Abstract. In this paper, we study the ratio of the $L_1$ and $L_2$ norms, denoted as $L_1/L_2$, to promote sparsity. Due to the non-convexity and non-linearity, there has been little attention to this scale-invariant metric. Compared to popular models in the literature such as the $L_p$ model for $p \in (0, 1)$ and the transformed $L_1$ (TL1), this ratio model is parameter free. Theoretically, we present a weak null space property (wNSP) and prove that any sparse vector is a local minimizer of the $L_1/L_2$ model provided with this wNSP condition. Computationally, we focus on a constrained formulation that can be solved via the alternating direction method of multipliers (ADMM). Experiments show that the proposed approach is comparable to the state-of-the-art methods in sparse recovery. In addition, a variant of the $L_1/L_2$ model to apply on the gradient is also discussed with a proof-of-concept example of MRI reconstruction.

Key words. Sparsity, $L_0$, $L_1$, null space property, alternating direction method of multipliers, MRI reconstruction.

AMS subject classifications. 90C90, 65K10, 49N45, 49M20

1. Introduction. Sparse signal recovery is to find the sparsest solution of $Ax = b$ where $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$ for $m \ll n$. This problem is often referred to as compressed sensing (CS) in the sense that the sparse signal $x$ is compressible. Mathematically, this fundamental problem in CS can be formulated as

$$\min_{x \in \mathbb{R}^n} \|x\|_0 \quad \text{s.t.} \quad Ax = b,$$

where $\|x\|_0$ is the number of nonzero entries in $x$. Unfortunately, (1.1) is NP-hard [29] to solve. A popular approach in CS is to replace $L_0$ by the convex $L_1$ norm, i.e.,

$$\min_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \quad Ax = b.$$  

Computationally, there are various $L_1$ minimization algorithms such as primal dual [8], forward-backward splitting [34], and alternating direction method of multipliers (ADMM) [4].

A major breakthrough in CS was the restricted isometry property (RIP) [6], which provides a sufficient condition of minimizing the $L_1$ norm to recover the sparse signal. Another sufficient condition is given in terms of null space of the matrix $A$, thus referred to as null space property (NSP). In particular, Zhang [48] proved that if a vector $x^*$ satisfies $Ax^* = b$ and

$$\sqrt{\|x^*\|_0} < \frac{1}{2} \min_{v} \left\{ \frac{\|v\|_1}{\|v\|_2} : v \in \ker(A) \setminus \{0\} \right\},$$

then $x^*$ is an optimal solution to both (1.1) and (1.2). Unfortunately, neither RIP nor NSP can be computed to verify for a given matrix [1, 39].
Alternatively, a computable condition for $L_1$’s exact recovery is based on coherence, which is defined as

\begin{equation}
\mu(A) := \max_{i \neq j} \frac{|a_i^T a_j|}{\|a_i\|\|a_j\|},
\end{equation}

for a matrix $A = [a_1, \cdots, a_N]$. Donoho-Elad [12] and Gribonval [16] proved independently that if

\begin{equation}
\|x\|_0 < \frac{1}{2} \left(1 + \frac{2}{\mu(A)} \right),
\end{equation}

then the $L_1$ minimization (1.2) yields the sparsest solution of (1.1). Clearly, the coherence $\mu(A)$ is bounded by $[0, 1]$. The inequality (1.5) implies that $L_1$ may not perform well for highly coherent matrices, i.e., $\mu(A) \sim 1$, as $\|x\|_0$ is then at most one (often not feasible).

Other than the popular $L_1$ norm, there are a variety of regularization functionals to promote sparsity, such as $L_p$ [9, 42, 21], $L_1$-$L_2$ [43, 23], capped $L_1$ (CL1) [47, 37], and transformed $L_1$ (TL1) [27, 45, 46]. Most of these models are nonconvex, leading to difficulties in proving exact recovery guarantees and algorithmic convergence, but they tend to give better empirical results compared to the convex $L_1$ approach. For example, it was reported in [43, 23] that $L_p$ gives superior results for incoherent matrices (i.e., $\mu(A)$ is small), while $L_1$-$L_2$ is the best for the coherent scenario. In addition, TL1 is always the second best no matter whether the matrix is coherent or not [45, 46].

In this paper, we study the ratio of $L_1$ and $L_2$ as a scale-invariant metric to approximate the desired $L_0$, which is scale-invariant itself. In one dimensional (1D) case (i.e., $n = 1$), the $L_1$/$L_2$ model is exactly the same as the $L_0$ model if we use the convention $\frac{0}{0} = 0$. The ratio of $L_1$ and $L_2$ was first proposed by Hoyer [18] as a sparseness measure and later highlighted in [19] as a scale-invariant metric. However, there has been little attention on it due to its computational difficulties arisen from being non-convex and non-linear. There are some theorems that establish the equivalence between the $L_1$/$L_2$ and the $L_0$ models, but only restricted to nonnegative signals [14, 43]. We aim to apply this ratio model to arbitrary signals. On the other hand, the $L_1$/$L_2$ metric has an intrinsic drawback that it tends to produce one erroneously large coefficient while suppressing the other non-zero elements, under which case the ratio is reduced. To compensate for this drawback, it is helpful to incorporate a box constraint, which will also be addressed in this paper.

Now we turn to a sparsity-related assumption that signal is sparse after a given transform, as opposed to signal itself being sparse. This assumption is widely used in image processing. For example, a natural image, denoted by $u$, is mostly sparse after taking gradient, and hence it is reasonable to minimize the $L_0$ norm of the gradient, i.e., $\|\nabla u\|_0$. To bypass the NP-hard $L_0$ norm, the convex relaxation replaces $L_0$ by $L_1$, where the $L_1$ norm of the gradient is the well-known total variation (TV) [36] of an image. A weighted $L_1$-$\alpha L_2$ model (for $\alpha > 0$) on the gradient was proposed in [24], which suggested that $\alpha = 0.5$ yields better results than $\alpha = 1$ for image denoising, deblurring, and MRI reconstruction. The ratio of $L_1$ and $L_2$ on the image gradient was used in deconvolution and blind deconvolution [20, 35]. We further adapt the proposed ratio model from sparse signal recovery to imaging applications, specifically focusing on MRI reconstruction.

The rest of the paper is organized as follows. In Section 2, we give a brief review of related works in CS including models and corresponding algorithms. Section 3 is devoted to theoretical analysis of the $L_1$/$L_2$ model. In Section 4, we apply the ADMM to minimize the ratio of $L_1$ and $L_2$ with two variants of incorporating a box constraint as well as applying on the image gradient. We conduct extensive experiments in Section 5 to demonstrate the performance of the proposed
The new formulation makes the objective function separable with respect to two variables $x$ and $y$. As a result, we form the augmented Lagrangian corresponding to (2.3) as

$$L_\rho(x, y; u) = \|x\|_1 + I(Ay - b) + \frac{\rho}{2} \|x - y\|_2^2,$$

for a positive parameter $\rho$ and a dual variable $u$. We aim to update $x, y$ in an alternating or sequential fashion, with a hope that each subproblem has a closed-form solution or can be calculated efficiently. This is a basic idea of the ADMM [4] that consists of three steps:

$$\begin{align*}
    x^{(k+1)} &= \arg\min_x L_\rho(x, y^{(k)}; u^{(k)}) = \arg\min_x \|x\|_1 + \frac{\rho}{2} \|x - y^{(k)} + u^{(k)}\|_2^2, \\
    y^{(k+1)} &= \arg\min_y L_\rho(x^{(k+1)}, y; u^{(k)}) = \arg\min_y I(Ay - b) + \frac{\rho}{2} \|x^{(k+1)} - y + u^{(k)}\|_2^2, \\
    u^{(k+1)} &= u^{(k)} + \rho(x^{(k+1)} - y^{(k+1)}).
\end{align*}$$

Both $x$ and $y$ subproblems have closed-form solutions. Specifically, the update for $x$ is given by

$$x^{(k+1)} = \text{shrink}\left(y^{(k)} - \frac{u^{(k)}}{\rho}, \frac{1}{\rho}\right),$$

where $\text{shrink}$ is often referred to as soft shrinkage,

$$\text{shrink}(v, \mu) = \text{sign}(v_i) \max(|v_i| - \mu, 0), \quad i = 1, 2, \ldots, n.$$

The update for $y$ is a projection to the affine space of $Ay = b$, i.e.,

$$y^{(k+1)} = (I - A^T(AA^T)^{-1}A)(x^{(k+1)} + \frac{u^{(k)}}{\rho}).$$
2.2. DCA Minimization. We describe a general framework of the difference of convex algorithm (DCA) [32, 33]. The DCA is applicable when the objective function has the form of difference of convex (DC) functions, i.e.,

\[
(2.9) \quad \min_{\mathbf{x} \in \mathbb{R}^n} g(\mathbf{x}) - h(\mathbf{x}).
\]

By linearizing the second term \(h(\cdot)\), the DCA iterates as follows,

\[
(2.10) \quad \mathbf{x}^{(k+1)} = \arg \min_{\mathbf{x} \in \mathbb{R}^n} g(\mathbf{x}) - \left\langle \mathbf{x}, \nabla h(\mathbf{x}^{(k)}) \right\rangle.
\]

If \(h(\cdot)\) is not differentiable at \(\mathbf{x}^k\), then the gradient \(\nabla h(\mathbf{x}^{(k)})\) can be replaced by any subgradient at \(\mathbf{x}^k\). Due to the flexibility of DC decomposition, \(g\) and \(h\) can be assumed to be strongly convex without loss of generality, in this case any limit point of the generated sequence is a critical point of (2.9) [32, 33]. On the other hand, there may still exist directions at critical points where the objective functions can be improved locally (e.g., one can find an univariate convex example [11, Example 2]). Some recent works focus on a sharper notion of d(irectional)-stationarity where no local improvement can be made [31, 25, 38], which is beyond the scope of the current paper.

We apply the DCA for two sparse promoting metrics: \(L_1-L_2\) [23, 44] and transformed \(L_1\) (TL1) [45, 46]. For practical performance we simply use a natural DC decomposition such that the convex subproblems in iterations are linear programs. In particular, the \(L_1-L_2\) model [23, 44] can be expressed as

\[
(2.11) \quad \min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{x}\|_1 - \|\mathbf{x}\|_2 \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{b}.
\]

We take \(g(\mathbf{x}) = \|\mathbf{x}\|_1 + I(A\mathbf{x} - \mathbf{b})\) and \(h(\mathbf{x}) = \|\mathbf{x}\|_2\). Since \(\nabla h(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|_2}\), the DCA goes as follows,

\[
(2.12) \quad \mathbf{x}^{(k+1)} = \arg \min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{x}\|_1 + I(A\mathbf{x} - \mathbf{b}) - \left\langle \mathbf{x}, \frac{\mathbf{x}^{(k)}}{\|\mathbf{x}^{(k)}\|_2} \right\rangle,
\]

which is an \(L_1\)-type of problem and can be solved by the ADMM. On the other hand, the TL1 model is defined as

\[
(2.13) \quad \min_{\mathbf{x} \in \mathbb{R}^n} \sum_{i=1}^{n} \rho_a(x_i) + I(A\mathbf{x} - \mathbf{b}), \quad \text{with} \quad \rho_a(t) = \frac{(1+a)|t|}{a + |t|},
\]

for \(a > 0\). TL1 approaches to \(L_0\) as \(a \to 0\) and approaches to \(L_1\) as \(a \to \infty\). TL1 can be solved by either DCA [46] or iterative thresholding [45], but DCA tends to give better results. Consequently, we consider the DC decomposition as \(g(\mathbf{x}) = \|\mathbf{x}\|_1 + I(A\mathbf{x} - \mathbf{b})\) and \(h(\mathbf{x}) = \sum_{i=1}^{n} \left(\frac{e^2}{a + |x_i|}\right)\). By substituting \(\nabla h(\mathbf{x}) = \frac{\mathbf{x}^{(2a+|x|)}}{(a+|x|)^2}\) into the general DCA iterations (2.10), we obtain a numerical scheme for minimizing the TL1. Due to the presence of the \(L_1\) term in \(g(\cdot)\), the DCA scheme for minimizing TL1 consists of a set of \(L_1\)-type subproblems that can be solved efficiently. Note that both TL1 and \(L_1-L_2\) do not require strongly convex DC decompositions to guarantee the convergence of DCA [46, 22], thanks to their specific forms.

3. Rationales of the \(L_1/L_2\) metric. We begin with a toy example to illustrate the advantages of \(L_1/L_2\) over other metrics, followed by some theoretical properties of the proposed model.
3.1. A toy example. Define a matrix $A$ as

$$A := \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \in \mathbb{R}^{5 \times 6},$$

and $b = (0, 0, 20, 40, 18)^T \in \mathbb{R}^5$. It is straightforward that any general solutions of $Ax = b$ have the form of $x = (t, t, t, 20 - 2t, 40 - 4t, 2(t - 9))^T$ for a scalar $t \in \mathbb{R}$. The sparsest solution occurs at $t = 0$, where the sparsity of $x$ is 3 and some local solutions include $t = 10$ for sparsity being 4 and $t = 9$ for sparsity being 5. In Figure 1, we plot various objective functions with respect to $t$, including $L_1$, $L_p$ (for $p = 1/2$), $L_1$-$L_2$, and TL1 (for $a = 1$ as suggested in [46]). Note that all these functions are not differentiable at the values of $t = 0, 9$, and 10, where the sparsity of $x$ is strictly smaller than 6. The sparsest vector $x$ corresponding to $t = 0$ can only be found by minimizing TL1 and $L_1$/$L_2$, while the other models find $t = 10$ as a global minimum. One drawback of TL1 is that the function is very narrow around the critical points, which implies it is sensitive to initial guess and difficult to find the global solution. In addition, TL1 has more local minimizers than our $L_1$/$L_2$ model (3 versus 2).
3.2. Theoretical properties. Our theoretical analysis is based on the null space property (NSP). To make the paper self-contained, we give a review of the NSP as a necessary and sufficient condition for the $L_1$ exact recovery. Then we present a weaker form that is related to local minimizers of the $L_1/L_2$ model.

**Definition 3.1** (null space property [10]). For any matrix $A \in \mathbb{R}^{m \times n}$, we say the matrix $A$ satisfies a null space property (NSP) of order $s$ if

$$\|v_S\|_1 < \|v_S\|_1, \quad v \in \ker(A) \setminus \{0\}, \quad \forall S \subset [n], \quad |S| \leq s.$$  

Donoho and Huo [13] proved that every $s$-sparse signal $x \in \mathbb{R}^n$ is the unique solution to the $L_1$ minimization (1.2) if and only if $A$ satisfies the NSP of order $s$. Note that NSP is no longer necessary if “every $s$-sparse vector” is relaxed. A sufficient condition for the exact $L_1$ recovery is given by (1.3) in terms of $L_1/L_2$, which will be examined numerically in Section 6.

Recently, Tran and Webster [40] generalized the NSP to deal with sparse promoting metrics that are symmetric, separable and concave, which unfortunately does not apply to $L_1/L_2$ (not separable), but this work motivates us to consider a weaker form of the NSP, as defined in Definition 3.2.

**Definition 3.2.** For any matrix $A \in \mathbb{R}^{m \times n}$, we say the matrix $A$ satisfies a weak null space property (wNSP) of order $s$ if

$$(s + 1) \|v_S\|_1 \leq \|v_S\|_1, \quad v \in \ker(A) \setminus \{0\}, \quad \forall S \subset [n], \quad |S| \leq s.$$  

Note that Definition 3.2 is weaker than the original NSP in Definition 3.1 in the sense that if a matrix satisfies wNSP then it also satisfies the NSP. The following theorem says that any $s$-sparse vector is a local minimizer of $L_1/L_2$ provided the matrix has the wNSP of order $s$. The proof is given in Appendix.

**Theorem 3.3.** Assume an $m \times n$ matrix $A$ satisfies the wNSP of order $s$, then any $s$-sparse solution of $Ax = b$ ($b \neq 0$) is a local minimum for $L_1/L_2$ in the feasible space of $Ax = b$. i.e., there exists a positive number $t^* > 0$ such that for every $v \in \ker(A)$ with $0 < \|v\|_2 \leq t^*$ we have

$$\frac{\|x\|_1}{\|x\|_2} \leq \frac{\|x + v\|_1}{\|x + v\|_2}.$$  

Finally, we show the optimal value of the $L_1/L_2$ subject to $Ax = b$ is upper bounded by the same ratio with $b = 0$; see Proposition 1.

**Proposition 1.** For any $A \in \mathbb{R}^{m \times n}, \ x \in \mathbb{R}^n$, we have

$$\inf_{z \in \mathbb{R}^n} \left\{ \frac{\|z\|_1}{\|z\|_2} \mid Az = Ax \right\} \leq \inf_{z \in \mathbb{R}^n} \left\{ \frac{\|z\|_1}{\|z\|_2} \mid z \in \ker(A) \setminus \{0\} \right\}.$$  

**Proof.** Denote

$$\alpha^* = \inf_{z \in \mathbb{R}^n} \left\{ \frac{\|z\|_1}{\|z\|_2} \mid Az = Ax \right\}.$$  

For every $v \in \ker(A) \setminus \{0\}$ and $t \in \mathbb{R}$, we have that

$$\alpha^* \leq \frac{\|x + tv\|_1}{\|x + tv\|_2},$$  

where $v$ is a $s$-sparse vector and $\alpha^*$ is the optimal value of the $L_1/L_2$ subject to $Ax = b$ with $b = 0$. Thus, the ratio $\frac{\|x\|_1}{\|x\|_2}$ is bounded by the ratio $\frac{\|x + tv\|_1}{\|x + tv\|_2}$ for any $t \in \mathbb{R}$, which completes the proof.
since \( A(x + tv) = b \). Then we obtain

\[
\lim_{t \to \infty} \frac{\|x + tv\|_1}{\|x + tv\|_2} = \lim_{t \to \infty} \frac{\|x/t + v\|_1}{\|x/t + v\|_2} = \frac{\|v\|_1}{\|v\|_2}
\]

Therefore, for every \( v \in \ker(A) \setminus \{0\} \),

\[
\alpha^* \leq \frac{\|v\|_1}{\|v\|_2},
\]

which directly leads to the inequality between these two infimums in (3.5).

Note that the left-hand-side of the inequality involves both the underlying signal \( x \) and the system matrix \( A \), which can be upper bounded by the minimum ratio that only involves \( A \). This relationship will be numerically checked in Section 6.

4. Numerical schemes. The proposed model is

\[
\min_{x \in \mathbb{R}^n} \left\{ \frac{\|x\|_1}{\|x\|_2} + I(Ax - b) \right\}.
\]

In Subsection 4.1, we detail the ADMM algorithm for minimizing (4.1), followed by a minor change to incorporate additional box constraint in Subsection 4.2. We discuss in Subsection 4.3 another variant of \( L_1/L_2 \) on the gradient to deal with imaging applications.

4.1. The \( L_1/L_2 \) minimization via ADMM. In order to apply the ADMM [4] to solve for (4.1), we introduce two auxiliary variables and rewrite (4.1) into an equivalent form,

\[
\min_{x, y, z, v, w} \left\{ \frac{\|z\|_1}{\|y\|_2} + I(Ax - b) \right\} \quad \text{s.t.} \quad x = y, \quad x = z,
\]

where \( I(\cdot) \) is the indicator function defined in (2.2). The augmented Lagrangian for (4.2) is

\[
L_{\rho_1, \rho_2}(x, y, z; v, w) = \frac{\|z\|_1}{\|y\|_2} + I(Ax - b) + \langle v, x - y \rangle + \frac{\rho_1}{2} \|x - y\|_2^2 + \langle w, x - z \rangle + \frac{\rho_2}{2} \|x - z\|_2^2
\]

\[
= \frac{\|z\|_1}{\|y\|_2} + I(Ax - b) + \frac{\rho_1}{2} \left\| x - y + \frac{1}{\rho_1} v \right\|_2^2 + \frac{\rho_2}{2} \left\| x - z + \frac{1}{\rho_2} w \right\|_2^2.
\]

The ADMM consists of the following five steps:

\[
\begin{cases}
    x^{(k+1)} := \arg \min_x L_{\rho_1, \rho_2}(x, y^{(k)}, z^{(k)}; v^{(k)}, w^{(k)}), \\
    y^{(k+1)} := \arg \min_y L_{\rho_1, \rho_2}(x^{(k+1)}, y, z^{(k)}; v^{(k)}, w^{(k)}), \\
    z^{(k+1)} := \arg \min_z L_{\rho_1, \rho_2}(x^{(k+1)}, y^{(k+1)}, z, v^{(k)}, w^{(k)}), \\
    v^{(k+1)} := v^{(k)} + \rho_1 (x^{(k+1)} - y^{(k+1)}), \\
    w^{(k+1)} := w^{(k)} + \rho_2 (x^{(k+1)} - z^{(k+1)}).
\end{cases}
\]

The update for \( x \) is a projection to the affine space of \( Ax = b \), similar to (2.8),

\[
x^{(k+1)} := \arg \min_x L_{\rho_1, \rho_2}(x, y^{(k)}, z^{(k)}; v^{(k)}, w^{(k)})
= \arg \min_x \left\{ \frac{\rho_1 + \rho_2}{2} \left\| x - f^{(k)} \right\|_2^2 \right\} \quad \text{s.t.} \quad Ax = b
= (I - AT(AAT)^{-1}A) f^{(k)} + A^T(AAT)^{-1} b,
\]
where

\[
(4.5) \quad f^{(k)} = \frac{\rho_1}{\rho_1 + \rho_2} \left( y^{(k)} - \frac{1}{\rho_1} v^{(k)} \right) + \frac{\rho_2}{\rho_1 + \rho_2} \left( z^{(k)} - \frac{1}{\rho_2} w^{(k)} \right).
\]

As for the \( y \)-subproblem, let \( c^{(k)} = \|z^{(k)}\|_1 \) and \( d^{(k)} = x^{(k+1)} + \frac{\gamma^{(k)}}{\rho_1} \) and the minimization subproblem reduces to

\[
(4.6) \quad y^{(k+1)} = \arg \min_y \frac{c^{(k)}}{\|y\|_2} + \frac{\rho_1}{2} \|y - d^{(k)}\|_2^2.
\]

If \( d^{(k)} = 0 \) then any vector \( y \) with \( \|y\|_2 = \sqrt[3]{\frac{c^{(k)}}{\rho_1}} \) is a solution to the minimization problem. If \( c^{(k)} = 0 \) then \( y = d^{(k)} \) is the solution. Now we consider \( d^{(k)} \neq 0 \) and \( c^{(k)} \neq 0 \). By taking derivative of the objective function with respect to \( y \), we obtain

\[
\left( -\frac{c^{(k)}}{\|y\|_2^2} + \rho_1 \right) y = \rho_1 d^{(k)}.
\]

As a result, there exists a positive number \( \tau^{(k)} \geq 0 \) such that \( y = \tau^{(k)} d^{(k)} \). Given \( \eta^{(k)} = \|d^{(k)}\|_2 \), \( \tau^{(k)} \) is a root of

\[
\tau^3 - \tau^2 - \frac{D^{(k)}}{F(\tau)} = 0.
\]

The cubic-root formula suggests that \( F(\tau) = 0 \) has only one real root, which can be found by the following closed-form solution.

\[
(4.7) \quad \tau^{(k)} = \frac{1}{3} + \frac{1}{3} \left( C^{(k)} + \frac{1}{C^{(k)}} \right), \quad \text{with} \quad C^{(k)} = \sqrt[3]{\frac{27D^{(k)} + 2 + \sqrt{(27D^{(k)} + 2)^2 - 4}}{2}}.
\]

In summary, we have

\[
(4.8) \quad y^{(k+1)} = \begin{cases} 
\epsilon^{(k)} & d^{(k)} = 0, \\
\tau^{(k)} d^{(k)} & d^{(k)} \neq 0,
\end{cases}
\]

where \( \epsilon^{(k)} \) is a random vector with the \( L_2 \) norm to be \( \sqrt[3]{\frac{c^{(k)}}{\rho_1}} \).

Finally, the ADMM update for \( z \) is given by soft shrinkage operator

\[
(4.9) \quad z^{(k+1)} = \text{shrink} \left( x^{(k+1)} + \frac{w^{(k)}}{\rho_2}, \frac{1}{\rho_2 \|y^{(k+1)}\|_2} \right),
\]

where \text{shrink} is defined in (2.7). We summarize the ADMM algorithm for solving the \( L_1/L_2 \) minimization problem in Algorithm 4.1.
Algorithm 4.1 The $L_1/L_2$ minimization via ADMM.

Input: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^{m \times 1}$, Max and $\epsilon \in \mathbb{R}$
while $k < \text{Max}$ or $\|x^{(k)} - x^{(k-1)}\|_2/\|x^{(k)}\| > \epsilon$ do
  $x^{(k+1)} = (I - A^T (AA^T)^{-1} A) f^{(k)} + A^T (AA^T)^{-1} b$
  $y^{(k+1)} = \begin{cases} e^{(k)} & d^{(k)} = 0 \\ \tau^{(k)} d^{(k)} & d^{(k)} \neq 0 \end{cases}$
  $z^{(k+1)} = \text{shrink} \left( x^{(k+1)} + \frac{w^{(k)}}{\rho_2} + \frac{1}{\rho_2} \|y^{(k+1)}\|_2 \right)$
  $v^{(k+1)} = v^{(k)} + \rho_1 (x^{(k+1)} - y^{(k+1)})$
  $w^{(k+1)} = w^{(k)} + \rho_2 (x^{(k+1)} - z^{(k+1)})$
  $k = k+1$
end while
return $x^{(k)}$

4.2. $L_1/L_2$ with box constraint. The $L_1/L_2$ model has an intrinsic drawback that tends to produce one erroneously large coefficient while suppressing the other non-zero elements, under which case the ratio is reduced. To compensate for this drawback, it is helpful to incorporate a box constraint, if we know lower/upper bounds of the underlying signal a priori. Specifically, we propose

$$
\text{(4.10)} \quad \min_{x \in \mathbb{R}^n} \left\{ \frac{\|x\|_1}{\|x\|_2} + I(Ax - b) \mid x \in [c, d] \right\},
$$

which is referred to as $L_1/L_2$-box. Similar to (4.2), we look at the following form that enforces the box constraint on variable $z$,

$$
\text{(4.11)} \quad \min_{x, y, z} \left\{ \frac{\|z\|_1}{\|y\|_2} + I(Ax - b) \right\} \quad \text{s.t. } x = y, \quad x = z, \quad z \in [c, d].
$$

The only change we need to make by adapting Algorithm 4.1 to the $L_1/L_2$-box is the $z$ update. The $z$-subproblem in (4.4) with the box constraint is

$$
\text{(4.12)} \quad \min_z \frac{1}{\|y^{(k+1)}\|_2} \|z\|_1 + \frac{\rho_2}{2} \|x^{(k+1)} - z + \frac{1}{\rho_2} w^{(k)}\|_2^2 \quad \text{s.t. } z \in [c, d].
$$

For a convex problem (4.12) involving the $L_1$ norm, it has a closed-form solution given by the soft shrinkage, followed by projection to the interval $[c, d]$. In particular, simple calculations show that

$$
\text{(4.13)} \quad z^{(k+1)}_i = \min \{ \max(\hat{z}_i, c), d \}, \quad i = 1, 2, \ldots, n,
$$

where $\hat{z} = \text{shrink}(r, \nu)$, $r = x^{(k+1)} + \frac{w^{(k)}}{\rho_2}$ and $\nu = \frac{1}{\rho_2 \|y^{(k+1)}\|_2}$. If the box constraint $[c, d]$ is symmetric, i.e., $c = -d$ and $d > 0$, it follows from [2] that the update for $z$ can be expressed as

$$
\text{(4.14)} \quad z^{(k+1)}_i = \text{sign}(v_i) \min \{ \max(|r_i| - \nu, 0), d \}, \quad i = 1, 2, \ldots, n.
$$
4.3. \( L_1/L_2 \) on the gradient. We adapt the \( L_1/L_2 \) model to apply on the gradient, which enables us to deal with imaging applications. Let \( u \in \mathbb{R}^{n \times m} \) be an underlying image of size \( n \times m \). Denote \( A \) as a linear operator that models a certain degradation process to obtain the measured data \( f \). For example, \( A \) can be a subsampling operator in the frequency domain and recovering \( u \) from \( f \) is called MRI reconstruction. In short, the proposed gradient model is given by

\[
\min_{u \in \mathbb{R}^{n \times m}} \frac{\|\nabla u\|_1}{\|\nabla u\|_2} \quad \text{s.t.} \quad Au = f,
\]

where \( \nabla \) denotes discrete gradient operator \( \nabla u := \{[u_{ij} - u_{(i+1)j}]_{i=1}^{n}\}_{j=1}^{m}, \{[u_{ij} - u_{ij+(j+1)}]_{i=1}^{m}\}_{j=1}^{n} \) with periodic boundary condition; hence the model is referred to as \( L_1/L_2\)-grad. To solve for (4.15), we introduce two auxiliary variables \( d \) and \( h \) to hold the value of \( \nabla u \), thus leading to

\[
\min_{u \in \mathbb{R}^{n \times m}} \frac{\|d\|_1}{\|h\|_2} \quad \text{s.t.} \quad Au = f, \quad d = \nabla u, \quad h = \nabla u.
\]

Note that we denote \( d \) and \( h \) in bold to indicate that they have two components corresponding to both \( x \) and \( y \) derivatives. The augmented Lagrangian is expressed as

\[
\mathcal{L}_{\lambda, \rho_1, \rho_2}(u, d, h; w, b, g) = \frac{\|d\|_1}{\|h\|_2} + \frac{\lambda}{2} \|Au - f - w\|_2^2 + \frac{\rho_1}{2} \|d - \nabla u - b\|_2^2 + \frac{\rho_2}{2} \|h - \nabla u - g\|_2^2.
\]

where \( w, b, g \) are dual variables and \( \lambda, \rho_1, \rho_2 \) are positive parameters. The updates for \( d, h \) are the same as Algorithm 4.1. Specifically for \( h \), we consider \( D(k) = \frac{\|d\|_1}{\rho_2\|\nabla u + g(k)\|_2^2} \) and hence \( \tau(k) \) is the root of the same polynomial as in (4.7). By taking derivative of (4.17) with respect to \( u \), we can obtain the \( u \)-update, i.e.,

\[
u^{(k+1)} = (\lambda A^T A - (\rho_1 + \rho_2) \Delta)^{-1} (\lambda A^T (f + w^{(k)}) + \rho_1 \nabla^T (d^{(k)} - b^{(k)}) + \rho_2 \nabla^T (h^{(k)} - g^{(k)})).
\]

Note for certain operator \( A \), the inverse in the \( u \)-update (4.18) can be computed efficiently via the fast Fourier transform (FFT). In summary, we present the ADMM algorithm for the \( L_1/L_2\)-grad model in Algorithm 4.2.

**Algorithm 4.2** The \( L_1/L_2\)-grad minimization via ADMM.

**Input:** \( f \in \mathbb{R}^{n \times m} \), \( A \), Max and \( \epsilon \in \mathbb{R} \).

**while** \( k < \text{Max} \text{ or } \|u^{(k)} - u^{(k-1)}\|_2 / \|u^{(k)}\| > \epsilon \) **do**

1. \( u^{(k+1)} = (\lambda A^T A - (\rho_1 + \rho_2) \Delta)^{-1} (\lambda A^T (f + w^{(k)}) + \rho_1 \nabla^T (d^{(k)} - b^{(k)}) + \rho_2 \nabla^T (h^{(k)} - g^{(k)})) \)

2. \( h^{(k+1)} = \begin{cases} e^{(k)} & \text{if } \nabla u^{(k+1)} + g^{(k)} = 0, \\ \tau^{(k)} (\nabla u^{(k+1)} + g^{(k)}) & \text{if } \nabla u^{(k+1)} + g^{(k)} \neq 0. \end{cases} \)

3. \( d^{(k+1)} = \text{shrink} \left( \nabla u^{(k+1)} + b^{(k)}, \frac{1}{\rho_1 \|h^{(k+1)}\|_2} \right) \)

4. \( b^{(k+1)} = b^{(k)} + \nabla u^{(k+1)} - d^{(k+1)} \)

5. \( g^{(k+1)} = g^{(k)} + \nabla u^{(k+1)} - h^{(k+1)} \)

6. \( w^{(k+1)} = u^{(k)} + f - Au^{(k+1)} \)

7. \( k = k + 1 \)

**end while**

**return** \( u^{(k)} \)
5. Numerical experiments. In this section, we carry out a series of numerical tests to demonstrate the performance of the proposed $L_1/L_2$ models together with its corresponding algorithms. All the numerical experiments are conducted on a standard desktop with CPU (Intel i7-6700, 3.4GHz) and MATLAB 9.2 (R2017a).

We consider two types of sensing matrices: one is called oversampled discrete cosine transform (DCT) and the other is Gaussian matrix. Specifically for the oversampled DCT, we follow the works of [15, 23, 44] to define $A = [a_1, a_2, \cdots, a_n] \in \mathbb{R}^{m \times n}$ with

$$a_j := \frac{1}{\sqrt{m}} \cos \left( \frac{2\pi w_j}{F} \right), \quad j = 1, \cdots, n,$$

where $w$ is a random vector uniformly distributed in $[0, 1]^m$ and $F \in \mathbb{R}_+$ controls the coherence in a way that a larger value of $F$ yields a more coherent matrix. In addition, we use $\mathcal{N}(0, \Sigma)$ (the multi-variable normal distribution) to generate Gaussian matrix, where the covariance matrix is $\Sigma = \{(1 - r) * I(i = j) + r\}_{i,j}$ with a positive parameter $r$. This type of matrices is used in the TL1 paper [46], which mentioned that a larger $r$ value indicates a more difficult problem in sparse recovery. Throughout the experiments, we consider sensing matrices of size $64 \times 1024$. The ground truth $x \in \mathbb{R}^n$ is simulated as $s$-sparse signal, where $s$ is the total number of nonzero entries. The support of $x$ is a random index set and the values of non-zero elements follow Gaussian normal distribution i.e., $(x_i)_i \sim \mathcal{N}(0, 1), \quad i = 1, 2, \cdots, s$. We then normalize the ground-truth signal to have maximum magnitude as 1 so that we can examine the performance of additional $[-1, 1]$ box constraint.

Due to the non-convex nature of the proposed $L_1/L_2$ model, the initial guess $x^{(0)}$ is very important and should be well-chosen. A typical choice is the $L_1$ solution (1.2), which is used here. Although the $L_1$ minimization can be solved by the ADMM (as described in Subsection 2.1), we adopt a commercial optimization software called Gurobi [30] to minimize the $L_1$ norm via linear programming for the sake of efficiency. The stop criterion is when the relative error of $x^{(k)}$ to $x^{(k-1)}$ is smaller than $10^{-8}$ or iterative number exceeds $10n$.

5.1. Algorithmic behaviors. We empirically demonstrate the convergence of the proposed ADMM algorithms in Figure 2. Specifically we examine the $L_1/L_2$ minimization problem (4.2), where the sensing matrix is an oversampled DCT matrix with $F = 10$ and ground-truth sparse vector has 12 non-zero elements. We also study the MRI reconstruction from 7 radical lines as a particular sparse gradient problem that involves the $L_1/L_2$-grad minimization of (4.15) by Algorithm 4.2.

There are two auxiliary variables $d, h$ in $L_1/L_2$ such that $x = y = z$, while two auxiliary variables $d, h$ are in $L_1/L_2$-grad for $\nabla u = d = h$. We show in the top row of Figure 2 the values of $\|x^{(k)} - y^{(k)}\|_2$ and $\|x^{(k)} - z^{(k)}\|_2$ as well as $\|\nabla u^{(k)} - d^{(k)}\|_2$ and $\|\nabla u^{(k)} - h^{(k)}\|_2$, all are plotted with respect to the iteration counter $k$. The bottom row of Figure 2 is for objective functions, i.e., $\|x^{(k)}\|_1 / \|x^{(k)}\|_2$ and $\|\nabla u^{(k)}\|_1 / \|\nabla u^{(k)}\|_2$ for $L_1/L_2$ and $L_1/L_2$-grad, respectively. All the plots in Figure 2 decrease rapidly with respect to iteration counters, which serves as heuristic evidence of algorithmic convergence. On the other hand, the objective functions in Figure 2 look oscillatory. This phenomenon implies difficulties in theoretically proving the convergence, as one key step in the convergence proof requires to show that objective function decreases monotonically [3, 41].

5.2. Comparison on various models. We now compare the proposed $L_1/L_2$ approach with other sparse recovery models: $L_1$, $L_p$ [9], $L_1-L_2$ [44, 23], and TL1 [46]. We choose $p = 0.5$ for $L_p$ and $a = 1$ for TL1. The initial guess for all the algorithms is the solution of the $L_1$ model. Both
Fig. 2. Plots of residual errors and objective functions for empirically demonstrating the convergence of the proposed algorithms.

$L_1$L_2$ and $T_1$L_1$ are solved via the DCA, as described in Subsection 2.2, with the same stop criterion as $L_1/L_2$, i.e., $||x^{(k)}-x^{(k-1)}||_2 \leq 10^{-8}$. As for $L_p$, we follow the default setting in [9].

We evaluate the performance of sparse recovery in terms of success rate, defined as the number of successful trials over the total number of trials. A success is declared if the relative error of the reconstructed solution $x^*$ to the ground truth $x$ is less than $10^{-3}$, i.e., $||x^*-x||_2/||x||_2 \leq 10^{-3}$. We further categorize the failure of not recovering the ground-truth as model failure and algorithm failure. In particular, we compare the objective function $F(\cdot)$ at the ground-truth $x$ and at the reconstructed...
solution $x^\ast$. If $F(x) > F(x^\ast)$, it means that $x$ is not a global minimizer of the model, in which case we call model failure. On the other hand, $F(x) < F(x^\ast)$ implies that the algorithm does not reach a global minimizer, which is referred to as algorithm failure. Although this type of analysis is not deterministic, it sheds some lights on which direction to improve: model or algorithm. For example, it was reported in [28] that $L_1$ has the highest model-failure rates, which justifies the need for nonconvex models.

In Figure 3, we examine two coherence levels: $F = 5$ corresponds to relatively low coherence and $F = 20$ for higher coherence. The success rates of various models reveal that $L_1/L_2$-box performs the best at $F = 5$ and is comparable to $L_1/L_2$ for the highly coherent case of $F = 20$. We look at Gaussian matrix with $r = 0.1$ and $r = 0.8$ in Figure 4, both of which exhibit very similar performance of various models. In particular, the $L_p$ model gives the best results for the Gaussian case, which is consistent in the literature [43, 23]. The proposed model of $L_1/L_2$-box is the second best for such incoherent matrices.

By comparing $L_1/L_2$ with and without box among the plots for success rates and model failures, we can draw the conclusion that the box constraint can mitigate the inherent drawback of the $L_1/L_2$ model, thus improving the recovery rates. In addition, $L_1/L_2$ is the second lowest in terms of model failure rates and simply adding a box constraint also increases the occurrence of algorithm failure compared to the none box version. These two observations suggest a need to further improve upon algorithms of minimizing $L_1/L_2$.

Finally, we provide the computation time for all the competing algorithms in Table 1 with the shortest time in each case highlighted in bold. The time for $L_1$ method is not included, as all the other methods use the $L_1$ solution as initial guess. It is shown that TL1 is the fastest for relatively lower sparsity levels and the proposed $L_1/L_2$-box is the most efficient at higher sparsity levels. The computational times for all these methods seem consistent with DCT and Gaussian matrices.

5.3. MRI reconstruction. As a proof-of-concept example, we study an MRI reconstruction problem [26] to compare the performance of $L_1$, $L_1/L_2$, and $L_1/L_2$ on the gradient. The $L_1$ on the gradient is the celebrated TV model [36], while $L_1/L_2$ on the gradient was recently proposed in [24].

We use a standard Shepp-Logan phantom as a testing image, as shown in Figure 5a. The data is obtained only along several radical lines in the frequency domain (after taking Fourier transform); an example of such sampling scheme using 7 lines is shown in Figure 5b. As this paper focuses on the constrained formulation, we do not consider noise, following the same setting as in the previous works [44, 24]. Since all the competing methods ($L_1$, $L_1/L_2$, and $L_1/L_2$) yield an exact recovery with 8 radical lines, with accuracy in the order of $10^{-8}$, we present the reconstructions results of 7 radical lines in Figure 5, which illustrates that the ratio model ($L_1/L_2$) gives much better results than the difference model ($L_1/L_2$). Figure 5 also includes quantitative measures of the performance by relative error (RE) between the reconstructed and ground-truth images, which shows significantly improvement of the proposed $L_1/L_2$-grad over a classic method in MRI reconstruction, called filter-back projection (FBP), and two recent works of using $L_1$ and $L_1/L_2$ on the gradient. Note that the state-of-the-art methods in MRI reconstruction are [17, 28] that have reported exact recovery from 7 lines, but both approaches require to tune model parameters, while our model (4.15) is parameter free.\footnote{There are three algorithmic parameters in (4.17), but they only affect the convergence speed of the algorithm.}

6. Empirical validations. A review article [7] indicated that two principles in CS are sparsity and incoherence, leading an impression that a sensing matrix with smaller coherence is easier for sparse recovery. However, we observe through numerical results [22] (also given in Figure 6b)
Fig. 3. Success rates, model failures, algorithm failures for 6 algorithms in the case of oversampled DCT matrices.
Fig. 4. Success rates, model failures, algorithm failures for 6 algorithms in the Gaussian matrix case.
Table 1

Computation time (sec.) in 5 algorithms.

(a) DCT matrix

| sparsity | 2  | 6  | 10 | 14 | 18 | 22 | mean |
|----------|----|----|----|----|----|----|------|
| TL1  | 0.049 | 0.050 | 0.066 | 0.207 | 0.218 | 0.795 | 0.208 |
| \(L_p\) | 0.061 | 0.137 | 0.209 | 0.355 | 0.515 | 0.565 | 0.307 |
| \(L_{1/L_2}\) | 0.049 | 0.050 | 0.071 | 0.260 | 0.550 | 0.625 | 0.267 |
| \(L_{1/L_2}\)-box | 0.102 | 0.183 | 0.247 | 0.313 | 0.325 | 0.332 | 0.324 |

(b) Gaussian matrix

| sparsity | 2  | 6  | 10 | 14 | 18 | 22 | mean |
|----------|----|----|----|----|----|----|------|
| TL1  | 0.048 | 0.069 | 0.092 | 0.330 | 0.654 | 0.755 | 0.325 |
| \(L_p\) | 0.094 | 0.254 | 0.423 | 0.472 | 0.530 | 0.534 | 0.385 |
| \(L_{1/L_2}\) | 0.049 | 0.070 | 0.093 | 0.272 | 0.598 | 0.677 | 0.293 |
| \(L_{1/L_2}\)-box | 0.090 | 0.179 | 0.239 | 0.301 | 0.324 | 0.322 | 0.314 |

that a more coherent matrix gives higher recovery rates. This contradiction motivates us to collect empirical evidence regarding to either prove or refuse whether coherence is relevant to sparse recovery. Here we examine one such evidence by minimizing the ratio of \(L_1\) and \(L_2\), which gives an upper bound for a sufficient condition of \(L_1\) exact recovery, see (1.3). To avoid the trivial solution of \(x = 0\) to the problem of min \(\{\|x\|_1 : A x = 0\}\), we incorporate a sum-to-one constraint. In other word, we define an expanded matrix \(\tilde{A} = [A; \text{ones}(n, 1)]\) (following Matlab’s notation) and an expanded vector \(\tilde{b} = [0; 1]\). We then adapt the proposed method to solve for min \(\{\|x\|_1 : \tilde{A} x = \tilde{b}\}\).

In Figure 6a, we plot the mean value of ratios from 50 random realizations of matrices \(A\) at each coherence level (controlled by \(F\)), which shows that the ratio actually decreases\(^2\) with respect to \(F\). As the \(L_0\) norm is bounded by the ratio (1.3), smaller ratio indicates it is more difficult to recover the signals. Therefore, Figure 6a is consistent with the common belief in CS.

\(^2\)We also observe that the ratio stagnates for larger \(F\), which is probably because of instability of the proposed method when matrix becomes more coherent.
We postulate that an underlying reason of more coherent matrices giving better results is minimum separation (MS), as formally introduced in [5]. In Figure 6b, we enforce the minimum separation of two neighboring spikes to be 40, following the suggestion of $2F$ in [15] (we consider $F$ up to 20). In comparison, we also give the success rates of the $L_1$ recovery without any restrictions on MS in Figure 6c. Note that we use the exactly same matrices in both cases (with and without MS). Figure 6c does not have a clear pattern regarding how coherence affects the exact recovery, which supports our hypothesis that minimum separation plays an important role in sparse recovery. It will be our future work to analyze it thoroughly.

Furthermore, it follows from Proposition 1 that the minimal ratios subject to a linear system $Ax = b$ is smaller than the one in the kernel space of $A$. Here, we empirically verify this proposition by calculating the ratio in $\min_x \left\{ \frac{||x||_1}{||x||_2} : Ax = 0 \right\}$ with 20 random realizations of the oversampled-DCT matrices $A$ at $F = 2$ and $F = 5$; the ratios are plotted in blue dots in Figure 7. On the
Fig. 6. The use of $\min_x \left\{ \frac{\|x\|_1}{\|x\|_2} : Ax = 0 \right\}$ as an upper bound for the $L_1$ recovery measured by success rates in the cases of with (b) and without (c) minimum separation.

other hand, we also compute $\min_x \left\{ \frac{\|x\|_1}{\|x\|_2} : Ax = b \right\}$ for each fixed $A$ with different sparsity levels: $s \in \{2, 4, 6, \ldots, 24\}$. We use a box plot\(^3\) to visualize the results. Each box in Figure 7 corresponds to one specific $A$ among 20 trials. We observe that most of blue dots are above the box when $F = 2$, which is consistent with Proposition 1. However, there exist many ratios subject to $Ax = b$ that appear higher than the ones of $Ax = 0$ at $F = 5$. This is due to our algorithmic limitation that we cannot guarantee the minimizing algorithm actually finds a global minimizer.

7. Conclusions and future works. In this paper, we have studied a novel metric $L_1/L_2$ to promote sparsity. Two main benefits of $L_1/L_2$ are scale invariant and parameter free. Two numerical algorithms based on the ADMM are formulated for the assumptions of sparse signals and sparse gradients, together with a variant of incorporating additional box constraint. The

\(^3\)The box plot in Matlab uses a red mark to indicate the median, bottom/top edges of the box to indicate the 25th and 75th percentiles, and the whiskers for the most extreme data points.
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experimental results demonstrate the performance of the proposed approaches in comparison to the state-of-the-art methods in sparse recovery and MRI reconstruction. As a by-product, minimizing the ratio also gives an empirical upper bound towards \( L_1 \)'s exact recovery, which motivates further investigations on exact recovery theories. Other future works include algorithmic improvement and convergence analysis. In particular, it is shown in Table 1, Figures 3 and 4 that \( L_1/L_2 \) is not as fast as competing methods in CS and also has certain algorithmic failures, which calls for a more robust and more efficient algorithm. In addition, we have provided heuristic evidence of the ADMM’s convergence and it will be interesting to analyze it theoretically.

Appendix: proof of Theorem 3.3. In order to prove Theorem 3.3, we need to introduce two lemmas:

**Lemma 7.1.** For any \( \mathbf{x}, \mathbf{v} \in \mathbb{R}^n \) and \( i \in [n] \), we have

\[
\begin{align*}
(7.1) & \quad n\|\mathbf{x}\|_2^2 - |x_i|\|\mathbf{x}\|_1 \geq (n-1) \left( \sum_{j \neq i} x_j^2 \right), \\
(7.2) & \quad n\|\mathbf{v}\|_1\|\mathbf{x}\|_2^2 \geq \|\mathbf{x}\|_1 \langle \mathbf{v}, \mathbf{x} \rangle.
\end{align*}
\]

Furthermore, if \( \|\mathbf{x}\|_0 = s \), then the constant \( n \) in the inequalities can be reduced to \( s \).
Proof. Simple calculations show that

\[ n\|\mathbf{x}\|_2^2 - |x_i|\|\mathbf{x}\|_1 = n \left( \sum_j x_j^2 \right) - |x_i| \left( \sum_j |x_j| \right) \]

\[ = (n - 1) \left( \sum_{j \neq 1} x_j^2 \right) + \sum_{j \neq i} x_j^2 + (n - 1)x_i^2 - \sum_{j \neq i} |x_i||x_j| \]

\[ = (n - 1) \left( \sum_{j \neq 1} x_j^2 \right) + \sum_{j \neq i} (|x_i| - |x_j|)^2 + |x_i||x_j| \]

\[ \geq (n - 1) \left( \sum_{j \neq 1} x_j^2 \right) \geq 0. \]

Therefore, we have

\[ \sum_i (n\|\mathbf{x}\|_2^2 - |x_i|\|\mathbf{x}\|_1)|v_i| \geq 0, \]

which implies that

\[ n\|\mathbf{v}\|_1\|\mathbf{x}\|_2^2 \geq \|\mathbf{x}\|_1 \left( \sum_i |x_i||v_i| \right) \geq \|\mathbf{x}\|_1 \langle \mathbf{v}, \mathbf{x} \rangle \]

\( \square \)

**Lemma 7.2.** Suppose that an s-sparse vector \( \mathbf{x} \) satisfies \( A\mathbf{x} = \mathbf{b} \) (\( \mathbf{b} \neq 0 \)) with its support on an index set \( S \) and the matrix \( A \) satisfies the wNSP of order \( s \). Define

\[ t_1 := \inf_{\mathbf{v}, t} \left\{ \frac{\|\sigma_t(\mathbf{v})\|_2^2 - \langle \mathbf{v}_S, \mathbf{x} \rangle \|\mathbf{x}\|_1}{\|\sigma_t(\mathbf{v})(\mathbf{v}_S, \mathbf{x}) - \|\mathbf{x}\|_1\|\mathbf{v}\|_2^2 \|} \mid \mathbf{v} \in \ker(A), \|\mathbf{v}\|_2 = 1, t \neq 0 \right\}, \]

where \( \sigma_t(\mathbf{v}) = (\sum_{i \in S} v_i \text{sign}(x_i)) + \text{sign}(t)|\mathbf{v}_S|_1 \). We can show that \( t_1 > 0 \).

**Proof.** For any \( \mathbf{v} \in \ker(A) \) and \( \|\mathbf{v}\|_2 = 1 \), it is straightforward that

\[ |\sigma_t(\mathbf{v})(\mathbf{v}_S, \mathbf{x})| - \|\mathbf{x}\|_1\|\mathbf{v}\|_2^2 \leq |\sigma_t(\mathbf{v})\|\mathbf{v}\|_2\|\mathbf{x}\|_2 + \|\mathbf{x}\|_1\|\mathbf{v}\|_2^2 \]

\[ \leq \|\mathbf{v}\|_1\|\mathbf{v}\|_2\|\mathbf{x}\|_2 + \|\mathbf{x}\|_1\|\mathbf{v}\|_2^2 \]

\[ = \|\mathbf{v}\|_1\|\mathbf{x}\|_2 + \|\mathbf{x}\|_1 \]

\[ \leq \sqrt{n}\|\mathbf{x}\|_2 + \|\mathbf{x}\|_1, \]

and

\[ |\sigma_t(\mathbf{v})| \geq |\text{sign}(t)|\mathbf{v}_S|_1| - |\sum_{i \in S} v_i |\text{sign}(x_i)| \geq \|\mathbf{v}_S\|_1 - \sum_{i \in S} |v_i| = \|\mathbf{v}_S\|_1 - \|\mathbf{v}_S\|_1. \]

It follows from the wNSP that \( \|\mathbf{v}_S\|_1 \geq (s + 1)|\mathbf{v}_S|_1 \), thus leading to the following two inequalities,

\[ |\sigma_t(\mathbf{v})| \geq \|\mathbf{v}_S\|_1 - \|\mathbf{v}_S\|_1 \geq s|\mathbf{v}_S|_1 \]

\[ |\sigma_t(\mathbf{v})| \geq \|\mathbf{v}_S\|_1 - \|\mathbf{v}_S\|_1 \geq (1 - \frac{1}{s + 1})\|\mathbf{v}_S\|_1 = \frac{s}{s + 1}\|\mathbf{v}_S\|_1. \]

Next we will discuss two cases: \( s > 1 \) and \( s = 1 \).
(i) For \( s > 1 \). Since \( \|\mathbf{v}\|_2 = 1 \), there exists an index \( j \in [n] \) such that \( |v_j| \geq \frac{1}{\sqrt{n}} \). Suppose such \( j \in S \), then we have that

\[
|\sigma_i(\mathbf{v})\|\mathbf{x}\|_2^2 - \langle \mathbf{v}_S, \mathbf{x} \rangle \|\mathbf{x}\|_1 | \geq |\sigma_i(\mathbf{v})\|\mathbf{x}\|_2^2 - \langle \mathbf{v}_S, \mathbf{x} \rangle \|\mathbf{x}\|_1 = \sum_{i \in S} |x_i| |v_i| \|\mathbf{x}\|_1 \\
\geq s \|\mathbf{v}_S\|_1 \|\mathbf{x}\|_2^2 - \left( \sum_{i \in S} |x_i| |v_i| \right) \|\mathbf{x}\|_1 \\
\geq \left( s \|\mathbf{x}\|_2^2 - |x_j| \|\mathbf{x}\|_1 \right) |v_j| \\
\geq \frac{1}{\sqrt{n}} \left( s \|\mathbf{x}\|_2^2 - |x_j| \|\mathbf{x}\|_1 \right) \\
\geq \frac{1}{\sqrt{n}} \left( s - 1 \right) \left( \sum_{i \neq j} x_i^2 \right) \\
\geq \frac{s - 1}{\sqrt{n}} \min_{j \in S} \left( \sum_{i \neq j} x_i^2 \right) > 0.
\]

(7.10)

Otherwise, we have that \( |v_j| < \frac{1}{\sqrt{n}} \forall j \in S \) and hence \( \|\mathbf{v}_S\|_1 < \frac{s}{\sqrt{n}} \). Since \( \|\mathbf{v}\|_1 \geq \|\mathbf{v}\|_2 = 1 \), then \( \|\mathbf{v}_S\|_1 \geq 1 - \frac{s}{\sqrt{n}} = \frac{\sqrt{n} - s}{\sqrt{n}} \). As a result, we get

\[
|\sigma_i(\mathbf{v})\|\mathbf{x}\|_2^2 - \langle \mathbf{v}_S, \mathbf{x} \rangle \|\mathbf{x}\|_1 | \geq |\sigma_i(\mathbf{v})\|\mathbf{x}\|_2^2 - \langle \mathbf{v}_S, \mathbf{x} \rangle \|\mathbf{x}\|_1 = \sum_{i \in S} |x_i| |v_i| \|\mathbf{x}\|_1 \\
\geq \frac{s}{s + 1} \|\mathbf{v}_S\|_1 \|\mathbf{x}\|_2^2 - \left( \sum_{i \in S} |x_i| \right) \|\mathbf{x}\|_1 \\
\geq \frac{s}{s + 1} \left( \frac{\sqrt{n} - s}{\sqrt{n}} \right) \|\mathbf{x}\|_2^2 - \left( \sum_{i \in S} |x_i| \right) \|\mathbf{x}\|_1 \\
= \left( \frac{\sqrt{n} - s}{s + 1} \right) \|\mathbf{x}\|_2^2 - \left( \sum_{i \in S} |x_i| \right) \|\mathbf{x}\|_1 \\
\geq \frac{\sqrt{n} - s}{s + 1} \|\mathbf{x}\|_2^2 - \frac{1}{\sqrt{n}} \|\mathbf{x}\|_2^2 \\
= \frac{\sqrt{n} - 1}{s + 1} \|\mathbf{x}\|_2^2 > 0.
\]

Therefore, we have

\[
(7.11) \quad t_1 = \frac{1}{2} \min \left\{ \frac{s - 1}{\sqrt{n}} \min_{j \in S} \left( \sum_{i \neq j} x_i^2 \right), \left( \frac{\sqrt{n} + 1}{s + 1} \right) \frac{s}{\sqrt{n}} \|\mathbf{x}\|_2^2 \right\} > 0.
\]

(ii) For \( s = 1 \). Suppose the only non-zero element is \( x_i \neq 0 \), and hence we have

\[
(7.12) \quad |\sigma_i(\mathbf{v})\|\mathbf{x}\|_2^2 - \langle \mathbf{v}_S, \mathbf{x} \rangle \|\mathbf{x}\|_1 | = |(v_i \text{sign}(x_i) + \text{sign}(t) \|\mathbf{v}_S\|_1) x_i^2 - (v_i x_i) x_i| = \|\mathbf{v}_S\|_1 x_i^2.
\]
If $|v_i| \geq \frac{1}{\sqrt{n}}$ then $\|v_S\|_1 \geq (s+1)|v_i| \geq \frac{s+1}{\sqrt{n}}$ and hence

$$\sum_{i \in S} t_i = \sum_{i \in S} (x_i + t v_i) = \sum_{i \in S} x_i \text{sign}(x_i) + \sum_{i \in S} t|v_i| = \sum_{i \in S} v_i \text{sign}(x_i) + \sum_{i \in S} t|v_i|$$

$$= \sum_{i \in S} v_i \text{sign}(x_i) + t \sum_{i \in S} |v_i| = \sum_{i \in S} v_i \text{sign}(x_i) + t \|v_S\|_1$$

$$= \|x\|_1 + t \left( \sum_{i \in S} v_i \text{sign}(x_i) + t \|v_S\|_1 \right)$$

$$= \|x\|_1 + t \sigma_t(v),$$

where

$$\sigma_t(v) = \sum_{i \in S} v_i \text{sign}(x_i) + t \|v_S\|_1.$$
Therefore, $g(t)$ is differentiable for $t \neq 0$. Simple calculations show that

\begin{equation}
\frac{\partial}{\partial t} g(t) = \frac{\partial}{\partial t} \left( \frac{\|x\|_1 + t\sigma_t(v)}{\|x\|^2_2 + 2t \langle v_S, x \rangle + t^2 \|v\|^2_2} \right)
= \frac{2\sigma_t(v) (\|x\|_1 + t\sigma_t(v)) (\|x\|^2_2 + 2t \langle v_S, x \rangle + t^2 \|v\|^2_2) - (2 \langle v_S, x \rangle + 2t \|v\|^2_2) (\|x\|_1 + t\sigma_t(v))^2}{(\|x\|^2_2 + 2t \langle v_S, x \rangle + t^2 \|v\|^2_2)^2}
\end{equation}

It follows from (7.16) that $\|x\|_1 + t\sigma_t(v) = \|x + tv\| > 0$ for $|t| < t_0$. Also there exists a positive number $t_1$ defined in lemma 2 and hence we have for $|t| < t_1$

$$\text{sign}\left[ (\sigma_t(v)\|x\|^2_2 - \langle v_S, x \rangle \|x\|_1 + (\sigma_t(v) \langle v_S, x \rangle - \|x\|_1 \|v\|^2_2) t \right]$$

Letting $t^* = \min\{t_0, t_1\}$, we have for any $t \in (0, t^*)$ that

$$\sigma_t(v) = \sum_{i \in S} v_i \text{sign}(x_i) + \text{sign}(t) \|v_S\|_1$$

where the first inequality is followed by Lemma 7.1. This implies that

\begin{equation}
\sigma_t(v)\|x\|^2_2 = \|\sigma_t(v)\|\|x\|^2_2 \geq \|\langle v_S, x \rangle \|\|x\|_1 \geq \langle v_S, x \rangle \|x\|_1
\end{equation}

where the first inequality is followed by Lemma 7.1. This implies that

\begin{equation}
\sigma_t(v)\|x\|^2_2 - \langle v_S, x \rangle \|x\|_1 \geq 0.
\end{equation}

As a result, we have $g'(t) \geq 0$ if $0 < t < t^*$. The function $g(t)$ is not differentiable at zero, but we can compute the sub-derivative as follows,

\begin{equation}
ge'(0^+) = \lim_{t \to 0^+} \frac{g(t) - g(0)}{t - 0} = \frac{2\|x\|_1 (\sigma_{-1}(v)\|x\|^2_2 - \langle v_S, x \rangle \|x\|_1)}{\|x\|^4_2} \geq 0.
\end{equation}

Similarly, we can get $g'(t) \leq 0$ if $-t^* < t < 0$ and $g'(0^-) \leq 0$. Therefore for any $|t| < t^*$ we have $g(0) \leq g(t)$, which implies that

\begin{equation}
\frac{\|x + tv\|_1}{\|x + tv\|_2} \geq \frac{\|x\|_1}{\|x\|_2}, \quad \forall |t| < t^*.
\end{equation}
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