SOME REMARKS ON TRÈVES’ CONJECTURE

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Abstract. I will give a discussion of the conditions involved in ‘Trèves’ conjecture on analytic hypoellipticity. I will discuss some microlocally characteristic sets and introduce a topology of monotropic functionals as suitable for solving the conjecture. The pseudodifferential operator representation is inspired by Cousin (cf. [5]).

1. Introduction

Trèves’ conjecture is existence of an involutive stratification equivalent with hypoellipticity. The concept of hypoellipticity is very sensitive to change of topology but there are geometric sets that are characteristic. We will discuss lineality and a set that relates to orthogonality. We will also consider three sets that occur in literature and that we consider as not characteristic. The first is relative representation of spectral function to a hypoelliptic operator (6.6). The third (section 11) relates to hypoelliptic operators as limits of operators dependent on a parameter. In the second we consider continuation of the contact transform to (T), which is considered as a Bäcklund transform. For this continuation, algebraicity is considered to be characteristic for hypoellipticity (11.4). It is necessary for hypoellipticity that the singularities have measure zero and in this study we assume parabolic singularities. The regular approximations are transversals and we only briefly discuss some possible generalizations.

The set of lineality is defined for a polynomial over a real (or complex) vector space $E\mathbb{R}$ (or $E\mathbb{C}$) is

$$\Delta(P) = \{ \eta \in E\mathbb{R}, P(\xi + it\eta) - P(\xi) = 0 \; \forall \xi \in E\mathbb{R} \; \forall t \in \mathbb{R} \}$$

It can be proved that $\Delta$ is standard complexified in the topology for $Exp$ (cf. [12]), why it is sufficient to consider purely imaginary translations as above. The set can be generalized to symbol classes where $\Delta$ has a locally algebraic definition or where the definition is locally algebraic modulo monotropy. The pseudo differential operators are realized from the symbol ideals using a representation derived from Cousin.

For constant coefficients polynomial differential operators, we note that the class of operators hypoelliptic in $D'$ is not radical. We can prove for the radical to the class, that the lineality is decreasing for iteration. For variable coefficients polynomial differential operators, we consider formally hypoelliptic operators, that is where the symbol is equivalent in strength with a constant coefficients polynomial operators, as the variable varies. We also assume that the real part of the symbol
is unbounded and does not change sign in the infinity, as the variable varies on a connected set.

The generalization to more general symbols will be using a lifting operator acting on a dynamical system, that maps into analytic symbols \( f(\xi) \in (I)(\Omega) \), where \( I \) is an ideal over a pseudo convex domain. We will mainly discuss operators \( A_\lambda \) on the form \( A_\lambda = P_\lambda + H_\lambda \), where \( P_\lambda \) is a polynomial for finite parameter values and \( H_\lambda \) is regularizing.

**Proposition 1.1.** Assume \( S \) a pseudo differential operator, self-adjoint and of exponentially finite type. Assume the symbol in \((I)(\Omega)\), where \( I \) is finitely generated and \( \Omega \) is pseudo convex. Assume the lineality to \( S \), \( \Omega_0 \) is decreasing for iteration. Assume singular points are mapped on to singular points in the dynamical system, with tangent determined (global pseudo base). Then, for \( u \in \mathcal{D}' \)

\[
WF_a(Su) = \tilde{\Omega}_0 \cup WF_a(u)
\]

Here \( \tilde{\Omega} \) is a set only dependent on \( \Omega \) and the symbol.

If \( \Omega_0 = \lim_j \Omega_j \), where \( \Omega_j \subset \Omega \) a pseudo convex set (and \( \Omega_j \) algebraic), then \( \Omega_0 \) must be an analytic set. Given that the level surfaces are of order 1, \( \Omega_0 \) has a locally algebraic definition through transversality. Conversely, if \( \Omega_0 \) has an algebraic definition and if we have a global pseudo base for \((I)\), then regular approximations are transversals and \( \Omega_0 \) is given by regular approximations. If \( \Omega_j \) are given by the lineality locally to \( A_\lambda = P_\lambda + H_\lambda \) and \( P_\lambda \sim P_\lambda \), then \( \Omega_0 \) is a set of lineality for the limit of \( P_\lambda \).

We will use monotropic functionals to study both the symbols to hypoelliptic operators and the equations in operator space. For the representations we consider, monotropy is microlocally indifferent, that is does not influence the geometry in a microlocally significant way. We will use the notation \( f \sim_m 0 \) explained as follows. Between the spaces \( \mathcal{B}(\mathbb{R}^n) \) and \( \mathcal{B}(\mathbb{R}^m) \), we consider over an \( \varepsilon \)-neighborhood of the real space, the space \( \mathcal{B}_m \) of \( C^\infty \)–functions bounded in the real infinity by a small constant with all derivatives. Thus, consider \( D^\alpha \phi - \mu_\alpha \to 0 \) in the real infinity, for all \( \alpha \) and \( \mu_\alpha \) constants. Obviously, the space of monotropic functionals \( \mathcal{B}_m \subset \mathcal{D}'_{L_1} \), why \( T \in \mathcal{B}_m \) has representation \( \sum_{|\alpha| \geq k} D^\alpha f_\alpha \) with \( f_\alpha \in L^1 \). If \( T \in \mathcal{D}'_{L_1} \) and \( \phi \in \mathcal{B}_m \), there is a \( S \in \mathcal{B}'_m \) such that \( S = T \) over \( \mathcal{B}_m \). We have that \( \mathbb{R}^n = \bigcup_{j=1}^\infty K_j \), for compact sets \( K_j \). Let \( \Phi_j,1 = (S-T)|_{K_j} \in \mathcal{E}' \subset \mathcal{B}'_m \) and \( \Phi_j,2 = \Phi - \Phi_j,1 \). We chose \( S \) such that \( \Phi_j,1 = 0 \) for all \( j \) and \( \Phi_j,2 \in \mathcal{B}'_m \). This gives existence of a functional \( S \) such that \( S(\phi) = \sum_\alpha f_\alpha(x)dx = \lim_{j \to \infty} T_j(\phi) \), where the limit is taken in \( \mathcal{D}'_{L_1} \).

Assume \( \Omega_0 \subset U \subset V \), where \( U \) is an open set. Assume \( U \) quasi-porteur for \( S \in H'(V) \), that is \( T = \imath(iu_S) \) for \( u_S \in H'(U) \), where \( i \) is the restriction homomorphism. Assume \( r'_T \) the transposed ramifier. Algebraicity for \( r'_T \) means that we can prove that the wave front-set is defined by \( b_T \) \cite{[20]} in \( H' \). Assume for the vorticity to the dynamical system \( \tilde{\omega}_0 \) changes sign finitely many times locally. On regions where \( \tilde{\omega}_0 \) has constant sign, we have isolated singularities in a sufficiently small neighborhood. The lift function \( F \) in \( f(\xi) = F(\gamma)(\xi) \), can be represented by \( \prod_p F_p \), relative a division in contingent regions.

**Proposition 1.2.** Assume \( F \) reduced and \( F_T \) algebraically dependent on \( T \). Then \( F_T \) is not regularizing.
Proof:
We have assumed conditions on the ramifier $r_T'$ such that we have existence of constants $c_1, c_2$ such that $c_1 \| \gamma \| | r_T' \gamma | \leq c_2 | \gamma |$ as $| \gamma | \to \infty$, that is the type $| F | = \limsup_{T \to \infty} \frac{1}{T} \log | F(\gamma) |$ is not dependent on $T$ in the $| \gamma | \to \infty$ and $| F_T | = | F |$. If for this reason $F$ is not of type $-\infty$, then the same holds for $F_T$. □

**Proposition 1.3.** The condition that $F(\gamma_T)$ is analytically hypoelliptic does not imply that $Re F(\gamma_T)$ or $Im F(\gamma_T)$ is analytically hypoelliptic.

Assume $Pr$ the pseudo-differential operator that corresponds to $F_T$ and that $P_T u = f_T$ in $H'(V)$, for an open set $V$, where we are assuming $f_T$ holomorphic, that $\lim_{T \to 0} f_T = f$ in $H'(V)$ and $\lim_{T \to 0} < u, P_T \varphi > = < f, \varphi >$, for $\varphi \in H$. We are assuming that $P_T$ maps $H \to H$ and that $D(P_T)$, the domain for $P_T$, has $D(P_T) \subset H(V)$.

**1.1. Paradoxal arguments.** First note that among parametrices to partially hypoelliptic differential operators, considered as Fredholm operators on $L^2$, there are examples of operators with non-trivial kernels. These can be proved to be hypoelliptic outside the kernel. If they are defined as regularizing on the kernel, they will not be hypoelliptic there. The class of partially hypoelliptic differential operators can be shown to be different from the class of hypoelliptic differential operators on $L^2$. The following argument for $C^\infty$-hypoellipticity is based on two fairly trivial observations,

i) The Dirac measure $\delta_0$ is not a $(C^\infty)$-hypoelliptic operator.

ii) If $E$ is a parametrix to a differential operator $P$ such that $PE - \delta_x \equiv 0$ in $V$, an open set in the real space (a neighborhood of $x$), then $P$ is not a $(C^\infty)$-hypoelliptic operator.

Proof of the observations:
For the first proposition, define a convolution operator on $E'$, $H(\varphi) = E_0 * \varphi$, where $E_0$ is a fundamental solution with singularities in $0$ to $P(D)$ and where $P(D)$ is a $(C^\infty)$-hypoelliptic differential operator with constant coefficients. If $\delta_0$ were $(C^\infty)$ hypoelliptic over $E'$, then sing supp $\delta_0 * \varphi =$ sing supp $\varphi$ and also sing supp $H(I P(D) \varphi) =$ sing supp $\varphi$, but since $E_0$ is regularizing outside the origin, $\varphi$ can have singularities that $H(\varphi)$ does not have.

For the second proposition, we use the notation $I_E(\varphi) = \int E(x,y) \varphi(y) dy$ and $I$ denotes the identity operator, that is an operator such that sing supp $I u =$ sing supp $u$ for all $u \in D'$. If $P(D)$ were $(C^\infty)$-hypoelliptic, then $I_E - I$ would be locally regularizing. If locally $I_{PE} = I$, we also have that locally $u - P u \in C^\infty$ for $u \in D$. But if $P(D)$ is $(C^\infty)$-hypoelliptic, then the same must hold for $P - 1$ and we have a contradiction. □

The first observation can immediately be adapted to analytic hypoellipticity. For the second we note that if $P$ is a differential operator, then $P - 1$ can not be regularizing and the proof is conclusive also for analytic hypoellipticity. As a consequence of this, the pseudo differential operators that we are studying will be assumed locally not regularizing.

The conclusions are as follows. If $f$ is the symbol to an analytically hypoelliptic pseudo differential operator in the class that we are studying, we have that all approximations $f_T$ can be chosen regular. The condition that the dependence of $T$ is algebraic for $f_T$ is necessary to avoid a paradox in the analogue to Weyl’s lemma. It is necessary to have symplecticity on each stratum. The involution is
used to guarantee existence of an inverse lifting function, since in this case $F_T$ can be chosen regular in $T$.

For a symbol in $\mathcal{B}_m$ over the real space (modulo regularizing operators), we again consider (modulo monotropy) locally algebraic symbols. For a constant coefficients polynomial operator a condition equivalent with hypoellipticity is that every distributional parametrix is very regular. These parametrices map $\mathcal{D}' \to \mathcal{D}'$, why it is necessary for the pseudo differential operator to be hypoelliptic, that the symbol is of real dominant type (orthogonal real and imaginary parts). The analysis is focused on the microlocal contribution from the lineality. The singular support is considered as a formal support in a ball of $\epsilon-$ radius.

Assume temporarily that the operator is not self-adjoint. Consider $E$, a parametrix to a homogeneously hypoelliptic, constant coefficients operator $P(D)$. Then, $I_E - I$ is regularizing in $\mathcal{D}'$ and thus is represented by a kernel in $C^\infty$, which has a regularizing action in $\mathcal{E}'$. However, it is not trivial to extend this action to $\mathcal{D}'$. Consider instead $C_{I_E} = I_E \varphi - \varphi I_E$, for some suitable real function in $C^\infty_0$ acting on $\mathcal{D}'$.

Since

$$C_{I_E} f = \varphi f - \varphi I_E f + I_E \varphi f - \varphi f, f \in \mathcal{D}'$$

This operator will be regularizing in $\mathcal{D}'$.

2. **Lineality**

2.1. **The lineality and the wavefront set.** The lineality $\Omega_0$ can be considered as the "boundary" to the frequency component. More precisely, assume $\Gamma$ a simple cone in $\Omega_0$ and $B_\Gamma = \lim_{t \to 0} A_{\Gamma t}$, where $A_{\Gamma t} = F^{-1} T^{\Gamma_0} H(E_{\mathbb{R}}) \to H'(E_{\mathbb{R}})$, for a real vector space $E_{\mathbb{R}}$. Assume $h_F$ the growth indicator to $B_\Gamma$ and that $g$ is growth indicator for the frequency component to $WF_0(u)$ (cf. [12] Ch.2, Theorem 4.3). As $h_F = g$ on $\Delta_0 = \Omega_0 \setminus 0$, we see that cones in $\Delta_0$ have indicator $\geq 0$. Let $W$ be the convex closure of the real support to $B_\Gamma$, that is $W = \{ y < y, \eta > \geq h_F(y) \ | \ \eta = 1 \}$.

2.2. **The lineality is standard complexified.** We can show that the lineality to a polynomial, is standard complexified in $\mathcal{E} \times \mathcal{P}$, why it is sufficient to consider completely imaginary translations of the real space. We shall now see that if we have lineality and if the lineality is locally algebraic, there is lineality in a complete disk (cf. [3]). Assume $0$ an essential singularity and $\Delta$ simply connected and closed $\ni 0$ with boundary $\Gamma' \cup \Gamma''$. If for a holomorphic function $f$, $|f|$ is bounded on $\Delta$ and $f(z) \to w$ as $\Gamma' \ni z \to 0$ and $\Gamma'' \ni z \to 0$, then $f(z) \to w$ uniformly as $z \to 0$ in $\Delta$. Conversely, if the limits on $\Gamma', \Gamma''$ are different, then $f$ can not be bounded on $\Delta$. Assume $\Delta$ with a algebraic definition locally, then given a sector $A0B$ where $f$ is assumed holomorphic, if $f \to w$ as $z \to 0$ on a line $0L$ in this sector, the same holds for any sector inner to $A0B$. Thus, if we have lineality on a line $OL$, we have lineality on the disk. The conclusion also holds for the several dimensional set of lineality, but since hypoellipticity can be derived from one dimensional translations, we do not prove this here.

2.3. **Remarks on hypoellipticity and symmetry.** An operator is considered as hypoelliptic, if its symbol is reduced in a neighborhood of the infinity, but for a holomorphic symbol it is not simultaneously reduced in a neighborhood of the
origin. Note also that if $f(z)$ is reduced as $z \to \infty$, then $\overline{f(z)}$ is not necessarily reduced as $z \to \infty$. If $f(\overline{z}) = \overline{f(z)}$, we have that $f(\overline{z})$ is not necessarily reduced, as $z \to \infty$. This property is consequently not symmetric with respect to the real axes. A necessary condition on a mapping $c$ to preserve reducedness, when $f(c(z)) = c(f(z))$, is that it is bijective.

In this context we consider the property $(P)$ for a continuous function $d$, that is $d(\overrightarrow{\alpha}) = \frac{d(\alpha)}{d(\beta)}$ as $T \to \infty$. For instance if $d$ is the distance function to the boundary, if there is no essential singularity in the infinity and if all singularities are isolated in the finite plane, then $d$ is globally reduced and $d$ has the property $(P)$.

2.4. The property $(P)$. Assume again that $f = e^w$ with $\varphi = e^\alpha$ and $L(e^w) = \hat{L}(\varphi) = e^{\hat{L}(\varphi)}$ and if $\hat{L}(-\alpha) = -\hat{L}(\alpha)$, we have $L(e^\hat{L}(\varphi)) = e^{\frac{1}{2}\hat{L}(\varphi)}$, which we denote property log $(P)$. If $L$ is algebraic, we have that it has the property log $(P)$. The property $(P)$ means that $\hat{L}(\varphi) + L(\varphi) \sim 0$ and if we have both the properties, we get $\hat{L}^{-1} = -\hat{L}$. If we assume $\hat{L} \sim_m \hat{W}$, where $W$ is algebraic in the infinity, then $e^{\hat{W}(-\varphi)} = W^{-1}(e^\varphi)$. We will consider $\hat{L} \to 0$, such that we have existence of an algebraic morphism $\hat{W}$ such that $\hat{L} \sim_m \hat{W}$ where $\hat{W}$ has the property $(P)$. We assume existence of $L^{-1}$ over an involutive set where we have a regular approximation. If $\varphi$ is a holomorphic function with $\varphi(\zeta) = \varphi_T(\zeta)$ and $\zeta_T \to \zeta_0$, as $T \to \infty$, then using Weierstrass theorem, we have existence of $s$ continuous, such that $s(\varphi_T + a) = \zeta_T$, for a constant $a$ and $s(a) = \zeta_0$. Further, $s$ can be approximated by polynomials of $1/(\varphi_T + a)$.

2.5. Linearity and the characteristic set. Treves’ conjecture is given for the characteristic set $\Sigma$ and our argument is given for the set of lineality. We will now argue that the conjecture can be derived from our result. Assume $\Sigma = \{ \zeta : f(\zeta) = 0 \}$ and $\Sigma = \Sigma_1 \cap \Sigma_2$, where $\Sigma_1 = \{ \text{Re } f(\zeta) = 0 \}$. Thus, if $\text{Im } f$ is algebraic, we have that $\Sigma_2$ is removable. The condition $\text{Re } f \perp \text{Im } f$ is considered as necessary for hypoellipticity. We note in this connection the well-known Weyl’s lemma (cf. [1]), if $w \in L^2(\{ z < 1 \})$ and for all $V \subset \mathbb{C}^n_0(\{ z < 1 \})$, we have $(w,dV) = (w,dV^\circ) = 0$, (harmonic conjugation) then $w$ is equivalent with a $C^1$ form.

Assume $(I) = (\ker h)$, where $h$ is a homomorphism and assume existence of a homomorphism $g$, such that $dh(f) = g(f)dz$. If $g$ is algebraic and $g^{-1}(0) = \text{const}$, we can define $\Delta$ as semi-algebraic. Note however that existence for a global base for $g$, does not imply existence of a global base for $h$. Let $C_1 = \{ f = c \}$ and assume $\Delta = V_1 \backslash C_1$, where $V_1 = \{ f_1 = 0 \}$ and $f_1 = \tau f - f$. Let $\Delta_0$ be $\Delta \backslash \{ x_0 \}$, where $x_0$ is the intersection point. We can choose $g(f_1) = 0$ on $C_1$ and $g(f_1) \neq 0$ on $\Delta$. Note that if $\Delta \cup C_1 = V_1$, then $I(V_1) = I(\Delta \cup C_1)$. Assume $\Delta_0 \cap C_1 = \emptyset$, then $g \in I(V_1)$ implies $g = pq$, where $p \in I(\Delta)$ and $q \in I(C_1)$. Assuming $C_1$ oriented, we can choose $\Delta$ as locally algebraic $p_q - q = g$, where $p_q$, have one-sided zerosets. If we assume $C_2 = \{ f = c \}$ and $I_2$ the ideal of non-constant functions and $NI_2 = V_1 \cup V_2$, then $I_2N_2 \sim \text{rad } I_2$. If $V_1 \cup V_2 = \emptyset$, we can write $g = g_1g_2 \in \text{rad } I_2$. Assume $\rho$ a measure such that $\rho(I(\Sigma_1)) = \rho(I(\Sigma_2))$ and correspondingly for $\rho_2$. If $\Sigma_1 \cap \Sigma_2 = \emptyset$, then the measures can not be absolute continuous with respect to each other. If we instead consider two ideals of analytic functions $I_1 = \{ f : dh(f) = 0 \}$ and $I_2 = \{ f : f = \text{const} \}$ and the corresponding measures $\rho_1(I) = \rho(I_1), \rho_2(I) = \rho(I_2)$. Then if $0 = \rho(I_1)$ implies $\rho(I_2) = 0$, we have $\rho_2$ is absolute continuous with respect to $\rho_1$. Thus, we have existence of $f_0$ Baire (cf. [3]), such that $\rho_1(f_0f) = \rho_2(f)$ and $f \in L^1(\rho_1)$. 

\[\text{Some Remarks on Trèves’ Conjecture} 5\]
Proposition 2.1. Given an analytic symbol with first surfaces $C$, the lineality can be studied locally as transversals. Conversely, given the lineality and a normal model, the lineality approximates the first surfaces to the symbol.

Existence of lineality can be seen as a proposition of possibility to continue the symbol on a set of infinite order, that is the symbol is not reduced with respect to analytic continuation. Assume for a measure $\rho$, $\rho(T \varphi) = \rho(\varphi^*)$, for $\varphi \in L^1$, on an algebraic set and $\rho(T \varphi) = 0$ implies $\rho(\varphi^*) = 0$, then we have existence of $\varphi_0$ Baire such that $\rho(T \varphi) = \rho(\varphi_0 \varphi^*)$

We know that (cf. [19]) every form $\sum_j B_j dx_j$ invariant relative closed contours, has the representation $\int \sum_j B_j dx_j = \int dW + \int \sum_j c_j dx_j$, where $dW$ is exact and the last integral is an absolute invariant. The argument can be repeated for our ramifier and $\int dV = \int B(dx_T, dy) - B(dx, dy)$ with $V(x, y) = W(r^*_T, x, y) - W(x, y)$ and $dV$ exact. We have assumed that the ramifier is close to translation and we have the following explanation of this. Assume $\sum_j F_j dx_j$ invariant in the sense that $\sum_j \int F_j(r^*_T x) dx_j = \int F_j(x) dx_j$ and assume that $T$ is translation. Let
\[
dK_T = \sum_j \left[ F_j(x)(r_T \zeta) - F_j(x)(\zeta) \right] dx_j
\]
\[
dL_T = \sum_j \left[ F_j(x)(\tau \zeta) - F_j(x)(\zeta) \right] dx_j
\]
We can prove that over regular approximations, we have that $\int dK_T + \int \sum_j C_{T,j} dx_j \sim \int dL_T + \int C'_{T,j} dx_j$, for constants $C_{T,j}, C'_{T,j}$.

3. Involution

3.1. Introduction. Given a multivalued surface, a canonical approximation is the spiral Puiseux approximation, but some results require a tangent determined, why we prefer transversal approximations. Sufficient conditions for existence of transversals are discussed in connection with the lifting principle.

We note that assuming polynomial right hand sides, for the associated dynamical system, monotropy is microlocally invariant. That is since a bounded set can not contribute as lineality, obviously $\epsilon$ translation does not affect this proposition. In this case monotropy (cf. [5]) corresponds exactly to adding a small constant (the value in the origin to a polynomial) to the symbol in the infinity. For analytic right hand sides, the two monotropy concepts are no longer equivalent, but the microlocal invariance can be proved for both separately.

3.2. Trèves curves. Assume $\langle,\rangle_1 = \text{Re } < y_T, y_T > -1$ for $\gamma_T \in \Gamma$. If $\Gamma_T$ describes a line, we have that $\gamma_T^\perp$ describes a line. Let $\Sigma = \{ \gamma_T : \frac{d}{d\gamma_T} F_T = F_T \}$ and $\Sigma_0 = \{ \gamma_T : \frac{d}{d\gamma_T} F_T = 0 \}$. Let $F_T'$ be the transposed operator to $F_T$ with respect to $\frac{d}{d\gamma_T}$, that is $F_T' = \frac{d}{d\gamma_T} F_T = F_T$ on $\Sigma$, $F(\frac{d}{d\gamma_T} \gamma_T = \gamma_T) = 0 \iff \frac{d}{d\gamma_T}(F_T' - F_T) = 0$. Let $A = T\Sigma = \{ \gamma_T : \frac{d^2}{d\gamma_T^2} F_T = \frac{d}{d\gamma_T} F_T \}$. If for every $\theta_T \in T\Sigma$, we have $\langle \frac{d}{d\gamma_T} \gamma_T, \theta_T \rangle > 1$ we have that $\frac{d}{d\gamma_T} F_T \in \text{bd } A$. Further, $\frac{d}{d\gamma_T} \frac{d}{d\gamma_T} F_T = \frac{d}{d\gamma_T} F_T' \frac{d}{d\gamma_T} F_T$ over $A$, why $F_T' = dF_T / d\gamma_T$. Thus, for instance $F_T' (\gamma_0) = 0$, where $F_T(\gamma_0)$ is constant. If $F_T'$ maps $A^\perp \to A^\perp$ over $\Sigma$, we have that $\langle \frac{d}{d\gamma_T} F_T(\gamma), \theta_T \rangle > 1$, $\frac{d}{d\gamma_T} F_T(\gamma) \in A^\perp$ and $\langle F_T(\gamma), \theta_T \rangle > 1$. A sufficient condition for $F_T$ to map $A^\perp \to A^\perp$ is that $\langle F_T(\gamma), \theta_T \rangle > 1 = \rho_T < \gamma_T, \theta_T > 1$, where $\rho_T$ is a function, not involving any differentials (a multiplier). The proposition is that $\gamma_T^\perp(\text{bd } A) \Rightarrow \gamma_T^\perp A$, which
can be fulfilled if A is on one side locally of a hyperplane. If the symbol ideal is symmetric and finitely generated over a pseudo convex domain, this can be assumed.

Assume $\Phi \perp \text{bd} \Sigma = \{ F_T(\eta) = c \}$, for a constant $c$, implies $\Phi \perp \{ \eta \geq c_1 \}$ (a semi algebraic characteristic set $\Sigma$). If $< F_T(\eta), \phi >= C_T < \eta, \phi >= 0$. Assume further that $F_T(\eta) = c \Leftrightarrow \eta = c_1$, for constants $c, c_1$, why $F_T$ maps $\Sigma \rightarrow \Sigma$. We know that if $\eta_T^* = y_T^*/x^*$ with $x^*, y_T^*$ polynomials and $\int_\Sigma \eta_T^* dxdx^* = 0$, then $\Sigma$ has measure zero. In the same manner for $\eta_T$. Assume bd $\Sigma = \{ \text{the set where } \eta \text{ changes sign} \}$ and where $\Sigma$ has measure $> 0$. If we have existence of $\gamma \perp \Sigma$ holomorphic, we must $\gamma \equiv 0$ on $\Sigma$, through Hurwitz theorem and we conclude that there can not exist an algebraic $\gamma \perp \Sigma$ with these conditions. The conclusion is that if $\gamma \perp \text{bd} \Sigma$, we can not have, for $\gamma$ algebraic, that $\Sigma$ stays locally on one side of a hyperplane. More precisely, if there are $2m$ characteristics through a singular point (cf. [3]), where $m$ is referring to the order of $X, Y$ in the associated dynamical system, and if the sign is changed passing the characteristics, then the set of for instance positive sign is not separated by a hyperplane. By giving the characteristics a direction however, the problem can be handled. Assume $\Sigma_3 = \Sigma_1 \cup \Sigma_3$, the domain for positive sign and that $\eta$ is an algebraic characteristic with $\int_\Sigma \eta dxdy = 0$, then $\eta = 0$ either on $\Sigma_1$ or $\Sigma_3$, depending on which direction $\eta$ has. We can thus have half-characteristics $\eta \perp \Sigma_1 \cup \Sigma_3$ with algebraic definition.

$F_T$ is said to be reduced for involution, if given existence of a regular approximation $G_T$ in (I) with $(I) = \ker H_T$, we have $\{ F_T, G_T \} = 0$ on $S_T$ implies $T = 0$. In this case there are no level surfaces to $F_T$ on $S_T$. Over reduced $x$, we have that $r_T x = x$ implies $r_T - id$ is only locally algebraic. If $r_T$ is algebraic in $T$ with minimally defined singularities, then $r_T - id \sim 0$ a polynomial.

**Boundary condition 3.1.** The boundary is characterized by the condition that $F_T$ is holomorphic in $T$, for $T \not\in \Sigma$ or $\frac{dF_T}{dT}$ holomorphic in $T$, for $T \not\in \Sigma$, where $\Sigma$ is given by locally isolated points and the regularity is close to the boundary.

More precisely, let $\Sigma = \{ \zeta_T : F_T = \text{const.} \}$. as previously $(I_1) = \{ \gamma_T : F_T(\gamma_T) \text{ is not constant } \}$, where $F_T$ is assumed holomorphically dependent on the one dimensional parameter $T$. Let $(I_2) = ((I_1) \cap (\frac{dF_T}{dT}(I_1)))$ and $N(I_2) = V_1 \cup V_2$, where $V_1 = \{ \zeta_T : F_T \text{ is not constant } \}$ and $V_2$ $= \{ \zeta_T : \frac{dF_T}{dT}$ is not constant $\}$. Using the Nullstellensatz, we can form $IN(I_2) \sim rad(I_2)$ and we claim that $(I_2)$ is radical. The condition can be generalized to higher order derivatives.

**Lemma 3.2.** The condition that $F_T$ is not reduced for involution means that there exist Trèves curves in $S_T$.

**Proof:**
Assume for this reason that $T \not= 0$ and that there exist $\gamma_T \subset S_T$ such that $\{ G_T, F_T \} = 0$ over $\gamma_T$, where $F_T$ is a lifting function and $G_T$ is a regular approximation of a singular point. Assume $G_T(\gamma) = G(\gamma_T)$ and $\frac{dF_T}{dT} G(\gamma_T) = G(\frac{dT}{d\gamma})$ with $G_1$ invertible over $\frac{dT}{d\gamma}$. Assume existence of $v_T$, a regular approximation of $T^\gamma$ with $\frac{dv_T}{d\gamma} = G_1(\frac{d\gamma}{d\gamma_T})$, then $\frac{dv_T}{d\gamma}, \theta_T \geq 1 > 0$ and if $G_1 : A^+ \rightarrow A^+$ (independence of $T$ at the boundary), we see that there exist Trèves curves for $F_T$ in $S_T$. □

**3.3. First surfaces.** Consider the system $\frac{dx}{X} = \frac{dy}{Y} = dt$ and the corresponding variation equations $\frac{dx}{T} = \frac{dx}{dx} x^* + \frac{dx}{dy} y^*$ and $\frac{dx}{dt} = \frac{dx}{dx} x^* + \frac{dx}{dy} y^*$. Assume $F_T(x, y, x^*, y^*)$ a first integral to the variation equations, algebraic in $x, y$ and homogeneous of order 1 in $x^*, y^*$. It is well known that $\int F_T(dx, dy)$ is invariant integral to the given system. Conversely, if $\int F_T(dx, dy)$ is invariant integral to the
system, then \( F_T(x^*, y^*) \) is integral to the variation equation. Assume \( V \) the Hamilton function to the system, that is \( \frac{dx}{dt} = \frac{dV}{dx^*}, \frac{dy}{dt} = -\frac{dV}{dy^*} \) and \( \frac{dx}{dt} = \frac{dV}{dt^*}, \frac{dy}{dt} = -\frac{dV}{dy^*} \).

Then given that \( \{ V, V_1 \} = 0 \), also \( V_1 \) is a Hamilton function. If \( V_2 \) is a Hamilton function, we have that \( \{ V, V_2 \} = 0 \) and \( V_1, V_2 \} = 0 \). We will consider an involutive set \( S_T \) such that for \( F_T \) a lifting function and \( V \) a Hamilton function, \( \{ V, F_T \} = 0 \) over \( S_T \). One of the most important problems in this approach seems to be existence of an inverse for \( F_T \). A sufficient condition is reducedness, but this is not suitable in connection with invariant integrals.

Assume \( V \) a Hamilton function and \( F_T \) a lifting function to the system \( \{ \gamma_T \} \) corresponding to the symbol. Further that \( G_T \) is a regular approximation (with respect to \( \frac{df}{d\gamma} \)) to the singularity in \( \{ \gamma_T \} \), not necessarily a lifting function. As \( \{ V, \cdot \} = H_V \) defines an ideal \( (I) \), we note that if \( F_T \in (I) = (I)(S_T) \) with \( \frac{dF_T}{dx} = \frac{dV}{dx} \) and \( \frac{dF_T}{dy} = \frac{dV}{dy} \) and if \( G_T \in (I) \), we have that \( \frac{dF_T}{dx} = \frac{G_T}{dT} \). We see that \( F_T \) is regular under these conditions. The proposition is that existence of \( G_T \) regular in \( (I)(S_T) \) and \( S_T \) involutive means that the lifting function with \( \{ \gamma \} \) is regular.

3.4. Continuation of the representation. Assume \( W \subset V \subset V' \) and \( A \) complex varieties and consider the mapping \( r^+ : V^+ \to A^+ \) with \( \ker r^+ = W^+ \). We then have, given \( T \in H'(A) \), there is a \( U \in H'(V') \) with \( F(U) = F(T) \) if and only if the \( F(U) \) is constant on \( W^+ \). Particularly, if \( F(T) \) has isolated singularities in the infinity, there is a continuation principle through the projection method.

Given a finitely generated system with polynomial right hand sides \( P, Q \). If the constant surface corresponding to \( P/Q \) is \( \{0\} \), then \( L_T \) is reduced with respect to contraction, that is

\[
-\frac{dL_T}{dy} \frac{dL_T}{dx} = \frac{dx}{dy} \Rightarrow T = 0
\]

. Particularly, consider \( F \) over \( \gamma \) with right hand sides \( P, Q \) and \( df_T = dF(\zeta + T) - dF(\zeta) \) for \( \zeta \in \Omega \). Then, over the lineality for \( Q/P \), \( \frac{df_T}{dT} = \frac{dy}{dx} = \frac{dy}{dx} = Q/P \).

In the same manner, if \( G \) is a different form to the same system and \( dG_T \) as above, if \( \{ f_T, g_T \} = 0 \) over \( V \), then \( df_T = dG_T \) over \( V \). If further \( \{ f_T, g_T \} \in \{ T(V) \} \), we have \( df_T = dG_T \) on \( V \), for a constant \( c_T \). Note that there is an ideal \( J \) such that \( \text{ran} J \sim_m \{ T(V) \} \). If \( f_T = F(x + iT, y) - F(x, y + iT) \) and \( g_T = f_T^* \), we have for an involutive set, that \( df_T = dG_T \). If we have \( f_T = g_T \) in \( L^2 \cap H \) we know that \( g_T = f_T^* \).

Obviously, \( H_T \) defines a functional in \( H^+ \). If \( \int_S H_T(f_T)dx\,dy = 0 \), we have either \( H_P(f_T) = 0 \) on \( \Sigma \) or \( \int_S dx\,dy = 0 \).

3.5. Continuous ramification. We are assuming the ramifier defines a regular covering \( (2) \), that is we are assuming \( \Psi : (I)(\Omega) \to (I')(\Omega), \) where the first is a Hausdorff space, \( \Psi \) is continuous, proper and almost injective (singular points are mapped onto a discrete set (subset of transversals) in \( \Omega \)). We write \( r_T' \) for \( \Psi \), \( N : (I)(\Omega) \to \Omega \) and the ramifier is the lift \( \Omega \to (I')(\Omega), \) such that \( r_T' I(\Omega) = I(r_T' \Omega) \). Denote the critical points to \( r_T \) with \( A \) and assume that they are parabolic. We assume \( \Psi \) such that \( I(r_T A) \) is nowhere dense in \( (I')(\Omega) \) and so that \( \Psi \) is locally a homeomorphism outside critical points. Finally, we are assuming that for all \( \gamma \in (I')(\Omega), \) there is a small neighborhood \( U_\gamma, \) open and arc-wise connected, such that the \( U_\gamma \setminus I(r_T A) \) is arc-wise connected. Wherever \( \Psi \) is holomorphically dependent on the parameter, the inverse \( r_T \) will be assumed continuous outside a discrete set.
If for instance \( A = \{ \zeta \mid d_{\zeta}f(\zeta) = f(\zeta) = 0 \} \), we are studying points \( \zeta_T \) that can be used to reach \( A \) from \( \{ f = c \} \), for a constant \( c \).

Assume \( \Omega_0 \subset \Omega \), where \( \Omega \) is assumed a pseudo-convex domain. Assume \( U \) an open set such that \( \Omega_0 \subset U \subset \Omega \). Assume \( T(= B_T) \) an analytic functional, \( T \in H'(\Omega) \), quasi portable by \( \Omega_0 \), that is we have existence of \( u \in H'(U) \) with \( T = i_{U,V}(u) \) (restriction homomorphism). Let \( \Omega_0 \subset \bigcup_{j=1}^{N} U_j \), for open sets \( U_j \subset \Omega \) and \( T = \sum_{j=1}^{N} T_j \), that is we can write \( T_j = i_{U_j,\Omega}(u_j) \) with \( u_j \in H'(U_j) \). Assume now the restriction homomorphism algebraic, then we have if \( \Omega_0 \) is complex analytic in a real analytic vector space, that \( T \) is portable by \( \Omega_0 \) (cf. [12] Ch.2, Section 2).

Assume \( h \) algebraic and let \( v_T(x) = h(r'_T x) - h(x) \), where \( r'_T \) is a continuous linear mapping \( d(r'_T f)/dx = df/dx \) and we write \( f(r_T \zeta) = r'_T f(\zeta) \). Let \( \Delta = \{ T \mid f(r_T \zeta) = f(\zeta) \ \forall \zeta \} \). Over an ideal, finitely generated and of Schwartz type topology with (weakly) compact translation (cf. [12]), there are given \( \{ r_T \zeta \} \) regular such that \( T(f(\zeta)) \zeta \) is algebraic in \( \zeta \), for constants \( 0 \neq f(\zeta) \to 0 \) as \( j \to \infty \). The sets \( \{ v_T = 0 \} \) will not contribute micro locally, however the sets \( \{ v_T = \text{const} \} \) contributes to invariance in the tangent space and gives a micro local contribution.

Assume \( L \) an analytic line, transversal in a first surface \( S_0 \) through \( p_0 \) and consider a neighborhood \( \Gamma \) of \( p_0 \) on \( L \). Denote \( \Sigma_T \) the set of points that can be joined with a point in \( \Gamma \), through a first surface to \( f \). We assume \( L \) transversal to every first surface through \( \Gamma \) of order 1. Transversality means existence of regular approximations. We will in this approach not assume minimally defined singularities. If for a first surface \( S \), we have \( S \cap \Sigma_T \neq \emptyset \), we have \( S' \subset \Sigma_T \), for all \( S' \sim S \) (conjugated in the sense of [15]). Thus for a generalization of the inhomogeneous Hange’s result, it is sufficient to consider the normal tube. That is if \( \Gamma \) gives a micro local contribution in \( p_0 \), then if \( S_{\perp} \) (transversal) has \( S_{\perp} \cap \Sigma_{\perp}(u) = \emptyset \), we have \( S' \subset \Sigma_T \), so that \( \Gamma \in S_{\perp} \). Note however that it is necessary for micro local contribution, that the set is not bounded globally.

### 3.6. The condition on involution

First a few notes on the lifting principle. Assume \( \gamma \in \mathcal{P} \), an analytic polyeder. It is not true that the lifting principle holds over every \( \mathcal{P} \), but by constructing a normal model \( \Sigma \) ([16]) to \( \mathcal{P} \), we have always (modulo monotropy) a lifting function. Let \( \Omega = \{ \zeta \mid r_T^* \gamma(\zeta) \in \Sigma \ \forall \zeta \in \mathcal{P} \} \). By the definition of the ramifier \( r_T \), \( \Omega = \{ r_T \zeta \mid \gamma(\zeta) \in \Sigma \ \forall \zeta \in \mathcal{P} \} \). We assume \( \gamma_T \) (real-) analytic on \( V \times \Omega \supseteq (T, \zeta) \). For \( \zeta \) fix in a neighborhood defined by \( T, F \) can be chosen holomorphic. Let \( \mathcal{P} = \{ \gamma(\zeta) \mid \gamma \text{ holomorphic in } \zeta \in \Omega \} \). Then, if we assume \( \mathcal{P} \) finitely generated over \( \Omega \) and \( r_T^* \mathcal{P} = \Sigma \), we get a corresponding \( \Omega = \{ r_T \Omega \} \) and \( f(r_T \zeta) = F(\gamma_T)(\zeta) \) for \( \zeta \) fix, can be extended to the domain for \( f \), in a neighborhood of a first surface. Thus, the construction is such that \( \Omega \) is a neighborhood of \( \{ T = 0 \} \) and \( \zeta \) in a first surface, why we have existence locally of a lifting function for a normal model.

The condition on involution gives existence of the inverse lifting function \( G_T = F_T^{-1} \). We are now interested in determining the domain where \( G_T \) is constant, algebraic, holomorphic etc. Note that if \( G_T(f) \) is algebraic in \( f \) and \( f \) is the symbol to a hypoelliptic operator, then in the real space, \( G_T(f) \in \mathcal{B}_{\mathbb{R}} \). Assume existence of \( G' \), derivative with respect to argument, then from the regularity conditions for the dynamical system, \( G_T(f) \) has isolated singularities and if \( G' \) holomorphic or constant, we must have isolated singularities for the symbol \( f_T \). Consider \( (I)_+ = \)}
\( \gamma_T \) \( F(\gamma_T) = F(\gamma) \) \( \text{Im } T > 0 \) and correspondingly \((I)\). Assume \( F_T \) algebraic in \( T \), then the signs will give an orientation to the first surfaces. Thus, \((I)_+ \) will correspond to conjugate classes of first surfaces (14). For instance in case \( F(\gamma_T) = F(\gamma_T) \), we have the same first surface in \((I)_+ \) but different orientations. We will assume the number of classes constant, when \( \text{Im } T \) is small (compare with the regularity conditions (3)). The regularity for \( G_T \) will now determine the character of the first surfaces. Regular first surfaces, for instance have only trivial conjugates, which will be the case if \( G_T \) is reduced. We have noted that all normal approximations can be chosen regular.

Consider the symbol \( F = PF_0 \) with \( P \) a polynomial, \( F_0 = \tilde{f}_0 \) and \( F_0 \) holomorphic or monotropic with a holomorphic function. Let 
\[
(I)_{\Lambda} = \{ f_0 \in (B_m)^{\prime} \mid PF_0 = 0 \text{ on } \Omega \}
\]
If \( \Lambda = Z_P \) (zero-set), we have that \( F_0 \upharpoonright \Lambda \) implies \( f_0 \in (I)_{\Lambda} \). Conversely, if the polynomial \( P \) is reduced and \( | PF_0 | < \epsilon \) at the boundary for a small number \( \epsilon \), then the Nullstellensatz (17) gives that \( F_0 \) is bounded by a small number at the boundary.

### 3.7. The lifting principle

Assume the right hand sides to the associated dynamical system \( X,Y \) are polynomials in \( \zeta \), then according to the lifting principle (cf. 17), we have on \( |X| \leq 1, |Y| \leq 1 \), existence of a function \( F \) holomorphic in \( x,y \), such that \( f(\zeta) = F(x,y)(\zeta) \). If \( \zeta \) is in a polynomially convex and compact set, \( f \) can be represented as a polynomial. Assume \( \vartheta = Y/X \) and \( \eta = y/x \), for polynomials \( X,Y \). Further, for constants, \( c,c' \), \( |\vartheta - \eta| > c \) and \( |\eta| < c' |\eta\vartheta-1| \) locally. We can determine \( w \) algebraic and locally maximal, such that \( |w\vartheta-1| < 1 \). For \( \eta\vartheta \sim_m w\vartheta \), we have existence of a holomorphic function \( F \), such that \( F(\eta)(\zeta) \sim_m f(\zeta) \) and \( F(\eta) = \text{const.} \Leftrightarrow \eta = \text{const.} \). If \( F \) is invariant for monotropy, the result \( F(\eta) = f \) follows directly from the lifting principle. Assume \( \mathcal{P} \) an analytic polyeder with separation condition (cf. 17), \( \mathcal{P} = (x,h(x)) \). Assume \( \Sigma = \{ |z_j| \leq 1 \} \) and \( \Delta^\epsilon = \{ |z_j| \leq 1 + \epsilon \} \), close to \( \Sigma \), for \( j = 1,2 \). Assume \( \Sigma = \Phi(\mathcal{P}) \), such that \( \Phi(\delta \mathcal{P}) \subseteq \delta \mathcal{P} \) (conformal) and that \( \Sigma \) is an analytic set with continuation in \( \Delta^\epsilon \). Then, \( \Sigma \) is a (normal) model for \( \mathcal{P} \). Assume \( f \) analytic on \( \mathcal{P} \), then we have existence of \( F \) holomorphic on \( \Sigma \), such that \( f(\zeta) = F(\Phi(\mathcal{P}))(\zeta) \). Note that we are assuming \( \zeta \) in a symmetric neighborhood of \( \{ T = 0 \} \). We can, according to Rouché's principle assume, \( |z_j - w_j| < \epsilon |z_j| \) and \( |w_j| \leq 1 \), for \( z_j \sim_m w_j \). For \( w_j \) polynomials, this is a proposition on \( F \) being invariant for monotropy.

### 3.8. Exactness and involution

We will use the Poisson bracket \( U,V = \sum_i \frac{\delta U}{\delta x_i} \frac{\delta V}{\delta y_i} - \frac{\delta V}{\delta x_i} \frac{\delta U}{\delta y_i} \). Assume \( V \) defined through \( \frac{\delta V}{\delta y_i} = \frac{\partial P}{\partial x_i}, \frac{\delta V}{\delta x_i} = -\frac{\partial P}{\partial y_i} \). Concerning the two possibilities for \((x_i,y_i) \) where \( i = 1,2 \), A) \((x,y,x^*,y^*)\) B) \((x,x^*,y,y^*)\) it does not appear to be important what representation we use.

Consider the sets \( \Phi_\vartheta = \{ e^{\vartheta} M = W \} \) and analogously for \( \Phi^* \). Thus, \( F^\vartheta(M) = F(e^{\vartheta} M) \). Assume over an involutive set that \( \exists F^{-1} \) and let \( G = F^{-1} F^\vartheta \) over \( M \). Then, \( G(M)/M = e^{\pm \vartheta} \). We will study the parabolic sets \( \pm \vartheta < 0 \), so that \( G(M) = \text{const.} M \). The spectrum is \( \{ e^{\vartheta} M = W \} \), then for a lifting operator \( F \), invertible and over \( \vartheta < 0 \), we have \( F^\vartheta(M) = \text{const.} F(M) \), if the constant is real, we have real eigenvectors. There will be a boundary in this approach, given by the set where \( \vartheta \) changes sign. Finally, we consider the sets where \( \vartheta > 0 \) (real and holomorphic). If the underlying sets in \( \Omega \) are simply connected, these sets
constitute neighborhoods of the constant surfaces. If we consider $F$ as an analytic functional, we have that $F$ has the closure of $\{ \theta < \alpha \}$ as semi-porteur if and only if the type for $\hat{F}$ is $\leq \alpha$, which particularly means that it is portable by any convex neighborhood of the semi-porteur.

3.9. Dependence of parameter. Given a closed trajectory, that does not end in a singular point $P$, that is the point $P$ stays inner to the trajectory. The point $P$ is called a center, if there are infinitely many closed trajectories, arbitrarily close to $P$, that circumscribes the point. We could say that the trajectory $\gamma_T \to \gamma_0 = P$, but does not reach it. We will assuming the boundary not $C^1$, but holomorphic and with only parabolic singularities, consider the problem of removing the center point as a Dirichlet problem.

There are certain conclusions on the singularities in $\Omega_1$, given the dependence of the parameter in $L^1$. We have the following weak form of minimally defined singularities. For $F = w \in B_m$, if $x, y \in B_m$ and $\int_I w_T(x, y) d\sigma \to \int_I w(x, y) d\sigma$, through a normal and regular approximation. Assume that the dependence of $T$ is holomorphic and $w$ algebraic in $(x, y)$. We have that $\{(x, y) \mid w_T(x, y) = w(x, y)\}$ has $\sigma$- measure zero. Assume that $\int |w|^2 d\sigma < \infty$ and $ww^* = w^* w$ and that $(x, y)$ is in the normal tube. Then we have normal and regular approximations, say $g_T$ of $\{x = \text{const.}, y = \text{const.}\}$. Assume $f_T \to 0$, as $T \to 0$ normally and regularly, such that $\frac{df_T}{dT}$ is holomorphic in $T$ (that is not a non-zero constant). If $\frac{df_T}{dT} = C \frac{dw_T}{dT}$, for a constant $C$ on a domain of positive measure, we still have a regular approximation. If $\gamma_T = (f_T, h(f_T))$ is the regular approximation and if the dependence of $T$ is algebraic in $\frac{df_T}{dT}$, then according to Hurwitz theorem, since polynomials never have zero-sets of infinite order, then the zero-set must have measure zero. Thus, given existence of regular and normal approximations, where we assume algebraic dependence of the parameter $T$, in the tangent space, then all normal approximations, algebraically dependent on the parameter in the tangent space, can be assumed regular (at least after adding a regular approximation).

**Proposition 3.3.** Assume $F_T$ with $L^1$- dependence in the parameter and existence of a normal and regular approximation algebraically dependent of the parameter in the tangent space, then all normal approximations, algebraically dependent of the parameter in the tangent space, can be chosen as regular.

Note that when the parameter is with respect to the ramifier, we assume algebraic dependence over transversals and tangents. There are numerous examples where $(dI)$ has a global (pseudo-)base, but not $(I)$. Finally, note that of $\Omega_1(dI) = \{T \mid r_T^1 F_1^1 = F_1^1, F_1^1 \in (dI)\}$ and $\Omega_2(dI) = \{T \mid r_T^1 F_2^2 = F_2^2, F_2^2 \in (dI)\}$, where $F \neq 0$ we have that $T \in \Omega_2 \Rightarrow T \in \Omega_1$ if $r_T^1$ is algebraic in the sense that it is geometrically equivalent with a polynomial. Assume all approximations of a parabolic singular point are on the form $\eta_T = \alpha_T e^{\tau_T}$, since we know that all normal approximations are regular, we can assume the singularities for $\alpha_T$ simple. Assume $\eta_T(x^1) \sim_m \eta_T^p(x)$, then it is sufficient to consider the case where $\eta_{1,T} = \frac{d\sigma_T}{dT}$ has isolated singularities. Since $\mid e^{-\sigma_T} \eta_T^p \mid < M$ as $\mid x^1 \mid \to \infty$ implies $\mid e^{-\sigma_T} \eta_T^p \mid < M$, as $\mid x^1 \mid \to \infty$. We will see that monotropy is a micro local invariant, this means that it is sufficient to consider parabolic approximations for $\eta^*$. Note that presence of lineality for the symbol, may result in $\text{Im} F$ in the space of hyperfunctions. We now note that if $F$ is symmetric, entire and of finite type in $E \chi \rho$, then the condition that $f$ represents a hypoelliptic operator, means that for some $\lambda$, $(\text{Im})^\lambda = \sum A_j F_j$ on a domain of holomorphy, for constant coefficients.
and a global pseudo-base representing the ideal of hypoelliptic operators. Thus, symbols to hypoelliptic operators do not have imaginary part outside the space of distributions and if hyperfunction representation is necessary, we must have contribution of lineality in the infinity.

3.10. A generalized Cousin integral. We denote with $\tilde{M} = -Y dx + X dy$ and correspondingly for $\tilde{W}$ Assume $\tilde{M}$ exact and $\tilde{W}$ closed, then the form corresponding to $\tilde{M}$ is exact after analytic continuation and in the same manner for $\tilde{W}$. Note however that the forms corresponding to $\tilde{M}$ and $\tilde{W}$ are not locally holomorphic, that is we do not have locally isolated singularities and the center case could appear.

Assume $\mu$ is a positively definite measure and consider

$$
\Phi_\mu(d\gamma) = \int_{P_\mu=0} d\mu(\gamma)
$$

where $P_\mu$ is a polynomial and gives a local definition of $\Delta$. Approximating a singular point through $d\gamma \to 0$, then either $\Phi_\mu(d\gamma) \to 0$ or we have existence of a point support measure $\mu'$ such that $[\Phi_\mu + \Phi_{\mu'}](d\gamma) \to 0$. Thus, for the measure corresponding to a hypoelliptic operator, we can choose $\mu$ with point support. Assume $\Phi_\mu(d(\gamma T - \gamma_0)) = \int_{\gamma T - \gamma_0} d\mu$. If $d\mu$ is a reduced measure, we must have $\gamma T = \gamma_0$. We know that if $d\mu$ is holomorphic (that is holomorphic coefficients), then $d\mu$ will be reduced, for $T$ close to 0. Assume $d\mu$ continuous and locally bounded, for all $T$ and that $d\mu = d\mu + d\mu_0$, where $d\mu_0$ is assumed with point support and $d\mu$ is holomorphic. Then $\int_{\gamma T - \gamma_0} d\mu = 0$ implies $\gamma T = \gamma_0$. Assume $\gamma T$ a closed contour and $\gamma_0$ a point, then for $T$ not close to 0, we have $\int_{\gamma T - \gamma_0} d\mu = 0$, implies $\gamma T \neq \gamma_0$.

This case includes the case with a center (cf. [3], Theorem 4).

4. Stratification

4.1. Introduction. If we consider a hypoelliptic analytic symbol $f$ as locally reduced, it is naturally necessary to use a stratification to define a globally hypoelliptic symbol. The model is centered around the set of lineality and we are always assuming the lineality locally is a subset of a domain of holomorphy, which means that its local complement set is analytic. We consider it to be necessary for the concept of hypoellipticity to have an approximation property for $\log f$. We will discuss an interpolation property. Further, it is necessary to have a concept of orthogonality between the real and imaginary parts of the symbol.

4.2. The arithmetic mean. For the arithmetic mean, we have that

$$
\lim_{c \to 0} \int_{C_c} MV dz(T) = MV(0)
$$

given that $MV$ is holomorphic, regularly that is without a porteur (cf. [12]). If for all closed contours $\int_{C_c} MV dz(T) = MV(0)$ implies $C_c = \{0\}$, then $MV$ is reduced for analytic continuation. If $\int_{C_c-0} MV dz(T) = 0$ for all closed contours in a leaf $\mathcal{L}$, then the form $MV dz(T)$ is closed in $\mathcal{L}$ and we have a mean value property above for the arithmetic mean in $\mathcal{L}$. Further, the closed contour $C_c \sim 0$ on $\mathcal{L}$.

4.3. The concept of stratification. Assume $X \subset Y$ are separable topological vectorspaces. We say that $Y$ is a stratifiable space if it has the property that to any open set $U$ we associate a sequence $\{U_j\}_{j=1}^\infty$ of open sets in $X$, such that

- $U_n \subset U$ for all $n$
- $U = \bigcup_{j=1}^\infty U_j$
- $U \subset V$ implies $U_n \subset V_n$ for all $n$
Further, (cf. [4]) given a topological vector space $X$ and with $Y$ as above, we can associate a topological vector space $Z(X)$, such that $X$ is closed in $Z(X)$. We say that $X$ is locally RA (retractible), if $X$ has a local extension property with respect to the stratification. Particularly, if $\Gamma$ is closed in $Y$ and $f$ is a continuous mapping $\Gamma \to X$, we have existence of $f$ that maps $Y \to Z(X)$.

4.4. A stratification using averages. A topological vector space $X$ is stratifiable, if for any open set $U$, there is a continuous mapping $f_U : X \to nbhd$, such that $f_U^{-1}(0) = X - U$ and if $U, V$ are open sets with $U \subset V$, we have $f_U \leq f_V$. We will for this reason study the averages $\mathcal{M}_1 \geq \mathcal{M}_2 \geq \ldots \geq \varphi$, where the boundary $\mathcal{M}_j = \varphi$ is common for all the averages and where $\mathcal{M}_j \to 0$, as $\varphi \to 0$. Let $F_1 = \{\mathcal{M}_1 \geq \varphi\}$ and let $f_1$ be a continuous function such that $\ker f_1 = \text{bd } F_1$ and $f_1 = \mathcal{M}(\varphi) - \varphi$. If $\mathcal{M}$ is holomorphic and $\mathcal{M}(x^0) = \lim_{r \to 0} \int_{C_r} \mathcal{M}(x)dx$ and if $C \subset \text{bd } F_1$ and $\varphi(x^0) = \lim_{r \to 0} \int_{C_r} \varphi(x)dx$. Further, $f_2(\varphi) = \mathcal{M}_2(\varphi) - \varphi$ with $\ker f_2 = \text{bd } F_2$, where $F_2 = \{f_2 \geq 0\}$ why $F_2 \subset F_1$ and $f_2 \leq f_1$, and so on.

A stratification of $\hat{B}$ can be mapped into a stratification of $B_m$, through $i_a : \hat{B} \to B_m$ and $i_a(\varphi) = \varphi + a$, for a constant $a$. This is a compact mapping with $i_a(\varphi_j - a) = i_a(\varphi_j) - a$. $(B_m)$ stratified in this manner with topology induced of Schwartz type is $\mathcal{FS}$, why the dual space $(B_m)'$ is $\mathcal{DS}$ (cf. [12]).

4.5. The arithmetic mean and duality in $L^1$. Assume $F_1 = \{\mathcal{M}(\varphi) \geq \varphi\}$ and $f_1 = \mathcal{M}(\varphi) - \varphi$ and $\Gamma = \{f_1 = 0\}$. If we assume $\mathcal{M}(\varphi)$ holomorphic, we must have that $\text{Flux}(\mathcal{M}(\varphi)) = 0$. Note that $\mathcal{M}(\varphi^\circ) = \mathcal{M}(\varphi)^\circ$ in $L^1$ and $\mathcal{M}(d\varphi) = d\mathcal{M}(\varphi)$. Thus, given a dynamical system with right hand sides harmonic conjugates, satisfying the regularity conditions, we see that the arithmetic mean satisfies a condition on vanishing flux. If $\Gamma$ is always reduced to a dynamical system considered in $L^1$, the boundary problem is solvable in $L^1$.

Assume now that the boundary value problem is solvable for $MV(\varphi)$, that is we assume $\Delta MV(\varphi) = 0$ on an open set $\Omega$. Using duality with respect to the scalar product in $L^1$, we consider

$$0 \to \phi \to MV(\phi) \to \Delta MV(\phi) \to 0$$

$$0 \leftarrow MV^{-1}(\Delta \sigma) \leftarrow \Delta \sigma \leftarrow \sigma \leftarrow 0$$

We are thus assuming $\Delta \sigma \in L^1$ with $\sigma \in L^1$. Let $E = \Delta L^1$ and $X = \{\phi \quad \text{such that } MV(\phi) \in E\}$, that is for a $f \in L^1$, we have $\text{supp } MV(\phi) = \Delta f$ in $L^1$. More precisely, we can describe $\Delta L^1 = E$ through the closure of $(M, W)$ with respect to $L^1$. Assume $\Phi \perp MV(W)^\perp \text{ and } \Phi \perp MV(W)$ in $L^1$. Assume $\Phi$ with support in a bounded neighborhood of the boundary (restriction to strata). The relations will then also hold in $L^2$ and we can apply Weyl’s lemma to conclude $\Phi \in C^3$ locally. Assume in a neighborhood of the boundary that $0 = |(\varphi, MV(W)) |$ | $(\varphi, W ) |$. Thus, if the problem is solvable for $MV$, it is solvable for $(M, W)$, given the inequality above. We now have $\varphi \in C^3$. The parametrix to the problem then has a trivial kernel and the problem is solvable.

**Proposition 4.1.** The arithmetic means applied to $f$ (and log $f$) form a stratification over $(B_m)'$ associated to $f$ in a finitely generated symmetric ideal of analytic functions over a pseudoconvex domain with transversals given by a locally algebraic ramifier. We have assumed parabolic singularities and no essential singularity in the infinity.
4.6. Reduction to tangent space. Assume $F \sim V_1 + iV_2$, and consider the condition

$$\frac{d}{dx} \log V_1 \text{ reduced and } \frac{d}{dy} \log V_1 \text{ reduced}$$

Given that $V_1 + iV_2$ is hypoelliptic with $\vartheta = \log V_1$, we have that if the property \([2]\) holds for $M_N(\vartheta)$, then it also holds for $\vartheta$. Note also that if $M \perp W$ with $TW = 0$, then we cannot conclude that $TV_2$ has vanishing flux. However, if the condition \([2]\) is satisfied for $V_1$ and $M \perp W$, we can conclude that $V_1 \perp V_2$. Let $\tilde{M} = Xdx + Ydy$ and $\tilde{W} = dF$. Then we can consider $F^\circ$ defined as $\frac{dF}{dx} = \frac{dF^\circ}{dy}$ and $\frac{dF}{dy} = -\frac{dF^\circ}{dy}$, so that $\tilde{M} = dF^\circ$. If the involution is taken over $F, V, G$, where $V$ is the Hamilton function, $F$ is the lifting function and $G$ is a regular approximation, then we can relate the involutive set to a condition $\int_{C_i} dF = 0$. Assume $F_N$ corresponds to $M_N(F)$ and $C_N$ is the corresponding contour, such that $\int_{C_N} dF = \int_{C_N} dF_N$ and $C_N \subset C_0 \subset C_{-N}$. Then, the conclusion is that the stratification of negative order is a covering of the involutive set.

4.7. Example. Assume for instance that $V = V_1 + iV_2 + \Delta$, such that $\frac{\partial}{\partial x} \Delta = \frac{\partial}{\partial y} \Delta = 0$ and where $\Delta$ is defined through involution and through the conditions $N(V_1 \perp V_2) = \frac{\partial}{\partial y} (V_1 \perp V_2)$. Hypoellipticity means that $\text{supp } \Delta = \{0\}$.

4.8. The lineality as closed contours. The lineality has a pre-image in the contour $C_T$ in the following manner. Let $f_T = e^{\vartheta_T}$ and assume $\vartheta_T - \vartheta \equiv 0$ and $\zeta_T \in \Delta$ locally (lineality). Assume $C_T = F^{-1}\{ \vartheta_T - \vartheta \}$ describes a simple contour with an analytic parametrization, then on $C_T, M_{-N}(\vartheta_T - \vartheta) \leq \vartheta_T - \vartheta$. Assume $\Delta = \{ \vartheta_T - \vartheta \equiv 0 \}$ locally analytic, then we have locally $I(\Delta) = \{ \vartheta_T - \vartheta \equiv 0 \}$ $N(\Delta) = \Delta$. This means that $C_T \cap 0 \equiv M_{-N}(\vartheta_T - \vartheta)$ has a point in common with $I(\Delta)$, that is $\{ \zeta_T \in C_T \} \cap N(\Delta)$.

Consider again the problem if the zero set has points in common with $C_T$. If $M_{-N} \in D^{(k)}_L$, we can assume that the restriction of a complex operator $P(\delta_T)$ to the real space, is such that $P(\delta_T)\sigma \mid R \sim M_{-N}$ in $D^{(k)}_L$, where $\sigma \in L^1(R)$. Extend the definition of $\sigma$ (standard complexify) to $L^1(R^2)$. We can then, in a neighborhood of the boundary corresponding to the symbol, assume that the parametrix to $P(\delta_T)$ is injective, $E(P(\delta_T)\sigma(\phi)) = \sigma(\phi) - r_T$, where $r_T$ is regularizing and $r_T \to 0$ as $T \to 0$. We must assume that $\sigma$ is not identically 0, but that $\sigma_T \equiv 0$ on $C_T$.

4.9. Further remarks on the stratification. Assume a global pseudo base in the tangent space and that $F(dz) = f(z)dz$, where $f$ is given by a locally reduced function. We are assuming $F$ has no linearity in the tangent space and that $\Delta$ can be given as a semi-algebraic set. If $F \in L^1$ in the parameter, then $\frac{dF}{dz} = f(z)$ a.e. A sufficient (and necessary) condition for equality, is that $F$ is absolute continuous. For example, if $df = f_0dy$, where $f_0$ corresponds to continuation. If $f$ is reduced with respect to analytic continuation (over strata) then $f_0$ is locally reduced. If
$df, dg$ are of type 0, then the same must hold for $f_0$. If $g = hf$, then over $dh = 0$, for $f_0$ to be reduced, we must have that $h$ is minimally defined. The first relation particularly means that $F$ preserves order of zero’s if $f$ is regular, particularly $F$ maps exponentials onto exponentials. If $f$ is absolute continuous, then zerosets are mapped onto zerosets. When $F(e^x) = e^{F(x)}$, if we assume $F(\overline{\tau}) = \overline{F(x)}$. If we only have $F(e^x) = \beta e^\phi$, where $\phi(\overline{\tau}) = \phi(x)$, then $e^{-\phi(x)} f(\overline{\tau}) = [\beta(\overline{\tau}) - \beta(x)] e^{-2i \Im \phi}$. Note that reducedness for $\beta$ is not necessarily symmetric.

4.10. Condition $(M_1)$ relative the stratification. We are assuming $\mathcal{T} : x \frac{dx}{x^\tau} \to x^\tau \frac{dx}{x^\tau}$ and $x \frac{dx}{x^\tau} \neq 0$ and that the systems $(M, W)\ldots$ are regular. In particular, we assume $r_T = \frac{dx}{dx_T} = \frac{dx}{dx_T}$ and in the same manner for $y$. Further, we are assuming that $F(X, Y) \sim R(M, W)$ and $F:0 \rightarrow \Delta$ and $F^{-1}:\Delta \rightarrow 0$ locally. We are thus assuming that the lineality is defined as “independent” of the system. The stratification is formed over $(B_m)^\prime$ and is relative $\mathcal{T} W = e^{\phi_T}$, where $\pm \phi_T > 0$ on each strata. We use the following concept of condition $(M_1)$. Let $L_N(\omega_T) = \frac{1}{|T|} \int_{C_N} M_N(\omega_T) dz(T)$, where $C_N$ is a closed contour, parameterized through $T$, such that $T \to C_N \downarrow \{0\}$ as $N \uparrow \infty$ and $\frac{1}{T} \to C_{-N} \uparrow \text{bd} C$, as $-N \uparrow \infty$ and where $\rho_T$ is the radius for $C_N$, with $\rho = \rho(N, T)$. The condition $(M_1)$ is that $\lim_{N \to \infty} L_N(\omega_T)$ is regular, that is that the functional corresponding to $M_N(\omega_T)$ is of real type. Note that if $\sigma_N \in L^1$, with $|\sigma_N| = 1$, we have the same argument for $M\ldots$ and $L\ldots$. We will now argue that if $\omega_T \equiv 0$, for $T$ small, then $M\ldots(\omega_T) \equiv 0$ on the contour $C\ldots$. Let $X_N = \{\omega_T \leq M_N(\omega_T)\}$, for $N \geq N_0$, where $M_N(\omega_T) \in L^1$ and algebraic in $T$, for $T$ small. If we extend the definition of $\sigma_N$, such that $\sigma_N$ is the evaluation functional on the boundary $\{\omega_T = M_N(\omega_T)\}$, with $\omega_T \in L^1$, then if $\omega_T \equiv 0$, we can choose $M\ldots(\omega_T) = \sigma_N$ on the inner of $X_N$, why $\omega_T \equiv M\ldots(\omega_T)$, for $T$ small. If $M\ldots - I$ is locally algebraic, then $\omega_T \equiv \Gamma = \Gamma$. We are assuming the Lagrange condition $\Gamma = I(\Delta) = I(\Delta) = \Gamma$.

Assume $\omega_T \equiv \omega_{\infty} ( \sim \omega_{1/T})$, for $T$ small, on a set with complex dimension, then we must have existence of $M\ldots$ as described above, such that for $N$ large, $\omega_T \equiv 0$, or equivalently $\omega_T \equiv 0$, $M\ldots(\omega_T) \equiv 0$, according to the conditions, we have that $M\ldots(\omega_T) \equiv 0$ on $C\ldots$, for $N$ large. We are assuming that $C\ldots$ includes the real infinity, as $N \rightarrow \infty$. Conversely, if $L_N(\omega_T) \neq 0$, as $N > N_0$ implies $\omega_T$ is not $\equiv \omega_{\infty}$, in the real infinity. The conclusion is that if the stratification has condition $(M_1)$ in the infinity, it is not possible to have lineality.

Consider the limit $L_N(\omega_T) = \int_{C_N} M_N(\omega_T - \omega) dz$, where $z(T) \in C_N$ a closed contour of radius $\delta$ and let $A_N$ be the porteur set to this limit considered in $H^\prime$. Obviously, we have $A_N \subseteq \Delta$, for $N \geq 0$. Consider the stratification of $(B_m)$ with $\{X_N\}$, that is a stratification using the means $M_N$. If $L_N$ are not regular, that is $A_N \neq \{0\}$, then we have on a connected set that $\omega_T - \omega \equiv 0$. (We are assuming Schwartz type topology for the symbol space). Conversely, consider the stratification of $(B_m)\prime$ and the contours $\{C_T\}$ that contribute to $\Delta$ through common points. In this case, if $M\ldots$ are of real type, there is no possibility of lineality. Thus, given an operator with lineality, we do not have condition $(M_1)$ for $(B_m)\prime$ in the stratification using $M\ldots$.

Proposition 4.1. If the stratification that we are considering has condition $(M_1)$, that is if all the means are of real type, then the symbol ideal is locally reduced and conversely.

We will discuss two other similar topological conditions in a later section. Since it is topological, we prefer the set of lineality to characterize hypoellipticity. The
condition \((M_1)\) at the boundary, means that the boundary behavior does not influence the microlocal behaviour in the infinity. A globally hypoelliptic operator is in this context a globally defined operator that is hypoelliptic in the infinity and for which the topology for the symbol space has condition \((M_1)\) (or a similar topological condition) at the boundary.

4.11. Reduction to real type. Assume \(F\) holomorphic and of finite exponential type. Further that \(F\) has finitely many zero’s on \(X \setminus U_0\), where \(X\) is assumed a bounded domain and \(U_0\) is a neighborhood of the infinity. Further, we assume that the zero’s \(P_1, \ldots, P_n\) are isolated and of finite order. Assume \(U_1\) is a neighborhood of \(P_1\) that does not contain any other zero’s. Then we have on \(X\) a holomorphic function \(F_1\), such that \(F - F_1\) is of type 0 on \(X\) and \(F_1\) is of type \(-\infty\) on \(U_1\). The remaining \(P_j\)’s are treated in the same way. Thus, \(F - \sum_j F_j\) is of type 0 on \(X\) and each \(F_j\) is of type \(-\infty\) on the corresponding \(U_j\).

4.12. Remarks on a spectral mapping problem. The definition of the mapping \(T\) starts with \(-Ydx + Xdy \rightarrow -\hat{Y}dx^* + \hat{X}dy^*\) and we are requiring \(\{W = 0\} \rightarrow \{\hat{W} = 0\}\). We consider the multipliers \(\chi X = Y, \chi^\delta \hat{X} = \hat{Y}, \lambda H = G\) and \(\lambda^\delta \hat{H} = \hat{G}\). We assume \(T: \chi \rightarrow \chi^\delta\) and \(\{\eta = \chi\} \rightarrow \{\eta^\delta = \chi^\delta\}\). We have that \(T\) preserves parabolic points, but is usually not a contact transform. If \(T\) has the property that it maps constants on constants and exponentials on exponentials, we know that \(T\) preserves parabolic approximations. Through the condition on vanishing flux, we can assume \((w, Tw)\) pure and that \(T\) preserves analyticity.

Consider \((J) = \{f \int_I f dx(t) = 0 \hat{V}\}\), where \(\hat{V}\) is a geometric set. One of the more difficult problems in our approach is to see that the spectral mapping result we use respects the stratification, that is if starting with a stratification of \((B_m)’\) and \(\hat{W}, \{X_j^\delta\}\), we have that the sets \(\{X_j^\delta\}\) where \(TX_j = X_j^\delta\) constitute a stratification. Consider \(\Phi^\delta = \{\vartheta^\delta, e^{-\vartheta^\delta} \chi^\delta = \text{const} \ \exists \vartheta^\delta\} \) and \(\Phi = \{\vartheta, e^{-\vartheta} \chi = \text{const} \ \exists \vartheta\}\). Consider the Legendre transform \(R\), according to \(\text{RE} < R(\chi), \chi > -1\). Let \(R(e^\vartheta) = \hat{R}(\vartheta) = \hat{I}R(\vartheta) = e^{\vartheta^*}\) and we note that \([\hat{R}, I] = [\hat{I}, R]\) implies that \(R\) is algebraic in \(H^*\) over \(\Phi^\delta \rightarrow \Phi^* \rightarrow \Phi\) and over a regular parabolic approximation, we can argue as in the spectral mapping theorem. For a hypoelliptic system, the continuation to \(T\) is algebraic and the stratification of \(X^*\) gives a stratification of \(X^\delta\). We can conversely argue that if these stratifications are equivalent, the system has no lineality.

5. Topology

5.1. Introduction. The concept of hypoellipticity is dependent on topology and we will use the monotropic functionals both for limits in the symbol space and for the equations in the operator space. The topological arguments are comparative and we compare with the more familiar hyperfunctions. However there are geometric sets that are characteristic for hypoellipticity, such as lineality and the set of orthogonality, for all topologies that we consider. Several parameters are necessary to define the class of hypoelliptic symbols. We give the approximation of the operator using operators dependent on a parameter and a second parameter is used to trace the transversal in determining microlocal contribution. Since this is an analytical study and not a geometrical, we do not attempt to minimize the number of parameters.
5.2. Topological fundamentals. The space $H(V)$, where $V$ is a complex analytic
variety, countable in the infinity, is the space of holomorphic functions with topology
of uniform convergence on compact sets. This is a Frechet type of space (FS) and
the dual space is denoted (DS). Given F-spaces $\{E_i\}$, if $i:E_{i+1} \to E_i$ the projective
space is (FS). If $i:E_i \to E_{i+1}$ compact, the inductive limit is compact. We start
with a topology of Schwartz type, that is given a separated space $E$, if $V$ is a
convex disc neighborhood of the origin in $E$, then we have existence of a convex
disc in $E$ that is a neighborhood of the origin such that $U \subset V$ and such that
$E_U \to E_V$ is compact, where $E_U$ is the completion of the normalized set $E_U$.
The topological arguments in this study are comparative. The symbols modulo
regularizing action are considered in $i$ neighborhood of the real space where we
come with monotropic functionals and the $\mathcal{D}_{L_p}^\prime$ spaces ($p = 1, 2$). We also give
a brief comparison with the hyperfunctions.

Proposition 5.1. If $(I)$ is an ideal of holomorphy with topology of Schwartz
type and a compact translation, consider $\text{rad}(I)$ with Schwartz type topology and
a weakly compact translation, if $\psi \sim_\infty 0$ in the $|\zeta| - \infty$, is in $\text{rad}(I)$, then
$\{d_1 \psi = \psi = 0\}$ is nowhere dense in $N(I)$.

Proposition 5.2. Assume $(J) = \text{ker} h$ a finitely generated ideal with topol-
yogy of Schwartz type and $r_T^\prime$ weakly compact. Assume for all $\psi \in (J)$, we have
$h(r_T^\prime \psi)/\psi \sim_\infty 0$ in the $\zeta - \infty$. If $\eta \neq \text{const.}$, we have that $\psi$ is in a bounded
set with respect to the origin.

Proof:
We can prove an estimate
$$|\eta - c_R| < 1/|\zeta| \quad |\zeta| > R$$
for a constant $c_R$ and $R$ sufficiently large. Thus, for $\phi \in (J)$
$$|\phi| < c_1/|\zeta| + c_2 |\eta(\phi)| \quad |\zeta| > R$$
for constants $c_1, c_2$. Symmetry follows from the conditions on $r_T^\prime$. □

5.3. Monotropic functionals. Assume $\mathcal{B}_m$ test functions, that is $C^\infty$ – functions,
bounded by a small constant in the infinity, such that $\mathcal{B} \subset \mathcal{B}_m \subset \mathcal{E}$ and $(\mathcal{B}_m)^\prime \subset
\mathcal{D}_{L_1}^\prime$. The Fourier transform over the real space is $P f_0$, where $P$ is a polynomial
and $f_0$ is a continuous function. We will modify $f_0$ to an $\epsilon$– neighborhood of the
real space as follows

i) $F_0$ is continuous on the real space and locally bounded on an $\epsilon$– neighbor-
hood of the real space.

ii) we have existence of $\lim_{-\epsilon_0} F_0(\xi + \Gamma)$, for any line $\Gamma$

iii) any line $\Gamma \subset \Delta(F_0)$ is such that $\Gamma \subset \Omega$, where $\Omega$ is a domain of holomorphy.

Note that the difference $\tau_T F_0 - F_0$, even when it is not holomorphic, will preserve
constant value over the lineality corresponding to $F_0$. Finally, assume

iv) $F_0 \sim_\infty W_0$, where $W_0$ is holomorphic and in $\text{Exp}$ of finite type.

We then have existence of $B_T$ (modulo monotropy). Assume further that the tran-
slation is algebraic over $P F_0$ and for $W_0$, that the lineality is quasi-porteur (cf. [12])
for $B_T$.

The first observation is that if $f_T \in L^1$ then $\mathcal{M}_N(\frac{d^n}{dz^n} f_T) = \sigma_T \in L^1$. Partic-
ularly, we have $f \in \mathcal{D}_{L_1}^\prime$ implies $\mathcal{M}(f) \in L^1$. As $d f = \alpha d x$, we have if
$d \mathcal{M}(f) = \beta d x$, then we must have $\beta = \mathcal{M}(\alpha)$, thus in $L^1$, $\alpha d x \mathcal{M}(\alpha) = \mathcal{M}(\frac{d^n}{dz^n} \alpha)$. We
now argue that $\mathcal{M}_{-N}$ is surjective in $B_m$. Consider for this reason $\mathcal{M}_N$ in $B_m$ and
assume that $(I)$ is defined by $f \in (I) \Leftrightarrow f \in B_m$ and $f \leq \mathcal{M}(f)$. We then have
\[ M(f - f_0) = 0 \text{ implies } f - f_0 = 0. \] Thus, we have that \( M_{-\mathcal{L}} \) is surjective over \((I)'\).

As \((I) \in B_m\), we must have \( B_m' \subset (I)'\), why the surjectivity follows for \( B_m'\). Note that 
\[ f = e^{\varphi} \text{ with } \varphi \text{ subharmonic, if we let } M(e^{\varphi}) = e^{M(\varphi)} \text{ and } M(\varphi - \varphi_0) = 0, \]
then \( \varphi - \varphi_0 = 0 \) implies \( f = f_0. \)

We have studied regular approximations according to \( F(\zeta + T_j) = F(\zeta) + c_j \) as \( 0 \neq c_j \to 0 \) as \( T_j \to T_0. \) Note that more generally, for \( dF(\zeta + T) - dF(\zeta) = dLT(\zeta) \) with for instance \( dLT \sim_0 0 \) as \( |\zeta| \to \infty. \) The projection method gives that 
\( f \sim_0 \) as \( |\zeta| \to \infty, \) means existence of \( g \) holomorphic such that \( g \to 0 \) as \( |\zeta| \to \infty. \) Thus we have existence of a polynomial \( P \) such that 
\[ |g(\zeta) - P(\zeta)| < \epsilon \text{ as } |\zeta| \to \infty, \] where we have assumed \( f = \tau g. \) Note that \( 1/\zeta = (1/\zeta_1, \ldots, 1/\zeta_n) \) and we can assume the condition in some variables and assume the others fixed and in the finite plane.

### 5.4. Algebraicity for exponential representations.

Consider the following problem, when for a continuous homomorphism \( L \) and \( \mathcal{L}' = \{ \exists! \eta \ L(e^{\psi}) = e^{<\eta, \psi>}\}, \) do we have \( L \in \mathcal{L}'. \) Let \( \mathcal{L}^0 = \{ \exists! \eta \ L(\psi) = <\eta, \psi>\}, \) where in this case \( L \) is assumed continuous and linear. If \( X = H(\Omega), \) for an open set \( \Omega, \) we assume 
\[ \tilde{X} = \{ e^{\psi} \in H(\Omega) \}. \]
If we have existence of \( \eta_\epsilon, \) for \( x \) fix, such that \( x \in \mathcal{L}^0(\tilde{X}), \) we have \( L \in \tilde{X}'. \) Assume for \( M(x) \), that \( M(\psi) = <\tilde{L}, \psi > \), then 
\[ L = \tilde{F}^{-1}M. \] When \( L \in \mathcal{L}' \) and if \( L \) is algebraic in \( e^{\psi}, \) that is linear in \( \psi, \) we have \( L(e^{\psi_1 + \psi_2}) = L(e^{\psi_1})L(e^{\psi_2}) = e^{<\eta_1, \psi_1>}e^{<\eta_2, \psi_2>}. \) Further if \( L_1, L_2 \in \mathcal{L} \), we have \( L_1L_2(e^{\psi}) = e^{<\eta_1 + \eta_2, \psi>} = e^{<\eta_1, \psi>}e^{<\eta_2, \psi>}. \) If \( \tilde{F} = \tilde{N_1} \tilde{N_2} = <\eta_1, \psi_1>, \) then \( \tilde{F}^{-1}(\tilde{N_1} \tilde{N_2}) = N_1N_2 \). Assume for \( M(x) \), that \( M(\psi) = <\eta, \psi > \), we assume the commutator \( C \) such that 
\[ C(\tilde{F}, N) = [\tilde{N_1}, \tilde{N_2}] \text{ and } \tilde{N_1} \tilde{N_2} = [\tilde{N}, \tilde{N}] \text{ and we say that } N \text{ is algebraic}. \]
Let \( <N(\psi), \theta> = <\psi, \tilde{N}(\theta)>. \) Then \( N(\psi) \in (\tilde{X}') \) implies \( \tilde{N}(\psi) \in X' \), for \( x \) fix and 
\[ \tilde{N} \tilde{F} = [\tilde{F}N]_1. \]
We have \( <\tilde{F}(\psi), \theta> = <\psi, \tilde{F}(\theta) > \) iff \( <e^{\psi}, \theta> = <\psi, \tilde{F}(\theta) > \).

Assume \( \Sigma \ni \gamma \to \gamma \in \mathcal{R} \) is on the form \( (e^{\varphi}, h(e^{\varphi})). \) Assume \( L \) within a constant is algebraic over \( \mathcal{R} \) (does not imply algebraic over \( \gamma \)). Note that if \( \pi \) is the projection \( \Sigma \to \mathcal{R} \) and \( \pi^{-1}(\Sigma) = \Sigma, \) then \( \Sigma \) may have points in the edge, even when \( \Sigma \) does not. Thus, there may be points in common for \( L_T \in (I_S), \) that are not present for \( L_T \in (I_S). \)

### 5.5. Some generalizations.

Assume \( V_1 \subset V \subset V_2, \) where \( V_1, V_2 \) are semi-algebraic and \( V \) analytic. Assume \( V_1 \subset \Omega \) a domain of holomorphy, such that the limit \( B_T \) is independent of starting-point, then \( V_1 \) is quasi-porteur to \( B_T \) and the same follows for \( V. \) Further, since \( V \) is analytic, \( V \) is porteur to \( B_T. \) Assume \( \tilde{V} \) the extension to full lines. Assume \( g_1 \) algebraic, such that if \( V_1 = \{ p_1(\gamma) = 0 \}, \) \( g_1p_1(\gamma) = \gamma \) locally. Then \( g_0(0) = 0 \) and \( V_1 \to \gamma \to \tilde{\gamma}, \) where the last mapping is into the wave front set, but regular approximations are assumed in \( V_1 \subset V_2. \) If \( V_1 \) is porteur to a functional \( T \in H', \) we can chose \( V_1 \) as a cone which through the topology can be assumed compact. Note that \( V_1^0 = \{ \gamma \to \gamma \}. \) Re \( \gamma \to \gamma \to \gamma \) \( \gamma \) \( \in \gamma \to \gamma \to \gamma \) \( \gamma \) \( \in \gamma \) \( \to \gamma \) \( \gamma \) \( \to \gamma \) \( \gamma \) \( \to \gamma \) if \( \varphi_0 \) corresponds to the orthogonal complement to \( bdV_1^0. \) If \( B_T \) has indicator \( h_0, \) we have for \( \Gamma \) a compact, convex cone \( \Gamma \ni 0 \) with inner points, that \( B_T \) is portable by \( \Gamma \Rightarrow h_0 \leq 1 \) on \( bdV^0. \)

If \( \gamma \) is defined by a homomorphism \( h \) such that \( h^N = 1, \) we have that all regular approximations of a singular point \( P, \gamma \to \gamma \) as \( t \to \infty, \) can be seen as on one side of a hyperplane \( \{(x, h(x)) \mid db(x) \geq \mu dx\}, \) for a constant \( \mu. \) More precisely, for a curve that reaches \( \Sigma \) as \( t \to \infty, \) if the part of the curve that is situated outside \( \Sigma \) is finitely generated, we claim that \( \gamma \) can be chosen locally on side of a hyperplane (cf. section on paradoxal arguments). We are assuming in the
following that $x$ is reduced. We have $\{p_1(x,y) = 0\} \sim \{\tilde{p}(\eta) = 0\}$, for a polynomial $\tilde{p}$ and $\eta = y/x$. Further, there are polynomials in $x, y, r_1, s_1$ such that $p_1(\gamma T) = \frac{d}{d\gamma} r_1(\gamma T) = s_1(\frac{d}{d\gamma} \gamma T)$. If $p_1$ is reduced, there is a polynomial $q_1$ such that $p_1 = q_1^2$.

Assume $F_T \to 0$ and $\frac{d}{d\gamma} F_T > 0$ over $\{p_1(\gamma T) > 0\}$. If we let $\frac{d}{d\gamma} F_T \sim p_1$, we are assuming $F_T$ conformal in $T$ and algebraic in $x, y$. Assume $V_1, V_2$ are extensions to full lines with indicators $h_1$ and $h_2$. It is then sufficient to prove that $h_1 \leq ch_2$, for a constant $c$, to have $V_1 \subset V_2$. Further, $c_1 h_1 \leq h_2 \leq c_2 h_1$, for constants $c_1, c_2$, gives $V_1 \sim V_2$. If $V_1$ is a semi-algebraic quasi-porteur to a functional for instance defining the first surface to the symbol with $V_1 \subset V$, where $V$ is analytic, then $V$ is porteur. Assume that $p_1, p_2$ have the same micro-local properties, in the sense that their sets of constant sign coincides. Assume $V_i = \{p_i(\gamma_T) \geq 0\}$ and $V_1 \subset V_2$ and $V = \{g = 0\}$ with $g$ analytic, then $\{p_2 < 0\} \subset \text{supp} \ g \subset \{p_1 < 0\}$. Thus, if singular points are in $\{g = 0\}$, then regular points will be in an "octant". We could say that the micro-local contribution from the symbol, is given by this "octant".

5.6. A comparison of hyperfunctions and monotropic functionals. If the symbol $F_T(\gamma)$ preserves a constant value in the $\gamma$-infinity, then $F_T \in B_m$, that is it is $C^\infty$ and bounded by a small constant in the infinity. In this case the Cauchy inequalities can be satisfied for a monotropic function, that is there is a $\varphi_T \in \mathfrak{A}$ (real-analytic functions) such that $F_T \sim_{m} \varphi_T$. If for $F_T(\gamma) = \sum_{\alpha} F_{\alpha T}/\alpha! \gamma^\alpha$ there is a number $\rho$ with $\rho < A$, for a constant $A$, such that $\rho^\alpha \sup \ F_{\alpha T} \to 0$ as $\alpha \to \infty$, then we have that $F_T$ is entire in $\gamma$ and of exponential type $A$ ([12]).

Note that a sufficient condition for existence of a global pseudo-base for the symbol ideal, is that it has an induced topology with Oka’s property.

If $\gamma_T$ is in $\mathfrak{A}(\Omega_\zeta)$, then $F_T \in B_K(\Omega_\zeta)$ that is hyperfunctions with compact support ([11]). In the case where $F$ is real-analytic, then so is $f$. If we assume instead that $G_{T,k} = \frac{d^k}{d\gamma^k} F_T$ has isolated singularities at the boundary and preserves constant value in the infinity, then intuitively we would have at worst algebraic singularities in the infinity. Assume $f \in B_m$ in $x, y$, then we have $f(x,y) = g(x,y) + P(\frac{1}{x}, \frac{1}{y})$, where $g$ is radial and bounded by a small number in the infinity and $P$ is polynomial. Further, if $f \in B_m$ there is a $\varphi \in \mathfrak{A}$ such that $|f - \varphi| < \epsilon$ at the boundary, for a small number $\epsilon$.

An important difference between $B_m$ and $\mathfrak{A}$ is the algebraic properties. A function $f$ is in $\mathfrak{A}$ if both its real and imaginary parts satisfy the Cauchy’s inequalities. More precisely, assume $L \in B(\Omega)$ where $L^2$ is defined by composition, such that $\phi \in \mathfrak{A}(\Omega)$ and $L^2(\phi) = L(J(D)\phi)$, where $J(D)$ is a local elliptic operator (cf. [9]) and where $L \in \mathfrak{L}(\Omega)$ such that $J(D)\phi \in E$. Then $\varphi \in \mathfrak{A}$ and also $J(D)\varphi \in \mathfrak{A}$, from the properties of $J(D)$. Any element in $\mathfrak{A}$ has a representation through $J(D)\varphi$ as above, why $L$ is defined on $\mathfrak{A}$ and $L \in B_K$. However we can have $1/f^N \to 0$ in the infinity, for some iterate $N$ without having $1/f \in B_m$, for instance if $f$ is the symbol to a self-adjoint operator partially hypoelliptic in $D$ with $\text{Re} f \ll \text{Im} f$. Thus we do not expect a radical behavior in the case of monotropic functionals.

Another important difference is the global property of the hyperfunctions ([11]), which is not present with the monotropic functionals. However, we can give the following argument. Let $F(T, \gamma) = F(\gamma T)$, where $\gamma = (x, y)$ and $T \in V = \bigcup_{j=1}^{N} V_j$, the parameter space. Assume $F$ algebraic in the parameter in the sense that $F(T_1, T_2, \gamma) = F(T_1, \gamma) F(T_2, \gamma)$.
Let\( \Omega \) be assumed the topology of Schwartz type and with (weakly) compact translation. \( P \) is a ps.d.o \( \gamma \) such that using the claim (1.1), we have that \( \mathcal{B}(\Omega) \) is hypoelliptic in the sense of monotropy. We assume the proposition \( \mathcal{B}(\Omega) \) is micro-locally invariant in the symbol-space. Assume for this reason \( P \) is a ps.d.o hypoelliptic in \( \mathfrak{A} \)-sense. Further, there is a regular sequence of \( \Gamma \) such that \( \Delta(P) = \Delta(P_1) \) and \( P \) is hypoelliptic in \( \mathfrak{A} \)-sense, for some constant \( \epsilon \) at the boundary. Define \( P \) such that \( \Pi \sim_m P \), that is \( \Pi \Pi \sim_m P \). The conclusion is that given a ps.d.o \( P \), hypoelliptic in \( \mathfrak{A} \)-sense, there is a ps.d.o \( P \) with the same lineality (= \{0\}), such that \( WF_\beta(P) = WF_\beta(u) \), why using the claim \( \mathcal{B}(\Omega) \), we have that \( P \) is hypoelliptic in \( B_m \).

For the discussion of the symbol space, we will use a topological argument. Assume the topology of Schwartz type and with (weakly) compact translation. Let \( \Omega_0^j = \{ \Gamma \ F(\gamma_j)(\zeta + \Gamma) = F(\gamma_j)(\xi + \Gamma) \} \), where \( F \) are assumed to satisfy the regularity conditions for the dynamical system. We can now prove for a sequence of \( \gamma_j \) that approximate a singularity, \( \Omega_0^j \downarrow \{0\} \) as \( j \uparrow \infty \). Let \( J \) be defined by \( N(J) = \Omega_0^j \), for some \( j \) and let \( T_0 = \{ \gamma \ F(\gamma)(\zeta + \Gamma) = F(\gamma)(\xi + \Gamma) \} \). Then we have existence of \( J \) as above with \( rad(J) \sim T_0 \). Further, there is a regular sequence of \( \Gamma_j \) such that \( F(\zeta + \Gamma_j) = F(\zeta) + C_j \), where \( 0 \neq C_j \rightarrow 0 \) as \( \Gamma_j \rightarrow 0 \). In the same manner, if the dependence of \( T \) is holomorphic for \( r_T \), we have existence of a regular sequence of \( T \) outside \( \Omega_0^j = \{ T \ F(\gamma_j)(\zeta) = F(\gamma_j)(\xi) \} \), such that \( 0 \neq C_j \rightarrow 0 \) as \( T \rightarrow \Omega_0^j \). This motivates why there is no loss of generality in assuming that \( r_T \) behaves locally as translation, in a regular approximation of \( \Omega_0^j \).

6. The mapping \( T \)

6.1. Introduction. We have seen that certain trace sets (clustersets) are characteristic for hypoellipticity, more precisely the absence of these sets is necessary. The mapping \( T \) which is derived from dynamical systems theory \( [3] \) will in this study be used to define and describe these sets. Characteristic for hypoellipticity, assuming the real and imaginary parts of the symbol are orthogonal, is that \( T \), given as a continuation of the contact transform (Legendre), is (topologically) algebraic.

6.2. Systems of multipliers. Consider the system with right hand sides \((X,Y)\) and \( \eta^T \hat{X} = \hat{Y} \). In the same manner, to the system \( (M,W) \), \( \gamma H = G \). This can be
seen as a multiplier problem. Note that if $\eta$ is a polynomial and the corresponding convolution equation is seen over $\mathcal{E}'$, then $\eta_0 \hat{X} = \eta^* \hat{X}$. Thus, if we assume $X, Y$ are holomorphic and of type 0, then $\eta^* = \hat{\vartheta}$. Assume using the Fourier-Borel transform, that $M = X x + Y y \to H$ and $W = Y x - X y \to G$, then the condition $\tilde{W} = 0$ is the condition that $x^* \frac{dx^*}{dx} = 0$, that is $\eta^* \tilde{X} = \tilde{Y}$. Further, $M_1 = H x + G y$ and $W_1 = G x - H y$, why the condition $\tilde{W}_1 = 0$ is a condition $\gamma^* \tilde{H} = \tilde{G}$, that is $x^* \frac{dx^*}{dx} = 0$. For $M_2 = (\tilde{X})_x + (\tilde{Y})_y$ and $W_2 = (\tilde{Y})_x - (\tilde{X})_y$, if we assume $\tilde{X}, \tilde{Y}$ are holomorphic of type 0, then $\eta_0 \hat{X} = 0$ is the condition that holomorphically bounded (in the real space). When have $\tilde{W} = 0$, we can prove that the mapping $T$ preserves order of zero.

Consider the following scheme

\[
(M, W) \to (H, G) \to (M_1, W_1)
\]

\[
(M_2, W_2) \to (\tilde{H}, \tilde{G}) \to (M_0, W_0)
\]

and the corresponding characteristic sets $\tilde{S} = \{(H, G) = 0\}$, $\tilde{S}_1 = \{(\tilde{M}_1, \tilde{W}_1) = 0\}$, $\tilde{S}_0 = \{(M_0, \tilde{W}_0) = 0\}$, $\tilde{S}_2 = \{(\tilde{M}_2, \tilde{W}_2) = 0\}$. Let $\Sigma \to T \Sigma_2, \Sigma_1 \to T \Sigma_0$. Let $\tilde{S}(M_2, W_2) = (M_1, W_1)$, The sets $\Sigma_1 \to \Sigma_2$ are connected through $SFS = \tilde{S}SF$ over $(X, Y)$.

Assume $g = \tilde{Y}/\tilde{X} - \eta^* > 0$, then we know that there exists a measure $v$, non-negative and slowly growing such that $g = \hat{v}$. The condition on positivity implies exactness over the tangent space (global pseudo-base). We can say that $T$ maps contingent regions on contingent regions, in the sense that the order of the regions are preserved, that is the number of defining functions is preserved. Let $\vartheta_2 = W_2/M_2$ and $\vartheta_1 = W_1/M_1$. Then, we have that $\vartheta_1$ changes sign as $(\vartheta_2 + \eta)/(1 + \eta \vartheta_2)$.

### 6.3. Degenerated points for the method.

The problem of determining $x \in L^2 \cap H$, so that $f = c_1 x + c_2 h(x)$ is trivial over $\{\eta = const.\}$. Consider a neighborhood where $\eta$ is quasi conformal, $|\eta(x) - x| < c$, locally for a constant $c$. Assume $\frac{\tilde{M}}{\tilde{W}} = \frac{x X + y Y}{x X - y Y}$, where $M = 0$ is the equation for mass conservation and $W$ is the vorticity. The Poincaré index counts the number of changes of sign $-\infty$ to $\infty$ and back, for this quotient, why if for instance $\{\eta + \vartheta \geq \mu\}$, for a positive constant $\mu$, the index is zero. For $\eta$ conformal we have $0 < |\vartheta - \eta|$. Assume $\eta$ algebraic over $\vartheta = \eta$. In a neighborhood of $\tilde{W} = 0$, if $\eta$ is constant, then $-\frac{\tilde{M}}{\tilde{W}}$ changes sign as $1/(\vartheta + \eta)$. In the same manner in a neighborhood of $\tilde{M} = 0$, the corresponding test is for $1/(\eta^{-1} + \vartheta)$. Note that conformal mappings do not preserve continuum or reducedness, unless they are bijective.

Consider the mapping $dx \to dv(x) \to dh(x)$, then $F(x, v(x))$ have isolated singularities. To describe the singularities to $F(x, h(x))$, we start with $(M, W)$ and consider $W = const.$ (not necessarily non-zero) or $W$ regular. We will use $\tilde{W} = G = Pf$, where $P$ is a polynomial and $f$ is a continuous function and uniformly bounded (in the real space). When have $\tilde{W} = 0$, equivalently $x^* \frac{dx^*}{dx} = 0$.
we have \( J \) algebraic. Thus, mapping case where \( \gamma \) \( \text{Pfaff's equation}, \ dc \)
assuming \( H \) \( \text{on } \Phi \) and \( \Psi \). Assume with \( c \)
Hamilton function corresponding to \( \Delta \) \( \text{that is if } \Phi \)
harmonic conjugation. Thus, if \( \Phi \) \( \Delta \) \( \omega \)
reduced) and \( \text{and } \sigma \)
of \( \xi \) \( \text{in } \mathbb{R}^n \), is left semibounded. If we let \( \omega = \chi_{(-\infty,\lambda]} \circ L \) and \( F(E\lambda u) = \omega F u, \) for the spectral projection \( E\lambda \) and \( F^{-1} \) \( w \in P_L^\prime \). For Baire functions we have the spectral mapping theorem, that is if \( A \) is harmonic conjugation, \( \sigma(T(A)) \subset \overline{T(\sigma(A))} \) and \( T \) is algebraic, we have equality.

6.5. The spectrum to multipliers. The spectral theory is set over \( \chi X = Y \) and \( \chi^\lambda \vec{X} = \vec{Y} \), that is \( \frac{d\chi(x)}{dx} = \chi \frac{dx}{dx} \) and \( \lambda^\lambda \vec{H} = G, \lambda \vec{H} = \vec{G} \). Let \( \Phi = \{ \lambda = \text{const.} \} \) and \( \Phi^\lambda = \{ \lambda^\lambda = \text{const.} \} \). Let \( \Psi \) and \( \Phi^\lambda \) be the constant sets to \( \chi \) and \( \chi^\lambda \). Let \( V \) be the Hamilton function corresponding to \( \Delta V = -W \). Assume that \( T: \lambda \rightarrow \lambda^\lambda \) is analytic on \( \Phi \) and \( \Psi \). Assume \( A \) the linear operator corresponding to \( A(M) = M^\lambda \), that is harmonic conjugation. Thus, if \( Y = \eta X \), then \( A(X) = \chi X \) and \( A(\vec{X}) = \chi^\lambda \vec{X} \). If \( Sp(T(A)) \subset \mathbb{R} \), then \( Sp(T(A)) \) is constant \( \subset \Phi^\lambda \).

We will also consider the sets \( \{ e^{-\theta} \chi^\lambda = \text{const.} \} \) and \( \{ e^{-\theta} \chi = \text{const.} \} \) as parabolic Riemann surfaces. Over these sets \( T \), when it acts as a Legendre transform,
we can consider it as algebraic and we can apply the spectral mapping theorem.

Note that through symplecticity we have that if $\chi^\Delta/\chi = \text{const.}$, then this holds in a point. If $\chi$ is algebraic in $T$, then $\chi^\Delta$ is algebraic $1/T$. We add the condition $T(0) = 0$ to the definition of $T$, corresponding to the condition that $P_b = P_0^\Delta$ exists, which is necessary for analytic continuation.

6.6. Conclusions concerning the trace formula. In the representation $e_\lambda(x, y) = \int_{\Delta(\xi)} e^{i(x-y)\xi}d\xi$, (cf. [13]) a trace in $(x-y)$ corresponds to a trace in $\xi$, considered as a functional in $H'$. Through Iversen’s result, the correspondent to $r_T$ in $\xi$, is a function, however multi-valued in most cases. If we reduce the situation to a real parameter, the continuity of $r_T$ means that there can be no trace in $\xi$ corresponding to the leaves, since the operator is elliptic. The only possibility is through change of leaves, but through the conditions that the covering is regular, we know that these trace sets do not have a measure.

6.7. Conjugation. Assume $\chi^\Delta H = G$ and consider the two FBI-transforms $F_G$ and $F_H$, where the kernels are harmonically conjugated. We are assuming $H = e^\Phi$, where $\Phi \geq 0$ over a parabolic surface and where $\Phi < 0$ implies $\Phi = \text{const.}$ If $\chi^\Delta = 1$, we have $F_G = F_H$ and we assume $F_{H/G} = F_1 = \delta_0$, the evaluation functional. Consider the problem of geometric equivalence. The mapping $T : \Psi = x^{\chi^\Delta} \rightarrow x^{\chi^\Delta}$, maps the set $\chi^\Delta = 0$ (or $\chi = \text{const}$) on to $T \Psi = 0$. Further $\frac{d\chi}{d\chi} = \frac{d\chi}{d\chi} = \frac{d\chi}{d\chi}$/ ($\chi$ quotient of polynomials) and we have $T T = \chi = \chi$. We can show that $T$ is not algebraic but if $T$ maps 0 onto 0, it has the property that $T \Psi = 0$ implies $\Psi = 0$. In the case where $\chi^\Delta$ is constant, we have not excluded the case where $G = 0$ (and $H = 0$). These points are singular and parabolic. Thus, if $\chi^\Delta = 1$, we have $T \Psi + T \Psi = 0$. In the case where $\chi^\Delta$ is algebraic, we have that $-H/G$ changes sign as $T \Psi / T \Psi$, a quotient of polynomials.

We are considering two types of conjugates (still referring to kernels), $F_G^\Delta = F_G$ and $F_H^\Delta = F_H$. We assume as above $F_1 = F_{T \Psi} / T \Psi$ corresponds to the proposition $F_{T \Psi} = F_{T \Psi}$, where as before $T \Psi = \vartheta^\Delta - \chi^\Delta$ and $\bar{T} \Psi = \vartheta^\Delta + \chi^\Delta$. Particularly, if $T \Psi = \rho \Psi$ and $T(\rho \Psi) = \rho T \Psi$, we have $T \Psi = -1/\rho T \Psi$. Further, $\frac{d\chi}{d\chi} = \frac{\rho d\chi}{d\chi}$. Now let $\chi^\Delta = i \rho^\Delta$, for a real $\rho^\Delta$ and assume $\vartheta^\Delta = \vartheta^\Delta$. Thus, $T \Psi = S \Psi$, why $S^2 = T T$. A differential $\Psi = w + iw^\Delta$ is said to be pure, if $\Psi^i = \psi^i$. We have $T(w + iw) = -i(w + iw)$ and $S(w + iw) = -i(w + iw)$ Finally, $S(u + iw) = -i(w + iw)$. Thus, $w + iw$ is pure and if $w$ is symmetric with respect to the origin, also $w + iw$ is pure. If $T \Psi = g$ and $g = \alpha g$ we have $\Psi = \alpha g$ which is pure, if $\alpha = 1$. Let $p = (w, w^\Delta)$ and consider $r = (w, T \Psi)$. Through the definition of $T$, we have $T \Psi^\Delta = (T \Psi)^\Delta$. Thus, if $(w, w^\Delta)$ is pure, the same holds for $(w + iw^\Delta) + (w + iw^\Delta)$ and it follows from the condition $T^2 = -I$ over $(w, w^\Delta)$ that $r$ is pure.

Assuming $w$ is pure, we have that $(w, w^\Delta + w^\Delta) = w - w^\Delta$, which is pure if $w \perp w^\Delta$, where we have used that $iw^\Delta = -w^\Delta$ and if $w^\Delta = iw$, then $(w + iw^\Delta) = w + iw$. If $(w^\Delta)^\circ = -iw$, then the form $(w + iw^\Delta)$ is pure.

6.8. Symplecticity and forms. Assume $u = adx + bdy$ and let $F(u) = F(a)dx + F(b)dy$. Then $F(u^\Delta) = -F(b)dx + F(a)dy = F(u)^\Delta$, assuming that $F(-b) = -F(b)$. Further, if $\frac{\partial F_1}{\partial x} = F_1\frac{\partial F}{\partial y}$, then $\Delta F_1 = F_1^\Delta$. Assume $F^\Delta$ a homomorphism and that it maps $0 \rightarrow 0$. We can as before write, $<F(M), \theta> =$
\( \rho_T(x,y) < M, \theta > \) and as \( F(M) = -F^\circ(W) = \Delta F^\circ(V) \), such that if \( \Delta V = 0 \), then \( F^\circ(W) = 0 \) and \( F(M) = 0 \).

Consider the mapping \( T: Xdx + Ydy \to \tilde{X}dx^* + \tilde{Y}dy^* \). We are assuming \( \gamma^0 \) in duality and write \( T\mathcal{M}(d\gamma) = M_2(d\gamma^0) \). \( T \) is first assumed an extension of a contact transform in the sense that \( T\mathcal{M} = \rho\mathcal{LM} \), where \( \rho \) is at least a Baire function. Assume \( T f_\tau / f_\tau = \text{const.} \Rightarrow \tau = 0 \) and \( dT f_\tau / df_\tau = \text{const.} \Rightarrow \tau = 0 \). Assume \( \alpha(w, w^\circ) \) a symplectic form over \( E = V \times V^\circ \) and consider \( E \times TE \). Assume \( S \) an involutive set with respect to \( \alpha \), in the sense that the sets \( \{ \chi = \frac{d}{dx} \chi = 0 \} \) and \( \{ \chi^0 = \frac{d}{dx^*} \chi^0 = 0 \} \) are both minimally defined and equivalent. That is formally, \( \alpha(Tw, Tw^\circ) = \alpha(\phi, w^\circ) + \varphi \sim \text{constant} \).

Consider the form \( (p, q)_{\sigma} = -(q, p)_{\sigma} \), where \( p = (w, w^\circ) \) and \( q = Tp \), where we are assuming \( (q, p)_{\sigma} = (T p, T^2 p)_{\sigma} = -(q, p)_{\sigma} \), that is skew-symmetric and bilinear, assuming the double transform in \( EX \) is equivalent with \( -I \) (after analytic continuation). Through the conditions \( (w_\tau)^\circ = (w^\circ)_{\tau} \), and the quotient \( T(w_\tau) / (\tau w_\tau) \) is never algebraic. We conclude that \( T \) under these conditions is symplectic for \( (\cdot)_{\sigma} \) and that the involutive set \( S \) has a corresponding extended involutive set with respect to \( (\cdot)_{\sigma} \).

**Proposition 6.1.** The mapping \( T \) when planar (pure) preserves analyticity.

**6.9. The reflection principle.** Consider \( \Phi^* = \{ \vartheta^* \to e^{-\vartheta^*} \chi = \text{const} \ \exists \theta^* \} \) and \( \Phi = \{ \vartheta \to e^{-\vartheta} \chi = \text{const} \ \exists \theta \} \). Consider the Legendre transform \( R \), according to \( \text{RE} < R(\chi), \chi > = -1 \).

Further, note that if \( \chi^* / \chi = f_0(\chi) \), where \( f_0 \) has slow growth like \( e^{-\varphi} \) as \( 0 \leq \varphi \to \infty \). Thus, \( R(e^{\vartheta}) = e^{-\varphi + \vartheta} \) and \( \vartheta^* = -\varphi + \vartheta^* \). Note that if \( T \) is considered as a continuous morphism on a Banach algebra \( A \) with \( T(e^{\vartheta}) = e^{\vartheta^*} \), then \( \vartheta^* \in A \). For instance if \( \theta_T \) is algebraic in \( T \) or a quotient of algebraic functions, for \( T \) close to 0, then the same holds for \( \theta_T \). The spectrum for \( R(f)/f \) contains under the conditions on \( f_0 \) both 0 and \( \infty \). If \( R \) preserves first surfaces, then we can extend the definition of \( T \) to \( T(0) = 0 \), that is a part of the boundary. Since \( T \) is pure if we assume the corresponding form closed, we have an analytic mapping.

Assume that the segments \( \gamma, \gamma^* \) have a point in common \( P_0 = P_0^* \). As \( \gamma_0 = f_T d\gamma = f_T d\gamma^* = \gamma_0^* \), where \( \Gamma \) is a closed contour \( \Gamma \sim \Gamma_0 \). We thus have a reflection principle for \( T \) expressed in preservation of the condition on flux. That is \( \text{Flux}(TW) = \int_T dTW \). Staring with the condition \( -H/G \sim TW / \bar{TW} \), we are assuming \( \text{Flux}(TW) = \text{Flux}(\bar{TW}) = 0 \), reflecting symmetry with respect to harmonic conjugates. Through the condition on parabolic singularities, there must be a point at the boundary, where \( TW = 0 \) such that \( W = 0 \). If the point is singular for the associated dynamical system \( (M, W) \), the condition must be symmetric, that is must have \( M = W = 0 \). Thus the condition on flux is necessary for regularity for this dynamical system. Consider \( T(\chi) = \chi^0 \) such that \( \gamma_1 \ldots \gamma_N \sim 1 \) and \( T(\gamma_1 \ldots \gamma_N) \sim 1 \), that is preserves the closed property. The proposition is thus that \( T \) preserves flux, but through symplecticity, that is \( T f = f \) implies \( f \) a point, it is not considered to be a normal mapping on a non-trivial set at the boundary.

**6.10. Codimension one singularities.** To determine if the symbol corresponds to a hypoelliptic operator, we must prove that every complex line, transversal to first surfaces, does not contribute to linearity. A complex line is considered as
transversal, if it intersects a first surface and the origin. We are assuming that $F$ is the lifting function with $\Delta F = -W$ on a domain $\Omega$ (and the associated equation $\Delta F^\circ = -M$ on $\Omega$). The boundary condition is assumed parabolic, that is $\Gamma$ is such that $P(b_T)F_T = 0$ implies $T = 0$ for a polynomial $P$ and $F_T = F$ in $T = 0$. We are thus assuming that the parametrix is invariant at the boundary, $E_T F_0 = F_0$ on $\Gamma$. Let $\phi_T = E_T F$. We are assuming in this approach that $W_T$ is defined by $\Delta \phi_T = E_T W + R_T$. Assume the operator $T$ is continuous at the boundary, with $T W = T W^\circ$, we then have $T W^\circ = \zeta T W$, for $\zeta \in \mathbb{C}$ and $|\zeta| = 1$. In a Puiseux-expansion, we have that the coefficient for $t$, $\chi^* = a_0 + a_1 t + \ldots$ is $\neq 0$ and $t = \sqrt[\chi]{\zeta}$. Thus, the order of the critical surface is one and we have singularities of codimension 1. If $T$ preserves the order of contact, we have the same conclusion for the multiplier $\chi$.

A co-dimensional one variety $S(p)$, is such that $S(p, x) = 0$ and $s_x \neq 0$, where $p = s_x$ is a characteristic variety, if $g(x, s_x) = 0, p dx = 0$ and $g_s dx + g_p dp = 0, dS = p dx$. As before, we have $T_T dx = (\nu_T p) dx$. $S(p)$ is involutive for $g, \gamma$, if $H_g(\gamma) = 0$ (the Poisson bracket). For parabolic singularities at the boundary, we are considering isolated singularities in higher order derivatives $\psi_N - 1 = \frac{1}{(\gamma \cdot \nu^{N-1})^2} = 0$ and $\psi_N \neq 0$. Assume $g$ is defined through $dS = \frac{dx}{dy} dx + \frac{dy}{dx} dy = \frac{dx}{dx} dx = \frac{dx}{dx} dx$. Symplecticity gives that $\psi_T^g / d\psi_T = \text{const.} \Rightarrow T = 0$. We have a canonical symplectic form, $d < p, dx > = 0$ on $S(p)$, where $p = dp / dx$ and $d(dp) = 0$. Note that we are assuming $T : x \frac{dx}{dx} \rightarrow x^* \frac{dx}{dx}$ and we must assume $x \frac{dx}{dx} > 0$ (Poincaré-index) implies $x \frac{dx}{dx} > 0$. If $T$ is algebraic, the spectral theorem can be applied with advantage and we must assume that $T$ does not have zero’s on the boundary, that is $x^* = \frac{dx}{dx}$ and $< x^* , x > = 0$. These sections correspond to $T : x^* , x \rightarrow x, x^* >$ (normal sections). We are not assuming the trajectories in a reflexive space, but it is sufficient to consider the Lagrange case $\Gamma^* = \Gamma$.

### 6.11. The mapping $T$ and parabolicity.

Consider $T$ over a set where it is algebraic in $H^\prime$, for instance Legendre. We have seen that there can be no closed contour in the infinity. On the other hand, if $T e^{\psi} = e^{\psi^* + \phi^*}$, where $\psi^*$ is related to $\theta$ through a Legendre transform, we have that $e^{\psi^*}$ can define a circle in the infinity. Further, if $\psi^* = P(\psi) \phi^*$ and $\theta_T = P(T) \phi$, taking the closure of the domain means that $\psi^* = e^{\alpha(\psi)} \phi^*$ and that $\psi_T / \phi_T = e^{\alpha(\psi)} \phi / \phi^*$ and we may well have that $\alpha(\psi) = \alpha(T) 

We are assuming that $T$ preserves the parabolic property for the stratification, that is it maps exponentials on exponentials. Note that we have a parabolic approximation if and only if for all functions $u$ harmonic in a neighborhood of the ideal boundary, with finite Dirichlet integral, we have vanishing flux (cf. [1]). Assume $T$ maps finite Dirichlet integrals on finite Dirichlet integrals. We have defined $T$ such that $\{ W = 0, W = 0 \}$, where $\Delta F = -W \rightarrow \Delta^e T F = -T W$ and we see that $T$ preserves parabolic approximations. Consider $T_1$ algebraic, in the sense that $T_1(e^{-\chi}) = T_1(e^{-\chi}) T_1(\chi)$. We then have that $T_1$ maps constants on constants, why if $\chi = e^\psi$, we have that $T_1(e^{-\chi}) = \text{const.}$ and $T_1(\chi) = T_1(e^\psi) = e^{\psi^*}$. Assume that $T(e^{-\chi}) = e^{\psi^*}$, that is $T$ maps constants on to exponentials and that $T_1(e^{-\chi}) = e^{\psi^*}$. Thus, if $\chi = T_1(\chi)$, we have that $e^{\psi^*} = e^{\psi^* + \psi^*}$. If $T = T_1$, there is no room for a closed contour, in the infinity. If we assume $T \sim T_1$, in the sense that $e^{\psi^*} = T(e^{-\chi}) = T_1(e^{-\chi})$ along two different paths, the result depends on
the property of monodromy for the stratification. That is if $T$ is locally injective with respect to path, we can write $T/I \sim I$.

6.12. The vanishing flux condition in phase space. Consider the linear functional $L(\phi) = \int_{\beta} d\phi$ and consider the difference $L(e^\phi) - e^{L(\phi)} = \widehat{[I,L]} = [\widehat{I},L]$. We note that the vanishing flux condition $L(\phi^\wedge) = 0$ does not imply $L(e^{\phi^\wedge}) = 0$. Assume that $(\beta)$ is a neighborhood of the origin and note

$$\int_\beta \bar{W}^\wedge = \int_{(\beta)} Wdxdy$$

where $\bar{W}^\wedge = Pdx + Qdy$. Immediately, we note that $\widehat{L}(\phi) = L(e^\phi) = \int_{\beta} de^\phi$ bounded in the infinity implies that $\widehat{L} \sim P(\frac{1}{T})$, as $T \to \infty$. Note that if the indicator for $L$ is $\alpha$, then the functional $L$ has support on a ball of radius $\alpha$. Starting with (3), if $W_T$ is algebraic in $T$, then the measure for $(\beta)$ is zero, that is $L(e^\phi)$ is of real type. Note also that $\int_{\beta} d\bar{T}W = 0$ does not imply $\int_{\beta} dW = 0$, however $\int_T (\bar{T}^{-1}W) = 0$. Particularly, if $e^\phi = e^\alpha$, we can consider $[L, [\widehat{I}, \widehat{I}]] = [\widehat{I}, [I,L]]$. For instance if $\int_{\beta} d\phi = -\infty$ then $L(e^\phi) = e^{L(\phi)}$ implies $L(e^\phi) = 0$. Assume $\phi_T$ algebraic in $T$, then $Flux(\phi_T) = 0$. In the same manner if $\phi_T$ is harmonic, then $Flux(\phi_T) = 0$, further if $\phi_T$ is algebraic in $T$, then $\int_{\beta} d\phi_T = 0$ implies $(\beta)$ has measure zero. Thus, if $L(e^{\phi_T}) = 0$ implies that the measure of $(\beta)$ is zero. We note the following result. Assume $\phi_T$ algebraic in $T$, then $d\phi_T$, $d\phi_T^\wedge$ are closed, which implies that $\phi_T$ is harmonic, why we have a real type operator. Note that if $\phi_T$ is harmonic on a disc, the mean is constant ($\equiv -\infty$) and the measure for the ideal boundary $(\beta)$ is zero.

7. Boundary conditions

7.1. Introduction. In the model, the singularities on the first surfaces to the symbol are mapped on to the boundary of the stratification, which is parabolic or more generally very regular. Hypoellipticity is a condition on behavior for the symbol in the infinity but the method using $T$ (basically the projection method) requires a discussion on the simultaneous behavior at the boundary. The boundary to the strata is defined by $\{M_N(f) = f\}$ but we also discuss the phase correspondent $\{M_N(\log f) = \log f\}$.

7.2. The $\bar{\delta}$-Neumann problem. We will deal with the following problem, given a regular approximation of a singular point in the boundary, $\Gamma$, determine $F_T$ such that

i) $F_T(\gamma)(\zeta) = f(\tau_T\zeta)$

ii) $F$ is holomorphic in $\gamma$ and algebraic in $T$.

We assume here that the boundary is finitely generated and in semi-algebraic neighborhood. The first part of $\text{ii)}$ is the lifting principle for a semi-algebraic domain. For $i)$ we note that as $f \in (I)$, a finitely generated ideal, there is no problem to determine $\gamma$, such that the formal series for $F_T$ converges. Consider now the second part of $\text{ii)}$. Given a regular approximation $U_T$ with algebraic dependence of the parameter $T$, we can use the $\bar{\delta}-$ Neumann problem to determine a lifting function $F_T$ such that $\delta_T F_T = \delta_T U_T$ and $F_T = U_T + L_T$. We assume the boundary finitely generated and we can in a suitable topology assume that given $F_T$, there is a domain of $\gamma$ such that the dependence of $T$ is as prescribed and $F_T(\gamma) = f(\tau_T\zeta)$. The
Note the symmetry condition that if for \( T \in V, \left[ LT \right]^* = r_T^*L^* \) is algebraic in \( T \) corresponding to a real coefficients polynomial in \( \overline{T} \), we have that \( r_T^* - c_T I \) holomorphic, that is \( r_T^* \) is holomorphic modulo monotropy. This means that for an algebraic dependence of \( T \), \( \delta_T F_T \) can have level surfaces.

### 7.3. Multipliers.
As the mapping \( x \to x^* \) preserves order of zero, we see that if \( x \) is locally reduced, then the same holds for \( x^* \). Consider the system of invariant curves \( \{ C_j \} = \{(x, h^j(x))\} \). Assume \( \eta_j(x) = \frac{h(x)}{x} \) such that \( \eta_j(x) = x^{j-1}\eta(x) = x\eta_{j-1}(x) \). Assume also that we identify using monotropy, the curves \( \{ C_j \} \sim \{(x, h(x))\} \). Assume \( < \eta X, \phi > = < \eta^* \tilde{X}, \varphi > = C < \tilde{X}, \varphi > \). Thus, \( \{ \eta = \text{const} \} \to \{ \eta^* = \text{const} \} \). The condition \( \eta^*_j = \text{const} \Leftrightarrow \eta^*_j = \text{const}/x^j \) and so on. Assume existence of an algebraic homomorphism \( w \), such that \( | w^{-1}\eta^*_j - \frac{1}{x^j} | < \epsilon \) as \( | x^* | \to \infty \). Then \( \eta^*(x^*) \sim \eta_j w(\frac{1}{x}) \) as \( | x^* | \to \infty \) and that \( \eta^* \) preserves constant value in the \( x^* \)-infinity, why the projection method can be applied to \( \eta^* \). It is of no significance what level surface we start with, that is \( \eta^*_j = \text{const} \Rightarrow \eta^* \) preserves a constant value in the infinity.

**Lemma 7.1.** Given a system of invariant curves \( \{ (x, h^j(x)) \} \) such that \( h(x^*) \sim \eta_j h^j(x) \), we have that \( \eta^* \) preserves a constant value in the infinity and the projection method can be applied.

Assume existence of a finite \( j \) such that for a lifting function \( F_T, \frac{\partial F_T}{\partial T} \) is algebraic in \( T \), that is \( F_T^j \sim \alpha_j(T)F \), in a neighborhood of \( T_0 \), where \( \alpha_j \) is a locally defined polynomial. Assume \( F_T^j(x, y) = G(e^x, y) \to 0 \) as \( | x | \to \infty \) for a \( G \in H' \) and \( y = h(x) \) finite in modulus. Thus, \( G(e^x, h(x)) = G(e^x, e^{h(x)}) \) is the representation we prefer. We have also assumed that \( F_T^j \) preserves a constant value in \( x \) as \( | x | \to \infty \) and \( y \) finite. If \( \frac{\partial F(\gamma_T)}{\partial T} = F_1(\frac{\partial \gamma_T}{\partial T}) \) and \( \theta_T \in T\Sigma \), we see that if \( F_1 \) is algebraic, then \( \theta_T = F^{-1} \frac{\partial}{\partial T} F(\gamma_T) \) for a \( \gamma_T \in \Sigma \). Further, \( \gamma_T, \theta_T > 1 \to 0 \) implies \( \gamma_T \in \{ B(\gamma_T) = \mu \} \), for a constant \( \mu \) and \( B \) algebraic. Assume \( B \) self-transposed and such that \( B(F(\gamma_T)) = \hat{F}B(\gamma_T) \). A sufficient condition for \( F \) to map \( \Gamma^\perp \to \Gamma^\perp \), given that \( \Gamma \) is symmetric, is that it maps constants on constants.

For a finitely generated boundary, we have the following result. Assume the singularity at the boundary, described by \( T \), that is \( F_T - \mu \to 0 \), for a small constant \( \mu \), such that \( F_1(T) \) is close to a polynomial as \( T \to 0 \). We are assuming \( \gamma_0 \) fix at the boundary and \( \gamma_T \in \text{a neighborhood of } \gamma_0 \) with \( F_T(\gamma_0) = F(\gamma_T) \) and where \( | w^{-1}F_T - 1/T | < \epsilon \), for an algebraic homomorphism \( w \), as \( T \to 0 \). Assume \( F_T \) invertible over regular approximations, such that \( \frac{\partial F_T}{\partial T} = F^{-1} \hat{\phi} T \neq 0 \) and \( \frac{\partial^2 F_T}{\partial T^2} = \frac{\partial^2}{\partial T^2} F^{-1} \hat{\phi} T \neq 0 \). Thus, if \( F_1 \) maps constants on constant, we have that \( F \) maps singular points on singular points and regular points on regular points.

### 7.4. The orthogonal to the boundary.
Assume \( F \) preserves a constant value as \( | x | \to \infty \) and \( | y | \to \infty \). Assume \( F(\eta X) = \eta^* FX \). Degenerate points are then on the form \( y^* dx^* - x^* dy^* = 0 \). If \( F \) is not algebraic, we are at least assuming that \( F^{-1} \) maps constants on constants or that \( F \sim T \) an algebraic function close to the boundary. Let \( < y^*, y > = \text{Re } < y^*, y > - 1 \), for \( y \in \Gamma \) and \( y^* \in \Gamma^0 \). Then if \( < y^*, y > > 0 \), \( \Gamma^0 \) is a line if \( \Gamma \) is a line. We write \( y^* \perp y \) for \( < y^*, y > > 0 \) and we define \( bdT^0 \) as the set where this relation holds over \( \Gamma \). If \( \Gamma^{00} = \Gamma \), then \( \Gamma^0 \subset \Gamma \). If \( < y^*, \cdot > > 1 \) is reduced, we have isolated singularities at the boundary.
We are considering $< y, y >_1 = 0$ for all $y \in \Gamma$, we have that $\Gamma \subset \text{bd } \Gamma^0$. Since $y \to y^*$ is a contact transform, we have $N(\Gamma) \subset N(\Gamma^0)$. Further, assuming $y^* \to y^{**}$ is a contact transform with $y^{**} \sim_0 y$, we get $\text{rad}(\Gamma) \sim \text{rad}(\Gamma^0)$ (equivalence in sense of ideals). Consider with these conditions $\Gamma_1 = \{ y < y^*, y >_1 = 0 \ y^* \in \Gamma^0 \}$, then also $\text{rad}(\Gamma_1) \sim \text{rad}(\Gamma^0)$. Note that $(T\Sigma^\perp)$ does not completely describe the micro-local contribution.

We have a few immediate results. Let $\Gamma_T = \{ \gamma_T \gamma = r_T^T \gamma \in \Gamma \}$ and assume the boundary condition $(S.1)$.

**Proposition 7.1.** $r_T^T - 1$ is locally algebraic if and only if $\Gamma_T \sim \Gamma^\perp$.

**Proof:**
We are assuming $r_T^T \gamma \perp \gamma$ for $\gamma \in \Gamma$, that is $< r_T^T \gamma, \gamma >_1 = 0$ for $\gamma \in \Gamma$ implies $\Gamma_T \subset \Gamma^\perp$. Assume further the ramifier symmetric, in the sense that $< r_T^T x, y >_1 = < x, r_T^T y >_1$. If $< \eta^*, \gamma >_1 = 0$ for $\gamma \in \Gamma$, we have existence of $\eta \in \Gamma$ such that $< r_T^T \eta, \gamma >_1 = 0$ why $\Gamma^\perp \subset \Gamma_T$. Thus, $\Gamma_T \sim \Gamma^\perp$. Assume $\gamma_T \in \Gamma_T \Rightarrow \gamma_T \in \Gamma_T$, then the first implication follows. The converse implication is obvious. □

The micro-local contribution that is given by $\{ r_T^T - 1 = 0 \}$ is thus a subset of the contribution given by $r_T^T - 1$ algebraic. We claim that it is necessary for the application to pseudo-differential operators, to assume the Lagrange case, $\Gamma \sim \Gamma^\perp$.

### 7.5. Trèves’ curves

We define $< \gamma_T, \theta_T >_1 = \text{Re } < \gamma_T, \theta_T > - 1$. Thus, $< F(\gamma_T), i \text{ Im } \theta_T >_1 = < F(\gamma_T), i \text{ Im } \theta_T >$ and if $\theta_T \in T\Sigma$ has the property that $\theta_T \in T\Sigma \Leftrightarrow \theta_T \in T\Sigma$, we have that $\text{Re } F(\gamma_T) \perp \text{Re } \theta_T$ implies $i F(\gamma_T) \perp \text{Im } \theta_T$. Conversely, if $T\Sigma$ is symmetric with respect to the origin, we have existence of $\gamma_T$ such that $\text{Re } F(\gamma_T) \perp \text{Re } \theta_T$ implies $\text{Im } F(\gamma_T) \perp \text{Im } \theta_T$ and analyticity for $\text{Im } F(\gamma_T)$ only means that $< F(\gamma_T), \theta_T > = < F(\gamma_T), i \text{ Im } \theta_T >$. Assume $\frac{\partial}{\partial T} \Gamma^\perp \to \Gamma^\perp$, such that $< \frac{\partial}{\partial T} F(\gamma_T), \theta_T > = 0$ implies $< \frac{\partial}{\partial T} F(\gamma_T), \theta_T > = 0$. Over $\{ F_T = \frac{\partial}{\partial T} F_T \}$, we have that if $F_T \perp \theta_T$, we have existence of Trèves curves. Conversely, given existence of $\gamma_T$ such that $< \frac{\partial}{\partial T} F(\gamma_T), \theta_T >_1 = 0$ implies $< \frac{\partial}{\partial T} F(\gamma_T), \theta_T >_1 = 0$, why $< \frac{\partial}{\partial T} F(\gamma_T) - \frac{\partial}{\partial T} F(\gamma_T), \theta_T >_1 = 0$ which is always true for real $T$. We conclude as has been noted before that the condition that $F_T$ is analytically hypoelliptic does not imply that the real and imaginary parts are analytically hypoelliptic.

Note that it is possible to have $(dH)$ has a global pseudo-base, when the pseudo-base for $(I)$ is only local. However,

**Proposition 7.1.** If $(J)$ is a finitely generated ideal of Schwartz type topology and with a compact ramification, such that $r_T^T \phi / \phi \sim_m 0$ in the $\zeta$- infinity. Then, $(I) \sim (r_T^T J)$ has a global pseudo base.

**Proof:**
We are considering $(I) \sim (r_T^T J)$, where $(J) = \ker h$ and as before $\eta(\phi) = h(\phi) / \phi$. Through the conditions, we can satisfy $| r_T^T h(\phi) | < c + | h(\phi) |$, for a constant $c$ and for $\phi \in (J)$, so $r_T^T$ is quasi conformal and $(I)$ is finitely generated, if $(J)$ is. Given that $h$ is algebraic and such that $h^N = 1$, we can show that $h(r_T^T \phi) \sim_m 0$ and by choosing $\phi$ reduced, we have that $\eta \sim_m 0$ over $(I)$. If $h$ is analytic, we assume locally $h$ is monotropic to an algebraic homomorphism. Thus, we can find an entire function $\gamma$ such that $\gamma \sim_m \eta$. □

### 7.6. Analytic set theory

Starting with the boundary condition in a higher, finite order derivative $F^{(j)}_T = \text{const}.$ implies $T = 0$. Consider the sets $\Sigma_1 = \{ \zeta_T F_T = \text{const.} \}, \Sigma_2 = \{ \zeta_T F_T = \text{const.}, F_T^{(1)} = \text{const.} \}$ and so on. This gives a finite
sequence $\Sigma_j \downarrow \{T = 0\}$. We can form the corresponding ideals in $\gamma$, such that $N(I_j) = V_1 \cup \ldots \cup V_j$, where $V_j = \{\zeta_T \quad F_T \text{ not constant}\}$. If we assume algebraic dependence of the parameter for $F_T - \text{const.} I$ and that we have a neighborhood of $\zeta_T$ that is a domain of holomorphy. Then the sets $V_j$ as geometric complements of algebraic sets, are analytic. We have noted examples where $V_1 \not\subset V_2$. We also note the following example, assume $F_T = \alpha_T/\beta_T$ such that $\alpha_T' = \gamma_T \beta_T'$, where $\gamma_T$ is assumed non-constant and regular holomorphic (not-Fuchs equation), then if $V_1, V_2$ are analytic, then since $V_2 \not\subset V_1$, the inclusion $V_1 \subset V_1 \cup V_2$ is strict and we have for the corresponding ideals $I_2 \subset I_1$. For a parabolic approximation, the set $\{\zeta_T \quad \theta_T(\zeta) > 0\}$ is the geometric complement to a first surface, which is with the conditions above an analytic set. Note that we may still have that the sets $V_i^\perp$ are first surfaces.

A different argument can be given using Trèves-curves. Assume $\Omega$ a domain of holomorphy with $\frac{\partial}{\partial T}\gamma_T \neq 0$ on $\Omega$. If $\int_{\Omega} (\frac{\partial}{\partial T}\gamma_T) \theta_TdT = 0$ and if we assume the integrand holomorphic, we must have $\theta_T = 0$ on $\Omega$, assuming it of positive measure. Assume $bd\Omega$ on one side locally of a hyperplane, then we have that $\gamma_T \neq 0$ on $\Omega$. Assume now $\Delta \subset \Omega$ where $\Delta$ is algebraic. Then, $\gamma_T$ has no zero’s on $\Delta$, but is not constant. We have assumed that constant functions are not holomorphic and we must also assume that they are not algebraic. For instance the complement to a first surface in a domain of holomorphy is not necessarily an analytic set. Note further that if $\Delta$ is algebraic, we can assume $\Delta^\perp$ is not algebraic and with the conditions under hand, it must be a first surface. Thus, we have that $\Omega \setminus \Delta$ is analytic and simultaneously $\Delta^\perp$ is a first surface.

7.7. A Tauberian problem. Let $V_N = \{\zeta_T \quad e^{M_N(\eta_T)} \quad M_N(\eta) \quad \text{subharmonic}\}$, which is a subset to $V_{N+1}$. Assume $V$ the set corresponding to $\eta$ subharmonic. Consider the complement set in $V_N$, $V^c = \{\zeta_T \quad \eta > M(\eta)\}$, then $V \subset V_N$, through the conditions and $V^c$ is analytic, if $V$ is analytic. We have that $\log X_1 \sim I_1 = \{\eta \geq 0\}$, In the same manner we consider $I_2 = \{\eta_2 \geq 0\} = \{\eta + M(\eta) \geq 0\}$, 

\[\ldots, I_N = \{\eta_N \geq 0\} = \{\sum_{j=0}^{N-1} M(j)(\eta) \geq 0\}.\]

Associated to these ideals, we consider $J_1 = \{\epsilon^\eta \quad \eta \geq 0\}$ and so on and we have if $e^{N_N}$ is analytic, that $N(J_N)$ contains a path $\zeta_T$ that is continuous. The proposition is thus that given a Tauberian condition, we have existence of a continuous approximation of a singular point. For instance if we have that $J_N$ is defined by an analytic function and the set $V^c$ above is analytic, then we have existence of a continuous path in $N(J_1)$.

7.8. The distance function. If $e^\theta$ represents the distance to isolated (essential) singularities, all situated on a finite distance from the origin, then this distance function is globally reduced. For a holomorphic function $u$, bounded in the infinity, we must have that the distance to essential singularities is finite. It is sufficient to consider points $P$ in a punctuated neighborhood of the origin. For a harmonic function $u$, we have that it is bounded in the finite plane, and we only have to apply Phragmén-Lindelöf’s theorem. Consider the representation $u = e^{\theta + m_1}$ harmonic, where $m_1$ is symmetric. Over a parabolic approximation where $-\theta - m_1$ is subharmonic, we assume $m_1 \to 0$ close to the boundary $\Gamma$. Assume now $d$ globally reduced and that $d \to 0$, as $P \to P_0 \in \Gamma$. Further, $\frac{1}{d}(P) \to \infty$, as $P \to \Gamma$ and $d(\frac{1}{d}(P)) \to \infty$, as $P \to \Gamma$. Then we can find $\epsilon$ small such that $|d(\frac{1}{d}(P)) - \frac{1}{d}(P)| < \epsilon$, as $P \to \Gamma$. If all singularities for $u$ are at a finite distance from the infinity, we have that $d(P) \to 0$, when $P \to \Gamma$ over the set where $\frac{1}{d}(P) > 0$ and on this set it is clear that in the limit $P = P_0$. The singularities at the boundary are assumed removable. Assume $P_0 \not\in 0$ and that, for instance $d(P) \sim |P - P_0|$, then $d(\frac{1}{d}(P)) \sim |\frac{1}{d} - \frac{1}{d_0}| \to 0$ implies that $P = P_0$, using reducedness for $d$ and we can conclude that the Dirichlet problem
\[ \Delta u = 0 \] on a set (it is sufficient to assume parabolic) with boundary value \( \frac{1}{T} \) is solvable, modulo monotropy.

**Proposition 7.1.** Assume the boundary holomorphic and only with parabolic singularities, then there is a regular approximation of a singular point that will reach the point.

### 7.9. Localization at the boundary.

Assume \( P(\delta_T) \) is the operator used to define the boundary condition, such that \( P^2(\delta_T) \) is hypoelliptic. \( C_T(\phi) = P(\delta_T)\phi - \phi P(\delta_T) \), where \( \phi \) is a real test function and where \( P \) is assumed such that \( P^2 \) is hypoelliptic. Thus, \( P(\delta_T)(\phi F_T) = \phi P(\delta_T)F_T + C_T(\phi) \). If \( P \) is hypoelliptic, we have that \( C_T \prec \prec \text{Re} P_T \). Otherwise, we will assume that \( P^2(\delta_T)(\phi) - \phi P(\delta_T)P(\delta_T) + P(\delta_T)\phi P(\delta_T) - \phi P^2(\delta_T) \sim \text{Im} P^2(\delta_T) \prec \prec \text{Re} P^2(\delta_T) \). Thus, if \( P(\delta_T)C_T(\phi) + C_T(\phi)P(\delta_T) \sim \text{Im} P^2(\delta_T) \). As \( T \to 0 \), we have that \( C_T(\phi) \to 0 \) (we assume \( \phi = 1 \) at the boundary). Using Nullstellensatz, that is \( PC_T + CT_P \prec \prec \epsilon \), for large \( T \) (and real). Let \( (PC_T)^* = C_T^*P^* \) and if \( P^* = P \) implies \( C_T^* \sim -C_T \). Symmetry with respect to \( * \) gives \( C_T \prec \prec \text{Re} P \) in the infinity, for \( P \) such that the square is hypoelliptic.

Assume now that the boundary condition is given by a differential operator (reduced) \( P(x, \frac{d}{dx}) \) such that there is a function \( g_N \in L^1 \) in the parameter close to the boundary, \( (g_N = M_N(f)) \) with \( P(x, \frac{d}{dx})g_N = I - \tau_N \), where \( \tau_N \) is regularizing as a pseudo differential operator. If we regard \( g_N \) as an operator \( L^1 \to D_{L^1} \), we can construct \( g_N \) as an operator with kernel \( G_N \in D_{L^1} \), that is a parametrix, \( g_N(\phi) = \int G_N(x,y)\phi(y)dy \). Given a parabolic boundary condition, we can assume \( P(x, \frac{d}{dx})g_N \in D_{L^1} \) for \( x \) in a neighborhood of a point \( x_0 \) at the boundary and with \( \text{sing supp } g_N = \{x_0\} \). Assume \( \phi \) a regular approximation of the singular point such that \( P(x, \frac{d}{dx})\phi = 0 \) implies \( x = x_0 \). For the parametrix, we then have \( m(N, P(x, \frac{d}{dx})\phi) >> 0 \) implies \( x = x_0 \) (modulo regularizing action), that is \( P(x, \frac{d}{dx})G_N \phi \) implies \( x = x_0 \).

### 7.10. Further remarks on the boundary.

If \( r_T^* \) is an algebraic homomorphism, then \( r_T^*e^\phi = e^{r_T^*\phi} \), and consequently \( \int_{(1)} r_T^*e^{r_T^*\phi}d(T) = \int_{(1)} e^{r_T^*\phi}dz(T) = 0 \), implies \( m(I) = 0 \) (measure zero set), using a result by Hurwitz. If \( e^{\psi_T^*} - 1 \) is holomorphic, we have either that \( e^{\psi_T^*} \equiv 1 \) or \( m(I) = 0 \).

**Lemma 7.2.** Assume \( e^{\psi_T^*} - 1 \) is holomorphic in the parameter \( T \) with \( \psi_T^* \) algebraic in \( T \). We then have that \( \int_{(1)}(e^{\psi_T^*} - 1)dz(T) = 0 \) implies \( m(I) = 0 \).

We also note the following consequence of the condition on vanishing flux, \( \int_{(C)} Mdzdy = \int_{(C)} M^{\phi} = 0 \), means that there is a trajectory \( \gamma \) such that \( M(\gamma) = 0 \) in \((C)\). Particularly, if \( f \) is such that \( \int_S M_N(d\hat{f})^{\phi} = 0 \) that is we have \( \int_{(S)} \Delta M_N(f)dzdy = 0 \), we must have that \( M_N(f) \) changes sign in points inner to \((S)\).

### 7.11. A very regular boundary.

The boundary is said to be very regular, if the singularities are located in a locally finite set of isolated points or segments of analytic curves (cf. [13]). Thus, we are assuming that if \( f_0 \) is a boundary element, then a very regular representation of the boundary preserves the locality of singularities, but not necessarily the order. Assume \( \Gamma = \{\Gamma_j\} \) is a locally finite set of analytic curves, where the set of common points is a discrete set. Given an element in \((B_m)^r \subset D_{L^1}\), we know for the real Fourier transform, that \( \hat{f} = P(\xi)f_0 \), where \( f_0 \) is a continuous function in the real space and \( P \) a polynomial. Extend \( f_0 \) to
a continuous function in a complex neighborhood of the real space and denote \( \tilde{f}_0 \) the function such that \( \tilde{f}_0 \) is the extended function. More precisely \( \tilde{f}_0 \) has a very regular representation at the boundary, with isolated singularities. Then \( f = 0 \) from \( P(\zeta) = 0 \), gives an extension of singularities to \( \Gamma \), locally algebraic segments. At the boundary, in a complex neighborhood of the real space, we are considering the symbol as \( F(D)f_0 \).

Consider in \( D^L_{\tilde{L}} \), \( R(\zeta)f_0(\zeta) \), where \( f_0 \) is the Fourier transform of a very regular operator, that is \( F(\gamma) \sim R(D)\tilde{f}_0 \), where \( \tilde{f}_0 \big|_{\mathbb{R}} = f_0 \) and \( \tilde{f}_0 \) is very regular. Consider \( \Gamma \to \Gamma^* \) through a simple Legendre transform. If we assume \( \Sigma \) discrete and that all approximations of \( \Sigma \) through \( \Gamma \) are regular (transversal intersection), then we can assume existence of a norm \( \rho \), such that \( \rho(z) \leq 1 \Leftrightarrow z \in \Gamma \). In conclusion, we are assuming a very regular boundary continued to \( \delta_\Gamma - \gamma_\delta \), that corresponds to a normal tube in \( \Omega \), thus that all singularities are situated on first surfaces. The parabolic singularities can be given by a very regular boundary, that \( \delta \Gamma \) is singular for \( f \geq 0 \) (possibly multi-valued). The condition on vanishing flux, \( \int_{\partial\delta}\langle dLf \rangle^h = \int_{\partial\delta} df^h = 0 \), since \( df^\delta = 0 \) for \( \delta = 0 \). That is if we have a “planar” reflection through the boundary, this is preserved by the boundary representation.

7.12. A very regular representation. For a boundary operator \( L \), a very regular representation is given by \( L(f) = f + \gamma_\delta(f) \), where \( \gamma_\delta \) is regularizing, for \( \delta > 0 \). Note that for a finite \( N \), the term \( M_{\delta} \gamma_\delta \) is still regularizing, for \( \delta > 0 \). We note that in this representation, the locality of singularities is not affected by the means, but the order of singularity is decreased by the mean and increased by the mean of negative order. Thus, given singularities of finite order, we must have that application of a finite order mean, decreases the set of singularities. Through the result from Iversen, we see that the set of singularities in a very regular boundary, must be assumed of measure zero. Note however, that if \( M_N(f) \) is locally injective, then the corresponding \( M_{\delta} \gamma(f) \) is locally surjective. We are assuming the neighborhood of \( \Gamma \) one-sided, that is \( \gamma_\delta(f) \geq 0 \) for \( f \geq 0 \). The condition on vanishing flux, \( \int_{\partial\delta} dL^h = 0 \), since \( d\gamma^\delta = 0 \) for \( \delta = 0 \). That is if we have a “planar” reflection through the boundary, this is preserved by the boundary representation.

7.13. The extended Dirac distribution. Assume \( \Sigma \) a set of common points for finitely many analytic curve segments, a discrete set without positive measure. The boundary condition corresponding to a very regular boundary can now be formulated, \( f_T \) is regular outside \( \Gamma_1 \cup \ldots \cup \Gamma_N \), that is at least one of the segments \( \Gamma_j \) is singular for \( f_T \). This means that if the boundary element is \( \delta_\Gamma - \gamma_\delta \), then \( [\delta_{\Gamma_1}, \delta_{\Gamma_2}] \neq [\delta_{\Gamma_2}, \delta_{\Gamma_1}] \). Only points in \( \Sigma \) give raise to a commutative system. Further, the system will be finitely generated in the sense that finitely many (sufficiently many) iterations of \( \delta_{\Gamma_1} \), will for different \( \Gamma_j \) produce regular points. More precisely, assume \( \Gamma_1 \) is a singular analytic curve, for \( \phi \) and \( \Gamma_2 \) is a regular analytic curve for \( \phi \) except for a point in \( \Sigma \) and in \( \Gamma_1 \cap \Gamma_2 \). Then \( \phi \big|_{\Gamma_2} \big|_{\Gamma_1} \) is the result of a regular approximation of a singular point, but \( \phi \big|_{\Gamma_1} \big|_{\Gamma_2} \) gives a singular approximation of a singular point. Compare Nishino’s concept of a normal tube (cf. [17]).

Consider \( \delta_{x_0} \to \delta_T \), where \( \Gamma \) has an analytic parametrization. This means \( \lim_{\Gamma \to \Gamma_0} \phi(x) = \phi(x_0) \). Assume the set \( \Sigma = \cap_{\Gamma} \Gamma_j \) discrete (compact), that is algebraic. If \( r_{\Gamma,\Sigma} \) is the restriction homomorphism \( H_1(V) \to H_\Sigma(V) \), for a domain \( V \) and if \( T \in H_1(V) \) and we have existence of \( U_T \in H_1(V) \) such that \( T = r_{\Gamma,\Sigma}(U_T) \), then we say that \( T \) has a continuation to \( \Gamma \). Thus, we can see \( \delta_T \) as
an operator $L$ to a very regular boundary. If $V$ is a domain of holomorphy and if the definition of a normal operator $L$ at the boundary, is not dependent on choice of $\Gamma$, we say that $\{\Gamma\}$ is a quasi porteur for $\delta \Gamma$. When $\Gamma$ is analytic, we say that it is a porteur.

8. **A Monodromy Condition**

8.1. **Introduction.** Since we are discussing the symbol in $(B_m)'$, there is no obvious monodromy concept that can be assumed. Assume condition $(M_1)$ is the condition that all means are of real type. For $f \in (B_m)'$ we have a local representation $f \sim P(D)f_0$, where close to the boundary $f_0$ is very regular. Assume the local condition $(M_2)$ is the condition that $P(D)$ is hypoelliptic. Finally, through a fundamental theorem in Fredholm theory, we have existence of a parametrix to $L_\lambda$ hypoelliptic in $L^1$. In presence of lineality, we will later argue that it is a porteur.

Assume $pr_1: L^1 \rightarrow R(L_\lambda)$, for a formally self-adjoint and hypo-elliptic differential operator. Assume $L_\lambda \in \phi(D'_L, L^1)$ (unbounded Fredholm-operator) then $D'_L = X_0 \otimes N(L_\lambda), L^1 = Y_0 \otimes R(L_\lambda)$ and the inverse $B_\lambda$ is bounded in $(L^1, D'_L)$.

Through a fundamental theorem in Fredholm theory, we have existence of $B_\lambda \in \Phi(L, D'_L)$ such that $B_\lambda L_\lambda = I$ on $X_0$ and $L_\lambda B_\lambda = I$ on $R(L_\lambda)$.

**Proposition 8.1.** Assume $E_\lambda$ a parametrix to $L_\lambda$ a hypoelliptic operator in $L^1$, then $(\delta \varepsilon - E_\lambda)$ is regularizing. Conversely, if $E_\lambda$ is regularizing, then $\delta \varepsilon - E_\lambda$ is hypoelliptic.

We give a short proof, if $R$ is regularizing then $|| \psi Ru ||_s \leq C$, for a constant $C = C(\psi, s)$ and $\psi \in C_0^\infty$ and a real $s$ and $u \in D'$. We can write $|| \psi u ||_s \leq C || \psi(A - R)u ||_s$, where $A = E - I$ is hypoelliptic. If $E$ is a parametrix $PEu - u \in C_\infty$, for a $u \in D'$ and if $P$ hypoelliptic in $L^1$, $PEu - Pu \in C_\infty$ implies $Eu - u \in C_\infty$, that is $(\delta \varepsilon - E)$ is regularizing. The result can be extended to $D'^r$, then $I_E - I$ is regularizing in the space of $D'^{r}$, but the result cannot be extended to $D'$. The condition $(M_3)$ is the condition that the symbol considered as a parametrix to a boundary operator has a trivial kernel.

8.2. **The (M)-conditions and orthogonality.** Assume $T_\gamma$ corresponds to analytic continuation. We will assume that $T_\gamma$ can be divided into translational movement and rotational movement, not necessarily independent. For $V = (V_1, V_2)$ the vorticity is given as $w = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$. Given that $\frac{\partial w}{\partial x}$ is bounded in the infinity, we have that $\frac{\partial w}{\partial x} = 1$ in the infinity iff $w = 0$ in the infinity, that is if $\frac{\partial w}{\partial x} = e^{-\varphi} \rightarrow 0$, in the infinity, is a “simple zero”. If $e^{-\varphi} \sim P(1/T) \sim c_0 + c_1/T + \ldots$, then we must have $c_0 = 0$ and $c_1 \neq 0$. If we compare with the global problem, the condition $r_T \varepsilon f - f = 0$ implies $T = 0$ is locally a condition $(M)$ and $r_T \log f - \log f = 0$ implies $T = 0$ is related to local parabolicity. The condition that $r_T \log f = \log r_T f$, means that the ramifier only contributes to the phase. If we assume $V_1 \perp V_2$, we can find a polynomial $P$ such that $\frac{1}{T} \sim_m \frac{1}{T}$, in the infinity.

Note in connection with the conditions $(M)$, if we assume $V_1 \perp V_2$, then for a hypoelliptic symbol both $V_1$ and $V_2$ will be unbounded in the infinity and thus the respective inverse is bounded in $\infty$. In presence of lineality, we will later argue that the imaginary part can be assumed bounded, where we assume $\text{Im } F_{1/T} \rightarrow 1/| \text{Im } F_T|$ preserves conditions $(M)$. Thus, in order to discuss the conditions $(M)$ using only bounded symbols for orthogonal parts, it is necessary to assume $F_T^{adj} - F_T$ bounded, where $F_T^{adj}$ is the adjoint symbol in $(B_m)'$. 

32  T. DAHN
8.3. A condition \((M_2)\) operator at the boundary. We now wish to define a formal condition \((M_2)\) operator on \((\mathcal{B}_m)^\prime\). This operator can be used to define a global condition \((M_2)\). For such an operator, we must have that \(\hat{T}_\gamma\) has representation with a point support measure \(\mu_0\). For a condition \((M_2)\) with respect to paths, the limit must be independent of starting point, why we must assume \(\mu_0 \in \mathcal{E}(0)\), that is of order 0. If we have the parabolic property, we must further assume the measure is positive in phase space. For a formal \((M_2)\)-operator, we only assume \(F \in L^1\) implies \(\hat{T}_\gamma F \in (\mathcal{B}_m)^\prime\). Consider now \(\hat{T}_\gamma\) modulo regularizing action, that is \(\hat{T}_\gamma \sim \delta_0 + \nu_\gamma\), where \(\nu_\gamma\) has support outside a point. If we assume \(\hat{T}_\gamma F \in \mathcal{B}_m\), we have \(\hat{\hat{T}}_\gamma F = Q f_0\). We write \(\hat{T}_\gamma F = Q(D) f_0\), where \(f_0\) is a regular operator. Assume \(F \in \mathcal{B}_m\), then we have \(F \sim Q_\lambda f_0\). We can in this context consider \(f_0\) as a global representation. Given that \(Q_\lambda\) is hypoelliptic and such that \(Q_\lambda \hat{T}_\gamma = \hat{T}_\gamma Q_\lambda\) we see that in the case with condition \((M_2)\), \(\hat{T}_\gamma\) preserves parametrices to \(Q_\lambda\). Conversely, given that \(\hat{T}_\gamma\) preserves very regular parametrices to \(Q_\lambda\) (that is we have condition \((M_2)\)) then we can derive that \(Q_\lambda\) is hypoelliptic. The conclusion is that with the conditions above, we have that hypoellipticity for \(Q_\lambda\) means that \(\hat{T}_\gamma\) defines condition \((M_2)\) and conversely. Note also that \(\hat{T}_\gamma\) can be globally represented in \((\mathcal{B}_m)^\prime\). Given \(Q\), we can define the possible continuations that preserve condition \((M_2)\), as \(\hat{T}_\gamma\) such that \(\hat{T}_\gamma F = Q(D) f_0\). For instance if \(\hat{T}_\gamma F = F + F_1\), we must assume \(Q F_1 = 0\), why \(F_1\) is regularizing. If on the other hand \(\hat{T}_\gamma F = c F\), for a \(c\) in the \(\infty\) and \(\hat{T}_\gamma Q = Q^*\), then for all real \(c\), we have \(c Q - Q c \prec Q\), that is \(\hat{T}_\gamma Q = Q^*\gamma\) is modulo regularizing action. Thus, localizing with a real \(c\) is possible and corresponds to localizing \(f_0\). Given that \(F\) is hypoelliptic and represented as \(Q f_0\) at the boundary, this property can be continued using \(\hat{T}_\gamma\), given that this is an algebraic argument. Assume for this reason, \((\hat{T}_\gamma F) \varphi = \int_{\Omega} k_\gamma \varphi d\sigma = 0\) implies \(Q \varphi = 0\) on \(\Omega\) or \(\sigma(\Omega) = 0\), that is in the case where \(\hat{T}_\gamma\) is algebraic, we have condition \((M_2)\). Note that \(\int_{\Omega} f_0 \varphi d\sigma = 0\) implies \(\sigma(\Omega) = 0\), which corresponds to "isolated zeros" to \(f_0\). In the case where \(\hat{T}_\gamma f_0(\varphi) = 0\), when \(\hat{T}_\gamma f_0\) is algebraic, we see that \(\sigma(\Omega) = 0\), that is we have condition \((M_2)\). Note that if \(\hat{T}_\gamma f_0 = 0\) along a line in \(\Omega\), this can be compared with the extension \(\delta_0\) to \(\delta_\gamma\), where condition \((M_2)\) is no longer with respect to a point.

**Lemma 8.2.** Assuming the symbol has a representation in \((\mathcal{B}_m)^\prime\) satisfying condition \((M_2)\), then this property is preserved if the continuation \(\hat{T}_\gamma\) as above is algebraic.

Condition \((M_2)\) does not however imply that \(\hat{T}_\gamma\) is algebraic.

8.4. The operator \(T\) and the conditions \((M)\). We have that \((w_\mu)^\delta = (w^\delta)_\mu\) but \((Tw^\mu) \neq T(w_\mu)^\delta\). Assume \(w = W/M \sim e^{\varphi} \rightarrow T e^{\varphi} = e^{\varphi}\). The problem is when \(e^{\varphi}\) respects the conditions \((M)\). Assume \(H(w) = w^\delta\) and \(T\) Legendre and that we have algebraic dependence of the parameter \(T\), then \(TH_\mu - T_\mu TH \sim_0 e^{P(\varphi)} - e^{P(T)\varphi}\), for a polynomial \(P\). The condition \((M_2)\) means that \(e^{P(\varphi)} - e^{P(T)\varphi}\) is not \(\sim 1\). Thus, the operator \(T\) does not necessarily preserve the condition \((M_2)\). The same holds for the conditions \((M_1),(M_3)\).

If the operator \(T\) is studied using \(T_\gamma\) acting on the Legendre transform, we see that an algebraic continuation in the infinity implies that we do not have linearity and further the conditions \((M_1)\) and \((M_2)\) in parameter infinity. If we only assume the continuation very regular on all strata, we still have conditions \((M_1)\), \((M_2)\) and \((M_3)\), but not necessarily an algebraic continuation. We can consider it to be algebraic modulo monotropy locally. Finally note that an algebraic continuation does not necessarily preserve condition \((M_1)\). However, by considering symbols modulo
regularizing action, we can restrict the representation to real type symbols. For this representation the corresponding functional is an infinite sequence of constant coefficients polynomials acting on a point support measure.

9. Further remarks on algebraicity

9.1. Introduction. An algebraic set is geometrically equivalent to a zero set of polynomials. Characteristic for an algebraic mapping $L$ is that $L(e^{p}) = e^{L(p)}$ and the zero set to an algebraic mapping is locally an algebraic set. The identity (evaluation) is considered as algebraic and we consider any operator that commutes with the identity in $H'$ as (topologically) algebraic.

9.2. Clustersets for multipliers. We will prove that given that $M \perp W$ implies $V_{1} \perp V_{2}$, we have that $\mathcal{T}V^{-N}$ algebraic in $\infty$ if we have hypoellipticity. Assume for this reason $V_{1} \perp V_{2}$ with $\frac{\nabla V_{1}}{V_{1}} = e^{-\vartheta^{*}} \to 0$, as $|\xi| \to \infty$ If $\mathcal{T}V_{1} = e^{\vartheta^{*}}\mathcal{T}V_{2}$, then we must have $e^{\vartheta^{*}} \to \infty$, as $|\xi| \to \infty$. For instance if $\vartheta^{k} = \vartheta^{*} + p$, where we assume $\frac{1}{p}$ is harmonic, in the sense that $p \to \infty$, as $|\xi| \to \infty$. Otherwise, if we assume $e^{\vartheta^{*}}$ bounded as $|\xi| \to \infty$, we have unbounded sublevel sets (cluster sets) for the multipliers. When $\mathcal{T}$ is the Legendre transform, this will disrupt the concept of monodromy in the infinity. We have noted that if $V_{T}$ is algebraic in $T$ and $F(\mathcal{L})(V) = 0$, we have that $\text{Im} V_{T} \equiv 0$. Note that if $\mathcal{T}V^{-N}$ is taken as the limit over strata, in case the symbol is not hypoelliptic, we have distributional limits in symbol space for the representation of $V$.

9.3. Example on algebraic mappings. Consider the mapping $A[a, b] = \tilde{a}$. If $E$ is a (topologically) algebraic mapping, and if $E(e^{\varphi}) = e^{E(\varphi)}$, then $\hat{E}$ is odd and also $\tilde{A}$. The proposition that $\hat{A}^{2}$ is odd, corresponds to $E(e^{-T(\varphi)}) = e^{-E(e^{T(\varphi)})}$, thus if $A$ is (topologically) algebraic, then $\hat{A}^{2}$ is (topologically) algebraic. Note that $E(e^{-\varphi}) = \frac{1}{E(e^{\varphi})}$. If $E \sim m E_{1}$ and $E_{1}$ is (topologically) algebraic, then $|E(e^{f}) - E_{1}(e^{1/f})| < \epsilon$ in $\infty$. If $E \in L^{1}$, then $\hat{E}(f) \to 0$, as $f \to \infty$. This means that $E \sim m E_{1}$, where $E_{1}$ is (topologically) algebraic, then $|E(f) - E_{1}(\hat{f})| \to 0$. We can start with a condition $\mathcal{M}_{N}(E) \in L^{1}$, which means that $\mathcal{M}_{N}(E) \sim m W$, where $\hat{W}$ is odd.

Further, we have $A^{3}[I, E](\varphi) = E(e^{\varphi})$ and in the same manner $A^{3}[E, I](\varphi) = e^{E(\varphi)}$. Thus, if $E(e^{\varphi}) = e^{\varphi(\varphi)}$, that is $\varphi(\varphi) = [\hat{I}, e](\varphi)$, then $\hat{E} \sim e$. If $E$ is assumed (topologically) algebraic and $\hat{E} \sim E$, then we should have that $\epsilon$ is (topologically) algebraic (modulo algebraic sets). Further, if $\epsilon$ is linear and $\hat{E} \to I$ at the boundary and if $\hat{E} \sim E$, then we must have $E - I \sim W$, where $W$ is algebraic at the boundary (with respect to concatenation of curve segments). More precisely $(E - I)(E + I) = E^{2} - I + R$, where $R$ is assumed to vanish. If $E - I \to I$ at the boundary and the same condition holds for $E + I$, we must have $E^{2} - I \to 0$, which is seen as $E$ being (topologically) algebraic at the boundary, that is $E - I \sim W(e^{-\varphi})$, where $W$ is algebraic in $T$ (compare the Lagrange condition). We are assuming $W(e^{\varphi})W(e^{-\varphi}) = W(1)$ Thus, if $W(1) = 1$, we have $W(e^{-\varphi}) = e^{W^{-1}(\varphi)}$ and the odd condition means that $W^{-1}(\varphi) = -W(\varphi)$.

9.4. The Legendre transform has removable singularities. Assume $\partial_{T}^{2}/\partial_{T} \to 0$, as $T \to \infty$ and further that $<\partial^{\varphi} \pm \vartheta, \partial^{\varphi} \pm \vartheta> = 0$ or formally $\partial^{\varphi}/\partial + \partial/\partial^{\varphi} \sim 1$. Assume $F_{T}^{2}.F_{T} \sim 1$, as $T \to \infty$ and the first condition, then the condition $v/\varphi \sim 1$
is impossible. If \( \vartheta^\circ - \vartheta \) real, then the condition contradicts symplecticity. Otherwise, we are assuming \( \text{Im} (\vartheta^\circ - \vartheta) \perp \text{Re} (\vartheta^\circ - \vartheta) \). The conclusion is that if \( \vartheta^\circ \) and \( \vartheta \) are related through a simple Legendre transform, then \( F_T^\circ \perp F \sim T \), as \( T \to \infty \) is not possible, that is the singularities are removable.

9.5. **Sufficient conditions for orthogonality.** Assume now \( M \perp W \) and consider the lift \( V(M, W) = V_1 + iV_2 \). The problem is now under what conditions we have that \( V_1 \perp V_2 \) (that is \( V_2 \ll V_1 \)). Assume \( V \) is the Hamilton function, that is \( M = \frac{\partial V}{\partial x} = \frac{\partial V}{\partial x} \) and \( W = -\frac{\partial V}{\partial y} = -\frac{\partial V}{\partial y} \). We write formally the condition that \( M \perp W \) as \( \frac{\partial V}{\partial x} (V_1 \perp V_2) \) and \( \frac{\partial V}{\partial y} (V_1 \perp V_2) \). The vanishing flux condition is \( \text{Flux}(\mathcal{T}W) = 0 \). The corresponding condition for \( \mathcal{T}V_2 \) is \( V_2 \ll V_1 \). Note that the lifting function \( F(X, Y) \) and the Hamilton function \( V(M, W) \) may have quite different algebraic properties, but are considered as related by involution and the condition on equal derivatives in the first order \( x^*, y^* \)-variables, with respect to \( (X, Y) \) arguments. For \( V_1 \) algebraic, we always have that \( \frac{\partial V}{\partial x} V_1 \perp V_1 \), that is \( \frac{\partial V}{\partial x} \log V_1 \to 0 \), as \( |T| \to \infty \). The condition \( \frac{\partial V}{\partial y} V_1 \perp V_2 \) is \( \left( \frac{\partial V}{\partial y} \log V_1 \right) \frac{\partial V}{\partial y} \to 0 \), as \( |T| \to \infty \) and if \( \frac{\partial V}{\partial y} \log V_2 \) is bounded in the infinity, then we could write the condition \( \frac{\partial V}{\partial x} V_1 \perp V_1 \) as \( \left( \frac{\partial V}{\partial y} \log V_2 \right) \frac{\partial V}{\partial y} \to 0 \), as \( |T| \to \infty \).

Let \( W/M = e^x \) and consider the iterated mean \( \mathcal{M}_N(\varphi) \), for \( N \) large. We can then assume \( \frac{\partial V}{\partial x} \mathcal{M}_N(\varphi) \) reduced at the boundary. Let \( V_1^{(N)}, V_2^{(N)} \) be the corresponding Hamilton function (locally). Consider the condition \( \left( \frac{\partial V}{\partial y} \log V_1^{(N)} \right) \frac{\partial V}{\partial y} \to 0 \), as \( T \to 0 \). This is obviously true for large \( N \). If \( \frac{\partial V}{\partial y} \log V_1^{(N)} \) is reduced at the boundary, then we have that \( V_2^{(N)}/V_1^{(N)} \to 0 \), as \( T \to \infty \). Further, \( \frac{\partial V}{\partial y} V_1^{(N)} \perp V_2^{(N)} \). A sufficient condition to conclude that \( V_1 \perp V_2 \) is that \( V \) corresponds to a reduced symbol. Note that we have assumed that \( V_1 \mid_1 \sim \frac{1}{V_1} \mid_1 \) and \( \frac{1}{V_2} \mid_1 \sim V_2 \mid_1 \).

9.6. **The orthogonal condition and the Dirichlet problem.** Assume instead of the condition \( \text{Im} F_T \ll \text{Re} F_T \), that

\[
\text{Re} F_T \mid \text{Im} F_T \to 0 \quad \text{as} \quad T \to \infty
\]

The condition is to be understood using \( T = (\text{Re} T, \text{Im} T) \in \mathbb{R}^n \), \( n = 2 \) and \( \text{Im} F_T \mid \text{Re} F_T \mid \to 0 \), as \( |T| \to \infty \), where we assume that the factors are not without support in a neighborhood of the infinity. Given that \( F_T \) is holomorphic in \( T \), we know that \( F_T \) can not be reduced in the origin and in the infinity simultaneously. Assume \( \text{Re} F_T \) reduced in the \( T^- \) infinity, we then have that \( \text{Im} F_T \) is bounded in the infinity. Assume \( F_T^\circ = F_T \), so that \( \text{Im} F_T \sim F_T^\circ - F_T \). The condition that \( F_T \) is algebraic in \( T \) close to the boundary, does not imply that \( \text{Im} F_T \) is algebraic in \( T \), close to the boundary. As we are assuming real dominance for the operator, we will assume the real part algebraically dependent on the parameter.

In the same manner, given that \( F_T \in \mathcal{D}'_{L^1} \), we do not necessarily have that \( F_T^\circ - F_T \in \mathcal{D}'_{L^1} \). Note that for the restriction to the real space, the Fourier transform to \( F_T \) is on the form \( P(\xi) f_0(\xi) \), why on the support of \( f_0 \), we necessarily have finite order of zero. We are assuming \( 0 \in \text{supp} f_0 \). We have that \( \text{Re} F_T \) can not be reduced in a neighborhood of parameter origin. If \( \text{Im} F_T \) were bounded in a neighborhood of the origin, the condition (1) could not be possible. The conclusion is that \( \text{Im} F_T \) must be unbounded at the boundary.

Conversely, if we assume \( \text{Re} F_T \) is reduced in a neighborhood of the parameter origin, then we have that \( \text{Im} F_T \) is bounded in a neighborhood of the boundary
and we have that \( \text{Im} F_T \) is necessarily unbounded in the infinity. This, does not contradict the condition \([4]\).

**Proposition 9.1.** Assume condition \([4]\). For a hypoelliptic operator, both the real part and the imaginary part are unbounded in the infinity. In presence of lineality for the real part the imaginary part is bounded in the infinity.

The conclusion is that given the condition \([4]\), if we further assume that \( \text{Re} F_T \) is reduced in the infinity, then we have that \( \text{Im} F_T \) can be assumed to be bounded at the boundary. The proposition that \( \text{Im} F_T \) is bounded in the infinity, means that it cannot be represented be a polynomial operator, unless the set of unboundedness is of measure zero. In this case if the set is normal (finite Dirichlet integral), we can give an algebraic representation for this part of the symbol. Sato gives a well known example of a (hyper-) function defined at the boundary, that is not a distribution in \( 0 \). Assume \( \vartheta_T \) algebraic in \( T \) and that \( \vartheta_T^\alpha - \overline{\vartheta_T} \sim \text{Im} \vartheta_T \), such that \( \text{Im} \vartheta_T \sim P(\alpha) \vartheta^\alpha - P(T) \vartheta \).

### 9.7. A necessary condition

Assume

\[
\frac{e^{\alpha(T)} - e^{\alpha(T)}}{e^{\alpha(T)}} \to 0 \quad \text{as} \quad T \to \infty
\]

We then have, \( e^{\alpha(T)} - \alpha(T) - e^{-2\text{Im} \alpha(T)} \to 0 \), as \( T \to \infty \). A sufficient condition for this is that \( \alpha(T) - \alpha(T) - 2\text{Im} \alpha(T) \to 0 \), where \( \alpha(T) \sim \alpha(T) \). Thus, the relationship between \( \alpha^\wedge(T) = \alpha(T) \) and \( \alpha(T) = \alpha(T) \), as a simple Legendre type condition, means that if \( e^{\alpha} \) is locally algebraic, then the same must hold for \( e^{\alpha(T)} \).

Formally

\[
\mathcal{T}(\text{Im} \vartheta_T) \sim e^{\alpha(T) \text{Im} \vartheta}_T
\]

where \( e^\vartheta_T \) is locally algebraic. Note that without the condition \([5]\), it is necessary to consider \( \mathcal{T} \) as acting on hyper-functions.

### 9.8. The complement sets to the first surfaces

Solvability in this context corresponds to the regularity conditions in dynamical systems and the corresponding conditions for first surfaces. The neighborhood \( \{ \varphi > 0 \} \sim \{ \vartheta = 0 \} \), that is \( \sum \varphi_j > 0 \) or equivalently \( \sum \vartheta_j \equiv 0 \), gives an analytic parametrization. If the set is not analytic, we do not have local solvability. \( \text{(The Fuchs condition)} \). If we do not have regular approximations of a first surface, the representation of the operator is not defined. Thus a local complement \( \{ f = c \} \) analytic, is necessary for solvability. A locally algebraic transversal is necessary for hypoellipticity. For example, if \( e^{-\vartheta_T} \to 0 \), as \( T \to \infty \), given an essential singularity in the infinity, we may have local solutions that are not analytic.

Assume \( \{ f = c \} \) a first surface to a holomorphic function. Given minimally defined singularities to \( f \), we know that the first surfaces are also analytic. This means that if \( f \in L^1 \), the complement in a domain of holomorphy is analytic. If we only assume \( f \in D'_{\text{L}^1} \), we do not have this strong result. Given that \( \Delta \) is analytic and \( \Delta \to \Delta^\perp \) is a Legendre transform, that is a contact transform, we should be able to prove \( \Omega \setminus \Delta \sim \Omega \setminus \Delta^\perp \), that is if \( \Delta^\perp \) is a first surface, it must be analytic. If \( \Delta \) is reduced, we have that \( \frac{1}{f} \log f \to 0 \) implies \( \frac{1}{f} f \to 0 \). Otherwise, if \( f \) is bounded, then the converse implication holds. Algebraic dependence implies that \( \frac{1}{f} \log f \to 0 \). Since \( \{ f = c \} \subset \{ \frac{1}{f} f = 0 \} \), we must have \( \Omega \setminus \{ f = c \} \) analytic implies \( \{ \frac{1}{f} f \neq 0 \} \) analytic. Obviously \( \frac{1}{f} \log f \neq 0 \) implies \( \frac{1}{f} f \neq 0 \) (assuming both not zero). Thus, \( \{ f \neq 0 \} \) is analytic and \( \{ f = c \} \) is analytic. In this case, if \( \{ f = c \} \) is analytic, then \( \{ \log f = c \} \) is analytic. If the complement to the
second set is analytic, the complement to the first set is analytic. The last condition implies solvability. Counterexamples can be given by \( f = \beta e^x \).

9.9. **A counterexample to solvability.** The problem that we start with is when the complement to a fist surface in a domain of holomorphy, is analytic? Assuming parabolicity, we make the approach \( e^{\phi} \), with \( \phi \) subharmonic. Assume the neighborhood of the first surface is given by \( \{ \phi \geq 0 \} \). We are now discussing \( \phi(x - y) = -\phi(y) + \psi(x, y) \). Through Tarski-Seidenberg’s result, we have if \( x, y \) in a semi-algebraic \( \{ P(x, y) \geq 0 \} \) and related through a Legendre transform, we have \( y \) in a semi-algebraic set \( \{ Q(y) \geq 0 \} \). The problem is now, if \( \{ Q(y) \geq 0 \} = \{ F_Q(y) = 0 \} \) locally, where \( F_Q \) is analytic. For instance \( R \) analytic with \( \bar{R}(x, y)F_P(x - y) = F_Q(y) \). Assume now \( x = 0 \), so \( \phi(-y) = C - \phi(y) \). If \( \phi(-y) \leq 0 \), then \( \phi(0) - \phi(y) \leq 0 \) and in the parabolic case \( \phi(-y) = C_1 \), a constant, as \( y \to \infty \). Thus, \( \phi(y) = C - C_1 \) constant. If \( \phi(-y) > 0 \), then \( C - \phi(y) > 0 \), that is \( \phi(y) \) is bounded, as \( y \to \infty \). In this case \( \{ y, \phi(y) < C \} \) is unbounded. The integral \( \int_{\phi < C} \) now corresponds to a distribution. In this case \( \{ \phi > 0 \} \) is not a semi-algebraic set. If \( \phi \) is hypoelliptic, then \( \phi \to \infty \), as \( y \to \infty \). If \( \phi \) is bounded, as \( y \to \infty \), then \( \phi \) must be not-hypoelliptic. In this case it is not possible to approximate \( \infty \) through \( \{ \phi > 0 \} \). The proposition in this case \( \int_{\phi < 0} \) can not be represented by an analytic function. Thus, if \( \phi \) has a parabolic and odd representation and if we do not have an essential singularity in \( \infty \), then there is a \( \varphi \) analytic and zero on \( \{ \phi > 0 \} \). If however, \( \phi(-y) = -\phi(y) + C \), we do not longer have representation with an analytic function.

We have that given an ideal with Schwartz type topology, finitely generated and symmetric over \( \Omega \), where \( \Omega \) is pseudo-convex, if the dependence of the parameter is in \( L^1 \) in the symbol space and algebraic in its tangent space, we can assume that we have (topologically) isolated singularities. In the case where \( \phi(-y) = -\phi(y) + C \), the approximation of the singularity only exists in a weak sense.

10. **Not hypoelliptic operators**

10.1. **Introduction.** To determine the class of hypoelliptic pseudo differential operators, we will first assume the operator has representation as an unbounded Fredholm operator with symbol in the radical to the ideal of hypoelliptic operators. In the last section we consider a different representation based on Cousin (cf. [5]).

10.2. **Pseudodifferential operators as Fredholm operators.** From the theory of linear Fredholm-operators, we know that any Fredholm operator \( A : E_1 \to E_2 \) between Banach spaces, has a twosided Fredholm inverse, that is there is a \( B : E_2 \to E_1 \), such that \( BA = I - P_1 \), \( AB = I - P_2 \), with \( P_1, P_2 \) finite rank operators, \( P_1 \) is the projection \( E_1 \to \ker A \) and \( (I - P_2) \) is the projection \( E_2 \to \im A \). Conversely, given an operator \( A \), continuous and linear \( E_1 \to E_2 \), such that we have operators \( B_1, B_2 \) with \( B_1A = I + R_1 \), \( AB_2 = I + R_2 \), \( R_1, R_2 \) compact operators, then \( A \) is Fredholm. Finally, the class of Fredholm operators is invariant to addition of a compact operator.

For our pseudodifferential operator \( A \), given existence of left- and right parametrices, the operator \( A \) is Fredholm and we have a Fredholm inverse. We have earlier noted that our pseudodifferential operator \( A \) can be compared with polynomial operators according to \( H_\lambda = A_\lambda - P_\lambda \), with \( H_\lambda \) regularizing. Any parametrix to \( P_\lambda \) can be considered as a parametrix to \( A_\lambda \). Assume now \( B_\lambda \) the Fredholm inverse to \( P_\lambda \), modified as in the preliminaries. We then know that \( B_\lambda - I \) is regularizing outside \( \ker B_\lambda \) as \( | \lambda | \to \infty \). We shall see below, that this is a
"radical" property, which means that for \( \varphi \notin \text{Ker} \, B_\lambda \) we have \( \text{sing supp}_{L^2}(\varphi) = \text{sing supp}_{L^2}(B_\lambda P_\lambda \varphi) = \text{sing supp}_{L^2}(P_\lambda \varphi) \). Naturally, \( A_\lambda \) has the same domain for hypoellipticity as \( P_\lambda \). Assume \( E_\lambda \) a left- and \( F_\lambda \) a right-parametrix to \( A_\lambda \) and \( B_\lambda \) the modified Fredholm parametrix to \( P_\lambda \). Then \( A_\lambda (E_\lambda - B_\lambda) = R_1 + P_2 \). \( R_1 \) regularizing and \( P_2 : E_2 \to \text{Im} \, P_\lambda \) continued with regularizing action outside \( \text{Ker} \, B_\lambda \). As \( A_\lambda \) is hypoelliptic outside \( \text{Ker} \, B_\lambda \), \( E_\lambda = B_\lambda + \Gamma_1 \), with \( \Gamma_1 \) regularizing. In the same manner \( F_\lambda = B_\lambda + \Gamma_2 \), with \( \Gamma_2 \) regularizing. The construction gives that \( \text{Ker} \, B_\lambda \) is a finite dimensional space and on this space any parametrix to \( A_\lambda \) is either regularizing or 0. We can assume \( \Gamma_j = 0 \) on \( \text{Ker} \, B_\lambda \).

**Proposition 10.1.** Assume \( B_\lambda \) the modified Fredholm inverse to \( P_\lambda \) as above, that is \( B_\lambda P_\lambda = I - P_1 \), where \( P_1 \) is regularizing and \( P_2 B_\lambda = I - P_2 \) with \( P_2 \) regularizing outside \( \text{Ker} \, B_\lambda \). Further \( A_\lambda \) a pseudodifferential operator so that \( H_\lambda = A_\lambda - P_1 \) with \( H_\lambda \) regularizing, then \( A_\lambda \) is hypoelliptic in \( L^2 \) if and only if \( K_\lambda = \{ 0 \} \).

The proposition can be read as follows, for a hypoelliptic pseudodifferential operator in our class \( A_\lambda \), we have that for \( \varphi \in L^2 \), \( B_\lambda \varphi = 0 \) implies \( \varphi = 0 \). The following Lemma is trivial.

**Lemma 10.2.** If \( P_\lambda^N \) is hypoelliptic, then \( B_\lambda^N - B_{N\lambda} \in C^\infty \).

By choosing \( \lambda \) appropriately, we can assume \( N = 2 \).

**Lemma 10.3.** For \( u \in L^2 \), \( \text{sing supp}_{L^2}(B_\lambda u - u) = \text{sing supp}_{L^2}(B_\lambda u + u) \).

Proof: Assume \( N = 2 \). We obviously have \( P_2\lambda - I \sim P_2\lambda + I \), which means that the singular supports for \( P_2\lambda(B_\lambda + I)u \) and \( P_2\lambda(I - B_\lambda)u \) coincide. Thus, the lemma holds for \( B_\lambda \). Finally, the singular support for \( (I + B_2\lambda + B_\lambda B_\lambda)u \) is the same as the one for \( (I + B_2\lambda - B_\lambda)u \). \( \square \)

According to Lemma 10.2 \( (B_\lambda - I)(B_\lambda + I) + I - B_2\lambda \) is regularizing and according to Lemma 10.3 this implies \( (B_\lambda - I)^2 + I - B_2\lambda \) is regularizing. So, if \( I - B_2\lambda \) is regularizing we have \( I - B_2\lambda \) is regularizing outside \( \text{Ker} \, B_\lambda \).

**Proposition 10.4.** If \( P_\lambda^N \) is hypoelliptic, then \( I - \sum_{j=1}^{N-1} B_\lambda^j \) is regularizing outside \( \text{Ker} \, B_\lambda \).

Proof: We have \( (B_\lambda - I) \sum_{j=0}^{\infty} B_\lambda^j = I \). Thus, \( (B_{N\lambda} - I) \sum_{j=0}^{\infty} B_\lambda^j + \sum_{j=1}^{N-1} B_\lambda^j \sim I \), where \( (B_{N\lambda} - I) \) is regularizing outside \( \text{Ker} \, B_\lambda \). \( \square \)

Assume one more time \( N = 2 \), then \( N(B_2\lambda) \subset N(B_\lambda) \) and if \( \varphi \in N(B_\lambda) \), according to Lemma 10.2 we have \( B_2\lambda \varphi \in C^\infty \). This gives good estimates using Hadamard’s lemma as we have, \( \| B_{N\lambda} \|_C \leq C_N \| \lambda \|^{-N} \). \( N \geq 2 \). Finally, since \( B_2\lambda \) is a \( L^2 \)-kernel of finite rank, we can find a canonical kernel \( K_{2\lambda} = \sum_{i,j} k_{ij} e_i \otimes e_j \) such that \( \| B_{2\lambda} - K_{2\lambda} \|= 0 \) and \( e_i \) some orthonormal system. That is \( K_{2\lambda} \varphi = \sum k_{ij}(\varphi, e_i)e_j \). If \( \varphi \in L^2 \) is such that \( B_\lambda \varphi = 0 \), using the theory of integral equations (cf. [21]) we can make an orthogonal decomposition of \( N(B_\lambda) \) according to

\[
S_{N\lambda} = R(B_{N\lambda}) \cap \overline{R(B_{N+1\lambda})}^\perp
\]

Thus, \( \varphi = B_{N\lambda} \varphi + N\lambda B_{(N+1)\lambda} \varphi \) on \( S_{N\lambda} \). If \( N > 2 \), say \( N_0 \), then \( S_{N\lambda} \subset C^\infty \) for \( M \geq N_0 \). On \( S_{N\lambda} \) and \( N < N_0 \) we have \( \varphi - B_{N\lambda} \varphi \subset C^\infty \) with \( K_{N\lambda} \varphi = \sum_{i,j} k_{ij}^N(\varphi, e_i)e_j \). These kernels \( B_{N\lambda} \) can thus be considered as hypoelliptic.

**10.3. Asymptotically hypoelliptic operators.** If \( B_\lambda \) is a parametrix to an operator \( L_1 \) in \( L^1 \) and \( B_\lambda L^1 \) - hypoelliptic on \( N(B_\lambda^N) \perp \), where \( N_0 \) is the minimal integer such that the zero-space is stable. Then, if \( N_0 = 1 \), we have that \( L_1 \) is hypoelliptic on \( L^1 \). Assume \( B_{\lambda,N} \) a \( L^1 \) - parametrix in \( L^1 \) to the iterated operator \( L_\lambda^N \).
then \( N(B_{\lambda,N_0}^N) = N(B_{\lambda,N_0}^B) \) implies \( N_1 = 1 \). and \( B_{\lambda,N_0}L_{\lambda}^{N_0} = L_{\lambda}^{N_0}B_{\lambda,N_0} = \delta_x - \gamma \) for some \( \gamma \in C^\infty \) where \( \gamma = 0 \) on \( N(E_{N_0}^N) \). Further, we have that \( L_{\lambda}^{N_0} \) is hypoelliptic on \( L^1 \). If \( B_{\lambda,N_0} \) is parametric to a \( L^1 \)-operator \( L_{\lambda}^{N_0} \) by adding a solution to the homogeneous equation \( H \neq 0 \) on \( N(B_{\lambda,N_0}) \), we get \( N(E_{N_0}^N + H) = \{0\} \) and \( B_{\lambda,N_0} = B_{\lambda,N_0}^N \in C^\infty \), that is the parametrix to \( L_{\lambda}^{N_0} \) is regularizing on \( N(E_{N_0}^N) \).

### 10.4. Iversen's condition and hypoellipticity.

We have seen that when \( T = \mathcal{L} \) (Legendre), we do not have micro local contribution. Consider for this reason the condition: If the phase space sequence \( v_\mu \) in

\[
0 < |Tv_\mu - \mathcal{L}v_\mu| < \epsilon \quad \mu \to 0
\]

gives a regular approximation \( e^{i\omega} \) of a singular point that can be continued analytically over the origin, then the corresponding operator does not have micro local contribution from this sequence. More precisely, if \( V(M,W) = V_1 + iV_2 \) is the lifting function, let \( V^N \) be the localization to \( X_N \). For \( V^N = P_N(T)f_0 \), for \( f_0 \) corresponding to a very regular operator, we assume \( \tilde{V}^N = (1/P_N)g_0 \), where \( g_0 \) is in the same class as \( f_0 \). We then have that for \( V \) corresponding to a hypoelliptic operator, we must have that the continuation from \( \tilde{V}^N \), \( (T - \mathcal{L})V^N \) is algebraic as \( T \to \infty \). Conversely, given that \( (T - \mathcal{L})V^N \) is algebraic, there is no room for linearity. In the terminology of Iversen (cf. [I]), if we can find a Jordan arc emanating from the origin, on which the limes inferior of the modulus of the closed contours corresponding to the strata have a limit, then there is a subsequence of \( \mu_n \) such that \( v_{\mu_n} \to v_0 \) on this arc. Assuming existence of a point in \( \mathfrak{B} \) where \( v_\mu \) is finite, we can find the arc \( g \) such that \( \lim_{\mu \in \mathfrak{B}} v_\mu = v_0 \). The conclusion is that we have analytic continuation in this case. Thus, assuming the conditions on \( V \) as above and that \( M \perp W \), we can use the path in Iversen’s proof to derive hypoellipticity.

**Proposition 10.5.** Given the representation \( f = V(M,W) \) with \( V_1 \perp V_2 \), assume \( V^N \) is the restriction to strata \( X_\perp^N \) and that the stratification of \( (B_m) \) has property (M2). Then we have that if \( (T - \mathcal{L})V^N \) is algebraic, as \( T \to \infty \), there is no closed contour contributing microlocally to the symbol.

### 10.5. Algebraic continuation and orthogonality.

The information on the closed contour giving microlocal contribution is in \( TV \), in the manner that if \( (T - \mathcal{L})V^N \) is algebraic, there is no possibility for presence of a closed contour that would contribute and we can conclude that \( V_2 \ll V_1 \). Assume now that \( V^N(I) \in (B_m)^' \subset \mathcal{D}'_{L^1} \), such that \( \tilde{V}^N = P_N(T)f_0 \), with \( f_0 \) continuous (very regular). We can now assume \( \tilde{V}^N = f_0 \), where \( g_0 \) has the same property as \( f_0 \). Our proposition is that if \( T(\vartheta) \sim_0 \alpha L \vartheta \), with \( \alpha \) holomorphic, then for a hypoelliptic operator, \( \alpha/P_N \) will be algebraic, as \( N \to \infty \). Presence of a contributing closed contour means that \( \alpha \) will be exponential. We will argue that for \( V \) corresponding to a hypoelliptic operator, we have that \( (T - \mathcal{L})V^N \) is algebraic in \( T^{-\infty} \). Assume \( \alpha = q_N + r_N \), where \( q_N \) is a polynomial. The condition on involution means that \( r_N/P_N \) behaves like the symbol, in the tangent space. If \( |r_N| < \epsilon \), then according to Nullstellensatz, we have that \( |\alpha|/P_N \ll \epsilon \) as \( T \to \infty \). Thus, the condition that \( P_N \) is reduced in the \( T^{-\infty} \) and \( r_N \) small, means that \( \alpha/P_N \) is algebraic in the \( T^{-\infty} \), as \( N \to \infty \). If we assume \( \alpha/P_N \) bounded in the \( T^{-\infty} \), then we have that also \( V^N \) has type 0. The conclusion is given that \( M \perp W \) and \( M = e^{i\varphi}W \), if \( F \) is algebraic, then we must have \( V_1 = e^{i\varphi}V_2 \) and that \( e^{-\varphi} \to 0 \) as \( T \to \infty \). This means for the continuation of \( F \) using \( T \), that the orthogonality is preserved.
10.6. Final remarks on hypoellipticity. We have seen that under the condition that \( M \bot W \) implies \( V_1 \bot V_2 \), then we have that absence of closed contour that contributes microlocally is equivalent with the proposition \((T - \xi)V^{-N}\) is algebraic. Through the condition we have that \( V \) preserves parabolic approximations.

It is known that for a symbol such that \( f^N \in (I_{HE}) \), for some integer \( N \), we have \( \Delta(f^2) \subset \Delta(f) \). We can give an interpolation problem for the iterates, \( \psi^j \psi'^{j/N} = M_N(\psi^j) \) and \( e^{\psi^j} = f^j \). As the kernel to the parametrix gets smaller as \( j \) increases, why \( \text{Im} f \equiv 0 \) on \( V^j \), for all \( N \) when \( j \) is large. More precisely, if we assume \( f(\xi)(\text{Im} F) \to \infty \), as \( \| \xi \| \to \infty \). Further, if we have \( f^N \in (I_{HE}) \), as \( \| \xi \| \to \infty \). Assume for simplicity, \( \| \xi \|^\delta \leq C \| f(\xi)^N \| \), for \( N \) positive and \( \| \xi \| \to \infty \). The problem is now if we can find \( \delta \) such that \( \| \xi \|^\delta \leq C' \| f(\xi) \| \), as \( \| \xi \| \to \infty \). In this case we can choose \( \delta = 1/N \). In this manner we can prove that given that the real part has lower bound with exponent \( \sigma \), then we can select \( \delta = \sigma/N \) as exponent for the lower bound to the imaginary part.

We have noted that presence of lineality for the symbol, may result in \( \text{Im} F \) in the space of hyperfunctions. We now note

**Proposition 10.6.** If \( F \) is symmetric, entire and of finite type in \( \text{Exp} \), then the condition that \( f \) represents a hypoelliptic operator, means that for some \( \lambda \), \( (\text{Im} f)^\lambda = \sum A_j F_j \) on a domain of holomorphy, for constant coefficients and a global pseudo-base \( F_j \) representing the ideal of hypoelliptic operators.

Thus, symbols to hypoelliptic operators do not have imaginary part outside the space of distributions and if hyper-function representation is necessary, we must have contribution of lineality in the infinity.

11. Examples

11.1. Introduction. There is a big number of examples published in the literature (cf. [22]) and we will deal with only some of them briefly here. We are assuming the pseudo differential operator \( P \) is defined as \( \lim_{\lambda \to 0} P_\lambda \), where the dependence of \( \lambda \) is locally algebraic in the symbol space. We are assuming a dependence of \( \xi(x) \) through reciprocal polars, in this context, that is \( x_T \to \xi_1/T \). We are assuming the limit unique, in the sense that \( \frac{dP}{dx} = 0 \) implies \( T = 0 \), for small \( T \) (regular approximations). However, we may have \( \frac{d^2}{dx^2}P_T = 0 \), even locally. An operator that is the regular limit of analytically hypoelliptic operators, with the conditions that we have, is analytically hypoelliptic. We can use Proposition 11.3 to construct an approximating sequence of symbols. The \( L^1 \)-dependence for parameter, means that the limit of \( P_\lambda \) in operator space is continuous.

11.2. Example 1. Consider \( P(x,y,\xi,\eta) = \xi^2 + x^2m\eta^2 \). This corresponds to an operator analytically hypoelliptic in \( \mathbb{R}^n \) with \( n = 2, m \geq 1 \). If, for a constant \( c_T \), \( \xi^2/T = x^2m\eta^2 - c_T \), then \( P_T \) is not analytically hypoelliptic, when \( c_T = 0 \). But, when \( c_T \neq 0 \), we have that \( P_T \) is analytically hypoelliptic and the limit \( P \) is a limit of analytically hypoelliptic operators.

11.3. Example 2. Consider \( P(x,y,z,\xi,\eta,\nu) = \xi^2 + x^2\eta^2 + \nu^2 \). This corresponds to an operator not analytically hypoelliptic in \( 0 \) in \( \mathbb{R}^n \), for \( n = 3 \). For this reason we consider \( \xi^2/T + x^2\eta^2 + \nu^2/T = c_T \), for a (possibly zero) constant \( c_T \) such that \( c_T \to 0 \) as \( T \to 0 \) and such that for a constant \( c'_T \), as \( T \to 0 \), \( x^2\eta^2 = c'_T(\xi^2/T + \nu^2/T) \). We now have constant surfaces through a suitable choice of \( \eta \), which means that the operator is not analytically hypoelliptic in \( 0 \).
11.4. Example 3. Consider $P(x, y, z, \xi, \eta, \nu) = \xi^2 + [\eta + (\frac{1}{3}x^2 + xy^2)v]^2 = \xi^2 + [\eta + x(\frac{1}{3}x^2 + y^2)]^2$. As we have assumed real arguments, it can be proved that $P$ is a regular limit of analytically hypoelliptic operators in $(x, \xi, y, \eta, z, \nu)$ and in combinations of these.

12. The mapping $(I) \to Op(I)$

Assume $T = \{y = F\}$ a transversal manifold, that is for a submersion $p$, ker $p = T_y \oplus L_x$, for a manifold $L$. For instance, assume $h$ such that $dh(f) = 0$ implies $f = const.$, then if $dh$ is locally algebraic (in the parameter), we must have where $f = const.$, that $dh(f) = const.$.. Assuming on an irreducible component in $\{f = const\}$, there is at least one point where $dh(f) = 0$ we can conclude that $f = const$ implies $dh(f) = 0$. A sufficient condition for this is that the tangent set $(I_\mu) = \{\zeta \mid h(f) = \mu f\}$ exists and has irreducible components. Note that under these conditions, the first surfaces have a locally algebraic definition and the complement sets are assumed locally analytic. We can thus assume that in a neighborhood of $df = 0$, we have that $dh(f) = 0$ gives a regular set. If we consider transversals on the form $dh(f) = \rho df$, for a locally regular function $\rho$, we can form the extended transversal as a Baire function.

Assume $\Gamma_A$ is the boundary given by $dF = 0$ and $\Gamma_\mu$ is given by $\{\zeta \mid \log dF = 0\}$. If $\gamma$ is transversal to $\Gamma_A$ ($= \Gamma_A \cap U$), we have

$$\Phi_{dF} = a_{\gamma}(z, \zeta) = \int_{\Gamma_A'} \frac{dzdF(z + \zeta)}{z - \zeta}.$$ 

We then have, if $\Phi_{dF}$ is the analytic continuation over transversals, that $\Phi_{dF} - \Phi_{dF} = dF = 0$ on $\Gamma_A$. Further, $d_{\zeta}(\Phi_{dF} - \Phi_{dF}) = 0$ at isolated points.

Now consider $F(\zeta, z)$ analytic for $\zeta$ bounded and $z$ large, such that $F(\zeta, z) \to 0$ as $z \to \infty$. We assume

$$F(\zeta, z) = \frac{d_zF(z + \zeta)}{d_zF(z)} - 1$$

Using Cousin (cf. 4, Ch.4), there is a polynomial $Q$ in $1/t$ such that $|F(\zeta, z)| < \epsilon + Q(1/t))$ on $\Delta_A$, a conical neighborhood of the linearity. We say that $Q$ preserves a constant value for $F(\zeta, z)$. Thus,

$$|d_zF(z + \zeta)| < (\epsilon + Q(1/t)) |d_zF(z)|$$

with $Q(1/t) \to 0$ as $t \to \infty$. Assume further $|dF| < |dF + dL|$, where ker $p = \{dF + dL\}$. Then formally $d_zF(z + \zeta) \sim_m (dF + dL)$ and $dL \sim_m 0$. Within monotropy, we thus have slow oscillation in the limit $z \to \infty$. We have that $d_zF$ is holomorphic with respect to $\{d_zF = 0\}$ according to $d_zF = tr \Phi_{dF}$, where $\Phi_{dF}$ corresponds to to $d_zL = L_z$ in the transversal decomposition. If $F$ is a minimally defining function of $N(J_h)$, we have $dF \neq 0$ on this set. If $\mu$ is such that $h(p) = \mu F$, we have that $\Phi_F - \Phi_F \to 0$ on $N(J_h)$, why $\Phi_F$ is analytic over the characteristic set $\{F = 0\}$. Note that

$$|\frac{d_zF(z + \zeta)}{dF(z)} + dL(z)| = \epsilon$$

as $z \to \infty$ along a transversal $\gamma$ emanating from the origin. Consider first $L_z$ and the corresponding $\Phi_{dF}$. This corresponds through the inverse Fourier-Borel transform, to an analytic functional that allows real support, why it is sufficient to consider the real space and $b_T$. Assume $u$ an entire function on a univalent domain and zero on points for multi-valentness and on $\Gamma_\mu$. We shall see that the corresponding
form defines a good contour for the associated pseudo differential operator. This is according to (cf. [20]) given by a realization with regularizing action on $D'$ why we have no loss of generality from the conditions on the zero-set in the approach. Consider now the form $\theta e^{\frac{2}{x}} \frac{dz}{dx} + i \tilde{H}(x-y)$ with $|x-y| \leq r$. Assume $y = x - \zeta$ and $|\zeta| \leq r$. We then have

$$e^{\lambda(\phi(x+\zeta)-\phi(x))-2\lambda \Re \frac{x^2}{u^2} \zeta e^{-|\zeta|^2} = \left[ u^2(x+\zeta) \right] e^{-2 \Re \left[ (\sum_{j=0}^{\infty} \frac{u^2}{\lambda^2} \zeta^2) \right] e^{-|\zeta|^2}}$$

The slow oscillation that we already established (within monotropy) implies that particularly $|\delta_j u^2|/u^2$ bounded, as $|x| \to \infty$

We conclude that for $|\zeta|$ bounded, as $|x| \to \infty$, have that the bracket tends to 1, as $|x| \to \infty$ and we have a good contour $\Gamma(x)$ for the form $\theta$. The pseudo differential operator can be realized through

$$\tilde{H}_\mu u_\Lambda(x) = C_\Lambda \int_{\Gamma(x)} e^{i\lambda(x-y)} \xi L_\mu^\mu(x,y,\xi)u_\Lambda(y)dyd\xi$$

where $L_\mu^\mu$ has compact support and the operator acts $D' \to C^\infty$. Finally, we have the case with $T_2$. Using Weierstrass theorem, we can find a polynomial $P_{\mu,c}$ which include the foliation in its zero’s. For any polynomial, we have that $\delta_j P_{\mu} \prec P_{\mu}$, why the second term is bounded, for $\lambda$ finite. The first term is bounded by slow oscillation as before. We have apparently a good contour also in this case.

We conclude that given the foliation for the symbol $\{f = c\}$ and the tangent spaces $(I_\mu)$, we can realize the pseudo differential operator as a locally polynomial operator where the polynomial part of the operator has zero’s on the foliation. This is a Levi decomposition of the operator

$$Au_\Lambda = \sum_{j,\mu} (P_{\mu,c_j} + \tilde{H}_{\mu,c_j})u_\Lambda$$

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