COUNTABLE TIGHTNESS AND THE GROTHENDIECK PROPERTY IN \( C_p \)-THEORY

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Abstract. The Grothendieck property has become important in research on the definability of pathological Banach spaces [CI], [HT], and especially [HT20]. We here answer a question of Arhangel’skiı˘ı by proving it undecidable whether countably tight spaces with Lindelöf finite powers are Grothendieck. We answer another of his questions by proving that PFA implies Lindelöf countably tight spaces are Grothendieck. We also prove that various other consequences of MA\(_{\omega_1}\) and PFA considered by Arhangelskiı˘ı, Okunev, and Reznichenko are not theorems of ZFC.

1. Introduction

For a topological space \( X \), \( C_p(X) \) is the set of continuous real-valued functions on \( X \), given the pointwise topology inherited from \( \mathbb{R}^X \). The classic theorem of Grothendieck [Gro52] states:

\[ \text{Proposition 1. Let } X \text{ be countably compact and let } A \subseteq C_p(X) \text{ be such that every infinite subset of } A \text{ has a limit point in } C_p(X). \text{ Then the closure of } A \text{ in } C_p(X) \text{ is compact.} \]

This theorem has many applications in Analysis. We became interested in it due to its applications in Model Theory (see [CI], [HT], and [HT20]). These involve questions of definability, especially of pathological Banach spaces. The upshot is that if certain topological spaces (type spaces) associated with a logic have the closure property \( X \) has in the above theorem, then these Banach spaces are not definable in that logic. We are therefore interested in what classes of topological spaces other than the countably compact ones satisfy the conclusion of Proposition 1. We restrict ourselves to only consider infinite completely regular spaces.

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Definition 1 [Arh98]. A $\subseteq X$ is **countably compact in** $X$ if every infinite subset of $A$ has a limit point in $X$. $X$ is a $g$-space if each $A \subseteq X$ which is countably compact in $X$ has compact closure. $X$ is a **Grothendieck space** (resp. **weakly Grothendieck space**) if $C_p(X)$ is a hereditary $g$-space (resp. a $g$-space).

Definition 2. $X$ is **countably tight** if whenever $A \subseteq X$ and $x \in \overline{A}$, there is a countable $B \subseteq A$ such that $x \in \overline{B}$. $X$ is **realcompact** if $X$ can be embedded as a closed subspace of a product of copies of the real line.

Theorem 2 [Arh98]. If $X$ is countably tight, then $X$ is weakly Grothendieck.

This is stated as “clear” in [Arh98]. Here is a proof:

Clearly,

Lemma 3. A closed subspace of a realcompact space is realcompact.

Lemma 4 [Eng89]. A completely regular space is compact if and only if it is realcompact and countably compact.

Definition 3. A space is $wD$ if whenever $\{d_n : n < \omega\}$ is a closed discrete subspace, there is an infinite $S \subseteq \omega$ and a discrete collection of open sets $\{U_n : n \in S\}$ with $d_n \in U_n$ for all $n \in S$.

Lemma 5 [Dou84], [Vau78]. Every realcompact space is $wD$.

Lemma 6 (folklore). Let $X$ be $wD$. Let $Y$ be countably compact in $X$. Then $\overline{Y}$ is countably compact.

Proof. Suppose not. Let $\{d_n : n < \omega\}$ be a closed discrete subspace of $\overline{Y}$. Let $\{U_n : n \in S\}$ be a discrete collection of open subsets of $X$, with $d_n \in U_n$ for every $n \in S$, where $S \subseteq \omega$ is infinite. Pick $e_n \in U_n \cap Y$. Then $\{e_n : n \in S\}$ is a closed discrete subspace of $Y$, contradiction. □

Lemma 7 [Arh92]. If $X$ is countably tight, then $C_p(X)$ is realcompact.

Proof of Theorem 2. Let $X$ be countably tight. Then $C_p(X)$ is realcompact and hence $wD$. Let $Y$ be countably compact in $C_p(X)$. Then $\overline{Y}$ is countably compact. But $\overline{Y}$ is realcompact, so $\overline{Y}$ is compact. □

2. Applications of the Proper Forcing Axiom

In [Arh98], Arhangel’skiĭ proved:

Proposition 8. MA + $\neg$CH implies that if $X$ is countably tight and $X^n$ is Lindelöf for all $n < \omega$, then $X$ is Grothendieck.
In fact, MA_{\omega_1} suffices.

A dramatic strengthening of Proposition 8 is

**Theorem 9.** PFA implies Lindelöf countably tight spaces are Grothendieck.

*Proof.* This actually follows easily from known results. First, a definition:

**Definition 4.** A space is *surlindelöf* if it is a subspace of \( C_p(X) \) for some Lindelöf \( X \).

Arhangel’skiĭ [Arh92] proved:

**Lemma 10.** PFA implies that every surindelöf compact space is countably tight.

Okunev and Reznichenko [OR07] proved:

**Lemma 11.** MA_{\omega_1} implies that every separable surindelöf compact countably tight space is metrizable.

**Definition 5.** A space \( X \) is *Fréchet-Urysohn* if whenever \( x \) is a limit point of \( Z \subseteq X \), there is a sequence in \( Z \) converging to \( x \).

It follows quickly that:

**Theorem 12.** PFA implies that every surindelöf compact space is Fréchet-Urysohn.

*Proof.* Metrizable spaces are clearly Fréchet-Urysohn. By countable tightness, if \( K \) is compact and \( L \subseteq K \) and \( p \in \overline{L} \), then there is a countable \( M \subseteq L \) such that \( p \in \overline{M} \). But \( \overline{M} \) is separable compact and so metrizable. \( \square \)

Arhangel’skiĭ proved:

**Lemma 13 [Arh98].** \( X \) is Grothendieck if and only if it is weakly Grothendieck and compact subspaces of \( C_p(X) \) are Fréchet-Urysohn.

This proves Theorem 9. \( \square \)

Okunev and Reznichenko [OR07] point out that the conclusions of Lemmas 10 and 11 can be simultaneously consistently achieved without large cardinals, so we have:

**Theorem 14.** If ZFC is consistent, so is ZFC plus “every Lindelöf countably tight space is Grothendieck”.

Lemmas 10 and 11 actually consistently solve several other problems of Arhangel’skiǐ:

**Problem 1 [Arh98].** If $X$ is separable and compact and $Y \subseteq C_p(X)$ is Lindelöf, does $Y$ have a countable network?

**Problem 2 [Arh92].** If $X$ is separable and compact and $C_p(X)$ is Lindelöf, must $X$ be hereditarily separable?

Notice that a positive answer to the first of these yields a positive answer to the second, since a space with a countable network is clearly hereditarily separable.

**Lemma 15 [Arh92, I.1.3].** $X$ has a countable network if and only if $C_p(X)$ does.

Okunev [Oku95] considers versions of Problem 1 with the additional hypothesis that finite powers of $Y$ are Lindelöf. He proves:

**Proposition 16.** MA + ¬CH implies that if $Y$ is a space with all finite powers Lindelöf and $X$ is a separable compact subspace of $C_p(Y)$, then $X$ is metrizable.

He states that this is a reformulation of

**Proposition 17.** MA + ¬CH implies that if $X$ is a separable compact space and $Y \subseteq C_p(X)$ has all finite powers Lindelöf, then $Y$ has a countable network.

Okunev and Reznichenko note that actually MA$_{\omega_1}$ suffices for these instead of MA + ¬CH. Okunev and Reznichenko also prove:

**Proposition 18 [OR07, 1.8].** PFA implies that every surlindelöf compact separable space is metrizable.

**Proposition 19 [OR07, 1.9].** PFA implies every surlindelöf compact space is $\aleph_0$-monolithic, where a space is $\aleph_0$-monolithic if the closure of every countable set has countable network weight.

We can use Lemmas 10 and 11 to prove:

**Theorem 20.** PFA implies that if $X$ is a separable compact space and $Y \subseteq C_p(X)$ is Lindelöf, then $Y$ has a countable network.

**Proof.** We closely follow part of the argument in [Oku95] for Proposition 17. He starts by recalling some material from [Arh92] (or see [Tka15]). Given
a continuous map $p : X \to Y$, the dual map $p^* : C_p(Y) \to C_p(X)$ is defined by $p^*(f) = f \circ p$, for all $f \in C_p(Y)$. The dual map is always continuous; it is an embedding if and only if $p$ is onto. If $Y \subseteq C_p(X)$, then the reflection map $\varphi_{XY} : X \to C_p(Y)$ is defined by $\varphi_{XY}(x)(y) = y(x)$, for all $x \in X$ and $y \in Y$. The reflection map is continuous.

Suppose $X$ is a separable compact space and $Y$ is a Lindelöf subspace of $C_p(X)$ which does not have a countable network. We consider the reflection map $\varphi_{XY} : X \to C_p(Y)$ and let $X_1 = \varphi_{XY}(X)$. Then $X_1$ is separable and compact. Next, consider the dual map $\varphi_{XY}^* : C_p(X_1) \to C_p(X)$. It’s an embedding, so $Y_1 = (\varphi_{XY}^*)^{-1}(Y)$ is a subspace of $C_p(X_1)$ homeomorphic to $Y$. Since $Y$ does not have a countable network, neither does $Y_1$. Then neither does $C_p(X_1)$, so neither does $X_1$. But by Lemmas 10 and 11, $X_1$ is metrizable. This is a contradiction, since compact metrizable spaces have a countable network. □

Let us mention some more open problems.

**Problem 3.** Are Lindelöf first countable spaces Grothendieck?

Although we can’t fully answer Problem 3, we can weaken the hypothesis of Theorem 9 in the first countable case:

**Theorem 21.** MA$_{\omega_1}$ implies that every Lindelöf first countable space is Grothendieck.

Before proving this, we need to mention some more general facts about $C_p$, taken from [Oku95].

**Lemma 22.** Let $Y \subseteq C_p(X)$. Let $x_1, x_2 \in X$. Let $x_1 \sim_Y x_2$ if $y(x_1) = y(x_2)$ for all $y \in Y$. Let $X_1$ be the set of equivalence classes and $\pi : X \to X_1$ the natural map. For any $y \in Y$, there is a $y' : X_1 \to \mathbb{R}$ such that $y = y' \circ \pi$. Give $X_1$ the weakest topology that makes all of the $y'$’s continuous. With this topology, $X_1$ is homeomorphic to $\varphi_{XY}(X)$. Then $(\varphi_{XY}^*)^{-1}(Y)$ is a subspace of $C_p(X_1)$ homeomorphic to $Y$.

In particular, this tells us that if $K$ is a separable compact subspace of $C_p(Y)$, then $Y$ is homeomorphic to a subspace of $C_p(K_1)$, where $K_1$ is a continuous image of $K$ and hence is separable and compact.

**Proof of Theorem 21.** Let $Y$ be Lindelöf and first countable. Let $K$ be a compact separable subspace of $Y$. Let $K_1$ be a continuous image of $K$, and $Y_1$ be a homeomorphic copy of $Y$ included in $C_p(K_1)$. We now invoke two applications of MA$_{\omega_1}$:

**Lemma 23** [OR07]. MA$_{\omega_1}$ implies that if $K$ is a compact separable space, then every Lindelöf subspace of $C_p(K)$ is hereditarily Lindelöf.
Lemma 24. \[ \text{MA}_{\omega_1} \] implies that every first countable hereditarily Lindelöf space is hereditarily separable.

But,

Lemma 25. \[ \text{Arh98 5.26} \]. Every hereditarily separable space is Grothendieck. □

A corollary of what we just proved is of interest.

Corollary 26. \[ \text{MA}_{\omega_1} \] implies that if \( K \) is a compact subspace of \( C_p(Y) \), where \( Y \) is Lindelöf and first countable, then \( K \) is metrizable.

Proof. In the previous proof, we showed \( Y \) was hereditarily separable. Arhangelskii proved:

Lemma 27. \[ \text{Arh97 3.13} \]. If \( Y \) is separable and \( K \) is a compact subspace of \( C_p(Y) \), then \( K \) is metrizable. □

In the spirit of Problem 3, one can ask:

Problem 4. If \( X \) is a separable compact space and \( Y \) is a Lindelöf first countable subspace of \( C_p(X) \), does \( Y \) have a countable network?

We have a partial answer:

Theorem 28. \[ \text{MA}_{\omega_1} \] implies that if \( X \) is a separable compact space and \( Y \) is a Lindelöf first countable subspace of \( C_p(X) \), then \( Y \) has a countable network.

This follows from what we have just done by the same argument as for Theorem 20.

Note that:

Theorem 29. If \( Y \) is a hereditary \( g \)-space, then countably compact subspaces of \( Y \) are compact.

Proof. Let \( Z \subseteq Y \) be countably compact. Then it is countably compact in itself and its closure in itself is compact. □

Problem 5. If countably compact subspaces of \( C_p(X) \) are compact, is \( X \) Grothendieck?

There is a necessary and sufficient condition on \( X \) so that \( C_p(X) \) is countably tight (see Lemma 34 below), and there is even a necessary and sufficient condition on \( X \) that ensures \( C_p(X) \) is Fréchet-Urysohn \[ \text{GN82} \], but these conditions are too onerous and entail more than we need.
Problem 6. Find a necessary and sufficient condition on $X$ such that compact subspaces of $C_p(X)$ are countably tight.

Definition 6. A sequence $\{x_\alpha : \alpha < \kappa\}$ is free if for all $\beta < \kappa$,

$$\{x_\alpha : \alpha < \beta\} \cap \{x_\alpha : \alpha \geq \beta\} = \emptyset.$$  

It is well-known that:

Lemma 30. If $X$ is Lindelöf and countably tight, then $X$ does not include an uncountable free sequence.

Proof. Suppose $F = \{x_\alpha : \alpha < \omega_1\}$ is free. Let $F_\beta = \{x_\alpha : \alpha < \beta\}$. Then $\{F_\beta : \beta < \omega_1\}$ is a decreasing family of closed subspaces of $X$. Since $X$ is Lindelöf, there is an $x \in \bigcap \{F_\beta : \beta < \omega_1\}$. Then $x \in F$ but $x \notin A$ for any countable $A \subseteq F$, contradicting countable tightness. □

Todorcevic proved:

Theorem 31 [Tod93]. PFA implies: if $X$ includes no uncountable free sequences, then every countably compact subspace of $C_p(X)$ is compact.

The hypothesis is weaker than that of Theorem 9, but so is the conclusion.

Problem 7. Does PFA imply that if $X$ includes no uncountable free sequences, then $X$ is (weakly) Grothendieck?

Todorcevic also proved:

Lemma 32 [Tod93]. Suppose every countably compact subspace of $C_p(X)$ is compact. Then every compact subspace of $C_p(X)$ is countably tight.

Proof. Suppose there is a $Z \subseteq C_p(X)$ such that there is a $y \in \overline{Z} - \bigcup \{\overline{Z_0} : Z_0 \subseteq Z$ is countable\}. But $\bigcup \{\overline{Z_0} : Z_0 \subseteq Z$ is countable\}$ is countably compact, hence compact, hence closed, a contradiction. □

Corollary 33 [Tod93]. PFA implies that if $X$ does not include any uncountable free sequences, then compact subspaces of $C_p(X)$ are countably tight.

3. Counterexamples

In [Arh98], Arhangel’skii asked whether the conclusion of Proposition 8 is true in ZFC. It is not:

Example 1. Assuming ♦ plus Kurepa’s Hypothesis, Ivanov [Iva78] constructs a compact space $Y$ of cardinality $2^\omega$ such that $Y^n$ is hereditarily separable for all $n < \omega$. $C_p(Y)$ is the required counterexample.
To see this, we require several results from the literature.

Lemma 34 [Arh92]. \( X^n \) is Lindelöf for every \( n < \omega \) if and only if \( C_p(X) \) is countably tight.

Lemma 35 [Arh92]. \( X \) embeds into \( C_p(C_p(X)) \).

Clearly, separable Fréchet-Urysohn spaces have cardinality \( \leq c \). Ivanov’s space \( Y \) is too big to be Fréchet-Urysohn, yet it embeds in \( C_p(C_p(Y)) \), so \( C_p(Y) \) cannot be Grothendieck, although it is weakly Grothendieck. \( (C_p(Y))^n \) is, however, (hereditarily) Lindelöf for all \( n < \omega \) by the Velichko-Zenor theorem:

Lemma 36 [Vel81], [Zen80]. If \( X^n \) is hereditarily separable for all \( n < \omega \), then \( (C_p(X))^n \) is hereditarily Lindelöf for all \( n < \omega \).

Ivanov’s space also provides counterexamples for various other propositions proved by Arhangel’skii, Okunev, and Reznichenko under MA_{\omega_1} or PFA. \( Y \) is surlindelöf, compact, countably tight, separable, but not metrizable. This violates the conclusion of Lemma [11]

Definition 7. An \( S \)-space is a hereditarily separable space that is not hereditarily Lindelöf. A strong \( S \)-space is an \( S \)-space with all finite powers hereditarily separable.

Lemma 37 [Tod89]. \( b = \aleph_1 \) implies there is a compact strong \( S \)-space.

\( b = \aleph_1 \) is weaker than CH, which is weaker than \( \diamond \). Todorcevic’s space will work for the purpose of violating the conclusion of Lemma [11] as well as \( Y \). (To be more precise, Todorcevic constructs a locally compact, locally countable strong \( S \)-space \( T \), but then its one-point compactification \( T^* = T \cup \{*\} \) is a compact strong \( S \)-space).

Both \( Y \) and \( T^* \), embedded in \( C_p(C_p(Y)) \) and \( C_p(C_p(T^*)) \) respectively, provide counterexamples to the conclusion of Proposition [18]

They also provide counterexamples to the conclusion of Proposition [19]. The point is that compact spaces with countable network weight are metrizable.

Any compact strong \( S \)-space \( Y \) refutes the conclusion of Proposition [17] \( (C_p(Y))^n \) will be hereditarily Lindelöf, but \( C_p(Y) \) does not have a countable network, else \( Y \) would, but then \( Y \) would be hereditarily Lindelöf.

The hypothesis for Example [1] seems too strong; \( \diamond \) ought to suffice. I conjecture that \( \diamond \) implies there is a “strong Ostaszewski space”, i.e. a strong \( S \)-space \( X \) which is countably compact, perfectly normal, but not compact. \( C_p(X) \) would then have finite products hereditarily Lindelöf, but would not
be Grothendieck since $X$ would be embedded in $C_p(C_p(X))$, violating Theorem 29. Notice we are not using perfect normality, so even CH might suffice.

**Remark.** [OR07] appears to have some misprints. I believe that their reference 2 should actually be our [Arh92]. Their Question 0.5 is asserted to be essentially the same as Problem IV.1.8 in [Arh92] but the latter problem apparently has nothing to do with the former, so this may be a misprint. They call “centered” what is normally called “linked” [KT79]. On the other hand, their use of “surlindelöf” for the concept [Arh92] calls “suplindelöf” is correct. Professor Arhangel’skii has informed me that “suplindelöf” was a mistranslation by the translator of the Russian original.
References

[Arh92] A. V. Arhangel'ski˘ı. Topological Function Spaces, volume 78 of Mathematics and its Applications (Soviet Series). Kluwer Academic Publishers Group, Dordrecht, 1992.

[Arh97] A. V. Arhangel'ski˘ı. On a theorem of Grothendieck in $C_p$-theory. Topology and its applications, 80:21–41, 1997.

[Arh98] A. V. Arhangel'ski˘ı. Embedding in $C_p$-spaces. Topology and its applications, 85:9–33, 1998.

[CI] P. Casazza and J. Jovino. On the undefinability of Tsirelson's space and its descendants. ArXiv preprint: 1710.05889.

[Dou84] E. K. van Douwen. The integers and topology. In K. Kunen and J. E. Vaughan, editors, Handbook of Set-theoretic Topology, pages 111–167. North-Holland, Amsterdam, 1984.

[Eng89] R. Engelking. General Topology. Heldermann Verlag, Berlin, 1989.

[GN82] J. Gerlits and Zs. Nagy. Some properties of $C(X)$, I. Topology and its Applications, 14:152–161, 1982.

[Gre52] A. Grothendieck. Critères de compacité dans les espaces fonctionnels généraux. American Journal of Mathematics, 74:168–186, 1952.

[HT] C. Hamel and F. D. Tall. $C_p$-theory for model theorists. Submitted.

[HT20] C. Hamel and F. D. Tall. Model theory for $C_p$-theorists. Topology and its applications, 2020. https://doi.org/10.1016/j.topol.2020.107197 Proceedings of a conference in honor of the 80th birthday of Professor A. V. Arhangel’ski˘ı.

[Iva78] A. V. Ivanov. On bicomputa all finite powers of which are hereditarily separable. Doklady Akademii Nauk SSSR, 243(5):1109–1112, 1978.

[KT79] K. Kunen and F. D. Tall. Between Martin’s axiom and Souslin’s hypothesis. Fund. Math., 102(3):173–181, 1979.

[Oku95] O. G. Okunev. On Lindelöf sets of continuous functions. Topology and its Applications, 63:91–96, 1995.

[OR07] O. Okunev and E. Reznichenko. A note on surlindelöf spaces. Topology Proceedings, 31(2):667–675, 2007.

[Sze80] Z. Szentmiklőssy. S-spaces and L-spaces under Martin’s axiom. In Colloquia Mathematica Societatis Janos Bolyai, volume 23, pages 1139–1145. North-Holland, 1980.

[Tka15] V. Tkachuk. A $C_p$-theory problem book. Problem Books in Mathematics. Vol. I-IV. Springer, 2011-2015.

[Tod89] S. Todorcevic. Partition problems in topology, volume 84 of Contemporary Mathematics. American Mathematical Society, Providence, RI, 1989.

[Tod93] S. Todorcevic. Sk&L combinatorics. In F. D. Tall, editor, The work of Mary Ellen Rudin, volume 705 of Ann. New York Acad. Sci., pages 130–167. New York Acad. Sci., 1993.

[Vau78] J. E. Vaughan. Discrete sequences of points. Topological Proceedings, 3:237–265, 1978.

[Vel81] N. V. Velichko. Weak topology of spaces of continuous functions. Mathematical notes of the Academy of Sciences of the USSR, 30:849–854, 1981.

[Zen80] P. Zenor. Hereditary m-separability and the hereditary m-Lindelöf property in product spaces and function spaces. Fundamenta Mathematicae, 106(3):175–180, 1980.