QUANTUM FLUCTUATIONS IN COSMOLOGY
AND HOW THEY LEAD TO A MULTIVERSE*

Alan H. Guth**

Center for Theoretical Physics,
Laboratory for Nuclear Science,
and Department of Physics,
Massachusetts Institute of Technology,
Cambridge, MA 02139, USA
**E-mail: guth@ctp.mit.edu
web.mit.edu

This article discusses density perturbations in inflationary models, offering a pedagogical description of how these perturbations are generated by quantum fluctuations in the early universe. A key feature of inflation is that rapid expansion can stretch microscopic fluctuations to cosmological proportions. I discuss also another important consequence of quantum fluctuations: the fact that almost all inflationary models become eternal, so that once inflation starts, it never stops.

I. INTRODUCTION

I have been asked to describe quantum fluctuations in cosmology, which I find a fascinating topic. It is a dramatic demonstration that the quantum theory that was developed by studying the hydrogen atom can be applied on larger and larger scales. Here we are applying quantum theory to the universe in its entirety, at time scales of order $10^{-36}$ second, and it all sounds incredibly fantastic. But the shocking thing is that it works, at least in the sense that it gives answers for important questions that agree to very good precision with what is actually measured. In addition to discussing the density perturbations that we can detect, however, I want to also discuss another important aspect of quantum fluctuations: specifically, quantum fluctuations in cosmology appear, in almost all our models, to lead to eternal inflation and an infinite multiverse. This is a rather mind-boggling concept, but given our success in calculating the fluctuations observed in the cosmic microwave background (CMB), it should make good sense to consider the other consequences of quantum fluctuations in the early universe. Thus, I think it is time to take the multiverse idea seriously, as a real possibility. The inhomogeneities that lead to eternal inflation are nothing more than the long-wavelength tail of the density perturbations that we see directly in the CMB.

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II. ORIGIN OF DENSITY PERTURBATIONS DURING THE INFLATIONARY ERA

The idea that quantum fluctuations might be the origin of structure in the universe goes back at least as far as a 1965 paper by Sakharov [1]. In the context of inflationary models, the detailed predictions are model-dependent, but a wide range of simple models give generic predictions which are in excellent agreement with observations. In this section I will give a pedagogical explanation of how these predictions arise, based on the time-delay formalism that was used in the paper I wrote with S.-Y. Pi [2]. This formalism, which we learned from Stephen Hawking, is the simplest to understand, and it is completely adequate for the dominant perturbations in single-field, slow-roll inflation. More sophisticated approaches are needed, however, to study multifield models or models that violate the slow-roll approximation, or to study extremely subdominant effects in single-field, slow-roll models. Even for multifield inflation, however, some of the simplicity of the time-delay formalism can be maintained by the use of the so-called $\delta N$ formalism [11, 12]. There are a number of reviews [12–14] and textbooks [15–18] that give a much more thorough discussion of density perturbations in inflationary models than is appropriate here.

Inflation [20–22] takes place when a scalar field has a large potential energy density. A

1 The original work on density perturbations arising from scalar-field-driven inflation centered around the Nuffield Workshop on the Very Early Universe, Cambridge, U.K., June-July 1982. Four papers came out of that workshop: Refs. [3], [2], [4], and [5]. Ref. [6] introduced a formalism significantly more general than the previous papers. These papers tracked the perturbations from their quantum origin through Hubble exit, reheating, and Hubble reentry. Earlier Mukhanov and Chibisov [6] had revived Sakharov’s idea in a modern context, studying the conformally flat perturbations generated during the inflationary phase of the Starobinsky model [8]. They developed a method of quantizing the metric fluctuations, a method more sophisticated than is needed for the simpler models of Refs. [2]–[5], and gave a formula (without derivation) for the final spectrum. For various reasons the calculations showing how the conformally flat fluctuations during inflation evolve to the conformally Newtonian fluctuations after inflation were never published, until the problem was reconsidered later in Refs. [9] and [10]. The precise answer obtained in Ref. [11], $Q(k) = \sqrt{24\pi G M \left(1 + \frac{1}{2} \ln \left(\frac{H}{k}\right)\right)}$, has not (to my knowledge) been confirmed in any modern paper. However, the fact that $Q(k)$ is proportional to $\ln(\text{const}/k)$ has been confirmed, showing that the 1981 paper by Mukhanov and Chibisov did correctly calculate what we now call $n_s$ (as was pointed out in Ref. [12]).
straightforward application of Noether’s theorem \[19\] gives the energy-momentum tensor of a canonically normalized scalar field as

$$T^\mu{}_{\nu} = pg^\mu{}_{\nu} + (p + \rho)u^\mu u^\nu ,$$

(1)

where

$$\rho = -\frac{1}{2}g^\mu{}_{\nu} \partial^\nu \varphi \partial^\mu \varphi + V(\varphi) ,$$

(2)

$$p = -\frac{1}{2}g^\mu{}_{\nu} \partial^\nu \varphi \partial^\mu \varphi - V(\varphi) ,$$

(3)

$$u^\mu = \left(-g^{\lambda\sigma} \partial^\lambda \varphi \partial^\sigma \varphi\right)^{-1/2} g^\mu{}_{\rho} \partial^\rho \varphi ,$$

(4)

where $\partial_\mu \equiv \partial/\partial x^\mu$. So, as long as the energy of the state is dominated by $V(\varphi)$, Eq. (3) guarantees that the pressure is large and negative. Einstein’s equations imply that negative pressure creates repulsive gravity, so any state whose energy is dominated by the potential energy of a scalar field will drive inflation. There are two basic scenarios — one where $\varphi$ starts at the top of a hill (new inflation \[21, 22\]), and one where it starts high on a hill and rolls down (chaotic inflation \[23\]); see Fig. (1). Either scenario is successful, and for the density perturbation calculation we can treat them at the same time. We use comoving coordinates, with a background metric describing a flat Friedmann-Robertson-Walker (FRW) universe:

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j ,$$

(5)

where $a(t)$ is the scale factor. Objects moving with the expansion of the universe are at rest in this coordinate system, with the expansion described solely by $a(t)$: when $a(t)$ doubles, all the distances in the universe double. In this metric the Klein-Gordon equation for a scalar field is given by

$$\ddot{\varphi} + 3H(\varphi, \dot{\varphi}) \dot{\varphi} - \frac{1}{a^2(t)} \nabla^2 \varphi = -\frac{\partial V(\varphi)}{\partial \varphi} ,$$

(6)

where an overdot indicates differentiation with respect to time $t$, and $\nabla^2$ is the Laplacian, $\sum_i \partial^2/\partial(x^i)^2$, with respect to the coordinates $x^i$. The equation is identical to the Klein-Gordon equation in Minkowski space, except that there is a drag term, $3H\dot{\varphi}$, which can be expected, since the energy density must fall if the universe is expanding. In addition, each spatial gradient is modified by $1/a(t)$, which converts the derivative to the current scale of spatial distance. The Hubble expansion rate $H \equiv \dot{a}/a$ is given by the Friedmann equation for a flat universe,

$$H^2 = \frac{8\pi}{3} G \left(\frac{1}{2} \dot{\varphi}^2 + V(\varphi)\right) .$$

(7)

In this language, the repulsive effect of the negative pressure that was mentioned above can be seen in the equation for the acceleration of the expansion,

$$\ddot{a} = -\frac{4\pi}{3} G(\rho + 3p)a .$$

(8)

If $p = -\rho$, as one finds when $V(\varphi)$ dominates, this equation gives $\ddot{a} = (8\pi/3)G\rho a$.

Using an assumption called the slow-roll approximation, which is valid for a large range of inflationary models, we can ignore the $\ddot{\varphi}$ term of Eq. (6) and the $\dot{\varphi}^2$ term in Eq. (7). In
addition, at sufficiently late times the Laplacian term can be neglected, since it is suppressed by $1/a^2(t)$. We are then left with a very simple differential equation,

$$3H(\varphi)\dot{\varphi} = -\frac{\partial V}{\partial \varphi},$$

which has a one-parameter class of solutions. That one parameter is itself trivial — it is a time offset. Given one solution $\varphi_0(t)$, the general solution can be written as $\varphi_0(t - \delta t)$, where $\delta t$ is independent of $t$. Since the differential equation (9) has no spatial derivatives, $\delta t$ can depend on position, so the most general solution can be written as

$$\varphi(\vec{x}, t) = \varphi_0(t - \delta t(\vec{x})).$$

Since we are interested in developing a first order perturbation theory, we can expand about $\varphi_0(t)$,

$$\varphi(\vec{x}, t) \equiv \varphi_0(t) + \delta \varphi(\vec{x}, t) = \varphi_0(t) - \dot{\varphi}_0(t) \delta t(\vec{x}),$$

so

$$\delta t(\vec{x}) = -\frac{\delta \varphi(\vec{x}, t)}{\dot{\varphi}_0(t)}.$$

Even though the numerator and denominator of the above expression both depend on time, the quotient does not. Thus at late times (within the inflationary era) — times late enough for Eq. (9) to be accurate — the nonuniformities of the rolling scalar field are completely characterized by a time-independent time delay.$^2$ It is useful to define a dimensionless measure of the time delay,

$$\delta N = H\delta t,$$

$^2$ The description of the perturbations at late times by a time-independent time delay $\delta t(\vec{x})$ is in fact much more robust than the approximation that $\ddot{\varphi}$ can be neglected. It is a consequence of the Hubble drag term, and will hold at sufficiently late times in any single-field model for which the slow-roll approximation is valid for more than a few $e$-folds. To see this, consider Eq. (6), with $H$ taken to be an arbitrary function of $\varphi$ and $\dot{\varphi}$. We will neglect the Laplacian term, since it is suppressed by $1/a^2(t)$, and we are interested in late times. Then, for each value of $\vec{x}$ there is a two-parameter class of solutions to this second order ordinary differential equation. To see the effect of the damping, suppose that we know the unperturbed solution, $\varphi_0(t)$, and a nearby solution, $\varphi_0(t) + \delta \varphi(t)$, where $\delta \varphi(t)$ is to be treated to first order. $\delta \varphi$ can depend on $\vec{x}$, but we suppress the argument because we consider one value of $\vec{x}$ at a time. We then find that $\dot{\delta \varphi}(t)$ and $\dot{\varphi}_0(t)$ obey the same differential equation. If we construct the Wronskian $W(t) \equiv \dot{\varphi}_0 \delta \varphi - \dot{\varphi}_0 \delta \varphi$, we find that

$$\dot{W} = -3 \left( H + \frac{\partial H}{\partial \dot{\varphi}} \dot{\varphi}_0 \right) W,$$

the solution to which is

$$W(t) = W_0 \exp \left\{ -3 \int_{t_0}^{t} dt \left( H + \frac{\partial H}{\partial \dot{\varphi}} \dot{\varphi}_0 \right) \right\}.$$

Thus $W(t)$ falls off roughly as $e^{-3Ht}$ or faster ($\dot{\varphi}_0 \partial H/\partial \dot{\varphi} > 0$), and so can be neglected after just a few $e$-folds of expansion. Then note that

$$\frac{d}{dt} \left( \frac{\delta \varphi}{\varphi_0} \right) = \frac{W(t)}{\varphi_0^2} \dot{\varphi}_0,$$

while in the slow-roll regime $\dot{\varphi}_0^2$ is approximately constant — from Eq. (9) one can show that the fractional change in $\dot{\varphi}_0^2$ during one Hubble time ($H^{-1}$) is approximately $2(\epsilon - \eta)$, as defined in Eqs. (14) and (15). Thus the time derivative of the ratio $\delta \varphi/\varphi_0$ falls off as $e^{-3Ht}$ or faster, implying that the time delay rapidly
which can be interpreted as the number of $e$-folds of inflation by which the field is advanced or retarded.

To justify the slow-roll approximation, we must adopt restrictions on the form of the potential energy function $V(\varphi)$. The slow-roll approximation is equivalent to saying that the field $\varphi$ evolves approximately at the drag-force limited velocity, where the drag force equals the applied force, with inertia playing only a negligible role. (This would be called the terminal velocity, except that it can change slowly with time.) From the first two terms of Eq. (6) one can see that the velocity approaches the drag-limited value with a time constant of order $H^{-1}$. Thus, for the field to evolve at the drag-limited velocity, it is essential that neither the drag coefficient nor the applied force changes significantly during a time of order $H^{-1}$. Thus we want to insist that $H^{-1}|\dot{H}| \ll H$, and that $H^{-1}|(\partial^2 V/\partial \varphi^2)\dot{\varphi}| \ll |\partial V/\partial \varphi|$. Using Eq. (9) to approximate $\dot{\varphi}$, these two conditions can be expressed in terms of the two slow-roll parameters \[\begin{align*}
\epsilon &\equiv \frac{1}{16\pi G} \left(\frac{V'}{V}\right)^2 \approx -\frac{\dot{H}}{H^2}, \quad 0 < \epsilon \ll 1, \\
\eta &\equiv \frac{1}{8\pi G} \frac{V''}{V} \approx -\frac{V''\dot{\varphi}}{HV'} \approx \epsilon - \frac{\ddot{H}}{2HH}, \quad |\eta| \ll 1,
\end{align*}\] where a prime denotes a derivative with respect to $\varphi$. Note that these slow-roll conditions do not by themselves guarantee that $\varphi$ will evolve at drag-limited velocity, because a large initial velocity will take time before it approaches the drag-limited value. But the slow-roll conditions do guarantee that for times long compared to $H^{-1}$, $\varphi$ will evolve at very nearly the drag-limited velocity.

To proceed, I will make two approximations that will simplify the problem enormously, but which are nonetheless extremely accurate for single-field slow-roll inflation. First, we will neglect all perturbations of the metric until the time when inflation ends. That is, until inflation ends we treat the scalar field as a quantum field in a fixed de Sitter space background. Thus, we will be ignoring the fluctuations in the energy-momentum tensor of the scalar field, since we are not allowing them to perturb the metric. However, we will calculate the fluctuations in the scalar field itself, as described by the time delay $\delta t(\vec{x})$. Since the scalar field is driving the inflation, the time delay $\delta t(\vec{x})$ measures the variation in the time at which inflation ends at different places in space. The amount of energy that is released at the end of inflation is much larger than the energy-momentum tensor fluctuations during inflation, so the spatial variation of the timing of this energy release becomes the dominant source of the density perturbations that persist at later times. To describe this release of energy, we make our second approximation. We will treat the ending of inflation...
as instantaneous. We will assume that the potential energy of the inflaton field is converted instantaneously into the thermal radiation of effectively massless particles, beginning the radiation-dominated era of cosmological history.

I will give heuristic justifications for these approximations, but I am not aware of a more rigorous justification that can be explained without developing an understanding of what happens when these approximations are avoided, and then the problem requires a much more detailed analysis. Such analyses have of course been done and are even described in textbooks. The answer that we will obtain agrees with the textbooks \[15, 16, 18\], for the single-field slow-roll case, down to the last factor of \(\sqrt{\pi}\). While the textbooks corroborate the answer that we will obtain, the methods are sufficiently different so that very little light is shed on the approximations described here. In a future publication\[25\], I will attempt a more detailed justification.

For a given theory we can calculate \(\varphi_0(t)\) by solving an ordinary differential equation, so Eq. (12) reduces the problem of calculating \(\delta t(\vec{x})\) to that of calculating the fluctuations of the scalar field, \(\delta \varphi(\vec{x}, t)\). This is a problem in quantum field theory, albeit quantum field theory in curved spacetime. The calculations closely resemble the familiar quantum field theory calculations for Minkowski space, but there is one point in the calculation where cosmology rears its head. While Minkowski space has a well-defined vacuum state, which is the starting point for most calculations, it is less clear what quantum state should be used to describe the fields evolving in de Sitter space, where the time dependence prevents the existence of a conserved total energy. In principle the quantum state is determined by the initial conditions for the universe, about which we know very little. However, while we do not know the quantum state of the early universe, there is a very natural choice,
corresponding at least locally to the concept of a vacuum state. To understand this choice, recall that we are interested in an exponentially expanding space, the de Sitter spacetime of inflation, so to a good approximation \( a(t) \propto e^{Ht} \), where \( H \) is constant. If we now treat the inflaton field \( \varphi(\vec{x}, t) \) as a quantum operator, we can as usual consider its Fourier transform:

\[
\varphi(\vec{x}, t) = \frac{1}{(2\pi)^3} \int d^3k e^{i\vec{k} \cdot \vec{x}} \tilde{\varphi}(\vec{k}, t). \tag{16}
\]

For a free quantum field theory in Minkowski spacetime, \( \tilde{\varphi}(\vec{k}, t) \) would be the sum of an annihilation operator term for particles of momentum \( \vec{k} \) and a creation operator term for particles of momentum \( -\vec{k} \), each corresponding to a de Broglie wavelength \( \lambda = 2\pi/|\vec{k}| \).

For an FRW spacetime, since the Fourier transform is defined in terms of the comoving coordinates \( \vec{x} \), the physical wavelength for a mode \( \vec{k} \) is not constant, but is given by

\[
\lambda_{\text{phys}}(t) = a(t) \frac{2\pi}{|\vec{k}|}. \tag{17}
\]

In other words, each mode is stretched as the universe expands. Thus, if we follow any mode backwards in time, it will have a shorter and shorter wavelength and a higher and higher frequency. The Hubble expansion rate \( H \) is approximately constant during inflation, so at very early times \( H \) is very small compared to the frequency, and hence is negligible. Thus, any given mode behaves at asymptotically early times exactly like a mode in Minkowski space, so the “natural” initial state is to simply start each mode in its Minkowski vacuum state in the asymptotic past. This is called the Bunch–Davies vacuum [26], and it is identical to what is also called the Gibbons–Hawking vacuum [27]. Gibbons and Hawking developed their description of the vacuum in a completely different formalism, based on the symmetries of the de Sitter spacetime (Eq. (16) with \( a(t) \propto e^{Ht} \)), but the two vacuum states are identical, as one would hope. The Bunch–Davies / Gibbons–Hawking vacuum is taken as the starting point for all standard calculations of density perturbations.

To discuss the spectrum of fluctuations of a spatially varying quantity such as \( \delta \varphi(\vec{x}, t) \), which is assumed to be statistically homogeneous and isotropic, cosmologists define a power spectrum \( P_{\delta \varphi}(k, t) \) by

\[
\langle \delta \tilde{\varphi}(\vec{k}, t) \delta \tilde{\varphi}(\vec{k}', t) \rangle \equiv (2\pi)^3 P_{\delta \varphi}(k, t) \delta^{(3)}(\vec{k} + \vec{k}') \, , \tag{18}
\]

or equivalently

\[
\langle \delta \varphi(\vec{x}, t) \delta \varphi(\vec{y}, t) \rangle = \frac{1}{(2\pi)^3} \int d^3k e^{i\vec{k} \cdot (\vec{x} - \vec{y})} P_{\delta \varphi}(k, t) \, . \tag{19}
\]

When \( \delta \varphi(\vec{x}, t) \) is a quantum field operator, the power spectrum is nothing more than the equal-time propagator, which can be calculated straightforwardly once the vacuum is specified, as described in the previous paragraph. From Eq. (16), \( \delta \tilde{\varphi}(\vec{k}, t) \) can be seen to obey the

\[3\] Conventions vary, but here we follow the conventions of Refs. [15] and [18]. The quantity \( \Delta f(\vec{k}) \) defined in Ref. [2] is related by \( \Delta f(\vec{k})^2 = k^3 P_f(k)/(2\pi)^3 \). Another common normalization, called \( P(k) \) in Ref. [15] and \( \Delta^2(k) \) in Ref. [18], is given by \( P(k) = \Delta^2(k) = k^3 P_f(k)/2\pi^2 \), so \( \Delta f(\vec{k})^2 = P(k)/4\pi \). According to Ref. [15], \( P(k) \) and \( P(k) \) are both called the spectrum. In the context of the curvature perturbation \( \mathcal{R} \), to be defined below, Ref. [28] defines yet another normalization that remains in common use, \( \delta_H \equiv \frac{\delta}{\dot{H}} \mathcal{R} \).
\[
\delta \ddot{\varphi} + 3H \dot{\delta \varphi} + \frac{k^2}{a^2} \delta \varphi = -\frac{\partial^2 V}{\partial \varphi^2} \delta \varphi.
\]

(20)

To make use of this equation, we consider its behavior around the time of Hubble exit, \(t_{ex}(k)\), when the wavelength is approximately equal to the Hubble length, defined more precisely by

\[
\frac{k^2}{a^2(t_{ex})} = H^2.
\]

(21)

We assume that the slow-roll conditions of Eqs. (14) and (15) are valid within several Hubble times \((H^{-1})\) of \(t_{ex}\) (but it is okay if they are violated later during the period of inflation, as described in footnote 2). For \(t \gtrsim t_{ex}\), the \(k^2/a^2\) term of Eq. (20) becomes insignificant.

From Eq. (15) we see that the right-hand-side of Eq. (20) has magnitude \(3 \eta H^2 \delta \varphi\), where \(\eta \ll 1\). Thus for times up to and including \(t_{ex}(k)\) and a little beyond, we can neglect the right-hand-side and treat \(\delta \varphi(\vec{x}, t)\) as a free, massless, minimally coupled field in de Sitter space. It is then straightforward to show that

\[
P_{\delta \varphi}(k, t) = \frac{H^2}{2k^3} \left[ 1 + \left( \frac{k}{a(t)H} \right)^2 \right].
\]

(22)

The time delay should be calculated at a time slightly beyond \(t_{ex}\), say by a few Hubble times, when \(k^2/a^2 \ll H^2\), when we can neglect the \(k^2/a^2\) terms in both Eqs. (20) and (22).

Using Eqs. (9), (12), and (14), one finds several useful expressions for \(P_{\delta N}(k)\):

\[
P_{\delta N}(k) = \frac{H^2}{\varphi_0^2} P_{\delta \varphi}(k) = \frac{H^4}{2k^3 \varphi_0^2} = \frac{2\pi G H^2}{k^3 \epsilon} = \frac{9}{2} \left( \frac{8\pi G}{3} \right)^3 \frac{V^3}{k^3 V^2}.\]

(23)

Since the time delay is evaluated a few Hubble times beyond \(t_{ex}(k)\), the quantities \(V, V',\) and \(H\) appearing in the above expressions should all be evaluated at \(\varphi_0(t)\) for \(t \approx t_{ex}(k)\). The distinction between \(t_{ex}(k)\) and a few Hubble times later is important only at higher order in the slow-roll approximation, since the change in \(\varphi_0\) over a Hubble time is of order \(\epsilon\).

Note that the fluctuations in \(\delta t\) are inversely proportional to \(\epsilon\), which is a consequence of the presence of \(\dot{\varphi}_0(t)\) in the denominator of Eq. (12). We are now in a position to test the consistency of the first of our approximations (see Fig. (II)), the approximation of neglecting the fluctuations in the scalar field energy-momentum tensor. The major source of these fluctuations is the fluctuation in potential energy caused by the fluctuations in \(\varphi\). Thus \(\delta \rho/\rho \approx V'\delta \varphi/V\), so

\[
P_{\delta \rho/\rho}(k) \approx \left( \frac{V'}{V} \right)^2 P_{\delta \varphi}(k) = \frac{8\pi G H^2 \epsilon}{k^3} \left[ 1 + \left( \frac{k}{a(t)H} \right)^2 \right].
\]

(24)

Thus, the fractional fluctuations in the mass density are of order \(\epsilon^2\) times the dimensionless fluctuations \(P_{\delta N}(k)\) of the time delay, so we expect that we can neglect the mass density fluctuations if we are interested in calculating the dominant term in the small \(\epsilon\) limit.\(^4\)\(^5\)\(^6\)

\(^4\) One source where from which this equation can be deduced is Ref. [26], but note that Eq. (3.6) is misprinted, and should read \(\psi_\xi(\eta) = \alpha^{-1}(\pi/4)^{1/2} \eta^{1/2} H_0^{(2)}(\eta)\).

\(^5\) One might worry that the second term in square brackets in Eq. (24) becomes large at early times,
III. EVOLUTION OF THE DENSITY PERTURBATIONS THROUGH THE END OF INFLATION

Thus, we have reduced the calculation of the density fluctuations to the schematic description of Fig. (II), with the power spectrum of the time delay given by Eq. (23). The role of quantum theory in this calculation is finished — it determined the power spectrum of the time delay. The rest of the calculation is general relativity and astrophysics. Here I will continue the derivation through the end of inflation, which carries it far enough to compare with the literature and to at least qualitatively understand the observational consequences.

To understand the implications of $\delta t(\vec{x})$ in the cosmic evolution as described by Fig. (II), it is convenient to switch to a new coordinate system in which the end of inflation happens at $t_0$ everywhere. Just define $\vec{x}' = \vec{x}$ and $t' = t - \delta t(\vec{x}')$. Clearly $dt = dt' + \partial_i \delta t dx^i$, so the transformation of the metric of Eq. (5) becomes $ds^2 = g'_{\mu \nu} dx'^\mu dx'^\nu$, where

$$
g'_{00} = -1 , \quad g'_{0i} = g'_{i0} = -\partial_i \delta t , \quad g'_{ij} = a^2(t' + \delta t(\vec{x}'))\delta_{ij} , \quad (25)
$$

with an inverse metric, to first order in $\delta t$, given by

$$
g''_{00} = -1 , \quad g''_{0i} = g''_{i0} = -\frac{1}{a^2} \partial_i \delta t , \quad g''_{ij} = \frac{1}{a^2(t' + \delta t(\vec{x}'))} \delta_{ij} . \quad (26)
$$

In the primed coordinates the phase transition happens sharply at $t' = t_0$, with a sudden change from $p = -\rho \equiv -\rho_{inf}$ in the inflationary phase to $p_{rad} = \frac{1}{3} \rho_{rad}$ in the radiation phase. During the inflationary phase the energy-momentum tensor is simply $T_{\mu \nu} = -\rho_{inf}\delta_{\mu \nu}$, while

$$
\begin{align*}
g_{00} &= -1 , \quad g_{0i} = 0 , \quad g_{ij} = a^2(t) \left[ (1 + A) \delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j} \right] . \\
\end{align*}
$$

Defining a new auxiliary field $\chi$ (not used in Ref. [16]) by

$$
\chi = \frac{1}{2} \left( 3 A + \nabla^2 B \right) - 3 \left[ \frac{\partial H}{\partial \varphi_0} \delta \varphi + \frac{\partial H}{\partial \dot{\varphi}_0} \delta \dot{\varphi} \right] , 
$$

the scalar field equation of motion in this gauge can be written as

$$
\ddot{\varphi} + 3H(\varphi_0, \dot{\varphi}_0)\delta \varphi + 3\dot{\varphi}_0 \left[ \frac{\partial H}{\partial \varphi_0} \delta \varphi + \frac{\partial H}{\partial \dot{\varphi}_0} \delta \dot{\varphi} \right] - \frac{1}{a^2} \nabla^2 \delta \varphi + V''(\varphi_0) \delta \varphi + \varphi_0 \chi = 0 .
$$

$\chi$ can be found from the source equation

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6 The fact that $P_{b/\rho}(k) \propto \epsilon$ is a strong argument to justify the neglect of metric fluctuations, but some experts may not be convinced without seeing a more complete formulation in which metric fluctuations can actually be calculated. In Ref. [24] I will show how the time-delay formalism can be embedded in a complete first-order calculation in synchronous gauge. (Synchronous gauge is most useful here, since the time delay is time-independent when measured in proper time. Other time coordinates will obscure the underlying simplicity.) Following the notation of Ref. [16], the metric is written as

$$
g_{00} = -1 , \quad g_{0i} = 0 , \quad g_{ij} = a^2(t) \left[ (1 + A) \delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j} \right] . \\
$$

Defining a new auxiliary field $\chi$ (not used in Ref. [16]) by

$$
\chi = \frac{1}{2} \left( 3 A + \nabla^2 B \right) - 3 \left[ \frac{\partial H}{\partial \varphi_0} \delta \varphi + \frac{\partial H}{\partial \dot{\varphi}_0} \delta \dot{\varphi} \right] , 
$$

the scalar field equation of motion in this gauge can be written as

$$
\ddot{\varphi} + 3H(\varphi_0, \dot{\varphi}_0)\delta \varphi + 3\dot{\varphi}_0 \left[ \frac{\partial H}{\partial \varphi_0} \delta \varphi + \frac{\partial H}{\partial \dot{\varphi}_0} \delta \dot{\varphi} \right] - \frac{1}{a^2} \nabla^2 \delta \varphi + V''(\varphi_0) \delta \varphi + \varphi_0 \chi = 0 .
$$

$\chi$ can be found from the source equation
after the transition it is given by Eq. (1). The energy-momentum tensor must be covariantly conserved, which means that
\[ D_\nu T^\nu \mu = \partial_\nu T^\nu \mu + \Gamma^\nu_\mu_\chi T^\chi \lambda - \Gamma^\lambda_\nu_\mu T^\nu \chi = 0 , \] (27)
where \( D_\nu \) denotes a covariant derivative. The affine connection coefficients \( \Gamma \) will not contain any \( \delta \)-functions, so \( \partial_\nu T^\nu \mu \) cannot contain any \( \delta \)-functions either; thus \( T^0_\mu \) must be continuous at \( t' = t_0 \). This implies that
\[ T^0_0 \text{rad} = p_\text{rad} + u^0_\text{rad} u_{0,\text{rad}} (\rho_\text{rad} + p_\text{rad}) = T^0_0 \text{inf} = -\rho_\text{inf} , \] (28)
\[ T^i_0 \text{rad} = (\rho_\text{rad} + p_\text{rad}) u_{i,\text{rad}} u^0_\text{rad} = T^i_0 \text{inf} = 0 . \] (29)
The second of these equations can only be satisfied if \( u_{i,\text{rad}} = 0 \), because \( u^0_\text{rad} \) cannot vanish, as \( u^\mu \) is timelike, and \( (\rho_\text{rad} + p_\text{rad}) = \frac{4}{3} \rho_\text{rad} \) cannot vanish without violating the first equation. Thus, the radiation fluid is necessarily at rest in the frame of reference in which the phase transition occurs simultaneously. Requiring \( u^2 = -1 \), one finds that
\[ u_{0,\text{rad}} = -1 , \quad u_{i,\text{rad}} = 0 ; \quad u^0_\text{rad} = 1 , \quad u^i_\text{rad} = \frac{1}{a^2} \partial_0 \delta t . \] (30)
Given the above equation, Eq. (28) leads immediately to \( p_\text{rad} = \rho_\text{inf} \); the energy density is conserved across the transition. Since the Einstein equations are partial differential equations that are second order in time derivatives, the metric and its first time derivative will be continuous across \( t' = t_0 \). So we have now found all the information needed to give a well-defined Cauchy problem for the evolution of the model universe, starting at the beginning of the radiation-dominated era. At this point the perturbations of interest have wavelengths
\[ \frac{\partial}{\partial t} (a^2 H \chi) = \dot{H} \nabla^2 \left( \frac{\delta \varphi}{\varphi_0} \right) . \]
The full metric can be recovered by using
\[ \nabla^2 A = 2 a^2 H \chi , \]
and then using the definition of \( \chi \) to find \( \dot{B} \). Note that the terms involving partial derivatives of \( H \) reproduce to first order the dependence of \( H \) on \( \varphi \) and \( \dot{\varphi} \) in Eq. (11), so these “metric perturbations” are taken into account by the time-delay calculation. The metric perturbations that are ignored are those proportional to \( \chi \), and they can be seen to be small in slow-roll inflation. The source term on the right is proportional to \( \dot{H} \), which is of order \( \epsilon \). The term enters the scalar field equation of motion with a prefactor of \( \varphi_0 \), which contributes another factor of \( \sqrt{\epsilon} \) to the suppression. At later times near the end of inflation, when the slow-roll condition might fail badly, \( \chi \) is strongly suppressed by the factor \( 1/a^2 \). For the slow-roll solution with \( \delta \varphi/\varphi_0 \approx -\delta t (\vec{x}) \), the source equation can be solved to give
\[ \chi(t) \approx \frac{H(t_{\text{ex}}) - H(t)}{a^2(t) H(t)} \nabla^2 \delta t (\vec{x}) , \]
where a constant of integration was chosen so that \( \chi(t) \approx 0 \) at Hubble exit. Thus, this formulation gives a solid underpinning to the intuitive idea that, at late times, each region can be treated as an independent Robertson-Walker universe. Each independent universe follows essentially the same history, differing from the other universes by only a time offset.
vastly larger than the Hubble length, but during the subsequent evolution the Hubble length will grow faster than the perturbation wavelength, so later the perturbations will come back inside the Hubble length. The description of the perturbations through the time of Hubble reentry was given in Refs. [2]–[5], and in many later sources, but for present purposes we will stop here.

At this point we can discuss the validity of the second of our key approximations, the approximation of an instantaneous phase transition. The actual transition, during which the scalar field rolls down the hill in the potential energy diagram and then oscillates about the minimum and reheats, very likely takes many Hubble times to complete. However, we need to keep in mind that the modes of interest exited the Hubble horizon some 50 or 60 Hubble times before the end of inflation, which means that at the end of inflation their physical wavelength is of order $e^{50}$ to $e^{60}$, or $10^{21}$ to $10^{26}$, times the Hubble length. Thus, even if the phase transition takes $10^{10}$ Hubble times, during this time light would be able to travel less than $10^{-10}$ wavelengths. Thus the phase transition is effectively instantaneous, on the time scale that is relevant for influencing a wave with the wavelengths under consideration. (Of course the reheat energy density that we calculated, $\rho_{\text{rad}} = \rho_{\text{inf}}$, was an artificiality of the instantaneous approximation. But the calculation can easily be adjusted to account for a lower reheat energy density, which can be found by doing a more accurate, homogeneous calculation. At the end of inflation we would still obtain, in the primed coordinate system, a radiation fluid that is at rest, with a uniform energy density.)

Note that the method used here depended crucially on the assumption that all parts of the universe would undergo the same sequence of events, so that the only difference from one place to another is an overall time offset. If there were more than one field for which the quantum fluctuations were relevant, then this would not be true, since a fluctuation of one field relative to the other could not be described as an overall time offset. Thus, multifield inflation requires a more sophisticated formalism.

### IV. SIMPLIFYING THE DESCRIPTION

The description we have given so far, specifying the metric and the matter content, is sufficient to calculate the rest of the history, but it is rather complicated. Furthermore, it is equivalent to many other complicated descriptions, related by coordinate transformations. It is therefore very useful to find a coordinate-invariant way of quantifying the density perturbations. One convenient approach is motivated by considering the Friedmann equation for a universe with spatial curvature, a universe which might be closed, open, or in the borderline case, flat:

$$H^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2}. \quad (31)$$

Here $k$ is a constant, where positive values describe a closed universe, and negative values describe an open one. We are interested in describing a perturbation of a homogeneous background universe that is flat, $k = 0$. Thus $\rho(\vec{x}, t)$ will on average equal $\rho_0(t)$, the value for the background universe, but it will fluctuate about this average. $H$ is normally thought of as part of the global description of the universe, but it has a locally defined analog given by

$$H_{\text{loc}} \equiv \frac{1}{3}D_\mu u^\mu, \quad (32)$$
where $D_\mu$ is the covariant derivative and $u^\mu$ is the fluid velocity. (Here we will deal only with a radiation fluid, but in a multicomponent fluid $u^\mu$ can be defined in terms of the total energy-momentum tensor, as described on p. 225 of Ref. [16].) We can then define

$$K(\vec{x}, t) \equiv a^2(t) \left[ \frac{8\pi}{3} G \rho(\vec{x}, t) - H_{\text{loc}}^2(\vec{x}, t) \right].$$ (33)

This quantity has a property called gauge invariance, which means that its value for any coordinate point $(\vec{x}, t)$ is not changed, to first order in the size of the perturbations, by any coordinate transformation that is itself of order of the size of the perturbations. In this case, the gauge invariance follows from the fact that, apart from the factor $a^2(t)$ which is irrelevant for this issue, $K(\vec{x}, t)$ a quantity that is coordinate-invariant, and which vanishes for the background universe. (Note that coordinate invariance by itself is not enough; if the quantity varied with time in the background solution, then its value at $(\vec{x}, t)$ would change if $t$ were redefined by a small amount.) $K$ also has the convenient property that it remains constant as long as spatial derivatives can be neglected, because it is exactly conserved for the case of a homogeneous universe.

We can calculate $K(\vec{x}, t)$ just after the end of inflation, using the primed coordinate system but dropping the primes. We have $H_{\text{loc}} = \frac{1}{3} D_\mu u^\mu = \frac{1}{3} g^{-1/2} \partial_\mu (g^{1/2} u^\mu)$, where $g \equiv -\det(g_{\mu\nu}) = a^6(t + \delta t(\vec{x}))$ and $u^\mu$ is given by Eq. (30). This gives $H_{\text{loc}} = H_{\text{inf}} + \nabla^2 \delta t / 3a^2$, which gives

$$K = -\frac{2}{3} H_{\text{inf}} \nabla^2 \delta t,$$ (34)

where $H_{\text{inf}}$ is the Hubble expansion rate during inflation, which we have treated as a constant. As in Eq. (6), $\nabla^2$ denotes the Laplacian operator with respect to the coordinates $x^i$. Then, given the power spectrum of Eq. (23) for $\delta N = H \delta t$, we find a power spectrum for $K$ given by

$$P_K(k) = \frac{4k^4}{9} P_{\delta N}(k) = \frac{2kH^4}{9\epsilon_0^2} = \frac{8\pi G H^2 k}{9\epsilon} = 2 \left( \frac{8\pi G}{3} \right)^3 V^3 k.$$ (35)

In the literature $K$ is seldom used, but instead it is much more common to use a variable called the curvature perturbation $R$, for which the usual definition is somewhat complicated. However, it is shown in Ref. [28] that

$$K = -\frac{2}{3} \nabla^2 R,$$ (36)

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7 At this point one might worry that the approximation of instantaneous reheating might be crucial to the answer we obtained. If we had used a more realistic picture of slow reheating which leads to a lower reheat energy density, we might expect that our method would give $K = -\frac{2}{3} H_{\text{reheat}} \nabla^2 \delta t$, which could be much smaller. Although it is not obvious, however, the reheat energy density does not affect $K$, which is conserved for wavelengths large compared to the Hubble length. Thus, the value we obtained here could have been calculated before reheating began, and is equal to the value that holds long after reheating, whether reheating is fast or slow. To understand the evolution of $K$ when $H$ changes, however, requires a more detailed calculation. Because of the factor of $a^2(t)$ in Eq. (33), the value of $K$ is in fact sensitive to quantities that are suppressed by factors of $1/a^2$. To see the conservation of $K$ for long wavelengths, one needs to include the contribution of the auxiliary field $\chi$ defined in footnote 6. This issue will be discussed in more detail in Ref. [25].

8 See, for example, p. 246 of Ref. [16].
so
\[ P_R(k) = \frac{9}{4k^4} P_K(k) = P_{\delta N}(k), \tag{37} \]
where \( P_{\delta N}(k) \) is given by Eq. \( \text{[23]} \). This answer agrees precisely with the answers obtained in Refs. \[13\], \[16\], \[18\], and \[2,9\].

The WMAP seven-year paper \[29\] quotes \( P_R(k_0) = (2.43 \pm 0.09) \times 10^{-9} \), where \( k_0 = 0.002 \text{ Mpc}^{-1} \) and \( \mathcal{P}_R(k) = k^3 P_R(k)/(2\pi^2) \). (The WMAP papers use \( \Delta^2(k) \) for \( \mathcal{P}_R(k) \).) If these fluctuations come from single-field slow-roll inflation, we can conclude from Eqs. \( \text{(37)} \) and \( \text{(23)} \) that at the time of Hubble exit,
\[ \frac{V^{3/2}}{M_{Pl}^3 V'} = 5.36 \times 10^{-4}, \tag{38} \]
where \( M_{Pl} = 1/\sqrt{8\pi G} \approx 2.44 \times 10^{18} \text{ GeV} \) is the reduced Planck mass.

V. DEDUCING THE CONSEQUENCES, COMPARING WITH OBSERVATION

From Eqs. \( \text{(37)} \) and \( \text{(23)} \) we can also deduce how the intensity of the fluctuations varies with \( k \), a relation which is parameterized by the scalar spectral index \( n_s(k) \), defined by
\[ n_s - 1 = \frac{d \ln P_R(k)}{d \ln k}. \tag{39} \]
To evaluate this expression for slow-roll inflation, we use the last expression in Eq. \( \text{(23)} \) for \( P_R(k) \); we recall that the expression is to be evaluated at \( t_{\text{ex}} \) (or some fixed number of Hubble times later), and find that Eq. \( \text{(21)} \) leads to \( d \ln k/dt_{\text{ex}} = H \). Then using Eq. \( \text{(9)} \) to write \( \dot{\varphi}_0 = -V'/3H \), these equations can be combined to give \[30\]
\[ n_s - 1 = -\frac{V'}{8\pi GV} \frac{d \ln (V^3/V'^2)}{d \varphi} = -6\epsilon + 2\eta \approx 4 \frac{\dot{H}}{H^2} - \frac{\ddot{H}}{HH}. \tag{40} \]
The WMAP seven-year paper \[29\] quotes \( n_s - 1 = -0.032 \pm 0.012 \). This result suggests that the slow roll parameters are indeed quite small, and furthermore they have very plausible values. The time of Hubble exit is typically of order 60 Hubble times before the end of inflation, depending mainly on the reheat temperature, which means that the natural time scale of variation is of order \( 60H^{-1} \). If each time derivative in the right-hand expressions of

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9 To compare with Ref. \[19\], note that \((2\pi)^3|\mathcal{R}_q|^2\) in this reference corresponds to \( P_R(q) \), and is given on pp. 482 and 491. To compare with Ref. \[13\], note that \( \mathcal{P}_\zeta(k) \), given on p. 406, corresponds to \( k^3 P_R(k)/(2\pi^2) \), as described on p. 89, and that \( M_{Pl} \) is the reduced Planck mass, \( 1/\sqrt{8\pi G} \). To compare with Ref. \[18\], use \( P_\zeta(k) = P_R(k) \), where \( P_\zeta(k) \) is given on p. 170. As explained in Ref. \[16\], \( \zeta \) and \( \mathcal{R} \) refer to slightly different quantities, but they agree for wavelengths long compared to the Hubble length. To compare with Ref. \[2\], note that \( S = K/(a^2 H^2) \), and that the equation for \( S(t' = 0) \) describes the conditions just after the end of inflation, with the scale factor \( R(t) = e^{\chi t}, \chi = H_{\text{inf}} \). The quantum fluctuations are quantified in this paper by \( \delta t = \delta \varphi/\dot{\varphi}_0 \), with \( \Delta \varphi^2 = k^3 P_{\delta \varphi}(k)/(2\pi)^3 = H^2/(16\pi^3) \), in agreement with Eq. \( \text{(22)} \) of the current paper.
Eqs. (14) and (15) is replaced by a factor of $H/60$, one sees that the slow roll parameters are plausibly of order $1/60$.

The case $n_s = 1$ is called scale-invariant, because it means that $P_R(k)$ is independent of $k$; that is, each mode has the same strength, at the time of Hubble exit, as any other mode. Since $P_R(k)$ is constant while the wavelength is long compared to the Hubble length, all modes also have the same strength at the time of Hubble reentry. Single-field inflation produces density fluctuations that are approximately scale-invariant, because all the modes that are visible today passed through Hubble exit during a small interval of time during inflation, so the conditions under which they were generated were very similar.

In addition to the nearly scale-invariant spectrum that we just calculated, there are two other key features of the density fluctuations that follow as a consequence of slow-roll single-field inflation. The first is that the fluctuations are adiabatic, which means that every component of the matter in the universe — the photons, the baryons, and the dark matter — fluctuate together. The temperature can be related to the density of photons, so it also fluctuates with the density baryons or dark matter. The reason for this feature is clear, because the time delay affects all properties of the matter in the universe the same way. Until the perturbations reenter the Hubble length (after which complicated things can happen), every region of space behaves just like any other region of space, except for a time offset. Thus the matter content of any one region can differ from that of some other region by at most an adiabatic compression or expansion. The WMAP team has tested this relation for the possibility of non-adiabatic fluctuations between photons and cold dark matter. By combining WMAP data with other data, they find at the 95% confidence level that the non-adiabatic component is at most 6% of the total in the case of “axion-type” perturbations, or 0.4% in the case of “curvaton-type” perturbations.

The other key prediction of slow-roll single-field inflation is that the perturbations should be Gaussian. Why are they Gaussian? They are Gaussian because $\delta \tilde{\varphi}(\vec{k}, t)$ is calculated in a quantum field theory. The perturbations are small so we expect accurate results at lowest order, which means that we are only calculating free-field-theory expectation values, and they are Gaussian. There are of course higher order corrections, which in a given model can also be calculated, but they are generically very small. So, to first approximation, we expect the answers to be Gaussian, which means in particular that the three-point correlation function should vanish. There has been a lot of effort to look for non-Gaussianity, but so far no convincing evidence for non-Gaussianity has been found.

The calculations shown here stop just after the end of inflation, but with a lot of work by many astrophysicists the calculations have been extended to make detailed predictions for the fluctuations that can be detected today in the cosmic microwave background. The success is beautiful. To process the data, the temperature pattern observed in the CMB is expanded in spherical harmonics, which is the spherical equivalent of Fourier transforming, providing information about how the intensity of the fluctuations varies with angular wavelength. Figure 3 shows the observed temperature fluctuations as a function of the multipole number $\ell$, using the 7-year WMAP data for $\ell < 800$, and ACBAR data for higher $\ell$. The red line is the theoretical curve that comes about by extending the inflationary predictions to the present day in a model with dark energy ($\Lambda$) and cold dark matter, using the best-fit parameters found by the WMAP team: $P_R(0.002 \text{ Mpc}^{-1}) = 2.42 \times 10^{-9}$, $n_s = 0.966$, $\Omega_\Lambda = 0.729$, $\Omega_{\text{dark matter}} = 0.226$, $\Omega_{\text{baryon}} = 0.045$, and $\tau = 0.085$, where $\tau$ is the optical depth experienced by the photons since the “recombination” of the primordial plasma at about 380,000 years after the big bang. While there are 6 free parameters, 4 of
FIG. 3: Comparison of the latest observational measurements of the temperature fluctuations in the CMB with several theoretical models, as described in the text. The temperature pattern on the sky is expanded in multipoles (i.e., spherical harmonics), and the intensity is plotted as a function of the multipole number $\ell$. Roughly speaking, each multipole $\ell$ corresponds to ripples with an angular wavelength of $360^\circ/\ell$.

them have values that are expected on the basis of theory ($n_s \approx 1$) or other observations ($\Omega_\Lambda$, $\Omega_{\text{dark matter}}$, and $\Omega_{\text{baryon}}$), and they agree well. One of the free parameters determines the overall height, so one should not be impressed that the height of the primary peak matches so well. But the location, shape, and relative heights of the peaks are really being predicted by the theory, so I consider it a spectacular success.

For comparison, the graph also shows predictions for several alternative theories, all of which are now ruled out by this data. The yellow line shows the expected curve for an open universe, with $\Omega_{\text{total}} = 0.30$. The green line shows an inflationary model with $\Omega_{\text{total}} = 1$, but with $\Omega_{\text{dark matter}} = 0.95$ and no dark energy. The magenta line shows the expectations for fluctuations generated by the formation of cosmic strings in the early universe, taken from Ref. [33]. Structure formation caused by cosmic strings or other “defects” was considered a viable possibility before this data existed, but now cosmic strings are completely ruled out as a major source of density fluctuations.

(There are possibly alternative ways to generate density perturbations with the same properties as those of inflation, but there is not yet a consensus about how easy it is to construct a plausible model. The cyclic ekpyrotic model [34, 38] was claimed to naturally produce such fluctuations, but these claims were disputed by a number of authors [39–42]. Now at least some of the founders of ekpyrosis [43, 44] agree that the original models do not give a nearly-scale invariant spectrum, as had been claimed. But these papers and others have proposed newer, more sophisticated versions of bouncing universes, generally...
involving either multiple fields, or settling for scale invariance for only a limited range of scales. Baumann, Senatore, and Zaldarriaga [45] have argued that any single-field model with attractor behavior has to be very close to de Sitter space to remain weakly coupled for at least the required $\sim 10$ $e$-folds needed to account for observations.

VI. OUTSTANDING QUESTIONS ABOUT DENSITY PERTURBATIONS

There are still a number of important, outstanding questions concerning density perturbations:

1. Will $B$-modes be found? Experiments are starting to measure the polarization of the CMB, for which the spherical harmonic expansion for the temperature pattern is replaced by an expansion in $E$-modes and $B$-modes [46]. The $E$-modes are those that can be expressed as gradients of scalar harmonic functions, and they are produced as a by-product of the density perturbations that we have been discussing. The $B$-modes are orthogonal to the $E$-modes; they cannot be expressed as gradients of scalar modes, and they cannot be produced by density perturbations. There can be foreground contamination, but the only known primordial source of $B$-modes is a background of gravitational waves. Thus, gravity waves might be discovered in the CMB before they can be seen directly. The discovery of a primordial gravity wave background would be very exciting, because it is the only thing that will give us a clue about the energy scale at which inflation happened. As far as we know now inflation might have happened anywhere from the electroweak scale up to the grand unified theory (GUT) scale, or a little beyond. The discovery of gravity waves would end the uncertainty, and would also give strong evidence for the inflationary picture. There are, however, many inflationary models for which the energy scale would be too low for the gravitational waves to be visible.

2. Can sub-Planckian physics influence the calculation of inflationary density perturbations? A typical GUT-scale inflationary model would include about 60 $e$-folds of inflation, expanding by a factor of $e^{60} \approx 10^{26}$. From the end of inflation to today the universe would expand by another factor of $\sim 10^{15} \text{GeV}/3 \text{K} \approx 10^{27}$. This means that a distance scale of 1 m today corresponds to a length of only about $10^{-53}$ m at the start of inflation, 18 orders of magnitude smaller than the Planck length ($\sim 10^{-35}$ m). With a little more than the minimal amount of inflation — which would be a certainty in the eternal inflation picture to be discussed below — even the largest scales of the visible universe would have been sub-Planckian at the start of inflation. So, it is relevant to ask whether inflation can possibly offer us a glimpse of sub-Planckian physics. There is of course no solid answer to this question, since there is no real understanding of how this process should be described. Kaloper, Kleban, Lawrence, and Shenker [47] have argued that the perturbations are determined primarily by local effective field theory on the scale of order $H$, so that sub-Planckian effects would be invisible except possibly in unconventional models for which the fundamental string scale is many orders of magnitude below the four-dimensional Planck mass, $\sim 10^{19}$ GeV. Some authors [48, 49] have reached similar conclusions, but other authors [50–53] have concluded that the effects might be much easier to see. The conclusions of Ref. [47] seem plausible to me, but certainly the role of sub-Planckian physics is not yet fully understood.
3. Will effects beyond the single-field slow-roll approximation be found? With multiple fields, or with unusual features in the potential for a single field, models can be constructed that predict significant non-Gaussianity, non-adiabaticity, or spectral distortions. There is an active industry engaged in studying models of this sort, and in looking for these nonstandard features in the data. The WMAP seven-year analysis [29] reports “no convincing deviations from the minimal model,” but we all await the data from the Planck mission, expected in less than a year, and the data from a variety of ground-based experiments.

VII. FLUCTUATIONS ON LARGER SCALES: ETERNAL INFLATION?

Since the density perturbation calculations have been incredibly successful, it seems to make sense to take seriously the assumptions behind these calculations, and follow them where they lead. I have to admit that there is no clear consensus among cosmologists, but to many of us the assumptions seem to be pointing to eternal inflation, and the multiverse.

The mechanism for eternal inflation is described most efficiently by separating the cases of the two types of potential functions shown in Fig. 1. For the new inflation case, that state for which the scalar field is poised on the top of the potential hill is a metastable state, often called a “false vacuum,” which decays by the scalar field rolling down the hill. This state decays exponentially, but in any working model of inflation the half-life of the decay is much longer than the doubling time associated with the exponential expansion. Thus, if we follow a region for a period of one half-life, at the end of the period only half of the original region would be still be inflating. However, the half that is still inflating will have a volume vastly larger than the volume of the entire region at the start, so the process will go on forever. Each decay will lead to the production of a “pocket” universe, and the creation of pocket universes will go on forever, as pieces of the ever-growing false vacuum region undergo decays. Once inflation starts, it never stops.\(^\text{10}\)

For the case of a chaotic-type potential, as in Fig. 1(b), naively one would think that the field would inexorably roll down the hill in some finite amount of time. However, Linde [57] discovered that when quantum fluctuations are taken into account, this need not be the case. To understand this, consider an inflating region of space of size \(H^{-1}\), with the inflaton field \(\varphi\) approximately uniform over this region, at some value \(\varphi_0\). After one Hubble time \((H^{-1})\) the region will have expanded by \(e^3 \approx 20\), and can be viewed as 20 Hubble-sized regions which will start to evolve independently. The average field \(\varphi\) in any one of these regions will usually be lower than \(\varphi_0\), due to the classical rolling down the hill, but the classical evolution will be modified by random quantum jumps, which can be estimated as \(\sim H/(2\pi)\). It is therefore possible that in one or more of these 20 regions, \(\varphi\) can equal or exceed \(\varphi_0\). A back-of-the-envelope calculation shows that if

\[
\frac{H^2}{|\dot{\varphi}|} \gtrsim 5 ,
\]

then the expectation value for the number of regions with \(\varphi > \varphi_0\) is greater than one. That implies that the number of Hubble-sized regions with \(\varphi > \varphi_0\) will grow exponentially

\(^\text{10}\) The first models of eternal new inflation were proposed by Steinhardt [54] and Linde [55]. Vilenkin [56] was the first to describe eternal inflation as a generic feature of new inflation.
with time, and the inflation becomes eternal. Note that $H^2/|\dot{\phi}| \approx \sqrt{P_R} \approx (GV)^{3/2}/|V'|$, so the eternally inflating behavior is really the large-$\varphi$, long-wavelength, tail of the density perturbation spectrum. Since $V^{3/2}/|V'|$ grows without bound as $\varphi \to \infty$ for most potentials under consideration, almost all models allow for eternal inflation.

There is certainly no proof that we live in a multiverse, but I will argue that there are three winds — that is, three independent scientific developments, arising from three different branches of science — which seem to be leading to the multiverse picture.

1. **Theoretical Cosmology: Eternal Inflation.** As I just described, almost all inflationary models are eternal into the future.

2. **String Theory: The Landscape.** String theory predicts that there is not just one kind of vacuum, but instead there are a colossal number of them: $10^{500}$ or maybe more [58, 59]. The underlying laws of physics would be the same everywhere, but nonetheless each type of vacuum would create an environment in which the low-energy laws of physics would be different. Thus, if there is a multiverse, it would be a varied multiverse, in which the different pocket universes would each appear to have their own laws of physics.

3. **Observational Astronomy: the Cosmological Constant.** The third “wind” has its roots in the fine-tuning that our universe appears to exhibit. In the past a minority of physicists argued that things such as the properties of ice or the energy levels of carbon-12 appeared to be fine-tuned for the existence of life, but not very many scientists found this convincing. If these properties were different, then maybe life would form some other way. However, a form of fine-tuning that many of us find much more convincing became evident starting in 1998, when two groups of astronomers [60, 61] announced that the expansion of the universe is not slowing down due to gravity, but is in fact accelerating. The simplest explanation is that the acceleration is caused by a nonzero energy density of the vacuum, also known as a cosmological constant. But that would mean that the vacuum energy density is nonzero, yet a full 120 orders of magnitude smaller than the Planck scale ($M_{Pl}^2$, where $M_{Pl} = 1/\sqrt{G}$), the scale that most theoretical physicists would consider natural. Physicists have struggled to find a physical explanation for this small vacuum energy density, but no generally accepted solution has been found. But if the multiverse is real, the problem could go away. With $10^{500}$ different types of vacuum, a small fraction, but nonetheless a large number of them, would be expected to have an energy density as small as what we observe. The smallness of the vacuum energy density would be explained, therefore, if we could explain why we should find ourselves in such an unusual part of the multiverse. But as pointed out by Weinberg and collaborators [62, 63] some time ago, there is a selection effect. If we assume that life requires the formation of galaxies, then one can argue that life in the multiverse would be concentrated in those pocket universes with vacuum energy densities in a narrow band about zero. Thus, while a typical vacuum energy density in the multiverse would be on the order of the Planck scale, almost all life in the multiverse would find a small value, comparable to what we see.

While the multiverse picture looks very plausible in the context of inflationary cosmology — at least to me — it raises a thorny and unsolved problem, known as the “measure problem.” Specifically, we do not know how to define probabilities in the multiverse. If the multiverse picture is right, then anything that can happen will happen an infinite number of times, so any distinction between common and rare events requires the comparison of
infinities. Such comparisons are not mathematically well-defined, so we must adopt a recipe, or “measure,” to define them. Since the advent of quantum theory essentially all physical predictions have been probabilistic, so probability is not a concept that we can dispense with. To date we do not understand the underlying physical basis for such a measure, but much progress has been made in examining proposals and ruling out many of them.\textsuperscript{11}

One might guess that this problem is easily handled by choosing a finite sample spacetime region in the multiverse, calculating the relative frequencies of different types of occurrences in the sample region, and then taking the limit as the region becomes infinite. This seems like a very reasonable approach, and in fact most of the measure proposals that have been discussed are formulated in this way. The problem is that the answers one obtains are found to depend sensitively on the method that is used to choose the sample region and to allow it to grow. The dependence on the method of sampling seems surprising, but it sounds plausible if we remember that the volume of the multiverse grows exponentially with time. A sample spacetime region will generally have some final time cutoff, and the spacetime volume will generally grow exponentially with the cutoff. But then, no matter how large the cutoff is taken, the volume will always be dominated by a region that is within a few time constants of the final time cutoff hypersurface. No matter how large the final time cutoff is taken, the statistics will never be dominated by the interior of the sample region, but instead will be dominated by the final time cutoff surface. For that reason, it is not surprising that the method of choosing this surface will always affect the answers.

There are a number of important questions, in the multiverse picture, that depend very crucially on the choice of measure. How likely is it that we observe a vacuum energy density as low as what we see? How likely is it that our universe has a mass density parameter $\Omega$ sufficiently different from 1 that we can hope to measure the difference? How likely is it that we might find evidence that our pocket universe collided with another sometime in its history? And, if there are many vacua in the landscape of string theory with low energy physics consistent with what we have measured so far, how likely is it that we will find ourselves in any particular one of them? If we live in a multiverse, then in principle all probabilities would have to be understood in the context of the multiverse, but it seems reasonable to expect that any acceptable measure would have to agree to good accuracy with calculations that we already know to be successful.

As discussed in Ref. \textsuperscript{[64]}, we can identify a class of measures that give reasonable answers. It seems plausible that the ultimate solution to this problem will give similar answers, but the underlying principles that might determine the right answer to this question remain very mysterious. Nonetheless, the success of inflation in explaining the observed properties of the universe, including the density perturbation predictions discussed here, provides strong motivation to expect that some solution to the measure problem will be found.

\textsuperscript{11}For a recent summary of a field that is in a state of flux, see Ref. \textsuperscript{[64]}. For a new proposal that was advanced since this summary, see Ref. \textsuperscript{[65]}.
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