A GENERALIZATION OF TOTAL GRAPHS OF MODULES

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Abstract. Let $R$ be a commutative ring, and let $M \neq 0$ be an $R$-module with a non-zero proper submodule $N$, where $N^* = N - \{0\}$. Let $\Gamma_{N^*}(M)$ denote the (undirected) simple graph with vertices $\{x \in M - N \mid x + x' \in N^* \text{ for some } x \neq x' \in M - N\}$, where distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in N^*$. We determine some graph theoretic properties of $\Gamma_{N^*}(M)$ and investigate the independence number and chromatic number.

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1. Introduction

Throughout, all rings are commutative with non-zero identity and all modules are unitary. Let $R$ be a ring, $M \neq 0$ an $R$-module, and $N$ a non-zero proper submodule of $M$. The total graph of a commutative ring $R$, denoted by $T(\Gamma(R))$, was introduced by Anderson and Badawi in [3], as the graph with all elements of $R$ as vertices, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$, where $Z(R)$ denotes the set of zero-divisors of $R$. The concept of total graphs is a great concept that is usually used in commutative algebra to obtain many interesting graphs in this field. In [1] and [2], A. Abbasi and S. Habibi, gave a generalization of the total graph. They studied in [2] the total graph $T(\Gamma_N(M))$ of a module $M$ over a commutative ring with respect to a proper submodule $N$. It is an undirected graph with the vertex set $M$, where two distinct vertices $m$ and $n$ are adjacent if and only if $m + n \in M(N)$, where $M(N) = \{m \in M \mid rm \in N \text{ for some } r \in R - (N : M)\}$. It is easy to see that $M(N)$ is closed under multiplication by scalars. However, $M(N)$ may not be an additive subgroup of $M$. Here we introduce a generalization of total graphs, denoted by $\Gamma_{N^*}(M)$, as the (undirected) simple graph with vertices $\{x \in M - N \mid x + x' \in N^* \text{ for some } x \neq x' \in M - N\}$, where distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in N^* = N - \{0\}$.

Let $G$ be a simple graph. If there is a path from any vertex to any other vertex of graph $G$, then $G$ is said to be connected, and $G$ is said to be totally disconnected.

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if there is no path connecting any pair of vertices. For vertices $x_1$ and $x_2$ of $G$, we define $d(x_1, x_2)$ to be the length of a shortest path between $x_1$ and $x_2$ ($d(x, x) = 0$ and $d(x_1, x_2) = \infty$ if there is no such path). The diameter of $G$ is $\text{diam}(G) = \sup\{d(x_1, x_2) \mid x_1 \text{ and } x_2 \text{ are vertices of } G\}$. The girth of $G$, denoted by $\text{gr}(G)$, is the length of its shortest cycle; $\text{gr}(G) = \infty$ if $G$ contains no cycles, in this case, $G$ is called an acyclic graph. A complete graph is one which every two vertices are adjacent. A complete graph with $n$ vertices is denoted by $K^n$. A bipartite graph $G$ is a graph whose vertex set $V(G)$ can be partitioned into disjoint subsets $U_1$ and $U_2$ in such a way that each edge of $G$ has one end vertex in $U_1$ and the other in $U_2$. In particular, if $E$ consists of all possible such edges, then $G$ is called a complete bipartite graph and is denoted by $K^{m,n}$ when $|U_1| = m$ and $|U_2| = n$. For a vertex $v$ of $G$, $\text{deg}(v)$ denotes the degree of $v$ and we set $\delta(G) := \min\{\text{deg}(v) : v \text{ is a vertex of } G\}$. A graph $G$ is called $k$-regular if every vertex has degree $k$. A subgraph of $G$ is the graph formed by a subset of the vertices and edges of $G$. Two subgraphs $G_1$ and $G_2$ of $G$ are said to be disjoint if $G_1$ and $G_2$ have no common vertices and no vertex of $G_1$ (resp., $G_2$) is adjacent (in $G$) to any vertex not in $G_1$ (resp., $G_2$). The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2$ whose vertex set is $V_1 \cup V_2$ and whose edge set is $E_1 \cup E_2$. A complete subgraph of $G$ is called a clique. The clique number, $\omega(G)$, is the greatest integer $n \geq 1$ such that $K^n$ is a subgraph of $G$, and $\omega(G) = \infty$ if $K^n \subseteq G$ for all $n \geq 1$. A matching in a graph $G$ is a set of edges such that no two have a vertex in common. A spanning matching of a graph is said to be a perfect matching. A star graph $S_k$ is the complete bipartite graph $K^{1,k}$. A Hamiltonian cycle is a cycle that visits each vertex exactly once. A graph that contains a Hamiltonian cycle is called a Hamiltonian graph. A walk is an alternating sequence of vertices and edges which are incident, that begins and ends with a vertex. A tour is a closed walk that traverses each edge at least once. An Eulerian tour in an undirected graph is a tour that traverses each edge exactly once. If such a tour exists, the graph is called Eulerian. A connected component (or just component) of an undirected graph is a maximal connected induced subgraph. An independent set is a set of vertices in a graph, no two of which are adjacent. That is, it is a set $S$ of vertices such that for every two vertices in $S$, there is no edge connecting the two. The vertex independence number of $G$, often called the independence number, is the cardinality of a largest independent vertex set, i.e., the maximum size among all independent vertex sets of $G$. The independence number is denoted by $\alpha(G)$. A vertex cover of $G$ is a set of vertices such that each edge of $G$ is incident to at least one vertex of the set. The vertex cover number is the minimum size
among all vertex covers in the graph, denoted by $\beta(G)$. A coloring of a graph is a proper (vertex) coloring with colors such that no two vertices sharing the same edge have the same color. A coloring using $k$ colors is called a (proper) $k$-coloring. The smallest number of colors needed to color the vertices of $G$ is called its chromatic number and is denoted by $\chi(G)$.

The main objective of this paper is to study some properties of the graph $\Gamma_{N^*}(M)$. We also investigate the independence number and chromatic number of the graph $\Gamma_{N^*}(M)$.

2. Properties of $\Gamma_{N^*}(M)$

In this section, we investigate some properties of $\Gamma_{N^*}(M)$. Throughout, $N$ is a non-zero proper submodule of a non-zero $R$-module $M$, where $R$ is commutative ring.

Definition 2.1. Let $R$ be a commutative ring, $M$ be an $R$-module, $N$ be a submodule of $M$, and let $N^* = N - \{0\}$. We define an undirected simple graph $\Gamma_{N^*}(M)$ with vertices $\{x \in M - N | x + x' \in N^* \text{ for some } x \neq x' \in M - N\}$, where distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in N^*$.

Remark 2.2. (1) $x \in V(\Gamma_{N^*}(M))$ if and only if $N(x) \neq \emptyset$, where $N(x) = \{x' \in V(\Gamma_{N^*}(M)) | x' \neq x, x + x' \in N^*\}$. So, there is no isolated vertex in $\Gamma_{N^*}(M)$. In particular, $\Gamma_{N^*}(M)$ is not totally disconnected.

(2) Let $x, y \in V(\Gamma_{N^*}(M))$ be adjacent with $x - y \in N^*$, then $x + y - x + y \in N$ and $x + y + x - y \in N$; so $2x, 2y \in N$.

(3) Let $x, y \in V(\Gamma_{N^*}(M))$ with $x \neq y$ and $N(x) \cap N(y) \neq \emptyset$. Then $x - y \in N^*$.

(4) $\Gamma_{N^*}(M)$ is a perfect matching if and only if for all $x, y \in V(\Gamma_{N^*}(M))$ with $x \neq y$, one has $N(x) \cap N(y) = \emptyset$ or $|N(x)| = |N(y)| = 1$.

Example 2.3. Let $M = \mathbb{Z}_{12}$ and $N = \{0, 4, 8\}$. Then $\Gamma_{N^*}(M)$ has the following form:

$$
\begin{array}{cccccc}
1 & \bullet & \cdots & \bullet & 3 \\
5 & \bullet & \cdots & \bullet & 7 \\
9 & \bullet & \cdots & \bullet & 11 \\
\end{array}
$$

Figure 1.

Theorem 2.4. If $x, y \in V(\Gamma_{N^*}(M))$ are distinct vertices connected by a path of length 3 with $x + y \neq 0$, then $x, y$ are adjacent.
Proof. Let $x, m_1, m_2, y$ be distinct vertices of $\Gamma_N(M)$ with a path $x \rightarrow m_1 \rightarrow m_2 \rightarrow y$. Since $x + m_1, m_1 + m_2$ and $m_2 + y \in N^*$, we have $x + y = (x + m_1) + (y + m_2) - (m_1 + m_2) \in N$. This yields $x + y \in N^*$, since $x + y \neq 0$; so $x$ and $y$ are adjacent. $\square$

Corollary 2.5. Let $e = xx'$ and $f = yy'$ be edges of $\Gamma_N(M)$, where the sum of each end point of $e$ and each end point of $f$ does not equal zero. If two end points of $e$ and $f$ are adjacent, then so are the other two.

Proof. Without loss of generality, we may assume that $x'$ and $y$ are adjacent; so there is a path $x \rightarrow x' \rightarrow y \rightarrow y'$ in $\Gamma_N(M)$. Therefore, $x$ and $y'$ are adjacent, by Theorem 2.4. $\square$

Remark 2.6. In Corollary 2.5, the condition “does not equal zero” is necessary. For instance, in Example 2.3, set $x = 5, y = 9, x' = 3, y' = 11$. Then $x$ and $y'$ are adjacent, but $x'$ and $y$ are not.

Theorem 2.7. If $x, y \in V(\Gamma_N(M))$ are distinct vertices connected by a path of length 4, then there exists a path of length 2 between them. In particular, there is $t \in V(\Gamma_N(M))$ with $t \in N(x) \cap N(y)$.

Proof. Let $x, m_1, m_2, m_3, y$ be distinct vertices of $\Gamma_N(M)$ with a path $x \rightarrow m_1 \rightarrow m_2 \rightarrow m_3 \rightarrow y$. If $x + m_3 \neq 0$ or $m_1 + y \neq 0$, then $x$ and $m_3$, or $y$ and $m_1$, are adjacent by Theorem 2.4, as we desired. So let $x = -m_3$ and $y = -m_1$. Then we have a path $x(= -m_3) \rightarrow m_1 \rightarrow m_2 \rightarrow m_3 \rightarrow y(= -m_1)$. Thus, $x(= -m_3) \rightarrow (-m_2) \rightarrow y(= -m_1)$ is a path of length 2. $\square$

Theorem 2.8. Let $\Gamma_N(M)$ be connected. If $t_1 - t_2 \in N^*$ for all adjacent vertices $t_1$ and $t_2$ of $\Gamma_N(M)$, then $\text{diam}(\Gamma_N(M)) \in \{1, 2\}$.

Proof. For every path of length 3 such as $x \rightarrow m_1 \rightarrow m_2 \rightarrow y$ in $\Gamma_N(M)$, if $x + y \neq 0$, then $x$ and $y$ are adjacent by Theorem 2.4 and $\text{diam}(\Gamma_N(M)) \leq 2$. Let $x = -y$. Then there is a path $x(= -y) \rightarrow m_1 \rightarrow m_2 \rightarrow y$. Our hypothesis yields $x \rightarrow m_2$, and we are done. $\square$

Theorem 2.9. $\text{diam}(\Gamma_N(M)) \in \{1, 2, 3, \infty\}$. In particular, if $\Gamma_N(M)$ is connected, then $\text{diam}(\Gamma_N(M)) \leq 3$.

Proof. By Theorem 2.7, we can reduce every path of length greater than 3 to a path of length at most 3. $\square$
Example 2.10. Let $M = \mathbb{Z}_8$ and $N = \{0, 2, 4, 6\}$. Then $\Gamma_{N^*}(M)$ has the following form:

![Figure 2.](image)

Theorem 2.11. Let $\Gamma_{N^*}(M)$ be connected. Then it is complete if and only if $2t = 0$ for every $t \in V(\Gamma_{N^*}(M))$.

Proof. Suppose that $\Gamma_{N^*}(M)$ is complete and $2t \neq 0$ for some $t \in V(\Gamma_{N^*}(M))$. Then $-t$ is a vertex and $0 = t + (-t) \in N^*$, which is a contradiction. Suppose $2t = 0$ for all vertices $t$. Then $\text{diam}(\Gamma_{N^*}(M)) \leq 3$ by Theorem 2.9. Let $x \rightarrow t \rightarrow y$ be a path in $\Gamma_{N^*}(M)$. Then by part (3) of Remark 2.2, $x + y \in N^*$ (since by assumption $2y = 0$ implies that $y = -y$). Let $d(x, y) = 3$. So there is a path $x \rightarrow t_1 \rightarrow t_2 \rightarrow y$ in $\Gamma_{N^*}(M)$. If $x + y = 0$, then $x = -y = y$ (since $2y = 0$); this contradicts our assumption, so $x + y \neq 0$. Hence, $x$ and $y$ are adjacent by Theorem 2.4. So, $\Gamma_{N^*}(M)$ is complete.

Theorem 2.12. If $2x \neq 0$ for every $x \in V(\Gamma_{N^*}(M))$, then $\text{gr}(\Gamma_{N^*}(M)) \in \{3, 4, 6, \infty\}$.

Proof. (1) It is clear that $\Gamma_{N^*}(M)$ has more than two vertices. For all $x \in V(\Gamma_{N^*}(M))$, let $|N(x)| = 1$. Then $\text{gr}(\Gamma_{N^*}(M)) = \infty$, since in this case $\Gamma_{N^*}(M)$ is just a perfect matching.

(2) Suppose there is $t \in V(\Gamma_{N^*}(M))$ such that $|N(t)| \geq 2$.

(1') If for all vertices $t$ with $|N(t)| \geq 2$, we have $|N(x)| = 1$ for every $x \in N(t)$, then there are not any cycles in $\Gamma_{N^*}(M)$.

(2') Suppose there exists $y \in N(t)$ such that $|N(y)| \geq 2$ and this condition is satisfied just for $y$. There is an $x \in N(t)$ such that $|N(x)| = 1$, since $|N(t)| \geq 2$. If $x \neq -y$, then by part 3 of Remark 2.2, one has $(-y) \rightarrow t \rightarrow y$ and if $x = -y$, then $(-y) \rightarrow t \rightarrow y \rightarrow m$ for some $m \in N(y)$. This implies that $(-m) \rightarrow (-y) \rightarrow t \rightarrow y \rightarrow m$, which contradicts $|N(x)| = 1$. So, we should have at least two vertices $x, y \in N(t)$ such that $|N(x)|, |N(y)| \geq 2$. If $x$ and $y$ are adjacent, then $x \rightarrow t \rightarrow y \rightarrow x$ and $\text{gr}(\Gamma_{N^*}(M)) = 3$. We assume that $x$ and $y$ are not adjacent for every $x, y \in N(t)$.

(a) Let $|N(x) \cap N(y)| = 1$. If $2t \in N^*$, then $x + t, y + t \in N^*$; so $x + y + 2t \in N$. This yields $x + y \in N$ and so $x + y = 0$ (since $x$ and $y$ are not adjacent). Therefore, $x = -y$ and there exists a path $(-y) \rightarrow t \rightarrow y \rightarrow (-t)$. This implies that $-y$ and
−t are adjacent. So, −t ∈ N(x) ∩ N(y) where contradicts our assumption, since |N(x) ∩ N(y)| = 1.

Now, assume that 2t /∈ N* and |N(t)| = 2. There is a path (−y) ---- x ---- t ---- y ---- (−x) in ΓN−(M) by part 3 of Remark 2.2; so (−t) ---- (−y) ---- x ---- t ---- y ---- (−x) and −t and −x are adjacent. Hence gr(ΓN−(M)) ≤ 6. If |N(t)| ≥ 3, there is a path m ---- t ---- y ---- (−x) for some vertex m /∈ x. Since m − x /∈ 0, one has gr(ΓN−(M)) ≤ 4, by Theorem 2.4. If −x, t are adjacent then there is a path (−x) ---- t ---- y ---- (−x) and gr(ΓN−(M)) = 3.

(b) Let |N(x) ∩ N(y)| ≥ 2. There is a path m ---- x ---- t ---- y ---- m. Hence ΓN−(M) contains a 4-cycle and gr(ΓN−(M)) ≤ 4.

□

**Corollary 2.13.** ΓN−(M) is an acyclic graph if and only if it is a disjoint union of some star components.

**Proof.** Suppose that graph ΓN−(M) is an acyclic graph. If ΓN−(M) has a non-star component, then there exists at least one path of length 3 as x ---- t1 ---- t2 ---- y in ΓN−(M). We assumed that ΓN−(M) is an acyclic graph, so x + y = 0, by Theorem 2.4. Hence, we have a path (−t2) ---- x ---- t1 ---- t2 ---- y. Theorem 2.7 yields there is a cycle in ΓN−(M), which contradicts our assumption. Hence all paths are of lengths 1 or 2. This implies that all components are in the form of stars. □

**Theorem 2.14.** The following statements hold for the clique number of ΓN−(M).

1. ω(ΓN−(M)) = 2 if N(x) ∩ N(y) = ∅ for every distinct x, y ∈ V (∆N−(M)).
2. If there exist adjacent vertices x and y in ΓN−(M) such that N(x) ∩ N(y) ≠ ∅, then ω(ΓN−(M)) ≥ 3.
3. If 2t = 0 for all t ∈ V (∆N−(M)) and there are adjacent vertices x and y in ΓN−(M) such that x' + y' ≠ 0 for some x' ∈ N(x) and y' ∈ N(y), then ω(ΓN−(M)) ≥ 4.

**Proof.** (1) It is clear, since ΓN−(M) is a perfect matching.

(2) It is clear, since there is a triangular cycle.

(3) There is a path x' ---- x ---- y ---- y' in ΓN−(M). In view of Theorem 2.4, x' and y' are adjacent. So, x', y and x, y' are adjacent by part 3 of Remark 2.2. Hence ω(ΓN−(M)) ≥ 4. □

**Definition 2.15.** (See [5, Definition 2.9]) Let m ∈ M − N. We call the subset m + N* a column of ΓN−(M). If 2m ∈ N* for every m ∈ M − N, then we call m + N* a connected column of ΓN−(M).

**Theorem 2.16.** Suppose ΓN−(M) contains at least one connected column and |N*| ≥ 4 with 2m ≠ 0 for every m ∈ N*. Then gr(ΓN−(M)) = 3.
**Proof.** Let $x + N^*$ be a connected column in $\Gamma_{N^*}(M)$. Then $2x \in N^*$. Let $n \neq 2x, -2x$ in such a way that $n \in N^*$. Then $x \xrightarrow{N^*} (x + n) \xrightarrow{N^*} (x - n) \xrightarrow{N^*} x$ is a cycle of length 3 in $\Gamma_{N^*}(M)$. \hfill $\square$

Recall that a vertex $x$ of a connected graph $G$ is called a cut-point of $G$ if there are vertices $u, w$ of $G$ such that $x$ lies on every path from $u$ to $w$ (with $x \neq u, x \neq w$). Equivalently, for a connected graph $G$, $x$ is called a cut-point of $G$ if $G - \{x\}$ is not connected.

**Theorem 2.17.** Let $\Gamma_{N^*}(M)$ be connected with $2x \neq 0$ for all $x \in V(\Gamma_{N^*}(M))$. Then $\Gamma_{N^*}(M)$ has no cut-points.

**Proof.** Assume the vertex $x$ of $\Gamma_{N^*}(M)$ is a cut-point. Then there exist vertices $u, w$ of $\Gamma_{N^*}(M)$ such that $x$ lies on every path from $u$ to $w$ (therefore, $x \neq u, w$).

By Theorem 2.9, the shortest path from $u$ to $w$ is of length 2 or 3.

**Case 1.** Suppose $u \xrightarrow{N^*} x \xrightarrow{N^*} w$ is a path of shortest length from $u$ to $w$. There is a path $(-w) \xrightarrow{N^*} u \xrightarrow{N^*} x \xrightarrow{N^*} (-w)$ in $\Gamma_{N^*}(M)$. So there exists a path $u \xrightarrow{N^*} (-w) \xrightarrow{N^*} (-x) \xrightarrow{N^*} (-u) \xrightarrow{N^*} w$ by part 3 of Remark 2.2, which contradicts our assumption.

**Case 2.** Suppose (without loss of generality) that $u \xrightarrow{N^*} x \xrightarrow{N^*} y \xrightarrow{N^*} w$ is a path of shortest length from $u$ to $w$ in $\Gamma_{N^*}(M)$. Therefore, $N(u) \cap N(w) = \emptyset$. Since $u$ and $w$ are not adjacent, by Theorem 2.4, we have $u + w = 0$ and $(-y) \xrightarrow{N^*} u(= -w) \xrightarrow{N^*} x \xrightarrow{N^*} y \xrightarrow{N^*} w$. So, there exists a path $u \xrightarrow{N^*} (-y) \xrightarrow{N^*} (-x) \xrightarrow{N^*} w$, which contradicts our assumption. \hfill $\square$

**Remark 2.18.** Suppose there is a path as $u \xrightarrow{N^*} t \xrightarrow{N^*} w$ in $\Gamma_{N^*}(M)$ such that $|N(u)| = |N(w)| = 1$. Then $\Gamma_{N^*}(M)$ has a cut-point.

**Theorem 2.19.** The degree of every vertex $x$ of $\Gamma_{N^*}(M)$ is either $|N^*|$ or $|N^*| - 1$. In particular, if $2m \in N^*$ for every vertex $m$ of $\Gamma_{N^*}(M)$, then $\Gamma_{N^*}(M)$ is a $|N^*| - 1$-regular graph.

**Proof.** Let $x \in V(\Gamma_{N^*}(M))$. If $x$ is adjacent to $y$, then $x + y = a \in N^*$ and hence, $y = x - a$ for some $a \in N^*$. There are two cases:

**Case 1.** Suppose that $2x \in N^*$. Then $x$ is adjacent to $a - x$ for every $a \in N^* - \{2x\}$. Thus the degree of $x$ is $|N^*| - 1$. In particular, if $2m \in N^*$ for every $m \in V(\Gamma_{N^*}(M))$, then $\Gamma_{N^*}(M)$ is a $|N^*| - 1$-regular graph.

**Case 2.** Suppose that $2x \notin N^*$. Then $x$ is adjacent to $a - x$ for all $a \in N^*$. Thus the degree of $x$ is $|N^*|$. \hfill $\square$
In general, it is not easy to determine when the graph $\Gamma_{N^*} (M)$ is Eulerian or Hamiltonian. Here we consider $M = \mathbb{Z}_n$, for some positive integer $n$, and investigate being Eulerian or Hamiltonian (or both) for $\Gamma_{N^*} (M)$.

**Lemma 2.20.** The followings hold.

1. [4, Theorem 3.4] If $G$ is a simple graph with $\nu \geq 3$ and $\delta \geq \nu/2$, where $\nu = |V(G)|$, then $G$ is Hamiltonian.
2. [4, Theorem 1.4] A connected graph $G$ is Eulerian if and only if it contains no vertices of odd degree.

**Example 2.21.** Let $M = \mathbb{Z}_n$ and $N = 2\mathbb{Z}_n$ with $n \geq 8$. Considering Theorem 2.19, $\Gamma_{N^*} (M)$ is $|N^*| - 1$-regular; so $\delta = |N^*| - 1 = |N| - 2 \geq N/2$, where $|N| = n/2 \geq 4$. Hence by Lemma 2.20, $\Gamma_{N^*} (M)$ is Hamiltonian.

**Remark 2.22.** If $\Gamma_{N^*} (M)$ is connected, $2x \notin N^*$ for every $x \in V(\Gamma_{N^*} (M))$, and if $|N^*|$ is an even integer, then $\Gamma_{N^*} (M)$ is Eulerian.

**Remark 2.23.** Let $M = \mathbb{Z}_n$ and $N = k\mathbb{Z}_n$ (so, $N = d\mathbb{Z}_n$ for $d = (k, n)$), and let $\Gamma_{N^*} (M)$ be connected.

1. If $n$ is an odd integer, then $|N^*|$ is even. Let $2x \in N^*$ for some $x \in V(\Gamma_{N^*} (M))$. Then $2x = td$ for some $t \in \mathbb{Z}$. Hence $x \in N$ is not a vertex. So $2x \notin N^*$ for every $x \in V(\Gamma_{N^*} (M))$. Hence, by Lemma 2.20 and Theorem 2.19, $\Gamma_{N^*} (M)$ is Eulerian.

2. Assume that $n$ and $k$ are even integers; then $d$ is an even integer. By Theorem 2.19, the degree of every vertex $x$ is $|N^*|$ or $|N^*| - 1$.

(i) Let $n = 2l$ for some $l \in \mathbb{N}$. If $d > 2$, then there exists at least one vertex $x$ such that $2x \notin N^*$. So, the degree of the vertex $x$ is $|N^*|$. Note that $|N^*|$ is an odd integer. Hence by Lemma 2.20, $\Gamma_{N^*} (M)$ is not Eulerian. If $d = 2$, then by Theorem 2.19, $\Gamma_{N^*} (M)$ is a $|N^*| - 1$-regular graph, so it is Eulerian.

(ii) Let $n = 2m$ for some $l, m \in \mathbb{N}$ such that $(2, m) = 1$. Since $d = 2m' \in N^*$ for some $m' \in V(\Gamma_{N^*} (M))$, the degree of vertex $m'$ is $|N^*| - 1$. Note that $n = td$ for some $t \in \mathbb{Z}$. If $t$ is an odd integer, then $|N^*|$ is even. So the degree of $m'$ is odd and $\Gamma_{N^*} (M)$ is not Eulerian. If $t$ is an even integer, then $|N^*|$ is odd. We have two cases.

(i') If $d > 2$, there exists at least one vertex $l$ such that $2l \notin N^*$. Therefore, by Theorem 2.19, $\deg (l) = |N^*|$, hence $\Gamma_{N^*} (M)$ is not Eulerian.

(ii') If $d = 2$, by Theorem 2.19, $\Gamma_{N^*} (M)$ is a $|N^*| - 1$-regular graph and so it is Eulerian.
(3) Let $n$ be an even integer and $k$ be an odd integer. Since $|N^*|$ is an odd integer and by part (1), $2x \notin N^*$ for every $x \in V(\Gamma_N(M))$, $\Gamma_N(M)$ is not Eulerian.

3. Independence number and chromatic number of $\Gamma_N(M)$

One of the interesting computing problems in graph theory is determining the independence number of a graph. Here we obtain the independence number of $\Gamma_N(M)$ with some special conditions. It is well-known that $\alpha(K^n) = 1$.

Lemma 3.1. [4, Theorem 1.7]

(1) A set is independent if and only if its complement is a vertex cover.

(2) The sum of the size of the largest independent set $\alpha(G)$ and the size of a minimum vertex cover $\beta(G)$ is equal to the number of vertices in the graph.

Theorem 3.2. Let $\Gamma_N(M)$ be connected and let $\nu = |V(G)|$.

(1) If $\text{diam}(\Gamma_N(M)) = 3$ and $d(m, m) = 3$ for every $m \in V(\Gamma_N(M))$, then $\alpha(\Gamma_N(M)) = \beta(\Gamma_N(M)) = \nu/2$.

(2) If $\text{diam}(\Gamma_N(M)) = 2$ and $2m \neq 0$ for every $m \in V(\Gamma_N(M))$, then $\alpha(\Gamma_N(M)) = 2$ and $\beta(\Gamma_N(M)) = \nu - 2$.

Proof. (1) Choose $x \in V(\Gamma_N(M))$. Put $A_x = \{-y \in V(\Gamma_N(M)) \mid y \text{ is adjacent to } x\}$, $A'_x = \{y \in V(\Gamma_N(M)) \mid y \neq -x \text{ and } y \text{ is not adjacent to } x\}$ and $P_x = A_x \cup A'_x$. For every $n \in V(\Gamma_N(M)) - \{x, -x\}$, $n \in P_x$ or $-n \in P_x$. Claim: $P_x$ is an independent set in $\Gamma_N(M)$.

By way of contradiction, suppose there exist $n_1, n_2 \in P_x$ that they are adjacent. Since $n_1, n_2 \in P_x$, so $n_1, n_2$ are not adjacent to $x$. We claim that for every vertex $t$ other than $x$ and $-x$, $t$ is adjacent to either $-x$ or $x$ (but not both of them, otherwise, $x \rightarrow t \rightarrow (-x)$, this yields $d(x, -x) = 2$). Hence, $n_1, n_2$ are not adjacent to $x$, which implies that $n_1, n_2$ are adjacent to $-x$.

Suppose there exists $t(\neq x, -x) \in V(\Gamma_N(M))$ such that $t$ is not adjacent to $x$ and $-x$. Since $d(x, -x) = 3$, there exists a path $x \rightarrow m_1 \rightarrow m_2 \rightarrow (-x)$ in $\Gamma_N(M)$. It is clear that $d(t, x) = d(t, -x) = 2$; otherwise, $t$ is adjacent to $x$ or $-x$. So, there exists $l \in V(\Gamma_N(M))$ such that $t \rightarrow l \rightarrow x$. There is a path $t \rightarrow l \rightarrow x \rightarrow m_1 \rightarrow m_2 \rightarrow (-x)$ such that $t$ is adjacent to $m_1$ (since $m_1$ and $x$ are adjacent and $d(t, -x) = 2$, so $t \neq -m_1$). Hence $t \rightarrow m_1 \rightarrow m_2 \rightarrow (-x)$ implies that $t$ and $-x$ are adjacent (since $t \neq x$) which is a contradiction. (Therefore, for every vertex $t$, all other vertices except $-t$ are adjacent to $t$ or $-t$.) Since $n_1, n_2$ are not adjacent to $x$, so $n_1, n_2$ are adjacent to $-x$ and $n_1 \rightarrow (-x) \rightarrow n_2 \rightarrow (-n_1)$ which by assumption $n_1$ and $n_2$ are adjacent. So, $d(n_1, -n_1) = 2$, a contradiction. This shows that $P_x$ is independent. On the other hand, for every $x \neq y \in V(\Gamma_N(M))$, one has
$|P_x| = |P_y|$. We have to show that $P_x$ is the largest independent set in $\Gamma_{N^*}(M)$. Suppose there exists an independent set $U$ in $\Gamma_{N^*}(M)$ such that $|U| > |P_x| = |\nu|/2$, where $\nu = |V(G)|$. So, there exists $g \in V(\Gamma_{N^*}(M))$ such that $g, -g \in U$. This implies that $U$ is not independent. Hence $P_x$ is the largest independent set in $\Gamma_{N^*}(M)$ and $\alpha(\Gamma_{N^*}(M)) = \beta(\Gamma_{N^*}(M)) = \nu/2$ by Lemma 3.1.

(2) Let $\Gamma_{N^*}(M)$ be connected with $\text{diam}(\Gamma_{N^*}(M)) = 2$ and let $2m \neq 0$ for every $m \in V(\Gamma_{N^*}(M))$. Put $G_x = \{x, -x\}$ for some $x \in V(\Gamma_{N^*}(M))$. We show that $G_x$ is the largest independent set in $\Gamma_{N^*}(M)$.

Claim: Every vertex $x$ is adjacent to every other vertex except $-x$.

By way of contradiction, assume that there is $m \in V(\Gamma_{N^*}(M))$ such that $d(m, x) = 2$ for some $x(\neq m) \in V(\Gamma_{N^*}(M))$. So there is a path $m \rightarrow t \rightarrow x$ in $\Gamma_{N^*}(M)$ for some $t \in V(\Gamma_{N^*}(M))$; therefore, $(-x) \rightarrow m \rightarrow t \rightarrow x$ such that $x$ and $-m$ are adjacent. Moreover, $d(x, -x) = 2$. Hence, there is a path $x \rightarrow l \rightarrow (-x)$ in $\Gamma_{N^*}(M)$ for some $l \in V(\Gamma_{N^*}(M))$. Now, the path $(-m) \rightarrow x \rightarrow l \rightarrow (-x)$ implies that $-m$ is adjacent to $-x$ by Theorem 2.4. So $m$ is adjacent to $x$, a contradiction. Therefore, $G_x$ is the largest independent set in $\Gamma_{N^*}(M)$. In this case $\alpha(\Gamma_{N^*}(M)) = 2$ and $\beta(\Gamma_{N^*}(M)) = \nu - 2$ where $\nu = |V(G)|$ by Lemma 3.1. □

One of the important aims in graph theory is determining the chromatic number of a given graph. Here we investigate the chromatic number of $\Gamma_{N^*}(M)$ in some special cases. It is well-known that $\chi(K^n) = n$ and $\chi(G) \geq \omega(G)$.

**Remark 3.3.** For all distinct $x, y \in V(\Gamma_{N^*}(M))$, if $N(x) \cap N(y) = \emptyset$, then it is obvious that $\Gamma_{N^*}(M)$ is a perfect matching and $\chi(\Gamma_{N^*}(M)) = 2$.

**Theorem 3.4.** Let $\Gamma_{N^*}(M)$ be connected and non-complete. If there exist two adjacent vertices $x$ and $y$ of $\Gamma_{N^*}(M)$ with $N(x) \cap N(y) \neq \emptyset$ or for every two non-adjacent vertices $x$ and $y$ of $\Gamma_{N^*}(M)$, $N(x) \cap N(y) \neq \emptyset$, then $\chi(\Gamma_{N^*}(M)) \geq 3$.

**Proof.** Since $\Gamma_{N^*}(M)$ is not complete, there exist non-adjacent vertices $x$ and $y$ in $\Gamma_{N^*}(M)$. By assumption, $N(x) \cap N(y) \neq \emptyset$; so $(-y) \rightarrow x \rightarrow t \rightarrow y$ and $l \rightarrow (-y) \rightarrow x \rightarrow t \rightarrow y \rightarrow l$ for some $l \in N(y) \cap N(-y)$. Thus $\chi(\Gamma_{N^*}(M)) \geq 3$. (It should be noted that if $y = -y$, then $\chi(\Gamma_{N^*}(M)) \geq 3$.) □

**Theorem 3.5.**

1. Let $\Gamma_{N^*}(M)$ be connected with $\text{diam}(\Gamma_{N^*}(M)) = 3$ and $d(m, -m) = 3$ for every $m \in V(\Gamma_{N^*}(M))$. Then $\chi(\Gamma_{N^*}(M)) = 3$.

2. Let $\Gamma_{N^*}(M)$ be connected with $\text{diam}(\Gamma_{N^*}(M)) = 2$ and let $2m \neq 0$ for every $m \in V(\Gamma_{N^*}(M))$. Then $\chi(\Gamma_{N^*}(M)) = \nu/2$, where $\nu = |V(G)|$.

**Proof.** (1) Let $x \in V(\Gamma_{N^*}(M))$. Considering our hypothesis and by the proof of part 1 of Theorem 3.2, for every vertex $t$ other than $x$ and $-x$, $t$ is adjacent to
either $-x$ or $x$ (but not to both of them, otherwise, $x \rightarrow t \rightarrow (-x)$, this implies that $d(x, -x) = 2$). If $t$ is adjacent to $x$, then $t$ is not adjacent to $-x$; so $t \in P_{-x}$, otherwise, $t \in P_x$. Hence $P_x \cup P_{-x} = V(\Gamma_{N^*}(M))$. Now we assign color $a$ to elements of $P_x$ and color $b$ to elements of $P_{-x}$. Therefore, $\chi(\Gamma_{N^*}(M)) = 2$.

(2) Let $l_1 \in V(\Gamma_{N^*}(M))$. Here, by the proof of part 2 of Theorem 3.2, every vertex $m$ is adjacent to all other vertices except $-m$. At first we assign color $1$ to $l_1$ and $-l_1$. Choose $l_i(\neq l_1, -l_1) \in V(\Gamma_{N^*}(M))$ where $i > 1$. Since $l_i$ is adjacent to $l_1$ and $-l_1$ and all of other vertices except $-l_i$, we assign color $i$ to $l_i$ and $-l_i$. Continuing in this manner for remaining vertices of $\Gamma_{N^*}(M)$, one has $\chi(\Gamma_{N^*}(M)) = \nu/2$.

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