FRÉCHET ALGEBRAS WITH A DOMINATING HILBERT ALGEBRA NORM

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ABSTRACT. Let \( \mathcal{L}^*(s) \) be the maximal \( \mathcal{O}^* \)-algebra of unbounded operators on \( \ell_2 \) whose domain is the space \( s \) of rapidly decreasing sequences. This is a noncommutative topological algebra with involution which can be identified, for instance, with the algebra \( \mathcal{L}(s) \cap \mathcal{L}(s') \) or the algebra of multipliers for the algebra \( \mathcal{L}(s', s) \) of smooth compact operators. We give a simple characterization of unital commutative Fréchet *-subalgebras of \( \mathcal{L}^*(s) \) isomorphic as a Fréchet spaces to nuclear power series spaces \( \Lambda_\infty(\alpha) \) of infinite type. It appears that many natural Fréchet *-algebras are closed *-subalgebras of \( \mathcal{L}^*(s) \), for example, the algebras \( C^\infty(M) \) of smooth functions on smooth compact manifolds and the algebra \( \mathcal{F}(\mathbb{R}^n) \) of smooth rapidly decreasing functions on \( \mathbb{R}^n \).

1. Introduction

Let \( s \) be the Fréchet space of rapidly decreasing complex sequences and let

\[
\mathcal{L}^*(s) := \{ x : s \to s : x \text{ is linear, } s \subset \mathcal{D}(x^*) \text{ and } x^*(s) \subset s \},
\]

where \( \mathcal{D}(x^*) \) is the domain of the adjoint of an unbounded operator \( x \) on \( \ell_2 \). The class \( \mathcal{L}^*(s) \) is known as the maximal \( \mathcal{O}^* \)-algebra with domain \( s \) and it can be seen as the largest *-algebra of unbounded operators on \( \ell_2 \) with domain \( s \) – for details see the book of Schmüdgen [13 Section I.2.1]. The *-algebra \( \mathcal{L}^*(s) \) can be topologised in several natural ways, as is shown in [13 Sections I.3.3 and I.3.5]. Here the space \( \mathcal{L}^*(s) \) is considered with – the best from the functional analysis point of view – locally convex topology \( \tau^* \) (for definition see Preliminaries and also Proposition 2.4). Indeed, standard tools of functional analysis, such as closed graph theorem, open mapping theorem or uniform boundedness principle, can be applied to \((\mathcal{L}^*(s), \tau^*)\) (see [8 Th. 4.5]). Furthermore, \( \mathcal{L}^*(s) \) is a topological *-algebra – i.e. multiplication is separately continuous and involution is continuous – but it is neither locally \( m \)-convex nor a \( Q \)-algebra. The algebra \( \mathcal{L}^*(s) \) is isomorphic as a topological *-algebra, for example, to the algebra \( \mathcal{L}(s) \cap \mathcal{L}(s') \), the algebra of multipliers for the algebra \( \mathcal{L}(s', s) \) of smooth compact operators and also to the matrix algebra

\[
\Lambda(\mathcal{A}) := \left\{ x = (x_{ij}) \in \mathbb{C}^{N \times N} : \forall N \in \mathbb{N} \exists n \in \mathbb{N} \sum_{i,j \in \mathbb{N}^2} |x_{ij}| \max \left\{ \frac{i^N}{j^n}, \frac{j^N}{i^n} \right\} < \infty \right\};
\]

for details and more information about topological and algebraic properties of \( \mathcal{L}^*(s) \) we refer the reader to [8].

The space \( s \) carries all the information about nuclear Fréchet (even locally convex) spaces. Indeed, by the Kōmura-Kōmura theorem, a Fréchet space is nuclear if and only if it is isomorphic to some closed subspace of \( s^N \) (see [14 Cor. 29.9]). What about closed subspaces of the space \( s \) itself? In [21] Vogt proved that a nuclear Fréchet space is isomorphic to a closed subspace of \( s \) if and only if it has the so-called property (DN). Moreover, quotients of \( s \) were characterised by Vogt and Wagner in [22] via the so-called property (\( \Omega \)). Consequently, we have the following characterization: a nuclear Fréchet space is isomorphic to a complemented subspace of \( s \) if and only if it has the properties (DN) and (\( \Omega \)). It is also well-known that a Fréchet space with (DN), (\( \Omega \)) and a Schauder basis is isomorphic to a power series space \( \Lambda_\infty(\alpha) \) of infinite type. However, it is still an open problem – a particular case of the famous Mityagin-Pełczyński problem – whether there is a complemented subspace of \( s \) without a basis.

In this paper, we are mainly interested in unital Fréchet algebras with involution which are isomorphic as Fréchet spaces to nuclear power series spaces of infinite type. We show that a large class of them –

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12010 Mathematics Subject Classification. Primary: 46J25. Secondary: 46A11, 46A63, 46E25, 46K15, 47L60.

Key words and phrases: Representations of commutative Fréchet algebras with involution, topological algebras of unbounded operators, nuclear Fréchet algebras of smooth functions, dominating norm, Hilbert algebras

The research of the author was supported by the National Center of Science, grant no. 2013/10/A/ST1/00091.
those algebras $E$ which admit a dominating Hilbert norm $\| \cdot \| = \sqrt{\langle \cdot , \cdot \rangle}$ such that
\begin{equation}
(\alpha y, z) = (y, \alpha^* z)
\end{equation}
for all $x, y, z \in E$ – can be embedded into $\mathcal{L}^*(s)$ as closed, even complemented, $^*$-subalgebras (see Theorem 3.3 and Remark 3.17). In the commutative case we even have the following characterization: a unital commutative Fréchet $^*$-algebra isomorphic as a Fréchet space to a nuclear power series space $\Lambda_{\infty}(\alpha)$ of infinite type is isomorphic as a Fréchet $^*$-algebra to a closed $^*$-subalgebra of $\mathcal{L}^*(s)$ if and only if it admits a dominating Hilbert norm satisfying condition \ref{cond} (see again Theorem 3.3). In Theorem 3.6 we also characterize commutative Fréchet unital $^*$-subalgebras of $\mathcal{L}^*(s)$ consisting of bounded operators on $\ell_2$ and isomorphic as Fréchet space to nuclear spaces $\Lambda_{\infty}(\alpha)$. It is worth noting that condition \ref{cond} appears in the definition of Hilbert algebras playing an important role in the theory of von Neumann algebras (see \cite{9} A.54).

The above-mentioned results may be seen as a step towards an analogue – in the context of nuclear power series spaces of infinite type – of the celebrated commutative Gelfand-Naimark theorem. In the separable case it states that there is one to one correspondence (given by isometric $^*$-isomorphisms) between Banach algebras $C(K)$ of continuous functions on compact Hausdorff metrizable spaces $K$ and closed unital commutative $^*$-subalgebras of the $C^*$-algebra $\mathcal{B}(\ell_2)$ of bounded operators on $\ell_2$.

Our results are applicable. In the last section we give concrete examples of Fréchet $^*$-algebras which can be represented in $\mathcal{L}^*(s)$ in the way described above. Among them there are: the algebras $C^\infty(M)$ of smooth functions on smooth compact manifolds, the algebras $\mathcal{C}(K)$ with Schauder basis of smooth Whitney jets on compact sets $K$ with the extension property, the algebra $\mathcal{S}(\mathbb{R}^n)$ of smooth rapidly decreasing functions on $\mathbb{R}^n$, nuclear power series algebras $\Lambda_{\infty}(\alpha)$ of infinite type and the noncommutative algebra $\mathcal{L}(s', s)$ of compact smooth operators. We also provide one counterexample. We show that the unital commutative Fréchet $^*$-algebra $A^\infty(\mathbb{D})$ of holomorphic functions on the open unit disc $\mathbb{D}$ with smooth boundary values is not isomorphic to any closed $^*$-subalgebra of $\mathcal{L}^*(s)$.

2. Preliminaries

The canonical $\ell_2$ norm and the corresponding scalar product will be denoted by $\| \cdot \|_{\ell_2}$ and $\langle \cdot , \cdot \rangle$, respectively.

For locally convex spaces $E$ and $F$, we denote by $\mathcal{L}(E, F)$ the space of all continuous linear operators from $E$ to $F$ and we set $\mathcal{L}(E) := \mathcal{L}(E, E)$. These spaces will be considered with the topology $\tau_{\mathcal{L}(E, F)}$ of uniform convergence on bounded sets.

By a topological $^*$-algebra $E$ we mean a topological vector space endowed with at least separately continuous multiplication and continuous involution which make $E$ a $^*$-algebra. A Fréchet $^*$-algebra is a topological $^*$-algebra whose underlying topological vector space is a Fréchet space (i.e. metrizable complete locally convex space). We do not require a Fréchet $^*$-algebra to be locally $m$-convex.

Let $\alpha = (\alpha_j)_{j \in \mathbb{N}}$ be a monotonically increasing sequence in $(0, \infty)$ such that $\lim_{j \to \infty} \alpha_j = \infty$. Then
\[
\Lambda_{\infty}(\alpha) := \{(\xi_j)_{j \in \mathbb{N}} \subset \mathbb{C}^\mathbb{N} : \| \xi_j \|_{\alpha, q}^2 := \sum_{j=1}^{\infty} |\xi_j|^2 e^{2\alpha_j q} < \infty \text{ for all } q \in \mathbb{N}_0\}
\]
equipped with the norms $\| \cdot \|_{\alpha, q}$, $q \in \mathbb{N}_0$, is a Fréchet space and it is called a power series space of infinite type. It appears that the space $\Lambda_{\infty}(\alpha)$ is nuclear if and only if $\sup_{j \in \mathbb{N}} \frac{\log j}{\alpha_j} < \infty$ (see e.g. \cite{14} Prop. 29.6). In particular, for the sequence $\alpha_j := \log j$, $j \in \mathbb{N}$, we obtain the space $s$ of rapidly decreasing sequences, i.e.
\begin{equation}
s := \{(\xi_j)_{j \in \mathbb{N}} \subset \mathbb{C}^\mathbb{N} : \| \xi_j \|_q^2 := \sum_{j=1}^{\infty} |\xi_j|^2 j^{2q} < \infty \text{ for all } q \in \mathbb{N}_0\}.
\end{equation}

By $s_n$ we denote the Hilbert space corresponding to the norm $\| \cdot \|_n$.

The strong dual of $s$ – i.e. the space of all continuous linear functionals on $s$ with the topology of uniform convergence on bounded subsets of $s$ (see e.g. \cite{14} Def. on p. 267) – is isomorphic to the space
\begin{equation}
s' := \{(\xi_j)_{j \in \mathbb{N}} \subset \mathbb{C}^\mathbb{N} : \| \xi_q \|_{-2q}^2 := \sum_{j=1}^{\infty} |\xi_j|^2 j^{-2q} < \infty \text{ for some } q \in \mathbb{N}_0\}
\end{equation}
of slowly increasing sequences equipped with the inductive limit topology for the sequence \((s_{-n})_{n \in \mathbb{N}_0}\) of Hilbert spaces corresponding to the norms \(\| \cdot \|_{-n}\). In other words, the locally convex topology on \(s'\) is given by the family \(\{ \| \cdot \|_B \mid B \in \mathcal{B} \}\) of seminorms, \(\| \cdot \|_B := \sup_{\eta \in B} \langle \eta, \xi \rangle\), where \(\mathcal{B}\) denotes the class of all bounded subsets of \(s\) and, recall, \((\cdot, \cdot)\) is the canonical scalar product on \(\ell_2\).

**Definition 2.1.** A Fréchet space \(E\) with a fundamental system \((\| \cdot \|_q)_{q \in \mathbb{N}_0}\) of seminorms

1. has the property (DN) (cf. [14, Def. on p. 359]) if there is a continuous norm \(\| \cdot \|\) on \(E\) – called a dominating norm – such that for all \(q \in \mathbb{N}_0\) there is \(r \in \mathbb{N}_0\) and \(C > 0\) such that

\[\|x\|_q^2 \leq C\|x\|_r \] 

for all \(x \in E\);

2. has the property (\(\Omega\)) (cf. [14, Def. on p. 367]) if for all \(p \in \mathbb{N}_0\) there is \(q \in \mathbb{N}_0\) such that for all \(r \in \mathbb{N}\) there are \(\theta \in (0, 1)\) and \(C > 0\) with

\[\|y\|_p^\theta \leq C\|y\|_r^{1-\theta}\|y\|^{\theta}_r\] 

for all \(y \in E'\), where \(E'\) is the topological dual of \(E\) and \(\|y\|_r^\theta := \sup\{ |y(x)| : \|x\|_r \leq 1 \}\).

The properties (DN) and (\(\Omega\)) are linear-topological invariants which play a key role in a structure theory of nuclear Fréchet spaces. The following Theorem is due to Vogt and Wagner.

**Theorem 2.2.** ([13, Ch. 31] and [22, 21]) A Fréchet space is isomorphic to:

(i) a closed subspace of \(s\) if and only if it is nuclear and has the property (DN);

(ii) a quotient of \(s\) if and only if it is nuclear and has the property (\(\Omega\));

(iii) a complemented subspace of \(s\) if and only if it is nuclear and has the properties (DN) and (\(\Omega\)).

We also cite another result of Vogt which will be crucial for our further considerations.

**Theorem 2.3.** ([13, Cor. 7.7]) Let \(E\) be a Fréchet space isomorphic to a power series space \(\Lambda_\infty(\alpha)\) of infinite type. Then for every dominating Hilbert norm \(\| \cdot \|\) on \(E\) there is an isomorphism \(u: E \to \Lambda_\infty(\alpha)\) such that \(\|u\|_2 = \|\xi\|\) for all \(\xi \in E\).

Let \(E\) be a Fréchet space with a continuous Hilbert norm \(\| \cdot \|\). Let \(H\) be the completion of \(E\) in the norm \(\| \cdot \|\) and let \((\cdot, \cdot)\) be the corresponding scalar product. Then we define

\[\mathcal{L}^*(E, \| \cdot \|) := \{ x : E \to E : x \text{ is linear}, E \subset \mathcal{D}(x^*) \text{ and } x^*(E) \subset E \},\]

where

\[\mathcal{D}(x^*) := \{ \eta \in H : \exists \zeta \in H \forall \xi \in E \ (x\xi, \eta) = (\xi, \zeta) \}\]

and \(x^*\eta := \zeta\) for \(\eta \in \mathcal{D}(x^*)\). In the case when \(E\) is a closed subspace of \(s\) or \(E = \Lambda_\infty(\alpha)\) write \(\mathcal{L}^*(E)\) instead of \(\mathcal{L}^*(E, \| \cdot \|_2)\). Since \(E\) is a dense linear subspace of \(H\), each \(x \in \mathcal{L}^*(E, \| \cdot \|)\) can be considered as a dense unbounded operator in \(H\) with domain \(\mathcal{D}(x) = E\), and thus it has the adjoint \(x^* : \mathcal{D}(x^*) \to H\). By definition, the operator \(x^*|_E\), for simplicity denoted again by \(x^*\), is in \(\mathcal{L}^*(E, \| \cdot \|)\), as well. Moreover, by definition,

\[\mathcal{D}(xy) := \{ \xi \in \mathcal{D}(x) : y\xi \in \mathcal{D}(x) \} = E\]

for all \(x, y \in \mathcal{L}^*(E, \| \cdot \|)\). This shows that \(\mathcal{L}^*(E, \| \cdot \|)\) is a *-algebra. In fact, the class \(\mathcal{L}^*(E, \| \cdot \|)\) can be seen as the largest *-algebra of unbounded operators on \(H\) with domain \(E\) and it is known as the maximal \(\mathcal{O}^*\)-algebra with domain \(E\) (see [15, 2.1] for details).

In the theory of maximal \(\mathcal{O}^*\)-algebras – and, more generally, of algebras of unbounded operators in Hilbert spaces – one consider the so-called graph topology ([15, Def. 2.1.1]). With \(E\) and \(\| \cdot \|\) as above, the graph topology of \(\mathcal{L}^*(E, \| \cdot \|)\) on \(E\) is, by definition, given by the system of seminorms \(\{ \| \cdot \|_a \mid a \in \mathcal{L}^*(E, \| \cdot \|) \}\), \(\| \xi \|_a := \| a\xi \|\) for \(\xi \in E\).

The following easy observation is kind of folklore – for completeness we present here the proof.

**Proposition 2.4.** Let \(E\) be a Fréchet space with a continuous Hilbert norm \(\| \cdot \|\). Then the graph topology of \(\mathcal{L}^*(E, \| \cdot \|)\) on \(E\) is weaker than the Fréchet space topology.
Proof. Let $(\cdot, \cdot)$ denote the scalar product corresponding to the Hilbert norm $\| \cdot \|$ and let $H$ be the completion of $E$ in the norm $\| \cdot \|$. We shall show that each $a \in \mathcal{L}^*(E, \| \cdot \|)$ is a continuous map from the Fréchet space $E$ to the Hilbert space $H$. Let $(\xi_j)_{j \in \mathbb{N}} \subset E$ be a sequence converging in the Fréchet space topology to $0$ and assume that $a\xi_j$ converges in the norm $\| \cdot \|$ to some $\eta \in H$. We have, for all $\zeta \in E$,

$$\lim_{j \to \infty} (a\xi_j, \zeta) = (\eta, \zeta)$$

and, on the other hand,

$$\lim_{j \to \infty} (a\xi_j, \zeta) = \lim_{j \to \infty} (\xi_j, a^* \zeta) = 0.$$

Hence, $(\eta, \zeta) = 0$ for all $\zeta \in E$, and thus $\eta = 0$. Consequently, by the closed graph theorem for Fréchet spaces (cf. [14, Th. 24.31]), the map $a: E \to H$ is continuous, which is the desired conclusion. \qed

Sometimes the initial Fréchet space topology and the graph topology coincide.

**Proposition 2.5.** Let $E$ be a Fréchet space isomorphic to a power series space $\Lambda_\infty(\alpha)$ of infinite type and let $\| \cdot \|$ be a dominating Hilbert norm on $E$. Then the graph topology of $\mathcal{L}^*(E, \| \cdot \|)$ on $E$ coincides with the Fréchet space topology.

Proof. Let $(\cdot, \cdot)$ denote the scalar product corresponding to the Hilbert norm $\| \cdot \|$. By [23, Cor. 7.7], there is an isomorphism $u: E \to \Lambda_\infty(\alpha)$ such that $\| u\xi \|_\ell_2 = \| \xi \|$ for all $\xi \in E$. Let $\| \xi \|_n := \| u\xi \|_{\alpha,n}$ for $\xi \in E$ and $n \in \mathbb{N}$. Then $(\| \cdot \|_n)_{n \in \mathbb{N}}$ is a fundamental sequence of dominating Hilbert norms on $E$. For $n \in \mathbb{N}$, we define the diagonal map $d_n: \Lambda_\infty(\alpha) \to \Lambda_\infty(\alpha)$, $d_n\xi := (e^{\alpha} \xi)_{j \in \mathbb{N}}$. Clearly, each $d_n$ is an automorphism of the Fréchet space $\Lambda_\infty(\alpha)$ and $\| d_n \xi \|_\ell_2 = \| \xi \|_{\alpha,n}$ for all $\xi \in \Lambda_\infty(\alpha)$. Now, for $n \in \mathbb{N}$, let $a_n: E \to E$, $a_n := u^{-1}d_n u$. We have

$$(a_n \xi, \zeta) = \langle u^{-1}d_n u\xi, \zeta \rangle = \langle d_n u\xi, u\zeta \rangle = \langle u\xi, d_n u\zeta \rangle = \langle \xi, u^{-1}d_n u\zeta \rangle = (\xi, a_n \zeta)$$

for all $\xi, \zeta \in E$, whence $a_n \in \mathcal{L}^*(E, \| \cdot \|)$. Consequently, since $\| \xi \|_n = \| a_n \xi \|$ for all $\xi \in E$, i.e. $\| \cdot \|_n = \| \cdot \|_{\alpha,n}$, the graph topology of $\mathcal{L}^*(E, \| \cdot \|)$ on $E$ is finer than the Fréchet space topology and thus, in view of Proposition 2.4 these topologies are equal. \qed

There are plenty natural topologies on the space $\mathcal{L}^*(E, \| \cdot \|)$ (see [15, Sect. 3.3, 3.5]). Here we are interested in the locally convex topology $\tau^*$ on $\mathcal{L}^*(E, \| \cdot \|)$ given by the seminorms

$$p^{a,B}(x) := \max \left\{ \sup_{\xi \in B} \| a\xi \|, \sup_{\xi \in B} \| a^*\xi \| \right\},$$

where $a$ and $B$ run over $\mathcal{L}^*(E, \| \cdot \|)$ and the class of all bounded subsets of $E$ equipped with the graph topology of $\mathcal{L}^*(E, \| \cdot \|)$, respectively (see [15, pp. 81–82]). It is well-known that $\mathcal{L}^*(E, \| \cdot \|)$ endowed with the topology $\tau^*$ is a topological $^*$-algebra (cf. [15, Prop. 3.3.15 (ii)]). If we, moreover, assume that $E$ is isomorphic to a power series space of infinite type and $\| \cdot \|$ is a dominating Hilbert norm on $E$, then, by Proposition 2.5 and [18, Prop. 3.3.15 (iv)], $(\mathcal{L}^*(E, \| \cdot \|), \tau^*)$ is complete. The following characterization of the topology $\tau^*$ is a direct consequence of Proposition 2.5.

**Proposition 2.6.** Let $E$ be a Fréchet space isomorphic to a power series space of infinite type and let $\| \cdot \|$ be a dominating Hilbert norm on $E$. Let $(\| \cdot \|_n)_{n \in \mathbb{N}}$ be a fundamental sequence of norms on $E$. Then the topology $\tau^*$ on $\mathcal{L}^*(E, \| \cdot \|)$ is given by the seminorms $(p^{a,B}_{n}(x))_{n \in \mathbb{N}, B \in \mathcal{B}_E}$,

$$(4) \quad p_{n,B}(x) := \max \left\{ \sup_{\xi \in B} \| x\xi \|_n, \sup_{\xi \in B} \| x^*\xi \|_n \right\},$$

where $\mathcal{B}_E$ denote the class of all bounded subsets of $E$.

3. Fréchet subalgebras of $\mathcal{L}^*(s)$

In this section we give abstract descriptions of two large classes of complemented commutative Fréchet $^*$-subalgebras of $\mathcal{L}^*(s)$ (Theorems 3.5 and 3.6). Moreover, we provide a criterion for the existence of a “nice” embedding in $\mathcal{L}^*(s)$ of not necessarily commutative Fréchet $^*$-algebras (see Remark 3.17).

Let us first recall the notion of Hilbert algebras.

**Definition 3.1.** (cf. [9, A.54]) A Hilbert algebra is a $^*$-algebra $E$ endowed with a Hilbert norm $\| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle}$ such that:
(α) \((xy, z) = (y, x^* z)\) for all \(x, y, z \in E\);
(β) for all \(x \in E\) there is \(C > 0\) such that \(\|xy\| \leq C\|y\|\) for all \(y \in E\), i.e. the left multiplication maps \(m_x : (E, \| \cdot \|) \to (E, \| \cdot \|), m_x(y) := xy\), are bounded;
(γ) \((y^*, x^*) = (x, y)\) for all \(x, y \in E\);
(δ) the linear span of the set \(E^2 := \{ab : a, b \in E\}\) is dense in \(E\).

Each norm \(\| \cdot \|\) satisfying conditions (α)–(δ) is called a Hilbert algebra norm.

**Remark 3.2.** If \(E\) is unital, then condition (δ) in the above definition is trivially satisfied. If, moreover, \(E\) is commutative, then (α) implies (γ). Hence, every Hilbert norm on a unital commutative *-algebra satisfying condition (α) and (β) is already a Hilbert algebra norm.

**Definition 3.3.** A Fréchet *-algebra is called a DN-algebra if it admits a Hilbert dominating norm satisfying condition (α) in Definition \ref{def:DN-algebra}. A DN-algebra is called a βDN-algebra if the corresponding Hilbert dominating norm satisfies conditions (α) and (β) simultaneously.

**Remark 3.4.** In \cite{13} Def. 1.5 M. Măntoiu and R. Purice defined a Fréchet-Hilbert algebra as a Fréchet *-algebra admitting a continuous Hilbert algebra norm (more precisely, in their definition the corresponding Hilbert algebra scalar product is predetermined). Hence, in view of Remark \ref{rem:DN-algebra} every unital commutative βDN-algebra is a Fréchet-Hilbert algebra.

Our main results read as follows.

**Theorem 3.5.** Let \(E\) be a unital commutative Fréchet *-algebra isomorphic as a Fréchet space to a nuclear power series space of infinite type. Then the following statements are equivalent.

1. \(E\) is isomorphic as a Fréchet *-algebra to a complemented *-subalgebra of \(\mathcal{L}^* (s)\).
2. \(E\) is isomorphic as a Fréchet *-algebra to a closed *-subalgebra of \(\mathcal{L}^* (s)\).
3. \(E\) is a DN-algebra.

**Theorem 3.6.** Let \(E\) be a unital commutative Fréchet *-algebra isomorphic as a Fréchet space to a nuclear power series space of infinite type. Then the following statements are equivalent.

1. \(E\) is isomorphic as a Fréchet *-algebra to a complemented *-subalgebra \(F\) of \(\mathcal{L}^* (s)\) such that \(F \subset \mathcal{L} (\ell_2)\).
2. \(E\) is isomorphic as a Fréchet *-algebra to a closed *-subalgebra \(F\) of \(\mathcal{L}^* (s)\) such that \(F \subset \mathcal{L} (\ell_2)\).
3. \(E\) is a βDN-algebra.

We divide the proof into a sequence of lemmas. As a by-product, we obtain also three results which are interesting enough to be stated as “corollaries”.

For every \(N, n \in \mathbb{N}_0\) we define the space

\[\mathcal{L}(s_n, s_N) \cap \mathcal{L}(s_{-N}, s_{-n}) := \{x \in \mathcal{L}(s_n, s_N) : \exists \bar{x} \in \mathcal{L}(s_{-N}, s_{-n}) \quad \bar{x}|_{s_n} = x\}\]

with the norm

\[r_{N,n}(x) := \max \left\{ \sup_{|\xi| \leq 1} |x\xi|_{s_N}, \sup_{|\xi| \leq 1} |\bar{x}\xi|_{-s_n} \right\} \]

Formally, the space \(\mathcal{L}(s_n, s_N) \cap \mathcal{L}(s_{-N}, s_{-n})\) is the projective limit of the Banach spaces \(\mathcal{L}(s_n, s_N)\) and \(\mathcal{L}(s_{-N}, s_{-n})\) with their standard norms, and thus is a Banach space itself. Since

\[\sup_{|\xi| \leq 1} |x^* \xi|_{s_N} = \sup_{|\xi| \leq 1} \sup_{|\eta| \leq 1} |(x^* \xi, \eta)| = \sup_{|\xi| \leq 1} \sup_{|\eta| \leq 1} |(\xi, \eta)| = \sup_{|\eta| \leq 1} |\bar{x}\eta|_{-n},\]

we have

\[r_{N,n}(x) = \max \left\{ \sup_{|\xi| \leq 1} |x\xi|_{s_N}, \sup_{|\xi| \leq 1} |x^* \xi|_{s_N} \right\} \]

where \(x^* \in \mathcal{L}(s_n, s_N)\) is the hilbertian adjoint of the operator \(\bar{x}\). Moreover, \(\mathcal{L}^* (s) = \mathcal{L}(s) \cap \mathcal{L}(s')\) (see, e.g., \cite{3} Prop. 3.7), hence

\[\mathcal{L}^* (s) = \{x : s \to s : x\text{ linear and } \forall N \in \mathbb{N}_0 \exists n \in \mathbb{N}_0 \quad r_{N,n}(x) < \infty\}\]

as sets. Therefore, we can endow \(\mathcal{L}^* (s)\) with the topology of the PLB-space (a countable projective limit of a countable inductive limit of Banach spaces)

\[\text{proj}_{N \in \mathbb{N}_0} \text{ind}_{n \in \mathbb{N}_0} \mathcal{L}(s_n, s_N) \cap \mathcal{L}(s_{-N}, s_{-n}).\]

It appears that the topology \(\tau^*\) and the PLB-topology on \(\mathcal{L}^* (s)\) coincide.
Lemma 3.7. We have
\[
\mathcal{L}^*(s) = \text{proj}_{N \in \mathbb{N}_0} \text{ind}_{n \in \mathbb{N}_0} \mathcal{L}(s_n, s_N) \cap \mathcal{L}(s_{-N}, s_{-n})
\]
as topological vector spaces.

Proof. By [8, Cor. 4.2], \(\mathcal{L}^*(s)\) is ultrabornological and \(\text{proj}_{N \in \mathbb{N}_0} \text{ind}_{n \in \mathbb{N}_0} \mathcal{L}(s_n, s_N) \cap \mathcal{L}(s_{-N}, s_{-n})\) is webbed as a PLB-space. Hence, by the open mapping theorem (see e.g. [14, Th. 24.30]), it is enough to show that the identity map
\[
i: \text{proj}_{N \in \mathbb{N}_0} \text{ind}_{n \in \mathbb{N}_0} \mathcal{L}(s_n, s_N) \cap \mathcal{L}(s_{-N}, s_{-n}) \to \mathcal{L}^*(s)
\]
is continuous. Let \(N \in \mathbb{N}_0\) and let \(B\) be a bounded subset of \(s\). For every \(m \in \mathbb{N}_0\) choose a constant \(\lambda_m > 0\) such that \(B \subset \{ \xi \in s : |\xi|_m \leq \lambda_m \}\). Then
\[
p_{N,B}(x) = \max \left\{ \sup_{\xi \in B} |x\xi|_N, \sup_{\xi \in B} |x^*\xi|_N \right\} \leq \lambda_m \max \left\{ \sup_{|\xi|_m \leq 1} |x\xi|_N, \sup_{|\xi|_m \leq 1} |x^*\xi|_N \right\} = \lambda_m r_{N,m}(x)
\]
for every \(m \in \mathbb{N}_0\) and \(x \in \mathcal{L}(s_m, s_N) \cap \mathcal{L}(s_{-m}, s_{-N})\), and thus \(i\) is continuous. \(\square\)

Lemma 3.8. For every Fréchet subspace \(F\) of \(\mathcal{L}^*(s)\) there is \(m \in \mathbb{N}_0\) such that \(F \subset \mathcal{L}(s_m, \ell_2) \cap \mathcal{L}(\ell_2, s_{-m})\) and, moreover, for each such \(m\),
\[
r_m : F \to [0, \infty), \quad r_m(x) := \max \left\{ \sup_{|\xi|_m \leq 1} ||x\xi||_{\ell_2}, \sup_{|\xi|_m \leq 1} ||x^*\xi||_{\ell_2} \right\},
\]
is a dominating norm on \(F\).

Proof. By the very definition of projective topology, the canonical embedding
\[
\text{proj}_{N \in \mathbb{N}_0} \text{ind}_{n \in \mathbb{N}_0} \mathcal{L}(s_n, s_N) \cap \mathcal{L}(s_{-N}, s_{-n}) \hookrightarrow \text{ind}_{n \in \mathbb{N}_0} \mathcal{L}(s_n, \ell_2) \cap \mathcal{L}(\ell_2, s_{-n})
\]
is continuous and thus the identity map \(\kappa : F \hookrightarrow \text{ind}_{n \in \mathbb{N}_0} \mathcal{L}(s_n, \ell_2) \cap \mathcal{L}(\ell_2, s_{-n})\) is continuous, as well. Hence, by Grothendieck’s factorization theorem [14, Th. 24.33], there is \(m \in \mathbb{N}\) such that \(\kappa(F) \subset \mathcal{L}(s_m, \ell_2) \cap \mathcal{L}(\ell_2, s_{-m})\). Since we can identify in a obvious way \(F\) with \(\kappa(F)\), we get the first part of the thesis.

Now, fix an arbitrary \(m \in \mathbb{N}_0\) such that \(F \subset \mathcal{L}(s_m, \ell_2) \cap \mathcal{L}(\ell_2, s_{-m})\). Then \(r_m\) is a continuous seminorm on \(F\). Since \(F\) is a Fréchet space, there is a sequence \((B_N)_{N \in \mathbb{N}}, B_N \subset B_{N+1}\), of bounded subsets of \(s\) such that \((p_N)_{N \in \mathbb{N}}\),
\[
p_N(x) := \max \left\{ \sup_{\xi \in B_N} |x\xi|_N, \sup_{\xi \in B_N} |x^*\xi|_N \right\}
\]
for \(x \in F\), is a fundamental sequence of seminorms on \(F\). Moreover, for every \(N \in \mathbb{N}\) there is \(\lambda_N > 0\) such that \(B_N \subset \{ \xi \in s : |\xi|_m \leq \lambda_N \}\). Hence, for \(x \in F\) and \(N \in \mathbb{N}\), we obtain
\[
p_N^2(x) = \max \left\{ \sup_{\xi \in B_N} |x\xi|_N^2, \sup_{\xi \in B_N} |x^*\xi|_N^2 \right\} \leq \max \left\{ \sup_{\xi \in B_N} (||x\xi||_{\ell_2} |x\xi|_{2N}), \sup_{\xi \in B_N} (||x^*\xi||_{\ell_2} |x^*\xi|_{2N}) \right\}
\]
\[
\leq \max \left\{ \sup_{\xi \in B_N} ||x\xi||_{\ell_2} \cdot \sup_{\xi \in B_N} |x\xi|_{2N}, \sup_{\xi \in B_N} ||x^*\xi||_{\ell_2} \cdot \sup_{\xi \in B_N} |x^*\xi|_{2N} \right\}
\]
\[
\leq \lambda_N \max \left\{ \sup_{|\xi|_m \leq 1} ||x\xi||_{\ell_2}, \sup_{|\xi|_m \leq 1} |x\xi|_{2N}, \sup_{|\xi|_m \leq 1} ||x^*\xi||_{\ell_2}, \sup_{|\xi|_m \leq 1} |x^*\xi|_{2N} \right\},
\]
where the first inequality follows from the Cauchy-Schwartz inequality. Finally, since
\[
\max\{ab, cd\} \leq \max\{a, c\} \cdot \max\{b, d\}
\]
for all \(a, b, c, d \geq 0\), we obtain
\[
p_N^2(x) \leq \lambda_N \max \left\{ \sup_{|\xi|_m \leq 1} ||x\xi||_{\ell_2}, \sup_{|\xi|_m \leq 1} |x^*\xi||_{\ell_2} \right\} \cdot \max \left\{ \sup_{\xi \in B_{2N}} |x\xi|_{2N}, \sup_{\xi \in B_{2N}} |x^*\xi|_{2N} \right\} = \lambda_N r_m(x)p_{2N}(x)
\]
for all \(x \in F\), and thus \(r_m\) is a dominating norm on \(F\). \(\square\)

Corollary 3.9. (i) Every Fréchet subspace of \(\mathcal{L}^*(s)\) is isomorphic to a closed subspace of \(s\).
(ii) Every Fréchet quotient of \(\mathcal{L}^*(s)\) is isomorphic to a quotient of \(s\).
Since, for all $m$

**Lemma 3.12.** Let $E$ is isomorphic to a quotient of $\pi$. Hence, $\pi$ is isomorphic to a quotient of $E$.

**Proof.** First note that every closed subspace and quotient of $\mathcal{L}^*(s)$ is nuclear because $\mathcal{L}^*(s)$ is nuclear itself (see [8] Prop. 3.8 & Cor. 4.2).

(i) This follows immediately from Lemma 3.8 and [1] Prop. 31.5.

(ii) Let $E$ be a Fréchet quotient of $\mathcal{L}^*(s)$. It follows from [8] Prop. 4.7 and [2] Cor. 1.2(a) and (c) that $E$, being isomorphic to a quotient of $\mathcal{L}^*(s)$, has the property (Ω). Therefore, by [13] Prop. 31.6, $E$ is isomorphic to a quotient of $s$.

(iii) This is a direct consequence of the previous items and [13] Prop. 31.7.

Let $e_j$ denote the $j$-th unit vector in $\mathbb{C}^N$. If $F$ is a Fréchet subspace of $\mathcal{L}^*(s)$ then, by Lemma 3.8 there is $m \in \mathbb{N}_0$ such that $F \subset \mathcal{L}(s_m, \ell_2) \cap \mathcal{L}(\ell_2, s_m)$ and $r_m$ is a continuous (dominating) norm on $F$. Since, for all $x \in F$, we have

$$[x]_m := \left( \sum_{j=1}^{\infty} ||xe_j||_{\ell_2}^2 j^{-2m-2} \right)^{1/2} \leq \left( \sum_{j=1}^{\infty} j^{-2} \right)^{1/2} \sup_{||\xi||_{\ell_2}} ||x\xi||_{\ell_2}$$

$$\leq \frac{\pi}{\sqrt{6}} \max \left\{ \sup_{||\xi||_{\ell_2}} ||x\xi||_{\ell_2}, \sup_{||\xi||_{\ell_2}} ||x^*\xi||_{\ell_2} \right\} = \frac{\pi}{\sqrt{6}} r_m(x),$$

the scalar product

$$(5) \quad [\cdot, \cdot]_m : F \times F \to \mathbb{C}, \quad [x, y]_m := \sum_{j=1}^{\infty} (xe_j, ye_j) j^{-2m-2},$$

is well-defined and $[\cdot, \cdot]_m = \sqrt{[\cdot, \cdot]_m}$ is a continuous Hilbert norm on $F$.

**Lemma 3.10.** Let $F$ be a commutative Fréchet *-subalgebra of $\mathcal{L}^*(s)$ and let $m \in \mathbb{N}_0$ be such that $F \subset \mathcal{L}(s_m, \ell_2) \cap \mathcal{L}(\ell_2, s_m)$. Then the norm $[\cdot, \cdot]_m$ defined by (5) is a Hilbert dominating norm on $F$ satisfying condition (α).

**Proof.** Since $F$ is commutative, we have

$$||x\xi||_{\ell_2} = ||x^*\xi||_{\ell_2}$$

for all $x \in E$ and all $\xi \in s$. Hence,

$$r_{m+2}(x) = \max \left\{ \sup_{||\xi||_{\ell_2}} ||x\xi||_{\ell_2}, \sup_{||\xi||_{\ell_2}} ||x^*\xi||_{\ell_2} \right\} = \sup_{||\xi||_{\ell_2}} ||x\xi||_{\ell_2}$$

$$= \sup_{||\xi||_{\ell_2}} \left( \sum_{j=1}^{\infty} ||xe_j||_{\ell_2} j^{-m} \cdot ||ye_j||_{\ell_2} \right) = \sup_{||\xi||_{\ell_2}} \left( \sum_{j=1}^{\infty} ||xe_j||_{\ell_2} j^{-m} \right)^{1/2} \cdot \left( \sum_{j=1}^{\infty} ||ye_j||_{\ell_2} j^{-m} \right)^{1/2} = \frac{\pi}{\sqrt{6}} [x]_m.$$

Therefore,

$$[\cdot, \cdot]_m \geq \frac{\sqrt{6}}{\pi} r_{m+2},$$

and, by Lemma 3.8 $[\cdot, \cdot]_m$ is a dominating norm on $F$. Moreover, we have

$$[xy, z]_m = \sum_{j=1}^{\infty} (xye_j, ze_j) j^{-2m-2} = \sum_{j=1}^{\infty} (ye_j, x^*ze_j) j^{-2m-2} = [y, x^*z]_m,$$

which completes the proof.

**Definition 3.11.** A closed subspace $E$ of the space $s$ is called orthogonally complemented in $s$ if there is a continuous projection $\pi$ in $s$ onto $E$ admitting the extension to the orthogonal projection in $\ell_2$. Then we call $\pi$ an orthogonal projection in $s$ onto $E$.

**Lemma 3.12.** Let $E$ be a Fréchet space isomorphic to a nuclear power series space of infinite type and let $||\cdot||$ be a dominating Hilbert norm on $E$. Then there is an orthogonally complemented subspace $G$ of $s$ and an isomorphism $w : E \to G$ of Fréchet spaces such that $||w\xi||_{\ell_2} = ||\xi||$ for all $\xi \in E$. 


Proof. Since $E$ is isomorphic to a nuclear power series space of infinite type, by Lemma 29.2(3) & Lemma 29.11(3)], $E$ has the properties (DN) and (Ω). Hence, by Prop. 31.7], $E$ is isomorphic to a complemented subspace of $s$. This means that there is a complemented subspace $F$ of $s$ with a continuous projection $π: s → F$ and a Fréchet space isomorphism $ψ: E → F$. Hence, $||·||_ψ: F → [0, ∞]$ defined by $||ξ||_ψ := ||ψ^{-1}ξ||$ is a dominating Hilbert norm on $F$. Since, $||·||_L$ is also a dominating Hilbert norm on $F$, by Cor. 7.7], there is an automorphism $u$ of $F$ such that $||uξ||_L = ||ξ||_ψ$ for all $ξ ∈ F$. Moreover, by Th. 7.2], there is an automorphism $v$ of $s$ such that $ρ := vπv^{-1}$ is the orthogonal projection in $s$ onto $G := v(F)$ and a simple analysis of the proof of Th. 7.2] shows that $||vξ||_L = ||ξ||_L$ for all $ξ ∈ F$. Therefore, the operator $w := vw$ has the desired properties.

Lemma 3.13. Let $E$ be a Fréchet space isomorphic to a nuclear power space of infinite type and let $||·||$ be a dominating Hilbert norm on $E$. Let $H$ denote the completion of $E$ in the norm $||·||$. Then there is a map $ϕ ∈ L(H, ℓ_2)$ and an orthogonally complemented subspace $G$ of $s$ such that

(i) $ϕ(E) = G$;
(ii) $ϕ^*(s) = E$;
(iii) $||ϕξ||_ℓ_2 = ||ξ||$ for all $ξ ∈ E$;
(iv) $ϕϕ^*$ is the orthogonal projection in $ℓ_2$ with $ϕϕ^*(s) = G$.

Moreover, the map

$Φ: L^*(E, ||·||) → L^*(s), \ x ↦ ϕxϕ^*$,

is a continuous injective *-algebra homomorphism with $im Φ = L^*(G)$ and the map

$P: L^*(s) → L^*(G), \ x ↦ ϕϕ^*xϕϕ^*$,

is a continuous projection onto $L^*(G)$.

Proof. By Lemma 3.12] there is an orthogonally complemented subspace $G$ of $s$ and an isomorphism $w: E → G$ of Fréchet spaces such that $||wξ||_ℓ_2 = ||ξ||$ for all $ξ ∈ E$. Let $ρ: s → s$ be the orthogonal projection onto $G$. The operators $w$, $ρ$ and the identity map $ι: G ↪ s$ can be extended to the continuous linear operators between Hilbert spaces (for simplicity denoted by the same symbols): $w: H → G$, $ρ: ℓ_2 → ℓ_2$ and $ι: G ↪ ℓ_2$, where $G$ is the closure of $G$ in $ℓ_2$. Therefore, the Hermitian adjoints $w^*$ and $ι^*$ of the operators $w$ and $ι$ are well-defined. We have thus the following commutative diagram of continuous linear maps between Fréchet and Hilbert spaces

and the diagram with the corresponding adjoint operators

It follows easily that $ι^*: ℓ_2 → G$ is the orthogonal projection onto $G$, whence $ι^*(s) = ρ(s) = G$. Moreover, $w^*(G) = E$. Indeed, if $(·, ·)$ denotes the scalar product on $E$ corresponding to the Hilbert norm $||·||$, then

$(w^*wξ, η) = ⟨wξ, wη⟩ = ⟨ξ, η⟩$

for all $ξ, η ∈ E$. Hence, $E$ being dense in $H$, $w^*w = id_H$, and so $w^*(G) = E$. Consequently, we have the following commutative diagram

It is easy to check that $ϕ := ωw$ satisfies conditions (i)–(iii) and a simple computation shows that $ϕϕ^*$ is a self-adjoint projection (and thus orthogonal) on $ℓ_2$ with $ϕϕ^*(s) = G$. In consequence, $Φ: L^*(E, ||·||) →
\( L^*(s), x \mapsto \varphi x \varphi^*, \) is an injective \(*\)-homomorphism with \( \text{im} \Phi = L^*(G) \) and, moreover, \( P : L^*(s) \to L^*(G), x \mapsto \varphi x \varphi^* \), is a projection.

Now, we shall prove the continuity of \( \Phi \). Let \( B \) be a bounded subset of \( s \) and let \( n \in \mathbb{N}_0 \). By the closed graph theorem, \( \varphi : E \to s \) and \( \varphi^* : s \to E \) are continuous maps between Fréchet spaces. Hence, there is a constant \( C > 0 \) and a continuous norm \( \| \cdot \|_E \) on \( E \) such that \( |\varphi \xi|_n \leq C \| \xi \|_E \) for \( \xi \in E \). Note also that the set \( \varphi^*(B) \) is bounded in the Fréchet space \( E \). Therefore,

\[
\max_{\xi \in B} \{ |(\Phi \xi)|_n, |(\Phi x)^*|_n \} = \max_{\xi \in B} \{ |\varphi x \varphi^*|_n, |\varphi^* \varphi^*|_n \} \leq C \max_{\eta \in \varphi^*(B)} \{ |x\eta|_E, |x^\ast \eta|_E \},
\]

which, by Proposition 2.6 gives the continuity of \( \Phi \). The continuity of \( P \) can be proved in a similar way.

**Proof.** This follows directly from Lemma 3.13.

---

**Corollary 3.14.** For every Fréchet space \( E \) isomorphic to a nuclear power series space of infinite type and a dominating Hilbert norm \( \| \cdot \| \) on \( E \) there is an orthogonally complemented subspace \( G \) of \( s \) such that \( L^*(E, \| \cdot \|) \approx L^*(G) \) as topological \(*\)-algebras. Moreover, the algebra \( L^*(G) \) is a complemented \(*\)-subalgebra of \( L^*(s) \).

**Proof.** By [8] Prop. 3.8 & Cor. 4.2, \( L^*(s) \) is a nuclear, ultrabornological PLS-space. These properties are inherited by complemented subspaces (see [12] Prop. 28.6), [17] Ch. II, 8.2, Cor. 1], [11] Prop. 1.2]. Hence, the desired conclusion follows from Corollary 3.13.

---

**Lemma 3.15.** Let \( E \) be a Fréchet space isomorphic to a nuclear power series space of infinite type and let \( \| \cdot \| \) be a dominating Hilbert norm on \( E \). Then the space \( L^*(E, \| \cdot \|) \) is a nuclear, ultrabornological PLS-space.

**Proof.** By [8] Prop. 3.8 & Cor. 4.2, \( L^*(s) \) is a nuclear, ultrabornological PLS-space. These properties are inherited by complemented subspaces (see [12] Prop. 28.6), [17] Ch. II, 8.2, Cor. 1], [11] Prop. 1.2]. Hence, the desired conclusion follows from Corollary 3.13.

---

**Lemma 3.16.** Let \( E \) be a unital DN-algebra isomorphic as a Fréchet space to a nuclear power series space of infinite type and let \( \| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle} \) be the corresponding Hilbert norm. Let

\[
\mathcal{M}_E := \{ m_x : x \in E \},
\]

where \( m_x : E \to E, m_x y := xy, \) denotes the left multiplication map for the element \( x \). Then \( \mathcal{M}_E \) is a complemented \(*\)-subalgebra of \( L^*(E, \| \cdot \|) \) and \( E \) is isomorphic as a Fréchet \(*\)-algebra to \( \mathcal{M}_E \).

**Proof.** By assumption,

\[
(m_x y, z) = (xy, z) = (y, x^\ast z) = (y, m_{x^\ast} z)
\]

for all \( x, y, z \in E \), hence \( E \subset \mathcal{D}(m_x^*) \) and \( (m_x^*)_E = m_{x^\ast}^* \). Consequently, \( m_x \in L^*(E, \| \cdot \|) \) for all \( x \in E \).

Define \( Q : L^*(E, \| \cdot \|) \to L^*(E, \| \cdot \|) \) by \( Q \varphi := m_{\varphi^\ast}(1) \), where \( 1 \) is the unit in \( E \). Clearly, \( Q \) is a projection onto \( \mathcal{M}_E \); we will show that \( Q \) is continuous. By Corollary 3.15 and the closed graph theorem (see e.g. [13] Th. 24.31), every linear map on \( L^*(E, \| \cdot \|) \) with closed graph is continuous. Assume that a net \( (\varphi_{\lambda})_{\lambda} \subset L^*(E, \| \cdot \|) \) converges to \( 0 \), \( (Q \varphi_{\lambda})_{\lambda} \) converges to \( \psi \) and both limits are taken in \( L^*(E, \| \cdot \|) \). Let us fix \( x \in E \). By the continuity of the multiplication in \( E \), there is \( C > 0 \) and a continuous norm \( \| \cdot \|_1 \) on \( E \) with \( \| yx \| \leq C \| y \|_1 \) for all \( y \in E \). Hence, we have

\[
\| \psi x \| \leq \| (\psi - Q \varphi_{\lambda}) x \| + \| (Q \varphi_{\lambda} x) \| \leq \| (\psi - Q \varphi_{\lambda}) x \| + C \| \varphi_{\lambda}(1) \|_1
\]

By assumption, \( \| (\psi - Q \varphi_{\lambda}) x \| \to 0 \) and \( \| \varphi_{\lambda}(1) \|_1 \to 0 \), which yields \( \psi x = 0 \). Consequently, \( \psi = 0 \) and \( Q \) is continuous.

Finally, we should to show that \( E \) is isomorphic as a topological \(*\)-algebra to \( \mathcal{M}_E \) - a complemented \(*\)-subalgebra of \( L^*(E, \| \cdot \|) \). Let us consider the map \( \Phi : E \to \mathcal{M}_E, \Phi x := m_x \). By the above, it is clear that \( \Phi \) is a \(*\)-algebra isomorphism. Let \( B \) be a bounded subset of \( E \) and let \( \| \cdot \|_0 \) be a continuous norm on \( E \). Since the multiplication on \( E \) is jointly continuous, there is \( C_1 > 0 \) and a continuous norm \( \| \cdot \|_1 \) on \( E \) such that \( \| xy \|_0 \leq C_1 \| x \|_1 \| y \|_1 \) for \( x, y \in E \). Moreover, by the continuity of the involution, there
is a constant $C_2 \geq 1$ and a continuous norm $\| \cdot \|_2$ on $E$ such that $\| \cdot \|_2 \geq \| \cdot \|_1$ and $\| x^* \|_1 \leq C_2 \| x \|_2$ for $x \in E$. Hence,

$$\max \{ \sup_{y \in B} \| m_x y \|_0, \sup_{y \in B} \| (m_x)^* y \|_0 \} = \max \{ \sup_{y \in B} \| xy \|_0, \sup_{y \in B} \| x^* y \|_0 \}$$

$$\leq C_1 \sup_{y \in B} \| y \|_1 \max \{ \| x \|_1, \| x^* \|_1 \} \leq C_1 C_2 C_3 \| x \|_2,$$

where $C_3 := \sup_{y \in B} \| y \|_1 < \infty$. This shows that $\Phi$ is continuous. Since $E$ and $M_E$ (as a complemented subspace of $L^*(E, \| \cdot \|)$, see also Corollary 3.15) satisfy assumptions of the open mapping theorem [13] Th. 24.30], the map $\Phi$ is an isomorphism of Fréchet $^*$-algebras, which completes the proof. 

Proof of Theorem 3.5. The implication (i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(iii): Let $F$ be a closed $^*$-subalgebra of $L^*(s)$ such that $E \cong F$ as Fréchet $^*$-algebras and let $T : E \to F$ be the corresponding isomorphism. By Lemma 3.8, there is $m \in \mathbb{N}_0$ such that $F \subset L(s_m, \ell_2) \cap L(\ell_2, -s_m)$ and thus, by Lemma 3.10, $\{ \cdot \}_m = \sqrt{\{ \cdot \}_m}$ is a dominating Hilbert norm on $F$ such that $\| xy, z \|_m = \| y, x^* z \|_m$ for all $x, y, z \in F$. Let $(\cdot, \cdot) : E \times E \to C, (x, y) := [Tx, Ty]_m$. Then, clearly, $(x, y, z) = (y, x^* z)$ for all $x, y, z \in E$ and $\| \cdot, \cdot \| := \sqrt{\{ \cdot, \cdot \}}$ is a dominating Hilbert norm on $E$, hence $E$ is a DN-algebra.

(iii)$\Rightarrow$(i): By Lemma 3.16, $E$ is isomorphic to a complemented $^*$-subalgebra of $L^*(E, \| \cdot \|)$ and, by Corollary 3.15, $L^*(E, \| \cdot \|)$ is isomorphic to a complemented $^*$-subalgebra of $L^*(s)$, which proves the theorem.

Proof of Theorem 3.6. Clearly, (i) implies (ii).

(ii)$\Rightarrow$(iii): Let $\Phi : E \to F$ be the isomorphism of the Fréchet $^*$-algebras $E$ and $F$. Since $F \subset L(\ell_2)$, by Lemma 3.10, $\{ \cdot \}_0$ is a Hilbert dominating norm on $F$ satisfying condition ($\alpha$). Consequently, $\| \cdot \| := [\Phi(\cdot)]_0$ is a Hilbert dominating norm on $E$ and it satisfies condition ($\alpha$).

Next, for all $x \in E$ there is $C > 0$ such that

$$\| (\Phi(xy))e_j \|_{\ell_2} = \| (\Phi x)(\Phi y)e_j \|_{\ell_2} \leq C\| (\Phi y)e_j \|_{\ell_2}$$

for all $y \in E$ and $j \in \mathbb{N}$. Hence, for all $x \in E$ there is $C > 0$ such that

$$\| xy \| = [\Phi(xy)]_0 = \left( \sum_{j=1}^{\infty} \| (\Phi(xy))e_j \|_{\ell_2}^2 j^{-2} \right)^{1/2} \leq C \left( \sum_{j=1}^{\infty} \| (\Phi y)e_j \|_{\ell_2}^2 j^{-2} \right)^{1/2}$$

$$= C [\Phi y]_0 = C \| y \|$$

for all $y \in E$, which gives condition ($\beta$) in Definition 3.1. This shows that $E$ is a $\beta$DN-algebra.

(iii)$\Rightarrow$(i): Let $H$ be the completion of $E$ in the norm $\| \cdot \|$ and let $M_E := \{ m_x : x \in E \}$, where $m_x : E \to E, m_x(y) := xy$. By Lemma 3.10, $M_E$ is a complemented $^*$-subalgebra of $L^*(E, \| \cdot \|)$ isomorphic to $E$. Moreover, by Lemma 3.13, $L^*(E, \| \cdot \|)$ is isomorphic to a complemented $^*$-subalgebra of $L^*(s)$ via the map

$$\Phi : L^*(E, \| \cdot \|) \to L^*(s), \hspace{1cm} x \mapsto \varphi x \varphi^*,$$

where $\varphi \in L(H, \ell_2)$ and its adjoint $\varphi^* \in L(\ell_2, H)$ satisfy conditions (i)–(iv) in Lemma 3.13. Hence, the assignent $x \mapsto \varphi m_x \varphi^*$ defines an isomorphism (of Fréchet $^*$-algebras) between $E$ and a complemented $^*$-subalgebra of $L^*(s)$. Now, it is left to show that $\varphi m_x \varphi^* \in L(\ell_2)$ for each $x \in E$. By assumption, for each $x \in E$, the map $m_x : (E, \| \cdot \|) \to (E, \| \cdot \|)$ is continuous. Consequently, for each $x \in E$ there are $C_1, C_2 > 0$ such that

$$\| \varphi m_x \varphi^* \xi \|_{\ell_2} = \| m_x \varphi^* \xi \| \leq C_1 \| \varphi^* \xi \| \leq C_2 \| \xi \|_{\ell_2}$$

for all $\xi \in \ell_2$, and the proof is complete. 

Remark 3.17. The proofs of implication (iii)$\Rightarrow$(i) in Theorems 3.5 and 3.6 work also when $E$ is not commutative. The implication (ii)$\Rightarrow$(iii) in Theorems 3.5 and 3.6 holds for any commutative Fréchet $^*$-algebra.
4. Examples

In this section we present several examples of classes of commutative Fréchet $^*$-algebras which can be embedded into $\mathcal{L}^s(s)$ as complemented $^*$-subalgebras consisting of bounded operators on $\ell_2$. In the case of unital algebras, in view of Theorem 5.3, it is enough to show that a given Fréchet $^*$-algebra is isomorphic to a complemented subspace of $s$ with Schauder basis (i.e. to a nuclear power series space of infinite type) and admits a Hilbert dominating norm satisfying conditions $(\alpha)$ and $(\beta)$ in Definition 5.1. Nonunital algebras will be extended in a natural way to unitals ones. At the end of this section we also give one interesting counterexample.

By $\mathcal{E}(K)$ we denote the space of (complex-valued) Whitney jets on a compact set $K \subset \mathbb{R}^n$,

$$\mathcal{E}(K) := \{(\partial^n x)_K : F \in C^\infty(\mathbb{R}^n)\}.$$  

The space $\mathcal{E}(K)$ thus consists of some special sequences $f = (f^{(\alpha)})_{\alpha \in \mathbb{N}_0^n}$ of continous functions on the set $K$. The Fréchet space topology on $\mathcal{E}(K)$ is given by the system of seminorms $(|| \cdot ||_m)_{m \in \mathbb{N}}$ defined, for example, in [12, Section 2]; here let us only note that

$$\sup\{||f^{(\alpha)}(x)|| : x \in K, |\alpha| \leq m\} \leq ||f||_m$$

for all $m \in \mathbb{N}_0$ and $f \in \mathcal{E}(K)$. The space $\mathcal{E}(K)$ is a Fréchet $^*$-algebra where the product $fg$ of $f, g \in \mathcal{E}(K)$ is defined by the Leibniz rule, i.e.,

$$(fg)^{(\alpha)} := \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} f^{(\beta)}g^{(\alpha-\beta)}$$

for $\alpha \in \mathbb{N}_0^n$ (see also [12, p. 133]). As involution we clearly take the pointwise conjugation, $f := (f^{(\alpha)})_{\alpha \in \mathbb{N}_0^n}$. We say that a compact set $K \subset \mathbb{R}^n$ has the extension property if there exists a continuous linear operator $E : \mathcal{E}(K) \to C^\infty(\mathbb{R}^n)$ such that $\partial^n x(Ef)|_K = f^{(\alpha)}$ for every $\alpha \in \mathbb{N}_0^n$. M. Tidten showed in [20, Folgerung 2.4] that a compact set $K \subset \mathbb{R}^n$ has the extension property if and only if $\mathcal{E}(K)$ has the property (DN).

All $^*$-algebras of smooth functions considered below are endowed with pointwise multiplication and conjugation. The algebra $H(\mathbb{C})$ of entire functions is endowed with pointwise multiplication and the involution $f \mapsto f^*$ defined by $f^*(z) := \overline{f(z)}$.

Let

$$\mathcal{K}_\infty := \{(x_{ij})_{i,j \in \mathbb{N}} \in C^{(\infty)} : \sup_{i,j \in \mathbb{N}} |x_{ij}|(ij)^n < \infty \text{ for all } n \in \mathbb{N}_0\}$$

be the algebra of rapidly decreasing matrices endowed with matrix multiplication and conjugation of the transpose as involution. The algebra $\mathcal{K}_\infty$ is isomorphic as the Fréchet $^*$-algebra to the algebra $\mathcal{L}^s(s)$ of compact smooth operators. Moreover, it is isomorphic as a Fréchet space to the space $s$. For further information concerning the algebra $\mathcal{K}_\infty$ we refer the reader to [15, 16, 17].

**Theorem 4.1.** The following Fréchet $^*$-algebras are isomorphic to some complemented $^*$-subalgebra of $\mathcal{L}^s(s)$ consisting of bounded operators on $\ell_2$.

(i) The algebras $C^\infty(M)$ of smooth functions on compact second-countable smooth manifolds $M$ without boundary.

(ii) The algebra $\mathcal{D}(K)$ of smooth functions on $\mathbb{R}^n$ with support contained in compact sets $K \subset \mathbb{R}^n$ such that $\text{int } K \neq \emptyset$.

(iii) The algebra $\mathcal{D}(\mathbb{R}^n)$ of smooth rapidly decreasing functions on $\mathbb{R}^n$.

(iv) The algebra $\mathcal{E}(K)$ of Whitney jets on compact sets $K \subset \mathbb{R}^n$ with the extension property admitting a Schauder basis.

(v) The algebra $H(\mathbb{C})$ of entire functions.

(vi) Nuclear power series algebras $\Lambda_\infty(\alpha)$ of infinite type with pointwise multiplication and conjugation.

(vii) The algebra $\mathcal{K}_\infty$ of rapidly decreasing matrices.

**Proof.** (i) We follow the general pattern of the reasoning of P. Michor [15]. Let us choose a Riemannian metric $g$ on $M$ and let $dV$ be the volume element (density) on $(M, g)$. Let $(\cdot, \cdot)_{L_2(M)}$ be the scalar product
on $L_2(M)$ defined by

$$(f,g)_{L_2(M)} := \int_M f\overline{g}dV$$

and let $||\cdot||_{L_2(M)}$ denote the corresponding Hilbert norm on $L_2(M)$. By the Sturm-Liouville decomposition [4], pp. 139–140], eigenfunctions $(u_k)_{k \in \mathbb{N}}$ of the Laplacian $\Delta$ induced by $g$ form an orthonormal basis of $L_2(M)$ and the sequence $(\lambda_k)_{k \in \mathbb{N}}$ of eigenvalues satisfies

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \text{ and } \lambda_k \to \infty.$$ 

Moreover, by the Weyl asymptotic formula [3], Note III.15, p. 184], there is a constant $C > 0$ depending only on $n$ and the choice of a Riemannian metric such that

$$(6) \quad \lambda_k \sim Ck^{\frac{2}{n}},$$

as $k \to \infty$.

We claim that $(u_k)_{k \in \mathbb{N}}$ is a Schauder basis of $C^\infty(M)$ whose coefficient space is equal to the space $s$ of rapidly decreasing sequences. Indeed, for each $r \in \mathbb{N}$, the operator

$$(I + \Delta)^r : H^{2r}(M) \to L_2(M)$$

is an isomorphism between the Sobolev space $H^{2r}(M)$ and $L_2(M)$. Therefore, since $a \in \mathcal{C}$ is a unique sequence $(a_k)_{k \in \mathbb{N}}$ of scalars such that $f = \sum_{k=1}^{\infty} a_k u_k$ and the series converges in the norm $||\cdot||_{L_2(M)}$, for each $r \in \mathbb{N}$ there is a unique sequence $(a_{k,r})_{k \in \mathbb{N}} \subset \mathbb{C}^n$ such that

$$(I + \Delta)^r f = \sum_{k=1}^{\infty} a_{k,r} u_k.$$ 

Since $(I + \Delta)^r$ is a symmetric unbounded operator on $L_2(M)$, we have

$$a_{k,r} = \langle (I + \Delta)^r f, u_k \rangle_{L_2(M)} = \langle f, (I + \Delta)^r u_k \rangle_{L_2(M)} = \langle f, (1 + \lambda_k)^r u_k \rangle_{L_2(M)} = a_k(1 + \lambda_k)^r,$$

whence

$$(I + \Delta)^r f = \sum_{k=1}^{\infty} a_k(1 + \lambda_k)^r u_k$$

and the series converges in the norm $||\cdot||_{L_2(M)}$. Therefore, $(a_k(1 + \lambda_k)^r)_{k \in \mathbb{N}} \in \ell_2$ for all $r \in \mathbb{N}$ and, by the Weyl asymptotic formula [3], $(a_k(1 + Ck^{\frac{2}{n}})^r)_{k \in \mathbb{N}} \in \ell_2$ for all $r \in \mathbb{N}$, which yields $(a_k)_{k \in \mathbb{N}} \in s$. Finally, it is a simple matter to show that for each $(a_k)_{k \in \mathbb{N}} \in s$ the series $\sum_{k=1}^{\infty} a_k u_k$ converges in $C^\infty(M)$. Hence, $(u_k)_{k \in \mathbb{N}}$ is a Schauder basis of $C^\infty(M)$ with the coefficient space equal to $s$ as claimed.

Now, $T : C^\infty(M) \to s$ defined by $Tu_k = e_k$ ($e_k$ denotes the $k$-th unit vector in $C^0 \cap C^\infty(M)$) is an isomorphism of Fréchet spaces such that $||Tf||_{\ell_2} = ||f||_{L_2(M)}$ for $f \in C^\infty(M)$. Therefore, since $||\cdot||_{\ell_2}$ is a dominating norm on $s$, $||\cdot||_{L_2(M)}$ is a dominating norm on $C^\infty(M)$. Clearly,

$$\langle fg, h \rangle_{L_2(M)} = \langle g, T(h) \rangle_{L_2(M)}$$

and $||fg||_{L_2(M)} \leq \sup_{x \in M} |f(x)||g||_{L_2(M)}$

for all $f, g, h \in C^\infty(M)$. Hence, by Theorem 3.6, $C^\infty(M)$ is isomorphic to a complemented *-subalgebra of $\mathcal{L}(s)$ consisting of bounded operators on $\ell_2$.

(ii) Choose $a \in \mathbb{R}^n$ and $r > 0$ such that $K$ is contained in the open ball $B(a, r)$ of radius $r$ centered at $a$. Then, clearly, $\mathcal{D}(K)$ is isomorphic as a Fréchet *-algebra to the algebra

$$\mathcal{D}(K, B(a, r)) := \{f \in C^\infty(B(a, r)) : \text{ supp}(f) \subset K\}$$

which is a closed subspace of the Fréchet space $C^\infty(B(a, r))$ of smooth functions on $B(a, r)$ with uniformly continuous partial derivatives (see [2] Ex. 28.9(5))). Let $1$ be the constant function on $B(a, r)$, everywhere equals 1. Let $\mathcal{D}(K, B(a, r))$ denote the linear span of $\mathcal{D}(K, B(a, r))$ and $1$. Then, clearly, $1$ is the unit in the Fréchet *-algebra $\mathcal{D}(K, B(a, r))$. By [2] Prop. 31.12, $\mathcal{D}(K, B(a, r))$ is isomorphic as a Fréchet space to $s$, and so is $\mathcal{D}(1, K, B(a, r))$. 

Now, we follow the proof of [14] Lemma 31.10. By [14] Prop. 14.27, \( (\| \cdot \|_m)_{m \in \mathbb{N}} \),

\[
\| f \|_m^2 := \sum_{|\alpha| \leq m} \int_{B(a,r)} |f^{(\alpha)}(x)|^2 \, dx = \sum_{|\alpha| \leq m} \| f^{(\alpha)} \|_{L^2(B(a,r))}^2,
\]

is a fundamental sequence of norms on \( C^\infty(\overline{B(a,r)}) \), and thus on \( \mathcal{D}_1(K, B(a,r)) \). Since \( f + \lambda 1^{(\alpha)} = f^{(\alpha)} \) for \( |\alpha| > 0 \), we have, by integration by parts and by the Cauchy-Schwartz inequality,

\[
\| (f + \lambda 1^{(\alpha)}) \|_{L^2(B(a,r))}^2 = \int_{B(a,r)} |(f + \lambda 1^{(\alpha)})|^2 \, dx = \int_{B(a,r)} (f + \lambda 1^{(\alpha)} (f + \lambda 1^{(\alpha)}) \, dx
\]

\[
= (-1)^{|\alpha|} \int_{B(a,r)} (f + \lambda 1)(f + \lambda 1)^{(2\alpha)} \, dx
\]

\[
\leq \| f + \lambda 1 \|_{L^2(B(a,r))} \| (f + \lambda 1^{(\alpha)}) \|_{L^2(B(a,r))}
\]

for all \( f + \lambda 1 \in \mathcal{D}_1(K, B(a,r)) \) and \( |\alpha| \leq m \). Hence, for all \( m \in \mathbb{N}_0 \) there is a constant \( C_m > 0 \) such that

\[
\| f + \lambda 1 \|_{m}^2 \leq C_m \| f + \lambda 1 \|_{L^2(B(a,r))} \| f + \lambda 1 \|_{2m}
\]

for all \( f + \lambda 1 \in \mathcal{D}_1(K, B(a,r)) \). Hence, \( (\| \cdot \|_{L^2(B(a,r))} \) is a dominating norm on \( \mathcal{D}_1(K, B(a,r)) \), and the corresponding scalar product satisfies condition \( (\alpha) \) in Definition 3.1. Futhermore,

\[
\| (f + \lambda 1)(g + \mu 1) \|_{L^2(B(a,r))} = \left( \int_{B(a,r)} |f(x) + \lambda |g(x) + \mu|^2 \, dx \right)^{1/2}
\]

\[
\leq \sup_{x \in B(a,r)} |f(x) + \lambda| \cdot \| g + \mu 1 \|_{L^2(B(a,r))}
\]

Hence, \( \mathcal{D}_1(K, B(a,r)) \) is a unital \( \beta \)D-algebra. By Theorem 3.6 \( \mathcal{D}_1(K, B(a,r)) \) is isomorphic to a complemented \( \ast \)-subalgebra of \( \mathcal{K}^\ast(s) \) consisting of bounded operators on \( \ell^2 \), and so is \( \mathcal{D}(K) \).

(iii) It is well-known that the map

\[
\Phi: \mathcal{S}(\mathbb{R}^n) \to \mathcal{D}\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^{n}\right), \quad (\Phi f)(x_1, \ldots, x_n) := (\tan x_1, \ldots, \tan x_2),
\]

is an isomorphism of Fréchet \( \ast \)-algebras (see [14] Ex. 29.5(3) and p. 402)), hence the conclusion follows from the previous example. Moreover, \( \| \cdot \|: \mathcal{S}(\mathbb{R}^n) \oplus \mathbb{C}1 \to [0, \infty) \),

\[
\| f + \lambda 1 \|^2 := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(\tan x_1, \ldots, \tan x_2) + \lambda|^2 \, dx_1 \cdots dx_n,
\]

is a Hilbert dominating norm on \( \mathcal{S}(\mathbb{R}^n) \oplus \mathbb{C}1 \) satisfying conditions \( (\alpha) \) and \( (\beta) \).

(iv) We shall show that there is a finite positive Borel measure \( \mu \) on \( K \) such that \( \| \cdot \|_{L^2(\mu)} \), \( \| f \|_{L^2(\mu)}^2 := \int_{K} |f|^2 d\mu \), is a dominating norm on \( \mathcal{D}(K) \). Then, since \( \| \cdot \|_{L^2(\mu)} \) is a Hilbert algebra norm, we would get our conclusion.

By [12] Th. 3.10, the norm \( \| \cdot \|_0 \), \( \| f \|_0 := \sup_{x \in K} |f(x)| \), is a dominating norm on \( \mathcal{D}(K) \). This means that for all \( k \) there is \( l \) and \( C > 0 \) such that

\[
\| f \|_k \leq C \| f \|_{l}^{1/2} f_{0}^{1/2}
\]

for all \( f \in \mathcal{D}(K) \). If there were a finite positive Borel measure \( \mu \) on \( K \), \( q \in \mathbb{N} \) and \( C > 0 \) such that

\[
\| f \|_0 \leq C \| f \|_{q}^{1/2} \| f \|_{L^2(\mu)}^{1/2}
\]

for all \( f \in \mathcal{D}(K) \) then, by (7), we would get

\[
\| f \|_k \leq C \| f \|_{l}^{1/2} \| f \|_{q}^{1/2} \| f \|_{L^2(\mu)}^{1/2} \leq C \| f \|_{\max\{l,q\}}^{1/2} \| f \|_{L^2(\mu)}^{1/2},
\]

and hence \( \| \cdot \|_{L^2(\mu)} \) would be a dominating norm on \( \mathcal{D}(K) \) (see [12] Remark 3.2(ii)). Our goal is thus to prove condition \( (\mathcal{R}) \).
In what follows, \( C \geq 1 \) denotes a constant which can vary from line to line but depends only on \( n \) and the set \( K \). By the last line of the proof of [1] Prop. 3.4], there is a positive Borel measure \( \mu \) on \( K \) – the so called Bernstein-Markov measure – such that
\[
|p|_0 \leq C\left(\begin{array}{c} n+j \\ j \end{array}\right)j^2||p||_{L_2(\mu)} \leq Cj^m||p||_{L_2(\mu)}.
\]
for all polynomials \( p \in \mathbb{C}[x_1, \ldots, x_n] \) with \( \text{deg}(p) \leq j \) and where \( m := n+2 \).

Let us fix \( f \in \mathcal{D}(K) \). By [12] Cor. 4.4(i), for all \( j \in \mathbb{N} \) there is a polynomial \( p_j \) with \( \text{deg}(p_j) \leq j \) such that
\[
|f - p_j|_0 \leq Cj^{-2m}||f||_{2m}.
\]
Applying this inequality twice and also inequality (9), we obtain
\[
|f - p_j|_0 \leq C(j^{-2m}||f||_{2m} + |p_j|_0) \leq C(j^{-2m}||f||_{2m} + j^m||p_j||_{L_2(\mu)})
\]

\[
\leq C(j^{-2m}||f||_{2m} + |f - p_j|_0 + ||f||_{L_2(\mu)}) \leq C(j^{-2m}||f||_{2m} + j^m||f||_{2m} + j^m||f||_{L_2(\mu)})
\]

for all \( j \in \mathbb{N} \). Therefore, taking the infimum over \( j \in \mathbb{N} \) in both sides of the above chain of inequalities and applying [12] Lemma 4.5, we obtain
\[
|f|_0 \leq C||f||_{2m}||f||_{L_2(\mu)}
\]
for all \( f \in \mathcal{D}(K) \), and (5) holds, as desired.

(v) We will show that the norm \( || \cdot ||_{L_2[-1,1]} \), which obviously satisfies conditions (\( \alpha \)) and (\( \beta \)) in Definition [5], is at the same time a dominating norm. Let us recall that the Fréchet space topology on \( H(\mathbb{C}) \) is given by the sequence of norms \( \langle || \cdot ||_r \rangle_{r \in \mathbb{N}} \), \( ||f||_r := \sup_{|z|=r} |f(z)| \). By Hadamard’s three circle theorem,
\[
||f||_r^2 \leq ||f||_1 ||f||_2
\]
for all \( r > 1 \) and \( f \in H(\mathbb{C} \setminus \{0\}) \), where \( H(\mathbb{C} \setminus \{0\}) \) is the space of holomorphic functions on \( \mathbb{C} \setminus \{0\} \). Let us consider the map \( \Psi: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \), \( \Psi(z) := \frac{1}{2}(z + \frac{1}{z}) \) and let \( E_r := \Psi(T_r) \), where \( T_r := \{ z \in \mathbb{C} : |z| = r \} \) for \( r > 0 \). For \( r > 1 \), one can show that \( E_r \) is the ellipse with the semi-axes \( a = \frac{1}{r}(r + r^{-1}) \) and \( b = \frac{1}{r}(r - r^{-1}) \). Moreover, \( E_1 = [-1,1] \) and \( E_{r-1} = E_r \) for \( r > 0 \). Since \( f \circ \Psi \in H(\mathbb{C} \setminus \{0\}) \) for all \( f \in H(\mathbb{C}) \), we have, by (10),
\[
||f \circ \Psi||_r^2 \leq ||f \circ \Psi||_1 ||f \circ \Psi||_{E_r}
\]
Hence,
\[
||f||_{E_r}^2 \leq ||f||_{[-1,1]} \||f||_{E_r}
\]
for all \( r > 1 \) and all \( f \in H(\mathbb{C}) \), where \( ||f||_E \) is the supremum norm on a subset \( E \) of \( \mathbb{C} \). Since the ellipses \( E_r \) contain arbitrary big circles, \( || \cdot ||_{[-1,1]} \) is a dominating norm on \( H(\mathbb{C}) \).

Let \( (Q_k)_{k \in \mathbb{N}} \) be the sequence of Legendre polynomials,
\[
Q_k(x) := \frac{(2k+1)}{2} \frac{1}{2k!} \frac{d^k}{dx^k}(x^2 - 1)^k
\]
for \( x \in [-1,1] \). Then, for each \( n \in \mathbb{N} \), \( (Q_k)_{0 \leq k \leq n} \) is an orthonormal basis of the space \( \mathcal{P}_n \) of complex-valued polynomials of degree at most \( n \) in one real variable endowed with the scalar product \( \langle p, q \rangle := \int_{-1}^{1} p(x)q(x)dx \). It is well-known that
\[
|Q_k(x)| \leq \left(\frac{2k+1}{2}\right)^{1/2}
\]
for all \( x \in [-1,1] \). Therefore, if \( p \in \mathcal{P}_n \), \( p = \sum_{k=0}^{n} c_k Q_k \) for some scalars \( c_k \)'s, then
\[
|p(x)| \leq \left(\sum_{k=0}^{n} |Q_k(x)|^2\right)^{1/2} \left(\sum_{k=0}^{n} |c_k|^2\right)^{1/2} \leq (n+1)||p||_{L_2[-1,1]}
\]
for every \( x \in [-1,1] \). Consequently,
\[
||p||_{[-1,1]} \leq (n+1)||p||_{L_2[-1,1]}
\]
for every $p \in \mathcal{P}_n$. This shows that the Lebesgue measure is a Bernstein-Markov measure on the interval $[-1, 1]$.

Now, let us take $f \in H(\mathbb{C})$, $f = \sum_{j=0}^{\infty} a_j z^j$. Let $p_n := \sum_{j=0}^{n} a_j z^j$ for $n \in \mathbb{N}_0$. Then, for every $n \in \mathbb{N}_0$ and $x \in [-1, 1]$, we have

$$|f(x) - p_n(x)| = \left| \sum_{j=n+1}^{\infty} a_j x^j \right| = \left| \sum_{j=n+1}^{\infty} a_j x^{j-n-1} \right| \cdot |x|^{n+1} \leq \frac{1}{(n+1)!} \sum_{j=0}^{\infty} (j + n + 1) \ldots (j + 1) a_{j+n+1} x^j = \frac{1}{(n+1)!} |f^{(n+1)}(x)| = \frac{1}{2^{n+1}} \int_{|z-x|=2} |f(z)| \leq \frac{1}{2^{n+1}} \sup_{|z| \leq 3} |f(z)|.$$

Hence,

$$|f - p_n|_{[-1, 1]} \leq \frac{1}{2^{n+1}}|f|_3.$$

Now, by applying computations from item (iv), we may derive from inequalities (11) and (12) that $\| \cdot \|_{L^2_{[-1, 1]}}$ is a dominating norm on $H(\mathbb{C})$ as claimed.

(vi) Let us define

$$|| \cdot || : \Lambda_\infty(\alpha) \oplus \mathbb{C}1 \rightarrow [0, \infty), \quad ||x + \lambda||^2 := \sum_{j=1}^{\infty} |x_j + \lambda|^2 j^{-2}$$

and

$$|| \cdot ||_t : \Lambda_\infty(\alpha) \oplus \mathbb{C}1 \rightarrow [0, \infty), \quad ||x + \lambda||_t := \max_{j \in \mathbb{N}} \left\{ \sup_{z \in \mathbb{C}} |x_j e^{i\alpha_j} + \lambda| \right\}$$

for $t \in \mathbb{N}$, where $1 := (1, 1, \ldots)$. By nuclearity, $(|| \cdot ||_t)_{t \in \mathbb{N}}$ is a fundamental sequence of norms on $\Lambda_\infty(\alpha) \oplus \mathbb{C}1$ and $|| \cdot ||$ is a well-defined continuous Hilbert norm on $\Lambda_\infty(\alpha) \oplus \mathbb{C}1$. Let $(\cdot, \cdot)$ denote the scalar product corresponding to the norm $|| \cdot ||$. An elementary computation shows that

$$(x + \lambda)(y + \mu, z + \nu) = (y + \mu), (x + \lambda)(z + \nu)$$

for all $x + \lambda, y + \mu, z + \nu \in \Lambda_\infty(\alpha) \oplus \mathbb{C}1$. Moreover, we have

$$||(x + \lambda)(y + \mu)|| = \left( \sum_{j=1}^{\infty} |(x_j + \lambda)(y_j + \mu)|^2 j^{-2} \right)^{1/2} \leq \sup_{j \in \mathbb{N}} |x_j + \lambda| ||y + \mu||$$

for all $x + \lambda, y + \mu \in \Lambda_\infty(\alpha) \oplus \mathbb{C}1$. Hence, the norm $|| \cdot ||$ satisfies conditions (a) and (b) in Definition 3.1.

We will show that $|| \cdot ||$ is a dominating norm on $\Lambda_\infty(\alpha) \oplus \mathbb{C}1$, i.e.

$$(13) \quad \forall s \in \mathbb{N} \exists \in \mathbb{N} \exists C > 0 \forall x + \lambda \in \Lambda_\infty(\alpha) \oplus \mathbb{C}1 \quad ||x + \lambda||^2 \leq C ||x + \lambda|| \cdot ||x + \lambda||_t.$$

Fix $s \in \mathbb{N}$, $x + \lambda \in \Lambda_\infty(\alpha) \oplus \mathbb{C}1$ and denote

$$R(t) := ||x + \lambda|| \cdot ||x + \lambda||_t$$

for $t \in \mathbb{N}$. Next, note that

$$\frac{\log(j+1)}{\alpha_j} \leq \frac{\log(\alpha_j^2)}{\alpha_j} = \frac{1}{\alpha_j} + 2 \frac{\log j}{\alpha_j}$$

for every $j \in \mathbb{N}$ and, by the nuclearity of $\Lambda_\infty(\alpha)$, there is a constant $\gamma \in \mathbb{N}$ such that $\frac{1}{\alpha_j} + 2 \frac{\log j}{\alpha_j} \leq \gamma$ for all $j \in \mathbb{N}$. Hence, $\frac{\log(j+1)}{\alpha_j} \leq \gamma$, and thus

$$e^{\gamma \alpha_j} \geq 1$$

for all $j \in \mathbb{N}$. We first claim that

$$(14) \quad ||\lambda||^2 \leq 4 R(\gamma).$$
Indeed, since \(|x_j + \lambda| \to |\lambda|\) as \(j \to \infty\), the set \(A := \{ j \in \mathbb{N} : |x_j + \lambda| < \frac{|\lambda|}{2}\}\) is finite or even empty. If \(A = \emptyset\), i.e. \(|x_j + \lambda| \geq \frac{|\lambda|}{2}\) for all \(j \in \mathbb{N}\), then

\[R(\gamma) \geq |x_1 + \lambda| \frac{|\lambda|}{2} \geq \frac{|\lambda|^2}{2}.
\]

Now, assume that \(A \neq \emptyset\) and let \(j_0 := \max\{ j \in \mathbb{N} : |x_j + \lambda| < \frac{|\lambda|}{2}\}\). Then, clearly, \(|x_{j_0 + 1} + \lambda| \geq \frac{|\lambda|}{2}\), which gives

\[||x + \lambda|| \geq |x_{j_0 + 1} + \lambda|(j_0 + 1)^{-1} \geq \frac{|\lambda|}{2}(j_0 + 1)^{-1}.
\]

Next, \(|x_{j_0} + \lambda| < \frac{|\lambda|}{2}\) implies that \(|x_{j_0}| > \frac{|\lambda|}{2}\), and thus

\[||x + \lambda||_t \geq |x_{j_0}| e^{t \gamma_{j_0}} > \frac{|\lambda|}{2} e^{t \gamma_{j_0}}
\]

for all \(t \in \mathbb{N}\). Consequently,

\[R(\gamma) \geq \frac{|\lambda|}{2}(j_0 + 1)^{-1} \frac{|\lambda|}{2} e^{t \gamma_{j_0}} = \frac{|\lambda|^2}{4} \frac{e^{\gamma_{j_0}}}{j_0 + 1} \geq \frac{|\lambda|^2}{4}
\]

as claimed.

We next claim that \(\sup_{j \in \mathbb{N}} |x_j|^2 e^{2s_{\alpha_j}} \leq 16R(2s + \gamma)\). Indeed, let \(t := 2s + \gamma\) and let us divide the set of natural numbers into 3 pieces:

\[A_1 := \left\{ j \in \mathbb{N} : |x_j| \leq \frac{|\lambda|}{2}\right\},
\]

\[A_2 := \left\{ j \in \mathbb{N} : \frac{|\lambda|}{2} < |x_j| \leq 2|\lambda|\right\},
\]

\[A_3 := \left\{ j \in \mathbb{N} : |x_j| > 2|\lambda|\right\}.
\]

First take \(k \in A_1\). Then \(|x_k + \lambda| \geq \frac{|\lambda|}{2}\), and so

\[R(t) \geq |x_k + \lambda| k^{-1} |x_k| e^{t \alpha_k} \geq \frac{|\lambda|}{2} |x_k| e^{t \alpha_k} \geq |x_k|^2 \frac{e^{t \alpha_k}}{k} \geq |x_k|^2 e^{(t-\gamma)\alpha_k} e^{\gamma_{\alpha_k}}
\]

\[\geq |x_k|^2 e^{(t-\gamma)\alpha_k} = |x_k|^2 e^{2s_{\alpha_k}}.
\]

Now, take \(k \in A_2\). Clearly the set \(A_2\) is finite. Let \(k_0 := \max\{ j \in \mathbb{N} : |x_j| > \frac{|\lambda|}{2}\}\). Then \(|x_{k_0 + 1}| \leq \frac{|\lambda|}{2}\), and thus \(|x_{k_0 + 1} + \lambda| \geq \frac{|\lambda|}{2}\). Consequently, we get

\[R(t) \geq |x_{k_0 + 1} + \lambda|(k_0 + 1)^{-1} |x_{k_0}| e^{t \gamma_{\alpha_k}} \geq \frac{|\lambda|}{2} (k_0 + 1)^{-1} \frac{|\lambda|}{2} e^{t \gamma_{\alpha_k}} = \frac{|\lambda|^2}{4} \frac{e^{t \gamma_{\alpha_k}}}{k_0 + 1}
\]

\[\frac{|x_{k_0}|^2}{16} e^{(t-\gamma)\alpha_k} e^{\gamma_{\alpha_k}} = \frac{|x_{k_0}|^2}{16} e^{2s_{\alpha_k}}.
\]

Finally, fix \(k \in A_3\). Then \(|x_k + \lambda| > \frac{|\lambda|}{2}\), and thus

\[R(t) \geq |x_k + \lambda| k^{-1} |x_k| e^{t \alpha_k} \geq \frac{|x_k|^2}{2} e^{t \alpha_k} \geq \frac{|x_k|^2}{2} e^{(t-\gamma)\alpha_k} e^{\gamma_{\alpha_k}}
\]

\[\geq \frac{|x_k|^2}{2} e^{2s_{\alpha_k}}.
\]

Combining (15) and (17), we get

\[\sup_{j \in \mathbb{N}} |x_j|^2 e^{2s_{\alpha_j}} \leq 16R(2s + \gamma)
\]

as claimed. Therefore, by (14),

\[||x + \lambda||^2 \leq 16||x + \lambda|| : ||x + \lambda||_{2s+\gamma},
\]

for all \(x + \lambda \in \Lambda_\infty(\alpha) \oplus \mathbb{C}1\), and \(||\cdot||\) is a dominating norm on \(\Lambda_\infty(\alpha) \oplus \mathbb{C}1\). Consequently, \((\Lambda_\infty(\alpha) \oplus \mathbb{C}1, (\cdot, \cdot))\) is a \(\beta\)DN-algebra.

Finally, by the above, \(\Lambda_\infty(\alpha) \oplus \mathbb{C}1\) has the property (DN) and, by [13] Ex. 1, Ch. 29, it has the property (Q). Moreover, by [13] Prop. 28.7, the space \(\Lambda_\infty(\alpha) \oplus \mathbb{C}1\) is nuclear. Consequently, by [13] Prop. 31.7, \(\Lambda_\infty(\alpha) \oplus \mathbb{C}1\) is isomorphic to a complemented subspace of \(s\) with Schauder basis, i.e. to a nuclear power series space of infinite type. Now, the thesis follows from Theorem 3.5 and from a simple observation that \(\Lambda_\infty(\alpha)\) is a complemented \((-1)\)-subalgebra of \(\Lambda_\infty(\alpha) \oplus \mathbb{C}1\).
(vii) Let $I$ be the identity matrix in $\mathbb{C}^{N^2}$ and let $e_j$ be the $j$-th unit vector in $\mathbb{C}^N$. We define on $\mathcal{X}_\infty \oplus \mathbb{C}I$ the Hilbert norm $\| \cdot \|$ by

$$\|x + \lambda\|^2 := \left( \sum_{j \in \mathbb{N}} \| (x + \lambda)e_j \|^2 \right)^{1/2}. $$

Clearly, $\| \cdot \|$ satisfies conditions (a) and (b) in Definition 5.1. Hence, by Theorem 3.4 and Remark 3.4, it is enough to show that $\| \cdot \|$ is a dominating norm, i.e.

$$\forall k \in \mathbb{N} \exists n \in \mathbb{N} \exists C > 0 \forall x + \lambda \in \mathcal{X}_\infty \oplus \mathbb{C}I \quad \|x + \lambda\|^2 \leq C\|x + \lambda\| \cdot \|x + \lambda\|_n,$$

where

$$\|x + \lambda\|_k := \max \{ \sup_{i,j \in \mathbb{N}} |x_{ij}|(ii)^k, |\lambda| \}.$$

Let us fix $x + \lambda \in \mathcal{X}_\infty \oplus \mathbb{C}I$. Note that

$$\|x + \lambda\|^2 = \|x + \lambda\|^2 = \left( \sum_{i \in \mathbb{N}, j \neq i} |x_{ij}|^2 + |x_{jj} + \lambda|^2 \right)j^{-2}. $$

for every $j \in \mathbb{N}$, and thus

$$\|x + \lambda\|^2 = \sum_{j \in \mathbb{N}} \left( \sum_{i \in \mathbb{N}, j \neq i} |x_{ij}|^2 + |x_{jj} + \lambda|^2 \right)j^{-2}.$$

We have

$$\sup_{i,j \in \mathbb{N}, i \neq j} |x_{ij}|^2(ij)^{2k} \leq \sup_{i,j \in \mathbb{N}, i \neq j} |x_{ij}| |(ij)^{2k+1} \cdot \sup_{i,j \in \mathbb{N}} |x_{ij}|(ij)^{2k+1} \leq \left( \sum_{i,j \in \mathbb{N}, i \neq j} |x_{ij}|^2 j^{-2} \right)^{1/2} \cdot \sup_{i,j \in \mathbb{N}} |x_{ij}|(ij)^{2k+1} \leq \left( \sum_{j \in \mathbb{N}} \left( \sum_{i \in \mathbb{N}, j \neq i} |x_{ij}|^2 + |x_{jj} + \lambda|^2 \right)j^{-2} \right)^{1/2} \cdot \sup_{i,j \in \mathbb{N}} |x_{ij}|(ij)^{2k+1}, |\lambda| \leq \|x + \lambda\| \cdot \|x + \lambda\|_m,$$

for all $k \in \mathbb{N}$. Moreover, from (3.3) applied to the algebra $s \oplus \mathbb{C}I$, it follows that for every $k \in \mathbb{N}$ there is $m \in \mathbb{N}$ and $C > 0$ such that

$$\max_{j \in \mathbb{N}} \{ |x_{jj}|^2(ij)^{4k}, |\lambda|^2 \} \leq C \left( \sum_{j \in \mathbb{N}} |x_{jj} + \lambda|^2 j^{-2} \right)^{1/2} \max_{j \in \mathbb{N}} |x_{jj}|(ij)^m, |\lambda| \leq C \left( \sum_{j \in \mathbb{N}} \left( \sum_{i \in \mathbb{N}, j \neq i} |x_{ij}|^2 + |x_{jj} + \lambda|^2 \right)j^{-2} \right)^{1/2} \cdot \sup_{i,j \in \mathbb{N}} |x_{ij}|(ij)^m, |\lambda| \leq C \|x + \lambda\| \cdot \|x + \lambda\|_m. $$

Therefore, for all $k \in \mathbb{N}$ there is $n \in \mathbb{N}$ and $C > 0$ such that

$$\|x + \lambda\|^2 \leq C \sum_{j \in \mathbb{N}} \{ |x_{jj}|^2(ij)^{4k}, |\lambda|^2 \} \leq C \|x + \lambda\| \cdot \|x + \lambda\|_n,$$

and thus $\| \cdot \|$ is a dominating norm on $\mathcal{X}_\infty \oplus \mathbb{C}I$, which completes the proof.

$\square$

Remark 4.2. (i) Every algebra $\mathcal{D}(K)$ is a closed *-subalgebra of $\mathcal{D}(L)$ for any closed ball $L$ containing $K$ and thus, by Theorem 4.1(ii), $\mathcal{D}(K)$ is automatically isomorphic to a closed *-subalgebra of $\mathcal{L}^*(s)$ consisting of bounded operators on $\ell_2$. In Theorem 4.1(iv), we prove that such a *-subalgebra can be choosen to be complemented.

(ii) By [8] Proposition 4.3, $\mathcal{L}^*(s)$ is isomorphic as a topological *-algebra to the matrix algebra

$$\Lambda(A) := \left\{ x = (x_{ij}) \in \mathbb{C}^{N^2} : \forall N \in \mathbb{N} \exists n \in \mathbb{N} \sum_{i,j \in \mathbb{N}} |x_{ij}| \max \left\{ \frac{1}{j^n}, \frac{1}{i^n} \right\} < \infty \right\}$$

for $\mathcal{X}_\infty \oplus \mathbb{C}I$. In Theorem 4.1(iv), we prove that such a *-subalgebra can be choosen to be complemented.
endowed with the so-called Köthe-type PLB-space topology. Clearly, $\mathcal{K}_\infty$ is a $^*$-subalgebra of $\Lambda(A)$. However, since $\mathcal{K}_\infty$ is dense in $\Lambda(A)$, this representation is not interesting for us.

We have also a similar phenomenon in the case of the algebra $s$. The diagonal matrices from $\Lambda(A)$ give – also in the topological sense – exactly the space $s'$. In particular, the $^*$-algebra $s$ has a simple representation in $\Lambda(A)$. But $s$ is dense in $s'$, so this representation does not work for us.

In the proof of Theorem 4.1, Examples (vi) and (vii), we see that the representations of $s$ and $\mathcal{K}_\infty$ in $\mathcal{L}^*(s)$ are much more sophisticated.

Finally, we shall give an example of a unital commutative Fréchet $^*$-algebra isomorphic as a Fréchet space to $s$ which is not a DN-algebra. Let $A^\infty(\mathbb{D})$ be the space of holomorphic functions on the open unit disc $\mathbb{D}$ which are smooth up to boundary. In other words,

$$A^\infty(\mathbb{D}) := \{ f \in A(\mathbb{D}) : f(k) \in A(\mathbb{D}) \text{ for all } k \in \mathbb{N} \},$$

where $A(\mathbb{D})$ is the disc algebra. The space $A^\infty(\mathbb{D})$ admits a natural Fréchet space topology given by the norms

$$||f||_p := \sup\{|f(j)(z)| : z \in \mathbb{D}, 0 \leq j \leq p\}, \quad p \in \mathbb{N}_0$$

and it is isomorphic to the space $s$ (cf. [19] Section 2). Moreover, $A^\infty(\mathbb{D})$ becomes a unital commutative Fréchet $^*$-algebra when endowed with the usual multiplication of functions and involution $f^*(z) := \overline{f(z)}$.

The algebra $A^\infty(\mathbb{D})$ is thus a (dense) $^*$-subalgebra of the disc algebra.

**Proposition 4.3.** The algebra $A^\infty(\mathbb{D})$ is not isomorphic to any closed $^*$-subalgebra of $\mathcal{L}^*(s)$.

**Proof.** By Theorem 3.15 it is enough to show that there is no DN-norm on $A^\infty(\mathbb{D})$.

Let $\{||\cdot||_p\}_{p \in \mathbb{N}}$ be the fundamental sequence of norms on the space $A^\infty(\mathbb{D})$ defined by (18). First, we will show that the norm $||\cdot||_{[-1,1]}$, $||f||_{[-1,1]} := \sup_{x \in [-1,1]} |f(x)|$, is not a dominating norm on $A^\infty(\mathbb{D})$.

Let $f_n(z) := (z^2 - 1)^n$ for $n \in \mathbb{N}$ and $z \in \mathbb{D}$. Let us fix $n \in \mathbb{N}$ and $0 \leq k \leq n$. Then, clearly, $||f_n||_{[-1,1]} = 1$ and $||f_n||_0 = 2^n$. Moreover, by the Leibniz rule, we obtain

$$|f_n^{(k)}(z)| = \left| \sum_{j=0}^{k} \binom{k}{j} \left( \frac{d}{dz} \right)^j (z-1)^n \left( \frac{d}{dz} \right)^{k-j} (z+1)^n \right|$$

$$= \left| \sum_{j=0}^{k} \binom{k}{j} \cdot n(n-1) \ldots (n-j+1)(z-1)^{n-j} \cdot n(n-1) \ldots (n-k+j+1)(z+1)^{n-k+j} \right|$$

$$\leq n^k 2^k \sup_{z \in \mathbb{D}, 0 \leq j \leq k} |z-1|^{n-j} |z+1|^{n-k+j}$$

$$= n^k 2^k 2^n \sup_{0 \leq \theta \leq \pi} (1 - \cos \theta)^{n-j} (1 + \cos \theta)^{n-k+j} \leq n^k 2^k 2^n$$

for all $z \in \mathbb{D}$, and thus

$$||f_n||_p \leq n^p 2^p 2^n$$

for every $p \in \mathbb{N}_0$. Consequently, for every $p \in \mathbb{N}$ and every $C > 0$ there is $n \in \mathbb{N}$ such that

$$||f_n||_p^2 = 4^n > C n^p 2^p 2^n$$

and therefore $||\cdot||_{[-1,1]}$ is not a dominating norm on $A^\infty(\mathbb{D})$.

Now, let $||\cdot|| = \sqrt{\langle \cdot, \cdot \rangle}$ be an arbitrary continuous norm on $A^\infty(\mathbb{D})$ satisfying condition (a) in Definition 3.1. We define a continuous linear functional $\Phi$ on $A^\infty(\mathbb{D})$ by $\Phi(f) := (f, 1)$, where $1$ is the identically one function. Then, by condition (a),

$$\Phi(f^* f) = \langle f^* f, 1 \rangle = ||f||^2 \geq 0$$

for every $f \in A^\infty(\mathbb{D})$, so $\Phi$ is a positive functional. In [16], there is an elementary proof of the fact that each positive linear functional on the disc algebra $A(\mathbb{D})$ has some simple integral representation. It appears that the same proof works in the case of the algebra $A^\infty(\mathbb{D})$, and thus there is a positive Borel measure $\mu$ on $[-1, 1]$ such that

$$\Phi(f) = \int_{-1}^{1} f(x) d\mu(x)$$
for every $f \in A^\infty(\mathbb{D})$. Hence,

$$\|f\|^2 = \int_{-1}^{1} |f(x)|^2 d\mu(x).$$

If $\|\cdot\|$ was a dominating norm on $A^\infty(\mathbb{D})$ then, since $\|\cdot\| \leq \mu([[-1,1]])\|\cdot\|_{[-1,1]}$, the norm $\|f\|_{[-1,1]}$ would be a dominating norm as well, contrary to our first claim. Hence, $A^\infty(\mathbb{D})$ is not a DN-algebra, which completes the proof. □

Acknowledgements. The author wishes to express his thanks to Leonhard Frerick for many stimulating conversations, especially during the stay at Trier University in June 2017. The author is also very endebted to his colleague Krzysztof Piszczek for valuables comments during the manuscript preparation.

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