WEAK SEQUENTIAL STABILITY FOR A NONLINEAR MODEL OF NEMATIC ELECTROLYTES

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To Alex Mielke, with friendship and admiration

Abstract. In this article we study a system of nonlinear PDEs modelling the electrokinetics of a nematic electrolyte material consisting of various ions species contained in a nematic liquid crystal.

The evolution is described by a system coupling a Nernst-Planck system for the ions concentrations with a Maxwell’s equation of electrostatics governing the evolution of the electrostatic potential, a Navier-Stokes equation for the velocity field, and a non-smooth Allen-Cahn type equation for the nematic director field.

We focus on the two-species case and prove apriori estimates that provide a weak sequential stability result, the main step towards proving the existence of weak solutions.

1. Introduction. In this paper we consider a version of the system derived in [4, (2.51)-(2.55)] describing the electrokinetics of a nematic electrolyte that consists of ions that diffuse and advect in a nematic liquid crystal environment.

The system can be written in terms of the following variables:

- the vector $n$ modelling the local orientation of the nematic liquid crystal molecules,
- the macroscopic velocity $v$ of the liquid crystal molecules,
- the pressure $p$,
- the electrostatic potential $\Phi$.

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• the concentrations \( c_k, k = 1, \ldots, N \), with valences \( z_k \in \{-1, 1\} \), of the families of charged ions present in the liquid crystal.

Actually, we consider a modified version of the system in [4], assuming certain simplifications commonly used in the mathematical literature on liquid crystals. More specifically we take equal elastic constants in the Oseen-Frank energy and use a Ginzburg-Landau configuration potential \( F \) of singular type (see below for more details) in order to avoid introducing the unit length constraint (cf. equation (2.56) of [4]) on \( n \) (and thus we can correspondingly drop the related Lagrange multiplier term \( \lambda n \) in the system in [4]). We will neglect the inertial effects, a standard assumption in the physical [16] and mathematical literature [19]. Furthermore neglecting body forces acting on the director field, we can write the resulting PDE system as follows:

\[
\frac{\partial c_k}{\partial t} + v \cdot \nabla c_k = \frac{1}{k_B \theta} \text{div} (c_k D_k \nabla \mu_k), \quad \text{for } k = 1, \ldots, N,
\]

\[
- \text{div}(\varepsilon_0 \varepsilon(n) \nabla \Phi) = \sum_{k=1}^{N} q z_k c_k,
\]

\[
\frac{\partial v}{\partial t} + (v \cdot \nabla) v + \nabla p = -K \text{div}(\nabla n \odot \nabla n) + \text{div} \sigma,
\]

\[
+ \varepsilon_0 \text{div} ((\nabla \Phi \otimes \nabla \Phi) \varepsilon(n)),
\]

\[
\text{div} v = 0,
\]

\[
\gamma_1 \left( \frac{\partial n}{\partial t} + (v \cdot \nabla) n - \Omega(v)n \right) + \gamma_2 D(v)n \in K \Delta n + \varepsilon_0 \varepsilon_a (\nabla \Phi \otimes \nabla \Phi) n - \frac{\partial F}{\partial n},
\]

where we use in (1.5) the inclusion sign because \( \partial F \) may be a multivalued operator (see below for more details), \( \mu_k \) are the electrochemical potentials of the ions associated to the various ions species \( c_k \), given by

\[
\mu_k := k_B \theta (\ln(c_k) + 1) + q z_k \Phi,
\]

\( k_B > 0 \) denotes the Boltzmann constant, \( \theta > 0 \) stands for the absolute temperature, assumed here constant, and \( q \) denotes the elementary charge.

Moreover, we have indicated by

\[
D(v) := \frac{1}{2}(\nabla v + \nabla v^t) \quad \text{and} \quad \Omega(v) := \frac{1}{2}(\nabla v - \nabla v^t)
\]

the symmetric and antisymmetric parts of the velocity gradient, respectively. The diffusion operator in (1.2) is ruled by the matrix

\[
\varepsilon(n) := \varepsilon_\perp \text{Id} + \varepsilon_a n \otimes n,
\]

with constants \( \varepsilon_\perp > 0 \) and \( \varepsilon_a \geq 0 \), \( \text{Id} \) denoting the identity matrix. Here \( \varepsilon_a = \varepsilon_{||} - \varepsilon_{\perp} \), where \( \varepsilon_{||} \) and \( \varepsilon_{\perp} \) denote the electric permittivity when the electric field \( \mathbf{E} = \nabla \Phi \) is parallel, respectively, perpendicular to \( n \). It must be pointed out that similar, more general expressions, allowing in particular biaxiality effects, for the electric permittivity tensor \( \varepsilon \) are provided in [1, 26].
The constant $\varepsilon_0 > 0$ stands for the vacuum dielectric permeability. The matrices $D_k$ are positive definite, i.e.,
\[
(D_k \xi) \cdot \xi > \alpha |\xi|^2
\]
for some $\alpha > 0$ and all $k = 1, \ldots, N$ and $\xi \in \mathbb{R}^3$. In the above we have denoted by $\nabla n \otimes \nabla n$ the $3 \times 3$ matrix whose $(i, j)$-component is $n_{k,i}n_{k,j}$ (here and in the sequel we assume summation over repeated indices and $n_{k,j} := \frac{\partial n}{\partial x_j}$ where $n = (n_1, n_2, n_3)$ and $x = (x_1, x_2, x_3)$. As customary, for $a, b \in \mathbb{R}^3$ we denote as $a \otimes b$ the $3 \times 3$ matrix with component $(i, j)$ given by $a_ib_j$.

The Nernst-Planck type equations (1.1) correspond to the continuity equation for ions with the electric potential $\Phi$ satisfying the Maxwell’s equation of electrostatics (1.2).

The Navier-Stokes equations (1.3), with the incompressibility constraint (1.4), rule the evolution of the liquid crystal flow. Note the Korteweg forces on the right-hand side being induced by the the director field and the effects of the electric field, respectively. As in [17], we assume for the total stress tensor the following general expression:
\[
\sigma = \alpha_1 (D(v)n \cdot n)n \otimes n + \alpha_2 n \otimes n + \alpha_3 n \otimes \dot{n} + \alpha_4 D(v) + \alpha_5 D(v)n \otimes n + \alpha_6 n \otimes D(v)n,
\]
where we have denoted $\dot{n} := \partial_t n + v \cdot \nabla n - \Omega(v)n$ the Lie derivative of $n$. Here the term $\alpha_4 D(v)$ represents the classical Newtonian stress tensor, while the other terms represent the additional stress produced by the interaction of the anisotropic liquid crystal molecules, see [9, 10].

As mentioned above, we avoid to insert the unit length constraint in (1.5) and instead require $|n| \leq 1$, in the spirit of the the variable length model proposed by J.L. Ericksen in [11]. Indeed, following an approach commonly used in the context of phase-transition models, we enforce the property $|n| \leq 1$ by means of the singular potential $\mathcal{F}$.

Namely, we assume $\mathcal{F} : \mathbb{R}^3 \to [0, +\infty]$ to be a $\lambda$-convex (i.e. $n \to \mathcal{F}(n) + \lambda |n|^2$ is convex for some $\lambda \geq 0$) and lower semicontinuous function whose effective domain (i.e., the set where it attains finite values) is assumed to coincide with the closed unit ball $\bar{B}_1$ of $\mathbb{R}^3$, with a reference choice being given by
\[
\mathcal{F}(n) = \frac{1}{2} F(|n|^2), \quad \text{where } F(r) = (1 - r) \log(1 - r), \quad (1.11)
\]
an expression similar to the Cahn-Hilliard logarithmic potential, but we point out that more general choices may be allowed. It is worth noting that, with the above choice, the minimum of $\mathcal{F}$ is attained for $|n| = 1 - 1/e$, whereas the states $n$ with $|n| > 1$ are in fact excluded (since $\mathcal{F}(n) = +\infty$ in that case). It is worth noting that, by rescaling, one may easily modify the expression of $\mathcal{F}$ so to assign the minimum energy configurations to the states with $|n| = 1$ (as one would expect from physics). On the other hand, we prefer to keep the expression in (1.11) because computationally simpler. The presence of a singular potential might seem unusual, yet it is but one of the most popular mathematical ways of dealing with the formidable difficulties and singularities imposed by having a unit-length constraint (which generates a Lagrange multiplier term with strong nonlinearities). One can relax this by using a non-singular potential of Ginzburg-Landau type (see for instance the review [21] and the references there) or a singular potential, as in the so-called Ericksen model of liquid crystals. Such an idea was introduced by J.L. Ericksen in [11, p. 109] in order to enforce the physicality of a scalar order parameter and
has already been applied to liquid crystal models in a number of papers (see for instance the review [19] and the references there, including [12, 13, 18, 24]).

It has the advantage that as soon as we have proved existence of a solution, then the constraint $|n| \leq 1$ is automatically satisfied. This helps in the estimates which actually could not be performed in this way in the case of a classical double-well potential.

Finally, in order to avoid complications due to the interaction with the boundary, we will settle the above system on the flat 3-dimensional torus

$$\mathcal{T}^3 = \left(\left[-\pi, \pi\right]|_{\{-\pi, \pi\}}\right)^3$$

so assuming periodic boundary conditions. These are a mathematical idealisation of the real system, a first step towards the more complicated setting presented in [4].

Our main aim here is show the existence of global-in-time weak solutions. These are usually obtained via three steps: ‘apriori estimates’, ‘compactness’, and suitable ‘approximation scheme’.

The apriori estimates are obtained on presumptive smooth solutions of the equation. Such estimates allow to control (in terms of initial data and fixed parameters of the system) certain norms, sufficiently strong, in order to allow to pass to the limit in a suitably designed approximation scheme.

The approximation scheme is usually designed such that one can obtain estimates for the approximating equations that are usually very close to the apriori estimates. The construction of such a scheme can be a highly tedious and non-trivial issue in presence of complex systems as we consider here (see for comparison our previous works on non-isothermal liquid crystals, with an approximation scheme [12] and without one, just with apriori estimates as in here [13]). To elucidate the main steps of the limit passage, it is customary to show first ‘compactness’ or ‘sequential stability’. This amounts to showing that any sequence of (regular) solutions to the problem that obeys the apriori bounds admits a subsequence converging to a weak solution of the same problem. Proving compactness is the main objective of the present paper, Theorem 2.2. Finally, we introduce an approximation scheme compatible with the apriori bounds that gives rise to global in time weak solutions for any finite energy initial data.

We actually focus on a simplified version of system (1.1)-(1.5), complemented with the initial conditions and with periodic boundary conditions in three dimensions of space and with no restrictions on the magnitude of the initial data. The precise simplifications will be introduced in the next section, but it is worth observing that, beyond setting some physical constants equal to one, the only effective reduction we are actually going to operate concerns the number of species $c_k$ which will be assumed to be equal to 2. Namely, we only take two species $c_p$ and $c_m$, which will then denote the density of positive and negative charges, respectively. Mathematically speaking, this ansatz simplifies the nature of the system (1.1)-(1.2), and in particular permits us to prove by means of very simple maximum principle arguments the uniform boundedness of $c_p$ and $c_m$, which is a key ingredient for obtaining the apriori estimates.

It is worth noting that we expect the same boundedness property to hold also in the general case of $N$-species, however the proof may be much more involved and require use of more technical results about invariant regions for evolutionary systems (see, e.g., [6]). We also expect that similar arguments could be applied in
the more complicated systems where one uses a tensorial order parameter, that is a matrix valued function, i.e. a $Q$-tensor in the LC terminology, instead of the vector-valued one, $n$, as done for instance in [26]. The current work is related to work done in certain simpler systems that can be regarded as subsets of our equations, such as Nernst-Planck-Navier-Stokes system (see for instance [8] and the references therein) and liquid crystal equations (see for instance the review [21]). It should be noted that there exists a large body of literature concerned with the Nernst-Planck-Navier-Stokes system, concerned with the fundamental issues of existence of weak and strong solutions (see for instance [8, 14, 25]), as well as regularity (see the recent work [15] and the references therein). The additional level of complexity generated by the addition of the liquid crystal equations (1.5) makes most of the approaches in the Nernst-Planck-Navier-Stokes literature largely inapplicable in a direct manner and generates additional challenging technical difficulties. Indeed, just the equation governing the evolution of the director fields has been long studied in articles on harmonic maps and their flows, see [20]. The addition to these equations of Navier-Stokes alone adds severe difficulties that are barely handled by weakening the unit-length constraint on the director through the use of a potential (regular or singular), see [21] for an overview of the problems and challenges posed in this context. The additional coupling with the Nernst-Planck system generates additional difficulties in both systems, in the Nernst-Planck part through the presence of the (low regularity) director dependence in the diffusion operator in (1.2), while in the Naver-Stokes part one also gets an additional (low regularity) stress tensor. Indeed, the low regularity of these coupling terms are the main technical challenges that the current work overcomes. The present article just opens the way for the treatment of the highly challenging situation in which one couples Nernst-Planck-Navier-Stokes with liquid crystal equations, emphasizing the main technical difficulties related to the low regularity of the coupling terms.

The main ingredients of the proofs are the following: first we perform an energy estimate which is mainly based on a key Lemma (cf. Lemma 3.2) providing sufficient conditions on the $\alpha_i$-coefficients such that the dissipation is non-negative. Then, via a maximum-principle technique, we prove pointwise bounds for $c_p$ and $c_m$. The $L^\infty$-estimate on the potential $\Phi$ follows instead by a Moser-iteration scheme proved in Lemma 3.4, while in Lemma 3.5 we state an $L^p$-regularity result for $n$. This result, based on an $L^p$-estimate for the potential $\partial F$, is in general new in the framework of non-smooth parabolic systems, while it is quite known in case of scalar equations (cf., e.g., [7]). Finally, an additional regularity result for $n$ (cf. Lemma 3.6) is shown in case the anisotropy coefficient $\varepsilon := \varepsilon_0 \varepsilon_a$ is sufficiently small. In the last Section 4 the weak sequential stability property result is proved for every $\varepsilon > 0$.

The plan of the paper is as follows: in the next Section 2 we introduce the simplified version of system (1.1)-(1.5) and state the precise formulation of our main theorem. Then, the basic apriori estimates are derived in Section 3. In Section 4 we will prove the stability result. Finally, in Section 5, we we provide the sketch of an approximation scheme that gives rise to a global-in-time weak solution for any finite energy initial data.

2. Main results. We start introducing some notation. Given a space of functions defined over $\Omega = T^3$, we will always use the same notation for scalar-, vector-, or tensor-valued function. For instance, we will indicate by the same letter $H$ the spaces $L^2(\Omega)$, $L^2(\Omega)^3$ and $L^2(\Omega)^{3 \times 3}$. Correspondingly, the norm in $H$ will be simply
denoted by $\| \cdot \|$. The notation actually subsumes the periodic boundary conditions. We also set $V = H^1(\Omega)$ (or $H^1(\Omega)^3$, or $H^1(\Omega)^3 \times \mathbb{R}$). For two $3 \times 3$ matrices $A$, $B$, we also set $A : B := A_{ij} B_{ij}$.

In view of the discussion carried out above, we now introduce the simplified system for which we shall prove existence of weak solutions. Namely, we assume that $\varepsilon_\perp, k_B \theta, K, \varepsilon_0, q, \gamma_1, \gamma_2$ are all set to one in their respective units and write $\varepsilon$ in place of $\varepsilon_a$. Moreover, we only take two species $c_p$ and $c_m$ with $z_p = 1$ and $z_m = -1$. Moreover we take, similarly in spirit as in [4], Section 3.1, the matrices $D_p = D_k = \text{Id} + \varepsilon n \otimes n$.\footnote{This simplification is not necessary for obtaining the energy law in Proposition 3.1, but essential in deriving the maximum principle in Proposition 3.3} We further assume that the system is non-dimensionalised (see for instance [4] for a suitable non-dimensionalisation) and in the non-dimensional setting we set most of the constants equal to one, except for those $\alpha_i, i \in \{1, \ldots, 6\}$ in the additional stress tensor because in the most natural subsequent studies it will be important to understand the relative magnitude of these terms. Then the simplified system takes the form

$$\begin{align*}
\frac{\partial c_p}{\partial t} + v \cdot \nabla c_p &= \text{div} \left( (\text{Id} + \varepsilon n \otimes n)(\nabla c_p + c_p \nabla \Phi) \right), \\
\frac{\partial c_m}{\partial t} + v \cdot \nabla c_m &= \text{div} \left( (\text{Id} + \varepsilon n \otimes n)(\nabla c_m - c_m \nabla \Phi) \right), \\
- \text{div} \left( (\text{Id} + \varepsilon n \otimes n) \nabla \Phi \right) &= c_p - c_m, \\
\frac{\partial v}{\partial t} + (v \cdot \nabla) v + \nabla p &= \alpha_4 \text{div} D(v) - \text{div}(\nabla n \otimes \nabla n) \\
&\quad + \text{div} \left( (\nabla \Phi \otimes \nabla \Phi)(\text{Id} + \varepsilon n \otimes n) \right) \\
&\quad + \text{div} \left( \alpha_1 (D(v)n \otimes n)n + \alpha_2 \hat{n} \otimes n + \alpha_3 n \otimes \hat{n} \right) \\
&\quad + \text{div} \left( \alpha_5 D(v)n \otimes n + \alpha_6 n \otimes D(v)n \right), \\
\text{div} v &= 0,
\end{align*}$$

(2.1 - 2.6)

Note that $\frac{\partial \mathcal{F}}{\partial n}$ denotes the subdifferential of $\mathcal{F}$ in the sense of convex analysis. Although one can use more general assumptions on the potential here we are assuming for definiteness that

$$\mathcal{F}(n) := \begin{cases} \
\frac{1}{2} F(|n|^2), & \text{if } |n| \leq 1 \\
+\infty, & \text{otherwise}, 
\end{cases}$$

(2.7)

where

$$F(r) := (1 - r) \log(1 - r) - F_*, \quad r \in (0, 1),$$

(2.8)

and $F_*$ is chosen such that $\min F(r) = F(1 - 1/e) = 0$.

It is worth observing from the very beginning that the choice (2.7) gives, for $|n| < 1$,

$$\frac{\partial \mathcal{F}(n)}{\partial n_i} = F'(|n|^2) n_i,$$

(2.9)

whence also

$$\frac{\partial^2 \mathcal{F}(n)}{\partial n_i \partial n_j} = F'(|n|^2) \delta_{ij} + 2 F''(|n|^2) n_i n_j,$$

(2.10)
where $\delta_{ij}$ is the Kronecker delta. Hence, if $F$ is given by (2.8), by a direct check one may verify that, for a suitable $\lambda > 0$, the matrix

$$
\frac{\partial^2 \mathcal{F}(n)}{\partial n_i \partial n_j} + \lambda \delta_{ij}
$$

is positive definite for every $n$ with $|n| < 1$. In other word, the functional $\mathcal{F}$ is $\lambda$-convex (i.e., convex up to a quadratic perturbation). As a consequence, we can compute the subdifferential of $\mathcal{F}$, noted below as $\partial \mathcal{F}$, in the sense of convex analysis. This is a $\lambda$-monotone operator (i.e., monotone up to a linear perturbation), whose domain is the set $\{|n| < 1\}$. Such an operator can be concretely represented through the partial derivatives computed in (2.9).

Moreover, in order to prove the energy estimate (cf. Lemma 3.2, see also [4, p. 2266]), let us suppose that there exists $\delta \in (0, 1)$ such that

$$
\alpha_4 > 0, \quad \alpha_4 - |\alpha_1| - |\alpha_5| - |\alpha_6| - \frac{1}{1-\delta} > 0.
$$

Finally, we assume the initial data to satisfy the following conditions, where $\bar{c} > 0$ is a given constant:

$$
\begin{align*}
&c_{p,0}, c_{m,0} \in L^\infty(\mathcal{T}^3), \quad 0 \leq c_{p,0}, c_{m,0} \leq \bar{c} \text{ a.e. in } \mathcal{T}^3, \\
v_0 \in L^2(\mathcal{T}^3), \quad \text{div } v_0 = 0, \\
n_0 \in H^1(\mathcal{T}^3), \quad |n_0(x)|^2 \leq 1, \forall x \in \mathcal{T}^3,
\end{align*}
$$

It is worth noting that the function $F(r)$ provided by (2.8) can be extended by continuity to $r \in [0, 1]$. On the other hand, in order to apply the theory of subdifferential operators to $\mathcal{F}$, $F$ has to be thought as a function defined on the whole $[0, +\infty)$ by assigning to it the value $+\infty$ as $r > 1$. In such a setting, it is clear that the last condition in (2.15) is equivalent to asking $\mathcal{F}(n) \in L^\infty(\mathcal{T}^3)$ or, also, $\mathcal{F}(n_0) \in L^1(\mathcal{T}^3)$. This last condition corresponds, physically speaking, to the finiteness of the configuration part of the energy functional defined in (3.3) below.

Let us now define the weak solutions, in a rather standard way, but emphasizing the spaces of functions used.

**Definition 2.1.** [Weak solutions] Assume hypotheses (2.8), (2.13)–(2.15). Then, the functions

$$
\begin{align*}
v \in L^\infty(0, T; H) \cap L^2(0, T; V), \\
n \in W^{1,p_0}(0, T; L^{p_0}(\mathcal{T}^3)) \cap L^{p_0}(0, T; W^{2,p_0}(\mathcal{T}^3)) \cap L^\infty(0, T; V) \cap L^\infty((0, T) \times \mathcal{T}^3), \\
\mathcal{F}(n) \in L^{p_0}(0, T; L^{p_0}(\mathcal{T}^3)) \quad \text{for some } p_0 > 1, \\
\Phi \in L^\infty(0, T; V) \cap L^\infty(0, T; L^\infty(\mathcal{T}^3)) \cap L^\infty(0, T; W^{1,p_M}(\mathcal{T}^3)) \quad \text{for some } p_M > 2, \\
c_p, c_m \in W^{1,4/3}(0, T; V) \cap L^2(0, T; V) \cap L^\infty((0, T) \times \mathcal{T}^3), \\
c_p, c_m \geq 0 \text{ a.e. in } \mathcal{T}^3 \times (0, T),
\end{align*}
$$
are a weak solution of (2.1)–(2.5) provided that
\[ \int_0^T \left( \frac{\partial c_p}{\partial t} \phi_p + \int_\Omega \nabla c_p \cdot \nabla \phi_p \right) = \int_0^T \int_\Omega \left( (\text{Id} + \varepsilon \nabla n)(\nabla c_p + c_p \nabla \Phi) \right) \cdot \nabla \phi_p, \] (2.22)
\[ \int_0^T \left( \frac{\partial c_m}{\partial t} \phi_m + \int_\Omega \nabla c_m \cdot \nabla \phi_m \right) = \int_0^T \int_\Omega \left( (\text{Id} + \varepsilon \nabla n)(\nabla c_m - c_m \nabla \Phi) \right) \cdot \nabla \phi_m, \] (2.23)
\[ \int_0^T \int_\Omega \left( \text{Id} + \varepsilon \nabla n \right) \nabla \Phi \cdot \nabla u = \int_0^T \int_\Omega (c_p - c_m) u, \] (2.24)
\[ \int_0^T \int_\Omega v \frac{\partial z}{\partial t} + (v \otimes v) : \nabla z = - \int_0^T v_0 z(0) \, dx + \int_0^T \left( \sigma : \nabla z - (\nabla v \otimes \nabla n) : \nabla z \right) \] 
[ + \int_0^T \int_\Omega ((\nabla \Phi \otimes \nabla \Phi)(\text{Id} + \varepsilon \nabla n) : \nabla z) \] (2.25)
\[ n_t + v \cdot \nabla n - \Omega(v)n + D(v)n \in \Delta n + \varepsilon (\nabla \Phi \otimes \nabla \Phi) n - \frac{\partial \mathcal{F}(n)}{\partial n} \quad \text{a.e. in } T^3 \times (0, T), \] (2.26)
with \( \sigma, D(v), \) and \( \Omega(v) \) defined as in (1.10) and (1.7), and holding true for every test functions \( \phi_p, \phi_m \in L^4(0, T; V), u \in L^2(0, T; V), z \in C^\infty(T^3 \times [0, T]), \) div \( z = 0 \)
and coupled with the initial conditions:
\[ c_p(0) = c_{p,0}, \ c_m(0) = c_{m,0}, \ \text{ in } V', \quad n(0) = n_0, \ v(0) = v_0, \ \text{ a.e. in } T^3. \] (2.27)

The weak sequential stability theorem we aim to prove is the following:

**Theorem 2.2.** Let us assume that there exists a family \((c_p^{(k)}, c_m^{(k)}, \Phi^{(k)}, v^{(k)}, n^{(k)})_{k \in \mathbb{N}}\)
of smooth solutions of the system (2.1)–(2.5) defined on the flat 3-dimensional torus \( T^3 \) and on the time interval \([0, T] ,\) and subject to the initial data
\[ c_p^{(k)}(0) = c_{p,0}^{(k)}, \ c_m^{(k)}(0) = c_{m,0}^{(k)}, \ v^{(k)}(0) = v_0^{(k)}, \ n^{(k)}(0) = n_0^{(k)}, \] (2.28)
with \((c_{p,0}^{(k)}, c_{m,0}^{(k)}, v_0^{(k)}) \in (C^\infty(T^3))^3, \) div \( v_0^{(k)} = 0. \) We furthermore assume that the conditions (2.7), (2.8), (2.13)–(2.15), (1.9) hold for the initial data specified above.

Moreover we assume that there exists a constant \( \tilde{C}, \) independent of \( k \in \mathbb{N}, \) such that
\[ \|c_p^{(k)}\|_{L^\infty} , \|c_m^{(k)}\|_{L^\infty} , \|n_0^{(k)}\|_{H^1(T^3)} , \|v_0^{(k)}\|_{H^1(T^3)} \leq \tilde{C} \]
\[ c_{k,0} \rightarrow c_{k,0}, \ i \in \{m, p\} \quad n_0^{(k)} \rightarrow n_0, \ v_0^{(k)} \rightarrow v_0 \] (2.29)
the latter convergence relations holding, e.g., in the sense of distributions.

Then there exists a (non-relabelled) sequence of the family \((c_p^{(k)}, c_m^{(k)}, \Phi^{(k)}, v^{(k)}, n^{(k)})\)
tending, in the sense explicated in relations (1.1)–(4.7) below, to a quintuple \((c_p, c_m, \Phi, v, n)\)
solving system (2.1)–(2.6) in the sense specified in Definition 2.1.

**Remark 2.3.** In fact one would need solutions which are not smooth but just ‘sufficiently regular’, but the precise minimal regularity needed is not of interest since in general the solutions obtained through approximations scheme are smooth. On the other hand, as one specifies a precise regularization method, the above proposition should be modified because the scheme may involve a regularization or truncation of some terms, in particular of the singular operator \( \mathcal{F} \) which may be replaced by a smooth functional \( \mathcal{F}_k, \) see Section 5 below for further details.
Once a precise approximation scheme compatible with the estimates is specified, as a consequence of Theorem 2.2 one can obtain the existence of global–in–time weak solutions:

**Theorem 2.4.** Let assumptions (1.9), (2.7), (2.8) hold and let the initial data satisfy (2.13)–(2.15). Then the problem (2.1)–(2.5) admits a weak solution in $(0,T) \times \mathbb{T}^3$ in the sense of Definition 2.1.

The rest of the paper is devoted to the proof of Theorem 2.2. In the final Section 5, we will then give the highlights of an approximation scheme compatible with the estimates. Specifying the details, one can then obtain a proof of Theorem 2.4.

3. **Apriori estimates.** We now prove a number of apriori estimates on the solutions of system (2.1)-(2.6). As noted above, we decided to perform the computations by directly working on the “original” equations without referring to any explicit regularization or approximation scheme. Of course, in such a setting, the procedure has just a formal character because the use of some test function as well as some integration by parts is not justified (this, for instance, surely happens in connection with the Navier-Stokes system (2.4)). On the other hand, the computations we are going to develop are not trivial and involve a certain number of subtlenesses; for this reason we believe that presenting them in the simplest possible setting might help comprehension. Actually, in the last part of the paper we will provide some hints about the construction of an approximation scheme being compatible with the estimates.

The first property we prove is the basic energy estimate resulting as a consequence of the variational nature of the model (cf. also [4, p. 2273]). We state it in the form of a

**Proposition 3.1 (Energy law).** Let $(c_m, c_p, \Phi, v, n) : \Omega \to \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ be a sufficiently smooth solution of system (2.1)-(2.6) on $\mathbb{T}^3 \times (0,T)$ complemented with the initial conditions (2.27) and satisfying the coefficient relations (2.12) (that ensure the non-negativity of the dissipation). Then there holds the energy inequality

\[
E(t) + \int_0^t \int_{\mathbb{T}^3} \left( \frac{1}{c_p} |\nabla c_p + c_p \nabla \Phi|^2 + \frac{1}{c_m} |\nabla c_m - c_m \nabla \Phi|^2 \right) dt
\]

\[+ \int_0^t \int_{\mathbb{T}^3} \left( \alpha_4 |D(v)|^2 + \alpha_1 (n \cdot D(v)n)^2 + 2(n \cdot D(v)n) + (\alpha_2 + \alpha_6) |D(v)n|^2 + |n|^2 \right) \geq 0 \]

\[\leq E(0) \] (3.2)

where the energy functional is defined as

\[E(t) = \int_{\mathbb{T}^3} \left( \frac{1}{2} |v|^2 + \frac{1}{2} |\nabla n|^2 + \mathcal{F}(n) + c_p \ln c_p + c_m \ln c_m + \frac{1}{2} (1 + \varepsilon n \otimes n) \nabla \Phi \cdot \nabla \Phi \right). \]

(3.3)

**Proof.** We multiply the equation (2.1) by $\ln c_p + \Phi$, integrate by parts using periodic boundary conditions to obtain (denoting $c'_i := \frac{dc_t}{dt} c_i$, $i \in\{m,p\}$):

\[\frac{d}{dt} \int_{\mathbb{T}^3} c_p (\ln c_p - 1) + \int_{\mathbb{T}^3} c'_p \Phi + \int_{\mathbb{T}^3} (v \cdot \nabla c_p) \Phi \]

\[+ \int_{\mathbb{T}^3} (\text{Id} + \varepsilon n \otimes n) (\nabla c_p + c_p \nabla \Phi) \cdot \left( \frac{\nabla c_p}{c_p} + \nabla \Phi \right) = 0, \] (3.4)
whence, by positive definiteness of the matrix $n \otimes n$,
\[
\frac{d}{dt} \int_{T^3} c_p(\ln c_p - 1) + \int_{T^3} c_p' \Phi + \int_{T^3} (v \cdot \nabla c_p) \Phi + \int_{T^3} \frac{1}{c_p} |\nabla c_p + c_p \nabla \Phi|^2 \leq 0. \tag{3.5}
\]

Similarly, testing (2.2) by $\ln c_m - \Phi$ we have
\[
\frac{d}{dt} \int_{T^3} c_m(\ln c_m - 1) - \int_{T^3} c_m' \Phi - \int_{T^3} (v \cdot \nabla c_m) \Phi + \int_{T^3} \frac{1}{c_m} |\nabla c_m - c_m \nabla \Phi|^2 \leq 0.
\tag{3.6}
\]

We now test (2.3) by $-\partial_t \Phi$ getting, after an integration by parts,
\[
- \int_{T^3} (\text{Id} + \varepsilon n \otimes n) \nabla \Phi \cdot \nabla \Phi_t + \int_{T^3} (c_p - c_m) \Phi_t = 0,
\tag{3.8}
\]
which can be expanded into
\[
- \frac{\varepsilon}{2} \frac{d}{dt} \int_{T^3} (n \nabla \Phi) \cdot \nabla \Phi - \int_{T^3} \nabla \Phi \cdot \nabla \Phi_t + \int_{T^3} (c_p - c_m) \Phi_t
\]
\[
= - \frac{\varepsilon}{2} \int_{T^3} \partial_t (n \nabla \Phi) \cdot \nabla \Phi.
\tag{3.9}
\]

Multiplying (2.3) by $-v \cdot \nabla \Phi$ and integrating by parts we get
\[
- \int_{T^3} (\text{Id} + \varepsilon n \otimes n) \nabla \Phi \cdot \nabla (v \cdot \nabla \Phi) + \int_{T^3} (c_p - c_m) v \cdot \nabla \Phi = 0.
\tag{3.10}
\]

Splitting the left-hand side and integrating by parts further, we obtain
\[
- \int_{T^3} (\nabla \Phi \otimes \nabla \Phi) : \nabla v - \varepsilon \int_{T^3} (n \nabla \Phi) \cdot \nabla (v \cdot \nabla \Phi) + \int_{T^3} (c_p - c_m) v \cdot \nabla \Phi = 0.
\tag{3.11}
\]

Multiplying (2.4) by $v$ and integrating by parts, we get
\[
\frac{1}{2} \frac{d}{dt} ||v||^2 + \alpha_4 ||D(v)||^2 = \int_{T^3} (\nabla n \otimes \nabla n) : \nabla v - \int_{T^3} \nabla \Phi \otimes \nabla \Phi : \nabla v
\]
\[
- \varepsilon \int_{T^3} ((\nabla \Phi \otimes \nabla \Phi)(n \otimes n)) : \nabla v
\]
\[
- \int_{T^3} (\alpha_1 (D(v)n \otimes n)n + \alpha_2 n \otimes n + \alpha_3 n \otimes \n + \alpha_2 D(v)n \otimes n + \alpha_5 n \otimes D(v)n) : \nabla v.
\tag{3.12}
\]
Finally, multiplying (2.6) by \( \dot{n} = n_t + v \cdot \nabla n \) we get
\[
\int_{T^3} \left( \dot{n} + D(v)n \right) \cdot \dot{n} + \frac{1}{2} \frac{d}{dt} \left\| \nabla n \right\|^2 + \frac{d}{dt} \int_{T^3} \mathcal{F}(n) + \int_{T^3} \nabla n \cdot (v \cdot \nabla n) \]
\[
= \varepsilon \int_{T^3} \nabla \Phi \otimes \nabla \Phi : n \otimes n_t + \varepsilon \int_{T^3} \left( (\nabla \Phi \otimes \nabla \Phi)n \right) \cdot (v \cdot \nabla n). \tag{3.13}
\]

We can now sum (3.5), (3.7), (3.9), (3.11), (3.12), (3.13). We combine a number of terms and may note several cancellations, namely
\[
\frac{d}{dt} \int_{T^3} \left( - \frac{1}{2} (\text{Id} + \varepsilon n \otimes n) \nabla \Phi \cdot \nabla \Phi + (c_p - c_m) \Phi \right) = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{13},
\]
\[
\mathcal{A}_{21} + \mathcal{A}_{22} = \mathcal{B}_2, \quad \mathcal{A}_3 = \mathcal{B}_3, \quad \mathcal{A}_4 = \mathcal{B}_6.
\]
The most delicate cancellation is \( \mathcal{A}_5 = \mathcal{B}_{51} + \mathcal{B}_{52} \), which amounts to
\[
- \int_{T^3} (n \otimes n \nabla \Phi) : \nabla (v \cdot \nabla \Phi) = - \int_{T^3} (\nabla \Phi \otimes \nabla \Phi)n \otimes n : \nabla v + \int_{T^3} (\nabla \Phi \otimes \nabla \Phi)n \cdot (v \cdot \nabla n),
\]
which, after expanding \( (n \otimes n \nabla \Phi) : \nabla (v \cdot \nabla \Phi) = (n \otimes n \nabla \Phi) \cdot (\nabla v \nabla \Phi) + (n \otimes n \nabla \Phi) \cdot (v \cdot \nabla) \nabla \Phi \), simplifies to
\[
- \int_{T^3} n_i n_j \partial_j \Phi n_k \partial_k \Phi = \int_{T^3} \partial_i \Phi \partial_j \Phi n_j n_k \partial_k n_i. \tag{3.14}
\]
Then, we integrate by parts the \( \partial_k \) derivative and note that no boundary terms appear due to the choice of periodic boundary conditions. Hence, using \( \partial_k n_k = 0 \) we obtain
\[
- \int_{T^3} n_i n_j \partial_j \Phi n_k \partial_k \Phi = \int_{T^3} n_i k n_j \partial_j \Phi \partial_i \Phi n_k \\
+ \int_{T^3} n_i n_j \partial_j \Phi \partial_k \Phi n_k + \int_{T^3} n_k n_j \partial_j \partial_k \Phi \partial_i \Phi n_k. \tag{3.15}
\]
We note that after permuting the indices the above turns into
\[
-2 \int_{T^3} n_i n_j \partial_j \Phi n_k \partial_k \Phi = 2 \int_{T^3} n_i k n_j \partial_j \Phi \partial_i \Phi n_k, \tag{3.16}
\]
which is exactly (3.14), thus proving the claimed cancellation \( \mathcal{A}_5 = \mathcal{B}_{51} + \mathcal{B}_{52} \).
Furthermore, as in [4] we have
\[
\mathcal{A}_7 + \mathcal{B}_7 = \alpha_1 (n \cdot D(v)n)^2 + 2(\dot{n} \cdot D(v)n) + \alpha_4 |D(v)|^2 + (\alpha_5 + \alpha_6) |D(v)n|^2 + |\dot{n}|^2 \tag{3.17}
\]
Collecting the above computations, and using also the charge conservation property
\[
\frac{d}{dt} \int_{T^3} (c_p + c_m) = 0, \tag{3.18}
\]
we finally arrive at
\[ \frac{d}{dt} \int_{\mathcal{T}^3} \left( \frac{1}{2} |v|^2 + \frac{1}{2} |\nabla n|^2 + \mathcal{F}(n) + c_p \ln c_p + c_m \ln c_m \right) - \frac{1}{2} (1 + \varepsilon n \otimes n) \nabla \Phi \cdot \nabla \Phi + (c_p - c_m) \Phi \] \]
\[ + \int_{\mathcal{T}^3} \left( \frac{1}{c_p} |\nabla c_p + c_p \nabla \Phi|^2 + \frac{1}{c_m} |\nabla c_m - c_m \nabla \Phi|^2 + \alpha_4 |D(v)|^2 \right) \]
\[ + \int_{\mathcal{T}^3} \left( \alpha_1 (n \cdot D(v)n)^2 + 2 (\dot{n} \cdot D(v)n)^2 + (\alpha_5 + \alpha_6) |D(v) n|^2 + |\dot{n}|^2 \right) \leq 0. \tag{3.19} \]
Let us now notice that, testing (2.3) by \( \Phi \) and integrating by parts, there follows
\[ \int_{\mathcal{T}^3} (c_p - c_m) \Phi = \int_{\mathcal{T}^3} (1 + \varepsilon n \otimes n) \nabla \Phi \cdot \nabla \Phi. \tag{3.20} \]
Replacing the above into (3.19), we obtain (3.2), which concludes the proof. \( \square \)

The energy estimate (3.2) implies a number of apriori bounds for the solutions of system (2.1)-(2.6), provided that the dissipation term is nonnegative. In our simplified setting (where we have set \( \gamma_1, \gamma_2 = 1 \)), this results as a restriction on the choice of the parameters \( \alpha_j \). Namely, we can observe the following

**Lemma 3.2.** If (2.12) holds true, then we have, for some \( \delta' > 0 \),
\[ \alpha_4 |D|^2 + \alpha_1 (n \cdot Dn)^2 + 2 (\dot{n} \cdot Dn) + (\alpha_5 + \alpha_6) |Dn|^2 + |\dot{n}|^2 \geq \delta' (|Dn|^2 + |\dot{n}|^2) \tag{3.21} \]
for arbitrary \( \dot{n} \in \mathbb{R}^3, n \in \mathbb{R}^3, D \in \mathbb{R}^{3 \times 3} \) with \( |n| \leq 1 \) and the matrix \( D \) symmetric and traceless.

**Proof.** Noting that we have (where we use that \( |n| \leq 1 \)):
\[ (n \cdot Dn)^2 \leq |n|^2 |Dn|^2 \leq |D|^2, \quad |2 (n \cdot Dn)| \leq 2 |\dot{n}||Dn| \leq (1 - \delta)|\dot{n}|^2 + \frac{1}{1 - \delta} |D|^2 \]
we immediately deduce that (2.12) implies the claimed (3.21).

In the sequel we shall always assume (2.12). In this way, as a consequence of the energy estimate (3.2), using also the positive definiteness of the matrix \( n \otimes n \) and (2.8), we can obtain a number of apriori bounds holding for any hypothetical solution of the system and independently of any eventual approximation or regularization parameter. Namely, we have
\[ \|v\|_{L^\infty(0,T;H)} + \|v\|_{L^2(0,T;V)} \leq c, \tag{3.22} \]
\[ \|n\|_{L^\infty(0,T;V)} \leq c, \quad |n| \leq 1 \quad \text{a.e. in} \ (0, T) \times \mathcal{T}^3, \tag{3.23} \]
\[ c_p, c_m \geq 0 \quad \text{a.e. in} \ (0, T) \times \mathcal{T}^3, \tag{3.24} \]
\[ \|\nabla \Phi\|_{L^\infty(0,T;H)} \leq c. \tag{3.25} \]
where \( c \) is a constant depending only on \( E(0) \) as defined in (3.3). Note that the second bound in (3.23) directly follows from our choice of the potential \( F \) and the positivity of \( c_p \) and \( c_m \) follows from the logarithmic term in (3.2), see also [25, Lemma 1, p. 1003]. In the following we provide a maximum principle (see also [25], Lemma 4 for a different argument).

**Proposition 3.3** (Maximum principle). Let \( c_p^0, c_m^0 : \mathcal{T}^3 \rightarrow \mathbb{R}_+ \) satisfy (2.27) and let \( v, n \) satisfy (3.22), (3.23). Then, if \( (c_p, c_m, \Phi) \) solve equations (2.1), (2.2), (2.3)
subject to periodic boundary conditions and initial data $c_p^0, c_m^0$ as above, then there follows

$$|c_p(x, t)|, |c_m(x, t)| \leq \bar{c}, \quad a.e. \ in \ (0, T) \times \mathcal{T}^3. \quad (3.26)$$

**Proof.** We multiply (2.1) by $(c_p - \bar{c})^+$ and integrate over $\mathcal{T}^3$ and by parts, to obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{T}^3} |(c_p - \bar{c})^+|^2 + \frac{1}{2} \int_{\mathcal{T}^3} v \cdot \nabla ((c_p - \bar{c})^+)^2$$

$$+ \int_{\mathcal{T}^3} (\text{Id} + \varepsilon n \otimes n) \nabla (c_p - \bar{c})^+ \cdot \nabla (c_p - \bar{c})^+$$

$$+ \int_{\mathcal{T}^3} (\text{Id} + \varepsilon n \otimes n) \nabla \Phi \cdot \nabla \left( \frac{1}{2} ((c_p - \bar{c})^+)^2 + \bar{c}(c_p - \bar{c})^+ \right) = 0. \quad (3.27)$$

Similarly, we get from (2.2)

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{T}^3} |(c_m - \bar{c})^+|^2 + \frac{1}{2} \int_{\mathcal{T}^3} v \cdot \nabla ((c_m - \bar{c})^+)^2$$

$$+ \int_{\mathcal{T}^3} (\text{Id} + \varepsilon n \otimes n) \nabla (c_m - \bar{c})^+ \cdot \nabla (c_m - \bar{c})^+$$

$$- \int_{\mathcal{T}^3} (\text{Id} + \varepsilon n \otimes n) \nabla \Phi \cdot \nabla \left( \frac{1}{2} ((c_m - \bar{c})^+)^2 + \bar{c}(c_m - \bar{c})^+ \right) = 0. \quad (3.28)$$

We now define

$$M(r) := \begin{cases} 
0 & \text{if } r \leq \bar{c}, \\
\frac{1}{2}(r - \bar{c})^2 + \bar{c}(r - \bar{c})^+ & \text{if } r \geq \bar{c},
\end{cases} \quad (3.29)$$

Then, summing (3.27) and (3.28) and using incompressibility, we deduce

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{T}^3} \left( |(c_p - \bar{c})^+|^2 + |(c_m - \bar{c})^+|^2 \right)$$

$$\leq - \int_{\mathcal{T}^3} (\text{Id} + \varepsilon n \otimes n) \nabla \Phi \cdot \nabla (M(c_p) - M(c_m)). \quad (3.30)$$

The integral on the right-hand side can be computed by using (2.3). This leads to

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{T}^3} \left( |(c_p - \bar{c})^+|^2 + |(c_m - \bar{c})^+|^2 \right) \leq - \int_{\mathcal{T}^3} (c_p - c_m)(M(c_p) - M(c_m)) \leq 0, \quad (3.31)$$

the inequality following from the monotonicity of the function $M$. Noting that (2.27) with the assumption (2.13) implies that the left-hand side is null at $t = 0$, we obtain the claimed estimate. 

In particular, we have obtained the additional bound

$$\|c_p\|_{L^\infty(0,T;L^\infty(\mathcal{T}^3))} + \|c_m\|_{L^\infty(0,T;L^\infty(\mathcal{T}^3))} \leq c. \quad (3.32)$$

where the constant $c$ depends just on the $L^\infty$ norm of $c_p(0)$ and $c_m(0)$. We can then test (2.1) by $c_p$ and (2.2) by $c_m$. Using once more the positive definiteness of the matrix $n \otimes n$, we may note that

$$\int_{\mathcal{T}^3} c_p \nabla \Phi \cdot \nabla c_p \leq \|c_p\|_{L^\infty(\mathcal{T}^3)} \|\nabla \Phi\|_H \|\nabla c_p\|_H \leq \varepsilon \|\nabla c_p\|_H \leq c + \frac{1}{2} \|\nabla c_p\|_H^2, \quad (3.33)$$

with an analogous relation holding for $c_m$ and where the constants $c > 0$ are independent of time in view of (3.25) and (3.32). Analogously we can estimate the term $-\int_{\mathcal{T}^3} \varepsilon (n \otimes n) c_p \nabla \Phi \cdot \nabla c_p$ by (3.23).
Then, it is not difficult to deduce the parabolic regularity estimate
\[ \|c_p\|_{L^2(0,T;V)} + \|c_m\|_{L^2(0,T;V)} \leq c. \] (3.34)
In view of the fact that \( \Phi \) is defined up to an additive constant, it is not restrictive to assume that
\[ \Phi_{\Omega} = \int_{T^3} \Phi(t) = 0 \text{ for a.e. } t \in (0,T). \] (3.35)
Of course, such a normalization property, joint with (3.25), implies
\[ \|\Phi\|_{L^\infty(0,T;V)} \leq c. \] (3.36)
We have, however, a better property which is given by the following

**Lemma 3.4** (Uniform boundedness of \( \Phi \)). *We have the additional estimate*
\[ \|\Phi\|_{L^\infty(0,T;L^\infty(T^3))} \leq c. \] (3.37)

**Proof.** The proof follows by applying a Moser iteration argument on equation (2.3) and using the uniform boundedness of the right-hand side following from estimate (3.32). Though the argument is very classical, in view of the fact that the equation has nonconstant coefficients we prefer to give some highlights for the reader’s convenience. As a general rule, we multiply equation (2.3) by \((\Phi)^{p-1} := |\Phi|^{p-1}\) sign \( \Phi \) where the exponent \( p \) will be taken larger and larger. This gives
\[ (p-1) \int_{T^3} (\text{Id} + \varepsilon n \otimes n)|\Phi|^{p-2}\nabla \Phi \cdot \nabla \Phi = \int_{T^3} (c_p - c_m)|\Phi|^{p-1}\text{sign } \Phi \]
\[ \leq c \int_{T^3} |\Phi|^{p-1} \leq c \int_{T^3} \left( \frac{1}{p} + \frac{p-1}{p} |\Phi|^p \right) \]
\[ \leq \frac{c}{p} + c \int_{T^3} |\Phi|^p. \] (3.38)
As a first step, we take \( p = p_0 = 6 \). Then, controlling the right-hand side by the Poincaré-Wirtinger inequality we deduce (cf. also (3.35))
\[ c \int_{T^3} |\Phi|^6 = c \|\Phi - \Phi_{\Omega}\|^6 \leq c \|\nabla \Phi\|^6 \leq c, \] (3.39)
the last inequality following from (3.25). Here and below, we are noting simply by \( \| \cdot \|_q \) the norm in \( L^q(T^3) \), \( 1 \leq q \leq \infty \), for notational simplicity. We also point out that all the estimates obtained in this proof are uniform with respect to the time variable, because so are (3.25) and (3.32) that serve as a starting point of the argument.

Hence, noting that
\[ (p-1) \int_{T^3} (\text{Id} + \varepsilon n \otimes n)|\Phi|^{p-2}\nabla \Phi \cdot \nabla \Phi \geq \frac{4(p-1)}{p^2} \int_{T^3} |\nabla (\Phi)^{p/2}|^2 \] (3.40)
at the first iteration, i.e. for \( p = 6 \), we deduce
\[ \|\nabla \Phi^3\|_2 \leq c, \] (3.41)
whence, recalling (3.25) and using Sobolev’s embeddings,
\[ \|\Phi\|^3_1 \leq c (\|\nabla \Phi^3\|_2 + \|\Phi\|^3_0) \leq c. \] (3.42)
Now, in order to take care of further iterations, we need to keep trace of the dependence on $p$ of the various constants. Let us, then, go back to (3.38) with a generic $p$ and combine it with (3.40) to deduce (for $p \geq 2$)

$$\int_{T^3} |\nabla (\Phi)^{p/2}|^2 \leq \frac{cp}{(p-1)} + \frac{cp^2}{p-1} \int_{T^3} |\Phi|^p \leq c + c(p+2) \int_{T^3} |\Phi|^p$$

where $c$ is independent of $p$.

Adding also $\|\Phi\|^p_p$ to both hand sides and using Sobolev’s embeddings, we then deduce

$$\|\Phi\|_{3p}^p = \|(\Phi)^{p/2}\|_6^2 \leq c\|\Phi\|^{p/2}_V$$

$$\leq c\|\Phi\|^{p/2}_V + c \int_{T^3} |\nabla (\Phi)^{p/2}|^2 \leq c + c(p+3)\|\Phi\|^p_p \leq c + cp\|\Phi\|^p_p, \quad (3.43)$$

where $c$ is still independent of $p$.

We define $b_p = \max(1,\|\Phi\|_p)$. Then, assuming without loss of generality that $c \geq 1$ the last inequality implies:

$$b_{3p}^p \leq c b_p^p$$

with $c > 1$ a constant independent of $p$. Then, since $\ln b_{3p} \leq \frac{\ln(c^p)}{p} + \ln b_p$, we get

$$\ln b_{3n} \leq \frac{\ln(c^{3(n-1)p})}{3n-1} + \ln b_{3n-1}$$

$$\leq \frac{\ln(c^{3(n-1)p})}{3n-1} + \frac{\ln(c^{3(n-2)p})}{3n-2} + \cdots + \ln b_p,$$

and hence

$$\ln b_{3n} \leq \sum_{k=1}^{n-1} \frac{\ln(c^{3k})}{3k^p} + \ln b_p.$$

Noting that constant $c$ is independent of $n$ and $p$ and letting $n \to \infty$ we obtain (3.37).

It is worth observing that the bounds derived up to this point are not sufficient for passing to the limit in (a suitable approximation) of system (2.1)-(2.6), the main trouble being represented by the quadratic terms in $\nabla \Phi$ and $\nabla n$. Indeed, at the moment such quantities are bounded only in $L^2$ with respect to space variables. Hence, at the limit we might expect occurrence of defect measures. Fortunately, this is not the case, because it is possible to improve a bit the regularity properties proved so far.

**Lemma 3.5 (Additional regularity estimate).** Let us assume that the initial data satisfy (2.13)-(2.15). Then the following additional regularity conditions hold:

$$\|\nabla \Phi\|_{L^\infty(0,T;L^p_{\text{PM}}(T^3))} \leq c_{p M_I}, \quad \text{for some } p M_I \geq 2 \quad (3.44)$$

$$\|n_t\|_{L^p(0,T;L^p(\mathbb{T}^3))} + \|\Delta n\|_{L^p(0,T;L^p(\mathbb{T}^3))} \leq c, \quad \text{for some } p_0 > 1 \quad (3.45)$$

$$\|\partial_{\mathcal{S}}(n)\|_{L^p(0,T;L^p(\mathbb{T}^3))} \leq c, \quad \text{for some } p_0 > 1. \quad (3.46)$$

**Proof.** The key point stands in the application of some refined elliptic regularity result to equation (2.3). Indeed, in view of the bound $|n| \leq 1$ and of the positive definiteness of $n \otimes n$, the matrix $\text{Id} + c n \otimes n$ is strongly elliptic and has bounded coefficients. Notice that in case we would make this argument rigorous we would need to truncate some coefficients at the approximated level (i.e. for the corresponding system with regular potential) when the bound $|n| \leq 1$ is not available. However
this is not the aim of this contribution where, indeed, our aim is that of presenting the highlights of the existence proof by focusing on actual mathematical difficulties rather than on technical details. Hence, since the right-hand side of (2.3) is uniformly bounded by (3.26), we can then apply the integrability result [22, Thm. 1, p. 198], which implies

\[ \| \nabla \Phi \|_{L^\infty(0,T;L^p(M(T^3)))} \leq c_{p_M} \quad \text{for some } p_M > 2. \]  

(3.47)

Note that, at least in three space dimensions, there is no quantitative control of \( p_M \). Nevertheless, we know that \( p_M > 2 \). As a consequence of (3.44), (2.6) can be rearranged in the form

\[ n_t - \Delta n + \partial F(n) = -v \cdot \nabla n + \Omega(v)n - D(v)n + \varepsilon (\nabla \Phi \otimes \nabla \Phi)n, \]  

(3.48)

where a simple check based on the previous estimates (3.22), (3.23) shows that, at least,

\[ v \cdot \nabla n + \Omega(v)n - D(v)n \in L^{\frac{3}{2}}(0,T;L^{\frac{3}{2}}(T^3)), \]

which together with (3.47) implies

\[ f \in L^p(0,T;L^p(T^3)). \]  

(3.49)

for all \( p \leq p_0 \) where

\[ p_0 := \min \left( \frac{3}{2}, \frac{p_M}{2} \right). \]  

(3.50)

Recalling (2.9), we observe that, componentwise, equation (3.48) takes the form

\[ \partial_t n_i - \Delta n_i + F'(|n|^2)n_i = f_i, \]  

(3.51)

where \( F' \) is monotone because \( F \) is convex.

This property, however, has to be a bit clarified. Indeed, the function \( F \) may be nonsmooth, and its subdifferential \( \partial F \) may be (and in fact has to be, in view of assumption (2.8)) a singular operator. Hence, here and below the use of \( F' \) to represent the subdifferential \( \partial F \) is formal and to make the procedure fully rigorous one should rather perform some regularization of \( F \) and then pass to the limit. Since this kind of argument is rather standard, we omit details for brevity.

Take from now on \( p := p_0 \) (for simplicity of notation). We then test (3.51) by the function \( G_i(n) = |F'(|n|^2)|^{p-1} \text{sign} F'(|n|^2)n_i \) to obtain

\[
\frac{1}{2} \int_{T^3} |F'(|n|^2)|^{p-1} \text{sign} F'(|n|^2) \frac{d}{dt}|n_i|^2 + \int_{T^3} |F'(|n|^2)|^{p} n_i^2 \\
+ \int_{T^3} |F'(|n|^2)|^{p-1} \text{sign} F'(|n|^2)|\nabla n_i|^2 + M_i = \int_{T^3} f_i \cdot F'(|n|^2)|^{p-1} \text{sign} F'(|n|^2)n_i, 
\]

(3.52)

where the “mixed” term \( M \) is given by

\[
M_i = (p - 1) \int_{T^3} |F'(|n|^2)|^{p-2} F''(|n|^2) n_i \nabla |n|^2 \cdot \nabla n_i \\
= \frac{(p - 1)}{2} \int_{T^3} |F'(|n|^2)|^{p-2} F''(|n|^2) |\nabla n|^2 \cdot \nabla n_i^2. 
\]

(3.53)

Let us sum (3.52) for \( i = 1, 2, 3 \). It is then easy to check that

\[
\sum_{i=1}^{3} M_i = \frac{(p - 1)}{2} \int_{T^3} |F'(|n|^2)|^{p-2} F''(|n|^2) |\nabla n|^2 \cdot \nabla n|^2 \geq 0 
\]

(3.54)
due to convexity of $F$. We split the term \( \int_{T^3} |F'(\|n\|^2)|^{p-1} \frac{\partial}{\partial t} \|n\|^2 \) over two subsets of $T^3$, namely

\[
T^3_+ := \left\{ x \in T^3, |n|^2(x) \geq 1 - \frac{1}{c} \right\}, \quad \text{respectively } T^3_- := \left\{ x \in T^3, |n|^2(x) < 1 - \frac{1}{c} \right\},
\]

where we neglect the dependence on $t$ due to convexity of $F$.

Then, taking into account that $F'(r) \geq 0$ for $r \in (1 - \frac{1}{c}, 1)$, neglecting the positive term $\int_{T^3} |F'(|n|^2)|^{p-1} \frac{\partial}{\partial t} |n|^2 \nabla |n|^2$ on the left-hand side, and using that $F'(|n|^2(x)) \in (-1, 0)$ for $x \in T^3_-$ we deduce:

\[
\begin{aligned}
\frac{1}{2} \int_{T^3} |F'(|n|^2)|^{p-1} \operatorname{sign} F'(|n|^2) \frac{d}{dt} |n|^2 + \int_{T^3} |F'(|n|^2)|^{p-1} |\nabla |n|^2 |
\leq \int_{T^3} F'(|n|^2) |\frac{\partial}{\partial t} |n|^2 | f \cdot n + \int_{T^3} \frac{d}{dt} |F'(|n|^2)|^{p-1} |\nabla |n|^2 |
\leq \| F'(|n|^2) \|_{L^p(T^3)} \| f \cdot n \|_p + \int_{T^3} |\nabla |n|^2 |
\leq c \| F'(|n|^2) \|_{L^p(T^3)} \| f \|_p + c_s \| f \|_p + c,
\end{aligned}
\]

where we also used H"older’s and Young’s inequalities and the apriori bounds (3.55).

Now, note that

\[
\frac{1}{2} \int_{T^3} |F'(|n|^2)|^{p-1} \operatorname{sign} F'(|n|^2) \frac{d}{dt} |n|^2 = \frac{d}{dt} \int_{T^3} \Gamma_p(|n|^2),
\]

where the function $\Gamma_p$ is defined by

\[
\Gamma_p(s) := \int_{1-s/e}^s |F'(r)|^{p-1} \operatorname{sign} F'(r)
\]

and is consequently nonnegative. Notice also that $\lim_{r \to 1^-} \Gamma_p(r) < +\infty$ and that

\[
\int_{T^3} |F'(|n|^2)|^{p-1} |\nabla |n|^2 | \geq 1 \int_{T^3} |F'(|n|^2)|^{p-1} - c
\]

(to see this, split the integral into the subregions $|n|^2 \leq 1/2$, where $F'$ is bounded and $|n| \geq 1/2$ which gives the control from below). Hence, taking $\sigma < 1/2$, we see that the first term on the right-hand side of (3.55) is controlled. On the other hand, integrating in time, we may note that the latter term in (3.55) is also controlled by (3.49). As a consequence, we obtain first

\[
\| F'(|n|^2) \|_{L^p((0,T) \times T^3)} \leq c
\]

and, as a consequence,

\[
\| \partial \mathcal{F}(n) \|_{L^p((0,T) \times T^3)} \leq c.
\]

Finally, comparing terms in (3.5) and applying elliptic regularity results of Agmon-Douglis-Nirenberg type, we get the bound

\[
\| n_t \|_{L^p(0,T;L^p(T^3))} + \| \Delta n \|_{L^p(0,T;L^p(T^3))} \leq c,
\]

where we also used the regularity $n_0 \in W^{1,2}(T^3)$ which is actually implied by our assumption (2.15).

In the case when the anisotropy coefficient $\varepsilon$ is small enough compared to the other parameters, we can prove some additional estimates. This is stated in the following
Lemma 3.6 (H^2-estimates). Let us assume that the initial data satisfy (2.13)–(2.15). Furthermore, let ε > 0 be small enough. Then, we have
\[ \|\Phi\|_{L^2(0,T;H^2(\Omega))} + \|n\|_{L^2(0,T;H^2(\Omega))} \leq c. \]  
(3.60)

Proof. We proceed in a natural way by testing (2.6) by −Δn. Then, we can preliminarily observe that, by λ-convexity of Φ (and consequent λ-monotonicity of the subdifferential),
\[ - \int_{\Omega} \partial\Phi(n) \cdot \Delta n \geq -\lambda \|\nabla n\|^2_H. \]  
(3.61)

As already noted before, this property, due to nonsmoothness of \( \partial\Phi \), may require an approximation argument to be proved rigorously.

That said, we arrive at the bound
\[ \frac{1}{2} \frac{d}{dt} \|\nabla n\|^2_H + \|\Delta n\|^2_H = \int_{\Omega} (v \cdot \nabla n) \cdot \Delta n - \int_{\Omega} (\Omega(v)n) \cdot \Delta n \]
\[ + \int_{\Omega} (D(v)n) \cdot \Delta n - \int_{\Omega} \varepsilon((\nabla \Phi \otimes \nabla \Phi)n) \cdot \Delta n + \lambda \|\nabla n\|^2_H =: \sum_{j=1}^5 I_j. \]  
(3.62)

and we need to estimate the terms \( I_j \) on the right-hand side. A key role will be played by the inequality
\[ \|\nabla z\|_{L^4(\Omega)} \leq c \|z\|_{L^\infty(\Omega)}^{1/2} \|z\|_{H^2(\Omega)}^{1/2}, \]  
(3.63)

holding for every \( z \in H^2(\Omega), \Omega \) being a smooth bounded domain of \( \mathbb{R}^3 \) (for instance \( \Omega = \mathcal{T}^3 \)). Then, integrating by parts and using (2.5) with the periodic boundary conditions, we have
\[ I_1 = - \int_{\Omega} (\nabla n \otimes \nabla n) \cdot \nabla v \leq \|\nabla n\|^2_{L^4(\Omega)} \|\nabla v\|_H \]
\[ \leq c \|n\|_{L^\infty(\Omega)} \left( \|n\|_H + \|\Delta n\|_H \right) \|\nabla v\|_H \]
\[ \leq c + \frac{1}{6} \|\Delta n\|^2_H + c \|\nabla v\|^2_H, \]  
(3.64)

where we used in an essential way the property \( |n| \leq 1 \) almost everywhere.

Next, it is clear that
\[ I_2 + I_3 \leq c \|n\|_{L^\infty(\Omega)} \|\nabla v\|_H \|\Delta n\|_H \leq \frac{1}{6} \|\Delta n\|^2_H + c \|\nabla v\|^2_H, \]  
(3.65)

and, finally,
\[ I_4 \leq c \varepsilon \|n\|_{L^\infty(\Omega)} \|\nabla \Phi\|^2_{L^2(\Omega)} \|\Delta n\|_H \leq c \varepsilon \|\Phi\|_{L^\infty(\Omega)} \|\Phi\|_{H^2(\Omega)} \|\Delta n\|_H \]
\[ \leq c \varepsilon^2 \|\Phi\|^2_{H^2(\Omega)} + \frac{1}{6} \|\Delta n\|^2_H, \]  
(3.66)

where for the last inequality we also used Lemma 3.4.

Taking (3.64)–(3.66) into account, (3.62) implies
\[ \frac{d}{dt} \|\nabla n\|^2_H + \|\Delta n\|^2_H \leq c + \lambda \|\nabla n\|^2_H + c \|\nabla v\|^2_H + c \varepsilon^2 \|\Phi\|^2_H + c \varepsilon^2 \|\Delta \Phi\|^2_H, \]  
(3.67)

where we point out that the constants \( c \), in particular the last one, may depend on the various parameters of the problem, but are independent of the coefficient \( \varepsilon \).
In order to control the last term, we apply elliptic regularity results to (2.3) (or, in other words, we test it by \(-\Delta \Phi\) to obtain
\[
\|\Delta \Phi\|_H \leq c(\|\nabla n\|_{L^4(\Omega)} \|n\|_{L^\infty(\Omega)} + \|\nabla \Phi\|_{L^4(\Omega)} + \|\partial n\|_{L^\infty(\Omega)} + 1)
\]
where in deducing (4.3) we also used the normalization \(\Phi \rightarrow 0\) uniformly with respect to the parameter \(k\). Of course this implies in particular \(\Phi\) in the sense specified in Definition 2.1.

Let us assume \(c_p, c_m, \Phi, v, n\) to be a family of approximating solutions complying with the estimates derived in the previous section uniformly with respect to the parameter \(k \in \mathbb{N}\). We will then prove that there exists a (non-relabelled) sequence of the above sequence tending, in a suitable way, to a quintuple \((c_p, c_m, \Phi, v, n)\) solving system (2.1)-(2.6) in the sense specified in Definition 2.1.

To this aim, we start deducing some convergence properties (as mentioned, we will always assume to hold up to the extraction of subsequences) arising as a consequence of the bounds (3.22)-(3.25), (3.32), (3.34), (3.36), (3.37), (3.44)-(3.46) and (3.60). Namely, there exists \(\xi \in L^{p_0}(0, T; L^{p_0}(\mathcal{T}^3))\) such that
\[
v_k \rightarrow v \quad \text{weakly star in } L^\infty(0, T; H) \cap L^2(0, T; V),
\]
\[
n_k \rightarrow n \quad \text{weakly star in } L^\infty(0, T; V) \cap L^\infty(0, T; L^\infty(\mathcal{T}^3)),
\]
\[
\Phi_k \rightarrow \Phi \quad \text{weakly star in } L^\infty(0, T; V) \cap L^\infty(0, T; L^\infty(\mathcal{T}^3)),
\]
\[
c_{p,k}, c_{m,k} \rightarrow c_p, c_m \quad \text{weakly star in } L^2(0, T; V) \cap L^\infty(0, T; L^\infty(\mathcal{T}^3)),
\]
\[
\nabla \Phi_k \rightarrow \nabla \Phi \quad \text{weakly star in } L^\infty(0, T; L^{p_M}(\mathcal{T}^3)),
\]
\[
\partial_t n_k, \Delta n_k, \Delta \mathcal{F}(n_k) \rightarrow n_t, \Delta n, \xi \quad \text{weakly in } L^{p_0}(0, T; L^{p_0}(\mathcal{T}^3)),
\]
where in deducing (4.3) we also used the normalization \((\Phi_k)_\Omega = 0\) and \(p_0, p_M\) are the exponents introduced in Lemma 3.5. Of course this implies in particular \(\Phi_\Omega = 0\). Let us notice that, in the limit, we preserve the boundedness conditions \(0 \leq c_p \leq \overline{c}\).
0 \leq c_m \leq \bar{c}, \ |n| \leq 1 \text{ almost everywhere in } \mathcal{T}^3. \text{ In addition to that, if } \varepsilon \text{ is sufficiently small (cf. Lemma 3.6), we also get:}
\begin{align}
\Phi_k, n_k \to \Phi, n \ & \text{ weakly in } L^2(0, T; H^2(\mathcal{T}^3)).
\end{align}
In the following we show how to treat the passing to the limit just for the most difficult terms. We first note that, by (3.22) and interpolation,
\begin{align}
\text{whence, using (3.34), there follows}
\|v_k\|_{L^4(0,T;L^3(\mathcal{T}^3))} \leq c,
\end{align}
whence, using (3.34), there follows
\begin{align}
\|v_k \cdot \nabla c_{p,k}\|_{L^{4/3}(0,T;L^{6/5}(\mathcal{T}^3))} + \|v_k \cdot \nabla c_{m,k}\|_{L^{4/3}(0,T;L^{6/5}(\mathcal{T}^3))} \leq c.
\end{align}
Then, using uniform boundedness of \(c_{p,k}, c_{m,k}\) as well as the bounds (3.25), (3.32) it is not difficult to deduce from (2.1), (2.2) that
\begin{align}
\|\partial_t c_{p,k}\|_{L^{4/3}(0,T;V')} + \|\partial_t c_{m,k}\|_{L^{4/3}(0,T;V')} \leq c.
\end{align}
Hence, taking also into account (3.45), the Aubin-Lions lemma with the uniform boundedness property gives
\begin{align}
c_{p,k}, c_{m,k}, n_k \to c_p, c_m, n \ & \text{ strongly in } L^q(0, T; L^q(\mathcal{T}^3)) \forall q \in [1, \infty).
\end{align}
Then, using (4.6), (4.11), the \(\lambda\)-monotonicity of \(\partial \mathcal{F}\), and the result [2, Prop. 1.1, p. 42], we get \(\xi = \partial \mathcal{F}(n)\). Moreover, by (4.3) and (4.4) we get
\begin{align}
\|c_{p,k} \nabla \Phi_k\|_{L^\infty(0,T;H)} + \|c_{m,k} \nabla \Phi_k\|_{L^\infty(0,T;H)} \leq c,
\end{align}
whence
\begin{align}
c_{p,k} \nabla \Phi_k \to c_p \nabla \Phi, \quad c_{m,k} \nabla \Phi_k \to c_m \nabla \Phi \ & \text{ weakly star in } L^\infty(0,T;H),
\end{align}
where we have used also (4.11). Then, interpolating between the spaces \(L^\infty(\mathcal{T}^3)\) and \(W^{2,p_0}(\mathcal{T}^3)\) at place 1/2 by means of the Gagliardo-Nirenberg inequality (cf. [23]), we deduce
\begin{align}
\|\nabla n_k\|_{L^{2p_0}(\mathcal{T}^3)} \leq c \|\Delta n_k\|_{L^{p_0}(\mathcal{T}^3)}^{1/2} \|n_k\|_{L^\infty(\mathcal{T}^3)}^{1/2} + c \|n_k\|_{L^{2}(\mathcal{T}^3)}^{1/2},
\end{align}
whence, taking the \((2p_0)\)-power and recalling (4.6) and the fact that \(|n_k| \leq 1\), we deduce that
\begin{align}
\|\nabla n_k \odot \nabla n_k\|_{L^{p_0}(0,T;L^{p_0}(\mathcal{T}^3))} \leq c,
\end{align}
where we recall that the exponent \(p_0 > 1\) is provided by relation (3.50) in Lemma 3.5 and its actual value cannot be determined explicitly because it depends on the unknown exponent \(p_M > 2\) coming from the use of Meyers’ argument. Finally, thanks also to the bound on \(\partial_t n_k\) in (4.6) and the Aubin-Lions lemma, we obtain the convergence
\begin{align}
\nabla n_k \odot \nabla n_k \to \nabla n \odot \nabla n \ & \text{ weakly in } L^{p_0}(0,T;L^{p_0}(\mathcal{T}^3)),
\end{align}
which is sufficient in order to conclude the passage to the limit as \(k \to \infty\) in order to obtain the claimed weak solutions.
5. Approximation scheme, global in time existence of weak solutions. We introduce an approximation scheme that is solvable and compatible with both the apriori bounds and the compactness arguments discussed above. Let \([n]_\omega, \omega > 0\) denote the standard regularization through a family of convolution kernels in the \(x\)–variable. Moreover, we introduce a sequence

\[ F_m \in C^1([0, \infty), F_m \text{ convex, } F_m(s) = F(s) \text{ for } s \leq 1 - \frac{1}{m}, F \text{ affine otherwise.} \]

Following [12], [13] we introduce a family of approximate problems:

\[
\begin{align*}
\frac{\partial c_p}{\partial t} + v \cdot \nabla c_p &= \text{div} \left( (\text{Id} + \varepsilon [n]_\omega \otimes [n]_\omega)(\nabla c_p + c_p \nabla \Phi) \right), \\
\frac{\partial c_m}{\partial t} + v \cdot \nabla c_m &= \text{div} \left( (\text{Id} + \varepsilon [n]_\omega \otimes [n]_\omega)(\nabla c_m - c_m \nabla \Phi) \right), \\
- \text{div} \left( (\text{Id} + \varepsilon [n]_\omega \otimes [n]_\omega)(\nabla \Phi) \right) &= c_p - c_m, \\
\partial_t v + v \cdot \nabla n - \Omega(v)n + D(v)n &= \Delta n + \varepsilon (\nabla \Phi \otimes \nabla \Phi) [n]_\omega - \frac{\partial \mathcal{F}_m(n)}{\partial n},
\end{align*}
\]

with the Galerkin approximation for the velocity field:

\[
\frac{d}{dt} \int_{\Omega} v \cdot \varphi \, dx - \int_{\Omega} \langle v \otimes v \rangle : \nabla \varphi \, dx = \int_{\Omega} [-\alpha_4 D(v) + (\nabla n \otimes \nabla n) : \nabla \varphi] \, dx
- \int_{\Omega} (\nabla \Phi \otimes \nabla \Phi)(\text{Id} + \varepsilon [n]_\omega \otimes [n]_\omega) : \nabla \varphi \, dx
- \int_{\Omega} (\alpha_1 (D(v)n \otimes n + \alpha_2 n \otimes n + \alpha_3 n \otimes \delta)) : \nabla \varphi \, dx
- \int_{\Omega} (\alpha_3 D(v)n \otimes n + \alpha_6 \circ D(v)n) : \nabla \varphi \, dx
- \omega \int_{\Omega} |v|^{q-2} v : \nabla \varphi \, dx, \quad q \in (3, 10/3),
\]

\(v \in C^1(\{0, T\}; X_N), \) for all \(\varphi \in X_N,\)

\(X_N = \text{span } \{ e_n \mid 1 \leq n \leq N \}, \ e_n \text{ orthonormal basis of the space } L^2(\Omega; R^3), \ \nabla \cdot e_n = 0. \)

Similarly to [12], the approximate solutions satisfying (5.1)–(5.5) can be obtained, first locally in time, by means of the standard Schauder fixed point argument. One notes that similarly as in Section 3 one will have cancellations of the worst terms, and for the terms which do not cancel exactly one will obtain lower-order, controllable reminders. For instance in order to obtain the most delicate cancellation \(\mathcal{A}_5 = \mathcal{B}_{51} + \mathcal{B}_{52}\) it suffices to multiply (5.4) by \(\partial_t [n]_\omega + v \cdot \nabla [n]_\omega,\) which gives rise to similar terms as \(\mathcal{A}, \mathcal{B}_{51}\) and \(\mathcal{B}_{52}\) but with \(n\) replaced by \([n]_\omega.\) This will also provide the terms \(\mathcal{A}_5 = \mathcal{B}_3.\) Of course, there will be other terms which will not cancel exactly but as mentioned they will cancel up to controllable reminders. In view of these considerations, we will obtain uniform bounds and thus the existence time can be extended to \([0, T].\) Note that positivity of the concentrations \(c_p, c_m\) follows from (5.1), (5.2), respectively, via the standard comparison principle.

Finally, adapting the arguments used in Section 4 to the approximation scheme (5.1)–(5.5), we may pass to the limit successively for

\[ m \to \infty, \ N \to \infty, \text{ and, finally } \omega \to 0 \]

to show the conclusion of Theorem 2.4, see [12] for details.
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