Likelihood Adaptively Modified Penalties

Yang Feng¹, Tengfei Li² and Zhiliang Ying¹

¹Department of Statistics, Columbia University, New York, NY, USA.
²Chinese Academy of Sciences, Beijing, China.

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Abstract

A new family of penalty functions, adaptive to likelihood, is introduced for model selection in general regression models. It arises naturally through assuming certain types of prior distribution on the regression parameters. To study stability properties of the penalized maximum likelihood estimator, two types of asymptotic stability are defined. Theoretical properties, including the parameter estimation consistency, model selection consistency, and asymptotic stability, are established under suitable regularity conditions. An efficient coordinate-descent algorithm is proposed. Simulation results and real data analysis show that the proposed method has competitive performance in comparison with existing ones.

1 Introduction

Classical work on variable selection dates back to Akaike (1973), who proposed to choose a model that minimizes the Kullback-Leibler (KL) divergence of the fitted model from the true model, leading to the well-known Akaike Information Criterion (AIC). Schwarz (1978) took a Bayesian approach by assuming prior distributions with nonzero probabilities on lower dimensional subspaces. He proposed what is known as the BIC method for model selection. Other types of $L_0$ penalties include $C_p$ (Mallows, 1973), AICC (Hurvich and Tsai, 1989), RIC (Foster and George, 1994) and EBIC (Chen and Chen, 2008), among others.
The $L_0$ regularization has a natural interpretation in the form of best subset selection. It also exhibits good sampling properties (Barron et al., 1999). However, in a high-dimensional setting, the combinatorial problem has NP-complexity, which is computationally prohibitive. As a result, numerous attempts have been made to modify the $L_0$ type regularization to alleviate the computational burden. They include bridge regression (Frank and Friedman, 1993), non-negative garrote (Breiman, 1995), LASSO (Tibshirani, 1996), SCAD (Fan and Li, 2001), elastic net (Zou and Hastie, 2005), adaptive LASSO or ALASSO (Zou, 2006), Dantzig selector (Candes and Tao, 2007), SICA (Lv and Fan, 2009), MCP (Zhang, 2010), among others.

To a certain extent, existing penalties can be classified into one of the following two categories: convex penalty and nonconvex penalty. Convex penalties, such as LASSO (Tibshirani, 1996), can lead to a sparse solution and are stable as the induced optimization problems are convex. Nonconvex penalties, such as SCAD (Fan and Li, 2001) and MCP (Zhang, 2010), can on the other hand lead to sparser solutions as well as the so-called oracle properties (the estimator works as if the identities of nonzero regression coefficients were known beforehand). However, the non-convexity of the penalty could make the entire optimization problem nonconvex, which in turn could lead to a local minimizer and the solution may not be as stable as the one if instead a convex penalty is used. Therefore, an important issue for nonconvex penalties is a good balance between sparsity and stability. For example, both SCAD and MCP have an extra tuning parameter which regulates the concavity of the penalty so that, when it exceeds a threshold, the optimization problem becomes convex.

It is well known that penalty functions have Bayesian interpretation. The classical $L_2$ penalty (ridge regression) is equivalent to the Bayesian estimator with a normal prior. The $L_1$-type penalties, such as LASSO, ALASSO etc., also have Bayesian counterparts; cf. Park and Casella (2008), Griffin and Brown (2010) and Hara and Sillanpä (2009).

Breiman (1996) initiated the discussion about the issue of stability in model selection. He demonstrated that many model selection methods are unstable but can be stabilized by perturbing the data and averaging over many predictors. Breiman (2001) introduced the random forest, providing a way to stabilize the selection process. Bühlmann and Yu (2002) derived theoretical results to analyze the variance reduction effect of bagging in hard decision problems. Meinshausen and Bühlmann (2010) proposed a stable selection, which combines the subsampling with high-dimensional variable
selection methods.

The main objective of the paper is to introduce a family of penalty functions for generalized linear models that can achieve a proper balance between sparsity and stability. Because for the generalized linear models, the loss function is often chosen to be the negative log-likelihood, it is conceivable to take into consideration the form of the likelihood in the construction of penalty functions. The Bayesian connection to the penalty construction and to the likelihood function make it natural to introduce penalty functions through suitable prior distributions. To this end, we introduce the family of negative absolute priors (NAP) and use it to develop what to be called likelihood adaptively modified penalties (LAMP). We define two types of asymptotic stability that cover a wide range of situations and provide a mathematical framework under which the issue of stability can be studied rigorously. We show that under suitable regularity conditions, the LAMP results in an asymptotically stable estimator.

The rest of the paper is organized as follows. Section 2 introduces the LAMP family with motivations from its likelihood and Bayesian connections. Specific examples are given for the commonly encountered generalized linear models. In Section 3, we introduce two types of asymptotic stability and study the asymptotic properties of LAMP family. The choice of the tuning parameters and an efficient algorithm are discussed in Section 4. In Section 5, we present simulation results and applied the proposed method to two real data sets. We conclude with a short discussion in Section 6. All the technical proofs are relegated to the Appendix.

## 2 Likelihood adaptively modified penalty

To introduce our approach, we will first focus on the generalized linear models (Nelder and Wedderburn, 1972). It will be clear that the approach also works for other types of regression models, including various nonlinear regression models. Indeed, our simulation results presented in Section 5 also include the probit model, which does not fall into the exponential family induced generalized linear models.

Throughout the paper, we shall use \((X, Y)\) and \((X_i, Y_i)\) to denote the iid random variables for \(i = 1, 2, \cdots, n\), where \(Y_i\) is the response observation of \(Y\) and \(X_i\) is the \(p+1\)-dimensional covariate observation of \(X\), \(X_i = (X_{i0}, X_{i1}, \cdots, X_{ip})^T\) with \(X_{i0} \equiv 1\). Let \(X' = (X_1, X_2, \cdots, X_n)^T\), the \(n \times (p+1)\) matrix...
of observed covariates. Following Nelder and Wedderburn (1972), we assume that the conditional density of \( Y_i \) given covariates \( X_i \) has the following form

\[
f(Y_i, \theta | X_i) = h(Y_i) \exp \left[ \frac{Y_i \xi_i - g(\xi_i)}{\varphi} \right],
\]

where \( g \) is a smooth convex function, \( \theta = (\alpha, \beta^T) = (\theta_0, \theta_1, \ldots, \theta_p)^T \) or \( (\theta_0, \beta_1, \ldots, \beta_p)^T \), \( \xi_i = X_i^T \theta \) and \( \varphi \) is the dispersion parameter. Then up to an affine transformation, the log-likelihood function is given by

\[
l(\theta) = \sum_{i=1}^{n} [Y_i\xi_i - g(\xi_i)].
\]

Note that the form of \( l \) is uniquely determined by \( g \) and vice versa.

For a given \( g \), we propose the induced penalty function

\[
p_\lambda(\beta) = \frac{\lambda^2}{g'(\alpha_1)\lambda_0}[g'(\alpha_1) - g'(\alpha_1 - \lambda_0 \beta)],
\]

which contains three parameters \( \alpha_1 \leq 0, \lambda_0 > 0 \) and \( \lambda > 0 \). The corresponding negative penalized log-likelihood function is

\[
\tilde{l}(\theta) = -l(\theta) + n \sum_{j=1}^{p} p_\lambda(|\beta_j|),
\]

Because the penalty function defined by (3) is likelihood specific, we will call such a penalty likelihood adaptively modified penalty (LAMP).

We clearly have \( p_\lambda(0) = 0 \) and \( p_\lambda'(0) = \lambda \). Furthermore, taking the first and second derivatives, we get

\[
p_\lambda'(\beta) = \lambda \frac{g'(\alpha_1 - \lambda_0 \beta)}{g'(\alpha_1)} \quad \text{and} \quad p_\lambda''(\beta) = -\lambda_0 \frac{g''(\alpha_1 - \lambda_0 \beta)}{g'(\alpha_1)}.
\]

Remark 1. The parameters have clear interpretations: \( \lambda \) is the usual tuning parameter to control the overall penalty level; \( \alpha_1 \) is a location parameter, which may be fixed as a constant; \( \lambda_0 \) controls the concavity of the penalty in the same way as \( a \) in SCAD (Fan and Li, 2001) and \( \gamma \) in MCP (Zhang, 2010).
Remark 2. The exponential family assumption given by (1) implies that $g'(\xi) = E(Y|X)$ and $g''(\xi) > 0$. When $Y$ is nonnegative, $g'(\xi) \geq 0$. Thus $p'_\lambda$ is negative and the corresponding LAMP must be concave.

Like many other penalty functions, the family of LAMPs also has a Bayesian interpretation. To see this, we introduce the following prior for any given $g$ that defines the exponential family (1).

**Definition 1.** Let $a_j \leq 0$, $b_j < 0$ and $c_j$, $j = 1, \ldots, p$ be constants. Define a prior density, to be called negative absolute prior (NAP),

$$p(\beta) \propto \prod_{j=1}^{p} \exp[-c_j \{g(a_j) - g(a_j + b_j|\beta_j)\}].$$

If we choose $a_j = \alpha_1$, $b_j = -\lambda_0/\lambda$ and $c_j = -\lambda^2/[g'(\alpha_1)\lambda_0]$, then the posterior mode is exactly the minimizer of (4), the penalty form of LAMP. We will show that such a choice will lead to good asymptotic properties.

**Remark 3.** The additive form for the penalty function suggests that the prior must be of the product form. For each $i$, hyperparameter $b_i$ scales $\beta_i$ while hyperparameter $c_i$ gives the speed of decay. For the $i$th parameter $\beta_i$, the larger the values of $b_i$ and $c_i$ are, the more information the prior has for $\beta_i$. Translating it into the penalty function, it means that values of $b_i$ and $c_i$ represent the level of penalty and they can be adjusted separately for each component.

**Remark 4.** The form of NAP is similar to that of a conjugate prior, in that the shape of the prior is adapted to that of the likelihood function. However, NAP also differs from the conjugate prior in several aspects: negative when $g'(\cdot) > 0$, absolute, from separate dimension, and with a LASSO term taken away. Unlike conjugate prior, which assumes additional samples, this can be seen as to take away the absolute value of redundant sample information from each dimension.

It is worth looking into the commonly encountered generalized linear models and examine properties of the corresponding LAMPs.

**Linear regression.** In this case, $g(\xi) = \xi^2/2$. Thus, LAMP reduces to the elastic net (Zou and Hastie, 2005).

$$p_\lambda(\beta) = \lambda\beta - \alpha_1^{-1}\lambda_0\beta^2.$$
Logistic regression. Here \( g(\xi) = \log(1 + e^\xi) \). Consequently, the penalty function

\[
p_\lambda(\beta) = \frac{\lambda^2 (1 + \rho)}{\lambda_0 \rho} \log \left[ \frac{1 + \rho}{1 + \rho \exp(-\lambda_0 / \lambda \beta)} \right],
\]

where \( \rho = \exp(\alpha_1) > 0 \). This penalty will be called sigmoid penalty.

Poisson regression. Here \( g(\xi) = e^\xi \) and we have

\[
p_\lambda(\beta) = \frac{\lambda^2}{\lambda_0} [1 - \exp(-\lambda_0 \beta)].
\]

This will be called the Poisson penalty.

Gamma regression. For the gamma regression, we have \( g(\xi) = -\log(-\xi) \). Then, the penalty has the following form

\[
p_\lambda(\beta) = \frac{\lambda^2 \alpha_1}{\lambda_0} [\log(-\alpha_1) - \log(\lambda_0 \beta - \alpha_1)].
\]

Inverse gaussian regression. For the inverse Gaussian, we have \( g(\xi) = -\sqrt{-2\xi} \). The resulting penalty

\[
p_\lambda(\beta) = \frac{2\lambda^2}{\lambda_0} \left[ (-\alpha_1)^{1/2} \{ (\frac{\lambda_0}{\lambda} \beta - \alpha_1)^{1/2} - (-\alpha_1)^{1/2} \} \right].
\]

Probit regression. As we mentioned earlier, LAMP approach can also accommodate regression models not in the form of naturally parametrized exponential families. In particular, it is applicable to the probit regression for binary outcomes. In this case, \( g(\xi) = -\log(\Phi(-\xi)) \), which leads to the following penalty form

\[
p_\lambda(\beta) = \frac{\lambda^2}{\lambda_0 \Phi(-\alpha_1)} \log[\Phi(-\alpha_1 + \frac{\lambda_0 \beta}{\lambda}) - \Phi(-\alpha_1)],
\]

where \( \Phi(\cdot) \) and \( \phi(\cdot) \) are respectively distribution and density functions of the standard normal random variable.

Remark 5. For the above examples, the effect of tuning parameters in the penalty function varies across settings. For example, \( \alpha_1 \) does not play a role in the Poisson regression. In addition, we have a natural choice for \( \alpha_1 \) for the penalty functions. Specifically, we can choose \( \alpha_1 = -1 \) for the cases of linear, gamma, and inverse gaussian, and \( \alpha_1 = 0 \) for the cases of logistic and probit.
As the examples show, the LAMP family is fairly rich. They also differ from the commonly used penalties. Figure 1 contains plots of penalty functions (left panel) along with their derivatives (right panel) that include LASSO, SCAD, MCP and two members of the LAMP family (sigmoid and Poisson). Here $\gamma = 1.1$ for MCP, $a = 2.1$ for SCAD, $\lambda_0 = 2/1.1 \approx 1.82$ for sigmoid and $1/1.1 \approx 0.91$ for Poisson penalty, and $\rho = 1$ for sigmoid penalty. The parameters are chosen to keep the maximum concavity of these penalties the same. Figure 1 shows that sigmoid and Poisson penalties lie between MCP and SCAD when the maximum concavity is the same. Also, from the graphs of the derivatives, it is easy to identify that the penalties of the LAMP family have continuous derivatives (actually they have continuous derivatives of any order for most common generalized linear models) as compared with the discontinuous ones for SCAD and MCP. It will be shown that this feature can make the optimization problem easier and the estimation more stable.

Remark 6. Consider a one-dimensional penalized logistic regression problem, $\min_\theta [-l(\theta) + p_\lambda(|\theta|)]$, where derivative of the penalty function $p_\lambda'(\cdot) > 0$. If the true parameter $\theta_0 \neq 0$, we need $p_\lambda'(|\theta_0|) \leq \sup_{\theta \in U(\theta_0; \epsilon)} l'(\theta)$, where $U(\theta_0; \epsilon)$ is a $\theta_0$-centered ball with radius $\epsilon > 0$ uniform in $n$, to get $\hat{\theta} \neq 0$, while for $\theta_0 = 0$, we need $p_\lambda'(0) \geq l'(0)$. Thus, to differentiate a nonzero $\theta_0$ from 0, we should make the difference between $p_\lambda'(|\theta_0|)$ for $\theta_0 \neq 0$ and $p_\lambda'(0)$ as large as possible. In addition, we need to control the second derivative
of the penalty function to make the penalized negative log-likelihood function globally convex. Figure 2 illustrates graphically how such dual objectives can be met for the logistic regression under the sigmoid penalty and MCP. The grey area represents the difference between \( p_\lambda'(|A|) \) and \( p_\lambda'(0) \) for the sigmoid penalty. A subset of the grey area, marked by darker color, represent the corresponding difference under MCP. The reason that MCP tends to have a smaller difference compared with sigmoid penalty is because the second derivative of the penalty needs to be controlled by that of the negative log-likelihood in order to maintain the global convexity.
3 Theoretical properties

3.1 Asymptotic stability

Recall that the negative log-likelihood considered here is convex, and the maximum likelihood estimation is uniquely defined and stable. By adding a nonconvex penalty, the convexity may be destroyed. To study stability of the penalized MLE, it is necessary to study the impact of the penalty on the concavity, especially when \( n \) is large.

For the negative penalized maximum log-likelihood estimation procedure \( \mathcal{H} \), if nonconvex penalties are used, the solution to the minimization problem may not be unique. Therefore, it is natural to study the behavior of the local maximizers in penalized likelihood estimates when the observations are perturbed. Here we introduce a new concept of asymptotic stability to describe the asymptotic performances of local minimizers in penalized likelihood estimates. Note that even for the negative penalized likelihood estimators with convex penalty where the unique minimizer exists, such asymptotic stability concept is still useful in characterizing the behavior of the global maximizer.

Suppose we want to minimize with respect to \( \theta \) a criterion function \( M_n(Z_n, \theta) \), where \( Z_n = (Z_1, \cdots, Z_n)^T \) and \( Z_i = (X_i^T, Y_i)^T \) is the \( i \)th observation of \( Z = (X, Y) \). Denote by \( S_Z, S_{Z_n} \) and \( \Theta \) the support for \( Z, Z_n \) and domain for \( \theta \), respectively. We say that \( \theta^* \) is a local minimizer if there exists a neighborhood within which \( M_n(Z_n, \cdot) \) attains its minimum. More precisely, the set of the local minimizers is defined as

\[
\text{arglmin}_{\theta} M_n(Z_n, \theta) \equiv \{ \theta^* \in \Theta | \exists \epsilon > 0, M_n(Z_n, \theta^*) = \min_{\|\theta - \theta^*\| \leq \epsilon} M_n(Z_n, \theta) \}.
\]

Throughout the paper, \( \| \cdot \| \) denotes the \( L_2 \)-norm of a vector or matrix (vectorized) while \( | \cdot | \) denotes the \( L_1 \) norm. For any \( \theta_* \in \Theta, r > 0 \) let \( U(\theta_*, r) \equiv \{ \theta \in \Theta | \|\theta - A\| < r \} \) the be \( \theta_* \)-centered ball with radius \( r \).

It is clear from the definition that the set of local maximizers is random. We characterize its asymptotic behavior in terms of whether or not the set converges to a single point as \( n \to \infty \). For a set \( A \), define its diameter as \( \text{diam}(A) \equiv \sup_{x, y \in A} \|x - y\| \).

**Definition 2** (Weak asymptotic stability). We say that the set of local min-
iminizers of \( M_n(Z_n, \cdot) \) satisfies weak asymptotic stability if \( \forall \delta > 0, \)
\[
\lim_{n \to \infty} P \left( \lim_{\epsilon \to 0} \text{diam} \left[ \bigcup_{\theta \in \arg\min \{M_n(Z_n + \mathcal{E}_n, \theta)\}} \left\{ \frac{\|\mathcal{E}_n\|}{\|Z_n + \mathcal{E}_n\|} \right\} > \delta \right] \right) = 0. \tag{7}
\]

The weak asymptotic stability characterizes the asymptotic behavior of local minimizers when the data are perturbed slightly. It shows that for large \( n \) and small perturbation, the local minimizers stay sufficiently close to each other with high probability. Defined below is a stronger version, which guarantees uniqueness of the minimizer.

**Definition 3 (Strong asymptotic stability).** We say that the set of local minimizers of \( M_n(Z_n, \cdot) \) satisfies strong asymptotic stability if
\[
\lim_{n \to \infty} P \left( \lim_{\epsilon \to 0} \text{diam} \left[ \bigcup_{\theta \in \arg\min \{M_n(Z_n + \mathcal{E}_n, \theta)\}} \left\{ \frac{\|\mathcal{E}_n\|}{\|Z_n + \mathcal{E}_n\|} \right\} = 0 \right] \right) = 1. \tag{8}
\]

**Remark 7.** Under the weak asymptotic stability, multiple minimizers, though shrinking to 0, may exist with high probability for any finite \( n \). The strong asymptotic stability, on the other hand, entails that for sufficiently large \( n \), the probability of having multiple minimizers must converge to 0, implying that there must be a unique minimizer with high probability.

**Remark 8.** \( L_0 \) penalties, though may have the weak asymptotic property if we adjust its tuning parameter like AIC, will never possess the strong asymptotic property, because then the solution of each submodel with the number of parameters constrained will be a local optimizer, and with no probability all solutions will coincide, as \( n \to \infty \).

We now consider the situation in which \( M_n(Z_n, \theta) \) can be approximated by an i.i.d. sum with a remainder term, i.e.,
\[
M_n(Z_n, \theta) = \frac{1}{n} \sum_{i=1}^{n} m(Z_i, \theta) + r_n(\theta). \tag{9}
\]

This general form includes the form of a negative log-likelihood plus a penalty. Let \( m_n(Z_n, \theta) = n^{-1} \sum_{i=1}^{n} m(Z_i, \theta) \). We assume throughout the rest of the paper that \( \Theta \) is compact with the true parameter value, denoted by \( \theta_0 \), lying in its interior and that \( \bar{m}(\theta) = \text{Em}(Z, \theta) \) is finite. We need the following regularity conditions.
(C1) For any $x \in S_Z, m(x, \theta)$ is convex as a function $\theta$, and $\bar{m}(\theta)$ is strictly convex.

(C2) There exists function $K : S_Z \mapsto \mathbb{R}$, such that $EK^2(Z) < \infty$ and, for any $\delta_1, \delta_2$ and $z$, such that $z + \delta_i \in S_Z, i = 1, 2$,

$$\sup_{\theta \in \Theta} |m(z + \delta_1, \theta) - m(z + \delta_2, \theta)| \leq K(z) \|\delta_1 - \delta_2\|.$$ 

(C3) The remainder term is asymptotically flat:

$$\lim_{n \to \infty} \sup \frac{|r_n(\theta_1) - r_n(\theta_2)|}{\|\theta_1 - \theta_2\|} = 0.$$ 

(C4) There exists a local minimizer of (9) that is a consistent estimator of $\theta_0$.

(C5) There exists $\delta_0 > 0$, such that

$$\lim_{n \to \infty} P(M_n(Z_n, \theta) \text{ is strictly convex within } U(\theta_0; \delta_0) \cap \Theta) = 1.$$ 

Remark 9. When $m$ and $r_n$ are both smooth functions, Conditions (C2) and (C3) can be guaranteed by assuming that the derivative of $m$ is bounded uniformly and that the derivative of $r_n$ tends to 0 uniformly. Condition (C5) is guaranteed by the convexity around the true parameters, uniformly in $n$.

The next two lemmas provide sufficient conditions for the two types of asymptotic stability.

Lemma 1. If Conditions (C1)-(C3) are satisfied, then we have weak asymptotic stability.

It is straightforward to verify that for generalized linear models, SCAD, MCP, sigmoid penalty and Poisson penalty all satisfy Conditions (C1)-(C3), leading to the weak asymptotic stability.

Lemma 2. If Conditions (C1)-(C5) are satisfied, then strong asymptotic stability holds.
3.2 Asymptotic properties

In this subsection, we study asymptotic properties for the proposed LAMP, including parameter estimation consistency, model selection consistency and asymptotic stability. Suppose $\beta_{j,0}$ or $\theta_{j,0}$ is the true value of $\beta_j$ or $\theta_j$, $\forall j \in \mathbb{N}$. Let $q \triangleq \sum_{j=1}^{p} I_{\beta_{j,0} \neq 0}$, the number of signals. Without loss of generalization, we assume $\beta = (\beta_1^T, \beta_2^T)^T$, where the true value $\beta_1$ has no zero element, and $\beta_2 = 0_{p-q}$, which indicates a $(p-q) \times 1$ vector with each element being 0. Denote $\zeta_1 = \inf \{|\beta_{1,0}|, \ldots, |\beta_{q,0}|\}$, $\zeta_2 = \sup \{|\beta_{1,0}|, \ldots, |\beta_{q,0}|\}$, then $[\zeta_1, \zeta_2]$ is the range of signal level. For any sequences $a_n, b_n$ we say $a_n \ll b_n$ if \(\lim_{n \to \infty} a_n/b_n = 0\). Here, we consider the setting of fixed $p, q, \zeta_1, \zeta_2$ when $n \to \infty$ though some results may be extended to the case of $p = o(\sqrt{n})$.

We first introduce certain regularity conditions which are needed for establishing asymptotic properties.

(C6) \[
\sup_{\theta \in \Theta} E(\|X\|^{k} | g^{(k)}(X^{T}\theta) |) < \infty, \inf_{\theta \in \Theta} \lambda_{\text{min}}(E XX^{T}|g^{(k)}(X^{T}\theta)) > 0,
\]
where $g^{(k)}$ denotes the $k$th derivative of $g$, and $\lambda_{\text{min}}(\cdot)$ of a matrix “.” denotes its minimum eigenvalue.

(C7) $g'(x) > 0$ at $x \in [-\infty, \alpha_1)$ and $\lim_{\xi \to -\infty} g'(\xi) = 0$; $g''(\xi)$ is increasing at $[-\infty, \alpha_1)$, where $\alpha_1 \leq 0$ is a constant.

(C8) $\lambda \to 0$, $\sqrt{n}\lambda \to +\infty$, and $\sqrt{n}\lambda g'(\alpha_1 - \lambda_0\zeta_1/\lambda) \to 0$.

(C9) \[
\lambda_{\text{min}}\{E[XX^{T}]\} > \frac{\lambda_0 g''(\alpha_1)}{|g'(\alpha_1)|} \frac{1}{g''(X^{T}\theta_0)}.
\]

**Theorem 1.** Suppose that Conditions (C6)-(C9) are satisfied. Then the penalized maximum likelihood estimator based on the LAMP family is consistent and asymptotically normal, and achieves model selection consistency and strong asymptotic stability.

**Lemma 3.** Suppose that (C6) and (C7) are satisfied and $\log[g'(-x)] = x^u L(x)$, where $u > 0$ is a constant and $L(x)$ is negative and slowly varying at $\infty$, i.e., for any $a > 0$, $\lim_{x \to \infty} L(ax)/L(x) = 1$, and $\lim_{x \to +\infty} L(x) \in [-\infty, 0)$. Then $n^{-1/2} \ll \lambda \ll \lambda_0 (\log n)^{-1/u} \ll 1$ implies (C8).
It can be verified that the condition on $\log[g'(−x)]$ is satisfied for the logistic, Poisson and probit regression models. To achieve model selection consistency, we can choose $\lambda_0 = o(1)$ and $\lambda = o((\log n)^{-1/u}\lambda_0)$.

We now present the corresponding results for the case of sigmoid penalty. Here, $\log(g'(−x)) = xL(x)$, where $L(x) = [-\log(1+e^x)/x]$ is a slowly varying function. The following conditions simplify (C6)–(C9).

(C6) $E\|X\|^3 < +\infty$.

(C7) Parameter $\rho = e^{\alpha_1}$ is a constant (does not depend on $n$) and $n^{-\frac{1}{2}} \ll \lambda_0/\log n \ll 1$.

(C8) $\lambda_0 \leq \lambda_{\min} \left[ E(XX^T \frac{e^{X^T \theta_0}}{(1 + e^{X^T \theta_0})^2}) (1 + \rho)/\rho \right].$

The following proposition shows that the results of Theorem 1 carry over to the penalized logistic regression when (C6)–(C8) are assumed.

**Corollary 1.** For the penalized maximum likelihood estimator of logistic regression with the sigmoid penalty (6), we have parameter estimation consistency, model selection consistency and strong asymptotic stability under conditions (C6)–(C8).

### 4 Algorithms

An important aspect of the penalized likelihood estimation method is the computational efficiency. For the LASSO penalty, Efron et al. (2004) proposed the path-following LARS algorithm. In Friedman et al. (2007), the coordinate-wise descent method was proposed. It optimizes a target function with respect to a single parameter at a time, cycling through all parameters until convergence is reached. For non-convex penalties, Fan and Li (2001) used the LQA approximation approach. In Zou and Li (2008), a local linear approximation type method was proposed for maximizing the penalized likelihood for a broad class of penalty functions. In Fan and Lv (2011), the coordinate-wise descent method was implemented for non-convex penalties as well. Yu and Feng (2013) proposed a hybrid approach of Newton-Raphson and coordinate descent for calculating the approximate path for penalized likelihood estimators with both convex and non-convex penalties.
4.1 Iteratively reweighted least squares

We apply quadratic approximation and use the coordinate descent algorithm similar to Keerthi and Shevade (2007) and Shevade and Keerthi (2003). Recall that our objective function is the negative penalized log-likelihood

\[
-\tilde{l}(\theta) = -l(\theta) + n \sum_{j=1}^{p} p_{\lambda}(|\beta_j|).
\]

Following Keerthi and Shevade (2007), let

\[
F_j = \frac{\partial l(\theta)}{\partial \theta_j},
\]

and define the violation function

\[
\text{viol}_j(\theta) \triangleq \begin{cases} 
|F_j|, & \text{if } j = 0, \\
\max\{0, -n\lambda - F_j, -n\lambda + F_j\}, & \text{if } \theta_j = 0, j > 0, \\
|F_j - n\text{sgn}(\theta_j)p'_{\lambda}(|\theta_j|)|, & \text{if } \theta_j \neq 0, j > 0.
\end{cases}
\]

We see that the objective function achieves its maximum value if and only if \(\text{viol}_j = 0\), for all \(j\). Thus we use \(\max_j\{\text{viol}_j\} < \tau\) as the stop condition of our iteration, with \(\tau > 0\) being the chosen tolerance threshold.

In each step, we use quadratic approximation to the log-likelihood function

\[
-\tilde{l}(\theta) \approx \frac{1}{2}(\tilde{Y} - X\theta)^T W (\tilde{Y} - X\theta) + \text{constant},
\]

where \(W\) and \(\tilde{Y}\) depend on the current value of \(\theta\). The algorithm is summarized as follows.

**Algorithm:** Set values for \(\tau > 0, \lambda, \lambda_0, \rho > 0\). Denote by \(X_{.j}\) the \((j+1)\)th column of \(X, j = 0, \cdots, p\).

**S1.** Standardize \(X_i, i = 1, 2, \cdots, n\).

**S2.** Initialize \(\theta = \theta^{(0)}\). Calculate \(\text{viol} = \max_j\{\text{viol}_j(\theta)\}\) and go to S3 if \(\text{viol} > \tau\) or else go to S5.

**S3.** Choose \(j^* \in \arg \max_j\text{viol}_j(\theta)\). Calculate \(W, v = \frac{1}{n}X'_{.j^*}WX_{.j^*}, z = \frac{1}{n}X'_{.j^*}W(\tilde{Y} - X\theta) + v\theta_{j^*}\). If \(j^* \neq 0\), let \(r = p'_{\lambda}(|\theta_{j^*}|)\); else \(r = 0\).

**S4.** If \(j^* = 0, \theta_{j^*} = z/v\); else \(\theta_{j^*} = \text{sgn}(z)(|z| - r)_+\).

Calculate \(\text{viol} = \max_j\{\text{viol}_j\}\) and go to S3 if \(\text{viol} > \tau\), else go to S5.

**S5.** Do the transformation of the coefficients \(\theta\) due to standardization.

For S2, the initial solution \(\theta^{(0)}\) can be taken as the zero solution or the MLE or the estimate calculated using a parameter \(\lambda^* \in U(\lambda; \epsilon)\) from the previous steps.

For S4, we first carry out the iterations for the variables in the current active set until convergence, then check whether additional variables should join the active set. Alternatively, we may speed up the calculation by using “warm start”. Readers are referred to Keerthi and Shevade (2007) and
For example, the logistic regression with sigmoid penalty is,
\[ \sum_{i=1}^{n} \log(1 + e^{-Y_iX_i^T\theta}) + n \sum_{j=1}^{p} p_{\lambda}(|\beta_j|), \]
where \( p_{\lambda}(\theta) = \frac{\lambda^2(1+\rho)}{\lambda_0} \log\left[1 + \frac{e^{\lambda_0\theta}}{\lambda_0\theta}\right] \).

We define \( r_i = \exp(-Y_iX_i^T\theta), i = 1, \cdots, n, \) and \( F_j(\theta) = \sum_{i=1}^{n} r_iY_iX_{i,j}/(1+r_i), j = 0, \cdots, d. \) In S3, we have the following two different approximation methods for updating.

1. Quadratic approximation from IRLS \cite{Munk2006}. Let \( W = \frac{1}{2} \text{diag}\{\tanh(\frac{\pi_1}{2})/\pi_1, \cdots, \tanh(\frac{\pi_n}{2})/\pi_n\} \) and \( \tilde{Y} = \frac{1}{2}W^{-1}Y \), where \( \pi_i = \frac{1}{1 + \exp(-Y_iX_i^T\theta(0))} \).

2. Quadratic approximation using Taylor expansion. Let \( W = \text{diag}\{\pi_1(1-\pi_1), \cdots, \pi_n(1-\pi_n)\} \) and \( \tilde{Y} = X\theta(0) + W^{-1}[Y \circ (1 - \pi)] \), where \( \pi_i = 1/[1 + \exp(-Y_iX_i^T\theta(0))] \) and \( \circ \) is the component-wise product operator.

For an initial estimator \( \theta(0) \), denote by \( a(\theta, \theta(0)) \) a quadratic approximation of \( -n^{-1}l(\theta) \) at \( \theta(0) \), i.e.,
\[ a(\theta(0), \theta(0)) = -n^{-1}l(\theta(0)), \quad \frac{\partial a(\theta, \theta(0))}{\partial \theta}|_{\theta=\theta(0)} = -n^{-1}l'(\theta(0)). \]

**Theorem 2.** Let \( C \subset \mathbb{R}^d \) be a closed set and the objective function \( M_n(\theta) = -n^{-1}l(\theta) + n \sum_{j=1}^{p} p_{\lambda}(|\beta_j|) \) is strictly convex. In addition, assume the quadratic approximation at \( \theta(0) \) satisfies \( a(\theta, \theta(0)) \geq -n^{-1}l(\theta) \) for all \( \theta \in C \). Then the algorithm constrained in \( C \) (minimization within \( C \)) converges to the true minimum \( \theta^* = \arg\min_{\theta} M_n(\theta) \). In addition, method 1 for the logistic regression satisfies the conditions on quadratic approximation.

### 4.2 Balance between stability and parsimony

An important issue is how to choose tuning parameters in the penalized likelihood estimation. For the LAMP family, there are three tuning parameters, i.e., \( \lambda, \lambda_0 \) and \( \alpha_1 \). Our numerical experiences show that the resulting estimator is not sensitive to the choice of \( \alpha_1 \). In most cases, we may simply take \( \alpha_1 = -1 \) or \( \alpha_1 = 0 \), depending on the type of regression (See Remark 5). For \( \lambda \) and \( \lambda_0 \), we recommend using cross-validation or BIC, so long as the solutions are stable enough. The \texttt{ncvreg} package described in \cite{Breheny2015}.
to determine a stable area or perform local diagnosis is recommended. There are two approaches to get a stable area: to control the smoothness of the \( \lambda \)-estimate curve and calculate the smallest eigenvalue of the penalized likelihood at each point of the path as stated in Theorem 1. Here, we take the second approach in all numerical analysis.

Our algorithm differs from \texttt{ncvreg} in the following two aspects: we use the “viol” function as the convergence criteria; we do not use the adaptive-scale. Both algorithms 1 and 2 use the linear approximation (suppose at \( \theta(0) \)) of the penalty term. \( p_\lambda(\|\theta\|) \approx p_\lambda(\|\theta(0)\|) + p'_\lambda(\|\theta(0)\|)(\|\theta\| - \|\theta_0\|) \). For algorithm 1, from concavity, we have \( p_\lambda(\|\theta\|) < p_\lambda(\|\theta(0)\|) + p'_\lambda(\|\theta(0)\|)(\|\theta\| - \|\theta_0\|) \), which naturally falls into the MM-algorithm framework.

To choose the \((\lambda_0, \lambda)\) pair, we use the hybrid approach introduced by Breheny and Huang (2011), i.e., combining BIC, cross-validation, and convexity diagnostics. For a path of solutions with a given value of \( \lambda_0 \) large enough, use AIC/BIC to select \( \lambda \) and use the convexity diagnostics to determine the locally convex regions of the solution path. If the chosen solution lies outside the stable region, we lower \( \lambda_0 \) to make the penalty more convex. After this process is iterated multiple times, we can find a value of \( \lambda_0 \) that produces a good balance between sparsity and convexity. Then we can fix \( \lambda_0 \) and use BIC or cross-validation to choose the best \( \lambda \).

5 Simulation results and examples

Simulation studies cover logistic, Poisson and probit cases. The performance of the LAMP family is compared with those of LASSO, SCAD and MCP. Particular attention is given to the logistic regression to demonstrate how sparsity and stability are properly balanced. Two classification examples from microarray experiments involving cancer patients are presented.

5.1 Logistic regression

We simulate from the logistic regression model with \( n = 200, p = 1000, \alpha = 0, \beta = (1.5, 1, -0.7, 0_{997})^T \), and \( X \sim N(0, \Sigma) \), where \( \Sigma_{i,j} = \rho + (1 - \rho)1_{i=j} \) with \( \rho = 0.5 \). The number of replications is 100 for this and all the subsequent simulations.

Table 1 reports true positive (TP), false positive (FP), proportion of correct fit (cf), proportion of over fit (of), proportion of under fit (uf), \( ||\hat{\beta} - \beta||_1 \)
Table 1: Simulation results for model selection and estimation under different penalties. Entries are mean values with the standard error in parentheses. Here “sig” represents the sigmoid penalty. The number inside the parentheses following by the penalty name represents the concavity parameter.

| Penalties | TP (SE) | FP (SE) | cf | of | uf | L1 | L2 |
|-----------|---------|---------|----|----|----|----|----|
| LASSO     | 1.87 (0.053) | 0.12 (0.041) | 0.05 | 0.01 | 0.94 | 2.49 | 1.46 |
| sig(.02)  | 1.87 (0.053) | 0.11 (0.040) | 0.05 | 0.01 | 0.94 | 2.48 | 1.45 |
| sig(.03)  | 1.90 (0.052) | 0.13 (0.042) | 0.06 | 0.01 | 0.93 | 2.41 | 1.41 |
| sig(.05)  | 1.95 (0.05)  | 0.15 (0.041) | 0.08 | 0.01 | 0.91 | 2.34 | 1.36 |
| sig(.07)  | 1.99 (0.048) | 0.20 (0.047) | 0.09 | 0.02 | 0.89 | 2.25 | 1.31 |
| sig(.09)  | 2.02 (0.047) | 0.20 (0.049) | 0.1  | 0.02 | 0.88 | 2.17 | 1.26 |
| sig(.15)  | 2.21 (0.057) | 1.86 (0.43)  | 0.1  | 0.19 | 0.71 | 3.18 | 1.36 |
| sig(.38)  | 2.61 (0.049) | 4.65 (0.40)  | 0.1  | 0.51 | 0.39 | 6.23 | 2.13 |
| SCAD(300) | 1.69 (0.073) | 0.14 (0.043) | 0.04 | 0.01 | 0.95 | 2.94 | 1.74 |
| SCAD(7)   | 1.69 (0.073) | 0.14 (0.043) | 0.04 | 0.01 | 0.95 | 2.94 | 1.74 |
| SCAD(5)   | 1.84 (0.077) | 3.20 (0.69)  | 0.03 | 0.11 | 0.86 | 2.94 | 1.74 |
| SCAD(4)   | 2.25 (0.089) | 12 (0.75)    | 0.01 | 0.45 | 0.54 | 91.9 | 23.5 |
| MCP(300)  | 1.69 (0.073) | 0.14 (0.043) | 0.04 | 0.01 | 0.95 | 2.94 | 1.74 |
| MCP(50)   | 1.70 (0.073) | 0.16 (0.047) | 0.04 | 0.01 | 0.95 | 2.94 | 1.74 |
| MCP(15)   | 1.75 (0.073) | 0.14 (0.043) | 0.06 | 0.01 | 0.93 | 2.45 | 1.45 |
| MCP(7)    | 1.79 (0.071) | 0.14 (0.040) | 0.07 | 0.01 | 0.92 | 2.21 | 1.31 |
| MCP(5)    | 1.91 (0.081) | 2.98 (0.69)  | 0.09 | 0.1  | 0.81 | 2.07 | 1.23 |
| MCP(4)    | 2.29 (0.092) | 11.89 (0.80) | 0.02 | 0.50 | 0.48 | 109.3 | 28.7 |

(L1 loss), and \( \| \hat{\beta} - \beta \|^2 \) (L2 loss). To compare performances among LASSO, SCAD, MCP and the sigmoid penalty, we use glmmnet to calculate the LASSO solution path, ncvreg to calculate the SCAD and MCP. For all penalties, we use EBIC (Chen and Chen, 2008) to choose the tuning parameter \( \lambda \) with other parameters fixed. The EBIC parameter \( \eta = 1 \).

From Table 1, it is clear that the sigmoid penalty outperforms SCAD and MCP in the sense that at a similar level of TP, the sigmoid penalty results in a smaller FP. Somewhat surprisingly the LASSO has a competitive performance, which may be attributed to the use of the EBIC selection criterion.
Table 2: Simulation results for TP and FP under different penalties with the tuning parameter selected by BIC. Mean values are presented.

| λ₀  | TP   | FP   | a   | TP   | FP   | γ   | TP   | FP   |
|-----|------|------|-----|------|------|-----|------|------|
| .04 | 1.86 | 0.28 | 1.1 | 1.89 | 0.21 | 1.1 | 1.90 | 0.21 |
| .05 | 1.81 | 0.22 | 7   | 1.89 | 0.21 | 7   | 1.90 | 0.21 |
| .06 | 1.84 | 0.24 | 14  | 1.89 | 0.21 | 14  | 1.90 | 0.21 |
| .08 | 1.89 | 0.22 | 20  | 1.89 | 0.21 | 20  | 1.90 | 0.21 |
| .10 | 1.91 | 0.22 | 27  | 1.88 | 0.23 | 27  | 1.89 | 0.22 |
| .11 | 1.89 | 0.21 | 34  | 1.83 | 0.23 | 34  | 1.86 | 0.24 |
| .13 | 1.89 | 0.21 | 41  | 1.83 | 0.24 | 41  | 1.83 | 0.23 |
| .15 | 1.89 | 0.21 | 54  | 1.83 | 0.28 | 54  | 1.83 | 0.29 |
| .16 | 1.89 | 0.21 | 60  | 1.86 | 0.32 | 60  | 1.86 | 0.32 |
| .18 | 1.89 | 0.21 | 67  | 1.88 | 0.35 | 67  | 1.88 | 0.36 |

5.2 Smoothness

The same logistic regression model as in Section 5.1 except \( \alpha_0 = -3, \beta_0 = (1.5, 1, -0.7, 0, 0, 0, 0, 0)^T \) and \( \Sigma_{i,j} = 0.5|i-j| \) is used. In Figure 3, we compare smoothness of solution paths generated from the sigmoid penalty, SCAD and MCP with the same concavity at 0. It can be seen that this choice will also lead to similar sparse level as in Table 2. SCAD and MCP use the same algorithm as sigmoid does (not adaptive-scale as in ncvreg package). The shorter vertical line is the BIC choice of \( \lambda \) and the longer one uses a 10-fold cross validation. To avoid variation due to the random division in the cross-validation, the result in Table 2 uses BIC to choose \( \lambda \) with the other tuning parameter fixed. The subfigures (c) and (d) for SCAD and MCP are both less smooth than subfigure (a) for the sigmoid penalty which is of similar smoothness as subfigure (b) for LASSO. The sigmoid penalty appears to outperform SCAD and MCP in terms of smoothness of the solution path.

5.3 Stability

The data are generated in the same way as in the preceding subsection. For each replication, we repeat 100 times cross-validation to select \( \lambda \) and calcu-
late its mean sample standard deviation. The box-plots for the mean sample standard deviations are generated. In addition, to evaluate the asymptotic stability as introduced in Section 5.3, we add a small random perturbation generated from $N(0, 0.1)$ to all the observations before conducting the analysis. The box-plot results are presented in Figure 4.

The result without the random error term evaluates the stability regarding the randomness of cross-validation for each penalty. The result with the random error evaluates the stability towards the random perturbation of the data. To ensure a fair comparison, we choose the same level of concavity at 0 for SCAD, MCP and the sigmoid penalty. It is seen from the box-plots
Figure 4: Box-plots for the mean standard deviations in Section 5.3. The left panel shows the box-plot without perturbation and the right one is that with perturbation.

that LASSO is the most stable one, while the sigmoid penalty outperforms SCAD and MCP, in terms of both the median and 75% quantile in the case without error, and 75% quantile in the case with the error added.

5.4 Poisson and probit

We simulate from the Poisson regression model with \( n = 250, \alpha = -1, \beta = (0.6, 0.4, 0, 0, 1, 0, 0)^T, \) and \( X \sim N(0, \Sigma) \), where \( \Sigma_{i,j} = \rho + (1 - \rho)1_{i=j} \) with \( \rho = 0.5 \). For probit regression, we simulate the \( x \) the same way and set \( \alpha = -2 \) and \( \beta = (3, 2, 1, 0, 0, 0, 0)^T \).

The results for the Poisson and probit regression models are summarized in Tables 3 and 4 respectively. For the Poisson regression, we also report the MRME (median of ratios of model error of a selected model to that of the ordinary maximum likelihood estimate under the full model). We observe similar selection performance using Poisson, SCAD and MCP. However, the Poisson penalty leads to the smallest MRME. For the probit model, all three nonconvex penalties lead to the same result for all criteria considered.
Table 3: Simulation results for Poisson regression under different penalties.

| Penalties | TP     | FP     | cf | of | uf | L1  | L2  | MRME |
|-----------|--------|--------|----|----|----|-----|-----|------|
| LASSO     | 2.98 (0.01) | 1.40 (0.13) | 0.26 | 0.72 | 0.02 | 0.63 | 0.34 | 0.49 |
| Poisson   | 2.80 (0.04) | 0.41 (0.08) | 0.65 | 0.15 | 0.20 | 0.49 | 0.30 | 0.19 |
| SCAD      | 2.78 (0.04) | 0.35 (0.07) | 0.54 | 0.24 | 0.22 | 0.47 | 0.30 | 0.20 |
| MCP       | 2.81 (0.04) | 0.38 (0.08) | 0.66 | 0.15 | 0.19 | 0.48 | 0.29 | 0.22 |

Table 4: Simulation results for probit regression under different penalties.

| Penalties | TP     | FP     | cf | of | uf | L1  | L2  | MRME |
|-----------|--------|--------|----|----|----|-----|-----|------|
| LASSO     | 3.00 (0)  | 2.13 (0.18) | 0.21 | 0.79 | 0  | 2.47 | 1.35 | 0.71 |
| probit    | 2.99 (0.01) | 0.20 (0.05) | 0.82 | 0.17 | 0.01 | 1.37 | 0.85 | 0.27 |
| SCAD      | 2.99 (0.01) | 0.20 (0.05) | 0.82 | 0.17 | 0.01 | 1.37 | 0.85 | 0.27 |
| MCP       | 2.99 (0.01) | 0.20 (0.05) | 0.82 | 0.17 | 0.01 | 1.37 | 0.85 | 0.27 |
Table 5: Classifications errors and selected model sizes under different penalties for lung cancer data.

| Penalties | $\lambda_0/a/\gamma$ | Training error | Test error | # of selected genes |
|-----------|-----------------------|----------------|------------|---------------------|
| LASSO     | —                     | 0/32           | 7/149      | 15                  |
| SCAD      | (3~6)                 | 0/32           | 6/149      | 13                  |
| MCP       | (3~6)                 | 0/32           | 7/149      | 3                   |
| sigmoid   | (.022~0.03)           | 0/32           | 7/149      | 3                   |
| sigmoid   | (.021)                | 0/32           | 6/149      | 5                   |

5.5 Examples

We apply the proposed LAMP to two gene expression datasets: lung cancer data (Gordon et al., 2002), and prostate cancer data (Singh et al., 2002). The two datasets are downloadable at [http://www.chestsurg.org](http://www.chestsurg.org) and [http://www.broad.mit.edu](http://www.broad.mit.edu). The response variable in each dataset is binary.

We aim to use the lung cancer data to classify malignant pleural mesothelioma (MPM) from adenocarcinoma (ADCA) of the lung. The data consists of 181 tissue samples, 32 of which are for training with remaining 149 for testing. Each sample is described by 12533 genes.

After the initial standardization of the predictors into mean zero and variance one, we apply LASSO, SCAD, MCP, and the sigmoid penalty, using glmnet for LASSO, ncvreg for SCAD and MCP. For each method, a 10-fold cross-validation is used to select the best $\lambda$. We repeat 10 times to make different divisions to calculate the cross-validation error. For SCAD and MCP, we evaluate the performance when $a, \gamma \in [3, 6]$ while for sigmoid penalty, $\lambda_0 \in (.005, 0.03)$. The results are summarized in Table 5. The result for sigmoid penalty is quite similar to MCP when $\lambda_0 \in (.022, 0.03)$. When $\lambda_0 = 0.021$, the sigmoid have 6 test errors with only 5 genes selected, which is better compared with SCAD.

For the prostate cancer data, the goal is to classify prostate tumor samples from the normal samples. There are 102 patient samples for training, and 34 patient samples for testing with 12600 genes in total. The result are reported in Table 6. Here we see the test errors are similar across methods, although the sigmoid penalty leads to the most sparse solution.
Table 6: Classifications errors and selected model sizes under different penalties for prostate cancer data.

| Penalties | $\lambda_0/a/\gamma$ | Training error | Test error | # of selected genes |
|-----------|-----------------------|----------------|------------|---------------------|
| LASSO     | —                     | 0/102          | 2/34       | 30                  |
| SCAD      | (20~25)               | 0/102          | 2/34       | 26                  |
| MCP       | (35~50)               | 0/102          | 2/34       | 24                  |
| sigmoid   | (.001)                | 0/102          | 1/34       | 26                  |
| sigmoid   | (.002)                | 0/102          | 2/34       | 23                  |

6 Discussion

Penalty based regularization methods have received much attention in recent years. This paper proposes a family of penalty functions (LAMP) that is adaptive and specific to the shapes of the log-likelihood functions. The proposed LAMP family is different from the well-known LASSO, SCAD and MCP. It can be argued that the new approach provides a good balance between sparsity and stability, two important aspects in model selection. It is shown that the resulting penalized estimation achieves model selection consistency and strong asymptotic stability, in addition to the usual consistency and asymptotic normality.

An important issue is how to choose the three parameters imbedded in a LAMP. The “location” parameter $\alpha_1$ can be chosen in an obvious way for the standard generalized linear models, while $\lambda$, which represents the penalty level, can be chosen through standard CV or information criteria. For $\lambda_0$, which controls the concavity level, it is computationally intensive to use CV. It is desirable to develop more effective ways to select $\lambda_0$. It is also important to study the stability of the solution path.

The LAMP approach can be modified to handle grouped variable selection. It will also be of interest to develop parallel results for semiparametric regression models.
A General results

We first state a general result about estimation consistency, model selection consistency and asymptotic normality.

Consider the penalized log-likelihood function \( \tilde{l}(\theta) \) as defined by (4). Let \( \alpha_0 \) be the true value of \( \alpha \). Recall \( \beta_{10} \) is the nonzero part of \( \beta_0 \) and \( \beta_{20} = 0 \). For notational simplicity, let

\[
Z, Z_i \text{ be i.i.d. with density } f(\cdot, \theta) \text{ that satisfies the following regularity conditions (see Fan and Li (2001)):
}

(C10) \( E_{\theta}[\partial \log f(Z, \theta) / \partial \theta_j] = 0, \)

\[
E_{\theta} \left[ \frac{\partial \log f(Z, \theta) \partial \log f(Z, \theta)}{\partial \theta_j \partial \theta_k} \right] = -E_{\theta} \left[ \frac{\partial^2 \log f(Z, \theta)}{\partial \theta_j \partial \theta_k} \right].
\]

where \( j, k = 0, 1, 2, \ldots, p; \)

(C11) \( 0 < E \left[ \frac{\partial \log f(Z, \theta)}{\partial \theta} |_{\theta = \theta_0} \right] \otimes 2 < \infty; \)

(C12) There exist functions \( M_{jkl}, \) a neighborhood \( \Theta_1 \) of \( \theta_0 \) such that \( E_{\theta_0} M_{jkl}(Z) < \infty \) and

\[
\left| \frac{\partial^3 \log f(Z, \theta)}{\partial \theta_j \partial \theta_k \partial \theta_l} \right| \leq M_{jkl}(Z)
\]

for all \( \theta \in \Theta_1, j, k \) and \( l. \)

Note that for the generalized linear models, (C10)-(C12) are satisfied under mild assumptions on covariates.

\[
p_{1,n} \triangleq \sup_{\beta \in [\zeta_1, \zeta_2], 1 \leq j \leq q} |p_{\lambda,j}'(\beta)|; \quad p_{2,n}(\beta) \triangleq \inf_{q < j \leq p} p_{\lambda,j}'(\beta); \quad p_{2,n} \triangleq p_{2,n}(0)
\]

\[
p_{3,n}(\beta) \triangleq \sup_{x \in (0, \beta), q < j \leq p} |p_{\lambda,j}''(x)|; \quad p_{4,n} \triangleq \sup_{\beta \in [\zeta_1, \zeta_2], 1 \leq j \leq q} |p_{\lambda,j}''(\beta)|;
\]

\[
p_{5,n} \triangleq \inf_{\beta \in [\zeta_1, \zeta_2], 1 \leq j \leq q} p_{\lambda,j}'''(\beta); \quad \Sigma_1 = \text{diag}\{p_{\lambda,1}''(|\beta_{10}|), \ldots, p_{\lambda,q}''(|\beta_{q0}|)\};
\]

\[
b = (0, p_{\lambda,1}''(|\beta_{10}|) \text{sgn}(\beta_{10}), \ldots, p_{\lambda,q}''(|\beta_{q0}|) \text{sgn}(\beta_{q0}))^T.
\]
Let \( \mathcal{R} = \begin{pmatrix} R_{00} & R_{01} & R_{02} \\ R_{10} & R_{11} & R_{12} \\ R_{20} & R_{21} & R_{22} \end{pmatrix} \) be the Fisher information matrix at \( \theta_0 \) partitioned by the intercept, the nonzero and zero parts and

\[
\Sigma_2 \triangleq \begin{pmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} + \Sigma_1 \end{pmatrix}.
\]

For \( i, j = 1, 2 \), define \( R_{ij}^* \triangleq R_{ij} - R_{i0} R_{0j}^{-1} R_{00} \). Let \( \|A_{m \times n}\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |A_{ij}| \) be the \( \infty \) norm of a matrix \( A \).

**Lemma 4.** Suppose that both \( -l(\theta) \) and \( -\text{El}(\theta) \) are strictly convex, satisfying (C10)-(C12), and that \( p_{\lambda,j}, j = 1, \ldots, p \) are continuous at 0. Assume there exist \( \epsilon_n^{(k)} < \zeta_1, k = 1, 2 \) such that \( n^{-\frac{k}{2}} \vee p_{1,n} = o(\epsilon_n^{(1)}) \), the \( k \)-th derivative of \( p_{\lambda,j}(\beta) \) is continuous for \( \theta \in (0, \epsilon_n^{(k)}) \cup [\zeta_1, \zeta_2] \) and exists at the limit \( \theta = 0_+ \),

\[
p'_{\lambda,j}(0_+), p''_{\lambda,j}(0_+) \in \overline{\mathbb{R}} = [-\infty, +\infty].
\]

Then, we have the following results.

1. *(Parameter estimation consistency)* If either of the following two conditions, (1.a) and (1.b), holds, then there exists a consistent local maximizer \( \hat{\theta}_n \).
   
   (1.a) \( p_{1,n} + p_{4,n} \to 0 \) as \( n \to \infty \).
   (1.b) \( p_{1,n} \to 0 \) and \( \lim_{n} p_{5,n} > 0 \).

2. *(Model selection consistency)* If any of the following three conditions, (2.a)-(2.c), holds, then any consistent local maximizer is model selection consistent.
   
   (2.a) As \( n \to \infty \), \( \min(n^{1/2}, p_{2,n}/p_{1,n}) \to \infty \) and for any \( u_n = O(p_{1,n} + 1/n) \), \( p_{2,n} = O(p_{2,n}(u_n)) \).
   (2.b) There exists \( C > 0 \) such that \( p_{2,n}/p_{1,n} \to C \) and \( \|R_{21}^* R_{11}^{-1}\|_\infty < C; \sqrt{n} p_{2,n} \to \infty \); and \( p_{3,n}(\epsilon_n^{(2)}) + p_{4,n} \to 0 \).
(2.c) For any \( u_n = O(p_{1,n} + 1/\sqrt{n}) \), \( \min(\sqrt{n} p_{2,n}(u_n), p_{2,n}(u_n)/p_{1,n}) \to \infty \).

3. (Asymptotic normality) Assume the estimator has the model selection consistency as stated in 2. If \( \sqrt{n} p_{1,n} \to 0 \), we have the asymptotic normality for the nonzero part,

\[
\sqrt{n} \Sigma_2 \left\{ \left( \hat{\alpha} - \alpha_{10} \right) \beta_{1} - \beta_{10} \right\} \to N(0, \begin{pmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{pmatrix}).
\]

Lemma 4 covers commonly used penalties including LASSO, SCAD, MCP, adaptive LASSO, hard thresholding, bridge and the LAMP family. Note that the conditions are satisfied for SCAD, and are slightly weaker than those imposed in Fan and Li (2001). It is also easy to verify that condition (2.a) is satisfied for SCAD and MCP, (2.b) for LASSO and (2.c) for bridge.

Suppose contrary to (2.a), \( \lim_{n} \sqrt{n} p_{2,n} < \infty \) and \( p_{3,n} + p_{4,n} \to 0 \). Then, the resulting estimator \( \hat{\beta}_n \) does not have the model selection consistency.

B Proofs

Proof of Lemma 1 For any \( \epsilon > 0 \), we have \( \|E_i\|/\sqrt{n} < \epsilon \), \( i = 1, 2 \). Select \( u_{i,n} \in \text{argmin} M_n(Z_n + E_i, \theta), i = 1, 2 \).

Define function \( f_i(u) = m_n(Z_n + E_i, u), i = 1, 2 \).

Throughout the proof, we use the following conventions: \( o(1) \) denotes a quantity approaching 0 as \( n \to \infty \) and \( \epsilon \to 0 \) simultaneously; \( o(1, n) \) for approaching 0 as \( n \to \infty \); \( o(1, \epsilon; n) \) for approaching 0 as \( \epsilon \to 0 \) with \( n \) fixed, and \( O(1, n) \) for a bounded sequence with \( n \to \infty \). If a subscript \( p \) is used, then the convergence (boundedness) is in probability.
First, from Condition (C2),

\[
\sup_{u \in \Theta} |f_2(u) - f_1(u)| \\
\leq \frac{1}{n} \sum_{i=1}^{n} K(Z_i)\|\epsilon_1 - \epsilon_2\| \\
\leq \frac{1}{\sqrt{n}} \left( \frac{1}{n} \sum_{i=1}^{n} K^2(Z_i)\|\mathcal{E}_1 - \mathcal{E}_2\| \right)^{1/2} \\
= O_p(1, n) o(1, \epsilon; n).
\]

Similarly we have

\[
\sup_{u \in \Theta} |m_n(Z_n, u) - f_1(u)| = O_p(1, n) o(1, \epsilon; n).
\]

Second, from Condition (C1), the \( f_i(u) \) are convex. We will show that for any \( r \in [0, 1] \), \( u_{2,n}' = ru_{1,n} + (1-r)u_{2,n} \), and \( i = 1, 2 \),

\[
f_i(u_{2,n}') \geq f_i(u_{i,n}) + o_p(1, n). \tag{10}
\]

Since \( u_{1,n} \) is a local minimum of \( f_1(u) + r_n(u) \), there exists \( \sigma_{1,n} \in (0, 1) \) such that

\[
f_1(u_{1,n}) + r_n(u_{1,n}) - r_n(u_{1,n} + \sigma_{1,n}(u_{2,n}' - u_{1,n})) \\
\leq f_1(u_{1,n} + \sigma_{1,n}(u_{2,n}' - u_{1,n})) \\
= f_1(\sigma_{1,n}u_{2,n}' + (1-\sigma_{1,n})u_{1,n}) \\
\leq \sigma_{1,n}f_1(u_{2,n}') + (1-\sigma_{1,n})f_1(u_{1,n}),
\]

where the last inequality follows from the convexity of \( f_1(\cdot) \). Therefore,

\[
f_1(u_{2,n}') \geq f_1(u_{1,n}) - \frac{r_n(u_{1,n} + \sigma_{1,n}(u_{2,n}' - u_{1,n})) - r_n(u_{1,n})}{\sigma_{1,n}\|u_{2,n}' - u_{1,n}\|} \|u_{2,n}' - u_{1,n}\|,
\]

which, in view of Condition (C3), implies (10) for \( i = 1 \). The case of \( i = 2 \) can be proved similarly.
Third, \( \text{Em}_n(Z_n, u'_{2,n}) - m_n(Z_n, u'_{2,n}) = o_p(1, n) \) from the law of large number. Note that

\[
\bar{m}(u'_{2,n}) - \bar{m}(u_{1,n}) = \left[ \text{Em}_n(Z_n, u'_{2,n}) - m_n(Z_n, u'_{2,n}) \right] + \left[ m_n(Z_n, u_{2,n}) - f_2(u_{2,n}) \right] - \left[ m_n(Z_n, u_{1,n}) - f_1(u_{1,n}) \right] + [f_2(u'_{2,n}) - f_1(u'_{2,n})]

\]

In other words, for any \( \delta > 0 \),

\[
\lim_{n \to \infty} P(\lim_{\epsilon \to 0} \bar{m}(u'_{2,n}) - \bar{m}(u_{1,n}) < -\delta) = 0.
\]

Similarly,

\[
\lim_{n \to \infty} P(\lim_{\epsilon \to 0} \bar{m}(u'_{2,n}) - \bar{m}(u_{2,n}) < -\delta) = 0.
\]

Combine these two together, we have, for any \( \delta > 0 \) and \( r \in [0, 1] \),

\[
\lim_{n \to \infty} P(\lim_{\epsilon \to 0} \bar{m}(u'_{2,n}) - [r \bar{m}(u_{1,n}) + (1 - r) \bar{m}(u_{2,n})] < -\delta) = 0. \tag{11}
\]

Define

\[
\Delta(u_1, u_2) \triangleq \max_{0 \leq r \leq 1} [r \bar{m}(u_1) + (1 - r) \bar{m}(u_2) - \bar{m}(ru_1 + (1 - r)u_2)],
\]

and

\[ C_\delta = \inf \{ C \geq 0 | \forall u_1, u_2 \in \Theta \text{ and } \|u_1 - u_2\| \geq C, \Delta(u_1, u_2) > \delta \}. \]

Since \( \Theta \) is compact, there exists \( \delta_1 > 0 \), such that \( C_\delta \) exists for any \( \delta \in (0, \delta_1] \). Since \( \bar{m} \) is strictly convex, \( C_\delta \to 0 \), as \( \delta \to 0 \), and \( C_\delta > 0 \). We thus conclude from (11) that for any \( \delta > 0 \)

\[
\lim_{n \to \infty} P(\lim_{\epsilon \to 0} \Delta(u_{1,n}, u_{2,n}) > \delta) = 0.
\]

By the definition of \( C_\delta \), for \( \delta' > \delta \),

\[
\{ \lim_{\epsilon \to 0} \|u_{1,n} - u_{2,n}\| > C_{\delta'} \} \subseteq \{ \lim_{\epsilon \to 0} \Delta(u_{1,n}, u_{2,n}) > \delta \}.
\]
Therefore, for any \( \delta' > \delta > 0 \),

\[
\lim_{n \to \infty} \mathbb{P}(\lim_{\epsilon \to 0} \|u_{1,n} - u_{2,n}\| > C_{\delta'}) = 0.
\]

Since \( \delta > 0 \) implies \( C_{\delta} > 0 \) and \( \delta \to 0 \) implies \( C_{\delta} \to 0 \), \( C_{\delta}' \) can be arbitrarily small. Thus, for any \( \eta > 0 \),

\[
\lim_{n \to \infty} \mathbb{P}(\lim_{\epsilon \to 0} \|u_{1,n} - u_{2,n}\| > \eta) = 0.
\]

\( \square \)

**Proof of Lemma 2** From Condition (C2) and similar to the proof of Lemma

\[
\sup_{r \in [0,1], \theta_i \in \Theta, i=1,2} \{[rM_n(Z_n + \mathcal{E}_n, \theta_1) + (1 - r)M_n(Z_n + \mathcal{E}_n, \theta_2) - M_n(Z_n, r\theta_1 + (1 - r)\theta_2)]
= O_p(1, n) o(1, \epsilon; n).
\]

In view of this and Condition (C5), there exists \( \epsilon_0 > 0 \), such that if \( \|\mathcal{E}_n\| \leq \sqrt{n\epsilon_0} \), then

\[
\lim_{n \to \infty} \mathbb{P}(M_n(Z_n + \mathcal{E}_n, \theta) \text{ is strictly convex within } U(\theta_0; \delta_0) \cap \Theta) = 1. \tag{12}
\]

On the other hand, (C4) and weak asymptotic stability imply

\[
\lim_{n \to \infty} \mathbb{P}(\lim_{\epsilon \to 0} \text{diam} \bigcup_{\|\mathcal{E}\| < \sqrt{n\epsilon}} \{\text{argmin}_{Z_n + \mathcal{E} \in S_{Z_n}} M_n(Z_n + \mathcal{E}, \theta)\} > \delta_0/2) = 0,
\]

\[
\lim_{n \to \infty} \mathbb{P}(\lim_{\epsilon \to 0} d(\theta_0, \bigcup_{\|\mathcal{E}\| < \sqrt{n\epsilon}} \{\text{argmin}_{Z_n + \mathcal{E} \in S_{Z_n}} M_n(Z_n + \mathcal{E}, \theta)\}) > \delta_0/2) = 0.
\]

Thus,

\[
\lim_{n \to \infty} \mathbb{P}(\lim_{\epsilon \to 0} \bigcup_{\|\mathcal{E}\| < \sqrt{n\epsilon}} \{\text{argmin}_{Z_n + \mathcal{E} \in S_{Z_n}} M_n(Z_n + \mathcal{E}, \theta)\} \subset U(\theta_0; \delta_0)) = 1. \tag{13}
\]
Denote by \( \tilde{\theta} \) and \( \theta^* \) the minimizers of \( M_n(Z_n + E_n, \theta) \) and \( M_n(Z_n, \theta) \) over \( \theta \in U(\theta_0; \delta_0) \cap \Theta \), respectively. We have
\[
0 \geq M_n(Z_n, \theta^*) - M_n(Z_n, \tilde{\theta}) = M_n(Z_n + E_n, \theta^*) - M_n(Z_n + E_n, \tilde{\theta}) + O_p(1) o(1, \epsilon; n) \geq O_p(1) o(1, \epsilon; n).
\]
Thus
\[
\lim_{\epsilon \to 0} \| M_n(Z_n, \theta^*) - M_n(Z_n, \tilde{\theta}) \| = 0,
\]
which, along with strict convexity of \( M_n(Z_n + E_n, \theta) \) as in (12), implies
\[
\lim_{\epsilon \to 0} \| \theta^* - \tilde{\theta} \| = 0.
\] (14)
Combining (13) and (14), we have
\[
\lim_{n \to \infty} P(\lim_{\epsilon \to 0} \text{diam} \bigcup_{\| E_n \| < \sqrt{n\epsilon}} \{ \text{argmin} M_n(Z_n + E_n, \theta) \} = 0) = 1.
\]

Proof of Lemma 3. First, it is obvious that \( \lambda \to 0 \) and \( \sqrt{n} \lambda \to +\infty \). Since \( \zeta_1 \) is fixed, it suffices to show
\[
\sqrt{n} \lambda \exp\left[\frac{\lambda_0}{\lambda} \zeta_1 - \alpha_1 \right] L\left(\frac{\lambda_0}{\lambda} \zeta_1 - \alpha_1 \right) \to 0.
\] (15)
From the monotonicity of \( \log(\cdot) \) in \((0, \infty)\), (15) is equivalent to
\[
\log(\sqrt{n} \lambda) + \left(\frac{\lambda_0}{\lambda} \zeta_1 - \alpha_1 \right) L\left(\frac{\lambda_0}{\lambda} \zeta_1 - \alpha_1 \right) \to -\infty.
\] (16)
Since \( \log(\sqrt{n} \lambda) \to \infty \) and \( L(\cdot) < 0 \) by assumption, we only need to show
\[
\log(\sqrt{n} \lambda) \ll \left(\frac{\lambda_0}{\lambda} \right)^u L\left(\frac{\lambda_0}{\lambda} \right).
\] (17)
Since \( L(\cdot) \) is a slowly-varying function, we have \( L(\cdot) = O(1) \) or \( L(\cdot) \to -\infty \). Thus, a sufficient condition will be \( \log(\sqrt{n} \lambda) \ll \left(\frac{\lambda_0}{\lambda} \right)^u \). Then taking \( 1/u \)-th power on both sides, we have \( \lambda_0 (\log(n))^{1/u} \ll \lambda_0 \), which can be guaranteed by the assumption \( \lambda (\log n)^{1/u} \ll \lambda_0 \).
Proof of Lemma 4. Part 1 (Parameter estimation consistency).

It suffices to show that for any \( \epsilon > 0 \), there exists \( C > 0 \), such that

\[
\lim_{n \to \infty} P[ \sup_{\|u\|=C} \bar{l}(\theta_0 + u(n^{-\frac{1}{2}} + p_{1,n})) - \bar{l}(\theta_0) < 0] > 1 - \epsilon.
\]  

(18)

This is because, under (18), there exists a local maximizer \( \hat{\theta} = (\hat{\theta}_0, \cdots, \hat{\theta}_p) \), such that \( \|\hat{\theta} - \theta_0\| = O_p(\alpha_n) \), where \( \alpha_n = n^{-\frac{1}{2}} + p_{1,n} \). Therefore, consistency holds as \( p_{1,n} \to 0 \).

To show (18), we apply the Taylor expansion to get, in view of the fact that \( p_{\lambda,j}(|\beta_{j,0}| + (n^{-\frac{1}{2}} + p_{1,n})u_j) \geq 0 = p_{\lambda,j}(|\beta_{j,0}|) \) for all \( j > q \),

\[
\begin{align*}
\bar{l}(\theta_0 + (n^{-\frac{1}{2}} + p_{1,n})u) - \bar{l}(\theta_0) & 
\leq (n^{-\frac{1}{2}} + p_{1,n})u^T l'(\theta_0) - \frac{1}{2}n u^T R u (n^{-\frac{1}{2}} + p_{1,n})^2 (1 + o_p(1)) \\
& - n \sum_{j=1}^{q} p'_{\lambda,j}(|\beta_{j,0}|) \text{sgn}(\beta_{j,0}) (n^{-\frac{1}{2}} + p_{1,n}) u_j \\
& + n \sum_{j=1}^{q} \frac{1}{2} p''_{\lambda,j}(r_j) (n^{-\frac{1}{2}} + p_{1,n})^2 u_j^2,
\end{align*}
\]

where \( |\beta_{j,0}| \leq r_j \leq |\beta_{j,0}| + (n^{-\frac{1}{2}} + p_{1,n})C \).

We now deal with the four terms on right-hand side of (19). From Condition (1.a), the first term is of order \( 1 + n^{\frac{1}{2}p_{1,n}} \) and the second term is of order \( (1 + n^{\frac{1}{2}p_{1,n}})^2 \). By the Cauchy inequality,

\[
\begin{align*}
n \sum_{j=1}^{q} p'_{\lambda,j}(|\beta_{j,0}|) \text{sgn}(\beta_{j,0}) (n^{-\frac{1}{2}} + p_{1,n}) u_j & \leq n \sqrt{q} p_{1,n} (n^{-\frac{1}{2}} + p_{1,n}) \|u\|.
\end{align*}
\]

Since \( q \) is fixed, it is also of the same or a smaller order compared to the second term. As \( p_{4,n} \to 0, \) \( p''_{\lambda,j}(\cdot) \) in the fourth term vanishes. Thus the fourth term is also controlled by the second term. Regarding the constant involving \( u \), the second term contains \( \|u\|^2 \), while both the first and third terms contain \( \|u\| \). Consequently, the whole expression is controlled by its second term as long as we choose a sufficiently large \( C \), noting that \( R \) is positive definite.

If Condition (1.b) holds, the fourth term is negative and of the same sign as the second term. Therefore, our conclusion still holds.
Part 2 (Model Selection Consistency).
First of all, from any of Condition (2.a), (2.b), and (2.c), we have for \( n \) sufficiently large, \( p_{2,n} > 0 \). Assume we have an \( \alpha_n \)-consistent local minimizer \( \hat{\theta}_n \). If the model selection consistency does not hold, then there exists a \( j \in \{q + 1, q + 2, \ldots, p\} \) such that \( \hat{\beta}_j \neq 0 \). This will result in contradiction if we can show that there exist \( \epsilon_n \ll \alpha_n \) and a neighborhood of \( \hat{\beta}_j, U(\hat{\beta}_j; \epsilon_n) \), within which the sign of \( \frac{\partial \tilde{l}(\theta)}{\partial \beta_j} \) does not change. Since \( \hat{\beta}_j \) is a solution, the sign of left derivative should be different from the sign of right derivative at this value. So the non-zero \( \hat{\beta}_j \) does not exist. Taking the Taylor expansion, we get

\[
\frac{\partial \tilde{l}(\theta)}{\partial \beta_j} = \frac{\partial \tilde{l}(\theta_0)}{\partial \beta_j} + \sum_{l=1}^{p} \frac{\partial^2 \tilde{l}(\beta_0)}{\partial \beta_l \partial \beta_j} (\beta_l - \beta_{l,0})(1 + o_p(1)) - np_{\lambda,j}(|\beta_j|)\text{sgn}(\beta_j),
\]  

for \( j = q + 1, q + 2, \ldots, p \). We see that the third term on the right-hand side of (20) does not depend on \( \beta_0 \) and is only related to the sign of \( \beta_j \), which remains constant in \( U(\hat{\beta}_j; \epsilon_n) \) since \( \hat{\beta}_j \neq 0 \). So if the first two terms are controlled by the third one, we can derive the sparsity using the method above. The coefficient of the sign function in the third term should be positive to control the direction of the derivative. That is, \( \inf_{q<j\leq p} p'_{\lambda,j}(0) > 0 \).

Under Condition (2.a), the orders of the three terms in (20) are \( \sqrt{n}, \sqrt{n} + np_{1,n}, p_{2,n}(u_n) \), where \( u_n \) is the sequence given in Condition (2.a). Condition (2.a) guarantees that the first two terms are controlled by the third one. Likewise, Condition (2.c) leads to the same conclusion.

Define

\[
h_n(v) \triangleq \left\{ \frac{1}{2} v^T R v + \sum_{j=1}^{q} \text{sgn}(\beta_{j,0}) \frac{p'_{\lambda,j}(|\beta_{j,0}|)}{p_{1,n}} v_j + \sum_{j=q+1}^{p} \frac{p'_{\lambda,j}(0)}{p_{1,n}} |v_j| \right\},
\]

and

\[
v_n = \frac{\hat{\theta} - \theta_0}{p_{1,n}} = \arg \max_v \tilde{l}(\theta_0 + p_{1,n} v).
\]

Since \( p_{3,n}(\epsilon_n^{(2)}) \rightarrow 0 \), we have \( \sup_{q<j\leq p} |p'_{\lambda,j}(0)| < \infty \) and, for any \( u_n \rightarrow 0 \),

\[
p_{2,n}(|u_n|) = p_{2,n} + o(1).
\]
The above results, together with \( p_{4,n} \to 0 \), imply that

\[
\tilde{l}(\theta_0 + p_{1,n}v) = \tilde{l}(\theta_0) + p_{1,n}v^T \frac{\partial l(\theta_0)}{\partial \theta} + \frac{1}{2} p_{1,n}^2 v^T \frac{\partial^2 l(\theta_0)}{\partial \theta \partial \theta^T} v(1 + o_p(1)) \\
- np_{1,n}(1 + o(1)) \sum_{j=1}^p \| \text{sgn}(\beta_{j,0}) v_j \| p'_{\lambda,j}(|\beta_{j,0}|) I_{\beta_{j,0} \neq 0} + \frac{p'_{\lambda,j}(|\beta_{j,0}|)}{p_{1,n}} |v_j| I_{\beta_{j,0} = 0}
\]

\[
= \tilde{l}(\theta_0) + np_{1,n}^2 \left( - \frac{1}{2} v^T \mathbb{R} v - \sum_{j=1}^q \| \text{sgn}(\beta_{j,0}) \| p'_{\lambda,j}(|\beta_{j,0}|) \| v_j \| \right) (1 + o_p(1)).
\]

As \( v_n \) is an maximizer given by (22), there exists \( \alpha_n \to 0 \), such that for any given \( \epsilon > 0 \), there exists a constant \( C > 0 \) such that

\[
P( \sup_{||v|| \leq C} \tilde{l}(\theta_0 + p_{1,n}(v_n + \alpha_n v)) - \tilde{l}(\theta_0 + p_{1,n}(v_n)) \leq 0) > 1 - \epsilon.
\]

From

\[
h_n(v_n + \alpha_n v) - h_n(v_n) = -(np_{1,n}^2)^{-1} \tilde{l}(\theta_0 + p_{1,n}(v_n + \alpha_n v)) - \tilde{l}(\theta_0 + p_{1,n}(v_n))(1 + o_p(1)),
\]

we get

\[
P( \sup_{||v|| \leq C} h_n(v_n + \alpha_n v) - h_n(v_n) \geq 0) > 1 - \epsilon.
\]

Thus we have

\[
\frac{\hat{\theta} - \theta_0}{p_{1,n}} - \arg \min_v h_n(v) \overset{p}{\to} 0. \tag{24}
\]

Now, we investigate the minimizer of \( h_n(v) \). Let \( v_n \overset{\Delta}{=} (v_{0,n}, v_{1,n}^T, v_{2,n}^T)^T \) where \( v_{j,n}^T \) is a \( q \times 1 \) vector. We see \( v_{2,n} \to v_2 = 0 \) in probability is a necessary condition for sparsity. From conclusions above,

\[
(v_{0,n}, v_{1,n}^T, v_{2,n}^T) - \arg \min_{v_{0},v_{1},v_{2}} \frac{1}{2}(v_{0}^2 \mathbb{R}_{00} + 2v_{0}\mathbb{R}_{01} v_{1} + 2v_{0}\mathbb{R}_{02} v_{2}) \tag{25}
+ v_{1}^T \mathbb{R}_{11} v_{1} + 2v_{1}^T \mathbb{R}_{12} v_{2} + v_{2}^T \mathbb{R}_{22} v_{2} + v_{1}^T \Lambda_1 a + |v_{2}| \Lambda_2 1 \overset{p}{\to} 0,
\]

where

\[
\Lambda_1 = p_{1,n}^{-1} \text{diag}\{p'_{\lambda,1}(|\beta_{10}|), p'_{\lambda,2}(|\beta_{20}|), \ldots, p'_{\lambda,q}(|\beta_{q0}|)\},
\]

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\( \Lambda_2 = p_{1,n}^{-1} \text{diag}\{ p'_{\lambda,q+1}(0), p'_{\lambda,q+2}(0), \ldots, p'_{\lambda,p}(0) \} \),

\( a = \text{sgn}(\beta_{1,0}) \) and \(|z| \leq 1 \),

KKT conditions lead to the following equations.

\[
R_{00} v_0 + R_{01} v_1 + R_{02} v_2 = 0, \\
R_{10} v_0 + R_{11} v_1 + R_{12} v_2 + \Lambda_1 a = 0, \\
R_{20} v_0 + R_{21} v_1 + R_{22} v_2 + \Lambda_2 z = 0,
\]

or

\[
R^*_{11} v_1 + R^*_{12} v_2 + \Lambda_1 a = 0, \\
R^*_{21} v_1 + R^*_{22} v_2 + \Lambda_2 z = 0,
\]

where \( z = (z_1, \ldots, z_{p-q})^T \) and

\[
z_j \in \{ z \mid \text{if } v_{q+j} \neq 0, z = \text{sgn}(v_{q+j}); \text{ else } |z| \leq 1 \}. 
\]

Therefore

\[
v_2 = 0 \iff R^*_{21} R^*_{11}^{-1} \Lambda_1 a + \Lambda_2 z = 0, |z| \leq 1 \\
\iff |\Lambda_2^{-1} R^*_{21} R^*_{11}^{-1} \Lambda_1 a| \leq 1,
\]

which is a necessary condition for sparsity. Now let

\[
\Lambda_2(v_2) \triangleq p_{1,n}^{-1} \text{diag}\{ p'_{\lambda,q+1}(v_{q+1}), p'_{\lambda,q+2}(v_{q+2}), \ldots, p'_{\lambda,p}(v_p) \}. 
\]

Then

\[
\frac{\partial \tilde{l}(\theta)}{\partial \beta_2} = \frac{\partial \tilde{l}(\theta_0)}{\partial \beta_2} + n p_{1,n} [(R_{20}, R_{21}, R_{22})(v_0, v_1^T, v_2^T)^T \\
(1 + o_p(1)) - \Lambda_2(p_{1,n} v_2) \text{sgn}(\beta_2)].
\]

From (23), \( \Lambda_2(p_{1,n} v_2) = \Lambda_2 + o(1) \). So with \( \sqrt{n p_{1,n}} \to \infty \) a sufficient condition for sparsity is that \( (R_{20}, R_{21}, R_{22})(v_0, v_1^T, v_2^T)^T \) is controlled by \( \Lambda_2 \) coordinatewisely. That is,

\[
R_{20} v_0 + R_{21} v_1 + R_{22} v_2 + \Lambda_2 z = 0, |z| < 1,
\]

which is similar to the last equation of (26) and equivalent to \( R^*_{21} R^*_{11}^{-1} \Lambda_1 a + \Lambda_2 z = 0, |z| < 1 \), which is implied by \(|\Lambda_2^{-1} R^*_{21} R^*_{11}^{-1} \Lambda_1 a| < 1 \). Furthermore, it can be guaranteed by

\[
\| R^*_{21} R^*_{11}^{-1} \|_\infty < (|\Lambda_2^{-1}|_\infty \| \Lambda_1 \|_\infty )^{-1} \to C. \quad (28)
\]

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From (27), it follows that the first two terms are controlled by the third one.
Part 3 (Asymptotic normality and oracle property).
Suppose \( \hat{\theta} = (\hat{\alpha}, \hat{\beta}_{1}^{T}, \mathbf{0}_{p-q}^{T})^{T} \) is the local minimizer, noting that \( \hat{\beta}_{2} = \mathbf{0}_{p-q} \) with probability tending to 1. Using parameter estimation consistency and model selection consistency property, for \( j = 0, 1, 2, \ldots, q \),

\[
0 = \frac{\partial l(\hat{\theta})}{\partial \theta_{j}} = \frac{\partial l(\hat{\theta})}{\partial \theta_{j}} - np_{\lambda,j}(|\hat{\beta}_{j}|)\text{sgn}(|\hat{\beta}_{j}|)I_{j \neq 0}
\]

\[
= \frac{\partial l(\theta_{0})}{\partial \theta_{j}} + \sum_{i=0}^{q} \left( \frac{\partial^{2} l(\theta_{0})}{\partial \theta_{j} \partial \theta_{i}} \right)(\theta_{i} - \theta_{i,0})(1 + o_{p}(1))
\]

\[- I_{j \neq 0}[np_{\lambda,j}(|\beta_{j,0}|)\text{sgn}(\beta_{j,0}) + np_{\lambda,j}(|\hat{\beta}_{j}|)(\hat{\beta}_{j} - \beta_{j,0})(1 + o_{p}(1))].
\]

Thus

\[
\sqrt{n} \Sigma_{2} \left[ \left( \frac{\hat{\alpha}}{\beta_{1}} - \alpha_{10} \right) + \Sigma_{2}^{-1} b \right] \longrightarrow N \left[ 0, \begin{pmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{pmatrix} \right].
\]

If the order of \( b \) is controlled by \( 1/\sqrt{n} \), the oracle property holds.

\[\square\]

**Proof of Theorem 1.** Using the notations of Lemma 4 and the form of the LAMP penalty, \( p_{1,n} = \lambda g'(\alpha_{1} - \frac{\lambda}{\lambda} \xi_{1})/g'(\alpha_{1}) \); \( p_{2,n} = \lambda; p_{3,n} = \lambda_{0} g''(\alpha_{1})/|g'(\alpha_{1})|; p_{4,n} = \lambda_{0} g''(\alpha_{1} - \frac{\lambda}{\lambda} \xi_{1})/|g'(\alpha_{1})| \).

From (C6), (C8), \( \lambda_{0}/\lambda \to \infty \), together with \( \lim_{x \to \infty} g'(x) = 0 \), \( \alpha_{1} \) is a constant and \( \lambda \to 0 \), we have

\[
\sup_{x \geq 0} p'_{\lambda}(x) = \sup_{x \geq 0} \lambda \frac{g'(\alpha_{1} - \frac{\lambda}{\lambda} x)}{g'(\alpha_{1})} \to 0.
\]

(C3) holds, and \( p_{1,n} \to 0, p_{4,n} \to 0 \).
From (C6)

\[
E \left[ \|X\|_2^2 g''(X^T \theta_0) + \|X\|_1^3 \sup_{\|\theta - \theta_0\| \leq \delta} g''(X^T \theta) \right] < \infty,
\]

which leads to (C10)-(C12) in Lemma 4. From \(p_{1,n} \to 0, p_{4,n} \to 0\), Condition (1.a) in Lemma 4 holds. (C4) holds.

Given \(u_n = O(p_{1,n} + \sqrt{p_{1,n}})\), from \(\sqrt{n}p_{1,n} \to 0\), we get \(u_n = O(1/\sqrt{n})\). Together with the smoothness of the function \(g\), \(p_{2,n}(u_n)/p_{2,n} \to 1\). From Condition (C8), it is obvious that \(p_{2,n} > 0, p_{2,n}/p_{1,n} \to \infty, \sqrt{n}p_{2,n} \to \infty\).

Condition (2.a) and conditions for asymptotic normality in Lemma 4 hold. Lastly, (C5) is implied by (C9).

In conclusion, with (C6)-(C9), the penalized maximum likelihood estimator based on the LAMP family is consistent and asymptotically normal, and achieves model selection consistency and strong asymptotic stability. □

**Proof of Theorem 2.** The idea of the proof is adapted from Munk et al. (2006) and Zou and Li (2008). For convenience, we define the following notations.

\[
M_n(\theta) \doteq M(1)(\theta) + M(2)(\theta),
\]

\[
M(1)(\theta) \doteq -\frac{1}{n} l(\theta), M(2)(\theta) \doteq \sum_{j=1}^{p} p_{\lambda}(|\theta_j|),
\]

\[
\theta^{(m)} \doteq (\theta_0^{(m_0)}, \theta_1^{(m_1)}, \ldots, \theta_p^{(m_p)})^T,
\]

\[
\theta^{(m)}_{(-j)}(x) \doteq (\theta_0^{(m_0)}, \theta_1^{(m_1)}, \ldots, \theta_{j-1}^{(m_{j-1})}, x, \theta_{j+1}^{(m_{j+1})}, \theta_p^{(m_p)})^T,
\]

\[
Q(x,j,\theta^{(m)}) \doteq a(\theta^{(m)}_{(-j)}(x), \theta^{(m)}),
\]

\[
\phi(x,j,\theta^{(m)}) \doteq \sum_{1 \leq k \leq p} p_{\lambda}(|\theta_k^{(m_k)}|) + \lambda^2 p_{\lambda}(|\theta_j^{(m_j)}|)(|x| - |\theta_j^{(m_j)}|),
\]

\[
R(x,j,\theta^{(m)}) \doteq Q(x,j,\theta^{(m)}) + \phi(x,j,\theta^{(m)}),
\]

where \(m_0, m_1, \ldots, m_p\) are the iteration times of \(\theta_0, \theta_1, \ldots, \theta_p\) respectively and \(m = (m_0, m_1, \ldots, m_p)\). From the concavity of the penalty on the positive part, we have

\[
M(2)(\theta^{(m+e)}) \leq \phi(\theta_j^{(m_j+1)}, j, \theta^{(m)}),
\]

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where \( e_j \) is a \((p + 1)\)-vector with the \( j \)-th element 1 and all the others 0; from conditions of the theorem, we see
\[
M(1)(\theta^{(m+e_j)}) \leq Q(\theta^{(m_j)}, j, \theta^{(m)}),
\]
and
\[
M(\theta^{(m+e_j)}) \leq R(\theta^{(m_j+1)}, j, \theta^{(m)}).
\]
In the algorithm,
\[
\theta^{(m_j+1)} = \arg \min_\theta R(\theta, j, \theta^{(m)}).
\]
Then
\[
M(\theta^{(m)}) = R(\theta^{(m)}, j, \theta^{(m)}) > R(\theta^{(m_j+1)}, j, \theta^{(m)}) \geq M(\theta^{(m+e_j)}).
\]
Since \( M \) decreases as iteration continues and has an lower bound, it converges.

By monotonicity of \( M(\theta^{(t)}) \) \((t \) represents the number of iterations), all points \( \theta^{(m)} \) are in a compact set
\[
\{ \theta \in C | M(\theta) \leq M(\theta^{(0_{p+1})}) \}.
\]
It is compact because \( M(\theta) \) is continuous and coercive. Then there exists a convergent subsequence \( \theta^{(t_i)}, \theta^* \in C, j_0 \in \{0, 1, \cdots, p\} \) such that
\[
\lim_{l \to \infty} \theta^{(t_i)} = \theta^*; j^{(t_i+1)} = j_0.
\]
Next let \( m_t^* \triangleq m_{j_0}^{(t+1)} \). For any \( v \in C \), we have:
\[
M(\theta^{(t+1)}) \leq M(\theta^{(t_i+1)}) \leq R(\theta_{j_0}^{(m_t^*)}, j_0|\theta^{(t_i)}) \leq R(v_{j_0}, j_0|\theta^{(t_i)}).
\]
Assume \( M(\theta^{(t+1)}) \to M(\theta^*) = R(\theta^*, j|\theta^*), j = 0, \cdots, p \). Taking limit \( l \to \infty \) on both sides of the equation above, we have
\[
M(\theta^*) \leq \lim_{l \to \infty} R(v_{j_0}, j_0|\theta^{(t_i)}) = R(v_{j_0}, j_0|\theta^*).
\]
Thus the subgradient of \( R(\cdot, j_0|\theta^*) \) at \( \theta^* \) is 0, which is exactly the derivative of \( M(\theta) \) with respect to \( \Theta_{j_0} \) at \( \theta^* \), because it can be easily verified that the smooth approximation keeps the first-order derivative the same.
From the algorithm and the definition of the “viol” function,

\[ \forall j \in \{0, \cdots, p\}, |\hat{R}(\cdot, j|\theta^{(t_i)})| \leq |\hat{R}(\cdot, j_0|\theta^{(t_i)})|. \]

Taking the derivative of both sides of the equation above, the subgradient of \( R(\cdot, j|\theta^*) \) at \( \theta^* \) is 0, which is exactly the same as the partial derivative of \( M(\theta) \) with respect to \( \Theta_j \), \( \forall j \in \{0, \cdots, p\} \). From strict convexity, \( \theta^* \) is the unique local minimum of \( M(\theta) \).

Method 1 for logistic regression uses Lemma 1 in Munk et al. (2006). That is,

\[
\log(1 + e^z) \leq \frac{z}{2} + \log(e^{-z_0/2} + e^{z_0/2}) + \frac{1}{2} \tanh(0.5z_0)(z_0^2 - z^2).
\]

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