Resolutions of p-stratifolds with isolated singularities

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Abstract  Recently M. Kreck introduced a class of stratified spaces called p-stratifolds [Kr3]. He defined and investigated resolutions of p-stratifolds analogously to resolutions of algebraic varieties. In this note we study a very special case of resolutions, so called optimal resolutions, for p-stratifolds with isolated singularities. We give necessary and sufficient conditions for existence and analyze their classification.

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1 Introduction

Roughly speaking, p-stratifolds are topological spaces which are constructed by attaching manifolds with boundary by a map to the already inductively constructed space. The attaching map has to fulfill some subtle properties. There is a more general notion of stratifolds introduced by M. Kreck [Kr3]. However, the only results concerning the resolution of stratifolds exist after going over to the subclass of p-stratifolds.

The situation simplifies very much, if we consider only p-stratifolds with isolated singularities, where the construction is done in two steps only. The first step is the choice of a countable number of points \( \{x_i\}_{i \in I} \subseteq \mathbb{N} \) which will become the isolated singularities. The second step is the choice of a smooth manifold \( N \) of dimension \( m \), together with a proper map \( g : \partial N \to \{x_i\}_{i \in I} \), where \( \{x_i\}_{i \in I} \) is considered as 0-dimensional manifold and the collection of boundary components \( f^{-1}(x_i) \) is equipped with a germ of collars. The p-stratifold is obtained by forming

\[
\mathcal{S} = N \cup_g \{x_i\}_{i \in I}.
\]

We reformulate this in a slightly different way.

Definition  An \( m \)-dimensional p-stratifold with isolated singularities is a topological space \( \mathcal{S} \) together with a proper map \( f : N \to \mathcal{S} \), where
\begin{itemize}
\item \(N\) is an \(m\)-dimensional manifold with boundary,
\item \(f|_N\) is a homeomorphism onto its image,
\item \(\mathcal{S} \smallsetminus f(\tilde{N})\) is a discrete countable set, denoted by \(\Sigma\), the singular set,
\item \(f^{-1}(x)\) is equipped with a germ of collars for all \(x \in \Sigma\),
\item \(U \subset \mathcal{S}\) is open if and only if \(U \cap \Sigma\) is open and \(f^{-1}(U)\) is open in \(N\).
\end{itemize}

The manifold \(f(\tilde{N})\) is called the top stratum, \(\Sigma\) is called the 0-stratum of \(\mathcal{S}\). Choose an identification \(\Sigma = \{x_i\}_{i \in I \subseteq \mathbb{N}}\) and denote the collection of boundary components mapped to a singular point \(L_i := f^{-1}(x_i)\) the link of \(\mathcal{S}\) at \(x_i \in \Sigma\).

A collar around \(L_i\) is a diffeomorphism \(c_i : L_i \times [0, \epsilon_i) \rightarrow U_i\), where \(U_i\) is an open neighbourhood of \(L_i\) in \(N\) and \(\epsilon_i > 0\), such that \(c_i|_{L_i \times \{0\}}\) is the identity map on \(L_i\). The germ is an equivalence class of collars, where two collars \(c_i : L_i \times [0, \epsilon_i) \rightarrow U_i\) and \(\tilde{c}_i : L_i \times [0, \tilde{\epsilon}_i) \rightarrow \tilde{U}_i\) are called equivalent if there is a positive \(\delta \leq \min\{\epsilon_i, \tilde{\epsilon}_i\}\), such that \(c_i|_{L_i \times [0, \delta]} \equiv \tilde{c}_i|_{L_i \times [0, \delta]}\). The role of the collars becomes clear if we define smooth maps from a smooth manifold to a p-stratifold.

**Definition** Let \(\mathcal{S}\) be a p-stratifold with isolated singularities and \(\rho : \mathcal{S} \rightarrow \mathbb{R}\) a continuous map. The map \(\rho\) is called smooth if \(\rho f|_N : \tilde{N} \rightarrow \mathbb{R}\) is smooth and there are representatives of the germ of collars \(c_i : L_i \times [0, \epsilon_i) \rightarrow N\) satisfying for all \(i:\]
\[
\rho f(c_i(x, t)) = \rho f(x) \quad \text{for all } x \in L_i.
\]

Let \(M\) be a smooth manifold. A continuous map \(g : M \rightarrow \mathcal{S}\) is called smooth, if for all smooth maps \(\rho : \mathcal{S} \rightarrow \mathbb{R}\) the composition \(\rho g\) is again smooth.

It is not hard to verify that the map \(g\) is smooth if and only if the restriction \(g|_{g^{-1}(\mathcal{S} - \Sigma)}\) is smooth.

The most important examples of p-stratifolds as defined above are algebraic varieties with isolated singularities.

**Example** (Algebraic varieties with isolated singularities)

Consider an algebraic variety \(V \subset \mathbb{R}^n\) with isolated singularities, i.e. the singular set \(\Sigma\) is zero-dimensional. Let \(s_i \in \Sigma\) be a singular point. There is nothing to do if \(s_i\) is open in \(V\). Otherwise consider the distance function \(\rho_i\) on \(\mathbb{R}^n\) given by \(\rho_i(x) := ||x - s_i||^2\). It is well known that there is an \(\epsilon_i > 0\) such that on \(V_{\epsilon_i}(s_i) := V \cap D_{\epsilon_i}(s_i)\) the restriction \(\rho_i|_{V_{\epsilon_i} - \{s_i\}}\) has no critical values. Here \(D_{\epsilon_i}(s_i)\) denotes the closed ball in \(\mathbb{R}^n\) of radius \(\epsilon_i\) centered at \(s_i\). Set \(\partial V_{\epsilon_i}(s_i) := V_{\epsilon_i}(s_i) \cap \partial D_{\epsilon_i}(s_i)\).
By following the integral curves of the gradient vector field of $\rho_i|_{V_{\varepsilon_i} - \{s_i\}}$, we obtain a diffeomorphism

$$h : \partial V_{\varepsilon_i}(s_i) \times [0, \varepsilon_i) \to V_{\varepsilon_i} - \{s_i\}$$

being the identity on $\partial V_{\varepsilon_i}(s_i) \times \{0\}$, see [H, §6.2]. We extend this map to a continuous map

$$\tilde{h} : \partial V_{\varepsilon_i}(s_i) \times [0, \varepsilon_i) \to V_{\varepsilon_i}.$$ 

Finally, we define the manifold $N$ (with obvious collar) by setting

$$N := V - (\cup_i \partial V_{\varepsilon_i}(s_i)) \cup \text{id} \partial V_{\varepsilon_i}(s_i) \times [0, \varepsilon_i].$$

The map $f = \text{id} \cup \tilde{h} : N \to V$ gives $V$ the structure of a p-stratifold with isolated singularities.

Since every complex algebraic variety is in particular a real one, we obtain the same result for a complex algebraic variety with isolated singularities.

From now on all p-stratifolds are p-stratifolds with isolated singularities. To simplify the notation combine the representatives of the collars $c_i : L_i \times [0, \varepsilon_i)$ to a single map $c : \cup_i L_i \times [0, \varepsilon_i) \to N$. Using this map the singular set $\Sigma$ is equipped with the germ of neighbourhoods $[U\Sigma]$ by taking $U\Sigma := f(\text{im} c) \cup (\Sigma - f(\partial N))$. The collars also give us a retraction $r : U\Sigma \to \Sigma$.

We also introduce the germ of closed neighbourhoods $[\overline{U}\Sigma]$ by setting $\overline{U}\Sigma := f(c(\cup_i L_i \times [0, \varepsilon_i/2])) \cup (\Sigma - f(\partial N))$. If we want to make the dependency on the representative of the germ of collars clear, we sometimes write $U\Sigma_c$ and $\overline{U}\Sigma_c$.

**Definition** Let $\mathcal{S}$ be an $n$-dimensional p-stratifold with top stratum $f(\tilde{\mathcal{N}})$. A **resolution** of $\mathcal{S}$ is a proper map $p : \tilde{\mathcal{S}} \to \mathcal{S}$ such that

- $\tilde{\mathcal{S}}$ is a smooth manifold;
- $p$ is a proper smooth map;
- the restriction of $p$ on $p^{-1}(f(\tilde{\mathcal{N}}))$ is a diffeomorphism on $f(\tilde{\mathcal{N}})$;
- $p^{-1}(f(\tilde{\mathcal{N}}))$ is dense in $\tilde{\mathcal{S}}$;
- the inclusion $\tilde{\Sigma} := p^{-1}(\Sigma) \hookrightarrow U\tilde{\Sigma} := p^{-1}(U\Sigma)$ is a homotopy equivalence for a representative of the neighbourhood $U\Sigma$ of $\Sigma$. 

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A resolution \( p : \hat{\mathcal{S}} \to \mathcal{S} \) is called optimal, if \( p|_{\hat{\Sigma}} : \hat{\Sigma} \to \Sigma \) is an \([n/2]\)-equivalence. In particular, it follows that \( p : \hat{\mathcal{S}} \to \mathcal{S} \) is an \([n/2]\)-equivalence as well.

If the manifold \( N \) is equipped with more structure, e.g. orientation or spin-structure, we introduce corresponding resolutions, which have more structure.

**Definition** Let \( \mathcal{S} = f(N) \cup \{x_i\}_i \) be a \( p \)-stratifold with oriented \( N \). A resolution \( p : \hat{\mathcal{S}} \to \mathcal{S} \) is called an oriented resolution, if \( \hat{\Sigma} \) is oriented and \( \hat{p}|_{\hat{\Sigma} \setminus f^{-1}(f(N))} \) is orientation preserving. Analogously, if \( N \) is spin, then \( p : \hat{\mathcal{S}} \to \mathcal{S} \) is called a spin resolution if \( \hat{\Sigma} \) is spin and \( \hat{p}|_{\hat{\Sigma} \setminus f^{-1}(f(N))} \) preserves the spin structure.

If \( V \) is an algebraic variety, Hironaka has shown [Hi] that there is a resolution of singularities in the sense of algebraic geometry. The above topological definition is modelled on the one from algebraic geometry. All conditions are analogous except the last one, which is always fulfilled in the context of algebraic geometry. As explained in [BR], a neighbourhood \( U \) of the singular set \( \Sigma \) of an algebraic variety \( V \) such that the inclusion \( \Sigma \hookrightarrow U \) is a homotopy equivalence can be obtained from a proper algebraic map \( \rho : V \to \mathbb{R} \) with \( \Sigma = \rho^{-1}(0) \) by taking \( U = \rho^{-1}(0, r) \), provided \( r > 0 \) is small enough. Thus for a resolution \( p : \hat{V} \to V \) the preimage \( \hat{U} := p^{-1}(U) \) is a neighbourhood of \( \hat{\Sigma} := p^{-1}(\Sigma) \) in \( \hat{V} \) obtained from \( \hat{\rho} := \rho \circ p \), hence the inclusion \( \hat{\Sigma} \hookrightarrow \hat{U} \) is a homotopy equivalence.

Note that \( \overline{\hat{U}} \setminus \hat{\Sigma} \) is a smooth manifold with boundary diffeomorphic to \( \partial N \). Consider the preimage of the neighbourhood of each singularity and set \( \overline{\hat{U}} \Sigma_i := p^{-1}(f(c_i(L_i \times [0, \varepsilon_i/2]))) \), where \( c_i \) is the restriction of the collar to \( L_i \times [0, \varepsilon_i/2] \).

It is not hard to verify that a resolution \( p : \hat{\mathcal{S}} \to \mathcal{S} \) is optimal if and only if the manifolds \( \overline{\hat{U}} \Sigma_i \) are \( ([n/2] - 1) \)-connected.

In contrast to algebraic varieties, resolutions of stratifolds in general do not exist, not even for isolated singularities. But in this case there is a simple necessary and sufficient condition, see [Kr3] and §6 for a proof.

**Theorem 1** An \( n \)-dimensional \( p \)-stratifold with isolated singularities admits a resolution if and only if each link of the singularity \( L_i \), vanishes in the bordism group \( \Omega_{n-1} \).

**Example** The \( p \)-stratifold \( \mathcal{S} = \mathbb{C}P^2 \times I \cup f \{x_0, x_1\} \) with the obvious stratification, such that \( f(\mathbb{C}P^2 \times \{0\}) = x_0 \) and \( f(\mathbb{C}P^2 \times \{1\}) = x_1 \), does not admit a resolution.
To give a feeling of the result concerning optimal resolution, we formulate the following special case which will be derived as Corollary 7 of Theorem 5 (cf. §2).

**Corollary** Let $\mathcal{S}$ be a p-stratifold with parallelizable links of singularities $L_i$. Assume $L_i$ is bounded by a parallelizable manifold, then $\mathcal{S}$ admits an optimal resolution.

We have shown above that every algebraic variety with isolated singularities admits a structure of a p-stratifold. One may ask the converse question. When does a p-stratifold with isolated singularities admit an algebraic structure? The following Theorem of Akbulut and King [AK, Thm. 4.1] clarifies the situation in the case of a real algebraic structure.

**Theorem 2** A topological space $X$ is homeomorphic to a real algebraic set with isolated singularities if and only if $X$ is obtained by taking a smooth compact manifold $M$ with boundary $\partial M = \bigcup_{i=1}^{r} L_i$, where each $L_i$ bounds, then crushing some $L_i$’s to points and deleting the remaining $L_i$’s.

Combining this result with Theorem 1 we immediately obtain:

**Corollary 3** A compact p-stratifold $\mathcal{S}$ with isolated singularities is homeomorphic to a real algebraic set with isolated singularities if and only if $\mathcal{S}$ admits a resolution.

**Example** (Resolutions of hypersurfaces with isolated singularities)

Let $p : \mathbb{R}^{n+1} \to \mathbb{R}$ be a polynomial with isolated singularities $\{s_i\}_i$, i.e. $s_i \in V := p^{-1}(0)$ and $s_i$ is an isolated critical point of $p$. Assume further that the points $s_i$ are not open. According to a previous example, the hypersurface $V$ admits a canonical structure of a p-stratifold. We have to investigate the link of the singularity, which is given by $\partial V_{\varepsilon_i}(s_i)$.

Choose a $\delta > 0$ such that all $c$ with $|c| \leq \delta$ are regular values of $p$ and take $c$ such that $p^{-1}(c) \neq \emptyset$. Then $p^{-1}(c)$ is a smooth manifold with trivial normal bundle. With the help of the gradient vector field we see that $p^{-1}(c) \cap S^n_{\varepsilon_i}(s_i)$ is diffeomorphic to $p^{-1}(0) \cap S^n_{\varepsilon_i}(s_i) = \partial V_{\varepsilon_i}(s_i)$. Thus, $\partial V_{\varepsilon_i}(s_i) \cong p^{-1}(c) \cap S^n_{\varepsilon_i}(s_i) \cong \partial(p^{-1}(c) \cap D^{n+1}_{\varepsilon_i}(s_i))$. We see that a resolution always exists, and since the bounding manifolds are automatically parallelizable, we even obtain an optimal resolution after choosing an appropriate bordism (compare with Figure 1).
In the case of a complex polynomial $p : \mathbb{C}^{n+1} \to \mathbb{C}$ $(n > 0)$, every deformation $p^{-1}(c)$ gives us an optimal resolution, provided $\|c\|$ is small enough. This follows from a result of Milnor [Mi4, Thm. 6.5] which states that $M := p^{-1}(c) \cap D^2 \cap \{s_i\}$ is homotopy equivalent to a wedge of $\mu \geq 1$ copies of $S^n$ and thus $(n - 1)$-connected.

Consider another interesting class of $p$-stratifolds with isolated singularities, namely those arising from a smooth group action.

**Definition** A smooth $S^1$-action on a smooth manifold $M$ is called *semi-free* if the action is free outside of the fixed point set, i.e. if $gx = x$ for a $g \in S^1, g \neq 1$ and $x \in M$, then $hx = x$ for all $h \in S^1$.

**Lemma 4** Let $M$ be a closed oriented manifold with semi-free $S^1$-action with only isolated fixed points. Then $M/S^1$ admits an oriented resolution if and only if $\dim M \equiv 0(\text{mod } 4)$.

**Proof** Let $\dim M = n$. There is nothing to show if the action is free. Thus let $x \in M$ be a fixed point. The differential of the action gives a representation of $S^1$ on $T_x M$ and there is an equivariant local diffeomorphism from $T_x M$ onto a neighbourhood of $x$ in $M$. According to [BT, Prop. (II.8.1)], every irreducible
representation of $S^1$ on $\mathbb{R}^n$ is equivalent to:

$$
\begin{pmatrix}
z^{n/2} & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & z^{n/2} & 0 & \vdots \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
z^{n/2} & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & z^{n/2}
\end{pmatrix}
$$

Since the action is semi-free and $x$ is an isolated fixed point we conclude that $\dim M$ is even. We can further assume $n_i = 1$ for all $i \in \{1, \ldots, n/2\}$. Let $\dim M = 2m$ and let $\{x_1, \ldots, x_k\}$ be the set of fixed points. Choose equivariant disks $D_{x_i}$ around $x_i$. In this situation we have

$$
M/S^1 = (M - \sqcup_i \hat{D}_{x_i})/S^1 \cup \{x_1, \ldots, x_k\}.
$$

The domain of the top stratum is then given by $N := (M - \sqcup_i \hat{D}_{x_i})/S^1$ and the singular set is $\Sigma := \{x_1, \ldots, x_k\}$. The links of singularities are given by $L_i \cong S^{2m-1}/S^1 = \mathbb{C}P^{m-1}$. Using Theorem 1 we conclude that the resolution exists if and only if $[\mathbb{C}P^{m-1}]$ vanishes in $\Omega^{SO}_{2m-2}$. For $m = 2l + 1$ the signature of $\mathbb{C}P^{m-1}$ is equal to 1, hence $\mathbb{C}P^{m-1}$ does not bound. In the case of an even $m = 2l + 2$ we have $\mathbb{C}P^{2l+1} = S^{4l+3}/S^1 = S(\mathbb{H}^{l+1})/S^1$, where $S(\mathbb{H}^{l+1})/S^1$ is the sphere bundle

$$
S^2 = S^3/S^1 \xrightarrow{C} S(\mathbb{H}^{l+1})/S^1
$$

and the associated disk bundle bounds.

As mentioned before, we are particularly interested in the classification of resolutions. Thus we have to decide when we are going to consider two resolutions as equivalent. We can restrict our attention to the resolving manifolds and introduce a relation on them, e.g. diffeomorphism, but in this case we completely ignore an important part of the resolution data, namely the resolving map. Hence, one can ask for diffeomorphisms between the resolving manifolds commuting with the resolving maps. This relation is very strong and, therefore, very hard to control. In the following definition, we combine these two ideas.
Definition Let $\mathcal{S}$ be a $p$-stratifold and $p : \hat{\mathcal{S}} \to \mathcal{S}$ and $p' : \hat{\mathcal{S}}' \to \mathcal{S}$ two resolutions of $\mathcal{S}$. We call the resolutions equivalent, if, for every representative of the neighbourhood germ $\bar{U}\Sigma_c$, there is a diffeomorphism $\varphi_c : \hat{\mathcal{S}} \to \hat{\mathcal{S}}'$ such that the following holds:

- $p'\varphi_c = p$ on $\hat{\mathcal{S}} - \bar{U}\Sigma_c$ and
- $rp'\varphi_c = rp$ on $\bar{U}\Sigma_c$, where $r : \bar{U}\Sigma_c \to \Sigma$ is the neighbourhood's retraction.

This means outside of an arbitrary small neighbourhood of the singularity, the diffeomorphism commutes with the resolving maps and near $\Sigma$ it only commutes after the composition with the retraction.

Observe that $\varphi_c$ gives a diffeomorphism $\partial\bar{U}\Sigma_c \to \partial\bar{U}\Sigma'_c$.

The classification of optimal resolutions is quite a difficult problem. For if $\hat{\mathcal{S}}$ is an optimal resolution of $\mathcal{S}$, then $\hat{\mathcal{S}}\sharp S$ is again optimal for an arbitrary homotopy sphere $S$. In particular, consider the sphere $S^n$ stratified as $D^n \cup \text{pt}$, then every homotopy sphere $S^n$ gives us a resolution of $S^n$. Thus, we weaken the problem and ask for the equivalence up to a homotopy sphere.

Definition Two resolutions $\hat{\mathcal{S}} \to \mathcal{S}$ and $\hat{\mathcal{S}}' \to \mathcal{S}$ are called almost equivalent if $\hat{\mathcal{S}}\sharp S$ is equivalent to $\hat{\mathcal{S}}'\sharp S$ for a homotopy sphere $S$.

A special case of the classification result is the following.

Corollary Let $\hat{\mathcal{S}} \to \mathcal{S}$ and $\hat{\mathcal{S}}' \to \mathcal{S}$ be two resolutions of a $2n$-dimensional $p$-stratifold $\mathcal{S}$ having $(n - 2)$-connected links of isolated singularities. Assume that $n \equiv 6 \pmod{8}$ and that $\bar{U}\Sigma_i$ and $\bar{U}\Sigma'_i$ are parallelizable with compatible parallelizations on the boundary. Let further $e(\bar{U}\Sigma_i) = e(\bar{U}\Sigma'_i)$ and $\text{sign}(\bar{U}\Sigma_i \cup \partial \bar{U}\Sigma'_i) = 0$. Then there is a $k \in \{0, 1\}$ such that $\hat{\mathcal{S}}\sharp k(S^n \times S^n)$ is almost equivalent to $\hat{\mathcal{S}}'\sharp k(S^n \times S^n)$.

2 Existence of optimal resolutions

Before proceeding with the existence of an optimal resolution we need to introduce some notation. For a topological space $X$ let $X(k)$ be the $k$-connected cover of $X$, which always comes with a fibration $p : X(k) \to X$. For further information see for example [Ba]. We take $X$ to be the classifying space $BO$ and denote by $\Omega_n^{BO(k)}$ the bordism group of closed $n$-dimensional manifolds together with a lift of the normal Gauss map, compare [St, Chap. I].
Theorem 5 An \( n \)-dimensional \( p \)-stratifold with isolated singularities admits an optimal resolution if and only if the normal Gauss map \( \nu_j : L_j \to BO \) admits a lift over \( BO([n/2] - 1) \), such that \( [L_j, \nu_j] = 0 \) in \( \Omega^{BO([n/2]-1)}_{n-1} \).

Proof Let \( \mathcal{S} \) be a \( p \)-stratifold with isolated singularities \( \{x_i\}_{i \in I} \) and \( \hat{\mathcal{S}} \) an optimal resolution of \( \mathcal{S} \). Set \( W_i := \Sigma_i \), then \( W_i \) is a smooth manifold with boundary \( L_i \) for \( i \in I \subseteq \mathbb{N} \). Consider the normal Gauss map, together with the \( ([n/2]-1) \)-connected cover over \( BO \).

\[
\begin{array}{ccc}
BO([n/2]-1) & \to & F \\
\nu_j & \searrow & \uparrow \\
& W_j & \to BO
\end{array}
\]

The obstructions for the existence of a lift lie in \( \tilde{H}^r(W_j, \pi_{r-1}(F)) \). Note that we can use global coefficients since the fibration \( BO([n/2] - 1) \to BO \) is simple.

Since the resolution is optimal the manifold \( W_i \) is \( ([n/2]-1) \)-connected, hence \( \tilde{H}^r(W_i) = 0 \) for \( r < [n/2] \).

Using the properties of the connected cover it follows from the long exact homotopy sequence that \( \pi_r(F) = 0 \) for \( r \geq [n/2] - 1 \).

Hence, there are no obstructions for the lifting of the normal Gauss map, thus \( [L_j, \nu_j] \) vanishes in \( \Omega^{BO([n/2]-1)}_{n-1} \).

The fact that the condition is also sufficient is an immediate consequence of the following result from [Kr1, Prop. 4].

Theorem 6 Let \( \xi : B \to BO \) be a fibration and assume that \( B \) is connected and has a finite \( [m/2] \)-skeleton. Let \( \bar{\nu} : M \to B \) be a lift of the normal Gauss map of an \( m \)-dimensional compact manifold \( M \). Then if \( m \geq 4 \), by a finite sequence of surgeries \( (M, \bar{\nu}) \) can be replaced by \( (M', \bar{\nu}') \) so that \( \bar{\nu}' : M' \to B \) is an \( [m/2] \)-equivalence.

For example, we obtain the following:

Corollary 7 Let \( \mathcal{S} \) be a \( p \)-stratifold with parallelizable links of singularities \( L_i \). Assume \( L_i \) is bounded by a parallelizable manifold, then \( \mathcal{S} \) admits an optimal resolution.
3 Classification of almost equivalent resolutions

Now we turn to the main result of this note. In this section we consider p-stratifolds with isolated singularities of dimension $2n > 4$ having $(n-2)$-connected links of singularities. The classification is based on the following result from [Kr1, Thm. 2].

**Theorem 8** For $n > 2$ let $W_1$ and $W_2$ be two compact connected $2n$-manifolds with normal $(n-1)$-smoothings in a fibration $B$. Let $g : \partial W_1 \to \partial W_2$ be a diffeomorphism compatible with the normal $(n-1)$-smoothings $\nu_1$ and $\nu_2$. Let further

- $e(W_1) = e(W_2)$,
- $\left[ W_1 \cup_g (-W_2), \tilde{\nu}_1 \cup \tilde{\nu}_2 \right] = 0 \in \Omega_{2n}(B)$.

Then $g$ can be extended to a diffeomorphism $G : W_1 \sharp k(S^n \times S^n) \to W_2 \sharp k(S^n \times S^n)$ for $k \in \mathbb{N}$. Moreover, if $B$ is 1-connected then $k \in \{0, 1\}$.

If $W_1$ is simply connected and $n$ is odd, then $k$ can be chosen as 0, i.e. we obtain diffeomorphism instead of stable diffeomorphism.

We have to explain some terms appearing in the last theorem. Let $B$ be a fibration over $BO$, a *normal $B$-structure* on a manifold $M$ is a lift $\bar{\nu}$ of the normal Gauss map $\nu : M \to BO$ to $B$.

**Definition** Let $B$ be a fibration over $BO$.

(1) A normal $B$-structure $\bar{\nu} : M \to B$ of a manifold $M$ in $B$ is a *normal $k$-smoothing*, if it is a $(k+1)$-equivalence.

(2) We say that $B$ is *$k$-universal* if the fiber of the map $B \to BO$ is connected and its homotopy groups vanish in dimension $\geq k + 1$.

Obstruction theory implies that if $B$ and $B'$ are both $k$-universal and admit a normal $k$-smoothing of the same manifold $M$, then the two fibrations are fiber homotopy equivalent. Furthermore, the theory of Moore-Postnikov decompositions implies that for each manifold $M$ there is a $k$-universal fibration $B^k$ over $BO$ admitting a normal $k$-smoothing, compare [Ba, §5.2]. Thus, the fiber homotopy type of the fibration $B^k$ over $BO$ is an invariant of the manifold $M$ and we call it the *normal $k$-type of $M$*.

There is an obvious bordism relation on closed $n$-dimensional manifolds with normal $B$-structures and the corresponding bordism group is denoted $\Omega_n(B)$.
Applying the theorem to our situation, we first have to determine the normal 
\((n-1)\)-type of an \((n-1)\)-connected \(2n\) manifold.

Consider a subgroup \(H\) of \(G := \pi_n(BO)\). Since the last group is always cyclic, 
the group \(H\) is determined by an integer \(k\), such that \(H = \langle kx \rangle\) where \(x\) is 
the generator of \(\pi_n(BO)\). We call this integer the \textit{index} of \(H\).

Every subgroup \(H\) of \(G = \pi_n(BO)\) gives us a fibration

\[
\begin{array}{ccc}
B & \longrightarrow & P(K(G/H,n)) \\
\downarrow p & & \downarrow \\
BO\langle n-1 \rangle & \longrightarrow & K(G/H,n)
\end{array}
\]

where the map \(\theta\) corresponds to the canonical epimorphism \(G \to G/H\). We 
denote the space \(B\) belonging to the index-
\(k\) group \(B_k\). The composition

\[
p_k : B_k \to BO\langle n-1 \rangle \to BO
\]

gives us a fibration over \(BO\) with fiber \(F_k\).

**Definition** A \(2n\)-dimensional manifold \(M\) is said to have the \textit{index} \(k\), if \(\nu_*\pi_n(M)\) is a subgroup of index \(k\) in \(\pi_n(BO)\).

**Theorem 9** The fibration \(B_k \to BO\) is the normal \((n-1)\)-type of an \((n-1)\)-
connected \(2n\)-dimensional manifold \(M\), if and only if \(M\) is of index \(k\).

The proof can be found in §6, which also contains proofs of the following two 
theorems.

Now we look for conditions implying an \((n-1)\)-connected \(2n\)-manifold to 
be bordant to a homotopy sphere. Note first that as an easy consequence 
from the universal coefficient theorem the first non-trivial homology group is 
free. The homological information of \(M\) is stored in the triple \((H_n(M), \Lambda, \nu)\) 
where \(\Lambda\) denotes the intersection product \(\Lambda : H_n(M) \to \mathbb{Z}\), we often simply 
write \(x \cdot y\) for \(\Lambda(x,y)\). The last data \(\nu\) is the normal bundle information, 
described in the following way. According to a theorem of Haefliger [Hae] 
every element of \(H_n(M)\) is represented by an embedding \(S^n \embed M\), and two 
embeddings corresponding to the same homotopy class are regular homotopic. 
Thus, assigning to an embedded sphere its normal bundle gives us a well defined 
map \(\nu : H_n(M) \to \pi_{n-1}(SO(n))\).

**Definition** An \((n-1)\)-connected \(2n\)-dimensional manifold \(M\) is called \textit{elementary} if \(H_n(M)\) admits a Lagrangian \(w.r.t.\ \Lambda\), such that \(\nu|_L \equiv 0\).
Theorem 10  Let $M$ be an $(n-1)$-connected manifold of dimension $2n$. Then $M$ is bordant to a homotopy sphere if and only if $M$ is elementary.

Our main result, based on the last two theorems is:

Theorem 11  For $n > 2$ let $\mathcal{J} \rightarrow \mathcal{J}$ and $\mathcal{J}' \rightarrow \mathcal{J}$ be two optimal resolutions of a $2n$-dimensional $p$-stratifold $\mathcal{S}$ with isolated singularities $\{x_i\}_{i \in I}$, such that each link $L_i$ is $(n-2)$-connected. Assume further that for a suitable representative $U\Sigma$ the following conditions hold for all $i \in I$:

1. $e(U\Sigma_i) = e(U\Sigma'_i)$;
2. $U\Sigma_i$ and $U\Sigma'_i$ have the same index $k_i$;
3. there exits normal $(n-1)$-smoothings $\nu_i$ and $\nu'_i$ of $U\Sigma_i$ and $U\Sigma'_i$ in the fibration $B_{k_i} \rightarrow BO$, such that $\nu_i|_{\partial U\Sigma_i} = \nu'_i|_{\partial U\Sigma'_i}$;
4. $U\Sigma_i \cup_{\partial} U\Sigma'_i$ is elementary.

If $n$ is odd, then $\mathcal{J}$ is almost equivalent to $\mathcal{J}'$.

If $n$ is even, then $\mathcal{J} \sharp k(S^n \times S^n)$ is almost equivalent to $\mathcal{J}' \sharp k(S^n \times S^n)$ for a $k \in \{0, 1\}$.

4  Algebraic invariants

In this section we will find algebraic invariants, which allow us to decide whether an $(n-1)$-connected closed $2n$-dimensional manifold is elementary or not ($n > 2$). Some proofs can be found in §6.7.

Recall the algebraic data corresponding to such a manifold $M$. We have a triple $(H, \Lambda, \nu_*)$, where $H = H_n(M)$ is a free $\mathbb{Z}$-module, $\Lambda : H \times H \rightarrow \mathbb{Z}$ is the intersection product and $\nu_* : H \rightarrow \pi_{n-1}(SO_n)$ is a normal bundle map, described in the previous section. The map $\nu_*$ is not a homomorphism, but satisfies the following equation:

$$\nu_*(x + y) = \nu_*(x) + \nu_*(y) + \partial \Lambda(x, y),$$  \hspace{1cm} (*)

where $\partial : \mathbb{Z} \cong \pi_n(S^n) \rightarrow \pi_{n-1}(SO_n)$ is the boundary map from the long exact homotopy sequence of the fibration $SO(n) \hookrightarrow SO(n+1) \rightarrow S^n$, see [W1].

Thus, we obtain an algebraic object, the set $T_n$ of triples $(H, \Lambda, \nu_*)$, where $H$ is a free $\mathbb{Z}$-module, $\Lambda : H \times H \rightarrow \mathbb{Z}$ is an $(-1)^n$-symmetric unimodular quadratic form and $\nu_* : H \rightarrow \pi_{n-1}(SO_n)$ is a map satisfying (*). We want to investigate
the assumptions under which an element \((H, \Lambda, \nu_s) \in T_n\) is elementary, i.e. when \(H\) possesses a Lagrangian \(L\) with respect to \(\Lambda\) such that \(\nu_s|_L \equiv 0\).

We begin with an observation that for a \(4k\)-dimensional manifold, the normal bundle information can be replaced by the stable normal bundle map.

**Lemma 12** Let \(n\) be even and let \(S^n \hookrightarrow M^{2n}\) be an embedding. The normal bundle \(\nu(S^n)\) of \(S^n\) in \(M\) is trivial if and only if \(\nu \oplus \mathbb{R}\) is trivial and the Euler class of \(\nu(S^n)\) vanishes.

Thus, instead of considering \((H, \Lambda, \nu_s) \in T_n\) we can go over to \((H, \Lambda, s\nu_s)\), where \(s\nu_s : H \rightarrow \pi_{n-1}(SO)\) corresponds to the stable normal bundle map. Since the Euler class of an embedded sphere representing \(x \in H\) can be identified with the self intersection class we conclude:

**Lemma 13** Let \(n\) be even. Then \((H, \Lambda, \nu_s) \in T_n\) is elementary if and only if \((H, \Lambda, s\nu_s)\) is elementary.

Let \(T_n^s\) denote the set of triples \((H, \Lambda, s\nu_s)\), with \(H\) and \(\Lambda\) as above and \(s\nu_s : H \rightarrow \pi_{n-1}(SO)\) a homomorphism. According to the different possibilities for \(\pi_{n-1}(SO)\) we distinguish 3 cases.

1. \(\pi_{n-1}(SO) = 0\).

**Claim** \((H, \Lambda, s\nu_s) \in T_n^s\) is elementary if and only if \(\text{sign}(\Lambda) = 0\), where sign denotes the signature of a quadratic form.

2. \(\pi_{n-1}(SO) = \mathbb{Z}\). Since \(\Lambda\) is unimodular it induces an isomorphism \(H \xrightarrow{\cong} H^*\), which we also denote by \(\Lambda\). The map \(s\nu_s\) gives an element of \(H^*\) and we consider \(\kappa_{s\nu_s} := \Lambda^{-1}(s\nu_s) \in H\).

**Claim** \((H, \Lambda, s\nu_s) \in T_n^s\) is elementary if and only if \(\text{sign}(\Lambda) = 0\) and \(\kappa_{s\nu_s}^2 = 0\).

3. \(\pi_{n-1}(SO) = \mathbb{Z}_2\). Let \((H, \Lambda, s\nu_s)\) be an element of \(T_n^s\) with vanishing signature and suppose \(\Lambda\) is of type \(II\), i.e. \(\Lambda(x, x) = 0 \pmod{2}\) for all \(x \in H\). Note that since \(n \neq 8\) in this case, an elementary element corresponding to a manifold always has a type \(II\) quadratic form. Thus, the dimension of \(H\) is even and according to [Mi1, Lem. 9] we can choose a basis \(\{\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k\}\) satisfying

\[
\Lambda(\lambda_i, \lambda_j) = 0, \quad \Lambda(\mu_i, \mu_j) = 0 \quad \text{and} \quad \Lambda(\lambda_i, \mu_j) = \delta_{ij}.
\]

Consider the set of all elements \(x \in H\) with \(\Lambda(x, x) = 0\) and denote its image under canonical projection on \(H \otimes \mathbb{Z}_2\) by \(H^0\). The class \(\Phi(H, \Lambda, s\nu_s) := \)
\[ \sum_{i=1}^{k} s\nu_*(\lambda_i)s\nu_*(\mu_i) \in \mathbb{Z}_2 \] is well-defined and is equal to the value \( s\nu_* \) takes most frequently on the finite set \( H^0 \), the class is called Arf invariant.

**Claim** An element (\( H, \Lambda, s\nu_* \)) \( \in T_n^s \) with type \( II \) form \( \Lambda \) is elementary if and only if \( \text{sign}(\Lambda) = 0 \) and \( \Phi(H, \Lambda, s\nu_*) = 0 \).

Consider now the case of an odd \( n \). The quadratic form is now skew symmetric. Depending on the values of \( \nu_* \) there are again three different cases (compare [Ke]), which were completely investigated in [W1].

(4) \( \pi_{n-1}(SO_n) = 0 \). In this case every element of \( T_n \) is elementary.

(5) \( \pi_{n-1}(SO_n) = \mathbb{Z}_2 \). As in (3), we can define the Arf invariant \( \Phi(H, \Lambda, \nu_*) = \sum_{i=1}^{k} \nu_*(\lambda_i)\nu_*(\mu_i) \in \mathbb{Z}_2 \), using a symplectic basis \( \{\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k\} \) of \( H \).

**Claim** An element (\( H, \Lambda, \nu_* \)) \( \in T_n \) is elementary if and only if \( \Phi(H, \Lambda, \nu_*) = 0 \) and \( \text{pr}_2\nu_*(\kappa) = 0 \), where \( \text{pr}_2 \) denotes the projection on the second component.

Knowing the algebraic description of elementary manifolds, we formulate a special case of Theorem 11.

**Corollary 14** Let \( \hat{\mathcal{S}} \rightarrow \mathcal{S} \) and \( \hat{\mathcal{S}}' \rightarrow \mathcal{S} \) be two resolutions of a 2n-dimensional p-stratifold \( \mathcal{S} \) having \( (n - 2) \)-connected links of isolated singularities. Assume that \( n \equiv 6 \) (mod 8) and that \( \overline{U}\Sigma_i \) and \( \overline{U}'\Sigma_i' \) are parallelizable with compatible parallelizations on the boundary. Let further \( e(\overline{U}\Sigma_i) = e(\overline{U}'\Sigma_i') \) and \( \text{sign}(\overline{U}\Sigma_i \cup_{\partial} \overline{U}'\Sigma_i') = 0 \). Then there is a \( k \in \{0, 1\} \) such that \( \mathcal{S}^k(S^n \times S^n) \) is almost equivalent to \( \hat{\mathcal{S}}^k(S^n \times S^n) \).

## 5 4-dimensional results

In this section we consider the exceptional case of a 4-dimensional p-stratifold and give a similar classification result in that situation. The proof of the main theorem can be found in §6.8.

For a 4-dimensional stratifold \( \mathcal{S} \), every link of the singularity \( L_i \) is a 3-dimensional manifold. According to the computation of \( \Omega_* \) by Thom [Th] we immediately obtain from Theorem 1:
Corollary 15 A four-dimensional p-stratifold with isolated singularities always admits a resolution.

If we further assume the links to be oriented we can use the following well-known result, which can be proved easily.

Proposition 16 Every orientable 3-manifold is parallelizable, hence in particular spin.

The normal 1-type of a simply connected 4-dimensional spin-manifold is given by $B\text{Spin}$. Since $\Omega^3_{\text{spin}} = 0$ (see [Mi3, Lem. 9]) we obtain the following corollary from Theorem 5.

Corollary 17 A four-dimensional p-stratifold with isolated singularities admits an optimal resolution if and only if all links of singularities are orientable.

We have to develop some notation in the topological category. Use $B\text{TOP}$ to denote the classifying space of topological vector bundles and let $B\text{TOPSpin}$ be the 2-connected cover over $B\text{TOP}$. Let $M$ be a simply connected 4-manifold.

Using the Wu-Formula we can explain the Stiefel-Whitney-classes of $M$. We call the topological manifold $M$ spin if $w_2(M)$ vanishes. One can show that the topological Gauss map of $M$ lifts to $B\text{TOPSpin}$ if and only if $M$ is spin. Note further that if such a lift exists, it is unique.

Using [Kr1, Thm. 2] and the h-cobordism-Theorem in dimension 4 [F, Thm. 10.3] we formulate:

Theorem 18 Let $M_1$ and $M_2$ be compact 4-dimensional topological spin manifolds with $e(M_1) = e(M_2)$ and let $g : \partial M_1 \rightarrow \partial M_2$ be a homeomorphism compatible with the induced spin-structures on the boundaries. If $M_1 \cup_g M_2$ vanishes in $\Omega^4_{B\text{TOPSpin}}$, then $g$ can be extended to a homeomorphism $G : M_1 \sharp k(S^2 \times S^2) \rightarrow M_2 \sharp k(S^2 \times S^2)$ for $k \in \{0, 1\}$.

We call two resolutions topologically equivalent if the diffeomorphism $\varphi_c$ in the definition of equivalent resolutions in §1 is replaced by a homeomorphism. Using this notation, we obtain the following classification result in dimension four:

Theorem 19 Let $\hat{\mathcal{S}} \rightarrow \mathcal{S}$ and $\hat{\mathcal{S}}' \rightarrow \mathcal{S}$ be two optimal resolutions of a 4-dimensional p-stratifold $\mathcal{S}$ with isolated singularities $\{x_i\}_{i \in I}$, such that each link $L_i$ is connected. Assume that both $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}'$ are spin and that for a suitable representative $\overline{U\Sigma}$ of the neighbourhood germ, the following conditions hold for all $i \in I$: 
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- \( e(\overline{U}\hat{\Sigma}_i) = e(\overline{U}\hat{\Sigma}_i') \),
- the spin-structures of \( \overline{U}\hat{\Sigma}_i \) and \( \overline{U}\hat{\Sigma}_i' \) coincide on the boundary,
- \( \text{sign}(\overline{U}\hat{\Sigma}_i \cup_\partial \overline{U}\hat{\Sigma}_i') = 0 \).

Then \( \hat{\mathcal{I}}^\sharp k(S^2 \times S^2) \) is topologically equivalent to \( \hat{\mathcal{I}}^\sharp k(S^2 \times S^2) \) for a \( k \in \{0, 1\} \).

6 Outline of the proofs

6.1 Proof of Theorem 1

Although the proof can be found in [Kr3], it is useful to understand its nature for the succeeding results.

One of the basic tools for constructing a resolving map is the following lemma, which can be proved with the help of Morse theory, cf. [Kr3].

**Lemma 20**  Let \( W \) be a smooth compact manifold with boundary. Then there is a codense compact subspace \( X \) of \( W \) and a continuous map \( f : \partial W \rightarrow X \) such that \( W \) is homeomorphic to \( \partial W \times [0, 1] \cup f X \), where on \( \partial W \times [0, 1) \) the homeomorphism can be chosen to be a diffeomorphism.

In other words, every smooth manifold with boundary arises from its collar by attaching a codense set. The notation *codense* stands for the complement of a dense subset. With this information we are ready to prove Theorem 1.

**Proof**  Let \( p : \hat{\mathcal{I}} \rightarrow \mathcal{I} \) be a resolution. Set \( W_i := \overline{U}\hat{\Sigma}_i \). Since \( W_i \) is a compact manifold with boundary \( L_i \), we obtain \([L_i] = 0\) in \( \Omega_{n-1} \).

Let on the other hand \( W_i \) be a compact manifold bounding \( L_i \) and let \( f(\bar{N}) \) be the top stratum of \( \mathcal{I} \). Set \( \hat{\mathcal{I}} := N \cup (\cup_i W_i) \) and construct with the help of the last lemma the following resolving map:

\[
\hat{\mathcal{I}} \cong N \cup_i (\partial W_i \times [0, \varepsilon_i] \cup f_i X_i) \xrightarrow{p} N \cup_i (\partial L_i \times [0, \varepsilon_i] \cup \{x_i\}) \cong \mathcal{I}.
\]
6.2 Proof of Theorem 9

We consider a \(2n\)-dimensional manifold \(M\), which is \((n - 1)\)-connected, and want to determine its \((n - 1)\) type. We begin with the classification up to fiber homotopy equivalence of fibrations \(p : B \to BO\), with a CW-complex \(B\), fulfilling

1. \(B\) is \((n - 1)\)-connected and
2. \(\pi_i(F) = 0\) for \(i \geq n\), where \(F\) is the fiber.

Compare such a fibration with the \((n - 1)\)-connected cover of \(BO\):

\[
\begin{array}{ccc}
BO(n - 1) & \xrightarrow{\pi_{n-1}} & F_{n-1} \\
\downarrow & & \downarrow \\
B & \xrightarrow{p} & BO
\end{array}
\]

Since all obstructions vanish, we obtain a lift \(\bar{p} : B \to BO\langle n - 1 \rangle\), which without loss of generality may be assumed to be a fibration. From the long exact homotopy sequence we see that the homotopy groups of the fiber vanish, except in dimension \((n - 1)\), where the group is \(\pi := \text{coker}(p_* : \pi_n(B) \to \pi_n(BO))\). Thus, \(\bar{p} : B \to BO\langle n - 1 \rangle\) is a fibration with fiber \(K(\pi, n - 1)\). Such fibrations are classified in [Ba, §5.2] as follows:

\[
\begin{array}{ccc}
[BO\langle n - 1 \rangle, K(\pi, n)]/[\text{Aut}(\pi)] & \overset{\cong}{\to} & F(K(\pi, n - 1), BO\langle n - 1 \rangle) \\
[\pi] & \mapsto & f^*(P(K(\pi, n)))
\end{array}
\]

Here \(F(K(\pi, n - 1), BO\langle n - 1 \rangle)\) denotes the set of all fibrations over \(BO\langle n - 1 \rangle\) with fiber \(K(\pi, n - 1)\) up to fiber homotopy equivalence. Thus \(\bar{p} : B \to BO\langle n - 1 \rangle\) is a pull back

\[
\begin{array}{ccc}
B & \xrightarrow{\bar{p}} & P(K(\pi, n)) \\
\downarrow & & \downarrow \\
BO\langle n - 1 \rangle & \xrightarrow{\theta} & K(\pi, n)
\end{array}
\]

with an appropriate map \(\theta\). The definition of \(\pi\) force the induced map \(\theta_* : \pi_n(BO\langle n - 1 \rangle) \to \pi_n(K(\pi, n)) = \pi\) to be surjective. Therefore we can assume \(\theta_*\) to be the canonical projection to the factor group \(\pi\). On the other hand, each factor group of \(\pi_n(BO)\) leads to a fibration with the claimed properties.

We summarize this discussion in

**Lemma 21** Fibrations with properties 1. and 2. are given by \(p_k : B_k \to BO\) up to fiber homotopy equivalence \((k \in \mathbb{N}, 0 \leq k < |\pi_n(BO)|)\).
Consider now the fibration \( p : B_k \to BO \) and ask for a lift:

\[
\begin{array}{c}
\nu : M \\
\downarrow
\end{array} \quad \begin{array}{c}
\nu : M \\
\downarrow \quad \downarrow
\end{array} \quad \begin{array}{c}
B_k \\
\downarrow \quad \downarrow
\end{array} \quad \begin{array}{c}
B_k \\
\downarrow \quad \downarrow
\end{array} \quad \begin{array}{c}
\to \quad \to \quad \to \quad \to
\end{array} \quad \begin{array}{c}
\to \quad \to \quad \to \quad \to
\end{array} \quad \begin{array}{c}
BO
\end{array}
\]

With the help of obstruction theory we see that such a lift exists if and only if

\[
\text{im} (\nu_* : \pi_n(M) \to \pi_n(BO)) \leq \langle kx \rangle,
\]
where \( \pi_n(BO) = \langle x \rangle \). Combining this discussion with Lemma 21, the statement of Theorem 9 follows immediately.

### 6.3 Surgery in the middle dimension

First we give a brief introduction in surgery, for more details compare \[ W2 \].

Surgery is a tool to eliminate homotopy classes in the category of manifolds. Let \( M \) be a compact \( m \)-dimensional manifold. One starts with an embedding \( f : S^n \times D^{m-r} \to M \) and define \( T := D^{r+1} \times D^{m-r} \cup_f (M \times I) \), where \( f \) is considered as a map to \( M \times 1 \). The corners of the manifold \( T \) can always be straighten, according to \[ CF \]. This construction is called attaching an \( (r+1) \) - handle and \( T \) the trace of a surgery via \( f \).

The boundary of \( T \) is \( M \cup (\partial M \times I) \cup M' \) and we call \( M' \) the result of a surgery of index \( r+1 \) via \( f \). It is not difficult to see that \( T \) can also be viewed as the trace of a surgery on \( M' \) via the obvious embedding of \( D^{r+1} \times S^{m-r-1} \) into \( M' \), compare \[ Mi2 \].

Since we are working in the category of manifolds with \( B \)-structures we have to ask, whether the result of surgery via an embedding \( f \) is equipped with a \( B \)-structure. For general results see \[ Kr1 \]. In our situation we are only looking at fibrations \( p_k : B_k \to B \) defined in \S 3.

**Lemma 22** Let \( M \) be a manifold of dimension \( m \in \{2n, 2n+1\} \) with \( B_k \)-structure and \( f : S^n \times D^{m-n} \to M \) an embedding. Then \( \nu : M \to B \) extends to a normal \( B_k \)-structure on \( T \), the trace of the surgery via \( f \).

**Proof** The embedding \( f : S^n \times D^{m-n} \to M \) induces a normal \( B_k \)-structure on \( S^n \times D^{m-n} \) denoted by \( f^* \nu \). There is a unique (up to homotopy) \( B_k \)-structure on \( D^{n+1} \times D^{m-n} \) and we have to show that its restriction to \( S^n \times D^{m-n} \) is
Let $F_k$ be the fiber of $p_k : B_k \to BO$. From the long exact homotopy sequence we see that the different $B_k$ structures on $S^n \times D^{m-n}$ are classified by $\pi_n(F_k)$. But the construction of $B_k$ implies that $\pi_n(F_k) = 0$, thus the restriction on $S^n \times D^{m-n}$ coincides with the given structure and we obtain a $B_k$-structure on $T$.

**Lemma 23** Let $M$ be an $(n-1)$-connected $2n$-dimensional manifold. Let $H_n(M)$ have a free basis $\{\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_r\}$ with

- $\lambda_i \cdot \lambda_j = 0$ for all $i, j$ and
- $\lambda_i \cdot \mu_j = \delta_{ij}$ for all $i, j$.

If further each embedded sphere representing a generator $\lambda_i$ has a trivial normal bundle, then by a finite sequence of surgeries the homology group $H_n(M)$ can be eliminated.

**Remark** The last two lemmas imply that the condition in Theorem 10 is sufficient.

**Proof** According to a result of Haefliger ([Hae]), every element of $H_n(M)$ can be represented by an embedding $S^n \hookrightarrow M$. Let $\lambda := \lambda_r$ and $\varphi_0 : S^n \hookrightarrow M$ an embedding representing $\lambda$. Since $\lambda$ is assumed to have a trivial normal bundle, the embedding can be extended to $\varphi : S^n \times D^n \hookrightarrow M$. Set $M_0 := M - \varphi((S^n \times D^n)^c)$ and let $M'$ denote the result of surgery via $\varphi$, i.e. $M' = M_0 \cup_{\varphi} (D^{n+1} \times S^{n-1})$. Combine now the long exact homology sequences of pairs $(M, M_0)$ and $(M', M_0)$ and obtain the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\partial} & H_n(M_0) \\
\downarrow & & \downarrow i_* \\
0 & \xrightarrow{1_{M_0}} & H_n(M) \\
\downarrow & & \downarrow \lambda \\
H_n(M') & \xrightarrow{\delta} & H_{n-1}(M_0) & \to & 0 \\
\downarrow & & & & \downarrow 0 \\
H_n(M') & & & & 0
\end{array}
\]

The excision, together with the surjectivity of $H_n(M) \xrightarrow{\lambda} \mathbb{Z}$, implies $M_0$ is $(n-1)$-connected, further we see $H_n(M_0) = \langle \lambda_1, \ldots, \lambda_r, \beta_1, \ldots, \beta_{r-1} \rangle$. 

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Thus $M'$ is $(n-1)$-connected as well and
\[ H_n(M') = \langle \bar{\lambda}_1, \ldots, \bar{\lambda}_{r-1}, \bar{\mu}_1, \ldots, \bar{\mu}_{r-1} \rangle, \]
where the generators are given by $\bar{\lambda}_i = \lambda_i + \lambda Z$ and $\bar{\mu}_i = \mu_i + \lambda Z$. We can always deform the embedding of the generator $\bar{\lambda}_i$ to $M_0$, such that it represents the class $\lambda_i + \lambda \in H_n(M_0)$. Thus we conclude $\bar{\lambda}_i \cdot \bar{\lambda}_j = 0$ und $\bar{\lambda}_i \cdot \bar{\mu}_j = \delta_{ij}$.

Since the intersection product $\lambda_i \cdot \lambda_r$ vanishes we obtain $\nu(\lambda_i + \lambda_r) = \nu(\lambda_i) + \nu(\lambda_r)$ (cf. [W1]). Now we proceed with the manifold $M'$ and inductively obtain the desired statement.

6.4 Surgery on odd-dimensional manifolds

Lemma 24  Let $T$ be a bordism in $\Omega_{2n}(B_k)$ between a manifold $M$ of index $k$ and a homotopy sphere $S$. Then $T$ is bordant in $\Omega_{2n}(B_k)$ rel. boundary to $T'$, such that its homology groups $H_n(T')$ and $H_{n+1}(T')$ are free and $H_i(T') = 0$ for $i \not\in \{0, n, n+1, 2n+1\}$.

Proof  According to Theorem 6, we can assume that $T$ is $(n-1)$-connected, further the Universal Coefficient Theorem implies that $H_{n+1}(T)$ is free and $\text{Tor}(H_n(T)) \cong \text{Tor}(H_n(T, M))$. We will show that the torsion of $H_n(T, M)$ can be eliminated by a finite sequence of surgeries.

From the long exact homology sequence of the pair $(T, M)$ we see, that every torsion element $\alpha' \in H_n(T, M)$ comes from an element $\alpha \in H_n(T)$. After possible correction of $\alpha$ by an element of $H_n(M) \cong \pi_n(B_k)$ we achieve $\nu(\alpha) = 0$.

Let $\varphi : S^n \times D^{n+1} \hookrightarrow T$ be an embedding representing $\alpha$. As in the previous proof, we set $T_0 = T - \varphi((S^n \times D^{n+1})^c)$ and $T' = T_0 \cup (D^{n+1} \times S^n)$. We combine now the exact triple sequences for $(T, T_0, M)$ and $(T', T_0, M)$ to obtain:
The element $\beta \in H_n(T')$ is given by the embedding $\psi : D^{n+1} \times S^n \hookrightarrow T'$ and corresponds to the homotopy class $[\psi|_{D^n \times S^n}]$.

We consider the two cases, where $\alpha$ is free or a torsion element modulo $i^*(H_n(\partial T))$, separately.

**Case 1** $\alpha$ is primitive (mod $i^*(H_n(\partial T))$).

In this case the Poincaré duality implies that the map $H_{n+1}(T, M) \xrightarrow{\alpha} \mathbb{Z}$ is surjective, therefore

$$H_n(T', M) \cong H_n(T, M) / \langle \alpha' \rangle.$$  

Hence the torsion group of $H_n(T', M)$ has been reduced.

**Case 2** $\alpha$ is torsion (mod $i^*(H_n(\partial T))$).

The map $H_{n+1}(T, M) \xrightarrow{\alpha} \mathbb{Z}$ is trivial now. Denote with $o(x)$ the order of a torsion element $x$. From the sequence above we see that $o(\alpha')d'(1) \subset \text{im } d$, thus there exists a $b' \in \mathbb{Z}$ such that

$$o(\alpha')d'(1) = b'd(1). \quad (*)$$

If $b' = 0$, then the element $\beta'$, corresponding to $\alpha'$, has infinite order, and the torsion rank again decreases.
If $b' \neq 0$, then $(\ker j'_*) \subset \text{Tor}(H_n(T_0, M))$, thus $j'_*$ is injective on the free part of $H_n(T_0, M)$, therefore the element $d'(1)$ has infinite order and $o(\beta') | |b'|$. We need a finer case differentiation.

**Claim** If $n$ is even and $\alpha$ a torsion element of order $a$ in $H_n(T)$, then $\beta \in H_n(T')$ is an element of infinite order.

Consider the pair sequences $(T, T_0)$ and $(T', T_0)$ and obtain the following diagram

\[
\begin{array}{cccccccc}
0 & \rightarrow & H_{n+1}(T_0) & \rightarrow & H_{n+1}(T) & \rightarrow & \mathbb{Z} & \rightarrow & 0 \\
& \downarrow & \downarrow & \downarrow d' & \downarrow l \rightarrow \alpha & \downarrow & \downarrow 1 \rightarrow \beta & \downarrow & \downarrow 0 \\
& & H_{n+1}(T') & & H_n(T_0) & & H_n(T) & & 0 \\
\end{array}
\]

As in the previous case, there exists a $b \in \mathbb{Z}$ such that

\[a \cdot d'(1) = b \cdot d(1) \quad \text{in } H_n(T_0)\]

or equivalently after the identification of the generators

\[(\varphi_0)_*(a \cdot ([S^n] \otimes 1) - b \cdot (1 \otimes [S^n])) = 0.\]

By computing the self intersection number of $a \cdot ([S^n] \otimes 1) - b \cdot (1 \otimes [S^n])$ we obtain $2ab([S^n] * \otimes [S^n]^*) \in \text{im } ((\varphi_0)^* : H^{2n}(W_0) \rightarrow H^{2n}(S^n \times S^n))$. Thus $2ab = 0$, and since $a \neq 0$ it follows $b = 0$. We conclude again that $\beta$ is of infinite order, as described previously.

**Claim** If $n$ is even and $\alpha$ has infinite order, then the element $\alpha'$ will be eliminated.
In this situation we have $\alpha = \tilde{\alpha} + i_*(\gamma)$, where $\tilde{\alpha}$ is a torsion element and $\gamma \in H_n(M)$. The normal bundle map is given by

$$\nu : H_n(T) \to \pi_{n-1}(SO(n+1)) = \pi_{n-1}(SO).$$

If $\pi_{n-1}(SO) = 0$ or $\mathbb{Z}$ the fact that the map $\nu : H_n(T) \to \pi_{n-1}(SO)$ is a homomorphism implies, that $\tilde{\alpha}$ already has a trivial normal bundle, this leads us in the situation of the last claim. It remains to study the situation $\pi_{n-1}(SO) = \mathbb{Z}/2$ with $\nu(\tilde{\alpha}) = 1$. Observe that without loss of generality we can assume $i_*(\gamma)$ to be primitive. Thus the map $H_{n+1}(T) \xrightarrow{\alpha} \mathbb{Z}$ from the sequence of the last claim is surjective and we obtain

$$H_n(T') \cong H_n(T)/\langle \alpha \rangle \Rightarrow H_n(T', M) \cong H_n(T, M)/\langle \alpha' \rangle.$$

**Claim** If $n$ is odd then the torsion group $\text{Tor}(H_n(T, M))$ can be reduced.

The group $\pi_n(SO_{n+1})$ acts on the trivializations via

$$(\omega, \varphi) \mapsto \psi : S^n \times D^{n+1} \to T$$

$$(x, y) \mapsto \varphi(x, \omega(x)y)$$

Note, that the change of trivialization does not affect the $B$-structure on $S^n \times D^{n+1}$ since the induced $B$-structures $\varphi^*\tilde{\nu}$ and $\psi^*\tilde{\nu}$ differ by an element from $\pi_n(F_k) = 0$.

Let $y_0$ be the basis point of $S^n$, then we see

$$\psi_0(x, y_0) = \varphi_0(x, \omega(x)(y_0)) = \varphi_0(x, pw(x)) = \varphi_0(\text{id} \times p\omega)\Delta(x),$$

where the map $p : SO(n+1) \to S^n$ is the canonical fiber bundle and $\Delta : S^n \to S^n \times S^n$ the diagonal. Denote by $i_l : S^n \to S^n \times S^n$ the inclusion in the $l$-th component and let $\iota$ be the generator of $\pi_n(S^n)$, set $i_l := (i_l)_*\iota$. We pass to homotopy and use the fact that the map $\pi_*\delta : \pi_{n+1}(S^{n+1}) \to \pi_n(S^n)$ from the long exact homotopy sequence for $p$ is given by multiplication with 2 [Ste]. Thus we compute

$$(\psi_0)_*(i_1) = (\varphi_0)_*(\text{id} \times p\omega)_*\Delta_*(\iota)$$

$$= (\varphi_0)_*(\text{id} \times p\omega)_*(i_1 + i_2)$$

$$= (\varphi_0)_*(i_1 + 2k \cdot i_2)$$

$$= (\varphi_0)_*(i_1) + 2k \cdot (\varphi_0)_*(i_2)$$

Since $M$ is $(n - 1)$-connected we obtain the corresponding statement in homology:

$$(\psi_0)_*([S^n] \otimes 1) = (\varphi_0)_*([S^n] \otimes 1) + 2k \cdot (\varphi_0)_*(1 \otimes [S^n])$$
The equality (*) now becomes
\[ b \cdot (\psi_0)_* (1 \otimes [S^n]) = a \cdot ((\psi_0)_* ([S^n] \otimes 1) - 2k(\psi_0)_* (1 \otimes [S^n])) \]
\[ a \cdot (\psi_0)_* ([S^n] \otimes 1) = (b + 2ka)(\psi_0)_* (1 \otimes [S^n]) \]
Since the case \( b + 2ka = 0 \) has already been treated, we assume \( b + 2ka \neq 0 \).
By choosing \( k \) appropriately we can achieve
\[ o(\beta') \leq o(\alpha'). \]
Let \( p \) be a prime number such that \( (\alpha'_p) \neq 0 \) in \( H_n(T, M; \mathbb{F}_p) \).
From the analogous sequences with \( \mathbb{F}_p \)-coefficients we conclude
\[ H_n(T', M; \mathbb{F}_p) \cong H_n(T_0, M; \mathbb{F}_p)/\text{im } d' \cong H_n(T, M; \mathbb{F}_p)/<\alpha'> \]
and with universal coefficient theorem
\[ |\text{Tor}(H_n(T', M))| < |\text{Tor}(H_n(T, M))|. \]
Combining all cases together we see, that a torsion element of \( H_n(T, M) \) can either be eliminated by a finite sequence of surgeries or can be replaced by an element of infinite order. Inductively we obtain the desired statement. \( \square \)

6.5 Proof of Theorem 10

**Proof** We only have to prove that every manifold normally \( B_k \) bordant to a homology sphere is elementary.

Let \( T \) be a bordism between \( M \) and a homology sphere \( S \). According to Theorem 6, we can without loss of generality assume that \( T \) is \((n-1)\)-connected and that the homology groups \( H_n(T) \) and \( H_{n+1}(T) \) are free. The long exact sequence of the pair \((T, M)\) together with Poincaré duality leads to the following commutative diagram:

\[ \begin{array}{ccc}
H_{n+1}(T) & \xrightarrow{\delta} & H_n(T, M) \xrightarrow{i_*} H_n(T) \xrightarrow{j_*} H_n(T, M) \\
\downarrow p-D. & & \downarrow p-D. \\
H^n(T, M) & \xrightarrow{d^*} & H^n(T) \xrightarrow{i^*} H^n(M) \xrightarrow{d} H^{n+1}(T, M)
\end{array} \]

From this we see
\[ \dim H_n(M) = 2(\dim \ker i_*). \]

Thus \( \ker i_* \) is a direct summand of \( H_n(M) \). It is not hard to verify, that \( \ker i_* \) is isotropic. Let now \( \lambda : S^n \hookrightarrow M \) be a representative of an element of \( \ker i_* \), then the map can be extended to \( \bar{\lambda} : D^n \rightarrow T \) and with the help of the Whitney-trick \([Hae]\) this map can be assumed to be an embedding. Thus the restriction of the normal bundle of \( D^n \) to the boundary gives us a trivialization of the normal bundle of \( \lambda \). \( \square \)
6.6 Proof of Theorem 11

Proof Using the statement of Theorem 10 we see, that the conditions of Theorem 6 are fulfilled for the manifolds $\tilde{U} \Sigma_i \sharp S$ and $\tilde{U} \Sigma'_i \sharp S$ for a homotopy sphere $S$. Thus, the diffeomorphism on the boundary can be extended to the the diffeomorphism $\tilde{U} \Sigma_i \sharp S \sharp k(S^n \times S^n) \rightarrow \tilde{U} \Sigma'_i \sharp S \sharp k(S^n \times S^n)$ for $k \in \{0, 1\}$. From the definition of resolutions and from construction the diffeomorphism on the boundaries we see that the obtained diffeomorphisms extend in the obvious way to a diffeomorphism $\tilde{U} \Sigma_i \sharp S \sharp k(S^n \times S^n) \rightarrow \tilde{U} \Sigma'_i \sharp S \sharp k(S^n \times S^n)$ having the desired properties.

It remains to show, that the conditions of the theorem are true for every pair of representatives of the neighbourhood germs $[\tilde{U} \Sigma_i]$ and $[\tilde{U} \Sigma'_i]$, once we have checked them on a single representative pair. If $c_i : L_i \times [0, \varepsilon_i/2] \rightarrow N$ and $d_i : L_i \times [0, \varepsilon'_i/2] \rightarrow N$ with $\varepsilon_i < \varepsilon'_i$ are two representatives of the germ of collars around $L_i$, then there exists a $\delta_i > 0$ such that $c_i$ coincides with $d'_i$ on $L_i \times [0, \delta_i]$. We choose a diffeomorphism $\eta_i : [0, \varepsilon_i/2] \rightarrow [0, \varepsilon'_i/2]$ with $\eta_i \equiv \text{id}$ on $[0, \delta_i]$. The map induces an isomorphism $\tilde{U} \Sigma_i c \rightarrow \tilde{U} \Sigma_i d$ making the following diagram commutative:

This completes the proof.

6.7 Algebraic invariants

ad (2) Let $(H, \Lambda, s\nu_\ast)$ be elementary and $\mathcal{L} = \langle \lambda_1, \ldots, \lambda_k \rangle$ a Lagrangian with $s\nu_\ast|\mathcal{L} \equiv 0$. Thus $\text{sign}(\Lambda) = 0$ and $0 = s\nu_\ast(\lambda_i) = \Lambda(\kappa_{s\nu_\ast}, \lambda_i)$ for all $1 \leq i \leq k$. Since $\mathcal{L}$ is maximal it follows that $\kappa_{s\nu_\ast} \in \mathcal{L}$ and therefore $s\nu_\ast(\kappa_{s\nu_\ast}) = 0$.

On the other hand let $\text{sign}(\Lambda) = 0$ and $s\nu_\ast(\kappa_{s\nu_\ast}) = 0$. Choose a basis $\{\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k\}$ of $H$ such that $\Lambda(\lambda_i, \lambda_j) = 0$ and $\Lambda(\lambda_i, \mu_j) = \delta_{ij}$.

There is nothing to show if $\kappa_{s\nu_\ast} = 0$. Otherwise we can without loss of generality assume that $s\nu_\ast(\lambda_i) = 0$ for all $i > 1$, since $s\nu_\ast$ is a homomorphism.
Recall the equality $s\nu_*(v) = \Lambda(\kappa_{s\nu_*}, v) \forall v \in H$. Since $\kappa_{s\nu_*} \in H$ there are $a_i, b_i \in \mathbb{Z}$ such that $\kappa_{s\nu_*} = \sum_{i=1}^{k}(a_i\lambda_i + b_i\mu_i)$. Consider a sub-Lagrangian $L' := \langle \lambda_2, \ldots, \lambda_k \rangle$. If $\kappa_{s\nu_*} \not\in L'$, build $L := \langle \lambda_2, \ldots, \lambda_k, \tilde{\kappa}_{s\nu_*} \rangle$, where $\tilde{\kappa}_{s\nu_*}$ is a primitive element of $H$ with $\kappa_{s\nu_*} \in \langle \tilde{\kappa}_{s\nu_*} \rangle$. This is a Lagrangian, satisfying $s\nu_*|_L \equiv 0$. In the case of $\kappa_{s\nu_*} \in L'$, the coefficients $a_1, b_1, \ldots, b_k$ have to be zero, thus $L = \langle \lambda_1, \ldots, \lambda_k \rangle$ is a Lagrangian with the desired property.

ad (3) The conditions are obviously necessary. To see that they are also sufficient choose a symplectic basis $\{\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k\}$ of $H$. Sort the generators in the following way

$$
\begin{align*}
\nu_*(\lambda_i) &= \nu_*(\mu_i) = 1 & \text{for } i \leq s, \\
\nu_*(\lambda_i) &= 0 & \text{for } i > s,
\end{align*}
$$

where $s$ is an integer between 0 and $k$. The assumption

$$
\Phi(H) = \sum_{i=1}^{k} s\nu_*(\lambda_i)s\nu_*(\mu_i) = 0
$$

implies that $s \equiv 0(\text{mod } 2)$. Construct a new basis $\{\lambda'_1, \ldots, \lambda'_k\}$ for $H$ by the substitution

$$
\begin{align*}
\lambda'_{2i-1} &= \lambda_{2i-1} + \lambda_{2i}, & \lambda'_{2i} &= \mu_{2i-1} - \mu_{2i}, \\
\mu'_{2i-1} &= \mu_{2i} & \mu'_{2i} &= \lambda_{2i}
\end{align*}
$$

for $2i \leq s$, and

$$
\begin{align*}
\lambda'_i &= \lambda_i & \mu'_i &= \mu_i
\end{align*}
$$

for $i > s$. This new basis is again symplectic and satisfies the condition

$$
\nu_*(\lambda'_1) = \cdots = \nu_*(\lambda'_k) = 0.
$$

### 6.8 Proof of Theorem 19

**Proof** As in the proof of Theorem 11 we conclude that it is enough to show that the diffeomorphism on the boundary $\partial \bar{U}_i \Sigma_i \longrightarrow \partial \bar{U}_i' \Sigma_i'$ can be extended to a homeomorphism on $\bar{U}_i \Sigma_i + k(S^2 \times S^2) \longrightarrow \bar{U}_i' \Sigma_i + k(S^2 \times S^2)$. Since the resolutions are optimal, the manifolds $\bar{U}_i \Sigma_i$ and $\bar{U}_i' \Sigma_i$ are 1-connected, hence $M := \bar{U}_i \Sigma_i \cup_{\partial} \bar{U}_i' \Sigma_i$ is again 1-connected. In order to apply Theorem 18 we have to show that the closed 4-dimensional spin manifold with vanishing signature is bordant to a homotopy sphere. Then we apply the topological 4-dimensional Poincaré conjecture proved by Freedman [F, Thm. 1.6] and obtain the desired statement.
We want to use surgery to prove that $M$ is bordant to a homotopy sphere. Since $M$ is spin and sign($M$) = 0, there is a basis $\{\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k\}$ of $H_2(M)$ satisfying
\[ \Lambda(\lambda_i, \lambda_j) = 0 \quad \Lambda(\mu_i, \mu_j) = 0 \quad \Lambda(\lambda_i, \mu_j) = \delta_{ij}. \]
We can not use the Haefliger’s embedding theorem in dimension 4, but according to [F, Thm. 3.1, 1.1] every generator $\lambda_i$ is represented by a topological embedding $S^2 \times D^2 \hookrightarrow M$. Knowing this we can proceed in exactly the same way as in the proof of Lemma 23, working in the category TOP.

Lemma 24 is still valid in dimension 4 as well as the arguments in §6.6.

To complete the proof observe that according to arguments from §6.6 it is enough to check the conditions for a single representative of the neighbourhood germ.

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