Nonclassicality generated by photon
annihilation-then-creation and
creation-then-annihilation operations

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We examine nonclassical properties of the field states generated by applying the photon annihilation-then-creation operation (AC) and creation-then-annihilation operation (CA) to the thermal and coherent states. Effects of repeated applications of AC and of CA are also studied. We also discuss experimental schemes to realize AC and CA with a cavity system using atom-field interactions. © 2009 Optical Society of America

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1. INTRODUCTION

In many applications of quantum information processing that employ radiation fields, it is often desirable that the fields possess nonclassical nature such as entanglement and squeezing. From a theoretical point of view, perhaps the simplest way of generating nonclassical field states is to apply the photon creation operation $\hat{a}^+$ to classical states such as the thermal and coherent states \[1\]. These nonclassical field states, referred to as the single-photon added coherent state \[2\] and the single-photon added thermal state \[3\], respectively, have been realized experimentally \[4–6\].

In this paper, we investigate nonclassical properties of the field states that result when the photon annihilation-then-creation operation (AC) $\hat{a}^+\hat{a}$ and the photon creation-then-annihilation operation (CA) $\hat{a}\hat{a}^+$ are applied to the thermal and coherent states. The noncommutativity of the photon operations, $[\hat{a}, \hat{a}^+] = 1$, breaks the symmetry between AC and CA. It thus is of interest to analyze and compare the
properties of the states that result after AC and CA. In fact, a direct observation of
the noncommutativity of the operators $\hat{a}$ and $\hat{a}^+$ has been achieved by experiment-
tally comparing such two states and proving that they differ from each other [7].
An experimental scheme to directly prove the commutation relation $[\hat{a}, \hat{a}^+] = 1$ has
also been proposed recently [8]. Here we look in particular at the possibility and the
degree of nonclassicality exhibited by the states produced by AC and CA. We also
investigate properties of the states obtained when AC and CA are applied repeatedly
to the thermal and coherent states.

The paper is organized as follows. In Sec. II we briefly review the effect of the
photon creation and annihilation operations $\hat{a}^+$ and $\hat{a}$. In Sec. III we study properties
of the states produced by AC and CA operated on the thermal state and the coherent
state. We also study the effect of repeated applications of AC and CA to the thermal
and coherent states. Experimental schemes to realize AC and CA with a cavity system
are described in Sec. IV. Finally, Sec. V presents a discussion of our results.

2. CREATION AND ANNIHILATION OPERATIONS

The creation (annihilation) operator $\hat{a}^+$ ($\hat{a}$) acting on a photon number state $|n\rangle$
results in an increase (a decrease) of the photon number by one. As the creation (an-
nihilation) operator is not a Hermitian, however, it cannot be expected to always play
the role of physically adding (subtracting) one photon. Let us consider an arbitrary
pure field state $|\psi\rangle = \sum_{n=0}^{\infty} C_n |n\rangle$, where $\sum_{n=0}^{\infty} |C_n|^2 = 1$. The state that results after
application of the operator $\hat{a}^+$ ($\hat{a}$) to the state $|\psi\rangle$ is $|\psi\rangle_C = N\{\hat{a}^+|\psi\rangle\} = \frac{\hat{a}^+|\psi\rangle}{\sqrt{\langle\psi|\hat{a}^+\hat{a}|\psi\rangle}}$

$\langle|\psi\rangle_A = N\{\hat{a}|\psi\rangle\} = \frac{\hat{a}|\psi\rangle}{\sqrt{\langle\psi|\hat{a}\hat{a}^+|\psi\rangle}}$, where $N$ signifies that normalization is to be performed on the unnormalized state inside the curly bracket. One easily finds

$$\langle\hat{n}\rangle_C - \langle\hat{n}\rangle = \frac{(\Delta\hat{n})^2}{\langle\hat{n}\rangle + 1} + 1 \quad (1)$$

$$\langle\hat{n}\rangle_A - \langle\hat{n}\rangle = \frac{(\Delta\hat{n})^2}{\langle\hat{n}\rangle} - 1 \quad (2)$$

where $\langle\hat{n}\rangle$, $\langle\hat{n}\rangle_C$ and $\langle\hat{n}\rangle_A$ refer to the mean photon number of the state $|\psi\rangle$, $|\psi\rangle_C$ and $|\psi\rangle_A$, respectively, and $(\Delta\hat{n})^2$ is the photon-number variance of the state $|\psi\rangle$ $[(\Delta\hat{n})^2 = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2]$. Eqs. (1) and (2) indicate that, only when the state $|\psi\rangle$ has zero photon-number variance, the creation (annihilation) operation increases (decreases) the mean photon number by one. The mean photon number increases at least by one under the action of the creation operator and decreases at most by one under the action of the annihilation operator. In fact, the mean photon number can even increase upon application of the annihilation operator, which occurs when $(\Delta\hat{n})^2 > \langle\hat{n}\rangle$. This tendency of the mean photon number to increase beyond one’s naive expectation has its origin in the relation $\hat{a}^+|n\rangle = \sqrt{n+1}|n+1\rangle$ and $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$; a large photon number carries a greater weight in the state obtained after the $\hat{a}^+$ or $\hat{a}$ operation than in the original state.

The operations $\hat{a}^+$ and $\hat{a}$ can have more profound effects upon the state they are applied to than simply changing the number of photons. In fact, the states that result after the operation $\hat{a}^+$ and $\hat{a}$ often exhibit significantly different physical properties
from the original state and from each other. It is well known that a creation operation on thermal and coherent states yields nonclassical states, while the states remain classical upon an annihilation operation.

3. ANNIHILATION-THEN-CREATION AND CREATION-THEN-ANNIHILATION OPERATIONS

In this section we consider the effect of the annihilation-then-creation operation (AC) $\hat{a}^+\hat{a}$ and the creation-then-annihilation operation (CA) $\hat{a}\hat{a}^+$ on thermal and coherent states. We further study the properties of the states that are obtained when AC and CA, respectively, are applied repeatedly. We look in particular at the possibility and the degree of nonclassicality exhibited by these states.

3.A. Initial thermal state

The thermal state is characterized by the density matrix

$$\rho = \sum_{n=0}^{\infty} \frac{\pi^n}{(1 + \pi)^{n+1}} |n\rangle\langle n|$$

(3)

where $\pi = \frac{1}{e^{\omega/kT} - 1}$. The mean photon number $\langle \hat{n} \rangle$ and the photon-number variance $(\Delta \hat{n})^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2$ for the state (3) are given respectively by $\langle \hat{n} \rangle = \pi$ and $(\Delta \hat{n})^2 = \pi^2 + \pi$, yielding the Mandel Q factor $[9] Q \equiv \frac{(\Delta \hat{n})^2}{\langle \hat{n} \rangle} - 1 = \pi$. Application of the
operation AC and CA, respectively, to the state (3) yields

\[
\rho_{AC} = N \{ \hat{a}^+ \hat{a} \rho \hat{a} \hat{a}^+ \} = \frac{\hat{a}^+ \hat{a} \rho \hat{a} \hat{a}^+}{\text{Tr} \{ \hat{a}^+ \hat{a} \rho \hat{a} \hat{a}^+ \}} = \frac{1}{(1 + 2\bar{n})(1 + \bar{n})\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n n^2 |n\rangle \langle n|}{(1 + \bar{n})^n}, \quad (4)
\]

\[
\rho_{CA} = N \{ \hat{a} \hat{a}^+ \rho \hat{a} \hat{a}^+ \} = \frac{\hat{a} \hat{a}^+ \rho \hat{a} \hat{a}^+}{\text{Tr} \{ \hat{a} \hat{a}^+ \rho \hat{a} \hat{a}^+ \}} = \frac{1}{(1 + 2\bar{n})(1 + \bar{n})^2} \sum_{n=0}^{\infty} \frac{\bar{n}^n (n + 1)^2 |n\rangle \langle n|}{(1 + \bar{n})^n}. \quad (5)
\]

The Mandel Q factor for the states (4) and (5) can be obtained through straightforward calculations. We show in Fig. 1 the calculated Q factor as a function of \( \bar{n} \), the mean photon number of the initial thermal state (3). It can be seen that, when the AC operation is applied to the thermal state, the Q factor becomes negative and thus the photon number distribution becomes sub-Poissonian for a sufficiently small initial mean photon number, i.e., for \( \bar{n} \leq 0.6 \). The sub-Poissonian photon statistics implies nonclassical properties of the state, although the reverse is not necessarily true. Under the CA operation, however, the photon statistics remains super-Poissonian regardless of \( \bar{n} \). The figure indicates also that the difference in the Mandel Q factor between the two states \( \rho_{AC} \) and \( \rho_{CA} \) is more significant when \( \bar{n} \) is smaller. The difference in the degree of nonclassicality between the two states (4) and (5) is illustrated in Fig. 2, in which the corresponding Wigner distribution functions [10,11] are plotted for the case \( \bar{n} = 0.57 \). Only the Wigner distribution for the state (4) takes on negative values in the vicinity of the origin.

We now consider the states produced by repeated applications of \( \hat{a}^+ \hat{a} \) or \( \hat{a} \hat{a}^+ \) to the
initial thermal state (3). A straightforward algebra yields

\[
\rho_{AC}^k = N\{(\hat{a}^+ \hat{a})^k \rho (\hat{a}^+ \hat{a})^k\}
\]

\[
= \frac{1}{Li_{-2k}(\frac{\pi}{1+n})} \sum_{n=0}^{\infty} \frac{\pi^n n^{2k}}{(1+n)^n} |n\rangle \langle n|, \tag{6}
\]

\[
\rho_{CA}^k = N\{(\hat{a} \hat{a}^+)^k \rho (\hat{a} \hat{a}^+)^k\}
\]

\[
= \frac{1}{Li_{-2k}(\frac{\pi}{1+n})} \sum_{n=0}^{\infty} \frac{\pi^{n+1} (n+1)^{2k}}{(1+n)^{n+1}} |n\rangle \langle n|, \tag{7}
\]

where \(k\) can be any positive integer and \(Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}\) is the polylogarithm. The mean photon number (\(\langle \hat{n} \rangle_{AC}^k\) and \(\langle \hat{n} \rangle_{CA}^k\)) and the variance (\([\langle (\Delta \hat{n})^2 \rangle]_{AC}^k\) and \([\langle (\Delta \hat{n})^2 \rangle]_{CA}^k\)) for the states (6) and (7), can also be obtained by straightforward calculations. We obtain

\[
\langle \hat{n} \rangle_{AC}^k = \frac{Li_{-2k-1}(\frac{\pi}{1+n})}{Li_{-2k}(\frac{\pi}{1+n})} = \langle \hat{n} \rangle_{CA}^k + 1, \tag{8}
\]

\[
[\langle (\Delta \hat{n})^2 \rangle]_{AC}^k = [\langle (\Delta \hat{n})^2 \rangle]_{CA}^k
\]

\[
= \frac{Li_{-2k-2}(\frac{\pi}{1+n})}{Li_{-2k}(\frac{\pi}{1+n})} - \left(\frac{Li_{-2k-1}(\frac{\pi}{1+n})}{Li_{-2k}(\frac{\pi}{1+n})}\right)^2. \tag{9}
\]

We thus see that the mean photon number of state (6) is always larger by exactly one than that of state (7) (Note, on the other hand, the operator identity \(\hat{a}^+ \hat{a} = \hat{a} \hat{a}^+ - 1\).) regardless of \(k\). We also see that the photon number variance is exactly the same for the two states (6) and (7), regardless of \(k\).

In general, as the number \(k\) of operations \(\hat{a}^+ \hat{a}\) or \(\hat{a} \hat{a}^+\) is increased from 1, the mean photon number of the state increases rapidly. This is due to the fact already mentioned in Sec. II that a larger photon number carries a greater weight in the state after the \(\hat{a}^+\)
or $\hat{a}$ operation than in the original state. It then is clear from Eqs. (8) and (9) that, for large $k$, the two states $\rho_{AC}^k$ and $\rho_{CA}^k$ possess almost identical photon-number statistics. In Fig. 3 we plot the Mandel Q factor as a function of $\bar{n}$ for the two states $\rho_{AC}^{k=20}$ and $\rho_{CA}^{k=20}$. The two curves are seen to be very close to each other. Close inspection reveals, however, that the curve for the state $\rho_{AC}^{k=20}$ lies slightly below that for the state $\rho_{CA}^{k=20}$, indicating that the state $\rho_{AC}^{k=20}$ shows slightly stronger nonclassicality than the state $\rho_{CA}^{k=20}$. For both states, the Mandel Q factor is negative and the photon-number distribution is sub-Poissonian if $\bar{n} \leq 0.57$. That the two states $\rho_{AC}^k$ and $\rho_{CA}^k$ approach each other as $k$ moves toward a large number can clearly be seen in Fig. 4, in which the fidelity between the two states, $F = \{ Tr[\sqrt{\rho_{AC}^k} \rho_{CA}^k \sqrt{\rho_{AC}^k}]^{1/2} \}^2$ [12] is plotted as a function of $k$, for the case $\bar{n} = 0.57$.

3.B. Initial coherent state

The coherent state $|\alpha\rangle$ has a Poissonian photon-number distribution, and thus $Q = 0$ for the coherent state. Upon application of AC and CA, respectively, the coherent state $|\alpha\rangle$ is transformed to the states

$$|\alpha\rangle_{AC} = N\{\hat{a}^+ \hat{a} |\alpha\rangle\} = \frac{\hat{a}^+ \hat{a} |\alpha\rangle}{\sqrt{\langle \alpha |\hat{a}^+ \hat{a}^+ \hat{a} \hat{a} |\alpha\rangle}}$$

(10)

$$|\alpha\rangle_{CA} = N\{\hat{a} \hat{a}^+ |\alpha\rangle\} = \frac{\hat{a} \hat{a}^+ |\alpha\rangle}{\sqrt{\langle \alpha |\hat{a} \hat{a}^+ \hat{a}^+ \hat{a} \alpha\rangle}}$$

(11)

It should be mentioned that, since the coherent state is an eigenstate of the annihilation operator $\hat{a}$, $|\alpha\rangle_{AC}$ is given simply by $|\alpha\rangle_{AC} = N\{\hat{a}^+ |\alpha\rangle\}$ and can be identified as the single-photon added coherent state.
Fig. 5 shows the Mandel Q factor for the states (10) and (11) as a function of $|\alpha|^2$, the mean photon number of the initial coherent state. It is seen that both states (10) and (11) have negative Q values regardless of $|\alpha|^2$. As before for the case of the initial thermal state, the AC operation produces stronger nonclassicality than the CA operation, with the difference more significant when $|\alpha|^2$ is smaller. In Fig. 6 we show the Wigner distributions for the states (10) and (11) for the case $|\alpha|^2 = 0.57$. One sees that both AC and CA operations lead to negative Wigner distributions, but the degree of negativity created by the AC operation is stronger.

Let us consider the states produced by repeated applications of AC and CA, respectively, to the coherent state, given by

$$|\alpha\rangle^k_{AC} = N\{(\hat{a}^+\hat{a})^k|\alpha\rangle\}, \quad (12)$$

$$|\alpha\rangle^k_{CA} = N\{(\hat{a}\hat{a}^+)^k|\alpha\rangle\}. \quad (13)$$

Shown in Fig. 7 is the Mandel Q factor for the state $|\alpha\rangle^k_{AC}^{k=20}$ and $|\alpha\rangle^k_{CA}^{k=20}$ as a function of $|\alpha|^2$. One sees here that both states exhibit negative Q factors, but the state $|\alpha\rangle^k_{AC}^{k=20}$ shows a slightly stronger sub-Poissonian photon distribution than $|\alpha\rangle^k_{CA}^{k=20}$. Comparing Fig. 5 and Fig. 7, one also sees that, as $k$ is increased, the two states $|\alpha\rangle^k_{AC}$ and $|\alpha\rangle^k_{CA}$ approach each other. In Fig. 8 we plot the fidelity between the two states, $F = |\langle \alpha |\alpha\rangle^k_{AC,CA}|^2$, as a function of $k$, for the case $|\alpha|^2 = 0.57$. The fidelity approaches 1 as $k$ is increased.
4. PHYSICAL IMPLEMENTATIONS

The photon creation operation $\hat{a}^+$ on the coherent and thermal states was realized experimentally by Zavatta et al. using an optical system based on parametric down-conversion in a nonlinear crystal followed by conditional detection of a single photon in the idler mode [4–6]. The same group later achieved an experimental realization of the photon annihilation-then-creation operation (AC) $\hat{a}^+\hat{a}$ and the photon creation-then-annihilation operation (CA) $\hat{a}\hat{a}^+$ on the thermal state using again an optical system, which provided direct observation of the noncommutativity of the operators $\hat{a}^+$ and $\hat{a}$ [7]. Here we briefly describe an experimental scheme to realize AC and CA using a cavity system along the line suggested by Sun et al. [13].

In order to realize AC, we pass two atoms, atom 1 prepared in the lower state $|g\rangle$ and atom 2 in the upper state $|e\rangle$, in order through a cavity. Let us assume an arbitrary pure state $|\psi\rangle_i = \sum_{n=0}^{\infty} C_n |n\rangle$ for the initial cavity field. (For simplicity we assume a pure state. Essentially the same analysis can be applied to obtain the same result even if the initial cavity field is in a mixed state.) A simple algebra yields that the state of the cavity field after the interaction of the cavity with the two atoms, conditioned upon the observation that atom 1 is found in state $|e\rangle$ and atom 2 in state $|g\rangle$ after the interaction, is given by

$$|\psi\rangle_1 = N \left\{ \sum_{n=1}^{\infty} \frac{\sin(\sqrt{ngt_1}) \sin(\sqrt{ngt_2}) C_n |n\rangle}{n} \right\}$$

(14)

where $g$ is the atom-field coupling constant, and $t_1$ and $t_2$ are, respectively, the inter-
action time between atom 1 and the cavity and between atom 2 and the cavity. The state (14) can approximately be identified as the state

$$|\psi\rangle_{AC} = N\{\hat{a}^+\hat{a}|\psi\rangle_i\} = N\{\sum_{n=1}^{\infty} nC_n|n\rangle\}$$

(15)

if the interaction times $t_1$ and $t_2$ are sufficiently short that $\sqrt{\langle n\rangle}g t_1 \ll 1$ and $\sqrt{\langle n\rangle}g t_2 \ll 1$, where $\langle n\rangle$ is the mean photon number of the initial field state $|\psi\rangle_i$.

Repeated operations of $AC$, $(\hat{a}^+\hat{a})^k$, can then be realized by letting $k$ pairs of atoms, each pair prepared in $|g\rangle$ and $|e\rangle$, interact with a cavity, and postselecting the case where each pair is found in $|e\rangle$ and $|g\rangle$, subjected to the condition that all interaction times are sufficiently short.

Similarly, the operation $CA$ can be realized by letting two atoms, atom 1 prepared in $|e\rangle$ and atom 2 in $|g\rangle$, interact with a cavity field, conditioned upon atom 1 being found in $|g\rangle$ and atom 2 in $|e\rangle$ after the interaction and the interaction times $t_1$ and $t_2$ being sufficiently short. In this case, the state of the cavity field after the interaction is

$$|\psi\rangle_2 = N\{\sum_{n=0}^{\infty} \sin(\sqrt{n+1}gt_1)\sin(\sqrt{n+1}gt_2)C_n|n\rangle\}$$

(16)

which, in the short time limit $\sqrt{\langle n\rangle} + 1gt_1 \ll 1$, $\sqrt{\langle n\rangle} + 1gt_2 \ll 1$, approaches the state

$$|\psi\rangle_{CA} = N\{\hat{a}\hat{a}^+|\psi\rangle_i\} = N\{\sum_{n=0}^{\infty} (n+1)C_n|n\rangle\}. \hspace{1cm} (17)$$

Repeated operations of $CA$, $(\hat{a}\hat{a}^+)^k$, can also be realized by using $k$ pairs of atoms.
The AC and CA operations can be realized with the cavity scheme described above with high fidelities $F_1 = |\langle \psi | \psi \rangle_{AC}|^2 \approx 1$ and $F_2 = |\langle \psi | \psi \rangle_{CA}|^2 \approx 1$, if the interaction times are sufficiently short. The problem, however, is that, in the short-time limit, the success probability of the scheme is low. Let $P_1 (P_2)$ be the success probability of the scheme to realize AC (CA), i.e., let $P_1 (P_2)$ be the probability to find atom 1 in $|e\rangle (|g\rangle)$ and atom 2 in $|g\rangle (|e\rangle)$ after the interaction. One can easily find that

$$P_1 = \sum_{n=1}^{\infty} \sin^2(\sqrt{n}gt_1) \sin^2(\sqrt{n}gt_2) |C_n|^2,$$

$$P_2 = \sum_{n=0}^{\infty} \sin^2(\sqrt{n+1}gt_1) \sin^2(\sqrt{n+1}gt_2) |C_n|^2.$$  

In the short-time limit, $P_1, P_2 \propto g^4t_1^2t_2^2 \ll 1$. For repeated operations $(\hat{a}^+\hat{a})^k$ and $(\hat{a}\hat{a}^+)^k$, the success probabilities are extremely small in the short-time limit as $P_1, P_2 \propto g^{4k}t_1^{2k}t_2^{2k}$.

Instead of taking the short interaction times, one can choose $t_1$ and $t_2$ in such a way that $\sqrt{\langle \hat{n} \rangle}gt_1 = \sqrt{\langle \hat{n} \rangle}gt_2 = \frac{\pi}{2}$ for AC and $\sqrt{\langle \hat{n} \rangle + 1}gt_1 = \sqrt{\langle \hat{n} \rangle + 1}gt_2 = \frac{\pi}{2}$ for CA, where $\langle \hat{n} \rangle$ is the mean photon number of the initial cavity field. This choice ensures that the desired state transitions $|g\rangle \leftrightarrow |e\rangle$ for both atoms 1 and 2 occur with high probabilities. When the atom makes a transition as desired, the cavity field gains or loses a photon as desired. One can thus hope that, with this choice of the interaction times, both the success probability and the fidelity may stay reasonably close to unity. Clearly, this strategy would work well if the initial cavity field has a sharp photon-number distribution centered at $\langle \hat{n} \rangle$, which is the case for the initial
coherent state but not for the initial thermal state. In Fig. 9 we plot the success probability $P_1$ and the fidelity $F_1$ we computed as a function of $|\alpha|^2$ for the case when the initial cavity field state is a coherent state, where $t_1$ and $t_2$ are chosen to satisfy the relation $\sqrt{\langle n \rangle} g t_1 = \sqrt{\langle n \rangle} g t_2 = \frac{\pi}{2}$. One can see clearly that both the success probability and the fidelity stay close to unity for $\langle \hat{n} \rangle \geq 50$. The success probability $P_2$ and the fidelity $F_2$ show similar behavior, the only significant difference being that, as $\langle \hat{n} \rangle \to 0$, $P_1 \to 0$ as can be seen from Fig 9 but $P_2 \to 1$. The result shown in Fig.9 suggests that, with this choice of the interaction times, even repeated operations of AC and CA can be realized experimentally with reasonably high success probability and fidelity, if the initial cavity field has a sufficiently sharp photon-number distribution.

To assess the feasibility of the suggested experiment, we take, as an example, $g \approx 10^7 Hz$ and $\langle \hat{n} \rangle = 100$. The interaction times $t_1$ and $t_2$ are then required to be $t_1 = t_2 = \frac{\pi/2}{\sqrt{\langle n \rangle} g} \approx 10^{-8} sec$. Since photon addition and subtraction should occur within these interaction times, we require the spontaneous emission decay rate $\gamma$ and the cavity field decay rate $\kappa$ be small compared with $\frac{1}{t_1} = \frac{1}{t_2}$, i.e., we require $\gamma \ll 10^8 Hz$ and $\kappa \ll 10^8 Hz$, an experimental condition that can be met without too much difficulty.

Photon annihilation-then-creation and creation-then-annihilation operations can be implemented experimentally using cavity-field atom interactions as described above as well as using optical means along the line of Refs. [4–6]. The cavity-QED method described here offers the flexibility of choosing interaction times in such a way that the desired states $|\psi\rangle_{AC}$ and $|\psi\rangle_{CA}$ can be generated with both high success probability
and high fidelity. The noncommutativity of the photon creation and annihilation operations can thus be verified with high success probability not easily achievable in the optical approach, if the cavity-QED method is adopted.

5. DISCUSSION

We have studied effects of the photon annihilation-then-creation operation (AC) $\hat{a}^+\hat{a}$ and the photon creation-then-annihilation operation (CA) $\hat{a}\hat{a}^+$, and their repeated applications on the thermal and coherent states. The common feature of AC and CA is that they both work to increase the mean photon number unless the initial state is the number state. That this is the case can be understood by noting that, due to the relation $\hat{a}^+\hat{a}|n\rangle = n|n\rangle$ and $\hat{a}\hat{a}^+|n\rangle = (n+1)|n\rangle$, a larger photon number carries a greater weight in the state obtained after the operation $\hat{a}^+\hat{a}$ or $\hat{a}\hat{a}^+$ is applied than in the initial state. Details of exactly how the weight of each photon number is changed, however, are different for the two operations. Close inspection of this difference shed light on one of the main observations of this work, namely the observation that AC in general produces stronger nonclassicality than CA. It should be noted, in particular, that the vacuum state, which is a part of the initial thermal or coherent state, disappears upon application of the $\hat{a}^+\hat{a}$ operation, whereas it still remains after the $\hat{a}\hat{a}^+$ operation. As a result, the state produced by the $\hat{a}^+\hat{a}$ operation has a significantly different photon number distribution from the initial “classical” distribution, especially if the vacuum state has a large weight in the initial state, i.e., if the ini-
tial thermal or coherent state has a small mean photon number. This explains why nonclassical properties are exhibited more strongly after AC than after CA and why the difference is more pronounced when the mean photon number of the initial state is small. If the operation $\hat{a}^+\hat{a}$ or $\hat{a}\hat{a}^+$ is applied repeatedly, the mean photon number of the state increases and accordingly the weight of the vacuum state becomes small. In such a case, the effect of the $\hat{a}^+\hat{a}$ operation will not be much different from the effect of the $\hat{a}\hat{a}^+$ operation. This explains why the two states produced by applying the $\hat{a}^+\hat{a}$ operation $k$ times and the $\hat{a}\hat{a}^+$ operation $k$ times approach each other as $k$ is increased.

In this work, we have used the Mandel Q factor and the Wigner distribution as an indicator of nonclassicality. In a recent work [14], nonclassicality of the field states generated by photon creation-then-annihilation operations were investigated using the nonclassical depth [15] as a criterion for nonclassicality. We further note that our investigation here focuses on nonclassical effects related to sub-Poissonian photon statistics. As photon-added thermal and coherent states have been shown to exhibit higher-order nonclassical effects beyond negativity of the Mandel Q factor and the Wigner distribution function [16], the states generated by photon creation-then-annihilation operations and by photon annihilation-then-creation operations should also exhibit such higher-order nonclassical effects.
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FIGURE CAPTIONS

Fig. 1: Mandel Q factor of the states $\rho_{AC}$ (dashed curve) and $\rho_{CA}$ (solid curve) vs. $\overline{n}$, the mean photon number of the initial thermal state.

Fig. 2: Wigner distribution of (a) the state $\rho_{AC}$ and (b) the state $\rho_{CA}$ for the case $\overline{n} = 0.57$. $x = \frac{1}{2}(\alpha + \alpha^*)$ and $y = \frac{1}{2i}(\alpha - \alpha^*)$.

Fig. 3: Mandel Q factor of the states $\rho_{AC}^{k=20}$ (dashed curve) and $\rho_{CA}^{k=20}$ (solid curve) vs. $\overline{n}$.

Fig. 4: Fidelity between the states $\rho_{AC}^k$ and $\rho_{CA}^k$ vs. $k$ for the case $\overline{n} = 0.57$.

Fig. 5: Mandel Q factor of the states $|\alpha\rangle_{AC}$ (dashed curve) and $|\alpha\rangle_{CA}$ (solid curve) vs. $|\alpha|^2$, the mean photon number of the initial coherent state.

Fig. 6: Wigner distribution of (a) the state $|\alpha\rangle_{AC}$ and (b) the state $|\alpha\rangle_{CA}$ for the case $|\alpha|^2 = 0.57$. $x = \frac{1}{2}(\alpha + \alpha^*)$ and $y = \frac{1}{2i}(\alpha - \alpha^*)$.

Fig. 7: Mandel Q factor of the states $|\alpha\rangle_{AC}^{k=20}$ (dashed curve) and $|\alpha\rangle_{CA}^{k=20}$ (solid curve) vs. $|\alpha|^2$.

Fig. 8: Fidelity between the states $|\alpha\rangle_{AC}^k$ and $|\alpha\rangle_{CA}^k$ vs. $k$ for the case $|\alpha|^2 = 0.57$.

Fig. 9: The success probability $P_1$ (solid curve) and the fidelity $F_1$ (dashed curve) vs. $|\alpha|^2$, the mean photon number of the initial coherent state. The interaction times $t_1$ and $t_2$ are chosen such that $|\alpha|gt_1 = |\alpha|gt_2 = \frac{\pi}{2}$. 
Fig. 1.
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