THE VOLUME OPERATOR IN
DISCRETIZED QUANTUM GRAVITY

R. Loll

Sezione INFN di Firenze
Largo E. Fermi 2
I-50125 Firenze, Italy

Abstract

We investigate the spectral properties of the volume operator in quantum gravity in the framework of a previously introduced lattice discretization. The presence of a well-defined scalar product in this approach permits us to make definite statements about the hermiticity of quantum operators. We find that the spectrum of the volume operator is discrete, but that the nature of its eigenstates differs from that found in an earlier continuum treatment.

---

1 Supported by the European Human Capital and Mobility program on “Constrained Dynamical Systems”
1 Introduction

One of the most active branches of research into the quantization of 3+1-dimensional gravity of the last few years has been the canonical, operator-based framework of the so-called loop approach. It is non-perturbative in the sense that it is not a priori restricted to the study of geometries close to flat Minkowski space. Its basic variables are (non-local) generalized Wilson loops of the $SL(2, \mathbb{C})$-valued Ashtekar connection. Also in the quantum theory the state space and operators are labelled by (equivalence classes of) closed curves in three-space, which has led to considerable progress in solving the quantum constraints of the theory. The first, formal solutions to all of the constraints, including the Wheeler-DeWitt equation, were found in this loop formulation [1].

Although since then many of the mathematical ingredients of loop representations have been scrutinized and better understood (see, for example, [2]), one is still lacking a rigorous control over the regularization procedure necessary for obtaining a well-defined quantum Hamiltonian. One difficulty is the absence of a natural background metric in the “fully diffeomorphism-invariant phase” of the theory. Secondly, since the basic variables are non-local, the definition of the quantum Hamiltonian $\hat{H}$ involves usually a shrinking of loop operators to points, which arguably is a rather ill-defined process. These problems, and the absence of a well-defined scalar product in the quantum representation, have hampered progress toward a better understanding of the “solutions to all the constraints” and of observables (which in the pure gravity theory are those operators commuting with the quantum Hamiltonian).

In a recent paper [3], we have proposed an alternative regularization for the loop approach, that does not involve a point-splitting for the definition of the Hamiltonian constraint. It is a lattice regularization of the type used in quantum chromodynamics [4], but with two important differences. Firstly, the lattice is considered as purely topological, and therefore the basic Wilson loop variables of the theory (with support on the links of the lattice) are both manifestly gauge- and spatial diffeomorphism-invariant. Secondly, since the “gauge group” $SL(2, \mathbb{C})$ is non-compact, we do not use the Haar measure to define the inner product, but a suitably defined measure on holomorphic $SL(2, \mathbb{C})$-valued wave functions, with respect to which the norm of holomorphic Wilson loop states is finite. Thus one may think of the construction as a finite approximation of the usual loop representation, where the support of loops has been restricted to a fixed cubic, topological lattice, and where the momenta are smeared out along lattice links. One expects this approximation to work ever better with increasing lattice size.

The main assets of this lattice model are its computational simplicity and the existence of
a well-defined scalar product. In a preliminary investigation of the quantum Hamiltonian we were able to find a large number of solutions to the Wheeler-DeWitt equation that have finite norm with respect to this inner product. Furthermore, questions about the selfadjointness of operators, and in particular observables, are now within reach. One test of this and other approaches is whether one can define physically interesting operators that are selfadjoint. (Reality, even in the classical theory, is a non-trivial requirement in the Ashtekar formulation, since the basic phase space variables are complex.) Since our choice of a scalar product corresponds to a choice of “reality conditions” at the quantum level, and moreover the setting is fully regularized, it makes sense to associate classical, real phase space functions with selfadjoint lattice operators.

In the present paper, we will be concerned with the so-called volume operator, introduced in [5]. Although it is not an observable of the pure theory, there are arguments suggesting it will become one once matter has been included. Rovelli and Smolin have presented a partial computation of its spectrum, based on a certain continuum regularization [6]. According to their arguments, the spectrum is both real and discrete, which by them is taken to indicate a fundamental discreteness of the theory at the Planck scale. This result is formal in the sense that it postulates the existence of the quantum operators involved (and the limiting procedure used to define them), and of a scalar product that makes the spectrum calculation meaningful.

Given the scalar product of the lattice model, one may in turn ask whether an analogue of the volume operator can be sensibly defined and whether its spectrum agrees with that found in the formal continuum calculation. We will show here that the answer to the first question is in the affirmative. The lattice regularizes in a natural way the terms cubic in momenta that appear in the definition of the volume operator, and its spectrum is again discrete. However, the nature of the eigenstates (to the extent they can be compared) disagrees with that found in [6]. In particular, we find that eigenstates of the volume operator are necessarily complex linear combinations of the Wilson loop states. We will also give a general argument for why trivalent spin network states are necessarily zero-eigenvectors of the volume operator 1. Finally, we will point out a difficulty that arises in requiring certain cubic operators (that are part of the definition of the volume operator) to be selfadjoint.

In the next section, we recall the construction of the classical volume function, and define a discretized form of the quantum volume operator in the holomorphic representation. In Sec. 3 we compute the local action, around a lattice vertex, of the volume operator on a

---

1 The authors of [6] have recently informed me that their derivation of the spectrum of these states contains a computational error.
number of Wilson loop states, and discuss the role of spin network states as eigenstates of this
operator. Finally, in Sec. 4 we compare our method and results with those of the continuum
approach.

2 Defining the volume operator

Let us first summarize the main ingredients of the lattice formulation introduced in [3].
The lattice is a cubic \( N \times N \times N \)-lattice, with periodic boundary conditions, i.e. its topology is
that of a three-torus. The basic operators associated with each lattice link \( l \) are a holomorphic
\( SL(2, \mathbb{C}) \)-link holonomy \( \hat{V}_{AB} \) and a canonical momentum operator \( \hat{p}_i \), with an adjoint index
\( i \). The wave functions are elements of \( \times_l L^2(SL(2, \mathbb{C}), d\nu^H) \), with the product taken over all
links. The measure is the heat kernel measure \( d\nu \), and the superscript \( H \) denotes the subset
of holomorphic \( L^2 \)-functions. The basic commutators are

\[
\begin{align*}
[\hat{V}^B_{A}(n, \hat{a}), \hat{V}^C_{D}(m, \hat{b})] &= 0 \\
[\hat{p}_i(n, \hat{a}), \hat{V}^A_{B}(m, \hat{b})] &= -\frac{i}{2} \delta_{nm} \delta_{\hat{a}\hat{b}} \tau_i \hat{V}^A_{B} \\
[\hat{p}_i(n, \hat{a}), \hat{p}_j(m, \hat{b})] &= i \delta_{nm} \delta_{\hat{a}\hat{b}} \epsilon_{ijk} \hat{p}_k,
\end{align*}
\]

Example of \( SL(2, \mathbb{C}) \)-invariant states are given by the Wilson loops (i.e. the traces of link
holonomies) \( Tr V(\gamma) = Tr V(l_1)V(l_2) \ldots V(l_n) \), where \( \gamma = l_1 \circ l_2 \circ \ldots \circ l_n \) is a closed lattice loop.
Recall that we do not have an explicit coordinate expression for the heat kernel measure \( d\nu \).
and therefore must use the holomorphic transform \( C_t : L^2(SU(2), dg) \to L^2(SL(2, \mathbb{F}), d\nu_t)^H \) and its inverse to compute scalar products in \( L^2(SL(2, \mathbb{F}), d\nu_t)^H \). It turns out that the operators \( \hat{p}_i \) are selfadjoint in the holomorphic representation (a fact that had been overlooked in [3]), i.e. they are the holomorphic transforms of the corresponding selfadjoint differential operators on \( L^2(SU(2), dg) \) (whose functional form coincides with (2.2), with the complex parameters \( \alpha_i \) substituted by real ones). The reason for this is basically that the \( \hat{p}_i \) map eigenspaces of the Laplacian \( \Delta = -4 \sum_i \hat{p}_i^2 \) with fixed eigenvalue \( k \) into themselves. Still, the operators \( \hat{V}_{AB} \) are not selfadjoint in the holomorphic representation, although multiplication by (real) \( V_{AB} \) is a selfadjoint operation in the \( SU(2) \)-representation.

The classical expression for the volume of a spatial region \( \mathcal{R} \) is given by

\[
\mathcal{V}(\mathcal{R}) = \int_{\mathcal{R}} d^3x \sqrt{\det g} = \int_{\mathcal{R}} d^3x \sqrt{\frac{1}{3!} [\epsilon_{abc} \epsilon^{ijk} E_a^i E_b^j E_c^k]},
\]

where \( E_a^i \) are the momenta of the canonically conjugate Ashtekar variables \( (A_a^i(x), E_a^i(x)) \). In the classical theory, \( \mathcal{V} \) is of course a real quantity. A natural lattice discretization of the \( \det g \)-term is

\[
D(n) := \epsilon_{abc} \epsilon^{ijk} p_i(n, \hat{a}) p_j(n, \hat{b}) p_k(n, \hat{c}),
\]

which in the continuum limit \( a \to 0 \) with respect to an arbitrary lattice spacing \( a \) goes over to \( a^6 \epsilon_{abc} \epsilon^{ijk} E_a^i E_b^j E_c^k + O(a^7) \). We may therefore take

\[
\mathcal{V}_{\text{latt}} = \sum_{n \in \mathcal{R}} \sqrt{|\epsilon_{abc} \epsilon^{ijk} p_i(n, \hat{a}) p_j(n, \hat{b}) p_k(n, \hat{c})|}
\]

as the lattice analogue of (2.3). The translation of this expression to the quantum theory is a priori not well defined, because of the presence of both the modulus and the square root. However, we are in the fortunate position that the operators

\[
\hat{D}(n) := \epsilon_{abc} \epsilon^{ijk} \hat{p}_i(n, \hat{a}) \hat{p}_j(n, \hat{b}) \hat{p}_k(n, \hat{c})
\]

are all self-adjoint. (Note that no operator ordering problem occurs in the definition of \( \hat{D}(n) \) since it contains an anti-symmetrization over spatial directions.) We may therefore go to a
basis of $\times_1L^2(SL(2,\mathbb{R}),\nu_t)^{\mathcal{H}}$ consisting of simultaneous eigenfunctions of all the $\hat{D}(n)$ and define the operator

$$\hat{V}_{\text{latt}} = \sum_n \sqrt{|\hat{D}(n)|}$$

(2.7)

through the square roots of the moduli of the eigenvalues of the $\hat{D}(n)$ in that basis. Thus, all we have to do to understand the regularized volume operator is to compute the spectra of the operators $\hat{D}(n)$.

As mentioned in the introduction, there already exists a (partial) spectrum calculation in the continuum [6] we can compare with. Its authors propose to work in terms of a basis of gauge- and spatially diffeomorphism-invariant states which is diagonal with respect to appropriate continuum analogues of the operators $\hat{D}(n)$ above (whose construction involves smearing $\det g$ over a small box and then letting the box shrink to zero). These states are given by so-called spin networks, constructed from trivalent (or n-valent) graphs whose edges are labelled by irreducible representations of $SU(2)$, and vertices by $SU(2)$-intertwining operators (see, for example, [7] and references therein). They can be thought of as certain (anti)symmetrized, real linear combinations of multiple Wilson loops with support on the graph. The difference with our discrete formulation is that one considers all possible graphs (with all possible labellings), whereas we keep the lattice fixed, and therefore the total number of degrees of freedom finite. Still, also the lattice approach allows for a similar construction of gauge-invariant states. Finding an efficient labelling for such states is a well-known problem in lattice gauge theory, and various methods have been used (see, for example, [8]), among them constructions reminiscent of the spin networks, generalized to cubic lattices (which have intersections of order higher than three). A problem that typically occurs is that such explicitly gauge-invariant bases are overcomplete, and that for doing computations one needs an efficient way of labelling a complete subset of independent states. For spin network states of valence higher than three, one encounters a similar problem. Whether one basis is better than another is determined by the dynamics of the basic operators of the theory, and may therefore be completely different for gravitational and gauge-theoretic applications.

When studying the volume operator it is indeed useful to consider a representation in which the lattice links are labelled by positive “occupation numbers”, which count the number of (unoriented) flux lines of basic spin-$\frac{1}{2}$ representations on the link (with the links contracted gauge-invariantly at the vertices). In essence this happens because the operators $\hat{D}(n)$ have a particularly simple structure: when acting on a multiple Wilson loop, they do not change its support (in terms of the flux line numbers), and only some finite-dimensional rearrangements
occur within the subset of states that share the same occupation numbers. In the context of the continuum theory, a similar observation was already made in [6]. In this respect, the volume operator is much simpler than the Hamiltonian operator, which (at least on the lattice) changes the support of a Wilson loop state it acts on [3].

The explicit part of the continuum spectrum calculation in [6] was made for trivalent spin networks. Although general gauge-invariant lattice states contain six-valent intersections, one can easily construct states that are only trivalent by assigning the occupation number zero to an appropriate subset of lattice links. (By contrast, in the continuum picture, assigning zero occupation number to an edge can be interpreted as creating a new, smaller graph.) The question therefore arises whether spin networks constructed from such trivalent states are also eigenstates in the lattice formulation. To answer it, it is sufficient to study the action of the operators $\hat{D}(n)$, as explained above. This will be the subject of the next section.

3 Computation of the spectrum

We will now present the results of the spectral computation for small occupation numbers around a single vertex $n$, which will be sufficient to illustrate our point; a complete construction will appear elsewhere [9]. It turns out that the spectrum of $\hat{D}(n)$ is discrete. This was not clear a priori, since the group $SL(2,\mathbb{C})$ is non-compact; it is a consequence of our choice of a scalar product. We will not speculate here whether this discreteness is of a fundamental nature or only an artifact of the regularization, that will disappear in an appropriately taken continuum limit.

Fig.1 illustrates the labelling of the link directions meeting at a vertex $n$. We will be interested in the behaviour of gauge-invariant states under the action of $\hat{D}(n)$. One
ingredient in the labelling of such a state is a 6-tuple \( \vec{j} \) of integers \( j_i \geq 0 \) giving the occupation numbers \((j_1, \ldots, j_6)\) of the links \(((n, \hat{1}), (n, \hat{2}), (n, \hat{3})); (n, -\hat{1}), (n, -\hat{2}), (n, -\hat{3})) \equiv ((n, \hat{1}), (n, \hat{2}), (n, \hat{3})); (n-\hat{1}, \hat{1}), (n-\hat{2}, \hat{2}), (n-\hat{3}, \hat{3}))\) intersecting at \( n \). We will call \( j := \sum_{i=1}^{6} j_i \) the order of a state (at \( n \)), which is an even integer. What remains to be specified is the way the \( j \) flux lines are joined pairwise at \( n \) to ensure gauge-invariance. By convention we allow a flux line coming in from the positive \( \hat{1} \)-direction, say, to be joined only to a flux line from one of the other five links, and not from the same link (i.e. we forbid “retracings”). This leads to a constraint on the occupation numbers: any \( j_i \) has to be equal to or smaller than the sum of the remaining \( j_k \), for example, \( j_6 \leq \sum_{i=1}^{5} j_i \).

Given \( \vec{j} \), the number of possible different contractions of flux lines at \( n \) is finite. Not all of them will lead to linearly independent Wilson loop states once the flux line configuration around \( n \) is extended to a gauge-invariant state of lattice loops. To understand this, think of a fixed (but arbitrary) such extension of the flux line configuration, so as to obtain a set of closed lattice curves. They may be thought of as a set \( \gamma_1, \gamma_2, \ldots, \gamma_k \) of lattice loops based at \( n \), and the corresponding multiple Wilson loop is the state \( \Psi = \text{Tr} V(\gamma_1) \text{Tr} V(\gamma_2) \ldots \text{Tr} V(\gamma_k) \).

Now, different contractions of the flux lines at \( n \) will lead to different Wilson loop states (with the same support), which in general are related by so-called Mandelstam constraints. For example, for the case of three lattice loops meeting at \( n \), one has the following identity [10]:

\[
\text{Tr} V(\gamma_1) \text{Tr} V(\gamma_2) \text{Tr} V(\gamma_3) = \text{Tr} V(\gamma_1) \text{Tr} V(\gamma_2 \circ \gamma_3) + \text{Tr} V(\gamma_2) \text{Tr} V(\gamma_1 \circ \gamma_3) + \text{Tr} V(\gamma_3) \text{Tr} V(\gamma_1 \circ \gamma_2) - \text{Tr} V(\gamma_1 \circ \gamma_2 \circ \gamma_3) - \text{Tr} V(\gamma_2 \circ \gamma_1 \circ \gamma_3).
\]

For the special case of a trivalent graph, Rovelli and Smolin have given a prescription for associating with each labelling of flux lines a unique quantum state, obtained by appropriately (anti)symmetrizing over all possible Wilson loop states sharing the same flux labels [6]: calling temporarily \( j(l) \) the occupation number of a link \( l \), the number of different multiloops one can associate with a given flux line labelling – by permuting the way individual flux lines are joined at vertices – is \( \prod_l j(l)! \), where the product is taken over all lattice links. The spin network state is then obtained by adding the corresponding \( \prod_l j(l)! \) Wilson loop states, with the weight \((-1)^{p+n})\), where \( p \) is the parity of the flux line permutation and \( n \) the number of closed loops in the multiloop.

The case of trivalent intersections turns out to be particularly simple, since there is only one way of contracting the (anti)symmetrized flux line configurations at each vertex. Translated to the lattice, and according to the reasoning at the end of the previous section, this means that \( \hat{D}(n)\Psi = d\Psi \) for any trivalent spin network state \( \Psi \) (i.e. a gauge-invariant
state on the lattice with at most trivalent intersections – at each vertex \( n \), at least three of the flux line labels of the adjacent links are zero). That is, \( \Psi \) is necessarily an eigenstate of \( \hat{D}(n) \), with eigenvalue \( d \) (where of course \( d \) depends on \( \Psi \)).

Let us now compute the action of the operator \( \hat{D}(n) \) on some trivalent lattice spin networks. Recall that the momenta in \( \hat{D}(n) \), (2.6), act non-trivially on the lattice links emanating from the vertex \( n \) in a positive direction. The simplest type of configuration is of order 4 and has \( \vec{J} = (2, 1, 1; 0, 0, 0) \), Fig.2a. There are two possible permutations of the flux lines, illustrated in Fig.2b. The corresponding spin network state \( \Psi \) is the sum of the two, \( \Psi = \psi_1 + \psi_2 \). One finds \( \hat{D}(n)\psi_1 = 0 \) and \( \hat{D}(n)\psi_2 = 0 \), and therefore \( \Psi \) is an eigenvector with eigenvalue zero.

At order \( j = 6 \) there are two admissible flux line labellings (up to a permutation of link labels). The first one is \( \vec{J} = (2, 2, 2; 0, 0, 0) \), where there are \( 2!2!2! = 8 \) flux line permutations, leading to Wilson loops states \( \psi_i, \ i = 1, \ldots, 8 \). One computes \( \hat{D}(n)\psi_i = 0, \forall i \), and the spin network state, which is again the weighted sum of the \( \psi_i \), is a zero-eigenvector of \( \hat{D}(n) \). Similarly, for the spin network state \( \Psi \) associated with \( \vec{J} = (3, 2, 1; 0, 0, 0) \), one finds \( \hat{D}(n)\Psi = 0 \).
We will explain shortly why indeed $\hat{D}(n)\Psi = 0$ for any trivalent spin network $\Psi$. Before doing so, let us look at a couple of examples that lead to non-trivial eigenstates of $\hat{D}(n)$. First, consider $\vec{j} = (1, 1, 1; 0, 0)$. The three possible contractions at $n$ are illustrated in Fig.3, where the dotted lines with arrows denote an arbitrary extension by other lattice links. The three possible Wilson loop states are $\psi_1 = \text{Tr} V(\alpha) V(\beta)$, $\psi_2 = \text{Tr} V(\alpha \circ \beta)$ and $\psi_3 = \text{Tr} V(\alpha \circ \beta^{-1})$, where we have assigned a definite orientation to the composite loops $\alpha$ and $\beta$. (Following [11], one may also write the Mandelstam constraints in a way that is independent of the loop extensions.) The $\psi_i$ are already spin networks in the sense that there are no flux line permutations to be taken into account. The action of $\hat{D}(n)$ yields

$$\hat{D}(n) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \frac{3i}{2} \begin{pmatrix} 0 & -1 & 1 \\ 2 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (3.2)$$

and its eigenvectors are easily computed,

$$\hat{D}(n)\Psi = 0, \quad \Psi = \psi_1 - \psi_2 - \psi_3$$

$$\hat{D}(n)\Psi = -\frac{3\sqrt{3}}{2} \Psi, \quad \Psi = \psi_1 - \frac{1}{4}(1 - i\sqrt{3})\psi_2 - \frac{1}{4}(1 + i\sqrt{3})\psi_3$$

$$\hat{D}(n)\Psi = \frac{3\sqrt{3}}{2} \Psi, \quad \Psi = \psi_1 - \frac{1}{4}(1 + i\sqrt{3})\psi_2 - \frac{1}{4}(1 - i\sqrt{3})\psi_3. \quad (3.3)$$

The presence of the zero-eigenvector is not surprising, since the three states $\psi_i$ are not independent from the outset, but rather obey the Mandelstam constraint $\psi_1 - \psi_2 - \psi_3 = 0$. There are only two linearly independent spin networks, and we could have removed the redundancy before looking for eigenstates of $\hat{D}(n)$. 
A non-trivial example of order 6 is given by $\vec{j} = (2, 1, 1; 1, 1, 0)$. We only sketch the result: there are 12 different configurations to start with, from contracting the flux lines at $n$. Symmetrization with respect to the two flux lines in the $\hat{1}$-direction leaves us with 6 spin network states. After using the Mandelstam constraints [10], only three linearly independent spin network states remain. Diagonalizing the action of $\hat{D}(n)$ on those states, one finds the three eigenvalues, $0$, $-3\sqrt{2}$ and $3\sqrt{2}$.

Finally, let us analyze the case $\vec{j} = (3, 1, 1; 1, 0, 0)$. There are 6 different ways of contracting the flux lines. After symmetrizing over the six permutations of the flux lines in the $\hat{1}$-direction, a single spin network state $\Psi$ is obtained, and one finds $\hat{D}(n)\Psi = 0$. Interestingly, if one looks at the Wilson loop states before the symmetrization, one can form two linear combinations that are non-trivial eigenstates of $\hat{D}(n)$, with eigenvalues $\pm\frac{3}{2}\sqrt{3}$. Thus it seems as if one were losing information about the gauge-invariant sector of the Hilbert space by looking only at the spin network states. However, this is presumably not the case, since the information may be contained in other spin networks, associated with a different labelling of flux lines.

The above examples show that it is possible to find eigenstates of spin network-type whose eigenvalues are non-zero. However, in all cases where at a vertex $n$ one can construct only a single spin network state (which therefore must be an eigenstate of $\hat{D}(n)$), its eigenvalue necessarily vanishes. As explained earlier, all trivalent vertices are of this type. This happens because the momenta $p_i$ in our representation are represented selfadjoinly and according to (2.2) contain each a factor of $i$ (or rather $i\hbar$), so that also $\hat{D}(n)$ is proportional to $i$. It is however easy to see that $\frac{1}{i}\hat{D}(n)$ maps a Wilson loop state $\text{Tr}_V(\gamma)$ into a real linear combination of such states. Therefore, if we have a spin network state $\Psi$ (that by construction is a real linear combination of Wilson loop states), its eigenvalue equation is $\hat{D}(n)\Psi = d\Psi$, with imaginary $d$. On the other hand, $\hat{D}(n)$ is a selfadjoint operator and its eigenvalues are real. We hence conclude that necessarily $d = 0$. Non-zero eigenvalues can only occur for intersections that are higher than trivalent, and the corresponding eigenvectors are always complex linear combinations of Wilson loop states, as illustrated by (3.3).

4 Discussion and conclusions

The calculations of the previous section took place around a single vertex $n$, but can be generalized immediately to lattice regions $\mathcal{R}$ containing several or even all of the lattice vertices, to obtain eigenstates of the volume operator $\hat{V}_{\text{latt}}(\mathcal{R})$. Although its spectrum is
obviously discrete, we have found that all quantum spin network states corresponding to trivalent graphs are eigenstates with eigenvalue zero. This disagrees with the continuum computation of the trivalent sector reported by Rovelli and Smolin [6], where a non-vanishing spectrum was found. It is therefore important to understand how the two regularization methods differ.

The presence of factors of $i$ in the definition of our momentum operators $\hat{p}_i$ can be traced back to the canonical commutators of the continuum Yang-Mills theory, $[\hat{A}_a^i(x), \hat{E}_b^j(y)] = i \delta_a^b \delta_i^j \delta^3(x - y)$, whose lattice analogues in the holomorphic representation are given by (2.1). However, we strictly speaking should be quantizing the classical Poisson brackets

$\{A_a^i(x), E_b^j(y)\} = i \delta_a^b \delta_i^j \delta^3(x - y)$ of the canonical Ashtekar variables [12], leading to canonical commutators

$$[\hat{A}_a^i(x), \hat{E}_b^j(y)] = \delta_a^b \delta_i^j \delta^3(x - y)$$

without a factor $i$. In [3], we quantized the commutators with the factor $i$, to facilitate comparison with the usual formalism of Hamiltonian lattice gauge theory. This was done with the understanding that the quantum commutators with and without $i$ can in a straightforward way be related by multiplying the canonical momenta by $i$. For instance, for the case of our lattice variables, defining new momenta $\hat{p}_i' := i \hat{p}_i$ leads to a version of the basic commutator algebra (2.1) without any factors of $i$ on the right-hand sides. In fact, a representation of the form $\hat{E} = \partial/\partial A$ for the canonical momentum operators has been used both in formal continuum formulations [1,13] and in previous lattice approaches to gravity [14]. Would the substitution $\hat{p}_i \to \hat{p}_i'$ change the results on the spectrum of the volume operator obtained in Sec.3? Obviously not: the operators $\hat{p}_i'$ and the corresponding composite operators $\hat{D}(n)'$ would become anti-hermitian, and their spectra purely imaginary. Otherwise, the spectra would of course remain discrete, and trivalent spin network states would still be eigenstates with zero eigenvalues, leaving our main results unaffected.

However, insisting that – for whatever physical reasons – the operators $\hat{D}(n)$ be self-adjoint, one seems forced to adopt a representation like $\hat{E} = \frac{1}{i} \partial/\partial A + \ldots$, where the dots stand for possible divergence terms. This is at least true if the representation is defined on a Hilbert space of states on $\mathcal{A}^{SL(2,\mathbb{R})}/G^{SL(2,\mathbb{R})}$ of connections modulo gauge, or an appropriate generalization thereof (our lattice Hilbert space is a discretized version of this space). This is an illustration of the well-known fact that requiring certain physical operators to be selfadjoint leads to restrictions on the possible quantum representations. One possible set of selfadjointness conditions for continuum gravity in the connection representation is to demand that
\[ \hat{E}_a \hat{E}^{ab} \] and \([\hat{H}, \hat{E}_a \hat{E}^{bi}]\] to be selfadjoint, which are the quantum counterparts of the reality conditions on the classical spatial three-metric \( q_{ab} \), \( \det q_{ab} = E_a^b E^{bi} \), and its evolution \( \{H, E_a^b E^{bi}\} \), where \( H \) denotes the Hamiltonian. Normally these “quantum reality conditions” are ill-defined, because they contain products of quantum operators at the same point. However, the basic variables \((V_A^B, p_i)\) of the lattice formulation are already regularized appropriately (recall also that the spatial diffeomorphisms have already been factored out), and it is therefore well-defined to require \( \hat{p}_i(n, \hat{a})\hat{p}_i(n, \hat{b}) \) to be selfadjoint. This is compatible with both selfadjoint and anti-selfadjoint momenta \( \hat{p}_i \). Demanding in addition the selfadjointness of \( \epsilon_{abc} \epsilon^{ijk} \hat{p}_i(n, \hat{a})\hat{p}_j(n, \hat{b})\hat{p}_k(n, \hat{c}) \) excludes the possibility of having anti-selfadjoint momenta with \( \hat{p}_i^\dagger = -\hat{p}_i \). Alternatively, if one is only interested in the selfadjointness of the volume operator \( \hat{V}_{\text{latt}}(R) \) (and not of the \( \hat{D}(n) \)), one may leave the momenta \( \hat{p}_i \) anti-selfadjoint, in which case the modulus in (2.7) takes care of turning the eigenvalues of \( \hat{V}_{\text{latt}}(R) \) into positive real numbers.

In the continuum treatment of [6], the definition of the volume operator involves a quantized version of the generalized Wilson loop variable with three momentum insertions (at loop parameters \( s, t \) and \( r \)),

\[
T^{abc}[\alpha](s, t, r) = Tr E^a(\alpha(s))V_\alpha(s, t)E^b(\alpha(t))V_\alpha(t, r)E^c(\alpha(r))V_\alpha(r, s),
\]

(4.2)

in the limit as the loop argument \( \alpha \) shrinks to a point. Note that it is only in this limit that the classical variable (4.2) becomes real, since the holonomies \( V_\alpha \) depend on the complex Ashtekar connections \( A_a \) – only for the “point loop” \( \alpha \), one has a real holonomy \( V_\alpha = 1 \). This problem does not arise if one interprets the calculations as taking place within the Euclidean theory (with the corresponding scalar product), where \( V_\alpha \) is real from the outset.

In the corresponding limiting procedure in the quantum theory one lets an auxiliary length variable \( L \) (measuring the edge length of a small box) go to zero and defines the volume operator as \( \hat{V}(R) := \lim_{L \to 0} \hat{V}_L(R) \) [6]. Unfortunately, as already noted there, it is difficult to make rigorous sense of this limit since for any \( L \neq 0 \), the operator \( \hat{V}_L \) is not even formally selfadjoint. As long as \( L \) is finite, acting with \( \hat{V}_L \) on a Wilson loop changes its support, so there is no obvious basis of loop states for which it is diagonal, and one cannot make sense of the square root operation. Thus, even if this construction does in the end lead to a finite operator, the status of the regularization procedure is still unclear. In any case, although the (left) spectrum of \( \hat{V}(R) \) is allegedly real, no scalar product was specified in [6], with respect to which its selfadjointness or otherwise could be established.

The differences between the continuum and lattice regularizations make a direct compar-
ison of the spectral computations difficult. Also it is not a priori clear to what extent they should agree, given that no continuum limit has yet been performed in the lattice formulation. Even if eventually an agreement on the vanishing of $\hat{V}_{\text{latt}}$ on trivalent states can be reached, it does not automatically follow that the non-zero parts of the spectra will coincide in both formalisms. The fact that in our approach the trivalent spin network states all “have zero volume” may be taken as an indication that they are degenerate from a physical point of view. In any case, it makes clear that considering trivalent states alone is not enough. – We are currently addressing the question of how to construct a non-overcomplete basis of holomorphic lattice states in terms of which the Hamiltonian and volume operators assume a simple form, which hopefully will lead to a complete spectral analysis of $\hat{V}_{\text{latt}}$ [9].

We also have pointed out that the requirement of having the $\epsilon_{abc} \epsilon^{ijk} \hat{E}_i^a \hat{E}_j^b \hat{E}_k^c$, or some suitably smeared versions thereof, represented by selfadjoint operators suggests a quantum representation $\hat{E}_i^a = \frac{1}{2} \partial / \partial A_i^a$ for the momenta, if the Hilbert space is a space of connections, as is usually the case. This is at odds with the commutation relations (4.1), if one represents $\hat{A}$ as the multiplication operator by $A$. Whether or not this distinction has any physical relevance remains to be seen; it is an example of how selfadjointness conditions on operators restrict the choice of possible quantum representations.

Acknowledgement. I would like to thank the participants of the Warsaw workshop on canonical and quantum gravity for discussion and comments.

References

[1] Rovelli, C. and Smolin, L.: Loop space representation of quantum general relativity, *Nucl. Phys.* B331 (1990) 80-152

[2] Ashtekar, A. and Isham, C.J.: Representations of the holonomy algebras of gravity and non-Abelian gauge theories, *Class. Quant. Grav.* 9 (1992) 1433-67; Ashtekar, A. and Lewandowski, J.: Representation theory of analytic holonomy $C^*$-algebras, in *Knots and quantum gravity*, ed. J. Baez, Clarendon Press (Oxford) 1994, 21-61; Ashtekar, A., Lewandowski, J., Marolf, D., Mourão, J. and Thiemann, T.: Quantization of diffeomorphism invariant theories of connections with local degrees of freedom, *preprint* Penn State U., Apr 1995, e-Print Archive: gr-qc 9504018
[3] Loll, R.: Non-perturbative solutions for lattice quantum gravity, to appear in *Nucl. Phys. B*, e-Print Archive: gr-qc 9502006

[4] Kogut, J. and Susskind, L.: Hamiltonian formulation of Wilson’s lattice gauge theories, *Phys. Rev.* D11 (1975) 395-408; Kogut, J.B.: The lattice gauge theory approach to quantum chromodynamics, *Rev. Mod. Phys.* 55 (1983) 775-836

[5] Smolin, L.: Recent developments in nonperturbative quantum gravity, in *Proc. Sant Feliu de Guixols 1991, Quantum gravity and cosmology*, World Scientific, Singapore, 1992, 3-84

[6] Rovelli, C. and Smolin, L.: Discreteness of area and volume in quantum gravity, *Nucl. Phys.* B442 (1995), in press

[7] Baez, J.B.: Spin network states in gauge theory, *preprint* UC Riverside, Nov 1994, e-Print Archive: gr-qc 9411007. Spin networks in nonperturbative quantum gravity, *preprint* UC Riverside, Apr 1995, e-Print Archive: gr-qc 9504036

[8] Furmanski, W. and Kolawa, A.: Yang-Mills vacuum: an attempt at lattice loop calculus, *Nucl. Phys.* B291 (1987) 594-628

[9] Loll, R.: *preprint* INFN Firenze, in preparation

[10] Loll, R.: Independent SU(2)-loop variables and the reduced configuration space of SU(2)-lattice gauge theory, *Nucl. Phys.* B368 (1992) 121-42; Yang-Mills theory without Mandelstam constraints, *Nucl. Phys.* B400 (1993) 126-44

[11] Rovelli, C. and Smolin, L.: Spin networks and quantum gravity, *preprint* U. Pittsburgh, Apr 1995, e-Print Archive: gr-qc 9505006

[12] Ashtekar, A.: New variables for classical and quantum gravity, *Phys. Rev. Lett.* 57 (1986) 2244-7; A new Hamiltonian formulation of general relativity, *Phys. Rev.* D36 (1987) 1587-1603; *Lectures on non-perturbative canonical gravity*, World Scientific, Singapore, 1991

[13] Jacobson, T. and Smolin, L.: Nonperturbative quantum geometries, *Nucl. Phys.* B299 (1988) 295-345; Brügmann, B.: Loop representations, in *Canonical gravity: from classical to quantum*, ed. J. Ehlers and H. Friedrich, Lecture Notes in Physics 434, Springer, Berlin, 1994
[14] Renteln, P. and Smolin, L.: A lattice approach to spinorial quantum gravity, *Class. Quant. Grav.* 6 (1989) 275-94; Renteln, P.: Some results of SU(2) spinorial lattice gravity, *Class. Quant. Grav.* 7 (1990) 493-502