MULTIPLE VACUA AND BOUNDARY CONDITIONS
OF SCHWINGER-DYSON EQUATIONS

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abstract

We discuss the relationship between the boundary conditions of the Schwinger-Dyson equations and the phase diagram of a bosonic field theory or matrix model. In the thermodynamic limit, many boundary conditions lead to the same solution, while other boundary conditions have no such limit. The list of boundary conditions for which a thermodynamic limit exists depends on the parameters of the theory. The boundary conditions of a physical solution may be quite exotic, corresponding to path integration over various inequivalent complex contours.

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Introduction

When attempting to solve a field theory or matrix model by using Schwinger–Dyson equations, one must address the problem that these equations do not possess a unique solution. This problem came to our attention when trying to numerically solve the equations for certain lattice field theories using a technique known as Source Galerkin [1], which is discussed elsewhere in these proceedings. It was found that to make the numerical method stable, one had to find some way of selecting the right boundary conditions. This led us to the more general question of how the boundary conditions are related to the phase diagram of a field theory or a matrix model. We shall summarize our work on this relation in this paper. Most of the details will appear elsewhere [2].

The naive resolution of the problem of how to select the boundary conditions is to simply pick the solution which corresponds to path integration over real fields. However, within certain phases of many theories, it can be shown that this solution is actually not the physical one. Furthermore in theories with actions unbounded below, such as certain matrix models and Euclidean Einstein gravity, the integral over real fields is not even convergent. This forces the consideration of “exotic” solutions of the Schwinger-Dyson equations, which have representations involving sums of integrals of the fields over various inequivalent contours in the complex plane. For theories with a local order parameter, symmetry breaking solutions are naturally generated by choosing a symmetry breaking set of contours. In the conventional approach to obtaining the broken phase, the real contour is chosen but a small symmetry breaking term is added to the action. This term is removed only after taking a thermodynamic limit, in which the number of degrees of freedom becomes infinite. In fact one can also obtain the broken phase by choosing a symmetry breaking boundary condition (contour) and then taking the thermodynamic limit directly. This is a simple example showing that the exotic solutions are not necessarily unphysical. Even for actions which are bounded below, it is not always possible to obtain all physical solutions from the integral over real fields. There are explicit matrix model examples of this. We conjecture that exotic solutions may also be physical for theories with a nonlocal order parameter, though this has yet to be demonstrated.

The difficulty in choosing the correct boundary conditions lies in the fact that there are so many of them. Furthermore since the Schwinger-Dyson equations satisfied by the partition function are linear, there naively appears to be a continuum of mixed phases, which does not make physical sense. Most of this problem is resolved by taking the thermodynamic limit. In this limit the solutions associated with many boundary conditions coalesce, while other boundary conditions do not lead to solutions with a thermodynamic limit. In general one is left with a discrete set of solutions, although sometimes a continuum of distinct solutions survives, labeled by a countable number of theta parameters. The set of solutions which survives in the thermodynamic limit may include both the physical vacua, and false vacua with complex free energy, as well as other unphysical solutions which must be discarded by hand.
However, most of the task of discarding solutions is done automatically in the thermodynamic limit. It can be shown that the set of boundary conditions for which this limit exists varies along paths in the space of coupling constants. This set may vary smoothly or it may be discontinuous at certain points along the path. We have found it easiest to study the behavior of this set in the context of one matrix models, while field theories have proven less tractable. However we suspect that our conclusions are quite general. One important conclusion is that the set of boundary conditions with a thermodynamic limit changes discontinuously as one crosses a phase boundary.

### Boundary Conditions In Zero Dimension

To illustrate the multiple solutions of the Schwinger-Dyson equations we begin with the simple example of a zero dimensional theory with a polynomial action, $S(\phi) = \frac{1}{n}g_n\phi^n$. The Generating functional of disconnected green’s functions $Z(J)$ satisfies the Schwinger–Dyson equation

$$\left(g_n\frac{\partial}{\partial J_{n-1}} - J\right)Z(J) = 0$$  \hfill (1)

The order of this equation is determined by the highest order term in the polynomial action. If the highest order term is $\phi^k$ then there is an $k - 1$ parameter space of solutions. One of these parameters is just the overall normalization of $Z$, so there is a $k - 2$ parameter space of solutions with distinct green’s functions. An integral representation of the solutions is

$$Z(J) = \int_{\Gamma} d\phi e^{-S(\phi)+J\phi}$$  \hfill (2)

It is easy to show that this satisfies the Schwinger-Dyson equation provided that

$$e^{-S(\phi)+J\phi}\big|_{\partial \Gamma} = 0$$  \hfill (3)

This condition is satisfied if $ReS(\phi) \to +\infty$ asymptotically on the contour $\Gamma$. For the polynomial action the condition becomes $Re g_k \phi^k \to +\infty$. Therefore there are $k$ wedge shaped domains in the complex plane in which the contour can run off to infinity. There are $k - 1$ independent contours satisfying (3), as expected from the order of the Schwinger-Dyson equation. In general the behavior of the action for large fields controls the solution set, even for many degrees of freedom. The condition (3) can be used to construct this set even when the action is non polynomial and the order of the Schwinger-Dyson equations is unclear. For example the action $S = \beta cos\phi$, (one plaquette QED), can be shown by this method to yield a two parameter class of solutions for $Z$. A basis set of solutions is given by the contours

$$\Gamma_1 = [-i\infty, +i\infty]$$  \hfill (4)
and
\[ \Gamma_2 = [-i\infty, 0] + [0, 2\pi] + [2\pi, 2\pi + i\infty] \] (5)

The difference between these two solutions is the usual solution in which the contour runs from 0 to \(2\pi\). Note that in general the exotic solutions for a gauge theory correspond to a complexification of the gauge group. In this simple case it is possible to verify the counting of solutions by coupling sources \(J\) and \(\bar{J}\) to the loop variables \(e^{i\phi}\) and \(e^{-i\phi}\). The Schwinger-Dyson equation is then,
\[ [\beta(\partial J - \partial \bar{J}) - 2(J\partial J - \bar{J}\partial \bar{J})]Z(J, \bar{J}) = 0 \] (6)

which when combined with the constraint,
\[ \frac{\partial}{\partial J} \frac{\partial}{\partial \bar{J}} Z = Z \] (7)

yields a two parameter class of solutions for \(Z\).

**Boundary Conditions in the General Case**

Let us consider the solution set for a lattice field theory. For theories in which the large field behavior of the action is dominated by independently variable local terms, such as \(g_k \phi^k(x)\), the construction of the space of solutions is a simple generalization of the zero dimensional case. One simply chooses one of the zero dimensional contours for each field at each lattice site. An arbitrary solution is obtained by summing solutions with a definite set of contours at each site. The solution set is then somewhat reduced by imposing the lattice symmetries, but is still very large. In a one matrix model, the lattice site label is replaced by an eigenvalue label, and since the interaction between eigenvalues is only logarithmic, the highest order term in the potential determines the allowable contours for each eigenvalue. For theories in which the large field behavior of the action is not dominated by independently variable local terms, the construction of the solution set is somewhat more complicated. This appears to be the situation for many lattice theories with a non-local order parameter.

For a generic action with a finite number of degrees of freedom, the number of independent solutions of the Schwinger-Dyson equations is exactly equal to the number of classical solutions, complex ones included. The exception to this rule occurs when the action has flat directions, or extrema with vanishing second derivative, in which case every term in the perturbative expansion about the classical solution diverges. For example in zero dimensions, the potential \(V = g\phi^4\) has only one classical solution, but there are three independent solutions of the Dyson Schwinger equations. Assuming that we are not considering such an exceptional case, the borel re-summation of the
perturbation series about any classical solution can be shown to yield one or more exact solutions of the Schwinger-Dyson equations with some exotic integral representation. One can see to which contours these solutions must correspond by taking the weak coupling limit. The contours, deformed so that $e^{-S}$ has constant phase, must avoid all the classical solutions of equal or lower action than the one whose perturbative expansion is being borel re-summed. There are often many choices of such contours. In terms of the inverse borel transform,

$$Z_i(g) = \int_{t=0}^{Ret=+\infty} e^{-t/g} B_i(t)$$

these choices correspond to various ways of avoiding positive real singularities in the Borel variable $t$. In the above equation the index $i$ labels a classical solution. Thus the complete set of solutions of the classical equations yields an over-complete set of solutions of the Schwinger-Dyson equations. An arbitrary solution is obtained by taking linear combinations of these solutions,

Thus away from the thermodynamic limit, there are a continuum of phases, resembling theta vacua. The resemblance to conventional theta vacua is actually quite strong. In the conventional approach to theta vacua, such as those in a Yang Mills theory, a particular theta vacuum is selected by adding a surface term to the action without any recourse to exotic contours. This alters the Schwinger Dyson equations only at the space time boundary, and therefore in the infinite volume limit the role of the surface term is to set a boundary condition for these equations. The surface term puts a different weight on the contribution to the generating function coming from fluctuations about different classical sectors. As we have seen above, this is equivalent to putting different weights on different complex contours, but without any modification of the action. Note however that it is not always easy or even possible to obtain all solutions of physical interest from the real integral, assuming it is convergent, by modifying the action with a surface term, or by small perturbations of the action which are removed after taking the thermodynamic limit. It is generally impossible to obtain false vacuum solutions in this way. Furthermore, there are examples of symmetry breaking solutions of matrix models which appear to be impossible to obtain from the real contour by any perturbation of the action [4]. The complete set of independent complex contours however accounts for all the phases of the theory.

To complete the classification of these phases one needs a rule for identifying solutions in the same phase at different values of the coupling constants. More precisely, one needs some set of first order differential equations in the coupling constants, which the partition function in a given phase should satisfy. A natural choice is given by the Schwinger action principle, which may be stated as,

$$(\frac{\partial}{\partial g} - < \frac{\partial S}{\partial g} >)Z = 0$$
or, for the example of the zero dimensional polynomial action, as

\[
\left( \frac{\partial}{\partial g_n} - \frac{1}{n} \frac{\partial}{\partial J^n} \right) Z = 0
\]

(10)

The action principle means that some set of contours and weights of contours associated with a solution is held fixed as one makes an infinitesimal change in the coupling constants. Note that there are classes of contours which lead to the same solution, either by being mutually deformable, or by leading to solutions which coalesce in the thermodynamic limit. However as one changes the couplings, these classes also change. Contours which were convergent may become non-convergent, and contours for which a thermodynamic limit existed may no longer have such a limit. Thus, the action principle does not allow one to fix the contours and weights globally in the space of coupling constants. There may be branch cuts in certain coupling constants, with branch points as phase boundaries. For instance in the zero dimensional $g\phi^4$ theory, if one rotates the phase of the coupling constant by $2\pi$ in accordance with the action principle, the contour of integration rotates by $\pi$ to maintain convergence, yielding an inequivalent solution. Thus the action principle fixes the phase only locally in the space of coupling constants.

Note that one could have chosen another set of first order differential equations in the couplings instead of the action principle. However the Schwinger action operators which annihilate the generating function commute with the Schwinger-Dyson operators which annihilate the generating function. If a continuum of solutions to the Schwinger-Dyson equations reduce to a discrete set in the thermodynamic limit, then in general the action principle is automatically induced. For instance matrix model solutions of the Schwinger-Dyson equation automatically satisfy the action principle in the planar limit. If theta parameters survive in a thermodynamic limit, then the action principle is not induced, but may certainly be consistently imposed.

**Vacuum Selection**

Having classified the solutions far from the thermodynamic limit, one needs some way of reducing the solution set. It is tempting to invoke such requirements as reality and positivity. Reality does not throw out all exotic solutions. If the coupling constants of the theory are real, one can always take linear combinations of exotic solutions with complex contours such that all Green’s functions become real. Thus reality is not so strong a constraint and also throws away false vacuum solutions which may be of physical interest. In seeking the true vacuum for for an action bounded below with real couplings, one might be tempted to invoke positivity, and only the integral over real fields is manifestly positive. However it is dangerous to invoke reality and positivity before taking a thermodynamic limit. The thermodynamic limit alone does most of
the job of reducing the solution set. This can be seen very explicitly in matrix models. Consider a model of an \( N \times N \) hermitian matrix \( M \),

\[
Z = \int dMe^{-N\text{tr}V(M)}
\]

(11)

\( M \) may be written as \( U\Lambda U^\dagger \) where \( \Lambda \) is diagonal, and then if the integral over \( U \) is performed one gets,

\[
Z = \int \prod_n d\Lambda_n \Delta^2[\Lambda]e^{-N\sum_n V(\Lambda_n)}
\]

(12)

where \( \Delta[\Lambda] \) is the Vandermonde determinant

\[
\Delta[\Lambda] = \prod_{n<m}(\Lambda_n - \Lambda_m)
\]

(13)

One can separately choose a contour for each eigenvalue by the condition \( \text{Re}V(\Lambda_n) \to +\infty \) for large \( \Lambda_n \). The boundary condition problem is then most easily understood if one first solves the model by the method of orthogonal polynomials. The reader is referred elsewhere for details [3]. To summarize the method very briefly, polynomials \( P_n(\lambda) = \lambda^n + ... \) are defined such that,

\[
\int d\lambda e^{-N\text{tr}V(\lambda)}P_n(\lambda)P_m(\lambda) = h_n\delta_{n,m}
\]

(14)

where the integral is over some permissible complex contour or sum of complex contours,

\[
\int d\lambda = \sum_i a_i \int_{\Gamma_i} d\lambda
\]

(15)

Note that it is assumed that the coefficients \( a_i \) of the integral contours are the same for each eigenvalue \( \Lambda_n \), therefore the orthogonal polynomial method actually does not include all possible boundary conditions of the Schwinger-Dyson equations. The \( P_n \) have the property that

\[
\lambda P_n(\lambda) = P_{n+1}(\lambda) + S_n P_n(\lambda) + R_n P_{n-1}(\lambda)
\]

(16)

The partition function and the Green’s functions may be written in terms of the coefficients \( R_n \) and \( S_n \). These coefficients satisfy a set of nonlinear recursion relations, known as the discrete string equations. A thermodynamic limit, or \( N \to \infty \) planar limit, is obtained if the \( R_n \) and \( S_n \) may be written in this limit in terms of smooth functions of \( x = \frac{n}{N} \) on the interval \([0,1]\). These functions are easily obtained as the local fixed points of the recursion relations. One can then look for the possible contours which give such limiting functions, using the fact the initial conditions of the discrete string equations, \( S_0 \) and \( R_1 \), are the one and two point connected greens functions of the zero
Some initial conditions will be attracted to one of the fixed point solutions, leading to
a thermodynamic limit, where as others will not. In this way it is easy to see that the
solutions associated with a very large number of boundary conditions either coalesce
or have no thermodynamic limit as $N \to \infty$. At certain critical values of the couplings,
such as the points about which a double scaling limit may be taken, the functions of $x$
which one obtains in the planar limit become complex within the interval $[0,1]$. This
means that as one crosses these critical domains, many of the boundary conditions
which give real solutions no longer have a thermodynamic limit. A large change in the
boundary conditions which lead to a thermodynamic limit is a general phenomenon
which occurs as one crosses a phase boundary.

There are some very exotic boundary conditions in matrix models which lead to
a physical thermodynamic limit. It has been shown [4] that in the planar limit the
potential $V(\lambda) = \frac{g}{4} \lambda^4 + \frac{\mu}{2} \lambda^2$ with $\mu < 0$ and $g$ positive has several solutions with varying
degrees of symmetry breaking, and that in the double scaling limit these extend to a
larger number of symmetries some of which differ perturbatively (beyond first order in
the string coupling) and others which differ only non-perturbatively. That one can
account for these various solutions with sums over different contour integrations has
also been shown [5] [6]. However it is a highly nontrivial problem to find the boundary
conditions associated with these solutions if one switches on a symmetry breaking
term, so that $V(\lambda) = \frac{g}{4} \lambda^4 + \frac{\mu}{2} \lambda^2 + \sigma \lambda$. The reason for the difficulty is as follows.
Consider an arbitrary set of boundary conditions (sums over contours) and consider
what happens to the initial conditions of the string equations, $S_0$ and $R_1$, as $N \to \infty$.
In this limit $S_0$ and $R_1$ are determined by the leading term in a saddle point expansion.
For $\sigma = 0$ there are two degenerate minima, and $S_0$ and $R_1$ can approach a continuum
of possible values, subject to the constraint $R_1 = \frac{\mu}{g} - S_0^2$. However for non-zero $\sigma$
there are no degenerate extrema, and for any set of contours the result for $S_0$ and $R_1$ is
dominated by the extremum with the lowest potential among those which the contours
pass through. Therefore as $N \to \infty$ one can only approach a discrete set of values
rather than a continuous set: $R_1 \to 0$ and $S_0 \to \lambda_i$ where $\lambda_i$ are the three extrema
of the potential $V(\lambda)$. These initial conditions lead to a smaller solution set than
was obtained in the case where the potential was degenerate. It appears that certain
solutions vanish, or rather collapse into other solutions, upon turning on the symmetry
breaking term. Naively that would seem to imply that the solutions which vanish have,
at $\sigma = 0$, singular Greens functions arising from differentiation of $Z$ with respect to
$\sigma$. But one can readily check that there are no such singularities. These solutions may
be smoothly evolved away from $\sigma = 0$ according to the action principle. Therefore
there should be some set of contours which yields these solutions both at $\sigma = 0$ and at
some small non-zero $\sigma$. We have just seen that the usual choice of a single measure for
every eigenvalue does not work. However there are many other boundary conditions
of the Schwinger-Dyson or loop equations for which orthogonal polynomial techniques
are not appropriate. For instance one may have staggered boundary conditions with
different contours of integration for different eigenvalues.

There are several phenomena which occur in the thermodynamic limit which appear to be disparate. One is the collapse of the space of boundary conditions, due to the fact that many boundary conditions do not lead to thermodynamic limit. The other is the appearance of new non-analyticity and phase boundaries in the coupling constants due to the accumulation of Lee-Yang zeroes. In fact these are not really disparate. We will give a heuristic argument below. The collapse of the space of boundary conditions means that as one varies the couplings, one must also vary the boundary conditions (contours), since the set of boundary conditions with a thermodynamic limit depends upon where one is in the space of coupling constants. It is then possible that by following a closed path in coupling constant space, one does not return to the boundary condition with which one started. This would mean that the thermodynamic limit has introduced a non-analyticity in the coupling constants, which in turn requires an accumulation of Lee-Yang zeroes. Note that there are simple zero dimensional analogues of this. Consider the polynomial action

$$S = \sum_{n=1}^{k} g_n \phi^n.$$  

The solutions are analytic in all but the highest coupling constant $g_k$. If one follows a closed path in the complex plane with any of the lower coupling constants, one does not have to change the contour to maintain convergence, and thus there is only a single Riemann sheet in the lower coupling constants. If however one tunes $g_k$ to zero, then the space of boundary conditions shrinks. If one now rotates the phase of $g_{k-1}$ by a large amount, then one must also rotate the contour of integration, whereas the contour could be held fixed for non-zero $g_k$. Green’s functions become multiply sheeted in $g_{k-1}$.

The conventional tool for studying the phase structure of theories with a local order parameter is the effective potential. It is interesting to note the consequences of our analysis of boundary conditions on the form of the effective potential. In fact, far from the thermodynamic limit there are many effective potentials. Consider again the zero dimensional $\phi^4$ theory. In terms of the function $\phi(J) = \frac{\partial}{\partial J} \ln Z(J)$, the Schwinger-Dyson equation is a nonlinear second order differential equation,

$$J = g\phi^3 + \mu\phi + g \frac{\partial^2}{\partial J^2} \phi(J) + 3g\phi(J) \frac{\partial}{\partial J} \phi(J)$$  \hspace{1cm} (17)

The effective potential $\Gamma(\phi)$ is defined by the relation $J = \frac{d\Gamma}{d\phi}$. Any effective potential can only correspond to a discrete set of solutions $\phi(J)$ out of the full continuous 2 parameter class. In fact in zero dimensions $Z(J)$ is analytic in $J$, and $\phi(J)$ has no branch cuts in $J$, so their is only one solution associated with each effective potential. Note that order by order in a loop expansion, $\phi(J)$ is multiply sheeted and the effective potential appears to possess multiple extrema which correspond to different boundary conditions. This is a spurious feature of the loop expansion in zero dimensions. Actually $\phi(J)$ has an infinite tower of poles in $J$ rather than branch cuts. In an appropriate thermodynamic limit these poles can coalesce, in which case several boundary conditions become described by the same exact effective potential. Many effective potentials associated with different boundary conditions coalesce in the ther-
modynamic limit, while effective potentials associated with other boundary conditions do not have a well defined thermodynamic limit.

Conclusion

The phase structure of bosonic field theories and matrix models appears to have a very natural interpretation in terms of the boundary conditions of Schwinger-Dyson equations. There are several interesting open questions of which we will list only a few. It is not known whether or how these ideas may be generalized to fermionic theories. Another question concerns the relation between the various boundary conditions and the phases of a theory with a nonlocal order parameter. As yet we have not been able to construct such a relation.

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