Quantum noise induced entanglement and chaos
in the dissipative quantum model of brain

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Abstract We discuss some features of the dissipative quantum model of brain in the
frame of the formalism of quantum dissipation. Such a formalism is based on the doubling of
the system degrees of freedom. We show that the doubled modes account for the quantum
noise in the fluctuating random force in the system-environment coupling. Remarkably, such
a noise manifests itself through the coherent structure of the system ground state. The
entanglement of the system modes with the doubled modes is shown to be permanent in
the infinite volume limit. In such a limit the trajectories in the memory space are classical
chaotic trajectories.

1 Introduction

In this paper we discuss chaos, entanglement and quantum noise in the dissipative quantum
model of brain [1, 2, 3, 4]. This model is an extension to dissipative dynamics of the quantum
field theory (QFT) model of brain originally formulated by Umezawa and Ricciardi in 1967
[5] and subsequently developed by Stuart, Takahashi and Umezawa [6], by Jibu and Yasue
[7] and by Jibu, Pribram and Yasue [8]. For a general account of the model see ref [2].

As we will see from the outcomes of our discussion, we will be lead to uncover some
consequences of this model which seem to be related, although the precise relation has not
yet been worked out in detail, to actual experimental findings in neurobiology [9].

The time scales of the working brain are such that it is extremely hard, if not impossible,
to think that they can be obtained in a classical approach: the configuration space of the
working brain is so large that only a quantum dynamics may account of the very short time
intervals needed to the brain to span it in such an efficient way as it does.

In the QFT model the memory storage is described in terms of the coherent Bose con-
densation process in the system lowest energy state (usually called the ground state or else
the vacuum state). Bose condensation occurs as a consequence of the action on the brain
of the external inputs. These break the symmetry of the quantum field dynamics. The
quantum fields are the dipole vibrational quantum fields associated with the water and other
bio-molecules endowed with static and/or radiative electric dipole moment.

According to the Goldstone theorem in QFT [10, 11], the spontaneous breakdown of
the symmetry implies the existence of long-range correlation modes (also called the Nambu-
Goldstone (NG) modes) in the ground state of the system. These modes are massless modes
in the infinite volume limit, but they may acquire a finite, non-zero mass due to boundary
or impurity effects [12]. In the quantum model of brain these modes are called dipole-
wave-quanta (DWQ). The density of their condensation in the ground states acts as a code
classifying the state and the memory there recorded. States with different code values are
unitarily inequivalent states, i.e there is no unitary transformation relating states of different
codes.
In formulating a mathematical model of brain we cannot avoid to take into consideration the dissipative character of its dynamics, since the brain is an intrinsically open system, continuously interacting with the environment. As elsewhere observed [1, 2], the very same fact of “getting an information” introduces a partition in the time coordinate, so that one may distinguish between before “getting the information” (the past) and after “getting the information” (the future): the arrow of time is in this way introduced. ...“Now you know it!” is the familiar warning to mean that now, i.e. after having received a certain information, you are not the same person as before getting it. It has been shown [3] that the psychological arrow of time (arising as an effect of memory recording) points in the same direction of the thermodynamical arrow of time (increasing entropy direction) and of the cosmological arrow of time (the expanding Universe direction) [13].

The canonical quantization procedure of a dissipative system requires to include in the formalism also the system representing the environment (or heat bath) in which the system is embedded. One possible way to do that is to depict the environment as the time-reversal image of the system [14]: the environment is thus described as the Double of the system in the time-reversed dynamics (the system image in the mirror of time).

Of course, the specific details of the system–environment coupling may be very intricate and changeable so that they are difficult to be measured and known. One possibility is to take into account the environmental influence on the brain by a suitable choice of the brain vacuum state among the infinitely many of them. Such a choice is triggered by the external input (breakdown of the symmetry), and it actually is the end point of the internal (spontaneous) dynamical process of the brain (self-organization). The chosen vacuum thus carries the signature (memory) of the reciprocal brain–environment influence at a given time under given boundary conditions. A change in the brain–environment reciprocal influence then would correspond to a change in the choice of the brain vacuum: the brain evolution through the vacuum states is thus the evolution of the coupling of the brain with the surrounding world.

Within this paper we will adopt, from the starting, the mathematical framework of QFT. This implies that the brain system will be described in terms of an infinite collection of damped harmonic oscillators $A_\kappa$ (the simplest prototype of a dissipative system) representing the DWQ [1].

The collection of damped oscillators is ruled by the Hamiltonian [1, 14]

$$H = H_0 + H_I,$$

$$H_0 = \sum_\kappa \hbar \Omega_\kappa (A^\dagger_\kappa A_\kappa - \tilde{A}^\dagger_\kappa \tilde{A}_\kappa), \quad H_I = i \sum_\kappa \hbar \Gamma_\kappa (A^\dagger_\kappa \tilde{A}_\kappa - A_\kappa \tilde{A}^\dagger_\kappa),$$

where $\Omega_\kappa$ is the frequency and $\Gamma_\kappa$ is the damping constant. The $\tilde{A}_\kappa$ modes are the “time-reversed mirror image” (the “mirror modes”) of the $A_\kappa$ modes. They are the doubled modes and represent the environment modes. $\kappa$ generically labels the mode degrees of freedom, e.g. spatial momentum (see [1, 14] for details).

Since the environment is described in the quantum dissipation formalism by the doubled degrees of freedom, from now on we will use, without further specification, the word environment meaning such a set of doubled degrees of freedom.

The paper is organized as follows. In Section 2 we show that the doubled $\tilde{A}$ modes actually account for the quantum noise in the fluctuating random force in the system-environment coupling. Section 3 contains a discussion of the entanglement between non–tilde and tilde mode sectors induced by quantum noise. Section 4 is devoted to show that we may have chaotic trajectories (in the sense of dynamical system theory) in the space of the memory states, a result which also suggests a possible connection with laboratory observations [9]. Concluding remarks are presented in Section 5. An extended report of the results obtained in this paper has been presented in [15].
It has been suggested that the doubled degrees of freedom may play some role in the discussion of consciousness mechanisms [1]. However, we leave out of the present paper the discussion on such a topic (the interested reader may consult refs. [1, 2] and the references there quoted).

2 Doubling and quantum noise

As a preliminary to the discussion presented in the following sections and in order to better understand the role played by the quantum noise in the dissipative brain model, in this section we shortly summarize the description of dissipative systems in the frame of the quantum Brownian motion as described by Schwinger [16] and by Feynman and Vernon [17].

We essentially refer to the results derived in refs. [18, 19], where the doubling of the phase-space degrees of freedom is discussed and the doubled variables are shown to account for the quantum noise effects in the fluctuating random force in the system-environment coupling. This result adds a new perspective to the doubling in the quantum model of brain. It seems to point to a relation with some experimental observations in the brain behavior [9]. However, we leave to a future work the deeper analysis which is needed in order to show the details of such a relation.

To be definite, we consider the damped harmonic oscillator (dho)

\[ m \ddot{x} + \gamma \dot{x} + \kappa x = 0, \tag{3} \]

as a simple prototype for dissipative systems. However, our results also apply to more general systems than the one represented in (3).

The damped oscillator Eq. (3) is a non-hamiltonian system and therefore the customary canonical quantization procedure cannot be followed. However, one can face the problem by resorting to well known tools such as the density matrix and the Wigner function.

It is instructive to consider first the special case of a particle in the absence of friction with Hamiltonian

\[ H = -\frac{\hbar^2}{2m} \left( \frac{\partial}{\partial x} \right)^2 + V(x). \tag{4} \]

The density matrix equation of motion is given by

\[ i\hbar \frac{d\rho}{dt} = [H, \rho]. \tag{5} \]

The density matrix function is

\[ \langle x + \frac{1}{2} y | \rho(t) | x - \frac{1}{2} y \rangle = \psi^*(x + \frac{1}{2} y, t) \psi(x - \frac{1}{2} y, t) \equiv W(x, y, t), \tag{6} \]

with the associated standard expression for the Wigner function [20, 21],

\[ W(p, x, t) = \frac{1}{2\pi\hbar} \int W(x, y, t)e^{\frac{i}{\hbar}py} dy. \tag{7} \]

In the coordinate representation, by introducing the notation

\[ x_\pm = x \pm \frac{1}{2} y, \tag{8} \]

Eq. (5) is written as

\[ i\hbar \frac{\partial}{\partial t} \langle x_+ | \rho(t) | x_- \rangle = \left\{ -\frac{\hbar^2}{2m} \left( \left( \frac{\partial}{\partial x_+} \right)^2 - \left( \frac{\partial}{\partial x_-} \right)^2 \right) + [V(x_+) - V(x_-)] \right\} \langle x_+ | \rho(t) | x_- \rangle, \tag{9} \]
and the equation for $W(p, x, t)$ is

$$i\hbar \frac{\partial}{\partial t} W(x, y, t) = \mathcal{H}_o W(x, y, t)$$

(10)

$$\mathcal{H}_o = \frac{1}{m} p_x p_y + V\left(x + \frac{1}{2} y\right) - V\left(x - \frac{1}{2} y\right),$$

(11)

$p_x = -i\hbar \frac{\partial}{\partial x}, \quad p_y = -i\hbar \frac{\partial}{\partial y}.$

(12)

The Hamiltonian (11) may be obtained from the Lagrangian

$$\mathcal{L}_o = m\dot{x}\dot{y} - V\left(x + \frac{1}{2} y\right) + V\left(x - \frac{1}{2} y\right).$$

(13)

We thus conclude that the density matrix and the Wigner function formalism requires, even in the non-dissipative case (zero mechanical resistance $\gamma$), the introduction of a “doubled” set of coordinates, $x_{\pm}$, or, alternatively, $x$ and $y$. One may understand this as related to the introduction of the “couple” of indices necessary to label the density matrix elements (cf. Eq. (9)).

Let us now consider the case of the particle interacting with a thermal bath at temperature $T$. Let $f$ denote the random force on the particle at the position $x$ due to the bath. The interaction Hamiltonian between the bath and the particle is written as

$$H_{int} = -fx.$$  

(14)

In the Feynman-Vernon formalism, the effective action for the particle is given by

$$A[x, y] = \int_{t_1}^{t_f} dt \mathcal{L}_o(\dot{x}, \dot{y}, x, y) + \mathcal{I}[x, y],$$

(15)

with $\mathcal{L}_o$ defined as in Eq.(13) and

$$e^{\frac{i}{\hbar} \mathcal{I}[x, y]} = \langle (e^{-\frac{i}{\hbar} \int_{t_1}^{t_f} f(t) x_{-}(t) dt})_+ (e^{\frac{i}{\hbar} \int_{t_1}^{t_f} f(t) x_{+}(t) dt})_+ \rangle.$$  

(16)

In Eq.(16), the symbol $\langle . . \rangle$ denotes average with respect to the thermal bath; “(.+)” and “(-.)” denote time ordering and anti-time ordering, respectively; the c-number coordinates $x_{\pm}$ are defined as in Eq.(8). If the interaction between the bath and the coordinate $x$ (i.e $H_{int} = -fx$) were turned off, then the operator $f$ of the bath would develop in time according to $f(t) = e^{iH_{\gamma}t/\hbar} f e^{-iH_{\gamma}t/\hbar}$ where $H_{\gamma}$ is the Hamiltonian of the isolated bath (decoupled from the coordinate $x$). $f(t)$ is then the force operator of the bath to be used in Eq.(16).

The interaction $\mathcal{I}[x, y]$ between the bath and the particle has been evaluated in ref. [18] for a linear passive damping due to thermal bath by following Feynman and Vernon [17], and Schwinger [16]. The final result is [18]:

$$\mathcal{I}[x, y] = \frac{1}{2} \int_{t_1}^{t_f} dt \left[ x(t) F_y^{ret}(t) + y(t) F_x^{adv}(t) \right]$$

$$+ \frac{i}{2\hbar} \int_{t_1}^{t_f} \int_{t_1}^{t_f} dt ds N(t-s) y(t) y(s),$$

(17)

where the retarded force on $y$ and the advanced force on $x$ are given in terms of the retarded and advanced Greens functions $G_{ret}(t-s)$ and $G_{adv}(t-s)$:

$$F_y^{ret}(t) = \int_{t_1}^{t_f} ds G_{ret}(t-s)y(s), \quad F_x^{adv}(t) = \int_{t_1}^{t_f} ds G_{adv}(t-s)x(s).$$

(18)
respectively. In Eq (17) $N(t-s)$ is the quantum noise in the fluctuating random force and it is given by

$$N(t-s) = \frac{1}{2} (f(t)f(s) + f(s)f(t)).$$  

The real and the imaginary part of the action are given by

$$\text{Re}A[x,y] = \int_{t_i}^{t_f} dt \mathcal{L},$$  

$$\mathcal{L} = m\dot{x}\dot{y} - \left[ V(x + \frac{1}{2}y) - V(x - \frac{1}{2}y) \right] + \frac{1}{2} \left[ x F_{y}^{ret} + y F_{x}^{adv} \right],$$

and

$$\text{Im}A[x,y] = \frac{1}{2\hbar} \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt ds N(t-s)y(t)y(s).$$

respectively. Eqs. (20), (21), and (22), are rigorously exact results for linear passive damping due to the bath.

They show that in the classical limit “$\hbar \to 0$” nonzero $y$ yields an “unlikely process” in view of the large imaginary part of the action implicit in Eq. (22) (cf. Eq. (17) ). Nonzero $y$, indeed, may lead to a negative real exponent in the evolution operator, which in the limit $\hbar \to 0$ may produce a negligible contribution to the probability amplitude. On the contrary, at quantum level nonzero $y$ accounts for quantum noise effects in the fluctuating random force in the system-environment coupling arising from the imaginary part of the action [18].

This is the conclusion we wanted to reach.

When in Eq.(21) we use $F_{y}^{ret} = \gamma \dot{y}$ and $F_{x}^{adv} = -\gamma \dot{x}$, we get

$$\mathcal{L}(\dot{x}, \dot{y}, x, y) = m\dot{x}\dot{y} - V(x + \frac{1}{2}y) + V(x - \frac{1}{2}y) + \gamma (x\dot{y} - y\dot{x}).$$

(23)

By using

$$V\left(x \pm \frac{1}{2}y\right) = \frac{1}{2} \kappa (x \pm \frac{1}{2}y)^2$$

(24)

in Eq. (23), the dho equation (3) and its complementary equation for the $y$ coordinate

$$m\ddot{y} - \gamma \dot{y} + \kappa y = 0.$$  

(25)

are derived. The $y$-oscillator is the time–reversed image of the $x$-oscillator (3).

From the manifold of solutions to Eqs. (3), (25) we could choose those for which the $y$ coordinate is constrained to be zero, then Eqs. (3) and (25) simplify to

$$m\ddot{x} + \gamma \dot{x} + \kappa x = 0, \quad y = 0.$$  

(26)

Thus we obtain the classical damped oscillator equation from a lagrangian theory at the expense of introducing an “extra” coordinate $y$, later constrained to vanish. Note that the constraint $y(t) = 0$ is not in violation of the equations of motion since it is a true solution to Eqs.(3) and (25).

We stress once more that the role of the “doubled” $y$ coordinate is absolutely crucial in the quantum regime since there it accounts for the quantum noise in the fluctuating random force in the system-environment coupling.

When one adopts the classical (legitimate) solution $y = 0$, the $x$ system appears to be open, “incomplete”; the knowledge of the details of the processes inducing the dissipation may not always be possible; these details may not be explicitly known and the dissipation mechanisms are sometimes globally described by such parameters as friction, resistance, viscosity etc.. In some sense, such parameters are introduced in order to compensate the information
loss caused by dissipation. Such a loss of information essentially amounts to neglecting the
bath variables which originate the damping and the fluctuations. Thus, by putting $x_+ = x_-$,
i.e. by choosing $y = 0$, the quantum features are washed out and one obtains the classical
limit (see also [19]).

In quantum mechanics canonical commutation relations are not preserved by time evolution
due to damping terms. The role of fluctuating forces is in fact the one of preserving the
canonical structure. According to our result, reverting from the classical level to the quantum
level, the loss of information occurring at the classical level due to dissipation manifests itself
in terms of “quantum” noise effects arising from the imaginary part of the action, to which
the $y$ contribution is indeed crucial.

Going back to the dissipative quantum model, it can be shown [14] that the Hamiltonian
Eq. (1), for each given $\kappa$, can be obtained by the canonical quantization procedure from the
Lagrangian (23) with the choice Eq. (24). Note that the classical equations for the dho $x$
and its time-reversal image $y$, Eqs. (3) and (25), are associated in the canonical quantization
procedure to the quantum operators $A$ and $\hat{A}$. When we consider the quantum field theory,
the $A$ and the $\hat{A}$ operators get labelled by the (continuously varying) suffix $\kappa$ and for each $\kappa$
value we have a couple of equations of the type (3) and (25) for the field amplitudes [14].

In the following sections we discuss some of the features of the dissipative quantum model
of brain related to the results presented above.

3 Quantum noise induced entanglement

We have seen that the doubled degrees of freedom account for the quantum noise in the
fluctuating random force in the system-environment coupling. On the other hand, the doubled
degrees of freedom fully characterizes the structure of the space of the states in the dissipative
quantum model of brain. In other words, the brain processes are intrinsically and inextricably
dependent on the quantum noise in the fluctuating random force in the brain-environment
coupling: there is a permanent brain-environment entanglement.

The study of the brain-environment entanglement (namely the entanglement between the
$A_\kappa$ and the $\hat{A}_\kappa$ modes) is the purpose of the present section. To do that we need to shortly
summarize some aspects of the state space of the dissipative quantum model.

Denote by $\{ N^\dagger_{A_\kappa}, N^\dagger_{\hat{A}_\kappa} \}$ the set of simultaneous eigenvectors of $\hat{N}_{A_\kappa} \equiv A^\dagger_{A_\kappa} A_\kappa$ and $\hat{N}_{\hat{A}_\kappa} \equiv \hat{A}^\dagger_{\hat{A}_\kappa} \hat{A}_\kappa$, with $N^\dagger_{A_\kappa}$ and $N^\dagger_{\hat{A}_\kappa}$ non-negative integers and denote by $|0\rangle_0 \equiv |N_{A_\kappa} = 0, N_{\hat{A}_\kappa} = 0\rangle$ the
state annihilated by $A_\kappa$ and by $\hat{A}_\kappa$: $A_\kappa |0\rangle_0 = 0 = \hat{A}_\kappa |0\rangle_0$ for any $\kappa$.

For definitiveness let us consider an initial time, say $t_0 = 0$. The memory state is defined
to be a zero energy eigenstate (the vacuum) of $H_0$. The form of $H_0$ (cf. Eq. (2)) then implies
that the memory state is a condensate of equal number of modes $A_\kappa$ and mirror modes $\hat{A}_\kappa$ for
any $\kappa$. Thus, we may have infinitely many memory states at $t_0$, each one corresponding to
different numbers $N_{A_\kappa}$ of $A_\kappa$ modes, for all $\kappa$, provided $N_{A_\kappa} - N_{\hat{A}_\kappa} = 0$ for all $\kappa$. We observe
that the commutativity of $H_0$ with $H_f$ ($[H_0, H_f] = 0$) ensures that the number ($N_{A_\kappa} - N_{\hat{A}_\kappa}$)
is a constant of motion for any $\kappa$. The $A_\kappa$ and $\hat{A}_\kappa$ modes are actually quasi-massless, i.e.
they have a non-zero effective mass, due to finite volume effects [1, 2, 12].

Denote by $|0\rangle_N$ the memory state with $N \equiv \{ N_{A_\kappa} = N_{\hat{A}_\kappa}, \forall \kappa, at \ t_0 = 0 \}$ the set of
integers defining the “initial value” of the condensate, namely the code associated to the
information recorded at time $t_0 = 0$.

The memory state is a two-mode ($SU(1, 1)$ generalized) coherent state (actually a two–
mode squeezed state [1]) and is generated, at finite volume $V$, by the action of the generator

$$G(\theta) = -i \sum_\kappa \theta_\kappa (A^\dagger_{A_\kappa} \hat{A}^\dagger_{\hat{A}_\kappa} - A_{A_\kappa} \hat{A}_{\hat{A}_\kappa})$$

(27)
upon the state $|0\rangle_0$:

$$|0\rangle_N = \exp(-iG(\theta))|0\rangle_0 = \prod_k \frac{1}{\cosh \theta_k} \exp\left(-\tanh \theta_k A_k A_k^\dagger \right) |0\rangle_0 . \quad (28)$$

The average number $N_{A_\kappa}$ is given by

$$N_{A_\kappa} = \langle 0|A_k A_k^\dagger |0\rangle_N = \sinh^2 \theta_\kappa , \quad (29)$$

which also relates the $N$-set, $N \equiv \{N_{A_\kappa} = \langle 0|A_k A_k^\dagger |0\rangle_N, \forall \kappa, at t_0 = 0\}$ to the $\theta$-set, $\theta \equiv \{\theta_\kappa, \forall \kappa, at t_0 = 0\}$. We also use the notation $N_{A_{\kappa}}(\theta) \equiv N_{A_{\kappa}}$ and $|0(\theta)\rangle \equiv |0\rangle_N$. In general we may refer to $N$ or, alternatively and equivalently, to the corresponding $\theta$, or vice versa.

We note that $|0\rangle_N$ is normalized to 1 for all $N$:

$$N\langle 0|0\rangle_N = 1 \quad \forall N . \quad (30)$$

The state spaces $\{0\rangle_N\}$ and $\{0\rangle_N\}$ (representations of the canonical commutation relations (CCR) of the operators $A_k$ and $A_k$) are each other unitarily inequivalent for different codes $N \neq N'$ in the infinite volume limit:

$$N\langle 0|0\rangle_N, \to 0 \quad \forall N \neq N' . \quad (31)$$

The whole space of states thus includes infinitely many unitarily inequivalent representations $\{0\rangle_N\}$, for all $N$‘s, of the CCR’s. The freedom introduced by the degeneracy among the vacua $|0\rangle_N$, for all $N$, solves the problem of memory capacity. A huge number of sequentially recorded memories may coexist without destructive interference since infinitely many vacua $|0\rangle_N$ are independently accessible. Recording information of code $N'$ does not necessarily produce destruction of previously printed information of code $N \neq N'$. In the non-dissipative case this could not happen. We thus realize the crucial role played in the state space structure by the doubled degrees of freedom, namely by the permanently present quantum noise in the brain-environment random coupling. Such a noise contribution, represented by the mirror modes, allows the possibility of introducing the $N$-coded “replicas” of the ground state, thus introducing a huge memory capacity.

The quantum noise, which remarkably manifests itself in the mentioned coherent structure of $|0\rangle_N$, is also responsible for its entangled nature. This goes as follows.

A two-mode state is an entangled state when it cannot be factorized into two single-mode states. Inspection of Eq. (28) shows that $|0\rangle_N$ can be written as

$$|0\rangle_N = \left( \prod_k \frac{1}{\cosh \theta_k} \right) \left( |0\rangle_0 \otimes |0\rangle_0 - \sum_k \tanh \theta_k \left( |A_k\rangle \otimes |A_k^\dagger\rangle \right) + \ldots \right) . \quad (32)$$

Here the tensor product between the tilde and non-tilde sectors has been explicitly expressed. Dots stand for higher power terms. It is clear that the second factor in the r.h.s. of the above equation cannot be reduced to the product of two single-mode components and therefore $|0\rangle_N$ can never be factorized into two single-mode states. The state $|0\rangle_N$ may be also written as [14]:

$$|0\rangle_N = \sum_{n=0}^{+\infty} \sqrt{W_n} \left( |n\rangle \otimes |\tilde{n}\rangle \right) , \quad (33)$$

$$W_n = \prod_k \frac{\sinh^{2n_k} \theta_k}{\cosh^2 (n_k+1) \theta_k} , \quad (34)$$

where $n$ and $\tilde{n}$ denote the sets $\{n_\kappa\}$ and $\{\tilde{n}_\kappa\}$, respectively, and with $0 < W_n < 1$ and $\sum_{n=0}^{+\infty} W_n = 1$. We have

$$N\langle 0|S_A|0\rangle_N = \sum_{n=0}^{+\infty} W_n ln W_n , \quad (35)$$
with:

\[
S_A \equiv -\sum_\kappa \left\{ A^\dag_\kappa A_\kappa \ln \sinh^2 \theta_\kappa - A_\kappa A^\dag_\kappa \ln \cosh^2 \theta_\kappa \right\}.
\]  

(36)

and thus \(S_A\) can be interpreted as the entropy operator \([1, 11, 14]\) (we might as well introduce \(S_{\tilde{A}}\) given by replacing \(A_\kappa\) and \(A^\dag_\kappa\) with \(\tilde{A}_\kappa\) and \(\tilde{A}^\dag_\kappa\), respectively, in (36)). Eq. (33) shows that it provides a measure of the degree of entanglement.

We stress that the entanglement is realized in the infinite volume limit. It is expressed by the unitary inequivalence relation with the vacuum \(|0_0\rangle\equiv |0_0\rangle \otimes |\tilde{0}_0\rangle\):

\[
\mathcal{N}(0)|0_0\rangle \underset{V \to \infty}{\longrightarrow} 0 \quad \forall \mathcal{N} \neq 0,
\]  

(37)

which is indeed verified only in the continuum \(\kappa\) limit, i.e. the infinite volume limit (see also Eq. (31)). The probability of having the component state \(|n, \tilde{n}\rangle\) in the memory state \(|0_\mathcal{N}\rangle\) is \(W_n\). Since \(W_n\) is a decreasing monotonic function of \(n\), the contribution of the states \(|n, \tilde{n}\rangle\) would be suppressed for large \(n\) at finite volume. In such a case, the tensor product of the tilde and non-tilde sectors would involve only a finite number of \(|n, \tilde{n}\rangle\) component states and \(|0_\mathcal{N}\rangle\) be different from the entangled state: a unitary transformation could disentangle the tilde and non-tilde sectors. Finite volume effects may thus spoil the entanglement. However, this is not the case in the infinite volume limit, where the summation extends to an infinite number of \(n\)-components.

We also note that the entanglement is generated by \(G(\theta)\), through which the brain is coupled with the environment. By considering the results of the previous section, we see that it is exactly the quantum noise in the random force, coupling the system (the brain) to the environment, the responsible for the entanglement. Dissipation is therefore the root of the entanglement and the robustness of the latter is in the non-unitary character of the former. A similar result has been also found at work in quite different physical contexts \([22]\).

Finally, we recall that the memory code \(\mathcal{N}\) is a macroscopic observable in the sense that it is not affected by quantum fluctuations (this is a general feature of systems in condensed matter physics where spontaneous symmetry breakdown occurs). The stability of the order parameter against quantum fluctuations is a virtue of the coherence of the DWQ boson condensation. The “change of scale” (from microscopic to macroscopic scale) is thus dynamically achieved through the boson condensation mechanism. The memory state \(|0_\mathcal{N}\rangle\) provides thus an example of “macroscopic quantum state” (other examples of macroscopic quantum states in condensed matter physics are the crystal state, the ferromagnetic state, the superconducting state, etc.).

In conclusion, the "brain (ground) state" may be represented as the collection (or the superposition) of the full set of entangled memory states \(|0_\mathcal{N}\rangle\), for all \(\mathcal{N}\).

In “the space of the representations of the CCR”, each representation \(|0_\mathcal{N}\rangle\) may be thought as a “point” labelled by a given \(\mathcal{N}\)-set (or \(\theta\)-set) and points corresponding to different \(\mathcal{N}\) (or \(\theta\)) sets are distinct points (do not overlap, cf. Eq. (31)). In other words, \(\mathcal{N}\) (or \(\theta\)) is a good code. We also refer to the space of the representations as to the “memory space”.

Till now our discussion has been limited to a given time \(t_0\). In the following section we analyze the time evolution of the memory states. We will see that trajectories in the memory space are chaotic trajectories.

### 4 Time evolution and chaotic trajectories in the memory space

In this section our task is to show that trajectories over the representations of the CCR in the memory space are chaotic trajectories. In order to see this, we will show that the requirements characterizing the chaotic behavior in non-linear dynamics are verified. These requirements can be formulated in a standard way \([23]\) as follows:
i) the trajectories are bounded and each trajectory does not intersect itself (trajectories are not periodic).

ii) there are no intersections between trajectories specified by different initial conditions.

iii) trajectories of different initial conditions are diverging trajectories.

We are referring to trajectories in the space of the representations and, as shown in [24, 25, 26], they are classical trajectories.

At finite volume \( V \), the time evolution of the memory state \( |0\rangle_N \) is given by [1]

\[
|0(t)\rangle_N = \exp \left(-it\frac{H}{\hbar}\right)|0\rangle_N
\]

\[
= \prod_k \frac{1}{\cosh(\Gamma_k t - \theta_k)} \exp \left(\tanh(\Gamma_k t - \theta_k)A_k^\dagger A_k^\dagger\right)|0\rangle_0 ,
\]

where the commutativity between \( H_I \) and \( G(\theta) \) has been used. Again, \( |0(t)\rangle_N \) is a \( SU(1,1) \) generalized coherent state. It is also an entangled state and, with the due changes, the discussion of the previous section on the entanglement also applies to it. For any \( t \)

\[
\mathcal{N}'(0(t)|0(t))_N = 1 ,
\]

and (for \( \int d^3\kappa \Gamma_\kappa \) finite and positive), in the infinite volume limit,

\[
\mathcal{N}'(0(t)|0(t'))_N \xrightarrow{V \to \infty} 0 \quad \forall t ,
\]

\[
\mathcal{N}'(0(t)|0(t'))_N \xrightarrow{V \to \infty} 0 \quad \forall t,t' , \quad t \neq t' .
\]

These relations express the unitary inequivalence of the states \( |0(t)\rangle_N \) (and of the associated Hilbert spaces \( \{ |0(t)\rangle_N \} \)) at different time values \( t \neq t' \) in the infinite volume limit: the non-unitarity of time evolution implied by damping is consistently recovered in the unitary inequivalence among the representations \( \{ |0(t)\rangle_N \} \) at different \( t \)'s in the infinite volume limit.

The non-unitary, damped time evolution is indeed manifest in the relation:

\[
\lim_{t \to \infty} \mathcal{N}'(0(t)|0(t))_N \propto \lim_{t \to \infty} \exp \left(-t \sum \kappa \Gamma_\kappa\right) = 0 ,
\]

which holds provided \( \sum \kappa \Gamma_\kappa > 0 \).

Time evolution of the memory state \( |0\rangle_N \) is thus represented as the (continuous) transition through the representations \( \{ |0(t)\rangle_N \} \) at different \( t \)'s, namely by the “trajectory” through the “points” \( \{ |0(t)\rangle_N \} \) in the space of the representations. The trajectory “initial condition” at \( t_0 = 0 \) is specified by the \( \mathcal{N} \)-set. As already mentioned, this is a classical trajectory [24, 25, 26]: transition between unitarily inequivalent representations would be strictly forbidden in a quantum dynamics.

We observe that the trajectories are bounded in the sense of Eq. (39), which shows that the “length” (the norm) of the “position vectors” (the state vectors at time \( t \)) in the representation space is finite (and equal to one) for each \( t \). We recall that \( SU(1,1) \) consists of all unimodular \( 2 \times 2 \) matrices leaving invariant the Hermitian form \( |z_1|^2 - |z_2|^2 \). Eq. (39) rests on such an invariance. Moreover, the set of points representing the coherent states \( |0(t)\rangle_N \) for any \( t \) can be shown to be isomorphic to the union of circles of radius \( r_\kappa^2 = \tanh^2(\Gamma_\kappa t - \theta_\kappa) \) for any \( \kappa \) [27].

Eqs. (40) and (41) express the fact that the trajectory of given \( \mathcal{N} \) does not crosses itself as time evolves (it is not a periodic trajectory): the “points” \( |0(t)\rangle_N \) and \( |0(t')\rangle_N \) through which the trajectory goes, for any \( t \) and \( t' \), with \( t \neq t' \), after the initial time \( t_0 = 0 \), never coincide. We thus conclude that the requirement i) is satisfied.
Eqs. (40) and (41) also hold for $\mathcal{N} \neq \mathcal{N}'$ in the infinite volume limit:

$$\mathcal{N} \langle 0(t) | 0\rangle_{\mathcal{N}, \mathcal{N}'} \rightarrow 0 \quad \forall t, \quad \forall \mathcal{N} \neq \mathcal{N}' \quad (43)$$

$$\mathcal{N} \langle 0(t) | 0(t')\rangle_{\mathcal{N}, \mathcal{N}'} \rightarrow 0 \quad \forall t, t', \quad \forall \mathcal{N} \neq \mathcal{N}' \quad (44)$$

Eqs. (43) and (44) have been obtained by using the fact that in the continuum limit, for given $t$ and $t'\,$ and for $\mathcal{N} \neq \mathcal{N}'$, $\cosh(\Gamma_{\kappa} t - \theta_{\kappa} + \theta'_{\kappa})$ and $\cosh(\Gamma_{\kappa} (t-t') - \theta_{\kappa} + \theta'_{\kappa})$, respectively, are never identically equal to 1 for all $\kappa$. Eq. (44) is true also for $t = t'$ for any $\mathcal{N} \neq \mathcal{N}'$. Eqs. (43) and (44) thus tell us that trajectories specified by different initial conditions ($\mathcal{N} \neq \mathcal{N}'$) never cross each other. Requirement ii) is thus satisfied.

The property ii) implies that no confusion (interference) arises among different memories, even as time evolves. In realistic situations of finite volume, states with different codes may have non–zero overlap (the inner products (43) and (44) are not zero). This means that some association of memories becomes possible. In such a case, a “crossing” point between two, or more than two, trajectories, one can switch from one of these trajectories to another one which there crosses. This may be felt indeed as association of memories or as “switching” from one information to another one. This reminds us of the “mental switch” occurring, for instance, during the perception of ambiguous figures [28], and, in general, while performing some perceptual and motor tasks [29, 30] as well as while resorting to free associations in memory tasks [31].

At each $t$, the average number of modes of type $A_{\kappa}$ is given by

$$\mathcal{N}_{A_{\kappa}}(\theta, t) \equiv \mathcal{N} \langle 0(t) | \hat{A}_{\kappa}^\dagger A_{\kappa} | 0(t) \rangle_{\mathcal{N}} = \sinh^2(\Gamma_{\kappa} t - \theta_{\kappa}) \quad (45)$$

and similarly for modes of type $\hat{A}_\kappa$. It can be shown that this number satisfies the Bose distribution. Eq. (45) is actually a statistical average and the memory state $| 0(t) \rangle_{\mathcal{N}}$ is found to be a thermal state [1].

Let us now study how the “distance” between trajectories in the memory space behave as time evolves. Let us consider two trajectories of different initial conditions, $\mathcal{N} \neq \mathcal{N}' (\theta \neq \theta')$. We want to compute the time evolution of the difference between the two codes. At time $t$, each component $\mathcal{N}_{A_{\kappa}}(t)$ of the code $\mathcal{N} \equiv \{ \mathcal{N}_{A_{\kappa}} = \mathcal{N}_{\hat{A}_{\kappa}}, \forall \kappa, \text{at } t_0 = 0 \}$ is given by the expectation value in the memory state of the number operator $A_{\kappa}^\dagger A_{\kappa}$. We have:

$$\Delta \mathcal{N}_{A_{\kappa}}(t) \equiv \mathcal{N}'_{A_{\kappa}}(0', t) - \mathcal{N}_{A_{\kappa}}(t) =$$

$$= \sinh^2(\Gamma_{\kappa} t - \theta_{\kappa} + \delta \theta) - \sinh^2(\Gamma_{\kappa} t - \theta_{\kappa}) \approx \sinh(2(\Gamma_{\kappa} t - \theta_{\kappa})) \delta \theta_{\kappa} \quad (46)$$

where $\delta \theta_{\kappa} \equiv \theta_{\kappa} - \theta'_{\kappa}$ (which, in full generality, may be assumed to be greater than zero), and the last equality holds for small $\delta \theta_{\kappa}$ (i.e. for a very small difference in the initial conditions of the two memory states). The time-derivative then gives

$$\frac{\partial}{\partial t} \Delta \mathcal{N}_{A_{\kappa}}(t) = 2\Gamma_{\kappa} \cosh(2(\Gamma_{\kappa} t - \theta_{\kappa})) \delta \theta_{\kappa} \quad (47)$$

From this we see that the difference between originally even slightly different $\mathcal{N}_{A_{\kappa}}$’s grows as a function of time. For large enough $t$, the modulus of the difference $\Delta \mathcal{N}_{A_{\kappa}}(t)$ and its time derivative diverge as $\exp(2\Gamma_{\kappa} t)$, for all $\kappa$’s. The quantity $2\Gamma_{\kappa}$, for each $\kappa$, appears thus to play a role similar to that of the Lyapunov exponent in chaos theory [23]. In conclusion, we see that trajectories in the memory space, differing by a small variation $\delta \theta$ in the initial conditions, diverge exponentially as time evolves. This may account for the high perceptive resolution in the recognition of the perceptual inputs.

Eq. (46) also shows that the difference between $\kappa$–components of the codes $\mathcal{N}$ and $\mathcal{N}'$ may become zero at a given time $t_{\kappa} = \frac{\theta_{\kappa}}{2\Gamma_{\kappa}}$. However, the difference between the codes $\mathcal{N}$ and $\mathcal{N}'$ does not necessarily become zero. The codes are made up by a large number (infinite in
the continuum limit) of $\mathcal{N}_A(\theta, t)$ components, and they are different even if a finite number of their components are equal. On the contrary, for $\delta \theta_\kappa \equiv \theta_\kappa - \theta'_\kappa$ very small, suppose that the time interval $\Delta t = \tau_{\text{max}} - \tau_{\text{min}}$, with $\tau_{\text{min}}$ and $\tau_{\text{max}}$ the minimum and the maximum, respectively, of $t_\kappa = \frac{\theta_\kappa}{\Gamma_\kappa}$, for all $\kappa$'s, be “very small”. Then the codes are “recognized” to be “almost” equal in such a $\Delta t$. Eq. (46) then expresses the “recognition” (or recall) process and we see how it is possible that “slightly different” $\mathcal{N}_A$–patterns (or codes) are “identified” (recognized to be the “same code” even if corresponding to slightly different inputs). Roughly, $\Delta t$ may be taken as a measure of the “recognition time”.

Let us finally recall that (see [1]) $\sum_\kappa E_\kappa \mathcal{N}_A(t) dt = \frac{1}{\beta} dS_A$, where $E_\kappa$ is the energy of the mode $A_\kappa$, $\beta = \frac{1}{k_B T}$, $k_B$ the Boltzmann constant. $dS_A$ is the entropy variation associated to the modes $A$ and $\mathcal{N}_A$ denotes the time derivative of $\mathcal{N}_A$. Eq. (47) leads then to the relation between the differences in the variations of the entropy and the divergence of trajectories of different initial conditions:

$$\Delta \sum_\kappa E_\kappa \mathcal{N}_A(t) dt = \sum_\kappa 2E_\kappa \Gamma_\kappa \cosh(2(\Gamma_\kappa t - \theta_\kappa)) \delta \theta_\kappa dt = \frac{1}{\beta}(dS_A' - dS_A).$$

(48)

An interesting question is the one of the relation between this last equation and the relation between the Lyapunov exponent and the Kolmogorov-Sinai entropy in the chaos theory [23]. We plan to study such a question in a future work.

In conclusion, also the requirement iii) is satisfied.

Trajectories in the representation space have thus chaotic behavior in the infinite volume limit. This result may have a connection with experimental observation of chaotic behavior in neural aggregates of the olfactory system of laboratory animals [9] (see also [32]-[35]). In the present paper, however, we do not analyze further such a possible connection.

5 Concluding remarks

The formalism of quantum dissipation is based on the doubling of the system degrees of freedom, and we have seen that the doubled modes $\hat{A}_\kappa$ account for the quantum noise in the fluctuating random force in the system-environment coupling. Remarkably, such a noise manifests itself through the coherent structure of the ground state $|0\rangle_N$. The entanglement of the $A_\kappa$ modes with the $\hat{A}_\kappa$ modes is permanent in the infinite volume limit and in such a limit the trajectories in the memory space are classical chaotic trajectories.

We have seen that the density matrix and the Wigner function formalism leads in a natural way to the doubling of the system modes. It is interesting that the Wigner function also enters in the analysis of the quantum aspects of chaotic neuron dynamics [36]. On the other hand, the chaotic behavior of the trajectories in the memory space suggests a possible connection, which, however, deserves to be further studied, with laboratory observations on the neuronal chaotic behavior [9].

From (45) we see that at at $t = \tau$, with $\tau$ the largest of the values $\tau_\kappa \equiv \frac{\theta_\kappa}{\Gamma_\kappa}$, the memory state $|0\rangle_N$ is reduced (decayed) to the "empty" vacuum $|0\rangle_0$: the information has been forgotten, the $N$ code is decayed. The time $t = \tau$ can be taken as the life-time of the memory of code $N$. For details on the life-time of the $\kappa$-modes see [3]. Considering time–dependent frequency for the DWQ, modes with higher momentum have been found to possess longer life–time. Since the momentum is proportional to the reciprocal of the distance over which the mode can propagate, this means that modes with shorter range of propagation survive longer. On the contrary, modes with longer range of propagation decay sooner. This mechanism may produce the formation of ordered domains of finite different sizes with different degree of stability: smaller domains would be the more stable ones [3]. On the other hand, since any value of the momentum is in principle allowed to the DWQ, we also see that a scaling law is
present in the domain formation (any domain size is possible in view of the momentum/size relation).

The memory state thus evolves into the "empty" vacuum $|0\rangle_0$ which acts as a sort of attractor state. However, as $t$ gets larger than $\tau$ we have

$$\lim_{t \to \infty} N(t) = \lim_{t \to \infty} \exp \left( -t \sum_{\kappa} \Gamma_{\kappa} \right) = 0 . \quad (49)$$

This shows that the state $|0(t)\rangle_N$ "diverges" away from the attractor state $|0\rangle_0$ with exponential law (we always assume $\sum_{\kappa} \Gamma_{\kappa} > 0$).

In order to avoid to fall into such an attractor, i.e. in order to not forget certain information, one needs to "restore" the $\mathcal{N}$ code by "refreshing" the memory by brushing up the subject (external stimuli maintained memory). One has to recover the whole $\mathcal{N}$-set, if the whole code is "corrupted", or "pieces" of the memory associated to those $\mathcal{N}_{\kappa}$, for certain $\kappa$'s, which have been lost at $t_{\kappa} = \theta_{\kappa} / \kappa$. Restoring the code is a sort of "updating the register" of the memories since it amounts to reset the memory clock to the (updated) initial time $t_0$. We also observe that even after the time $\tau$ is passed by, the code $\mathcal{N}$ may be recovered provided $t$ is not much larger than $\tau$, namely, as far as the approximation $\cosh(\Gamma_{\kappa} t - \theta_{\kappa}) \approx \exp(-t \sum_{\kappa} \Gamma_{\kappa})$ does not hold, cf. Eq. (49).

We note that the mentioned commutativity between $H_I$ (actually $H$) and $G(\theta)$ guarantees that evolution in time does not affect the "measurement" of the code $\mathcal{N}$; i.e. at each instant of time $t$ during the time evolution, provided $t$ is smaller than $\tau$, one may know the "initial conditions" of the trajectory under consideration.

We observe that Eq. (45) may be also rewritten as [1]

$$\mathcal{N}_{A_{\kappa}}(\theta, t) \equiv \mathcal{N}(0(t)|A_{\kappa}^\dagger A_{\kappa}|0(t))_{\mathcal{N}} = \exp(-\beta E_{\kappa}(1 - \tanh^2(\Gamma_{\kappa} t - \theta_{\kappa})))$$

Eq. (50) expresses the average number of modes $A_{\kappa}$ in terms of the Boltzmann distribution $\exp(-\beta E_{\kappa})$ times a factor describing the process of creation or annihilation, respectively, of one couple $A_{\kappa}A_{\kappa}^\dagger$ in the state $|0(t)\rangle_N$. The creation and the annihilation of the couple of modes $A_{\kappa}A_{\kappa}^\dagger$ are indeed equivalent processes in the quasi-equilibrium approximation (the limit of stationary free energy) where $\tanh^2(\Gamma_{\kappa} t - \theta_{\kappa}) = \exp(-\beta E_{\kappa})$, see [1].

Eq. (50) shows that, in the quasi-equilibrium approximation, the "measure" of $\mathcal{N}_{A_{\kappa}}(\theta, t)$ (its expectation value in the memory state at time $t$) is weighted by the Boltzmann factor, which, for given $\beta$, is larger for smaller $E_{\kappa}$, i.e. for smaller $\kappa$ (and, vice versa, is smaller for larger $\kappa$). Since smaller $\kappa$ modes are the short lived ones, the probability of reading out the memory code, i.e. of recalling, is larger for short-lived memory (and, vice versa smaller for long-lived memory). This is true also for the memory printing process, since it corresponds to the creation in the memory state of the couples of $A_{\kappa}A_{\kappa}^\dagger$ modes. Eq. (50) thus provides a possible description of the fact that short-lived memories are easily stored and easily recalled (which is of crucial importance in our relational life with the external world), and that, on the contrary, long-lived memories may require much more work to be stored, according indeed to our familiar experience. An interesting question is the one concerning those extremely stable memories controlling, e.g., vital functions. These memories are "protected" from being recalled: in general they "cannot be recalled" under the influence of any external input. According to the discussion presented above, it would seem that they are associated with codes whose components are characterized by very high values of $\kappa$, so that the probability weighting the process of annihilation or creation of the associated couples $A_{\kappa}A_{\kappa}^\dagger$ in their memory state is near to zero.

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