LONG-TIME DYNAMICS OF COMPLETELY INTEGRABLE
SCHRÖDINGER FLOWS ON THE TORUS

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Abstract. In this article, we are concerned with long-time behaviour of solutions to a semi-classical Schrödinger-type equation on the torus. We consider time scales which go to infinity when the semi-classical parameter goes to zero and we associate with each time-scale the set of semi-classical measures associated with all possible choices of initial data. We emphasize the existence of a threshold: for time-scales below this threshold, the set of semi-classical measures contains measures which are singular with respect to Lebesgue measure in the "position" variable, while at (and beyond) the threshold, all the semi-classical measures are absolutely continuous in the "position" variable.

1. Introduction

1.1. The Schrödinger equation in the large time and high frequency régime.

This article is concerned with the dynamics of the linear equation

\[
\begin{cases}
    i\hbar \partial_t \psi_h(t,x) = H(hD_x)\psi_h(t,x), & (t,x) \in \mathbb{R} \times \mathbb{T}^d, \\
    \psi_h|_{t=0} = u_h,
\end{cases}
\]

on the torus \( \mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d \), with \( H \) a smooth real-valued function on \( (\mathbb{R}^d)^* \) (the dual of \( \mathbb{R}^d \)), and \( h > 0 \). In other words, \( H \) is a function on the cotangent bundle \( T^* \mathbb{T}^d = \mathbb{T}^d \times (\mathbb{R}^d)^* \) that does not depend on the first variable, and thus gives rise to a completely integrable Hamiltonian flow. We are interested in the simultaneous limits \( h \to 0^+ \) (high frequency limit) and \( t \to +\infty \) (large time evolution). Our results give a description of the limits of sequences of "position densities" \( |\psi_h(t_h,x)|^2 \) at times \( t_h \) that tend to infinity as \( h \to 0^+ \).

To be more specific, let us denote by \( S_h^t \) the propagator associated with \( H(hD_x) \):

\[
S_h^t := e^{-i\frac{t}{\hbar}H(hD_x)}.
\]

Fix a time scale, that is, a function

\[
\tau : \mathbb{R}_+^* \to \mathbb{R}_+^*, \\
\hbar \mapsto \tau_h,
\]

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1For the sake of simplicity, we shall assume that \( H \in C^\infty(\mathbb{R}^d) \). However the smoothness assumption on \( H \) can be relaxed to \( C^k \), where \( k \) large enough, in most results of this article.
such that \( \liminf_{h \to 0^+} \tau_h > 0 \) (actually, we shall be mainly concerned in functions that go to \(+\infty\) as \( h \to 0^+ \)). Consider a sequence of initial conditions \( (u_h) \), normalised in \( L^2(\mathbb{T}^d) \):

\[
\|u_h\|_{L^2(\mathbb{T}^d)} = 1 \quad \text{for} \quad h > 0, \quad \text{and} \quad h\text{-oscillating in the terminology of} \quad [15, 17], \quad \text{i.e.}:
\]

\[
(2) \quad \limsup_{h \to 0^+} \|1_{[0,R]} (-h^2 \Delta) u_h\|_{L^2(\mathbb{T}^d)} \to 0,
\]

where \( 1_{[0,R]} \) is the characteristic function of the interval \([0, R]\). Our main object of interest is the density \( |S_h^t u_h|^2 \), and we introduce the probability measures on \( \mathbb{T}^d \)

\[
\nu_h (t, dx) := |S_h^t u_h(x)|^2 \, dx;
\]

the unitary character of \( S_h^t \) implies that \( \nu_h \in \mathcal{C} (\mathbb{R}; \mathcal{P} (\mathbb{T}^d)) \). To study the long-time behaviour of the dynamics, we rescale time by \( \tau_h \) and look at the time-scaled probability densities:

\[
(3) \quad \nu_h (\tau_h t, dx) .
\]

When \( t \neq 0 \) is fixed and \( \tau_h \) grows too rapidly, it is a notoriously difficult problem to obtain a description of the limit points (in the weak-* topology) of these probability measures as \( h \to 0^+ \), for rich enough families of initial data \( u_h \). See for instance \([31, 30]\) in the case where the underlying classical dynamics is chaotic, the \( u_h \) are a family of lagrangian states, and \( \tau_h = h^{-2+\epsilon} \). In completely integrable situations, such as the one we consider here, the problem is of a different nature, but rapidly leads to intricate number theoretical issues \([25, 24, 26] \).

We soften the problem by considering the family of probability measures \([3]\) as elements of \( L^\infty (\mathbb{R}; \mathcal{P} (\mathbb{T}^d)) \). Our goal will be to give a precise description of the set \( \mathcal{M} (\tau) \) of their accumulation points in the weak-* topology for \( L^\infty (\mathbb{R}; \mathcal{P} (\mathbb{T}^d)) \), obtained as \( (u_h) \) varies among all possible sequences of initial data \( h\text{-oscillating and normalised in} \quad L^2 (\mathbb{T}^d) \).

The compactness of \( \mathbb{T}^d \) ensures that \( \mathcal{M} (\tau) \) is non-empty. Having \( \nu \in \mathcal{M} (\tau) \) is equivalent to the existence of a sequence \( (h_n) \) going to 0 and of a normalised, \( h_n\text{-oscillating sequence} \quad (u_{h_n}) \) in \( L^2 (\mathbb{T}^d) \) such that:

\[
(4) \quad \lim_{n \to +\infty} \frac{1}{\tau_{h_n}} \int_{\tau_{h_n} a}^{\tau_{h_n} b} \int_{\mathbb{T}^d} \chi (x) |S_{h_n}^t u_{h_n} (x)|^2 \, dx \, dt = \int_{a}^{b} \int_{\mathbb{T}^d} \chi (x) \nu (t, dx) \, dt,
\]

for every real numbers \( a < b \) and every \( \chi \in \mathcal{C} (\mathbb{T}^d) \). In other words, we are averaging the densities \( |S_h^t u_h(x)|^2 \) over time intervals of size \( \tau_n \). This averaging, as we shall see, makes the study more tractable.

If case \([4]\) occurs, we shall say that \( \nu \) is obtained through the sequence \( (u_{h_n}) \). To simplify the notation, when no confusion can arise, we shall simply write that \( h \to 0^+ \) to mean that we are considering a (discrete) sequence \( h_n \) going to \( 0^+ \), and we shall denote by \( (u_h) \) (instead of \( (u_{h_n}) \)) the corresponding family of functions.

\[\text{In what follows, } \mathcal{P} (X) \text{ stands for the set of Radon probability measures on a Polish space } X.\]
Remark 1.1. When the function $\tau$ is bounded, the convergence of $\nu_h(\tau_h t, \cdot)$ to an accumulation point $\nu(t, \cdot)$ is locally uniform in $t$. Moreover, $\nu$ can be completely described in terms of semiclassical defect measures of the corresponding sequence of initial data $(u_h)$, transported by the classical Hamiltonian flow $\phi_s : T^*\mathbb{T}^d \rightarrow T^*\mathbb{T}^d$ generated by $H$, which in this case is completely integrable. Explicitly,

$$\phi_s(x, \xi) := (x + sdH(\xi), \xi).$$

This is nothing but a formulation of Egorov’s theorem (see, for instance, [11]). Consider for instance, the case where the initial data $u_h$ are coherent states: fix $\rho \in C^\infty_c(\mathbb{R}^d)$ with $\|\rho\|_{L^2(\mathbb{R}^d)} = 1$, fix $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$, and let $u_h(x)$ be the $2\pi\mathbb{Z}^d$-periodization of the following coherent state:

$$\frac{1}{h^{d/4}} \rho \left( \frac{x - x_0}{\sqrt{h}} \right) e^{i\xi_0 \cdot x}.$$  

Then $\nu_h(t, \cdot)$ converges, for every $t \in \mathbb{R}$, to:

$$\delta_{x_0 + tdH(\xi_0)}(x).$$

When the time scale $\tau_h$ is unbounded, the $t$-dependence of elements $\nu \in \mathcal{M}(\tau)$ is not described by such a simple propagation law. From now on we shall only consider the case where $\tau_h \rightarrow +\infty$.

The problem of describing the elements in $\mathcal{M}(\tau)$ for some time scale $(\tau_h)$ is related to several aspects of the dynamics of the flow $S^t_h$ such as dispersive effects and unique continuation. In [4, 22] the reader will find a description of these issues in the case where the propagator $S^t_h$ is replaced by the semiclassical Schrödinger flow $e^{it\Delta}$ corresponding to the Laplacian on an arbitrary compact Riemannian manifold (corresponding to $H(hD_x) = -h^2\Delta$ in the case of flat tori). In that setting, the time scale $\tau_h = 1/h$ appears in a natural way, since it transforms the semiclassical propagator into the non-scaled flow $e^{it\Delta}$. The possible accumulation points of sequences of probability densities of the form $|e^{it\Delta}u_h|^2$ depend on the nature of the dynamics of the geodesic flow in the manifold under consideration. Even in the case that the geodesic flow is completely integrable, different type of concentration phenomena may occur, depending on fine geometrical issues (compare the situation in Zoll manifolds [20] and on flat tori [21, 3]). When the geodesic flow has the Anosov property, the results in [5] rule out concentration on sets of small dimensions, by proving lower bounds on the Kolmogorov-Sinai entropy of semiclassical defect measures.

1.2. Semiclassical defect measures. Our results are more naturally described in terms of Wigner distributions and semiclassical measures (these are the semiclassical version of the microlocal defect measures [16, 32], and have also been called microlocal lifts in the recent literature about quantum unique ergodicity, see for instance the celebrated paper [19]). The Wigner distribution associated to $u_h$ (at scale $h$) is a distribution on the cotangent bundle $T^*\mathbb{T}^d$, defined by

$$\int_{T^*\mathbb{T}^d} a(x, \xi) w^h_{u_h}(dx, d\xi) = \langle u_h, \text{Op}_h(a)u_h \rangle_{L^2(\mathbb{T}^d)}, \quad \text{for all } a \in C^\infty_c(T^*\mathbb{T}^d),$$
where $\text{Op}_h(a)$ is the operator on $L^2(\mathbb{T}^d)$ associated to $a$ by the Weyl quantization. The reader not familiar with these objects can consult the appendix of this article. For the moment, just recall that if $\chi$ is a smooth function on $T^*\mathbb{T}^d = \mathbb{T}^d \times (\mathbb{R}^d)^*$ that depends only on the first coordinate, then
\begin{equation}
\int_{T^*\mathbb{T}^d} \chi(x)w^h_{u_h}(dx, d\xi) = \int_{\mathbb{T}^d} \chi(x)|u_h(x)|^2 dx.
\end{equation}

The main object of our study will be the (time-scaled) Wigner distributions corresponding to solutions to (1):
\[w_h(t, \cdot) := w^h_{S^*_h t u_h}.
\]

The map $t \mapsto w_h(t, \cdot)$ belongs to $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$, and is uniformly bounded in that space as $h \to 0^+$ whenever $(u_h)$ is normalised in $L^2(\mathbb{T}^d)$. Thus, one can extract subsequences that converge in the weak-\* topology on $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$. In other words, after possibly extracting a subsequence, we have
\[
\int_{\mathbb{R}} \int_{T^*\mathbb{T}^d} \varphi(t) a(x, \xi) w_h(t, dx, d\xi) dt \xrightarrow{h \to 0} \int_{\mathbb{R}} \int_{T^*\mathbb{T}^d} \varphi(t) a(x, \xi) \mu(t, dx, d\xi) dt
\]
for all $\varphi \in L^1(\mathbb{R})$ and $a \in C^\infty_c(T^*\mathbb{T}^d)$, and the limit $\mu$ belongs to $L^\infty(\mathbb{R}; \mathcal{M}_+(T^*\mathbb{T}^d))$.\footnote{$\mathcal{M}_+(X)$ denotes the set of positive Radon measures on a Polish space $X$.}

The set of limit points thus obtained, as $(u_h)$ varies among normalised sequences, will be denoted by $\widetilde{\mathcal{M}}(\tau)$. We shall refer to its elements as (time-dependent) semiclassical measures.

Moreover, if $(u_h)$ is $h$-oscillating, it follows that $\mu \in L^\infty(\mathbb{R}; \mathcal{P}(T^*\mathbb{T}^d))$ and identity (1) is also verified in the limit: if $\nu(t, \cdot)$ is the image of $\mu(t, \cdot)$ under the projection map $(x, \xi) \mapsto x$, then
\[
\int_a^b \int_{\mathbb{T}^d} \chi(x)|S^*_h t u_h(x)|^2 dx dt \xrightarrow{h \to 0} \int_a^b \int_{T^*\mathbb{T}^d} \chi(x) \mu(t, dx, d\xi) dt,
\]
for every $a < b$ and every $\chi \in C^\infty(\mathbb{T}^d)$. Therefore, $\mathcal{M}(\tau)$ coincides with the set of projections onto $x$ of semiclassical measures in $\widetilde{\mathcal{M}}(\tau)$ corresponding to $h$-oscillating sequences (see \cite{15,17}).

It is also shown in the appendix that the elements of $\widetilde{\mathcal{M}}(\tau)$ are measures that are invariant by the Hamiltonian flow $\phi_s$:
\[
\int_{T^*\mathbb{T}^d} a \circ \phi_s(x, \xi) \mu(t, dx, d\xi) = \int_{T^*\mathbb{T}^d} a(x, \xi) \mu(t, dx, d\xi), \quad \forall \mu \in \widetilde{\mathcal{M}}(\tau), \forall s \in \mathbb{R}, \text{a.e. } t \in \mathbb{R}^d.
\]

1.3. Regularity of semiclassical measures. The main results in this article are aimed at obtaining a precise description of the elements in $\widetilde{\mathcal{M}}(\tau)$ (and, as a consequence, of those of $\mathcal{M}(\tau)$). We first present a regularity result which emphasises the critical character of the time scale $\tau_h = 1/h$ in situations in which the Hessian of $H$ is non-degenerate, definite (positive or negative).
**Theorem 1.2.** (1) If \( \tau_h \ll 1/\hbar \) then \( \mathcal{M}(\tau) \) contains elements that are singular with respect to the Lebesgue measure \( dt \, dx \). Besides, \( \mathcal{M}(\tau) \) contains all uniform orbit measures of \( \phi_s \).

(2) Suppose \( \tau_h \sim 1/\hbar \) or \( \tau_h \gg 1/\hbar \). Assume that the Hessian \( d^2H(\xi) \) is definite for all \( \xi \).

Then

\[
\mathcal{M}(\tau) \subseteq L^\infty(\mathbb{R}; L^1(\mathbb{T}^d)),
\]

i.e. the elements of \( \mathcal{M}(\tau) \) are absolutely continuous with respect to \( dt \, dx \).

**Remark 1.3.** Theorem 1.2(2) applies in particular when the data \( (u_h) \) are eigenfunctions of \( H(\hbar D_x) \), and shows (assuming the Hessian of \( H \) is definite) that the weak limits of the probability measures \( |u_h(x)|^2 \, dx \) are absolutely continuous.

**Remark 1.4.** The conclusion of Theorem 1.2(2) may fail if the condition on the Hessian of \( H \) is not satisfied. We give here two counter-examples.

Fix \( \omega \in \mathbb{R}^d \) and take \( H(\xi) = \xi \cdot \omega \). Let \( \mu_0 \) be an accumulation point in \( \mathcal{D}'(T^*\mathbb{T}^d) \) of the Wigner distributions \( (w_{s\hbar}^h)_{s\hbar} \) of the Wigner distributions \( (w_{s\hbar}^h)_{s\hbar} \) defined in \([3]\), associated to the initial data \( (u_h) \). Let \( \mu \in \mathcal{M}(\tau) \) be the limit of \( w_{s\hbar}^h \) as \( s \to \infty \) in \( L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d)) \). Then an application of Egorov’s theorem (actually, a particularly simple adaptation of the proof of Theorem 4 in \([20]\)) gives the relation, valid for any time scale \( (\tau_h) \):

\[
\int_{T^*\mathbb{T}^d} a(x,\xi) \mu(t, dx, d\xi) = \int_{T^*\mathbb{T}^d} \langle a(x,\xi) \rangle \mu_0(dx, d\xi),
\]

for any \( a \in C^\infty_c(T^*\mathbb{T}^d) \) and a.e. \( t \in \mathbb{R} \). Here \( \langle a \rangle \) stands for the average of \( a \) along the Hamiltonian flow \( \phi_s \), that is in our case

\[
\langle a \rangle(x,\xi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T a(x + s\omega,\xi) \, ds.
\]

Hence, as soon as \( \omega \) is resonant (in the sense of \([2,1]\) and \( \mu_0 = \delta_{x_0} \otimes \delta_{\xi_0} \) for some \( (x_0,\xi_0) \in T^*\mathbb{T}^d \), the measure \( \mu \) will be singular with respect to \( dt \, dx \).

It is also easy to provide counter-examples where the Hessian of \( H \) is non-degenerate, but not definite. On the two-dimensional torus \( \mathbb{T}^2 \), consider for instance \( H(\xi) = \xi_1^2 - \xi_2^2 \), where \( \xi = (\xi_1,\xi_2) \). Take for \( (u_h(x_1, x_2)) \) the periodization of

\[
\frac{1}{(2\pi\hbar)^{1/2}} \rho \left( \frac{x_1 - x_2}{\hbar} \right)
\]

where \( \rho \in C^\infty_c(\mathbb{R}) \) satisfies \( \|\rho\|_{L^2(\mathbb{R})} = 1 \). Then the functions \( u_h \) are eigenfunctions of \( H(\hbar D_x) \) and the measures \( |u_h(x_1, x_2)|^2 \, dx_1 \, dx_2 \) obviously concentrate on the diagonal \( \{x_1 = x_2\} \).

Note that statement (2) of Theorem 1.2 has already been proved in the case \( H(\xi) = |\xi|^2 \) and \( \tau_h = 1/\hbar \) in \([\mathcal{X}]\) and \([\mathcal{Y}]\) with different proofs. However, the extension of the methods in these references to more general \( H \) is not straightforward, even in the case where \( H(\xi) = \xi \cdot A\xi \), where \( A \) is a symmetric linear map : \( (\mathbb{R}^d)^* \to \mathbb{R}^d \) (i.e. the Hessian of \( H \) is constant), the difficulty arising when \( A \) has irrational coefficients.
The proof in [3] extends to the \((t, x)\)-dependent Hamiltonian \(|\xi|^2 + \hbar^2 V(t, x)\) with \(V\) continuous except for points forming a set of zero-measure. Recently, this has been extended in [7] to more general perturbations of the Laplacian (allowing for potentials \(V \in L^\infty(\mathbb{R} \times \mathbb{T}^d)\)) by means of an abstract argument that uses the result in [6, 3] for \(V = 0\) as a black-box. In fact, the proof of the result in [7] applies to our context.

**Remark 1.5.** Theorem 1.2 and [7] imply that statement (2) of Theorem 1.2 also holds for sequences of solutions to the Schrödinger equation corresponding to the perturbed Hamiltonian \(H(\hbar D_x) + \hbar \tau V(t)\) where \(V \in L^\infty(\mathbb{R}; L(L^2(\mathbb{T}^d)))\). The size \(\hbar \tau^{-1}\) of the perturbation is in some sense optimal; in Section 4.3 we present an example communicated to us by J. Wunsch showing that absolute continuity of the elements of \(\mathcal{M}(1/\hbar)\) may fail in the presence of a subprincipal symbol of order \(\hbar^\beta\) with \(\beta \in (0, 2)\) even in the case \(H(\xi) = |\xi|^2\).

Theorem 1.2(2) admits a microlocal refinement, which allows us to deal with more general Hamiltonians \(H\) whose Hessian is not necessarily definite at every \(\xi \in \mathbb{R}^d\). Given \(\mu \in \tilde{\mathcal{M}}(\tau)\) we shall denote by \(\tilde{\mu}\) the image of \(\mu\) under the map \(\pi_2 : (x, \xi) \mapsto \xi\). It is shown in the appendix that \(\tilde{\mu}\) does not depend on \(t\) (it can be obtained as \(\tilde{\mu} = (\pi_2)_* \mu_0\), where the measure \(\mu_0\) is an accumulation point in \(\mathcal{D}'(T^* \mathbb{T}^d)\) of the sequence \((u_{h_n})\)).

**Theorem 1.6.** Let \(\mu \in \tilde{\mathcal{M}}(1/\hbar)\) and denote by \(\mu_\xi(t, \cdot)\) the disintegration of \(\mu(t, \cdot)\) with respect to the variable \(\xi\), i.e. for every \(\theta \in L^1(\mathbb{R})\) and every bounded measurable function \(f\):

\[
\int_\mathbb{R} \theta(t) \int_{\mathbb{T}^d \times \mathbb{R}^d} f(x, \xi) \mu(t, dx, d\xi) dt = \int_\mathbb{R} \theta(t) \int_{\mathbb{R}^d} \left( \int_{\mathbb{T}^d} f(x, \xi) \mu_\xi(t, dx) \right) \bar{\mu}(d\xi) dt.
\]

Then for \(\bar{\mu}\)-almost every \(\xi\) where \(d^2H(\xi)\) is definite, the measure \(\mu_\xi(t, \cdot)\) is absolutely continuous.

Let us introduce the closed set

\[
C_H := \{ \xi \in \mathbb{R}^d : d^2H(\xi) \text{ is not definite} \}.
\]

The following consequence of Theorem 1.6 provides a refinement on Theorem 1.2(2), in which the global hypothesis on the Hessian of \(H\) is replaced by a hypothesis on the sequence of initial data.

**Corollary 1.7.** Suppose \(\nu \in \mathcal{M}(1/\hbar)\) is obtained through an \(h\)-oscillating sequence \((u_h)\) having a semiclassical measure \(\mu_0\) such that \(\mu_0(\mathbb{T}^d \times C_H) = 0\). Then \(\nu\) is absolutely continuous with respect to \(dt dx\).

We show in Section 4.2 that absolute continuity may fail for the elements of \(\mathcal{M}(1/\hbar)\) when \(H(\xi) = |\xi|^{2k}, k \in \mathbb{N} \text{ and } k > 1\); a situation where the Hessian is degenerate at \(\xi = 0\).

**1.4. Second-microlocal structure of the semiclassical measures.** Theorem 1.6 is a consequence of a more detailed result on the structure of the elements of \(\mathcal{M}(1/\hbar)\). We follow here the strategy of [3] that we adapt to a general Hamiltonian \(H(\xi)\). The proof relies on a decomposition of the measure associated with the primitive submodules of \((\mathbb{Z}^d)^*\). Before stating it, we must introduce some notation.
We note that

\[ I_\Lambda := \{ \xi \in (\mathbb{R}^d)^* : dH (\xi) \cdot k = 0, \; \forall k \in \Lambda \} \, . \]

We define also a submodule \( \Lambda \subset (\mathbb{Z}^d)^* \) primitive if

\[ \langle \Lambda \rangle \cap (\mathbb{Z}^d)^* = \Lambda \] (here \( \langle \Lambda \rangle \) denotes the linear subspace of \( (\mathbb{R}^d)^* \) spanned by \( \Lambda \)). Given such a submodule we define:

\[ \langle \alpha \rangle_\Lambda (\cdot, \xi) \]

We define also \( L^p (\mathbb{T}^d, \Lambda) \) for \( p \in [1, \infty] \) to be the subspace of \( L^p (\mathbb{T}^d) \) consisting of the functions \( u \) such that \( \widehat{u} (k) = 0 \) if \( k \notin (\mathbb{Z}^d)^* \setminus \Lambda \) (\( \widehat{u} (k) \) stand for the Fourier coefficients of \( u \)). Given \( a \in C^\infty_c (T^* \mathbb{T}^d) \) and \( \xi \in \mathbb{R}^d \), denote by \( \langle a \rangle_\Lambda (\cdot, \xi) \) the orthogonal projection of \( a (\cdot, \xi) \) on \( L^2 (\mathbb{T}^d, \Lambda) \):

\[ \langle a \rangle_\Lambda = \sum_{k \in \Lambda} \tilde{a}_k (\xi) \frac{\text{e}^{ijkx}}{2\pi^d} \]

We denote by \( m_\langle a \rangle_\Lambda (\xi) \) the operator acting on \( L^2 (\mathbb{T}^d, \Lambda) \) by multiplication by \( \langle a \rangle_\Lambda (\cdot, \xi) \).

**Theorem 1.8.** Let \( \mu \in \widetilde{\mathcal{M}} (1/h) \). For every primitive submodule \( \Lambda \subset (\mathbb{Z}^d)^* \), there exist a positive measure \( \mu_\Lambda \in C (\mathbb{R}; \mathcal{M}_+(T^* \mathbb{T}^d)) \) supported on \( \mathbb{T}^d \times I_\Lambda \) and invariant by the Hamiltonian flow \( \phi_s \), such that: for every \( a \in C^\infty_c (T^* \mathbb{T}^d) \) that vanishes on \( \mathbb{T}^d \times C_H \) and every \( \theta \in L^1 (\mathbb{R}) \):

\[ \int_{\mathbb{R}} \theta (t) \int_{T^* \mathbb{T}^d} a (x, \xi) \mu (t, dx, d\xi) \, dt = \sum_{\Lambda \subset \mathbb{Z}^d} \int_{\mathbb{R}} \theta (t) \int_{\mathbb{T}^d \times I_\Lambda} a (x, \xi) \mu_\Lambda (t, dx, d\xi) \, dt, \]

the sum being taken over all primitive submodules of \( (\mathbb{Z}^d)^* \).

In addition, there is a measure \( \rho_\Lambda \) on \( I_\Lambda \), taking values on the set of non-negative, symmetric, trace-class operators acting on \( L^2 (\mathbb{T}^d, \Lambda) \), such that the following holds:

\[ \int_{\mathbb{T}^d \times I_\Lambda} a (x, \xi) \mu_\Lambda (t, dx, d\xi) = \int_{I_\Lambda} \text{Tr} \left( e^{-\frac{1}{2} i \xi (H (\sigma) D_x D_\sigma) m_\langle a \rangle_\Lambda (\sigma)} e^{\frac{1}{2} i \xi (H (\sigma) D_x D_\sigma) \rho_\Lambda (d\sigma)} \right) \, . \]

When the Hessian of \( H \) is definite, formulae (9), (10) hold for every \( a \in C^\infty_c (T^* \mathbb{T}^d) \) and therefore completely describe \( \mu \).

Theorem 1.8 has been proved for \( H (\xi) = |\xi|^2 \) in [21] for \( d = 2 \) and in [3] for arbitrary dimension (there, a formula similar to (10) is proved for the \( x \)-dependent Hamiltonian \( |\xi|^2 + h^2 V (x) \)).

We see that Theorem 1.8 allows to describe the dependence of \( \mu \) on the parameter \( t \).

As was noticed in [20], [21], the semiclassical measures of the sequence of initial data \( (u_h) \) do not determine uniquely the time dependent semiclassical measure \( \mu \). On the other hand, these are fully determined by the “measures” \( \rho_\Lambda \), which are two-microlocal objects determined by the initial data \( (u_h) \). The contents of (10) is that the measure \( \mu_\Lambda (t, dx, d\xi) \) can be obtained by transporting \( \rho_\Lambda \) by a certain Schrödinger flow, and then tracing out

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4 Later in the paper, we will tend to identify both by working in the canonical bases of \( \mathbb{R}^d \).
certain directions. The $\rho_\Lambda$ are obtained by a process of successive two-microlocalizations along nested sequences of submanifolds in frequency space; this process gives an explicit construction of $\mu$ in terms of the initial data. This two-microlocal construction is in the spirit of that done in \[29, 12, 13\] in Euclidean space. We also refer the reader to the articles \[33, 34, 35\] for related work regarding the study of the wave-front set of solutions to semiclassical integrable systems.

**Remark 1.9.** The arguments in Section 6.1 of \[3\] show that Theorem 1.6 is a consequence of Theorem 1.8. Therefore, in this article only the proof of Theorem 1.8 will be presented.

**Remark 1.10.** Theorem 1.8 holds for the time scale $\tau_h = 1/h$. If $\tau_h \ll 1/h$, the elements of $\tilde{M}(\tau)$ can be described by a similar result (see Section 3.3) involving expression (9). However, the propagation law appearing in the formula replacing (10) involves classical transport rather than propagation along a Schrödinger flow, and as a result Theorem 1.2(2) does not hold for $\tau_h \ll 1/h$.

When the Hessian of $H$ is constant Theorem 1.8 gives a complement to the results announced in \[3\] (where the argument was only valid when the Hessian has rational coefficients). The statement is as follows.

**Corollary 1.11.** Suppose $H(\xi) = \xi \cdot A\xi$ where $A : (\mathbb{R}^d)^* \rightarrow \mathbb{R}^d$ is a symmetric definite linear map. Given $\nu \in \mathcal{M}(1/h)$ there exists, for each primitive module $\Lambda \subseteq (\mathbb{Z}^d)^*$, a non-negative, self-adjoint, trace-class operator $\Sigma_\Lambda$ acting on $L^2(\mathbb{T}^d, \Lambda)$ such that, for $b \in C(\mathbb{T}^d)$ and $\theta \in L^1(\mathbb{R})$:

$$\int_{\mathbb{R}} \theta(t) \int_{\mathbb{T}^d} b(x) \nu(t, dx) dt = \sum_{\Lambda \subseteq \mathbb{Z}^d} \int_{\mathbb{R}} \theta(t) \text{Tr} \left( m_{(b)}\Lambda e^{-itH(D_x)} \Sigma_\Lambda e^{itH(D_x)} \right) dt.$$  

In fact, $\Sigma_\Lambda := \rho_\Lambda(I_\Lambda)$, where $\rho_\Lambda$ is given by Theorem 1.8.

Comparing the special case (11) to the general case (9), (10), we see that in the former case the propagation law involve the constant propagator $e^{-itH(D_x)}$, whereas in the latter case we need a “superposition” of propagators $e^{-\frac{it}{2}d^2H(\sigma)D_x \cdot D_y}$ depending on $\sigma \in I_\Lambda$.

**1.5. Hierarchy of time scales.** In this section, we present a more detailed discussion on the dependence of the set $\mathcal{M}(\tau)$ on the time scale $\tau$; we shall also clarify the link between the time-dependent Wigner distributions and those associated with eigenfunctions. Eigenfunctions are the most commonly studied objects in the field of quantum chaos, however, we shall see that they do not necessarily give full information about the time-dependent Wigner distributions.

A particular case of our problem is when the initial data $(u_h)$ are eigenfunctions of $H(hD_x)$. We note that the spectrum of $H(hD_x)$ coincides with $H(h\mathbb{Z}^d)$; given $E_h \in H(h\mathbb{Z}^d)$ the corresponding normalised eigenfunctions are of the form:

$$u_h(x) = \sum_{H(hk) = E_h} c^h_k e^{ik \cdot x}, \text{ with } \sum_{k \in \mathbb{Z}^d} |c^h_k|^2 = \frac{1}{(2\pi)^d}.$$
In addition, one has:
\[
\nu_h (\tau_h t, \cdot) = |S_h^{\tau_h t} u_h|^2 = |u_h|^2,
\]
independently of \((\tau_h)\) and \(t\). Let us denote by \(\mathcal{M} (\infty)\) the set of accumulation points in \(\mathcal{P} (\mathbb{T}^d)\) of sequences \(|u_h|^2\) where \((u_h)\) varies among all possible \(h\)-oscillating sequences of normalised eigenfunctions \((12)\). Denote by \(\mathcal{M}_{av} (\tau)\) the subset of \(\mathcal{P} (\mathbb{T}^d)\) consisting of measures of the form:
\[
\mathcal{C}_{\mathcal{M}} \int_0^1 \nu (t, \cdot) \, dt, \quad \text{where } \nu \in \text{Conv} \mathcal{M} (\tau).
\]

**Proposition 1.12.** Suppose \((\tau_h)\) and \((\tau'_h)\) are time scales tending to infinity and such that \(\tau'_h \ll \tau_h\). Then:
\[
\mathcal{M} (\infty) \subseteq \mathcal{M} (\tau) \subseteq L^\infty (\mathbb{R}; \mathcal{M}_{av} (\tau')).
\]

**Remark 1.13.** As a consequence of Theorem 1.2(2) we obtain that all eigenfunction limits \(\mathcal{M} (\infty)\) are absolutely continuous under the definiteness assumption on the Hessian of \(H\).

A time scale of special importance is the one related to the minimal spacing of eigenvalues: define
\[
\tau_h^H := h \sup \left\{ \left| E^1_h - E^2_h \right|^{-1} : E^1_h \neq E^2_h, \ E^1_h, E^2_h \in H (h\mathbb{Z}^d) \right\}.
\]

It is possible to have \(\tau_h^H = \infty\): for instance, if \(H (\xi) = |\xi|^\alpha\) with \(0 < \alpha < 1\) or \(H (\xi) = \xi \cdot A\xi\) with \(A\) a real symmetric matrix that is not proportional to a matrix with rational entries, in some other situations, such as \(H (\xi) = |\xi|^\alpha\) with \(\alpha > 1\), \((13)\) is finite: \(\tau_h^H = h^{1-\alpha}\).

**Proposition 1.14.** If \(\tau_h \gg \tau_h^H\) one has:
\[
\mathcal{M} (\tau) = \text{Conv} \mathcal{M} (\infty).
\]

This result is a consequence of the more general results presented in Section 5.

Note that Proposition 1.14 allows to complete the description of \(\mathcal{M} (\tau)\) in the case \(H (\xi) = |\xi|^2\) as the time scale varies.

**Remark 1.15.** Suppose \(H (\xi) = |\xi|^2\), or more generally, that \(\tau_h^H \sim 1/h\) and the Hessian of \(H\) is definite. Then:
- if \(\tau_h \ll 1/h\), \(\exists \nu \in \mathcal{M} (\tau)\) such that \(\nu \perp dt\, dx\);
- if \(\tau_h \sim 1/h\), \(\mathcal{M} (\tau) \subseteq L^\infty (\mathbb{R}; L^1 (\mathbb{T}^d))\);
- if \(\tau_h \gg 1/h\), \(\mathcal{M} (\tau) = \text{Conv} \mathcal{M} (\infty)\).

Note that in this case the regularity of semiclassical measures can be precised. The elements in \(\mathcal{M} (\infty)\) are trigonometric polynomials when \(d = 2\), as shown in [18]; and in general they are more regular than merely absolutely continuous, see [1, 18, 28]. The same phenomenon occurs with those elements in \(\mathcal{M} (1/h)\) that are obtained through sequences whose corresponding semiclassical measures do not charge \(\{\xi = 0\}\), see [2].

---

5Conv \(X\) stands for the closed convex hull of a set \(X \subset L^\infty (\mathbb{R}; \mathcal{P} (\mathbb{T}^d))\) with respect to the weak-* topology.

6This is the content of the Oppenheim conjecture, settled by Margulis [9, 23].
1.6. Organisation of the paper. The key argument of this article is a second microlocalisation on primitive submodules which is the subject of Section 2.2 and leads to Theorems 2.4 and 2.5. In Section 3, successive microlocalisations allow to prove Theorem 1.8 and Theorem 1.2(2) when $\tau_h \sim 1/h$. Examples are developed in Section 4 in order to prove Theorem 1.2(1). Finally, the results concerning hierarchy of time-scales are proved in Section 5 (and lead to Theorem 1.2 for $\tau_h \gg 1/h$).

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2. Two-microlocal analysis of integrable systems on $\mathbb{T}^d$

2.1. Invariant measures and a resonant partition of phase-space. As in [3], the first step in our strategy to characterise the elements in $\widehat{\mathcal{M}}(\tau)$ consists in introducing a partition of phase-space $T^*\mathbb{T}^d$ according to the order of “resonance” of $\xi$, that induces a decomposition of the measures $\mu \in \widehat{\mathcal{M}}(\tau)$. We say that a measure $\mu \in \mathcal{M}_+ (T^*\mathbb{T}^d)$ is a positive $H$-invariant measure on $T^*\mathbb{T}^d$ whenever $\mu$ is invariant under the action of the hamiltonian flow

\begin{equation}
(\phi_s)_* \mu = \mu, \quad \text{with} \quad (\phi_s)_*(x, \xi) = (x + sdH(\xi), \xi).
\end{equation}

Recall that $\mathcal{L}$ is the family of all primitive submodules of $(\mathbb{Z}^d)^*$ and that with each $\Lambda \in \mathcal{L}$, we associate the set $I_\Lambda$ defined in (7): if $\Lambda^\perp \subseteq \mathbb{R}^d$ is the orthogonal to $\Lambda$ with respect to the duality in $(\mathbb{R}^d)^* \times \mathbb{R}^d$ then $I_\Lambda = dH^{-1}(\Lambda^\perp)$. Denote by $\Omega_j \subset \mathbb{R}^d$, for $j = 0, \ldots, d$, the set of resonant vectors of order exactly $j$, that is:

$$
\Omega_j := \left\{ \xi \in (\mathbb{R}^d)^* : \text{rk} \Lambda_\xi = d - j \right\},
$$

where

$$
\Lambda_\xi := \left\{ k \in (\mathbb{Z}^d)^* : k \cdot dH(\xi) = 0 \right\}.
$$

Note that the sets $\Omega_j$ form a partition of $(\mathbb{R}^d)^*$, and that $\Omega_0 = dH^{-1}(\{0\})$; more generally, $\xi \in \Omega_j$ if and only if the Hamiltonian orbit $\{\phi_s(x, \xi) : s \in \mathbb{R}\}$ issued from any $x \in \mathbb{T}^d$ in the direction $\xi$ is dense in a subtorus of $\mathbb{T}^d$ of dimension $j$.

The set $\Omega := \bigcup_{j=0}^{d-1} \Omega_j$ is usually called the set of resonant momenta, whereas $\Omega_d = (\mathbb{R}^d)^* \setminus \Omega$ is referred to as the set of non-resonant momenta.

Finally, write

$$
R_\Lambda := I_\Lambda \cap \Omega_{d-\text{rk} \Lambda}.
$$

Saying that $\xi \in R_\Lambda$ is equivalent to any of the following statements:

(i) for any $x_0 \in \mathbb{T}^d$ the time-average $\frac{1}{T} \int_0^T (\phi_{x_0+tdH(\xi)}) (x) \ dt$ converges weakly, as $T \to \infty$, to the Haar measure on the torus $x_0 + \mathbb{T}_{A^\perp}$. Here, we have used the notation $\mathbb{T}_{A^\perp} := \Lambda^\perp / (2\pi \mathbb{Z}^d \cap \Lambda^\perp)$;
(ii) $\Lambda_\xi = \Lambda$.

Moreover, if $\text{rk} \Lambda = d - 1$ then $R_\Lambda = dH^{-1} \left( \Lambda^\perp \setminus \{0\} \right) = I_\Lambda \setminus \Omega_0$. Note that,

$$
(\mathbb{R}^d)^* = \bigsqcup_{\Lambda \in \mathcal{L}} R_\Lambda,
$$

that is, the sets $R_\Lambda$ form a partition of $(\mathbb{R}^d)^*$. As a consequence, any measure $\mu \in \mathcal{M}_+(T^*\mathbb{R}^d)$ decomposes as

$$
\mu = \sum_{\Lambda \in \mathcal{L}} \mu\big|_{T^d \times R_\Lambda}.
$$

Therefore, the analysis of a measure $\mu$, reduces to that of $\mu\big|_{T^d \times R_\Lambda}$ for all primitive submodule $\Lambda$. Given an $H$-invariant measure $\mu$, it turns out that $\mu\big|_{T^d \times R_\Lambda}$ are utterly determined by the Fourier coefficients of $\mu$. Indeed, define the complex measures on $\mathbb{R}^d$:

$$
\widehat{\mu}_k := \int_{T^d} e^{-ik \cdot x} \frac{\mu(dx, \cdot)}{(2\pi)^{d/2}}, \quad k \in \mathbb{Z}^d,
$$

so that, in the sense of distributions,

$$
\mu(x, \xi) = \sum_{k \in \mathbb{Z}^d} \widehat{\mu}_k(\xi) \frac{e^{ik \cdot x}}{(2\pi)^{d/2}}.
$$

Then, the following Proposition holds.

**Proposition 2.1.** Let $\mu \in \mathcal{M}_+ \left( T^*\mathbb{T}^d \right)$ and $\Lambda \in \mathcal{L}$. The distribution:

$$
\langle \mu \rangle_\Lambda(x, \xi) := \sum_{k \in \Lambda} \widehat{\mu}_k(\xi) \frac{e^{ik \cdot x}}{(2\pi)^{d/2}}
$$

is a finite, positive Radon measure on $T^*\mathbb{T}^d$.

Moreover, if $\mu$ is a positive $H$-invariant measure on $T^*\mathbb{T}^d$, then every term in the decomposition (16) is a positive $H$-invariant measure, and

$$
\mu\big|_{T^d \times R_\Lambda} = \langle \mu \rangle_\Lambda\big|_{T^d \times R_\Lambda}.
$$

Besides, identity (17) is equivalent to the fact that $\mu\big|_{T^d \times R_\Lambda}$ is invariant by the translations

$$(x, \xi) \mapsto (x + v, \xi), \quad \text{for every } v \in \Lambda^\perp.$$
2.2. Second microlocalization on a resonant submanifold. Let \((u_h)\) be a bounded sequence in \(L^2(\mathbb{T}^d)\) and suppose (after extraction of a subsequence) that its Wigner distributions \(w_h(t) = w_{\frac{t}{h\tau_h}}\) converge to a semiclassical measure \(\mu \in L^\infty(\mathbb{R}; \mathcal{M}_+(T^\ast \mathbb{T}^d))\) in the weak-* topology of \(L^\infty(\mathbb{R}; \mathcal{D}'(T^\ast \mathbb{T}^d))\).

From now on, we shall assume that the time scale \((\tau_h)\) satisfies:

\[
\langle \rho \rangle \quad \text{is a bounded sequence.} 
\]

Given \(\Lambda \in \mathcal{L}\), the purpose of this section is to study the measure \(\mu_{|T^d \times R \Lambda}\) by performing a second microlocalization along \(I_\Lambda\) in the spirit of \([12, 13, 14, 29, 27]\) and \([3, 21]\). By Proposition \(2.1\) it suffices to characterize the action of \(\mu_{|T^d \times R \Lambda}\) on test functions having only \(x\)-Fourier modes in \(\Lambda\). With this in mind, we shall introduce two auxiliary “distributions” which describe more precisely how \(w_h(t)\) concentrates along \(T^d \times I_\Lambda\). They are actually not mere distributions, but lie in the dual of the class of symbols \(S^1\) that we define below.

In what follows, we fix \(\xi_0 \in \mathcal{R}_\Lambda\) such that \(dH(\xi_0)\) is definite and, without loss of generality\(^7\), we restrict our discussion to normalised sequences of initial data \((u_h)\) that satisfy:

\[
\hat{u_h}(k) = 0, \quad \text{for } hk \in \mathbb{R}^d \setminus B(\xi_0; \epsilon/2),
\]

where \(B(\xi_0, \epsilon/2)\) is the ball of radius \(\epsilon/2\) centered at \(\xi_0\). The parameter \(\epsilon > 0\) is taken small enough, in order that

\[
d^2H(\xi) \text{ is definite for all } \xi \in B(\xi_0, \epsilon);\]

this implies that \(I_\Lambda \cap B(\xi_0, \epsilon)\) is a submanifold of dimension \(d - \text{rk}\Lambda\), everywhere transverse to \(\langle \Lambda \rangle\), the vector subspace of \((\mathbb{R}^d)\) generated by \(\Lambda\). By eventually reducing \(\epsilon\), we have

\[
B(\xi_0, \epsilon/2) \subset (I_\Lambda \cap B(\xi_0, \epsilon)) \oplus \langle \Lambda \rangle,
\]

by which we mean that any element \(\xi \in B(\xi_0, \epsilon/2)\) can be decomposed in a unique way as \(\xi = \sigma + \eta\) with \(\sigma \in I_\Lambda \cap B(\xi_0, \epsilon)\) and \(\eta \in \langle \Lambda \rangle\). We thus get a map

\[
F : B(\xi_0, \epsilon/2) \longrightarrow (I_\Lambda \cap B(\xi_0, \epsilon)) \times \langle \Lambda \rangle
\]

\[\xi \mapsto (\sigma(\xi), \eta(\xi))\]

With this decomposition of the space of frequencies, we associate two-microlocal test-symbols. We denote by \(S^1\) the class of smooth functions \(a(x, \xi, \eta)\) on \(T^\ast \mathbb{T}^d \times \langle \Lambda \rangle\) that are:

(i) compactly supported on \((x, \xi) \in T^\ast \mathbb{T}^d, \xi \in B(\xi_0, \epsilon/2),\)

(ii) homogeneous of degree zero at infinity w.r.t. \(\eta \in \langle \Lambda \rangle\), i.e. such that there exist \(R_0 > 0\) and \(a_{\text{hom}} \in C^\infty_c(T^\ast \mathbb{T}^d \times S(\langle \Lambda \rangle))\) with

\[
a(x, \xi, \eta) = a_{\text{hom}} \left( x, \xi, \frac{\eta}{|\eta|} \right), \quad \text{for } |\eta| > R_0 \text{ and } (x, \xi) \in T^\ast \mathbb{T}^d
\]

\(^7\)This can be made by applying a cut-off in frequencies to the data.

\(^8\)Note that this is achieved under the weaker hypothesis that \(d^2H(\xi)\) is non-singular and defines a definite bilinear form on \(\langle \Lambda \rangle \times \langle \Lambda \rangle\).
(we have denoted by $S(\Lambda)$ the unit sphere in $\langle \Lambda \rangle \subseteq (\mathbb{R}^d)^*$);

(iii) such that their non vanishing Fourier coefficients (in the $x$ variable) correspond to frequencies $k \in \Lambda$:

$$a(x, \xi, \eta) = \sum_{k \in \Lambda} \hat{a}_k(\xi, \eta) \frac{e^{ik \cdot x}}{(2\pi)^{d/2}}.$$  

We will also express this fact by saying that $a$ has only $x$-Fourier modes in $\Lambda$.

Let $\chi \in C^\infty_c(\langle \Lambda \rangle)$ be a nonnegative cut-off function that is identically equal to one near the origin. For $a \in S^1_\Lambda$, $R > 1$, $\delta < 1$, we decompose $a$ into:

$$a(x, \xi, \eta) = \sum_{j=1}^3 a_j(x, \xi, \eta)$$

with

\begin{align*}
    a_1(x, \xi, \eta) &= a(x, \xi, \eta) \left( 1 - \chi \left( \frac{\eta}{R} \right) \right) \left( 1 - \chi \left( \frac{\eta(\xi)}{\delta} \right) \right), \\
    a_2(x, \xi, \eta) &= a(x, \xi, \eta) \left( 1 - \chi \left( \frac{\eta}{R} \right) \right) \chi \left( \frac{\eta(\xi)}{\delta} \right), \\
    a_3(x, \xi, \eta) &= a(x, \xi, \eta) \chi \left( \frac{\eta}{R} \right). 
\end{align*}

(20)

(21)

This induces a decomposition of the Wigner distribution:

$$w_h(t) = w_{I_{\Lambda}}^h(t) + w_{I_{\Lambda},h,R}(t) + w_{I_{\Lambda},h,R,\delta}(t)$$

when testing against functions $a$ with Fourier modes in $\Lambda$, where:

$$\langle w_{I_{\Lambda}}^h(t), a \rangle := \int_{T^d \times T^d} a_1(x, \xi, \tau_h \eta(\xi)) \, w_h(t) \, (dx, d\xi),$$

\begin{equation}
\langle w_{I_{\Lambda},h,R}(t), a \rangle := \int_{T^d \times T^d} a_2(x, \xi, \tau_h \eta(\xi)) \, w_h(t) \, (dx, d\xi),
\end{equation}

and

\begin{equation}
\langle w_{I_{\Lambda},h,R,\delta}(t), a \rangle := \int_{T^d \times T^d} a_3(x, \xi, \tau_h \eta(\xi)) \, w_h(t) \, (dx, d\xi),
\end{equation}

that we shall analyse in the limits $h \to 0^+$, $R \to +\infty$ and $\delta \to 0$ (taken in that order). One sees that

$$\lim_{\delta \to 0^+} \lim_{R \to +\infty} \lim_{h \to 0} \theta(t) \left\langle w_{I_{\Lambda},h,R,\delta}^h(t), a \right\rangle dt = \int_{\mathbb{R}} \int_{T^d \times T^d} \theta(t) a_\infty \left( x, \xi, \frac{\eta(\xi)}{\eta(\xi)} \right) \mu(t, dx, d\xi) |_{\tau_h \Lambda} dt$$

where $\mu \in \hat{\mathcal{M}}(\tau_h)$ is the semiclassical measure obtained through the sequence $(u_h)$. The restriction of the measure thus obtained to $T^d \times R_\Lambda$ vanishes, and we do not need to further analyse the term involving the distribution $w_{I_{\Lambda},h,R,\delta}^h(t)$.

For $a \in S^1_\Lambda$, we introduce the notation

$$\text{Op}_h^{I_{\Lambda}}(a(x, \xi, \eta)) := \text{Op}_h \left( a(x, \xi, \tau_h \eta(\xi)) \right)$$
so that the distributions $w_{I_h,R,\delta}^\Lambda(t)$ and $w_{I_h,\delta,R}(t)$ can be expressed for all $t \in \mathbb{R}$ by
\[
\langle w_{I_h,R,\delta}^\Lambda(t), a \rangle = \langle u_h, \mathcal{S}^{-\tau_h t}_h \text{Op}_h^\Lambda(a_2) \mathcal{S}^{\tau_h t}_h u_h \rangle_{L^2(\mathbb{T}^d)}, \\
\langle w_{I_h,\delta,R}(t), a \rangle = \langle u_h, \mathcal{S}^{-\tau_h t}_h \text{Op}_h^\Lambda(a_3) \mathcal{S}^{\tau_h t}_h u_h \rangle_{L^2(\mathbb{T}^d)}.
\]
Notice that, for all $\beta \in \mathbb{N}^d$,
\[
\left\| \partial_\xi^\beta (a(x,h\xi,\tau_h \eta(h\xi))) \right\|_{L^\infty} \leq C_\beta (\tau_h h)^{|eta|}.
\]
The Calderón-Vaillancourt theorem (see [8] or the appendix of [3] for a precise statement) therefore ensures that there exist $N \in \mathbb{N}$ and $C_N > 0$ such that
\[
\forall a \in \mathcal{S}_1^\Lambda, \quad \left\| \text{Op}_h^\Lambda(a) \right\|_{L^1(\mathbb{R}^d)} \leq C_N \sum_{|\alpha| \leq N} \left\| \partial_{x,\xi}^\alpha a \right\|_{L^\infty},
\]
since $(h \tau_h)$ is bounded.

As a consequence of (24), both $w_{I_h,R,\delta}^\Lambda$ and $w_{I_h,\delta,R}$ are bounded in $L^\infty(\mathbb{R};(\mathcal{S}_1^\Lambda)')$. After possibly extracting subsequences, we have for every $\varphi \in L^1(\mathbb{R})$ and $a \in \mathcal{S}_1^\Lambda$,
\[
\int_{\mathbb{R}} \varphi(t) \langle \tilde{\mu}_\Lambda(t,\cdot),a \rangle \, dt := \lim_{\delta \to 0} \lim_{R \to \infty} \lim_{h \to 0^+} \int_{\mathbb{R}} \varphi(t) \langle w_{I_h,R,\delta}^\Lambda(t),a \rangle \, dt,
\]
and
\[
\int_{\mathbb{R}} \varphi(t) \langle \tilde{\mu}_\Lambda(t,\cdot),a \rangle \, dt := \lim_{R \to \infty} \lim_{h \to 0^+} \int_{\mathbb{R}} \varphi(t) \langle w_{I_h,\delta,R}(t),a \rangle \, dt.
\]

**Remark 2.2.** When $\tau_h \ll 1/h$ the quantization of our symbols generates a semi-classical pseudodifferential calculus with gain $h \tau_h$. The operators $\text{Op}_h^\Lambda(a)$ are semiclassical both in $\xi$ and $\eta$. This implies that the accumulation points $\tilde{\mu}_\Lambda$ and $\tilde{\mu}_\Lambda$ are positive measures (see for instance [4]).

Because of the existence of $R_0 > 0$ and of $a_{\text{hom}} \in C^\infty_c(T^*\mathbb{T}^d \times S(\Lambda))$ such that
\[
a(x,\xi,\eta) = a_{\text{hom}} \left(x,\xi,\frac{\eta}{|\eta|}\right), \quad \text{for } |\eta| \geq R_0,
\]
for $R$ large enough, the value $\langle w_{I_h,R,\delta}^\Lambda(t),a \rangle$ only depends on $a_{\text{hom}}$. Therefore, the limiting object $\tilde{\mu}_\Lambda(t,\cdot) \in (\mathcal{S}_1^\Lambda)'$ is zero-homogeneous in the last variable $\eta \in \mathbb{R}^d$, supported at infinity, and, by construction, it is supported on $\xi \in I_\Lambda$. This can be also expressed as the fact that $\tilde{\mu}_\Lambda$ is a “distribution” on $\mathbb{T}^d \times I_\Lambda \times \overline{\langle \Lambda \rangle}$ (where $\overline{\langle \Lambda \rangle}$ is the compactification of $\langle \Lambda \rangle$ by adding the sphere $S(\Lambda)$ at infinity) supported on $\{\eta \in S(\Lambda)\}$. Besides, the distribution $\tilde{\mu}_\Lambda$ is supported on $\mathbb{T}^d \times I_\Lambda \times \langle \Lambda \rangle$. Indeed, we have for all $t$,
\[
\langle w_{I_h,\delta,R}(t), a(x,\xi,\eta) \rangle = \langle w_{I_h,\delta,R}(t), a(x,\sigma(\xi),\eta) \rangle + O(\tau_h^{-1})
\]
since, by (24),
\[
\text{Op}_h^\Lambda(a_3(x,\xi,\eta)) = \text{Op}_h^\Lambda(a(x,\sigma(\xi) + \tau_h^{-1}\eta,\eta)\chi(\eta/R))
\]

\[
= \text{Op}_h^\Lambda(a(x,\sigma(\xi),\eta)\chi(\eta/R)) + O(\tau_h^{-1})
\]
where the $O(\tau_h^{-1})$ term is understood in the sense of the operator norm of $L(L^2(\mathbb{R}^d))$ and depends on $R$ (the fact that we first let $h$ go to $0^+$ is crucial here).

From the decomposition $w_h(t) = w_{1,h,R,h}^I(t) + w_{\Delta, h,R}^I(t) + w_{h,R,R,h}^I(t)$ (when testing against symbols having Fourier modes in $\Lambda$), it is immediate that the measure $\mu(t, \cdot)_{|_{T^d \times R}}$ is related to $\tilde{\mu}^A$ and $\bar{\mu}_A$ according to the following Proposition.

**Proposition 2.3.** Let

$$\nu^A(t, \cdot) := \int_{(\Lambda)} \tilde{\nu}^A(t, \cdot, d\eta)_{|_{T^d \times R}}, \quad \nu_A(t, \cdot) := \int_{(\Lambda)} \tilde{\nu}_A(t, \cdot, d\eta)_{|_{T^d \times R}}.$$

Then both $\nu^A(t, \cdot)$ and $\nu_A(t, \cdot)$ are $H$-invariant positive measures on $T^*\mathbb{T}^d$ and satisfy:

$$\mu(t, \cdot)_{|_{T^d \times R}} = \nu^A(t, \cdot) + \nu_A(t, \cdot).$$

This proposition motivates the analysis of the structure of the accumulation points $\bar{\nu}_A(t, \cdot)$ and $\tilde{\nu}^A(t, \cdot)$. It turns out that both $\tilde{\nu}^A$ and $\bar{\nu}_A$ have some extra regularity in the variable $x$, although for two different reasons. Our next two results form one of the key steps towards the proof of Theorem 1.2.

Let us first deal with $\tilde{\nu}^A(t, \cdot)$. We define, for $(x, \xi, \eta) \in T^*\mathbb{T}^d \times ((\Lambda) \setminus \{0\})$ and $s \in \mathbb{R},$

$$\phi_s^0(x, \xi, \eta) := (x + sdH(\xi), \xi, \eta),$$

$$\phi_s^1(x, \xi, \eta) := \left( x + sd^2H(\sigma(\eta)) \frac{\eta}{|\eta|}, \xi, \eta \right).$$

This second definition extends in an obvious way to $\eta \in S(\Lambda)$ (the sphere at infinity). On the other hand, the map $(x, \xi, \eta) \mapsto \phi_s^1(x, \xi, \eta)$ extends to $\eta = 0$.

**Theorem 2.4:** $\tilde{\nu}^A(t, \cdot)$ is a positive measure on $\mathbb{T}^d \times I_\Lambda \times \langle \Lambda \rangle$ supported on the sphere at infinity $S(\Lambda)$ in the variable $\eta$. Besides, for a.e. $t \in \mathbb{R}$, the measure $\tilde{\nu}^A(t, \cdot)$ satisfies the invariance properties:

$$\phi_s^0 \ast \tilde{\nu}^A(t, \cdot) = \tilde{\nu}^A(t, \cdot), \quad \phi_s^1 \ast \tilde{\nu}^A(t, \cdot) = \tilde{\nu}^A(t, \cdot), \quad s \in \mathbb{R}.$$

Note that this result holds whenever $\tau_h \ll 1/h$ or $\tau_h = 1/h$. This is in contrast with the situation we encounter when dealing with $\bar{\nu}_A(t, \cdot)$. The regularity of this object indeed depends on the properties of the scale.

**Theorem 2.5.** (1) The distributions $\bar{\nu}_A(t, \cdot)$ are supported on $\mathbb{T}^d \times I_\Lambda \times \langle \Lambda \rangle$ and are continuous with respect to $t \in \mathbb{R}$. Moreover, they satisfy the following propagation law:

$$\forall t \in \mathbb{R}, \quad \bar{\nu}_A(t, x, \xi, \eta) = (\phi_{i|\eta}) \ast \bar{\nu}_A(0, x, \xi, \eta).$$

(2) If $\tau_h \ll 1/h$ then $\bar{\nu}_A(t, \cdot)$ is a positive measure. When $\tau_h = 1/h$, the projection of $\bar{\nu}_A(t, \cdot)$ on $T^*\mathbb{T}^d$ is a positive measure, whose projection on $\mathbb{T}^d$ is absolutely continuous with respect to the Lebesgue measure.

**Remark 2.6.** For $\tau_h = 1/h$ the propagation law satisfied by distributions $\bar{\nu}_A(t, \cdot)$ can be interpreted in terms of a Schrödinger flow type propagator. The precise statement can be found in Proposition 2.13 in Section 2.4.
Remark 2.7. Note that for all \( \xi \in \mathbb{R}^d \setminus C_H \) (recall that \( C_H \) stands for the points where the Hessian \( d^2H(\xi) \) is not definite) we have \( \mathbb{R}^d = \Lambda^\perp \oplus d^2H(\xi) \langle \Lambda \rangle \). Therefore, the flows \( \phi_s^0 \) and \( \phi_s^1 \) are independent on \( \mathbb{T}^d \times \left( R_\Lambda \setminus C_H \right) \times \langle \Lambda \rangle \).

Remark 2.8. If \( \text{rk} \Lambda = 1 \) then (28) implies that, for a.e. \( t \in \mathbb{R} \), and for any \( \nu \in \langle \Lambda \rangle \), the measure \( \tilde{\mu}^A (t, \cdot) \mid_{\mathbb{T}^d \times R_\Lambda \times \langle \Lambda \rangle} \) is invariant under

\[
(x, \sigma, \eta) \mapsto (x + d^2H(\sigma) \cdot \nu, \sigma, \eta).
\]

On the other hand, the invariance by the Hamiltonian flow and Proposition 2.7 imply that \( \tilde{\mu}^A (t, \cdot) \mid_{\mathbb{T}^d \times R_\Lambda \times \langle \Lambda \rangle} \) is also invariant under

\[
(x, \sigma, \eta) \mapsto (x + v, \sigma, \eta)
\]

for every \( v \in \Lambda^\perp \). Using Remark 2.7 and the fact that the Hessian \( d^2H(\sigma) \) is definite on the support of \( \tilde{\mu}^A (t, \cdot) \mid_{\mathbb{T}^d \times R_\Lambda \times \langle \Lambda \rangle} \), we conclude that the measure \( \tilde{\mu}^A (t, \cdot) \mid_{\mathbb{T}^d \times R_\Lambda \times \langle \Lambda \rangle} \) is constant in \( x \in \mathbb{T}^d \) in this case.

Remark 2.9. Consider the decomposition \( \mu(t, \cdot) = \sum_{\Lambda \in \mathcal{L}} \mu^A(t, \cdot) + \sum_{\Lambda \in \mathcal{L}} \mu^A(t, \cdot) \). given by Proposition 2.3. When \( \tau_h = 1/h \), Theorem 2.3 implies that the second term defines a positive measure whose projection on \( \mathbb{T}^d \) is absolutely continuous with respect to the Lebesgue measure.

We now give the proof of Theorems 2.4 and 2.5.

2.3. Invariance properties of \( \tilde{\mu}^A \). In this section, we prove Theorem 2.4. The positivity of \( \tilde{\mu}^A(t, \cdot) \) can be deduced following the lines of [14] §2.1, or those of the proof of Theorem 1 in [16]; see also the appendix of [3].

Let us now check the invariance property (28). We use the following Lemma which gives approximate transport equations by the flow \( \phi^0 \).

Lemma 2.10. For every \( a \in \mathcal{S}^1_\Lambda \) and \( \varphi \in C_0^\infty (\mathbb{R}) \), we have

\[
\int_\mathbb{R} \varphi(t) \langle u_h, S_h^{-\tau_h t} \quad \text{Op}_h^A (a) S_h^{\tau_h t} \quad u_h \rangle_{L^2(\mathbb{T}^d)} dt = \int_\mathbb{R} \varphi(t) \langle u_h, \quad \text{Op}_h^A (a \circ \phi^{0}_{\tau_h t}) \quad u_h \rangle_{L^2(\mathbb{T}^d)} dt + o(1).
\]

Remark 2.11. Consider the \( h \)-dependent flow

\[
\phi_{s h}^1 (x, \xi, \eta) := (x + s \tau_h (dH(\xi) - dH(\sigma(\xi))), \xi, \eta),
\]

then if \( a \) has only Fourier modes in \( \Lambda \), we have

\[
a \circ \phi_{s h}^1 (x, \xi, \eta) = \sum_{k \in \Lambda} \hat{a}_k(\xi, \eta) e^{ik \cdot (x + s \tau_h (dH(\xi) - dH(\sigma(\xi)))}
\]

\[
= \sum_{k \in \Lambda} \hat{a}_k(\xi, \eta) e^{ik \cdot (x + s \tau_h dH(\xi))} = a \circ \phi_{\tau_h s}^0 (x, \xi, \eta).
\]

This comes from the fact that for every \( k \in \Lambda \) and \( \xi \in \mathbb{R}^d \), one has \( k \cdot dH(\sigma(\xi)) = 0 \).
We postpone the proof of Lemma 2.10 to the end of this section and we start proving (28). The invariance by the “geodesic flow” \( \phi^0 \) is standard and can be proved following the lines of the proof of property 3 in the appendix. Using (20), we have

\[
\begin{align*}
\int_{\mathbb{R}} \varphi(t) \langle \tilde{\mu}^A(t, \cdot) , a \rangle dt &= \lim_{\delta \to 0} \lim_{R \to +\infty} \lim_{h \to 0} \int_{\mathbb{R}} \varphi(t) \langle w^{1^h}_{h,R,\delta} (t) , a \rangle dt \\
&= \lim_{\delta \to 0} \lim_{R \to +\infty} \lim_{h \to 0} \int_{\mathbb{R}} \varphi(t) \langle w_h(t) , a_2 (x, \xi, \tau_h \eta(\xi)) \rangle dt
\end{align*}
\]

(30) (along subsequences). Notice that the symbol

\[
a_2 \circ \phi^1_s(x, \xi, \eta) = a_2 \left( x + sd^2 H(\sigma(\xi)) \frac{\eta}{|\eta|}, \xi, \eta \right),
\]

is a well-defined element of \( S^1_A \), since, for fixed \( R \), \( a_2 \) is identically equal to zero near \( \eta = 0 \); moreover

\[
\forall \omega \in S(\Lambda), \quad (a_2 \circ \phi^1_s)_{\text{hom}}(x, \xi, \omega) = a_{\text{hom}}(x + sd^2 H(\sigma(\xi)) \omega, \xi, \omega);
\]

therefore,

\[
\int_{\mathbb{R}} \varphi(t) \langle \tilde{\mu}^A(t, \cdot) , a \circ \phi^1_s \rangle dt = \lim_{\delta \to 0} \lim_{R \to +\infty} \lim_{h \to 0} \int_{\mathbb{R}} \varphi(t) \langle w^{1^h}_{h,R,\delta} (t) , a \circ \phi^1_s \rangle dt.
\]

(31) In order to relate (31) to (30) we note that the symbol:

\[
a_2 \circ \phi^1_{s/|\eta|} (x, \xi, \eta),
\]

satisfies:

\[
a_2 \circ \phi^1_{s/|\eta|} (x, \xi, \eta) = a_2 \left( x + \frac{s}{|\eta(\xi)|} \left( dH(\xi) - dH(\sigma(\xi)), \xi, \tau_h \eta(\xi) \right) \right)
\]

\[
= a_2 \left( x + sd^2 H(\sigma(\xi)) \frac{\eta(\xi)}{|\eta(\xi)|} + \frac{s}{|\eta(\xi)|} G(\xi)[\eta(\xi), \eta(\xi)], \xi, \tau_h \eta(\xi) \right)
\]

\[
(32)
\]

we have used that, on supp \( a_2 \), we have \( |\eta(\xi)| \leq C \delta \) and the function

\[
G(\xi) = \int_0^1 d^3 H(\sigma(\xi) + t\eta(\xi))(1 - t) dt,
\]

(33) is uniformly bounded. On the other hand, by Lemma 2.10 and Remark 2.11, we have

\[
\langle w^{1^h}_{h,R,\delta} (t) , a \rangle = \langle w_h(t) , a_2 (x, \xi, \tau_h \eta(\xi)) \rangle = \left\langle w_h(0) , a_2 \circ \phi^1_{t} (x, \xi, \tau_h \eta(\xi)) \right\rangle + o_h(1).
\]

(34) Therefore, combining (32) and (34) we obtain:

\[
\int_{\mathbb{R}} \varphi(t) \langle w^{1^h}_{h,R,\delta} (t) , a \circ \phi^1_s \rangle dt = \int_{\mathbb{R}} \varphi(t) \langle w_h(0) , a_2 \circ \phi^1_s \circ \phi^1_{t} (x, \xi, \tau_h \eta(\xi)) \rangle dt + O(\delta) + o_h(1)
\]

\[
= \int_{\mathbb{R}} \varphi \left( t - \frac{s}{|\eta|} \right) \langle w_h(0) , a_2 \circ \phi^1_{t} (x, \xi, \tau_h \eta(\xi)) \rangle dt + O(\delta) + o_h(1).
\]

(35)
Since $|\eta| > R$ on the support of $a_2$, we have for all $K \in \mathbb{N}$

$$\lim_{R \to +\infty} \int_{\mathbb{R}} \sup_{x, \xi, \eta} \sup_{|\alpha| \leq K} |\partial^\alpha_{x, \xi, \eta} \left[ (\varphi(t - s/|\eta|) - \varphi(t)) \left( a_2 \circ \phi^1_t \right) (x, \xi, \eta) \right]| \, dt = 0,$$

which implies, in view of (35):

$$\lim_{\delta \to 0} \lim_{R \to +\infty} \int_{\mathbb{R}} \varphi(t) \left( w^{A}_{h, R, \delta}(t), a \circ \phi^1_{s} \right) dt = \lim_{\delta \to 0} \lim_{R \to +\infty} \int_{\mathbb{R}} \varphi(t) \left( w_{h}(0), a \circ \phi^1_{t} \right) (x, \xi, \tau_{h}(\xi)) dt.$$

Applying again (34) to the left hand side of the above identity concludes the proof of Theorem 2.4.

Let us now prove Lemma 2.10.

**Proof of Lemma 2.10** Write

(36)

$$\int_{\mathbb{R}} \varphi(t) \langle u_{h}, S^{-\tau_{h}}_{h, a} S^{\tau_{h}}_{h, a} u_{h} \rangle_{L^{2}(\mathbb{T} \mathbb{R})} \, dt$$

$$= \sum_{k, j \in \mathbb{Z}^{d}, k - j \in \mathbb{A}} \hat{\varphi} \left( \tau_{h} \left( H(hk) - H(hj) \right) \right) \hat{u}_{h}(k) \hat{u}_{h}(j) \hat{a}_{j-k} \left( h \frac{k + j}{2}, \tau_{h} \right) \left( h \frac{k + j}{2} \right)$$

where $a(x, \xi, \eta) = \sum_{k \in \mathbb{A}} \hat{a}_{k}(\xi, \eta)e^{ikx}$. Notice that,

$$H(hk) - H(hj) = h \frac{k + j}{2} \cdot (k - j) + h^{3} \tau_{h} \left( h \frac{k + j}{2}, k - j \right),$$

where $r_{h}(\xi, \ell)$ satisfies, for every $K \subset \mathbb{R}^{d}$ compact and convex and for every $\beta \in \mathbb{N}^{d}$,

$$|\partial^\beta r_{h}(\xi, \ell)| \leq C_{K, \beta} |\ell|^{3}, \quad \xi, h \ell \in K.$$ 

Therefore, since $\hat{\varphi}$ is uniformly Lipschitz, we have, for $hk, hj \in K$

$$\hat{\varphi} \left( \tau_{h} \left( H(hk) - H(hj) \right) \right) - \hat{\varphi} \left( \tau_{h} \frac{k + j}{2} \cdot (k - j) \right) + h^{2} \tau_{h} M_{h} \left( h \frac{k + j}{2}, k - j \right)$$

with

$$M_{h}(\xi, \ell) = r_{h}(\xi, \ell) \int_{0}^{1} \hat{\varphi}' \left( \tau_{h} \frac{k + j}{2} \cdot (k - j) \right) + h^{2} \tau_{h} r_{h}(\xi, \ell) ds.$$ 

Plugging this equation in (36) we obtain:

$$\int_{\mathbb{R}} \varphi(t) \langle u_{h}, S^{-\tau_{h}}_{h, a} S^{\tau_{h}}_{h, a} u_{h} \rangle_{L^{2}(\mathbb{T} \mathbb{R})} \, dt$$

$$= \int_{\mathbb{R}} \varphi(t) \langle u_{h}, \text{Op}_{h}^{A} \left( a(x + \tau_{h}tdH(\xi, \xi, \eta)) u_{h} \right) \rangle_{L^{2}(\mathbb{T} \mathbb{R})} dt + h^{2} \tau_{h} \langle u_{h}, \text{Op}_{h}^{A} \left( R_{a}^{h} u_{h} \right) \rangle_{L^{2}(\mathbb{T} \mathbb{R})},$$
where the symbol \( R^h \) is characterized by its Fourier coefficients:
\[
\widehat{R}^h(\ell, \xi, \eta) = M_h(\xi, \ell) \hat{a}_\ell(\xi, \eta).
\]
Since the function \( a \) is compactly supported in \((x, \xi)\), we deduce from the properties of \( M_h \) and \( r_h \) that for all \( \beta \in \mathbb{N}^d \), there exists \( C_\beta > 0 \) such that
\[
\left| \partial^\beta_x R_h(x, h\xi, \tau_h \eta(h\xi)) \right| \leq C_\beta, \quad (x, \xi) \in T^* \mathbb{T}^d.
\]
Passing to the limit \( h \to 0^+ \) allows to conclude the proof of the Lemma.

\[\Box\]

2.4. Propagation and regularity of \( \tilde{\mu}_\Lambda \). This section is devoted to proving Theorem 2.5 and showing that, when \( \tau_h = 1/h \), the distribution \( \tilde{\mu}_\Lambda \) satisfies a propagation law that involves the family of unitary propagators
\[
e^{-\frac{i}{h}d^2H(\sigma)D_t}D_y, \quad \sigma \in I_\Lambda.
\]
We start by proving Theorem 2.5(1). The statement on the support of \( \tilde{\mu}_\Lambda \) was already proved in Section 2.2. The propagation law (and hence, the continuity with respect to \( t \)) comes from the following result.

Proposition 2.12. For every \( a \in S^1_\Lambda \) and every \( t \in \mathbb{R} \) the following holds:
\[
\langle w_{I_\Lambda, h, R}(t), a \rangle = \langle w_{I_\Lambda, h, R}(0), a \circ \tilde{\phi}^1_h \rangle + o_h(1),
\]
where
\[\text{(37)} \quad \tilde{\phi}^1_h(x, \xi, \eta) := (x + td^2H(\xi)\eta, \xi, \eta) .\]
Proof. We use Lemma 2.10 and Remark 2.11 to conclude:
\[
\langle w_{I_\Lambda, h, R}(t), a \rangle = \langle w_h(0), a_3 \circ \tilde{\phi}^1_h(x, \xi, \tau_h \eta(\xi)) \rangle + o_h(1),
\]
where \( a_3 \) is defined by (21). By definition of \( \tilde{\phi}^1_h \) and by Taylor expansion, we obtain
\[
a_3 \circ \tilde{\phi}^1_h(x, \xi, \tau_h \eta(\xi)) = a_3 \left( x + t \tau_h(dH(\xi) - dH(\sigma(\xi)), \xi, \tau_h \eta(\xi)) \right)
= a_3 \left( x + td^2H(\sigma(\xi))\tau_h \eta(\xi) + t \tau_h G(\xi)[\eta(\xi), \eta(\xi)], \xi, \tau_h \eta(\xi) \right)
=: b_h(t, x, \xi, \tau_h \eta(\xi)),
\]
where \( G \) is defined by (33) and is bounded and smooth on the support of \( a_3 \). Therefore, (38)
\[
b_h(t, x, \xi, \eta) = a_3 \left( x + td^2H(\sigma(\xi))\eta + t \tau_h^{-1}G(\xi)[\eta(\xi), \eta], \xi, \eta \right)
= a_3 \left( x + td^2H(\sigma(\xi))\eta, \xi, \eta \right) + O(\tau_h^{-1}),
\]
from which the result follows.

Let us now focus on statement (2) of Theorem 2.5. The result concerning time scales \( \tau_h \ll 1/h \) was already discussed in Remark 2.2.

From now on, suppose \( \tau_h = 1/h \). Let us introduce some notations. We consider the set \( L^2(\mathbb{T}^d, \Lambda) \subseteq L^2(\mathbb{T}^d) \) consisting of functions having only Fourier modes in \( \Lambda \). For any
For $\Phi \in L^2(\mathbb{T}^d, \Lambda)$,  
\[ \text{Op}_1(a_\sigma(y, \eta))\Phi = \sum_{\lambda, v \in \Lambda} \hat{a}_\lambda \left( \sigma, v + \frac{\lambda}{2} \right) \hat{\Phi}(v) e^{i(v+\lambda)y}, \]
where $a(x, \xi, \eta) = \sum_{k \in \Lambda} \hat{a}_k(\xi, \eta) e^{ikx}$. Thus Op$_1(a_\sigma(y, \eta))$ is the Weyl quantization of the symbol $a_\sigma(y, \eta) := a(y, \sigma, \eta)$. In particular, for $a = a(x, \xi)$ independent of the variable $\eta$, Op$_1(a_\sigma(y))$ is the multiplication operator
\[ \text{Op}_1(a_\sigma(y))\Phi = a(y, \sigma)\Phi \]
we will simply denote by $a_\sigma(y)$ this multiplication operator. We have the expression similar to formula (52) in the appendix,
\[ \langle \Phi, \text{Op}_1(a_\sigma(y, \eta))\Phi \rangle_{L^2(\mathbb{T}^d, \Lambda)} = \frac{1}{(2\pi)^{d/2}} \sum_{\nu, \nu' \in \Lambda} \hat{a}_{\nu'-\nu} \left( \sigma, \frac{\nu + \nu'}{2} \right) \hat{\Phi}(\nu) \hat{\Phi}(\nu'). \]
Then, the last statement of Theorem 2.5(2) is a consequence of the following Proposition.

**Proposition 2.13.** There exists $M \in \mathcal{M}_+ \left( I_\Lambda; \mathcal{L}^1 \left( L^2(\mathbb{T}^d, \Lambda) \right) \right)$ such that for all $a \in C_0^\infty(T^*\mathbb{T}^d)$ with Fourier modes in $\Lambda$ and all $\varphi \in L^1(\mathbb{R})$,
\[ \int_\mathbb{R} \varphi(t) \langle \tilde{\mu}_\Lambda (t, \cdot), a \rangle dt = \int_\mathbb{R} \varphi(t) \int_{I_\Lambda} \text{Tr} \left( e^{-\frac{i\nu}{2} d^2 H(\sigma) D_\varphi D_y} a_\sigma(y) e^{\frac{i\nu}{2} d^2 H(\sigma) D_\varphi D_y} M(d\sigma) \right) dt. \]

**Remark 2.14.** (i) The operator-valued measure $M$ is globally defined, it describes the limit of $\langle w_{I_\Lambda \cdot h, R}(t), a \rangle$ for symbols $a = a(x, \xi)$. For symbols $a = a(x, \xi, \eta) \in S^1_\Lambda$, one cannot build such a global measure (see Remark 2.13 which emphasizes the technical obstruction).

(ii) Note that for any given $\sigma$, the operator $e^{\frac{i\nu}{2} d^2 H(\sigma) D_\varphi D_y}$ obviously preserves $L^2(\mathbb{T}^d, \Lambda)$.

The proof of Proposition 2.13 relies on three steps:

1. We first define an operator $U_h$ which maps $(2\pi\mathbb{Z}^d)$-periodic functions on $(2\pi\mathbb{Z}^d)$-periodic functions with Fourier frequencies only in $\Lambda$.
2. Then, we express $w_{I_\Lambda \cdot h, R}(t)$ in terms of $U_h$ and the operators $e^{\frac{i\nu}{2} d^2 H(\sigma) D_\varphi D_y}$.
3. We then conclude by passing to the limit when $h \to 0$ and $R \to +\infty$.

**First Step: Construction of the operator $U_h$.** We introduce an auxiliary lattice $\tilde{\Lambda} \subset \mathbb{Z}^d$ such that $\Lambda^\perp \oplus \tilde{\Lambda} = \mathbb{Z}^d$ (recall that $\Lambda^\perp$ is the orthogonal of $\Lambda$ in the duality sense). We denote by $\alpha$ the projection on $\langle \tilde{\Lambda} \rangle$, in the direction of $\Lambda^\perp$. We have $\alpha(\mathbb{Z}^d) = \tilde{\Lambda} \subset \mathbb{Z}^d$. For $\sigma \in (\mathbb{R}^d)^*$, we shall denote by $\sigma^\alpha \in \langle \Lambda \rangle$ the linear form $\sigma^\alpha(y) = \sigma \cdot \alpha(y)$. We fix a bounded fundamental domain $D_\Lambda$ for the action of $\Lambda$ on $\langle \Lambda \rangle$. For $\eta \in \langle \Lambda \rangle$, there is a unique \{\eta\} $\in D_\Lambda$ (the “fractional part” of $\eta$) such that $\eta - \{\eta\} \in \Lambda$. Finally, take $b \in C_0^\infty((\mathbb{R}^d)^*)$

---

9Recall that given a Hilbert space $H$, $\mathcal{L}^1(H)$ stands for the space of bounded trace-class operators acting on $H$ and, for a Polish space $X$, $\mathcal{M}_+(X; \mathcal{L}^1(H))$ denotes the set of positive measures taking values on $\mathcal{L}^1(H)$.  

---
supported in the ball $B(\xi_0, \epsilon) \subset (\mathbb{R}^d)^*$, and identically equal to 1 on $B(\xi_0, \epsilon/2)$. We set for $f \in L^2(\mathbb{T}^d)$, $\sigma \in I_\Lambda$, $y \in \mathbb{T}^d$,

$$U_h f(\sigma, y) = (2\pi)^{-\frac{d}{2}} e^{i\frac{\alpha^\sigma}{2} y} \int_{x \in \mathbb{T}^d} f(x) \sum_{\eta \in \langle \Lambda \rangle, (\sigma, \eta) \in F(h\mathbb{Z}^d)} b(\sigma + \eta)e^{\frac{\eta y}{h}} e^{-\frac{1}{h}(\sigma + \eta) \cdot x} dx$$

$$= (2\pi)^{-\frac{d}{2}} e^{i\frac{\alpha^\sigma}{2} y} \sum_{\eta \in \langle \Lambda \rangle, (\sigma, \eta) \in F(h\mathbb{Z}^d)} b(\sigma + \eta) \hat{f} \left( \frac{\sigma + \eta}{h} \right) e^{\frac{\eta y}{h}}.$$

Recall that $F$ is the local coordinate system defined in (19), with the property that if $F(\xi) = (\sigma(\xi), \eta(\xi))$ then $\xi = \sigma(\xi) + \eta(\xi)$. Note that $U_h f(\sigma, y) = 0$ if $(\sigma, \eta) \notin F(h\mathbb{Z}^d)$ for every $\eta \in \langle \Lambda \rangle$ (since the sum has an empty index set). The role of the term $e^{i\frac{\alpha^\sigma}{2} y}$ becomes clear in the following lemma.

**Lemma 2.15.** If $f$ is $(2\pi\mathbb{Z})^d$-periodic, then $U_h f$ is $(2\pi\mathbb{Z})^d$-periodic and has only frequencies in $\Lambda$. Therefore, $U_h$ maps $L^2(\mathbb{T}^d)$ into the subspace $L^2(\mathbb{T}^d, \Lambda)$ of $L^2(\mathbb{T}^d)$.

**Proof.** It is enough to show that for any $\sigma \in I_\Lambda$, $\eta \in \langle \Lambda \rangle$ such that $\sigma + \eta \in h\mathbb{Z}^d$,

$$\frac{\eta}{h} + \left\{ \frac{\sigma^\alpha}{h} \right\} \in \Lambda.$$

By definition, $\frac{\eta}{h} + \left\{ \frac{\sigma^\alpha}{h} \right\} \in \langle \Lambda \rangle$, and we want to prove that for any $k \in 2\pi\mathbb{Z}^d$, $(\frac{\eta}{h} + \left\{ \frac{\sigma^\alpha}{h} \right\}) \cdot k \in 2\pi\mathbb{Z}$. We write

$$(\frac{\eta}{h} + \left\{ \frac{\sigma^\alpha}{h} \right\}) \cdot k = (\sigma + \eta) \cdot k - \sigma \cdot k + \frac{\sigma^\alpha}{h} \cdot k$$

and we know that $\frac{\sigma^\alpha}{h} \cdot k \in 2\pi\mathbb{Z}$. We then use the fact that there exists $\lambda \in \Lambda$ such that $\left\{ \frac{\sigma^\alpha}{h} \right\} = \frac{\sigma^\alpha}{h} + \lambda$, and write

$$\left\{ \frac{\sigma^\alpha}{h} \right\} \cdot k - \sigma \cdot k = \frac{\sigma}{h} \cdot \alpha(k) + \lambda \cdot k = \frac{\sigma}{h} \cdot (\alpha(k) - k) + \lambda \cdot k.$$

Since $\sigma + \eta = hl$ for some $l \in \mathbb{Z}^d$ and since $k \in (2\pi\mathbb{Z})^d$, we obtain $\frac{\sigma}{h} \cdot (\alpha(k) - k) = l \cdot (\alpha(k) - k)$ with $\alpha(k) - k \in \Lambda^\perp \cap (2\pi\mathbb{Z})^d$ (since $\alpha(\mathbb{Z}^d) = \Lambda \subset \mathbb{Z}^d$). Finally, we get $\frac{\sigma}{h} \cdot (\alpha(k) - k) \in 2\pi\mathbb{Z}$ and we also have $\lambda \cdot k \in 2\pi\mathbb{Z}$, which concludes the proof. \hfill \Box

Note that if $f$ is $(2\pi\mathbb{Z})^d$-periodic, the Fourier coefficients of $U_h f$ satisfy

$$\forall \eta \in \Lambda, \quad \hat{U_h f}(\sigma, \eta) = b(\sigma + h\eta - h \left\{ \frac{\sigma^\alpha}{h} \right\}) \hat{f} \left( \frac{\sigma}{h} + \eta - \left\{ \frac{\sigma^\alpha}{h} \right\} \right)$$

and we have the following Plancherel-type formula.

**Lemma 2.16.** If $f$ is $(2\pi\mathbb{Z})^d$-periodic, then we have

$$\forall \sigma \in I_\Lambda, \quad \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)b(hk)|^2 = \sum_{\sigma \in h(\mathbb{Z}^d)} \int_{\mathbb{T}^d} |U_h f(\sigma, y)|^2 dy.$$
Proof. We have for all \((2\pi\mathbb{Z}^d)\)-periodic function \(f\),
\[
\sum_{k \in \mathbb{Z}^d} |\hat{f}(k)b(hk)|^2 = \sum_{\sigma \in I, \eta \in (\Lambda)} |b(\sigma + \eta)|^2 \left| \frac{\hat{f}(\sigma + \eta)}{h} \right|^2
\]
\[
= \frac{1}{(2\pi)^d} \sum_{\sigma \in (h\mathbb{Z}^d)} \int_{T^d} \sum_{\sigma + \eta + \eta' \in h\mathbb{Z}^d} b(\sigma + \eta)\overline{b(\sigma + \eta')} \frac{\hat{f}(\sigma + \eta)}{h} \frac{\hat{f}(\sigma + \eta')}{h} \\
\times \exp\left(\frac{i}{h} (y \cdot (\eta - \eta'))\right) dy
\]
\[
= \sum_{\sigma \in (h\mathbb{Z}^d)} \int_{T^d} \left| U_h f(\sigma, y) \right|^2 dy.
\]
\[\square\]

Second step: Link between \(w_{I, h, R}\) and \(U_h\). It is in this step that we really see the relevance of the objects introduced previously. It comes from the two following lemmas:

**Lemma 2.17.** For any \(a \in S^1_L\),
\[
\int_{T^d} a_3 \left( x, \xi, \frac{\eta(\xi)}{h} \right) u^h_{u_h}(dx, d\xi)
\]
\[
= \sum_{\sigma \in (h\mathbb{Z}^d)} \left\langle U_h u_h(\sigma, y), \text{Op}_1 \left( a_{\sigma} \left( y, \eta - \left\{ \frac{\sigma^\alpha}{h} \right\} \right) \chi \left( \frac{\eta - \left\{ \frac{\sigma^\alpha}{h} \right\}}{R} \right) \right) \right\rangle_{L^2(T^d)} U_h u_h(\sigma, y) + O(h).
\]

Proof. We have by \((26)\),
\[
\int_{T^d} a_3 \left( x, \xi, \frac{\eta(\xi)}{h} \right) u^h_{u_h}(dx, d\xi) = \int_{T^d} a_3 \left( x, \sigma(\xi), \frac{\eta(\xi)}{h} \right) u^h_{u_h}(dx, d\xi) + O(h).
\]
Then, using \((52)\)
\[
\int_{T^d} a_3 \left( x, \sigma(\xi), \frac{\eta(\xi)}{h} \right) u^h_{u_h}(dx, d\xi)
\]
\[
= \frac{1}{(2\pi)^d} \sum_{k - k' \in \Lambda} \tilde{u}_h(k)\overline{\tilde{u}_h(k')} \tilde{a}_{k - k'} \left( \frac{h}{k + k'} \right) \frac{1}{h} \eta \left( \frac{h}{k + k'} \right) \chi \left( \frac{1}{hR} \eta \left( \frac{h}{k + k'} \right) \right).
\]
We write \(hk = \sigma + h\eta, \ hk' = \sigma + h\eta'\), \((\sigma, h\eta), (\sigma, h\eta') \in F(h\mathbb{Z}^d)\),
using the fact that \( k' - k \in \Lambda \). In particular, \( \sigma (h^{k+k'}) = \sigma \). Then,

\[
\int_{T^*T^d} a_3 \left( x, \sigma (\xi), \frac{\eta(\xi)}{h} \right) w_h \, (dx, d\xi)
\]

\[
= \frac{1}{(2\pi)^{d/2}} \sum_{\sigma + h\eta, \sigma + h\eta' \in h\mathbb{Z}^d} \hat{u}_h \left( \frac{1}{h} \sigma + \eta \right) \hat{u}_h \left( \frac{1}{h} \sigma + \eta' \right) \hat{a}_{\eta' - \eta} \left( \sigma, \frac{\eta + \eta'}{2} \right) \chi \left( \frac{\eta + \eta'}{2R} \right)
\]

\[
= \frac{1}{(2\pi)^{d/2}} \sum_{\sigma \in \sigma (h\mathbb{Z}^d)} \sum_{\eta, \eta' \in \Lambda} \widehat{U_h u_h} (\sigma, \eta + \{ \frac{\sigma}{h} \}) \widehat{U_h u_h} (\sigma, \eta' + \{ \frac{\sigma}{h} \}) \hat{a}_{\eta' - \eta} \left( \frac{\eta + \eta'}{2} \right) \chi \left( \frac{\eta + \eta'}{2R} \right)
\]

which is the desired expression. \( \square \)

To simplify the notation in the computations that follow, we set:

\[
A (\sigma, \eta) := \frac{1}{2} d^2 H(\sigma) \eta \cdot \eta.
\]

**Lemma 2.18.** For any \( a \in S^1_\Lambda \), for any \( t \in \mathbb{R} \),

\[
\int_{T^*T^d} a_3 \left( x, \xi, \frac{\eta(\xi)}{h} \right) w_h (t, dx, d\xi)
\]

\[
= \sum_{\sigma \in \sigma (h\mathbb{Z}^d)} \langle e^{-itA(\sigma, D_y)} U_h u_h (\sigma, y) \rangle \chi \left( \frac{\eta - \{ \frac{\sigma}{h} \}}{R} \right) \left( \frac{1}{R} \right) \left( \frac{\eta - \{ \frac{\sigma}{h} \}}{R} \right)
\]

\[
+ o(1)
\]

**Proof.** We use Proposition 2.12 and apply Lemma 2.17 to the symbol \( b_0 (t, x, \xi, \eta) \) which was defined in the proof of Proposition 2.12 (see (38) with \( h = 0 \)). Then, the result follows by using the fundamental property of the Weyl quantization, namely that, since \( A (\sigma, \eta) \) is quadratic in \( \eta \) and does not depend on \( y \):

\[
\text{Op}_1 \left( b_0 (t, y, \sigma, \eta) \right) = e^{-itA(\sigma, D_y)} \text{Op}_1 \left( a_\sigma \left( y, \eta - \left\{ \frac{\sigma}{h} \right\} \right) \chi \left( \frac{\eta - \{ \frac{\sigma}{h} \}}{R} \right) \right) e^{itA(\sigma, D_y)}.
\]

\( \square \)
Third step: Passing to the limit. If \( a \in \mathcal{S}_1^\Lambda \) is compactly supported in \( \eta \), the map \( \sigma \mapsto \text{Op}_1 (a_\sigma (y, \eta)) \) belongs to the Banach space \( \mathcal{C}_c (I_\Lambda; \mathcal{K}(L^2(T^d, \Lambda))) \). The dual of this space is \( \mathcal{M}_\Lambda := \mathcal{M}(I_\Lambda; L^1(L^2(T^d, \Lambda))) \), the space of trace-class operator valued measures. Let us consider the element \( \rho^h \in \mathcal{M}_\Lambda \), defined by letting

\[
\langle \rho^h, K \rangle = \sum_{\sigma \in \sigma(hZ^d)} \langle U_h u_h (\sigma, y), K (\sigma) U_h u_h (\sigma, y) \rangle
\]

for all \( K \in \mathcal{C}_c (I_\Lambda; \mathcal{K}(L^2(T^d, \Lambda))) \). Lemma \([2.16]\) implies that \( \langle \rho^h \rangle \) is bounded in \( \mathcal{M}_\Lambda \) if \( \langle u_h \rangle \) is bounded in \( L^2(T^d) \). Besides, each \( \rho^h \) is positive (meaning that \( \langle \rho^h, K \rangle \geq 0 \) if \( K (\sigma) \geq 0 \) for all \( \sigma \)). We consider \( M(\sigma) \) a weak-* limit of the family \( \rho^h \) in \( \mathcal{M}_\Lambda \) and we now restrict our attention to symbols \( a \in \mathcal{S}_1^\Lambda \) that do not depend on \( \eta \). We write

\[
\chi \left( \frac{\eta - \{ \sigma^h / h \}}{R} \right) - \chi \left( \frac{\eta}{R} \right) = - \frac{1}{R} \int_0^1 d\chi \left( \frac{\eta}{R} - \frac{s}{R} \left\{ \frac{\sigma}{h} \right\} \right) \cdot \left\{ \frac{\sigma^h}{h} \right\} ds,
\]

and we obtain

\[
\text{Op}_1 (b_0 (t, y, \sigma, \eta)) = e^{-itA(\sigma, D_y)} \text{Op}_1 (a_\sigma (y) \chi (\eta / R)) e^{itA(\sigma, D_y)} + O(1/R).
\]

whence

\[
\lim_{R \to +\infty} \lim_{h \to 0} \int_{T^d \times \Omega} a_3 (x, \xi, \eta / h) w_h (t, dx, d\xi) = \lim_{R \to +\infty} \left\langle M(\sigma), e^{-itA(\sigma, D_y)} \text{Op}_1 (a_\sigma (y) \chi (\eta / R)) e^{itA(\sigma, D_y)} \right\rangle.
\]

Remark 2.19. The arguments used to get rid of the term \( \left\{ \frac{\sigma^h}{h} \right\} \) crucially exploit the presence of a factor \( 1/R \) in front of it. Such an argument cannot be used for symbols \( a \in \mathcal{S}_1^\Lambda \) which depends non trivially of the variable \( \eta \) as in Lemma \([2.17]\). In such a situation, by working in \( L^2(\mathbb{R}^d) \), one can define locally an operator-valued measure \( M(\sigma) \); however, this object cannot be globally defined on the torus.

3. An iterative procedure for computing \( \mu \)

3.1. First step of the construction. What was done in the previous section can be considered as the first step of an iterative procedure that allows to effectively compute \( \mu(t, \cdot) \) solely in terms of the sequence of initial data \( (u_h) \). Recall that we assumed in \([2.2]\) without loss of generality, that the projection on \( \xi \) of \( \mu(t, \cdot) \) was supported in a ball contained in \( \mathbb{R}^d \setminus C_H \). We have decomposed this measure as a sum

\[
\mu(t, \cdot) = \sum_{\Lambda \in \mathcal{L}} \mu_\Lambda (t, \cdot) + \sum_{\Lambda \in \mathcal{L}} \mu^\Lambda (t, \cdot),
\]

where \( \Lambda \) runs over the set of primitive submodules of \( \mathbb{Z}^d \), and where

\[
\mu_\Lambda (t, \cdot) = \int_{\langle \Lambda \rangle} \tilde{\mu}_\Lambda (t, \cdot, d\eta) |_{T^d \times R_\Lambda}, \quad \mu^\Lambda (t, \cdot) = \int_{\langle \Lambda \rangle} \tilde{\mu}^\Lambda (t, \cdot, d\eta) |_{T^d \times R_\Lambda}.
\]

\([\text{Here } \mathcal{K}(H) \text{ denotes the space of compact operators acting on a Hilbert space } H.\])
From Theorem 2.5, the distributions $\tilde{\mu}_\Lambda$ have the following properties:

1. $\tilde{\mu}_\Lambda(t, dx, d\xi, d\eta)$ is in $C(\mathbb{R}; (S^1_\Lambda)'$) and all its $x$-Fourier modes are in $\Lambda$; with respect to the variable $\xi$, $\tilde{\mu}_\Lambda(t, dx, d\xi, d\eta)$ is supported in $I_\Lambda$;

2. if $\tau_h \ll 1/h$ then for every $t \in \mathbb{R}$, $\tilde{\mu}_\Lambda(t, \cdot)$ is a positive measure and:

$$\tilde{\mu}_\Lambda(t, \cdot) = \left(\tilde{\phi}^1_t\right)_* \tilde{\mu}_\Lambda(0, \cdot),$$

where:

$$\tilde{\phi}^1_t : (x, \xi, \eta) \mapsto (x + sd^2 H(\sigma(\xi))\eta, \xi, \eta);$$

3. if $\tau_h = 1/h$ then $\int_{\langle \Lambda \rangle} \tilde{\mu}_\Lambda(t, \cdot, d\eta)$ is in $C(\mathbb{R}; \mathcal{M}_+(T^*_\mathbb{R} \mathbb{T}))$ and $\int_{\mathbb{R}^d \times \langle \Lambda \rangle} \tilde{\mu}_\Lambda(t, \cdot, d\xi, d\eta)$ is an absolutely continuous measure on $\mathbb{T}$. In fact, with the notations of Section 2.3, we have, for every $a \in C^\infty_c(T^*_\mathbb{R} \mathbb{T})$ with Fourier modes in $\mathcal{L}$,

$$\int_{T^d \times I_\Lambda \times \langle \Lambda \rangle} a(x, \xi) \tilde{\mu}_\Lambda(t, dx, d\xi, d\eta) = \int_{I_\Lambda} \text{Tr} \left( a_\sigma e^{-i\frac{sd^2 H(\sigma)}{\eta}d\nu\cdot D_y M(\sigma) d\sigma} e^{i\frac{sd^2 H(\sigma) D_y \cdot D_y}{\eta}} \right)$$

where $M \in \mathcal{M}_+(I_\Lambda; C^1 \left(L^2(\mathbb{T}, \Lambda)\right))$ and $a_\sigma$ is the multiplication operator by $a(\cdot, \sigma)$, acting on $L^2(\mathbb{T}, \Lambda)$.

On the other hand, the measures $\tilde{\mu}_\Lambda$ satisfy:

1. for $a \in S^1_\Lambda$, $\langle \tilde{\mu}_\Lambda(t, dx, d\xi, d\eta), a(x, \xi, \eta) \rangle$ is obtained as the limit of

$$\langle w^I_{h,R,\delta}(t), a \rangle = \int_{T^d} \chi \left( \frac{\eta(\xi)}{\delta} \right) \left( 1 - \chi \left( \frac{\tau_h \eta(\xi)}{R} \right) \right) \frac{a(x, \xi, \tau_h \eta(\xi))}{w_h(t)(dx, d\xi)},$$

in the weak-$*$ topology of $L^\infty(\mathbb{R}, (S^1_\Lambda)'$), as $h \longrightarrow 0^+, R \longrightarrow +\infty$ and then $\delta \longrightarrow 0^+$ (possibly along subsequences);

2. $\tilde{\mu}_\Lambda(t, dx, d\xi, d\eta)$ is in $L^\infty(\mathbb{R}, \mathcal{M}_+(T^*_\mathbb{R} \mathbb{T} \times \langle \Lambda \rangle))$ and all its $x$-Fourier modes are in $\Lambda$. With respect to the variable $\eta$, the measure $\tilde{\mu}_\Lambda(t, dx, d\xi, d\eta)$ is 0-homogeneous and supported at infinity: we see it as a measure on the sphere at infinity $S(\Lambda)$. With respect to $\xi$ it is supported on $\{\xi \in I_\Lambda\};$

3. $\tilde{\mu}_\Lambda$ is invariant by the two flows,

$$\phi^0_s : (x, \xi, \eta) \mapsto (x + sdH(\xi), \xi, \eta), \quad \text{and} \quad \phi^1_s : (x, \xi, \eta) \mapsto (x + sd^2 H(\sigma(\xi))\frac{\eta}{|\eta|}, \xi, \eta).$$

This is the first step of an iterative procedure; the next step is to decompose the measure $\mu^\Lambda(t, \cdot)$ according to primitive submodules of $\Lambda$. We need to adapt the discussion of [3]; to this aim, we introduce some additional notation.

Fix a primitive submodule $\Lambda \subseteq \mathbb{Z}^d$ and $\sigma \in I_\Lambda \setminus C_H$. For $\Lambda_2 \subsetneq \Lambda_1 \subsetneq \Lambda$ primitive submodules of $(\mathbb{Z}^d)^*$, for $\eta \in \langle \Lambda_1 \rangle$, we denote

$$\Lambda_\eta(\sigma, \Lambda_1) := \left( \Lambda_1^\perp \oplus \mathbb{R} d^2 H(\sigma) \cdot \eta \right)^\perp \cap (\mathbb{Z}^d)^*$$

$$= \left( \mathbb{R} d^2 H(\sigma) \cdot \eta \right)^\perp \cap \Lambda_1,$$
where the orthogonal is always taken in the sense of duality. We note that $\Lambda_{\eta}(\sigma, \Lambda_1)$ is a primitive submodule of $\Lambda_1$, and that the inclusion $\Lambda_{\eta}(\sigma, \Lambda_1) \subset \Lambda_1$ is strict if $\eta \neq 0$ since $d^2H(\sigma)$ is definite. We define:

$$R_{\Lambda_2}^{\Lambda_1}(\sigma) := \{ \eta \in \langle \Lambda_1 \rangle, \Lambda_{\eta}(\sigma, \Lambda_1) = \Lambda_2 \}.$$ 

Because $d^2H(\sigma)$ is definite, we have the decomposition $(\mathbb{R}^d)^* = (d^2H(\sigma).\Lambda_2)^\perp \oplus \langle \Lambda_2 \rangle$. We define $P_{\Lambda_2}^\sigma$ to be the projection onto $\langle \Lambda_2 \rangle$ with respect to this decomposition.

3.2. Step $k$ of the construction. In the following, we set $\Lambda = \Lambda_1$, corresponding to step $k = 1$. We now describe the outcome of our decomposition at step $k (k \geq 1)$; we will indicate in §3.3 how to go from step $k$ to $k + 1$, for $k \geq 1$.

At step $k$, we have decomposed $\mu(t, \cdot)$ as a sum

$$\mu(t, \cdot) = \sum_{1 \leq l \leq k} \sum_{\Lambda_1 \supset \Lambda_2 \supset \ldots \supset \Lambda_l} \mu_{\Lambda_1}^{\Lambda_2 \ldots \Lambda_l-1}(t, \cdot) + \sum_{\Lambda_1 \supset \Lambda_2 \supset \ldots \supset \Lambda_k} \mu_{\Lambda_1}^{\Lambda_2 \ldots \Lambda_k}(t, \cdot),$$

where the sums run over the strictly decreasing sequences of primitive submodules of $(\mathbb{Z}^d)^*$ (of lengths $l \leq k$ in the first term, of length $k$ in the second term). We have

$$\mu_{\Lambda_1}^{\Lambda_2 \ldots \Lambda_l-1}(t, x, \xi) = \int_{R_{\Lambda_2}^{\Lambda_1}(\xi) \times \ldots \times R_{\Lambda_l}^{\Lambda_{l-1}}(\eta)} \mu_{\Lambda_1}^{\Lambda_2 \ldots \Lambda_l-1}(t, x, \xi, d\eta_1, \ldots, d\eta_l) |_{\mathbb{T}^d \times R_{\Lambda_1}};$$

$$\mu_{\Lambda_1}^{\Lambda_2 \ldots \Lambda_k}(t, x, \xi) = \int_{R_{\Lambda_2}^{\Lambda_1}(\xi) \times \ldots \times R_{\Lambda_k}^{\Lambda_{k-1}}(\xi) \times \langle \Lambda_k \rangle} \mu_{\Lambda_1}^{\Lambda_2 \ldots \Lambda_k}(t, x, \xi, d\eta_1, \ldots, d\eta_k) |_{\mathbb{T}^d \times R_{\Lambda_1}}.$$

The distributions $\mu_{\Lambda_1}^{\Lambda_2 \ldots \Lambda_l-1}$ have the following properties:

1. $\mu_{\Lambda_1}^{\Lambda_2 \ldots \Lambda_l-1} \in \mathcal{C}(\mathbb{R}, \mathcal{D}'(T^*\mathbb{T}^d \times \mathbb{S}(\Lambda_1) \times \ldots \times \mathbb{S}(\Lambda_{l-1}) \times \langle \Lambda_l \rangle))$ and all its $x$-Fourier modes are in $\Lambda_l$; with respect to $\xi$ it is supported in $I_{\Lambda_l}$;

2. for every $t \in \mathbb{R}$, $\mu_{\Lambda_1}^{\Lambda_2 \ldots \Lambda_l-1}(t, \cdot)$ is invariant under the flows $\phi^j_s$ ($j = 0, 1, \ldots, l - 1$) defined by

$$\phi^0_s(x, \xi, \eta_1, \ldots, \eta_l) = (x + sdH(\xi), \xi, \eta_1, \ldots, \eta_{l-1}, \eta_l);$$

$$\phi^j_s(x, \xi, \eta_1, \ldots, \eta_l) = (x + sd^2H(\xi) \frac{\eta_j}{|\eta_j|}, \xi, \eta_1, \ldots, \eta_l);$$

3. if $\tau_h \ll 1/h$ then for every $t \in \mathbb{R}$, $\mu_{\Lambda_1}^{\Lambda_2 \ldots \Lambda_l-1}(t, \cdot)$ is a positive measure and

$$\mu_{\Lambda_1}^{\Lambda_2 \ldots \Lambda_l-1}(t, \cdot) = \left( \frac{\tau_h}{\phi^l_s} \right)_{s} \mu_{\Lambda_1}^{\Lambda_2 \ldots \Lambda_l-1}(0, \cdot),$$

where, for $(x, \xi, \eta_1, \ldots, \eta_l) \in T^*\mathbb{T}^d \times \mathbb{S}(\Lambda_1) \times \ldots \times \mathbb{S}(\Lambda_{l-1}) \times \langle \Lambda_l \rangle$ we define:

$$\phi^l_s : (x, \xi, \eta_1, \ldots, \eta_l) \mapsto (x + sd^2H(\xi)\eta_l, \xi, \eta_1, \ldots, \eta_l);$$
Finally, we define the space $x$ where the sum runs over all primitive submodules $\Lambda$.

Section 2.2, we consider test functions in $T^{\ast}\mathbb{T}$.

On the other hand

On the other hand $\hat{\mu}_{A_{1}A_{2}\cdots A_{k}}$ is a positive operator valued measure on $I_{A_{1}}\times S(\Lambda_{1}) \times \cdots \times S(\Lambda_{k})$ taking values in $L^{1}(L^{2}(\mathbb{T}^{d}, \Lambda))$.

On the other hand $\hat{\mu}_{A_{1}A_{2}\cdots A_{k}}$ satisfy:

(1) $\hat{\mu}_{A_{1}A_{2}\cdots A_{k}}$ is in $L^{\infty}(\mathbb{R}, \mathcal{M}_{+}(T^{\ast}\mathbb{T}^{d}\times S(\Lambda_{1}) \times \cdots \times S(\Lambda_{k})))$ and all its $x$-Fourier modes are in $\Lambda_{k}$;

(2) $\hat{\mu}_{A_{1}A_{2}\cdots A_{k}}$ is invariant by the $k + 1$ flows, $\phi_{0}^{\sigma} : (x, \xi, \eta) \mapsto (x + s dH(\xi), \xi, \eta, \ldots, \eta)$, and $\phi_{l}^{\sigma} : (x, \xi, \eta_{1}, \ldots, \eta_{k}) \mapsto (x + s d^{2}H(\sigma(\xi))_{|\eta_{1}}^{\eta_{l}}, \xi, \eta_{1}, \ldots, \eta_{k})$ (where $l = 1, \ldots, k$).

Finally, we define the space $S_{\Lambda_{k}}^{k}$ which is the class of smooth functions $a(x, \xi, \eta_{1}, \ldots, \eta_{k})$ on $T^{\ast}\mathbb{T}^{d} \times (\Lambda_{1}) \times \cdots \times (\Lambda_{k})$ that are

(i) smooth and compactly supported in $(x, \xi) \in T^{\ast}\mathbb{T}^{d}$;

(ii) homogeneous of degree 0 at infinity in each variable $\eta_{1}, \ldots, \eta_{k}$;

(iii) such that their non-vanishing $x$-Fourier coefficients correspond to frequencies in $\Lambda_{k}$.

3.3. From step $k$ to step $k + 1$ ($k \geq 1$). After step $k$, we leave untouched the term

$\sum_{1 \leq l \leq k} \sum_{\Lambda_{1} \supset \cdots \supset \Lambda_{k}} \hat{\mu}_{A_{1}A_{2}\cdots A_{k}}$ and decompose further $\sum_{\Lambda_{1} \supset \cdots \supset \Lambda_{k}} \hat{\mu}_{A_{1}A_{2}\cdots A_{k}}$. Using the positivity of $\hat{\mu}_{A_{1}A_{2}\cdots A_{k}}$, we use the procedure described in Section 2.1 to write

$$\hat{\mu}_{A_{1}A_{2}\cdots A_{k}}(\sigma, \cdot) = \sum_{\Lambda_{k+1} \subseteq \Lambda_{k}} \hat{\mu}_{A_{1}A_{2}\cdots A_{k}}\big|_{\eta_{k} \in R_{\Lambda_{k+1}}^{\Lambda_{k}}(\sigma)},$$

where the sum runs over all primitive submodules $\Lambda_{k+1}$ of $\Lambda_{k}$. Moreover, by Proposition 2.1, all the $x$-Fourier modes of $\hat{\mu}_{A_{1}A_{2}\cdots A_{k}}\big|_{\eta_{k} \in R_{\Lambda_{k+1}}^{\Lambda_{k}}(\sigma)}$ are in $\Lambda_{k+1}$. To generalize the analysis of Section 2.2, we consider test functions in $S_{\Lambda_{k+1}}^{k+1}$. We let

$$u_{h,R_{1},\ldots,R_{k+1}}^{A_{1}A_{2}\cdots A_{k+1}}(t, x, \xi, \eta_{1}, \ldots, \eta_{k+1}) := \left(1 - \chi \left(\frac{\eta_{k+1}}{R_{k+1}}\right)\right)$$

$$\times u_{h,R_{1},\ldots,R_{k}}^{A_{1}A_{2}\cdots A_{k}}(t, x, \xi, \eta_{1}, \ldots, \eta_{k}) \otimes \delta_{\hat{\mu}_{\Lambda_{k+1}}(\eta_{k})}(\eta_{k+1}),$$
and
\[ w_{\Lambda_k+1, h, R_1, ..., R_{k+1}}^\Lambda(t, x, \xi, \eta_1, \ldots, \eta_{k+1}) := \chi \left( \frac{\eta_{k+1}}{R_{k+1}} \right) \times w_h^\Lambda \left( t, x, \xi, \eta_1, \ldots, \eta_k \right) \otimes \delta_{\rho_{k+1}^\Lambda (\eta_k)}(\eta_{k+1}). \]

By the Calderón-Vaillancourt theorem, both \( w_{\Lambda_k+1, h, R_1, ..., R_{k+1}}^\Lambda \) and \( w_h^\Lambda \) are bounded in \( L^\infty(\mathbb{R}, (S_{\Lambda_{k+1}}^k)' ) \). After possibly extracting subsequences, we can take the following limits:
\[
\begin{align*}
\lim_{R_{k+1} \to +\infty} \ldots \lim_{R_1 \to +\infty} \lim_{h \to 0} & \langle w_h^{\Lambda_1 \Lambda_2 \ldots \Lambda_k}(t), \alpha \rangle =: \langle \mu_{\Lambda_1 \Lambda_2 \ldots \Lambda_k+1}(t), \alpha \rangle,
\end{align*}
\]
and
\[
\begin{align*}
\lim_{R_{k+1} \to +\infty} \ldots \lim_{R_1 \to +\infty} \lim_{h \to 0} & \langle w_h^{\Lambda_1 \Lambda_2 \ldots \Lambda_k}(t), \alpha \rangle =: \langle \mu^{\Lambda_1 \Lambda_2 \ldots \Lambda_k}(t), \alpha \rangle.
\end{align*}
\]
Then the properties listed in the preceding subsection are a direct generalisation of Theorems 2.4 and 2.5 (see also [3], Section 4) and writing
\[
(41) \quad \hat{\mu}_{\Lambda_1 \Lambda_2 \ldots \Lambda_k}(t, \sigma) = \int_{\langle \Lambda_{k+1} \rangle} \hat{\mu}_{\Lambda_1 \Lambda_2 \ldots \Lambda_{k+1}}(t, \sigma, d\eta_{k+1}) + \int_{\langle \Lambda_{k+1} \rangle} \hat{\mu}_{\Lambda_1 \Lambda_2 \ldots \Lambda_k}(t, \sigma, d\eta_{k+1}).
\]

Remark 3.1. By construction, if \( \Lambda_{k+1} = \{0\} \), we have \( \hat{\mu}_{\Lambda_1 \Lambda_2 \ldots \Lambda_{k+1}} = 0 \), and the induction stops. Similarly to Remark 2.8, one can also see that if \( \text{rk} \Lambda_{k+1} = 1 \), the invariance properties of \( \hat{\mu}_{\Lambda_1 \Lambda_2 \ldots \Lambda_{k+1}} \) imply that it is constant in \( x \).

Remark 3.2. Note that in the preceding definition of \( k \)-microlocal Wigner transform for \( k \geq 1 \), we did not use a parameter \( d \) tending to 0 as we did when \( k = 0 \) in order to isolate the part of the limiting measures supported above \( R_{\Lambda_{k+1}}^\Lambda(\sigma) \). This comes directly from the restrictions made in [40] and [41].

3.4. Proof of Theorem 1.8. This iterative procedure allows to decompose \( \mu \) along decreasing sequences of submodules. In particular, when \( \tau_h \sim 1/h \), it implies Theorem 1.8. Indeed, to end the proof of Theorem 1.8, we let
\[
\mu_{\Lambda}(t, \cdot) = \sum_{0 \leq k \leq d} \sum_{\Lambda_1 \supset \cdots \supset \Lambda_k \supset \Lambda} \mu_{\Lambda}^{\Lambda_1 \Lambda_2 \ldots \Lambda_k}(t, \cdot)
\]
and
\[
\rho_{\Lambda}(\sigma) = \sum_{0 \leq k \leq d} \sum_{\Lambda_1 \supset \cdots \supset \Lambda_k \supset \Lambda} \int_{R_{\Lambda_1}^\Lambda(\xi) \times \cdots \times R_{\Lambda_k}^\Lambda(\xi) \times \langle \Lambda \rangle} \hat{\rho}_{\Lambda}^{\Lambda_1 \Lambda_2 \ldots \Lambda_k}(\sigma, d\eta_1, \ldots, d\eta_k) |_{\sigma \in R_{\Lambda_1}},
\]
where \( \Lambda_1, \ldots, \Lambda_k \) run over the set of strictly decreasing sequences of submodules ending with \( \Lambda \). We know that \( \mu_{\Lambda}^{\Lambda_1 \Lambda_2 \ldots \Lambda_k} \) is supported on \( \{ \xi \in I_{\Lambda_1} \} \), and since \( \Lambda \supset \Lambda_1 \) we have \( I_{\Lambda_1} \subset I_\Lambda \). We also let
\[
\rho_{\Lambda}(\sigma) = \sum_{0 \leq k \leq d} \sum_{\Lambda_1 \supset \cdots \supset \Lambda_k \supset \Lambda} \int_{R_{\Lambda_1}^\Lambda(\xi) \times \cdots \times R_{\Lambda_k}^\Lambda(\xi)} \hat{\rho}_{\Lambda}^{\Lambda_1 \Lambda_2 \ldots \Lambda_k}(\sigma, d\eta_1, \ldots, d\eta_k) |_{\sigma \in R_{\Lambda_1}},
\]
where the $\tilde{\rho}_\Lambda^{\Lambda_1,\Lambda_2,\ldots,\Lambda_k}$ are the operator-valued measures appearing in (32).

As already mentioned, Theorem 1.8 implies Theorem 1.2 in the case $\tau_h \sim 1/h$. The proof of Theorem 1.2 in the case $\tau_h \ll 1/h$ is discussed in Section 4 and in the case $\tau_h \gg 1/h$, in Section 5.

4. Some Examples of Singular Concentration

4.1. Singular concentration for time scales $\tau_h \ll 1/h$. In this section, we focus on the case $\tau_h \ll 1/h$ and prove Theorem 1.2(1).

Consider $\rho \in \mathcal{S}(\mathbb{R}^d)$ with $\|\rho\|_{L^2(\mathbb{R}^d)} = 1$ and such that the Fourier transform $\hat{\rho}$ is compactly supported. Let $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ and $(\varepsilon_h)$ a sequence of positive real numbers that tends to zero as $h \to 0^+$. Form the wave-packet:

\begin{equation}
(v_h)(x) := \frac{1}{(\varepsilon_h)^{d/2}} \rho \left( \frac{x - x_0}{\varepsilon_h} \right) e^{i \xi_0 \cdot x} .
\end{equation}

Define

\begin{equation}
(u_h) := P v_h ,
\end{equation}

where $P$ denotes the periodization operator $P v(x) := \sum_{k \in \mathbb{Z}^d} v(x + 2\pi k)$. Since $\rho$ is rapidly decreasing, we have $\|u_h\|_{L^2(T^d)} \to 1$. It is not hard to check that $(u_h)$ is $h$-oscillatory.

Theorem 1.2(1) is a consequence of our next result.

Proposition 4.1. Let $(\tau_h)$ be such that $\lim_{h \to 0^+} h \tau_h = 0$; suppose that $\varepsilon_h \gg h \tau_h$. Then the Wigner distributions of the solutions $S_h^{\tau_h} u_h$ converge weakly-$*$ in $L^\infty(\mathbb{R}; \mathcal{D}(T^*\mathbb{T}^d))$ to $\mu_{(x_0,\xi_0)}$, defined by:

\begin{equation}
\int_{T^*\mathbb{T}^d} a(x, \xi) \mu_{(x_0,\xi_0)}(dx, d\xi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T a(x_0 + tdH(\xi_0), \xi_0) dt , \quad \forall a \in \mathcal{C}_c(T^*\mathbb{T}^d).
\end{equation}

Proof. Start noticing that the sequence $(u_h)$ has the unique semiclassical measure $\mu_0 = \delta_{x_0} \otimes \delta_{\xi_0}$. Using property (4) in the appendix, we deduce that the image $\overline{\mu}$ of $\mu(t, \cdot)$ by the projection from $T^d \times \mathbb{R}^d$ onto $\mathbb{R}^d$ satisfies:

\begin{equation}
\overline{\mu} = \sum_{\Lambda \in \mathcal{L}} \overline{\mu_\Lambda} = \delta_{\xi_0}.
\end{equation}

Since for every primitive module $\Lambda \subset \mathbb{Z}^d$ the positive measure $\overline{\mu_\Lambda}$ is supported on $R_\Lambda$, and these sets form a partition of $\mathbb{R}^d$, we conclude that $\overline{\mu_\Lambda} = 0$ unless $\Lambda = \Lambda_{\xi_0}$ and therefore $\mu = \mu_{\Lambda_{\xi_0}}$. Therefore, in order to characterize $\mu$ it suffices to test it against symbols with Fourier coefficients in $\Lambda_{\xi_0}$. Let $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$ be such a symbol; we can restrict our attention to the case where $a$ is a trigonometric polynomial in $x$. Let $\varphi \in L^1(\mathbb{R})$. Recall that, by Lemma 2.10, the Wigner distributions $w_h(t)$ of $S_h^{\tau_h} u_h$ satisfy

\begin{equation}
\int_{\mathbb{R}} \varphi(t) \langle w_h(t), a \rangle dt = \int_{\mathbb{R}} \varphi(t) \langle w_h(0), a \circ \phi_{\tau h} \rangle dt + o(1) ;
\end{equation}

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moreover the Poisson summation formula ensures that the Fourier coefficients of \( u_h \) are given by:

\[
\hat{u}_h(k) = \frac{(\varepsilon_h)^{d/2}}{(2\pi)^{d/2}} \hat{\rho} \left( \frac{\varepsilon_h}{h} (hk - \xi_0) \right) e^{-i(k-\xi_0/h \cdot x_0)}.
\]

Combining this with the explicit formula (52) for the Wigner distribution presented in the appendix we get:

\[
(44) \quad \int_\mathbb{R} \varphi(t) \langle w_h(t), a \rangle dt = \frac{(\varepsilon_h)^d}{(2\pi)^{3d/2}} \sum_{k-j \in \Lambda_{\xi_0}} \hat{\varphi} \left( \tau_h dH \left( \frac{h - j}{2} \right) \cdot (k - j) \right) \hat{a}_{j-k} \left( \frac{h - j}{2} \right)
\]

\[
\hat{\rho} \left( \frac{\varepsilon_h}{h} (hk - \xi_0) \right) \hat{\rho} \left( \frac{\varepsilon_h}{h} (hj - \xi_0) \right) e^{-i(k-j \cdot x_0) + o(1)}.
\]

Now, since \( k - j \in \Lambda_{\xi_0} \) we can write:

\[
\left| dH \left( \frac{h + j}{2} \right) \cdot (k - j) \right| = \left| \left[ dH \left( \frac{h + j}{2} \right) - dH (\xi_0) \right] \cdot (k - j) \right|
\]

\[
\leq C \left| \frac{h + j}{2} - \xi_0 \right| |k - j|.
\]

By hypothesis, both \( \hat{\rho} \) and \( k \mapsto \hat{a}_k(\xi) \) are compactly supported, and hence the sum (44) only involves terms satisfying:

\[
\left| \frac{\varepsilon_h}{h} \middle| \frac{h - j}{2} - \xi_0 \right| \leq R, \left| \frac{\varepsilon_h}{h} \middle| \frac{h - j}{2} - \xi_0 \right| \leq R \text{ and } |j - k| \leq R
\]

for some fixed \( R \). This in turn implies

\[
\left| \tau_h dH \left( \frac{h + j}{2} \right) \cdot (k - j) \right| \leq CR^2 \frac{\tau_h}{\varepsilon_h}.
\]

This shows that the limit of (44) as \( h \to 0^+ \) coincides with that of:

\[
\left( \frac{\varepsilon_h}{h} \right)^d \sum_{k-j \in \Lambda_{\xi_0}} \hat{\varphi}(0) a_{j-k} \left( \frac{h + j}{2} \right) \hat{\rho} \left( \frac{\varepsilon_h}{h} (hk - \xi_0) \right) \hat{\rho} \left( \frac{\varepsilon_h}{h} (hj - \xi_0) \right) e^{-i(k-j \cdot x_0)}
\]

\[
= \hat{\varphi}(0) \langle w_h(0), a \rangle,
\]

which is precisely:

\[
\hat{\varphi}(0) a(x_0, \xi_0) = \hat{\varphi}(0) \lim_{T \to \infty} \frac{1}{T} \int_0^T a(x_0 + tdH(\xi_0), \xi_0) dt,
\]

since \( a \) has only Fourier modes in \( \Lambda_{\xi_0} \).

We next present a slight modification of the previous example in order to illustrate the two-microlocal nature of the elements of \( \mathcal{M} (\tau) \). Define now, for \( \eta_0 \in \mathbb{R}^d \):

\[
u_h(x) = P \left[ v_h(x) e^{i\eta_0/(h \tau_h)} \right],
\]
where $v_h$ was defined in (42).

**Proposition 4.2.** Suppose that $\lim_{h \to 0^+} h\tau_h = 0$ and $\varepsilon_h \gg h\tau_h$. Suppose moreover that $d^2 H(\xi_0)$ is definite and that $\eta_0 \in \Lambda_{\xi_0}$. Then the Wigner distributions of $S_h^{\tau} u_h$ converge weakly*- in $L^\infty(\mathbb{R}; D'(T^*\mathbb{T}^d))$ to the measure:

$$\mu(t, \cdot) = \mu_{(x_0 + td^2 H(\xi_0)\eta_0, \xi_0)}, \quad t \in \mathbb{R},$$

where $\mu_{(x_0, \xi_0)}$ is the uniform orbit measure defined in (4.4).

**Proof.** The same argument we used in the proof of Proposition 4.1 gives $\mu = \mu_{\Lambda_{\xi_0}}$. We claim that $w_{I_{\Lambda_{\xi_0}}, h, R}(0)$ converges to the measure:

$$\tilde{\mu}_{\Lambda_{\xi_0}}(0, x, \xi, \eta) = \mu_{(x_0, \xi_0)}(x, \xi) \delta_{\eta_0}(\eta).$$

Assume this is the case, then Proposition 2.12 implies:

$$\tilde{\mu}_{\Lambda_{\xi_0}}(t, x, \xi, \eta) = \mu_{(x_0 + td^2 H(\xi_0)\eta_0, \xi_0)}(x, \xi) \delta_{\eta_0}(\eta), \quad \forall t \in \mathbb{R},$$

and, since $\tilde{\mu}_{\Lambda_{\xi_0}}(t, \cdot)$ are probability measures, it follows from Proposition 2.3 that $\tilde{\mu}_{\Lambda_{\xi_0}} = 0$ and:

$$\mu_{\Lambda_{\xi_0}}(t, \cdot) = \int_{\Lambda_{\xi_0}} \tilde{\mu}_{\Lambda_{\xi_0}}(t, \cdot, d\eta) = \mu_{(x_0 + td^2 H(\xi_0)\eta_0, \xi_0)}.$$ 

Let us now prove the claim. Set

$$\tilde{u}_h(x) = v_h(x) e^{i\eta_0/(h\tau_h)}.$$

Consider $h_0 > 0$ and $\chi \in C^\infty_0(\mathbb{R}^d)$ such that $\chi \tilde{u}_h = \tilde{u}_h$ for all $h \in (0, h_0)$ and $P \chi^2 \equiv 1$. We now take $a \in S^1_A$ and denote by $\tilde{a}$ the smooth compactly supported function defined on $\mathbb{R}^d$ by $\tilde{a} = \chi^2 a$. Using the fact that the two-scale quantization admits the gain $h\tau_h$ (see Remark 2.2),

$$\langle u_h , \text{Op}^{\Lambda_{\xi_0}}_{h}(a)u_h \rangle_{L^2(\mathbb{T}^d)} = \langle u_h , \text{Op}^{\Lambda_{\xi_0}}(\tilde{a})u_h \rangle_{L^2(\mathbb{R}^d)} = \langle \tilde{u}_h , \text{Op}^{\Lambda_{\xi_0}}(a)\tilde{u}_h \rangle_{L^2(\mathbb{R}^d)} + O(h\tau_h).$$

Therefore, it is possible to lift the computation of the limit of $w_{I_{\Lambda_{\xi_0}}, h, R}(0)$ to $T^*\mathbb{R}^d \times \Lambda_{\xi_0}$ and, in consequence, replace sums by integrals. A direct computation gives:

$$\langle \tilde{u}_h , \text{Op}^{\Lambda_{\xi_0}}(a)\tilde{u}_h \rangle_{L^2(\mathbb{R}^d)} = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} e^{i\xi \cdot (x - y)} P(x) \rho(y) \times a \left( x_0 + \frac{\varepsilon_h (x + y)}{2}, \xi_0 + \frac{1}{\tau_h} \eta_0 + \frac{h}{\varepsilon_h} \xi, \tau_h \eta_0 + \frac{h}{\varepsilon_h} \xi_0 + \frac{h}{\varepsilon_h} \xi \right) dx dy d\xi.$$

Note that if $F(\xi) = (\sigma, \eta)$, then

$$\forall k \in \Lambda, \quad F(\xi + k) = (\sigma, \eta + k) = F(\xi) + (0, k),$$
which implies that \( dF(\xi)k = (0, k) \) and \( d\eta(\xi)k = k \) for all \( k \in \Lambda_{\xi_0} \). We deduce \( d\eta(\xi_0)\eta_0 = \eta_0 \) since \( \eta_0 \in \langle \Lambda_{\xi_0} \rangle \) and, in view of \( \eta(\xi_0) = 0 \), a Taylor expansion of \( \eta(\xi) \) around \( \xi_0 \) gives
\[
\tau_h \eta \left( \xi_0 + \frac{1}{\tau_h} \eta_0 + \frac{h}{\varepsilon_h} \xi \right) = \eta_0 + o(1).
\]
Therefore, as \( h \) goes to 0,
\[
\langle \tilde{u}_h, \text{Op}_h^{\Lambda_{\xi_0}}(a)\tilde{u}_h \rangle \to a(x_0, \xi_0, \eta_0) = \langle \tilde{\mu}_{\Lambda_{\xi_0}}, a \rangle.
\]

**4.2. Singular concentration for Hamiltonians with critical points.** We next show by a quasimode construction that for Hamiltonians having a degenerate critical point (of order \( k > 2 \)) and for time scales \( \tau_h \ll 1/h^{k-1} \), the set \( \tilde{\mathcal{M}}(\tau) \) always contains singular measures.

Suppose \( \xi_0 \in \mathbb{R}^d \) is such that:
\[
dH(\xi_0), d^2H(\xi_0), ..., d^{k-1}H(\xi_0) \quad \text{vanish identically.}
\]
The Hamiltonian \( H(\xi) = |\xi|^k \) (\( k \) an even integer) — corresponding to the operator \( (-\Delta)^{\frac{k}{2}} \) — provides such an example (with \( \xi_0 = 0 \)). Let \( u_h = \text{P}v_h \), where \( v_h \) is defined in (42). If \( \varepsilon_h \gg h \) it is not hard to see that
\[
\|H(hD_x)u_h - H(\xi_0)u_h\|_{L^2(\mathbb{T}^d)} = O\left(h^k/(\varepsilon_h)^k\right).
\]
Therefore,
\[
\left\| S_h^* u_h - e^{-i\frac{1}{2}H(\xi_0)} u_h \right\|_{L^2(\mathbb{T}^d)} = tO\left(h^{k-1}/(\varepsilon_h)^k\right),
\]
and, it follows that, for compactly supported \( \varphi \in L^1(\mathbb{R}) \) and \( a \in C^\infty_0(T^*\mathbb{T}^d) \),
\[
\int_{\mathbb{R}} \varphi(t)\langle u_h(t), a \rangle dt = \int_{\mathbb{R}} \varphi(t)\langle u_h, \text{Op}_h(a)u_h \rangle_{L^2(\mathbb{T}^d)} dt + O\left(\tau_h h^{k-1}/(\varepsilon_h)^k\right).
\]
Choosing \( (\varepsilon_h) \) tending to zero and such that \( \varepsilon_h \gg (\tau_h h^{k-1})^{1/k} \), the latter quantity converges to \( a(x_0, \xi_0)\|\varphi\|_{L^1(\mathbb{R})} \) as \( h \to 0^+ \). In other words,
\[
dt \otimes \delta_{x_0} \otimes \delta_{\xi_0} \in \tilde{\mathcal{M}}(\tau),
\]
whence \( dt \otimes \delta_{x_0} \otimes \delta_{\xi_0} \in \mathcal{M}(\tau) \).

**Remark 4.3.** In the special case of \( H(\xi) = |\xi|^k \) (\( k \) an even integer), we know that the threshold \( \tau_h^H \) is precisely \( h^{1-k} \). From the discussion of (55) and previously known results about eigenfunctions of the laplacian, we know that the elements of \( \mathcal{M}(\tau) \) are absolutely continuous for \( \tau_h \gg 1/h^{k-1} \). In the case of \( \tau_h = 1/h^{k-1} \), one can still show that elements of \( \mathcal{M}(\tau) \) are absolutely continuous. This requires some extra work which consists in checking that all our proofs still work in this case for \( \tau_h = 1/h^{k-1} \) and \( \xi \) in a neighbourhood of \( \xi_0 = 0 \), replacing the Hessian \( d^2H(\xi_0) \) by \( d^kH(\xi_0) \), and the assumption that the Hessian is definite by the remark that \( [d^kH(\xi_0), \xi^k] = 0 \implies \xi = 0 \).
In the general case of a Hamiltonian having a degenerate critical point, the existence of such a threshold, and its explicit determination, is by no means obvious.

4.3. The effect of the presence of a subprincipal symbol. Here we present some remarks concerning how the preceding results may change when the Hamiltonian $H(hDx)$ is perturbed by a small potential $h^\beta V(t,x)$. Suppose $V \in L^\infty(\mathbb{R} \times \mathbb{T}^d)$ and define

$$ P_{\beta,h} := H(hDx) + h^\beta V(t,x), \text{ with } \beta > 0, $$

and denote by $S^t_{\beta,h}$ the corresponding propagator (starting at $t = 0$):

$$ S^t_{\beta,h} := e^{-i\frac{\tau}{h} P_{\beta,h}}. $$

Let us fix a time scale $\tau = (\tau_h)$ that tends to infinity as $h \to 0^+$. Define $\widetilde{M}_{\beta,V}(\tau)$ to be the set of accumulation points of the time-scaled Wigner distributions

$$ w^\beta_h(t,\cdot) = w^h_{S^t_{\beta,h},u_h}, $$

as $(u_h)$ varies among all normalised sequences in $L^2(\mathbb{T}^d)$. For the sake of simplicity, from now on we shall fix the time scale $\tau_h = 1/h$. The discussion that follows can be easily adapted to more general time scales by changing the ranges of values of $\beta$.

1) $\beta > 2$. In this case it can be easily shown that $w^\beta_h$ and $w_h$ have the same weak-* accumulation points in $L^\infty(\mathbb{R}; D'(\mathbb{T}^d \times \mathbb{T}^d))$. Therefore, the potential is a negligible perturbation and, in particular, for every $V \in L^\infty(\mathbb{R} \times \mathbb{T}^d)$,

$$ \widetilde{M}_{\beta,V}(1/h) = \widetilde{M}(1/h). $$

2) $\beta = 2$. When $H(\xi) = |\xi|^2$, the question has been addressed in \[3\]\[11\]. It turns out that whenever $V$ is not constant,

$$ \widetilde{M}_{2,V}(1/h) \neq \widetilde{M}(1/h). $$

In fact, the structure of $\widetilde{M}_{2,V}(1/h)$ is similar to that of $\widetilde{M}(1/h)$, but the propagation law that replaces \[10\] involves the propagator associated to the averaged Hamiltonian $|\xi|^2 + (V\chi)(t,x)$.

3) $\beta < 2$. In this case, it is possible to find potentials $V$ for which Theorem \[12\] fails, i.e. such that there exist $\mu \in \widetilde{M}_{\beta,V}(1/h)$ such that the projection of $\mu$ on $x$ is not absolutely continuous with respect to $dt dx$. The following example is due to Jared Wunsch. On the 2-dimensional torus, take $H(\xi) = |\xi|^2$ and $V(x_1,x_2) := W(x_2)$ such that $W(x_2) = (x_2)^2/2$ in $\{|x_2| < 1/2\}$. Take $\varepsilon \in (0,1)$ and

$$ u_h(x,y) := \frac{1}{\pi^{1/4} h^{3/4}} e^{\frac{-x_1^2}{2h}} e^{-\frac{(x_2)^2}{h^{2\varepsilon}}} \chi(y), $$

where $\chi$ is a smooth function that is equal to one in $\{|x_2| < 1/4\}$ and identically equal to 0 in $\{|x_2| > 1/2\}$. One checks that

$$ (-h^2 \Delta + h^{2(1-\varepsilon)} V - 1) u_h = h^{2-\varepsilon} u_h + O(h^\infty). $$

\[11\]In that work, it is assumed that the set of discontinuity points of $V$ has measure zero.
It follows that for $\varphi \in L^1(\mathbb{R})$ and $a \in C^\infty_c(T^*\mathbb{T}^2)$,

$$
\lim_{h \to 0^+} \int_{\mathbb{R}} \varphi(t) \left\langle S_{2(1-\epsilon),h}^{t/h}, \text{Op}_{h}(a)S_{2(1-\epsilon),h}^{t/h}u_h \right\rangle_{L^2(T^2)} dt = \lim_{h \to 0^+} \int_{\mathbb{R}} \varphi(t) \langle u_h, \text{Op}_{h}(a)u_h \rangle_{L^2(T^2)} dt = \left( \int_{\mathbb{R}} \varphi(t) dt \right) \int_{T^*T^2} a(x, \xi) \mu(dx, d\xi),
$$

and it is not hard to see that $\mu$ is concentrated on $\{x_2 = 0, \xi_1 = 1, \xi_2 = 0\}$. In particular the image of $\mu$ by the projection to $T^2$ is supported on $\{x_2 = 0\}$.

5. Hierarchies of time scales

The following result makes explicit the relation between the sets $\widetilde{M}(\tau)$ as the time scale $(\tau_h)$ varies.

**Proposition 5.1.** Let $(\tau_h)$ and $(\sigma_h)$ be time scales tending to infinity as $h \to 0^+$ such that $\lim_{h \to 0^+} \sigma_h/\tau_h = 0$. Then for every $\mu \in \widetilde{M}(\tau)$ and almost every $t \in \mathbb{R}$ there exist $\mu^t \in \text{Conv} \widetilde{M}(\sigma)$ such that

$$(45) \quad \mu(t, \cdot) = \int_0^1 \mu^t(s, \cdot) ds.$$  

Before presenting the proof of this result, we shall need two auxiliary lemmas.

**Lemma 5.2.** Let $(\sigma_h)$ be a time scale tending to infinity as $h \to 0^+$. Let $c_h^{(n)}_{\sigma_h} > 0, n \in \mathbb{N}$, be a normalised family in $L^2(T^d)$ and define:

$$w_h^{(n)}(t, \cdot) := w^{\sigma_h}_{c_h^{(n)}_{\sigma_h}}(t, \cdot).$$

Let $c_h^{(n)} \geq 0, n \in \mathbb{N}$, be such that $\sum_{n \in \mathbb{N}} c_h^{(n)} = 1$. Then, every weak-* accumulation point in $L^\infty(\mathbb{R}; D'(T^*T^d))$ of

$$(46) \quad \sum_{n \in \mathbb{I}_h} c_h^{(n)} w_h^{(n)}(t, \cdot)$$

belongs to $\text{Conv} \widetilde{M}(\sigma)$.

**Proof.** Suppose $\mu_h$ possesses an accumulation point $\tilde{\mu} \in L^\infty(\mathbb{R}; M_+(T^*T^d))$ that does not belong to $\text{Conv} \widetilde{M}(\sigma)$. By the Hahn-Banach theorem applied to the convex sets $\{\tilde{\mu}\}$ and $\text{Conv} M(\sigma)$ we can ensure the existence of $\varepsilon > 0$, $a \in C^\infty_c(T^*T^d)$ and $\theta \in L^1(\mathbb{R})$ such that:

$$\int_{\mathbb{R}} \theta(t) \langle \tilde{\mu}(t, \cdot), a \rangle dt < -\varepsilon < 0,$$

and,

$$(47) \quad \int_{\mathbb{R}} \theta(t) \langle \mu(t, \cdot), a \rangle dt \geq -\frac{\varepsilon}{3}, \quad \forall \mu \in \text{Conv} \widetilde{M}(\sigma).$$
Suppose that \( \tilde{\mu} \) is attained through a sequence \((h_k)\) tending to zero. For \( k > k_0 \) big enough, 
\[
\int_{\mathbb{R}} \theta(t) \sum_{n \in L_{h_k}} c_{h_k}^{(n)} \left\langle w_{h_k}^{(n)}(t, \cdot), a \right\rangle \, dt \leq -\frac{3}{2} \varepsilon,
\]
which implies that there exists \( n_k \in \mathbb{N} \) such that:
\[
(48) \quad \int_{\mathbb{R}} \theta(t) \left\langle w_{h_k}^{(n_k)}(t, \cdot), a \right\rangle \, dt \leq -\frac{3}{2} \varepsilon.
\]
Therefore, every accumulation point of \( \left( w_{h_k}^{(n_k)} \right) \) also satisfies (48) which contradicts (47).

**Lemma 5.3.** Let \( \tau, \sigma \) and \( \mu \) be as in Proposition 5.1. For every \( \alpha < \beta \) there exists \( \mu_{\alpha,\beta} \in \text{Conv } \mathcal{M}(\sigma) \) such that
\[
\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mu(t, \cdot) \, dt = \int_{0}^{1} \mu_{\alpha,\beta}(t, \cdot) \, dt.
\]

**Proof.** Let \( \mu \in \mathcal{M}(\tau) \). Then there exist an \( h \)-oscillating, normalised sequence \((u_h)\) such that, for every \( \theta \in L^1(\mathbb{R}) \) and every \( a \in C_c^\infty(T^d \mathbb{T}^d) \):
\[
\lim_{h \to 0^+} \int_{\mathbb{R}} \theta(t) \left\langle S_h^{\tau_h t} u_h, O_{\mathcal{P}}(a) S_h^{\tau_h t} u_h \right\rangle \, dt = \int_{\mathbb{R}} \theta(t) \left\langle \mu(t, \cdot), a \right\rangle \, dt.
\]
Write \( N_h := \tau_h / \sigma_h \); by hypothesis \( N_h \to \infty \) as \( h \to 0^+ \). Let \( \alpha < \beta \), define \( L := \beta - \alpha \) and put:
\[
\delta_h := \left\lfloor L N_h \right\rfloor, \quad t_n^h := \alpha N_h + n \delta_h,
\]
where \( \lfloor LN_h \rfloor \) is the integer part of \( LN_h \). Then,
\[
\frac{1}{L} \int_{\alpha}^{\beta} \left\langle S_h^{\tau_h t} u_h, O_{\mathcal{P}}(a) S_h^{\tau_h t} u_h \right\rangle_{L^2(\mathbb{T}^d)} \, dt = \frac{1}{L N_h} \int_{\alpha N_h}^{\beta N_h} \left\langle S_h^{\tau_h t} u_h, O_{\mathcal{P}}(a) S_h^{\tau_h t} u_h \right\rangle_{L^2(\mathbb{T}^d)} \, dt
\]
\[
= \frac{1}{L N_h} \sum_{n=1}^{\lfloor LN_h \rfloor} \int_{t_{n-1}^h}^{t_n^h} \left\langle S_h^{\tau_h t} u_h, O_{\mathcal{P}}(a) S_h^{\tau_h t} u_h \right\rangle_{L^2(\mathbb{T}^d)} \, dt
\]
\[
= \frac{1}{L N_h} \sum_{n=1}^{\lfloor LN_h \rfloor} \int_{0}^{\delta_h} \left\langle S_h^{\tau_h t} v_h^{(n)}, O_{\mathcal{P}}(a) S_h^{\tau_h t} v_h^{(n)} \right\rangle_{L^2(\mathbb{T}^d)} \, dt,
\]
where the functions \( v_h^{(n)} := S_h^{\tau_h t} u_h \) form, for each \( n \in \mathbb{Z} \), a normalised sequence indexed by \( h > 0 \). The result then follows by Lemma 5.2 and using the fact that \( \delta_h \to 1 \) as \( h \to 0^+ \). \( \square \)

**Proof of Proposition 5.1.** Let \( \mu \in \mathcal{M}(\tau) \); an application of the Lebesgue differentiation theorem gives the existence of a countable dense set \( S \subset C_c^\infty(T^d \mathbb{T}^d) \) and a set \( N \subset \mathbb{R} \) of
measure zero such that, for \( a \in S \) and \( t \in \mathbb{R} \setminus N \),
\[
\lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \int_{T^d} a(x,\xi) \mu(s, dx, d\xi) \, ds = \int_{T^d} a(x,\xi) \mu(t, dx, d\xi). \tag{49}
\]
Fix \( t \in \mathbb{R} \setminus N \); then, for any \( \varepsilon > 0 \) there exist \( \mu^\varepsilon \in \text{Conv} \tilde{\mathcal{M}}(\sigma) \) such that, for every \( a \in C^\infty_c(T^d) \),
\[
\frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \int_{T^d} a(x,\xi) \mu(s, dx, d\xi) \, ds = \int_0^1 \int_{T^d} a(x,\xi) \mu^\varepsilon(s, dx, d\xi) \, ds. \tag{50}
\]
Note that \( \text{Conv} \tilde{\mathcal{M}}(\sigma) \) is sequentially compact for the weak-* topology, therefore, there exist a sequence \( (\varepsilon_n) \) tending to zero and a \( \mu^t \in \text{Conv} \tilde{\mathcal{M}}(\sigma) \) such that \( \mu^\varepsilon_n \) converges weakly-* to \( \mu^t \). Identities (49) and (50) ensure that \( \mu(t, \cdot) = \int_0^1 \mu^t(s, \cdot) \, ds. \)

**Remark 5.4.** Projecting on \( x \) in identity (49) we deduce that given \( \nu \in \mathcal{M}(\tau) \) there exist \( \nu^t \in \mathcal{M}(\sigma) \) such that:
\[
\nu(t, \cdot) = \int_0^1 \nu^t(s, \cdot) \, ds.
\]
This, together with the fact that elements of \( \mathcal{M}(1/h) \) are absolutely continuous imply the conclusion of Theorem 1.2(2) when \( \tau_h \gg 1/h \).

Denote by \( \tilde{\mathcal{M}}(\infty) \) the set of weak-* limit points of sequences of Wigner distributions \( (w_{uk}) \) corresponding to sequences \( (u_h) \) consisting of normalised eigenfunctions of \( H(hD_x) \). We now focus on a family of time scales \( \tau \) for which the structure of \( \tilde{\mathcal{M}}(\tau) \) can be described in terms of the closed convex hull of \( \tilde{\mathcal{M}}(\infty) \). Given a measurable subset \( O \subseteq \mathbb{R}^d \), we define:
\[
\tau_h^H(O) := h \sup \{|H(hk) - H(hj)|^{-1} : H(hk) \neq H(hj), hk, hj \in h\mathbb{Z}^d \cap O\}.
\]
Note that the scale \( \tau_h^H \) defined in the introduction coincides with \( \tau_h^H(\mathbb{R}^d) \). The following holds.

**Proposition 5.5.** Let \( O \subseteq \mathbb{R}^d \) be an open set such that \( \tau_h^H(O) \) tends to infinity as \( h \to 0^+ \). Suppose \( (\tau_h) \) is a time scale such that \( \lim_{h \to 0^+} \tau_h^H(O)/\tau_h = 0 \). If \( \mu \in \tilde{\mathcal{M}}(\tau) \) is obtained through a sequence whose semiclassical measure satisfies \( \mu_0(\mathbb{T}^d \times (\mathbb{R}^d \setminus O)) = 0 \) then \( \mu \in \text{Conv} \tilde{\mathcal{M}}(\infty) \).

**Proof.** As in [36], for \( a \in C^\infty_c(T^d) \) and \( \theta \in L^1(\mathbb{R}) \), we write:
\[
\int_{\mathbb{R}} \theta(t) \langle w_h(t), a \rangle \, dt = \frac{1}{(2\pi)^{d/2}} \sum_{h,j \in \mathbb{Z}^d} \hat{\theta} \left( \tau_h \frac{H(hk) - H(hj)}{h} \right) \hat{u}_h(k) \hat{u}_h(j) \hat{a}_{j-k} \left( \frac{h}{2}(k + j) \right).
\]
Our assumptions on the semiclassical measure of the initial data implies that, for a.e. \( t \in \mathbb{R} \):

\[
\mu \left( t, \mathbb{T}^d \times (\mathbb{R}^d \setminus O) \right) = 0.
\]

Suppose that \( \mu \) is obtained through the normalised sequence \((u_h)\). Suppose that \( a \in C^\infty_c (\mathbb{T}^d \times O) \) and that \( \text{supp} \hat{\theta} \) is compact. For \( 0 < h < h_0 \) small enough,

\[
\tau_h \frac{H(hk) - H(hj)}{h} \notin \text{supp} \hat{\theta}, \quad \forall hk, hj \in O \text{ such that } H(hk) \neq H(hj).
\]

Therefore, for such \( h \), \( a \) and \( \theta \),

\[
\int_{\mathbb{R}} \theta(t) \langle w_h(t), a \rangle \, dt = \hat{\theta}(0) \sum_{E_h \in H(h\mathbb{Z}^d) \cap H(O)} c_h^{E_h} \langle P_{E_h} u_h, \text{Op}_h(a) P_{E_h} u_h \rangle_{L^2(\mathbb{T}^d)},
\]

where \( P_{E_h} \) stands for the orthogonal projector onto the eigenspace associated to the eigenvalue \( E_h \). This can be rewritten as:

\[
\int_{\mathbb{R}} \theta(t) \langle w_h(t), a \rangle \, dt = \hat{\theta}(0) \sum_{E_h \in H(h\mathbb{Z}^d) \cap H(O)} c_h^{E_h} \langle w_h^{E_h}, a \rangle,
\]

where

\[
v_h^{E_h} := \frac{P_{E_h} u_h}{\|P_{E_h} u_h\|_{L^2(\mathbb{T}^d)}}, \quad \text{and} \quad c_h^{E_h} := \|P_{E_h} u_h\|_{L^2(\mathbb{T}^d)}^2.
\]

Note that \( v_h^{E_h} \) are eigenfunctions of \( H(hD_x) \) and the fact that \((u_h)\) is normalised implies:

\[
\sum_{E_h \in H(h\mathbb{Z}^d) \cap H(O)} c_h^{E_h} = 1.
\]

We conclude by applying (a straightforward adaptation of) Lemma 5.2 to \( v_h^{E_h} \) and \( c_h^{E_h} \). \( \square \)

**Corollary 5.6.** Suppose \( \tau^H := \tau_h^H (\mathbb{R}^d) \rightarrow \infty \) as \( h \rightarrow 0^+ \) and that \((\tau_h)\) is a time scale such that \( \tau^H_h \ll \tau_h \). Then

\[
\widehat{\mathcal{M}}(\tau) = \text{Conv} \widehat{\mathcal{M}}(\infty).
\]

**Proof.** The inclusion \( \widehat{\mathcal{M}}(\tau) \subseteq \text{Conv} \widehat{\mathcal{M}}(\infty) \) is a consequence of the previous result with \( O = \mathbb{R}^d \). The converse inclusion can be proved by reversing the steps of the proof of Proposition 5.5. \( \square \)

**Remark 5.7.** Proposition 1.14 is a direct consequence of this result.
6. Appendix: Basic Properties of Wigner Distributions and Semi-Classical Measures

In this Appendix, we review basic properties of Wigner distributions and semiclassical measures. Recall that we have defined $w^h_{uh}$ for $u_h \in L^2(\mathbb{T}^d)$ as:

$$\int_{T^*\mathbb{T}^d} a(x,\xi)w^h_{uh}(dx,d\xi) = \langle u_h, \text{Op}_h(a)u_h \rangle_{L^2(\mathbb{T}^d)}, \quad \text{for all } a \in C_c^\infty(T^*\mathbb{T}^d),$$

(51)  

Start noticing that (51) admits the more explicit expression:

$$\int_{T^*\mathbb{T}^d} a(x,\xi)w^h_{uh}(dx,d\xi) = \frac{1}{(2\pi)^{d/2}} \sum_{k,j \in \mathbb{Z}^d} \hat{u}_h(k)\hat{a}_j \left( \frac{h}{2} (k + j) \right),$$

(52)  

where $\hat{u}_h(k) := \int_{\mathbb{T}^d} u_h(x)e^{-ik \cdot x} dx$ and $\hat{a}_j(\xi) := \int_{\mathbb{T}^d} a(x,\xi)e^{-ik \cdot x} dx$ denote the respective Fourier coefficients of $u_h$ and $a$, with respect to the variable $x \in \mathbb{T}^d$.

By the Calderón-Vaillancourt theorem [5], the norm of $\text{Op}_h(a)$ is uniformly bounded in $h$: indeed, there exists an integer $K_d$, and a constant $C_d > 0$ (depending on the dimension $d$) such that, if $a$ is a smooth function on $T^*\mathbb{T}^d$, with uniformly bounded derivatives, then

$$\|\text{Op}_1(a)\|_{L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)} \leq C_d \sum_{\alpha \in \mathbb{N}^{2d}, |\alpha| \leq K_d} \sup_{T^*\mathbb{T}^d} |\partial^\alpha a| =: C_d M(a).$$

A proof in the case of $L^2(\mathbb{R}^d)$ can be found in [10]. As a consequence of this, equation (51) gives:

$$\left| \int_{T^*\mathbb{T}^d} a(x,\xi)w^h_{uh}(dx,d\xi) \right| \leq C_d \|u_h\|^2_{L^2(\mathbb{T}^d)} M(a), \quad \text{for all } a \in C_c^\infty(T^*\mathbb{T}^d).$$

Therefore, if $w_h(t,\cdot) := w^h_{S^h_{\tau_h} u_h}$ for some function $h \mapsto \tau_h \in \mathbb{R}_+$ and $(u_h)$ is bounded in $L^2(\mathbb{T}^d)$ one has that $(w_h)$ is uniformly bounded in $L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$. Let us consider $\mu \in L^\infty(\mathbb{R}; \mathcal{D}'(T^*\mathbb{T}^d))$ an accumulation point of $(w_h)$ for the weak-$*$ topology.

It follows from standard results on the Weyl quantization that $\mu$ enjoys the following properties:

1. $\mu \in L^\infty(\mathbb{R}; \mathcal{M}_{+}(T^*\mathbb{T}^d))$, meaning that for almost all $t$, $\mu(t,\cdot)$ is a positive measure on $T^*\mathbb{T}^d$.
2. The unitary character of $S^h_t$ implies that $\int_{T^*\mathbb{T}^d} \mu(t, dx,d\xi)$ does not depend on $t$; from the normalization of $u_h$, we have $\int_{T^*\mathbb{T}^d} \mu(\tau, dx,d\xi) \leq 1$, the inequality coming from the fact that $T^*\mathbb{T}^d$ is not compact, and that there may be an escape of mass to infinity. Such escape does not occur if and only if $(u_h)$ is $h$-oscillating, in which case $\mu \in L^\infty(\mathbb{R}; \mathcal{P}(T^*\mathbb{T}^d))$.
3. If $\tau_h \to \infty$ as $h \to 0^+$ then the measures $\mu(t,\cdot)$ are invariant under $\phi_s$, for almost all $t$ and all $s$.
4. Let $\tilde{\mu}$ be the measure on $\mathbb{R}^d$ image of $\mu(t,\cdot)$ under the projection map $(x,\xi) \mapsto \xi$. Then $\tilde{\mu}$ does not depend on $t$. Moreover, if $\overline{\mu}$ stands for the image under the same
projection of any semiclassical measure corresponding to the sequence of initial data \((u_h)\) then \(\bar{\mu} = \overline{\mu_0}\).

For the reader’s convenience, we next prove statements (3) and (4) (see also [20] for a proof of these results in the context of the Schrödinger flow \(e^{ith}\) on a general Riemannian manifold). Let us begin with the invariance through the Hamiltonian flow. We set

\[ a_s(x, \xi) := a(x + s dH(\xi)), \]

The symbolic calculus for Wey’s quantization implies:

\[
\frac{d}{ds} S^s_h \text{Op}_h(a_s) S^{-s}_h = S^s_h \text{Op}_h(\partial_s a_s) S^{-s}_h - i \frac{\hbar}{\hbar} S^s_h \left[ H(hD) , \text{Op}_h(a_s) \right] S^{-s}_h
\]

Therefore,

\[
\text{Op}_h(a_s) S^{-s}_h = \text{Op}_h(a) + \mathcal{O}(\hbar^2) \quad \text{and for } \theta \in L^1(\mathbb{R}),
\]

\[
\int_{\mathbb{R}} \theta(t) \langle w_h(t), a \rangle \, dt = \int_{\mathbb{R}} \theta(t) \langle u_h, S^{-\tau_h t} \text{Op}_h(a) S^{\tau_h t} u_h \rangle \, dt
\]

\[
= \int_{\mathbb{R}} \theta(t) \langle u_h, S^{-\tau_h t/s} \text{Op}_h(a \circ \phi_s) S^{\tau_h t/s} u_h \rangle \, dt + \mathcal{O}(\hbar^2)
\]

\[
= \int_{\mathbb{R}} \theta(t + s/\tau_h) \langle u_h, S^{-\tau_h t} \text{Op}_h(a \circ \phi_s) S^{\tau_h t} u_h \rangle \, dt + \mathcal{O}(\hbar^2)
\]

\[
= \int_{\mathbb{R}} \theta(t + s/\tau_h) \langle w_h(t), a \circ \phi_s \rangle \, dt + \mathcal{O}(\hbar^2).
\]

Since \(\|\theta(\cdot + s/\tau_h) - \theta\|_{L^1} \rightarrow 0\) (recall that we have assumed that \(\tau_h \rightarrow \infty\) as \(h \rightarrow 0^+\)) we obtain

\[
\int_{\mathbb{R}} \theta(t) \langle w_h(t), a \rangle \, dt - \int_{\mathbb{R}} \theta(t) \langle w_h(t), a \circ \phi_s \rangle \, dt \rightarrow 0, \quad \text{as } h \rightarrow 0^+,
\]

whence the invariance under \(\phi_s\).

Let us now prove property (4). Consider \(\overline{\mu}\) the image of \(\mu\) by the projection \((x, \xi) \mapsto \xi\), we have for \(a \in C_0^\infty(\mathbb{R}^d)\):

\[
\langle w_h(t), a(\xi) \rangle - \langle w_{w_h}, a(\xi) \rangle = \int_0^t \frac{d}{ds} \langle w_h(s), a(\xi) \rangle \, ds
\]

\[
= \int_0^t \langle u_h, \frac{d}{ds} \left( S^{-\tau_h s} \text{Op}_h(a) S^{\tau_h s} \right) u_h \rangle \, ds
\]

\[
= 0,
\]

as \(\frac{d}{ds} S^{-\tau_h s} \text{Op}_h(a(\xi)) S^{\tau_h s} = 0\) (for \(a\) only depending on \(\xi\) we have \(\text{Op}_h(a) = a(hD_x)\), which commutes with \(H(hD_x)\)). Therefore, taking limits we find, for every \(\theta \in L^1(\mathbb{R})\):

\[
\int_{\mathbb{R}} \theta(t) \int_{T^*\mathbb{T}^d} \mu(\xi, dx, d\xi) = \left( \int_{\mathbb{R}} \theta(t) \, dt \right) \int_{T^*\mathbb{T}^d} \mu_0(dx, d\xi),
\]
where $\mu_0$ is any accumulation point of $(w^h_{u_0})$. As a consequence of this, we find that $\overline{\mu}$ does not depend on $t$ and:

$$\overline{\mu}(\xi) = \int_{\mathbb{T}^d} \mu_0(dy, \xi).$$

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