Dugundji’s Canonical Covers, asymptotic and covering dimension

Jesús P. Moreno-Damas

Abstract

Given a nowheredense closed subset \( X \) of a metrizable compact space \( \tilde{X} \), we characterize the dimension of \( X \) in terms of the multiplicity of the canonicals covers of the complementary of \( X \), specially in some particular cases, like when \( \tilde{X} \) is the Hilbert cube or the finite dimensional cube and \( X \), a Z-set of \( \tilde{X} \). In this process, we solve some questions in the literature.

1 Introduction

This paper reinforces the relation between the canonical covers, used by Dugundji in [9] to prove his famous Extension Theorem, and the way to describe the dimension of certain subsets of compact spaces in terms of the corresponding complement. Such a relation was treated in [5] in the special case when a compact metrizable space is embedded as a Z-set in the Hilbert cube. Concretely, in [5], the authors proved that a compact Z-subset of the Hilbert cube has finite topological dimension if and only if there exists a canonical cover of finite order in its complement (for us, the order of a cover \( \alpha \) is 1 less the supremum of the number elements of \( \alpha \) with non empty intersection, that is, 1 less the multiplicity of \( \alpha \).

The above kind of relations are in the core of so called Higson-Roe functor with allows us to compare coarse properties of a coarse structure with topological properties of the corresponding Higson-Roe corona (see [19] and [4] for a specific example). In particular, asymptotic dimensional properties with covering dimensional properties. Several works go in this way, for example [2], [3], [7] and [8].

When the coarse structure is the topological one attached to a metrizable compactification, Grave proved in [11] and [12] that the asymptotic dimension of that coarse structure exceeds just by 1 the covering dimension of the corona. That result allowed the authors in [4] to prove that the gap between the \( C_0 \)-asymptotic dimension of the complement of a Z-set in the Hilbert and the topological dimension of the corresponding Z-set is precisely 1.

As proved in [5], the particularity of the canonical covers in the complementary is that
every one has information about the dimension of the Z-set. In fact, what was really proved in that article is that, if a compact Z-set in the Hilbert cube has dimension \( n \), then the order of any canonical cover of the complement is greater than or equal to \( n \) and there exists a canonical cover which order is less than or equal to \( 2n + 1 \).

As pointed out in that article, the definition of canonical cover is in the core of the \( C_0 \) coarse geometry (see [19], [22] and [23] for definitions). It is natural to ask if, for every canonical cover, the gap between its order and the dimension of the Z-set is at least 1 and, as it was did in [5] page 3713, if one can always find a canonical cover whose gap is just 1.

In this paper, we use canonical covers to strength and improve relations pointed out in [5]. In particular, we are able to work in a more general framework and to answer positively the questions mentioned above. We also describe accurately the relations between the canonical covers and the \( C_0 \) coarse structure. Finally, we use our results here to give an easier proof of a Grave’s result in [11] and [12], relating the topological dimension of the corona with the corresponding asymptotic dimension.

In section 2 we introduce the basic needed definitions and notation and state an easy criterium to detect coarse equivalences.

Section 3, which is the core of the paper, contains the main technical tools and results in this paper. Firstly, we relate the canonical covers with the coarse structures, getting that a canonical cover is precisely an open and locally finite cover which is uniform for the topological coarse structure attached to a compactification. In our case, that compactification is metrizable and that means that the cover is uniform for the \( C_0 \) coarse structure. That states the relation pointed out in [5].

Taking it into account, we are able to relate the multiplicity of canonical covers with the asymptotic dimension of the \( C_0 \) coarse spaces, using the Grave’s result mentioned above, to obtain:

(A) If \( \tilde{X} \) is a nowheredense closed subset of a metrizable compact space \( \tilde{\tilde{X}} \), then \( \dim \tilde{X} \leq n \) if and only if for every canonical cover \( \alpha \) of \( \tilde{\tilde{X}} \setminus \tilde{X} \), there exists a canonical cover \( \beta \) which has \( \alpha \) as a refinement and whose order is less than or equal to \( n + 1 \).

To improve this equivalence when \( \tilde{X} \) is the Hilbert cube and \( X \) is a Z-set of \( \tilde{X} \), we define a more general category, the cylindrical subsets. This allows us to extends results which work for the Z-sets of the Hilbert cube, to other cases, like the Z-sets of the finite dimensional cube \([0,1]^n\). We get:

(B) If \( X \) is a cylindrical, nowheredense and closed subset of a metrizable compact space \( \tilde{\tilde{X}} \) (in particular, if \( X \) is a Z-set of \( \tilde{\tilde{X}} \) and \( \tilde{\tilde{X}} \) is the Hilbert Cube or the finite dimensional cube), then \( \dim \tilde{X} \leq n \) if and only if there exists a canonical cover of \( \tilde{\tilde{X}} \setminus \tilde{X} \) whose order is less than or equal to \( n + 1 \).

Actually, (B) is satisfied not only for canonical covers, but also for open and \( C_0 \)-uniform covers. Consequently, the order of every open and uniform cover is at least the
$C_0$-asymptotic dimension. That means that, when $X$ is a cylindrical subset, from the asymptotic dimensional point of view, the open and uniform covers are big enough. That suggests a natural question: In the general case, when is a uniform cover big enough from the asymptotic dimensional point of view? We finish the section by answering that.

In section 4 we recover Grave’s result in an easier way. To do it, we give some results in Topological Dimension Theory (one of them, developed by us in [16]).

2 Preliminaries: Basic definitions and notations

We say that compactification pack is a vector $(X, \hat{X}, \tilde{X})$ such that $\tilde{X}$ is a compact Hausdorff space, $X$ is a nowhere dense closed subset of $\tilde{X}$ and $\hat{X} = \tilde{X} \setminus X$. Observe that $\tilde{X}$ is a compactification of $\hat{X}$ and $X$ is its corona.

For us, a cover $\alpha$ of a set $Z$ is a collection of subsets of $Z$ whose union is $Z$.

If $Z$ is a topological space and $\alpha$ is a family of subsets of $Z$, we say that $\alpha$ is open if every $U \in \alpha$ is open. If $Z$ is a metric space, we say that mesh $\alpha = \sup \{\text{diam} U : U \in \alpha\}$.

If and $A \subset Z$ and $\alpha$ and $\beta$ are families of subsets of $Z$, we denote $\alpha(A) = \bigcup_{U \in A} U$.

By $\beta \prec \alpha$ we mean that $\beta$ is a refinement of $\alpha$. The multiplicity of $\alpha$ is $\text{mult} \alpha = \sup \{\#A : A \subset \alpha, \bigcap_{U \in A} U \neq \emptyset\}$ (where, for every set $B$, $\#B$ means the cardinal of $B$).

The concept of canonical cover was used by Dugundji in [9]. In our language, a canonical cover $\alpha$ of a compactification pack $(X, \hat{X}, \tilde{X})$ is an open and locally finite cover of $\tilde{X}$ such that for every $x \in X$ and every neighborhood $V_x$ of $x$ in $\tilde{X}$ there exists a neighborhood $W_x$ of $x$ in $\tilde{X}$ with $\alpha(W_x) \subset V_x$.

Let us give some definitions of coarse geometry. For more information, see [19]. Let $E, F \subset Z \times Z$, let $x \in Z$ and let $K \subset Z$. The product of $E$ and $F$, denoted by $E \circ F$, is the set $\{(x, z) : \exists y \in Z \text{ such that } (x, y) \in E, (y, z) \in F\}$, the inverse of $E$, denoted by $E^{-1}$, is the set $E^{-1} = \{(y, x) : (x, y) \in E\}$, the diagonal, denoted by $\Delta$, is the set $\{(z, z) : z \in Z\}$. If $x \in Z$, the $E$-ball of $x$, denoted by $E_x$ is the set $E_x = \{y : (y, x) \in E\}$ and, if $K \subset Z$, $E(K)$ is the set $\{y : \exists x \in K \text{ such that } (y, x) \in E\}$. We say that $E$ is symmetric if $E = E^{-1}$.

A coarse structure $\mathcal{E}$ over a set $Z$ is a family of subsets of $Z \times Z$ which contains the diagonal and is closed under the formation of products, finite unions, inverses and subsets. The elements of $\mathcal{E}$ are called controlled sets. $B \subset Z$ is said to be bounded if there exists $x \in Z$ and $E \in \mathcal{E}$ with $B = E_x$ (equivalently, $B$ is bounded if $B \times B \in \mathcal{E}$).

A map $f : (Z, \mathcal{E}) \to (Z', \mathcal{E}')$ between coarse spaces is called coarse if $f \times f(E)$ is controlled for every controlled set $E$ of $Z$ and $f^{-1}(B)$ is bounded for every bounded subset $B$ of $Z'$. We say that $f$ is a coarse equivalence if $f$ is coarse and there exists a coarse map $g : (Z', \mathcal{E}') \to (Z, \mathcal{E})$ such that $\{(g \circ f(x), x) : x \in Z\} \in \mathcal{E}$ and $\{(f \circ g(y), y) : y \in Z'\} \in \mathcal{E}'$. In this case, $g$ is called a coarse inverse of $f$.

A subset $A \subset Z$ is coarse dense if $Z = E(A)$ for some $E \in \mathcal{E}$.
Given $E \subset Z \times Z$, we denote $\mathcal{K}(E) = \{E_x : x \in Z\}$. If $\alpha$ is a family of subsets of $Z$, we say that $\mathcal{D}(\alpha) = \bigcup_{U \in \alpha} U \times U$ (observe that $(x, y) \in \mathcal{D}(\alpha)$ if and only if $x, y \in U$ for some $U \in \alpha$).

A family of subsets $\alpha$ of $Z$ is called uniform if $\mathcal{D}(\alpha) \in \mathcal{E}$. Note that, if $E \in \mathcal{E}$, then $\mathcal{K}(E)$ is uniform. Dydak and Hoffland showed in [10] that the coarse structures can be described in terms of the uniform covers.

Intuitively, the uniform families of subsets of $(Z, \mathcal{E})$ behave like the controlled sets. For example, suppose that $f : (Z, \mathcal{E}) \to (Z', \mathcal{E}')$ is a coarse map between are coarse spaces, $E$ is a controlled set of $(Z, \mathcal{E})$, $\beta$ is a family of subsets of $(Z, \mathcal{E})$ and $\gamma$ is a family of subsets of $Z$ such that $\beta < \alpha$. Then, $E(\alpha)$ and $\beta$ are uniform families of subsets of $(Z, \mathcal{E})$ and $f(\alpha) = \{f(U) : U \in \alpha\}$ is a uniform family of subsets of $(Z', \mathcal{E}')$. For every $B \subset Z$, $B$ is bounded if and only if there exists a uniform family of subsets $\gamma$ of $(Z, \mathcal{E})$ such that $B \in \gamma$ (equivalently, if $\{B\}$ is uniform).

If $Z$ is a topological space and $E \subset Z \times Z$, we say that $E$ is proper if $E(K)$ and $E^{-1}(K)$ are relatively compact for every relatively compact subset $K \subset Z$. If $\mathcal{E}$ is a coarse structure over $Z$, we say that $(Z, \mathcal{E})$ is a proper coarse space if $Z$ is Hausdorff, locally compact and paracompact, $\mathcal{E}$ contains a neighborhood of the diagonal in $Z \times Z$ and the bounded subsets of $(Z, \mathcal{E})$ are precisely the relatively compact subsets of $Z$.

By $Q$ we denote the Hilbert cube $[0, 1]^n$. Let $\tilde{X} = [0, 1]^n$ with $n \in \mathbb{N}$ or $\tilde{X} = Q$ and suppose that $d$ is a metric in $\tilde{X}$. $X$ is a $Z$-set of $\tilde{X}$ if it is closed and for every $\varepsilon > 0$ there exists a continuous function $f : \tilde{X} \to \tilde{X}$ such that $d'(f, Id) < \varepsilon$ —where $d'$ is the supreme metric— and $f(\tilde{X}) \cap X = \emptyset$ (the definition of $Z$-set given in [3], chapter I-3, pag. 2, is equivalent in this context).

For us, an increasing function between two ordered sets $f : (X, <) \to (X', <')$ is a function such that $x < y$ implies $f(x) \leq f(y)$.

The following identities will be useful along this article. Let $Z, Z'$ be sets, suppose $x \in Z$, $x' \in Z'$, $A, B \subset Z$ and $E, F \subset Z \times Z$. Consider a family of subsets $\alpha$ of $Z$ and a map $f : Z \to Z'$. Then:

$$\mathcal{D}(\alpha)(A) = \alpha(A)$$
$$\mathcal{(E \circ F)}_x = E(F_x)$$
$$E(A) = \bigcup_{a \in A} E_a$$
$$E(A) \cap B \neq \emptyset \quad \text{if and only if} \quad A \cap E^{-1}(B) \neq \emptyset.$$  
$$E(B) \subset A \quad \text{if and only if} \quad E \cap (Z \setminus A) \times B = \emptyset$$

$$f(x) \times f(E))_{x'} = \bigcup_{a \in f^{-1}(x')} f(E_a) = f(E(f^{-1}(x')))$$

The following proposition provides a criterion to detect coarse equivalences:
Proposition 1. If $f : (X, \mathcal{E}) \to (Y, \mathcal{F})$ is a map between coarse spaces, then

a) $f$ is a coarse equivalence.

b) There exists $g : Y \to X$ such that $f \times f(E) \in \mathcal{F}$ for every $E \in \mathcal{E}$, $g \times g(F) \in \mathcal{E}$ for every $F \in \mathcal{F}$, and $(x, g \circ f(x)) : x \in X \in \mathcal{E}$ and $(y, f \circ g(y)) : y \in Y \in \mathcal{F}$.

c) $f \times f(E) \in \mathcal{F}$ for every $E \in \mathcal{E}$, $(f \times f)^{-1}(F) \in \mathcal{E}$ for every $F \in \mathcal{F}$ and there exists $g : Y \to X$ such that $(y, f \circ g(y)) : y \in Y \in \mathcal{F}$.

d) $f \times f(E) \in \mathcal{F}$ for every $E \in \mathcal{E}$, $(f \times f)^{-1}(F) \in \mathcal{E}$ for every $F \in \mathcal{F}$ and $f(X)$ is coarse dense in $(Y, \mathcal{F})$.

are equivalent. Moreover, if $g$ is the map of b) or c), then $g$ is a coarse inverse of $f$.

Proof. Throughout the proof, every time we use a map called $g : Y \to X$, we will denote by $G$ and $H$ the sets

$$G = \{(x, g \circ f(x)) : x \in X\}$$

$$H = \{(y, f \circ g(y)) : y \in Y\}$$

It is obvious that a) implies b).

To see that b) implies a), it is sufficient to show that $f^{-1}(U)$ is bounded for every bounded subset $U$ of $X$ and $g^{-1}(V)$ is bounded for every bounded subset $V$ of $Y$. But it is easily deduced from:

$$f^{-1}(V) \times f^{-1}(V) \subset G \circ (g \times g(V \times V)) \circ G^{-1} \in \mathcal{E}$$

$$g^{-1}(U) \times g^{-1}(U) \subset H \circ (f \times f(U \times U)) \circ H^{-1} \in \mathcal{F}$$

Moreover, $g$ is a coarse inverse of $f$.

To see that b) implies c) it is sufficient to show that $(f \times f)^{-1}(F) \in \mathcal{E}$ for every $F \in \mathcal{F}$. But it follows from:

$$(f \times f)^{-1}(F) \subset G \circ (g \times g(F)) \circ G^{-1} \in \mathcal{E}$$

To see that c) implies b), it is sufficient to show that $g \times g(F) \in \mathcal{E}$ for every $F \in \mathcal{F}$ and that $G \in \mathcal{E}$. But it follows from:

$$g \times g(F) \subset (f \times f)^{-1}(H^{-1} \circ F \circ H) \in \mathcal{E}$$

$$G \in (f \times f)^{-1}(H) \in \mathcal{E}$$

Moreover, since $g$ satisfies b), $g$ is a coarse inverse of $f$.

To see that c) implies d), it suffices to show that $f(X)$ is coarse dense in $(Y, \mathcal{F})$. And it happens, since $Y = H(f(X))$. To see that, take $M \in \mathcal{F}$ such that $Y = M(f(X))$. By \[ 3 \], $Y = \bigcup_{y \in f(X)} M_y$, so there
exists a partition of \( Y \{ P_y : y \in f(X) \} \) such that \( P_y \subset M_y \) for every \( y \in f(X) \). Choose \( x_y \in f^{-1}(y) \) for every \( y \in f(X) \). Let \( g : Y \to X \) be the map such that, for every \( y \in f(X) \) and every \( y' \in P_y, g(y') = x_y \).

Observe that \( (y', f \circ g(y')) = (y', f(x_y)) = (y', y) \in P_y \times \{ y \} \subset M_y \times \{ y \} \subset M \). Then, \( H \subset M \in \mathcal{F} \) and \( H \in \mathcal{F} \).

**Corollary 2.** If \( f : (X, \mathcal{E}) \to (Y, \mathcal{F}) \) is a bijective coarse equivalence, then \( f^{-1} \) is a coarse inverse of \( f \).

**Proof.** Since \( f \) is a coarse equivalence, it satisfies property d) of Proposition 1. Then, \( f \) and \( f^{-1} \) satisfy property c) and hence, \( f^{-1} \) is a coarse inverse of \( f \).

**Corollary 3.** If \( f : (X, \mathcal{E}) \to (Y, \mathcal{F}) \) is a coarse equivalence and \( \alpha \) is a uniform cover of \( (Y, \mathcal{F}) \), then \( f^{-1}(\alpha) \) is a uniform cover of \( X \).

**Proof.** Since \( D(f^{-1}(\alpha)) = (f \times f)^{-1}(D(\alpha)) \) is a controlled set of \( (X, \mathcal{E}) \), it follows that \( f^{-1}(\alpha) \) is uniform.

### 3 Canonical Covers, Asymptotic Dimension and Dimension

#### 3.1 Canonical Covers and coarse structures

Firstly, we relate the canonical covers with the \( C_0 \) coarse structures.

Recall the following definition, see [19]:

**Definition 4.** Let \( (X, \hat{X}, \check{X}) \) be a compactification pack. The topological coarse structure \( \mathcal{E} \) over \( \hat{X} \) attached to the compactification \( \check{X} \) is the collection of all \( E \subset \hat{X} \times \hat{X} \) satisfying any of the following equivalent properties:

a) \( \text{Cl}_{\hat{X} \times \hat{X}} E \) meets \( \hat{X} \times \hat{X} \setminus \hat{X} \times \hat{X} \) only in the diagonal of \( X \times X \).

b) \( E \) is proper and for every net \( (x_\lambda, y_\lambda) \subset E \), if \( \{ x_\lambda \} \) converges to a point \( x \) of \( X \), then \( y_\lambda \) converges also to \( x \).

c) \( E \) is proper and for every point \( x \in X \) and every neighborhood \( V_x \) of \( x \) in \( \check{X} \) there exists a neighborhood \( W_x \) of \( x \) in \( \check{X} \) such that \( E(W_x) \subset V_x \).

**Remarks 5.**

- In this coarse structure, the bounded subsets of \( \mathcal{E} \) are precisely the relatively compact subsets of \( \hat{X} \). Moreover, if \( \hat{X} \) is metrizable, then \( \mathcal{E} \) is proper.
- The definition above is Proposition 2.27 and Definition 2.28 of [19] (pags. 26-27) together with the author’s correction in [20]. The property c) above is not the one of that proposition, but it is equivalent (see [5]).
Proposition 6. Let \((X, \hat{X}, \tilde{X})\) be a compactification pack. Suppose that \(\mathcal{E}\) is the topological coarse structure over \(\hat{X}\) attached to the compactification \(\tilde{X}\) and let \(E \subset \hat{X} \times \tilde{X}\). Then, \(E \in \mathcal{E}\) if and only if, for every \(x \in X\) and every neighborhood \(V_x\) of \(x\) in \(\tilde{X}\), there exist a neighborhood \(W_x\) of \(x\) in \(\tilde{X}\) such that \((E \cup E^{-1})(W_x) \subset V_x\).

Proof. If \(E \in \mathcal{E}\), then also \(E \cup E^{-1} \in \mathcal{E}\), thus \(E \cup E^{-1}\) satisfies property c) of definition 4 and we get the necessity.

Let us see the sufficiency. To prove that \(E\) satisfies property c) of definition 4 it suffices to show that \(E\) is proper, that is, \(E(K) \cup E^{-1}(K)\) is relatively compact in \(X\) for every relatively compact subset \(K\) of \(\hat{X}\).

Take a relatively compact subset \(K\) of \(\hat{X}\). Fix \(x \in X\). Since \(\tilde{X} \setminus K\) is a neighborhood of \(x\) in \(\tilde{X}\), there exists an open neighborhood \(W_x\) of \(x\) in \(\tilde{X}\) such that \((E \cup E^{-1})(W_x) \subset \tilde{X} \setminus K\), i.e. \(K \cap (E \cup E^{-1})(W_x) = \emptyset\). By (4), it is equivalent to \((E \cup E^{-1})(K) \cap W_x = \emptyset\). Since \(\bigcup_{x \in X} W_x\) is an open neighborhood of \(X\) in \(\tilde{X}\) with \((E \cup E^{-1})(K) \cap \bigcup_{x \in X} W_x = \emptyset\), we have that \((E \cup E^{-1})(K) = E(K) \cup E^{-1}(K)\) is relatively compact in \(\hat{X}\).

Corollary 7. Let \((X, \hat{X}, \tilde{X})\) be a compactification pack and suppose that \(\mathcal{E}\) is the topological coarse structure over \(\hat{X}\) attached to the compactification \(\tilde{X}\). If \(\alpha\) is a family of subsets of \(\hat{X}\), then \(\alpha\) is uniform if and only if for every \(x \in X\) and every neighborhood \(V_x\) of \(x\) in \(\tilde{X}\), there exists a neighborhood \(W_x\) of \(x\) in \(\tilde{X}\) such that \(\alpha(W_x) \subset V_x\).

Particularly, \(\alpha\) is a canonical cover of \((X, \hat{X}, \tilde{X})\) if and only if it is an open and locally finite cover of \(\hat{X}\) which is controlled for \(\mathcal{E}\).

Proof. \(\alpha\) is uniform if and only if \(\mathcal{D}(\alpha) \in \mathcal{E}\). By Proposition 6 and the symmetry of \(\mathcal{D}(\alpha)\), that is equivalent to say that for every \(x \in X\) and every neighborhood of \(x\) in \(\tilde{X}\) there exists a neighborhood \(W_x\) of \(x\) in \(\tilde{X}\) such that \(\mathcal{D}(\alpha)(W_x) \subset V_x\), i.e. \(\alpha(W_x) \subset V_x\).

Recall the following definition of Wright in 22 or 23 (see also of example 2.6 of 19, pag. 22):

Definition 8. Let \((\hat{X}, d)\) be a metric space. The \(C_0\) coarse structure is the collection of all subsets \(E \subset \hat{X} \times \hat{X}\) such that for every \(\varepsilon > 0\) there exists a compact subset \(K\) of \(\hat{X}\) such that \(d(x, y) < \varepsilon\) whenever \((x, y) \in E \setminus K \times K\).

We have the following using the same argument as in 4:

Proposition 9. Let \((X, \hat{X}, \tilde{X})\) be a metrizable compactification pack and let \(d\) be a metric on \(\hat{X}\). Then, the topological coarse structure over \(\hat{X}\) attached to the compactification \(\tilde{X}\) is the \(C_0\) coarse structure over \(\hat{X}\) attached to \(d\).

Proof. The proof of Proposition 6 of 4 is valid in this context. But, using Proposition 6 we get the following shorter proof:

Denote by \(\mathcal{E}\) and \(\mathcal{E}_0\) the topological coarse structure attached to \(\tilde{X}\) and the \(C_0\) coarse structure attached to \(d\) respectively.
Let $E \in \mathcal{E}_0$. Observe that $E \cup E^{-1} \in \mathcal{E}_0$. Fix $x \in X$ and a neighborhood $V_x$ of $x$ in $\hat{X}$. Choose an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset V_x$ and a compact subset $K$ of $\hat{X}$ such that $d(x, y) < \frac{\varepsilon}{2}$ whenever $(x, y) \in (E \cup E^{-1}) \setminus K \times K$. Take $\delta = \min \left\{ \frac{\varepsilon}{2}, \frac{d(K, X)}{2} \right\}$ and $W_x = B(x, \delta)$. Pick a point $y \in (E \cup E^{-1})(W_x)$ and take $z \in W_x$ such that $(y, z) \in E \cup E^{-1}$. Since $z \not\in K$, it follows that $y, z \in (E \cup E^{-1}) \setminus K \times K$ and $d(y, z) < \frac{\varepsilon}{2}$. Hence $d(y, x) \leq d(y, z) + d(z, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and, consequently, $(E \cup E^{-1})(W_x) \subset B(x, \varepsilon) \subset V_x$. Therefore, $E \in \mathcal{E}_0$. 

Take now $E \in \mathcal{E}$ and fix $\varepsilon > 0$. For every $x \in X$, consider a neighborhood $W_x$ of $x$ contained in $B(x, \frac{\varepsilon}{2})$ such that $(E \cup E^{-1})(W_x) \subset B(x, \frac{\varepsilon}{2})$. Let $K = \hat{X} \setminus \bigcup_{x \in X} W_x$ and suppose $(y, z) \in E \setminus K \times K$. Observe that neither $y \in K$ nor $z \in K$. If $z \not\in K$, then there exists $x \in X$ with $z \in W_x$ and $y \in E(W_x)$. If $y \not\in K$, then there exists $x \in X$ such that $y \in W_x$ and $z \in E^{-1}(W_x)$. In both cases, $y, z \in W_x \cup (E \cup E^{-1})(W_x) \subset B\left( x, \frac{\varepsilon}{2} \right)$ and hence $d(y, z) \leq d(y, x) + d(x, z) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore, $E \in \mathcal{E}_0$. □

From now on, we will use the following notation:

**Definition 10.** Let $(X, \hat{X}, \tilde{X})$ be a metrizable compactification pack. The $C_0$ coarse structure over $\hat{X}$ attached to $(X, \hat{X}, \tilde{X})$, denoted by $\mathcal{E}_0(X, \hat{X}, \tilde{X})$ or by $\mathcal{E}_0$ when no confusion can arise, is the topological coarse structure attached to the compactification $\tilde{X}$, i.e. the $C_0$ coarse structure attached to any metric of $\tilde{X}$ restricted to $\tilde{X}$.

We use this notation because working with a metric in $\tilde{X}$ simplifies the calculus and is easier to see geometrically (see figure 1).

![Figure 1. (X, \hat{X}, \tilde{X}) and uniform cover of \mathcal{E}_0(X, \hat{X}, \tilde{X})](image)

**Proposition 11.** Let $(\hat{X}, d)$ be a locally compact metric space and let $\mathcal{E}_0$ be the $C_0$ coarse structure. Consider a family of subsets $\alpha$ of $\hat{X}$. Then, $\alpha$ is uniform for $\mathcal{E}_0$ if and only if:

- $\alpha$ is proper, i.e. for every relatively compact subset $K$ of $\hat{X}$, $\alpha(K)$ is relatively compact in $\hat{X}$.

and

- For every $\varepsilon > 0$ there exists a compact subset $K$ of $\hat{X}$ such that $\operatorname{diam} U < \varepsilon$ for every $U \in \alpha$ with $U \cap K = \emptyset$. 

8
Proof. Suppose that \( \alpha \) is uniform. Since \( D(\alpha) \) is controlled and hence proper, \( Q \) shows that \( \alpha(K) = D(\alpha)(K) \) is relatively compact for every relatively compact \( K \) of \( \hat{X} \).

Let \( \varepsilon > 0 \) and consider a compact subset \( K \) of \( \hat{X} \) such that \( d(x, y) < \varepsilon \) for every \( (x, y) \in D(\alpha) \setminus K \times K \). Take \( U \in \alpha \) with \( U \cap K = \emptyset \) and \( x, y \in U \). Since \( x, y \notin K \), \( (x, y) \in D(\alpha) \setminus K \times K \) and hence \( d(x, y) < \varepsilon \). Thus, \( \text{diam } U \leq \varepsilon \).

For the reciprocal, fix \( \varepsilon > 0 \) and consider a compact subset \( K \) of \( \hat{X} \) with \( \text{diam } U < \varepsilon \) for every \( U \in \alpha \) such that \( U \cap K = \emptyset \). Since \( \alpha \) is proper, \( \alpha(K) \) is relatively compact. Consider \( K' = \alpha(K) \) and pick \( (x, y) \in D(\alpha) \setminus K' \times K' \). Observe that neither \( x \in K \) nor \( y \in K \). Suppose, without loss of generality, that \( x \notin K' \). Let \( U \in \alpha \) be such that \( x, y \in U \). Since \( x \in U \) and \( x \notin \alpha(K) = \bigcup_{U \in \alpha} U \cap K \neq \emptyset \), we have that \( U \cap K = \emptyset \). Then, \( d(x, y) \leq \text{diam } U < \varepsilon \). Hence, \( D(\alpha) \in \mathcal{E}_0 \).

Lemma 12. Let \( (X, \hat{X}, \tilde{X}) \) be a metrizable compactification pack and consider \( (\hat{X}, \mathcal{E}_0) \).

Fix a sequence of open subsets \( \{W_n\}_{n=0}^{\infty} \) of \( \tilde{X} \) with \( W_0 = \tilde{X} \), \( W_0 \supset W_1 \supset W_2 \supset \ldots \) and \( X = \bigcap_{n=0}^{\infty} W_n \).

Let \( \{\beta_n\}_{n=0}^{\infty} \) be a sequence of families of open subsets of \( \tilde{X} \) with \( \beta_0 = \{\tilde{X}\} \), \( X \subset \bigcup_{U \in \beta_n} U \) for every \( n \) and \( \lim_{m,n \to \infty} \text{mesh}\{U \cap W_m : U \in \beta_n\} = 0 \).

Consider \( \alpha(\{\beta_n\}, \{W_n\}) = \{U \cap (W_{n+2} \setminus W_n) : U \in \beta_n, n \geq 0\} \). Then:

a) \( \alpha(\{\beta_n\}, \{W_n\}) \) is an open and uniform family of subsets of \( \hat{X} \).

b) If each \( \beta_n \) is finite, \( \alpha(\{\beta_n\}, \{W_n\}) \) is locally finite.

c) If \( \gamma \) is a uniform family of subsets of \( \hat{X} \), then there exists a subsequence \( \{n_k\}_{k=0}^{\infty} \) with \( n_0 = 0 \) such that \( \gamma \prec \alpha(\{\beta_k\}, \{W_{n_k}\}) \).
Figure 2. Cover $\alpha(\{\beta_n\}, \{W_n\})$

Proof. Observe that if $K$ is a compact subset of $\hat{X}$, then $\{K \cap \overline{W}_n\}_{n=0}^\infty$ is a sequence of nested compact subsets such that $\bigcap_{n=0}^\infty K \cap \overline{W}_n = \varnothing$ and, hence there exists $N \in \mathbb{N}$ with $K \cap \overline{W}_N = \varnothing$. Equivalently, if $U$ is an open subset of $\hat{X}$ containing $X$, then there exist $N \in \mathbb{N}$ such that $W_N \subset U$.

For short, let us denote $\alpha(\{\beta_n\}, \{W_n\})$ by $\alpha$. Obviously, $\alpha$ is open.

Let us see that $\alpha$ is uniform, i.e. $\mathcal{D}(\alpha) \in \mathcal{E}_0$. Fix $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that for every $m, n \geq N$, mesh$\{U \cap W_m : U \subset \beta_n\} < \varepsilon$. Put $K = \hat{X} \setminus \overline{W}_{N+2}$, pick $(x, y) \in \mathcal{D}(\alpha) \setminus K \times K$ and take $V \in \alpha$ with $x, y \in V$. Let $n > 0$ and $U \in \beta_n$ be such that $V = U \cap \overline{W}_n \setminus \overline{W}_{n+2}$. Since $(x, y) \notin K \times K$, neither $x \in K$ nor $y \in K$.

Suppose, without loss of generality, that $x \notin K$, or equivalently, that $x \in W_{N+2}$. Since $x \in V \subset \hat{X} \setminus \overline{W}_{N+2}$, we have that $W_{N+2} \setminus \{x\} \neq \varnothing$ and, consequently, $n > N$. Thus, $d(x, y) \leq \text{diam } U \cap W_n < \varepsilon$ and $\mathcal{D}(\alpha) \in \mathcal{E}_0$.

Fix $x \in \hat{X}$ and take $n \geq 0$ such that $x \in W_n \setminus \overline{W}_{n+1}$. Consider the neighborhood of $x$ $W_n \setminus \overline{W}_{n+2}$. It is easy to check that $\{V \in \alpha : V \cap W_n \setminus \overline{W}_{n+2} \neq \varnothing\} \subset \{U \cap (W_k \setminus \overline{W}_{k+2}) : U \in \beta_k, k = n - 1, n, n + 1\}$. Then, if each $\beta_n$ is finite, $\alpha$ is locally finite.

Let $\gamma$ be a uniform family of subsets of $(\hat{X}, \mathcal{E}_0)$. Let us construct a subsequence $\{n_k\}_{k=0}^\infty$ with $n_0 = 0$ such that $\gamma < \alpha(\{\beta_k\}, \{W_{n_k}\})$. We will use the characterization of uniform families of subsets given in Lemma [1].

Let $L_n = \sup\{\text{diam } V : V \in \gamma, V \subset W_n\}$ for every $n \in \mathbb{N} \cup \{0\}$. Let us see that $L_n \to 0$. Fix $\varepsilon > 0$ and suppose that $K$ is a compact subset of $\hat{X}$ such that $\text{diam } V < \varepsilon$ for every $V \in \alpha$ with $V \cap K = \varnothing$. Choose $N \in \mathbb{N}$ with $K \cap W_N = \varnothing$. Fix $n \geq N$ and $V \in \gamma$ with $V \subset W_n$. Since $V \cap K = \varnothing$, we have that $\text{diam } V < \varepsilon$. Consequently, $L_n \leq \varepsilon$ for every $n \geq N$.

Put $n_0 = 0$. Assuming that $n_0, \ldots, n_{k-1}$ are defined, let us define $n_k$:

Choose $m > 0$ such that $\overline{W}_m \subset \bigcup_{U \in \beta_k} U$. Consider the cover of $\hat{X}$ $\beta_k \cup \{\hat{X} \setminus \overline{W}_m\}$ and let $L$ be its Lebesgue number. Take $m'$ such that $L_n < L$ for every $n \geq m'$.

Since $\gamma$ is a uniform cover of $\mathcal{E}_0$, it is proper. Since $\hat{X} \setminus \overline{W}_{n_{k-1}}$ is relatively compact in $\hat{X}$, it follows that $\gamma(\hat{X} \setminus \overline{W}_{n_{k-1}})$ is relatively compact in $\hat{X}$. Choose $m''$ with $\overline{W}_{m''} \cap \gamma(\hat{X} \setminus \overline{W}_{n_{k-1}}) = \varnothing$.

Let $n_k = \max\{n_{k-1} + 1, m + 1, m', m''\}$. By definition of $m''$:

$$\gamma(\hat{X} \setminus \overline{W}_{n_{k-1}}) \subset \hat{X} \setminus \overline{W}_{n_k}$$

(7)

Let us see that:

$$\{V \in \gamma : V \subset W_{n_k}\} \prec \beta_k$$

(8)

Fix $V \in \gamma$ with $V \subset W_{n_k}$. Since $n_k \geq m'$, $\text{diam } V \leq L_{n_k} < L$. Then, there exists $U \in \beta_k \cup \{\hat{X} \setminus \overline{W}_m\}$ such that $V \subset U$. Necessarily $U \in \beta_k$, because $V \cap (X \setminus \overline{W}_m) = \varnothing$. 10
Let us see now that $\gamma < \alpha(\{\beta_k\}, \{W_n\})$. Fix $V \in \gamma$. Since $V$ is bounded and hence relatively compact, there exists $N$ with $W_n \cap V = \emptyset$. Then, $V \cap (\Xi \setminus W_n) \neq \emptyset$ for every $n \geq N$. Let $k = \min\{k' : V \cap (\Xi \setminus W_{n_{k'}}) \neq \emptyset\}$. By (7),
\[ V \subset \gamma(\Xi \setminus W_{n_k}) \subset \Xi \setminus W_{n_{k+1}} \tag{9} \]

If $k = 0$, then (9) implies that $V \subset \Xi \setminus W_{n_1} \subset \Xi \cap (W_{n_0} \setminus W_{n_2}) \in \alpha(\{\beta_k\}, \{W_{n_k}\})$.

Suppose $k \geq 1$. By definition of $k$, $V \cap (\Xi \setminus W_{n_{k-1}}) = \emptyset$, that is, $V \subset W_{n_{k-1}}$. By (9), $V \subset W_{n_{k-1}} \setminus W_{n_{k+1}}$.

Since $V \subset W_{n_{k-1}}$, by applying (8) to $k - 1$, we get that there exists $U \in \beta_{k-1}$ such that $V \subset U$. Therefore, $V \subset U \cap (W_{n_{k-1}} \setminus W_{n_{k+1}}) \in \alpha(\{\beta_k\}, \{W_{n_k}\})$.

\[ \square \]

**Proposition 13.** Let $(X, \hat{X}, \hat{\gamma})$ be a metrizable compactification pack and consider $(\hat{X}, E_0)$. For every uniform family $\gamma$ of subsets of $\hat{X}$, there exists a canonical cover $\alpha$ with $\gamma < \alpha$.

**Proof.** Set $W_0 = \hat{X}$ and $\beta_0 = \{\hat{X}\}$. Let $k = \sup_{x \in \hat{X}} d(x, X)$. For every $n \in \mathbb{N}$, put $W_n = B\left(X, \frac{k}{2n}\right)$. By the compactness of $X$, for every $n \in \mathbb{N}$ we may choose be a finite subfamily $\beta_n$ of $\{B(x, \frac{k}{2n}) : x \in X\}$ such that $X \subset \bigcup_{U \in \beta_n} U$.

By Lemma 12, there exists a subsequence $\{n_j\}$ with $n_0 = 0$ such that $\gamma \cup \{x : x \in \hat{X}\} < \alpha(\{\beta_j\}, \{W_n\})$. Moreover, $\alpha$ is open, locally finite and uniform for $E_0$.

Since $\gamma \cup \{x : x \in X\}$ is a cover of $\hat{X}$, $\alpha$ is a cover too. By Corollary 7, $\alpha(\{\beta_j\}, \{W_n\})$ is a canonical cover.

\[ \square \]

Now, given a metrizable compactification pack $(X, \hat{X}, \hat{X})$, we relate the dimension of $X$ with the order of the canonical covers in $\hat{X}$.

**Definition 14.** Let $Z$ be a set, suppose $x \in Z$, $V \subset Z$, $E \subset Z \times Z$ and that $\alpha$ is a family of subsets of $Z$. Then:

- $\text{mult}_x \alpha = \#\{U \in \alpha : x \in V\}$
- $\text{mult}_V \alpha = \#\{U \in \alpha : V \cap U \neq \emptyset\}$
- $\text{mult}_E \alpha = \sup\{\text{mult}_E \alpha : x \in Z\}$

**Definition 15.** Let $(Z, E)$ be a coarse space. We say that $\text{asdim}(Z, E) \leq n$ if $(Z, E)$ satisfies any of the following equivalent properties:

a) For every uniform cover $\beta$ there exists a uniform cover $\alpha \succ \beta$ with $\text{mult} \alpha \leq n + 1$.

b) For every controlled set $E$ there exists a uniform cover $\alpha$ such that $\text{mult}_E \alpha \leq n + 1$.

The equivalence of the properties above is given in [11] or [12] (Chapter 3.2 or [11], p. 35-39).

From [11] or [12] we take the following theorem (Theorem 5.5 or [11], pag. 59), adapted to our language:

11
Theorem 16 (Grave’s theorem). If \((X, \hat{X}, \check{X})\) be a metrizable compactification pack, then
\[
\text{asdim}(\hat{X}, \mathcal{E}_0) = \dim X + 1
\]

Lemma 17. If \(\alpha\) is a family of subsets of a set \(Z\) and \(E, F \subset Z\), then:
\[
\text{mult}_F E(\alpha) \leq \text{mult}_{E^{-1} \circ F} \alpha
\]
\[
\text{mult} E(\alpha) \leq \text{mult}_{E^{-1}} \alpha
\]

Proof. Let \(U \subset Z\). From \([2]\) and \([4]\) we get the equivalences: \(U \cap (E^{-1} \circ F)_x \neq \emptyset \Leftrightarrow U \cap E^{-1}(F_x) \neq \emptyset \Leftrightarrow E(U) \cap F_x \neq \emptyset\).

Then, \(\text{mult}_{(E^{-1} \circ F)_x} \alpha = \#\{U \in \alpha : U \cap (E^{-1} \circ F)_x \neq \emptyset\} = \#\{U \in \alpha : E(U) \cap F_x \neq \emptyset\} = \text{mult}_F E(\alpha)\). Taking supreme over \(x\) we get \(\text{mult}_{E^{-1} \circ F} \alpha \geq \text{mult}_F E(\alpha)\).

The second inequality is deduced from the first one taking into account that \(\text{mult} E(\alpha) = \text{mult}_\Delta E(\alpha)\). \(\square\)

Proposition 18. Let \((X, \hat{X}, \check{X})\) be a metrizable compactification pack and consider \((\hat{X}, \mathcal{E}_0)\). Let \(n \in \mathbb{N} \cup \{0\}\). Then,
\[
\begin{align*}
&\text{a)} \quad \text{asdim}(\hat{X}, \mathcal{E}_0) \leq n. \\
&\text{b)} \quad \text{For every uniform cover } \beta \text{ of } \hat{X} \text{ there exists an open, locally finite and uniform cover } \alpha \text{ of } \hat{X} \text{ such that } \beta \prec \alpha \text{ and } \text{mult} \alpha \leq n + 1.
\end{align*}
\]

are equivalent.

Proof. Taking into account Corollary \([7]\) it is obvious that b) implies a). Let us see the reciprocal.

Consider a uniform cover \(\beta\) of \(\hat{X}\). Let \(E\) be an open, symmetric and controlled neighborhood of the diagonal. Since \(\text{asdim}(\hat{X}, \mathcal{E}_0) \leq n\), there exists a uniform cover \(\gamma\) of \(\hat{X}\) such that \(\text{mult}_{D(\beta)_0 E^2}(\gamma) \leq n + 1\). Let \(\alpha = E \circ D(\beta)(\gamma)\). Then, \(\text{mult}_E \alpha = \text{mult}_E (E \circ D(\beta)(\gamma)) \leq \text{mult}_{E^2} D(\beta)(\gamma) \leq \text{mult}_{D(\beta)_0 E^2}(\gamma) \leq n + 1\). Particulary, \(\text{mult} \alpha \leq \text{mult}_E \alpha \leq n + 1\).

Clearly, \(\alpha\) is uniform. To prove that \(\alpha\) is open, fix \(V \in \alpha\) and take \(U \in \gamma\) such that \(V = E(D(\beta)(U))\). Since \(V = \bigcup_{x \in D(\beta)(U)} E_x\) (see \([3]\)) and each \(E_x\) is open, we have that \(V\) is open. Moreover, for every \(x \in \hat{X}\), \(\text{mult}_{E_x} \alpha \leq \text{mult}_E \alpha \leq n + 1 < \infty\) and we get that \(\alpha\) is locally finite.

Finally, fix \(W \in \beta\) and take \(U \in \gamma\) with \(W \in U \neq \emptyset\). Then, by \([1]\), \(W \subset \bigcup_{U \cap W' \neq \emptyset} W' = \beta(U) = D(\beta)(U) \subset E \circ D(\beta)(U) \in \alpha\). Thus, \(\beta \prec \alpha\).

Therefore, \(\alpha\) is the desired cover. \(\square\)

Theorem 19. Let \((X, \hat{X}, \check{X})\) be a metrizable compactification pack and consider \((\hat{X}, \mathcal{E}_0)\). Let \(n \in \mathbb{N} \cup \{0\}\). Then,
a) \( \dim X \leq n \).

b) \( \asdim(X, E_0) \leq n + 1 \).

c) For every uniform cover \( \beta \) of \( \hat{X} \) there exists a canonical cover \( \alpha \succ \beta \) which multiplicity is less than or equal to \( n + 2 \).

d) For every canonical cover \( \beta \) of \( \hat{X} \) there exists canonical cover \( \alpha \succ \beta \) which multiplicity is less than or equal to \( n + 2 \).

are equivalent.

Proof. Take into account Corollary 7. The equivalence between a) and b) is Grave’s theorem. b) implies c) due to Proposition 18. The implication c) \( \Rightarrow \) d) is obvious. d) implies b) because of Proposition 13.

Implications a) \( \Rightarrow \) b), a) \( \Rightarrow \) c) and a) \( \Rightarrow \) d) are also a consequence of Proposition 49.

3.2 Canonical covers on special spaces

Let \( (X, \hat{X}, \tilde{X}) \) be a metrizable compactification pack. If \( \tilde{X} \) is the Hilbert cube \( Q \) or of the finite dimensional cube \( [0, 1]^n \), Theorem 19 proves (B) partially, but we have to show that the multiplicity of every canonical cover is greater than \( \dim X + 2 \), and not just for the bigger ones, as stated.

In general, just one canonical cover can not say anything about the dimension of \( X \). For example, suppose that \( \hat{X} \) is countable —to be more specific, let \( Y \) be a metrizable compact set, let \( \{y_n\} \) be a dense sequence in \( Y \) and put \( X = Y \times \{0\}, \hat{X} = (\bigcup_{n \in \mathbb{N}} \{y_1, \ldots, y_n\} \times \{\frac{1}{n}\}) \) and \( \tilde{X} = X \cup \hat{X} \), all of them with the topology induced by \( Y \times [0, 1] \). \( \{\{x\} : x \in \hat{X}\} \) is a canonical cover of \( \hat{X} \) whose multiplicity is 1, independently on the dimension of \( X \).

For proving (A), we need the special topological properties of the Z-sets of \( Q \) or \( [0, 1]^n \). By this reason, we will define the cylindrical subsets, a class of subsets with involve the Z-sets of the Hilbert cube or the finite dimensional cube, which have the properties we need to prove (A).

Definition 20. A subset \( X \) of a topological space \( \tilde{X} \) is said to be cylindrical if there exists an embedding \( j : X \times [0, 1] \hookrightarrow \tilde{X} \) such that \( j(x, 0) = x \) for every \( x \in X \).

Remark 21.

- If \( X \) has a tubular neighborhood in \( \tilde{X} \), then \( X \) is cylindrical in \( \tilde{X} \), but the reciprocal is false (take for example \( X = \{0\} \) and \( \tilde{X} = [-1, 1] \)).

Definition 22. We say that a compactification pack \( (X, \hat{X}, \tilde{X}) \) is cylindrical if \( X \) is a cylindrical subset of \( \tilde{X} \).

Lemma 23. Every Z-set of the Hilbert cube is a cylindrical subset.
Lemma 26. \[ \times \]

Proof. Anderson’s theorem (see [6], Theorem II-11.1 or [1]) states that every homeomorphism between two Z-sets of \( Q \) can be extended to a homeomorphism of \( Q \) onto itself.

Let \( X \) be a Z-set of \( Q \). Since \( X \times \{0\} \) and \( X \) are Z-sets of \( Q \times [0,1] \approx Q \) and \( Q \) respectively and \( g : X \times \{0\} \to X \), \((x,0) \to x\) is a homeomorphism, there exists a homeomorphism \( h : Q \times [0,1] \to Q \) which extends \( g \). Particulary, \( h|_{X \times [0,1]} : X \times [0,1] \to Q \) is an embedding and \( X \) is a cylindrical subset of \( Q \).

\[ \blacklozenge \]

Remarks 24.

- If \( h \) is the function in Lemma 25’s proof and \( f = h^{-1} \), then \( f : Q \to Q \times [0,1] \) is a homeomorphism such that \( f(X) \subset Q \times \{0\} \). Since the reciprocal is obvious, we have an easy proof of the known result in infinite dimensional topology: A closed subset \( X \) of \( Q \) is a Z-set of \( Q \) if and only if there exists a homeomorphism \( f : Q \to Q \times [0,1] \) such that \( f(X) \subset Q \times \{0\} \).

- In \( Q \), to be cylindrical is stronger than to be nowhere dense (see \( \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\} \times Q \) in \([0,1] \times Q\) and weaker than to be a Z-set (see \( \left\{ \frac{1}{2} \right\} \times Q \) in \([0,1] \times Q\).

Using example VI 2 of [13], it follows easily that:

Lemma 25. Let \( n \in \mathbb{N} \). The Z-sets of \([0,1]^n\) are precisely the closed subsets of \([0,1]^n \setminus (0,1)^n\)

Lemma 26. Let \( n \in \mathbb{N} \). Every Z-set of \([0,1]^n\) is a cylindrical subset.

Proof. Consider the copy \([-1,1]^n\) of \([0,1]^n\). Suppose that \( X \) is a Z-set of \([-1,1]^n\). According with Lemma 25, \( X \subset [-1,1]^n \setminus (-1,1)^n \). Consider the continuous maps \( j_1 : X \times [0,1] \to X \times \left[ \frac{1}{2}, 1 \right], \ (x,t) \to (x, \frac{1+t}{2}) \) and \( j_2 : X \times \left[ \frac{1}{2}, 1 \right] \to [-1,1]^n, \ (x,t) \to t \cdot x \). It is easy to check that \( j_2 \circ j_1 : X \times [0,1] \to [-1,1]^n \) is an embedding such that \( j_2 \circ j_1(x,0) = x \) for every \( x \in X \).

\[ \blacklozenge \]

Lemma 27. Let \( X \) be a compact metric space with \( \dim X \geq n < \infty \). Then, there exists an \( \varepsilon > 0 \) such that every open and finite cover \( \alpha \) of \( X \times [0,1] \) with

a) \( \text{mesh}\{\pi_1(U) : U \in \alpha\} < \varepsilon \) (where \( \pi_1 : X \times [0,1] \to X \) is the projection).

b) there does not exist any \( U \in \alpha \) which intersect both \( X \times \{0\} \) and \( X \times \{1\} \).

satisfies \( \text{mult} \alpha \geq n + 2 \).

Proof. Remember that if \( X \) is a compact Hausdorff space, then \( \dim X \times [0,1] = \dim X + 1 \). It is a corollary of [13], page 194. Also, from Theorem 7, Theorem 8 or Theorems 4-6 of [18].

Consider on \( X \times [0,1] \) the supremum metric. Since \( \dim X \times [0,1] = \dim X + 1 \geq n + 1 \), there exists \( \varepsilon > 0 \) such that, for every open cover \( \beta \) of \( X \times [0,1] \) with \( \text{mesh} \beta < \varepsilon \), we have \( \text{mult} \beta \geq n + 2 \).
Let α be an open cover of X satisfying properties a) and b) for ε. Take k ∈ \mathbb{N} with \frac{1}{2k} < \varepsilon. Let us construct on X × [0, 2k] a cover γ₀ like in the figure 3.

Let α′ be the symmetric cover of α on X × [0, 1] given by α′ = \{φ(U) : U ∈ α\}, where φ : X × [0, 1] → X × [0, 1] is the symmetry φ(x, t) = (x, 1 − t). Pull forward the cover α to the intervals [2j, 2j + 1], for j = 0, \ldots, k − 1, by means of the translations f_j : X × \mathbb{R} → X × \mathbb{R}, f_j(x, t) = (x, t + 2j) and pull forward the cover α′ to the intervals [2j + 1, 2j + 2], for j = 0, \ldots, k − 1, by means of the translations f'_j : X × \mathbb{R} → X × \mathbb{R}, f'_j(x, t) = (x, t + 2j + 1). Let γ₀ be the cover of X × [0, 2k] given by the union of those covers joining every pulled U ∈ α which meets X × \{i\}, for i = 1, \ldots, 2k − 1, with their reflections on the pulled subsets of α′.

More accurately,

$$γ₀ = \{f_j(U) : U ∈ α, 0 ≤ j ≤ k − 1, f_j(U) ∩ X × \{i\} = \emptyset ∀ i = 1, \ldots, 2k − 1\} \cup \{f'_j ∘ φ(U) : U ∈ α, 0 ≤ j ≤ k − 1, f'_j ∘ φ(U) ∩ X × \{i\} = \emptyset ∀ i = 1, \ldots, 2k − 1\} \cup \{f_j(U) ∪ f'_j ∘ φ(U) : U ∈ α, U ∩ X × \{1\} ≠ \emptyset, 0 ≤ j ≤ k − 2\} \cup \{f'_j ∘ φ(U) ∪ f_{j+1}(U) : U ∈ α, U ∩ X × \{0\} ≠ \emptyset, 1 ≤ j ≤ k − 1\}$$

Let γ be the cover of X × [0, 1] given by γ = \{ψ(U) : U ∈ γ₀\}, where ψ : X × [0, 2k] → X × [0, 1] is the homothety ψ(x, t) = (x, \frac{1}{2k} t). It is easy to check that γ is an open cover of X × [0, 1] with mesh γ < ε which has the same multiplicity as α. By definition of ε, we get mult α = mult γ ≥ n + 2.

**Remark 28.**

- We get an easy generalization of Lemma [27] by changing “there exists ε > 0” by “there exists an open and finite cover of X α₀” and “mesh π₁(α) < ε” by “π₁(α) ≪ α₀”. Hint: if X and Y are compact spaces and α is an open and finite cover of X × Y, then there exist two open and finite covers β₁ and β₂ of X and Y respectively such that \{U × V : U ∈ β₁, V ∈ β₂\} ≪ α.

The following result is based on Theorem 5.9 of [11] (pag. 59):
Lemma 29. Let $X$ be a compact metric space. Consider the compactification pack $(X \times \{0\}, X \times (0, 1], X \times [0, 1])$ and consider $(X \times (0, 1], E_0)$. If $\alpha$ is an open and uniform cover of $X \times (0, 1]$, then $\dim X \leq \mult \alpha - 2$. Particularly, it happens if $\alpha$ is a canonical cover.

Proof. If $\mult \alpha = \infty$, the result is obvious. Suppose now that $\mult \alpha = n < \infty$. To get a contradiction, assume that $\dim X \geq n - 1$. By Lemma 27 there exists $\varepsilon > 0$ such that every open cover $\beta$ of $X \times [0, 1]$ satisfying properties a) and b) of that lemma, also satisfies $\mult \beta \geq n + 1$.

Take into account the characterization of uniform covers of Proposition 11. Consider on $X \times [0, 1]$ the supremum metric. Since $\alpha$ is uniform, there exists a compact subset of $X \times (0, 1]$ such that $\text{diam } U < \varepsilon$ for every $U \in \alpha$ with $K \cap U = \emptyset$. Take $\delta_1 > 0$ such that $K \subset X \times (\delta_1, 0]$.

Since $\alpha$ is proper, $\alpha(X \times \{\delta_1\})$ is relatively compact. Take $\delta_2 > 0$ such that $\alpha(X \times \{\delta_1\}) \subset X \times (\delta_2, 1]$. Let $\gamma_0 = \{U \cap X \times [\delta_2, \delta_1] : U \in \alpha\}$ and consider the cover $\gamma$ of $X \times [0, 1]$ given by $\gamma = \phi^{-1}(\gamma_0)$, where $\phi : X \times [0, 1] \to X \times [\delta_0, \delta_1]$ is the homeomorphism $\phi(x, t) = (x, t\delta_2 + (1 - t)\delta_1)$.

Clearly, $\gamma$ is an open cover of $X \times [0, 1]$ such that $\mult \gamma \leq \mult \alpha \leq n$, mesh $\pi_1(\gamma) < \varepsilon$ and no $V \in \gamma$ intersects both $X \times \{0\}$ and $X \times \{1\}$. By Lemma 27, $\mult \gamma \geq n + 1$, in contradiction with $\mult \gamma = \mult \gamma_0 \leq \mult \alpha \leq n$. Hence, $\dim X \leq n - 2$. □

Proposition 30. Let $(X, \hat{X}, \hat{X})$ be a metrizable cylindrical compactification pack. If $\alpha$ is an open and uniform cover of $(\hat{X}, E_0)$, then $\dim X \leq \mult \alpha - 2$.

Proof. Consider the compactification pack $(X \times \{0\}, X \times (0, 1], X \times [0, 1])$ and suppose $E'_0 = E_0(X \times \{0\}, X \times (0, 1], X \times [0, 1])$.

Let $d$ be a metric on $\hat{X}$ and let $j : X \times [0, 1] \to \hat{X}$ be an embedding such that $j(x, 0) = x$ for every $x \in X$. Consider on $X \times [0, 1]$ the metric $d'$ given by $d'(a, b) = d(j(a), j(b))$.

Let $\beta = j^{-1}([X \times (0, 1]) \alpha \right)$. From the continuity of $j$ we get that $\beta$ is an open cover over $X \times [0, 1]$ and, from the injectivity of $j$, that $\mult \beta \leq \mult \alpha$.

![Figure 4. A part of a uniform cover $\alpha$ or $(\hat{X}, E_0)$, the induced cover in $j(X \times (0, 1])$ and $\beta = j^{-1}([X \times (0, 1]) \alpha)$.](image-url)
Let us see that $\beta$ is uniform for $E'_0$. Let $\varepsilon > 0$. Since $\alpha$ is uniform for $E_0$, there exists a compact subset $K$ of $\hat{X}$ such that $d(x, y) < \varepsilon$ whenever $(x, y) \in D(\alpha) \setminus K \times K$.

Let $K' = j^{-1}(K)$. Observe that $K' \subset j^{-1}(\hat{X}) = X \times (0, 1]$. Moreover, $K'$ is compact, because it is a closed subset of $X \times [0, 1]$. Finally, for every $(a, b) \in D(\beta) \setminus K' \times K'$, we have that $(j(a), j(b)) \in D(\alpha) \setminus K \times K$ and, consequently, $d'(a, b) = d(j(a), j(b)) < \varepsilon$.

Since $\beta$ is an open and uniform cover of $(X \times (0, 1], E'_0)$, Lemma \ref{lem:asdim} shows that $\dim X \leq \mult \alpha - 2$.

**Proposition 31.** Let $(X, \hat{X}, \tilde{X})$ be a cylindrical metrizable compactification pack (in particular, if $\tilde{X}$ is $Q$ or $[0, 1]^n$, with $n \in \mathbb{N}$, and $X$ is a Z-set of $\tilde{X}$). Consider $(\hat{X}, E_0)$.

Then, the following properties are equivalent:

a) $\dim X \leq n$.

b) $\asdim(\hat{X}, E_0) \leq n + 1$.

c) For every uniform cover $\beta$, there exists a canonical cover $\alpha$ such that $\beta \prec \alpha$ and $\mult \alpha \leq n + 2$.

d) For every canonical cover $\beta$, there exists a canonical cover $\alpha$ such that $\beta \prec \alpha$ and $\mult \alpha \leq n + 2$.

e) There exists a canonical cover $\alpha$ with $\mult \alpha \leq n + 2$.

f) There exists an open and uniform cover $\alpha$ of $(X, \hat{X}, \tilde{X})$ with $\mult \alpha \leq n + 2$.

**Proof.** The equivalences between a), b), e) and d) are given in Proposition \ref{prop:asdim}. The implications d) $\Rightarrow$ e) $\Rightarrow$ f) are obvious. f) implies a) because of Proposition \ref{prop:mult}. \qed

### 3.3 Canonical covers on general spaces

Let us start with the following lemma to get a corollary from Proposition \ref{prop:asdim}.

**Lemma 32.** If $\alpha$ and $\beta$ are families of subsets of a set $Z$ and there is a surjective map $\phi: \alpha \rightarrow \beta$ such that $U \supset \phi(U)$ for every $U \in \alpha$, then $\mult \beta \leq \mult \alpha$.

**Proof.** $\mult \beta = \sup \{\# B : B \subset \beta, \bigcap_{B \in \beta} V \neq \emptyset\} \leq \sup \{\# \phi^{-1}(B) : B \subset \beta, \bigcap_{B \in \phi^{-1}(B)} U \neq \emptyset\} \leq \sup \{\# A : A \subset \alpha, \bigcap_{A \in \alpha} U \neq \emptyset\} = \mult \alpha$. \qed

**Corollary 33.** Let $(X, \hat{X}, \tilde{X})$ be a cylindrical metrizable compactification pack and consider $(\hat{X}, E_0)$. Then,

a) If $\alpha$ is a uniform cover $\hat{X}$ which has an open refinement, then $\mult \alpha \geq \asdim(\hat{X}, E_0) + 1$.

b) If $E$ is an open and controlled neighborhood of the diagonal of $\hat{X} \times \tilde{X}$, then $\mult_E \alpha \geq \asdim(\hat{X}, E_0) + 1$ for every uniform cover $\alpha$ of $\hat{X}$.
Proof. To prove a), take an open refinement \( \beta \) of \( \alpha \). Let \( \mathcal{T} \) be the topology of \( \hat{X} \) and consider the map \( \phi : \alpha \to \mathcal{T} \) given by \( \phi(V) = \bigcup_{U \in \beta} U \). Observe that \( U \supset \phi(U) \) for each \( U \). Let \( \gamma = \phi(\alpha) \). By Lemma \( [32] \) \( \text{mult} \gamma \leq \text{mult} \alpha \).

Since \( \gamma \prec \alpha \), we have that \( \gamma \) is uniform. Moreover,

\[
\bigcup_{W \in \gamma} W = \bigcup_{V \in \alpha} \phi(V) = \bigcup_{V \in \alpha} \bigcup_{U \in \beta} U = \bigcup_{U \in \beta} U = \hat{X}
\]

Since \( \gamma \) is an open and uniform cover of \( \hat{X} \), Proposition \( [29] \) and Grave’s theorem show that \( \text{mult} \alpha \geq \text{mult} \gamma \geq \dim X + 2 = \text{asdim}(\hat{X}, \mathcal{E}_0) + 1 \).

Now, let us see b). \( E(\alpha) \) is uniform because \( E \) is controlled and \( \alpha \), uniform. By \( [3] \), \( E(\alpha) = \{ E(V) : V \in \alpha \} = \{ \bigcup_{x \in V} E_x : V \in \alpha \} \). Since each \( E_x \) is open, we get that \( E(\alpha) \) is open and, since \( \alpha < E(\alpha) \), that \( E(\alpha) \) is a cover of \( \hat{X} \).

From Lemmas \( [17] \) and \( [30] \) and Grave’s theorem, we get \( \text{mult}_E \alpha \geq \text{mult} E(\alpha) \geq \dim X + 2 \geq \text{asdim}(\hat{X}, \mathcal{E}_0) + 1 \).

Let \( (X, \hat{X}, \tilde{X}) \) be a cylindrical and metrizable compactification pack and consider \( (\hat{X}, \mathcal{E}_0) \). Corollary \( [33] \) means that, from the asymptotic dimensional point of view, an open and uniform cover \( \alpha \) is a controlled and open neighborhood of the diagonal are big enough.

This observation suggests a question. In the general case when \( (X, \hat{X}, \tilde{X}) \) is not necessary cylindrical, when is a cover \( \alpha \) or a controlled set \( E \) big enough from the asymptotic dimensional point of view? The following results will answer this question.

**Proposition 34.** Let \( (X, \hat{X}, \tilde{X}) \) be a metrizable compactification pack and suppose that \( d \) is a metric on \( \hat{X} \). If \( k = \sup_{x \in \hat{X}} d(x, X) \), then the map \( h : (0, k] \to (0, k], t \to \sup_{x \in X} d(x, \hat{X} \setminus B(X, t)) \) is increasing and satisfies \( \lim_{t \to 0} h(t) = 0 \).

Proof. If \( t \leq t' \), then \( \hat{X} \setminus B(X, t') \supset \hat{X} \setminus B(X, t) \) and, for every \( x \in X \), \( d(x, \hat{X} \setminus B(X, t)) \leq d(x, \hat{X} \setminus B(X, t')) \). Taking supreme over \( x \), we get \( h(t) \leq h(t') \).

Fix \( \varepsilon > 0 \). Since \( X \) is compact and \( X \subset \bigcup_{x \in X} B(X, \frac{\varepsilon}{2}) \), there exists \( x_1, \ldots, x_r \in X \) such that \( X \subset \bigcup_{j=1}^r B(x_j, \frac{\varepsilon}{2}) \). For every \( j \), pick a point \( y_j \in \tilde{X} \cap B(x_j, \frac{\varepsilon}{2}) \). Let \( \delta = \min_{1 \leq j \leq r} d(y_j, X) \).

Fix \( t < \delta \). For every \( x \in X \), there is \( j = 1, \ldots, r \) such that \( x \in B(x_j, \frac{\varepsilon}{2}) \). Observe that \( y_j \in \tilde{X} \setminus B(X, t) \) and thus, \( d(x, \tilde{X} \setminus B(X, t)) \leq d(x, y_j) \leq d(x, x_j) + d(x_j, y_j) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \). Taking supreme over \( x \), we get \( h(t) \leq \varepsilon \). \( \square \)

The following proposition implies Corollary \( [36] \). This corollary has been proved independently by Grave in \( [11] \) or \( [12] \), by us in \( [16] \) and \( [17] \) and by Mine and Yamashita in \( [15] \). We add this proposition here because we need the explicit functions used there. Moreover, we get an easy proof of Corollary \( [36] \).
Proposition 35. Let \((X, \hat{X}, \tilde{X})\) be a metrizable compactification pack. Consider the compactification pack \((X \times \{0\}, X \times (0,1], X \times [0,1])\) and the coarse structures \(\mathcal{E}_0 = \mathcal{E}_0(X, \hat{X}, \tilde{X})\) and \(\mathcal{E}_0' = \mathcal{E}_0(X \times \{0\}, X \times (0,1], X \times [0,1])\). Let \(d_0\) be a metric on \(\hat{X}\). Consider the metric \(d = \frac{1}{k}d_0\) on \(\tilde{X}\), where \(k = \sup_{x \in \tilde{X}} d_0(x, X)\).

Then, there exist an \(f : (\hat{X}, \mathcal{E}_0) \to (X \times (0,1], \mathcal{E}_0')\) and a \(g : (X \times (0,1], \mathcal{E}_0') \to (\hat{X}, \mathcal{E}_0)\) satisfying

- For every \(x \in \hat{X}\), \(f(x) = (z, t)\), with \(t = d(x, X)\), \(z \in X\) and \(d(x, z) = t\).
- For every \((z, t) \in X \times (0,1], g(z, t) = y\) with \(y \in \tilde{X} \setminus B(X, t)\) and \(d(y, \hat{X} \setminus B(X, t)) = d(z, \tilde{X} \setminus B(X, t))\).

in which case, they are coarse equivalences, the one inverse of the other.

Proof. Since \(\sup_{x \in \tilde{X}} d(x, X) = 1\), such \(f\) and \(g\) do exist. To check their coarse equivalences, it suffices to show that they satisfy property c) of Proposition 1. Consider on \(X \times [0,1]\) the metric \(d'((x, t), (z, s)) = d(x, z) + |t - s|\).

Fix \(E \in \mathcal{E}_0\) and let us see that \(f \times f(E) \in \mathcal{E}_0'\). Let \(\varepsilon > 0\) and suppose that \(K\) is a compact subset of \(\tilde{X}\) such that \(d(x, x') < \frac{\varepsilon}{5}\) whenever \((x, x') \in E \setminus K \times K\).

Let \(\delta = \min\{\frac{\varepsilon}{5}, d(K, X)\}\) and consider \(K' = X \times [0,1]\). Pick \(((z, t), (z', t')) \in f \times f(E) \setminus K' \times K'\) and take \((x, x') \in E\) such that \(f(x) = (z, t)\) and \(f(x') = (z', t')\). Suppose, without loss of generality, that \(t \leq t'\).

Neither \((z, t) \in K'\) nor \((z', t') \in K'\), that is, either \(t < \delta\) or \(t' < \delta\). Then, \(d(x, X) = t < \delta \leq d(K, X)\) and thus, \(x \notin K\). Hence \((x, x') \in E \setminus K \times K\) and \(d(x, x') < \frac{\varepsilon}{5}\). Moreover, \(t' = d(x', X) \leq d(x', z) \leq d(x', x) + d(x, z) = d(x', x) + t < d(x', x) + \delta\). Therefore,

\[
d'(\{(x, t), (z, s)\}) = d(z, z') + |t - t'| \leq d(z, x) + d(x, x') + d(x', z') + t' - t = t + d(x, x') + t' - t = d(x, x') + 2t' \leq 3d(x, x') + 2\delta < 3\frac{\varepsilon}{5} + 2\frac{\varepsilon}{5} = \varepsilon
\]

and we get \(f \times f(E) \in \mathcal{E}_0'\).

Fix \(F \in \mathcal{E}_0'\) and let us see that \((f \times f)^{-1}(F) \in \mathcal{E}_0\). Let \(\varepsilon > 0\) and suppose that \(K'\) is a compact subset of \(X \times (0,1]\) such that \(d'((z, t), (z', t')) < \frac{\varepsilon}{3}\) whenever \(((z, t), (z', t')) \in F \setminus K' \times K'\). Take \(\delta_0\) such that \(K' \subset X \times [\delta_0, 1]\). Put \(\delta = \min\{\delta_0, \frac{\varepsilon}{3}\}\) and put \(K = \tilde{X} \setminus B(X, \delta)\). Pick \((x, x') \in (f \times f)^{-1}(F) \setminus K \times K\) and take \(((z, t), (z', t')) \in F\) such that \(f(x) = (z, t)\) and \(f(x') = (z', t')\). Suppose, without loss of generality, that \(t \leq t'\).

Neither \(x \in K\) nor \(x' \in K\), that is, either \(t = d(x, X) < \delta\) or \(t' = d(x', X) < \delta\). Then, \(t < \delta \leq \delta_0\) and thus \((z, t) \notin K'\). Hence, \(((z, t), (z', t')) \in E' \setminus K' \times K'\) and \(d'((z, t), (z', t')) < \frac{\varepsilon}{3}\). Therefore,

\[
d(x, x') \leq d(x, z) + d(z, z') + d(z', x') = t + d(z, z') + t' = 2t + d(z, z') + |t - t'| < 2\delta + d'((z, t), (z', t')) < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]
and we get \((f \times f)^{-1}(F) \in \mathcal{E}_0\).  

Let \(G = \{(z, t) : (z, t) \in X \times (0, 1]\} \) and let us see that \(G \in \mathcal{E}_0'\). Fix \(\varepsilon > 0\). Consider the function \(h : (0, 1] \to (0, 1], t \to \sup_{x \in X} d(x, \tilde{X} \setminus B(X, t))\). By Proposition \([34]\), \(\lim_{t \to 0} h(t) = 0\), so there exists \(\delta > 0\) such that \(h(t) < \frac{\varepsilon}{3}\) when \(t < \delta\).

Let \(K = X \times [\delta, 1]\) and pick \((x, t), (z, s)\) \(\in G \setminus K \times K\). Then, either \(t < \delta\) or \(s < \delta\). Observe that \((z, s) = f \circ g(x, t)\). Put \(y = g(x, t)\), so we have \((z, s) = f(y)\). Since \(y \in \tilde{X} \setminus B(X, t)\), it follows that \(s = d(y, X) \geq t\) and thus, \(t < \delta\). Therefore,

\[
d'(x, t), (z, s)) = d(x, z) + |t - s| \leq d(x, y) + d(y, z) + s - t = d(x, y) + 2d(y, z) - t < d(x, y) + 2d(y, X) \leq 3d(x, y) = 3d(x, \tilde{X} \setminus B(X, t)) \leq 3h(t) < \frac{3\varepsilon}{3} = \varepsilon
\]

and we get \(G \in \mathcal{E}_0'\). Therefore, \(f\) is a coarse equivalence and \(g\) is its coarse inverse. □

**Corollary 36.** If \((X_1, \tilde{X}_1, \tilde{X}_1)\) and \((X_2, \tilde{X}_2, \tilde{X}_2)\) are two metrizable compactification packs such that \(X_1\) and \(X_2\) are homeomorphic, then \((\tilde{X}_1, \mathcal{E}_0')\) and \((\tilde{X}_2, \mathcal{E}_0')\) are coarse equivalent where, for \(i = 1, 2\), \(\mathcal{E}_0' = \mathcal{E}_0(X_i, \tilde{X}_i, \tilde{X}_i)\).

**Proof.** For \(i = 1, 2\), consider the compactification pack \((X_i \times \{0\}, X_i \times (0, 1], X_i \times [0, 1])\) and suppose \(\mathcal{E}_0' = \mathcal{E}_0(X_i \times \{0\}, X_i \times (0, 1], X_i \times [0, 1])\).

Let \(h : X_1 \to X_2\) be a homeomorphism and let \(d\) be a metric on \(X_1\). Consider on \(X_1 \times [0, 1]\) and \(X_2 \times [0, 1]\) the metrics \(d_1\) and \(d_2\) respectively, given by:

\[
d_1((x, t), (y, s)) = d(x, y) + |t - s| \\
d_2((x, t), (y, s)) = d(h^{-1}(x), h^{-1}(y)) + |t - s|
\]

Using \(d_1\) and \(d_2\), it is easy to check that the map \(h' : X_1 \times (0, 1] \to X_2 \times (0, 1], (x, t) \to (h(x), t)\) satisfies property \(d)\) of Proposition \([1]\) and hence, \(h'\) is a coarse equivalence. Finally, from Proposition \([35]\) we get

\[
(\tilde{X}_1, \mathcal{E}_0') \approx (X_1 \times (0, 1], \mathcal{E}_0') \approx (X_2 \times (0, 1], \mathcal{E}_0') \approx (\tilde{X}_2, \mathcal{E}_0')
\]

□

**Proposition 37.** Let \((X, \tilde{X}, \bar{X})\) be a compactification pack, let \(d\) be a metric on \(\tilde{X}\), let \(E \subseteq \tilde{X} \times \tilde{X}\) and let \(k = \sup_{x \in \tilde{X}} d(x, X)\). Then, \(E \in \mathcal{E}_0\) if and only if there exists \(\phi : (0, k] \to \mathbb{R}^+\) with \(\lim_{t \to 0} \phi(t) = 0\) such that

\[
E \subseteq \{(x, y) \in X : d(x, y) < \phi(\min\{d(x, X), d(y, X)\})\}
\]

**Proof.** Suppose such \(\phi\) exists. Fix \(\varepsilon > 0\) and take \(\delta > 0\) such that \(\phi(t) < \varepsilon\) for every \(t < \delta\). Let \(K = \tilde{X} \setminus B(X, \delta)\) and pick \((x, y) \in E \setminus K \times K\). Then, neither \(x \in K\) nor \(y \in K\). In
any case, \( \min\{d(x, X), d(y, X)\} < \delta \). Hence, \( d(x, y) \leq \phi(\min\{d(x, X), d(y, X)\}) < \varepsilon \) and \( E \in \mathcal{E}_0 \).

Now, assume that \( E \in \mathcal{E}_0 \). For every \( t \in (0, k] \), let \( K_t = \hat{X} \setminus \overline{B}(X, t) \) and let \( \phi(t) = t + \sup_{(x, y)}d(x, y) : (x, y) \in E \setminus (K_t \times K_t) \) (if \( E \setminus (K_t \times K_t) \) is empty, put \( \phi(t) = t \)). Let us see that \( \lim_{t \to 0} \phi(t) = 0 \). Fix \( \varepsilon > 0 \), consider a compact subset \( K \) of \( \hat{X} \) such that \( d(x, y) < \frac{\varepsilon}{2} \) whenever \( (x, y) \in E \setminus K \times K \) and set \( \delta = d(x, K) \). Fix \( t < \delta \) and pick \( (x, y) \in E \setminus K_t \times K_t \). Since \( K \subset K_t \) and hence \( (x, y) \in E \setminus K_t \times K_t \), we have that \( d(x, y) < \frac{\varepsilon}{2} < \frac{\varepsilon}{2} \) and \( \phi(t) < \varepsilon \).

Pick a point \((x, y) \in E \) and put \( t_0 = \min\{d(x, X), d(y, Y)\} \). Since \((x, y) \in E \setminus K_{t_0} \times K_{t_0} \), we have that \( d(x, y) < \phi(t_0) = \phi(\min\{d(x, X), d(y, X)\}) \). \( \square \)

**Proposition 38.** Let \((X, \hat{X}, \tilde{X}) \) be a compactification pack, let \( d \) be a metric on \( \tilde{X} \), let \( E \subset \tilde{X} \times \tilde{X} \) and let \( k = \sup_{x \in \tilde{X}}d(x, X) \). Then, \( E \) is a neighborhood of the diagonal if and only if there exists an increasing function \( \lambda : (0, k] \to \mathbb{R}^+ \) such that

\[
\{(x, y) \in X : d(x, y) < \lambda(\min\{d(x, X), d(y, X)\})\} \subset E
\]

**Proof.** Suppose such \( \lambda \) exists. Let \( \lambda_0 : (0, k] \to \mathbb{R}^+ \) be an increasing and continuous function such that \( \lambda_0(t) \leq \lambda(t) \) for every \( t \). For example, we may define \( \lambda_0 \) as follows: set \( \lambda_0\left(\frac{k}{n}\right) = \lambda\left(\frac{k}{n+1}\right) \) for every \( n \in \mathbb{N} \) and extend \( \lambda_0 \) linearly to every interval \( [\frac{k}{n}, \frac{k}{n+1}] \).

Let \( F = \psi^{-1}(\mathbb{R}^+) \), where \( \psi \) is the continuous function \( \psi : \tilde{X} \times \tilde{X} \to \mathbb{R}, (x, y) \to \lambda_0(\min\{d(x, X), d(y, X)\}) - d(x, y) \). Then, \( F \) is an open subset of \( \tilde{X} \times \tilde{X} \) containing the diagonal such that \( F \subset \{(x, y) \in X : d(x, y) < \lambda(\min\{d(x, X), d(y, X)\})\} \subset E \).

Now, suppose \( E \) is a neighborhood of the diagonal. Consider the supremum metric \( d_\infty \) on \( \tilde{X} \times \tilde{X} \). Take an open subset \( F \subset E \) containing the diagonal. By the closeness of \( \tilde{X} \times \tilde{X} \setminus F \), we may define the map \( f : X \to \mathbb{R}^+, x \to d_\infty((x, x), \tilde{X} \times \tilde{X} \setminus F) \) and it is continuous. For every \( k \in (0, k] \), \( K_t \) is compact and then, \( f \) has a minimum in \( K_t \), so we may define the map \( \lambda : (0, k] \to \mathbb{R}^+ \), \( t \to \min_{x \in K_t} f(t) \).

For every \( t' \leq t \), \( K_t \supset K_{t'} \), hence \( \lambda(t) \leq \lambda(t') \) and we get that \( \lambda \) is increasing. Let \((x, y) \in \hat{X} \times \hat{X} \) be such that \( d(x, y) < \lambda(\min\{d(x, X), d(y, X)\}) \). Suppose, without loss of generality, that \( d(x, X) \leq d(y, X) \) and put \( t = d(x, X) \). Since \( x \in K_t \), we have

\[
d_\infty((x, x), (x, y)) < \lambda(t) \leq d_\infty((x, x), \hat{X} \times \hat{X} \setminus F)
\]

Then, \((x, y) \notin \hat{X} \times \hat{X} \setminus F \) and we conclude that \((x, y) \in F \subset E \). \( \square \)

**Lemma 39.** Let \( f : Z \to Z' \) be a map between two sets, let \( E \subset Z \times Z \), and let \( \alpha \) be a family of subsets of \( Z' \). Then, \( \multi_{E,f^{-1}}(\alpha) \leq \multi_{f \times f(E)}(\alpha) \).

**Proof.** Fix \( x \in Z \) and take \( U \in \alpha \) such that \( f^{-1}(U) \) meets \( E_x \). Then, by (6), \( \emptyset \neq f(f^{-1}(U) \cap E_x) = U \cup f(E_x) \subset U \cup \bigcup_{z \in f^{-1}(f(x))} f(E_x) = U \cup (f \times f(E))_f(x) \).

Hence, \( \multi_{E_x, f^{-1}}(\alpha) = \#\{f^{-1}(U) : U \in \alpha, f^{-1}(U) \cap E_x \neq \emptyset\} \leq \#\{U : U \in \alpha, U \cap (f \times f(E))_f(x) \neq \emptyset\} \leq \multi_{(f \times f(E))_f(x)}(\alpha) \leq \multi_{f \times f(E)}(\alpha) \). Taking supreme over \( x \) we get the inequality. \( \square \)
Proposition 40. Let \((X, \hat{X}, \overline{X})\) be a metrizable compactification pack, let \(d\) be a metric on \(X\), let \(k = \sup_{x \in X} d(x, X)\) and let \(\lambda : (0, k] \to \mathbb{R}^+\) be an increasing and continuous function such that \(\lim_{t \to 0} \lambda(t) = 0\).

Consider the maps \(h : \mathbb{R}^+ \to (0, k]\) and \(\phi : (0, k] \to \mathbb{R}^+\) given by:

\[
h(t) = \begin{cases} 
\sup_{x \in X} d(x, \hat{X} \setminus B(X, t)) & \text{if } t \leq k \\
\sup_{x \in X} d(x, \hat{X} \setminus B(X, t)) & \text{if } t \geq k
\end{cases}
\]

\[
\phi(t) = h(t) + \lambda(t) + h(t + \lambda(t))
\]

and consider the set:

\[
E_{d, \lambda} = \{(x, y) \in \hat{X} \times \hat{X} : d(x, y) < \phi(\min\{d(x, X), d(y, X)\})\}
\]

Then, \(E_{d, \lambda}\) is a controlled subset of \((\hat{X}, \mathcal{E}_0)\) such that \(\text{mult}_{E_{d, \lambda}} \alpha \geq \dim X + 2\) for every uniform cover \(\alpha\) of \((\hat{X}, \mathcal{E}_0)\).

Proof. By Proposition 34, \(\phi(t) \to 0\) when \(t \to 0\). Hence, from Proposition 37, we get \(E_{d, \lambda} \in \mathcal{E}_0\). Let \(\alpha\) be a uniform cover of \((\hat{X}, \mathcal{E}_0)\).

Consider the compactification pack \((X \times \{0\}, X \times (0, 1], X \times [0, 1])\) and the coarse space \((\hat{X}, \mathcal{E}_0')\), where \(\mathcal{E}_0' = \mathcal{E}_0(X \times \{0\}, X \times (0, 1], X \times [0, 1])\). Consider on \(\tilde{X}\) the metric \(\overline{d} = \frac{1}{k}d\) and, on \(X \times [0, 1]\), the metric \(\overline{d}_{\infty}((x, t), (x', t')) = \max\{d(x, x'), |t - t'|\}\).

By Proposition 35, there is a coarse equivalence \(g : X \times (0, 1] \to \hat{X}\) such that for every \((x, t), g(x, t) = y\) where \(y \in \hat{X} \setminus B_\overline{d}(X, t)\) is such that \(\overline{d}(x, y) = \overline{d}(x, \hat{X} \setminus B_\overline{d}(X, t))\). Consider the function \(\overline{\lambda} : (0, 1] \to \mathbb{R}^+\) such that for every \(t\),

\[
\overline{\lambda}(t) = \frac{1}{k} \lambda(kt)
\]

(10)

and consider the set \(F_0 = \{((x, t), (x', t')) \in (X \times (0, 1])^2 : \overline{d}_{\infty}((x, t), (x', t')) < \overline{\lambda}(\min\{t, t'\})\}\).

Since, for each \((x, t)\), we have \(t = \overline{d}_{\infty}((x, t), X \times \{0\})\), Proposition 37 shows \(F_0 \in \mathcal{E}_0\) and Proposition 38 that \(F_0\) is a neighborhood of the diagonal. Let \(F \subset F_0\) be a controlled and open neighborhood of the diagonal and let \(\beta = F(g^{-1}(\alpha))\).

Let us see that:

i) \(\beta\) is an open and uniform cover of \((X \times (0, 1], \mathcal{E}_0')\).

ii) \(g \times g(F) \subset E_{d, \lambda}\).

For each \(V \in \beta\), \(V = F(U) = \bigcup_{x \in U} F_x\) for some \(U \in g^{-1}(\alpha)\). Since each \(F_x\) is open, \(V\) is open. Moreover, \(g^{-1}(\alpha) \prec \beta\) and hence \(\beta\) is a cover of \(X \times (0, 1]\). Finally, by Corollary 3, \(\beta = F(g^{-1}(\alpha))\) is uniform for \(\mathcal{E}_0'\) and we get i).

Let us consider:

- the map \(\overline{h} : \mathbb{R}^+ \to (0, 1]\) such that \(\overline{h}(t) = \sup_{x \in X} \overline{d}(x, \hat{X} \setminus B_\overline{d}(X, t))\) when \(t \leq 1\) and \(\overline{h}(t) = 1\) when \(t \geq 1\)
\begin{itemize}
  \item the map $\overline{\phi} : (0, 1] \to \mathbb{R}^+$, given by $\overline{\phi}(t) = \overline{h}(t) + \overline{X}(t) + \overline{h}(t + \overline{X}(t))$
  \item the set $E_{\overline{d}, \overline{X}} = \{(x, y) \in X : \overline{d}(x, y) < \overline{\phi}(\min\{\overline{d}(x, X), \overline{d}(y, X)\})\}$
\end{itemize}

By Proposition 34, $\overline{h}$ and $\overline{\phi}$ are increasing. It is easy to check that, for every $t$, $\overline{h}(t) = \frac{1}{k}\overline{h}(kt)$ and $\overline{\phi}(t) = \frac{1}{k}\overline{\phi}(kt)$. Using those equalities and (10), it follows easily that $E_{\overline{d}, \overline{X}} = E_{d, \lambda}$. Then, to prove ii), it suffices to show that $g \times g(F) \subseteq E_{\overline{d}, \overline{X}}$.

Pick $(y, y') \in g \times g(F)$ and take $((x, t), (x', t')) \in F$ such that $g(x, t) = y$ and $g(x', t') = y'$. Suppose, without loss of generality, that $t \leq t'$. Observe that $t' = t + |t - t'| < t + \overline{d}_\infty((x, t), (x', t')) \leq t + \overline{X}(t)$. Then:

$$\begin{align*}
\overline{d}(y, y') &\leq \overline{d}(y, x) + \overline{d}(x, x') + \overline{d}(x', y') \\
&\leq \overline{d}(x, \hat{X} \setminus B_{\overline{\pi}}(X, t)) + \overline{d}_\infty((x, t), (x', t')) + \overline{d}(x', \hat{X} \setminus B_{\overline{\pi}}(X, t')) < \\
&\overline{h}(t) + \overline{X}(t) + \overline{h}(t') \leq \overline{h}(t) + \overline{X}(t) + \overline{h}(t + \overline{X}(t)) = \overline{\phi}(t)
\end{align*}$$

(11)

Since $y \in \hat{X} \setminus B_{\overline{\pi}}(X, t)$ and $y' \in \hat{X} \setminus B_{\overline{\pi}}(X, t')$, it follows that $\overline{d}(y, X) \geq t$ and $\overline{d}(y', X) \geq t' \geq t$. Since $\overline{\phi}$ is increasing, from (11) we get:

$$\overline{d}(y, y') < \overline{\phi}(t) \leq \overline{\phi}(\min\{\overline{d}(y, X), \overline{d}(y', X)\})$$

Therefore, $(y, y') \in E_{\overline{d}, \overline{X}}$ and we get ii).

Applying i), ii), Proposition 30 and Lemmas 17 and 39, we conclude:

$$\dim X + 2 \leq \mult \beta = \mult F(g^{-1}(\alpha)) \leq \mult_F g^{-1}(\alpha) \leq \mult_{g \times g(F)} \alpha \leq \mult_{E_{d, \lambda}} \alpha$$

\square

**Corollary 41.** Let $(X, \hat{X}, \overline{X})$ be a compactification pack and consider $(\hat{X}, \mathcal{E}_0)$. Let $E$ be the controlled set $E_{d, \lambda}$ of Proposition 40. Then:

a) $\mult_E \alpha \geq \asdim(\hat{X}, \mathcal{E}_0) + 1$ for every uniform cover $\alpha$ of $\hat{X}$.

b) $\mult \alpha \geq \asdim(\hat{X}, \mathcal{E}_0) + 1$ for every uniform cover $\alpha$ of $\hat{X}$ such that $\mathcal{K}(E) < \alpha$.

**Proof.** Let $\alpha$ be a uniform cover of $\hat{X}$. By Proposition 40 and Grave’s theorem, $\mult_E \alpha \geq \dim X + 2 = \asdim(\hat{X}, \mathcal{E}_0) + 1$, so we get a).

To see b), suppose $\mathcal{K}(E) < \alpha$. For every $U \in \alpha$, let $V_U = \{x \in \hat{X} : E_x \subset U\}$ and let $\gamma = \{V_U : U \in \alpha\}$. Observe that, by (3), $E(V_U) = \bigcup_{x \in V_U} E_x \subset U$ for every $U$. Hence, $\gamma \prec \alpha$ and we get that $\gamma$ is uniform. For all $x \in \hat{X}$, there is $U \in \alpha$ such that $E_x \subset U$ and, consequently, $x \in V_U$. Then, $\gamma$ is a cover of $\hat{X}$.

Fix $x \in \hat{X}$ and let $U \in \alpha$ be such that $E_x \cap V_U \neq \emptyset$. Let $y \in E_x \cap V_U$. Since $E$ is symmetric, $x \in E_y$ and, since $y \in V_U$, we have that $E_y \subset U$. Then, $x \in U$. Hence,
mult_E \gamma = \# \{ V_U : U \in \alpha, E_x \cap V_U \neq \emptyset \} \leq \# \{ U : U \in \alpha, x \in U \} = \text{mult}_x \alpha. Taking supreme over x and applying a) we get:
\[
\text{mult}_E \gamma \leq \text{mult} \alpha \leq \text{asdim}(\hat{X}, E_0) + 1
\]

Corollary 41 means that, from the point of view of the asymptotic dimension, the set \( E_{d, \lambda} \) of Proposition 40 and the cover \( \mathcal{K}(E_{d, \lambda}) \) are big enough.

**Proposition 42.** Let \( (X, \hat{X}, \tilde{X}) \) be a metrizable compactification pack and consider \( (\hat{X}, E_0) \). Then, the following are equivalent:

a) \( \dim X \leq n \).

b) \( \text{asdim}(\hat{X}, E_0) \leq n + 1 \).

c) For every uniform cover \( \beta \) there exists a canonical cover \( \alpha \) such that \( \beta \prec \alpha \) and \( \text{mult} \alpha \leq n + 2 \).

d) For every canonical cover \( \beta \) there exists a canonical cover \( \alpha \) such that \( \beta \prec \alpha \) and \( \text{mult} \alpha \leq n + 2 \).

e) There exists a uniform cover \( \alpha \) (a canonical cover, respectively) such that \( \text{mult}_E \alpha \leq n + 1 \), where \( E \) is the subset \( E_{d, \lambda} \) of Proposition 40.

**Proof.** It is a consequence of Corollary 7 and Propositions 19 and 40.

4 An easier proof of Grave’s theorem

Let \( (X, \hat{X}, \tilde{X}) \) be a compactification pack. In order to prove Grave’s theorem, we have to see that:
\[
\text{asdim}(\hat{X}, E_0) \geq \dim X + 1 \tag{12}
\]
\[
\text{asdim}(\hat{X}, E_0) \leq \dim X + 1 \tag{13}
\]

**Proof 1 of (12).** It is a consequence of Proposition 40.

But it is not the natural way to prove \( (12) \). The natural way (more or less the way used by Grave applying other result instead of Proposition 29) is the following:

**Proof 2 of (12).** Consider \( (X \times (0, 1], E'_0) \), where \( E'_0 = E_0(X \times \{0\}, X \times (0, 1], X \times [0, 1]) \).

Since \( X \) and \( X \times \{0\} \) are homeomorphic, by Corollary 36, it follows that \( (\hat{X}, E_0) \) and \( (X \times (0, 1], E'_0) \) are coarse equivalent. Therefore, \( \text{asdim}(\hat{X}, E_0) = \text{asdim}(X \times (0, 1], E'_0) \geq \dim X + 1 \), where the last inequality is given by Propositions 13 and 29.
In Theorem 2.9 of [5] (pag. 3712), the authors defined a canonical cover similar to 
\( \alpha(\{\beta_n\}, \{W_n\}) \) of Lemma [12]. They used it to prove that if \( X \) has finite dimension, then there exists a canonical cover with finite multiplicity. They bounded the multiplicity of that canonical cover by \( 2 \dim X + 2 \).

The construction of that canonical cover induces an implicit problem in the Topological Dimension Theory: To decrease the bound of the multiplicity of that canonical cover to the minimum (that is \( \dim X + 2 \)) we need the sequence \( \{\beta_n\} \) to satisfy some special dimensional properties. Does this special sequence exist?

We solved that topological problem in dimension theory in [16] —we quote it in Theorem [48]—. With this techniques, we will be able to give a new proof of (13) and, at the same time, define a canonical cover with minimal multiplicity.

**Definition 43.** Let \( \alpha_1, \ldots, \alpha_m \) be families of subsets of a set \( Z \). The common multiplicity of \( \alpha_1, \ldots, \alpha_m \) is:

\[
\text{mult}(\alpha_1, \ldots, \alpha_m) = \sup_{x \in Z} \text{mult}_x \alpha_1 + \cdots + \text{mult}_x \alpha_m = \\
\sup \left\{ \sum_{i=1}^{m} \#A_i : A_i \subset \alpha_i \forall i, \bigcap_{i=1}^{m} U \neq \emptyset \bigcap_{U \in A_i} \right\}
\]

**Remark 44.**

- \( \text{mult}(\alpha_1, \ldots, \alpha_m) \) is a multiplicity greater than or equal to \( \text{mult}(\alpha_1 \cup \cdots \cup \alpha_m) \) which is equal when \( \alpha_1, \ldots, \alpha_r \) are pairwise disjoint.

**Proposition 45.** Let \( (\tilde{X}, d) \) be a metric space and consider \( X \subset \tilde{X} \). Consider the topology \( T \) of \( \tilde{X} \) attached to \( d \) and the map

\[ v : T|_X \to T, U \to \{x \in \tilde{X} : d(x, U) < d(x, X \setminus U)\} \]

(assuming that \( d(x, \emptyset) = \infty \) for every \( x \in \tilde{X} \)). Then:

a) For every \( U \in T|_X \), \( v(U) \cap X = U \).

b) \( v(X) = \tilde{X} \) and \( v(\emptyset) = \emptyset \).

c) For every \( U_1, U_2 \in T|_X \), \( U_1 \subset U_2 \) if and only if \( v(U_1) \subset v(U_2) \).

d) For every \( U_1, \ldots, U_r \in T|_X \), \( U_1 \cap \cdots \cap U_r \neq \emptyset \) if and only if \( v(U_1) \cap \cdots \cap v(U_r) \neq \emptyset \).

e) For every \( U \in T|_X \), \( U = \emptyset \) if and only if \( v(U) = \emptyset \).

f) For every \( U_1, U_2 \in T|_X \), \( v(U_1 \cap U_2) = v(U_1) \cap v(U_2) \).

**Proof.** a)-c) are easy to check. e) is a consequence of a) and b). d) is a consequence of e) and f). It suffices to prove f).

Since \( U_1 \cap U_2 \subset U_1 \), c) shows that \( v(U_1 \cap U_2) \subset v(U_1) \). By the same reason, \( v(U_1 \cap U_2) \subset v(U_2) \). Hence, \( v(U_1 \cap U_2) \subset v(U_1) \cap v(U_2) \).
Fix \( x \in v(U_1) \cap v(U_2) \). Then, \( d(x, U_1) < d(x, X \setminus U_1) \) and \( d(x, U_2) < d(x, X \setminus U_2) \). Choose \( \varepsilon > 0 \) such that \( d(x, X \setminus U_1) - d(x, U_1) > \varepsilon \) and \( d(x, X \setminus U_2) - d(x, U_2) > \varepsilon \).

Take \( y \in X \) such that \( d(y, X) - d(y, x) > d(x, X \setminus U_1) - d(x, U_1) + \frac{\varepsilon}{2} > \varepsilon \). For every \( z \in X \setminus U_1 \), we have \( d(y, z) \geq d(y, x) - d(x, z) - d(x, y) > d(x, X \setminus U_1) - d(x, X) + \frac{\varepsilon}{2} \geq d(x, X \setminus U_1) - d(x, U_1) - \frac{\varepsilon}{2} > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} \).

Then, \( d(y, X \setminus U_1) > \frac{\varepsilon}{2} \) and hence, \( y \in U_1 \). By the same reason, \( y \in U_2 \). Therefore:

\[
y \in X \text{ and } d(y, x) < d(x, X) + \frac{\varepsilon}{2} \Rightarrow y \in U_1 \cap U_2
\]

From (14) we deduce

\[
d(x, X \setminus (U_1 \cap U_2)) \geq d(x, X) + \frac{\varepsilon}{2}
\]

Fix \( \delta > 0 \) and take \( y' \in X \) such that \( d(y, y') < d(x, X) + \min \{ \delta, \frac{\varepsilon}{2} \} \). By (14), \( y' \in U_1 \cap U_2 \). Hence,

\[
d(x, X) \leq d(x, U_1 \cap U_2) < d(x, X) + \delta \text{ for every } \delta > 0
\]

and we get \( d(x, U_1 \cap U_2) = d(x, X) \). We conclude from (15) that \( d(x, U_1 \cap U_2) < d(x, X \setminus (U_1 \cap U_2)) \), hence that \( x \in v(U_1 \cap U_2) \) and finally that \( v(U_1) \cap v(U_2) \subset v(U_1 \cap U_2) \). \( \Box \)

Remarks 46.

- The function \( v \) defined above is called “Ext” in [21], pag 125.
- The function described in Proposition 2.7 of [9], pag 3711, satisfies properties a)-d) of proposition.

Proposition 47. Let \( (X, \hat{X}, \tilde{X}) \) be a metrizable compactification pack, let \( T \) be the topology of \( \tilde{X} \) and consider \( (\hat{X}, \mathcal{E}_0) \).

Suppose that \( \{W_n\}_{n=0}^{\infty} \) is a sequence of open neighborhoods of \( X \) such that \( W_0 = \tilde{X} \), \( W_0 \supset \tilde{W}_1 \supset W_1 \supset \tilde{W}_2 \supset W_2 \supset \ldots \) and \( \bigcap_{n=0}^{\infty} W_n = X \).

Suppose that \( \{\alpha_n\}_{n=0}^{\infty} \) is a family of open covers of \( X \) and let, for every \( n \), \( \beta_n = \{v(U) : U \in \alpha_n\} \), where \( v : T|_X \to T \) is a map satisfying properties a)-d) of Proposition 46.

Consider the cover \( \alpha(\{\beta_n\},\{W_n\}) \) defined in Proposition 46. Then,

a) \( \text{mult } \alpha(\{\beta_n\},\{W_n\}) \leq \sup_{n \in \mathbb{N} \cup \{0\}} \text{mult}(\alpha_n, \alpha_{n+1}) \).

b) If \( \text{mesh } \alpha_n \to 0 \), then \( \lim_{m,n \to 0} \text{mesh } \{V \cap W_m : V \in \beta_n\} = 0 \) and, consequently, \( \alpha(\{\beta_n\},\{W_n\}) \) is a uniform cover of \( (\hat{X}, \mathcal{E}_0) \).

Proof. For short, denote \( \alpha(\{\beta_n\},\{W_n\}) \) by \( \alpha \). Property a) of Proposition 46 states that, for every \( n \), \( v : \alpha_n \to \beta_n \) is bijection and property d) states that, for every \( n_1, \ldots, n_r \):

\[
\text{mult}(\beta_{n_1}, \ldots, \beta_{n_r}) = \sup \left\{ \sum_{k=1}^{r} \#B_k : B_k \subset \beta_{n_k} \forall k, \bigcap_{k=1}^{r} \bigcup_{V \in B_k} V \neq \emptyset \right\}
\]

26
\[ \sup \left\{ \sum_{k=1}^{r} \# \{ V(U) : U \in A_k \} : A_k \subset \alpha_{nk} \forall k, \bigcap_{k=1}^{r} U \in A_k, v(U) \neq \emptyset \right\} = \mu \text{mult}(\alpha_{n1}, \ldots, \alpha_{nr}) \]

Fix \( x \in \tilde{X} \). It is easy to check that if \( G \in \alpha \) with \( x \in G \), then \( G = V \cap (W_k \setminus W_{k-2}) \), where \( V \in \beta_k \), \( k \in \{N, N+1\} \) and \( N = \max \{ n \in \mathbb{N} : x \in W_n \} \). Then, \( \mu \text{mult}_x \alpha = \{ G \in \alpha : x \in G \} \leq \#V \cap (W_N \setminus W_{N-2}) : V \in \beta_N, x \in V \} + \#V \cap (W_{N+1} \setminus W_{N-1}) : V \in \beta_N, x \in V \} \leq \#V : V \in \beta_N, x \in V \} + \#V : V \in \beta_N, x \in V \} \leq \text{mult}(\beta_N, \beta_{N+1}) = \mu \text{mult}(\alpha_N, \alpha_{N+1}) \leq \sup_{n \in \mathbb{N}\setminus\{0\}} \mu \text{mult}(\alpha_n, \alpha_{n+1}) \). Taking supreme over \( x \) we get a).

Assume mesh \( \alpha_n \rightarrow 0 \). Suppose mesh \( \{ V \cap W_m : V \in \beta_n \} \neq 0 \) when \( m, n \to \infty \). Then, there exists \( \varepsilon > 0 \) and two subsequences \( \{ n_k \} \) and \( \{ m_k \} \) such that mesh\( \{ V \cap W_{m_k} : V \in \beta_{n_k} \} \geq \varepsilon \). For every \( k \), take \( U_k \in \alpha_k \) with \( \text{diam} v(U_k) \cap W_{m_k} > \varepsilon \) and take \( x_k, y_k \in v(U_k) \cap W_{m_k} \) with \( \text{d}(x_k, y_k) > \varepsilon \).

Since \( \tilde{X} \) is compact, we may suppose, by taking subsequences if necessary, that \( x_k \to x \) and \( y_k \to y \) for some \( x, y \in \tilde{X} \). Then,

\[ \text{d}(x, y) = \lim \text{d}(x_k, y_k) \geq \varepsilon \]

For every \( i \) and every \( k \geq i \), we have \( x_k, y_k \in W_{m_k} \subset W_k \subset W_i \) and hence, \( x, y \in W_i \).

Thus, \( x, y \in \bigcap_{l \in \mathbb{N}} W_i = X \).

Let \( N \in \mathbb{N} \) with mesh \( \alpha_N < \frac{\varepsilon}{3} \). Choose \( U_x, U_y \in \alpha_N \) such that \( x \in U_x \) and \( y \in U_y \). Observe that \( v(U_x) \) and \( v(U_y) \) are two neighborhoods of \( x \) and \( y \) respectively. Since mesh \( \alpha_{n_k} \to 0 \), \( x_k \to x \) and \( y_k \to y \), it follows that there is \( k' \) such that mesh \( \alpha_{k'} < \frac{\varepsilon}{3} \), \( x_{k'} \in v(U_x) \) and \( y_{k'} \in v(U_y) \). Since \( v(U_{k'}) \cap v(U_x) \supset \{ x \} \neq \emptyset \) and \( v(U_{k'}) \cap v(U_y) \supset \{ y \} \neq \emptyset \), we have that \( U_{k'} \cap U_x \neq \emptyset \) and \( U_{k'} \cap U_y \neq \emptyset \). Take \( x' \in U_{k'} \cap U_x \) and \( y' \in U_{k'} \cap U_y \) to get:

\[ \text{d}(x, y) \leq \text{d}(x, x') + \text{d}(x', y') + \text{d}(y', y) \leq \text{diam} U_x + \text{diam} U_{k'} + \text{diam} U_y \leq \text{mesh} \alpha_N + \text{mesh} \alpha_{n_{k'}} + \text{mesh} \alpha_N < 3\frac{\varepsilon}{3} = \varepsilon \]

in contradiction with [17].

Then, \( \lim_{m,n \to \infty} \text{mesh} \{ V \cap W_m : V \in \beta_n \} = 0 \) and hence, by Proposition [12], \( \alpha \) is a uniform cover. \( \square \)

From [16] we take the following result:

**Theorem 48.** Let \((X, d)\) be a compact metric space with \( \text{dim} X \leq n < \infty \) and suppose \([\varepsilon_i]_{i=1}^{\infty} \subset \mathbb{R}^+\). Then, there exists a sequence of open and finite covers of \( X \) \([\alpha_i]_{i=0}^{\infty}\) such that:

a) \( \alpha_0 = \{ X \} \) and \( \text{mesh} \alpha_i < \varepsilon_i \) for every \( i \in \mathbb{N} \).
b) \( \text{mult}(\alpha_i, \alpha_{i+1}) \leq n + 2 \) for every \( i \in \mathbb{N} \cup \{0\} \)

Proof. It is a consequence of Theorems 74, 81, 104 or 154 of [16]. \qed

**Proposition 49.** Let \((X, \hat{X}, \tilde{X})\) be a compactification pack, consider \((\hat{X}, \mathcal{E}_0)\) and let \(\gamma\) be a uniform cover of \(\hat{X}\). Then, there exists an open, locally finite and uniform cover \(\alpha\) (i.e. a canonical cover) such that \(\gamma \prec \alpha\) and \(\text{mult} \alpha \leq \dim X + 2\).

Moreover, given a sequence of open subsets of \(\hat{X}\) \(\{W_i\}_{i=0}^\infty\) with \(W_0 = \hat{X}\), \(W_0 \supset W_1 \supset W_2 \supset \ldots\) and \(\bigcap_{i=0}^\infty W_i = X\), we can construct such \(\alpha\) by letting \(\alpha = \alpha(\{\beta_k\}, \{W_{i_k}\})\) (as defined in Proposition 12), where \(\{\beta_k\}_{k=0}^\infty\) is a sequence of open and finite families of subsets of \(\hat{X}\) such that \(\beta_0 = \{\hat{X}\}\), \(X \subset \bigcup_{V \in \beta_i} V\) for every \(i\) and \(\lim_{i,j \to \infty} \text{mesh}\{V \cap W_j : V \in \beta_i\} = 0\) and \(\{i_k\}_{k=0}^\infty\) is a subsequence with \(i_0 = 0\).

Proof. Let \(\{W_i\}_{i=0}^\infty\) be as above (for example, consider \(W_0 = \hat{X}\) and, for every \(i \in \mathbb{N}\), \(W_i = B(X, \frac{k}{2^i})\), where \(k = \sup_{x \in \hat{X}} d(x, X)\)).

If \(\dim X = \infty\), this proposition is a consequence of Propositions 13 and 12. If \(\dim X = n < \infty\), by Theorem 12 there exist a sequence of open and finite covers \(\{\alpha_i\}_{i=0}^\infty\) of \(X\) with \(\alpha_0 = \{X\}\), \(\lim_{i \to \infty} \text{mesh} \alpha_i = 0\) and \(\text{mult}(\alpha_i, \alpha_{i+1}) \leq n + 2\) for every \(i \in \mathbb{N} \cup \{0\}\).

![Figure 5. Covers \(\{\alpha_i\}_{i=0}^\infty\) and \(\{\alpha_i \cup \alpha_{i+1}\}_{i=0}^\infty\) of \([0,1]\) with \(\text{mult} \alpha_i \leq 2\) and \(\text{mult}(\alpha_i, \alpha_{i+1}) \leq 3\) \(\forall i \in \mathbb{N} \cup \{0\}\)](image)

Let \(\mathcal{T}\) be the topology of \(\hat{X}\), consider a map \(v : \mathcal{T}|_X \to \mathcal{T}\) satisfying properties a)-d) of Proposition 46 and suppose \(\beta_i = \{v(U) : U \in \alpha_i\}\) for every \(i \in \mathbb{N} \cup \{0\}\).

By Proposition 47 \(\lim_{i,j \to \infty} \text{mesh}\{V \cap W_j : V \in \beta_i\} = 0\) and, by Lemma 12 there
exists a subsequence \( \{i_k\}_{k=0}^\infty \) with \( i_0 = 0 \) such that \( \gamma \cup \{\{x\}\}_{x \in \hat{X}} < \alpha(\{\beta_k\}, \{W_{i_k}\}) \).

\[ \gamma \cup \{\{x\}\}_{x \in \hat{X}} \text{ is a cover of } \hat{X}, \text{ so it is } \alpha(\{\beta_k\}, \{W_{i_k}\}). \]

According to \( \alpha(\{\beta_k\}, \{W_{i_k}\}) \) is also an open, uniform and locally finite cover of \( \hat{X} \) such that \( \text{mult} \alpha(\{\beta_k\}, \{W_{i_k}\}) \leq \sup_{\alpha \in \mathbb{N} \cup \{0\}} \text{mult}(\alpha_i, \alpha_{i+1}) \leq n + 2. \) Particulary, it is a canonical cover of \((X, \hat{X}, \tilde{X})\) (see Lemma 7). □

Proof of (13). It is a consequence of Proposition 49 □

References

[1] Anderson, R. D. Topological properties of the Hilbert cube and the infinite product of open intervals Trans. Amer. Math. Soc. 126 (1967) 200-216

[2] G. Bell, A. Dranishnikov, Asymptotic dimension in Bedlewo. Topology Proc. 38 (2011), 209-236

[3] G. Bell, A. N. Dranishnikov, Asymptotic dimension. Topology Appl. 155 (2008), no. 12, 1265-1296

[4] E. Cuchillo Ibáñez, J. Dydak, A. Kodama y M. A. Morón, \( C_0 \) coarse geometry of complements of Z-sets in the Hilbert cube. Trans. Amer. Math. Soc. 360 (2008), no. 10, 5229–5246

[5] E. Cuchillo Ibáñez, M. A. Morón, Canonical covers and dimension of Z-sets in the Hilbert cube. Proc. Amer. Math. Soc. 136 (2008), no. 10, 3709–3716

[6] T. A. Chapman, Lectures on Hilbert Cube Manifolds. American Mathematical society, 1976. Regional Conference Series in Mathematics Vol. 28.
[7] A. N. Dranishnikov, *Asymptotic topology*. Russian Math. Surveys 55 (2000), no. 6, 1085–1129

[8] A. N. Dranishnikov, J. Smith *On asymptotic Assouad-Nagata dimension*. Topology Appl. 154 (2007), no. 4, 934-952.

[9] James Dugundji, *An extension of Tietze’s theorem*, Pacific Journal of Mathematics 1 (1952) 353-367. MR00444116

[10] J. Dydak and S. Hoffland, *An alternative definition of coarse structures*. Topology Appl. 155 (2008), no. 9, 1013–1021

[11] Bernd Grave, *Coarse geometry and asymptotic dimension*. Phd Thesis.

[12] Bernd Grave *Asymptotic dimension of coarse spaces*. New York J. Math. 12 (2006)

[13] Hurewicz-Wallman, *Dimension Theory*. Princeton University Press, Princeton, NJ, 1941.

[14] W. Hurewicz, *Sur la dimension des produits Cartésiens*, Annals of Mathematics (2), vol. 36 (1935), pp. 194-197

[15] Kotaro Mine, Atsushi Yamashita *C₀ coarse structures and smirnov compactifications* Preprint (2011) arXiv:1106.1672

[16] Jesús P. Moreno Damas, *Geometría de recubrimientos: Dimensión Topológica y Estructuras Coarse C₀* PhD thesis. Universidad Complutense de Madrid, 2012.

[17] Jesús P. Moreno Damas, *Propiedades de la geometría C₀ a gran escala con compactificación de Higson metrizable* Trabajo para la obtención del DEA. Universidad Complutense de Madrid, 2007.

[18] K. Morita, *On the Dimension of Product Spaces*, American Journal of Mathematics, Vol. 75, No. 2, 205-223, Apr. 1953

[19] John Roe, *Lectures on Coarse Geometry*. American Mathematical Society, 2003. University Lecture Series Vol. 31.

[20] John Roe, *Corrections to Lectures on Coarse Geometry*, http://www.math.psu.edu/roe/writings/correction.pdf

[21] Sze-Tsen Hu *Theory of Retracts*. Wayne State University Press, 1965

[22] Nick Wright, *C₀ Coarse Geometry*. PhD thesis. Penn State University 2002.

[23] Wright, Nick *C₀ coarse geometry and scalar curvature*. J. Funct. Anal. 197 (2003), no. 2, 469-488.