A natural extension of a left invariant lower semi-continuous weight

J. Kustermans
Institut for Matematik og Datalogi
Odense Universitet
Campusvej 55
5230 Odense M
Denmark

April 1997

Abstract

In this paper, we describe a natural method to extend left invariant weights on $C^*$-algebraic quantum groups. This method is then used to improve the invariance property of a left invariant weight. We also prove some kind of uniqueness result for left Haar weights on $C^*$-algebraic quantum groups arising from algebraic ones.

Introduction

Possibly the most important object associated to locally compact groups is the left Haar measure. So it is not a great surprise that a major role in the $C^*$-algebraic approach to quantum groups is played by so called left Haar weights. However, there are some different ways to expres the left invariance. We will discuss one of them in this introduction.

Therefore, consider a Hilbert space $H$ and a non-degenerate sub-$C^*$-algebra $B$ of $B(H)$. Let $v, w \in H$, then $\omega_{v,w}$ denotes the element in $B^*$ such that $\omega_{v,w}(x) = \langle x v, w \rangle$ for every $x \in B$.

Consider moreover a comultiplication $\Delta$ on $B$, this is a non-degenerate $^*$-homomorphism from $B$ into $M(B \otimes B)$ which is coassociative: $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.

Probably the weakest interesting form of left invariance is the following one. Let $\varphi$ be a densely defined lower semi-continuous weight on $B$, then we call $\varphi$ weakly left invariant if and only if we have for every $x \in B^+$ with $\varphi(x) < \infty$ and every $v \in H$ that

$$\varphi((\omega_{v,v} \otimes \iota)\Delta(x)) = \langle v, v \rangle \varphi(x).$$

It seems to be an interesting question whether it is reasonable to require that a left invariant weight satisfies also some kind of converse:

Let $\varphi$ be a densely defined lower semi-continuous weight on $B$, then we call $\varphi$ left invariant if and only if $\varphi$ satisfies the following two conditions:

1. $\varphi$ is weakly left invariant

2. Let $x$ be an element in $B^+$. If $\varphi((\omega_{v,v} \otimes \iota)\Delta(x)) < \infty$ for every $v \in H$, then we have that $\varphi(x) < \infty$.

\(^{1}\)Research Assistant of the National Fund for Scientific Research (Belgium)
We prove in the first section that under some extra conditions on \((B, \Delta)\), any weakly left invariant weight has a smallest extension which is left invariant. We would like to mention that C*-algebraic quantum groups according to an upcoming definition of Masuda, Nakagami and Woronowicz will satisfy these extra conditions.

In the last section, we apply the results of the first section to prove two results concerning left invariant weights on C*-algebraic quantum groups arising from algebraic ones (see [7] and [13]):

- We show that the canonical left Haar weight is automatically left invariant in the sense above. As a consequence, we get also that the form of left invariance mentioned above is equivalent with an ever stronger form of left invariance involving the slice C*-valued weight \(\iota \otimes \varphi\).

- We prove that left invariant weights are uniquely determined under some relative invariance condition. A similar result for quantum E(2) can be found in [1].

It seems that the techniques used in this last section can also be applied to the more general case of C*-algebraic quantum groups according to Masuda, Nakagami & Woronowicz.

We should mention that the used terminology of left invariance and weak left invariance is by no means standard. It was only introduced to be able to talk about left invariance.

We end this introductions with some basic notations and results.

For any Hilbert space \(H\), the set of bounded operators on \(H\) will be denoted by \(B(H)\) whereas the set of compact operators on \(H\) will be denoted by \(B_0(H)\).

Consider a C*-algebra \(B\). A norm continuous one-parameter group \(\alpha\) on \(B\) is a group homomorphism from \(\mathbb{R}\) into the group of *-automorphisms on \(B\) such that the mapping \(\mathbb{R} \to B : b \mapsto \alpha_t(b)\) is norm continuous for every \(b \in B\).

Using the theory of analytic functions, we can define for every complex number \(z\) a closed linear operator \(\alpha_z\) from within \(B\) into \(B\). (see e.g. [3])

Let \(\varphi\) be a densely defined lower semi-continuous weight on a C*-algebra \(B\). We will use the following notations:

- \(M^{+}_\varphi = \{ b \in B^+ \mid \varphi(b) < \infty \}\)
- \(\mathcal{N}_\varphi = \{ b \in B \mid \varphi(b^*b) < \infty \}\)
- \(M_\varphi = \text{span } M^{+}_\varphi = \mathcal{N}_\varphi^\ast \mathcal{N}_\varphi\).

A GNS-construction of \(\varphi\) is by definition a triple \((H_\varphi, \pi_\varphi, \Lambda_\varphi)\) such that

- \(H_\varphi\) is a Hilbert space
- \(\Lambda_\varphi\) is a linear map from \(\mathcal{N}_\varphi\) into \(H_\varphi\) such that
  1. \(\Lambda_\varphi(\mathcal{N}_\varphi)\) is dense in \(H_\varphi\)
  2. We have for every \(a, b \in \mathcal{N}_\varphi\) that \(\langle \Lambda_\varphi(a), \Lambda_\varphi(b) \rangle = \varphi(b^*a)\)

Because \(\varphi\) is lower semi-continuous, \(\Lambda_\varphi\) is closed.

- \(\pi_\varphi\) is a non-degenerate representation of \(B\) on \(H_\varphi\) such that \(\pi_\varphi(a) \Lambda_\varphi(b) = \Lambda_\varphi(ab)\) for every \(a \in M(B)\) and \(b \in \mathcal{N}_\varphi\). (The non-degeneracy of \(\pi_\varphi\) is a consequence of the lower semi-continuity of \(\varphi\).)
1 A natural extension of left invariant weights

In this section, we consider a Hilbert space $H$, a non-degenerate sub-$C^*$-algebra $B$ of $B(H)$ and a non-degenerate $^*$-homomorphism $\Delta$ from $B$ into $M(B \otimes B)$ satisfying the following properties:

1. $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$

2. Let $b$ be an element in $M(B)$. If $\Delta(b) = b \otimes 1$, then $b$ belongs to $\mathfrak{U}$.1.

Furthermore, we assume that $(B, \Delta)$ arises from a unitary $W$ on $H \otimes H$, i.e. $W$ is a unitary operator on $H \otimes H$ such that

1. $\Delta(x) = W^*(1 \otimes x)W$ for every $x \in B$.

2. $B$ is the closure of the set $\{(\iota \otimes \omega)(W) \mid \omega \in B_0(H)^*\}$ in $B(H)$.

Consider $v, w \in H$. Then $\omega_{v,w}$ will denote the element in $B^*$ such that $\omega_{v,w}(x) = \langle xv, w \rangle$ for every $x \in B$.

We will moreover assume the existence of a densely defined lower semi-continuous weight $\varphi$ on $B$ such that for every $x \in \mathcal{M}_+^\varphi$ and every $v \in H$ that $(\omega_{v,v} \otimes \iota)\Delta(x)$ belongs to $\mathcal{M}_+^\varphi$ and

$$\varphi((\omega_{v,v} \otimes \iota)\Delta(x)) = \varphi(x) \langle v, v \rangle.$$  

The purpose of this section is to arrive at a natural extension of this weight $\varphi$. We use this extension to prove some interesting results about $C^*$-algebraic quantum groups in the last section.

Notice that the left invariance property implies also that $(\omega_{v,v} \otimes \iota)\Delta(x)$ belongs to $B$ for every $x \in B$ and every $v, w \in H$.

The proof of the following lemma is essentially the same as the proof of lemma 1.4 of [3]. We include it for the sake of completeness.

**Lemma 1.1** Consider $b \in B^+$ such that $(\omega_{v,v} \otimes \iota)\Delta(b)$ belongs to $\mathcal{M}_+^\varphi$ for every $v \in H$. Then there exists a unique element $T$ in $B(H)^+$ such that $\langle Tv, v \rangle = \varphi((\omega_{v,v} \otimes \iota)\Delta(b))$ for every $v \in H$.

**Proof**: By polarisation, we have for every $v, w \in H$ that $(\omega_{v,v} \otimes \iota)\Delta(b)$ belongs to $\mathcal{M}_+^\varphi$. This allows us to define a semi-innerproduct $( , )$ on $H$ such that $\langle v, w \rangle = \varphi((\omega_{v,v} \otimes \iota)\Delta(b))$ for every $v, w \in H$.

Define the subspace $N = \{ v \in H \mid (v, v) = 0 \}$ and turn $H_N$ into a innerproduct space in the usual way. Define $K$ to be the completion of $H_N$.

Next, we denote the quotient mapping from $H$ into $H_N$ by $F$, so $F$ is a linear mapping from $H$ into $K$ such that $\langle F(v), F(w) \rangle = \varphi((\omega_{v,v} \otimes \iota)\Delta(b))$ for every $v, w \in H$. In the next part, we will prove that $F$ is continuous.

Choose a sequence $(v_n)_{n=1}^{\infty}$ in $H$, $v \in H$ and $w \in K$ such that $(v_n)_{n=1}^{\infty} \rightarrow v$ and $(F(v_n))_{n=1}^{\infty} \rightarrow w$.

Take $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\|F(v_n) - F(v_m)\| \leq \varepsilon$ for every $m, n \in \mathbb{N}$ with $m, n \geq n_0$.

This implies that

$$\varphi((\omega_{v_n-v_m,v_n-v_m} \otimes \iota)\Delta(b)) = \|F(v_n) - F(v_m)\|^2 \leq \varepsilon^2 \quad (*)$$

for every $m, n \in \mathbb{N}$ with $m, n \geq n_0$.

Take $l \in \mathbb{N}$ with $l \geq n_0$. Because $(\omega_{v_l-v_m,v_l-v_m} \otimes \iota)\Delta(b))_{m=1}^{\infty}$ converges to $(\omega_{v_l-v_l,v_l-v_l} \otimes \iota)\Delta(b)$, the lower semi-continuity of $\varphi$ and the previous inequality imply that

$$\varphi((\omega_{v_l-v_l,v_l-v_l} \otimes \iota)\Delta(b)) \leq \varepsilon^2.$$
Consequently, \(\|F(v_t) - F(v)\| \leq \varepsilon\).
So we have proven that \((F(v_n))_{n=1}^\infty\) converges to \(F(v)\), which implies that \(F(v) = w\).

Hence, we see that \(F\) is closed implying that \(F\) is continuous by the closed graph theorem.

Define \(T = F^* F\), then \(T\) belongs to \(B(H)^+\) and \(\langle Tv, v \rangle = \langle F(v), F(v) \rangle = \varphi((\omega_{v,v} \otimes \iota)\Delta(b))\) for every \(v \in H\).

We will use the technique of the previous lemma once more in the proof of the next lemma.

**Lemma 1.2** Consider \(b \in B^+\) such that \((\omega_{v,v} \otimes \iota)\Delta(b)\) belongs to \(\mathcal{M}_\varphi^+\) for every \(v \in H\). Let \(T\) be the unique element in \(B(H)^+\) such that \(\langle Tv, v \rangle = \varphi((\omega_{v,v} \otimes \iota)\Delta(b))\) for every \(v \in H\). Then we have for every \(u \in H \otimes H\) that \((\omega_{u,u} \otimes \iota)(\Delta(b)_{23})\) belongs to \(\mathcal{M}_\varphi^+\) and \(\varphi((\omega_{u,u} \otimes \iota)(\Delta(b)_{23})) = \langle (1 \otimes T) u, u \rangle\).

**Proof:** We have by polarisation for every \(v, w \in H\) that \((\omega_{v,w} \otimes \iota)\Delta(b)\) belongs to \(\mathcal{M}_\varphi\) and

\[\varphi((\omega_{v,w} \otimes \iota)\Delta(b)) = \langle Tv, w \rangle.\]

Using this, it is not so difficult to check for every \(x \in H \otimes H\) that \((\omega_{x,x} \otimes \iota)(\Delta(b)_{23})\) belongs to \(\mathcal{M}_\varphi^+\) and

\[\varphi((\omega_{x,x} \otimes \iota)(\Delta(b)_{23})) = \langle (1 \otimes T) x, x \rangle \quad (*)\]

Choose \(y \in H \otimes H\). Then there exists a sequence \((y_n)_{n=1}^\infty\) in \(H \otimes H\) such that \((y_n)_{n=1}^\infty\) converges to \(y\). Then the sequence \(( (\omega_{y_n,y_n} \otimes \iota)(\Delta(b)_{23}) )_{n=1}^\infty\) converges to \((\omega_{y,y} \otimes \iota)(\Delta(b)_{23})\) so the lower semi-continuity of \(\varphi\) implies that

\[\varphi((\omega_{y,y} \otimes \iota)(\Delta(b)_{23})) \leq \liminf_{n \to \infty} \varphi((\omega_{y_n,y_n} \otimes \iota)(\Delta(b)_{23})) = \liminf_{n \to \infty} \langle (1 \otimes T) y_n, y_n \rangle = \langle (1 \otimes T) y, y \rangle,
\]

where we used equality (*) in the second last equality. So we see that \((\omega_{y,y} \otimes \iota)(\Delta(b)_{23})\) belongs to \(\mathcal{M}_\varphi^+.\)

By the discussion above, we know already that \((\omega_{u,u} \otimes \iota)(\Delta(b)_{23})\) belongs to \(\mathcal{M}_\varphi^+.\)

Take a sequence \((u_n)_{n=1}^\infty\) in \(H \otimes H\) such that \((u_n)_{n=1}^\infty\) converges to \(u\).

Choose \(\varepsilon > 0\). Then there exists \(n_0 \in \mathbb{N}\) such that \(\langle (1 \otimes T) (u_n - u_m), u_n - u_m \rangle \leq \varepsilon^2\) for every \(m, n \in \mathbb{N}\) with \(m, n \geq n_0\).

Take \(l \in \mathbb{N}\) with \(l \geq n_0\). By (*), we have for every \(m \in \mathbb{N}\) with \(m \geq n_0\) that

\[\varphi((\omega_{u_l-u_m,u_l-u_m} \otimes \iota)(\Delta(b)_{23})) = \langle (1 \otimes T) (u_l - u_m), u_l - u_m \rangle \leq \varepsilon^2.\]

Because \(( (\omega_{l,l} - u_m,u_l - u_m) \otimes \iota)(\Delta(b)_{23}) )_{m=1}^\infty\) converges to \((\omega_{u_l-u_m,u_l-u_m} \otimes \iota)(\Delta(b)_{23})\), the lower semi-continuity of \(\varphi\) and the previous inequality imply that

\[\varphi((\omega_{u_l-u_m,u_l-u_m} \otimes \iota)(\Delta(b)_{23})) \leq \varepsilon^2.\]

Using the fact that the mapping \(H \otimes H \to \mathbb{R}^+: y \mapsto \varphi((\omega_{y,y} \otimes \iota)(\Delta(b)_{23}))^\frac{1}{2}\) is a semi-norm on \(H \otimes H\), this last inequality implies that

\[|\varphi((\omega_{u_l-u_m} \otimes \iota)(\Delta(b)_{23}))^\frac{1}{2} - \varphi((\omega_{u,u} \otimes \iota)(\Delta(b)_{23}))^\frac{1}{2}| \leq \varphi((\omega_{u_l-u_m,u_l-u_m} \otimes \iota)(\Delta(b)_{23}))^\frac{1}{2} \leq \varepsilon.\]

Therefore, we get that the sequence \(( \varphi((\omega_{u,u} \otimes \iota)(\Delta(b)_{23})) )_{n=1}^\infty\) converges to \(\varphi((\omega_{u,u} \otimes \iota)(\Delta(b)_{23})).\)
By (*), we have for every \( n \in \mathbb{N} \) that
\[
\varphi((\omega_{u_n,u_n} \otimes i)(\Delta(b)_{23})) = \langle (1 \otimes T) u_n, u_n \rangle
\]
which implies that \( \left( \varphi((\omega_{u_n,u_n} \otimes i)(\Delta(b)_{23})) \right)_{n=1}^{\infty} \) converges to \( \langle (1 \otimes T) u, u \rangle \). Comparing these two results, we see that
\[
\varphi((\omega_{u,u} \otimes i)(\Delta(b)_{23})) = \langle (1 \otimes T) u, u \rangle.
\]

\[ \square \]

**Lemma 1.3** Consider \( b \in B^+ \) such that \( (\omega_{v,v} \otimes i)\Delta(b) \) belongs to \( \mathcal{M}_\varphi^+ \) for every \( v \in H \). Then there exists a unique positive number \( r \) such that \( \varphi((\omega_{v,v} \otimes i)\Delta(b)) = r \langle v, v \rangle \) for every \( v \in H \).

**Proof:** Call \( T \) the unique element in \( B(H)^+ \) such that \( \varphi((\omega_{v,v} \otimes i)\Delta(b)) = \langle Tv, v \rangle \) for every \( v \in H \). Take \( v_1, v_2, w_1, w_2 \in H \). By the previous lemma, we know that the element
\[
(\omega_{W(v_1 \otimes w_1),W(v_2 \otimes w_2)} \otimes i)(\Delta(b)_{23})
\]
belongs to \( \mathcal{M}_\varphi \) and
\[
\varphi((\omega_{W(v_1 \otimes w_1),W(v_2 \otimes w_2)} \otimes i)(\Delta(b)_{23}))
= \langle (1 \otimes T)W(v_1 \otimes w_1),W(v_2 \otimes w_2) \rangle
= \langle W^*(1 \otimes T)W(v_1 \otimes w_1),v_2 \otimes w_2 \rangle \quad (a)
\]
Using the fact that \( \Delta(x) = W^*(1 \otimes x)W \) for every \( x \in B \), we get that
\[
(\omega_{W(v_1 \otimes w_1),W(v_2 \otimes w_2)} \otimes i)(\Delta(b)_{23})
= (\omega_{v_1 \otimes w_1,v_2 \otimes w_2} \otimes i)(W^*_{12}\Delta(b)_{23}W_{12})
= (\omega_{v_1 \otimes w_1,v_2 \otimes w_2} \otimes i)((\Delta \otimes i)\Delta(b))
= (\omega_{v_1,v_2} \otimes \omega_{w_1,w_2} \otimes i)((i \otimes \Delta)\Delta(b))
= (\omega_{\omega_{w_1,w_2} \otimes i} \Delta((\omega_{v_1,v_2} \otimes i)\Delta(b))) \quad (b)
\]
By assumption, we have that \( (\omega_{v_1,v_2} \otimes i)\Delta(b) \) belongs to \( \mathcal{M}_\varphi \), so the left invariance of \( \varphi \) implies that \( (\omega_{u_1,u_2} \otimes i)\Delta((\omega_{v_1,v_2} \otimes i)\Delta(b)) \) belongs to \( \mathcal{M}_\varphi \) and
\[
\varphi((\omega_{W(v_1 \otimes w_1),W(v_2 \otimes w_2)} \otimes i)(\Delta(b)_{23})) = \langle w_1, w_2 \rangle \varphi((\omega_{v_1,v_2} \otimes i)\Delta(b)).
\]
Using (b), this implies that
\[
\varphi((\omega_{W(v_1 \otimes w_1),W(v_2 \otimes w_2)} \otimes i)(\Delta(b)_{23})) = \langle w_1, w_2 \rangle \varphi((\omega_{v_1,v_2} \otimes i)\Delta(b))
= \langle w_1, w_2 \rangle \langle Tv_1, v_2 \rangle = \langle (T \otimes 1)(v_1 \otimes w_1), v_2 \otimes w_2 \rangle.
\]
Comparing this equality with equality (a), we see that
\[
\langle W^*(1 \otimes T)W(v_1 \otimes w_1),v_2 \otimes w_2 \rangle = \langle (T \otimes 1)(v_1 \otimes w_1), v_2 \otimes w_2 \rangle.
\]
So we arrive at the conclusion that \( W^*(1 \otimes T)W = T \otimes 1 \).

This implies that \( (1 \otimes T)W = W(T \otimes 1) \). So we have for every \( \omega \in B_0(H)^* \) that \( (i \otimes \omega)(W)T = (\omega \otimes i)(W) \). Because we assumed that \( B \) is the closure of the set \( \{ (i \otimes \omega)(W) | \omega \in B_0(H)^* \} \) in \( B(H) \), we arrive at the conclusion that \( BT \subseteq B \). The selfadjointness of \( T \) implies that also \( BT \subseteq B \), so we see that \( T \) is an element of \( M(B) \).

Furthermore, the equation \( W^*(1 \otimes T)W = T \otimes 1 \) implies that \( \Delta(T) = T \otimes 1 \). By assumption, we get the existence of an element \( r \in \mathbb{R}^+ \) such that \( T = r1 \). The lemma follows.

\[ \square \]

This lemma justifies the following definition.
Definition 1.4 We define the mapping \( \theta \) from \( B^+ \) into \([0, \infty]\) such that we have for every \( b \in B^+ \) that

- If \( (\omega_{v, v} \otimes \iota) \Delta(b) \) belongs to \( \mathcal{M}_\varphi^+ \) for every \( v \in H \), we define \( \theta(b) \) to be the unique positive number such that \( \varphi((\omega_{v, v} \otimes \iota) \Delta(b)) = \theta(b) \langle v, v \rangle \) for every \( v \in H \).
- If there exists \( v \in H \) such that \((\omega_{v, v} \otimes \iota) \Delta(b) \) does not belong to \( \mathcal{M}_\varphi^+ \), we define \( \theta(b) = \infty \).

Proposition 1.5 The mapping \( \theta \) is a densely defined lower semi-continuous weight on \( B \) which extends \( \varphi \).

Proof: It is easy to check that \( \theta \) is a weight on \( B \) which extends \( \varphi \). We turn to the lower semi-continuity. Choose \( \alpha \in \mathbb{R}^+ \). Take a sequence \( (b_n)_{n=1}^\infty \) in \( B^+ \) and \( b \in B^+ \) such that \( (b_n)_{n=1}^\infty \to b \) and \( \theta(b_n) \leq \alpha \) for every \( n \in \mathbb{N} \).

Take \( w \in H \). Choose \( m \in \mathbb{N} \). Because \( \theta(b_m) \leq \alpha \), we have by definition that \( (\omega_{w, w} \otimes \iota) \Delta(b_m) \) belongs to \( \mathcal{M}_\varphi^+ \) and

\[
\varphi((\omega_{w, w} \otimes \iota) \Delta(b_m)) = \theta(b_m) \langle w, w \rangle \leq \alpha \langle w, w \rangle.
\]

Because \(( (\omega_{w, w} \otimes \iota) \Delta(b_n))_{n=1}^\infty \) converges to \((\omega_{w, w} \otimes \iota) \Delta(b) \) and \( \varphi \) is lower semi-continuous, this implies that \( \varphi((\omega_{w, w} \otimes \iota) \Delta(b)) \leq \alpha \langle w, w \rangle \). In particular, we get that \( (\omega_{w, w} \otimes \iota) \Delta(b) \) belongs to \( \mathcal{M}_\varphi^+ \).

Using the definition of \( \theta \), we see that

\[
\theta(b) \langle v, v \rangle = \varphi((\omega_{v, v} \otimes \iota) \Delta(b)) \leq \alpha \langle v, v \rangle
\]

for every \( v \in H \). Consequently, \( \theta(b) \leq \alpha \).

From this all, the lower semi-continuity of \( \theta \) follows.

Because \( \theta \) is an extension of \( \varphi \), we get immediately the following result.

Proposition 1.6 Consider \( b \in \mathcal{M}_\varphi^+ \) and \( v \in H \). Then \((\omega_{v, v} \otimes \iota) \Delta(b) \) belongs to \( \mathcal{M}_\varphi^+ \) and

\[
\theta((\omega_{v, v} \otimes \iota) \Delta(b)) = \theta(b) \langle v, v \rangle.
\]

We can even do better.

Proposition 1.7 Consider \( b \in B^+ \) such that \((\omega_{v, v} \otimes \iota) \Delta(b) \) belongs to \( \mathcal{M}_\varphi^+ \) for every \( v \in H \). Then \( b \) belongs to \( \mathcal{M}_\varphi^+ \).

Proof: For every \( u_1, u_2 \in H \otimes H \), we define the element \( f_{u_1, u_2} \in B^* \) such that \( f_{u_1, u_2}(x) = \langle \Delta(x) u_1, w_2 \rangle \) for every \( x \in B \). By lemma 1.3 (applied to \( \theta \) instead of \( \varphi \)), we have the existence of a positive number \( r \) such that \( \theta((\omega_{v_1, w_1} \otimes \iota) \Delta(b)) = r \langle v, w \rangle \) for every \( v, w \in H \).

Take \( v_1, v_2, w_1, w_2 \in H \). By supposition, we have that \((\omega_{v_1, w_1} \otimes \iota) \Delta(b) \) belongs to \( \mathcal{M}_\varphi \). This implies by the definition of \( \theta \) that \((\omega_{v_2, w_2} \otimes \iota) \Delta((\omega_{v_1, w_1} \otimes \iota) \Delta(b)) \) belongs to \( \mathcal{M}_\varphi \) and

\[
\varphi((\omega_{v_2, w_2} \otimes \iota) \Delta((\omega_{v_1, w_1} \otimes \iota) \Delta(b))) = \theta((\omega_{v_1, w_1} \otimes \iota) \Delta(b)) \langle v_2, w_2 \rangle = r \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle = r \langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle.
\]

Using the coassociativity of \( \Delta \), we have also that

\[
(\omega_{v_2, w_2} \otimes \iota) \Delta((\omega_{v_1, w_1} \otimes \iota) \Delta(b)) = ((\omega_{v_1, w_1} \otimes \omega_{v_2, w_2}) \Delta \otimes \iota) \Delta(b) = (f_{v_1 \otimes v_2, w_1 \otimes w_2} \otimes \iota) \Delta(b).
\]
So we see that \((f_{v_1 \otimes w_1, v_2 \otimes w_2} \otimes \iota)\Delta(b)\) belongs to \(\mathcal{M}_\varphi\) and that
\[
\varphi((f_{v_1 \otimes w_1, v_2 \otimes w_2} \otimes \iota)\Delta(b)) = r \langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle
\]
This implies for every \(u \in H \otimes H\) that \((f_{u,u} \otimes \iota)\Delta(b)\) belongs to \(\mathcal{M}_\varphi^+\) and \(\varphi((f_{u,u} \otimes \iota)\Delta(b)) = r \langle u, u \rangle\).
This allows, just as in the proof of lemma [2] to conclude for every \(u \in H \otimes H\) that \((f_{u,u} \otimes \iota)\Delta(b)\) belongs to \(\mathcal{M}_\varphi^+\) and \(\varphi((f_{u,u} \otimes \iota)\Delta(b)) = r \langle u, u \rangle\).

Let us now choose \(v \in H\). Take \(w \in H\) such that \(\langle w, w \rangle = 1\). By the preceding result, we know that \((f_{W^*(w \otimes v), W^*(w \otimes v)} \otimes \iota)\Delta(b)\) belongs to \(\mathcal{M}_\varphi^+\). Using the fact that \(\Delta(x) = W^*(1 \otimes x)W\) for every \(x \in B\), it is easy to check that \(f_{W^*(w \otimes v), W^*(w \otimes v)} = \omega_{v,v}\) so we get that \((\omega_{v,v} \otimes \iota)\Delta(b)\) belongs to \(\mathcal{M}_\varphi^+\). By definition, we get that \(b\) belongs to \(\mathcal{M}_\varphi^+\).

## 2 A useful result concerning KMS-weights

We will use the following definition of a KMS-weight (see [3]):

**Definition 2.1** Consider a C*-algebra \(B\) and a densely defined lower semi-continuous weight \(\varphi\) on \(B\) such that there exist a norm-continuous one parameter group \(\sigma\) on \(B\) such that

1. We have that \(\varphi \sigma_t = \varphi\) for every \(t \in \mathbb{R}\).
2. We have that \(\varphi(x^*x) = \varphi(\sigma_{\frac{t}{2}}(x)\sigma_{\frac{t}{2}}(x)^*)\) for every \(x \in D(\sigma_{\frac{t}{2}})\).

Then \(\varphi\) is called a KMS-weight on \(B\) and \(\sigma\) is called a modular group for \(\varphi\).

This is not the usual definition of a KMS-weight (see [3]) but we prove in [3] that this definition is equivalent to the usual one.

A rather useful proposition concerning KMS-weights is the following one (see [3]):

**Proposition 2.2** Consider a C*-algebra \(B\) and let \(\varphi\) be a KMS-weight on \(B\) with modular group \(\sigma\). Consider moreover a GNS-construction \((H_\varphi, \Lambda_\varphi, \pi_\varphi)\) for \(\varphi\). Then:

- There exists a unique anti-unitary operator \(J\) on \(H\) such that \(J\Lambda_\varphi(x) = \Lambda_\varphi(\sigma_{\frac{t}{2}}(x)^*)\) for every \(x \in \mathcal{N}_\varphi \cap D(\sigma_{\frac{t}{2}})\)

- We have for every \(x \in \mathcal{N}_\varphi\) and every \(a \in D(\sigma_{\frac{t}{2}})\) that \(xa\) belongs to \(\mathcal{N}_\varphi\) and
\[
\Lambda_\varphi(xa) = J\pi_\varphi(\sigma_{\frac{t}{2}}(a))^*J\Lambda_\varphi(x) .
\]

We now prove that a KMS-weight has no proper extensions which are relatively invariant under its modular group.

**Proposition 2.3** Consider a KMS-weight \(\varphi\) on a C*-algebra \(B\) with modular group \(\sigma\). Let \(\eta\) be a lower semi-continuous weight on \(B\) which is an extension of \(\varphi\) and such that \(\eta\) is relatively invariant under \(\sigma\). Then \(\varphi = \eta\).

**Proof:** Take a GNS-construction \((H_\varphi, \Lambda_\varphi, \pi_\varphi)\) for \(\varphi\) and a GNS-construction \((H_\eta, \Lambda_\eta, \pi_\eta)\) for \(\eta\). Because \(\eta\) is relatively invariant under \(\sigma\), we get the existence of a strictly positive number \(\lambda\) such that \(\eta \sigma_t = \lambda^t \eta\) for every \(t \in \mathbb{R}\).
So we get the existence a positive injective operator $T$ in $H_\eta$ such that $T^{it}\Lambda_\eta(a) = \lambda^{-\frac{t}{2}} \Lambda_\eta(\sigma_t(a))$ for every $a \in \mathcal{N}_\eta$ and $t \in \mathbb{R}$.

Choose $y \in \mathcal{N}_\eta$.

Define for every $n \in \mathbb{N}$ the element

$$y_n = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 \int^t) \sigma_t(y) dt$$

which is clearly analytic with respect to $\sigma$. We have also that $(y_n)_{n=1}^\infty$ converges to $y$.

By [8], we have for every $n \in \mathbb{N}$ that $y_n$ belongs to $\mathcal{N}_\eta$ and

$$\Lambda_\eta(y_n) = \frac{n}{\sqrt{\pi}} \int \exp(-n^2 \int^t) \lambda^{\frac{t}{2}} T^{it}\Lambda(y) dt .$$

This implies immediately that $(\Lambda_\eta(y_n))_{n=1}^\infty$ converges to $\Lambda_\eta(y)$.

We can also take an approximate unit $(e_i)_{i \in I}$ for $B$ in $\mathcal{N}_\varphi$. Then $(e_i y_n)_{(i,n) \in I \times \mathbb{N}}$ converges to $y$.

We have also for every $i \in I$ and $n \in \mathbb{N}$ that $e_i y_n$ belongs to $\mathcal{N}_\eta$ and $\Lambda_\eta(e_i y_n) = \pi_\eta(e_i) \Lambda_\eta(y_n)$. Consequently, the net $(\Lambda_\eta(e_i y_n))_{(i,n) \in I \times \mathbb{N}}$ converges to $\Lambda_\eta(y)$.

We have for every $i \in I$ and $n \in \mathbb{N}$ that $e_i$ belongs to $\mathcal{N}_\varphi$ and $y_n$ belongs to $D(\sigma_{\frac{t}{2}})$ implying that $e_i y_n$ belongs to $\mathcal{N}_\varphi$ by the previous proposition.

Because $\varphi \subseteq \eta$, we have moreover for every $i, j \in I$ and $m, n \in \mathbb{N}$ that

$$\|\Lambda_\varphi(e_i y_n) - \Lambda_\varphi(e_j y_m)\| = \|\Lambda_\eta(e_i y_n) - \Lambda_\eta(e_j y_m)\| .$$

This last equality implies that the net $(\Lambda_\varphi(e_i y_n))_{i,n \in I \times \mathbb{N}}$ is Cauchy and hence convergent in $H_\varphi$. Therefore, the closedness of $\Lambda_\varphi$ implies that $y$ is an element of $\mathcal{N}_\varphi$. The proposition follows.

\section*{3 Improving the invariance of left Haar weights}

In [8], we constructed the reduced C$^*$-algebraic quantum group out of an algebraic one. We constructed a left invariant weight on this C$^*$-algebraic quantum group. In this section, we will improve the invariance property of this left Haar weight. We will also prove a uniqueness property. It is very well possible that the techniques used in this section can be useful in the more case general case of C$^*$-algebraic quantum groups according to Masuda, Nakagami & Woronowicz (upcoming paper).

We will start this section with a small overview concerning algebraic quantum groups and the C$^*$-algebraic quantum groups arising from them. For a more detailed treatment, we refer to [8].

So consider an algebraic quantum group $(A, \Delta)$ according to A. Van Daele (see [13]). This means that $(A, \Delta)$ satisfies the following properties:

The object $A$ is a non-degenerate $^*$-algebra and $\Delta$ is a non-degenerate $^*$-homomorphism

$$\Delta : A \to M(A \otimes A)$$

satisfying the following properties:

1. $(\Delta \circ \iota) \Delta = (\iota \circ \Delta) \Delta$.

2. The linear mappings $T_1, T_2$ from $A \otimes A$ into $M(A \otimes A)$ such that

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \quad \text{and} \quad T_2(a \otimes b) = \Delta(a)(b \otimes 1)$$

for all $a, b \in A$, are bijections from $A \otimes A$ into $A \otimes A$. 


We assume furthermore the existence of a non-trivial positive linear functional $\varphi$ on $A$ such that we have for all $a, b \in A$

\[(\iota \circ \varphi)(\Delta(a)(b \otimes 1)) = \varphi(a)b .\]

Next, we take a GNS-pair $(H, \Lambda)$ of the left Haar functional $\varphi$ on $A$. This means that $H$ is a Hilbert space and that $\Lambda$ is a linear mapping from $A$ into $H$ such that

1. The set $\Lambda(A)$ is dense in $H$.
2. We have for every $a, b \in A$ that $\langle \Lambda(a), \Lambda(b) \rangle = \varphi(b^*a)$.

As usual, we can associate a multiplicative unitary to $(A, \Delta)$:

**Definition 3.1** We define $W$ as the unique unitary element in $B(H \otimes H)$ such that $W(\Lambda \circ \Lambda)(\Delta(b)(a \otimes 1)) = \Lambda(a) \otimes \Lambda(b)$ for every $a, b \in A$. The element $W$ is called the fundamental unitary associated to $(A, \Delta)$.

The GNS-pair $(H, \Lambda)$ allows us to represent $A$ by bounded operators on $H$:

**Definition 3.2** We define $\pi$ as the unique $\ast$-homomorphism from $A$ into $B(H)$ such that $\pi(a)\Lambda(b) = \Lambda(ab)$ for every $a, b \in A$. We have also that $\pi$ is injective.

Notice also that it is not immediate that $\pi(x)$ is a bounded operator on $H$ (because $\varphi$ is merely a functional, not a weight), but the boundedness of $\pi(x)$ is connected with the following equality:

We have for every $a, b \in A$ that

\[\pi((\iota \circ \varphi)(\Delta(b^*)(1 \otimes a))) = (\iota \circ \omega_{\Lambda(a), \Lambda(b)})(W)\] (1)

The mapping $\pi$ makes it possible to define our reduced C*-algebra:

**Definition 3.3** We define $A_r$ as the closure of $\pi(A)$ in $B(H)$. So $A_r$ is a non-degenerate sub-C*-algebra of $B(H)$.

Equation (1) implies that

\[A_r = \text{ closure of } \{ (\iota \circ \omega)(W) \mid \omega \in B_0(H)^\ast \} \text{ in } B(H) .\]

As usual, we use the fundamental unitary to define a comultiplication on $A_r$. We will denote it by $\Delta_r$.

**Definition 3.4** We define the mapping $\Delta_r$ from $A_r$ into $B(H \otimes H)$ such that $\Delta_r(x) = W^*(1 \otimes x)W$ for all $x \in A_r$. Then $\Delta_r$ is an injective $\ast$-homomorphism.

It is not so difficult to show that $\Delta_r$ on the C*-algebra level is an extension of $\Delta$ on the $\ast$-algebra level:

**Result 3.5** We have for all $a \in A$ and $x \in A \otimes A$ that $(\pi \circ \pi)(x)\Delta_r(\pi(a)) = (\pi \circ \pi)(x\Delta(a))$ and $\Delta_r(\pi(a))(\pi \circ \pi)(x) = (\pi \circ \pi)(\Delta(a) x)$.

Using the above result, it is not so hard to prove the following theorem:

**Theorem 3.6** We have that $A_r$ is a non-degenerate sub-C*-algebra of $B(H)$ and $\Delta_r$ is a non-degenerate injective $\ast$-homomorphism from $A_r$ into $M(A_r \otimes A_r)$ such that:
There exists a unique weight \( \phi \), so we see that the weight is determined by the following theorem (see \([7]\), theorem 6.12, proposition 6.2 and the remarks after it):

**Theorem 3.8** There exists a unique closed linear map \( \Lambda_r \) from within \( A_r \) into \( H \) such that \( \pi(A) \) is a core for \( \Lambda_r \) and \( \Lambda_r(\pi(a)) = \Lambda(a) \) for every \( a \in A \).

There exists moreover a unique weight \( \phi_r \) on \( A_r \) such that \((H, \Lambda_r, \iota)\) is a GNS-construction for \( \phi_r \).

We have also that \( \pi(A) \subseteq M_{\phi_r} \) and that \( \phi_r(\pi(a)) = \varphi(a) \) for every \( a \in A \).

**Proposition 3.9** The weight \( \varphi_r \) is a faithful KMS-weight. We denote the modular group of \( \phi_r \) by \( \sigma \).

This weight satisfies the following left invariance property (corollary 6.14 of \([7]\)).

**Proposition 3.10** Consider \( a \in M_{\phi_r} \) and \( \omega \in A_r^+ \). Then \((\omega \otimes \iota) \Delta_r(a) \) belongs to \( M_{\phi_r} \) and \( \varphi_r((\omega \otimes \iota) \Delta_r(a)) = \omega(1) \varphi_r(a) \).

We have even a stronger form of left invariance condition, but we only need this one.

So we see that the weight \( \varphi_r \) satisfies the requirements of the beginning of section \([\ref{sec:3}]\).

Now we have enough information to prove some results.

The first theorem improves the left invariance property of \( \varphi_r \).

**Theorem 3.11** Consider \( a \in A_r^+ \) such that \((\omega_{v,v} \otimes \iota) \Delta_r(a) \) belongs to \( M_{\phi_r}^+ \) for every \( v \in H \). Then \( a \) belongs to \( M_{\phi_r}^+ \).

**Proof:** It is clear that we can apply the results from section \([\ref{sec:3}]\) to the weight \( \varphi_r \). So, call \( \theta \) the weight on \( A_r \) arising from \( \varphi_r \) as described in definition \([\ref{def:6}]\). Hence, \( \theta \) is a densely defined lower semi-continuous weight on \( A_r \) which extends \( \varphi_r \).

Choose \( b \in M_{\phi_r}^+ \) and \( t \in \mathbb{R} \). Take \( v \in H \).

In definition 5.1 of \([7]\), we defined a \(*\)-automorphism \( \tau_t \) on \( A_r \) which was implemented by a positive injective (generally unbounded) operator \( M \), i.e. \( \tau_t(x) = M^{it}xM^{-it} \) for every \( x \in A_r \). In proposition 5.7 of \([7]\), we proved the commutation \( \Delta_r, \sigma_t = (\tau_t \otimes \sigma_t) \Delta_r \). Therefore,

\[
(\omega_{v,v} \otimes \iota) \Delta_r(\sigma_t(b)) = (\omega_{v,v} \otimes \iota)((\tau_t \otimes \sigma_t)\Delta_r(b)) = \sigma_t((\omega_{M^{-it}v,M^{-it}v} \otimes \iota)\Delta_r(b)). \tag{*}
\]
By the definition of \( \theta \), we have that \((\omega_{M^{-it}v,M^{-it}v} \otimes \iota)\Delta_r(b)\) belongs to \(\mathcal{M}^+_\varphi \) and

\[
\varphi_r((\omega_{M^{-it}v,M^{-it}v} \otimes \iota)\Delta_r(b)) = \langle M^{-it}v, M^{-it}v \rangle \theta(b) = \langle v, v \rangle \theta(b).
\]

This implies that \(\sigma((\omega_{M^{-it}v,M^{-it}v} \otimes \iota)\Delta_r(b))\) belongs to \(\mathcal{M}^+_\varphi \) and

\[
\varphi_r((\omega_{M^{-it}v,M^{-it}v} \otimes \iota)\Delta_r(b)) = \varphi_r((\omega_{M^{-it}v,M^{-it}v} \otimes \iota)\Delta_r(b)) = \langle v, v \rangle \theta(b).
\]

Therefore, equation (*) implies that \((\omega_{v,v} \otimes \iota)\Delta_r(\sigma_t(b))\) belongs to \(\mathcal{M}^+_\varphi \) and

\[
\varphi_r((\omega_{v,v} \otimes \iota)\Delta_r(\sigma_t(b))) = \langle v, v \rangle \theta(b).
\]

We get by the definition of \( \theta \) that \(\sigma_t(b)\) belongs to \(\mathcal{M}^+_\theta \) and that

\[
\langle v, v \rangle \theta(\sigma_t(b)) = \varphi_r((\omega_{v,v} \otimes \iota)\Delta_r(\sigma_t(b))) = \langle v, v \rangle \theta(b)
\]

for every \( v \in H \). So we see that \(\theta(\sigma_t(b)) = \theta(b)\).

Consequently, we have proven that \( \theta \) is invariant with respect to \( \sigma \). Combining this with the fact that \( \theta \) is an extension of \( \varphi_r \) and using proposition 2.3, we see that \( \varphi_r = \theta \). The theorem follows.

Next, we prove some results about the uniqueness of the left Haar weight on the C*-algebra level. In both cases, the uniqueness follows from some left invariance property and some relative invariance property.

We first start with a weaker form of uniqueness. The proof strongly resembles the proof of proposition 7.1 of [7].

**Remark 3.12** If we look at the proof of proposition 6.9 of [7], we see that also the following weaker form is true:

Consider a dense left ideal \( N \) in \( A_r \) such that \((\omega_{v,v} \otimes \iota)\Delta_r(a)\) belongs to \( N \) for all \( a \in N \) and all \( v \in H \). Then \( \pi(A) \) is a subset of \( N \).

**Proposition 3.13** Consider a densely defined lower semi-continuous weight \( \eta \) on \( A_r \) such that we have for every \( a \in \mathcal{M}^+_\eta \) and every \( v \in H \) that \((\omega_{v,v} \otimes \iota)\Delta_r(a)\) belongs to \(\mathcal{M}^+_\eta \) and

\[
\eta((\omega_{v,v} \otimes \iota)\Delta_r(a)) = \langle v, v \rangle \eta(a).
\]

Then there exists a positive number \( r \) such that \( r \varphi_r \subseteq \eta \).

**Proof**: We know that \( N_\eta \) is a dense left ideal in \( A \). Choose \( a \in N_\eta \) and \( v \in H \), then

\[
[(\omega_{v,v} \otimes \iota)\Delta_r(a)]^* [(\omega_{v,v} \otimes \iota)\Delta_r(a)] \leq \|\omega_{v,v}\| (\omega_{v,v} \otimes \iota)\Delta_r(a^*a).
\]

Because \( a^*a \) belongs to \( \mathcal{M}^+_\eta \), we have by assumption that \((\omega_{v,v} \otimes \iota)\Delta_r(a^*a)\) belongs to \(\mathcal{M}^+_\eta \). Therefore, the previous inequality implies that the element \([\omega_{v,v} \otimes \iota]_r(a^*a)\) belongs to \(\mathcal{M}^+_\eta \). Hence, \((\omega_{v,v} \otimes \iota)_r(a)\) belongs to \( N_\eta \).

By the remark before this proposition, we can conclude from this all that \( \pi(A) \) is a subset of \( N_\eta \). Because \( A^*A = A \), we get that \( \pi(A) \) is a subset of \( \mathcal{M} \).

So we can define the positive linear functional \( \phi \) on \( A \) such that \( \phi(a) = \eta(\pi(a)) \) for every \( a \in A \).
Choose $a, b \in A$. Take $c \in B$. Then
\[
\varphi(c^*(\iota \circ \phi)(\Delta(a)(b \otimes 1))) = \phi((\varphi \circ \iota)((c^* \otimes 1)\Delta(a)(b \otimes 1)))
\]
\[
= \eta((\pi \circ \iota)((c^* \otimes 1)\Delta(a)(b \otimes 1)))
\]
\[
= \eta((\omega_{\Lambda(b),\Lambda(c) \otimes \iota})\Delta_r(\pi(a))).
\]
Because $\pi(a)$ is an element of $\mathcal{M}_\eta$, we get by assumption that $((\omega_{\Lambda(b),\Lambda(c) \otimes \iota})\Delta_r(\pi(a)))$ belongs to $\mathcal{M}_\eta$ and
\[
\eta((\omega_{\Lambda(b),\Lambda(c) \otimes \iota})\Delta_r(\pi(a))) = \eta(\pi(a)) (\Lambda(b), \Lambda(c)) = \phi(a) \varphi(c^*b).
\]
Therefore, $\phi$ is a left invariant functional on $A$. By the uniqueness of the Haar functional on the $^*$-algebra level (see theorem 3.7 of [13]), we know that there exists a unique norm continuous one-parameter group $\Phi$ on $\mathcal{A}$ such that $\Phi(A) \subseteq \mathcal{M}_\eta$ and $\eta(x) = r \varphi_r(x)$ for every $x \in \mathcal{A}$.

Take a GNS-construction $(H_\eta, \Lambda_\eta, \pi_\eta)$ for $\eta$.

Choose $y \in \mathcal{N}_\eta$. Because $\pi(A)$ is a core for $\Lambda_r$ (see theorem 3.8), there exists a sequence $y_n \in \pi(A)$ such that $(y_n)_{n=1}^\infty$ converges to $y$ and $(\Lambda_r(y_n))_{n=1}^\infty$ converges to $\Lambda_r(y)$. By the first part of the proof, we know already for every $n \in \mathbb{N}$ that $y_n$ belongs to $\mathcal{N}_\eta$. Furthermore, we have for every $m, n \in \mathbb{N}$ that
\[
||\Lambda_\eta(y_m) - \Lambda_\eta(y_n)||^2 = \eta((y_m - y_n)^* (y_m - y_n))
\]
\[
= r \varphi_r((y_m - y_n)^* (y_m - y_n)) = r ||\Lambda_r(y_m) - \Lambda_r(y_n)||^2.
\]
This implies that $(\Lambda_\eta(y_n))_{n=1}^\infty$ is Cauchy and hence convergent. So the closedness of $\Lambda_\eta$ implies that $y$ belongs to $\mathcal{N}_\eta$ and that $(\Lambda_\eta(y_n))_{n=1}^\infty$ converges to $\Lambda_\eta(y)$.

We have for every $n \in \mathbb{N}$ that
\[
\langle \Lambda_\eta(y_n), \Lambda_\eta(y_n) \rangle = \eta(y_n^* y_n) = r \varphi_r(y_n^* y_n) = r \langle \Lambda_r(y_n), \Lambda_r(y_n) \rangle
\]
which implies that $\langle \Lambda_\eta(y), \Lambda_\eta(y) \rangle = r \langle \Lambda_r(y), \Lambda_r(y) \rangle$. The proposition follows.

Under stronger conditions, we can prove a real uniqueness result which will be shown below.

From proposition 3.15 of [13], we know that there exists a unique norm continuous one-parameter group $K$ on $\mathcal{A}_r$ such that $(\sigma_t \otimes K_t)\Delta_r = \Delta_r, \sigma_t$ for every $t \in \mathbb{R}$. This one parameter group is used to formulate some uniqueness result.

**Theorem 3.14** Consider a non-zero densely defined lower semi-continuous weight $\eta$ on $\mathcal{A}_r$ such that

1. We have for every $a \in \mathcal{M}_\eta^+$ and $v \in H$ that $(\omega_{v,v} \otimes \iota)\Delta_r(a)$ belongs to $\mathcal{M}_\eta^+$ and
\[
\eta((\omega_{v,v} \otimes \iota)\Delta_r(a)) = \eta(a) \langle v, v \rangle.
\]

2. The weight $\eta$ is relatively invariant under $K$.

Then there exists a unique strictly invariant positive number $r$ such that $\eta = r \varphi_r$.

**Proof:** By assumption, there exists a unique strictly positive number $\lambda$ such that $\eta K_t = \lambda^t \eta$ for every $t \in \mathbb{R}$. The previous proposition implies the existence of a positive number $r$ such that $\eta$ is an extension of $r \varphi_r$. We automatically have that $r \neq 0$. (If $r$ would be 0, then $\eta$ would be 0 on the set $\pi(A)$ so the lower semi-continuity of $\eta$ would imply that $\eta = 0$.)
Because $\varphi_r$ is invariant under $\sigma$, we have also the existence of an injective positive operator $\nabla$ in $H$ such that $\nabla^{it} \Lambda_r(x) = \Lambda_r(\sigma_t(x))$ for every $t \in \mathbb{R}$ and $x \in \mathcal{N}_{\varphi_r}$. Then $\sigma_t(x) = \nabla^{-it} x \nabla^{-it}$ for every $x \in A_r$ and $t \in \mathbb{R}$.

Again, the requirements of the beginning of section 1 are satisfied and we get a weight $\eta$ according to definition 1.4. Hence, $\theta$ is a densely defined lower semi-continuous weight on $A_r$ which extends $\eta$. So $\theta$ will also extend $r \varphi_r$.

Choose $b \in M^+_{\theta}$ and $t \in \mathbb{R}$. Take $v \in H$. By the remarks before this theorem, we get that

$$(\omega_{v,v} \otimes t) \Delta_r(\sigma_t(b)) = (\omega_{v,v} \otimes t)((\sigma_t \otimes K_t) \Delta_r(b))$$

$$= K_t((\omega_{v,-it,v} \otimes t) \Delta_r(b)) . \quad (*)$$

By the definition of $\theta$, we have that $(\omega_{v,-it,v} \otimes t) \Delta_r(b)$ belongs to $M^+_{\theta}$ and

$$\eta((\omega_{v,-it,v} \otimes t) \Delta_r(b)) = \langle \nabla^{-it} v, \nabla^{-it} v \rangle \theta(b) = \langle v, v \rangle \theta(b) .$$

This implies that $K_t((\omega_{v,-it,v} \otimes t) \Delta_r(b))$ belongs to $M^+_{\theta}$ and

$$\eta(K_t((\omega_{v,-it,v} \otimes t) \Delta_r(b))) = \lambda^t \eta((\omega_{v,-it,v} \otimes t) \Delta_r(b)) = \lambda^t \langle v, v \rangle \theta(b) .$$

Therefore, equation $(*)$ implies that $(\omega_{v,v} \otimes t) \Delta_r(\sigma_t(b))$ belongs to $M^+_{\theta}$ and

$$\eta((\omega_{v,v} \otimes t) \Delta_r(\sigma_t(b))) = \lambda^t \langle v, v \rangle \theta(b) .$$

Therefore we get by the definition of $\theta$ that $\sigma_t(b)$ belongs to $M^+_{\theta}$ and that

$$\langle v, v \rangle \theta(\sigma_t(b)) = \eta((\omega_{v,v} \otimes t) \Delta_r(\sigma_t(b))) = \lambda^t \langle v, v \rangle \theta(b)$$

for every $v \in H$. So we see that $\theta(\sigma_t(b)) = \lambda^t \theta(b)$.

Consequently, we have proven that $\theta$ is relatively invariant with respect to $\sigma$. Combining this with the fact that $\theta$ is an extension of $r \varphi_r$ and using proposition 2.3, we see that $\theta = r \varphi_r$. Because $r \varphi_r \subseteq \eta \subseteq \theta$, we get that $\eta = r \varphi_r$. \hfill \blacksquare

Of course, also the following uniqueness result is valid. It follows immediately from propositions 3.13 and 2.3 (without using the extension $\theta$).

**Theorem 3.15** Consider a non-zero densely defined lower semi-continuous weight $\eta$ on $A_r$ such that

1. We have for every $a \in M^+_{\theta}$ and $v \in H$ that $(\omega_{v,v} \otimes t) \Delta_r(a)$ belongs to $M^+_{\theta}$ and

$$\eta((\omega_{v,v} \otimes t) \Delta_r(a)) = \eta(a) \langle v, v \rangle .$$

2. The weight $\eta$ is relatively invariant under $\sigma$.

Then there exists a unique strictly positive number $r$ such that $\eta = r \varphi_r$.

**References**

[1] S. Bajaj, Représentation régulière du groupe quantique des déplacements de Woronowicz. Preprint.

[2] S. Bajaj & G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croisés de $C^*$-algèbres. *Ann. scient. Éc. Norm. Sup.*, 4e série, t. 26 (1993), 425–488.
[3] I. Ciorănescu and L. Zsidó, Analytic generators for one-parameter groups. Tôhoku Math. Journal 28 (1976), 327–362.

[4] F. Combes, Poids associé à une algèbre hilbertienne à gauche. Compos. Math. 23 (1971), 49–77.

[5] M. Enock and J.-M. Schwartz, Kac Algebras and Duality of Locally Compact Groups. Springer-Verlag, Berlin (1992).

[6] U. Haagerup, Operator-valued weights in von Neumann algebras I. J. Funct. Anal. 32 (1979), 175–206.

[7] J. Kustermans and A. Van Daele, C*-algebraic quantum groups arising from algebraic quantum groups. (1996) To appear in International Journal of Mathematics.

[8] J. Kustermans, A construction procedure for KMS-weights on C*-algebras. In preparation.

[9] J. Kustermans, Universal C*-algebraic quantum groups arising from algebraic quantum groups. Preprint Odense Universitet (1997).

[10] T. Masuda and Y. Nakagami, A von Neumann Algebra Framework for the Duality of Quantum Groups. Publications of the RIMS Kyoto University 30 (1994), 799–850.

[11] S. Stratila and L. Zsidó, Lectures on von Neumann algebras. Abacus Press, Tunbridge Wells, England (1979).

[12] M. Takesaki, Theory of Operator Algebras I. Springer-Verlag, New York (1979).

[13] A. Van Daele, An Algebraic Framework for Group Duality. (1996) To appear in Advances of Mathematics.

[14] A. Van Daele, Multiplier Hopf Algebras. Trans. Am. Math. Soc. 342 (1994), 917–932.

[15] J. Verding, Weights on C*-algebras. Phd-thesis. K.U. Leuven (1995).

[16] S.L. Woronowicz, From multiplicative unitaries to quantum groups. Preprint Warszawa (1995).

[17] S.L. Woronowicz, Pseudospaces, pseudogroups and Pontriagin duality. Proceedings of the International Conference on Mathematical Physics, Lausanne (1979), 407–412.