Low-energy Universality and the
New Charmonium Resonance at 3870 MeV

Eric Braaten and Masaoki Kusunoki

Physics Department, Ohio State University, Columbus, Ohio 43210
(Dated: March 26, 2022)

Abstract

The recently-discovered narrow charmonium resonance near 3870 MeV is interpreted as a hadronic molecule whose constituents are the charm mesons \( D^0 \) and \( \bar{D}^{*0} \) or \( D^0 \) and \( D^{*0} \). Because of an accidental fine-tuning of the molecule to very near the \( D^0 \bar{D}^{*0} \) threshold, it has some universal properties that are completely determined by the unnaturally large \( D^0 \bar{D}^{*0} \) scattering length \( a \). Its narrow width can be explained by the suppression by a factor of \( 1/a \) of decay modes other than the decay of a constituent \( \bar{D}^{*0} \) or \( D^{*0} \). Its production rates are also suppressed by a factor of \( 1/a \). A particularly predictive mechanism for generating the large scattering length is the accidental fine-tuning of a P-wave charmonium state to the \( D^0 \bar{D}^{*0} \) threshold.

PACS numbers: 12.38.-t, 12.38.Bx, 13.20.Gd, 14.40.Gx
I. INTRODUCTION

The discovery of the $J/\psi$ and other charmonium resonances in 1974 played a crucial role in the construction of the gauge theories of the strong and electroweak forces that constitute the Standard Model of particle physics. Subsequent studies of the spectroscopy, decays, and production of both charmonium and bottomonium resonances have played important roles in the development of quantum chromodynamics (QCD), the gauge theory of the strong interactions. The recent unexpected discovery by the Belle collaboration of a narrow charmonium resonance near 3.87 GeV \cite{Ref1} has presented a new challenge to our understanding of QCD.

The new charmonium state $X(3870)$ was discovered in electron-positron collisions through the $B$-meson decay $B^\pm \rightarrow K^\pm X$ followed by the decay $X \rightarrow J/\psi\pi^+\pi^-$. Its mass was measured to be $M_X = 3872.0 \pm 0.6 \pm 0.5$ MeV \cite{Ref1}. It is narrow compared to other charmonium states above the threshold for decay into two charm mesons: the upper bound on the width is $\Gamma_X < 2.3$ MeV. The discovery has been confirmed by the CDF collaboration who observed $X$ through $J/\psi\pi^+\pi^-$ events in proton-antiproton collisions and measured its mass to be $M_X = 3871.4 \pm 0.7 \pm 0.4$ MeV \cite{Ref2}.

There have been several recent papers discussing the possible interpretations of the $X(3870)$ \cite{Ref3, Ref4, Ref5, Ref6, Ref7, Ref8}. The most conventional interpretations are previously undiscovered states in the charmonium spectrum, such as one of the lowest D-wave states with spin/parity/charge-conjugation quantum numbers $J^{PC} = 2^{--}$ or $2^{--}$ or one of the first excited P-wave states with $J^{PC} = 1^{++}$ or $1^{--}$. A more exotic possibility is a “hybrid charmonium” state in which a gluonic mode has been excited. Another possibility, a $D^0\bar{D}^{*0}$ or $\bar{D}^0D^{*0}$ molecule, is motivated by the fact that the $X(3870)$ is extremely close to the threshold $3871.2 \pm 0.7$ MeV for decay into the charmed mesons $D^0D^{*0}$. The possibility of hadronic molecules formed from charm mesons was suggested long ago \cite{Ref9, Ref10}. The most favorable channels for forming a molecule from the pion-exchange interaction are the P-wave channel with $J^{PC} = 0^{+-}$ and the S-wave channel with $J^{PC} = 1^{++}$ \cite{Ref3}.

In this paper, we explore the consequences of identifying $X(3870)$ as an S-wave $D^0\bar{D}^{*0}/\bar{D}^0D^{*0}$ molecule. The tiny binding energy of the molecule implies that the $D^0\bar{D}^{*0}$ scattering length $a$ is unnaturally large. The molecule therefore has properties that depend on $a$ but are insensitive to other details of the interactions of $D^0$ and $\bar{D}^{*0}$, a phenomenon called “low-energy universality.” In Section II we discuss the implications of low-energy universality for the wavefunction of $X$ and describe two possible mechanisms for generating the large $D^0\bar{D}^{*0}$ scattering length. In Section III we discuss the implications of low-energy universality for decays of $X$. One mechanism for generating the large $D^0\bar{D}^{*0}$ scattering length, the fine-tuning of the energy of a P-wave charmonium state to the $D^0\bar{D}^{*0}$ threshold, gives a particularly distinctive pattern of branching fractions. Low-energy universality gives highly nontrivial predictions for 3-body systems, such as $D^0\bar{D}^0\bar{D}^{*0}$. Unfortunately, as shown in Section IV the spectacular possibility of shallow $D^0\bar{D}^0\bar{D}^{*0}$ molecules called Efimov states can be excluded. In Section V we present a nonrelativistic effective field theory that illustrates the two mechanisms for generating a large $D^0\bar{D}^{*0}$ scattering length. Our results are summarized in Section VI.
II. LOW-ENERGY UNIVERSALITY

We will assume that the closeness of $M_X$ to the $D^0\bar{D}^{*0}$ threshold is no accident and that $X(3870)$ is indeed a hadronic molecule whose constituents are the charm mesons $D^0$ and $\bar{D}^{*0}$ or $\bar{D}^0$ and $D^{*0}$. What makes this molecule unique among all the hadrons that can be interpreted as 2-body bound states of other hadrons is its extremely small binding energy. If the low-energy interaction between two hadrons is mediated by pion exchange, the natural scale for the binding energy of a molecule composed of the two hadrons is $m^2_\pi/(2m_{\text{red}})$, where $m_{\text{red}}$ is their reduced mass. The natural energy scale for a $D^0\bar{D}^{*0}$ molecule is about 10 MeV. The binding energy of the $X(3870)$ (which is positive by definition) has been measured to be $B_2 = -0.5 \pm 0.9$ MeV. Thus it is likely to be less than 0.4 MeV, which is much smaller than the natural energy scale. The only other two-body bound state of hadrons whose binding energy is known to be small compared to the natural energy scale is the deuteron. Its binding energy is 2.4 MeV, which is small compared to the natural energy scale of 20 MeV for a $pn$ molecule.

We will further assume that $X(3870)$ is an S-wave bound state of $D^0\bar{D}^{*0}$ or $\bar{D}^0 D^{*0}$, because this has particularly interesting implications. Since the constituents have $J^P$ quantum numbers 0$^-$ and 1$^-$, the $J^{PC}$ quantum numbers of the molecule must be 1$^{++}$ or 1$^{+-}$. The interaction between $D^0$ and $\bar{D}^{*0}$ at energies less than $m^2_\pi/m_D \approx 10$ MeV is dominated by the S-wave channel and can be described by a single parameter: the S-wave $D^0\bar{D}^{*0}$ scattering length $a$. A shallow S-wave bound state implies an S-wave scattering length that is large compared to the natural length scale $1/m_\pi$. The low-energy few-body observables for nonrelativistic particles with short-range interactions and a large scattering length have universal features that are insensitive to the details of the mechanism that generates the large scattering length. This phenomenon is called low-energy universality. If $a > 0$, the simplest universal prediction is that there is a shallow 2-body bound state whose binding energy $B_2$ for sufficiently large $a$ approaches

$$B_2 \to \frac{1}{2m_{\text{red}} a^2},$$

where $m_{\text{red}}$ is the reduced mass of the two constituents. If the binding energy of the $X(3870)$ were measured, the $D^0\bar{D}^{*0}$ scattering length $a$ could be predicted using (1) with $m_{\text{red}} = m_{D^0}m_{D^{*0}}/(m_{D^0} + m_{D^{*0}})$. For example, if the binding energy of $X(3870)$ were 0.5 MeV or 0.1 MeV, the scattering length would be 6.3 fm or 14.2 fm, respectively. These are both much larger than the natural length scale $1/m_\pi = 1.5$ fm.

Low-energy universality has other implications for the interpretation of $X(3870)$ as a $D^0\bar{D}^{*0}/\bar{D}^0 D^{*0}$ molecule. There is a universal prediction for the $D^0\bar{D}^{*0}$ or $\bar{D}^0 D^{*0}$ wavefunction:

$$\psi(r) \to (2\pi a)^{-1/2}\exp(-r/a)/r.$$

Voloshin has exploited this universal wavefunction to calculate the momentum distributions for the decays $X \to D^0\bar{D}^{*0} \pi^0$ and $X \to D^0\bar{D}^0 \gamma$. There are also components of the wavefunction that correspond to other hadronic states with the same $J^{PC}$ quantum numbers. If $J^{PC} = 1^{++}$, they include the P-wave charmonium states $\chi_{c1}(1P)$ and $\chi_{c1}(2P)$. If $J^{PC} = 1^{+-}$, they include the P-wave charmonium states $h_c(1P)$ and $h_c(2P)$. In either case, they also include $D^+ D^{*-}/D^- D^{*+}$ states and $D^{*0} \bar{D}^{*0}$ states. In an appropriate hadronic basis,
the quantum state for $X(3870)$ can be written

$$|X\rangle = Z_{DD^*}^{1/2} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}(p) \frac{1}{\sqrt{2}} (|D^0(p)D^{*0}(-p)| + |\bar{D}^0(p)\bar{D}^{*0}(-p)|),$$

$$+ \sum_H Z_H^{1/2} |H\rangle.$$  \hspace{1cm} (3)

where $\tilde{\psi}(p)$ is the Fourier transform of the $D^0\bar{D}^{*0}/\bar{D}^0D^{*0}$ wavefunction and the sign $\pm$ is determined by the charge conjugation quantum number $C = \pm 1$. The other hadronic states $H$ can be discrete states, such as $\chi_{c1}(2P)$ or $h_c(2P)$, or continuum states, such as $D^+(p)D^{*-}(-p)$. The probability factors $Z_{DD^*}$ and $Z_H$ are real and positive, and they add up to 1. Low-energy universality implies that as the scattering length $a$ increases, the probabilities for components of the wavefunction other than $D^0\bar{D}^{*0}$ or $\bar{D}^0D^{*0}$ decrease as $1/a$ and the $D^0\bar{D}^{*0}/\bar{D}^0D^{*0}$ wavefunction approaches (2). In the limit $a \to \infty$, the state becomes a pure $D^0\bar{D}^{*0}/\bar{D}^0D^{*0}$ molecule.

A scattering length that is large compared to the natural length scale necessarily requires a fine-tuning. In the case of the $D^0\bar{D}^{*0}$ molecule, the fine-tuning parameters can be identified with the up and down quark masses $m_u$ and $m_d$. The masses of $D^0$ and $\bar{D}^{*0}$ are sensitive to $m_u$, because these hadrons contain an up quark as a constituent. The $D^0\bar{D}^{*0}$ potential is sensitive to $m_u$ and $m_d$ through the pion mass. There are two distinct mechanisms for generating a large $D^0\bar{D}^{*0}$ scattering length. The first mechanism is a fine-tuning of parameters that have a large effect on the $D^0\bar{D}^{*0}/\bar{D}^0D^{*0}$ channel without significantly affecting other channels. This could be a fine-tuning of the range and depth of the $D^0\bar{D}^{*0}$ potential so that there is a bound state very close to threshold and thus a large scattering length. Equivalently, it could be a fine-tuning of the masses of the $D^0$ and $\bar{D}^{*0}$ to obtain a bound state very close to threshold in the $D^0\bar{D}^{*0}$ potential. This mechanism requires the quantum numbers of $X$ to be $J^{PC} = 1^{++}$, because this is the only S-wave channel for which the potential due to pion exchange is sufficiently attractive to produce a bound state \[. \hspace{1cm} \]

In the limit $a \to \infty$, the probabilities for components of the wavefunction other than $D^0\bar{D}^{*0}$ or $\bar{D}^0D^{*0}$ scale as $1/a$. This will be illustrated in Section \[ using an explicit field theory model. If the energy gap $\nu_H$ between the state $H$ in (3) and the $D^0\bar{D}^{*0}$ threshold is much greater than the natural energy scale $m_\pi^2/m_{\text{red}}$, a more complete estimate of the dimensionless suppression factor in $Z_H$ is $m_\pi^2/(m_{\text{red}}\nu_H^2a)$. For most channels, the energy gap $\nu_H$ is much larger than the natural energy scale. For example, the energy gap for $\chi_{c1}(1P)$ is $\nu_H = -360$ MeV. If $am_\pi \gg 1$, the suppression factor $m_\pi^2/(m_{\text{red}}\nu_H^2a)$ is $1/1600$ for $B_2 = 0.5$ MeV and $1/3500$ for $B_2 = 0.1$ MeV. If this is the correct mechanism for generating the large scattering length, we can probably neglect all components of the wavefunction other than $D^0\bar{D}^{*0}/\bar{D}^0D^{*0}$ and set $Z_{DD^*} \approx 1$.

A second mechanism for a large $D^0\bar{D}^{*0}$ scattering length is an accidental fine-tuning of one of the P-wave charmonium states $\chi_{c1}(2P)$ or $h_c(2P)$ to the $D^0\bar{D}^{*0}$ threshold. $X(3870)$ will have the same $J^{PC}$ quantum numbers as the charmonium state: $1^{++}$ in the case of $\chi_{c1}(2P)$ and $1^{+-}$ in the case of $h_c(2P)$. This mechanism is analogous to the Feshbach resonances \[ that can be used to control the scattering lengths for atoms by adjusting the magnetic field \[. \hspace{1cm} \]

Feshbach resonances are currently being used to tune the scattering lengths for atoms to arbitrarily large values in order to study Bose-Einstein condensates of bosonic atoms and degenerate gases of fermionic atoms in the strongly-interacting regime. In the case of the $D^0\bar{D}^{*0}$ molecule, the fine-tuning parameter can be identified as $m_u$, which can shift the $D^0$ and $\bar{D}^{*0}$ masses, thus changing the energy gap $\nu$ between the $\chi_{c1}(2P)$ or $h_c(2P)$
and the $D^0\bar{D}^{*0}$ threshold. In potential models, which ignore the coupling of charmonium states to continuum channels such as $DD$ and $DD^*$, the estimates of the energy gap for $\chi_{c1}(2P)$ or $h_c(2P)$ are both $\nu \approx 90$ MeV [13, 14]. The predictions of these models for the mass of the 1D state $\psi(3770)$ with $J^{PC} = 1^{--}$ are too large by about 50 MeV, so they are also likely to overpredict the masses of the $\chi_{c1}(2P)$ and $h_c(2P)$. The error presumably arises mostly from the neglect of coupled-channel effects, which are sensitive to the light quark masses. If the coupled-channel effects shift the 2P charmonium states down by about 90 MeV relative to the $DD^*$ threshold, they could fortuitously tune the energy gap $\nu$ for $\chi_{c1}(2P)$ or $h_c(2P)$ to be smaller than the natural low-energy scale $m_c^2/m_D \approx 10$ MeV associated with pion exchange. In this case, a resonant interaction between the $\chi_{c1}(2P)$ or $h_c(2P)$ and $D^0\bar{D}^{*0}$ states generates a large $D^0\bar{D}^{*0}$ scattering length $a$ that increases as $1/\nu$. If $a > 0$, there is a shallow bound state whose binding energy approaches $|\Gamma_{PC}| = 1$ in the limit $a \to \infty$, the probability $Z_X$ for $\chi_{c1}(2P)$ or $Z_h$ for $h_c(2P)$ scales as $1/a$, as do the probabilities $Z_H$ for all other channels besides $D^0\bar{D}^{*0}/\bar{D}^0D^{*0}$. This will be illustrated in Section V using an explicit field theory model. The probability $Z_X$ or $Z_h$ includes a dimensionless suppression factor $1/(am_c)$, whose value is about 1/4.3 for $B_2 = 0.5$ MeV and 1/9.7 for $B_2 = 0.1$ MeV. There may also be further suppression from a small numerical coefficient associated with Zweig’s rule, because the processes $\chi_{c1}(2P) \to D^0\bar{D}^{*0}$ and $h_c(2P) \to D^0\bar{D}^{*0}$ require the creation of a light quark-antiquark pair. If this is the correct mechanism for generating the large scattering length, we can probably neglect all components of the wavefunction other than $D^0\bar{D}^{*0}/\bar{D}^0D^{*0}$ and $\chi_{c1}(2P)$ or $h_c(2P)$. We can then set $Z_{DD^*} \approx 1 - Z_X$ in the case $J^{PC} = 1^{++}$ and $Z_{DD^*} \approx 1 - Z_h$ in the case $J^{PC} = 1^+$. 

III. DECAYS

An important requirement for any interpretation of the $X(3870)$ is that it provide an explanation for its narrow width. The upper bound $\Gamma_X < 2.3$ MeV implies that the width of $X$ is more than an order of magnitude smaller than that of the D-wave state $\psi(3770)$. According to our interpretation, $X$ is below the $D^0\bar{D}^{*0}$ threshold and it therefore cannot decay into $D^0\bar{D}^{*0}$. Its quantum numbers $J^P = 1^+$ forbid a decay into $D^0\bar{D}^0$ or $D^+D^-$. It can however decay into $D^0\bar{D}^0\pi^0$ or $D^0\bar{D}^0\gamma$ by the decay of a constituent $D^{*0}$ or $\bar{D}^{*0}$ of the molecule. It can also decay into a lighter charmonium state by a radiative or hadronic transition or into light hadrons via a process in which the charm quark and antiquark annihilate.

We first consider the contribution to the width $\Gamma_X$ from the decay of a constituent $D^{*0}$ into $D^0\pi^0$ or $D^0\gamma$. The width $\Gamma[D^{*0} \to D^0\pi^0]$ can be deduced from the measured width of the $D^{*+}$, the branching fraction for $D^{*+} \to D^0\pi^+ + D^0\pi^0$, and isospin symmetry: $\Gamma[D^{*0} \to D^0\pi^0] = 31 \pm 7$ keV. We have treated the difference between the branching fraction for $D^{*+} \to D^0\pi^+$ and twice the branching fraction for $D^{*+} \to D^+\pi^0$ as a systematic error associated with isospin symmetry breaking. The width $\Gamma[D^{*0} \to D^0\gamma]$ can then be deduced from the measured branching fractions for $D^{*0} \to D^0\pi^0$ and $D^{*0} \to D^0\gamma$: $\Gamma[D^{*0} \to D^0\gamma] = 19 \pm 5$ keV. The contributions to the width $\Gamma_X$ from these decays are

$$\Gamma[X \to D^0\bar{D}^0\pi^0] = Z_{DD^*}C_\pi\Gamma[D^{*0} \to D^0\pi^0],$$  \hspace{1cm} \tag{4a}$$
$$\Gamma[X \to D^0\bar{D}^0\gamma] = Z_{DD^*}C_\gamma\Gamma[D^{*0} \to D^0\gamma].$$  \hspace{1cm} \tag{4b}$$
The factors $C_\pi$ and $C_\gamma$ take into account interference between the decay of $D^{*0}$ from the $D^0\bar{D}^{*0}$ component of the wavefunction and the decay of $D^{*0}$ from the $D^0\bar{D}^{*0}$ component. If the charge conjugation quantum number of $X$ is $C = +1$, there is constructive interference in the decay $X \to D^0\bar{D}^0\pi^0$ and destructive interference in the decay $X \to D^0\bar{D}^0\gamma$. If $C = -1$, the pattern is reversed. If the binding energy $B_2$ is 0.5 MeV to about 2.2 (or 0.56) if $B_2 = 0.1$ MeV and to 2 if $B_2 = 0$. The coefficient $C_\gamma$ ranges from about 0.58 (or 3.42) if $B_2 = 0.5$ MeV to about 1.36 (or 2.64) if $B_2 = 0.1$ MeV and to 2 if $B_2 = 0$. The probability factor $Z_{DD^*}$ is close to 1. Thus the lower bound on the width provided by the sum of (4a) and (4b) ranges from 0.58 (or 3.42) if the binding energy $B$ is 0.13) if the binding energy $B$.

All these contributions to the decay rate are suppressed by a factor of $1/|\tilde{\chi}|$.

The remaining decay channels of the $X(3870)$ are radiative transitions to lower charmonium states such as $X \to \psi(2S)\gamma$ or $X \to \eta_c(2S)\gamma$, hadronic transitions to lower charmonium states such as the discovery mode $X \to J/\psi\pi^+\pi^-$, and annihilation decays into light hadrons such as $X \to p\bar{p}$. These decays proceed through the short-distance part of the $D^0\bar{D}^{*0}$ wavefunction or through other components of the wavefunction, such as $\chi_{c1}(2P)$ or $h_c(2P)$. All these contributions to the decay rate are suppressed by a factor of $1/a$ and go to 0 as the binding energy of $X$ goes to 0.

If the large $D^0\bar{D}^{*0}$ scattering length arises from an accidental fine-tuning within the $D^0\bar{D}^{*0}/\bar{D}^0\bar{D}^{*0}$ channel with $J^{PC} = 1^{++}$, the probability $Z_H$ for a component of the wavefunction with a large energy gap $|\nu_H| \gg m_\pi^2/m_{red}$ is suppressed by $m_\pi^2/(m_{red}^2a)$. Thus the radiative transitions, hadronic transitions, and annihilation decays of $X(3870)$ are dominated by the $D^0\bar{D}^{*0}/\bar{D}^0\bar{D}^{*0}$ component of the wavefunction. For example, the amplitudes for radiative and hadronic transitions to $J/\psi$ can be approximated by

$$A[X \to J/\psi + \gamma] \approx Z_{DD^*}^{1/2} \int \frac{d^3p}{(2\pi)^3} \bar{\psi}(p)\sqrt{2}A[D^0(p)\bar{D}^{*0}(\bar{p}) \to J/\psi + \gamma], \quad (5a)$$

$$A[X \to J/\psi + h] \approx Z_{DD^*}^{1/2} \int \frac{d^3p}{(2\pi)^3} \bar{\psi}(p)\sqrt{2}A[D^0(p)\bar{D}^{*0}(\bar{p}) \to J/\psi + h], \quad (5b)$$

where $h$ consists of light hadrons. An example of a light hadronic state $h$ is the discovery channel $\pi^+\pi^-$. The light hadronic state $h$ must be odd under charge conjugation. Since $\bar{D}^0$ and $D^{*0}$ have isospin $\frac{1}{2}$ and $J/\psi$ has isospin 0, $h$ can have isospin 0 or 1. For $p \ll m_\pi$, the momentum space wavefunction $\psi(p)$ can be approximated by its universality limit:

$$\psi(p) \to (2\pi a)^{-1/2} \frac{1}{p^2 + 1/a^2}. \quad (6)$$

The momentum integrals in (5) are cut off at large momentum by the $p$-dependence of the transition amplitude for $D^0\bar{D}^{*0} \to J/\psi + h$. For this transition to occur, the heavy $c$ and $\bar{c}$ in the $J/\psi$ must be in the same momentum states as the $c$ and $\bar{c}$ in the $D^0$ and $\bar{D}^{*0}$. Thus
the transition amplitude includes a factor of the momentum space wavefunction \( \tilde{\psi}_{J/\psi}(p) \) for the \( J/\psi \). The momentum scale associated with this wavefunction is \( m_c v \approx 700 \text{ MeV} \), where \( v \) is the typical velocity of the charm quark in charmonium. The amplitude on the right side of (5) therefore includes the overlap factor \( \int d^3p \tilde{\psi}(p) \tilde{\psi}_{J/\psi}(p) \). The explicit factor of \( a^{-1/2} \) in the universal wavefunction in (2) combines with a factor of \( (m_c v)^{-1/2} \) from the integral to give a dimensionless suppression factor. Thus the decay rate scales as \( 1/(am_c v) \) as \( a \to \infty \).

If a phenomenological framework for calculating radiative and hadronic transitions of \( D^0 \bar{D}^{*0} \) to quarkonium were available, the rates for radiative and hadronic transitions of \( X \) could be calculated using equations analogous to (5).

If the large \( D^0 \bar{D}^{*0} \) scattering length arises from an accidental fine-tuning of the \( P \)-wave charmonium state \( \chi_{c1}(2P) \) to the \( D^0 \bar{D}^{*0} \) threshold, the radiative and hadronic transitions and the annihilation decays of \( X(3870) \) can also proceed through the \( \chi_{c1}(2P) \) component of the wavefunction. These contributions to the decay rate are suppressed by the probability factor \( Z_X \), which scales as \( 1/(am_c) \). Although this suppression factor has the same power of \( a \) as in the \( D^0 \bar{D}^{*0} \) contribution, the \( \chi_{c1}(2P) \) contribution may be numerically larger because of a factor of \( m_c v/m_\pi \) in the ratio of the suppression factors. Thus decays of \( X \) into modes that are possible final states of the decay of \( \chi_{c1}(2P) \) are likely to be dominated by the \( \chi_{c1}(2P) \) component of the wavefunction. The rate for these decays will be given by the rate for the corresponding decays of the \( \chi_{c1}(2P) \) in the absence of the fine-tuning multiplied by the probability factor \( Z_X \). For example, the decay rates for radiative transitions to \( J/\psi \) and for hadronic transitions to \( J/\psi \) via the emission of a light hadronic state with total isospin 0 are

\[
\Gamma[X \to J/\psi + \gamma] \approx Z_X \Gamma[\chi_{c1}(2P) \to J/\psi + \gamma], \tag{7a}
\]
\[
\Gamma[X \to J/\psi + h_{I=0}] \approx Z_X \Gamma[\chi_{c1}(2P) \to J/\psi + h_{I=0}]. \tag{7b}
\]

The last factors in (7a) and (7b) are the decay rates for \( \chi_{c1}(2P) \) assuming that it has mass \( m_X \) but ignoring the resonant interaction with \( D^0 \bar{D}^{*0} \). Hadronic transitions in which the light hadronic state \( h \) has total isospin 1, such as \( X \to J/\psi + \rho^0 \), cannot proceed through the \( \chi_{c1}(2P) \) component of the wavefunction. They must therefore be dominated by the \( D^0 \bar{D}^{*0}/D^0 \bar{D}^{*0} \) component. The amplitude for such a transition can be approximated by (5). There are well-developed phenomenological frameworks for calculating the radiative and hadronic transition rates [16, 17] for charmonium states such as \( \chi_{c1}(2P) \). If the rates for several radiative or hadronic transitions with isospin 0 were measured and found to be all suppressed relative to the predictions for \( \chi_{c1}(2P) \) decays by a common factor \( Z_X \), it would be strong evidence in favor of this fine-tuning mechanism. The total width of \( \chi_{c1}(2P) \) in the absence of the resonant interaction with \( D^0 \bar{D}^{*0} \) must be significantly larger than the width of \( \chi_{c1}(1P) \), which is about 1 MeV, since \( \chi_{c1}(2P) \) has more decay channels and the decays have larger phase space. However, the suppression from the probability factor \( Z_X \) could reduce this contribution to the width \( \Gamma_X \) so that it is comparable to or even smaller than the contribution from the decay of \( D^{*0} \) or \( D^{*0} \).

If the large \( D^0 \bar{D}^{*0} \) scattering length arises from an accidental fine-tuning of the \( P \)-wave charmonium state \( h_{c}(2P) \) to the \( D^0 \bar{D}^{*0} \) threshold, the radiative and hadronic transitions and the annihilation decays of \( X \) can also proceed through the \( h_{c}(2P) \) component of the wavefunction. The decay rates for radiative transitions and for hadronic transitions with total isospin 0 would be given by the corresponding decay rates of \( h_{c}(2P) \) multiplied by a probability factor \( Z_h \).

The identification of \( X(3870) \) as a shallow S-wave molecule also has implications for its
production rate in high-energy collisions. As the $D^0\bar{D}^{*0}$ scattering length $a$ increases, the production rate decreases as $1/a$. If the large value of $a$ arises from an accidental fine-tuning within the $D^0\bar{D}^{*0}$ channel, the production will proceed primarily through the creation of $D^0$ and $\bar{D}^{*0}$ (or $\bar{D}^0$ and $D^{*0}$) with small relative momentum of order $1/a$. In this case, the suppression factor of $1/a$ in the production rate comes from the factor $a^{-1/2}$ in the universal wavefunction $|2\rangle$. If the large value of $a$ arises from an accidental fine-tuning of the P-wave charmonium state $\chi_{c1}(2P)$ to the $D^0\bar{D}^{*0}$ threshold, the production rate may be dominated by production of the $\chi_{c1}(2P)$. In this case, the suppression factor of $1/a$ comes from the probability $Z_\chi$ for the $\chi_{c1}(2P)$ component of $X$. Similarly, if the large value of $a$ arises from an accidental fine-tuning of $h_c(2P)$ to the $D^0\bar{D}^{*0}$ threshold, the production rate is suppressed by a factor of $1/a$ from the probability $Z_{h_c}$ for the $h_c(2P)$ component of $X$.

IV. ABSENCE OF EFIMOV STATES

The most remarkable predictions of low-energy universality, which were discovered by Efimov [18], occur in the 3-body sector. At sufficiently low energies, the effective interaction between three nonrelativistic particles with short-range forces can be described by an effective potential $V_{\text{eff}}(R)$ that depends only on the hyperspherical radius $R$, which is a weighted average of the separations of the three particles [19]. If the scattering length is large compared to the range $\ell$ of the force, the effective potential in the region $\ell \ll R \ll |a|$ is scale-invariant. In the case of identical particles of mass $m$, the hyperspherical radius is just the root-mean-square separation of the three pairs of particles and the scale-invariant potential is

$$V_{\text{eff}}(R) \approx -\frac{4 - \lambda_0}{2mR^2}, \quad (8)$$

where $\lambda_0$ is the minimum of the nontrivial solutions to

$$\sqrt{3}\lambda^{1/2} \cos(\pi \lambda^{1/2}/2) = 8 \sin(\pi \lambda^{1/2}/6). \quad (9)$$

The minimum solution is $\lambda_0 = -s_0^2$, where $s_0 \approx 1.00624$. In the resonant limit $a \to \infty$ in which the scattering length is tuned to be infinitely large, the 2-body bound state has zero binding energy and there are infinitely many arbitrarily-shallow 3-body bound states called Efimov states. If the particles are identical bosons, the ratio of the binding energies of adjacent states approaches a universal constant $e^{2\pi/s_0} \approx 515.03$. The 3-body spectrum in the resonant limit has an asymptotic discrete scaling symmetry with discrete scaling factor $e^{\pi/s_0} \approx 22.7$. This symmetry is related to an infrared renormalization group limit cycle [19]. A limit that is more relevant to a physical problem with a large but finite scattering length is the scaling limit defined by $\Lambda \to \infty$ with $a$ fixed, where $\Lambda$ is the natural momentum scale set by the inverse of the range of the interaction. In the scaling limit, the binding energies $B_3$ and $B'_3$ of the shallowest and next-to-shallowest Efimov states for identical bosons are in the intervals $B_2 < B_3 < 6.75B_2$ and $6.75B_2 < B'_3 < 1406B_2$, where $B_2 = 1/(ma^2)$ is the 2-body binding energy [21]. If these binding energies are smaller than the natural energy scale $\Lambda^2/m$, these Efimov states should appear as real states in the spectrum. Thus, there should be at least one Efimov state if $\Lambda^2/m > 6.75B_2$ and at least two if $\Lambda^2/m > 1406B_2$. As an illustration, we consider helium atoms, which have a large scattering length [21]. The helium dimer is very shallow: its binding energy $B_2 \approx 1.3$ mK is smaller than the natural
low-energy scale $\Lambda^2/m \approx 400$ mK by about a factor of 300. Thus we would expect either one or two Efimov states. There are in fact two helium trimers: a ground state and an excited state with binding energies $B_3' \approx 130$ mK and $B_3 \approx 2$ mK. Both can be interpreted as Efimov states [$21$].

The large $D^0 \bar{D}^{*0}$ scattering length raises the exciting possibility of shallow $D^0 D^0 \bar{D}^{*0}$ molecules within 10 MeV of the $D^0 D^0 \bar{D}^{*0}$ threshold generated by the Efimov effect. Unfortunately, this possibility can be excluded. The $D^0 D^0 \bar{D}^{*0}$ sector involves only two identical bosons and only two of the three pairs of particles have a resonant interaction with a large scattering length. Furthermore a $D^0 \bar{D}^{*0}$ pair can fluctuate into a $\bar{D}^0 \bar{D}^{*0}$ pair, and the other $D^0$ has no resonant interaction with this component of the wavefunction. Low-energy interactions in the 3-body sector can again be described by an effective potential which in the region $m_{\pi}^{-1} \ll R \ll |a|$ has the scale-invariant form [8]. The form of the potential can be derived from results given in Ref. [20]. If we ignore the 8% mass difference between the $D^0$ and $\bar{D}^{*0}$, the only difference is that the equation for $\lambda_0$ is

$$\sqrt{3}\lambda^{1/2} \cos(\pi\lambda^{1/2}/2) = 2\sin(\pi\lambda^{1/2}/6).$$

Of the factor of 4 difference with [9], one factor of 2 comes from there being only two identical bosons instead of three and the other factor of 2 comes from the 3-body system being a superposition of a $D^0 D^0 \bar{D}^{*0}$ molecule and a $\bar{D}^0 D^0 \bar{D}^{*0}$ molecule. The minimum nontrivial solution to [10] is $\lambda_0 \approx 0.3533$. Since this is positive, the Efimov effect does not arise and there are no shallow 3-body bound states.

V. A FIELD THEORY MODEL

If we consider only momenta small compared to the natural momentum scale $m_{\pi}$, hadrons such as $D^0$ and $\bar{D}^{*0}$ can be treated as point particles with pointlike interactions and can therefore be described by a local nonrelativistic field theory. In Section II, we discussed two fine-tuning mechanisms that can generate a large scattering length for $D^0 D^{*0}/D^0 \bar{D}^{*0}$. Only one of these mechanism is capable of producing a large scattering length in the $1^-$ channel, but either one is capable of producing a large scattering length in the $1^+$ channel. We will therefore focus on the possibility $J^{PC} = 1^+$ for the quantum numbers of $X(3870)$. A model that can describe either of the two fine-tuning mechanisms is a nonrelativistic field theory with local fields $D$, $\bar{D}$, $\mathbf{D}$, $\mathbf{\bar{D}}$, and $\chi$ for the $D^0$, $\bar{D}^0$, $D^{*0}$, $\bar{D}^{*0}$, and $\chi_{cl}(2P)$. The hamiltonian density is the sum of mass terms, kinetic terms, and interaction terms:

$$\mathcal{H}_{\text{mass}} = m_{D^0} (D^\dagger D + \bar{D}^\dagger \bar{D}) + m_{D^{*0}} (D^\dagger \cdot D + \bar{D}^\dagger \cdot \bar{D})$$

$$+ (m_{D^0} + m_{D^{*0}} + \nu_0) \chi^\dagger \cdot \chi$$

$$\mathcal{H}_{\text{kin}} = -\frac{1}{2} m_{D^0}^{-1} (D^\dagger \nabla^2 D + \bar{D}^\dagger \nabla^2 \bar{D}) - \frac{1}{2} m_{D^{*0}}^{-1} (D^\dagger \cdot \nabla^2 D + \bar{D}^\dagger \cdot \nabla^2 \bar{D})$$

$$- \frac{1}{2} (m_{D^0} + m_{D^{*0}})^{-1} \chi^\dagger \cdot \nabla^2 \chi.$$ 

$$\mathcal{H}_{\text{int}} = \lambda_0 (\mathbf{D} \cdot \mathbf{D})^\dagger \cdot (\mathbf{D} \cdot \mathbf{D})$$

$$+ g_0 \left[ \chi^\dagger \cdot (\mathbf{D} \cdot \mathbf{D}) + (\mathbf{D} \cdot \mathbf{D})^\dagger \cdot \chi \right],$$

where $\lambda_0$, $g_0$, and $\nu_0$ are bare parameters that require renormalization. A similar field theory has been used to describe the behavior of cold atoms near a Feshbach resonance [22]. If we impose an ultraviolet cutoff $\Lambda$ on loop momenta and drop terms that decrease as inverse
powers of $\Lambda$, the cutoff dependence can be removed by eliminating $\lambda_0$, $g_0$, and $\nu_0$ in favor of renormalized parameters $\lambda$, $g$, and $\nu$ defined by

\begin{align}
\lambda &= Z^{-1}_\lambda \lambda_0, \\
g &= Z^{-1}_g g_0, \\
\nu &= \nu_0 + [Z^{-1}_\lambda - 1]g_0^2/\lambda_0,
\end{align}

(12a) (12b) (12c)

where the renormalization constant $Z_\lambda$ is

\[ Z_\lambda = 1 + \frac{2}{\pi^2} m_{\text{red}} \lambda \Lambda \]

(13)

and $m_{\text{red}} = m_{D^0} m_{D^{*0}}/(m_{D^0} + m_{D^{*0}})$ is the reduced mass. Note that the combinations $g_0/\lambda_0 = g/\lambda$ and $\nu_0 - g_0^2/\lambda_0 = \nu - g^2/\lambda$ are renormalization invariants.

The natural scale for the ultraviolet momentum cutoff is $\Lambda \sim m_\pi$. The natural magnitude for the bare coupling constant $\lambda_0$ can be deduced by dimensional analysis: $|\lambda_0| \sim 1/(m_{\text{red}} m_\pi)$. This can be made evident by writing the renormalization condition (12a) in the form

\[ \frac{1}{\lambda} = \frac{1}{\lambda_0} + \frac{2}{\pi^2} m_{\text{red}} \lambda \Lambda. \]

(14)

If the renormalized coupling constant $\lambda$ is fixed and $\Lambda$ is sufficiently large, $\lambda_0$ must scale like $(m_{\text{red}} \Lambda)^{-1}$ to compensate for the effect of the ultraviolet cutoff. The natural magnitude for $g_0$ is $\zeta m_\pi^{1/2}/m_{\text{red}}$, where the factor of $m_\pi^{1/2}/m_{\text{red}}$ comes from dimensional analysis and $\zeta$ is a numerical suppression factor associated with the violation of Zweig’s rule by the process $\chi_{c1}(2P) \to D^0 \bar{D}^{*0}$, which requires the creation of a light quark-antiquark pair. The renormalization condition (12b) implies that the numerical suppression factor $\zeta$ is stable under renormalization and does not require fine-tuning. There is no natural magnitude for the bare parameter $\nu_0$: it is completely adjustable. In the absence of fine-tuning, the renormalization constant $Z_\lambda$ in (13) is comparable to 1. The renormalization conditions (12a), (12b), and (12c) then imply that the natural magnitudes of the renormalized coupling constants are $|\lambda| \sim (m_{\text{red}} m_\pi)^{-1}$, $|g| \sim \zeta m_\pi^{1/2}/m_{\text{red}}$, and $|\nu| \sim \max(|\nu_0|, \zeta^2 m_\pi^2/m_{\text{red}})$.

The 2-body scattering amplitude in this model can be calculated analytically. The scattering length is

\[ a = \frac{m_{\text{red}}}{\pi} \left( \lambda - \frac{g^2}{\nu} \right). \]

(15)

The natural magnitude for $|a|$ is $1/m_\pi$. The scattering length can be made unnaturally large either by making $\lambda$ sufficiently large, which corresponds to tuning the potential between $D^0$ and $\bar{D}^{*0}$, or by making $\nu$ sufficiently small, which corresponds to tuning the energy gap between the $\chi_{c1}(2P)$ and the $D^0 \bar{D}^{*0}$ threshold. In either case, low-energy universality implies that as $a$ increases, the binding energy of the molecule approaches (11) and the $D^0 \bar{D}^{*0}$ or $\bar{D}^0 D^{*0}$ wavefunction approaches (2).

The first mechanism for generating a large scattering length is to make $\lambda$ unnaturally large: $|\lambda| \gg |\lambda_0|$. This can be accomplished by tuning $\lambda_0$ towards the critical value $-\pi^2/(2m_{\text{red}} \Lambda)$, so that there is a near cancellation between the two terms on the right side of (14). This fine-tuning makes the renormalization constant $Z_\lambda$ much less than 1. The renormalization condition (12b) implies that this fine-tuning also increases the strength of
the effective coupling constant between $\chi$ and $\mathcal{DD}$: $|g| \gg |g_0|$. This is also evident from the fact that $g/\lambda = g_0/\lambda_0$ is a renormalization invariant. There is a limit to how large the scattering length can be made using this mechanism. When $Z_\chi$ becomes smaller than $g_0^2/|\lambda_0\nu_0|$, the $g_0^2/\lambda_0$ term in $[12c]$ begins to dominate over the $\nu_0$ term. In this case, both terms in the scattering length (15) become large and they tend to cancel each other. Thus, with this mechanism, the maximum magnitude of the scattering length is of order $(\lambda_0/g_0)^2m_{\text{red}}|\nu_0|$ which is of order $\zeta^{-2}m_{\text{red}}|\nu_0|/m_\pi^3$.

The second mechanism for generating a large scattering length is to make $\nu$ sufficiently small. This can be accomplished by tuning $\nu_0$ towards the critical value $-|Z_\lambda^{-1} - 1|g_0^2/\lambda_0$ for which there is a near cancellation between the two terms on the right side of (12c). The scattering length can be made arbitrarily large using this mechanism.

The calculation of the binding energy $B_2$ of $X$ and of the probability $Z_\chi$ for the $\chi_{\text{cl}}(2P)$ component of the wavefunction can both be reduced to the solution of a cubic polynomial. The binding momentum $\kappa$ defined by $B_2 = \kappa^2/(2m_{\text{red}})$ satisfies the cubic equation

$$\kappa^2 + 2m_{\text{red}}\nu = \frac{m_{\text{red}}}{\pi} \lambda \kappa \left[ \kappa^2 + 2m_{\text{red}} \left( \nu - g^2/\lambda \right) \right]. \quad (16)$$

In either of the two limits $\lambda \to \infty$ or $\nu \to 0$, one of the three roots of this polynomial has the limiting behavior $\kappa \to 1/a$. If $a > 0$, the probability $Z_\chi$ for the $\chi_{\text{cl}}(2P)$ component of the wavefunction is

$$Z_\chi = \frac{1}{2\pi} \left( \frac{g^2/\lambda^2}{\kappa^2 + 2m_{\text{red}}(\nu - g^2/\lambda)^2} + \frac{1}{4\pi\kappa} \right)^{-1} \frac{\kappa - \pi/(m_{\text{red}}\lambda)}{\kappa^2 + 2m_{\text{red}}(\nu - g^2/\lambda)}. \quad (17)$$

After expressing the observables $B_2$ and $Z_\chi$ as functions of $a$ and the renormalization invariants $g/\lambda$ and $\nu - g^2/\lambda$, they can be expanded in powers of $1/a$:

$$B_2 \approx \frac{1}{2m_{\text{red}}a} \left( 1 - \frac{\pi(g/\lambda)^2}{m_{\text{red}}^2(\nu - g^2/\lambda)^2a} + \ldots \right), \quad (18a)$$

$$Z_\chi \approx \frac{\pi(g/\lambda)^2}{m_{\text{red}}^2(\nu - g^2/\lambda)^2a} + \ldots. \quad (18b)$$

For any fine-tuning that produces a large scattering length, the bare coupling constants approach definite limiting values and therefore the renormalization invariants $g/\lambda$ and $\nu - g^2/\lambda$ approach definite limiting values. Thus the probability $Z_\chi$ decreases like $1/a$. This illustrates our assertion that with the probability for states other than $D^{*0}\bar{D}^{*0}$ or $\bar{D}^{*0}D^{*0}$ scales as $1/a$.

We proceed to discuss how the decays of $X(3870)$ could be described within this effective field theory. In the decay $X \to D^0\bar{D}^0\pi^0$, which is dominated by the decay of a constituent $D^{*0}$ or $\bar{D}^{*0}$, the typical momentum of the final $D^0$ or $\bar{D}^0$ is 40 MeV, which is much smaller than the natural momentum scale $m_\pi$. Thus this decay can be described within the effective theory by introducing a $\pi^0$ field into the lagrangian with an interaction term that allows the decay $\bar{D}^{*0} \to D^0\pi^0$. In the decay $X \to D^0\bar{D}^0\gamma$, which is also dominated by the decay of a constituent $D^{*0}$ or $\bar{D}^{*0}$, the typical momentum of the recoiling $D^0$ or $\bar{D}^0$ is 140 MeV, which is comparable to the natural momentum scale $m_\pi$. Thus this decay need not be described accurately within an effective theory in which hadrons are treated as point particles with pointlike interactions.

The radiative and hadronic transitions and the annihilation decays of $X(3870)$ produce particles with momenta larger than the $m_\pi$. They therefore cannot be described explicitly
within an effective theory in which hadrons are treated as point particles with pointlike interactions. The inclusive effects of these decays can however be taken into account implicitly through local terms in the hamiltonian density. The inclusive effects of transitions of $D^0\bar{D}^{*0}$ or $\bar{D}^0D^{*0}$ to charmonium states and of their annihilation into light hadrons can be taken into account through an imaginary part of the bare coupling constant $\lambda_0$. The inclusive effects of transitions of $\chi_{c1}(2P)$ to other charmonium states and of its decays into light hadrons can be taken into account through an imaginary part of the bare parameter $\nu_0$: $\text{Im}\lambda_0 = -\frac{1}{2}\Gamma_{\chi_{c1}(2P)}$. The imaginary part of $\nu_0$ can take into account interference effects associated with transitions of $D^0\bar{D}^{*0}$ and $\chi_{c1}(2P)$ to the same final states. If the parameters $\lambda_0$, $g_0$, and $\nu_0$ have small imaginary parts, the scattering length (15) is complex-valued with a small imaginary part. If a fine-tuning makes the real part of $a$ large, the binding energy of $X$ is given by the real part of the expression (1). Its imaginary part should be interpreted as $\frac{1}{2}\Delta\Gamma_X$, where $\Delta\Gamma_X$ is the contribution to the width from the decays whose effects are taken into account through $\text{Im}\lambda_0$, $\text{Im}g_0$, and $\text{Im}\nu_0$. At first order in the imaginary parts of $\lambda_0$, $g_0$, and $\nu_0$, the contribution to the width is

$$\Delta\Gamma_X = \frac{2}{\pi^2a^4} \left[ (1 - 2a\Lambda/\pi)^2 (-\text{Im}\lambda_0) + 2\frac{g}{\nu}(1 - 2a\Lambda/\pi)\text{Im}g_0 + \frac{g^2}{\nu^2}(-\text{Im}\nu_0) \right]. \quad (19)$$

If we express $g/\nu$ in terms of $a$ and the renormalization invariants, we see that it increases linearly with $a$: $g/\nu = a(g/\lambda)/(\nu - g^2/\lambda)$. Thus all three terms in (19) scale as $1/a$ in the limit $a \to \infty$. This scaling behavior is in agreement with that deduced in Section III.

VI. SUMMARY

We have explored the implications of low-energy universality for the identification of $X(3870)$ as an S-wave $D^0\bar{D}^{*0}/D^0\bar{D}^{*0}$ molecule. Its shallow binding energy requires some fine-tuning mechanism to generate a large $D^0\bar{D}^{*0}$ scattering length $a$. Two possible mechanisms are an accidental fine-tuning of parameters associated with the $D^0\bar{D}^{*0}$ sector and an accidental fine-tuning of a P-wave charmonium state to the $D^0\bar{D}^{*0}$ threshold. A field theory model that illustrates both of these mechanisms was presented. With either mechanism, the probabilities for components of the wavefunction other than $D^0\bar{D}^{*0}$ or $\bar{D}^0D^{*0}$ are suppressed by a factor of $1/a$. The decay rates into modes other than those associated with decay of a constituent $\bar{D}^{*0}$ or $D^{*0}$ are also suppressed by a factor of $1/a$. The assumption that the large scattering length arises from the fine-tuning of $\chi_{c1}(2P)$ or $h_c(2P)$ to the $D^0\bar{D}^{*0}$ threshold is particularly predictive. The decay rates for radiative transitions and for hadronic transitions via emission of light hadrons with total isospin 0 should differ from the corresponding decay rates of $\chi_{c1}(2P)$ or $h_c(2P)$ in the absence of the fine-tuning by a common suppression factor. Low-energy universality also has nontrivial predictions for 3-body systems, such as $D^0\bar{D}^{*0}\bar{D}^{*0}$, although the spectacular possibility of Efimov states can be excluded. In conclusion, if the $X(3870)$ is indeed an S-wave $D^0\bar{D}^{*0}/\bar{D}^0D^{*0}$ molecule, it will provide a beautiful example of the remarkable phenomenon of low-energy universality.

This research was supported in part by the Department of Energy under grant DE-FG02-91-ER4069.

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