Existence of High Energy-Positive Solutions for a Class of Elliptic Equations in the Hyperbolic Space

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Abstract
We study the existence of positive solutions for the following class of scalar field problem on the hyperbolic space:

\[-\Delta_{B^N} u - \lambda u = a(x)|u|^{p-1}u \text{ in } B^N, \quad u \in H^1(B^N),\]

where \( B^N \) denotes the hyperbolic space, \( 1 < p < 2^* - 1 := \frac{N+2}{N-2} \), if \( N \geq 3 \); \( 1 < p < +\infty \), if \( N = 2 \), \( \lambda < \frac{(N-1)^2}{4} \), and \( 0 < a \in L^\infty(B^N) \). We prove the existence of a positive solution by introducing the min–max procedure in the spirit of Bahri–Li in the hyperbolic space and using a series of new estimates involving interacting hyperbolic bubbles.

Keywords Hyperbolic space · Hyperbolic bubbles · Palais–Smale decomposition · Semilinear elliptic problem

Mathematics Subject Classification Primary · 35J20 · 35J60 · 58E30

1 Introduction

In this paper, we investigate the existence of positive solutions for the following semilinear elliptic problem on the hyperbolic space \( B^N \):

\[-\Delta_{B^N} u - \lambda u = a(x)|u|^{p-1}u \text{ in } B^N, \quad u \in H^1(B^N),\]
\[ (P_\lambda) \quad \left\{ -\Delta_{\mathbb{B}^N} u - \lambda u = a(x) |u|^{p-1} u, \quad u \in H^1 \left( \mathbb{B}^N \right) \right\}, \]

where \(1 < p < 2^* - 1 := \frac{N+2}{N-2};\) if \(N \geq 3; 1 < p < +\infty,\) if \(N = 2,\) \(\lambda < \frac{(N-1)^2}{4},\)
\(H^1 (\mathbb{B}^N)\) denotes the Sobolev space on the disk model of the hyperbolic space \(\mathbb{B}^N,\)
\(\Delta_{\mathbb{B}^N}\) denotes the Laplace Beltrami operator on \(\mathbb{B}^N,\) \(\frac{(N-1)^2}{4}\) being the bottom of the \(L^2-\) spectrum of \(-\Delta_{\mathbb{B}^N},\) and \(a(x) \in L^\infty (\mathbb{B}^N).\) Moreover, we investigate the existence of solutions under the following hypotheses:

(a1) \(a(x) > 0, \forall x \in \mathbb{B}^N\) and \(\lim_{d(x,0) \to \infty} a(x) = 1,\)
(a2) there exist some positive constants \(C\) and \(\delta > 0\) such that
\[ a(x) \geq 1 - C \exp(-\delta d(x,0)) \quad \forall d(x,0) \to \infty, \]

where \(d\) is the hyperbolic distance.

The problem, \((P_\lambda)\) when posed in all of \(\mathbb{R}^N,\) with \(\lambda = -1,\) has been the subject of intense research in the past few decades, starting from the seminal papers by Berestycki–Lions [9, 10], Bahri–Berestycki [5], Bahri–Li [6], and Bahri–Lions [7]. Apart from purely mathematical interest, such semilinear elliptic equations also arise in several physical phenomena, e.g., nonlinear Schrödinger and Klein–Gordan equations (see [20, 30, 31]). The main difficulty in studying such equations in all of \(\mathbb{R}^N\) comes from the lack of compactness due to the unboundedness of the domain \(\mathbb{R}^N,\) i.e., through the vanishing of mass in the sense of concentration compactness of Lions [25].

Non-compact subcritical problems have been studied thoroughly and brought to a high level of sophistication by many authors. We refer to the papers [1–4, 15–17, 26, 28, 29, 32] and references quoted therein without any claim of completeness. The asymptotic of \(a(x)\) in (a1) suggests that the corresponding “limiting problem,” i.e.,
\[ -\Delta_{\mathbb{R}^N} u + u = u^p \quad \text{in} \quad \mathbb{R}^N, \quad u \in H^1 (\mathbb{R}^N), \quad u > 0 \quad \text{in} \quad \mathbb{R}^N, \quad \text{(1.1)} \]

will play an essential role. In fact, the proof of the existence of positive solutions introduced by Bahri–Li in [6] essentially depends on the uniqueness (up to translation) and decay estimate of the unique solution, and it is worth mentioning that Bahri–Li’s solution to the equation \((P_\lambda)\) does not correspond to the mountain pass critical point. More precisely, consider the energy functional \(E_a : H^1 (\mathbb{R}^N) \to \mathbb{R}\) defined by
\[ E_a(u) := \int_{\mathbb{R}^N} \left( |\nabla u|^2 + u^2 \right) \, dx \left( \int_{\mathbb{R}^N} a(x) |u|^{p+1} \, dx \right)^{\frac{2}{p+1}}. \]

and it was shown in [6] that the Palais-Smale condition holds for \(E_a\) in the range \((S_1, 2 \frac{p-1}{p+1} S_1),\) where
\[ S_1 := \inf_{u \in H^1 (\mathbb{R}^N) \setminus \{0\}} E_a(u). \]
We note that if $a(x) \geq 1$ in $\mathbb{R}^N$, then the mountain pass level for $E_a$ is strictly less than $S_1$ and hence achieved (see [1]). But assumptions (a1) and (a2), in general, do not ensure $a(x) \geq 1$. So Bahri–Li constructed solutions in higher energy level by using energy estimates involving interacting (translated) solutions to the limiting problem.

In recent years, many efforts have been made to generalize these scalar field-type equations for non-local operators in the Euclidean space or in domains after their connection with various physical problems. We refer [11, 13, 18, 19] without claim of any completeness. In particular, the authors in [13] studied a similar equation for the fractional Laplacian in the Euclidean space although there were not successful to obtain a high-energy solution in the spirit of Bahri–Li. The main hurdle for them is that the corresponding limiting problem does not have global uniqueness (see [21, 22]).

As concerns the analogous equation in the hyperbolic space, Sandeep–Mancini, in their seminal paper [27], have shown the existence and uniqueness of finite energy-positive solutions of the following homogeneous elliptic equation:

$$-\Delta_{\mathbb{B}^N} u - \lambda u = u^p, \quad u \in H^1(\mathbb{B}^N),$$  \hspace{1cm} (1.2)

where $\lambda \leq \frac{(N-1)^2}{4}, \; 1 < p \leq \frac{N+2}{N-2}$ if $N \geq 3$; $1 < p < \infty$ if $N = 2$. Among many other results, they established in the subcritical case, and for $p > 1$, if $N = 2$ and $1 < p < 2^* - 1$ if $N \geq 3$, the problem (1.2) has a positive solution if and only if $\lambda < \left( \frac{N-1}{4} \right)^2$. These positive solutions are also shown to be unique up to hyperbolic isometries, except possibly for $N = 2$ and $\lambda > \frac{2(p+1)}{(p+3)^2}$. The question of finite energy solutions was fully resolved in [27] and subsequently, the existence of sign-changing solutions, compactness, and non-degeneracy were studied in [12, 23, 24] and on the other hand, authors in [8, 14] showed the existence of infinite energy solutions of (1.2) and determined the exact asymptotic behavior of wide classes of finite and infinite energy solutions.

One should notice that for the existence of finite energy solutions, one looks for the energy functional associated with (1.2) which is defined as follows:

$$E_{\lambda}(u) = \frac{1}{2} \int_{\mathbb{B}^N} \left( |\nabla_{\mathbb{B}^N} u|^2 - \lambda u^2 \right) \, dV_{\mathbb{B}^N} - \frac{1}{p+1} \int_{\mathbb{B}^N} |u|^{p+1} \, dV_{\mathbb{B}^N}.$$  

In the subcritical case, the variational problem lacks compactness because of the hyperbolic translation (see Sect. 3 for more details), and so it cannot be solved by the standard minimization method. Moreover, in [12], a detailed analysis of the Palais–Smale decomposition is performed. One can easily see that if $U$ is a solution of (1.2), then

$$u := U \circ \tau, \quad \text{for} \; \tau \in I(\mathbb{B}^N),$$

where $I(\mathbb{B}^N)$ is the group of isometries on the hyperbolic space, is also a solution, and hence, if we define a sequence by varying $\tau_n \in I(\mathbb{B}^N)$, then $u_n$ is also a Palais–Smale (PS) sequence for $E_{\lambda}$. In fact, it was shown in [12, Theorem 3.3] that in the subcritical
case, i.e., when \(1 < p < \frac{N+2}{N-2}\), non-compact PS sequences are made of finitely many sequences of the form \(u_n\). We call this a hyperbolic bubble (See Sect. 3 below for more details). In our analysis, these hyperbolic bubbles will play a major role.

Inspired by the above-mentioned papers, our main goal in this article is to study whether positive solutions can still exist for a perturbed problem as in \((P_\lambda)\). Moreover, we seek solutions in the higher energy range following Bahri–Li in [6]. In particular, we prove the following theorem.

**Theorem 1.1** Assume \(a(x)\) satisfies (a1)–(a2). Then \((P_\lambda)\) admits a positive solution for all \(\lambda\) in the range:

\[
\lambda \in \begin{cases} 
(\infty, \frac{2(p+1)}{(p+3)^2})^\frac{2}{p+3}, & N = 2, \\
(\infty, \frac{(N-1)^2}{4}), & N \geq 3.
\end{cases}
\]

### 1.1 Main Novelty and the Strategy of the Proof

As discussed earlier, we follow the approach of Bahri–Li. We first show that the energy functional associated with \((P_\lambda)\) satisfies the Palais–Smale condition in the range \((S_{1,\lambda}, S_{2,\lambda})\) where \(S_{i,\lambda}, i = 1, 2\) are defined in (4.3), and a detailed PS decomposition is provided in Proposition 3.1. The proof of proposition 3.1 is a straightforward adaptation of [12, Theorem 3.3] to the problem \((P_\lambda)\). In the next step, we prove the key energy estimates. The main novelty lies in this step, where we compute the energy estimates involving the convex combination of translated hyperbolic bubbles with two different centers. To this end, we need precise estimates on the interacting hyperbolic bubbles, and since the hyperbolic volume grows exponentially, i.e.,

\[
dV_{\mathbb{B}^N} \propto e^{(N-1)r}, \quad r \to \infty,
\]

where \(r\) denotes the geodesic distance \(d(x,0)\), and \(dV_{\mathbb{B}^N}\) denotes the hyperbolic volume form, and to compensate for the exponential volume growth of the hyperbolic space, one requires a new way to tackle the integrals involving interacting bubbles. Unlike in the Euclidean space, where solutions to (1.1) decays exponentially and the euclidean volume grows polynomially. We compute the effect of interacting bubbles in the integral in a novel way by breaking the integral in different sub-regions, and using the precise asymptotic estimate on the solution of (1.2) obtained in [27]. We establish a series of new estimates for interacting terms, which are crucial in proving the corresponding energy of the convex combination is strictly less than \(S_{2,\lambda}\). Finally, the existence of a positive solution is proved using the min–max procedure of Bahri–Li by suitably defining the center of mass for functions in \(H^1(\mathbb{B}^N)\) on the hyperbolic space. We refer Sect. 5 below for more details.

The paper is organized as follows: In Sect. 2, we introduce some of the notations and some geometric definitions and preliminaries concerning hyperbolic space. In Sect. 3, we state and prove the Palais–Smale decomposition theorem associated with \((P_\lambda)\). Sect. 4 contains the proof of key estimates on the energy, while Sect. 5 contains the proof of the main Theorem 1.1.
2 Notations and Functional Analytic Preliminaries

In this section, we will introduce some of the notations and definitions used in this paper and also recall some of the embeddings related to the Sobolev space on the hyperbolic space.

We will denote by $B^N$ the disk model of the hyperbolic space, i.e., the unit disk equipped with the Riemannian metric $g_{B^N} := \sum_{i=1}^{N} \left( \frac{2}{1-|x|^2} \right)^2 dx^2$. The Euclidean unit ball $B(0, 1) := \{ x \in \mathbb{R}^n : |x|^2 < 1 \}$ equipped with the Riemannian metric

$$ds^2 = \left( \frac{2}{1-|x|^2} \right)^2 dx^2$$

constitute the ball model for the hyperbolic $n$-space, where $dx$ is the standard Euclidean metric and $|x|^2 = \sum_{i=1}^{n} x_i^2$ is the standard Euclidean length. To simplify our notations, we will denote $g_{B^N}$ by $g$. The corresponding volume element is given by $dV_{B^N} = \left( \frac{2}{1-|x|^2} \right)^N dx$, where $dx$ denotes the Lebesgue measure on $\mathbb{R}^N$.

**Hyperbolic distance on $B^N$.** The hyperbolic distance between two points $x$ and $y$ in $B^N$ will be denoted by $d(x, y)$. For the hyperbolic distance between $x$ and the origin, we write

$$\rho := d(x, 0) = \int_0^r \frac{2}{1-s^2} ds = \log \frac{1+r}{1-r},$$

where $r = |x|$, which in turn implies that $r = \tanh \frac{\rho}{2}$. Moreover, the hyperbolic distance between $x, y \in B^N$ is given by

$$d(x, y) = \cosh^{-1} \left( 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)} \right).$$

It easily follows that a subset $S$ of $B^N$ is a hyperbolic sphere in $B^N$ iff $S$ is a Euclidean sphere in $\mathbb{R}^N$ and contained in $B^N$, probably with a different center and different radius, which can be computed. Geodesic balls in $B^N$ of radius $a$ centered at the origin will be denoted by

$$B(0, a) := \{ x \in B^N : d(x, 0) < a \}.$$

We also need some information on the isometries of $B^N$. Below we recall the definition of a particular type of isometry, namely the hyperbolic translation. For more details on the isometry group of $B^N$, we refer [33].

**Hyperbolic Translation.** For $b \in B^N$, define

$$\tau_b(x) = \frac{(1-|b|^2)x + (|x|^2 + 2x.b + 1)b}{|b|^2|x|^2 + 2x.b + 1},$$

(2.1)
then \( \tau_b \) is an isometry of \( \mathbb{B}^N \) with \( \tau_b(0) = b \). The map \( \tau_b \) is called the hyperbolic translation of \( \mathbb{B}^N \) by \( b \). It can also be seen that \( \tau_{-b} = \tau_b^{-1} \).

The hyperbolic gradient \( \nabla_{\mathbb{B}^N} \) and the hyperbolic Laplacian \( \Delta_{\mathbb{B}^N} \) are given by

\[
\nabla_{\mathbb{B}^N} = \left( 1 - \frac{|x|^2}{2} \right)^2 \nabla, \quad \Delta_{\mathbb{B}^N} = \left( 1 - \frac{|x|^2}{2} \right)^2 \Delta + (N - 2) \frac{1 - |x|^2}{2} \langle x, \nabla \rangle.
\]

**A sharp Poincaré-Sobolev inequality.** (see \[27\])

**Sobolev Space** We will denote by \( H^1(\mathbb{B}^N) \) the Sobolev space on the disk model of the hyperbolic space \( \mathbb{B}^N \), equipped with norm \( \|u\| = \left( \int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u|^2 \right)^{\frac{1}{2}} \), where \( |\nabla_{\mathbb{B}^N} u| \) is given by \( |\nabla_{\mathbb{B}^N} u| := \langle \nabla_{\mathbb{B}^N} u, \nabla_{\mathbb{B}^N} u \rangle_{\mathbb{B}^N}^{\frac{1}{2}} \).

For \( N \geq 3 \) and every \( p \in \left( 1, \frac{N + 2}{N - 2} \right) \) there exists an optimal constant \( S_{N, p} > 0 \) such that

\[
S_{N, p} \left( \int_{\mathbb{B}^N} |u|^{p+1} \, dV_{\mathbb{B}^N} \right)^{\frac{2}{p+1}} \leq \int_{\mathbb{B}^N} \left[ |\nabla_{\mathbb{B}^N} u|^2 - \frac{(N - 1)^2}{4} u^2 \right] \, dV_{\mathbb{B}^N},
\]

for every \( u \in C_0^\infty(\mathbb{B}^N) \). If \( N = 2 \), then any \( p > 1 \) is allowed.

A basic information is that the bottom of the spectrum of \( -\Delta_{\mathbb{B}^N} \) on \( \mathbb{B}^N \) is

\[
\frac{(N - 1)^2}{4} = \inf_{u \in H^1(\mathbb{B}^N) \setminus \{0\}} \frac{\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u|^2 \, dV_{\mathbb{B}^N}}{\int_{\mathbb{B}^N} |u|^2 \, dV_{\mathbb{B}^N}}. \tag{2.2}
\]

**Remark 2.1** A consequence of (2.2) is that if \( \lambda < \frac{(N-1)^2}{4} \), then

\[
\|u\|_{H_{\lambda}} := \|u\|_{\lambda} := \left[ \int_{\mathbb{B}^N} \left( |\nabla_{\mathbb{B}^N} u|^2 - \lambda u^2 \right) \, dV_{\mathbb{B}^N} \right]^{\frac{1}{2}}, \quad u \in C_0^\infty(\mathbb{B}^N)
\]

is a norm, equivalent to the \( H^1(\mathbb{B}^N) \) norm and the corresponding inner product is given by \( \langle u, v \rangle_{H_{\lambda}} \).

### 3 Palais–Smale Characterization

In this section, we study the Palais–Smale sequences (PS sequences) corresponding to the problem \((P_{\lambda})\). To be precise, define the associated energy functional \( I_{\lambda, a} \) as

\[
I_{\lambda, a}(u) = \frac{1}{2} \int_{\mathbb{B}^N} \left[ |\nabla_{\mathbb{B}^N} u|^2_{\mathbb{B}^N} - \lambda u^2 \right] \, dV_{\mathbb{B}^N} - \frac{1}{p+1} \int_{\mathbb{B}^N} a(x) |u|^{p+1} \, dV_{\mathbb{B}^N}.
\]

We say a sequence \( u_n \in H^1(\mathbb{B}^N) \) is a Palais–Smale sequence for \( I_{\lambda, a} \) at a level \( d \) if \( I_{\lambda, a}(u_n) \to d \) and \( I'_{\lambda, a}(u_n) \to 0 \) in \( H^{-1}(\mathbb{B}^N) \). One can easily see that PS sequences are bounded. Throughout this section, we assume \( a(x) \to 1 \) as \( d(x, 0) \to \infty \).
3.1 Compactness and Palais–Smale Sequences

Before introducing the Palais–Smale characterization, we shall remark on the compactness properties of \((P_\lambda)\) posed in the hyperbolic space. To see this, let \(u \in H^1(B^N)\) and \(b_n \in B^N\) such that \(b_n \to \infty\), and let \(\tau_n\) be the hyperbolic translation given by (2.1) such that \(\tau_n(0) = b_n\). Define \(u_n = u \circ \tau_n\), then \(\|u_n\|_{H^1(B^N)} = \|u\|_{H^1(B^N)}\) but \(u_n \to 0\) weakly in \(H^1(B^N)\). This shows that the embedding \(H^1(B^N) \hookrightarrow L^p(B^N)\) is not compact for any \(2 < p < \frac{2N}{N-2}\) even in the subcritical case.

It is worth noticing that if \(v\) is a solution of \((P_\lambda)\) with \(a(x) \equiv 1\), then \(v \circ \tau\) is also a solution for any \(\tau \in I(B^N)\), where \(I(B^N)\) is an isometry group. In particular, if we define

\[
v_n = v \circ \tau_n,
\]

where \(\tau_n\) as defined in (2.1) with \(\tau_n(0) \to \infty\), then \(v_n\) is a PS sequence converging weakly to zero. Thus, in the limiting case, i.e., \(a(x) \to 1\), as \(d(x, 0) \to \infty\), the functional \(I_{\lambda, a}\) for the Palais–Smale sequences exhibits sequences of the form \(v_n\). To be precise, we state the following proposition.

**Proposition 3.1** Let \(\{u_n\} \subset H^1(B^N)\) be a PS sequence for \(I_{\lambda, a}\) at a level \(d \geq 0\). Then there exists a subsequence (still denoted by \(\{u_n\}\)) for which the following holds: there exists an integer \(m \geq 0\), sequences \(\tau^i_n \in I(B^N)\), functions \(\bar{u}, w_i \in H^1(B^N)\) for \(1 \leq i \leq m\) such that

\[
\begin{align*}
-\Delta_{B^N} \bar{u} - \lambda \bar{u} &= a(x)|\bar{u}|^{p-1} \bar{u} \text{ in } H^{-1}(B^N), \\
-\Delta_{B^N} w_i - \lambda w_i &= |w_i|^{p-1} w_i \text{ in } H^{-1}(B^N), w_i \neq 0, \\
u_n - \left(\bar{u} + \sum_{i=1}^{m} w_i \left(\tau^i_n(\bullet)\right)\right) &\to 0 \text{ as } n \to \infty, \\
I_{\lambda, a}(u_n) &\to I_{\lambda, a}(\bar{u}) + \sum_{i=1}^{m} I_{\lambda, 1}(w_i) \text{ as } n \to \infty, \\
\tau^i_n(0) &\to \infty, \quad d(\tau^i_n(0), \tau^j_n(0)) \to \infty \text{ as } n \to \infty, \quad \text{for } 1 \leq i \neq j \leq m.
\end{align*}
\]

To prove the above proposition, we need the following auxiliary lemmas.

**Lemma 3.1** Let \(\{u_n\}\) be a bounded sequence in \(H^1(B^N)\) such that

\[
\sup_{y \in B^N} \int_{B(y,r)} |u_n|^q dV_{B^N} \to 0 \quad \text{as} \quad n \to \infty,
\]

for some \(r > 0\) and \(2 \leq q < 2^*\). Then \(u_n \to 0\) strongly in \(L^p(B^N)\) for all \(p \in (2, 2^*)\). In addition, if \(u_n\) satisfies

\[
-\Delta_{B^N} u_n - \lambda u_n - a(x)|u_n|^{p-1} u_n \to 0 \text{ in } H^{-1}(B^N), \quad (3.1)
\]
then \( u_n \to 0 \) strongly in \( H^1(\mathbb{B}^N) \).

**Proof** For any \( s \in (q, 2^*) \), we have from the interpolation inequality:

\[
\|u_n\|_{L^s(B(y,r))} \leq \|u_n\|_{L^q(B(y,r))}^{\frac{s-q}{s-2}} \|u_n\|_{L^{2^*}(B(y,r))}^{\frac{2^*-q}{s-2}} \tag{3.2}
\]

where \( \lambda = \frac{s-q}{s-2} \). The Sobolev inequalities on geodesic balls in the hyperbolic space imply that there exists a positive constant \( C > 0 \) independent of \( y \in \mathbb{B}^N \) such that

\[
\int_{B(y,r)} |u_n|^s \, dV_{\mathbb{B}^N} \leq C \|u_n\|_{L^q(B(y,r))}^{s(1-\lambda)} \left( \int_{B(y,r)} (|u_n|^2 + |\nabla_{\mathbb{B}^N} u_n|^2) \, dV_{\mathbb{B}^N} \right)^{\frac{s}{2}}. \tag{3.3}
\]

Using (3.2) and (3.3), we get

\[
\int_{B(y,r)} |u_n|^s \, dV_{\mathbb{B}^N} \leq C \sup_{y \in \mathbb{B}^N} \|u_n\|_{L^q(B(y,r))}^{s(1-\lambda)} \int_{\mathbb{B}^N} (|u_n|^2 + |\nabla_{\mathbb{B}^N} u_n|^2) \, dV_{\mathbb{B}^N}.
\]

Now covering \( \mathbb{B}^N \) by balls of radius \( r \), in such a way that each point of \( \mathbb{B}^N \) is contained inside at most \( N_0 \) balls, we find

\[
\int_{\mathbb{B}^N} |u_n|^s \, dV_{\mathbb{B}^N} \leq C \sup_{y \in \mathbb{B}^N} \|u_n\|_{L^q(B(y,r))}^{s(1-\lambda)} \int_{\mathbb{B}^N} (|u_n|^2 + |\nabla_{\mathbb{B}^N} u_n|^2) \, dV_{\mathbb{B}^N}.
\]

Therefore, utilizing the lemma hypothesis, we get for the sequence \( \{u_n\} \) that \( \|u_n\|_{L^s(\mathbb{B}^N)} \to 0 \) for all \( s \in (q, 2^*) \). This proves the first part of the lemma if \( q = 2 \); otherwise, if \( q > 2 \), then again, one can argue similarly by choosing \( s \in (2, q) \). In addition, if (3.1) is satisfied, then we obtain

\[
\left| -\Delta_{\mathbb{B}^N} u_n - \lambda u_n - a(x)|u_n|^{p-1} u_n, u_n \right|_{H^1} = o(1) \|u_n\|_{\lambda}. \tag{3.4}
\]

Since \( \{u_n\} \) is bounded in \( H^1(\mathbb{B}^N) \), the RHS is \( o(1) \). On the other hand, for the LHS, we notice that because \( u_n \) is bounded in \( H^1(\mathbb{B}^N) \) and \( u_n \to 0 \) strongly in \( L^r(\mathbb{B}^N) \), for \( r \in (2, 2^*) \), we must have \( u_n \to 0 \) weakly in \( H^1(\mathbb{B}^N) \). Moreover, by the previous part, \( u_n \to 0 \) strongly in \( L^{p+1}(\mathbb{B}^N) \). Thus, by (3.4), we get \( u_n \to 0 \) strongly in \( H^1(\mathbb{B}^N) \).

\( \square \)

**Lemma 3.2** Let \( \phi_k \rightharpoonup \phi \) weakly in \( H^1(\mathbb{B}^N) \), then we have

\[
a |\phi_k|^{p-1} \phi_k - a |\phi|^{p-1} \phi \to 0 \quad \text{in} \quad H^{-1}(\mathbb{B}^N).
\]

**Proof** Defining \( \psi_k = \phi_k - \phi \), gives \( \psi_k \rightharpoonup 0 \) weakly in \( H^1(\mathbb{B}^N) \). Particularly, \( \{\psi_k\} \) is bounded in \( H^1(\mathbb{B}^N) \). Thus, up to a subsequence, \( \psi_k \to 0 \) strongly in \( L^q_{\text{loc}}(\mathbb{B}^N) \).

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for all $2 < q < 2^*$ and $\psi_k \to 0$ a.e.. As a result, $a |\phi + \psi_k|^{p-1} (\phi + \psi_k) - a|\phi|^{p-1} \phi \to 0$ a.e.. Therefore, it follows from Vitaly’s convergence theorem that $a |\phi + \psi_k|^{p-1} (\phi + \psi_k) - a|\phi|^{p-1} \phi \to 0$ strongly in $L^{p+1}_{\text{loc}} (\mathbb{B}^N)$. Also, we can see that for every $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that

$$
\left| a|\phi + \psi_k|^{p-1} (\phi + \psi_k) - a|\phi|^{p-1} \phi \right|^{\frac{p+1}{p}} \leq \varepsilon |\psi_k|^{p+1} + K_\varepsilon |\phi|^{p+1}. \quad (3.5)
$$

Furthermore, as $\psi_k \to 0$ weakly in $H^1(\mathbb{B}^N)$ implies $\psi_k$ is uniformly bounded in $L^{p+1}(\mathbb{B}^N)$ and the fact that $|\phi|^{p+1} \in L^1(\mathbb{B}^N)$, it can be immediately observed from (3.5) that given $\varepsilon > 0$, there exists $R > 0$ for which

$$
\int_{\mathbb{B}^N \setminus B(0,R)} |a|\phi + \psi_k|^{p-1} (\phi + \psi_k) - a|\phi|^{p-1} \phi |^{\frac{p+1}{p}} \, dV_{\mathbb{B}^N} < \varepsilon.
$$

Consequently, $a |\phi + \psi_k|^{p-1} (\phi + \psi_k) - a|\phi|^{p-1} \phi \to 0$ strongly in $L^{p+1}(\mathbb{B}^N)$. Hence, as $H^1(\mathbb{B}^N)$ is continuously embedded in $L^{p+1}(\mathbb{B}^N)$, which is the dual space of $L^\frac{p+1}{p}(\mathbb{B}^N)$, it follows that $a |\phi + \psi_k|^{p-1} (\phi + \psi_k) - a|\phi|^{p-1} \phi \to 0$ strongly in $H^{-1}(\mathbb{B}^N).$ \hfill \Box

Lemma 3.3 For each $c_0 \geq 0$, there exists $\delta > 0$ such that if $v \in H^1(\mathbb{B}^N)$ solves

$$
-\Delta_{\mathbb{B}^N} v - \lambda v = |v|^{p-1} v \text{ in } H^{-1}(\mathbb{B}^N),
$$

and $\|v\|_{H^1(\mathbb{B}^N)} \leq c_0, \|v\|_{L^2(\mathbb{B}^N)} \leq \delta$, then $v \equiv 0$.

Proof Taking $v$ as a test function yields

$$
C' \|v\|_{H^1(\mathbb{B}^N)}^2 \leq \|v\|_{H^1(\mathbb{B}^N)}^2 \geq \int_{\mathbb{B}^N} |v|^{p+1} \, dV_{\mathbb{B}^N} \leq \|v\|_{L^2(\mathbb{B}^N)}^p \left( \int_{\mathbb{B}^N} |v|^{\frac{2N}{N-2}} \, dV_{\mathbb{B}^N} \right)^{\frac{N-2}{N-2}}
$$

$$
\leq C \delta^\alpha \|v\|_{H^1(\mathbb{B}^N)}^\gamma,
$$

with $\gamma = \frac{2N}{N-2} \beta$, by Hölder and Sobolev inequalities, where $C', C$ denote non-negative constants independent of $c_0, \delta$ and $\alpha = \left( \frac{2N}{N-2} - (p + 1) \right) \left( \frac{2N}{N-2} - 2 \right)^{-1}$, $\beta = (p - 1) \left( \frac{2N}{N-2} - 2 \right)^{-1}$. Now, if $p \geq 1 + \frac{4}{N}$, then $\gamma \geq 2$ and we conclude easily if $\delta$ is small enough. On the other hand, if $p < 1 + \frac{4}{N}$, we deduce

$$
\|v\|_{H^1(\mathbb{B}^N)} \leq C \delta^k \text{ for } k = \alpha(2 - \gamma)^{-1} > 0.
$$

Therefore, choosing $\delta > 0$ small enough, we can conclude the lemma. \hfill \Box
Proof of Proposition 3.1} The boundedness of a PS sequence in $H^1(\mathbb{R}^N)$ follows from the standard arguments. To be precise, $I_{\lambda,a}(u_n) = d + o(1)$ and $I'_{\lambda,a}(u_n)(u_n) = o(1)\|u_n\|_{H^1(\mathbb{R}^N)}$, evaluating $I_{\lambda,a}(u_n) - \frac{1}{p+1}I'_{\lambda,a}(u_n)(u_n)$, we obtain

$$\|u_n\|_{H^1(\mathbb{R}^N)}^2 \leq C + o(1)\|u_n\|_{H^1(\mathbb{R}^N)} ,$$

and hence, boundedness follows. As a result, we can assume $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ up to a subsequence. Moreover, $|I'_{\lambda,a}(u_n)(v)| \to 0$ as $n \to \infty \quad \forall v \in H^1(\mathbb{R}^N)$ implies

$$\int_{\mathbb{R}^N} \nabla u_n \cdot \nabla v \, dV_{\mathbb{R}^N} - \lambda \int_{\mathbb{R}^N} u_n v \, dV_{\mathbb{R}^N} - \int_{\mathbb{R}^N} a(x)|u_n|^{p-1}u_nv \, dV_{\mathbb{R}^N} \to 0,$$

as $n \to \infty$ for all $v \in H^1(\mathbb{R}^N)$.

Furthermore, using 3.2, we deduce that weak limit $u$ satisfies

$$-\Delta u - \lambda u = a(x)|u|^{p-1}u \quad \text{in} \quad H^{-1}(\mathbb{R}^N).$$

Next, we show that $u_n - u$ is a PS sequence for $I_{\lambda,a}$ at the level $d - I_{\lambda,a}(u)$. First of all, applying Brezis-Lieb lemma, we obtain the following equations:

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dV_{\mathbb{R}^N} - \int_{\mathbb{R}^N} |\nabla u|^2 \, dV_{\mathbb{R}^N} = \int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 \, dV_{\mathbb{R}^N} + o(1),$$

$$\int_{\mathbb{R}^N} |u_n|^2 \, dV_{\mathbb{R}^N} - \int_{\mathbb{R}^N} |u|^2 \, dV_{\mathbb{R}^N} = \int_{\mathbb{R}^N} |u_n - u|^2 \, dV_{\mathbb{R}^N} + o(1),$$

$$\int_{\mathbb{R}^N} a(x)|u_n|^{p+1} \, dV_{\mathbb{R}^N}(x) - \int_{\mathbb{R}^N} a(x)|u|^{p+1} \, dV_{\mathbb{R}^N}(x) = \int_{\mathbb{R}^N} a(x)|u_n - u|^{p+1} \, dV_{\mathbb{R}^N}(x) + o(1).$$

Thus, applying the above equations, we obtain

$$I_{\lambda,a}(u_n - u) = \frac{1}{2}\left(\|u_n\|_2^2 - \|u\|_2^2\right) - \frac{1}{p+1}\left(\int_{\mathbb{R}^N} a(x)|u_n|^{p+1} \, dV_{\mathbb{R}^N}(x) - \int_{\mathbb{R}^N} a(x)|u|^{p+1} \, dV_{\mathbb{R}^N}(x)\right) + o(1)$$

$$\to d - I_{\lambda,a}(u) \quad \text{as} \quad n \to \infty.$$

Now for $\phi \in H^1(\mathbb{R}^N)$,

$$I'_{\lambda,a}(u_n - u)(\phi) = \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla \phi \, dV_{\mathbb{R}^N} - \lambda \int_{\mathbb{R}^N} (u_n - u) \phi \, dV_{\mathbb{R}^N} - \int_{\mathbb{R}^N} a(x)|u_n - u|^{p-1}(u_n - u) \phi \, dV_{\mathbb{R}^N}.$$
Using (3.6), we get

\[
I'_{\lambda,a}(u_n - u), (\phi)
= \int_{\mathbb{B}^N} a(x) \left[ |u_n|^{p-1} u_n - |u|^{p-1} u - |u_n - u|^{p-1} (u_n - u) \right] \phi \, dV_{\mathbb{B}^N} + o(1).
\]

Applying the Hölder inequality, the above quantity can be estimated as

\[
|\phi|_{L^2(\mathbb{B}^N)} \left[ \int_{\mathbb{B}^N} a(x) \left( |u_n|^{p-1} u_n - |u|^{p-1} u - |u_n - u|^{p-1} (u_n - u) \right)^{\frac{2N}{N+2}} \, dV_{\mathbb{B}^N} \right]^\frac{N+2}{2N}.
\]

Now, we can observe that the term inside the bracket is of \( o(1) \), which follows from the standard arguments using Vitali’s convergence theorem. Hence, we get \( I'_{\lambda,a}(u_n - u) = o(1) \).

Therefore, in view of the Lemma 3.1, we have, either \( u_n - u \to 0 \) strongly in \( H^1(\mathbb{B}^N) \), in that case we are done or there exists \( \alpha > 0 \), such that up to a subsequence

\[
Q_n(1) := \sup_{y \in \mathbb{B}^N} \int_{B(y,1)} |u_n - u|^2 \, dV_{\mathbb{B}^N} > \alpha > 0.
\]

Consequently, we can find a sequence \( \{y_n\} \subset \mathbb{B}^N \) such that

\[
\int_{B(y_n,1)} |u_n - u|^2 \, dV_{\mathbb{B}^N} \geq \alpha. \tag{3.7}
\]

Now define \( v_n(x) := (u_n - u)(T_n(x)) \), where \( T_n(x) = \tau_{y_n}(x) \) and \( \tau_{y_n} \) is the hyperbolic translation of \( \mathbb{B}^N \) by \( y_n \). Now, as \( \tau_{y_n} \) is an isometry, so \( v_n \) will form a PS sequence at the same level as \( u_n - u \) which implies \( v_n \) is also bounded in \( H^1(\mathbb{B}^N) \), and hence, converges weakly in \( H^1(\mathbb{B}^N) \) upto a subsequence to say \( v \). The compact embedding of \( H^1(B(0,1)) \) into \( L^2(B(0,1)) \) and (3.7) implies \( v \neq 0 \). Also, as \( u_n - u \to 0 \) weakly in \( H^1(\mathbb{B}^N) \) and using (3.7) it follows that

\[
\tau_{y_n}(0) \to \infty \quad \text{as} \quad n \to \infty.
\]

Let us now define,

\[
w_n := v_n - v.
\]

Clearly, the fact that \( v_n \to v \) implies \( w_n \to 0 \) weakly in \( H^1(\mathbb{B}^N) \). Applying this and Lemma 3.2, in the definition of \( I'_{\lambda,1}(w_n) \) results in

\[
I'_{\lambda,1}(w_n) = o(1) \quad \text{in} \quad H^{-1}(\mathbb{B}^N),
\]
i.e.,

$$-\Delta_{\mathbb{R}^N} w_n - \lambda w_n - |w_n|^{p-1} w_n \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N).$$

Next, we show that \( v \) satisfies

$$-\Delta_{\mathbb{R}^N} v - \lambda v = |v|^{p-1} v \quad \text{in } H^{-1}(\mathbb{R}^N), \quad v \in H^1(\mathbb{R}^N), \quad (3.8)$$

To prove this, let \( \tilde{v} \in C_0^\infty(\mathbb{R}^N) \). Since, \( v_n \rightharpoonup v \) weakly in \( H^1(\mathbb{R}^N) \), we estimate as follows:

\[
(v, \tilde{v})_{H_\lambda} = \lim_{n \to \infty} (v_n, \tilde{v})_{H_\lambda} = \lim_{n \to \infty} \int_{\mathbb{R}^N} \nabla_{\mathbb{R}^N} v_n \nabla_{\mathbb{R}^N} \tilde{v} \, dV_{\mathbb{R}^N} - \lambda \int_{\mathbb{R}^N} v_n \tilde{v} \, dV_{\mathbb{R}^N} = \lim_{n \to \infty} \int_{\mathbb{R}^N} \nabla_{\mathbb{R}^N} (u_n - u) (T_n(x)) \nabla_{\mathbb{R}^N} \tilde{v} \, dV_{\mathbb{R}^N} - \lambda \int_{\mathbb{R}^N} (u_n - u) (T_n(x)) \tilde{v} \, dV_{\mathbb{R}^N}.
\]

Also,

\[
\left| \int_{\mathbb{R}^N} a(T_n(x)) |v_n(x)|^{p-1} v_n(x) \tilde{v}(x) \, dV_{\mathbb{R}^N} - \int_{\mathbb{R}^N} |v|^{p-1} v \tilde{v} \, dV_{\mathbb{R}^N} \right| \\
\leq \left| \int_{\mathbb{R}^N} a(T_n(x)) \left( |v_n|^{p-1} v_n - |v|^{p-1} v \right) \tilde{v} \, dV_{\mathbb{R}^N} \right| \\
+ \left| \int_{\mathbb{R}^N} (a(T_n(x)) - 1) |v|^{p-1} v \tilde{v} \, dV_{\mathbb{R}^N} \right| \\
:= I_n^1 + J_n^1.
\]

Since \( T_n(0) \to \infty \), \( |v|^{p-1} v \tilde{v} \in L^1(\mathbb{R}^N) \), \( a \in L^\infty(\mathbb{R}^N) \) and \( a(x) \to 1 \) as \( d(x, 0) \to \infty \), the dominated convergence theorem yields

\[
\lim_{n \to \infty} J_n^1 = 0.
\]

On the other hand, since \( \tilde{v} \) has a compact support; applying Vitaly’s convergence theorem gives

\[
\lim_{n \to \infty} I_n^1 \leq \lim_{n \to \infty} \|a\|_{L^\infty(\mathbb{R}^N)} \int_{\text{supp } v} \left| |v_n|^{p-1} v_n - |v|^{p-1} v \right| |\tilde{v}| \, dV_{\mathbb{R}^N} = 0.
\]

The above two estimates help us to conclude that \( v \) satisfies (3.8).
Further, applying Brezis–Lieb Lemma, we get
\[
\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} v_n|^2 \, dV_{\mathbb{B}^N} - \int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} v|^2 \, dV_{\mathbb{B}^N} - \int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} w_n|^2 \, dV_{\mathbb{B}^N} \to 0,
\]
\[
\int_{\mathbb{B}^N} |v_n|^2 \, dV_{\mathbb{B}^N} - \int_{\mathbb{B}^N} |v|^2 \, dV_{\mathbb{B}^N} - \int_{\mathbb{B}^N} |w_n|^2 \, dV_{\mathbb{B}^N} \to 0,
\]
as \( n \to \infty \). In view of the above steps, we rerun the procedure for the Palais–Smale (PS) sequence \( v_n - v \) to end up in either of the two cases. If it converges to zero, we stop, or else we repeat the steps. But it is worth noticing that this process has to terminate in finitely many steps, and we obtain \( v_1, v_2, \ldots, v_n \) which denotes the limit solutions of (3.8) obtained through the procedure, we have
\[
\sum_{i=1}^n \int_{\mathbb{B}^N} |v_i|^2 \, dV_{\mathbb{B}^N} \leq \liminf_{n \to \infty} \int_{\mathbb{B}^N} |u_n - u|^2 \, dV_{\mathbb{B}^N}.
\]
Thus in view of Lemma 3.3, \( n \) cannot approach infinity.

### 4 Key Lemma: Energy Estimates Involving Hyperbolic Bubbles

In this section, we derive key energy estimates of the functional associated with \( (P_\lambda) \) involving hyperbolic bubbles. To this end, let us recall the asymptotic estimates of positive solutions to the corresponding homogeneous problem:
\[
-\Delta_{\mathbb{B}^N} w - \lambda w = |w|^{p-1} w, \quad w > 0 \text{ in } \mathbb{B}^N, \quad w \in H^1(\mathbb{B}^N),
\]
Then by elliptic regularity, any solution, \( w \in H^1(\mathbb{B}^N) \), is also in \( C^\infty \) and satisfies the decay property (See [27, Lemma 3.4]): for every \( \varepsilon > 0 \), there exist positive constants \( C_1^\varepsilon \) and \( C_2^\varepsilon \) such that there holds
\[
C_1^\varepsilon e^{-(c(N,\lambda)+\varepsilon) d(x,0)} \leq w(x) \leq C_2^\varepsilon e^{-(c(N,\lambda)-\varepsilon) d(x,0)}, \quad \text{for all } x \in \mathbb{B}^N,
\]
where \( c(N, \lambda) = \frac{1}{2}(N - 1 + \sqrt{(N - 1)^2 - 4\lambda}) \). This decay property will play a pivotal role in the subsequent analysis.

We define the functionals \( J, J_\infty : H^1(\mathbb{B}^N) \to \mathbb{R} \) as
\[
J(u) := \frac{\|u\|_{\lambda}^2}{\left(\int_{\mathbb{B}^N} a(x)|u(x)|^{p+1} \, dV_{\mathbb{B}^N}(x)\right)^{\frac{2}{p+1}}},
\]
\[
J_\infty(u) := \frac{\|u\|_{\lambda}^2}{\left(\int_{\mathbb{B}^N} |u(x)|^{p+1} \, dV_{\mathbb{B}^N}(x)\right)^{\frac{2}{p+1}}}.
\]
and the energy levels

\[ S_{1,\lambda} := \inf_{u \in H^1(B^N) \setminus \{0\}} J_\infty(u), \quad S_{m,\lambda} := m^{\frac{p-1}{p+1}} S_{1,\lambda}, \quad m = 2, 3, 4, \ldots \]  

(4.3)

Let us recall a convex inequality.

**Lemma 4.1** Let \( p > 1 \) be any real number. Then for any non-negative real numbers \( a, b \), there holds

\[
(a + b)^{p+1} \geq a^{p+1} + b^{p+1} + p(a^p b + a b^p).
\]

For the proof of the above lemma, we refer [15].

**Lemma 4.2** Let \( a(x) \) satisfies (a1)–(a2) and let \( w \) be a unique radial solution of (4.1). Then, there exists a large number \( R_0 \), such that for any \( R \geq R_0 \), \( d(x_1, 0) \geq R^\alpha, \ d(x_2, 0) \geq R^\alpha \), \( R^\alpha \leq d(x_1, x_2) \leq R^\alpha - \alpha \min\{d(x_1, 0), \ d(x_2, 0)\} \), where \( \alpha > \alpha' > 1 \), it holds

\[
J\left( tu_1 + (1 - t)u_2 \right) < S_{2,\lambda},
\]

(4.4)

where \( 0 \leq t \leq 1 \) and \( u_i = w\left(\tau_{-x_i}(\cdot)\right), \ i = 1, 2 \).

**Proof 1** : In this step, we will prove (4.4) for \( t = \frac{1}{2} \). Set,

\[ A := \|w\|_{H_\lambda}^2. \]

Then by taking \( w \) as a test function in (4.1), we obtain

\[ A = \int_{B^N} w(x)^{p+1} \, dV_{B^N}. \]

Therefore,

\[
J\left( \frac{1}{2} (u_1 + u_2) \right) = J\left( u_1 + u_2 \right) \\
\leq \frac{\|u_1\|_{H_\lambda}^2 + \|u_2\|_{H_\lambda}^2 + 2 \langle u_1, u_2 \rangle_{H_\lambda}}{(\int_{B^N} (u_1 + u_2)^{p+1} \, dV_{B^N} - \int_{B^N} (1 - a(x)) (u_1 + u_2)^{p+1} \, dV_{B^N})^{\frac{2}{p+1}}} \\
\leq \frac{2A + 2 \langle u_1, u_2 \rangle_{H_\lambda}}{(\int_{B^N} (u_1 + u_2)^{p+1} \, dV_{B^N} - \int_{B^N} (1 - a(x)) (u_1 + u_2)^{p+1} \, dV_{B^N})^{\frac{2}{p+1}}}. 
\]

(4.5)
Firstly, we estimate the integral, \( \int_{\mathbb{B}_N} u_1^p u_2 \, dV_{\mathbb{B}_N} \). Since \( w \) is positive, radially symmetric, symmetric decreasing and smooth, we get

\[
\int_{\mathbb{B}_N} u_1^p u_2 \, dV_{\mathbb{B}_N} = \int_{\mathbb{B}_N} w \left( d(\tau_{-x_1}(x), 0) \right)^p w \left( d(\tau_{-x_2}(x), 0) \right) \, dV_{\mathbb{B}_N}(x)
\]

\[
\geq \int_{d(x, x_1) \leq 1} w \left( d(\tau_{-x_1}(x), 0) \right)^p w \left( d(\tau_{-x_2}(x), 0) \right) \, dV_{\mathbb{B}_N}(x)
\]

\[
\geq C \int_{d(x, x_1) \leq 1} w \left( d(\tau_{-x_2}(x), 0) \right) \, dV_{\mathbb{B}_N}(x)
\]

\[
= \int_{d(x, x_1) \leq 1} w \left( d(x, x_2) \right) \, dV_{\mathbb{B}_N}(x).
\]

In the above calculations, we have used the following observations:

\[
d(x, x_1) \leq 1 \implies d(x, x_2) \leq d(x, x_1) + d(x_1, x_2) \leq 1 + d(x_1, x_2).
\]

Consequently, \( w \left( d(x, x_2) \right) \geq w \left( 1 + d(x_1, x_2) \right) \). Therefore, for \( d(x_1, x_2) \gg 1 \),

\[
\int_{\mathbb{B}_N} u_1^p u_2 \, dV_{\mathbb{B}_N} \geq C_\varepsilon w \left( 1 + d(x_1, x_2) \right) \geq C_\varepsilon e^{-c(N, \lambda) + \varepsilon + d(x_1, x_2)},
\]

where \( C_\varepsilon \) is a positive constant independent of \( R \), and for \( \gamma \in \mathbb{R}, \gamma^+ \) (respectively, \( \gamma^- \)) stands for any \( \gamma + \delta \) (respectively, \( \gamma - \delta \)) with \( \delta > 0 \). Next, we estimate

\[
\int_{\mathbb{B}_N} (1 - a)_+ (u_1 + u_2)^{p+1} \, dV_{\mathbb{B}_N} \leq C \sum_{i=1}^{2} \int_{\mathbb{B}_N} (1 - a)_+ u_i^{p+1} \, dV_{\mathbb{B}_N}
\]

\[
\leq C \sum_{i=1}^{2} \int_{\mathbb{B}_N} w \left( d(\tau_{-x_i}(x), 0) \right)^{p+1} \frac{e^{\delta d(x, 0)}}{dV_{\mathbb{B}_N}} \, dV_{\mathbb{B}_N}
\]

\[
= C \sum_{i=1}^{2} \int_{d(x, x_i) > d(x_i, 0)} \frac{w \left( d(\tau_{-x_i}(x), 0) \right)^{p+1}}{e^{\delta d(x, 0)}} \, dV_{\mathbb{B}_N} + \int_{d(x, x_i) \leq d(x_i, 0)} \frac{w \left( d(\tau_{-x_i}(x), 0) \right)^{p+1}}{e^{\delta d(x, 0)}} \, dV_{\mathbb{B}_N}.
\]

Now we shall estimate \( I_1^i + I_2^i \) with \( \delta \in (Kc(N, \lambda) + (N - 1), (p + 1)c(N, \lambda)) \), and choose \( \varepsilon > 0 \) such that \( \delta < (p + 1)(c(N, \lambda) - \varepsilon) \), where \( K \) is fixed, and
is such that $0 < K < (p + 1) - \frac{N-1}{c(N, \lambda)}$. Before moving further, we note that $\frac{c(N, \lambda)}{N-1} > \frac{1}{2}$ and $p + 1 > 2$ implies $(p + 1) - \frac{N-1}{c(N, \lambda)} > 0$. Let us first consider $I_1^i$:

$$I_1^i = \int_{d(x, x_i) > d(x_i, 0)} \frac{w(d(\tau_{-x_i}(x), 0))}{e^{\delta d(x, 0)}} \, dV_B(x)$$

$$\leq \int_{d(x, x_i) > d(x_i, 0)} \frac{w(d(x_i, 0))}{e^{\delta d(x, 0)}} \, dV_B(x)$$

$$\leq C_N \int_{d(x, x_i) > d(x_i, 0)} e^{-\delta d(x, 0)} \, dV_B(x)$$

$$\leq C_N \int_{d(x, x_i) > d(x_i, 0)} e^{-\delta d(x, 0)} \, dV_B(x)$$

$$= C_N \int_{d(x, x_i) > d(x_i, 0)} \int_{r=0}^{\infty} e^{-\delta r} e^{(N-1) r} \, dr$$

$$\leq C_N e^{-\delta d(x_i, 0)} e^{(N-1) d(x_i, 0)}.$$

where passing from the second inequality to third, we have used $d(x, x_i) > d(x_i, 0)$ which in turn implies, $d(x, 0) \geq d(x, x_i) - d(x_i, 0) > 0$, and the last step follows from the choice of $\delta$ as $\delta > K c(N, \lambda) + (N - 1) > N - 1$. Now, we estimate $I_2^i$ for $\delta < (p + 1) (c(N, \lambda) - \varepsilon)$, and using triangle inequality $d(x_i, 0) \leq d(x_i, x) + d(x, 0)$,

$$I_2^i = \int_{d(x, x_i) \leq d(x_i, 0)} \frac{w(d(\tau_{-x_i}(x), 0))}{e^{\delta d(x, 0)}} \, dV_B(x)$$

$$\leq C_N \int_{d(x, x_i) \leq d(x_i, 0)} e^{-(c(N, \lambda) - \varepsilon) d(x, x_i)} \frac{e^{-\delta d(x, 0)}}{e^{\delta d(x, 0)}} \, dV_B(x)$$

$$\leq C_N \int_{d(x, x_i) \leq d(x_i, 0)} e^{-\delta d(x, 0)} \, dV_B(x)$$

$$\leq C_N \int_{d(x, x_i) \leq d(x_i, 0)} e^{-\delta d(x, 0)} \, dV_B(x)$$

$$\leq C_N \int_{d(x, x_i) \leq d(x_i, 0)} \int_{d(x, 0) \leq d(x_i, 0)} e^{-\delta d(x, 0)} \, dV_B(x)$$

$$\leq C_N e^{-\delta d(x_i, 0)} e^{(N-1) d(x_i, 0)}.$$

Combining the above estimates results in

$$\int_{B_N} (1 - a) (u_1 + u_2)^{p+1} \, dV_B(x)$$

$$\leq C_N \sum_{i=1}^{2} \left( e^{-\delta d(x, 0)} (u_1, 0) + e^{-\delta d(x_i, 0)} e^{(N-1) d(x_i, 0)} \right)$$
\[ \leq C_\varepsilon \sum_{i=1}^{2} \left( e^{-(\delta-(N-1)d(x_i,0))} \right) \text{ since } \delta < (p + 1)(c(N, \lambda) - \varepsilon) \]
\[ \leq C_\varepsilon e^{\left( -\frac{Kc(N,\lambda)d(x_1,x_2)}{K + R^{d-\alpha}} \right)} \]
\[ \leq C_\varepsilon e^{\left( -\frac{c(N,\lambda)+cKd(x_1,x_2)}{1 + \frac{1}{K} R^{d-\alpha}} \right)} \]
\[ \leq C_\varepsilon e^{-(c(N,\lambda)+cK)d(x_1,x_2)} \]
\[ \leq o(1) \left( \int_{B_N} u_1^{p} u_2 \, dV_{B_N} \right) \]

for large \( R > 0 \). \hfill (4.6)

In the above calculations, we have used our hypothesis, \( d(x_1,x_2) \leq (K + R^{d-\alpha})d(x_i,0) \) for \( i = 1, 2 \). Further, as \( p > 1 \), we can use Lemma 4.1 to get
\[ \int_{B_N} (u_1 + u_2)^{p+1} \, dV_{B_N} \geq \int_{B_N} \left( u_1^{p+1} + u_2^{p+1} \right) \, dV_{B_N} \]
\[ + p \int_{B_N} \left( u_1^{p} u_2 + u_1 u_2^{p} \right) \, dV_{B_N}. \hfill (4.7) \]

Since, \( u_1, u_2 \) solves (4.1) that implies
\[ \langle u_1, u_2 \rangle_{H_k} = \int_{B_N} u_1^{p} u_2 \, dV_{B_N} = \int_{B_N} u_2^{p} u_1 \, dV_{B_N}, \| u_i \|_{H_k} \]
\[ = \int_{B_N} u_i^{p+1} \, dV_{B_N}, \ i = 1, 2. \hfill (4.8) \]

Therefore, using (4.8) in (4.7), we obtain
\[ \int_{B_N} (u_1 + u_2)^{p+1} \, dV_{B_N} \geq 2A + 2p \langle u_1, u_2 \rangle_{H_k}. \hfill (4.9) \]

Combining (4.9) and (4.6) with (4.5) yields
\[ J(u_1 + u_2) \leq \frac{2A + 2 \langle u_1, u_2 \rangle_{H_k}}{\left( \int_{B_N} (u_1 + u_2)^{p+1} \, dV_{B_N} - \int_{B_N} (1 - a) + (u_1 + u_2)^{p+1} \, dV_{B_N} \right)^{\frac{2}{p+1}}} \]
\[ \leq \frac{2A + 2 \langle u_1, u_2 \rangle_{H_k}}{(2A + (2p - o(1)) \langle u_1, u_2 \rangle_{H_k})^{\frac{2}{p+1}}} \]
\[ = \frac{(2A)^{\frac{p-1}{p+1}} \left( 1 + \frac{1}{A} \langle u_1, u_2 \rangle_{H_k} \right)^{\frac{2}{p+1}}}{\left( 1 + \frac{p-o(1)}{A} \langle u_1, u_2 \rangle_{H_k} \right)^{\frac{2}{p+1}}} \]

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From [27], it is known that $S_{1,\lambda}$ is achieved by $w$, which is a solution of (4.1). This in turn implies $S_{1,\lambda} = A^{\frac{p-1}{p+1}}$ and $S_{2,\lambda} = (2A)^{\frac{p-1}{p+1}}$. Hence,

$$
J(u_1 + u_2) \leq S_{2,\lambda} \frac{1 + \frac{1}{A} \langle u_1, u_2 \rangle_{H_\delta}}{\left(1 + \frac{p-o(1)}{A} \langle u_1, u_2 \rangle_{H_\delta}\right)^{\frac{2}{p+1}}}
$$

$$
\leq S_{2,\lambda} \frac{1 + \frac{1}{A} \langle u_1, u_2 \rangle_{H_\delta}}{\left(1 + \frac{2p-o(1)}{(p+1)A} \langle u_1, u_2 \rangle_{H_\delta}\right)^{\frac{2}{p+1}}}.
$$

Now, $\frac{1}{A} < \frac{2p-o(1)}{(p+1)A}$ for $R$ large and hence, $J(u_1 + u_2) < S_{2,\lambda}$.

**Step 2:** We will complete the proof of the lemma in this step. Suppose,

$$
v_1 = tu_1, \quad v_2 = (1-t)u_2, \quad \text{where} \quad t \in [0, 1].
$$

Doing a straight forward computation as in (4.5), it is easy to check that

$$
J(v_1 + v_2) \leq \left(\int_{\mathbb{B}^N} \frac{(t^2 + (1-t)^2) A + 2t(1-t) \langle u_1, u_2 \rangle_{H_\delta}}{\left(\int_{\mathbb{B}^N} |v_1 + v_2|^{p+1} \text{d}V_{\mathbb{B}^N} - \int_{\mathbb{B}^N} (1-a) \langle v_1 + v_2 \rangle^{p+1} \text{d}V_{\mathbb{B}^N}\right)^{\frac{2}{p+1}}} \right)^{\frac{p+1}{2}}.
$$

Observing that for $t$ or $1-t$ tending to zero, $v_1 + v_2$ tends to $u_2$ or $u_1$, and consequently, $J(v_1 + v_2)$ converges to $S_{1,\lambda}$. Therefore, there exists some $\delta' > 0$ such that any $\min\{t, 1-t\} \leq \delta'$, then $J(v_1 + v_2) < S_{2,\lambda}$, and this $\delta'$ is independent of (large) $R$. Moreover, $J(\frac{u_1 + u_2}{2}) < S_{2,\lambda}$, i.e., $J(tu_1 + (1-t)u_2) < S_{2,\lambda}$ for $t = \frac{1}{2}$. Therefore, there exists a neighborhood, say $N(\frac{1}{2})$ of $t = \frac{1}{2}$ such that $J(tu_1 + (1-t)u_2) < S_{2,\lambda}$ for all $t \in N(\frac{1}{2})$. From here onward, we will assume $\min\{t, 1-t\} \geq \delta'$ and $t \in ([0, 1] \setminus N(\frac{1}{2}))$. Using (4.6), we can get the subsequent inequality

$$
\int_{\mathbb{B}^N} (1-a) \langle v_1 + v_2 \rangle^{p+1} \text{d}V_{\mathbb{B}^N} = \int_{\mathbb{B}^N} (1-a) \langle tu_1 + (1-t)u_2 \rangle^{p+1} \text{d}V_{\mathbb{B}^N}
$$

$$
\leq \int_{\mathbb{B}^N} (1-a) \langle u_1 + u_2 \rangle^{p+1} \text{d}V_{\mathbb{B}^N}
$$

$$
\leq o(1) \langle u_1, u_2 \rangle_{H_\delta}. \quad (4.10)
$$

\footnote{Springer}
Using Lemma 4.1 and equations in (4.8), we obtain

\[
\int_{\mathbb{B}^N} |v_1 + v_2|^{p+1} \, dV_{\mathbb{B}^N} = \int_{\mathbb{B}^N} |tu_1 + (1-t)u_2|^{p+1} \, dV_{\mathbb{B}^N} \\
\geq t^{p+1} \int_{\mathbb{B}^N} u_1^{p+1} \, dV_{\mathbb{B}^N} + (1-t)^{p+1} \int_{\mathbb{B}^N} u_2^{p+1} \, dV_{\mathbb{B}^N} \\
+p \int_{\mathbb{B}^N} \left[ t^p (1-t)u_1^p u_2 + t(1-t)^p u_1 u_2^p \right] \, dV_{\mathbb{B}^N} \\
= \left( t^{p+1} + (1-t)^p \right) A + p \left\{ t^p (1-t) + t(1-t)^p \right\} \langle u_1, u_2 \rangle_{H^\lambda}.
\]

We claim that

\[
\langle u_1, u_2 \rangle_{H^\lambda} = \int_{\mathbb{B}^N} u_1^p u_2 \, dV_{\mathbb{B}^N} \to 0 \quad \text{as} \quad R \to \infty.
\]

\[
\int_{\mathbb{B}^N} u_1^p u_2 \, dV_{\mathbb{B}^N} = \int_{\mathbb{B}^N} \left( d(\tau-x_1(x),0) \right)^p w \left( d(\tau-x_2(x),0) \right) \, dV_{\mathbb{B}^N}(x) \\
= \int_{\mathbb{B}^N} \left( d(x,0) \right)^p w \left( d(x, \tau-x_1(x_2)) \right) \, dV_{\mathbb{B}^N}(x) \\
= \int_{d(x,0) \leq \frac{d(\bar{x},0)}{2}} \left( d(x,0) \right)^p w \left( d(x, \tau-x_1(x_2)) \right) \, dV_{\mathbb{B}^N}(x) \\
+ \int_{d(x,0) > \frac{d(\bar{x},0)}{2}} \left( d(x,0) \right)^p w \left( d(x, \tau-x_1(x_2)) \right) \, dV_{\mathbb{B}^N}(x)
\]

where \( \bar{x} = \tau-x_1(x_2) \)

\[:= I_1 + I_2. \]

Now,

\[I_1 := \int_{d(x,0) \leq \frac{d(\bar{x},0)}{2}} \left( d(x,0) \right)^p w \left( d(x, \tau-x_1(x_2)) \right) \, dV_{\mathbb{B}^N}(x). \]

As \( d(\bar{x},x) \geq d(\bar{x},0) - d(x,0) \geq d(\bar{x},0) - \frac{d(\bar{x},0)}{2} = \frac{d(\bar{x},0)}{2} \)

Consequently, \( w \left( d(\bar{x},x) \right) \leq w \left( \frac{d(\bar{x},0)}{2} \right) \). Therefore,

\[I_1 \leq \int_{d(x,0) \leq \frac{d(\bar{x},0)}{2}} \left( d(x,0) \right)^p w \left( \frac{d(\bar{x},0)}{2} \right) \, dV_{\mathbb{B}^N}(x) \\
\leq C \epsilon e^{-c(N,\lambda-\epsilon) \frac{d(\bar{x},0)}{2}} \int_0^{\frac{d(\bar{x},0)}{2}} e^{-c(N,\lambda-\epsilon)(p+1) + (N-1)r} \, dr \\
\leq C \epsilon \left[ e^{-c(N,\lambda-\epsilon)(p+1) + (N-1)} \frac{d(\bar{x},0)}{2} - e^{-c(N,\lambda-\epsilon) \frac{d(\bar{x},0)}{2}} \right] = o(1), \]

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since \(c(N, \lambda)(p + 1) > N - 1\), we can choose \(\varepsilon\) such that \((c(N, \lambda) - \varepsilon)(p + 1) > N - 1\) and where \(o(1)\) tends to 0 as \(R \to \infty\). Next, we shall estimate \(I_2\) to estimate \(I_2\) we distinguish into the following two cases. Case 1 : 1 < \(p < 2\).

\[
I_2 : = \int_{d(x,0) > \frac{d(\bar{c},0)}{2}} w \left( d(x, 0) \right)^p w \left( d(x, \tau_{x_1}(x_2)) \right) dV_{B_N}(x)
\]

\[
\leq \left( \int_{d(x,0) > \frac{d(\bar{c},0)}{2}} w^2 \left( d(x, 0) \right) dV_{B_N}(x) \right)^{\frac{p}{2}}
\]

\[
\leq \left( \int_{d(x,0) > \frac{d(\bar{c},0)}{2}} w^2 \left( d(x, \tau_{x_1}(x_2)) \right) dV_{B_N}(x) \right)^{\frac{2-p}{2}}
\]

\[
\leq C_\varepsilon \left( \int_{d(x,0) > \frac{d(\bar{c},0)}{2}} e^{-(c(N,\lambda) - \varepsilon)2r} e^{(N-1)r} dr \right)^{\frac{p}{2}}
\]

\[
\leq C \int_{0}^{\infty} e^{-(c(N,\lambda) - \varepsilon)\frac{2}{2-p}r} e^{(N-1)r} dr = o(1)
\]

where \(o(1)\) tends to 0 as \(R \to \infty\), i.e., \(d(x_1, x_2) \to \infty\), in the second last step we have used the fact that \(\frac{2}{2-p} > 2\) whenever \(1 < p < 2\), and choose \(\varepsilon\) such that \((c(N, \lambda) - \varepsilon)\frac{2}{2-p} > N - 1\) and \((c(N, \lambda) - \varepsilon)2 > N - 1\). Case 2 : \(p \geq 2\)

\[
I_2 : = \int_{d(x,0) > \frac{d(\bar{c},0)}{2}} w \left( d(x, 0) \right)^p w \left( d(x, \tau_{x_1}(x_2)) \right) dV_{B_N}(x)
\]

\[
\leq \left( \int_{d(x,0) > \frac{d(\bar{c},0)}{2}} \left( w \left( d(x, 0) \right) \right)^{\frac{2Np}{N+2}} dV_{B_N}(x) \right)^{\frac{N+2}{2N}}
\]

\[
\times \left( \int_{B_N} w^{\frac{2}{N+2}} \left( d(x, \tau_{x_1}(x_2)) \right) dV_{B_N}(x) \right)^{\frac{1}{N+2}}
\]

\[
\leq C_\varepsilon \left( \int_{d(\bar{c},0)}^{\infty} e^{-(c(N,\lambda) - \varepsilon)\frac{2Np}{N+2}+(N-1)r} dr \right)^{\frac{N+2}{2N}} = o(1),
\]
where in the last step, we used the fact that \( c(N, \lambda) - \epsilon \frac{2^N p}{N + 2} - (N - 1) > 0 \) for \( p \geq 2 \). Now by choosing minimum among all finitely many \( \epsilon \), (in step 1 and step 2), we can make all the above constants uniform. Therefore, using (4.10) and (4.11), we can evaluate the following

\[
J (v_1 + v_2) \leq \frac{(t^2 + (1 - t)^2) A + 2t(1 - t) \langle u_1, u_2 \rangle_{H_k}}{\left[ (t^{p+1} + (1 - t)^{p+1}) A + p (t^p(1 - t) + t(1 - t)^p - o(1)) \langle u_1, u_2 \rangle_{H_k} \right]^\frac{2}{p+1}}
\]

\[
= \frac{2t(1 - t) \langle u_1, u_2 \rangle_{H_k}}{(t^{p+1} + (1 - t)^{p+1}) A}\frac{2}{p+1}
\]

\[
\times \left( 1 + p \left( \frac{(t^p(1 - t) + t(1 - t)^p)}{(t^{p+1} + (1 - t)^{p+1}) A} - o(1) \right) \right) \langle u_1, u_2 \rangle_{H_k}\frac{2}{p+1}.
\]

Since \( \frac{A}{A_2} = S_{l, \lambda} = 2^{1-p} S_{2, \lambda} \) we have,

\[
J (v_1 + v_2) \leq S_{l, \lambda} \times \frac{t^2 + (1 - t)^2}{(t^{p+1} + (1 - t)^{p+1}) A}\frac{2}{p+1}
\]

\[
+ \frac{2t(1 - t) \langle u_1, u_2 \rangle_{H_k}}{(t^{p+1} + (1 - t)^{p+1}) A}\frac{2}{p+1}
\]

\[
\times \left( 1 + p \left( \frac{(t^p(1 - t) + t(1 - t)^p)}{(t^{p+1} + (1 - t)^{p+1}) A} - o(1) \right) \right) \langle u_1, u_2 \rangle_{H_k}\frac{2}{p+1}
\]

\[
= S_{l, \lambda} \times \frac{t^2 + (1 - t)^2}{(t^{p+1} + (1 - t)^{p+1}) A}\frac{2}{p+1}
\]

\[
+ \frac{2t(1 - t) \langle u_1, u_2 \rangle_{H_k}}{(t^{p+1} + (1 - t)^{p+1}) A}\frac{2}{p+1}
\]

\[
\times \left( 1 - \frac{2p}{p+1} \left( \frac{(t^p(1 - t) + t(1 - t)^p)}{(t^{p+1} + (1 - t)^{p+1}) A} \right) \langle u_1, u_2 \rangle_{H_k} + o(1) \right)
\]

\[
= S_{l, \lambda} \times \frac{t^2 + (1 - t)^2}{(t^{p+1} + (1 - t)^{p+1}) A}\frac{2}{p+1}
\]

\[
+ C g(t) \langle u_1, u_2 \rangle_{H_k} + o(\langle u_1 u_2 \rangle_{H_k}),
\]

where \( g(t) \) is a bounded function and \( C > 0 \) is a constant. Thus, we obtain

\[
J (v_1 + v_2) \leq S_{l, \lambda} \times \frac{t^2 + (1 - t)^2}{(t^{p+1} + (1 - t)^{p+1}) A}\frac{2}{p+1}
\]

\[
+ g(t) \langle u_1, u_2 \rangle_{H_k} + o(\langle u_1 u_2 \rangle_{H_k}).
\]
Moreover, we have
\[ \max_{t \in [0, 1]} \frac{t^2 + (1 - t)^2}{(t^{p+1} + (1 - t)^{p+1})^{\frac{2}{p+1}}} < 2^{\frac{p-1}{p+1}}, \quad \forall t \in (0, 1). \]

Therefore, using the above fact and letting \( R \to \infty \), we conclude
\[ J(v_1 + v_2) < 2^{\frac{p-1}{p+1}} S_{1,\lambda} = S_{2,\lambda}. \]

This completes the proof of Lemma 4.2.

\[ \square \]

5 Proof of Theorem 1.1

In this section, we shall show the existence of a solution by employing the min–max procedure in the spirit of Bahri–Li [6]. Before going further, let us define some notations and functional.

\[ \Sigma := \left\{ u \in H^1(\mathbb{B}^N) : \| u \|_\lambda = 1 \right\}, \quad \Sigma^+ := \left\{ u \in \Sigma : u \geq 0 \text{ a.e. in } \mathbb{B}^N \right\}. \]

We shall now define the center of mass corresponding to \( \Sigma \) on the hyperbolic space. To this end, it is worth noticing that
\[ \frac{|x|}{d(x, 0)} := \frac{|x|}{\log \left( \frac{1+|x|}{1-|x|} \right)} \rightarrow \begin{cases} \frac{1}{2}, & x \to 0 \\ 0, & x \to 1. \end{cases} \]

Moreover, the function is bounded in \( \mathbb{B}^N \), i.e., the unit disk in \( \mathbb{R}^N \). Therefore, let \( M := \sup \left\{ \frac{|x|}{d(x, 0)} : x \in \mathbb{R}^N \text{ with } |x| < 1 \right\} < \infty \). However, we cannot define center of mass in terms of \( \frac{|x|}{d(x, 0)} \), since at infinity, the ratio tends to 0. To overcome this difficulty, we define a function \( k(x) > 0 \) such that \( d(\frac{x}{k(x)}, 0) = 1 \), i.e., \( k(x) := k = \frac{|x|}{\tanh(\frac{1}{2})} \).

Now we are in a situation to define the following quantity:
Let \( m : \Sigma \to B(0, 1) \) defined as follows:
\[ m(u) := \frac{1}{\| u \|_{L^{p+1}(\mathbb{B}^N)}} \int_{\mathbb{B}^N} \frac{x}{k} |u|^{p+1} \ dV_{\mathbb{B}^N}. \]

Clearly \( |m(u)| < \tanh(\frac{1}{2}) \). As discussed in Sect. 2, and using elementary computation we can also conclude that \( d(m(u), 0) < 1 \). Moreover, \( m \) is continuous from \( \Sigma \) to \( \mathbb{B}^N \).

We can now prove Theorem 1.1 as follows:
**Proof** Let $J$ as defined in (4.2), we define

$$I_z := \inf_{m(u) = z, \ u \in \Sigma} J(u) \text{ for } z \in B^N \text{ such that } d(z, 0) < 1.$$  

It is straightforward to note that if $\inf_{u \in \Sigma} J(u) < S_{1, \lambda}$, then exploiting the standard variational arguments (see [34]), the existence of a positive solution of $(P_\lambda)$ can be proven, which in fact, is a constant multiple of the global minimum of the functional $J$.

Henceforth, we are left with the case when $\inf_{u \in \Sigma} J(u) \geq S_{1, \lambda}$.

If $I_z = S_{1, \lambda}$ for some $z \in B^N$ with $d(z, 0) < 1$ then there exists some $u \in \Sigma^+, m(u) = z$, such that $J(u) = S_{1, \lambda}$. The theorem, under this condition, follows from the concentration compactness principle established in [34] in the spirit of [6], with minor modifications needed for the hyperbolic space and again; in this case, the solution obtained is a constant multiple of the global minimum of $J$.

Thus we are only left with the following possibility

$$I_z > S_{1, \lambda} \text{ for every } z \in B^N \text{ with } d(z, 0) < 1.$$  

We now claim the existence of some positive constant $R_1$ for which the following holds:

$$J(w(\tau_{-y}(\bullet))) \leq S_{1, \lambda} \text{ for every } y \in B^N \text{ with } d(y, 0) \geq R_1.$$  

To prove the above claim, we can observe the following by utilizing (a2):

$$\liminf_{d(y, 0) \to \infty} a(\tau_y(x)) \geq 1 \text{ for every } x \in B^N.$$  

Further, applying Fatou’s lemma and the above equation yields

$$\int_{B^N} |w(x)|_{p+1} \, dV_{B^N} \leq \int_{B^N} \liminf_{d(y, 0) \to \infty} a(\tau_y(x)) |w(x)|_{p+1} \, dV_{B^N} \leq \liminf_{d(y, 0) \to \infty} \int_{B^N} a(\tau_y(x)) |w(x)|_{p+1} \, dV_{B^N}. \quad (5.1)$$

Thus, we have

$$\lim_{d(y, 0) \to \infty} J(w(\tau_{-y}(\cdot))) = \frac{\|w\|_{\lambda}^2}{\liminf_{d(y, 0) \to \infty} \left( \int_{B^N} a(\tau_y(x)) |w(x)|_{p+1} \, dV_{B^N} \right)^{\frac{2}{p+1}}} \leq \frac{\|w\|_{\lambda}^2}{\left( \int_{B^N} |w(x)|_{p+1} \, dV_{B^N} \right)^{\frac{2}{p+1}}} = J_\infty(w) = S_{1, \lambda},$$

where, we have used (5.1) to get the last inequality. Hence, the claim follows.
Fixing any \( z \in \mathbb{B}^N \) such that \( d(z, 0) < 1 \) and \( I_z > S_{1,\lambda} \). Consequently, \( S_{1,\lambda} < \frac{1}{2} \left( I_z + S_{1,\lambda} \right) < I_z \). Thus, using above the claim, there exists some positive constant \( \tilde{R}_1 \), such that

\[
I_z > \frac{1}{2} \left( I_z + S_{1,\lambda} \right) > S_{1,\lambda} \geq J(w(\tau_{-y}(\bullet))) \quad \forall y \quad \text{with} \quad d(y, 0) \geq \tilde{R}_1.
\]

Assume \( \alpha, \alpha' \) and \( R > R_0 \) be the same constants as were in lemma 4.2. Choose \( R_2 > \max \{ R^\alpha, \tilde{R}_1 \} \) be very large and

\[
x_2 = \left( 0, 0, \ldots, \tanh \left( \frac{R_2 - R^\alpha}{2} \right) \right).
\]

Define a map \( h_0 : \partial B(0, R_2) \to \Sigma^+ \) as

\[
h_0(x_1) = \frac{w(\tau_{-x_1}(\cdot))}{\|w(\tau_{-x_1}(\cdot))\|_\lambda}, \quad \text{where} \quad x_1 \in \partial B(0, R_2).
\]

By the choice of \( R_2 \), we have \( R_2 > \tilde{R}_1 \), so we get

\[
J(h_0(x_1)) < \frac{1}{2} \left( I_z + S_{1,\lambda} \right) < I_z \quad \text{for} \quad x_1 \in \partial B(0, R_2).
\]

Now we define another map \( h^* : B(0, R_2) \to \Sigma^+ \) by

\[
h^*(tx_1 + (1-t)x_2) = \frac{tw(\tau_{-x_1}(\cdot)) + (1-t)w(\tau_{-x_2}(\cdot))}{\|tw(\tau_{-x_1}(\cdot)) + (1-t)w(\tau_{-x_2}(\cdot))\|_\lambda},
\]

for \( 0 \leq t \leq 1, \ x_1 \in \partial B(0, R_2) \). It can be observed that \( h^*|_{\partial B(0, R_2)} = h_0 \).

Also, from Lemma 4.2, we obtain

\[
J(h^*(y)) < S_{2,\lambda} \quad \text{for all} \quad y \in B(0, R_2). \quad (5.2)
\]

Next, we define some min–max value. Let

\[
\Gamma := \{ h : B(0, R_2) \to \Sigma^+ : h \text{ is continuous,} \ h|_{\partial B(0, R_2)} = h_0 \}
\]

and

\[
\mu_0 := \inf_{h \in \Gamma} \max_{y \in B(0, R_2)} J(h(y)).
\]

We now claim that the following holds:

\[
S_{1,\lambda} < I_z \leq \mu_0 < S_{2,\lambda}. \quad (5.3)
\]
Using (5.2), it is evident that $\mu_0 < S_{2,\lambda}$. Furthermore, consider the map

$$m \circ h : B(0, R_2) \to \mathbb{B}^N,$$ where $h \in \Gamma$.

Then one can see that $\tau_{x_1}(y) \to x_1$ as $x_1 \to \infty$, and for fixed $y \in \mathbb{B}^N$, and hence, we have

$$\lim_{R_2 \to \infty} m \circ h_0(x_1) = \frac{x_1}{|x_1|} \tanh(1/2) \text{ uniformly for } x_1 \in \partial B(0, R_2).$$

Thus, the above convergence immediately implies that for any $z \in \mathbb{B}^N$ with $d(z, 0) < d(m \circ h_0(x_1), 0)$ for $x_1 \in \partial B(0, R_2)$, $z \not\in m \circ h(\partial B(0, R_2))$, and hence, $\deg(m \circ h, B(0, R_2), z)$ is well defined.

Let us define $I : \tilde{B}(0, R_2) \to B(0, 1)$ by

$$I(x) := \frac{x}{\tanh(R_2)} \tanh(1/2).$$

Moreover, for $R_2$ large enough, $m \circ h(\partial B(0, R_2)) = I(\partial B(0, R_2))$. Therefore, applying degree theory, for $R_2$ large enough, we have

$$\deg(m \circ h, B(0, R_2), z) = \deg(I, B(0, R_2), z) = 1.$$ 

In particular, using the solution property of degree, we can guarantee the existence of some $y \in B(0, R_2)$, such that, $m \circ h(y) = z$. Hence, $\mu_0 \geq I_\Sigma > S_{1,\lambda}$.

From Proposition 3.1, we can conclude that $I_{\lambda,a}$ satisfies PS condition for $\frac{p-1}{(p+1)} A < c < \frac{p-1}{p+1} A$. Equivalently, $J|_{\Sigma^+}$ satisfies PS condition for $S_{1,\lambda} < c < S_{2,\lambda}$, i.e., $A^{\frac{p-1}{p+1}} < c < (2A)^{\frac{p-1}{p+1}}$. Therefore, by (5.3), $J|_{\Sigma^+}$ satisfies PS condition at $\mu_0$. As a result, using deformation lemma, $\mu_0$ is a critical value of $J|_{\Sigma^+}$ with some corresponding critical point $0 \neq u \geq 0$, i.e., $J'|_{\Sigma^+}(u) = 0$. Hence, we obtain a non-negative solution of $(P_\lambda)$ by scaling $u$. Furthermore, $u$ is a positive solution to $(P_\lambda)$ follows from the maximum principle.

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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