NEW KÄHLER METRIC ON QUASIFUCHSIAN SPACE AND
ITS CURVATURE PROPERTIES

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Abstract. Let \( QF(S) \) be the quasifuchsian space of a closed surface \( S \) of genus \( g \geq 2 \). We construct a new mapping class group invariant Kähler metric on \( QF(S) \). It is an extension of the Weil-Petersson metric on the Teichmüller space \( \mathcal{T}(S) \subset QF(S) \). We also calculate its curvature and prove some negativity for the curvature along the tautological directions.

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Introduction

Teichmüller space \( \mathcal{T}(S) \) carries a natural mapping class group invariant Kähler metric, called a Weil-Petersson metric \( g_{WP} \). There have been active studies on the properties of this metric since its birth. More recently, some new Kähler metrics with more desirable properties such as Kähler hyperbolicity have been

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found where the Kähler hyperbolicity means that the Kähler metric is complete with bounded curvatures and it has a bounded Kähler primitive. Such Kähler hyperbolic metrics are studied by McMullen [14] and Liu-Sun-Yau [13].

In Kleinian group theory, the quasifuchsian space $QF(S)$ is a quasiconformal deformation space of the Fuchsian space $F(S)$ which can be identified with $\mathcal{T}(S)$. By Bers’ simultaneous uniformization theorem, $QF(S)$ can be naturally identified with $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ where $\bar{S}$ is a surface with an orientation reversed. With this identification, the mapping class group acts diagonally on $QF(S)$ and $F(S) = \mathcal{T}(S)$ sits diagonally on $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$. But this diagonal embedding is totally real. Hence if one gives a product Kähler metric on $QF(S)$, this metric is not an extension of a Kähler metric on $F(S)$. There have been several attempts to extend a Kähler metric of $\mathcal{T}(S)$ to $QF(S)$. Bridgeman and Taylor [4] described a quasi-metric which extends the Kähler metric of $\mathcal{T}(S)$ but it vanishes along the pure bending deformation vectors [5].

In this article, we give a completely new mapping class group invariant Kähler metric on $QF(S)$ which extends any Kähler metric on $\mathcal{T}(S)$. Indeed such a metric is already defined in the paper [11] a few years ago. The metric is defined by a Kähler potential which is a combination of $L^2$ norm of a fiber and a Kähler potential on the base $\mathcal{T}(S)$. We will see that $QF(S)$ can be embedded, via Bers embedding using the complex projective structures, in the holomorphic bundle over $\mathcal{T}(S)$ with fibers being quadratic holomorphic differentials as a bounded open neighborhood of the zero section. The Kähler metric we construct is the restriction to this open neighborhood. We choose then the Weil-Petersson metric on $\mathcal{T}(S)$ and show that the new Kähler metric on $QF(S)$ has similar properties such as its Kähler form has a bounded primitive and the curvature has non-positivity for some directions.

**Theorem 0.1.** There exists a mapping class group invariant Kähler metric on $QF(S)$ which extends the Weil-Petersson metric on $\mathcal{T}(S) \subset QF(S)$. Furthermore the curvature of the metric is non-positive when evaluated on the tautological section (and vanishes along vertical directions), its Ricci curvature is bounded from above by $-\frac{1}{\pi(g-1)}$ when restricted to Teichmüller space, and its Kähler form has a bounded primitive.

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1. Preliminaries

1.1. Quasifuchsian space. Recall that the isometry group of the hyperbolic 3-space $\mathbb{H}^3$ can be identified with $PSL(2, \mathbb{C})$. We use the unit ball in $\mathbb{R}^3$ as a realization of $\mathbb{H}^3$. The ideal boundary is then $S^2$ and is further identified with $\mathbb{C}P^1$ such that the action of $PSL(2, \mathbb{C})$ on $S^2$ is the natural extension of its isometric action on $\mathbb{H}^3$. 
The Teichmüller space $T(S)$ is realized as the space of Fuchsian representations, i.e., discrete and faithful representations $\rho : \pi_1(S) \to PSL(2, \mathbb{R})$ up to conjugacy. Let $\Gamma_\rho$ be the image of $\rho$, whence $\Gamma_\rho$ acts on $S^2$ by Möbius map preserving the equator. Then any quasiconformal map $f$ from $S^2$ into itself induces a quasiconformal deformation $\rho_f$ defined by

$$\rho_f(\gamma) = f \circ \rho(\gamma) \circ f^{-1}.$$ 

If furthermore $\rho_f(\gamma)$ is an element of $PSL(2, \mathbb{C})$ for any $\gamma \in \pi_1(S)$ then it defines a representation of $\pi_1(S)$ in $PSL(2, \mathbb{C})$. Collection of such quasiconformal deformations of Fuchsian representations is denoted $QF(S)$ and is identified with an open set of a character variety $\chi(\pi_1(S), PSL(2, \mathbb{C}))$. Hence it has a natural induced complex structure from $\chi(\pi_1(S), PSL(2, \mathbb{C}))$.

If $\phi : \pi_1(S) \to PSL(2, \mathbb{C})$ is a quasifuchsian representation, then $\mathcal{M}_\phi = \mathbb{H}^3/\phi(\pi_1(S))$ is a quasifuchsian hyperbolic 3-manifold which is homeomorphic to $S \times \mathbb{R}$. Then two ideal boundaries of $\mathcal{M}_\phi$ define a pairs of points $(X, Y) \in T(S) \times T(S)$. This is known as Bers' simultaneous uniformization of $QF(S)$ [2]. In this case, we denote $\mathcal{M}_\phi$ by $Q(X, Y)$. In this identification, a Fuchsian representation $\rho : \pi_1(S) \to PSL(2, \mathbb{R})$ whose quotient $X = \mathbb{H}^2/\rho(\pi_1(S))$ is a point in $T(S)$ gets identified with $(X, \bar{X})$.

The mapping class group $Mod(S)$ acts on the space of representations $\rho : \pi_1(S) \to PSL(2, \mathbb{C})$ by pre-composition $\phi \rho = \rho \circ \phi_e$ where $\phi \in Mod(S)$ and $\phi_e$ is the induced homomorphism on $\pi_1(S)$. Then $Mod(S)$ acts on $QF(S) = T(S) \times T(\bar{S})$ diagonally

$$\phi \rho = \phi(X, Y) = (\phi X, \phi Y).$$

1.2. Space of complex projective structures on surface. A complex projective structure on $S$ is a maximal atlas $\{(\phi_i, U_i), \phi_i : U_i \to S^2\}$ whose transition maps $\phi_i \circ \phi_i^{-1}$ are restrictions of Möbius maps. Then the developing map $dev : \bar{S} \to S^2$ gives rise to a holonomy representation $\rho : \pi_1(S) \to PSL(2, \mathbb{C})$. We denote the space of marked complex projective structures on $S$ by $\mathcal{P}(S)$. Since Möbius transformations are holomorphic, a projective structure determines a complex structure on $S$. In this way we obtain a forgetful map

$$\pi : \mathcal{P}(S) \to T(S).$$

Obviously a Fuchsian representation $\rho : \pi_1(S) \to PSL(2, \mathbb{R}) \subset PSL(2, \mathbb{C})$ preserving the equator of $S^2$ gives rise to an obvious projective structure by identifying $\mathbb{H}^2$ with the upper and lower hemisphere of $S^2$. This gives an embedding

$$\sigma_0 : T(S) \to \mathcal{P}(S).$$

More generally, for $X \in T(S)$ and $Z \in \pi^{-1}(X) := P(X)$, by conformally identifying $\bar{X} = \mathbb{H}^2$, we obtain a developing map $dev : \mathbb{H}^2 \to S^2 = \mathbb{C}P^1$ for $Z$. Hence the developing map can be regarded as a meromorphic function $f = dev$.
on $\mathbb{H}^2$. Then the Schwarzian derivative

$$S(f) = \left[ \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \right] \, dz^2$$

descends to $X$ as a holomorphic quadratic differential. It is known that for any element in holomorphic quadratic differentials $Q(X)$ on $X$, one can show that there exists a complex projective structure over $X$ by solving Schwarzian linear ODE equation.

In this way, $P(S)$ can be identified with a holomorphic vector bundle $Q(S)$ over $T(S)$ whose fiber over $X$ is $Q(X)$. In particular this identifies $P(X)$ with $Q(X)$ as affine spaces, [7], and the choice of a base point $Z_0$ in $P(X)$ gives an isomorphism $Z \to Z - Z_0$. Hence we will choose $Z_0 = \sigma_0(X)$ and $T(S)$ will be identified with zero section on $Q(S)$.

### 1.3. Embedding of quasifuchsian space into the space of complex projective structures.

Recall that given $X \in T(S), Y \in \bar{T}(\bar{S})$ the Bers’ uniformization determines the quasifuchsian manifold $Q(X, Y)$. Then $Q(X, Y)$ has domain of discontinuity $\Omega_+ \cup \Omega_-$ with $\Omega_+/Q(X, Y) = X, \Omega_-/Q(X, Y) = Y$ where $Q(X, Y)$ is viewed as a quasifuchsian representation into $PSL(2, \mathbb{C})$.

As a quotient of a domain in $\mathbb{C}P^1$ by a discrete group in $PSL(2, \mathbb{C}), \Omega_-/Q(X, Y)$ is a marked projective surface $\Sigma_Y(X)$. Then for a fixed $Y$, we obtain a quasifuchsian section, called a Bers’ embedding

$$\beta_Y : T(S) \to P(Y) \subset P(\bar{S}).$$

It is known that this map

$$Q(X, Y) \to \Omega_-/Q(X, Y)$$

is a homeomorphism onto its image in $P(\bar{S})$; see e.g. [7]. Under the identification of $P(\bar{S})$ with $Q(\bar{S})$ such that $\sigma_0(T(S))$ is a zero section,

$$Q(X, Y) \to \Omega_-/Q(X, Y) - \sigma_0(Y),$$

this embedding includes zero section which is the image of $T(S)$.

The space $Q(Y)$ of quadratic differentials is also equipped with $L^\infty$-norm defined by

$$||\phi||_\infty = \sup_Y \rho^{-2}||\phi(z)||$$

where $\rho(z)|dz|$ is a hyperbolic metric on $Y$. Then by Nehari’s bound [14] we get

**Theorem 1.1.** The above embedding of $Q(X, Y)$ into $Q(Y)$ is contained in a ball of radius $\frac{3}{2}$ in $Q(Y)$ where the norm is the $L^\infty$-norm on quadratic differentials.

**Corollary 1.2.** The quasifuchsian space $QF(S)$ embeds into a neighborhood of a zero section in $Q(\bar{S})$ which is contained in a ball of radius $9\pi(g-1)$ in $L^2$-norm on each fiber $Q(Y)$. 

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Proof. The $L^2$-norm of a quadratic differential $\phi(z)dz^2$ is given by
\[
\int_Y |\phi(z)|^2 \rho(z)^{-4} \rho(z)^2 |dz|^2 \leq ||\phi||^2_{\infty} 2\pi (2g - 2) \leq 9\pi (g - 1).
\]

\[\square\]

1.4. Vector bundle isomorphism between quadratic differentials and Beltrami differentials. The holomorphic tangent bundle of Teichmüller space $\mathcal{T}(S)$ is a holomorphic vector bundle over Teichmüller space whose fiber over $X$ is the set of harmonic Beltrami differentials $B(X)$. For a harmonic Beltrami differential $\mu(z)dz$ over $X$ with a hyperbolic metric $g = \rho(z)|dz|$, the $L^2$-norm defines the Weil-Petterson metric
\[
\|\mu\|^2_{WP} = \int_X |\mu(z)|^2 \rho(z)^2 |dz|^2
\]
on the tangent space of $\mathcal{T}(S)$. The set of harmonic Beltrami differentials $B(X)$ and $Q(X)$ are vector bundle isomorphic by the natural identification of differential forms with tangent vectors via the metric,
\[
\Phi = \phi(z)dz^2 \rightarrow \beta = \beta_\Phi = \frac{\phi(z)}{\rho^2(z)} \frac{d\bar{z}}{dz}.
\]
The $L^2$-norms are by definition preserved,
\[
\|\beta\|^2 = \|\beta\|^2_{WP} = \int_X \frac{|\phi(z)|^2}{\rho^2(z)} \rho^2(z)|dz|^2 = ||\Phi||^2.
\]

By Corollary 1.2, we get

Corollary 1.3. Under this isomorphism between cotangent bundle and holomorphic tangent bundle of $\mathcal{T}(S)$, the quasifuchsian space $QF(S)$ embeds into a neighborhood of a zero section in the holomorphic tangent bundle of $\mathcal{T}(S)$ which is contained in a ball of radius $9\pi (g - 1)$ in $L^2$-norm on each fiber $B(X)$.

2. Griffiths negativity and Kähler metric on the holomorphic vector bundles

2.1. Griffiths negativity. As elaborated above the space $QF(S)$ can be realized as an open set in the tangent bundle of $\mathcal{T}(S)$, and we shall construct metrics on $QF(S)$ using some general constructions. For that purpose we recall the notion of Griffiths positivity. Identifying $\mathcal{P}(S)$ with the holomorphic vector bundle $\mathcal{Q}(S)$ whose fiber over $Y \in \mathcal{T}(S)$ is $Q(Y)$, one can give a mapping class group invariant Kähler metric on $\mathcal{Q}(S)$ as follows. By a theorem of Berndtsson [3], one can show that $\mathcal{Q}(S)$ is Griffiths positive. See [11] for a proof. Hence its dual bundle $\mathcal{B}(S) = \mathcal{Q}^*(S)$, which is a tangent bundle of Teichmüller space whose fiber is the set of Beltrami differentials, is Griffiths negative. We fix in the
rest of the paper this realization of $QF(S)$ as a subset in $Q^*(S)$. The $L^2$-norm of a Beltrami differential $w = \mu(v) \frac{dv}{dv}$ is given by

$$||w||^2 = (w, w) = \int_Y |\mu(v)|^2 \rho(v)^2 |dv|^2$$

where $v$ is a local holomorphic coordinate on $Y$ and $\rho(v)|dv|$ is a hyperbolic metric on $Y$. Here $(,)$ denotes the $L^2$ inner product over each fiber and $|| \cdot ||$ denotes its associated norm.

The Kähler metric depending on a constant $k > 0$ and a Kähler metric on $\mathcal{T}(S)$, is constructed on $\mathcal{B}(S) = Q^*(S)$ via Kähler potential

$$\Phi(w) = ||w||^2 + k \pi^* \psi(w),$$

where $w$ is an element in the fiber, $\psi$ is a Kähler potential on $\mathcal{T}(S)$ and $\pi : \mathcal{B}(S) \rightarrow \mathcal{T}(S)$ is a projection.

In local holomorphic coordinates $(z, \bar{z})$ around $w_0$, where $z = (z_1, \cdots, z_{3g-3})$ is local holomorphic coordinates around $\pi(w_0) = z_0$, and $w = \sum x^\alpha e_\alpha(z)$ with respect to local holomorphic sections $e_\alpha$, for a holomorphic tangent vector at $w$ $T = u + v$ with a canonical decomposition into $\mathcal{T}(S)$ direction $u$ and vertical fiber direction $v$, the norm of $T$ with respect to the Kähler metric $g$ defined by the Kähler potential $\Phi$ is given by

$$||T||^2_T = \tilde{\partial}_T \partial_T \Phi(w) = -(R(u, \bar{u})w, w) + (D_u w + v, D_u w + v) + k \partial_u \partial_u \psi > 0,$$

where $R$ is a curvature of the Chern connection $\nabla$ on $\mathcal{B}(S)$ and $\nabla = D + \bar{D}$ is a decomposition into $(1,0)$ and $(0,1)$ part of the connection. See [11] for details.

Since this construction is general, we treat this construction as general as possible in the following subsection.

2.2. Kähler metrics on Griffiths negative vector bundles. Let $\pi : E \rightarrow M$ be a holomorphic vector bundle of rank $r$ over a complex manifold $M$, $\dim M = n$. Let $\{e_i\}_{i=1}^r$ be a local holomorphic frame of $E$ and $\{z^\alpha\}_{\alpha=1}^n$ be local coordinates of $M$. Let $G$ be a Hermitian metric on $E$ with Griffiths negative curvature, that is

$$R_{i\bar{j}\alpha\beta}v^i\bar{v}^j\xi^\alpha \bar{\xi}\bar{\beta} < 0$$

for any non-zero vectors $v = v^i e_i \in E$ and $\xi = \xi^\alpha \frac{\partial}{\partial z^\alpha} \in TM$. Here

$$R_{i\bar{j}\alpha\beta} = -\partial_\alpha \partial_\beta G_{ij} + G^{kl} \partial_\alpha G_{il} \partial_\beta G_{kj}$$

denotes the Chern curvature tensor of the Hermitian metric $G$. With respect to the local holomorphic frame $\{e_i\}_{i=1}^r$, the complex manifold $E$ is equipped with the following holomorphic coordinates

$$(z; v) = (z^1, \cdots, z^n, v^1, \cdots, v^r),$$
representing the point \( v = v^i e_i \in E \). The Hermitian metric \( G \) also gives the norm square function on \( E \). By abuse of notation, we also denote it by \( G \), i.e. the function

\[
v \in E \mapsto G(v) = G(v, \bar{v}) = G_{ij} v^i \bar{v}^j.
\]

Then \( \partial \bar{\partial} G \) is a \((1,1)\)-form on \( E \).

**Lemma 2.1.** Denote \( \delta v^i := dv^i + G_{ai} \bar{G}^{li} dz^\alpha \). Then

\[
\partial \bar{\partial} G = -R_{ij\alpha\beta} v^i \bar{v}^j dz^\alpha \wedge d\bar{z}^\beta + G_{ij} \delta v^i \wedge \delta \bar{v}^j.
\]

**Proof.** This follows by a direct computation,

\[
-R_{ij\alpha\beta} v^i \bar{v}^j dz^\alpha \wedge d\bar{z}^\beta + G_{ij} \delta v^i \wedge \delta \bar{v}^j
\]

\[
= -(-\partial_\alpha \partial_\beta G_{ij} + G^{lk} \partial_\alpha G_{il} \partial_\beta G_{kj}) v^i \bar{v}^j dz^\alpha \wedge d\bar{z}^\beta
\]

\[
+ G_{ij} (dv^i + G_{ai} \bar{G}^{li} dz^\alpha) \wedge (d\bar{v}^j + G_{\beta k} \bar{G}^{j} d\bar{z}^\beta)
\]

\[
= G_{ij} \delta v^i \wedge \delta \bar{v}^j + G_{ij} \delta v^i \wedge d\bar{v}^j + G_{ij} dv^i \wedge d\bar{z}^j + G_{ij} dv^i \wedge d\bar{v}^j
\]

\[
= \partial \bar{\partial} G.
\]

\( \square \)

Now we assume \((M, \omega = \sqrt{-1} g_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta)\) is a Kähler manifold and \((E, G)\) is Griffiths negative. Define the \((1,1)\)-form

\[
(2.1) \quad \Omega := \pi^* \omega + \sqrt{-1} \partial \bar{\partial} G.
\]

Lemma 2.1 then implies that \( \Omega \) is a Kähler metric on \( E \). In terms of local coordinates \( \Omega \) is

\[
(2.2) \quad \Omega = \sqrt{-1} \Omega_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta + \sqrt{-1} G_{ij} \delta v^i \wedge \delta \bar{v}^j,
\]

where

\[
(2.3) \quad \Omega_{\alpha\beta} := -R_{ij\alpha\beta} v^i \bar{v}^j + g_{\alpha \bar{\beta}}
\]

is a positive definite matrix. The differential \( \partial G \) of \( G \) is a globally defined one-form on \( E \), and its norm square is \( G \). Indeed,

\[
\partial G = G_\alpha dz^\alpha + G_i dv^i = G_i (dv^i + G_{ai} \bar{G}^{li} dz^\alpha) = G_i \delta v^i.
\]

Its norm square with respect to the metric \( \Omega \) is

\[
\| \partial G \|^2 = G_i G_j \bar{G}^{ji}.
\]

Since \( G = G_{ij} v^i \bar{v}^j \), so \( G_i = G_{ij} \bar{v}^j \) and

\[
G_i G_j G^{ji} = G_{il} \bar{v}^l G_{kj} v^k G^{ji} = G_{kl} v^k \bar{v}^l = G,
\]

which yields that

\[
\| \partial G \|^2 = G_i G_j \bar{G}^{ji} = G,
\]

which is independent of the metric \( \omega \).
**Proposition 2.2.** The norm of the one-form $\partial G$ with respect to $\Omega$ is given by
\[ \|\partial G\|^2 = G \]
for any metric $\omega$ on $M$. In particular,
\[ \|\partial G\|^2 < R \]
on the disk bundle $S_R = \{(z,v) \in E | G(z,v) < R \}$.

As a corollary, we obtain

**Corollary 2.3.** If $\omega$ is $d$-bounded, $\omega = d\beta$ for some (locally defined) bounded one-form $\beta$, then $\Omega$ is also $d$-bounded on any bounded domain of $E$ with $\Omega = d(\partial G + \pi^*\beta)$ and bounded one-form $\partial G + \pi^*\beta$.

Specifying to the space $QF(S)$ we find that new Kähler metric on $QF(S)$ has a bounded primitive if the Kähler form on $T(S)$ has a bounded primitive.

### 3. Curvature of the new Kähler metric

In this section, we will calculate the curvature of the Kähler metric $\Omega$ (2.1). By [6, Section 2], the Hermitian metric $G$ gives a decomposition on the tangent bundle $TE$ of $E$, i.e.

\[ TE = H \oplus V, \]

where the horizontal subbundle $H$ and vertical subbundle $V$ are given by

\[ H = \text{Span}_\mathbb{C} \left\{ \frac{\delta}{\delta z^\alpha} := \frac{\partial}{\partial z^\alpha} - G_{\alpha j} \bar{\partial} \frac{\partial}{\partial v^i}, 1 \leq \alpha \leq n \right\}, \quad V = \text{Span}_\mathbb{C} \left\{ \frac{\partial}{\partial v^i}, 1 \leq i \leq r \right\}. \]

By duality, the cotangent bundle $T^*E = H^* \oplus V^*$ with

\[ H^* = \text{Span}_\mathbb{C} \{dz^\alpha, 1 \leq \alpha \leq n \}, \quad V^* = \text{Span}_\mathbb{C} \left\{ \delta v^i = dv^i + G_{\alpha i} \bar{\partial} dz^\alpha, 1 \leq i \leq r \right\}. \]

Let $\nabla = \nabla' + \bar{\partial}$ denote the Chern connection of $\Omega$ and

\[ R^\Omega = \nabla^2 = \nabla' \circ \bar{\partial} + \bar{\partial} \circ \nabla' \in A^{1,1}(E, \text{End}(TE)) \]
denote the Chern curvature of $\nabla$. Then

\[
\nabla' \left( \frac{\delta}{\delta z^\alpha} \right) = \left\langle \nabla' \left( \frac{\delta}{\delta z^\alpha} \right), \frac{\delta}{\delta z^\beta} \right\rangle \Omega^\beta_\gamma \frac{\delta}{\delta z^\gamma} + \left\langle \nabla' \left( \frac{\delta}{\delta z^\alpha} \right), \frac{\partial}{\partial v^i} \right\rangle G^\beta_i \frac{\partial}{\partial v^i}
\]

\[
= \left( \partial \Omega_{\alpha \beta} - \left\langle \frac{\delta}{\delta z^\alpha}, \bar{\partial} \left( \frac{\delta}{\delta z^\beta} \right) \right\rangle \right) \Omega^\beta_\gamma \frac{\delta}{\delta z^\gamma} + \partial \Omega_{\alpha \beta} \Omega^\beta_\gamma \frac{\delta}{\delta z^\gamma},
\]

(3.1)
where the last equality holds since $\bar{\partial} \left( \frac{\delta}{\delta z^\alpha} \right)$ is vertical, and
\[
\nabla' \left( \frac{\partial}{\partial v^i} \right) = \left\langle \nabla' \left( \frac{\partial}{\partial v^i} \right), \frac{\delta}{\delta z^\beta} \Omega^\beta \delta \frac{\delta}{\delta z^\gamma} + \left\langle \nabla' \left( \frac{\partial}{\partial v^i} \right), \frac{\partial}{\partial v^k} \right\rangle G^j_k \frac{\partial}{\partial v^k} \right\rangle
\]
\[
= - \left\langle \nabla' \left( \frac{\partial}{\partial v^i}, \bar{\partial} \left( \frac{\delta}{\delta z^\beta} \right) \right), \Omega^\beta \delta \frac{\delta}{\delta z^\gamma} + \bar{\partial} G_{ij} G^{jk} \frac{\partial}{\partial v^k} \right\rangle
\]
\[
= G_{ij} \partial(G_{k \bar{\beta}} G^{jk}) \Omega^\beta \delta \frac{\delta}{\delta z^\gamma} + \bar{\partial} G_{ij} G^{jk} \frac{\partial}{\partial v^k}
\]
\[
= G_{ij} \partial(G_{k \bar{\beta}} G^{jk}) \Omega^\beta \delta \frac{\delta}{\delta z^\gamma} \otimes \frac{\delta}{\delta z^\gamma} + \partial \partial G_{ij} G^{jk} \frac{\partial}{\partial v^k}
\]
where the last equality follows from the fact $G_{ij} = 0$ since $G_{ij}$ is a metric along vertical directions. From (3.1) and (3.2), the curvature $R^\Omega$ is
\[
R^\Omega \left( \frac{\delta}{\delta z^\alpha} \right) = \left( \nabla' \circ \bar{\partial} + \bar{\partial} \circ \nabla' \right) \left( \frac{\delta}{\delta z^\alpha} \right)
\]
\[
= \nabla' \left( - \bar{\partial}(G_{ai} G^{\bar{i}}) \frac{\partial}{\partial v^i} \right) + \bar{\partial} \left( \partial_{\alpha \beta} \Omega^\beta \delta \frac{\delta}{\delta z^\gamma} \right)
\]
\[
= \left( - \partial \bar{\partial}(G_{ai} G^{\bar{i}}) - \partial G_{ai} G^{\bar{i}} \wedge \bar{\partial}(G_{ai} G^{\bar{i}}) + \partial \partial_{\alpha \beta} \Omega^\beta \wedge \bar{\partial}(G_{ai} G^{\bar{i}}) \right) \frac{\partial}{\partial v^k}
\]
\[
+ \left( \bar{\partial}(\partial_{\alpha \beta} \Omega^\beta) - \partial(G_{ai} G^{\bar{i}}) G_{ij} \Omega^\beta \wedge \bar{\partial}(G_{ai} G^{\bar{i}}) \right) \frac{\delta}{\delta z^\gamma},
\]
and
\[
R^\Omega \left( \frac{\partial}{\partial v^i} \right) = \bar{\partial} \circ \nabla' \left( \frac{\partial}{\partial v^i} \right)
\]
\[
= \bar{\partial} \left( G_{ij} \partial(G_{k \bar{\beta}} G^{jk}) \Omega^\beta \delta \frac{\delta}{\delta z^\gamma} + \partial G_{ij} G^{jk} \frac{\partial}{\partial v^k} \right)
\]
\[
= \bar{\partial}(G_{ij} \partial(G_{k \bar{\beta}} G^{jk}) \Omega^\beta \delta \frac{\delta}{\delta z^\gamma})
\]
\[
+ \left( G_{ij} \partial(G_{k \bar{\beta}} G^{jk}) \Omega^\beta \wedge \bar{\partial}(G_{ai} G^{\bar{i}}) + \bar{\partial}(G_{ij} G^{\bar{i}}) \right) \frac{\partial}{\partial v^k}.
\]
Therefore, we obtain

**Proposition 3.1.** The Chern curvature $R^\Omega$ satisfies

(i) $\left\langle R^\Omega \left( \frac{\partial}{\partial v^i} \right), \frac{\partial}{\partial v^j} \right\rangle = \left( R_{i \alpha \beta \sigma} R_{k \gamma \gamma \sigma} v^k \bar{v}^\gamma \Omega^\beta \gamma + R_{i \alpha \sigma} \right) d z^\alpha \wedge d z^\sigma$.

(ii) $\left\langle R^\Omega \left( \frac{\delta}{\delta z^\alpha} \right), \frac{\delta}{\delta z^\beta} \right\rangle = \partial(\partial \partial_{\alpha \beta} \Omega^\beta) \Omega^\gamma - R_{p \gamma \gamma \beta} R_{q \alpha \sigma} \bar{v}^p d z^\gamma \wedge d z^\sigma$.

**Proof.** (i) We compute the inner product according to (3.4),
\[
\left\langle R^\Omega \left( \frac{\partial}{\partial v^i} \right), \frac{\partial}{\partial v^j} \right\rangle = G_{iq} \partial(G_{k \bar{\beta}} G^{jk}) \Omega^\beta \gamma \wedge \bar{\partial}(G_{ai} G^{\bar{i}}) G_{k \bar{\beta}} + \bar{\partial}(G_{iq} G^{\bar{i}}) G_{k \bar{\beta}}.
\]
Note that $G_{ij} G^{jk} = \delta^k_i$, hence
\[
\partial \partial_{\alpha}(G_{k \bar{\beta}} G_{ai}) = 0 = \partial_{\alpha}(G_{k \bar{\beta}} G_{ai}) G_{i \bar{\gamma}} + G_{k \bar{\beta}} G_{ai \bar{\gamma}},
\]
Then
\[
G_{i\bar{q}}\partial(G_{\bar{k}\bar{q}} G^{k\bar{q}}) = \left[G_{i\bar{q}}(\partial_{\alpha}(G_{\bar{k}\bar{q}})G^{k\bar{q}}) + G_{i\bar{q}}(G_{\bar{k}\bar{q}} \partial_{\alpha}(G^{k\bar{q}}))\right] dz^\alpha
= (G_{i\bar{q}} G_{\alpha\bar{k}} G^{k\bar{q}} - G_{\bar{k}\bar{q}} G^{k\bar{q}} G_{\alpha\bar{i}}) dz^\alpha
= (G_{\alpha\bar{i}} G_{\beta\bar{k}} G^{k\bar{q}} G_{\alpha\bar{i}} G^{k\bar{q}}) dz^\alpha.
\]

But \( R_{i\bar{a}\bar{b}} = -\partial_{\alpha} \partial_{\beta} G_{i\bar{a}} + G^{k\bar{j}} \partial_{\alpha} G_{i\bar{j}} \partial_{\beta} G_{k\bar{l}} \) and \( G = G(z, v) = G_{ij}(z)v^i \bar{v}^j \), hence
\[G_i = G_{ij} \bar{v}^j, G_{\alpha\bar{i}} = G_{\alpha\bar{i}} \bar{v}^i,\]
and
\[R_{i\bar{a}\bar{b}} \bar{v}^i = -G_{\alpha\bar{i}} G_{\beta\bar{j}} G_{\alpha\bar{j}} G_{\beta\bar{k}} G_{\bar{k}\bar{l}} \]

Finally we get
\[
(3.5) \quad G_{i\bar{q}}\partial(G_{\bar{k}\bar{q}} G^{k\bar{q}}) = \left(G_{\alpha\bar{k}} G^{k\bar{q}} G_{i\bar{q}} G^{\bar{q}k}\right) dz^\alpha = -R_{i\bar{a}\bar{b}} dz^\alpha.
\]

Similar calculations give
\[
\left\langle R^\Omega \left(\frac{\partial}{\partial v^i} , \frac{\partial}{\partial v^j}\right)\right\rangle = \left(R_{i\bar{a}\bar{b}} R_{k\bar{j}\gamma\delta} v^k \bar{v}^j \Omega^{\gamma\delta} + R_{i\bar{j}\alpha\beta} dz^\alpha \wedge dz^\beta\right).
\]

(ii) Using (3.5) we compute
\[
\left\langle R^\Omega \left(\frac{\delta}{\delta z^\alpha} , \frac{\delta}{\delta z^\beta}\right)\right\rangle
= \left\langle \left(\partial(\partial_{\alpha\beta} \Omega^{\gamma\delta}) - \partial(G_{\alpha\beta} G^{\gamma\delta}) G_{ij} \Omega^{\gamma\delta} \wedge \partial(G_{\alpha\beta} G^{\gamma\delta})\right) \frac{\delta}{\delta z^\gamma} \wedge \frac{\delta}{\delta z^\delta}\right\rangle
= \partial(\partial_{\alpha\beta} \Omega^{\gamma\delta}) \Omega^{\gamma\beta} - R_{p\gamma\beta} R_{k\delta\alpha\beta} v^k \bar{v}^l G^{\gamma\delta} dz^\gamma \wedge dz^\delta.
\]

Remark 3.2. From (i), when evaluated on a vertical vector, \( \left\langle R^\Omega \left(\frac{\partial}{\partial v^i} , \frac{\partial}{\partial v^j}\right)\right\rangle \) vanishes, that is \( \left\langle R^\Omega \left(\frac{\partial}{\partial v^i} , \frac{\partial}{\partial v^j}\right)\right\rangle = 0.\)

There exists a canonical holomorphic section of \( \mathcal{V} \), that is
\[P = v^i \frac{\partial}{\partial v^i} \in \mathcal{O}_E(\mathcal{V})\]
which is called the tautological section (see e.g. [1, Section 3]). Denote
\[\psi_{\alpha\beta} = -R_{i\bar{a}\bar{b}} \bar{v}^i \bar{v}^j.\]
From Proposition 3.1 the $(1,1)$-form $(R^\Omega(P), P)$ is
\[
\langle R^\Omega(P), P \rangle = \left\langle R^\Omega \left( \frac{\partial}{\partial v^i} \right), \frac{\partial}{\partial v^j} \right\rangle v^i \bar{v}^j
\]
\[
= (R_{\bar{l}a\bar{\beta}}R_{\bar{k}j\bar{\gamma}a}v^k \bar{v}^j \Omega_{\bar{l}j} \gamma + R_{ij\alpha\beta}) v^i \bar{v}^j dz^\alpha \wedge d\bar{z}^\sigma
\]
\[
= \left( \Psi_{\bar{a}\beta} \Psi_{\bar{\gamma}\alpha} \Omega_{\bar{a}\beta} - \Psi_{\bar{a}\sigma} \right) dz^\alpha \wedge d\bar{z}^\sigma.
\]

For any point $(z, v)$ outside the zero section, i.e. in the set
\[
E^o := \{(z, v) \in E; v \neq 0\},
\]
the vector $P(z, v) \neq 0$. So $(\Psi_{\bar{a}\beta})$ is a positive definite matrix on $E^o$ by Griffiths negativity. Since
\[
\Omega_{\alpha\beta} - \Psi_{\alpha\beta} = g_{\alpha\beta}
\]
is positive definite, so
\[
\langle \sqrt{-1} R^\Omega(P), P \rangle = \sqrt{-1} \left( \Psi_{\alpha\beta} \Psi_{\gamma\alpha} \Omega_{\alpha\beta} - \Psi_{\alpha\sigma} \right) dz^\alpha \wedge d\bar{z}^\sigma
\]
\[
\leq \sqrt{-1} \left( \Psi_{\alpha\beta} \Psi_{\gamma\alpha} \Omega_{\alpha\beta} - \Psi_{\alpha\sigma} \right) dz^\alpha \wedge d\bar{z}^\sigma = 0.
\]
Thus, we obtain

**Proposition 3.3.** $\langle \sqrt{-1} R^\Omega(P), P \rangle$ is a non-positive $(1,1)$-form on $E$.

**Remark 3.4.** Moreover, $\langle \sqrt{-1} R^\Omega(P), P \rangle$ is a strictly negative $(1,1)$-form on $E^o$ along the horizontal directions, that is
\[
\langle R^\Omega(\xi, \bar{\xi})(P), P \rangle < 0
\]
for any nonzero vector $\xi = \xi^\alpha \frac{\delta}{\delta z^\alpha} \in \mathcal{H}(z, v)$, $(z, v) \in E^o$. In fact, from (3.7),
\[
\langle R^\Omega(\xi, \bar{\xi})(P), P \rangle = 0 \text{ if and only if}
\]
\[
(\Psi_{\bar{\gamma}\alpha} - \Omega_{\bar{\gamma}\alpha}) (\Psi_{\alpha\beta} \xi^\alpha) (\Psi_{\gamma\sigma} \bar{\xi}^\sigma) = 0.
\]

Since $(\Psi_{\bar{\gamma}\alpha} - \Omega_{\bar{\gamma}\alpha})$ is positive definite on $E^o$, so (3.8) is equivalent to
\[
\Psi_{\alpha\beta} \xi^\alpha = 0.
\]

On the other hand, $(\Psi_{\alpha\beta})$ is positive definite on $E^o$ by Griffiths negativity of $(E, G)$, which implies that $\xi = 0$.

The Ricci curvature of the Kähler metric is
\[
Ric^\Omega := \text{Tr} (R^\Omega) = G^{ji} \left\langle R^\Omega \left( \frac{\partial}{\partial v^i} \right), \frac{\partial}{\partial v^j} \right\rangle + \Omega^{\beta\gamma} \left\langle R^\Omega \left( \frac{\delta}{\delta z^\alpha} \right), \frac{\delta}{\delta z^\beta} \right\rangle
\]
\[
= G^{ji} (R_{\bar{l}a\bar{\beta}}R_{\bar{k}j\bar{\gamma}a}v^k \bar{v}^j \Omega_{\bar{l}j} \gamma + R_{ij\alpha\beta}) dz^\alpha \wedge d\bar{z}^\sigma
\]
\[
+ \Omega^{\beta\gamma} (\bar{\partial} (\partial \Omega_{\alpha\beta}) \sigma^\gamma) \Omega_{\bar{\gamma}\beta} - R_{\bar{p}\bar{\gamma}\beta}R_{\bar{k}\bar{q}\alpha\sigma}v^k \bar{v}^q G^{\bar{p}\bar{q}} dz^\gamma \wedge d\bar{z}^\sigma
\]
\[
= \bar{\partial} \partial \log \det(G_{ij}) + \bar{\partial} \partial \log \det(\Omega_{\alpha\beta})
\]
\[
= \bar{\partial} \partial \log \left( \det(G_{ij}) \cdot \det(\Omega_{\alpha\beta}) \right).
\]
Denote by \( \iota : M \to E \) the natural embedding (as the zero section of \( E \)), then
\[
\iota^*(\text{Ric}\,\Omega) = \iota^*(\bar\partial\bar\partial \log (\det(G_{i\bar{j}}) \cdot \det(\Omega_{\alpha\bar{\beta}}))) = \bar\partial\bar\partial \log (\det(G_{i\bar{j}}) \cdot \det(g_{\alpha\bar{\beta}}))
\]
is the \((1,1)\)-form on \( M \).

In particular, consider \( E = TM \) and \((M,\omega)\) is Teichmüller space with Weil-Petersson metric, that is, \((G_{i\bar{j}}) = (g_{\alpha\bar{\beta}})\) is Weil-Petersson metric. For any unit vector \( \xi \in E = TM \), i.e. \( \|\xi\|^2 = 1 \), then
\[
(3.10) \quad \iota^*(\text{Ric}\,\Omega)(\xi,\bar{\xi}) = \bar\partial\bar\partial \log (\det(g_{\alpha\bar{\beta}}) \cdot \det(g_{\alpha\bar{\beta}})) (\xi,\bar{\xi}) = 2\text{Ric}(\xi,\bar{\xi}),
\]
where \( \text{Ric} := \bar\partial\bar\partial \log \det(g_{\alpha\bar{\beta}}) \) denotes the Ricci curvature of Weil-Petersson metric. From [15, Lemma 4.6 (i)], the Ricci curvature of Weil-Petersson metric satisfies
\[
(3.11) \quad \text{Ric}(\xi,\bar{\xi}) \leq -\frac{1}{2\pi(g-1)}.
\]
where \( g \) denotes the genus of Riemann surfaces. Substituting (3.11) into (3.10), we obtains
\[
(3.10) \quad \iota^*(\text{Ric}\,\Omega)(\xi,\bar{\xi}) \leq -\frac{1}{\pi(g-1)}.
\]
Thus

**Proposition 3.5.** Let \((M,\omega)\) be Teichmüller space with the Weil-Petersson metric, and let \( E = TM \) be the holomorphic tangent bundle. When restricting to \( M \), the Ricci curvature of \( \Omega \) is bounded from above by \(-\frac{1}{\pi(g-1)}\).

Now we begin to prove our main theorem:

**Proof of Theorem 0.1.** From Corollary 1.3, the quasifuchsian space \( QF(S) \) embeds into a neighborhood of a zero section in the holomorphic tangent bundle of \( \mathcal{T}(S) \) which is contained in a ball of radius \( 9\pi(g-1) \) in \( L^2 \)-norm on each fiber \( B(X) \). Since the tangent bundle \( B(S) \) of \( \mathcal{T}(S) \) with the Weil-Petersson metric \( \omega_{WP} \) is Griffiths negative, so it defines a norm \( G \) on \( B(S) \). Denote \( \pi : B(S) \to \mathcal{T}(S) \), then the following \((1,1)\)-form
\[
\Omega = \pi^*\omega_{WP} + \sqrt{-1}\bar{\partial}\bar{\partial}G
\]
defines a mapping class group invariant Kähler metric on \( B(S) \) (see [11]). From (2.2), one sees that \( \Omega \) is an extension of the Weil-Petersson metric \( \omega_{WP} \). From [14, Theorem 1.5], the Weil-Petersson metric \( \omega_{WP} \) has a bounded primitive with respect to Weil-Petersson metric. By Corollary 2.3, the Kähler metric \( \Omega \) also has a bounded primitive with respect to \( \Omega \). And by Propositions 3.3, 3.5, the Chern curvature \( R^\Omega \) of \( \Omega \) is non-positive when evaluated on the tautological section \( P \), and its Ricci curvature is bounded from above by \(-\frac{1}{\pi(g-1)}\) when restricted to Teichmüller space. \( \square \)
Remark 3.6. Finally to put our results in perspective we remark that the space $\mathcal{P}(S)$ of marked complex projective structures is identified with the cotangent bundle of $\mathcal{T}(S)$ and the natural holonomy map $\mathcal{P}(S) \to \chi = \chi(\pi_1(S), PSL(2, \mathbb{C}))$ to the character variety is a local biholomorphic mapping by the results of Earle-Hejhal-Hubbard [8, 9, 10] (see also [7, Theorem 5.1]). Thus our constructions and results are also valid for $\mathcal{P}(S)$ and for its image in $\chi$. The space $QF(S)$ of quasifuchsian representations is also an open subset of $\chi$, $\mathcal{T}(S) \subset QF(S) \subset \chi$, and it might be interesting to understand the geometry of character variety $\chi$ using our metric on these open subsets.

The above remark applies also to the Hitchin component for any real split simple Lie group $G$ of real rank two, namely $G = SL(3, R), Sp(2, \mathbb{R}), G_2$. Indeed Labourie [12] generalized the construction in [11] of Kähler metric for $SL(3, \mathbb{R})$ to the above $G$. In this case the Hitchin component is proved to be a bundle over Teichmüller space with fiber being space of holomorphic differentials of degree $3, 4, 6$, respectively. Hence we obtain

Corollary 3.7. The curvature of the Kähler metric on the Hitchin component for real split simple Lie groups of real rank 2 vanishes along vertical directions, and non-positive along tautological sections.

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