Symmetric spaces associated to classical groups with even characteristic

Junbin Dong, Toshiaki Shoji and Gao Yang

Abstract. Let $G = GL(V)$ for an $N$-dimensional vector space $V$ over an algebraically closed field $k$, and $G^\theta$ the fixed point subgroup of $G$ under an involution $\theta$ on $G$. In the case where $G^\theta = O(V)$, the generalized Springer correspondence for the unipotent variety of the symmetric space $G/G^\theta$ was described in [SY], assuming that $\text{ch} \ k \neq 2$. The definition of $\theta$ given there, and of the symmetric space arising from $\theta$, make sense even if $\text{ch} \ k = 2$. In this paper, we discuss the Springer correspondence for those symmetric spaces with even characteristic. We show, if $N$ is even, that the Springer correspondence is reduced to that of symplectic Lie algebras in $\text{ch} \ k = 2$, which was determined by Xue. While if $N$ is odd, the number of $G^\theta$-orbits in the unipotent variety is infinite, and a very similar phenomenon occurs as in the case of exotic symmetric space of higher level, namely of level $r = 3$.

0. Introduction

0.1 Let $G$ be a connected reductive group over an algebraically closed field $k$, and $\theta : G \to G$ an involutive automorphism of $G$. Let $G^\theta = \{g \in G \mid \theta(g) = g\}$ be the fixed point subgroup of $G$ by $\theta$. It is known by Vust [V] that $G^\theta$ is a reductive group if $\text{ch} \ k \neq 2$. Let $\mathfrak{g}$ be the Lie algebra of $G$. Then $\theta$ induces a linear automorphism on $\mathfrak{g}$, which we also denote by $\theta$. Put $\mathfrak{g}^\theta = \{x \in \mathfrak{g} \mid \theta(x) = x\}$. It is known (e.g., [Spr, Th. 5.4.4]) that

\begin{equation}
\text{(0.1.1)} \quad \text{Lie}(G^\theta) = \mathfrak{g}^\theta \text{ if } \text{ch} \ k \neq 2.
\end{equation}

In general, we have $\text{Lie}(G^\theta) \subset \mathfrak{g}^\theta$, but the equality not necessarily holds if $\text{ch} \ k = 2$. Let $G^\theta_{\text{uni}} = \{g \in G \mid \theta(g) = g^{-1}\}$ be the set of $\theta$-fixed points in $G$, where $\iota : G \to G$ is the anti-automorphism $g \mapsto g^{-1}$. Let $G_{\text{uni}}$ be the set of unipotent elements in $G$, and put $G^\theta_{\text{uni}} = G^\theta \cap G_{\text{uni}}$. The conjugation action of $G^\theta$ on $G$ preserves $G^\theta_{\text{uni}}$ and $G^\theta_{\text{uni}}$. In the case where $\text{ch} \ k \neq 2$, it is known by Richardson [R] that $G^\theta_{\text{uni}} = \{g^\theta(g)^{-1} \mid g \in G\}$ coincides with the connected component of $G^\theta$ containing the unit element, and the map $g \mapsto g^\theta(g)^{-1}$ gives an isomorphism $G/G^\theta_{\text{uni}} \simeq G^\theta_{\text{uni}}$. Thus we can regard $G^\theta_{\text{uni}}$ as a symmetric space $G/G^\theta$.

In the Lie algebra case, with $\text{ch} \ k \neq 2$, the symmetric space is defined as $\mathfrak{g}^{-\theta} = \{x \in \mathfrak{g} \mid \theta(x) = -x\}$. Let $\mathfrak{g}_{\text{nil}}$ be the set of nilpotent elements in $\mathfrak{g}$, and put $\mathfrak{g}^\theta_{\text{nil}} = \mathfrak{g}^{-\theta} \cap \mathfrak{g}_{\text{nil}}$. It is known that $G^\theta_{\text{uni}} = \mathfrak{g}^\theta_{\text{nil}}$, and the isomorphism is compatible with the action of $G^\theta$. It is also known by [R] that

\begin{equation}
\text{(0.1.2)} \quad \text{The number of } G^\theta \text{-orbits in } G^\theta_{\text{uni}} \text{ is finite if } \text{ch} \ k \neq 2.
\end{equation}

0.2. We consider $G = GL(V)$, where $V$ is an $N$-dimensional vector space over $k$ with $\text{ch} \ k \neq 2$. Let $\theta$ be the involutive automorphism such that $G^\theta$ is the symplectic group $Sp(V)$ or the orthogonal group $O(V)$. First assume that $G^\theta = Sp(V)$, which
we denote by $H$. In this case, we also consider the exotic symmetric space $G^\theta \times V$, on which $H$ acts diagonally. It is known by Kato [K1] that $G^\theta_{\text{uni}} \times V$ has finitely many $H$-orbits (actually he considered the Lie algebra case, $\mathfrak{g}_{\text{nil}}^\theta \times V$, the so-called "Kato’s exotic nilcone"). Put $N = 2n$, and let $W_n$ be the Weyl group of type $G_n$. He established in [K1] the Springer correspondence between the $H$-orbits in $\mathfrak{g}_{\text{nil}}^\theta \times V$ and irreducible representations of $W_n$, based on the Ginzburg theory ([CG]) on Hecke algebras. After that in [SS], the Springer correspondence was also proved for $G^\theta_{\text{uni}} \times V$, based on the theory of the generalized Springer correspondence due to Lusztig [L1]. (We remark that a straightforward generalization of the classical theory of the Springer correspondence does not hold for the symmetric space $G^\theta_{\text{uni}}$ or $\mathfrak{g}_{\text{nil}}^\theta$ itself).

The special feature in the exotic nilcone is that for any $x \in \mathfrak{g}_{\text{nil}}^\theta \times V$, the stabilizer of $x$ in $G^\theta$ is connected, and so only the constant sheaves appear in the Springer correspondence. This type of phenomenon also appears in the nilpotent orbits of the Lie algebra $\mathfrak{sp}(V)$ in characteristic 2. Noting this, in [K2], Kato proved that the Springer correspondence for the exotic nilcone $\mathfrak{g}_{\text{nil}}^\theta \times V$ (for any ch $k$) is the same as the Springer correspondence for the ordinary nilcone $\mathfrak{sp}(V)_{\text{nil}}$ (for ch $k = 2$), by using a certain deformation argument of schemes over $\mathbb{Z}$.

0.3. As a generalization of the exotic symmetric space $G^\theta \times V$, we consider the variety $G^\theta_{\text{uni}} \times V^r - 1$ for an integer $r \geq 1$, on which $H$ acts diagonally. We consider its unipotent part $\mathcal{X}_r = G^\theta_{\text{uni}} \times V^r - 1$ with diagonal $H$-action. We call $\mathcal{X}_r$ the (unipotent) exotic symmetric space of level $r$. The crucial difference from the exotic case (i.e., $r = 2$) is that the number of $H$-orbits in $\mathcal{X}_r$ is infinite if $r \geq 3$. Hence the discussion in [SS] can not be applied to this case. Nevertheless, it was shown in [Sh] that a generalization of the Springer correspondence still holds in the following sense; we consider the complex reflection group $W_{n,r} = S_n \times (\mathbb{Z}/r\mathbb{Z})^n$ (hence $W_{n,1} \simeq S_n$ and $W_{n,2} \simeq W_n$). Then for each $\rho \in W_{n,r}^\wedge$, one can construct a smooth, irreducible, $H$-stable, locally closed subvariety $X_\rho$ of $\mathcal{X}_r$ and we have a natural correspondence $\rho \leftrightarrow \text{IC}(\overline{X_\rho}, \mathbb{Q}_l)$. Moreover, for an element $z \in X_\rho$, one can define the Springer fibre $\mathcal{B}_z$ as a closed subset of the flag variety of $H$, and the cohomology $H^i(\mathcal{B}_z, \mathbb{Q}_l)$ has a structure of $W_{n,r}$-module (Springer representation of $W_{n,r}$). Then for a generic $z \in X_\rho$, the top cohomology $H^{2d_z}(\mathcal{B}_z, \mathbb{Q}_l)$ gives the irreducible representation $\rho$ (here $d_z = \dim \mathcal{B}_z$). In this way, any irreducible representation of $W_{n,r}$ is realized as the top cohomology of the Springer fibre.

0.4. Next we consider the case where $G^\theta = O(V)$ with ch $k \neq 2$. We put $H = SO(V)$. In this case, the stabilizer of $x \in G^\theta_{\text{uni}}$ in $H$ is not necessarily connected, and we need to consider twisted local systems on $H$-orbits. Moreover the Springer correspondence is not enough to cover all the $H$-orbits in $G^\theta_{\text{uni}}$. In [SY], the generalized Springer correspondence for $G^\theta_{\text{uni}}$ was established. Actually it was shown there that Lusztig’s theory of the generalized Springer correspondence for reductive groups fits very well to our setting under a suitable modification. For example, if $N = 2n + 1$ or $N = 2n$, the Springer correspondence is given by the correspondence $\rho \leftrightarrow \text{IC}(\overline{\mathcal{O}_\rho}, \mathbb{Q}_l)$, where $\rho \in S_n^\wedge$ and $\mathcal{O}_\rho$ is a certain $H$-orbit corresponding to $\rho$ (note that twisted local systems do not appear in this part).
0.5. In this paper, we consider \( \theta : G \to G \), defined in a similar way as \( \theta \) in 0.4, but under the condition that \( \text{ch } k = 2 \). The definition of \( G^\theta \) given in 0.4 makes sense even if \( \text{ch } k = 2 \) (see 1.2). We call \( G^\theta \) the symmetric space over a field of characteristic 2. The aim of this paper is to establish the Springer correspondence for the \( G^\theta \)-orbits in \( G_{\text{uni}}^\theta \).

First assume that \( N = 2n \) is even. In that case, we can show (Proposition 1.7) that \( G^\theta = Sp(V) \), the symplectic group in characteristic 2, and there exists an isomorphism \( G_{\text{uni}}^\theta \simeq g_{\text{nil}}^\theta \), compatible with the action of \( Sp(V) \), where \( g^\theta = sp(V) \) is the Lie algebra of \( Sp(V) \). Hence considering the Springer correspondence for the symmetric space \( G_{\text{uni}}^\theta \) is essentially the same as considering the same problem for the ordinary nilcone \( sp(V)_{\text{nil}} \). Put \( H = Sp(V) \) and \( h = sp(V) \). As was explained in 0.2, the Springer correspondence for \( h_{\text{nil}} = sp(V)_{\text{nil}} \) was determined by [K2] in connection with the exotic symmetric space. After that Xue [X1, X2] established the Springer correspondence for the Lie algebras of classical type in characteristic 2, based on the Lusztig’s theory of the generalized Springer correspondence. Note that the difficulty in the Lie algebras of even characteristic is that the regular semisimple elements not necessarily exist. (Recall, in the case of reductive groups \( G \) with Weyl group \( W \), that the strategy of proving the Springer correspondence is first to construct a semisimple perverse sheaf \( K \) on \( G \), equipped with \( W \)-action, by making use of the finite Galois covering arising from the set of regular semisimple elements in \( G \), then restrict it to the unipotent variety \( G_{\text{uni}} \), to obtain the correspondence \( \rho \leftrightarrow IC(\overset{\cdot}{\theta}, \overset{\cdot}{\mathcal{E}}) \), where \( \rho \) is the irreducible representation of \( W \), and \( \overset{\cdot}{\mathcal{E}} \) is a certain unipotent class in \( G \), \( \overset{\cdot}{\mathcal{E}} \) is a local system on \( \overset{\cdot}{\mathcal{E}} \).) In our situation, \( h \) does not have regular semisimple elements. In order to overcome this defect, he replaced \( H \) and \( h \) by a bigger group \( \overset{\cdot}{H} \) and its Lie algebra \( \overset{\cdot}{h} \), where \( \overset{\cdot}{H} \) is an extension of \( H \) by a connected center, so that \( \overset{\cdot}{h} \) has regular semisimple elements. Then following the above procedure, he could prove the Springer correspondence for \( \overset{\cdot}{h}_{\text{nil}} = h_{\text{nil}} \), in which case it gives a bijection \( \rho \leftrightarrow IC(\overset{\cdot}{\mathcal{E}}, \overset{\cdot}{Q}_{\mathfrak{t}}) \) between irreducible representations of \( W_n \), and nilpotent orbits in \( h_{\text{nil}} \). Hence our problem of the Springer correspondence for \( G_{\text{uni}}^\theta \) is easily solved as a corollary of Xue’s result.

0.6. Next assume that \( N = 2n + 1 \) is odd. We write \( G = GL(V') \) where \( V' \) is an \( N \)-dimensional vector space. Then we can show (Proposition 1.13) that \( G^\theta \simeq Sp(V) \), where \( V \) is an \( 2n \)-dimensional subspace of \( V' \). Moreover \( G_{\text{uni}}^\theta \simeq g_{\text{nil}}^\theta \), compatible with the action of \( G^\theta \). Put \( H = Sp(V) \) and \( h = sp(V) \) as in 0.5. We can show that there is an embedding \( h \hookrightarrow g^\theta \), and \( g^\theta \) is isomorphic to \( h \times V \times k \) as algebraic varieties, where the action of \( H \) on \( g^\theta \) corresponds to the diagonal action of \( H \) on \( h \times V \), together with the trivial action of \( H \) on \( k \). Hence the situation is very similar to the exotic symmetric space mentioned in 0.2. But note that the structure of the nilcone is different. In the exotic case, \( g_{\text{nil}}^\theta \times V \) is considered as the nilcone. In our case, \( g_{\text{nil}}^\theta = (h_{\text{nil}} \times V) \cap g_{\text{nil}} \), where \( g = g(\overset{\cdot}{V}') \) is the Lie algebra of \( G \), which is a certain \( H \)-stable subset of \( h_{\text{nil}} \times V \).

More interesting thing is that the number of \( H \)-orbits in \( g_{\text{nil}}^\theta \) is infinite. Hence this gives a counter-example for (0.1.2) in the case where \( \text{ch } k = 2 \). (Also this gives a counter-example for (0.1.1) since \( G^\theta = H \) and \( g^\theta \neq h \).) We remark that \( g_{\text{nil}}^\theta \) should be understood as an analogue of the exotic symmetric space of higher level.
as discussed in 0.3 rather than the exotic symmetric space itself. In fact, following
Kato’s observation ([K2]), the Springer correspondence for $\mathfrak{h}_{\text{nil}}$ corresponds to that
for the exotic symmetric space $G_{\text{uni}}^{d} \times V$. Hence it is natural to expect that the
Springer correspondence for $\mathfrak{h}_{\text{nil}} \times V$ should correspond to that for $(G_{\text{uni}}^{d} \times V) \times V =
G_{\text{uni}}^{d} \times V^{2}$, and the correspondence is dominated by $W_{n,3}$ (the complex reflection
group for $r = 3$) as the special case of [Sh].

We show in Theorem 9.7 and Proposition 9.10 that this certainly holds. It is
proved that a similar result as in 0.3 holds for $\mathfrak{g}_{\theta}^{\text{nil}}$ with respect to $W_{n,3}$; namely, there
exists a smooth, irreducible, locally closed subvariety $X_{\rho}$ of $\mathfrak{g}_{\theta}^{\text{nil}}$ for each $\rho \in W_{n,3}^{\wedge}$
such that $\rho \leftrightarrow \text{IC}(\mathcal{X}_{\rho}, \mathbb{Q}_{l})$ gives the Springer correspondence, namely, for $z \in X_{\rho}$,
the Springer fibre $\mathcal{B}_{z}$ is defined as a closed subset of the flag variety of $H$, and
$H^{i}(\mathcal{B}_{z}, \mathbb{Q}_{l})$ gives rise to a $W_{n,3}$-module. For a generic $z \in X_{\rho}$, $H^{2d_{z}}(\mathcal{B}_{z}, \mathbb{Q}_{l})$ gives
the irreducible representation $\rho$ of $W_{n,3}$. Any irreducible representation of $W_{n,3}$ is
realized in this way on the top cohomology of the Springer fibre.

0.7. The proof of the Springer correspondence for $\mathfrak{g}_{\theta}^{\text{nil}}$ is basically done by
modifying the arguments in [Sh] for the case of exotic symmetric spaces of higher
level. But in order to apply the discussion in [Sh] to our case, it is necessary to
construct a representation of $W_{n} = W_{n,2}$ on a certain semisimple perverse sheaf $K$
on $\mathfrak{h}$. Note that a similar perverse sheaf $\tilde{K}$ on $\mathfrak{h}$ equipped with $W_{n}$-action is already
constructed in 0.5. Here we need to determine its restriction $\tilde{K}|_{\mathfrak{h}} = K$ on $\mathfrak{h}$. In order
to do this, we construct a representation of $W_{n}$ on $K$ directly, without referring $\tilde{h}$, just using the subregular semisimple elements in $\mathfrak{h}$ which are open dense in the set
of semisimple elements in $\mathfrak{h}$ (though in one step we need to apply the result for $\tilde{h}$). This process is quite similar to the procedure in [SS], which is regarded as a
reflection of Kato’s deformation argument between the Springer correspondence for
$G_{\text{uni}}^{d} \times V$ and for $\mathfrak{h}_{\text{nil}}$.

Note that in [Sh] (and in [SS]) the proof of the main result is reduced to the case
where $r = 1$, namely the case $G_{\text{uni}}^{d}$, by induction on $r$. But the discussion in
[Sh] can not cover this case since the Springer correspondence does not exist for
$G_{\text{uni}}^{d}$, and we had to refer the result of Henderson [Hen], which he proved by using
the Fourier-Deligne transform of perverse sheaves on the Lie algebra. So, in some
sense, the discussion in [Sh] is unsatisfactory from the view point of the strategy in
0.5. In the case of $\mathfrak{g}_{\theta}^{\text{nil}}$, this unpleasant situation does not occur since the Springer
correspondence exists for $\mathfrak{h}_{\text{nil}}$ (though we cannot avoid to use $\tilde{h}$).

Some notations:

For any finite group $\Gamma$, we denote by $\Gamma^{\wedge}$ the set of isomorphism classes of
irreducible representations of $\Gamma$ over $\mathbb{Q}_{l}$.

Let $G$ be an algebraic group and $\mathfrak{g}$ its Lie algebra. $G$ acts on $\mathfrak{g}$ by the adjoint
action. We denote this action simply by $\text{Ad}(g)x = g \cdot x$ for $g \in G, x \in \mathfrak{g}$.

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1. Symmetric spaces in characteristic 2

1.1. Let $V$ be an $N$-dimensional vector space over an algebraically closed field $k$. We assume that $\text{ch } k \neq 2$. Let $(\cdot, \cdot)$ be the non-degenerate symmetric bilinear form on $V$. The orthogonal group $O(V)$ associated to the form $(\cdot, \cdot)$ is defined as

\begin{equation}
O(V) = \{ g \in GL(V) \mid (gv, gw) = (v, w) \text{ for any } v, w \in V \}.
\end{equation}

If we define the quadratic from $Q(v)$ on $V$ by $Q(v) = (v, v)$, (1.1.1) is equivalent to

\begin{equation}
O(V) = \{ g \in GL(V) \mid Q(gv) = Q(v) \text{ for any } v \in V \}.
\end{equation}

We fix a basis $e_1, \ldots, e_N$ of $V$, and identify $GL(V)$ with $G = GL_N$ by using this basis. If we define the matrix $J \in GL_N$ by $J = (\langle e_i, e_j \rangle)$, (1.1.1) is also written as

\begin{equation}
O(V) = \{ g \in G \mid {}^t g J g = J \}.
\end{equation}

We define a map $\theta : G \to G$ by $\theta(g) = J^{-1}( {}^t g^{-1} ) J$. Then $\theta$ gives rise to an involutive automorphism on $G$, and we have $G^\theta = O(V)$.

In the above discussion, if we replace the symmetric bilinear form by the non-degenerate skew-symmetric bilinear form $(\cdot, \cdot)$ on $V$ with even $N$, we can define the symplectic group $Sp(V)$ in a similar way as $O(V)$ by using (1.1.1). The matrix $J \in GL_N$ is defined similarly, and by using an involutive automorphism $\theta : G \to G$ defined by $\theta(g) = J^{-1}( {}^t g^{-1} ) J$, we obtain an analogue of (1.1.3) for $Sp(V)$. Hence in this case also $G^\theta = Sp(V)$.

1.2. Hereafter, throughout the paper, we assume that $\text{ch } k = 2$. Put $G = GL_N$. Consider an involutive automorphism $\theta : G \to G$ defined by $\theta(g) = J^{-1}( {}^t g^{-1} ) J$, where

\begin{equation}
J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_n \\ 0 & 1_n & 0 \end{pmatrix}
\end{equation}

if $N = 2n + 1$,

\begin{equation}
J = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}
\end{equation}

if $N = 2n$, where

\begin{equation}
\text{Mat}(n, k) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A, B, C, D \in k^{n \times n}, B = C \right\}.
\end{equation}
with 1_a the identity matrix of degree n. Thus we can consider the fixed point subgroup G^θ and the symmetric space G^uθ. If ch k ≠ 2, then G^θ ≃ O(V), and the generalized Springer correspondence with respect to G^uθ was discussed in [SY]. The aim of this paper is to consider a similar problem for G^θ in the case where ch k = 2.

1.3. We consider θ : G → G as in 1.2. Since J = J^{-1} = tJ, we have

\begin{equation}
G^θ = \{ g ∈ G \mid J^{-1}(t^gJ) = g^{-1} \} = \{ g ∈ G \mid t^gJ = Jg \}.
\end{equation}

Let g = gl_N be the Lie algebra of G, and θ : g → g be the linear automorphism induced from θ : G → G. Since ch k = 2, θ is given as x → -J(t^x)J = J(t^x)J for x ∈ g. Hence

\begin{equation}
g^θ = \{ x ∈ g \mid J(t^x)J = x \} = \{ x ∈ g \mid t^J(x) = Jx \}.
\end{equation}

Put g^θ-nil = g^θ ∩ g-nil. By comparing (1.3.1) and (1.3.2), we have

Lemma 1.4. The map g ↦ g - 1 gives an isomorphism G^θ-nil ∼ G^θ-nil, which is compatible with the conjugate action of G^θ.

1.5. In order to study G^θ and G^uθ, we need to consider the orthogonal group over the field of characteristic 2. Since (1.1.1) and (1.1.2) are not equivalent in the case where ch k = 2, we have to define the orthogonal group by using the quadratic form Q(v) on V. Let Q(v) be a quadratic form on V. We define an associated bilinear form \langle , \rangle by

\begin{equation}
\langle v, w \rangle = Q(v + w) - Q(v) - Q(w).
\end{equation}

Here we give the quadratic form Q(v) explicitly as follows. First assume that N = 2n, and fix a basis e_1, ..., e_n, f_1, ..., f_n of V. For v = \sum_i x_i e_i + \sum_i y_i f_i, define

Q(v) = \sum_{i=1}^n x_i y_i.

Then the associated bilinear form is given by \langle v, w \rangle = \sum_i (x_i y'_i + x'_i y_i) for v = \sum_i x_i e_i + \sum_i y_i e_i, w = \sum_i x'_i e_i + \sum_i y'_i f_i. Next assume that N = 2n + 1, and fix a basis e_0, e_1, ..., e_n, f_1, ..., f_n of V. For v = \sum_i x_i e_i + \sum_i y_i f_i, define

Q(v) = x_0^2 + \sum_{i=1}^n x_i y_i.

Then the associated bilinear form is given by \langle v, w \rangle = \sum_{i≥1} (x_i y'_i + x'_i y_i) for v = \sum_{i≥0} x_i e_i + \sum_{i≥1} y_i f_i, w = \sum_{i≥0} x'_i e_i + \sum_{i≥1} y'_i f_i.
We define an orthogonal group $O(V)$ as in (1.1.1). If $g \in O(V)$, $g$ leaves the form $\langle \cdot, \cdot \rangle$ invariant by (1.5.1). It follows that

$$(1.5.2) \quad O(V) \subset \{ g \in GL(V) \mid \langle gv, gw \rangle = \langle v, w \rangle \text{ for any } v, w \in V \}.$$ 

1.6. Assume that $N = 2n$. Since $\text{ch } k = 2$, the symmetric bilinear form on $V$ coincides with the skew-symmetric bilinear form on $V$. The definition of the symplectic group given in 1.1 makes sense even if $\text{ch } k = 2$, which we denote by $Sp(V)$. Thus the right hand side of (1.5.2) coincides with $Sp(V)$ with respect to this form, and we have

$$(1.6.1) \quad O(V) \subset Sp(V).$$

By using the explicit description of the associated symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $V$ given in 1.5, we see that $Sp(V)$ can be written as

$$(1.6.2) \quad Sp(V) = \{ g \in G \mid ^t g J g = J \},$$

where $J$ is as in (1.2.2). In particular, we have $G^\theta = Sp(V)$.

Let $\mathfrak{sp}(V)$ be the Lie algebra of $Sp(V)$. It follows from (1.6.2), we have

$$\mathfrak{sp}(V) \subseteq \{ x \in \mathfrak{gl}(V) \mid \langle x v, w \rangle + \langle v, x w \rangle = 0 \text{ for any } v, w \in V \}$$

$$= \{ x \in \mathfrak{gl}(V) \mid ^t x J + J x = 0 \}$$

$$= \mathfrak{g}^\theta.$$ 

The last equality follows from (1.3.2). Here $\dim \mathfrak{sp}(V) = \dim Sp(V) = 2n^2 + n$. The dimension of the space of symmetric matrices in $V$ is equal to $n(2n + 1)$. Thus the equality holds in the above formulas. We have

$$(1.6.3) \quad \mathfrak{sp}(V) = \mathfrak{g}^\theta.$$ 

Summing up the above arguments, together with Lemma 1.5, we have the following.

**Proposition 1.7.** Assume that $N = 2n$. Then $G^\theta = Sp(V)$, and $G^\theta_{\text{uni}} \cong \mathfrak{g}^\theta_{\text{nil}} = \mathfrak{sp}(V)_{\text{nil}}$. The behaviour of $G^\theta$-orbits on $G^\theta_{\text{uni}}$ is the same as that of $Sp(V)$-orbits on $\mathfrak{sp}(V)_{\text{nil}}$.

**Remark 1.8.** Let $W_n$ be the Weyl group of type $C_n$. It is known by Hesselink [Hes] and Spaltenstein [Spa] that the number of $Sp(V)$-orbits in $\mathfrak{sp}(V)_{\text{nil}}$ is finite, and those orbits are parametrized by $W_n^\wedge$.

**Remark 1.9.** In the case where $\text{ch } k = 2$ and $N$ is even, it is not known whether $O(V)$ is realized as $G^\theta$ for a certain involutive automorphism $\theta : G \to G$.

1.10. Assume that $N = 2n + 1$. By changing the notation, we consider the vector space $V'$ with basis $e_0, e_1, \ldots, e_n, f_1, \ldots, f_n$, and let $V$ be the subspace of $V'$...
spanned by \( e_1, \ldots, e_n, f_1, \ldots, f_n \). We identify \( Sp(V) \) with \( Sp_{2n} \) by using this basis. \( Sp_{2n} \) can be explicitly written as

\[ Sp_{2n} = \left\{ \begin{pmatrix} f & g \\ h & k \end{pmatrix} \in GL_{2n} \mid f, g, h, k \in \mathcal{M}_n, (\ast) \right\}, \tag{1.10.1} \]

where \( \mathcal{M}_n \) is the set of square matrices of degree \( n \), and the condition (\ast) is given by

\[ t^* h f = t^* f h, \quad t^* g k = t^* k g, \quad t^* h g + t^* f k = 1. \tag{1.10.2} \]

We have the following result.

**Proposition 1.11.** Assume that \( N = 2n + 1 \), and \( G = GL_N \). Then

\[ G^\theta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \mid y \in Sp_{2n} \right\}. \tag{1.11.1} \]

In particular, \( G^\theta \simeq SO(V') \simeq Sp(V) \).

**Proof.** Take \( x \in GL_N \), and write it as

\[ x = \begin{pmatrix} a & b & c \\ \, t^* d & f & g \\ \, t^* e & h & k \end{pmatrix}, \]

where \( b = (b_1, \ldots, b_n) \), \( c = (c_1, \ldots, c_n) \) and \( d = (d_1, \ldots, d_n) \), \( e = (e_1, \ldots, e_n) \), together with \( f, g, h, k \in \mathcal{M}_n \). Note that \( G^\theta = \{ x \in G \mid t^* x J x = J \} \). The condition \( t^* x J x = J \) implies, in particular, that

\[ a^2 + e \cdot \, t^* d + d \cdot t^* e = 1, \]
\[ \, t^* b \cdot b + \, t^* h f + \, t^* f h = 0, \]
\[ \, t^* c \cdot c + \, t^* k g + \, t^* g k = 0. \]

Since \( e \cdot \, t^* d + d \cdot t^* e = 2 \sum_{i=1}^n d_i e_i = 0 \), the first equality implies that \( a = 1 \). Since the diagonal entries of \( \, t^* h f + \, t^* f h \) are all zero, the \( ii \)-entry of the matrix \( \, t^* b \cdot b + \, t^* h f + \, t^* f h \) is equal to \( b_i^2 \). Hence \( b = 0 \) by the second equality. Similar argument by using the third shows that \( c = 0 \). Now the condition \( t^* x J x = J \) is equivalent to the condition that

\[ \begin{cases} e f + d h = 0, \\
e g + d k = 0, \\
\, t^* h g + \, t^* f k = 1, \\
\, t^* h f = \, t^* f h, \\
\, t^* k g = \, t^* g k. \end{cases} \tag{1.11.2} \]
By comparing (1.11.2) with (1.10.2), we can write as

\[
    x = \begin{pmatrix}
        1 & 0 & 0 \\
        t^d & f & g \\
        t^e & h & k
    \end{pmatrix} \quad \text{with} \quad y = \begin{pmatrix}
        f & g \\
        t^h & k
    \end{pmatrix} \in Sp_{2n}.
\]

Since \(y\) is non-degenerate, the relation \((e, d)y = 0\) implies that \(d = e = 0\). This proves (1.11.1). Now by 1.5, one can check that \(G^\theta \subset O(V')\). We have \(\dim G^\theta = \dim Sp_{2n} = \dim SO_{2n+1} = 2n^2 + n\). Since \(Sp(V)\) is connected, we conclude that \(G^\theta = SO(V')\). The proposition is proved. \(\square\)

1.12. We determine \(g^\theta\) in the case where \(N = 2n + 1\). By (1.3.2), we have \(\dim g^\theta = (n + 1)(2n + 1)\). Since \(\dim G^\theta = \dim Sp_{2n} = 2n^2 + n\), we see that (1.12.1) \(\text{Lie} G^\theta \not\subset g^\theta\), namely (0.1.1) does not hold for \(G^\theta\).

More precisely, by the direct computation, we have the following.

\[
    g^\theta = \{ x = \begin{pmatrix}
        a & b & c \\
        t^c & f & g \\
        t^b & h & t^f
    \end{pmatrix} \mid a \in k, f, g, h \in \mathcal{A}_n, t^h = h, t^g = g \},
\]

\[\text{Lie} G^\theta = \{ x \in g^\theta \mid a = b = c = 0 \} \simeq \text{sp}(V).\]

Summing up the above arguments, together with Lemma 1.4, we have the following.

**Proposition 1.13.** Assume that \(N = 2n + 1\). Then \(G^\theta \simeq Sp(V)\), and \(G^\theta \simeq g^\theta\). Under the embedding \(\text{sp}(V)_{\text{nil}} \hookrightarrow g^\theta_{\text{nil}}\), \(\text{sp}(V)_{\text{nil}}\) is a \(G^\theta\)-stable subset of \(g^\theta_{\text{nil}}\), and the action of \(G^\theta\) on \(\text{sp}(V)_{\text{nil}}\) coincides with the conjugate action of \(Sp(V)\) on it.

1.14. We write \(H = Sp(V)\) and \(h = \text{sp}(V)\). We identify \(H\) with \(G^\theta\), and \(h\) with a subspace of \(g^\theta\). Then \(g^\theta\) can be written as \(g^\theta = h \oplus g_V\), where \(g_V\) is a subspace of \(g^\theta\) consisting of \(x \in g^\theta\) such that \(f = g = h = 0\) (notation in (1.13.2)). We express \(x \in g_V\) as \(x = (a, b, c)\). Put \(\mathfrak{z} = \{(a, 0, 0) \mid a \in k\}\), and let \(g_V\) be the subspace of \(g_V\) consisting of \(x = (0, b, c)\). Then \(g_V' = g_V \oplus \mathfrak{z}\). \(g_V\) is \(H\)-stable, and \(H\) acts trivially on \(\mathfrak{z}\). We identify \(g_V\) with \(V\) under the correspondence \((0, b, c) \in g_V \iff \sum_{i=1}^{n} c_i e_i + \sum_{i=1}^{n} b_i f_i \in V\) for \(b = (b_1, \ldots, b_n), c = (c_1, \ldots, c_n)\). Then we can identify \(h \oplus g_V\) with \(h \times V\), where the natural action of \(H\) on \(g^\theta\) preserves \(h \oplus g_V\), which corresponds to the diagonal action of \(H\) on \(h \times V\).

**Remark 1.15.** The above discussion shows that considering the action of \(H\) on \(g^\theta\) is the same as considering the diagonal action of \(H\) on \(h \times V\). Hence the situation is quite similar to the case of exotic symmetric spaces studied in [K1], [SS]. Recall that the exotic symmetric space (in the Lie algebra case) is defined as \(g^{-\theta} \times V\) for \(G^\theta = Sp(V)\) with \(\text{ch} k \neq 2\), together with the diagonal action of \(G^\theta\). But note that the structure of the nilpotent variety is different. In the exotic case, as the nilpotent variety, we have considered \(g^\theta_{\text{nii}} \times V\) (Kato's exotic nilpotent cone [K1]). In the present case, we consider \(g^\theta_{\text{nii}} = (h \oplus g_V) \cap g_{\text{nii}}\), which corresponds to a certain subset of \(h_{\text{nii}} \times V\).
1.16. We follow the notation in 1.14. In the remainder of this section, we shall show that \( g_{\text{nil}}^\theta \) has infinitely many \( H \)-orbits. Let \( B' \) be the subgroup of \( G = GL_N \) consisting of elements

\[
(1.16.1) \quad \begin{pmatrix} a & 0 & c \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ 0 & 0 & k; \end{pmatrix}
\]

where \( a \in k, c, d \in k^n \) (row vectors), \( f, g, k \in \mathcal{M}_n \), and \( f \) is upper triangular, \( k \) is lower triangular. Then \( B' \) is a \( \theta \)-stable Borel subgroup of \( G \). Put \( b' = \text{Lie} B' \), and let \( n' \) be the nilpotent radical of \( b' \). Then \( g^\theta \cap n' \subset g_{\text{nil}}^\theta \), and we have

\[
(1.16.2) \quad g^\theta \cap n' = \left\{ \begin{pmatrix} 0 & 0 & c \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ 0 & 0 & \cdot \cdot \cdot \end{pmatrix} \mid f : \text{strict upper triangular}, \ t g = g \right\}.
\]

The following result gives a counter-example for (0.1.2) in the case where \( \text{ch} k = 2 \).

Proposition 1.17. Let \( \mathcal{O}_0 \) be the regular nilpotent orbit in \( gl_N \). Then \( \mathcal{O}_0 \cap g^\theta \) has infinitely many \( H \)-orbits. In particular, \( g_{\text{nil}}^\theta \) has infinitely many \( H \)-orbits.

Proof. Assume that \( f \in \mathcal{M}_n \) corresponds to the transformation on the subspace \( \langle e_1, \ldots, e_n \rangle \) of \( V \) defined by

\[
f : e_n \mapsto e_{n-1} \mapsto \cdots \mapsto e_1 \mapsto 0,
\]

and put \( g = \text{Diag}(0, \ldots, 0, 1) \in \mathcal{M}_n \). Then \( y = \begin{pmatrix} f & g \\ 0 & t f \end{pmatrix} \) gives an element in \( \mathfrak{h} = \mathfrak{sp}_{2n} \), which is a regular nilpotent element in \( \mathfrak{sp}_{2n} \). Let \( c = (0, 0, \ldots, 0, \xi) \) with \( \xi \in k \), and put \( z = (0, 0, c) \in \mathfrak{g}_V \). Then \( x = y + z \in g_{\text{nil}}^\theta \) by (1.16.2), which we denote by \( x(\xi) \). Since the operation of \( x(\xi) \) on \( V' \) is given by

\[
\begin{align*}
f_1 &\mapsto f_2 \mapsto \cdots \mapsto f_{n-1} \mapsto f_n, \\
f_n &\mapsto e_n + \xi e_0, \\
e_0 &\mapsto \xi e_n, \\
e_n &\mapsto e_{n-1} \mapsto \cdots \mapsto e_1 \mapsto 0,
\end{align*}
\]

\( x(\xi) \in \mathcal{O}_0 \) for any \( \xi \in k^* \). In order to prove the proposition, it is enough to see that (1.17.1) \( x(\xi) \) and \( x(\xi') \) are not conjugate under \( H \) if \( \xi \neq \xi' \).

We show (1.17.1). Assume there exists \( h \in H \) such that \( h(x(\xi)) = x(\xi') \). Write \( x(\xi) = y + z, x(\xi') = y + z' \). Since \( H \) leaves the decomposition \( g^\theta = \mathfrak{h} \oplus \mathfrak{g}_V \) invariant, we must have \( h(y) = y \) and \( h(z) = z' \). Hence \( h \in Z_H(y) \). Since \( y \in g^\theta \cap n' \) is regular nilpotent, \( Z_H(y) \) satisfies the property

\[
(1.17.2) \quad Z_H(y) \subset \left\{ \begin{pmatrix} f_1 & g_1 \\ 0 & f_1^{-1} \end{pmatrix} \in Sp_{2n} \mid f_1 : \text{upper unitriangular} \right\}.
\]
On the other hand, the action of $H$ on $g_V$ is given as in 1.14. By (1.17.2), we have
\[ h \cdot t^{(0, \ldots, 0, \xi, 0, 0, \ldots, 0)} = t^{(c_1, \ldots, c_n, 0, 0, \ldots, 0)} \]
for some $c_1, \ldots, c_n$ with $c_n = \xi$. Since $h(z) = z'$, we have $\xi = \xi'$. Thus (1.17.1) holds. The proposition is proved. \qed

Remark 1.18. In later discussion, we use the Jordan decomposition of Lie algebras. It is known (see [Spr, 4.4.20]) that if $g$ is the Lie algebra of an algebraic group $G$, the Jordan decomposition works. So, in the setting of 1.15, we can apply the Jordan decomposition for $h = \text{Lie} H$, but not for $g^\theta$.

2. Intersection cohomology related to $\mathfrak{sp}(V)_{sr}$

2.1. Let $H = Sp(V), \mathfrak{h} = \mathfrak{sp}(V)$, and we follow the notation in 1.6. Let $W_n$ be the Weyl group of $H$. As pointed out in Remark 1.8, nilpotent orbits in $\mathfrak{h}$ are parametrized by $W_n^\wedge$. The Springer correspondence between the set of nilpotent orbits and $W_n^\wedge$ was first considered by Kato [K2]. Then Xue [X1], [X2] established the general theory of the Springer correspondence for classical Lie algebras in characteristic 2.

The main difficulty in considering $H$ and $\mathfrak{h}$ relies on the fact that the regular semisimple elements do not exist for $\mathfrak{h}$. In order to overcome this defect, Xue replaced $H$ and $\mathfrak{h}$ by bigger ones $\widetilde{H}$ and $\widetilde{\mathfrak{h}}$, extension by connected center, so that $\widetilde{\mathfrak{h}}$ has regular semisimple elements, and established the Springer correspondence by using the bijection $\widetilde{\mathfrak{h}}_{\text{nil}} \simeq \mathfrak{h}_{\text{nil}}$. (Actually Xue considers the adjoint group $\widetilde{H}_{\text{ad}}$ and its Lie algebra $\widetilde{\mathfrak{h}}_{\text{ad}}$, but the theory of the Springer correspondence for them is essentially the same as that for $\widetilde{H}$ and $\widetilde{\mathfrak{h}}$.)

In this paper, for later applications to the case where $N = 2n + 1$, we discuss the Springer correspondence for $\mathfrak{h}$ more directly, without using the regular semisimple elements (though we need to use $\widetilde{\mathfrak{h}}$). In the discussion below, we borrowed the idea to use $\mathfrak{D}$ from [SY]. Note that those discussions have strong resemblance with the case of exotic symmetric spaces associated to symplectic groups with $\text{ch} \mathbf{k} \neq 2$ ([SS, 3]), as explained in the Introduction.

2.2. Let $T \subset B$ be subgroups of $G = GL_N$ with $N = 2n$ given by
\[ B = \left\{ \begin{pmatrix} a & b \\ 0 & t^{-1}a^{-1} \end{pmatrix} \mid a, b \in \mathcal{M}_n, a : \text{upper triangular}, t^b = a^{-1}b' a \right\}, \]
\[ T = \{ \text{Diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \mid t_i \in \mathbf{k}^* \}. \]

By (1.10.1), $B$ is a Borel subgroup of $H$ and $T$ is a maximal torus of $H$ contained in $B$. Let $\mathfrak{t}$ be the Lie algebra of $T$. Since $\text{ch} \mathbf{k} = 2$, we have
\[ \mathfrak{t} = \{ \text{Diag}(t_1, \ldots, t_n, t_n, \ldots, t_1) \mid t_i \in \mathbf{k} \}. \]
We define a subset $t_{sr}$ of $t$ by

$$
(2.2.2) \quad t_{sr} = \{ s = \text{Diag}(t_1, \ldots, t_n, t_1, \ldots, t_n) \mid t_i \neq t_j \text{ for } i \neq j \}.
$$

$t_{sr}$ is open dense in $t$, and for any $s \in t_{sr}$, $Z_H(s) \cong SL_2 \times \cdots \times SL_2$ ($n$-times). Put $h_{sr} = \bigcup_{g \in H} g(t_{sr})$. Then $h_{sr}$ is the set of semisimple elements in $h$ such that all the eigenspaces have dimension 2.

Recall that $s \in h$ is called regular semisimple if $Z^0_H(s)$ is a maximal torus of $H$. For any $s \in t$, dim $Z_H(s) \geq 3n$. Hence $t$ does not contain regular semisimple elements. This implies that $h$ does not contain regular semisimple elements (see Lemma 2.3). An element $s \in h_{sr}$ is said to be subregular semisimple.

Let $U$ be the unipotent radical of $B$. Let $b$ be the Lie algebra of $B$, and $n = \text{Lie } U$. $n$ is written as

$$
(2.2.3) \quad n = \left\{ \begin{pmatrix} a & b \\ 0 & t_a \end{pmatrix} \mid a : \text{strict upper triangular, } t^b = b \right\}
$$

Let $\Phi^+ \subset \text{Hom}(T, k^*) \cong Z^n$ be the set of positive roots of type $C_n$,

$$
\Phi^+ = \{ \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq n \},
$$

where $\varepsilon_1, \ldots, \varepsilon_n$ is a basis of $\text{Hom}(T, k^*)$ given by $\varepsilon_i : \text{Diag}(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}) \mapsto t_i$. The set of positive long (resp. short) roots $\Phi^+_l$, (resp. $\Phi^+_s$) is given as

$$
\Phi^+_l = \{ 2\varepsilon_i \mid 1 \leq i \leq n \},
$$

$$
\Phi^+_s = \{ \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq n \}.
$$

The root space decomposition of $n$ with respect to $T$ is given as

$$
n = \bigoplus_{\alpha \in \Phi^+} g_{\alpha},
$$

where $g_{\alpha} = \{ x \in n \mid s \cdot x = \alpha(s)x \text{ for any } s \in T \}$. Let $d\alpha \in \text{Hom}(t, k)$ be the differential of $\alpha \in \text{Hom}(T, k^*)$. Then $t$ acts on $g_{\alpha}$ by $[s, x] = d\alpha(s)x$ for $s \in t$. Since $\text{ch } k = 2$, the weight space decomposition of $n$ with respect to $t$ is given as

$$
n = \mathfrak{D} \oplus \bigoplus_{\alpha \in \Phi^+_l} g_{\alpha} = \mathfrak{D} \oplus n_{s},
$$

where $n_{s} = \bigoplus_{\alpha \in \Phi^+_s} g_{\alpha}$, and $\mathfrak{D} = \bigoplus_{\alpha \in \Phi^+_l} g_{\alpha}$ is the weight space of weight 0. Explicitly, $\mathfrak{D}$ is given as follows;

$$
(2.2.4) \quad \mathfrak{D} = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b : \text{diagonal} \right\}.
$$
In particular, we have

\[(2.2.5) \quad [t, D] = 0.\]

According to (2.2.4), we express an element in $D \cong k^n$ as $b = (b_1, \ldots, b_n) \in D$ for $b = \text{Diag}(b_1, \ldots, b_n)$. For $k = 0, \ldots, n$, put $D_k = \{b \in D \mid b_i = 0 \text{ for } i > k\}$. Then $D_k$ is a $T$-stable subspace of $D$. We also put $D_0^0 = \{b \in D_k \mid b_i \neq 0 \text{ for } 1 \leq i \leq k\}$, which is an open dense subset of $D_k$.

We need a lemma.

**Lemma 2.3.**

(i) Assume that $x \in h$ is semisimple. Then there exists $g \in H$ such that $gx \in t$.

(ii) Assume that $x \in b$ is semisimple. Then there exists $g \in B$ such that $gx \in t$.

**Proof.** Assume that $x \in h$ is semisimple. Consider the eigenspace decomposition $V = W_1 \oplus \cdots \oplus W_a$ of $x$. Then $W_1, \ldots, W_a$ are mutually orthogonal with respect to the form $\langle \cdot, \cdot \rangle$. Then the restriction of the form on $W_i$ gives a non-degenerate skew-symmetric bilinear form. In particular, $\dim W_i$ is even. We can find a basis $e'_1, \ldots, e'_n, f'_1, \ldots, f'_n$ of $V$ such that $\{e'_j, f'_j \mid j \in I_i\}$ gives a symplectic basis of $W_i$, where $[1, n] = \coprod_{1 \leq i \leq a} I_i$ is a partition of $[1, n]$. We define a map $g : V \to V$ by $g(e_j) = e'_j, g(f_j) = f'_j$ for each $j$. Then $g \in H$, and $x' = g^{-1}x \in t$. This proves (i).

Next assume that $x \in b$ is semisimple. Let $s \in t$ be the projection of $x \in b$. We consider the eigenspace decomposition of $s$ on $V$, $V = V_1 \oplus \cdots \oplus V_a$, where $V_i$ is the eigenspace of $s$ with respect to the eigenvalue $\mu_i$. This defines a partition $[1, n] = \coprod I_i$ such that $\{e_j, f_j \mid j \in I_i\}$ gives a symplectic basis of $V_i$. Since $x$ is semisimple, and has the same eigenvalues $\mu_1, \ldots, \mu_a$, we have a decomposition $V = V_1 \oplus \cdots \oplus V_a$ into eigenspaces of $x$, where $W_i$ is the eigenspace with respect to the eigenvalue $\mu_i$. Here $\dim W_i = \dim V_i$. As before, we can find a symplectic basis $e'_1, \ldots, e'_n, f'_1, \ldots, f'_n$ of $V$, and $g \in H$ associated to this basis such that $x' = g^{-1}x \in t$. We show that there exists a choice of $e'_1, \ldots, e'_n, f'_1, \ldots, f'_n$ such that $g \in B$, i.e., the choice such that the subspace spanned by $e_1, \ldots, e_k$ coincides with that by $e'_1, \ldots, e'_k$ for $k = 1, \ldots, n$. Since $e_1$ is an eigenvector for $x$, we can put $e'_1 = e_1$. We may assume that $e'_1 \in W_1$. Let $\overline{V} = \langle e_1 \rangle / \langle e_1 \rangle$. Then $\overline{V}$ has a natural symplectic basis $\overline{e}_j, \overline{f}_j (2 \leq j \leq n)$, and $x$ induces $\overline{x} \in \mathfrak{sp}(\overline{V})$. By induction on $n$, one can find the required basis $\overline{e}'_j, \overline{f}'_j (2 \leq j \leq n)$ of $\overline{V}$. This produces vectors $e'_1, \ldots, e'_n, f'_1, \ldots, f'_n$, and finally we can choose $f'_1 \in W_1$ by the condition that $\langle e'_1, f'_1 \rangle = 1$ and $f'_1$ is orthogonal for all other vectors $e'_j, f'_j$. Thus we obtain the basis $e'_1, \ldots, e'_n, f'_1, \ldots, f'_n$ as required, and $g \in B$. (ii) holds. The lemma is proved. \hfill \Box

### 2.4

We consider the varieties

\[
\tilde{X} = \{(x, gB) \in h \times H/B \mid g^{-1}x \in b\},
\]

\[X = \bigcup_{g \in H} g(b),\]
and define a map $\pi : \tilde{X} \to X$ by $(x, gB) \mapsto x$. $\pi$ is a proper map onto $X$, and so $X$ is irreducible, closed in $\mathfrak{h}$. (Later in Lemma 2.9, it will be shown that $X = \mathfrak{h}$.)

For $0 \leq k \leq n$, put $\mathfrak{N}_{k, \text{sr}} = \bigcup_{g \in B} g(t_{\text{sr}} + \mathfrak{D}_k)$. We define varieties

$$Y_k = \bigcup_{g \in H} g(\mathfrak{N}_{k, \text{sr}}) = \bigcup_{g \in H} g(t_{\text{sr}} + \mathfrak{D}_k),$$

and define a map $\psi^{(k)} : \tilde{Y}_k \to Y_k$ by $(x, gB) \to x$. In the case where $k = n$, we write $\tilde{Y}_k, Y_k$ and $\psi^{(k)}$ simply by $\tilde{Y}, Y$ and $\psi$. We have a lemma.

**Lemma 2.5.** $\tilde{Y} = \pi^{-1}(Y)$, and $\psi$ coincides with the restriction of $\pi$ on $\pi^{-1}(Y)$. In particular, $\psi : \tilde{Y} \to Y$ is a proper surjective map.

**Proof.** It is enough to show that $Y \cap \mathfrak{b} = \mathfrak{N}_{n, \text{sr}}$. Assume that $y \in Y \cap \mathfrak{b}$. Since $y \in Y$, there exists $g \in H, s \in t_{\text{sr}}, z \in \mathfrak{D}$ such that $y = g(s) + g(z) \in \mathfrak{b}$, where $g(s)$: semisimple, $g(z)$: nilpotent. Moreover since $[s, z] = 0$ by (2.2.5), we have $[g(s), g(z)] = 0$. By Lemma 2.3, replacing $y$ by its $B$-conjugate, we may assume that $g(s) \in t_{\text{sr}}, g(z) \in \mathfrak{n}$. Then there exists $\tilde{w} \in N_H(T)$ with $w \in S_n \subset W_n$ such that $g\tilde{w}(s) = s$. Since $\tilde{w}$ leaves $\mathfrak{D}$ invariant, by replacing $g$ by $g\tilde{w}$ and $z$ by $\tilde{w}^{-1}z$, we may further assume that $g(s) = s, g(z) \in \mathfrak{n}$ with $s \in t_{\text{sr}}, z \in \mathfrak{D}$. Hence $g \in Z_H(s) \simeq SL_2 \times \cdots \times SL_2$ (n-factors). If we write $g = (g_1, \ldots, g_n)$ with $g_i \in SL_2$, and $z = (z_1, \ldots, z_n) \in \mathfrak{D}$, the action of $g$ on $z \in \mathfrak{D}$ corresponds to the conjugate action of $g_i$ on the matrix $\begin{pmatrix} 0 & z_i \\ 0 & 0 \end{pmatrix}$ for $i = 1, \ldots, n$. Now the condition $g(z) \in \mathfrak{n}$ implies that, if $z_i \neq 0$ then $g_i$ is upper triangular. It follows that $g(z) \in \mathfrak{D}$, and so $y \in t_{\text{sr}} + \mathfrak{D}$, up to $B$-conjugate. We have $Y \cap \mathfrak{b} \subset \mathfrak{N}_{n, \text{sr}}$. The other inclusion is obvious. The lemma is proved. \hfill $\Box$

2.6. For $s \in t_{\text{sr}}$, we have $Z_H(s) = Z_H(t_{\text{sr}}) \simeq SL_2 \times \cdots \times SL_2$ (n-factors), and $B \cap Z_H(t_{\text{sr}}) \simeq B_2 \times \cdots \times B_2$, where $B_2$ is the Borel subgroup of $SL_2$ consisting of upper triangular matrices. The action of $Z_H(t_{\text{sr}})$ on $\mathfrak{D}$ is described as in the proof of Lemma 2.5. In particular, $B \cap Z_H(t_{\text{sr}})$ leaves $\mathfrak{D}_k$ invariant for each $k$. Then $\tilde{Y}_k$ can be expressed as

$$\tilde{Y}_k \simeq H \times^B \mathfrak{N}_{k, \text{sr}} \simeq H \times^{B \cap Z^h(t_{\text{sr}})} (t_{\text{sr}} + \mathfrak{D}_k). \tag{2.6.1}$$

Hence $\tilde{Y}_k$ is smooth and irreducible. $\tilde{Y} \simeq H \times^{B \cap Z^h(t_{\text{sr}})} (t_{\text{sr}} + \mathfrak{D})$ is a locally trivial fibration over $H/(B \cap Z^h(t_{\text{sr}}))$ with fibre isomorphic to $t_{\text{sr}} + \mathfrak{D}$, and $\tilde{Y}_k$ is a subbundle of $\tilde{Y}$ corresponding to a closed subset $t_{\text{sr}} + \mathfrak{D}_k$ of $t_{\text{sr}} + \mathfrak{D}$. Hence $\tilde{Y}_k$ is a closed subset of $\tilde{Y}$ for each $k$. The map $\psi^{(k)}$ is the restriction of $\psi$ on $\tilde{Y}_k$. Since $\psi : \tilde{Y} \to Y$ is proper, $Y_k = \psi(\tilde{Y}_k)$ is an irreducible closed subset in $Y$. $\psi^{(k)} : \tilde{Y}_k \to Y_k$ is a proper surjective map.
The following relation can be verified by a similar argument as in the proof of Lemma 2.5.

\[(2.6.2)\quad Y_k \cap (t_{sr} + \mathcal{D}) = \bigcup_{w \in S_n} \hat{w}(t_{sr} + \mathcal{D}_k), \quad (0 \leq k \leq n).\]

Put \(Y_k^0 = Y_k - Y_{k-1}\). By (2.6.2), we have

\[(2.6.3)\quad Y_k^0 = \bigcup_{g \in H} g(t_{sr} + \mathcal{D}_k^0).\]

### 2.7.

For any subset \(I \subset [1, n]\), put

\[(2.7.1)\quad \mathcal{D}_I = \{b \in \mathcal{D} \mid b_i \neq 0 \text{ for } i \in I, b_i = 0 \text{ for } i \notin I\}.\]

Note that if \(I = [1, k]\), \(\mathcal{D}_I\) coincides with \(\mathcal{D}_k^0\). Since the action of \(B \cap Z_H(t_{sr})\) on \(\mathcal{D}\) is given by the action of its \(T\)-part, \(t_{sr} + \mathcal{D}_I\) is \(B \cap Z_H(t_{sr})\)-stable. We define a locally closed subvariety \(\overline{Y}_I\) of \(\overline{Y}\) by

\[(2.7.2)\quad \overline{Y}_I \simeq H \times_{B \cap Z_H(t_{sr})} (t_{sr} + \mathcal{D}_I),\]

and a map \(\psi_I : \overline{Y}_I \to Y\) by \(g \cdot x \mapsto gx\), where \(g \cdot x\) is the image of \((g, x) \in H \times (t_{sr} + \mathcal{D}_I)\) on its quotient. Then \(\text{Im } \psi_I = \bigcup_{g \in H} (t_{sr} + \mathcal{D}_I)\) coincides with \(Y_k^0\) for \(k = |I|\) by (2.6.3), which depends only on \(k\).

For \(I \subset [1, n]\), we define a parabolic subgroup \(Z_H(t_{sr})_I\) of \(Z_H(t_{sr})\) by the condition that the \(i\)-th factor is \(B_2\) if \(i \in I\), and is \(SL_2\) otherwise. Since \(Z_H(t_{sr})_I\) stabilizes \(\mathcal{D}_I\), one can define

\[(2.7.3)\quad \tilde{Y}_I = H \times_{Z_H(t_{sr})_I} (t_{sr} + \mathcal{D}_I).\]

Then \(\psi_I\) factors through \(\tilde{Y}_I\),

\[(2.7.4)\quad \psi_I : \overline{Y}_I \xrightarrow{\xi_I} \tilde{Y}_I \xrightarrow{\eta_I} Y_k^0,\]

for \(|I| = k\), where \(\xi_I\) is the natural surjection, and \(\eta_I\) is given by \(g \cdot x \mapsto gx\) (similar notation as \(\psi_I\)). Then \(\xi_I\) is a locally trivial fibration with fibre isomorphic to

\[Z_H(t_{sr})_I/(B \cap Z_H(t_{sr})) \simeq (SL_2/B_2)^I' \simeq P_1',\]

where \(I'\) is the complement of \(I\) in \([1, n]\).

Let \(S_I\) be the symmetric group of letters in \(I \subset [1, n]\), hence \(S_I \simeq S_k\) for \(|I| = k\). Then \(\mathcal{W}_I = N_H(Z_H(t_{sr})_I)/Z_H(t_{sr})_I \simeq S_I \times S_{I'}\). \(\mathcal{W}_I\) acts on \(\overline{Y}_I\) and \(\tilde{Y}_I\) since \(t_{sr} + \mathcal{D}_I\) is stable by \(N_H(Z_H(t_{sr})_I)\). Now the map \(\eta_I : \tilde{Y}_I \to Y_k^0\) turns out to be a finite Galois
covering with Galois group \( \mathcal{W}_I \),

\[
\eta_I : \tilde{Y}_I \rightarrow \tilde{Y}_I/\mathcal{W}_I \simeq Y_k^0.
\]

2.8. For \( 0 \leq k \leq n \), we define \( \tilde{Y}_k^+ \) as \( \psi^{-1}(Y_k^0) \). Then \( \tilde{Y}_k^+ = \coprod_I \tilde{Y}_I \), where \( I \) runs over all the subsets \( I \subset [1, n] \) such that \( |I| = k \). (The disjointness follows from (2.6.2)). \( \tilde{Y}_I \) is smooth, irreducible by (2.7.2), and \( \tilde{Y}_I \) form the connected components of \( \tilde{Y}_k^+ \). Since \( Y = \coprod_{0 \leq k \leq n} Y_k^0 \), we have

\[
\tilde{Y} = \coprod_{0 \leq k \leq n} \tilde{Y}_k^+.
\]

In the case where \( I = [1, k] \), we denote \( \tilde{Y}_I \) by \( \tilde{Y}_k^0 \). Then \( \tilde{Y}_k^0 \) is an open dense subset of \( \tilde{Y}_k \). By (2.6.1), \( S_n \simeq N_H(Z_H(t_{nr}))/Z_H(t_{sr}) \) acts on \( \tilde{Y} \), which leaves \( \tilde{Y}_k^+ \) stable for any \( k \). We have

\[
\tilde{Y}_k^+ = \coprod_{I \subset [1, n]} \tilde{Y}_I = \coprod_{w \in S_n/(S_k \times S_{n-k})} w(\tilde{Y}_k^0).
\]

We have the following lemma.

**Lemma 2.9.** Let the notations be as before.

(i) \( X = \bigcup_{g \in H} g(b) = \mathfrak{h} \).

(ii) \( Y_k \) is an irreducible closed subset in \( Y \). Hence \( Y_k^0 \) is open dense in \( Y_k \).

(iii) \( \dim \tilde{Y}_k = \dim H - n + k \).

(iv) \( \dim Y_k = \dim \tilde{Y}_k - (n - k) = (\dim H - 2n) + 2k \).

(v) \( Y = \coprod_{0 \leq k \leq n} Y_k^0 \) gives a stratification of \( Y \) by smooth strata \( Y_k^0 \), and the map \( \psi : \tilde{Y} \rightarrow Y \) is semismall with respect to this stratification.

**Proof.** (ii) is already given in 2.6. By (2.6.1),

\[
\dim \tilde{Y}_k = \dim H - \dim(B \cap Z_H(t_{nr})) + \dim(t_{sr} + \mathfrak{D}_k)
\]

\[
= \dim H - 2n + (n + k)
\]

\[
= \dim H - n + k,
\]

since \( \dim(B \cap Z_H(t_{nr})) = \dim(B_2 \times \cdots \times B_2) = 2n \). Thus (iii) holds. Since \( \tilde{Y}_I \) is smooth, irreducible, and \( \eta_I \) is a finite Galois covering, \( Y_k^0 = \eta_I(\tilde{Y}_I) \) is smooth, irreducible. By using the decomposition \( \psi_I = \eta_I \circ \xi_I \) for \( I = [1, k] \), we see that \( \dim \tilde{Y}_k = \dim Y_k + (n - k) \). Hence (iv) holds. It follows from (iv), \( \dim Y = \dim Y_n = \dim H \). Since \( \dim \psi^{-1}(x) = n - k \) for any \( x \in Y_k^0 \) by (2.7.1) and by \( \psi_I = \eta_I \circ \xi_I \), we have \( \dim \psi^{-1}(x) = (\dim Y - \dim Y_k^0)/2 \) by (iv). Thus (v) holds. Since \( Y_k^0 \) is open dense in \( X \), \( \dim X = \dim H \). Since \( X \) is irreducible closed in \( \mathfrak{h} \), we obtain (i). \( \square \)
2.10. Let \( \psi_k : \tilde{\mathcal{Y}}^+_k \to Y_k^0 \) be the restriction of \( \psi \) on \( \psi^{-1}(Y_k^0) \). Since \( \psi \) is proper, \( \psi_k \) is also proper. Let \( \overline{Q}_I \) be the constant sheaf on \( \tilde{\mathcal{Y}}^+_k \). Since \( \tilde{\mathcal{Y}}_I \) is a connected component for any \( I \) by (2.8.1), we have

\[
(2.10.1) \quad (\psi_k)_* \overline{Q}_I \cong \bigoplus_{I \subseteq [1,n]} (\psi_I)_! \overline{Q}_I.
\]

On the other hand, since \( \eta_I : \tilde{\mathcal{Y}}_I \to Y_k^0 \) is a finite Galois covering with group \( \mathcal{U}_I \), we have

\[
(2.10.2) \quad (\eta_I)_* \overline{Q}_I \cong \bigoplus_{\rho \in (\mathcal{U}_I)^\wedge} \rho \otimes \mathcal{L}_\rho,
\]

where \( \mathcal{L}_\rho = \text{Hom}(\rho, (\eta_I)_! \overline{Q}_I) \) is a simple local system on \( Y_k^0 \) corresponding to \( \rho \in (\mathcal{U}_I)^\wedge \).

Since \( \psi_k \) is proper, and \( \tilde{\mathcal{Y}}_I \) is a closed subset of \( \tilde{\mathcal{Y}}^+_k \), \( \psi_I \) is proper. As \( \psi_I = \eta_I \circ \xi_I \), \( \xi_I \) is also proper. By a similar discussion as in [Sh, (1.6.1)], we see that \( R^i(\xi_I)_! \overline{Q}_I \) is a constant sheaf for any \( i \geq 0 \). Since \( \xi_I \) is a \( P'_1 \) bundle, we have

\[
(2.10.3) \quad (\xi_I)_! \overline{Q}_I \cong (\xi_I)_!(\xi_I)^* \overline{Q}_I \cong H^\bullet(P'_1) \otimes \overline{Q}_I,
\]

where \( H^\bullet(P'_1) \) denotes \( \bigoplus_{i \geq 0} H^{2i}(P'_1, \overline{Q}_I) \), which we regard as a complex of vector spaces \( (K_i) \) with \( K_{odd} = 0 \). It follows that

\[
(2.10.4) \quad (\psi_I)_! \overline{Q}_I \cong (\eta_I)_!(\xi_I)_! \overline{Q}_I \cong H^\bullet(P'_1) \otimes (\eta_I)_! \overline{Q}_I.
\]

Note that \( P_1 \) is the flag variety of \( SL_2 \), and \( Z/2Z \) is the Weyl group of \( SL_2 \). We define an action of \( Z/2Z \) on \( H^\bullet(P_1) \) as the Springer representation of \( Z/2Z \), i.e., \( Z/2Z \) acts non-trivially on \( H^2(P_1) = \overline{Q}_I \), and acts trivially on \( H^0(P_1) = \overline{Q}_I \). We define an action of \( (Z/2Z)^{[1,n]} \) on \( H^\bullet(P'_1) \) by the Springer action of the factor \( Z/2Z \) corresponding to \( I' \), and the trivial action of the factor \( Z/2Z \) corresponding to \( I \). Note that \( W_n = S_n \times (Z/2Z)^n \) and \( \mathcal{U}_I \cong S_I \times S_{I'} \). Let \( W_I = S_I \times (Z/2Z)^I \) be the Weyl group of type \( C_{|I|} \), and define \( W_{I'} \) similarly. For \( \rho \in (\mathcal{U}_I)^\wedge = (S_I \times S_{I'})^\wedge \), we consider the action of \( W_I \times W_{I'} \) on \( H^\bullet(P'_1) \otimes \rho \), where \( (Z/2Z)^{[1,n]} \) acts trivially on \( \rho \). In particular, for \( I = [1,k] \), we obtain \( (W_k \times W_{n-k}) \) acting on \( \rho \). Note that \( S_n / (S_k \times S_{n-k}) \cong W_n / (W_k \times W_{n-k}) \). By (2.8.1), (2.10.1) and (2.10.4), we have

\[
(2.10.5) \quad (\psi_k)_! \overline{Q}_I \cong \bigoplus_{\rho \in (S_k \times S_{n-k})^\wedge} \text{Ind}^{W_n}_{W_k \times W_{n-k}} (H^\bullet(P_1^{n-k}) \otimes \rho) \otimes \mathcal{L}_\rho,
\]

where \( \mathcal{L}_\rho = \text{Hom}(\rho, (\eta_I)_! \overline{Q}_I) \) is the simple local system on \( Y_k^0 \) with \( I = [1,k] \).
2.11. For each $k$, let $\bar{\psi}_k$ be the restriction of $\psi$ on $\psi^{-1}(Y_k)$. Then $\bar{\psi}_k : \psi^{-1}(Y_k) \to Y_k$ is a proper map. Assume that $k \geq 1$. We have $Y_k - Y_{k-1} = Y^0_k$. Since $\bar{\psi}_k$ is proper, $(\bar{\psi}_k)_{!} \hat{Q}_l$ is a semisimple complex on $Y_k$. We note the following.

(2.11.1) Assume that $(\bar{\psi}_{k-1})_{!} \hat{Q}_l$ is equipped with an action of $W_n$. Then the $W_n$-action can be extended to a $W_n$-action on $(\bar{\psi}_k)_{!} \hat{Q}_l$.

In fact, let $j : Y^0_k \hookrightarrow Y_k$ be the open immersion. By (2.10.5), $(\psi_k)_{!} \hat{Q}_l$ has an action of $W_n$. This induces an action of $W_n$ on $(j \circ \psi_k)_{!} \hat{Q}_l$, and on its perverse cohomology $pH^i((j \circ \psi_k)_{!} \hat{Q}_l)$. On the other hand, by the assumption, $pH^i((\bar{\psi}_{k-1})_{!} \hat{Q}_l)$ is equipped with $W_n$-action. We consider the long exact sequence of the perverse cohomology obtained from the distinguished triangle $(j_{!}(\psi_k)_{!} \hat{Q}_l, (\bar{\psi}_k)_{!} \hat{Q}_l, (\bar{\psi}_{k-1})_{!} \hat{Q}_l)$. By (2.10.5), $(\psi_k)_{!} \hat{Q}_l$ is a semisimple complex which is a sum of various $L_{\rho}[2i]$. Hence $pH^i((j \circ \psi_k)_{!} \hat{Q}_l) = 0$ for odd $i$. By induction, we have $pH^i((\bar{\psi}_k)_{!} \hat{Q}_l) = 0$ for odd $i$. Since $(\bar{\psi}_k)_{!} \hat{Q}_l$ is a semisimple complex, the $W_n$-action of $pH^i((\bar{\psi}_{k-1})_{!} \hat{Q}_l)$ and on $pH^i((j \circ \psi_k)_{!} \hat{Q}_l)$ determines the $W_n$-action on $pH^i((\bar{\psi}_k)_{!} \hat{Q}_l)$, uniquely. (2.11.1) is proved.

2.12. We have a natural bijection

\[
\prod_{0 \leq k \leq n} (S_k \times S_{n-k})^\wedge \simeq W_n^\wedge, \quad \rho \longmapsto \widehat{\rho}
\]

satisfying the following properties; take $\rho = \rho' \boxtimes \rho'' \in (S_k \times S_{n-k})^\wedge$, where $\rho' \in S_k^\wedge, \rho'' \in S_{n-k}^\wedge$. We extend $\rho'$ to $\widehat{\rho}' \in W_n^\wedge$ so that $(\mathbb{Z}/2\mathbb{Z})^k$ acts trivially on it. On the other hand, we extend $\rho''$ to $\widehat{\rho}'' \in W_{n-k}^\wedge$ so that each factor $\mathbb{Z}/2\mathbb{Z}$ of $(\mathbb{Z}/2\mathbb{Z})^{n-k}$ acts non-trivially on $\widehat{\rho}''$. Put

\[
\widehat{\rho} = \text{Ind}_{W_k \times W_{n-k}}^{W_n} (\rho' \boxtimes \rho'').
\]

Then $\widehat{\rho} \in W_n^\wedge$, and the correspondence $\rho \mapsto \widehat{\rho}$ gives the required bijection.

We show the following proposition. Put $d_k = \dim Y_k$.

Proposition 2.13. $\psi_l \hat{Q}_l[d_n]$ is a semisimple perverse sheaf on $Y$, equipped with $W_n$-action, and is decomposed as

\[
\psi_l \hat{Q}_l[d_n] \simeq \bigoplus_{0 \leq k \leq n} \bigoplus_{\rho \in (S_k \times S_{n-k})^\wedge} \widehat{\rho} \otimes IC(Y_k, \mathcal{L}_\rho)[d_k],
\]

where $\mathcal{L}_\rho$ is a simple local system on $Y^0_k$ defined in (2.10.2).

Proof. The formula (2.10.5) can be rewritten as

\[
(\psi_k)_{!} \hat{Q}_l \simeq \left( \bigoplus_{\rho \in (S_k \times S_{n-k})^\wedge} \widehat{\rho} \otimes \mathcal{L}_\rho \right)[-2(n-k)] + \mathcal{M}_k,
\]

where $\mathcal{M}_k$ is a sum of various $\mathcal{L}_{\rho'}[-2i]$ for $\rho' \in (S_k \times S_{n-k})^\wedge$ with $0 \leq i < n-k$.

For $0 \leq m \leq n$, let $\bar{\psi}_m$ be as in 2.10. We consider the following formula.
(2.13.3) \((\psi_m)_t \mathcal{Q}_t \simeq \bigoplus_{0 \leq k \leq m} \bigoplus_{\rho \in (S_k \times S_{n-k})^\wedge} \hat{\rho} \otimes \text{IC}(Y_k, \mathcal{L}_{\rho})[-2(n - k)] + \mathcal{M}_m,\)

where \(\mathcal{M}_m\) is a \(\mathbb{Z}\)-linear combination of various \(\text{IC}(Y_k, \mathcal{L}_{\rho})[-2i]\) for \(0 \leq k \leq m\) and \(\rho' \in (S_k \times S_{n-k})^\wedge\) with \(i < n - k\). We note that (2.13.3) will imply the proposition. In fact, \(\psi_n = \psi\) for \(k = n\), and \(d_n - d_k = 2n - 2k\) by Lemma 2.9. But since \(d_n - 2i > d_k\) if \(i < n - k\), \(\text{IC}(Y_k, \mathcal{L}_{\rho})[d_n - 2i]\) is not a perverse sheaf. Since \(\psi\) is semismall, \(\psi \mathcal{Q}_t\) is a semisimple perverse sheaf. Thus \(\mathcal{M}_n = 0\) and (2.13.1) follows.

We show (2.13.3) by induction on \(m\). If \(m = 0\), then \((\psi_m)_t \mathcal{Q}_t\) coincides with \((\psi_m)_t \mathcal{Q}_t\). Hence (2.13.3) holds by (2.13.2) applied for \(k = 0\). We assume that (2.13.3) holds for any \(k < m\). Recall that \(Y_m^0 = Y_m - Y_{m-1}\). Since \((\psi_m)_t \mathcal{Q}_t\) is a semisimple complex, it is a direct sum of the form \(A[s]\) for a simple perverse sheaf \(A\). Suppose that the support \(\text{supp} A\) of \(A\) is not contained in \(Y_{m-1}\). Then \((\text{supp} A) \cap Y_m^0 \neq \emptyset\), and \(A|_{Y_m^0}\) is a simple perverse sheaf on \(Y_m^0\). The restriction of \((\psi_m)_t \mathcal{Q}_t\) on \(Y_m^0\) is isomorphic to \((\psi_m)_t \mathcal{Q}_t\), and it is described as in (2.13.2). It follows that \(A|_{Y_m^0} = \mathcal{L}_{\rho}\) (up to shift) for some \(\rho \in (S_m \times S_{n-m})^\wedge\). This implies that \(A = \text{IC}(Y_m, \mathcal{L}_{\rho})[d_m]\), and the direct sum of \(A[s]\) appearing in \((\psi_m)_t \mathcal{Q}_t\) such that \((\text{supp} A) \cap Y_m^0 \neq \emptyset\) is given by

\[
K_1 = \bigoplus_{\rho \in (S_m \times S_{n-m})^\wedge} \hat{\rho} \otimes \text{IC}(Y_m, \mathcal{L}_{\rho})[-2(n - m)] + \mathcal{M}_m',
\]

where \(\mathcal{M}_m'\) is a \(\mathbb{Z}\)-linear combination of \(\text{IC}(Y_m, \mathcal{L}_{\rho'})[-2i]\) for \(\rho' \in (S_m \times S_{n-m})^\wedge\) with \(0 \leq i < n - m\).

If \(\text{supp} A\) is contained in \(Y_{m-1}\), \(A[s]\) appears as a summand of \((\psi_{m-1})_t \mathcal{Q}_t\). By induction, \((\psi_{m-1})_t \mathcal{Q}_t\) is described as in (2.13.3) by replacing \(m\) by \(m - 1\). Thus if we exclude the contribution from the restriction of \(K_1\) on \(Y_{m-1}\), such \(A[s]\) is determined from \((\psi_{m-1})_t \mathcal{Q}_t\). Note that, by induction, we can construct an action of \(W_n\) on \((\psi_m)_t \mathcal{Q}_t\) by (2.11.1). We consider the restriction of \(K_1\) on \(Y_{m-1}\). Since each simple component of \(\mathcal{M}_m'\) is contained in \(\mathcal{M}_m\), we can ignore this part. Let \(K_1'\) be the direct sum part of \(K_1\). Then the restriction of \(K_1'\) on \(Y_{m-1}\) affords the representation of \(W_n\) corresponding to a sum of various \(\hat{\rho}\) for \(\rho \in (S_m \times S_{n-m})^\wedge\). But by (2.13.3) applied for \(m - 1\), the direct sum part of \((\psi_{m-1})_t \mathcal{Q}_t\) affords the representation of \(W_n\). The irreducible representations appearing there is of the form \(\hat{\rho}\), which are different from \(\hat{\rho}\) for \(K_1'|_{Y_{m-1}}\). Since each component of \(\mathcal{M}_{m-1}\) is contained in \(\mathcal{M}_m\), we see that the restriction of \(K_1\) on \(Y_{m-1}\) has no overlapping with \((\psi_{m-1})_t \mathcal{Q}_t\) modulo \(\mathcal{M}_m\). Thus (2.13.3) holds for \(m\). The proposition is proved. \(\square\)

3. Intersection cohomology on \(csp(V)\)

3.1. We follow the notation in Section 2. The conformal symplectic group \(CSp_N\) is defined by

\[
CSp_N = \{ g \in GL_N \mid g^\ast Jg = \lambda_g J \text{ for some } \lambda_g \in k^\ast \},
\]
which we denote by $\widetilde{H}$. By fixing the basis of $V$ as before, we also write it as $\widetilde{H} = CSp(V)$. $\widetilde{H}$ is a connected group with connected center $\widetilde{Z}$, where

$$\widetilde{Z} = \{\lambda 1_N \mid \lambda \in k^*\},$$

and contains $H$ as a closed subgroup. $\widetilde{H}/\widetilde{Z}$ is the adjoint symplectic group $\widetilde{H}_{ad}$. Let $\mathfrak{h} = \text{cap}(V)$ be the Lie algebra of $\widetilde{H}$. $\mathfrak{h}$ contains the center $\mathfrak{j} \simeq k$, and we put $\mathfrak{h}_{ad} = \mathfrak{h}/\mathfrak{j}$, which is the Lie algebra of $\widetilde{H}_{ad}$, and is called the adjoint Lie algebra. In [X1, Lemma 6.2], Xue proved that $\mathfrak{h}_{ad}$ has regular semisimple elements, and established the Springer correspondence for $\mathfrak{h}_{nil}$ by making use of the intersection cohomology on $\mathfrak{h}_{ad}$. Considering $\mathfrak{h}_{ad}$ is essentially the same as considering $\mathfrak{h}$. In this section, we shall connect Xue’s result with ours discussed in Section 2.

\[3.2.\] Put

\[
\hat{T} = \{\text{Diag}(t_1, \ldots, t_n, \lambda t_1^{-1}, \ldots, \lambda t_n^{-1}) \mid t_i \in k^*, \lambda \in k^*\},
\]

\[
\hat{t} = \{\text{Diag}(t_1, \ldots, t_n, t_1 + \lambda, \ldots, t_n + \lambda) \mid t_i \in k, \lambda \in k\}.
\]

Then $\hat{T}$ is a maximal torus of $\tilde{H}$ containing $T$, and $\text{Lie} \hat{T} = \hat{t} \supset \mathfrak{t}$. We denote an element in $\hat{t}$ as $\hat{\xi} = (s, \lambda) = (t_1, \ldots, t_n, \lambda t_1^{-1}, \ldots, \lambda t_n^{-1})$ for $s = (t_1, \ldots, t_n) \in k^n$ and $\lambda \in k^*$. Let $\hat{B} = \hat{T}U$ be the Borel subgroup of $\hat{H}$ containing $\hat{B}$. Put $\hat{b} = \text{Lie} \hat{B}$. We have $\hat{b} = \hat{t} \oplus \mathfrak{n}$. Put

\[
\hat{t}_{\text{reg}} = \{(s, \lambda) \in \hat{t} \mid t_i \neq t_j \text{ for } i \neq j, \lambda \in k^*\}.
\]

Then for $\hat{\xi} \in \hat{t}_{\text{reg}}$, we have $Z_{\hat{H}}(\hat{\xi}) = \hat{T}$. Thus $\hat{\xi}$ is a regular semisimple element in $\hat{t}$. $\hat{t}_{\text{reg}}$ is open dense in $\hat{t}$. Put $\hat{h}_{\text{reg}} = \bigcup_{g \in \hat{H}} g(\hat{t}_{\text{reg}})$. By using the conjugacy of maximal tori in $\hat{H}$, we see that $\hat{h}_{\text{reg}}$ coincides with the set of regular semisimple elements in $\hat{h}$. $\hat{h}_{\text{reg}}$ is open dense in $\hat{h}$ since it is the intersection with regular semisimple elements in $\mathfrak{g}l_N$. Put $\hat{b}_{\text{reg}} = \hat{h}_{\text{reg}} \cap \hat{b}$. We consider the varieties

\[
\check{Y}^b = \{(x, g \hat{B}) \in \hat{h} \times \hat{H}/\hat{B} \mid g^{-1}x \in \hat{b}_{\text{reg}}\},
\]

\[
Y^b = \bigcup_{g \in \hat{H}} g(\hat{b}_{\text{reg}}) = \hat{h}_{\text{reg}},
\]

and define a map $\psi^b : \check{Y}^b \to Y^b$ by $(x, g \hat{B}) \mapsto x$. Then

\[
\check{Y}^b \simeq \hat{H} \times \hat{b}_{\text{reg}} \simeq \hat{H} \times \hat{t}_{\text{reg}},
\]

and $\psi^b$ is a finite Galois covering with Galois group $W_n$. 
Let $\tilde{\mathcal{Q}}_l$ be the constant sheaf on $\tilde{Y}^\phi$. Then $(\psi^\phi)_!\tilde{\mathcal{Q}}_l$ is a semisimple local system on $Y^\phi$, equipped with $W_n$-action, and is decomposed as

\[(3.2.3) \quad (\psi^\phi)_!\tilde{\mathcal{Q}}_l \simeq \bigoplus_{\rho \in W_n^\wedge} \rho \otimes \mathcal{L}_\rho^\phi,
\]

where $\mathcal{L}_\rho^\phi = \text{Hom}(\rho, (\psi^\phi)_!\tilde{\mathcal{Q}}_l)$ is a simple local system on $Y^\phi$.

### 3.3

We consider the varieties

$$X^\phi = \{(x, g\tilde{B}) \in \tilde{h} \times \tilde{\mathcal{H}}/\tilde{B} \mid g^{-1}x \in \tilde{b}\},$$

and define a map $\pi^\phi: \tilde{X}^\phi \to X^\phi$ by $(x, g\tilde{B}) \mapsto x$. $\pi^\phi$ is a proper map onto $X^\phi$. Hence $X^\phi$ is irreducible and closed in $\tilde{h}$. Since $\tilde{h}_{\text{reg}} \subset X^\phi$, we have $X^\phi = \tilde{h}$.

We consider the complex $K = (\pi^\phi)_!\tilde{\mathcal{Q}}_l$ on $X^\phi = \tilde{h}$. We can define a similar map $\pi^\phi_{\text{ad}}: \tilde{X}^\phi_{\text{ad}} \to X^\phi_{\text{ad}} = \tilde{h}_{\text{ad}}$, by replacing $\tilde{X}^\phi, X^\phi, \pi^\phi$ by $\tilde{X}^\phi_{\text{ad}}, X^\phi_{\text{ad}}, \pi^\phi_{\text{ad}}$, respectively. We consider the complex $K_{\text{ad}} = (\pi^\phi_{\text{ad}})_!\tilde{\mathcal{Q}}_l$ on $\tilde{h}_{\text{ad}}$. Let $\phi: \tilde{h} \to \tilde{h}_{\text{ad}}$ be the natural projection. By the base change theorem, we have $\phi^*K_{\text{ad}} \simeq K$. It is known by [X1, Prop. 6.6] that $K_{\text{ad}}$ coincides with the intersection cohomology $\text{IC}(\tilde{h}_{\text{ad}}, \mathcal{L}_{\text{ad}})$ for a certain semisimple local system $\mathcal{L}_{\text{ad}}$ on the set of regular semisimple elements in $\tilde{h}_{\text{ad}}$. Since $\phi$ is smooth with connected fibre, $K$ is also expressed by an intersection cohomology on $\tilde{h}$. Since $K|_{Y^\phi} \simeq (\psi^\phi)_!\tilde{\mathcal{Q}}_l$, we have $K \simeq \text{IC}(\tilde{h}, (\psi^\phi)_!\tilde{\mathcal{Q}}_l)$. Thus by (3.2.3), the following result holds.

**Proposition 3.4.** $(\psi^\phi)_!\tilde{\mathcal{Q}}_l[\text{dim} \tilde{h}]$ is a semisimple perverse sheaf on $\tilde{h}$, equipped with $W_n$-action, and is decomposed as

\[(3.4.1) \quad (\psi^\phi)_!\tilde{\mathcal{Q}}_l[\text{dim} \tilde{h}] \simeq \bigoplus_{\rho \in W_n^\wedge} \rho \otimes \text{IC}(\tilde{h}, \mathcal{L}_\rho^\phi)[\text{dim} \tilde{h}],
\]

where $\mathcal{L}_\rho^\phi$ is a simple local system on $\tilde{h}_{\text{reg}}$ given in (3.2.3).

### 3.5

The set of nilpotent elements $\tilde{h}_{\text{nil}}$ in $\tilde{h}$ coincides with $h_{\text{nil}}$. The subvariety $(\pi^\phi)^{-1}(h_{\text{nil}})$ of $\tilde{X}^\phi$ can be identified with

\[\tilde{X}_{\text{nil}} = \{(x, gB) \in \tilde{h} \times G/B \mid g^{-1}x \in n\},\]

and the restriction of $\pi^\phi$ on $\tilde{X}_{\text{nil}}$ coincides with the map $\pi_1: \tilde{X}_{\text{nil}} \to h_{\text{nil}}$. Note that $\pi_1$ is surjective since $\bigcup_{g \in H} g(n) = h_{\text{nil}}$ by Lemma 2.9 (i). Since $(\pi^\phi)_!\tilde{\mathcal{Q}}_l$ has a natural action of $W_n$ by Proposition 3.4, $(\pi_1)_!\tilde{\mathcal{Q}}_l$ has also an action of $W_n$. The following result gives the Springer correspondence for $\mathfrak{sp}(V)$, which is essentially due to Xue [X1].
Theorem 3.6. (i) $(\pi_1)_! \mathcal{Q}_t[\dim \mathfrak{h}_{\text{nil}}]$ is a semisimple perverse sheaf on $\mathfrak{h}_{\text{nil}}$, equipped with $W_n$-action, and is decomposed as

$$(\pi_1)_! \mathcal{Q}_t[\dim \mathfrak{h}_{\text{nil}}] \simeq \bigoplus_{\rho \in W_n^\wedge} \rho \otimes \text{IC}(\mathcal{O}_\rho, \mathcal{Q}_t)[\dim \mathcal{O}_\rho],$$

where $\mathcal{O}_\rho$ is an $H$-orbit in $\mathfrak{h}_{\text{nil}}$, and the map $\rho \mapsto \mathcal{O}_\rho$ gives a bijective correspondence between $W_n^\wedge$ and the set of $H$-orbits in $\mathfrak{h}_{\text{nil}}$.

(ii) For each $\rho \in W_n^\wedge$, we have

$$\text{IC}(\widetilde{\mathfrak{h}}, \mathcal{L}_\rho^b)|_{\mathfrak{h}_{\text{nil}}} \simeq \text{IC}(\mathcal{O}_\rho, \mathcal{Q}_t), \quad (\text{up to shift}).$$

In fact, Xue proved in [X1, Prop. 6.4] the corresponding formula for $(\widetilde{\mathfrak{h}}_{\text{ad}})_{\text{nil}}$, the nilpotent variety of $\widetilde{\mathfrak{h}}_{\text{ad}}$. Since $\phi^{-1}((\widetilde{\mathfrak{h}}_{\text{ad}})_{\text{nil}}) \simeq \mathcal{J} \times \widetilde{\mathfrak{h}}_{\text{nil}}$, his result can be translated to the formula in $\mathfrak{h}_{\text{nil}}$, which induces our formula for $\mathfrak{h}_{\text{nil}}$. Note that $\phi$ gives a bijective map $\mathfrak{h}_{\text{nil}} \to (\widetilde{\mathfrak{h}}_{\text{ad}})_{\text{nil}}$, compatible with the action of $\widetilde{H}$ and $\widetilde{\mathfrak{h}}_{\text{ad}}$. Also note that it is known by [Spa] that for any $x \in \mathfrak{h}_{\text{nil}}$, $Z_H(x)$ is connected. Hence only the constant sheaf $\mathcal{Q}_t$ appears as a local system on $\mathcal{O}_\rho$ in the Springer correspondence. The explicit correspondence was described in [X2] by making use of (a generalization of) Lusztig's symbols.

3.7. Let $\mathcal{B} = \tilde{H}/\tilde{B}$ be the flag variety of $\tilde{H}$, and $W = N_{\tilde{B}}(\tilde{T})/\tilde{T} \simeq W_n$ be the Weyl group of $\tilde{H}$. For each $x \in \mathfrak{h}$, put $\mathcal{B}_x = \{ gB \in \mathcal{B} \mid g^{-1}x \in \mathfrak{b} \}$. We consider the structure of $\mathcal{B}_x$. Let $x = s + z$ be the Jordan decomposition of $x$ in $\mathfrak{h}$, where $s$: semisimple and $z$: nilpotent such that $[s, z] = 0$. We assume that $s \in \mathfrak{t}$. Put $C = Z^0_B(s)$ and $\mathfrak{c} = \text{Lie } C$. Then $z \in \mathfrak{c}_{\text{nil}}$. $B_C = B \cap C$ is a Borel subgroup of $C$ containing $\tilde{T}$, and we consider the flag variety $\mathcal{B}^C = C/\tilde{B}_C$, and its subvariety $\mathcal{B}_z^C$. For each $w \in W$, $\tilde{B}_{C,w} = w\tilde{B}w^{-1} \cap C$ is a Borel subgroup of $C$ containing $\tilde{T}$, and one can consider the flag variety $\mathcal{B}_z^{C,w} = C/\tilde{B}_{C,w}$ of $C$. Let $W_s$ be the stabilizer of $s$ in $W$. Then $W_s$ is the Weyl group of $C$. Put

$$\tilde{\mathcal{M}} = \{ g \in \tilde{H} \mid g^{-1}s \in \mathfrak{b} \},$$

$$\mathcal{M} = \{ g \in \tilde{H} \mid g^{-1}s \in \mathfrak{t} \}.$$

Then $C \times \tilde{B}$ acts on $\tilde{\mathcal{M}}$, by $(h, b) \cdot y = hyb^{-1}$, and similarly $C \times \tilde{T}$ acts on $\mathcal{M}$. Put $\Gamma = C \setminus \tilde{H}/\tilde{T}$, $\hat{\Gamma} = C \setminus \tilde{H}/B$. The natural map $\Gamma \to \hat{\Gamma}$ gives a bijection $\Gamma \simeq \hat{\Gamma}$, and we can identify $\Gamma$ with a set of representatives in $W$ of the cosets $W_s \setminus W$. This implies that $\bigsqcup_{w \in \Gamma} \mathcal{B}_z^{C,w} \simeq B_s$ by $h(\tilde{B}_{C,w}) \mapsto hwB$, and we have

$$\mathcal{B}_x \simeq \prod_{w \in \Gamma} \mathcal{B}_z^{C,w} \simeq \prod_{w \in W_s \setminus W} \mathcal{B}_z^{C}.$$
Recall that \( K = (\pi^\flat)_{*} \mathcal{Q} \) is a complex with \( W \)-action by Proposition 3.4. Hence for any \( x \in \tilde{B} \), \( H_{x}^{i} K \simeq H_{x}^{i}(\mathcal{B}_{x}, \mathcal{Q}_{t}) \) has a structure of \( W \)-module (Springer representation of \( W \)). For \( C \), we can also consider \( \pi_{C} : C \times \tilde{B}_{C} B_{C} \rightarrow c \), similarly to \( \pi^{\flat} \), where \( \tilde{b}_{C} = \text{Lie} B_{C} \). Since \( c \) has regular semisimple elements, the previous discussion can be applied, and \( (\pi_{C_{x}})_{*} \mathcal{Q}_{t} \) turns out to be a complex with \( W_{s} \)-action. It follows that \( H_{x}^{i}(\mathcal{B}_{x}, \mathcal{Q}_{t}) \) is a \( W_{s} \)-module (Springer representation of \( W_{s} \)). Then (3.7.1) can be interpreted by Springer representations as follows.

**Theorem 3.8.** \( H_{x}^{i}(\mathcal{B}_{x}, \mathcal{Q}_{t}) \simeq \text{Ind}_{W_{s}}^{W} H_{x}^{i}(\mathcal{B}_{x}, \mathcal{Q}_{t}) \) as \( W \)-modules.

**Proof.** This formula corresponds to the special case of Lusztig’s character formula for generalized Green functions \([L_{2}, \text{Thm.} 8.5]\). Concerning the proof, essentially the same argument can be applied to our setting (but ignoring the \( F_{q} \)-structure). We give an outline of the proof below for the sake of completeness.

We fix \( s \in \tilde{t}, z \in n \) such that \([s, z] = 0\). For \( x \in \tilde{b} \), let \( x_{s} \) be its semisimple part. As in \([L_{2}, \text{Lemma} 8.6]\), one can find an open dense subset \( \mathcal{U} \) of \( c = \text{Lie} Z_{H}^{0}(s) \) satisfying the following properties:

(3.8.1) \( \mathcal{U} \) contains 0, and

(i) \( g(\mathcal{U}) = \mathcal{U} \) for any \( g \in C \),
(ii) \( x \in \mathcal{U} \) if and only if \( x_{s} \in \mathcal{U} \),
(iii) If \( x \in \mathcal{U}, g \in C, g^{-1}(s + x) \in \tilde{b} \), then \( g^{-1}x_{s} \in \tilde{b} \) and \( g^{-1}s \in \tilde{b} \).
(iv) If \( x \in \mathcal{U}, g \in C, g^{-1}(s + x) \in \tilde{t} \), then \( g^{-1}x_{s} \in \tilde{t} \) and \( g^{-1}s \in \tilde{t} \).

Note that \( \mathcal{U} \) contains \( c_{\text{nil}} \) by (ii). We define a subvariety \( \tilde{X}^{\flat}_{\mathcal{U}} \) of \( \tilde{X}^{\flat} \) by

\[
\tilde{X}^{\flat}_{\mathcal{U}} = \{ s + x, g\tilde{B} \in \tilde{X}^{\flat} \mid x \in \mathcal{U} \}.
\]

Let \( \gamma \in \tilde{\Gamma} \) be a double coset in \( \tilde{H} \), and consider the variety

\[
\tilde{X}^{\flat}_{\mathcal{U}, \gamma} = \{ (s + x, g\tilde{B}) \in \tilde{X}^{\flat}_{\mathcal{U}} \mid g \in \gamma \}
\]

\[
= \{ (s + x, g\tilde{B}) \in (s + \mathcal{U}) \times \tilde{H}/\tilde{B} \mid g \in \gamma, g^{-1}(s + x) \in \tilde{b} \}.
\]

Then one can show that

(3.8.2)

\[
\tilde{X}^{\flat}_{\mathcal{U}} = \coprod_{\gamma \in \tilde{\Gamma}} \tilde{X}^{\flat}_{\mathcal{U}, \gamma},
\]

where \( \tilde{X}^{\flat}_{\mathcal{U}, \gamma} \) is an open and closed subset of \( \tilde{X}^{\flat}_{\mathcal{U}} \) for \( \gamma \in \tilde{\Gamma} \).

Let \( \gamma \in \Gamma \) be the double coset in \( \tilde{H} \) corresponding to \( \gamma \in \tilde{\Gamma} \). Take \( g_{\gamma} \in \gamma \) and put \( B_{\gamma} = g_{\gamma}\tilde{B}g_{\gamma}^{-1} \cap C \). By definition, \( s \in g_{\gamma}(\tilde{t}) \), and \( B_{\gamma} \) is a Borel subgroup of \( C \) containing a maximal torus \( T_{\gamma} = \tilde{T} \). By replacing \( \tilde{H}, \tilde{B}, \tilde{T} \) by \( C = Z_{H}^{0}(s), B_{\gamma}, T_{\gamma} \), we can define \( \psi_{\gamma} : \tilde{Y}_{\gamma} \rightarrow Y_{\gamma} = c_{\text{reg}}, \pi_{\gamma} : \tilde{X}_{\gamma} \rightarrow X_{\gamma} = c \) corresponding to \( \psi^{\flat} : \tilde{Y}^{\flat} \rightarrow \ldots \)
$Y^b = \tilde{t}_{\text{reg}}, \pi^b : \tilde{X}^b \to X^b = \tilde{t}$. Put

$$\tilde{X}_{\mathcal{U},\gamma} = \pi_{\gamma}^{-1}(\mathcal{U}) \subset \tilde{X}_{\gamma}.$$ 

By using th property (iv) in (3.8.1), one can show that

(3.8.3) The map $(x, zB_{\gamma}) \mapsto (s + x, zg_{\gamma}\tilde{B})$ gives an isomorphism $\tilde{X}_{\mathcal{U},\gamma} \cong \tilde{X}_{\mathcal{U},\gamma}^b$.

Replacing $\tilde{X}^b$ by $\tilde{Y}^b$, a similar discussion works. Put

$$\tilde{Y}_{\mathcal{U},\gamma} = \{(x, g\tilde{T}) \in \tilde{Y}_{\mathcal{U}}^b \mid g \in \gamma\},$$

$$\tilde{Y}_{\mathcal{U},\gamma} = (\psi_{\gamma})^{-1}(\mathcal{U}) \subset \tilde{Y}_{\gamma},$$

where $\tilde{Y}_{\mathcal{U},\gamma}^b = (\psi)^{-1}(\tilde{t}_{\text{reg}} \cap (s + \mathcal{U}))$. Let $\gamma_0$ be the orbit in $\Gamma$ corresponding to $W_s \subset W$. Now $W$ acts on $\tilde{Y}_{\mathcal{U}}^b$ by $w : (x, g\tilde{T}) \mapsto (x, gw^{-1}\tilde{T})$. Then $W$ permutes the subsets $\tilde{Y}_{\mathcal{U},\gamma_0}^b$, which corresponds to the right action of $W$ on $\Gamma$. In particular, the stabilizer of $\tilde{Y}_{\mathcal{U},\gamma_0}^b$ in $W$ coincides with $W_s$. As an analogue of (3.8.2), we have

(3.8.4)

$$\tilde{Y}_{\mathcal{U}}^b = \prod_{\gamma \in \Gamma} \tilde{Y}_{\mathcal{U},\gamma}^b \cong \prod_{w \in W/W_s} w(\tilde{Y}_{\mathcal{U},\gamma_0}),$$

where $\tilde{Y}_{\mathcal{U},\gamma}^b$ is a non-empty, open and closed subset for each $\gamma \in \Gamma$.

Combining it with (3.8.3), the following holds.

(3.8.5) The map $\pi^b : \tilde{X}_{\mathcal{U},\gamma}^b \to \mathfrak{h}$ is a proper map onto $s + \mathcal{U}$. $\tilde{X}_{\mathcal{U},\gamma}^b$ is irreducible.

$\psi^b(\tilde{Y}_{\mathcal{U},\gamma}^b) = (s + \mathcal{U}) \cap \tilde{b}_{\text{reg}}$, and $\tilde{Y}_{\mathcal{U},\gamma}^b = (\pi^b)^{-1}((s + \mathcal{U}) \cap \tilde{b}_{\text{reg}}) \cap \tilde{X}_{\mathcal{U},\gamma}^b$.

Let $\mathcal{V} = s + (\mathfrak{c}_{\text{reg}} \cap \mathcal{U})$, and define

$$\tilde{Y}_{\mathcal{V}}^b = \{(x, g\tilde{T}) \in \tilde{Y}^b \mid x \in \mathcal{V}\},$$

$$(\tilde{Y}_{\gamma})_{s + \mathcal{V}} = \{(x, gT_{\gamma}) \in \tilde{Y}_{\gamma} \mid s + x \in \mathcal{V}\}.$$ 

Then we have the following commutative diagram 

$$(3.8.6)$$

$$\begin{array}{ccc}
\tilde{Y}_{\mathcal{V}}^b & \xleftarrow{\alpha} & \prod_{\gamma \in \Gamma} (\tilde{Y}_{\gamma})_{s + \mathcal{V}} \\
\downarrow & & \downarrow \\
\mathcal{V} & \xleftarrow{\beta} & -s + \mathcal{V},
\end{array}$$

where $\alpha : (x, zT_{\gamma}) \mapsto (s + x, zg_{\gamma}\tilde{T})$, $\beta : x \mapsto s + x$, and the vertical maps are projections to the first factor. It is shown that $\alpha$ turns out to be an isomorphism. $\beta$ is also an isomorphism.
The $\widetilde{Y}_V^g$ is invariant under the action of $W$ on $\widetilde{Y}_V^g$. In view of (3.8.4), it is decomposed as

$$
(3.8.7) \quad \widetilde{Y}_V^g = \prod_{\gamma \in \Gamma} \widetilde{Y}_{V,\gamma}^g \cong \prod_{w \in W/W_s} w(\widetilde{Y}_{V,\gamma}^g),
$$

where $\widetilde{Y}_{V,\gamma} = \{(x, gT) \in \widetilde{Y}_V \mid x \in V, g \in \Gamma\}$. $\alpha$ gives an isomorphism $(\widetilde{Y}_V^g)_{\gamma} \cong \widetilde{Y}_{V,\gamma}^g$ for each $\gamma \in \Gamma$. $W_s$ acts naturally on $(\widetilde{Y}_V^g)_{\gamma}$, and the map $(\widetilde{Y}_V^g)_{\gamma} \rightarrow \widetilde{Y}_{V,\gamma}^g$ is $W_s$-equivariant.

Now (3.8.6) and (3.8.7) imply that

$$
(3.8.8) \quad \beta^*((\psi^\beta)_!Q_s)|_\gamma \cong \bigoplus_{\gamma \in \Gamma} (\psi_{\gamma})_!Q_s|_{-s+\gamma} \cong \text{Ind}^W_{W_s}((\psi_{\gamma})_!Q_s|_{-s+\gamma})
$$

as local systems equipped with $W$-action. Here we consider $(\psi_{\gamma})_!Q_s$ as a local system with $W_s$-action. Put $K = (\pi^\beta)_!Q_s$, and $K_\gamma = (\pi_{\gamma})_!Q_s$. Since $K|_{\gamma Y} = (\psi^\beta)_!Q_s$, $K_{\gamma |_{\gamma Y}} \cong (\psi_{\gamma})_!Q_s$, (3.8.8) can be rewritten as

$$
\beta^* (K|_{\gamma Y}) \cong \bigoplus_{\gamma \in \Gamma} K_{\gamma |_{-s+\gamma}}.
$$

Note that $\gamma$ is an open subset of $s + \gamma$. Then the above isomorphism can be extended to an isomorphism

$$
(3.8.9) \quad \beta^* (K|_{s+\gamma}) \cong \bigoplus_{\gamma \in \Gamma} (K_{\gamma |_{\gamma Y}}).
$$

$K|_{s+\gamma}$ has a natural $W$-action induced from the $W$-action on $(\psi^\beta)_!Q_s$, which coincides with the $W$-action restricted from the $W$-action on $K$. Similarly, the $W_s$-action on $K_{\gamma |_{\gamma Y}}$ induced from that on $(\psi_{\gamma})_!Q_s$ coincides with the $W_s$-action restricted from that on $K_{\gamma |_{\gamma Y}}$. Thus (3.8.9) can be written, as complexes with $W$-action,

$$
\beta^* (K|_{s+\gamma}) \cong \text{Ind}^W_{W_s}(K_{\gamma |_{\gamma Y}}).
$$

Now taking the stalk at $s + z \in s + \gamma$ of the $i$-th cohomology sheaf on both sides, we have an isomorphism of $W$-modules.

$$
\mathcal{H}^i_{s+z} K \cong \text{Ind}^W_{W_s} (\mathcal{H}^i_z K_{\gamma |_{\gamma Y}}).
$$

The theorem follows from this. \qed

3.9. Recall that $Y$ is an open dense subset of $h$. We regard $Y$ as a locally closed subset of $\tilde{h}$. We consider the restriction $(\pi^\beta)_!Q_s|_Y$ of $(\pi^\beta)_!Q_s$ on $Y$, which inherits the $W$-module structure from $(\pi^\beta)_!Q_s$. By applying Theorem 3.8, we shall prove the following result.
Proposition 3.10. \((\pi^b)_!\tilde{Q}_t|_{\mathcal{Y}}\) is isomorphic to \(\psi_!\tilde{Q}_t\) as complexes equipped with \(W\)-action. In particular, for \(\rho \in (S_k \times S_{n-k})^\wedge\), we have

\[
(3.10.1) \quad \text{IC}(\mathfrak{h}, \mathcal{L}_\rho^b)|_{\mathcal{Y}} \simeq \text{IC}(Y_k, \mathcal{L}_\rho)[d_k - d_n].
\]

**Proof.** The isomorphism \((\pi^b)_!\tilde{Q}_t|_{\mathcal{Y}} \simeq \psi_!\tilde{Q}_t\) follows from the base change theorem. We show that this isomorphism is compatible with \(W\)-action. For \(0 \leq k \leq n\), we can consider \(Y_k^0\) as a locally closed subvariety of \(\mathfrak{h}\). By the base change theorem, we have \((\pi^b)_!\tilde{Q}_t|_{Y_k^0} \simeq (\psi_!\tilde{Q}_t|_{Y_k^0})\) has a \(W\)-action inherited from that on \((\pi^b)_!\tilde{Q}_t\). We compare this \(W\)-action with that of \((\psi_!\tilde{Q}_t|_{Y_k^0})\). For this, we investigate the \(W\)-module structure of the stalk at \(x \in Y_k^0\) of both cohomology sheaves. We apply Theorem 3.8 in the case where \(x = s + z \in t_{sp} + D_k^0\). In this case, \(C = Z_0 H(s) \simeq SL_2 \times \cdots \times SL_2\) in the notation in 2.6, and \(c \simeq 3 + (\mathfrak{s}l_2 \oplus \cdots \oplus \mathfrak{s}l_2). \ z \in c_{nil}\) is written as \(z = \sum_{i=1}^{n} z_i\), where \(z_i \in (\mathfrak{s}l_2)_{nil}\) is such that \(z_i \neq 0\) for \(1 \leq i \leq k\) and \(z_i = 0\) for \(i > k\). Thus \(\mathcal{B}^C \simeq P^n_1\) and \(\mathcal{A}^C \simeq P^{n-k}_1\). We have

\[
(3.10.2) \quad H^\bullet(P_1^{n-k}) = (Q_t[-2] \oplus Q_t)^{\otimes (n-k)} \simeq \bigoplus_{J \subset [k+1, n]} Q_t[-2|J].
\]

Since \(W_s\) is the Weyl group of \(C\), we have \(W_s \simeq (\mathbb{Z}/2\mathbb{Z})^n\), and the Springer representation of \(W_s\) on \(H^\bullet(\mathcal{B}^C_x)\) is given by \(\varphi_J\) on each factor \(Q_t[-2|J]\), where \(\varphi_J\) is a one-dimensional representation of \((\mathbb{Z}/2\mathbb{Z})^n\) such that the \(i\)-th factor \(\mathbb{Z}/2\mathbb{Z}\) acts non-trivially for \(i \in J\), and acts trivially for \(i \in [1, n] - J\). It follows, for \(|J| = k'\), that

\[
(3.10.3) \quad \text{Ind}^W_{W_s} \varphi_J = \bigoplus_{\rho \in (S_{n-k'} \times S_{k'})^\wedge} \hat{\rho} \otimes Q_t^\dim \rho.
\]

In particular, by applying \(J = [k + 1, n]\), we have

\[
(3.10.4) \quad (\pi^b)_!(\tilde{Q}_t)_x \simeq H^\bullet(\mathcal{B}_x, \tilde{Q}_t) \simeq \bigoplus_{\rho \in (S_k \times S_{n-k})^\wedge} \hat{\rho} \otimes \tilde{Q}_t^\dim \rho[-2(n-k)] \oplus \mathcal{N}_x,
\]

where \(\mathcal{N}_x\) is a sum of complexes \(Q_t[-2i]\) with \(i < n - k\).

On the other hand, by taking the stalk at \(x \in \mathfrak{h}\) in both sides of (2.13.2), and by taking into account the action of \(W\) given in (2.10.5), we have

\[
(3.10.5) \quad ((\psi_!\tilde{Q}_t)_x \simeq \bigoplus_{\rho \in (S_k \times S_{n-k})^\wedge} \hat{\rho} \otimes \tilde{Q}_t^\dim \rho[-2(n-k)] + \mathcal{N}'_x,
\]

where \(\mathcal{N}'_x\) is a sum of complexes of the form \(\tilde{Q}_t[-2i]\) with \(i < n - k\). By comparing (3.10.4) and (3.10.5), we obtain the following.
The $W$-module structure of $(\psi_k)_{|Y_k}$ coincides with the $W$-module structure of $(\pi^\flat)_{|Y_k}$, up to a sum of $\mathcal{L}[-2i]$ with $i < n - k$ for various local systems $\mathcal{L}$ on $Y_k$.

Now the proof of Proposition 2.14 shows that the $W$-module structure of $\psi_{\dot{Q}}_i$ is completely determined from the $W$-module structure of $(\psi_k)_{|Y_k}$ ignoring the part $\mathcal{N}_k$. A similar discussion holds also for $(\pi^\flat)_{|Y_k}$, the $W$-module structure of $(\pi^\flat)_{|Y_k}$ ignoring the part $\mathcal{L}[-2i]$ with $i < n - k$. This proves the first assertion of the proposition. (3.10.1) then follows by comparing (2.14.1) and (3.4.1). The proposition is proved. □

4. THE VARIETY OF SEMISIMPLE ORBITS

4.1. Recall the map $\pi : \tilde{X} \to X = \mathfrak{h}$ as in 2.4. In this section, we consider $\mathcal{B}$ as $H/B$ (not as $\tilde{H}/\tilde{B}$), and for each $x \in \mathfrak{h}$ put $\mathcal{B}_x = \{gB \in H/B \mid g^{-1}x \in \mathfrak{b}\}$. Then $\mathcal{B}_x \simeq \pi^{-1}(x)$. Let $\mathcal{O}_x$ be the $H$-orbit of $x$ in $\mathfrak{h}$. Put $\nu_H = \dim U$. We have the following.

\begin{equation}
\dim \mathcal{B}_x \leq \nu_H - \frac{1}{2} \dim \mathcal{O}_x = \frac{1}{2}(\dim Z_H(x) - \dim T).
\end{equation}

In fact, this formula was proved by Xue [X1, Lemma 6.1] in the case where $x \in \mathfrak{h}$ is nilpotent. Although he assumes that the group is of adjoint type, the proof works for $H$ and $\mathfrak{h}$. In the general case, write $x = s + z$, where $s$: semisimple, $z$: nilpotent such that $[s, z] = 0$. Put $C = Z^0_H(s)$, and consider the variety $C^0_x$ defined similarly to $C_x$, where $C^0_x$ is the flag variety for $C$, and $z \in \text{Lie}C$ is nilpotent. By a similar discussion as in (3.7.1), we have $\dim \mathcal{B}_x = \dim C^0_x$. Then (4.1.1) follows from the corresponding formula for the nilpotent case.

4.2. In this section we put $W = N_H(T)/T \simeq W_n$. For $x \in \mathfrak{h}$, let $x = x_s + x_n$ be the Jordan decomposition of $x$, where $x_s$ is semisimple, $x_n$ is nilpotent such that $[x_s, x_n] = 0$. By Lemma 2.3, the set of semisimple orbits in $\mathfrak{h}$ is in bijection with $\Xi = \mathfrak{t}/S_n$, under the natural action of $S_n \subset W_n = W$ on $\mathfrak{t}$. Since $\Xi$ is identified with the set of closed orbits in $\mathfrak{h}$, the Steinberg map $\omega : \mathfrak{h} \to \Xi$ is defined by associating the $H$-orbit of $x$ for $x$ (see [X1, 6.4]).

For a semisimple element $s \in \mathfrak{h}$, let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_a$ be the eigenspace decomposition of $s$, with $\dim V_i = n_i :$ even. Then $Z_H(s) \simeq S_{p_{n_1}} \times \cdots \times S_{p_{n_a}}$. Put $\mathfrak{n} = \{n_1, \ldots, n_a\}$. Then $\mathfrak{n}$ determines the structure of $Z_H(s)$ up to isomorphism, which we denote by $C(\mathfrak{n})$. Let $\mathcal{O}$ be a nilpotent orbit in $\text{Lie}C(\mathfrak{n})$. We define

$$X_{\mathfrak{n}, \mathcal{O}} = \{x \in \mathfrak{h} \mid x = x_s + x_n, Z_H(x_s) \simeq C(\mathfrak{n}), x_n \in \mathcal{O}\}.$$

Then $X_{\mathfrak{n}, \mathcal{O}} = X_{\mathfrak{n}', \mathcal{O}'}$ if they are not disjoint, and $\mathfrak{h} = \bigcup_{\mathcal{O}} X_{\mathfrak{n}, \mathcal{O}}$ gives a partition of $\mathfrak{h}$. By considering the Steinberg map, for $x \in X_{\mathfrak{n}, \mathcal{O}}$, we have

$$\dim X_{\mathfrak{n}, \mathcal{O}} = \dim \mathcal{O}_x + \dim \{s \in \mathfrak{t} \mid Z_H(s) \simeq C(\mathfrak{n})\}.$$
In particular, for \( x \in X_{n,\mathcal{O}} \),
(4.2.1) \[ \dim X_{n,\mathcal{O}} \leq \dim \mathcal{O}_x + \dim t. \]

We have a lemma.

**Lemma 4.3.** The map \( \pi : \tilde{X} \to X \) is semismall.

**Proof.** We know that \( \tilde{X} \) is smooth and \( \pi \) is proper. For \( x \in X_{n,\mathcal{O}} \), by (4.1.1) and (4.2.1),
\[
\dim \mathcal{B}_x \leq \frac{1}{2}(\dim X - (\dim \mathcal{O}_x + \dim t)) \\
\leq \frac{1}{2}(\dim X - \dim X_{n,\mathcal{O}}).
\]

The lemma follows. \( \square \)

**4.4.** We consider the Steinberg variety
\[
Z = \{ (x, gB, g'B) \in \mathfrak{h} \times \mathcal{B} \times \mathcal{B} \mid g^{-1}x \in \mathfrak{b}, g'^{-1}x \in \mathfrak{b} \}.
\]

We denote by \( \varphi : Z \to \mathfrak{h} \) the projection \( (x, gB, g'B) \mapsto x \). We have a commutative diagram
\[
\begin{array}{ccc}
Z & \xrightarrow{\alpha} & t \\
\downarrow \varphi & & \downarrow \omega_1 \\
\mathfrak{h} & \xrightarrow{\omega} & \Xi,
\end{array}
\]
where \( \alpha : (x, gB, g'B) \mapsto p_1(g^{-1}x) \) \( (p_1 \) is the projection \( \mathfrak{b} \to \mathfrak{t} \), \( \omega_1 \) is the restriction of \( \omega \) on \( \mathfrak{t} \). Note that \( \omega_1 \) is a finite morphism. Put \( \sigma = \omega_1 \circ \alpha \), and \( d' = \dim \mathfrak{h} - \dim \mathfrak{t} \).

We define a constructible sheaf \( \mathcal{T} \) on \( \Xi \) by
(4.4.1) \[ \mathcal{T} = \mathcal{H}^{2d'}(\sigma_!\mathbb{Q}_l) = R^{2d'}\sigma_!\mathbb{Q}_l. \]

Recall the notion of perfect sheaves in [L1, (5.4.4)]. A constructible sheaf \( \mathcal{E} \) on an irreducible variety is called a perfect sheaf if it satisfies the following two condition; (a) there exists an open dense smooth subset \( V_0 \) of \( V \) such that \( \mathcal{E}|_{V_0} \) is locally constant, and that \( \mathcal{E} = \text{IC}(V, \mathcal{E}|_{V_0}) \), (b) the support of any non-zero constructible subsheaf of \( \mathcal{E} \) has support \( V \).

Perfect sheaves enjoy the following properties; if \( \pi : V' \to V \) is a finite morphism with \( V' \) smooth, and \( \mathcal{E}' \) is a locally constant sheaf on \( V' \), then \( \mathcal{E} = \pi_*\mathcal{E}' \) is a perfect sheaf. Moreover, if \( 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0 \) is an exact sequence of constructible sheaves on \( V \) such that \( \mathcal{E}_1, \mathcal{E}_3 \) are perfect, then \( \mathcal{E}_2 \) is perfect.

We have the following lemma.

**Lemma 4.5.** The sheaf \( \mathcal{T} \) is a perfect sheaf on \( \Xi \).
Proof. The lemma can be proved by a similar argument as in the proof of Theorem 5.5 in [L1]. For the sake of completeness, and for fixing the notations, we will give the proof below. Let \( p : Z \to B \times B \) be the projection \((x, gB, g'B) \mapsto (gB, g'B)\). \( B \times B \) is decomposed into \( H \)-orbits, \( \tilde{B} \times \tilde{B} = \bigsqcup_{w \in W} \mathcal{O}_w \), where \( \mathcal{O}_w \) is the \( H \)-orbit containing \((B, wB)\). Put \( Z_w = p^{-1}(\mathcal{O}_w) \) for each \( w \in W \). \( Z_w \to \mathcal{O}_w \) is a locally trivial fibration with fibre isomorphic to \( B \cap wB \). Let \( \alpha_w \) be the restriction of \( \alpha \) on \( Z_w \). Then \( \alpha_w \) is a locally trivial fibration with fibre isomorphic to

\[
H \times (B \cap wB)^{-1} (n \cap wn),
\]

where \( n = \text{Lie} \, U \). In particular, \( \dim \alpha_w^{-1}(s) = d^f \) for any \( s \in t \). Moreover, each fibre is an irreducible variety. Let \( \sigma_w \) be the restriction of \( \sigma \) on \( Z_w \), and put \( \mathcal{F}_w = \mathcal{H}^{2d^f}((\sigma_w), Q_l) \). It follows from the above remark that

\[
R^{2d^f}(\alpha_w)_! \bar{Q}_l \simeq \bar{Q}_l.
\]

Since \( \omega_1 \) is a finite morphism, we have

\[
R^{2d^f}(\sigma_w)_! \bar{Q}_l \simeq R^0(\omega_1)_! R^{2d^f}(\alpha_w)_! \bar{Q}_l \simeq (\omega_1)_! \bar{Q}_l.
\]

It follows that \( \mathcal{F}_w \) is a perfect sheaf since \( \omega_1 \) is finite. By (4.5.1), \( \alpha_w^{-1}(s) \) is a vector bundle over \( \mathcal{O}_w \), and \( \mathcal{O}_w \) is a vector bundle over \( B \). It follows that \( H^i(\alpha_w^{-1}(s), Q_l) = 0 \) for odd \( i \). This implies that \( R^i(\sigma_w)_! Q_l = 0 \) for odd \( i \). Now we have a filtration \( Z = \bigsqcup_{w \in W} Z_w \) by locally closed subvarieties \( Z_w \). For an integer \( m \geq 0 \), let \( Z_m \) be the union of \( Z_w \) such that \( \dim \mathcal{O}_w = m \), and put \( Z_{\leq m} = \bigsqcup_{m' \leq m} Z_{m'} \). Then \( Z_{\leq m} \) is closed in \( Z \), and \( Z_m \) is open in \( Z_{\leq m} \). Let \( \sigma_m \) be the restriction of \( \sigma \) on \( Z_m \), and define \( \sigma_{\leq m} \) similarly. Since \( R^i(\sigma_w)_! \bar{Q}_l = 0 \) for odd \( i \), we have \( R^i(\sigma_m)_! \bar{Q}_l = 0 \) for odd \( i \). Thus we have an exact sequence

\[
0 \longrightarrow R^{2d^f}(\sigma_m)_! \bar{Q}_l \longrightarrow R^{2d^f}(\sigma_{\leq m})_! \longrightarrow R^{2d^f}(\sigma_{\leq m-1})_! \bar{Q}_l \longrightarrow 0.
\]

Since \( R^{2d^f}(\sigma_m)_! \bar{Q}_l = \bigoplus_{\dim \mathcal{O}_w = m} \mathcal{F}_w \) is a perfect sheaf on \( \Xi \), by induction on \( m \), we see that \( R^{2d^f} \sigma_l \bar{Q}_l \) is a perfect sheaf. The lemma is proved.

\[\square\]

**Proposition 4.6.** \( \mathcal{F} \simeq \bigoplus_{w \in W} \mathcal{F}_w \) as sheaves on \( \Xi \).

**Proof.** Recall \( t_{sr} \) in (2.2.2), and put \( \Xi_{sr} = \omega_1(t_{sr}) \). Then \( \Xi_{sr} \) is an open dense subset of \( \Xi \). Since \( \mathcal{F} \) and \( \bigoplus_{w \in W} \mathcal{F}_w \) are perfect sheaves on \( \Xi \), it is enough to show that their restrictions on \( \Xi_{sr} \) are isomorphic. Put \( Z_0 = \sigma^{-1}(\Xi_{sr}) \). Then \( Z_0 \backsimeq \tilde{Y} \times_Y \tilde{Y} \), where \( \psi : \tilde{Y} \to Y \) is as in 2.4. Let \( \sigma_0 \) be the restriction of \( \sigma \) on \( Z_0 \), which is the composite of the natural map \( Z_0 \to Y \) with the map \( Y \to \Xi_{sr} \). (Note that for any \( s \in t_{sr} \), \( \omega^{-1}(\omega(s)) = \bigcup_{g \in H} g(s + D) = Y \). The restriction of \( \mathcal{F} \) on \( \Xi_{sr} \) is isomorphic to \( R^{2d^f}(\sigma_0)_! \bar{Q}_l \). Recall that \( \tilde{Y}^k_+ = \bigsqcup I \tilde{Y}_I \) for \( 0 \leq k \leq n \) in (2.8.1). For any \( I \subset [1, n] \) such that \( |I| = k \), we consider the map \( \psi_I : \tilde{Y}_I \to Y^0_k \) as in 2.7. Let \( Y_I \) be as in (2.7.3). Then \( \psi_I \) is decomposed as \( \psi_I = \eta_I \circ \xi_I \), where \( \eta_I \) is a finite Galois covering with Galois group \( W_I \), and \( \xi_I \) is a locally trivial fibration with fibre isomorphic to...
For each subset \(I, J\) of \([1, n]\) such that \(|I| = |J| = k\), put \(\tilde{Z}_{I,J} = \tilde{Y}_I \times_{\tilde{Y}_k} \tilde{Y}_J\) under the inclusion \(Y^0_k \hookrightarrow Y\). We have a partition \(Z_0 = \bigsqcup_{I,J} \tilde{Z}_{I,J}\) by locally closed subsets \(\tilde{Z}_{I,J}\). We define \(\tilde{Z}_{I,J} = \tilde{Y}_I \times_{\tilde{Y}_k} \tilde{Y}_J\). The natural map \(\varphi_{I,J} : \tilde{Z}_{I,J} \to Y^0_k\) is decomposed as \(\varphi_{I,J} = \eta_{I,J} \circ \xi_{I,J}\), where \(\eta_{I,J} : \tilde{Z}_{I,J} \to Y^0_k\) is a finite Galois covering with Galois group \(\mathcal{W}_I \times \mathcal{W}_J\), and \(\xi_{I,J} : \tilde{Z}_{I,J} \to \tilde{Z}_{I,J}\) is a locally trivial fibration with fibre isomorphic to \(P^{n'}_1 \times P^{n'}_1\).

Let \(\alpha_0 : Z_0 \to t\) be the restriction of \(\alpha\) on \(Z_0\), and \(\alpha^w_0\) the restriction of \(\alpha_0\) on \(Z_0 \cap Z_w\). For \(s \in t_{sr}\), we have a partition of \(\tilde{Z}_{I,J}\),

\[
\tilde{Z}_{I,J} \cap \alpha_0^{-1}(s) = \bigsqcup_{w \in W} (\alpha^w_0)^{-1}(s) \cap \tilde{Z}_{I,J}.
\]

By using the property of \(\varphi_{I,J} = \xi_{I,J} \circ \eta_{I,J}\) mentioned above, we see that \((\alpha^w_0)^{-1}(s) \cap \tilde{Z}_{I,J}\) is an open and closed subset of \(\tilde{Z}_{I,J}\) for any \(w \in W\). Moreover, the odd cohomology of \((\alpha^w_0)^{-1}(s) \cap \tilde{Z}_{I,J}\) vanishes. It follows that

\[
(4.6.1) \quad (\alpha_{I,J})!Q_\ell \simeq \bigoplus_{w \in W} (\alpha^w_{I,J})!Q_\ell,
\]

where \(\alpha_{I,J}\) is the restriction of \(\alpha_0\) on \(\tilde{Z}_{I,J}\), and \(\alpha^w_{I,J}\) is the restriction of \(\alpha^w_0\) on \(Z_w \cap \tilde{Z}_{I,J}\). Moreover, we have \(R^i(\alpha^w_{I,J})!Q_\ell = 0\) for odd \(i\). By considering the long exact sequence arising from the filtration \(Z_0 = \bigsqcup_{I,J} \tilde{Z}_{I,J}\), (4.6.1) implies that

\[
(4.6.2) \quad R^{2d'}(\alpha_0)!Q_\ell \simeq \bigoplus_{w \in W} R^{2d'}(\alpha^w_0)!Q_\ell.
\]

By applying \(R^0(\omega)_!\) on both sides of (4.6.2), we have

\[
R^{2d'}(\sigma_0)!Q_\ell \simeq \bigoplus_{w \in W} R^{2d'}(\sigma^w_0)!Q_\ell,
\]

where \(\sigma^w_0\) is the restriction of \(\sigma_0\) on \(Z_0 \cap Z_w\). This shows that the restrictions of \(\mathcal{F}\) and of \(\bigoplus_w \mathcal{F}_w\) on \(\Xi_{sr}\) are isomorphic. The proposition is proved.

4.7. By the Künneth formula, we have \(\varphi_!Q_\ell \simeq \pi_1Q_\ell \otimes \pi_1Q_\ell\). Since \((\pi^w)_!Q_\ell\) is a complex with \(W\)-action by Proposition 3.4, \(\pi_1Q_\ell = (\pi^w)_!Q_\ell|_b\) has the action of \(W\) inherited from \((\pi^w)_!Q_\ell\). Hence \(\varphi_!Q_\ell\) has a natural action of \(W \times W\). It follows that \(\mathcal{F} = \mathcal{H}^{2d'}(\sigma_0!Q_\ell) \simeq \mathcal{H}^{2d'}(\omega_!(\varphi_!Q_\ell))\) is a sheaf equipped with \(W \times W\)-action. We note that under the decomposition of \(\mathcal{F}\) in Proposition 4.6, the action of \(W \times W\) has the following property; for each \(w_1, w_2 \in W\),

\[
(4.7.1) \quad (w_1, w_2) \cdot \mathcal{F}_w = \mathcal{F}_{w_1 w_2^{-1}}.
\]
In fact, since $\mathcal{T}$ is a perfect sheaf by Lemma 4.5, it is enough to check the relation (4.7.1) for the restriction of $\mathcal{T}$ on $\Xi_{sr}$. The action of $\mathcal{W}_I \times \mathcal{W}_J$ on $(\varphi_{IJ})!\bar{Q}_l$ is extended to the action of $\tilde{\mathcal{W}}_I \times \tilde{\mathcal{W}}_J$ (here $\mathcal{W}_I = \mathcal{W}_I \ltimes (\mathbb{Z}/2\mathbb{Z})^n$) so that the $i$-th factor $\mathbb{Z}/2\mathbb{Z}$ of $\tilde{\mathcal{W}}_I$ acts trivially if $i \in I$ and acts non-trivially if $i \in I'$, and similarly for $\tilde{\mathcal{W}}_J$. This action induces an action of $W \times W$ on $(\varphi_{0})!\bar{Q}_l$, which is nothing but the action of $W \times W$ inherited from the action of $W$ on $\pi_!\bar{Q}_l$ by Proposition 3.10. Then a similar relation as (4.7.1) for $(\varphi_{IJ})!\bar{Q}_l$ under the decomposition $(\varphi_{IJ})!\bar{Q}_l = \bigoplus_{w \in W} (\varphi_{wIJ})!\bar{Q}_l$, where $\varphi_{wIJ}$ is the restriction of $\varphi_{IJ}$ on $\tilde{Z}_{IJ} \cap Z_w$, can be verified directly by using the decomposition $\varphi_{IJ} = \varphi_{1IJ} \circ \xi_{IJ}$. Thus (4.7.1) is proved.

We consider the cohomology group $H^2_c(\Xi, \mathcal{T})$. The following fact holds.

**Proposition 4.8.** $H^2_c(\Xi, \mathcal{T})$ has a structure of $W \times W$-module, which is isomorphic to the two-sided regular representation of $W$.

**Proof.** Since $\mathcal{T}$ is a sheaf with $W \times W$-action, $H^i_c(\Xi, \mathcal{T})$ has a structure of $W \times W$-module. By Proposition 4.6, we have a decomposition

$$H^2_c(\Xi, \mathcal{T}) \simeq \bigoplus_{w \in W} H^2_c(\Xi, \mathcal{T}_w).$$

By (4.5.2)

$$H^2_c(\Xi, \mathcal{T}_w) \simeq H^2_c(\Xi, (\omega_1)!\bar{Q}_l) \simeq H^2_c(t, \bar{Q}_l) = \bar{Q}_l$$

since $\dim t = n$. The proposition then follows from (4.7.1). \qed

The following lemma, originally due to Lusztig [L1, Lemma 6.7], was proved in [Sh, Lemma 7.6]. Note that in [Sh] it is stated for the unipotent variety, but it works for any variety. Actually this result is proved in [L2, (7.4.2)], in a full generality.

**Lemma 4.9.** Let $A, A'$ be simple perverse sheaves on $X = \mathfrak{h}$. Then we have

$$\dim H^0_c(X, A \otimes A') = \begin{cases} 1 & \text{if } A' \simeq D(A), \\ 0 & \text{otherwise}, \end{cases}$$

where $D(A)$ is the Verdier dual of $A$.

**4.10.** Recall that $\pi : \tilde{X} \to X$ is semismall by Lemma 4.3, and so $K = \pi_!\bar{Q}_l[d]$ is a semisimple perverse sheaf on $X = \mathfrak{h}$, where $d = \dim X = \dim \mathfrak{h}$. We can write it as

$$K = \bigoplus_A V_A \otimes A,$$

where $A$ is a simple perverse sheaf and $V_A = \text{Hom}(K, A)$ is the multiplicity space for $A$. We have the following.
Proposition 4.11. Put $m_A = \dim V_A$ for each $A$. Then we have
\[ \sum_A m_A^2 = |W|. \]

Proof. We have
\[ H_{2n}^c(\Xi, \mathcal{T}) = H_{2n}^c(\Xi, R^{2d'}\omega_l(\pi_l\bar{Q}_l \otimes \pi_l\bar{Q}_l)). \]

Consider the spectral sequence
\[ H_c^i(\Xi, R^l\omega_l(\pi_l\bar{Q}_l \otimes \pi_l\bar{Q}_l)) \Rightarrow H^{i+j}_c(X, \pi_l\bar{Q}_l \otimes \pi_l\bar{Q}_l). \]

Since $\dim X = d' + n = d$, $\dim \Xi = n$, we have
\[ H_{2n}^c(\Xi, \mathcal{T}) \simeq H_{2d}^c(X, \pi_l\bar{Q}_l \otimes \pi_l\bar{Q}_l) \simeq H_0^c(X, K \otimes K). \]

Hence by (4.10.1), we have
\[ \dim H_{2n}^c(\Xi, \mathcal{T}) = \sum_{A, A'} m_A m_{A'} \dim H_0^c(X, A \otimes A'). \]

By Lemma 4.9, $H_0^c(X, A \otimes A') \neq 0$ only when $D(A) \simeq A'$, in which case, we have $\dim H_0^c(X, A \otimes A') = 1$. But since $K$ is self-dual, $m_A = m_{D(A)}$ for each $A$. It follows that $\dim H_{2n}^c(\Xi, \mathcal{T}) = \sum_A m_A^2$. On the other hand, by Proposition 4.8, $\dim H_{2n}^c(\Xi, \mathcal{T}) = |W|$. The proposition is proved. \qed

5. INTERSECTION COHOMOLOGY ON $\mathfrak{sp}(V)$

5.1. In Section 3, we have considered the intersection cohomologies on $\widetilde{\mathfrak{h}}$. In this section, we consider their restriction on $\mathfrak{h}$. We follow the notation in 2.2. In particular, $n_s = \bigoplus_{\alpha \in \Phi^+} g_\alpha$ and $\mathcal{D} = \bigoplus_{\alpha \in \Phi^+} g_\alpha$ are subspaces of $\mathfrak{h}$ such that $n = n_s \oplus \mathcal{D}$. Put $n_0 = \bigcup_{g \in B} g(t)$. Let $\overline{n_0}$ be the closure of $n_0$ in $\mathfrak{h}$. We show the following lemma.

Lemma 5.2. (i) $\overline{n_0} = t \oplus n_s$. In particular, $t \oplus n_s$ is $B$-stable.

(ii) Let $\mathfrak{h}_{ss}$ be the set of semisimple elements in $\mathfrak{h}$. Then
\[ \overline{\mathfrak{h}_{ss}} = \bigcup_{g \in H} g(t \oplus n_s). \]

Moreover, $\dim \overline{\mathfrak{h}_{ss}} = \dim \mathfrak{h} - 2n$. 
Proof. First we show that
\begin{equation}
\bigcup_{g \in B} g(t) \subset t \oplus n_s.
\end{equation}

Any \( g \in B \) can be written by (1.10.1) as
\[ g = \begin{pmatrix} b & c \\ 0 & t_b^{-1} \end{pmatrix}, \]
where \( b, c \) are square matrices of degree \( n \), with \( b \) non-singular upper triangular, and \( c \) satisfies the condition that \( t_c = b^{-1}c'tb \). Then \( g^{-1} \) can be written as
\[ g^{-1} = \begin{pmatrix} b^{-1} & -b^{-1}c'tb \\ 0 & tb \end{pmatrix} = \begin{pmatrix} b^{-1}t_c & 0 \\ 0 & t_b \end{pmatrix}. \]

Thus, for a diagonal matrix \( s \) of degree \( n \), we have
\[ x = g \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} g^{-1} = \begin{pmatrix} bsb^{-1} & bs'tc + cs'tb \\ 0 & t^{-1}s'tb \end{pmatrix}. \]

Since \( bse'tc + cs'tb \) is of the form \( A + t'A \) for a squar matrix \( A \), its diagonal entries are all zero, hence \( x \in t \oplus n_s \). (5.2.2) holds.

Recall that, for \( k = 0 \), \( \widetilde{Y}_0 \simeq H \times B \mathfrak{m}_{0,sr} \), where \( \mathfrak{m}_{0,sr} = \bigcup_{g \in B} g(t_{sr}) \). Hence
\[ \dim \widetilde{Y}_0 = \dim H - \dim B + \dim \mathfrak{m}_{0,sr}. \]
Since \( \dim \widetilde{Y}_0 = \dim H - n \) by Lemma 2.9 (iii), we see that \( \dim \mathfrak{m}_{0,sr} = \dim B - n \). We have
\[ \mathfrak{m}_{0,sr} \subset \mathfrak{m}_{0} \subset t \oplus n_s \]
by (5.2.2). Since \( \dim t \oplus n_s = \dim B - n \), we have \( \dim \mathfrak{m}_{0} = \dim t \oplus n_s \). Since \( \mathfrak{m}_{0} \) is irreducible, (i) holds.

Put \( \tilde{X}_0 = H \times B \) \( t \oplus n_s \) and \( X_0 = \bigcup_{g \in H} g(t_{sr}) \). The map \( \pi^{(0)} : \tilde{X}_0 \to \mathfrak{h} \) is proper and \( \text{Im } \pi^{(0)} = X_0 \). Hence \( X_0 \) is a closed subset of \( \mathfrak{h} \). Recall that \( Y_0 = \bigcup_{g \in H} g(t_{sr}) \), then \( Y_0 \subset \bigcup_{g \in G} g(t) \subset X_0 \). Since \( Y_0 \) is open dense in \( \mathfrak{h}_{ss}, Y_0 = \mathfrak{h}_{ss} \). Hence \( \mathfrak{h}_{ss} \subset X_0 \).

On the other hand, \( \bigcup_{g \in B} g(t) = \mathfrak{m}_{0} \) is contained in \( \mathfrak{h}_{ss} \). Hence \( t \oplus n_s \subset \mathfrak{h}_{ss} \) by (i), and so \( X_0 \subset \mathfrak{h}_{ss} \). It follows that \( X_0 = \mathfrak{h}_{ss} \). (5.2.1) is proved. The last assertion follows from Lemma 2.9 (iv) since \( \dim \mathfrak{h}_{ss} = \dim Y_0 \).

### Remark 5.3.
In the case of reductive Lie algebras \( \mathfrak{g} \) with \( p \neq 2 \), clearly the closure of \( \bigcup_{g \in B} g(t) \) coincides with \( \mathfrak{b} \), and \( \mathfrak{h}_{ss} = \mathfrak{g} \). Lemma 5.2 gives a special phenomenon occurring in the case where \( p = 2 \) and regular semisimple elements do not exist.
We shall generalize Lemma 5.2 in connection with $N_{k,sr}$. For $k = 0, 1, \ldots, n$, put

\begin{align}
N_k &= \bigcup_{g \in B} g(t + D_k).
\end{align}

Note that for $k = 0$, $N_k$ coincides with the previous notation. Let $\overline{N}_k$ be the closure of $N_k$ in $\mathfrak{h}$. We define varieties

\begin{align}
\overline{X}_k &= \{(x, gB) \in \mathfrak{h} \times H/B \mid g^{-1}x \in \overline{N}_k\},
X_k &= \bigcup_{g \in H} g(\overline{N}_k),
\end{align}

and define a map $\pi^{(k)}: \overline{X}_k \to \mathfrak{h}$ by $(x, gB) \mapsto x$. Then $\pi^{(k)}$ is proper, and $\text{Im} \pi^{(k)} = X_k$. Hence $X_k$ is closed in $\mathfrak{h}$. We show a lemma.

**Lemma 5.5.** (i) $\overline{N}_k = t \oplus n_s \oplus D_k$ for each $k$. In particular, $t \oplus n_s \oplus D_k$ is $B$-stable.

(ii) $X_0 = \overline{h}_{ss}$ and $X_n = \mathfrak{h}$. In particular, the map $\pi^{(n)}: \overline{X}_n \to X_n$ coincides with the map $\pi: \overline{X} \to X$ given in 2.4.

**Proof.** First we show

\begin{align}
\bigcup_{g \in B} g(D_k) \subset \bigoplus_{\alpha \in \Phi^+ \atop \alpha = \epsilon_i + \epsilon_j} \mathfrak{g}_\alpha \oplus D_k.
\end{align}

In fact, $x \in D_k$ is written as $x = \begin{pmatrix} 0 & d & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $d = \text{Diag}(d_1, \ldots, d_n)$ is a diagonal matrix with $d_i = 0$ for $i > k$. Thus under the notation in the proof of Lemma 5.2, for $g \in B$, we have

\begin{align}
xgx^{-1} = \begin{pmatrix} 0 & bd^t b \\ 0 & 0 \end{pmatrix} \in \bigoplus_{\alpha = \epsilon_i + \epsilon_j} \mathfrak{g}_\alpha \oplus D.
\end{align}

Put $y = (y_{ij}) = bd^t b$, and $b = (b_{ij})$. Then $y_{ii} = \sum_{j \geq i} b_{ij}^2 d_j$. It follows that $y_{ii} = 0$ for $i > k$, and (5.5.1) holds.

Since we know $\bigcup_{g \in B} g(t) \subset t \oplus n_s$ by Lemma 5.2, we have

\begin{align}
N_k &= \bigcup_{g \in B} g(t + D_k) \subset t \oplus n_s \oplus D_k.
\end{align}

On the other hand, since $N_{k,sr} \subset N_k$ and $\overline{X}_k \simeq H \times B \overline{N}_k$, $\overline{Y}_k$ is regarded as a subvariety of $\overline{X}_k$. It follows, by (5.5.2), that

\begin{align}
\dim \overline{Y}_k \leq \dim \overline{X}_k
\end{align}
\[ \dim \tilde{Y}_k = \dim H - n + k \]

We know \( \dim \tilde{Y}_k = \dim H - n + k \) by Lemma 2.9. Hence the inequalities in (5.5.3) are actually equalities, and we have \( \dim \mathfrak{N}_k = \dim \mathfrak{t} \oplus \mathfrak{n}_s \oplus \mathfrak{D}_k \). Since \( \mathfrak{N}_k \) is irreducible, we conclude that \( \mathfrak{N}_k = \mathfrak{t} \oplus \mathfrak{n}_s \oplus \mathfrak{D}_k \). This proves (i).

For (ii), we know \( X_0 = \tilde{\mathfrak{n}}_s \mathfrak{a} \) by Lemma 5.2. Since \( \mathfrak{N}_n = \mathfrak{b} \) by (i), we have \( X_n = X \). Thus (ii) holds. The lemma is proved. \( \square \)

As a corollary, we have the following.

**Proposition 5.6.** For \( k = 0, 1, \ldots, n \), we have

(i) \( \tilde{X}_k \) is a smooth, irreducible variety.

(ii) \( X_k \) is a closed subset of \( \mathfrak{h} \), with \( X_k = \bigcup_{g \in H} g(\mathfrak{t} \oplus \mathfrak{n}_s \oplus \mathfrak{D}_k) \).

(iii) \( \tilde{Y}_k \) is open dense in \( \tilde{X}_k \), and \( Y_k \) is open dense in \( X_k \).

(iv) \( \dim \tilde{X}_k = \dim H - n + k \), \( \dim X_k = \dim H - 2n + 2k \).

**Proof.** By Lemma 5.5, \( \tilde{X}_k \simeq H \times B (\mathfrak{t} \oplus \mathfrak{n}_s \oplus \mathfrak{D}_k) \). Hence \( \tilde{X}_k \) is smooth, irreducible. This proves (i). (ii) also follows from Lemma 5.5. We have \( \tilde{Y}_k \simeq H \times B \mathfrak{N}_k, \mathfrak{s} \), and \( \mathfrak{N}_k, \mathfrak{s} \) is open dense in \( \mathfrak{t} \oplus \mathfrak{n}_s \oplus \mathfrak{D}_k \). Hence \( \tilde{Y}_k \) is open dense in \( \tilde{X}_k \). \( \tilde{Y}_k \) coincides with the subset of \( X_k \) consisting of \( x \in X_k \) such that its semisimple part \( x_s \) is contained in \( Y_0 = \bigcup_{g \in H} g(\mathfrak{t}_s) \). Since \( Y_0 \) is open dense in \( \mathfrak{h}_s \), \( Y_k \) is open dense in \( X_k \). This proves (iii). Then (iv) follows from Lemma 2.9. \( \square \)

We shall prove the following theorem.

**Theorem 5.7.** \( \pi_1 Q_t \mathcal{L}_[d_\mathfrak{l}] \) is a semisimple perverse sheaf on \( X = \mathfrak{h} \), equipped with the action of \( W_n \), and is decomposed as

\[
\pi_1 Q_t \mathcal{L}_[d_\mathfrak{l}] \simeq \bigoplus_{0 \leq k \leq n} \bigoplus_{\rho \in (\mathfrak{S}_k \times \mathfrak{S}_{n-k})^\wedge} \hat{\rho} \otimes \text{IC}(\pi_1 \mathcal{X}_k, \mathcal{L}_\rho)[d_k].
\]

**5.8.** As in the case of \( Y_0^0 \), put \( X_k^0 = X_k - X_{k-1} \). By (2.6.3), \( Y_k^0 \subset X_k^0 \). Hence \( Y_k^0 \) is open dense in \( X_k^0 \). For each \( k \), we define a locally closed subvariety \( \tilde{X}_k^+ \) of \( \tilde{X} \) by \( \tilde{X}_k^+ = \pi^{-1}(X_k^0) \). Let \( \pi_k : \tilde{X}_k^+ \rightarrow X_k^0 \) be the restriction of \( \pi \) on \( \tilde{X}_k^+ \).

Take \( s \in \mathfrak{t} \) and consider the decomposition \( V = V_1 \oplus \cdots \oplus V_n \) into eigenspaces of \( s \), with \( \dim V_i = 2n_i \). Then \( Z_H(s) \simeq Sp(V_1) \times \cdots \times Sp(V_n) \). Put \( H_i = Sp(V_i) \) and \( \mathfrak{h}_i = \mathfrak{sp}(V_i) \). We consider the corresponding decomposition \( \mathfrak{D} = \mathfrak{D}_1 \oplus \cdots \oplus \mathfrak{D}_n \) with \( \mathfrak{D}_i \subset \mathfrak{h}_i \). The subvariety \( X_{H_i,0}^k \) of \( \mathfrak{h}_i \) is defined similarly to \( X_k^0 \) by replacing \( H, \mathfrak{h}, \mathfrak{D}, \) etc. by \( H_i, \mathfrak{h}_i, \mathfrak{D}_i, \) etc.

For each \( (x, gB) \in \tilde{X}_k^+ \), we associate a subset \( I \subset [1, n] \) such that \( |I| = k \) as follows. By definition, \( g^{-1}x \in X_k^0 \cap \mathfrak{b} \). In the case where \( g^{-1}x \in \mathfrak{n} \), put \( I = [1, k] \).

In general, by replacing \( g^{-1}x \in \mathfrak{b} \) by its \( B \)-conjugate if necessary, we may assume that \( g^{-1}x = s + z \), where \( s \in \mathfrak{t}, z \in \mathfrak{n} \) with \( [s, z] = 0 \), hence \( z \in \text{Lie} Z_H(s) \). Then \( z \) can be written as \( z = \sum_i z_i \) with \( z_i \in \mathfrak{h}_i \). There exists \( k_i \) such that \( z_i \in X_{H_i,0}^{k_i} \) for
Proof. It is clear from the definition that \( \tilde{\mathcal{X}}_I \) for the nilpotent case for \( H_i \). Now \( V = V_1 \oplus \cdots \oplus V_a \) gives a partition \( [1, n] = \bigcup J_i \) such that \( |J_i| = n_i \). Under the correspondence \( J_i \leftrightarrow [1, n_i] \), \( I' \) gives a subset \( I_i \subset J_i \) (the first \( k_i \) letters in \( J_i \)). Put \( I = \bigcup I_i \). Thus \( I \) is a subset of \( [1, n] \) such that \( |I| = k \). Note that \( I \) depends only on the \( B \)-conjugates of \( g^{-1}x \). Thus \( I \) is determined by \( (x, gB) \in \tilde{X}_k^+ \). We denote this assignment by \( (x, gB) \mapsto I \). For each \( I \) with \( |I| = k \), we define a subset \( \tilde{X}_I \) of \( \tilde{X}_k^+ \) by

\[
(5.8.1) \quad \tilde{X}_I = \{(x, gB) \in \tilde{X}_k^+ \mid (x, gB) \mapsto I \}.
\]

We show the following lemma.

Lemma 5.9. \( \tilde{X}_k^+ \) is decomposed as

\[
(5.9.1) \quad \tilde{X}_k^+ = \bigcap_{I \subset [1, n]} \tilde{X}_I,
\]

where \( \tilde{X}_I \) is an irreducible component of \( \tilde{X}_k^+ \) for each \( I \).

Proof. It is clear from the definition that \( \tilde{X}_I \) are mutually disjoint, and gives a partition (5.9.1) of \( \tilde{X}_k^+ \). We show that \( \tilde{X}_I \) is irreducible. In fact, \( \tilde{X}_k \cap \mathfrak{h}_{nil} = \bigcup \mathfrak{g}_{\mathfrak{h}} g(\mathfrak{n}_a \oplus \mathfrak{O}_k) \) is irreducible. The set of \( s \in t \) such that the eigenspace decomposition of \( V \) gives a fixed partition \( [1, n] = \bigcup J_i \) is irreducible. Hence the set of \( x = s + z \in X_k^0 \cap b \) with \( s \) above is irreducible. Since \( \tilde{X}_I \) is the set of \( H \)-conjugates of \( (x, B) \) with \( x \) as above, \( \tilde{X}_I \) is irreducible. Now \( \tilde{Y}_I \) is a subset of \( \tilde{X}_I \), which consists of \( (x, gB) \mapsto I \) such that \( g^{-1}x = s + z \), with \( s \) above. Thus \( \tilde{Y}_I \) is an open dense subset of \( \tilde{X}_I \). Since \( \tilde{Y}_I^+ = \bigcup \tilde{Y}_I \), with \( \tilde{Y}_I \) irreducible, \( \tilde{X}_k^+ = \bigcup \tilde{Y}_I \) gives a decomposition into irreducible components, where \( \tilde{Y}_I \) is the closure of \( \tilde{Y}_I \) in \( \tilde{X}_k^+ \). In order to prove the lemma, it is enough to see that \( \tilde{X}_I \) is closed. But the set \( Z_I = \bigcup Z_{I'} \) is closed in \( \tilde{X}_I \), where \( I' \) runs over all subsets of \( [1, n] \) such that \( I' \subset I \). Hence \( \tilde{X}_I \) is irreducible.

\( \tilde{X}_I \) as above, and gives a subset \( \tilde{Y}_I \) of \( \tilde{X}_I \) such that the eigenspace decomposition of \( V \) gives a fixed partition \( [1, n] = \bigcup \bigcup J_i \) is irreducible. Hence the set of \( x = s + z \in X_k^0 \cap b \) with \( s \) above is irreducible. Since \( \tilde{X}_I \) is the set of \( H \)-conjugates of \( (x, B) \) with \( x \) as above, \( \tilde{X}_I \) is irreducible. Now \( \tilde{Y}_I \) is a subset of \( \tilde{X}_I \), which consists of \( (x, gB) \mapsto I \) such that \( g^{-1}x = s + z \), with \( s \) above. Thus \( \tilde{Y}_I \) is an open dense subset of \( \tilde{X}_I \). Since \( \tilde{Y}_I^+ = \bigcup \tilde{Y}_I \), with \( \tilde{Y}_I \) irreducible, \( \tilde{X}_k^+ = \bigcup \tilde{Y}_I \) gives a decomposition into irreducible components, where \( \tilde{Y}_I \) is the closure of \( \tilde{Y}_I \) in \( \tilde{X}_k^+ \). In order to prove the lemma, it is enough to see that \( \tilde{X}_I \) is closed. But the set \( Z_I = \bigcup Z_{I'} \) is closed in \( \tilde{X}_I \), where \( I' \) runs over all subsets of \( [1, n] \) such that \( I' \subset I \). Hence \( \tilde{X}_I = Z_I \cap \tilde{X}_k^+ \) is closed in \( \tilde{X}_k^+ \). The lemma is proved. \( \Box \)

5.10. For each \( x \in X_k^0 \), we associate an \( x \)-stable isotropic subspace \( W_x \subset V \) with \( \dim W_x = n \) as follows. Assume that \( x \in X_k^0 \cap b \). Up to \( B \)-conjugate, we can write \( x = s + z \) with \( s \in t, z \in n, [s, z] = 0 \). In the case where \( x \) is \( s = 0 \), let \( W_x \) be the subspace of \( V \) spanned by \( e_1, \ldots, e_k \). Then \( W_x \) is an \( n \)-stable isotropic subspace of \( V \), and is \( x \)-stable. For \( x = x + z \in b \) in general, as in the discussion in 5.8, \( x \) can be written as \( x = \sum_{i=1}^{a} z_i \) with \( z_i \in X_{k_i}^H(t) \) for some \( k_i \) such that \( 0 \leq k_i \leq n_i \) and that \( \sum k_i = k \). We define an isotropic subspace \( W_x \subset V_i \) for each \( i \), by applying the above discussion to the nilpotent element \( z_i \in b_i \), and put \( W_x = W_1 \oplus \cdots \oplus W_a \). Then \( W_x \) is an \( (B \cap Z_H(s)) \)-stable isotropic subspace of \( V \) with \( \dim W_x = k \). In general, for \( x \in X_k^0 \), one can find \( g \in H \) such that \( g^{-1}x \in b \), and that \( g^{-1}x \) can be written as \( g^{-1}x = x' = s + z \) as above. If we fix such \( s \), the
choice of \( g \) is unique up to \((B \cap Z_H(s))\)-conjugate. We define \( W_{x'} \) as above, and put \( W_x = g(W_{x'}) \). This \( W_x \) satisfies the required property.

\( W_x^\perp/W_x \) has a natural symplectic structure, and put \( H_x = Sp(W_x^\perp/W_x) \). The action of \( x \) on \( W_x \) induces \( x|_{W_x^\perp/W_x} \in \text{Lie} \ H_x \). One can check that \( x|_{W_x^\perp/W_x} \) is contained in \( X_0^H_x \), where \( X_0^H_x \) is defined similarly to \( X_0 \) by replacing \( H \) by \( H_x \).

**5.11.** For \( i = 1, \ldots, n \), let \( M_i \) be the isotropic subspace of \( V \) spanned by \( e_1, \ldots, e_i \). Put \( \overline{M_i} = M_i^\perp/M_i \). \( \overline{M_i} \) has a natural symplectic structure. We fix \( k \), and consider \( G_1 = GL(M_k) \), \( H_2 = Sp(\overline{M}_k) \). Also put \( g_1 = \text{Lie} \ G_1 \), \( h_2 = \text{Lie} \ H_2 \). Let \( X_0^0 \) be the subvariety of \( X' = h_2 \) defined similarly to \( X_0^0 \) by replacing \( H \) by \( H_2 \). We consider \( X_0^0 = X_0' \) for \( k' = 0 \).

We define a variety \( \mathcal{G}_k \) by

\[
\mathcal{G}_k = \{(x, \phi_1, \phi_2) \mid x \in X_k^0, \phi_1 : W_x \to M_k, \phi_2 : W_x^\perp/W_x \to \overline{M}_k (\text{symplectic isom.})\}.
\]

We consider the diagram

\[
\begin{array}{ccc}
g_1 \times X_0' & \xleftarrow{\sigma} & \mathcal{G}_k & \xrightarrow{q} & X_k^0 \\
q : (x, \phi_1, \phi_2) \mapsto x, & & & & \\
\sigma : (x, \phi_1, \phi_2) \mapsto (\phi_1(x|_{W_x})\phi_1^{-1}, \phi_2(x|_{W_x^\perp/W_x})\phi_2^{-1}).
\end{array}
\]

\( H \times (G_1 \times H_2) \) acts on \( \mathcal{G}_k \) by

\[
(g, (h_1, h_2)) : (x, \phi_1, \phi_2) \mapsto (gx, h_1\phi_1g^{-1}, h_2\phi_2g^{-1})
\]

for \( g \in H, h_1 \in G_1, h_2 \in H_2 \). Moreover, \( \sigma \) is \( H \times (G_1 \times H_2) \)-equivariant with respect to the adjoint action of \( G_1 \times H_2 \) and the trivial action of \( H \) on \( g_1 \times X_0' \). By a standard argument, one can check

\[
\text{(5.11.3) The map } q \text{ is a principal bundle with fibre isomorphic to } G_1 \times H_2. \text{ The map } \sigma \text{ is a locally trivial fibration with smooth, connected fibre of dimension dim } H.
\]

**Remark 5.12.** The variety \( \mathcal{G}_k \) introduced here is a different type of the variety \( \mathcal{G}_k \) discussed in [SS, 4.5]. The discussion below has some similarity with the discussion in [Sh, Section 2].

**5.13.** Let \( B_1 \) be the Borel subgroup of \( G_1 \) which is the stabilizer of the flag \((M_i)_{0 \leq i \leq k}\) in \( G_1 \), and \( B_2 \) the Borel subgroup of \( H_2 \) which is the stabilizer of the flag \((M_{k+i}/M_i)_{k \leq i \leq n}\) in \( H_2 \). Put

\[
\tilde{g}_1 = \{(x, gB_1) \in g_1 \times G_1/B_1 \mid g^{-1}x \in \text{Lie} \ B_1\},
\]

and define \( \pi^1 : \tilde{g}_1 \to g_1 \) by \( (x, gB_1) \mapsto x \). We define \( \pi^2 : \tilde{X}' \to X' = h_2 \) similarly to \( \pi : \tilde{X} \to X \), by replacing \( H \) by \( H_2 \). We put \( \tilde{X}'_0 = \tilde{X}'_0^+ = (\pi^2)^{-1}(X'_0) \), and let \( \pi^2_0 \) be
the restriction of \( \pi^2 \) on \( \tilde{X}'_0 \). We define a variety

\[
(5.13.1) \quad \tilde{Z}^+_k = \{(x, gB, \phi_1, \phi_2) \mid (x, gB) \in \tilde{X}'_k, \\
\phi_1 : W_x \simeq M_k, \phi_2 : W^+_x / W_x \simeq \overline{M}_k \}
\]

and define a map \( \bar{q} : \tilde{Z}^+_k \to \tilde{X}'_k \) by the natural projection. We define a map \( \bar{\sigma} : \tilde{Z}^+_k \to \tilde{g}_1 \times \tilde{X}_0' \) as follows; take \((x, gB, \phi_1, \phi_2) \in \tilde{Z}^+_k \). Let \( s \) be the semisimple part of \( x \). Then \( B_s = Z_H(s) \cap gBg^{-1} \) is a Borel subgroup of \( Z_H(s) \) such that \( x \in \text{Lie} B_s \). By 5.10, \( W_x \) is a \( B_s \)-stable subspace of \( V \), and is decomposed as \( W_x = W_1 \oplus \cdots \oplus W_a \) according to the decomposition \( V = V_1 \oplus \cdots \oplus V_a \) into eigenspaces of \( s \). Then \( \prod_i (GL(W_i) \cap B_s) \) is a Borel subgroup of \( \prod_i GL(W_i) \), and there exists a unique Borel subgroup \( B_x^1 \) of \( GL(W_x) \) containing it. We see that \( x|_{W_x} \in \text{Lie} B^1_x \). We denote by \( g_1B_1 \subset G_1 / B_1 \) the element corresponding to the Borel subgroup \( \phi_1(B_x^1)\phi^{-1}_1 \) of \( G_1 \). On the other hand, \( W_i \) is \( B_s \)-stable, and we have a homomorphism \( B_s \to Sp(W^+_i / W_i) \). If we denote by \( B_{s,i} \) the image of this map, then \( \prod_i B_{s,i} \) is a Borel subgroup of \( \prod_i Sp(W^+_j / W_j) \), and there exists a unique Borel subgroup \( B_x' \) of \( Sp(W^+_x / W_x) \) containing it. We see that \( x|_{W_x^+ / W_x} \in B_x' \). We denote by \( g_2B_2 \subset H_2 / B_2 \) the element corresponding to the Borel subgroup \( \phi_2(B^j_x)\phi^{-2}_2 \) of \( H_2 \). We now define \( \bar{\sigma} : \tilde{Z}^+_k \to \tilde{g}_1 \times \tilde{X}_0' \) by

\[
\bar{\sigma} : (x, gB, \phi_1, \phi_2) \mapsto ((\phi_1(x|_{W_x})\phi^{-1}_1, g_1B_1), (\phi_2(x|_{W_x^+ / W_x})\phi^{-2}_2, g_2B_2)).
\]

We define a map \( \bar{\pi}_k : \tilde{Z}^+_k \to \tilde{g}_1 \times \tilde{X}_0' \) by \((x, gB, \phi_1, \phi_2) \mapsto (x, \phi_1, \phi_2)\). Then we have the following commutative diagram extending (5.11.2).

\[
\begin{array}{ccc}
\tilde{g}_1 \times \tilde{X}_0' & \xleftarrow{\bar{\sigma}} & \tilde{Z}^+_k \xrightarrow{\bar{\pi}_k} \tilde{X}'_k \\
\pi^1 \times \pi^2 \downarrow & & \downarrow \pi_k \\
\bar{g}_1 \times X_0' & \xleftarrow{\sigma} & \bar{g}_k \xrightarrow{\bar{q}} X^0_k.
\end{array}
\]

5.14. Let \( g_{1,\text{reg}} \) be the set of regular semisimple elements in \( g_1 \). Let \( \psi^1 \) be the restriction of \( \pi^1 : \bar{g}_1 \to g_1 \) to \((\pi^1)^{-1}(g_{1,\text{reg}})\). Then \( \psi^1 \) is a finite Galois covering with Galois group \( S_k \), and \((\psi^1)_* \mathcal{Q}_l \) is decomposed as

\[
(\psi^1)_* \mathcal{Q}_l \simeq \bigoplus_{\rho_1 \in S_k^\wedge} \rho_1 \otimes \mathcal{L}^1_{\rho_1},
\]

where \( \mathcal{L}^1_{\rho_1} \) is a simple local system on \( g_{1,\text{reg}} \) corresponding to \( \rho_1 \). \( g_{1,\text{reg}} \) is an open dense subset of \( g_1 \), and it is well-known that \((\pi^1)_* \mathcal{Q}_l[\dim g_1] \) is a semisimple perverse sheaf, equipped with \( S_k \)-action, and is decomposed as

\[
(5.14.1) \quad (\pi^1)_* \mathcal{Q}_l[\dim g_1] \simeq \bigoplus_{\rho_1 \in S_k^\wedge} \rho_1 \otimes \text{IC}(g_1, \mathcal{L}^1_{\rho_1})[\dim g_1].
\]
We put $A_{\rho_1} = IC(g_1, \mathcal{L}_{\rho_1}^1)[\dim g_1]$.

On the other hand, the varieties $Y_i^{0\prime} / \tilde{Y}_i^{0\prime}$ and the map $\psi_i^2 : \tilde{Y}_i^{0\prime} \to Y_i^{0\prime}$ are defined similarly to $Y_i^0 / \tilde{Y}_i$ and $\psi_i : \tilde{Y}_i^{0\prime} \to Y_i^{0\prime}$, where $H$ by $H_2$. In particular, in the case where $i = 0$, we have, by (2.10.5),

\begin{equation}
(\psi_0^2) : \tilde{Q}_i \simeq \bigoplus_{\rho_2 \in \mathcal{S}^\wedge_{n-k}} H^\ast(P_{1\prime}^{n-k}) \otimes \rho_2 \otimes \mathcal{L}_{\rho_2}^2;
\end{equation}

where $\mathcal{L}_{\rho_2}^2$ is a simple local system on $Y_0' = Y_0^0$ corresponding to $\rho_2$. Since $Y_0'$ is an open dense smooth subset of $X_0'$, we can consider the intersection cohomology $A_{\rho_2} = IC(X_0', \mathcal{L}_{\rho_2}^2)[\dim X_0']$ on $X_0'$.

Now $A_{\rho_1} \boxtimes A_{\rho_2}$ is a $(G_1 \times H_2)$-equivariant simple perverse sheaf on $g_1 \times X_0'$. By (5.11.3), there exists a unique simple perverse sheaf $A_\rho$ on $X_0'$ such that

\begin{equation}
q^\ast A_\rho[\beta_2] \simeq \sigma^\ast(A_{\rho_1} \boxtimes A_{\rho_2})[\beta_1],
\end{equation}

where $\beta_1 = \dim H$ and $\beta_2 = \dim(G_1 \times H_2)$. (Here we put $\rho = \rho_1 \boxtimes \rho_2 \in (S_k \times S_{n-k})^\wedge$.)

We have the following lemma.

**Lemma 5.15.** Let $\mathcal{L}_\rho$ be a simple local system on $Y_k^0$ given in (2.10.5). Then we have

$$A_\rho \simeq IC(X_k^0, \mathcal{L}_\rho)[d_k].$$

**Proof.** Since $A_\rho$ is a simple perverse sheaf on $X_k^0$, in order to prove the lemma, it is enough to see that

\begin{equation}
\mathcal{H}^{-d_k} A_\rho|_{Y_k^0} \simeq \mathcal{L}_\rho.
\end{equation}

We consider, for $I = [1, k], I' = \emptyset$, the following commutative diagram.

\begin{equation}
\begin{array}{c}
\tilde{g}_{1, \text{reg}} \times \tilde{Y}_I' \xleftarrow{\tilde{\sigma}_0} \tilde{Z}_I' \xrightarrow{\tilde{q}_0} \tilde{Y}_I \\
\xi^1 \times \xi_2' \downarrow \quad \downarrow \xi_l \quad \downarrow \xi_l \\
(G_1/T_1 \times t_{1,\text{reg}}) \times \tilde{Y}_I' \xleftarrow{\tilde{\sigma}_0} \tilde{Z}_I \xrightarrow{\tilde{q}_0} \tilde{Y}_I \\
\eta^1 \times \eta_2' \downarrow \quad \downarrow \eta_l \quad \downarrow \eta_l \\
g_{1, \text{reg}} \times Y_0' \xleftarrow{\sigma_0} g_{k, sr} \xrightarrow{q_0} Y_k^0,
\end{array}
\end{equation}

where $g_{k, sr} = q^{-1}(Y_k^0), \tilde{Z}_I' = \tilde{q}^{-1}(\tilde{Y}_I')$, and $\tilde{Z}_I$ is the quotient of $\tilde{Z}_I'$ by the natural action of the group $Z_T/t_{sr} / (Z_T/t_{sr}) \cap B$. The maps $\tilde{q}_0, q_0, \tilde{\sigma}_0, \sigma_0$ are defined as the restriction of the corresponding maps $\tilde{q}, q, \tilde{\sigma}, \sigma$.

Now the map $\eta^1 \times \eta_2'$ is a finite Galois covering with Galois group $S_k \times S_{n-k}$. Since the bottom squares in the diagram (5.15.2) are cartesian, this Galois covering is compatible with the Galois covering $\eta_I$. Hence for any $\rho_1 \in S_{n-k}, \rho_2 \in S^\wedge_{n-k}$, we have

$$\sigma^\ast(\mathcal{L}_{\rho_1}^1 \boxtimes \mathcal{L}_{\rho_2}^2) \simeq q^\ast \mathcal{L}_\rho$$
for \( \rho = \rho_1 \boxtimes \rho_2 \in (S_k \times S_{n-k})^\wedge \). (5.15.1) follows from this. The lemma is proved. \( \square \)

We can now state the following result.

**Proposition 5.16.** \((\pi_k)_! \bar{Q}_I\) is decomposed as

\[
(\pi_k)_! \bar{Q}_I \simeq H^* (P_1^{n-k}) \otimes \bigoplus_{\rho \in (S_k \times S_{n-k})^\wedge} \hat{\rho} \otimes \text{IC}(X^0_k, \mathscr{L}_\rho),
\]

where \( \hat{\rho} \) is regarded as a vector space, ignoring the \( W_n \)-action.

**5.17.** We prove Proposition 5.16 and Theorem 5.7 simultaneously, by induction on \( n \). We assume that the theorem and the proposition holds for \( n' < n \). First we show

**Lemma 5.18.** Proposition 5.16 holds for \( k \neq 0 \).

**Proof.** For any \( I \subset \{1, n\} \) with \(|I| = k\), put \( \tilde{Z}_I = \tilde{q}^{-1} (\tilde{X}_I) \). We have the following commutative diagram

\[
\begin{array}{ccc}
\bar{g}_1 \times \tilde{X}_0' & \longrightarrow & \tilde{Z}_I \\
\downarrow \pi_1^1 \times \pi_2^1 & & \downarrow \pi_I \\
g_1 \times X'_0 & \leftarrow & \mathscr{G}_k \rightarrow X^0_k,
\end{array}
\]

where \( \pi^1, \pi^2 \) are as in (5.13.2). \( \pi_I \) is the restriction of \( \pi_k : \tilde{X}_k^+ \rightarrow X^0_k \). Since \( \tilde{X}_k \) is closed in \( \tilde{X}_k^+ \), \( \pi_I \) is proper. We note that both squares are cartesian squares. We show the following.

(5.18.2) Any simple summand (up to shift) of the semisimple complex \((\pi_I)_! \bar{Q}_I\) is contained in the set \( \{ A_\rho \mid \rho \in (S_k \times S_{n-k})^\wedge \} \).

Put \( K_1 = (\pi_1)_! \bar{Q}_I \) and \( K_2 = (\pi_2)_! \bar{Q}_I \). We have

\[
K_1 \simeq \bigoplus_{\rho_1 \in S_k^\wedge} \rho_1 \otimes \text{IC}(\mathfrak{g}_1, \mathscr{L}^1_{\rho_1}),
\]

\[
K_2 \simeq H^* (P_1^{n-k}) \otimes \bigoplus_{\rho_2 \in S_{n-k}^\wedge} \hat{\rho}_2 \otimes \text{IC}(X'_0, \mathscr{L}^2_{\rho_2}).
\]

In fact, the first formula follows from (5.14.1), the second formula follows from Proposition 5.16, by applying the induction hypothesis to the case \( n' = n - k < n \) and \( k' = 0 \). Since both squares in (5.18.1) are cartesian, we have \( \sigma^* (K_1 \boxtimes K_2) \simeq q^* (\pi_I)_! \bar{Q}_I \), up to shift. Then (5.18.2) follows from (5.14.3).

Now by Lemma 5.9, we have \( (\pi_k)_! \bar{Q}_I \simeq \bigoplus (\pi_I)_! \bar{Q}_I \). Hence (5.18.2) implies, by Lemma 5.15, that

(5.18.3) Any simple summand (up to shift) of the semisimple complex \((\pi_k)_! \bar{Q}_I\) is contained in the set \( \{ \text{IC}(X^0_k, \mathscr{L}_\rho) \mid \rho \in (S_k \times S_{n-k})^\wedge \} \).
(5.18.3) implies, in particular, that any simple summand of \( K = (\pi_k)!\bar{Q}_t \) has its support \( X^0_k \). Since the restriction of \( K \) on \( Y^0_k \) coincides with \( K_0 = (\psi_k)!\bar{Q}_t \), the decomposition of \( K \) into simple summands is determined by the decomposition of \( K_0 \). Hence the lemma follows from (2.10.5). \( \square \)

5.19. We consider the semisimple complex \( (\pi_0)!\bar{Q}_t \) for \( \pi_0 : \bar{X}_0 \to X_0 = \bar{h}_{ss} \). In this case, the induction hypothesis can not be applied. But \( (\pi_0)!\bar{Q}_t|_{Y_0} \simeq (\psi_0)!\bar{Q}_t \), and by (2.10.5) we have

\[
(\psi_0)!\bar{Q}_t \simeq H^\bullet(\mathbb{P}_1^n) \otimes \bigoplus_{\rho \in S^\wedge_n} \hat{\rho} \otimes \mathcal{L}_\rho,
\]

by ignoring the \( W_n \)-module structure. It follows that \( (\pi_0)!\bar{Q}_t \) can be written as

\[(5.19.1) \quad (\pi_0)!\bar{Q}_t \simeq H^\bullet(\mathbb{P}_1^n) \otimes \bigoplus_{\rho \in S^\wedge_n} \hat{\rho} \otimes \text{IC}(X_0, \mathcal{L}_\rho) + \mathcal{N}_0.\]

Here \( \mathcal{N}_0 \) is a sum of various complexes of the form \( A[i] \), where \( A \) is a simple perverse sheaf such that \( \dim \text{supp } \ A < \dim X_0 \).

For each \( 0 \leq m \leq n \), let \( \overline{\pi}_m \) be the restriction of \( \pi \) on \( \pi^{-1}(X_m) \). The following formula can be proved by a similar argument as in the proof of (2.13.3), by using Lemma 5.18 and (5.19.1) instead of (2.10.5).

\[
(5.19.2) \quad (\overline{\pi}_m)!\bar{Q}_t[d_m] \simeq \bigoplus_{0 \leq k \leq m} \bigoplus_{\rho \in (S_k \times S_{n-k})^\wedge} \hat{\rho} \otimes \text{IC}(X_k, \mathcal{L}_\rho)[d_m - 2(n - k)] + \mathcal{M}_m + \mathcal{N}_0.
\]

where \( \mathcal{M}_m \) is a sum of various \( \text{IC}(X_k, \mathcal{L}_\rho)[d_m - 2i] \) for \( 0 \leq k \leq m \) and \( \rho \in (S_k \times S_{n-k})^\wedge \) with \( i < n - k \).

Note that, if \( k > 0 \), then all the simple perverse sheaves \( A \) appearing in the decomposition of \( (\pi_k)!\bar{Q}_t \) (up to shift) have support \( X_k \) by Lemma 5.18. This is also true for a simple perverse sheaf \( A \) appearing in the first term of \( (\pi_0)!\bar{Q}_t \) in (5.19.1). By Lemma 2.9, we have \( \dim X_k \geq \dim X_0 \) for any \( k \). Hence the above perverse sheaves \( A \) have the property that \( \dim \text{supp } \ A \geq \dim X_0 \). Since any perverse sheaf \( A' \) appearing in \( \mathcal{N}_0 \) has the property that \( \dim \text{supp } A' < \dim X_0 \), there is no interaction between \( \mathcal{N}_0 \) and other parts in the computation of \( (\pi_m)!\bar{Q}_t \). Thus \( \mathcal{N}_0 \) appears in (5.19.2) without change (up to shift).

We consider the case where \( m = n \). In this case, \( (\pi_n)!\bar{Q}_t[d_n] = \pi_!\bar{Q}_t[d] \) is a semisimple perverse sheaf by Lemma 4.3. This implies that \( \mathcal{M}_n = 0 \), and we have

\[(5.19.3) \quad \pi_!\bar{Q}_t[d] \simeq \bigoplus_{0 \leq k \leq n} \bigoplus_{\rho \in (S_k \times S_{n-k})^\wedge} \hat{\rho} \otimes \text{IC}(X_k, \mathcal{L}_\rho)[d_k] + \mathcal{N}_0.\]

We now apply Proposition 4.11. Since \( \sum_{\hat{\rho} \in W^\wedge_n} (\dim \hat{\rho})^2 = |W_n| \), we must have \( \mathcal{N}_0 = 0 \). This proves Proposition 5.16 in the case where \( k = 0 \). Hence Proposition
5.16 holds for any \( k \) by Lemma 5.18. The theorem now follows from (5.19.3). It remains to consider the case where \( n = 1 \). But in this case, \( X_0 = t \) coincides with the center of \( h = \mathfrak{sl}_2 \), and the proposition is easily verified. This completes the proof of Theorem 5.7 and Proposition 5.16.

As a corollary to Theorem 5.7, we have the following.

**Corollary 5.20.** Assume that \( \rho \in (S_k \times S_{n-k})^\wedge \), and let \( \tilde{\rho} \in W_n^\wedge \) be as in (2.12.1). Then

\[
\begin{align*}
(5.20.1) & \quad IC(\tilde{\mathfrak{h}}, \mathcal{L}_\rho)|_h \simeq IC(X_k, \mathcal{L}_\rho) \quad (\text{up to shift}), \\
(5.20.2) & \quad IC(X_k, \mathcal{L}_\rho)|_{h_{nil}} \simeq IC(\overline{\mathcal{O}}_{\tilde{\rho}}, \overline{Q}_t) \quad (\text{up to shift}).
\end{align*}
\]

**Proof.** (5.20.1) is obtained by comparing Theorem 5.7 with Proposition 3.4. By comparing Proposition 3.4 and Theorem 3.6, we have \( IC(\tilde{\mathfrak{h}}, \mathcal{L}_\rho)|_h \simeq IC(\overline{\mathcal{O}}_{\tilde{\rho}}, \overline{Q}_t) \), up to shift. Hence by (5.20.1),

\[
IC(X_k, \mathcal{L}_\rho)|_{h_{nil}} \simeq IC(\tilde{\mathfrak{h}}, \mathcal{L}_\rho)|_{h_{nil}} \simeq IC(\overline{\mathcal{O}}_{\tilde{\rho}}, \overline{Q}_t),
\]

up to shift. Thus (5.20.2) holds. \qed

### 6. Intersection Cohomology on \( g^\theta \)

**6.1.** From this section until the end of this paper, we discuss about \( G^\theta \) with \( N : \text{odd} \). So, assume that \( N = 2n + 1 \), and \( G = GL_N \). We follow the notation in 1.10. We have \( G^\theta \simeq SO(V') \simeq Sp(V) \), and \( G^\theta \simeq \mathfrak{g}^\theta \) by Proposition 1.11 and 1.13. Put \( H = Sp(V) \) and \( \mathfrak{h} = \mathfrak{sp}(V) \). As in 1.14, \( \mathfrak{g}^\theta = \mathfrak{h} \oplus \mathfrak{g}_V, \mathfrak{h} \oplus \mathfrak{g}_V \oplus \mathfrak{j} \). \( H \) acts trivially on \( \mathfrak{j} \simeq \mathfrak{k} \), and the action of \( H \) on \( \mathfrak{h} \oplus \mathfrak{g}_V \) can be identified with the diagonal action of \( H \) on \( \mathfrak{h} \times V \). Moreover, \( G_{\text{uni}}^\theta \simeq \mathfrak{g}^\theta_{\text{nil}} = (\mathfrak{h} \oplus \mathfrak{g}_V)_{\text{nil}} \). Hence considering \( G^\theta \) with \( G^\theta \)-action is essentially the same as considering the variety \( \mathfrak{h} \times V \) with diagonal action of \( H \) (but see Remark 1.15). For \( H \) and \( \mathfrak{h} \), we use the same notation as in the previous sections.

**6.2.** Put

\[
\mathcal{D}_{n,3} = \{ \mathbf{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3_{\geq 0} \mid \sum m_i = n \}.
\]

Recall the definition of \( \mathfrak{N}_k \) in (5.4.1) and \( \mathfrak{N}_{k,\text{sr}} \) in 2.4. For \( i = 1, \ldots, n \), let \( M_i \) be the subspace of \( V \) spanned by \( e_1, \ldots, e_i \). For \( \mathbf{m} \in \mathcal{D}_{n,3} \), we define varieties

\[
\begin{align*}
\widetilde{\mathcal{X}}_{\mathbf{m}} & = \{(x, v, gB) \in \mathfrak{h} \times V \times H/B \mid g^{-1}x \in \overline{\mathfrak{N}}_{p_2}, g^{-1}v \in M_{p_1}\} \\
\mathcal{X}_{\mathbf{m}} & = \bigcup_{g \in H} g(\overline{\mathfrak{N}}_{p_2} \times M_{p_1}),
\end{align*}
\]

where we put $p_1 = m_1, p_2 = m_1 + m_2$. We define a map $\pi^{(m)} : \widetilde{Y}_m \to \mathcal{X}_m$ by $(x, v, gB) \mapsto (x, v)$. $\pi^{(m)}$ is a proper surjective map. Since

$$\widetilde{Y}_m \cong H \times B (\mathfrak{N}_{p_2} \times M_{p_1})$$

$\widetilde{Y}_m$ is smooth, irreducible by Lemma 5.5. We also define varieties

$$\widetilde{Y}_m = \{(x, v, gB) \in \mathfrak{h} \times V \times H/B \mid g^{-1}x \in \mathfrak{N}_{p_2, sr}, g^{-1}v \in M_{p_1}\},$$

$$\mathcal{Y}_m = \bigcup_{g \in H} g(\mathfrak{N}_{p_2, sr} \times M_{p_1}),$$

and a map $\psi^{(m)} : \widetilde{Y}_m \to \mathcal{Y}_m$ by $(x, v, gB) \mapsto (x, v)$. In the case where $m = (n, 0, 0)$, we write $\mathcal{X}_m, \mathcal{Y}_m, \pi^{(m)}$ and $\widetilde{Y}_m, \mathcal{Y}_m, \psi^{(m)}$ simply as $\mathcal{X}, \mathcal{Y}, \pi$ and $\widetilde{Y}, \mathcal{Y}, \psi$. Note that $\mathcal{X} = \bigcup_{g \in H} g(b \times M_n)$ is a closed subset of $\mathfrak{h} \times V$, and $\widetilde{Y}_m$ is a closed subset of $\mathcal{X}$ for any $m$. Also note that in the case where $m_1 = 0$, $\mathcal{X}_m, \mathcal{Y}_m$, etc. coincide with $X_{m_2}, Y_{m_2}$, etc. in previous sections.

As in (2.6.1), one can write $\widetilde{Y}_m$ as

$$\widetilde{Y}_m \cong H \times B (\mathfrak{N}_{p_2, sr} \times M_{p_1})$$

$$\cong H \times B \cap Z_H(t_{sr}) ((t_{sr} + D_{p_2}) \times M_{p_1}).$$

For each $I \subset [1, n]$, let $M_I$ be the subset of $M_n$ consisting of $v = \sum_{i \in I} c_i e_i$ with $c_i \neq 0$. For $m \in D_{n, 3}$, we define $\mathcal{I}(m)$ as the set of $I = (I_1, I_2, I_3)$ such that $[1, n] = \bigcup_{1 \leq i \leq 3} I_i$ with $|I_i| = m_i$. For $I \in \mathcal{I}(m)$, put $D_I = D_{I_2} + \overline{D}_{I_1}$, where $\overline{D}_{I_1}$ is the closure of $D_{I_1}$ in $D$. We define a variety

$$(6.2.1) \quad \widetilde{Y}_I = H \times B \cap Z_H(t_{sr}) ((t_{sr} + D_{I_1}) \times M_{I_1}).$$

Since the actions of $B \cap Z_H(t_{sr})$ on $D$ and on $M_n$ are given by the actions of the torus part $T$, $(t_{sr} + D_{I_1}) \times M_{I_1}$ is $B \cap Z_H(t_{sr})$-stable. Hence $\widetilde{Y}_I$ is well-defined. Let $\psi_I : \widetilde{Y}_I \to \mathcal{Y}$ be the map defined by $g \ast (x, v) \mapsto (gx, gv)$, where $g \in H, (x, v) \in (t_{sr} + D_{I_1}) \times M_{I_1}$. Then $\text{Im} \psi_I$ is independent of the choice of $I \in \mathcal{I}(m)$, which we denote by $\mathcal{Y}^0_m$. We have, for any $I \in \mathcal{I}(m)$,

$$\mathcal{Y}^0_m = \bigcup_{g \in H} g((t_{sr} + D) \times M_{I_1}).$$

For $I \in \mathcal{I}(m)$, we define a parabolic subgroup $Z_H(t_{sr})_I$ by the condition that the $i$-th factor is $SL_2$ if $i \in I_3$ and is $B_2$ otherwise (a generalization of $Z_H(t_{sr})_I$ in 2.7). Since $Z_H(t_{sr})_I$ stabilizes $D_{I_1}$ and $M_{I_1}$, one can define

$$(6.2.2) \quad \widetilde{Y}_I = H \times Z_H(t_{sr})_I ((t_{sr} + D_{I_1}) \times M_{I_1}).$$
The map $\psi_1$ factors through $\widetilde{\mathcal{Y}}_1$,

$$\psi_1 : \widetilde{\mathcal{Y}}_1 \xrightarrow{\xi_1} \widetilde{\mathcal{Y}}_{1'} \xrightarrow{\eta_1} \mathcal{Y}_m^0,$$

where $\xi_1$ is the natural projection, and $\eta_1$ is the map defined by $g \ast (x, v) \mapsto (gx, gv)$. $\xi_1$ is the locally trivial fibration with fibre isomorphic to

$$Z_H(t_{sr})/(B \cap Z_H(t_{sr})) \simeq (SL_2/B_2)^{I_3} \simeq P^{I_3}.$$

Let $S_I \simeq S_{I_1} \times S_{I_2} \times S_{I_3}$ be the stabilizer of $I = (I_1, I_2, I_3)$ in $S_n$. Now $N_H(t_{sr})/Z_H(t_{sr}) \simeq S_n$, and $S_n$ acts on $Z_H(t_{sr}) \simeq SL_2 \times \cdots \times SL_2$ as the permutation of factors. Since $S_I$ stabilizes $M_{I_1}$ and $\mathcal{Y}_1$, $S_I$ acts on $\mathcal{Y}_I$ and on $\mathcal{Y}_{I'}$. Now the map $\eta_1$ is a finite Galois covering with Galois group $S_I$.

For each $m$, we define $I(m) = ([1, p_1], [p_1 + 1, p_2], [p_2 + 1, n])$, and put $\widetilde{\mathcal{Y}}_{I(m)} = \mathcal{Y}_m^{0}$, $S_I(m) = S_m$. Note that $\mathcal{Y}_m^{0}$ is an open dense subset of $\mathcal{Y}_m$, hence irreducible. Put $\psi^{-1}(\mathcal{Y}_m^{0}) = \mathcal{Y}_m^{+}$. $S_n$ acts naturally on $\mathcal{Y}$, and the map $\psi$ is $S_n$-equivariant with respect to the trivial action of $S_n$ on $\mathcal{Y}$. Hence it preserves the subset $\mathcal{Y}_m$ of $\mathcal{Y}$, and the stabilizer of $\mathcal{Y}_m^{0}$ in $S_n$ coincides with $S_m$. One can check that

$$(6.2.3) \quad \mathcal{Y}_{m}^{+} = \coprod_{I \in I(m)} \mathcal{Y}_I = \coprod_{w \in S_n/S_m} w(\mathcal{Y}_m^{0}),$$

where $\mathcal{Y}_I$ is an irreducible component, hence is a connected component.

As in [Sh, 1.3], we define a partial order on $\mathcal{Y}_{n,3}$ by $m' \leq m$ if $p_i' \leq p_i$ for $i = 1, 2$, where $(p_1', p_2')$ are defined for $m'$ similarly to $(p_1, p_2)$ for $m$. Then $\mathcal{Y}_{m'} \subset \mathcal{Y}_m$ and $\mathcal{Y}_{m'} \subset \mathcal{Y}_m$ if $m' \leq m$. One can check that

$$(6.2.4) \quad \mathcal{Y}_m^0 = \mathcal{Y}_m - \bigcup_{m' \leq m} \mathcal{Y}_{m'}.$$

Thus $\mathcal{Y}_m^0$ is an open dense subset of $\mathcal{Y}_m$, and we have a partition $\mathcal{Y}_m = \bigsqcup_{m' \leq m} \mathcal{Y}_{m'}. It follows that $\mathcal{Y}_{m'} \subset \mathcal{Y}_m$ if and only if $m' \leq m$. Also we have $\mathcal{Y} = \bigsqcup_{m \in \mathcal{A}_{n,3}} \mathcal{Y}_m^0$.

The following lemma can be proved in a similar way as Lemma 1.4 in [Sh] (the special case where $r = 3$).

**Lemma 6.3.** Let $m \in \mathcal{A}_{n,3}$.

(i) $\mathcal{Y}_m$ is open dense in $\mathcal{X}_m$, and $\mathcal{Y}_m$ is open dense in $\mathcal{X}_m$.

(ii) $\dim \mathcal{X}_m = \dim \mathcal{Y}_m = 3n^2 + 2m_1 + m_2$.

(iii) $\dim \mathcal{Y}_m = \dim \mathcal{Y}_m = 2n^2 + 2m_1 + m_2 - m_3$.

(iv) $\mathcal{Y} = \bigsqcup_{m \in \mathcal{A}_{n,3}} \mathcal{Y}_m^0$ gives a stratification of $\mathcal{Y}$ by smooth strata $\mathcal{Y}_m^0$, and the map $\psi : \mathcal{Y} \rightarrow \mathcal{Y}$ is semismall with respect to this stratification.

**6.4.** Let $\psi_m : \mathcal{Y}_m^+ \rightarrow \mathcal{Y}_m^0$ be the restriction of $\psi$ on $\mathcal{Y}_m^+$. Then $\psi_m$ is $S_n$-equivariant with respect to the natural action of $S_n$ on $\mathcal{Y}_m^+$ and the trivial action of
For a positive integer \( m \), By (6.2.3), we have

\[
(\psi_m)_!\hat{Q}_! \simeq \bigoplus_{I \in \mathcal{I}(m)} (\psi_1)_!\hat{Q}_!.
\]

Since \( \eta_\mathcal{I} : \hat{\mathcal{Y}}_\mathcal{I} \rightarrow \mathcal{Y}_\mathcal{I}^{0} \) is a finite Galois covering with Galois group \( S_\mathcal{I} \), \( (\eta_\mathcal{I})!\hat{Q}_! \) is a semisimple local system on \( \mathcal{Y}_\mathcal{I}^{0} \), and is decomposed as

\[
(\eta_\mathcal{I})!\hat{Q}_! \simeq \bigoplus_{\rho \in S_\mathcal{I}^1} \rho \otimes \mathcal{L}_\rho,
\]

where \( \mathcal{L}_\rho = \text{Hom}(\rho, (\eta_\mathcal{I})!\hat{Q}_!) \) is a simple local system on \( \mathcal{Y}_\mathcal{I}^{0} \).

Now by a similar argument as in 2.10, we have

\[
(\psi_\mathcal{I})!\hat{Q}_! \simeq (\eta_\mathcal{I})!((\xi_\mathcal{I})!\hat{Q}_!) \simeq H^*_w(P_1^m) \otimes (\eta_\mathcal{I})!\hat{Q}_!.
\]

For a positive integer \( r \), let \( W_{n,r} = S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n \) be the complex reflection group. Hereafter we assume that \( r = 3 \), and consider \( W_{n,r} = W_{n,3} \). Since \( S_m \) is a subgroup of \( S_n \), we can consider \( W_{m,r} = S_m \ltimes (\mathbb{Z}/r\mathbb{Z})^n \) as a subgroup of \( W_{n,r} \). Let \( \zeta \) be a primitive \( r \)-th root of unity in \( \hat{Q}_! \), and define a linear character \( \tau_i : \mathbb{Z}/r\mathbb{Z} \rightarrow \hat{Q}_! \) by \( \tau_i(\alpha) = \zeta^{i-1} \), where \( \alpha \) is a fixed generator of \( \mathbb{Z}/r\mathbb{Z} \). Let \( \rho \in \mathcal{S}_m \). Since \( S_m \simeq \prod_i S_{m_i} \), \( \rho \) is written as \( \rho = \rho_1 \boxtimes \rho_2 \boxtimes \rho_3 \), with \( \rho_i \in S_{m_i}^\Lambda \). Here \( W_{m,r} \simeq \prod_i W_{m_i,r} \). We extend the irreducible \( S_m \)-module \( \rho_i \) to an irreducible \( W_{m,r} \)-module \( \tilde{\rho}_i \) by defining the action of \( (\mathbb{Z}/r\mathbb{Z})^{m_i} \) via \( \tau_i^{\otimes m_i} \), and put \( \tilde{\rho} = \tilde{\rho}_1 \boxtimes \tilde{\rho}_2 \boxtimes \tilde{\rho}_3 \in W_{m,r}^\Lambda \). We also define \( \tilde{\rho}' = \tilde{\rho}_1 \boxtimes \tilde{\rho}_2 \boxtimes \tilde{\rho}_3' \in W_{m,r}^\Lambda \), where \( \tilde{\rho}_3' \) is the trivial extension of \( \rho_3 \) to \( W_{m,r} \). Put \( \tilde{\rho} = \text{Ind}_{W_{m,r}}^{W_{m,r}} \tilde{\rho}_3 \). Then \( \tilde{\rho} \) is an irreducible \( W_{n,r} \)-module, and any irreducible representation of \( W_{n,r} \) is obtained in this way from \( \rho \in S_{m_i}^\Lambda \) for various \( m_i \).

In view of (6.4.1), (6.4.2) and (6.4.3), similarly to the discussion in 2.10, \( (\psi_m)_!\hat{Q}_! \) can be written as

\[
(\psi_m)_!\hat{Q}_! \simeq \bigoplus_{\rho \in S_m^\Lambda} \text{Ind}_{S_m}^{S_n} (H^*(P_1^{m_3}) \otimes \rho) \otimes \mathcal{L}_\rho.
\]

We define an action of \( \mathbb{Z}/r\mathbb{Z} \) on \( H^*(P_1) = H^2(P_1) \oplus H^0(P_1) \) by \( \tau_r \oplus \tau_1 \), and define an action of \( (\mathbb{Z}/r\mathbb{Z})^{m_3} \) on \( H^*(P_1^{m_3}) \simeq H^*(P_1) \otimes \mathcal{S}_{m_3}^{m_3} \) as its tensor product. Thus we can extend \( H^*(P_1^{m_3}) \otimes \rho \) to a complex of \( W_{m,r} \)-modules \( H^*(P_1^{m_3}) \otimes \tilde{\rho}' \). Thus by (6.4.4), we can define a \( W_{n,r} \)-action on \( (\psi_m)_!\hat{Q}_! \),

\[
(\psi_m)_!\hat{Q}_! \simeq \bigoplus_{\rho \in S_m^\Lambda} \text{Ind}_{W_{m,r}}^{W_{n,r}} (H^*(P_1^{m_3}) \otimes \tilde{\rho}') \otimes \mathcal{L}_\rho.
\]

Note that (6.4.5) can be rewritten as

\[
(\psi_m)_!\hat{Q}_! \simeq \left( \bigoplus_{\rho \in S_m^\Lambda} \tilde{\rho} \otimes \mathcal{L}_\rho \right) [-2m_3] + \mathcal{L}_m,
\]
where $\mathcal{N}_m$ is a sum of various $\mathcal{L}_\rho[-2i]$ for $\rho \in S_m^\wedge$ with $0 \leq i < m$.

6.5. For each $m \in D_{n,3}$, let $\psi_m$ be the restriction of $\psi$ on $\psi^{-1}(\mathcal{Y}_m)$. Put $d_m = \dim \mathcal{Y}_m$. Let $D_{n,3}^0$ be the set of $m = (m_1, m_2, m_3) \in D_{n,3}$ such that $m_3 = 0$. Take $m = (m_1, m_2, 0) \in D_{n,3}^0$. For $0 \leq k \leq m_2$, define $m(k) \in D_{n,3}$ by $m(k) = (m_1, k, m_2 - k)$. The following result can be proved in a similar way as Proposition 1.7 in [Sh]. It is a special case where $r = 3$ of [loc. cit.]. (See also the proof of Proposition 2.13).

**Proposition 6.6.** Assume that $m \in D_{n,3}^0$. Then $(\psi_m)_! \bar{Q}_l[d_m]$ is a semisimple perverse sheaf on $\mathcal{Y}_m$, equipped with $W_{n,3}$-action, and is decomposed as

$$
(\psi_m)_! \bar{Q}_l[d_m] \simeq \bigoplus_{0 \leq k \leq m_2, \rho \in S_m^\wedge} \rho \otimes \text{IC}(\mathcal{Y}_m(k), \mathcal{L}_\rho)[d_m(k)].
$$

6.7. For $m = (m_1, m_2, 0) \in D_{n,3}^0$, let $W_{n,m}^\natural$ be the subgroup of $W_n$ defined by $W_{n,m}^\natural \simeq S_{m_1} \times W_{m_2}$, where $S_{m_1}$ is the group of permutations for $[1, m_1]$ and $W_{m_2} = W_{m_2, 2}$ is the group of signed permutations for $[m_1 + 1, n]$. Let $m(k) = (m_1, k, k')$ with $k + k' = m_2$. For $\rho = \rho_1 \boxtimes \rho_2 \boxtimes \rho_3 \in S_n(m(k))$, we define $\rho^\natural \in W_n^\natural$ by $\rho^\natural = \rho_1 \boxtimes \rho_2^\prime$, where $\rho_2^\prime \in W_{m_2}^\wedge$ is given by $\rho_2^\prime = \text{Ind}_{W_{n,2}^\wedge}^{W_n} (\bar{\rho}_2 \boxtimes \bar{\rho}_3)$. (Here $\bar{\rho}_2$ is the trivial extension of $\rho_2$ to $W_k$, $\bar{\rho}_3$ is the extension of $\rho_3$ to $W_{k'}$ with non-trivial action of $\mathbb{Z}/2\mathbb{Z}$ for each factor.)

Recall the map $\psi^{(m)} : \mathcal{Y}_m \to \mathcal{Y}_m$ given in 6.2. The following result is a variant of Proposition 6.6, and is proved by a similar argument (see also Proposition 3.5 in [Sh]).

**Proposition 6.8.** Assume that $m = (m_1, m_2, 0) \in D_{n,3}^0$. $\psi^{(m)}_! \bar{Q}_l[d_m]$ is a semisimple perverse sheaf on $\mathcal{Y}_m$, equipped with $W_{n,m}^\natural$-action, and is decomposed as

$$
\psi^{(m)}_! \bar{Q}_l[d_m] \simeq \bigoplus_{0 \leq k \leq m_2, \rho \in S_m^\wedge} \rho^\natural \otimes \text{IC}(\mathcal{Y}_m(k), \mathcal{L}_\rho)[d_m(k)].
$$

6.9. For each $m \in D_{n,3}$, we define $\pi_m$ the map $\pi^{-1}(\mathcal{X}_m) \to \mathcal{X}_m$ as the restriction of $\pi$ to $\pi^{-1}(\mathcal{X}_m)$. Thus $\pi_m$ is a proper surjective map to $\mathcal{X}_m$. The following result is a generalization of Theorem 5.7 (note that if $m = (0, m_2, 0)$, this coincides with Theorem 5.7.)

**Theorem 6.10.** Assume that $m \in D_{n,3}^0$. Then $(\pi_m)_! \bar{Q}_l[d_m]$ is a semisimple perverse sheaf on $\mathcal{X}_m$, equipped with $W_{n,3}$-action, and is decomposed as

$$
(\pi_m)_! \bar{Q}_l[d_m] \simeq \bigoplus_{0 \leq k \leq m_2, \rho \in S_m^\wedge} \hat{\rho} \otimes \text{IC}(\mathcal{X}_m(k), \mathcal{L}_\rho)[d_m(k)].
$$

6.11. The theorem can be proved in an almost parallel way as the proof of Theorem 2.2 in [Sh], the special case where $r = 3$, once formulated appropriately.
Also the proof is quite similar to the proof of Lemma 5.18. So in the following, we give an outline of the proof.

For \((x,v) \in \mathcal{X}\), we define an \(x\)-stable isotropic subspace \(W = W(x,v)\) as follows: If \(x\) is nilpotent, put \(W = k[x]v\). Since \((x,v) \in g(n \times M_n)\) for some \(g \in H\), \(W\) is isotropic. For general \(x = s + z\), where \(s\) is semisimple, \(z\) is nilpotent such that \([s,z] = 0\), we consider the decomposition of \(V\) into eigenspaces of \(s\), \(V = V_1 \oplus \cdots \oplus V_a\) as in 5.8. Then \(Z_H(s) \simeq Sp(V_1) \times \cdots \times Sp(V_a)\), and \(z \in h_{nil}\) can be written as \(z = \sum_{i=1}^a z_i\), where \(z_i \in sp(V_i)_{nil}\). We write \(v = \sum_{i=1}^a v_i\) with \(v_i \in V_i\). Then \((z_i,v_i) \in \mathcal{X}^{(i)}\), where \(\mathcal{X}^{(i)}\) is defined similarly to \(\mathcal{X}\) by replacing \(H\) by \(Sp(V_i)\).

We define a subspace \(W_i = W(z_i,v_i)\) of \(V_i\) as above, and put \(W = \bigoplus_i W_i\). Then \(W = W(x,v)\) satisfies the required properties. We consider \(\mathcal{X}_m\) for \(m \in \mathcal{D}_{n,3}\). Note that

\[
(6.11.1) \quad \dim W(x,v) \leq m_1 \text{ if } (x,v) \in \mathcal{X}_m.
\]

It follows from the construction of \(W\), we have

\[
(6.11.2) \quad x|_W \in \mathfrak{gl}(W) \text{ is a regular element.}
\]

Put \(W = W(x,v)\) for \((x,v) \in \mathcal{X}_m\). Then \(V' = W^+/W\) has a natural symplectic structure, and we define \(H' = Sp(V'), \mathfrak{h}' = \text{Lie } H'. \) \(x\) induces an endomorphism \(x'\) on \(V'\), and we have \(x' \in \mathfrak{h}'\). Let \(X_k, X_k^0, Y_k, etc.\) be the varieties defined for \(H'\), similarly to \(X_k, X_k^0, Y_k, etc.\) defined for \(H\). It is easy to see that if \((x,v) \in \bigcap_{m_1+m_2} M_{m_1} \times M_{m_2}\) with \(W(x,v) = M_{m_1}\), then \(x' \in \bigcap_{m_2} M_{m_2}\). It follows that

\[
(6.11.3) \quad x' \in X_{m_2}' \text{ if } (x,v) \in \mathcal{X}_m \text{ and } \dim W(x,v) = m_1.
\]

6.12. Assume that \(m \in \mathcal{D}_{n,3}\). As in the case of \(\mathcal{Y}_m^0\), we define an open subset \(\mathcal{X}_m^0\) of \(\mathcal{X}_m\) as

\[
\mathcal{X}_m^0 = \mathcal{X}_m - \bigcup_{m' < m} \mathcal{X}_{m'}.
\]

It follows from (6.11.1) and (6.11.3) that

\[
(6.12.1) \quad \mathcal{X}_m^0 = \{(x,v) \in \mathcal{X} \mid \dim W(x,v) = m_1 \text{ and } x' \in X_{m_2}'\}.
\]

We define \(\mathcal{X}^+_m = \pi^{-1}(\mathcal{X}_m^0)\), and let \(\pi_m : \mathcal{X}^+_m \to \mathcal{X}_m^0\) be the restriction of \(\pi\) on \(\mathcal{X}_m^+\). To \((x,v,gb) \in \mathcal{X}^+_m\), we associate \(I \in \mathcal{I}(m)\) as follows; assume that \((x,v) \in \mathfrak{b} \times M_n\), and \(x = s + z\) be the Jordan decomposition of \(x \in \mathfrak{b}\). By replacing \((x,v)\) by \(B\)-conjugate, we assume that \(s \in t, z \in \mathfrak{n}\). Then the decomposition \(V = V_1 \oplus \cdots \oplus V_a\) gives a decomposition \(M_n = M_{n,1} \oplus \cdots \oplus M_{n,a}\). Here \(M_n\) has a basis \(\{e_1, \ldots, e_n\}\), and \(M_{n,i}\) has a basis \(\{e_j \mid j \in J_i\}\), which gives a partition \([1,n] = \bigsqcup_{1 \leq i \leq a} J_i\). \((x,v)\) determines \((z_i,v_i)\), and put \(q_i = \dim W(z_i,v_i)\). Clearly \(q_i \leq \dim M_{n,i}\), and put \(J'_i \subset J_i\) as the set of first \(q_i\) letters in \(J_i\). We define \(I_1 = \bigsqcup_{1 \leq i \leq a} J'_i\). Since \(\dim W(x,v) = m_1\), we have \(|I_1| = m_1\) by (6.12.1). Here \(x' \in X_{m_2}'\) again by (6.12.1). Thus by 5.8, one can associate to \(x'\) a subset \(I_2\) of \([m_1+1,n]\) such that \(|I_2| = m_2\). We have \(I_1 \cap I_2 = \emptyset\), and \(I = (I_1, I_2, I_3)\) gives an element in \(\mathcal{I}(m)\) \((I_3)\) is the complement.
of $I_1 \cup I_2$.) The assignment $(x, v) \mapsto I$ does not depend on the $B$-conjugates of $(x, v)$, thus we obtain a well-defined map $(x, v, gB) \mapsto I$.

We define a subvariety $\widetilde{X}_I$ of $X_m^+$ by

\[(6.12.2) \quad \widetilde{X}_I^+ = \{(x, v) \in \widetilde{X}_m^+ \mid (x, v, gB) \mapsto I\}.\]

The following lemma can be proved in a similar way as Lemma 5.9 (see also Lemma 2.4 in [Sh]).

**Lemma 6.13.** $\widetilde{X}_I^+$ is decomposed as

\[\widetilde{X}_I^+ = \coprod_{I \in \mathcal{I}(m)} \widetilde{X}_I.\]

6.14. We fix $m = (m_1, m_2, m_3) \in \mathcal{Z}_{n, 3}$. We consider the space $V_0 = M_{m_1}$ and $\overline{V}_0 = V_0^{\perp}/V_0$. As in 5.11, we put $G_1 = GL(V_0)$, $H_2 = Sp(\overline{V}_0)$. We consider $X_{m_2}^+ \subset X'$ with respect to $H_2$. Put $\mathfrak{g}_1 = \text{Lie} G_1$, and let $\mathfrak{g}_1^0$ be the set of regular elements in $\mathfrak{g}_1$. For $\xi = (x, v) \in X$, put $W_\xi = W(x, v)$. As a variant of (5.11.1) (see also [Sh, 2.5]) we define a variety

\[(6.14.1) \quad K_m = \{(x, v, \phi_1, \phi_2) \mid \xi = (x, v) \in X_m^0, \quad \phi_1 : W_\xi \cong V_0, \phi_2 : W_\xi^{\perp}/W_\xi \cong V_0^{\perp} (\text{symplectic isom.})\}\]

with morphisms

\[q : K_m \to X_m^0, \quad (x, v, \phi_1, \phi_2) \mapsto (x, v),\]

\[\sigma : K_m \to \mathfrak{g}_1^0 \times X_{m_2}^{0}, \quad (x, v, \phi_1, \phi_2) \mapsto (\phi_1(x|_{W_\xi})\phi_1^{-1}, \phi_2(x|_{W_\xi^{\perp}/W_\xi})\phi_2^{-1})\]

$H \times (G_1 \times H_2)$ acts on $K_m$ by

\[(g, (h_1, h_2)) : (x, v, \phi_1, \phi_2) \mapsto (gx, gv, h_1\phi_1g^{-1}, h_2\phi_2g^{-1}).\]

Moreover, $\sigma$ is $H \times (G_1 \times H_2)$-equivariant with respect to the adjoint action of $G_1 \times H_2$ and the trivial action of $H$ on $\mathfrak{g}_1^0 \times X_{m_2}^{0}$. Similarly to (5.11.3), we have the following. (The proof is similar to ([Sh, 2.5]).

\[(6.14.2) \quad \text{The map } q \text{ is a principal bundle with fibre isomorphic to } G_1 \times H_2. \quad \text{The map } \sigma \text{ is a locally trivial fibration with smooth connected fibre of dimension } \dim H + m_1.\]

We define a Borel subgroup $B_1$ of $G_1$ and $B_2$ of $H_2$ as in 5.13, by replacing $k$ by $m_1$, and define $\pi^1 : \mathfrak{g}_1 \to \mathfrak{g}_1$ similarly. Put $\mathfrak{g}_1^0 = (\pi^1)^{-1}(\mathfrak{g}_1^0)$, and let $\varphi^1 : \mathfrak{g}_1^0 \to \mathfrak{g}_1^0$ be the restriction of $\pi^1$. We define $X'$ by using $H_2/B_2$, and let $\pi_{m_2}^2 : X_{m_2}^{0} \to X_{m_2}^{0}$ be
the corresponding map. We define a variety

\[ \tilde{Z}_m^+ = \{ (\xi, gB, \phi_1, \phi_2) \mid (\xi, gB) \in \tilde{X}_m^+, \phi_1 : W_\xi \cong V_0, \phi_2 : W_\xi/W_\xi \cong V_0 \}; \]

and define a map \( \tilde{q} : \tilde{Z}_m^+ \to \tilde{X}_m^+ \) by the natural projection. We define a map \( \tilde{\sigma} : \tilde{X}_m^+ \to \tilde{g}_1^0 \times \tilde{X}_m^+ \) as follows; take \( (x, v, B, \phi_1, \phi_2) \in \tilde{Z}_m^+ \). Since \( \xi = (x, v) \in \tilde{X}_m^+ \), \( W_\xi \) coincides with \( g(M_m) \). Let \( g_1 B_1 \) be the element corresponding to the flag \( \phi_1(g(M_i))_{0 \leq i \leq m_1} \), and \( g_2 B_2 \) the element corresponding to the isotropic flag \( \phi_2(g(M_i)/g(M_m))_{m_1 \leq i \leq n} \). Then

\[ \tilde{\sigma} : (x, v, gB, \phi_1, \phi_2) \mapsto ((\phi_1(x|W_\xi)\phi_1^{-1}, g_1 B_1), (\phi_2(x|W_\xi/W_\xi)\phi_2^{-1}, g_2 B_2)). \]

We also define a map \( \tilde{\pi}_m : \tilde{Z}_m^+ \to K_m \) by \( (x, v, gB, \phi_1, \phi_2) \mapsto (x, v, \phi_1, \phi_2) \). Then we have the following commutative diagram (compare this with (5.13.2));

\[ \begin{array}{ccc}
\tilde{g}_1^0 \times \tilde{X}_m^+ & \xleftarrow{\tilde{s}} & \tilde{Z}_m^+ \\
\varphi^1 \times \pi^2 & \downarrow & \tilde{\pi}_m \\
\tilde{g}_1^0 \times X_{m_2}^0 & \xleftarrow{\sigma} & K_m \\
\end{array} \]

(6.14.3)

6.15. By (5.14.1), \((\pi^1)_!\tilde{Q}_l\) can be written as \((\pi^1)_!\tilde{Q}_l \simeq IC(g_1, \mathcal{L})\) for a semisimple local system \( \mathcal{L} \) on \((g_1)_{\text{reg}}\); the set of regular semisimple elements in \( g_1 \). Since \( g_1^0 \) is an open dense subset of \( g_1 \) containing \((g_1)_{\text{reg}}\), \((\varphi^1)_!\tilde{Q}_l\) can be written as

\[ (\varphi^1)_!\tilde{Q}_l \simeq \bigoplus_{\rho_1 \in S_{\rho_1}} \rho_1 \otimes IC(g_1^0, \mathcal{L}_{\rho_1}^1). \]

By applying Proposition 5.16 to the map \( \tilde{\pi}_{m_2}^2 : \tilde{X}_{m_2}^+ \to X_{m_2}^0 \), we have

\[ (\pi_{m_2}^2)_!\tilde{Q}_l \simeq H^*(P_{m_1}^m) \otimes \bigoplus_{\rho_2 \in (S_{m_2} \times S_{m_3})^\wedge} \tilde{\rho}_2 \otimes IC(X_{m_2}^0, \mathcal{L}_{\rho'}^\vee). \]

Put \( A_{\rho_1} = IC(g_1^0, \mathcal{L}_{\rho_1})[\dim g_1] \) and \( A_{\rho_2} = IC(X_{m_2}^0, \mathcal{L}_{\rho_2}^\vee)[\dim X_{m_2}^0] \). Then \( A_{\rho_1} \boxtimes A_{\rho_2} \) is a \((G_1 \times H_2)\)-equivariant simple perverse sheaf on \( g_1^0 \times X_{m_2}^0 \). By a similar argument as in 5.14, thanks to (6.14.2), one can construct a simple perverse sheaf \( A_\rho \) on \( X_{m_2}^0 \), where \( \rho = \rho_1 \boxtimes \rho_2 \in S_{m_2}^\wedge \), satisfying the following property.

\[ q^* A_\rho [\beta] \simeq \sigma^*(A_{\rho_1} \boxtimes A_{\rho_2})[\beta_1], \]

where \( \beta_1 = \dim H + m_1 \) and \( \beta_2 = \dim (G_1 \times H_2) \). The following lemma can be proved in a similar way as [Sh, Lemma 2.7] (see also the proof of Lemma 5.15).
Lemma 6.16. Let $\mathcal{L}_\rho$ be a simple local system on $\mathcal{Y}_m^0$ as given in (6.4.4). Then we have

$$A_\rho \simeq IC(\mathcal{X}_m^0, \mathcal{L}_\rho)[d_m].$$

By using Lemma 6.16, we can prove the following.

**Proposition 6.17.** Under the notation of Lemma 6.16, $(\pi_m)_! \overline{Q}_l$ is decomposed as

$$(\pi_m)_! \overline{Q}_l \simeq H^\bullet(\mathcal{P}_1^m) \otimes \bigoplus_{\rho \in S_m^\wedge} \hat{\rho} \otimes IC(\mathcal{X}_m^0, \mathcal{L}_\rho),$$

where $\hat{\rho}$ is regarded as a vector space, ignoring the $W_{n,3}$-action.

**Proof.** We fix $I = (I_1, I_2, I_3) \in \mathcal{I}(m)$. Then the following commutative diagram is obtained from (6.14.3).

\[
\begin{array}{ccc}
\mathcal{Y}_1^0 \times \mathcal{X}_I^0 & \leftarrow & \mathcal{Z}_I^0 \\
\varphi^1 \times \varphi^2 & & \downarrow \pi_1 \\
\mathcal{Y}_1^0 \times \mathcal{X}_{m_2}^0 & \leftarrow & \mathcal{K}_m \\
\end{array}
\]

where $\mathcal{Z}_I = \overline{\mathcal{Z}}_I^0$. Note that $\pi_1$ is proper, and the both squares are cartesian squares. Then the proposition can be proved in a similar way as in the proof of Proposition 2.8 in [Sh]. Also see the discussion in the proof of Lemma 5.18. We omit the details. \qed

**Remark 6.18.** The proof of [Sh, Prop. 2.8] uses the induction on $r$, and depends on the Henderson’s result [Hen] for the case where $r = 1$, which was proved by making use of the Fourier-Deligne transform on perverse sheaves on Lie algebras. In turn, in the proof of Lemma 5.18, we needed to assume that $k \neq 0$. But thanks to Proposition 5.16, we don’t need any restriction in the proof of Proposition 6.17. Note that Proposition 5.16 was proved by making use of the $W_n$-actions on perverse sheaves on $\mathfrak{h}$.

6.19. Now the theorem is proved by a similar argument as in 2.10 in [Sh]. See also the discussion in 5.19. Note that in our situation, $N_0$ does not appear thanks to Proposition 5.16. In 5.19, if $\mathcal{N}_0 = 0$, one can construct a representation of $W_n$ on $(\pi_m)_! \overline{Q}_l$ by a similar method as in the case of $(\overline{\psi}_m)_! \overline{Q}_l$ (see the discussion in 2.11). A similar argument works for $\pi_m$ also.

6.20. Recall the map $\pi^{(m)}_m : \mathcal{Y}_m \to \mathcal{X}_m$ given in 6.2, and $W_m^2$ given in 6.7. The following result is a variant of Theorem 6.10, and is proved in a similar way as Theorem 3.2 in [Sh]. Note that if $m = (m_1, m_2, 0)$, then $W_m^2 = S_{m_1} \times W_{m_2,2}$. In the special case where $m_1 = 0, m_2 = n$, Theorem 6.21 coincides with Theorem 5.7.

**Theorem 6.21.** Assume that $m = (m_1, m_2, 0) \in \mathcal{P}_0^0$. $\pi^{(m)}_1 \overline{Q}_l[d_m]$ is a semisimple perverse sheaf on $\mathcal{X}_m^0$, equipped with $W_m^2$-action, and is decomposed as

$$\pi^{(m)}_1 \overline{Q}_l[d_m] \simeq \bigoplus_{0 \leq k \leq m_2} \bigoplus_{\rho \in S_{m_2}^\wedge} \rho^k \otimes IC(\mathcal{X}_m^0, \mathcal{L}_\rho)[d_{m_2}]$$
7. Nilpotent variety for $\mathfrak{g}^\theta$

7.1. For each $m \in \mathcal{D}_{n,3}$ with $p_1 = m_1, p_2 = m_1 + m_2$, we define varieties

$$\tilde{\mathcal{X}}_{m,\text{nil}} = \{(x, v, gB) \in \mathfrak{h} \times V \times H/B \mid g^{-1}x \in \mathfrak{n}_s \oplus \mathcal{D}_{p_2}, g^{-1}v \in M_{p_1}\},$$

$$\mathcal{X}_{m,\text{nil}} = \bigcup_{g \in H} g((\mathfrak{n}_s \oplus \mathcal{D}_{p_2}) \times M_{p_1}).$$

We write $\mathcal{X}_{m,\text{nil}}$ as $\mathcal{X}_{\text{nil}}$ if $m = (n, 0, 0)$. Note that $\mathfrak{n} \times M_n \simeq \mathfrak{n} \oplus (M_n)^g$ is contained in the nilpotent radical of a Borel subalgebra of $\mathfrak{g}l_{2n+1}$ (here $(M_n)^g$ is the corresponding subspace of $V_\mathfrak{g}$). Hence

$$\mathcal{X}_{m,\text{nil}} \subset \mathcal{X}_{\text{nil}} \subset \mathfrak{g}^\theta_{\text{nil}}.$$  

(7.1.1)

In this paper, we are only concerned with $\mathcal{X}_{\text{nil}}$, not with $\mathfrak{g}^\theta_{\text{nil}}$. However it is likely that $\mathfrak{g}^\theta_{\text{nil}} = \mathcal{X}_{\text{nil}}$.

We define a map $\pi_1^{(m)} : \tilde{\mathcal{X}}_{m,\text{nil}} \to \mathcal{X}_{m,\text{nil}}$ by $(x, v, gB) \mapsto (x, v)$. It is clear that $\tilde{\mathcal{X}}_{m,\text{nil}}$ is smooth and irreducible, and $\pi_1^{(m)}$ is a proper map onto $\mathcal{X}_{m,\text{nil}}$. Since

$$\tilde{\mathcal{X}}_{m,\text{nil}} \simeq H \times^B ((\mathfrak{n}_s \oplus \mathcal{D}_{p_2}) \times M_{p_1}),$$

we have

$$\dim \tilde{\mathcal{X}}_{m,\text{nil}} = \dim H - (n + m_3) + m_1$$

$$= 2n^2 + m_1 - m_3.$$  

(7.1.2)

7.2 For an integer $r \geq 1$, let $\mathcal{P}_{n,r}$ be the set of $r$-tuples of partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ such that $\sum_{1 \leq i \leq r} |\lambda^{(i)}| = n$. In the case where $r = 1$, we simply denote by $\mathcal{P}_n$ the set of partitions of $n$. It is well-known that $S_n^\lambda$ is in bijection with $\mathcal{P}_n$. We denote by $\rho_\lambda$ the irreducible representation of $S_n$ corresponding to $\lambda \in \mathcal{P}_n$. We normalize the parametrization so that the unit representation $1_{S_n}$ corresponds to $\lambda = (n)$. We consider the complex reflection group $W_{n,r}$. As explained in 6.4, the irreducible representation of $W_{n,r}$ is constructed from irreducible representations of symmetric groups. By this construction, we have a natural parametrization between $W_{n,r}^\lambda$ and $\mathcal{P}_{n,r}$. We denote by $\rho_\lambda$ the irreducible representation of $W_{n,r}$ corresponding to $\lambda \in \mathcal{P}_{n,r}$.

We consider the nilpotent orbits in $\mathfrak{h} = \mathfrak{sp}(V)$. By Theorem 3.6, nilpotent orbits $\mathcal{O}$ are parametrized by $W_{n,2}^\lambda$, as $\mathcal{O} = \mathcal{O}_\rho$ for $\rho \in W_{n,2}^\lambda$. We denote by $\mathcal{O}_\lambda$ the nilpotent orbit $\mathcal{O}_\rho$ in $\mathfrak{h}$ corresponding to $\rho = \rho_\lambda$ with $\lambda \in \mathcal{P}_{n,2}$.

7.3 Let $V_1$ be an $n$-dimensional vector space over $k$, and $G_1 = GL(V_1)$. Put $\mathfrak{g}_1 = \text{Lie} G_1$. We consider the diagonal action of $G_1$ on $\mathfrak{g}_1 \times V_1$. $G_1$ acts on $(\mathfrak{g}_1)_{\text{nil}} \times V_1$. It is known by [AH], [T] that the set of $G_1$-orbits in $(\mathfrak{g}_1)_{\text{nil}} \times V_1$ is parametrized by $\mathcal{P}_{n,2}$. We denote by $\mathcal{O}_\lambda$ the $G_1$-orbit corresponding to $\lambda \in \mathcal{P}_{n,2}$. According to [AH], the explicit correspondence is given as follows; take $(x, v) \in (\mathfrak{g}_1)_{\text{nil}} \times V_1.$
Put \( E^x = \{ y \in \text{End}(V_1) \mid xy = yx \} \). \( E^x \) is a subalgebra of \( \text{End}(V_1) \), stable by the multiplication of \( x \). If we put \( W = E^x v \), \( W \) is an \( x \)-stable subspace of \( V_1 \). We denote by \( \lambda^{(1)} \) the Jordan type of \( x|_W \) and by \( \lambda^{(2)} \) the Jordan type of \( x|_{V_1/W} \). Then the Jordan type of \( x \) is \( \lambda^{(1)} + \lambda^{(2)} \), and \( \lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \mathfrak{P}_{n,2} \). \( \mathcal{O}_\lambda \) is defined as the \( G_1 \)-orbit containing \((x,v)\). This gives the required labelling of \( G_1 \)-orbits in \((\mathfrak{g}_1)_{\text{nil}} \times V_1 \).

Note that if \((x,v) \in \mathcal{O}_\lambda \), the Jordan type of \( x \) is \( \lambda^{(1)} + \lambda^{(2)} \) for \( \lambda = (\lambda^{(1)}, \lambda^{(2)}) \).

For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \in \mathfrak{P}_n \), put \( n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i \). Let \( \mathcal{O}_\lambda \) be the \( G_1 \)-orbit in \((\mathfrak{g}_1)_{\text{nil}} \) consisting of \( x \) of Jordan type \( \lambda \). It is well-known that

\[
(7.3.1) \quad \dim \mathcal{O}_\lambda = n^2 - n - 2n(\lambda).
\]

Let \( \mathcal{O}_{\lambda} \) be as above. The following formula was proved in [AH, Prop. 2.8]. Put \( \nu = \lambda^{(1)} + \lambda^{(2)} \), and define \( n(\lambda) = n(\lambda^{(1)}) + n(\lambda^{(2)}) \). Then

\[
(7.3.2) \quad \dim \mathcal{O}_\lambda = \dim \mathcal{O}_\nu + |\lambda^{(1)}| = n^2 - n - 2n(\lambda) + |\lambda^{(1)}|.
\]

7.4. For \( \lambda \in \mathfrak{P}_{n,3} \), we shall define a variety \( X_\lambda \subset \mathfrak{h} \times V \). Put \( \lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) \), and \( m_i = |\lambda^{(i)}| \) for \( i = 1, 2, 3 \). Let \( P \) be the parabolic subgroup of \( H \), which is the stabilizer of the subspace \( V_0 = M_m \), and \( L \) the Levi subgroup of \( P \) containing \( T \). Then \( L \simeq GL(V_0) \times Sp(V_0) \), where \( V_0 = V_0^+ / V_0 \). Put \( G_1 = GL(V_0) \) and \( \mathfrak{g}_1 = \text{Lie} G_1 \). \( G_1 \) acts on \( \mathfrak{g}_1 \times V_0 \), as the restriction of the action of \( H \) on \( \mathfrak{h} \times V \).

Let \( \mathcal{O}_1 \) be the \( G_1 \)-orbit in \((\mathfrak{g}_1)_{\text{nil}} \times V_0 \) corresponding to \( \mathcal{O}_\lambda \) with \( \lambda = (\lambda^{(1)}, \emptyset) \in \mathfrak{P}_{m_1,2} \). Put \( H_2 = Sp(V_0) \) and \( \mathfrak{h}_2 = \text{Lie} H_2 \). We denote by \( \mathcal{O}_2 \) the \( H_2 \)-orbit \( \mathcal{O}_\lambda \) in \((\mathfrak{h}_2)_{\text{nil}} \) corresponding to \( \lambda' = (\lambda^{(2)}, \lambda^{(3)}) \in \mathfrak{P}_{m_2 + m_3,2} \). Put \( \text{Lie} P = \mathfrak{p} \). We define a set \( \mathcal{M}_\lambda \subset \mathfrak{p}_{\text{nil}} \times V_0 \) by

\[
(7.4.1) \quad \mathcal{M}_\lambda = \{ (x, v) \in \mathfrak{p}_{\text{nil}} \times V_0 \mid (x|_{V_0}, v) \in \mathcal{O}_1, x|_{V_0} \in \mathcal{O}_2 \},
\]

and define \( X_\lambda = \bigcup_{g \in H} g(\mathcal{M}_\lambda) \). Let \( n_P \) be the nilpotent radical of \( \mathfrak{p} \). We define a variety

\[
(7.4.2) \quad \bar{X}_\lambda = H \times^P ((\mathcal{O}_1 + \mathcal{O}_2) + n_P)
\]

and a map \( \pi_\lambda : \bar{X}_\lambda \to \mathfrak{X}_{\text{nil}} \) by \( g \ast x \mapsto gx \). Then \( \pi_\lambda \) is proper, and \( \text{Im} \pi_\lambda = \bigcup_{g \in H} g(\mathcal{O}_1 + \mathcal{O}_2 + n_P) \) is closed in \( \mathfrak{X}_{\text{nil}} \).

We show that

**Lemma 7.5.** Under the notation above,

(i) \( X_\lambda \) is an \( H \)-stable, smooth, irreducible, locally closed subvariety of \( \mathfrak{X}_{\text{nil}} \). Moreover, \( \text{Im} \pi_\lambda = \overline{X}_\lambda \).

(ii) \( \dim \bar{X}_\lambda = \dim X_\lambda = 2 \dim U_P + \dim \mathcal{O}_1 + \dim \mathcal{O}_2 \).
(iii) \( X_\lambda \) gives a partition of \( \mathcal{X}_{\text{nil}} \), namely,

\[
\mathcal{X}_{\text{nil}} = \bigsqcup_{\lambda \in \mathcal{P}_{n,3}} X_\lambda.
\]

**Proof.** Take \((x, v) \in L \times M_n\), with \(x = (x_1, x_2)\) such that \((x_1, v) \in \mathcal{O}_1\) and \(x_2 \in \mathcal{O}_2\). Then we can write as

(7.5.1) \[ X_\lambda \simeq H \times Z_L(x, v)U_P \left( (x + n_P) \times \{v\} \right). \]

Thus by (7.4.2), we see that \( \dim \mathcal{X}_\lambda = \dim X_\lambda \). (ii) is proved.

We have \( X_\lambda \subset \text{Im} \pi_\lambda \). As a variant of \( \pi_\lambda \), it is possible to define \( \pi'_\lambda \) by replacing \( \mathcal{O}_i \) by any orbits \( \mathcal{O}'_i \subset \mathcal{O}_i \) for \( i = 1, 2 \). In that case, \( \text{Im} \pi'_\lambda \) is also closed. This implies that \( X_\lambda \) is an open dense subset of \( \text{Im} \pi_\lambda \), hence \( X_\lambda \) is locally closed in \( \mathcal{X}_{\text{nil}} \), and \( \text{Im} \pi_\lambda = \overline{X}_\lambda \). This proves (i).

We show (iii). Take \((x, v) \in \mathcal{X}_{\text{nil}}\). Up to \( H \)-conjugate, we may assume that \((x, v) \in \mathfrak{h}_{\text{nil}} \times M_n\). Let \( x' \) be the restriction of \( x \) on \( M_n \). Let \( W = E x' v \) for \( V_1 = M_n \) under the notation in 7.3. Let \( \lambda^{(1)} \) be the Jordan type of \( x'|_W \). Then \((x'|_W, v) \in \mathcal{O}^{(\lambda^{(1)})}_{\text{nil}} \) in \( W \). Let \( P \) be the stabilizer of \( W \). Then \( x'' = x|_W \) gives an element in \( \mathfrak{sp}(W)_{\text{nil}} \). Assume that \( x'' \in \mathcal{O}_\lambda \) with \( \lambda'' = (\lambda^{(2)}, \lambda^{(3)}) \). Then \((x, v) \in X_\lambda\) with \( \lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) \) in \( \mathcal{P}_{n,3} \). This implies that \( \mathcal{X}_{\text{nil}} = \bigcup_{\lambda} X_\lambda \). The above discussion shows that the \( H \)-conjugates of \((x, v)\) determines \( \lambda \) uniquely. Hence they are disjoint, and (iii) holds.

**Remark 7.6.** It follows from the construction, \( X_\lambda \) is a single \( H \)-orbit if \( m_1 = 0 \). Since \( \mathfrak{g}^0 \) contains infinitely many \( H \)-orbits by Proposition 1.8, some of \( X_\lambda \) contains infinitely many \( H \)-orbits.

**7.7.** For \( m \in \mathcal{P}_{n,3} \), we define \( \lambda = \lambda(m) \in \mathcal{P}_{n,3} \) by \( \lambda^{(i)} = (m_i) \) for \( i = 1, 2, 3 \). Also we define a subset \( \mathcal{P}(m) \) of \( \mathcal{P}_n \) as the set of \( \lambda \in \mathcal{P}_{n,3} \) such that \( |\lambda^{(i)}| = m_i \) for \( i = 1, 2, 3 \). For \( m \in \mathcal{P}^0_{n,3} \), put

(7.7.1) \[
\mathcal{P}(m) = \bigsqcup_{0 \leq k \leq m_2} \mathcal{P}(m(k)).
\]

We have the following result.

**Proposition 7.8.** Assume that \( m \in \mathcal{P}^0_{n,3} \). Then we have

(i) \( \mathcal{X}_{m,\text{nil}} = \overline{X}_{\lambda(m)} \).

(ii) \( \dim \mathcal{X}_{m,\text{nil}} = \dim \mathcal{X}_{m,\text{nil}} = 2n^2 + m_1 \).
(iii) For \( \mu \in \tilde{\mathcal{P}}(m) \), we have \( X_\mu \subset \mathcal{X}_{m,\text{nil}} \).

\textbf{Proof.} We show (iii). Since \( m_3 = 0 \), we have \( \mathcal{D}_{m_1 + m_2} = \mathcal{D} \); hence we can write as

\[ \mathcal{X}_{m,\text{nil}} = \bigcup_{g \in H} g(n \times M_{m_1}). \]

Take \((x, v) \in X_\mu\). Up to \( H\)-conjugate, we may assume that \((x, v) \in \mathcal{M}_\mu\), where \( \mathcal{M}_\mu \) is as in (7.4.1). Assume that \( \mu \in \tilde{\mathcal{P}}(m) \). Let \( P \) be the stabilizer of \( W = M_{m_1} \), and \( L \simeq GL(W) \times Sp(\overline{W}) \). We may further assume that \((x, v)\) is of the form \((x_1 + x_2 + z, v) \in \mathfrak{h}_{\text{nil}} \times V\), where \((x_1, v) \in \mathcal{O}_1, x_2 \in \mathcal{O}_2, z \in \mathfrak{n}_P\) in the notation of 7.4. Hence, up to \( H\)-conjugate, we can take \( v \in M_{m_1} \) and \( x \in \mathfrak{n} \). It follows that \( X_\mu \subset \mathcal{X}_{m,\text{nil}} \). This proves (iii). Now by (iii), \( X_{\Lambda(m)} \subset \mathcal{X}_{m,\text{nil}} \). By Lemma 7.5 (ii), \( \dim X_{\Lambda(m)} = 2 \dim U_P + \dim \mathcal{O}_1 + \dim \mathcal{O}_2 \). Since \( \mathcal{O}_1 \) corresponds to \(((m_1), -)\), we have \( \dim \mathcal{O}_1 = m_1^2 \) by (7.3.2). On the other hand, since \( \mathcal{O}_2 \) corresponds to \(((m_2), -)\), which is the regular nilpotent orbit in \((\mathfrak{h}_2)_{\text{nil}}\). Thus dim \( \mathcal{O}_2 = \dim H_2 - m_2 \). It follows that

\[
\dim X_{\Lambda(m)} = (\dim H - \dim G_1 - \dim H_2) + m_1^2 + (\dim H_2 - m_2) = 2n^2 + m_1.
\]

By the previous discussion, we have

\[
\dim X_{\Lambda(m)} \leq \dim \mathcal{X}_{m,\text{nil}} \leq \dim \tilde{\mathcal{X}}_{m,\text{nil}}.
\]

By (7.1.2) (note that \( m_3 = 0 \)), the above inequalities are actually equalities. Since \( X_{\Lambda(m)} \) is irreducible, we conclude that \( \overline{X}_{\Lambda(m)} = \mathcal{X}_{m,\text{nil}} \). Hence (i) holds. (ii) also follows from this. \( \square \)

7.9. Let \( P = LU_P \) be a parabolic subgroup of \( H \), where \( L \) is a Levi subgroup, and \( U_P \) is the unipotent radical of \( P \). Let \( \pi_P : P \to L \) be the natural projection. Let \( \mathcal{O}' \) be an \( L\)-orbit in \((\text{Lie } L)_{\text{nil}}\). Let \( \mathcal{O} \) be an \( H\)-orbit in \( \mathfrak{h}_{\text{nil}} \) such that \( \mathcal{O} \cap \pi_P^{-1}(\mathcal{O}') \) is open dense in \( \pi_P^{-1}(\mathcal{O}') \). For any \( x \in \mathcal{O} \), consider the variety

\[
\mathcal{P}_x = \{ gP \in H/P \mid g^{-1}x \in \pi_P^{-1}(\mathcal{O}') \}.
\]

Then clearly \( \mathcal{P}_x \neq \emptyset \). The following result can be proved in a similar way as [Sh, Prop. 6.3].

\textbf{Proposition 7.10.} Under the setting in 7.9, assume that \( x \in \mathcal{O} \).

(i) \( \mathcal{P}_x \) consists of one point.
(ii) \( \dim Z_H(x) = \dim Z_L(x') \) for \( x' \in \mathcal{O}' \).
(iii) Let \( x_1 \in \pi_P^{-1}(\mathcal{O}') \) be such that \( \dim Z_H(x_1) = \dim Z_H(x) \). Then \( x_1 \in \mathcal{O} \).
(iv) Take \( x_1 \in \mathcal{O} \cap \pi_P^{-1}(\mathcal{O}') \) and put \( x' = \pi_P(x_1) \). Let \( Q = Z_P(x') \). Then \( \dim Z_Q(x_1) = \dim Z_H(x_1) \). In particular,

\[
Z_H(x_1) = Z_P(x_1) = Z_Q(x_1).
\]
(v) $P$ acts transitively on $\mathcal{O} \cap \pi_p^{-1}(\mathcal{O}')$, and $Q$ acts transitively on $\mathcal{O} \cap \pi_p^{-1}(\mathcal{O}')$.

Proof. For the sake of completeness, we give an outline of the proof below. First we show that

$$\dim \mathcal{P}_x = 0.$$  \hspace{1cm} (7.10.1)

Replacing by $H$-conjugate, we may assume that $x \in \mathcal{O} \cap \pi_p^{-1}(\mathcal{O}')$. Put $x' = \pi_p(x) \in \mathcal{O}'$. We have $\dim \pi_p^{-1}(x) = \dim U_p$. Put $c = \dim \mathcal{O}, c' = \dim \mathcal{O}'$. By [L1, Prop.1.2] (actually a similar argument works also for the Lie algebra case, see [X2, Prop. 3.1]), we have $\dim (\mathcal{O} \cap \pi_p^{-1}(x')) \leq (c - c')/2$. Since $\mathcal{O} \cap \pi_p^{-1}(\mathcal{O}')$ is open dense in $\pi_p^{-1}(\mathcal{O}')$, $\mathcal{O} \cap \pi_p^{-1}(x')$ is open dense in $\pi_p^{-1}(x')$. It follows that

$$\dim U_p \leq (c - c')/2.$$  

On the other hand, by Proposition 3.1 (ii) in [X2] (or [L1, Prop. 1.2 (b)]), we have

$$\dim \mathcal{P}_x \leq (\dim U - c/2) - (\dim U_L - c'/2)$$

$$= \dim U_p - (c - c')/2,$$

where $U_L = U \cap L$ is a maximal unipotent subgroup in $L$. Hence $\dim \mathcal{P}_x \leq 0$. Since $\mathcal{P}_x \neq \emptyset$, we obtain (7.10.1). We also have $c - c' = 2 \dim U_p$. This implies that $\dim Z_H(x) = \dim Z_L(x')$. Hence (ii) holds. Now consider (iv). Based on the above computation, in a similar way as in [Sh, Prop. 6.3], we can show dim $Z_H(x_1) = \dim Z_Q(x_1)$. Since it is known that $Z_H(x_1)$ is connected ([X2, 2.14]) in the case of characteristic 2, we have $Z_H(x_1) = Z_Q(x_1)$. Hence $Z_H(x_1) = Z_P(x_1)$, which proves (iv). Put

$$\mathcal{U} = \{(x_1, gP) \in \mathfrak{h}_{\text{nil}} \times H/P \mid g^{-1}x_1 \in \pi_p^{-1}(\mathcal{O}')\}.$$  

Then $\mathcal{U} \simeq H \times P \pi_p^{-1}(\mathcal{O}')$ and so $\mathcal{U}$ is irreducible. Let $f : \mathcal{U} \rightarrow \mathfrak{h}_{\text{nil}}$ be the first projection, and put $\mathcal{U}_\phi = f^{-1}(\phi')$. Then $\mathcal{U}_\phi \simeq H \times P (\mathcal{O} \cap \pi_p^{-1}(\mathcal{O}'))$, and so $\mathcal{U}_\phi$ is also irreducible. For any $x \in \mathcal{O}$, $\dim f^{-1}(x) = 0$ by (7.10.1). Thus $\dim \mathcal{U}_\phi = \dim \mathcal{O}$. Since $f : \mathcal{U}_\phi \rightarrow \mathcal{O}$ is an $H$-equivariant surjective map, for any $\xi \in \mathcal{U}_\phi$, the $H$-orbit $H\xi$ is open dense in $\mathcal{U}_\phi$. It follows that

$$H \text{ acts transitively on } \mathcal{U}_\phi.$$  \hspace{1cm} (7.10.3)

Take $x, x_1 \in \mathcal{O} \cap \pi_p^{-1}(\mathcal{O}')$. Since $(x, P), (x_1, P) \in \mathcal{U}_\phi$, there exists $g \in P$ such that $gx = x_1$. This proves the first statement of (v). Now assume that $x_1, x_2 \in \mathcal{O} \cap \pi_p^{-1}(\mathcal{O}')$. Then there exists $g \in P$ such that $gx_1 = x_2$. But since $\pi_p$ is $P$-equivariant, $g \in Z_P(x) = Q$. This proves the second statement of (v).

We show (i). We may assume that $x \in \mathcal{O} \cap \pi_p^{-1}(\mathcal{O}')$. Then $P \in \mathcal{P}_x$. Assume that $gP \in \mathcal{P}_x$. Then $(x, P), (x, gP) \in \mathcal{U}_\phi$, and so there exists $h \in Z_H(x)$ such that $gP = hP$ by (7.10.3). But by (iv), $h \in P$, and so (i) holds. (iii) is proved in the same way as in [Sh]. The proposition is proved. 

7.11. We return to the original setting. For later application, we shall consider some open dense subvariety $X_0^0$ of $X_\lambda$. As in 7.4, let $\mathcal{O}_1$ be a $G_1$-orbit in $(\mathfrak{g}_1)_{\text{nil}} \times W$ and $\mathcal{O}_2$ an $H_2$-orbit in $(\mathfrak{h}_2)_{\text{nil}}$. We denote by $\mathcal{O}_1'$ the $G_1$-orbit in $(\mathfrak{g}_1)_{\text{nil}}$ which is the
projection of $\mathcal{O}_1$ to $(g_1)_{\text{nil}}$, hence $\mathcal{O}_1' = \mathcal{O}_\lambda(1)$ (see 7.3). We define a subset $\mathcal{M}_\lambda^0$ of $\mathcal{M}_\lambda$ as the set of $(x, v)$ such that the orbit $\mathcal{O}$ containing $x$ satisfies the property that $\mathcal{O} \cap \pi^{-1}_P(\mathcal{O}_1' \times \mathcal{O}_2)$ is open dense in $\pi^{-1}_P(\mathcal{O}_1' \times \mathcal{O}_2) = (\mathcal{O}_1' \times \mathcal{O}_2) + n_P$. Clearly, $\mathcal{M}_\lambda^0$ is open dense in $\mathcal{M}_\lambda$. We define

$$(7.11.1) \quad X_\lambda^0 = \bigcup_{g \in H} g(\mathcal{M}_\lambda^0).$$

Let $\pi_\lambda : \tilde{X}_\lambda \to \overline{X}_\lambda$ be as in 7.4. We define

$$\tilde{X}_\lambda^0 = H \times^P \mathcal{M}_\lambda^0.$$ 

Since $\tilde{X}_\lambda \simeq H \times^P \mathcal{M}_\lambda$, and $\mathcal{M}_\lambda$ is open dense in $\mathcal{M}_\lambda$, $\tilde{X}_\lambda^0$ is open dense in $\tilde{X}_\lambda$, and one can check that $\pi^{-1}_\lambda(\tilde{X}_\lambda^0) = \tilde{X}_\lambda^0$. Since $\pi_\lambda$ is proper, $X_\lambda^0$ is open dense in $X_\lambda$. Let $\pi_\lambda : \tilde{X}_\lambda \to X_\lambda$ be the restriction of $\pi_\lambda$ on $\tilde{X}_\lambda^0$. We have a lemma.

**Lemma 7.12.** $\pi_\lambda^0$ gives an isomorphism $\tilde{X}_\lambda^0 \simeq X_\lambda^0$.

**Proof.** $\pi_\lambda^0$ is $H$-equivariant and surjective. Take $(x, v) \in \mathcal{M}_\lambda^0$, and put $x' = \pi_P(x) = (x_1, x_2)$ with $(x_1, v) \in \mathcal{O}_1, x_2 \in \mathcal{O}_2$. Furthermore, by our assumption, $\mathcal{O} \cap \pi^{-1}_P(\mathcal{O}_1')$ is open dense in $\pi^{-1}_P(\mathcal{O}_1')$, where $\mathcal{O}$ is the $H$-orbit containing $x$, and $\mathcal{O}_1' = \mathcal{O}_1' \times \mathcal{O}_2$. Hence, under the notation of (7.9.1), we have

$$\pi^{-1}_\lambda(x, v) \simeq \{gP \in H/P \mid g^{-1}(x, v) \in \mathcal{M}_\lambda^0 \} \subset \mathcal{P}_x.$$ 

By Proposition 7.10, we have $\mathcal{P}_x = \{P\}$. It follows that $\pi^{-1}_\lambda(x, v) = \{P\}$. This shows that $\pi_\lambda^0$ gives a bijective morphism from $\tilde{X}_\lambda^0$ onto $X_\lambda^0$. The map $g(x, v) \mapsto gP$ gives a well-defined morphism $X_\lambda^0 \to \tilde{X}_\lambda^0$, which is the inverse of $\pi_\lambda$. \hfill $\square$

**7.13.** From now on, we fix $m \in \mathfrak{D}_{n, 3}$. Put $\mathcal{B} = H/B$. For any $z = (x, v) \in \mathcal{X}_{m, \text{nil}}$, define

$$\mathcal{B}_z = \{gB \in \mathcal{B} \mid g^{-1}x \in \mathfrak{n}, g^{-1}v \in M_n\},$$

$$\mathcal{B}_z^{(m)} = \{gB \in \mathcal{B} \mid g^{-1}x \in \mathfrak{n}, g^{-1}v \in M_{m_1}\}.$$ 

Hence $\mathcal{B}_z^{(m)} \subset \mathcal{B}_z$ are closed subvarieties of $\mathcal{B}$.

For each integer $d \geq 0$, we define a subset $X(d)$ of $\mathcal{X}_{m, \text{nil}}$ by

$$(7.13.1) \quad X(d) = \{z \in \mathcal{X}_{m, \text{nil}} \mid \dim \mathcal{B}_z^{(m)} = d\}.$$ 

Then $X(d)$ is a locally closed subvariety of $\mathcal{X}_{m, \text{nil}}$, and $\mathcal{X}_{m, \text{nil}} = \bigsqcup_{d \geq 0} X(d)$. We consider the Steinberg varieties $\mathcal{Z}_d^{(m)}$ and $\mathcal{Z}_d^{(1)}$, which are a generalization of the Steinberg variety considered in 4.4.
\[ \mathcal{Z}^{(m)} = \{(z, gB, g'B) \in \mathcal{X} \times \mathcal{B} \times \mathcal{B} \mid (z, gB) \in \mathcal{X}_m, (z, g'B) \in \mathcal{X}_m \} \]

\[ \mathcal{Z}_1^{(m)} = \{(z, gB, g'B) \in \mathcal{X}_{\text{nil}} \times \mathcal{B} \times \mathcal{B} \mid (z, gB) \in \mathcal{X}_{\text{nil}, m}, (z, g'B) \in \mathcal{X}_{\text{nil}, m} \}. \]

Recall, for \( m = (m_1, m_2, 0) \in \mathcal{Q}_{n,3} \), that \( W_{m}^{2} = S_{m_1} \times W_{m_2} \). We show the following lemma.

**Lemma 7.14.** Under the notation in 7.13,

(i) \( \dim \mathcal{Z}_1^{(m)} = \dim \mathcal{X}_{\text{nil}, m} \). The set of irreducible components of \( \mathcal{Z}_1^{(m)} \) with maximal dimension is parametrized by \( w \in W_{m}^{2} \).

(ii) \( \dim \mathcal{Z}^{(m)} = \dim \mathcal{Z}_1^{(m)} + n \). The set of irreducible components of \( \mathcal{Z}^{(m)} \) with maximal dimension is parametrized by \( w \in W_{m}^{2} \).

(iii) Assume that \( X(d) \neq \emptyset \). For any \( z \in X(d) \), we have

\[ \dim \mathcal{B}_2^{(m)} \leq \frac{1}{2} (\dim \mathcal{X}_{\text{nil}, m} - \dim X(d)). \]

In particular, \( \pi_1^{(m)} : \mathcal{X}_{\text{nil}, m} \to \mathcal{X}_{\text{nil}, m} \) is semismall with respect to the stratification \( \mathcal{X}_{\text{nil}, m} = \bigsqcup_d X(d) \).

**Proof.** Let \( p_1 : \mathcal{Z}_1^{(m)} \to \mathcal{B} \times \mathcal{B} \) be the projection to the second and third factors. For each \( w \in W_n \), let \( \mathcal{O}_w \) be the \( H \)-orbit of \((B, wB)\) in \( \mathcal{B} \times \mathcal{B} \). We have \( \mathcal{B} \times \mathcal{B} = \bigsqcup_{w \in W_n} \mathcal{O}_w \). Put \( Z_w = p_1^{-1}(\mathcal{O}_w) \). Then \( \mathcal{Z}_1^{(m)} = \bigsqcup_{w \in W_n} Z_w \). Here \( Z_w \) is a vector bundle over \( \mathcal{O}_w \cong H/(B \cap wBw^{-1}) \) with fibre isomorphic to

\[ (n \cap w(n)) \times (M_{m_1} \cap w(M_{m_1})). \]

(Note that \( \mathcal{Y}_{p_2} = n \oplus \mathcal{O}_{m_1+m_2} = n \) since \( m_3 = 0 \).) We have

\[ \dim Z_w = \dim H - \dim T + \dim (M_{m_1} \cap w(M_{m_1})). \]

Here \( \dim (M_{m_1} \cap w(M_{m_1})) \leq m_1 \), and the equality holds if and only if \( w(M_{m_1}) = M_{m_1} \), namely \( w \in W_{m}^{2} \). It follows that \( \dim Z_w \leq 2n^2 + m_1 \) and the equality holds if and only if \( w \in W_{m}^{2} \). Hence by Proposition 7.8, \( \dim Z_w = \dim \mathcal{X}_{\text{nil}, m} \) if \( Z_w \) has the maximal dimension. This implies that \( \{Z_w \mid w \in W_{m}^{2}\} \) gives the set of irreducible components of \( \mathcal{Z}_1^{(m)} \) with maximal dimension. This proves (i). For (ii), we consider \( \mathcal{Z}_w = p^{-1}(\mathcal{O}_w) \), where \( p : \mathcal{Z}^{(m)} \to \mathcal{B} \times \mathcal{B} \) is the projection to the second and third factors. Then \( \mathcal{Z}_w \) is a vector bundle over \( \mathcal{O}_w \), with fibre isomorphic to

\[ t \times (n \cap w(n)) \times (M_{m_1} \cap w(M_{m_1})). \]

Hence (ii) is proved in a similar way as (i).
We show (iii). Let \( q_1 : \mathcal{Z}_{\mathbf{m}}^{(1)} \to \mathcal{R}_{\mathbf{m}, \text{nil}} \) be the projection on the first factor. For each \( z \in \mathcal{R}_{\mathbf{m}, \text{nil}}, q_1^{-1}(z) \simeq \mathcal{R}_{2}^{(z)} \times \mathcal{R}_{2}^{(z)}. \) By (7.13.1), we have
\[
\dim q_1^{-1}(X(d)) = \dim X(d) + 2d.
\]
Since \( \dim q_1^{-1}(X(d)) \leq \dim \mathcal{Z}_{\mathbf{m}}^{(1)} = \dim \mathcal{R}_{\mathbf{m}, \text{nil}}, \) we see that \( 2d \leq \dim \mathcal{R}_{\mathbf{m}, \text{nil}} - \dim X(d). \) This proves (iii). The lemma is proved. \( \square \)

8. Springer correspondence for \( g^0 \)

8.1. In this section we shall prove the Springer correspondence for \( g^0. \) In [Sh], the Springer correspondence was established for the exotic symmetric space of level \( r \) for arbitrary \( r \geq 1. \) Once Theorem 6.10 and Theorem 6.21 are proved, a similar discussion as in [Sh, 7] can be applied to our situation as the special case where \( r = 3. \)

Assume that \( \mathbf{m} \in \mathcal{D}_{n,3}. \) We consider the variety \( \mathcal{Z}^{(\mathbf{m})} \) as in 7.13. We denote by \( \varphi : \mathcal{Z}^{(\mathbf{m})} \to \mathcal{R}_{\mathbf{m}} \) the map \( (z, gB, g'B) \mapsto z. \) Let \( \alpha : \mathcal{Z}^{(\mathbf{m})} \to \mathfrak{t} \) be the map defined by \( (x, v, gB, g'B) \mapsto p_1(g^{-1}x), \) similarly to 4.4. As in 4.4, we have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{Z}^{(\mathbf{m})} & \xrightarrow{\alpha} & \mathfrak{t} \\
\varphi \downarrow & & \downarrow \omega_1 \\
\mathcal{R}_{\mathbf{m}} & \xrightarrow{\sim} & \Xi,
\end{array}
\]
where \( \sim \) is the composite of the projection \( \mathcal{R}_{\mathbf{m}} \to \mathfrak{h} \) and \( \omega. \) Put \( \sigma = \omega_1 \circ \alpha, \) and \( d'_m = \dim \mathcal{R}_{\mathbf{m}, \text{nil}} = 2n^2 + m_1. \) We define a constructible sheaf \( \mathcal{T} \) on \( \Xi \) by
\[
(8.1.1) \quad \mathcal{T} = \mathcal{H}^{2d'_m}(\sigma_1 \mathbf{Q}_l) = R^{2d'_m} \sigma_1 \mathbf{Q}_l.
\]

Note that if \( m_1 = 0, \mathcal{Z}^{(\mathbf{m})} \) coincides with \( Z \) in 4.4, and \( \mathcal{T} \) is nothing but the sheaf \( \mathcal{F} \) defined in (4.4.1). Also \( \mathcal{F} \) corresponds to the special case where \( r = 3 \) of the sheaf \( \mathcal{F} \) defined in [Sh, 7.1]. The following properties of \( \mathcal{F} \) are obtained by similar arguments as in [Sh], so we omit the proof. The discussion in the proof of Lemma 4.5 can be generalized to this case (see also [Sh, Lemma 7.2]), and we obtain

Lemma 8.2. The sheaf \( \mathcal{F} \) is a perfect sheaf on \( \Xi. \)

Let \( p : \mathcal{Z}^{(\mathbf{m})} \to \mathcal{B} \times \mathcal{B} \) be the projection to the second and third factors. We put \( \mathcal{Z}_w^{(\mathbf{m})} = p^{-1}(\mathcal{O}_w) \) for \( w \in W_n, \) where \( \mathcal{O}_w \) is as in the proof of Lemma 4.5. Let \( \sigma_w \) be the restriction of \( \sigma \) on \( \mathcal{Z}_w^{(\mathbf{m})}, \) and put \( \mathcal{F}_w = \mathcal{H}^{2d'_m}(\sigma_w) \mathbf{Q}_l. \) In a similar way as in the proof of Proposition 4.6 (see also [Sh, Prop. 7.3], here the role of \( W_n \) is replaced by \( W_3 \mathfrak{g} \)), we can prove

Proposition 8.3. \( \mathcal{F} \simeq \bigoplus_{w \in W_3 \mathfrak{g}} \mathcal{F}_w. \)
8.4. By the Künneth formula, we have \( \varphi_lQ_t \simeq \pi_1^{(m)}Q_t \otimes \pi_1^{(m)}Q_t \). By Theorem 6.21, \( \pi_1^{(m)}Q_t \) has a natural structure of \( W_m^2 \)-module. Hence \( \varphi_lQ_t \) has a structure of \( W_m^2 \times W_m^2 \)-module. It follows that \( \mathcal{F} = \mathcal{H}^{2d_m}(\pi_1Q_t) \simeq \mathcal{H}^{2d_m}(\bar{\omega}_1(\varphi_lQ_t)) \) has a natural action of \( W_m^2 \times W_m^2 \). Under the decomposition of \( \mathcal{F} \) in Proposition 8.3, the action of \( W_m^2 \times W_m^2 \) has the following property (see the discussion in 4.7). For each \( w_1, w_2 \in W_m^2 \),

\[
(w_1, w_2) \cdot \mathcal{I}_w = \mathcal{I}_{w_1w_2^{-1}}. \tag{8.4.1}
\]

Let \( a_0 \) be the element in \( \Xi \) corresponding to the \( S_n \)-orbit of \( 0 \in t \), and \( \mathcal{I}_{a_0} \) be the stalk of \( \mathcal{I} \) at \( a_0 \in \Xi \). By Proposition 8.3, we have a decomposition

\[
\mathcal{I}_{a_0} = \bigoplus_{w \in W_m^2} (\mathcal{I}_w)_{a_0}, \tag{8.4.2}
\]

where \((\mathcal{I}_w)_{a_0}\) is the stalk of \( \mathcal{I}_w \) at \( a_0 \). \( W_m^2 \times W_m^2 \) acts on \( \mathcal{I}_{a_0} \) following (8.4.1). In a similar way as in (4.5.2), one can show that \( \mathcal{I}_w \simeq (\omega_1)_!Q_t \). Since \( \omega_1^{-1}(a_0) = \{0\} \in t \), \( (\mathcal{I}_w)_{a_0} \simeq H^0_c(\omega_1^{-1}(a_0), Q_t) \sim Q_t \). Thus we have proved the following result, which is the stalk version of Proposition 4.8.

**Proposition 8.5.** \( \mathcal{I}_{a_0} \) has a structure of \( W_m^2 \times W_m^2 \)-module, which coincides with the two-sided regular representation of \( W_m^2 \).

8.6. We consider the map \( \pi_1^{(m)} : \tilde{\mathcal{I}}_{m,\text{nil}} \rightarrow \mathcal{I}_{m,\text{nil}} \), and put \( K_{m,1} = (\pi_1^{(m)})_!Q_t[d'_m] \). By Lemma 7.14 (iii), \( \pi_1^{(m)} \) is semismall. Hence \( K_{m,1} \) is a semisimple perverse sheaf on \( \mathcal{I}_{m,\text{nil}} \), and is decomposed as

\[
K_{m,\text{nil}} \simeq \bigoplus_A V_A \otimes A, \tag{8.6.1}
\]

where \( A \) is a simple perverse sheaf which is isomorphic to a direct summand of \( K_{m,1} \), and \( V_A = \text{Hom}(K_{m,1}, A) \) is the multiplicity space for \( A \). The following result is a counter part of Proposition 4.11 to the case of nilpotent variety (see also Proposition 7.8 in [Sh]).

**Proposition 8.7.** Under the notation above, put \( m_A = \dim V_A \) for each direct summand \( A \). Then we have

\[
\sum_A m_A^2 = |W_m^2|. \]

**Proof.** By 8.4, we have

\[
\mathcal{I}_{a_0} \simeq \mathcal{H}^{2d_m}(\tilde{\mathcal{I}}_{m,\text{nil}}, \pi_1^{(m)}Q_t \otimes \pi_1^{(m)}Q_t) \\
\simeq \mathcal{H}^0_c(\tilde{\mathcal{I}}_{m,\text{nil}}, K_{m,1} \otimes K_{m,1}).
\]
By applying (8.6.1), we have

$$\dim \mathcal{T}_{a_0} \simeq \sum_{A, A'} (m_A m_{A'}) \dim \mathcal{H}^0_c(\mathcal{X}_{m, \text{nil}}, A \otimes A').$$

Apply Lemma 4.9 for $X = \mathcal{X}_{m, \text{nil}}$. Then $\mathcal{H}^0_c(\mathcal{X}_{m, \text{nil}}, A \otimes A') \neq 0$ only when $D(A) = A'$, in which case $\dim \mathcal{H}^0_c(\mathcal{X}_{m, \text{nil}}, A \otimes A') = 1$. But since $K_{m,1}$ is self-dual, $m_A = m_{D(A)}$ for each $A$. It follows that $\dim \mathcal{T}_{a_0} = \sum_A m_A^2$. On the other hand, by Proposition 8.5, we have $\dim \mathcal{T}_{a_0} = |W^2_m|$. This proves the proposition. \(\square\)

8.8. We consider the map $\pi^{(m)} : \widetilde{\mathcal{X}}_m \to \mathcal{X}_m$. By Theorem 6.21, $\pi^{(m)}_i \mathcal{Q}_i[d_m]$ is a semisimple perverse sheaf on $\mathcal{X}_m$, equipped with $W^2_m$-action, and is decomposed as

$$\pi^{(m)}_i \mathcal{Q}_i[d_m] \simeq \bigoplus_{\rho \in (W^2_m)^\wedge} \rho \otimes K_{\rho},$$

where $K_{\rho}$ is a simple perverse sheaf on $\mathcal{X}_m$, and $d_m = \dim \mathcal{X}_m$. More precisely, it is given as $K_{\rho} = \text{IC}(\mathcal{X}_{m(k)}; \mathcal{L}_{\rho^l})|_{d_m(k)}$ if $\rho = \rho^l$ for $\rho^l \in S^\wedge_{m(k)}$. We consider the complex $(\pi^{(m)}_i \mathcal{Q}_i[d'_m])$ for the map $\pi^{(m)}_i : \widetilde{\mathcal{X}}_{m, \text{nil}} \to \mathcal{X}_{m, \text{nil}}$, where $d'_m = \dim \mathcal{X}_{m, \text{nil}}$. The following result gives the Springer correspondence for $W^2_m$. (Compare Theorem 8.9 and Corollary 8.11 with Theorem 7.12 and Corollary 7.14 in [Sh]).

**Theorem 8.9** (Springer correspondence for $W^2_m$). Assume that $m \in \mathcal{D}_{n,3}$. Then $(\pi^{(m)}_i \mathcal{Q}_i[d'_m])$ is a semisimple perverse sheaf on $\mathcal{X}_{m, \text{nil}}$, equipped with $W^2_m$-action, and is decomposed as

$$\pi^{(m)}_i \mathcal{Q}_i[d'_m] \simeq \bigoplus_{\rho \in (W^2_m)^\wedge} \rho \otimes L_{\rho},$$

where $L_{\rho}$ is a simple perverse sheaf on $\mathcal{X}_{m, \text{nil}}$ such that

$$K_{\rho}|_{\mathcal{X}_{m, \text{nil}}} \simeq L_{\rho}[d_m - d'_m].$$

**Proof.** As discussed in (8.6.1), $K_{m,1} = (\pi^{(m)}_i \mathcal{Q}_i[d'_m])$ is a semisimple perverse sheaf. Since $K_{m,1}$ is the restriction of $\pi^{(m)}_i \mathcal{Q}_i$ on $\mathcal{X}_{m, \text{nil}}$, we have a natural homomorphism

$$\alpha : \mathcal{Q}_i[W^2_m] \simeq \text{End}(\pi^{(m)}_i \mathcal{Q}_i) \to \text{End} K_{m,1}.$$ 

In order to prove (8.9.1), it is enough to show that $\alpha$ is an isomorphism. By Proposition 8.7, we have $\dim \text{End} K_{m,1} = |W^2_m|$. Thus we have only to show that $\alpha$ is injective. Note that $\mathcal{T}_{a_0} = \mathcal{H}^0_c(\mathcal{X}_{m, \text{nil}}, K_{m,1} \otimes K_{m,1})$, and $K_{m,1}$ is decomposed as $K_{m,1} = \bigoplus_{\rho} \rho \otimes (K_{\rho}|_{\mathcal{X}_{m, \text{nil}}})$ by (8.8.1). This decomposition determines the $W^2_m \times W^2_m$-module structure of $\mathcal{T}_{a_0}$. But by Proposition 8.5, $\mathcal{T}_{a_0}$ is isomorphic to the two-sided regular representation of $W^2_m$. This implies, in particular, $K_{\rho}|_{\mathcal{X}_{m, \text{nil}}} \neq 0$ for any
\( \rho \in W_m^\natural \). Hence \( \alpha \) is injective, and so (8.9.1) holds. Now (8.9.2) follows by comparing (8.8.1) and (8.9.1). The theorem is proved. \( \square \)

**8.10.** For each \( \mathbf{m} \in \mathcal{D}_{n,3}^0 \), we denote by \( (W^\wedge_{n,3})_m \) the set of irreducible representations \( \hat{\rho} \in W^\wedge_{n,3} \) corresponding to \( \rho \in S^\wedge_{m(k)} \) for various \( 0 \leq k \leq m_2 \). Then we have

\[
(8.10.1) \quad W^\wedge_{n,3} = \prod_{m \in \mathcal{D}_{n,3}^0} (W^\wedge_{n,3})_m.
\]

For each \( \rho \in S^\wedge_{m(k)} \), we can construct \( \rho^\natural \in (W^\wedge_{m})^\wedge \) as in 6.7, and the map \( \rho \mapsto \rho^\natural \) gives a bijective correspondence

\[
(8.10.2) \quad \prod_{0 \leq k \leq m_2} S^\wedge_{m(k)} \simeq (W^\wedge_{m})^\wedge.
\]

It follows that the correspondence \( \hat{\rho} \leftrightarrow \rho \leftrightarrow \rho^\natural \) gives a bijective correspondence

\[
(8.10.3) \quad (W^\wedge_{n,3})_m \simeq (W^\wedge_{m})^\wedge, \quad \hat{\rho} \leftrightarrow \rho^\natural.
\]

We consider the map \( \overline{\pi}_m : \pi^{-1}(\mathcal{X}_m) \to \mathcal{X}_m \). Then by Theorem 6.10, \( (\overline{\pi}_m) : \mathcal{Q}[d_m] \) is a semisimple perverse sheaf, equipped with \( W_{n,3} \)-action, and is decomposed as

\[
(8.10.4) \quad (\overline{\pi}_m) : \mathcal{Q}[d_m] \simeq \bigoplus_{\hat{\rho} \in (W^\wedge_{n,3})_m} \hat{\rho} \otimes K_{\rho^\natural},
\]

where \( K_{\rho^\natural} \) is a simple perverse sheaf on \( \mathcal{X}_m \) as defined in (8.8.1). Let \( \overline{\pi}_{m,1} : \pi^{-1}(\mathcal{X}_{m,\text{nil}}) \to \mathcal{X}_{m,\text{nil}} \) be the restriction of \( \overline{\pi}_m \) on \( \mathcal{X}_{m,\text{nil}} \). By applying (8.9.2), we see that \( (\overline{\pi}_{m,1}) : \mathcal{Q}[d'_m] \) is a semisimple perverse sheaf. As a corollary to Theorem 8.9, we obtain the Springer correspondence for \( W_{n,3} \).

**Corollary 8.11** (Springer correspondence for \( W_{n,3} \)). Assume that \( \mathbf{m} \in \mathcal{D}_{n,3}^0 \). Then \( (\overline{\pi}_{m,1}) : \mathcal{Q}[d'_m] \) is a semisimple perverse sheaf on \( \mathcal{X}_{m,\text{nil}} \), equipped with \( W_{n,3} \)-action, and is decomposed as

\[
(8.11.1) \quad (\overline{\pi}_{m,1}) : \mathcal{Q}[d'_m] \simeq \bigoplus_{\hat{\rho} \in (W^\wedge_{n,3})_m} \hat{\rho} \otimes L_{\rho^\natural},
\]

where \( L_{\rho^\natural} \) is the simple perverse sheaf on \( \mathcal{X}_{m,\text{nil}} \) as given in Theorem 8.9.
9. Determination of the Springer correspondence

9.1. In this section, we shall determine $L_\rho$ appearing in the Springer correspondence explicitly. Let $m = (m_1, m_2, 0) \in \mathcal{D}_{n,3}^0$. We define a variety $G_m$ by

$$G_m = \{(x, v, W_1) \mid (x, v) \in X_m, W_1: \text{isotropic,}
\quad \dim W_1 = m_1, x(W_1) \subset W_1, v \in W_1\}.$$ 

Let $\zeta: G_m \to X_m$ be the projection to the first two factors. Then the map $\pi^{(m)}: \tilde{\mathcal{X}}_m \to \mathcal{X}_m$ is factored as

$$\pi^{(m)}: \tilde{\mathcal{X}}_m \xrightarrow{\phi} G_m \xrightarrow{\zeta} \mathcal{X}_m;$$

where $\phi$ is defined by $(x, v, gB) \mapsto (x, v, gM_{m_1})$. $\phi$ is surjective since there exists an $x$-stable maximal isotropic subspace containing $W_1$. $\zeta$ is also surjective since $\pi^{(m)}$ is surjective. Since $m \in \mathcal{D}_{n,3}^0$, we have $\dim \tilde{\mathcal{X}}_m = \dim \mathcal{X}_m$. It follows that $\dim G_m = \dim X_m$.

In the case where $m_1 = 0$, $\pi^{(m)}: \tilde{\mathcal{X}}_m \to \mathcal{X}_m$ coincides with the map $\pi: \tilde{\mathcal{X}} \to \mathcal{X} = \mathfrak{h}$, where $\pi = \varphi, \zeta = \text{id}$.

By modifying the definition of $K_m$ in 6.14, we define a variety $\mathcal{H}_m$ by

$$\mathcal{H}_m = \{(x, v, W_1, \phi_1, \phi_2) \mid (x, v, W_1) \in \mathcal{G}_m,
\quad \phi_1: W_1 \cong V_0, \phi_2: W_1^\perp/W_1 \cong \nabla_0 \text{ (symplectic isom.)}\},$$

where $V_0 = M_{m_1}$ and $\nabla_0 = V_0^\perp/V_0$. We also define a variety $\tilde{\mathcal{Z}}_m$ by

$$\tilde{\mathcal{Z}}_m = \{(x, v, gB, \phi_1, \phi_2) \mid (x, v, gB) \in \tilde{\mathcal{X}}_m,
\quad \phi_1: gM_{m_1} \cong V_0, \phi_2: (gM_{m_1})^\perp/gM_{m_1} \cong \nabla_0\}.$$ 

As in 6.14, we consider $G_1 = GL(V_0), H_2 = Sp(\nabla_0)$, and $\mathfrak{h}_2 = \text{Lie } H_2$. The maps $\pi^2: \tilde{\mathcal{X}}' \to \mathcal{X}' = \mathfrak{h}_2, \pi^1: \tilde{\mathfrak{g}}_1 \to \mathfrak{g}_1$ are given as in 6.14. We have the following commutative diagram

$$\begin{array}{ccc}
\tilde{\mathfrak{g}}_1 \times \tilde{\mathcal{X}}' & \xleftarrow{\tilde{\sigma}} & \tilde{\mathcal{Z}}_m \\
\pi^1 \times \pi^2 \downarrow & & \tilde{\phi} \downarrow \phi \\
\mathfrak{g}_1 \times \mathcal{X}' & \xleftarrow{\sigma} & \mathcal{H}_m \\
\downarrow \zeta & & \downarrow \zeta \\
& \mathcal{X}_m, &
\end{array}$$

(9.1.3)
where morphisms are defined as
\[
q : (x, v, W_1, \phi_1, \phi_2) \mapsto (x, v, W_1), \\
\sigma : (x, v, W_1, \phi_1, \phi_2) \mapsto (\phi_1(x|_{W_1})\phi_1^{-1}, \phi_2(x|_{W_1^+/W_1})\phi_2^{-1}), \\
\tilde{\varphi} : (x, v, gB, \phi_1, \phi_2) \mapsto (x, v, (gM_{m_1}), \phi_1, \phi_2).
\]

\(\tilde{\sigma}, \tilde{q}\) are defined naturally.

One can check that both squares are cartesian squares. Moreover, it is easy to see that

(9.1.4) \(q\) is a principal bundle with fibre isomorphic to \(G_1 \times H_2\), and \(\sigma\) is a locally trivial fibration with smooth connected fibre of dimension \(\dim H + m_1\).

9.2. For a fixed \(k\), we consider the variety \(\tilde{\mathcal{Y}}_{m(k)}^+ = (\psi^{(m)})^{-1}(\mathcal{Y}_{m(k)}^0)\) as in 6.2, and let \(\mathcal{G}_{m(k),sr} = \zeta^{-1}(\mathcal{Y}_{m(k)}^0)\) be the locally closed subvariety of \(\mathcal{G}_{m}\). Here the varieties \(Y^0_k, \tilde{Y}^+_k\) are defined similarly as in Section 2 by replacing \(X\) by \(X'\). As the restriction of (9.1.3), we have the following commutative diagram

\[
\begin{array}{cccccc}
\bar{g}_{1,reg} \times \tilde{Y}_k & \leftrightarrow & \tilde{Z}_{m(k)}^+ & \longrightarrow & \tilde{\mathcal{Y}}_{m(k)}^+ \\
\downarrow & & \downarrow & & \varphi_0 \\
\bar{g}_{1,reg} \times Y_k^0 & \leftrightarrow & \mathcal{H}_{m(k),sr} & \longrightarrow & \mathcal{G}_{m(k),sr} \\
\downarrow & & & & \zeta_0 \\
& & \mathcal{G}_{m(k)}^0,
\end{array}
\]

where \(\mathcal{H}_{m(k),sr} = q^{-1}(\mathcal{G}_{m(k),sr})\) and \(\tilde{Z}_{m(k)}^+ = \tilde{q}^{-1}(\tilde{\mathcal{Y}}_{m(k)}^+),\) and \(\varphi_0, \zeta_0\) are restrictions of \(\varphi, \zeta\), respectively. The following result can be proved in a similar way as [Sh, (8.2.4)].

(9.2.2) The map \(\zeta_0\) gives an isomorphism \(\mathcal{G}_{m(k),sr} \simeq \mathcal{Y}_{m(k)}^0\).

Take \(\rho \in (W_{m}^2)^\wedge\). Then by (8.10.2), there exists an integer \(k\) and \(\rho_0 \in S_{m(k)}^\wedge\) such that \(\rho = \rho_0^2\). Thus \(K_\rho\) in (8.8.1) is given by \(K_\rho = IC(\mathcal{X}_{m(k)}, L_{\rho_0})[d_{m(k)}]\), where \(L_{\rho_0}\) is a simple local system on \(\mathcal{Y}_{m(k)}^0\). By (9.2.2), we can regard \(L_{\rho_0}\) as a simple local system on \(\mathcal{G}_{m(k),sr}\). We put \(A_\rho = IC(\mathcal{G}_{m(k),sr}, L_{\rho_0})[d_{m(k)}]\). Then \(A_\rho\) is an \(H\)-equivariant simple perverse sheaf on \(\mathcal{G}_{m(k),sr}\), which we regard as a perverse sheaf on \(\mathcal{G}_{m}\) by extension by zero.

We show the following result.

**Proposition 9.3.** Under the setting in 9.2, we have

(i) \(\varphi_{\mathcal{Q}_l}[d_{m}]\) is a semisimple perverse sheaf, equipped with \(W_{m}^2\)-action, and is decomposed as

\[
\varphi_{\mathcal{Q}_l}[d_{m}] \simeq \bigoplus_{\rho \in (W_{m}^2)^\wedge} \rho \otimes A_\rho.
\]

(9.3.1)
(ii) \( \zeta_! A_\rho \simeq K_\rho \).

**Proof.** By (5.14.1), we can write as

\[
(9.3.2) \quad (\pi_1)_! \mathcal{Q}_l[\dim g_1] \simeq \bigoplus_{\rho_1 \in S_{m_1}} \rho_1 \otimes A_{\rho_1}
\]

where \( A_{\rho_1} = IC(g_1, \mathcal{L}_\rho_1)[\dim g_1] \) is a simple perverse sheaf on \( g_1 \). On the other hand, by Theorem 5.7, we can write as

\[
(9.3.3) \quad (\pi_2)_! \mathcal{Q}_l[\dim h_2] \simeq \bigoplus_{\rho' \in W_{m_2}} \rho' \otimes A_{\rho'},
\]

where \( A_{\rho'} = IC(X'_k, \mathcal{L}_\rho'[d_k]) \) for some \( k \) and \( \rho_2 \in (S_k \times S_{m_2-k})^\wedge \) such that \( \rho' = \hat{\rho}_2 \), which is a simple perverse sheaf on \( X' \). By applying a similar argument as in 6.14 to the diagram (9.1.3), together with (9.1.4), one can find an \( H \)-equivariant simple perverse sheaf \( \tilde{A}_\rho \) on \( G_m \) such that

\[
(9.3.4) \quad q^* \tilde{A}_\rho[\beta_2] \simeq \sigma^*(A_{\rho_1} \boxtimes A_{\rho'})[\beta_1],
\]

where \( \beta_1 = \dim H + m_1 \), and \( \beta_2 = \dim G_1 + \dim H_2 \). Moreover, \( \rho \in W_{m_2}^\wedge \) is given by \( \rho = \rho_1 \boxtimes \rho' \in (S_{m_1} \times W_{m_2,2})^\wedge \). By using a similar argument as in (Sh, 8.2), based on the diagram (9.2.1), one can show that the restriction of \( \tilde{A}_\rho \) on \( G_{m(k),sr} \) coincides with \( \mathcal{L}_{\rho_0} \). Hence we have \( \tilde{A}_\rho = A_\rho \).

Put \( K_1 = (\pi_1)_! \mathcal{Q}_l[\dim g_1], K_2 = (\pi_2)_! \mathcal{Q}_l[\dim h_2] \), and also put \( K = \varphi_! \mathcal{Q}_l[d_m] \). Since both squares in (9.1.3) are cartesian, we have

\[
q^* K[\beta_2] \simeq \sigma^*(K_1 \boxtimes K_2)[\beta_1].
\]

In particular, \( K \) is a semisimple perverse sheaf. It follows from the discussion based on the diagram (9.2.1), we see that \( K \) has a natural action of \( W_m^2 \). Then by using (9.3.2), (9.3.3) and (9.3.4), we obtain (9.3.1). This proves (i).

Next we show (ii). Since \( \zeta \) is proper, \( \zeta_! A_\rho \) is a semisimple complex on \( \mathcal{X}_m \). Since \( \zeta K = \pi_1^{(m)} \mathcal{Q}_l[d_m] \) is a semisimple perverse sheaf, \( \zeta_! A_\rho \) is also a semisimple perverse sheaf by (i). By applying \( \zeta_! \) on both sides of (9.3.1), we have

\[
(9.3.5) \quad \pi_1^{(m)} \mathcal{Q}_l[d_m] \simeq \bigoplus_{\rho \in (W_m^2)^\wedge} \rho \otimes \zeta_! A_\rho.
\]

By using the diagram (9.2.1), one can show that the \( W_m^2 \)-module structure of \( \pi_1^{(m)} \mathcal{Q}_l[d_m] \) induced from \( \zeta_! \) coincides with the \( W_m^2 \)-structure given in the formula (8.8.1). Thus by comparing (9.3.5) with (8.8.1), we obtain (ii). The proposition is proved. \( \square \)
9.4. For each \( m \in \mathcal{Z}_n \), put \( G_{m,\text{nil}} = \zeta^{-1}(\mathcal{H}_{m,\text{nil}}) \). Then the map \( \pi_1^{(m)} \) is factored as
\[
\pi_1^{(m)} : \mathcal{H}_{m,\text{nil}} \xrightarrow{\varphi_1} \mathcal{G}_{m,\text{nil}} \xrightarrow{\zeta_1} \mathcal{H}_{m,\text{nil}},
\]
where \( \varphi_1, \zeta_1 \) are restrictions of \( \varphi, \zeta \). Since \( \pi^{(m)} \) is surjective, \( \varphi_1 \) is surjective. Put \( \mathcal{H}_{m,\text{nil}} = q^{-1}(\mathcal{G}_{m,\text{nil}}) \). The inclusion map \( \mathcal{G}_{m,\text{nil}} \hookrightarrow \mathcal{G}_m \) is compatible with the diagram (9.1.3), and we have a commutative diagram
\[
\begin{array}{ccc}
g_1 \times X' & \xleftarrow{\sigma} & \mathcal{H}_m \\
\uparrow & & \uparrow \\
(g_1)_{\text{nil}} \times X'_{\text{nil}} & \xleftarrow{\sigma_1} & \mathcal{H}_{m,\text{nil}} \\
\end{array}
\]
(9.4.1)
where \( \sigma_1, q_1 \) are restrictions of \( \sigma, q \), respectively, and vertical maps are natural inclusions. A similar property as (9.1.4) still holds for \( \sigma_1, q_1 \), and both squares are cartesian squares.

For each \( \lambda \in \mathcal{P}(m(k)) \), we define a subset \( \mathcal{G}_\lambda \) of \( \mathcal{G}_{m,\text{nil}} \) as follows. Write \( \lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) \), where \( |\lambda^{(1)}| = m_1, |\lambda^{(2)}| = m_2 - k, |\lambda^{(3)}| = m_2 - k \). Let \( \mathcal{H} = \mathcal{H}_{\lambda^{(1)}} \) be the \( G_1 \)-orbit in \( (g_1)_{\text{nil}} \) and \( \mathcal{H}_2 = \mathcal{H}_{\lambda^{(2)},\lambda^{(3)}} \) be the \( H_2 \)-orbit in \( X'_{\text{nil}} = (h_2)_{\text{nil}} \) (see the notation in 7.2). Put \( \mathcal{G}_\lambda = q_1(\sigma_1^{-1}(\mathcal{H} \times \mathcal{H}_2)) \). Then \( \mathcal{G}_\lambda \) is an \( H \)-stable, irreducible smooth subvariety of \( \mathcal{G}_{m,\text{nil}} \).

Let \( \overline{\mathcal{G}}_\lambda \) be the closure of \( \mathcal{G}_\lambda \) in \( \mathcal{G}_{m,\text{nil}} \). Recall the map \( \pi_\lambda : \overline{X}_\lambda \to \overline{X}_\lambda \) defined in 7.4, and let \( \overline{X}_\lambda^0 \) be as in 7.11. It follows from the construction that \( \overline{X}_\lambda^0 \) is a closed subset of \( \mathcal{G}_{m,\text{nil}} \). We show a lemma.

**Lemma 9.5.**

(i) \( \overline{\mathcal{G}}_\lambda \) coincides with \( \overline{X}_\lambda \). In particular, \( \overline{X}_\lambda^0 \) is an open dense subset of \( \overline{\mathcal{G}}_\lambda \).

(ii) \( \zeta_1(\overline{\mathcal{G}}_\lambda) = \overline{X}_\lambda \), and \( \zeta_1^{-1}(X_\lambda^0) = \overline{X}_\lambda^0 \). Hence the restriction of \( \zeta_1 \) on \( \zeta_1^{-1}(X_\lambda^0) \) gives an isomorphism \( \zeta_1^{-1}(X_\lambda^0) \cong \overline{X}_\lambda^0 \).

**Proof.** It follows from the construction that \( \overline{X}_\lambda^0 \subset \mathcal{G}_\lambda \). Hence \( \overline{X}_\lambda \subset \overline{\mathcal{G}}_\lambda \). By Lemma 7.5, \( \dim \overline{X}_\lambda = 2 \dim U_P + \dim \mathcal{H}_1 + \dim \mathcal{H}_2 \). On the other hand,
\[
\dim \mathcal{G}_\lambda = \dim \mathcal{H}_1 + \dim \mathcal{H}_2 + \beta_1 - \beta_2
= 2 \dim U_P + \dim \mathcal{H}_1 + \dim \mathcal{H}_2.
\]
(Here \( \beta_1, \beta_2 \) are as in (9.3.4), and \( \dim \mathcal{H}_1 = \dim \mathcal{H}_1 + m_1 \) by (7.3.1).) Since \( \overline{X}_\lambda \) is irreducible and closed, we have \( \overline{X}_\lambda = \overline{\mathcal{G}}_\lambda \). This proves (i). Then the restriction of \( \zeta_1 \) on \( \overline{\mathcal{G}}_\lambda \) coincides with the map \( \pi_\lambda : \overline{X}_\lambda \to \overline{X}_\lambda \). Thus (ii) follows from Lemma 7.12. The lemma is proved.

9.6. Recall the set \( \mathcal{P}(m) \) in (7.7.1) for each \( m \in \mathcal{Z}_n \). By (8.10.2), the set \( (W_m^2)^{\text{\#}} \) is parametrized by \( \mathcal{P}(m) \). We denote by \( \rho_\lambda^m \) the irreducible representation of \( W_m^2 \) corresponding to \( \lambda \in \mathcal{P}(m) \). On the other hand, we denote by \( \hat{\rho}_\lambda \) the
irreducible representation of $W_{n,3}$ belonging to $(W_{n,3}^\wedge)_m$ under the correspondence (8.10.3). By (8.10.1), we have a parametrization

$$W_{n,3}^\wedge \simeq \coprod_{m \in \mathcal{D}_{n,3}} \mathcal{P}(m).$$

The following result determines the Springer correspondence explicitly (compare with [Sh, Thm. 8.7]).

**Theorem 9.7.** Assume that $m \in \mathcal{D}_{n,3}$.

(i) Let $L_\rho$ be as in Theorem 8.9. Assume that $\rho = \rho_\lambda^\wedge \in (W_m^\wedge)^\wedge$ for $\lambda \in \mathcal{P}(m)$. Then we have

$$L_\rho \simeq \text{IC}(X_\lambda, \mathcal{Q}_l)[\dim X_\lambda].$$

(ii) (Springer correspondence for $W_m^\wedge$)

$$(\pi_1^{(m)}|_{\mathcal{Q}_l}[d'_m] \simeq \bigoplus_{\lambda \in \mathcal{P}(m)} \rho_\lambda^\wedge \otimes \text{IC}(X_\lambda, \mathcal{Q}_l)[\dim X_\lambda].$$

(iii) (Springer correspondence for $W_{n,3}$)

$$(\pi_m|_{\mathcal{Q}_l}[d'_m] \simeq \bigoplus_{\lambda \in \mathcal{P}(m)} \hat{\rho}_\lambda \otimes \text{IC}(X_\lambda, \mathcal{Q}_l)[\dim X_\lambda].$$

**Proof.** By Proposition 9.3, we know that $\zeta_1 A_\rho = K_\rho$ under the notation in 9.2. Hence by the base change theorem, $(\zeta_1)(A_\rho|_{\mathcal{G}_{m,\text{nil}}}) \simeq K_\rho|_{\mathcal{G}_{m,\text{nil}}}$. For $\lambda \in \mathcal{P}(m(k))$, put $\rho = \rho_\lambda^\wedge$. We define a simple perverse sheaf $B_\rho$ on $\mathcal{G}_{m,\text{nil}}$ as follows. Let $\mathcal{O}_1, \mathcal{O}_2$ be as in 9.4. Put $B_{\rho_1} = \text{IC}(\mathcal{O}_1, \mathcal{Q}_l)[\dim \mathcal{O}_1]$ for $\rho_1 = \rho_{\lambda(1)}^\wedge \in S_m^\wedge$, and $B_{\rho_2} = \text{IC}(\mathcal{O}_2, \mathcal{Q}_l)[\dim \mathcal{O}_2]$ for $\rho_2 = \hat{\rho}_2 \in W_{m,2,2}^\wedge$ with $\rho_2 \in (S_k \times S_{m_2 - k})^\wedge$. By a similar construction as in the proof of Proposition 9.3, there exists a unique simple perverse sheaf $B_\rho$ on $\mathcal{G}_{m,\text{nil}}$ satisfying the relation

$$q_1^* B_\lambda[\beta_2] \simeq \sigma^*(B_{\rho_1} \boxtimes B_{\rho_2})[\beta_1].$$

We know that $A_{\rho_1}|_{\mathcal{G}_{m,\text{nil}}} \simeq B_{\rho_1}$, up to shift. On the other hand, by Corollary 5.20, we have $A_{\rho_2}|_{\mathcal{G}_{m,\text{nil}}} \simeq B_{\rho_2}$, up to shift. Thus by comparing (9.7.2) and (9.3.4), we see that the restriction of $A_\rho$ on $\mathcal{G}_{m,\text{nil}}$ coincides with $B_\rho$, up to shift. Also by (9.7.2), the restriction of $B_\rho$ on $\mathcal{G}_\lambda$ is a constant sheaf $\mathcal{Q}_l$. In particular, $\text{supp} B_\rho = \overline{\mathcal{G}_\lambda}$. By Lemma 9.5, the support of $(\zeta_1)_! B_\lambda$ coincides with $\overline{\mathcal{G}_\lambda}$. By Theorem 8.9, we know that the restriction of $K_\rho$ on $\mathcal{G}_{m,\text{nil}}$ is a simple perverse sheaf $L_\rho$. Hence in order to show (9.7.1), it is enough to see that $L_\rho|_{\mathcal{X}_\lambda^0}$ is a constant sheaf $\mathcal{Q}_l$. By Lemma 9.5 (ii), $\zeta^{-1}(\mathcal{X}_\lambda^0) = \tilde{X}_\lambda^0 \subset \mathcal{G}_\lambda$, and $\zeta^{-1}(\mathcal{X}_\lambda^0) \subset \mathcal{X}_\lambda^0$. It follows that $(\zeta_1)_! B_\lambda|_{\mathcal{X}_\lambda^0}$ coincides with $\mathcal{Q}_l$, up to shift. This proves (9.7.1). (ii) and (iii) then follows from Theorem 8.9 and Corollary 8.11. The theorem is proved. \qed
9.8. For each \( z \in \mathcal{X}_{m,\text{nil}} \), we consider the Springer fibres \( \mathcal{B}_z = \pi^{-1}(z) \) and \( \mathcal{B}_z^{(m)} = (\pi^{(m)})^{-1}(z) \) as in 7.13. We have \( \mathcal{B}_z^{(m)} \subset \mathcal{B}_z \). The cohomology group \( H^i(\mathcal{B}_z^{(m)}, \mathcal{Q}_t) \) has a structure of \( W_m^2 \)-module, and \( H^i(\mathcal{B}_z, \mathcal{Q}_t) \) has a structure of \( W_{n,3} \)-module. For \( \lambda \in \mathcal{P}(m) \), put

\[
(9.8.1) \quad d_{\lambda} = \frac{1}{2}(\dim \mathcal{X}_{m,\text{nil}} - \dim X_{\lambda})
\]

We have a lemma.

**Lemma 9.9.** Assume that \( \lambda \in \mathcal{P}(m) \).

(i) For any \( z \in X_{\lambda} \), \( \dim \mathcal{B}_z^{(m)} \geq d_{\lambda} \). The set of \( z \in X_{\lambda} \) such that \( \dim \mathcal{B}_z^{(m)} = d_{\lambda} \) forms an open dense subset of \( X_{\lambda} \).

(ii) For any \( z \in X_{\lambda} \), \( H^{2d_{\lambda}}(\mathcal{B}_z^{(m)}, \mathcal{Q}_t) \) contains an irreducible \( W_m^2 \)-module \( \rho_{\lambda}^{\mathcal{B}_z} \).

**Proof.** First we show (ii). For any \( z \in \mathcal{X}_{m,\text{nil}} \), Theorem 9.7 (ii) implies that

\[
(9.9.1) \quad H^i(\mathcal{B}_z^{(m)}, \mathcal{Q}_t) \simeq \bigoplus_{\mu \in \mathcal{P}(m)} \rho_{\mu}^{\mathcal{B}_z} \otimes \mathcal{H}_z^{i-d_{\mu}+\dim X_{\mu}} IC(X_{\mu}, \mathcal{Q}_t)
\]

as \( W_m^2 \)-modules. Assume that \( z \in X_{\lambda} \) and put \( i = 2d_{\lambda} \). Since \( \mathcal{H}^0 IC(X_{\lambda}, \mathcal{Q}_t) = \mathcal{Q}_t \), \( H^{2d_{\lambda}}(\mathcal{B}_z^{(m)}, \mathcal{Q}_t) \) contains \( \rho_{\lambda}^{\mathcal{B}_z} \). This proves (ii).

(ii) implies, in particular, that \( \dim \mathcal{B}_z^{(m)} \geq d_{\lambda} \). Put \( d = \dim(\pi^{(m)})^{-1}(X_{\lambda}) - \dim X_{\lambda} \). Let \( X(d) \) be as in (7.13.1). Then \( X(d) \cap X_{\lambda} \) is open dense in \( X_{\lambda} \). Hence \( \dim X_{\lambda} \leq \dim X(d) \). By Lemma 7.14 (iii), we have, for any \( z \in X(d) \cap \mathcal{B}_z^{(m)} \),

\[
\dim \mathcal{B}_z^{(m)} \leq \frac{1}{2}(\dim \mathcal{X}_{m,\text{nil}} - \dim X(d)) \leq \frac{1}{2}(\dim \mathcal{X}_{m,\text{nil}} - \dim X_{\lambda}) = d_{\lambda}.
\]

Hence \( \dim \mathcal{B}_z^{(m)} = d_{\lambda} \) and \( d = d_{\lambda} \). This proves (i). \( \square \)

We show the following result (compare with [Sh, Prop. 8.16]).

**Proposition 9.10.** Take \( z \in X_{\lambda}^0 \), and assume that \( \lambda \in \mathcal{P}(m) \).

(i) \( \dim \mathcal{B}_z^{(m)} = d_{\lambda} \), and \( H^{2d_{\lambda}}(\mathcal{B}_z^{(m)}, \mathcal{Q}_t) \simeq \rho_{\lambda}^{\mathcal{B}_z} \) as \( W_m^2 \)-modules.

(ii) \( \dim \mathcal{B}_z = d_{\lambda} \), and \( H^{2d_{\lambda}}(\mathcal{B}_z, \mathcal{Q}_t) \simeq \rho_{\lambda}^{\mathcal{B}_z} \) as \( W_{n,3} \)-modules. Hence the map \( z \mapsto H^{2d_{\lambda}}(\mathcal{B}_z, \mathcal{Q}_t) \) gives a canonical bijection

\[
\{ X_{\lambda}^0 \mid \lambda \in \mathcal{P}_{n,3} \} \sim W_{n,3}^\wedge.
\]

**Proof.** We prove (i). We consider the diagram as in (9.1.3), restricted to the nilpotent variety as in 9.4. Write \( \lambda = (\lambda^{(1)}, \lambda') \) with \( \lambda' = (\lambda^{(2)}, \lambda^{(3)}) \). Put

\[
\begin{align*}
d_{\lambda^{(1)}} &= (\dim (g_1)_{\text{nil}} - \dim G'_1)/2, \\
d_{\lambda'} &= (\dim X'_{\text{nil}} - \dim G_2)/2.
\end{align*}
\]
We note that

$$d_{\lambda} = d_{\lambda^{(1)}} + d_{\lambda^{r}}. \tag{9.10.1}$$

In fact, by Proposition 7.8 and Lemma 7.5,

$$d_{\lambda} = ((2n^2 + m_1) - (2 \dim U_1 + \dim \mathcal{O}_1 + \dim \mathcal{O}_2))/2$$

$$= (\dim G_1 + \dim H_2 - n - \dim \mathcal{O}_1' - \dim \mathcal{O}_2)/2$$

$$= (\dim(g)_\text{nil} + \dim(h)_\text{nil} - \dim \mathcal{O}_1' - \dim \mathcal{O}_2)/2.$$

Thus (9.10.1) holds.

Take $z \in X^0_\lambda$. Assume that $\lambda \in \tilde{\mathcal{D}}(m)$. By Lemma 9.5 (ii), $\zeta_1$ gives an isomorphism $\zeta_1^{-1}(X^0_\lambda) \to X^0_\lambda$. Hence there exists a unique $z_* \in \zeta_1^{-1}(X^0_\lambda)$ such that $\zeta_1(z_*) = z$. Since $\zeta_1^{-1}(X^0_\lambda) \subset \mathcal{G}_\lambda$, by using the diagram (9.1.3) and its restriction on the nilpotent variety (9.4.1), one can find $(x_1, x_2) \in \mathcal{O}_1' \times \mathcal{O}_2$ such that $\sigma_1^{-1}(x_1, x_2) = q_1^{-1}(z_*)$. Note that dim $\mathcal{B}_z^{1} = d_{\lambda^{(1)}}$, dim $\mathcal{B}_z^{2} = d_{\lambda^{r}}$, where $\mathcal{B}_z$ is the flag variety for $G_1$ and $\mathcal{B}_z^{1}$ is the flag variety for $H_2$. Thus by (9.1.3) together with (9.10.1), we have

$$\text{(9.10.2) } (R^{2d_{\lambda^{(1)}}}(\pi^{1}_z \mathcal{Q}_t))_{x_1} \otimes (R^{2d_{\lambda^{r}}} \pi^{2}_z \mathcal{Q}_t)_{x_2} \simeq \left( R^{2d_{\lambda}}(\pi^{1}_z \mathcal{Q}_t) \right)_{z_*},$$

where $\xi$ is an element in $\sigma_1^{-1}(x_1, x_2) = q_1^{-1}(z_*)$. Since $\zeta_1^{-1}(X^0_\lambda) \simeq X^0_\lambda$, we have

$$\text{(9.10.3) } H^{2d_{\lambda}}(\mathcal{B}_z^{(m)}, \mathcal{Q}_t) \simeq \left( R^{2d_{\lambda}}(\pi^{1}_z \mathcal{Q}_t) \right)_{z} \simeq \left( R^{2d_{\lambda}}(\pi^{1}_z \mathcal{Q}_t) \right)_{z_*}.$$

We already know, from the Springer correspondence for $g_1$ and $h_2$,

$$\dim \left( R^{2d_{\lambda^{(1)}}}(\pi^{1}_z \mathcal{Q}_t) \right)_{x_1} = \dim H^{2d_{\lambda^{(1)}}}(\mathcal{B}_z^{1}, \mathcal{Q}_t) = \rho_{\lambda^{(1)}},$$

$$\dim \left( R^{2d_{\lambda^{r}}} \pi^{2}_z \mathcal{Q}_t \right)_{x_2} = \dim H^{2d_{\lambda^{r}}}(\mathcal{B}_z^{2}, \mathcal{Q}_t) = \rho_{\lambda^{r}}.$$

Then (9.10.2) and (9.10.3) show that $\dim H^{2d_{\lambda}}(\mathcal{B}_z^{(m)}, \mathcal{Q}_t) = \dim \rho_{\lambda}^z$. By Lemma 9.9 (ii), $H^{2d_{\lambda}}(\mathcal{B}_z^{(m)}, \mathcal{Q}_t)$ contains $\rho_{\lambda}^z$. It follows that $H^{2d_{\lambda}}(\mathcal{B}_z^{(m)}, \mathcal{Q}_t) \simeq \rho_{\lambda}^z$ as $W_{\hat{m}}^z$-modules. (9.10.2) also shows that $\dim \mathcal{B}_z^{(m)} = d_{\lambda}$. This proves (i).

Next we show (ii). We consider the decomposition of $H^i(\mathcal{B}_z^{(m)}, \mathcal{Q}_t)$ in (9.9.1). By Theorem 9.7 (iii), we have a similar decomposition

$$H^i(\mathcal{B}_z, \mathcal{Q}_t) \simeq \bigoplus_{\mu \in \mathcal{T}(m)} \tilde{\rho}_{\mu} \otimes \mathcal{H}^{i-d_{\mu}+\dim X_{\mu}} \text{IC}(\mathcal{X}_{\mu}, \mathcal{Q}_t). \tag{9.10.4}$$

Since $H^i(\mathcal{B}_z^{(m)}, \mathcal{Q}_t) = 0$ for $i > 2d_{\lambda}$, (9.9.1) implies that $\mathcal{H}^{i-d_{\mu}+\dim X_{\mu}} \text{IC}(\mathcal{X}_{\mu}, \mathcal{Q}_t) = 0$ for any choice of $\mu \in \mathcal{T}(m)$ and of $i > 2d_{\lambda}$. It follows, by (9.10.4) that $H^i(\mathcal{B}_z, \mathcal{Q}_t) = 0$ for $i > 2d_{\lambda}$. Since $\mathcal{B}_z^{(m)} \subset \mathcal{B}_z$, dim $\mathcal{B}_z \geq$ dim $\mathcal{B}_z^{(m)} = d_{\lambda}$. Hence dim $\mathcal{B}_z = d_{\lambda}$. By (i) and (9.9.1), we see that $\mathcal{H}^{2d_{\lambda}-d_{\mu}+\dim X_{\mu}} \text{IC}(\mathcal{X}_{\mu}, \mathcal{Q}_t) = 0$ for any
\( \mu \neq \lambda \), and is equal to \( \bar{Q}_l \) for \( \mu = \lambda \). Hence by (9.10.4), we have \( H^{2\lambda}(\mathcal{R}_z, \bar{Q}_l) \simeq \hat{\rho}_\lambda \) as \( W_{n,3} \)-modules. This proves (ii). The proposition is proved.

\[ \square \]

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J. Dong
School of Mathematical Sciences, Tongji University
1239 Siping Road, Shanghai 200092, P. R. China
E-mail: dongjunbin1990@126.com
T. Shoji  
School of Mathematical Sciences, Tongji University  
1239 Siping Road, Shanghai 200092, P. R. China  
E-mail: shoji@tongji.edu.cn

G. Yang  
School of Mathematical Sciences, Tongji University  
1239 Siping Road, Shanghai 200092, P. R. China  
E-mail: yanggao_izumi@foxmail.com