Liouville Theory on the Pseudosphere:
Bulk-Boundary Structure Constant

Bénédicte Ponsot

International School for Advanced Studies (SISSA),
Via Beirut 2-4, 34014 Trieste, Italy

Abstract

Liouville field theory on the pseudosphere is considered (Dirichlet conditions). We compute explicitly the bulk-boundary structure constant with two different methods: first we use a suggestion made by Hosomichi in JHEP 0111 (2001) that relates this quantity directly to the bulk-boundary structure constant with Neumann conditions, then we do a direct computation. Agreement is found.

PACS: 11.25.Hf

1 Introduction

We introduce the action density of the Liouville field theory

$$\mathcal{L}(z, \bar{z}) = \frac{1}{4\pi} \left( \partial_a \phi(z, \bar{z}) \right)^2 + \mu e^{2b\phi(z, \bar{z})},$$

where $\phi$ is the Liouville field, $\mu$ is called the cosmological constant and the parameter $b$ is the coupling constant. LFT is a conformal field theory with central charge

$$c_L = 1 + 6Q^2,$$

where $Q = b + 1/b$ is called the background charge. In what follows we will consider the so called weak coupling regime with $c_L \geq 25$. We note the conformal primary fields $V_\alpha(z, \bar{z})$. These fields

1 ponsot@fm.sissa.it
are primaries with respect to the energy momentum tensor

\[ T(z) = -\left( \partial \phi \right)^2 + Q \partial^2 \phi, \]
\[ \bar{T}(\bar{z}) = -\left( \bar{\partial} \phi \right)^2 + Q \bar{\partial}^2 \phi, \]

and have conformal weight \( \Delta_{\alpha} = \bar{\Delta}_{\alpha} = \alpha(Q - \alpha) \). One identifies the operator \( V_{\alpha} \) with its reflected image \( V_{Q-\alpha} \):

\[ V_{\alpha}(z, \bar{z}) = S(\alpha) V_{Q-\alpha}(z, \bar{z}) \tag{1} \]

where we introduced the bulk reflection amplitude [1, 2]

\[ S(\alpha) = \frac{(\pi \mu \gamma(b^2))^{(Q-2\alpha)/b}}{b^2} \frac{\gamma(2\alpha b - b^2)}{\gamma(2 - 2\alpha/b + 1/b^2)}, \]

which is unitary: \( S(\alpha)S(Q - \alpha) = 1 \), and as usual \( \gamma(x) = \Gamma(x)/\Gamma(1 - x) \).

An important set among the primaries are the fields \( V_{-\frac{nb}{2}} \), \( n \in \mathbb{N} \), which are degenerate with respect to the conformal symmetry algebra and satisfy linear differential equations [3]. For example, the first non trivial case consists of \( \alpha = -\frac{b}{2} \), and the corresponding operator satisfies

\[ \left( \frac{1}{b^2} \partial^2 + T(z) \right) V_{-b/2} = 0, \]

as well as the complex conjugate equation.

It follows from these equations that when one performs the operator product expansion of one of these degenerate operators with a generic operator, then the OPE is truncated [3]. For example:

\[ V_{-b/2} V_{\alpha} = c_+ V_{\alpha - b/2} + c_- V_{\alpha + b/2}. \]

The structure constants \( c_\pm \) are special cases of the bulk three point function, and can be computed perturbatively as Coulomb gas (or screening) integrals [4, 5]. One can take \( c_+ = 1 \), as in this case there is no need of insertion of interaction, whereas \( c_- \) requires one insertion of the Liouville interaction \( -\mu \int e^{2\phi} d^2 z \). It is given by the expression

\[ c_- = -\mu \frac{\pi \gamma(2b\alpha - 1 - b^2)}{\gamma(-b^2)\gamma(2b\alpha)}. \]

Similarly, there exists also a dual series of degenerate operators \( V_{-\frac{m}{2b}} \) with the same properties.

Liouville field theory on a pseudosphere was considered in [6]: the pseudosphere geometry can be realized as the disk \( |z| < 1 \) with metric

\[ ds^2 = e^{\phi(z)} |dz|^2, \]

and

\[ e^{\phi(z)} = \frac{4R^2}{(1 - z\bar{z})^2}, \]

where \( R \) is interpreted as the radius of the pseudosphere. It was found in [6] that there is a whole variety of vacua considered as boundary conditions at the absolute that are labelled by

\[ ^2 \text{See also prior works on the subject [7, 8].} \]
two positive integers \((m,n)\), in one to one correspondence with the degenerate representations of the Virasoro algebra. Then, the (finite) content of boundary operators is simply determined by the fusion algebra, like in the rational case. We will need later the expression for the one point function of a primary field \(V_\alpha\) [6]:

\[
U_{m,n}(\alpha) = \frac{\sin(\pi b^{-1} Q) \sin(\pi mb^{-1}(2\alpha - Q)) \sin(\pi bQ) \sin(\pi nb(2\alpha - Q))}{\sin(\pi mb^{-1}Q) \sin(\pi b^{-1}(2\alpha - Q)) \sin(\pi bQ) \sin(\pi nb(2\alpha - Q))} U_{1,1}(\alpha)
\]  

(2)

where the \((1,1)\) one point function reads

\[
U_{1,1}(\alpha) = \frac{(\pi \mu \gamma(b^2))^{-\alpha/b} \Gamma(bQ) \Gamma(Q/b) Q}{\Gamma(bQ - 2b\alpha) \Gamma(Q/b - 2\alpha/b)(Q - 2\alpha)}
\]  

(3)

This expression satisfies the reflection property [1].

2 Hosomichi’s proposal [9]

It was first noticed in [6] that the one point function in the pseudosphere geometry (2) is related to the one point function with Neumann boundary conditions computed in [10]. The latter result has the following expression:

\[
U_s(\alpha) = \frac{2}{b} (\pi \mu \gamma(b^2))^{(Q-2\alpha)} \Gamma(2b\alpha - b^2) \Gamma(2b^{-1} \alpha - b^{-2} - 1) \cosh[2\pi(2\alpha - Q)s],
\]

where the boundary parameter \(s\) is related to the boundary cosmological constant [10]:

\[
\cosh(2\pi bs) = \frac{\mu B}{\sqrt{\mu}} \sqrt{\sin(\pi b^2)}.
\]

Let us note that (4) satisfies the reflection property [1]. If one first perform a Fourier transform w.r.t the parameter \(s\) on the one point function:

\[
\tilde{U}(\alpha, p) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{4\pi sp} U_s(\alpha) \, ds
\]  

(4)

and then one does the transformation:

\[
\int_{-i\infty}^{+i\infty} \sin(2\pi npb) \sin(2\pi mb^{-1}) \tilde{U}(\alpha, p) \, dp,
\]

(5)

this reproduces, up to the factor

\[
-(\pi \mu \gamma(b^2))^{-Q/2b} \frac{\sin(\pi b^{-1}Q) \sin(\pi bQ)}{\sin(\pi mb^{-1}Q) \sin(\pi nbQ)} \Gamma(bQ) \Gamma(Q/b) Q
\]

(6)

the expression (2). It was then proposed in [9] that this relation also holds at the level of the bulk-boundary structure constant, which is what we are going to check.
The bulk-boundary structure constant with Neumann boundary conditions was calculated in [9]; it has the form of a \( b \)-deformed hypergeometric function in the Barnes representation:

\[
R_b(\alpha, \beta) = 2\pi(\mu \gamma(b^2))^{2-2b^2} \frac{1}{4}(Q-2\alpha-\beta) \\
\times \frac{\Gamma_b(Q-\beta)\Gamma_b(2\alpha-\beta)\Gamma_b(2Q-2\alpha-\beta)}{\Gamma_b(Q)\Gamma_b(Q-2\beta)\Gamma_b(2\alpha)\Gamma_b(Q-2\alpha)} \\
\times \int_{-i\infty}^{i\infty} dt \frac{S_b(t+\beta/2+\alpha-Q/2)S_b(t+\beta/2-\alpha+Q/2)}{S_b(t-\beta/2-\alpha+3Q/2)S_b(t-\beta/2+\alpha+Q/2)}.
\]

(7)

We introduced the Barnes’ Double Gamma function:

\[
\log \Gamma_2(s|\omega_1, \omega_2) = \left( \frac{\partial}{\partial t} \sum_{n_1,n_2=0}^{\infty} (s + n_1\omega_1 + n_2\omega_2)^{-t} \right)_{t=0}
\]

and by definition \( \Gamma_b(x) \equiv \frac{\Gamma_2(x|b^{-1})}{\Gamma_2(Q|b^{-1})} \). This function satisfies the functional relation \( \Gamma_b(x+b) = \sqrt{2\pi x}^{b-\frac{1}{2}} \Gamma_b(x) \), as well as the dual relation with \( b \) replaced by \( b^{-1} \). \( \Gamma_b(x) \) is a meromorphic function of \( x \), whose poles are located at \( x = -nb - mb^{-1}, n, m \in \mathbb{N} \). The \( S_b(x) \) function is related to the \( \Gamma_b(x) \) function: \( S_b(x) \equiv \frac{\Gamma_b(x)}{\Gamma_b(Q-x)} \).

The integration contour of the integral (7) is located to the right of the poles:

\[
t = -\beta/2 + Q/2 - \nu b - \mu b^{-1}, \quad t = -\beta/2 + \alpha - Q/2 - \nu b - \mu b^{-1}, \quad \mu, \nu \in \mathbb{N},
\]

and to the left of the poles:

\[
t = \beta/2 + \alpha - Q/2 + \nu b + \mu b^{-1}, \quad t = \beta/2 - \alpha + Q/2 + \nu b + \mu b^{-1}, \quad \mu, \nu \in \mathbb{N}.
\]

Let us consider now a degenerate boundary operator \( B_{\beta}^{\sigma \sigma} \) with spin \( \beta = -ub - vb^{-1}, u \) and \( v \) being positive integers. What should be seen is that for this value of \( \beta \), we have to pick up residues at poles\(^3\) that are located at

\[
t = \pm(\alpha + \beta/2 - Q/2 + kb + lb^{-1}), \quad k = 0, \ldots, u, \quad l = 0, \ldots, v.
\]

(8)

It is convenient to introduce at this point the truncated \( b \)-deformed hypergeometric series:

\[
\phi(A = -ub, B; C; -ix) = \sum_{k=0}^{\nu} e^{2\pi i k x} \prod_{i=0}^{k-1} \frac{\sin \pi b(B + ib) \sin \pi b(A + ib)}{\sin \pi b(C + ib) \sin \pi b(Q + ib)},
\]

and define

\[
\phi_{\nu}^\beta(x) \equiv \phi(-ub, 2\alpha - ub - Q; 2\alpha; -ix).
\]

\(^3\)This computation does not hold for the case \( 2\alpha = \beta \); as it was noticed in [11], the \( b \)-deformed hypergeometric function degenerates for this value.
The evaluation of the residues \((\mathbf{3})\) gives

\[
\frac{1}{b} (\pi \mu \gamma(b^2))^{\frac{1}{2} (Q-2\alpha-\beta)} \Gamma(2b\alpha - b^2) \Gamma(2b^{-1} \alpha - 1 - b^{-2}) b^{b^2-b^{-2}} \times \\
\prod_{i=0}^{u-1} \frac{\Gamma(2b\alpha - bQ + b\beta + ib^2)}{\Gamma(2b\alpha + ib^2)} \prod_{j=0}^{v-1} \frac{\Gamma(\frac{2\alpha}{b} - \frac{Q\alpha}{b\beta} + \frac{\beta}{b\gamma} + u)}{\Gamma(\frac{2\alpha}{b} + \frac{\beta}{b\gamma} + u)} \Gamma(\frac{Q\alpha}{b} - \frac{\beta}{b\gamma} + u) \\
\times \left( e^{2\pi s(2\alpha+\beta-Q)} \phi^\alpha_b(2s) \phi^\beta_{1/b}(2s) + s \to -s \right). \tag{9}
\]

Then, after performing transformations \((\mathbf{11})\), \((\mathbf{5})\) and multiplying by \((\mathbf{1})\), we obtain, for \(2\alpha \neq \beta\):

\[
R_{m,n}(\alpha, \beta) = -\frac{(\pi \mu \gamma(b^2))^{\frac{1}{2} (2\alpha-\beta)} U_{m,n}(\alpha)}{4 \sin \pi mb(2\alpha - Q) \sin \pi mb^{-1}(2\alpha - Q)} b^{b^2-b^{-2}} \times \\
\prod_{i=0}^{u-1} \frac{\Gamma(2b\alpha - bQ + b\beta + ib^2)}{\Gamma(2b\alpha + ib^2)} \prod_{j=0}^{v-1} \frac{\Gamma(\frac{2\alpha}{b} - \frac{Q\alpha}{b\beta} + \frac{\beta}{b\gamma} + u)}{\Gamma(\frac{2\alpha}{b} + \frac{\beta}{b\gamma} + u)} \Gamma(\frac{Q\alpha}{b} - \frac{\beta}{b\gamma} + u) \\
\times \left( e^{\sqrt{-1} \pi mb(2\alpha+\beta-Q)} \phi^\alpha_b(\sqrt{-1} mb) - n \to -n \right) \left( e^{\sqrt{-1} \pi mb^{-1}(2\alpha+\beta-Q)} \phi^\beta_{1/b}(\sqrt{-1} mb^{-1}) - m \to -m \right). \tag{10}
\]

One can check that this expression satisfies the reflection property \((\mathbf{11})\). It is not difficult to see that for \(n = 1\) and \(\beta = -b\) as well as for \(n = 1, 2\) and \(\beta = -2b\), the term in the first parenthesis vanishes identically. We have no doubt, although we did not do it in general, that our coefficient \(R_{m,n}(\alpha, -ub - vb^{-1})\) is zero\(^4\) whenever the fusion rules for the degenerate representations corresponding to the boundary conditions and the boundary operator are not satisfied, as expected from the results of \([6]\).

### 3 Direct computation

We use the trick of \([12]\) and consider an auxiliary bulk two point function including a degenerate operator \(V_{\beta/2}\) and a generic operator \(V_\alpha\). For the sake of simplicity, we shall consider \(\beta = -ub\). The two point function can be factorized equivalently in the \(s\)- and \(t\)-channels. A straightforward generalization of the case \(u = 1\) already studied in \([6]\) leads to the following equation for \(R_{m,n}(\alpha, \beta)\)\(^5\): \(^6\):

\[
\sum_{k=0}^{n} C(\alpha, \beta/2, Q - \alpha - \beta/2 - kb) U_{m,n}(\alpha + \beta/2 + kb) F_{\alpha+\beta/2+kb,\beta} \begin{bmatrix} \beta/2 \\ \alpha \end{bmatrix} \begin{bmatrix} \beta/2 \\ \alpha \end{bmatrix} = R_{m,n}(\alpha, \beta) R_{m,n}(\beta/2, \beta) D_{m,n}(\beta)
\]

where we introduced:

\(^4\)Vanishing of the structure constant will always be due to the terms into parenthesis.

\(^5\)Such an equation was first obtained in \([13]\) for the case of A-type Virasoro minimal models.

\(^6\)When the bulk operator \(V_{-ub/2}\) approaches the boundary, it gives rise to primary boundary operators \(B_0, B_{-b}, \cdots, B_{-ub}\). We consider here the contribution of \(B_{-ub} \equiv B_\beta\).
• The bulk three point function $C(\alpha_1, \alpha_2, \alpha_3)$. In the case where $\alpha_1 + \alpha_2 + \alpha_3 = Q - kb$, its value can be found in [5]:

$$C(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{-\pi \mu}{\gamma(-b^2)}\right)^k \prod_{i=0}^{k-1} \frac{\gamma(-ib^3)}{\gamma(2b\alpha_1 + ib^2)\gamma(2b\alpha_2 + ib^2)\gamma(2b\alpha_3 + ib^2)}.$$ 

• $U_{m,n}(\alpha)$ is defined as in equation (2).

• $D_{m,n}(\beta)$ is the boundary two point function of two degenerate boundary operators with spin $\beta$. It is usual in conformal field theory to normalize the two point function of primaries to one.

• $F_{\alpha+\beta/2+kb,\beta}^{\alpha,\beta/2,\alpha/2}$ is a special case of the fusion matrix, which expresses the change of basis between the $s$-channel conformal block and the $t$-channel conformal block. The fusion matrix was built for generic spins in [14]; we recall its expression:

$$F_{\alpha_1,\alpha_3,\alpha_2}^{\alpha_4,\alpha_1} = \frac{\Gamma_b(2Q - \alpha_3 - \alpha_2 - \alpha_3)\Gamma_b(\alpha_3 + \alpha_2 + \alpha_3)}{\Gamma_b(Q - \alpha_2 - \alpha_3 + \alpha_2\alpha_3)\Gamma_b(2Q - \alpha_3)} \frac{1}{\Gamma_b(2Q - 2\alpha_1)} \int_{-\infty}^{\infty} dt \frac{S_b(U_1 + t)S_b(U_2 + t)S_b(U_3 + t)S_b(U_4 + t)}{S_b(V_1 + t)S_b(V_2 + t)S_b(V_3 + t)S_b(Q + t)}$$

where:

$$\begin{align*}
U_1 &= \alpha_2 + \alpha_1 - \alpha_2 \\
U_2 &= Q + \alpha_2 - \alpha_1 \\
U_3 &= \alpha_2 + \alpha_4 - Q \\
U_4 &= \alpha_2 - \alpha_1 + \alpha_4 \\
V_1 &= Q + \alpha_1 - \alpha_2 + \alpha_4 \\
V_2 &= \alpha_2 + \alpha_4 - \alpha_2 \\
V_3 &= 2\alpha_21
\end{align*}$$

In the case of degenerate Virasoro representations, the generic fusion coefficient develops poles; the relevant quantity is given by the residue at these poles. In our case the pole is located at $t = Q - \alpha - \beta - kb$. We find:

$$F_{\alpha+\beta/2+kb,\beta}^{\alpha,\beta/2,\alpha/2} = \prod_{l=1}^{u} \Gamma(bQ - b\beta + (l - 1)b^2) \prod_{l=k}^{u-1} \frac{\Gamma(2b\alpha + b\beta + (l + k)b^2)}{\Gamma(2b\alpha + lb^2)\Gamma(bQ + lb^2)} \times \prod_{i=1}^{k} \frac{\Gamma(bQ + (i - 1)b^2)\Gamma(2bQ - 2b\alpha - b\beta - 2kb^2 + (i - 1)b^2)}{\Gamma(bQ - b\beta - ib^2)\Gamma(2bQ - 2b\alpha - b\beta - ib^2)}.$$ 

We checked that the expression found for $R_{m,n}(\alpha, \beta = -ub)$ with this method indeed coincides with [10], provided $R_{m,n}(\beta/2, \beta)$ is normalized to one for those values of $m, n$ which make the bulk-boundary coefficient $R_{m,n}(\alpha, \beta)$ non-vanishing.
Acknowledgments

Work supported by the Euclid Network HPRN-CT-2002-00325.

References

[1] H. Dorn and H.-J. Otto, “Two and three point functions in Liouville theory”, Nucl. Phys. B429 (1994) 375, hep-th/9403141

[2] A.B. Zamolodchikov and Al.B. Zamolodchikov, "Structure Constants and Conformal Bootstrap in Liouville Field Theory", Nucl. Phys. B477 (1996) 247, hep-th/9506136

[3] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, “Infinite conformal symmetry in 2D quantum field theory”, Nucl. Phys. B241 (1984) 333

[4] B.L. Feigin, D.B. Fuchs, Representation of the Virasoro algebra, in: A. M. Vershik, D. P. Zhelobenko (Eds), Representations of Lie groups and related topics, Gordon and Breach, London, 1990.

[5] V.S. Dotsenko and V.A. Fateev, ”Four point correlations functions and the operator algebra in the two dimensional conformal invariant theories with the central charge c < 1” Nucl. Phys. B251 (1985) 691

[6] A.B. Zamolodchikov and Al.B. Zamolodchikov, “Liouville Field Theory on a Pseudosphere”, hep-th/0101152

[7] E. D’Hoker and R. Jackiw, “Liouville field theory”, Phys. Rev. D26 (1982) 3517.

[8] E. D’Hoker and R. Jackiw, “Space translation breaking and compactification in the Liouville theory”, Phys. Rev. Lett. 50 (1983) 1719.

[9] K. Hosomichi, ”Bulk-Boundary Propagator in Liouville Theory on a Disc”, JHEP 0111 (2001) 044, hep-th/0108093

[10] V.A. Fateev, A.B. Zamolodchikov and Al.B. Zamolodchikov, “Boundary Liouville field theory I. Boundary State and Boundary two point Function”, hep-th/0001012

[11] I.K. Kostov, B. Ponsot and D. Serban, “Boundary Liouville Theory and 2D Quantum Gravity”, hep-th/0307189

[12] J. Teschner, ”On the Liouville three point function”, Phys. Lett. B363 (1995) 65, hep-th/9507109

[13] J.L. Cardy and D.C. Lewellen, “Bulk and boundary operators in conformal field theory”, Phys. Lett. B259 (1991) 274

[14] B. Ponsot, J. Teschner, “Liouville bootstrap via harmonic analysis on a noncompact quantum group”, hep-th/9911110