ON STABILIZATION OF SOLUTIONS OF HIGHER ORDER EVOLUTION INEQUALITIES

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Abstract. We obtain sharp conditions guaranteeing that every non-negative weak solution of the inequality
\[
\sum_{|\alpha|=m} \partial^\alpha a_\alpha(x, t, u) - u_t \geq f(x, t)g(u) \quad \text{in } \mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty), \quad m, n \geq 1,
\]
stabilizes to zero as \( t \to \infty \). These conditions generalize the well-known Keller-Osserman condition on the grows of the function \( g \) at infinity.

1. Introduction

We study non-negative solutions of the inequality
\[
\sum_{|\alpha|=m} \partial^\alpha a_\alpha(x, t, u) - u_t \geq f(x, t)g(u) \quad \text{in } \mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty), \quad m, n \geq 1, \tag{1.1}
\]
where \( a_\alpha \) are Caratheodory functions such that
\[
|a_\alpha(x, t, \zeta)| \leq A\zeta^p, \quad |\alpha| = m, \tag{1.2}
\]
with some constants \( A > 0 \) and \( p \geq 1 \) for almost all \((x, t) \in \mathbb{R}_+^{n+1}\) and for all \( \zeta \in [0, \infty) \). In addition, it is assumed that \( f \) is a measurable function on the set \( \mathbb{R}_+^{n+1} \), \( g(\zeta^{1/p}) \) is a non-decreasing convex function on the closed interval \([0, \infty)\), and \( g(\zeta) > 0 \) for all \( \zeta > 0 \). As is customary, by \( \alpha \) we mean a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with \( |\alpha| = \alpha_1 + \ldots + \alpha_n \) and \( \partial^\alpha = \partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_n}_{x_n}, \quad x = (x_1, \ldots, x_n) \).

Definition 1.1. A non-negative function \( u \in L^p_{loc}(\mathbb{R}_+^{n+1}) \) is called a weak solution of (1.1) if \( f(x, t)g(u) \in L^1_{loc}(\mathbb{R}_+^{n+1}) \) and, moreover, for any non-negative function \( \varphi \in C_0^\infty(\mathbb{R}_+^{n+1}) \) the following inequality holds:
\[
\int_{\mathbb{R}_+^{n+1}} \sum_{|\alpha|=m} (-1)^m a_\alpha(x, t, u) \partial^\alpha \varphi \, dxdt + \int_{\mathbb{R}_+^{n+1}} w \varphi_t \, dxdt \geq \int_{\mathbb{R}_+^{n+1}} f(x, t)g(u)\varphi \, dxdt. \tag{1.3}
\]

Questions treated in this paper were earlier investigated mainly for differential operators of the second order [1–14]. The case of higher order operators has been studied much less [15–16]. Our aim is to obtain sufficient stabilization conditions for weak solution of inequality (1.1). In so doing, no initial conditions on solutions of (1.1) are imposed. We even admit that \( u(x, t) \) can tend to infinity as \( t \to +0 \). We also impose no ellipticity conditions on the coefficients \( a_\alpha \) of the differential operator. Thus, our results can be applied to both parabolic and so-called anti-parabolic inequalities.

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2. Main results

Theorem 2.1. Let
\[ \int_{1}^{\infty} g^{-1/m}(\zeta)\zeta^{p/m-1} d\zeta < \infty \]  
(2.1)
and
\[ \lim_{t \to \infty} \text{ess inf } f = \infty \]  
(2.2)
for any compact set \( K \subset \mathbb{R}^n \). Then every non-negative weak solution of (1.1) stabilizes to zero as \( t \to \infty \) in the \( L_1 \) norm on an arbitrary compact set \( K \subset \mathbb{R}^n \), i.e.
\[ \lim_{t \to \infty} \text{ess sup } \int_{K} u(x, \tau) \, dx = 0. \]  
(2.3)

The proof of Theorem 2.1 is given in Section 3.

Remark 2.1. Since \( u \in L_{1,\text{loc}}(\mathbb{R}^{n+1}) \), the integral on the left in (2.3) is defined for almost all \( \tau \in (0, \infty) \).

Example 2.1. Consider the inequality
\[ \Delta^{m/2} u - u_t \geq f(x, t)u^\lambda \quad \text{in } \mathbb{R}^{n+1}, \]  
(2.4)
where \( m \) is a positive even integer and \( \lambda \) is a real number. By Theorem 2.1 if
\[ \lambda > 1 \]  
(2.5)
and (2.2) is valid, then every non-negative weak solution of (2.4) stabilizes to zero as \( t \to \infty \) in \( L_1 \) norm on an arbitrary compact subset of \( \mathbb{R}^n \).

Now, let us consider the inequality
\[ \Delta^{m/2} u - u_t \geq f(x, t)u \ln^\nu(1 + u) \quad \text{in } \mathbb{R}^{n+1}, \]  
(2.6)
where \( \nu \) is a real number. In other words, we examine the case of the critical exponent \( \lambda = 1 \) in the right-hand side of (2.4) “spoiled” by the logarithm. As before, we assume that \( m \) is a positive even integer.

It can easily be seen that (2.1) is equivalent to the condition
\[ \nu > m. \]  
(2.7)
Thus, if (2.2) and (2.7) are valid, then Theorem 2.1 implies that every non-negative weak solution of (2.6) stabilizes to zero as \( t \to \infty \) in \( L_1 \) norm on an arbitrary compact subset of \( \mathbb{R}^n \).

Condition (2.7) is the best possible. Really, let us show that there exists a positive function \( f \in C(\mathbb{R}^n \times [0, \infty)) \) for which (2.2) holds and the inequality
\[ \Delta^{m/2} u - u_t \geq f(x, t)u \ln^m(1 + u) \quad \text{in } \mathbb{R}^{n+1}, \]  
(2.8)
has a classical solution satisfying the bound \( u(x, t) \geq e \) for all \( (x, t) \in \mathbb{R}^{n+1} \). It is obvious that this solution is also a solution of (2.6) for all \( \nu \leq m \). We shall seek it in the form
\[ u(x, t) = e^{w(x, t)}, \]
where
\[ w(x, t) = (t + 1) \sum_{i=1}^{n} e^{x_i}. \]
By direct differentiation, one can verify that

$$\Delta^{m/2} u \geq \sum_{i=1}^{n} (t+1)^m e^{mx_i} e^{mu(x,t)} u + \sum_{i=1}^{n} (t+1) e^{-x_i} e^{w(x,t)} u$$

for all \((x,t) \in \mathbb{R}^{n+1}_+\). Since

$$u_t = \sum_{i=1}^{n} e^{x_i} e^{w(x,t)} u,$$

this yields

$$\Delta^{m/2} u - u_t \geq \sum_{i=1}^{n} (t+1)^m e^{mx_i} e^{mu(x,t)} u \tag{2.9}$$

for all \((x,t) \in \mathbb{R}^{n+1}_+\). It can be seen that

$$e^{w(x,t)} = \ln u \geq \frac{1}{2} \ln(1 + u)$$

for all \((x,t) \in \mathbb{R}^{n+1}_+\). Thus, (2.9) implies inequality (2.8) with

$$f(x,t) = \frac{1}{2m} \sum_{i=1}^{n} (t+1)^m e^{mx_i}.$$

Note that, along with (2.7), we have established the exactness of condition (2.5). In fact, any solution of (2.8) satisfying the inequality \(u \geq e\) on the set \(\mathbb{R}^{n+1}_+\) is also a solution of (2.4) for all \(\lambda \leq 1\).

Remark 2.2. If \(m = 2\) and \(p = 1\), then (2.1) takes the form

$$\int_{1}^{\infty} (g(t) t)^{-1/2} dt < \infty. \tag{2.10}$$

It is easy to see that (2.10) is equivalent to the well-known Keller-Osserman condition

$$\int_{1}^{\infty} \left( \int_{1}^{t} g(s) ds \right)^{-1/2} dt < \infty \tag{2.11}$$

on the grows of the function \(g\) at infinity [17, 18] which plays an important role in the theory of semilinear elliptic and parabolic equations (see, for instance, [14] and references therein). Really, since \(g\) is a non-decreasing positive function on the interval \((0, \infty)\), we have

$$\int_{1}^{t} g(s) ds \geq \int_{t/2}^{t} g(s) ds \geq t \frac{1}{2} g\left( \frac{t}{2} \right), \quad t > 2.$$

Hence, (2.10) implies (2.11). On the other hand,

$$\int_{1}^{t} g(s) ds \leq t g(t), \quad t > 1;$$

therefore, (2.10) follows from (2.11).

We can in this context call (2.1) as a generalized Keller-Osserman condition. In Example 2.1 it is shown that this condition is the best possible. We put forward a hypothesis that (2.1) is also a necessary stabilization condition for solutions of inequality (1.1).
3. Proof of Theorem 2.1

Below, it is assumed that \( u \) is a non-negative weak solution of (1.1). By \( C \) we mean various positive constants that can depend only on \( m, n, \) and \( p \).

Let us use the following notations. We denote \( B_r = \{ x \in \mathbb{R}^n : |x| < r \} \) and \( Q_r^{t_1,t_2} = \{ (x,t) \in \mathbb{R}^{n+1} : |x| < r, t_1 < t < t_2 \} \). Further, let \( \omega \in C^\infty(\mathbb{R}) \) be a non-negative function such that \( \supp \omega \subset (-1,1) \) and

\[
\int_{-\infty}^{\infty} \omega \, dt = 1.
\]

We need the Steklov-Schwartz averaging kernel

\[
\omega_h(t) = \frac{1}{h} \omega \left( \frac{t}{h} \right), \quad h > 0.
\] (3.1)

**Lemma 3.1.** Let \( 0 < r_1 < r_2 \) and \( 0 < h < \tau_1 < \tau_2 < \tau \) be some real numbers with \( r_2 \leq 2r_1 \). If \( f(x,t) \geq 0 \) for almost all \( (x,t) \in Q_{r_2}^{\tau_2 - \tau + h} \), then

\[
\frac{1}{(r_2 - r_1)^m} \int_{Q_{r_2}^{\tau_2 - \tau + h} \setminus Q_{r_1}^{\tau_2 - \tau + h}} u^p \, dx \, dt + \frac{1}{\tau_2 - \tau_1} \int_{Q_{r_1}^{\tau_1 - \rho - h}} u \, dx \, dt
\]

\[
\geq C \left( \int_{Q_{r_1}^{\tau_1 - \rho - h}} f(x,t) g(u) \, dx \, dt + \int_{Q_{r_1}^{\tau_1 - \rho - h}} \omega_h(\tau - t) u \, dx \, dt \right).
\] (3.2)

**Proof.** We take a non-decreasing function \( \varphi_0 \in C^\infty(\mathbb{R}) \) such that

\[
\varphi_0|_{(-\infty,0]} = 0 \quad \text{and} \quad \varphi_0|_{[1,\infty)} = 1.
\]

Also let

\[
\eta(t) = \int_t^\infty \omega_h(\tau - \xi) \, d\xi.
\]

Using

\[
\varphi(x,t) = \varphi_0 \left( \frac{r_2 - |x|}{r_2 - r_1} \right) \varphi_0 \left( \frac{t - \tau + \tau_2}{\tau_2 - \tau_1} \right) \eta(t)
\]

as a test function in (1.3), we obtain

\[
\int_{\mathbb{R}^{n+1}} \sum_{|\alpha| = m} (-1)^m a_\alpha(x,t,u) \partial^\alpha \varphi_0 \left( \frac{r_2 - |x|}{r_2 - r_1} \right) \varphi_0 \left( \frac{t - \tau + \tau_2}{\tau_2 - \tau_1} \right) \eta(t) \, dx \, dt
\]

\[
+ \int_{\mathbb{R}^{n+1}} u \varphi_0 \left( \frac{r_2 - |x|}{r_2 - r_1} \right) \frac{\partial \varphi_0}{\partial t} \left( \frac{t - \tau + \tau_2}{\tau_2 - \tau_1} \right) \eta(t) \, dx \, dt
\]

\[
\geq \int_{\mathbb{R}^{n+1}} f(x,t) g(u) \varphi_0 \left( \frac{r_2 - |x|}{r_2 - r_1} \right) \varphi_0 \left( \frac{t - \tau + \tau_2}{\tau_2 - \tau_1} \right) \eta(t) \, dx \, dt
\]

\[
+ \int_{\mathbb{R}^{n+1}} u \varphi_0 \left( \frac{r_2 - |x|}{r_2 - r_1} \right) \varphi_0 \left( \frac{t - \tau + \tau_2}{\tau_2 - \tau_1} \right) \omega_h(\tau - t) \, dx \, dt.
\] (3.3)

Condition (1.2) allows us to assert that

\[
\frac{1}{(r_2 - r_1)^m} \int_{Q_{r_2}^{\tau_2 - \tau + h} \setminus Q_{r_1}^{\tau_2 - \tau + h}} u^p \, dx \, dt
\]

\[
\geq C \int_{\mathbb{R}^{n+1}} \left| \sum_{|\alpha| = m} a_\alpha(x,t,u) \partial^\alpha \varphi_0 \left( \frac{r_2 - |x|}{r_2 - r_1} \right) \varphi_0 \left( \frac{t - \tau + \tau_2}{\tau_2 - \tau_1} \right) \eta(t) \right| \, dx \, dt.
\]
In so doing, we obviously have
\[
\frac{1}{\tau_2 - \tau_1} \int_{Q_{\tau_2}^{\tau_2,\tau - \tau_1}} u \, dx dt \geq C \int_{\mathbb{R}^{n+1}_+} u \varphi_0 \left( \frac{r_2 - |x|}{r_2 - r_1} \right) \frac{\partial \varphi_0}{\partial t} \left( \frac{t - \tau + \tau_2}{\tau_2 - \tau_1} \right) \eta(t) \, dx dt
\]
and
\[
\int_{\mathbb{R}^{n+1}_+} f(x,t)g(u) \varphi_0 \left( \frac{r_2 - |x|}{r_2 - r_1} \right) \varphi_0 \left( \frac{t - \tau + \tau_2}{\tau_2 - \tau_1} \right) \eta(t) \, dx dt \geq \int_{Q_{\tau_1}^{\tau_1,\tau - h}} f(x,t)g(u) \, dx dt.
\]
Finally, since \( \omega_h(\tau - t) = 0 \) for all \( t \not\in (\tau - h, \tau + h) \) and \( \varphi_0((t - \tau + \tau_2)/(\tau_2 - \tau_1)) = 1 \) for all \( t > \tau - h \), the second summand in the right-hand side of (3.3) can be estimated as follows:
\[
\int_{\mathbb{R}^{n+1}_+} u \varphi_0 \left( \frac{r_2 - |x|}{r_2 - r_1} \right) \varphi_0 \left( \frac{t - \tau + \tau_2}{\tau_2 - \tau_1} \right) \omega_h(\tau - t) \, dx dt \geq \int_{Q_{\tau_1}^{\tau_1,\tau - h}} \omega_h(\tau - t)u \, dx dt.
\]
Thus, combining (3.3) with the last four inequalities, we deduce (3.2). \( \square \)

For any real number \( R > 0 \) we define the function
\[
J_R(r,\tau) = \frac{1}{\text{mes} Q_{2R}^{r-2mR^m,\tau}} \int_{Q_{r}^{r,m,\tau}} u^p \, dx dt,
\]
where \( 0 < r \leq 2R \) and \( \tau > 2^m R^m \). Also put
\[
G(\zeta) = g(\zeta^{1/p}), \quad \zeta \geq 0.
\]

**Lemma 3.2.** Let \( R > 0 \) and \( \tau > 0 \) be some real numbers such that \( \tau > 2^m R^m \) and \( f(x,t) \geq 0 \) for almost all \( (x,t) \in Q_{2R}^{r-2mR^m,\tau} \). Then for all \( r_1 \) and \( r_2 \) satisfying the condition \( R \leq r_1 < r_2 \leq 2R \) at least one of the following two inequalities is valid:
\[
J_R(r_2,\tau) - J_R(r_1,\tau) \geq C(r_2 - r_1)^m G(J_R(r_1,\tau)) \essinf_{Q_{2R}^{r-2mR^m,\tau}} f, \quad (3.4)
\]
\[
J_R(r_2,\tau) - J_R(r_1,\tau) \geq CR^{mp-1}(r_2 - r_1)G^p(J_R(r_1,\tau)) \essinf_{Q_{2R}^{r-2mR^m,\tau}} f^p. \quad (3.5)
\]

**Proof.** Inequality (3.2) of Lemma 3.1 with \( \tau_1 = r_{1}^m \) and \( \tau_2 = r_{2}^m \) yields
\[
\frac{1}{(r_2 - r_1)^m} \int_{Q_{r_2}^{r_2,m,\tau+h}|Q_{r_2}^{r_2,m,\tau+h}} u^p \, dx dt + \frac{1}{r_2^m - r_1^m} \int_{Q_{r_1}^{r_1,m,\tau-h}} u \, dx dt \geq C \int_{Q_{r_1}^{r_1,m,\tau-h}} f(x,t)g(u) \, dx dt
\]
for all sufficiently small \( h > 0 \). Note that the second summand in the right-hand side of (3.2) is non-negative; therefore, it can be dropped. Passing to the limit as \( h \to +0 \) in the last estimate, we obviously obtain
\[
\frac{1}{(r_2 - r_1)^m} \int_{Q_{r_2}^{r_2,m,\tau}|Q_{r_2}^{r_2,m,\tau}} u^p \, dx dt + \frac{1}{r_2^m - r_1^m} \int_{Q_{r_2}^{r_2,m,\tau-h}} u \, dx dt \geq C \int_{Q_{r_1}^{r_1,m,\tau}} f(x,t)g(u) \, dx dt. \quad (3.6)
\]
Assume that
\[
\frac{1}{(r_2 - r_1)^m} \int_{Q_{r_2}^{r_2^m, \tau} \setminus Q_{r_1}^{r_1^m, \tau}} u^p \, dxdt \geq \frac{1}{r_2^m - r_1^m} \int_{Q_{r_2}^{r_2^m, \tau - r_1^m}} u \, dxdt. \tag{3.7}
\]

Then (3.6) implies the inequality
\[
\int_{Q_{r_2}^{r_2^m, \tau} \setminus Q_{r_1}^{r_1^m, \tau}} u^p \, dxdt \geq C(r_2 - r_1)^m \int_{Q_{r_1}^{r_1^m, \tau}} f(x, t) g(u) \, dxdt. \tag{3.8}
\]

We have
\[
Q_{r_2}^{r_2^m, \tau} \setminus Q_{r_1}^{r_1^m, \tau} \subset Q_{r_2}^{r_2^m, \tau} \setminus Q_{r_1}^{r_1^m, \tau},
\]

therefore,
\[
J_R(r_2, \tau) - J_R(r_1, \tau) = \frac{1}{\text{mes} \, Q_{2R}^{r_2^m R^m, \tau}} \int_{Q_{r_2}^{r_2^m, \tau} \setminus Q_{r_1}^{r_1^m, \tau}} u^p \, dxdt \\
\geq \frac{1}{\text{mes} \, Q_{2R}^{r_2^m R^m, \tau}} \int_{Q_{r_2}^{r_2^m, \tau} \setminus Q_{r_1}^{r_1^m, \tau}} u^p \, dxdt.
\]

Combining this with (3.8), we obtain
\[
J_R(r_2, \tau) - J_R(r_1, \tau) \geq \frac{C(r_2 - r_1)^m}{\text{mes} \, Q_{2R}^{r_2^m R^m, \tau}} \int_{Q_{r_1}^{r_1^m, \tau}} f(x, t) g(u) \, dxdt. \tag{3.9}
\]

Let us establish the validity of the estimate
\[
\int_{Q_{r_1}^{r_1^m, \tau}} f(x, t) g(u) \, dxdt \geq G(J_R(r_1, \tau)) \text{ mes} \, Q_{r_1}^{r_1^m, \tau} \, \text{ess inf}_f. \tag{3.10}
\]

Really, if
\[
\text{ess inf}_f = 0,
\]

then (3.10) is evident; therefore, it can be assumed without loss of generality that
\[
\text{ess inf}_f > 0.
\]

In this case, we have
\[
\int_{Q_{r_1}^{r_1^m, \tau}} g(u) \, dxdt < \infty,
\]

since
\[
\int_{Q_{r_1}^{r_1^m, \tau}} f(x, t) g(u) \, dxdt < \infty.
\]

In so doing, we obviously obtain
\[
\int_{Q_{r_1}^{r_1^m, \tau}} f(x, t) g(u) \, dxdt \geq \text{ess inf}_f \int_{Q_{r_1}^{r_1^m, \tau}} g(u) \, dxdt. \tag{3.11}
\]

Since \( G \) is a non-decreasing convex function, one can assert that
\[
\frac{1}{\text{mes} \, Q_{r_1}^{r_1^m, \tau}} \int_{Q_{r_1}^{r_1^m, \tau}} g(u) \, dxdt = \frac{1}{\text{mes} \, Q_{r_1}^{r_1^m, \tau}} \int_{Q_{r_1}^{r_1^m, \tau}} G(u^p) \, dxdt \\
\geq G\left( \frac{1}{\text{mes} \, Q_{r_1}^{r_1^m, \tau}} \int_{Q_{r_1}^{r_1^m, \tau}} u^p \, dxdt \right) \geq G(J_R(r_1, \tau)),
\]
whence in turn it follows that
\[ \int_{Q_{r_1}^{r_1} \setminus Q_{r_1}^{r_1}} g(u) \, dx \, dt \geq G(J_R(r_1, \tau)) \, \operatorname{mes} Q_{r_1}^{r_1} \tau. \]

In view of (3.11), this implies (3.10).

Further, it does not present any particular problem to verify that
\[ A_1 R^{m+n} \leq \operatorname{mes} Q_{r_1}^{r_1} \tau \leq A_2 R^{m+n} \quad (3.12) \]
for all \( R \leq r \leq 2R \), where the constants \( A_1 > 0 \) and \( A_2 > 0 \) depend only on \( m \) and \( n \). Thus, combining (3.9) and (3.10), we arrive at (3.4).

Now, assume that the opposite inequality to (3.7) holds, i.e.
\[ \frac{1}{(r_2 - r_1)^m} \int_{Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1}} u^p \, dx \, dt < \frac{1}{r_2 - r_1^m} \int_{Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1}} u \, dx \, dt. \]

Combining this with (3.6), we have
\[ \int_{Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1}} u \, dx \, dt \geq C(r_2^m - r_1^m) \int_{Q_{r_1}^{r_1}} f(x, t) g(u) \, dx \, dt. \quad (3.13) \]

By the Hölder inequality, one can show that
\[ \int_{Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1}} u \, dx \, dt \leq \left( \operatorname{mes} Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1} \right)^{(p-1)/p} \left( \int_{Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1}} u^p \, dx \, dt \right)^{1/p}, \]
whence, taking into account the fact that
\[ \operatorname{mes} Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1} \leq C(r_2^m - r_1^m) R^m, \]
we obtain
\[ \int_{Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1}} u \, dx \, dt \leq C(r_2^m - r_1^m)^{(p-1)/p} R^{m(p-1)/p} \left( \int_{Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1}} u^p \, dx \, dt \right)^{1/p}. \]

Thus, (3.13) implies the estimate
\[ \int_{Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1}} u^p \, dx \, dt \geq \frac{C(r_2^m - r_1^m)}{R^{m(p-1)}} \left( \int_{Q_{r_1}^{r_1}} f(x, t) g(u) \, dx \, dt \right)^p. \quad (3.14) \]

Due to the obvious inequality
\[ r_2^m - r_1^m \geq C(r_2 - r_1) R^{m-1} \]
and relationship (3.10) estimate (3.14) yields
\[ \int_{Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1}} u^p \, dx \, dt \geq C R^{m(p+1)+n-1} (r_2 - r_1) G^p(J_R(r_1, \tau)) \, \operatorname{ess inf} \, f^p. \quad (3.15) \]

Since
\[ Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1} \subset Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1} \tau, \]
we have
\[ J_R(r_2, \tau) - J_R(r_1, \tau) = \frac{1}{\operatorname{mes} Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1}} \int_{Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1}} u^p \, dx \, dt \]
\[ \geq \frac{1}{\operatorname{mes} Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1}} \int_{Q_{r_2}^{r_2} \setminus Q_{r_1}^{r_1}} u^p \, dx \, dt. \]
In view of (3.12), this implies the estimate
\[
J_R(r_2, \tau) - J_R(r_1, \tau) \geq \frac{C}{R^{m+n}} \int_{Q_R^{r_2-r_1}} u^p \, dx \, dt
\]
from which, taking into account (3.15), we obtain (3.5).

Lemma 3.3. Let \( R > 0 \) and \( \tau > 0 \) be some real numbers such that \( \tau > 2^m R^m \) and
\[
\int_{Q_R^{R^m, \tau}} u^p \, dx \, dt > 0.
\] (3.16)

If \( f(x, t) \geq 0 \) for almost all \((x, t) \in Q_{2R}^{r-2^m R^m, \infty}, \) then
\[
\int_{J_R(R, \tau)}^{\infty} G^{-1/m}(\zeta/2) \zeta^{1/m-1} \, d\zeta + \int_{J_R(R, \tau)}^{\infty} \frac{d\zeta}{G(\zeta/2)} + \int_{J_R(R, \tau)}^{\infty} \frac{d\zeta}{G'p(\zeta/2)} \geq C \min \left\{ \frac{R}{\ln R} \text{ ess inf}_{Q_{2R}^{r-2^m R^m, \tau}} f^{1/m}, \frac{R}{\ln R} \text{ ess inf}_{Q_{2R}^{r-2^m R^m, \tau}} f^p \right\}.
\] (3.17)

Proof. We construct a finite sequence of real numbers \( \{r_i\}_{i=0}^l \) as follows. Let us take \( r_0 = R. \) Assume further that \( r_i \) is already defined. If \( r_i \geq 3R/2, \) then we put \( l = i \) and stop; otherwise we take
\[
r_{i+1} = \sup \{ r \in [r_i, 2R] : J_R(r, \tau) \leq 2J_R(r_i, \tau) \}.
\]
Since \( u \in L_{\text{p, loc}}(\mathbb{R}^{n+1}) \), this procedure must terminate at a finite step. It follows from (3.10) that \( J_R(r_i, \tau) > 0 \) for all \( 0 \leq i \leq l; \) therefore, \( \{r_i\}_{i=0}^l \) is a strictly increasing sequence. It can also be seen that
\[
J_R(r_{i+1}, \tau) = 2J_R(r_i, \tau) \quad \text{for all } 0 \leq i \leq l - 2
\]
and
\[
J_R(r_i, \tau) \leq 2J_R(r_{l-1}, \tau).
\]
Moreover, if \( J_R(r_i, \tau) < 2J_R(r_{l-1}, \tau), \) then \( r_l = 2R \) and \( r_i - r_{l-1} \geq R/2. \)

In view of Lemma 3.2 for any \( 0 \leq i \leq l - 1 \) at least one of the following two inequalities is valid:
\[
J_R(r_{i+1}, \tau) - J_R(r_i, \tau) \geq C(r_{i+1} - r_i)^m G(J_R(r_i, \tau)) \text{ ess inf}_{Q_{2R}^{r-2^m R^m, \tau}} f,
\] (3.18)
\[
J_R(r_{i+1}, \tau) - J_R(r_i, \tau) \geq CR^{mp-1}(r_{i+1} - r_i)G''p(J_R(r_i, \tau)) \text{ ess inf}_{Q_{2R}^{r-2^m R^m, \tau}} f^p.
\] (3.19)

By \( \Xi_1 \) we denote the set of integers \( 0 \leq i \leq l - 1 \) for which (3.18) is valid. In so doing, let \( \Xi_2 = \{0, \ldots, l - 1\} \setminus \Xi_1. \)

We claim that
\[
\int_{J_R(r_i, \tau)}^{\infty} G^{-1/m}(\zeta/2) \zeta^{1/m-1} \, d\zeta + \int_{J_R(r_i, \tau)}^{\infty} \frac{d\zeta}{G(\zeta/2)} + \int_{J_R(r_i, \tau)}^{\infty} \frac{d\zeta}{G'p(\zeta/2)} \geq C(r_{i+1} - r_i) \min \left\{ \text{ ess inf}_{Q_{2R}^{r-2^m R^m, \tau}} f^{1/m}, \text{ ess inf}_{Q_{2R}^{r-2^m R^m, \tau}} f^p \right\}
\] (3.20)
for all \( i \in \Xi_1. \) Really, (3.18) implies the inequality
\[
\left( \frac{J_R(r_{i+1}, \tau) - J_R(r_i, \tau)}{G(J_R(r_i, \tau))} \right)^{1/m} \geq C(r_{i+1} - r_i) \text{ ess inf}_{Q_{2R}^{r-2^m R^m, \tau}} f^{1/m}.
\]
If $J_R(r_{i+1}, \tau) = 2J_R(r_i, \tau)$, then due to monotonicity of the function $G$ we have
\[
\int_{J_R(r_i, \tau)}^{J_R(r_{i+1}, \tau)} G^{-1/m}(\zeta/2)\zeta^{1/m-1} d\zeta \geq C \left(\frac{J_R(r_{i+1}, \tau) - J_R(r_i, \tau)}{G(J_R(r_i, \tau))}\right)^{1/m}.
\]
Combining the last two inequalities, we get
\[
\int_{J_R(r_i, \tau)}^{J_R(r_{i+1}, \tau)} G^{-1/m}(\zeta/2)\zeta^{1/m-1} d\zeta \geq C(r_{i+1} - r_i) \quad \text{ess inf}_{Q_{2r_i}^{2m-R_m,\tau}} f^{1/m},
\]
whence (3.20) follows at once.

In turn, if $J_R(r_{i+1}, \tau) < 2J_R(r_i, \tau)$ for some $i \in \Xi_1$, then $i = l - 1$ and, moreover, $r_{i+1} - r_i \geq R/2$. Thus, (3.13) implies the estimate
\[
\frac{J_R(r_{i+1}, \tau) - J_R(r_i, \tau)}{G(J_R(r_i, \tau))} \geq CR^{m-1}(r_{i+1} - r_i) \quad \text{ess inf}_{Q_{2r_i}^{2m-R_m,\tau}} f.
\]
Since
\[
\int_{J_R(r_i, \tau)}^{J_R(r_{i+1}, \tau)} \frac{d\zeta}{G(\zeta/2)} \geq \frac{C(J_R(r_{i+1}, \tau) - J_R(r_i, \tau))}{G(J_R(r_i, \tau))},
\]
this yields
\[
\int_{J_R(r_i, \tau)}^{J_R(r_{i+1}, \tau)} \frac{d\zeta}{G(\zeta/2)} \geq CR^{m-1}(r_{i+1} - r_i) \quad \text{ess inf}_{Q_{2r_i}^{2m-R_m,\tau}} f,
\]
whence we again derive (3.20).

In a similar way, it can be shown that
\[
\int_{J_R(r_i, \tau)}^{J_R(r_{i+1}, \tau)} \frac{d\zeta}{G_p(\zeta/2)} \geq CR^{mp-1}(r_{i+1} - r_i) \quad \text{ess inf}_{Q_{2r_i}^{2m-R_m,\tau}} f^p
\]
for all $i \in \Xi_2$. Really, taking into account (3.19), we have
\[
\frac{J_R(r_{i+1}, \tau) - J_R(r_i, \tau)}{G_p(J_R(r_i, \tau))} \geq CR^{mp-1}(r_{i+1} - r_i) \quad \text{ess inf}_{Q_{2r_i}^{2m-R_m,\tau}} f^p.
\]
In view of the inequality
\[
\int_{J_R(r_i, \tau)}^{J_R(r_{i+1}, \tau)} \frac{d\zeta}{G_p(\zeta/2)} \geq \frac{C(J_R(r_{i+1}, \tau) - J_R(r_i, \tau))}{G_p(J_R(r_i, \tau))},
\]
this obviously implies (3.21).

Further, summing (3.20) over all $i \in \Xi_1$, we obtain
\[
\int_{J_R(r_0, \tau)}^{\infty} G^{-1/m}(\zeta/2)\zeta^{1/m-1} d\zeta + \int_{J_R(r_0, \tau)}^{\infty} \frac{d\zeta}{G(\zeta/2)} \geq C \min \left\{ \frac{\text{ess inf}_{Q_{2r_i}^{2m-R_m,\tau}} f^{1/m}}{R^{m-1}}, \frac{\text{ess inf}_{Q_{2r_i}^{2m-R_m,\tau}} f}{R^{m-2m-R_m,\tau}} \right\} \sum_{i \in \Xi_1} (r_{i+1} - r_i).
\]
Analogously, (3.21) yields
\[
\int_{J_R(r_0, \tau)}^{\infty} \frac{d\zeta}{G_p(\zeta/2)} \geq CR^{mp-1} \frac{\text{ess inf}_{Q_{2r_i}^{2m-R_m,\tau}} f^p}{R^{m-2m-R_m,\tau}} \sum_{i \in \Xi_2} (r_{i+1} - r_i).
\]
Thus, summing the last two inequalities, we conclude that
\[
\int_{J_R(r_0,\tau)} G^{-1/m}(\zeta/2)\zeta^{1/m-1} \, d\zeta + \int_{J_R(r_0,\tau)} \frac{d\zeta}{G(\zeta/2)} + \int_{J_R(r_0,\tau)} \frac{d\zeta}{G^p(\zeta/2)}
\geq C \min \left\{ \text{ess inf}_{Q_{2R}^{r_0}} f^{1/m}, R^{m-1} \text{ess inf}_{Q_{2R}^{r_0}} f, R^{mp-1} \text{ess inf}_{Q_{2R}^{r_0}} f^p \right\} (r_l - r_0).
\]
To complete the proof, it remains to note that \( r_l - r_0 \geq R/2 \) by the construction of the sequence \( \{r_i\}_{i=0}^\infty \).

We need the following known assertion.

\textbf{Lemma 3.4} (see [19, Lemma 2.3]). Let \( \psi : (0, \infty) \to (0, \infty) \) and \( \gamma : (0, \infty) \to (0, \infty) \) be measurable functions satisfying the condition
\[
\gamma(\zeta) \leq \text{ess inf}_{\{\zeta/\theta, \theta \zeta\}} \psi \quad \text{for almost all } \zeta \in (0, \infty).
\]
with some real number \( \theta > 1 \) for almost all \( \zeta \in (0, \infty) \). Also assume that \( 0 < \alpha \leq 1, M_1 > 0, M_2 > 0, \) and \( \nu > 1 \) are some real numbers with \( M_2 \geq \nu M_1 \). Then
\[
\left( \int_{M_1}^{M_2} \gamma^{-\alpha}(\zeta)\zeta^{\alpha-1} \, d\zeta \right)^{1/\alpha} \geq A \int_{M_1}^{M_2} \frac{d\zeta}{\psi(\zeta)},
\]
where the constant \( A > 0 \) depends only on \( \alpha, \nu, \) and \( \theta \).

\textbf{Lemma 3.5.} Let the hypotheses of Theorem 2.1 be valid, then
\[
\int_{Q_{1-R^{m},r}^{2R}} u^p \, dx \, dt \to 0 \quad \text{as } \tau \to \infty \quad (3.22)
\]
for any real number \( R > 0 \).

\textit{Proof.} It can easily be seen that (2.1) is equivalent to the condition
\[
\int_{1}^{\infty} G^{-1/m}(\zeta/2)\zeta^{1/m-1} \, d\zeta < \infty.
\]
By Lemma 3.4 we obtain
\[
\left( \int_{1}^{\infty} G^{-1/m}(\zeta/2)\zeta^{1/m-1} \, d\zeta \right)^m \geq C \int_{1}^{\infty} \frac{d\zeta}{G(\zeta)}.
\]
This allows us to assert that
\[
\int_{1}^{\infty} \frac{d\zeta}{G(\zeta)} < \infty.
\]
Since \( G \) is a non-decreasing positive function on the interval \( (0, \infty) \), we have \( G^p(\zeta) \geq G^{p-1}(1)G(\zeta) \) for all \( \zeta \geq 1 \). Hence, we can also assert that
\[
\int_{1}^{\infty} \frac{d\zeta}{G^p(\zeta)} < \infty.
\]
Assume that \( R > 0 \) is some given real number. In view of (2.2), \( f \) is a non-negative function almost everywhere on \( Q_{2R}^{r_2-2m,R^{m},\infty} \) for all sufficiently large \( \tau \). In so doing, it is obvious that the right-hand side of (3.17) tends to infinity as \( \tau \to \infty \). Thus, applying Lemma 3.3 we obtain
\[
J_R(R, \tau) \to 0 \quad \text{as } \tau \to \infty,
\]
whence (3.22) follows an once.
Proof of Theorem 2.1. Let $K$ be a compact subset of $\mathbb{R}^n$ and $R > 0$ be a real number such that $K \subset B_{R/2}$. We denote

$$U(t) = \int_{B_{R/2}} u(x, t) \, dx.$$  \hspace{1cm} (3.23)

Since $u \in L_{1,\text{loc}}(\mathbb{R}^{n+1})$, the right-hand side of (3.23) is defined for almost all $t \in (0, \infty)$ and, moreover, $U \in L_{1,\text{loc}}(0, \infty)$. Let us put

$$U_h(\tau) = \int_0^\infty \omega_h(\tau - t)U(t) \, dt, \quad \tau > h > 0,$$

where $\omega_h$ is given by (3.1). We have

$$\|U_h - U\|_{L_1(H)} \to 0 \quad \text{as} \quad h \to +0$$

for any compact set $H \subset (0, \infty)$. Hence, there exists a sequence of positive real numbers $\{h_i\}_{i=1}^\infty$ such that $h_i \to 0$ as $i \to \infty$ and

$$\lim_{i \to \infty} U_{h_i}(\tau) = U(\tau)$$

for almost all $\tau \in (0, \infty)$. Also let $\tau_0 > R^m$ be a real number such that $f$ is a non-negative function almost everywhere on $Q_{R}^{\tau_0 - R^m, \infty}$. In view of (2.2), such a real number obviously exists.

Lemma 3.1 with $r_1 = R/2$, $r_2 = R$, $\tau_1 = (R/2)^m$, and $\tau_2 = R^m$ yields

$$\frac{1}{R^m} \int_{Q_{R}^{\tau - R^m, \tau + h_i}} u^p \, dx \, dt + \frac{1}{R^m - (R/2)^m} \int_{Q_{R}^{\tau - R^m, \tau - (R/2)^m}} u \, dx \, dt \geq C \int_{Q_{R/2}^{\tau - h_i, \tau + h_i}} \omega_{h_i}(\tau - t)u \, dx \, dt$$  \hspace{1cm} (3.24)

for all $\tau > \tau_0$ and for all $i$ such that $h_i < (R/2)^m$. Note that the first summand in the right-hand side of (3.2) is non-negative; therefore, it can be dropped.

Since

$$\int_{Q_{R/2}^{\tau - h_i, \tau + h_i}} \omega_{h_i}(\tau - t)u \, dx \, dt = U_{h_i}(\tau), \quad \tau > h_i,$$

passing to the limit as $i \to \infty$ in (3.24), we obtain

$$\frac{1}{R^m} \int_{Q_{R}^{\tau - R^m, \tau}} u^p \, dx \, dt + \frac{1}{R^m - (R/2)^m} \int_{Q_{R}^{\tau - R^m, \tau - (R/2)^m}} u \, dx \, dt \geq CU(\tau)$$  \hspace{1cm} (3.25)

for almost all $\tau \in (\tau_0, \infty)$. By the Hölder inequality,

$$\int_{Q_{R}^{\tau - R^m, \tau - (R/2)^m}} u \, dx \, dt \leq \left( \int_{Q_{R}^{\tau - R^m, \tau}} u^p \, dx \, dt \right)^{1/p} \left( \int_{Q_{R}^{\tau - R^m, \tau}} u \, dx \, dt \right)^{1/(1-p)}.$$

Thus, (3.25) implies the estimate

$$R^{-m} \int_{Q_{R}^{\tau - R^m, \tau}} u^p \, dx \, dt + R^{n-(m+n)/p} \left( \int_{Q_{R}^{\tau - R^m, \tau}} u^p \, dx \, dt \right)^{1/p} \geq CU(\tau)$$

for almost all $\tau \in (\tau_0, \infty)$. To complete the proof, it remains to use Lemma 3.5. \hfill $\Box$
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