3-MANIFOLDS THAT BOUND NO DEFINITE 4-MANIFOLD

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Abstract. We produce a rational homology 3-sphere that does not smoothly bound either a positive or negative definite 4-manifold. Such a 3-manifold necessarily cannot be rational homology cobordant to a Seifert fibered space or any 3-manifold obtained by Dehn surgery on a knot. The proof requires an analysis of short characteristic covectors in bimodular lattices.

1. Introduction

By a definite 4-manifold we mean a 4-manifold whose intersection form is positive or negative definite. Many familiar classes of 3-manifolds are known to bound either positive or negative definite 4-manifolds. These include, for example, any Seifert fibered rational homology sphere or any 3-manifold obtained by nonzero surgery on a knot in $S^3$. Lens spaces and those 3-manifolds which are the double covers of $S^3$ branched over an alternating knot are examples of 3-manifolds that bound positive and negative definite 4-manifolds.

However, one can often argue that a given 3-manifold cannot smoothly bound both positive and negative definite 4-manifolds (all manifolds in this paper are assumed to be smooth and oriented). For example, consider the Poincaré homology sphere $P$, oriented as the link of the Brieskorn singularity. Since its Frøyshov $h$-invariant (or equivalently, the Heegaard Floer correction term $d$) is positive, it cannot bound a positive definite 4-manifold (see [4, Theorem 3] or [18, Corollary 9.8]). On the other hand, $P$ does bound negative definite 4-manifolds, e.g., either as $-1$-surgery on the left-handed trefoil or as the boundary of the negative $E_8$-plumbing.

To obstruct a 3-manifold from bounding either positive or negative definite 4-manifolds is rather more difficult. Frøyshov has announced the first examples of integral homology spheres that bound no definite 4-manifolds, provided that the first homology of the putative definite 4-manifold does not contain 2-torsion [3]. His argument depends on the fact that the $h$-invariant and the $q_2$-invariant arising from instanton homology are linearly independent. Recent work of Nozaki, Sato, and Taniguchi [16], using filtered instanton Floer homology, gives examples of integral homology spheres that bound no definite 4-manifolds, without any restrictions on the torsion in homology. Any rational homology sphere rationally cobordant to one of the examples of [16] evidently has the same property.

The main result of this paper is the following.

Theorem 1.1. There exist rational homology spheres that are not cobordant to an integral homology sphere and that bound no definite 4-manifold.

The authors understand that Mathew Hedden has produced similar (unpublished) examples. As far as we understand, such a result is not currently accessible with the
techniques of [16], since filtered instanton Floer homology is only defined for integral homology spheres.

The proof of our theorem instead combines an inequality of Ozsváth and Szabó [18] relating correction terms and the squares of first Chern classes of spin$^c$ structures together with an analysis of short characteristic covectors in bimodular lattices, using work of Elkies [2] on unimodular lattices.

Let us give a specific example: let $N$ denote the 3-manifold $N := 3P \# Y$, i.e. the sum of three copies of the Poincaré sphere $P$ and the Seifert fibered space $Y = Y(2; \frac{15}{13}, \frac{17}{3}, \frac{23}{22})$. (See Figure 1.) Note that $H_1(N) \cong \mathbb{Z}/2\mathbb{Z}$, so $N$ cannot be homology cobordant to an integral homology sphere. We will prove below that $N$ cannot bound a definite 4-manifold.

One can obtain other examples satisfying various properties by constructing 3-manifolds that are integral or rational homology cobordant to $N$. For example, work of Livingston [11] implies that $N$ is integral homology cobordant to an irreducible 3-manifold, which can further be assumed to be hyperbolic by work of Myers [13]. From what we discussed above we get the following immediate corollaries.

**Corollary 1.2.** $N$ is not rational homology cobordant to any Seifert fibered space.

Examples of integral homology spheres that are not integral homology cobordant to any Seifert fibered space have been produced by Stoffregen [21] and Frøyshov [3]; previously, rational homology spheres that are not integral homology cobordant to any Seifert fibered space appeared in [1].

**Corollary 1.3.** $N$ is not rational homology cobordant to any 3-manifold obtained by Dehn surgery on a knot in $S^3$.

It is often quite difficult to prove that a 3-manifold cannot be obtained by surgery on a knot (see, for example, [7]), and most of the known obstructions are not preserved under rational homology cobordism. However, in [6] examples of 3-manifolds $Y$ with $b_1(Y) = 1$ are constructed that are not rational homology cobordant to 0-surgery on a knot.

In Section 4 we will give a more direct proof that the 3-manifold $\overline{Y}$ is not obtained as Dehn surgery along a knot in $S^3$, and use $\overline{Y}$ to produce another example of a spineless 4-manifold (see [10] and [5] for previous work on the subject—Hayden and Piccirillo’s results in particular are much stronger than ours).
Characteristic covectors. In Section 3 we describe an obstruction for a rational homology sphere $Y$ with $H_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$ to bound a definite 4-manifold. In Section 4 we show that the manifold $N$ described above satisfies the conditions of the obstruction from Section 3.

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2. Characteristic covectors of bimodular lattices

In this section, a lattice $\Lambda$ will be a subset $\Lambda \subset \mathbb{R}^m$ isomorphic to $\mathbb{Z}^m$, and such that, with respect to the Euclidean scalar product on $\mathbb{R}^m$ one has $v \cdot w \in \mathbb{Z}$ for each $v, w \in \Lambda$. A lattice is said to be minimal if it contains no vector of square 1.

We denote with $\Lambda'$ the dual lattice of $\Lambda$, i.e. the set of vectors in $\mathbb{R}^m$ that pair integrally with each element in $\Lambda$. Note that $\Lambda \subset \Lambda'$, and that $\Lambda'$ is not a lattice according the definition above, unless $\Lambda = \Lambda'$. (There are always vectors in $\Lambda'$ that square to a rational if the containment is strict.) If $\Lambda = \Lambda'$ we say that $\Lambda$ is unimodular (or self-dual). Even if $\Lambda$ and $\Lambda'$ both live in $\mathbb{R}^m$ and we use the scalar product on $\mathbb{R}^m$ for the pairing, we use the notation $\langle \xi, v \rangle$ to denote the pairing of a covector $\xi \in \Lambda'$ and a vector $v \in \Lambda$.

We denote with $\text{Char}(\Lambda)$ the subset of $\Lambda'$ comprising all $\xi$ such that $\langle \xi, v \rangle \equiv v^2 \pmod{2}$. Each such $\xi$ is a characteristic covector. (Again, when $\Lambda$ is unimodular $\xi$ is actually an element in $\Lambda$, but we prefer to talk about covectors to emphasize that we are thinking about the dual.) Throughout the section, the letter $L$ will always denote a positive definite integral lattice of rank $n$ and determinant 2 (a bimodular lattice), $A$ will be an auxiliary lattice of determinant 2, and $M_A$ will be the lattice $L \oplus A$. Note that $M'_A/M_A \cong L'/L \oplus \Lambda'/A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ has a unique metabolizer (i.e. a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$ that is isotropic with respect to the induced $\mathbb{Q}/\mathbb{Z}$-valued bilinear form induced by the product on $M_A$), so that $M_A$ is an index-2 sublattice of a unimodular lattice $U_A$. (See, for example, [9] for a topologically-minded treatment.) In fact, $U_A$ is simply $L \oplus A \cup (L' \setminus L) \times (A' \setminus A) \subset L' \oplus A'$. With a slight abuse of notation, we view $L$ and $A$ as subsets of $U_A$. It is easy to see that $L = A^\perp$ and $A = L^\perp$. Since neither $L$ nor $A$ have a metabolizer, note that $L \otimes \mathbb{Q} \cap U_A = L$ and $A \otimes \mathbb{Q} \cap U_A = A$. Note that $U_A$ is uniquely determined by both $L$ and $A$, but we do not make the dependency on $L$ explicit in the notation.

Dualizing, $U'_A \cong U_A$ is an index-2 subset of $M'_A = L' \oplus A'$. Moreover, the restriction maps $U'_A \to L'$ and $U'_A \to A'$ are both onto. In the same way, the restriction maps $\text{Char}(U_A) \to \text{Char}(L)$ and $\text{Char}(U_A) \to \text{Char}(A)$ are onto.

Two choices stand out. We can choose $A = L$, or $A = A_1$, where $A_1$ is the rank-1 lattice generated by a vector of square 2 (i.e. a root). We write $U$ and $M$ instead of $U_{A_1}$ and $M_{A_1}$. Call $r$ one of the two generators of the auxiliary lattice $A = A_1$. By construction, $U$ contains $r$, a vector of norm 2.
Lemma 2.1. Let $\xi \in \text{Char}(L)$ be a characteristic covector. Then $\xi^2$ is an integer, and $\xi^2 \equiv n \pm 1 \pmod{8}$.

Proof. Choose a characteristic covector $\xi_U \in \text{Char}(U)$ that extends $\xi$, and call $\xi_A \in \text{Char}(A_1)$ its restriction to $A_1$. Then $\xi_U^2 \equiv \text{rk}U = n + 1 \pmod{8}$, and $\xi_U^2 = \xi^2 + \xi_A^2$. Since $\xi_A^2 \equiv 0, 2 \pmod{8}$ by direct verification, the lemma follows. \hfill \square

We call $\text{Char}_\pm(L)$ the set of characteristic covectors $\xi$ of $L$ for which $\xi^2 \equiv n \pm 1 \pmod{8}$. Let us go back to the case of $A$ arbitrary. Since $A$ is a determinant-2 lattice, $\text{Char}_\pm(A)$ are defined, too.

Lemma 2.2. $\xi_u \in U'_A$ is a characteristic covector of $U_A$ if and only if there exists a sign $s$ such that $\xi_u|_L \in \text{Char}_s(L)$ and $\xi_u|_A \in \text{Char}_{-s}(A)$.

Proof. Call $\xi_{\ell} = \xi_u|_L$ and $\xi_a = \xi_u|_A$. The ‘only if’ direction is clear, since $\xi_{\ell}^2 + \xi_a^2 = \xi_u^2 \equiv n + \text{rk}A \pmod{8}$.

Let us look at the ‘if’ direction. Now, let $C := \text{Char}_+(L) \times \text{Char}_-(A) \cup \text{Char}_+(A) \cup \text{Char}_+(A) \subset M$. By the ‘only if’ direction above, $C$ contains $\text{Char}(U_A)$, which is an affine space over $2U'_A$. On the other hand, $\text{Char}_+(L) \times \text{Char}_+(A)$ is also affine over $2L + 2A = 2M_A$, and, by the remark below the statement, if $v \in U_A \setminus (L \oplus A) = (L \setminus L) \times (A \setminus A)$ and $\xi \in \text{Char}_+(L) \times \text{Char}_+(A)$ then $\xi + 2v \in \text{Char}_+(L) \times \text{Char}_+(A)$, so that $C$ is affine over $2U'_A$, too. In particular, $C$ and $\text{Char}(U_A)$ are both affine subspaces over $2U'_A$, and since $\text{Char}(U_A) \subset C$ then they are equal. \hfill \square

We denote with $I^m$ the lattice $\mathbb{Z}^m$ with the diagonal intersection form (i.e. with an orthonormal basis). The lattice $\Delta_n := I^{n-1} \oplus A_1$ is the unique bimodular lattice of rank $n$ whose intersection form is diagonal.

Lemma 2.3. The lattice $U$ is diagonal if and only if $L$ is.

Proof. Suppose that $U$ is diagonal, with base $e_0, \ldots, e_n$. Then the generator $r$ of $A$ is a root in $U$. In particular, up to re-indexing the generators of $U$ (and possibly flipping signs), $r = e_0 - e_1$. Now, $L \cong \langle r \rangle^\perp$, but $\langle r \rangle^\perp$ is spanned by $e_0 + e_1, e_2, \ldots, e_n$, and in particular it is isomorphic to $\Delta_n$.

If $L = \Delta_n$, then $L$ embeds in $I^{n+1}$ as we have just seen; however, $U$ is uniquely determined by $L$, so $U \equiv I^{n+1}$. \hfill \square

We recall a result of Elkies on characteristic covectors in unimodular lattices.

Theorem 2.4 ([2]). Let $\Lambda$ be unimodular lattice of rank $m$. Then

$$\min_{\xi \in \text{Char}(\Lambda)} \xi^2 \leq m,$$

and the equality is attained if and only if $\Lambda$ is the diagonal lattice $I^m$. 
We find it convenient to introduce the notation for the defect of a lattice. The defect $d(\Lambda)$ of $\Lambda$ is:

$$d(\Lambda) = \min_{\xi \in \text{Char}(\Lambda)} \frac{\xi^2 - \text{rk}\Lambda}{4}.$$  

Note that Elkies’ theorem can be rephrased as saying that a unimodular lattice has non-positive defect, and that the defect is 0 if and only if the lattice is diagonal.

When $L$ is bimodular, using Lemma 2.1 we can identify two defects, denoted with $d_\pm(L)$:

$$d_\pm(L) = \min_{\xi \in \text{Char}_\pm(L)} \frac{\xi^2 - \text{rk}L}{4}.$$  

Note that $d_\pm(L) \equiv \pm \frac{1}{4} \pmod{2}$. The main result of this section is a bimodular version of of Elkies’ theorem and gives another characterization of $\Delta_n$ (see [17] for a different characterization).

**Theorem 2.5.** For every bimodular lattice $L$, $d_+(L) + d_-(L) \leq 0$. Moreover, if equality is attained, $L$ is diagonal, and in particular $d_\pm(L) = \pm \frac{1}{4}$.

As mentioned above, a root in a lattice $\Lambda$ is a vector of square 2; we say that $\Lambda$ is a root lattice if it is rationally spanned by its set of roots. To a collection $R \subset \Lambda$ of $\mathbb{Q}$-linearly independent roots we associate an edge-weighted graph $G(R)$ as follows: the set of vertices of $G(R)$ is $R$, and there is an edge joining $r, s \in R$ with weight $r \cdot s$ if $r \cdot s \neq 0$. Note that if $r, s \in R$ are distinct, they are linearly independent, so by Cauchy–Schwarz $r \cdot s \in \{-1, 0, 1\}$.

We will make the choice $A = L$ in the proof. To make the notation lighter, we denote $U_L$ by $D$, and we call it the double of $L$. To distinguish between the two summands in $M_A \cong L \oplus L$, we still call $A$ its second summand; however, we drop the dependency on $A$ from the notation. In summary, we have $M = L \oplus A$ as an index-2 sublattice of $D$, and $L$ and $A$ are viewed as a pair of orthogonal sublattices in $M$ and in $D$.

We call $\pi_\ell : M \to L$ and $\pi_a : M \to A$ the two orthogonal projections, and $\rho_\ell : D' \to L'$ and $\rho_a : D' \to A'$ the two restriction maps. In fact, $\pi_\ell$ and $\rho_\ell$ are both restrictions of a linear map $D \otimes \mathbb{Q} \to L \otimes \mathbb{Q}$ (and similarly for $A$).

**Proof.** We can assume, without loss of generality, that $L$ is minimal (i.e. it contains no vectors of norm 1); indeed, it is easy to verify that $d_\pm(L) = d_\pm(L \oplus I^m)$, and $L$ is diagonal if and only if $L \oplus I^m$.

Call $n$ the rank of $L$. Consider now $D$, the double of $L$. $D$ is a unimodular lattice, so $d(D) \leq 0$. By Lemma 2.2, $d(D) = \min\{d_+(L) + d_-(A), d_-(L) + d_+(A)\} = d_+(L) + d_-(L)$, which proves the first assertion.

Let us now suppose that $d_+(L) + d_-(L) = 0$; again by Elkies’ theorem, this implies that $D$ is the diagonal lattice $I^{2n}$. Call $e_1, \ldots, e_{2n}$ an orthonormal basis of $D$.

We claim that $L$ is a root lattice.

Since $L$ is minimal, $e_i \in D \setminus M$. However, since $M$ has index 2 in $D$, $2e_i \in M$. We also know that $2e_i \notin L \cup A$: indeed, as mentioned at the beginning of the section, $L \otimes \mathbb{Q} \cap D = L$, so if $2e_i \in L$, then also $e_i \in L$. (By symmetry, this proves the statement for $A$ as well.)
This implies that \( r_i := \pi_\ell(2e_i) \) and \( \pi_\ell(2e_i) \) are two non-zero vectors whose squares sum to \( (2e_i)^2 = 4 \); since \( L \) is minimal, they both have square 2. Since the collection \( \{2e_i\} \) is a rational basis of \( D \), the collection \( \{r_i\} \) is a set of roots that rationally spans \( L \), which proves the claim.

If \( d_\pm(L) = \pm \frac{1}{4} \), then, since \( d_\pm(A_1) = \pm \frac{1}{4} \), \( d(U) = 0 \) and by Elkies’ theorem \( U \) is diagonal (recall that \( U \) is the unimodular overlattice of \( L \oplus A_1 \)). By Lemma 2.3 \( L \) is diagonal.

Suppose now \( d_\pm(L) \neq \pm \frac{1}{4} \), so that in particular \( |d_+(L)| = |d_-(L)| =: d \geq \frac{7}{4} \). Consider the characteristic covector \( \xi_0 = e_1 + \cdots + e_{2n} \in \text{Char}(D) \), and, for each \( i \), the characteristic covector \( \xi_i = \xi_0 - 2e_i \in \text{Char}(D) \). Note that \( \xi_i \) is norm-minimizing among all characteristic covectors in \( D \) for \( i = 0, \ldots, 2n \), and so that its restrictions \( \lambda_i = \rho_\ell(\xi_i) \in L' \) and \( \alpha_i = \rho_\ell(\xi_i) \) are characteristic and they minimize the norm in their congruence class. That is, if \( \lambda_i \in \text{Char}_+(L) \), then \( \lambda_i \) minimizes the norm among all elements in \( \text{Char}_+(L) \); in this case, \( \alpha_i \in \text{Char}_-(A) \) and \( \alpha_i \) minimizes the norm in \( \text{Char}_-(A) \).

Without loss of generality, let us suppose that \( \lambda_0 \in \text{Char}_+(L) \). The key observation is that \( \lambda_i \in \text{Char}_-(L) \) for each \( i = 1, \ldots, 2n \). This follows from the fact, observed above, that \( e_i \notin L \), so that \( \pi_\ell(2e_i) \in 2L' \setminus 2L \), and in particular \( \pi_\ell(2e_i) \) swaps \( \text{Char}_+(L) \) and \( \text{Char}_-(L) \).

Now, since \( \lambda_i \in \text{Char}_-(L) \) for each \( i > 0 \) is a norm-minimizer in its class:

\[
|\lambda_0^2 - \lambda_i^2| = 8d \geq 14.
\]

However,

\[
\lambda_0^2 - \lambda_i^2 = 2\langle \lambda_0, r_i \rangle - r_i^2,
\]

so that for each \( i > 0 \):

\[
|\langle \lambda_0, r_i \rangle| \geq 4d - 1 \geq 3.
\]

Pick a subset \( J \subset \{1, \ldots, 2n\} \) of indices such that \( R = \{r_j \mid j \in J\} \) is a rational basis for \( L \); this in particular means that \( |J| = n \). Up to relabelling, let us assume \( J = \{1, \ldots, n\} \). We claim that \( G(R) \) is bicolorable.

To see this, we will prove that all cycles in \( G(R) \) have even length, and in fact they all have length 4. Assume by contradiction that there is a cycle \( C \subset G(R) \) of length \( k \geq 3, k \neq 4 \). Up to another relabelling, let us assume that \( C \) comprises \( r_1, \ldots, r_k \) in this order. Up to replacing \( r_i \) with \( -r_i \) for some values of \( i \), we can assume that all edges \( (r_1, r_2), \ldots, (r_{k-1}, r_k) \) are labelled with \(-1\). Under this assumption, \( (r_k, r_1) \) has to be labelled by \(+1\), for otherwise \( (r_1 + \cdots + r_k)^2 = 0 \), which would contradict the fact that \( R \) is a linearly independent set. Now recall that \( R \subset L \subset D \cong \mathbb{I}^{2n} \) comprises elements of square 2. So there is a basis \( f_1, \ldots, f_{2n} \) of \( D \) such that \( r_1 = f_1 - f_2, \ldots, r_{k-1} = f_{k-1} - f_k \), and \( r_k = f_k + f_1 \); but then \( r_1 + \cdots + r_k = 2f_1 \in L \), which implies \( f_1 \in L \) since \( L \otimes \mathbb{Q} \cap D = L \), and this contradicts the minimality of \( L \).

Since \( G(R) \) is bicolorable there is a subset \( R' \subset R \), indexed by \( J' \subset J \), containing \( \lceil \frac{n}{2} \rceil \) roots that are pairwise orthogonal.

Now, by Bessel’s inequality:

\[
\lambda_0^2 \geq \sum_{j \in J'} \frac{\langle \lambda_0, r_j \rangle^2}{r_j^2} \geq \left\lfloor \frac{n}{2} \right\rfloor \cdot \frac{9}{2} > 2n,
\]
which contradicts the fact that $\lambda_0^3 \leq \lambda_0^2 + \alpha_0^2 = \zeta_0^2 = 2n$.  \qedhere

Remark 2.6. Note that the assumption that the length of the cycle is not 4 is in fact used: for the 4-cycle as above, the embedding given by $(f_1 - f_2, f_2 - f_3, -f_1 - f_2, -f_1 + f_4)$ has components that sum to $-f_1 - f_2 - f_3 + f_4$.

3. The obstruction

In this section we discuss a topological application of Theorem 2.5. We start with an algebraic topology lemma.

Lemma 3.1. Let $X$ be a compact, oriented 4-manifold with boundary $Y$, a closed 3-manifold with $H_1(Y)$ finite of square-free order. Then $|\det Q_X| = |H_1(Y)|$ and all spin$^c$ structures on $Y$ extend to $X$.

Proof. Let us look at the long exact sequence for the pair $(X, Y)$:

$$0 \rightarrow H^2(X, Y) \rightarrow H^2(X) \rightarrow H^2(Y) \rightarrow H^3(X, Y) \rightarrow H^3(X) \rightarrow 0.$$ 

All spin$^c$ structures on $Y$ extend if and only if the restriction map $H^2(X) \rightarrow H^2(Y)$ is onto. Since $H^2(Y)$ is finite, $H^3(X, Y)$ and $H^3(X)$ have the same rank, $b_3$; call $B$ and $A$ their torsion subgroups, respectively. For the same reason, $H^2(X, Y)$ and $H^2(X)$ have the same rank, $b_2$; by the universal coefficient theorem and Poincaré–Lefschetz duality, their torsion subgroups are isomorphic to $A$ and $B$, respectively. Since torsion can only map to torsion, call $\tau_i$ the map obtained by restricting $\pi_i^*: H^i(X, Y) \rightarrow H^i(X)$ to the torsion subgroup, and then projecting the target to the torsion subgroup; we regard $\tau_2$ as a map $\tau_2: A \rightarrow B$, and $\tau_3$ as a map $\tau_3: B \rightarrow A$. Note that, since $\pi_2^*$ is onto and $H^2(Y)$ is torsion, the induced map on the quotient $H^3(X, Y)/B \rightarrow H^3(X)/A$ is injective (in fact, an isomorphism). Finally, the map $H^2(X, Y)/A \rightarrow H^2(X)/B$ is represented by the intersection form $Q_X$ of $X$.

With this in place, we can then apply the nine lemma to

$$0 \rightarrow B \quad H^3(X, Y) \quad H^3(X, Y)/B \quad 0$$

$$0 \rightarrow A \quad H^3(X) \quad H^3(X)/A \quad 0$$

and

$$0 \rightarrow A \quad H^2(X, Y) \quad H^2(X, Y)/A \quad 0$$

$$0 \rightarrow B \quad H^2(X) \quad H^2(X)/B \quad 0$$

to obtain that $\ker \pi_3^*$ is a group $G$ of order $|\ker \tau_3| = |B|/|A|$, and that $\coker \pi_2^*$ is a group $H$ of order $|\coker \tau_2| \cdot |\coker Q_X| = (|B|/|A|) \cdot |\det Q_X|$.

It follows that we can extract a short exact sequence of finite groups:

$$0 \rightarrow H \rightarrow H_1(Y) \rightarrow G \rightarrow 0,$$
from which
\[ |H_1(Y)| = |H| \cdot |G| = \frac{|B|^2}{|A|^2} |\det Q_X|. \]
Since we assumed that \(H_1(Y)\) has square-free order, we conclude that \(|A| = |B|\) and \(|H_1(Y)| = |\det Q_X|\); each of these conclusions imply that \(H^2(X) \to H^2(Y)\) is onto. \(\square\)

To a closed, oriented, spin\(^c\) rational homology 3-sphere \((Y, t)\), Ozsváth and Szabó [20] associate a family of invariants, collectively called Heegaard Floer homology; we will work with two of these, namely \(HF^+(Y, t)\) and \(HF^\infty(Y, t)\), and we will recall a few results from [18]. For convenience, we will work over the field of two elements.

Recall that from Heegaard Floer homology we can extract a rational number \(d(Y, t)\), called the correction term of \((Y, t)\), that is an invariant under spin\(^c\) rational homology cobordism [18], and reduces modulo 2 to the rho-invariant \(\rho(Y, t)\). The correction term \(d(Y, t)\) is defined to be the minimal grading of any non-zero element in the image of the map \(HF^\infty(Y, t) \to HF^+(Y, t)\).

In the remainder of the section \(Y\) will always denote a closed, oriented 3-manifold with \(H_1(Y) \cong \mathbb{Z}/2\mathbb{Z}\); we say that \(Y\) is a homology \(\mathbb{RP}^3\). Hence \(Y\) has exactly two spin\(^c\) structures.

Our argument will depend on the value of these correction terms. First we pin down their value modulo 2.

**Proposition 3.2.** We can label the two spin\(^c\) structures on \(Y\) as \(t_+\) and \(t_-\), so that \(d(Y, t_\pm) \equiv \pm \frac{1}{2} \pmod{2}\).

We start with a preliminary lemma.

**Lemma 3.3.** Let \(W\) be a cobordism from an integral homology sphere \(Z\) to a homology \(\mathbb{RP}^3\), \(Y\). Then both spin\(^c\) structures on \(Y\) extend to \(W\).

**Proof.** Carve an open neighborhood of a path from \(Z\) to \(Y\) into \(W\), to obtain a 4-manifold \(X\) with boundary \(Y \# (-Z)\). The statement is equivalent to the fact that both spin\(^c\) structures on \(\partial X = Y \# (-Z)\) extend to \(X\), which follows from Lemma 3.3 \(\square\)

Fix a spin\(^c\) structure \(s\) on \(Y\) and a simply-connected 4-manifold \(X\) with spin\(^c\) structure \(s\), such that \(\partial X = Y\) and \(s|_{Y} = t\). Recall now that the d-invariant \(d(Y, t)\) reduces modulo 2 to the rho-invariant \(\rho(Y, t) \in \mathbb{Q}/2\mathbb{Z}\). The rho-invariant is defined as \(\rho(Y, t) \equiv \frac{c_1(s)^2 - c_1(X)}{4} \pmod{2}\); it follows from the definition that if \((W, s)\) is a spin\(^c\) cobordism from \((Y, t)\) to \((Y', t')\), then \(\rho(Y', t') - \rho(Y, t) \equiv \frac{c_1(s)^2 - c_1(W)}{4} \pmod{2}\). In particular, integral homology spheres \(Z\) have \(\rho(Z, t) = 0\), since they bound spin manifolds (which have signature divisible by 8, by the van der Blij Lemma [12] Section II.5) and which have a spin\(^c\) structure with trivial first Chern class).

**Proof of Proposition 3.2.** Since \(\rho(Y, t)\) lifts to \(d(Y, t)\), the statement clearly reduces to showing that \(d(Y, t_\pm) \equiv \pm \frac{1}{2} \pmod{2}\). This is what we set out to prove, by finding a suitable cobordism from \(Y\) to an integral homology sphere.

Pick a knot \(K\) in \(Y\) such that \([K] \not\in H_1(Y)\). Then there exists a slope \(\gamma\) such that the result of Dehn surgery along \(K\) with slope \(\gamma\) is an integral homology sphere \(Z_0 := Y_\gamma(K)\). Let \(K_0 \subset Z_0\) denote the dual knot. Since \(H_1(Y) \cong \mathbb{Z}/2\mathbb{Z}\) and \(Z_0\) is an
integer homology sphere, the surgery on $K_0$ that returns $Y$ must have slope $2/q$ for some odd integer $q$.

We can write $2/q$ as a (negative) continued fraction $2/q = [0, -\frac{q+1}{2}, -2]$, so that $Y$ can be represented by the surgery diagram in Figure 2. It is easy to see that the 3-manifold obtained by doing surgery on the 0- and $-\frac{q+1}{2}$-framed components is again an integral homology sphere, which we denote by $Z$, and that the cobordism $W$ from $Z$ to $Y$ given by the $-2$-framed 2-handle is negative definite. Moreover, since $W$ is obtained by attaching a single 2-handle to an integral homology sphere, $H_1(W) = H_3(W) = 0$; in particular, both spin$^c$ structures on $Y$ extend to $W$.

Since $W$ is the trace of a 2-handle attachment along a knot in an integral homology sphere with framing $-2$, the two spin$^c$ structures $s_+$ and $s_-$, with Chern classes 0 and $2\gamma \in H^2(W;\mathbb{Z}) \equiv \mathbb{Z} \cdot \gamma$ respectively, have $c_1(s_+)^2 = 0$, $c_1(s_-)^2 = -2$. By the cobordism formula mentioned above, letting $t_\pm$ be the restriction of $s_\pm$ to $Y$, we get:

$$\rho(Y, t_+) = \frac{c_1(s_+)^2 - \sigma(W)}{4} + \rho(Z, t) \equiv \frac{1}{4} \pmod{2},$$

$$\rho(Y, t_-) = \frac{c_1(s_-)^2 - \sigma(W)}{4} + \rho(Z, t) \equiv -\frac{1}{4} \pmod{2},$$

thus concluding the proof. \hfill \Box

Proposition 3.2 justifies the following definition.

**Definition 3.4.** For $Y$ a homology $\mathbb{R}P^3$, we set $d_{\pm 1/4}(Y) = d(Y, t_\pm)$.

Note that the labelling is chosen so that $d_{\pm 1/4}(Y) \equiv \pm \frac{1}{4} \pmod{2}$; observe also that since $d(Y, t) = -d(-Y, t)$, we have that $d_{\pm 1/4}(-Y) = -d_{\mp 1/4}(Y)$.

Now suppose that $Y$ bounds a *positive definite* 4-manifold $W$. In this context we have the following inequality.
Theorem 3.5 ([18]). For each spin$^c$ structure $s$ on $W$ with $s|_Y = t$, we have
\[
\frac{c_1(s)^2 - b_2(W)}{4} \geq d(Y,t).
\]
Moreover, the two sides of the inequality are congruent modulo 2.

We are now ready to give a topological translation of Theorem 2.5.

Proposition 3.6. Let $Y$ be a homology $\mathbb{RP}^3$. If $Y$ bounds a positive definite 4-manifold, then $d_{1/4}(Y) + d_{-1/4}(Y) \leq 0$. Moreover, if equality is attained, then $d_{\pm 1/4}(Y) = \pm \frac{1}{4}$.

Proof. Suppose that $Y$ bounds a positive definite 4-manifold $W$, and let $L$ be the lattice $(H_2(W; \mathbb{Z})/\text{Tor} \cdot Q_W)$. By Lemma 3.3, $L$ is a positive definite lattice of determinant 2, and the first Chern class gives a surjection $c_1: \text{Spin}^c(W) \to \text{Char}(L)$. Call $n = b_2(W) = \text{rk}L$.

By the last statement in Theorem 3.5, using the labelling of Proposition 3.2, we see that $s \in \text{Spin}^c(W)$ restricts to $t_\pm$ if and only if $c_1(s) \in \text{Char}_\pm(L)$. Let $\xi_+ \in \text{Char}_+(L)$ and $\xi_- \in \text{Char}_-(L)$ be characteristic covectors with minimal square; note that there exist spin$^c$ structures $s_\pm$ on $W$ such that $c_1(s_\pm) = \xi_\pm$, and that $s_\pm$ restricts to $t_\pm$.

Then using Theorem 3.5 and Theorem 2.5 we get
\[
d_{1/4}(Y) + d_{-1/4}(Y) \leq \frac{\xi_+^2 - n}{4} + \frac{\xi_-^2 - n}{4} = d_+(L) + d_-(L) \leq 0,
\]
proving the first part of the theorem. Furthermore, if $d_{1/4}(Y) + d_{-1/4}(Y) = 0$, then the above inequality forces $d_+(L) + d_-(L) = 0$, and so by Theorem 2.5 we get that $d_\pm(L) = \pm \frac{1}{4}$. This, in turn, together with Theorem 3.5 forces $d_{\pm 1/4}(Y) = \pm \frac{1}{4}$. \qed

4. The example

Recall that we defined $\overline{Y}$ as the Seifert fibered space $Y(2;\frac{15}{13},\frac{17}{15},\frac{23}{22})$ and $N = 3P\#\overline{Y}$, where $P$ is the Poincaré homology sphere, oriented as the boundary of the negative $E_8$-plumbing; equivalently, $P$ is the Brieskorn sphere $\Sigma(2,3,5)$.

We start by computing the correction terms of $\overline{Y}$.

Proposition 4.1. $d_{-1/4}(\overline{Y}) = -\frac{17}{4}$ and $d_{1/4}(\overline{Y}) = -\frac{31}{4}$.

Proof. Since $-\overline{Y}$ is the boundary of a negative definite plumbing with a single bad vertex, we can compute these correction terms using Çağrı Karakurt’s implementation [3] of Némethi’s formula [14, Section 11.13], which, in turn, is a generalization of Ozsváth and Szabó’s algorithm from [19]. \qed

Remark 4.2. Némethi computes the $d$-invariant of a Seifert fibered space as a sum of two terms; the first summand is expressed in terms of Dedekind–Rademacher sums associated to the Seifert parameters [14, Section 11.9], while the second depends on the minimum of a certain function $\tau: \mathbb{Z}_{>0} \to \mathbb{Z}$. The function is eventually increasing, and the minimum is contained in a bounded interval $[0, N]$, where $N$ can be chosen to be the product of the multiplicities of the fibers.

Furthermore, in principle the computation of the correction terms of $\overline{Y}$ could be done in other ways: either by computing the minimal squares in the lattice associated
to the canonical negative plumbing of $-\overline{Y}$ [19, Corollary 1.5] or by following the entire algorithm in [19].

We are now ready to prove our main result; more precisely, we will prove that $N$ does not bound a definite 4-manifold.

Proof of Theorem 1.1. By additivity of correction terms, and since $d(P, t) = 2$ for the unique spin\(^c\) structure $t$ on $P$, we know that $d_{\pm 1/4}(N) = 3d(P, t) + d_{\pm 1/4}(\overline{Y})$. By the previous proposition, we get $d_{\pm 1/4}(N) = \mp \frac{7}{4}$.

Proposition 3.6 now implies that $N$ cannot bound a positive definite 4-manifold. Reversing orientation and again applying Proposition 3.6 shows that $N$ cannot bound a negative definite 4-manifold either.

We conclude with two observations about $\overline{Y}$ and spineless 4-manifolds.

Proposition 4.3. Let $\Sigma$ be an integral homology sphere. The 3-manifold $\overline{Y} \# \Sigma$ is not integral homology cobordant to a 3-manifold obtained as Dehn surgery along a knot in $S^3$.

Proof. If we let $Y = S^{3/2}_{q}(K)$ with $q > 0$, then, by [15, Proposition 1.4 and Lemma 2.4]:

$$d_{1/4}(Y) \in \left\{ -2V_0(K) + \frac{1}{4}, -2V_1(K) + \frac{1}{4} \right\} \implies d_{1/4}(Y) \geq -2V_0(K) + \frac{1}{4},$$

so that in particular

$$d_{1/4}(Y) - d_{-1/4}(Y) \geq \frac{1}{2}.$$ However

$$d_{1/4}(-\overline{Y} \# \Sigma) - d_{-1/4}(-\overline{Y} \# \Sigma) = d_{1/4}(\overline{Y} \# \Sigma) - d_{-1/4}(\overline{Y} \# \Sigma) = -\frac{7}{2},$$

so $\pm(\overline{Y} \# \Sigma)$ cannot be integrally homology cobordant to a positive surgery along a knot in $S^3$. \hfill \Box

The following remark was suggested to the authors by Adam Levine.

Remark 4.4. Note that the previous proposition implies that, for any integral homology sphere $\Sigma$, $\overline{Y} \# \Sigma$ cannot bound a simply-connected spineless 4-manifold, i.e. a 4-manifold $W$ that is homotopy equivalent to $S^2$ but such that the generator of $H_2(W)$ is not represented by a PL-sphere.

Closely following Levine and Lidsman’s approach [10], we produce a homotopy $S^2$ whose boundary is $\overline{Y} \# \Sigma$ for some homology sphere $\Sigma$, which is necessarily going to be spineless. We sketch the construction, which is very similar to [10].

The key observation is that there is an integral homology sphere $-\Sigma$ such that $\overline{Y}$ is obtained as integral surgery along a knot in $-\Sigma$. For example, we can choose $\Sigma$ to be the Brieskorn sphere $\Sigma(15, 17, 181)$. Indeed, the negative plumbing graph of $\Sigma(15, 17, 181)$ is obtained by adding a single vertex to the negative plumbing graph of $-\overline{Y}$, which exhibits $\overline{Y}$ as surgery along a singular fiber of $-\Sigma(15, 17, 181)$.

By [10, Lemma 3.2 and Proposition 3.1], the 4-manifold obtained from the trace of this surgery and carving a path in $\Sigma \times I$ is a homotopy $S^2$ whose boundary is $\overline{Y} \# \Sigma$. 


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