Dirac operator on a noncommutative Toeplitz torus

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Abstract
We construct a $1^+$-summable regular even spectral triple for a noncommutative torus defined by a C*-subalgebra of the Toeplitz algebra.

1 Introduction
In noncommutative geometry [1], a noncommutative topological space is presented by a noncommutative C*-algebra. Usually definitions of such C*-algebras are motivated by imitating some features of the classical spaces. For instance, a noncommutative version of any compact two-dimensional surface without boundary can be found in [2], where the corresponding C*-algebras are defined as subalgebras of the Toeplitz algebra.

The metric aspects of a noncommutative space are captured by the notation of a spectral triple [2]. Given a unital C*-algebra $A$, a spectral triple $(A, \mathcal{H}, D)$ for $A$ consists of a dense *-subalgebra $\mathcal{A} \subset A$, a Hilbert space $\mathcal{H}$ together with a faithful *-representation $\pi : A \to B(\mathcal{H})$, and a self-adjoint operator $D$ on $\mathcal{H}$, called Dirac operator, such that

$[D, \pi(a)] \in B(\mathcal{H})$ for all $a \in \mathcal{A}$,

$(D + i)^{-1} \in K(\mathcal{H})$.

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Here $K(\mathcal{H})$ denotes the set of compact operators on $\mathcal{H}$.

The purpose of the present paper is the construction of a spectral triple for the noncommutative torus from [7]. The noncommutative torus was chosen because the self-adjoint operator $D$ from the spectral triple has a similar structure to the Dirac operator on a classical torus with a flat metric. Our main theorem shows that this spectral triple is even, regular, and $1^+$-summable.

For the convenience of the reader, we recall the definitions of the just mentioned properties of a spectral triple (see [4]). By a slight abuse of notation, we will not distinguish between a densely defined closable operator and its closure. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be even, if there exists a grading operator $\gamma \in B(\mathcal{H})$ satisfying

$$\gamma^* = \gamma, \quad \gamma^2 = 1, \quad \gamma D = -D \gamma, \quad \gamma \pi(a) = \pi(a) \gamma \text{ for all } a \in \mathcal{A}. \quad (3)$$

We call $(\mathcal{A}, \mathcal{H}, D)$ regular, if $\delta^k(a) \in B(\mathcal{H})$ and $\delta^k([D, a]) \in B(\mathcal{H})$ for all $a \in \mathcal{A}$ and $k \in \mathbb{N}$, where $\delta(x) := [[D], x]$ for $x \in B(\mathcal{H})$. The term $1^+$-summable means that $(1 + |D|)^{-(1+\epsilon)}$ is a trace class operator for all $\epsilon > 0$ but $(1 + |D|)^{-1}$ is not a trace class operator.

Consider the polar decomposition $D = F|D|$ of the Dirac operator. The grading operator $\gamma$ gives rise to a decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ such that $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & F_{+-} \\ F_{-+} & 0 \end{pmatrix}$. If the spectral triple satisfies the properties of the previous paragraph, then $F_{+-}$ and $F_{-+}$ are Fredholm operators and one defines $\text{ind}(D) := \text{ind}(F_{+-})$. The operator $F$ is called the fundamental class of $D$ and it is said to be non-trivial if $\text{ind}(D) \neq 0$.

### 2 Noncommutative Toeplitz torus

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc and $\bar{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ its closure in $\mathbb{C}$. Consider the Hilbert space $L_2(\mathbb{D})$ with respect to the standard Lebesgue measure and its closed subspace $A_2(\mathbb{D})$ consisting of all $L_2$-functions which are holomorphic in $\mathbb{D}$. We denote by $P$ the orthogonal projection from $L_2(\mathbb{D})$ onto $A_2(\mathbb{D})$. For all $f \in C(\bar{\mathbb{D}})$, the Toeplitz operator $T_f \in B(A_2(\mathbb{D}))$ is defined by

$$T_f(\psi) := P(f \psi), \quad \psi \in A_2(\mathbb{D}) \subset L_2(\mathbb{D}),$$

and the Toeplitz algebra $\mathcal{T}$ is the $C^*$-algebra generated by all $T_f$ in $B(A_2(\mathbb{D}))$.

It is well known (see e.g. [6]) that the compact operators $K(A_2(\mathbb{D}))$ belong to $\mathcal{T}$ and that the quotient $\mathcal{T}/K(A_2(\mathbb{D})) \cong C(S^1)$ gives rise to the $C^*$-algebra extension

$$0 \longrightarrow K(A_2(\mathbb{D})) \longrightarrow \mathcal{T} \overset{\sigma}{\longrightarrow} C(S^1) \longrightarrow 0, \quad (4)$$

where $\sigma : \mathcal{T} \longrightarrow C(S^1)$ is given by $\sigma(T_f) = f|_{S^1}$ for all $f \in C(\bar{\mathbb{D}})$.

There are alternative descriptions for the Toeplitz algebra. For instance, consider the Hilbert space $L_2(S^1)$ with respect to the Lebesgue measure on $S^1$ and the orthonormal basis $\{\frac{1}{\sqrt{2\pi}} e^{ik} : k \in \mathbb{Z}\}$, where $u \in C(S^1) \subset L_2(S^1)$ is the unitary
function given by \( u(\zeta) = \zeta, \ \zeta \in S^1 \). Let \( P_+ \) denote the orthogonal projection from \( L_2(S^1) \) onto \( \text{span}\{u^n : n \in \mathbb{N}\} \cong \ell_2(\mathbb{N}) \). For all \( f \in C(S^1) \), define \( \hat{T}_f \in B(\ell_2(\mathbb{N})) \) by
\[
\hat{T}_f(\phi) := P_+(f \phi), \quad \phi \in \text{span}\{u^n : n \in \mathbb{N}\} \subset L_2(S^1).
\]
Then \( T \) is isomorphic to the C*-subalgebra of \( B(\ell_2(\mathbb{N})) \) generated by the operators \( \{\hat{T}_f : f \in C(S^1)\} \), and the C*-algebra extension \((5)\) becomes
\[
0 \longrightarrow K(\ell_2(\mathbb{N})) \longrightarrow T \longrightarrow C(S^1) \longrightarrow 0 \quad (6)
\]
with \( \sigma(\hat{T}_f) = f \).

Let us also mention that \( T \) may be considered as a deformation of the C*-algebra of continuous functions on the closed unit disc \( \bar{\Delta} \) (see \( \text{(5)} \)). From this point of view, the equivalent C*-algebra extensions \((8)\) and \((6)\) correspond to the exact sequence
\[
0 \longrightarrow C_0(\mathbb{D}) \longrightarrow C(\bar{\Delta}) \longrightarrow C(S^1) \longrightarrow 0, \quad (7)
\]
where \( \tau(f) = f|_{S^1} \).

Recall that the torus \( T^2 \) can be constructed as a topological manifold by dividing the boundary \( S^1 = \partial \mathbb{D} \) into four quadrants and gluing opposite edges together. Then the C*-algebra of continuous functions on \( T^2 \) is isomorphic to
\[
C(T^2) := \{ f \in C(\bar{\Delta}) : f(e^{it}) = f(-ie^{-it}), \ f(e^{-it}) = f(ie^{it}), \ t \in [0, \pi] \}. \quad (8)
\]

Motivated by \((8)\) and the analogy between \((7)\) and \((1)\) (or \((6)\)), we state the following definition of the noncommutative Toeplitz torus:

**Definition 1.** The C*-algebra of the noncommutative Toeplitz torus is defined by
\[
C(T^2_q) := \{ a \in T : \sigma(a)(e^{it}) = \sigma(a)(e^{-it}), \ \sigma(a)(e^{-it}) = \sigma(a)(e^{it}), \ t \in [0, \pi] \}.
\]

That \( C(T^2_q) \) is a C*-subalgebra of \( T \) follows from the fact that \( \sigma \) is a C*-algebra homomorphism. Note that gluing the point \( e^{it} \in S^1 \) to \( -ie^{-it} \in S^1 \) and the point \( e^{-it} \in S^1 \) to \( ie^{it} \in S^1 \) for all \( t \in [0, \pi] \) yields a topological space homeomorphic to the wedge sum \( S^1 \vee S^1 \) of two pointed circles. Setting
\[
C(S^1 \vee S^1) := \{ f \in C(S^1) : f(e^{it}) = f(-ie^{-it}), \ f(e^{-it}) = f(ie^{it}), \ t \in [0, \pi] \}, \quad (9)
\]
we can write
\[
C(T^2_q) = \{ a \in T : \sigma(a) \in C(S^1 \vee S^1) \}. \quad (10)
\]
Moreover, \((5)\) and \((1)\) yield the C*-algebra extension
\[
0 \longrightarrow K(\ell_2(\mathbb{N})) \longrightarrow C(T^2_q) \longrightarrow C(S^1 \vee S^1) \longrightarrow 0.
\]

3
3 Spectral triple on the noncommutative Toeplitz torus

The Dirac operator on a local chart in two dimensions with the flat metric, see \[3\], up to constant and change of orientation is given by

\[
D = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ -\frac{\partial}{\partial \bar{z}} & 0 \end{pmatrix}, \quad \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).
\]

(11)

Since \(\frac{\partial}{\partial z}\) acts on \(A_2(\mathbb{D})\) in the obvious way, we want to use the same structure to define a spectral triple for the noncommutative Toeplitz torus. Clearly, one can construct a noncommutative version of any (orientable) compact surface without boundary by choosing appropriate boundary conditions in Definition 4 (see \[7\]). However, by the Gauss–Bonnet theorem, only the (classical) torus admits a dense local chart with a flat metric, therefore we restrict here our discussion to the quantum analogue of the torus.

Our principal aim is to find a dense \(*\)-subalgebra \(\mathcal{A} \subset C(T^2)\) and an operator \(\partial_z\), which should be closely related to \(\frac{\partial}{\partial z}\) from (11), such that \([\partial_z, a]\) is bounded for all \(a \in \mathcal{A}\). Recall that an orthonormal basis for \(A_2(\mathbb{D})\) is given by \(\{\varphi_n : n \in \mathbb{N}\}\), where \(\varphi_n := \frac{\sqrt{n+1}}{\sqrt{2\pi}} z^n\) [6]. Complex differentiation yields \(\frac{\partial}{\partial z}(\varphi_n) = \sqrt{n(n+1)}\varphi_{n-1}\). If we define an operator \(\partial_z\) on \(A_2(\mathbb{D})\) by \(\partial_z(\varphi_n) := n\varphi_{n-1}\), then \(\frac{\partial}{\partial z} - \partial_z\) extends to a bounded operator on \(A_2(\mathbb{D})\) since the coefficients \(\sqrt{n(n+1)} - n\) are uniformly bounded. As a consequence, the commutators \([\frac{\partial}{\partial z}, a]\) are bounded for all \(a \in \mathcal{A}\) if and only if the commutators with \(\partial_z\) are bounded.

In order to simplify the notation, we will use the description of the Toeplitz algebra on \(\ell_2(\mathbb{N}) \cong \text{span}\{u^n : n \in \mathbb{N}\} \subset L_2(S^1)\). For \(m \in \mathbb{Z}\), set \(e_m := \frac{1}{\sqrt{2\pi}} u^m\) and let \(\partial_z\) be defined by

\[
\partial_z(e_n) := n e_{n-1} \quad \text{on } \text{dom}(\partial_z) := \{ \sum_{n \in \mathbb{N}} \alpha_n e_n \in \ell_2(\mathbb{N}) : \sum_{n \in \mathbb{N}} n^2 |\alpha_n|^2 < \infty \}.
\]

(12)

Moreover, consider the number operator \(N\) on \(\ell_2(\mathbb{N})\) determined by

\[
N(e_n) := n e_n \quad \text{on } \text{dom}(N) := \text{dom}(\partial_z).
\]

(13)

Let \(S\) be the unilateral shift operator on \(\ell_2(\mathbb{N})\) so that we have

\[
S(e_n) = e_{n+1}, \quad n \in \mathbb{N}, \quad S^* (e_n) = e_{n-1}, \quad n > 1, \quad S^* (e_0) = 0.
\]

(14)

Since \(N\) is a self-adjoint positive operator on \(\text{dom}(N) = \text{dom}(\partial_z)\) and since \(S^*\) is a partial isometry such that \(\ker(S^*) = \text{Ran}(N)^{\perp}\), it follows that \(\partial_z = S^* N\) is the polar decomposition of the closed operator \(\partial_z\). Clearly, \(\partial_z^* = NS\), so

\[
\partial_z^* (e_n) = (n+1) e_{n+1} \quad \text{and } \text{dom}(\partial_z^*) = \text{dom}(N).
\]

(15)

Under the unitary isomorphism \(A_2(\mathbb{D}) \cong \ell_2(\mathbb{N})\) given by \(\varphi_n \mapsto e_n\) on the bases described above, the operator \(\partial_z\) on \(\ell_2(\mathbb{N})\) is unitary equivalent to a bounded
perturbation of the Cauchy-Riemann operator $\frac{\partial}{\partial z}$ on $A_2(D)$. Therefore we take $\partial_z$ on $\ell_2(\mathbb{N})$ as a replacement for $\frac{\partial}{\partial z}$ on $A_2(D)$.

Note that, in the commutative case and with functions represented by multiplication operators, one has $[\frac{\partial}{\partial z}, f] = \frac{\partial f}{\partial z}$ for all $f \in C^1(\mathbb{D})$ but clearly not all continuous functions are differentiable. In the following, we will single out a dense $*$-subalgebra $A \subset C(T^2_\mathbb{Q}) \subset B(\ell_2(\mathbb{N}))$ which can be viewed as an algebra of infinitely differentiable functions. With $C(S^1 \cup S^1) \subset C(S^1)$ defined in [9], set $C^\infty(S^1 \cup S^1) := C(S^1 \cup S^1) \cap C^\infty(S^1)$ and let

$$A_0 := \{ \hat{T}_f : f \in C^\infty(S^1 \cup S^1) \} \subset C(T^2_\mathbb{Q}).$$

Using the obvious embedding $\text{End}(\text{span}\{e_1, \ldots, e_n\}) \subset K(\ell_2(\mathbb{N})) \subset C(T^2_\mathbb{Q})$, consider

$$\mathcal{F}_0 := \bigcup_{n \in \mathbb{N}} \text{End}(\text{span}\{e_1, \ldots, e_n\}) \subset C(T^2_\mathbb{Q}).$$

We will take $A$ to be the $*$-subalgebra of $C(T^2_\mathbb{Q})$ generated by the elements of $A_0$ and $\mathcal{F}_0$, i.e.,

$$A := \ast\text{-alg}(A_0 \cup \mathcal{F}_0) \subset C(T^2_\mathbb{Q}). \quad (16)$$

**Lemma 2.** The algebra $A$ defined in (16) is dense in $C(T^2_\mathbb{Q})$ and its elements admit bounded commutators with $\partial_z$ and $\partial_z^*$. Furthermore, $\delta_N^k(a)$, $\delta_N^k([\partial_z, a])$ and $\delta_N^k([\partial_z^*, a])$ are bounded for all $a \in A$ and $k \in \mathbb{N}$, where $\delta_N(x) := [N, x]$ for $x \in B(\ell_2(\mathbb{N}))$.

**Proof.** The set $\mathcal{F}_0$ contains all finite operators on $\text{span}\{e_n : n \in \mathbb{N}\}$, therefore it is dense in $K(\ell_2(\mathbb{N}))$. As a consequence, all compact operators $K(\ell_2(\mathbb{N}))$ belong to the closure of $A$. From [7], it follows that $\|T_f\| \leq \|f\|_\infty$. By the Stone-Weierstrass theorem, $C^\infty(S^1 \cup S^1)$ is dense in $C(S^1 \cup S^1)$ with respect to the norm $\|\cdot\|_\infty$. Thus each $\hat{T}_f \in C(T^2_\mathbb{Q})$ can be approximated by elements from $A_0$. Let $a \in C(T^2_\mathbb{Q})$. Writing $a = a - \hat{T}_{\sigma(a)} + \hat{T}_{\sigma(a)}$, where $\hat{T}_{\sigma(a)} \in C(T^2_\mathbb{Q})$ and $a - \hat{T}_{\sigma(a)} \in K(\ell_2(\mathbb{N}))$, we conclude that $a$ lies in the closure of $A$, so $A$ is dense in $C(T^2_\mathbb{Q})$.

By the Leibniz rule $[A, BC] = [A, B]C + B[A, C]$ for the commutator $[\cdot, \cdot]$, it suffices to prove the boundedness of the commutators for the elements belonging to the generating set $A_0 \cup \mathcal{F}_0$. From the definitions of $\mathcal{F}_0$ and $N$, it follows that $Na \in \mathcal{F}_0$ and $aN \in \mathcal{F}_0$ for all $a \in \mathcal{F}_0$. This immediately that implies $\delta_N^k(a) \in B(\ell_2(\mathbb{N}))$ for all $k \in \mathbb{N}$ since each term of the iterated commutators belongs to $\mathcal{F}_0 \subset B(\ell_2(\mathbb{N}))$. Note also that $aS^* \in \mathcal{F}_0$ and $S^*a \in \mathcal{F}_0$ for all $a \in \mathcal{F}_0$, therefore $[\partial_z, a] = S^*(Na) - (aS^*)N \in \mathcal{F}_0$. In particular, $[\partial_z, a]$ and $\delta_N^k([\partial_z, a])$ are bounded for all $k \in \mathbb{N}$.

Next consider $\hat{T}_f \in A_0$. To determine the action of $\hat{T}_f$ on $\ell_2(\mathbb{N})$, we represent $f$ by its Fourier series $f = \sum_{k \in \mathbb{Z}} \hat{f}(k)u^k$, where $\hat{f}(k) \in \mathbb{C}$. Since multiplication by $u^k$ yields $u^ke_m = e_{m+k}$, one obtains from (5)

$$\hat{T}_f(e_m) = P_+\left(\sum_{k \in \mathbb{Z}} \hat{f}(k)u^ke_m\right) = P_+\left(\sum_{k \in \mathbb{Z}} \hat{f}(k)e_{m+k}\right) = \sum_{n \in \mathbb{N}} \hat{f}(n-m)e_n. \quad (17)$$
If \( f \in C^\infty(\mathbb{S}^1) \), then partial integration shows that \( f' \in C(\mathbb{S}^1) \) has the Fourier series \( f' = \sum_{k \in \mathbb{Z}} ik\hat{f}(k) u^k \). Therefore, for all \( m \in \mathbb{N} \),

\[
\begin{align*}
[N, \hat{T}_f](e_m) &= \sum_{n \in \mathbb{N}} n \hat{f}(n-m) e_n - \sum_{n \in \mathbb{N}} m \hat{f}(n-m) e_n = \sum_{n \in \mathbb{N}} (n-m) \hat{f}(n-m) e_n \\
&= -i P_+ \left( \sum_{k \in \mathbb{Z}} i(k-m) \hat{f}(k-m) e_k \right) = -i \hat{T}_{f'}(e_m) \quad (18)
\end{align*}
\]

by (17) and the Fourier series of \( f' \). Similarly,

\[
\begin{align*}
[\partial_z, \hat{T}_f](e_m) &= \sum_{n \in \mathbb{N}} n \hat{f}(n-m) e_{n-1} - \sum_{n \in \mathbb{N}} m \hat{f}(n-(m-1)) e_n \\
&= \sum_{n \in \mathbb{N}} (n-m+1) \hat{f}(n-m+1) e_n = -i P_+ \left( \bar{u} \sum_{k \in \mathbb{Z}} i(k-m) \hat{f}(k-m) e_k \right) \\
&= -i \hat{T}_{\bar{u}f'}(e_m). \quad (19)
\end{align*}
\]

This yields \([\partial_z, \hat{T}_f] = -i \hat{T}_{\bar{u}f'} \in B(\ell_2(\mathbb{N}))\), \( \delta^{S}_{\partial_z}(\hat{T}_f) = (-i)^k \hat{T}_{f^k} \in B(\ell_2(\mathbb{N}))\), and \( \delta^{S}_{\partial_z}(\hat{T}_f) = (-i)^{k+1} \hat{T}_{(\bar{u}f')^k} \in B(\ell_2(\mathbb{N}))\), the latter because \( \bar{u}f' \) is a \( C^\infty \) function. The statement for \( \partial_z^* \) can be proven analogously or by using \([\partial_z^*, a] = -[\partial_z, a^*] \) together with \( a^* \in F_0 \) for all \( a \in F_0 \) and \( \hat{T}_f = \hat{T}_{f'} \) for all \( f' \in C(\mathbb{S}^1) \). \( \Box \)

Now we are in a position to construct our spectral triple and describe its fundamental properties.

**Theorem 3.** Let \( A \) denote the dense \(*\)-subalgebra of \( C(\mathbb{T}^2_\theta) \) from Lemma 2. Set \( \mathcal{H} : = \ell_2(\mathbb{N}) \oplus \ell_2(\mathbb{N}) \) and define a \(*\)-representation \( \pi : A \to B(\ell_2(\mathbb{N}) \oplus \ell_2(\mathbb{N})) \) by \( \pi(a) := a \oplus a \). Consider the self-adjoint operator

\[
D := \begin{pmatrix} 0 & \partial_z^* \\ \partial_z & 0 \end{pmatrix} \quad \text{on} \quad \text{dom}(D) := \text{dom}(N) \oplus \text{dom}(N).
\]

Then \((A, \mathcal{H}, D)\) is a \( 1^+ \)-summable regular even spectral triple for \( C(\mathbb{T}^2_\theta) \) with grading operator \( \gamma : = \text{id} \oplus (-\text{id}) \). The Dirac operator \( D \) has discrete spectrum \( \text{spec}(D) = \mathbb{Z} \), each eigenvalue \( k \in \text{spec}(D) \) has multiplicity 1, and a complete set of eigenvectors \( \{ b_k : k \in \mathbb{Z} \} \) satisfying \( Db_k = kb_k \) is given by

\[
b_k := \frac{1}{\sqrt{2}} (e_{k-1} \oplus e_k), \quad b_{-k} := \frac{1}{\sqrt{2}} (-e_{k-1} \oplus e_k), \quad k > 0, \quad b_0 := 0 \oplus e_0.
\]

Its fundamental class \( F = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} \) is non-trivial and \( \text{ind}(D) = 1 \).

**Proof.** We have already mentioned that the operator \( \partial_z = S^* N \) is closed. Hence \( D \) is self-adjoint by its definition. Since \([D, \pi(a)]\) has \([\partial_z, a]\) and \([\partial_z^*, a]\) as its non-zero matrix entries, the boundedness of these commutators for all \( a \in A \) follows from Lemma 2. As \( \partial_z = S^* N \) and \( \partial_z^* = NS = S(N+1) \), the polar decomposition of \( D \) reads as

\[
D = F[D] = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} \begin{pmatrix} N+1 & 0 \\ 0 & N \end{pmatrix}.
\]

(20)
In particular, the entries of the commutators with $|D|$ are given by commutators with $N$, thus the regularity can easily be deduced from Lemma 2. Clearly, $\gamma$, $D$ and $\pi(a)$ satisfy (3), so the spectral triple is even. From (12) and (15), it follows immediately that $D(b_k) = kb_k$ for all $k \in \mathbb{Z}$. Since $\{b_k : k \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{H}$, we have $\text{spec}(D) = \mathbb{Z}$ and each eigenvalue has multiplicity 1. The $1^\epsilon$-summability follows from the convergence behavior of the series $\sum_{k\in\mathbb{Z}} (1 + |k|)^{-(1+\epsilon)}$, $\epsilon \geq 0$. Finally, by the polar decomposition given in (20), $\text{ind}(D) = \text{ind}(S^*) = 1$.

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