Lanczos potentials and a definition of gravitational entropy for perturbed Friedman–Lemaitre–Robertson–Walker spacetimes

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Abstract

We give a prescription for constructing a Lanczos potential for a cosmological model which is a purely gravitational perturbation of a Friedman–Lemaitre–Robertson–Walker spacetime. For the radiation equation of state, we find the Lanczos potential explicitly via Fourier transforms. As an application, we follow up a suggestion of Penrose (1979 Singularities and time-asymmetry General Relativity: An Einstein Centenary Survey ed S W Hawking and W Israel (Cambridge: Cambridge University Press)) and propose a definition of gravitational entropy for these cosmologies. With this definition, the gravitational entropy initially is finite if and only if the initial Weyl tensor is finite.

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1. Introduction

While there is no generally agreed definition of gravitational entropy in general relativity (GR), it was conjectured some years ago by Penrose [20] that it should be related to the clumping of matter, therefore to the degree of inhomogeneity and anisotropy of a spacetime and therefore associated with the Weyl or conformal curvature. Specifically, Penrose suggested that a measure for the gravitational entropy should involve an integral of a quantity derived from the Weyl tensor, and that the definition of the scalar product on the Hilbert space of one-particle states for linearized GR in flat space might provide guidance [20]. This definition, which we review below, is written in terms of potentials for the (linearized) Weyl spinor$^3$.

$^3$ For a different approach to implementing Penrose’s suggestion via graviton number in FLRW cosmologies, see [17]. For more on potentials in linear theory and electromagnetism see e.g. chapter 6 of [21].
There have subsequently been several attempts to construct gravitational entropy measures using polynomial invariants of the Weyl and Ricci tensors (see, e.g., [5, 10, 19]) but with no complete success. From the point of view of Penrose’s suggestion these have the wrong ‘differential order’ being constructed algebraically from the Weyl tensor rather than from potentials for it. Another approach has been to note that, via the Bianchi identities, the Weyl tensor is related to the density gradient, which is a natural physical measure of inhomogeneity. There are proposals for a definition of gravitational entropy based on density contrast functions which are covariant, non-local and work globally for dust cosmologies, provided the initial singularity is isotropic [12, 16]. This is encouraging, as part of the motivation of [20] was to connect the notion of low gravitational entropy to a restriction on the nature of the initial singularity.

It is unclear how any of these measures could relate to the established notion of black hole entropy or whether there should be a relation between the entropy in gravitational waves and black hole entropy at all.

Our purpose in this paper is to return to Penrose’s suggestion and take it as far as possible, using the scalar product from linear theory in flat space to motivate a definition in curved space. This requires a potential for the Weyl tensor or Weyl spinor and we shall use the Lanczos potential ([14], see also [1, 3]). It is a general result of Illge [13] that any spinor field with the symmetries of the Weyl spinor locally has a Lanczos potential which is (uniquely) determined by its value at a space-like hypersurface \(S\). Furthermore, for a vacuum spacetime there exists a potential for the Lanczos potential, a second or superpotential for the Weyl spinor, again determined by its value at \(S\) [13] (see also [1]).

Apart from Illge’s result, there is no general prescription for obtaining a Lanczos potential for a given spacetime. A general expression for a Lanczos potential in the case of perfect fluid spacetimes with zero shear and vorticity was given in [18]. More recently, this result has been extended by Holgersson [11] to Bianchi I perfect-fluid spacetimes. There are also several examples in the literature of Lanczos potentials for particular exact solutions, including Gödel, Schwarzschild, Taub and Kerr [4, 6, 7, 15, 18]. Since there is freedom in the choice of the Lanczos potential, these examples are usually made subject to symmetry assumptions.

In this paper, we consider the Lanczos potential and superpotential for linearly perturbed Friedman–Lemaitre–Robertson–Walker (FLRW) spacetimes. We obtain wave equations for tensors defining invariant parts of the Lanczos potential, and solve them by Fourier transforms to give explicit solutions. Then, as an application, we closely follow Penrose’s idea [20] in order to propose a measure of the gravitational entropy and apply it to linearly perturbed FLRW cosmologies. The measure is defined as far as possible in such a way as to carry over to more general situations. Thus given a choice of ‘time’, in the sense of a space-like hypersurface, we construct a Lanczos potential with data at that hypersurface, and a tensor which obeys the equations for a second potential only at that hypersurface. Then we define a complex structure at the hypersurface on the space of potentials. From this, we construct a measure of gravitational entropy \(S\) which is now a functional of the (linearized) Weyl tensor.

Our definition is of course speculative. It has a reasonably good motivation, but does not have any obvious monotonicity property. It has at least one good property: the main positive result of this part of the paper is that \(S\) is finite at the initial singularity only for those linearized Weyl tensors which are finite at the initial singularity. Here the Weyl tensor initially is understood to be finite if the metric (background plus perturbation) can be rescaled so as to extend conformally through the singularity, in other words if the initial singularity is still isotropic [10]. If the initial Weyl tensor is singular, in the sense understood here, then the initial gravitational entropy is infinite. Thus finite initial gravitational entropy, as defined here, requires finite initial Weyl tensor.
The plan of the paper is as follows. To end this section we review linearized GR, introducing the scalar product on the space of solutions and explain how this motivates Penrose’s suggestion. In the next section, we review the \((1 + 3)\)-formalism for cosmological models and apply it to perturbations of FLRW cosmologies. Then in section 3, we obtain an expression for a Lanczos potential for these perturbations, solving the wave equations which arise by Fourier transforms. Finally in section 4, we use the analysis from earlier sections, together with a definition of complex structure on the space of potentials, to suggest a definition of gravitational entropy for these cosmologies.

In linearized GR, we start by perturbing the flat metric \(\eta_{ab}\) according to the equation

\[
g_{ab} = \eta_{ab} + \Phi_{ab}.
\]

For simplicity, we shall assume that the perturbation \(\Phi_{ab}\) is subject to the following gauge conditions:

\[
\Phi_{a}^{a} = 0 = \nabla^{a} \Phi_{ab}.
\]

We obtain the linearized connection (in the sense of the perturbation of the Ricci rotation coefficients) as the tensor \(L_{abc} = L_{[abc]}\) defined by

\[
L_{abc} = \nabla_{[a} \Phi_{bc]}.
\]

Note that

\[
L_{ab}^{b} = 0 = \eta^{abcd} L_{abc} = \nabla^{c} L_{abc}.
\]

With the convention

\[
(\nabla_{c} \nabla_{d} - \nabla_{d} \nabla_{c}) V^{a} = R^{a}_{abcd} V^{b},
\]

the linearized Riemann tensor is

\[
R_{ab}^{cd} = -\nabla_{[a} L_{cd]}^{b} = -\nabla^{c} L_{ab}^{d}.
\]

The field equation is the linearization of the Einstein vacuum equation and is equivalent to the wave equation on \(\Phi_{ab}\):

\[
R_{a}^{\ c} := R_{ab}^{\ c} = \frac{1}{2} \Box \Phi_{ab} = 0.
\]

In spinors we write \(\Phi_{ab} = \Phi_{ABAB'}B'\) (not to be confused with the Ricci spinor) and

\[
L_{abc} = L_{ABCC'} \epsilon_{A'B'B} + c.c.
\]

for a symmetric spinor \(L_{ABCC'}\). Then (2) and (4), taking account of (1), are

\[
\nabla_{C}^{b} \Phi_{ABC'D'} = -2L_{ABCC'},
\]

\[
\nabla_{D}^{C} L_{ABCC'} = \psi_{ABCD},
\]

where \(\psi_{ABCD}\) is the Weyl spinor.

We define the symplectic form on the space of solutions by

\[
\Omega(\Phi, \tilde{\Phi}) = \int_{S} (L_{abc} \Phi^{bc} - \tilde{L}_{abc} \Phi^{bc}) \ dS^{a},
\]

where the integral is over a space-like hypersurface \(S\). Then this is independent of surface by virtue of the field equation, and is also gauge-invariant. For the scalar product on the Hilbert space of classical solutions, we need the complex structure \(J\) on the space of solutions, which is usually defined by the splitting into positive and negative frequencies: if \(\Phi = \Phi_{+} + \Phi_{-}\) is that splitting then

\[
J \Phi = i(\Phi_{+} - \Phi_{-}),
\]

and then the inner product is

\[
\langle \Phi, \tilde{\Phi} \rangle = \Omega(\Phi, J \tilde{\Phi}).
\]

This can be seen to be positive-definite by writing it in terms of the Fourier transform.
This construction motivates Penrose’s [20] suggested guide to a definition of gravitational entropy: given a solution to a classical field theory, the coherent state built on that classical state can be thought of as the most closely corresponding quantum state; the expectation value of the number operator in the quantum state is then a measure of the ‘number of particles’ underlying the classical state, which in turn is a measure of the entropy; but this expectation value is just the norm of the classical state in the scalar product. If it were possible to find a definition like (8) above in a curved spacetime, it would therefore be a candidate for a definition of gravitational entropy. (Again, this might be just the entropy in gravitational waves. Black hole entropy could be another story.)

This is the idea we pursue here. We replace the linearized connection $L_{abc}$ by the Lanczos potential, which by Illge [13] always exists satisfying (6). The second or superpotential $\Phi_{ab}$ derived from $L_{abc}$ always exists for vacuum but we shall be concerned with cosmological solutions when its existence is problematic, as is the correct definition of $J$. On the other hand, we do not expect a definition which is independent of surface or, equivalently, independent of time. Thus our aim will be to mimic (8) as closely as possible but basing the construction on a choice of hypersurface.

2. 1 + 3 Formalism and tensor perturbations of FLRW cosmologies

We consider a spacetime with a distinguished time-like direction given by the velocity vector field $u^a$ of the fluid, and use the formalism of [8, 9, 22], with $g_{ab}u^au^b = -1$. We introduce the tensor which, at each point, projects into the space orthogonal to $u^a$ by

$$h_{ab} = g_{ab} + u_au_b.$$  \hfill (9)

Then

$$h^a_ah^b_b = h^b_b, \quad h^b_au_b = 0, \quad h^a_a = 3.$$  \hfill (10)

The covariant derivative of $u_a$ can be written, as usual, as

$$\nabla_a u_b = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab} - \dot{u}_b u_a.$$  \hfill (11)

where

$$\sigma_{ab} = \sigma_{(ab)}, \quad \sigma^a_a = 0, \quad \sigma_{ab}u^b = 0, \quad \omega_{ab} = \omega_{(ab)}, \quad \omega_{ab}u^b = 0.$$  \hfill (12)

Then $\dot{u}^a$ is the acceleration, $\omega_{ab}$ is the vorticity tensor, $\sigma_{ab}$ is the shear and $\theta$ is the expansion.

The stress–energy tensor for perfect fluids is

$$T_{ab} = \rho u_au_b + p h_{ab}.$$  \hfill (13)

where $\rho$ is the energy density and $p$ is the isotropic pressure of the fluid.

We shall be principally concerned with the case of vanishing vorticity. Then the fluid flow is orthogonal to space-like hypersurfaces $S_t$, which can be labelled by proper-time $t$ along the flow, $h_{ab}$ is the (Riemannian) metric on these hypersurfaces and its Levi-Civita covariant derivative, say $D_a$, is defined by projection: if $V_{b...c}$ is a tensor orthogonal to $u^a$ on all indices then

$$D_a V_{b...c} = h^d_a h^e_b \cdots h^f_c \nabla_d V_{e...f}.$$  

A useful operation below will be curl, defined for a symmetric tensor $X_{ab}$ orthogonal to $u^a$ by

$$(\text{curl } X)^{ab} := \eta^{cd(a} D_c X^{b)}_d,$$  \hfill (14)

where $\eta_{abcd}u^a$ is the volume form of $S_t$, and $\eta_{abcd}$ is the spacetime volume form. It will frequently be convenient to omit the brackets and write just curl $X^ab$ or curl $X_{ab}$.
The Weyl tensor can be decomposed into its electric and magnetic parts, $E_{ab}$ and $H_{ab}$ relative to the velocity vector $u^a$ as

$$E_{ab} = C_{acbd} u^c u^d, \quad H_{ab} = C^*_{acbd} u^c u^d, \quad (15)$$

where the dual $C^*_{acbd}$ is

$$C^*_{acbd} = \frac{1}{2} \eta_{acbd} C_{stbd}. \quad (16)$$

From their definition, $E_{ab}$ and $H_{ab}$ are symmetric, trace-free and orthogonal to $u^a$.

The Bianchi identities for the Weyl tensor in the case of a twist-free perfect fluid which also has vanishing acceleration can be written as the following system (see, e.g., [22]):

$$\dot{E}_{ab} = -\theta E_{ab} - \frac{1}{2} (\rho + p) \sigma_{ab} + \text{curl} H_{ab} + 3 \sigma_c^{(a} E_{b)c} - \sigma_{cd} E_{cd} H_{ab}, \quad (17)$$

$$\dot{H}_{ab} = -\theta H_{ab} - \text{curl} E_{ab} + 3 \sigma_c^{(a} H_{b)c} - \sigma_{cd} H_{cd} E_{ab}, \quad (18)$$

$$D_a E_{ab} = \eta^{bcd} \sigma_{ce} H_{e}^d + \frac{1}{3} D^b \rho, \quad (19)$$

$$D_a H_{ab} = -\eta^{bcd} \sigma_{ce} E_{e}^d, \quad (20)$$

where the dot is $u^a \nabla_a$.

Now we use this formalism to consider perturbations of FLRW cosmologies which are purely gravitational. The background is conformally-flat, so that $E_{ab} = H_{ab} = 0$ and the fluid flow is geodesic, shear-free and twist-free so that $\dot{u}_a = \omega_{ab} = \sigma_{ab} = 0$.

We consider the FLRW metric $g_{ab}$ linearly perturbed with $\delta g_{ab} = \Phi_{ab}$. Following [23], for purely gravitational perturbations we may consistently impose the gauge conditions

$$\delta \Phi_{1ab} u^b = \delta \Phi_{1a} = 0, \quad (21)$$

For the linearized field equation, we characterize the perturbation as purely gravitational by requiring that the perturbation in the Ricci tensor in the form $\delta R^b_a$ vanish:

$$\delta R^b_a = 0. \quad (22)$$

This implies that $\delta \rho = \delta p = 0$, and with the gauge conditions (21) also $\delta u^a = \delta u_a = 0$, so that $\delta T^b_a = 0$ for the stress–energy–momentum tensor.

For the perturbation in the kinematic quantities it easily follows, for example in the formalism of [22], that

$$\delta \theta = 0 = \delta \omega_{ab} = \delta \Omega_a$$

while, for the shear, we introduce the notation

$$\Sigma_{ab} := \delta \sigma_{ab} = \frac{1}{2} \delta \Phi_{ab}. \quad (23)$$

For the Weyl tensor, which is zero in the background, we find (from the equations in, e.g., [22])

$$E_{ab} = -\Sigma_{ab} - \frac{1}{2} \theta \Sigma_{ab}, \quad (24)$$

$$H_{ab} = \text{curl} \Sigma_{ab}. \quad (25)$$

Now the field equation (22) is

$$\Box \Phi_{ab} = \frac{1}{2} \rho \Phi_{ab}, \quad (26)$$

(compare, e.g., [23]). We note the following identities for trace-free, symmetric tensors $\chi_{ab}$ orthogonal to $u^a$:
\[ D^a \text{curl } \chi_{ab} = 0, \quad (27) \]
\[ \nabla^a \chi_{ab} = D^a \chi_{ab}, \quad (28) \]
\[ (\nabla^a \chi_{ab}) = \nabla^a \chi_{ab}, \quad (29) \]
\[ (\text{curl} \chi_{ab}) = \text{curl} \chi_{ab} - \frac{i}{2} \theta \text{curl} \chi_{ab}, \quad (30) \]
\[ \text{curl} \chi_{ab} = -\chi_{ab} - \Box \chi_{ab} - \theta \chi_{ab} + \left(\rho - \frac{1}{2} \theta^2\right) \chi_{ab}. \quad (31) \]

Then, from (21), (23), (28), and (29)
\[ D^a \Sigma_{ab} = 0, \quad (32) \]
and from (23) and (26) we calculate
\[ \Box \Sigma_{ab} = \frac{2}{3} \theta \Sigma_{ab} + \left(\frac{1}{2} \rho - \frac{4}{3} p + \frac{1}{6} \theta^2\right) \Sigma_{ab}. \quad (33) \]

It is now easy to check that, neglecting second-order terms, (17)–(20) are satisfied with \( E_{ab} \) and \( H_{ab} \) as in (24) and (25). Specifically (18) follows from (30), (19) from (29) and (32), (20) from (27), and finally (17), which is the hardest, from (26), (31), (33) and the Raychaudhuri equation
\[ \dot{\theta} + \frac{\theta^2}{3} + \frac{1}{2}(\rho + 3p) = 0. \quad (34) \]

3. The Lanczos potential

The Lanczos potential is a tensor \( L_{abc} = -L_{bac} \), connected to the Weyl tensor by the equation:
\[ C_{ab}^{cd} = -\nabla^c L_{ab}^{d} - \nabla^d L_{ab}^{c} - \text{traces}, \quad (35) \]
which should be compared with (4) (in general, we follow [11] but our definition of \( L_{abc} \) is twice the usual definition, in order to maintain (2)). There is gauge freedom in \( L_{abc} \) satisfying (35), which can be reduced by imposing the Lanczos gauge conditions:
\[ L_{ab}^b = 0 = \eta^{abcd} L_{abc} = \nabla_c L_{ab}^c, \]
the same conditions as in (3). When these are imposed, the ‘− traces’ term in (35) is \( -2\delta^{[c}_{[a} Q_{b]} d] \) where
\[ Q_{ac} = \nabla^b L_{abc}, \]
which is symmetric and trace-free by virtue of the gauge conditions on \( L_{abc} \). This term vanishes in the Minkowski space version of the theory described in section 1, but does not necessarily vanish in curved space.

The algebraic gauge conditions ensure that \( L_{abc} \) can be expressed in terms of a symmetric spinor field \( L_{ABCC'} \) as
\[ L_{abc} = L_{ABCC'} \epsilon_{AXB'B'} + L_{A'B'C'C} \epsilon_{AB}, \]
and the differential gauge condition then implies
\[ \nabla_{C} L_{ABCC'} = 0. \quad (36) \]
Now (35) takes the spinor form
\[ \nabla^{C} L_{ABCC'} = \psi_{ABCD}, \quad (37) \]
just as in (6) but where \( \psi_{ABCD} \) is now the full (nonlinear) Weyl spinor. There is no need to symmetrize in (37) because of (36). Illge [13] shows that (37) has a unique solution given
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on a space-like surface, but one cannot in general find the second potential as in (5) as this equation has a curvature obstruction from the Ricci tensor: given (6), (5) implies

$$\psi^{C'D'}_{, A} \Theta_{B C D'} = 0,$$

(38)

where $\psi_{A B A'}$ is the Ricci spinor.

Holgersson [11] gave a useful decomposition of the Lanczos potential into irreducible parts in the (1 + 3)-formalism as follows:

$$L_{abc} = 2 [a A b] [c - A_a h b c] - 2 [a C b] c + \eta_{ab} S_d c + u_a [\eta_{b d}] p_d - u_c \eta_{ab} p_d,$$

(39)

where $A_a$ and $P_a$ are orthogonal to $u^a$ and $S_{ab}$ and $C_{ab}$ are trace-free, symmetric and orthogonal to $u^a$. This gives 16 components for $L_{abc}$ (three each for $A$ and $P$, five each for $S$ and $C$) which agrees with the eight complex components for $L_{ABCC'}$. Holgersson [11] also gave formulae for the electric and magnetic parts of the Weyl tensor which are useful below.

We want to calculate a Lanczos potential for a perturbed FLRW spacetime with some given perturbation $\Phi_{ab}$ in the metric, as considered in section 2. Since the perturbation is characterized by a trace-free, symmetric tensor orthogonal to $u^a$, we seek a Lanczos potential as in (39) with vector parts zero: $A_a = P_a = 0$. Then from (39) and (35), or quoting from [11] we find

$$E_{ab} = \frac{1}{2} (\text{curl } S_{ab} - C_{ab}),$$

$$H_{ab} = \frac{1}{2} (\text{curl } C_{ab} + S_{ab}).$$

(40)

(41)

Equating these to (24) and (25) we have equations for $C_{ab}$ and $S_{ab}$. Now, if a superpotential $\phi_{ab}$ also existed for all times with

$$L_{abc} = \nabla [a \phi_{b c}],$$

(42)

then, from (39), we would have

$$C_{ab} = \frac{1}{2} \left( \phi_{ab} + \frac{\theta}{3} \phi_{ab} \right), \quad S_{ab} = \frac{1}{2} \text{curl } \phi_{ab},$$

(43)

but this is incompatible with the Bianchi identities (17)–(20). This incompatibility is a consequence of the obstruction (38). However, (43) suggests another ansatz, namely

$$C_{ab} = \frac{1}{2} \left( \psi_{ab} + \frac{\theta}{3} \phi_{ab} \right), \quad S_{ab} = \frac{1}{2} \text{curl } \phi_{ab}$$

(44)

in terms of another unknown tensor $\psi_{ab}$. (This is simply an ansatz, in that we express two unknown tensors, $C$ and $S$, in terms of two other unknown tensors, $\phi$ and $\psi$; the justification of the ansatz is the simplification which results, for example, in (50) and (51)).

Using (30) we find from (41) and (44)

$$H_{ab} = \frac{1}{4} \text{curl } (\phi + \psi)_{ab}.$$ 

(45)

Comparing this equation with (25) we can choose

$$\Sigma_{ab} = \frac{1}{4} (\phi_{ab} + \psi_{ab}),$$

(46)

so that $\psi_{ab}$ is known once $\phi_{ab}$ has been found. Then, using (40)

$$E_{ab} = \frac{1}{4} \left( - \psi_{ab} - \frac{\dot{\theta}}{3} \phi_{ab} - \frac{\theta}{3} \phi_{ab} + \text{curl } \phi_{ab} \right),$$

(47)

and combining this with (24), (31) and (46) we get

$$\Box \phi_{ab} + \frac{4}{3} \theta \phi_{ab} + \left( \frac{\dot{\theta}}{3} + \frac{\theta^2}{9} - \rho \right) \phi_{ab} = \frac{8}{3} \theta \Sigma_{ab},$$

(48)
which is a wave equation for $\phi_{ab}$. Note that this wave-equation is not (26). In fact, if we introduce $X_{ab} = \phi_{ab} - \Phi_{ab}$ then $X_{ab}$ satisfies
\[
\Box X_{ab} + \frac{4}{3} \theta X_{ab} + \left( \frac{\dot{\theta}}{3} + \frac{\theta^2}{9} - \rho \right) X_{ab} = \frac{1}{2} \left( \rho + p \right) \Phi_{ab},
\] (49)
using the Raychaudhuri equation (34) again. Equation (49) is not satisfied by zero, so that $\phi_{ab}$ cannot be taken to be the perturbed metric $\Phi_{ab}$. For later use, we note that, in terms of $X_{ab}$ and $\Phi_{ab}$:
\[
\phi_{ab} = \Phi_{ab} + X_{ab},
\] (50)
\[
\psi_{ab} = \Phi_{ab} - X_{ab}.
\] (51)
Given initial data $(\phi_{ab}(x, t_0), \dot{\phi}_{ab}(x, t_0))$ or equivalently $(\phi_{ab}(x, t_0), \psi_{ab}(x, t_0))$, in terms of spatial coordinates $x$ at some time $t_0$, a solution to (48) exists and is unique. We therefore have a complete prescription to determine a unique $L_{abc}$ for linearly perturbed FLRW, subject to these data. We can achieve (43), and therefore (42), at a given instant by choosing the data to be
\[
\phi_{ab}(x, t_0) = \Phi_{ab}(x, t_0)
\]
\[
\dot{\phi}_{ab}(x, t_0) = \Phi_{ab}(x, t_0)
\] (52)
at that instant, or equivalently
\[
X_{ab}(x, t_0) = 0 = \dot{X}_{ab}(x, t_0),
\] (53)
but this will not then be true at other times.

We summarize our results so far in the following proposition:

**Proposition:** Given a perturbed FLRW spacetime and a choice of time $t_0$, a Lanczos potential $L_{abc}$, in the Lanczos gauge, may be uniquely specified by (39) with (44), (46) and (48), subject to the data (52). We may define a superpotential $\phi_{ab}$ such that (42) holds at $t_0$ but this will not hold at other times.

To obtain explicit solutions for $\phi$ and $X$, we first recall some details of the FLRW metrics. For simplicity, we shall assume $k = 0$, the spatially-flat case, so that the metric is
\[
g = R(t)^2(d\mathbf{x}^2 + dy^2 + dz^2) - dt^2,
\]
or, introducing conformal time $\tau$,
\[
g = \tilde{R}(\tau)^2(d\mathbf{x}^2 + dy^2 + dz^2) - d\tau^2),
\]
where $R(t) = \tilde{R}(\tau)$ and $dt/R(t) = d\tau$. The overdot will consistently stand for $d/dt$, and $d/d\tau$ will always be written explicitly.

We choose, as spatial coordinates, $x = (x^i)$ for $i = 1, 2, 3$, and note that $dS = R^3d^3x$ is the volume element on the hypersurfaces of constant $t$. We shall assume an equation of state of the form $p = (\gamma - 1)\rho$, and then $\theta = 3\dot{R}/R$, while the conservation equation implies that $\rho = \rho_0R^{-3\gamma}$ for constant $\rho_0$. The Friedmann equation reduces to
\[
\dot{R}^2 = \frac{1}{2} \rho R^2,
\] (54)
(with the convention that $8\pi G/c^2 = 1$) and, without loss of generality, the solution is $R = t^n$ where $n = \frac{2}{\gamma}$.

In coordinate components, for a tensor $\chi_{ab}$ orthogonal to $u^a$ the d’Alembertian is
\[
\Box \chi_{ij} = \frac{1}{R^2} \Delta_0 \chi_{ij} - \dddot{\chi}_{ij} + \frac{\dddot{R}}{R} \chi_{ij} + 2 \left( \frac{\dot{R}}{R} + \frac{\dddot{R}}{R^2} \right) \chi_{ij}
\] (55)
where \( \Delta_0 \) is the flat Laplacian in \( x \), and for the time evolution we find, in components:

\[
(a^i \nabla_i x) y_j = \dot{x}_j - \frac{2R}{R} \delta_{ij}.
\]

Define the Fourier transform \( \hat{\Phi}_{ij}(t, q) \) of the coordinate components \( \Phi_{ij}(t, x) \) of \( \Phi_{ab} \) in the usual way as

\[
\hat{\Phi}_{ij}(t, q) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \Phi_{ij}(t, x) \exp(i q \cdot x) \, d^3x,
\]

then, suppressing indices for clarity, (26) with the aid of (55), becomes

\[
\ddot{\Phi} - \frac{\dot{R}}{R} \dot{\Phi} - \left( \frac{2R}{R} - \frac{|q|^2}{R^2} \right) \Phi = 0,
\]

which, as a check, is equation 15.10.39 of [23] (when making the comparison, recall that \( \Phi \) represents the two polarization states, i.e. they are symmetric, trace-free matrices of a standard form, orthogonal to \( q \) and suitably normalized. Then (57) gives

\[
\frac{d^2 \hat{H}}{d\tau^2} + \frac{1}{\tau} \frac{d\hat{H}}{d\tau} + \left( |q|^2 - \frac{\nu^2}{\tau^2} \right) \hat{H} = 0,
\]

where \( \nu = \frac{(n-1)}{2(1-n)} \), which is Bessel’s equation of order \( \nu \) in \( |q| \tau \), so that each of \( \hat{H}^\tau \) and \( \hat{H}^\times \) is a \( q \)-dependent linear combination of Bessel functions \( J_{\nu}(|q| \tau) \).

Now we need the Fourier transform of (49), in coordinate components, taking account of (54), and again suppressing indices for clarity. This is

\[
\ddot{X} - \frac{5\dot{R}}{R} \dot{X} - \left( \frac{3\dot{R}}{R} - \frac{9R^2}{R^2} - \frac{|q|^2}{R^2} \right) X = -\frac{1}{2} (\rho + p) \Phi,
\]

Analogously to (57), put

\[
\ddot{X}_{ij} = \tau^\beta (h^\tau_{ij}(q) \dot{G}^\tau(\tau, q) + h^\times_{ij}(q) \dot{G}^\times(\tau, q)),
\]

with \( \beta = \frac{5(\nu+1)}{2(1-n)} \) to find

\[
\frac{d^2 \hat{G}^\tau}{d\tau^2} + \frac{1}{\tau} \frac{d\hat{G}^\tau}{d\tau} + \left( |q|^2 - \frac{1}{4\tau^2} \right) \hat{G}^\tau = \frac{-n}{(1-n)^2} \tau^{a-\beta-2} \dot{H}^\times,
\]

and similarly for \( G^\times \) in terms of \( H^\tau \).

We may solve (60) by variation of parameters: suppose \( t = t_0 \) corresponds to \( \tau = a \), then we want \( \hat{G}(\alpha) = \hat{G}(\alpha) = 0 \) by (53). The homogeneous equation is Bessel’s equation with \( \nu = 1/2 \), and the solution for \( \hat{G}^\tau \) is

\[
\hat{G}^\tau(\tau) = \frac{-n\tau^{-1/2}}{(1-n)^2 |q|} \int_\sigma^\tau \sigma^{a-\beta-3/2} \sin(|q|(\tau - \sigma)) \dot{H}^\times(\sigma) \, d\sigma,
\]

and similarly for \( \hat{G}^\times \).

To summarize, (57), (59) and (61) determine \( X \) and \( \Phi \) and then (50) and (51) determine \( \phi \) and \( \psi \); finally (44) and (39) determine the Lanczos potential.
4. Gravitational entropy for perturbed FLRW

We want to recapitulate as much as possible of the argument in section 1, using the Lanczos potential in place of the $L$ used there. In place of (7), given two solutions of (26), the Einstein equations for gravitational perturbations of FLRW, we shall define

$$\Omega(\Phi_1, \Phi_2) = \frac{1}{2} \int S_t \left( L_{abc} \phi_{(1)ab} - L_{abc} \phi_{(2)ab} \right) dS,$$

with $\phi_{(i)}$ related to $L_{abc}$ as in (42) at a particular choice of $t_0$. Then from (39) and (44) we find

$$\Omega(\Phi_1, \Phi_2) = \frac{1}{2} \int S_t \left( \psi_{(1)ab} \phi_{(2)ab} - \phi_{(1)ab} \psi_{(2)ab} \right) dS.$$

Unlike the case of linear theory, this integral is not independent of time, since

$$\nabla_a \left( L_{abc} \phi_{(1)ab} - L_{abc} \phi_{(2)ab} \right) \neq 0.$$

From (63), using (50) and (51)

$$\Omega(\Phi_1, \Phi_2) = \frac{1}{2} \int S_t \left( \dot{\psi}_{(1)ab} \phi_{(2)ab} - \dot{\phi}_{(1)ab} \psi_{(2)ab} \right) dS - \frac{1}{2} \int S_t \left( \dot{X}_{(1)ab} X_{(2)ab} - \dot{X}_{(2)ab} X_{(1)ab} \right) dS.$$

Define the current

$$j^a_{\phi} := \phi_{(1)ab} \nabla^a \phi_{(2)bc} - \phi_{(2)ab} \nabla^a \phi_{(1)bc},$$

then by (26) $j^a_{\phi}$ is conserved:

$$\nabla_a j^a_{\phi} = 0.$$

Therefore the integral

$$\int S_t u_a j^a_{\phi} dS = - \int S_t \left( \dot{\phi}_{(1)ab} \phi_{(2)ab} - \dot{\phi}_{(2)ab} \phi_{(1)ab} \right) dS$$

is constant in time, i.e. the first integral in (64) is necessarily constant in time, but the others will not be. (It might be argued that one should use just (65) as the definition of the symplectic form, precisely because it is independent of time; however this is a definition which would not be available in more general settings, whereas (62) would.)

Our strategy now is the following: we seek a definition of gravitational entropy based on (8); we do not have a symplectic form which is independent of time, nor do we expect to obtain a definition of entropy which is independent of time; however we do have a unique definition of first and second potentials for the Weyl tensor given a choice of time; therefore we shall make a natural choice of complex structure, given a choice of time, and use (8) at that time to define the entropy. By using (63) precisely at $t_0$ we simplify (64) greatly, since $X_{ab}$ vanishes there.

Let us for simplicity restrict to the radiation equation of state, so that $\gamma = 4/3$, and then $n = 1/2, \alpha = 3/2, \beta = 7/2$ and $\nu = 1/2$, and we can take $R = t^{1/2} = \tau/2$ and $\rho R^4 = 3/4$.

The Bessel functions $J_{\pm 1/2}$ can be written in terms of elementary functions, and as solutions for $\tilde{H}^{(i)}$ we can take

$$\tilde{H}^{(i)} = \tau^{-1/2} \left( A^{(i)}(q) \sin(|q| \tau) + B^{(i)}(q) \cos(|q| \tau) \right).$$

Reality of $\Phi_{ab}$ implies that

$$\tilde{\Phi}(t, q) = \tilde{\Phi}(t, -q)$$

and then from (57) and (66) the same holds for $A^{(i)}$ and $B^{(i)}$. 

Note that $A$ parametrizes a `growing mode' and $B$ a `decaying mode' in the standard terminology, and note also that a growing mode is a perturbation of the conformal metric which is finite at the initial singularity while a decaying mode is not (essentially because for the former $\Phi = O(t)$ as $t \to 0$, just like the unperturbed spatial metric, while for the latter $\Phi = O(t^{1/2})$, which diverges by comparison with the spatial metric). Thus perturbations with nonzero $B$ have infinite initial Weyl tensor while perturbations with $B$ zero but $A$ nonzero have finite initial Weyl tensor. This will be important below.

Introduce

$$K(q) = A(q)B(q) - B(q)A(q)$$

then the first integral in (64) is

$$\frac{1}{2} \int S_t (\dot{\Phi}_{(1)}^a \Phi^{ab}_{(2)} - \Phi_{(2)ab} \Phi^a_{(1)}) dS = \frac{2}{(2\pi)^{3/2}} \int |q| K(q) d^3q,$$  (69)

which, as expected, is constant in time.

For the complex structure on the space of solutions of (26), we follow the discussion in [2]. There, for Klein–Gordon fields, the symplectic form is taken to be

$$\Omega((\phi, \pi), (\bar{\phi}, \bar{\pi})) = \int S_t (\pi \dot{\bar{\phi}} - \dot{\phi} \bar{\pi}) dS,$$

with $\pi = \dot{\phi}$, and the complex structure is defined by

$$J\phi = (\Theta)^{-1/2}\pi; \quad J\pi = - (\Theta)^{-1/2}\phi.$$  (70)

Here $\Theta = -D_\alpha D^\alpha$, where $D_\alpha$ is the intrinsic covariant derivative on $S_t$. We seek to follow this prescription, with (63) as the symplectic form so that $\psi$ then takes the role of $\pi$. Also $\Theta = -R^{-2}\Delta_0$ where, as before, $\Delta_0$ is the flat, three-dimensional Laplacian. In terms of the Fourier transform, $\Theta = |q|^2 R^{-2}$ so that $J$ becomes

$$J\hat{\phi} = \frac{R}{|q|} \hat{\psi}; \quad J\hat{\psi} = - \frac{|q|}{R} \hat{\phi}.$$  (71)

We use (57) and (66) to translate (70) into an action on $A$ and $B$, all evaluated at $\tau = a$. First for the relation of $\hat{\phi}$ and $\hat{\psi}$ to $A$ and $B$ from (57) and (66) we have

$$\begin{pmatrix} \hat{\phi} \\ \hat{\psi} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

where

$$M_{11} = a \sin a|q|, \quad M_{12} = a \cos a|q|,$$

$$M_{21} = \frac{2}{a} (\sin a|q| + a|q| \cos a|q|),$$

$$M_{22} = \frac{2}{a} (\cos a|q| - a|q| \sin a|q|).$$

Then, conjugating $J$ defined by (70) with $M$ and introducing $\alpha := a|q|$, we find the action of $J$ on $A$ and $B$ to be

$$\begin{pmatrix} JA \\ JB \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

where

$$J_{11} = - \frac{1}{\alpha} \cos 2\alpha - \frac{1}{2\alpha^2} \sin 2\alpha, \quad J_{12} = - \frac{1}{\alpha^2} (\cos \alpha - \alpha \sin \alpha)^2 - \cos^2 \alpha,$$

$$J_{21} = \frac{1}{\alpha^2} (\sin \alpha + \alpha \cos \alpha)^2 + \sin^2 \alpha, \quad J_{22} = -J_{11}.$$  (71)
We note the behaviour as $a \to 0$, when
\begin{align*}
J_{11} &= -\frac{2}{\alpha} + O(\alpha), & J_{12} &= -\frac{1}{\alpha^2} + O(1), \\
J_{21} &= 4 + O(\alpha^2), & J_{22} &= -J_{11}.
\end{align*}
(72)

In (68), following the analogy of (8), we set $A^{(2)} = JA^{(1)}$ and $B^{(2)} = JB^{(1)}$, to find
\begin{align*}
K(q, a) &= J_{21}A^{(1)}(q)A^{(1)}(q) + J_{22}A^{(1)}(q)B^{(1)}(q) - J_{11}B^{(1)}(q)A^{(1)}(q) - J_{12}B^{(1)}(q)B^{(1)}(q),
\end{align*}
(73)

where the choice of $J$ has introduced an explicit dependence on $a$. Substituting this into (69) gives our definition of gravitational entropy at proper time $t_0$, corresponding to conformal time $\tau = a$:
\begin{align*}
S_g(\Phi; t_0) &= \frac{2}{(2\pi)^{3/2}} \int |q| K(q, a) d^3q,
\end{align*}
(74)

with $K(q, a)$ as in (73).

We deduce the following properties of our definition.

- The entropy at time $t_0$ is determined by the Weyl tensor at $t_0$, but not locally (since e.g. the operator $(\Theta)^{-1/2}$ in (70) is non-local). In particular, it is not the integral of a scalar invariant of curvature.
- From the specific form (71) of $J$ (though it follows more generally) $S_g$ can be seen to be positive definite.
- However, for small $a$ we note from (72) that if $B \neq 0$ then $K = O(a^{-2})$, which diverges on the approach to the initial singularity, while if $B = 0$ but $A \neq 0$ then $K = O(1)$, which is finite. In other words, with this definition the initial gravitational entropy is finite if the initial Weyl tensor is finite, and infinite otherwise. This is an important success for the definition.
- Necessarily, for a decaying mode the gravitational entropy will decay from its infinite initial value and tend to a positive constant in the remote future (for this, from (73), we note that $J_{12}$ in (71) tends to a constant as $t \to \infty$). But $J_{12}$ is not monotonic in time, which makes it unlikely that $S_g$ is.
- From (71), $J_{31}$ runs from a value of $4$ at $\alpha = 0$ to a limit of $1$ at large $\alpha$ and is also not monotonic (though it is positive). Thus for a purely growing mode, the gravitational entropy runs from a finite initial value to a finite but lower final value, and again there is no reason to expect it to be monotonic. This is perhaps less successful for the definition, but this is after all a result from linear theory, though in a non-flat background—without nonlinearity, there is no gravitational clumping.

The definition has been framed in such a way that it should extend to more general situations, and it remains to be seen whether the property of being initially finite only for an isotropic singularity persists.

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