Dynamical entanglement entropy with angular momentum and U(1) charge

Paweł Caputa$^{a,b}$, Gautam Mandal$^b$ and Ritam Sinha$^{b,3}$

$^a$NITheP, Department of Physics and Centre for Theoretical Physics
   University of the Witwatersrand, Wits, 2050, South Africa
$^b$Department of Theoretical Physics
   Tata Institute of Fundamental Research, Mumbai 400005, India.

Abstract

We consider time-dependent entanglement entropy (EE) for a 1+1 dimensional CFT in the presence of angular momentum and U(1) charge. The EE saturates, irrespective of the initial state, to the grand canonical entropy after a time large compared with the length of the entangling interval. We reproduce the CFT results from an AdS dual consisting of a spinning BTZ black hole and a flat U(1) connection. The apparent discrepancy that the holographic EE does not \textit{a priori} depend on the U(1) charge while the CFT EE does, is resolved by the charge-dependent shift between the bulk and boundary stress tensors. We show that for small entangling intervals, the entanglement entropy obeys the first law of thermodynamics, as conjectured recently. The saturation of the EE in the field theory is shown to follow from a version of quantum ergodicity; the derivation indicates that it should hold for conformal as well as massive theories in any number of dimensions.
1 Introduction and Summary

Entanglement entropy (EE) of a quantum system has turned out to be a useful observable in many areas of physics; see reviews [1, 2, 3]. In this paper, we will primarily use EE as a dynamical tool, especially to describe equilibration in 1+1 dimensional quantum field theories. Time-dependent EE in 1+1 dimensional CFT has been studied in detail in [4, 2]. Let us consider a CFT with an infinite spatial direction; the EE for a single interval of length $l$, is found to saturate, according to the formula

$$S_{\text{ent}}(t, l|\psi) \xrightarrow{t \gg l} l s_{\text{eqm}}(E), \quad s_{\text{eqm}}(E) = \sqrt{2\pi cE/3} \quad (1.1)$$

In the above equation, the LHS is the EE of an interval of length $l$, computed in the state $|\psi\rangle$ at time $t$; we will denote the energy density of the state as $E$. $s_{\text{eqm}}(E)$ is the equilibrium

---

4In [4, 2], the entropy density is given by $s_{\text{eqm}} = c\pi/(3\beta)$, where $\beta$ is the inverse temperature of a canonical ensemble equivalent to a microcanonical ensemble at energy E, given by $\beta = \sqrt{\pi c/6E}$. With this, we recover the RHS of (1.1).
entropy density in the microcanonical ensemble as a function of energy density \( E \). It is assumed here that length of the interval \( l \) is greater than the characteristic length scale \( 1/\sqrt{E} \) associated with the state \( |\psi\rangle \) (this condition will play an important role in Section 8). \[5\) showed that the time-development in (1.1) can be interpreted holographically in terms of a BTZ black hole, and derived (1.1) using the Ryu-Takayanagi definition of holographic EE \[6\] (see \[7, 8, 9\] for other recent works on holographic thermalization using dynamic EE). The linear growth in time was given an intuitive explanation in terms of oppositely moving entangled pair of excitations \[4\]; one of the objectives of this paper is to explain the saturation value in terms of quantum ergodicity.

An important point to note about (1.1) is the *information loss* aspect of this equation: on the RHS of (1.1), all information about the specific state \( |\psi\rangle \) appears to be lost, other than energy \( E \) of the state. We will elaborate on this further in Section 2.3. Indeed, the above statement of equilibration is similar in spirit to the following statement of quantum ergodicity (see \[10\], Section III-A)

\[
\text{Tr} (\rho_{\text{pure}} O) \xrightarrow{t \to \infty} \text{Tr} (\rho_{\text{mc}} O), \quad \rho_{\text{pure}} = |\psi\rangle \langle \psi|, \quad \rho_{\text{mc}} = \frac{1}{\Omega(E)} \sum_{i \in \mathcal{H}_E} |i\rangle \langle i|, \quad (1.2)
\]

which is believed to be true for a class of “macroscopic” observables \( O \). Here, \( \rho_{\text{mc}} \) defines a microcanonical ensemble at energy \( E \); \( \mathcal{H}_E \) denotes the subspace of states with this energy, and \( \Omega(E) \) is the dimension of \( \mathcal{H}_E \). We will, in fact, derive (1.1) from (1.2), modulo some assumptions, in Section 7. Thus, in the time-development described in (1.1) not only is the memory of the initial state lost, the RHS is given in terms of a mixed state. It is worth noting that in (1.2) no mention is made about the time scale of change; thus, the time-development in (1.1) provides a time scale for equilibration.

In this paper, we will show that the above statement of equilibration also holds in the presence of additional conserved charges, besides the energy \( E \). In particular, we will show that if the initial state \( |\psi\rangle \) has a non-zero angular momentum \( J \) and a U(1) charge \( Q \), we have \[6\]

\[
S_{\text{ent}}(t, l | \psi) \xrightarrow{t \gg l} l s_{\text{eqm}}(E, J, Q) \quad (1.3)
\]

Here \( s_{\text{eqm}}(E, J, Q) \) equals the equilibrium entropy density in a microcanonical ensemble, described in Eqs. (2.19) and (4.7), which give the detailed form of the time-dependence. Eq. (1.3) is the main result of our paper; it is presented here in the limit in which the excitation energy of the initial state is much higher than \( 1/t, 1/l \). The precise version of this statement as well as the exact expression for the LHS without this restriction is given in Sections 2 and 4. We derive (1.3) also from a holographic set-up (Sections 3 and 5). The holographic dual consists of a spinning BTZ black hole plus a U(1) gauge field described by a CS theory.

\[5\] In this paper, we will mostly be concerned with a non-compact spatial direction, so \( J \) is actually a linear momentum. However, we regard this non-compact direction as arising in the limit of a large circle (the bulk dual is a BTZ black string which can be regarded as the limit of a BTZ black hole), and will continue to call \( J \) an ‘angular momentum’.

\[6\] The divergent piece \( S_{\text{div}} \) is the same as in the previous literature, including in \[4, 5\]. We do not have anything new to add regarding this term; for a recent discussion, see \[11\].
Eq. (1.3) leads us to the following natural conjecture for an integrable 1+1 dimensional CFT. Suppose the initial state $|\psi\rangle$ has an infinite number of non-zero conserved charges $Q_i, i = 1, ..., \infty$ (including energy). We conjecture that the long time behaviour of the EE in this case is given by

$$S_{\text{ent}}(t, L | \psi) \xrightarrow{t \gg l} s_{\text{eqm}}(\{Q_i\})$$

$$s_{\text{eqm}}(\{Q_i\}) = s_{\text{GGE}}(\{\mu_i\})$$ (1.4)

where $\mu_i$ are values of chemical potentials conjugate to the infinite number of charges $Q_i$ carried by the quantum state $|\psi\rangle$. In the second line, we have used the equivalence between the microcanonical ensemble (with infinite number of charges) and the generalized Gibbs ensemble (GGE). The corresponding generalization of (1.2) to 2-dimensional integrable systems has already been proved [12, 13]. A natural speculation about a holographic dual of the above involves higher spin black holes [14] (see Section 5.2 for a brief discussion).

A few remarks are in order:

1. We reproduce the CFT results in this paper from AdS in two ways: (a) by an explicit evaluation of the Ryu-Takayanagi (RT) formula for holographic EE and matching with CFT, and (b) showing that the RT formula follows from a conventional AdS/CFT dual of the CFT correlators of twist fields in a double scaling limit. Method (b) constitutes a ‘proof’ of the RT prescription for 1+1 dimensional CFT (see Section 3.1 for details).

2. We encounter a puzzle in applying the RT prescription for holographic EE in the presence of the U(1) charge. The U(1) charge we consider is implemented in the AdS dual by a U(1) Chern-Simons (CS) theory, and addition of a U(1) charge does not change the metric. Therefore the RT holographic EE is independent of the U(1) charge, which seems to be in conflict with the CFT expressions which clearly depend on this charge. We resolve this puzzle in Section 5.1.

This work was first presented in the Seventh Crete regional meeting in string theory [33]. After the work was complete and it was ready to be submitted, the preprints [34] appeared which have some overlap with our paper, especially in Sections 5 and 6.

## 2 Entanglement Entropies with spin: CFT

### 2.1 Thermofield double

Let us consider a 2D CFT at a finite temperature, which is represented by a thermofield double consisting of two identical copies of the CFT [15]. Consider the following initial (pure) state, belonging to the thermofield double, on the time slice $t = 0$:

$$|\psi\rangle = C \sum_i \exp[-\beta(E_i + \Omega J_i)/2] \langle i | \otimes | i \rangle$$ (2.1)

The normalization constant $C$ is given in terms of the partition function

$$|C|^{-2} = Z(\beta, \Omega) = \text{Tr} e^{-\beta(H + \Omega J)} = \text{Tr} e^{-\beta_+ L_0 - \beta_- \bar{L}_0},$$ (2.2)
and we use the following identifications
\[ H = L_0 + \bar{L}_0, \quad J = L_0 - \bar{L}_0, \quad \beta_{\pm} = \beta (1 \pm \Omega) \] (2.3)
The index \( i \) of the sums goes over a complete set of states of \( H_1 \) (equivalently \( H_2 \)). The definition implies that the expectation values \( \langle E \rangle, \langle J \rangle \) in this state are non-zero, and are related to the inverse temperature \( \beta \) and the ‘angular velocity’ \( \Omega \) (which is essentially a chemical potential for the conserved angular momentum).\(^7\) In a Euclidean spacetime, \( \Omega \) must be chosen to be purely imaginary:
\[ \Omega = i \Omega_E \] (2.4)
Now, consider two identical entangling regions \( A \subset \mathbb{R} \) in both copies of the field theory and compute the time evolution of the EE. First, we take \( A \) to be a semi-infinite line. Following the prescription in \([16]\) (see review \([2]\)) the entanglement Renyi entropy (ERE) is given by the CFT functional integral over a Riemann surface obtained by gluing \( n \) copies of a cylinder \(^8\) along two semi-infinite cuts from \( z_1 = \bar{z}_1 = 0 \) and from \( z_2 = i\beta_+/2, \bar{z}_2 = -i\beta_-/2 \), both running off to infinity (see Figure 1). Such a partition function, in turn, boils down to the two-point function of twist fields \( \phi_{\pm} \) \([16]\) at the two branch points. Thus, the ERE is given by
\[ S^{(n)}(\phi) = \frac{1}{1 - n} \log \langle \phi^+(z_1, \bar{z}_1) \phi^-(z_2, \bar{z}_2) \rangle \] (2.5)

![Figure 1: The lower branch cuts on left and right represent the (holomorphic and antiholomorphic coordinates) of the entangling interval in the first copy of the CFT. The upper branch cuts represent the second CFT.](image)

To obtain this two-point function we first map the cylinder to the plane with coordinates given by
\[ w(z) = \exp \left( \frac{2\pi z}{\beta_+} \right), \quad \bar{w}(\bar{z}) = \exp \left( \frac{2\pi \bar{z}}{\beta_-} \right) \] (2.6)

\(^7\)As mentioned in footnote \([5]\) for a non-compact spatial direction, \( J \) is actually a linear momentum; \( \Omega \) is the corresponding chemical potential.

\(^8\)We will use complex coordinates \((z, \bar{z}) = \sigma_1 \pm i \sigma_2 \) on the cylinder, with \(-H \) and \( iP \) as the generator of translation along \( \sigma_2 \) and \( \sigma_1 \). The formula (2.2) implies the following twisted identification \((\sigma_1, \sigma_2) \equiv (\sigma_1 - \beta \Omega_E, \sigma_2 + \beta) \); in terms of the complex coordinates \( z \equiv z + i\beta_+, \bar{z} \equiv \bar{z} - i\beta_- \), where we have used (2.3) and (2.4).
The two-point function on the plane of an operator $O$ with conformal dimensions $(h, \bar{h})$ is given by

$$\langle O(w_1, \bar{w}_1)O(w_2, \bar{w}_2) \rangle = \frac{1}{(w_2 - w_1)^{2h}(\bar{w}_2 - \bar{w}_1)^{2\bar{h}}}$$ (2.7)

Now, under a conformal mapping $(w, \bar{w}) \rightarrow (z, \bar{z})$, correlators transform as

$$\langle O(z_1, \bar{z}_1)O(z_2, \bar{z}_2) \rangle = \prod_i \left( \frac{dw_i}{dz_i} \right)^h \left( \frac{d\bar{w}_i}{d\bar{z}_i} \right)^{\bar{h}} \langle O(w_1, \bar{w}_1)O(w_2, \bar{w}_2) \rangle$$ (2.8)

The ERE (2.5) can be obtained by using these results and the fact that for the twist fields $\phi_{\pm}$ of order $n$

$$h = \bar{h} = \frac{c}{24} \left( n - \frac{1}{n} \right)$$ (2.9)

As explained in [16], the EE is obtained by taking the $n \rightarrow 1$ limit. This gives

$$S_{EE} = S^{(1)} = \frac{c}{6} \log \left( \frac{\beta_+ \beta_-}{\pi^2 \epsilon^2} \sinh \frac{\pi (z_2 - z_1)}{\beta_+} \sinh \frac{\pi (\bar{z}_2 - \bar{z}_1)}{\beta_-} \right)$$ (2.10)

The cut-off $\epsilon$ is used to regularize the expression, as in [16]. To explicitly evaluate (2.10) we substitute the values of $(z_{1,2}, \bar{z}_{1,2})$ mentioned above (2.5), and obtain the divergent value

$$S_{EE} = S_{EE,0} = \frac{c}{6} \log \left( \frac{\beta_+ \beta_-}{\pi^2 \epsilon^2} \right)$$ (2.11)

Here the subscript zero indicates that the EE is computed at $t = 0$.

### 2.1.1 Time-dependent EE

We will now consider the (Lorentzian) time-evolution of the thermofield state (2.1):

$$|\psi(t)\rangle = \exp[-iHt]|\psi\rangle = C \sum_i \exp[-\beta(E_i + \Omega J_i)/2 - i2E_i t] |i\rangle \otimes |i\rangle$$ (2.12)

and will compute the time-dependent ERE and EE based on this time-dependent state. In the notation of footnote 8, the total evolution operator in (2.12) translates $(\sigma_1, \sigma_2) = (0, 0) \rightarrow (\sigma_1, \sigma_2) = (-\beta \Omega E/2, \beta/2 + 2it)$. This implies the following analytically continued location of the two branch points

$$z_1 = \bar{z}_1 = 0, \quad z_2 = -2t + i \beta_+/2, \quad \bar{z}_2 = 2t - i \beta_-/2$$ (2.13)

Note that $\bar{z}_2 \neq z_2^*$; this happens because $\sigma_2$ is now complex. By using the new locations of the branch points in (2.10)

$$S_{EE} = \frac{c}{6} \log \left( \frac{\beta_+ \beta_-}{\pi^2 \epsilon^2} \cosh \frac{2\pi t}{\beta_+} \cosh \frac{2\pi t}{\beta_-} \right)$$ (2.14)

---

9This equation appears in Ref. [20], where it signifies the equilibrium EE of a finite interval of length $|z_2 - z_1|$. 

5
Clearly at large $t \gg \beta, \beta \Omega_E$, the $\cosh$ terms can be replaced by exponentials, which show that the finite part grows linearly with time:

$$S_{EE}(t) = S_{EE,0} + t (2s_{eqm}), \quad s_{eqm} = \frac{\pi c}{3\beta(1 - \Omega^2)} \quad t \gg \beta, \beta \Omega_E \quad (2.15)$$

where $s_{eqm}$ is the equilibrium entropy density, further elaborated below (2.19). $S_{EE,0}$ is already defined in (2.11).

### 2.1.2 Finite interval

Now, take $A$ be a finite interval of length $l$. In this case, we need to consider a functional integral over the cylinder with two finite cuts. The locations of the branch points ($z_i, \bar{z}_i$) are

$$z_1 = z_1 = 0, \quad z_2 = \bar{z}_2 = l,$$
$$z_3 = l - 2t + i \frac{\beta_+}{2}, \quad \bar{z}_3 = l + 2t - i \frac{\beta_+}{2}, \quad z_4 = -2t + i \frac{\beta_+}{2}, \quad \bar{z}_4 = 2t - i \frac{\beta_+}{2} \quad (2.16)$$

As described in [4, 5], the entanglement Renyi entropy is given by the four-point correlator of the twist fields

$$S_n = \frac{1}{1 - n} \log \langle \phi^+(z_1, \bar{z}_1)\phi^-(z_2, \bar{z}_2)\phi^+(z_3, \bar{z}_3)\phi^-(z_4, \bar{z}_4) \rangle \quad (2.17)$$

As before, a way to compute this would be by mapping the points to the plane using (2.6), computing the correlator there and transforming back to the cylinder by using (2.8). The details of this calculation are similar to the $\Omega = 0$ case discussed in [5]. The 4-point function on the plane depends on the cross-ratio

$$x = \frac{w_{12}w_{34}}{w_{13}w_{24}} = \frac{2\sinh^2 \frac{\pi t}{\beta_+}}{\cosh \frac{2\pi t}{\beta_+} + \cosh \frac{4\pi}{\beta_+}}, \quad \bar{x} = \frac{\bar{w}_{12}\bar{w}_{34}}{\bar{w}_{13}\bar{w}_{24}} = \frac{2\sinh^2 \frac{\pi t}{\beta_-}}{\cosh \frac{2\pi t}{\beta_-} + \cosh \frac{4\pi}{\beta_-}} \quad (2.18)$$

where $w_{ij} = w(z_i) - w(z_j)$ and similarly for $\bar{w}$. Let us assume that $l, t \gg \beta, \beta \Omega_E$. We then have: $x \sim (1 + \exp[\frac{4\pi}{\beta_+}(t - l/2)])^{-1}$ up to $O(\exp[-t/\beta_+], \exp[-l/\beta_-])$.

Case (i): For $t < l/2$ [10], we then have $x \to 1$, which, in terms of the original coordinates, implies $z_2 \to z_3$ and hence a factorization $\langle 1 4 \rangle \langle 2 3 \rangle$. Once we realize this, we can go back to (2.17) and evaluate the four-point function as

$$\langle \phi^+(z_1, \bar{z}_1)\phi^-(z_4, \bar{z}_4)\rangle \langle \phi^-(z_2, \bar{z}_2)\phi^+(z_3, \bar{z}_3) \rangle$$

Case (ii): For $t > l/2$ [11] by similar reasonings, we have $x \to 0$, which implies the other factorization for the 4-point function

$$\langle \phi^+(z_1, \bar{z}_1)\phi^-(z_2, \bar{z}_2)\rangle \langle \phi^+(z_3, \bar{z}_3)\phi^-(z_4, \bar{z}_4) \rangle$$

---

[10] Strictly speaking, we need here $(l/2 - t) \gg \beta_+, \beta_-$, to ensure $x \to 1$.
[11] We actually need $(t - l/2) \gg \beta_+$. See footnote [10].
Using our results from the previous subsections about the two-point function, we find the following behaviour of the EE:

\[
S_{EE} = \begin{cases} 
2t \left( 2s_{eqm} \right) + S_{div} & t \leq l/2 \\
(2s_{eqm}) + S_{div} & t \geq l/2
\end{cases}
\]

\[
s_{eqm} = \frac{\pi c}{3\beta(1 - \Omega^2)} = \sqrt{\frac{\pi c}{6}(E + J)} + \sqrt{\frac{\pi c}{6}(E - J)}
\]

(2.19)

Clearly the EE saturates after time \(t = l/2\). Here \(s_{eqm}\) is the equilibrium entropy density of (either copy of) the CFT; the first expression on the second line gives its value in the canonical ensemble and the second expression gives the microcanonical value (see Section A for the relation between the two ensembles). For \(\Omega = 0\) the results of this section correctly reduce to those derived in [5].

The divergent part \(S_{div}\) is the same as \(S_{EE,0}\) is (2.11) and (2.15).

We have thus proved (1.3) starting from a rather special pure state of the form (2.1). We will now present a more general derivation starting from an arbitrary initial state.

### 2.2 Single CFT, arbitrary state

Let us now consider a single CFT, defined on a cylinder (with coordinates described in footnote 8). We start with a pure state \(|B\rangle\) at time \(\sigma_2 = 0\), and evolve it by (i) translating in \(\sigma_2\) by \(\beta/4\) (as in [5]) and (ii) in \(\sigma_1\) by \(-\beta\Omega E\); this leads to another pure state

\[
|\psi\rangle = \exp[-\beta(H + \Omega J)/4]|B\rangle,
\]

(2.20)

We will regard this as the initial state for further, Lorentzian, time evolution, and compute the time-dependent EE for a single interval in the state

\[
|\psi(t)\rangle = \exp[-iHt]|\psi\rangle = \exp[-(\beta/4 + it)H - \beta\Omega J/4]|B\rangle.
\]

(2.21)

By choosing \(|B\rangle\) arbitrarily, we can obtain an arbitrary initial state \(|\psi\rangle\). We will comment in Section 2.3 on the independence of the EE with respect to the choice of this initial state.

Let us first consider the case where the interval is a half-line. Suppose at \(\sigma_2 = 0\), the half-line ends at \(\sigma_1 = 0\). Then after the evolution, this point is translated to \(\sigma_1 = -\beta\Omega E/4, \sigma_2 = \beta/4 + it\) \(\text{[13]}\), or, in terms of the \(z, \bar{z}\) coordinates (see footnote 8), to the point

\[
z_1 = t + i\beta_+/4, \quad \bar{z}_1 = -(t + i\beta_-/4)
\]

(2.22)

The computation of the time-dependent EE, in part similar to that described above, involves a generalization of the techniques in [4, 5]. The entanglement Renyi entropy involves computing the one-point function of the twist fields \(\phi^+(z_1, \bar{z}_1)\) (of (2.5)) on an (analytically

12 The expression for the time-dependent entanglement entropy for the finite interval, in (2.19), is twice that for the half-line (2.15) as observed in [5] for \(\Omega = 0\). We will encounter the same feature for a single CFT, as well as for the bulk duals.

13 Recall that in (2.21), \(-H, iJ\) are, respectively, the translation operators in \(\sigma_2, \sigma_1\), and \(\Omega = i\Omega_E\). See below (2.12).
and \( \sigma \) the entanglement entropy (see Fig 2.2) where \( \langle \phi^+(w_1, \bar{w}_1) \rangle |_{UHP} = (w_1 - \bar{w}_1)^{-h-h} \) (2.23)

which equals the two-point function of \( \phi^+ \) at \( w_1 \) with its image \( \phi^- \) at \( \bar{w}_1 \) in the full plane. The original one-point function on the strip is now obtained by (2.6)

\[
\langle \phi^+(z_1, \bar{z}_1) \rangle = \left( \frac{\beta_+}{\pi} \sinh \left( \frac{\pi z_1}{\beta_+} - \frac{\pi \bar{z}_1}{\beta_-} \right) \right)^{-h} \left( \frac{\beta_-}{\pi} \sinh \left( \frac{\pi z_1}{\beta_+} - \frac{\pi \bar{z}_1}{\beta_-} \right) \right)^{-h}
\] (2.24)

By putting the values (2.22), we can compute the Renyi entropy. Taking the \( n \to 1 \) limit, we obtain the EE

\[
S_{EE} = \frac{c}{6} \log \left( \cosh \left( \frac{2\pi t}{\beta(1-\Omega^2)} \right) \right) + \frac{c}{12} \log \frac{\beta^2(1-\Omega^2)}{\pi^2 \epsilon^2}
\] (2.25)

For large \( t \gg \beta, \beta \Omega \), the EE evolves linearly with a coefficient equal to half of the one for the thermo-field double. For \( \Omega = 0 \) we recover the result of [5].

In the case of finite interval, let us suppose that the interval stretches from \( \sigma_1 = -l/2 \) and \( \sigma_1 = l/2 \) at \( \sigma_2 = 0 \). At time \( t \) these end-points are translated to \( (z_1, \bar{z}_1) \) and \( (z_2, \bar{z}_2) \), where

\[
z_1 = -\frac{l}{2} + \frac{\beta_+}{4} - t, \quad \bar{z}_1 = -\frac{l}{2} + \frac{\beta_-}{4} + t, \quad z_2 = \frac{l}{2} + \frac{\beta_+}{4} - t, \quad \bar{z}_2 = \frac{l}{2} - \frac{\beta_+}{4} + t
\]

The computation of the EE follows by using a slight modification of [4, 5]. The Renyi entropy is given in terms of a two-point function on the above-mentioned strip which can be obtained, from the UHP result

\[
\langle \phi_+(w_1) \phi_-(w_2) \rangle \sim \left( \frac{|w_1 - \bar{w}_2||w_2 - \bar{w}_1|}{|w_1 - w_2||\bar{w}_1 - \bar{w}_2||w_1 - \bar{w}_1||w_2 - \bar{w}_2|} \right)^{h+h}
\] (2.26)

using the conformal transformation (2.6). The EE turns out to be

\[
S_{EE} = \frac{c}{6} \log \left( \frac{\beta_+ \beta_-}{\pi^2 \epsilon^2} \left( \cosh \frac{2\pi \Omega l}{\beta(1-\Omega^2)} + \cosh \frac{4\pi t}{\beta(1-\Omega^2)} \right) \sinh \frac{\pi t}{\beta(1+\Omega)} \sinh \frac{\pi t}{\beta(1-\Omega)} \right)
\] (2.27)

As in case of the thermo-field double, we again have, for large \( t/\beta \) and \( l/\beta \), two cases (depending on the relative magnitude of \( t \) and \( l/2 \)), that clearly illustrate the saturation of the entanglement entropy (see Fig 2.2)

\[
S_{EE} = \begin{cases} 
2t \, s_{eqm} + S_{div} & t \leq l/2 \\
ls_{eqm} + S_{div} & t \geq l/2
\end{cases}
\] (2.28)
where $s_{\text{eqm}}$ is the equilibrium entropy density given in (2.19). Note that the saturation value of the entanglement entropy for the single CFT is expectedly half of that in the case of the thermofield double given by (2.19). Also note that the saturation value depends on the angular momentum (see Fig 2.2).

The above equation (2.28) is again of the form of (1.3). Thus, we have now proved this equation starting from an arbitrary initial state (2.20).

2.3 Information loss

We wish to mention a rather remarkable feature of the EE described in this subsection. By choosing the state $|B\rangle$ in (2.20) appropriately, we can make the initial state $|\psi\rangle$ completely arbitrary (contrast this with the state (2.1) which is fixed by the choice of $\beta, \Omega$); however, the entanglement entropy of an interval in any such state is independent of the choice of the state (this statement is even true for EE at any finite time). The feature of the calculation that makes this happen is the following. Recall that the choice of $|\psi\rangle$ corresponds to the choice of a boundary condition for the two-dimensional CFT (in an appropriate coordinate system, the state specifies a boundary condition on the boundary of the upper half plane (UHP)). As has been shown in [4], as long as the state $|\psi\rangle$ is a conformally invariant boundary state, the correlation function of twist fields in the UHP, involved in computing the Reny entropy boils down to correlators on the plane involving the original twist fields and their images in the lower half plane. This result is universal and is independent of the choice of the specific conformal boundary state, of which there is an infinite tower (the so-called Ishibashi states). Furthermore, as emphasized in [4], even if our initial state is not one of the conformally invariant boundary states, RG flow takes it to the nearby Ishibashi state; thus, for sufficiently large length scales/time scales the result becomes completely universal. From the holographic viewpoint, the universality is encapsulated by the fact that the bulk is given by a BTZ black hole geometry. These features have already appeared in the work of [5]. Such universalities with respect to the initial state have also been remarked upon in [7, 9].
3 EE with spin: holographic calculation

As shown in [19, 39] (see also [20, 21, 35]) the above CFT calculations find natural duals in BTZ geometries. For non-zero angular momentum $J$, the holographic dual of the thermofield double involves (a Euclidean continuation of) the eternal (2+1)-dimensional BTZ black hole [22], given by

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} dt^2 + \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 \left( d\phi - \frac{r_+ r_-}{2 r^2} dt \right)^2$$ (3.1)

Here $\phi \sim \phi + 2\pi$ for the BTZ black hole, and $\phi \in \mathbb{R}$ for the BTZ black string. [4] The mass $M$, angular momentum $J$, temperature $T$ and angular velocity $\Omega$ are determined by the inner ($r_-$) and outer ($r_+$) horizons, as follows:

$$M = r_+^2 + r_-^2, \quad J = 2 r_+ r_-, \quad T = \frac{1}{\beta} = \frac{2\pi r_+}{r_+^2 - r_-^2}, \quad \Omega = \frac{r_-}{r_+}$$ (3.2)

The BTZ metric can be mapped into the Poincare patch of Euclidean AdS$_3$ ($ds^2 = (dy^2 + dw_+ dw_-)/y^2$) via

$$w_\pm = \frac{\sqrt{r^2 - r_+^2}}{r^2 - r_-^2} e^{2\pi u_\pm/\beta}, \quad y = \sqrt{\frac{r_+^2 - r_-^2}{r^2 - r_-^2}} e^{\pi(u_+/\beta + u_-/\beta_-)}$$ (3.3)

where $u_\pm = \phi \pm t$, and $\beta = \beta(1 + \Omega)$ (see, e.g., [23], Eq. (21)). The Euclidean continuation of the above geometry (3.1) is given by $t \to it$, $u_\pm \to (z, \bar{z})$, $w_\pm \to (w, \bar{w})$, $\Omega \to i\Omega_E$. Note that in the limit $r \gg r_+$, the (Euclidean continuation of the) map (3.3) precisely reduces to the transformation (2.6) in the CFT, as it must for consistency with holography.

We will now compute the holographic EE (hEE) for an interval $A$ by using the Ryu-Takayanagi (RT) proposal [6], or more precisely the generalization in [20] for computing covariant EE, according to which the hEE is given by the length of the extremal geodesic(s) that connects the boundary of $A$. The precise formula reads

$$S_{hEE} = \frac{\mathcal{L}(\gamma)}{4G_N}$$ (3.4)

where, in our case, $\mathcal{L}(\gamma)$ is the length computed with the metric (3.1) and $G_N$ is the Newton constant in 3 dimensions. Following along the lines of [5], it is easy to verify that (3.4) indeed reproduces the CFT results (2.19) and (2.28). We will skip the explicit expressions since they are a straightforward generalization of [5] (we will make some more remarks on the geodesic lengths below in Section 3.2), and prefer to include an alternative ‘derivation’, which is more closely related to standard AdS/CFT arguments.

14 The BTZ string can be obtained from the BTZ black hole by scaling ($r, r_\pm, t, \phi$) with a parameter $\lambda$ such that, as $\lambda \to \infty$, $\lambda r, \lambda r_\pm, t/\lambda, \phi/\lambda$ are held fixed. In this paper, we will mostly be concerned with the black string, since the dual CFT has non-compact space. For the black string, the angular momentum $J$ actually becomes the linear momentum; however, as declared in footnote 5, we continue using the notation $J$ and the misnomer ‘angular momentum’.

15 The original RT prescription faces a subtlety for Lorentzian backgrounds. Namely, in general, geodesics that connect the boundaries do not lie on fixed time slices. In these cases EE is given by the area of the extremal surface given by the saddle point of the area action [20].
3.1 A ‘proof’ of Ryu-Takayanagi formula for 1+1 dimensional CFT

Recall that, in CFT, Entanglement Renyi Entropy (ERE) of a single interval \([u, v]\) is computed by the two-point function of the twist fields \(\langle \Phi_+(u)\Phi_-(v) \rangle\) with dimension \(h\):

\[
h = \frac{c}{24} \left( n - \frac{1}{n} \right) = \frac{1}{16G_N} \left( n - \frac{1}{n} \right)
\]

More precisely the ERE, from CFT, is given by

\[
S_{[u,v]}^{(n)} = \frac{1}{1 - n} \ln \langle \phi_+(u)\phi_-(v) \rangle
\]

We will now show how in a double scaling limit fixed, the expression (3.6) reduces to the Ryu-Takayanagi expression (3.4) through more or less standard AdS/CFT arguments.

Our strategy of computing (3.6) would be to compute the CFT two-point function holographically. We need to take the limit \(c \to \infty\) to ensure semiclassical gravity. The CFT two-point function will be given in terms of a bulk propagator of a dual scalar field whose mass \(m\) (see the paragraph around (3.9) for subtle assumptions involved in the existence of such dual scalar fields), by the standard mass-dimension formula for large \(c\), will be \(m = h\). The bulk propagator between two points, on the other hand, is given in terms of the geodesic length of a particle of mass \(m\) connecting the two points. Using these results, the CFT two-point function in (3.6) boils down to

\[
\lim_{c \to \infty} \langle \phi_+(u)\phi_-(v) \rangle \xrightarrow{\text{AdS/CFT}} e^{-2h L([u, v])}
\]

In the second equality we inserted the Brown-Henneaux relation \(R_{\text{AdS}}/G_N = 2c/3\). In our formulas \(R_{\text{AdS}} = 1\).
where $L(\gamma_{u,v})$ is the length of the geodesic, in the BTZ geometry, connecting the two boundary points $u$ and $v$. Using this and (3.5), the holographic entanglement Renyi entropy (3.6) reduces to

$$S^{(n)}_{[u,v]} = \frac{2hL(\gamma_{u,v})}{n-1} \frac{L(\gamma_{u,v})}{8G_N} \left(1 + 1/n\right)$$

(3.8)

Taking the $n \to 1$ limit, we recover the formula prescribed by Ryu and Takayanagi (3.4).

The steps mentioned in the above ‘proof’ are symbolically represented in the ‘commutative diagram’ in Figure 3.1.

The importance of double scaling: In order to have a semiclassical gravity dual, we must take the limit $c \to \infty$. Now, the conventional relation between two-point functions of CFT primary fields and two-point functions of the corresponding bulk duals assumes that dimensions of the CFT fields do not scale with the central charge (this ensures the use of linear response under deformation of the CFT by these fields). This assumption appears to be, a priori, violated by the twist operators with scaling dimensions (3.5). However, we should recall that eventually we are interested in the entanglement entropy which involves taking the limit $n \to 1$. What we propose here is that we should take a judicious combination of the $c \to \infty$ and $n \to 1$ limits; to be precise, let us define the following double scaling limit

$$c \to \infty, n \to 1, c(n-1) = \text{fixed}$$

(3.9)

It is easy to see that in this limit the dimension $\hbar$ of the twist operator remains finite; hence computation of its two-point function by the method described above should be justified. In terms of the commutative diagram Fig. 3.1, the above remarks justify the use of AdS/CFT in the left vertical arrow for $n = 1 + \epsilon$.

A remark: it would be interesting to understand the connection of the above argument with that of Lewkowycz and Maldacena in [37]. If one took the $c \to \infty$ limit without the concurrent $n \to 1$, the bottom left of the commutative diagram in Fig. 3.1 would be represented by the back-reacted conical geometry sourced by the world-line of the very massive quantum of the scalar field described above [38]; this is, at least qualitatively, similar to the picture of [37]. However, in view of the above discussion, it appears that in the double scaling limit described above, the bulk partition function with the conical geometry is given in terms of a propagator of the scalar particle in the undeformed geometry. We hope to come back to this interesting issue in the future.

3.2 Conclusion of this section

In the light of our arguments above, the holographic computation and its agreement with results of Section 2 become very transparent. In fact, to reproduce the CFT results now, we only need to know the length of a massive geodesic (with mass equal to $\hbar$) between two points at the boundary of the spinning BTZ background. This length was found in [23] (formula (34))\(^{17}\). Using it in our algorithm precisely reproduces the CFT two point function

\(^{17}\)see also [24] for more discussion on the geodesic length and AdS/CFT correlators
from bulk:
\[
\langle \phi^+(z_1, \bar{z}_1)\phi^-(z_2, \bar{z}_2) \rangle = \left( \frac{\beta_+}{\pi \epsilon} \sinh \frac{\pi l}{\beta_+} \right)^{-2h} \left( \frac{\beta_-}{\pi \epsilon} \sinh \frac{\pi l}{\beta_-} \right)^{-2h}
\]

(3.10)

Since the CFT two-point function itself is reproduced, we get the same entanglement entropy as in the CFT.

For two finite intervals (which appear here for the thermofield double) we use the same arguments since the four point correlator factorizes into a product of two-point functions (see also [25]), which are then computed using the bulk propagator, as above. Similarly, the holographic EE for the pure B-state is just the half of the full space answer, as found earlier in the \( \Omega = 0 \) case in [5].

We have thus holographically derived (2.19) and (2.28) and hence holographically proved (1.3) for the case with non-zero angular momentum.

## 4 EE for a charged state: CFT

In this section, we will suppose that the CFT has a global \( U(1) \) charge, and that the initial state has a non-zero value of this charge. For simplicity, we will first consider the case of zero angular momentum. In this case, the counterpart of (2.1) will be given by

\[
|\psi\rangle = C \sum_i \exp\left[\frac{-\beta(E_i - \mu Q_i)/2}{2}\right] |i\rangle \otimes |i\rangle
\]

(4.1)

The \( U(1) \) symmetry implies that the CFT has a \( U(1) \) Kac-Moody algebra

\[
J(z)J(0) = k/(2z^2) + \text{regular terms}
\]

(4.2)

plus its antiholomorphic counterpart. The Kac-Moody currents have the usual OPE with the stress tensor \( T_{zz}, \bar{T}_{\bar{z}\bar{z}} \).

It is well-known, e.g. in the context of \( \mathcal{N} = 2 \) superconformal field theories, that the Kac-Moody and Virasoro algebras admit an automorphism called “spectral flow”. By choosing a flow parameter \( \eta = \mu/2 \), (using the conventions of [26], Eq. (2.7)) we find the following expression for the automorphism

\[
L_0 \rightarrow L_0^{(\mu)} = L_0 - \frac{\mu}{k}Q/2 + \frac{\mu^2}{4k}, \quad Q \rightarrow Q^{(\mu)} = Q - \mu/k
\]

(4.3)

where \( k \) is the level of the \( U(1) \) Kac-Moody algebra, defined in (1.2). Although it is perhaps best studied in the context of \( \mathcal{N} = 2 \) superconformal theories, the phenomenon of spectral flow is very generic; it exists for simple systems such as free massless charged fermions (see Section A.1) for which half-integral spectral flows connect the NS and R sectors; indeed in [27] arguments have been presented for its appearance under rather general circumstances.

With this proviso, we will assume that the charged models we have possess a spectral flow. It is easy, then, to see that the CFT calculations in previous sections can be simply generalized by using the unitary transformation implementing the spectral flow. For example, consider the Renyi entropy for the CFT on the plane which has the generic form

\[
S_{\text{Renyi}}^{(n)} = Z_n / Z_1^n
\]

(4.4)
where $Z_n$ is a partition function of an appropriate $n$-sheeted surface and $Z_1$ is the partition function on the plane. Now note that, by spectral flow (using $H = 2L_0$ for $J = 0$),

$$\text{Tr} \exp[-\beta(H - \mu Q + k\mu^2/2)] = \text{Tr} \exp[-\beta H], \quad (4.5)$$

which trivially leads to

$$\text{Tr} \exp[-\beta(H - \mu Q)] = \text{Tr} \exp[-\beta(H - \mu^2/4k)] \quad (4.6)$$

Thus, the effect of adding the $\mu Q$ term is equivalent, in the partition function and hence in (4.4), to the universal shift (4.6) to the Hamiltonian. Using this line of reasoning, it is easy to show that the time-dependent EE is given by applying this shift to the energy $E$ in the expression for $s_{\text{eqm}}$. The generalization to non-zero $J$ is straightforward, in that the same shift again applies to the energy $E$. Using this shift to (2.19), and the relation between $\mu$ and $Q$ as in Section A (this relation is also discussed in [26, 27]), we now get the general result for the dynamical EE for non-zero $E, J$ and $Q$

$$S_{\text{ent}}(t, l|\psi) = \begin{cases} \frac{t}{2} s_{\text{eqm}}(E, J, Q) + S_{\text{div}}, & t \leq l/2 \\ \frac{t}{2} s_{\text{eqm}}(E, J, Q) + S_{\text{div}}, & t > l/2 \end{cases}$$

$$s_{\text{eqm}} = \sqrt{\frac{\pi c}{6} (E + J - \frac{\pi}{2k} Q^2)} + \sqrt{\frac{\pi c}{6} (E - J - \frac{\pi}{2k} Q^2)} \quad (4.7)$$

The dynamical EE for the pure state and single CFT with non-zero $Q$ follows similarly and it is given by half of the above result for the thermofield double.

We have thus derived the form of the dynamical EE, (1.3), for arbitrary $E, J, Q$.

## 5 Holographic EE with charge: BTZ plus CS U(1)

The bulk dual of the above CFT has been described in various places; in particular, we will follow the account given in [26]. The bulk dual consists of AdS gravity plus a bulk $U(1)$ CS gauge field. The metric is given, as in Sec 3, by a spinning black hole (3.1); in addition, there is a bulk gauge field solution given by the flat connection

$$A = \frac{\mu}{k} (dz + d\bar{z}) \quad (5.1)$$

Before proceeding, we now encounter an obvious puzzle.

### 5.1 A puzzle

In the previous section (Section 4) we found that the entropy density clearly depends on the charge $Q$. The grand canonical expression of such an entropy density is, therefore, expected to be of the form $s(\beta, \Omega, \mu)$. In the bulk dual, as we just mentioned, $\mu$ appears only in the gauge field solution (5.1) and not in the metric which retains the $\mu = 0$ form, (3.1). The Ryu-Takayanagi prescription, therefore, will give the time-dependent EE as in the uncharged case. In particular, we will again obtain (2.19) and (2.28). This clearly appears to contradict (1.7) which we derived in the CFT.
Indeed, rather than for the EE, one could ask the same question about the BH entropy. By the black hole area law,

$$s = s(\beta) = \frac{\pi c}{(3\beta(1-\Omega^2))}$$  \hspace{1cm} (5.2)

which is clearly independent of the chemical potential $\mu$ for the charge. Indeed, in terms of the microcanonical ensemble, the above entropy density precisely agrees with (2.19). The puzzle is, how can we get the entropy density in (4.7) from gravity?

The resolution of this puzzle can be described in two equivalent ways, one in the language of the microcanonical ensemble and the second in the language of the (equivalent) grand canonical ensemble.

- **‘Microcanonical’ resolution:**

  Although the U(1) CS action in the bulk is topological and hence does not couple to the metric, it has a boundary term of the form $A^2$ (see, e.g. [26]). This leads to an additional contribution to $T_{zz}$ at the boundary, resulting in the following shift

$$L_0 = L_{0,\text{bulk}} + Q^2/(4k), \quad \bar{L}_0 = \bar{L}_{0,\text{bulk}} + Q^2/(4k),$$  \hspace{1cm} (5.3)

This shift, in fact, is the bulk equivalent of (4.6). See also Sec 6 for another application of this shift.

- **‘Grand Canonical’ resolution:**

  The microcanonical expression for the entropy density in (4.7) can be converted into the grand canonical form using the formulae in Section A. Surprisingly, from that expression, the $\mu$-dependence drops out, leaving the expression (5.2)! The temperature and the two ‘chemical potentials’ $\Omega, \mu$ are of course the same in the CFT and in the AdS dual; hence we get agreement between the bulk and boundary expressions.

**Summary:** We have thus proved (1.3) holographically in the presence of both angular momentum and charge.

### 5.2 Higher Spin

It is natural to speculate how to extend the above calculations to the case of further additional charges. A natural setting for this is to consider higher spin black hole backgrounds whose CFT dual corresponds to coset models [28] (this speculation was made earlier in [13]). There exist limits of the parameter space of this duality, which are described by free fermions which describe a particularly simple form of an integrable CFT. A similar integrable system of free fermions was recently discussed in [13] where a version of (1.2) was found to be true in the framework of the generalized Gibbs ensemble, and it was speculated there that the equilibrium configuration of the Gaberdiel-Gopakumar free fermions could be given by the higher spin black holes. This makes it rather natural to conjecture that (1.4) should be true in this case, where the bulk dual geometry should be that of a higher spin black hole. See the most recent progress in this direction [34].
6 Universal limits

By now, there is a lot of evidence that in the limit of a small entangling region $A$, EE obeys the analogue of the first law of thermodynamics \cite{29, 30}

$$\Delta E_A = T_{\text{ent}} \Delta S_A$$

(6.1)

Here, the increase of energy in an interval $A = l$ is computed by integrating the holographic energy-momentum tensor $T_{tt}$ over the entangling interval, $\Delta S_A$ is a leading-$l$ difference between the EE computed in an excited state and that in the vacuum, and $T_{\text{ent}}$ is a universal constant that depends on the number of dimensions. More explicitly

$$\Delta E_A = \int_{A=l} dx T_{tt} = \frac{M R l}{16 \pi G_N}$$

(6.2)

where $R$ is the radius of asymptotically $AdS_3$ background with time component of the metric given by $f(z)^{-1} \sim 1 + M z^2$.

For spinning BTZ solution \cite{3.1}, the increase EE of a single interval of length $l$ \cite{20} is given by

$$\Delta S_l = \frac{c}{6} \log \left( \frac{\beta_- \beta_+}{\pi^2 \epsilon^2} \sinh \frac{\pi l}{\beta_+} \sinh \frac{\pi l}{\beta_-} \right) - \frac{c}{3} \log \epsilon \sim \frac{c \pi^2 (1 + \Omega^2) T^2 l^2}{18 (1 - \Omega^2)^2}$$

(6.3)

Using the relation between the mass and the temperature

$$M = (2 \pi T)^2 \frac{1 + \Omega^2}{(1 - \Omega^2)^2}$$

(6.4)

the first law relation for EE becomes

$$\Delta E_l = \frac{3}{\pi l} \Delta S_l$$

(6.5)

in agreement with \cite{29}.

On the other hand, in the limit of large $l$ or large temperature $T$, the EE reaches the extensive form given by the thermal entropy of the system (see the review \cite{2})

$$S_l \approx l s_{\text{eqm}}(\beta)$$

(6.6)

For spinning BTZ, $s_{\text{eqm}}$ can be read off from \cite{2.19}. Thus,

$$S_l \approx \frac{c \pi l}{3 \beta (1 - \Omega^2)}$$

(6.7)

The result depends only on the central charge of the CFT and on $\beta$ and $\Omega$, and is, therefore, a universal limiting value.

\textsuperscript{18}Or in the limit of $\beta \to \infty$
Let us now look how the universal limits incorporate the presence of the U(1) CS fields. As explained in the previous section, U(1) gauge fields give an additional boundary contribution to the energy-momentum tensor

\[ T_{tt} = T_{tt}^{\text{grav}} + T_{tt}^{\text{gauge}} = \frac{MR}{16\pi G_N} + \frac{\mu^2}{2\pi} = \frac{\pi c}{6\beta^2} + \frac{\mu^2}{4\pi k} \equiv E_{\text{bulk}} + \frac{\mu^2}{4\pi k} \] (6.8)

This is consistent with (5.4) noting \( T_{tt} = L_0/\pi \). This way, using the spectral flow argument, we have, as in (6.9)

\[ E_{\text{bulk}} = E_{\text{bdry}} - \frac{\mu^2}{4\pi k} \] (6.9)

and the first law-like relation remains the same.

The value of the thermal entropy at which EE saturates can be expressed in terms of microcanonical energy density \( E \) and potential \( \mu \). However, in the grand canonical ensemble it is only a function of \( \beta \) that matches the holographic prescription that is “blind” to the gauge fields, as we noted in the previous section.

7 Relation with quantum ergodicity

The idea of ergodicity is that given sufficient time, the “time average of various properties of a system”, evolving from some initial state \( \mathcal{S}_0 \) can be equated to an “ensemble average of those properties”, where the ensemble is constructed out of all possible states \( \mathcal{S} \) of the system which have the same conserved charges as \( \mathcal{S}_0 \). In the classical version, the initial state is a point in the phase space; ergodicity says that under dynamical evolution the point moves “democratically” in the submanifold \( M \) of the phase space, allowed by conservation laws, so that the time average of a phase space function \( f(q,p) \) can be equated to an average of \( f \) taken over \( M \) with uniform weight. In quantum mechanics, under the usual conservation law of energy, the statement boils down to (see, e.g., the review in [10])

\[ \frac{1}{T} \int_0^T dt \ \text{Tr} \ (\rho_{\text{pure}}(t) \ O) \xrightarrow{T \to \infty} \text{Tr} \ (\rho_{\text{micro}} \ O), \quad \rho_{\text{pure}} = |\psi\rangle\langle\psi|, \]

\[ \rho_{\text{micro}} = \frac{1}{N} \sum_{i=1}^N |i\rangle\langle i| \approx \rho_{\text{thermal}} = \frac{1}{Z} \exp[-\beta H] \] (7.1)

Sometimes an alternative statement (Eq. (11.2)) is made [10]

\[ \text{Tr} \ (\rho_{\text{pure}} \ O) \xrightarrow{t \to \infty} \text{Tr} \ (\rho_{\text{thermal}} \ O) \] (7.2)

with respect to a certain class of “macroscopic” observables, for which the time averaging in (7.1) is not necessary. Let us now consider a partition \( A \cup B \) of space, say for quantum field theory, or for spins on a lattice. Consider a basis of states \( |i_A\rangle|n_B\rangle \) where the states \( |i_A\rangle \) are supported entirely on \( A \), and the states \( |n_B\rangle \) are supported entirely on \( B \) (it could consist of spins in \( A \) and \( B \). Let us consider the projection operator \( P_B = \sum_n |n_B\rangle\langle n_B| \) onto the states of \( B \). To make connection with the discussion above, we choose \( O = P_B |i_A\rangle\langle j_A| \)
in (1.2), assuming that this is an appropriate “macroscopic” operator. Eq. (1.2) then gives us the following limiting value of the matrix element of the reduced density matrix $\rho_A = \text{Tr}(P_B \rho_{\text{pure}})$

$$\langle i_A|\rho_A|j_A\rangle \xrightarrow{t \to \infty} \langle i_A|\rho_{A,\beta}|j_A\rangle, \quad \rho_{A,\beta} \equiv \text{Tr}(P_B \exp[-\beta H]/Z) \quad (7.3)$$

The asymptotic value of the time-dependent EE, then would be

$$S_{\text{EE}} \xrightarrow{t \to \infty} S[\rho_{A,\beta}] \equiv -\text{Tr}(\rho_{A,\beta} \ln \rho_{A,\beta}) \quad (7.4)$$

The latter entropy measures the von Neumann entropy of the reduced density matrix in an overall mixed state. Now we expect that for a large enough $l = l_A$

$$S[\rho_{A,\beta}]/l = s(\beta) \quad (7.5)$$

where $s(\beta)$ is the thermal entropy density at an inverse temperature $\beta$. In Section B we present a proof of this statement using a discrete system and assuming the equivalence between microcanonical and grand canonical ensembles. Additionally, in Section 8 we explicitly verify (7.5) in the case of a massive, charged scalar field.

We have therefore proved

$$S_{\text{EE}} \xrightarrow{t \to \infty} l s \quad (7.6)$$

This is the same as (1.1), where a time scale of saturation is set by $l$.

The proof of saturation outlined above holds in principle for any field theory and in any number of dimensions. Hence, we expect the behaviour (7.6) to be valid quite generally.

For integrable systems and the generalized Gibbs ensemble, the story of quantum ergodicity is less developed, although we still expect an equation of the form (7.2) to hold for a suitable class of “macroscopic” observables (see [10]).

8 Non-CFT: EE for charged, massive scalar field

In this section, we consider an a priori calculation of EE for a charged, massive scalar field. The motivation for this calculation is to have an additional evidence for (7.3). Our CFT calculations for the saturation value of the time-dependent EE already provide indirect evidence for this formula. However, in this section we consider a non-conformal system and perform a direct computation of the EE using the methods of [36].

The charged scalar field is described by a Hamiltonian

$$H = \int dx \left( \pi^\dagger \pi + (\nabla \phi)^\dagger (\nabla \phi) + m^2 \phi^\dagger \phi \right) \quad (8.1)$$

and a conserved U(1) charge

$$Q = i \int dx (\pi^\dagger \phi - \phi^\dagger \pi) \quad (8.2)$$

These observables typically display some non-locality; however, see [35] for the behaviour of local observables.
We will suppose that the full system (with spatial partition $A$ and its complement $B$) is in a grand canonical ensemble

$$\rho_{\text{total}} = \frac{\exp[-\beta(H - \mu Q)]}{\text{Tr}[\exp[-\beta(H - \mu Q)]]} \quad (8.3)$$

We are interested in computing the reduced density matrix $\rho_A = \text{Tr}_B \rho_{\text{total}}$, and the EE $S_A = -\text{Tr}_A \rho_A \ln \rho_A$. We will proceed using a generalization of the formalism described in [36], and describe only the essentially new features. Note that formalism in [36] permits a straightforward generalization from the case of a single scalar $\phi$ to multiple flavours $\phi_a, a = 1, 2, ..., N_f$, with the $C$-matrix generalized to

$$(C^2)_{ij}^{ab} = \sum_{k=1}^{n} <\phi_i^a \phi_k^c > <\pi_k^c \pi_j^b> \quad (8.4)$$

The EE is given by (using the notation from (7.3))

$$S[\rho_{A,\beta}] = \text{Tr}[(C + 1/2) \log(C + 1/2) - (C - 1/2) \log(C - 1/2)] \quad (8.5)$$

where the trace is now over both the $\{i, j\}$ and $\{a, b\}$ indices. For the free complex scalar at

Figure 4: Plot of $S[\rho_{A,\beta}]/l$ (the LHS of (7.3) vs $\beta \mu$, for $\beta m = 10$. The plot marked with squares has $l/\beta = 10$; the plot marked with triangles has $l/\beta = 25$. The solid line corresponds to the RHS of (7.5), viz. the thermal entropy density, which is obtained using standard formulae, and Mathematica. It is clear that (7.5) holds to a good accuracy. The agreement is better for small $\beta \mu$ than for large $\beta \mu$; however, this could be due to some numerical instability.
hand, \( N_f = 2 \), and \( C_{ij}^{ab} = C_{ij} \delta^{ab} \). We compute \( C^2 \) using (8.4) and the following ingredients:

\[
< \phi_j \phi_k^\dagger > = \frac{1}{2N} \sum_{a=0}^{N-1} \frac{1}{2\epsilon \omega_i} [\coth(\frac{\omega_i + \mu}{2T}) + \coth(\frac{\omega_i - \mu}{2T})] \cos[\frac{2\pi}{N} a(j - k)]
\]

\[
< \pi_j \pi_k^\dagger > = \frac{1}{2N} \sum_{a=0}^{N-1} \frac{\epsilon \omega_i}{2} [\coth(\frac{\omega_i + \mu}{2T}) + \coth(\frac{\omega_i - \mu}{2T})] \cos[\frac{2\pi}{N} a(j - k)]
\]

\[
< \pi_j \pi_k^\dagger > = < \pi_j^\dagger \pi_k >, \quad < \phi_j \phi_k^\dagger > = < \phi_j^\dagger \phi_k > \tag{8.6}
\]

The computation of (8.5) is performed numerically. We reproduce a representative plot in Fig. 4.

**Acknowledgments**

It is a pleasure to thank Pallab Basu, Justin David, Abhishek Dhar, Matthew Headrick, Sachin Jain, Nilay Kundu, Shiraz Minwalla, Takeshi Morita, Johannes Oberreuter, Aninda Sinha, Nilanjan Sircar and Sandip Trivedi for very useful discussions and comments. We are grateful to Dileep Jatkar, Mukund Rangamani, Aninda Sinha and Spenta Wadia for important feedbacks on earlier versions of the preprint. GM would like to thank the organizers and participants of (a) the ICTS meeting “US-India advanced studies institute on thermalization: from glasses to black holes” held at IISc, Bangalore (June 10-21, 2013), and (b) the “Seventh Crete Regional Meeting on String Theory” held at Kolyambari, Crete, Greece (June 16-22, 2013), for hospitality and stimulating discussions during the finishing stage of this work. GM would also like to thank the organizers of the latter meeting for an opportunity to present this work. The work of PC is based upon research supported by the South African Research Chairs Initiative of the Department of Science and Technology and National Research Foundation. PC would also like to thank Tata Institute for Fundamental Research in Mumbai for hospitality and support during this project.

**A Microcanonical vs grand canonical quantities**

Consider a grand canonical ensemble given by the following density matrix and partition function

\[
\rho = \frac{1}{Z} \exp[-\beta(H + \Omega P - \mu Q)], \quad Z = \text{Tr} \exp[-\beta(H + \Omega P - \mu Q)] \tag{A.1}
\]

The partition function can be written as

\[
Z \equiv \exp[-\beta G] = \sum_{E,J,Q} \exp[S(E, Q, J) - \beta(E + \Omega P - \mu Q)] \tag{A.2}
\]

If the summand in the last function has a single sharp maximum around a unique set of values \( E, P, Q \), the distribution essentially becomes equivalent to that of a microcanonical ensemble, where we have

\[
\frac{\partial S}{\partial E} = \beta, \quad \frac{\partial S}{\partial J} = -\beta \Omega, \quad \frac{\partial S}{\partial Q} = -\beta \mu \tag{A.3}
\]
This gives us the grand canonical parameters in terms of the microcanonical ones. The converse relations are also easy to derive:

\[
\frac{\partial (\beta G)}{\partial \beta} = E - \Omega J - \mu Q, \quad -\frac{1}{\beta} \frac{\partial (\beta G)}{\partial \Omega} = J, \quad -\frac{1}{\beta} \frac{\partial (\beta G)}{\partial \mu} = Q
\]  

(A.4)

This gives us the following relation

\[
S = \beta^2 \frac{\partial G}{\partial \beta}
\]  

(A.5)

Using the above relations, we can prove that if the microcanonical entropy density is given by

\[
S = \sqrt{aE + bJ - dQ^2} + \sqrt{aE - bJ - dQ^2}
\]  

(A.6)

the grand canonical expression for the entropy is

\[
s = \frac{2a}{\beta (1 - a^2 \Omega^2 / b^2)}
\]  

(A.7)

Surprisingly, \( S(\beta, \Omega, \mu) = S(\beta, \Omega) \), which is independent of \( \beta \).

**Notation:** In the body of the paper, we have used the notations \( E, J, Q \) as the energy density, \( J \) as the angular momentum density and \( Q \) as the charge density whereas \( s \) denotes the entropy density. Equations (A.6) and (A.7) hold for the densities with trivial modifications.

### A.1 Free massless charged fermion in 1+1

We consider free massless charged fermions in 1+1 dimension, at a temperature \( 1/\beta \) and chemical potential \( \mu \). Explicit calculation gives the following Gibbs free energy

\[
g(\beta, \mu) = -\left( \frac{\pi}{6\beta^2} + \frac{\mu^2}{2\pi} \right)
\]  

(A.8)

The energy density \( e \) and the charge density \( q \) are given by

\[
e = \frac{\pi}{6\beta^2} + \frac{\mu^2}{2\pi}, \quad q = \frac{\mu}{\pi}
\]  

(A.9)

For this system the grand canonical entropy density is given by

\[
s(\beta, \mu) = s(\beta) = \frac{\pi}{3\beta}
\]  

(A.10)

and the microcanonical entropy density is given by

\[
s(e, q) = \sqrt{\frac{2\pi}{3} (e - \frac{\pi}{2} q^2)}
\]  

(A.11)
**B Proof of (7.5)**

Consider a basis of the Hilbert space \( \{|i,A\rangle, |k,B\rangle\} \), \( i = 1, \ldots, N_A; k = 1, \ldots, N_B \), where \( N_A, N_B \) denote the number of independent states \( |i,A\rangle \) belonging to the partition A, and similarly for \( N_B \). A microcanonical density matrix \( \rho_{mc} \) is given by

\[
\rho_{mc} = \frac{1}{N_A N_B} \sum_{i,A,k,B} |i,A\rangle |k,B\rangle \langle k,B| \langle i,A|
\]

(B.1)

By tracing over the B states, we get

\[
\rho_{A,mc} \equiv \text{Tr}_B \rho_{mc} = \frac{1}{N_A} \sum_{i,A} |i,A\rangle \langle i,A|
\]

The von Neumann entropy of this density matrix is given by

\[
S_A = \ln N_A
\]

Now imagine that our system is a lattice of \( n \)-level ‘spins’ (\( n = 2 \) is Ising), and that there are \( l_A \) spins in the partition A; then

\[
S_A = l_A \ln n
\]

Now we can easily show that the von Neumann entropy of (B.1) is

\[
S = (l_A + l_B) \ln n
\]

Hence the entropy density is

\[
s = \ln n
\]

Now by denoting \( l_A \) as \( l \), and assuming the equivalence between the microcanonical ensemble in (B.1) and the canonical ensemble as in (7.3) we obtain (7.5).

**References**

[1] P. Calabrese, J. Cardy, and B. Doyon, “Entanglement entropy in extended systems”, J. Phys. A 42, 500301 (2009).

[2] P. Calabrese and J. Cardy, “Entanglement entropy and conformal field theory,” J. Phys. A 42 (2009) 504005 [arXiv:0905.4013 [cond-mat.stat-mech]].

[3] J. Eisert, M. Cramer and M. B. Plenio, “Area laws for the entanglement entropy - a review,” Rev. Mod. Phys. 82, 277 (2010) [arXiv:0808.3773 [quant-ph]].

[4] P. Calabrese and J. L. Cardy, “Evolution of entanglement entropy in one-dimensional systems,” J. Stat. Mech. 0504, P04010 (2005) [cond-mat/0503393].
[5] T. Hartman and J. Maldacena, “Time Evolution of Entanglement Entropy from Black Hole Interiors,” arXiv:1303.1080 [hep-th].

[6] S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,” Phys. Rev. Lett. 96, 181602 (2006) [hep-th/0603001].

[7] V. Balasubramanian, A. Bernamonti, N. Copland, B. Craps and F. Galli, “Thermalization of mutual and tripartite information in strongly coupled two dimensional conformal field theories,” Phys. Rev. D 84, 105017 (2011) [arXiv:1110.0488 [hep-th]].

[8] J. Abajo-Arrastia, J. Aparicio and E. Lopez, “Holographic Evolution of Entanglement Entropy,” JHEP 1011, 149 (2010) [arXiv:1006.4090 [hep-th]], T. Albash and C. V. Johnson, “Evolution of Holographic Entanglement Entropy after Thermal and Electromagnetic Quenches,” New J. Phys. 13, 045017 (2011) [arXiv:1008.3027 [hep-th]].

[9] H. Liu and S. J. Suh, “Entanglement Tsunami: Universal Scaling in Holographic Thermalization,” arXiv:1305.7244 [hep-th].

[10] A. Polkovnikov, K. Sengupta, A. Silva and M. Vengalattore, “Nonequilibrium dynamics of closed interacting quantum systems,” Rev. Mod. Phys. 83, 863 (2011) [arXiv:1007.5331 [cond-mat.stat-mech]].

[11] H. Liu and M. Mezei, “A Refinement of entanglement entropy and the number of degrees of freedom,” arXiv:1202.2070 [hep-th].

[12] M. Rigol, V. Dunjko, V. Yurovsky and M. Olshanii, “Relaxation in a Completely Integrable Many-Body Quantum System: An Ab Initio Study of the Dynamics of the Highly Excited States of 1D Lattice Hard-Core Bosons,” Phys. Rev. Lett. 98, 050405 (2007). P. Calabrese, F.H.L. Essler and M. Fagotti, “Quantum quench in the transverse field Ising chain: I. Time evolution of order parameter correlators”, Journal of Statistical Mechanics: Theory and Experiment (2012) P07016.

[13] G. Mandal and T. Morita, “Quantum quench in matrix models: Dynamical phase transitions, Selective equilibration and the Generalized Gibbs Ensemble,” arXiv:1302.0859 [hep-th].

[14] M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, “Black holes in three dimensional higher spin gravity: A review,” arXiv:1208.5182 [hep-th].

[15] Y. Takahashi and H. Umezawa, Collective Phenomena 2 (1975) 55.

[16] P. Calabrese and J. L. Cardy, “Entanglement entropy and quantum field theory,” J. Stat. Mech. 0406, P06002 (2004) [hep-th/0405152].

[17] M. Headrick, “Entanglement Renyi entropies in holographic theories,” Phys. Rev. D 82, 126010 (2010) [arXiv:1006.0047 [hep-th]].

[18] J. L. Cardy, “Conformal Invariance and Surface Critical Behavior,” Nucl. Phys. B 240, 514 (1984).
[19] W. Israel, “Thermo field dynamics of black holes,” Phys. Lett. A 57, 107 (1976).

[20] V. E. Hubeny, M. Rangamani and T. Takayanagi, “A Covariant holographic entanglement entropy proposal,” JHEP 0707 (2007) 062 [arXiv:0705.0016 [hep-th]].

[21] M. Cadoni and M. Melis, “Holographic entanglement entropy of the BTZ black hole,” Found. Phys. 40, 638 (2010) [arXiv:0907.1559 [hep-th]].

[22] M. Banados, C. Teitelboim and J. Zanelli, “The Black hole in three-dimensional space-time,” Phys. Rev. Lett. 69, 1849 (1992) [hep-th/9204099].

[23] E. Keski-Vakkuri, “Bulk and boundary dynamics in BTZ black holes,” Phys. Rev. D 59 (1999) 104001 [hep-th/9808037].

[24] J. Louko, D. Marolf and S. F. Ross, “On geodesic propagators and black hole holography,” Phys. Rev. D 62, 044041 (2000) [hep-th/0002111].

[25] T. Hartman, “Entanglement Entropy at Large Central Charge,” arXiv:1303.6955 [hep-th].

[26] P. Kraus and F. Larsen, “Partition functions and elliptic genera from supergravity,” JHEP 0701, 002 (2007) [hep-th/0607138].

[27] M. R. Gaberdiel, T. Hartman and K. Jin, “Higher Spin Black Holes from CFT,” JHEP 1204, 103 (2012) [arXiv:1203.0015 [hep-th]].

[28] M. R. Gaberdiel and R. Gopakumar, “An AdS3 Dual for Minimal Model CFTs,” Phys. Rev. D 83, 066007 (2011) [arXiv:1011.2986 [hep-th]].

[29] J. Bhattacharya, M. Nozaki, T. Takayanagi and T. Ugajin, “Thermodynamical Property of Entanglement Entropy for Excited States,” Phys. Rev. Lett. 110 (2013) 091602 [arXiv:1212.1164 [hep-th]].

[30] M. Nozaki, T. Numasawa, A. Prudeniati and T. Takayanagi, “Dynamics of Entanglement Entropy from Einstein Equation,” arXiv:1304.7100 [hep-th]. • D. Allahbakhshi, M. Alishahiha and A. Naseh, “Entanglement Thermodynamics,” arXiv:1305.2728 [hep-th]. • G. Wong, I. Klich, L. A. Pando Zayas and D. Vaman, “Entanglement Temperature and Entanglement Entropy of Excited States,” arXiv:1305.3291 [hep-th].

[31] M.M. Wolf, F. Verstraete, M.B. Hastings, J.I. Cirac, “Area laws in quantum systems: mutual information and correlations,” arXiv:0704.3906 [quant-ph]; Phys. Rev. Lett. 100, 070502 (2008).

[32] P. Kraus and T. Ugajin, “An Entropy Formula for Higher Spin Black Holes via Conical Singularities,” arXiv:1302.1583 [hep-th].

[33] Talk by G. Mandal in the Seventh Crete regional meeting in string theory, June 18, 2013, Kolyambari, Crete, Greece (see http://hep.physics.uoc.gr/mideast7/program.html).
[34] J. de Boer and J. I. Jottar, “Entanglement Entropy and Higher Spin Holography in AdS$_3$,” arXiv:1306.4347 [hep-th].

[35] M. Ammon, A. Castro and N. Iqbal, “Wilson Lines and Entanglement Entropy in Higher Spin Gravity,” arXiv:1306.4338 [hep-th].

[36] Jean-Sebastien Caux and Fabian H. L. Essler, “Time evolution of local observables after quenching to an integrable model”, arXiv:1301.3806 [cond-mat.stat-mech].

[37] C. P. Herzog and M. Spillane, “Tracing Through Scalar Entanglement,” Phys. Rev. D 87, 025012 (2013) [arXiv:1209.6368 [hep-th]].

[38] A. Lewkowycz and J. Maldacena, “Generalized gravitational entropy,” arXiv:1304.4926 [hep-th].

[39] S. Deser and R. Jackiw, “Three-Dimensional Cosmological Gravity: Dynamics of Constant Curvature,” Annals Phys. 153, 405 (1984).

[40] J. M. Maldacena, “Eternal black holes in anti-de Sitter,” JHEP 0304, 021 (2003) [hep-th/0106112].