Restriction of representations of $\text{GL}(n+1, \mathbb{C})$ to $\text{GL}(n, \mathbb{C})$ and action of the Lie overalgebra

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1 The statement

Abstract. Consider a restriction of an irreducible finite dimensional holomorphic representation of $\text{GL}(n+1, \mathbb{C})$ to the subgroup $\text{GL}(n, \mathbb{C})$. We write explicitly formulas for generators of the Lie algebra $\mathfrak{gl}(n+1)$ in the direct sum of representations of $\text{GL}(n, \mathbb{C})$. Nontrivial generators act as differential-difference operators, the differential part has order $n$, the difference part acts on the space of parameters (highest weights) of representations. We also formulate a conjecture about unitary principal series of $\text{GL}(n, \mathbb{C})$.

1.1. The Gelfand–Tsetlin formulas. It is well known that restrictions of finite dimensional holomorphic representations of the general linear group $\text{GL}(n, \mathbb{C})$ to the subgroup $\text{GL}(n-1, \mathbb{C})$ is multiplicity free. Considering a chain of restrictions

$$\text{GL}(n, \mathbb{C}) \supset \text{GL}(n-1, \mathbb{C}) \supset \text{GL}(n-2, \mathbb{C}) \supset \cdots \supset \text{GL}(1, \mathbb{C})$$

we get a canonical decomposition of our representation into a direct sum of one-dimensional subspaces. Taking a vector in each line we get a basis of the representation. In [7] Gelfand and Tsetlin announced formulas for action of generators of the Lie algebra $\mathfrak{gl}(n)$ in this basis. It turns out that the Lie algebra $\mathfrak{gl}(n)$ acts by difference operators in the space of functions on a certain convex polygon in the lattice $\mathbb{Z}^{(n-1)/2}$. In particular, this gives an explicit realization of representations of the Lie algebra $\mathfrak{gl}(n)$.

For various proofs of the Gelfand–Tsetlin formulas and constructions of the bases, see [3], [28], [2], [11], [29], [19], [17], [18] (see more references in [17]), for other classical groups, see [8], [17], [26], for applications to infinite-dimensional representations, see [25], [13], [9], for formulas on the group level, see [8], [9]. There are many other continuations of this story. However, our standpoint is slightly different.

1.2. Actions of overalgebras in the spectral decompositions. We can write images of many operators under the classical Fourier transform. It is commonly accepted that Plancherel decompositions of representations are higher analogs of the Fourier transform.

Consider a group $G$ and its subgroup $H$. Restrict an irreducible unitary representation of $G$ to $H$. Generally, an explicit spectral decomposition of the restriction seems hopeless problem. However, there is a collection of explicitly

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solvable problems of this type. In [20] it was conjectured that in such cases the action of the Lie algebra \( g \) can be written explicitly as differential-difference operators. In fact, in [20] there was considered the tensor product \( V_s \otimes V_s^\ast \) of a highest weight and a lowest weight unitary representations of \( \text{SL}(2, \mathbb{R}) \) (i.e., \( G \simeq \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \) and \( H \simeq \text{SL}(2, \mathbb{R}) \) is the diagonal). This representation is a multiplicity free integral over the principal series of \( \text{SL}(2, \mathbb{R}) \). It appears that the action of all generators of the Lie algebra \( \mathfrak{sl}(2, \mathbb{R}) \) in the spectral decomposition can be written explicitly as differential-difference operators. The differential part of these operators has order two and the difference operators are difference operators in the imaginary direction.3

Molchanov [12]–[15] solved several problems of this type related to rank-one symmetric spaces4. In all his cases the Lie algebra acts by differential-difference operators; difference operators act in the imaginary dimension. However, the order of the differential part in some cases is 4.

In the present paper, we consider restrictions of holomorphic finite-dimensional representations \( \text{GL}(n+1, \mathbb{C}) \to \text{GL}(n, \mathbb{C}) \) and write explicit formulas for the action of the overalgebra in the spectral decomposition. Also, we formulate a conjecture concerning restrictions of unitary principal series.

1.3. Notation. The group \( \text{GL}(n, \mathbb{C}) \) and its subgroups. Denote by \( \text{GL}(n, \mathbb{C}) \) the group of invertible complex matrices of size \( n \). By \( \mathfrak{gl}(n) \) we denote its Lie algebra, i.e. the Lie algebra of all matrices of size \( n \). Let \( \text{U}(n) \subset \text{GL}(n, \mathbb{C}) \) be the subgroup of unitary matrices.

By \( E_{ij} \) we denote the matrix whose \( ij \)'s entry is 1 and other entries are 0. The unit matrix is denoted by 1 or \( 1_n \), i.e. \( 1 = \sum_j E_{jj} \). Matrices \( E_{ij} \) can be regarded as generators of the Lie algebra \( \mathfrak{gl}_n \).

Denote by \( N_+^+ = N_+^+ \subset \text{GL}(n, \mathbb{C}) \) the subgroup of all strictly upper triangular matrices of size \( n \), i.e., matrices of the form

\[
Z = \begin{pmatrix}
1 & z_{12} & \cdots & z_{1(n-1)} & z_{1n} \\
0 & 1 & \cdots & z_{2(n-1)} & z_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & z_{(n-1)n} \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}.
\]

By \( N_n^- \) we denote the group of strictly lower triangular matrices.

2Also, some explicitly solvable spectral problems in representation theory can be regarded as special cases of the restriction problem. In particular, a decomposition of \( L^2 \) on a classical pseudo-Riemannian symmetric space \( G/H \) can be regarded as a special case of the restriction of a Stein type principal series of a certain overgroup \( \tilde{G} \) to the symmetric subgroup \( G \). [21]. So, for \( L^2 \) on symmetric spaces the problem of action of the overalgebra discussed below makes sense.

3On Sturm–Liouville in imaginary direction, see [23] and further references in that paper, see also [10]. The most of known appearances of such operators are related to representation theory and spectral decompositions of unitary representations.

4In particular, he examined the restrictions of maximally degenerate principal series for the cases \( \text{GL}(n+1, \mathbb{R}) \supset \text{O}(n, 1) \), \( \text{O}(p, q) \supset \text{O}(p-1, q) \).
By $B^{-}_n \subset \text{GL}(n, \mathbb{C})$ we denote the subgroup of lower triangular matrices, $g \in B^{-}_n$ if $g_{ij} = 0$ if $i < j$. Let $\Delta$ be the subgroup of diagonal matrices, we denote its elements as $\delta = \text{diag}(\delta_1, \ldots, \delta_n)$.

Next, we need a notation for sub-matrices. For a matrix $X$ denote by $[X]_{\alpha\beta}$ the left upper corner of $X$ of size $\alpha \times \beta$. Let $I : 0 < i_1 < i_2 < \ldots < i_\alpha \leq \alpha$, $J : 1 \leq j_1 \leq \ldots \leq j_\beta$ be sets of integers. Denote by $[X(IJ)] = [X(i_1, \ldots, i_\alpha, j_1, \ldots, j_\beta)]$ (1.1)

the matrix composed of entries $x_{i\mu,j\nu}$. By $[X(I)] = [X(i_1, \ldots, i_\alpha)]$ (1.2)

we denote matrix composed of $i_1, \ldots, i_\alpha$-th rows of $X$ (the order is not necessary increasing, also we allow coinciding rows).

1.4. Holomorphic representations of $\text{GL}(n, \mathbb{C})$. Recall that irreducible finite dimensional holomorphic representations $\rho$ of the group $\text{GL}(n, \mathbb{C})$ are enumerated by collections of integers (signatures) $p : p_1 \geq \ldots \geq p_n$. This means that there is a cyclic vector $v$ (a highest weight vector) such that

$$\rho(Z)v = v \quad \text{for each } Z \in N^+_n,$$

$$\rho(\delta)v = \prod_{j=1}^{n} \delta_j^{p_j} v, \quad \text{where } \delta = \text{diag}(\delta_1, \ldots, \delta_n).$$

Denote such representation by $\rho_p = \rho^p$. Denote the set of all signatures by $\Lambda_n$.

The dual (contragradient) representation to $\rho_p$ has the signature $p^* := (-p_n, \ldots, -p_1)$.

1.5. Realizations of representations by differential operators. Recall a model for irreducible finite dimensional holomorphic representations for details, see Section 2). We consider the space $\text{Pol}(N^+_n)$ of polynomials in the variables $z_{ij}$. The generators of the Lie algebra $\mathfrak{gl}(n)$ act in this space via differential operators

$$E_{kk} = -\sum_{i<k} z_{ik}\partial_{ik} + p_k + \sum_{j>k} z_{kj}\partial_{kj}, \quad (1.4)$$

$$E_{k(k+1)} = \partial_{k(k+1)} + \sum_{i<k} z_{ik}\partial_{i(k+1)}, \quad (1.5)$$

$$E_{(k+1)k} = \sum_{j<k} z_{j(k+1)}\partial_{j(k+1)} + (p_{k+1} - p_k)z_{k(k+1)} - z_{k(k+1)}\sum_{j>k} z_{kj}\partial_{kj} +$$

$$+ \sum_{m>k+1} \det \begin{pmatrix} z_{k(k+1)} & z_{km} \\ 1 & z_{(k+1)m} \end{pmatrix} \partial_{(k+1)m}, \quad (1.6)$$

3
where
\[ \partial_{kl} := \frac{\partial}{\partial z_{kl}}. \]

All other generators can be expressed in the terms of \( E_{kk}, E_{k(k+1)}, E_{(k+1)k} \) (also it is easy to write explicit formulas as it is explained in Section 2). In this way, we get a representation of the Lie algebra \( \mathfrak{gl}(n) \).

There exists a unique finite dimensional subspace \( V_p \) invariant with respect to such operators. The representation \( \rho_p \) is realized in this space. The highest weight vector is \( f(Z) = 1 \). In the next subsection we describe this space more explicitly.

Remark. This approach arises to [6]. Of course, this model is a coordinatization of the constructions of representations of \( \text{GL}(n, \mathbb{C}) \) in sections of line bundles over the flag space \( B^-_n \setminus \text{GL}(n, \mathbb{C}) \) (as in the Borel–Weil–Bott theorem). The space \( N^+_n \simeq \mathbb{C}^{n(n-1)/2} \) is an open dense chart on this space, the terms with \( \partial \) in (1.4)–(1.5) correspond to vector fields on the flag space, the zero order terms are corrections corresponding to the action in the bundle. Elements of the space \( V^n_p \) are precisely polynomials, which are holomorphic as section of bundle on the whole flag space. However, we need explicit formulas and prefer a purely coordinate language.

1.6. Descriptions of the space \( V_p \). Zhelobenko operators.

a) Description-1. Denote by \( d\hat{Z} \) the standard Lebesgue measure on \( N^+_n \)
\[ d\hat{Z} = \prod_{k<l} d\text{Re}z_{kl} d\text{Im}z_{kl}. \]

Denote by \( d\mu(Z) \) the measure on \( N^+_n \) given by the formula
\[ d\mu(Z) = d\mu_p^n(Z) = \prod_{j=1}^{n-1} \text{det}([Z]_{jn}([Z]_{jn})^*)^{-(p_j-p_j-1)-2} d\hat{Z}. \] (1.7)

Proposition 1.1

a) \( V^n_p = L^2(N^+_n, \mu_p^n) \cap \text{Pol}(N^+_n) \).

b) The \( L^2 \)-inner product in \( V^n_p \) is \( \text{U}(n) \)-invariant.

Proof is given in Section 2.

Description-2. We define the Zhelobenko operators by
\[ R_{km} := \partial_{km} + \sum_{j>m} z_{mj} \partial_{kj}, \quad m > k. \] (1.8)

Theorem 1.2 The space \( V^n_p \) consists of polynomials satisfying the conditions
\[ \left\{ R_{pj(j+1)}^{p_j-p_{j+1}+1} f(Z) = 0. \right\} \]

See Zhelobenko, [28], [27], §6, Theorem 2, [30], Theorem 48.7.

Description-3. There is one’s more description of the space \( V^n_p \), which can be used with coordinate language.
**Proposition 1.3** The space $V_p^n$ coincides with the space determined by the reproducing kernel

$$
\prod_{j=1}^{p-1} \det \left( \left[Z\right]_j \left[Z\right]_j^* \right)^{p_j - p_j - 1}.
$$

On Hilbert spaces determined by reproducing kernels, see, e.g., [22], Sect. 7.1. The proposition is more-or-equivalent to the Borel–Weil theorem.

1.7. The restriction of representations $GL(n+1, \mathbb{C})$ to $GL(n, \mathbb{C})$. Consider the representation $\rho_{n+1}^r$ of $GL(n+1, \mathbb{C})$ with a signature $r = (r_1, \ldots, r_{n+1})$. It is well-known (see Gelfand, Tsetlin [7]) that the restriction of $\rho_{n+1}^r$ to $GL(n, \mathbb{C})$ is multiplicity free and is a direct sum of all representations $\rho_q$ of $GL(n, \mathbb{C})$ with signatures satisfying the interlacing conditions

$$
r_1 \geq q_1 \geq r_2 \geq q_2 \geq \ldots \geq q_n \geq r_{n+1}. \tag{1.9}
$$

Our purpose is to write explicitly the action of Lie algebra $gl(n+1)$ in the space $\bigoplus V_q^n$.

1.8. Normalization. First, we intend to write a $GL(n, \mathbb{C})$-invariant pairing

$$
J_{p,q} : V_{p+1}^n \times V_q^n \to \mathbb{C},
$$
as an integral

$$
J_{p,q}(\varphi, \psi) = \int_{N^+_{n+1} \times N^+_{n}} L_{p,q}(U,Z) \varphi(Z) \psi(U) \, d\mu_{p+1}^n(Z) \, d\mu_n^p(U), \tag{1.10}
$$
where $Z$ ranges in $N^+_{n+1}$, $U$ ranges in $N^+_{n}$, and the kernel

$$
L_{p,q}(U,Z) \in V_{p+1}^n \otimes V_q^n
$$
is a polynomial in the variables $z_{ij}$, $u_{kl}$.

Denote $Z^{\text{ext}} := [Z]_{(n+1)n}$, this matrix is obtained from $Z$ by cutting of the last column. Denote by $U^{\text{ext}}$ the $n \times (n+1)$-matrix obtained from $U$ by adding the zero last column,

$$
U^{\text{ext}} = \begin{pmatrix}
1 & u_{12} & \ldots & u_{1n} & 0 \\
0 & 1 & \ldots & u_{2n} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}. \tag{1.11}
$$

Consider the $(n+1) \times (n+1)$-matrix composed from the first $(n+1 - \alpha)$ rows of the matrix $Z$ and the first $\alpha$ rows of the matrix $U^{\text{ext}}$. Denote by $\Phi_\alpha$ the determinant of this matrix:

$$
\Phi_\alpha(Z, U) = \det \begin{pmatrix}
[Z]_{(n+1-\alpha)(n+1)} \\
[U^{\text{ext}}]_{\alpha(n+1)}
\end{pmatrix}, \tag{1.12}
$$
Consider the $n \times n$-matrix composed of the first $n - \alpha$ rows of the matrix $Z^{\text{cut}}$ and the first $\alpha$ rows of the matrix $U$, denote by $\Psi_{\alpha}$ its determinant:

$$
\Psi_{\alpha}(Z, U) = \det \left( \begin{array}{c|c}
Z^{\text{cut}}_{(n-\alpha)n} & U_{\alpha n}\end{array} \right).
$$

(1.13)

**Proposition 1.4** Consider signatures $p = (p_1, \ldots, p_{n+1})$, $q = (q_1, \ldots, q_n)$ such that $p$ and $q^*$ are interlacing. Then the expression (1.10) with the kernel

$$
L_{p, q}(U, Z) = \Phi_{1}^{p_1+q_1} \Phi_{2}^{p_2+q_2} \Phi_{3}^{p_3+q_3} \cdots \Phi_{n}^{p_n+q_n}.
$$

(1.14)
determines a $\text{GL}(n, \mathbb{C})$-invariant nonzero pairing between $V_{p}^{n+1}$ and $V_{q}^{n}$.

For instance, for $n = 3$ we get

$$
L_{p, q}(U, Z) = \det \begin{pmatrix}
1 & z_{12} & z_{13} & z_{14} \\
0 & 1 & z_{23} & z_{24} \\
0 & 0 & 1 & z_{34} \\
u_{12} & u_{13} & 0
\end{pmatrix}^{p_3+q_1} \det \begin{pmatrix}
1 & z_{12} & z_{13} & -p_3 \cdot q_2 \\
0 & 1 & z_{23} & 0 \\
u_{12} & u_{13} & 1
\end{pmatrix} \times \det \begin{pmatrix}
1 & z_{12} & z_{13} & z_{14} \\
0 & 1 & z_{23} & z_{24} \\
u_{12} & u_{13} & 0
\end{pmatrix}^{p_2+q_2} \det \begin{pmatrix}
1 & z_{12} & z_{13} & -p_2 \cdot q_3 \\
0 & 1 & z_{23} & 0 \\
u_{12} & u_{13} & 1
\end{pmatrix} \times \det \begin{pmatrix}
1 & z_{12} & z_{13} & z_{14} \\
0 & 1 & z_{23} & z_{24} \\
u_{12} & u_{13} & 0
\end{pmatrix}^{p_1+q_3}.
$$

(1.15)

Next we pass to the dual signature

$$
(r = p^*)
$$

(below $p$ and $q$ are rigidly linked by this restraint), and represent $L_{p, q}$ in the form

$$
L_{r, q}(U, Z) := L_{p, q}(U, Z) = \Phi_{1}^{q_1-r_2} \Phi_{2}^{q_2-r_3} \Phi_{3}^{q_3-r_4} \cdots \Phi_{n}^{q_n-r_{n+1}}.
$$

(1.16)

**1.9. Action of the overalgebra.** Fix a signature $r = p^* \in \Lambda_{n+1}$ as in the previous subsection. Consider the space

$$
\mathfrak{U}_{r} := \bigoplus_{q_1, \ldots, q_n \in \mathbb{Z}} V_{q}^{n}.
$$

(1.16)

We can regard elements of this space as 'expressions'

$$
F(U, q)
$$

of continuous variables $u_{kl}$, where $1 \leq k < l \leq n$, and of integer variables $q_1, \ldots, q_n$. Integer variables range in the domain $r_j \geq q_j \geq r_{j-1}$. More precisely, for a fixed $q$ the expression $F(U, q)$ is a polynomial in the variables $u_{kl}$, moreover, this polynomial is contained in $V_{q}$.
Our family of forms $L_{p,q}$ determines a duality between $V_r$ and the space $\mathfrak{U}_r$, hence we get a canonical identification of $V_r$ and $\mathfrak{U}_r$. Therefore, we have a canonically defined action of the Lie algebra $\mathfrak{gl}(n+1)$ in $\mathfrak{U}_r$. We preserve the notation $E_{kl}$ for operators in $V_p$ and denote operators in $\mathfrak{U}_r$ by $F_{kl}$. For $1 \leq k, l \leq n$ the operators $F_{kl}$ act in the space $\mathfrak{U}_r$ by the first order differential operators in $U$ with coefficients depending on $q$ according the standard formulas (see (1.4)–(1.6)). For instance,

\begin{equation}
F_{kk} = - \sum_{i<k} u_{ik} \partial_{ik} + q_k + \sum_{j>k} u_{kj} \partial_{kj}, \quad (1.17)
\end{equation}

\begin{equation}
F_{k(k+1)} = \partial_{k(k+1)} + \sum_{i<k} u_{ik} \partial_{i(k+1)}, \quad (1.18)
\end{equation}

\begin{equation}
F_{(k+1)k} = \sum_{j<k} u_{(k+1)j} \partial_{kj} + (q_{k+1} - q_k)u_{k(k+1)} - u_{k(k+1)} \sum_{j>k} u_{kj} \partial_{kj} + \sum_{m>k+1} \det \begin{pmatrix} u_{k(k+1)} & u_{km} \\ 1 & u_{(k+1)m} \end{pmatrix} \partial_{(k+1)m}, \quad (1.19)
\end{equation}

where $\partial_{kl} := \frac{\partial}{\partial u_{kl}}$.

The purpose of this work is to present formulas for the generators $F_{1(n+1)}$, $F_{(n+1)n}$. Together with (1.17)–(1.20) they generate the Lie algebra $\mathfrak{gl}(n)$, formulas for the remaining generators $F_{j(n+1)}$, $F_{(n+1)j}$ consist of similar aggregates as below, but are longer.

Denote by $T_j^\pm$ the following difference operators

$$T_j^\pm F(\ldots, q_j, \ldots) = F(\ldots, q_j \pm 1, \ldots)$$

(the remaining variables do not change). We will write expressions, which are polynomial in $u_{kl}$, $\partial_{kl}$, linear in $T_j^\pm$ and rational in $q_j$. These expressions satisfy the commutation relations in $\mathfrak{gl}(n)$ and preserve the space $\mathfrak{U}_r$.

For $1 \leq k < l \leq n$ denote by $[k,l]$ the set

$$[k,l] := \{k, k+1, \ldots, l\}.$$

For a set

$$I : 1 \leq i_1 < i_2 < \cdots < i_m$$

we write

$$I \triangleleft [k,l] \quad \text{if} \quad i_1 = k, \ i_m = l. \quad (1.21)$$

More generally, if $I \subset J \subset [1,n]$, we write $I \triangleleft J$ if the minimal (resp. maximal) element of $I$ coincides with the minimal (resp. maximal) element of $J$.

For any $I \triangleleft [1,n]$ we define the operator

$$R_I := R_{i_1i_2}R_{i_2i_3} \cdots R_{i_{m-1}i_m}.$$
Theorem 1.5  

a) The generator $E_{1(n+1)}$ acts by the formula

$$F_{1(n+1)} = \sum_{m=1}^{n} A_m \left( \sum_{l \in [1,m]} \prod_{t \in [1,m] \setminus l} (l - m + q_t - q_m) \cdot R_l \right) T_m^r, \quad (1.22)$$

where coefficients $A_m$ are given by

$$A_m = \frac{\prod_{j=m+1}^{n+1} (q_m - r_j + j - m - 1)}{\prod_{\alpha \neq m} (q_m - q_\alpha + \alpha - m)}. \quad (1.23)$$

b) The generator $E_{(n+1)n}$ is given by the formula

$$F_{(n+1)n} = \sum_{k=1}^{n} B_m \left( \left( \sum_{l \in [1,m]} \prod_{t \in [m,n] \setminus l} (q_m - q_t + l - m + 1) \cdot R_l \right) T_m^r, \quad (1.23)$$

where

$$B_m = \frac{\prod_{j=n-k}^{n} (q_k - r_{n+2-j} - n + j)}{\prod_{j \neq k} (r_j - r_k - k + j).} \quad (1.23)$$

1.10. Further structure of the paper. Section 2 contains preliminaries on holomorphic representations of $GL(n, \mathbb{C})$. In Section 3, we verify the formula for the kernel $L(Z, U)$. Our main statement is equivalent to a verification of differential-difference equations for the kernel $L(Z, U)$, this is done in Section 4. In Section 5, we formulate a conjecture about analog of our statement for unitary representations.

Acknowledgements. Fifteen years ago the topic of the paper was one of aims of a joint project with M. I. Graev, which was not realized in that time (I would like to emphasis his nice paper [9]). I am grateful to him and also to V. F. Molchanov for discussions of the problem.

2 Holomorphic representations of $GL(n, \mathbb{C})$.

2.1. Realization in the space of functions on $GL(n, \mathbb{C})$. For details and proof, see [28]. We say that a function $f(g)$ on $GL(n, \mathbb{C})$ is a polynomial function if it can be expressed as a polynomial expression in matrix elements $g_{ij}$ and $\det(g)^{-1}$. Denote the space of polynomial functions by $\mathbb{C}[GL(n, \mathbb{C})]$. The group $GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ acts in $\mathbb{C}[GL(n, \mathbb{C})]$ by the left and right shifts:

$$\lambda_{left-right}(h_1, h_2) f(g) = f(h_1^{-1} g h_2).$$

This representation is a direct sum

$$\lambda \simeq \oplus_{p \in \Lambda_n} \rho_p^* \otimes \rho_p.$$ 

A summand $V_p^* \otimes V_p$ is the $GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$-cyclic span of the vector

$$\Delta[p] = \prod_{j=1}^{n-1} \det[g]^{p_j - p_{j+1}^{-1}} \cdot (\det g)^{p_n}.$$
If \( h_1 \in N_n^- \) and \( h_2 \in N_n^+ \), then \( \lambda(h_1, h_2) \Delta[p] = \Delta[p] \).

Next, consider the space of polynomial functions invariant with respect to left shifts on elements of \( N_n^- \)

\[
f(h^{-1}g) = f(g).
\] (2.1)

Equivalently, we consider the space of polynomial functions on \( \mathbb{C}[N_n^- \setminus \text{GL}(n, \mathbb{C})] \).

Each \( \rho_p \) has a unique \( N_n^- \)-invariant vector, therefore \( \mathbb{C}[N_n^- \setminus \text{GL}(n, \mathbb{C})] \) is a multiplicity free direct sum \( \oplus \rho_p \).

Fix a signature \( p \) and consider the space of \( N_n^- \)-invariant functions \( H_p \) such that for any diagonal matrix \( \delta \), we have

\[
f(\delta g) = f(g) \prod_{j=1}^{n} \delta_j^{p_j}.
\]

The group \( \text{GL}(n, \mathbb{C}) \) acts in \( H_p \) by the right shifts,

\[
\lambda_{\text{right}}(h)f(g) = f(gh).
\] (2.2)

This representation is irreducible and equivalent to \( \rho_p \). The vector \( \Delta_p \) is its highest weight vector.

2.2. Realization in the space of functions on \( N_n^- \). An element \( g \in \text{GL}(n, \mathbb{C}) \) satisfying the condition

\[
\det[g]_{jj} \neq 0 \quad \text{for all } j
\]

admits a unique Gauss decomposition

\[
g = bZ, \quad \text{where } b \in B_n^-, \ Z \in N_n^+.
\]

Notice that

\[
b_{jj} = \frac{\det[g]_{jj}}{\det[g]_{(j-1)(j-1)}},
\] (2.3)

\[
z_{ij} = \frac{\det \left[ \begin{array}{cccc} 1 & 2 & \ldots & i-1 \\ 1 & 2 & \ldots & i \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \ldots & i \\ \end{array} \right]}{\det \left[ \begin{array}{cccc} 1 & 2 & \ldots & i \\ 1 & 2 & \ldots & i \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \ldots & i \\ \end{array} \right]}.
\] (2.4)

Restrict a function \( f \in H_p \) to the subgroup \( N_n^+ \). We get a polynomial in the variables \( z_{ij} \), where \( i < j \). By the Gauss decomposition, \( f \) is uniquely determined by this restriction. Therefore the space \( V_p \) can be regarded as a subspace of the space of polynomial in \( z_{ij} \). The description of this space is given by the Zhelobenko Theorem 1.2.

Denote the factors in the Gauss decomposition of \( Zg \) in the following way

\[
Zg =: b(Z, g) \cdot Z[g], \quad \text{where } b(Z, g) \in B^-_n, \ Z[g] \in N^+_n.
\] (2.5)
Then
\[ \rho_p(g) f(Z) = f(Z^{[g]}) \prod_{j=1}^{n} b_{jj}(Z, g)^{p_j}, \]  
(2.6)

where \( b_{jj}(Z, g) \) are the diagonal matrix elements of the lower triangular matrix \( b(Z, g) \).

We present formulas for transformations \( Z \mapsto Z^{[g(t)]} \), where \( g(t) \) are standard one-parametric subgroups in \( \text{GL}(n, \mathbb{C}) \).

1) Let \( g(t) = \exp(1 + tE_{kl}) \), where \( k < l \). Then we have transformation
\[ z_{kl} \mapsto z_{kl} + t; \]
\[ z_{ml} \mapsto z_{ml} + tz_{kl}, \quad m < k; \]
\[ z_{ij} \mapsto z_{ij} \quad \text{for other pairs } i, j. \]

2) Let \( g(t) = \exp(tE_{kk}) \). We get
\[ z_{kj} \mapsto e^t z_{kj}, \quad k < j; \]
\[ z_{ik} \mapsto e^{-t} z_{ik}, \quad i < k; \]
\[ z_{ij} \mapsto z_{ij} \quad \text{for other pairs } i, j. \]

3) Let \( g(t) = \exp(tE_{(k+1)k}) \). Then
\[ z_{ik} \mapsto z_{ik} + tz_{(k+1)}, \quad i < k; \]
\[ z_{kj} \mapsto \frac{z_{kj}}{1 + tz_{(k+1)}}, \quad j > k; \]
\[ z_{(k+1)m} \mapsto z_{(k+1)m} + t \det \begin{pmatrix} z_{k(k+1)} & z_{km} \\ 1 & z_{(k+1)m} \end{pmatrix}, \quad m > k + 1; \]
\[ z_{ij} \mapsto z_{ij} \quad \text{for other pairs } i, j. \]

4) Let \( g(t) = \exp(tE_{1n}) \), then
\[ z_{1n} \mapsto z_{1n} + t, \]
\[ z_{ij} \mapsto z_{ij} \quad \text{for other pairs } i, j. \]

Expressions (1.4)–(1.6) for the action of the Lie algebra \( \text{gl}(n) \) easily follows from these formulas. Also, we get
\[ E_{1n} = \partial_{1n}. \]  
(2.7)

2.3. The Jacobian.

Lemma 2.1 The complex Jacobian \( J(Z, g) \) of a transformation \( Z \mapsto [Z]^{[g]} \) is
\[ J(Z, g) = \prod_{j=1}^{n-1} \det([Z]_{jj})^{-2} = \prod_{j=1}^{n-1} \det([Z]_{jn}[Z]_{nj})^{-2}. \]  
(2.8)
Proof. We say that a function 
\[ c : \mathbb{N}^n_+ \times \text{GL}(n, \mathbb{C}) \to \mathbb{C} \]
satisfies the **chain identity** if
\[ c(Z, gh) = c(Z, g)c(Z^g, h). \] (2.9)

The Jacobian satisfies this identity. Let us verify that the right hand side of (2.8) also satisfies it. It suffices to consider one factor \( \det[Z^g]_{jj}^{-2} \).

Let us evaluate \( \det[Z^gh]_{jj} \). Represent \( Z, g, h \) etc. as block \( j \times (n-j) \)-matrices,
\[
Z = \begin{pmatrix}
Z_{11} & Z_{12} \\
0 & Z_{22}
\end{pmatrix}, \quad g := \begin{pmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{pmatrix}, \quad g := \begin{pmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{pmatrix}. \] (2.10)

Represent \( Zg \) as \( C^{-1}U \), where \( U \in \mathbb{N}^n_+ \), \( C \in B^\_n \). Then
\[
\begin{pmatrix}
C_{11} & 0 \\
C_{21} & C_{22}
\end{pmatrix}
\begin{pmatrix}
Z_{11}G_{11} + Z_{12}G_{21} & Z_{11}G_{12} + Z_{12}G_{22} \\
Z_{22}G_{21} & Z_{22}G_{22}
\end{pmatrix} = \begin{pmatrix}
U_{11} & U_{12} \\
0 & U_{22}
\end{pmatrix}.
\]

Since \( \det U_{11} = 1 \), we have
\[
\det C_{11} = \det(Z_{11}G_{11} + Z_{12}G_{21})^{-1}.
\]

Next,
\[
\det[Z^gh]_{jj} = \det[U]_{jj} = \det[U_{11}H_{11} + U_{12}H_{21}] = \det C_{11} \cdot \det(\begin{pmatrix} Z_{11}G_{11} + Z_{12}G_{21}H_{11} + (Z_{11}G_{12} + Z_{12}G_{22})H_{21} \end{pmatrix}) = \frac{\det(Z_{11}G_{11} + Z_{12}G_{21})}{\det(Z_{11}G_{11} + Z_{12}G_{21})} = \det[Z^gh]_{jj},
\]
this proves the desired statement.

Since the both sides of (2.8) satisfy the chain identity (2.9), it suffices to verify the identities for a system of generators of \( \text{GL}(n, \mathbb{C}) \). Thus we can verify (2.8) for elements one-parametric subgroups \( \exp(tE_{k(k+1)}), \exp(tE_{kk}), \exp(tE_{(k+1)k}) \).

Formulas for the corresponding transformations are present in the previous subsection. Only the case \( \exp(tE_{(k+1)k}) \) requires a calculation. In this case, the Jacobian matrix is triangular. Its diagonal values are
- \( (1 + tz_{k(k+1)})^{-2} \) for \( z_{k(k+1)} \);
- \( (1 + tz_{k(k+1)})^{-1} \) for \( z_{km} \), where \( m > k + 1 \);
- \( (1 + tz_{(k+1)k}) \) for \( z_{(k+1)m} \), where \( m > k + 1 \).

The Jacobian \( (1 + tz_{k(k+1)})^{-2} \).

On the other hand, in the product (2.8) the \( j \)-th factor is \( (1 + tz_{k(k+1)})^{-2} \), other factors are 1.

**2.4. Proof of Proposition 1.1**
The \( U(n) \)-invariance of the \( L^2(N_n^+, \mu_\mathbf{p}) \)-inner product. We must check the identity
\[
\int_{N_n^+} |(\rho_\mathbf{p}(g))(Z)|^2 \, d\mu_\mathbf{p}(Z) = \int_{N_n^+} |\varphi(Z)|^2 \, d\mu_\mathbf{p}(Z).
\] (2.11)

Fix \( g \in U(n) \). Substituting \( Z = U^{[g]} \) to the right-hand side, we get
\[
\int_{N_n^+} |\varphi(U^{[g]})|^2 \prod_{j=1}^{n-1} \det \left([U^{[g]}]_{jn}(U^{[g]}_{jn})^* \right)^{-(p_j-p_{j+1})^{-2}} \, d(U^{[g]}).
\]

In notation (2.10) we have
\[
Ug = \begin{pmatrix}
U_{11}G_{11} + U_{12}G_{21} & U_{11}G_{12} + U_{12}G_{22} \\
* & *
\end{pmatrix}.
\]

Therefore \([U^{[g]}]_{jn}\) has the form
\[
(C_{11}(U_{11}G_{11} + U_{12}G_{21}) \quad C_{11}(U_{11}G_{12} + U_{12}G_{22}))
\]
where \( C_{11} \) is a lower-triangular matrix, and
\[
\det(C_{11}(U_{11}G_{11} + U_{12}G_{21})) = 1.
\]

Hence
\[
\det \left([U^{[g]}]_{jn}(U^{[g]}_{jn})^* \right) = \\
= \det \left([U_{jn}]_{jn}^* \right) \det(U_{11}G_{11} + U_{12}G_{21})^{-1} \det(U_{11}G_{11} + U_{12}G_{21})^{*^{-1}}.
\]

Next,
\[
([U]_{jn}([U]_{jn})^* = \\
= (U_{11}G_{11} + U_{12}G_{21})(G_{12}U_{11}^* + G_{22}U_{12}^*) + (U_{11}G_{12} + U_{12}G_{22})(G_{12}U_{11}^* + G_{22}U_{12}^*) = \\
= U_{11}(G_{11}G_{11} + G_{12}G_{12})U_{11} + U_{11}(G_{11}G_{21} + G_{12}G_{22})U_{12} + \\
+ U_{12}(G_{21}G_{12} + G_{22}G_{22})U_{11} + U_{12}(G_{21}G_{21} + G_{22}G_{22})U_{12}.
\]

The matrix \( g \) is unitary, \( gg^* = 1 \). Therefore the last expression equals to
\[
U_{11} \cdot 1 \cdot U_{11}^* + U_{11} \cdot 0 \cdot U_{12}^* + U_{12} \cdot 0 \cdot U_{11}^* + U_{12} \cdot 1 \cdot U_{12}^*.
\]

Thus
\[
\det \left([U^{[g]}]_{jn}(U^{[g]}_{jn})^* \right) = \det \left([U]_{jn}([U]_{jn})^* \right) |\det(U_{11}G_{11} + U_{12}G_{21})|^{-2}.
\]

Keeping in mind
\[
d(U^{[g]}) = |\det(U_{11}G_{11} + U_{12}G_{21})|^{-4} d\hat{Z},
\]

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we get
\[
\prod_{j=1}^{n-1} \det \left( [U^{[i]}]_{jn} (U^{[i]}]_{jn})^* \right)^{-1} (p_j - p_{j+1})^{-2} \ d(U^{[i]}) =
\]
\[
= \sigma(U, g) \prod_{j=1}^{n-1} \det \left( [U]_{jn} (U]_{jn})^* \right) \ dU,
\]
where
\[
\sigma(U, g) = \left| \prod_{j=1}^{n-1} \det([Ug]_{jj})^{(p_j - p_{j+1})} \right|^2.
\]

Let us represent $Ug$ in the form $DW$, where $D \in B_n^-$, $g \in N_n^+$. Then
\[
\det([Ug]_{kk}) = \prod_{i=1}^{k} b_{ii}(U, g), \quad k \leq n - 1,
\]
(2.12)
\[
\det(g) = \prod_{i=1}^{n} b_{ii}(U, g),
\]
(2.13)
here $b_{ii}(U, g)$ are the diagonal elements of $Ug$. Therefore,
\[
\sigma(U, g) = \left| \prod_{j=1}^{n-1} b_{jj}(U, g)^{p_j - p_{j+1}} \right|^2,
\]

Since $g \in U(n)$, we have $|\det(g)| = 1$. Keeping in mind (2.13) we come to
\[
\sigma(U, g) = \left| \prod_{j=1}^{n} b_{jj}(U, g)^{p_j} \right|^2.
\]
Therefore the right-hand side of (2.11) equals
\[
\int_{N_n^+} \left| f(U^{[i]}) \prod_{j=1}^{n} b_{jj}(U, g)^{p_j} \right|^2 \ dU,
\]
i.e., coincides with the left-hand side.

Thus the group $U(n)$ acts in $L^2(N_n^+, d\mu_p)$ by the unitary operators (2.6).

Intersection of the space of $L^2$ and the space of polynomials. Denote this intersection by $W$.

First, $W$ is $U(n)$-invariant. Indeed, for $\psi \in W$ we have $\rho_p(g)\psi \in L^2$. By definition, $\rho_p(g)\psi$ is a rational holomorphic function. Represent it as an irreducible fraction $\alpha(Z)/\beta(Z)$. Let $Z_0$ be a non-singular point of the manifold $\beta(Z) = 0$, let $O$ be neighborhood of $Z_0$. It is easy to see that
\[
\int_O |\rho_p(g)\psi|^2 \ d\tilde{Z} = \infty.
\]
Therefore $\rho_p(g)\psi$ is a polynomial.

Second, the space $W$ is non-zero. For instance it contains a constant function. Indeed, $\int_{N^+} d\mu_p$ were evaluated in [24] and it is finite.

The third, $W$ is finite-dimensional. Indeed, our measure has the form $|r(Z)|^{-2} d\hat{Z}$, where $r(Z, \overline{Z})$ is a polynomial in $Z, \overline{Z}$. A polynomial $\varphi \in W$ satisfies the condition

$$\int_{N^+} \left| \frac{\varphi(Z)}{r(Z, \overline{Z})} \right|^2 d\hat{Z} < \infty.$$ 

Clearly, degree of $\varphi$ is uniformly bounded.

Next, the operators (2.6) determine a unitary representation in $L^2(N^+_0)$ and this representation is an element of the principal non-unitary series (see, e.g., [6] or [28], Addendum). But a representation of the principal series cannot have more than one finite-dimensional subrepresentation. □

3 The formula for the kernel

In this section we prove Proposition 1.4.

3.1. Functions $\Phi[\ldots], \Psi[\ldots]$. Consider the matrices $Z, Z^{cut}, U, U^{ext}$ as in Subs. 1.8. We use notation (1.2) for their sub-matrices. Let $\alpha = 1, \ldots, n-1$. Let $I, J$ be sets of integers,

$$I: 0 < i_1 < \cdots < i_{n+1-\alpha} \leq n+1, \quad J: 0 < j_1 < \cdots < j_\alpha \leq n.$$ 

Denote

$$\Phi_\alpha[I; J] = \Phi_\alpha[i_1, \ldots, i_{n+1-\alpha}; j_1, \ldots, j_\alpha] := \det \left( \begin{bmatrix} Z(I) \\ U^{ext}(J) \end{bmatrix} \right). \quad (3.1)$$

Let $I, J$ be sets of integers,

$$I: 0 < i_1 < \cdots < i_{n-\alpha} \leq n+1, \quad J: 0 < j_1 < \cdots < j_\alpha \leq n.$$ 

Denote

$$\Psi_\alpha[I; J] = \Psi_\alpha[i_1, \ldots, i_{n-\alpha}; j_1, \ldots, j_\alpha] := \det \left( \begin{bmatrix} Z^{cut}(I) \\ U(J) \end{bmatrix} \right). \quad (3.2)$$

We have two collections of variables $z_{ij}, u_{ij'}$. Zhelobenko operators $R_{kl}$ in $z$ and $u$ we denote by

$$R_{kl}^z, \quad R_{kl}^u.$$ 

Lemma 3.1

$$R_{kl}^z \Phi_\alpha[I; J] = \begin{cases} 0, & \text{if } k \notin I; \\ 0, & \text{if } k \in I, l \notin I; \\ (-1)^{\theta(I,k,l)} \Phi_\alpha[I^c; J], & \text{if } k \in I, l \notin I. \end{cases} \quad (3.3)$$
where $I^\circ$ is obtained from $I$ by replacing $k$ by $l$, and the corresponding change of order. If $k = i_s$ and $i_t < l < i_{t+1}$, when

$$I^\circ = (i_1, \ldots, i_{s-1}, i_{s+1}, \ldots, i_{t-1}, l, i_{t+1}, \ldots, i_{n+1-\alpha})$$

and

$$\theta(I, k, l) = t - s.$$

The same properties (with obvious modifications) hold for

$$R^u_{kl} \Phi_\alpha[I; J], \quad R^u_{kl} \Psi_\alpha[I; J], \quad R^w_{kl} \Psi_\alpha[I; J].$$

**Proof.** Recall that

$$R^z_{kl} = \partial_{kl} + \sum_{j > k} z_{kj} \partial_{kj}.$$

If $k \notin I$, then $\Phi_\alpha[I, J]$ does not depend on variables $z_{km}$. Therefore we get 0. If $k \in I$, these variables are present only in one row. Application of $R^z_{kl}$ is equivalent to a change of the row

$$\begin{pmatrix} 0 & \ldots & 0 & 1 & \ldots & z_{k(k+1)} & \ldots & z_{kl} & \ldots & z_{k(n+1)} \end{pmatrix}$$

of the matrix $\begin{pmatrix} [Z(I)] \\ [U_{ext}(J)] \end{pmatrix}$ by the row

$$\begin{pmatrix} 0 & \ldots & 0 & 0 & \ldots & 1 & \ldots & z_{l(l+1)} & \ldots & z_{k(n+1)} \end{pmatrix}.$$

If $l \in I$, then this row is present in the initial matrix, and again det = 0. Otherwise, we come to $\Phi_\alpha[I^\circ; J]$ up to the order of rows. \(\square\)

**Corollary 3.2** For any $k$, $l$, $\alpha$,

$$\begin{align*}
(R^z_{kl})^2 \Phi_\alpha[I; J] &= 0, \\
(R^u_{kl})^2 \Phi_\alpha[I; J] &= 0,
\end{align*}$$

$$\begin{align*}
(R^z_{kl})^2 \Psi_\alpha[I; J] &= 0, \\
(R^u_{kl})^2 \Psi_\alpha[I; J] &= 0.
\end{align*}$$

**Corollary 3.3**

$$R^z_{kl} R^z_{lm} \Phi_\alpha[I; J] = \begin{cases} -R^z_{km} \Phi_\alpha[I; J], & \text{if } k \in I, \ l \notin I, \ m \notin I; \\
0, & \text{otherwise.}
\end{cases} \quad (3.4)$$

The same property holds for operators $R^u$ and for $\Psi_\alpha[I; J]$.

3.2. Verification of the Zhelobenko conditions.

**Lemma 3.4** $L_{p,q} \in V_p \otimes V_q.$
PROOF. By the interlacing conditions, $L_{p,q}$ is a polynomial. We must verify the identities

\[(R_{k(k+1)}^z)^{p_k-p_{k+1}+1}L_{p,q} = 0, \quad (R_{k(k+1)}^u)^{q_k-q_{k+1}+1}L_{p,q} = 0. \quad (3.5)\]

To be definite verify the first equality,

\[
(R_{k(k+1)}^z)^{p_k-p_{k+1}+1}L_{p,q} = (R_{k(k+1)}^z)^{p_k-p_{k+1}+1} \left( \Psi_{n-k}^{-p_{k+1}-q_{n+1-k}} \Phi_{n+1-k}^{-p_k+q_{n+1-k}} \cdot \text{remaining factors} \right) = (R_{k(k+1)}^z)^{p_k-p_{k+1}+1} \left( \Psi_{n-k}^{-p_{k+1}-q_{n+1-k}} \Phi_{n+1-k}^{-p_k+q_{n+1-k}} \cdot \text{remaining factors} \right)
\]

The sum of exponents (the both exponents are positive) is

\[
(-p_{k+1} - q_{n+1-k}) + (p_k + q_{n+1-k}) = p_k - p_{k+1},
\]

and we obtain the desired 0. If $k = n$ the factor $\Psi_{n-k}$ is absent, we have

\[
(R_{n(n+1)}^z)^{p_n-p_n+1} (\Phi_n^{p_n+q_1}) \cdot \text{remaining factors}.
\]

By the interlacing condition $-p_{n+1} \geq q_1$ and we again obtain 0. \qed

3.3. Invariance of the kernel. Consider matrices $\Phi_\alpha$, $\Psi_\alpha$ defined by (1.13).

Lemma 3.5 Let $g \in \text{GL}(n, \mathbb{C})$. Denote $\bar{g} = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}(n+1, \mathbb{C})$. Then

\[
\Phi(Z^{[\bar{g}]}, U^{[\bar{g}]}) = 
= \Phi(Z, U) \det(g) \det([Z\bar{g}]_{(n+1-\alpha)(n+1-\alpha)})^{-1} \det([Ug]_{\alpha\alpha})^{-1}. \quad (3.6)
\]

\[
\Psi(Z^{[\bar{g}]}, U^{[\bar{g}]}) = 
= \Psi(Z, U) \det(g) \det([Z\bar{g}]_{(n-\alpha)(n-\alpha)})^{-1} \det([Ug]_{\alpha\alpha})^{-1}. \quad (3.7)
\]

PROOF. We write the Gauss decompositions of $Z\bar{g}$, $Ug$,

\[
Z\bar{g} = BP, \quad Ug = CQ.
\]

Then the Gauss decomposition of $U^{ext}$ (see (1.11)) is

\[
U^{ext}g = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix},
\]

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denote factors in the right-hand side by \( \tilde{C}, \tilde{Q} \). Then

\[
\begin{pmatrix}
[Z_{(n+1-\alpha)(n+1)}] \\
[\text{ext}(\tilde{g})]_{\alpha(n+1)}
\end{pmatrix} = \begin{pmatrix}
[P_{(n+1-\alpha)(n+1)}] \\
[\tilde{Q}]_{\alpha(n+1)}
\end{pmatrix} = \\
\begin{pmatrix}
(\tilde{B}_{(n+1-\alpha)(n+1)})^{-1} & 0 \\
0 & ([C]_{\alpha})^{-1}
\end{pmatrix} \begin{pmatrix}
[Z_{(n+1-\alpha)(n+1)}] \\
[U]_{\alpha(n+1)}
\end{pmatrix} \cdot g
\]

We pass to determinants in the left-hand side and the right-hand side. Keeping in mind

\[
\det(\tilde{B}_{(n+1-\alpha)(n+1)}) = \det(\tilde{Z}g)_{(n+1-\alpha)(n+1-\alpha)},
\]

\[
\det(\tilde{C})_{\alpha} = \det(\text{ext}g)_{\alpha} = \det(Ug)_{\alpha},
\]

we come to (3.6). Proof of (3.7) is similar. □

Proof of Proposition 1.4. Applying the lemma and (2.12), we get

\[
L_{p,q}(Z, g) = L_{p,q}(Z, U) \prod_{j=1}^{n} b_{\lambda_j}(Z, g)^{-q_j-p_j} \prod_{j=1}^{n} b_{\lambda_j}(U, g)^{-p_j-q_j} \det(g)^{p_1+q_1}.
\]

Since

\[
b_{(n+1)(n+1)}(Z, g) = 1, \quad \det(g) = \prod_{j=1}^{n} b_{\lambda_j}(Z, g) = \prod_{j=1}^{n} b_{\lambda_j}(U, g),
\]

we come to

\[
L_{p,q}(Z, g) = L_{p,q}(Z, U) \prod_{j=1}^{n} b_{\lambda_j}(Z, g)^{-p_j} \prod_{j=1}^{n} b_{\lambda_j}(U, g)^{-q_j}.
\]

Therefore,

\[
(\rho_{p}(g) \otimes \rho_{q}(g))L_{p,q} = L_{p,q}.
\]

Thus \( L_{p,q} \) is a non-zero GL\((n, \mathbb{C})\)-invariant vector in \( V_{p} \otimes V_{q} \). We also know that such a vector is unique up to a scalar factor. □

4 The calculation

Below \( q \in \Lambda_n, \ r = p^* \in \Lambda_{n+1} \) are signatures. The signatures \( r \) and \( q \) are interlacing. The kernel \( L \) is the same as above (1.14),

\[
L := L_{q}^{r}(Z, U) = \Phi_{1}^{q_1-r_2} \Phi_{2}^{q_2-r_3} \Phi_{3}^{q_3-r_4} \ldots \Phi_{n}^{q_n-r_{n+1}}.
\]

Below

\[
R_{kl} := R_{kl}^n.
\]
We must verify the differential-difference equations

\[ E_{1(n+1)L} = F_{1(n+1)L}, \quad E_{(n+1)nL} = F_{(n+1)nL}, \]

\( F_{1(n+1)} \), \( F_{(n+1)n} \) are given by (1.22)–(1.22). Evaluations of the left-hand sides is easy, it is contained in the next subsection. Evaluation of the right-hand sides is tricky and occupies the remaining part of the section.

4.1. Evaluation of \( E_{1(n+1)L} \).

Lemma 4.1

\[ E_{1(n+1)L} = \left( -\frac{q_1 - r_2}{\Phi_1} + \sum_{\alpha=2}^{n} \frac{(q_\alpha - r_{\alpha+1}) R_{1\alpha} \Psi_{\alpha-1}}{\Phi_\alpha} \right) \cdot L. \]  \hspace{1cm} (4.1)

Remark. For \( \alpha = 1 \) we immediately get \( \partial \Phi_1 / \partial z_{1(n+1)} = -1 \). On the other hand, taking formally \( \alpha = 1 \) in the sum in (4.1) we get \( R_{11} \Psi_0 / \Phi_1 \). At first glance, we must assume that \( \Psi_0 = 1 \) and that \( R_{11} \) is the identical operator. Under this assumption, \( R_{11} \Psi_0 / \Phi_1 = 1 / \Phi_1 \). However, this gives an incorrect sign. ⊠

In the following two proofs we need manipulations with determinants. To avoid huge matrices or compact notation, which are difficult for reading, we expose calculations for matrices having a minimal size that allows to visualize picture.

Proof. We have (see (2.7))

\[ E_{1(n+1)L} = \frac{\partial L}{\partial z_{1(n+1)}} = \left( \sum_{\alpha=1}^{n} (q_\alpha - r_{\alpha+1}) \frac{\partial \Phi_\alpha}{\partial z_{1(n+1)}} \cdot \Phi^{-1}_\alpha \right) \cdot L. \]

Take \( n = 3 \) and \( \alpha = 2 \),

\[ \Phi_2 = \det \begin{pmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & 1 & z_{23} & z_{24} \\ 1 & u_{12} & u_{13} & 0 \\ 0 & 1 & u_{23} & 0 \end{pmatrix}. \]  \hspace{1cm} (4.2)

Then

\[ \frac{\partial \Phi_2}{\partial z_{14}} = - \det \begin{pmatrix} 0 & 1 & z_{23} \\ 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \end{pmatrix} = \det \begin{pmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 0 & 1 & u_{23} \end{pmatrix} = R_{12} \Psi_1. \]

Lemma 4.2

\[ E_{(n+1)nL} = \left( z_{n(n+1)}(r_1 - q_1) - \sum_{j=1}^{n-1} \frac{(r_{\alpha+1} - q_{\alpha+1}) R_{(\alpha+1)n} \Phi_{\alpha+1}}{\Psi_\alpha} \right) \cdot L. \]
PROOF. Split $E_{(n+1)n}$ as a sum

$$E_{(n+1)n} = D + (r_1 - r_2)z_{n(n+1)}, \quad D := \sum_{j=1}^{n-1} z_{j(n+1)} \partial_{jn} - z_{n(n+1)}^2 \partial_{n(n+1)}.$$  

Obviously,

$$E_{(n+1)n} L =$$

$$= \left((r_1 - r_2)z_{n(n+1)} + \sum_{\alpha=1}^{n} \frac{(q_{\alpha} - r_{\alpha+1})D\Phi_{\alpha}}{\Phi_{\alpha}} + \sum_{\beta=1}^{n-1} \frac{(r_{\beta+1} - q_{\beta+1})D\Psi_{\beta}}{\Psi_{\beta}}\right) L$$

Let us evaluate all $D\Phi_{\alpha}$, $D\Psi_{\alpha}$.

1) $D\Phi_{\alpha} = 0$ for all $\alpha \neq 1$. Take $n = 3$, $\alpha = 2$, 

$$D\Phi_{2} = D \det \begin{pmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & 1 & z_{23} & z_{24} \\ 1 & u_{12} & u_{13} & 0 \\ 0 & 1 & u_{23} & 0 \end{pmatrix} = \det \begin{pmatrix} 1 & z_{12} & z_{14} \\ 0 & 1 & z_{24} \\ 1 & u_{12} & 0 \end{pmatrix} = 0.$$  

Variables $z_{jn}$ (in our case $z_{34}$) are present only in the next-to-last column. We apply the operator $D$ to this column and come to a matrix with coinciding columns.

2) $D\Phi_{1} = -z_{n(n+1)}\Phi_{1}$. Take $n = 3$, 

$$D\Phi_{1} = D \det \begin{pmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & 1 & z_{23} & z_{24} \\ 0 & 0 & 1 & z_{34} \\ 1 & u_{12} & u_{13} & 0 \end{pmatrix} =$$

$$= \det \begin{pmatrix} 1 & z_{12} & z_{14} \\ 0 & 1 & z_{24} \\ 1 & u_{12} & 0 \end{pmatrix} + z_{34}^2 \det \begin{pmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 1 & u_{12} & u_{13} \end{pmatrix} =$$

$$= -z_{34} \det \begin{pmatrix} 1 & z_{12} & z_{14} \\ 0 & 1 & z_{24} \\ 1 & u_{12} & 0 \end{pmatrix} + z_{34}^2 \det \begin{pmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 1 & u_{12} & u_{13} \end{pmatrix} =$$

$$= -z_{34} \det \begin{pmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & 1 & z_{23} & z_{24} \\ 0 & 0 & 1 & 0 \\ 1 & u_{12} & u_{13} & 0 \end{pmatrix} - z_{34} \det \begin{pmatrix} 1 & z_{12} & z_{13} & 0 \\ 0 & 1 & z_{23} & 0 \\ 0 & 0 & 1 & z_{34} \\ 1 & u_{12} & u_{13} & 0 \end{pmatrix} =$$

$$= -z_{34} \det \begin{pmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & 1 & z_{23} & z_{24} \\ 0 & 0 & 1 & z_{34} \\ 1 & u_{12} & u_{13} & 0 \end{pmatrix} = -z_{34} \Phi_{1}.$$
3) \( D\Psi_{\alpha} = -R_{(\alpha+1)n}\Phi_{\alpha+1} \). Take \( n = 4, \alpha = 1 \).

\[
D\Psi_1 = D\det \begin{pmatrix}
1 & z_{12} & z_{13} & z_{14} \\
0 & 1 & z_{24} & z_{24} \\
0 & 0 & 1 & z_{34} \\
1 & u_{12} & u_{13} & u_{14}
\end{pmatrix} = \det \begin{pmatrix}
1 & z_{12} & z_{13} & z_{15} \\
0 & 1 & z_{24} & z_{25} \\
0 & 0 & 1 & z_{35} \\
1 & u_{12} & u_{13} & 0
\end{pmatrix} = \det \begin{pmatrix}
1 & z_{12} & z_{13} & z_{14} & z_{15} \\
0 & 1 & z_{24} & z_{24} & z_{25} \\
0 & 0 & 1 & z_{35} & z_{35} \\
1 & u_{12} & u_{13} & u_{14} & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix} = -R_{24}\Phi_2.
\]

Thus we know all \( D\Phi_{\alpha}, D\Psi_{\alpha} \). This gives us the desired statement. \( \square \)

4.2. Quadratic relations. Now we wish to write a family of quadratic relations between different functions of type \( R_{kl}\Phi_{\alpha}, R_{kl}\Psi_{\beta} \). We introduce notation

\[
R_{kk}\Phi_{k-1} := +\Phi_{k-1}, \quad R_{kk}\Psi_{k-1} = -\Psi_{k-1}, \quad R_{11}\Psi_0 = -\Psi_0 = 1.
\]

We do not introduce operators \( R_{kk} \) and \( R_{kk}\Phi_{k-1}, R_{kk}\Psi_{k-1} \) are only symbols used in formulas. In particular, formula (4.1) now can be written the form

\[
E_{1(n+1)}L = L \cdot \sum_{\alpha=1}^{n} \frac{(q_{\alpha} - r_{\alpha+1})R_{1\alpha}\Psi_{\alpha-1}}{\Phi_{\alpha}}
\]

without a term with abnormal sign. Below it allows to avoid numerous anomalies in the formulas, duplications of formulas and branchings of calculations.

**Lemma 4.3** a) Let \( m < \alpha \leq \beta \),

\[
\sum_{j=\alpha+1}^{\beta+1} R_{mj}\Phi_{\alpha} \cdot R_{j(\beta+1)}\Psi_{\beta} = -\Phi_{\alpha} \cdot R_{m(\beta+1)}\Psi_{\beta} + \Phi_{\beta+1} \cdot R_{ma}\Psi_{\alpha-1}.
\]

b) Let \( m < \alpha < \beta \). Then

\[
\sum_{j=\alpha+1}^{\beta+1} R_{mj}\Psi_{\alpha} \cdot R_{j(\beta+1)}\Psi_{\beta} = -\Psi_{\alpha} \cdot R_{m(\beta+1)}\Psi_{\beta}.
\]

c) Let \( m < \alpha < \beta \). Then

\[
\sum_{j=\alpha+1}^{\beta} R_{mj}\Phi_{\alpha} \cdot R_{j(\beta+1)}\Phi_{\beta} = -\Phi_{\alpha} \cdot R_{m(\beta+1)} - R_{m(\beta+1)}\Phi_{\alpha} \cdot \Phi_{\beta}.
\]
Proof. Denote by $\xi$ the row $\begin{pmatrix} Z \\ U^{ext} \end{pmatrix}$. Let us enumerate $z$-rows of these matrix by 1, 2, 3, \ldots and $u$-rows by marks $\overline{1}$, $\overline{2}$, $\overline{3}$, \ldots. Consider the following minors of $\Delta$:

$$\Phi_\alpha = \det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}]$$

$$R_{m(\beta + 1)}\Psi_\beta = (-1)^{\beta - m} \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi]; \quad (4.7)$$

Notice that we $\xi$ in the last row allows to replace the last determinant by

$$\det \left[ \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] \right]_{n \pi}.$$ 

and this proves (4.7).

We wish to apply one of the Plücker identities to the product of these minors (see, e.g., [4], Section 9.1). We take $\overline{m}$ from the collection $(1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m})$, exchange it with an element of the collection $(1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi)$ and consider the product of the corresponding minors. Next, we take the sum of all such products of $\Delta$. In this way, we obtain

$$\det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}] \cdot \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] = \Phi_\alpha = \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] = (4.8)$$

$$\times \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] + \cdots +$$

$$+ \det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] \times \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] = \Phi_\alpha = \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] = (4.9)$$

$$+ \det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] \times \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] + \cdots +$$

$$+ \det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] \times$$

$$\times \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] + \cdots +$$

$$+ \det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] \times$$

$$\times \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] + \cdots +$$

$$+ \det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] \times$$

$$\times \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] + \cdots +$$

$$+ \det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] \times$$

$$\times \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] + \cdots +$$

$$+ \det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] \times$$

$$\times \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] + \cdots +$$

$$+ \det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] \times$$

$$\times \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] + \cdots +$$

$$+ \det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] \times$$

$$\times \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] + \cdots +$$

$$+ \det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] \times$$

$$\times \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] + \cdots +$$

$$+ \det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] \times$$

$$\times \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] + \cdots +$$

$$+ \det \Delta[1, \ldots, n + 1 - \alpha; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] \times$$

$$\times \det \Delta[1, \ldots, n - \beta; \overline{1}, \ldots, \overline{m}, \ldots, \beta + 1; \xi] + \cdots +$$

$$= \Phi_\alpha \cdot R_{m(\beta + 1)}\Psi_\beta.$$
Next, let us watch signs. Denote by $(i_1 i_2 \ldots i_k)$ a cycle in a substitution. Denote by $\sigma(\cdot)$ the parity of a substitution.

Examine summands (4.13)–(4.16). For $j = \alpha + 1, \ldots, \beta + 1$, we have

$$\det \Delta[1, \ldots, n+1-\alpha; \overline{1}, \ldots, \overline{m-1}, j, m+1, \ldots, \overline{\alpha}] = \sigma(j(m+1) \ldots \alpha) R_{mj} \Phi_\alpha = (-1)^{\alpha - m} R_{mj} \Phi_\alpha$$

For $j = \alpha + 1, \ldots, \beta$ we have

$$\det \Delta[1, \ldots, n-\beta; \overline{1}, \ldots, \hat{m}, \ldots, \overline{j-1}, m, j+1, \ldots, \beta + 1; \xi] = \sigma(m \ldots (j-1)) \cdot \sigma((j+1) \ldots (\beta+1)) R_{j(\beta+1)} \Psi_\beta = (-1)^{\beta - m + 1} R_{j(\beta+1)} \Psi_\beta.$$ 

Next, in (4.17) we have

$$\det \Delta[1, \ldots, n-\beta; \overline{1}, \ldots, \hat{m}, \ldots, \overline{j}, m; \xi] = (-1)^{\beta - m} \Psi_\beta$$

and this gives the anomaly of a sign mentioned above.

Examine the last summand (4.19)–(4.20). We get

$$\det \Delta[1, \ldots, n+1-\beta; \overline{1}, \ldots, \hat{m}, \ldots, \overline{\beta+1}, \overline{m}; \xi] = \sigma(m \ldots (\beta+1)) \Phi_{\beta+1} = (-1)^{\beta+1-m} \Phi_{\beta+1}.$$ 

Combining all signs we get the desired identity.

**Proof of the statement b.** We compose a new matrix $\Delta = \begin{pmatrix} Z^{cut} \\ U \end{pmatrix}$, consider the following minors of $\Delta$,

$$\Psi_\alpha = \det \Delta[1, \ldots, n-\alpha; \overline{1}, \ldots, \overline{\alpha}],$$

$$R_{m(\beta+1)} \Psi_\beta = (-1)^{\beta - m} \det \Delta[1, \ldots, n-\beta; \overline{1}, \ldots, \hat{m}, \ldots, \overline{\beta}, \overline{\beta+1}],$$

and write the Plucker identity exchanging the $\overline{m}$-th row of $\Psi_\alpha$ with rows of $R_{m(\beta+1)} \Psi_\beta$.

**Proof of the statement c.** We compose a matrix $\Delta := \begin{pmatrix} Z \\ U^{ext} \end{pmatrix}$, take the minors

$$\Phi_\alpha = \det \Delta[1, \ldots, n+1-\alpha; \overline{1}, \ldots, \overline{\alpha}],$$

$$R_{m(\beta+1)} \Phi_\beta = (-1)^{\beta - m} \det \Delta[1, \ldots, n+1-\beta; \overline{1}, \ldots, \hat{m}, \ldots, \overline{\beta}, \overline{\beta+1}].$$
and write the Plucker identities exchanging $\mathfrak{m}$-th of $\Phi_\alpha$ with rows of the second minor.

4.3. A recurrence formula. Our next purpose is to derive a formula for $F_{1(n+1)}L$. Let us evaluate one of summands of this expression,

$$
\Theta_{1n} := \left( \sum_{I \subseteq [1,n], I \notin [1,n] \setminus I} \prod_{l \in [1,n] \setminus I} (l - n + q_l - q_n) \cdot R_l \right) T_n^{-n} L.
$$

(4.21)

Recall that the notation $J \triangledown \{1, 2, \ldots, n\}$ means that $J$ is a subset in $\{1, 2, \ldots, n\}$ containing 1 and $n$, see Subsection 1.9.

Remark. By definition, the right-hand side is a rational expression of the form

$$
\frac{F}{\prod \Phi_\alpha^s \prod \Psi_\beta^t}.
$$

It is easy to observe (see below) that all the exponents $s_\alpha, t_\beta$ are equal 1. In a calculation below we decompose this function as a sum of ‘prime fractions’ with denominators $\Phi_\alpha, \Psi_\beta$ and finally get an unexpectedly simple expression.

Let $1 \leq k < l \leq n$, $\mu < n$. Denote

$$
W_{kl}^\mu := \frac{(q_\mu - r_\mu - 1)R_{kl} \Phi_\mu}{\Phi_\mu^\mu} + \frac{(r_{\mu+1} - q_{\mu+1})R_{kl} \Psi_\mu}{\Psi_\mu^\mu};
$$

(4.22)

$$
w_{kl}^\mu := (-1)^{l-k-1} \prod_{j=k+1}^{\mu} (q_\mu - q_j + j - n - 1) \prod_{j=\mu+1}^{l-1} (q_\mu - q_j + j - n); \quad (4.23)
$$

$$
\zeta_{kl} = \sum_{\mu=k}^{l-1} w_{kl}^\mu W_{kl}^\mu + w_{k(n-1)}^{n-1} R_{kn} \Psi_{n-1}^{-1} \Phi_n.
$$

(4.24)

Notice that the additional term in the last row is 0 if $l \neq n$.

Lemma 4.4

$$
R_{kl}L = L \cdot \sum_{\mu: k \leq \mu < l} W_{kl}^\mu;
$$

(4.25)

$$
R_{kl}T_n^{-n}L = L \cdot \left( \sum_{\mu: k \leq \mu < n} W_{kl}^\mu \right) \cdot \frac{\Psi_{n-1}^{-1}}{\Phi_n}, \quad \text{for } l < n;
$$

(4.26)

$$
R_{kn}T_n^{-n}L = L \cdot \left( \sum_{\mu: k \leq \mu < n} W_{kl}^\mu + \frac{R_{kn} \Psi_{n-1}^{-1}}{\Psi_{n-1}^{-1}} \right) \cdot \frac{\Psi_{n-1}^{-1}}{\Phi_n}.
$$

(4.27)

Proof. Obviously,

$$
R_{kl}L = L \left( \sum_{j=1}^{n} \frac{(q_\mu - r_{\mu+1})R_{kl} \Phi_\mu}{\Phi_\mu} + \sum_{j=1}^{n} \frac{(r_{\mu+1} - q_{\mu+1})R_{kl} \Psi_\mu}{\Psi_\mu} \right).
$$
By Lemma 3.1 only terms with \( k \leq \mu < l \) give a nonzero contribution. This gives the first row. Next,

\[
T_n^{-1} L = L \cdot \frac{\Psi_{n-1}}{\Phi_n}, \quad R_{kl} T_n^{-1} L = (R_{kl} L) \cdot \frac{\Psi_{n-1}}{\Phi_n} + L \cdot R_{kl} \left( \frac{\Psi_{n-1}}{\Phi_n} \right)
\]

The second term in the right-hand side is 0 for \( l < n \). For \( l = n \) this term equals \( R_{kn} \Psi_{n-1}/\Phi_n \).

\[\text{Lemma 4.5}\]

\[
\Theta_{1n} = \left( \sum_{J = \{j_1 < j_2 < \cdots < j_s\} \in \{1,2,\ldots,n\}} \zeta_{j_1j_2j_3\cdotsj_{s-1}j_s} \right) L \cdot \frac{\Psi_{n-1}}{\Phi_n}. \quad \text{(4.28)}
\]

**Proof.** First, we evaluate \( R_I (T_n^{-1} L) \cdot L^{-1} \).

Each \( R_I \) is a product \( R_{i_1i_2}\cdots R_{i_{t-1}i_t} \). Each \( R_{i_t} \) is a first order differential operator without term of order zero, therefore we can expand \( R_I L \) according the Leibniz rule.

Many summands of this expansion are zero by a priori reasons. Indeed, \( R_{kl} \Phi_{\mu} \), \( R_{kl} \Psi_{\mu} \) are nonzero only if \( k \leq \mu < l \). Also, \( R_{kl} R_{ab} \Phi_{\mu} = 0 \) if \( l < a \). In the case \( l = a \) we have \( R_{kl} R_{lb} \Phi_{\mu} = -R_{kb} \Phi_{\mu} \).

Therefore the expression \( (R_I T_n^{-1} L)/L \) is a sum of products of the following type

\[
A_{m_1} A_{m_2} A_{m_3} \cdots A_{m_{t-1}} A_{m_t} \cdot \frac{\Psi_{n-1}}{\Phi_n}, \quad \text{(4.29)}
\]

where \( M = \{m_1, \ldots, m_t\} \prec I \). and each factor \( A_{m_{\tau+1}} \) has a form

\[
(q_\mu - r_{\mu+1}) R_{i_1i_{a+1}} R_{i_{a+1}i_{a+2}} \cdots R_{i_{t-1}i_t} \Phi_{\mu} \quad \text{or} \quad (r_{\mu+1} - q_\mu+1) R_{i_1i_{a+1}} R_{i_{a+1}i_{a+2}} \cdots R_{i_{t-1}i_t} \Psi_{\mu},
\]

where \( i_{c-1} \leq \mu < i_c \). These expressions are equal correspondingly

\[
\frac{(-1)^{c-a+1}(q_\mu - r_{\mu+1})}{\Phi_{\mu}} R_{i_1i_{c}} \Phi_{\mu} \quad \text{and} \quad \frac{(-1)^{c-a+1}(r_{\mu+1} - q_\mu+1)}{\Psi_{\mu}} R_{i_1i_{c}} \Psi_{\mu},
\]

Now fix \( M \) and consider the sum \( S[I,M] \) of all summands (4.29) with fixed \( M \). It is easy to see that

\[
S[I,M] = (-1)^{c-a+1} \prod_{\gamma=1}^{t-1} \left( \sum_{i_a \in I : m_\gamma \leq i_a < m_{\gamma+1}} \sum_{\mu : i_a \leq \mu < i_c} W_\mu^{i_a m_{\gamma+1}} \right) \cdot L.
\]
Next, we represent $\Theta_{1n}$ as

$$\Theta_{1n} = \left( \sum_{I \subseteq \llbracket 1, n \rrbracket} \prod_{l \in I} (l - n + q_l - q_n) \cdot \sum_{M \subseteq I} S[I, M] \right) \cdot L \cdot \frac{\Psi_{n-1}}{\Phi_n} =$$

$$= \sum_{M \subseteq \llbracket 1, n \rrbracket} \left( \sum_{I \supseteq M} \prod_{l \in \llbracket 1, n \rrbracket \setminus I} (l - n + q_l - q_n) \cdot S[I, M] \right) \cdot L \cdot \frac{\Psi_{n-1}}{\Phi_n}.$$

Fix $M = \{m_1 < \cdots < m_{\ell}\}$. Each summand in the big brackets splits into a product of the form

$$\prod_{\gamma = 1}^{t-1} H[m_\gamma, m_{\gamma+1}],$$

where $H[m_\gamma, m_{\gamma+1}]$ is an expression, which depend on $m_\gamma, m_{\gamma+1}$ and does not depend on other $m_j$. Since the coefficients also are multiplicative in the same sense, the whole sum in the big brackets also splits in a product

$$\prod_{\gamma = 1}^{t-1} Z[m_\gamma, m_{\gamma+1}],$$

where each factor $Z[m_\gamma, m_{\gamma+1}]$ depends only on $m_\gamma, m_{\gamma+1}$ and not on the remaining elements of a set $M$.

It remains to evaluate factors $Z[\cdot]$

$$Z[k, l] = \sum_{\mu = k}^{l} v_{kl}^\mu W_{kl}^\mu,$$

where

$$v_{kl}^\mu = \sum_{J} \left( - \prod_{m \in [k, \mu] \setminus J} (-1) (m - n + q_n - q_m) \cdot \prod_{m=\mu+1}^{\ell-1} (m - n + q_n - q_m) \right) =$$

$$= \prod_{m=\mu+1}^{\ell-1} (m - n + q_n - q_m) \cdot \left( - \sum_{J} \prod_{m \in [k, \mu] \setminus J} (-1) (m - n + q_n - q_m) \right),$$

the summation here is taken other all subsets $J \subset [k, \mu]$, containing $k$. In the brackets we get

$$\prod_{m=\mu+1}^{\mu} (-1) \left[ 1 - (m - n + q_n - q_m) \right]$$

Thus $Z_{kl}$ equals to $\zeta_{kl}$. \qed

Let $\zeta_{kl}$ be as above (1.24). Denote

$$\Theta_{mn} := \left( \sum_{J = \{j_1 < j_2 < \cdots < j_s\} \subseteq \{m, 2 \ldots n\}} \zeta_{j_1 j_2} \zeta_{j_2 j_3} \cdots \zeta_{j_{s-1} j_s} \right) L \cdot \frac{\Psi_{n-1}}{\Phi_n}.$$

By the previous lemma, this notation is compatible with the earlier notation $\Theta_{1n}$. Also, $\Theta_{nn} = \frac{\Psi_{n-1}}{\Phi_n}$. 25
Lemma 4.6 The $\Theta_{mn}$ satisfies the following recurrence relation,

$$\Theta_{mn} = \zeta_{m(m+1)} \Theta_{(m+1)n} + \zeta_{m(m+2)} \Theta_{(m+2)n} + \cdots + \zeta_{mn} \Theta_{nn}.$$ 

This statement is obvious.

4.4. Evaluation of $\Theta_{1n}$. Denote

$$s^\mu_{m\tau} = \prod_{j=m}^{\mu-1} (q_\tau - q_j + j - n)$$

$$\sigma^\mu_\tau = (q_\mu - r_{\mu+1}) \prod_{i=\mu+2}^r (q_\tau - r_i - n + 1 + i).$$

Lemma 4.7

$$\Theta_{mn} L^{-1} = - \sum_{\mu=m}^n s^\mu_{mn} \sigma^\mu_n R_{mn} \Psi_{\mu-1} \Phi_\mu.$$ 

In particular, this gives an explicit expression for $\Theta_{1n}$.

Proof. We prove our statement by induction. Assume that for $\Theta_{nn}$, $\Theta_{(n-1)n}$, \ldots, $\Theta_{(m+1)n}$ the formula is correct. We must derive the equality

$$\Theta_{mn} \cdot L^{-1} = \sum_{\gamma=m+1}^n \zeta_{m\gamma} \Theta_{(m+1)n} =$$

$$= \sum_{\gamma=m+1}^n \left( \sum_{\mu=m}^{\gamma-1} \sum_{\nu=\gamma}^n w^\mu_{m\gamma} W^\mu_{m\gamma} \left( \sum_{\nu=\gamma}^n s^{\nu}_{\gamma\nu} \sigma^{\nu}_n R_{\gamma\nu} \Psi_{\nu-1} \Phi_\nu \right) \right).$$

We have

$$s^\mu_{m\gamma} s^n_{\gamma\nu} = \prod_{j=m+1}^{\mu} (q_n - q_j + j - n + 1) \prod_{j=m+1}^{\gamma-1} (q_n - q_j + j - n + 1) \prod_{j=\gamma}^{\nu-1} (q_n - q_j + j - n + 1) =$$

$$= \prod_{j=m+1}^{\gamma-1} (q_n - q_j + j - n + 1) \prod_{j=\mu+1}^{\nu-1} (q_n - q_j + j - n + 1) =: \xi_{mn\nu}.$$ 

We stress that the expression $\xi_{mn\nu}$ does not depend on $\gamma$. Our sum transforms to

$$\sum_{\mu, \nu; \mu < \nu} \xi_{mn\nu} \sigma^\nu_n \left[ \sum_{\gamma=\mu+1}^{\nu} W^\mu_{m\gamma} R_{\gamma\nu} \Psi_{\nu-1} \Phi_\nu \right] + \frac{\Psi_{n-1}}{\Phi_n}.$$
Denote by $B_{\mu}^{\mu}$ the expression in the square brackets and write it explicitly:

$$B_{\mu}^{\mu} = \sum_{\gamma=\mu+1}^{\nu} \left( \frac{(r_{\mu+1} - q_{\mu+1})R_{m\gamma} \Psi_{\mu}}{\Psi_{\mu}} + \frac{(q_{\mu} - r_{\mu+1})R_{m\gamma} \Phi_{\mu}}{\Phi_{\mu}} \right) \frac{R_{\gamma\nu} \Psi_{\nu-1}}{\Phi_{\nu}} =$$

$$= (r_{\mu+1} - q_{\mu+1}) \sum_{\gamma=\mu+1}^{\nu} \frac{R_{m\gamma} \Psi_{\mu}}{\Psi_{\mu}} \frac{R_{\gamma\nu} \Psi_{\nu-1}}{\Phi_{\nu}} +$$

$$+ (q_{\mu} - r_{\mu+1}) \sum_{\gamma=\mu+1}^{\nu} \frac{R_{m\gamma} \Phi_{\mu}}{\Phi_{\mu}} \frac{R_{\gamma\nu} \Psi_{\nu-1}}{\Phi_{\nu}}. \quad (4.30)$$

Next, we apply the quadratic relations (4.4) and (4.5) and transform the last expression to

$$= (r_{\mu+1} - q_{\mu+1}) \sum_{\gamma=\mu+1}^{\nu} \frac{R_{m\gamma} \Psi_{\mu}}{\Psi_{\mu}} \frac{R_{\gamma\nu} \Psi_{\nu-1}}{\Phi_{\nu}} + (q_{\mu} - r_{\mu+1}) \frac{R_{m\nu} \Psi_{\nu-1}}{\Phi_{\nu}}.$$

Thus,

$$\Theta_{mn} := \sum_{\mu, \nu < \mu < \nu} \xi_{mn}^{\mu} \sigma_{n}^{\nu} \left( (q_{\mu+1} - q_{\mu}) \frac{R_{mn} \Psi_{\nu-1}}{\Phi_{\nu}} + (q_{\mu} - r_{\mu+1}) \frac{R_{mn} \Psi_{\nu-1}}{\Phi_{\mu}} \right). \quad (4.31)$$

We collect similar terms and get

$$\Theta_{mn} := \sum_{\nu} \left( S_{1} + S_{2} \right) \frac{R_{mn} \Psi_{\nu-1}}{\Phi_{\nu}},$$

where

$$S_{1} := \sum_{\mu=1}^{\nu-1} \xi_{\mu\nu} \sigma_{n}^{\nu} (q_{\mu+1} - q_{\mu}), \quad (4.32)$$

$$S_{2} := \sum_{\kappa=\nu+1}^{n} \xi_{\kappa\nu} \sigma_{n}^{\nu} (q_{\kappa} - r_{\nu+1}). \quad (4.33)$$

First, we transform $S_{2}$,

$$S_{2} = (q_{\nu} - r_{\nu+1}) \cdot \prod_{j=m+1}^{\nu} (q_{n} - q_{j} - n + j - 1) \times$$

$$\times \sum_{\kappa=\nu+1}^{n} \left( \prod_{j=\nu+1}^{\kappa-1} (q_{n} - q_{j} - n + j) \cdot \prod_{j=2}^{n-\kappa} (q_{n} - r_{n+2-j} - j) \right). \quad (4.34)$$

The sum in the second line is evaluated in the following lemma.
Lemma 4.8  Let $n - k \geq \nu + 1$. Then

$$\sum_{x=n-k+1}^{n} \prod_{j=\nu+1}^{x-1} (q_n - q_j + j - n) \cdot (q_x - r_{x+1}) \cdot \prod_{j=2}^{n-x} (q_n - r_{n+2-j} - j) = \prod_{\nu+1}^{n-k} (q_n - q_j + j - n) \cdot \prod_{j=2}^{k} (q_n - r_{n+2-j} - k + 1). \quad (4.35)$$

In particular,

$$\sum_{x=\nu+1}^{n} \ldots = \prod_{j=2}^{n-\nu} (q_n - r_{n+2-j} - k + 1),$$

and

$$(q_{\nu} - r_{\nu+1}) \sum_{x=\nu+1}^{n} \ldots = \sigma_{n}^{\nu}. \quad (4.36)$$

Proof of Lemma 4.8 We prove the statement by induction. Let for a given $k$ the statement hold. Then

$$\begin{align*}
\sum_{x=n-k}^{n} \sum_{x=n-k+1}^{n} &+ \left\{ n-k \text{-th term} \right\} = \\
&= \prod_{\nu+1}^{n-k} (q_n - q_j + j - n) \cdot \prod_{j=2}^{k} (q_n - r_{n+2-j} - k + 1) + \\
&+ \prod_{j=\nu+1}^{n-k-1} (q_n - q_j + j - n) \cdot (q_n - r_{n-k+1}) \cdot \prod_{j=2}^{k} (q_n - r_{n+2-j} - j) = \\
&= \prod_{j=\nu+1}^{n-k-1} (q_n - q_j + j - n) \prod_{j=2}^{k} (q_n - r_{n+2-j} - k + 1) \cdot \left( (q_n - q_{n-k}) + (q_n - r_{n-k+1}) \right)
\end{align*}$$

and the big bracket joins to the product $\prod_{j=2}^{k}$. □

Let us return to our calculation. We must evaluate $S_1 + S_2$,

$$S_1 + S_2 = \sigma_n^{\nu} \sum_{\mu=1}^{\nu-1} \xi_{\mu \nu} (q_{\mu+1} - q_\mu) + S_2 = \sigma_n^{\nu} \sum_{\mu=1}^{\nu-1} \left\{ \prod_{j=m+1}^{\mu} (q_n - q_j + j - n - 1) \cdot \prod_{j=\mu+1}^{\nu-1} (q_n - q_j + j - n) \cdot (q_\mu - q_{\mu+1}) \right\} + \sigma_n^{\nu} \prod_{j=m+1}^{\nu} (q_n - q_j + j - n - 1). \quad (4.37)$$
An evaluation of this sum is similar to the proof of Lemma 4.8. We verify the following identity by induction,
\[
\sigma_\nu^n \sum_{\nu-k}^{\nu-1} \xi_{\mu\nu} (q_{\mu+1} - q_{\mu}) + S_2 = \prod_{j=m+1}^{\nu-k+1} (q_{n} - q_{j} + j - 1 - n) \prod_{j=\nu-k+1}^{\nu-1} (q_{n} - q_{j} + j - n).
\]
Substituting \( k = \nu - 1 \) we get the desired coefficient in (4.31).

It remains to notice that formulas for the coefficient in the front of \( \Phi_n \) are slightly different. In this case, the sum \( S_2 \) is absent, but \( \zeta_{mn} \) has an extra term \( \frac{1}{\Phi_n} \). Starting this place, the calculation is the same, the extra term replaces the additional term in (4.37).

This completes a proof of Lemma 4.7.

4.5. The extension of calculations. Next, we must evaluate other summands in \( F_1(n+1)L \). Denote
\[
\Theta_1 \tau := \sum_{I \subset [1, \tau]} \prod_{l \in [1, \tau] \setminus I} (l - \tau + q_l - q_\tau) \cdot R_I T_{m-1} L.
\]

**Lemma 4.9**

\[
\Theta_1 \tau = - \sum_{\mu=1}^{m} s_{1\tau}^{\mu} R_{1\mu} \Psi_{\mu-1} - \Phi_{\mu}.
\]

**Proof.** We decompose \( T_{\tau} L \) as
\[
T_{\tau} L = \left( \Phi_1^{-r_2+q_1} \Psi_1^{-q_2} \ldots \Psi_{\tau-1}^{-r_{\tau-1}+q_{\tau-1}} \Phi_{\tau}^{-r_{\tau}+1+q_{\tau}} \right) \times \left( \Psi_{m-1}^{-q_m} \Phi_{\tau}^{-r_{\tau+1}+q_{\tau}} \ldots \Phi_{n}^{-r_{n+1}+q_{n}} \right).
\]
Factors \( R_{kl} \) of operators \( R_I \) act as zero on the factors of the second bracket. Therefore this bracket can be regarded as a constant. After this, we get the same calculations as for \( \Theta_1 n \) but \( n \) is replaced by \( \tau \). \( \square \)

4.6. The summation.

**Lemma 4.10**

\[
F_1(n+1)L + E_1(n+1)L = 0.
\]

**Proof.**
\[
F_1(n+1)L = \sum_{\tau=1}^{n} A_\tau \sum_{\alpha=1}^{\tau} s_{1\tau}^{\alpha} \sigma_{\tau}^{\alpha} R_{1\alpha} \Psi_{\alpha-1} - \Phi_{\alpha}.
\]

We must verify the following identities
\[
\sum_{\tau=1}^{n} A_\tau s_{1\tau}^{\alpha} \sigma_{\tau}^{\alpha} = -(q_\alpha - r_{\alpha+1}).
\]
A summand in the left-hand side equals
\[
A_r s_1^\alpha \sigma_1^\alpha = \prod_{j=1}^{n-\tau+1} (q_m - r_{n+2-j} + n - \tau + 1 - j) \times \prod_{j \neq \tau} (q_j - q + j - m) \\
\times (q_\alpha - r_{\alpha+1}) \prod_{i=1}^{\alpha-1} (q_m - q_i + i - m) \cdot \prod_{k=n-m+2}^{n-\alpha} (q_m - r_{n+2-k} + n - \alpha - k + 1).
\]

The factor \(\prod_{i=1}^{\alpha-1} (q_m - q_i + i - m)\) cancels, the factors \(\prod_{j=1}^{n-\tau+1}\) and \(\prod_{k=n-m+2}^{n-\alpha}\) join together and we come to
\[
A_r s_1^\alpha \sigma_1^\alpha = (q_\alpha - r_{\alpha+1}) \prod_{j=1}^{n-\sigma} (q_m - r_{n+2-j} + n - \tau + 1 - j) \prod_{j \neq \tau, j > \sigma} (q_j - q + j - m).
\]

Next, we write a rational function
\[
V_\alpha(x) := \prod_{j=1}^{n-\sigma} (x + r_{n+2-j} + n - j + 1) \prod_{j=\sigma}^{n} (x - q_j + j)
\]
and evaluate the sum of its residues,
\[
\sum_{j=\sigma}^{m} \text{Res}_{x=q_j-j} V_\alpha(x) = -\text{Res}_{x=\infty} V_\alpha(x) \quad (4.40)
\]
Multiplying this identity by \((q_\alpha - r_{\alpha+1})\) we get \((4.39)\). \(\square\)

4.7. Invariance of the space \(\mathfrak{F}_r\).

**Lemma 4.11** The space \(\mathfrak{F}_r\) (see \((1.16)\)) is invariant with respect to the operator \(F_{1(n+1)}\).

**PROVE.** Define elements \(\ell_Z(U) \in \mathfrak{F}_r\), where \(Z\) ranges in \(T_{n+1}^+\), by
\[
\ell_Z(U) = L(Z, U).
\]
First, functions \(\ell_Z(U)\) generate the space \(\mathfrak{F}_r\) (because the pairing \(V_p \times \mathfrak{F}_r \to \mathbb{C}\) determined by \(L\) is nondegenerate). Next,
\[
F_{1(n+1)} \ell_Z(U) = F_{1(n+1)} L(Z, U) = -E_{1(n+1)} L(Z, U) = -\frac{\partial}{\partial z_{1(n+1)}} (\ell_Z(U)).
\]
Differentiating a family of elements of \(\mathfrak{F}_r\) with respect to a parameter we get elements of the same space \(\mathfrak{F}_r\). \(\square\)

4.8. Formula for \(F_{(n+1)n}\). Here a calculation is more-or-less the same, we omit details (in a critical moment we use the quadratic identities \((4.4)\), \((4.6)\)).
5 Infinite dimensional case. A conjecture

5.1. Principal series. Now consider two collections of complex numbers $(p_1,\ldots,p_n)$ and $(p'_1,\ldots,p'_n)$ such that

$$p_j - p'_j \in \mathbb{Z}, \quad \text{Re}(p_j + p'_j) = -2(j - 1).$$

Consider a representation $\rho_{p|p'}^n$ of the group $\GL(n,\mathbb{C})$ in the space $L^2(N^+_n)$ determined by the formula

$$\rho_{p|p'}^n(g)\varphi(g) = \varphi(Z^{|g|}) \cdot \prod_{j=1}^n b_{jj}(Z,g)^{p_j} b_{jj}(Z,g)^{p'_j}. \quad (5.1)$$

We get the unitary (nondegenerate) principal series of representations of $\GL(n,\mathbb{C})$ (see [6]). Denote by $\Lambda_n^{\text{unitary}}$ the space of all parameters $p, p'$. 

**Remark.** In formula (5.1), we have complex numbers in complex powers. We understand them in the following way:

$$b_{jj}^{p_j} b_{jj}^{p'_j} = |b_{jj}|^{2p_j} (b_{jj}/b_{jj})^{p_j - p'_j}.$$ 

In the right hand side, the first factor has a positive base of the power, the second factor has an integer exponent. Hence the product is well defined. ☑

**Remark.** Formula (5.1) makes sense if $p_j - p'_j \in \mathbb{Z}$, and gives a non-unitary principal series of representations. The construction of holomorphic representation discussed above corresponds to $p \in \Lambda_n, p' = 0$. If both $p, p' \in \Lambda_n$ (5.2)

then our representation contains a finite-dimensional (nonholomorphic) representation $\rho_p \otimes \overline{\rho_p}$, where $\overline{\rho_p}$ denotes the complex conjugate representation. ☑

5.2. Restriction to the smaller subgroup. Consider a representation $\rho_{p|p'}^{n+1}$ of the group $\GL(n + 1,\mathbb{C})$. According [1], the restriction of $\rho_{p|p'}^n$ to the subgroup $\GL(n,\mathbb{C})$ is a multiplicity free integral of all representations of $\rho_{p|p'}^n$ of unitary nondegenerate principal series of $\GL(n,\mathbb{C})$. Moreover, the restriction has Lebesgue spectrum. Thus the restriction $U_{p|p'}^n$ can be realized in the space $L^2(N^+_n \times \Lambda_n^{\text{unitary}})$ by the formula

$$U_{p|p'}(g)\psi(U,q|q') = \psi(Z^{|g|}, q|q') \cdot \prod_{j=1}^n b_{jj}(Z,g)^{p_j} b_{jj}(Z,g)^{p'_j}. \quad (5.2)$$

5.3. Intertwining operator. Next, we write an integral operator

$$L^2(N^+_{n+1}) \to L^2(N^+_n \times \Lambda_n^{\text{unitary}})$$

by the formula

$$J\varphi(U,q|q') = \int_{N^+_{n+1}} \varphi(Z)L_{p,q}(Z,U) L_{p',q'}(Z,U) d\tilde{Z}. \quad (5.3)$$
5.4. Additional operators. Denote the operators (1.22), (1.22) by $F_{1(n+1)}^p$, $F_{(n+1)n}^p$. Denote by $F_{1(n+1)}^{p}$, $F_{(n+1)n}^{p}$ the operators obtained from $F_{1(n+1)}^p$, $F_{(n+1)n}^p$ by replacing

$$\frac{\partial}{\partial z_{kl}} \mapsto \frac{\partial}{\partial z_{kl}^o}, \quad p_j \mapsto p_j^o, \quad q_k \mapsto q_k^o.$$

Define the operators

$$F_{1(n+1)}^{p|p} = F_{1(n+1)}^p + F_{1(n+1)}^{p^o},$$
$$F_{(n+1)n}^{p|p} = F_{(n+1)n}^p + F_{(n+1)n}^{p^o}.$$

Conjecture 5.1 The operators $F_{1(n+1)}^{p|p}$, $F_{(n+1)n}^{p|p}$ are images of the operators $E_{1(n+1)}$, $E_{(n+1)n}$ under the integral transform (5.3).

The statement seems doubtless, since we have an analytic continuation of our finite-dimensional formulas from the set (5.2). However, this is not an automatic corollary of our result. In particular, it is necessary to find the Plancherel measure on $\Lambda_{n}^{unitary}$ (i.e. a measure making the operator $J$ unitary).

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