1] \) such that

\[
\| T_{m_{t_N}}(f, g, h) \|_{L_{2/3}^2} \geq C N^{1/2}
\]

with \( C \) being independent of \( N \).

Define:

\[
m(\xi, \eta, \delta) = \sum_{j \geq 1} \sum_{k \geq 1} \sum_{l \geq 1} v_{j+k+l} \sigma_{j+k+l} \psi(\xi_j - j) \psi(\eta_k - k) \psi(\delta_l - l) \prod_{r \geq 2} \psi(\xi_r - 1) \psi(\eta_r - 1) \psi(\delta_r - 1)
\]

with \( \sigma_l = s_l(t_N) \) if \( N \geq 2 \) and \( 2^{N+1} \leq l \leq 2^{N+3} \). Then a quick calculation shows that,

\[
T_m(f^N, g^N, h^N)(x) = T_{m_{t_N}}(f^N, g^N, h^N)(x),
\]

which means, from 5.8,

\[
\| T_m(f^N, g^N, h^N) \|_{L_{2/3}} \approx N^{1/2}.
\]

Hence \( T_m \) is unbounded on \( L^2 \times L^2 \times L^2 \rightarrow L^{2/3} \). On the other hand with the definition, \( m \) is a smooth function with bounded derivatives and moreover,

\[
\| m \|_{L^q} \approx q \left( \sum_{j \geq 3} v_j^q (l - 1)^2 \right)^{1/q} = \left( \sum_{j \geq 3} (\log(l - 1))^q (l - 1)^{2-q} \right)^{1/q}
\]

which, by comparison with the divergent series \( \sum_{n \geq 3} (1/n) \), shows that \( m \in \bigcap_{q \geq 3} L^q \) and \( m \notin L^q \) if \( q \leq 3 \).
6 Remark on extension to $n > 3$

Let $n \geq 3$. Let $m(\xi_1, \ldots, \xi_n)$ be a function on $\mathbb{R}^{d \otimes n}$. Define an operator $T$ as,

$$T_m(f_1, \ldots, f_n)(x) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} m(\xi_1, \ldots, \xi_n) f_1(\xi_1) \cdots f_n(\xi_n) e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_n)} d\xi_1 \cdots d\xi_n.$$  

If one follows the steps taken in 4.4.6 closely, one can also deduce similarly that for $n \geq 3$,

$$\|T_m\|_{L_2^d \times \cdots \times L^2 \rightarrow L^{2/n}} < \infty,$$

if $m \in L^q(\mathbb{R}^{d \otimes n}) \cap C^{M_1}(\mathbb{R}^{d \otimes n})$, with $M_q = \left\lfloor \frac{4M}{qA(n)} \right\rfloor + 1$ for some $1 \leq q < 4$ and $A(n) = \frac{2n}{n-1}$, and some constant $A_q$ and if

$$\|\partial^\alpha m\|_{L^2} \leq C_0$$

for all $|\alpha| \leq M_q$. In fact,

$$\|T_m\|_{L_2^d \times \cdots \times L^2 \rightarrow L^{2/n}} \lesssim C_{q,d,q} \|m\|_{q/A(n)}.$$  

To see this, for instance, one can define similarly as in Section 4 that,

$$U_r = \{ (k_1, \ldots, k_n) \in \mathbb{Z}^{d \otimes n} : 2^{-r} \|b\|_{\ell^r} < |b_{(k_1, \ldots, k_n)}| \leq 2^{-r} \|b\|_{\ell^r} \},$$

$$U_{1,r} = \{ (k_1, \ldots, k_n) \in U_r : \text{card} \{ (k_2, \ldots, k_n) : (k_1, \ldots, k_n) \in U_r \} \geq 2^{q/n} \|b\|_{\ell^q} \|b\|_{\ell^{q/n}} \}$$

and

$$U_{2,r} = \{ (k_1, \ldots, k_n) \in U_r : \text{card} \{ (k_2, \ldots, k_n) : (k_1, \ldots, k_n) \in U_r \} < 2^{q/n} \|b\|_{\ell^q} \|b\|_{\ell^{q/n}} \}.$$  

One can follow the steps taken in 4.4, and in particular in the second to last step, the quantity

$$2^{-2rz/n} \left( \sum_{(k_2, \ldots, k_n) : (k_1, \ldots, k_n) \in U_{2,r}} 1 \right)^{1/n}$$

gives the power $2^{r(q(n-1)/2n-1)}$ - after taking $n/2$th power both sides of the inequality, which implies that $q < 2n/(n-1)$ in order for the expression for $\|T_m\|_{L_2^d \times \cdots \times L^2 \rightarrow L^{2/n}}$ to be summable in $r$. The same can be said for the case in 4.6. Hence $q < 2n/(n-1)$ is sufficient. One can argue similarly for the derivation of the dominant $\|m\|_{q/A(n)}$. Since the extension is mechanic, further details are omitted.

As discussed, the lack of duality theory prevents further extension from this approach to any other parts of the Banach range $1/p_1 + \cdots + 1/p_n = 1/r$ for the operator norm $\|T_m\|_{L^{1/p_1} \times \cdots \times L^{1/p_n} \rightarrow L^r}$.

It might be possible to construct systematically a counterexample multiplier $m \in \bigcap_{q>A(n)} L^q$ for every $n$ for the sufficiency direction, to show that indeed $q = A(n)$ is the optimal Lebesgue exponent in the case of $n$-linear multiplier operator. The author is currently not concerned with this direction.

7 Appendix

The following utilizes the material in [9]. First one should see the definition of the function spaces $F_{q,p}^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ [9]. The important point here is,

$$F_{q,p}^0 = L^q.$$  

See the remark 1.65 in [9]. One also has the following fact, which is a rephrasing of parts of Theorem 1.64 in [9], using the notations already introduced in this paper:

**Theorem A.** Let $\Phi_n^{j,G}$ be as in 3.1 with sufficient smoothness (see Remark 3). Let $f \in F_{q,p}^s(\mathbb{R}^d)$. Then $f$ can be represented as,

$$f = \sum_{(j,G,n)} b_n^{j,G} \Phi_n^{j,G}.$$  

Furthermore, this representation is unique and the map

$$I : f \mapsto \{ b_n^{j,G} \}$$  

is.
is an isomorphic map of $F_{qp}^s$ onto $f_{qp}^s$. The latter is a sequence space whose elements $\lambda = \{\lambda_j\}_{j,n}^G$ are given the following semi-norm:

$$
\|\lambda\| = \left( \sum_{(j,G,n)} |2^{jn/2} \lambda_j^G \chi_{Q,n}|^p \right)^{1/q} \quad \text{in} \mathcal{L}^q(\mathbb{R}^d). 
$$ (7.2)

**Theorem A, 7.1, 7.2 imply 4.1.**

### 7.1 Lemma 5

It was mentioned in [7] that the proof of Lemma 5 is essentially done in Lemma 7 of [6]. Since the proof in [6] is quite involved, only its ideas will be presented here. This discussion aims to guarantee that although the version of the lemma stated in [7] was stated for $2d$, it’s immaterial to change it to $3d$ or any $nd$.

First, a small point here is that, if one follows the argument there then because of the change in dimensions one should arrive at

$$
|\langle \Phi_{n,j}^G, m \rangle| \leq C_0 \ 2^{-(K+1+3d/2)j}
$$

for the conclusion. But clearly, $2^{-3d/2} \leq 2^{-d}$.

If the multiplier had any decay of the form,

$$
|\partial^\alpha m(\vec{x})| \leq (1 + |\vec{x}|)^M_1
$$

for $|\alpha| \leq K$, for some large $K, M_1$, then one can utilize this and the moment cancellation properties $\Phi_{n,j}^G$ ($j \neq 0$) and apply the material in the Appendix B.2 of [4] to get the desired decay 3.4 of the wavelet transforms of $m$. The moment cancellation properties of $\Phi_{n,j}^G$ when $j \neq 0$ come from those of $\phi_M$ in its definition. There are no cancellation assumptions when $j = 0$, but then one can use the decay 7.1 and those of the wavelets to apply the material in Appendix B.1 of [4] instead.

Our $m$ is assumed of no decay 7.3. Hence one can decompose $m$ into parts,

$$
m = \sum_i m_i
$$

with $m_i$ being defined as in [6]. Then $m_i = m_0(2^i \cdot)$. Then $m_0$ does possesses the decay 7.3 [6]. One can arrive at 3.4 for $m_0$. Now for $i < 0$, one has through change of variables,

$$
\langle \Phi_{n,j}^G, m_0(2^i \cdot) \rangle = 2^{ic} \langle \Phi_{n,j+i}^G, m_0 \rangle = 2^{-c|i|} \langle \Phi_{n,j+i}^G, m_0 \rangle.
$$ (7.4)

Here $c = c(d) = d/2$ for any dimension $d$ of $\vec{x}$. Then one can take $j \geq 0$ in 7.4 large enough so that $k = j + i \in \mathbb{N}_0$. For $i > 0$, one uses,

$$
\langle \Phi_{n,j}^G, m_0(2^i \cdot) \rangle = 2^{-c|i|} \langle \Phi_{n,|i|}^G, m_0 \rangle.
$$

Putting all these back, one gets the desired conclusion for $m$. None of the tools mentioned here is particular to any dimension $d$. Hence the conclusion 3.4 holds for $3d$ and for any $nd$.

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