Some Questions Around The Hilbert 16th Problem

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Hilbert sixteenth problem asks for existence of uniform upper bound $H(n)$ for the number of limit cycles of a planar polynomial vector field of degree $n$. It is proved in [3] that every polynomial vector field of degree 2 has at most 4 limit cycles. It is well known that the number of limit cycles of a polynomial vector field on the plane is finite, see [5].

Using polar change of coordinates we replace the concept of limit cycle with $2\pi$-periodic solution $(r(t), \theta(t))$ for the corresponding vector field in $(r, \theta)$.

More generally consider the equation

$$Z' = a_n(t)Z^n + a_{n-1}(t)Z^{n-1} + \ldots$$

where $Z' = dZ/dt$ and $a_i(t)$ are $c^1$ functions of real variable $t$.

For a fixed $t$, put $U(t)$ for the set of all $z$ in $\mathbb{C}$ with the property that $\phi_t$ can be defined at $(Z,0)$, where $\phi$ is the flow of corresponding autonomous system, adding $t' = 1$ to the equation. $U(t)$ is an open simply connected subset of $\mathbb{C}$. Furthermore $\phi_t$, as a map from $U(t)$ to $\mathbb{C}$, is a holomorphic function. For if $a_i$ is not analytic we approximate it with analytic functions by some Weierstrass type approximation theorem. Then we note that the uniform limit of holomorphic functions is necessarily a holomorphic function. This shows that for $n > 1$ the above system is not a complete Vector field. Because any one to one entire function must be linear $\phi_t(Z) = a(t)Z + b(t)$.

Differentiating in $t$ implies that $n = 1$ Similarly if $\phi_t$ is a mobious function then $n$ can be at most two This shows for $n > 2$ the argument of Smale described in [8] is not applicable. Since every continuous function from sphere to sphere is holomorphic if it is holomorphic on an open and dense subset of sphere whose complement has zero Lesbegue measure. On the other hand every fixed point of a holomorphic function has non negative Lefschetz number. In fact for $n = 3$ there are examples of autonomous equation $z' = f(z)$ for which there are two isochronous centers with different
period $T_1$ and $T_2$ so $\phi_T$ can not be extended even continuously to whole sphere.

The "complexification" of the Hilbert 16th problem is an elegant and subtle idea but in some cases is not effective. In this note we suggest some different points for consideration of limit cycle problem: 

1) Let $[X, Y] = 0$ and $\gamma$ be a limit cycle for $X$ then $\gamma$ must be invariant under $Y$, namely $X$ and $Y$ share on limit cycles. Since for every positive function $f$, the two vector fields $X$ and $fX$ have the same trajectories, it is natural and interesting to compare $C(X)$ with $C(fX)$. By $C(X)$, centralizer of $X$, one means all vector fields $Z$ with $[X, Z] = 0$. Note that locally around a non singular point of $X$, $C(X)$ and $C(fX)$ are isomorphic lie algebras. This local fact is no longer true globally (for a non vanishing Vector field on arbitrary surface), and it may be false around a singularity of a vector field. For example, put $f(x, y) = x^2 + y^2 + 1$ and vector field $X$ as follow 

$$
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x,
\end{align*}
$$

$X$ is a non vanishing vector field on $\mathbb{R}^2 \setminus \{0\}$, on the other hand $X$ is a vector field with singularity at origin. In both cases, two Lie algebras $C(X)$ and $C(fX)$ are not isomorph since the operation of lie bracket is zero in $C(fX)$, but it is not the case in $C(X)$.

It is also interesting that one look at the Hilbert sixteen problem in a non analytic but smooth manner, for example consider the following question:

Let $L$ be the Lienard polynomial vector field

$$
\begin{align*}
\dot{x} &= y - F(x) \\
\dot{y} &= -x,
\end{align*}
$$

without center and $S$ be a smooth vector field with $[L, S] = 0$, is it necessarily $S = kL$ for some constant $k$?

**Remark** Non triviality of centralizer of non integrable vector field $X$ with components $(P, Q)$ is equivalent to complete integrability of Hamiltonian $zP + wQ$ in $\mathbb{R}^4$
Example 1 Consider vector fields $X$ and $Y$ as follows:

$$(X) \begin{cases} \dot{x} = y + x(x^2 + y^2 - 1) \\ \dot{y} = -x + y(x^2 + y^2 - 1), \end{cases}$$

and

$$(Y) \begin{cases} \dot{x} = 2y + x(x^2 + y^2 - 1) \\ \dot{y} = -2x + y(x^2 + y^2 - 1), \end{cases}$$

$X$ and $Y$ are independent out of circle $x^2 + y^2 = 1$. This circle is a hyperbolic limit cycle for $X$ and $Y$ while we have $[X,Y] = 0$.

2) We choose an arbitrary homogeneous vector field of degree $n + 1$. For example:

$$\begin{cases} \dot{x} = y(x^2 + y^2)^{\frac{n}{2}} \\ \dot{y} = -x(x^2 + y^2)^{\frac{n}{2}}, \end{cases}$$

The cyclicity of origin under perturbation of this vector field among polynomial vector fields of degree at most $n$ is not less than $H(n)$. Now put $X$ for above vector field and $Y$ for the following linear center

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x, \end{cases}$$

$[X,Y] = 0$. Is it possible that for any perturbation $X_\epsilon$ of $X$, one have perturbation $Y_\epsilon$ with $[X_\epsilon,Y_\epsilon] = 0$?

Note that linear center has finite cyclicity by analytic perturbation with finite parameter. (The cyclicity of a singularity of a vector field among a family of vector fields is the maximum number of limit cycles which can be produced around the singularity with small perturbation of the vector field in the family, see [7].)

3) Let we have a vector field in $\mathbb{R}^n$ whose two first components depend only on $x$ and $y$. Actually we have a planar vector field with these two components. Existence of invariant compact submanifold of codimension at most 2 could lead to existence of closed orbit in planar system. Since projection of such submanifold on two first components can not be a single point.
4) Any Vector field on a surface defines a singular foliation of dimension one and according to definition in [9] every limit cycle is considered as a separatrix. It would be interesting to produce some $C^*$ algebraic invariants depending only on degree of polynomial Vector fields in the plane.

5) In [2] it is given a uniform upper bound, depending only on $n$, for the length of a closed orbit of a polynomial vector field of degree $n$. This computation is based on the Riemannian metrics of the upper half sphere. On the other hand it is well known that the number of closed geodesics of a surface with negative curvature is uniformly bounded by the length of the closed geodesics. Inspired by these two concepts we ask: Can we equip the phase portrait of a certain polynomial vector field with an appropriate Riemannian metric whose curvature is negative out of a finite number of analytic curves such that the trajectories of our vector field would be unparameterized geodesics. In particular is it possible to equip the punctured plane to a Riemannian metric with negative curvature such that the trajectories of the vander pol equation would be unparameterized geodesics. If the answer to the latter question is positive then we could give another proof for the fact that the van der pol equation can not have more than one limit cycle, see [4] for information on the limit cycle of the vander pol equation. For a subtle relation between limit cycles and complex geometry, see [6].

6) A possible relation to operator theory: In the following arxiv note we interpreted the number of limit cycles of the Lienard vector field $L$ in terms of codimension of the range of functional operator defined by $L$ with $L(g) = L.g$, the derivative of $g$ along the trajectories of $L$. See Counting limit Cycles via Index theory. In fact if this operator happen to be bounded and has a closed range with respect to an appropriate norm, then we would have a "Fredholm index" interpretation for the number of limit cycles of our vector field. What Banach Functional space is appropriate for the domain of the operator $L(g) = L.g$, such that the operator would be a Fredholm operator whose index is equal to the number of limit cycles? Can we equip the space of smooth or analytic maps on the plane with the structure of a topological vector space such that the corresponding operator would be a Fredholm operator and the same Fredholm index interpretation would be still valid? (I thank A. Zeghib for his suggestion for consideration of TVS as a possible resolution to this problem). Is the generalization of the theory of "Fredholm Operators on Banach space" for TVS a trivial problem?

Finally we give two questions related to subject 1:
Question 1  For an analytic Vector field $X$ on the plane or sphere let $C_\omega(X)$ be the space of all real analytic vector fields $Y$ with $[X,Y] = 0$. Assume that $C_\omega(X)$ is an infinite dimensional Lie algebra. Does this imply that there is a non constant analytic function $f$ globally defined on the plane or sphere with $X.f = 0$?

Question 2  For every non vanishing vector field $X$ on the plane and positive smooth function $f$, is the centralizer $C(X)$ of $X$ isomorph to $C(fX)$. If the the answer is affirmative we could actually assign a unique (up to isomorphism) Lie algebra to every smooth foliation of the plane: We choose a generating vector field $X$ then we introduce $C(X)$ as the required Lie algebra.

Example 2  Let an analytic vector field $X$ on sphere has a center and a limit cycle simultaneously. Then its analytic centralizer $C(X)$ consisting of all analytic vector field $Y$ with $[Y,X] = 0$ has dimension one. Let $V$ be the Van der pol then $C(V)$ has at most two dimension. (What is the exact dimension of centralizer of Van der pol vector field?). In the example 1 $C(X)$ is a 2 dimensional lie algebra while the centralizer of the following vector field(as a vector field on the plane or sphere) is a 4 dimensional space

$$\begin{cases}
\dot{x} = x \\
\dot{y} = y,
\end{cases}$$

Complex dilatation and Limit cycle theory

In this current version of our note we add this section which is essentially the same as the materials in the following Mathoveflow post:

Let $X = P \partial_x + Q \partial_y$ be a vector field on $\mathbb{R}^2$. Assume that we have

$$P_x P_y + Q_x Q_y = 0 \quad (1)$$

Does this imply that the vector field $X$ is a divergence-free vector field with respect to a Riemannian metric defined on $\mathbb{R}^2 \setminus S$ where $S$ is the set of singularities of $X$? Obviously this would imply that any $X$ with $P_x P_y + Q_x Q_y = 0$ can not have any limit cycle. But is it really the case? Namely:

Question 3  Does the equality $P_x P_y + Q_x Q_y = 0$ imply that $X = P \partial_x + Q \partial_y$ does not have any limit cycle?
In the following we shall introduce some motivations for the above question:

For a complex function \( f = U + iV : \mathbb{C} \to \mathbb{C} \) we recall the definition of the operators \( f \mapsto \partial f, \ f \mapsto \bar{\partial} f \) and the dilatation \( \mu(f) \) as follows:
\[
\bar{\partial} f = (U_x - V_y) + i(U_y + V_x), \quad \partial f = (U_x + V_y) + i(V_x - U_y) \quad \mu(f) = \frac{\partial f}{\bar{\partial} f}.
\]

To our vector field \( X = P \partial_x + Q \partial_y \) we associate the complex function \( X = P(z) + iQ(z) \). Then we consider the complex dilatation of \( X \) with \( \mu(z) = \bar{\partial}X(z) / \partial X(z) \).

Let \( \phi_t \) be the flow of the vector field \( X \). So with a similar argument for proof of the standard variational equation in [4, page 299] and commutativity of operators \( \partial \phi_t \) and \( \bar{\partial} \phi_t \) with \( d/dt \) we arrive at the following differential equations
\[
\begin{align*}
(\partial \phi_t)' &= \partial X(\phi_t(x)) \partial \phi_t + \bar{\partial} X(\phi_t(x)) \bar{\partial} \phi_t \\
(\bar{\partial} \phi_t)' &= \bar{\partial} X(\phi_t(x)) \partial \phi_t + \partial X(\phi_t(x)) \bar{\partial} \phi_t 
\end{align*}
\]

This imply that \( \mu(\phi_t)'(x) = \frac{\partial X(\phi_t(x))}{(\partial \phi_t)^2} \text{exp} \int_0^t \text{div} X(\phi_t(x)) dt \)

**Proposition** For a vector field \( X = P \partial_x + Q \partial_y \) if the dilatation \( \mu(X(z)) = \lambda \) is a constant map in \( z \) with \( |\lambda| = 1 \) then \( X \) does not have any limit cycle.

**Proof** Assume that \( \mu(X(z)) = \lambda \) for a fixed complex number \( \lambda \) with \( |\lambda| = 1 \). So after a linear change of coordinate \( H(z) = (\beta)z \) where \( \beta \) is a complex number with \( \beta^2 \lambda = 1 \) then we have \( H^* X(z) = \beta X(z/\beta) \). Since \( H \) is a linear function then it is equal to its linear part. Since both \( H \) and \( H^{-1} \) are holomorphic maps then we have
\[
\mu(H^* (X))(z) = \mu(H^{-1} \circ X \circ H)(z) = \mu(X)(z) \times \left( \frac{H'(z)}{|H'(z)|} \right)^2 = \lambda \cdot 1 / \beta^2 = 1
\]

(2)

For the dilatation of functions after right and left composition with holomorphic maps see [1, page 9]

Note that \( X \) and \( H^* (X) \) are orbitally equivalent vector fields. So if we prove that every vector field \( Y \) with \( \mu(Y(z)) = 1 \) can not have any limit cycle then the proof of the proposition would be completed. But it is an obvious fact: let \( Y = P \partial_x + Q \partial_y \) be a vector field whose dilatation function is identically equal to 1. this implies that \( P_y = Q_y = 0 \). So \( P, Q \) are functions in \( x \). It is an standard fact in the theory of ordinary differential
equation that a one dimensional autonomous vector field can not posses a periodic orbit. We apply this to $x' = P(x)$. So $Y$ has no periodic orbit. An alternative Proof for non existence of periodic orbit for the planar vector field $Y(x,y) = P(x)\partial_x + Q(x)\partial_y$ is the following: We have $[Y,\partial/\partial y] = 0$. If $\gamma$ is a periodic orbit of $Y$, then there is a point $A = (x_A, y_A)$ on $\gamma$ whose $x$-coordinate $x_A$ is maximum on $\gamma$. Then $Y(A)$ is a vertical vector, i.e. $Y(A)$ is parallel to the $y$-axis. So $P(A) = 0$. Then $P(x_A, y) = 0, \forall y \in \mathbb{R}$. This shows that the vertical line $x = x_A$ is invariant under the flow of $Y$. This contradicts the fact that the periodic orbit $\gamma$ intersects this invariant line. In fact this situation violates the uniqueness of solutions of ordinary differential equation.

So a natural question is the following:

**Question 4** Is the above Proposition true without the assumption $|\lambda| = 1$? According to the very process in 2 we may reformulate this question as follows; let we have a vector field $X$ whose dilatation $\mu(X) = \lambda$ where $\lambda$ is a constant real number. Can such a vector field $X$ have any limit cycle? Obviously the columns of the jacobian of any vector field $X$ with the above property are orthogonal to each other. So this situation is a motivation for question 3.

**Some examples showing that questions 3 has possibly an affirmative answer:**

Each of the following vector fields satisfy \([\mathbf{1}]\) and non of them have any limit cycle:

1. A vector field in the form $p(x)\partial_x + Q(y)\partial_y$

2. Every vector field in the form $P(y)\partial_x + Q(x)\partial_y$

3. Every vector field in the form $Z' = f(z)$ where $f = P + iQ$ is a holomorphic map on $\mathbb{C}$

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