A covariant Stinespring theorem

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Abstract

We prove a finite-dimensional covariant Stinespring theorem for compact quantum groups. Let $G$ be a compact quantum group, and let $T := \text{Rep}(G)$ be the rigid $C^*$-tensor category of finite-dimensional continuous unitary representations of $G$. Let $\text{Mod}(T)$ be the rigid $C^*$-2-category of cofinite semisimple finitely decomposable $T$-module categories. We show that finite-dimensional $G$-$C^*$-algebras can be identified with equivalence classes of 1-morphisms out of the object $T$ in $\text{Mod}(T)$. For 1-maps $X : T \rightarrow M_1$, $Y : T \rightarrow M_2$, we show that covariant completely positive maps between the corresponding $G$-$C^*$-algebras can be ‘dilated’ to isometries $\tau : X \rightarrow Y \otimes E$, where $E : M_2 \rightarrow M_1$ is some ‘environment’ 1-morphism. Dilations are unique up to partial isometry on the environment; in particular, the dilation minimising the quantum dimension of the environment is unique up to a unitary. When $G$ is a compact group this recovers previous covariant Stinespring-type theorems.

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1 Introduction

Stinespring theorems. In finite-dimensional (f.d.) quantum and classical physics we identify systems with f.d. $C^*$-algebras and dynamics with completely positive trace-preserving (CPTP) linear maps (called channels). To formulate an expressive physical theory it is useful to place symmetry restrictions on dynamics; one therefore introduces a compact group $G$ and identifies systems with f.d. $C^*$-algebras with a $G$-action (called $G$-$C^*$-algebras or $C^*$-dynamical systems), and dynamics with covariant channels intertwining the $G$-actions. Recently these notions have been generalised to compact quantum groups \cite{Wan98, Sol10, DC17}.

An essential tool in the study of finite-dimensional quantum physics, particularly quantum information theory, is the finite-dimensional Stinespring theorem \cite{Hol07} Thm. 2 \cite{Sti55} Thm. 1, which characterises channels between f.d. $C^*$-algebras $A$, $B$. In general (a special case of \cite[Sza10] Thm. 15), the theorem implies that for any completely positive linear map $f : A \to B$ there exists an f.d. right Hilbert $B$-module $\mathcal{E}$, a multiplicative $*$-homomorphism $\Phi : A \to B^*(\mathcal{E})$ (where $B^*(\mathcal{E})$ are the adjointable operators on $\mathcal{E}$) and an adjointable $B$-module map $V : B \to \mathcal{E}$ such that $f(x) = V^\dagger \Phi(x)V$. The completely positive map $f^\dagger$ is trace-preserving if and only if $V$ is an isometry. This reduces the study of dynamics between indecomposable f.d. $C^*$-algebras to the study of isometric maps between Hilbert modules.

A covariant version of the Stinespring theorem, applicable to covariant channels between $G$-$C^*$-algebras for a compact group $G$, has also appeared \cite[Thm. 1]{Sza79} \cite[Thm. 2.1]{Pan82}. The statement is similar to the non-covariant case, except that $\mathcal{E}$ is now a $G$-equivariant Hilbert module, and $V$ is an intertwiner of representations. This result reduces dynamics between f.d. matrix $G$-$C^*$-algebras to isometric intertwiners between equivariant Hilbert modules.

In this work we prove a covariant Stinespring theorem which extends to the case where $G$ is any compact quantum group and holds for maps between any pair of $G$-$C^*$-algebras. We also show uniqueness of the dilation up to a partial isometry on the environment. (In fact, our results are in fact somewhat more general than this — the theory works in any rigid $C^*$-tensor category, not just in the category of f.d. continuous unitary representations of a compact quantum group.)

Our results can be interpreted as showing that the theory of finite-dimensional $G$-$C^*$-algebras and completely positive maps, which is formulated in the rigid $G$-$C^*$-tensor category $\mathcal{T} := \text{Rep}(G)$, is the Morita-theoretical ‘shadow’ of a theory whose dynamics are given by isometric 2-morphisms in the semisimple $C^*$-2-category $\text{Mod}(\mathcal{T})$ (which is equivalent to the 2-category of f.d. $G$-$C^*$-algebras, finitely generated $G$-equivariant Hilbert bimodules, and equivariant bimodule morphisms). From a utilitarian standpoint, one can use the 2-category $\text{Mod}(\mathcal{T})$ to construct, study and manipulate covariant channels. More foundationally, however, the result suggests that the more fundamental theory may be one which identifies systems with 1-morphisms and dynamics with isometric 2-morphisms in $\text{Mod}(\mathcal{T})$. In the non-covariant case, such a theory has already been proposed \cite[Chap. 8]{Vic12a, Vic12b} \cite[Chap. 8]{HV19}; there, objects were identified with classical information, 1-morphisms with classically controlled quantum systems, and isometric 2-morphisms with classically controlled dynamics. In the covariant case, there are new phenomena (such as inequivalent simple objects) which require interpretation. We here limit ourselves to the presentation of the theorem, leaving questions of interpretation for future work.

A categorical formulation of covariant physics. We now explain what we mean by Morita theory. For any simple object $r$ of a semisimple $C^*$-2-category $\mathcal{C}$, the endomorphism category $\text{End}(r)$ is a rigid $C^*$-tensor category. Morita theory describes how objects, 1-morphisms and 2-morphisms in $\mathcal{C}$ appear as certain algebraic structures in $\text{End}(r)$.

The category $\text{Rep}(G)$ of f.d. continuous unitary representations of a compact quantum group $G$ is a rigid $C^*$-tensor category. (In fact, every rigid $C^*$-tensor category $\mathcal{T}$ with a faithful unitary $\mathcal{C}$-linear tensor functor $\mathcal{T} \to \text{Hilb}$ (called a fibre functor), where $\text{Hilb}$ is the category of finite-dimensional Hilbert spaces and linear maps, is equivalent to $\text{Rep}(G)$ for some $G$.) The theory of $G$-$C^*$-algebras and covariant channels admits a natural formulation in terms of algebraic structures in the category $\text{Rep}(G)$. In brief, every finite-dimensional $G$-$C^*$-algebra $A$ possesses a canonical $G$-invariant functional
\(\phi : A \rightarrow \mathbb{C}\), such that, with the inner product \(\langle x | y \rangle = \phi(x^* y)\), \(A\) becomes a f.d. continuous unitary \(G\)-representation carrying an algebra structure. This gives a correspondence between \(G\)-\(C^*\)-algebras and\n
separable standard Frobenius algebras (SSFAs) \(A, B, \ldots\) in \(\text{Rep}(G)\). Likewise, CP maps between \(G\)-\(C^*\)-algebras correspond to \(CP\) morphisms in \(\text{Rep}(G)\) \(A \rightarrow B\); channels are CP morphisms preserving the counit. (See Section 4.1 for a more detailed summary and references.) More generally, one may also consider SSFAs and CP morphisms in rigid \(C^*\)-tensor categories \(T\) not possessing a fibre functor, although in this case there is no obvious way to identify the SSFAs and CP morphisms with concrete \(C^*\)-algebras and linear maps. The theory of covariant finite-dimensional physics can therefore be identified with the theory of SSFAs and CP morphisms in a rigid \(C^*\)-tensor category \(T\).

For these SSFAs and CP morphisms to arise by Morita theory, we need to define a rigid \(C^*\)-2-category in which \(T\) embeds as an endomorphism category. Just as we ‘unpacked’ a compact quantum group \(G\) to obtain a rigid \(C^*\)-tensor category \(\text{Rep}(G)\) with fibre functor in which our physical theory is formulated, we want to ‘unpack’ \(\text{Rep}(G)\) further to obtain a rigid \(C^*\)-2-category. We here consider two ways to ‘unpack’ a rigid \(C^*\)-tensor category \(T\) which are \(C^*\)-adaptations of well-known constructions. One is the (strict) 2-category \(\text{Mod}(T)\), whose objects are semisimple cofinite finitely decomposable left \(T\)-module categories, whose 1-morphisms are unitary \(T\)-module functors, and whose 2-morphisms are \(T\)-module functors — in this 2-category, \(T\) embeds as the endomorphism category \(\text{End}_T(\mathcal{P})\) of \(\mathcal{P}\) considered as a \(T\)-module category. The other is the 2-category \(\text{Bimod}(T)\), whose objects are SSFAs in \(T\), whose 1-morphisms are dagger bimodules, and whose 2-morphisms are bimodule homomorphisms — in this 2-category, \(T\) embeds as the endomorphism category \(\text{End}(\mathcal{I})\) of the trivial SSFA.

**Semisimple \(C^*\)-2-categories.** In fact, \(\text{Bimod}(T)\) and \(\text{Mod}(T)\) are equivalent semisimple \(C^*\)-2-categories.

Following [DR] we say that a rigid \(C^*\)-2-category is furthermore presemisimple if it is locally semisimple and additive, and every object decomposes as a finite direct sum of simple objects. The missing ingredient for semisimplicity is splitting of dagger idempotents at the 1-morphism level, which has no parallel in the theory of rigid \(C^*\)-tensor categories. We therefore need to propose a definition of a dagger idempotent 1-morphism. In [DR], which treated the non-unitary case, idempotent 1-morphisms were defined as separable algebras in endomorphism categories. In the unitary \(C^*\)-setting, we do not want to work with all separable algebras, and so need to tighten this definition. We propose that the relevant idempotents in the \(C^*\) setting are SSFAs in endomorphism categories. This is motivated physically by the fact that these idempotents can be identified with \(G\)-\(C^*\)-algebras.

Our definition of splitting of such an idempotent is Morita-theoretical in nature. It is well-known (e.g. [Lau05]) that, in a 2-category with duals, every 1-morphism \(X : r \rightarrow s\) out of \(r\) induces a Frobenius algebra (the ‘pair of pants’ algebra) on the object \(X \otimes X^*\) of the endomorphism category \(\text{End}(r)\). In a presemisimple \(C^*\)-2-category, this construction can be normalised (Proposition 2.24) to produce SSFAs in the rigid \(C^*\)-multitensor category \(\text{End}(r)\) from separable 1-morphisms out of \(r\). (Separability of a 1-morphism is a sort of nondegeneracy condition (Definition 2.22.) We say that an SSFA \(A\) in the endomorphism category \(\text{End}(r)\) splits if it is isomorphic to the pair of pants \(X \otimes X^*\) for some separable 1-morphism \(X : r \rightarrow s\). We say that a rigid \(C^*\)-tensor category is semisimple if it is presemisimple and all SSFAs in all endomorphism categories split.

As we already mentioned, we show that \(\text{Bimod}(T)\) and \(\text{Mod}(T)\) are semisimple \(C^*\)-2-categories (Proposition 3.12, Corollary 3.26), and that there is an equivalence \(\text{Bimod}(T) \simeq \text{Mod}(T)\) (Theorem 3.21). These 2-categories can be seen as higher idempotent completions of the rigid \(C^*\)-tensor category \(T\). We say that a semisimple \(C^*\)-2-category is connected if the Hom-category between any pair of nonzero objects is nonzero; we observe that every connected semisimple \(C^*\)-2-category \(\mathcal{C}\) is equivalent to \(\text{Mod}(T)\) for some rigid \(C^*\)-tensor category \(T\) (Proposition 3.25). (Here \(T\) can be chosen as the endomorphism category of any simple object in \(\mathcal{C}\).)
A classification of $G$-$C^*$-algebras. These results already yield a Morita-theoretical characterisation of f.d. $G$-$C^*$-algebras; or, more generally, of SSFAs in a rigid $C^*$-tensor category $\mathcal{T}$. Indeed, by semisimplicity of $\text{Mod}(\mathcal{T})$, every $F$ in $\mathcal{T}$ arises as a pair of pants algebra $X \otimes X^*$ for some separable 1-morphism $X : \mathcal{T} \rightarrow \mathcal{M}$ in $\text{Mod}(\mathcal{T})$.

This results in a classification that has appeared elsewhere $[\text{DCY12}, \text{Nes13}, \text{NY18}]$, although we have not found the notion of equivalence up to a phase in $\text{End}_\mathcal{T}(\mathcal{M})$ in other works. We say that an SSFA in $\mathcal{T}$ is simple if it cannot be decomposed as a nontrivial direct sum. We say that two SSFAs in $\mathcal{T}$ are Morita equivalent if they are equivalent as objects of $\text{Bimod}(\mathcal{T})$. Let $\mathcal{M}$ be a cofinite semisimple indecomposable right $\mathcal{T}$-module category, for some rigid $C^*$-tensor category $\tilde{\mathcal{T}}$; we say that two objects $X_1, X_2$ of $\mathcal{M}$ are equivalent up to a phase in $\tilde{\mathcal{T}}$ if there is an object $\theta$ of $\tilde{\mathcal{T}}$ with unit dimension such that $X_1$ is unitarily isomorphic to $X_2 \otimes \theta$. We then have the following theorem:

**Theorem (Theorem 4.6).** Let $\mathcal{T}$ be a rigid $C^*$-tensor category. There is a bijective correspondence between:

- Morita equivalence classes of simple SSFAs in $\mathcal{T}$.
- Equivalence classes of cofinite semisimple indecomposable left $\mathcal{T}$-module categories.

Let $\mathcal{M}$ be a cofinite semisimple indecomposable left $\mathcal{T}$-module category. Since $\mathcal{M}$ is indecomposable, the category $\text{End}_\mathcal{T}(\mathcal{M})$ of $\mathcal{T}$-module endofunctors on $\mathcal{M}$ is a rigid $C^*$-tensor category with a right action on $\mathcal{M}$. There is a bijective correspondence between:

- Isomorphism classes of simple SSFAs in the corresponding Morita class.
- Isomorphism classes of objects in $\mathcal{M}$, up to a phase in $\text{End}_\mathcal{T}(\mathcal{M})$.

In particular, the connected (a.k.a ergodic) $G$-$C^*$-algebras of e.g. $[\text{BDRV05}, \text{DCY12}, \text{ADC15}]$ are those arising from simple objects of $\mathcal{M}$ (Proposition 4.8).

The covariant Stinespring theorem. We then consider dynamics; that is, CP morphisms and channels between SSFAs in $\mathcal{T}$. We embed $\mathcal{T}$ as the endomorphism category $\text{End}_\mathcal{T}(\mathcal{M})$ in $\text{Mod}(\mathcal{T})$. By semisimplicity, every SSFA in $\mathcal{T}$ is isomorphic to a pair of pants algebra $X \otimes X^*$ for a separable 1-morphism $X : \mathcal{T} \rightarrow \mathcal{M}$. In this context, we now state our main theorem. The equations below use the diagrammatic calculus for pivotal dagger 2-categories (Section 2.3); the main point is that $f$ can be expressed entirely in terms of the dilation $\tau$ and the rigid structure of the 2-category $\text{Mod}(\mathcal{T})$.

**Theorem (Theorem 4.11).** Let $X : \mathcal{T} \rightarrow \mathcal{M}_1$, $Y : \mathcal{T} \rightarrow \mathcal{M}_2$ be separable 1-morphisms in $\text{Mod}(\mathcal{T})$, and let $f : X \otimes X^* \rightarrow Y \otimes Y^*$ be a CP morphism between the corresponding SSFAs in $\text{End}_\mathcal{T}(\mathcal{T}) \simeq \mathcal{T}$.

Then there exists a 1-morphism $E : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ (the ‘environment’) and a 2-morphism $\tau : X \rightarrow Y \otimes E$ such that the following equation holds:

\[
\begin{array}{ccc}
Y & \rightarrow & Y^* \\
\downarrow & & \downarrow \\
X & \otimes & X^* \\
\downarrow f & \searrow & \nwarrow \\
Y & \rightarrow & Y^* \\
\end{array}
\]

(1)
We say that $\tau$ is a dilation of $f$. The morphism

![Diagram](image)

is an isometry if and only if $f$ is a channel. (Here $n_X$ and $n_Y$ are normalising factors associated with the canonical trace (Definition 2.22).)

In the other direction, for any 1-morphism $E : \mathcal{M}_2 \to \mathcal{M}_1$ and 2-morphism $\tau : X \to Y \otimes E$, the morphism $f : X \otimes X^* \to Y \otimes Y^*$ defined by (1) is CP, and a channel if and only if (2) is an isometry.

Different dilations for a CP morphism $f : X \otimes X^* \to Y \otimes Y^*$ are related by a partial isometry on the environment. Specifically, let $\tau_1 : X \to Y \otimes E_1$, $\tau_2 : X \to Y \otimes E_2$ be two dilations of $f$. Then there exists a partial isometry $\alpha : E_1 \to E_2$ such that

$$(\text{id}_Y \otimes \alpha) \circ \tau_1 = \tau_2 \quad (\text{id}_Y \otimes \alpha^*) \circ \tau_2 = \tau_1$$

In particular, the dilation minimising the quantum dimension of the environment $d(E)$ is unique up to unitary $\alpha$. (A concrete construction of the minimal dilation from any other dilation is specified in the last paragraph of the proof.)

This theorem recovers the aforementioned previous results in the literature: the f.d. noncovariant Stinespring theorem follows from setting $T := \text{Hilb}$, and the f.d. covariant Stinespring theorem for a compact group $G$ follows from setting $T := \text{Rep}(G)$ (Example 4.12).

We finish by reiterating the proposal that, rather than identifying systems with SSFAs in $T$ and dynamics with CP morphisms, we might identify systems with 1-morphisms in $\text{Mod}(T)$ and dynamics with isometries. We can then see how the 2-categorical theory extends the algebraic theory: SSFAs and CP morphisms in $T$ correspond to 1-morphisms $X : T \to \mathcal{M}$ in $\text{Mod}(T)$ and isometries of type $X \to Y \otimes E$, whereas the 2-categorical theory encompasses all 1-morphisms, and all isometries. In fact, the 2-categorical theory unites the theories of SSFAs and CP morphisms in all rigid $C^*$-tensor categories categorically Morita equivalent to $T$, which appear as the endomorphism categories of simple objects in $\text{Mod}(T)$. However, the physical interpretation of this extended theory is still unclear.

1.1 Related work

Categorical Morita equivalence of compact quantum groups. In [NY18] a notion of a Morita-Galois object was introduced. This is a $G_1$-$G_2$-$C^*$-algebra whose category of equivariant right Hilbert $A$-modules is an invertible $\text{Rep}(G_1)$-$\text{Rep}(G_2)$-bimodule category; such an algebra can be reconstructed from any $\text{Rep}(G_1)$-$\text{Rep}(G_2)$-bimodule category, which includes the Hom-category $\text{Hom}(\mathcal{M}_1, \mathcal{M}_2)$ between simple objects in any semisimple $C^*$-category with fibre functor. We also note that, in the language of this work, two compact quantum groups (or more generally two rigid $C^*$-tensor categories $\mathcal{T}_1, \mathcal{T}_2$) are categorically Morita equivalent precisely when there is an equivalence $\text{Mod}(\mathcal{T}_1) \simeq \text{Mod}(\mathcal{T}_2)$.

Previous covariant Stinespring theorems. As we have mentioned, our result recovers [Hol07] Thm. 2], the finite-dimensional cases of [Sti55 Thm. 1] [Scu79 Thm. 1] [Pau82 Thm. 2.1], and the special case of [Sza10 Thm. 15] applying to f.d. unital $C^*$-algebras. Since the category $\text{Mod}(\text{Rep}(G))$ is equivalent to the 2-category of equivariant finitely generated Hilbert bimodules over f.d. $G$-$C^*$-algebras [NY18 P.13], it may also be interesting to consider recent work on completely positive maps between Hilbert $C^*$-modules [Asa09] [BRS10] [Joi10].
Module categories. The 2-category \( \text{Mod}(\mathcal{T}) \) defined here is as a semisimple \( C^* \)-version of the 2-category of exact module categories over a tensor category \( \mathcal{E} \).\( \text{EnO16} \) Rem. 7.12.15,\( \text{Ost03} \). Our proofs demonstrate that when working in the \( C^* \)-setting there is no need to use abelian category theory; all one needs is linearity, the \( C^* \)-axioms and idempotent splitting. We hope this will make module category theory more accessible to researchers in the quantum information community.

Torsion-freeness for rigid \( C^* \)-tensor categories. Several works considered torsion-freeness for compact quantum groups \( \text{Mey08} \) \( \text{Voi11} \) and more generally for rigid \( C^* \)-tensor categories \( \text{ADC15} \). It follows from our results that a rigid \( C^* \)-tensor category \( \mathcal{T} \) is torsion-free precisely when the 2-category \( \text{Mod}(\mathcal{T}) \) has a single simple object up to equivalence (which is necessarily \( \mathcal{T} \) itself). In this way we can already characterise \( \text{Mod}(\mathcal{T}) \) for connected compact groups with torsion-free fundamental group \( \text{Mey08} \) §7.2, quantum groups \( SU_q(2) \) for \( q \in (-1, 1) \)\( \text{Voi11} \) Prop. 3.2, free orthogonal quantum groups \( \text{Voi11} \) Cor. 7.7, and free unitary quantum groups \( \text{ADC15} \) Cor. 2.9.

Irreducibly covariant channels and Temperley-Lieb channels. A number of recent works have studied irreducibly covariant channels for compact groups from a quantum information-theoretical perspective, e.g. \( \text{MSD17} \) \( \text{SC18} \) \( \text{Nwu14} \). In our language, these are channels between connected matrix \( G-C^* \)-algebras, i.e. \( G-C^* \)-algebras corresponding to simple 1-morphisms \( X : \mathcal{T} \to \mathcal{T} \) in \( \text{Mod}(\mathcal{T}) \).

Recent work has also considered Temperley-Lieb channels covariant for actions of the free orthogonal quantum groups \( O^*_F \) \( \text{BC16} \) \( \text{BCLY20} \). Since these compact quantum groups are torsion-free \( \text{Voi11} \) Cor. 7.7, all simple \( O^*_F-C^* \)-algebras are matrix \( O^*_F-C^* \)-algebras \( X \otimes X^* \) for some \( X \in \text{Rep}(G) \). In this setting, it is observed that covariant channels \( X \otimes X^* \to Y \otimes Y^* \) may be constructed from isometries \( X \to Y \otimes E \) in \( \text{Rep}(O^*_F) \). In fact, the covariant Stinespring theorem proven here implies that every covariant channel may be constructed in this way, and also characterises when two such isometries produce equivalent channels.

Operator algebras and CP maps in rigid \( C^* \)-tensor categories. In \( \text{JP17a} \) the authors give a fully general definition of a \( C^* \)-algebra in a rigid \( C^* \)-tensor category \( \mathcal{T} \). (From the results in \( \text{Vic11} \), in the finite-dimensional case these probably correspond to SSFAs in \( \mathcal{T} \); indeed, for connected SSFAs this was shown in \( \text{JP17b} \).) Whereas we focus on the finite-dimensional case, where Morita equivalence classes of f.d. \( G-C^* \)-algebras correspond to cofinite \( \text{Rep}(G) \)-module categories, in \( \text{JP17a} \) infinite-dimensional \( G-C^* \)-algebras are treated, which corresponds to dropping the cofiniteness condition on the module categories. A Stinespring theorem is also proposed in \( \text{JP17a} \), but from quite a different perspective. We hope to generalise the results of this paper to the infinite-dimensional setting in future work.

After the completion of this paper we were made aware of the recent work \( \text{HP20} \) §5.3. There a CP map between pair of pants algebras in a rigid \( C^* \)-2-category is defined by \( \text{GPJP21} \); it is shown that this definition matches the definition of a CP map given in \( \text{JP17a} \). The present work relates this definition to the usual Stinespring’s theorem for \( G-C^* \)-algebras, as well as proving uniqueness of a dilation up to partial isometry.

Q-system completion for \( C^* \) 2-categories. In two recent works \( \text{CPJP21} \) \( \text{GY20} \) a notion of idempotent completion of a \( C^* \)-2-category has been defined. We discuss the relationship between these definitions and the Bimod construction defined here in Remark 2.18.

Standard duals for rigid \( C^* \)-2-categories. In \( \text{GL19} \) a notion of standard duality for rigid \( C^* \)-2-categories with finite-dimensional centres was introduced; we make use of this here, in the special case of presemisimple \( C^* \)-2-categories. In this presemisimple case we find that the characterisation of standard duals using the equivalence \( \text{Mat}(\mathcal{C}) \simeq \mathcal{C} \) (Remark 6.4) is useful for calculations.
1.2 Summary

In Section 2 we review necessary background material on 2-category theory, covering 2-categories and their diagrammatic calculus (Section 2.1), 2-functors and icons (Section 2.2), pivotal dagger 2-categories (Section 2.3), rigid $C^*$-2-categories (Section 2.4) and semisimplicity (Section 2.5).

In Section 3.1 we define the 2-category $\text{Bimod}(\mathcal{T})$ for a rigid $C^*$-tensor category $\mathcal{T}$, and prove that it is a semisimple $C^*$-2-category. In Section 3.2 we define the 2-category $\text{Mod}(\mathcal{T})$. In Section 3.3 we show the equivalence $\text{Bimod}(\mathcal{T}) \simeq \text{Mod}(\mathcal{T})$, and observe that every connected semisimple $C^*$-2-category is equivalent to $\text{Mod}(\mathcal{T})$ for some $\mathcal{T}$.

In Section 4 we classify $G$-$C^*$-algebras (Section 4.1) and prove the covariant Stinespring theorem (Section 4.2) and covariant Choi theorem (Section 4.3).

2 Background on 2-category theory

In this section we will review some definitions and results about 2-categories.

2.1 Diagrammatic calculus for 2-categories

Sometimes the noun ‘2-category’ is taken to indicate a strict 2-category. In this work, by contrast, when we say ‘2-category’ we mean the general, fully weak notion, which is sometimes called a bicategory. We will explicitly use the adjective ‘strict’ to distinguish strict 2-categories. We assume that the reader is familiar with the definition of a 2-category; see e.g. [JY21, Def. 2.1.3].

The abstract definition of a 2-category is important for checking whether some collection of data does or does not constitute a 2-category. When working concretely inside a given 2-category it is more convenient to use the diagrammatic calculus for 2-categories, which takes account of the coherence implied by the pentagon and triangle equalities.

We assume that the reader is familiar with this calculus, which is summarised in, for example, [Mar14], [HV19, §8.1.2]. It is a straightforward extension of the graphical calculus for monoidal categories.

- The objects $r, s, \ldots$ of the 2-category are represented by labelled regions. (To avoid having too many letters in the diagrams, we will often use black-and-white pattern shading rather than letter labels in order to show which regions correspond to which objects.)
- The 1-morphisms $X, Y, \cdots : r \to s$ are represented by edges, separating the region $r$ on the left from the region $s$ on the right. Identity 1-morphisms are invisible in the diagrammatic calculus.
- The 2-morphisms $f, g, \cdots : X \to Y$ are represented by labelled boxes with an $X$-edge entering from below and a $Y$-edge leaving above. Identity 2-morphisms are invisible in the diagrammatic calculus, as are the components of the associators and L/R unitors and their inverses.
- Vertical and horizontal composition of 2-morphisms are represented by vertical and horizontal juxtaposition respectively. We represent vertical and horizontal composition by $\circ$ and $\otimes$ respectively.

We do not keep track of identity 1-morphisms and bracketing of 1-morphism composites in the diagrammatic calculus. However, 2-categorical coherence implies that we do not need to keep track of this information while performing our calculations. Indeed, once a choice of bracketing and identity 1-morphisms is specified for the source and target of a diagram (we call such a choice a parenthesis scheme), the diagram represents a unique 2-morphism, however it is interpreted.

Proposition 2.1 ([Bar08, Prop. 4.1]). After specifying a parenthesis scheme for the source and target 1-morphisms of a 2-morphism diagram, the 2-morphism represented by the diagram is independent of the choice of parentheses, associators and unitors used to interpret the interior of the diagram.
2.2 Diagrammatic calculus for 2-functors

Here we use the term 2-functor for a functor between 2-categories whose coherence constraints are isomorphisms; this is sometimes called a pseudofunctor. We assume the reader is familiar with this notion; see e.g. [JY21, Def. 4.1.2]. We remark that if \( C, C' \) are monoidal categories (i.e. one-object 2-categories), then a 2-functor \( C \to C' \) is simply a monoidal functor.

In order to represent 2-functors within the diagrammatic calculus, we use the calculus of functorial boxes [Mel06]. In this calculus we represent the functor \( F_{r,s} \) by drawing a coloured box around 1- and 2-morphisms in \( C(r,s) \). The calculus is summarised in [Ver22, §2.2].

Notation 2.2. In this work we will use shading in diagrams for two reasons: firstly to distinguish which regions in a diagram correspond to which objects, and secondly to indicate functorial boxes. To reduce confusion we use colour only for functorial boxes. That is, regions corresponding to objects are pattern-shaded in black and white, whereas functorial boxes are in colour.

Finally, we assume that the reader is familiar with the definition of an icon between 2-functors [JY21, Def. 4.6.2][Lac10]. We say that an icon is invertible if all its 2-morphism components are invertible.

2.3 Pivotal dagger 2-categories

A pivotal dagger 2-category is a straightforward horizontal categorification of a pivotal dagger category [Sel10, Sec. 7.3][HV19, Def. 3.51]. We assume the reader is familiar with the notion of duality for 1-morphisms in a 2-category (see e.g. [JY21, Def. 6.1.1]; what they call the right adjoint we call the right dual, and what they call the triangle equations we call the snake equations).

Let \( X : r \to s \) be a 1-morphism in a 2-category, and suppose that \( [X^* : s \to r, \eta : \text{id}_s \to X^* \otimes X, \epsilon : X \otimes X^* \to \text{id}_r] \) is a right dual for \( X \). In order to represent duality in the graphical calculus, we draw an upward-facing arrow on the \( X \)-wire and a downward-facing arrow on the \( X^* \)-wire, and draw \( \eta \) and \( \epsilon \) as a cup and a cap, respectively. Then the snake equations become purely topological:

\[
\begin{align*}
\eta_{X^*Y} & \quad \epsilon_{X \otimes Y} \\
X^*Y & \quad YX
\end{align*}
\]

**Proposition 2.3** ([HV19, Lemmas 3.6, 3.7]). If \( [X^*, \eta_X, \epsilon_X] \) and \( [Y^*, \eta_Y, \epsilon_Y] \) are right duals for \( X : r \to s \) and \( Y : s \to t \) respectively, then \( [Y^* \otimes X^*, \eta_{X \otimes Y}, \epsilon_{X \otimes Y}] \) is a right dual for \( X \otimes Y \), where \( \eta_{X \otimes Y} \) and \( \epsilon_{X \otimes Y} \) are defined as follows:

\[
\begin{align*}
\eta_{X \otimes Y} & \quad \epsilon_{X \otimes Y}
\end{align*}
\]

Moreover, for any object \( r \), \( [\text{id}_r, \text{id}_{id_r}, \text{id}_{id_r}] \) is right dual to \( \text{id}_r \).

**Proposition 2.4** ([HV19, Lem. 3.4]). Let \( X : r \to s \) be a 1-morphism, and let \( [X^*, \eta, \epsilon], [X'^*, \eta', \epsilon'] \) be right duals. Then there is a unique invertible 2-morphism \( \alpha : X^* \to X'^* \) such that:

\[
\begin{align*}
X^* & \quad X'^*
\end{align*}
\]
Explicitly, $\alpha$ is defined as follows:

\[
\begin{array}{c}
X' \\
\uparrow^\eta \\
\uparrow\epsilon \\
X
\end{array}
\quad
\begin{array}{c}
\alpha
\end{array}
\quad
\begin{array}{c}
X \\
\downarrow^\eta \\
\downarrow\epsilon \\
X'
\end{array}
\]  \tag{6}

Using duality, we can define a notion of transposition for 2-morphisms.

**Definition 2.5.** Let $X, Y: r \to s$ be 1-morphisms with chosen right duals $[X^*, \eta_X, \epsilon_X]$ and $[Y^*, \eta_Y, \epsilon_Y]$. For any 2-morphism $f: X \to Y$, we define its right transpose (a.k.a. mate) $f^*: Y^* \to X^*$ as follows:

\[
\begin{array}{c}
X' \\
\uparrow^f \\
Y
\end{array}
\quad
\begin{array}{c}
\alpha
\end{array}
\quad
\begin{array}{c}
Y' \\
\downarrow^f \\
X
\end{array}
\]  \tag{7}

A choice of a right dual for every 1-morphism in $C$ thus defines a contravariant 2-functor $\ast: C \to C$, the dual functor, whose multiplicators and unitors are defined using Proposition 2.3 and Proposition 2.4.

**Definition 2.6.** We say that a 2-category $C$ with chosen right duals is pivotal if there is an invertible icon $\iota: \ast \ast \to \text{id}$ from the double duals 2-functor $\ast \circ \ast: C \to C$ to the identity 2-functor. The invertible icon is called a pivotal structure.

We assume the reader is familiar with the notion of a dagger 2-category and a unitary (a.k.a. dagger) 2-functor (see e.g. [HK16]). Here are some basic definitions we will use throughout.

**Definition 2.7.** Let $C$ be a dagger 2-category. We say that a 2-morphism $f: X \to Y$ is:

- An isometry if $f^1 \circ f = \text{id}_X$.
- A coisometry if $f \circ f^1 = \text{id}_Y$.
- Unitary if it is an isometry and a coisometry.
- A partial isometry if $(f^1 \circ f)^2 = f^1 \circ f$ (or equivalently $(f \circ f^1)^2 = f \circ f^1$).
- Positive if $X = Y$ and there exists some 2-morphism $g: X \to X'$ such that $f = g^1 \circ g$.

**Definition 2.8.** Let $C$ be a dagger 2-category. We say that a 1-morphism $X: r \to s$ is an equivalence if there exists a 1-morphism $X^{-1}: s \to r$ and unitary 2-morphisms $\alpha: \text{id}_s \to X^{-1} \otimes X$ and $\beta: X \otimes X^{-1} \to \text{id}_r$. We sometimes write that $[X, X^{-1}, \alpha, \beta]: r \to s$ is an equivalence. If an equivalence $X: r \to s$ exists we say that the objects $r$ and $s$ are equivalent.

The following lemma is common knowledge.

**Lemma 2.9.** Let $[X, X^{-1}, \alpha, \beta]$ be an equivalence in a dagger 2-category. Then there exists an equivalence $[X, X^{-1}, \alpha, \beta']$ such that $[X^{-1}, \alpha, \beta']$ is a right dual for $X$ (we call such an equivalence an adjoint equivalence).

Following [Pen18], we say that a choice of right duals on a dagger 2-category is a unitary duals functor if the associated duals functor is a dagger 2-functor. Given a unitary duals functor, there is a canonical associated pivotal structure [Sel10 §7.3] (for which, in particular, all of the 2-morphism components of the pivotal structure are unitary [Pen18 Cor. 3.10]). In this case one may define left cups and caps as the daggers of the right cups and caps, which satisfy snake equations analogous to (3).

**Definition 2.10.** We call a dagger 2-category equipped with a unitary duals functor a pivotal dagger 2-category.
We use the following useful notation to represent morphisms in a pivotal dagger 2-category. Let \( f : X \to Y \) be a 2-morphism. We first make the box for the 2-morphism \( f \) asymmetric by tilting the right vertical edge. We now represent the transpose \( f^* : Y^* \to X^* \) by rotating the box, as though we had ‘yanked’ both ends of the wire in the RHS of (7):

\[
\begin{array}{c}
\phantom{X^*} \\
X^* \\
\downarrow \\
\hline
f^* \\
\uparrow \\
Y^*
\end{array} := \begin{array}{c}
\phantom{X} \\
X \\
\downarrow \\
\hline
f \\
\uparrow \\
Y
\end{array}
\]

(8)

We represent the dagger \( f^\dagger : Y \to X \) by reflection in a horizontal axis, preserving the direction of any arrows:

\[
\begin{array}{c}
\phantom{X} \\
X \\
\downarrow \\
\hline
f \\
\uparrow \\
Y
\end{array} := \begin{array}{c}
\phantom{Y} \\
Y \\
\downarrow \\
\hline
f^\dagger \\
\uparrow \\
X
\end{array}
\]

(9)

Finally, we represent the conjugate \( f_* := (f^*)\dagger = (f^\dagger)^* \) by reflection in a vertical axis:

\[
\begin{array}{c}
\phantom{X} \\
\phantom{X} \\
\phantom{X} \\
\hline
f^* \\
\uparrow \\
Y \\
\hline
\phantom{X} \\
\phantom{X} \\
\phantom{X}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f^\dagger \\
\uparrow \\
X \\
\hline
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y}
\end{array}
\]

(10)

Using this notation, 2-morphisms now freely slide around cups and caps.

**Proposition 2.11** ([HV19, Lemma 3.12, Lemma 3.26]). Let \( C \) be a pivotal dagger 2-category and \( f : X \to Y \) a 2-morphism. Then:

\[
\begin{array}{c}
\phantom{X} \\
\phantom{X} \\
\phantom{X} \\
\hline
f^* \\
\uparrow \\
Y \\
\hline
\phantom{X} \\
\phantom{X} \\
\phantom{X}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f^\dagger \\
\uparrow \\
X \\
\hline
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f \\
\uparrow \\
Y \\
\hline
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f \\
\uparrow \\
X \\
\hline
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f \\
\uparrow \\
X \\
\hline
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f^* \\
\uparrow \\
Y \\
\hline
\phantom{X} \\
\phantom{X} \\
\phantom{X}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f^\dagger \\
\uparrow \\
X \\
\hline
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f \\
\uparrow \\
Y \\
\hline
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f \\
\uparrow \\
X \\
\hline
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f^* \\
\uparrow \\
Y \\
\hline
\phantom{X} \\
\phantom{X} \\
\phantom{X}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f^\dagger \\
\uparrow \\
X \\
\hline
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f \\
\uparrow \\
Y \\
\hline
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f \\
\uparrow \\
X \\
\hline
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y}
\end{array} = \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f^* \\
\uparrow \\
Y \\
\hline
\phantom{X} \\
\phantom{X} \\
\phantom{X}
\end{array}
\]

(11)

**Definition 2.12.** Let \( X : r \to s \) be an 1-morphism and let \( f : X \to X \) be a 2-morphism in a pivotal dagger 2-category \( C \). We define the right trace of \( f \) to be the following 2-morphism \( \text{Tr}_R(f) : \text{id}_r \to \text{id}_r \):

\[
\begin{array}{c}
\phantom{X} \\
\phantom{X} \\
\phantom{X} \\
\hline
f^* \\
\uparrow \\
Y \\
\hline
\phantom{X} \\
\phantom{X} \\
\phantom{X}
\end{array} := \begin{array}{c}
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y} \\
\hline
f^\dagger \\
\uparrow \\
X \\
\hline
\phantom{Y} \\
\phantom{Y} \\
\phantom{Y}
\end{array}
\]

(10)

Using this notation, 2-morphisms now freely slide around cups and caps.

We define the right dimension \( \text{dim}_R(X) \) of an 1-morphism \( X : r \to s \) to be \( \text{Tr}_R(\text{id}_X) \). The left trace \( \text{Tr}_L(f) : \text{id}_s \to \text{id}_s \) and left dimension \( \text{dim}_L(X) \) are defined analogously using the right cup and left cap.

**2.4 Rigid \( C^* \)-2-categories**

We assume the reader is familiar with the notion of a rigid \( C^* \)-tensor category (see e.g. [NT13, §2.1]). We assume that our rigid \( C^* \)-tensor categories are semisimple, but we do not assume that the endomorphism algebra of the tensor unit is one-dimensional.

We will now review the notion of a (presemisimple) \( C^* \)-2-category.
Definition 2.13. We say that a dagger 2-category is $\mathbb{C}$-linear if:

- For any 1-morphisms $X_1, X_2 : r \to s$, the Hom-set $\text{Hom}(X_1, X_2)$ is a complex vector space.
- Horizontal and vertical composition induce linear maps on Hom-spaces, and the dagger induces antilinear maps.

We say that an $\mathbb{C}$-linear dagger 2-category is furthermore a $C^*$-2-category if:

- The vector spaces of 2-morphisms are Banach spaces, and $||f \circ g|| \leq ||f|| ||g||$.
- $||f^\dagger \circ f|| = ||f||^2$ for any 2-morphism $f : X \to Y$; in particular, for any 1-morphism $X$ the $\ast$-algebra $\text{End}(X)$ is a $C^*$-algebra.
- For any 2-morphism $f : X \to Y$, the 2-morphism $f^\dagger \circ f$ is a positive element of the $C^*$-algebra $\text{End}(X)$.

A $C^*$-category can be defined in the obvious analogous way, so that the Hom-categories of a $C^*$-2-category are all $C^*$-categories. We say that a 2-functor or functor is $\mathbb{C}$-linear if it induces linear maps on morphism spaces.

We say that a $C^*$-2-category is rigid if it has duals for 1-morphisms.$^1$

Remark 2.14. We observe that the Hom-categories of a rigid $C^*$-2-category are $W^*$-categories in the sense of [GLRS5 Def. 2.1], since the Hom-spaces are finite-dimensional. This gives us a polar decomposition [GLRS5 Cor. 2.7]. Indeed, for any 2-morphism $f : X \to Y$, we define $|f| := (f^\dagger \circ f)^{1/2}$, where this is the positive square root in the f.d. $C^*$-algebra $\text{End}(X)$. Then there exists a unique partial isometry $u : X \to Y$ such that:

$$f = u \circ |f| \quad u^\dagger \circ u = s(|f|) \quad u \circ u^\dagger = s(|f^\dagger|)$$

Here $s(|f|)$ is the support of $|f|$, i.e. the least projection of all the projections $p$ in $\text{End}(X)$ such that $p \circ |f| = |f| \circ p = |f|$ [Sak12 Def. 1.10.3].

We recall the following definitions for $C^*$-1-categories:

- A direct sum of two objects $X_1, X_2$ is an object $X_1 \oplus X_2$ together with isometries $i_1 : X_1 \to X_1 \oplus X_2$, $i_2 : X_2 \to X_1 \oplus X_2$ such that $i_1 \circ i_1^\dagger + i_2 \circ i_2^\dagger = \text{id}_{X_1 \oplus X_2}$.
- A zero object is an object $\mathbf{0}$ such that $\text{Hom}(\mathbf{0}, \mathbf{0})$ is the zero-dimensional vector space.
- We say that the category is additive if it has a zero object and pairwise direct sums.
- For any object $X$, we say that a morphism $f \in \text{End}(X)$ is a dagger idempotent if $f = f^\dagger = f \circ f$. We say that a splitting of the dagger idempotent is an object $V$ together with an isometry $\iota_f : V \to X$ such that $f = \iota_f \circ \iota_f^\dagger$. We say that the category is idempotent complete if every dagger idempotent has a splitting.
- We say that the category is semisimple if it is additive and idempotent complete, and the $C^*$-algebra $\text{End}(X)$ is finite-dimensional for every object $X$. In a semisimple category every object is a finite direct sum of simple objects, i.e. objects $X_i$ such that $\text{End}(X_i) \cong \mathbb{C}$.

We say that a $C^*$-2-category $\mathcal{C}$ is locally additive, locally semisimple, etc. if all its Hom-categories are.

The following definitions are obvious unitary adaptations of those from [DR §1].

Definition 2.15. Let $\mathcal{C}$ be a locally additive $C^*$-2-category.

$^1$This definition of rigidity for $C^*$-2-categories is only really satisfactory when $\text{End}(\text{id}_r)$ is finite-dimensional for all objects $r$ of $\mathcal{C}$; the problem is that a unitary dual functor is not known to exist in general [Zit07]. We only work with presemisimple $C^*$-2-categories, which all satisfy this condition.
• We say that a zero object in \( C \) is an object \( 0 \) such that the category \( \text{End}(0) \) is the terminal 1-category.

• We say that a direct sum of two objects \( r_1, r_2 \) in \( C \) is an object \( r_1 \oplus r_2 \) with inclusion and projection 1-morphisms \( \iota_i : r_i \to r_1 \oplus r_2, \rho_i : r_1 \oplus r_2 \to r_i \) such that:
  
  – \( \iota_1 \otimes \rho_1 \) is unitarily isomorphic to \( \text{id}_{r_1} \).
  
  – \( \iota_1 \otimes \rho_2 \in \text{Hom}(r_1, r_2) \) and \( \iota_2 \otimes \rho_1 \in \text{Hom}(r_2, r_1) \) are zero 1-morphisms.
  
  – \( \text{id}_{r_1 \oplus r_2} \) is a direct sum of \( \rho_1 \otimes \iota_1 \) and \( \rho_2 \otimes \iota_2 \).

• We say that \( C \) is additive if it has a zero object and direct sums.

In order to define semisimplicity for \( C^* \)-2-categories we will need a notion of idempotent completeness which will be introduced in Section 2.5. However, following [DR, §1] we can already define the following weaker notion.

**Definition 2.16.** An additive \( C^* \)-2-category is presemisimple if it is locally semisimple, rigid, and every object is a finite direct sum of objects \( \{ r_i \} \) with simple identity, i.e. \( \text{id}_{r_i} \) is a simple object of \( \text{End}(r_i) \).

In a presemisimple \( C^* \)-2-category an object has simple identity if and only if it is not decomposable as a nontrivial direct sum. We call such objects simple.

It is easy to check that zero objects and direct sums in presemisimple \( C^* \)-2-categories are unique up to equivalence and preserved under \( C \)-linear unitary 2-functors.

In order to perform computations in presemisimple \( C^* \)-2-categories we will make use of a convenient equivalence that categorifies matrix notation for morphisms in semisimple 1-categories [HV19, §2.2.4]. These results are certainly known to experts [HV19, Chap. 8] [RV19, §2.1], although we have not seen proofs elsewhere. We provide a summary in Appendix 6.1.

Every presemisimple \( C^* \)-2-category \( C \) has a canonical unitary dual functor. This follows immediately from the more general result in [GL19]; indeed, presemisimplicity implies finite-dimensional centres, in the language of that work. For the following proposition, we observe that for any object \( r \) in \( C \), the \( C^* \)-algebra \( \text{End}((d_r) \) is commutative. In particular, there is a unique trace mapping each of the minimal orthogonal projections to 1, which we call \( \text{Tr}_{r} : \text{End}((d_r) \to \mathbb{C} \).

**Proposition 2.17 ([GL19, Prop. 7.3.3]).** Let \( X : r \to s \) be a 1-morphism in a presemisimple \( C^* \)-2-category \( C \) and let \( [X^*, \eta, \epsilon] \) be a right dual. Define a map \( \phi_X : \text{End}(X) \to \mathbb{C} \) as follows:

\[
\phi_X(T) := \text{Tr}_s[\eta^\dagger \circ (\text{id}_{X^*} \otimes T) \circ \eta]
\]

Define a second map \( \psi_X : \text{End}(X) \to \mathbb{C} \) as follows:

\[
\psi_X(T) := \text{Tr}_r[\epsilon \circ (T \otimes \text{id}_{X^*}) \circ \epsilon^\dagger]
\]

We say that \( [X^*, \eta, \epsilon] \) is a standard dual for \( X \) precisely when \( \phi_X = \psi_X \). In this case the map \( \phi_X = \psi_X \) is tracial, positive and faithful, and does not depend on the choice of standard dual.

A standard dual exists for every object [GL19, Def. 7.29]. It is straightforward to show (following the same approach as in the 1-categorical case [NT13, Thm. 2.2.21]) that a choice of standard duals for every object defines a unitary dual functor on \( C \). Different choices of standard duals are related by a unitary isomorphism (Proposition 2.4). The tensor product of standard duals (Proposition 2.3) is standard.
2.5 Semisimplicity

To define semisimplicity of a rigid $C^*$-2-category we need a notion of idempotent splitting at the level of 1-morphisms. In [DR §1.3] it was proposed that categorified idempotents in the non-unitary setting correspond to separable monads (i.e. separable algebras in endomorphism categories). Semisimplicity corresponds to splitting of these algebras (we will explain what this means shortly).

In the unitary $C^*$-setting, we do not want to work with all separable algebras, and so need to tighten this definition of an idempotent. We propose that the relevant idempotents in a presemisimple $C^*$-2-category are standard separable Frobenius algebras in endomorphism categories. There is a physical motivation for this definition: as we will see in Section 4.1, in the category of representations of a compact quantum group $G$, Frobenius algebras correspond to pairs of a finite-dimensional $G$-$C^*$-algebra (a.k.a. $C^*$-dynamical system) and a $G$-invariant linear functional. There is a unique choice of linear functional on a $G$-$C^*$-algebra such that the corresponding Frobenius algebra is standard and separable.

**Remark 2.18.** The notion of idempotent splitting in the $C^*$-setting has already been considered in previous works; we mention now how our assumptions of standardness and separability compare. In [CPJP21], the $Q$-system completion of a $C^*$-2-category is defined. These $Q$-systems are separable Frobenius algebras, but they are not standard, since there is no assumption of rigidity on the $C^*$-2-category. Because there is no assumption of rigidity of the original $C^*$-2-category, the question of rigidity of the $Q$-system completion does not arise in their work. Here our additional standardness assumption is used to show rigidity of the idempotent completion.

However, in [GY20], an idempotent completion on a rigid $C^*$-2-category was studied, and it was stated there that, even without the standardness assumption, the completion is rigid. Therefore, it seems that it is possible to drop the standardness assumption on the Frobenius algebras, although we do not do this here.

We remark that the idempotent completions in both these works are more general than the one we define here, since they complete a general 2-category rather than just a tensor category. It would not be hard to extend our completion to a 2-category, but we did not need this for our purposes.

2.5.1 Standard separable Frobenius algebras

In this section, let $\mathcal{T}$ be a rigid $C^*$-tensor category.

**Definition 2.19.** An algebra $[A, m, u]$ in $\mathcal{T}$ is an object $A$ with multiplication and unit morphisms, depicted as follows:

\[
m : A \otimes A \to A \quad u : 1 \to A
\]

These morphisms satisfy the following associativity and unitality equations:

\[
\begin{align*}
\quad & = \quad = \\
\quad & = \quad =
\end{align*}
\]  

Analogously, a coalgebra $[A, \delta, \epsilon]$ is an object $A$ with a comultiplication $\delta : A \to A \otimes A$ and a counit $\epsilon : A \to 1$ obeying the following coassociativity and counitality equations:

\[
\begin{align*}
\quad & = \quad = \\
\quad & = \quad =
\end{align*}
\]
The dagger of an algebra \([A, m, u]\) is a coalgebra \([A, m^\dagger, u^\dagger]\). A algebra \([A, m, u]\) in \(\mathcal{T}\) is called Frobenius if the algebra and adjoint coalgebra structures are related by the following Frobenius equation:

\[
\begin{array}{c}
\begin{tikzpicture}[scale=0.6]
\draw[fill=white] (0,0) circle (0.2);
\draw[fill=white] (2,0) circle (0.2);
\draw[fill=white] (1,-1) circle (0.2);
\draw (0,0) -- (1,-1) -- (2,0);
\end{tikzpicture}
\end{array}
= \begin{array}{c}
\begin{tikzpicture}[scale=0.6]
\draw[fill=white] (0,0) circle (0.2);
\draw[fill=white] (2,0) circle (0.2);
\draw[fill=white] (1,-1) circle (0.2);
\draw (0,0) -- (1,-1) -- (2,0);
\end{tikzpicture}
\end{array}
\tag{15}
\]

**Definition 2.20.** Frobenius algebras are canonically self-dual. Indeed, it is easy to check that for any Frobenius algebra \(A\) the following cup and cap fulfil the snake equations \([3]\):

\[
\begin{array}{c}
\begin{tikzpicture}[scale=0.6]
\draw[fill=white] (0,0) circle (0.2);
\draw[fill=white] (2,0) circle (0.2);
\draw[fill=white] (1,-1) circle (0.2);
\draw (0,0) -- (1,-1) -- (2,0);
\end{tikzpicture}
\end{array}
:= \begin{array}{c}
\begin{tikzpicture}[scale=0.6]
\draw[fill=white] (0,0) circle (0.2);
\draw[fill=white] (2,0) circle (0.2);
\draw[fill=white] (1,-1) circle (0.2);
\draw (0,0) -- (1,-1) -- (2,0);
\end{tikzpicture}
\end{array}
\tag{16}
\]

If the cup and cap \([16]\) are a standard duality for \(A\) (in the sense of Proposition \(2.17\)), we say that the Frobenius algebra is standard.

A Frobenius algebra is separable (a.k.a. special) if the following additional equation is satisfied:

\[
\begin{array}{c}
\begin{tikzpicture}[scale=0.6]
\draw[fill=white] (0,0) circle (0.2);
\draw[fill=white] (2,0) circle (0.2);
\draw[fill=white] (1,-1) circle (0.2);
\draw (0,0) -- (1,-1) -- (2,0);
\end{tikzpicture}
\end{array}
= \begin{array}{c}
\begin{tikzpicture}[scale=0.6]
\draw[fill=white] (0,0) circle (0.2);
\draw[fill=white] (2,0) circle (0.2);
\draw[fill=white] (1,-1) circle (0.2);
\draw (0,0) -- (1,-1) -- (2,0);
\end{tikzpicture}
\end{array}
\tag{17}
\]

From now on we will be concerned with separable standard Frobenius algebras (SSFAs).

**Definition 2.21.** Let \(A, B\) be SSFAs in \(\mathcal{T}\). We say that a morphism \(f : A \to B\) is a \(\ast\)-homomorphism if it obeys the following equations:

\[
\begin{array}{c}
\begin{tikzpicture}[scale=0.6]
\draw[fill=white] (0,0) circle (0.2);
\draw[fill=white] (2,0) circle (0.2);
\draw[fill=white] (1,-1) circle (0.2);
\draw (0,0) -- (1,-1) -- (2,0);
\end{tikzpicture}
\end{array}
= \begin{array}{c}
\begin{tikzpicture}[scale=0.6]
\draw[fill=white] (0,0) circle (0.2);
\draw[fill=white] (2,0) circle (0.2);
\draw[fill=white] (1,-1) circle (0.2);
\draw (0,0) -- (1,-1) -- (2,0);
\end{tikzpicture}
\end{array}
\tag{18}
\]

We say that it is a \(\ast\)-cohomomorphism if it obeys the following equations:

\[
\begin{array}{c}
\begin{tikzpicture}[scale=0.6]
\draw[fill=white] (0,0) circle (0.2);
\draw[fill=white] (2,0) circle (0.2);
\draw[fill=white] (1,-1) circle (0.2);
\draw (0,0) -- (1,-1) -- (2,0);
\end{tikzpicture}
\end{array}
= \begin{array}{c}
\begin{tikzpicture}[scale=0.6]
\draw[fill=white] (0,0) circle (0.2);
\draw[fill=white] (2,0) circle (0.2);
\draw[fill=white] (1,-1) circle (0.2);
\draw (0,0) -- (1,-1) -- (2,0);
\end{tikzpicture}
\end{array}
\tag{19}
\]

Clearly the dagger of a \(\ast\)-homomorphism is a \(\ast\)-cohomomorphism.

If \(f\) is a \(\ast\)-homomorphism and is additionally unitary, we say that it is a unitary \(\ast\)-isomorphism. (It is easy to check that a unitary \(\ast\)-isomorphism is also a \(\ast\)-cohomomorphism.)

**2.5.2 Idempotent splitting**

Let \(\mathcal{C}\) be a presemisimple \(C^\ast\)-2-category, with its canonical unitary duals functor. Recall the definition of the dimension and trace in a pivotal dagger 2-category (Definition \(2.12\)).

Let \(X : r \to s\) be a 1-morphism, and let \([X^\ast, \eta, \epsilon]\) be the right dual defined by the unitary duals functor. By the \(C^\ast\)-axioms, \(\dim_L(X) = \eta^\dagger \circ \eta\) is a positive element of the commutative \(C^\ast\)-algebra \(\text{End}(\text{id}_s)\).

**Definition 2.22.** We call a 1-morphism \(X : r \to s\) in \(\mathcal{C}\) separable if \(\dim_L(X)\) is invertible. We write \(n_X := \sqrt{\dim_L(X)}\) for the positive square root and \(n_X^{-1}\) for its (positive) inverse.
Remark 2.23. In matrix notation (Remark 6.4) there is a $\ast$-isomorphism $\operatorname{End}(\text{id}_s) \cong \operatorname{End}(\text{id}_\vec{\tau})$ for some object $\vec{\tau}$ of $\operatorname{Mat}(\mathcal{C})$; up to permutation of the factors this isomorphism maps $\dim_L(X)$ to the matrix

$$\operatorname{diag}([\sum_k d(M_{k1}), \ldots, \sum_k d(M_{kn})])$$

where $M_{jk}$ are the entries of the 1-morphism matrix $M$ corresponding to $X$ under the equivalence $\Phi : \operatorname{Mat}(\mathcal{C}) \cong \mathcal{C}$. We see that $\dim_L(X)$ is invertible precisely when the matrix $M$ has no columns of zeros.

In the following diagrams we leave regions corresponding to the object $r$ unshaded and shade regions corresponding to the object $s$ with wavy lines.

Proposition 2.24. Let $r, s$ be objects of $\mathcal{C}$ and let $X : r \to s$ be a separable 1-morphism.

We define a pair of pants algebra on the object $X \otimes X^*$ of the rigid $\mathcal{C}^*$-tensor category $\operatorname{End}(r)$ by the following multiplication $m : (X \otimes X^*) \otimes (X \otimes X^*) \to X \otimes X^*$ and unit $u : \text{id}_r \to X \otimes X^*$:

This algebra is a SSFA in $\operatorname{End}(r)$.

Proof. That this is a Frobenius algebra is very easy to check (it just comes down to snake equations and isotopy) and we leave it to the reader. Separability is also clear:

For standardness, we require (Proposition 2.17) that for any morphism $T : X \otimes X^* \to X \otimes X^*$, with respect to the Frobenius cup and cap (16) the left trace is equal to the right trace. This comes down to the following equation:

For this we observe that the Frobenius cup and cap is simply the tensor product cup and cap on $X \otimes X^*$ (Proposition 2.3) which is standard.
We now define semisimplicity. This precisely corresponds to [CPJP21, Def. 3.34], except that our Frobenius algebras are standard.

**Definition 2.25.** Let $C$ be a presemisimple $C^*$-2-category. Let $r$ be an object of $C$. We say that a SSFA $A$ in $\text{End}(r)$ **splits** if there exists an object $s$ of $C$ and a separable 1-morphism $X : r \to s$ such that $A$ is unitarily $*$-isomorphic to the pair of pants algebra $X \otimes X^*$.

We say that $C$ is **semisimple** if, for every object $r$ of $C$, every SSFA in $\text{End}(r)$ splits.

### 3 Two semisimple completions of a rigid $C^*$-tensor category

Let $\mathcal{T}$ be a rigid $C^*$-tensor category. We are about to define two semisimple $C^*$-2-categories in which $\mathcal{T}$ embeds as the endomorphism category of a fixed object. We will then show that these two 2-categories are equivalent.

#### 3.1 The 2-category $\text{Bimod}(\mathcal{T})$

The following construction is identical to the constructions in [CPJP21, Def. 3.17][GY20, Notation 2.16], except that our Frobenius algebras are standard as well as separable.

**3.1.1 Definition**

In what follows let $\mathcal{T}$ be a rigid $C^*$-tensor category.

**Definition 3.1.** Let $A$ and $B$ be SSFAs in $\mathcal{T}$. A **left dagger $A$-module** is an object $M$ in $\mathcal{T}$ together with a morphism $\rho : A \otimes M \to M$ (the **left action**) fulfilling the following equations:

\[
\rho = \rho \\
\rho = \rho ^\dagger = \rho (23)
\]

A **right dagger $B$-module** is defined similarly, with a **right action** $\rho : M \otimes B \to M$ and the analogous equations. An $A \dashv B$-dagger bimodule is an object $M$ which is a left dagger $A$-module and a right dagger $B$-module, such that the left and right actions commute:

\[
\rho = : = (24)
\]

Every SSFA $A$ has a trivial $A \dashv A$-dagger bimodule $A_A$:

\[
\rho = = : = : = (25)
\]

**Definition 3.2.** A **bimodule homomorphism** $A_M B \to A_N B$ is a morphism $f : M \to N$ that commutes with the $A$-$B$ action:

\[
\rho = (26)
\]
Two dagger bimodules are \textit{(unitarily) isomorphic} if there is a (unitary) invertible bimodule homomorphism \( A_M B \to A_N B \).

Given two SSFAs \( A, B \), the \( A-B \) dagger bimodules and bimodule homomorphisms form a category which we write as \( A\text{-Mod-}B \). Left dagger \( A \)-modules and right dagger \( A \)-modules likewise form categories which we write as \( A\text{-Mod} \) and \( \text{Mod-}A \) respectively.

Since dagger idempotents in \( \mathcal{T} \) split, we can compose dagger bimodules \( A_M B \) and \( B_N C \) to obtain an \( A-C \)-dagger bimodule \( A_M \otimes_B N_C \), as follows. First we observe that the following endomorphism is a dagger idempotent (for this, we use that the Frobenius algebra \( B \) is separable):

\begin{equation}
\begin{align*}
\end{align*}
\end{equation}

The \textit{relative tensor product} \( A_M \otimes_B N_C \), or \textit{tensor product of bimodules}, is defined as the object obtained by splitting this idempotent. We depict the isometry \( i : M \otimes_B N \to M \otimes N \) as a downwards pointing triangle:

\begin{equation}
\begin{align*}
\end{align*}
\end{equation}

For dagger bimodules \( A_M B \) and \( B_N C \), the relative tensor product \( M \otimes_B N \) is itself an \( A-C \)-dagger bimodule with the following action \( A \otimes (M \otimes_B N) \otimes C \to M \otimes_B N \):

\begin{equation}
\begin{align*}
\end{align*}
\end{equation}

The relative tensor product is also defined on morphisms of bimodules. Let \( A_M B, A_M' B \) and \( B_N C, B_N' C \) be dagger bimodules and let \( f : A_M B \to A_M' B \) and \( g : B_N C \to B_N' C \) be bimodule homomorphisms. Then the relative tensor product \( f \otimes_B g : A_M \otimes_B N_C \to A_M' \otimes_B N_C' \) is a bimodule homomorphism defined as follows:

\begin{equation}
\begin{align*}
\end{align*}
\end{equation}

\textbf{Definition 3.3.} Let \( \mathcal{T} \) be a rigid \( C^* \)-tensor category. We define a \( C^* \)-2-category \( \text{Bimod}(\mathcal{T}) \) as follows:

- \textit{Objects}. Standard separable Frobenius algebras \( A, B, \ldots \) in \( \mathcal{T} \).
- \textit{Hom-categories}. \( \text{Hom}(A, B) := A\text{-Mod-}B \). (The \( C^* \)-norm is that of \( \mathcal{T} \).)
• **Horizontal composition.** Relative tensor product.

• **Associator.** For $M : A \to B$, $N : B \to C$, $O : C \to D$ the associator component $\alpha_{M,N,O}$ is defined as follows:

\[
\alpha_{M,N,O} : (M \otimes B)(N \otimes O) \to (M \otimes (N \otimes O))
\]

\[
\alpha_{M,N,O} = (\alpha_{M,N,O})_{M,N,O}
\]

\[
\alpha_{M,N,O} = (\alpha_{M,N,O})_{M,N,O}
\]

(31)

• **Identity 1-morphisms.** We define $\text{id}_A : A \to A$ to be the dagger bimodule $A_A$.

• **Unitors.** For $M : A \to B$ the left and right unitor components $\lambda_M$ and $\rho_M$ are defined as follows:

\[
\lambda_M : M \otimes A \to M
\]

\[
\rho_M : A \otimes M \to M
\]

(32)

We leave to the reader the straightforward checks that $\text{Bimod}(\mathcal{T})$ is indeed a well-defined $C^*$-2-category.

We observe that $\mathcal{T}$ embeds in $\text{Bimod}(\mathcal{T})$ as an endomorphism category.

**Proposition 3.4.** Let $\mathbb{1}$ be the trivial SSFA in $\mathcal{T}$. There is a unitary isomorphism of $C^*$-tensor categories $F : \mathcal{T} \cong \text{End}(\mathbb{1})$ defined as follows:

- Every object of $\mathcal{T}$ is taken to itself considered as a bimodule over the trivial SSFA.

- Every morphism of $\mathcal{T}$ is taken to itself considered as a bimodule homomorphism with respect to the actions of the trivial SSFA.

3.1.2 Semisimplicity

We will now show that $\text{Bimod}(\mathcal{T})$ is rigid. The definition of the right duals here is from [Yam04], which deals with the non-unitary case.

**Definition 3.5.** Let $A, B$ be SSFAs in $\mathcal{T}$ and let $A_B B$ be a dagger bimodule. We define the dual dagger bimodule $A_B(M^*)_A$ as follows. The underlying object of the bimodule is the dual object $M^*$ of $M$ in the rigid $C^*$-tensor category $\mathcal{T}$. The left $B$-action is defined as follows:

\[
\alpha_{M,N,O} : (M \otimes B)(N \otimes O) \to (M \otimes (N \otimes O))
\]

\[
\alpha_{M,N,O} = (\alpha_{M,N,O})_{M,N,O}
\]

(33)

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The right $A$-action is defined as follows:

The following lemma shows that one can equally well express these actions in terms of the left cup and cap in the pivotal dagger category $T$.

**Lemma 3.6.** Let $A, B$ be SSFA's in $C$ and let $AM_B$ a dagger bimodule. Then the following equations hold:

\[
B = B
\]

\[
A = A
\]

**Proof.** We show the second equation; the proof of the first is similar. Since $A$ is standard, by Proposition 2.4 there exists a unitary $U: A^* \rightarrow A$ such that the following equation is satisfied:

\[
U = U
\]

We then have the following sequence of equalities (where we offset the edge of the module action box in order to make the transpose visible):

\[
A = A
\]

We now prove that these maps are indeed dagger module actions.

**Proposition 3.7.** The maps $(33)$ and $(34)$ give $M^*$ the structure of a left dagger $B$-module and a right dagger $A$-module respectively.
Proof. We provide the proof for the right $A$-action; the proof for the left $B$-action is similar.

\[ B B \rightarrow B (M^*) \otimes_A M_B \]

and a cap 2-morphism $\epsilon_{A_M_B} : A_M \otimes_B (M^*)_A \rightarrow A A_A$ witnessing the duality:

\[ \text{Proposition 3.8. Bimod}(T) \text{ is rigid; in particular, } [B(M^*)_A, \eta_{A_M_B}, \epsilon_{A_M_B}] \text{ is a right dual for } A M_B. \]

Proof. It is straightforward to check that the cup and cap (41) are bimodule homomorphisms. We also need to check that the snake equations (3) are satisfied. We show the second of those equations

\[ (38) \]

\[ (39) \]

\[ (40) \]
Here the first equality is by definition of the action for the dual bimodule; the second equality is by
snake equations, the dagger bimodule equations and separability of the Frobenius algebras $A$ and $B$;
and the third equality is by commutativity of the left and right module actions, the dagger module
equations and separability of the Frobenius algebras $A$ and $B$.

Remark 3.9. In (42) we omitted the triangles (28) in order to keep the size of the diagrams reasonable;
the reader may insert them, and will observe that they cancel using separability of the Frobenius algebra
and the following equations:

We prove the first equation of (43), the other is shown similarly:

Here the last equality is by two snake equations, the dagger module equations and separability of the
Frobenius algebra $A$.

We now show the other aspects of semisimplicity.

Definition 3.10. We say that an SSFA $A$ in $\mathcal{T}$ is simple if the $C^*$-algebra $\text{End}(AA)$ of bimodule
endomorphisms of the identity $A - A$ bimodule is one-dimensional.
Lemma 3.11. Every SSFA in $\mathcal{T}$ may be decomposed as a direct sum of simple SSFAs.

Proof. Let $A$ be a SSFA. Take a complete family of minimal orthogonal projections $\{p_i\}$ in the commutative finite-dimensional $C^*$-algebra $\text{End}(A_A)$, split the idempotents to obtain factors $\{A_i\}$ with isometries $\iota_i : A_i \to A$ such that $\iota_i \circ \iota_i^\dagger = p_i$, then define the structure of an SSFA on $A_i$ by:

$$m_i = \iota_i^\dagger \circ m \circ (\iota_i \otimes \iota_i) \quad \quad u_i = \iota_i^\dagger \circ u$$

It is easy to check that the $A_i$ are Frobenius algebras and that $A \cong \bigoplus_i A_i$; simplicity of the $A_i$ follows from minimality of the projections $\{p_i\}$. This was already shown in [NY18, Lem. 2.8]. We now need only show that the $A_i$ are standard and separable. For separability:

$$m_i \circ m_i^\dagger = \iota_i^\dagger \circ m \circ (p_i \otimes \epsilon) \circ m^\dagger \circ \iota_i = \iota_i^\dagger \circ m \circ m^\dagger \circ \iota_i = \text{id}_{A_i}$$

Here the second equality uses the fact that $p_i$ is a bimodule morphism to pull the $p_i$ through the multiplication $m$ and cancel it with $\iota_i^\dagger$; the third equality uses separability of $A$ and the fact that $\iota_i$ is an isometry.

Finally we show that the $A_i$ are standard. For this, we need to show that, for any $f \in \text{End}(A_i)$:

$$\text{Tr}[u_i^\dagger \circ m_i \circ (f \otimes \text{id}_{A_i}) \circ m_i^\dagger \circ u_i] = \text{Tr}[u_i^\dagger \circ m_i \circ (\text{id}_{A_i} \otimes f) \circ m_i^\dagger \circ u_i]$$

We show this as follows:

$$\text{Tr}[u_i^\dagger \circ m_i \circ (f \otimes \text{id}_{A_i}) \circ m_i^\dagger \circ u_i] = \text{Tr}[u_i^\dagger \circ m_i \circ (\iota_i f \iota_i^\dagger \otimes \epsilon) \circ m^\dagger \circ \iota_i \circ u]$$

$$= \text{Tr}[u_i^\dagger \circ m \circ (\iota_i f \iota_i^\dagger \otimes \text{id}_{A_i}) \circ m^\dagger \circ \iota_i \circ u]$$

$$= \text{Tr}[u_i^\dagger \circ m \circ (\iota_i \otimes \iota_i f \iota_i^\dagger) \circ m^\dagger \circ u]$$

Here the first equality uses the definition of the multiplication and unit of $A_i$; the second equality uses that $p_i$ is a bimodule morphism to bring the $p_i$ next to the $\iota_i$ and $\iota_i^\dagger$, where they disappear; the third equality is of standardness of $A$; and for the final equality one simply repeats the process in the opposite direction.

Proposition 3.12. $\text{Bimod}(\mathcal{T})$ is semisimple.

Proof. We showed in Proposition 3.8 that $\text{Bimod}(\mathcal{T})$ is rigid.

There is clearly a direct sum of bimodules and a zero bimodule yielding local additivity. For an $A$-$B$ bimodule $X$ and $p \in \text{End}(X)$ it is straightforward to define an $A$-$B$ bimodule structure on the splitting of the idempotent $p$; local idempotent completeness follows. Then observe that every endomorphism algebra in $\text{Bimod}(\mathcal{T})$ is finite-dimensional, since it is a subalgebra of an endomorphism algebra in $\mathcal{T}$. $\text{Bimod}(\mathcal{T})$ is therefore locally semisimple.

It is straightforward to check that the direct sum of SSFAs in $\mathcal{T}$ is a direct sum of objects in $\text{Bimod}(\mathcal{T})$ in the sense of Definition 2.15. The existence of a zero object is clear. That every object can be decomposed as a finite direct sum of simple objects is the content of Lemma 3.11.

Finally, idempotent splitting is shown in [CPJP21, Cor. 3.37].

Remark 3.13. Let $A_M$ be a bimodule. In this section we defined a right dual bimodule $\rho(M^*)_A$. With the cup and cap specified in [41], this is in general not a standard right dual in $\text{Bimod}(\mathcal{T})$. However, it is straightforward to define a normalised cup and cap $\eta, \epsilon$ so that $[\rho(M^*)_A, \eta, \epsilon]$ is a standard right dual for $A_M$. We leave the details to the reader.
3.2 The 2-category $\text{Mod}(\mathcal{T})$

We now define the second semisimple $C^*$-2-category in which $\mathcal{T}$ embeds.

**Definition 3.14.** A semisimple left $\mathcal{T}$-module category is a semisimple $C^*$-category $\mathcal{M}$ together with:

- A unitary linear bifunctor $\tilde{\otimes} : \mathcal{T} \times \mathcal{M} \to \mathcal{M}$.
- Unitary natural isomorphisms $l_X : 1 \tilde{\otimes} X \cong X$ and $m_{U,V,X} : (U \otimes V) \tilde{\otimes} X \cong U \tilde{\otimes} (V \tilde{\otimes} X)$ satisfying analogues of the pentagon and triangle equations [Ost03, Def. 6].

A semisimple right $\mathcal{T}$-module category can be defined analogously.

Following [ADC15, NY18], we say that the module category $\mathcal{M}$ is cofinite (a.k.a. proper) if for any $X,Y \in \mathcal{M}$ we have $\text{Hom}_\mathcal{M}(X,U_i \tilde{\otimes} Y) = 0$ for all but finitely many $i$, where $\{U_i\}$ are representatives of the isomorphism classes of simple objects in $\mathcal{T}$.

**Definition 3.15.** Let $\mathcal{M}_1, \mathcal{M}_2$ be semisimple left $\mathcal{T}$-module categories. A unitary $\mathcal{T}$-module functor $\mathcal{M}_1 \to \mathcal{M}_2$ is a unitary linear functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ together with a unitary natural isomorphism $c_{U,X} : F(U \tilde{\otimes} X) \to U \tilde{\otimes} F(X)$; the $\{c_{U,X}\}$ must satisfy certain coherence equations [Ost03, Def. 7].

**Definition 3.16.** Let $F,G : \mathcal{M}_1 \to \mathcal{M}_2$ be unitary $\mathcal{T}$-module functors. We say that a natural transformation $\eta : F \to G$ is a morphism of $\mathcal{T}$-module functors if the following diagram commutes for any $U \in \mathcal{T}, X \in \mathcal{M}_1$:

$$
\begin{array}{ccc}
F(U \tilde{\otimes} X) & \xrightarrow{c_{U,X}} & U \tilde{\otimes} F(X) \\
\downarrow{\eta_{U\tilde{\otimes}X}} & & \downarrow{\text{id}_U \otimes \eta_X} \\
G(U \tilde{\otimes} X) & \xrightarrow{c_{U,X}} & U \tilde{\otimes} G(X)
\end{array}
$$

It is straightforward to define a notion of direct sum for cofinite semisimple $\mathcal{T}$-module categories (see e.g. [EGNO16, Prop. 7.3.4]).

**Definition 3.17.** We say that a $\mathcal{T}$-module category is indecomposable if it is not equivalent to a nontrivial direct sum of $\mathcal{T}$-module categories. We say that a $\mathcal{T}$-module category is finitely decomposable if it is equivalent to a finite direct sum of indecomposable module categories.

**Definition 3.18.** The (strict) 2-category $\text{Mod}(\mathcal{T})$ is defined as follows:

- **Objects.** Cofinite semisimple finitely decomposable left $\mathcal{T}$-module categories.
- **1-morphisms.** Unitary $\mathcal{T}$-module functors.
- **2-morphisms.** Morphisms of $\mathcal{T}$-module functors.

It is straightforward to show that $\text{Mod}(\mathcal{T})$ is an additive $\mathbb{C}$-linear dagger 2-category in the sense of Section 2.4. Semisimplicity and rigidity will follow from Proposition 3.12 and Theorem 3.21.

3.3 Equivalence of the completions

We now observe that the 2-categories we have defined are equivalent.

**Definition 3.19.** We define a unitary $\mathbb{C}$-linear 2-functor $\Psi : \text{Bimod}(\mathcal{T}) \to \text{Mod}(\mathcal{T})$ as follows:
• **On objects:** The SSFA \( A \) is mapped to its category of right dagger bimodules \( \text{Mod}^\mathcal{T}A \), considered as a left \( \mathcal{T} \)-module category under the following action:

\[
U \tilde{\otimes} X_A := U \otimes X_A \\
(f \tilde{\otimes} g) := f \otimes g
\]

• **On 1-morphisms:** A dagger bimodule \( \alpha M_B \) is mapped to the unitary \( \mathcal{T} \)-module functor \( \text{Mod}^\mathcal{T}A \to \text{Mod}^\mathcal{T}B \) given by relative tensor product, i.e.

\[
X_A \mapsto X \otimes \alpha M_B \\
(f : X_A \to Y_A) \mapsto f \otimes \text{id}_{\alpha M_B}
\]

The unitary natural isomorphism \( \{ c_{U,X} \} \) is defined using the isometries of the relative tensor product (28).

• **On 2-morphisms:** A bimodule homomorphism \( f : \alpha M_B \to \alpha N_B \) is mapped to a natural isomorphism of module functors whose components are as follows:

\[
(id_X \otimes \alpha f) : X \otimes \alpha M_B \to X \otimes \alpha N_B
\]

• **Multiplicator and unitor:** Defined using the associator and right unitor of \( \text{Bimod}(\mathcal{T}) \).

**Remark 3.20.** We leave to the reader the straightforward checks that \( \Psi \) is indeed a well-defined unitary \( \mathbb{C} \)-linear 2-functor. It is necessary to show in particular that the left \( \mathcal{T} \)-module category \( \text{Mod}^\mathcal{T}A \) is semisimple and cofinite. Semisimplicity follows from local semisimplicity of \( \text{Bimod}(\mathcal{T}) \) (Proposition 3.12). For cofiniteness observe that rigidity of \( \text{Bimod}(\mathcal{T}) \) implies a linear isomorphism between the vector spaces \( \text{Hom}_{\text{Mod}^\mathcal{T}A}(X_A, U_i \otimes Y_A) \) and \( \text{Hom}_\mathcal{T}(X \otimes A Y^*, U_i) \); cofiniteness follows by semisimplicity of \( \mathcal{T} \). (This was already observed in the proof of [NY18, Thm. 3.2].)

**Theorem 3.21.** The 2-functor \( \Psi \) is an equivalence.

**Proof.** This result is well-known in the case of a fusion category \( \mathcal{T} \) (where there are finitely many simple objects) and a proof was sketched in [CPJP21, Ex. 3.39]. The proof in the general case is very similar. For the reader’s convenience we provide the proof in full in Appendix 6.2.

**Corollary 3.22.** The category \( \text{Mod}(\mathcal{T}) \) is a semisimple \( \mathbb{C}^* \)-2-category.

It also follows that \( \mathcal{T} \) embeds in \( \text{Mod}(\mathcal{T}) \) as an endomorphism category, just as for \( \text{Bimod}(\mathcal{T}) \) (Proposition 3.4).

**Corollary 3.23.** Let \( \text{End}_\mathcal{T}(\mathcal{T}) \) be the category of endomorphisms of the \( \mathcal{T} \)-module category \( \mathcal{T} \). There is an equivalence of \( \mathbb{C}^* \)-tensor categories \( \mathcal{T} \to \text{End}_\mathcal{T}(\mathcal{T}) \) defined by composing the equivalence \( F \) of Proposition 3.4 with the equivalence \( \Psi_{1,1} : 1 \text{-Mod} 1 \to \text{End}_\mathcal{T}(\mathcal{T}) \).

Finally, we observe that this result allows us to characterise semisimple \( \mathbb{C}^* \)-2-categories in general.

**Definition 3.24.** We say that a semisimple \( \mathbb{C}^* \)-2-category is **connected** if the Hom-category between any pair of nonzero objects is not the terminal category.

**Proposition 3.25** ([GY20, Lem. 2.2.3]). For any object \( r \) of a connected semisimple \( \mathbb{C}^* \)-2-category \( \mathcal{C} \), there is an equivalence \( \mathcal{C} \simeq \text{Bimod}(\text{End}(r)) \).

**Proof.** The equivalence \( \Delta : \mathcal{C} \to \text{Bimod}(\text{End}(r)) \) is defined as follows.

• **On objects.** For every nonzero object \( s \) of \( \mathcal{C} \), pick a separable 1-morphism \( P_s : r \to s \) in \( \mathcal{C} \). Then define \( \Delta(s) := P_s \otimes (P_s)^* \), where \( P_s \otimes (P_s)^* \) is the pair of pants SSFA in \( \text{End}(r) \) corresponding to \( P_s \).
• **On 1-morphisms.** For every 1-morphism \( X : s \to t \) in \( C \), define \( \Delta(X) := P_s \otimes X \otimes (P_t)^* \), which is an \( \Delta(s) \cdot \Delta(t) \) dagger bimodule with the following action (here and throughout the proof we leave regions corresponding to the object \( r \) unshaded, we shade regions corresponding to the object \( s \) with wavy lines, and we shade regions corresponding to the object \( t \) with polka dots):

\[
\begin{array}{ccc}
& & \Delta(t) \\
\Delta(s) & - & \Delta(t) \\
& & \Delta(s)
\end{array}
\]

\[ (45) \]

• **On 2-morphisms.** For every 2-morphism \( f : X \to Y \) in \( C \), we define \( \Delta(f) := \text{id}_{P_s} \otimes f \otimes \text{id}_{(P_t)^*} \).

• **Multiplier.** Let \( X : s \to t \), \( Y : t \to u \) be 1-morphisms in \( C \). Then we define the multiplicator component \( \mu_{X,Y} : \Delta(X) \otimes \Delta(Y) \to \Delta(X \otimes Y) \) as the following bimodule homomorphism (we shade regions corresponding to the object \( u \) with a checkerboard effect):

\[
\begin{array}{ccc}
& & \Delta(u) \\
\Delta(s) & - & \Delta(u) \\
& & \Delta(s)
\end{array}
\]

\[ (46) \]

• **Unitor.** Trivial (up to unitors/associators in \( C \)).

It is straightforward to check that \( \Delta \) is a \( C \)-linear unitary 2-functor. It was shown in [GY20, Lem. 2.2.3] that it is furthermore a local equivalence. The only additional thing we must prove is essential surjectivity on objects. For any SSFA \( A \) in \( \text{End}(r) \), we must show that there exists an object \( s \) of \( C \) such that \( \Delta(s) \) is Morita equivalent to \( A \). Since \( C \) is semisimple, there certainly exists an object \( s \) and a 1-morphism \( X : r \to s \) such that \( A \cong (X \otimes X^*) \). Now \( \Delta(s) = P_s \otimes (P_s)^* \), where \( P_s : r \to s \) is the 1-morphism chosen in the definition of the pseudofunctor \( \Delta \). We claim that \( X \otimes X^* \) is Morita equivalent to \( P_s \otimes (P_s)^* \). This follows from [MRV19, Thm. A.1], which implies that two 1-morphisms \( r \to s, r \to t \) produce Morita equivalent SSFAs in \( \text{End}(r) \) iff \( s \) and \( t \) are equivalent objects in \( C \).[2]

**Corollary 3.26.** Every connected semisimple \( C^* \)-2-category is equivalent to \( \text{Mod}(T) \) for some rigid \( C^* \)-tensor category \( T \).

4 A covariant Stinespring theorem

In the last section we defined two equivalent semisimple 2-categories in which a rigid \( C^* \)-tensor category \( T \) embeds as the endomorphisms of a fixed object. We will now apply this to the study of finite-dimensional \( G \)-\( C^* \)-algebras and covariant completely positive maps.

[2] To be precise, the cited theorem classifies morphisms into \( r \) rather than morphisms out of \( r \). However, the proof works equally well for morphisms out of \( r \); just read the diagrams from left to right.
4.1 A classification of finite-dimensional $G$-$C^*$-algebras

We will first briefly recall how finite-dimensional $G$-$C^*$-algebras (a.k.a. $C^*$-dynamical systems) for a compact quantum group $G$ may be identified with SSFAs in the category Rep($G$) of finite-dimensional continuous unitary representations of $G$. This characterisation already appeared in [NY18]; other relevant works include [Vic11, BKLR15, Ban99]. See [Ver20a, §2.1.4] for a more thorough summary.

We consider first of all the familiar notion of a finite-dimensional $G$-$C^*$-algebra, or $C^*$-dynamical system, for an ordinary compact group $G$. Let $A$ be a finite-dimensional $C^*$-algebra. An action of a compact group $G$ on $A$ is a continuous homomorphism $\tau : G \to \text{Aut}(A)$, where $\text{Aut}(A)$ is the group of $*$-automorphisms of $A$. For any such action there is a canonical invariant trace $\phi : A \to \mathbb{C}$ which is preserved under the $G$-action in the sense that $\phi(\tau(g)(x)) = \phi(x)$ for all $x \in A$.

The canonical invariant trace $\phi$ induces an inner product $\langle x|y \rangle := \phi(x^*y)$ on the finite-dimensional complex vector space underlying the $C^*$-algebra $A$, which thus acquires the structure of a Hilbert space. It is not hard to show (see e.g. [Ver20a, §3.1]) that the $C^*$-algebra structure of $A$ further induces the structure of an SSFA on the Hilbert space $A$; the multiplication and unit of the SSFA are precisely the multiplication and unit of the $C^*$-algebra, and the counit of the SSFA is the trace $\phi$. Since $\phi$ is preserved under the action $\tau$ of $G$, this action induces a continuous unitary representation of $G$ on the Hilbert space $A$ such that the structure morphisms of the SSFA — the multiplication $m : A \otimes A \to A$ and the unit $u : \mathbb{C} \to A$ — are intertwiners. From a $G$-$C^*$-algebra for a compact group $G$ we have therefore constructed an SSFA in the rigid $C^*$-tensor category Rep($G$) of finite-dimensional continuous unitary representations of $G$. In the other direction, every SSFA in Rep($G$) has a natural involution such that the resulting $*$-algebra is a $G$-$C^*$-algebra. These constructions are inverse and set up a bijective correspondence between $*$-isomorphism classes of $G$-$C^*$-algebras and unitary $*$-isomorphism classes of SSFAs in Rep($G$). An SSFA in Rep($G$) is therefore a $G$-$C^*$-algebra equipped with its canonical invariant trace.

We extend the notion of symmetry by generalising from representation categories of compact groups to rigid $C^*$-tensor categories $T$ with simple unit object equipped with a faithful unitary linear functor $F : T \to \text{Hilb}$, called a fibre functor. By Tambara-Krein-Woronowicz (T-K-W) duality [NT13 Thm. 2.3.2], these are precisely the categories Rep($G$) of finite-dimensional continuous unitary representations of compact quantum groups $G$, equipped with their canonical fibre functor. We define a finite-dimensional $G$-$C^*$-algebra for a compact quantum group $G$ to be an SSFA in the rigid $C^*$-tensor category Rep($G$). To recover a concrete $C^*$-algebra from such an SSFA $A$ one considers the object $F(A)$, which is a Hilbert space with the structure of a separable Frobenius algebra; this algebra possesses a natural involution with a $C^*$-norm. We remark that, in this more general case, the canonical invariant functional on $A$ (that is, the counit of the SSFA) may not be tracial as a concrete functional on the $G$-$C^*$-algebra $F(A)$; in fact, it is only tracial when $d(A) = \dim(F(A))$ [Ver20a, Thm. 5.3]. This causes no problems, provided one is content to move from completely positive trace-preserving maps to completely positive functional-preserving maps in this more general setting.

Finally, we could generalise still further and consider SSFAs in general rigid $C^*$-tensor categories. In this case there is no obvious way to identify these SSFAs with concrete $C^*$-algebras, since they do not have an associated vector space in general; however, the results we are about to obtain apply at this level of generality.

Before considering channels, we will draw one straightforward consequence of what has already been proven: namely, a classification of finite-dimensional $G$-$C^*$-algebras for a compact quantum group $G$, which amounts to a classification of SSFAs in Rep($G$). In fact, we will work in the most general setting and classify SSFAs in $T$ for a rigid $C^*$-tensor category $T$, whether or not a fibre functor exists.\footnote{For a very elementary algebraic perspective on the definition of a compact quantum group and its finite-dimensional unitary representation theory, see [Ver20a, §2.1.4] (which borrows heavily from the more sophisticated presentations in [Tim08, NTR13]. Here we will not even need to define a compact quantum group, since we already know what a rigid $C^*$-tensor category with fibre functor is.}

\footnote{There are other, less abstract definitions of a $G$-$C^*$-algebra, e.g. [Van98]; this definition is equivalent, as was observed in [Ban99, NY18]. Indeed, given an SSFA $A$, using the canonical fibre functor $F : \text{Rep}(G) \to \text{Hilb}$ one recovers (by T-K-W duality) a coaction of the Hopf $*$-algebra $A_G$ associated to $G$ on the f.d. $C^*$-algebra $F(A)$ [Ver20a, Prop. 3.2.4].}
This classification is certainly not new (see e.g. [DCY12, Nes13, NY18]), although our version seems to offer some additional precision regarding the equivalence relation on objects of a module category corresponding to unitary ∗-isomorphism of the associated SSFAs.

**Definition 4.1.** We say that SSFAs in $T$ are **Morita equivalent** if they are equivalent as objects of $\text{Bimod}(T)$.

**Lemma 4.2.** Let $A, \tilde{A}$ be SSFAs in $T$. Following Proposition 3.4, we embed $T \cong \text{End}(1)$ in $\text{Bimod}(T)$. Then $A$ is Morita equivalent to $\tilde{A}$ if and only if $A$ is unitarily ∗-isomorphic to a pair of pants algebra $M \otimes M^*$ for some separable 1-morphism $M : 1 \to \tilde{A}$ in $\text{Bimod}(T)$.

**Proof.** Only if. We will show that from a Morita equivalence $A \cong \tilde{A}$ we can construct a satisfactory 1-morphism $M : 1 \to A$. In the following diagrams we leave the regions corresponding to the object 1 unshaded; we shade the regions corresponding to the object $A$ with wavy lines; and we shade the regions corresponding to the object $\tilde{A}$ with polka dots.

Let $[E, E^{-1}, \alpha_E, \beta_E]$ be the data associated to an adjoint equivalence $E : A \to \tilde{A}$ in $\text{Bimod}(T)$. We will first consider the relationship between the right dual $[E^{-1}, \alpha_E, \beta_E]$ for $E$ and the standard right dual $[E^*, \eta_E, \epsilon_E]$. Let $v : E^* \to E^{-1}$ be the isomorphism relating the right duals $E^*$ and $E^{-1}$ by Proposition 2.4, i.e.:

\begin{align}
\alpha_E &= v \\
\beta_E &= v^{-1}
\end{align}

(Here we drew the $E^{-1}$ wire with a triangular downwards-pointing arrow.) We first observe that

\begin{equation}
v^\dagger = v^{-1} \otimes \text{dim}_R(E),
\end{equation}

which can be seen by the following equation:

\begin{align}
\alpha_E &= v \\
\beta_E &= v^{-1}
\end{align}

In the same way it can be shown that

\begin{equation}
(v^{-1})^\dagger = \text{dim}_L(E) \otimes v,
\end{equation}

and therefore that $\text{dim}_L(E) \otimes \text{id}_{E^{-1}} \otimes \text{dim}_R(E) = \text{id}_{E^{-1}}$. It follows that $E$ is a separable 1-morphism. We also make the following further observation for later:

\begin{align}
\alpha_E &= v \\
\beta_E &= v^{-1}
\end{align}
Let $1_A : \mathbb{I} \to A$ be the 1-morphism in Bimod($C$) corresponding to $A$ considered as a $\mathbb{I}$-$A$ bimodule, and let $A_1 : A \to \mathbb{I}$ be the 1-morphism corresponding to $A$ considered as an $A$-$\mathbb{I}$ bimodule. We will show that $A_1$ is a standard right dual for $1_A$ with a cup and cap we will now define.

Recall from Lemma 3.11 that $A \cong \bigoplus_k A_k$, where $A_k$ are simple SSFAs, and let $\iota_k : A_k \to A$ be the isometric injections. In the following diagrams we draw the $\iota_k$ as downward-pointing triangles, the $\iota_k^*$ as upward-pointing triangles, and the structure morphisms of the algebra $A_k$ as white circles with a $k$ next to them. We define the cup and cap $\eta$, $\epsilon$ as follows (on the LHS of the following definitions is the cup/cap as it appears in Bimod($T$), and on the RHS is the definition as a concrete bimodule homomorphism in $T$):

\[
\begin{align*}
\eta &:= \sum_k \frac{d(A_k)^{1/4}}{4} A_1 \\
\epsilon &:= \sum_k \frac{d(A_k)^{-1/4}}{2} A \otimes A
\end{align*}
\]  

(51)

It is straightforward to check by considering the underlying morphisms in $T$ that this cup and cap obey the snake equations (3). We will now show that the duality is standard. Let $T \in \text{End}(1_A)$.

Clearly $\epsilon \circ (T \otimes \text{id}_{A_1}) \circ \epsilon^*$ is the following concrete morphism in $T$:

\[
\begin{align*}
\sum_k d(A_k)^{-1/2}
\end{align*}
\]  

(52)

In the notation of Proposition 2.17 we therefore see that $\psi_{1_A}(T) = \sum_k d(A_k)^{-1/2} \psi_{A_k}[\iota_k^* \circ T \circ \iota_k]$, where $\psi_{A_k} : \text{End}(A_k) \to \mathbb{C}$ is the trace defined by the standard duality on $A_k$ in $T$.

On the other hand, it is also clear that $\tilde{\eta}^* \circ (\text{id}_{A_1} \otimes T) \circ \tilde{\eta}$ is the following concrete morphism in $T$:

\[
\begin{align*}
\sum_k d(A_k)^{1/2}
\end{align*}
\]  

(53)
Since the $A_k$ are simple, as an $A_k$-$A_k$ bimodule endomorphism, $m_k \circ (\text{id}_{A_k} \otimes (\iota_k^1 \circ T \circ \iota_k)) \circ m_k = \alpha_{k,T} \text{id}_{A_k}$ for some scalars $\alpha_{k,T}$, so, in the notation of Proposition 2.17, $\phi_{1,A_A}(T) = \sum_k d(A_k)^{1/2} \alpha_{k,T}$. But now, taking the trace of $\alpha_{k,T} \text{id}_{A_k}$ in $T$, we obtain the following equation:

$$\alpha_{k,T} \cdot d(A_k) = \psi_{A_k}[\iota_k^1 \circ T \circ \iota_k].$$  \hfill (54)

Here for the second equality we used separability and standardness of $A_k$. It follows that $\phi_{1,A_A}(T) = \sum_k d(A_k)^{1/2} \alpha_{k,T} = \sum_k d(A_k)^{-1/2} \psi_{A_k}[\iota_k^1 \circ T \circ \iota_k] = \psi_{A_A}(T)$,

so the right dual $[A_A, \eta, \epsilon]$ is indeed standard. We observe in particular that $\text{dim}_L(1_{A_A}) = \sum_k d(A_k)^{1/2} (\iota_k \circ \iota_k^1)$; $1_{A_A}$ is therefore a separable 1-morphism, with $n_{1_{A_A}} = \sum_k d(A_k)^{1/4} (\iota_k \circ \iota_k^1)$.

Now we set $M := 1_{A_A} \otimes E : 1 \to \hat{A}$. Using the fact that the tensor product of standard duals (Proposition 2.3) is standard it is easy to see that

$$n_M = n_E \otimes (\alpha_E^1 \circ (\text{id}_{E^{-1}} \otimes n_{1_{A_A}} \otimes \text{id}_E) \circ \alpha_E).$$

It follows that $M$ is a separable 1-morphism.

We claim that there is a unitary $\ast$-isomorphism between $A$ and the pair of pants SSFA $M \otimes M^\ast$. It does not matter which standard right dual $M^\ast$ we pick in defining the pair of pants algebra $M \otimes M^\ast$ — they will all produce unitarily $\ast$-isomorphic SSFAs — so we pick the tensor product dual $E^\ast \otimes A_{A_1}$.

This yields an SSFA on the object $(1_{A_A} \otimes E) \otimes (E^\ast \otimes A_{A_1})$ with the following multiplication and unit:

$$n_{1_{A_A}} \circ \iota_{E^\ast}$$  \hfill (55)

We will show that the following map $f : (1_{A_A} \otimes E) \otimes (E^\ast \otimes A_{A_1}) \to A$ is a unitary $\ast$-isomorphism:

$$n_{1_{A_A}} \circ \iota_{E^\ast}$$  \hfill (56)

Here the 2-morphism $\hat{m} : 1_{A_A} \otimes A_{A_1} \to A$ represented by a white circle is concretely the following morphism in $T$, where this time the white circle represents the multiplication $m : A \otimes A \to A$ of the
Frobenius algebra $A$, as usual:

It is easy to check that $\tilde{m}$ is unitary by separability of the Frobenius algebra, the Frobenius equation and (28).

We first need to show that $f$ is a $\ast$-homomorphism. We will begin with multiplicativity (the first equation of (18)). We will need the following equation in $\text{Bimod}(\mathcal{T})$, which can be straightforwardly checked by considering the underlying morphisms in $\mathcal{T}$:

We also need the following equation, which can again be straightforwardly checked by considering the underlying morphisms in $\mathcal{T}$:

(Here the morphism $1_{AA} \to A \otimes A$ represented by a white circle is concretely just the comultiplication of the Frobenius algebra $A$.)

Now we prove multiplicativity:

30
Here the first equality is clear; the second equality is by (58); the third equality is by unitarity of $\hat{m}$; the fourth equality is by (59); the fifth equality is by the pivotal dagger structure on Bimod($\mathcal{T}$); and the sixth equality is by (50).

Unitality (the second equation of (18)) is shown by the following equalities:

Here the first equality can be seen by inserting $v^{-1} \circ v$ on the $E^*$-wire and using (49). The second equality can straightforwardly be seen by considering the underlying morphisms in $\mathcal{T}$.

For $*$-preservation (the third equation of (18)), we have the following equalities:
Here the first equality is by the pivotal dagger structure on $\text{Bimod}(\mathcal{T})$; the second equality is by (58) (or more precisely, (58) precomposed by $\tilde{m} \otimes \tilde{m}^\dagger$); and the final equality is by a snake equation for $\tilde{A}$. We have shown that $f$ is a $\ast$-homomorphism. Now we need only show that $f$ is unitary:

\begin{align*}
(66)
= & \\
= & \\
(67)
= & \\
(68)

Here in the first line the first equality is by unitarity of $\tilde{m}$ and the second equality is by (50); in the second line the first equality is by the same argument as in (64) and the second equality is by unitarity of $\tilde{m}$. The proof of the ‘only if’ direction is complete.

If. We now show the opposite implication: if there exists a separable 1-morphism $1_{\tilde{M}} : 1 \to \tilde{A}$ and a unitary $\ast$-isomorphism $f : 1_{\tilde{M}} \otimes (1_{\tilde{M}})^\ast \to A$, then $A$ and $\tilde{A}$ are Morita equivalent.

We first observe that the right $A$-dagger module $M_{\tilde{A}}$ is in fact an $A$-$\tilde{A}$-dagger bimodule by the following left $A$-action:

\begin{align*}
(69)

32
We therefore obtain a 1-morphism \( A\tilde{M}_A : A \to \tilde{A} \) in \( \text{Bimod}(\mathcal{T}) \). We will show that it is an equivalence with weak inverse \((A\tilde{M}_A)^*\), proving that \( A \) and \( \tilde{A} \) are Morita equivalent. For this we need to produce unitary 2-morphisms \( A\tilde{A} \to A\tilde{M}_A \otimes (A\tilde{M}_A)^* \) and \( \tilde{A}A \to (A\tilde{M}_A)^* \otimes A\tilde{M}_A \).

The following equalities show that \( f^\dagger : A \to 1\tilde{M}_A \otimes (1\tilde{M}_A)^* \) is in fact an \( A\)-\( A \) bimodule morphism:

\[
\begin{align*}
\text{Here in the first diagram the right } A \text{-action on } (1\tilde{M}_A)^* \text{ is defined in terms of the left } A \text{-action on } 1\tilde{M}_A \text{ as in } \text{(35)}, \text{ using the fact, communicated in Remark 3.13, that } \tilde{A}(M^*)_A \text{ is a standard right dual bimodule for } A\tilde{M}_A. \text{ For the first equality we used isotopy of the diagram, and for the second equality we used the first } \ast \text{-homomorphism condition } (18). \text{ We have therefore found the first desired unitary 2-morphism, } f^\dagger : A \to A\tilde{M}_A \otimes (A\tilde{M}_A)^*. \text{ We will now obtain the second. Let } x := \sqrt{\dim L(A\tilde{M}_A)} \in \text{End}(id_{\tilde{A}\tilde{A}}), \text{ and let } \epsilon : A\tilde{M}_A \otimes (A\tilde{M}_A)^* \to AA_A \text{ be the cap of the standard duality in } \text{Bimod}(\mathcal{T}). \text{ The following equations show that the 2-morphism } \\
\epsilon \circ (id_{A\tilde{M}_A} \otimes x \otimes id_{(A\tilde{M}_A)^*}) \text{ is unitary:}
\end{align*}
\]

\[
\begin{align*}
\text{Here for the last equality of } (72) \text{ we used } (71). \text{ We now show that } \dim L(A\tilde{M}_A), \text{ and therefore also } x, \text{ is invertible. Indeed, by assumption, } \\
\dim L(1\tilde{M}_A) = \dim L(1A_A \otimes A\tilde{M}_A) \text{ is invertible. By Remark 6.4 up to permutation of the factors we have the following expressions for left dimensions in the commutative } C^\ast \text{-algebra } \text{End}(id_{\tilde{A}}):
\end{align*}
\]

\[
\begin{align*}
\dim L(1A_A \otimes A\tilde{M}_A) &= \left[ \sum_i d(A_i)^1/2 d(M_{i1}), \ldots, \sum_i d(A_i)^1/2 d(M_{i\nu_r}) \right] \\
\dim L(A\tilde{M}_A) &= \left[ \sum_i d(M_{1i}), \ldots, \sum_i d(M_{\nu_r}) \right]
\end{align*}
\]

\[
\text{Here } \tilde{M} : \sigma \to \tilde{\sigma} \text{ is the matrix of 1-morphisms corresponding to } A\tilde{M}_A \text{ under the equivalence } \Phi : \\
\text{Mat}(\text{Bimod}(\mathcal{T})) \xrightarrow{\sim} \text{Bimod}(\mathcal{T}). \text{ Since all the } d(A_i) \text{ are nonzero, an entry in the vector } \dim L(A\tilde{M}_A) \text{ can}
\]
be zero only if the corresponding entry in the vector $\dim L(1_A \otimes _A \tilde{M}_A)$ is zero; therefore invertibility of $\dim L(1_M \otimes _A \tilde{M}_A)$ implies invertibility of $\dim L(1_M \otimes _A \tilde{M}_A)$.

Let $\eta : \tilde{A} \tilde{A} \rightarrow (A \tilde{M}_A)^* \otimes _A \tilde{M}_A$ be the cup of the standard duality in $\text{Bimod}(T)$. We will now show that $\eta \circ x$ is a unitary 2-morphism $\tilde{A} \tilde{A} \rightarrow (A \tilde{M}_A)^* \otimes _A \tilde{M}_A$, finishing the proof:

\begin{align*}
0 &= 0 \\
0 &= 0 \quad \text{(73)}
\end{align*}

Here for the second equality of (73) we used (72). \hfill \square

In Lemma 3.11 we showed that every SSFA in $T$ is a direct sum of simple SSFAs. We therefore need only classify the simple SSFAs. In Lemma 4.2 we showed that two SSFAs in $T$ are Morita equivalent precisely when one can be expressed as a pair of pants algebra over the other in $\text{Bimod}(T)$. By Theorem 3.21 we can rephrase this in terms of $\text{Mod}(T)$. We also observe that all nonzero 1-morphisms between simple objects of $\text{Mod}(T)$ are separable. We may therefore construct all simple SSFAs in $T$ as follows:

- Obtain representatives $\{M_i\}_{i \in I}$ of equivalence classes of semisimple cofinite indecomposable left $T$-module categories.
- For each $i \in I$ and for each unitary $T$-module functor $X : T \rightarrow M_i$ in $\text{Bimod}(T)$, construct the pair of pants SSFA $X \otimes X^*$ in $\text{End}_T(T) \simeq T$.

To turn this into a classification, we need to determine when two 1-morphisms $C \rightarrow M_i$ in $\text{Mod}(T)$ give rise to the same SSFA.

It is clear that certain nonisomorphic 1-morphisms will give rise to unitarily $*$-isomorphic SSFAs. For instance, set $T = \text{Rep}(G)$ for some ordinary compact group $G$, and let $\theta$ be a nontrivial one-dimensional representation. For any other representation $X$, clearly $X$ and $X \otimes \theta$ are nonisomorphic objects in $\text{End}_T(T) \simeq \text{Rep}(G)$. However, since $\theta \otimes \theta^* \cong 1$, there is a unitary $*$-isomorphism $X \otimes X^* \cong (X \otimes \theta) \otimes (X \otimes \theta)^*$. In fact, we will now see that this is all that can go wrong; two 1-morphisms will produce the same SSFA if and only if they are ‘equivalent up to a phase’ in this way. For this we use the following theorem.

**Definition 4.3.** We say that two 1-morphisms $X : r \rightarrow s$ and $Y : r \rightarrow t$ in a dagger 2-category are **equivalent** when there exists an equivalence $E : t \rightarrow s$ and a unitary 2-morphism $\tau : X \rightarrow Y \otimes E$.

**Theorem 4.4 (Ver20, Thm. 5.7).** Let $C$ be a $\mathbb{C}$-linear pivotal dagger 2-category with split dagger idempotents. Let $s, t$ be simple objects, and let $X : r \rightarrow s$ and $Y : r \rightarrow t$ be 1-morphisms. Then $X$ and $Y$ are equivalent in $C$ if and only if the separable Frobenius algebras $X \otimes X^*$ and $Y \otimes Y^*$ in $\text{End}(r)$ are unitarily $*$-isomorphic.
Applying Theorem 4.4 in $\Mod(T)$, we obtain the following classification of simple SSFAs in a rigid $C^*$-tensor category.

**Definition 4.5.** Let $T$ be a rigid $C^*$-tensor category. We say that an object $\theta$ in $T$ is a phase if $\theta \otimes \theta^* \cong 1 \cong \theta^* \otimes \theta$; or, equivalently, if $d(\theta) = 1$.

Let $\mathcal{M}$ be a right $T$-module category. We say that two objects $X_1, X_2$ of $\mathcal{M}$ are equivalent up to a phase in $T$ if there is a unitary isomorphism $X_1 \cong X_2 \otimes \theta$ for a phase $\theta$ in $T$.

**Theorem 4.6** (Classification of SSFAs in a rigid $C^*$-tensor category). Let $T$ be a rigid $C^*$-tensor category. There is a bijective correspondence between:

- Morita equivalence classes of simple SSFAs in $T$.
- Equivalence classes of cofinite semisimple indecomposable left $T$-module categories.
- Unitary $*$-isomorphism classes of simple SSFAs in the corresponding Morita class.
- Isomorphism classes of objects in $\mathcal{M}$, up to a phase in $\End_T(\mathcal{M})$.

**Proof.** The first correspondence has already been explained.

For the second correspondence, by Theorem 4.4 there is a bijective correspondence between unitary $*$-isomorphism classes of SSFAs in the corresponding Morita class and isomorphism classes of objects in $\Hom_T(\mathcal{T}, \mathcal{M})$ up to a phase in $\End_T(\mathcal{M})$, where $\End_T(\mathcal{M})$ acts on the right by postcomposition. There is a left $T$-module action on $\Hom_T(\mathcal{T}, \mathcal{M})$ induced by the local equivalence $\Psi_{1,1} : T \to \End_T(T)$. We claim that $\Hom_T(\mathcal{T}, \mathcal{M})$ is equivalent to $\mathcal{M}$ as a $T$-$\End_T(\mathcal{M})$ bimodule category. Indeed, by essential surjectivity of $\Psi$, there exists an SSFA $A$ and an equivalence of $T$-$\End_T(\mathcal{M})$ bimodule categories $E : \mathcal{M} \sim \Mod_A$ (where the right action of $\End_T(\mathcal{M})$ is given by the equivalence $\tilde{E} : \End_T(\mathcal{M}) \to \End_T(\Mod_A) : F \mapsto E^{-1} \otimes F \otimes E$.) The equivalence $\Psi$ also induces an equivalence of $T$-$\End_T(\Mod_A)$ bimodule categories $\Psi_{1,A} : \Mod_A \sim \Hom_T(\mathcal{T}, \Mod_A)$, where the right action of $\End_T(\Mod_A)$ on $\Hom_T(\mathcal{T}, \Mod_A)$ is given by postcomposition; this can be extended to a morphism of $T$-$\End_T(\mathcal{M})$ bimodule categories using $E$. Finally, there is an equivalence of left $T$-$\End_T(\mathcal{M})$ bimodule categories $\Hom_T(\mathcal{T}, \Mod_A) \sim \Hom_T(\mathcal{T}, \mathcal{M})$ given by postcomposition with $E^{-1}$. □

By what was already said at the beginning of this section, to obtain a classification of finite-dimensional $G$-$C^*$-algebras for a compact quantum group $G$, simply set $T = \Rep(G)$ in Theorem 4.6.

Before moving on we make a brief remark about how connectedness (a.k.a. ergodicity) of SSFAs (considered in e.g. [BDRV05, DCY12, ADC15]) relates to the above classification.

**Definition 4.7.** Let $T$ be a rigid $C^*$-tensor category. We say that an simple SSFA $A$ in $T$ is connected if $\Hom(1, A)$ (i.e. the Hom-space between these objects in $T$) is one-dimensional.

**Proposition 4.8.** Let $A$ be an SSFA in $T$, let $\mathcal{M}$ be the $T$-module category representing its Morita class, and let $X$ be an object of $\mathcal{M} \simeq \Hom_T(\mathcal{T}, \mathcal{M})$ such that $X \otimes X^* \cong A$. Then $A$ is connected precisely when $X$ is a simple object in $\mathcal{M}$.

**Proof.** Rigidity of $\Mod(T)$ induces a linear isomorphism between the vector spaces $\Hom(1, X \otimes X^*)$ in $T \simeq \End_T(T)$ and $\Hom(X)$ in $\Hom_T(\mathcal{T}, \mathcal{M})$. □
4.2 A covariant Stinespring theorem

We now consider covariant channels between $G$-$C^*$-algebras.

Let us consider the case without symmetry first. Let $A, B$ be two finite-dimensional $C^*$-algebras. As explained in Section 4.1, using the canonical trace on these $C^*$-algebras we define an inner product giving rise to SSFAs $A, B$ in Hilb. The standard notion of a physical transformation, or *channel*, is a completely positive trace-preserving linear map. It was shown in [CHK16, HV19] that complete positivity of a linear map $A \to B$ as a morphism in Hilb can be expressed in terms of the Frobenius algebra structures on $A, B$. To this end we consider the following definition, which makes sense in any rigid $C^*$-tensor category.

**Definition 4.9.** Let $\mathcal{T}$ be a rigid $C^*$-tensor category and let $A, B$ be SSFAs in $\mathcal{T}$. Let $f : A \to B$ be a morphism. We say that $f$ satisfies the CP condition, or is a CP morphism, when there exists an object $S$ of $\mathcal{T}$ and a morphism $g : A \otimes B \to S$ such that the following equation holds:

$$\begin{array}{c}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
A \otimes B & \xrightarrow{g} & S \\
\end{array}$$

(75)

In other words, the morphism on the LHS of (75) is positive as a morphism in $\text{Rep}(G)$.

It is shown in [HV19 Thm. 7.18] that a linear map $A \to B$ is completely positive precisely when it obeys the CP condition as a morphism in Hilb. To complete the definition of a channel, we observe that, since the canonical trace on a finite-dimensional $C^*$-algebra is precisely the counit of the corresponding SSFA, trace-preservation corresponds to counit-preservation.

**Definition 4.10.** We say that a CP morphism $f : A \to B$ is a channel when it satisfies the second equation of (75).

This characterisation extends straightforwardly to $G$-$C^*$-algebras for a compact quantum group $G$. In Section 4.1, we saw how an SSFA $A$ in $\text{Rep}(G)$ corresponds to a $G$-$C^*$-algebra equipped with its canonical $G$-invariant functional; the concrete $C^*$-algebra is obtained as the image $F(A)$ of the SSFA under the canonical fibre functor $F : \text{Rep}(G) \to \text{Hilb}$, and the concrete $A_G$-coaction is obtained by T-K-W duality. The canonical fibre functor maps a morphism $A \to B$ in $\text{Rep}(G)$ to a covariant linear map $F(A) \to F(B)$ (that is, an intertwiner of $G$-representations). It is known (see e.g. [Ver20a Prop. 3.22]) that, for SSFAs $A, B$ in $\text{Rep}(G)$, for any covariant completely positive map $f : F(A) \to F(B)$ there is a unique CP morphism $\tilde{f} : A \to B$ in $\text{Rep}(G)$ such that $F(\tilde{f}) = f$. Preservation of the canonical $G$-invariant functional precisely corresponds to counit preservation. Completely positive maps/channels between $G$-$C^*$-algebras can therefore be identified with CP morphisms/channels between SSFAs in $\text{Rep}(G)$.

We can further generalise by considering CP morphisms and channels between SSFAs in $\mathcal{T}$, where $\mathcal{T}$ is a general rigid $C^*$-tensor category. Without a fibre functor there is no obvious way to identify $G$-$C^*$-algebras or morphisms with linear maps; however, the theory holds in this general setting.

We now state the result. Let $\mathcal{T}$ be a rigid $C^*$-tensor category. We saw in Corollary 3.23 that $\mathcal{T}$ embeds as the endomorphism category $\text{End}_r(\mathcal{T})$ in the semisimple $C^*$-2-category $\text{Mod}(\mathcal{T})$. By semisimplicity, for every SSFA $A$ in $\mathcal{T} \simeq \text{End}_r(\mathcal{T})$ there exists an object $M$ of $\text{Mod}(\mathcal{T})$ and a separable 1-morphism $X : \mathcal{T} \to M$ such that $A \cong X \otimes X^*$. This is the context for the following theorem.

**Theorem 4.11** (Covariant Stinespring theorem). Let $\mathcal{C}$ be a semisimple $C^*$-2-category and let $r$ be any object. Let $X : r \to s$, $Y : r \to t$ be separable 1-morphisms, and let $f : X \otimes X^* \to Y \otimes Y^*$ be a CP morphism between the corresponding SSFAs in $\text{End}(r)$.
Then there exists a 1-morphism \( E : t \to s \) (the ‘environment’) and a 2-morphism \( \tau : X \to Y \otimes E \) such that the following equation holds:

\[
\text{(76)}
\]

We say that \( \tau \) is a dilation of \( f \). The morphism

\[
\text{(77)}
\]

is an isometry if and only if \( f \) is a channel.

In the other direction, for any 1-morphism \( E : t \to s \) and 2-morphism \( \tau : X \to Y \otimes E \), the morphism \( f : X \otimes X^* \to Y \otimes Y^* \) defined by (76) is CP, and a channel if and only if (77) is an isometry.

Different dilations for a CP morphism \( f : X \otimes X^* \to Y \otimes Y^* \) are related by a partial isometry on the environment. Specifically, let \( \tau_1 : X \to Y \otimes E_1 \), \( \tau_2 : X \to Y \otimes E_2 \) be two dilations of \( f \). Then there exists a partial isometry \( \alpha : E_1 \to E_2 \) such that

\[
(id_Y \otimes \alpha) \circ \tau_1 = \tau_2 \quad (\text{id}_Y \otimes \alpha^\dagger) \circ \tau_2 = \tau_1
\]

In particular, the dilation minimising the quantum dimension of the environment \( d(E) \) is unique up to unitary \( \alpha \). (A concrete construction of the minimal dilation from any other dilation is specified in the last paragraph of the proof.)

Proof. The fact that a morphism between SSFAs is CP iff it admits a representation (76) was shown in [HP20, Lem. 5.12]. It is straightforward to see that (77) is an isometry if and only if \( f \) is a channel:

\[
\text{(78)}
\]

We now show that different dilations are related by a partial isometry. Let \( \tau_1 : X \to Y \otimes E_1 \), \( \tau_2 : X \to Y \otimes E_2 \) be two dilations of the same CP morphism. For each \( i \in \{1, 2\} \) we define the
following morphism $\tilde{\tau}_i : Y^* \otimes X \to E_i$:

$$\begin{array}{c}
\tilde{\tau}_1 \circ \tilde{\tau}_2 = \tilde{\tau}_2 \circ \tilde{\tau}_1
\end{array}$$

(79)

The fact that $\tau_1$ and $\tau_2$ are dilations of the same CP morphism comes down to the following equation:

$$\begin{array}{c}
\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1
\end{array}$$

(80)

In inline notation, this is:

$\tilde{\tau}_1 \circ \tilde{\tau}_2 = \tilde{\tau}_2 \circ \tilde{\tau}_1$

(81)

We perform the polar decomposition (Remark 2.14) on $\tilde{\tau}_1$ and $\tilde{\tau}_2$. Observe that

$$|\tilde{\tau}| := |\tilde{\tau}_1| = (\tilde{\tau}_1^\dagger \circ \tilde{\tau}_1)^{1/2} = (\tilde{\tau}_2^\dagger \circ \tilde{\tau}_2)^{1/2} = |\tilde{\tau}_2|,$$

where the second equality is by (81). We therefore have

$$\tilde{\tau}_1 = u_1 \circ |\tilde{\tau}|$$

$$\tilde{\tau}_2 = u_2 \circ |\tilde{\tau}|$$

where $u_i : Y^* \otimes X \to E_i$ is a partial isometry such that $u_i^\dagger \circ u_i = s(|\tilde{\tau}|)$ and $u_2 \circ u_2^\dagger = s(|\tilde{\tau}_2|)$.

Now we define $\alpha := u_2 \circ u_1^\dagger$. To see that $\alpha : E_1 \to E_2$ is a partial isometry:

$$\alpha^\dagger \circ \alpha = u_1 \circ u_2^\dagger \circ u_2 \circ u_1^\dagger = u_1 \circ s(|\tilde{\tau}|) \circ u_1^\dagger = u_1 \circ u_1^\dagger \circ u_1 \circ u_1^\dagger = s(|\tilde{\tau}|)$$

(82)

The fact that $(\text{id}_Y \otimes \alpha) \circ \tau_1 = \tau_2$ follows immediately from the following equation (simply transpose the $Y$-wire):

$$\alpha \circ (\tilde{\tau}_1) = u_2 \circ u_1^\dagger \circ u_1 \circ |\tilde{\tau}| = u_2 \circ s(|\tilde{\tau}|) \circ |\tilde{\tau}| = u_2 \circ |\tilde{\tau}| = \tilde{\tau}_2$$

The proof that $(\text{id}_Y \otimes \alpha^\dagger) \circ \tau_2 = \tau_1$ is similar.

To see that the dilation minimising the quantum dimension of the environment is unique up to a unitary, suppose that $\tau_1$ and $\tau_2$ are minimal dilations related by a partial isometry $\alpha : E_1 \to E_2$. Then $\tau_1 = (\text{id}_Y \otimes (\alpha^\dagger \circ \alpha)) \circ \tau_1$. Let us split the dagger idempotent $\alpha^\dagger \circ \alpha$ to obtain an isometry $i : \tilde{E}_1 \to E_1$. Then $(\text{id}_Y \otimes i^\dagger) \circ \tau_1$ is also a dilation, and $d(\tilde{E}_1) \leq d(E_1)$ with equality iff $i$ is unitary; unitarity of $i$ therefore follows by minimality of $\tau_1$, and it follows that $\alpha$ is an isometry. Making the same argument for $\alpha \circ \alpha^\dagger$ and $\tau_2$ we obtain that $\alpha$ is a coisometry, and therefore $\alpha$ is unitary.

Finally, to construct a minimal dilation from a given dilation $\tau : X \to Y \otimes E$, take the projection $s(|\tilde{\tau}|)$. Split this idempotent to obtain an isometry $i : \tilde{E} \to E$ and define the minimal dilation as $(\text{id}_Y \otimes i^\dagger) \circ \tau$. It follows from (82) and the definition of $s(|\tilde{\tau}|)$ that, for any other dilation $\tau' : X \to Y \otimes E'$, the partial isometry $\alpha : E \to E'$ relating it to the minimal dilation will be a genuine isometry; therefore $d(E) \leq d(E')$ with equality iff $\alpha$ is unitary.

\[\square\]
We now show how this theorem recovers previous results in the literature.

**Example 4.12 (Finite-dimensional Stinespring theorem).** Let us show how Theorem 4.11 implies the standard f.d. covariant Stinespring’s theorem (e.g. [Hol07, Thm. 2][Sti55 Thm. 1][Sza10 Thm. 15][Scu79 Thm. 1][Pau82 Thm. 2.1]).

Let $A, B$ be $G$-$C^*$-algebras. There is an equivalence between the 2-category $\text{Mod}(\text{Rep}(G))$ and the 2-category whose objects are finite-dimensional $G$-$C^*$-algebras, whose 1-morphisms are $G$-equivariant finitely generated Hilbert bimodules, and whose 2-morphisms are equivariant bimodule maps. This equivalence takes the object $\text{Rep}(G)$ onto the trivial $G$-$C^*$-algebra $C$. Therefore $A \cong X \otimes X^*$ and $B \cong Y \otimes Y^*$, where $X$ and $Y$ are equivariant right Hilbert $A$- and right Hilbert $B$-modules respectively, considered as 1-morphisms $C \to A$ and $C \to B$. In Theorem 4.11 we characterised CP maps $f : X \otimes X^* \to Y \otimes Y^*$. Making a slight conventional change (which is clearly equivalent by bending the $E$-wire and scaling $\tau$), the theorem says that $f$ is completely positive if and only if there exists a Hilbert $A$-$B$-bimodule $E : A \to B$ and an equivariant bimodule map $\tau : X \otimes E \to Y$ such that the following equation holds:

\[
\begin{align*}
\begin{array}{cc}
Y^* & \tau \\
Y & f \\
X & X^*
\end{array}
\end{align*}

= \begin{array}{cc}
\tau & E \\
\tau \otimes E^* & X \\
X^* & n_X
\end{array}
\]

(83)

Here the regions corresponding to $A$ are shaded with wavy lines and regions corresponding to $B$ with polka dots. We observe that $X \otimes E$ is an equivariant right Hilbert $B$-module; this is the Hilbert $B$-module $E$ in [Sza10 Eq. 22]. The pair of pants algebra $(X \otimes E) \otimes (E^* \otimes X^*)$ is the $*$-algebra $B^*(E)$. Now it is easy to check that the 2-morphism

\[
\begin{array}{cc}
\tau & E \\
\tau \otimes E^* & X \\
X^* & n_X
\end{array}
\]

(84)

is a $*$-homomorphism; we thus obtain the equivariant $*$-homomorphism $\Phi : A \to B^*(E)$. As a map $B^*(E) \to B$, the 2-morphism

\[
\begin{array}{cc}
\tau & E \\
\tau \otimes E^* & X \\
X^* & n_X
\end{array}
\]

(85)

can be expressed as $x \mapsto \hat{\tau}x^\dagger$, where $\hat{\tau} := \tau \otimes (n_Y^{-1/2} \circ n_X^{1/2})$. (Here the normalisation comes from the choice of functional on the algebra.) We therefore set $\tau^\dagger : Y \to X \otimes E$ to be the equivariant module map $V$ of [Sza10 Eq. 22]. We thus obtain the characterisation of completely positive maps $A \to B(H)$ given in that theorem. Another common statement (not actually given in [Sti55]) is that $f$ is unital if and only if $V$ is an isometry. But $f$ is unital if and only if $f^\dagger$ is trace-preserving, and so
we require that

\[
\begin{array}{c}
\tau \\
X \otimes E \\
\end{array}
\]

is trace-preserving. By Theorem 4.11 we see that (86) preserves the canonical separable trace if and only if \(\tau^\dagger \otimes (n_{Y^{-1/2}} \circ n_{X \otimes E}^{1/2}) = V\) is an isometry.

4.3 A covariant Choi theorem

We finish by observing the following corollary of the covariant Stinespring theorem.

Theorem 4.13 (Covariant Choi theorem). Let \(C\) be a semisimple \(C^*\)-2-category and let \(r\) be any object. Let \(X : r \to s\), \(Y : r \to t\) be separable 1-morphisms, and let \(X \otimes X^*\) and \(Y \otimes Y^*\) be the corresponding SSFA in \(\text{End}(r)\). Then there is a bijective correspondence (in fact, an isomorphism of convex cones, in the sense that it preserves positive linear combinations) between:

- \(CP\) morphisms \(f : X \otimes X^* \to Y \otimes Y^*\).
- Positive elements \(\tilde{f} \in \text{End}(Y^* \otimes X)\).

The correspondence is given as follows:

\[
\begin{array}{c}
f \\
x \\
\end{array}
= \begin{array}{c}
f \\
y \\
\end{array}
\]

\[
(87)
\]

Proof. Let \(\tau : X \to Y \otimes E\) be a dilation of \(f\), then:

\[
\begin{array}{c}
f \\
x \\
\end{array}
= \begin{array}{c}
f \\
y \\
\end{array}
\]

\[
(88)
\]

The last diagram is clearly the composition of a 2-morphism with its dagger and is therefore positive.

In the other direction, let \(\tilde{f}\) be positive. Then we can choose \(m\) such that \(\tilde{f} = m^\dagger \circ m\), and transposing the relevant wires we obtain a dilation for \(f\).

Remark 4.14. To recover the usual Choi’s theorem for matrix \(C^*\)-algebras [Cho75], let \(\mathcal{T} = \text{Hilb}\) and \(X,Y\) be 1-morphisms \(\text{Hilb} \to \text{Hilb}\) in \(\text{Mod(Hilb)}\), i.e. Hilbert spaces. Then Theorem 4.13 says precisely that CP maps \(B(X) \to B(Y)\) correspond to positive elements of \(B(Y^* \otimes X)\).
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6 Appendices

6.1 Matrix notation for presemisimple 2-categories

Definition 6.1. Let $\mathcal{C}$ be a presemisimple $C^*$-2-category, and let $\{r_\sigma\}_{\sigma \in \Sigma}$ be representatives of equivalence classes of simple objects in $\mathcal{C}$, with index set $\Sigma$. We define a presemisimple $C^*$-2-category $\text{Mat}(\mathcal{C})$ as follows:

- Objects: Finite-length vectors $\vec{\sigma} = [\sigma_1, \ldots, \sigma_n]$ of elements $\sigma_i \in \Sigma$ (including the empty vector, which is a zero object).

- 1-morphisms $[\sigma_1, \ldots, \sigma_{n_1}] \to [\tau_1, \ldots, \tau_{n_2}]$: $n_1 \times n_2$ matrices $M$ whose $i, j$-th entry $M_{ij}$ is a 1-morphism $r_{\sigma_i} \to r_{\tau_j}$.

- 2-morphisms $M \to N$: $n_1 \times n_2$ matrices $f$ whose $i, j$-th entry $f_{ij}$ is a 2-morphism $M_{ij} \to N_{ij}$.

- Composition of 1-morphisms: $(M \otimes N)_{ik} := \oplus_j M_{ij} \otimes M_{jk}$.

- Identity 1-morphisms: $(\text{id}_{[\sigma_1, \ldots, \sigma_n]})_{ij} := \delta_{ij} \mathbb{1}_i$ (where $\delta_{ij}$ indicates the zero 1-morphism $r_{\sigma_i} \to r_{\sigma_j}$ if $i \neq j$, and $\mathbb{1}_i := \text{id}_{r_{\sigma_i}}$).

- Horizontal composition of 2-morphisms: $(f \otimes g)_{km} := \sum_i i_l \circ (f_{kl} \otimes g_{lm}) \circ i_l^\dagger$, where $i_l : M_{kl} \otimes M_{lm} \to \oplus_l M_{kl} \otimes M_{lm}$ is the injection isometry of the direct sum.
• Vertical composition of 2-morphisms: \((g \circ f)_{ij} := g_{ij} \circ f_{ij}\).

• Dagger on 2-morphisms: \((f^\dagger)_{ij} := (f_{ij})^\dagger\).

• \(C\)-linear structure on 2-morphisms: \((\lambda f)_{ij} = \lambda f_{ij}\).

• Associators: Matrix entries \((\alpha_{M,N,O})_{ij}\) are the unitary natural isomorphisms \(\oplus_k (\oplus_j M_{ij} \otimes N_{jk}) \otimes O_{kl} \cong \oplus_j M_{ij} \otimes (\oplus_k N_{jk} \otimes O_{kl})\) in \(C\).\[\]

• Unitors: The matrix entries (modulo \(0 \otimes X \cong X \cong X \otimes 0\)) are given by the unitors \(\text{id}_{r_{i_1}} \otimes M_{ij} \cong M_{ij} \cong M_{ij} \otimes \text{id}_{r_{i_j}}\) in \(C\).

• Additive structure on 1-morphisms: \((M \oplus N)_{ij} := M_{ij} \oplus N_{ij}\).

We now show that \(\text{Mat}(C)\) has duals. In fact, it is straightforward to show that all right duals in \(\text{Mat}(C)\) are of the following form. Let \(M : [\sigma_1, \ldots, \sigma_n] \to [\tau_1, \ldots, \tau_n]\) be a 1-morphism in \(\text{Mat}(C)\). Pick a dual \(\{(M_{kl})^*, \eta_{kl}, \epsilon_{kl}\}\) for each \(M_{kl}\). We now define a dual 1-morphism \(M^*\) as the ‘conjugate transpose’ matrix, i.e.:

\[
(M^*)_{kl} := (M_{lk})^*
\]

We observe that:

\[
(M^* \otimes M)_{km} = \bigoplus_l (M_{lk})^* \otimes M_{lm} \quad \quad \quad (M \otimes M^*)_{km} = \bigoplus_l M_{kl} \otimes (M_{ml})^*
\]

We then define a right cup and cap \(\eta : \text{id}_{[\tau_1, \ldots, \tau_n]} \to M^* \otimes M\) and \(\epsilon : M \otimes M^* \to \text{id}_{[\sigma_1, \ldots, \sigma_n]}\) as the following diagonal matrices of 2-morphisms:

\[
\eta := \text{diag}([\eta_{11}, \ldots, \eta_{1n_2}]) \quad \quad \epsilon := \text{diag}([\epsilon_{11} \circ \iota_{11}, \ldots, \epsilon_{1n_1} \circ \iota_{1n_1}])
\]

Here \(i_{lk} : (M_{lk})^* \otimes M_{lk} \to \bigoplus_l (M_{lk})^* \otimes M_{lk}\) and \(\iota_{kl} : M_{kl} \otimes (M_{kl})^* \to \bigoplus_l M_{kl} \otimes (M_{kl})^*\) are the isometric injections of the direct sums. We will show one of the snake equations for \([M^*, \eta, \epsilon]\); the other is shown similarly. Let \(s = (\text{id}_{M^*} \otimes \epsilon) \circ (\eta \otimes \text{id}_{M^*})\). Let \(\{\rho_{\sigma,i}, \iota_{\sigma,i}\}\) and \(\{\rho_{\tau,i}, \iota_{\tau,i}\}\) be the dual projection and injection 1-morphisms of the direct sum decompositions \(\sigma = \bigoplus_i [\sigma_i]\), \(\tau = \bigoplus_i [\tau_i]\). We need to show that

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\[ s = \text{id}_{M^*}; \text{or equivalently that } \text{id}_{i_{\tau,j}} \otimes s \otimes \text{id}_{\rho_{\sigma,i}} = \text{id}_{i_{\tau,j}} \otimes \text{id}_{M^*} \otimes \text{id}_{\rho_{\sigma,i}} = \text{id}_{(M_{ij})^*}, \text{for all } i,j. \] Now:

\[ \eta \varepsilon \begin{bmatrix} \sigma_i \\ \tau_j \end{bmatrix} \]

\[ \begin{bmatrix} \sigma_i \\ \tau_j \end{bmatrix} \]

\[ \begin{bmatrix} \sigma_i \\ \tau_j \end{bmatrix} \]

\[ \begin{bmatrix} \sigma_i \\ \tau_j \end{bmatrix} \]

\[ \begin{bmatrix} \sigma_i \\ \tau_j \end{bmatrix} \]

Here the first equality uses duality of \( \epsilon \) and \( \rho \), the second equality follows straightforwardly from the definitions \((89,91)\), and the final equality follows since \([\begin{bmatrix} (M_{ij})^* \\ \eta_{ij} \\ \epsilon_{ij} \end{bmatrix} \]

\[ \text{is a right dual for } M_{ij} \text{ in } \mathcal{C}.\]

We now define an equivalence between \( \text{Mat}(\mathcal{C}) \) and \( \mathcal{C} \).

**Definition 6.2.** The unitary \( \mathcal{C} \)-linear 2-functor \( \Phi : \text{Mat}(\mathcal{C}) \rightarrow \mathcal{C} \) is defined as follows:

- **On objects:** \( \Phi([\sigma_1, \ldots, \sigma_n]) := r_{\sigma_1} \oplus \cdots \oplus r_{\sigma_n} \).

- **On 1-morphisms:** Let \( M : [\sigma_1, \ldots, \sigma_n] \rightarrow [\tau_1, \ldots, \tau_m] \). Let \( \{i_{\tau,j}, \rho_{\sigma,i}\} \) and \( \{\tau_{i,j}, \rho_{\tau,k}\} \) be the injection and projection 1-morphisms for the direct sums \( r_{\sigma_1} \oplus \cdots \oplus r_{\sigma_n} \) and \( r_{\tau_1} \oplus \cdots \oplus r_{\tau_m} \). Then define \( \Phi(M) := \oplus_{ij} ((\rho_{\sigma,i} \otimes M_{ij}) \otimes \tau_{i,j}). \)

- **On 2-morphisms:** Let \( f : M \rightarrow N \) be a 2-morphism. Then we define \( \Phi(f) := \sum_{kl} i_{kl} \circ ((\rho_{\sigma,k} \otimes \id_{i_{\tau,l}}) \circ i_{kl}^t) \), where \( i_{kl} : ((\rho_{\sigma,k} \otimes M_{kl}) \otimes \tau_{i,l}) \rightarrow \oplus_{kl} ((\rho_{\sigma,k} \otimes M_{kl}) \otimes \tau_{i,l}) \) is the isometric injection of the direct sum.

- **Multiplicators:** Given by the natural unitary isomorphisms \( \Phi(M) \otimes \Phi(N) = (\oplus_{ij} ((\rho_{\sigma,i} \otimes M_{ij}) \otimes \tau_{i,j}) \otimes (\oplus_{kl}((\rho_{\sigma,k} \otimes N_{kl}) \otimes \tau_{i,l}))) \cong \oplus_{ijkl} (((\rho_{\sigma,i} \otimes M_{ij}) \otimes (\tau_{i,j} \otimes \rho_{\tau,k})) \otimes (N_{kl} \otimes \tau_{i,l})) \cong \oplus_{i} (\rho_{\sigma,i} \otimes (\oplus_{j} (M_{ij} \otimes N_{ji})) \otimes \tau_{i,l}) = \Phi(M \otimes N) \).

Here for the first isomorphism we used the associators and distributivity of the direct sum over tensor product in \( \mathcal{C} \), and for the second isomorphism we used \( \tau_{i,j} \otimes \rho_{\tau,k} \cong \delta_{jk} \id_{X_{\tau,j}} \).

- **Unitors:** Given by the unitary isomorphisms \( \id_{[\sigma_1, \ldots, \sigma_n]} \cong \oplus_i (i_i \otimes \rho_i) \cong \Phi(\id_{[\sigma_1, \ldots, \sigma_n]}) \).

**Proposition 6.3.** The 2-functor \( \Phi : \text{Mat}(\mathcal{C}) \rightarrow \mathcal{C} \) is an equivalence.

**Proof.**

- **Essentially surjective on objects.** By presemisimplicity every object is a direct sum of simples, and direct sums are unique up to equivalence.
• Essentially surjective on 1-morphisms. Let $X : \bigoplus_i r_{\sigma_i} \to \bigoplus_j r_{\tau_j}$ be a 1-morphism. Now we have
\[
X \cong \left( \bigoplus_i (\rho_{\sigma,i} \otimes t_{\sigma,i}) \otimes X \right) \otimes \left( \bigoplus_j (\rho_{\tau,j} \otimes t_{\tau,j}) \right)
\cong \bigoplus_{i,j} \left( \rho_{\sigma,i} \otimes \left( (t_{\sigma,i} \otimes X) \otimes \rho_{\tau,j} \right) \right) \otimes t_{\tau,j}
\cong \Phi(M)
\]
where $M : [\sigma_1, \ldots, \sigma_{n_1}] \to [\tau_1, \ldots, \tau_{n_2}]$ is defined by $M_{ij} := (t_{\sigma,i} \otimes X) \otimes \rho_{\tau,j}$.

• Full on 2-morphisms. Let $f : \Phi(M) = \bigoplus_{i,j} \left( \rho_{\sigma,i} \otimes (\rho_{\tau,i} \otimes t_{\sigma,i}) \otimes t_{\tau,j} \right) \to \bigoplus_{i,j} \left( \rho_{\sigma,i} \otimes \left( (t_{\sigma,i} \otimes X) \otimes \rho_{\tau,j} \right) \right) \cong \Phi(N)$ be a 2-morphism in $C$. Let $\kappa_{\sigma} : id_{\bigoplus_i r_{\sigma_i}} \Rightarrow \bigoplus_i \rho_{\sigma,i} \otimes t_{\sigma,i}$ and $\lambda_{\sigma,i} : t_i \otimes \rho_i \Rightarrow id_{r_{\sigma_i}}$ be the unitary isomorphisms in the definition of the direct sum $\bigoplus_i r_{\sigma_i}$, and define $\kappa_{\tau}$ and $\lambda_{\tau,i}$ similarly. Let $\nabla_{M,i,j} : ((\rho_{\sigma,i} \otimes M_{ij}) \otimes t_{\tau,j}) \to \bigoplus_{i,j} \left( (\rho_{\sigma,i} \otimes M_{ij}) \otimes t_{\tau,j} \right)$, $\nabla_{N,i,j} : ((\rho_{\sigma,i} \otimes N_{ij}) \otimes t_{\tau,j}) \to \bigoplus_{i,j} \left( (\rho_{\sigma,i} \otimes N_{ij}) \otimes t_{\tau,j} \right)$, $\nabla_{\sigma,i} : \rho_{\sigma,i} \otimes t_{\sigma,i} \to \bigoplus_i \rho_{\sigma,i} \otimes t_{\sigma,i}$, $\nabla_{\tau,i} : \rho_{\tau,i} \otimes t_{\tau,i} \to \bigoplus_i \rho_{\tau,i} \otimes t_{\tau,i}$ be the isometric injections of the various direct sums of 1-morphisms; we depict these as labelled downwards-pointing triangles in the diagram, and their daggers as labelled upwards-pointing triangles. Then we have the following sequence of equalities in $C$:

\[
f = \sum_{i,j,k,l,m,n} \kappa_{\sigma} \kappa_{\tau} \nabla_{M,i,j} \nabla_{N,k,l} \nabla_{\sigma,i} \nabla_{\tau,i} (\rho_{\sigma,i} \otimes \rho_{\tau,j} \otimes \rho_{\sigma,j} \otimes \rho_{\tau,i} \otimes \rho_{\sigma,k} \otimes \rho_{\tau,l} \otimes \rho_{\sigma,m} \otimes \rho_{\tau,n}) (93)
\]
\[
\sum_{i,n} \rho
\]
Here the first equality is by $\kappa_t^\dagger \circ (\sum_i \nabla_{\sigma,i} \circ \nabla^\dagger_{\sigma,i}) \circ \kappa_\sigma = \text{id}_{\text{id}_{\text{Mat}^{\dagger}}_{\tau_{r,s}}} \circ \kappa_t^\dagger \circ (\sum_n \nabla_{\tau,n} \circ \nabla^\dagger_{\tau,n}) \circ \kappa_\tau = \text{id}_{\text{id}_{\text{Mat}^{\dagger}}_{\tau_{r,s}}} \circ \sum_{l,m} \nabla_{M,l,m} \circ \nabla^\dagger_{M,l,m} = \text{id}_{\Phi(M)}$ and $\sum_{j,k} \nabla_{N,j,k} \circ \nabla^\dagger_{N,j,k} = \text{id}_{\Phi(N)}$. The second equality is by $\text{id}_{\text{id}_{\tau_{r,s}} \circ \rho_{r,s}} = \delta_i(\lambda_{\tau,r,s}^\dagger \circ \lambda_{\tau,r,s})$, $\text{id}_{\text{id}_{\tau_{r,s}} \circ \rho_{r,s}} = \delta_i(\lambda_{\tau,r,s}^\dagger \circ \lambda_{\tau,r,s})$, $\text{id}_{\text{id}_{\tau_{r,s}} \circ \rho_{r,s}} = \delta_i(\lambda_{\tau,r,s}^\dagger \circ \lambda_{\tau,r,s})$ and $\text{id}_{\text{id}_{\tau_{r,s}} \circ \rho_{r,s}} = \delta_i(\lambda_{\tau,r,s}^\dagger \circ \lambda_{\tau,r,s})$, which allows us to use $\kappa_t^\dagger \circ (\sum_i \nabla_{\sigma,i} \circ \nabla^\dagger_{\sigma,i}) \circ \kappa_\sigma = \text{id}_{\text{id}_{\text{Mat}^{\dagger}}_{\tau_{r,s}}}$ and $\kappa_t^\dagger \circ (\sum_n \nabla_{\tau,n} \circ \nabla^\dagger_{\tau,n}) \circ \kappa_\tau = \text{id}_{\text{id}_{\text{Mat}^{\dagger}}_{\tau_{r,s}}}$ again.

In the last diagram we see $\Phi(\tilde{f})$ where $\tilde{f}_{in} : M_{in} \to N_{in}$ is the morphism in the dashed box.

- **Faithful on 2-morphisms.** Let $f, g : M \to N$ be 2-morphisms in $\text{Mat}(\mathcal{C})$. Then:

$$\phi(f) = \phi(g) \Leftrightarrow \sum_{ij} \nabla_{N,i,j} \circ (\text{id}_{\rho_{r,s}} \otimes f_{ij} \otimes \text{id}_{\tau_{r,s}}) \circ \nabla^\dagger_{M,i,j} = \sum_{ij} \nabla_{N,i,j} \circ (\text{id}_{\rho_{r,s}} \otimes g_{ij} \otimes \text{id}_{\tau_{r,s}}) \circ \nabla^\dagger_{M,i,j}$$

$$\Rightarrow \text{id}_{\rho_{r,s}} \otimes f_{ij} \otimes \text{id}_{\tau_{r,s}} = \text{id}_{\rho_{r,s}} \otimes g_{ij} \otimes \text{id}_{\tau_{r,s}} \quad \forall i, j$$

$$\Rightarrow (\kappa^t_{\sigma} \otimes \text{id} \otimes \kappa^t_{\tau}) \circ (\text{id} \otimes g_{ij} \otimes \text{id} \otimes \kappa^t_{\tau}) = (\kappa^t_{\sigma} \otimes \text{id} \otimes \kappa^t_{\tau}) \circ (\text{id} \otimes g_{ij} \otimes \text{id} \otimes \kappa^t_{\tau}) \quad \forall i, j$$

$$\Leftrightarrow f_{ij} = g_{ij} \quad \forall i, j$$

\[\square\]

**Remark 6.4** (Matrix notation for standard duals and traces). Let $X : r \to s$ be any 1-morphism in $\mathcal{C}$. By Proposition 6.3 for any objects $r, s$ of $\mathcal{C}$ there exist adjoint equivalences $[E_r, E^*_r, \alpha_r, \beta_r] : r \to \Phi([\sigma_1, \ldots, \sigma_n])$ and $[E_s, E^*_s, \alpha_s, \beta_s] : s \to \Phi([\tau_1, \ldots, \tau_n])$, and unitary isomorphisms $u : E^*_r \otimes X \otimes E_s \to \Phi(M)$ and $\check{u} : E^*_s \otimes X^* \otimes E_r \to \Phi(M^*)$ for some morphisms $M : [\sigma_1, \ldots, \sigma_n] \to [\tau_1, \ldots, \tau_n]$ and $M^* : [\tau_1, \ldots, \tau_n] \to [\sigma_1, \ldots, \sigma_n]$ in $\text{Mat}(\mathcal{C})$.
In the following diagrams we draw the wires for $E_r, E_r^*, E_s, E_s^*$ in blue, and draw $\alpha_r, \alpha_s$ as cups and $\beta_r, \beta_s$ as caps. We shade the functorial boxes corresponding to the equivalence $\Phi : \text{Mat}(\mathcal{C}) \to \mathcal{C}$ in a lighter blue. Using fullness and faithfulness of $\Phi$ we define 2-morphisms $\tilde{\eta} : \text{id}_\mathcal{C} \to M^* \otimes M$ and $\tilde{\epsilon} : M \otimes M^* \to \text{id}_\mathcal{C}$ in $\text{Mat}(\mathcal{C})$ as follows:

$$
\eta X^* X := \eta u u M^* M X^* X E_r E_s E_s^* E_r^*
$$

$$
\epsilon X^* X := \eta u u M^* M X^* X E_s X E_r E_r^*
$$

It is easy to check that $[M^*, \tilde{\eta}, \tilde{\epsilon}]$ is a right dual for $M$ in $\text{Mat}(\mathcal{C})$. By the discussion following Definition 6.1 there exist standard right duals $[(M_{ij})^*, \tilde{\eta}_{ij}, \tilde{\epsilon}_{ij}]$ for each of the 1-morphisms $M_{ij}$, so that $M^*$ is defined as in (90) and $\tilde{\eta}, \tilde{\epsilon}$ as in (91).

We will now consider the trace $\psi_X = \phi_X : \text{End}(X) \to \mathbb{C}$. Let $T \in \text{End}(X)$. In the notation just defined, we have the following expression for $\eta^t \circ (\text{id}_{X^*} \otimes T) \circ \eta$:

$$
\eta T X^* X := \eta u u M^* M X^* X E_r E_s E_s^* E_r^*
$$

$$
\eta T X^* X := \eta u u M^* M X^* X E_s X E_r E_r^*
$$

For the first equality we used (97) and unitarity of $\bar{u}, \beta_r, \beta_s$. For the second equality we used fullness and faithfulness of the equivalence $\Phi$ and unitarity of the multiplicator $\mu_{M^*, M}$; here $T : M \to M$ is a uniquely defined 2-morphism in $\text{Mat}(\mathcal{C})$. 

50
It is straightforward to check that \( x \mapsto \beta_s \circ (\text{id}_{E_s} \otimes x \otimes \text{id}_{E_s^*}) \circ \beta_s^* \) defines a \( * \)-isomorphism \( \nu_1 : \text{End}(\Phi(\bar{\tau})) \rightarrow \text{End}(s) \), and likewise \( x \mapsto \nu_2 \circ x \circ \nu_2 \) defines a \( * \)-isomorphism \( \nu_2 : \text{End}(\bar{\tau}) \rightarrow \text{End}(\Phi(\bar{\tau})) \) (recall \( \nu_2 \) is our notation for the unitor of the 2-functor \( \Phi \)). Since these are \( * \)-isomorphisms of commutative f.d. \( C^* \)-algebras they must preserve the canonical trace in the sense that \( \text{Tr}(\nu_1(x)) = \text{Tr}(x) \).

In the rightmost diagram of (98) we see \((\nu_1 \circ \nu_2)(\eta^1 \circ (\text{id}_{M^*} \otimes \bar{T}) \circ \eta)\) and so it follows that:

\[
\phi_X(T) = \text{Tr}_{\bar{T}}[\eta^1 \circ (\text{id}_{M^*} \otimes \bar{T}) \circ \eta]
\]

But it is straightforward to calculate in \( \text{Mat}(\mathcal{C}) \) that:

\[
\tilde{\eta}^1 \circ (\text{id}_{M^*} \otimes \bar{T}) \circ \eta = \text{diag}([\sum_{i} (\tilde{\eta}_i^1) \circ (\text{id}_{(M_i)^*} \otimes \bar{T}_i) \circ \tilde{\eta}_i, \ldots , \sum_{i} (\tilde{\eta}_n^1) \circ (\text{id}_{(M_n)^*} \otimes \bar{T}_n) \circ \tilde{\eta}_n])
\]

Using this together with a similar argument for \( \psi_X \), we see that:

\[
\phi_X(T) = \sum_{i,j} \tilde{\eta}^1_{ij} \circ (\text{id}_{(M_{ij})^*} \otimes \bar{T}_{ij}) \circ \tilde{\eta}_{ij} = \sum_{i,j} \epsilon_{ij} \circ (\bar{T}_{ij} \otimes \text{id}_{(M_{ij})^*}) \circ \epsilon^1_{ij} = \psi_X(T) \quad (99)
\]

### 6.2 Proof of Theorem 3.21

**Theorem.** The 2-functor \( \Psi \) is an equivalence.

**Proof.** We prove the result now.

- **Essentially surjective on objects.** We use [NY18 Thm. A.1], which shows that for any nonzero cofinite semisimple indecomposable left \( \mathcal{C} \)-module category \( \mathcal{M} \) there exists an SSFA \( A \) in \( \mathcal{T} \) such that \( \mathcal{M} \) is equivalent to \( \text{Mod}-A \). Since direct sums of objects are preserved by linear 2-functors, essential surjectivity follows.

- **Essentially surjective on 1-morphisms.** We need to show that for any SSFAs \( A, B \), the local functor \( \Psi_{A,B} : \text{Mod}-A \rightarrow \text{Hom}_\mathcal{T}(\text{Mod}-A, \text{Mod}-B) \) is essentially surjective. In other words, for any module functor \( F : \text{Mod}-A \rightarrow \text{Mod}-B \) there exists some dagger bimodule \( A \cdot M_B \) such that \( F \) is unitarily isomorphic to \( \Psi(A \cdot M_B) \).

We first consider the special case where \( A = B = 1 \). Here the left \( \mathcal{T} \)-module action on \( \text{Mod}-1 = \mathcal{T} \) is given by tensor product on the left, and the functor \( \Psi_{1,1} : \mathcal{T} \rightarrow \text{End}_\mathcal{T}(\mathcal{T}) \) is given by tensor product on the right. We will show that \( \Psi_{1,1} : \mathcal{T} \rightarrow \text{End}_\mathcal{T}(\mathcal{T}) \) is an equivalence, implying essential surjectivity in this case. For this, observe that \( \mathcal{T} \) is an invertible \( \mathcal{T} \)-\( \mathcal{T} \) bimodule category in the sense of [NY18 Def. 3.1] (consider the two-object rigid \( C^* \)-2-category \( \mathcal{C} \) with the set \( \{0, 1\} \) of objects, where \( \mathcal{C}_{00} = \mathcal{C}_{01} = \mathcal{C}_{10} = \mathcal{C}_{11} = \mathcal{T} \), composition is by tensor product keeping track of the objects, and the dual functor takes a 1-morphism in \( \mathcal{C}_{01} \) to its dual in \( \mathcal{C}_{10} \) and vice versa). By [NY18 Thm. 3.2 (c)], it follows that the functor \( \Psi_{1,1} \) is an equivalence\(^7\).

We will now show that this is enough to imply essential surjectivity for the other \( \text{Hom} \)-categories. Indeed, let \( F : \text{Mod}-A \rightarrow \text{Mod}-B \) be a module functor. Now \( \Psi(1_A) \otimes F \otimes \Psi(B_B) \in \text{End}_\mathcal{T}(\mathcal{T}) \); therefore, by the special case just proven, there exists some object \( O_F \) of \( \mathcal{T} \) and a unitary isomorphism \( U : \Psi(1_A) \otimes F \otimes \Psi(B_B) \cong \Psi(O_F) \).

\(^6\)Strictly speaking the cited theorem proves this for right \( \mathcal{T} \)-module categories, but this is just a matter of convention. Indeed, a right \( \mathcal{T} \)-module category is just a left module category over the category \( \mathcal{T}^{\text{op}} \) with opposite tensor product. But \( \dagger \circ * : \mathcal{T} \rightarrow \mathcal{T}^{\text{op}} \) is an equivalence, where \( \dagger \) is the dagger functor and \( * \) is the right duals functor; this induces an equivalence \( E : \text{Mod}(\mathcal{T}) \rightarrow \text{Mod}(\mathcal{T}^{\text{op}}) \). Then it suffices to observe that there is an equivalence of right \( \mathcal{T} \)-module categories \( E(\text{Mod}-A) \cong A-\text{Mod} \) (which takes a right \( A \)-module to its dual left \( A \)-module and a right \( A \)-module morphism to its conjugate).

\(^7\)Strictly speaking, this cited theorem also uses the opposite convention and shows that the functor from \( \mathcal{T} \) acting on the left to the endomorphism category of \( \mathcal{T} \) as a right \( \mathcal{T} \)-module category is an equivalence. To get round this, run the argument for \( \mathcal{T}^{\text{op}} \) rather than \( \mathcal{T} \).
In what follows we use light blue shading for the functorial boxes of the 2-functor $\Psi$, and we label regions corresponding to objects of $\text{Mod}(\mathcal{T})$ with the name of the corresponding object. We first observe that $O_F$ naturally has the structure of an $A$-$B$ dagger bimodule. Indeed, we define the following morphism $\Psi(A \otimes O_F \otimes B) \to \Psi(O_F)$ in $\text{End}_\mathcal{T}(\mathcal{T})$:

Using fullness and faithfulness of the equivalence $\Psi_{1,1} : \mathcal{T} \to \text{End}_\mathcal{T}(\mathcal{T})$, we can pull this back uniquely to obtain a preimage $\alpha : A \otimes O_F \otimes B \to O_F$ in $\mathcal{T}$. Using faithfulness of $\Psi_{1,1}$ we now show that $\alpha$ is a dagger bimodule action. For the first condition of (23):

$$
\alpha

\alpha

\alpha

A A

\alpha

\alpha

A A

\alpha

\alpha

A A

(100)

(101)
Here for the first equality we used unitarity of the unitor $\nu_1$ of the 2-functor $\Psi$; for the second equality we used the definition of $\alpha$ as the pullback of the morphism (100); for the third equality we used unitarity of the unitor $\nu_1$ and unitarity of $U$; for the fourth equality we used associativity of the SSFAs $A$ and $B$; and for the final equality we used the definition of $\alpha$ again. By faithfulness of $\Psi_{1,1}$ we can remove the functorial boxes and so we obtain the desired equality.
For the second condition of (23):

\[ \alpha \]

Here the first equality is by definition of \( \alpha \); the second equality is by unitality of the Frobenius algebras \( A \) and \( B \); and the final equality is by unitarity of \( \nu_1 \), manipulation of functorial boxes, and unitarity of \( U \). Again, by faithfulness of \( \Psi_{1,1} \) we can remove the functorial boxes and so obtain the desired equality.

For the third condition of (23):

\[ \alpha \]

(106)
Here the first equality is by definition of $\alpha$ and unitarity of $\Psi$; the second equality is by the Frobenius equation for $A$ and $B$ and manipulation of functorial boxes; the third equality is by unitarity of $v_1$; the fourth equality is by definition of $\alpha$; and the final equality is by unitarity of $v_1$. Again, by faithfulness of $\Psi_{1,1}$ we can remove the functorial boxes and so obtain the desired equality.

Having defined an $A$-$B$ dagger bimodule structure on $O_F$, we now claim that $F$ is unitarily isomorphic to $\Psi(A(O_F)B)$. To prove this, we will define a dagger idempotent in $\text{Hom}_T(\text{Mod-}A, \text{Mod-}B)$ for which the objects $F$ and $\Psi(A(O_F)B)$ are both valid splittings. Since the splitting of a dagger idempotent is unique up to a unitary isomorphism, the result follows.

The dagger idempotent will be an endomorphism of $\Psi(AA \otimes OF \otimes BB)$. We make some preliminary remarks. First, it is obvious that the comultiplication morphisms of the SSFAs $A$ and $B$ are $A$-$A$ and $B$-$B$ bimodule homomorphisms. In $\text{Bimod}(T)$ we depict these as follows (recall that, as the unit objects in $A$-$\text{Mod-}A$ and $B$-$\text{Mod-}B$, the bimodules $AA$ and $BB$ are not depicted in the diagrammatic calculus of $\text{Bimod}(T)$):

\[
(m_A)^\dagger: AA \rightarrow AA \otimes AA
\]

\[
(m_B)^\dagger: BB \rightarrow BB \otimes BB
\]

To avoid any confusion we remark that these are not the same as the cups of (51). By separability of $A$ and $B$, these 2-morphisms in $\text{Bimod}(T)$ are isometries.

\[\text{(107)}\]

\[\text{(108)}\]

\[\text{(109)}\]

\[\text{(110)}\]
We now define the following 2-morphism \( \iota_F : F \to \Psi(A \otimes O \otimes B) : \)

\[
\begin{array}{c}
\begin{array}{c}
\text{Mod-A} \\
\hline
\text{Mod-B}
\end{array}
\end{array}
\]

Clearly \( \iota_F \) is an isometry:

\[
\begin{array}{c}
\begin{array}{c}
\text{Mod-A} \\
\hline
\text{Mod-B}
\end{array}
\end{array}
\]

Here the first equality is by unitarity of \( \upsilon_1 \) and of \( U \), and the second equality is by separability of \( A \) and \( B \) and unitarity of \( \upsilon_A \) and \( \upsilon_B \).

It follows that \( \pi := \iota_F \circ \iota^*_F \in \text{End}(\Psi(A \otimes O \otimes B)) \) is a dagger idempotent which is split by \( F \).

To show that \( \Psi(A(O_F)_B) \) also splits \( \pi \) we need to define an isometry \( \iota : \Psi(A(O_F)_B) \to \Psi(A \otimes O \otimes B) \) such that \( \iota \circ \iota^* = \pi \). Consider the action \( \alpha : A \otimes O \otimes B \to O \) defining the \( A-B \) bimodule structure on \( A(O_F)_B \). By the first condition of \( (23) \), \( \alpha \) is a bimodule homomorphism \( A \otimes O \otimes B \to A(O_F)_B \). We therefore define \( \iota \) as follows:

\[
\begin{array}{c}
\begin{array}{c}
\text{Mod-A} \\
\hline
\text{Mod-B}
\end{array}
\end{array}
\]

To see that \( \iota \) is an isometry it suffices to show that \( \alpha \circ \alpha^* = \text{id}_{O_F} \). But this follows straightforwardly.

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wardly from the dagger bimodule equations and separability of $A$ and $B$:

\[
\begin{align*}
\alpha & = \alpha \\
\alpha & = \alpha
\end{align*}
\]

Here the first equality is by the third equation of (23); the second equality is by the first equation of (23); the third equality is separability of $A$ and $B$; and the final equality is by the second equation of (23).

To finish we need to show that $\iota \circ \iota^\dagger = \pi$. We first observe the following equation for $\alpha^\dagger \circ \alpha$:

\[
\begin{align*}
\alpha & = \alpha \\
\alpha & = \alpha
\end{align*}
\]

Here the first equality is by the third equation of (23); the second equality is by the first equation of (23); and the final equality is by the Frobenius equation (15).

We also observe that by the Frobenius equation we have the following equation in $\text{Bimod}(T)$ (the analogous equation for $B$ also holds):

\[
\begin{align*}
\alpha & = \alpha \\
\alpha & = \alpha
\end{align*}
\]

We then have the following equation for $\iota \circ \iota^\dagger$:

\[
\begin{align*}
\alpha & = \alpha \\
\alpha & = \alpha
\end{align*}
\]

(117)
Here the first equality is by (115); the second equality is by unitarity of $\nu_1$; the third equality is by definition of $\alpha$; and the final equality is by (116). In the last diagram of (117) we indeed see $\pi = \iota_f \circ \iota_f^\dagger$ and so essential surjectivity on 1-morphisms is proven.

- **Faithful on 2-morphisms.** Let $f, g : A_MB \to A_NB$ be bimodule homomorphisms. Faithfulness is the statement that $\Psi(f) = \Psi(g) \Rightarrow f = g$. Now suppose $\Psi(f) = \Psi(g)$ and consider the component of this natural transformation on the object $A_A$ of Mod-$A$ as a morphism in $\mathcal{T}$:

\[
\begin{align*}
A_A \otimes_A A_NB & \Rightarrow A_A \otimes_A A_MB \\
(119)
\end{align*}
\]

Here the first implication is by pre- and postcomposition on both sides of the equality; the second implication is by definition of the idempotent (27); and the final implication is by unitality,
counting and separability of $A$, the Frobenius equation for $A$, the fact that $f$ and $g$ are bimodule homomorphisms, and \[23\].

- **Full on 2-morphisms.** We must show that for any morphism of module functors $f : \Psi(A_M B) \to \Psi(A M_B)$ there exists a bimodule morphism $\phi : A_M \to A N_B$ such that $\Psi(\phi) = f$. This is seen as follows:

\[
\begin{align*}
\Psi(\phi) &= f \\
\Psi(\phi) &= f \\
\Psi(\phi) &= f
\end{align*}
\]

(121)

\[
\begin{align*}
\Psi(\phi) &= f \\
\Psi(\phi) &= f \\
\Psi(\phi) &= f
\end{align*}
\]

(122)

\[
\begin{align*}
\Psi(\phi) &= f \\
\Psi(\phi) &= f \\
\Psi(\phi) &= f
\end{align*}
\]

(123)

Here the first equality is by unitarity of $v_A$, $v_B$ and $v_1$, and separability of $A$ and $B$; the second equality is by unitarity of $v_A$ and $v_B$; the third equality is by fullness and faithfulness of the equivalence $\Psi_{1,1}$ (here $\hat{f} : A \otimes A M_B \to A \otimes A N \otimes B B$ is the unique preimage of the morphism $\Psi(A \otimes_A M \otimes_B B) \to \Psi(A \otimes_A N \otimes_B B)$ contained in the dashed box); and the final equality is by unitarity of $v_1$ and manipulation of functorial boxes.