The Key Equation for One-Point Codes

Michael E. O’Sullivan, Maria Bras-Amorós

July 23, 2008

Abstract

For Reed-Solomon codes, the key equation relates the syndrome polynomial —computed from the parity check matrix and the received vector—to two unknown polynomials, the locator and the evaluator. The roots of the locator polynomial identify the error positions. The evaluator polynomial, along with the derivative of the locator polynomial, gives the error values via the Forney formula. The Berlekamp-Massey algorithm efficiently computes the two unknown polynomials.

This chapter shows how the key equation, the Berlekamp-Massey algorithm, the Forney formula, and another formula for error evaluation due to Horiguchi all generalize in a natural way to one-point codes. The algorithm presented here is based on Kötter’s adaptation of Sakata’s algorithm.

Published as M. E. O’Sullivan, M. Bras-Amorós, The Key Equation for One-Point Codes, Chapter 3 of Advances in Algebraic Geometry Codes, World Scientific, E. Martínez-Moro, C. Munuera, D. Ruano (eds.), vol. 5, pp. 99-152, 2008. ISBN 978-981-279-400-0.

*San Diego State University, mosullivan@sdsu.edu.
†Universitat Rovira i Virgili, maria.bras@urv.cat.
## Contents

1 Introduction 3

2 The key equation for Reed-Solomon codes 4  
2.1 Reed-Solomon codes 4  
2.2 Polynomials for decoding 5  
2.3 The key equation and the Berlekamp-Massey algorithm 7  
2.4 Error evaluation without the evaluator polynomial 9  
2.5 Connections to the Euclidean algorithm 10

3 The key equation for Hermitian codes 12  
3.1 The Hermitian curve 12  
3.2 Hermitian codes 13  
3.3 Polynomials for decoding 14  
3.4 Another basis for $\mathbb{F}_{q^2}(x, y)$ 17  
3.5 The key equation 19  
3.6 Solving the key equation 22  
3.7 Error evaluation without the error evaluator polynomials 25  
3.8 An example 26

4 The key equation for one-point codes 29  
4.1 Curves, function fields and differentials 29  
4.2 One-point codes and their duals 31  
4.3 The trace and a dual basis 32  
4.4 Polynomials for decoding 36  
4.5 The key equation and its solution 38  
4.6 Error evaluation without the error evaluator polynomials 40

5 Bibliographical notes 43
1 Introduction

For Reed-Solomon codes, the key equation relates the syndrome polynomial—computed from the parity check matrix and the received vector—to two unknown polynomials, the locator and the evaluator. The exact formulation of the key equation has evolved since Berlekamp’s introduction of the term [2]. There are also key equations for other algorithms, such as Sugiyama et al. [42], and Berlekamp-Welch [44]. The goal of this chapter is to show that the key equation, the Berlekamp-Massey algorithm and the error evaluation formulas of Forney and Horiguchi [18] all generalize to one-point codes. An important aspect of the generalization is to treat the ideal of error locator polynomials as a module over a polynomial ring in one variable, which is essentially the approach Kötter used in his version of the Berlekamp-Massey-Sakata algorithm [21]. The chapter is divided into three main sections, Reed-Solomon codes, Hermitian codes, and one-point codes. We have attempted to make each section as self-contained as possible, and to minimize the mathematical background required.

The section on Reed-Solomon codes gives a concise treatment of the key-equation, the Berlekamp-Massey algorithm, and the error evaluation formulas in a manner that will generalize easily to one-point codes. Two aspects of our approach are atypical, though certainly not new. First, the locator polynomial vanishes at the error positions—as opposed to the usual definition which uses the reciprocals of the positions—because this is more natural in the context of algebraic geometry codes. Second, the syndrome is a rational polynomial—rather than a polynomial—because this is in accord with the duality of codes on algebraic curves. Theorem 1 gives a very formal statement of the properties satisfied by the intermediate polynomials computed in the Berlekamp-Massey algorithm. Analogous results are established in the later sections for Hermitian and one-point codes. At the end of the section on Reed-Solomon codes we briefly discuss the usual formulation of the key equation—see for example [37, 3]—and the connections with the Euclidean algorithm and the algorithm of Sugiyama et al. There are also interesting connections to the Berlekamp-Welch algorithm and to the list decoding algorithm of Lee and O’Sullivan [24], but these are not developed here.

The section on Hermitian codes requires little if any background in algebraic geometry, and only minimal familiarity with the algebra of polynomial rings and Gröbner bases. The presentation of this section closely parallels that of the section on Reed-Solomon codes, so that overall similarity between the two as well as the new complexities are as clear as possible. The locator polynomial is replaced with the ideal of polynomials vanishing at the error locations, and the problem is to find several locator polynomials of minimal degree, one for each congruence class modulo $q$, where the field size is $q^2$. The syndrome is again a rational polynomial, and the property of a locator is that its product with the syndrome eliminates the denominator, giving a polynomial. The product of the locator and the syndrome also may be used for error evaluation. Kötter’s algorithm is essentially $q$ Berlekamp-Massey algorithms operating in parallel, and the only place in which the algebra of the curve is used is in the computation
of recursions of candidate locator polynomials with the syndrome. The Forney formula and Horiguchi formula for error evaluation are simple, but not obvious, generalizations of those for Reed-Solomon codes.

The section on one-point codes shows that the decoding algorithms and formulas for Hermitian codes need only minor modification to apply to general one-point codes. The focus of this section is not reproving the decoding results in the more general setting; instead, it is to establish the algebraic structure that makes the algorithms work. In particular, we will need to use differentials, residues of differentials, and duality with respect to the residue map. This section does require the theory of curves and algebraic function fields, but we have tried to build the exposition using a small number of key results as a base. The treatment is based on O'Sullivan [32, 33], with, we hope, improvements in exposition.

2 The key equation for Reed-Solomon codes

In this section, we briefly discuss Reed-Solomon codes, set up the decoding problem and introduce the locator and evaluator polynomials. The syndrome is defined as a rational polynomial, but it may also be seen as a power series. We then present the key equation and the Berlekamp-Massey algorithm in a form that we will generalize to codes from algebraic curves. We derive Horiguchi’s formula for error evaluation, which removes the need to compute the error evaluator polynomial. Finally, we explore connections with the Euclidean algorithm.

2.1 Reed-Solomon codes

Let \( \mathbb{F}_q \) be the finite field of \( q \) elements. Given \( n \) different elements \( \alpha_1, \ldots, \alpha_n \) of \( \mathbb{F}_q \), define the map \( \text{ev} : \mathbb{F}_q[x] \to \mathbb{F}_q^n, f \mapsto (f(\alpha_1), \ldots, f(\alpha_n)) \). The generalized Reed-Solomon code \( \text{GRS}(\bar{\alpha}, k) \), where \( \bar{\alpha} = (\alpha_1, \ldots, \alpha_n) \), is defined as the image by \( \text{ev} \) of the polynomials in \( \mathbb{F}_q[x] \) with degree at most \( k-1 \). It has generator matrix

\[
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\alpha_2 & \alpha_2 & \ldots & \alpha_n \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_n^{k-1} & \alpha_n^{k-1} & \ldots & \alpha_n^{k-1}
\end{pmatrix}
\]

It is well known (see for instance [37 §5.1]) that the parity check matrix of \( \text{GRS}(\bar{\alpha}, k) \) is then

\[
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\alpha_2 & \alpha_2 & \ldots & \alpha_n \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_n^{k-1} & \alpha_n^{k-1} & \ldots & \alpha_n^{k-1}
\end{pmatrix}
\begin{pmatrix}
\beta_1 & 0 & \ldots & 0 \\
0 & \beta_2 & \ldots & 0 \\
0 & \ldots & \ddots & \vdots \\
0 & \ldots & 0 & \beta_n
\end{pmatrix}
\]

4
for some $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{F}_q$. That is, $(c_1, c_2, \ldots, c_n)$ is in $GRS(\bar{\alpha}, k)$ if and only if $(c_1\beta_1, c_2\beta_2, \ldots, c_n\beta_n)$ is in $GRS^\perp(\bar{\alpha}, n-k)$.

If the field size is $q$ and $n = q - 1$ then it is said to be a conventional Reed-Solomon code or just Reed-Solomon code and we denote it by $RS(k)$. In this case it can be proven that $\beta_i = \alpha_i$. So the parity check matrix is

$$
\begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_n \\
\alpha_2^2 & \alpha_2 & \cdots & \alpha_n^2 \\
\alpha_3^2 & \alpha_3 & \cdots & \alpha_n^3 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n-k} & \alpha_{n-k} & \cdots & \alpha_{n-k}^n
\end{pmatrix},
$$

2.2 Polynomials for decoding

Suppose that a word $c \in GRS^\perp(\bar{\alpha}, n-k)$ is transmitted and that the vector $u$ is received. The vector $e = u - c$ is the error vector. We assume that $e$ has $t \leq \frac{n-k}{2}$ non-zero positions. We will use $c, u, e$ and $t$ throughout this section. The decoding task is to recover $e$ from $u$ and thereby get $c = u - e$.

We define the error locator polynomial associated to $e$ as

$$f_e = \prod_{j: e_j \neq 0} (x - \alpha_j)$$

and the error evaluator polynomial as

$$\varphi_e = \sum_{j: e_j \neq 0} e_j \prod_{k: c_k \neq 0, k \neq j} (x - \alpha_k).$$

The utility of the error locator polynomial and the error evaluator polynomial is that the error positions can be identified as the indices $j$ such that $f_e(\alpha_j) = 0$ and the error values can be computed by the so-called Forney formula given in the next lemma, whose verification is straightforward.

Lemma 1. If $e_j \neq 0$ then $e_j = \varphi_e(\alpha_j) f_e'(\alpha_j)$.

Another useful fact about $f_e$ and $\varphi_e$ is that from the received vector we know the first coefficients of the power series in $\frac{1}{x}$ obtained when dividing $\varphi_e$ by $f_e$. This is shown in the next lemma.

Lemma 2. $\frac{\varphi_e}{f_e} = \frac{1}{x} \left( s_0 + \frac{s_1}{x} + \frac{s_2}{x^2} + \cdots \right)$, where $s_n = \sum_{j=1}^n e_j \alpha_j^n$. In particular, for $a \leq n-k-1$, $s_a = \sum_{j=1}^n u_j \alpha_j^a$. 

5
Proof.

$$\frac{\varphi e}{f e} = \sum_{j=1}^{n} \frac{e_j}{x - \alpha_j} = \frac{1}{x} \sum_{j=1}^{n} \frac{e_j}{1 - \frac{\alpha_j}{x}} = \frac{1}{x} \sum_{j=1}^{n} e_j \sum_{a=0}^{\infty} \left( \frac{\alpha_j}{x} \right)^a = \frac{1}{x} \sum_{a=0}^{\infty} \frac{1}{x^a} \sum_{j=1}^{n} e_j \alpha_j^a = \frac{1}{x} \sum_{a=0}^{\infty} s_a \frac{x^a}{x^a}$$

Looking at the parity check matrix of $GRS(\bar{\alpha}, n - k)^{\perp}$, it can be deduced that for $a \leq n - k - 1$, $\sum_{j=1}^{n} c_j \alpha_j^a = 0$. Hence, $s_a = \sum_{j=1}^{n} e_j \alpha_j^a = \sum_{j=1}^{n} (u_j - c_j) \alpha_j^a = \sum_{j=1}^{n} u_j \alpha_j^a$.

\[ \square \]

Definition. For a vector $e$, the syndrome of $e$ is $S = \frac{\varphi e}{f e}$. The syndrome of order $a$ is $s_a = \sum_{j=1}^{n} u_j \alpha_j^a$.

Just as any element of the field $\mathbb{F}_q(x)$ may be written as a Laurent series in $x$, any $h \in \mathbb{F}_q(x)$ also may be written as a Laurent series in $1/x$, $h = \sum_{a \leq d} h_a x^a$ for some $d \in \mathbb{Z}$. If $h_d$ is nonzero in this expression, we say the degree of $h$ is $d$, and if $h_d = 1$ we say that $h$ is monic. Notice that $h \in \mathbb{F}_q[x]$ if and only if $h_a = 0$ for all $a < 0$ and that our definition of degree coincides with the usual one on $\mathbb{F}_q[x]$. Henceforth, we will not use the form for $h$ given above. Instead we will write Laurent series in $1/x$ in the form $h = \frac{1}{x} \sum_{a=0}^{d} h_a x^{-a}$. It is understood that the sum is over all integers $a \geq -d - 1$ where $d$ is the degree of $h$. In this form, $h$ is a polynomial when $h_a = 0$ for all $a \geq 0$. As an example, the syndrome is $S = \frac{1}{x} \sum_{a=0}^{\infty} s_a x^{-a}$. Its degree is $-1$, unless $s_0 = 0$.

Lemma 3. Let $f$ be a polynomial and let $\alpha \in \mathbb{F}_q$. If the Laurent series in $\frac{1}{x}$ given by $\frac{f(x)}{x - \alpha}$ has no term of degree $-1$ then $f(\alpha) = 0$.

Proof. There exists $g \in \mathbb{F}_q[x]$ such that $f(x) = f(\alpha) + (x - \alpha)g(x)$. Then

$$\frac{f(x)}{x - \alpha} = \frac{f(\alpha)}{x - \alpha} + g(x) = g(x) + \frac{f(\alpha)}{x} \left( 1 + \frac{\alpha}{x} + \left( \frac{\alpha}{x} \right)^2 + \cdots \right) = g(x) + \frac{f(\alpha)}{x} + \frac{\alpha f(\alpha)}{x^2} + \frac{\alpha^2 f(\alpha)}{x^3} + \cdots$$

If the term of degree $-1$ is zero, then $f(\alpha) = 0$. \[ \square \]
Proposition 1. If \( fS \) has no terms of degrees \(-1, -2, \ldots, -t \) then \( f \) is a multiple of \( f^c \). In particular, if \( fS \) is a polynomial then \( f \) is a multiple of \( f^c \).

Proof. Suppose \( fS \) has no terms of degrees \(-1, -2, \ldots, -t \). Suppose \( e_j \neq 0 \) and let

\[
g(x) = \prod_{k : e_k \neq 0, k \neq j} (x - \alpha_k).
\]

Note that \( \deg g = t - 1 \) and so \( fgS \) has no term of degree \(-1\). Now,

\[
f gS = \sum_{k : e_k \neq 0} \frac{e_k fg}{x - \alpha_k}
\]

\[
= e_j \frac{fg}{x - \alpha_j} + \sum_{k : e_k \neq 0, k \neq j} \frac{e_k f g}{x - \alpha_k}.
\]

Since \( fgS \) has no term of degree \(-1\) and the right term in the previous sum is a polynomial, we deduce that \( \frac{fg}{x - \alpha_j} \) has no term of degree \(-1\). By the previous lemma, \( x - \alpha_j \) must divide \( f \). Since \( j \) was chosen arbitrarily such that \( e_j \neq 0 \), we conclude that \( f^c \) must divide \( f \). \( \square \)

2.3 The key equation and the Berlekamp-Massey algorithm

We now present the version of the Berlekamp-Massey algorithm that will be our model for generalization to codes from algebraic curves. The Berlekamp-Massey algorithm finds the minimal solution to the key equation.

Definition. We will say that polynomials \( f, \varphi \) satisfy the key equation for syndrome \( S \) when \( fS = \varphi \).

The Berlekamp-Massey Algorithm

Initialize: \[
\begin{pmatrix}
  f^{(0)} \\
  g^{(0)} \\
  \varphi^{(0)} \\
  \psi^{(0)}
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Algorithm: For \( m = 0 \) to \( n - k - 1 \),

\[
d = \deg f^{(m)}
\]

\[
\mu = \sum_{a=0}^{d} f^{(m)}_a s_{a + (m - d)}
\]

\[
p = 2d - m - 1
\]

\[
U^{(m)} = \begin{cases}
  \begin{pmatrix} 1 & -\mu x^p \\ 0 & 1 \end{pmatrix} & \text{if } \mu = 0 \text{ or } p \geq 0 \\
  \begin{pmatrix} x^{-p} & -\mu \\ 1/\mu & 0 \end{pmatrix} & \text{otherwise.}
\end{cases}
\]

7
\[
\begin{pmatrix}
  f^{(m+1)} \\
g^{(m+1)}
\end{pmatrix}
\begin{pmatrix}
  \varphi^{(m+1)} \\
  \psi^{(m+1)}
\end{pmatrix}
= U^{(m)}
\begin{pmatrix}
  f^{(m)} \\
g^{(m)}
\end{pmatrix}
\begin{pmatrix}
  \varphi^{(m)} \\
  \psi^{(m)}
\end{pmatrix}
\]

**Output:** \( f^{(n-k)}, \varphi^{(n-k)} \).

Notice that this algorithm uses only the syndromes of order up to \( n-k-1 \) and these are exactly the syndromes that can be computed from the received vector. We may think of \( f^{(m)}, \varphi^{(m)} \) and also \( g^{(m)}, \psi^{(m)} \) as approximate solutions of the key equation. The algorithm takes a linear combination of two approximate solutions to create a better approximation.

**Theorem 1.** For all \( m \geq 0 \),

1. \( f^{(m)} \) is monic of degree at most \( m \).
2. \( \deg (f^{(m)} S - \varphi^{(m)}) \leq -m + \deg f^{(m)} - 1 \). In particular, \( f^{(m)} S \) has no terms in degrees \(-1, -2, \ldots, -m + \deg f^{(m)} \).
3. \( g^{(m)} S - \psi^{(m)} \) is monic of degree \(-\deg f^{(m)}\).
4. \( \deg (g^{(m)}) \leq m - \deg f^{(m)} \).

**Proof.** We will proceed by induction on \( m \). It is easy to verify the case \( m = 0 \). Assume the statements are satisfied at step \( m \). Let \( d = \deg f^{(m)} \). Notice that \( d \leq m \) by item (1), and \( \mu \) is the coefficient of \( x^{d-m-1} \) in \( f^{(m)} S \). Furthermore, since \( d - m - 1 < 0 \), and \( \varphi^{(m)} \) is a polynomial, \( \mu \) is the coefficient of \( x^{d-m-1} \) in \( f^{(m)} S - \varphi^{(m)} \).

If \( \mu = 0 \), then the algorithm retains the polynomials from the \( m \)th iteration, e.g., \( f^{(m+1)} = f^{(m)} \). The induction hypothesis immediately gives items (1), (3), and (4) of the theorem, and item (2) follows from \( \mu = 0 \).

Consider the case when \( p = 2d - m - 1 \geq 0 \) and \( \mu \neq 0 \). The algorithm sets \( f^{(m+1)} = f^{(m)} - \mu x^d g^{(m)} \). By the induction hypothesis,

\[
\deg (x^p g^{(m)}) \leq 2d - m - 1 + m - d = d - 1,
\]
so \( \deg (f^{(m+1)}) = \deg (f^{(m)}) = d < m \) and \( f^{(m+1)} \) is monic, so item (1) holds.

Now,

\[
f^{(m+1)} S - \varphi^{(m+1)} = (f^{(m)} S - \varphi^{(m)}) - \mu x^p (g^{(m)} S - \psi^{(m)}).
\]

The degree of each term is \( d - m - 1 \) and the coefficients of \( x^{d-m-1} \) cancel. Thus \( \deg(f^{(m+1)} S - \varphi^{(m+1)}) \leq -(m + 1) + \deg(f^{(m+1)}) - 1 \), as required. This proves item (2). Items (3) and (4) are trivial in this case, since \( g^{(m+1)} = g^{(m)} \) and \( \varphi^{(m+1)} = \varphi^{(m)} \).

Finally, consider the case when \( p = 2d - m - 1 < 0 \), in which \( f^{(m+1)} = x^{-p} f^{(m)} - \mu g^{(m)} \). By computing the degrees of each summand, one can see that \( f^{(m+1)} \) is monic of degree \( m + 1 - d \leq m + 1 \) as claimed in item (1). We have

\[
f^{(m+1)} S - \varphi^{(m+1)} = x^{-p} (f^{(m)} S - \varphi^{(m)}) - \mu (g^{(m)} S - \psi^{(m)}).
\]

The degree of each term is \(-d\) and the coefficients cancel. Thus \( \deg(f^{(m+1)} S - \varphi^{(m+1)}) < -d \). We can see that item (2) holds since \(-(m+1)+\deg(f^{(m+1)})-1 =
\]
In particular, when $\phi$ holds since $(m + 1) - \deg(f(m)) = \deg(g(m))$. Item 3 holds since $g(m)S - \psi(m) = \mu^{-1}(f(m)S - \phi(m))$, which has degree exactly $d - m - 1 = -\deg(f(m))$ and it is monic. □

The next few results show that the algorithm produces the minimal solution to the key equation, $f^e$ and $f^e S$.

**Lemma 4.** For all $m$, $\deg f(m) \leq t$.

*Proof.* Consider $f^e g(m)S - f^e \psi(m)$. This is a polynomial since $f^e S, g(m), \psi(m)$ and $f^e$ are. Since the degree of $f^e$ is $t$, we have $\deg(f^e g(m)S - f^e \psi(m)) = t - \deg f(m)$, using item 3 in Theorem 1. Thus $t - \deg f(m) \geq 0$. □

**Lemma 5.** When $m \geq 2t$, $f^e(m) = f^e$ and $\varphi(m) = \varphi^e$.

*Proof.* Theorem 1 tells us that $f^e(m)S$ has no terms of degree $-1, \ldots, -m + \deg(f(m))$. From the previous lemma, if $m \geq 2t$ then $-m + \deg(f(m)) \leq -2t + t = -t$. Thus, $f^e(m)S$ has no terms of degree $-1, \ldots, -t$. By Proposition 1 $f^e(m)$ must be a multiple of $f^e$; by Theorem 1 it is monic; and, by the preceding lemma, its degree is at most $t$. Thus, it must be equal to $f^e$.

On the other hand, $\deg f^e S - \varphi(m) \leq -m + t - 1 \leq -t - 1 < 0$. Since both $f^e S$ and $\varphi(m)$ are polynomials, this means $\varphi(m) = f^e S = \varphi^e$. □

**Proposition 2.** If $t \leq \frac{d - 1}{2}$ then the previous algorithm outputs $f^e$ and $\varphi^e$.

*Proof.* If $t \leq \frac{d - 1}{2}$ then $n - k \geq d - 1 \geq 2t$ and the result follows from Lemma 5. □

### 2.4 Error evaluation without the evaluator polynomial

We now derive a formula for error evaluation that does not use the error evaluator polynomial, and thereby removes the need for computing it. It is called the Horiguchi-Kötter algorithm in [3] and appears in [13] [20].

From the algorithm it is clear that

\[
\begin{pmatrix}
 f(m) \\
g(m) \\
\varphi(m) \\
\psi(m)
\end{pmatrix} = U^{(m-1)}U^{(m-2)}\ldots U^{(1)}U^{(0)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Taking determinants, since each $U^{(m)}$ has determinant 1, we get

\[
f(m)\varphi(m) - g(m)\varphi^e = -1. \quad (1)
\]

In particular, when $m \geq 2t$,

\[
f^e \psi(m) - g^e \varphi = -1.
\]

Let $j$ be such that $e_j \neq 0$. Evaluating at $\alpha_j$ we get $g^e(\alpha_j)\varphi^e(\alpha_j) = 1$ and so $\varphi^e(\alpha_j) = (g^e(\alpha_j))^{-1}$. Using Lemma 1 we can establish the following proposition.
Proposition 3. For $m \geq 2t$ and $g^{(m)}$ as in the Berlekamp-Massey algorithm, if $e_j \neq 0$ then

$$e_j = (f^e(\alpha_j)g^{(m)}(\alpha_j))^{-1}.$$  

The last proposition tells us that in the Berlekamp-Massey algorithm we do not need to multiply $U^{(m)}$ by all the matrix

$$
\begin{pmatrix}
  f^{(m)} & \varphi^{(m)} \\
  g^{(m)} & \psi^{(m)}
\end{pmatrix}
$$

but by the vector

$$
\begin{pmatrix}
  f^{(m)} \\
  g^{(m)}
\end{pmatrix}.
$$

Then the initialization step will be

$$
\begin{pmatrix}
  f^{(0)} \\
  g^{(0)}
\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
$$

and the updating step will be

$$
\begin{pmatrix}
  f^{(m+1)} \\
  g^{(m+1)}
\end{pmatrix} = U^{(m)} \begin{pmatrix}
  f^{(m)} \\
  g^{(m)}
\end{pmatrix}.
$$

2.5 Connections to the Euclidean algorithm

Suppose that all the $\alpha_i$ defining the GRS code are $n$th roots of unity. In particular, we could demand that none of the $\alpha_i$ are zero and take $n = q - 1$. From the definition of $S$ it is easy to see that $s_a = s_{n+a}$ for all $a \geq 0$, and consequently,

$$S(x^n - 1) = s_0 x^{n-1} + s_1 x^{n-2} + \cdots + s_{n-2} x + s_{n-1}.$$  

Call this polynomial $\mathfrak{S}$. We might alter our definition of the key equation to say that $f$, and $\varphi$ are solutions when $\mathfrak{S} = \varphi(x^n - 1)$. That is

$$f(s_0 x^{n-1} + s_1 x^{n-2} + \cdots + s_{n-2} x + s_{n-1}) = \varphi(x^n - 1). \tag{2}$$

Of course, the solution set is the same as for our original equation, and $f^e$, $\varphi^e$ are the minimal degree solutions such that $f^e$ is monic. The result analogous to Theorem $\|$ states that $\deg(f^{(m)}S - \varphi^{(m)}) \leq n - 1 - m + \deg f^{(m)}$ and $g^{(m)}S - \psi^{(m)}$ is monic of degree $n - \deg f^{(m)}$. When the weight of $e$ is $t$, $f^{(2t)} = f^e$ and $\varphi^{(2t)} = \varphi^e$ give the least common multiple of $\mathfrak{S}$ and $x^n - 1$; the lcm is $f^e \mathfrak{S} = \varphi^e(x^n - 1)$.

For a linear algebra perspective, write $f = f_0 + f_1 x + \cdots + f_t x^t$. The key equation requires that $\sum_{i=0}^t f_i s_{i+a} = 0$ for all $0 \leq a \leq n - t - 1$. Setting $f_t = 1$, there are $t$ unknowns, $f_0, \ldots, f_{t-1}$, so the $t$ equations where $a = 0, \ldots, t - 1$, are enough to determine the coefficients of $f$. Thus we need to know the syndromes $s_0$ to $s_{2t-1}$ to compute $f^e$. This verifies that the Berlekamp-Massey algorithm used with a code of redundancy $2t$ can correct $t$ errors.
Equation (2) leads to a relationship between the Berlekamp-Massey algorithm and the Euclidean algorithm. Let \( m_0, m_1, \ldots, m_r \) be the iterations of the algorithm in which \( p^{(m)} < 0 \) and let \( m_{t+1} = 2t \) where \( t \) is the weight of \( e \). One can check that \( p^{(m)} < -p^{(m_e)} \) for all \( m_e < m \leq m_{e+1} \). For each \( \ell = 0, \ldots, r \), let
\[
V^{(\ell)} = U^{(m_{\ell+1})} \ldots U^{(m_{r+1})} U^{(m_r)}.
\]
Then
\[
V^{(\ell)} = \left( \frac{q_\ell}{(\mu^{(m_e)})^{-1}} - \mu^{(m_\ell)} \right)
\]
where \( q_\ell \) is a monic polynomial of degree \( p^{(m_\ell)} \). Define recursively,
\[
\begin{align*}
A_0 &= \left( \frac{\mathcal{S}}{x^n - 1} \right) \\
B_0 &= (x^n - 1) \\
A_{t+1} &= V^{(\ell)} (A_\ell) \\
B_{t+1} &= V^{(\ell)} (B_\ell)
\end{align*}
\]
so that \( A_{t+1} = -\mu^{(m_\ell)} B_\ell + q_\ell A_\ell \) and \( B_{t+1} = (\mu^{(m_\ell)})^{-1} A_\ell \). Rearranging, we get
\[
B_{t+1} = -\frac{\mu^{(m_\ell-1)}}{\mu^{(m_\ell)}} \left( B_{t-1} - q_{t-1} B_\ell \right).
\]
This is a variant of the classical Euclidean algorithm for computing the greatest common divisor with the modification that the remainders are all monic. We will sketch the main points and leave verification of the details to the reader.

Notice that \( A_\ell = f^{(m_{\ell-1})}\mathcal{S} - \varphi^{(m_{\ell-1})}(x^n - 1) \) and similarly \( B_\ell = g^{(m_{\ell-1})}\mathcal{S} - \varphi^{(m_{\ell-1})}(x^n - 1) \). From the discussion after (2), \( \deg B_\ell = n - \deg f^{(m_{\ell-1})} \).

Referring to the Berlekamp-Massey algorithm, \( \deg f^{(m_{\ell+1})} = \deg f^{(m_\ell)} < \deg f^{(m_{\ell-1})} \) so we have \( \deg B_{\ell+1} < \deg B_\ell \) and the sequence of \( B_\ell \) does indeed satisfy the requirements of the Euclidean algorithm with monic quotients.

At the final iteration, \( m_{r+1} = 2t \), \( f^{(m_{r+1})} = f^e \) and \( \varphi^{(m_{r+1})} = \varphi^e \) so that \( A_{r+1} = f^e \mathcal{S} - \varphi^e(x^n - 1) = 0 \). As noted earlier, \( f^e \mathcal{S} \) is a constant multiple of the lcm of \( \mathcal{S} \) and \( x^n - 1 \). We also have \( B_{r+1} = g^{(2t)} \mathcal{S} - \varphi^{(2t)}(x^n - 1) \) is the monic greatest common divisor of \( \mathcal{S} \) and \( (x^n - 1) \), namely \( \prod_{\ell \in \sigma} (x - \alpha_\ell) \).

Thus we see that the Berlekamp-Massey algorithm breaks each division of this version of the Euclidean algorithm into several steps, one for each subtraction of a monomial multiple of the divisor. The Berlekamp-Massey algorithm is also more efficient than the Euclidean algorithm, because it never computes the \( B_\ell \). It takes advantage of the fact that \( B_0 = x^n - 1 \) is very sparse, and just computes the critical coefficients \( \mu^{(m)} \) via the polynomials \( f^{(m)} \) and \( \mathcal{S} \).

Berlekamp’s formulation of the key equation was different from the one presented here. To obtain his formulation, let
\[
\sigma^e = x^t f^e \left( \frac{1}{x} \right) = \prod_{k: \sigma_k \neq 0} (1 - \alpha_k x)
\]
\[
\omega^e = x^{t-1} \varphi \left( \frac{1}{x} \right) = \sum_{j: \sigma_j \neq 0} e_j \prod_{k: \sigma_k \neq 0, k \neq j} (1 - \alpha_k x).
\]
These polynomials are $\Lambda(x)$ and $\Gamma(x)$ respectively in $[3, 37]$. Then
\[x^{n+t-1} e x \left( \frac{1}{x} \right) = x^{n+t-1} \varphi_x \left( \frac{1}{x} \right) \left( \frac{1}{x} \right)^n - 1 \]
\[\sigma^e \left( s_0 + s_1 x + \cdots + s_{n-1} x^{n-1} \right) = \omega^e \left( 1 - x^n \right) \]
\[\sigma^e \left( s_0 + s_1 x + \cdots + s_{2t-1} x^{2t-1} \right) \equiv \omega^e \mod x^{2t} \]

This is essentially the key equation in $[2, 3, 37]$, modulo minor changes due to different choices of parity check matrix.

The algorithm of Sugiyama et al [42] is based on the equation
\[\sigma^e \left( s_0 + s_1 x + \cdots + s_{2t-1} x^{2t-1} \right) + x^{2t} T = \omega^e \]

One can run the Euclidean algorithm on $R_0 = x^{2t}$ and $R_1 = s_0 + s_1 x + \cdots + s_{2t-1} x^{2t-1}$ until the remainder has degree less than $t$. Sugiyama et al showed that the resulting combination of $(s_0 + s_1 x + \cdots + s_{2t-1} x^{2t-1})$ and $x^{2t}$ obtained is $\omega^e$ and that the coefficient of $(s_0 + s_1 x + \cdots + s_{2t-1} x^{2t-1})$ is $\sigma^e$. The article [42] actually treats the more general situation of Goppa codes and error-erasure decoding.

### 3 The key equation for Hermitian codes

The most widely studied algebraic geometry codes are those from Hermitian curves. One reason for the interest in Hermitian curves is that they are maximal curves, meeting the Weil bound on the number of points for a given genus. They also have a very simple formula, and a great deal of symmetry, which leads to lots of structure that makes them useful in coding. The short articles of Stichtenoth [40] and Tiersma [43], and Stichtenoth’s book [41] are good references for information on Hermitian curves and codes.

In this section we will derive the key equation and the algorithm for solving it in a manner that parallels the section on Reed-Solomon codes. We will not discuss one very important issue: The syndromes computed from the received vector are insufficient for exploiting the full error correction capability of the Berlekamp-Massey-Sakata decoding algorithm. The majority voting algorithm of Feng-Rao [10] and Duursma [7] is required to compute more syndrome values. We will not discuss majority voting. Instead, we simply deal with the problem solved by the BMS algorithm, computing the error locator ideal from the syndrome of the error vector. A detailed treatment of majority voting may be found in the chapter on algebraic geometry codes by Høholdt et al [17]. The conditions ensuring success in the majority voting algorithm are best understood in terms of the “footprint” of the error vector, which is discussed below, and can lead to decoding beyond the minimum distance [4].

#### 3.1 The Hermitian curve

Let $q$ be a prime power. We will use the following equation for the Hermitian curve over $\mathbb{F}_{q^2}$,
\[X^{q+1} = Y^q + Y.\]
For each $\alpha \in \mathbb{F}_q^2$, $\alpha^{q+1}$ is the norm of $\alpha$ with respect to the extension $\mathbb{F}_q^2/\mathbb{F}_q$, so $\alpha^{q+1}$ belongs to $\mathbb{F}_q$. On the other hand, $\beta^q + \beta$ is the trace of $\beta$ with respect to $\mathbb{F}_q^2/\mathbb{F}_q$, so $\beta^q + \beta$ also belongs to $\mathbb{F}_q$. Each element $\gamma \in \mathbb{F}_q$ has $q$ preimages under the trace map, and $\gamma$ has $q + 1$ preimages under the norm map (unless $\gamma = 0$ when there is one). Thus there are $n = q + (q - 1)(q + 1) = q^3$ points on the curve. We label them $P_1 = (\alpha_1, \beta_1), P_2 = (\alpha_2, \beta_2), \ldots, P_n = (\alpha_n, \beta_n)$.

Let $\mathbb{F}_q[x, y]/(x^{q+1} - y - 1) = \mathbb{F}_q[x, y]$, where $x$ is the image of $X$ in the quotient and $y$ is the image of $Y$. Since $y^q = x^{q+1} - y$, each element $f$ in $\mathbb{F}_q[x, y]$ can be expressed in a unique way as a sum $f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots + f_{q-1}(x)y^{q-1}$. That is, $\{1, y, \ldots, y^{q-1}\}$ is a basis of $\mathbb{F}_q[x, y]$ as an $\mathbb{F}_q[x]$-module. Also, $\mathcal{M} = \{x^ay^b : 0 \leq a, 0 \leq b < q\}$ is a basis of $\mathbb{F}_q[x, y]$ as a $\mathbb{F}_q$-vector space.

We wish to introduce a function on $\mathbb{F}_q[x, y]$ akin to the degree function on $\mathbb{F}_q[x]$. Notice that any weighted degree in $\mathbb{F}_q[x, y]$ such that $X^{q+1}$ and $Y^q - Y$ have equal weights is obtained by assigning to $X$ a weight $kq$ and to $Y$ a weight $k(q + 1)$ for some non-negative integer $k$. Letting $k = 1$, we define the order function $\rho$ by $\rho(x^ay^b) = \deg_{q(q+1)}(X^aY^b) = aq + b(q + 1)$ and for $f = \sum_{a, b > 0} f_{a, b}x^ay^b$ we define $\rho(f) = \max_{f_{a, b} \neq 0} \rho(x^ay^b)$.

One can see that $x^ay^b$ and $x^{a'}y^{b'}$ in $\mathcal{M}$ satisfy $\rho(x^ay^b) = \rho(x^{a'}y^{b'})$ if and only if $a = a'$ and $b = b'$ and that $\rho(\mathbb{F}_q[x, y]) = \rho(\mathcal{M})$. Define $\Lambda = \rho(\mathbb{F}_q[x, y]) = q\mathbb{N}_0 + (q + 1)\mathbb{N}_0$. The map $\rho : \mathbb{F}_q[x, y] \to \Lambda$ satisfies $\rho(fg) = \rho(f) + \rho(g)$. This suggests extending it to the quotient field of $\mathbb{F}_q[x, y]$, which we will write $\mathbb{F}_q(x, y)$, by defining $\rho(f/g) = \rho(f) - \rho(g)$. Now the image of $\rho$ is all of $\mathbb{Z}$.

### 3.2 Hermitian codes

We define the evaluation map

$$\text{ev} : \mathbb{F}_q[x, y] \to \mathbb{F}_q^n$$

$$f \mapsto (f(\alpha_1, \beta_1), f(\alpha_2, \beta_2), \ldots, f(\alpha_n, \beta_n)).$$

The Hermitian code $H(m)$ over $\mathbb{F}_q^2$ is the linear code generated by $(\{f(P_1), \ldots, f(P_n)\} : f \in \mathcal{M}, \rho(f) \leq m)$. It is shown in [10] (see also [19]) that $H(m) = \mathbb{F}_q^m$ when $m \geq q^3 + q^2 - q - 1$ and that for $m < q^3 + q^2 - q - 1$ the dual of $H(m)$ is $H(q^3 + q^2 - q - 2 - m)$. Clearly, the monomials $\ell x^ay^b$ such that $0 \leq b < q$ and $aq + b(q + 1) \leq m$ are a basis for the space $\{f \in \mathbb{F}_q^2 : \rho(f) \leq m\}$, so they may be used to create a generating matrix for $H(m)$. Since $x^ay^b - x$ vanishes on all points $P_\ell$, we should not use monomials $x^ay^b$ with $a \geq q^2$ in the generating matrix. This is only an issue when $m \geq q^3$. Thus for $m \in \Lambda$ and $m = aq + b(q + 1)$, with $b < q$, a generator matrix of $H(m)$ is obtained by evaluating monomials $x^ay^b$
whose weighted degree is at most $m$ and such that $a' < q^2$.

\[
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\beta_1 & \beta_2 & \ldots & \beta_n \\
\alpha_1^2 & \alpha_2^2 & \ldots & \alpha_n^2 \\
\alpha_1 \beta_1 & \alpha_2 \beta_2 & \ldots & \alpha_n \beta_n \\
\beta_1^2 & \beta_2^2 & \ldots & \beta_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^a \beta_1^b & \alpha_2^a \beta_2^b & \ldots & \alpha_n^a \beta_n^b
\end{pmatrix}
\]

### 3.3 Polynomials for decoding

Suppose that a word $c \in H(m)^\perp$ was transmitted and that a vector $u$ is received. The vector $e = u - c$ is the error vector. Let $t$ be the weight of $e$. Define the error locator ideal of $e$ as

\[I_e = \{ f \in F_{q^2}[x,y] : f(\alpha_k, \beta_k) = 0 \text{ for all } k \text{ with } e_k \neq 0 \}\]

and the syndrome for $e$ as

\[S = \sum_{k=1}^{n} e_k \frac{x^{q+1} - \alpha_k^{q+1}}{(x - \alpha_k)(y - \beta_k)} = \sum_{k=1}^{n} e_k \frac{y^q + y - \beta_k^q - \beta_k}{(x - \alpha_k)(y - \beta_k)}. \tag{3}\]

Notice that the order of each term in the summand is $q^2 - q - 1$.

We will give three justifications for this definition of the syndrome in the lemmas below. We note first that for any $(\alpha, \beta) \in F_{q^2}^2$ on the Hermitian curve,

\[
\frac{x^{q+1} - \alpha^{q+1}}{x - \alpha} = \alpha^q \left( \frac{x}{\alpha} \right)^{q+1} - 1 = \alpha^q \left( \frac{x}{\alpha} \right)^{q+1} + \frac{x}{\alpha} + 1 \]

and

\[
\frac{y^q + y - \beta^q - \beta}{y - \beta} = 1 + \frac{y^q - \beta^q}{y - \beta} = 1 + y^{q-1} + \beta y^{q-2} + \ldots + \beta^{q-2} y + \beta^{q-1}
\]

We will use these identities several times during this presentation.

The first lemma gives a nice relationship between $I_e$ and $S$. We will show later that the converse also holds.

**Lemma 6.** If $f \in I_e$ then $fS \in F_{q^2}[x, y]$. 

Proof. If $f \in I^c$ and $P_k, \ldots, P_t$ are the error positions then there exist $g_k, \ldots, g_t$ and $h_k, \ldots, h_t$ in $\mathbb{F}_{q^2}[x, y]$ such that

\[
f = g_k(x - \alpha_k) + h_k(y - \beta_k) = g_k(x - \alpha_k) + h_k(y - \beta_k) \\
\vdots
\]

\[
= g_t(x - \alpha_t) + h_t(y - \beta_t)
\]

Hence

\[
fS = \sum_{k_i; e_{k_i} \neq 0} e_{k_i} \left( g_{k_i} \frac{y^q + y - \beta_{k_i}^q - \beta_{k_i}}{y - \beta_{k_i}} + h_{k_i} \frac{x^{q+1} - \alpha_{k_i}^{q+1}}{x - \alpha_{k_i}} \right)
\]

\[
= \sum_{k_i; e_{k_i} \neq 0} e_{k_i} g_{k_i} \left( 1 + y^{q-1} + \beta_{k_i} y^{q-2} + \cdots + \beta_{k_i}^{q-2} y + \beta_{k_i}^{q-1} \right)
\]

\[
+ \sum_{k_i; e_{k_i} \neq 0} e_{k_i} h_{k_i} \left( x^q + \alpha_{k_i} x^{q-1} + \cdots + \alpha_{k_i}^{q-1} x + \alpha_{k_i}^{q} \right)
\]

which belongs to $\mathbb{F}_{q^2}[x, y]$. □

The next lemma shows that for $f \in I^c$, the product $fS$ may be used for error evaluation. We will need the derivative of $y$ with respect to $x$. Since $q = 0$ in $\mathbb{F}_{q^2}$, and $d(y^q + y)/dx = d(x^{q+1})/dx$, we deduce that $dy/dx = x^q$. We say that $f$ has a simple zero at a point $P$ when $f(P) = 0$ but $f'(P) \neq 0$.

**Lemma 7.** If $f \in I^c$ and $P_k$ is an error position then $e_k f'(P_k) = fS(P_k)$. If $f$ has a simple zero at $P_k$ then

\[
e_k = \frac{fS(P_k)}{f'(P_k)}.
\]

Proof. The rational function

\[
\frac{x^{q+1} - \alpha_{k_i}^{q+1}}{(x - \alpha_{j})(y - \beta_{j})} = \frac{y^q + y - \beta_{j}^q - \beta_{j}}{(x - \alpha_{j})(y - \beta_{j})}
\]

gives a well defined value at any point different from $(\alpha_j, \beta_j)$, so when $j \neq k$,

\[
f \left( \frac{x^{q+1} - \alpha_{k_i}^{q+1}}{(x - \alpha_{j})(y - \beta_{j})} \right) (P_k) = 0.
\]

Consequently,

\[
fS(P_k) = e_k \left( f \left( \frac{x^{q+1} - \alpha_{k_i}^{q+1}}{(x - \alpha_{k_i})(y - \beta_{k_i})} \right) \right) (P_k).
\]

Since $f(P_k) = 0$, there are $g, h \in \mathbb{F}_{q^2}[x, y]$ such that $f = (x - \alpha_k) g + (y - \beta_k) h$.
When $f$ has a simple zero at $P_k$,

$$e_k = fS(P_k) / f'(P_k).$$

As our final justification for our definition of $S$, we show that the syndrome values for the vector $e$, that is the products ev($x^ay^b$) · $e$, appear as coefficients in a particular expansion of $S$.

**Lemma 8.** Let $s_{a,b} = \sum_{k=1}^{n} c_k \alpha_k^a \beta_k^b$ and let $\delta_b$ be 1 when $b = 0$ and 0 otherwise.

$$S = \frac{1}{x} \sum_{b=0}^{q-1} \sum_{a=0}^{\infty} s_{a,b} x^{-a}(y^{q-1-b} + \delta_b)$$

**Proof.**

$$\frac{y^q + y - \beta_k^q - \beta_k}{(x - \alpha_k)(y - \beta_k)} = \left(1 + \frac{y^q - \beta_k^q}{y - \beta_k}\right) \frac{1}{x} \left(\frac{1}{1 - \alpha_k x}\right)$$

$$= (1 + y^{q-1} + \beta_k y^{q-2} + \ldots + \beta_k^{q-1}) \left(\frac{1}{x} + \frac{\alpha_k}{x^2} + \frac{\alpha_k^2}{x^3} + \ldots\right)$$

$$= \sum_{0 \leq a \leq b < q} \alpha_k^a \beta_k^b x^{-a-1}(y^{q-1-b} + \delta_b).$$

Hence,

$$S = \sum_{k=1}^{n} \sum_{0 \leq a \leq b < q} c_k \alpha_k^a \beta_k^b x^{-a-1}(y^{q-1-b} + \delta_b)$$

$$= \sum_{0 \leq a \leq b < q} \sum_{k=1}^{n} c_k \alpha_k^a \beta_k^b x^{-a-1}(y^{q-1-b} + \delta_b)$$

$$= \frac{1}{x} \sum_{0 \leq a \leq b < q} s_{a,b} x^{-a}(y^{q-1-b} + \delta_b)$$
3.4 Another basis for $\mathbb{F}_{q^2}(x, y)$

The final result of the previous section suggests that we introduce a new basis for $\mathbb{F}_{q^2}(x, y)$ over $\mathbb{F}_{q^2}(x)$. For $0 \leq b < q$, let

$$z_b^* = \begin{cases} y^{q-1} + 1 & \text{if } b = 0 \\ y^{q-1-b} & \text{otherwise.} \end{cases}$$ \hfill (4)

Notice that $\rho(z_b^*) = (q + 1)(q - 1 - b) = q^2 - 1 - b(q + 1)$. We will call $\{z_b^* : b = 0, \ldots, q - 1\}$ the $*$-basis. We will write the syndrome, and products of the syndrome with polynomials in $x, y$, using the $*$-basis. An element $f \in \mathbb{F}_{q^2}[x, y]$ is monic in the $*$-basis when its leading term, say $f_{a,b}x^ay^b$, has $f_{a,b} = 1$. The following two lemmas show how this basis is useful for decoding.

**Lemma 9.** The coefficient of $z_c^*$ in $y^bz_c^*$ is 1 if $b = c$ and is 0 otherwise.

*Proof.* One can prove by a straightforward computation that for $0 \leq b, c < q$,

$$y^bz_c^* = \begin{cases} z_0^* & \text{if } b = c = 0 \\ z_0^* - z_{q-1}^* & \text{if } b = c \neq 0 \\ x^qz_{q-b}^* & \text{if } b > c = 0 \\ x^qz_{q-c-b}^* - z_{q-1-c-b}^* & \text{if } b > c > 0 \\ z_{c-b}^* & \text{if } c > b \end{cases}$$

For example, if $b > c > 0$ then

$$y^bz_c^* = y^{q-1+b-c}$$

$$= y^{b-c-1}(x^q + 1 - y^b)$$

$$= x^{q+1}z_{q+c-b}^* - z_{q-1+c-b}^*$$

Notice that $2 \leq q + c - b \leq q - 1$, so that each of the indices in this case is between 1 and $q - 1$. Thus the coefficient of $z_0^*$ in $y^bz_c^*$ is 0 when $0 \leq b \leq c$. Similar arguments apply to the other cases. \hfill $\square$

Any element of $\mathbb{F}_{q^2}(x, y)$ may be expressed uniquely as $\sum_{b=0}^{q-1} h_bz_b^*$ for $h_b \in \mathbb{F}_{q^2}(x)$. We will write $h_b$ in the form used for the syndrome in Section 2.1

$$h_b = \frac{1}{x} \sum_a h_{a,b}x^{-a},$$

where it is understood that $a$ varies over all integers larger than some unspecified bound. For example, we will write the syndrome as

$$S = \frac{1}{x} \sum_{b=0}^{q-1} \sum_a s_{a,b}x^{-a}z_b^*,$$

where it is understood that $s_{a,b} = 0$ for $a < 0$.

**Lemma 10.** Let $f \in \mathbb{F}_q[x, y]$, let $a \in \mathbb{Z}$ and let $b$ satisfy $0 \leq b < q$. The coefficient of $x^{-a-1}z_b^*$ in $fS$ equals the coefficient of $z_0^*/x$ in $x^ay^bfS$. 

17
More precisely, expand \( \tilde{f} = y^b f \), \( S \), and \( fS \) as follows.

\[
\tilde{f} = \sum_{c=0}^{q-1} \tilde{f}_c y^c = \sum_{c=0}^{q-1} \sum_{a} \tilde{f}_{a,c} x^a y^c
\]

\[
S = \frac{1}{x} \sum_{c=0}^{q-1} s_c z_c^* = \frac{1}{x} \sum_{c=0}^{q-1} \sum_{a} s_{a,c} x^{-a} z_c^*
\]

\[
fS = \frac{1}{x} \sum_{c=0}^{q-1} t_c z_c^* = \frac{1}{x} \sum_{c=0}^{q-1} \sum_{a} t_{a,c} x^{-a} z_c^*
\]

Here \( s_c = \sum_{a} s_{a,c} x^{-a} \) and similar definitions hold for \( t_c \) and \( \tilde{f}_c \). Then

\[
t_b = \sum_{c=0}^{q-1} \tilde{f}_c s_c \quad \text{and} \quad t_{a,b} = \sum_{c=0}^{q-1} \sum_{i} \tilde{f}_{i,c} s_{i+a,c}.
\]

**Proof.** From the previous lemma, the coefficient of \( z_0^* \) in

\[
y^b(fS) = \frac{1}{x} \sum_{c=0}^{q-1} t_c y^b z_c^*
\]

is \((1/x)t_b\). On the other hand, \( y^b f = \tilde{f} \), so

\[
(y^b f)S = \left( \sum_{c=0}^{q-1} \tilde{f}_c y^c \right) \left( \frac{1}{x} \sum_{d=0}^{q-1} s_d z_d^* \right)
= \frac{1}{x} \sum_{c=0}^{q-1} \sum_{d=0}^{q-1} \tilde{f}_c s_d z_d^*.
\]

Applying the previous lemma, the coefficient of \( z_0^* \) is \((1/x) \sum_{c=0}^{q-1} \tilde{f}_c s_c\). We conclude that \( t_b = \sum_{c=0}^{q-1} \tilde{f}_c s_c\). Writing \( \tilde{f}_c = \sum_{i} \tilde{f}_{i,c} x^i \) and \( s_c = \sum_{j} s_{j,c} x^{-j} \) we have

\[
\tilde{f}_c s_c = \sum_{i} \tilde{f}_{i,c} x^i \sum_{j} s_{j,c} x^{-j}
= \sum_{a} x^{-a} \sum_{i} \tilde{f}_{i,c} s_{i+a,c}.
\]

This sum is finite since \( \tilde{f}_c \) has finite support, and it gives the formula for \( t_{a,b} \). \( \square \)

This lemma tells us that to identify the coefficient of \( x^{-a-1} z_b^* \) in \( fS \), we write \( \tilde{f} = y^b f \) in the standard basis and then compute the recursion \( t_{a,b} = \sum_{c=0}^{q-1} \sum_{i} \tilde{f}_{i,c} s_{i+a,c} \).
3.5 The key equation

We now define the key equation and approximate solutions to the key equation. We establish some simple lemmas that show basic properties of approximate solutions and how two approximate solutions can be combined to get a better approximation.

Definition. We say that \( f, \varphi \in \mathbb{F}_{q^2}[x, y] \) solve the key equation for syndrome \( S \) when \( fS = \varphi \).

For a nonzero \( f \in \mathbb{F}_{q^2}[x, y] \), writing \( fS = \sum_{a=0}^{q-1} t_{a,b} x^{-a} z_b^* \), we see that \( f, \varphi \) satisfy the key equation when \( t_{a,b} = 0 \) for \( a \geq 0 \) and \( \varphi = \sum_{b=0}^{q-1} t_{a,b} x^{-a} z_b^* \).

Definition. We say that \( f \) and \( \varphi \) in \( \mathbb{F}_{q^2}[x, y] \), with \( f \) nonzero, solve the \( K \)th approximation of the key equation for syndrome \( S \) (or the \( K \)th key equation, for short) when the following two conditions hold.

1. \( \rho(fS - \varphi) \leq q^2 - q - 1 - K \),
2. \( \varphi \), written in the \(*\)-basis, is a sum of terms whose order is at least \( q^2 - q - K \).

We will also say that \( 0 \) and \( x^{-a-1} z_b^* \), for \( a < 0 \), solve the \( K \)th key equation.

Notice that \( \rho(x^{-a-1} z_b^*) = q^2 - q - 1 - (aq + b(q + 1)) \), and when \( a < 0 \), we have \( x^{-a-1} z_b^* \in \mathbb{F}_{q^2}[x, y] \). Thus, for \( 0, x^{-a-1} z_b^* \), condition (1) holds with \( K = aq + b(q + 1) \), but condition (2) is not satisfied. It is convenient to make this pair a solution to the key equation, so we have included the special case in the definition.

For \( f \neq 0 \), (1) means that \( fS - \varphi \) has only terms \( x^{-a-1} z_b^* \) with \( aq + b(q + 1) \geq K \), while (2) means that \( \varphi \) has only terms \( x^{-a-1} z_b^* \) with \( aq + b(q + 1) < K \) and with \( a < 0 \) because \( \varphi \) is a polynomial. Consequently, using the expression for \( fS \) above, \( f, \varphi \) solve the \( K \)th key equation if and only if

\[
t_{a,b} = 0 \text{ whenever } a \geq 0 \text{ and } aq + b(q + 1) < K, \text{ and}
\]

\[
\varphi = \frac{1}{x} \sum_{b=0}^{q-1} \sum_{a<0} \sum_{aq+b(q+1)<K} t_{a,b} x^{-a} z_b^*.
\]

Example 1. The pair \( y^b, 0 \) satisfies the \(-b(q+1)\) key equation. We have

\[
\rho(y^b S) \leq b(q + 1) + q^2 - q - 1 = q^2 - q - 1 - (-b(q + 1)).
\]

The pair \( 0, z_b^* \) satisfies the \( b(q + 1) - q \) key equation.

\[
\rho(z_b^*) = (q - 1 - b)(q + 1) = q^2 - q - 1 - (b(q + 1) - q)
\]

The following technical lemmas will be used to simplify the proof of Theorem 2 which establishes the properties of the decoding algorithm.
Lemma 11. Suppose that \( f \neq 0 \) and that \( f, \varphi \) satisfy the \( K \)th key equation for syndrome \( S \). Let \( g \in \mathbb{F}_q[x, y] \) with \( \rho(g) < K \). Then, in the \( \ast \)-basis expansion of \( gfS \) the coefficient of \( z_0^i/x \) is 0. Consequently, if \( g \) and \( h \) are both monic of order \( K \) then the coefficients of \( z_0^i/x \) in \( gfS \) and \( hfS \) are equal.

Proof. It is sufficient to establish this result for a monomial, \( g = x^aqy^b \), with \( aq + b(q + 1) < K \). By Lemma \[10\] the coefficient of \( z_0^i/x \) in \( x^aqy^bfS \) is equal to the coefficient of \( x^{-a-1}z_0^i \) in \( fS \). Expanding \( fS \) as in Lemma \[10\] this coefficient is \( t_{a,b} \). The discussion after the definition of approximate solutions to the key equation shows that \( t_{a,b} = 0 \) for \( a \geq 0 \) and \( aq + b(q + 1) < K \). The final statement of the lemma follows from \( \rho(g - h) < K \).

Lemma 12. Suppose that \( f, \varphi \) satisfy the \( K \)th key equation. For any nonnegative integer \( i \), the \( K - iq \) key equation is satisfied by \( x^if, x^i\varphi \).

Proof. It is trivial to check the lemma for the case when \( f = 0 \) and \( \varphi = x^{-a-1}z_0^* \). For \( f \neq 0 \), we certainly have \( x^if, x^i\varphi \in \mathbb{F}_q[x, y] \). The terms in \( \varphi \) have order at least \( q^2 - q - K \), so the terms in \( x^i\varphi \) have order at least \( q^2 - q - K + iq = q^2 - q - (K - iq) \). We also assume \( \rho(fS - \varphi) \leq q^2 - q - 1 - K \), so \( \rho(x^i(fS - \varphi)) \leq q^2 - q - 1 - (K - iq) \).

Notice that an analogous result does not hold for multiplication by \( y \). The example above shows that \( 0, z_0^* \) solves the \( K = 1 \) key equation. Yet, \( 0, yz_1^* \) does not solve the key equation of order \( K - (q + 1) = -q \). Indeed, \( yz_1^* = z_0^* - z_{q-1}^* \) and the \( -z_{q-1}^* \) term violates the requirements of the definition.

Lemma 13. Suppose that \( f, \varphi \) and \( g, \psi \) satisfy the \( K \)th key equation where \( K = aq + b(q + 1) \). Suppose in addition that \( f \neq 0 \) and \( gS - \psi \) is monic of order \( q^2 - q - 1 - K \). Let the coefficient of \( x^{-a-1}z_0^* \) in \( fS \) be \( \mu \). Then \( f - \mu g, \varphi - \mu \psi \) satisfy the \( (K + 1) \)th key equation.

Proof. By assumption, \( \rho(fS - \varphi) \leq q^2 - q - 1 - K \) and \( \varphi \) has terms of order \( q^2 - q - K \) or larger. Furthermore, \( \mu \) is the coefficient of \( x^{-a-1}z_0^* \) in \( fS \). Since \( \rho(x^{-a-1}z_0^*) = q^2 - q - 1 - K \), when \( \mu = 0 \) the inequality above is strict, and \( f, \varphi \) solve the \( (K + 1) \)th key equation. Suppose \( \mu \neq 0 \). Then both \( fS - \varphi \) and \( gS - \psi \) have order \( q^2 - q - 1 - K \). Since \( gS - \psi \) is monic, \( \rho((fS - \varphi) - \mu(gS - \psi)) < q^2 - q - 1 - K \). Furthermore, \( \psi \) has terms of order at least \( q^2 - q - 1 - K \) (allowing for the case in which \( g = 0 \)) so \( \varphi - \mu \psi \) has terms of order at least \( q^2 - q - (K + 1) \), as required for the \( (K + 1) \)th key equation.

Proposition 14 below is a generalization of Proposition 1 to Hermitian curves. It also gives the converse of Lemma 10. First, we need a lemma.

Lemma 14. Let \( f \in \mathbb{F}_q[x, y] \) and let \( (\alpha, \beta) \in \mathbb{F}_q^2 \) be a point on the Hermitian curve. Then \( f(\alpha, \beta) \) is the coefficient of \( z_0^i/x \) in the \( \ast \)-basis expansion of \( f_{x^{q+1} - \alpha y^{q+1}}(x - \alpha)(y - \beta) \).
Proof. We know that \( f(x, y) = f(\alpha, \beta) + (x - \alpha)g + (y - \beta)h \) for some \( g, h \in \mathbb{F}_q[x, y] \). Thus \( f(x, y) \) has a polynomial part plus \( f(\alpha, \beta) \) and let \( \Delta \in \mathbb{P} \) satisfy the hypotheses of the proposition. Let \( \Lambda \) be the maximal element of \( \Delta \) such that \( \rho(\Delta) = \Lambda - \rho(I) \). The quotient ring \( F_\alpha[x, y]/I \) is a \( t \)-dimensional vector space. A basis for this space is obtained by taking the classes of \( x^ay^b \) for \( \rho(x^ay^b) \in \Delta \), so \(|\Delta| = t\).

Since \( F_\alpha[x] \) is a principal ideal domain, for any ideal \( I \) with \( F_\alpha[x, y]/I \) finite dimensional over \( F_\alpha \), the ideal \( I \) is a free module over \( F_\alpha[x] \) of rank \( q \). For each \( i \) with \( 0 \leq i \leq q - 1 \) let \( f_i \) be such that \( \rho(f_i) \) is minimal among \( \{f \in I : \rho(f) \equiv i \mod q\} \). Then \( \{f_i : 0 \leq i \leq q - 1\} \) is a Gröbner basis for \( I \). By reducing \( f_i \) by multiples of \( f_j \), for \( j \neq i \) we may assume that all nonzero terms of \( f_i \), except the leading term, have order in \( \Delta = \Lambda - \rho(I) \).

**Proposition 4.** If the expansion of \( fS \) in the \( \ast \)-basis has zero coefficients for all \( x^{-a}z^*_h \) such that \( aq + b(q + 1) \in \Delta \), then \( f \in I^\ast \). In particular, let \( K \) be the maximal element of \( \Delta \). If \( f, \varphi \) satisfy the \((K + 1)\)th key equation then \( f \in I^\ast \).

Proof. Let \( f \) satisfy the hypotheses of the proposition. Let \( e_k \neq 0 \) and let \( P_k = (\alpha_k, \beta_k) \). We will prove that \( f(P_k) = 0 \). Consider the ideal \( I' = \{h \in \mathbb{F}_q[x, y] : h(P_j) = 0 \text{ for all } j \text{ with } e_j \neq 0 \text{ and } j \neq k\} \) and let \( \Delta' = \Delta \setminus \{\rho(f) : f \in I'\} \). Notice that \(|\Delta'| = t - 1\), so there exists some \( g \in I' \) such that \( \rho(g) \in \Delta \setminus \Delta' \). Reducing \( g \) modulo a reduced Gröbner basis for \( I' \), we can ensure that every monomial in \( g \) has order in \( \Delta \).

As in Lemma 10 write \( fS = 1 + \sum_{a=0}^{q-1} \sum_{b=0}^{q-1} t_{a,b}x^{-a}z^*_h \). Lemma 10 shows that \( t_{a,b} \) is the coefficient of \( z^*_h / x \) in \( x^ay^b fS \). For \( aq + b(q + 1) \in \Delta \), the hypothesis of this lemma is that \( t_{a,b} = 0 \), so the coefficient of \( z^*_h / x \) in \( x^ay^b fS \) is 0. Since \( g \) is a linear combination of monomials with order in \( \Delta \), the coefficient of \( z^*_h / x \) in \( gfS \) is 0.

On the other hand, Lemma 14 and the definition of the syndrome, 9, shows that the coefficient of \( z^*_h / x \) in \( gfS \) is

\[
\sum_{j=1}^{n} e_jg(P_j)f(P_j) = e_kg(P_k)f(P_k).
\]
Here we have used $g(P_j) = 0$ for $j \neq k$ since $g \in I'$. Since $g \not\in I^e$ we must have $g(P_k) \neq 0$, so we conclude that $f(P_k) = 0$.

The final statement of the proposition follows immediately from the observation following the definition of approximate solutions to the key equation. If $f, \varphi$ satisfy the $K$th key equation for $K = 1 + \max \Delta^c$, then the coefficient of $x^{-a-1}z_b^*$ is 0 for any $a \geq 0$ and $aq + b(q + 1) < K$.

3.6 Solving the key equation

As noted in the introduction, this algorithm is based on Kötter’s version of Sakata’s generalization of the Berlekamp-Massey algorithm. The algorithm uses the algebra of $\mathbb{F}_q[x,y]$ in only one place, the computation of $\tilde{f}$, otherwise all the computations involve polynomials in $x$, which are easily implementable using shift-registers.

The value $M$ determining the final iteration of the algorithm is given in Proposition 5 below.

Decoding algorithm for Hermitian codes

Initialize: For $i = 0$ to $q-1$, set

$$
\begin{pmatrix}
  f_i^{(0)} & \varphi_i^{(0)} \\
  g_i^{(0)} & \psi_i^{(0)}
\end{pmatrix} =
\begin{pmatrix}
  y^i & 0 \\
  0 & -z_i^*
\end{pmatrix}
$$

Algorithm: For $m = 0$ to $M$, and for each pair $i, j$ such that $m \equiv i + j \mod q$, set

$$
d_i = \rho(f_i^{(m)})
$$

$$
r_i = \frac{m - d_i - j(q+1)}{q}
$$

$$
\hat{f}_i = y^i f_i
$$

$$
\mu_i = \sum_{c=0}^{q-1} \sum_a (\hat{f}_i)_{a,c}s_{a+r_i,c}
$$

$$
p = \frac{d_i + d_j - m}{q} - 1
$$

The update for $j$ is analogous to the one for $i$ given below.

$$
U_i^{(m)} = \begin{cases}
  \begin{pmatrix}
    1 & -\mu_i x^p \\
    0 & 1 \\
    x^{-p} & -\mu_i \\
    \frac{1}{\mu_i} & 0
  \end{pmatrix} & \text{if } \mu_i = 0 \text{ or } p \geq 0 \\
  \begin{pmatrix}
    1 & 0 \\
    0 & 1 \\
    x^{-p} & -\mu_i \\
    \frac{1}{\mu_i} & 0
  \end{pmatrix} & \text{otherwise.}
\end{cases}
$$

$$
\begin{pmatrix}
  f_i^{(m+1)} \\
  g_i^{(m+1)}
\end{pmatrix} = U_i^{(m)} \begin{pmatrix}
  f_i^{(m)} \\
  g_i^{(m)}
\end{pmatrix}
$$

Output: $f_i^{(M+1)}, \varphi_i^{(M+1)}$ for $0 \leq i < q$. 

22
Remark 1. The monomial $x^r y^j$ used to define $\tilde{f}_i$ is the shift necessary so that $x^r y^j \bar{f}^{(m)}_i$ has leading term of order $m$. Indeed, $\rho(x^r y^j \bar{f}^{(m)}_i) = r \cdot q + f(q + 1) + d_i = m$. Lemma 14 says that $\mu_i$ is the coefficient of $x^{-r_i - 1}z_j^*$ in $f^{(m)}_i \mathcal{S}$.

Theorem 2. For $m \geq 0$,

1. $f^{(m)}_i$ is monic and $\rho(f^{(m)}_i) \equiv i \mod q$.
2. $f^{(m)}_i, \varphi_i^{(m)}$ satisfy the $m - \rho(f^{(m)}_i)$ approximation of the key equation.
3. $g^{(m)}_i, \psi_i^{(m)}$ satisfy the $\rho(f^{(m)}_i) - q$ approximation of the key equation and $g^{(m)}_i \mathcal{S} - \psi_i^{(m)}$ is monic of order $q^2 - 1 - \rho(f^{(m)}_i)$.
4. $\rho(g^{(m)}_i) < m - \rho(f^{(m)}_i) + q$.

Proof. We will proceed by induction on $m$. Example 11 establishes the base step, $m = 0$.

Assume that the statements of the theorem are true for $m$, we will prove them for $m + 1$. It is sufficient to consider a pair $i, j$ with $0 \leq i, j < q - 1$ satisfying $i + j \equiv m \mod q$. Let $d_i, r_i, \mu_i, p$ be as defined in the algorithm.

The induction hypothesis says that $f^{(m)}_i, \varphi_i^{(m)}$ satisfy the $m - d_i$ key equation. By Lemma 10, $\mu_i$ is the coefficient of $x^{-r_i - 1}z_j^*$ in $f^{(m)}_i \mathcal{S}$. A simple computation shows $\rho(x^{-r_i - 1}z_j^*) = q^2 - q - 1 - (m - d_i)$. Consequently, if $\mu_i = 0$, then $f^{(m)}_i, \varphi_i^{(m)}$ solve the $(m + 1 - d_i)$ key equation. In this case, the algorithm retains the data from the iteration $m$, e.g. $f^{(m + 1)} = f^{(m)}$. It is easy to verify that the properties of the theorem hold.

If $\mu_i \neq 0$, we consider two cases. First, suppose $p \geq 0$. The algorithm sets $f^{(m + 1)}_i = f^{(m)}_i - \mu_i x^p g^{(m)}_i$. Notice that $\rho(\mu_i x^p g^{(m)}_i) < (d_i + d_j - m - q) + (m - d_j + q) = d_i$. This shows that $\rho(f^{(m + 1)}_i) = d_i$ and $f^{(m + 1)}_i$ is monic, as claimed in item 1. By the induction hypothesis and Lemma 12, $x^p g^{(m)}_i, x^p \psi^{(m)}_i$ satisfy the $d_j - q - pq = m - d_i$ key equation and $x^p (g^{(m)}_j \mathcal{S} - \psi^{(m)}_j)$ is monic of order $q^2 - q - 1 - (m - d_i)$. Lemma 13 shows that $f^{(m + 1)}_i - \mu_i x^p g^{(m)}_i, \varphi^{(m)}_i - \mu_i x^p \psi^{(m)}_i$ solve the $(m + 1 - d_i)$ key equation. Since $d_i = \rho(f^{(m + 1)}_i)$, we have established item 2 of the theorem. Items 3 and 4 follow because $g^{(m + 1)}_i = g^{(m)}_i$ and $\psi^{(m + 1)}_i = \psi^{(m)}_i$.

Suppose now that $p < 0$ and $\mu_i \neq 0$. In this case, $f^{(m + 1)}_i = x^{-p} f^{(m)}_i - \mu_i g^{(m)}_j$. A simple computation shows $\rho(x^{-p} f^{(m)}_i) = m - d_j + q$ while $\rho(g^{(m)}_j) < m - d_i + q$. Thus, $f^{(m + 1)}_i$ is monic, and

$$\rho(f^{(m + 1)}_i) = \rho(x^{-p} f^{(m)}_i) = m - d_j + q \equiv i \mod q.$$  

From Lemma 12, $x^{-p} f^{(m)}_i, x^{-p} \varphi^{(m)}_i$ satisfy the key equation of order $m - d_i + pq = d_j - q$. By the induction hypothesis, $g^{(m)}_j, \psi^{(m)}_j$ satisfy the key equation of the same order. Furthermore, $\mu_i$ is the coefficient of $x^{-p - r_i - 1}z_j^*$ in $x^{-p} f^{(m)}_i \mathcal{S}$. 

23
Noting that \( q(p + r_i) + j(q + 1) = d_j - q \) we may apply Lemma 13 to obtain that \( f_i^{(m+1)}, \varphi_i^{(m+1)} \) satisfy the key equation of order \( d_j - q + 1 = m + 1 - \rho(f_i^{(m+1)}) \). This proves item (2).

To prove items (4) and (3), we first establish that \( \mu_j = \mu_i \). We claim that each is the coefficient of \( z_0^\ast \) in \( x^{-r_i-1}f_j^{(m)}f_i^{(m)}S \). We know \( \mu_i \) is the coefficient of \( x^{-r_i-1}z_0^\ast \) in \( f_i^{(m)}S \), which by Lemma 10 is the coefficient of \( z_0^\ast / x \) in \( x^{-r_i}f_j^{(m)}S \). Since \( x^{-r_i-1}f_j^{(m)} \) and \( x^{-r_i}f_j^{(m)} \) are both monic of order \( m - d_i \), Lemma 11 says the coefficients of \( z_0^\ast / x \) in \( x^{-r_i}f_j^{(m)}S \) and \( x^{-r_i-1}f_j^{(m)}f_i^{(m)}S \) are equal. A similar argument works for \( j \), which establishes the claim.

Since \( \mu_j = \mu_i \neq 0 \), the algorithm sets \( g_i^{(m+1)} = \mu_i^{-1}f_j^{(m)} \) and \( \psi_i^{(m)} = \mu_i^{-1}\varphi_j^{(m)} \). We can verify item (4),

\[
(m + 1) - \rho(f_i^{(m+1)}) + q = m + 1 - (m - d_j + q) + q
\]

\[
= d_j + 1
\]

\[
> \rho(g_i^{(m+1)})
\]

Item (3) also follows since \( g_i^{(m+1)}, \psi_i^{(m+1)} \) satisfy the \( m - d_j \) key equation and \( m - d_j = \rho(f_i^{(m+1)}) - q \). Furthermore, \( g_i^{(m+1)}S - \psi_i^{(m+1)} = \mu_j^{-1}(f_j^{(m)}S - \varphi_j^{(m)}) \) is monic.

The next results establish the iteration number \( M \) at which the algorithm may be terminated. This depends on the footprint of \( e, \Delta^e \), introduced earlier as well as the orders of the Gröbner basis for \( I^e \). For \( i = 0 \) up to \( q - 1 \) define \( \sigma_i = \min\{\rho(f^e) : f \in I^e \} \) and \( \rho(f) \equiv i \mod q \).

**Lemma 15.** For all \( m \) and for all \( i, \rho(f_i^{(m)}) \leq \sigma_i. \)

**Proof.** Let \( f_i^e \in I^e \) have pole order \( \sigma_i \) and consider \( f_i^e g_i^{(m)}S - f_i^e \psi_i^{(m)} \). By Theorem 2(3), we have \( \rho(f_i^e g_i^{(m)}S - f_i^e \psi_i^{(m)}) = \sigma_i + q^2 - 1 - \rho(f_i^{(m)}) \). This must be an element of \( \Lambda \) because \( f_i S, g_i^{(m)} \) and \( \psi_i^{(m)} \) are all in \( \mathbb{F}_q[x, y] \). Since \( \sigma_i - \rho(f_i^{(m)}) \) is a multiple of \( q \), and \( q^2 - q - 1 \not\in \Lambda, \) we must have \( \sigma_i - \rho(f_i^{(m)}) \geq 0. \)

**Proposition 5.** Let \( \sigma_{\max} = \max\{\sigma_i : 0 \leq i \leq q - 1\} \) and let \( \delta_{\max} = \max\{c \in \Delta^e \}. \) For \( m > \sigma_{\max} + \delta_{\max}, \) each of the polynomials \( f_i^{(m)} \) belongs to \( I^e \). Let \( M = \sigma_{\max} + \max\{\delta_{\max}, q^2 - q - 1\} \). Each of the pairs \( f_i^{(M+1)}, \varphi_i^{(M+1)} \) satisfies the key equation.

**Proof.** By Theorem 2 \( f_i^{(m)}, \varphi_i^{(m)} \) satisfy the \( m - \rho(f_i^{(m)}) \) key equation. If \( m > \sigma_{\max} + \delta_{\max}, \) then \( m - \rho(f_i^{(m)}) > \delta_{\max}, \) so the result follows from Lemma 4. For \( M = \sigma_{\max} + \max\{\delta_{\max}, q^2 - q - 1\}, \) we have

\[
\rho(f_i^{(M+1)}S - \varphi_i^{(M+1)}) \leq q^2 - q - 1 - (M + 1 - \rho(f_i^{(M+1)})) < 0.
\]

Since \( f_i^{(M+1)} \) is a locator, \( \varphi_i^{(M+1)} \) must equal \( f_i^{(M+1)}S. \)
3.7 Error evaluation without the error evaluator polynomials

In this section we generalize the error evaluation formula in Proposition 3 that uses just the error locator polynomial $f$ and the update polynomial $g$ to determine error values. The main result is Theorem 3, which is readily derived from Proposition 6. Unfortunately, the proposition requires a result that takes some work to establish: In the algorithm, when $i+j \equiv m \mod q$, $\mu_i = \mu_j$. This was shown for $p < 0$ in the proof of Theorem 2, but in order to show it for $p \geq 0$ we need a rather technical result, Proposition 14. Since the result is easier to state using the language of residues—instead of referring to the coefficient of $z_0^s/x$—we have deferred it to the section on general one-point codes.

**Proposition 6.** Let $B_i^{(m)} = \begin{pmatrix} f_i^{(m)} & \phi_i^{(m)} \\ g_i^{(m)} & \psi_i^{(m)} \end{pmatrix}$. Then for all $m$,

$$\sum_{i=0}^{q-1} \det B_i^{(m)} = -\sum_{i=0}^{q-1} y^i z_i^s = -1 \quad (7)$$

**Proof.** We proceed by induction. The case $m = 0$ is a simple calculation. Assume that the statement of the theorem is true for $m$: we will prove it for $m+1$. It is sufficient to show that $\det B_i^{(m+1)} = \det B_i^{(m)}$ if $2i \equiv m \mod q$ and $\det B_i^{(m+1)} + \det B_j^{(m+1)} = \det B_i^{(m)} + \det B_j^{(m)}$ if $i+j \equiv m \mod q$ and $i \neq j$.

If $2i \equiv m \mod q$, then $B_i^{(m+1)} = U_i^{(m)} B_i^{(m)}$, where

$$U_i^{(m)} = \begin{cases} \begin{pmatrix} 1 & -\mu_i x^p \\ 0 & 1 \end{pmatrix} & \text{if } \mu_i = 0 \text{ or } p \geq 0 \\ \begin{pmatrix} x^{-p} & -\mu_i \\ 1/\mu_i & 0 \end{pmatrix} & \text{otherwise.} \end{cases}$$

Since $\det U_i^{(m)} = 1$ in either case, we have $\det B_i^{(m+1)} = \det B_i^{(m)}$.

Assume now that $i+j \equiv m \mod q$ and that $i \neq j$. Proposition 14 shows that $\mu_i = \mu_j$, so, from the algorithm

$$B_i^{(m+1)} = \begin{cases} \begin{pmatrix} f_i^{(m)} - \mu_i x^p g_j^{(m)} & \phi_i^{(m)} - \mu_i x^p \psi_j^{(m)} \\ g_i^{(m)} & \psi_i^{(m)} \end{pmatrix} & \text{if } \mu_i = 0 \text{ or } p \geq 0 \\ \begin{pmatrix} x^{-p} f_i^{(m)} - \mu_i g_j^{(m)} & x^{-p} \phi_i^{(m)} - \mu_i \psi_j^{(m)} \\ \mu_i^{-1} f_j^{(m)} & \mu_i^{-1} \psi_j^{(m)} \end{pmatrix} & \text{otherwise.} \end{cases}$$

The two cases lead respectively to

$$\det B_i^{(m+1)} = \begin{cases} f_i^{(m)} \psi_i^{(m)} - g_i^{(m)} \phi_i^{(m)} - \mu_i x^p (g_j^{(m)} \psi_i^{(m)} - f_j^{(m)} \phi_j^{(m)}) & \text{or} \\ f_j^{(m)} \psi_j^{(m)} - g_j^{(m)} \phi_j^{(m)} - \mu_i x^p (g_i^{(m)} \phi_i^{(m)} - f_i^{(m)} \psi_i^{(m)}) \end{cases}$$
To obtain $\det B_{i}^{(m+1)}$ one simply switches $i$ and $j$ in these formulas. When we take the sum of $\det B_{i}^{(m+1)}$ and $\det B_{j}^{(m+1)}$, the final terms cancel, so $\det B_{i}^{(m+1)} + \det B_{j}^{(m+1)} = \det B_{i}^{(m)} + \det B_{j}^{(m)}$.

We now take $M$ as in Proposition 5, so that the algorithm of the previous section has produced solutions to the key equation. Let

$$f_{i} = f_{i}^{(M+1)} \quad \varphi_{i} = \varphi_{i}^{(M+1)}$$

$$g_{i} = g_{i}^{(M+1)} \quad \psi_{i} = \psi_{i}^{(M+1)}$$

Then the $f_{i}$ are a basis for $I^{e}$ as a module over $\mathbb{F}_{q^{2}}[x,y]$ and $f_{i}, \varphi_{i}$ satisfy the key equation.

**Theorem 3.** If $P_{k}$ is an error position.

$$e_{k} = \left( \sum_{i=0}^{q-1} f_{i}^{e}(P_{k})g_{i}(P_{k}) \right)^{-1}$$

(8)

**Proof.** From the preceding lemma,

$$\sum_{i=0}^{q-1} (f_{i}\psi_{i} - g_{i}\varphi_{i}) = -1$$

Evaluating at an error position $P_{k}$ we have $-\sum_{i=0}^{q-1} g_{i}\varphi_{i}(P_{k}) = -1$. Apply Lemma 7 to get $\sum_{i=0}^{q-1} g_{i}(P_{k})e_{k}f_{i}^{e}(P_{k}) = 1$. Solving for $e_{k}$ gives the formula. □

3.8 An example

Consider the Hermitian curve associated to the field extension $\mathbb{F}_{9} = \mathbb{F}_{3}[\alpha]$ where $\alpha^{2} = \alpha + 1$. Let $x$ and $y$ be the classes of $X$ and $Y$ in the quotient $\mathbb{F}_{9}[X,Y]/(X^{4} - Y^{3} - Y)$. The basis monomials are $x^{a}y^{b}$ for $a \geq 0$ and $0 \leq b \leq 2$. The order of $x$ is 3 and the order of $y$ is 4. So,

$$\Lambda = \{0, 3, 4, 6, 7, 8, 9, 10, \ldots \}.$$ 

The Hermitian curve in this case has 27 points, which we take in the following order: $(1, \alpha), (1, \alpha^{2}), (1, 2), (\alpha, 1), (\alpha, \alpha^{5}), (\alpha^{2}, \alpha), (\alpha^{2}, \alpha^{3}), (\alpha^{3}, 1), (\alpha^{3}, \alpha^{5}), (\alpha^{3}, \alpha^{7}), (2, \alpha), (2, \alpha^{3}), (2, 2), (\alpha^{5}, 1), (\alpha^{5}, \alpha^{5}), (\alpha^{5}, \alpha^{7}), (\alpha^{6}, \alpha), (\alpha^{6}, \alpha^{3}), (\alpha^{6}, 2), (\alpha^{7}, 1), (\alpha^{7}, \alpha^{5}), (\alpha^{7}, \alpha^{7}), (0, \alpha^{2}), (0, \alpha^{6}), (0, 0)$.

Let us consider correction of two errors. There are two choices for $\Delta^{e}$ when the weight of $e$ is two, $\{0, 4\}$ when the points are on a vertical line, and $\{0, 3\}$ when they are not. Following [4], we will call the latter case “generic” and former “non-generic.” In either case $\sigma_{\text{max}} = 8$. From Proposition 5 the computation of all error locators and evaluators is complete after iteration number $M = \Delta^{e}$.
\( \sigma_{\text{max}} + \max\{\delta_{\text{max}}, q^2 - q - 1\} \). Thus, for an algorithm to correct either of the two errors we terminate the algorithm with iteration \( M = 8 + 5 = 13 \), and take the data for superscript 14. We will explain in detail the first steps in the generic case. All the computations are summarized in Table 3.8. The computations in the non-generic case are summarized in Table 3.8.

For the generic error vector we take error values \( \alpha^2 \) at the point \((\alpha, 1)\) and \( \alpha^7 \) at the point \((\alpha^6, \alpha^3)\), so the error vector is

\[
e = (000\alpha^2 00000000000000000\alpha^7 0000000).
\]

The associated syndromes are

| \( s_{a,0} \) | \( s_{a,1} \) | \( s_{a,2} \) | \( s_{2,0} \) | \( s_{2,1} \) | \( s_{2,2} \) | \( s_{3,0} \) | \( s_{3,1} \) | \( s_{3,2} \) |
|---|---|---|---|---|---|---|---|---|
| \( \alpha \) | \( \alpha^2 \) | \( \alpha^3 \) | \( 0 \) | \( 1 \) | \( 2 \) | \( \alpha^6 \) | \( \alpha^5 \) | \( \alpha^4 \) |

To initialize \( f, g, \varphi, \psi \) we take

\[
\begin{align*}
f_0 &= 1 & g_0 &= 0 & \varphi_0 &= 0 & \psi_0 &= 2y^2 + 2 \\
f_1 &= y & g_1 &= 0 & \varphi_1 &= 0 & \psi_1 &= 2y \\
g_2 &= 0 & f_2 &= y^2 & \varphi_2 &= 0 & \psi_2 &= 2
\end{align*}
\]

We start with \( m = 0 \). The pairs \( i, j \) with \( i + j \equiv m \mod 3 \) are 0, 0 and 1, 2.

The data computed in the algorithm is,

\[
\begin{align*}
r_0 &= 0 & f_0 &= 1 & \mu_0 &= s_{0,0} &= \alpha^5, \\
r_1 &= -4 & f_1 &= x^4 + 2y & \mu_1 &= s_{0,0} + 2s_{-4,1} &= \alpha^5, \\
r_2 &= -4 & f_2 &= x^4 + 2y & \mu_2 &= s_{0,0} + 2s_{-4,1} &= \alpha^5.
\end{align*}
\]

For the pair 0, 0, \( p = -1 \), and for the pair 1, 2, \( p = 3 \), so

\[
U_0^{(0)} = \left( \begin{array}{c} x \\ \alpha^3 \\ 0 \end{array} \right) \quad \text{and} \quad U_1^{(0)} = U_2^{(0)} = \left( \begin{array}{c} 1 \\ \alpha x^3 \\ 1 \end{array} \right).
\]

As a result,

\[
\begin{align*}
f_0^{(1)} &= x & r_0^{(1)} &= \alpha^5 y^2 + \alpha^5 & y_0^{(1)} &= \alpha^3 & \psi_0^{(1)} &= 0, \\
f_1^{(1)} &= y & r_1^{(1)} &= \alpha^5 x^3 & y_1^{(1)} &= 0 & \psi_1^{(1)} &= 2y \\
f_2^{(1)} &= y^2 & r_2^{(1)} &= \alpha^5 x^3 y & y_2^{(1)} &= 0 & \psi_2^{(1)} &= 2
\end{align*}
\]

For \( m = 1 \), the pairs \( i, j \) with \( i + j \equiv m \mod 3 \) are 0, 1 and 2, 2, and,

\[
\begin{align*}
r_0 &= -2 & f_0 &= xy & \mu_0 &= s_{-1,1} &= 0, \\
r_1 &= -1 & f_1 &= y & \mu_1 &= s_{-1,1} &= 0, \\
r_2 &= -5 & f_2 &= x^4 y + 2y^2 & \mu_2 &= s_{-1,1} + 2s_{-5,2} &= 0.
\end{align*}
\]

This means that

\[
U_0^{(1)} = U_1^{(1)} = U_2^{(1)} = \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right)
\]

and \( f, \varphi, g, \psi \) remain unchanged.

For \( m = 2 \), the pairs \( i, j \) with \( i + j \equiv m \mod 3 \) are 0, 2 and 1, 1, and,

\[
\begin{align*}
r_0 &= -3 & f_0 &= xy^2 & \mu_0 &= s_{-2,2} &= 0 \\
r_1 &= -2 & f_1 &= y^2 & \mu_1 &= s_{-2,2} &= 0 \\
r_2 &= -2 & f_2 &= y^2 & \mu_2 &= s_{-2,2} + 2s_{-2,2} &= 0
\end{align*}
\]
Again, this means \( U_0^{(2)} = U_1^{(2)} = U_2^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), and \( f_i, \varphi_i, g_i, \psi_i \) remain unchanged.

For \( m = 3 \), the pairs \( i,j \) with \( i + j \equiv m \mod 3 \) are 0, 0 and 1, 2. Now, 
\[
\begin{align*}
 r_0 &= 0 \quad f_0 = x \quad \mu_0 = s_{1,0} = \alpha^2 \\
 r_1 &= -3 \quad f_1 = x^4 + 2y \quad \mu_1 = s_{1,0} + 2s_{-3,1} = \alpha^2 \\
 r_2 &= -3 \quad f_2 = x^4 + 2y \quad \mu_2 = s_{1,0} + 2s_{-3,1} = \alpha^2 
\end{align*}
\]
For the pair 0, 0 we have \( p = 0 \) and for the pair 1, 2 we have \( p = 2 \) so,
\[
U_0^{(3)} = \begin{pmatrix} 1 & \alpha^6 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U_1^{(3)} = U_2^{(3)} = \begin{pmatrix} 1 & \alpha^6x^2 \\ 0 & 1 \end{pmatrix}
\]

Consequently,
\[
\begin{align*}
 f_0^{(4)} &= x + \alpha \quad \varphi_0^{(4)} = \alpha^5y^2 + \alpha^5 \quad g_0^{(4)} = \alpha^3 \quad \psi_0^{(4)} = 0 \\
 f_1^{(4)} &= y \quad \varphi_1^{(4)} = \alpha^5x^3 + \alpha^2x^2 \quad g_1^{(4)} = 0 \quad \psi_1^{(4)} = 2y \\
 f_2^{(4)} &= y^2 \quad \varphi_2^{(4)} = \alpha^5x^3y + \alpha^2x^2y \quad g_2^{(4)} = 0 \quad \psi_2^{(4)} = 2
\end{align*}
\]
The subsequent steps are summarized in Table 8. Let \( f_i = f_i^{(14)} \) and similarly for the other data. The locators and associated derivatives are (using \( \frac{d^2}{dx} = x^3 \)),
\[
\begin{align*}
 f_0 &= x^2 + x + \alpha^7 \quad (f_0)' = 2x + 1 \\
 f_1 &= y + \alpha^5x + \alpha \quad (f_1)' = x^3 + \alpha^5 \\
 f_2 &= y^2 + \alpha^7x^2 + \alpha^7x + \alpha^3 \quad (f_2)' = 2x^3y + \alpha^3x + \alpha^7
\end{align*}
\]
The points where \( f_0, f_1, \) and \( f_2 \) vanish are exactly \( P_1 = (\alpha, 1) \) and \( P_{21} = (\alpha^6, 2) \), coinciding with the error positions. The polynomials \( \varphi \) are:
\[
\begin{align*}
 \varphi_0 &= \alpha^5xy^2 + y^2 + 2xy + \alpha x + 2, \\
 \varphi_1 &= \alpha^5x^3 + \alpha^2x^2 + \alpha^2xy + \alpha^5x + \alpha^6, \\
 \varphi_2 &= \alpha^5x^3y + 2xy^2 + \alpha^2x^2y + 2x^3 + \alpha^7y^2 + \alpha^2xy + x^2 + \alpha^7x + 1
\end{align*}
\]
The error values at these positions can be computed using the formula in Lemma 7. For example,
\[
\begin{align*}
 e_4 &= \frac{\varphi_1(P_4)}{(f_1)'(P_4)} = \frac{\alpha^5}{\alpha^3} = \alpha^2 \quad e_{21} = \frac{\varphi_1(P_{21})}{(f_1)'(P_{21})} = \frac{\alpha^6}{\alpha^7} = \alpha^7.
\end{align*}
\]
The same error values could have been obtained using \( f_0, \varphi_0 \) instead of \( f_1, \varphi_1 \). However, we could not have used \( f_2, \varphi_2 \), because the zero of \( f_2 \) at \( P_{21} \) is not simple.

By Theorem 3, the error values can also be obtained using \( g_0, g_1, g_2 \) instead of \( \varphi_0, \varphi_1, \varphi_2 \). Since \( g_0 = \alpha^6x + \alpha^7 \), and \( g_1 = g_2 = 0 \) there is only one term to compute.
\[
\begin{align*}
 e_4 &= (f_0'(P_4)g_0(P_4))^{-1} \quad e_{21} = ((f_0)'(P_{21})g_0(P_{21}))^{-1} \\
 &= (\alpha^3 \cdot \alpha^3)^{-1} \quad = (\alpha^7 \cdot \alpha^2)^{-1} \\
 &= \alpha^2 \quad = \alpha^7.
\end{align*}
\]
An example of a non-generic error vector is 
\[
(0000000a^20a^7000000000000000000).
\]

28
The error positions correspond to the points $P_7 = (\alpha^2, \alpha)$ and $P_9(\alpha^2, 2)$ which lie on the line $x = \alpha^2$. The associated syndromes are 

\[
\begin{array}{cccccccc}
  & s_{0,c} & s_{1,c} & s_{2,c} & s_{3,c} & s_{4,c} & s_{5,c} & s_{6,c} & s_{7,c} & s_{8,c} \\
 s_{a,0} & \alpha^0 & \alpha^1 & \alpha & \alpha^3 & \alpha^4 & \alpha & \alpha^3 & \alpha^5 & \alpha^3 \\
 s_{a,1} & \alpha^7 & \alpha & \alpha^5 & \alpha^7 & \alpha & \alpha^3 & \alpha^5 & \alpha^7 & \alpha^7 \\
 s_{a,2} & \alpha^2 & 2 & \alpha^6 & 1 & \alpha^2 & 2 & \alpha^6 & 1 & \alpha^2 \\
\end{array}
\]

The steps of the algorithm are summarized in Table 3.8. Notice that after step $m = 4$, $f_0 = x - \alpha^2 = x + \alpha^6$ is already a locator.

4 The key equation for one-point codes

Sakata’s generalization of the Berlekamp-Massey algorithm was originally designed for a monomial ordering on a polynomial ring in several variables [38]. It has been adapted to the more general setting of a ring with an order function [17], which corresponds to an algebraic variety (curve, surface or higher dimensional object) and a choice of valuation on the variety [31]. In the case of a curve $C$, one takes the ring $R$ of functions having poles only at a single point $Q$ on $C$, and the pole order function. The one-point codes defined by $C$ and $Q$ are obtained by evaluating functions in $R$ at rational points $P_1, P_2, \ldots, P_n$ that are distinct from $Q$.

In this section we show that the results in the Hermitian codes section, with very minor modifications, apply to one-point codes. The main challenge is to establish the dual bases in which we write the locator polynomial and the evaluator, which is now a differential. Once this foundation is set, the decoding material falls in place via the same arguments as were used for Hermitian codes. We simply state the results here and leave verification to the reader. The section starts with a quick tour of the main properties of uniformizing parameters, differentials, residues, and other topics that are needed to establish the algorithms and formulas for decoding. Our primary reference for this section is Stichtenoth’s book [11], but another valuable resource is Pretzel’s book [36].

4.1 Curves, function fields and differentials

Let $K$ be a function field of transcendence degree one over $\mathbb{F}_q$. Let $C$ be the smooth curve over $\mathbb{F}_q$ defined by $K$. We assume that $\mathbb{F}_q$ is algebraically closed in $K$, which is equivalent to $C$ being absolutely irreducible. Let $Q$ be a rational point of $C$ and let $v_Q$ be the associated valuation of $K$. Let $L(mQ)$ be the space of functions on $C$ having poles only at $Q$ and of order at most $m$ there. Each $L(mQ)$ contains $L((m-1)Q)$, and is either equal to it, when we say $m$ is a gap, or of dimension one larger, when $m$ is a nongap. Let $\Lambda$ be the set of nongaps and let $\Lambda^c$ be its complement in $\mathbb{Z}$. $\Lambda$ is called the Weierstrass semigroup of $C$ at $Q$. The union of the $L(mQ)$ is a ring,

\[ R = \bigcup_{m=0}^{\infty} L(mQ) \]
For $f \in R$, we define $\rho(f) = -\nu_Q(f)$ to be the pole order of $f$ at $Q$. Formally, we set $\rho(0) = -\infty$.

Let $\kappa$ be the smallest positive element of $\Lambda$. For each $b = 0, \ldots, \kappa - 1$, let $\lambda_b$ be the smallest element of $\Lambda$ congruent to $b$ modulo $\kappa$. Any integer may be written in a unique way as $\lambda_b + ak$ for some $b \in \{0, \ldots, \kappa - 1\}$ and $a \in \mathbb{Z}$. Elements of $\Lambda$ have $a \geq 0$ and elements of $\Lambda^c$ have $a < 0$. The set $\lambda_1, \ldots, \lambda_{\kappa-1}$ is usually known as the Apéry set of $\Lambda$ (named so after [1]). Let $x \in R$ have pole order $\kappa$, and for each $b$, let $z_b$ have pole order $\lambda_b$. We also assume that some uniformizing parameter $u_Q$ at $Q$ has been selected, and that $x$, and $z_b$ are monic with respect to $u_Q$. That is, when either $x$ or $z_b$ is written as a power series in $u_Q$ the initial term has coefficient 1. In particular, $z_0 = 1$.

**Proposition 7.** With the notation above, $R$ is a free module over $\mathbb{F}_q[x]$ with basis $\{z_b\}^{\kappa-1}_{b=0}$. This is also a basis for $K$ over $\mathbb{F}_q(x)$.

**Proof.** Let $y \in R$ satisfy $\rho(y) \equiv b \mod \kappa$. Since $\lambda_b$ is the smallest element of $\Lambda$ congruent to $b$, there is some nonnegative $a$ such that $\rho(y) = \lambda_b + ak$. Now $\rho(y) = \rho(x^a z_b)$ so there is some $\beta \in \mathbb{F}_q$ such that $\rho(y - \beta x^a z_b) \equiv \rho(y) \mod \kappa$. Continuing in this manner, we find that for some $g_j \in \mathbb{F}_q[x]$, the pole order of $y - \sum_j g_j z_j$ is negative. Since $y - \sum_j g_j z_j \in R$, the pole order must be $-\infty$; that is $y - \sum_j g_j z_j = 0$.

On the other hand, no nontrivial combination $\sum_j g_j z_j$ can equal 0. If $g_j \neq 0$ then $\rho(g_j z_j) \equiv j \mod \kappa$. Thus $\rho(\sum_j g_j z_j) = \max_{j: g_j \neq 0} \{\rho(g_j z_j)\}$ which is not $-\infty$. Thus, $R$ is free over $\mathbb{F}_q[x]$ with basis $\{z_j\}^{\kappa-1}_{j=0}$. The argument for linear independence holds for $g_j \in \mathbb{F}_q(x)$ as well. Since $x$ has only one pole, and that of order $\kappa$, the dimension of $K$ over $\mathbb{F}_q(x)$ is $\kappa$. [11, I.4.11]. Thus $\{z_j\}^{\kappa-1}_{j=0}$ is a basis for $K$ over $\mathbb{F}_q(x)$.

There are parallel constructions for differentials. The module of differentials of $K$ over $\mathbb{F}_q$, which we denote $\Omega$, is a one-dimensional vector space over $K$. For any separating element $u \in K$, in particular for a uniformizing parameter, $du$ is a basis for $\Omega$. If $u_P$ is a uniformizing parameter at a point $P$, then any $\omega \in \Omega$ may be written in the form $\sum_{i=1}^{\infty} c_i u_P^i du_P$ with $c_i \in \mathbb{F}_q$ and $c_r \neq 0$. One defines $\nu_P(\omega) = r$ and $\text{res}_P(\omega) = c_{r-1}$ (or $\text{res}_P(\omega) = 0$ if $r > -1$). These definitions are independent of the choice of uniformizing parameter. We will say that $\omega$ is monic, relative to $u_P$, when $c_r = 1$. The divisor of $\omega$ is $\omega = \sum_P \nu_P(\omega)$, where the sum is over all points of $\mathcal{C}$. For any divisor $D$, $\Omega(D)$ is the space of differentials such that $(\omega) \geq D$. Thus, $\Omega(mQ)$ is the space of differentials which have valuation at least $m$ at $Q$ and which have nonnegative valuation elsewhere.

Let

$$\Omega(-\infty Q) = \bigcup_{m=0}^{\infty} \Omega(-mQ)$$

It is evident that $\Omega(-\infty Q)$ is a module over $R$.

The most fundamental invariant of the curve $\mathcal{C}$ is its genus, $g$. We will use the following fundamental results about divisors and the genus.

- The degree of any differential is $2g - 2$. 


• For the point \( Q \), the number of positive gaps, \(|N \setminus \Lambda|\), is \( g \).

• \( \Omega(-\infty Q) \) is isomorphic to \( R \) when \( (2g-2)Q \) is a canonical divisor.

• The Riemann-Roch theorem: For any divisor \( D \),
\[
\dim L(D) - \dim \Omega(D) = m + 1 - g.
\]

• The residue theorem: For any differential \( \omega \), \( \sum_P \text{res}_P(\omega) = 0 \), where the sum is over all points of \( C \).

4.2 One-point codes and their duals

Let \( P_1, P_2, \ldots, P_n \) be distinct rational points on \( C \), each different from \( Q \), and let \( D = P_1 + P_2 + \cdots + P_n \). We define the evaluation map \( \text{ev} \) as follows.
\[
\text{ev} : R \longrightarrow \mathbb{F}_q^n
\]
\[
f \mapsto (f(P_1), f(P_2), \ldots, f(P_n))
\]

Similarly, we have the residue map
\[
\text{res} : \Omega(-\infty Q - D) \longrightarrow \mathbb{F}_q^n
\]
\[
\omega \mapsto (\text{res}_{P_1}(\omega), \text{res}_{P_2}(\omega), \ldots, \text{res}_{P_n}(\omega))
\]

Restricting the evaluation map to \( L(mQ) \) and the residue map to \( \Omega(mQ - D) \) we get exact sequences.
\[
0 \longrightarrow L(mQ - D) \longrightarrow L(mQ) \longrightarrow \mathbb{F}_q^n
\]
\[
0 \longrightarrow \Omega(mQ) \longrightarrow \Omega(mQ - D) \longrightarrow \mathbb{F}_q^n
\]

The image codes are \( C_L(D, mQ) = \text{ev}(L(mQ)) \) and \( C_\Omega(D, mQ) = \text{res}(\Omega(mQ - D)) \).

**Proposition 8.** The codes \( C_L(D, mQ) \) and \( C_\Omega(D, mQ) \) are dual.

**Proof.** For \( f \in L(mQ) \) and \( \omega \in \Omega(mQ - D) \), the poles of \( f\omega \) are supported on \( D \). From the residue theorem,
\[
\text{ev}(f) \cdot \text{res}(\omega) = \sum_{k=1}^n \text{res}_{P_k}(f\omega) = -\text{res}_Q(f\omega) = 0
\]

The Riemann-Roch theorem says
\[
\dim L(mQ) - \dim \Omega(mQ) = m + 1 - g
\]
\[
\dim L(mQ - D) - \dim \Omega(mQ - D) = m - n + 1 - g
\]
Taking the difference,

\[(\dim L(mQ) - \dim L(mQ - D)) + (\dim \Omega(mQ - D) - \dim \Omega(mQ)) = n\]

Thus, the codes are of complementary dimension and are orthogonal, so they are dual codes.

One consequence of the proposition is that the code \(C_\Omega(D, -Q)\) is the whole space \(F_q^n\). In a later section we will identify a differential, \(h_{P_k} dx \in \Omega(-Q - D)\), whose image under \(\text{res}\) is 1 in position \(k\) and 0 elsewhere. The syndrome of an error vector \(e\) will be \(\sum_{k=1}^n e_k h_{P_k} dx\).

We will consider the family of codes \(C_\Omega(D, mQ)\). The check matrix is constructed by taking rows of the form \(ev(x^a z_b)\) for \(a \kappa + \lambda_b \leq m\), arranged by increasing pole order. As in earlier sections, we assume \(c \in C_\Omega(D, mQ)\) is sent, the vector \(u \in F_q^n\) is received, and \(e = u - c\), the error vector, has weight \(t\).

4.3 The trace and a dual basis

We have identified a basis for \(K\) over \(F_q(x)\); we now seek a dual basis for \(\Omega\). The dual basis is constructed using the intimate relationship between differentials and the trace map of an extension of function fields (see [41, II.4, IV.3], or [36, 13.12-13]). Let \(\text{Tr}\) be the trace map from \(K\) to \(F_q(x)\). Recall that the dual basis to \(\{z^b\}_{b=0}^{\kappa-1}\) is the unique set of elements of \(K\), \(z_0, \ldots, z_{\kappa-1}\) such that \(\text{Tr}(z^b z^*_j)\) is 1 if \(b = j\) and 0 otherwise.

We will use a result that appears as Proposition 8 in Ch. X of [22]: Let \(F\) be a separable finite extension of \(k(x)\) and let \(Q_1, \ldots, Q_r\) be the distinct points over a point \(P\) of \(k(x)\). Let \(y\) be an element of \(F\). Then

\[
\sum_{i=1}^r \text{res}_{Q_i}(y dx) = \text{res}_{P}(\text{Tr}(y) dx)
\]

The theorem assumes \(k\) is an algebraically closed field. It is also true if \(k\) is not algebraically closed provided \(P, Q, P_i\) are rational points since the residues are defined for rational points and unchanged when one passes to the algebraic closure. In our case, let \(\infty\) be the point on the projective line where \(x\) has a pole. On \(C\), \(x\) will also have a pole at any point mapping to \(\infty\). Since the only pole of \(x\) is \(Q\), the formula says \(\text{res}_Q(y dx) = \text{res}_\infty(\text{Tr}(y) dx)\) for any \(y \in K\).

**Proposition 9.** For each \(b \in \{0, \ldots, \kappa - 1\}\), \(z^*_b dx\) is an element of \(\Omega(-\infty Q)\), \(-z^*_b dx\) is monic, relative to \(u_Q\), and \(v_Q(z^*_b dx) = \lambda_b - \kappa - 1\). Additionally,

\[
\text{res}_Q(z^*_b z^*_j x^a dx) = \begin{cases} 
-1 & \text{when } a = -1 \text{ and } j = b \\
0 & \text{otherwise}
\end{cases}
\]

**Proof.** We will prove the residue formula first. Using the formula for the residue
at \( Q \) and the property of the dual basis,

\[
\text{res}_Q(z_j z_h^* x^a dx) = \text{res}_\infty(x^a \text{Tr}(z_j z_h^*)dx) = \begin{cases} 
\text{res}_\infty(x^a dx) & \text{when } j = b \\
0 & \text{otherwise}
\end{cases}
\]

For the case \( j = b \), note that \( u = 1/x \) is a uniformizing parameter at \( \infty \), and \( x^a dx = u^{-a}(-u^{-2}du) = -u^{-a-2}du \). The residue is \(-1\) when \( a = -1 \) and is zero otherwise.

Now let \( j \equiv \nu_Q(z_b^* dx) + 1 \mod \kappa \) and let \( a \) be such that 

\[
\nu_Q(z_b^* dx) = \lambda_j - 1 - (a + 1)\kappa
\]

Then 

\[
\nu_Q(z_j z_h^* x^{-a-1} dx) = -\lambda_j + (\lambda_j - 1 - (a + 1)\kappa) + (a + 1)\kappa = -1
\]

Therefore, \( \text{res}_Q(z_j z_h^* x^{-a-1} dx) \neq 0 \). By what we proved earlier, this can only be true when \( j = b \) and \( a = 0 \). Therefore, \( \nu_Q(z_b^* dx) = \lambda_b - 1 - \kappa \). Furthermore, since \( \text{res}(z_b z_h^* x^{-1} dx) = -1 \), and \( z_b \) is monic, \(-z_h^* dx \) is also monic (relative to \( u_Q \)).

Finally, we show \( z_b^* dx \in \Omega(-\infty Q) \). From the residue formula we can see that for each \( z_j \) and any \( h_j \in \mathbb{F}_q[x] \), \( \text{res}_Q(h_j z_j z_h^*) = 0 \). Since any element of \( R \) can be expressed in the form \( \sum_{j=0}^{\kappa-1} h_j z_j \), we conclude that \( \text{res}_Q(f z_h^* dx) = 0 \) for any \( f \in R \). Now suppose that \( z_b^* dx \) has a pole at some point \( P \neq Q \). By the strong approximation theorem, we may choose \( f \in R \) to eliminate any other poles of \( z_b^* dx \) away from \( P \) and \( Q \) and we may also ensure that \( \nu_P(f z_h^* dx) = -1 \). Then \( \text{res}_Q(f z_b^* dx) = -\text{res}_P(f z_b^* dx) \neq 0 \), which contradicts what was shown above. Thus \( z_b^* dx \) can have a pole only at \( Q \).

**Proposition 10.** With the notation above, \( \Omega(-\infty Q) \) is a free module over \( \mathbb{F}_q[x] \) with basis \( \{ z_b^* dx \}_{b=0}^{\kappa-1} \). This is also a basis for \( \Omega \) over \( \mathbb{F}_q(x) \).

**Proof.** Let \( l(mQ) = \dim L(mQ) \) and \( i(mQ) = \dim \Omega(mQ) \). From the Riemann-Roch theorem one can show

\[
l((m-1)Q) - l(mQ) = i((m-1)Q) - i(mQ) - 1
\]

If \( m \in \Lambda \), the left hand side is \(-1\), so \( i((m-1)Q) = i(mQ) \). Conversely, if \( m \in \Lambda^c \) then the left hand side is \( 0 \), so \( i((m-1)Q) = i(mQ) + 1 \) and there is some \( \omega \in \Omega(-\infty Q) \) such that \( \nu_Q(\omega) = m - 1 \). Thus

\[
\{ \nu_Q(\omega) + 1 : \omega \in \Omega(-\infty Q) \} = \Lambda^c = \bigcup_{b=0}^{\kappa-1} \{ \lambda_b - a\kappa : a > 0 \}
\]

We now proceed as in Proposition 7. Let \( \omega \in \Omega(-\infty Q) \) and let \( i \) and \( a > 0 \) be such that \( \nu_Q(\omega) = \lambda_b - a\kappa - 1 \). There is some \( \alpha \in \mathbb{F}_q \) such that
\( \nu_Q(\omega - \alpha x^{a-1}z_i^*dx) > \lambda_b - ak - 1 \). Continuing in this manner, there exist \( g_b \in F_q[x] \) such that \( \omega - \sum_{b=0}^{\kappa-1} g_b z_i^*dx \) has valuation at \( Q \) larger than \((2g - 2)\). It is also in \( \Omega(-\infty Q) \), so it has no poles away from \( Q \). Thus \( \omega - \sum_{b=0}^{\kappa-1} g_b z_i^*dx = 0 \), for otherwise it would have degree greater than \((2g - 2)\). This shows any \( \omega \in \Omega(-\infty Q) \) is a combination of \( z_i^*dx \) with coefficients in \( F_q \).

Uniqueness and the extension to \( \Omega \) are shown as in Proposition 7.

The next result is required to derive the error evaluation formula that is analogous to Theorem 3.

**Proposition 11.** Let \( M/L \) be a finite separable field extension and let \( \text{Tr} \) be the trace map from \( M \) to \( L \). Let \( z_1, \ldots, z_n \) be a basis for \( M \) over \( L \) and let \( z_1^*, \ldots, z_n^* \) be the dual basis. Then

\[
\sum_{i=1}^{n} z_i z_i^* = 1
\]

**Proof.** Since \( M \) is finite and separable over \( L \) there is some \( y \in M \) such that \( M = L(y) \). We will show the result first for the basis \( 1, y, y^2, \ldots, y^{n-1} \). Let \( F(T) \in L[T] \) be the minimal polynomial of \( y \) and let \( F'(T) \) be its formal derivative. Let

\[
C(T) = \frac{F(T)}{T - y} = c_{n-1}T^{n-1} + c_{n-2}T^{n-2} + \cdots + c_1 T + c_0
\]

where \( c_i \in M \) and \( c_{n-1} = 1 \). The proof of [41 III.5.10] (or [23 VI.5.5]) shows that the dual basis to \( 1, y, y^2, \ldots, y^{n-1} \) is \( c_0/F'(y), \ldots, c_{n-1}/F'(y) \). For this basis, the sum in (9) is

\[
\sum_{i=1}^{n} y^i F'(y) = \frac{1}{F'(y)} C(y)
\]

In some algebraic closure of \( M \), let \( y_1, y_2, \ldots, y_{n-1} \) be the roots of \( F \) that are distinct from \( y \) and let \( y_n = y \). Then \( C(y) = \prod_{i=1}^{n-1} (y - y_i) \). Since \( F'(T) = \sum_{i=1}^{n} \prod_{j \neq i} (T - y_j) \), \( F'(y) = \prod_{i=1}^{n-1} (y - y_i) = C(y) \), so the sum in (10) is 1 as claimed.

Now suppose \( \{z_i\} \) is another basis let \( \{z_i^*\} \) be its dual basis, and let \( \{y_i^*\} \) be the dual basis to \( \{y_i\} \). Let \( M \) be the change of basis matrix from the \( z \)-basis to the \( y \)-basis: \( z = \sum_{i=1}^{n} m_{a,i} y^i \). The change of basis matrix \( M \) from the \( z^* \)
basis to the $y^*$ basis is $(M^T)^{-1}$, as the following computation shows.

$$\delta_{a,b} = \text{Tr}(z_az_b^*) = \text{Tr} \left( \sum_{i=1}^{n} m_{a,i} y^i \sum_{j=1}^{n} \overline{m}_{b,j} y_j^* \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} m_{a,i} \overline{m}_{b,j} \text{Tr}(y^i y_j^*)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} m_{a,i} \overline{m}_{b,i}$$

A similar computation shows $\sum_{a=1}^{n} z_a z_a^* = 1$,

$$\sum_{a=1}^{n} z_a z_a^* = \sum_{a=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{a,i} y^i \overline{m}_{a,j} y_j^*$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} y^i y_j^* \sum_{a=1}^{n} m_{a,i} \overline{m}_{a,j}$$

$$= \sum_{i=1}^{n} y^i y_i^* = 1$$

\[\square\]

**Example 2.** A natural generalization of Hermitian codes is the norm-trace codes, which were studied in [13]. Consider the field extension, $\mathbb{F}_{q^r}/\mathbb{F}_q$. Let $N$ be the norm function and $\text{Tr}$ the trace function for this extension. The norm-trace curve is $\text{Tr}(y) = N(x)$, that is

$$\sum_{i=0}^{r-1} y^i = x^{\frac{q^r-1}{q-1}}$$

In the function field of this curve, $y$ is a solution to the polynomial $F(T) \in \mathbb{F}_q(x)[T]$, $F(T) = \sum_{i=0}^{r-1} T^i - x^{\frac{q^r-1}{q-1}}$. Dividing by $T - y$ and substituting
We also have $F'(T) = 1$. Thus the dual basis to $1, y, \ldots, y^{q^r - 1}$ is $y^∗_0, \ldots, y^∗_{q^r - 1}$ where $y^*_j = \sum_{i=[\log_q(j+1)]}^{q^r - 1} y^{q^r - 1-j}$.

### 4.4 Polynomials for decoding

Define the error locator ideal of $e$ to be

$$I^e = \{ f \in R : f(P_k) = 0 \text{ for all } k \text{ with } e_k \neq 0 \}$$

For a point $P$, let

$$h_P = \frac{1}{x - x(P)} \sum_{b=0}^{\kappa-1} z_b(P) z^*_b.$$ 

We define the syndrome of $e$ to be

$$S = \sum_{k=1}^{n} e_k h_{P_k}.$$ 

As we did with Hermitian codes, we will give three justifications for this definition of the syndrome. The first is that the coefficients of $S$ are the products $ev(x^a z_b) \cdot e$.

**Lemma 16.** Let $s_{a,b} = \sum_{k=1}^{n} e_k (x(P_k))^a (z(P_k))^b$. Then

$$S = \frac{1}{x} \sum_{b=0}^{\kappa-1} \sum_{a=0}^{\infty} s_{a,b} x^{-a} z^*_b.$$ 

36
Proof. Writing \((x - x(P))^{-1}\) as a series in \(1/x\) we have

\[
h_P = \frac{1}{x} \left( \sum_{a=0}^{\infty} \left( \frac{x(P)}{x} \right)^a \right) \left( \sum_{b=0}^{\kappa-1} z_b(P) z_b^* \right)
\]

\[= \sum_{b=0}^{\kappa-1} \sum_{a=0}^{\infty} (x(P))^a z_b(P) x^{-a} z_b^*
\]

Thus

\[
S = \frac{1}{x} \sum_{k=1}^{n} e_k \sum_{a=0}^{\kappa-1} \sum_{b=0}^{\infty} (x(P))^a z_b(P) x^{-a} z_b^*
\]

\[= \frac{1}{x} \sum_{b=0}^{\kappa-1} \sum_{a=0}^{\infty} x^{-a} z_b^* \sum_{k=1}^{n} e_k (x(P))^a z_b(P)
\]

\[= \frac{1}{x} \sum_{b=0}^{\kappa-1} \sum_{a=0}^{\infty} s_{a,b} x^{-a} z_b^*
\]

\[\text{(12)}
\]

37
The connection between the error locator ideal and the syndrome is now clear.

**Lemma 18.** For \( f \in R, f \in I^e \) if and only if \( f Sdx \in \Omega(-\infty Q) \).

*Proof.* From the previous lemma, \( Sdx \) has a simple pole at each \( P_k \) where \( e_k \) is nonzero. Thus \( f Sdx \in \Omega(-\infty Q) \) if and only if \( \nu_{P_k}(f) \geq 1 \) whenever \( e_k \neq 0 \). This is just saying \( f \in I^e \).

Finally, we show that for \( f \in I^e \), \( fS \) may be used for error evaluation.

**Lemma 19.** Let \( P_k \) be an error position and let \( u_k \) be a uniformizing parameter at \( P_k \). If \( f \) is an error locator and \( \varphi = f Sdx \), then

\[
\frac{e_k df}{du_k}(P_k) = \frac{\varphi}{du_k}(P_k)
\]

*Proof.* Since \( f \) vanishes at \( P_k \) we can write \( f = a_1 u_k + a_2 u_k^2 + \cdots \). Each \( h_{P_k} \) has a simple pole at \( P_k \) and no pole at \( P_j \) for \( j \neq k \), so from the definition of \( S \),

\[
Sdx = (e_k u_k^{-1} + c_0 + c_1 u_k + \cdots) du_k
\]

Thus

\[
\frac{f Sdx}{du_k} = e_k a_1 + \cdots
\]

On the other hand,

\[
\frac{df}{du_k} = a_1 + 2a_2 u_k + \cdots
\]

Evaluating the two at \( P_k \) amounts to setting \( u_k = 0 \), which gives the result.

To compute \( e_k \) using this formula, we need \( f \) to have a simple zero at \( P_k \). The formula simplifies when \( x - x(P_k) \) itself is a uniformizing parameter at \( P_k \),

\[
\epsilon_k \frac{df}{dx}(P_k) = f S(P_k)
\]

### 4.5 The key equation and its solution

The key equation and the algorithm for solving it are little changed from those for Hermitian codes. We use \( \kappa \) instead of \( q \) in the indexing of \( z \) and \( z^* \). The key equation uses differentials, not just polynomials. The key equation for Hermitian codes can be derived from the one in this section by dividing by \( dx \), whose divisor is \( (2g-2)Q \), and thereby shifting the pole order by \( 2g - 2 = q^2 - q - 2 \). We will simply state the main results, and leave adaptations of the proofs in the previous section to the reader.
Definition. We say that \( f \in \mathbb{R} \) and \( \varphi \in \Omega(-\infty Q) \) solve the key equation for syndrome \( S \) when \( fSdx = \varphi \). We say that a nonzero \( f \in \mathbb{R} \) and \( \varphi \in \Omega(-\infty Q) \) solve the \( K \)-th approximation of the key equation for syndrome \( S \) when the following two conditions hold.

1. \( \rho(fSdx - \varphi) \leq 1 - K \),
2. \( \varphi \), written in the \( * \)-basis, is a sum of terms whose order is at least \( 2 - K \).

We will also say that \( 0 \) and \( x^{-a-1}z_b^* dx \), for \( a < 0 \), solve the \( \alpha + \lambda_b \) key equation.

One could also express this definition in terms of the valuation \( \nu_Q \), \( f \) and \( \varphi \) solve the \( K \)-th key equation when \( \nu_Q(fSdx - \varphi) \geq K - 1 \). Since each \( hPdx \) has a simple pole at \( Q \), \( \nu_Q(Sdx) \geq -1 \). Therefore, \( \rho(z_bSdx) \leq 1 - \lambda_b \), so the pair \( 0, z_b^* \) satisfies the \( \lambda_b - \kappa \) key equation.

Here are the three lemmas used in the proof that the decoding algorithm works.

**Lemma 20.** Suppose that \( f \neq 0 \) and that \( f, \varphi \) satisfy the \( K \)th key equation for syndrome \( S \). If \( g \) and \( h \) are both monic of order \( K \) then \( \text{res}_Q(gfS) = \text{res}_Q(hfS) \).

**Lemma 21.** Suppose that \( f, \varphi \) satisfy the \( K \)th key equation. For any nonnegative integer \( i \), \( x^i f, x^i \varphi \) satisfy the \( K - i \kappa \) key equation.

**Lemma 22.** Suppose that \( f, \varphi \) and \( g, \psi \) satisfy the \( K \)th key equation where \( K = \alpha \kappa + \lambda_b \). Suppose in addition that \( f \neq 0 \) and \( gSdx - \psi \) is monic of order \( 1 - K \). Let the coefficient of \( x^{-a-1}z_b^* \) in \( fSdx \) be \( \mu \). Then \( f - \mu g, \varphi - \mu \psi \) satisfy the \( (K + 1) \)th key equation.

The decoding algorithm has only minor changes: \( \kappa \) replaces \( q \), \( z_i \) replaces \( y^i \) and \( \lambda_i \) replaces \( i(q + 1) \).

**Decoding algorithm for one-point codes**

Initialize: For \( i = 0 \) to \( \kappa - 1 \), set 
\[
\begin{pmatrix}
  f_i^{(0)} \\
  g_i^{(0)} \\
  \varphi_i^{(0)} \\
  \psi_i^{(0)}
\end{pmatrix} = \begin{pmatrix}
  z_i \\
  0 \\
  0 \\
  -z_i^* dx
\end{pmatrix}
\]

Algorithm: For \( m = 0 \) to \( M \), and for each pair \( i, j \) such that \( m \equiv i + j \mod \kappa \), set
\[
\begin{align*}
d_i &= \rho(f_i^{(m)}) \\
r_i &= \frac{m - d_i - \lambda_i}{\kappa} \\
f_i &= z_j f_i \\
\varphi_i &= \sum_{a=0}^{\kappa-1} \sum_{c=0}^{a} (\tilde{f}_i)_{a,c} s_{a+r,i,c} \\
p &= \frac{d_i + d_j - m}{\kappa} - 1
\end{align*}
\]

The update for \( j \) is analogous to the one for \( i \) given below.
Theorem 4. For and that this is the coefficient of \( x^\mu \) analogous to the one in Lemma 10 one can show that

\[
U_i^{(m)} = \begin{cases} 
(1, -\mu_i x^p) & \text{if } \mu_i = 0 \text{ or } p \geq 0 \\
(0, 1) & \text{otherwise.} 
\end{cases}
\]

Output: \( f_i^{(m+1)}, \varphi_i^{(M+1)} \) for \( 0 \leq i < \kappa \).

One can check that at iteration \( m \), \( \rho(x^\tau z_j f_i^{(m)}) = m \). Using an argument analogous to the one in Lemma 10 one can show that \( \mu_i = \text{res}_Q(x^\tau z_j f_i^{(m)} Sdx) \) and that this is the coefficient of \( x^{-r_i} z_j^\tau \) in \( f_i^{(m)} S \).

Theorem 4. For \( m \geq 0 \),

1. \( f_i^{(m)} \) is monic and \( \rho(f_i^{(m)}) \equiv i \mod \kappa \).
2. \( f_i^{(m)}, \varphi_i^{(m)} \) satisfy the \( m - \rho(f_i^{(m)}) \) approximation of the key equation.
3. \( g_i^{(m)}, \psi_i^{(m)} \) satisfy the \( \rho(f_i^{(m)}) - \kappa \) approximation of the key equation and \( g_i Sdx - \psi_i^{(m)} \) is monic of order \( 1 + \kappa - \rho(f_i^{(m)}) \).
4. \( \rho(g_i^{(m)}) < m - \rho(f_i^{(m)}) + \kappa \).

The iteration at which the algorithm can terminate depends on the set \( \Delta^e = \Lambda - \rho(I^e) \) and the values \( \sigma_i = \min \{ \rho(f^\tau) : f \in I^e \text{ and } \rho(f^\tau) \equiv i \mod \kappa \} \).

Proposition 12. If \( \text{res}_Q(x^\alpha z_j f Sdx) = 0 \) for all \( a, b \) such that that \( a \alpha + \lambda b \in \Delta^e \) then \( f \in I^e \). In particular, if \( f, \phi \) satisfy the max \( \Delta^e \) key equation, then \( f \in I^e \).

Proposition 13. Let \( \sigma_{\max} = \max \{ \sigma_i : 0 \leq i \leq \kappa - 1 \} \) and let \( \delta_{\max} = \max \{ c \in \Delta^e \} \).

For \( m > \sigma_{\max} + \delta_{\max} \), each of the polynomials \( f_i^{(m)} \) belongs to \( I^e \). Let \( M = \sigma_{\max} + \max \{ \delta_{\max}, 2g - 1 \} \). Each of the pairs \( f_i^{(M+1)}, \varphi_i^{(M+1)} \) satisfies the key equation.

4.6 Error evaluation without the error evaluator polynomials

The error evaluation formula that we derived for Hermitian codes carries over to one-point codes. We have to stipulate that \( x - x(P_k) \) has a simple zero at \( P_k \), though it may be possible to remove this restriction. As was mentioned in the section on Hermitian codes, the derivation of the formula depends on the fact that at iteration \( m \), and for \( i + j \equiv m \mod \kappa \), \( \mu_i = \mu_j \) in the decoding algorithm. This is proven in Proposition 14 below.

In the proof of the proposition we will use the Cauchy-Binet Theorem. Let \( B, C \) be \( n \times 2 \) matrices and let \( T \) be an \( n \times n \) matrix such that \( C = TB \).
$I, J$ two-element subsets of $\{1, \ldots, n\}$, let $C_I$ be the two rows of $C$ indexed by $I$ and let $T_J^I$ be the $2 \times 2$ submatrix of $T$ consisting of entries from the rows in $I$ and the columns in $J$. The Cauchy-Binet theorem says that

$$\det C_I = \sum_J \det T_J^I \det B_J$$

where the sum runs over all two-element subsets $J$ of $\{1, \ldots, n\}$.

**Proposition 14.** In the $m$th iteration of the algorithm, $\mu_i = \mu_j$ for $i + j \equiv m \mod \kappa$. Furthermore for $i \neq j$, the coefficient of $z_0^*$ in the $\star$-basis expansion of each of the following determinants is 0:

$$\det \begin{pmatrix} f_i^{(m)} & \varphi_i^{(m)} \\ f_j^{(m)} & \varphi_j^{(m)} \end{pmatrix}, \quad \det \begin{pmatrix} g_i^{(m)} & \psi_i^{(m)} \\ g_j^{(m)} & \psi_j^{(m)} \end{pmatrix}, \quad \det \begin{pmatrix} f_i^{(m)} & \varphi_i^{(m)} \\ g_j^{(m)} & \psi_j^{(m)} \end{pmatrix}.$$

The coefficient of $z_0^*$ is $-dx$ in

$$\det \begin{pmatrix} f_i^{(m)} & \varphi_i^{(m)} \\ g_i^{(m)} & \psi_i^{(m)} \end{pmatrix}.$$  \hspace{1cm} (14)

The formulas may also be expressed using residues via Proposition 9. The coefficient of $z_0^*$ in $D$ is 0 if and only if $\text{res}_Q (x^a D) = 0$ for all $a$. Equation (14) is equivalent to saying that

$$\text{res}_Q \left( x^a \det \begin{pmatrix} f_i^{(m)} & \varphi_i^{(m)} \\ g_i^{(m)} & \psi_i^{(m)} \end{pmatrix} \right) = \begin{cases} 1 & \text{if } a = -1 \\ 0 & \text{otherwise} \end{cases}.$$

**Proof.** The proof proceeds by induction. The determinental conditions are readily verified for $m = 0$. The inductive step has two parts. First, we show that if the determinental conditions hold for $m$, then $\mu_i = \mu_j$ for $i + j \equiv m \mod \kappa$ in the $m$th iteration of the algorithm. Then we show that the determinental conditions hold for $m + 1$.

Assume the determinental conditions hold for $m$. Let $i + j \equiv m \mod \kappa$ and let $\mu_i, \mu_j, r_i, r_j$ be as in the algorithm. We will suppress the superscript $(m)$ on $f_i^{(m)}$ and the other data. We will show below that

$$\mu_i = \text{res}_Q \left( x^{-p-1} f_j f_i S dx - x^{-p-1} f_j \varphi_i \right). \hspace{1cm} (15)$$

One of the hypotheses of the lemma is that the coefficient of $z_0^*$ in $f_j \varphi_i - f_i \varphi_j$ is 0. Thus, we may substitute $f_i \varphi_j$ for $f_j \varphi_i$ in (15) to say that

$$\mu_i = \text{res}_Q \left( x^{-p-1} f_j f_i S dx - x^{-p-1} f_j \varphi_i \right). \hspace{1cm} (16)$$

The right hand side of this formula is the analogue of (15) with $j$ and $i$ switched. This shows that $\mu_j = \mu_i$. 

41
To establish (15), we apply item (2) of Theorem 4 to obtain

\[ \rho(f_i S dx - \varphi_i) \leq 1 + \rho(f_i) - m. \]

As we noted before Theorem 4, \( \mu_i \) is the coefficient of \( x^{-r_i} z_j^* \) in \( f_i S \). Thus,

\[ \rho(f_i S dx - \varphi_i - \mu_i x^{-r_i} z_j^* dx) < 1 + \rho(f_i) - m. \]

Multiplying by \( x^{-p-1} f_j \) we have

\[ \rho(x^{-p-1} f_j f_i S dx - x^{-p-1} f_j \varphi_i - \mu_i x^{-p-1} f_j x^{-r_i-1} z_j^* dx) < 1. \]

Equivalently, the valuation of the expression is nonnegative. This shows that

\[ \text{res}_Q(x^{-p-1} f_j f_i S dx - x^{-p-1} f_j \varphi_i) = \text{res}_Q(\mu_i x^{-p-2} f_j z_j^* dx). \]

The expression on the right has valuation \(-1\), and residue \( \mu_i \), which establishes (15).

We now prove that the determinantal conditions of the lemma hold for \( m+1 \).

Let

\[ B^{(m)}_i = \left( \begin{array}{c} f_i^{(m)} \\ g_i^{(m)} \\ \varphi_i^{(m)} \\ \psi_i^{(m)} \end{array} \right), \quad \text{and let} \quad B^{(m)} = \begin{pmatrix} B_0^{(m)} \\ B_1^{(m)} \\ B_2^{(m)} \\ \cdots \\ B_{\kappa-1}^{(m)} \end{pmatrix}. \]

Let \( T \) be the update matrix for the \( m \)th iteration, so \( B^{(m+1)} = TB^{(m)} \). We want to show that for \( I \subseteq \{1, \ldots, 2\kappa\} \) and \( B^{(m+1)}_I \) the appropriate \( 2 \times 2 \) submatrix, the coefficient of \( z_i^* \) in \( \det B^{(m+1)}_I \) is 0 unless \( I \) is a consecutive pair of the form \( \{2i+1, 2i+2\} \) for \( i = 0, \ldots, \kappa-1 \). From the inductive hypotheses, the coefficient of \( z_i^* \) in \( \det B^{(m)}_I \) is only nonzero for these \( I \). Consequently, from the Cauchy-Binet theorem

\[ \text{res}_Q(x^a \det B^{(m+1)}_I) = \sum_j \text{res}_Q(x^a \det T^j_I \det B^{(m)}_j) \quad (17) \]

where the sum runs over all \( J \) of the form \( \{2j+1, 2j+2\} \).

From the algorithm, for \( i + j \equiv m \mod \kappa \) and \( i \neq j \),

\[ \begin{pmatrix} f_i^{(m+1)} \\ g_i^{(m+1)} \\ \varphi_i^{(m+1)} \\ \psi_i^{(m+1)} \\ f_j^{(m+1)} \\ g_j^{(m+1)} \\ \varphi_j^{(m+1)} \\ \psi_j^{(m+1)} \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 0 & 0 & -\mu x^p \\ 0 & 1 & 0 & 0 \\ 0 & -\mu x^p & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_i^{(m)} \\ g_i^{(m)} \\ \varphi_i^{(m)} \\ \psi_i^{(m)} \end{pmatrix}, & \text{if } \mu = 0 \text{ or } p \geq 0 \\ \begin{pmatrix} x^p & 0 & 0 & -\mu \\ 0 & 1/\mu & 0 & 0 \\ 0 & -\mu & x^p & 0 \\ 1/\mu & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_i^{(m)} \\ g_i^{(m)} \\ \varphi_i^{(m)} \\ \psi_i^{(m)} \end{pmatrix}, & \text{otherwise.} \end{cases} \quad (18) \]
Notice that we have used $\mu = \mu_i = \mu_j$. Of course, if $i = j$, i.e. $2i = m$ mod $\kappa$, then the formula is simpler, $B_i^{(m+1)} = U_i^{(m)}B_i^{(m)}$, with $U_i^{(m)}$ from the algorithm.

In the formula (17), we consider two cases for $I$. If there is no $i, j$ with $i + j \equiv m$ mod $\kappa$ such that $I \subseteq \{2j + 1, 2j + 2, 2i + 1, 2i + 2\}$ then for all $J$ of the form $\{2j + 1, 2j + 2\}$, $T_J^f$ has a row that is all zeros, and $\det T_J^f = 0$. For such $I$, we therefore have $\text{res}_Q \left( x^a \det B_I^{(m+1)} \right) = 0$.

Now consider $I \subseteq \{2j + 1, 2j + 2, 2i + 1, 2i + 2\}$ with $i + j \equiv m$ mod $\kappa$. Similar reasoning shows that

$$\text{res}_Q \left( x^a \det B_I^{(m+1)} \right) = \text{res}_Q \left( x^a \det T_J^f \det B_J^{(m)} \right) + \text{res}_Q \left( x^a \det T_J^f \det B_J^{(m)} \right)$$

where $J = \{2j + 1, 2j + 2\}$ and $J = \{2i + 1, 2i + 2\}$. There are $\binom{4}{2}$ choices of $I$ to check for each of the two possible update matrices. If $I = J$ or $I = \overline{J}$, then either $\det T_J^f = 1$ and $\det T_J^f = 0$ or vice-versa depending on the matrix. Thus the induction hypothesis shows that the coefficient of $z_0^5$ in $\det B_I^{(m)}$ is $-dx$ as desired. For $I = \{2i + 1, 2j + 2\}$ or $\{2i + 2, 2j + 2\}$, and for either update matrix, $\det T_J^f = \det T_J^f = 0$. Thus the coefficient of $z_0^5$ in $\det B_J^{(m)}$ is $0$ as desired. Finally, for $I = \{2i + 1, 2j + 1\}$, and for either update matrix, $\det T_J^f = -\det T_J^f$ and this is a monomial in $x$.

$$\text{res}_Q \left( x^a \det B_I^{(m+1)} \right) = \text{res}_Q \left( x^a \det T_J^f \left( \det B_J^{(m)} - \det B_J^{(m)} \right) \right)$$

The induction hypothesis says that the coefficient of $z_0^5$ is the same in $\det B_J^{(m)}$ and $\det B_J^{(m)}$. Thus the coefficient of $z_0^5$ in $\det B_I^{(m+1)}$ is $0$ as desired. \hfill $\square$

**Proposition 15.** Let $B_i^{(M)} \equiv \begin{pmatrix} f_i^{(m)} \\ g_i^{(m)} \\ \psi_i^{(m)} \end{pmatrix}$. Then for all $m$,

$$\sum_{i=0}^{\kappa-1} \det B_i^{(m)} = -dx \left( \sum_{i=0}^{\kappa-1} z_iz_i^* \right) = -dx \quad (19)$$

**Theorem 5.** Suppose that $x - x(P_k)$ is a uniformizing parameter at an error position $P_k$. Let $f' = df/dx$. Then

$$e_k = \left( \sum_{i=0}^{\kappa-1} f_i'(P_k)g_i(P_k) \right)^{-1} \quad (20)$$

### 5 Bibliographical notes

The history of the key equation may be divided into three stages. In the first stage there is the key equation and iterative solution of it in Berlekamp’s book [24], and a more implementation oriented approach in Massey’s article [27].
These articles build on the Peterson-Gorenstein-Zierler decoding algorithm [34, 14] and Forney’s improvements [12], which use matrices and are less efficient.

The second stage includes two new algorithms. Sugiyama et al [42] define a key equation and give an efficient solution to it using the Euclidean algorithm. The Welch-Berlekamp algorithm [44] is related to the rational interpolation problem and has its own key equation. A number of articles explore the algebraic formulation of these algorithms, efficient implementation, or the relationship between the different algorithms. Among these we mention Fitzpatrick’s article on the key equation [11], comparisons of the Euclidean and Berlekamp-Massey algorithms by Dornstetter [6] and Heydtmann and Jensen [16], and comparisons of key equations in Moon and Gunther [28], Morii and Kasahara [29], and Yaghoobian and Blake [45]. A more extensive discussion and bibliography may be found in Roth’s textbook [37, Ch. 6].

A third stage concerns the extension of the key equation and decoding algorithms to algebraic geometry codes. The key breakthrough was Sakata’s algorithm for finding linear recurrence relations for higher dimensional arrays [38]. We are using K"otter’s version of the algorithm for algebraic curves [21], in which the ring of functions is treated as a module over a polynomial ring. The Forney formula is generalized for one-point codes in Hansen et al [15] and in Leonard [25, 26]. Several generalizations of the key equation have appeared. Chabanne and Norton [5] work with a polynomial ring in several variables and express the syndrome as a power series. The key equation is generalized to arbitrary codes on curves by Ehrhard [8], Porter, Shen and Pellikaan [35], and by Farrán [9]. A later paper by Shen and Tzeng [39], deals with one-point codes. There are elements of all these approaches in this chapter, but we have maintained the focus on one-point codes, where the generalizations are particularly simple, and the treatment is based on the articles of O’Sullivan [30, 32, 33].

Acknowledgment

This work was partly supported by the Spanish Ministry of Education through projects TSI2007-65406-C03-01 “E-AEGIS” and CONSOLIDER CSD2007-00004 “ARES”, and by the Government of Catalonia under grant 2005 SGR 00446.

References

[1] Roger Apéry. Sur les branches superlinéaires des courbes algébriques. C. R. Acad. Sci. Paris, 222:1198–1200, 1946.

[2] Elwyn R. Berlekamp. *Algebraic coding theory*. McGraw-Hill Book Co., New York, 1968.

[3] Richard Blahut. *Algebraic codes for data transmission*. Cambridge University Press, Cambridge, UK, 2003.
[4] Maria Bras-Amorós and Michael E. O'Sullivan. The correction capability of the Berlekamp-Massey-Sakata algorithm with majority voting. *Appl. Algebra Engrg. Comm. Comput.*, 17(5):315–335, 2006.

[5] Hervé Chabanne and Graham H. Norton. The $n$-dimensional key equation and a decoding application. *IEEE Trans. Inform. Theory*, 40(1):200–203, 1994.

[6] Jean-Louis Dornstetter. On the equivalence between Berlekamp’s and Euclid’s algorithms. *IEEE Trans. Inform. Theory*, 33(3):428–431, 1987.

[7] Ivan M. Duursma. Majority coset decoding. *IEEE Trans. Inform. Theory*, 39(3):1067–1070, 1993.

[8] Dirk Ehrhard. Decoding algebraic-geometric codes by solving a key equation. In *Coding theory and algebraic geometry (Luminy, 1991)*, volume 1518 of *Lecture Notes in Math.*, pages 18–25. Springer, Berlin, 1992.

[9] José-Ignacio Farrán. Decoding algebraic geometry codes by a key equation. *Finite Fields Appl.*, 6(3):207–217, 2000.

[10] Gui Liang Feng and Thammavarapu R. N. Rao. Decoding algebraic-geometric codes up to the designed minimum distance. *IEEE Trans. Inform. Theory*, 39(1):37–45, 1993.

[11] Patrick Fitzpatrick. On the key equation. *IEEE Trans. Inform. Theory*, 41(5):1290–1302, 1995.

[12] G. David Forney, Jr. On decoding BCH codes. *IEEE Trans. Inform. Theory*, IT-11:549–557, 1965.

[13] Olav Geil. On codes from norm-trace curves. *Finite Fields Appl.*, 9(3):351–371, 2003.

[14] Daniel Gorenstein and Neal Zierler. A class of error-correcting codes in $p^m$ symbols. *J. Soc. Indust. Appl. Math.*, 9:207–214, 1961.

[15] Johan P. Hansen, Helge Elbrønd Jensen, and Ralf Kötter. Determination of error values for algebraic-geometry codes and the Forney formula. *IEEE Trans. Inform. Theory*, 44(5):1881–1886, 1998.

[16] Agnes E. Heydtmann and Jørn M. Jensen. On the equivalence of the Berlekamp-Massey and the Euclidean algorithms for decoding. *IEEE Trans. Inform. Theory*, 46(7):2614–2624, 2000.

[17] Tom Høholdt, Jacobus H. van Lint, and Ruud Pellikaan. Algebraic geometry of codes. In *Handbook of coding theory, Vol. I, II*, pages 871–961. North-Holland, Amsterdam, 1998. Vol. I.
[18] Toshio Horiguchi. High-speed decoding of BCH codes using a new error-evaluation algorithm. *Electronics and Communications in Japan*, 72(12):63–71, 1989.

[19] Jørn Justesen and Tom Høholdt. *A course in error-correcting codes*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2004.

[20] Ralf Kötter. On the determination of error values for codes from a class of maximal curves. In *Proc. 35-th Allerton Conference on Communication, Control, and Computing*, pages 44–53, 1997.

[21] Ralf Kötter. A fast parallel implementation of a Berlekamp-Massey algorithm for algebraic-geometric codes. *IEEE Trans. Inform. Theory*, 44(4):1353–1368, 1998.

[22] Serge Lang. *Introduction to algebraic geometry*. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1972. Third printing, with corrections.

[23] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[24] Kwankyu Lee and Michael E. O'Sullivan. List decoding of Reed-Solomon codes from a Gröbner basis perspective. *Journal of Symbolic Computation*, to appear.

[25] Douglas A. Leonard. A generalized Forney formula for algebraic-geometric codes. *IEEE Trans. Inform. Theory*, 42(4):1263–1268, 1996.

[26] Douglas A. Leonard. Efficient Forney functions for decoding AG codes. *IEEE Trans. Inform. Theory*, 45(1):260–265, 1999.

[27] James L. Massey. Shift-register synthesis and BCH decoding. *IEEE Trans. Inform. Theory*, IT-15:122–127, 1969.

[28] Todd K. Moon and Jacob H. Gunther. On the equivalence of two Welch-Berlekamp key equations and their error evaluators. *IEEE Trans. Inform. Theory*, 51(1):399–401, 2005.

[29] Masakatu Morii and Masao Kasahara. Generalized key-equation of remainder decoding algorithm for Reed-Solomon codes. *IEEE Trans. Inform. Theory*, 38(6):1801–1807, 1992.

[30] Michael E. O’Sullivan. Decoding of Hermitian codes: the key equation and efficient error evaluation. *IEEE Trans. Inform. Theory*, 46(2):512–523, 2000.

[31] Michael E. O’Sullivan. New codes for the Berlekamp-Massey-Sakata algorithm. *Finite Fields Appl.*, 7(2):293–317, 2001.
[32] Michael E. O'Sullivan. The key equation for one-point codes and efficient error evaluation. *J. Pure Appl. Algebra*, 169(2-3):295–320, 2002.

[33] Michael E. O'Sullivan. On Koetter's algorithm and the computation of error values. *Des. Codes Cryptogr.*, 31(2):169–188, 2004.

[34] W. Wesley Peterson. Encoding and error-correction procedures for the Bose-Chaudri codes. *IRE Transactions on Information Theory*, 6:459–470, 1960.

[35] Sidney C. Porter, Ba-Zhong Shen, and Ruud Pellikaan. Decoding geometric Goppa codes using an extra place. *IEEE Trans. Inform. Theory*, 38(6):1663–1676, 1992.

[36] Oliver Pretzel. *Codes and algebraic curves*, volume 8 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998.

[37] Ron Roth. *Introduction to coding theory*. Cambridge University Press, Cambridge, 2006.

[38] Shojiro Sakata. Extension of Berlekamp-Massey algorithm to $n$ dimensions. *Inform. and Comput.*, 84(2):207–239, 1990.

[39] Ba-Zhong Shen and Kenneth K. Tzeng. Decoding geometric Goppa codes up to designed minimum distance by solving a key equation in a ring. *IEEE Trans. Inform. Theory*, 41(6, part 1):1694–1702, 1995. Special issue on algebraic geometry codes.

[40] Henning Stichtenoth. A note on Hermitian codes over $GF(q^2)$. *IEEE Trans. Inform. Theory*, 34(5, part 2):1345–1348, 1988.

[41] Henning Stichtenoth. *Algebraic function fields and codes*. Universitext. Springer-Verlag, Berlin, 1993.

[42] Yasuo Sugiyama, Masao Kasahara, Shigeichi Hirasawa, and Toshihiko Namekawa. A method for solving key equation for decoding Goppa codes. *Information and Control*, 27:87–99, 1975.

[43] Herman J. Tiersma. Remarks on codes from Hermitian curves. *IEEE Trans. Inform. Theory*, 33(4):605–609, 1987.

[44] Lloyd R. Welch and Elwyn R. Berlekamp. Error correction for algebraic block codes, 1983. US Patent 4 633 470.

[45] Tomik Yaghoobian and Ian F. Blake. Two new decoding algorithms for Reed-Solomon codes. *Appl. Algebra Engrg. Comm. Comput.*, 5(1):23–43, 1994.
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|----|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

Table 2: Steps for correct two groups in positions on a vertical line.