Analytical Result for Dimensionally Regularized Massless Master Double Box with One Leg off Shell

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Abstract

The dimensionally regularized massless double box Feynman diagram with powers of propagators equal to one, one leg off the mass shell, i.e. with non-zero $q^2 = p_i^2$, and three legs on shell, $p_i^2 = 0$, $i = 2, 3, 4$, is analytically calculated for general values of $q^2$ and the Mandelstam variables $s$ and $t$. An explicit result is expressed through (generalized) polylogarithms, up to the fourth order, dependent on rational combinations of $q^2, s$ and $t$, and a one-dimensional integral with a simple integrand consisting of logarithms and dilogarithms.

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1 Introduction

Massless four point Feynman diagrams contribute to many important physical amplitudes. They are much more complicated than two- and three-point diagrams because depend on many parameters: the Mandelstam variables $s$ and $t$ and the values of the external momenta squared, $p_i^2$, $i = 1, 2, 3, 4$. In the most general case, when all the legs are off the mass shell, $p_i^2 \neq 0$, there exists an explicit analytical result \cite{1} for the master (i.e. with powers of the propagators equal to one) double box diagram (see Fig. 1) strictly in four dimensions. Still no similar results are available for pure off shell four point diagrams with ultraviolet, infrared and/or collinear divergences.

In the opposite case, when all the end-points are on shell, i.e. for $p_i^2 = 0$, $i = 1, 2, 3, 4$, the problem of the analytical evaluation of such diagrams, in expansion in $\epsilon = (4 - d)/2$ in the framework of dimensional regularization \cite{2} with the space-time dimension $d$ as a regularization parameter, was completely solved during last year in \cite{3, 4, 5}. Among intermediate situations, when some legs are on shell and the rest of them off shell, the case of one leg off shell, $q^2 = p_1^2 \neq 0$ and three legs on shell is very important because of the relevance to the process $e^+e^- \rightarrow 3\text{jets}$ (see, e.g., \cite{6}). The purpose of this paper is to analytically evaluate the master double box diagram of such type, as a function of $q^2, s$ and $t$, and thereby demonstrate that the NNLO analytical calculations for this process are indeed possible.

One of the ways to evaluate the four point diagrams with one leg off shell is to expand them in the limit $q^2 \rightarrow 0$ and compute as many terms of the resulting expansion as possible. We explain how to do this, following the strategy of regions \cite{7, 8}, in the next section and present the leading power term in this expansion which provides a very non-trivial check of the subsequent analytical result.

To analytically evaluate the considered diagram we straightforwardly apply the method of ref. \cite{3}: we start from the alpha-representation of the double box and, after expanding some of the involved functions in Mellin–Barnes (MB) integrals, arrive at a six-fold MB integral representation with gamma functions in the integrand. Then we use a standard procedure of taking residues and shifting contours to resolve the structure of singularities in the parameter of dimensional regularization, $\epsilon$. This procedure leads to the appearance of multiple terms where Laurent expansion in $\epsilon$ becomes possible. Resulting integrals in all the MB parameters but the last two are evaluated explicitly in gamma functions and their derivatives. The last two-fold MB integral is evaluated by closing an initial integration contour in the complex plane to...
the right, with an explicit summation of the corresponding series. A final result is expressed through (generalized) polylogarithms dependent on rational combinations of $q^2, s$ and $t$ and a one-dimensional integral with a simple integrand consisting of logarithms and dilogarithms.

2 Expansion in the limit $q^2 \to 0$

The dimensionally regularized master massless double box Feynman integral with one leg off shell, $q^2 = p_1^2 \neq 0$, and three legs on shell, $p_i^2 = 0$, $i = 2, 3, 4$, can be written as

$$F(s, t, q^2; \epsilon) = \int \int \frac{d^d k d^d l}{(k^2 + 2p_1 k + q^2)(k^2 - 2p_2 k)k^2(k - l)^2} \times \frac{1}{(l^2 + 2p_1 l + q^2)(l^2 - 2p_2 l)(l + p_1 + p_3)^2},$$

(1)

where $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, and $k$ and $l$ are respectively loop momenta of the left and the right box. Usual prescriptions, $k^2 = k^2 + i0$, $s = s + i0$, etc are implied. To expand the given diagram in the limit $q^2 \to 0$ one can apply the so-called strategy of regions [7, 8] based on the analysis of various regions in the space of the loop integration momenta, Taylor expanding the integrand in the parameters that are considered small in the given region and extending resulting integrations to the whole integration domain in the loop momenta. When applying this strategy all integrals without scale are by definition put to zero.

Let us choose, for convenience, the external momenta as follows:

$$p_1 = \bar{p}_1 - \frac{q^2}{Q^2} \bar{p}_2, \quad p_2 = \bar{p}_2, \quad \bar{p}_{1,2} = (+Q/2, 0, 0, Q/2),$$

where $s = -Q^2$. The given limit $|q^2| \ll |s|, |t|$ is closely related to the Sudakov limit so that it is reasonable to consider each loop momentum to be one of the following types:

- **hard (h):** $k \sim Q \sim \sqrt{-t}$,
- **1-collinear (1c):** $k_+ \sim q^2/Q$, $k_- \sim Q$, $\tilde{k} \sim \sqrt{-q^2}$,
- **2-collinear (2c):** $k_+ \sim Q$, $k_- \sim q^2/Q$, $\tilde{k} \sim \sqrt{-q^2}$.

Here $k_\pm = k_0 \pm k_3$, $\tilde{k} = (k_1, k_2)$. We mean by $k \sim Q$, etc. that any component of $k_\mu$ is of order $Q$.

It turns out that the (h-h), (1c-h) and (1c-1c) are the only non-zero contributions to the leading power behaviour in the limit $q^2 \to 0$. Any term originating from the (h-h) contribution is given by the expansion of the integrand in Taylor series in $q^2$ and expressed through on-shell double boxes in shifted dimensions and can be analytically
evaluated by the algorithm presented in [4]. The (1c-1c) contribution is obtained by expanding propagators number 2, 4 and 7 in a special way. In particular, propagators number 2 and 4 are expanded, respectively, in $l^2$ and $k^2$. (See [8] for instructive 2-loop examples of expansions in limits of the Sudakov type.)

The (1c-h) and (1c-1c) contributions are evaluated with the help of a two-fold (respectively, one-fold) MB representation. Still this program of the evaluation of a large number of terms of the expansion looks very complicated because one needs, for phenomenological reasons, the values of $q^2$ greater than $s$ and $t$ so that a reliable summation of a resulting series, using Padé approximants, requires the knowledge of at least first 20–30 terms. Such a great number of terms can be hardly evaluated since a lot of irreducible structures appear. This asymptotic expansion is however very useful for comparison with the explicit result derived below.

The leading power terms of the asymptotic expansion calculated in expansion in $\epsilon$, up to a finite part, are

$$F(s, t, q^2; \epsilon) = \left(\frac{i\pi^{d/2}e^{-\gamma_E}}{(-s)^{2+2\epsilon}}\right)^{2} \sum_{i=0}^{4} \frac{g_i(X, Y)}{\epsilon^i} + O(q^2 \ln^3(q^2/s)) + O(\epsilon), \quad (2)$$

where $X = q^2/s$, $Y = t/s$ and

$\begin{align*}
g_4(X, Y) &= -1, \\
g_3(X, Y) &= -2(\ln X - \ln Y), \\
g_2(X, Y) &= \frac{11\pi^2}{12} + 3 \ln X \ln Y - \frac{3}{2} \ln^2 Y, \\
g_1(X, Y) &= 2 \ln Y \text{Li}_2(-Y) - 2 \text{Li}_3(-Y) + \frac{2}{3} \ln^3 X - \frac{3}{2} \ln^2 X \ln Y - \frac{1}{2} \ln X \ln^2 Y \\
&\quad - \frac{1}{6} \ln^3 Y + \ln^2 Y \ln(1 + Y) + \pi^2 \left[\frac{3}{2} \ln X - \frac{19}{6} \ln Y + \ln(1 + Y)\right] + \frac{49\zeta(3)}{6}, \\
g_0(X, Y) &= 26 \text{Li}_4(-Y) - 2S_{2,2}(-Y) - 2(\ln X + 6 \ln Y + \ln(1 + Y)) \text{Li}_3(-Y) \\
&\quad + 2 \ln Y \text{Li}_3\left(\frac{Y}{1 + Y}\right) + (\ln^2 Y + 2 \ln X \ln Y + 4\pi^2) \text{Li}_2(-Y) \\
&\quad - \frac{1}{2} \ln^4 X + \frac{1}{2} \ln^3 X \ln Y + \frac{1}{4} \ln^2 X \ln^2 Y - \frac{1}{2} \ln X \ln^3 Y + \frac{7}{8} \ln^4 Y \\
&\quad + \ln(1 + Y) \left[\ln X \ln^2 Y - \frac{5}{3} \ln^3 Y + \frac{1}{2} \ln^2 Y \ln(1 + Y) - \frac{1}{3} \ln Y \ln^2(1 + Y)\right] \\
&\quad + \pi^2 \left[-\frac{2}{3} \ln^2 X - \frac{7}{3} \ln X \ln Y + \frac{25}{6} \ln^2 Y + \ln X \ln(1 + Y) - 2 \ln Y \ln(1 + Y) \\
&\quad + \frac{1}{2} \ln^2(1 + Y)\right] + \zeta(3) \left[\frac{19}{3} \ln X - \frac{34}{3} \ln Y + 2 \ln(1 + Y)\right] + \frac{83\pi^4}{180}. \quad (3)\end{align*}$

Here $\text{Li}_a(z)$ is the polylogarithm [9] and

$$S_{a,b}(z) = \frac{(-1)^{a+b-1}}{(a-1)!b!} \int_{0}^{1} \frac{\ln^{a-1}(t) \ln^{b}(1-zt)}{t} \, dt \quad (4)$$
the generalized polylogarithm [10].

3 From alpha parameters through MB representation to analytical result

The alpha representation of the double box looks like:

$$F(s, t, q^2; \epsilon) = -\Gamma(3 + 2\epsilon) \left( i\pi^{d/2} \right)^2 \int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_7 \delta \left( \sum \alpha_i - 1 \right) D^{1 + 3\epsilon} A^{-3 - 2\epsilon},$$

where

$$D = (\alpha_1 + \alpha_2 + \alpha_7)(\alpha_3 + \alpha_4 + \alpha_5) + \alpha_6(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7),$$

$$A = [\alpha_1\alpha_2(\alpha_3 + \alpha_4 + \alpha_5) + \alpha_3\alpha_4(\alpha_1 + \alpha_2 + \alpha_7) + \alpha_6(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4)](-s) + \alpha_5\alpha_6\alpha_7(-t) + \alpha_5[(\alpha_1 + \alpha_3)\alpha_6 + \alpha_3(\alpha_1 + \alpha_2 + \alpha_7)](-q^2).$$

As it is well-known, one can choose a sum of an arbitrary subset of $\alpha_i, i = 1, \ldots, 7$ in the argument of the delta function in (5), and we use the same choice as in [5].

Starting from (5) we perform the same change of variables as in [3] and apply seven times the MB representation

$$\frac{1}{(X + Y)^\nu} = \frac{1}{\Gamma(\nu)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dw \frac{Y^w}{X^w} \Gamma(\nu + w) \Gamma(-w)$$

in order to separate terms in the functions involved to make possible an explicit parametric integration. The two extra MB integrations arise form the extra term with $q^2$. After such integrations we are left with a 7-fold MB integral of a ratio of gamma functions. Fortunately, one of the integrations can be explicitly taken using the first Barnes lemma and we arrive at the following nice 6-fold MB integral:

$$F(s, t, q^2; \epsilon) = -\left( i\pi^{d/2} \right)^2 \Gamma(-1 - 3\epsilon)(-s)^{3 + 2\epsilon} (2\pi i)^6 \int dv dw dz_1 \left( \frac{q^2}{s} \right)^v \left( \frac{t}{s} \right)^w$$

$$\times \Gamma(1 + w) \Gamma(1 + v + w) \Gamma(-v) \Gamma(-w) \Gamma(1 - w_3 + v)$$

$$\times \Gamma(w_2) \Gamma(-1 - 2\epsilon - w - w_2) \Gamma(w_3 - v) \Gamma(-1 - 2\epsilon - w - w_3)$$

$$\times \Gamma(1 - w_2 + z_1) \Gamma(1 - w_3 + z_1) \Gamma(\epsilon + w + w_2 + w_3 - z_1) \Gamma(-z_1)$$

$$\times \Gamma(1 + w + w_2 + w_3) \Gamma(-1 - 4\epsilon - w - w_2 - w_3) \Gamma(1 - w_3)$$

$$\times \Gamma(1 - \epsilon + z) \Gamma(2 + 2\epsilon + w + w_2 + z - z_1) \Gamma(2 + 2\epsilon + w + w_3 + z - z_1)$$

$$\times \frac{(-2 - 3\epsilon - w - w_2 - w_3 + z_1 - z) \Gamma(z_1 - z)}{\Gamma(3 + 2\epsilon + w + z)}.$$

It differs from its analog for $q^2 = 0$ by the additional integration in $v$. This variable enters only four gamma functions in the integrand. The integral is evaluated in
expansion in $\epsilon$, up to a finite part, by resolving singularities in $\epsilon$ absolutely by the same strategy as in the case $q^2 = 0$ [3]. Note that the infrared and collinear poles are a little bit softer than in the pure on-shell case, the integration variable $v$ playing the role of an infrared regulator. The two key gamma functions that are responsible for the generation of poles in $\epsilon$ are the same as in the previous case:

$$\Gamma(\epsilon + w + w_2 + w_3 - z_1) \Gamma(-2 - 3\epsilon - w - w_2 - w_3 + z_1 - z).$$

The labeling of resulting terms is therefore the same: the initial integral is decomposed as $J = J_{00} + J_{01} + J_{10} + J_{11}$, etc. (Only arguments of some gamma functions are shifted by $v$.) The applied strategy makes it possible to perform all the integrations apart from the last two, in $v$ and $w$. We obtain four groups of terms with 26 terms in each group: the terms without MB integration, with MB integration in $v$ or $w$ and, finally, with a two-fold integration in $v$ and $w$. The one-fold integrals are explicitly evaluated by closing contour and summing up series, using formulae from [11].

The contribution of the resulting two-fold MB integral takes the form

$$2 \left( \frac{i \pi^{d/2}}{(-s^3)^{d/2}} \right) \int \frac{dv dw}{1 + w} \left( \frac{q^2}{s} \right)^{v} \left( \frac{t}{s} \right)^{w} \left( \frac{1}{s} \right)^{v} \Gamma(1 + v + w) \Gamma(-v) \Gamma(1 + w) \Gamma(-w)^2$$

$$\times \left[ \Gamma(1 + v + w) \Gamma(-v - w) \left( \frac{1}{\epsilon - \gamma_E} - 2 \ln(-s) - \frac{5}{1 + w} - \frac{1}{1 + v + w} \right) \right.$$

$$\left. + \psi(1 + v) - 2\psi(-v - w) - 3\psi(-w) + 2\psi(1 + w) + \psi(1 + v + w) \right]$$

$$\left. - \Gamma(1 + v) \Gamma(-v) \Gamma(1 + w) \Gamma(-w) \right].$$

The integration contours are straight lines along imaginary axes with $-1 < \text{Re } v, \text{Re } w, \text{Re } v + w < 0$. By closing contours it is possible to convert this integral into a two-fold series where each term is identified as a derivative of the Appell function $F_2$ in parameters, up to the third order. The $1/\epsilon$ part is then explicitly summed up with a result in terms of polylogarithms. (In fact, it is proportional to the $\epsilon$ part of the master one-loop box.)

The so obtained result can be transformed into a one-dimensional integral with a simple integrand. To present the final result let us turn to the variables $x = s/q^2$ and $y = t/q^2$ keeping in mind typical phenomenological values of the involved parameters relevant to the process $e^+ e^- \rightarrow 3\text{jets}$:

$$F(s, t, q^2; \epsilon) = \frac{\left( i \pi^{d/2} e^{-\gamma_E \epsilon} \right)^2}{-s^2 t(-q^2)^2} \sum_{i=0}^{4} \frac{f_i(x, y)}{\epsilon^i} + O(\epsilon).$$

We obtain

$$f_4(x, y) = -1,$$

$$f_3(x, y) = 2(\ln x + \ln y),$$

(12)

(13)
\[ f_2(x, y) = 3 \text{Li}_2(x) + \text{Li}_2(y) - 2(\ln x + \ln y)^2 \]
\[ + 3 \ln(1 - x) \ln x + \ln(1 - y) \ln y - \frac{5\pi^2}{12}, \]
(14)
\[ f_1(x, y) = 2 \left[ \text{Li}_3 \left( \frac{-x}{1-x-y} \right) + \text{Li}_3 \left( \frac{-y}{1-x-y} \right) - \text{Li}_3 \left( \frac{-xy}{1-x-y} \right) \right] \]
\[ - \ln x \text{Li}_2 \left( \frac{y}{1-x} \right) - \ln y \text{Li}_2 \left( \frac{x}{1-y} \right) + 2 \ln(1-x-y) \]
\[ \times \left[ -\frac{1}{6} \left( \ln^2(1-x-y) + \pi^2 \right) + \ln(1-x) \ln x + \ln(1-y) \ln y - \ln x \ln y \right] \]
\[ + 3 \text{Li}_3(x) - 8 \text{Li}_3(y) + 4 \text{Li}_3 \left( \frac{-x}{1-x} \right) - 2 \text{Li}_3 \left( \frac{-y}{1-y} \right) \]
\[ - (3 \ln x + 4 \ln y) \text{Li}_2(x) + 3 \ln y \text{Li}_2(y) + \frac{4}{3} \ln^3 x - \frac{2}{3} \ln^3(1-x) + \ln^2(1-x) \ln x \]
\[ - \frac{9}{2} \ln(1-x) \ln^2 x + \pi^2 \frac{2}{6} \left( 5 \ln x - 4 \ln(1-x) \right) + \frac{4}{3} \ln^3 y + \frac{1}{3} \ln^3(1-y) \]
\[ - 2 \ln^2(1-y) \ln y - \ln(1-y) \ln^2 y + \pi^2 \frac{6}{6} \left( 5 \ln y + 2 \ln(1-y) \right) \]
\[ + 4 \ln x \ln y \left( \ln x - \ln(1-x) + \ln y \right) + \frac{25\zeta(3)}{6}. \]
(15)

The \( \epsilon^0 \) part involves a one-dimensional integral:

\[ f_0(x, y) = \int_0^1 dz \left\{ z^{-1} \ln(1-z) (4 \ln^2(1-x-yz) - \ln^2(1-y-xz)) \right\} \]
\[ - \frac{4y}{1-x-yz} \left[ \ln(1-yz) (\ln(1-z) \ln(1-yz) - 2 \text{Li}_2(z)) \right] \]
\[ - 2 \left( \ln(1-z) - \ln z \right) \text{Li}_2 \left( -(1-x-yz)/x \right) \]
\[ - \frac{x}{1-y-xz} \left[ \ln(1-xz) \left( 3 \ln^2(1-z) - 6 \ln(1-z) \ln(1-xz) + 2 \text{Li}_2(z) \right) \right] \]
\[ + 2 \left( 6 \ln(1-z) - \ln z \right) \text{Li}_2 \left( -(1-y-xz)/y \right) \} \}
\[ - 5 \text{Li}_4(x) + 14 \text{Li}_4 \left( \frac{x}{1-y} \right) - 2 \text{Li}_4 \left( \frac{-x}{1-x} \right) - 6 \text{Li}_4 \left( \frac{xy}{(1-x)(1-y)} \right) \]
\[ + 8 \text{Li}_4 \left( \frac{-x}{1-x-y} \right) + 24 \text{Li}_4(y) - 2 \text{Li}_4(1-y) + 8 \text{Li}_4 \left( \frac{-y}{1-y} \right) - 2 \text{Li}_4 \left( \frac{y}{1-x} \right) \]
\[ - 8 \text{Li}_4(1-x) - 8 \text{Li}_4 \left( \frac{-y}{1-x-y} \right) - 20 \text{Li}_4 \left( \frac{1-x-y}{1-y} \right) + 10 \text{Li}_4 \left( \frac{1-x-y}{1-x} \right) \]
\[ - 3S_{2,2}(x) - 8S_{2,2}(y) - 6S_{2,2} \left( \frac{x}{1-y} \right) + (2 \ln y - 2 \ln x - 3 \ln(1-x)) \text{Li}_3(x) \]
\[ + 2(16 \ln(1-y) - 11 \ln y - 2(1-x-y) + \ln x) \text{Li}_3 \left( \frac{x}{1-y} \right) \]
\[-(8 \ln y + 2 \ln(1 - x) + 3 \ln x) \operatorname{Li}_3 \left( \frac{-x}{1 - x} \right)\] 
\[-2(4 \ln(1 - y) - 4 \ln y - \ln(1 - x) + \ln x) \operatorname{Li}_3 \left( \frac{xy}{(1-x)(1-y)} \right)\] 
\[+(14 \ln(1 - y) - 18 \ln y + 4 \ln(1 - x - y)) \operatorname{Li}_3 \left( \frac{-x}{1 - x - y} \right) + 2 \ln y \operatorname{Li}_3 \left( \frac{-xy}{1 - x - y} \right)\] 
\[+(7 \ln y - 8 \ln(1 - y)) \operatorname{Li}_3(y) + (8 \ln(1 - y) + \ln y + 2 \ln x) \operatorname{Li}_3 \left( \frac{-y}{1 - y} \right)\] 
\[-4(2 \ln y + 7 \ln(1 - x) + 2 \ln(1 - x - y) - 8 \ln x) \operatorname{Li}_3 \left( \frac{y}{1 - x} \right)\] 
\[-2(\ln y + 5 \ln(1 - x) + 8 \ln(1 - x - y) - 7 \ln x) \operatorname{Li}_3 \left( \frac{-y}{1 - x - y} \right)\] 
\[-\frac{1}{2} \left( \operatorname{Li}_2(x) \right)^2 - \left( \operatorname{Li}_2 \left( \frac{x}{1 - y} \right) \right)^2 - \frac{3}{2} \left( \operatorname{Li}_2(y) \right)^2 + 4 \left( \operatorname{Li}_2 \left( \frac{y}{1 - x} \right) \right)^2\] 
\[+ \left[ \ln^2(1 - x) - 4 \ln^2 y + 2 \ln y \left( 4 \ln(1 - x) - \ln x \right) - 2 \ln y \ln x - 3 \ln(1 - x) \ln x\right.\] 
\[+ \frac{7}{2} \ln^2 x + \frac{3}{2} \ln^2 y\left. \right] \operatorname{Li}_2(x) + \left[ 12 \ln^2(1 - y) + 15 \ln^2 y + 2 \ln y \ln(1 - x - y) + \ln x\right.\] 
\[+ 9 \ln y + 2 \ln y \ln(1 - x - y) - 4 \ln(1 - x) + 4 \ln x\] 
\[-2(\ln^2(1 - x - y) + \ln^2 x) \operatorname{Li}_2 \left( \frac{x}{1 - y} \right) + \left[ -4 \ln^2(1 - y) - 11 \ln^2 y\right.\] 
\[+ 2 \ln y \left( 4 \ln(1 - x) - 3 \ln x \right) + \ln(1 - y) \left( 5 \ln y - 2 \ln x + \ln^2 x - \frac{\pi^2}{3} \right) \operatorname{Li}_2(y)\] 
\[+ \left[ 8 \ln^2 y - 8 \ln y \ln(1 - x) - 10 \ln^2(1 - x) + 8 \ln^2(1 - x - y) - 8 \ln y \ln(1 - x - y) \right.\] 
\[-8 \ln(1 - x) \ln(1 - x - y) - 2 \ln x) + 2 \ln(1 - y) \ln x + \ln^2 x - \frac{2 \pi^2}{3} \right] \operatorname{Li}_2 \left( \frac{y}{1 - x} \right)\] 
\[\left. + \left[ \ln^2(1 - x) - 4 \ln^2(1 - y) - 8 \ln^2 y + 2 \ln y \left( 4 \ln(1 - x) - 3 \ln x\right.\right.\] 
\[+ 2 \ln(1 - y) \left( 4 \ln y - \ln x \right) - 2 \ln(1 - x) \ln x + 2 \ln^2 x \operatorname{Li}_2 \left( \frac{xy}{(1-x)(1-y)} \right)\] 
\[+ 2 \ln^4(1 - x - y) + \frac{1}{3} \ln^3(1 - x - y) \left[ 2 \ln y - 3 \ln(1 - y) - 9 \ln x - 11 \ln(1 - x) \right]\] 
\[+ \ln^2(1 - x - y) \left[ \pi^2 - 3 \ln^2(1 - y) + 3 \ln^2 y + 6 \ln^2(1 - x) - \ln(1 - y) \ln y - 10 \ln x\right.\] 
\[+ 4 \ln(1 - x) \ln x - 2 \ln^2 x - \ln y \left( 5 \ln(1 - x) + \ln x \right) \] 
\[+ \frac{1}{3} \ln(1 - x - y) \left[ 7 \ln^3(1 - y)\right.\] 
\[\left. - 2 \ln(1 - y) \left( 5 \pi^2 + 12 \ln^2 y + 7 \pi^2 \ln(1 - x) + \pi^2 \ln x + 6 \ln^2 y \ln x - 4 \ln^3(1 - x) \right.\right.\] 
\[+ 15 \ln^2(1 - y) \left( \ln y - 2 \ln x \right) - 21 \ln^2(1 - x) \ln x + 3 \ln(1 - x) \ln^2 x\] 
\[+ \ln y \left( 2 \pi^2 + 9 \ln^2 x + 15 \ln^2(1 - x) - 6 \ln(1 - x) \ln x \right) \]
\[-\frac{5}{6} \ln^4(1-x) - \frac{2}{3} \ln^4 x + \frac{23}{6} \ln^3(1-x) \ln x - \frac{17}{4} \ln^2(1-x) \ln^2 x + \frac{7}{2} \ln(1-x) \ln^3 x\]

\[-\frac{\pi^2}{6} \left(3 \ln^2(1-x) - 10 \ln(1-x) \ln x + 5 \ln^2 x\right) - \ln^4(1-y) - \frac{2}{3} \ln^4 y\]

\[-\frac{19}{6} \ln^3(1-y) \ln y + 5 \ln^2(1-y) \ln^2 y + \frac{2}{3} \ln(1-y) \ln^3 y\]

\[+\frac{\pi^2}{6} \left(9 \ln^2(1-y) - \ln(1-y) \ln y - 5 \ln^2 y\right)\]

\[+\frac{1}{3} \ln(1-x) \ln(1-y)(\ln^2(1-x) - 4 \ln^2(1-y)) - \frac{8}{3}(\ln^2 x + \ln^2 y) \ln x \ln y\]

\[+3 \ln^3(1-y) \ln x + \ln^2(1-y) \ln y(4 \ln(1-x) + \ln x) - 2 \ln(1-x) \ln x\]

\[+\frac{1}{3} \ln y \left[-\ln^3(1-x) - 9 \ln^2(1-x) \ln x - 12 \ln y \ln^2 x\right]\]

\[+6 \ln(1-x) \ln x(2 \ln y + 3 \ln x) - \ln(1-y) \left[8 \ln^2 y \ln(1-x)\right]\]

\[+\ln(1-x) \ln x(\ln(1-x) - 2 \ln x) + \ln y(-8 \ln^2(1-x) + 6 \ln(1-x) \ln x + \ln^2 x)\]

\[+\frac{\pi^2}{3} \ln y(4 \ln(1-x) - 5 \ln x) - \ln(1-y) \ln x\]

\[+\zeta(3) \left[12(\ln(1-x) - y) - \ln(1-y)) + 13 \ln(1-x) - \frac{25}{3}(\ln x + \ln y)\right] + \frac{23\pi^4}{180}. \quad (16)\]

One may hope that the one-dimensional integral that is left can also be evaluated in terms of polylogarithms. To do this it is necessary to complete the table of integrals derived in [4].

This result is in agreement with the leading power behaviour when \(q^2 \to 0\) [5]. When performing this comparison it is reasonable to start with (10), take minus residue at \(v = 0\) (the first pole of \(\Gamma(-v)\)), integrate in \(w\) by closing the contour to the right, and take into account the three other contributions (without MB integration, and with integration in \(v\) or \(w\)) that were not presented above. Eqs. (12–16) also agree with results based on numerical integration in the space of alpha parameters [12] (where the 1% accuracy for the \(1/\epsilon\) and \(\epsilon^0\) parts is guaranteed).

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