Hunting for Statistical Anisotropy in Tensor Modes with B-mode Observations

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We investigate a possibility of constraining statistical anisotropies of the primordial tensor perturbations by using future observations for the Cosmic Microwave Background (CMB) B-mode polarization. By parameterizing a statistically-anisotropic tensor power spectrum as $P_h(k) = P_h(k) \sum_n g_n \cos^n \theta_k$, where $\theta_k$ is an angle of the direction of $\hat{k} = k/k$ from a preferred direction, we find that it would be possible for future B-mode observations such as CMB-S4 to detect the tensor statistical anisotropy at the level of $g_n \sim \mathcal{O}(0.1)$.

I. INTRODUCTION

The detection of the B-mode polarization signal in Cosmic Microwave Background (CMB) is one of the most important challenges in cosmology, because it is sensitive to the primordial gravitational waves (PGWs) generated during inflation. In standard inflationary scenario, the amplitude of the PGWs generated from vacuum fluctuations during inflation is expected to depend only on the energy density of the inflation, $\rho_{\text{inf}}$, and hence its detection was considered as a direct probe for the scale of unknown physics. The current constraint on the amplitude of the PGWs is $r \lesssim 0.07$, where $r$ represents the ratio of the power spectrum of the PGWs to that of the primordial curvature perturbations $\Delta^2_{\text{curv}}$. In 2020s, next generation CMB experiments such as LiteBIRD [4] and CMB-S4 [5] are expected to achieve higher sensitivities reaching $r \sim 10^{-3}$ which corresponds to $\rho_{\text{inf}} \approx 6 \times 10^{15}$ GeV in the conventional vacuum fluctuation case. Recently, other mechanisms of generating PGWs during inflation by introducing some matter fields (e.g. gauge fields) have been proposed, [6] and then this means that the vacuum PGWs are no longer the unique target of the B-mode observation. Interestingly, the PGWs generated in these new mechanisms have not only the different relations between $r$ and $\rho_{\text{inf}}$ but also observable signatures distinct from the vacuum one, e.g., non-Gaussianity [8, 9]. Among them, statistical anisotropies of the PGWs should be useful to distinguish the generation mechanisms and to extract richer information on the early universe from the B-mode observation.

The statistical anisotropy has been pursued mainly in the power spectrum of the curvature perturbation $P_\zeta$. This is because the anisotropic inflation and solid inflation models predict a quadrupole anisotropy in the curvature perturbation, $P_\zeta(k) = P_\zeta(k) (1 + g_2 \cos^2 \theta_k)$ [10-17]. Furthermore, recent studies [18, 19] argue that higher spin fields generate statistical anisotropies beyond quadrupole in $P_\zeta$ during inflation. Indeed, these kinds of anisotropies imprint interesting signatures in the CMB angular power spectrum. While in the standard picture CMB power spectra have only diagonal components in the angular multipole space due to a rotational invariance, statistical anisotropies can generate specific nonzero off-diagonal correlations between temperature and polarization in CMB data. Several works have been discussed to test such kind of correlations due to the statistically-anisotropic curvature perturbation [20-24]. So far, however, there is no evidence of the quadrupole anisotropy in $P_\zeta$ and we have an upper bound $|g_2| \lesssim 10^{-2}$ [1, 25].

In this paper, we study the statistical anisotropy in the power spectrum of the PGWs. Although little attention has been paid to the statistically-anisotropic PGWs, recent study [26] has proposed a model where large statistical anisotropies in $P_h$ can be generated. In this model, U(1) gauge field is kinematically coupled to a spectator scalar field and gains a large background expectation value which breaks the isotropy of the universe, and then due to the anisotropy of the universe the perturbations of the spectator field and the gauge field could source the anisotropic tensor modes. Remarkably, the higher-order statistical anisotropies beyond quadrupole in $P_h$ can be predicted irrespective of the model parameters. A similar prediction is also obtained when 2-form field takes over the role of the U(1) gauge field [27]. Other than this type of model, several works have suggested the generation of testable statistical anisotropies in tensor modes [21, 28, 29]. Inspired by these predictions, we explore a possibility to test these higher statistical anisotropies of tensor modes through the B-mode angular power spectrum. We model the tensor statistical anisotropies as $P_h(k) = P_h(k) \sum_n g_n (k/k_0)^n \cos^n \theta_k$ and evaluate detectabilities of the coefficients $g_n$ in future missions. Compared with the previous study [28], we further investigate the sensitivities of $g_n$ up to $n = 6$.

This paper is organized as follows. In Sec. II, we describe basic equations for our Fisher analysis. In Sec. III we obtain 1σ uncertainties of the anisotropic parameters, $g_n$ and $g_{LM}$. We conclude in Sec. IV.
II. BASIC EQUATIONS

A. Anisotropies

Harmonic coefficients of B-mode anisotropies induced by the primordial tensor perturbations \( h_{\ell2} (k) \) can be written in terms of the transfer function \( T_{\ell}^{(B)} (k) \) (see e.g. Ref. [22]),

\[
a^{(B)}_{\ell m} = 4\pi (-i)^{\ell} \int \frac{d^3 k}{(2\pi)^3} h_{\ell} (k) T_{\ell}^{(B)} (k) Y_{\ell m}^* (\hat{k}),
\]

with \( Y_{\ell m} \) being the spin-\( s \) spherical harmonics. The power spectrum of the tensor perturbations is defined as

\[
\langle h_{+2} (k_1) h_{-2} (k_2) \rangle = \frac{1}{(2\pi)^3} P_h (k_1) \delta^{(3)} (k_1 - k_2),
\]

where we have used \( h_{-2} (k) = h_{+2} (k) \). If the rotational invariance is broken, the power spectrum could have the directional dependence, which can be parameterized as [30],

\[
P_h (k) = P_h (k) \sum_{LM} Q_{LM} (k) Y_{LM} (\hat{k}),
\]

with \( L \) running over even numbers, 0, 2, 4, . . . , where \( P_h (k) \) is the isotropic (monopole) part, and \( \hat{k} := k/k \). Taking into account the directional dependence, we obtain the correlation of the harmonic coefficients [28],

\[
C_{\ell_1 m_1; \ell_2 m_2}^{BB} := \langle a^{(B)}_{\ell_1 m_1} a^{(B)}_{\ell_2 m_2}^* \rangle = \frac{2}{\pi} \delta_{\ell_2 - \ell_1} (-1)^{m_1 + m_2} \sum_{LM} \mathcal{G}_{\ell_1 \ell_2 L}^{-m_1 m_2 M; -220} \times \int dk k^2 P_h (k) Q_{LM} (k) T_{\ell_1}^{(B)} (k) T_{\ell_2}^{(B)} (k),
\]

where \( \delta_{\ell_2 - \ell_1} \) is 1 if \( a \) is even, and 0 otherwise, and \( \mathcal{G}_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3: s_1 s_2 s_3} \) is the spin-weighted Gaunt integral that is written in terms of the product of Wigner’s 3-j-symbols,

\[
\mathcal{G}_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3: s_1 s_2 s_3} := \int d\Omega \tilde{Y}_{\ell_1 m_1} (\Omega) Y_{\ell_2 m_2} (\Omega) \tilde{Y}_{\ell_3 m_3} (\Omega) = \left[ \frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{4\pi} \right]^{\frac{1}{2}} \times \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} -s_1 & -s_2 & -s_3 \end{array} \right).
\]

In the present study, we assume that the scale dependence of the anisotropic parameter is given as [28],

\[
Q_{LM} (k) = q_{LM} \left( \frac{k}{k_0} \right)^{\gamma},
\]

with constants \( q_{LM} \) and \( \gamma \). Finally, Eq. (4) reads

\[
C_{\ell_1 \ell_2}^{BB} (\gamma) = \frac{2}{\pi} \int dk k^2 P_h (k) T_{\ell_1}^{(B)} (k) T_{\ell_2}^{(B)} (k) \left( \frac{k}{k_0} \right)^{\gamma},
\]

where

\[
C_{\ell_1 \ell_2}^{BB} (\gamma) := \frac{2}{\pi} \int dk k^2 P_h (k) T_{\ell_1}^{(B)} (k) T_{\ell_2}^{(B)} (k) \left( \frac{k}{k_0} \right)^{\gamma}.
\]

Note that for the case with \( q_{LM} = \delta_{LM} \delta_{\ell_0 \ell_0} \), that is, statistically-isotropic power spectrum, one can find that the above expression is equivalent to the standard form of the angular power spectrum.

In the theoretical models which predict the statistical anisotropy in primordial tensor modes, the statistical anisotropy is often parameterized in terms of the power series of the cosine function (e.g., [1, 25]), as

\[
P_h (k) = P_h (k) \sum_{n=even}^N g_n (k/k_0)^{\gamma} \cos^n \theta_k,
\]

where \( \theta_k \) measures the angle of the direction of \( \hat{k} \) from a preferred direction. Thus, it should be useful to give a relation between the parameters \( g_n \) and \( q_{LM} \), and according to Eq. (A11) their relations are given by

\[
q_{0M} = 2\sqrt{\pi} \left( g_0 + g_2 + g_4 + \frac{g_6}{5} \right) \delta_{LM},
\]

\[
q_{2M} = 4\sqrt{\pi} \left( \frac{g_2}{3} + \frac{g_4}{7} + \frac{g_6}{21} \right) \delta_{LM},
\]

\[
q_{4M} = 16\sqrt{\pi} \left( \frac{g_4}{105} + \frac{g_6}{47} \right) \delta_{LM},
\]

\[
q_{6M} = \frac{32}{231} \sqrt{\pi} \frac{g_6}{13} \delta_{LM}.
\]

B. Fisher information matrix

To quantify the 1σ uncertainties of the anisotropic parameters, \( \{q_{LM}\} \) or \( \{g_n\} \), we use the Fisher information matrix. The details of the computation of the Fisher information matrix in our study are provided in Appendix A. Here we consider only the B-mode in the full expression in Eq. (AS) or Eq. (A10) with Eq. (A9), and it reads

\[
F_L^{BB} = \frac{f_{sky}}{4\pi} \sum_{\ell_1 \ell_2} (2\ell_1 + 1)(2\ell_2 + 1) \left( \begin{array}{ccc} \ell_1 & \ell_2 & L \\ 2 & 2 & 0 \end{array} \right)^2 \times \frac{\left( C_{\ell_1 \ell_2}^{BB} \right)^2}{C_{\ell_1 \ell_2}^{BB} C_{\ell_1 \ell_2}^{BB}},
\]

where \( C_{\ell_1 \ell_2}^{BB} \) is the total angular power spectrum of B-mode polarization defined in Eq. (AS). Using Eq. (AS)
or Eq. (A10), we can estimate the uncertainties of the measurement of the anisotropic parameters,
\[ \sigma^2_{qLM} = (F_{LM;LM})^{-1}, \quad \sigma^2_{q_n} = (F_{nn})^{-1}. \]  
(15)

A noise model we adopt in the present study is
\[ N_{qBB}^{fBB} = N_{fBB}^{fBB} \varepsilon^2 \sigma_n^2, \]  
(16)

in which we assume the detector noise \( N_{fBB}^{fBB} \) and the beam effect \( \sigma_b \) are parameterized as
\[ N_{fBB}^{fBB} = \left( \frac{\pi}{10800} \frac{w_{BB}^{-1/2}}{\mu\text{K.arcmin}} \right)^2 \mu\text{K}^2 \text{str}, \]  
(17)
\[ \sigma_b = \frac{\pi}{10800} \frac{\theta_{\text{FWHM}}}{\text{arcmin}} \frac{1}{\sqrt{8 \ln 2}}. \]  
(18)

with \( \theta_{\text{FWHM}} \) being the full width at half maximum (FWHM) of the beam in the unit of arcmin. Although we do not take into account neither the lensing effect from the E-mode induced by scalar perturbations nor the foreground noises sourced by dust emission, it is possible to emulate the cases including them by increasing the noise parameter \( w_{BB}^{-1/2} \).

III. DETECTABILITY OF STATISTICAL ANISOTROPIES OF TENSOR PERTURBATIONS

We use camb2nd\(^1\) to compute the transfer function of B-mode with the cosmological parameters from the Planck 2015 results (TT,TE,EE+lowP+lensing+ext in Ref. [35]), which are tabulated in Table I. We assume a 0.5 degree FWHM beam (designed in LiteBIRD [4]) and the noise level with \( w_{BB}^{-1/2} = 1.0, 5.0, 63.1 \mu\text{K.arcmin} \) which correspond to CMB-S4 [5], LiteBIRD [4], and Planck [34], respectively. In addition to them, we also compute the cosmic-variance-limited (CVL) case with \( w_{BB}^{-1/2} = 0 \).

The 1\( \sigma \) uncertainties of the measurements of \( q_n \) and \( q_{LM} \) with even numbers of \( n \) and \( L \) up to \( n, L \leq 6 \) are summarized in Tables III and IV which are computed with fixed \( \gamma; \gamma = 0 \) (Table III), \( \gamma = -1, -1/2 \) (Table III) and \( \gamma = 1/2, 1 \) (Table IV). Throughout this paper, the isotropic part of angular power spectrum is supposed to be \( C_{BB}^{\text{obs}} := C_{BB}^\gamma (\gamma = 0) \) with \( r = 0.01 \) at \( k = k_{\text{pivot}} \).

Hence the observed signal is given as
\[ C_{\ell_1 \ell_2 \gamma_{m_1} \gamma_{m_2}}^{\text{obs}} (\gamma) = C_{\ell_1 \ell_2}^{BB} \delta_{\gamma_{m_1} \gamma_{m_2}} + C_{\ell_1 \ell_2}^{BB} (\gamma = 0). \]  
(19)

In the case with \( \gamma = 0 \) (Table III), we fix \( g_0 = 1 \) or \( q_{00} = 1 \) and vary \( g_n \) (\( q_{LM} \)) for \( n \geq 2 \) (\( L \geq 2 \)), whereas in the cases with \( \gamma \neq 0 \) (Table III and IV), we vary also \( g_0 \) or \( q_{00} \).

The tables, we show the results with \( f_{\text{sky}} = 1 \). One can obtain those with \( f_{\text{sky}} < 1 \) by multiplying the values by \( 1/\sqrt{f_{\text{sky}}} \).

Note that \( \sigma_{g_n} \), with \( n \leq N \) has a strong dependence on the number of parameters \( N \) due to the non-vanishing off-diagonal components of the Fisher information matrix, whereas \( \sigma_{q_{LM}} \) is independent of \( N \) since the corresponding Fisher information matrix is diagonal. Hence, as for \( \sigma_{g_n} \), we compute their uncertainties for \( N = 2, 4, 6 \), respectively.

In Table III we find that, when we take into account both \( g_4 \) and \( g_6 \), their uncertainties are greater than the unity even in the case with \( w_{BB}^{-1/2} = 1 \mu\text{K.arcmin} \), which leads to the difficulty of measurement of such higher-order anisotropies. On the other hand, when we take into account up to \( g_4 \), it is implied that the hexadecapole anisotropy with \( g_4 = \mathcal{O}(1) \) can be detected by an observatory whose specification is similar to CMB-S4.

\begin{table*}[ht]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{parameter} & \textbf{CVL} & 1.0 & 5.0 & 63.1 \\
\hline\hline
amplitude of curvature perturbation & \( P_{\mathcal{R}0} \) & \( 2.384 \times 10^{-9} \) & & & \\
\hline
tensor-to-scalar ratio & \( r \) & 0.01 & & & \\
\hline
pivot scale & \( k_{\text{pivot}} \) & 0.002 Mpc\(^{-1} \) & & & \\
\hline
spectral index & \( n_s \) & 0.9667 & & & \\
\hline
reduced Hubble parameter & \( h \) & 0.6774 & & & \\
\hline
dark matter fraction & \( h^2\Omega_{\text{DM}} \) & 0.1188 & & & \\
\hline
baryon fraction & \( h^2\Omega_{\text{b}} \) & 0.02230 & & & \\
\hline
effective number of neutrinos & \( N_{\text{ eff}} \) & 3.046 & & & \\
\hline
photon’s temperature & \( T_{\gamma,0} \) & 2.7255 K & & & \\
\hline
optical depth & \( \tau \) & 0.066 & & & \\
\hline
Helium abundance & \( Y_p \) & 0.24667 & & & \\
\hline
\end{tabular}
\caption{Cosmological parameters used in the present study. All parameters except for the tensor-to-scalar ratio are provided by Planck 2015 results \( \text{TT,TE,EE+lowP+lensing+ext in Ref. [33]} \). The amplitude of curvature perturbation and the tensor-to-scalar ratio are evaluated at \( k = k_{\text{pivot}} \) and we assume \( n_{s,0.002} = n_{s,0.05} \) in the notation of the Planck 35.}
\end{table*}

\begin{table*}[ht]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{parameter} & \textbf{value} & & & \\
\hline\hline
\hline
amplitude of curvature perturbation & \( P_{\mathcal{R}0} \) & \( 2.384 \times 10^{-9} \) & & \\
\hline
tensor-to-scalar ratio & \( r \) & 0.01 & & \\
\hline
pivot scale & \( k_{\text{pivot}} \) & 0.002 Mpc\(^{-1} \) & & \\
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\hline
Helium abundance & \( Y_p \) & 0.24667 & & \\
\hline
\end{tabular}
\caption{\( \sigma_{g_n} \) for \( w_{BB}^{-1/2} = 63.1 \), 5.0, 1.0\( \mu\text{K.arcmin} \) and the CVL case with \( \gamma = 0 \) and \( f_{\text{sky}} = 1 \).}
\end{table*}

\(^1\)This Boltzmann code is not public yet, but we have confirmed that the transfer functions obtained from this precisely agree with those from CAMB (https://camb.info/). See also Ref. [32] in which we used the same code.
In Table III and IV, we estimate the uncertainties with various γ. In the red-tilted cases, our results even with γ = −1 indicate the possibility to detect $g_2$ by LiteBIRD and CMB-S4, while it is fairly difficult to get a signal of $g_4$ even with CMB-S4. On the other hand, in the blue-tilted cases, we can marginally detect $g_4$, since much power is induced to the angular power spectrum on large $\ell$. Note that, in Ref. 28, the authors reported the uncertainties with γ = −2, $\sigma_{g_{0M}} = 30$ and $\sigma_{g_{2M}} = 58$ in our notations. In the present study, we obtained $\sigma_{g_{0}} = 20.6$ and $\sigma_{g_{2}} = 56.4$ with γ = −2 in the CVL case, which are well consistent to the previous results.

| $\gamma$ | CVL | 1.0 | 5.0 | 63.1 |
|---------|-----|-----|-----|-----|
| $g_0$  | $5.28 \times 10^{-2}$ | $9.67 \times 10^{-2}$ | $1.31 \times 10^{-1}$ | $5.93 \times 10^{-1}$ |
| $g_2$  | $1.43 \times 10^{-1}$ | $2.61 \times 10^{-1}$ | $3.58 \times 10^{-1}$ | $1.64$ |

| $\gamma$ | CVL | 1.0 | 5.0 | 63.1 |
|---------|-----|-----|-----|-----|
| $g_0$  | $1.23 \times 10^{-2}$ | $8.26 \times 10^{-2}$ | $1.96 \times 10^{-1}$ | $1.42$ |
| $g_2$  | $3.36 \times 10^{-2}$ | $2.18 \times 10^{-1}$ | $5.26 \times 10^{-1}$ | $3.96$ |

Table III: $\sigma_{g_{\gamma}}$ for $w_{BB}^{-1/2} = 63.1, 5.0, 1.0 \mu$K-arcmin and the CVL case with $\gamma = −1$ (left), −1/2 (right) and $k_0 = k_{\text{pivot}}$ and $f_{\text{sky}} = 1$.

| $\gamma$ | CVL | 1.0 | 5.0 | 63.1 |
|---------|-----|-----|-----|-----|
| $g_0$  | $2.00 \times 10^{-4}$ | $2.31 \times 10^{-2}$ | $8.37 \times 10^{-2}$ | $5.37$ |
| $g_2$  | $5.47 \times 10^{-4}$ | $5.94 \times 10^{-2}$ | $2.08 \times 10^{-1}$ | $1.48 \times 10^{1}$ |

| $\gamma$ | CVL | 1.0 | 5.0 | 63.1 |
|---------|-----|-----|-----|-----|
| $g_0$  | $2.11 \times 10^{-3}$ | $1.10 \times 10^{-2}$ | $4.19 \times 10^{-2}$ | $4.01$ |
| $g_2$  | $5.76 \times 10^{-5}$ | $2.80 \times 10^{-2}$ | $1.02 \times 10^{-1}$ | $1.02 \times 10^{1}$ |

Table IV: $\sigma_{g_{\gamma}}$ for $w_{BB}^{-1/2} = 63.1, 5.0, 1.0 \mu$K-arcmin and the CVL case with $\gamma = 1/2$ (left), 1 (right) and $k_0 = k_{\text{pivot}}$ and $f_{\text{sky}} = 1$.

Let us compare our result with theoretical predictions. The model in Ref. 26 predicts $g_0 = 1, g_2 = −1, g_4 = 1, g_6 = −1$ irrespective of the model parameters, while γ $\lesssim −1/2$ is required to produce a detectable amplitude of the sourced PGW (i.e. $r_{\text{source}} \gtrsim 10^{-3}$). In the case of γ $= −1, −1/2$ (Table III), these result show that the predicted $g_0$ and $g_2$ are marginally detectable, whereas it is challenging to measure $g_4$ and $g_6$ even with the CMB-S4 experiment at $1\sigma$ level. Let us also consider the discrimination between the models. The prediction in Ref. 27 is $g_0 = 1, g_2 = 1, g_4 = −2, g_6 = 1$, and the sign of $g_2$ is flipped from that of Ref. 26. This difference is originated in the distinction between the particle types which generate the PGWs (i.e. U(1) gauge field or 2-form field). Therefore, once $g_n (n \geq 2)$ is detected, we may gain an insight what type of particle plays an important role in the primordial universe.

Note that, although we fix $k_0 = k_{\text{pivot}}$ in Tables III and
IV. CONCLUSION

We investigated the detectability of the statistical anisotropies of the primordial tensor power spectrum using the Fisher information matrix assuming the observations by CMB-S4, LiteBIRD and Planck. We parameterize the primordial tensor power spectrum in Eq. (8) with Eq. (9) and Eq. (10), and estimate the 1σ-uncertainties of \( q_{LM} \) and \( g_n \) given in Eq. (11) with the fiducial values \( q_{LM} = g_n = 0 \) for \( L \) (or \( n \)) \( \geq 2 \) in the case of \( \gamma = 0 \) and \( q_{LM} = g_n = 0 \) for \( L \) (or \( n \)) \( \geq 0 \) in the case of \( \gamma \neq 0 \).

Our results are tabulated in Tables I, II, and III in Table IV, we find that a relatively large statistical anisotropy \( \sigma \) to the inverse of Eq. (8).

In the actual observations, the signal converted from E-mode through the gravitational interaction, into account. In the actual observations, the detectability of the anisotropies depends on how well we can remove the contamination, namely, delensing. Roughly speaking, this effect can be included in our result by increasing \( w_{BB} \) defined in Eq. (17). If the delensing is not perfectly performed, the detectability of the anisotropies by CMB-S4 would be worse.

\[
\mathbf{F}_{ij} = \frac{f_{\text{sky}}}{2} \sum_{\ell_1 m_1} \sum_{\ell_2 m_2} \text{Tr} \left[ C^{-1}_{\ell_1} \frac{\partial C_{\ell_1 m_1; \ell_2 m_2}}{\partial \theta_i} C^{-1}_{\ell_2} \frac{\partial C_{\ell_2 m_2; \ell_1 m_1}}{\partial \theta_j} \right]
\]

\[
= f_{\text{sky}} \sum_{XY} \sum_{\ell_1 m_1} \sum_{\ell_2 m_2} \frac{\partial C^X_{\ell_1 m_1; \ell_2 m_2}}{\partial \theta_i} (q_{\ell_1 \ell_2}^{-1})_{XY} \frac{\partial C^Y_{\ell_2 m_2; \ell_1 m_1}}{\partial \theta_j}
\]

where \( f_{\text{sky}} \) denotes the fraction of the sky covered, \( X, Y = TT, TE, EE, BB \) and

\[
\mathbf{F}_{LM, LM'} = \delta_{LL'} \delta_{MM'} \mathbf{F}_L,
\]

with \( \mathcal{N}^X \) being the noises for the detection of \( X = TT, TE, EE, BB \). The angular power spectrum \( C^X_\ell \) with \( X = BB \) is given by Eq. (4) with \( \ell_1 = \ell_2 = \ell \) and \( \gamma = 0 \), and those of \( X = TT, TE, EE \) can also be calculated by replacing the transfer functions in Eq. (4) with the corresponding ones. If we choose \( \{ \theta_i \} = \{ q_{LM} \} \), the Fisher matrix becomes

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where

\[ F_L = \frac{f_{\text{sky}}}{4\pi} \sum_{s_1 s_2 s_3 s_4} (-1)^{s_1 + s_3} \sum_{\ell_1 \ell_2} (2l_1 + 1)(2l_2 + 1) \times \left( \frac{l_1}{s_1} \frac{l_2}{s_2} \frac{L}{s_3-s_2} \right) \times \sum_{XY} C_{\ell_1 s_2}^{s_1 X} (\epsilon_{\ell_1 \ell_2}) \chi X C_{\ell_2 s_3}^{s_2 Y} \right), \tag{A9} \]

Alternatively, if we choose \( \{\theta_i\} = \{g_n\} \), we have

\[ F_{mn} = \sum_{LM} \frac{\partial q_{LM}}{\partial g_m} \sum_{LM} \frac{\partial q_{LM}}{\partial g_n} L_{XY} \tag{A10} \]

The coefficients in the right-hand side are found to be

\[ \frac{\partial q_{LM}}{\partial g_n} = \delta_{MN} \sqrt{2L+1} \sqrt{\pi} \int_{-1}^{1} \mu \nu P_L(\mu) d\mu, \tag{A11} \]

where \( P_L(\mu) \) is the Legendre polynomials.

\[ \text{where} \]

\[ F_L = \frac{f_{\text{sky}}}{4\pi} \sum_{s_1 s_2 s_3 s_4} (-1)^{s_1 + s_3} \sum_{\ell_1 \ell_2} (2l_1 + 1)(2l_2 + 1) \times \left( \frac{l_1}{s_1} \frac{l_2}{s_2} \frac{L}{s_3-s_2} \right) \times \sum_{XY} C_{\ell_1 s_2}^{s_1 X} (\epsilon_{\ell_1 \ell_2}) \chi X C_{\ell_2 s_3}^{s_2 Y} \right), \tag{A9} \]

Alternatively, if we choose \( \{\theta_i\} = \{g_n\} \), we have

\[ F_{mn} = \sum_{LM} \frac{\partial q_{LM}}{\partial g_m} \sum_{LM} \frac{\partial q_{LM}}{\partial g_n} L_{XY} \tag{A10} \]

The coefficients in the right-hand side are found to be

\[ \frac{\partial q_{LM}}{\partial g_n} = \delta_{MN} \sqrt{2L+1} \sqrt{\pi} \int_{-1}^{1} \mu \nu P_L(\mu) d\mu, \tag{A11} \]

where \( P_L(\mu) \) is the Legendre polynomials.
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