The universal $\eta$-invariant for manifolds with boundary

Ulrich Bunke

March 11, 2014

Abstract

We extend the theory of the universal $\eta$-invariant to the case of relative bordism groups of manifolds with boundaries. This allows the construction of secondary descendants of the universal $\eta$-invariant. We obtain an interpretation of Laures’ $f$-invariant as an example of this general construction.

Contents

1 Introduction

2 The topological universal $\eta$-invariant

3 Cycles for relative bordism theory

4 The group $Q_n^R(M(B,A))$

5 The doubling construction

6 The analytic universal $\eta$-invariant

7 The index theorem

8 Relative differential $K$-theory and cycles

9 Geometrizations

10 The intrinsic formula

11 Tertiary invariants

12 Laures’ $f$-invariant

* NWF I - Mathematik, Universität Regensburg, 93040 Regensburg, GERMANY, ulrich.bunke@mathematik.uni-regensburg.de
1 Introduction

In this paper we investigate the question how elements in bordism groups of manifolds with boundary can be detected using spectral invariants of Dirac operators, namely the $\eta$-invariant of Atiyah-Patodi-Singer \cite{3}. The corresponding problem for bordism groups of closed manifolds has been thoroughly studied in \cite{11} and led to the introduction of the universal $\eta$-invariant. The purpose of the present paper is to extend this theory from closed manifolds to manifolds with boundary.

The $\eta$-invariant of a Dirac operator $D_M$ on a closed manifold $M$ was defined in \cite{3} as its $\zeta$-regularized signature

$$\eta(D_M) := \sum_{\lambda \in \text{spec}(D_M) \setminus \{0\}} \text{mult}(\lambda) \frac{\text{sign}(\lambda)}{|\lambda|^s} \bigg|_{s=0}.$$  \hfill (1)

The sum converges if $\text{Re}(s)$ is large, and the value at $s = 0$ is defined as the evaluation of the meromorphic continuation of the sum. The $\eta$-invariant $\eta(D_M)$ is one term in the APS index formula \cite{3} for the index of a Dirac operator $D_W$ with so-called APS boundary conditions extending $D_M$ over a zero bordism $W$ of $M$:

$$\text{index}(D_W)_{\text{APS}} = \int_W \text{index density} - \frac{\eta(D_M) + \dim(\ker(D_M))}{2}.$$  \hfill (2)

This formula is the starting point for the construction of bordism invariants of $M$. These topological invariants are derived not just from a single $\eta$-invariant but from the relation between the $\eta$-invariants of the twisted Dirac operators $\eta(D_M \otimes V)$ for various geometric vector bundles $V$ (a shorthand for a triple $V = (V, h^V, \nabla^V)$ consisting of a complex vector bundle with hermitean metric and metric connection) on $M$. The rough idea is to form suitable linear combinations of $\eta$-invariants such that the integral of the index density (which encodes the continuous dependence of the $\eta$-invariant) drops out, and to consider equivalence classes of the values in $\mathbb{R}/\mathbb{Z}$ in order to get rid of the contribution of the index. A typical example of this idea is the well-known construction of the $\rho$-invariant

$$[\eta(D_M \otimes V) - \dim(V)\eta(D_M)] \in \mathbb{R}/\mathbb{Z}$$

for a flat geometric bundle $V$. The universal invariant of this kind is the analytic version of the universal $\eta$-invariant $\eta^{an}$ introduced in \cite{11}. One of the main results in that paper is a complete description of $\eta^{an}$ in terms of homotopy theory. To this end we introduce a topological version $\eta^{top}$ of the universal $\eta$-invariant defined in terms Thom spectra and their $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$-versions, and we show a secondary index theorem stating that $\eta^{an} = \eta^{top}$.

There are various ways to extend the definition of $\eta(D_M)$ to manifolds with boundaries. In order to extend \cite{11} one must define a selfadjoint extension of $D_M$ by choosing suitable
boundary conditions. Another possibility would be to attach an infinite cylinder. After attaching the cylinder the operator $\mathcal{D}_M$ has a natural selfadjoint extension. If $M$ is compact one can interpret the sum (1) as a trace of a function of $\mathcal{D}_M$. In the case with boundaries completed by cylinders the corresponding trace does not exist on the nose but can be defined using a regularization procedure, e.g. employing the $b$-calculus of Melrose or [10]. Finally, one can avoid non-compact manifolds or boundaries at all by forming doubles. In the present paper we prefer this last method whose details are given in Section 5.

The natural domain for the generalization of the universal $\eta$-invariant $\eta^an$ to manifolds with boundary is a relative bordism group. It can be defined as a homotopy group of a relative bordism spectrum. The data for such a spectrum consist of a map $B \to B\text{Spin}^c$ and a subspace $A \subseteq B$. The relative bordism spectrum $M(B, A)$ will be defined in Section 3. The elements of $\pi_n(M(B, A))$ can be interpreted as bordism classes of $n$-dimensional $B$-manifolds with boundary, where the $B$-structure is refined to an $A$-structure.

The topological version of the universal $\eta$-invariant $\eta^{top}$ was defined for every spectrum in [11] (see Section 2). In particular it can be applied to $M(B, A)$. We know that it can detect torsion elements in $\pi_n(M(B, A))$ which survive $K$-localization.

In the present paper (Sections 4, 5, and 6) we generalize the construction of the analytic version $\eta^an$ to $M(B, A)$. Extending the main result of [11] we show the secondary index Theorem 7.1 stating that $\eta^{top} = \eta^an$.

The simplest definition of $\eta^an$ given in Definition 6.1 does not involve $\eta$-invariants on manifolds with boundary but $\mathbb{R}/\mathbb{Z}$-valued indexes on zero bordisms. In order to give a formula in terms of $\eta$-invariants (called the intrinsic formula) we need to choose a further structure called geometrization [11]. In a certain sense the notion of a geometrization generalizes the notion of a connection on a principal bundle. Following the lines of [11] in Section 9 we extend the notion of a geometrization to the relative case. In Theorem 10.3 we provide the corresponding intrinsic formulas for $\eta^an$.

The notion of a geometrization involves differential $K$-theory and the construction of differential $K$-theory classes from geometric vector bundles. This construction is usually called the cycle map. For manifolds with boundary we must introduce the relative version of differential $K$-theory for pairs $(M, N)$ of a manifold $M$ and a submanifold $N$. As a technical ingredient of independent interest we construct the cycle map for relative differential $K$-theory in Section 8. It associates a relative differential $K$-theory class to a pair $(V, \rho)$ of a geometric bundle $V$ on $M$ together with a trivialization $\rho$ of geometric bundles of the restriction $V|_N$.

In a certain sense the universal $\eta$-invariant is a secondary invariant for the index of Dirac operators. The novelty of the case with boundaries is that it allows to define tertiary descendants of the index which are secondary for the universal $\eta$-invariant (see Section 11). A first example of such an invariant has been studied in [16] and identified with Laures’ $f$-invariant [21], [22]. This invariant can detect certain elements in the stable homotopy groups of spheres. Our purpose in the present paper (Section 12) is to show how this fits into the general framework of the universal $\eta$-invariant and how general properties of the universal $\eta$-invariant imply (already known) features of the $f$-invariant.
Motivated by recent work [19] in Section 13 we discuss a second example, a Spin-bordism version $f^{\text{Spin}}$ of the $f$-invariant. In this case, as an illustration, we specialize our general theory and provide a geometrization and an intrinsic formula for $f^{\text{Spin}}$.

We try to keep this paper short and refer [11] for many details of the language and some arguments.

Acknowledgment: I thank Fei Han and Weiping Zhang for the interesting discussion on their recent paper [18]. One of the purposes of the present paper, in particular of Section 13, is to answer some questions asked in [18].

2 The topological universal $\eta$-invariant

In this section we introduce the topological version of the universal $\eta$-invariant. Let $E$ denote a spectrum, $n$ be an integer, and $\pi_n(E)_{\text{tors}} \subseteq \pi_n(E)$ denote the torsion subgroup of the homotopy group of $E$ in degree $n$. The universal $\eta$-invariant introduced in [11] is a homomorphism of abelian groups

$$\eta^{\text{top}} : \pi_n(E)_{\text{tors}} \to Q^R_n(E) .$$

In the following we first describe the target group $Q^R_n(E)$ and then the construction of $\eta^{\text{top}}$.

For an abelian group $G$ and a spectrum $E$ we let $\text{Moore}(G)$ denote the Moore spectrum of $G$ (see [8, Sec. 2]) and abbreviate $EG := E \wedge \text{Moore}(G)$. At various places we will use the fibre sequence of spectra

$$\Sigma^{-1}E_R \to \Sigma^{-1}E_R/\mathbb{Z} \to E \to E_R \to E_R/\mathbb{Z} .$$

Let $K$ denote the complex $K$-theory spectrum.

**Definition 2.1.** We define the abelian group

$$Q^R_n(E) := \frac{\text{Hom}^{\text{cont}}(K^0(E), \pi_{n+1}(K\mathbb{R}/\mathbb{Z}))}{U^R_n} ,$$

where the subgroup $U^R_n$ will be explained below.

The subgroup $U^R_n \subseteq \text{Hom}^{\text{cont}}(K^0(E), \pi_{n+1}(K\mathbb{R}/\mathbb{Z}))$ consists of all homomorphisms determined by elements $y \in \pi_{n+1}(E_R)$ as compositions

$$K^0(E) \ni \phi \mapsto \left( \Sigma^{n+1}S \xrightarrow{y} E_R \xrightarrow{\phi} K\mathbb{R} \to K\mathbb{R}/\mathbb{Z} \right) \in \pi_{n+1}(K\mathbb{R}/\mathbb{Z}) .$$

In order to talk about continuous homomorphisms we equip the group $K^0(E)$ with the profinite topology [7, Def. 4.9], and the group $\pi_{n+1}(\mathbb{R}/\mathbb{Z})$ with the discrete topology.
We now construct the homomorphism $\eta^{\text{top}}$. We will describe the value $\eta^{\text{top}}(x) \in Q^R_n(E)$ for $x \in \pi_n(E)_{\text{tors}}$. The fibre sequence (4) induces a long exact sequence of abelian groups
\[
\pi_{n+1}(E R) \to \pi_{n+1}(E R / \mathbb{Z}) \to \pi_n(E) \to \pi_n(E R).
\]
Since it is torsion the image of $x$ in $\pi_n(E R)$ vanishes. Therefore we can find a lift $\tilde{x} \in \pi_{n+1}(E R / \mathbb{Z})$ of $x$ which is unique up to elements coming from $\pi_{n+1}(E R)$. The element $\tilde{x}$ determines a continuous homomorphism
\[
K^0(E) \ni \phi \mapsto (\Sigma^{n+1} S \tilde{x} \to E R / \mathbb{Z} \xrightarrow{\phi} K R / \mathbb{Z}) \in \pi_{n+1}(K R / \mathbb{Z}).
\] (5)

**Definition 2.2.** We define the topological universal $\eta$-invariant as the map (3) such that $\eta^{\text{top}}(x)$ is represented by the composition (5).

The latter is well-defined independently of the choice of $\tilde{x}$ exactly since we take the quotient by $U^R_n$ in the definition of $Q^R_n(E)$.

We refer to [11, Sec. 2.4] for an analysis of $\eta^{\text{top}}$ in terms of stable homotopy theory.

3 Cycles for relative bordism theory

A map between spaces $B \to B Spin^c$ gives rise to a Thom spectrum $MB$. The associated generalized homology theory is the bordism theory of $B$-manifolds. A map $i : A \to B$ induces a map of Thom spectra which we will extend to a fibre sequence
\[
\Sigma^{-1}M(B, A) \to MA \to MB \to M(B, A)
\] (6)
in order to define the spectrum $M(B, A)$ as the cofibre. We can and will assume that $i : A \to B$ is an embedding of a subspace and a cofibration.

By the Thom-Pontrjagin construction the generalized cohomology theory represented by the spectrum $M(B, A)$ can be described as the bordism theory of $B$-manifolds with boundary, on which the $B$-structure is refined to an $A$-structure. In the following we give a more detailed description of the cycles $(M, N, f)$ for classes in $\pi_n(M(B, A))$ and the equivalence relation.

1. $M$ is a compact $n$-dimensional Riemannian manifold with boundary $N$. We assume that the Riemannian metric has a product structure near the boundary of $M$.

2. $f : M \to B$ is a continuous map which is refined by a stable normal $B$-structure [11, Def. 3.1]. By definition such a refinement consists of a lift $\hat{f}$ and a homotopy filling the diagram
\[
\begin{array}{ccc}
M & \xrightarrow{f} & B \\
\downarrow \hat{f} & & \downarrow \\
BSpin^c(k) & & BSpin^c
\end{array}
\] (7)
together with the choice of an isomorphism of real vector bundles

\[ TM \oplus \hat{f}^* \xi^k \cong \mathbb{R}^{n+k}_M. \quad (8) \]

The right-hand side is the notation for the trivial \( n + k \)-dimensional real vector bundle, and \( \xi^k \) is the universal \( k \)-dimensional real vector bundle with a Spin\(^c\)-structure on \( B\text{Spin}^c(k) \).

3. The restriction \( f|_N \) has values in the subspace \( A \). Note that the stable normal \( A \)-structure on \( N \) is given by the restriction \( \hat{f}|_N \) and the isomorphism

\[ TN \oplus (s \circ \hat{f}|_N)^* \xi^{k+1} \cong TN \oplus \mathbb{R}_N \oplus \hat{f}|_N^* \xi^k \cong TM|_N \oplus \hat{f}|_N^* \xi^k \cong \mathbb{R}^{n+k}_N, \quad (9) \]

where \( s : B\text{Spin}^c(k) \to B\text{Spin}^c(k+1) \) denotes the stabilization, the second isomorphism uses the outer normal vector field at \( N \), and the last isomorphism is given by restriction of (8) to the boundary.

A zero bordism \((W, F)\) of such a cycle is given by the following data:

1. A compact \( n + 1 \)-dimensional Riemannian manifold \( W \) with corners of codimension 2 and a partition of the boundary \( \partial W = \partial_A W \cup \partial_B W \) such that \( \partial_A W \cap \partial_B W \) is the codimension two stratum. We assume a product structure of the Riemannian metric near all boundary faces and that the faces meet at a right angle in the corner.

2. \( F : W \to B \) is a map which is refined to a normal \( B \)-structure.

3. The restriction of \( F|_{\partial_A W} \) has values in \( A \).

4. There is an isomorphism of cycles

\[ \partial_B(W, F) \cong (M, N, f). \]

In detail this means that \( M \cong \partial_B W \) equipped with the induced Riemannian metric, \( F|_N = f \), and the refinement of \( f \) to a normal \( B \)-structure is induced from that of \( F \) similarly as in (9).

**Remark 3.1.** Before we can talk about geometric cycles we must fix once and for all models for the spaces \( B\text{Spin}^c(k) \) and bundles \( \xi^k \), for the maps \( s : B\text{Spin}^c(k) \to B\text{Spin}^c(k+1) \), and for the isomorphisms \( s^* \xi^{k+1} \cong \mathbb{R}_{B\text{Spin}^c(k)} \oplus \xi^k \).

In this definition of cycles and relations we use the Riemannian metric in order to define normal vector fields which in turn are required to define the boundary restriction of a \( B \)-structure. Later we will also use the Levi-Civita connection in order to define geometric differential operators.
4 The group \( Q^R_n(M(B, A)) \)

In Definition 2.1 we have introduced the group \( Q^R_n(E) \) for an arbitrary spectrum \( E \). In the case of a relative bordism spectrum \( E = M(B, A) \) we can rewrite the definition of \( Q_n(M(B, A)) \) in terms of the pair of spaces \( (B, A) \). This alternative picture of \( Q_n(M(B, A)) \) will be used in the definition of \( \eta^{an} \).

We consider the Eilenberg-MacLane spectrum \( HP_R := H[R, b, b^{-1}] \) with \( \deg(b) = -2 \). It represents two-periodic real cohomology. The Chern character induces an isomorphism \( \text{ch} : K^R \cong HP_R \).

We now use that Thom spectra defined through maps to \( B Spin^c \) (as apposed to \( BO \)) are \( K \)-oriented. In particular, we have a Thom isomorphism \( \text{Thom}^K : K^0(B, A) \cong K^0(M(B, A)) \).

Assume now that \( n \) is an odd integer. Then we have an isomorphism
\[
\mathbb{Z} \cong \pi_0(K) \xrightarrow{b^{-\frac{n+1}{2}}} \pi_{n+1}(K)
\]
which induces an isomorphism \( \mathbb{R}/\mathbb{Z} \cong \pi_{n+1}(K^R/\mathbb{Z}) \).

Using the Thom isomorphism and this identification we get an identification
\[
Q^R_n(M(B, A)) \cong \frac{\text{Hom}^{cont}(K^0(B, A), R/\mathbb{Z})}{\tilde{U}^R_n} \quad (10)
\]

In order to describe the subgroup \( \tilde{U}^R_n \) we rewrite the definition of \( U^R_n \) using the Riemann-Roch formula. We consider the projection
\[
p_{n+1} : \pi_{n+1}(HP^R) \cong \mathbb{R} \to \mathbb{R}/\mathbb{Z}.
\]
The elements of \( \tilde{U}^R_n \) are homomorphisms determined by elements \( y \in H[R, n+1(B, A) \) by the formula
\[
K^0(B, A) \ni \phi \mapsto p_{n+1}(\langle y, \text{Td}^{-1} \cup \text{ch}(\phi) \rangle) \quad (11)
\]
where \( \text{Td}^{-1} \in H[P^0(B) \) is the universal Todd class pulled back from \( B Spin^c \).

5 The doubling construction

5.1 The double of a \( B \)-manifold

In this section we describe a doubling construction. It will be used in order to get rid of boundary components of type \( A \) in order to simplify the analytic arguments later. We consider a zero bordism \((W, F)\) as in Section 3. In this situation we form the double
\[
DW := W \cup_{\partial A W} W^{op},
\]
where we use the subscript \( \text{op} \) in order to indicate that the right copy has the opposite orientation. The double \( D W \) is a Riemannian manifold whose boundary is again a double
\[
\partial(DW) \cong D(\partial W).
\]
The double \( D W \) has an induced \( B \)-structure. Its underlying map \( DF : DW \to B \) given by \( F \) on both copies of \( W \). Furthermore, the analog of the isomorphism (8) is given on \( W^{\text{op}} \) by
\[
TW^{\text{op}} \oplus \hat{F}^* \xi^k \cong R^{n+k} \cong R^{n+k}
\]
where \( \epsilon \) flips the \( n \)th basis vector, and the first isomorphism is (8) for the normal \( B \)-structure of the left copy of \( W \). This map glues with the normal \( B \)-structure on the left copy in view of the fact, that the glueing for the tangent bundle \( T DW \) is given by the isomorphism
\[
TW|_{\partial W} \cong T\partial_A W \oplus R_{\partial_A W^{\text{op}}} \cong (U^+ \oplus U^-)|_{\partial_A W^{\text{op}}}
\]

5.2 The double of the spinor bundle

A refinement of the normal \( B \)-structure of \( W \) to a geometric tangential \( Spin^c \)-structure [11, 3.2] induces a Dirac bundle \( S_W \). There is a natural construction of a geometric tangential \( Spin^c \)-structure on the double \( D W \). Since we are only interested in the associated Dirac operators we will describe this double on the level of Dirac bundles, see [23, Sec. II, Def. 5.2].

The opposite Dirac bundle \( S_W^{\text{op}} \) is obtained from \( S_W \) by replacing the Clifford multiplication \( c : TW \times S_W \to S_W \) by its negative. We obtain the Dirac bundle on \( D W \) by glueing \( S_W \) on \( W \) with \( S_W^{\text{op}} \) on \( W^{\text{op}} \) using the isomorphism
\[
S_W|_{\partial W} \xrightarrow{c^{(\nu)}} S_W^{\text{op}}|_{\partial W^{\text{op}}}
\]
given by the Clifford multiplication with the outer normal vector field \( \nu \).

5.3 The double of relative geometric bundles

We now consider a \( \mathbb{Z}/2\mathbb{Z} \)-graded geometric bundle \( U \) on \( W \) with a product structure near the boundary and geometry preserving isomorphism
\[
\sigma : U^+_{|\partial_A W} \xrightarrow{\sim} U^-_{|\partial_A W}.
\]
In this situation we define the double \( D(U, \sigma) \), a geometric bundle on \( D W \), by glueing \( U \) on \( W \) with \( U^- \oplus U^- \) on \( W^{\text{op}} \) using the isomorphism
\[
U_{|\partial_A W} \xrightarrow{\text{(\sigma,14)}} (U^- \oplus U^-)|_{\partial_A W^{\text{op}}}.
\]

Remark 5.1. Note, that in contrast to the double \( D W \) and the spinor bundle the double \( D(U, \sigma) \) has no reflection symmetry. One should rather think of \( D(U, \sigma) \) as representing a \( K \)-theory class in \( K^0(DW, W^{\text{op}}) \) which corresponds to the class of \((U, \sigma)\) in \( K^0(W, \partial_A W) \) under excision.
6 The analytic universal $\eta$-invariant

In this section we define the analytic version of the universal $\eta$-invariant

$$\eta^\text{an}: \pi_n(M(B, A))_{\text{tors}} \to \mathbb{Q}_R^\text{Q}(M(B, A))$$

in the relative case. Let $(M, N, f)$ by a cycle representing a relative bordism class $x \in \pi_n(M(B, A))_{\text{tors}}$ as explained in Section 3. Then there exists a non-vanishing integer $\ell$ such that $\ell x = 0$. Hence there exists a zero bordism $(W, F)$ such that $\partial_B(W, F) \cong \ell(M, N, f)$.

Let $\phi \in K^0(B, A)$. Note that $f : (M, N) \to (B, A)$ is a map of pairs. We can choose a pair $(V, \rho)$ of a $\mathbb{Z}/2\mathbb{Z}$-graded geometric bundle $V$ on $M$ and a geometry preserving isomorphism $\rho : V^+_N \to V^-_N$ such that $[V, \rho] = f^*\phi \in K^0(M, N)$. We assume that the geometry has a product structure near $N$. We can further assume that there exists a pair $(U, \sigma)$ as in Section 5.3 such that $(U, \sigma)|_{\partial B} \cong \ell(V, \rho)$.

The twisted Dirac operator $\mathcal{D}_\mathcal{D} \otimes \mathcal{D}(U, \sigma)$ on the double $\mathcal{D}W$ with APS-boundary conditions at $\partial\mathcal{D}W$ is a Fredholm operator $(\mathcal{D}_\mathcal{D} \otimes \mathcal{D}(U, \sigma))_{\text{APS}}$ which is odd with respect to the grading $z$ given by the Clifford multiplication with the volume form. As usual, we define its index $\text{index}(\mathcal{D}_\mathcal{D} \otimes \mathcal{D}(U, \sigma))_{\text{APS}} \in \mathbb{Z}$ as the $z$-graded dimension of its kernel.

We now consider the quantity

$$e := \left[ \frac{1}{\ell} \text{index}(\mathcal{D}_\mathcal{D} \otimes \mathcal{D}(U, \sigma))_{\text{APS}} \right] \in \mathbb{R}/\mathbb{Z}.$$ 

(13)

One checks the following properties in a similar manner as in [11, Prop 3.4]

1. Using the continuous dependence of $e$ on the geometric data we get independence of $e$ from the geometric structures on $M, W, V$ and $U$. Furthermore $e$ only depends on the homotopy class of the isomorphisms $\sigma, \rho$.

2. We now consider $e$ as a function of the pair $(V, \rho)$. It follows from additivity of the construction of $e$ under sums of bundles that $e$ induces a continuous homomorphism $\tilde{e} : K^0(B, A) \to \mathbb{R}/\mathbb{Z}$.

3. The class $[\tilde{e}] \in Q^R_n(M(B, A))$ (using the picture [10] of $Q^R_n(M(B, A))$) of the homomorphism $\tilde{e}$ does not depend on the choice of the integer $\ell$ and the zero bordism $(W, F)$.

4. The class $[\tilde{e}] \in Q^R_n(M(B, A))$ only depends on the bordism class $x$.

Definition 6.1. We define the value of the analytic version of the universal $\eta$-invariant on $x$ by

$$\eta^\text{an}(x) := [\tilde{e}].$$
7 The index theorem

Let $\eta^{\text{top}}$ be the topological universal $\eta$-invariant defined in Definition 2.2 for $E = M(B, A)$, and $\eta^{\text{an}}$ be the analytical universal $\eta$-invariant defined in Definition 6.1.

**Theorem 7.1** (Secondary index theorem). We have the equality $\eta^{\text{an}} = \eta^{\text{top}}$.

**Proof.** We adapt the proof given for the absolute case in [11]. The remainder of the present section is devoted to the proof of this theorem.

7.1 A geometric cycle for $\tilde{x}$

We define the pointed space $C_\ell$ as the cofibre of $\ell$-fold covering $S^1 \to S^1 \to C_\ell$. (14)

It is a Moore space and related with the Moore spectrum of $\mathbb{Z}/\ell\mathbb{Z}$ by an equivalence of spectra $\Sigma^\infty C_\ell \cong \Sigma^{-1}\text{Moore}(\mathbb{Z}/\ell\mathbb{Z})$. We use the equivalence of spectra $M(B, A) \wedge C_\ell \cong M(B \times C_\ell, A \times C_\ell \cup B \times *_{C_\ell})$ in order to interpret elements in the homotopy of $M(B, A) \wedge C_\ell$ geometrically as in Section 3.

We consider the cofibre sequence of spectra obtained by forming the smash product of the cofibre sequence (14) with $M(B, A)$. It induces a long exact sequence in homotopy. We consider the following segment of this sequence:

$$
\pi_n(M(B, A) \wedge C_\ell) \xrightarrow{\partial} \pi_n(M(B, A)) \to \pi_n(M(B, A)) \ .
$$

Let $x \in \pi_n(M(B, A))$ be an $\ell$-torsion element. Then we can choose a lift $\tilde{x}_\ell \in \pi_n(M(B, A) \wedge C_\ell)$ of $x$. It induces a choice of $\tilde{x} \in \pi_{n+1}(M(B, A)\mathbb{R}/\mathbb{Z})$ used in the definition of $\eta^{\text{top}}(x)$ in Section 2 via the map $M(B, A) \wedge C_\ell \cong \Sigma^{-1}M(B, A)\mathbb{Z}/\ell\mathbb{Z} \to \Sigma^{-1}M(B, A)\mathbb{R}/\mathbb{Z}$.

We form the Riemannian manifold with boundary

$$
\tilde{W} := (S^1 \times W) \cup_{S^1 \times \partial_b W \equiv \partial(S^1 \times M)^{op}} (S^2_\ell \times M)^{op} \ ,
$$

where $S^2_\ell$ is a two-sphere with $\ell$ disjoint open discs deleted (see [11, Sec. 3.5]) equipped with a Riemmanain metric with product structure which induces the standard metric on the $\ell$ copies of $S^1$ in its boundary. We define a map $\tilde{F} : \tilde{W} \to B$ such that its restrictions to the summands are given by

$$
S^1 \times W \overset{pr_W}{\to} W \to B \ , \quad S^2_\ell \times M \overset{pr_M}{\to} M \overset{f}{\to} B .
$$
Note that the restriction of $F$ to the boundary of $\tilde{W}$ factorizes over $A$. We use the stable framings of $S^1$ and $S^2_\ell$ and the normal $B$-structures on $f$ and $F$ in order to refine the restrictions of $\tilde{F}$ to the left and right pieces to normal $B$-structures. We refer to [11, Sec. 3.5] for more details. The two refinements can be glued to a normal $B$-structure for $\tilde{F}$ by a similar construction as for the double in Section 5.

We furthermore define a map $\tilde{G} : \tilde{W} \to C_\ell$ such that its restrictions to the summands are given by

$$S^1 \times W \overset{pr_1}{\to} S^1 \to C_\ell, \quad S^2_\ell \times M \overset{pr_2}{\to} S^2_\ell \to C_\ell,$$

where $g$ is defined as in [11, (40)]. The geometric cycle $(\tilde{W}, \partial \tilde{W}, (\tilde{F}, \tilde{G}))$ represents an element $\tilde{x}_\ell \in \pi_{n+1}(M(B, A) \wedge C_\ell)$.

**Lemma 7.2.** We have $\partial \tilde{x}_\ell = x$.

**Proof.** This is shown exactly as in the proof of [11, Lemma 3.7].

### 7.2 An analytic picture of the pairing $\langle \text{Thom}^K(\phi), \varepsilon(\tilde{x}_\ell) \rangle$.

Let $\varepsilon : S \to K$ be the unit of the ring spectrum $K$. We get a class

$$\varepsilon(\tilde{x}_\ell) \in K_{n+2}(M(B, A) \wedge C_\ell).$$

For $\phi \in K^0(B, A)$ we consider the pairing $\langle \text{Thom}^K(\phi), \varepsilon(\tilde{x}_\ell) \rangle \in K_{n+2}(C_\ell)$. The goal of this subsection is the construction of a geometric representative of this $K$-homology class.

In the following we use the notation $(U, \sigma)$ and $(V, \rho)$ as in Section 6. We let $\tilde{V}$ be the $\mathbb{Z}/2\mathbb{Z}$-graded geometric bundle on $\tilde{W}$ which is naturally given by $pr_{\tilde{W}}^*U$ on $S^1 \times W$, and by $pr_{\tilde{M}}^*V$ on $S^2_\ell \times M$. It comes with a natural isomorphism $\tilde{\rho} : \tilde{V}^+|_{\partial \tilde{W}} \to \tilde{V}^-|_{\partial \tilde{W}}$ induced by the isomorphisms $\sigma$ and $\rho$.

The twisted Dirac operator on $\partial_{D(\tilde{W})} \otimes D(\tilde{V}, \tilde{\rho})$ gives rise to a Kasparov $K$-theory class

$$[\partial_{D(\tilde{W})} \otimes D(\tilde{V}, \tilde{\rho})] \in KK_{n+2}(C(D\tilde{W}), \mathbb{C})$$

and thus to a $K$-homology class

$$D\tilde{G}_*[\partial_{D(\tilde{W})} \otimes D(\tilde{V}, \tilde{\rho})] \in KK_{n+2}(C(C_\ell), \mathbb{Z}) \cong K_{n+2}(C_\ell).$$

Here for pointed space $(X, *)$ we let $C(X)$ denote the algebra of complex-valued continuous functions vanishing at $*$.

**Proposition 7.3.** We have an equality

$$D\tilde{G}_*[\partial_{D(\tilde{W})} \otimes D(\tilde{V}, \tilde{\rho})] = \langle \text{Thom}^K(\phi), \varepsilon(\tilde{x}_\ell) \rangle.$$
Proof. Under the Thom isomorphism

\[ \text{Thom}_K : K_{n+2}(M(B, A) \wedge C_\ell) \cong K_{n+2}(B/A \wedge C_\ell) \]

the class \( \epsilon(x_\ell) \) corresponds to a class

\[ \text{Thom}_K(\epsilon(x_\ell)) \in K_{n+2}(B/A \wedge C_\ell) \cong KK_{n+2}(C(B/A \wedge C_\ell), \mathbb{C}) . \]

We first represent this class in terms of Dirac operators.

In order to define \( K \)-homology classes associated to Dirac operators on manifolds with boundary we get rid of boundary components by implicitly completing the manifolds with infinite cylinders. The \( B \)-structure on \( \tilde{W} \) induces a \( \text{Spin}^c \)-structure. We choose an extension of the Levi-Civita connection on \( \tilde{W} \) to a \( \text{Spin}^c \)-connection with a product structure at the boundary. The \( \text{Spin}^c \)-Dirac operator on \( \tilde{W} \) then gives rise to a class \( [D / \tilde{W}] \in KK_{n+2}(C(\tilde{W}/\partial\tilde{W}), \mathbb{C}) \).

The element \( \text{Thom}_K(\epsilon(x_\ell)) \) is given by

\[ (\tilde{F}, \tilde{G})^* : C(B/A \wedge C_\ell) \rightarrow C(\tilde{W}/\partial\tilde{W}) \] .

The argument is similar to that \cite{11} Lemma 3.8 using that \( [D / \tilde{W}] \) is the relative \( K \)-theory fundamental class of the \( \text{Spin}^c \)-manifold with boundary \( (\tilde{W}, \partial\tilde{W}) \).

As in \cite{11} Sec. 3.5 one now checks that that

\[ \langle \text{Thom}_K(\epsilon(\tilde{x}_\ell)), \epsilon(\tilde{x}_\ell) \rangle = \tilde{G}_*([D / \tilde{W}] \cap \tilde{F}^*\phi) \in KK_{n+2}(C(C_\ell), \mathbb{C}) . \quad (15) \]

We have the equality \([\tilde{V}, \tilde{\rho}] = \tilde{F}^*\phi \in K^0(\tilde{W}, \partial\tilde{W})\). Under the isomorphism \( K^0(\tilde{W}, \partial\tilde{W}) \cong KK(\mathbb{C}, C(\tilde{W}/\partial\tilde{W})) \) the class \([\tilde{V}, \tilde{\rho}]\) is represented by the Kasparov module \((C(\tilde{W}, \tilde{V}), F_\tilde{\rho})\), where \( F_\tilde{\rho} \in \Gamma(\tilde{W}, \text{End}(\tilde{V})) \) is any extension of \( \tilde{\rho} \) to all of \( \tilde{W} \). The cap product in \( (15) \) is represented by the Kasparov product \([C(\tilde{W}, \tilde{V}), F_\tilde{\rho}] \otimes_{C(\tilde{W}/\partial\tilde{W})} [D_\tilde{W}] \) which can be represented by the Callias type operator \cite{9}

\[ [D_\tilde{W} \otimes \tilde{V} + F_\tilde{\rho}] \in KK_{n+2}(C(\tilde{W}/\partial\tilde{W}), \mathbb{C}) . \]

It is now a consequence of the relative index theorem that

\[ \tilde{G}_*[D_\tilde{W} \otimes \tilde{V} + F_\tilde{\rho}] = \mathcal{D}\tilde{G}_*[D_{\mathcal{D}\tilde{W}} \otimes \mathcal{D}(\tilde{V}, \tilde{\rho})] . \]

\[ \square \]
7.3 The final step

In the final step of the proof of Theorem 7.1 we must show that

$$\left[ \frac{1}{\ell} \index(\partial_{DW} \otimes D(U, \sigma))_{APS} \right] \in \mathbb{R}/\mathbb{Z}$$

is equal to the image of $D\hat{G}_s[\partial_{DW} \otimes D(\tilde{V}, \tilde{\rho})] \in K_{n+2}(C_{\ell})$ under the natural map $K_{n+2}(C_{\ell}) \cong \mathbb{Z}/\ell\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. But this is exactly the fact shown in at the end of the proof of [11, Theorem 3.6].

8 Relative differential $K$-theory and cycles

The definition of a geometrization involves differential $K$-theory, in particular the functor $\hat{K}^0$. More precisely, it employs the Hopkins-Singer version of differential $K$-theory. We refer to [11, Sec. 4.2] and the discussion below for a review of the relevant structures. By now there are various constructions of this version of differential $K$-theory. First of all we have the Hopkins-Singer construction [20]. Other, more geometric models are based on families of Dirac operators [14] or structured vector bundles [24]. All of them give an equivalent functor $\hat{K}^0$ by [15].

In the present paper we need the relative version of differential $K$-theory. Ad-hoc constructions of relative differential cohomology theories have been considered e.g. in [17] or [6]. But if one represents differential cohomology in terms of sheaves of spectra on the site of smooth manifolds with open covering topology, then the definition of the relative groups becomes completely natural. Therefore we will use this set-up which was developed in detail in [12], see also [13]. In particular we have a sheaf of spectra $\hat{K}$ (which we will describe in (16) below) representing differential $K$-theory in the sense that

$$\hat{K}^0(M) = \pi_0(\hat{K}(M)).$$

The evaluation of the periodic de Rham complex $\Omega P$ on a manifold $M$ is defined by

$$\Omega P(M) := \Omega(M)[b, b^{-1}],$$

where $\deg(b) = -2$. By

$$\sigma^{\geq 0}\Omega P(M) \subset \Omega P(M)$$

we denote its (stupid) truncation which just neglects the part of negative total degree. Using the Eilenberg-MacLane functor $H$ from chain complexes to spectra we can define a sheaf of spectra $H(\sigma^{\geq 0}\Omega P)$. The sheaf $\hat{K}$ is now defined as the pull-back of sheaves of spectra

$$\xymatrix{ \hat{K} \ar[r]^-{R} & H(\sigma^{\geq 0}\Omega P) \ar[d]^-{ch} \\ K \ar[r]^-{ch} & HP_{\mathbb{R}} }$$

(16)
Here the lower horizontal map is the map of constant sheaves of spectra induced by the Chern character $\text{ch} : K \to HP_{\mathbb{R}}$. Furthermore, the right vertical map is the composition of the map obtained by applying $H$ to the embedding $\sigma^{\geq 0}\Omega P \hookrightarrow \Omega P$ with a version of the de Rham isomorphism $H(\Omega P) \cong HP_{\mathbb{R}}$. We refer to [12] for the technical details.

The short exact sequence of sheaves of complexes

$$0 \to \sigma^{\geq 0}\Omega P \to \Omega P \to \sigma^{\leq -1}\Omega P \to 0$$

induces a fibre sequence of sheaves of spectra

$$\Sigma^{-1} H(\sigma^{\leq -1}\Omega P) \xrightarrow{\partial} H(\sigma^{\geq 0}\Omega P) \to H(\Omega P) \to H(\sigma^{\leq -1}\Omega P).$$

We have natural isomorphisms

$$\pi_0(\Sigma^{-1} H(\sigma^{\leq -1}\Omega P)(M)) \cong \Omega P^{-1}(M)/\text{im}(d), \quad \pi_0(H(\sigma^{\geq 0}\Omega P)(M)) \cong \Omega P^0_d(M),$$

where $\Omega P^0_d(M) \subseteq \Omega P^0(M)$ is the subspace of closed forms of total degree zero. Under these isomorphisms the boundary operator $\partial$ induces, after application of $\pi_0$, the de Rham differential

$$d : \Omega P^{-1}(M)/\text{im}(d) \to \Omega P^0_d(M).$$

The maps $R$ and $I$ in [16] induce, after applying $\pi_0$, the curvature map and the underlying class map

$$R : \hat{K}^0(M) \to \Omega P^0_d(M), \quad I : \hat{K}^0(M) \to K^0(M),$$

where the target of the latter is identified using the natural isomorphism $\pi_*(\hat{K}(M)) \cong K^{-*}(M)$. Furthermore, since [16] is cartesian, the fibres of the left and right vertical maps coincide. We thus obtain a fibre sequence of sheaves of spectra

$$\Sigma^{-1} H(\sigma^{\leq -1}\Omega P) \xrightarrow{\partial} \hat{K} \to K \to H(\sigma^{\leq -1}\Omega P) \quad (17)$$

which, after applying $\pi_0$ and using that $\pi_0(H(\sigma^{\leq -1}\Omega P)(M)) = 0$, gives the exact sequence

$$K^{-1}(M) \xrightarrow{\text{ch}} \Omega P^{-1}(M)/\text{im}(d) \xrightarrow{\partial} \hat{K}^0(M) \to K^0(M) \to 0. \quad (18)$$

We now generalize these calculations to the relative case. We consider an embedding of a submanifold $i : N \to M$. Then we define the relative differential $K$-theory group by

$$\hat{K}^0(M, N) := \pi_0(\text{fibre}(\hat{K}(M) \to \hat{K}(N))). \quad (19)$$

From the long exact sequence in homotopy we get a natural isomorphism

$$\pi_0(\text{fibre} : H(\sigma^{\geq 0}\Omega P)(M) \to H(\sigma^{\geq 0}\Omega P)(N)) \cong \Omega P^0_d(M, N),$$

where $\Omega P^0_d(M, N) \subseteq \Omega P^0_d(M)$ the subspace of all closed forms whose restriction to $N$ vanishes. We conclude that in the relative case the curvature becomes a map

$$R : \hat{K}^0(M, N) \to \Omega P^0_d(M, N).$$

In order to generalize the exact sequence [18] to the relative case we calculate, term by term, the fibre of the evaluation of (17) on the inclusion $N \to M$. 

14
1. The homotopy group $\pi_*$ of the fibre of $K(M) \to K(N)$ is the relative $K$-theory group $K^{-*}(M,N)$.

2. $\pi_0$ of the fibre of $\hat{K}(M) \to \hat{K}(N)$ is, by definition, $\hat{K}^0(M,N)$.

3. We represent the fibre of $H(\sigma^{\leq -1}\Omega P)(M) \to H(\sigma^{\leq -1}\Omega P)(N)$ by
   $$\Sigma^{-1}H(\text{Cone}(\sigma^{\leq -1}\Omega P(M) \to \sigma^{\leq -1}\Omega P(N)))$$

   Explicitly, this cone is the complex
   $$\sigma^{\leq -1}\Omega^*(M) \oplus \sigma^{\leq -1}\Omega^*(N), \quad d(\alpha, \beta) := (d\alpha, \alpha|_N - d\beta).$$

   In particular, its cohomology in degree $-1$ is the group
   $$A^0 := \{(\alpha, \beta) \in \Omega P^{-1}_d(M) \oplus \Omega P^{-2}(N) \mid \alpha|_N = d\beta\}.$$

   We denote by $[\alpha, \beta]$ the class in $A^0$ represented by the pair $(\alpha, \beta)$. The following Lemma is now an immediate consequence of these calculations.

   **Lemma 8.1.** We have an exact sequence
   $$K^{-1}(M,N) \xrightarrow{\text{ch}} A^0 \xrightarrow{\alpha} \hat{K}^0(M,N) \xrightarrow{I} K^0(M,N) \to 0.$$  \hfill (20)

   Furthermore, for $[\alpha, \beta] \in A^0$ have
   $$R(a([\alpha, \beta])) = d\alpha.$$  \hfill (21)

   In the remainder of this section we discuss the cycle map. We consider a pair $(V, \rho)$ of a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundle over $M$ and a geometry preserving isomorphism $\rho : V^+_|N \to V^-|N$. This pair represents a relative $K$-theory class $[V, \rho] \in K^0(M,N)$. We want to refine this class to a differential $K$-theory class.

   In the following lemma functoriality means that the construction commutes with pull-backs along smooth maps between manifolds. Note that $\text{ch}(\nabla V) \in \Omega P^0_d(M,N)$.

   **Lemma 8.2.** There exists a functorial construction of a class $\hat{\mathbf{c}}(\nabla V) \in \hat{K}^0(M,N)$ such that $I(\hat{\mathbf{c}}(\nabla V)) = [V, \rho]$ and $R(\hat{\mathbf{c}}(\nabla V)) = \text{ch}(\nabla V)$.

   **Proof.** This was exercise [12, Ex. 4.180]. Here is the solution.

   We use the sheaf $\hat{\text{ku}}^V$ of spectra on smooth manifolds introduced in Section [13 Sec. 6]. It is constructed by group-completing the nerve $\mathbb{N}(\text{Iso}(\text{Vect}_\nabla))$ of the symmetric monoidal stack of vector complex bundles with connections $\text{Iso}(\text{Vect}_\nabla)$. It is universal for additive characteristic classes for vector bundles with connection. In particular, in [13 Sec. 6.1] we have constructed a map of sheaves of spectra
   $$\hat{r} : \hat{\text{ku}}^V \to \hat{K}.$$
We start with the construction of the cycle map in the absolute case. To this end we consider a geometric bundle $V$ on a manifold $M$ as an object of $\text{Vect}_\odot(M)$ and therefore as a point in $\Omega^\infty\text{ku}^\nabla(M)$. We let $\left[V\right]_{\text{ku}^\nabla} \in \pi_0(\text{ku}^\nabla(M))$ be the class of its connected component. Then $\left[\hat{V}\right] := \hat{r}(\left[V\right]_{\text{ku}^\nabla}) \in \hat{K}^0(M)$ is the Hopkins-Singer differential $K$-theory class of the bundle $V$. The association $V \mapsto \left[\hat{V}\right]$ is called the cycle map.

Remark 8.3. In the models of differential $K$-theory developed in [14] or [24] the differential $K$-theory class of a geometric vector bundle is tautologically defined. In the present paper we need the detour over $\text{ku}^\nabla$ since we use a different homotopy theoretic definition of Hopkins-Singer differential $K$-theory in terms of the sheaf $\text{K}$ which is not immediately related to vector bundles. Recall that the use of sheaves of spectra was essential for the definition of relative differential $K$-theory in (19).

We now extend the cycle map to the relative case. We again first construct a class $\left[V, \rho\right]_{\text{ku}^\nabla} \in \pi_0(\text{ku}^\nabla(M,N))$ and then set $\left[\hat{V}, \rho\right] := \hat{r}(\left[V, \rho\right]_{\text{ku}^\nabla})$.

We can consider the isomorphism $\rho : \text{V}_N^+ \to \text{V}_N^-$ as a path in $\mathfrak{N}(\text{Iso}(\text{Vect}_{\odot}^\nabla(N)))$. We now apply the group completion map $c : \mathfrak{N}(\text{Iso}(\text{Vect}^\nabla)) \to \Omega^\infty\text{ku}^\nabla$ and obtain a path in $\Omega^\infty\text{ku}^\nabla(N)$. We can consider the pair $\left(c(\text{V}^+) - c(\text{V}^-), c(\rho) - c(\text{V}_N^-)\right)$ of a point and a path as a point in the standard model of the homotopy fibre $\Omega^\infty\text{ku}^\nabla(M,N)$ of the restriction map $\Omega^\infty\text{ku}^\nabla(M) \to \Omega^\infty\text{ku}^\nabla(N)$. By definition, this point represents the class $\left[\hat{V}, \rho\right]_{\text{ku}^\nabla} \in \pi_0(\text{ku}^\nabla(M,N))$. 

9 Geometrizations

The main ingredient of the intrinsic formula for the universal $\eta$-invariant to be discussed in Section 10 is the notion of a geometrization. This new concept was introduced in [11, Definition 4.3]. In the following we extend the notion of a geometrization to the relative case.

We consider a quadruple $(M,N,f,\nabla^TM)$, where $(M,N,f)$ is as in Section 3 We further choose a tangential $\text{Spin}^c$-structure and let $\nabla^TM$ be a $\text{Spin}^c$-extension of the Levi-Civita connection. It gives rise to the form $\text{Td}(\nabla^TM) \in \Omega^\infty\text{d}(M)$ representing the class $f^*\text{Td}^{-1}$.

If $G : K^0(B,A) \to \hat{K}^0(M,N)$ is a continuous map, then by the same argument as in the proof [11, Lemma 4.2] there exists a unique continuous map $c_G$ called the cohomological character which completes
Definition 9.1. A geometrization of \((M, N, f, \tilde{\nabla}^TM)\) is a continuous map \(G : K^0(B, A) \to \hat{K}^0(M, N)\) such that the cohomological character preserves the \(b\)-degree.

The construction of geometrizations is a non-trivial matter. Here we demonstrate such a construction in the example where \(A = \ast\), \(B = BSpin\), and where \(B \to BSpin^c\) is the canonical map. This example will be employed in Section 13. Thus we consider a Riemannian manifold \(M\) with an embedded submanifold \(N\) and a map \(f : M \to BSpin\) such that \(f|_N\) is constant with value \(\ast\), and which is refined by a normal \(B\)-structure. We assume that the Riemannian metric has a product structure near \(N\). Our reason for considering this more general situation where \(N\) is not necessarily the boundary of \(M\) is that we want to include a case where Lemma 9.2 below gives a non-trivial result. We can assume that \(\hat{f}\) in \((7)\) factors over a map \(\hat{f}_{Spin} : M \to BSpin(k)\) for some \(k \in \mathbb{N}\), and that \(\hat{f}_{Spin|N}\) is constant. We let \(P \to M\) be the \(Spin(k)\)-principal bundle classified by \(\hat{f}_{Spin}\) and form the associated \(Spin^c(k)\) principal bundle \(\tilde{P} := P \times_{Spin(k)} Spin^c(k)\). We choose a tangential \(Spin^c\)-structure, i.e. an isomorphism

\[
Q \otimes \tilde{P} \cong Spin^c(n + k)_M,
\]

(22)

where \(Q \in Spin^c(TM)\), and we use \([8]\) in order to view the left-hand side as a \(Spin^c\)-structure on the trivial \(n + k\)-dimensional bundle. The connection \(\tilde{\nabla}^TM\) is an extension of the Levi-Civita connection to \(Q\).

The bundle \(P|_N\) is trivialized. We choose a connection \(\nabla^P\) on \(P\) which restricts to the trivial connection on \(N\). It further induces a connection \(\nabla^{\tilde{P}}\) on \(\tilde{P}\).

If \((\theta, V_\theta)\) is a complex, finite-dimensional representation of \(Spin(k)\), then we can define a geometric bundle \(P(\theta)\) by forming the associated bundle \(P(\theta) := P \times_{Spin(k)} V_\theta\) with the induced connection \(\nabla^{P(\theta)}\). If \(\iota : V_\theta \to \mathbb{C}^m\) is an isomorphism of Hilbert spaces, then using the trivialization of \(P|_N\) we get an isomorphism of geometric bundles \(P(\iota) : P(\theta)|_N \to \mathbb{C}^m_N\). We define the \(\mathbb{Z}/2\mathbb{Z}\)-graded bundle \(\tilde{P}(\theta)\) such that its even part is \(P(\theta)\), and its odd part is \(\mathbb{C}^m_M\). By Lemma 8.2 we then get a class

\[
[\tilde{P}(\theta), P(\iota)] \in \hat{K}^0(M, N).
\]

In the following we show that the class \([\tilde{P}(\theta), P(\iota)]\) depends on \(\iota\) in a non-trivial way. Because of this the construction of geometrization along the lines of [11] Prop. 5.13 has to
be modified as will be explained below. Assume that we have chosen a second isomorphism \( \iota' \). Then we can write
\[ \iota' = \exp(L) \circ \iota \]
for some Lie algebra element \( L \in u(\dim(V_\theta)) \). Let \( \text{PD}[N] \in H^1(M, N; \mathbb{R}) \) denote the dual class of the orientation class \([N] \in H_{n-1}(M; \mathbb{R})\).

**Lemma 9.2.** We have
\[ [\tilde{\text{P}}(\theta), P(\iota')] - [\tilde{\text{P}}(\theta), P(\iota)] = a(\frac{\text{Tr}(L)}{2\pi i} b \text{PD}[N]) . \]

**Proof.** We are going to use the homotopy formula. On \([0,1] \times M\) we define the \( \mathbb{Z}/2\mathbb{Z} \)-graded bundle
\[ \hat{P} := \text{pr}_M^* \text{P}(\theta) \oplus \bigoplus_{[0,1] \times M} \).

On its restriction to \( N \) we consider the isomorphism \( \rho(t) := \exp(tL) \circ \text{pr}_N^* \text{P}(\iota) \), where \( t \) is the coordinate of the interval. Then \( \rho \) interpolates between \( \text{P}(\iota) \) and \( \text{P}(\iota') \). In order to turn \( \hat{P} \) into a geometric bundle \( \hat{P} \) we equip its even part with the connection \( \text{pr}_M^* \nabla^{P(\theta)} \) and the odd part with the connection
\[ \nabla^{\text{triv}} - \chi(r)Ldt , \]
where \( r : M \rightarrow [0, 1) \) is the normal coordinate near \( N \), \( \chi(r) \) is a cut-off function which is equal to 1 near \( r = 0 \) and vanishes for \( r > 1/2 \). Then \( \rho : \hat{P}_{|[0,1] \times N}^+ \rightarrow \hat{P}_{|[0,1] \times N}^- \) is an isomorphism of geometric bundles. We have
\[ R([\tilde{\text{P}}, \rho]) = \text{pr}_M^* R[\tilde{\text{P}}(\theta), P(\iota)] + dt \wedge d\chi(r) \frac{\text{Tr}(L)}{2\pi i} b . \]

By the homotopy formula for differential \( K \)-theory we have
\[ [\tilde{\text{P}}(\theta), P(\iota')] - [\tilde{\text{P}}(\theta), P(\iota)] = a([d\chi(r) \frac{\text{Tr}(L)}{2\pi i} b, 0]) , \]
where we use the mapping cone notation \([-,-]\) for forms which was introduced before Lemma (8.1). Finally note that \([d\chi(r), 0] \in H^1(M, N; \mathbb{R})\) is the Poincaré dual class of \([N] \in H_{n-1}(M; \mathbb{R})\). \( \square \)

**Corollary 9.3.** If \( N \) is the boundary of \( M \), then the class \([\tilde{\text{P}}(\theta), P(\iota)]\) does not depend on the choice of \( \iota \).

**Proof.** We have \([N] = 0\). \( \square \)

Let \( \tilde{R}(\text{Spin}(k)) \subset R(\text{Spin}(k)) \) be the ideal of the representation ring of \( \text{Spin}(k) \) of elements with vanishing dimension. The associated bundle construction induces a homomorphism
\[ \text{ass} : \tilde{R}(\text{Spin}(k)) \rightarrow K^0(B\text{Spin}(k), *) . \]

It follows from the completion theorem [4] that \( \text{ass} \) is injective and has a dense range.
We choose a basis \((\theta_i)_{i \in I}\) of the free \(\mathbb{Z}\)-module \(\tilde{R}(Spin(k))\). We consider the element \(\theta_i \in \tilde{R}(BSpin(k))\) as a \(\mathbb{Z}/2\mathbb{Z}\)-graded representation of \(BSpin(k)\). For each \(i \in I\) we further choose a Hilbert space isomorphism \(\iota_i\) between the even and odd parts of \(\theta_i\). We can define a continuous map

\[
G_0 : K^0(BSpin(k), *) \to \hat{K}^0(M, N)
\]

by the prescription

\[
G_0(\text{ass}(\theta_i)) := [\hat{P}(\theta_i), P(\iota_i)]
\]

for all \(i \in I\). Then we clearly have the identity \(I \circ G_0 = f^*\). The cohomological character of \(G_0\) is fixed by

\[
c_{G_0}(Td^{-1} \cup ch(\text{ass}(\theta_i))) = Td(\hat{\nabla}_M) \land ch(\hat{P}(\theta_i)) \in \Omega^0(M, N).
\]

The map \(c_{G_0}\) only preserves the \(b\)-degree if the equality of Todd forms \(Td(\hat{\nabla}_M) = Td(\hat{P})^{-1}\) holds true. In general this is not the case, and in order to turn \(G_0\) into a geometrization, we must add a correction term. Using the tangential \(Spin^c\)-structure \([22]\) we can define the transgression

\[
\delta := \hat{\text{Td}}(\hat{\nabla}_M) \oplus \hat{\nabla}^{triv}) \in \Omega P^{-1}(M)/\text{im}(d).
\]

Then

\[
d(\delta \land \text{Td}(\hat{\nabla}^{-1})) = \text{Td}(\hat{\nabla}^{-1}) - \text{Td}(\hat{\nabla}^{triv})^{-1}.
\]

Let \(i : BSpin(k) \to BSpin\) be the inclusion. We define the continuous map

\[
\mathcal{G} : K^0(BSpin, *) \to \hat{K}^0(M, N)
\]

by

\[
\mathcal{G}(\phi) := G_0(i^* \phi) - a([\delta \land c_{G_0}(Td^{-1} \cup ch(i^* \phi))]/\text{Td}(\hat{\nabla}_M)2, 0)]
\]

One easily checks that its cohomological character satisfies preserves \(b\)-degree. Therefore \(\mathcal{G}\) is a geometrization.

In contrast to the absolute case this geometrization not only depends on \(\nabla^P\), but also on the choice of the isomorphisms \(\iota_i\). Nevertheless the construction is sufficiently canonical so that if \((M, N, f, \hat{\nabla}^T_M)\) is obtained by taking the boundary \(\partial_B\) of a zero bordism \((W, F, \hat{\nabla}^T_W)\), then the geometrization extends to \(W\).

\section{The intrinsic formula}

From now on we consider the notation as in Section \([6]\). We are going to express the quantity \([13]\) solely in terms of data on a cycle \((M, N, f)\) for \(x \in \pi_n(M(B, A))_{\text{tors}}\) such that \(\ell x = 0\). Recall that we have fixed a tangential \(Spin^c\)-structure and a \(Spin^c\)-extension...
\[ \nabla^{TM} \text{ of the Levi-Civita connection. Let } \mathcal{G} \text{ be a geometrization of } (M, N, f, \nabla^{TM}) \text{ as in Definition } 9.1. \text{ Let } \phi \in K^0(B, A) \text{ and } (V, \rho) \text{ be as in Section } 6 \text{ such that } [V, \rho] = f^*\phi. \text{ Then by (20) there exists a class} \\
\gamma_\phi := [\alpha_\phi, \beta_\phi] \in \Omega^{P-1}_{\text{cl}}(M) \oplus \Omega^{P-2}(N) \text{ such that} \\
\hat{[V, \rho]} = G(\phi) - a(\gamma_\phi). \quad (27) \]

Here we abuse the \([-,-]-notation for elements in } A^0 \text{ and use it in order to write elements in } A^0/\text{im}(\text{ch}).$

**Assumption 10.1.** We assume that there exists a geometrization \( \mathcal{G}_{W, \partial A W} \) which induces the geometrization \( \mathcal{G} \) on the \( \ell \) copies of \( (M, N) \) in the boundary of \( \partial_B W \).

**Remark 10.2.** This assumption is non-trivial. In general not every geometrization on \( (M, N) \) can be obtained as such a restriction. We refer to [11] for a detailed discussion. But note that the example of a geometrization constructed in Section 9 has this property.

For a Dirac operator \( \mathcal{D} \) we define the reduced \( \eta \)-invariant by

\[ \xi(\mathcal{D}) := \left[ \frac{\eta(\mathcal{D}) + \dim(\ker(\mathcal{D}))}{2} \right] \in \mathbb{R}/\mathbb{Z}. \]

Let \( \mathcal{D}_{DM} \otimes \mathcal{D}(V, \rho) \) the Spin\(^c\)-Dirac operator on the double \( \mathcal{D} M \) twisted by the double of the bundle \( (V, \rho) \).

**Theorem 10.3** (Intrinsic formula). The element \( \eta^a(x) \in Q_n(M(B, A)) \) is represented by the homomorphism

\[ K^0(B, A) \ni \phi \mapsto \left[ -\int_M \text{Td}(\nabla^{TM}) \wedge \alpha_\phi - \int_N \text{Td}(\nabla^{TN}) \wedge \beta_\phi \right]_{\mathbb{R}/\mathbb{Z}} - \xi(\mathcal{D}_{DM} \otimes \mathcal{D}(V, \rho)) \in \mathbb{R}/\mathbb{Z}. \]

**Proof.** First note that the first term on the right-hand side is well-defined since for \((\alpha_\phi, \beta_\phi) \in \text{im}(\text{ch})\) the sum of the two integrals yields an integer. We start with the APS index theorem [3] (compare with (2)):

\[ \left[ \frac{1}{\ell} \text{index}(\mathcal{D}_{DW} \otimes \mathcal{D}(U, \sigma))_{\text{APS}} \right] = \left[ \frac{1}{\ell} \int_{\mathcal{D} W} \text{Td}(\nabla^{TDW}) \wedge \text{ch}(\nabla^{D(U, \sigma)}) \right] - \xi(\mathcal{D}_{DM} \otimes \mathcal{D}(V, \rho)). \]

Using the odd \( \mathbb{Z}/2\mathbb{Z} \)-symmetry of \( \mathcal{D}(U, \sigma) \) on \( W^{\text{op}} \subset \mathcal{D} W \), (27), (21) and Stokes’ theorem we calculate

\[ \int_{\mathcal{D} W} \text{Td}(\nabla^{TDW}) \wedge \text{ch}(\nabla^{D(U, \sigma)}) = \int_W \text{Td}(\nabla^{TW}) \wedge R(\mathcal{G}_{W, \partial A W(\phi)}) - \ell \int_M \text{Td}(\nabla^{TM}) \wedge \alpha_\phi - \int_{\partial A W} \text{Td}(\nabla^{\partial A W}) \wedge \alpha_{\phi|\partial A W} \]
We now observe that the homomorphism

$$K^0(B, A) \ni \phi \mapsto \left[ \frac{1}{\ell} \int_W \text{Td}((\tilde{\nabla} T W) \wedge R(G_W, \partial A W(\phi))) \right] \in \mathbb{R}/\mathbb{Z}$$

factorizes over the cohomological character $c_{G_W, \partial A W}$. In view of (11) it therefore belongs to the subgroup $U_{\mathbb{R}}$. We conclude that $\eta^a n(x)$ is represented by the asserted map. \qed

11 Tertiary invariants

In this section we describe the construction of an invariant $\kappa^{\text{top}}$ which is a secondary version of the universal $\eta$-invariant and may detect elements in the homotopy of $MA$ which become trivial when mapped to the homotopy of $MB$. Special cases will be discussed in the subsequent Sections 12 and 13.

We let $n$ be an even integer. In order to simplify matters we make the assumption that $\pi_{n-1}(M(B, A))$ is a torsion group. We consider the diagram

\[
\begin{array}{ccc}
\pi_{n-1}(MB) & \longrightarrow & \pi_{n-1}(M(B, A)) \\
\downarrow \alpha & & \downarrow \eta^{\text{top}} \\
Q_{n-1}(M(B, A)) & \longrightarrow & \ker(i_*) \\
\downarrow p & & \downarrow \kappa^{\text{top}} \\
Q_{n-1}(B, A) & & \end{array}
\]

(28)

The upper line is a segment of the long exact sequence associated to the fibre sequence (6). The map $\alpha$ is the obvious composition. We define the abelian group

$$Q_{n-1}^\mathbb{R}(B, A) := \frac{Q_{n-1}^\mathbb{R}(M(B, A))}{\text{im}(\alpha)}.$$ 

Finally, we define the homomorphism $\kappa^{\text{top}}$ by the following diagram chase. Consider an element $y \in \ker(i_*)$. Then we choose a lift $x \in \pi_{n-1}(M(B, A))$. By assumption it is torsion and therefore in the domain of the universal $\eta$-invariant. By construction, the image $p(\eta^{\text{top}}(x)) \in Q_{n-1}^\mathbb{R}(B, A)$ is independent of the choice of the lift.

Definition 11.1. We define the map

$$\kappa^{\text{top}} : \ker(i_*) \rightarrow Q_{n-1}^\mathbb{R}(B, A)$$

(29)

such that $\kappa^{\text{top}}(y) := p(\eta^{\text{top}}(x))$.

Remark 11.2. We consider the universal $\eta$-invariant $\eta^{\text{top}} : \pi_{n-1}(MA)_{\text{tors}} \rightarrow Q_{n-1}^\mathbb{R}(MA)$ as a secondary invariant of the $K$-orientation $MA \rightarrow B\text{Spin}^c \rightarrow K$. In this sense $\kappa^{\text{top}}$ is a tertiary invariant.
12 Laures’ \( f \)-invariant

In this section we discuss an example for the tertiary invariant defined in 11.1 which has already been studied intensively. We consider the case where

\[
A = \ast, \quad B = BU,
\]

and where \( BU \to B\text{Spin}^c \) is the canonical map. Then \( MA \cong S \) is the sphere spectrum, \( MU \cong MB \), and the corresponding cohomology theories are called framed bordism and complex bordism. In particular, the tertiary invariant detects elements in the stable homotopy groups of the sphere. The usual notation for the relative bordism spectrum is

\[
\overline{MU} := M(BU, \ast).
\]

The main problem is to define a map out of the group \( Q_{n-1}^\mathbb{R}(BU, \ast) \) which is able to detect interesting elements.

The construction of the desired evaluation on \( Q_{n-1}^\mathbb{R}(BU, \ast) \) employs an elliptic cohomology theory. To this end we fix an integer \( D \in \mathbb{N} \) with \( N \geq 2 \) and a \( D \)’th root of unity \( \zeta_D \). Then there exists a Landweber exact elliptic cohomology theory over the ring of modular forms

\[
\mathcal{MF}^E_* := \mathcal{MF}^{\Gamma_1(D)}_*[D^{-1}, \zeta_D^{-1}]
\]

for the group \( \Gamma_1(D) \) whose \( q \)-expansions have coefficients in the ring \( \mathbb{Z}[D^{-1}, \zeta_D^{-1}] \) (see e.g. [21 Thm. 1.2.1]). This cohomology theory is represented by a spectrum \( E \) which fits into a sequence of maps

\[
MU \to E \to K[D^{-1}, \zeta_D][[q]],
\]

(31)

where \( MU \to E \) is the complex orientation of the cohomology theory \( E \), and the map \( E \to K[D^{-1}, \zeta_D][[q]] \) is induced by the evaluation at the Tate curve.

From now on we assume that \( n \) is an even integer. The space of \( q \)-expansions

\[
\mathcal{MF}^E_{n/2}[[q]] \subseteq \mathbb{Z}[D^{-1}, \zeta_D][[q]] \cong \pi_n(K[D^{-1}, \zeta_D][[q]])
\]

is the image of \( \pi_n(E) \) under this evaluation.

We extend the composition \( (31) \) to a composition of maps of vertical fibre sequences

\[
\begin{array}{ccccccc}
S & \xrightarrow{1} & S & \xrightarrow{e} & K[D^{-1}, \zeta_D] & \\
& \downarrow & & \downarrow & & \\
& MU & \xrightarrow{} & E & \xrightarrow{} & K[D^{-1}, \zeta_D][[q]] & \\
& & \downarrow & & \downarrow & & \\
& & \overline{MU} & \xrightarrow{} & E & \xrightarrow{} & K[D^{-1}, \zeta_D][[q]]/K[D^{-1}, \zeta_D] \xrightarrow{\exists} \prod_{i=1}^\infty q^iK[D^{-1}, \zeta_D] \\
& & & & \phi & & \\
\end{array}
\]
The map $\phi$ is defined as the natural factorization. We interpret $\phi$ as a sequence of classes $\phi_i \in K[D^{-1}, \zeta_D]^{i}(MU)$ defined for all positive $i \in \mathbb{N}$. For even $n$ the evaluation against $\phi$ induces a map

$$\tilde{\ev}_\phi : \text{Hom}_{\text{cont}}(K^0(MU), \pi_n(\mathbb{R}/\mathbb{Z})) \to \frac{\mathbb{C}[[q]]}{\mathbb{Z}[D^{-1}, \zeta_D][[q]] + q^0\mathbb{C} + \mathcal{MF}_{n/2}^E[[q]]},$$

$$\tilde{\ev}_\phi(h) := \left[ \sum_{i=1}^{\infty} \ev_{\phi_i}(h)q^i \right].$$

We now observe that for $y \in U_{n-1}$ we have $\tilde{\ev}_\phi(y) \in \mathcal{MF}_{n/2}^E[[q]] + q^0\mathbb{C}$. Therefore $\tilde{\ev}_\phi$ descends to a homomorphism

$$\ev_\phi : Q_{n-1}^R(MU) \to \frac{\mathbb{C}[[q]]}{\mathbb{Z}[D^{-1}, \zeta_D][[q]] + q^0\mathbb{C} + \mathcal{MF}_{n/2}^E[[q]]}.$$

Since $n$ is even we have $\pi_{n-1}(MU) \cong 0$ and there are no interesting elements which can be detected by the evaluation of the universal $\eta$-invariant for $MU$ using this evaluation. On the other hand, this fact implies that the evaluation $\ev_\phi$ actually further descends to a homomorphism

$$\ev_\phi : Q_{n-1}^R(BU, *) \to \frac{\mathbb{C}[[q]]}{\mathbb{Z}[D^{-1}, \zeta_D][[q]] + q^0\mathbb{C} + \mathcal{MF}_{n/2}^E[[q]]}.$$

We now assume in addition that $n$ satisfies $n \neq 2$. Then we have the equality

$$\ker(i_* : \pi_{n-2}(S) \to \pi_{n-2}(MU)) = \pi_{n-2}(S).$$

**Definition 12.1.** The $f$-invariant is defined to be the homomorphism

$$f := \ev_\phi \circ \kappa_{MU}^\top : \pi_{n-2}(S) \to \frac{\mathbb{C}[[q]]}{\mathbb{Z}[D^{-1}, \zeta_D][[q]] + q^0\mathbb{C} + \mathcal{MF}_{n/2}^E[[q]]}.$$

Here $\kappa_{MU}^\top$ is the specialization of $\kappa_{\text{top}}$ in (28) to the present case. The name $f$-invariant is justified by the fact verified in [16] that $f$ is indeed Laures’ $f$ invariant introduced in [21]. For explicit calculations we refer to [21], [22] and [25]. In particular, the $f$-invariant is non-trivial.

Using our knowledge on the universal $\eta$-invariant we can reproduce the following information on Laures’ $f$-invariant. The relative bordism spectrum $MU$ is the main constituent of the Adams tower

$$S \leftarrow \Sigma^{-1}MU \leftarrow \Sigma^{-1}MU \wedge \Sigma^{-1}MU \leftarrow \Sigma^{-1}MU \wedge \Sigma^{-1}MU \wedge \Sigma^{-1}MU \leftarrow \ldots$$

(33)

The Adams tower induces a decreasing filtration $F_m^MU\pi_m(W)$ of the $m$th homotopy group of a spectrum $W$ for every $m \in \mathbb{Z}$. In order to define this filtration we consider the smash
product of the tower \( (33) \) with \( W \). For \( k \in \mathbb{N} \) we define \( F_{MU}^k \pi_m(W) \subseteq \pi_m(W) \) to be the subgroup of elements which lift to \( W \wedge MU^k \).

For all integers \( k \geq 0 \) and even \( n \) we have the obvious isomorphisms

\[
F_{MU}^k \pi_{n-2}(S) \cong F_{MU}^{k-1} \pi_{n-1}(MU) .
\]

In particular we have

\[
\pi_{n-2}(S) \cong F_{MU}^2 \pi_{n-2}(S) \cong F_{MU}^1 \pi_{n-1}(MU) .
\]

The complex orientation \( MU \rightarrow K \) of \( K \)-theory induces a map

\[
F_{MU}^3 \pi_{n-2}(S) \cong F_{MU}^2 \pi_{n-2}(MU) \rightarrow F_{K}^2 \pi_{n-1}(MU) ,
\]

where the filtration \( F_{K}^* \) is defined similarly replacing \( MU \) by \( K \) in the construction of the Adams tower. By [11, Prop. 2.7] the image of this map is annihilated by \( \eta_{top} \). Since \( K_{*}(MU) \) is torsion-free we conclude that \( \kappa_{top}^{MU} \) has a factorization over an injective map

\[
\kappa_{top}^{MU} : \text{Gr}_{MU}^2 \pi_{n-2}(S) \rightarrow Q_{n-1}(BU, \ast) .
\]

It has been shown in [21] that \( \overline{eV}_\phi \) detects the image of \( \kappa_{top}^{MU} \), i.e. that the \( f \)-invariant induces an injective map

\[
\bar{f} : \text{Gr}_{MU}^2 \pi_{n-2}(S) \rightarrow Q_{n-1}(BU, \ast) \rightarrow \mathbb{C}[[q]] /
\mathbb{Z}[D^{-1}, \zeta_D][[q]] + q^n \mathbb{C} + \mathcal{MF}^E_{n/2}[[q]] .
\]

Remark 12.2. In [10] we gave an intrinsic formula for the \( f \)-invariant in terms of a sequence of \( \eta \)-invariants of Dirac operators twisted by bundles derived from the tangent bundle. In order to interpret this as an example of the intrinsic formula [10,3] it would be necessary to translate this construction to a construction with a complementary bundle. This would require to extend the theory of geometrizations to \( K \)- and differential \( K \)-theory with coefficients in \( \mathbb{Z}[D^{-1}, \zeta_D] \). We think that this is possible but that the details are not very enlightening. We will demonstrate the intrinsic formula in a second example in Section 13.

13 A Spin-version of the \( f \)-invariant

In this section we discuss a \( Spin \)-version \( \kappa_{top}^{Spin} \) of the tertiary invariant defined in 11.1. We will see in Proposition 13.3 that it is non-trivial. Similarly as in Section 12 we define an evaluation leading to a \( Spin \)-version \( f^{Spin} \) of the \( f \)-invariant. We make the intrinsic formula for \( f^{Spin} \) explicit.

We consider the case where

\[
A = \ast , \quad B = BSpin ,
\]

24
and we choose the canonical map $B\text{Spin} \to B\text{Spin}^c$. In analogy to (30) we set

$$\overline{M\text{Spin}} := M(B\text{Spin}, \ast).$$

In the following we construct a sequence of $K$-theory classes $\phi_i \in K^0(B\text{Spin}, \ast)$, $i \geq 1$, which will be used to define an evaluation on $Q_{n-1}^{\mathbb{R}}(B\text{Spin}, \ast)$. For a real vector bundle $V$ on a manifold or space $M$ we consider the $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundle

$$\tilde{V} := V \otimes \mathbb{C} \oplus \mathbb{C}^{\text{op}}_M$$

and define the formal power series of $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundles

$$\Theta(V, q^{1/2}) := \bigotimes_{u=1}^{\infty} S_{q^u}(\tilde{V}) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{-q^v - \frac{1}{2}}(\tilde{V}).$$

To be precise, in this definition we expand the tensor products and sort the terms according to the powers of $q^{1/2}$. In particular, we interpret a summand $-W$ as $W^{\text{op}}$, where $W^{\text{op}}$ is obtained from $W$ by flipping the grading. In this way we get a series of complex vector bundles

$$\Theta(V, q^{1/2}) = \sum_{i=0}^{\infty} q^{i/2}\Theta_i(V),$$

where the $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundles $\Theta_i(V)$ are constructed from $V$ in a functorial way using operations of the tensor calculus. The power series is multiplicative in the sense that

$$\Theta(V \oplus V', q^{1/2}) \cong \Theta(V, q^{1/2}) \otimes \Theta(V', q^{1/2}).$$

Furthermore, since

$$[\Theta(V, q^{1/2})] = 1 + O(q^{1/2}) \in K[[q^{1/2}]]^0(M),$$

the $K$-theory class $[\Theta(V, q^{1/2})]$ is a multiplicative unit. Therefore we can extend the association $V \mapsto [\Theta(V, q^{1/2})]$ to $K$-theory classes, i.e. we can define a $K$-theory class $\Theta(\xi, q^{1/2}) \in K[[q^{1/2}]]^0(M)$ for a $K$-theory class $\xi \in K^0(M)$.

Since the construction $V \mapsto \Theta_i(V)$ is functorial, a trivialization $\rho : V \cong \mathbb{R}^k$ induces a sequence of isomorphisms

$$\Theta(\rho)_i : \Theta_i(V)^+ \to \Theta_i(V)^-, \quad i \geq 1,$$

Let $\xi \in K^0(B\text{Spin})$ be the class of the normalized universal bundle. Then we get a class

$$\Theta(\xi, q^{1/2}) \in K[[q^{1/2}]]^0(B\text{Spin}).$$

We have a preferred trivialization $\iota$ of $\xi|\ast$ so that we get a sequence of relative classes

$$[\Theta_i(\xi), \Theta_i(\iota)] \in K^0(B\text{Spin}, \ast), \quad i \geq 1.$$
The series
\[ 1 + \sum_{i=1}^{\infty} q^{i/2}[\Theta_i(\xi), \Theta_i(\nu)] \in 1 + q^{1/2}K[[q^{1/2}]](BSpin, \ast) \]
is invertible. We define a sequence of classes \( \phi_i \in K[[q^{1/2}]](BSpin, \ast) \) for all positive integers \( i \) uniquely such that
\[ 1 + \sum_{i=1}^{\infty} q^{i/2} \phi_i = \left( 1 + \sum_{i=1}^{\infty} q^{i/2}[\Theta_i(\xi), \Theta_i(\nu)] \right)^{-1}. \] (34)

For \( k \in \mathbb{Z} \) let
\[ \mathcal{M}F_{k}^{\Gamma_0(2), \mathbb{R}}[[q^{1/2}]] \subseteq \mathbb{R}[[q^{1/2}]] \]
denote the groups of \( q^{1/2} \)-expansions of holomorphic modular forms of weight \( k \) for the congruence group \( \Gamma_0(2) \subseteq SL(2, \mathbb{Z}) \) with real Fourier coefficients at the cusp at \( \infty \).

We now assume that \( n \in \mathbb{Z} \) is even. The evaluation against the classes \( \phi_i \) induces a map
\[ \tilde{\ev}_{\phi} : \text{Hom}^{cont}(K^0(MSpin), \pi_n(\mathbb{R}/\mathbb{Z})) \to \frac{\mathbb{R}[[q^{1/2}]]}{\mathbb{Z}[[q^{1/2}]] + q^0\mathbb{R} + \mathcal{M}F_{n/2}^{\Gamma_0(2), \mathbb{R}}[[q^{1/2}]]}, \]
\[ \tilde{\ev}_{\phi}(h) := \left[ \sum_{i=1}^{\infty} \ev_{\phi_i}(h)q^{i/2} \right]. \]

If \( y \in U_{n-1}^{\mathbb{R}} \), then by [19] we know that \( \tilde{\ev}_{\phi}(y) \in \mathcal{M}F_{n/2}^{\Gamma_0(2), \mathbb{R}}[[q^{1/2}]] + q^0\mathbb{R} \). Therefore \( \tilde{\ev}_{\phi} \) descends to a homomorphism
\[ \ev_{\phi} : Q_{n-1}^\mathbb{R}(MSpin) \to \frac{\mathbb{R}[[q^{1/2}]]}{\mathbb{Z}[[q^{1/2}]] + q^0\mathbb{R} + \mathcal{M}F_{n/2}^{\Gamma_0(2), \mathbb{R}}[[q^{1/2}]]}. \]

**Lemma 13.1.** If \( n \equiv 0(4) \), then the universal \( \eta \)-invariant
\[ \eta^{\text{top}} : \pi_{n-1}(MSpin) \to Q_{n-1}^\mathbb{R}(MSpin) \]
is trivial.

**Proof.** One checks using the calculation of \( MSpin \) by [1] and the results of [2] that the \( K \)-localization map \( \pi_{n-1}(MSpin) \to \pi_{n-1}(MSpin_K) \) is trivial. The assertion now follows from the fact that \( \eta^{\text{top}} \) factors over the \( K \)-localization [11] Lemma 2.8. \( \Box \)

From now on we assume that \( n \equiv 0(4) \). Lemma [13.1] implies that \( \alpha : \pi_{n-1}(MSpin) \to Q_{n-1}^\mathbb{R}(MSpin) \) vanishes, where \( \alpha \) is defined in (28). Therefore the evaluation \( \ev_{\phi} \) actually factorizes over
\[ \ev_{\phi} : Q_{n-1}^\mathbb{R}(BSpin, \ast) \to \frac{\mathbb{R}[[q^{1/2}]]}{\mathbb{Z}[[q^{1/2}]] + q^0\mathbb{R} + \mathcal{M}F_{n/2}^{\Gamma_0(2), \mathbb{R}}[[q^{1/2}]]}. \]

If \( n \geq 6 \), then \( \ker(i_* : \pi_{n-2}(S) \to \pi_{n-2}(MSpin)) = \pi_{n-2}(S) \) so that \( \kappa_{MSpin}^{\text{top}} \) is defined by [11.1] on the whole stable homotopy group.
Definition 13.2. For every integer \( n \) with \( n \equiv 0(4) \) and \( n \geq 6 \) we define the Spin-version of the \( f \)-invariant by

\[
f^{\text{Spin}} := \mathfrak{ev}_\phi \circ \kappa^{\text{top}}_{\text{MSpin}} : \pi_{n-2}(S) \rightarrow \frac{\mathbb{R}[[q^{1/2}]]}{\mathbb{Z}[[q^{1/2}]] + q^0 \mathbb{R} + \mathcal{M}_{n/2} \mathcal{F}[[q^{1/2}]]}.
\]

At the moment we have the following information about \( f^{\text{Spin}} \).

Proposition 13.3. 1. The Spin version \( f^{\text{Spin}} \) of the \( f \)-invariant vanishes 2-locally.

2. The tertiary invariant \( \kappa^{\text{top}}_{\text{MSpin}} \) is non-trivial and detects, at least, the odd torsion in \( \text{Gr}^2_{MU} \pi_{n-2}(S) \).

Proof. The two-local case

In the following all spectra a two-localized. We have the ABP-splitting \([1]\)

\[
\text{MSpin} \cong F \vee G,
\]

where \( F \) is a wedge of truncated copies of \( \text{ko} \) and \( G \) is a wedge of shifted copies of \( H\mathbb{Z}/2\mathbb{Z} \).

We have a factorization of the unit \( S \rightarrow \text{MSpin} \) as

\[
S \rightarrow F \rightarrow F \vee G.
\]

For a spectrum \( X \) Bousfield localization at \( K \) leads to a fibre sequence

\[
X^K \rightarrow X \rightarrow X_K \rightarrow \Sigma X^K.
\]

By \([2]\) we have \( G_K \cong 0 \) and therefore \( G^K \cong G \). We apply \( K \)-localization to the sequence

\[
\Sigma^{-1} \text{MSpin} \rightarrow S \rightarrow \text{MSpin}.
\]

We get the following web of horizontal and vertical fibre sequences

\[
\begin{array}{c}
\Sigma^{-1} \text{MSpin}_K \rightarrow S^K \rightarrow F^K \vee G \\
\downarrow \quad \downarrow \quad \downarrow \\
\Sigma^{-1} \text{MSpin} \rightarrow S \rightarrow F \vee G \\
\downarrow \quad \downarrow \quad \downarrow \\
\Sigma^{-1} \text{MSpin}_K \rightarrow S_K \rightarrow F_K
\end{array}
\]

The right upper map factorizes as \( S^K \rightarrow F^K \rightarrow F^K \vee G \). Recall that we assume that \( n \equiv 0(4) \). We have an exact sequence

\[
\pi_{n-1}(F) \rightarrow \pi_{n-1}(F_K) \rightarrow \pi_{n-2}(F^K) \rightarrow \pi_{n-2}(F) \xrightarrow{(i)} \pi_{n-2}(F_K).
\]

27
We now use that the canonical map $ko[a..\infty] \to KO$ becomes the $K$-localization map after 2-completion. In view of the structure of $F$ this implies that the map $(i)$ is injective, and that $\pi_{n-1}(F_K) \cong 0$. We conclude that $\pi_{n-2}(F^K) = 0$.

Let $y \in \pi_{n-2}(S)$. The image of $y$ in $\pi_{n-2}(S_K)$ vanishes. We choose a lift $z \in \pi_{n-2}(S^K)$. Its image in $\pi_{n-2}(F^K \vee G)$ vanishes to that we can find a further lift $\tilde{z} \in \pi_{n-1}(MSpin^K)$. The image $x \in \pi_{n-1}(MSpin)$ of $\tilde{z}$ can serve as lift of $y$ in the construction of $k_{MSpin}^{top}$, see Section 11. But since the universal $\eta$-invariant factorizes over $K$-localization we see that $k_{MSpin}^{top}(y) = 0$ since the image of $x$ in $\pi_{n-1}(MSpin_K)$ vanishes.

The odd torsion

We now localize all spectra at an odd prime. Then we have $MSpin \cong MSp$. Furthermore, by a result of Baker-Morava [5] we know that $MSp$ is a summand of $MU$. It follows that the sequence

$$\Sigma^{-1}MSpin \rightarrow S \rightarrow MSpin$$

is a summand of the sequence

$$\Sigma^{-1}MU \rightarrow S \rightarrow MU.$$ 

This implies a commutative diagram

$$\begin{array}{ccc}
\pi_{n-2}(S) & \xrightarrow{k_{MSpin}^{top}} & \pi_{n-1}(MSpin) \\
\downarrow & & \downarrow \\
\pi_{n-2}(S) & \xrightarrow{k_{M^U}^{top}} & \pi_{n-1}(MU)
\end{array}$$

$$\begin{array}{ccc}
\pi_{n-2}(S) & \xrightarrow{\eta_{top}} & Q_{n-1}(MSpin) \\
\downarrow & & \downarrow \\
\pi_{n-2}(S) & \xrightarrow{\eta_{top}} & Q_{n-1}(MU)
\end{array}$$

where the first map is the construction $y \mapsto x$. Since we know that Laures’ $f$-invariant detects $Gr_{M^U}^{2} \pi_{n-2}(S)$ we conclude that

$$k_{MSpin}^{top} : Gr_{M^U}^{2} \pi_{n-2}(S) \rightarrow Q_{n-1}(MSpin)$$

is injective, too.

We now derive an intrinsic formula for $f_{Spin}^{top}$. Let $y \in \pi_{n-2}(S)$ and consider a cycle $(M, N, f, \nabla^{TM})$ for a lift $x \in \pi_{n-1}(MSpin)$ of $y$. We use the notation introduced in Section 9 and assume that $G$ is a geometrization as constructed there.

Using a geometric series and forming the analog of (34) on the level of bundles we define a sequence of $\mathbb{Z}/2\mathbb{Z}$-graded vector bundles $\Phi_i := \Phi_i(f_{Spin}^{top}k)$ and isomorphisms $\iota_i : \Phi_i^{+} \rightarrow \Phi_i^{-}$ such that

$$f^{*}\phi_i = [\Phi_i, \iota_i] , \quad i \geq 1 .$$
Note again, that the bundles $\Phi_i$ are obtained from $f_{Spin}^*\xi^k$ in a functorial way using the operations of tensor calculus. In particular, the connection $\nabla P$ induces a connection on $f_{Spin}^*\xi^k$ which is compatible with the trivialization at $N$, and therefore connections $\nabla^{\Phi_i}$ on $\Phi_i$ which are compatible with $\iota_i$ for all $i \geq 1$. We therefore get geometric bundles $(\Phi_i, \iota_i)$, and by Lemma 8.2 differential $K$-theory classes

$$\hat{\phi}_i := [\Phi_i, \iota_i] \in \hat{K}^0(M, N).$$

Since $N$ is the boundary of $M$, by Corollary 9.3 the classes $\hat{\phi}_i$ do not depend on $\iota_i$ and we thus have in $\hat{K}^0(M, N)$:

$$\hat{\phi}_i = G_0(\phi_i).$$

We define the correction forms

$$\gamma_i = [\alpha_i, \beta_i] = \frac{\Omega P^{-1}(M) \oplus \Omega P^{-2}(N)}{im(ch)}$$

such that

$$\hat{\phi}_i = G(\phi_i) - a(\gamma_i).$$

From (26) and (24) we see that we can take

$$\gamma_i = \left[ \frac{\delta \wedge \text{c}_0(Td^{-1} \cup ch(\phi_i))}{Td(\nabla T M)^2}, 0 \right] = \left[ \frac{\delta \wedge ch(\nabla^{\Phi_i})}{Td(\nabla T M)}, 0 \right],$$

where $\delta$ is defined in (25).

By specializing Theorem 10.3 we get the desired intrinsic formula for $f_{Spin}(y)$.

**Proposition 13.4.** The class $f_{Spin}(y) = ev_\phi(\eta^{\top}(x))$ is represented by the series

$$\sum_{i=1}^{\infty} q^{i/2} \left( \left[ -\int_M \delta \wedge ch(\nabla^{\Phi_i}) \right] - \xi(D_{DM} \otimes D(\Phi_i, \iota_i)) \right).$$

**Remark 13.5.** The analogy with Laures’ $f$-invariant lets us strongly believe, that the $Spin$-version $f_{Spin}$ is non-trivial, too. Since we know that $\kappa_{MSpin}^{top}$ is non-trivial, the remaining question is whether the evaluation $ev_\phi$ is strong enough to detect some non-trivial elements in the image of $\kappa_{MSpin}^{top}$. The first case to check would be the evaluation of $\kappa_{MSpin}^{top}(\beta)$ for a non-trivial three-torsion element $\beta \in \pi_1(S) \cong \mathbb{Z}/6\mathbb{Z}$. At the moment the calculation of examples seems to be a non-trivial matter.

**Remark 13.6.** This intrinsic formula again works with bundles associated to a geometric normal $Spin^c$-structure. For esthetic reasons it would be interesting to have a formula which uses geometric bundles associated to the tangent bundle.
References

[1] D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson. The structure of the Spin cobordism ring. Ann. of Math. (2), 86:271–298, 1967.

[2] D. W. Anderson and L. Hodgkin. The $K$-theory of Eilenberg-MacLane complexes. Topology, 7:317–329, 1968.

[3] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry. I. Math. Proc. Cambridge Philos. Soc., 77:43–69, 1975.

[4] M. F. Atiyah and G. B. Segal. Equivariant $K$-theory and completion. J. Differential Geometry, 3:1–18, 1969.

[5] Andrew Baker and Jack Morava. Msp localized away from 2 and odd formal group laws. Glasgow University Mathematics Department preprint no. 93/55, 2005.

[6] C. Becker. Relative differential cohomology. ArXiv e-prints, October 2013.

[7] J.M. Boardman. Handbook of Algebraic Topology, chapter Stable Operations in Generalized Cohomology. Elsevier (Amsterdam), 1995.

[8] A. K. Bousfield. The localization of spectra with respect to homology. Topology, 18(4):257–281, 1979.

[9] U. Bunke. A $K$-theoretic relative index theorem and Callias-type Dirac operators. Math. Ann., 303(2):241–279, 1995.

[10] U. Bunke. Index theory, eta forms, and Deligne cohomology. Mem. Amer. Math. Soc., 198(928):vi+120, 2009.

[11] U. Bunke. On the topological contents of eta invariants. ArXiv e-prints, March 2011.

[12] U. Bunke. Differential cohomology. ArXiv e-prints, August 2012.

[13] U. Bunke, T. Nikolaus, and M. Völkl. Differential cohomology theories as sheaves of spectra. ArXiv e-prints, November 2013.

[14] U. Bunke and T. Schick. Smooth $K$-theory. Astérisque, (328):45–135 (2010), 2009.

[15] U. Bunke and Th. Schick. Uniqueness of smooth extensions of generalized cohomology theories. J. Topol., 3(1):110–156, 2010.

[16] Ulrich Bunke and Niko Naumann. The $f$-invariant and index theory. Manuscripta Math., 132(3-4):365–397, 2010.

[17] F. Ferrari Ruffino. Relative (generalized) differential cohomology. ArXiv e-prints, January 2014.
[18] F. Han and W. Zhang. $\eta$-invariant and Modular Forms. ArXiv e-prints, December 2013.

[19] Fei Han and Weiping Zhang. Modular invariance, characteristic numbers and $\eta$ invariants. J. Differential Geom., 67(2):257–288, 2004.

[20] M.J. Hopkins and I.M. Singer. Quadratic functions in geometry, topology, and M-theory. J. Differential Geom., 70(3):329–452, 2005.

[21] Gerd Laures. The topological $q$-expansion principle. Topology, 38(2):387–425, 1999.

[22] Gerd Laures. On cobordism of manifolds with corners. Trans. Amer. Math. Soc., 352(12):5667–5688 (electronic), 2000.

[23] H. B. Lawson, Jr. and M.-L. Michelsohn. Spin geometry, volume 38 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1989.

[24] J. Simons and D. Sullivan. Structured vector bundles define differential $K$-theory. In Quanta of maths, volume 11 of Clay Math. Proc., pages 579–599. Amer. Math. Soc., Providence, RI, 2010.

[25] H. von Bodecker. On the geometry of the f-invariant. ArXiv e-prints, August 2008.