κ-Minkowski spacetime as the result of Jordanian twist deformation

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Abstract

Two one-parameter families of twists providing κ–Minkowski *-product deformed spacetime are considered: Abelian and Jordanian. We compare the derivation of quantum Minkowski space from two perspectives. The first one is the Hopf module algebra point of view, which is strictly related with Drinfeld’s twisting tensor technique. The other one relies on an appropriate extension of ”deformed realizations” of nondeformed Lorentz algebra by the quantum Minkowski algebra. This extension turns out to be de Sitter Lie algebra. We show the way both approaches are related. The second path allows us to calculate deformed dispersion relations for toy models ensuing from different twist parameters. In the Abelian case one recovers κ–Poincaré dispersion relations having numerous applications in doubly special relativity. Jordanian twists provide a new type of dispersion relations which in the minimal case (related to Weyl-Poincaré algebra) takes an energy-dependent linear mass deformation form.

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I. INTRODUCTION

Noncommutative geometry has found many applications in physical theories in recent years. It has been suggested as a description of spacetime at the Planck scale and proposed as a background for the unification of gravity and quantum field theory. The first idea of noncommuting coordinates was suggested as long ago as the 1940s by Snyder [1]. More recently, deformed coordinate spaces based on algebraic relations 

\[ [x^\mu, x^\nu] = i \theta^{\mu\nu} \]  

(with \( \theta^{\mu\nu} \) constant) have been introduced in Ref. [2] as a consequence to quantum gravity [2]-[4]. These presently known as "canonical" spacetime commutation relations have been the subject of many investigations (see, e.g., [2]-[13]). Alternatively, this type of noncommutative coordinates was introduced in string theory as coordinates on the spacetime manifold attached to the ends of an open string in a particular gauge field background [5]. The same authors have also introduced the so-called Seiberg-Witten map relating gauge theories on commutative and noncommutative (NC) spaces. This research was a beginning of extensive studies on quantum field theories (QFT) defined over NC spaces [6]-[11], particularly with the twisted (\( \theta \)-deformed) Poincaré invariance of NC QFT in Refs. [9, 10]. A growing number of investigations concern gauge theories [14]-[17] and, moreover, NC gravity [18]-[21].

The Lie-algebraic type of noncommutativity has also been widely investigated. Inspired by the \( \kappa \)-deformed Poincaré algebra [22, 23], the \( \kappa \)-Minkowski spacetime has been introduced in Refs. [24, 25], together with dimensionfull masslike deformation parameter \( \kappa \) (usually connected with Planck mass \( M_P \)). It has been further used by many authors [22]-[31] as a starting point to construct quantum field theories and then to discuss Planck scale physics. The past few years have experienced an ever-growing interest in \( \kappa \)-deformed spacetime motivated by so-called doubly special relativity (DSR) [30, 31] which includes, besides the velocity of light, a second invariant mass parameter (\( \kappa \)). In the framework of DSR, the kinematic consequences of deformed spacetime have been examined; see, e.g., [32]-[34].

Lie-algebraic quantum deformations [19] are the most physically appealing from a bigger set of quantum deformations [14] in the Hopf-algebraic framework of quantum groups [35]-[38]. It appears that quantum deformations of Lie algebras are controlled by classical \( r \)-matrices satisfying the classical Yang-Baxter (YB) equation: homogeneous or inhomogeneous. Particularly, an effective tool is provided by the so-called twisted deformations [35] which classical \( r \)-matrices satisfy the homogeneous YB equation and can be applied to both Hopf algebra (coproduct) as well as related Hopf module algebra (\( * \) product) [39]. Two types of explicit examples of twisting two tensors are the best known and investigated in the literature: Abelian [40] and Jordanian [41] as well as the extended Jordanian one [42] (see also [43, 44]). The \( \kappa \)-deformation of Poincaré algebra is characterized by the inhomogeneous classical YB equation, which implies that one should not expect to get \( \kappa \)-Minkowski space from a Poincaré twist. However, twists belonging to extensions
of the Poincaré algebra are not excluded [45, 46]. An interesting result has been found in Ref. [47], where starting from nonstandard (Jordanian) deformed $D = 4$ conformal algebra the $\kappa$–Minkowski space has been obtained in two ways: by applying the Fadeev-Reshetikin-Takhtajan technique [38] or exploiting a bialgebra structure generated by a classical $\mathfrak{so}(4, 2)$ $r$ matrix.

As a final remark, let us remind the reader that the classification of quantum deformations strongly relies on classification of the classical $r$ matrices. In a case of relativistic Lorentz and Poincaré symmetries, such a classification has been performed some time ago in Ref. [48]. A passage from the classical $r$-matrix to twisting two-tensor and the corresponding Hopf-algebraic deformation is a nontrivial task. Explicit twists for Zakrzewski’s list have been provided in Ref. [49] as well as their superization [44, 50]. Systematic quantizations of the corresponding Lorentz and Poincaré Hopf algebras are carried on in Ref. [51]. In particular, the noncommutative spacetimes described by three types of Abelian Poincaré twists have been calculated in Ref. [52].

In this article we shall work with a flat spacetime $\mathbb{M} \equiv \mathbb{R}^n$ of arbitrary dimension $n \geq 2$. The paper is organized as follows: in Sec. II, we establish formalism and notation. We also show how to associate to a given $\ast$-product the corresponding left and right operator realizations of noncommutative coordinates by means of generalized differential operators. Section III describes two families of twists providing $\kappa$-Minkowski deformed spacetime with deformed coproducts and antipodes. These are Abelian and Jordanian families. The Abelian family has been previously investigated [45]. One finds out, in an explicit form, the operator realization of the $\kappa$-Minkowski coordinates $\hat{x}^\mu$ for each value of twist parameters. In this section we also point out the smallest possible subalgebras, containing Poincaré algebra, to which one can reduce the deformation procedure. Lie algebra comprising the $\kappa$-deformed Minkowski algebra and deformed realizations of the standard Lorentz algebra is introduced in Sec. IV. This turns out to be $\mathfrak{so}(n, 1)$ algebra. Similarly as in Ref. [53] one introduces deformed generators $M_{\mu\nu}$, d’Alambert operator $\hat{\Box}$, and Dirac derivatives $D_\mu$ in the Lorentzian case. It should be noted that in Ref. [53] it was Euclidean algebra. The generalized d’Alambert operator allows one to calculate, in explicit form, dispersion relations for plane wave solutions. A number of toy models based on different values of twist parameters are considered. Appendix A collects improved ordinary differential equations (ODE) for functions describing deformed generators. Appendix B refers to Weyl-Poincaré algebra as to minimal algebra containing both the Poincaré subalgebra as well as the twist element.
II. PRELIMINARIES AND NOTATION

It is well-known that representations of Lie algebras (Lie groups) bring these objects into the broader context of operator algebras which are essential for quantum theories. In a case when representation space is, like in any field theory, a space of functions (fields), this context is provided by an algebra of (partial) differential operators on an underlying spacetime manifold. In this case the algebra of functions itself becomes automatically a Hopf module algebra over Hopf algebra of differential operators. Twisted deformations in this broader context are a step forward in the geometrization of the traditional Drinfeld scheme [35]. It has been argued in a seminal paper [18] (see also [13, 39, 54]) that such a framework is very useful in noncommutative geometry and deformed field theories and gives a hope to describe gravity at the Planck scale 1.

As an example, let us consider Lie algebra $\mathfrak{gl}(n, \mathbb{R}) = \mathfrak{gl}(n, \mathbb{R}) \supset t^n$ of the inhomogeneous general linear group as a semidirect product of $\mathfrak{gl}(n, \mathbb{R})$ with translations $t^n$. We choose a basis $L_{\mu \nu}, P_\lambda$ in $\mathfrak{gl}(n, \mathbb{R}) \supset t^n$ with the following standard set of commutation relations:

$$[L_{\mu \nu}, L_{\rho \lambda}] = \delta_\rho^\nu L_{\mu \lambda} - \delta_\lambda^\mu L_{\mu \rho}, \quad [L_{\mu \nu}, P_\lambda] = -\delta_\mu^\lambda P_\nu$$

(1)

$\mu, \nu, \ldots = 0, \ldots, n - 1$, and $n$ -denotes a dimension of spacetime $M = \mathbb{R}^n$ which is not yet provided with any metric structure. Nevertheless, for the sake of future applications, we shall use “relativistic” notation with spacetime indices $\mu$ and $\nu$ running $0, \ldots, n - 1$ and space indices $j, k = 1, \ldots, n - 1$.

This Lie algebra contains several interesting subalgebras, e.g., $\mathfrak{isl}(n, \mathbb{R}) = \mathfrak{isl}(n, \mathbb{R}) \supset t^n$ of the inhomogeneous special linear transformations. In this case instead of diagonal generators $L_{\mu \mu}$ we shall use its traceless counterparts:

$$L_\mu = L_{\mu \mu} - \frac{1}{n} L, \quad \text{where} \quad L = \sum L_{\mu \mu}$$

(2)

denotes a central element in $\mathfrak{gl}(n, \mathbb{R}),^2$ with $\sum L_\mu = 0$. Therefore, we have a basis $\{ L_{\nu \mu}, \mu \neq \nu, L_k, k = 1, \ldots, n - 1, P_\lambda \}$ of $n^2 + n - 1$ elements in $\mathfrak{isl}(n, \mathbb{R})$.

For any (constant of arbitrary signature) metric tensor $g = (g_{\mu \nu})$: $g_{\mu \nu} = g_{\nu \mu}$ on $\mathbb{R}^n$, one can associate a subalgebra of the inhomogeneous orthogonal transformations $\mathfrak{iso}(g; n) = \mathfrak{iso}(g; n) \supset t^n \subset \mathfrak{isl}(n, \mathbb{R})$, $^3$ which is defined by the following set of commutation relations:

$$[M_{\mu \nu}, M_{\rho \lambda}] = g_{\nu \rho} M_{\mu \lambda} + g_{\mu \rho} M_{\nu \lambda} - g_{\nu \lambda} M_{\mu \rho} - g_{\mu \lambda} M_{\nu \rho}$$

(3)

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1 See, e.g., [21] for a different framework.
2 However, it is not central in $\mathfrak{isl}(n, \mathbb{R})$.
3 We shall write $\mathfrak{iso}(n - p, p)$ whenever the signature $p$ will become important.
\[ [M_{\mu\nu}, P_\lambda] = g_{\nu\lambda} P_\mu - g_{\mu\lambda} P_\nu \] (4)

where \( M_{\mu\nu} = -M_{\nu\mu} \) is defined by the embedding

\[ M_{\mu\nu} = g_{\mu\lambda} L_\lambda - g_{\nu\lambda} L_\mu. \] (5)

The algebra \( \mathfrak{igl}(n, \mathbb{R}) \), as well as its classical subalgebras, acts on the algebra \( \mathcal{A}_M \) of smooth (complex-valued) functions on the spacetime manifold \( M = \mathbb{R}^n \) via first-order differential operators (derivations \( \equiv \) vectors fields): this is defined by natural representation - the so-called Schwinger realization:

\[ L_\mu = x^\mu \partial_\nu, \quad P_\lambda = \partial_\lambda \] (6)

of \( \mathfrak{igl}(n, \mathbb{R}) \) into infinite dimensional Lie algebra of complex-valued vector fields \( \mathcal{X}M = \text{Der}(\mathcal{A}_M) \). For the purpose of deformations, one needs to work over a (algebraically closed) field of complex numbers \( \mathbb{C} \). In what follows, we shall use complex Lie algebras \( \mathfrak{igl}(n) \equiv \mathfrak{igl}(n, \mathbb{C}) \), etc., instead their real counterparts. A real Lie algebra structure can be eventually encoded in a corresponding reality structure (involutive antiautomorphism) but we shall not focus on this point here.

In order to simplify the notation, we shall use the same letter to denote an abstract element in \( \mathfrak{igl}(n) \) and its (first-order) differential operator realization in \( \mathcal{X}M \). This induces an embedding of the corresponding enveloping algebras

\[ U_{\mathfrak{igl}(n)} \hookrightarrow U_{\mathcal{X}M} \] (7)

as an embedding of Hopf algebras with primitive coproducts

\[ \Delta(X) = X \otimes 1 + 1 \otimes X \] (8)

for \( X \in \mathcal{X}M \). Nondeformed counit and antipode maps read as \( \varepsilon(X) = 0, \varepsilon(1) = 1, S(X) = -X, \) and \( S(1) = 1 \). Let us emphasize that the enveloping algebra \( U_{\mathcal{X}M} \) is simultaneously an algebra of linear (complex-valued, partial) differential operators over \( M \). It can be equipped with a natural Hermitian involution defined on generators by \( (x^\mu)^* = x^\mu, (\partial_\mu)^* = -\partial_\mu \). The embedding (7) provides a real (anti-Hermitian) realization \( (Y^* = -Y, \text{ for } Y \in \mathfrak{g} ) \) only for the subalgebra \( \mathfrak{isl}(n, \mathbb{R}) \) and its subalgebras like \( \text{iso}(g; n) \). In these cases Hermitian conjugation is compatible with the corresponding reality structures. The action via derivations of \( \mathcal{X}M \) on \( \mathcal{A}_M \) extends to a Hopf module algebra action of \( U_{\mathcal{X}M} \) on the algebra \( \mathcal{A}_M \) (for details concerning Hopf module algebras, see, e.g., [13, 18]).

Since \( \mathcal{A}_M \) is a Hopf module algebra over \( U_{\mathcal{X}M} \), it becomes automatically a Hopf module algebra over its sub-Hopf algebras, too, particularly over \( U_{\mathfrak{igl}(n)} \) as well as its subsequent Hopf subalgebras. This can be
further deformed, by a suitable twisting element $\mathcal{F}$, to achieve deformed Hopf module algebra $(\mathcal{A}^F, U^F_{XM})$, where the algebra $\mathcal{A}^F_{M}$ is equipped with a twisted star-product

$$f \star g = \mu \circ \mathcal{F}^{-1}(f \otimes g) = \bar{\mu}(f)\bar{\mu}(g)$$

(9)

Hereafter the twisting element $\mathcal{F}$ is symbolically written in the following form:

$$\mathcal{F} = f^\alpha \otimes f^\alpha \quad \text{and} \quad \mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}^\alpha$$

(10)

Quantized Hopf algebra has nondeformed algebraic sector (commutators), while coproducts and antipodes are subject to deformation:

$$\Delta^\mathcal{F}(\cdot) = \mathcal{F}\Delta(\cdot)\mathcal{F}^{-1}, \quad S^\mathcal{F}(\cdot) = u S(\cdot) u^{-1}$$

(11)

where $u = f^\alpha S(f^\alpha)$. Let us recall that twisting two-tensor $\mathcal{F}$ is an invertible element in $U_{XM} \otimes U_{XM}$ which fulfills the 2-cocycle and normalization conditions [35, 55]:

$$\mathcal{F}_{12}(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta)\mathcal{F}, \quad (\epsilon \otimes id)\mathcal{F} = 1 = (id \otimes \epsilon)\mathcal{F}$$

(12)

Its relation with a corresponding classical $r$-matrix $r$ satisfying the classical Yang-Baxter equation is via a universal (quantum) $r$-matrix $\mathcal{R}$:

$$\mathcal{R} = \mathcal{F}^{21}\mathcal{F}^{-1} = 1 + a r \mod(a^2)$$

(13)

where $a$ denotes the deformation parameter. As it is well-known from the general framework of quantum deformations [35], a twisted deformation requires a topological extension of the enveloping algebra $U_g$ of some Lie algebra $g$ into an algebra of formal power series $U[[a]]$ in the formal parameter $a$ (see, e.g., [43, 55]). For the purpose of the present paper, we shall call $U_{XM}[[a]]$ an algebra of formal (or generalized) differential operators. Accordingly, the Hopf module algebra $\mathcal{A}^F_M$ has to be extended to $\mathcal{A}^F_M[[a]]$ as well. Particularly, deformed algebra $\mathcal{A}^F_{M}$ can be represented by deformed $\star$-commutation relations

$$[x^\mu, x^\nu]_\star \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = \theta^{\mu\nu}(x) \equiv \theta^{\mu\nu} + \theta^{\mu\nu}_A x^A + \ldots$$

(14)

replacing the nondeformed (commutative) one

$$[x^\mu, x^\nu] = 0$$

(15)

where the coordinate functions $(x^\mu)$ play a role of generators for the corresponding algebras: deformed and nondeformed. The action of differential operators on functions induced by derivations (vector fields) remains the same in deformed and nondeformed cases. Moreover, the twisted star product enables us to
introduce two operator realizations of the algebra \( A_F^\mathcal{M} \) in terms of (formal) differential operators on \( \mathcal{M} \). The so-called left-handed and right-handed realizations are naturally defined by

\[
\tilde{x}_L^\mu(f) = x^\mu \star f \quad \text{and} \quad \tilde{x}_R^\mu(f) = f \star x^\mu
\]  

(16)

with \( \tilde{x}_L^\mu, \tilde{x}_R^\mu \in U_{\mathcal{M}}[[a]] \) satisfying the operator commutation relations

\[
[\tilde{x}_L^\mu, \tilde{x}_L^\nu] = \theta^{\mu\nu}(\tilde{x}) \quad \text{and} \quad [\tilde{x}_R^\mu, \tilde{x}_R^\nu] = -\theta^{\mu\nu}(\tilde{x})
\]  

(17)

correspondingly. In other words, the above formulas describe embedding of \( A_F^\mathcal{M} \) into \( U_{\mathcal{M}}[[a]] \). These operator realizations are particular cases of the so-called Weyl map (see, e.g., [56] and references therein for more details). They allow us to calculate the commutator:

\[
[x^\mu, f]_\star = \tilde{x}_L^\mu(f) - \tilde{x}_R^\mu(f)
\]  

(18)

It has been argued in Ref. [15] (see also [20]) that any Lie-algebraic star product (when generators satisfy the Lie algebra structure)

\[
[x^\mu, x^\nu]_\star = i\theta^{\mu\nu} x^A
\]  

(19)

can be obtained by twisting an element in the form

\[
\mathcal{F} = \exp\left(\frac{1}{2} x^A g_A(\partial_y, \partial_z)\right) \big|_{y \to x, \ z \to x}
\]  

(20)

The star product

\[
f(x) \star g(x) = \exp\left(\frac{1}{2} x^A g_A(\partial_y, \partial_z)\right)f(y)g(z) \big|_{y \to x, \ z \to x}
\]  

(21)

would imply

\[
[x^\mu, f]_\star = i\theta^{\mu\nu} x^A \partial_\nu f
\]  

(22)

i.e., a vector field action on \( f \). The last formula is particularly important for obtaining a Seiberg-Witten map for noncommutative gauge theories (see [15, 20]). We are going to show using explicit examples that this formula is not satisfied for an arbitrary twist in the form of (20). However, we shall find an explicit twist for \( \kappa \)-deformed Minkowski spacetime which belongs to the class described by (22).

Before proceeding further, let us comment on some important differences between the canonical and Lie-algebraic cases. In the former, related to the Moyal product case

\[
[x^\mu, x^\nu]_{, M} = i\theta^{\mu\nu}
\]  

(23)
with a constant antisymmetric matrix $\theta^{\mu\nu}$; cf. (14), one finds that
\[ [g, f]_{\star, M} = \partial_{\nu} \zeta^\nu(f, g) \] (24)
is a total derivative, e.g.,
\[ [x^\mu, f]_{\star, M} = \partial_{\nu}(i\theta^{\mu\nu}f) \] (25)
This further implies the following tracial property of the integral:
\[ \int d^n x [g, f]_{\star, M} = 0 \] (26)
which is rather crucial for a variational derivation of Yang-Mills field equations (see [17]). In contrast to (25), Eq. (22) rewritten under the form
\[ [x^\mu, f]_{\star} = \partial_{\nu}(i\theta^{\mu\nu}x^\nu f) - i\theta^{\mu\nu}_\nu f \] (27)
indicate obstructions to the tracial property (26) provided that $\theta^{\mu\nu}_\nu \neq 0$. 4

III. $\kappa$–MINKOWSKI SPACETIME FROM TWIST: HOPF MODULE ALGEBRA POINT OF VIEW

Our first task is to find explicit twists in order to achieve a twisted star-product realization of the well-known $\kappa$–deformed Minkowski spacetime algebra [24, 25]:
\[ [x^0, x^m]_{\star} = iax^m \equiv \frac{t}{\kappa}x^m \] (28)
for $m = 1, \ldots, n - 1$ with remaining elements commuting. Here $\alpha$ is the above-mentioned formal parameter and $\kappa = \frac{1}{a}$ has the mass dimension. Strictly speaking, formulas (28) mean that the corresponding algebra of functions $\mathcal{A}_{\text{ind}}[[\alpha]]$ has been provided with a twisted star product (9) which leads to the commutation relations (28). Of course, as we shall see later on, different twisted star products may lead to the same commutation relations (28). In what follows, we shall present explicit results for two one-parameter families of twists providing quantum $\kappa$–Minkowski spacetime.

Jordanian family

To this aim we shall consider a one-parameter family of two-dimensional Borel subalgebras $\mathfrak{b}^2(r) = \{J_r, P_r\} \subset \mathfrak{gl}(n, \mathbb{R})$:
\[ [J_r, P_r] = P_r, \quad J_r = -x^0 \partial_0 + \frac{1}{r}x^k \partial_k, \quad P_r = r \partial_0 \] (29)

4 For the $\kappa$–deformation [see (28) and (42) below], one has $\theta^{\mu\nu}_\nu = (n - 1)a$. 

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with a numerical factor \( r \neq 0 \). In terms introduced before the \( \mathfrak{sl}(n) \) basis, the element \( J_r \) has the form 
\[
J_r = \left( \frac{\sigma_1}{n} - 1 \right) L - \frac{\sigma_1}{n} L_0.
\]
The corresponding one-parameter family of Jordanian twists\(^5\) can be expressed as (see [41]–[44] for Jordanian twists):
\[
\tilde{Z}_r = \exp (J_r \otimes \sigma_r)
\]
(30)
where \( \sigma_r = \ln(1 + iarP_0) \) with formal parameter \( a = \frac{1}{\kappa} \). Direct calculations show that, regardless of the value of \( r \), twisted commutation relations (14) take the form of that for \( \kappa \)-Minkowski spacetime (28).

The twists (30) can be used to deform the entire \( U_{XMM} \) Hopf algebra. For generic \( r \neq 0 \), the smallest subalgebra containing simultaneously the Borel subalgebra (29) and one of the orthogonal subalgebras \( \mathfrak{iso}(g; n) \) [e.g., Poincaré subalgebra \( \mathfrak{iso}(n - 1, 1) \)] is \( \mathfrak{isl}(n) \). However, there are three exceptions.

(A) For \( r = n - 1 \) in \( n \)-dimensional spacetime, the smallest subalgebra is \( \mathfrak{isl}(n) \).

(B) For \( r = -1 \) (\( J_{-1} = -L \)) in an arbitrary dimension, the smallest subalgebra is Weyl-orthogonal algebra \( \mathfrak{iwso}(g; n) \). It contains a central extension of any orthogonal algebra \( \mathfrak{so}(g; n) \).\(^6\) In this case, the commutation relation (3-4) should be supplemented by
\[
[M_{\mu\nu}, L] = 0, \quad [P_\mu, L] = P_\mu
\]
(31)
Of course, for physical applications we will choose the Weyl-Poincaré algebra. This minimal one-generator extension of the Poincaré algebra \( \mathfrak{iso}(n - 1, 1) \) has been used in Ref. [47] (cf. Appendix B).

(C) \( r = 1 \) in \( n = 2 \) dimensions, \( J_1 = M_{10} \) is a boost generator for nondiagonal metric \( g_{00} = g_{11} = 0 \), \( g_{01} = g_{10} = 1 \) with the Lorentzian signature. This corresponds to the so-called light-cone deformation of the Poincaré algebra \( \mathfrak{iso}(1, 1) \).\(^5\)

Since in the generic case we are dealing with \( \mathfrak{isl}(n) \) Lie algebra, we shall write deformed coproducts and antipodes in terms of its generators \( \{L_\mu, P_\mu\} \). The deformed coproducts read as follows:
\[
\Delta_r (P_0) = 1 \otimes P_0 + P_0 \otimes e^{i\sigma_r}, \quad \Delta_r (P_k) = 1 \otimes P_k + P_k \otimes e^{-\frac{i}{\kappa} \sigma_r}
\]
\[
\Delta_r (L_\mu^m) = 1 \otimes L_\mu^m + L_\mu^m \otimes 1, \quad \Delta_r (L_0^k) = 1 \otimes L_0^k + L_0^k \otimes e^{\frac{i}{\kappa} \sigma_r}
\]
(32)
(33)

\(^5\) The Borel subalgebra commutation relation leads to the validity of the cocycle condition (12) which in turn guarantees associativity of the corresponding star product.

\(^6\) The signature of the metric \( g \) is irrelevant from an algebraic point of view.

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\[ \Delta_r \left( L_k^0 \right) = 1 \otimes L_k^0 + L_k^0 \otimes e^{\frac{i \alpha r}{x}} - i a r J_r \otimes P_k e^{-\frac{i \alpha r}{x}} \]  

\[ \Delta_r \left( L_0^0 \right) = 1 \otimes L_0^0 + L_0^0 \otimes 1 - i a r J_r \otimes P_0 e^{-\frac{i \alpha r}{x}} \]

Here

\[ e^{\beta r} = (1 + iar P_0)^\beta = \sum_{m=0}^{\infty} \frac{d!}{m!} \beta^m (iar P_0)^m \]

and \( \beta_m = \beta(\beta - 1) \ldots (\beta - m + 1) \) denotes the so-called falling factorial. The antipodes are:

\[ S_r \left( P_0 \right) = -P_0 e^{-\frac{i \alpha r}{x}}, \quad S_r \left( P_k \right) = -P_k e^{\frac{i \alpha r}{x}} \]  

\[ S_r \left( L_0^k \right) = -L_0^k e^{-\frac{2i \alpha r}{x}}, \quad S_r \left( L_0^0 \right) = -\left( L_0^0 + iar J_r P_k \right) e^{\frac{i \alpha r}{x}} \]  

\[ S_r \left( L_0^0 \right) = -L_0^0 - iar J_r P_0, \quad S_r \left( L_k^m \right) = -L_k^m \]

The Jordanian one-parameter family of twists (30) generates, of course, left- and right-hand representations, respectively, which give a realization of (16) and (17) in the following form:

I. Left-handed representations:

\[ \hat{x}_{L,r}^i = x^i \left( 1 + r A \right)^{-\frac{1}{4}}, \quad \text{and} \quad \hat{x}_{L,r}^0 = x^0 \left( 1 + r A \right) \]

Hereafter one introduces for convenience a Hermitian operator \( A = ia \partial_0 \equiv \frac{i}{\kappa} \partial_0 \). \(^7\)

II. Right-handed representations (Hermitian for \( r = n - 1 \)):

\[ \hat{x}_{R,r}^i = x^i, \quad \text{and} \quad \hat{x}_{R,r}^0 = x^0 \left( 1 + r A \right) - i ax^k \partial_k \]

Particularly, using (13), we obtain

\[ [x^i, f]_{\ast,r} = x^i \left( 1 + r A \right)^{-\frac{1}{4}} - 1 \right) f, \quad [x^0, f]_{\ast,r} = i a x^k \partial_k f \]

which is different from (19). However for \( r = -1 \) one obtains the desired commutator

\[ [x^\mu, f]_{\ast} = i \left( a^\nu x^\mu - a^\mu x^\nu \right) \partial_\nu f \]

providing the \( \kappa \)-deformed Minkowski spacetime, i.e., \( a^\mu = (a, 0, \ldots, 0) \).

\(^7\) This notation will be particularly convenient and utilized in the subsequent section.
Abelian family

κ–Minkowski spacetime can be also implemented by the one-parameter family of Abelian twists \([45, 46]\) (with \(s\) being a numerical parameter):

\[
\mathfrak{A}_s = \exp \left[ -ia \left( s \partial_0 \otimes x^k \partial_k - (1 - s) x^k \partial_k \otimes \partial_0 \right) \right]
\] (43)

All are \(U_{\text{ig}(\omega)}\) twists in a sense explained before. A special case \(s = \frac{1}{2}\) has been treated in Ref. [45]. Thus one gets the following:

I. Left-handed representation (Hermitian for \(s = 0\)):

\[
\hat{x}^i_{L,s} = x^i e^{(s-1)A}, \quad \hat{x}^0_{L,s} = x^0 + iax^k \partial_k
\] (44)

II. Right-handed representation (Hermitian for \(s = 1\)):

\[
\hat{x}^i_{R,s} = x^i e^{sA}, \quad \hat{x}^0_{R,s} = x^0 + ias x^k \partial_k
\] (45)

This implies

\[
[x^i, f]_{\star,s} = x^i \left( e^{(s-1)A} - e^{sA} \right) f, \quad [x^0, f]_{\star,s} = iax^k \partial_k f
\] (46)

which is different from \(19\) for any value of the parameter \(s\). The deformed coproducts read as follows (cf. \([45]\)):

\[
\Delta_s (P_0) = 1 \otimes P_0 + P_0 \otimes 1, \quad \Delta_s (P_k) = e^{iasP_0} \otimes P_k + P_k \otimes e^{-ia(1-s)P_0}
\] (47)

\[
\Delta_s \left( L^m_k \right) = 1 \otimes L^m_k + L^m_k \otimes 1, \quad \Delta_s \left( L^k_0 \right) = e^{-iasP_0} \otimes L^k_0 + L^k_0 \otimes e^{ia(1-s)P_0}
\] (48)

\[
\Delta_s \left( L^0_k \right) = e^{iasP_0} \otimes L^0_k + L^0_k \otimes e^{-ia(1-s)P_0} - iasP_k \otimes D + ias(1-s)D \otimes P_k
\] (49)

\[
\Delta_s \left( L^0_0 \right) = 1 \otimes L^0_0 + L^0_0 \otimes 1 + iasP_0 \otimes D - ias(1-s)D \otimes P_0
\] (50)

Here \(D = x^k \partial_k\). The antipodes are

\[
S_s (P_0) = -P_0, \quad S_s (P_k) = -P_k e^{-iaP_0}
\] (51)

\[
S_s \left( L^m_k \right) = -L^m_k, \quad S_s \left( L^k_0 \right) = -L^k_0 e^{iaP_0}
\] (52)

\[
S_s \left( L^0_0 \right) = -L^0_0 - ias(1 - 2s)DP_0
\] (53)
\[ S_s \left( L_0^0 \right) = -e^{-iasP_0} L_0^0 e^{ias(1-s)P_0} - ia \left[ sP_1 D e^{-iasP_0} - (1-s)DP_k e^{ias(1-s)P_0} \right] \] (54)

The above relations are particularly simple for \( s = 0, 1, \frac{1}{2} \).

Before closing this section, let us point out that all twists considered here correspond to the same classical \( r \)-matrix (Poisson bi-vector on \( \mathbb{H} \)):

\[ r = (x^k \partial_k) \wedge \partial_0 \in \mathcal{X} \mathbb{M} \wedge \mathcal{X} \mathbb{M} \] (55)

IV. LIE ALGEBRA OF \( \kappa \)-DEFORMED LORENTZIAN SPACETIME.

The noncommutative \( \kappa \)-Minkowski space can be realized in quantized relativistic phase space \([58]\) or in a Schrödinger representation \((x^\mu, p_\mu = -i\partial_\mu)\) in terms of generalized differential operators.

In Refs. \([46, 53, 59]\), Meljanac et al., following an earlier development \([28, 60]\), have found an interesting realization of the noncommutative coordinates \( \hat{x}^\mu \) in terms of generalized differential operators. Their approach assumes the following ansatz:

\[ \hat{x}^i = x^i \phi(A), \quad \hat{x}^0 = x^0 \psi(A) + iax^k \partial_k \gamma(A) \] (56)

for noncommutative coordinates \( \hat{x}^\mu \in U \mathcal{X} \mathbb{M}[[a]] \), where as before \( A = ia\partial_0 \). Functions \( \phi, \psi, \) and \( \gamma \) are taken to be real analytic; however generalization to complex analytic is straightforward and will not be discussed here. These functions obey initial conditions \( \phi(0) = 1 \) and \( \psi(0) = 1 \), and \( \gamma(0) \) has to be finite in order to ensure a proper classical limit at \( a \to 0 \). The operators \( \hat{x}^i \) are automatically Hermitian while Hermiticity of \( \hat{x}^0 \) requires the additional assumption that

\[ \psi' + (n - 1)\gamma = 0 \] (57)

where \( \psi' = \frac{d\psi}{dA} \) and \( n \) denotes spacetime dimension. As we will see later on, this condition will be satisfied only in a few exceptional cases. Apparently, the ansatz (56) could be guessed from our twisted realizations (39), (40), (44), (45).

Now, the \( \kappa \)-Minkowski commutation relations [cf. (28)]

\[ [\hat{x}^0, \hat{x}^k] = i\tau \hat{x}^k \] (58)

are equivalent to the property that functions \( \phi, \psi, \) and \( \gamma \) do satisfy the ODE \([53]\):

\[ (\ln \phi)' \psi = \gamma - \tau \] (59)

hereafter \( \tau = \pm 1 \) for convenience [cf. (17)] \(^8\).

\(^8\) The sign convention \( \tau = \pm 1 \) is due to the difference between left- and right-handed realizations; see (17). In fact, it can be eliminated by rescaling \( \hat{x}^0 \mapsto \tau \hat{x}^0 \).
Throughout this paper we shall be interested in solutions of (59) for $\gamma = \text{constant}$. In this case, for any $\psi$ one easily gets

$$\phi = \Psi^{\gamma - \tau} \quad \text{where} \quad \Psi = \exp \left( \int_0^A \frac{dA'}{\psi(A')} \right)$$  \hspace{1cm} (60)

Specializing further to the linear case $\psi = 1 + rA$, one finds

$$\Psi = e^A \quad \text{for} \quad r = 0$$  \hspace{1cm} (61)

and

$$\Psi = (1 + rA)^{\frac{1}{r}} \quad \text{for} \quad r \neq 0$$  \hspace{1cm} (62)

The Hermiticity restriction (57) is satisfied only for $\psi = 1$ and $\gamma = 0$ and in $n$-dimensions for $\psi = 1 \pm (n - 1)A, \gamma = \mp 1$ subcases. All of our twisted products realizations turn out to be special cases of the above more general formulas (60)-(62).

I.) One finds that the first case (61), i.e., $\psi = 1$, corresponds to the Abelian twists with $\gamma = s, \tau = +1$ for left-handed realizations [cf. (44)] and $\gamma = s - 1, \tau = -1$ for right-handed ones. Two subcases $s = 0, \tau = 1$ and $s = 1, \tau = -1$ give rise to Hermitian representations.

II.) The case $r \neq 0$ (62) is related to the Jordanian family under rather restricted values of $\gamma$: for left-handed representation $\gamma = 0, \tau = 1$, while $\gamma = \tau = -1$ for right-handed ones [cf. (40)]. We do not know twist realizations for generic $\gamma$ and $r \neq 0$. Among Jordanian twists only the case $\gamma = \tau = -1$ with $r = n - 1$ is Hermitian in a spacetime of dimension $n$ and can be reduced to the subgroup $isl(n)$.

Now, following the general method developed in Ref. [53] (see also [60]), one can try to covariantly incorporate the $\kappa$–Minkowski algebra (58) into the extension of undeformed orthogonal algebra (5) by assuming

$$[M_{\mu \nu}, M_{\rho \lambda}] = \eta_{\nu \rho} M_{\mu \lambda} + \eta_{\mu \lambda} M_{\nu \rho} - \eta_{\nu \lambda} M_{\mu \rho} - \eta_{\mu \rho} M_{\nu \lambda}$$  \hspace{1cm} (63)

$$[M_{\mu \nu}, \hat{x}_\lambda] = \eta_{\nu \lambda} \hat{x}_\mu - \eta_{\mu \lambda} \hat{x}_\nu - i a_\mu M_{\nu \lambda} + i a_\nu M_{\mu \lambda}$$  \hspace{1cm} (64)

where $\hat{x}_\lambda = \eta_{\lambda \nu} \hat{x}^\nu$ and $a_\mu = \eta_{\mu \nu} a^\nu : (a^\nu) = (\tau a, 0, \ldots, 0)$. The main point is the ansatz

$$M_{i0} = x_i \partial_0 F_1 - x_0 \partial_i F_2 + i a x_i \Delta F_3 + i a x^k \partial_k \partial_i F_4$$  \hspace{1cm} (65)

where $F_p \equiv F_p(A), p = 1, 2, 3, 4$ are (real) analytic functions to be determined from (63)-(64). In turn, generators

$$M_{ij} = x_i \partial_j - x_j \partial_i$$
remain undeformed, i.e., in the Schwinger realization: $M_{ik}^* = -M_{ik}$. Formula (65) describes the deformed realization of the "boost" generators $M_{i0}$ together with the initial conditions $F_1(0) = F_2(0) = 1$. A difference between our and the original approach [53] is that we do not apriori assume the Euclidean signature for the metric $\eta_{\mu\nu}$. One should notice that the Lorentzian signature $\eta_{\mu\nu} = (-1, 1, \ldots, 1)$ is more natural and expected in this context: the commutation relations (58) distinguish one of the variables $x^0$. In contrast, the Euclidean signature puts all variables on equal footing.

In order to keep under control the difference between the Euclidean and Lorentzian cases, we shall temporarily introduce a coefficient $\epsilon = \eta_{00} = \pm 1$. (Notice that $x_0 = \epsilon x^0$ and $[\partial_0, x_0] = \epsilon$.)

Inserting ansatz (65) into the algebra (63), we obtain the following equations:

\[ F_1 F_2 + AF_1' F_2 + \epsilon AF_1 F_4 - 2\epsilon AF_1 F_3 = 1 \]  
(66)

\[ 2F_2^2 - \epsilon F_3' F_2 + F_3 F_4 = 0 \]  
(67)

where $\epsilon = 1$ (Euclidean case) or $\epsilon = -1$ (Lorentzian case). Substituting (56) and (65) into

\[ [M_{i0}, \hat{x}_i] = \epsilon \hat{x}_i + i\epsilon \tau M_{i0} \]  
(68)

\[ [M_{i0}, \hat{x}_j] = -\delta_{ij} \hat{x}_0 + i\epsilon \tau M_{ij} \]  
(69)

one obtains some overdetermined system of ODE; see Appendix A for details. Its solutions can be recast into the form (here $\phi = \Psi^{\gamma - \tau}$):

\[ F_2 = \frac{\psi}{\phi}; \quad F_3 = \frac{\eta \tau}{2\phi}; \quad F_4 = -\frac{\epsilon \gamma}{\phi} \]  
(70)

and

\[ F_1 \psi + AF_1' \psi - AF_1 (\gamma + \tau) - \phi = 0 \]  
(71)

The last equation is consistent with (59), (66, 70) for both values $\epsilon = \pm 1$. Its solution

\[ F_1(A) = \Psi^\gamma \Psi - \Psi^{-1} 2A \equiv \Psi^\gamma \Psi^{\tau} - \Psi^{-\tau} 2A \tau \]  
(72)

together with (70) determines generators $M_{i0}$ completely. It is worth noticing that the Hermticity of $\hat{x}_0$ automatically implies reality for the boost generators: $M_{i0}^* = -M_{i0}$. Since the Euclidean case has been already studied in Refs. [53, 59], further on, until the end of the present paper, we shall use only Lorentzian signature $\epsilon = -1$. 

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Analogously, following Ref. [53] we have also obtained (with some sign corrections) a generalized d’Alambert operator \( \square = \Delta H_1 - \partial_0^2 H_2 \):

\[
[M_{\mu\nu}, \square] = 0 \tag{73}
\]

under the form \([H_1(0) = H_2(0) = 1]\)

\[
H_1(A) = \Psi^{-2\gamma}, \quad H_2(A) = (\Psi + \Psi^{-1} - 2)A^{-2} \tag{74}
\]

where \( \Delta = \partial_k \partial^k \) above denotes a \( n-1 \) dimensional, spacelike, Laplace operator. We are now in a position to define Dirac derivatives \( D_\mu \) as

\[
[\tilde{\square}, \hat{x}_\mu] = 2D_\mu, \quad [\tilde{\square}, D_\mu] = 0 \tag{75}
\]

This implies

\[
D_0 = \partial_0 G_2 + ia\Delta G_3, \quad D_i = \partial_i G_1 \tag{76}
\]

where (see Appendix A)

\[
G_1 = H_1 \phi = \Psi^\gamma, \quad G_2 = G_1 F_1 = \frac{\Psi - \Psi^{-1}}{2A}, \quad G_3 = -\frac{\tau}{2} H_1 = -\frac{\tau}{2} \Psi^{-2\gamma} \tag{77}
\]

Let us remark that only formulas for \( G_2 \) and \( H_2 \) are universal in the sense that they do not depend on the parameters \( \gamma \) and \( \tau \).

Direct calculations performed on solutions with constant \( \gamma \) give rise to

\[
D_\mu D^\mu = \tilde{\square} \left( 1 + \frac{a^2}{4} \tilde{\square} \right), \quad \sqrt{1 + a^2 D_\mu D^\mu} = 1 + \frac{a^2}{2} \tilde{\square} \tag{78}
\]

and \((\tau = \pm 1)\)

\[
\Psi^{-\tau} = -ia\tau D_0 + \sqrt{1 + a^2 D_\mu D^\mu}. \tag{79}
\]

These relations allow us to enlarge the Lorentz - \( \kappa \)–Minkowski algebra [53], [63], [64] by the following commutation relations:

\[
[D_\mu, D_\nu] = 0, \quad [M_{\mu\nu}, D_3] = \eta_{\nu\lambda} D_\mu - \eta_{\mu\lambda} D_\nu \tag{79}
\]

\[
[D_k, \hat{x}_0] = 0, \quad [D_k, \hat{x}_j] = \delta_{jk} \left( -ia\tau D_0 + \sqrt{1 + a^2 D_\mu D^\mu} \right) \tag{80}
\]

\[
[D_0, \hat{x}_j] = -ia\tau D_j, \quad [D_0, \hat{x}_0] = -\sqrt{1 + a^2 D_\mu D^\mu} \tag{81}
\]

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in order to include the Dirac derivatives. In this way one has obtained a new (non Lie-algebraic) quantum extension of the Lorentz algebra which contains \( \frac{1}{2} n(n+3) \) generators \((M_{\mu\nu}, \hat{x}_\mu, D_\mu)\) in \( n \)-dimensional spacetime \( \mathbb{M}^9 \). Its algebraic structure is completely described by the commutation relations (58), (63), (64), (79) - (80) and does not depend on a particular differential operator realization which had been used for its construction. Particularly, it does not depend on a twisting tensor itself. It contains undeformed Poincaré algebra \( \text{iso}(n - 1, 1): (M_{\mu\nu}, D_\mu) \); cf. (79). Another Lie-algebraic part \((M_{\mu\nu}, \hat{x}_\mu)\) splits into Lorentzian sub-algebra \( \text{so}(n - 1, 1) \) generated by \((M_{\mu\nu})\) combined with the quantum \( \kappa \)-Minkowski space \((\hat{x}^\mu)\): (58), (63), (64). It contains one free (formal) parameter \( a \) and sign convention \( \tau \equiv \pm 1 \). From the algebraic point of view this dependence can be removed by rescaling \( \hat{x}_0 \leftrightarrow \frac{a}{\tau} \hat{x}_0 \) or equivalently by setting \( \tau a = 1 \). It makes this algebra isomorphic by substituting

\[
\hat{x}_0 = ia \tau M_{0i}, \quad \hat{x}_i \sim M_{0i} - M_{ii}
\]

(82)
to the (nondeformed) simple \( \text{so}(n, 1) \) Lie algebra for a "frozen" value of the parameter \( a \). This observation can be helpful for studying representations for such systems.

From the physical point of view, however, a dimensionful constant \( \kappa = \frac{1}{a} \) can be related with some fundamental constant of nature, similarly like in DSR \(^{10}\). Assuming for a moment that twist has a physical meaning, we will see how its parameters enter the so-called dispersion relations obtained from plane wave solutions of the corresponding Klein-Gordon equations with a deformed operator \( \tilde{\Box} \). Before doing that, let us emphasize that the presented formalism has a well-defined classical limit \( a \to 0 \) which reconstructs standard Minkowski spacetime together with the Poincaré group acting on it and Heisenberg-type relations between position and momenta operators. Particularly, \( \tilde{\Box} \) becomes a standard d’Alambert operator \( \Box \).

**Dispersion relations**

To this aim let us consider as a toy model involving d’Alambert operator

\[
\tilde{\Box} = \triangle H_1(A) - \partial_0^2 H_2(A)
\]
in specific realizations (74). One is looking for a plane wave solution of the deformed Klein-Gordon equation

\[
\left( \Box - m_0^2 \right) \omega_k = 0
\]

(83)

\(^9\) It is a relativistic version of the Euclidean algebra obtained already in [53].

\(^{10}\) See [33] for interrelations between DSR and de Sitter group \( SO(4, 1) \).
where \( \omega_k = \exp(i k_{\mu} x^\mu) \) represents the plane wave with the covariant wave vector \( k = (k_{\mu}) \); \( m_0 \) denotes a mass. Straightforward and simple calculations give rise to the general form of deformed "dispersion relation"

\[
m_0^2 = k_0^2 H_2 \left( -\frac{k_0}{\kappa} \right) - k^2 H_1 \left( -\frac{k_0}{\kappa} \right)
\]

where \( k^2 = k^i k_i \). Further specialization to the case when \( \psi = 1 \) which corresponds to the Abelian twists give rise to

\[
m_0^2 = \left[ 2\kappa \sinh \left( \frac{k_0}{2\kappa} \right) \right]^2 - k^2 \exp \left[ (2s - 1) \frac{k_0}{\kappa} \right]
\]

(85)

For the case \( s = 1 \), the above expression is in agreement with the formula known from DSR and which had been originated in a \( \kappa \)-Poincaré algebra deformed Casimir operator. However, for the Hermitian case \( s = 0 \), the dispersion formula is different. For Jordanian twists (\( \psi = 1 + rA \), \( \tau - 2\gamma = 1 \)), one gets instead

\[
m_0^2 = \kappa^2 \left[ \left( 1 - \frac{k_0}{\kappa} \right)^{\frac{1}{2}} + \left( 1 - \frac{k_0}{\kappa} \right)^{-\frac{1}{2}} - 2 \right] - k^2 \left[ 1 - \frac{k_0}{\kappa} - \left( \frac{k_0}{\kappa} \right)^2 - \ldots \right]
\]

(86)

Particularly, the Hermitian case in \( n = 4 \) dimensions requires \( r = 3 \) and gives

\[
m_0^2 = k_0^2 \left[ 1 + 3 \frac{k_0}{\kappa} + \frac{25}{3} \left( \frac{k_0}{\kappa} \right)^2 + \ldots \right] - k^2 \left[ 1 - \frac{k_0}{\kappa} - \left( \frac{k_0}{\kappa} \right)^2 - \ldots \right]
\]

(87)

In contrast, for our special \( r = -1 \) case, the dispersion formula takes a very simple linear "mass renormalization" form

\[
m_0^2 \left( 1 + \frac{k_0}{\kappa} \right) = k_0^2 - k^2
\]

(88)

Its mathematical structure is similar to that in Ref. [32]. Particularly, for (free) massless fields, e.g., in electrodynamics, wave equations remain unchanged.

V. CONCLUSION

We have obtained \( \kappa \)-deformed Minkowski spacetime by twisting \( i\mathfrak{g}(n, \mathbb{R}) \) with a one-parameter family of Jordanian and Abelian twists. In both cases, we have introduced deformed coproducts, antipodes, and (generalized) differential operator realizations of noncommutative coordinates \( \hat{x}^\mu \). Hermitian representations are found for both families of twists.

Deformed generators \( M_{\mu\nu}, \hat{x}_\mu, \) d’Alambert operator \( \hat{\Box} \), and Dirac derivatives \( D_\mu \) are found, following the method developed in Refs. [53, 59, 60], to form a new (quantum) algebra of non-Lie type which contains
undeformed Poincaré algebra $\text{iso}(n-1,1)$. Its Lie-algebraic part $(M_{\mu\nu}, \xi_\mu)$ is shown to be isomorphic to the classical simple Lie algebra $\text{so}(n,1)$.

As a physical application, we have calculated dispersion relations ensuing from the plane wave solutions of the generalized Klein-Gordon equation. One recovers a standard DSR dispersion formula as well as a new one. Although such twist dependence of dispersion relations seems to be unsatisfactory from a physical point of view, in principle one does not know what to expect at the Planck scale. On the other hand, a similar situation is encountered in quantum mechanics: a spectrum of the Schrödinger operator depends on a concrete realization (e.g., potential).

The special case of the Jordanian family gives the minimal symmetry algebra: Weyl-Poincaré algebra which is studied in more details in Appendix B. The dispersion formula related to this minimal case has a new and interesting form of linear mass deformation. In addition, the commutation relations (14) take a form (42) which is necessary in NC Yang-Mills theory.

We summarize the paper with a claim that the role of twist has to be reconsidered: although infinitely many twists accomplish the same $\kappa$–Minkowski spacetime together with the same $\text{so}(n,1)$ algebra and its nonlinear extension, the physical properties of a concrete model are twist-dependent. One possible answer may be done by relating twist with some interaction. In order to solve this problem, some further studies and discussions are needed.

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Appendix A

Following the same methods as in Ref. [53] and taking into account the Lorentzian signature for the spacetime metric, one obtains a partially modified set of ODEs for functions $G_i$ determining generators:

$$D_i = \partial_{i}G_1 (A); \quad D_0 = \partial_{0}G_2 (A) + i a \Delta G_3 (A);$$
and for functions $H_i$ introduced in the ansatz for the d’Alambert operator:

\[ \tilde{\Box} = \Delta H_1(A) - \partial_0^2 H_2(A) \]

This set of equations reads

\[
G_1 F_1 = G_2 ; \quad F_3 G_1 = G_3 \tag{89}
\]

\[
G'_1 F_2 + F_4 G_1 = 0 \tag{90}
\]

\[
AG'_2 F_2 + F_2 G_2 + 2A F_1 G_3 - G_1 = 0 \tag{91}
\]

\[
G'_3 F_2 + 2G_3 (F_3 + F_4) = 0 \tag{92}
\]

\[
AH'_2 F_2 + 2F_2 H_2 - 2F_1 H_1 = 0 \tag{93}
\]

\[
H'_1 F_2 + 2(F_3 + F_4) H_1 = 0 \tag{94}
\]

Appendix B

Minimal case: Weyl-Poincaré algebra

The minimal case $r = -1$ in physical $n = 4$ dimensions deserves special attention. It is related to research in Ref. [47], where, however, the $\kappa$-Minkowski spacetime has been obtained with different techniques (see the introduction). Below we shall present coproducts and antipodes for all 11 generators in "physical" basis $(M_k, N_k, L, P_\mu)$ of the Poincaré-Weyl algebra containing the Lorentz subalgebra of rotation $M_k = -\frac{1}{2} \epsilon_{klm} M_{lm}$ and boost $N_k = i M_{k0}$ generators:

\[
[M_i, M_j] = i \epsilon_{ijk} M_k , \quad [M_i, N_j] = i \epsilon_{ijk} N_k , \quad [N_i, N_j] = -i \epsilon_{ijk} M_k \tag{95}
\]

Abelian four-momenta $P_\mu = -i \partial_\mu$ ($\mu = 0, \ldots, 3, k = 1, 2, 3$)

\[
[M_j, P_k] = i \epsilon_{jkl} P_l , \quad [M_j, P_0] = 0 , \tag{96}
\]

\[
[N_j, P_k] = -i \delta_{jk} P_0 , \quad [N_j, P_0] = -i P_j \tag{97}
\]

and dilatation generator $L = x^\mu \partial_\mu$ as before:

\[
[M_k, L] = [N_k, L] = 0 \quad [P_\mu, L] = P_\mu \tag{98}
\]
The deformed coproducts are

\[
\tilde{\Delta}(P_\mu) = 1 \otimes P_\mu + P_\mu \otimes e^{\tilde{\sigma}}, \quad \tilde{\Delta}(M_k) = 1 \otimes M_k + M_k \otimes 1
\]

(99)

\[
\tilde{\Delta}(N_k) = 1 \otimes N_k + N_k \otimes 1 + \frac{1}{\kappa} L \otimes P_k e^{-\tilde{\sigma}}
\]

(100)

\[
\tilde{\Delta}(L) = 1 \otimes L + L \otimes 1 + \frac{1}{\kappa} L \otimes P_0 e^{-\tilde{\sigma}}
\]

(101)

Here \(e^{-\tilde{\sigma}} = \left(1 + \frac{1}{\kappa} P_0 \right)^{-1}\) and \(L = -J_{-1}\). The antipodes are

\[
\tilde{S}(P_\mu) = -P_\mu e^{-\tilde{\sigma}}, \quad \tilde{S}(M_k) = -M_k
\]

(102)

\[
\tilde{S}(N_k) = -N_k + \frac{1}{\kappa} LP_k, \quad \tilde{S}(L) = -L + \frac{1}{\kappa} LP_0
\]

(103)

The corresponding natural left- and right-handed representations of noncommutative coordinates \(\hat{x}_\mu\) and deformed Lorentz generators \(M_{\mu\nu}\), Dirac derivatives \(D_\mu\) and d’Alembert operator \(\tilde{\Box}\) are determined by functions \(\psi, \phi, F_i, G_i,\) and \(H_i\).

In this case one gets (formulas below are spacetime dimension-independent) the following:

I. For left-handed representation \((\tau = 1, \gamma = 0, r = -1)\)

\[
\psi = \phi = 1 - A, \quad F_1(A) = \frac{2 - A}{2(1 - A)}
\]

(104)

\[
F_2 = 1, \quad F_3 = -\frac{1}{2} \psi^{-1}, \quad F_4 = 0
\]

(105)

\[
G_1 = 1, \quad G_2 = \frac{A - 2}{2(A - 1)}, \quad G_3 = -\frac{1}{2} \psi^{-1}
\]

(106)

\[
H_1 = H_2 = \psi^{-1}
\]

(107)

II. For right-handed representation \((\tau = \gamma = r = -1)\)

\[
\psi = 1 - A, \quad \phi = 1, \quad F_1(A) = 1 - \frac{1}{2} A
\]

(108)

\[
F_2 = 1 - A, \quad F_3 = \frac{1}{2}, \quad F_4 = -1
\]

(109)

\[
G_2 = \frac{A - 2}{2(A - 1)}, \quad G_3 = \frac{1}{2} \psi^{-1}, \quad G_1 = H_1 = H_2 = \psi^{-1}
\]

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