Regularity criteria and Liouville theorem for 3D inhomogeneous Navier–Stokes flows with vacuum

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Abstract. In this paper, we investigate the 3D inhomogeneous Navier–Stokes flows with vacuum, and obtain regularity criteria and Liouville type theorems in the Lorentz space if a smooth solution \((\rho, u)\) satisfies suitable conditions.

Mathematics Subject Classification. 35Q35, 76D03.

Keywords. Inhomogeneous Navier–Stokes flows, Regularity criteria, Liouville theorem, Lorentz space.

1. Introduction. We consider the existence of solutions \((\rho, u) : Q_T \to \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \) to the inhomogeneous Navier–Stokes flows

\[
\begin{aligned}
\partial_t \rho + u \cdot \nabla \rho &= 0, \\
\rho u_t - \Delta u + \rho(u \cdot \nabla)u + \nabla \pi &= 0, \\
\text{div } u &= 0,
\end{aligned}
\tag{1.1}
\]

Here \(\rho\) is the density function, \(u\) is the flow velocity vector, and \(\pi\) is the pressure function. We consider the initial value problem of (1.1) with the initial data

\[
\begin{aligned}
\rho(x, 0) = \rho_0(x) \quad \text{and} \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^3.
\end{aligned}
\tag{1.2}
\]

Kazhikov [9] proved that the inhomogeneous Navier–Stokes equations (1.1)–(1.2) have at least one global weak solution in the energy space for smooth data with no vacuum. After that, Ladyzhenskaya and Solonnikov [11] established the unique solvability for the system with smooth initial data that has no vacuum, in short, the global well-posedness in two dimensions and local well-posedness in three dimensions is established. Moreover, if the initial data is small enough, then global well-posedness is true for a smooth solution to (1.1)–(1.2). Simon [19] constructed global weak solutions to the system with finite energy with the statues containing vacuum (see e.g. Danchin [5,6] for
the almost critical Sobolev spaces and Abidi et al. [1] for axi-symmetric initial data and Mucha et al. [16] and the references therein).

On the other hands, regarding the regularity criteria for the system (1.1)–(1.2), Kim [10] established the following Serrin type condition

\[ u \in L^t(0,T; L^{s,\infty}(\mathbb{R}^3)), ~ \frac{3}{s} + \frac{2}{t} = 1, ~ s \in (3, \infty), \]

in the framework of Lorentz space (see e.g. Ye and Zhang [22], Sun and Qian [20] for Besov space). We recall the definition and property of Lorentz spaces. Let \( m(\varphi, t) \) be the Lebesgue measure of the set \( \{ x \in \mathbb{R}^3 : |\varphi(x)| > t \} \), i.e.,

\[ m(\varphi, t) := m \{ x \in \mathbb{R}^3 : |\varphi(x)| > t \}. \]

We denote by \( L^{p,q}(\mathbb{R}^3) \) the Lorentz space with \( 1 \leq p, q \leq \infty \) with the norm

\[
\| \varphi \|_{L^{p,q}(\mathbb{R}^3)} = \begin{cases} 
\left( \int_0^\infty t^{q} \left( m(\varphi, t) \right)^{q/p} \frac{dt}{t} \right)^{1/q} < \infty & \text{for } 1 \leq q < \infty, \\
\sup_{t \geq 0} \left\{ t \left( m(\varphi, t) \right)^{1/p} \right\} < \infty & \text{for } q = \infty.
\end{cases}
\]

Note that \( L^r(\mathbb{R}^3) = L^{r,r}(\mathbb{R}^3) \subset L^{r,q}(\mathbb{R}^3) \) for \( 1 < r < q \leq +\infty \).

Since Kim’s work, recently, Bosia et al. [2] established the following weak-L^p Prodi–Serrin type regularity criterion for the standard Navier–Stokes equations, (i.e., \( \rho \equiv 0 \))

\[ u \in L^{t,\infty}(0,T; L^{s,\infty}(\mathbb{R}^3)), ~ \frac{3}{s} + \frac{2}{t} = 1, ~ s \in (3, \infty). \]

In this direction, first result is stated as

**Theorem 1.1.** Let \((\rho, u)\) be the unique local strong solution in the time interval \([0,T)\) to the system (1.1)–(1.2) and \(\rho \geq 0\). Then there exists a positive constant \(\varepsilon\) such that \((\rho, u)\) is a regular solution on \([0,T]\) provided that one of the following two conditions holds

(A) \( u \in L^{q,\infty}(0,T; L^{p,\infty}(\mathbb{R}^3)) \) and

\[ \| u \|_{L^{q,\infty}(0,T; L^{p,\infty}(\mathbb{R}^3))} \leq \varepsilon, \quad \text{with} \quad \frac{3}{p} + \frac{2}{q} = 1, \quad 3 < p < \infty; \]

(B) \[ \| \nabla u \|_{L^{q}(0,T; L^{p,\infty}(\mathbb{R}^3))} < \infty, \quad \text{with} \quad \frac{3}{p} + \frac{2}{q} = 2, \quad \frac{3}{2} < p < \infty. \]

**Corollary 1.1.** Let \((\rho, u)\) be the unique local strong solution in the time interval \([0,T)\) to the system (1.1)–(1.2) and \(\rho \geq 0\). Suppose that \(u\) satisfies the following conditions

\[ \| u \|_{L^{q}(0,T; L^{p,\infty}(\mathbb{R}^3))} < \infty, \quad \text{with} \quad \frac{3}{p} + \frac{2}{q} = 1, \quad 3 < p < \infty, \]

then the solution pair \((\rho, u)\) can be extended beyond the time \(T > 0\).
Remark 1.1. Theorem 1.1 and Corollary 1.1 seem to hold for bounded domains with smooth boundary under Dirichlet or slip type boundary conditions.

On the other hand, regarding Liouville type theorems for the stationary fluid flows contains the density function, in particular, compressible Navier–Stokes equations, Chae [3] showed if the smooth solution \((\rho, u)\) satisfies \(\|\rho\|_{L^\infty(\mathbb{R}^3)} + \|\nabla u\|_{L^2(\mathbb{R}^3)} + \|u\|_{L^{3/2}(\mathbb{R}^3)} < \infty\), then \(u \equiv 0\) and \(\rho = \text{constant}\) (see Li and Yu [14]). In the Lorentz framework, very recently, Li and Niu [12] proved \((\rho, u)\) to be a smooth solution to the stationary density-dependent Navier–Stokes equations, Chae [3] showed if the smooth solution \((\rho, u)\) is fully predictable and thus in this paper, we just give a sketch of proof.

Theorem 1.2. Let \((\rho, u)\) be a smooth solution to the following stationary density-dependent Navier–Stokes equations

\[
\begin{aligned}
& u \cdot \nabla \rho = 0, \\
& -\Delta u + \rho(u \cdot \nabla)u + \nabla \pi = 0, \\
& \text{div } u = 0.
\end{aligned}
\]  

(1.3)

If \(\rho \in L^\infty(\mathbb{R}^3), \nabla u \in L^2(\mathbb{R}^3), \) and \(u \in L^{2,q}(\mathbb{R}^3)\) with \(\frac{9}{2} \leq q < \infty\), then we have \(u \equiv 0\) in \(\mathbb{R}^3\).

Remark 1.2. In the light of the proof in Jarrín [8], the result in Theorem 1.2 is fully predictable and thus in this paper, we just give a sketch of proof.

Remark 1.3. We obtained Theorem 1.2 independently of the results of Liu and Liu [13].

2. Preliminaries. In this section, we introduce the definition of functional space and its properties. For \(1 \leq q \leq \infty\), we denote the usual Sobolev spaces by \(W^{k,q}(\mathbb{R}^3) = \{ u \in L^q(\mathbb{R}^3) : D^p u \in L^q(\mathbb{R}^3), 0 \leq |p| \leq k \}\). When \(q = 2\), we write \(W^{k,2}(\mathbb{R}^3)\) as \(H^k(\mathbb{R}^3)\). We review some useful estimates for the proof of the theorems.

We need the Hölder inequality in Lorentz spaces (see [17]).

Lemma 2.1. Assume \(1 \leq p_1, p_2 \leq \infty, 1 \leq q_1, q_2 \leq \infty, \) and \(u \in L^{p_1,q_1}(\Omega), v \in L^{p_2,q_2}(\Omega)\). Then \(uv \in L^{p_3,q_3}(\Omega)\) with \(\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}\) and \(\frac{1}{q_3} \leq \frac{1}{q_1} + \frac{1}{q_2}\), and the inequality

\[
\|uv\|_{L^{p_3,q_3}(\Omega)} \leq C\|u\|_{L^{p_1,q_1}(\Omega)}\|v\|_{L^{p_2,q_2}(\Omega)}
\]

is valid.

Recall the following useful Gronwall lemma required in our proof, which is first shown by [2] (see e.g. [15,18]).
**Lemma 2.2.** Let $\phi$ be a measurable positive function defined on the interval $[0, T]$. Suppose that there exists $\kappa_0 > 0$ such that for all $0 < \kappa < \kappa_0$ and a.e. $t \in [0, T]$, $\phi$ satisfies the inequality

$$ \frac{d}{dt} \phi \leq \mu \lambda^{1-\kappa} \phi^{1+2\kappa}, $$

where $0 < \lambda \in L^{1,\infty}(0, T)$ and $\mu > 0$ with $\mu \|\lambda\|_{L^{1,\infty}(0, T)} < \frac{1}{2}$. Then $\phi$ is bounded on $[0, T]$.

**3. Proof of the theorems.**

**Proof of Theorem 1.1.** By the $L^2$-energy estimate, we know

$$ \frac{d}{dt} \int_{\mathbb{R}^3} \rho \|u(\cdot, s)\|_{L^2}^2 \, ds \leq 0, $$

which implies

$$ u \in L^\infty(0, T; L^2(\mathbb{R}^3)). \quad (3.1) $$

**Proof of (A).** Taking $u_t$ to the first equation in (1.1), testing by $u_t$, and integrating over the whole space, we easily derive

$$ \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^3} \rho |u_t|^2 \, dx \leq \int_{\mathbb{R}^3} |\rho^{1/2} u \cdot \nabla u \cdot \rho^{1/2} u_t| \, dx := \mathcal{I}. \quad (3.2) $$

Applying the Hölder inequality and Lemma 2.1, we obtain

$$ \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^3} \rho |u_t|^2 \, dx \leq C \|u\|_{L^{p, \infty}(\mathbb{R}^3)} \|\nabla u\|_{L^{\frac{2p}{p-2}, \infty}(\mathbb{R}^3)} \|\sqrt{\rho} u_t\|_{L^{2, \infty}(\mathbb{R}^3)} $$

$$ \leq C \|u\|_{L^{p, \infty}(\mathbb{R}^3)} \|\nabla u\|_{L^{\frac{2p}{p-2}, \infty}(\mathbb{R}^3)}^2 + \varepsilon \|\sqrt{\rho} u_t\|_{L^{2, \infty}(\mathbb{R}^3)}^2 $$

$$ \leq C \|u\|_{L^{p, \infty}(\mathbb{R}^3)} \|\nabla u\|_{L^{2, \infty}(\mathbb{R}^3)}^{2(1-\frac{2}{p})} \|\nabla^2 u\|_{L^{2, \infty}(\mathbb{R}^3)}^\frac{6}{p} + \varepsilon \|\sqrt{\rho} u_t\|_{L^{2, \infty}(\mathbb{R}^3)}^2 $$

$$ \leq C \|u\|_{L^{p, \infty}(\mathbb{R}^3)} \|\nabla u\|_{L^{2, \infty}(\mathbb{R}^3)}^2 + \varepsilon (\|\nabla^2 u\|_{L^{2, \infty}(\mathbb{R}^3)}^2 + \|\sqrt{\rho} u_t\|_{L^{2, \infty}(\mathbb{R}^3)}^2). \quad (3.3) $$

Here, $\varepsilon > 0$ is later determined. By using the classical regularity theory to the second equation in (1.1) and Lemma 2.1, it immediately follows that

$$ \|\nabla^2 u\|_{L^{2, \infty}(\mathbb{R}^3)}^2 \leq C (\|\sqrt{\rho} u_t\|_{L^{2, \infty}(\mathbb{R}^3)}^2 + \|\rho \cdot \nabla u\|_{L^{2, \infty}(\mathbb{R}^3)}^2) $$

$$ \leq C (\|\sqrt{\rho} u_t\|_{L^{2, \infty}(\mathbb{R}^3)}^2 + \|\rho\|_{L^{2, \infty}(\mathbb{R}^3)} \|u\|_{L^{p, \infty}(\mathbb{R}^3)} \|\nabla u\|_{L^{\frac{2p}{p-2}, \infty}(\mathbb{R}^3)}^2) $$

$$ \leq C (\|\sqrt{\rho} u_t\|_{L^{2, \infty}(\mathbb{R}^3)}^2 + C \|u\|_{L^{p, \infty}(\mathbb{R}^3)} \|\nabla u\|_{L^{2, \infty}(\mathbb{R}^3)}^2 + \frac{1}{16} \|\sqrt{\rho} u_t\|_{L^{2, \infty}(\mathbb{R}^3)}^2). \quad (3.4) $$

Inserting (3.4) into (3.3) and choosing $\varepsilon > 0$ such that $C \varepsilon < \frac{1}{8}$, we see that

$$ \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^{2, \infty}(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \rho |u_t|^2 \, dx \leq C \|u\|_{L^{p, \infty}(\mathbb{R}^3)} \|\nabla u\|_{L^{2, \infty}(\mathbb{R}^3)}. $$

For $\kappa > 0$, let $q_\kappa = q + \kappa(4-s)$ and $p_\kappa$ be such that $(p_\varepsilon, q_\kappa)$ satisfies $\frac{3}{p_\varepsilon} + \frac{2}{q_\kappa} = 1$. Through applying interpolation, we have

$$ \|u\|_{L^{p_\kappa, \infty}(\mathbb{R}^3)}^q \leq \|u\|_{L^{p_\kappa, \infty}(\mathbb{R}^3)}^{q(1-\kappa)} \|u\|_{L^{5, \infty}(\mathbb{R}^3)} \leq C \|u\|_{L^{p_\kappa, \infty}(\mathbb{R}^3)}^{q(1-\kappa)} \|\nabla u\|_{L^2(\mathbb{R}^3)}. $$
Then it yields that
\[
\frac{d}{dt} \| \nabla u \|^2_{L^2(\mathbb{R}^3)} \leq C \| u \|^q(1-\kappa)_{L^p,\infty(\mathbb{R}^3)} \| \nabla u \|^{2(1+2\kappa)}_{L^2(\mathbb{R}^3)}.
\]

Now, invoke Lemma 2.2 with \( \phi = \| \nabla u \|_{L^2(\mathbb{R}^3)} \) (see [15, p.6] for a detailed proof) and therefore, the proof is complete.

**Proof of (B).** This proof is almost the same as that of part (A). Indeed, from the estimate (3.2), we know
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u \|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \rho |u_t|^2 \, dx \leq \int_{\mathbb{R}^3} |\rho^{1/2}u \cdot \nabla u \cdot \rho^{1/2}u_t| \, dx := I.
\]

Applying the Hölder inequality and Lemma 2.1, we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u \|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \rho |u_t|^2 \, dx \leq C \| \nabla u \|_{L^p,\infty(\mathbb{R}^3)} \| u \|^{2p}_{L^{2p/3}(\mathbb{R}^3)} \| \sqrt{\rho}u_t \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq C \| \nabla u \|_{L^p,\infty(\mathbb{R}^3)} \| \nabla u \|^{2(2-\frac{2}{3})}_{L^2(\mathbb{R}^3)} \| \nabla^2 u \|^{2(\frac{2}{3} - 1)}_{L^2(\mathbb{R}^3)} + \varepsilon \| \sqrt{\rho}u_t \|^2_{L^2(\mathbb{R}^3)}
\]
\[
\leq C \| u \|^{2p}_{L^{2p/3}(\mathbb{R}^3)} \| \nabla u \|^2_{L^2(\mathbb{R}^3)} + \varepsilon (\| \nabla^2 u \|^2_{L^2(\mathbb{R}^3)} + \| \sqrt{\rho}u_t \|^2_{L^2(\mathbb{R}^3)}).
\]

By using the classical regularity theory, it immediately follows that
\[
\| \nabla^2 u \|^2_{L^2(\mathbb{R}^3)} \leq C (\| \sqrt{\rho}u_t \|^2_{L^2(\mathbb{R}^3)} + \| pu \cdot \nabla u \|^2_{L^2(\mathbb{R}^3)})
\]
\[
\leq C (\| \sqrt{\rho}u_t \|^2_{L^2(\mathbb{R}^3)} + C \| u \|^{2p}_{L^{2p/3}(\mathbb{R}^3)} \| \nabla u \|^2_{L^2(\mathbb{R}^3)} + \frac{1}{16} \| \nabla^2 u \|^2_{L^2(\mathbb{R}^3)}).
\]

Inserting (3.6) into (3.5), we see that
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u \|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \rho |u_t|^2 \, dx \leq C \| u \|^{2p}_{L^{2p/3}(\mathbb{R}^3)} \| \nabla u \|^2_{L^2(\mathbb{R}^3)}.
\]

By Gronwall’s inequality, we obtain the desired result, and thus the proof is finally complete. \( \square \)

Next, to obtain the Liouville type result, we start by introducing the test functions \( \varphi_R \) and \( \omega_R \) as follows: for a fixed \( R > 1 \), we define first the function \( \varphi_R \in C^0(\mathbb{R}^3) \) by \( 0 \leq \varphi_R \leq 1 \) such that for \( |x| \leq \frac{R}{2} \), we have \( \varphi_R(x) = 1 \), for \( |x| \geq R \), we have \( \varphi_R(x) = 0 \), and
\[
\| \nabla \varphi_R \|_{L^\infty} \leq \frac{c}{R}.
\]

From [7, Lemma III. 3.1], we recall that

**Lemma 3.1.** Let \( \varphi_R(x) \) be defined as above, then there exist a vector-valued function \( \omega_R \) and a constant \( C(p) \) such that
\[
\text{div}(\omega_R) = \nabla \varphi_R \cdot u \quad \text{over} \ B_R, \quad \text{and} \quad \omega_R = 0 \quad \text{over} \ \partial B_R \cup \partial B_{\frac{R}{2}}.
\]

where \( \partial B_r := \{ x \in \mathbb{R}^3 : |x| = r \} \) and \( \| \omega_R \|_{W^{1,p}(B_r)} \) with \( \| \nabla \omega_R \|_{L^p} \leq C(p) \| \nabla \varphi_R \cdot u \|_{L^p} \) for \( 1 < p < +\infty \).
For the second result, we recall the following Caccioppoli type estimate:

**Proposition 3.1.** Let $C(R/2, R) = \{x \in \mathbb{R}^3 : R/2 < |x| < R\}$. If the solution $u$ satisfies $u \in L^p_{loc}(\mathbb{R}^3)$ and $\nabla u \in L^2_{loc}(\mathbb{R}^3)$ with $3 \leq p < +\infty$, then for all $R > 1$, we have

$$
\int_{B_{R/2}} |\nabla u|^2 dx \lesssim \left( \int_{C(R/2, R)} |\nabla u|^\frac{2}{p} dx + \int_{C(R/2, R)} |u|^p dx \right)^{\frac{\frac{2}{p}}{\frac{3}{p}}}
$$

$$
R^{2-\frac{2}{p}} \left( \int_{C(R/2, R)} |u|^p dx \right)^{\frac{1}{p}}. \tag{3.9}
$$

**Proof.** The proof is almost the same as that of [8], we give a sketch of the proof for the convenience of the readers. We start by introducing the test functions $\varphi_R$ and $W_R$ as follows: we have $W_R \in W^{1,q}(B_R)$ with $\text{supp}(W_R) \subset C(R/2, R)$ and

$$
\|\nabla W_R\|_{L^q(C(R/2, R))} \leq c\|\nabla \varphi_R \cdot u\|_{L^q(C(R/2, R))}. \tag{3.10}
$$

Once we have defined the functions $\varphi_R$ and $W_R$ above, we consider now the function $\varphi_R u - W_R$ and we write

$$
\int_{B_R} (-\Delta u + (\rho u \cdot \nabla) u + \nabla \pi) \cdot (\varphi_R u - W_R) dx = 0.
$$

First of all, since $W_R$ is a solution of problem (3.8) and $u$ is a divergence free vector, note that

$$
\int_{B_R} \nabla \pi \cdot (\varphi_R u - W_R) dx = 0.
$$

Thus by this inequality and the identity above, we can write the following estimate:

$$
\int_{B_{R/2}} |\nabla u|^2 dx = -\sum_{i,j=1}^3 \int_{B_R} \partial_j u_i (\partial_j \varphi_R) u_i dx + \sum_{i,j=1}^3 \int_{B_R} (\partial_j u_i) \partial_j (W_R)_i dx
$$

$$
+ \int_{B_R} (\rho u \cdot \nabla) u \cdot (\varphi_R u - W_R) dx := \sum_{i=1}^3 K_i. \tag{3.11}
$$

The term $K_2$ in (3.11) is estimated by

$$
K_2 = \sum_{i,j=1}^3 \int_{B_R} (\partial_j u_i) \partial_j (W_R)_i dx = \sum_{i,j=1}^3 \int_{C(R/2, R)} (\partial_j u_i) \partial_j (W_R)_i dx.
$$
Applying Hölder’s inequality and (3.10), we have

\[
\mathcal{K}_2 \lesssim \left( \int_{C(R/2,R)} |\nabla u|^{\frac{p}{2}} \, dx \right)^{\frac{2}{p}} \frac{1}{R} \left( \int_{C(R/2,R)} |u|^q \, dx \right)^{\frac{1}{q}}
\]

\[
\lesssim \left( \int_{C(R/2,R)} |\nabla u|^{\frac{p}{2}} \, dx \right)^{\frac{2}{p}} \left( \int_{C(R/2,R)} |\nabla \varphi_R \cdot u|^q \, dx \right)^{\frac{1}{q}}
\]

\[
\lesssim \left( \int_{C(R/2,R)} |\nabla u|^{\frac{p}{2}} \, dx \right)^{\frac{2}{p}} R^{2-\frac{q}{p}} \left( \int_{C(R/2,R)} |u|^p \, dx \right)^{\frac{1}{p}}
\]

Now we study each term in sequence. Due to the property for the function \( \partial_i \varphi_R \), we know that if \( |x| > R \), then we have \( \text{supp} (\nabla \varphi_R) \subset C(R/2,R) \), and thus it is rewritten as

\[
\mathcal{K}_1 = - \sum_{i,j=1}^{3} \int_{C(R/2,R)} \partial_j u_i (\partial_j \varphi_R) u_i dx.
\]

Applying the Hölder inequalities, we have

\[
\mathcal{K}_1 \lesssim \left( \int_{C(R/2,R)} |\nabla u|^{\frac{p}{2}} \, dx \right)^{\frac{2}{p}} \left( \int_{C(R/2,R)} |u|^q \, dx \right)^{\frac{1}{q}}
\]

\[
\lesssim \left( \int_{C(R/2,R)} |\nabla u|^{\frac{p}{2}} \, dx \right)^{\frac{2}{p}} R^{2-\frac{q}{p}} \left( \int_{C(R/2,R)} |u|^p \, dx \right)^{\frac{1}{p}}
\]

For \( \mathcal{K}_3 \), by integrating by parts, we know

\[
\mathcal{K}_3 = - \sum_{i,j=1}^{3} \int_{B_R} (\rho u_i u_j) \left( (\partial_j \varphi_R) u_i + \varphi_R \partial_j u_i - \partial_j (W_R)_i \right) dx
\]

\[
= - \sum_{i,j=1}^{3} \int_{B_R} (\rho u_i u_j) \left( (\partial_j \varphi_R) u_i dx - (u_i u_j) \varphi_R (\partial_j u_i) + (u_i u_j) \partial_j (W_R)_i \right) dx
\]

\[
= J_1 + J_2 + J_3.
\]

Again, we estimate each term separately. In term \( J_1 \), in the previous way, it is easily checked that

\[
J_1 \lesssim \left( \int_{C(R/2,R)} |u|^p \, dx \right)^{\frac{2}{p}} R^{2-\frac{q}{p}} \left( \int_{C(R/2,R)} |u|^p \, dx \right)^{\frac{1}{p}}
\]
where we use $\|\rho\|_{L^\infty} < \infty$. Next, to estimate the term $J_2$, by integration by parts and the divergence free condition for $u$, we know

$$J_2 = -\frac{1}{2} \sum_{i,j=1}^{3} \int_{B_R} u_j \varphi_R \partial_j (u_i^2) \, dx = \frac{1}{2} \sum_{i,j=1}^{3} \int_{C(R/2, R)} u_i^2 (\partial_j \varphi_R) u_j \, dx.$$ 

In the same way as for $J_1$, it follows that

$$J_2 \lesssim \left( \int_{C(R/2, R)} |u|^p \, dx \right)^{\frac{2}{p}} R^{2 - \frac{2}{p}} \left( \int_{C(R/2, R)} |u|^p \, dx \right)^{\frac{1}{p}}.$$ 

Similarly, we have

$$J_3 \lesssim \left( \int_{C(R/2, R)} |u|^p \, dx \right)^{\frac{2}{p}} R^{2 - \frac{2}{p}} \left( \int_{C(R/2, R)} |u|^p \, dx \right)^{\frac{1}{p}}.$$ 

Summing up $J_1 - J_3$, we have

$$K_3 \lesssim \left( \int_{C(R/2, R)} |u|^p \, dx \right)^{\frac{2}{p}} R^{2 - \frac{2}{p}} \left( \int_{C(R/2, R)} |u|^p \, dx \right)^{\frac{1}{p}}.$$ 

Again, collecting up $K_1 - K_3$, we finally obtain the desired result. \qed 

**Proof of Theorem 1.2.** For $1 < p < r \le q < +\infty$, we have

$$\int_{B_R} |u|^p \, dx \le c \, R^{3(1 - \frac{p}{r})} \|u\|^p_{L^r, \infty} \le c \, R^{3(1 - \frac{p}{r})} \|u\|^p_{L^r, q}, \quad R > 1, \quad (3.11)$$

see [4, Proposition 1.1.10]. From (3.9), we know that through Lemma 3.1, we write for all $R > 1$,

$$\int_{B_{R/2}} |\nabla u|^2 \, dx \lesssim (G_u + S_u) P_u, \quad (3.12)$$

where

$$G_u := c \left( R^\frac{p}{2} \left( \frac{1}{R^3} \int_{C(R/2, R)} |\nabla u|^\frac{p}{2} \, dx \right) \right)^{\frac{2}{p}},$$

$$S_u := R^\frac{p}{2} \left( \frac{1}{R^3} \int_{C(R/2, R)} |u|^p \, dx \right)^{\frac{3}{p}},$$

$$P_u := R^{2 - \frac{p}{2}} \left( R^\frac{3}{p} \left( \frac{1}{R^3} \int_{C(R/2, R)} |u|^p \, dx \right) \right)^{\frac{1}{p}}.$$
For this, we introduce the cut-off function $\theta_R \in C_0^\infty(\mathbb{R}^3)$ such that $\theta_R = 1$ on $C(R/2, R)$, $\text{supp}(\theta_R) \subset C(R/4, 2R)$, and $\|\nabla \theta_R\|_{L^\infty} \leq \frac{c}{R}$. With the same arguments as in [8], (3.12) yields

$$\int_{B_{\frac{R}{2}}} |\nabla u|^2 dx \leq c \left( \|\theta_R(\nabla u)\|_{L^\frac{5}{2}}^2 + \|\theta_R u\|_{L^{r,q}}^2 \right) R^{2 - \frac{2}{r}} \|u\|_{L^{r,q}(C(R/4,R))}.$$  

Taking $p = 4$ and $r = 9/2$, it follows $u \equiv 0$ in $\mathbb{R}^3$ as $R \to \infty$. The proof of Theorem 1.2 is complete. $\square$

Acknowledgements. We would like to appreciate the anonymous referee for valuable comments. Jae-Myoung Kim was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2020R1C1C1A01006521).

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Received: 6 December 2022

Revised: 18 February 2023

Accepted: 3 April 2023