Distance-based and continuum Fano inequalities with applications to statistical estimation

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Abstract

In this technical note, we give two extensions of the classical Fano inequality in information theory. The first extends Fano’s inequality to the setting of estimation, providing lower bounds on the probability that an estimator of a discrete quantity is within some distance $t$ of the quantity. The second inequality extends our bound to a continuum setting and provides a volume-based bound. We illustrate how these inequalities lead to direct and simple proofs of several statistical minimax lower bounds.

1 Introduction

Fano’s inequality is a central tool in information theory, serving as a key ingredient in the proofs of many converse results [12, 9]. In mathematical statistics, it also plays a central role in minimax theory—specifically, in proving lower bounds on achievable rates of convergence for estimators. This line of work in statistics dates back to the seminal work of Hasminskii and Ibragimov [14, 15], and continues through a variety of works to the present day (e.g., [4, 5, 18, 19, 20, 21]).

Let us begin by stating Fano’s inequality, which provides a lower bound on the error in a multi-way hypothesis testing problem. Let $V$ be a random variable taking values in a finite set $V$ with cardinality $|V| ≥ 2$. If we define the binary entropy function $h_2(p) = -p \log p - (1-p) \log (1-p)$, Fano’s inequality takes the following form [e.g. 9, Chapter 2]:

\[
\text{Lemma 1 (Fano). For any Markov chain } V \to X \to \hat{V}, \text{ we have}
\]

\[
h_2(\mathbb{P}(\hat{V} \neq V)) + \mathbb{P}(\hat{V} \neq V) \log(|V| - 1) \geq H(V | \hat{V}).
\]  

A standard simplification of Lemma 1 is to note that $h_2(p) ≤ \log 2$ for any $p ∈ [0, 1]$, so that if $V$ is uniform on the set $V$ and hence $H(V) = \log |V|$, then

\[
\mathbb{P}(\hat{V} \neq V) ≥ 1 - \frac{I(V; X) + \log 2}{\log |V|}.
\]  

Note that the probability $\mathbb{P}(\hat{V} \neq V)$ can be interpreted as the error in a $|V|$-ary hypothesis testing problem, where $X$ represents the observation, $V$ represents the latent class label, and $\hat{V}$ represents a testing function.

When applied to derive statistical minimax bounds on an estimation problem, the usual procedure is to “reduce” the estimation problem to a testing problem before applying the usual form of Fano’s inequality. See the papers of Yu [21] and Yang and Barron [20] for a description of this standard
reduction. In this note, we provide extensions of inequalities (1) and (2) that directly yield bounds on the estimation error, thereby allowing this step to be avoided. More specifically, suppose that we have the distance-like function $\rho$ on $V$ and are interested in bounding the estimation error $\rho(\hat{V}, V)$. We begin by providing analogues of the lower bounds (1) and (2) that replace the testing error with the tail probability $P(\rho(\hat{V}, V) > t)$. By Markov’s inequality, such control directly yields bounds on the expectation $E[\rho(\hat{V}, V)]$. We then extend these bounds to continuous spaces, providing a natural volume-based analogue of inequality (2). As we show in Section 3 these distance-based Fano inequalities allow direct and simple proofs of various minimax bounds without the need for computing metric entropy as in standard arguments.

2 Two distance-based Fano inequalities

In this section, we provide the two main results of this note—namely, Proposition 1, which applies to estimation in discrete problems, and Proposition 2 for continuous-valued variables.

2.1 Discrete problems

We begin with the distance-based analogue of the usual discrete Fano inequality in Lemma 1. Let $V$ be a random variable supported on a finite set $V$ with cardinality $|V| \geq 2$, and let $\rho : V \times V \to \mathbb{R}$ be a symmetric function defined on $V \times V$. In the usual setting, the function $\rho$ is a metric on the space $V$, but our theory applies to general functions. For a given scalar $t \geq 0$, the maximum and minimum neighborhood sizes at radius $t$ are given by

$$N_{t}^{\text{max}} := \max_{v \in V} \{ \text{card}\{v' \in V \mid \rho(v, v') \leq t\} \} \quad \text{and} \quad N_{t}^{\text{min}} := \min_{v \in V} \{ \text{card}\{v' \in V \mid \rho(v, v') \leq t\} \}.$$  

Defining the error probability $P_t = P(\rho(\hat{V}, V) > t)$, we then have the following generalization of Fano’s inequality:

**Proposition 1.** For any Markov chain $V \to X \to \hat{V}$, we have

$$h_2(P_t) + P_t \log \frac{|V| - N_{t}^{\min}}{N_{t}^{\max}} + \log N_{t}^{\max} \geq H(V \mid \hat{V}).$$  

(3)

Before proving the proposition, it is informative to note that it reduces to the standard form of Fano’s inequality (1) in a special case. Suppose that we take $\rho$ to be the 0-1 metric, meaning that $\rho(v, v') = 0$ if $v = v'$ and 1 otherwise. Setting $t = 0$ in Proposition 1 we have $P_0 = P[\hat{V} \neq V]$ and $N_{0}^{\min} = N_{0}^{\max} = 1$, whence inequality (3) reduces to inequality (1). Other weakenings allow somewhat clearer statements:

**Corollary 1.** If $V$ is uniform on $V$ and $(|V| - N_{t}^{\min}) > N_{t}^{\max}$, then

$$P(\rho(\hat{V}, V) > t) \geq 1 - \frac{I(V; X) + \log 2}{\log |\frac{|V|}{N_{t}^{\max}}|}.  \quad (4)$$

Inequality (4) is the natural analogue of the classical mutual-information based form of Fano’s inequality (2), and it provides a qualitatively similar bound. The main difference is that the usual cardinality $|V|$ is replaced by the ratio $|V|/N_{t}^{\max}$. This quantity serves as a rough measure of the
number of possible “regions” in the space \( \mathcal{V} \) that are distinguishable—that is, the number of subsets of \( \mathcal{V} \) for which \( \rho(v, v') > t \) when \( v \) and \( v' \) belong to different regions. Such sets are known as packing sets, and their construction is a standard step in the usual reduction from testing to estimation [21, 20]. Our bound allows us to skip the packing set construction and directly compute \( I(V; X) \) where \( V \) takes values over the full space, as opposed to computing the mutual information \( I(V'; X) \) for a \( V' \) uniformly distributed over a packing set contained within \( \mathcal{V} \). In some cases, the former calculation can be simpler, as illustrated in examples to follow. We note that inequality (4) is similar in spirit to an inequality of Aeron et al. [1, Lemma IV-1], who show an inequality that asymptotically bounds the error of tensorized distortion measures in terms of a rate distortion function. In contrast, our bound is non-asymptotic, and it applies to general functions \( \rho \) and sets \( \mathcal{V} \) without quantization.

We now prove the corollary:

**Proof.** First, by the information-processing inequality [e.g. 9, Chapter 2], we have
\[
I(V; \hat{V}) \leq I(V; X),
\]
and hence
\[
H(V \mid X) \leq H(V \mid \hat{V}).
\]
Since \( h_2(P_t) \leq \log 2 \), inequality (3) implies that
\[
H(V \mid X) - \log N_t^{\max} \leq H(V \mid \hat{V}) - \log N_t^{\max} \leq \mathbb{P}(\rho(\hat{V}, V) > t) \log \frac{|\mathcal{V}| - N_t^{\min}}{N_t^{\max}} + \log 2.
\]
Rearranging the preceding equations yields
\[
\mathbb{P}(\rho(\hat{V}, V) > t) \geq \frac{H(V \mid X) - \log N_t^{\max} - \log 2}{\log \frac{|\mathcal{V}| - N_t^{\min}}{N_t^{\max}}}.
\] (5)

Note that his bound holds without any assumptions on the distribution of \( V \).

By definition, we have \( I(V; X) = H(V) - H(V \mid X) \). When \( V \) is uniform on \( \mathcal{V} \), we have \( H(V) = \log |\mathcal{V}| \), and hence \( H(V \mid X) = \log |\mathcal{V}| - I(V; X) \). Substituting this relation into the bound (5) yields the inequality
\[
\mathbb{P}(\rho(\hat{V}, V) > t) \geq \frac{\log |\mathcal{V}|}{\log \frac{|\mathcal{V}| - N_t^{\min}}{N_t^{\max}}} - \frac{I(V; X) + \log 2}{\log \frac{|\mathcal{V}| - N_t^{\min}}{N_t^{\max}}} \geq 1 - \frac{I(V; X) + \log 2}{\log \frac{|\mathcal{V}|}{N_t^{\max}}},
\]
as claimed.

We complete this subsection by proving Proposition 1 using an argument that parallels that of the classical Fano inequality [3].

**Proof.** Letting \( Z \) be a \{0, 1\}-valued indicator variable for the event \( \rho(\hat{V}, V) \leq t \), we compute the entropy \( H(Z, V \mid \hat{V}) \) in two different ways. On one hand, by the chain rule for entropy, we have
\[
H(Z, V \mid \hat{V}) = H(V \mid \hat{V}) + H(Z \mid V, \hat{V}),
\] (6)
where the final term vanishes since \( Z \) is \((V, \hat{V})\)-measurable. On the other hand, we also have
\[
H(Z, V \mid \hat{V}) = H(Z \mid \hat{V}) + H(V \mid Z, \hat{V}) \leq H(Z) + H(V \mid Z, \hat{V}),
\] (7)
using the fact that conditioning reduces entropy. Applying the definition of conditional entropy yields

\[ H(V \mid Z, \hat{V}) = P(Z = 0)H(V \mid Z = 0, \hat{V}) + P(Z = 1)H(V \mid Z = 1, \hat{V}), \]

and we upper bound each of these terms separately. For the first term, we have

\[ H(V \mid Z = 0, \hat{V}) \leq \log(|V| - N_t^{\min}), \]

since conditioned on the event \( Z = 0 \), the random variable \( V \) may take values in a set of size at most \(|V| - N_t^{\min}\). For the second, we have

\[ H(V \mid Z = 1, \hat{V}) \leq \log N_t^{\max}, \]

since conditioned on \( Z = 1 \), or equivalently on the event that \( \rho(\hat{V}, V) \leq t \), we are guaranteed that \( V \) belongs to a set of cardinality at most \( N_t^{\max} \).

Combining the pieces and noting \( P(Z = 0) = P_t \), we have proved that

\[ H(Z, V \mid \hat{V}) \leq H(Z) + P_t \log (|V| - N_t^{\min}) + (1 - P_t) \log N_t^{\max}. \]

Combining this inequality with our earlier equality (6), we see that

\[ H(V \mid \hat{V}) \leq H(Z) + P_t \log (|V| - N_t^{\min}) + (1 - P_t) \log N_t^{\max}. \]

Since \( H(Z) = h_2(P_t) \), the claim (3) follows.

### 2.2 Continuous problems

Thus far, we have considered problems in which the random variable \( V \) takes values in a discrete set of finite cardinality. In this section, we show how to extend Proposition 1—specifically, its mutual-information based form (4)—to non-discrete domains. This extension has applications to many problems in statistical decision theory, as sketched in Section 3.2, allowing quick proofs of several results. It may prove useful in other domains as well, though for brevity we focus only on a few examples.

The proposition requires a bit of additional notation and a few assumptions. First, we assume that the set \( V \subset \mathbb{R}^d \) has a volume (Lebesgue measure) that is non-zero and finite. Define the “ball”\( \mathbb{B}_\rho(t, v) = \{ v' \in \mathbb{R}^d : \rho(v, v') \leq t \} \)of “radius” \( t \) centered at \( v \). Here we use ball and radius with quotation marks, since \( \rho \) may not be a metric, let alone symmetric or even positive. For a set \( S \) in \( d \)-dimensions, we let \( \partial S \) denote its boundary, and define the volume of \( \partial S \) via Lebesgue measure in \((d-1)\)-dimensional space. With this notation, we assume the volumes of the two surface areas \( \text{Vol}(\partial \mathbb{V}) \) and \( \sup_{v \in V} \text{Vol}(\partial \mathbb{B}_\rho(t, v) \cap V) \) are both finite. Under this regularity condition on the pair \((\rho, V)\), we have the following result:

**Proposition 2.** If \( V \) is uniform over \( \mathcal{V} \), then for any Markov chain \( V \rightarrow X \rightarrow \hat{V} \), we have

\[
\mathbb{P}(\rho(\hat{V}, V) \geq t) \geq 1 - \frac{I(V; X) + \log 2}{\log \frac{\text{Vol}(\mathcal{V})}{\sup_{v \in \mathcal{V}} \text{Vol}(\mathbb{B}_\rho(t, v) \cap \mathcal{V})}}.
\]
See Appendix B for the proof.

Given the generalized Fano inequality (4), the form of Proposition 2 is not surprising: the volume ratio in the bound (8) is the continuous analog of the ratio $\frac{|V|}{N}$ in the discrete case. The bound (8) is related to some recent results by Ma and Wu [16]. In their work, the volume ratio is introduced as a lower bound on packing entropy, following the usual reduction from estimation to testing. Consequently, unlike the bound (8), it does not allow the random vector $V$ to take continuous values. The main advantage of Proposition 2 is that it provides direct bounds on estimation error for continuous random vectors $V$ in terms of mutual information, without the need for performing discretization and bounding metric entropy.

3 Consequences for statistical minimax theory

In this section, we develop some consequences of the previous results for statistical minimax theory. Accordingly, we begin by setting up the standard framework for minimax bounds in statistics. Let $P$ be a family of distributions on a sample space $X$, and let $\theta : P \rightarrow \Theta$ be a function mapping $P$ to some parameter space $\Theta$. Given a sample $X^n = (X_1, \ldots, X_n) \in X^n$ of size $n$ drawn i.i.d. from a distribution $P \in P$, let $\hat{\theta} = \hat{\theta}(X^n)$ be a measurable function of $X^n$, which we view as an estimate of the unknown quantity $\theta(P)$. The quality of this estimator can be measured in terms of the risk

$$E_P \left[ \Phi \left( \rho(\hat{\theta}(X^n), \theta(P)) \right) \right],$$

where $\rho : \Theta \times \Theta \rightarrow \mathbb{R}_+$ is a (semi)-metric on the parameter space, and $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non-decreasing loss function. For instance, for Euclidean space with $\rho(\hat{\theta}, \theta) = \|\hat{\theta} - \theta\|_2$ and $\Phi(t) = t^2$, the error measure corresponds to the usual mean-squared error of an estimator. In terms of this criterion (9), the minimax risk for the family $P$ is given by

$$M_n(\theta(P), \Phi \circ \rho) := \inf_{\hat{\theta}} \sup_{P \in P} E_P \left[ \Phi \left( \rho(\hat{\theta}(X^n), \theta(P)) \right) \right],$$

where the infimum ranges over all measurable functions $\hat{\theta}$ of the observed sample $X^n$.

3.1 Consequences of Proposition 1

We begin by showing how Proposition 1 can be used to derive lower bounds on the minimax risk. As previously noted, Proposition 1 is a generalization of the classical Fano inequality (1), so it leads naturally to a generalization of the classical Fano lower bound on minimax error, which we describe here.

Consider a family of distributions $\{P(\cdot | v)\}_{v \in V} \subset P$ indexed by a finite set $V$. This family induces an associated collection of parameters $\{\theta_v := \theta(P(\cdot | v))\}_{v \in V} \subset \Theta$. Given a function $\rho_V : V \times V \rightarrow \mathbb{R}$ and a scalar $t$, we define the separation $\delta(t)$ of this set relative to the metric $\rho$ on $\Theta$ via

$$\delta(t) := \sup \{ \delta : \rho(\theta_v, \theta_w) \geq \delta \text{ for all } v, w \in V \text{ such that } \rho_V(v, w) > t \}.$$  

As a special case, when $t = 0$ and $\rho_V$ is the discrete metric, this definition reduces to that of a packing set: we are guaranteed that $\rho(\theta_v, \theta_w) \geq \delta(0)$ for all distinct pairs $v \neq w$. This type of packing construction underlies the classical Fano approach to minimax lower bounds [15, 4, 21, 20].
On the other hand, allowing for \( t > 0 \) lends greater flexibility to the construction, since only certain pairs \( \theta_v \) and \( \theta_w \) are required to be well-separated.

Given a set \( \mathcal{V} \) and associated separation function \( \delta(t) \), we assume the canonical estimation setting: nature chooses a vector \( V \in \mathcal{V} \) uniformly at random, and conditioned on this choice \( V = v \), a sample \( X^v_i \) of size \( n \) is drawn i.i.d. from the distribution \( P(\cdot \mid v) \). We then have the following corollary of Proposition 1:

**Corollary 2.** Given \( V \) uniformly distributed over \( \mathcal{V} \) with separation function \( \delta(t) \), we have

\[
\mathcal{M}_n(\theta(P), \Phi \circ \rho) \geq \Phi\left( \frac{\delta(t)}{2} \right) \left( 1 - \frac{I(X^v_1; V) + \log 2}{\log \frac{1}{N_{\max}}} \right) \quad \text{for all } t.
\]

(12)

See Appendix A for the proof.

The classical form of the Fano lower bound on the minimax risk can be recovered as a special case of the lower bound (12). Indeed, if we set \( t = 0 \) and let \( \rho_{\mathcal{V}} \) be the discrete metric, then \( N^\max_0 = 1 \) and \( |\mathcal{V}| \) is the cardinality of an \( \epsilon := \delta(0) \) packing set, which we denote by \( M(\epsilon) \). Consequently, we obtain

\[
\mathcal{M}_n(\theta(P), \Phi \circ \rho) \geq \Phi\left( \frac{\epsilon}{2} \right) \left( 1 - \frac{I(X^v_1; V) + \log 2}{\log M(\epsilon)} \right),
\]

(13)

which is a well-known lower bound on the minimax risk \([15, 21, 20]\). In general, however, Corollary 2 gives somewhat more flexibility than the classical Fano risk bound (13), because it allows \( V \) to be uniformly distributed on a set \( \mathcal{V} \) that may not induce a well-separated packing (i.e. \( \epsilon = \delta(0) \) may be small, weakening the classical bound (13)), and it allows more careful choice for the separation \( t \). We now give two illustrative applications of Corollary 2 that highlight these aspects of the bound.

We first show how Corollary 2 can be exploited to obtain sharp lower bounds for the problem of sparse mean estimation. Suppose that our goal is to estimate the mean \( \theta \) of a Gaussian distribution on \( \mathbb{R}^d \), where \( \theta \in \mathbb{R}^d \) has at most \( s \) non-zero entries (written as \( \|\theta\|_0 \leq s \)). We consider the minimax risk in squared Euclidean norm over the family

\[
\mathcal{P}_{s,d} := \{ \mathcal{N}(\theta, \sigma^2 I_{d \times d}) \mid \|\theta\|_0 \leq s \},
\]

(14)

where we observe \( X_i \) drawn i.i.d. as \( \mathcal{N}(\theta, \sigma^2 I) \). For this family, we have the following lower bound:

**Corollary 3.** For the \( s \)-sparse Gaussian location family (14), there is a universal constant \( c > 0 \) such that

\[
\mathcal{M}_n(\theta(\mathcal{P}_{s,d}), \|\cdot\|_2^2) \geq c \frac{\sigma^2 s \log(d)}{n}.
\]

(15)

This lower bound is sharp up to a constant factor (cf. Donoho and Johnstone \([10]\)). One way of proving the lower bound (15) is by first constructing a \( \delta \)-packing of the set of \( s \)-sparse mean vectors, and then applying the classical Fano bound (13). See the paper \([18]\) for an instance of this proof in the setting of fixed design regression, which includes the normal location model as a particular case. Corollary 2 however, allows a proof without such a packing construction.
Proof. Consider the set $\mathcal{V} = \{ v \in \{-1,0,1\}^d \mid \|v\|_0 = s \}$, which satisfies $|\mathcal{V}| = 2^s \binom{d}{s}$. If we define $\theta_v = \epsilon v$ for some $\epsilon > 0$, then the separation function is lower bounded as $\delta(t) > \max\{\sqrt{t},1\} \epsilon$. Consequently, for $V$ uniformly distributed on $\mathcal{V}$, Corollary 2 implies the lower bound
\[
\mathcal{M}_n(\theta(\mathcal{P}_{s,d}), \|\cdot\|_2^2) > \frac{(t \lor 1) \epsilon^2}{4} \left(1 - \frac{I(V; X_{1}^\top) + \log 2}{\log(d) - \log N_{t}^{\max}}\right).
\]
By scaling it is no loss of generality to assume that $\sigma^2 = 1$. For $V$ uniform on $\mathcal{V}$, we have $\mathbb{E}[\|V\|_2^2] = s$, and thus for $V,W$ independent and uniform on $\mathcal{V}$,
\[
I(V; X) \leq n \frac{1}{|\mathcal{V}|^2} \sum_{v \in \mathcal{V}} \sum_{w \in \mathcal{V}} D_{\text{kl}}(N(\epsilon v, I) \mid N(\epsilon w, I)) = \frac{n\epsilon^2}{2} \mathbb{E} \left[\|V - W\|_2^2\right] = ns \epsilon^2.
\]
Taking $t = \lfloor s/4 \rfloor$, we find that $N_{t}^{\max} \leq \lfloor s/4 \rfloor \lfloor d / (s/4) \rfloor$ and
\[
\log \frac{|\mathcal{V}|}{N_{t}^{\max}} \geq \log 2^{s} \binom{d}{s} \log \lfloor s/4 \rfloor \frac{d}{(s/4)} \geq \log \frac{(s/4)!(d - (s/4))!}{s!(d - s)!} \geq cs \log \frac{d}{s}
\]
for a numerical constant $c$. Combining the pieces yields the bound
\[
\mathcal{M}_n(\theta(\mathcal{P}_{s,d}), \|\cdot\|_2^2) > \frac{(\lfloor s/4 \rfloor \lor 1) \epsilon^2}{4} \left(1 - \frac{n\epsilon^2 s + \log 2}{cs \log \frac{d}{s}}\right),
\]
and setting $\epsilon^2 \asymp \log \frac{d}{s} / n$ yields the claim (15).

In other applications, the covariance structure of a random vector $V$ uniformly distributed over $\mathcal{V}$ plays an important role in giving sharp minimax lower bounds. Some examples include recent work on privacy preserving statistical analysis [11], results obtaining sharp lower bounds for compressive sensing [6, 17, Chapter 6], and in distributed estimation problems [22]. In such settings, the classical Fano minimax bound (13) requires rather delicate constructions of packing sets (requiring probabilistic arguments using matrix concentration, among other techniques); Corollary 2 allows this complication to be side-stepped. Here we illustrate this use of Corollary 2 for the problem of estimating a sparse vector from linear measurements.

More precisely, suppose that we observe the pair $(Y, X) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ linked via the linear regression model $Y = X\theta + \varepsilon$, where the noise vector $\varepsilon \in \mathbb{R}^n$ has i.i.d. $N(0, \sigma^2)$ entries. We assume that $\theta$ is an $s$-sparse vector, meaning that $\|\theta\|_0 \leq s$. In this case, our family of distributions is
\[
\mathcal{P}_{X,s} := \left\{ Y \sim N(X\theta, \sigma^2 I_{n \times n}) \mid \theta \in \mathbb{R}^d, \|\theta\|_0 \leq s \right\}.
\]
Candès and Davenport [6, Theorem 1] derived a lower bound for this model based on constructing a packing set with particular covariance properties. Here we derive such a result as a direct consequence of Corollaries 1 and 2.

**Corollary 4.** For the $s$-sparse compressed sensing family (15), there is a universal constant $c > 0$ such that
\[
\mathcal{M}_n(\theta(\mathcal{P}_{X,s}), \|\cdot\|_2^2) \geq c \frac{\sigma^2 s d \log(d/s)}{\|X\|_{\text{Fr}}^2}.
\]
For appropriate matrices $X$, the minimax risk is also upper bounded by the right-hand side of equation (17). Indeed, when the entries of $X \in \mathbb{R}^{n \times d}$ are chosen as i.i.d. Gaussian random variables, setting $\hat{\theta}$ to be the minimizer of $\|X\theta - Y\|_2^2 + \lambda \|\theta\|_1$ for appropriate $\lambda$ attains this rate \cite{8}. For more general random $X$, $\ell_1$-minimization techniques such as the Lasso or Dantzig selector \cite{7, 2} achieve these rates with the quantity $\log(d/s)$ replaced by $\log d$. We now turn to the proof of the lower bound in the claim (17).

Proof. As in the proof of Corollary 3 we let the set $V = \{v \in \{-1, 0, 1\}^d \mid \|v\|_0 = s\}$, and we define $\theta_v = \epsilon v$ for some $\epsilon > 0$. The separation function is then lower bounded as $\delta(t) > \max\{\sqrt{7}, 1\} \epsilon$, and for $V$ uniformly distributed on $V$, Corollary 2 implies the lower bound

$$\mathfrak{m}_n(\theta(P_{X,s}), \|\cdot\|_2^2) > \frac{(t \vee 1) \epsilon^2}{4} \left(1 - \frac{I(V; Y) + \log 2}{\log (\theta) - \log N_{\text{max}}^t}\right).$$

We have $\text{Cov}(V) = (s/d)I_{d \times d}$ for $V$ uniform on $V$, so for $V, W$ independent and uniform on $V$,

$$I(V; Y) \leq \frac{1}{|V|^2} \sum_{v \in V} \sum_{w \in V} D_{kl}(N(X\theta_v, \sigma^2 I)|N(X\theta_w, \sigma^2 I)) = \frac{\epsilon^2}{2\sigma^2} \mathbb{E} \left[\|XY - XV\|_2^2\right] = \frac{se^2\|X\|_{F^2}}{d}.$$

By taking $t = \lfloor s/4 \rfloor$, we find that $\log (\theta) \geq cs \log d/s$ for a numerical constant $c$ as was the case for Corollary 3. Combining the pieces yields the bound

$$\mathfrak{m}_n(\theta(P_{s,d}), \|\cdot\|_2^2) > \frac{(s/4) \vee 1) \epsilon^2}{4} \left(1 - \frac{se^2\|X\|_{F^2}}{s/(d\sigma^2) + \log 2}{\frac{cs \log d/s}{\log (\theta)}\right),$$

and setting $\epsilon^2 \approx \sigma^2 d \log d/s/\|X\|_{F^2}^2$ yields the claim (17). \hfill \square

### 3.2 Applications of Proposition 2

The final set of applications of the results in this note concerns the direct application of the continuous version of Fano’s inequality, Proposition 2. In applications of the continuous Fano inequality, we no longer require a reduction from the original estimation problem to a discrete estimation problem as in Corollary 2 or the classical minimax Fano bound \cite{13}; Proposition 2 provides an immediate lower bound.

We first consider normal mean estimation, where $P = \{N(\theta, \sigma^2 I_{d \times d}) \mid \theta \in \mathbb{R}^d\}$ and our goal is to estimate the mean $\theta$ given $n$ i.i.d. observations $X^n_\theta$ from $N(\theta, \sigma^2 I)$. Proposition 2 implies the following corollary, which follows by an integration argument.

**Corollary 5.** For the $d$-dimensional normal location family with $d \geq 2$,

$$\mathfrak{m}_n(\theta(P), \|\cdot\|_2^2) \geq \frac{(d - 1)^2 \log 2}{4d^2} \frac{\sigma^2 d}{n} - \frac{\sigma^2 d}{n}.$$

**Proof.** Let $V$ be uniform on the $\ell_2$-ball of radius $r$ centered at 0, and conditioned on $V = v$, let $X^n_\theta = (X_1, \ldots, X_n)$ be an i.i.d. sample of size $n$ from the multivariate Gaussian distribution $N(v, \sigma^2 I_{d \times d})$. Then for $0 \leq t \leq r$, the volume ratio is given by $\text{Vol}(V)/\text{Vol}(B_{\mathbb{R}}^2(t)) = (r/t)^d$, and Proposition 2 implies that

$$\mathbb{P}(\|\hat{V} - V\|_2 \geq t) \geq 1 - \frac{I(V; X^n_\theta) + \log 2}{d \log \frac{r}{t}}.$$

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Setting \( r = 2t \), the denominator is equal to \( d \log 2 \), so it only remains to upper bound the mutual information.

Since \( V \) is uniform on the \( \ell_2 \)-ball, we have \( \mathbb{E}[X_i] = 0 \). Consequently, from the independence of samples and the maximum entropy property of the Gaussian, we have

\[
I(V; X^n_1) = h(X^n_1) - h(X^n_1 \mid V) \leq \frac{n}{2} \log \frac{\det(\text{Cov}(X_1))}{\det(\sigma^2 I_{d\times d})},
\]

using the fact that \( X_1 \) is Gaussian conditioned on \( V \). Since \( V \) is uniform on the \( \ell_2 \)-ball of radius \( 2t \), we have \( \text{Cov}(X_1) = \sigma^2 I_{d \times d} + \mathbb{E}[VV^T] \leq (\sigma^2 + 4t^2)I_{d \times d} \). Putting together the pieces yields

\[
I(V; X^n_1) \leq \frac{n}{2} \log \left( 1 + \frac{4t^2}{\sigma^2} \right). \tag{18}
\]

We provide two arguments based on inequality \([15]\): the first is simpler, while the second provides sharper constants. For the first argument, the concavity of the log function immediately implies \( I(V; X^n_1) \leq 2nt^2/\sigma^2 \), since \( \log(1 + x) \leq x \). With this mutual information bound, we obtain

\[
\mathbb{P}(\|\hat{V}(X^n_1) - V\|_2 \geq t) \geq 1 - \frac{\log 2}{d \log 2} - \frac{I(V; X^n_1)}{d \log 2} = \frac{d - 1}{d} - \frac{I(V; X^n_1)}{d \log 2} \geq \frac{1}{2} \cdot \frac{2nt^2}{d \sigma^2 \log 2}.
\]

Setting \( t^2 = d\sigma^2 \log 2/4n \) implies the estimation lower bound \( \mathbb{P}(\|\hat{V} - V\|_2^2 \geq t^2) \geq 1/4 \) directly from the volume bound in Proposition \([2]\).

To obtain the sharper inequality claimed in the corollary, we apply an integration argument. For any positive random variable \( Y \), \( \mathbb{E}[Y] = \int_0^\infty \mathbb{P}(Y \geq t) dt \), and integrating \( \mathbb{P}(\|\hat{V} - V\|_2 \geq t) \) bounds the expected error. As a consequence, we find from our information bound \([18]\) and the continuous Fano inequality \([8]\) that

\[
\int_0^\infty \mathbb{P}(\|\hat{V} - V\|_2 \geq \sigma \sqrt{t/2}) dt \geq \int_0^\infty \left[ \frac{d - 1}{d} - \frac{n \log(1 + t)}{2d \log 2} \right] dt
= \frac{n}{2d \log 2} \left[ \exp\left( \frac{2(d - 1) \log 2}{n} \right) - 1 - \frac{2(d - 1) \log 2}{d} \right] \geq \frac{(d - 1)^2 \log 2}{dn} \tag{19}
\]

(see Appendix [C] for the computation of the integral). Rewriting inequality \([19]\), we have

\[
\frac{4}{\sigma^2} \mathbb{E} \left[ \|\hat{V} - V\|_2^2 \right] = \int_0^\infty \mathbb{P}(\|\hat{V} - V\|_2^2 \geq \sigma^2 t/4) dt \geq \frac{(d - 1)^2 \log 2}{d} \frac{1}{n},
\]

which is the claimed inequality of the corollary.

As our second example application of the volume-based Fano inequality in Proposition \([2]\) we consider the standard fixed-design linear regression model \( Y = X\theta + \varepsilon \), where \( \varepsilon \in \mathbb{R}^n \) is i.i.d. \( \text{N}(0, \sigma^2) \) and \( X \in \mathbb{R}^{n \times d} \), where the goal is to estimate \( \theta \). In this case, our family of distributions is

\[
\mathcal{P}_X := \left\{ \text{N}(X\theta, \sigma^2 I_{n \times n}) \mid \theta \in \mathbb{R}^d \right\} = \left\{ Y = X\theta + \varepsilon \mid \varepsilon \sim \text{N}(0, \sigma^2 I_{n \times n}), \theta \in \mathbb{R}^d \right\}. \tag{20}
\]

We make the simplifying assumption that \( X \in \mathbb{R}^{n \times d} \) is of full column rank, and that \( d \geq 9 \). Letting \( \gamma_{\text{max}}(X) \) denote the maximum singular value of \( X \), we have the following corollary to Proposition \([2]\):
Corollary 6. For the standard linear regression model \([20]\), we have
\[
\mathcal{M}_n(\theta(\mathcal{P}_X), \|\cdot\|_2^2) \geq \frac{1}{12} \cdot \frac{1}{\gamma_{\max}^2(X/\sqrt{n})} \cdot \frac{d\sigma^2}{n}.
\]

Proof. Letting \(\theta = V \in \mathbb{R}^d\) be uniform on the \(\ell_2\)-ball of radius \(r\), we have
\[
I(V; Y) \leq \int D_{\text{kl}}(\mathcal{N}(Xv, \sigma^2 I_{n\times n}) \| \mathcal{N}(Xw, \sigma^2 I_{n\times n})) \, d\mu(v)d\mu(w)
= \frac{1}{2\sigma^2} \mathbb{E} \left[ \|X(V - W)\|_2^2 \right] = \frac{d}{d + 2} \frac{\text{tr}(X^T X)}{\sigma^2} r^2 = \frac{d}{d + 2} \frac{\|X\|_{\text{Fr}}^2}{\sigma^2} r^2,
\]
where \(\mu\) is uniform on the \(\ell_2\)-ball of radius \(r\), as are the independent random vectors \(V\) and \(W\). By setting \(r = 2t\), Proposition 2 implies that any estimator \(\hat{\theta}\) of \(\theta\) must satisfy
\[
\mathbb{P}(\|\hat{\theta} - \theta\|_2 \geq t) \geq \frac{d - 1}{d} - \frac{4t^2 \|X\|_{\text{Fr}}^2}{\sigma^2 d(d + 2) \log 2}.
\]

Applying an integration argument as in Corollary 5, we use the identity \(\int_0^\infty [c_1 - c_2 t]_+ \, dt = c_1^2/2c_2\) to show that for any estimator \(\hat{\theta}\) we have
\[
\mathbb{E} \left[ \|\hat{\theta}(X^n) - \theta\|_2^2 \right] \geq \int_0^\infty \left[ \frac{d - 1}{d} - \frac{4t \|X\|_{\text{Fr}}^2}{\sigma^2 d(d + 2) \log 2} \right] \, dt \geq \frac{(d - 1)^2}{d^2} \cdot \frac{d(d + 2)\sigma^2 \log 2}{8 \|X\|_{\text{Fr}}^2}.
\]

To slightly simplify the above bound, we note that \(\|X/\sqrt{n}\|_{\text{Fr}}^2 \leq d\gamma_{\max}^2(X/\sqrt{n})\), which gives the minimax lower bound
\[
\mathcal{M}_n(\theta(\mathcal{P}), \|\cdot\|_2^2) \geq \frac{\log 2}{8} \cdot \frac{(d - 1)^2(d + 2)}{d^3} \cdot \frac{1}{\gamma_{\max}^2(X/\sqrt{n})} \cdot \frac{d\sigma^2}{n} \geq \frac{1}{12} \cdot \frac{1}{\gamma_{\max}^2(X/\sqrt{n})} \cdot \frac{d\sigma^2}{n}
\]
the last inequality following for \(d \geq 9\).

4 Conclusion

We have provided two quantitative Fano inequalities, in the forms of Propositions 1 and 2, that allow direct proofs of several statistical minimax lower bounds. It would be interesting to see if these inequalities can be used in more classical information-theoretic applications, either to give simpler proofs of existing converse results or to prove new converse results. We hope to explore these questions and other applications in future work.

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A Proof of Corollary \[2\]

For any \( \epsilon \geq 0 \) and any estimator \( \hat{\theta} \) of \( \theta \), the non-decreasing nature of \( \Phi \) implies that

\[
E[\Phi(\rho(\hat{\theta}, \theta))] \geq E\left[\Phi(\delta) \mathbb{1}\left\{ \rho(\hat{\theta}(X^n_1), \theta) \geq \epsilon \right\}\right] = \Phi(\epsilon) \mathbb{P}\left(\rho(\hat{\theta}(X^n_1), \theta) \geq \frac{\epsilon}{2}\right).
\]

Consequently, setting \( \epsilon = \delta(t) \), the lower bound \((12)\) in the corollary will follow from Proposition \[1\] if we can show that

\[
\mathbb{P}\left(\rho(\hat{\theta}(X^n_1), \theta_V) \geq \frac{\delta(t)}{2}\right) \geq \mathbb{P}\left(\rho_V(\hat{\theta}(X^n_1), V) > t\right). \tag{21}
\]

In order to establish this claim, we define the testing function

\[
\hat{v}(X^n_1) := \underset{v \in \mathcal{V}}{\text{argmin}} \rho(\theta_v, \hat{\theta}(X^n_1)). \tag{22}
\]

Now assume that \( \rho(\hat{\theta}, \theta_w) < \frac{\delta(t)}{2} \). Then for any \( w \) with \( \rho_V(w, v) > t \), we have

\[
\rho(\hat{\theta}, \theta_w) \geq \rho(\theta_v, \theta_w) - \rho(\hat{\theta}, \theta_v) > \delta(t) - \frac{\delta(t)}{2} = \frac{\delta(t)}{2}.
\]

As a consequence, we have \( \rho(\hat{\theta}, \theta_w) > \rho(\hat{\theta}, \theta_v) \) for all \( w \) with \( \rho_V(w, v) > t \), and thus the choice \((22)\) must give a \( \hat{v} \) such that \( \rho_V(\hat{v}, v) \leq t \). In particular, the event \( \rho(\hat{\theta}, \theta_v) < \delta(t)/2 \) implies that \( \rho_V(\hat{v}, v) \leq t \), and by conditioning on \( V = v \), we find that for any \( t \in \mathbb{R} \),

\[
\mathbb{P}(\rho(\hat{\theta}(X^n_1), \theta_V) \geq \delta(t)/2 \mid V = v) \geq \mathbb{P}(\rho_V(\hat{v}, V) > t \mid V = v).
\]

Averaging over all \( v \in \mathcal{V} \) and taking an infimum over all tests \( \hat{v} \) yields the claim \((21)\).

B Proof of Proposition \[2\]

Throughout this proof, we use \( A + B \) to denote Minkowski addition of two sets \( A \) and \( B \) in \( \mathbb{R}^d \).

The proof is based on a sequence of partitions of the space \( \mathcal{V} \), each of which allows us to apply Proposition \[1\]. Let \( \epsilon_n = 2^{-n} \), and consider a grid of \( \mathbb{R}^d \) into boxes whose vertices are at points \( \epsilon_n z \) for \( z \in \mathbb{Z} \), where each block is of width \( \epsilon_n \), i.e. the boxes are given by translations of \([-\epsilon_n/2, \epsilon_n/2]^d\).

Let \( \mathcal{W}^{(n)} \) denote the partition of \( \mathcal{V} \) into these boxes, where we abuse notation and let \(|\mathcal{W}^{(n)}|\) denote the number of boxes whose intersection with \( \mathcal{V} \) has non-zero volume. Assign an arbitrary indexing to the blocks (and partial blocks) \( \mathcal{W}^{(n)} \), and for a vector \( v \in \mathcal{V} \), let \( [v]_{\mathcal{W}^{(n)}} \) be an arbitrary (but fixed) point \( w \) of the box in \( \mathcal{W}^{(n)} \) into which \( v \) falls (break ties at the boundaries arbitrarily; the boundaries have Lebesgue measure zero so this is insignificant). In addition, let

\[
N_t(\mathcal{W}^{(n)}) := \sup_{v \in \mathcal{V}} \{ \text{card}\{[v]_{\mathcal{W}^{(n)}} \mid v' \in \mathcal{V}, \rho(v, v') \leq t\}\}
\]

be the maximum number of blocks touched in a radius \( t \) of some point \( v \in \mathcal{V} \). See Figure \[1\] for a visual representation of our construction.
With this notation, we may provide the proof. For $V$ uniform on $\mathcal{V}$, define the random variable $V^{(n)} = [V]_{W^{(n)}}$. By Proposition 1, we have

$$\log 2 + \log \frac{|W^{(n)}|}{N_t(W^{(n)})} \mathbb{P}(\rho(\hat{V}, V^{(n)}) > t) \geq H(V^{(n)} \mid X) - \log N_t(W^{(n)})$$

$$= H(V^{(n)} \mid X) - H(V^{(n)}) + H(V^{(n)}) - \log N_t(W^{(n)}). \quad (23)$$

Inspecting inequality (23), it suffices to show that, for any $v \in \mathcal{V}$, the following inequalities are valid:

$$\liminf_n \mathbb{P}(\rho(\hat{V}, V^{(n)}) > t) \leq \mathbb{P}(\rho(\hat{V}, V) \geq t), \quad (24a)$$

$$\liminf_n \log \frac{|W^{(n)}|}{N_t(W^{(n)})} \leq \log \frac{\text{Vol}(\mathcal{V})}{\text{Vol}(\mathbb{B}_\rho(t, v) \cap \mathcal{V})}, \quad \text{and} \quad (24b)$$

$$\liminf_n \left[ H(V^{(n)}) - \log N_t(W^{(n)}) \right] \geq \log \frac{\text{Vol}(\mathcal{V})}{\sup_{v \in \mathcal{V}} \text{Vol}(\mathbb{B}_\rho(t, v) \cap \mathcal{V})}. \quad (24c)$$
To see the sufficiency, suppose that the conditions (24) hold. Since \( \mathcal{W}^{(n+1)} \) partitions \( \mathcal{W}^{(n)} \), we have that for all \( n \),

\[
I(V^{(n)}; X) \leq I(V^{(n+1)}; X) \leq I(V; X)
\]

(see [13, Chapter 5]). As a consequence, we find that under conditions (24),

\[
\log 2 + \log \frac{\text{Vol}(V)}{\text{Vol}(B_{\rho}(t, v) \cap V)} \mathbb{P}(\rho(\hat{V}, V) \geq t) \geq \liminf_n \left\{ \log 2 + \log \frac{|\mathcal{W}^{(n)}|}{N_t(\mathcal{W}^{(n)})} \mathbb{P}(\rho(\hat{V}, V^{(n)}) > t) \right\}
\]

\[
\geq \liminf_n \left\{ -I(V^{(n)}; X) + H(V^{(n)}) - \log N_t(\mathcal{W}^{(n)}) \right\}
\]

\[
\geq -I(V; X) + \log \sup_{v \in V} \text{Vol}(V) \text{vol} \frac{\text{Vol}(V)}{\sup_{v \in V} \text{Vol}(B_{\rho}(t, v) \cap V)}
\]

where inequality (i) follows from inequalities (24a) and (24c), (ii) from the consequence (23) of Proposition 1, and (iii) is a consequence of inequality (24b). We thus complete the proof of Proposition 2 by proving each of the inequalities (24).

**Inequality (24a)** The simplest is inequality (24a), which follows by the Portmanteau theorem on convergence in distribution (e.g. [3, Section 2]). Since \( V^{(n)} \rightarrow V \), the continuity of \( \rho \) implies \( \rho(\hat{V}, V^{(n)}) \rightarrow \rho(\hat{V}, V) \). That the set \( \{ z \in \mathbb{R} \mid z \geq t \} \) is closed implies that

\[
\liminf_n \mathbb{P}(\rho(\hat{V}, V^{(n)}) > t) \leq \limsup_n \mathbb{P}(\rho(\hat{V}, V^{(n)}) \geq t) \leq \mathbb{P}(\rho(\hat{V}, V) \geq t),
\]

the last inequality following from the Portmanteau theorem.

**Inequality (24b)** For inequalities (24b) and (24c), we provide volume counting arguments. We begin with two inequalities that will prove useful. Let \( A \) be any set with finite surface area and volume and let \( \epsilon \geq 0 \). Then, denoting \((d-1)\)-dimensional Lebesgue measure by \( \lambda^{d-1} \),

\[
\text{Vol}(A \setminus (\partial A + [-\epsilon, \epsilon]^d)) \geq \text{Vol}(A) - \int_{\partial A} \text{Vol}([-\epsilon, \epsilon]^d) d\lambda^{d-1} = \text{Vol}(A) - (2\epsilon)^d \text{Vol}(\partial A)
\]

\[
\text{Vol}(A + [-\epsilon, \epsilon]^d) \leq \text{Vol}(A) + \int_{\partial A} \text{Vol}([-\epsilon, \epsilon]^d) d\lambda^{d-1} = \text{Vol}(A) + (2\epsilon)^d \text{Vol}(\partial A).
\]

(25)

Now we turn to our counting arguments. For the set \( \mathcal{W}^{(n)} \), we have

\[
\frac{1}{\epsilon_n^d} \text{Vol}(V \setminus (\partial V + [-\epsilon_n, \epsilon_n]^d) \leq |\mathcal{W}^{(n)}| \leq \frac{1}{\epsilon_n^d} \text{Vol}(V + [-\epsilon_n, \epsilon_n]^d).
\]

(26)

By construction of the set \( \mathcal{W}^{(n)} \), we additionally have for any \( v \in V \) that

\[
\frac{1}{\epsilon_n^d} \text{Vol}(B_{\rho}(t, v) \cap V \setminus (\partial (B_{\rho}(t) \cap V) + [-\epsilon_n, \epsilon_n]^d)) \leq N_t(\mathcal{W}^{(n)}) \leq \sup_{v \in V} \frac{1}{\epsilon_n^d} \text{Vol}(B_{\rho}(t, v) \cap V + [-\epsilon_n, \epsilon_n]^d).
\]

(27)
As a consequence of the upper and lower volume bounds (26) and (27), we may prove inequality (24b): for any \( v \in \mathcal{V} \) we have

\[
\liminf_n \log \frac{|\mathcal{W}(n)|}{N_t(\mathcal{W}(n))} \leq \liminf_n \frac{\text{Vol}(\mathcal{V} + [-\varepsilon_n, \varepsilon_n]^d)}{\text{Vol}(\mathcal{B}_\rho(t, v) \cap \mathcal{V} \setminus (\partial \mathcal{B}_\rho(t, v) \cap \mathcal{V}) + [-\varepsilon_n, \varepsilon_n]^d)}
\]

\[
= \log \frac{\text{Vol}(\mathcal{V})}{\text{Vol}(\mathcal{B}_\rho(t, v) \cap \mathcal{V})}
\]

by the assumed finiteness of \( \text{Vol}(\partial \mathcal{V}) \) (recall the inequalities (25)). This gives inequality (24b).

**Inequality (24c):** It remains to prove inequality (24c). For this, we note that since \( \mathcal{V} \) is uniform on \( \mathcal{V} \), there are at least

\[
H(\mathcal{V}(n)) \geq \frac{\varepsilon_n^d \text{Vol}(\mathcal{V} \setminus (\partial \mathcal{V} + [-\varepsilon_n, \varepsilon_n]^d))}{\varepsilon_n^d \text{Vol}(\mathcal{V})} \log \frac{\text{Vol}(\mathcal{V})}{\varepsilon_n^d} = \frac{\text{Vol}(\mathcal{V} \setminus (\partial \mathcal{V} + [-\varepsilon_n, \varepsilon_n]^d))}{\text{Vol}(\mathcal{V})} \log \frac{\text{Vol}(\mathcal{V})}{\varepsilon_n^d},
\]

which implies (with the counting estimate (27)) that

\[
H(\mathcal{V}(n)) - \log N_t(\mathcal{W}(n))
\]

\[
\geq \frac{\text{Vol}(\mathcal{V} \setminus (\partial \mathcal{V} + [-\varepsilon_n, \varepsilon_n]^d))}{\text{Vol}(\mathcal{V})} \log \frac{\text{Vol}(\mathcal{V})}{\varepsilon_n^d} - \sup_{v \in \mathcal{V}} \log \frac{\text{Vol}(\mathcal{B}_\rho(t, v) \cap \mathcal{V} + [-\varepsilon_n, \varepsilon_n]^d)}{\varepsilon_n^d}
\]

\[
= \left( \frac{\text{Vol}(\mathcal{V} \setminus (\partial \mathcal{V} + [-\varepsilon_n, \varepsilon_n]^d))}{\text{Vol}(\mathcal{V})} - 1 \right) \log \frac{\text{Vol}(\mathcal{V})}{\varepsilon_n^d} + \log \frac{\text{Vol}(\mathcal{V})}{\sup_{v \in \mathcal{V}} \text{Vol}(\mathcal{B}_\rho(t, v) \cap \mathcal{V} + [-\varepsilon_n, \varepsilon_n]^d)}.
\]

(28)

Now we use the regularity assumption on \( \mathcal{V} \). By inequalities (25), the first ratio in the preceding display is \( \text{Vol}(\mathcal{V} \setminus (\partial \mathcal{V} + [-\varepsilon_n, \varepsilon_n]^d))/\text{Vol}(\mathcal{V}) = 1 + o(\log |\varepsilon_n|^{-1}) \), so

\[
\liminf_n \left( \frac{\text{Vol}(\mathcal{V} \setminus (\partial \mathcal{V} + [-\varepsilon_n, \varepsilon_n]^d))}{\text{Vol}(\mathcal{V})} - 1 \right) \log \frac{\text{Vol}(\mathcal{V})}{\varepsilon_n^d} = 0.
\]

The second term in the display (28) similarly satisfies (again by the inequalities (24))

\[
\sup_{v \in \mathcal{V}} \text{Vol}(\mathcal{B}_\rho(t, v) \cap \mathcal{V}) \leq \sup_{v \in \mathcal{V}} \left( \text{Vol}(\mathcal{B}_\rho(t, v) \cap \mathcal{V} + [-\varepsilon_n, \varepsilon_n]^d) \right)
\]

\[
\leq \sup_{v \in \mathcal{V}} \left\{ \text{Vol}(\mathcal{B}_\rho(t, v) \cap \mathcal{V}) + 2^d \varepsilon_n^d \text{Vol}(\partial \mathcal{B}_\rho(t, v) \cap \mathcal{V}) \right\}.
\]

By the regularity assumptions on \( \mathcal{V} \) and \( \rho \), the second term in expression (28) thus has limit \( \log \frac{\text{Vol}(\mathcal{V})}{\sup_{v \in \mathcal{V}} \text{Vol}(\mathcal{B}_\rho(t, v) \cap \mathcal{V})} \) as \( n \to \infty \). This, in turn, completes the proof of inequality (24c).
C Proof of inequality (19)

We wish to compute the integral \( \int_{0}^{\infty} \left[ \frac{d-1}{d} - \frac{n \log(1+t)}{2d \log 2} \right] + dt \). We first use the fact that the indefinite integral of \( \log(1 + x) \) is \((x + 1) \log(x + 1) - x\) and that for \( t \geq 0 \) we have

\[
\frac{n \log(1 + t)}{2d \log 2} \leq \frac{d-1}{d} \quad \text{if and only if} \quad t \leq t_{\text{max}} := \exp\left( \frac{2(d-1) \log 2}{n} \right) - 1.
\]

As a consequence, we find that

\[
\int_{0}^{\infty} \left[ \frac{d-1}{d} - \frac{n \log(1 + t)}{2d \log 2} \right] + dt = t_{\text{max}} \frac{d-1}{d} - \frac{n}{2d \log 2} \left[ (t_{\text{max}} + 1) \log(t_{\text{max}} + 1) - t_{\text{max}} \right]
\]

\[
= \frac{d-1}{d} \left[ \exp\left( \frac{2(d-1) \log 2}{n} \right) - 1 - \exp\left( \frac{2(d-1) \log 2}{n} \right) \right] + \frac{n}{2d \log 2} \left[ \exp\left( \frac{2(d-1) \log 2}{n} \right) - 1 \right]
\]

\[
= \frac{n}{2d \log 2} \left[ \exp\left( \frac{2(d-1) \log 2}{n} \right) - 1 - \frac{2(d-1) \log 2}{n} \right] \geq \frac{(d-1)^2 \log 2}{dn},
\]

where the final inequality follows by the Taylor expansion of \( \exp(\cdot) \), since \( \exp(x) \geq 1 + x + x^2/2 \).

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