MUTUAL SELECTION IN NETWORK EVOLUTION: THE ROLE OF THE INTRINSIC FITNESS

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We propose a new mechanism leading to scale-free networks which is based on the presence of an intrinsic character of a vertex called fitness. In our model, a vertex \( i \) is assigned a fitness \( x_i \), drawn from a given probability distribution function \( f(x) \). During network evolution, with rate \( p \) we add a vertex \( j \) of fitness \( x_j \) and connect to an existing vertex \( i \) of fitness \( x_i \) selected preferentially to a linking probability function \( g(x_i, x_j) \) which depends on the fitnesses of the two vertices involved and, with rate \( 1 - p \) we create an edge between two already existed vertices with fitnesses \( x_i \) and \( x_j \), with a probability also preferential to the connection function \( g(x_i, x_j) \). For the proper choice of \( g \), the resulting networks have generalized power laws, irrespective of the fitness distribution of vertices.

Keywords: Complex networks; scale-free networks; fitness.

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Complex networks are powerful tools to describe a large variety of biological, social, and technical networks. A network is a mathematical object which consists of vertices connected by edges. Despite differences in their nature, many real-world networks are characterized by similar topological properties, in contrast to those obtained by traditional random graphs. One of the most interesting phenomena is the scale-free (SF) behavior, which means a power-law distribution of connectivity, \( P(k) \sim k^{-\gamma} \), where \( P(k) \) is the probability that a vertex in the network is of degree \( k \) and \( \gamma \) is a positive real number determined by the given network. In order to understand how SF networks arise, much work has been done in the past decade. It has been shown that growth and preference seem to be the principal mechanisms for SF behavior.

The exploring the preference can be directed in two classes. The first class of research is based on the rich-get-richer rule, which was implemented by newcomers preferential connecting to old vertices with certain topological characteristics. In the best known Barabási-Albert (BA) Model, the net-
work grows at a constant rate and new vertices attach to old ones with probability
\( \Pi(i) \sim k_i \). In this way, vertices of high degree are more likely to receive further
dges from newcomers. In fact, this extreme assumption is not always available for
many networks when their sizes are huge. The similar mechanism was also used
in weighted networks, for instance, the network grows at a constant rate and new
vertices attach to old ones with probability \( \Pi(i) \sim s_i \), where \( s_i = \sum_{j \in U(i)} w_{ij} \)
is the strength of vertex \( i \) and the sum runs over the set \( U(i) \) of neighbors of
\( i \).4 The second class of research utilizes the fit-get-richer mechanism, which was
carried out by newcomers preferential connecting to old vertices with high intrinsic
fitnesses.7, 8, 9, 10, 11, 12 This is better adapted to model certain networks where topo-
logical properties are essentially determined by “physical” information intrinsically
related to the role played by each vertex in the network, such as the ability of an
individual, the content of a web page, or the innovation of a scientific article.

Caldarelli et al recently introduced a varying vertex fitness model9 where they
consider an undirected graph of \( N \) vertices. At every vertex \( i \) a fitness \( x_i \), which is
a real number measuring its importance or rank, is assigned. Fitnesses are random
numbers taken from a given probability distribution \( f(x) \). For every couple of ver-
tices, \( i \) and \( j \), an edge is created with probability \( g(x_i, x_j) \) (a symmetric function
of its arguments) depending on the “importance” of both vertices, i.e., on \( x_i \) and
\( x_j \). Actually this is a natural generalization of the classic Erdős-Rényi graph.13 Al-
though it is a static model, the network recovers the power-law behavior of degree,
betweenness, and clustering coefficient.9 On the other hand, Bedogne and Rodgers
proposed a growing network with intrinsic vertex fitnesses.12 Besides employing
the edge-created mechanism suggested in Ref.9 they also considered two cases of
new vertices connecting to old ones, uniform or degree-preferential. The interplay
between the fitness linking mechanism and uniform attachment results in an ex-
ponential degree distribution for any fixed fitness \( x \), while the degree-preferential
attachment instead induces that the degree distribution decays as a power law.12

Models of the first class often present us such an evolution picture: old vertices
are passively attached by newcomers according to the degree- (strength-) preferen-
tial mechanism. On the contrary, models belongs to the second class pay much at-
tention to the creation and reinforcement of internal connections. Combining above
two aspects, we argue that the connection between two vertices is the result of their
mutual affinity and attachment. Not only for interactions among new vertices and
old ones, but also for that among old vertices, which we call “mutual selection”. Mo-
tivated by this, we suggest an evolving network model ruled by the fitness-dependent
selection dynamics. The generated network has a good right-skewed distribution of
degrees.

The present model starts from an initial \( m \) isolated seeds and each vertex \( i \) is
endowed with a fitness \( x_i \geq 0 \), drawn from a given probability distribution \( f(x) \).
At each time step, we perform either of the following two operations. (i) With
rate \( p \in (0, 1) \) we add a new vertex \( j \) of fitness \( x_j \in f(x) \) to the network. The
new vertex connects to an existing vertex \( i \) of fitness \( x_i \) selected preferentially to
a linking probability function \( g(x_i, x_j) \) which is symmetric and dependent on the associativity of the both vertices. (ii) With rate \( 1 - p \) we create an edge between two vertices, \( i \) and \( j \), already presented in the network with the probability also preferential to their integration \( g(x_i, x_j) \). After \( t \) time steps, this scheme generates a network of \( m + pt \) vertices and \( t \) links. Notice that either process is chosen in the network growth, only one edge is added to the system at each time step (duplicate and self-connected edges are forbidden), however, this is not essential.

In our model, each vertex is assigned a fitness, either initial seeds or subsequent newcomers. Denoting \( N_k(x, t) \) the average number of vertices with degree \( k \) and fitness \( x \) at time \( t \), we can write out the rate equation for network evolution

\[
\frac{\partial N_k(x, t)}{\partial t} = p \int_0^\infty f(x')g(x, x')[N_{k-1}(x, t) - N_k(x, t)]dx' 
+ p\delta_{k,1}f(x) 
+ 2(1-p)[N_{k-1}(x, t) - N_k(x, t)] 
\times \sum_{k=0}^\infty \int_0^\infty \int_0^\infty g(x, x')N_k(x', t)dx' N_k(x, t)dx.
\]  

The first term on the right hand side (rhs) of Eq. (1) represents the change in the average number of the vertices with degree \( k \) and fitness \( x \) due to process (i). The second term on the rhs accounts for the continuous introduction, with rate \( p \), of new vertices with fitnesses drawn from the probability distribution \( f(x) \). The last term on the rhs represents the change in the average number of the vertices with degree \( k \) and fitness \( x \) due to process (ii). We also define

\[
N(x, t) = \sum_{k=0}^\infty N_k(x, t),
\]

and

\[
N(t) = \int_0^\infty N(x, t)dx,
\]

as the average number of the vertices of fitness \( x \) at time \( t \) and the average number of the vertices at time \( t \), respectively. Summing Eq. (1) over \( k \) we obtain

\[
\frac{\partial N(x, t)}{\partial t} = pf(x),
\]

which yields

\[
N(x, t) = pf(x)t + mf(x).
\]

Integrating Eq. (5) we find as expected \( N(t) = pt + m \), and therefore Eq. (5) can be rewritten as

\[
N(x, t) = f(x)N(t).
\]
Now we can rewrite the integrals in the third term on the rhs of Eq. (1) in terms of $f$ and $g$

$$\sum_{k=0}^{\infty} \int_{0}^{\infty} g(x, x')N_k(x', t)dx'$$

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \int_{0}^{\infty} g(x, x')N_k(x', t)dx'N_l(x, t)dx$$

$$= \frac{1}{N(t)} \int_{0}^{\infty} \int_{0}^{\infty} g(x, x')f(x')dx'$$

$$= \frac{A(x)}{N(t)},$$

where

$$A(x) = \frac{\int_{0}^{\infty} g(x, x')f(x')dx'}{\int_{0}^{\infty} g(x, x')f(x')dx'}. (8)$$

Furthermore, we assume that

$$B(t) = \int_{0}^{\infty} f(x') \sum_{k=0}^{\infty} \int_{0}^{\infty} g(x, x')N_k(x, t)dx dx'$$

and the differential of which reads

$$\frac{\partial B(t)}{\partial t} = p \int_{0}^{\infty} f(x') \int_{0}^{\infty} g(x, x')f(x)dx dx' = C, (10)$$

where $C$ is a constant. Thus $N_k(x, t)$ grows linearly with $t$, and we introduce the function $D_k(x)$ such that

$$N_k(x, t) = D_k(x)t. (11)$$

The degree distribution of vertices can be estimated from $D_k(x)$ instead. Substituting Eqs. (7) and (11) into Eq. (1) gives the recursive equation for $D_k(x)$

$$D_k(x) = \frac{p \int_{0}^{\infty} f(x')g(x, x')dx' + \frac{2(1-p)}{p}A(x)}{p \int_{0}^{\infty} f(x')g(x, x')dx' + \frac{2(1-p)}{p}A(x) + 1}D_{k-1}(x)$$

$$+ \frac{p \delta_{k,1}f(x)}{p \int_{0}^{\infty} f(x')g(x, x')dx' + \frac{2(1-p)}{p}A(x) + 1}. (12)$$

By defining

$$H(x) = \frac{p}{c} \int_{0}^{\infty} f(x')g(x, x')dx' + \frac{2(1-p)}{p}A(x), (13)$$

we can rewrite Eq. (12) as

$$D_k(x) = \frac{H(x)}{H(x) + 1}D_{k-1}(x) + \frac{p \delta_{k,1}f(x)}{H(x) + 1}, (14)$$

which can be solved recursively

$$D_k(x) = \frac{pf(x)H^{k-1}(x)}{[H(x) + 1]^{k}}. (15)$$
The result demonstrates that for every fixed $x$, the degree distribution of the generated network should follow the right-skewed behavior. Moreover, the mutual selection rule presented here brings on the proportionality of the vertex degree to its fitness, which means that $H(x)$ is an implicit function of $k$. Thus, given proper forms of the linking probability function $g(x_i, x_j)$, one can construct networks with power-law degree distributions.

Fig. 1. (color online) Degree distributions of vertices of the generated networks for different fitness distribution functions: uniform (a), exponential $f(x) = e^{-x}$ (b), and power-law $f(x) = x^{-3}$ (c). The linking probability function is $g(x_i, x_j) = x_i x_j$. Each plot corresponds to one experiment of network generation with parameters $N = 10^5$ and $m = 10$. 
To test above argument, we present computer simulations of the model, as shown in Fig. 1. We choose the simplest case $g(x_i, x_j) = x_i x_j$ and plot degree distributions for three kinds distribution functions of vertex fitnesses: uniform, exponential, and power-law. Even for this basic form of $g$, one can still notice the generalized power laws of the degree distribution in all cases, in agreement with analytical predictions.

In summary, we have presented an simple model to justify the ubiquity of SF networks in nature, which results from the mutual selection rule based on a symmetric linking probability function $g(x_i, x_j)$ dependent on the affinity of the intrinsic fitnesses of the involved vertices, $i$ and $j$. We found that it is always possible to find a proper form of $g$ so that the generated network is SF in spite of the fitness distribution. In case that the values of vertex degrees are not available, we believe that the present model is relatively suitable.

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