STRESS TENSORS
FOR INSTANTANEOUS VACUA
IN 1+1 DIMENSIONS *

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Abstract

The regularized expectation value of the stress-energy tensor for a massless bosonic or fermionic field in 1+1 dimensions is calculated explicitly for the instantaneous vacuum relative to any Cauchy surface (here a spacelike curve) in terms of the length \( L \) of the curve (if closed), the local extrinsic curvature \( K \) of the curve, its derivative \( K' \) with respect to proper distance \( x \) along the curve, and the scalar curvature \( R \) of the spacetime:

\[
T_{00} = -\frac{\epsilon \pi}{6L^2} - \frac{K^2}{24\pi}, \quad T_{01} = -\frac{K'}{12\pi}, \quad T_{11} = -\frac{\epsilon \pi}{6L^2} - \frac{K^2}{24\pi} + \frac{R}{24\pi},
\]

in an orthonormal frame with the spatial vector parallel to the curve. Here \( \epsilon = 1 \) for an untwisted (i.e., periodic in \( x \)) one-component massless bosonic field or for a twisted (i.e., antiperiodic in \( x \)) two-component massless fermionic field, \( \epsilon = -\frac{1}{2} \) for a twisted one-component massless bosonic field, and \( \epsilon = -2 \) for an untwisted two-component massless fermionic field. The calculation uses merely the energy-momentum conservation law and the trace anomaly (for which a very simple derivation is also given herein, as well as the expression for the Casimir energy of bosonic and fermionic fields twisted by an arbitrary amount in \( R^{D-1} \times S^1 \)). The two coordinate and conformal invariants of a quantum state that are (nonlocally) determined by the stress-energy tensor are given. Applications to topologically modified deSitter spacetimes, to a flat cylinder, and to Minkowski spacetime are discussed.

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I. INTRODUCTION AND MAIN RESULT

In a globally hyperbolic spacetime of 1+1 dimensions (one timelike and one spacelike), the trace anomaly \[1–17\] and the conservation law for the regularized expectation value of the stress-energy-momentum tensor of a massless field determine this stress tensor throughout the spacetime if it is given on a Cauchy surface (a spacelike surface, here a one-dimensional curve or line, that each inextendible nonspacelike curve intersects once and only once; the existence of a Cauchy surface is equivalent to global hyperbolicity) \[4, 7, 10, 12, 15\]. However, the stress tensor on a Cauchy line depends on the quantum state.

Here we consider states which are the instantaneous vacua relative to Cauchy lines. That is, for any Cauchy line, we consider the state which is the vacuum with respect to the instantaneous normal-ordered Hamiltonian at the line for the evolution of the massless field off the line in a gaussian normal coordinate system in which the time \(t\) is the proper time from the Cauchy line along a timelike geodesic (say of constant comoving spatial coordinate \(x\)) that intersects the Cauchy line orthogonally.

In this gaussian normal coordinate system, the spacetime metric in the neighborhood of the Cauchy line where the coordinate system is nonsingular (e.g., in the region where the orthogonal timelike geodesics of different \(x\) have not crossed each other) has the standard form

\[
ds^2 = -dt^2 + a^2(t,x)dx^2,
\]

where the Cauchy line is at \(t = 0\). I choose the comoving spatial coordinate \(x\) to be proper distance along the Cauchy line (with an unspecified origin), so on the Cauchy line \(t = 0\) we have \(a = 1\). I shall assume that the topology of the spacetime is \(R^1 \times S^1\), where the \(R^1\) is the time \(t\), so that the Cauchy line is a closed curve \(S^1\) in \(x\), and its total proper length (the period of \(x\)) will be denoted by \(L\), meaning that \((t,x)\) is identified with \((t, x + L)\). (For a spacetime with topology \(R^2\) in which the Cauchy line has the topology of \(R^1\) and is infinitely long, one may simply set \(L = \infty\) in the resulting formulas.)

A one-dimensional connected manifold (with metric), such as the Cauchy line, has no intrinsic curvature, so its only intrinsic geometric property is its total length \(L\) and its connectivity (whether it is topologically \(S^1\), as assumed here unless \(L = \infty\), or whether it is \(R^1\)). However, its imbedding in the spacetime has a local extrinsic curvature \(K(x)\) at each point \(x\), the logarithmic rate of change of the
proper distance separation of the orthogonal geodesics (here the comoving timelike geodesics of constant \(x\)). In the gaussian coordinate system above,

\[ K = \frac{\dot{a}}{a} \quad (2) \]

for any line of constant \(t\), where the overdot denotes a derivative with respect to the proper time \(t\) along the comoving geodesics (which, by the construction of gaussian normal coordinates, are orthogonal to all \(t = \text{const.}\) lines). Along the Cauchy line itself, where I have chosen coordinates so that \(a = 1\) there, one simply has \(K(x) = \dot{a}\).

Any two-dimensional spacetime has its Riemann curvature tensor algebraically determined purely by the metric tensor \(g_{\mu\nu}\) and by the scalar curvature \(R\) as

\[ R_{\mu\nu\rho\sigma} = \frac{1}{2}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \quad (3) \]

and in the gaussian normal coordinates above one can readily calculate that the scalar curvature of the spacetime is

\[ R = \frac{2\ddot{a}}{a} = 2(\dot{K} + K^2). \quad (4) \]

(I am using the sign conventions of MTW \[18\], except that the sign of the extrinsic curvature is here chosen to be opposite that of MTW in order that expanding comoving geodesics correspond to positive extrinsic curvature.)

The instantaneous Hamiltonian on the Cauchy line for a (one-component) real massless scalar field \(\phi\) is simply

\[ H = \int_0^L \frac{1}{2}(\pi_\phi^2 + \phi'^2)dx, \quad (5) \]

where \(\pi_\phi\) is the momentum conjugate to \(\phi\) and the prime denotes a spatial derivative with respect to the proper distance \(x\) along the line. (For the instantaneous Hamiltonian on another \(t = \text{const.}\) line, the prime must mean \(a^{-1}d/dx\) in order that it be the derivative with respect to proper distance there, and \(dx\) must be replaced by \(a\,dx\) in order that it be the proper distance interval, so when \(a\) is time-dependent the instantaneous Hamiltonian depends on the time of the line.) The instantaneous Hamiltonian on the Cauchy line depends only on the intrinsic properties of the line and not on its embedding in the spacetime or on the curvature of the spacetime (though of course the proper distance along the line, and thus also the length of the line, is induced from the metric of the spacetime). Thus, on the Cauchy line itself, the instantaneous vacuum of the normal-ordered Hamiltonian depends only on the intrinsic properties of that line.

In fact, a short calculation shows that this instantaneous vacuum, for an un-twisted scalar field (one with periodic boundary conditions in the periodic spatial
coordinate \( x \), has the configuration-space representation wavefunctional

\[
\Psi [\phi(x)] \propto \exp \left\{ \frac{1}{4\pi} \int_0^L dx \int_0^L d\tilde{x} \ln \left[ 4 \sin^2 \frac{\pi(x - \tilde{x})}{L} \right] \phi'(x)\phi'(\tilde{x}) \right\},
\]

(6)

where the prime on \( \phi'(\tilde{x}) \) has the obvious meaning of \( d/d\tilde{x} \). The zero-mode, \( \phi(x) = \text{const} \), is not damped and is not normalizable, but this is precisely analogous to the fact that the ground state of a free nonrelativistic particle in infinite flat space is not normalizable, because its position is completely indeterminate \[20, 21, 7\]. With the zero-mode being in its nonnormalizable zero-momentum ground state, it makes no contribution to the stress tensor, though it does make the expectation value of the two-point function \( \phi(x)\phi(\tilde{x}) \) divergent. Only quantities invariant under a constant shift of \( \phi \), such as the stress tensor, can be well defined and finite. However, we shall not need the explicit form of the wavefunctional here, which is only given to show directly that the instantaneous vacuum state on a Cauchy line does not depend on the extrinsic curvature of the line or the curvature of the spacetime.

Now that the notation and situation have been established, the main result, to be established below, can be simply stated: On the Cauchy line, and in the frame in which \( d/dt \) is the unit timelike vector and \( d/dx \) is the orthonormal spacelike vector, the regularized stress-energy tensor expectation value \( T_{\mu\nu} \equiv \langle \hat{T}_{\mu\nu} \rangle \) (not bothering with angular brackets around this expectation value for simplicity, since I will rarely need a symbol for the regularized stress-energy tensor operator \( \hat{T}_{\mu\nu} \) itself) for \( N_b \) massless bosonic field components (e.g., \( N_b \) real scalar fields, or \( \frac{1}{2}N_b \) complex scalar fields; \( N_b \) counts the number of one-particle states for a given momentum) and for \( N_f \) massless fermionic field components (e.g., \( N_f \) real Majorana-Weyl spinor fields or \( \frac{1}{2}N_f \) two-component complex fermion-antifermion spinor fields) has the orthonormal covariant components of energy density

\[
T_{tt} = -\frac{N_b + \frac{1}{2}N_f}{24\pi} \left( K^2 + \epsilon \frac{4\pi^2}{L^2} \right),
\]

(7)

energy flux or momentum density

\[
T_{tx} = T_{xt} = -\frac{N_b + \frac{1}{2}N_f}{24\pi} 2K',
\]

(8)

and pressure

\[
T_{xx} = -\frac{N_b + \frac{1}{2}N_f}{24\pi} \left( K^2 - R + \epsilon \frac{4\pi^2}{L^2} \right),
\]

(9)

where \( \epsilon = 1 \) for untwisted (i.e., periodic in \( x \)) massless bosonic fields or for a standard twisted (i.e., antiperiodic in \( x \)) massless fermionic fields, \( \epsilon = -\frac{1}{2} \) for a twisted massless bosonic field, \( \epsilon = -2 \) for an untwisted massless fermionic field, \( \epsilon = 1 - 6\chi + 6\chi^2 \) for complex bosonic fields \( \phi \) with the boundary condition

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\( \phi(x + L) = e^{2\pi i \chi \phi(x)} \) with \( 0 \leq \chi \leq 1 \), and \( \epsilon = -1 + 6\chi - 6\chi^2 \) for complex fermionic fields \( \psi \) with the boundary condition \( \psi(x + L) = e^{2\pi i \chi \psi(x)} \), again with \( 0 \leq \chi \leq 1 \).

(The ordinary untwisted fields correspond to \( \chi = 0 \) or \( \chi = 1 \), and the standard twisted fields correspond to \( \chi = \frac{1}{2} \).

In the theory of closed strings [24], the Ramond boundary condition [25] gives untwisted fermionic fields, and the Neveu-Schwarz boundary condition [26] gives standard twisted fermionic fields on the string world-sheet. Twisting with \( 2\chi \) nonintegral occurs in orbifolds and in four-dimensional fermionic string constructions [27, 28, 29], or to coupling a charged field to a flat \( U(1) \) connection that is topologically nontrivial over the \( S^1 \) corresponding to the periodic \( x \).) If there are \( N_b \) bosonic field components with various twistings \( \chi_i \) and \( N_f \) field components with various twistings \( \chi_j \), the general formula for \( \epsilon \) is

\[
\epsilon = \left( N_b + \frac{1}{2}N_f \right)^{-1} \left[ \sum_{i=1}^{N_b} \left( 1 - 6\chi_i + 6\chi_i^2 \right) - \sum_{j=1}^{N_f} \left( 1 - 6\chi_j + 6\chi_j^2 \right) \right].
\]

Since the \( \chi \)'s are restricted to lie between 0 and 1 inclusive, one can immediately see that \( \epsilon \) is restricted to lie between \(-2\) (its value for purely untwisted fermionic fields) and 1 (its value for purely untwisted bosonic fields and/or standard twisted fermionic fields).

This stress tensor expectation value is nonlocally determined, because the quantum state (the instantaneous vacuum) is nonlocally defined, but in 1+1 dimensions the nonlocal dependence is on the only nonlocal intrinsic geometric property of the Cauchy line, namely its length \( L \) (along with the type of boundary condition for the field). The dependence on the local quantities of the extrinsic curvature of the line and the scalar curvature of the spacetime arise from the definition of the regularized stress tensor operator. In higher dimensions, the regularized stress tensor operator would remain local, but the instantaneous vacuum state would in general have a much more complicated nonlocal dependence on the intrinsic geometry of the spatial Cauchy hypersurface.

II. DERIVATION OF THE TRACE ANOMALY

In order to derive the regularized stress tensor expectation values given above, we use the energy-momentum conservation law and the trace anomaly. For the benefit of those who, like me, may have difficulty in remembering the precise formula for the trace anomaly even in the simple case of 1+1 dimensions, let me give a simple derivation of it from certain easily-remembered facts that I will not derive. (Others may wish to skip this digression, though it gives a simple case of the method to be used to calculate the stress tensor on a general Cauchy line. After coming up with it, I found that somewhat similar simple derivations had been given in [8, 12, 13], and
doubtless every other element of this derivation is also somewhere in the literature, though I have not seen all the parts put together previously in such a simple way that the value of the trace anomaly can literally be worked out in one’s head in the middle of the night.)

The trace of the regularized stress-energy tensor of a massless field in 1+1 dimensions is a local covariant analytic scalar function of the metric and its derivatives that has no dimensional parameters in it and which vanishes in flat spacetime. Its dimensionality of inverse length squared then forces it to be $\alpha R$ for some pure number $\alpha$ [6], which might in principle depend on the number and types of fields, but which does not depend on the global boundary conditions of the fields. (The dimensionality requires the function to have two derivatives in it, and multiples of the scalar curvature are the only local covariant scalar functions of the metric and its derivatives with precisely two derivatives. If one used a covariant expression involving more derivatives of the metric, such as $R_{\mu\nu}$, one would have to take a root to get the dimension right, but this would not be analytic in the limit of flat spacetime.)

Consider a spacetime which is flat $S^1 \times R^1$ for $t < 0$ and the unit deSitter metric for $t > 0$, with metric
\[ ds^2 = -dt^2 + [\theta(-t) + \theta(t) \cosh^2 t]dx^2 \] (11)
(where $\theta$ is the step function that is 0 for negative argument, $\frac{1}{2}$ for zero argument, and 1 for positive argument) and with $x$ identified with period $L = 2\pi$. The instantaneous vacuum for the Cauchy line $t = 0$ is obviously the same as for any line of constant $t < 0$ and gives the static vacuum for the flat region of the spacetime. (It also is the same as the instantaneous vacuum for any line with constant $t > 0$, but I will not need this fact here.) This vacuum has, in the flat region $t < 0$, a constant Casimir energy density $\rho_C$ and an equal pressure $P_C = \rho_C$ (since the trace is zero in the flat region). There is no energy flux in this situation, which is symmetrical under $x \to -x$.

When one crosses the Cauchy line to $t$ slightly positive, the trace anomaly $\alpha R$ can change the pressure. However, the energy-momentum conservation law prevents the energy density from changing discontinuously across the Cauchy line. (It would jump, as we shall find in the more general case below, if the pressure had a term that were a delta function in time, but this could arise only if the curvature, and hence the trace anomaly, had a delta function in it. This would occur if the intrinsic curvature were not the same on the two sides of the Cauchy line, but it is zero on both sides in the metric above, since the Cauchy line is a geodesic for both metrics.) Thus the energy density is $\rho_C$ also on the deSitter side of the Cauchy line.

The static vacuum for the flat region of the spacetime ($t \leq 0$), evaluated on the geodesic Cauchy line at $t = 0$, is given by a path integral over field configurations, on the $\tau \leq 0$ half of the Euclidean (i.e., positive-definite-metric) cylinder obtained by
setting \( t = -i\tau \) and analytically continuing \( \tau \) to real negative values, which match the argument of the wavefunctional on the \( \tau = 0 \) Cauchy line and which vanish asymptotically as \( \tau \to -\infty \). Similarly, the deSitter-invariant vacuum for \( t \geq 0 \), evaluated on the geodesic Cauchy line at \( t = 0 \), is given by a path integral over the hemisphere, bounded by this line, that is the Euclidean analytic continuation of the deSitter Lorentzian spacetime metric one gets by setting \( t = i(\frac{\pi}{2} - \theta) \) and taking \( \theta \) to be a real spherical polar angle in the range between 0 (where the hemisphere closes off at its ‘north pole’) and \( \frac{\pi}{2} \) (at the equator where the match is made to the Lorentzian deSitter metric at its geodesic Cauchy line). (This is only true if \( x \) has period \( L = 2\pi \), which is the period one gets by the standard construction of the unit deSitter metric as the metric induced on a unit timelike hyperboloid in flat 2+1 dimensional spacetime. Regularity of fields at the north pole requires a bosonic field to be untwisted and a fermionic field to be twisted in order to give a deSitter-invariant vacuum by this construction, since increasing \( x \) by its period of \( 2\pi \) corresponds to a rotation of \( 2\pi \) radians about the pole. One might have expected this even from the Lorentzian 1+1 dimensional deSitter spacetime, viewed as a timelike hyperboloid embedded in flat 2+1 dimensional spacetime, since going once around the hyperboloid corresponds to a rotation of \( 2\pi \) in the enveloping flat spacetime, and a rotation of \( 2\pi \) reverses the sign of fermionic, but not bosonic, fields.)

The metric on this hemispheric analytic continuation of the deSitter spacetime is conformally related to the metric on the flat Euclidean half-cylinder. Furthermore, field configurations on the hemisphere that are regular at the north pole correspond to field configurations on the cylinder that vanish exponentially as \( \tau \to -\infty \). Therefore, since the action of a massless field in a two-dimensional spacetime is conformally invariant, the path integrals over the hemisphere and half-cylinder give the same values or amplitudes [given by Eq. (3) above for an untwisted scalar field] for the same field configurations on the bounding Cauchy lines. Thus the instantaneous vacuum of an untwisted bosonic or a twisted fermionic field on the \( t = 0 \) Cauchy line, which is the static vacuum for \( t < 0 \), is also the deSitter invariant vacuum for \( t > 0 \).

One can readily check this by a one-page calculation, which I shall not bother repeating here (and which I could not do in my head in the middle of the night), showing that if one takes any Cauchy line of uniform extrinsic curvature, \( K' = 0 \) (including, of course, the cases \( K = 0 \) in which the Cauchy lines are geodesic, but not restricting oneself to the \( K' = 0 \) lines of constant \( t \)), in the complete 1+1 deSitter spacetime in which \( a = \cosh t \) for all negative as well as positive time, then the resulting instantaneous vacuum is the same, as it should be in order to have the boost and conformal invariance of the deSitter spacetime. (There is no mixing of positive and negative frequency modes between any two Cauchy lines of uniform extrinsic curvature in 1+1 dimensional deSitter spacetime.)
Therefore, the instantaneous vacuum (for a periodic bosonic field or an antiperiodic fermionic field) in the $t > 0$ region of the metric (11) has the same invariance as the deSitter spacetime, and its regularized stress-energy tensor must be simply proportional to the metric. This gives a trace of twice the negative of the energy density, namely $-2\rho_C$. Since the scalar curvature of the unit-scale deSitter region of the metric above is $R = 2$, the coefficient of $R$ in the trace anomaly is $\alpha = -\rho_C$, minus the Casimir energy density in the periodic space of length $2\pi$ for $t < 0$.

Next, one remembers that the Casimir energy density in a space periodic in one dimension can readily be obtained from a corresponding thermal energy density and pressure [30]: The Casimir effect can be obtained by a path integral over the Euclidean space of topology $R^1 \times S^1$ obtained by making the Lorentzian time imaginary (i.e., $\tau = it$ real), as discussed above. If the coordinates $\tau$ and $x$ are interchanged, this Euclidean space is precisely the same as the $S^1 \times R^1$ space that one obtains by assigning an imaginary period of magnitude $2\pi$ to time in flat Lorentzian spacetime and then taking this time to be imaginary. This is the space on which the path integral gives the thermal energy density $\rho_T$ of flat spacetime at a temperature $T = \frac{1}{2\pi}$, the inverse of the period of the imaginary time [31–37]. The antiperiodicity in $x$ that we needed above for the fermionic field becomes precisely the antiperiodicity in imaginary time needed for a thermal state of fermionic fields. Because of making the opposite coordinates of the Euclidean $R^1 \times S^1$ imaginary in going to the unbounded Lorentzian thermal case in contrast to going to the periodic Lorentzian (Casimir) vacuum case, and because the two Euclidean orthonormal components $T_{\tau\tau}$ and $T_{xx}$ have equal magnitudes but opposite signs in order to give a trace of zero, one gets $\rho_C = -\rho_T$ and hence $\alpha = \rho_T$. Finally, an elementary calculation gives, for a one-component bosonic field (e.g., a real scalar field) at temperature $T = \frac{1}{2\pi}$,

$$\alpha_s = \rho_{Ts} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{|k|}{e^{|k|/T} - 1} = \frac{\pi T^2}{6} = \frac{1}{24\pi}.$$  \hspace{1cm} (12)

One gets the same result [14] for a two-component massless fermionic field (e.g., neutrinos plus anti-neutrinos), with the integral for each component being half as large:

$$\alpha_f = \rho_{T_f} = 2\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{|k|}{e^{|k|/T} + 1} = \frac{\pi T^2}{6} = \frac{1}{24\pi}.$$  \hspace{1cm} (13)

Since the trace anomaly is a local quantity, it does not depend on the periodicity properties of the fields. Thus it is $\frac{R}{24\pi}$ for either a one-component bosonic field (untwisted or twisted) or a two-component fermionic field (twisted or untwisted). Of course, for a two-component bosonic field, such as a complex scalar field, it is twice as large. Therefore, for $N_b$ bosonic components and $N_f$ fermionic components, the trace anomaly is

$$T^\mu_\mu = \alpha R = \frac{1}{24\pi} (N_b + \frac{1}{2} N_f) R.$$  \hspace{1cm} (14)
III. CASIMIR ENERGY DENSITY OF TWISTED FIELDS IN ARBITRARY DIMENSIONS

For use below, it is also of interest to calculate the Casimir energy density for twisted bosonic and untwisted fermionic fields. In a periodic one-dimensional space of length $L$, it has been given by Isham \[22\] as $\frac{\pi}{12L^2}$ for a standard twisted scalar field $[\phi(x + L) = -\phi(x)]$, which is $-\frac{1}{2}$ the value one gets by the argument above for an untwisted scalar field (after scaling the length from $2\pi$ to $L$). Citing results from \[38, 22, 39\] in 1+3 dimensions, Avis and Isham \[40\] note that “twice the (regularized) self-energy of a scalar field plus the spinor’s self-energy sums to zero: a typical supersymmetry result.” This indeed gives $\frac{\pi}{12L^2}$ for a twisted massless real scalar field, and it also gives $-2$ times the value for an untwisted scalar field, or $\frac{\pi}{6L^2}$, for an untwisted massless two-component fermionic field in 1+1 dimensions, which is what I shall take here, though Davies and Unruh \[14\] seem to state a value half as large, and Birrell and Davies \[16\] seem to state a value twice as large (possibly from assuming a different number of components for the fermionic field).

Another possibility one can consider is the case in which one has a complex field which, when one goes once around the period of $x$, returns to its original value multiplied by the phase $e^{2\pi i \chi}$, with $0 \leq \chi \leq 1$. (Alternatively, the complex field may be left periodic but coupled to a flat but nontrivial $U(1)$ gauge field whose integral around the period of $x$, when multiplied by the charge coupling constant of the field, gives a phase of $2\pi i \chi$, up to an integral multiple of $2\pi$ that has no effect.) For a complex scalar field with this partial twisting in 1+1 dimensions, Dowker and Banach \[39\] find that the Casimir energy density is $2(1 - 6\chi + 6\chi^2)$ times the untwisted ($\chi = 0$) density of $-\frac{\pi}{6L^2}$ a single real scalar field. (The factor of 2 is for the two real components of the complex scalar field.) If the supersymmetry result stated by Isham \[22\] is valid when extended to this case, that the fermionic Casimir energy density is the opposite of that of the same number of bosonic components with the same periodicity conditions, then a two-component fermionic field which gets multiplied by the phase $e^{2\pi i \chi}$ when one goes around the loop given by the coordinate $x$ of proper length $L$ would have a Casimir energy density of $2\pi(1 - 6\chi + 6\chi^2)/(6L^2)$. This does give the same values as above in the untwisted case ($\chi = 0$) and in the standard twisted case ($\chi = \frac{1}{2}$), so in this paper I shall assume that it is correct. Then the Casimir energy density in 1+1 dimensions for $N_b$ bosonic field components with various twistings $\chi_i$ and $N_f$ field components with various twistings $\chi_j$ is

$$\rho_C = -\frac{4\pi^2 \alpha \epsilon}{L^2} = -\frac{\pi}{6L^2} \left[ \sum_{i=1}^{N_b} (1 - 6\chi_i + 6\chi_i^2) - \sum_{j=1}^{N_f} (1 - 6\chi_j + 6\chi_j^2) \right]. \quad \text{(15)}$$

Incidentally, if one combines the supersymmetry result \[40\] with the trick of Toms \[30\] for getting the Casimir energy density in a periodic space of $S^1$ length $L$. 
(with periodic boundary conditions for bosonic fields and with antiperiodic boundary conditions for fermionic fields) as the negative of the thermal pressure in an infinite space with temperature $1/L$, then one can readily give it for both untwisted and standard twisted bosonic and fermionic fields in a flat spacetime of dimension $D$ and topology $S^1 \times R^{D-1}$:

$$
\rho_C = \pi^{-D/2} \Gamma(D/2) \zeta(D) L^{-D} [-N_{ub} + N_{uf} + (1 - 2^{1-D})(N_{tb} - N_{tf})],
$$

(16)

where $\Gamma(D/2)$ is the ordinary gamma function,

$$
\zeta(D) \equiv \sum_{n=1}^{\infty} \frac{1}{n^D}
$$

(17)

is the ordinary Riemann zeta function, $L$ is the (constant) length of the periodic $S^1$ spatial dimension, $N_{ub}$ is the number of untwisted (periodic in the $S^1$) bosonic field components (degrees of freedom or number of one-particle states for a given momentum), $N_{uf}$ is the number of untwisted fermionic field components, $N_{tb}$ is the number of twisted (antiperiodic in the $S^1$) bosonic field components, and $N_{tf}$ is the number of twisted fermionic field components.

Furthermore, one can readily generalize the results of Dowker and Banach [39] (say by using the method of images for the Green function [38]) to get the Casimir energy density of $N_b$ bosonic and $N_f$ fermionic field components that are paired into complex components that are twisted by the arbitrary phase angles $2\pi \chi_i$:

$$
\rho_C = -\pi^{-D/2} \Gamma(D/2) L^{-D} \sum_{i=1}^{N_b+N_f} (-1)^{F_i} \sum_{n=1}^{\infty} \frac{\cos 2\pi n \chi_i}{n^D},
$$

(18)

where $F_i = 0$ if the $i$th component is bosonic and $F_i = 1$ if the $i$th component is fermionic. This essentially just replaces the Riemann zeta function in Eq. (16) with the final sum in Eq. (18), which obviously is $\zeta(D)$ for $\chi_i = 0$ or $\chi_i = 1$ and is $-(1 - 2^{1-D}) \zeta(D)$ for $\chi_i = \frac{1}{2}$. When the total number of dimensions of the spacetime is even, $D = 2m$ for some integer $m$, one may evaluate this sum explicitly [41, 42] to get

$$
\rho_C = \frac{(-2\pi)^m}{2m(2m-1)!! L^D} \sum_{i=1}^{N_b+N_f} (-1)^{F_i} B_{2m}(\chi_i)
$$

(19)

$$
= \frac{(-2\pi)^m}{2m(2m-1)!! L^D} \sum_{i=1}^{N_b+N_f} (-1)^{F_i} \sum_{k=0}^{2m} \left( \begin{array}{c} 2m \\ k \end{array} \right) B_k \chi_i^{2m-k}
$$

(20)

in terms of Bernoulli polynomials $B_{2m}(\chi_i)$ or of Bernoulli numbers $B_k$, for $0 \leq \chi_i \leq 1$. This of course reduces to Eq. (15) for $m = 1$ ($D = 2$).
Of course, given the uniform Casimir energy density $\rho_C$ in a periodic flat spacetime, the entire stress-energy may be obtained immediately by Toms’ thermal correspondence [30] as diagonal in the appropriate orthonormal frame, with the pressure in the periodic direction being $(D-1)\rho_C$ and the pressure in each of the $D-2$ transverse directions being $-\rho_C$.

**IV. DERIVATION OF THE STRESS TENSOR OF AN INSTANTANEOUS VACUUM**

Having completed this digression of calculating the trace anomaly of $N_b$ bosonic and $N_f$ fermionic massless field components in 1+1 dimensions and the Casimir energy density in a spacetime of any dimension that is static and periodic in one of the spatial dimensions, we may return to calculating the stress tensor expectation value of the instantaneous vacuum on an arbitrary Cauchy line in an arbitrary 1+1 dimensional spacetime. For this we perform a similar trick of replacing the part of the spacetime below the line ($t < 0$) with a flat spacetime such that the extrinsic curvature is zero on that side. That is, in the metric (1) we keep the correct form for $a(t, x)$ for $t > 0$ but set $a(t, x) = 1$ for $t < 0$. This keeps the intrinsic geometry, $ds^2 = dx^2$, the same on both sides of the Cauchy line, but it gives a discontinuity in the extrinsic curvature $K$ and hence a delta-function contribution in the scalar curvature $R$ of the spacetime. Thus we can no longer argue that the energy density is continuous across the Cauchy line, but we can still readily use the energy-momentum conservation law, along with the trace anomaly, to calculate what the stress tensor is just above the Cauchy line.

This law, $T^{\mu\nu} = 0$, is simplest to solve in null coordinates in which the metric has the form

$$ds^2 = -e^{2\sigma} du dv,$$

(21)

with $\sigma$ being a function of the two null coordinates $u$ and $v$. In terms of $\sigma$, the only nonzero Christoffel symbols in the $(u, v)$ coordinate basis are

$$\Gamma^u_{uu} = 2\sigma_u, \quad \Gamma^v_{vv} = 2\sigma_v,$$

(22)

and the scalar curvature is

$$R = -2\sigma^{;\mu}_{;\mu} = 8e^{-2\sigma}\sigma_{,uv},$$

(23)

where the comma denotes a partial derivative, so the trace anomaly is

$$T^\mu_\mu = -4e^{-2\sigma} T_{uv} = \alpha R = \frac{1}{24\pi}(N_b + \frac{1}{2}N_f)R = \frac{1}{3\pi}(N_b + \frac{1}{2}N_f)e^{-2\sigma}\sigma_{,uv}. \quad (24)$$
Now an explicit calculation of $T_{\mu\nu} = 0$ in these null coordinates, making use of the trace anomaly, readily shows that

$$T_{uu} = 2\alpha [\sigma_{,uu} - \sigma_{,u}\sigma_{,u} + U(u)],$$

(25)

$$T_{uv} = T_{vu} = -2\alpha \sigma_{,uv},$$

(26)

$$T_{vv} = 2\alpha [\sigma_{,vv} - \sigma_{,v}\sigma_{,v} + V(v)],$$

(27)

where $U$ is an arbitrary function of the one null coordinate $u$, and $V$ is an arbitrary function of the other null coordinate $v$.

In the flat spacetime for $t < 0$, we can choose $u = t - x$ and $v = t + x$, so $\sigma = 0$ there. The instantaneous vacuum on the Cauchy line is the static vacuum for $t < 0$, which has $T_{uv} = T_{vu} = 0$ and $\rho_C = T_{uu} + T_{vv} = 4\alpha U(u) = 4\alpha V(v) = -4\pi^2 \alpha \epsilon/L^2$, with $\epsilon$ being defined by Eq. (10) for a general set of massless fields with various periodicity conditions, or by the preceding discussion in special cases. Then Eqs. (25)–(27) gives the stress tensor in the region $t > 0$ in terms of the derivatives of $\sigma$ there.

To calculate the result explicitly, one needs to transform the metric (1) from the $(t,x)$ coordinate system to the null coordinate system $(u,v)$ in which the metric has the form (21). To the order needed, one finds that

$$u \approx t - \frac{1}{2} K t^2 - x,$$

(28)

$$v \approx t - \frac{1}{2} K t^2 + x,$$

(29)

$$\sigma \approx \ln a \approx Kt + (-\frac{1}{2} K^2 + \frac{1}{4} R) t^2.$$

(30)

Here $K$ means $K(x)$, the value of $K(t,x) = \dot{a}/a$ at $t = 0$ [i.e., $K$ as a function purely of $x \approx (v - u)/2$ at $u + v = 0$], rather than the value of $K(t,x)$ at the point of interest.

Now, remembering the implicit $x$ (but not $t$) dependence of $K$ in Eqs. (28)–(30) when one differentiates $\sigma$, one gets the following expressions [with $K'$ denoting $dK(x)/dx$] on the Cauchy line $t = 0$ (or $u + v = 0$):

$$\sigma_{,uu} = \frac{1}{8} R - \frac{1}{2} K',$$

(31)

$$\sigma_{,uv} = \sigma_{,vu} = \frac{1}{8} R,$$

(32)

$$\sigma_{,vv} = \frac{1}{8} R + \frac{1}{2} K'.$$

(33)
Inserting these back into Eqs. (25)–(27), along with \( U(u) = V(v) = -\pi^2 \epsilon / L^2 \), gives the following components of the stress-energy tensor on the Cauchy line in the null coordinate system:

\[
T_{uu} = \alpha \left( \frac{1}{4} R - \frac{1}{2} K^2 - K' - \frac{\epsilon^2}{L^2} \right),
\]
(34)

\[
T_{uv} = T_{vu} = -\frac{1}{4} \alpha R,
\]
(35)

\[
T_{vv} = \alpha \left( \frac{1}{4} R - \frac{1}{2} K^2 + K' - \frac{\epsilon^2}{L^2} \right).
\]
(36)

When these components are transformed back to the \((t, x)\) coordinates (which are orthonormal coordinates on the Cauchy line), and when one uses Eq. (14) for the coefficient \( \alpha \) of \( R \) in the trace anomaly, one gets the stress-energy tensor components listed above in Eqs. (34)–(36). This completes the derivation of the main result.

V. COVARIANT FORMS FOR THE STRESS TENSOR AND CONFORMAL INVARIANTS

It is of interest to write the stress-energy tensor in three explicitly covariant forms, in terms of the latter two of which one may find two nonlocal conformal invariants of the quantum state. For example, the general expressions (25)–(27) for the components of a conserved stress-energy tensor with trace \( \alpha R \) in 1+1 dimensions can be written in terms of an auxiliary scalar field \( \Phi \) in the covariant form

\[
T_{\mu\nu} = \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} g_{\mu\nu} (\nabla \Phi)^2 + 2\kappa (\Phi_{,\mu\nu} - g_{\mu\nu} \Box \Phi),
\]
(37)

where \( \kappa \equiv \sqrt{\alpha / 2} \), \( (\nabla \Phi)^2 \equiv \Phi^{\mu\nu} \Phi_{,\mu\nu} \), and

\[
\Box \Phi \equiv \Phi^{,\mu}_{,\mu} = -\kappa R.
\]
(38)

Here I have chosen the normalization of \( \Phi \) so that if one sets \( \kappa = 0 \) (or, more realistically, considers a field \( \Phi \) with large derivatives so that the quadratic terms in \( \Phi \) in Eq. (37) dominate over the linear terms in \( \Phi \)), then the stress-energy tensor (37) has the standard form for a classical scalar field \( \phi = \Phi \).

When the metric is written in the form of Eq. (21), one may write

\[
\Phi = 2\kappa (\sigma + f),
\]
(39)

where \( \Box f \equiv -4e^{-2\sigma} f_{,uv} = 0 \), so \( f \) must have the form of a function of the null coordinate \( u \) plus another function of the null coordinate \( v \), \( f(u, v) = p(u) + q(v) \).
Then one can easily calculate that in Eqs. (25) and (27),
\[ U(u) = f_{uu} + f_u f_u = \frac{d^2p}{du^2} + \left(\frac{dp}{du}\right)^2 = e^{-p} \frac{d^2}{du^2} e^p, \]  
\[ (40) \]
\[ V(v) = f_{vv} + f_v f_v = \frac{d^2q}{dv^2} + \left(\frac{dq}{dv}\right)^2 = e^{-q} \frac{d^2}{dv^2} e^q. \]  
\[ (41) \]

Although a given $\Phi$ determines a unique stress-energy tensor by Eq. (37), the reverse is not true, in that $\Phi$ is not uniquely determined by $T_{\mu\nu}$, at least locally. That is, for fixed functions $U(u)$ and $V(v)$ (uniquely determining and uniquely determined by $T_{\mu\nu}$ once the null coordinates $u$ and $v$, and hence the conformal factor $e^{2\sigma}$ in the metric (21), are fixed for a given geometry), there are two-parameter local solutions of Eqs. (40) and (41) for both $p(u)$ and $q(v)$. [One of the resulting four parameters cancels when one adds $p(u)$ to $q(v)$ to get $f$, and one of the remaining three parameters corresponds to the obvious invariance of the stress-energy tensor (37) under adding a constant to $\Phi$.] In particular, for the vacuum (Casimir) values $U(u) = V(v) = -\pi^2 \epsilon / L^2$, one has the general local solution for $f$ being
\[ f = f_0 + \ln \sin \frac{\pi \sqrt{\epsilon} (u - u_0)}{L} + \ln \sin \frac{\pi \sqrt{\epsilon} (v - v_0)}{L}, \]  
\[ (42) \]
with the three arbitrary constants $f_0$, $u_0$, and $v_0$. Note that for $\epsilon < 1$, $f$ cannot have the periodicity $(u, v) \equiv (u - L, v + L)$ of the spacetime unless $u_0$ and $v_0$ are taken to $\pm i\infty$, in which case one gets the one-parameter sets of global solutions $f = \pm i\pi \sqrt{\epsilon} (u + v + c)/L$ for the single constant $c$. For $\epsilon = 1$, the general local three-parameter solution (12) has the same periodicity as the coordinates and so is a global solution, but even there one cannot have $u_0$ or $v_0$ real if the solution is to be finite everywhere, and so, just as for other positive values of $\epsilon$, one needs $f$ and hence $\Phi$ to be complex, a consequence of the negativity of the Casimir energy density in that case.

A second covariant way to get a conserved stress-energy tensor with trace $\alpha R$ in 1+1 dimensions is to start with a nowhere null or vanishing closed divergenceless one-form $\omega = \omega_\mu dx^\mu$, i.e., one obeying $\omega \cdot \omega = 0$, $\omega \equiv 0$ (or $\omega_{\mu;\nu} = 0$), and $\delta \omega \equiv *d*\omega = 0$ (or $\omega_{\mu}^{;\mu} = 0$), so that locally, but not necessarily globally, $\omega = dh$ for some scalar field $h$ obeying $\Box h = 0$. In terms of null coordinates $(u, v)$, any closed divergenceless one-form can be written in terms of two functions, $P$ and $Q$, each of one of the two null coordinates:
\[ \omega = P(u) du + Q(v) dv. \]  
\[ (43) \]
scalar field

\[ F = \ln (c \omega \cdot \omega) \equiv \ln (c \omega_\mu \omega^\mu) = -2\sigma + \ln P(u) + \ln Q(v) - \ln (-4c) \]  

(44)  

for an arbitrary nonzero constant \( c \) [e.g., \( c = \text{sgn}(\omega \cdot \omega) \)]. This scalar field automatically obeys \( \Box F = R \). Now one can write the stress-energy tensor as

\[ T_{\mu \nu} = \alpha \left( 2\omega_\mu \omega_\nu - F_{;\mu} - \frac{1}{2} F_{;\mu} F_{;\nu} + [-\omega \cdot \omega + \Box F - \frac{1}{4} (\nabla F)^2] g_{\mu \nu} \right). \]  

(45)  

A comparison with Eqs. (25) and (27) shows that now

\[ U(u) = P^2 + \frac{3}{4P^2} \left( \frac{dP}{du} \right)^2 - \frac{1}{2P} \frac{d^2 P}{du^2} = y^{-4} + y^{-1} \frac{d^2 y}{du^2}; \]  

(46)  

\[ V(v) = Q^2 + \frac{3}{4Q^2} \left( \frac{dQ}{dv} \right)^2 - \frac{1}{2Q} \frac{d^2 Q}{dv^2} = z^{-4} + z^{-1} \frac{d^2 z}{dv^2}; \]  

(47)  

where \( y(u) = P^{-1/2} \) and \( z(v) = Q^{-1/2} \), so

\[ \omega = \frac{du}{y^2(u)} + \frac{dv}{z^2(v)}. \]  

(48)  

Since \( P(u) \) and \( Q(v) \) are nowhere zero, \( y \) and \( z \) are everywhere finite, though they are imaginary if \( P \) and \( Q \) are negative, or complex if \( P \) and \( Q \) themselves are pure imaginary, which would also give a real stress-energy tensor (45).

Analogously to the situation with the stress-energy tensor (37), so the stress-energy tensor (45) is uniquely determined by the closed divergenceless one-form \( \omega \) but does not uniquely determine it, at least locally. For example, \( P(u) \) and \( Q(v) \) uniquely determine \( U(u) \) and \( V(v) \), but there is a two-parameter set of local solutions of Eqs. (45) and (46) for each of them for given \( U(u) \) and \( V(v) \). In particular, for the vacuum (Casimir) values \( U(u) = V(v) = -\pi^2 \epsilon/L^2 \), one has the general local solutions for \( P \) and \( Q \) being

\[ P(u) = \frac{\pm i\pi \sqrt{\epsilon}/L}{\cosh \psi_0 + \sinh \psi_0 \sin \left[ 2\pi \sqrt{\epsilon}(u - u_0)/L \right]}, \]  

(49)  

\[ Q(v) = \frac{\pm i\pi \sqrt{\epsilon}/L}{\cosh \tilde{\psi}_0 + \sinh \tilde{\psi}_0 \sin \left[ 2\pi \sqrt{\epsilon}(v - v_0)/L \right]}, \]  

(50)  

where \( u_0, \psi_0, v_0, \text{and} \tilde{\psi}_0 \) are four arbitrary constants (possibly complex), and where each of the two arbitrary signs may be chosen independently. However, if the spacetime has the periodicity \((u, v) \equiv (u - L, v + L) \) that we have been assuming, so that \( U(u + L) = U(u) \) and \( V(v + L) = V(v) \), then unless \( \epsilon = 1 \), we must have \( \psi_0 = 0 \) and \( \tilde{\psi}_0 = 0 \), so that one has only the four discrete (zero-parameter) solutions

\[ P(u) = \pm Q(v) = \pm i\pi \sqrt{\epsilon}/L, \]  

(51)
again with the two signs independent.

For a spacetime of topology \( R^1 \times S^1 \), the one-form \( \omega \) that determines the stress-energy tensor (15) gives two conformal invariants of the quantum state, each a nonlocal functional of the geometry and of the stress-energy tensor field:

\[
E_u = \epsilon + \left( \frac{1}{\pi} \int_{u_0}^{u_0+L} \omega_u du \right)^2, \quad (52)
\]

\[
E_v = \epsilon + \left( \frac{1}{\pi} \int_{v_0}^{v_0+L} \omega_v dv \right)^2, \quad (53)
\]

where the integral for \( E_u \) is taken along the null line \( v = \text{const.} \), and the integral for \( E_v \) is taken along the null line \( u = \text{const.} \) (Of course I could have simply written each integrand above as the one-form \( \omega \), but the coordinate forms above shows more explicitly which null coordinate is to varied for the corresponding integral.)

It is important to note that in the cases in which the two-parameter set of local solutions for either \( P \) or \( Q \) are also global solutions in the topology \( R^1 \times S^1 \) (which occurs only when \( \epsilon = 1 \) and only when either the left-moving modes or the right-moving modes, respectively, are in their ground state in some coordinate and conformal frame), then any choice out of the possible solutions of Eq. (46) or (47) for \( P \) or \( Q \) gives the same value for the corresponding conformal invariant, namely \( E_u = 0 \) or \( E_v = 0 \). That is, in these special cases in which there are many global choices for \( \omega = P(u) du + Q(v) dv \) that lead to the same stress-energy tensor (15), each of these choices gives the same values in Eqs. (52) and (53) for the invariants \( E_u \) and \( E_v \). This ambiguity in \( \omega \) occurs for the vacuum state of untwisted bosonic fields and standard twisted fermionic fields in deSitter spacetime, which is the instantaneous vacuum for any geodesic Cauchy line through any point: one gets different choices for \( \omega \) by setting \( P(u) \) and \( Q(v) \) to have the constant value given by Eq. (0) in each of various different null coordinate systems \( (u,v) \) such that \(-u = v = x \) along the corresponding geodesic Cauchy line with proper distance \( x \), and these choices, when transformed back to a single coordinate system, give \( P(u) \) and \( Q(v) \) of the forms given by Eqs. (19) and (31).

\( E_u \) and \( E_v \) are invariants not only under a change of the null coordinates \( u \) and \( v \) to new null coordinates \( \tilde{u}(u) \) and \( \tilde{v}(v) \) and hence a corresponding shift in the metric (21) of \( \sigma \) to \( \tilde{\sigma} = \sigma - \frac{1}{2} \ln \frac{du}{\tilde{du}} - \frac{1}{2} \ln \frac{dv}{\tilde{dv}} \), but also under an arbitrary conformal transformation (Weyl rescaling that may vary arbitrarily with position) that changes \( \sigma(u,v) \) to an arbitrary new function of \( u \) and \( v \). I have chosen the constant terms in Eqs. (52) and (53) so that \( E_u \) and \( E_v \) are both zero if and only if the quantum state is the instantaneous vacuum of a set of quantum fields [with \( \epsilon \) determined by Eq. (10) from the boundary conditions of how twisted the various fields are] for at least one Cauchy line (which can be put into the form \( \tilde{u} + \tilde{v} = 0 \) for some choice of
null coordinates $\tilde{u}$ and $\tilde{v}$) and for at least one choice of the conformal factor in the metric (in particular, for the choices that gives $\tilde{\sigma} = \text{const.}$). For a state which is not the instantaneous vacuum for any choice of Cauchy line and for any conformally transformed metric, $\mathcal{E}_u$ and/or $\mathcal{E}_v$ are positive. [Neither one can be negative for the stress-energy tensor arising from a quantum state of a set of fields with the corresponding value of $\epsilon$, though one can certainly choose an (imaginary) $\omega$, and hence an artificial conserved stress-energy tensor ($^{[13]}$) with the correct trace but with $\mathcal{E}_u$ and/or $\mathcal{E}_v$ arbitrarily negative. Analogously, one can write down in Minkowski spacetime a covariantly constant (and hence conserved) artificial stress-energy tensor that has a constant negative energy density everywhere, but one cannot get such a tensor as the expectation value of the regularized stress-energy tensor operator in any quantum state.] I have arbitrarily chosen the normalization of $\mathcal{E}_u$ and $\mathcal{E}_v$ so that for any combination of untwisted massless bosonic fields and the standard twisted massless fermionic fields (giving $\epsilon = 1$ and a negative Casimir energy density for the ground state on a spacetime geometry that has a static flat metric on $R^1 \times S^1$), the excited states with zero stress-energy tensor on this geometry give $\mathcal{E}_u = 1$ and $\mathcal{E}_v = 1$.

Of course, there are an infinite number of invariants (under coordinate and conformal or Weyl transformations of the metric) of a quantum state of massless fields in 1+1 dimensions, but the only independent ones that are determined purely by the stress-energy tensor are $\mathcal{E}_u$ and $\mathcal{E}_v$ given above. One can readily see this, because if one chooses null coordinates $\tilde{u} \propto \int \omega_u du$ and $\tilde{v} \propto \int \omega_v dv$ but normalized so that the periodicity is $(\tilde{u}, \tilde{v}) \equiv (\tilde{u} - L, \tilde{v} + L)$, and then makes a conformal or Weyl transformation so that $\tilde{\sigma} = 0$ in this coordinate system, then the stress-energy tensor has only the two constant components $T_{\tilde{u}\tilde{u}}$ and $T_{\tilde{v}\tilde{v}}$, and these two constants determine only two invariants. In terms of these two constants, the invariants given above are

$$\mathcal{E}_u = \epsilon + \frac{L^2}{2\alpha \pi^2} T_{\tilde{u}\tilde{u}},$$  

$$\mathcal{E}_v = \epsilon + \frac{L^2}{2\alpha \pi^2} T_{\tilde{v}\tilde{v}}.$$  

In particular, if one has a static state for a single untwisted bosonic field (one with $\epsilon = 1$ and $\alpha = \frac{1}{24\pi}$) in which $T_{\tilde{u}\tilde{u}} = T_{\tilde{v}\tilde{v}} = \frac{L}{2\pi}$, and one further sets the length of the $S^1$ to be $L = 2\pi$, then $\mathcal{E}_u = \mathcal{E}_v = 1 + 12E$, with the ground state having the energy $E = -\frac{1}{12}$.

Other independent coordinate and conformal invariants of the quantum state do not affect the stress-energy tensor. For example, in the static flat geometry on $R^1 \times S^1$, the quantum state which is the vacuum excited by precisely two particles in the left-moving mode $e^{-2\pi i u/L}$ with wavelength $L$ has the same stress-energy tensor as the quantum state which is the vacuum excited by precisely one particle in the
left-moving mode \( e^{-4\pi i u/L} \) with wavelength \( L/2 \), though these states are surely not equivalent to each other under any coordinate and/or conformal transformations.

A third and very similar covariant way to get a conserved stress-energy tensor with trace \( \alpha R \) in 1+1 dimensions is to start with a nowhere null or vanishing conserved traceless symmetric second-rank tensor \( \tau \) (i.e., a tensor with covariant components \( \tau_{\mu\nu} \) obeying \( Tr\tau^2 \equiv \tau_{\alpha\beta}\tau_{\alpha\beta} \neq 0 \), \( \tau_{\mu\nu} \neq 0 \), \( Tr\tau \equiv \tau_{\alpha} = 0 \), and \( \tau_{\mu\nu} = \tau_{\nu\mu} \)), which in terms of null coordinates \((u,v)\) must have the form

\[
\tau = \tau_{uu}(u)du \otimes du + \tau_{vv}(v)dv \otimes dv
\]

where \( \tau_{uu}(u) \) and \( \tau_{vv}(v) \) are each a function only of the corresponding null coordinate and are each everywhere nonzero. Then construct the scalar function

\[
\mathcal{F} = \frac{1}{4} \ln [Tr(\tau^2)^2] \equiv \frac{1}{4} \ln [(\tau_{\alpha\beta}\tau_{\alpha\beta})^2]
\]

and the stress-energy tensor

\[
T_{\mu\nu} = \tau_{\mu\nu} + \alpha \left\{ \frac{1}{2} \mathcal{F}_{;\mu}\mathcal{F}_{;\nu} - \mathcal{F}_{;\mu\nu} + [\Box \mathcal{F} - \frac{1}{4} (\nabla \mathcal{F})^2]g_{\mu\nu} \right\}.
\]

If one compares this to Eq. (45), one sees that

\[
\tau = \alpha(2\omega \otimes \omega - \omega \cdot \omega g),
\]

or, in component form,

\[
\tau_{\mu\nu} = \alpha(2\omega_{\mu}\omega_{\nu} - \omega^\alpha\omega_\alpha g_{\mu\nu}),
\]

and that \( \mathcal{F} \) is the same as \( \mathcal{F} \), up to an additive constant that drops out of the derivatives appearing in Eqs. (43) and (58). Furthermore, one can readily calculate that, analogous to Eqs. (56) and (57), \( \tau_{uu}(u) \) and \( \tau_{vv}(v) \) obey the nonlinear second-order differential equations

\[
4\alpha \tau_{uu}\tau_{uu;uu} - 5\alpha \tau_{uu;u}\tau_{uu;u} - 8\alpha \tau_{uu}\tau_{uu}\tau_{uu} + 8T_{uu}\tau_{uu}\tau_{uu} = 0,
\]

\[
4\alpha \tau_{vv}\tau_{vv;vv} - 5\alpha \tau_{vv;v}\tau_{vv;v} - 8\alpha \tau_{vv}\tau_{vv}\tau_{vv} + 8T_{vv}\tau_{vv}\tau_{vv} = 0,
\]

One might consider \( \tau_{\mu\nu} \) to be the “classical” (traceless) stress-energy tensor corresponding to the full quantum stress-energy tensor \( T_{\mu\nu} \) (by which is meant here, as throughout this paper, the expectation value \( \langle \hat{T}_{\mu\nu} \rangle \) of the regularized stress-energy tensor operator \( \hat{T}_{\mu\nu} \) in the quantum state of the fields), though the second-order character of Eqs. (61) and (62) mean that \( \tau_{\mu\nu} \) is not locally uniquely determined by \( T_{\mu\nu} \). However, in a spacetime with the topology \( R^1 \times S^1 \), as we have been assuming in this paper, \( \tau_{\mu\nu} \) is globally uniquely determined by \( T_{\mu\nu} \) if \( \epsilon \neq 1 \) or if both the invariants \( \mathcal{E}_u \) and \( \mathcal{E}_v \) are nonzero.
We can also readily write these coordinate and conformal invariants in terms of the tensor $\tau_{\mu\nu}$ in the following way: Construct the auxiliary flat metric

$$d\tilde{s}^2 = \tau_{\mu\nu} dx^\mu dx^\nu = \tau_{uu}(u) du^2 + \tau_{vv}(v) dv^2,$$

which can have any signature if $\epsilon > 0$ (but which is necessarily positive definite if $\epsilon < 0$), and using it evaluate the squared spacetime interval $\tilde{s}_u^2$ (the square of the geodesic distance if the points are spacelike separated, or minus the square of the proper time separation if the points are timelike separated) between the points $(u, v)$ and $(u + L, v)$, and the squared spacetime interval $\tilde{s}_v^2$ between the points $(u, v)$ and $(u, v + L)$. Then a comparison with Eqs. (52) and (53) shows that

$$E_u = \epsilon + \frac{\tilde{s}_u^2}{2\alpha^2 \pi^2},$$

$$E_v = \epsilon + \frac{\tilde{s}_v^2}{2\alpha^2 \pi^2}. \tag{65}$$

These formulas show that it might have been more natural to define these invariants to be a factor of $2\alpha^2 \pi^2$ larger than what I have defined, which would have also made them scale linearly with the stress-energy tensor if one increased the number of fields (and hence $\alpha$), but I have chosen a normalization so that if $\epsilon = 1$, states with zero energy flux and nonpositive Casimir energy density in a spatially-periodic static flat spacetime have the invariants between 0 (the value for the ground state) and 1 (the value when the energy density is zero).

VI. EXAMPLES OF INSTANTANEOUS VACUUM STRESS TENSORS IN SIMPLE SPACETIMES

Now we can apply the formulas (7)–(9) or (34)–(36) for the stress-energy tensor of an instantaneous vacuum (which necessarily has $E_u = E_v = 0$) to a few simple examples. We start with the 1+1 dimensional deSitter spacetime and one type of modification of it. For example, if we apply the formulas above to the unit deSitter metric of Eq. (11) (with $R = 2$) for a Cauchy line of constant $t > 0$, we find that $L = 2\pi \cosh t$, $K = \tanh t$, $K' = 0$, so for the untwisted one-component bosonic or twisted two-component fermionic field one gets the deSitter-invariant stress tensor $T_{\mu\nu} = \frac{1}{16\pi} R g_{\mu\nu}$ as the stress tensor of the instantaneous vacuum corresponding to any Cauchy line of constant $K$. There is a unique state giving this stress tensor, the deSitter invariant state of the massless field (provided one allows the homogeneous mode to be in its nonnormalizable state of zero conjugate momentum [24, 21, 7]).
However, if we apply the stress tensor calculation to a modified deSitter metric with the same local metric components as Eq. (11), but with $x$ having a period $2\pi \lambda$ different from $2\pi$ for a constant $\lambda$ different from one, then the length of the $t = 0$ Cauchy line is $L = 2\pi \lambda$, and the stress tensor of the instantaneous untwisted bosonic or twisted fermionic field vacuum of this line is not deSitter invariant (at least for for $\epsilon = 1$). For example, the null radiation components are

$$T_{uu} = T_{vv} = \frac{1}{2} \alpha (1 - \epsilon \lambda^{-2}),$$

and these are not invariant under boosts, unless $\epsilon = \lambda^2$. (For $N_b$ bosonic field components and $N_f$ fermionic field components all partially twisted by the angle $2\pi \chi$, evaluating $\epsilon$ by Eq. (10) gives the null radiation components as

$$T_{uu} = T_{vv} = \frac{N_b (\lambda^2 - 1 + 6\chi - 6\lambda^2) + N_f (\frac{1}{2} \lambda^2 + 1 - 6\chi + 6\lambda^2)}{48\pi \lambda^2},$$

which can be made zero for any positive $\lambda < 1$ by two particular choices of $\chi$, symmetrically arranged about $\chi = \frac{1}{2}$, but for $\lambda > 1$ there is no real choice of $\chi$ which will make the null radiation components zero for $N_b$ and/or $N_f$ positive.)

If one writes the unit local deSitter metric given by the $t > 0$ part of Eq. (25) in null coordinates $u = gd t - x \equiv \sin^{-1} \tanh t - x$ and $v = gd t + x \equiv \sin^{-1} \tanh t + x$, where $gd$ is the Gudermannian function, the metric becomes

$$ds^2 = \frac{-du dv}{\cos^2 [(u + v)/2]}.$$  

The modification of the global deSitter structure to give $x$ period $2\pi \lambda$, rather than $2\pi$, means that $(u, v)$ is identified with $(u - 2\pi \lambda, v + 2\pi \lambda)$. Then Eqs. (25)–(27) imply that the null radiation components given by Eq. (66) or (67) are the same in this coordinate system over the entire modified deSitter spacetime. These are also the radiation components of the instantaneous vacuum with respect to any $t = \text{const}$ Cauchy line, since all these vacua are the same. [However, one must note that the null coordinates defined by Eqs. (28) and (29) near a $t = \text{const} \neq 0$ Cauchy line are scaled to be a factor $\cosh t$ larger than the global null coordinates in Eq. (58).] The constancy of these covariant null components of the stress tensor does not imply the constancy of the orthonormal components, and indeed the trace of the square of the traceless or radiation part of the stress-energy tensor,

$$\tilde{T}_{\mu \nu} \equiv (T_{\mu \nu} - \frac{1}{2} T_\rho g_{\mu \nu}),$$

depends on the time $t = gd^{-1}[(u + v)/2] \equiv \tanh^{-1} \sin [(u + v)/2]$: 

$$\tilde{T}^{\mu \nu} \tilde{T}_{\mu \nu} = 2\alpha^2 (1 - \epsilon \lambda^{-2})^2 \cosh^{-4} t = 2\alpha^2 (1 - \epsilon \lambda^{-2})^2 \cos^4 [(u + v)/2].$$  

20
Thus the instantaneous vacuum of any of the $t = \text{const}$. Cauchy surfaces has a stress tensor which is not only frame dependent (not invariant under boosts) but is also inhomogeneous (in time), unless $\epsilon = \lambda^2$. (Remember that $\epsilon \leq 1$.)

For a generic $\lambda$ this is not at all surprising, since the period (in $x$) of the oscillations of $t$ of the spatial geodesics $t = \tanh^{-1} [\gamma \sin (x - x_0)]$ for arbitrary $\gamma$ is $2\pi$, so if $x$ does not have a period that is a integer multiple of this, then the geodesics (except for the one with $\gamma = 0$) do not close in one period of $x$, and the global structure of the spacetime does not have the $SO(2,1)$ invariance of the standard deSitter spacetime. Nevertheless, for $\lambda$ equal to an integer $n$ (larger than one, in order not to be the standard deSitter spacetime), one simply has an $n$-fold covering spacetime of the standard deSitter spacetime, so the geodesics do close in one period of $x$ and one has a spacetime with the same $SO(2,1)$ invariance as the standard deSitter spacetime. Yet the stress-energy tensor of any of the instantaneous vacua corresponding to these geodesics does not share this $SO(2,1)$ invariance. [In fact the vacua corresponding to geodesics of different $\gamma$ in a fixed coordinate system, while being related by $SO(2,1)$ transformations, are not identical but have stress tensors with the null components $T_{uu}$ and $T_{uv}$ at a common intersection point ($t = 0, x = x_0$) scaled down and up respectively by the squared boost factor $\frac{1 + \gamma^2}{1 - r^2}$.]

One may define an ‘instantaneous radiation energy’ $\tilde{E}$ on a Cauchy line to be the integral over the line of the local energy density $\tilde{T}_{tt}$ of the traceless or radiation part of the stress-energy tensor expectation value in an arbitrary quantum state:

$$
\tilde{E} = \int_0^L dx \tilde{T}_{tt} = \int_0^L dx \frac{1}{2} (T_{tt} + T_{xx}) = \langle : H : \rangle - \frac{4\pi^2 \alpha \epsilon}{L} + \alpha \int_0^L dx \left( \frac{1}{2} R - K^2 \right),
$$

(71)

the expectation value $\langle : H : \rangle$ of the normal-ordered instantaneous Hamiltonian $H$ of Eq. (4), shifted by a function of the intrinsic and extrinsic properties of the line and the enveloping spacetime. [Here $\tilde{T}_{tt}$ is the time-time component in the orthonormal frame given by the line, i.e., in a coordinate system in which the metric near the line has the form given by Eq. (4) with $a = 1$ on the line.]

Since $\langle : H : \rangle$ is bounded below by its value of zero for the instantaneous vacuum for the corresponding Cauchy line, so is the instantaneous radiation energy $\tilde{E}$ bounded below by its value $\tilde{E}_0$ for the instantaneous vacuum:

$$
\tilde{E} = \langle : H : \rangle + \tilde{E}_0 \geq \tilde{E}_0 = -\frac{4\pi^2 \alpha \epsilon}{L} + \alpha \int_0^L dx \left( \frac{1}{2} R - K^2 \right).
$$

(72)

For a geodesic Cauchy line in the unit modified deSitter spacetime with $x$ having period $2\pi n$ for some integer $n > 1 \geq \epsilon$, one gets

$$
\tilde{E}_0 = 2\pi \alpha \left( \frac{n^2 - \epsilon}{12n} \right) > 0.
$$

(73)
Thus all states in one of the \( n \)-fold coverings of the standard deSitter spacetime have positive instantaneous radiation energy \( \tilde{E} \). But since the components of the traceless or radiation part of the stress-energy tensor are not invariant under boosts (at least if they are nonzero and finite), there is no quantum state of a massless field in this covering of deSitter spacetime with a finite deSitter-invariant stress-energy tensor expectation value. (One might think that one could get one by averaging the instantaneous vacua for the geodesic Cauchy lines of different \( \gamma \) over the group of boosts that transform one into another, but this group is noncompact, and the resulting state, if well defined at all, would have an infinite stress tensor.)

After writing this section, I found that it is simply a rediscovery of what Davies and Fulling had pointed out twenty years ago [4], with only the rather trivial extension here to arbitrarily twisted fields and the addition of the demonstration that there is no deSitter-invariant state with finite stress tensor in the \( n \)-fold covering of deSitter spacetime. However, a reminder of these facts may be useful to a forgetful person such as myself.

A second simple example to which we can apply our formulas is (for concreteness) an untwisted one-component bosonic field or a twisted two-component fermionic field in the flat \( (R = 0) \) metric on \( S^1 \times R^1 \),

\[
ds^2 = -dt^2 + d\varphi^2,
\]

in which \( \varphi \) is given a period \( 2\pi \) [i.e., the point given by the coordinates \((t, \varphi)\) is identified with the point given by the coordinates \((t, \varphi + 2\pi)\)]. Here if we take the Cauchy line to be one of the closed geodesic lines \( t = \text{const.} \), the instantaneous vacuum is the standard static vacuum with energy density and pressure given by the Casimir value \( \rho_C = \frac{1}{24\pi} \), as derived above.

But suppose we instead take the Cauchy line to be a smooth extrinsically curved spacelike line \( t = t(\varphi) \) with

\[
\tilde{t} \equiv \frac{dt}{d\varphi}
\]

having a magnitude less than unity everywhere (but a nonzero magnitude over at least a nonzero interval of \( \varphi \), so that the line is indeed not geodesic), and with \( t(2\pi) = t(0) \) and \( \tilde{t}(2\pi) = \tilde{t}(0) \). The proper length along the line is then \( dx = \sqrt{1 - \tilde{t}^2} d\varphi \), so the total length of the Cauchy line is

\[
L = \int_0^{2\pi} d\varphi \sqrt{1 - \tilde{t}^2} < 2\pi.
\]

The extrinsic curvature of the line is

\[
K = (1 - \tilde{t}^2)^{-3/2} \tilde{t} \equiv \left[ 1 - \left( \frac{dt}{d\varphi} \right)^2 \right]^{-3/2} \frac{d^2t}{d\varphi^2}.
\]
Since the length of the Cauchy line is less than that of the closed geodesics \( t = \text{const} \), and since the extrinsic curvature contributes negatively to the local energy density of the instantaneous vacuum of the line, the energy density is everywhere lower than the Casimir value \( \rho_C = -\frac{1}{24\pi} \). Thus the corresponding instantaneous radiation energy (or instantaneous total energy, since here where the trace is zero the traceless or radiation part of the stress tensor is the entire stress tensor),

\[
\tilde{E}_0 = -\frac{\pi}{6L} - \frac{1}{24\pi} \int_0^{2\pi} K^2 \sqrt{1 - \tilde{t}^2} d\varphi,
\]

is lower than the Casimir energy, \(-\frac{1}{12}\), of a geodesic Cauchy line.

Because the static vacuum, the instantaneous vacuum for a geodesic Cauchy line in this case, is supposed to be the ground state of this system, it might seem surprising that a different state, namely the instantaneous vacuum of a curved Cauchy line, could have a lower instantaneous radiation or total energy. However, the resolution of this apparent paradox is that the integral for the instantaneous energy is adding up different components of the stress-energy tensor (namely, those orthogonal to the curved Cauchy line) than those that go into the Casimir energy corresponding to the static Killing vector \( \xi = \frac{d}{dt} \). The energy corresponding to \( \xi \) for the instantaneous vacuum of a Cauchy line is

\[
E = -\int \xi^\mu T_{\mu\nu} \epsilon_{\nu\varnothing} dx^\varnothing = \int_0^L \xi^\mu T_{\mu0} dx \\
= \int_0^{2\pi} [(1 - \tilde{t}^2)^{-1/2} T_{00} - \tilde{t}(1 - \tilde{t}^2)^{-1/2} T_{10}] \sqrt{1 - \tilde{t}^2} d\varphi \\
= -\frac{\pi^2}{3L^2} - \frac{1}{24\pi} \int_0^{2\pi} (K^2 + 2\tilde{t}K') d\varphi = -\frac{\pi^2}{3L^2} + \frac{1}{24\pi} \int_0^{2\pi} K^2 d\varphi \\
= \frac{1}{12} \left\{ -\left( \int_0^{2\pi} \frac{d\varphi}{2\pi} \right) \left[ 1 - \left( \frac{dt}{d\varphi} \right)^2 \right] \right\}^{-2} \left[ \int_0^{2\pi} \frac{d\varphi}{2\pi} \left[ 1 - \left( \frac{dt}{d\varphi} \right)^2 \right] \left( \frac{d^2t}{d\varphi^2} \right)^2 \right]^{-3},
\]

using 0 and 1 for the timelike and spacelike orthonormal components, respectively, in the frame given by the Cauchy line, and performing an integration by parts after remembering that the prime on \( K' \) is a derivative with respect to the proper length \( x \) rather than with respect to the angular coordinate \( \varphi \) that has period \( 2\pi \).

Although it is not obvious to me directly from this expression that the Killing energy \( E \) for a general instantaneous vacuum is bounded below by the Casimir value \(-\frac{1}{12}\) of the static vacuum, as it must be by general arguments, one can see that to lowest order in \( \tilde{t} \) it is. A Fourier expansion of \( t(\varphi) \) shows that to quadratic order all terms in \( E + \frac{1}{12} \) are positive, except the \( m = \pm 1 \) terms, which vanish to quadratic order but give a positive fourth-order contribution.

Nevertheless, the fact that one can make the instantaneous radiation energy density arbitrarily negative over the whole Cauchy line (by making the length of
the line arbitrarily short; the $K^2$ term also contributes negatively but must be zero somewhere on a smooth closed line in this flat spacetime, since the $x$-integral of $K$ itself is zero by the trivial holonomy of the closed curves) translates into the fact that one can make even the $\varphi$-density of the Killing energy arbitrarily negative over all but an arbitrarily small range of the $\varphi$ integration. That is, even though the total Killing energy must be no less than $-\frac{1}{12}$ for the flat $S^1 \times R^1$ spacetime in which the period of the $S^1$ is $2\pi$ (or no less than $-\frac{1}{12\lambda}$ if the period of the $S^1$ were $2\pi\lambda$), the excess energy can be concentrated into an arbitrarily small region, leaving almost all of the space with lower energy density than the ground state.

As a third example, consider the flat Minkowski spacetime on the topology $R^2$, with the metric

$$ds^2 = -dt^2 + dz^2, \quad (80)$$

where both $t$ and $z$ range between $-\infty$ and $+\infty$. Again, for concreteness and simplicity, take the standard untwisted one-component bosonic field or twisted two-component fermionic field. In this case a Cauchy line $t(z)$ across the spacetime would have infinite length $L$, unless it consisted of an infinite number of segments of opposite sign of

$$\tilde{t} \equiv \frac{dt}{dz} \quad (81)$$

that are sufficiently nearly null $[\tilde{t}$ sufficiently near $\pm 1$, such as $\tilde{t} = \tanh (z^2 \sin z)]$ to make the total length finite, but then the integral of $K^2$ in the total Killing energy would diverge. When the total length $L$ is indeed infinite, the contribution to the Killing energy for the section of the Cauchy line with $z_1 \leq z \leq z_2$ is

$$E(z_1, z_2) = \frac{1}{24\pi} \int_{z_1}^{z_2} dz \left[ \frac{(dp)}{(dz)} \right]^2 - \frac{d^2}{dz^2} p^2 \right], \quad (82)$$

where

$$p \equiv \frac{\tilde{t}}{\sqrt{1 - \tilde{t}^2}} \equiv \left[ 1 - \left( \frac{dt}{dz} \right)^2 \right]^{-1/2} \frac{dt}{dz}. \quad (83)$$

The integrand can be negative for an arbitrarily long range of $z$, for example by having $K \equiv \frac{dp}{dz}$ constant, but if this range in which the integrand is negative is extended to $z = \pm \infty$, then the line $t(z)$ cannot be a Cauchy line, as can be shown by some calculations summarized in the following paragraph:

For a spacelike line $t(z)$ to be a Cauchy line in the flat Minkowski spacetime $\{SU\}$, it must obviously have $z$ extend over the entire domain $-\infty < z < \infty$, but this necessary condition is not sufficient. One must also have both of the standard null coordinates $u$ and $v$ extend over their infinite range $-\infty < u < \infty, -\infty < v < \infty$, etc.
on the line. In terms of $p$ and $z$, one can readily calculate that along the line $t = t(z)$,

\begin{align}
    u \equiv t - z &= t(0) - \int_0^z dz \left( 1 - \frac{p}{\sqrt{1 + p^2}} \right), \\
    v \equiv t + z &= t(0) + \int_0^z dz \left( 1 + \frac{p}{\sqrt{1 + p^2}} \right).
\end{align}

If $p$ oscillates indefinitely about 0 or remains bounded, both $u$ and $v$ will generally have an infinite range (though sufficiently asymmetric oscillations which become unbounded can preclude this), and the line $t(z)$ will be a Cauchy line as $z$ covers the entire real axis. However, if $p$ monotonically becomes arbitrarily large as $z$ tends to $\pm \infty$, one needs the integral of $1/p^2$ to diverge. If $p$ has a asymptotic power-law dependence on $z$, then the exponent must be no larger than $\frac{1}{2}$ in order that the line be a Cauchy line. On the other hand, the minimum asymptotic exponent in order that the energy integrand in Eq. (82) remain nonpositive is $\frac{2}{3}$, so the integrand cannot be positive over an entire infinite range of $z$ if the line $t = t(z)$ is a Cauchy line.

In fact, one can show further that if $z_1$ is taken to $-\infty$ and $z_2$ is taken to $+\infty$ along a Cauchy line, then the Killing energy between these two limits cannot remain negative for the instantaneous vacuum corresponding to this line. This result might appear to follow simply from the nonnegativity of the total energy in the unbounded Minkowski spacetime, but there is the logical possibility that for a quantum state not in the same Fock sector as the Minkowski vacuum, the energy in any finite region might be negative and only compensated by a positive energy (probably necessarily infinite) that is entirely at spatial infinity. This possibility does indeed occur in 1+3 dimensional Minkowski spacetime, in which a regular state of a conformally invariant scalar field with negative energy density everywhere has been found by Brown, Ottewill, and Siklos [44]. However, the instantaneous vacuum corresponding to a Cauchy line does not give a possible way of constructing such a state, if one exists, for a massless field in 1+1 dimensional Minkowski spacetime.

Neither does the 1+1 dimensional analogue of the 1+3 dimensional negative-energy-density state [44] give negative energy density everywhere; instead, it gives precisely the ordinary vacuum state of zero energy density everywhere in that case. Some preliminary calculations suggest to me (though I have do not have a rigorous proof) that in flat unbounded Minkowski spacetime with 1+1 dimensions, unlike the case in 1+3 dimensions, there are no regular states in which the first integral of Eq. (79) gives a negative total integrated ‘energy’ $E$, even when one ignores a possible contribution entirely at spatial infinity.
VII. CONCLUSIONS

For a massless bosonic or fermionic field (possibly with arbitrarily twisted boundary conditions) in 1+1 dimensional curved spacetime, an extension of an elementary derivation of the trace anomaly given herein shows that the instantaneous vacuum relative to any Cauchy line has a very simple form for its stress-energy tensor on that line [Eqs. (7)–(9), or (34)–(36)]. This form depends on the local properties of the line (its extrinsic curvature and its derivative along the line) and of the spacetime there (its scalar curvature) and on only one nonlocal quantity, a quadratic expression in the twisting of the field, divided by the square of the length of the Cauchy line (assuming it is closed; for an infinitely long Cauchy line this nonlocal quantity is zero). The explicit form of the resulting stress-energy tensor can readily be used to deduce various properties of instantaneous vacua in certain simple spacetimes, such as the nonexistence of any quantum state with a finite deSitter-invariant stress-energy tensor in covering spaces of the 1+1 dimensional deSitter spacetime.

As asides to the basic calculations, the Casimir energy density for arbitrarily twisted massless fields in static spacetimes that are periodic in one spatial dimension is given [Eq. (18)], and various covariant forms of the generic stress-energy tensor of a massless quantum field in 1+1 dimension are presented [Eqs. (37), (45), and (58)], in terms of which one may find two nonlinear functionals [Eqs. (52)–(53) or (64)–(65)] that are coordinate and conformal invariants of the quantum state.

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