Abstract. We describe the structure of the automorphism groups of algebras Morita equivalent to the first Weyl algebra $A_1$. In particular, we give a geometric presentation for these groups in terms of amalgamated products, using the Bass-Serre theory of groups acting on graphs. A key role in our approach is played by a transitive action of the automorphism group of the free algebra $\mathbb{C}\langle x, y \rangle$ on the Calogero-Moser varieties $C_n$ defined in [BW]. Our results generalize well-known theorems of Dixmier and Makar-Limanov on automorphisms of $A_1$, answering an old question of Stafford (see [S]). Finally, we propose a natural extension of the Dixmier Conjecture for $A_1$ to the class of Morita equivalent algebras.

Let $A_1 := \mathbb{C}\langle x, y \rangle/(xy - yx - 1)$ be the first Weyl algebra over $\mathbb{C}$ with canonical generators $x$ and $y$. In his classic paper [D], Dixmier described the group $\text{Aut} A_1$ of automorphisms of $A_1$: specifically, he proved that $\text{Aut} A_1$ is generated by the following transformations

1. $\Phi_p : (x, y) \mapsto (x, y + p(x)),$ $\Psi_q : (x, y) \mapsto (x + q(y), y),$ where $p(x) \in \mathbb{C}[x]$ and $q(y) \in \mathbb{C}[y].$ Using this result of Dixmier, Makar-Limanov (see [ML1, ML2]) showed that $\text{Aut} A_1$ is isomorphic to the group $G_0 \subset \text{Aut} \mathbb{C}\langle x, y \rangle$ of ‘symplectic’ (i.e. preserving $\omega = xy - yx$) automorphisms of the free algebra $\mathbb{C}\langle x, y \rangle$: the corresponding isomorphism

2. $G_0 \cong \text{Aut} A_1$

is induced by the canonical projection $\mathbb{C}\langle x, y \rangle \rightarrow A_1$. On the other hand, the results of [ML1] (see, e.g., [C]) also imply that $G_0$ is given by the amalgamated free product

3. $G_0 = A *_U B,$

where $A$ is the subgroup of symplectic affine transformations

4. $(x, y) \mapsto (ax + by + e, cx + dy + f),$ $a, b, \ldots, f \in \mathbb{C},$ $ad - bc = 1,$

$B$ is the subgroup of triangular (Jonquières) transformations

5. $(x, y) \mapsto (ax + q(y), a^{-1}y + h),$ $a \in \mathbb{C}^*, h \in \mathbb{C},$ $q(y) \in \mathbb{C}[y],$

and $U$ is the intersection of $A$ and $B$ in $G_0$:

6. $(x, y) \mapsto (ax + by + e, a^{-1}y + h),$ $a \in \mathbb{C}^*, b, e, h \in \mathbb{C}.$

Combining (2) and (3), we thus get decomposition $\text{Aut} A_1 \cong A *_U B$, which completely describes the structure of $\text{Aut} A_1$ as a discrete group (cf. [A]).

The aim of the present paper is to generalize the above results to the case when $A_1$ is replaced by a noncommutative domain $D$, Morita equivalent to $A_1$ as a $\mathbb{C}$-algebra. This question was originally posed by Stafford in [S] (see loc. cit., p. 636). To explain why it is natural, we recall that the algebras $D$ are classified,
up to isomorphism, by a single integer \( n \geq 0 \); the corresponding isomorphism classes are represented by the endomorphism rings \( D_n := \text{End}_A M_n \) of certain distinguished right ideals of \( A_1 \) and can be realized geometrically as algebras of global differential operators on rational singular curves (see [K] [BW1] and [BW4] for a detailed exposition). Thus the Dixmier group \( \text{Aut} A_1 = \text{Aut} D_0 \) appears naturally as the first member in the family \( \{ \text{Aut} D_n : n \geq 0 \} \). Our aim is to describe the ‘higher’ groups in this family: in particular, to give a presentation of \( \text{Aut} D_n \) for arbitrary \( n \geq 0 \) in terms of amalgamated products.

The groups \( \text{Aut} D_n \) for \( n \geq 1 \) can be naturally identified with subgroups of \( \text{Aut} D_0 \). To be precise, let \( \text{Pic} D \) denote the (noncommutative) Picard group of a \( \mathbb{C} \)-algebra \( D \). By definition, \( \text{Pic} D \) is the group of \( \mathbb{C} \)-linear Morita equivalences of the category of \( D \)-modules; its elements can be represented by the isomorphism classes of invertible \( D \)-bimodules \([P]\) (see, e.g., [B]). There is a natural group homomorphism \( \omega_D : \text{Aut} D \to \text{Pic} D \), taking \( \sigma \in \text{Aut} D \) to the class of the bimodule \([1_i D_0]\), and if \( D' \) is a ring Morita equivalent to \( D \), with progenerator \( M \), then there is a group isomorphism \( \alpha_M : \text{Pic} D' \sim \text{Pic} D \) given by \([P] \mapsto [M^* \otimes_D P \otimes_D M] \). Thus, in our situation, for each \( n \geq 0 \) we have the following diagram

\[
\begin{align*}
\text{Aut} D_n & \xrightarrow{\omega_{D_n}} \text{Pic} D_n \\
i_n : \text{Aut} D_0 & \xrightarrow{\omega_{D_0}} \text{Pic} D_0 \\
\alpha_{M_n} \quad \text{where the vertical map } \alpha_{M_n} \text{ is an isomorphism and the two horizontal maps are injective.} \quad \text{Moreover, since } D_0 = A_1, \text{ a theorem of Stafford (see [S]) implies that } \omega_{D_0} \text{ is actually an isomorphism.}
\end{align*}
\]

Inverting this isomorphism, we define the embedding \( i_n : \text{Aut} D_n \hookrightarrow \text{Aut} D_0 \), which makes (7) a commutative diagram.

Recall that we defined \( G_0 \) to be the automorphism group of the free algebra \( \mathbb{C}(x, y) \) preserving \([x, y]\). Now, for \( n > 0 \), we introduce the groups \( G_n \) geometrically, in terms of a natural action of \( G_0 \) on the Calogero-Moser spaces (see [W])

\[
C_n := \{ (X, Y) \in \text{Mat}_n(\mathbb{C}) \times \text{Mat}_n(\mathbb{C}) : \text{rk}([X, Y] + I_n) = 1 \}/\text{PGL}_n(\mathbb{C}) ,
\]

where \( \text{PGL}_n(\mathbb{C}) \) operates on matrices \((X, Y)\) by simultaneous conjugation. The action of \( G_0 \) on \( C_n \) is given by

\[
(\sigma^{-1}(X), \sigma^{-1}(Y)) , \quad \sigma \in G_0 ,
\]

where \( \sigma^{-1}(X) \) and \( \sigma^{-1}(Y) \) are the noncommutative polynomials \( \sigma^{-1}(x) \in \mathbb{C}(x, y) \) and \( \sigma^{-1}(y) \in \mathbb{C}(x, y) \) evaluated at \((X, Y)\). It is known that \( C_n \) is a smooth affine algebraic variety of dimension \( 2n \), equipped with a natural symplectic structure, and it is easy to check that \( G_0 \) preserves that structure. Now, a theorem of Wilson and the first author (see [BW]) implies that (5) is a transitive action for all \( n \geq 0 \).

We define the groups \( G_n \) to be the stabilizers of points of \( C_n \) under this action: precisely, for each \( n \geq 0 \), we fix a basepoint \((X_0, Y_0) \in C_n \), with

\[
X_0 = \sum_{k=1}^{n-1} E_{k+1,k} , \quad Y_0 = \sum_{k=1}^{n-1} (k - n) E_{k,k+1} ,
\]

where \( E_{i,j} \) stands for the elementary matrix with \((i, j)\)-entry 1, and let

\[
G_n := \text{Stab}_{G_0}(X_0, Y_0) , \quad n \geq 0 .
\]
The following result can be viewed as a generalization of the above-mentioned theorem of Makar-Limanov; in a slightly different form, it has already appeared in [BW4] (cf. loc. cit., p. 120; see also [W2]).

**Theorem 1.** There is a natural isomorphism of groups $G_n \cong Aut D_n$.

Specifically, we have group homomorphisms

$$G_n \hookrightarrow G_0 \cong Aut A_1 \stackrel{i_n}{\rightarrow} Aut D_n,$$

where the first map is the canonical inclusion, the second is the Makar-Limanov isomorphism [2] and $i_n$ is the embedding defined by (7). We claim that the image of $i_n$ coincides with the image of $G_n$, which gives the required isomorphism.

Theorem 1 is a simple consequence of the main results of [BW]: in fact, it is shown in [BW] that there is a natural $G_0$-equivariant bijection (called the Calogero-Moser correspondence) between $\bigsqcup_{n \geq 0} C_n$ and the space of isomorphism classes of right ideals of $A_1$. Under this bijection, the points $(X_0, Y_0) \in C_n$ correspond precisely to the classes of the ideals $M_n$.

We will use Theorem 1 to give a geometric presentation for the groups $Aut D_n$. To this end, we associate to each space $C_n$ a graph $\Gamma_n$ consisting of orbits of certain subgroups of $G_0$ and identify $G_n$ with the fundamental group $\pi_1(\Gamma_n, *)$ of a graph of groups $\Gamma_n$ defined by the stabilizers of points of those orbits in $\Gamma_n$. The Bass-Serre theory of groups acting on graphs [Sc] will give then an explicit formula for $\pi_1(\Gamma_n, *)$ in terms of generalized amalgamated products (see [10] below).

To define the graph $\Gamma_n$, we take the subgroups $A$, $B$ and $U$ of $G_0$ defined by the transformations (4), (5) and (6). Restricting the action of $G_0$ on $C_n$ to these subgroups, we let $\Gamma_n$ be the oriented bipartite graph, with vertex and edge sets

$$\text{Vert}(\Gamma_n) := (A \setminus C_n) \sqcup (B \setminus C_n), \quad \text{Edge}(\Gamma_n) := U \setminus C_n,$$

and the incidence maps $\text{Edge}(\Gamma_n) \to \text{Vert}(\Gamma_n)$ given by the canonical projections $i : U \setminus C_n \to A \setminus C_n$ and $\tau : U \setminus C_n \to B \setminus C_n$. Since the elements of $A$ and $B$ generate $G_0$ and $G_0$ acts transitively on each $C_n$, the graph $\Gamma_n$ is connected.

Now, on each orbit in $A \setminus C_n$ and $B \setminus C_n$ we choose a basepoint and elements $\sigma_A \in G_0$ and $\sigma_B \in G_0$ moving these basepoints to the basepoint $(X_0, Y_0)$ of $C_n$. Next, on each $U$-orbit $O_U \in U \setminus C_n$ we also choose a basepoint and an element $\sigma_U \in G_0$ moving this basepoint to $(X_0, Y_0)$ and such that $\sigma_U \in \sigma_A A \cap \sigma_B B$, where $\sigma_A$ and $\sigma_B$ correspond to the (unique) $A$- and $B$-orbits containing $O_U$. Using a standard construction in the Bass-Serre theory (see [Sc], Sect. 5.4), we then assign to the vertices and edges of $\Gamma_n$ the stabilizers $A_\sigma = G_\sigma \cap \sigma A \sigma^{-1}$, $B_\sigma = G_\sigma \cap \sigma B \sigma^{-1}$, $U_\sigma = G_\sigma \cap \sigma U \sigma^{-1}$ of the corresponding elements $\sigma$ in the graph of right cosets of $G_0$ under the action of $G_n$. These data together with natural group homomorphisms $a_\sigma : U_\sigma \to A_\sigma$ and $b_\sigma : U_\sigma \to B_\sigma$ define a graph of groups $\Gamma_n$ over $\Gamma_n$, and its fundamental group $\pi_1(\Gamma_n, T)$ relative to a maximal tree $T \subseteq \Gamma_n$ has canonical presentation (see [Sc], Sect. 5.1):

$$\pi_1(\Gamma_n, T) = \langle A_\sigma * U_\sigma * B_\sigma * \ldots * \langle \text{Edge}(\Gamma_n \setminus T) \rangle \rangle \quad (e^{-1} a_\sigma(g) e = b_\sigma(g) \quad \forall e \in \text{Edge}(\Gamma_n \setminus T), \forall g \in U_\sigma).$$

In (10), the amalgams $A_\sigma * U_\sigma * B_\sigma * \ldots$ are taken along the stabilizers of edges of the tree $T$, while $\langle \text{Edge}(\Gamma_n \setminus T) \rangle$ denotes the free group based on the set of edges of $\Gamma_n$ in the complement of $T$.

Our main observation is the following.
Theorem 2. For each \( n \geq 0 \), the group \( G_n \) is isomorphic to \( \pi_1(\Gamma_n, T) \). In particular, \( G_n \) has an explicit presentation of the form (10).

Proof. One can prove Theorem 2 using the standard Bass-Serre theory (as exposed in [Sc], Ch I, Sect. 5, or [DD], Ch. I, Sect. 9). However, it seems that the more economic and intuitively clearer proof is based on topological arguments: namely, an abstract version of Van Kampen’s Theorem, which we are now going to explain.

Let \( \mathcal{G}_n := \mathcal{C}_n \rtimes G_0 \) denote the (discrete) transformation groupoid corresponding to the action of \( G_0 \) on \( \mathcal{C}_n \). The canonical projection \( p : \mathcal{G}_n \to G_0 \) is then a connected covering of groupoids,[1] which maps identically the vertex group of \( \mathcal{G}_n \) at \((X_n, Y_0) \in \mathcal{C}_n\) to the subgroup \( G_n \subseteq G_0 \). Now, each of the subgroups \( A, B \) and \( U \) of \( G_0 \) can be lifted to \( \mathcal{G}_n \): \( p^{-1}(A) = \mathcal{G}_n \rtimes G_0 A, p^{-1}(B) = \mathcal{G}_n \rtimes G_0 B \) and \( p^{-1}(U) = \mathcal{G}_n \rtimes G_0 U \), and these fibre products are naturally isomorphic to the subgroupoids \( \mathfrak{A}_n := \mathfrak{C}_n \rtimes A, \mathfrak{B}_n := \mathfrak{C}_n \rtimes B \) and \( \mathfrak{U}_n := \mathfrak{C}_n \rtimes U \) of \( \mathcal{G}_n \), respectively. Since the coproducts in the category of groups coincide with coproducts in the category of groupoids and the latter can be lifted through coverings (see [O], Lemma 3.1.1), the decomposition (3) implies

\[
\mathcal{G}_n = \mathfrak{A}_n \ast \mathfrak{U}_n \ast \mathfrak{B}_n, \quad \forall n \geq 0.
\]

Note that, unlike \( \mathcal{G}_n \), the groupoids \( \mathfrak{A}_n, \mathfrak{B}_n \) and \( \mathfrak{U}_n \) are not transitive (if \( n \geq 1 \)), so (11) can be viewed as an analogue of the Seifert-Van Kampen Theorem for non-connected spaces (see, e.g., [Gr], Ch. 6, Appendix). As in the topological situation, computing the fundamental (vertex) group from (11) amounts to contracting the connected components (orbits) of \( \mathfrak{A}_n \) and \( \mathfrak{B}_n \) to points (vertices) and \( \mathfrak{U}_n \) to edges. This defines a graph which is exactly \( \Gamma_n \). Now, choosing basepoints in each of the contracted components and assigning the fundamental groups at these basepoints to the corresponding vertices and edges defines a graph of groups (see [HMM], p. 46). By loc. cit., Theorem 3, this graph of groups is (conjugate) isomorphic to the graph \( \Gamma_n \) described above, and our group \( G_n \) is isomorphic to \( \pi_1(\Gamma_n, T) \). \( \square \)

Theorems 1 and 2 reduce the problem of describing the groups \( \text{Aut} D_n \) to a purely geometric problem of describing the structure of the orbit spaces of \( A \) and \( B \) and \( U \) on the Calogero-Moser varieties \( \mathcal{C}_n \). Using the earlier results of [W] and [BW] and some basic invariant theory, one can obtain much information about these orbits and (thence about the groups \( G_n \)). In particular, the graphs \( \Gamma_n \) can be completely described for small \( n \); it turns that \( \Gamma_n \) is a finite tree for \( n = 0, 1, 2 \), but has infinitely many cycles for \( n \geq 3 \) (see examples below).

We now explain the origin of \( \Gamma_n \). It turns out that these graphs can be realized as quotient graphs of a certain ‘universal’ tree \( \Gamma \) on which all the groups \( \text{Aut} D_n \) naturally act. Our construction of \( \Gamma \) is motivated by algebraic geometry: specifically, a known application of the Bass-Serre theory in the theory of surfaces (see, e.g., [GD], [W]). In that approach, the automorphism group of an affine surface \( S \) is described via its action on a tree whose vertices correspond to certain (admissible) projective compactifications of \( S \). Following the standard (by now) philosophy in noncommutative geometry (see, e.g., [SV]), we may think of our algebra \( D \) as the coordinate ring of a ‘noncommutative affine surface’; a ‘projective compactification’ of \( D \) is then determined by a choice of filtration. Thus, we will define \( \Gamma \) by taking as its vertices a certain class of filtrations on the algebra \( D \). It turns

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1We refer to [M], Ch. 3, for the theory of coverings of groupoids.
out that these filtrations can be naturally parametrized by an infinite-dimensional adelic Grassmannian \( \text{Gr}^{ad} \) introduced in [W1] and studied in [W BW BW3] (in particular, we rely heavily on results of [BW3]). Our construction is close in spirit to Serre’s classic application of Bruhat-Tits trees for computing arithmetic subgroups of \( \text{SL}_2(\mathbb{K}) \) over the function fields of smooth curves (see [Se], Chap. II, §2); however, at the moment, we are not aware of any direct connection.

We begin by briefly recalling the definition of \( \text{Gr}^{ad} \). Let \( \mathbb{C}[z] \) be the polynomial ring in one variable \( z \). For each \( \lambda \in \mathbb{C} \), we choose a \( \lambda \)-primary subspace in \( \mathbb{C}[z] \), that is, a \( \mathbb{C} \)-linear subspace \( V_\lambda \subseteq \mathbb{C}[z] \) containing a power of the maximal ideal \( \mathfrak{m}_\lambda \) at \( \lambda \). We suppose that \( V_\lambda = \mathbb{C}[z] \) for all but finitely many \( \lambda \)'s. Let \( V = \bigcap_\lambda V_\lambda \) (such a subspace \( V \) is called primary decomposable in \( \mathbb{C}[z] \)) and, finally, let

\[
W = \prod_\lambda (z - \lambda)^{-n_\lambda} V \subset \mathbb{C}(z),
\]

where \( n_\lambda \) is the codimension of \( V_\lambda \) in \( \mathbb{C}[z] \). By definition, \( \text{Gr}^{ad} \) consists of all subspaces \( W \subset \mathbb{C}(z) \) obtained in this way. For each \( W \in \text{Gr}^{ad} \) we set

\[
A_W := \{ f \in \mathbb{C}[z] : fW \subseteq W \}.
\]

Taking Spec of \( A_W \) gives then a rational curve \( X \), the inclusion \( A_W \to \mathbb{C}[z] \) corresponds to normalization \( \pi : \mathbb{A}^1 \to X \) (which is set-theoretically a bijective map), and the \( A_W \)-module \( W \) defines a rank 1 torsion-free coherent sheaf \( \mathcal{L} \) over \( X \). In this way, the points of \( \text{Gr}^{ad} \) correspond bijectively to isomorphism classes of triples \( (\pi, X, \mathcal{L}) \) (see [W1]).

Now, following [BW], for \( W \in \text{Gr}^{ad} \) we define\(^2\)

\[
D(W) := \{ D \in \mathbb{C}(z)[\partial_z] : DW \subseteq W \},
\]

where \( \mathbb{C}(z)[\partial_z] \) is the ring of rational differential operators in the variable \( z \). This last ring carries two natural filtrations: the standard filtration, in which both \( z \) and \( \partial_z \) have degree 1, and the differential filtration, in which \( \deg(z) = 0 \) and \( \deg(\partial_z) = 1 \). These filtrations induce two different filtrations on the algebra \( D(W) \), which we denote by \( \{D^s_\bullet(W)\} \) and \( \{D^{\partial}_\bullet(W)\} \) respectively.

Now, let \( D \) be a fixed domain Morita equivalent to \( A_1 \). Following [BW3], we consider the set \( \text{Gr}^{ad}(D) \) of all algebra isomorphisms \( \sigma_W : D(W) \to D \), where \( W \in \text{Gr}^{ad} \) (more precisely, \( \text{Gr}^{ad}(D) \) is the set of all pairs \( (W, \sigma_W) \), where \( W \in \text{Gr}^{ad} \) and \( \sigma_W \) is an isomorphism as above). Each \( \sigma_W \in \text{Gr}^{ad}(D) \) maps the two distinguished filtrations \( \{D^s_\bullet(W)\} \) and \( \{D^{\partial}_\bullet(W)\} \) into the algebra \( D \): we call their images the admissible filtrations on \( D \) of type \( A \) and type \( B \), respectively. Let \( \mathbb{P}_A(D) \) and \( \mathbb{P}_B(D) \) denote the sets of all such filtrations coming from various \( \sigma_W \in \text{Gr}^{ad}(D) \). By definition, we have then two natural projections

\[
\mathbb{P}_A(D) \xleftarrow{\pi_A} \text{Gr}^{ad}(D) \xrightarrow{\pi_B} \mathbb{P}_B(D).
\]

We say that \( (W, \sigma_W) \) and \( (W', \sigma_W') \) are equivalent in \( \text{Gr}^{ad}(D) \) if their images under \( \pi_A \) and \( \pi_B \) coincide. Writing \( \text{Gr}^{ad}(D) / \sim \) for the set of equivalence classes

\(^2\)In geometric terms, \( D(W) \) can be thought of as the ring \( D_\mathcal{L}(X) \) of twisted differential operators on \( X \) with coefficients in \( \mathcal{L} \).

\(^3\)More generally, we may think of \( \text{Gr}^{ad} \) as a groupoid, in which the objects are the \( W \)'s and the arrows are given by the algebra isomorphisms \( D(W) \to D(W') \). For \( D = D(W) \), the set \( \text{Gr}^{ad}(D) \) is then a costar in \( \text{Gr}^{ad} \), consisting of all arrows with target at \( W \). In [BW3], this set was denoted by \( \text{Grad} D \).
under this relation, we define an oriented graph $\Gamma$ by

$$\text{Vert}(\Gamma) := \mathbb{P}_A(D) \bigsqcup \mathbb{P}_B(D), \quad \text{Edge}(\Gamma) := \text{Gr}^{\text{ad}}(D)/\sim,$$

with incidence maps $\text{Edge}(\Gamma) \to \text{Vert}(\Gamma)$ induced by the projections \(\text{loc. cit.}\). Observe that the group $\text{Aut} D$ acts naturally on the set $\text{Gr}^{\text{ad}}(D)$ (by composition), and this action induces an action of $\text{Aut} D$ on the graph $\Gamma$ via \(\text{loc. cit.}\). We write $D \sqcap \Gamma$ for the corresponding quotient graph.

**Theorem 3.** (a) $\Gamma$ is a tree, which is independent of $D$ (up to isomorphism).

(b) For each $n \geq 0$, the graph $D_n \sqcap \Gamma$ is naturally isomorphic to $\Gamma_n$.

Theorem 2 can be viewed as a generalization of the main results of [BW3]. Indeed, this last paper is concerned with a description of the maximal abelian ad-nilpotent (mad) subalgebras of $D_n$: its main theorems (see loc. cit., Theorem 1.5 and Theorem 1.6) say that the space $\text{Mad}(D_n)$ of all mad subalgebras of $D_n$ is independent of $D_n$ and its quotient modulo the natural action of $\text{Aut} D_n$ is isomorphic to the orbit space $B \sqcap \mathcal{C}_n$. Now, it is easy to see that every mad subalgebra defines an admissible filtration on $D_n$ of type $B$, and conversely the zero degree component of every filtration of type $B$ is a mad subalgebra of $D_n$. Thus, we have a natural bijection $\mathbb{P}_B(D_n) \cong \text{Mad}(D_n)$, which is equivariant under the action of $\text{Aut} D_n$. This implies that $\mathbb{P}_B(D_n)$ does not depend on $D_n$, which is part of Theorem 2(a), and

$$\text{Aut} D_n \sqcap \mathbb{P}_B(D_n) \cong \text{Aut} D_n \sqcap \text{Mad}(D_n) \cong B \sqcap \mathcal{C}_n,$$

which is part of Theorem 2(b). In fact, the entire Theorem 2 can be proved using the techniques of [BW3]. We should also mention that for $D = A(k)$ our construction of the tree $\Gamma$ agrees with the one given in [A].

We now look at examples of the graphs $\Gamma_n$ and groups $G_n$ for small $n$. For $n = 0$, the space $\mathcal{C}_0$ is just a point, and so are a fortiori its orbit spaces. The graph $\Gamma_0$ is thus a segment, and the corresponding graph of groups $\Gamma_0$ is given by \(A \xrightarrow{U} B\). Formula (10) then says that $G_0 = A *_U B$, which agrees, of course, with the Makar-Limanov isomorphism (3).

For $n = 1$, we have $\mathcal{C}_1 \cong \mathbb{C}^2$, with $(X_0, Y_0)$ corresponding to the origin. Since each of the groups $A$, $B$ and $U$ contains translations $(x, y) \mapsto (x+a, y+b)$, $a, b \in \mathbb{C}$, they act transitively on $\mathcal{C}_1$. So again $\Gamma_1$ is just the segment, and $\Gamma_1$ is given by \(A_1 \xrightarrow{U_1} B_1\), where $A_1 := G_1 \cap A$, $B_1 := G_1 \cap B$ and $U_1 := G_1 \cap U$. Since, by definition, $G_1$ consists of all $\sigma \in G_0$ preserving $(0, 0)$, the groups $A_1$, $B_1$ and $U_1$ are obvious:

$$A_1 : \quad (x, y) \mapsto (ax + by, cx + dy), \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1,$$

$$B_1 : \quad (x, y) \mapsto (ax + q(y), a^{-1}y), \quad a \in \mathbb{C}^*, \quad q \in \mathbb{C}[y], \quad q(0) = 0,$$

$$U_1 : \quad (x, y) \mapsto (ax + by, a^{-1}y), \quad a \in \mathbb{C}^*, \quad b \in \mathbb{C}.$$

It follows from (10) that $G_1 = A_1 *_{U_1} B_1$.

For $n = 2$, the situation is already more interesting. A simple calculation shows that $U$ has three orbits in $\mathcal{C}_2$: two closed orbits of dimension 3 and one open orbit of dimension 4. Moreover, the $B$-orbits coincide with the $U$-orbits. Combinatorially, this means that the group $A$ acts transitively, and the graph $\Gamma_2$ is a tree with one nonterminal and three terminal vertices corresponding to the $A$-orbit and the
\( B \)-orbits, respectively. In this case, the graph of groups \( \Gamma_2 \) is given by

\[
\begin{align*}
G_{2,y} \rtimes \mathbb{C}^* \quad & \quad \mathbb{C}^* \\
\mathbb{C}^* \quad & \quad \mathbb{Z}_2 \quad G_{2,y}^{(1)} \rtimes \mathbb{Z}_2 \\
G_{2,x} \rtimes \mathbb{C}^* \quad & \quad \mathbb{C}^*
\end{align*}
\]

where \( G_{2,x} \) and \( G_{2,y} \) are the subgroups of \( G_0 \) consisting of all transformations \( \Phi_p \) and \( \Psi_q \) (see (1)), with \( p \in \mathbb{C}[x] \) and \( q \in \mathbb{C}[y] \) satisfying \( p(0) = p'(0) = 0 \) and \( q(0) = q'(0) = 0 \) respectively, and \( G_{2,y}^{(1)} := \{ \Phi_x \Psi_q \Phi_x \in G_0 : q \in \mathbb{C}[y], q(\pm 1) = 0 \} \).

Formula (10) yields the presentation

\[ G_2 = (G_{2,x} \rtimes \mathbb{C}^*) \ast_{\mathbb{C}^*} (G_{2,y} \rtimes \mathbb{C}^*) \ast_{\mathbb{Z}_2} (G_{2,y}^{(1)} \rtimes \mathbb{Z}_2). \]

In particular, \( G_2 \) is generated by its subgroups \( G_{2,x}, G_{2,y}, G_{2,y}^{(1)} \) and \( \mathbb{C}^* \).

Now, for \( n = 3 \), the structure of the graph \( \Gamma_3 \) and the group \( G_3 \) is much more complicated. The graph \( \Gamma_3 \) is not a tree: in fact, it has infinitely many circuits. Nevertheless, the group \( G_3 \) can still be described explicitly:

\[ G_3 = \pi_1(T_3, G_3) \ast \langle E_+ (\Gamma_3 \setminus T_3) \rangle \]

where \( T_3 \) is a (maximal) tree in \( \Gamma_3 \) given in Figure 2. \( \pi_1(T_3, G_3) \) is the corresponding tree product of stabilizer groups, and the complement graph \( \Gamma_3 \setminus T_3 \) is shown in Figure 2.

We would like to end this paper with some questions and conjectures.

1. By [ML1], it is known that \( G_0 \) is isomorphic to the group \( \text{SAut} \mathbb{A}_2^2 \) of symplectic automorphisms of the affine plane \( \mathbb{A}_2^2 \) (as in the case of the Weyl algebra, the isomorphism \( G_0 \cong \text{SAut} \mathbb{A}_2^2 \) is induced by the canonical projection \( \mathbb{C}(x, y) \to \mathbb{C}[x, y] \)). Thus, the groups \( G_n \) can be naturally identified with subgroups of \( \text{Aut} \mathbb{A}_2^2 \). Do these last subgroups have a geometric interpretation?

2. In this paper, we have described the structure of \( G_n \) and \( \text{Aut} D_n \) as discrete groups. However, these two groups carry natural algebraic structures and can be viewed as infinite-dimensional algebraic groups (in the sense of Shafarevich [SH]). Despite being isomorphic to each other as discrete groups, they are not isomorphic as algebraic groups (for \( n = 0 \), this phenomenon was observed in [BW]). A natural question is to explicitly describe the algebraic structures on \( G_n \) and \( \text{Aut} D_n \); in particular, to compute the corresponding (infinite-dimensional) Lie algebras. The last question was an original motivation for our work. For \( G_0 \), the answer is known (see [G]).

3. Compute the homology of the groups \( G_n \) for all \( n \). Again, for \( n = 0 \), the answer is known (see [Al]): \( H_*(G_0, \mathbb{Z}) \cong H_*(\text{SL}_2(\mathbb{C}), \mathbb{Z}) \). One may wonder whether the groups \( H_*(G_n, \mathbb{Z}) \) are strong enough invariants to distinguish the algebras \( D_n \) up to isomorphism. Unfortunately, the answer is ‘no’: in fact, it follows from our description of \( G_1 \) that \( H_*(G_1, \mathbb{Z}) \cong H_*(\text{SL}_2(\mathbb{C}), \mathbb{Z}) \). However, for \( n \geq 2 \), it seems
that the groups $H_*(G_n, \mathbb{Z})$ are neither isomorphic to $H_*(\text{SL}_2(\mathbb{C}), \mathbb{Z})$ nor to each other, so they may provide interesting invariants.

4. Finally, we would like to propose an extension of the well-known Dixmier Conjecture for $A_1$ (see [D], Problème 11.1) to the class of Morita equivalent algebras. We recall that if $D$ is a domain Morita equivalent to $A_1$, then there is a unique integer $n \geq 0$ such that $D \cong D_n$, where $D_n$ is the endomorphism ring of the right ideal $M_n = x^n A_1 + (y + nx^{-1}) A_1$. For two unital $\mathbb{C}$-algebras $A$ and $B$, we denote by $\text{Hom}(A, B)$ the set of all unital $\mathbb{C}$-algebra homomorphisms $A \to B$.

**Conjecture 1.** For all $n, m \geq 0$, we have

$$\text{Hom}(D_n, D_m) = \begin{cases} \emptyset & \text{if } n \neq m \\ \text{Aut} D_n & \text{if } n = m \end{cases}$$

Formally, Conjecture 1 is a strengthening of the Dixmier Conjecture for $A_1$: in fact, in our notation, the latter says that $\text{Hom}(D_0, D_0) = \text{Aut} D_0$. Does actually the Dixmier Conjecture imply Conjecture 1?

**Acknowledgments.** We are grateful to J. Alev, V. Bavula, O. Chalykh, K. Vogtmann, D. Wright, G. Wilson and E. Zelmanov for interesting discussions, questions and comments. We would also like to thank D. Wright for providing us with reference [Wr], which turned out to be very useful, and G. Wilson for sending us a copy of Quillen’s private notes on trees and amalgams. This work was partially supported by NSF grant DMS 09-01570.

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Figure 1. Maximal Tree $T_3$ (above); Graph $\Gamma_3 \setminus T_3$ (below)
Figure 2. Maximal Tree of Groups $T_3$