TRAVELING WAVE FRONTS IN A DIFFUSIVE AND COMPETITIVE LOTKA-VOLTERRA SYSTEM

ZENQI D*, SHULING YAN AND KAIGE ZHUANG
School of Mathematics and Statistics, Jiangsu Normal University
Xuzhou, Jiangsu 221116, China

ABSTRACT. In this paper, we consider a two-species competitive and diffusive system with nonlocal delays. We investigate the existence of traveling wave fronts of the system by employing linear chain techniques and geometric singular perturbation theory. The existence of the traveling wave fronts analogous to a bistable wavefront for a single species is proved by transforming the system with nonlocal delays to a six-dimensional system without delay.

1. Introduction. Nonlinear phenomena is one of the most common phenomena in the nature. As we known, it is a common phenomenon that many species compete for limited resources. Therefore, it is very important to study multi-species competition models. One of the famous models for population dynamics is Lotka-Volterra competitive system which has received great attention [15]. Since the individuals may be at the different location in their history, in population dynamics, time delay and spatial dispersal or migration from one location to another seem to be inevitable. Britton [2] considered comprehensively these two factors and introduced the so-called spatio-temporal delay or nonlocal delay. Nowadays, models with spatio-temporal delay or nonlocal delay attracted much attention due to its significant sense in mathematical theory and practical fields [9, 10, 20, 22]. There are some methods which have been used to prove the existence of traveling wave solutions [1, 24]. Lv and Wang [20] studied the traveling wave solutions of the following competitive and diffusive Lotka-Volterra system with discrete delays

\[
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t) \left[1 - a_1 u_1(x,t - \tau_1) - b_1 u_2(x,t - \tau_2)\right], \\
\frac{\partial u_2(x,t)}{\partial t} &= d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t) \left[1 - a_2 u_1(x,t - \tau_3) - b_2 u_1(x,t - \tau_4)\right],
\end{align*}
\]

(1)

by using the upper-lower solution technique and the monotone iteration scheme, they showed that for every \( c > c^* \), system (1) has a traveling wave fronts with speed \( c \) that connects \((0, 1)\) and \((1, 0)\). Provided that \( \tau_1 \) and \( \tau_4 \) are sufficiently small. Huang and Zou [12] considered the following cooperative Lotka-Volterra

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* The corresponding author.
system with delays
\[
\begin{cases}
\frac{\partial u_1(x,t)}{\partial t} = d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t) \left[ 1 - a_1 u_1(x,t - \tau_1) + b_1 u_2(x,t - \tau_2) \right], \\
\frac{\partial u_2(x,t)}{\partial t} = d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t) \left[ 1 + b_2 u_1(x,t - \tau_3) - a_2 u_2(x,t - \tau_4) \right].
\end{cases}
\] (2)

Since system (1) is a competitive system and system (2) is a cooperative system, they need to construct different upper and lower solutions to prove the existence of traveling wave solutions.

Gourley and Ruan [9] have studied the traveling wave solutions of the following competition diffusion system with nonlocal delays
\[
\begin{cases}
\frac{\partial u_1(x,t)}{\partial t} = d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t) \left[ 1 - a_1 u_1(x,t) - b_1 (g_1 * u_2)(x,t) \right], \\
\frac{\partial u_2(x,t)}{\partial t} = d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t) \left[ 1 - a_2 u_2(x,t) - b_2 (g_2 * u_1)(x,t) \right],
\end{cases}
\] (3)

where \((g_1 * u_2)(x,t)\) and \((g_2 * u_1)(x,t)\) are defined by
\[
\begin{cases}
(g_1 * u_2)(x,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{t} g_1(x-y,t-s)u_2(y,s)dsdy, \\
(g_2 * u_1)(x,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{t} g_2(x-y,t-s)u_1(y,s)dsdy,
\end{cases}
\]

with
\[
g_1(y,s) = \frac{1}{\tau_1} e^{-\frac{s}{\tau_1}} \frac{1}{\sqrt{4d_2 \pi s}} e^{-\frac{y^2}{4d_2 s}},
\]
and
\[
g_2(y,s) = \frac{1}{\tau_2} e^{-\frac{s}{\tau_2}} \frac{1}{\sqrt{4d_1 \pi s}} e^{-\frac{y^2}{4d_1 s}},
\]
in which the kernel functions \(g_1, g_2\) are the so-called weak kernel. By employing linear chain techniques and geometric singular perturbation theory, they investigated the existence of traveling wave fronts solutions for the competitive system (3).

Geometric singular perturbation theory based on previous work by Fenichel [7] has received a great deal of interest and has been used by many researchers to obtain the traveling wave fronts of different equations [4, 5, 13, 11, 21, 25, 26]. Li and Zhu [17] studied the limit cycles bifurcated from the limit periodic sets of the predator-prey systems with Holling types response functions, as well as the multiplicity of such limit cycles by applying the geometric singular perturbation theory developed by Dumortier, Roussarie and Maesschalck [3, 6]. Du et al. [4, 5] used geometric singular perturbation theory to investigate the existence of solitary wave and traveling waves solutions of delayed Camassa-Holm equations and nonlinear Belousov-Zhabotinskii systems, respectively. Liu [19] considered a class of three-dimensional singularly perturbed predator-prey systems having two predators competing exploitatively for the same prey in a constant environment. By using dynamical systems techniques and the geometric singular perturbation theory, they gave precise conditions which guarantee the existence of stable relaxation oscillations. Hek [11] explained and explored geometric singular perturbation theory and its use in biological practice.
In this paper, we consider the diffusive and competitive Lotka-Volterra system with nonlocal delays as follows:

\[
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t) \left[ 1 - a_1(g_1 \ast u_1)(x,t) - b_1(g_2 \ast u_2)(x,t) \right], \\
\frac{\partial u_2(x,t)}{\partial t} &= d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t) \left[ 1 - b_2(g_3 \ast u_1)(x,t) - a_2(g_4 \ast u_2)(x,t) \right],
\end{align*}
\]

(4)

where \( r_1, r_2, a_1, a_2, b_1, b_2 \) are positive constants and

\[
(g_j \ast u_i)(x,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{t} g_j(x-y, t-s) u_i(y,s) ds dy,
\]

\( j = 1, 2, 3, 4, i = 1, 2 \), where

\[
g_j(y,s) = \frac{1}{\tau_j} e^{-\frac{s}{\tau_j}} \int_{-\infty}^{t} e^{-\frac{y^2}{4d_j s}} dy, j = 1, 2,
\]

and

\[
g_j(y,s) = \frac{1}{\tau_j} e^{-\frac{s}{\tau_j}} \int_{-\infty}^{t} e^{-\frac{y^2}{4d_{j-2} s}} dy, j = 3, 4.
\]

We will investigate the existence of traveling wave solutions of system (4) by employing linear chain techniques and geometric singular perturbation theory.

It is necessary to point out that, when the parameters are taken as different values, some celebrated equations can be derived from system (4), such as, when \( g_j(t,x) = \delta(t-\tau_j) \delta(x) \), where \( \delta \) denotes Dirac’s delate function, system (4) reduces to system (1). When \( g_j(t,x) = \delta(t) \delta(x) \), then system (4) becomes the diffusive and competitive model

\[
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t) \left[ 1 - a_1 u_1(x,t) - b_1 u_2(x,t) \right], \\
\frac{\partial u_2(x,t)}{\partial t} &= d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t) \left[ 1 - a_2 u_2(x,t) - b_2 u_1(x,t) \right],
\end{align*}
\]

(5)

which was studied by Wang and Li [24]. By constructing a pair of upper-lower solutions and using monotone iterative techniques, they obtained the existence of a traveling wave solution connecting two steady states. When parameters \( b_1 \) and \( b_2 \) are varied into \(-b_1\) and \(-b_2\), the competitive system (4) will become the following cooperative Lotka-Volterra system with nonlocal delays

\[
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t) \left[ 1 - a_1(g_1 \ast u_1)(x,t) + b_1(g_2 \ast u_2)(x,t) \right], \\
\frac{\partial u_2(x,t)}{\partial t} &= d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t) \left[ 1 + b_2(g_3 \ast u_1)(x,t) - a_2(g_4 \ast u_2)(x,t) \right],
\end{align*}
\]

(6)

which considered by Li and Wang [16]. Systems (4) and (6) are different models, since (6) is a cooperative system and (4) is a competitive system. Recently, Han et al. [10] studied the traveling wave solutions of system (4) by using different methods, respectively. They proved that there exists traveling wave solutions of the system connecting equilibrium \((0,0)\) to some unknown positive steady state for wave speed \( c > c^* \) where \( d = \frac{d_2}{d_1} \) and \( r = \frac{r_2}{r_1} \). We not only obtain the existence of traveling wave solutions, but also establish the existence results of monotone and bistable wavefronts for system (4).
In this paper, we will consider the traveling wave solution of the competitive system (4) with nonlocal delays provided that delays are sufficiently small by employing geometric singular perturbation theory. The rest of this paper is organized as follows. In section 2, we introduce geometric singular perturbation theory which is important to obtain our main results. In section 3, based on the relation between traveling wave and heteroclinic orbit of the equilibria for the associated ordinary differential equations, we establish the existence of traveling wave fronts of system (4) by using geometric singular perturbation theory. In section 4, we obtain the existence of bistable wavefronts of system (4) by using the spectral method which is similar to [18].

2. Preliminaries. In the section, in order to present our results on the existence of traveling wave solutions, we first introduce the geometric singular perturbation theory and some definitions.

Lemma 1. ([7, 13]) For the system

\[
\begin{aligned}
x'(t) &= f(x, y, \varepsilon), \\
y'(t) &= \varepsilon g(x, y, \varepsilon),
\end{aligned}
\]

where \( x \in \mathbb{R}^n, y \in \mathbb{R}^l \) with \( n, l \geq 1 \) in general and \( \varepsilon \) is a real parameter, \( f, g \) are \( C^\infty \) on the set \( V \times I \) where \( V \in \mathbb{R}^{n+l} \) and \( I \) is an open interval containing 0. If when \( \varepsilon = 0 \), the system has a compact, normally hyperbolic manifold of critical points \( M_0 \), which is contained in the set \( \{ f(x, y, 0) = 0 \} \). Then for any \( 0 < r < +\infty \), if \( \varepsilon > 0 \), but sufficiently small, there exists a manifold \( M_\varepsilon \), satisfying

(i): which is locally invariant under the flow of (7);
(ii): which is \( C^r \) in \( x, y \) and \( \varepsilon \);
(iii): \( M_\varepsilon = \{(x, y) : x = h^\varepsilon(y)\} \) for some \( C^r \) function \( h^\varepsilon(y) \) and \( y \) in some compact \( K \);
(iv): there exist locally invariant stable and unstable manifolds \( W^s(M_\varepsilon) \) and \( W^u(M_\varepsilon) \) that lie within \( o(\varepsilon) \), and are diffeomorphic to \( W^s(M_0) \) and \( W^u(M_0) \).

Definition 1. ([13]) The manifold \( M_0 \) is said to be normally hyperbolic if the linearization of system (7) at each point in \( M_0 \) has exactly \( l \) eigenvalues on the imaginary axis \( Re(\lambda) = 0 \).

Definition 2. ([13]) A set \( M \) is locally invariant under the flow from (7) if it has neighborhood \( V \) so that no trajectory can leave \( M \) without also leaving \( V \). In other words, it is locally invariant if for all \( x \in M \), \( x: [0, t] \subset V \) implies that \( x: [0, t] \subset M \), similarly with [0, t] replaced by \([t, 0]\) when \( t < 0 \), where \( x: t \) denotes the application of a flow after time \( t \) to the initial condition \( x \).

Definition 3. ([18]) A traveling wave solution of (4) is a solution \( u(x, t) = \Phi(x + ct) \) for some wave speed \( c \in \mathbb{R} \) and wave profile \( \Phi(t) \in C^2(\mathbb{R}, \mathbb{R}^N) \). In addition, if \( \Phi(t) \) is monotone in \( t \in \mathbb{R} \), then it is called a traveling wavefront solution.

3. Existence of traveling fronts solutions. Notice that system (4) has a trivial equilibrium \( E_0 = (0, 0) \), two semitrivial spatially homogeneous equilibria \( E_1 = (\frac{1}{a_1}, 0), E_2 = (0, \frac{1}{a_2}) \) and a positive spatially homogeneous equilibrium \( E^* = (k_1, k_2) \), where

\[
k_1 = \frac{a_2 - b_1}{a_1 a_2 - b_1 b_2}, \quad k_2 = \frac{a_1 - b_2}{a_1 a_2 - b_1 b_2},
\]
provided that \( a_1 a_2 \neq b_1 b_2 \) and \( \text{sgn}(a_1 a_2 - b_1 b_2) = \text{sgn}(a_2 - b_1) = \text{sgn}(a_1 - b_2) \).

The stability of the semi-trivial equilibrium \( E_i \) means that the \( i \)-th competitor \((i = 1, 2)\) wins the competition. Since the two semi-trivial steady-state solutions possible transition to each other, which has aroused the research interest of many scholars. In fact, in this paper, we shall prove that when the coexistence equilibrium \( E^* \) is absent, a transition can occur between \( E_1 \) and \( E_2 \) in the form of a traveling wave fronts solution.

In order to prove the existence of traveling wave fronts, we need to introduce some new variables.

Let
\[
\begin{align*}
    w_1(x, t) &= (g_1 * u_1)(x, t), \quad w_2(x, t) = (g_2 * u_2)(x, t), \\
    w_3(x, t) &= (g_3 * u_1)(x, t), \quad w_4(x, t) = (g_4 * u_2)(x, t).
\end{align*}
\]

It is straightforward to see that system (4) is equivalent to
\[
\begin{align*}
    \frac{\partial u_1}{\partial t} &= d_1 \frac{\partial^2 u_1}{\partial x^2} + r_1 u_1 \left( 1 - a_1 w_1 - b_1 w_2 \right), \\
    \frac{\partial u_2}{\partial t} &= d_2 \frac{\partial^2 u_2}{\partial x^2} + r_2 u_2 \left( 1 - b_2 w_3 - a_2 w_4 \right), \\
    \frac{\partial w_1}{\partial t} &= d_1 \frac{\partial^2 w_1}{\partial x^2} + \frac{1}{\tau_1} u_1 - \frac{1}{\tau_1} w_1, \\
    \frac{\partial w_2}{\partial t} &= d_2 \frac{\partial^2 w_2}{\partial x^2} + \frac{1}{\tau_2} u_2 - \frac{1}{\tau_2} w_2, \\
    \frac{\partial w_3}{\partial t} &= d_1 \frac{\partial^2 w_3}{\partial x^2} + \frac{1}{\tau_3} u_1 - \frac{1}{\tau_3} w_3, \\
    \frac{\partial w_4}{\partial t} &= d_2 \frac{\partial^2 w_4}{\partial x^2} + \frac{1}{\tau_4} u_2 - \frac{1}{\tau_4} w_4.
\end{align*}
\]

Assume that system (8) has a traveling wave solution in the form
\[
\begin{align*}
    u_i(x, t) &= u_i(z), \quad z = x + ct, \quad i = 1, 2, \\
    w_j(x, t) &= w_j(z), \quad z = x + ct, \quad j = 1, 2, 3, 4,
\end{align*}
\]

where \( c > 0 \) is the wave speed, then we get the corresponding traveling-wave system
\[
\begin{align*}
    cu_1' &= d_1 u_1'' + r_1 u_1 \left( 1 - a_1 w_1 - b_1 w_2 \right), \\
    cu_2' &= d_2 u_2'' + r_2 u_2 \left( 1 - b_2 w_3 - a_2 w_4 \right), \\
    cw_1' &= d_1 w_1'' + \frac{1}{\tau_1} u_1 - \frac{1}{\tau_1} w_1, \\
    cw_2' &= d_2 w_2'' + \frac{1}{\tau_2} u_2 - \frac{1}{\tau_2} w_2, \\
    cw_3' &= d_1 w_3'' + \frac{1}{\tau_3} u_1 - \frac{1}{\tau_3} w_3, \\
    cw_4' &= d_2 w_4'' + \frac{1}{\tau_4} u_2 - \frac{1}{\tau_4} w_4,
\end{align*}
\]

where ‘ denotes differentiation with respect to \( z \).

Let
\[
\begin{align*}
    v_1 &= d_1 u_1', \quad v_2 = d_2 u_2', \quad v_3 = d_1 w_1', \\
    v_4 &= d_2 w_2', \quad v_5 = d_1 w_3', \quad v_6 = d_2 w_4'.
\end{align*}
\]

Since we need delays \( \tau_1, \tau_2, \tau_3 \) and \( \tau_4 \) are sufficient small, so we replace \( \tau_1, \tau_2, \tau_3 \) and \( \tau_4 \) with \( c^2 \tau_1, c^2 \tau_2, c^2 \tau_3 \) and \( c^2 \tau_4 \), respectively. Then the system can be
written as

\[
\begin{align*}
  u'_1 &= \frac{1}{d_1} v_1, \\
  v'_1 &= \frac{c}{d_1} v_1 - r_1 u_1 \left[ 1 - a_1 w_1 - b_1 w_2 \right], \\
  u'_2 &= \frac{1}{d_2} v_2, \\
  v'_2 &= \frac{1}{d_2} v_2 - r_2 u_2 \left[ 1 - b_2 w_3 - a_2 w_4 \right], \\
  w'_1 &= \frac{1}{d_1} v_3, \\
  c^2 v'_3 &= \frac{c^2}{d_1} v_3 - \frac{1}{\tau_1} u_1 + \frac{1}{\tau_1} w_1, \\
  w'_2 &= \frac{1}{d_2} v_4, \\
  c^2 v'_4 &= \frac{c^2}{d_2} v_4 - \frac{1}{\tau_2} u_2 + \frac{1}{\tau_2} w_2, \\
  w'_3 &= \frac{1}{d_1} v_5, \\
  c^2 v'_5 &= \frac{c^2}{d_2} v_5 - \frac{1}{\tau_3} u_1 + \frac{1}{\tau_3} w_3, \\
  w'_4 &= \frac{1}{d_2} v_6, \\
  c^2 v'_6 &= \frac{c^2}{d_2} v_6 - \frac{1}{\tau_4} u_2 + \frac{1}{\tau_4} w_1.
\end{align*}
\]

(9)

In the following, we introduce new state variables

\[
\begin{align*}
  \bar{u}_i &= u_i, \bar{w}_j = w_j, j = 1, 2, 3, 4, \\
  \bar{v}_i &= v_i, i = 1, 2, \bar{v}_i = \epsilon v_i, i = 3, 4, 5, 6,
\end{align*}
\]

and then drop the tildes, then system (9) becomes

\[
\begin{align*}
  u'_1 &= \frac{1}{d_1} v_1, \\
  v'_1 &= \frac{c}{d_1} v_1 - r_1 u_1 \left[ 1 - a_1 w_1 - b_1 w_2 \right], \\
  u'_2 &= \frac{1}{d_2} v_2, \\
  v'_2 &= \frac{1}{d_2} v_2 - r_2 u_2 \left[ 1 - b_2 w_3 - a_2 w_4 \right], \\
  \epsilon w'_1 &= \frac{1}{d_1} v_3, \\
  \epsilon v'_3 &= \frac{\epsilon c}{d_1} v_3 - \frac{1}{\tau_1} u_1 + \frac{1}{\tau_1} w_1, \\
  \epsilon w'_2 &= \frac{1}{d_2} v_4, \\
  \epsilon v'_4 &= \frac{\epsilon c}{d_2} v_4 - \frac{1}{\tau_2} u_2 + \frac{1}{\tau_2} w_2, \\
  \epsilon w'_3 &= \frac{1}{d_1} v_5, \\
  \epsilon v'_5 &= \frac{\epsilon c}{d_1} v_5 - \frac{1}{\tau_3} u_1 + \frac{1}{\tau_3} w_3, \\
  \epsilon w'_4 &= \frac{1}{d_2} v_6, \\
  \epsilon v'_6 &= \frac{\epsilon c}{d_2} v_6 - \frac{1}{\tau_4} u_2 + \frac{1}{\tau_4} w_1.
\end{align*}
\]

(10)
When $\epsilon = 0$, system (10) is reduced to the equations (four-order ODE). And the traveling wave solutions of the non-delayed problem have been studied by previous investigators [8, 14]. In this degenerate case the system is four-dimensional, but for $\epsilon > 0$, the existence of a traveling front solution of system (4) between $E_1$ and $E_2$ is equivalent to the existence of a heteroclinic orbit connection between the equilibrium points of the twelve-dimensional system (10). We shall still denote these equilibria by $E_1$ and $E_2$. However, for system (10) they are given by

\[
E_1 = \left( \frac{1}{a_1}, 0, 0, 0, \frac{1}{a_1}, 0, 0, 0, \frac{1}{a_1}, 0, 0, 0 \right),
\]

\[
E_2 = \left( 0, 0, \frac{1}{a_2}, 0, 0, 0, \frac{1}{a_2}, 0, 0, 0, \frac{1}{a_2}, 0 \right).
\]

Our purpose is to apply the geometric singular perturbation theory described in [7]. When $\epsilon > 0$ is sufficiently small, system (10) is referred to as the slow system. Note that when $\epsilon = 0$, (10) dose not define a dynamic system in $\mathbb{R}^{12}$, by introducing a new independent variable $\eta$ defined by $z = \epsilon \eta$, system (10) can be transformed into

\[
\begin{aligned}
\dot{u}_1 &= \frac{1}{a_1} v_1, \\
\dot{v}_1 &= \epsilon \left( \frac{1}{a_1} v_1 - r_1 u_1 \left[ 1 - a_1 w_1 - b_1 w_2 \right] \right), \\
\dot{u}_2 &= \frac{1}{a_2} v_2, \\
\dot{v}_2 &= \epsilon \left( \frac{1}{a_2} v_2 - r_2 u_2 \left[ 1 - b_2 w_3 - a_2 w_4 \right] \right), \\
\dot{w}_1 &= \frac{1}{a_1} v_3, \\
\epsilon \dot{v}_3 &= \frac{1}{a_1} v_3 - \frac{1}{\tau_1} u_1 + \frac{1}{\tau_1} w_1, \\
\dot{w}_2 &= \frac{1}{a_2} v_4, \\
\dot{v}_4 &= \frac{1}{a_2} v_4 - \frac{1}{\tau_2} u_2 + \frac{1}{\tau_2} w_2, \\
\dot{w}_3 &= \frac{1}{a_1} v_5, \\
\dot{v}_5 &= \frac{1}{a_1} v_5 - \frac{1}{\tau_3} u_1 + \frac{1}{\tau_3} w_3, \\
\dot{w}_4 &= \frac{1}{a_2} v_6, \\
\dot{v}_6 &= \frac{1}{a_2} v_6 - \frac{1}{\tau_4} u_2 + \frac{1}{\tau_4} w_4,
\end{aligned}
\]

where dots denote differentiation with respect to $\eta$. System (11) is called the fast system. The slow system and fast system are equivalent when $\epsilon > 0$. When $\epsilon = 0$, the slow system (10) does not define a dynamical system in the whole of $\mathbb{R}^{12}$ and the dynamics take place only on the set

\[
M_0 = \{ (u_1, v_1, u_2, v_2, w_1, v_3, w_2, v_4, w_3, v_5, w_4, v_6) \in \mathbb{R}^{12} : v_3 = v_1 = v_5 = v_6 = 0, w_1 = w_2 = u_1, w_2 = u_4 = w_4 = u_2 \},
\]

which is a four-dimensional invariant manifold of system (10). If this invariant manifold is normally hyperbolic, then we can obtain an invariant manifold $M_\epsilon$ of system (10) for $\epsilon > 0$, which is close to $M_0$. The restriction of system (10) to this invariant manifold $M_\epsilon$ yields a four-dimensional system.
From Fenichel [7], we know that verifying normal hyperbolicity of \( M_0 \) involves checking that the linearization of the fast system (11), restricted to \( M_0 \), has precisely \( \dim M_0 \) eigenvalues on the imaginary axis, with the remainder of the spectrum being hyperbolic. The linearization of system (11) restricted to \( M_0 \) is given by the matrix

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\tau_1} & 0 & 0 & 0 & \frac{1}{\tau_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\tau_2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\tau_2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\tau_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\tau_4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\tau_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\tau_4} \\
\end{pmatrix},
\]

which has eigenvalues \( \{0, 0, 0, \pm \frac{1}{\sqrt{\tau_1 d_1}}, \pm \frac{1}{\sqrt{\tau_2 d_2}}, \pm \frac{1}{\sqrt{\tau_4 d_4}}\} \). Obviously, normal hyperbolicity is verified and there exists an invariant manifold \( M_\epsilon \). It is close to \( M_0 \) for \( \epsilon > 0 \) sufficiently small. And the orbit has to stay on the slow manifold when perturbed. In fact, \( M_\epsilon \) can be expressed in the form

\[
M_\epsilon = \{(u_1, v_1, u_2, v_2, w_1, v_3, w_2, v_4, w_3, v_5, w_4, v_6) \in \mathbb{R}^{12},
\]

\[
v_3 = h_1(u_1, v_1, u_2, v_2, \epsilon), v_4 = h_2(u_1, v_1, u_2, v_2, \epsilon),
\]

\[
v_5 = h_3(u_1, v_1, u_2, v_2, \epsilon), v_6 = h_4(u_1, v_1, u_2, v_2, \epsilon),
\]

\[
w_1 = u_1 + g_1(u_1, v_1, u_2, v_2, \epsilon), w_2 = u_2 + g_2(u_1, v_1, u_2, v_2, \epsilon),
\]

\[
w_3 = u_1 + g_3(u_1, v_1, u_2, v_2, \epsilon), w_4 = u_2 + g_4(u_1, v_1, u_2, v_2, \epsilon)\},
\]

with functions \( h_i \) and \( g_i \) satisfy

\[
h_i(u_1, v_1, u_2, v_2, 0) = 0,
\]

\[
g_i(u_1, v_1, u_2, v_2, 0) = 0, \ i = 1, 2, 3, 4.
\]

The slow system (10) restricted to \( M_\epsilon \) is

\[
\begin{cases}
u_1' = \frac{1}{\tau_1} v_1, \\
u_2' = \frac{1}{\tau_2} v_2, \\
v_3' = \frac{1}{\tau_1} v_1 - r_1 u_1 [1 - a_1(u_1 + g_1) - b_1(u_2 + g_2)], \\
v_4' = \frac{1}{\tau_2} v_2 - r_2 u_2 [1 - b_2(u_1 + g_3) - a_2(u_2 + g_4)],
\end{cases}
\]

(12)

when \( \epsilon = 0 \), system (12) again reduces to the system of the non-delayed equations, and the traveling wave solution has been studied previously. For \( \epsilon > 0 \) and sufficiently small, system (12) still possesses equilibrium points, we also write it as

\[
E_1 = \left(\frac{1}{a_1}, 0, 0, 0\right), E_2 = \left(0, 0, \frac{1}{a_2}, 0\right),
\]

and that it falls within the class of systems studied by Gardner [8]. Indeed, Gardner [8] studied the following competition systems, which can be described as traveling
wave form

\[
\begin{aligned}
&u'_1 = \frac{1}{d_1} v_1 , \\
v'_1 = \frac{\epsilon}{d_1} v_1 - u_1 M(u_1, u_2) , \\
u'_2 = \frac{1}{d_2} v_2 , \\
v'_2 = \frac{1}{d_2} v_2 - u_2 N(u_1, u_2).
\end{aligned}
\]

(13)

Thus, Gardner’s results can be applicable to system (12). The functions \(g_1, g_2, g_3, g_4\) in system (12) would need to involve \(u_1\) and \(u_2\) only. We shall show that this is indeed the case, up to order \(\epsilon^2\).

Note that the functions \(h_i\) and \(g_i\) can be expanded into the form of Taylor series about \(\epsilon\)

\[
\begin{aligned}
&h_i(u_1, v_1, u_2, v_2, \epsilon) = \epsilon h_i^{(1)}(u_1, v_1, u_2, v_2) + \epsilon^2 h_i^{(2)}(u_1, v_1, u_2, v_2) + \cdots , \\
g_i(u_1, v_1, u_2, v_2, \epsilon) = \epsilon g_i^{(1)}(u_1, v_1, u_2, v_2) + \epsilon^2 g_i^{(2)}(u_1, v_1, u_2, v_2) + \cdots ,
\end{aligned}
\]

where \(i = 1, 2, 3, 4\). The restriction of system (10) to \(M_\epsilon\) is a four-dimensional system. We first determine the functions \(h_i\) and \(g_i\). Note that \(M_\epsilon\) is the invariant manifold for the flow of system (10). More precisely, \(h_1\) and \(g_1\) satisfy

\[
\begin{aligned}
\epsilon \left\{ \frac{1}{d_1} \frac{\partial g_1}{\partial u_1} v_1 + \frac{\partial g_1}{\partial v_1} \left( \frac{c}{d_1} v_1 - r_1 u_1 (1 - a_1 u_1 - b_1 u_2 - a_1 g_2 - b_1 g_2) \right) + \\
\frac{1}{d_2} \frac{\partial g_1}{\partial v_2} \frac{\partial g_1}{\partial u_2} + \frac{\partial g_1}{\partial v_2} \left( \frac{c}{d_2} v_2 - r_2 u_2 (1 - b_2 u_1 - a_2 u_2 - b_2 g_3 - a_2 g_4) \right) \right\} = \frac{1}{d_1} \epsilon h_1 ,
\end{aligned}
\]

and

\[
\begin{aligned}
\epsilon \left\{ \frac{1}{d_1} \frac{\partial h_1}{\partial u_1} + \frac{\partial h_1}{\partial v_1} \left( \frac{c}{d_1} v_1 - r_1 u_1 (1 - a_1 u_1 - b_1 u_2) \right) + \frac{\partial h_1}{\partial u_1} v_2 + \\
\frac{\partial h_1}{\partial v_2} \left( \frac{c}{d_2} v_2 - r_2 u_2 (1 - b_2 u_3 - a_2 u_4) \right) \right\} = \frac{1}{d_1} \epsilon^2 c + \frac{1}{\tau_1} g_1 .
\end{aligned}
\]

In fact, by complex calculating, we obtain

\[
\begin{aligned}
h_1^{(1)} &= v_1, & h_2^{(1)} &= v_2, & h_3^{(1)} &= v_1, & h_4^{(1)} &= v_2, \\
h_1^{(2)} &= 0, & h_2^{(2)} &= 0, & h_3^{(2)} &= 0, & h_4^{(2)} &= 0, \\
g_1^{(1)} &= 0, & g_2^{(1)} &= 0, & g_3^{(1)} &= 0, & g_4^{(1)} &= 0, \\
g_1^{(2)} &= -\tau_1 r_1 u_1 (1 - a_1 u_1 - b_1 u_2), & g_2^{(2)} &= -\tau_2 r_2 u_2 (1 - b_2 u_1 - a_2 u_2), \\
g_3^{(2)} &= -\tau_3 r_1 u_1 (1 - a_1 u_1 - b_1 u_2), & g_4^{(2)} &= -\tau_4 r_2 u_2 (1 - b_2 u_1 - a_2 u_2).
\end{aligned}
\]
On the basis of above tedious calculations, system (12) becomes, to order \( \epsilon^2 \),

\[
\begin{align*}
    v_1' &= \frac{1}{\sigma_1} v_1, \\
    u_1' &= \frac{1}{\sigma_1} v_1 - r_1 u_1 [1 - a_1 u_1 - b_1 u_2 + \epsilon^2 \tau_1 r_1 a_1 u_1 (1 - a_1 u_1 - b_1 u_2) + \epsilon^2 \tau_2 r_2 b_1 u_2 (1 - b_2 u_1 - a_2 u_2)], \\
    u_2' &= \frac{1}{\sigma_2} v_2, \\
    v_2' &= \frac{1}{\sigma_2} v_2 - r_2 u_2 [1 - b_2 u_1 - a_2 u_2 + \epsilon^2 \tau_3 r_1 b_2 u_1 (1 - a_1 u_1 - b_1 u_2) + \epsilon^2 \tau_3 r_2 a_2 u_2 (1 - b_2 u_1 - a_2 u_2)],
\end{align*}
\]

which has the structure of the system (13). Also,

\[
E_1 = \left( \frac{1}{a_1}, 0, 0, 0 \right), \quad E_2 = \left( 0, 0, \frac{1}{a_2}, 0 \right),
\]

are the equilibria of the above equation. Therefore, by using the results in [8], there exists a heteroclinic connection between the equilibria \( E_1 \) and \( E_2 \) of the above equation. It is implies that system (4) exist a traveling wave fronts.

We have the following result on the existence of a traveling wave fronts for system (4) with nonlocal delays.

**Theorem 1.** Assume that \( a_1 a_2 \neq b_1 b_2 \) and either \( b_1 < a_2, b_2 < a_1 \) or \( b_1 > a_2, b_2 > a_1 \), then system (4) has a traveling wave fronts connects \( E_1 = \left( \frac{1}{a_1}, 0 \right) \) and \( E_2 = \left( 0, \frac{1}{a_2} \right) \), provided that \( \tau_i, i = 1, 2, 3, 4 \) are sufficiently small.

4. Existence of bistable wavefronts. In this section, we will prove the existence of bistable wavefronts of system (4). The method used in this section is similar to [18]. We first recall some known results on traveling wavefronts of bistable system without time delay. Then we will prove the existence of bistable wavefronts of system (4).

Let consider the following reaction-diffusion systems:

\[
\frac{\partial u(x, t)}{\partial t} = d \frac{\partial^2 u(x, t)}{\partial x^2} + f(u(x, t)), \quad (14)
\]

where \( x \in \mathbb{R}, d = \text{diag}(d_1, ..., d_n) \) with \( d_i > 0, i = 1, ..., n, t > 0, u \in \mathbb{R}^n, f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous function.

We denote the traveling wave coordinate \( x + ct \) by \( t \) and derive from (14) that

\[
c \Phi'(t) = d \Phi''(t) + f(\Phi(t)), \quad t \in \mathbb{R} \quad (15)
\]

under the boundary value conditions

\[
\lim_{t \to -\infty} \Phi(t) = \Phi_-, \quad \lim_{t \to \infty} \Phi(t) = \Phi_+,
\]

with \( \Phi_- < \Phi_+ \) and \( f(\Phi_+) = 0 \).

Similarly, a traveling wave solution \( \Phi(t) = (\phi_1(t), \phi_2(t))^T \) of (4) satisfies

\[
\begin{align*}
    d_1 \phi_1''(t) - c \phi_1'(t) + r_1 \phi_1 [1 - a_1 (g_1 * \phi_1)(x, t) - b_1 (g_2 * \phi_2)(x, t)] &= 0, \\
    d_2 \phi_2''(t) - c \phi_2'(t) + r_2 \phi_2 [1 - b_2 (g_3 * \phi_1)(x, t) - a_2 (g_4 * \phi_2)(x, t)] &= 0,
\end{align*}
\]

(16)
where \((g_j \ast \phi)(t), j = 1, 2, 3, 4\) are defined by

\[
(g_j \ast \phi)(t) = \int_{-\infty}^{+\infty} \int_0^{+\infty} g_j(y, s) \phi(t - y - cs) ds dy,
\]

with \(g_i(y, s)\) are given as

\[
g_i(y, s) = \frac{1}{\tau_i} e^{\frac{y}{\tau_i}} \frac{1}{\sqrt{4d_i\pi s}} e^{-\frac{y^2}{4d_i\pi s}}, \quad i = 1, 2.
\]

and

\[
g_i(y, s) = \frac{1}{\tau_i} e^{\frac{y}{\tau_i}} \frac{1}{\sqrt{4d_i-2\pi s}} e^{-\frac{y^2}{4d_i-2\pi s}}, \quad i = 3, 4.
\]

We need the following existence theorem to prove the existence of traveling wavefronts of (16).

**Lemma 2** ([18, 23]). Let the vector-valued function \(f(\Phi)\) vanish at a finite number of points in the interval \([\Phi_-, \Phi_+]\) with \(\Phi_- < \Phi_1, ..., \Phi_m < \Phi_+\), where \(\Phi = (\Phi_1, ..., \Phi_m)\). We assume that all the eigenvalues of the matrices \(f'(\Phi_-)\) and \(f'(\Phi_+)\) lie in the left half-plane and that there exist vectors \(p_k \geq 0\) such that

\[
p_k f'(\Phi_k) > 0, \quad k = 1, 2, ..., m.
\]

Then there exists a monotone traveling wave, i.e., a constant \(c\) and a twice continuously differentiable vector-valued function \(\Phi(t)\), monotonically decreasing, satisfying system (15) and the conditions (16).

In order to prove the existence of bistable wavefronts connecting the semitrivial equilibria \(E_1 = \left(\frac{1}{a_1}, 0\right)\) and \(E_2 = \left(0, \frac{1}{a_2}\right)\), we also let

\[
w_1 = (g_1 * u)(x, t), \ w_2 = (g_2 * u_2)(x, t), \ w_3 = (g_3 * u_1)(x, t), \ w_4 = (g_4 * u_2)(x, t).
\]

Then it is straightforward to see that system (4) is equivalent to

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= d_1 \frac{\partial^2 u_1}{\partial x^2} + r_1 u_1 \left[1 - a_1 w_1 - b_1 w_2\right], \\
\frac{\partial u_2}{\partial t} &= d_2 \frac{\partial^2 u_2}{\partial x^2} + r_2 u_2 \left[1 - b_2 w_3 - a_2 w_4\right], \\
\frac{\partial w_1}{\partial t} &= d_1 \frac{\partial^2 w_1}{\partial x^2} + \frac{1}{\tau_1} u_1 - \frac{1}{\tau_1} w_1, \\
\frac{\partial w_2}{\partial t} &= d_2 \frac{\partial^2 w_2}{\partial x^2} + \frac{1}{\tau_2} u_2 - \frac{1}{\tau_2} w_2, \\
\frac{\partial w_3}{\partial t} &= d_1 \frac{\partial^2 w_3}{\partial x^2} + \frac{1}{\tau_3} u_1 - \frac{1}{\tau_3} w_3, \\
\frac{\partial w_4}{\partial t} &= d_2 \frac{\partial^2 w_4}{\partial x^2} + \frac{1}{\tau_4} u_2 - \frac{1}{\tau_4} w_4.
\end{align*}
\]

(17)

Now let us introduce new state variables

\[
u_1 = \frac{1}{a_1} u_1, \ u_2 = \frac{1}{a_1} u_2, \ v_1 = \frac{1}{a_1} w_1, \ v_2 = \frac{1}{a_1} w_2, \ v_3 = \frac{1}{a_1} w_3, \ v_4 = \frac{1}{a_1} w_4.
\]
and drop the star, then system (17) is equivalent to
\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= d_1 \frac{\partial^2 u_1}{\partial x^2} + r_1 \left[ \frac{1}{a_1} - u_1 \right] [-a_1 w_1 + b_1 w_2], \\
\frac{\partial u_2}{\partial t} &= d_2 \frac{\partial^2 u_2}{\partial x^2} + r_2 u_2 \left[ 1 - b_2 + b_2 w_3 - a_2 w_4 \right], \\
\frac{\partial w_1}{\partial t} &= d_1 \frac{\partial^2 w_1}{\partial x^2} + \frac{1}{\tau_1} u_1 - \frac{1}{\tau_1} w_1, \\
\frac{\partial w_2}{\partial t} &= d_2 \frac{\partial^2 w_2}{\partial x^2} + \frac{1}{\tau_2} u_2 - \frac{1}{\tau_2} w_2, \\
\frac{\partial w_3}{\partial t} &= d_1 \frac{\partial^2 w_3}{\partial x^2} + \frac{1}{\tau_3} u_3 - \frac{1}{\tau_3} w_3, \\
\frac{\partial w_4}{\partial t} &= d_2 \frac{\partial^2 w_4}{\partial x^2} + \frac{1}{\tau_4} u_4 - \frac{1}{\tau_4} w_4.
\end{align*}
\]

And the corresponding wave system of (18) reduces to
\[
\begin{align*}
c\phi_1' &= d_1 \phi_1'' + r_1 \left[ \frac{1}{a_1} - \phi_1 \right] [-a_1 \varphi_1 + b_1 \varphi_2], \\
c\phi_2' &= d_2 \phi_2'' + r_2 \phi_2 \left[ 1 - b_2 + b_2 \varphi_3 - a_2 \varphi_4 \right], \\
c\varphi_1' &= d_1 \varphi_1'' + \frac{1}{\tau_1} \phi_1 - \frac{1}{\tau_1} \varphi_1, \\
c\varphi_2' &= d_2 \varphi_2'' + \frac{1}{\tau_2} \phi_2 - \frac{1}{\tau_2} \varphi_2, \\
c\varphi_3' &= d_1 \varphi_3'' + \frac{1}{\tau_3} \phi_3 - \frac{1}{\tau_3} \varphi_3, \\
c\varphi_4' &= d_2 \varphi_4'' + \frac{1}{\tau_4} \phi_4 - \frac{1}{\tau_4} \varphi_4.
\end{align*}
\]

with the following asymptotical boundary conditions
\[
\begin{align*}
\lim_{t \to -\infty} \left( \phi_1, \phi_2, \varphi_1, \varphi_2, \varphi_3, \varphi_4 \right) &= (0, 0, 0, 0, 0, 0), \\
\lim_{t \to \infty} \left( \phi_1, \phi_2, \varphi_1, \varphi_2, \varphi_3, \varphi_4 \right) &= \left( \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_1}, \frac{1}{a_2} \right).
\end{align*}
\]

Define
\[
\Phi_- = (0, 0, 0, 0, 0, 0), \quad \Phi_+ = \left( \frac{1}{a_1}, \frac{1}{a_1}, \frac{1}{a_1}, \frac{1}{a_1}, \frac{1}{a_1}, \frac{1}{a_1} \right).
\]

By using Lemma 2 to system (18), then we have the following existence result.

**Theorem 2.** Assume that \(a_1 < b_2\) and \(a_2 < b_1\) holds. Then there exists a monotone vector-valued function \(\Phi(t) = (\phi_1, \phi_2, \varphi_1, \varphi_2, \varphi_3, \varphi_4) \in C^2(\mathbb{R}^2, \mathbb{R}^4)\) satisfying (19) and (20). Furthermore system (4) has a monotone and bistable traveling wavefront connecting equilibrium \(E_1\) with equilibrium \(E_2\).

**Proof.** We need to prove that all the eigenvalues of \(f'(\Phi_-)\) and \(f'(\Phi_+)\) have negative real parts, whereas there exist vectors \(p_1 \geq 0, p_2 > 0\) satisfying \(p_1 f'(\Phi_1) > 0, p_2 f'(\Phi_2) > 0\), where \(\Phi_1\) and \(\Phi_2\) are given by
\[
\Phi_1 = \left( \frac{1}{a_1} - k_1, \frac{1}{a_1} - k_2, \frac{1}{a_1} - k_1, k_2, \frac{1}{a_1} - k_1, k_2 \right), \quad \Phi_2 = \left( \frac{1}{a_1}, 0, \frac{1}{a_1}, 0, \frac{1}{a_1}, 0 \right).
\]
We have the matrices of $f'(\Phi_-)$ and $f'(\Phi_+)$ by direct calculation

$$f'(\Phi_-) = \begin{pmatrix} 0 & 0 & \frac{1}{\tau_1} & 0 & \frac{1}{\tau_3} & 0 \\ 0 & r_2[1 - \frac{b_2}{a_1}] & 0 & \frac{1}{\tau_2} & 0 & \frac{1}{\tau_4} \\ -r_1 & 0 & -\frac{1}{\tau_1} & 0 & 0 & 0 \\ r_1 \frac{b_2}{a_1} & 0 & 0 & -\frac{1}{\tau_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\tau_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\tau_4} \end{pmatrix},$$

$$f'(\Phi_+) = \begin{pmatrix} r_1[1 - \frac{b_2}{a_2}] & 0 & \frac{1}{\tau_1} & 0 & \frac{1}{\tau_3} & 0 \\ 0 & 0 & 0 & \frac{1}{\tau_2} & 0 & \frac{1}{\tau_4} \\ 0 & 0 & -\frac{1}{\tau_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\tau_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\tau_3} & 0 \\ 0 & -r_2 b_2 \frac{a_2}{a_2} & 0 & 0 & -\frac{1}{\tau_4} \\ 0 & -r_2 a_2 k_{2x} & 0 & 0 & -\frac{1}{\tau_4} \end{pmatrix}. $$

It is easy to see that all the eigenvalues of the matrices $f'(\Phi_-)$ and $f'(\Phi_+)$ are negative.

Direct calculation shows that $f'(\Phi_1)$ is given by

$$f'(\Phi_1) = \begin{pmatrix} 0 & 0 & \frac{1}{\tau_1} & 0 & \frac{1}{\tau_3} & 0 \\ 0 & 0 & 0 & \frac{1}{\tau_2} & 0 & \frac{1}{\tau_4} \\ -r_1 a_1 k_1 & 0 & -\frac{1}{\tau_1} & 0 & 0 & 0 \\ r_1 b_1 k_1 & 0 & 0 & -\frac{1}{\tau_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\tau_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\tau_2} \end{pmatrix}. $$

Furthermore, for any $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) > 0$, we know that

$$(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) f'(\Phi_1) > 0,$$

is equivalent to

$$\xi_4 b_1 > \xi_3 a_1, \xi_5 b_2 > \xi_6 a_2, \xi_1 > \xi_3, \xi_2 > \xi_4, \xi_1 > \xi_5, \xi_2 > \xi_6. \quad (21)$$

In view of $a_1 < b_2$ and $a_2 < b_1$, then there exists $\xi$ such that (21) is true.

For $\Phi_2$, similar to the above discussion, we obtain

$$f'(\Phi_2) = \begin{pmatrix} r_1 & 0 & \frac{1}{\tau_1} & 0 & \frac{1}{\tau_3} & 0 \\ 0 & r_2 & 0 & \frac{1}{\tau_2} & 0 & \frac{1}{\tau_4} \\ 0 & 0 & -\frac{1}{\tau_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\tau_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\tau_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\tau_4} \end{pmatrix}. $$

If $\xi_1 > \xi_3, \xi_1 > \xi_3$ and $\xi_2 > \xi_4, \xi_2 > \xi_6$ hold, then one has

$$(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) f'(\Phi_2) = \left( r_1 \xi_1, r_2 \xi_2, \frac{\xi_1 - \xi_3}{\tau_1}, \frac{\xi_2 - \xi_4}{\tau_2}, \frac{\xi_1 - \xi_5}{\tau_3}, \frac{\xi_2 - \xi_6}{\tau_4} \right) > 0.$$

Therefore, all the conditions of Lemma 2 are satisfied, (18) has a traveling wavefront satisfies (19) and (20). From the above discussion, system (4) has a monotone and bistable traveling wavefront connecting equilibrium $E_1$ with equilibrium $E_2$. □
Remark 1. In section 3, we establish the existence of traveling wave solution of system (4) when $\tau_1$, $\tau_2$, $\tau_3$ and $\tau_4$ are sufficiently small. However, in section 4, the same $\tau_1$, $\tau_2$, $\tau_3$ and $\tau_4$ in system (4) do not affect the existence of bistable wavefronts. In other words, the delays appeared in the interaction terms of system (4) are not sensitive to the existence of bistable wavefronts.

5. Conclusion. This paper studies a two-species competition and diffusion system with nonlocal delays. Based on the relation between traveling wave solution and heteroclinic orbit of the equilibria for the associated ordinary differential equations. By applying geometric singular perturbation theory and linear chain techniques, we establish the existence result of traveling wave fronts for the system (4) when delays are sufficiently small. We also obtain the existence of bistable wavefronts of system (4) by using the spectral method.

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REFERENCES

[1] J. Al-Omari and S. A. Gourley, Monotone traveling fronts in an age-structured reaction-diffusion model of a single species, J. Math. Biol., 45 (2002), 294–312.
[2] N. F. Britton, Spatial structures and periodic traveling waves in an integro-differential reaction diffusion population model, SIAM J. Appl. Math., 50 (1990), 1663–1688.
[3] P. De Maesschalck and F. Dumortier, Canard solutions at non-generic turning points, Trans. Amer. Math. Soc., 358 (2006), 2291–2334.
[4] Z. Du, J. Li and X. Li, The existence of solitary wave solutions of delayed Camassa-Holm equation via a geometric approach, J. Funct. Anal., 275 (2018), 988–1007.
[5] Z. Du and Q. Qiao, The dynamics of traveling waves for a nonlinear Belousov-Zhabotinskii system, J. Differential Equations, 269 (2020) 7214–7230.
[6] F. Dumortier and R. Roussarie, Multiple canard cycles in generalized Liénard equations, J. Differential Equations, 174 (2001), 1–29.
[7] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, J. Differential Equations, 31 (1979), 53–98.
[8] R. A. Gardner, Existence and stability of travelling wave solutions of competition models: A degree theoretic approach, J. Differential Equations, 44 (1982), 343–364.
[9] S. A. Gourley and S. Ruan, Convergence and traveling fronts in functional differential equations with nonlocal terms: A competition model, SIAM J. Math. Anal., 35 (2003), 806–822.
[10] B.-S. Han, Z.-C. Wang and Z. Du, Traveling waves for nonlocal Lotka-Volterra competition systems, Discrete Contin. Dyn. Syst. Ser. B, 25 (2020), 1959–1983.
[11] G. Hek, Geometric singular perturbation theory in biological practice, J. Math. Biol., 60 (2010), 347–386.
[12] J. Huang and X. Zou, Travelling wave fronts in diffusive and cooperative Lotka-Volterra system with delays, J. Math. Anal. Appl., 271 (2002), 455–466.
[13] C. K. R. T. Jones, Geometric Singular Perturbation Theory, in: R Johnson(Ed.), Dynamical Systems, Lecture Notes in Math., Springer, New York, 1609 (1995), 44–118.
[14] Y. Kan-On, Parameter dependence of propagation speed of travelling waves for competition diffusion equations, SIAM J. Math. Anal., 26 (1995), 340–363.
[15] X. Li and X. Miao, Population dynamical behavior of non-autonomous Lotka-Volterra competitive system with random perturbation, Discrete Contin. Dyn. Syst., 24 (2009), 523–545.
[16] W.-T. Li and Z.-C. Wang, Traveling fronts in diffusive and cooperative Lotka-Volterra system with nonlocal delays, Z. Angew. Math. Phys., 58 (2007), 571–591.
[17] C. Li and H. Zhu, Canard cycles for predator-prey systems with Holling types of functional response, J. Differential Equations, 254 (2013), 879–910.
[18] G. Lin and W.-T. Li, Bistable wavefronts in a diffusive and competitive Lotka-Volterra type system with nonlocal delays, J. Differential Equations, 244 (2008), 487–513.
[19] W. Liu, One-dimensional steady-state Poisson-Nernst-Planck systems for ion channels with multiple ion species, *J. Differential Equations*, **246** (2009), 428–451.

[20] G. Lv and M. X. Wang, Travelling wave fronts in diffusive and competitive Lotka-Volterra system with delays, *Nonlinear Anal. Real World Appl.*, **11** (2010), 1323–1329.

[21] M. B. A. Mansour, A geometric construction of traveling waves in a generalized nonlinear dispersive-dissipative equation, *J. Geom. Phys.*, **69** (2013), 116–122.

[22] R. H. Martin Jr. and H. L. Smith, Reaction-diffusion systems with the time delay: Monotonicity, invariance, comparison and convergence, *J. Reine Angew. Math.*, **413** (1991), 1–35.

[23] A. I. Volpert, Vitaly A. Volpert and Vladimir A. Volpert, *Traveling Wave Solutions of Parabolic Systems*, Transl. Math. Monomer., vol. 140, AMS, Providence, RI, 1994.

[24] Z.-C. Wang, W.-T. Li and S. Ruan, Traveling wave fronts in reaction-diffusion systems with spatio-temporal delays, *J. Differential Equations*, **222** (2006), 185–232.

[25] Y. Xu, Z. Du and L. Wei, Geometric singular perturbation method to the existence and asymptotic behavior of traveling waves for a generalized Burgers-KdV equation, *Nonlinear Dynam.*, **83** (2016), 65–73.

[26] Z. Zhao and Y. Xu, Solitary waves for Korteweg-de Vries equation with small delay, *J. Math. Anal. Appl.*, **368** (2010), 43–53.

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E-mail address: duzengji@163.com,zjdu@jsnu.edu.cn(Du)
E-mail address: yanshuling20111@163.com(Yan)
E-mail address: 610833859@qq.com(Zhuang)