APPROXIMATIONS BY DISJOINT SUBCONTINUA AND A POSITIVE ENTROPY CONJECTURE

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Abstract. In 1983, E.D. Tymchatyn constructed a hereditarily locally connected continuum which is the closure of a first category ray. We show the example reopens a conjecture of G.T. Seidler and H. Kato on positive entropy homeomorphisms. Additionally, we show that every indecomposable semi-continuum can be approximated by a sequence of disjoint subcontinua, and no composant of an indecomposable continuum can be embedded into a Suslinian continuum.

1. Introduction

In 1990, G.T. Seidler proved every homeomorphism on a regular curve has zero topological entropy [14, Theorem 2.3]. H. Kato strengthened this result in 2004 by showing every monotone map on a regular curve has zero entropy [7, Corollary 1.2].

In his 1990 paper, Seidler conjectured: Every homeomorphism on a rational curve has zero topological entropy [14, Conjecture 3.4]. In 1993, Kato asked a similar question: If \( f : X \to X \) is a homeomorphism of a continuum \( X \), and the topological entropy of \( f \) is positive, is \( X \) non-Suslinian? [5, Question 1]. A positive answer to the latter implies the former, because every rational continuum is Suslinian.

In 2016, a positive answer to Kato’s question was announced [11, Corollary 27]. Unfortunately, the proof in [11] relies on a generalization of the statement [12, Theorem 30]:

Suppose that \( \{X_n\}_{n=1}^{\infty} \) is a collection of disjoint subcontinua of a continuum \( Y \) such that

\[
\lim_{n \to \infty} d_H(X_n, Y) = 0.
\]

Then \( Y \) is non-Suslinian.

In Section 3 of this paper, we will present an example by E.D. Tymchatyn of a Suslinian continuum which can be approximated by a sequence of mutually disjoint arcs. The example shows that [12, Theorem 30] and its generalization [11, Theorem 17] are false. Seidler’s conjecture thus remains an open problem. The full impact will be summarized in Section 3.2.

In Section 4 of the paper, we will focus on approximations in another context. We will show that every indecomposable semi-continuum \( X \) can be approximated by a sequence of disjoint continua. As indicated above, this does not automatically
implies that $X$ does not embed into a Suslinian continuum. The situation is further complicated by the fact that there exists an indecomposable connected subset of the plane which can be embedded into a rational continuum [8, Example 2]. But, on a positive note, we are able to show that composants of indecomposable continua cannot be embedded into Suslinian continua.

2. Terminology

All spaces under consideration are separable and metrizable.

A continuum is a compact connected space with more than one point.

A continuum $Y$ is:
- **regular** provided $Y$ has a basis of open sets with finite boundaries;
- **hereditarily locally connected** if every subcontinuum of $Y$ is locally connected;
- **rational** provided $Y$ has a basis of open sets with countable boundaries;
- **Suslinian** if every collection of pairwise disjoint subcontinua of $Y$ is countable [10].

Note that regular $\equiv$ rim-finite, and rational $\equiv$ rim-countable. It is well-known that:

regular $\Rightarrow$ hereditarily locally connected $\Rightarrow$ rational $\Rightarrow$ Suslinian.

Suslinian continua are sometimes called *curves* because they are 1-dimensional.

A space $Y$ is approximated by a sequence of continua $X_0, X_1, \ldots \subseteq Y$ if

$$\lim_{n \to \infty} d_H(X_n, Y) = 0.$$  

Here, $d_H$ denotes the Hausdorff distance generated by a metric $d$ on $Y$.

A *ray* is a continuous one-to-one image of $[0, \infty)$.

An *arc* is any space homeomorphic to $[0, 1]$.

A connected set $X$ is indecomposable if $X$ cannot be written as the union of two proper closed connected subsets. This is equivalent to saying every proper closed connected subset of $X$ is nowhere dense.

A *semi-continuum* is a continuum-wise connected space.

A *composant* of a continuum $Y$ is defined to be the union of all proper subcontinua of $Y$ that contain a given point. Note that each composant of a continuum is a semi-continuum. The class of spaces which are homeomorphic to composants of indecomposable continua includes all singular dense meager composants; see [9].

3. Tymchatyn’s Counterexample

In this section, we present [15, Example 3], and show it is a counterexample to [12, Theorem 30]. More precisely, we show the example is a Suslinian continuum which can be approximated by a sequence of disjoint arcs.

3.1. The counterexample. The following is taken directly from [15, Example 3].

Let $[0, 1]$ denote a unit segment on the $z$-axis in Euclidean 3-space. Let $C_1, C_2, \ldots$ be a sequence of Cantor sets in $[0, 1]$ such that for each $n = 1, 2, \ldots$:

1 We corrected two typos in the definitions of $A_n$ and $\sim$. 


the components of \([0, 1] \setminus C_n\) have diameter less than \(1/n\);
- if \(n\) is even then \(C_n \cap C_{n-1} = \{b_n\}\), where \(b_n = \sup C_{n-1} = \sup C_n\);
- if \(n > 1\) is odd then \(C_n \cap C_{n-1} = \{a_n\}\), where \(a_n = \inf C_{n-1} = \inf C_n\);
- \(C_n \cap (C_1 \cup \ldots \cup C_{n-2}) = \emptyset\).

The \(C_n\)'s may of course be recursively defined. At step \(n\) one must make sure that \(a_n < 1/(n + 1)\) or \(b_n > 1 - 1/(n + 1)\), depending on whether \(n\) is even or odd.

If \(C\) is a Cantor set in \([0, 1]\), \(x\) and \(y\) two points of \(C\) are said to be consecutive endpoints of \(C\) if \(x\) and \(y\) are the two endpoints of the closure of a component of \([0, 1] \setminus C\).

For each natural number \(n\) let \(P_n\) be the plane in Euclidean 3-space which contains the \(z\)-axis and the point \((1, n, 0)\). If \(x\) and \(y\) are consecutive endpoints of \(C_n\), let \(\overline{xy}\) be the semicircle in \(P_n\) with endpoints \(x\) and \(y\). For each \(n\) let

\[A_n = C_n \cup \bigcup \{\overline{xy} : x\) and \(y\) are consecutive endpoints of \(C_n\}\}.

Then each \(A_n\) is an arc in \(P_n\).

Let \(X = [0, 1] \cup A_1 \cup A_2 \cup \ldots\). "

Apparently, \(X\) is a continuum and \(R := A_1 \cup A_2 \cup \ldots\) is a dense ray in \(X\).

In the example, Tymchatyn subsequently defines an equivalence relation on \(X\) by \(x \sim y\) if and only if \(x = y\) or \(\{x, y\} \subseteq z_1 \cup z_2\) for some consecutive endpoints \(z_1\) and \(z_2\) of some \(C_n\). He notes that:

(i) \(Y := X/ \sim\) is an upper semi-continuous decomposition of \(X\);
(ii) \(R/ \sim\) is a ray which is dense and first category in \(Y\);
(iii) \(Y\) is hereditarily locally connected.

Item (i) guarantees that \(Y\) is metrizable. To verify it, first observe that the semicircles in \(X\) can be enumerated so that their diameters converge to 0. So for any \(y \in Y\) and \(X\)-open set \(U \supseteq y\), the set \(K := \bigcup \{z \in Y : z \setminus U \neq \emptyset\}\) is compact. Thus \(y\) is contained in the smaller \(X\)-open set \(U \setminus K\) which is a union of elements of \(Y\). This shows the decomposition is upper semi-continuous. By normality of \(X\) we can now see \(Y\) is Hausdorff, and therefore metrizable.

Item (ii) follows from the fact that \(R\) is a dense ray and each \(A_n/ \sim\) is an arc nowhere dense in \(Y\). It implies there is a sequence of disjoint arcs in \(R/ \sim\) that converges to \(Y\) in the Hausdorff distance. Simply parametrize \(R/ \sim\) with a one-to-one continuous surjection \(f : [0, \infty) \to R/ \sim\) and use the fact that each tail \(f[n, \infty)\) is dense in \(Y\).

Item (iii), the hereditarily locally connected property of \(Y\), holds because \(X\) is hereditarily locally connected (\(X\) contains no convergence continuum), and \(Y\) is a monotone decomposition of \(X\).
We remark that every hereditarily locally connected continuum is Suslinian, and in this particular example the Suslinian property is easy to detect. For observe that each (non-degenerate) subcontinuum of $Y$ must contain a semicircle element of $Y$, and there are only countably many semicircles. Moreover, $Y$ is homeomorphic to the space of irrationals plus the countable point set of semicircles. Each subcontinuum of $Y$ must intersect that countable point set.

3.2. Impact. The example above shows [12, Theorem 30] and [11, Theorem 17] are false. We note that [12, Lemma 29] and [11, Lemma 15] seem to correctly imply the respective statements of [12, Theorem 30] and [11, Theorem 17]. But we identified at least one error in each lemma proof. For instance, in the proof of [12, Lemma 29], a component of

$$(f^{i_1}_{i_1} |_{A_{i_1}})^{-1}(U)$$

is mistaken for a component of $(f^{i_1}_{i_1})^{-1}(U)$. It will actually only be a component of $A_{i_1} \cap (f^{i_1}_{i_1})^{-1}(U)$.

We conclude the following.

False: [12, Theorem 30] and [11, Theorem 17];
Likely false: [12, Lemma 29] and [11, Lemma 15];
Questionable: [11, Theorem 25], [11, Corollary 26], [11, Corollary 27] (all proofs rely on [11, Theorem 17]);
Still open: [14, Conjecture 3.4] and [5, Question 1];
Also questionable: [6, Theorem 2.8] (proof relies on [11, Corollary 27]);
Miscellaneous: Introductions in [3, Section 1] and [9, Section 1]. In the latter, we remarked that a continuum in which some meager composant is singular dense must be non-Suslinian. Part of our justification was [12, Theorem 30]. The remark is correct nevertheless by [13, Corollary 6.4].

4. INDECOMPOSABLE SEMI-COUNTINUOUS

Following [1, Definition 4.5], if $X$ is a semi-continuum, $K \subseteq X$, and $\mathcal{U}$ is a finite collection of open subsets of $X$, then we say $K$ disrupts $\mathcal{U}$ if no continuum in $X \setminus K$ intersects each member of $\mathcal{U}$.

Lemma 4.1. If $X$ is an indecomposable semi-continuum, then no finite collection of non-empty open subsets of $X$ is disrupted by (the union of) finitely-many proper continua

$$K_0, K_1, ..., K_{n-1} \subseteq X.$$ 

Proof. Let $X$ be an indecomposable semi-continuum, and let $K_0, K_1, ..., K_{n-1} \subseteq X$ be continua. Suppose for a contradiction that $K := \bigcup \{K_i : i < n\}$ disrupts a finite collection of non-empty open sets.

Let $l$ be the least positive integer with the property that some collection of non-empty open sets of size $l$ is disrupted by $K$. That is,

$$l = \min \{|\mathcal{U}| : \mathcal{U} \text{ is a collection of non-empty open subsets of } X, \text{ and } K \text{ disrupts } \mathcal{U}\}.$$
Since \( K \) is nowhere dense, \( l \geq 2 \). Let \( \mathcal{V} = \{ V_0, V_1, \ldots, V_{l-1} \} \) be a collection of non-empty open sets such that \( K \) disrupts \( \mathcal{V} \). By minimality and finiteness of \( l \), the set

\[
N := \bigcup \{ M \subseteq X \setminus K : M \text{ is a continuum and } M \cap V_j \neq \emptyset \text{ for each } j \geq 1 \}
\]

contains a dense subset of \( V_1 \).

We claim that every constituent \( M \subseteq N \) is contained in a semi-continuum \( S \subseteq N \) such that \( S \) intersects some \( K_i \). To see this, fix \( p \in M \) and \( q \in V_0 \). Since \( X \) is a semi-continuum, there is a continuum \( L \subseteq X \) such that \( \{ p, q \} \subseteq L \). The assumption \( K \) disrupts \( \mathcal{V} \) implies \( (M \cup L) \cap K \neq \emptyset \), whence \( L \cap K \neq \emptyset \). Boundary bumping [4, Lemma 6.1.25] in \( L \) now shows that for each \( n < \omega \) there is a continuum \( L_n \subseteq L \setminus K \) such that \( p \in L_n \) and \( d(L_n, K) < 2^{-n} \). The semi-continuum \( S := \bigcup \{ M \cup L_n : n < \omega \} \) is contained in \( N \), and \( S \cap K \neq \emptyset \) by compactness of \( K \).

We conclude that \( N' := \bigcup \{ S : S \text{ is a maximal semi-continuum in } N \} \) has at most \( n \) connected components. As \( V_1 \subseteq \overline{N'} \), this implies some component \( C \) of \( N' \) is dense in a non-empty open subset of \( V_1 \). Then \( \overline{C} \) is a closed connected subset of \( X \setminus V_0 \) with non-empty interior. This violates indecomposability of \( X \).

Repeated applications of Lemma 4.1 will show:

**Theorem 4.2.** If \((X, d)\) is an indecomposable semi-continuum, then there is a sequence of pairwise disjoint continua \( K_0, K_1, \ldots \subseteq X \) such that \( d_H(K_n, X) \to 0 \), where \( d_H \) is the Hausdorff metric generated by \( d \).

This generalizes part of [9, Corollary 1.2].

**Theorem 4.3.** If \( X \) is homeomorphic to a composant of an indecomposable continuum, then \( X \) does not embed into any Suslinian continuum. Moreover, every compactification of \( X \) contains \( c = |\mathbb{R}| \) pairwise disjoint dense semi-continua.

**Proof.** Suppose \( X \) is homeomorphic to a composant of indecomposable continuum \( I \). Let \( Y = \gamma X \) be any compactification of \( X \).

Let \( \iota : X \hookrightarrow I \) be a homeomorphic embedding such that \( \iota(X) \) is a composant of \( I \). Let \( Z \) be the closure of the diagonal \( \langle \iota(x), \gamma(x) \rangle \) in the product \( I \times Y \). More precisely, define a homeomorphism \( \xi : X \hookrightarrow I \times Y \) by \( \xi(x) = \langle \iota(x), \gamma(x) \rangle \), and let \( Z = \overline{\xi(X)} \). In the proof of [9, Theorem 1.1], it was shown that \( Z \) is an indecomposable continuum and \( \xi[X] \) is a composant of \( Z \).

By Lavrentiev’s Theorem [4, Theorem 4.3.21], the homeomorphism \( \pi_Y \upharpoonright \xi[X] \) extends to a homeomorphism between \( G_\delta \)-sets \( Z' \subseteq Z' \) with \( \xi[X] \subseteq Z' \) and \( Y' \subseteq Y \). By [2, Theorem 9], \( Z' \) contains \( c \)-many composants of \( Z \). Thus \( Y' \) contains \( c \) pairwise disjoint semi-continua which are dense in \( Y \).

5. Questions

In Seidler’s conjecture, we consider replacing “rational” with the stronger condition “hereditarily locally connected”.

**Question 1.** Is Seidler’s conjecture true for hereditarily locally connected continua?
The answer is yes for hereditarily locally connected \textit{plane} continua, because these continua are known to be \textit{finitely Suslinian}. See [10, §1.5].

**Question 2.** Is it true that no indecomposable semi-continuum can be embedded into a rational continuum?

We note that the first category ray in Tymchatyn’s example does not provide a negative answer Question 2. The ray is not indecomposable because every connected subset of a hereditarily locally connected continuum is locally connected.

**Question 3.** \(X \subseteq \mathbb{R}^2\) be a first category plane ray. Is \(\text{cl}_{\mathbb{R}^2} X\) necessarily non-rational? Non-Suslinian?

Question 3 may be related to Question 2, because we have conjectured that every first category plane ray is indecomposable.

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