AT THE EDGE OF A ONE-DIMENSIONAL JELLIUM

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Abstract. We consider a one-dimensional classical Wigner jellium, not necessarily charge neutral, for which the electrons are allowed to exist beyond the support of the background charge. The model can be seen as a one-dimensional Coulomb gas in which the external field is generated by a smeared background on an interval. It is a true one-dimensional Coulomb gas and not a one-dimensional log-gas. The system exists if and only if the total background charge is greater than the number of electrons minus one. For various backgrounds, we show convergence to point processes, at the edge of the support of the background. In particular, this provides asymptotic analysis of the fluctuations of the right-most particle. Our analysis reveals that these fluctuations are not universal, in the sense that depending on the background, the tails range anywhere from exponential to Gaussian-like behavior, including for instance Tracy–Widom-like behavior. We also obtain a Rényi-type probabilistic representation for the order statistics of the particle system beyond the support of the background.

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1. Introduction

Introduced by Wigner in [Wig34, Wig38] for modeling electrons in metals, the jellium is a Coulomb gas of like-signed equally charged particles for which an external potential is induced by a background of smeared charge with opposite sign. The model was inspired by the Hartree–Fock model of quantum mechanics. This model and its variants go by many other names, including the one-component plasma or uniform electron gas, see for instance [LLS18]. Typically, one imposes the constraint that all charged particles live in some compact region which is equivalent to imposing an infinite external potential on the complement of this compact region. Charge neutrality is also usually assumed, in other words the total charge of the background matches the number of particles. These restrictions ensure that the system exists and the mathematical interest typically focuses on the limiting system as the volume of the compact set (the background) goes to infinity (thermodynamic limit). The classical one-dimensional jellium has been rigorously studied by Baxter [Bax63] who found the partition function exactly, by Kunz [Kun74] who showed the Wigner lattice (crystalization) exists for all temperatures, by Aizenman and Martin [AM80], and by Aizenman, Goldstein, and Lebowitz [AGL01], among others. In the quantum case, Brascamp and Lieb [BL02] represented the partition function exactly and showed crystallization when the inverse temperature parameter $\beta$ is large enough, while the proof of crystallization for all temperatures is obtained in [JJ14] (see also [KCZ+16] for thermal effects on crystallization).

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In this work, we consider the classical jellium but we do not assume that the particles are restricted to live on the region where there is background charge, nor do we assume that the whole system is charge neutral. The system is well-defined if and only if the total background charge is greater than the number of particles minus one. We will assume that this condition is satisfied. The limiting behavior of the particles in the bulk does not change by allowing the particles to leave the background region, thus we focus our attention on the edge of the system, near where the background charge ends. One may then view the system as a jellium on a half-space. More importantly, similar extremal analysis has been carried out for many similar models such as the one-dimensional log-gas and two-dimensional unconfined jellium. Thus it is natural, in the above described setting, to analyze the asymptotic location of the particles farthest away from the origin.

The backgrounds we consider below have a finite total charge which is allowed to grow as $n \to \infty$. Edge statistics for a related model one-dimensional jellium, with infinite background charge, were analyzed in [DKM17, DKM18]. In the case of a uniform background with support growing “fast enough” we obtain a system behaving similarly to theirs; however, we will see that in general one may obtain a range of varying behaviors at the edge.

Let us now describe our model. The Coulomb kernel in dimension $d = 1$ is

$$g(x) = -\frac{|x|}{2}, \quad x \in \mathbb{R}, \quad (1.1)$$

which is the fundamental solution of the Poisson equation $\Delta g = -\delta_0$ in the sense of distributions. More precisely for all smooth and compactly supported $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\int_{\mathbb{R}} g(x) \frac{d^2}{dx^2} \varphi(x) dx = -\varphi(0).$$

Let $\mu = \mu_+ - \mu_-$ be a possibly signed measure on $\mathbb{R}$ with finite first absolute moment, namely $g \in L^1(|\mu|)$ where $|\mu| = \mu_+ + \mu_-$. The Coulomb potential generated at the point $x \in \mathbb{R}$ by $\mu$ is

$$U_\mu(x) = (g * \mu)(x) = -\int \frac{|x-y|}{2} \mu(dy), \quad (1.2)$$

which satisfies $\Delta U_\mu = -\mu$ in the sense that for all smooth and compactly supported $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\int_{\mathbb{R}} U_\mu(x) \frac{d^2}{dx^2} \varphi(x) dx = -\int \varphi(d\mu), \quad (1.3)$$

see for instance [Hel14, Lan72]. The Coulomb (self-interaction) energy of $\mu$ is defined by

$$\mathcal{W}(\mu) = \frac{1}{2} \iint g(x-y) \mu(dx)\mu(dy) = \frac{1}{2} \int U_\mu(x) \mu(dx). \quad (1.4)$$

The electric field generated at the point $x \in \mathbb{R}$ by a (possibly signed) measure $\mu$ on $\mathbb{R}$ is

$$E_\mu(x) = -\frac{d}{dx} U_\mu(x) = \frac{1}{2} \int \text{sign}(x-y) \mu(dy). \quad (1.5)$$

For all $n \geq 1$, we consider $n$ unit negatively charged particles (electrons) at positions $x_1, \ldots, x_n$ in $\mathbb{R}$, lying in a positive background of total charge $\alpha > 0$. The background is smeared according to a probability measure $\rho$ on $\mathbb{R}$ with finite Coulomb energy $\mathcal{W}(\rho)$. We could alternatively suppose that the particles are positively charged (cations) and the background is negatively charged; this reversed choice would not affect the analysis of the model. The total energy of the system is

$$\frac{1}{2} \sum_{i<j} |x_i - x_j| - \alpha \sum_{i=1}^n U_\rho(x_i) + \alpha^2 \mathcal{W}(\rho).$$

However, the term $\alpha^2 \mathcal{W}(\rho)$ will not be that important for our analysis so we set

$$H_n(x_1, \ldots, x_n) = \frac{1}{2} \sum_{i<j} |x_i - x_j| - \alpha \sum_{i=1}^n U_\rho(x_i). \quad (1.6)$$

The one-dimensional Coulomb model comes with remarkable identities such as, for all $x \in \mathbb{R}$,

$$\int \frac{|x-y|}{2} d\rho(y) = \frac{x}{2} + \int (y-x) 1_{(x, \infty)} d\rho(y) - \int \frac{y}{2} d\rho(y). \quad (1.7)$$
In the same spirit, Baxter’s combinatorial identity \cite{Bax63} states that for all \((x_1, \ldots, x_n) \in \mathbb{R}^n\),

\[
- \sum_{i<j} |x_i - x_j| = \sum_{i<j} (x_{(j)} - x_{(i)}) = \sum_{k=1}^{n} \frac{2k-n-1}{2} x_{(k)},
\]

(1.8)

where \(x_{(n)} \leq \cdots \leq x_{(1)}\) is the reordering of \(x_1, \ldots, x_n\); in particular,

\[
x_{(n)} = \min_{1 \leq i \leq n} x_i \quad \text{and} \quad x_{(1)} = \max_{1 \leq i \leq n} x_i,
\]

(1.9)

which allows to rewrite the energy as

\[
H_n(x_1, \ldots, x_n) = \sum_{k=1}^{n} \left[ \frac{2k-n-1}{2} x_{(k)} - \alpha_n U_{\rho}(x_{(k)}) \right].
\]

(1.10)

For simplicity, we assume in the whole text that \(\rho\) is absolutely continuous with respect to Lebesgue measure, with a density function still denoted \(\rho\) by a slight abuse of notation. We say that the system is neutral in charge when \(\alpha = n\), and that the background is uniform when \(\rho\) is the uniform distribution on an interval \([a, b]\). For all \(\beta > 0\), we set

\[
\mathcal{Z}_n = \int_{\mathbb{R}^n} e^{-\beta H_n(x_1, \ldots, x_n)} \, dx_1 \cdots dx_n \in [0, \infty].
\]

(1.11)

It can be checked that \(\mathcal{Z}_n < \infty\) if and only if \(\alpha > n - 1\), see \cite[Lemma 2.1]{CGZJ20b}.

When \(\alpha > n - 1\), we can then define the Boltzmann–Gibbs probability measure \(P_n\) on \(\mathbb{R}^n\) by

\[
dP_n(x_1, \ldots, x_n) = \frac{e^{-\beta H_n(x_1, \ldots, x_n)}}{\mathcal{Z}_n} \, dx_1 \cdots dx_n.
\]

(1.12)

This is called a Coulomb gas with external potential \(V = -\frac{\alpha}{n} U_{\rho}\). We are dealing with electrostatics in the sense that the charges do not move. In our setting, this external potential arises from a smeared background with distribution \(\rho\), however, one can define general Coulomb gases for any confining external potential \(V\) for which \(\frac{\alpha}{n} \Delta V\) may not necessarily be a probability measure. Let

\[
\mathbf{X}^{(n)} = (X_1^{(n)}, \ldots, X_n^{(n)}) \sim P_n.
\]

(1.13)

Example 1.1 (Coulomb gas with quadratic external field). Let us consider the example for which \(\rho\) is the uniform probability measure on an interval \([a, b]\) with \(a < b\). Then, for all \(x \in \mathbb{R}\),

\[
- U_{\rho}(x) = \frac{1}{2(b-a)} \int_{a}^{b} |x - y| \, dy = \begin{cases} \frac{|x - \frac{a+b}{2}|}{2} & \text{if } x \notin [a, b] \\ \frac{(x - \frac{a+b}{2})^2 + (b-a)^2}{2(b-a)} & \text{if } x \in [a, b] \end{cases}.
\]

(1.14)

The potential \(V = -\frac{\alpha}{n} U_{\rho}\) then behaves quadratically on \([a, b]\) and is affine outside \([a, b]\). Conditioned on all the particles lying inside \([a, b]\), it is possible to interpret \(P_n\) as a conditioned Gaussian law. Using Baxter’s identity \cite{Bax63} together with \cite{Bax63}, we have that when \(\{x_1, \ldots, x_n\} \subset [a, b]\),

\[
H_n(x_1, \ldots, x_n) = \sum_{k=1}^{n} \frac{2k-n-1}{2} x_{(k)} + \frac{\alpha}{2(b-a)} \sum_{i=1}^{n} \left( x_{(i)} - \frac{a+b}{2} \right)^2 + \frac{n \alpha (b-a)}{8}.
\]

(1.15)

This formula shows then that \(\mathbf{X}^{(n)} \sim P_n\) is conditionally Gaussian in the sense that

\[
\text{Law}\left( (X_1^{(n)}, \ldots, X_{(1)}) \mid \{X_1, \ldots, X_n\} \subset [a, b]\right) = \text{Law}\left( (Y_n, \ldots, Y_1) \mid a \leq Y_n \leq \cdots \leq Y_1 \leq b \right)
\]

(1.16)

where \(Y_1, \ldots, Y_n\) are independent real Gaussian random variables with

\[
EY_k = \frac{a+b}{2} + \frac{b-a}{2 \alpha} (n + 1 - 2k) \quad \text{and} \quad \text{Var}(Y_k) = \frac{b-a}{\alpha^2}.
\]

This was already noted in \cite{Bax63}. Now if we consider the limit \(a \to -\infty, b \to \infty\) with \(a/(b-a) \to c > 0\), then \(P_n\) can be interpreted as a Coulomb gas for which the potential is quadratic everywhere, namely \(V = \frac{\alpha}{\pi a^2} |x|^2\). Since the second derivative of \(V\) is a constant, this can also be seen as a jellium.
with a background equal to a multiple of Lebesgue measure on the whole of \( \mathbb{R} \). Note that this jellium is not neutral, but rather, has an infinite charge imbalance for every \( n \). Under the scaling \( x_i = \sqrt{n} y_i \), see Remark 1.2, this limiting case matches the model studied by [DKM+18] Equation (14) but note that their \( \alpha \) plays the role of our \( c \) up to a dilation. This Coulomb gas model with quadratic external field in one dimension is analogous to the complex Ginibre ensemble which is a Coulomb gas in two dimensions.

**Remark 1.2 (Scale invariance).** The model (1.12) has a scale invariance which comes from the homogeneity of the one-dimensional Coulomb kernel. More precisely, if we denote by \( \text{dil}_\sigma \) the law of the random vector \( \sigma X \) when \( X \sim \mu \), then, for all \( \sigma > 0 \), dropping the \( n \) subscript on \( P \),

\[
\text{dil}_\sigma(P^{\alpha, \beta, \rho}) = P^{\alpha, \beta, \rho} \text{dil}_\sigma, \nonumber
\]

In other words, if \( X^{(n)} \sim P^{\alpha, \beta, \rho} \) then \( \sigma X^{(n)} \sim P^{\alpha, \beta, \rho} \text{dil}_\sigma \). This is useful in the asymptotic analysis of the model as \( n \to \infty \), and reveals the special role played by \( \alpha \) as a shape parameter. Here the inverse temperature \( \beta \) is a scale parameter, in contrast with the situation for log-gases.

**Structure of the paper.**

- Section 2 states our main results: Theorems 2.1, 2.4, 2.6 and Corollary 2.8.
- Section 3 proves our main results. This is done in two steps. In Section 3.1, we first discuss the right-hand sides of equations (2.3) and (2.1) contained in our main results. In Section 3.2, we complete the proofs by showing that the left-hand sides of these two equations converge to the right-hand sides, and we also prove the behavior of the single right-most particle as described in Corollary 2.8.
- Appendices A and B give results about tail asymptotics and stochastic domination used in the proofs of our main results.
- Our main results concern point processes with infinitely many particles. If one only needs conditional results about the right-most particle (or finitely many particles), the proofs can be greatly simplified. Appendix C illustrates this simplification.

## 2. Main results

As a prerequisite to studying the edge asymptotics, one should first verify that, at the macroscopic level, the limiting equilibrium measure is equal to \( \rho \) as one would naturally expect. For those interested in precise details of such global asymptotics, we refer to [CGZ19b, Theorem 2.2].

Our main results show that in the limit, as \( n \to \infty \), one can obtain an infinite point process at the edge of the jellium. These point processes can be interpreted as infinite-volume Gibbs measures when the background region is expanded in only one direction (leaving the other end of the background fixed). It will be apparent in our proofs, that the limits are indeed Gibbsian, in the sense that the limit does not depend on how one takes these infinite limits, as long as one end is fixed. Our first Theorem 2.1 does this in the natural setting of a growing uniform background \( \rho_n \), with an asymptotically neutral system similar to the original model of [Ba93]. Since our results concern the edge of the jellium, it is natural to require \( \rho_n \) to be supported on \((-\infty, 0]\). Under the conditions of the next theorem, \( \frac{1}{n} \sum_{i=1}^{n} \delta_{n-1} X^{(n)} \) converges to the uniform measure on \([-1, 0]\).

**Theorem 2.1 (Point process at the edge, asymptotically neutral regime).** Suppose

- \( \beta > 0 \) is fixed;
- \( \alpha_n - (n - 1) = 2\lambda \in (0, \infty) \);
- \( \rho_n \) is uniform on \([-\alpha_n, 0]\).

If \( X^{(n)} \sim P_n \) as in (1.13), then

\[
\lim_{n \to \infty} \text{Law}(X^{(n)}, \ldots, X^{(n)}(1)) = \lim_{n \to \infty} \text{Law}(Y_k, \ldots, Y_1 \mid Y_0 < \cdots < Y_1) \tag{2.1}
\]

where \( \{Y_i\}_{i \geq 1} \) are independent random variables such that \( Y_i \) has a density proportional to

\[
\exp\left(-\beta\left((i - 1 + \lambda)x + \frac{x^2}{2}\right)(-\infty, 0]\right)\right).\nonumber
\]

The result should also hold when \( \lim_{n \to \infty} (\alpha_n - (n - 1)) = 2\lambda \in (0, \infty) \), however, for simplicity, we have only considered the special case \( \alpha_n - (n - 1) = 2\lambda \). It seems that the more general case amounts to proving the right continuity, with respect to \( \lambda \), of the limiting process.
Remark 2.2 (Relaxing uniformity of background). Since we are only interested in the edge behavior, one should be able to relax the assumption that $\rho_n$ is uniform at the left of 0 by requiring that $\lim_{n \to \infty} \alpha_n \rho_n$ in the vague sense is the Lebesgue measure restricted to $(-\infty, 0]$. Indeed, due to crystallization and translation symmetry breaking of the one-dimensional jellium [AM80, AJJ10, JJ14], one is not allowed to continuously increase the background charge when taking the thermodynamic limit, but rather, the increases must be in (roughly) integer steps.

We next consider a (generalized) model similar to that studied in [DKM17, DKM18, ADK19] for the quadratic Coulomb gas model of Example 1.1. In our case, we will use a background with finite total charge, but we allow the total background charge $\alpha_n$ which grows at a rate faster than $n$. This has a similar effect to first growing the background to obtain a quadratic external potential, and then taking the number of particles to infinity. We will see that the special case where $\rho_n$ is uniform (with $\alpha_n$ growing faster than $n$) shares the same features as the Coulomb gas with external quadratic potential [DKM18, Equation (14)]; however this sort of behavior is not seen in general. In order to get such Gaussian behavior it is necessary for the background charge to extend uniformly beyond the region where the extremal particles live. This is the special case $\gamma = 2$, in our next result. For general nonneutral systems, one may interpolate between exponential and Gaussian tails, and even beyond, by varying the decay of the background at the edge of its support. Under the conditions of the next theorem, $\frac{1}{n} \sum_{i=1}^{\delta n-1} \delta X_1^{(n)}$ converges to the uniform measure on $[-1, 0]$ as in Theorem 2.1.

**Theorem 2.4** (Point process at the edge, nonneutral regime). Suppose that

- $\beta > 0$ is fixed;
- $\alpha_n$ is such that $\lim_{n \to \infty} (\alpha_n - (n - 1)) = \infty$;
- $\rho_n$ is such that for some fixed $\gamma > 1,
  \alpha_n \rho_n(x) = 1_{[-\alpha_n^{-1}, 0]}(x)dx + (\gamma - 1)x^{\gamma - 2}1_{[0, \gamma^{1/(\gamma - 1)}]}(x)dx.$

If $X^{(n)} \sim P_n$ as in [1,13], then (2.1) holds except now $\{Y_i\}_{i \geq 1}$ are independent random variables such that $Y_i$ has a density proportional to

$$\exp\left(-\beta\left[i \cdot \frac{1}{2} x + \frac{x^2}{2} 1_{(-\infty, 0)} + \frac{x^\gamma}{\gamma} 1_{(0, \infty)}\right]\right).$$

When $\gamma \in (1, 2)$ there is an (integrable) singularity in the background density $\rho_n$ at 0. This singularity is not important to the edge behavior, but rather serves only to give the density of $Y_i$ a clean form. One could smooth out this density at the cost of complicating the density of $Y_i$.

**Remark 2.5** (Gaussian and Tracy-Widom like cases). When $\gamma = 2$, then Theorem 2.4 is about the model of Example 1.1 and represents the particle locations as Gaussian variables conditioned on a convex simplex. On the other hand, if $\gamma = 3/2$ in Corollary 2.8, one obtains the tail behavior of the Tracy–Widom distribution $\text{TW}_\beta$ where we recall from [RRV11, DV13, BN12],

$$\mathbb{P}(M > t) = e^{-\frac{1}{2} \beta t^{3/2}(1 + \alpha_n - (n - 1))} \quad \text{for} \quad M \sim \text{TW}_\beta.$$  

The final situation we consider is to let $\rho$ be fixed. Since the total background is $\alpha_n \rho$, this amounts to growing the background vertically on the support of $\rho$. Under the conditions of the next theorem, $\frac{1}{n} \sum_{i=1}^{\delta n} \delta X_1^{(n)}$ converges to the measure $\rho$.

**Theorem 2.6** (Point process at the edge, neutral regime with fixed background). Suppose that

- $\beta > 0$ and fixed;
- $\alpha_n$ is such that $\lim_{n \to \infty} (\alpha_n - (n - 1)) = 2\lambda \in (0, \infty)$;
- $\rho$ is supported inside $(-\infty, 0]$ and the support contains the origin 0.

If $X^{(n)} \sim P_n$ as in [1,13], then for all $m \geq 1$ the order statistics satisfy

$$\left(X^{(n)}_m, \ldots, X^{(n)}_1\right) \sim \text{Law} \left(\frac{2}{\beta} \sum_{i=m}^{\infty} \frac{Z_i}{i(2\lambda - 1 + i)}, \ldots, \frac{2}{\beta} \sum_{i=1}^{\infty} \frac{Z_i}{i(2\lambda - 1 + i)}\right),$$

where $\{Z_i\}_{i \geq 1}$ is a sequence of independent exponential random variables of unit mean.
The shape of $\rho$ plays no role in the above theorem as long as the support contains the point 0.

**Remark 2.7** (Point process between a left-side and right-side background). In the spirit of the two-dimensional analysis in [BGZIS, BGZNW20], one may also consider the jellium where the support of $\rho$ is contained in $(-\infty, a] \cup [b, \infty)$ for some real numbers $a$ and $b$ in the support of $\rho$ with $a < b$. Set $\lambda_n = [\alpha_n - (n - 1)]/2$, and choose $\eta \in \mathbb{C}$ with $|\eta| = 1$. Suppose that $\alpha_n$ satisfies

$$\frac{\alpha_n}{n} \xrightarrow{n \to \infty} 1 \quad \text{and} \quad \exp(2\pi i(\lambda_n - \alpha_n(-\infty, a])) = \eta.$$  

Then using similar arguments to those in this work, we could show that $\{X_1^{(n)}, \ldots, X_n^{(n)}\} \cap [a, b]$ converges to a point process depending only on the parameter $\eta$, as in [AMS0]. Here $\eta$ parametrizes the possible limit point processes.

We finally consider the single right-most particle of the gas:

$$M_n = \max_{1 \leq i \leq n} X_i^{(n)} = X_{(1)}^{(n)}.$$  

(2.3)

**Corollary 2.8** (Tail asymptotics at the right).

- Under the assumptions of Theorem 2.4 or Theorem 2.6

$$\lim_{n \to \infty} \mathbb{P}(M_n > t) = e^{-\beta M(1+c_\gamma)} \quad \text{for all } t, \text{ with } \lim_{t \to \infty} c_\gamma = 0.$$  

(2.4)

- Under the assumptions of Theorem 2.3

$$\lim_{n \to \infty} \mathbb{P}(M_n > t) = e^{-\frac{c_\gamma}{\beta}}(1+c_\gamma) \quad \text{for all } t, \text{ with } \lim_{t \to \infty} c_\gamma = 0.$$  

(2.5)

3. PROOFS OF MAIN RESULTS

3.1. Infinite Coulomb gases focused at an edge. As a preliminary to the proofs, in this subsection, we give a description of the limiting objects in Theorems 2.1, 2.4, 2.5, 2.6 as Coulomb gases with an infinite number of particles. By (1.8) and in the spirit of (1.16), this will be related to the notion of conditioning an infinite number of particles to have a specific ordering.

Let $\mu$ be a locally finite measure on $\mathbb{R}$ such that $\mu(-\infty, t] = \infty$ for every $t \in \mathbb{R}$. The notion of an infinite Coulomb gas on $\mathbb{R}$ associated to $\mu$ at inverse temperature $\beta$ is as follows. Take $V$ such that $\Delta V = \mu$. Since $\mu(-\infty, t] = \infty$ for every $t \in \mathbb{R}$, we know that $\lim_{x \to -\infty} V(x)/|x| = \infty$. Since $V$ is convex, by possibly adding a linear term to $V$, we may assume that there exists $t_0 \in \mathbb{R}$ such that $V|_{[t_0, \infty)}$ is non-decreasing. Let $\lambda > 0$ and $\{Y_k\}_{k \geq 1}$ be independent random variables such that $Y_k$ has a density proportional to

$$\exp\left(-\beta \left[(k-1+\lambda)x + V(x)\right]\right).$$

Take a random vector $(\xi_1^{(n)}, \ldots, \xi_n^{(n)})$ such that

$$\text{Law}(\xi_1^{(n)}, \ldots, \xi_n^{(n)}) = \text{Law}(Y_n, \ldots, Y_1 \mid Y_n \leq \cdots \leq Y_1).$$  

(3.1)

We will be interested in the limit, in $n$, of the point processes

$$\{\xi_1^{(n)}, \ldots, \xi_n^{(n)}\}.$$  

We see this limit as an infinite Coulomb gas at inverse temperature $\beta$ since if $x_1 \geq \cdots \geq x_n$ then

$$\sum_{i=1}^n [(i-1+\lambda)x_i + V(x_i)] = -\frac{1}{2} \sum_{i<j} (x_i - x_j) + \sum_{i=1}^n \left(V(x_i) + \frac{2\lambda+n-1}{2} x_i\right).$$

Hence the total potential energy contains both a two-body Coulomb interaction portion as well as a confining potential given by $V(x) + \frac{2\lambda+n-1}{2} x$, whose Laplacian is $\mu$. By this construction we obtain a family of infinite Coulomb gases indexed by $\lambda > 0$.

We will be interested here in the following three cases.

- Case where for some $\gamma > 1$

$$V(x) = \frac{x^2}{2} 1_{x<0} + \frac{x^\gamma}{\gamma} 1_{x \geq 0}$$  

(3.2)
• Case
\[ V(x) = \frac{x^2}{2} \mathbf{1}_{x < 0} \]  

• Infinite half-well case
\[ V(x) = \begin{cases} \infty, & \text{if } x < 0 \\ 0, & \text{if } x \geq 0 \end{cases} \]  

The infinite half-well (or hard wall) case \((3.4)\) admits the following simple description.

**Proposition 3.1** (Infinite Coulomb gas in an infinite half-well). Let \(\lambda > 0\) and \(\beta > 0\), and take a sequence \(\{Y_k\}_{k \geq 1}\) of independent random variables such that \(Y_k\) has a density proportional to
\[ x \mapsto \exp \left( -\beta (k - 1 + \lambda) x \right) \mathbf{1}_{x \geq 0}. \]

If \(\{Z_i\}_{i \geq 1}\) is a sequence of independent exponential random variables of unit mean, then
\[ \lim_{n \to \infty} \text{Law}(Y_k \mid Y_1 \leq \cdots \leq Y_k) = \text{Law} \left( \frac{2}{\beta} \sum_{i=k}^{\infty} \frac{Z_i}{i(2\lambda - 1 + i)} \right). \]

Moreover, we have for all \(m \geq 1\),
\[ \lim_{n \to \infty} \text{Law}(Y_m, \ldots, Y_1 \mid Y_m \leq Y_{m-1} \leq \cdots \leq Y_1) = \text{Law} \left( \frac{2}{\beta} \sum_{i=m}^{\infty} \frac{Z_i}{i(2\lambda - 1 + i)} \right). \]

As \(\lambda \to 0\), we lose a particle to infinity, and we recover then the point process for \(\lambda = 1\).

**Proof.** Let \(k\) be fixed. Let \(T^{(n)}_k, \ldots, T^{(n)}_1\) be random variables such that
\[ \text{Law}(T^{(n)}_k, \ldots, T^{(n)}_1) = \text{Law}(Y_n, \ldots, Y_k \mid Y_n \leq \cdots \leq Y_k). \]

Then \(T^{(n)}_n, \ldots, T^{(n)}_k\) have a joint density proportional to
\[ (x_n, \ldots, x_k) \mapsto \exp \left[ -\beta \sum_{j=k}^{n} (j - 1 + \lambda) x_j \right] \mathbf{1}_{x_n \leq \cdots \leq x_k}. \]

We can perform the change of variables
\[ z_i = x_i - x_{i+1} \text{ if } i \in \{k, \ldots, n - 1\} \quad \text{and} \quad z_n = x_n \]

or equivalently \(x_j = \sum_{i=j}^{n} z_i \) for any \(i \in \{k, \ldots, n\}\) to obtain a density proportional to
\[ (x_n, \ldots, z_k) \mapsto \prod_{i=k}^{n} \exp \left[ -\beta \frac{1}{2} (2\lambda + i - 1) z_i \right] \mathbf{1}_{z_i \geq 0}. \]

This tells us that, if \(\{Z_i\}_{i \geq k}\) is a sequence of independent exponential random variables of unit mean, we have that the law of \((T^{(n)}_n, \ldots, T^{(n)}_k)\) is the same as the law of
\[ \frac{2}{\beta} \frac{Z_n}{n(2\lambda + n - 1)}, \sum_{i=n-1}^{n} \frac{Z_i}{i(2\lambda + i - 1)}, \ldots, \sum_{i=k}^{n} \frac{Z_i}{i(2\lambda + i - 1)}. \]

In particular,
\[ \text{Law}(T^{(n)}_k) = \text{Law} \left( \frac{2}{\beta} \sum_{i=k}^{n} \frac{Z_i}{i(2\lambda + i - 1)} \right) \]

so that, by taking \(n \to \infty\), we obtain that
\[ T^{(n)}_k \xrightarrow{n \to \infty} \frac{2}{\beta} \sum_{i=k}^{\infty} \frac{W_i}{i(2\lambda + i - 1)}. \]

Finally, if \(k = 1\),
\[ \text{Law}(\xi^{(n)}_m, \ldots, \xi^{(n)}_1) = \text{Law}(T^{(n)}_m, \ldots, T^{(n)}_1) \]

and we obtain \((3.5)\) by taking \(n \to \infty\) in \((3.7)\).
Remark 3.2 (Gumbel limit). If \( \frac{2}{\beta} = \chi = 2\lambda - 1 \), let us show that the final coordinate of the right-hand side of (3.3) has a Gumbel limit as \( \chi \to \infty \). Indeed, this coordinate can be written

\[
M_\chi = \sum_{i=1}^{\infty} \frac{\chi}{k(\chi + k)} Z_k
\]

where \( \{Z_k\}_{k \geq 1} \) are independent exponential random variables of unit mean. It turns out that

\[
M_\chi \to \mathbb{E}[M_\chi] \quad \text{law} \to G - \gamma \text{ where } \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right)
\]

is the Euler–Mascheroni constant and \( G \) is a standard Gumbel random variable. Indeed, we could use characteristic functions or Fourier transform and start by noting that for any \( u \in \mathbb{R} \),

\[
\mathbb{E}[e^{-iuM_\chi}] = \prod_{k=1}^{\infty} \left( 1 + \frac{iu\chi}{k(\chi + k)} \right)^{-1}.
\]

On the other hand,

\[
\prod_{k=1}^{\infty} \exp\left( \frac{iu\chi}{k(\chi + k)} \right) \left( 1 + \frac{iu\chi}{k(\chi + k)} \right)^{-1} \to \prod_{k=1}^{\infty} e^{\frac{iu}{k}} = e^{iu/G}(iu) = e^{iu}(iu + 1)
\]

where we have used Weierstrass’s formula \( \Gamma(z) = z^{-1}e^{-\gamma z} \prod_{k=1}^{\infty} e^{k^{-1}z(1+k^{-1}z)}^{-1} \) and the identity \( z\Gamma(z) = \Gamma(z + 1) \). It remains to note that if \( Z \) is an exponential random variable of unit mean so that \( G = -\log(Z) \) is a standard Gumbel random variable, then for all \( u \in \mathbb{R} \),

\[
\mathbb{E}[e^{-iuG}] = \int_{0}^{\infty} e^{-r}e^{iu\log r} dr \int_{0}^{\infty} e^{-r}e^{iu} dr = \Gamma(iu + 1).
\]

We now state the existence of more general infinite Coulomb gases focused at an edge in the following proposition where, for simplicity, we take \( t_0 = 0 \).

Proposition 3.3 (Infinite Coulomb gases at the edge). Let \( V : \mathbb{R} \to \mathbb{R} \) be a continuous function such that

\[
\lim_{x \to -\infty} \frac{V(x) - x}{|x|} = \infty \quad \text{and such that } \left| V \right|_{[0, \infty)} \text{ is non-decreasing.}
\]

For \( \lambda, \beta > 0 \) consider \( \{Y_k\}_{k \geq 1} \) and \( \{(\xi^{(n)}_1, \ldots, \xi^{(n)}_n)\} \) as in [5.1]. Then, for any \( k \geq 1 \), the limit

\[
\Theta_k = \lim_{n \to \infty} \text{Law}(Y_k \mid Y_n \leq \cdots \leq Y_k)
\]

exists.

Moreover, if we take \( \tilde{Y}_k \sim \Theta_k \) independent of \( (Y_1, \ldots, Y_{k-1}) \) we have that

\[
\lim_{n \to \infty} \text{Law}(\xi^{(n)}_k, \ldots, \xi^{(n)}_1) = \text{Law}(\tilde{Y}_k, Y_{k-1}, \ldots, Y_1 \mid \tilde{Y}_k \leq Y_{k-1} \leq \cdots \leq Y_1)
\]

(3.8)

Proof. Let \( k \) be fixed and \( \{\tilde{Y}^{(n)}_k\}_{k \geq n} \) be a sequence of random variables such that

\[
\text{Law}(\tilde{Y}^{(n)}_k) = \text{Law}(Y_k \mid Y_n \leq \cdots \leq Y_k).
\]

By Proposition [B.4] the sequence \( \{\tilde{Y}^{(n)}_k\}_{n \geq 1} \) is stochastically increasing, namely

\[
\mathbb{P}(\tilde{Y}^{(n+1)}_k \leq t) \leq \mathbb{P}(\tilde{Y}^{(n)}_k \leq t)
\]

(3.9)

for every \( t \in \mathbb{R} \) and \( n \geq 1 \). It is enough to control \( \tilde{Y}^{(n)}_k \) from above to know that it converges in law. We begin by noticing that, again by Proposition [B.3]

\[
\mathbb{P}(Y_k \leq t \mid Y_n \leq \cdots \leq Y_k) \leq \mathbb{P}(Y_k \leq t \mid Y_n \leq \cdots \leq Y_k).
\]

But \( \text{Law}(Y_k \mid 0 \leq Y_n \leq \cdots \leq Y_k) \) can be equivalently described by taking independent random variables \( W_n, \ldots, W_k \) such that \( W_i \) has a density proportional to

\[
x \mapsto \exp(-\beta \left[ (i-1 + \lambda)x + V(x) \right]) 1_{x \geq 0}
\]

and noticing that

\[
\text{Law}(Y_k \mid 0 \leq Y_n \leq \cdots \leq Y_k) = \text{Law}(W_k \mid W_n \leq \cdots \leq W_k).
\]
Let \( \{B_i\}_{i \geq k} \) be a sequence of independent random variables such that the random variable \( B_i \) has a density proportional to
\[
x \mapsto \exp(-\beta(i-1+\lambda)x)1_{x \geq 0}.
\]
By Proposition [B.3], we know that
\[
P(B_k \leq t \mid B_n \leq \cdots \leq B_k) \leq P(W_k \leq t \mid W_n \leq \cdots \leq W_k).
\]
Let \( \varepsilon > 0 \). By Proposition [B.3] \( \text{Law}(B_k \mid B_n \leq \cdots \leq B_k) \) converges as \( n \to \infty \) so that there exists \( T > 0 \) such that
\[
1 - \varepsilon < P(B_k \leq T \mid B_n \leq \cdots \leq B_k)
\]
for every \( n \). This implies that the sequence \( \{\tilde{Y}_k^{(n)}\}_{n \geq k} \) is tight and, since (3.9) shows it is stochastically increasing, it has a limit that we shall call \( \tilde{Y}_k \).

The right-hand side of (3.8) is well-defined since we may find open intervals \( I_1, \ldots, I_k \) such that
\[
P\left(\tilde{Y}_k \leq Y_{k-1} \leq \cdots \leq Y_1\right) > 0.
\]
To prove the second statement of the proposition we notice that, by Lemma [3.2]
\[
\text{Law} \left(\xi_k^{(n)}, \ldots, \xi_1^{(n)}\right) = \text{Law} \left(\tilde{Y}_k^{(n)}, Y_{k-1}, \ldots, Y_1 \mid \tilde{Y}_k^{(n)} \leq Y_{k-1} \leq \cdots \leq Y_1\right)
\]
where we are supposing \( \tilde{Y}_k^{(n)} \) is independent of \( Y_1, \ldots, Y_{k-1} \). In particular, we have that
\[
\lim_{n \to \infty} P \left(\tilde{Y}_k^{(n)} \leq Y_{k-1} \leq \cdots \leq Y_1\right) = P \left(\tilde{Y}_k \leq Y_{k-1} \leq \cdots \leq Y_1\right)
\]
since \( Y_1, \ldots, Y_{k-1} \) do not have atoms and thus
\[
P\left(Y_i = Y_j\right) = 0 \quad \text{for } i \neq j \quad \text{and} \quad P\left(Y_i = \tilde{Y}_k\right) = 0 \quad \text{for } i \in \{1, \ldots, k-1\}.
\]
Finally, for any closed set \( A \subset \mathbb{R}^k \),
\[
\limsup_{n \to \infty} P \left((Y_1, \ldots, Y_{k-1}, \tilde{Y}_k^{(n)}) \in A \quad \text{and} \quad \tilde{Y}_k^{(n)} \leq Y_{k-1} \leq \cdots \leq Y_1\right)
\]
\[
\leq P \left((Y_1, \ldots, Y_{k-1}, \tilde{Y}_k) \in A \quad \text{and} \quad \tilde{Y}_k \leq Y_{k-1} \leq \cdots \leq Y_1\right),
\]
so that
\[
\limsup_{n \to \infty} P \left((Y_1, \ldots, Y_{k-1}, \tilde{Y}_k^{(n)}) \in A \quad \text{and} \quad \tilde{Y}_k^{(n)} \leq Y_{k-1} \leq \cdots \leq Y_1\right)
\]
\[
\leq \frac{P \left((Y_1, \ldots, Y_{k-1}, \tilde{Y}_k) \in A \quad \text{and} \quad \tilde{Y}_k \leq Y_{k-1} \leq \cdots \leq Y_1\right)}{P \left(\tilde{Y}_k \leq Y_{k-1} \leq \cdots \leq Y_1\right)}.
\]
This is one of the equivalent conditions of weak convergence, thus the proof is complete. \qed

**Remark 3.4** (Limiting point process). The convergence (3.8) defines a probability measure on the space of sequences of real numbers \( \mathbb{R}^{Z_{\geq 1}} \). More precisely, if we let, for any \( k \geq 1 \), \( \pi_k : \mathbb{R}^{Z_{\geq 1}} \to \mathbb{R}^k \) be the projection onto the first \( k \) coordinates and \( Y_1, \ldots, Y_{k-1}, \tilde{Y}_k \) be as in Proposition 3.3 then the infinite Coulomb gas is the probability measure \( \Gamma \) on \( \mathbb{R}^{Z_{\geq 1}} \) such that, for every \( k \geq 1 \),
\[
(\pi_k)_{\Gamma} = \text{Law}(\tilde{Y}_k, Y_{k-1}, \ldots, Y_1 \mid \tilde{Y}_k \leq Y_{k-1} \leq \cdots \leq Y_1),
\]
where \( (\pi_k)_{\Gamma} \) denotes the image measure of \( \Gamma \) by \( \pi_k \). In particular, for any continuous \( f : \mathbb{R} \to \mathbb{R} \) whose support is bounded from below, we have that
\[
\sum_{i=1}^{n} f(\xi_i^{(n)}) \xrightarrow{\text{Law}} \sum_{i=1}^{\infty} f(W_i) \quad \text{where} \quad (W_i)_{i \geq 1} \sim \Gamma.
\]
This statement is in fact equivalent to (3.8) for every \( k \geq 1 \).
3.2. Proofs of Theorems 2.6, 2.1, 2.4 Corollary 2.8 The following proofs share the same first steps which we explain now. We define

$$V_i^{(n)}(x) = \frac{2i - 1 - n}{2} x - \alpha_n U_{\rho_n}(x).$$  \hfill (3.10)

For every $n \geq 1$, we consider $n$ independent random variables $Y_1^{(n)}, \ldots, Y_n^{(n)}$ such that $Y_i^{(n)}$ has a density proportional to $x \in \mathbb{R} \mapsto e^{-\beta \alpha_i V_i^{(n)}(x)}$. By (1.10), the Coulomb gas $X_1^{(n)}, \ldots, X_k^{(n)}$ satisfies

$$\text{Law} \left( X_1^{(n)}, \ldots, X_k^{(n)} \right) = \text{Law} \left( Y_1^{(n)}, \ldots, Y_1^{(n)} \mid Y_1^{(n)} \leq \cdots \leq Y_k^{(n)} \right).$$

We fix $k \geq 1$ and we study $(X_k^{(n)}, \ldots, X_1^{(n)})$. By Lemma 3.2, there is a simple description of this vector by considering a random variable $\tilde{Y}_k^{(n)}$ independent of $Y_1^{(n)}, \ldots, Y_k^{(n)}$ and such that

$$\text{Law}(\tilde{Y}_k^{(n)}) = \text{Law} \left( Y_k^{(n)} \mid Y_1^{(n)} \leq \cdots \leq Y_k^{(n)} \right).$$

Namely, we have that

$$\text{Law} \left( X_k^{(n)}, \ldots, X_1^{(n)} \right) = \text{Law} \left( \tilde{Y}_k^{(n)}, Y_{k-1}^{(n)}, \ldots, Y_1^{(n)} \mid \tilde{Y}_k^{(n)} \leq Y_{k-1}^{(n)} \leq \cdots \leq Y_1^{(n)} \right).$$

This suggests dividing the argument into two parts: first, understand the limiting law of the random vector $(\tilde{Y}_k^{(n)}, Y_{k-1}^{(n)}, \ldots, Y_1^{(n)})$ and next, perform the conditioning. Since $\tilde{Y}_k^{(n)}, Y_{k-1}^{(n)}, \ldots, Y_1^{(n)}$ are independent, we need only understand the limit of each one separately. Our goal is to obtain the infinite Coulomb gas discussed in Section 3.2. We first take the limits

$$Y_i^{(n)} \xrightarrow{n \to \infty} Y_i,$$  \hfill (3.11)

where the law of $Y_i$ is proportional to

$$\exp \left( - \beta \left[ (\lambda + i - 1) x + V(x) \right] \right) dx,$$

and $V$ is one of the potentials in (3.2), (3.3) or (3.4). Next, we take the limits

$$\tilde{Y}_k^{(n)} \xrightarrow{n \to \infty} \tilde{Y}_k,$$  \hfill (3.12)

where the law of $\tilde{Y}_k$ is the limit as $n \to \infty$ of

$$\text{Law} \left( Y_k \mid Y_m \leq \cdots \leq Y_k \right).$$

which exists due to Proposition 3.3 and 3.1. After proving (3.11) and (3.12), a standard conditioning argument such as the one at the end of the proof of Proposition 3.3 completes the proof. We proceed now to prove (3.11) and (3.12) for each of the cases. We begin with Theorem 2.6

Proof of Theorem 2.6 We begin by noticing that, by (1.7),

$$V_i^{(n)}(x) = \frac{2i - 1 - n}{2} x - \alpha_n U_{\rho_n}(x) = \left( i - 1 + \frac{\alpha_n - (n - 1)}{2} \right) x + \alpha_n \int_{[x,0]} (s - x) d\rho(s) - \frac{\alpha_n}{2} \int_{\mathbb{R}} s d\rho(s).$$

Since $\frac{\alpha_n}{2} \int_{\mathbb{R}} s d\rho(s)$ is independent of $x$ we may redefine

$$V_i^{(n)}(x) = \left( i - 1 + \frac{\alpha_n - (n - 1)}{2} \right) x + \alpha_n \int_{[x,0]} (s - x) d\rho(s).$$

In this case $V$ will be the one in (3.4) so that $Y_i$ is proportional to

$$\exp \left( - \beta (i - 1 + \lambda) x \right) 1_{[0,\infty)}(x) dx,$$

To get (3.11) it is enough to notice that

$$\lim_{n \to \infty} V_i^{(n)}(x) = \begin{cases} (i - 1 + \lambda) x & \text{if } x \geq 0 \infty \\ (i - 1 + \lambda) x & \text{if } x < 0 \end{cases}$$

so that, since for any $\varepsilon \in (0, \lambda)$ and $N > i$

$$V_i^{(n)}(x) > (i - 1 + \lambda - \varepsilon) x + N \int_{[x,0]} s d\mu(s) - P(x).$$
for $n$ large enough, dominated convergence gives (3.11). What is left to prove is (3.12). By Proposition 3.1, the limit of $\text{Law}(Y_k \mid Y_m \leq \cdots \leq Y_k)$ as $m \to \infty$ is

$$\frac{2}{\beta} \sum_{j=k}^{\infty} \frac{Z_j}{j(2\lambda - 1 + j)},$$

where $\{Z_j\}_{j \geq k}$ is a sequence of independent exponential random variables of unit mean. Now, to study the sequence $(\tilde{Y}_k^{(n)})_{n \geq k}$, we shall consider two new random variables. The first one will be an upper bound for $\tilde{Y}_k^{(n)}$ and will be denoted by $U^{(n)}$. It is defined so that it satisfies

$$\text{Law}(U^{(n)}) = \text{Law} \left( Y_k^{(n)} \mid 0 \leq Y_k^{(n)} \leq \cdots \leq Y_k \right).$$

For the second one we fix an integer $M > k$. The random variable will be a lower bound of $\tilde{Y}_k^{(n)}$ and will be denoted by $L_M^{(n)}$. We ask $L_M^{(n)}$ to satisfy

$$\text{Law}(L_M^{(n)}) = \text{Law} \left( Y_k^{(n)} \mid Y_M^{(n)} \leq \cdots \leq Y_k \right).$$

By Proposition 3.1, we have the stochastic domination

$$\mathbb{P} \left( U^{(n)} \leq t \right) \leq \mathbb{P} \left( \tilde{Y}_k^{(n)} \leq t \right) \leq \mathbb{P} \left( L_M^{(n)} \leq t \right).$$

To understand $U^{(n)}$ we notice that the density of

$$\text{Law} \left( Y_k^{(n)} \mid 0 \leq Y_k^{(n)} \leq \cdots \leq Y_k \right)$$

is proportional to

$$(y_1, \ldots, y_k) \mapsto \prod_{i=k}^{n} \exp \left[ -\frac{\beta}{2}(\alpha_n - (n - 1) + 2(i - 1))y_i \right] 1_{0 \leq y_k \leq \cdots \leq y_k}.$$ 

As in the proof of Proposition 3.1 under the change of variables (3.6), we can see that

$$\text{Law}(U^{(n)}) = \frac{2}{\beta} \sum_{j=k}^{n} \frac{Z_j}{j(\alpha_n - (n - 1) + j - 1)}.$$ 

Since $\alpha_n - (n - 1) \xrightarrow{n \to \infty} 2\lambda > 0$, we can bound each term in the sum by $\frac{Z_j}{j(2\lambda + j - 1)}$ for $n$ large enough. Since $\sum_{j \geq 1} \frac{Z_j}{j(\alpha_n - (n - 1) + j - 1)}$ converges almost surely and since $\frac{Z_j}{j(2\lambda + j - 1)}$ converges to $\frac{Z_j}{j(2\lambda + j - 1)}$, we can use the dominated convergence theorem to obtain that

$$\frac{2}{\beta} \sum_{j=k}^{n} \frac{Z_j}{j(\alpha_n - (n - 1) + j - 1)} \xrightarrow{a.s.} \frac{2}{\beta} \sum_{j=k}^{\infty} \frac{Z_j}{j(2\lambda + j - 1)}.$$ 

To understand $L_M^{(n)}$ we take the limit of $(Y_M^{(n)}, \ldots, Y_k^{(n)})$ which is $(Y_M, \ldots, Y_k)$ by (3.11). Since the random variables involved have no atoms, taking the limit commutes with conditioning so that

$$L_M^{(n)} \xrightarrow{n \to \infty} L_M,$$

where

$$\text{Law}(L_M) = \text{Law} \left( Y_k \mid Y_M \leq \cdots \leq Y_k \right).$$

As before, we can see that

$$\text{Law}(L_M) = \frac{M}{\beta} \sum_{j=k}^{\infty} \frac{Z_j}{j(\lambda + (j - 1))},$$

where $\{Z_j\}_{j \geq k}$ is a sequence of independent exponential random variables of unit mean. By taking upper and lower limits in

$$\mathbb{P}(U^{(n)} \leq t) \leq \mathbb{P}(\tilde{Y}_k^{(n)} \leq t) \leq \mathbb{P}(L_M^{(n)} \leq t),$$
we get that
\[
\mathbb{P}\left( \frac{2}{\beta} \sum_{j=k}^{\infty} \frac{W_j}{j(x + (j - 1))} \leq t \right) \leq \lim \inf_{n \to \infty} \mathbb{P}(\tilde{Y}_k^{(n)} \leq t) \leq \lim \sup_{n \to \infty} \mathbb{P}(\tilde{Y}_k^{(n)} \leq t) \leq \mathbb{P}\left( \frac{2}{\beta} \sum_{j=k}^{M} \frac{W_j}{j(x + (j - 1))} \leq t \right).
\]

By taking \( M \to \infty \) we obtain \((3.12)\). \hfill \Box

**Proof of Theorem 2.1.** We have \( \alpha_n \rho_n(x) = 1_{[-\alpha_n,0]}(x) \). Since \( \alpha_n \to \infty \), every subsequence has an increasing (sub)sequence so that we may assume without loss of generality that \( \alpha_n \) is increasing. Let us define
\[
V(x) = \frac{x^2}{2} 1_{x \leq 0}.
\]
We have \( \Delta V(x) = 1_{(-\infty,0]}(x) \) for \( x \neq 0 \). To understand \( V_i^{(n)} \), which is \( \frac{2x-n-1}{2} - \alpha_n U_{\rho_n}(x) \) by \((3.10)\), we notice that \( -\alpha_n U_{\rho_n}(x) = V(x) + \frac{2x}{\alpha_n} \) if \( x \in [-\alpha_n,\infty) \), and for \( x \in (-\alpha_n,-\alpha_n] \), it is affine such that it is differentiable everywhere. One may see this by direct calculation using \((1.7)\), or by noticing that the Laplacians of both \( -\alpha_n U_{\rho_n} \) and \( V \) are the same inside \( [-\alpha_n,\infty) \), and then by calculating the derivative of \( -\alpha_n U_{\rho_n} \) at \( -\alpha_n \). Therefore, setting
\[
V_i(x) = (i - 1 + \frac{\alpha_n - (n-1)}{2})x + V(x) = (i - 1 + \lambda) x + V(x)
\]
we have that
\[
V_i^{(n)}(x) = V_i(x) \quad \text{if} \quad x \in [-\alpha_n,\infty)
\]
and that it is extended in an affine and differentiable way at the left of this interval. We also have that \( Y_i \) has a density proportional to \( e^{-\beta V_i(x)} \). By dominated convergence, since \( \{V_i^{(n)}\}_{n \geq i} \) is an increasing sequence of functions that converges to \( V_i \), we obtain \((3.11)\).

We now prove \((3.12)\). For this, we shall consider two new random variables. The first one will bound \( \tilde{Y}_k^{(n)} \) from above and will be denoted by \( U^{(n)} \). It is defined so that it satisfies
\[
\text{Law}(U^{(n)}) = \text{Law}(Y_k | Y_n \leq \cdots \leq Y_k).
\]
For the second one we fix an integer \( M > k \). The random variable will be a lower bound of \( \tilde{Y}_k^{(n)} \) and will be denoted by \( L_M^{(n)} \). We ask \( L_M^{(n)} \) to satisfy
\[
\text{Law}(L_M^{(n)}) = \text{Law}(Y_k^{(n)} | Y_M^{(n)} \leq \cdots \leq Y_k^{(n)}).
\]
By Lemma \((B.3)\) we have the stochastic domination
\[
\mathbb{P}(U^{(n)} \leq t) \leq \mathbb{P}(\tilde{Y}_k^{(n)} \leq t)
\]
while, by Proposition \((B.4)\)
\[
\mathbb{P}(\tilde{Y}_k^{(n)} \leq t) \leq \mathbb{P}(L_M^{(n)} \leq t).
\]
By definition,
\[
U^{(n)} \xrightarrow{\text{law}} \tilde{Y}_k,
\]
so that we also have
\[
\lim_{n \to \infty} \mathbb{P}(U^{(n)} \leq t) = \mathbb{P}(\tilde{Y}_k \leq t)
\]
when \( t \) is a point of continuity of \( \mathbb{P}(\tilde{Y}_k \leq t) \). Since, we also have
\[
L_M^{(n)} \xrightarrow{\text{law}} U^{(M)},
\]
we obtain that
\[
\mathbb{P}(\tilde{Y}_k \leq t) \leq \lim \inf_{n \to \infty} \mathbb{P}(\tilde{Y}_k^{(n)} \leq t) \leq \lim \sup_{n \to \infty} \mathbb{P}(\tilde{Y}_k^{(n)} \leq t) \leq \mathbb{P}(U^{(M)} \leq t).
\]
Taking \( M \to \infty \) proves \((3.12)\) thus completing the proof. \hfill \Box
Proof of Theorem [2.4]. We have that
\[ \alpha_n \rho_n(x) = 1_{[-\infty,0]}(x) + (\gamma - 1)x^{\gamma - 2} 1_{(0,1)}(\alpha_n (x - 1)^{\frac{1}{\gamma - 1}}). \]
Since \( \alpha_n \to \infty \), every subsequence has an increasing (sub)sequence so that we may assume \( \alpha_n \)

is increasing. Let us define
\[ V(x) = \frac{x^2}{2} 1_{x \leq 0} + \frac{x^\gamma}{\gamma} 1_{x > 0} \]
whose Laplacian is
\[ \Delta V(x) = 1_{(-\infty,0]}(x) + (\gamma - 1)x^{\gamma - 2} 1_{(0,\infty)}. \]
Notice that
\[ -\alpha_n U_{\rho_n}(x) = V(x) + \frac{n}{2}x \quad \text{if} \quad x \in \left[ -\frac{\alpha_n + n}{2}, \left( \frac{\alpha_n - n}{2} \right)^{1/(\gamma - 1)} \right] \]
and that it is affine at the left and at the right of this interval with the derivatives at the endpoints coinciding. One may obtain this by a direct calculation using [1.7], or by noticing that the Laplacians of both functions, \( -\alpha_n U_{\rho_n} \) and \( V \), are the same inside that interval and the derivatives of \( -\alpha_n U_{\rho_n} \) should be \( -\alpha_n/2 \) at the left and \( \alpha_n/2 \) at the right of the interval. If we define
\[ V_i(x) = \left( i - \frac{1}{2} \right)x + V(x) \]
we have that
\[ V_i^{(n)}(x) = V_i(x) \quad \text{if} \quad x \in \left[ -\frac{\alpha_n + n}{2}, \left( \frac{\alpha_n - n}{2} \right)^{1/(\gamma - 1)} \right] \]
and that it is extended in an affine and differentiable way outside this interval. In this case we have the law of \( Y_i \) is proportional to
\[ x \mapsto e^{-\beta V_i(x)}. \]
By dominated convergence, since \( \{V_i^{(n)}\}_{n \geq 1} \) is an increasing sequence of functions that converges to \( V_i \), we obtain [3.11], namely
\[ Y_i^{(n)} \overset{\text{law}}{\longrightarrow} Y_i. \]
Now fix \( k \geq 1 \). To prove [3.12] we consider a random vector \( \left( \tilde{Y}_k^{(n)}, \ldots, \tilde{Y}_n^{(n)} \right) \) such that
\[ \text{Law} \left( \tilde{Y}_k^{(n)}, \ldots, \tilde{Y}_n^{(n)} \right) = \text{Law} \left( Y_k^{(n)}, \ldots, Y_n^{(n)} \mid Y_k^{(n)} \leq \cdots \leq Y_k^{(n)} \right). \]
Consider the event
\[ A_n = \left\{ -\frac{\alpha_n + n}{2} \leq \tilde{Y}_k^{(n)} \quad \text{and} \quad \tilde{Y}_k^{(n)} \leq \left( \frac{\alpha_n - n}{2} \right)^{1/(\gamma - 1)} \right\}. \]
We will see that
\[ P(A_n) \overset{n \to \infty}{\longrightarrow} 1, \tag{3.13} \]
so that, given a bounded continuous function \( f : \mathbb{R} \to \mathbb{R} \) and by writing
\[ E[f(\tilde{Y}_k^{(n)})] = E[f(\tilde{Y}_k^{(n)}) \mid A_n] P(A_n) + E[f(\tilde{Y}_k^{(n)}) \mid A_n^c] P(A_n^c), \]
we see that
\[ \lim_{n \to \infty} E[f(\tilde{Y}_k^{(n)})] = \lim_{n \to \infty} E[f(\tilde{Y}_k^{(n)}) \mid A_n] \tag{3.14} \]
assuming one of these limits exists. To see that the right-side limit exists, we repeat the same argument except with the sequence \( \{Y_i\}_{i \geq k} \) instead of \( \{Y_i^{(n)}\}_{n \geq k} \). If we show the analogue of [3.13], for \( \{Y_i\}_{i \geq k} \), then we also have the analogue of [3.14]. But note that for \( \{Y_i\}_{i \geq k} \), the left-hand side of [3.14] is known to exist by Proposition 3.3 and the right-hand side coincides with the one associated to \( \{Y_i^{(n)}\}_{n \geq k} \) since \( V_i = V_i^{(n)} \) in \( \left( -\frac{\alpha_n + n}{2}, \left( \frac{\alpha_n - n}{2} \right)^{1/(\gamma - 1)} \right) \). Thus we need only show [3.13] with respect to both \( \{Y_i^{(n)}\}_{n \geq k} \) and \( \{Y_i\}_{i \geq k} \).

By Proposition [2.4], we know that
\[ P(Y_k^{(n)} \leq t \mid 0 \leq Y_k^{(n)} \leq \cdots \leq Y_k^{(n)} \leq \cdots \leq Y_k^{(n)}) \leq P(Y_k^{(n)} \leq t \mid Y_k^{(n)} \leq \cdots \leq Y_k^{(n)}). \]
Let \( \{B_i\}_{i \geq k} \) be positive independent random variables such that \( B_i \) has a density proportional to
\[ x \in (0, \infty) \mapsto \exp \left[ -\beta \left( i - \frac{1}{2} \right)^2 \right]. \]
Since $V_i^{(n)}(x) - (i - 1/2)x$ is increasing for $x > 0$, by Lemma B.3 we have that
\[
P(B_k \leq t \mid B_n \leq \cdots \leq B_k) \leq P(Y_n^{(n)} \leq t \mid 0 \leq Y_n^{(n)} \leq \cdots \leq Y_k^{(n)}).
\]
But, by Proposition 3.1, we know that $\inf_{n \geq k} P(B_k \leq t \mid B_n \leq \cdots \leq B_k)$ converges to 1 as $t \to \infty$, so that the same happens to $\inf_{n \geq k} P(Y_k^{(n)} \leq t)$ and, in particular,
\[
P\left(\frac{\alpha_n - n}{2} \leq \frac{n}{2}\right)^{1/(\gamma-1)} \xrightarrow{n \to \infty} 1.
\]
To prove that
\[
P\left(\frac{-\alpha_n + n}{2} \leq \frac{\alpha_n - n}{2}\right)^{1/(\gamma-1)} \xrightarrow{n \to \infty} 1
\]
we consider the system seen from $-n$. More precisely, we use that
\[
P(Y_n^{(n)} \geq t \mid Y_n^{(n)} \leq \cdots \leq Y_k^{(n)} \leq -n) \leq P(Y_n^{(n)} \geq t \mid Y_n^{(n)} \leq \cdots \leq Y_k^{(n)})
\]
and we focus on the left-hand side of this inequality that can be rewritten
\[
P(-n - Y_n^{(n)} \leq -n - t \mid 0 \leq -n - Y_k^{(n)} \leq \cdots \leq -n - Y_n^{(n)}).
\]
The same argument as before applied to $-n - Y_1^{(n)}, \ldots, -n - Y_k^{(n)}$ instead of $Y_1^{(n)}, \ldots, Y_k^{(n)}$ gives
\[
P\left(-n - Y_n^{(n)} \leq \frac{\alpha_n - n}{2} \mid 0 \leq -n - Y_k^{(n)} \leq \cdots \leq -n - Y_n^{(n)}\right) \xrightarrow{n \to \infty} 1
\]
which implies (3.15). The same arguments work for the sequence $\{Y_i^{(n)}\}_{i \geq k}$ instead of $\{Y_i^{(n)}\}_{n \geq k}$ which completes the proof.

**Proof of Corollary 3.8** By Theorems 2.6, 2.1 and 2.4 we know that $M_n$ converges in law. The limiting law is $\Theta_1$ from Proposition 3.3 by choosing $V$ properly, and $\Theta_1$ can be also described as
\[
\Theta_1 = \text{Law}(Y_1 \mid \tilde{Y}_2 \leq 1)
\]
where $\tilde{Y}_2$ and $Y_1$ are independent random variables such that $\tilde{Y}_2 \sim \Theta_2$ and $Y_1$ has a density proportional to $\rho(x) = e^{-\beta(x+V)}$. In particular, $\Theta_1$ has a density proportional to
\[
x \in \mathbb{R} \mapsto \rho(x)\mathbb{P}(\tilde{Y}_2 \leq x).
\]
In fact, by repeating the same kind of procedure with $\Theta_2$ instead of $\Theta_1$ and using induction on $k$ it can be proved that the regularity of the density coincides with the regularity of $V$ but this will not be needed here. Since $\Theta_1$ has no atoms we have that
\[
\lim_{n \to \infty} \mathbb{P}(M_n > t) = \Theta_1(t, \infty)
\]
and we may conclude by Proposition A.2.

**Appendix A. Tail asymptotics**

In this appendix, we prove a proposition regarding the tail asymptotics of the right-most particles of our infinite Coulomb gases at the edge. We first need a short lemma to prove the proposition.

**Lemma A.1.** If $Y_1$ and $Y_2$ are independent random variables, then
\[
\mathbb{P}(Y_1 \geq t, Y_2 \leq Y_1) \sim \mathbb{P}(Y_1 \geq t).
\]

**Proof.** By independence of $Y_1$ and $Y_2$,
\[
\mathbb{P}(Y_1 \geq t, Y_2 \leq Y_1) = \int_{\mathbb{R}} \mathbb{P}(Y_1 \geq t \vee y) d\mathbb{P}_{Y_2}(y),
\]
where $\mathbb{P}_{Y_2}$ denotes the law of $Y_2$. Since $\mathbb{P}(Y_1 \geq t \vee y) d\mathbb{P}_{Y_2}(y)$ is increasing in $t$ we may use the monotone convergence theorem to obtain
\[
\frac{\mathbb{P}(Y_1 \geq t, Y_2 \leq Y_1)}{\mathbb{P}(Y_1 \geq t)} = \int_{\mathbb{R}} \frac{\mathbb{P}(Y_1 \geq t \vee y) d\mathbb{P}_{Y_2}(y)}{\mathbb{P}(Y_1 \geq t)} \xrightarrow{t \to \infty} \int_{\mathbb{R}} d\mathbb{P}_{Y_2}(y) = 1.
\]

□
Proposition A.2 (Tail asymptotics at the right). Using the notation of Proposition 3.3, if $X \sim \Theta_1$ is the right-most particle of the infinite Coulomb gas, we have that

$$\log P(X \geq t) = \log \int_t^\infty e^{-\beta (\lambda x + V(x))} dx + O(1).$$

In particular, if $V$ is (3.2) for some $\gamma > 1$, we have that

$$\frac{1}{t} \log P(X \geq t) \xrightarrow{t \to \infty} -\frac{\beta}{\gamma},$$

and if $V$ is (3.3) or (3.4) we have that

$$\frac{1}{t} \log P(X \geq t) \xrightarrow{t \to \infty} -\beta \lambda.$$

Proof. Let $Y_1$ have a density proportional to $e^{-\beta (\lambda x + V(x))}$ and let $Y_2 \sim \Theta_2$ from Proposition 3.3 be independent of $Y_1$. By (3.8), $P(X \geq t) = P(Y_1 \geq t | Y_2 \leq Y_1)$. By Lemma A.1

$$\log P(X \geq t) = \log P(Y_1 \geq t) + O(1) = \log \int_t^\infty e^{-\beta (\lambda x + V(x))} dx + O(1).$$

Suppose that $V(x) = \frac{x^2}{2}, x \geq 0, \gamma > 1$. Then, for every $\varepsilon > 0$ there is $T > 0$ such that for $t \geq T$,

$$\log \int_t^\infty e^{-\beta (1+\varepsilon) V(x)} dx \leq \log \int_t^\infty e^{-\beta (\lambda x + V(x))} dx \leq \log \int_t^\infty e^{-\beta V(x)} dx.$$ 

Thus, we only need to show that

$$\frac{1}{t} \log \int_t^\infty e^{-\beta (1+\varepsilon) V(x)} dx \xrightarrow{t \to \infty} -\frac{\beta (1+\varepsilon)}{\gamma},$$

for every $\varepsilon > 0$. This can be obtained by the change of variables $s = x/t$ and by using Laplace’s method since the minimum of $\beta (1+\varepsilon) x^\gamma$ for $s \in [1, \infty)$ is attained at $s = 1$. For $V$ chosen as in (3.3) or (3.4) we only need to use that $V(x) = 0$ for $x \geq 0$, and that

$$\int_t^\infty e^{-\beta \lambda x} dx = \frac{e^{-\beta \lambda t}}{\beta \lambda}.$$ 

\[\square\]

Appendix B. Stochastic domination and conditioning

Note that stochastic domination is also known as stochastic monotonicity. Throughout this appendix, we use “density” to refer to a Radon–Nikodym derivative.

Lemma B.1 (Dominion from non-decreasing density). Let $\mu$ and $\nu$ be two probability measures for which there exists a non-decreasing measurable function $\rho : \mathbb{R} \to [0, \infty)$ such that $d\nu = \rho d\mu$. If $X \sim \mu$ and $Y \sim \nu$ then, for every $t \in \mathbb{R}$,

$$P(Y \leq t) \leq P(X \leq t).$$

Proof. Since $\rho$ is non-decreasing there exists $a \in [-\infty, \infty]$ such that $\rho(a) \leq 1$ for $x \in (-\infty, a)$ and $\rho(a) \geq 1$ for $x \in (a, \infty)$.

If $t \leq a$ we have that $P(Y \leq t) = \int_{-\infty}^t \rho d\mu \leq \int_{-\infty}^t d\mu = P(X \leq t)$.

If $t > a$ we have that $P(Y > t) = \int_{(t, \infty)} \rho d\mu \geq \int_{(t, \infty)} d\mu = P(X > t)$.

Hence $P(Y \leq t) = 1 - P(Y > t) \leq 1 - P(X > t) = P(X \leq t)$.

\[\square\]

Lemma B.2 (Conditioning by steps). Let $X_1, \ldots, X_n$ be independent random variables such that $P(X_n \leq \cdots \leq X_1) > 0$. Fix $k \in \{1, \ldots, n\}$ and consider a random variable $Y_k$ such that, by possibly enlarging the probability space,

$$\text{Law}(Y_k) = \text{Law}(X_k | X_n \leq \cdots \leq X_k)$$

and $Y_k$ is independent of $(X_1, \ldots, X_{k-1})$. Then, $P(Y_k \leq X_{k-1} \leq \cdots \leq X_1) > 0$ and

$$\text{Law}(X_k, \ldots, X_1 | X_n \leq \cdots \leq X_1) = \text{Law}(Y_k, X_{k-1}, \ldots, X_1 | Y_k \leq X_{k-1} \leq \cdots \leq X_1).$$
Proof. Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and some \(A \in \mathcal{F}\) such that \(\mathbb{P}(A) > 0\), let us use the notation \(\mathbb{P}_A(C) = \mathbb{P}(C \cap A)/\mathbb{P}(A)\). For any \(A, B \in \mathcal{F}\) such that \(\mathbb{P}(A \cap B) > 0\), we have that
\[
\mathbb{P}_A(B) > 0 \quad \text{and} \quad (\mathbb{P}_A)_B = \mathbb{P}_{A \cap B}.
\] (B.1)

In our setting, we consider \(\Omega = \mathbb{R}^n\) with \(\mathbb{P}\) given by the law of \((X_n, \ldots, X_1)\). We have
\[
A = \{(x_n, \ldots, x_1) \in \mathbb{R}^n : x_n \leq \cdots \leq x_k\} \quad \text{and} \quad B = \{(x_n, \ldots, x_1) \in \mathbb{R}^n : x_k \leq \cdots \leq x_1\}.
\]

Let \(Y_k, \ldots, Y_n\) be random variables such that
\[
\text{Law}(Y_n, \ldots, Y_k) = \text{Law}(X_n, \ldots, X_k | X_n \leq \cdots \leq X_k).
\]

Then, \(\mathbb{P}_A\) is the law of \((Y_n, \ldots, Y_k, X_{k-1}, \ldots, X_1)\) and the left-hand side of (B.1) tells us that
\[
\mathbb{P}(Y_k \leq X_{k-1} \leq \cdots \leq X_1) = \mathbb{P}_A(B) > 0
\]
while the right-hand side of (B.1) tells us that
\[
\text{Law}(Y_k, X_{k-1}, \ldots, X_1 | Y_k \leq X_{k-1} \leq \cdots \leq X_1) = \text{Law}(X_k, \ldots, X_1 | X_n \leq \cdots \leq X_1).
\]

In particular, we have that
\[
\text{Law}(Y_k, X_{k-1}, \ldots, X_1 | Y_k \leq X_{k-1} \leq \cdots \leq X_1) = \text{Law}(X_k, \ldots, X_1 | X_n \leq \cdots \leq X_1)
\]
which is the sought property. \(\square\)

The non-decreasing density condition of Lemma B.1 is preserved under order conditioning as the following proposition states.

**Proposition B.3** (Preservation of non-decreasing densities under ordering). Let \(X_1, \ldots, X_n, Y_1, \ldots, Y_n\) be independent real random variables such that \(X_i \sim \mu_i\) and \(Y_i \sim \nu_i\), where \(\nu_i\) is absolutely continuous with respect to \(\mu_i\) with a non-decreasing density \(\rho_i = d\nu_i/d\mu_i\). If we define
\[
\tilde{\mu} = \text{Law}(X_1 | X_n \leq \cdots \leq X_1) \quad \text{and} \quad \tilde{\nu} = \text{Law}(Y_1 | Y_n \leq \cdots \leq Y_1)
\]
then \(\tilde{\nu}\) is absolutely continuous with respect to \(\tilde{\mu}\), with non-decreasing density \(\tilde{\rho} = d\tilde{\nu}/d\tilde{\mu}\).

**Proof.** By Lemma B.2 if we take random variables \(\tilde{X}_2\) and \(\tilde{Y}_2\) such that
\[
\text{Law}(\tilde{X}_2) = \text{Law}(X_2 | X_n \leq \cdots \leq X_2) \quad \text{and} \quad \text{Law}(\tilde{Y}_2) = \text{Law}(Y_2 | Y_n \leq \cdots \leq Y_2)
\]
and such that \(\tilde{X}_2\) is independent of \(X_1\) and \(\tilde{Y}_2\) is independent of \(Y_1\), we have
\[
\tilde{\mu} = \text{Law}(X_1 | \tilde{X}_2 \leq X_1) \quad \text{and} \quad \tilde{\nu} = \text{Law}(Y_1 | \tilde{Y}_2 \leq Y_1).
\]

So, by induction, it is enough to prove the lemma for \(n = 2\). In this case, \(\tilde{\mu}\) has a density with respect to \(\mu_1\) proportional to \(x \in \mathbb{R} \mapsto \mu_2(-\infty, x]\) and \(\tilde{\nu}\) has a density with respect to \(\mu_1\) proportional to
\[
x \in \mathbb{R} \mapsto \mu_1(x) \nu_2(-\infty, x] = \mu_1(x) \int_{(-\infty, x]} \mu_2 \, d\mu_2.
\]

In particular, \(\tilde{\nu}\) is absolutely continuous with respect to \(\tilde{\mu}\) and
\[
d\tilde{\nu} \propto \frac{\mu_1(x) \int_{(-\infty, x]} \mu_2 \, d\mu_2}{\mu_2(-\infty, x]} \, d\tilde{\mu}.
\]

The proof is completed once we show that
\[
G(x) = \int_{(-\infty, x]} \frac{\mu_2 \, d\mu_2}{\mu_2(-\infty, x]}
\]
is non-decreasing. Note that \(G\) is well-defined since if the denominator is zero then the numerator, which would be an integral over a measure zero set, is also zero.

We must show that if \(x \leq y\) then \(G(x) \leq G(y)\). We know that \(\mu_2(t) \leq \mu_2(s)\) for any \(t \in (-\infty, x]\) and \(s \in (x, y]\). By integrating over \((t, s) \in (-\infty, x] \times (x, y]\), with respect to \(\mu_2 \otimes \mu_2\), we obtain
\[
\left( \int_{(-\infty, x]} \mu_2(t) \, d\mu_2(t) \right) \mu_2(x, y] \leq \left( \int_{(x, y]} \mu_2(s) \, d\mu_2(s) \right) \mu_2(-\infty, x].
\]
Proposition B.4. Let \( \mu_2(-\infty, y] \) which, after dividing by \( \mu_2(-\infty, x] \mu_2(-\infty, y] \), completes the proof.

We next provide two applications of Proposition B.3 that are useful our context.

**Proposition B.4.** Let \( X_1, \ldots, X_n \) be independent random variables with \( \mathbb{P}(X_n \leq \cdots \leq X_1) > 0 \). Then, for every \( 1 \leq k \leq m \leq n \) and \( t \in \mathbb{R} \),

\[
\mathbb{P}(X_k \leq t \mid X_n \leq \cdots \leq X_1) \leq \mathbb{P}(X_k \leq t \mid X_m \leq \cdots \leq X_1).
\]

**Proof.** By the same reasoning as in Lemma B.2 if we take a random variable \( \tilde{X}_k \) such that

\[
\mathbb{P}(\tilde{X}_k = \mathbb{P}(X_k \mid X_k \leq \cdots \leq X_1)
\]

and independent of \( X_1, \ldots, X_{k-1} \), then

\[
\mathbb{P}(\tilde{X}_k = \mathbb{P}(X_k \mid X_n \leq \cdots \leq X_{k+1} \leq \tilde{X}_k)
\]

and

\[
\mathbb{P}(\tilde{X}_k = \mathbb{P}(X_k \mid X_m \leq \cdots \leq X_{k+1} \leq \tilde{X}_k).
\]

Thus it suffices to consider the case \( k = 1 \). It is enough to prove that

\[
\mathbb{P}(X_1 \leq t \mid X_{p+1} \leq \cdots \leq X_1) \leq \mathbb{P}(X_1 \leq t \mid X_p \leq \cdots \leq X_1)
\]

for every \( p \in \{m, \ldots, n-1\} \). For this, let \( Y_p \sim \mathbb{P}(X_p \mid X_{p+1} \leq X_p) \) be independent of \( X_1, \ldots, X_{p-1} \). Then, by Lemma B.2 we have

\[
\mathbb{P}(X_1 \mid X_{p+1} \leq \cdots \leq X_1) = \mathbb{P}(X_1 \mid Y_p \leq X_{p-1} \leq \cdots \leq X_1).
\]

Since \( Y_p \) has, with respect to the law of \( X_p \), the non-decreasing density

\[
y \mapsto \frac{\mathbb{P}(X_{p+1} \leq y)}{\mathbb{P}(X_{p+1} \leq X_p)},
\]

by Proposition B.3 \( \mathbb{P}(X_1 \mid Y_p \leq X_{p-1} \leq \cdots \leq X_1) \) has a non-decreasing density with respect to the law \( \mathbb{P}(X_1 \mid X_p \leq X_{p-1} \leq \cdots \leq X_1) \). It remains finally to use Lemma B.1.

**Lemma B.5.** Let \( t_0 \in [-\infty, \infty) \) and let \( X_1, \ldots, X_n, Y_1, \ldots, Y_n \) be independent real random variables taking values on \([t_0, \infty)\) such that \( X_i \) has density \( e^{-g_i} \) and \( Y_i \) has density \( e^{-h_i} \), and such that \( g_i - h_i \) is non-decreasing. Then, for every \( t \in \mathbb{R} \),

\[
\mathbb{P}(Y_1 \leq t \mid Y_n \leq \cdots \leq Y_1) \leq \mathbb{P}(X_1 \leq t \mid X_n \leq \cdots \leq X_1).
\]

**Proof.** Since \( Y_i \) has the non-decreasing density \( e^{g_i - h_i} \), with respect to the law of \( X_i \), we may use Proposition B.3 and Lemma B.1 to complete the proof.

**Appendix C. Conditional law of right-most particle**

The main results in this work concern point processes with infinitely many particles. If one is concerned only with the right-most particle (or finitely many particles) in a conditional setting, the proofs can be greatly simplified. We illustrate this in the present appendix.

More specifically, Proposition C.1 is non-asymptotic, and gives the location of the \( k \) right-most particles of the gas, under conditioning, for an arbitrary background of compact support. Up to the conditioning, this can be seen as some sort of one-dimensional analog of a similar phenomenon for two-dimensional Coulomb gases due to Kostlan [Kos92] and considered, for instance, in [HKPV09, CP13, BGZ18, CZ21, CGZ20a]. The proposition is reminiscent of a classical representation theorem due to Alfred Rényi [R65], which states that if \( \{Z_{ij}\}_{1 \leq i \leq k} \) are independent and identically distributed exponential random variables of unit mean, and if \( \{\tilde{Z}_{ij}\}_{1 \leq i \leq k} \) are the order statistics (max to min), then the joint distribution of \( \{\tilde{Z}_{ij}\}_{1 \leq i \leq k} \) is given by the identity in distribution

\[
\mathbb{P}(Z_{(k)} \leq \cdots \leq Z_{(1)}) = \mathbb{P}(Z_{(1)} \leq \cdots \leq Z_{(1)}).
\]

We write \( \text{Card} S \) to denote the cardinality of a set \( S \).
Proposition C.1 (Right-most particles outside the background). Suppose that
  - $\beta > 0$ is fixed;
  - $\alpha > n - 1$;
  - $\rho$ is supported inside $(-\infty, 0]$.

Let us denote $X(n)_{i(1)}$ instead of $X_{i(1)}^{(n)}$, and define $N_n = \text{Card}\{1 \leq i \leq n : X_i > 0\}$.

Then, for all $1 \leq k \leq n$,

$$\text{Law} \left( X_{k(1)}, \ldots, X_{(1)} | N_n = k \right) = \text{Law} \left( Y_{k(1)}, \ldots, Y_{(1)} | Y_k < \cdots < Y_1 \right)$$

where $(Y_{i})_{1 \leq i \leq k}$ are independent exponential random variables with $EY_i = \frac{2}{\beta(\alpha - n - 1 + 2 \gamma)}$.

Alternatively, for all $1 \leq k \leq n$,

$$\text{Law} \left( X_{k(1)}, \ldots, X_{(1)} | N_n = k \right) = \text{Law} \left( Z_1, Z_1 + Z_2, \ldots, Z_1 + \cdots + Z_k \right)$$

where $(Z_i)_{1 \leq i \leq k}$ are independent exponential random variables with $EZ_i = \frac{2}{\beta(\alpha - n + 1)}$.

Proof. By (1.10) we know that

$$\text{Law} \left( X^{(n)}(n), \ldots, X^{(n)}_{(1)} \right) = \text{Law} \left( Y^{(n)}_{1}, \ldots, Y^{(n)}_{1} | Y^{(n)}_{1} \leq \cdots \leq Y^{(n)}_{1} \right)$$

where $Y^{(n)}_{n}, \ldots, Y^{(n)}_{1}$ are independent random variables and $Y^{(n)}_{1}$ has a density proportional to

$$y \in \mathbb{R} \mapsto \exp \left( -\beta \left[ \frac{2k - n - 1}{2} y - \alpha U_\rho(y) \right] \right).$$

Then, we have that

$$\text{Law} \left( X^{(n)}_{n}, \ldots, X^{(n)}_{1} | N_n = k \right)$$

$$= \text{Law} \left( Y^{(n)}_{n}, \ldots, Y^{(n)}_{1} | Y^{(n)}_{n} \leq \cdots \leq Y^{(n)}_{k+1} \leq 0 < Y^{(n)}_{k} \leq \cdots \leq Y^{(n)}_{1} \right).$$

Then, by the independence of $\left( Y_{n}^{(n)}, \ldots, Y_{k+1}^{(n)} \right)$ and $\left( Y_{k}^{(n)}, \ldots, Y_{1}^{(n)} \right)$, we have that

$$\text{Law} \left( X^{(n)}_{k(1)}, \ldots, X_{(1)}^{(n)} | N_n = k \right) = \text{Law} \left( Y_{k}^{(n)}, \ldots, Y_{1}^{(n)} | 0 < Y_{k}^{(n)} \leq \cdots \leq Y_{1}^{(n)} \right).$$

By (1.1) we can see that

$$U_\rho(y) = -\frac{y}{2} + \frac{1}{2} \int \sigma \, d\rho(\sigma),$$

for $y > 0$ so that

$$\text{Law} \left( Y^{(n)}_{k}, \ldots, Y^{(n)}_{1} \right) \bigg| 0 < Y^{(n)}_{k} \leq \cdots \leq Y^{(n)}_{1} \bigg) = \text{Law} \left( Y_{k}, \ldots, Y_{1} | Y_{k} \leq \cdots \leq Y_{1} \right)$$

where $Y_i$ follows the law of $Y^{(n)}_{1}$ conditioned to be positive which has a density proportional to

$$y \in (0, \infty) \mapsto e^{-\frac{1}{2} (\alpha - n + 1 + 2 \gamma) y}.$$ 

Then $\text{Law} (X^{(n)}_{k(1)}, \ldots, X_{(1)}^{(n)} | N_n = k)$ has a joint density proportional to

$$(x_k, \ldots, x_1) \mapsto \exp \left[ -\frac{\beta}{2} \sum_{j=1}^{k} (\alpha - n + 1 + 2j) x_j \right] \mathbb{1}_{0 \leq x_k \leq \cdots \leq x_1}.$$

We can perform the change of variables

$$z_i = x_i - x_{i+1} \text{ if } i \in \{1, \ldots, k - 1\} \quad \text{and} \quad z_k = x_k$$

or equivalently $x_j = \sum_{i=j}^{k} z_i$ for any $i \in \{1, \ldots, k\}$ to obtain a density proportional to

$$(z_k, \ldots, z_1) \mapsto \prod_{i=1}^{k} \exp \left[ -\frac{\beta}{2} (\alpha - n + i) z_i \right] \mathbb{1}_{z_i \geq 0}$$

which implies the second assertion of the proposition. \qed

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