Bourgain’s Entropy Estimates Revisited

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Abstract

This serves as a near complete set of notes on Bourgain’s well-known paper *Almost sure convergence and bounded entropy* [2]. The two entropy results are treated, as is one of the applications. These notes are designed to be independent of Bourgain’s paper and self-contained. There are, at times, differences between Bourgain’s notation and my own. The same goes for organization. However, the proofs herein are essentially his.

1 Preliminaries

Our setting will be a probability space \((X, \mathcal{F}, \mu)\). We are interested in certain sequences of operators. In particular, given a sequence of operators on \(L^2(\mu)\) we want to make some uniform estimate on their entropy. What kind of conditions must this sequence satisfy? We are led to the following definitions.

**Definition.** Given a pseudo-metric space \((Y, d)\) (that is, \(d\) need not separate points) and a subset \(S \subset Y\), define the \(\delta\)-entropy number of \(S\) to be the minimal number of (closed) \(\delta\)-balls in the \(d\) pseudo-metric needed to cover \(S\). We denote this by \(N(S, d, \delta)\).

**Notation.** Denote by \(\mathcal{M}(X)\) the set of measurable functions \(f : X \to \mathbb{R}\). By \(L^p(\mu)\), it will always be meant the subset of \(\mathcal{M}(X)\) which has finite \(L^p\)-norm. That is, \(L^p(\mu)\) consists of real-valued functions only.

**Definition.** Let \(T_j : \mathcal{M}(X) \to \mathcal{M}(X), j \in \mathbb{N}\), be a sequence of linear operators. We say \((T_j)\) is a Bourgain sequence if the following are satisfied:

1. \(T_j : L^1(\mu) \to L^1(\mu)\) are bounded,
2. \(T_j : L^2(\mu) \to L^2(\mu)\) are isometries,
3. \(T_j\) are positive, i.e., if \(f \geq 0\) a.s.[\(\mu\)] then \(T_j(f) \geq 0\) a.s.[\(\mu\)],
4. \(T_j(1) = 1\) (here, 1 refers to the constant function 1),
5. $T_j$ satisfy a mean ergodic condition, i.e.,
\[
\frac{1}{J} \sum_{j=1}^{J} T_j f \to \int_X f(x) \, d\mu(x) \quad \text{in } L^1(\mu)-\text{norm for all } f \in L^1(\mu).
\]

The mean ergodic condition will prove useful as is. The first four assumptions lead to the following properties.

**Lemma 1.** Let $T : \mathcal{M}(X) \to \mathcal{M}(X)$ be a linear operator which satisfies assumptions (1)-(4) above. Then, $T : L^\infty(\mu) \to L^\infty(\mu)$ with $\|T f\|_{L^\infty(\mu)} \leq \|f\|_{L^\infty(\mu)}$. Further, $T(f^2) = T(f)^2$ a.s.$[\mu]$ for all $f \in L^2(\mu)$.

*Proof.* By assumption 4, $T(c) = c$ for all constant functions $c$. By assumption 3, $T(g) \leq T(h)$ a.s.$[\mu]$ whenever $g \leq h$ a.s.$[\mu]$. Let $f \in L^\infty(\mu)$. Then, $f \leq \|f\|_{L^\infty(\mu)}$ a.s.$[\mu]$, so that $T(f) \leq T(\|f\|_{L^\infty(\mu)}) = \|f\|_{L^\infty(\mu)}$ a.s.$[\mu]$. Similarly, $-T(f) = T(-f) \leq T(\|f\|_{L^\infty(\mu)}) = \|f\|_{L^\infty(\mu)}$ a.s.$[\mu]$. Thus, $|T(f)| \leq \|f\|_{L^\infty(\mu)}$ a.s.$[\mu]$, giving the first statement.

We now approach the second statement. First, suppose $A \in \mathcal{F}$. As $0 \leq \chi_A \leq 1$, we have $0 \leq T(\chi_A) \leq 1$. By assumption 2, $\mu(A) = \|\chi_A\|^2_{L^2(\mu)} = \|T(\chi_A)\|^2_{L^2(\mu)} = \int_X T(\chi_A)^2 \, d\mu$. On the other hand, $\mu(A) = 1 - \|\chi_A^c\|^2_{L^2(\mu)} = 1 - \|1 - \chi_A\|^2_{L^2(\mu)} = 1 - \|T(1 - \chi_A)\|^2_{L^2(\mu)} = 1 - \|T(\chi_A)\|^2_{L^2(\mu)} = 1 - \int_X (1 - T(\chi_A))^2 \, d\mu = \int_X 2T(\chi_A) - T(\chi_A)^2 \, d\mu$. Setting these two expressions of $\mu(A)$ equal, we have $\int_X T(\chi_A) \, d\mu = \int_X T(\chi_A)^2 \, d\mu$. As $0 \leq T(\chi_A) \leq 1$ a.s.$[\mu]$, it must be that $T(\chi_A) = 0, 1$ a.s.$[\mu]$.

Namely, $T$ takes indicator functions to a.s. indicator functions.

Now suppose $A, B \in \mathcal{F}$ are disjoint. Then, $\chi_A \chi_B = 0$. So, $0 = \int_X \chi_A \chi_B \, d\mu = \langle \chi_A, \chi_B \rangle = \langle T(\chi_A), T(\chi_B) \rangle = \int_X T(\chi_A) T(\chi_B) \, d\mu$. As the integrand is necessarily nonnegative a.s.$[\mu]$, we have $T(\chi_A) T(\chi_B) = 0$ a.s.$[\mu]$.

Let $s = \sum_{i=1}^n c_i \chi_{A_i}$ be a simple function, where $A_i$ are pairwise disjoint. Then, $s^2 = \sum_i c_i^2 \chi_{A_i}$. Now, $T(s) = \sum_i c_i T(\chi_{A_i})$, which gives $T(s)^2 = \sum_i \sum_k c_i c_k T(\chi_{A_i}) T(\chi_{A_k}) = \sum_i c_i^2 T(\chi_{A_i})^2 = \sum_i c_i^2 T(\chi_{A_i}) = T(s^2)$ a.s.$[\mu]$.

Let $f \in L^2(\mu)$ and $\epsilon > 0$. Denote by $\|T\|$ the operator norm of $T$ on $L^1(\mu)$. Choose a simple function $s$ so that $|s| \leq |f|$, $\|s - f\|_{L^2(\mu)} < \epsilon/(4\|f\|_{L^2(\mu)})$ and $\|s^2 - f^2\|_{L^1(\mu)} < \epsilon/(2\|T\|)$. Then,

\[
\|T(f^2) - T(f)^2\|_{L^1(\mu)} \\
\leq \|T(f^2) - T(s^2)\|_{L^1(\mu)} + \|T(s^2) - T(s)^2\|_{L^1(\mu)} + \|T(s)^2 - T(f)^2\|_{L^1(\mu)} \\
= \|T(f^2 - s^2)\|_{L^1(\mu)} + \|T(f - T(s))(T(f) + T(s))\|_{L^1(\mu)} \\
\leq \|T\| \|f - s\|_{L^2(\mu)}^2 + \|T(f + s)\|_{L^2(\mu)} \|T(f - s)\|_{L^2(\mu)} \\
< \epsilon/2 + 2\|f\|_{L^2(\mu)} \|f - s\|_{L^2(\mu)} \\
< \epsilon.
\]

As $\epsilon$ is arbitrary, we have $T(f^2) = T(f)^2$ a.s.$[\mu]$. □

Our results will focus on a more general sequence. Let $S_n : \mathcal{M}(X) \to \mathcal{M}(X)$, $n \in \mathbb{N}$, be a sequence of linear operators where each $S_n : L^2(\mu) \to L^2(\mu)$ is bounded (not necessarily uniformly). We have the following Banach principle-type statements.
Theorem 2. Let $S_n : L^2(\mu) \to L^2(\mu)$ be bounded. Suppose that for some $2 \leq p < \infty$, $\sup_n |S_n f|$ is finite a.s. [\mu] for all $f \in L^p(\mu)$. Then, there is a finite-valued function $\theta(\epsilon)$ so that

$$\mu \left\{ x \in X : \sup_n |S_n f(x)| > \theta(\epsilon) \right\} < \epsilon$$

for every $\|f\|_{L^p(\mu)} \leq 1$.

Theorem 3. Let $S_n : L^2(\mu) \to L^2(\mu)$ be bounded. Suppose $\sup_n f$ converges a.s. [\mu] for all $f \in L^\infty(\mu)$. Then, for each $\epsilon > 0$ and $\eta > 0$, there exists $\rho(\epsilon, \eta) > 0$ such that

$$\mu \left\{ x \in X : \sup_n |S_n f(x)| > \eta \right\} < \epsilon$$

for all $\|f\|_{L^\infty(\mu)} \leq 1$ and $\|f\|_{L^1(\mu)} \leq \rho(\epsilon, \eta)$.

Theorem 2 is a classical result, and Theorem 3 is due to Bellow and Jones [1]. We postpone the proofs of these two theorems until Section 5. The principal assumption we make on the sequence $(S_n)$ is that it commutes with a Bourgain sequence $(T_j)$, that is, $S_n T_j = T_j S_n$ for all $n, j$. More on this later.

2 Normal Random Variables

The proofs of the two entropy results rely heavily on the theory of Gaussian (or normal) random variables and Gaussian processes. It is advantageous at this point to review a few fundamental results. Let $(\Omega, \mathcal{B}, P)$ be another probability space.

**Definition.** We say a random variable $g : \Omega \to \mathbb{R}$ is normal (or Gaussian) with mean $m$ and variance $\sigma^2$ if it has the density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left( -\frac{(x-m)^2}{2\sigma^2} \right),$$

i.e., $P(g \in A) = \int_A f(x) \, dx$ for all Borel sets $A$. A normal random variable with mean 0 and variance 1 is called a standard normal random variable.

Recall, for a random variable $g$ with mean 0, the variance is given by $\sigma^2 = E(g^2) = \|g\|_{L^2(P)}^2$, where $E(\cdot)$ is expectation. Also, if $g$ is a normal random variable with mean 0, then it is centered, that is, $P(g > 0) = P(g < 0) = 1/2$. The following results are well-known in probability theory, and we present them without proof.

**Lemma 4.** If $g$ is a random variable on $\Omega$, then

$$\int_\Omega |g(\omega)| \, P(d\omega) = \int_0^\infty P(|g| > t) \, dt \quad \text{and} \quad \int_\Omega |g(\omega)|^2 \, P(d\omega) = 2 \int_0^\infty tP(|g| > t) \, dt.$$
Lemma 5. Let $g$ be a normal random variable with mean 0. Then, the moment generating function is given by

$$\int_{\Omega} e^{\lambda g(\omega)} P(d\omega) = e^{\lambda^2 \sigma^2 / 2}, \quad \lambda \in \mathbb{R}.$$  

Lemma 6. For each $1 \leq p < \infty$, there exists a constant $C_p > 0$ so that $\|g\|_{L^p(P)} \leq C_p \|g\|_{L^2(P)}$ for all normal random variables $g$ with mean 0.

Lemma 7. Let $g_1, g_2, \ldots, g_m$ be independent standard normal random variables. Then, for any constants $a_i, \sum a_i g_i$ is a normal random variable with mean 0 and variance $\sum a_i^2$.

Lemma 4 is proven by two simple applications of Fubini’s Theorem. The proof of Lemma 5 is a standard result and is found in most probability texts. Lemma 7 follows immediately from independence. Only Lemma 6 is a somewhat deep result. In fact, something stronger is true; the $L^p$ and $L^q$ norms of a normal random variable are uniformly equivalent for any $1 \leq p, q < \infty$. We need only the case $q = 2$. A proof of the general result can be found in [7] (Corollary 3.2).

We now state an important estimate for normal random variables [3]. The proof is postponed until Section 6.

Theorem 8. Let $G_1, \ldots, G_N$ be normal random variables each with mean 0. If for some constant $s$ we have $P\{\omega \in \Omega : \sup_n |G_n(\omega)| \leq s\} \geq 1/2$, then $\| \sup_n |G_n| \|_{L^1(P)} \leq 6s$.

We now turn our attention to Gaussian processes.

Definition. Let $T$ be a countable indexing set. We say a collection of random variables $(G_t : t \in T)$ is a Gaussian process if each $G_t$ has mean 0 and all finite linear combinations $\sum_t a_t G_t$ are normal random variables.

Note that this definition is not entirely standard, in particular the requirement that each $G_t$ have mean 0. It is added here because throughout all Gaussian processes we deal with have this property, and it makes life simpler later on.

If each $G_t$ is itself a finite linear combination of mean 0 normal random variables, then $(G_t : t \in T)$ is trivially a Gaussian process. Define a pseudo-metric on $T$ by $d_G(s, t) = \|G_s - G_t\|_{L^2(P)}$. Denote the entropy number of $T$ by $N(T, d_G, \delta)$. The following fundamental result is Sudakov’s inequality.

Theorem 9. There exists a universal constant $R$ such that if $(G_t : t \in T)$ is a Gaussian process, then

$$\sup_{\delta > 0} \delta \sqrt{\log N(T, d_G, \delta)} \leq R \left\| \sup_{t \in T} |G_t| \right\|_{L^1(P)}.$$

For the remainder of the paper, the notation $R$ is fixed on this constant. A proof of Sudakov’s inequality can be found in [7] (Theorem 3.18).
3 The First Entropy Result

Recall, we consider a sequence \( S_n : L^2(\mu) \to L^2(\mu) \) of bounded operators. For \( f \in L^2(\mu) \), define a pseudo-metric on \( \mathbb{N} \) by \( d_f(n, n') = \| S_nf - S_{n'}f \|_{L^2(\mu)} \). For \( \delta > 0 \), let \( N_f(\delta) := N(\mathbb{N}, d_f, \delta) \), that is, the \( \delta \)-entropy number of the set \( \{ S_nf : n \in \mathbb{N} \} \) in \( L^2(\mu) \). We now state and prove the first of Bourgain’s entropy results.

**Proposition 1.** Let \( S_n : L^2(\mu) \to L^2(\mu) \) be bounded (not necessarily uniformly), and assume \( (S_n) \) commutes with a Bourgain sequence \( (T_j) \). Suppose that for some \( 1 \leq p < \infty \), \( \sup_n |S_nf| < \infty \) a.s.\([\mu]\) for all \( f \in L^p(\mu) \). Then, there exists a constant \( C > 0 \) such that \( \delta \)\((\log N_f(\delta))^{1/2} \leq C\| f \|_{L^2(\mu)} \) for all \( \delta > 0 \) and \( f \in L^2(\mu) \).

**Proof.** As \( (X, \mu) \) is a probability space, \( L^p(\mu) \supset L^q(\mu) \) when \( p < q \). So, if \( p < 2 \), then \( \sup_n |S_nf| < \infty \) a.s.\([\mu]\) for all \( f \in L^p(\mu) \supset L^2(\mu) \) for any \( q \geq 2 \). Therefore, assume without loss of generality that \( p \geq 2 \).

For \( M \in \mathbb{N} \), let \( \mathcal{M} = \{1, 2, \ldots, M\} \). Note, \( \sup_M N(\mathcal{M}, d_f, \delta) = N_f(\delta) \). Therefore, it suffices to find \( C \), independent of \( M \), such that \( \delta \)\((\log N(\mathcal{M}, d_f, \delta))^{1/2} \leq C\| f \|_{L^2(\mu)} \) for all \( f, \delta \). Fix \( M \in \mathbb{N} \).

Suppose we could show \( \delta \)\((\log N(\mathcal{M}, d_f, \delta))^{1/2} \leq C\| f \|_{L^2(\mu)} \) for all \( f \in L^\infty(\mu) \) and all \( \delta > 0 \). Let \( f \in L^2(\mu) \) and \( \delta > 0 \). Let \( D = \max\{\|S_1\|, \ldots, \|S_M\|\} \) be the maximum of the \( L^2(\mu) \) operator norms. Choose \( f_1 \in L^\infty(\mu) \) with \( |Sf_1| \leq |f| \) and \( \|f - f_1\|_{L^2(\mu)} < \delta/(2D) \). Then, \( \|S_nf - S_nf_1\|_{L^2(\mu)} < \delta/2 \) for all \( n \in \mathcal{M} \) and \( \|N(\mathcal{M}, d_f, \delta) \leq N(\mathcal{M}, d_{f_1}, \delta/2) \). Hence, \( \delta \)\((\log N(\mathcal{M}, d_{f_1}, \delta))^{1/2} \leq 2C\|f_1\|_{L^2(\mu)} \leq 2C\|f\|_{L^2(\mu)} \), and we have the desired estimate with \( 2C \). Therefore, it suffices to prove the result for all \( L^\infty(\mu) \) functions. Fix \( f \in L^\infty(\mu) \).

We will fix \( J \) at some large integer. By the mean ergodic condition on \( T_j \), we have

\[
J^{-1} \sum_{j=1}^J T_j(f^2) \to \|f^2\|_{L^1(\mu)} = \|f\|_{L^2(\mu)}^2
\]

in \( L^1(\mu) \)-norm. But, \( |J^{-1} \sum_{j=1}^J T_j(f^2)| \leq \|f\|_{L^\infty(\mu)}^2 \) a.s.\([\mu]\) by Lemma 1. Of course, if a sequence of functions \( h_n \) converges to a constant \( c \) in \( L^1(\mu) \)-norm and \( |h_n| \leq B \) a.s.\([\mu]\) for all \( n \), it follows \( h_n \to c \) in \( L^q(\mu) \)-norm for all \( 1 \leq q < \infty \). Hence, as \( p \geq 2 \), choose \( J \) so large that

\[
\left\| \frac{1}{J} \sum_{j=1}^J T_j(f^2) \right\|_{L^p/2(\mu)} \leq 2\|f\|_{L^2(\mu)}^2.
\]

Similarly, for each pair \( n, n' \in \mathcal{M} \), \( J^{-1} \sum_{j=1}^J T_j(S_nf - S_{n'}f)^2 \to \|S_nf - S_{n'}f\|_{L^2(\mu)}^2 = d_f(n, n')^2 \) in \( L^1(\mu) \)-norm, and thus in probability. So, for \( J \) large enough,

\[
\mu(\mathcal{C}_{n,n'}) := \mu \left\{ x \in X : J^{-1} \sum_{j=1}^J T_j(S_nf - S_{n'}f)^2(x) \geq \frac{1}{4} d_f(n, n')^2 \right\} > 1 - \frac{1}{16M^2}.
\]

Choose \( J \) big enough so that this holds for each pair \( n, n' \).

Let \( g_1, \ldots, g_J \) be a sequence of independent standard normal random variables on a probability space \((\Omega, \mathcal{B}, P)\). Define the functions \( F, F^* \) on the product space \( X \times \Omega \) by
\[
F(x, \omega) = J^{-1/2} \sum_{j=1}^{J} g_j(\omega) T_j f(x) \quad \text{and} \quad F^*(x, \omega) = \sup_{n \in \mathbb{M}} |S_n F(x, \omega)|.
\]

By the commutativity assumption, \(S_n F(x, \omega) = J^{-1/2} \sum_{j=1}^{J} g_j(\omega) T_j S_n f(x)\). Note, for each fixed \(x\) such that \(T_j S_n f(x)\) is finite for all \(j, n\), \((S_n F(x, \cdot) : n \in \mathbb{M})\) is a Gaussian process. The focus of the proof will be finding the “correct” \(x\) to fix.

We define four sets \(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \subset X\). First, let

\[
\mathcal{A} = \left\{ x \in X : |T_j S_n f(x)| < \infty \text{ for all } 1 \leq j \leq J, n \in \mathbb{M} \right\}.
\]

As each \(T_j S_n f \in L^2(\mu)\), it is clear that \(\mu(\mathcal{A}) = 1\). Set

\[
\mathcal{B} = \left\{ x \in X : T_j (S_n f - S_{n'} f)^2(x) = T_j (S_n f - S_{n'} f)(x)^2 \text{ for all } 1 \leq j \leq J, n, n' \in \mathbb{M} \right\}
\]

By Lemma II, \(\mu(\mathcal{B}) = 1\). Let \(\mathcal{C} = \bigcap_{n, n'} \mathcal{C}_{n, n'}\), i.e.,

\[
\mathcal{C} = \left\{ x \in X : \left( J^{-1} \sum_{j=1}^{J} T_j (S_n f - S_{n'} f)^2(x) \right)^{1/2} \geq \frac{1}{2} d_f(n, n') \text{ for all } n, n' \in \mathbb{M} \right\}.
\]

Now,

\[
\mu\left( \bigcup_{n, n' \in \mathbb{M}} \mathcal{C}_{n, n'} \right) \leq \sum_{n, n'} \mu(\mathcal{C}_{n, n'}) < \sum_{n, n'} \frac{1}{16 M^2} = \frac{1}{16}.
\]

Equivalently, \(\mu(\mathcal{C}) > 15/16\).

Finally, let \(R' = 5 \sqrt{2C_p} \theta(1/4)\), where where \(\theta\) is the finite-valued function from Theorem 2 and \(C_p\) is the constant from Lemma 6. Note, \(R'\) does not depend on \(f\) or \(M\). Set

\[
\mathcal{D} = \left\{ x \in X : P\{ \omega \in \Omega : F^*(x, \omega) \leq R' \| f \|_{L^2(\mu)} \} \geq 1/2 \right\}.
\]

Now, by Lemmas II and 7 and II, we have
\[
\int \|F(\cdot, \omega)\|_{L^p(\mu)} P(d\omega) = \int \Omega \left( \int_X |F(x, \omega)|^p \mu(dx) \right)^{1/p} P(d\omega) \\
\leq \left( \int \Omega \int_X |F(x, \omega)|^p \mu(dx) P(d\omega) \right)^{1/p} \\
= \left( \int_X \int_\Omega |x| J^{-1/2} g_j(\omega) T_j f(x) \right)^p P(d\omega) \mu(dx) \right)^{1/p} \\
= \left( \int_X \left| \sum_{j=1}^J J^{-1/2} g_j(\cdot) T_j f(x) \right|^{p} \mu(dx) \right)^{1/p} \\
\leq \left( \int_X C_p^p \left| \sum_{j=1}^J J^{-1/2} g_j(\cdot) T_j f(x) \right|^{p} \mu(dx) \right)^{1/p} \\
= C_p \left( \int_X \left| \sum_{j=1}^J (T_j f)^2 \right|^{p/2} \mu(dx) \right)^{1/p} \\
\leq C_p \sqrt{2} \|f\|_{L^2(\mu)}.
\]

By Chebyshev’s inequality, \( P\{ \omega : \|F(\cdot, \omega)\|_{L^p(\mu)} > 5\sqrt{2} C_p \|f\|_{L^2(\mu)} \} \leq 1/5 < 1/4 \), or equivalently,

\[
P\{ \omega \in \Omega : \|F(\cdot, \omega)\|_{L^p(\mu)} \leq 5\sqrt{2} C_p \|f\|_{L^2(\mu)} \} > 3/4.
\]

Fix an \( \omega \) in the above set. Then, \( F(\cdot, \omega) \in L^p(\mu) \). By Theorem 2,

\[
\mu \left\{ x \in X : \sup_{n \in \mathbb{N}} |S_n F(x, \omega)| \leq \|F(\cdot, \omega)\|_{L^p(\mu)} \theta(1/4) \right\} > 3/4.
\]

Of course, \( F^*(x, \omega) \leq \sup_n |S_n F(x, \omega)| \). Further, we have a bound on \( \|F(\cdot, \omega)\|_{L^p(\mu)} \) by the choice of \( \omega \). Thus,

\[
\mu \left\{ x \in X : F^*(x, \omega) \leq 5\sqrt{2} C_p \theta(1/4) \|f\|_{L^2(\mu)} \right\} > 3/4.
\]

As this holds for all such \( \omega \), we have

\[
P\{ \omega \in \Omega : \mu \{ x \in X : F^*(x, \omega) \leq R \|f\|_{L^2(\mu)} \} > 3/4 \} > 3/4.
\]

We now apply Fubini’s theorem. In particular,
3/4 < \int_\Omega \chi_{\mu(F^*(x,\omega) \leq R' ||f||_{L^2(\mu)}) > 3/4}(\omega) P(d\omega) \leq \int_\Omega \frac{4}{3} \mu\{x : F^*(x,\omega) \leq R' ||f||_{L^2(\mu)}\} P(d\omega)
= \frac{4}{3} \int_X \int_\Omega \chi_{F^*(x,\omega) \leq R' ||f||_{L^2(\mu)}}(x,\omega) \mu(dx) P(d\omega)
= \frac{4}{3} \int_X \int_\Omega \chi_{F^*(x,\omega) \leq R' ||f||_{L^2(\mu)}}(x,\omega) P(d\omega) \mu(dx)
= \frac{4}{3} \int_X P\{\omega : F^*(x,\omega) \leq R' ||f||_{L^2(\mu)}\} \mu(dx)
= \frac{4}{3} \left[ \int_D P\{F^*(x,\omega) \leq R' ||f||_{L^2(\mu)}\} \mu(dx) + \int_{D^c} P\{F^*(x,\omega) \leq R' ||f||_{L^2(\mu)}\} \mu(dx) \right]
\leq \frac{4}{3} (\mu(D) + 1/2).

The last line follows from the definition of \( D \). This gives \( \mu(D) > 1/16 \).

The estimates \( \mu(\mathcal{A}) = 1, \mu(\mathcal{B}) = 1, \mu(\mathcal{C}) > 15/16, \) and \( \mu(D) > 1/16 \) together imply that \( \mu(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} \cap D) > 0 \). Fix \( \overline{\omega} \in \mathcal{A} \cap \mathcal{B} \cap \mathcal{C} \cap D \). Define \( G_n(\omega) = S_n F(\overline{\omega}, \omega) = J^{-1/2} \sum g_j(\omega) T_j S_n f(\overline{\omega}) \). As \( \overline{\omega} \in \mathcal{A}, (G_n : n \in \mathcal{M}) \) is a Gaussian process. By Sudakov’s inequality and Theorem \( \mathbb{5} \) and because \( \overline{\omega} \in D \), we see that

\[
\sup_{\delta > 0} \delta (\log N(\mathcal{M}, d_G, \delta))^{1/2} \leq R \int_\Omega \sup_{n \in \mathcal{M}} |G_n(\omega)| P(d\omega)
= R \int_\Omega F^*(\overline{\omega}, \omega) P(d\omega) \leq 6RR' ||f||_{L^2(\mu)}.
\]

On the other hand, each \( G_n - G_{n'} \) is a linear combination of independent standard normal random variables. It follows from Lemma \( \mathbb{7} \) again and because \( \overline{\omega} \in \mathcal{B} \cap \mathcal{C} \) that

\[
d_G(n, n') = ||G_n - G_{n'}||_{L^2(p)} = \left( \frac{1}{J} \sum_{j=1}^J \left( T_j S_n f(\overline{\omega}) - T_j S_{n'} f(\overline{\omega}) \right)^2 \right)^{1/2}
= \left( \frac{1}{J} \sum_{j=1}^J T_j (S_n f - S_{n'} f)^2(\overline{\omega}) \right)^{1/2}
\geq \frac{1}{2} d_f(n, n').
\]

This implies \( N(\mathcal{M}, d_f, \delta) \leq N(\mathcal{M}, d_G, \delta/2) \) for all \( \delta > 0 \). Hence, \( \delta (\log N(\mathcal{M}, d_f, \delta))^{1/2} \leq 12RR' ||f||_{L^2(\mu)} \). We note that \( 12RR' \) is universal, and does not depend on \( M \) or \( f \). As \( f \in L^\infty(\mu) \) was arbitrary, this holds all a.s. bounded functions. By our earlier note, \( \delta (\log N(\mathcal{M}, d_f, \delta))^{1/2} \leq 24RR' ||f||_{L^2(\mu)} =: C||f||_{L^2(\mu)} \) for all \( f \in L^2(\mu) \) and all \( \delta > 0 \). Taking the supremum over \( M \), \( \delta (\log N_f(\delta))^{1/2} \leq C||f||_{L^2(\mu)}. \) \( \square \)
4 The Second Entropy Result

It will now be necessary to assume the \((S_n)\) are uniformly bounded. Of course, by dividing out a constant, we may assume each \(S_n\) is an \(L^2(\mu)\)-contraction, i.e., \(\|S_n f\|_{L^2(\mu)} \leq \|f\|_{L^2(\mu)}\) for all \(n\).

**Proposition 2.** Let \(S_n\) be a sequence of \(L^2(\mu)\) contractions that commute with a Bourgain sequence \((T_j)\). Suppose \(S_n f\) converges a.s.\([\mu]\) for all \(f \in L^\infty(\mu)\). Then, there exists a finite-valued function \(C(\delta)\) such that \(N_f(\delta) \leq C(\delta)\) for all \(\delta > 0\) and \(\|f\|_{L^2(\mu)} \leq 1\).

**Proof.** Suppose we can show the uniform entropy estimate for all \(f \in L^\infty(\mu), \|f\|_{L^2(\mu)} \leq 1, \delta > 0\). Choose \(f_1 \in L^\infty(\mu), |f_1| \leq |f|\) such that \(\|f - f_1\|_{L^2(\mu)} < \delta/2\). Then, \(\|S_n f - S_n f_1\|_{L^2(\mu)} < \delta/2\) for all \(n\) and \(N_f(\delta) \leq N_{f_1}(\delta/2) \leq C(\delta/2)\). As \(\delta\) and \(f\) are arbitrary, we have the uniform estimate with the function \(C_0(\delta) = C(\delta/2)\). It therefore suffices to prove the result for a.s. bounded functions.

We proceed by contradiction. Suppose not, i.e., suppose there is some \(\delta > 0\) such that \(N_f(\delta)\) is unbounded over all such \(f\). Define the constant \(R' = \frac{\delta}{2\gamma}\). As per Theorem 3, pick the constant \(\rho(1/10, R'/10)\). Choose \(K \in \mathbb{N}, K > 1\) big enough so that \(\frac{1}{\delta}(R' - 1000(\log K)^{-1/2}) > R'/10\) and \(2(\log K)^{-1/2} < \rho(1/10, R'/10)\).

Now, by our assumption, there is some \(f \in L^\infty, \|f\|_{L^2(\mu)} \leq 1\) such that \(N_f(\delta) > K\). In particular, there is a subset \(I \subset \mathbb{N}\) with \(|I| = K\) (cardinality) and \(\|S_n f - S_n' f\|_{L^2(\mu)} > \delta\) for all \(n \neq n' \in I\).

As before, we will need to choose an appropriately large \(J\). First, denote \(B = \|f\|_{L^\infty(\mu)}\). Fix a number \(T > 0\) such that \(T > 3\sqrt{\log K}\) and \(\exp(\frac{T^2}{2B'}) \leq \frac{7}{2R'B'}\). Note, the quantity \(e^{\lambda^2(2-B^2)} - e^{\lambda^2(1-B^2)}\) is strictly positive for all \(\lambda \in [\sqrt{\log K}, T/3]\). Let \(\gamma > 0\) be the minimum value of this quantity for \(\lambda\) in this interval. As \(J^{-1}\sum J_j(f^2) \to \|f^2\|_{L^1(\mu)} = \|f\|_{L^2(\mu)}^2\) in \(L^1(\mu)\)-norm, it converges in probability. So, pick \(J\) large enough so that

\[
\mu(Y) := \mu\left\{x \in X : \frac{1}{J}\sum_{j = 1}^{J} T_j(f^2)(x) - \|f\|_{L^2(\mu)}^2 > 1\right\} < \gamma. \tag{2}
\]

Recall from the proof of Proposition 1, \(J^{-1}\sum J_j(S_n f - S_n' f)^2 \to \|S_n f - S_n' f\|_{L^2(\mu)}^2\) in probability for each pair \(n, n' \in I\). So, just as we did before, take \(J\) big enough so that if

\[
Z_1 = \left\{x \in X : \left(J^{-1}\sum_{j = 1}^{J} T_j(S_n f - S_n' f)^2(x)\right)^{1/2} \geq \frac{1}{2}\|S_n f - S_n' f\|_{L^2(\mu)} \text{ for all } n, n' \in I\right\}
\]

then \(\mu(Z_1) > 4/5\). \tag{3}

Again, define \(F(x, \omega) = J^{-1/2}\sum_{j = 1}^{J} g_j(\omega)T_j f(x)\). Write \(F(x, \omega) = \varphi(x, \omega) + H(x, \omega)\) where

\[
\varphi(x, \omega) = F(x, \omega)\chi_{\{|F(x, \omega)| \leq 6\sqrt{\log K}\}} \quad \text{and} \quad H(x, \omega) = F(x, \omega)\chi_{\{|F(x, \omega)| > 6\sqrt{\log K}\}}.
\]

Define three subsets of \(\Omega\) by
\[ \mathcal{A} = \left\{ \omega \in \Omega : \int_{X} \sup_{n \in I} |S_n H(x, \omega)| \mu(dx) \leq 90 \right\}, \]
\[ \mathcal{B} = \left\{ \omega \in \Omega : \mu \left\{ x \in X : \sup_{n \in I} |S_n F(x, \omega)| > R'(\log K)^{1/2} \right\} > 1/5 \right\}, \]
\[ \mathcal{C} = \left\{ \omega \in \Omega : \int_{X} |\varphi(x, \omega)| \mu(dx) \leq 12 \right\}. \]

Suppose for the moment that we could choose \( \overline{\omega} \in \mathcal{A} \cap \mathcal{B} \cap \mathcal{C} \). Define \( \psi(x) = \frac{1}{\psi}(\log K)^{-1/2} \varphi(x, \overline{\omega}) \). Simply from the definition of \( \varphi \), it follows
\[ |\psi| \leq 1. \]

As \( \overline{\omega} \in \mathcal{C} \),
\[ \int_{X} |\psi(x)| \mu(dx) \leq 2(\log K)^{-1/2} < \rho(1/10, R'/10). \]

By Theorem [3], we have that
\[ \mu \left\{ x \in X : \sup_{n} \left| S_n \psi(x) \right| > R'/10 \right\} < 1/10. \]

(4)

On the other hand, as \( \overline{\omega} \in \mathcal{A} \), we have by Chebyshev that
\[ \mu \left\{ x \in X : \sup_{n \in I} \left| S_n H(x, \overline{\omega}) \right| > 1000 \right\} \leq 90/1000 < 1/10, \]

or equivalently
\[ \mu \left\{ x \in X : \sup_{n \in I} \left| S_n H(x, \overline{\omega}) \right| \leq 1000 \right\} > 9/10. \]

(5)

As \( \overline{\omega} \in \mathcal{B} \),
\[ \mu \left\{ x \in X : \sup_{n \in I} \left| S_n F(x, \overline{\omega}) \right| > R'(\log K)^{1/2} \right\} > 1/5. \]

(6)

Now, \( \sup_n |S_n \varphi| \geq \sup_I |S_n \varphi| \geq \sup_I |S_n F| - \sup_I |S_n H| \). So, taking the intersection of the sets in (5) and (6), we have
\[ \mu \left\{ x \in X : \sup_{n} \left| S_n \varphi(x, \overline{\omega}) \right| \geq R'(\log K)^{1/2} - 1000 \right\} > 1/10. \]

Applying the definition of \( \psi \) and the choice of \( K \),
\[ \mu \left\{ x \in X : \sup_{n} \left| S_n \psi(x) \right| > R'/10 \right\} > 1/10. \]

This clearly contradicts (4). Therefore, it suffices to find such an \( \overline{\omega} \).
Estimate of $A$.

Fix $\lambda \in [\log K, T/3]$. By considering $F(x, \omega)$ as a normal random variable (in $\omega$) with variance $J^{-1} \sum T_j f(x)^2$, it follows from Lemma 5 that

$$\int_X \int_\Omega \exp(\lambda F(x, \omega)) P(d\omega) \mu(dx) = \int_X \exp \left( \frac{\lambda^2}{2J} \sum_{j=1}^J T_j f(x)^2 \right) \mu(dx)$$

$$= \int_X \exp \left( \frac{\lambda^2}{2J} \sum_{j=1}^J T_j (f^2)(x) \right) \mu(dx).$$

Now, by (2), we see

$$J^{-1} \sum_{j=1}^J f(x)^2 \leq \|f\|_{L^2(\mu)}^2 + 1 \leq 2 \text{ on } Y^c.$$  

On the other hand, we have

$$J^{-1} \sum_{j=1}^J f(x)^2 \leq B^2 \text{ a.s.}[\mu].$$  

So,

$$\int_X \int_\Omega \exp(\lambda F(x, \omega)) P(d\omega) \mu(dx) = \int_X \exp \left( \frac{\lambda^2}{2J} \sum_{j=1}^J T_j (f^2)(x) \right) \mu(dx)$$

$$\leq \int_{Y^c} e^{\lambda^2} \mu(dx) + \int_Y e^{\lambda^2 B^2} \mu(dx)$$

$$\leq e^{\lambda^2} + \gamma e^{\lambda^2 B^2}$$

$$\leq e^{\lambda^2} + (e^{\lambda^2(2-B^2)} - e^{\lambda^2(1-B^2)})e^{\lambda^2 B^2} = e^{2\lambda^2}.$$  

Define $\mu_t(\omega) = \mu\{x \in X : |F(x, \omega)| > t\}$. Then, for all $t > 0$, we have from above that

$$e^{\lambda t} \int_\Omega \mu_t(\omega) P(d\omega) = e^{\lambda t} \int_X P\{|F(x, \omega)| > t\} \mu(dx)$$

$$= 2e^{\lambda t} \int_X P\{F(x, \omega) > t\} \mu(dx)$$

$$= 2 \int_X \int_{\{F(x, \omega) > t\}} e^{\lambda t} P(d\omega) \mu(dx)$$

$$\leq 2 \int_X \int_{\{F(x, \omega) > t\}} e^{\lambda F(x, \omega)} P(d\omega) \mu(dx)$$

$$\leq 2 e^{2\lambda t}.$$  

Set $t = 3\lambda$. Then, $\int_\Omega \mu_t(\omega) P(d\omega) \leq 2e^{-t^2/9}$, and this holds for all $t \in [3\sqrt{\log K}, T]$. On the other hand, $J^{-1} \sum T_j (f^2) \leq B^2 \text{ a.s.}[\mu]$. So, for all $t, \lambda > 0$,
$e^{\lambda t} \int_{\Omega} \mu_t(\omega) P(d\omega) \leq 2 \int_X \int_{\Omega} e^{\lambda F(x,\omega)} P(d\omega) \mu(d \omega)$

$$= 2 \int_X \exp \left( \frac{\lambda^2}{2J} \sum_{j=1}^{J} T_j(f^2)(x) \right) \mu(dx)$$

$$\leq 2 \int_X \exp \left( \frac{\lambda^2 B^2}{2} \right) \mu(dx)$$

$$= 2 e^{\lambda^2 B^2/2}.$$ 

Set $t = \lambda B^2$ to see $\int_X \mu_t(\omega) P(d\omega) \leq 2 e^{-\lambda^2 B^2/2} = 2 e^{-t^2/(2B^2)}$ for all $t > 0$.

From Lemma 4,

$$\int_{\Omega} \int_X |H(x,\omega)|^2 \mu(dx) P(d\omega) = 2 \int_{\Omega} \int_0^\infty t \mu\{|H(x,\omega)| > t\} dt P(d\omega)$$

$$= 2 \int_{\Omega} \int_0^{3 \sqrt{\log K}} t \mu\{|H(x,\omega)| > t\} dt P(d\omega)$$

$$+ 2 \int_{\Omega} \int_{3 \sqrt{\log K}}^\infty t \mu\{|H(x,\omega)| > t\} dt P(d\omega) \quad (7)$$

By definition of $H$, and an application of Chebyshev,

$$2 \int_{\Omega} \int_0^{3 \sqrt{\log K}} t \mu\{|H(x,\omega)| > t\} dt P(d\omega)$$

$$= 2 \int_{\Omega} \int_0^{3 \sqrt{\log K}} t \mu\{|H(x,\omega)| > 6 \sqrt{\log K}\} dt P(d\omega)$$

$$\leq 2 \int_{\Omega} \int_0^{3 \sqrt{\log K}} t \left( \frac{1}{6 \sqrt{\log K}} \right)^2 \left( \int_X |H(x,\omega)|^2 \mu(dx) \right) dt P(d\omega) \quad (8)$$

$$= \frac{1}{4} \int_X \int_{\Omega} |H(x,\omega)|^2 \mu(dx) P(d\omega)$$

$$\leq \frac{1}{2} \int_X \int_{\Omega} |H(x,\omega)|^2 \mu(dx) P(d\omega).$$

As $|H| \leq |F|$ everywhere, $\mu\{|H(x,\omega)| > t\} \leq \mu_t(\omega)$ for all $t$ and $\omega$. So,
\begin{align*}
2 \int_{\Omega} \int_{3\sqrt{\log K}}^\infty t \mu\{|H(x, \omega)| > t\} \, dt \, P(d\omega) \\
&\leq 2 \int_{\Omega} \int_{3\sqrt{\log K}}^\infty t \mu_t(\omega) \, dt \, P(d\omega) \\
&= 2 \int_{\Omega} \int_T^{\infty} t \mu_t(\omega) \, dt \, P(d\omega) + 2 \int_{\Omega} \int_T^{\infty} t \mu_t(\omega) \, dt \, P(d\omega) \\
&\leq 4 \int_T^{\infty} t e^{-t^2/9} \, dt + 4 \int_T^{\infty} t e^{-t^2/(2B^2)} \, dt \\
&= 18K^{-1} - \frac{9}{2} e^{-T^2/9} + 4B^2 e^{-T^2/(2B^2)} \\
&\leq 32K^{-1}
\end{align*}

by the choice of $T$. Combining (7), (8), and (9),

\[
\frac{1}{2} \int_{\Omega} \int_X |H(x, \omega)|^2 \mu(dx) \, P(d\omega) \leq 32K^{-1}.
\]

Stated another way, \( \int_{\Omega} \|H(\cdot, \omega)\|_{L^2(\mu)}^2 \, P(d\omega) \leq 64/K \). This gives

\[
\int_{\Omega} \int_X \sup_{n \in I} |S_n H(x, \omega)| \mu(dx) \, P(d\omega) = \int_{\Omega} \left\| \sup_{n \in I} |S_n H(\cdot, \omega)| \right\|_{L^1(\mu)} \, P(d\omega) \\
\leq \int_{\Omega} \| \sup_{n \in I} |S_n H(\cdot, \omega)| \|_{L^2(\mu)} \, P(d\omega) \\
= \int_{\Omega} \left( \int_X \| \sup_{n \in I} |S_n H(x, \omega)| \|^2 \mu(dx) \right)^{1/2} \, P(d\omega) \\
\leq \int_{\Omega} \left( \int_X \sum_{n \in I} |S_n H(x, \omega)|^2 \mu(dx) \right)^{1/2} \, P(d\omega) \\
= \int_{\Omega} \left( \sum_{n \in I} \|S_n H(\cdot, \omega)\|_{L^2(\mu)}^2 \right)^{1/2} \, P(d\omega) \\
\leq \int_{\Omega} \left( \sum_{n \in I} \|H(\cdot, \omega)\|_{L^2(\mu)}^2 \right)^{1/2} \, P(d\omega) \\
= K^{1/2} \int_{\Omega} \|H(\cdot, \omega)\|_{L^2(\mu)} \, P(d\omega) \\
\leq K^{1/2} \left( \int_{\Omega} \|H(\cdot, \omega)\|_{L^2(\mu)}^2 \, P(d\omega) \right)^{1/2} \\
\leq K^{1/2}(64/K)^{1/2} = 8.
\]

It follows by Chebyshev that \( P\{\int_X \sup_{n \in I} |S_n H(x, \omega)| \mu(dx) > 90\} \leq 8/90 < 1/10 \), or equivalently, \( P(A) > 9/10 \).

Estimate of $B$. 

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Denote $F^*(x, \omega) = \sup_{n \in I} |S_n F(x, \omega)|$. Define the sets $Z_2, Z_3 \subset X$ by

$$Z_2 = \{ x \in X : |T_j S_n f(x)| < \infty \text{ for all } 1 \leq j \leq J, \ n \in I \},$$

$$Z_3 = \{ x \in X : T_j (S_n f - S_{n'} f)^2(x) = T_j (S_n f - S_{n'} f)(x)^2 \text{ for all } 1 \leq j \leq J, \ n, n' \in I \}.$$ 

As in the proof of Proposition 1, $\mu(Z_2) = \mu(Z_3) = 1$. Let $Z = Z_1 \cap Z_2 \cap Z_3$, so that $\mu(Z) > 4/5$ by (4).

Fix $x \in Z$. Define a Gaussian process by $G_n(\omega) = S_n F(x, \omega)$ and let $d_G(n, n') = \|G_n - G_{n'}\|_{L^2(P)}$ as before. By the definition of $Z$ and the original hypothesis,

$$d_G(n, n') = \left( J^{-1} \sum_{j=1}^J T_j (S_n f - S_{n'} f)(x)^2 \right)^{1/2} \geq \frac{1}{2} \|S_n f - S_{n'} f\|_{L^2(\mu)} > \delta/2$$

for all $n \neq n' \in I$. So, $N(I, d_G; \delta/4) = |I| = K$. By Sudakov’s inequality,

$$\int_{\Omega} F^*(x, \omega) P(d\omega) \geq \frac{\delta}{R^4} \left[ \log N(I, d_G; \delta/4) \right]^{1/2} = 6 \left( \frac{\delta}{24R} \right) (\log K)^{1/2} > 6R'(\log K)^{1/2}. $$

It follows from Theorem 8 that $P\{ \omega : F^*(x, \omega) \leq R'(\log K)^{1/2} \} < 1/2$, or equivalently, $P\{ \omega : F^*(x, \omega) > R'(\log K)^{1/2} \} > 1/2$. As this holds for all $x \in Z$, we have

$$\mu\left\{ x \in X : P\{ \omega \in \Omega : F^*(x, \omega) > R'(\log K)^{1/2} \} > 1/2 \right\} > 4/5.$$ 

By the same Fubini trick as in the proof of Proposition 1, we see that

$$4/5 < \int_X \int_{\Omega} 2X\{F^*(x, \omega) > R'(\log K)^{1/2}\} P(d\omega) \mu(dx) = \int_X \int_{\Omega} 2X\{F^*(x, \omega) > R'(\log K)^{1/2}\} \mu(dx) P(d\omega)$$

$$= 2 \left[ \mu\{ x : F^*(x, \omega) > R'(\log K)^{1/2} \} P(d\omega) + \int_{B^c} \mu\{ x : F^*(x, \omega) > R'(\log K)^{1/2} \} P(d\omega) \right] \leq 2(P(B) + 1/5),$$

which implies $P(B) > 1/5$.

**Estimate of C.**

Now,
Thus, the hypothesis. So, there is some

from Chebyshev, we see

Recall, our setting is a probability space \((\Omega, \mathcal{F}, \mu)\).

Proof of Theorem 2.

Theorem 2 is a well-known result. The proof is as follows.

Fix \(\epsilon > 0\). Let \(f \in L^p(\mu)\). Then, \(S^*f := \sup_n |S_nf| < \infty \text{ a.s.}[\mu]\), by the hypothesis. So, there is some \(n \in \mathbb{N}\) (depending on \(f\)), so that \(\mu\{x : S^*f(x) > n\} \leq \epsilon/3\). Thus,

\[
L^p(\mu) = \bigcup_{n=1}^{\infty} \left\{ f \in L^p(\mu) : \mu\{S^*f > n\} \leq \epsilon/3 \right\}.
\]

Denote \(B_n = \{ f \in L^p(\mu) : \mu\{S^*f > n\} \leq \epsilon/3 \}\). Let \(S^*_nf(x) := \sup\{|S_nf(x)| : 1 \leq n \leq N\}\). It follows that

\[
B_n \cap B_{n+1} = \bigcap_{N=1}^{\infty} \left\{ f \in L^p(\mu) : \mu\{S^*_nf > n\} \leq \epsilon/3 \right\}.
\]
Denote $B_n^\infty = \{ f \in L^p(\mu) : \mu\{ S_N^* f > n \} \leq \varepsilon/3 \}$. Fix $n, N$. Now, it is clear that $\| S_N^* f \|_{L^2(\mu)} \leq \| \sum_{k=1}^N |S_k| f \|_{L^2(\mu)} \leq (\sum_{k=1}^N \| S_k \| |f|_{L^2(\mu)}) =: C \| f \|_{L^2(\mu)}$ for all $f \in L^2(\mu)$, where $\| S_k \|$ refers to the operator norm on $L^2(\mu)$. Suppose $(f_k) \in B_n^\infty$ and $f_k \to f$ in $L^p(\mu)$-norm. Let $r \in \mathbb{N}$. Then, by Chebyshev,

$$\mu\{ S_N^* f > n + 1/r \} \leq \mu\{ |S_N^* f - S_N^* f_k| > 1/r \} + \mu\{ S_N^* f_k > n \}$$

$$\leq r^2 \| S_N^* (f - f_k) \|_{L^2(\mu)}^2 + \varepsilon/3$$

$$\leq C^2 r^2 \| f - f_k \|_{L^2(\mu)}^2 + \varepsilon/3$$

$$\leq C^2 r^2 \| f - f_k \|_{L^p(\mu)}^2 + \varepsilon/3 \to \varepsilon/3.$$

Because $\bigcup\{ S_N^* f(x) > n + 1/r \} = \{ S_N^* f(x) > n \}$ and these sets are nested, we have the limit $\mu\{ S_N^* f > n + 1/r \} \to \mu\{ S_N^* f > n \}$. Namely, $\mu\{ S_N^* f > n \} \leq \varepsilon/3$ and $f \in B_n^\infty$. Thus, $B_n^\infty$ is closed (in $L^p(\mu)$), which implies $B_n$ is closed. It follows from the Baire Category Theorem, because $L^p(\mu) = \bigcup B_n$, that one of $B_n$ contains an open set. That is, there exists some $n \in \mathbb{N}$, $\delta > 0$, and $f_0 \in L^p(\mu)$ such that $f \in B_n$ for all $\| f - f_0 \|_{L^p(\mu)} \leq \delta$. In particular, $\mu\{ S^*(f_0 + \delta g) > n \} \leq \varepsilon/3$ for all $\| g \|_{L^p(\mu)} \leq 1$. Thus, for all $\| g \|_{L^p(\mu)} \leq 1$, we have

$$\mu\left\{ S^* g > \frac{2n}{\delta} \right\} \leq \mu\{ S^* (f_0 + \delta g) > n \} + \mu\{ S^* f_0 > n \} \leq 2\varepsilon/3 < \varepsilon.$$

If we set $\theta(\varepsilon) = 2n/\delta$, we have the desired result.

Theorem 3 and its proof are taken directly from Bellow and Jones [1]. It is included here only for completeness.

Denote the space $Y_0 = \{ f \in L^\infty(\mu) : \| f \|_{L^\infty(\mu)} \leq 1 \}$. We will be concerned with the $L^2(\mu)$-norm on $Y_0$. It is well-known that $Y_0$ is complete under $\| \cdot \|_{L^2(\mu)}$. Denote $\delta$-balls in $Y_0$ by $B_\delta(f) = \{ g \in Y_0 : \| f - g \|_{L^2(\mu)} < \delta \}$. The first step is to prove the following lemma.

**Lemma 10.** If $f_0 \in Y_0$ and $\delta > 0$, then $B_\delta(0) \subseteq B_\delta(f_0) - B_\delta(f_0) = \{ g_1 - g_2 : g_1, g_2 \in B_\delta(f_0) \}$.

**Proof.** Let $g \in B_\delta(0)$, so that $\| g \|_{L^2(\mu)} < \delta$. Define

$$u_1(x) = \begin{cases} 
    g(x) & \text{if } f_0(x), g(x) \text{ are both finite, and } f_0(x)g(x) \leq 0, \\
    0 & \text{otherwise},
\end{cases}$$

$$u_2(x) = \begin{cases} 
    -g(x) & \text{if } f_0(x), g(x) \text{ are both finite, and } f_0(x)g(x) > 0, \\
    0 & \text{otherwise}.
\end{cases}$$

Let $g_1 = f_0 + u_1$ and $g_2 = f_0 + u_2$. Now, it is clear that $g = u_1 - u_2 = (f_0 + u_1) - (f_0 + u_2) = g_1 - g_2$ for a.s.$[\mu] \ x \in X$. It is also clear that $\| f_0 - g_1 \|_{L^2(\mu)} = \| u_1 \|_{L^2(\mu)} \leq \| g \|_{L^2(\mu)} < \delta$. Similarly, $\| f_0 - g_2 \|_{L^2(\mu)} < \delta$. So, we will be done if we can show $g_1, g_2 \in Y_0$.

Fix an $x$ such that $f_0(x), g(x)$ are both finite and $f_0(x)g(x) \leq 0$, i.e., $f_0(x)$ and $g(x)$ have opposite signs. Then, $u_1(x) = g(x)$ and $|g_1(x)| = |f_0(x) + u_1(x)| = |f_0(x) + g(x)|$. As
they have opposite signs, \(|f_0(x) + g(x)| \leq \max\{|f_0(x)|, |g(x)|\}. Now suppose \(x\) is such that \(f_0(x), g(x)\) are finite and \(f_0(x)g(x) > 0\). Then, \(u_1(x) = 0\) and \(|g_1(x)| = |f_0(x)|\). Hence, \(|g_1| \leq \max\{|f_0|, |g|\}\) for a.s.\([\mu] \ x\), which implies \(\|g_1\|_{L^\infty(\mu)} \leq \max\{\|f_0\|_{L^\infty(\mu)}, \|g\|_{L^\infty(\mu)}\} \leq 1\). Namely, \(g_1 \in Y_0\). Precisely the same argument shows \(g_2 \in Y_0\).

We can now proceed to the proof of Theorem 3. This proof also relies on the Baire Category Theorem.

**Proof of Theorem 3.** Fix \(\epsilon, \eta > 0\). Choose \(0 < \alpha < 1/2\) so that \(\alpha < \epsilon/3\) and \(\eta > 2\alpha\). For \(N \in \mathbb{N}\) define

\[
F_N(\alpha) = \left\{ f \in Y_0 : \mu \left\{ x \in X : \sup_{m \geq N} |S_N f(x) - S_m f(x)| \leq \alpha \right\} \geq 1 - \alpha \right\}.
\]

For \(M > N\), define

\[
F_{N,M}(\alpha) = \left\{ f \in Y_0 : \mu \left\{ x \in X : \sup_{N \leq m \leq M} |S_N f(x) - S_m f(x)| \leq \alpha \right\} \geq 1 - \alpha \right\}.
\]

Fix \(N\) and \(M > N\). We wish to show \(F_{N,M}(\alpha)\) is closed, with respect to \((Y_0, \| \cdot \|_{L^2(\mu)})\). Let \((f_k) \in F_{N,M}(\alpha)\) and \(f_k \to f \in Y_0\) in \(L^2(\mu)\)-norm. Let \(g_k(x) = \sup_{N \leq m \leq M} |S_N f_k(x) - S_m f_k(x)|\) and \(g(x) = \sup_{N \leq m \leq M} |S_N f(x) - S_m f(x)|\). By \(\|S_n\|\) it is meant the operator norm on \(L^2(\mu)\). Now,

\[
\|g - g_k\|_{L^2(\mu)} \leq \left( \|S_N\| + \sum_{k=N}^M \|S_k\| \right) \|f - f_k\|_{L^2(\mu)} \to 0.
\]

For \(n \in \mathbb{N}\), we have that

\[
\mu \left\{ x \in X : g(x) > \alpha + \frac{1}{n} \right\} \leq \mu \left\{ x \in X : g_k(x) > \alpha \right\} + \mu \left\{ x \in X : |g - g_k| > \frac{1}{n} \right\} \leq \alpha + n^2 \|g - g_k\|_{L^2(\mu)}^2 \to \alpha.
\]

As \(\{g > \alpha\} = \bigcup_n \{g > \alpha + \frac{1}{n}\}\), and this is a nested sequence, we see \(\mu\{g > \alpha\} \leq \alpha\). But this says \(f \in F_{N,M}(\alpha)\), and \(F_{N,M}(\alpha)\) is closed.
Now, as the relevant sets are nested and decreasing, it is easy to see that $F_N(\alpha) = \bigcap_{M > N} F_{N,M}(\alpha)$. So, each $F_N(\alpha)$ is closed. Let $g \in Y_0$. By hypothesis, $S_n g$ converges a.s. $\mu$. By Egoroff’s Theorem, there is a set $E$ with $\mu(X - E) < \alpha$ so that $S_n g$ converges uniformly on $E$. Then, there is some $N$ such that $|S_N g(x) - S_n g(x)| \leq \alpha$ whenever $n \geq N$ and $x \in E$. This implies $g \in F_N(\alpha)$. Hence, $Y_0 = \bigcup_N F_N(\alpha)$. As $Y_0$ is complete under $\| \cdot \|_{L_2(\mu)}$, it follows by the Baire Category Theorem that at least one of $F_N(\alpha)$ contains an open set. That is, there is some $N_0$, some $f_0 \in Y_0$, and some $\delta > 0$ such that $B_\delta(f_0) \subset F_{N_0}(\alpha)$.

Let $S_{N_0}^* f = \sup_{1 \leq n \leq N_0} |S_n f|$ and $S^* f = \sup_n |S_n f|$. Note, for each $g \in Y_0$, $\|S_n g\|_{L_2(\mu)} \leq \|S_n\| \|g\|_{L_2(\mu)}$, so that each $S_n$ is continuous at 0 in $(Y_0, \| \cdot \|_{L_2(\mu)})$. Hence, $S_{N_0}^* f$ is continuous at 0, because $S_{N_0}^* (f) \leq \sum_{n=1}^{N_0} |S_n f|$. So, there is some $0 < \delta' < \delta$ such that $\|S_{N_0}^* f\|_{L_2(\mu)}^2 < \alpha(\eta - 2\alpha)^2$ when $f \in B_{\delta'}(0)$.

Fix $f \in B_{\delta'}(0)$. Now, by Lemma 10, we see $B_{\delta'}(0) \subset B_\delta(f_0) \subset B_{\delta}(f_0)$. Thus, there are $g_1, g_2 \in B_\delta(f_0) \subset F_{N_0}(\alpha)$ such that $f = g_1 - g_2$ a.s. $\mu$. By definition of $F_{N_0}(\alpha)$, we see that

$$
\mu\left\{ x \in X : \sup_{m \geq N_0} |S_{N_0} g_1(x) - S_m g_1(x)| \leq \alpha \right\} \geq 1 - \alpha,
$$

$$
\mu\left\{ x \in X : \sup_{m \geq N_0} |S_{N_0} g_2(x) - S_m g_2(x)| \leq \alpha \right\} \geq 1 - \alpha.
$$

Now, except for a set of probability 0,

$$
\sup_{m \geq N_0} |S_{N_0} f(x) - S_m f(x)| = \sup_{m \geq N_0} \left| S_{N_0} g_1(x) - S_m g_1(x) - S_{N_0} g_2(x) + S_m g_2(x) \right|
\leq \sup_{m \geq N_0} \left| S_{N_0} g_1(x) - S_m g_1(x) \right| + \sup_{m \geq N_0} \left| S_{N_0} g_2(x) - S_m g_2(x) \right|.
$$

Therefore,

$$
\mu\left\{ x \in X : \sup_{m \geq N_0} |S_{N_0} f(x) - S_m f(x)| \leq 2\alpha \right\}
\geq \mu\left( \left\{ \sup_{m \geq N_0} |S_{N_0} g_1 - S_m g_1| \leq \alpha \right\} \cap \left\{ \sup_{m \geq N_0} |S_{N_0} g_2 - S_m g_2| \leq \alpha \right\} \right)
\geq 1 - 2\alpha.
$$

Define

$$
C = \left\{ x \in X : S^* f(x) > \eta \right\},
$$

$$
D = \left\{ x \in X : S^* f(x) \leq S_{N_0}^* f(x) + 2\alpha \right\},
$$

$$
E = \left\{ x \in X : \sup_{m \geq N_0} |S_{N_0} f(x) - S_m f(x)| \leq 2\alpha \right\}.
$$
Then, \( \mu(E) \geq 1 - 2\alpha \) by above. Now, for any \( x \in X \) and \( m \leq N_0 \) it is clear that

\[
|S_m f(x)| \leq S^*_N f(x).
\]

On the other hand, for \( x \in E \) and any \( m \geq N_0 \) we have

\[
|S_m f(x)| \leq |S^*_N f(x)| + 2\alpha \leq S^*_N f(x) + 2\alpha.
\]

Namely, for every \( x \in E \) we see \( S^*_N f(x) \leq S^*_N f(x) + 2\alpha \), or \( x \in D \). Thus, \( \mu(D) \geq \mu(E) \geq 1 - 2\alpha \). Finally,

\[
\mu(C) = \mu(C \cap D) + \mu(C \cap D^c)
\leq \mu\{x \in X : S^*_N f(x) > \eta - 2\alpha\} + \mu(D^c)
\leq \|S^*_N f\|_{L^2(\mu)}^2 \frac{1}{(\eta - 2\alpha)^2} + 2\alpha
< 3\alpha < \epsilon.
\]

This holds for all \( f \in B^\delta(0) \). If we set \( \rho(\epsilon, \eta) = \delta' \), we have the desired result.

\[\square\]

6 Proof of Theorem 8

For this section, it will be convenient to discuss normal random vectors, that is, measurable maps \( G : \Omega \to \mathbb{R}^N \) where \( G = (G_1, \ldots, G_N) \) and each \( G_j \) is a normal random variable. We will say \( G \) has mean 0 if each \( G_j \) has mean 0.

The concepts of distribution and independence are easily extended to this case. We say two random vectors \( G \) and \( H \) have the same distribution if \( \mathbb{P}(G \in E) = \mathbb{P}(H \in E) \) for all measurable sets \( E \subset \mathbb{R}^N \). We say \( G \) and \( H \) are independent if \( \mathbb{P}(G \in E) \mathbb{P}(H \in F) = \mathbb{P}(G \in E, H \in F) \) for all \( E, F \subset \mathbb{R}^N \).

**Lemma 11.** Let \( G \) and \( H \) be mean 0 normal random vectors which are independent and have the same distribution. Then, for any \( \theta \in \mathbb{R} \), the normal random vector \((G, H) : \Omega \to \mathbb{R}^{2N}\) has the same distribution as \( (G \sin \theta + H \cos \theta, G \cos \theta - H \sin \theta) \).

**Proof.** A normal random vector (and its distribution) is completely determined by its covariance matrix \( \text{cov}(s, t) = E(G_s G_t) \). But, it is clear that for all \( s, t \)

\[
E(G_s G_t) = E((G_s \sin \theta + H_s \cos \theta)(G_t \sin \theta + H_t \cos \theta)),
E(H_s H_t) = E((G_s \cos \theta - H_s \sin \theta)(G_t \cos \theta - H_t \sin \theta)),
E(G_s H_t) = 0 = E((G_s \sin \theta + H_s \cos \theta)(G_t \cos \theta - H_t \sin \theta)).
\]

Hence, the covariance matrices of \((G, H)\) and \((G \sin \theta + H \cos \theta, G \cos \theta - H \sin \theta)\) are the same. \[\square\]
The above lemma is, in some sense, the statement that mean 0 normal random vectors are rotation invariant.

Now, for a mean 0 normal random vector \( G \) on \( (\Omega, \mathcal{B}, P) \), we can always find normal random vectors \( H \) and \( K \) on some probability space \( (\Omega', \mathcal{B}', P') \) such that \( H, K \) are independent, and \( H, K \) have the same distribution as \( G \). That is \( P(G \in E) = P'(H \in E) = P'(K \in E) \) for all measurable sets \( E \subset \mathbb{R}^N \). We say \( H, K \) are independent copies of \( G \). We now prove Theorem 8. This proof is taken from [3].

Proof of Theorem 8 Write \( G = (G_1, \ldots, G_N) \) as a normal random vector. Let \( H, K \) be independent copies of \( G \) on some probability space \( (\Omega', \mathcal{B}', P') \). Define \( S(G) : \Omega \to [0, \infty) \) by \( S(G)(\omega) = \max_n |G_n(\omega)| \). Define \( S(H), S(K) \) similarly. It follows \( S(G), S(H), S(K) \) have the same distribution and that \( S(H), S(K) \) are independent. Then, for \( t \geq s \)

\[
2P\{S(G) \leq s\}P\{S(G) > t\} = P'\{S(H) \leq s\}P'\{S(H) > t\} + P'\{S(K) \leq s\}P'\{S(K) > t\} = P'\{S(H) \leq s, S(K) > t\} + P'\{S(K) \leq s, S(H) > t\}.
\]

Apply Lemma 11 to \( H, K \) with \( \theta = \pi/4 \). Then, \( (H, K) \) and \( (H + K, H - K) \) have the same distribution. It follows \( P'\{S(H) \leq s, S(K) > t\} = P'\{S(H + K) \leq s, S(H - K) > t\} \). Similarly for the second term, so that

\[
2P\{S(G) \leq s\}P\{S(G) > t\} = P\left\{ S\left( \frac{H + K}{\sqrt{2}} \right) \leq s, S\left( \frac{H - K}{\sqrt{2}} \right) > t \right\} + P\left\{ S\left( \frac{H - K}{\sqrt{2}} \right) \leq s, S\left( \frac{H + K}{\sqrt{2}} \right) > t \right\} = P\left( \sqrt{2}(H + K) \leq s\sqrt{2}, (H - K) > t\sqrt{2} \right) \cup \left( S(H - K) \leq s\sqrt{2}, (H + K) > t\sqrt{2} \right),
\]

where the last equality follows as the sets are clearly disjoint. Consider the first set in the union, \( S(H + K) \leq s\sqrt{2}, S(H - K) > t\sqrt{2} \). In this set \( \sqrt{2}(t - s) < S(H + K) - S(H - K) \leq S(2K) = 2S(K) \) or \( \sqrt{2}S(K) > t - s \). Similarly, \( \sqrt{2}(t - s) < S(H + K) - S(K - H) \leq S(2H) = 2S(H) \) or \( \sqrt{2}S(H) > t - s \). The same calculations work in the other set. So, we have

\[
2P\{S(G) \leq s\}P\{S(G) > t\} \leq P\{\sqrt{2}S(H) > t - s, \sqrt{2}S(K) > t - s\} = P\{\sqrt{2}S(G) > t - s\}^2.
\] (10)

We now define a sequence \( (t_n) \). Set \( t_0 = s \) and \( t_{n+1} = t_n \sqrt{2} + s \). It is easily checked by induction that \( t_n = (\sqrt{2} + 1)(2^{(n+1)/2} - 1)s \). Define \( q = 2P\{S(G) \leq s\} \) and \( x_n = q^{-1}P\{S(G) > t_n\} \). By the hypothesis, \( q \geq 1 \). By construction and from (10),

\[
q^2 x_{n+1} = qP\{S(G) > t_{n+1}\} = 2P\{S(G) \leq s\}P\{S(G) > t_{n+1}\} \leq P\left( S(G) > \frac{t_{n+1} - s}{\sqrt{2}} \right)^2 = P\{S(G) > t_n\}^2 = q^2 x_n^2,
\]
or \( x_{n+1} \leq x_n^2 \). This implies that \( x_n \leq x_0^{2^n} \). It is easily seen that \( x_0 = \frac{2q}{2q} \leq 1/2 \) as \( q \geq 1 \). Hence, \( P\{S(G) > t_n\} = q x_n \leq q x_0^{2^n} \leq q 2^{-2^n} \). By Lemma 4,

\[
\|S(G)\|_{L^1(P)} = \int_0^{\infty} P\{S(G) > t\} \, dt = \int_0^s P\{S(G) > t\} \, dt + \sum_{n=0}^{\infty} \int_{t_n}^{t_{n+1}} P\{S(G) > t\} \, dt \\
\leq s + \sum_{n=0}^{\infty} \int_{t_n}^{t_{n+1}} P\{S(G) > t_n\} \, dt \\
\leq s + \sum_{n=0}^{\infty} (t_{n+1} - t_n)q 2^{-2^n}.
\]

Of course, \( q = 2P\{S(G) \leq s\} \leq 2 \) trivially. From the induction characterization of \( t_n \), we see \( t_{n+1} - t_n = s 2^{(n+1)/2} \). Therefore,

\[
\|S(G)\|_{L^1(P)} \leq s + s 2^{3/2} \sum_{n=0}^{\infty} 2^{n/2} 2^{-2^n} \leq s + s 2^{3/2} \sum_{n=0}^{\infty} 2^{n/2} 2^{-2^n} \\
= s + s 2^{3/2} \sum_{n=0}^{\infty} 2^{-3n/2} = s + s \frac{2^{3/2}}{1 - 2^{-3/2}} \leq 6s.
\]

\[\square\]

7 An Application

Let \( T = \mathbb{R}/\mathbb{Z} \). A function \( f \) on \( \mathbb{R} \) with period 1 can be viewed as a function on \( T \). Let \( m \) be Lebesgue measure, and consider the probability space \((T, m)\). Let \((a_j)\) be any non-zero sequence of real numbers which converge to 0. For \( f : T \to \mathbb{R} \), consider the operators

\[
S_n f(x) = \frac{1}{n} \sum_{j=1}^{n} f(x + a_j).
\]

Bellow asked whether \( S_n f \) converges to \( f \) a.s. for all \( f \in L^1(m) \). The answer to this turns out to be no. In fact, it is not even true for all \( f \in L^\infty(m) \).

We will prove this using the second entropy result. In this case, it is beneficial to consider complex-valued functions temporarily. Here, the operators \( S_n \) make perfectly good sense applied to complex-valued functions. In fact, we also have the nice property that \( S_n(\text{Re } f) = \text{Re}(S_n f) \) and similarly for the imaginary part. We will take advantage of this. First, we need two technical lemmas.

Lemma 12. Let \((a_j)\) be a sequence of non-zero real numbers converging to 0. Then, given any \( r \in \mathbb{N} \), there exist integers \( J_1 < J_2 < \ldots < J_r \) satisfying the following: if \( \vec{\alpha} = (\alpha_1, \ldots, \alpha_r) \) is a vector of 0’s and 1’s, then there is an integer \( n(\vec{\alpha}) \) such that

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\[|1 - J_s^{-1} \sum_{j \leq J_s} e^{2\pi i a_j n(\bar{a})}| < \frac{1}{10} \quad \text{if } \alpha_s = 0,\]
\[|1 - J_s^{-1} \sum_{j \leq J_s} e^{2\pi i a_j n(\bar{a})}| > \frac{1}{2} \quad \text{if } \alpha_s = 1,\]

for all \(1 \leq s \leq r\).

**Lemma 13.** Let \((Y, S, \nu)\) be a measure space and define \(\|h\| = (\int_Y |h|^p \, d\nu)^{1/p}\) for the measurable functions from \(Y\) to \(\mathbb{C}\), where \(1 \leq p < \infty\). Let \(r_0 \in \mathbb{N}\) and \(r = 4r_0^2 + 2r_0 \in \mathbb{N}\). Suppose \(\{h_1, \ldots, h_r\}\) is a collection of complex-valued functions on \(Y\) such that \(\|h_j - h_k\| > \alpha\) for all \(j \neq k\). Then, there exists a set \(I \subset \{1, \ldots, r\}\), \(|I| = r_0\) such that either \(\|\text{Re} h_j - \text{Re} h_k\| > \alpha/4\) for all \(j \neq k \in I\) or \(\|\text{Im} h_j - \text{Im} h_k\| > \alpha/4\) for all \(j \neq k \in I\).

We temporarily postpone the proofs of these lemmas and proceed to the solution of Bellow’s question.

**Theorem 14.** Let \((a_j)\) be any real sequence of numbers which converge to 0 and \(a_j \neq 0\) for all \(j\). Then, there exists \(f \in L^\infty(m)\) such that \(S_n f\) does not converge a.s.\([m]\).

**Proof.** Let \(T_j f(x) = f(x + b_j)\) for some real sequence \((b_j)\). Then, the fact that \(T_j(1) = 1\) and \(T_j\) are positive is obvious. By periodicity, \(T_j\) is an isometry on \(L^1(m)\) and \(L^2(m)\). Also, \(\|S_n f\|_{L^2(m)} \leq \frac{1}{n} \sum \|f\|_{L^2(m)} = \|f\|_{L^2(m)}\) and \(T_jS_n = S_nT_j\).

Let \(w\) be an irrational number and \(b_j = (j - 1)w\). It then follows from the equidistribution theorem (or a special case of Birkhoff’s Ergodic Theorem) that \((T_j)\) satisfies the mean ergodic condition. Thus, \((S_n)\) commutes with a Bourgain sequence, and the second entropy result can be applied to \((S_n)\).

It suffices to show that for some \(\delta > 0\) we have \(\sup\{N_f(\delta) : \|f\|_{L^2(m)} \leq 1\} = \infty\). In fact, we will do this with \(\delta = 1/40\). Let \(r_0 \in \mathbb{N}\) and \(r = 4r_0^2 + 2r_0\). By Lemma [12], choose integers \(J_1 < \ldots < J_r\). Define a complex-valued function \(g : T \to \mathbb{C}\) by

\[g(x) = 2^{-r/2} \sum_{\alpha \in \{0, 1\}^r} e^{2\pi i n(\bar{\alpha})x}.\]

As we said before, we can consider \(S_n\) acting on complex-valued functions. Although the second entropy result cannot be applied in this case, we will use \(g\) to manufacture an appropriate real-valued function. Note, \(\|g\|_2^2 = \int_0^1 g(x)\bar{g}(x) \, dx = 2^{-r} \sum_\alpha 1 = 1\) by orthogonality. (The notation \(\|\cdot\|_2\) has the obvious meaning, where we make the distinction here from \(L^2(m)\) and \(\|\cdot\|_{L^2(m)}\) which implies real-valued functions). Also,

\[S_{J_s} g(x) = 2^{-r/2} \sum_{\alpha \in \{0, 1\}^r} \beta_{s, \alpha} e^{2\pi in(\bar{\alpha})x},\]

where
\[
\beta_{s,\bar{\alpha}} = J_s^{-1} \sum_{j \leq J_s} e^{2\pi i a_j n(\bar{\alpha})}.
\]

Fix an \(\bar{\alpha}\) and suppose \(\alpha_s = 1\) and \(\alpha_t = 0\). By Lemma \[12\] we have
\[
|\beta_{s,\bar{\alpha}} - \beta_{t,\bar{\alpha}}| \geq |\beta_{s,\bar{\alpha}} - 1| - |1 - \beta_{t,\bar{\alpha}}| > \frac{1}{2} - \frac{1}{10} = \frac{2}{5}
\]

By symmetry, this holds so long as \(\alpha_s \neq \alpha_t\). Hence, by orthogonality and above,
\[
\|S_{J_s}g - S_{J_t}g\|_2 = 2^{-r/2} \left( \sum_{\bar{\alpha}} |\beta_{s,\bar{\alpha}} - \beta_{t,\bar{\alpha}}|^2 \right)^{1/2} \\
\geq 2^{-r/2} \left( \sum_{\bar{\alpha}: \alpha_s \neq \alpha_t} |\beta_{s,\bar{\alpha}} - \beta_{t,\bar{\alpha}}|^2 \right)^{1/2} \\
> 2^{-r/2}(2/5) \left( \sum_{\bar{\alpha}: \alpha_s \neq \alpha_t} 1 \right)^{1/2} \\
= 2^{-r/2}(2/5)2^{(r-1)/2} \\
> 1/5.
\]

This holds for all \(s \neq t\).

Apply Lemma \[13\] to the set \(\{S_{J_s}g, \ldots, S_{J_r}g\}\) to find a subset \(I \subset \{J_1, \ldots, J_r\}\), \(|I| = r_0\) such that either \(\|\text{Re}S_{J_s}g - \text{Re}S_{J_t}g\|_2 > 1/20\) or \(\|\text{Im}S_{J_s}g - \text{Im}S_{J_t}g\|_2 > 1/20\) for all \(J_s \neq J_t \in I\). If it is the first, set \(f = \text{Re}g\), and if it is the second, set \(f = \text{Im}g\). Then, \(\|f\|_{L^2(m)} \leq \|g\|_2 = 1\). Further, \(\|S_{J_s}f - S_{J_t}f\|_{L^2(m)} > 1/20\) for all \(J_s \neq J_t \in I\). As no two such \(S_{J_s}f\) could be contained in the same \(1/40\)-ball in \(L^2(m)\), we see \(N_f(1/40) \geq |I| = r_0\).

As \(r_0\) is arbitrary, \(\sup N_f(1/40) = \infty\).

To conclude, we need only establish Lemmas \[12\] and \[13\]. First, we recall three simple results in complex arithmetic.

**Claim.** Let \(a, b \in \mathbb{C}\), with \(|a|, |b| \leq 1\), and \(\lambda \in \mathbb{R}\). Then,

1. \(|ab - 1| \leq |a - 1| + |b - 1|\),
2. \(\text{Re}(ab) \leq |a - 1| + \text{Re}(b)\),
3. \(|1 - e^{2\pi i \lambda}| \leq 2\pi|\lambda|\).

**Proof.** First, \(|ab - 1| = |ab - a + a - 1| \leq |ab - a| + |a - 1| \leq |b - 1| + |a - 1|\). Second, \(\text{Re}(ab) = \text{Re}(ab - b) + \text{Re}(b) \leq |ab - b| + \text{Re}(b) \leq |a - 1| + \text{Re}(b)\). Third, recall \(1 - \cos(2x) \leq 2x^2\) for all real \(x\). So, \(|1 - e^{2\pi i \lambda}|^2 = (1 - \cos(2\pi \lambda))^2 + \sin^2(2\pi \lambda) = 2(1 - \cos(2\pi \lambda)) \leq 4\pi^2\lambda^2\).
Proof of Lemma 12. Fix \( r \in \mathbb{N} \). If \( r = 1 \), then set \( J_1 = 1 \) and choose \( n(\bar{\alpha}) \) accordingly. Assume \( r > 1 \). For each \( 1 \leq s \leq r \), we will construct integers \( m_s \) simultaneously as \( J_s \).

Set \( J_1 = 1 \) and choose \( m_1 \) so that \( |1 - e^{2\pi i a_1 m_1}| > 3/4 \). Assume \( J_s \) and \( m_t \) are known for all \( t < s \). Let \( M_s = \sum_{t<s} |m_t| \). As \( a_j \to 0 \), there is a \( L_s > 0 \) such that \( \sup_{J_s< L_s/100} |a_j| \leq (400M_s\pi)^{-1} \). Further, we can choose \( T_s > 0 \) (depending on \( a_j \) for \( j \leq J_{s-1} \)) such that for each \( z \in \mathbb{Z} \) there is a corresponding \( t \in \mathbb{Z} \), \( |t| \leq T_s \) satisfying \( |e^{2\pi i a_j z} - e^{2\pi i t}| < 1/50r \) for all \( j \leq J_{s-1} \).

As \( a_j \to 0 \), we can choose \( J_s \) such that \( J_s > L_s, J_s > J_{s-1}, \) and \( J_s^{-1} \sum_{j \leq J_s} |a_j| < (100T_s)^{-1} \). Also, as \( a_j \neq 0 \) for all \( j \), note that

\[
\lim_{R \to \infty} \left| \frac{1}{R} \int_0^R \left( \frac{1}{J_s} \sum_{j \leq J_s} e^{2\pi i a_j x} \right) dx \right| = \lim_{R \to \infty} \left| \frac{1}{R} \left( \frac{1}{J_s} \sum_{j \leq J_s} \frac{1}{2\pi i a_j} (e^{2\pi i a_j R} - 1) \right) \right| \\
\leq \lim_{R \to \infty} \frac{1}{R} J_s^{-1} \sum_{j \leq J_s} \frac{1}{|a_j|} = 0.
\]

It follows there is \( y_s > 0 \) such that \( \text{Re}(J_s^{-1} \sum_{j \leq J_s} e^{2\pi i a_j y_s}) < 1/10 \); otherwise, this limit could not be 0. Set \( z_s \) to be the integer part of \( y_s \), and take \( |t_s| \leq T_s \) as prescribed above. Set \( m_s = z_s - t_s \). Define all \( J_s \) and \( m_s \) in this manner.

Then, for each \( 1 < s \leq r \), by second and third statements of the above claim and by construction, we have

\[
\text{Re} \left( J_s^{-1} \sum_{j \leq J_s} e^{2\pi i a_j m_s} \right) = J_s^{-1} \sum_{j \leq J_s} \text{Re}(e^{2\pi i a_j y_s} e^{2\pi i a_j (m_s - y_s)}) \\
\leq J_s^{-1} \sum_{j \leq J_s} \left( \text{Re}(e^{2\pi i a_j y_s}) + |1 - e^{2\pi i a_j (m_s - y_s)}| \right) \\
= \text{Re} \left( J_s^{-1} \sum_{j \leq J_s} e^{2\pi i a_j y_s} \right) + J_s^{-1} \sum_{j \leq J_s} |1 - e^{2\pi i a_j (z_s - t_s)}| \\
\leq \frac{1}{10} + 2\pi J_s^{-1} \sum_{j \leq J_s} |a_j|(|z_s - y_s| + |t_s|) \\
\leq \frac{1}{10} + \frac{2\pi (T_s + 1)}{100 T_s} \leq \frac{1}{10} + \frac{4\pi}{100} < 1/4.
\]

This gives

\[
J_s > L_s, \\
|1 - e^{2\pi i a_j m_s}| = |e^{2\pi i a_j z_s} - e^{2\pi i t_s}| < 1/50r \quad \text{for all} \quad j \leq J_{s-1}, \\
1 - J_s^{-1} \sum_{j \leq J_s} e^{2\pi i a_j m_s} \geq 1 - \text{Re} \left( J_s^{-1} \sum_{j \leq J_s} e^{2\pi i a_j m_s} \right) > 3/4.
\]
But, $J_1 = 1$ and $m_1$ was chosen so the last condition is also true for $s = 1$. Further, if we rewrite the second condition, we have

$$\sup_{j \leq J_{t-1}} |1 - e^{2\pi ia_j m_t}| < 1/50r \quad \text{for all } 2 \leq t \leq r,$$

$$\left|1 - J_s^{-1} \sum_{j \leq J_s} e^{2\pi ia_j m_s}\right| > 3/4 \quad \text{for all } 1 \leq s \leq r.$$

Fix $\bar{\alpha} \in \{0, 1\}^r$. Define $n_s = \alpha_s m_s$ and $n(\bar{\alpha}) = n_1 + \ldots + n_r$. Fix $1 \leq s \leq r$. Then, by the first statement in the claim,

$$\left|J_s^{-1} \sum_{j \leq J_s} e^{2\pi ia_j n(\bar{\alpha})} - J_s^{-1} \sum_{j \leq J_s} e^{2\pi ia_j n_s}\right| \leq J_s^{-1} \sum_{j \leq J_s} |e^{2\pi ia_j n(\bar{\alpha})} - e^{2\pi ia_j n_s}|$$

$$= J_s^{-1} \sum_{j \leq J_s} \left|\exp\left(2\pi ia_j \left(\sum_{t<s} n_t\right)\right) \exp\left(2\pi ia_j \left(\sum_{t>s} n_t\right)\right) - 1\right|$$

$$\leq J_s^{-1} \sum_{j \leq J_s} \left|\exp\left(2\pi ia_j \left(\sum_{t<s} n_t\right)\right) - 1\right| + J_s^{-1} \sum_{j \leq J_s} \left|\exp\left(2\pi ia_j \left(\sum_{t>s} n_t\right)\right) - 1\right| =: I + II.$$

If $s = 1$, then $I = 0$. If $s > 1$, then as $J_s > L_s$,

$$I = J_s^{-1} \sum_{j < J_s/100} \left|\exp\left(2\pi ia_j \left(\sum_{t<s} n_t\right)\right) - 1\right| + J_s^{-1} \sum_{J_s/100 < j \leq J_s} \left|\exp\left(2\pi ia_j \left(\sum_{t<s} n_t\right)\right) - 1\right|$$

$$\leq J_s^{-1} \left(\sum_{j < J_s/100} 2\right) + J_s^{-1} \left(\sum_{J_s/100 < j \leq J_s} 2\pi |a_j| \left|\sum_{t<s} n_t\right|\right)$$

$$\leq \frac{1}{50} + 2M_s \pi J_s^{-1} (J_s - J_s/100) \sup_{j > J_s/100} |a_j|$$

$$\leq \frac{1}{50} + 2M_s \pi \sup_{j > L_s/100} |a_j| < \frac{1}{50} + \frac{1}{200} = \frac{1}{40}.$$ 

On the other hand, for all $s$, we have by applying the first statement of the claim several times

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Proof of Lemma 13. Suppose not. Then, there exist subsets $\alpha \mid \| \mathcal{J} \| < r_0$ so that for any $j \notin \mathcal{J}$ there is an $m_1 \in M_1$ such that $\| \Re h_j - \Re h_{m_1} \| \leq \alpha / 4$, and for any $j \notin \mathcal{J}$ there is an $m_2 \in M_2$ such that $\| \Im h_j - \Im h_{m_2} \| \leq \alpha / 4$. Define $M = M_1 \cup M_2$. Then, $|M| = m < 2r_0$.

For each $j \notin \mathcal{J}$, we can associate a pair $(a, b)$ with $a, b \in M$ such that $\| \Re h_j - \Re h_a \| \leq \alpha / 4$ and $\| \Im h_j - \Im h_b \| \leq \alpha / 4$. Associate one such pair to each $j \notin \mathcal{J}$. Now, there are $m^2$ distinct such pairs. But, there are $r - m$ points $j \notin \mathcal{J}$. As $m < 2r_0$ and by the definition of $r$, we have $r - m > m^2$. Therefore, two distinct points $j, k \notin \mathcal{J}$ must be assigned the same pair, say $(a, b)$. Namely,

\[
II = J_s^{-1} \sum_{j \leq J_s} \left| \exp \left( 2\pi i a_j \left( \sum_{t > s} n_t \right) \right) - 1 \right|
\leq \sup_{j \leq J_s} \left| \exp \left( 2\pi i a_j \left( \sum_{t > s} n_t \right) \right) - 1 \right|
= \sup_{j \leq J_s} \left| \prod_{t > s} e^{2\pi i a_j n_t} - 1 \right|
\leq \sum_{t > s} \sup_{j \leq J_s} \left| e^{2\pi i a_j n_t} - 1 \right|
\leq \sum_{t > s} \frac{1}{50r} \leq \frac{1}{20}.
\]

Hence, we have that for all $s$

\[
\left| J_s^{-1} \sum_{j \leq J_s} e^{2\pi i a_j n(\bar{a})} - J_s^{-1} \sum_{j \leq J_s} e^{2\pi i a_j n_s} \right| < \frac{1}{40} + \frac{1}{50} < \frac{1}{20}.
\]

Now, if $\alpha_s = 0$, then $n_s = 0$ and

\[
\left| J_s^{-1} \sum_{j \leq J_s} e^{\pi i a_j n(\bar{a})} - 1 \right| < 1/20 < 1/10.
\]

If $\alpha_s = 1$, then $n_s = m_s$ and

\[
\left| J_s^{-1} \sum_{j \leq J_s} e^{\pi i a_j n(\bar{a})} - 1 \right| \geq \left| J_s^{-1} \sum_{j \leq J_s} e^{\pi i a_j m_s} - 1 \right| - \left| J_s^{-1} \sum_{j \leq J_s} e^{\pi i a_j n(\bar{a})} - J_s^{-1} \sum_{j \leq J_s} e^{\pi i a_j m_s} \right|
> 3/4 - 1/20 > 1/2.
\]

This completes the proof. 

Proof of Lemma 13. Suppose not. Then, there exist subsets $M_1, M_2 \subset \{1, \ldots, r\}$ with $|M_1|, |M_2| < r_0$ so that for any $j \notin M_1$ there is an $m_1 \in M_1$ such that $\| \Re h_j - \Re h_{m_1} \| \leq \alpha / 4$, and for any $j \notin M_2$ there is an $m_2 \in M_2$ such that $\| \Im h_j - \Im h_{m_2} \| \leq \alpha / 4$. Define $M = M_1 \cup M_2$. Then, $|M| = m < 2r_0$.

For each $j \notin M$, we can associate a pair $(a, b)$ with $a, b \in M$ such that $\| \Re h_j - \Re h_a \| \leq \alpha / 4$ and $\| \Im h_j - \Im h_b \| \leq \alpha / 4$. Associate one such pair to each $j \notin M$. Now, there are $m^2$ distinct such pairs. But, there are $r - m$ points $j \notin M$. As $m < 2r_0$ and by the definition of $r$, we have $r - m > m^2$. Therefore, two distinct points $j, k \notin M$ must be assigned the same pair, say $(a, b)$. Namely,
\[ \|h_j - h_k\| \leq \|\text{Re} h_j - \text{Re} h_k\| + \|\text{Im} h_j - \text{Im} h_k\| \]
\[ \leq \|\text{Re} h_j - \text{Re} h_a\| + \|\text{Im} h_j - \text{Im} h_a\| + \|\text{Re} h_a - \text{Re} h_k\| + \|\text{Im} h_a - \text{Im} h_k\| \]
\[ \leq \alpha. \]

This contradicts the hypothesis.

8 Comments

1. The only significant difference between the proof of the first entropy result here and in [2] is the use of Theorem 8. Bourgain uses a different statement, namely that there is some \( c > 0 \) so that

\[ P \left\{ \omega : \mu \left\{ x : F^*(x, \omega) \geq c \int \Omega F^*(x, \omega') P(d\omega') \right\} > c \right\} > c \]

for all \( N, J \in \mathbb{N} \) big enough and \( f \in L^\infty(\mu), \|f\|_{L^2(\mu)} \leq 1 \). Theorem 8 is used in an almost identical way to this statement in the proofs of both entropy results.

2. Aside from the above comment, the proofs of the second entropy result here and in [2] differ in only one other place, and only slightly. Bourgain states and uses a different Banach principle for \( L^\infty(\mu) \). In particular, he states that if \( S_n f \) converges almost surely for all \( f \in L^\infty(\mu) \), then there is some function \( \delta(\epsilon) \), which goes to 0 as \( \epsilon \to 0 \), such that \( \int_X \sup_n |S_n f| \, d\mu < \delta(\epsilon) \) whenever \( \|f\|_{L^\infty(\mu)} \leq 1 \) and \( \|f\|_{L^1(\mu)} < \epsilon \). This result can be used in the same manner as Theorem 3. I have included the result from Bellow and Jones instead, because their proof is so easily understood.

3. For more on this and related topics, see work by Roger Jones [4, 5], Michael Lacey [6], and Michel Weber (with Mikhail Lifshits and Dominique Schneider) [8 - 19].

4. I will be happy to field any questions or concerns via e-mail. I would also appreciate being alerted to any typos. Be sure to use a descriptive subject, as I get lots of spam.

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