On the lattice of the $\sigma$-permutable subgroups of a finite group

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Abstract

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$, $G$ a finite group and $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$.

A set $\mathcal{H}$ of subgroups of $G$ is said to be a complete Hall $\sigma$-set of $G$ if every member $\neq 1$ of $\mathcal{H}$ is a Hall $\sigma$-subgroup of $G$ for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall $\sigma$-subgroup of $G$ for every $\sigma_i \in \sigma(G)$. A subgroup $A$ of $G$ is said to be $\sigma$-permutable in $G$ if $G$ possesses a complete Hall $\sigma$-set and $A$ permutes with each Hall $\sigma$-subgroup $H$ of $G$, that is, $AH = HA$ for all $i \in I$.

We characterize finite groups with distributive lattice of the $\sigma$-permutable subgroups.

1 Introduction

Throughout this paper, $G$ always denotes a finite group. Moreover, we use $\mathcal{L}(G)$ to denote the lattice of all subgroups of $G$. $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. As usual, $\pi(G)$ is the set of all primes dividing the order $|G|$ of $G$. The subgroups $A$ and $B$ of $G$ are said to be permutable if $AB = BA$. In this case they also say that $A$ permutes with $B$. If $A$ permutes with all Sylow subgroups of $G$, then $A$ is called $S$-permutable in $G$ [1]. Recall also that an element $a$ of the lattice $\mathcal{L}$ is called meet-distributive [2, p. 136] if $a \wedge (b \lor c) = (a \wedge b) \lor (a \wedge c)$ for all $b, c \in \mathcal{L}$.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

A set $\mathcal{H}$ of subgroups of $G$ is a complete Hall $\sigma$-set of $G$ [3] if every member $\neq 1$ of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$ for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $i$ such that $\sigma_i \cap \pi(G) \neq \emptyset$. $G$ is said to be $\sigma$-full if it possesses a complete Hall $\sigma$-set.

Definition 1.1. We say that a subgroup $A$ of $G$ is said to be $\sigma$-quasinormal or $\sigma$-permutable in $G$ [4] if $G$ is $\sigma$-full and $A$ permutes with each Hall $\sigma_i$-subgroup $H$ of $G$ for all $i \in I$.

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Remark 1.2. (i) If $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$, then $A$ is $\sigma$-permutable in $G$ (see Proposition 3.1 below).

(ii) $G$ is called $\sigma$-decomposable (Shemetkov [3]) or $\sigma$-nilpotent (Guo and Skiba [6]) if $G = H_1 \times \cdots \times H_t$, where $\{H_1, \ldots, H_t\}$ is a complete Hall $\sigma$-set of $G$. It is not difficult to show that $G$ is $\sigma$-nilpotent if and only if every subgroup of $G$ is $\sigma$-permutable in $G$.

(iii) In the classical case when $\sigma = \sigma^0 = \{\{2\}, \{3\}, \ldots\}$: $G$ is $\sigma^0$-nilpotent if and only if $G$ is nilpotent; a subgroup $A$ of $G$ is $\sigma^0$-permutable in $G$ if and only if it is $S$-permutable in $G$.

(iv) In the other classical case when $\sigma = \sigma^\pi = \{\pi, \pi'\}$: $G$ is $\sigma^\pi$-nilpotent if and only if it is $\pi$-decomposable, that is, $G = O_\pi(G) \times O_{\pi'}(G)$; a subgroup $A$ of a $\pi$-separable group $G$ is $\sigma^\pi$-permutable in $G$ if and only if $A$ permutes with all Hall $\pi$-subgroups and with all Hall $\pi'$-subgroups of $G$.

(v) If $G$ is $\sigma$-full, the set $\mathcal{L}_{\sigma_{\text{per}}}(G)$ of all $\sigma$-permutable subgroups of $G$ is partially ordered with respect to set inclusion. Moreover, $\mathcal{L}_{\sigma_{\text{per}}}(G)$ is a lattice since $1 \in \mathcal{L}_{\sigma_{\text{per}}}(G)$ and, by Lemma 2.1 below, for any $A_1, \ldots, A_n \in \mathcal{L}_{\sigma_{\text{per}}}(G)$ the subgroup $\langle A_1, \ldots, A_n \rangle$ is the least upper bound for $\{A_1, \ldots, A_n\}$ in $\mathcal{L}_{\sigma_{\text{per}}}(G)$.

The conditions under which the lattice $\mathcal{L}_{\text{sn}}(G)$ of all subnormal subgroups of $G$ is modular or distributive are known (see [2 Theorems 9.2.3, 9.2.4]). It is well-known also that the lattice $\mathcal{L}_n(G)$ of all normal subgroups of $G$ is modular if and only if in every factor group $G/R$, any two $G/R$-isomorphic normal subgroups coincide (see [7] and [2 Theorem 9.1.6]). Kegel proved [8] that the set $\mathcal{L}_S(G)$ of all $S$-permutable subgroups of $G$ forms a sublattice of the lattice $\mathcal{L}_{\text{sn}}(G)$. Since $\mathcal{L}_n(G) \subseteq \mathcal{L}_S(G) \subseteq \mathcal{L}_{\text{sn}}(G)$, where both inclusions in general are strict, it seems natural to ask: Under what conditions the lattice $\mathcal{L}_S(G)$ is modular or distributive? Moreover, in view of Remark 1.2(v), it makes sense to consider the following general

Question 1.3 (See Questions 6.10 and 6.11 in [3]). Under what conditions the lattice $\mathcal{L}_{\sigma_{\text{per}}}(G)$ is modular or distributive?

Note that if $K \trianglelefteq H$ and $K, H \in \mathcal{L}_{\sigma_i}(G)$, where $\mathcal{L}_{\sigma_i}(G)$ is the set of all $\sigma$-permutable $\sigma_i$-subgroups of $G$, then $O^{\sigma_i}(G)$ normalizes both subgroups $K$ and $H$ (see Lemma 2.4(1) below) and hence we can consider $O^{\sigma_i}(G)$ as a group of operators for $H/K$ (assuming, as usual, that $(hK)^a = h^aK$ for all $hK \in H/K$ and $a \in O^{\sigma_i}(G)$).

We do not know under what conditions on $G$ the lattice $\mathcal{L}_{\sigma}(G)$ is modular. Nevertheless, we give a full answer to the second part of Question 1.3.

Theorem A. Suppose that $G$ is $\sigma$-full. Let $D = G^{\text{nil}}$ and $\mathcal{L} = \mathcal{L}_{\sigma}(G)$. Then the lattice $\mathcal{L}$ is distributive if and only if the following conditions hold:

(i) Every two members of $\mathcal{L}$ are permutable.

(ii) The lattice $\mathcal{L}_n(G)$ of all normal subgroups of $G$ is distributive.

(iii) $G/D$ is cyclic and $D$ is a meet-distributive element of $\mathcal{L}$. 

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In every factor group $\bar{G} = G/R$, any two $O^{\sigma_i}(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}, \bar{L} \in \mathcal{L}_{\sigma}(\bar{G})$ for some $i$, coincide.

In this theorem $G^{\sigma_N}$ denotes the $\sigma$-nilpotent residual of $G$, that is, the intersection of all normal subgroups $N$ of $G$ with $\sigma$-nilpotent quotient $G/N$.

Theorem A remains to be new for each special partition of $\mathbb{P}$. In particular, in the case when $\sigma = \sigma^0$ we get from Theorem A the following

**Corollary 1.4.** Let $D = G^{\sigma_0}$ be the nilpotent residual of $G$ and $\mathcal{L} = \mathcal{L}_S(G)$. Then the lattice $\mathcal{L}$ is distributive if and only if the following conditions hold:

1. Conditions (i), (ii) and (iii) in Theorem A hold for $G$.
2. In every factor group $\bar{G} = G/R$, any two $O^{\sigma}(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}, \bar{L} \in \mathcal{L}_{\sigma}(\bar{G})$ and $p$ is a prime, coincide.

In this corollary $\mathcal{L}_{\sigma}(\bar{G})$ denotes the set of all $S$-permutable $p$-subgroups of $\bar{G}$.

In the case when $\sigma = \sigma^\pi$ (see Remark 1.2(iv)) we get from Theorem A the following fact.

**Corollary 1.5.** Let $D$ be the $\pi$-decomposable residual of $G$, that is, the smallest normal subgroup of $G$ with $\pi$-decomposable quotient $G/D$. Suppose that $G$ is $\pi$-separable and let $\mathcal{L} = \mathcal{L}_{\sigma}(G)$. Then the lattice $\mathcal{L}$ is distributive if and only if the following conditions hold:

1. Conditions (i), (ii) and (iii) in Theorem A hold for $G$.
2. In every factor group $\bar{G} = G/R$, any two $O^{\sigma}(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}, \bar{L} \in \mathcal{L}_{\sigma}(\bar{G})$ and $\pi$ is a prime, coincide.
3. In every factor group $\bar{G} = G/R$, any two $O^{\sigma}(\bar{G})$-isomorphic sections $\bar{H}/\bar{K}$ and $\bar{L}/\bar{K}$, where $\bar{K}, \bar{H}, \bar{L} \in \mathcal{L}_{\sigma}(\bar{G})$ and $\pi'$ are $\sigma$-permutable $\pi'$-subgroups of $\bar{G}$, coincide.

The proof of Theorem A consists of many steps and the next theorems are two of them.

**Theorem B.** Suppose that $G$ is $\sigma$-full. Then $\mathcal{L}_{\sigma}(G)$ is a sublattice of the lattice $\mathcal{L}(G)$.

**Corollary 1.6** (Kegel [8]). The set $\mathcal{L}_S(G)$ of all $S$-permutable subgroups of $G$ forms a sublattice of the lattice $\mathcal{L}(G)$.

There are at least three different proofs of Corollary 1.6 (see, for example, [8, 9, 14]). One more, the shortest one, gives the proof of Theorem B.

**Theorem C.** A $\sigma$-nilpotent subgroup $A$ of $G$ is $\sigma$-permutable in $G$ if and only if each characteristic subgroup of $A$ is $\sigma$-permutable in $G$.

**Corollary 1.7** (See [9] or [14] Theorem 1.2.17]). Let $A$ be a nilpotent subgroup of $G$. Then the following statements are equivalent:

(i) $A$ is $S$-permutable in $G$.

(ii) Each Sylow subgroup of $A$ is $S$-permutable in $G$. 

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(iii) Each characteristic subgroup of A is S-permutable in G.

2 Proof of Theorems B and C

Lemma 2.1 (See [10, A, Lemma 1.6]). Let A, B and H be subgroups of G. If \(AH = HA\) and \(BH = HB\), then \(⟨A, B⟩H = H⟨A, B⟩\).

A subgroup \(A\) of \(G\) is called \(\sigma\)-subnormal in \(G\) \([1]\) if there is a subgroup chain \(A = A_0 \leq A_1 \leq \cdots \leq A_t = G\) such that either \(A_{i-1} \leq A_i\) or \(A_i/(A_{i-1})A_i\) is \(\sigma\)-primary for all \(i = 1, \ldots, t\).

The importance of this concept is related to the following result.

Lemma 2.2 (See [1] Theorem B). Let \(A\) be a subgroup of \(G\). If \(G\) possesses a complete Hall \(\sigma\)-set \(\mathcal{K}\) such that \(AH^x = H^xA\) for all \(H \in \mathcal{K}\) and all \(x \in G\), then \(A\) is \(\sigma\)-subnormal in \(G\).

Lemma 2.3 (See Lemma 2.6 in [4]). Let \(A, K\) and \(N\) be subgroups of \(G\). Suppose that \(A\) is \(\sigma\)-subnormal in \(G\) and \(N\) is normal in \(G\).

1. \(A \cap K\) is \(\sigma\)-subnormal in \(K\).
2. If \(|G : A|\) is a \(\sigma\)-number, then \(O^{\sigma_i}(A) = O^{\sigma_i}(G)\).
3. \(AN/N\) is \(\sigma\)-subnormal in \(G/N\).
4. If \(A\) is a \(\sigma\)-group, then \(A \leq O_{\sigma_i}(G)\).
5. If \(H \neq 1\) is a Hall \(\sigma_i\)-subgroup of \(G\), then \(A \cap H \neq 1\) is a Hall \(\sigma_i\)-subgroup of \(A\).
6. If \(K\) is \(\sigma\)-subnormal in \(G\) and the subgroups \(A\) and \(K\) are \(\sigma\)-nilpotent, then \(⟨A, K⟩\) is \(\sigma\)-nilpotent.

Proof of Theorem B. In fact, in view of Lemmas 2.1 and 2.2, it is enough to show that if \(A\) and \(B\) are \(\sigma\)-subnormal subgroups of \(G\) such that for a Hall \(\sigma\)-subgroup \(H\) of \(G\) we have \(AH = HA\) and \(BH = HB\), then \(A \cap H = H(\cap A)\). Assume that this is false and let \(G\) be a counterexample of minimal order. Then \(G\) is not a \(\sigma\)-group, since otherwise we have \(H = G\) and so \(G = (A \cap B)H = H(A \cap B)\).

Let \(E = AH \cap BH\). Then \(A \cap E\) and \(B \cap E\) are \(\sigma\)-subnormal subgroups in \(E\) by Lemma 2.3(1). Moreover, \(AH \cap E = H(A \cap E) = (A \cap E)H\). Similarly, \((B \cap E)H = H(B \cap E)\). Hence the hypothesis holds for \((A \cap E, B \cap E, H, E)\). Assume that \(E \leq G\). Then the choice of \(G\) implies that \(A \cap B = (A \cap E) \cap (B \cap E)\) is permutable with \(H\). Hence \(E = G\), so \(G = AH = BH\). Thus \(|G : A|\) and \(|G : B|\) are \(\sigma\)-numbers. Hence by Lemma 2.3(2) we have \(O^{\sigma_i}(A) = O^{\sigma_i}(G) = O^{\sigma_i}(B)\). Therefore, since \(G\) is not a \(\sigma\)-group, it follows that \(V = A_G \cap B_G \neq 1\). Moreover, \(A/V\) and \(B/V\) are \(\sigma\)-subnormal subgroups of \(G/V\) by Lemma 2.3(3). Also we have

\[(A/V)(HV/V) = AH/V = HA/V = (HV/V)(A/V)\]

and \((B/V)(HV/V) = (HV/V)(B/V)\), where \(HV/V\) is a Hall \(\sigma_i\)-subgroup of \(G/V\). Hence the choice
of $G$ implies that
\[(A \cap B/V)(HV/V) = ((A/V) \cap (B/V))(HV/V) =
\]
\[= (HV/V)((A/V) \cap (B/V)) = (HV/V)(A \cap B/V).
\]
But then
\[(A \cap B)H = (A \cap B)HV = HV(A \cap B) = H(A \cap B).
\]
This contradiction completes the proof of the result.

**Lemma 2.4** (See Lemmas 2.8, 3.1 and Theorem B in [4]). Let $A$ and $B$ be subgroups of $G$. Suppose that $G$ possesses a complete Hall $\sigma$-set $\mathfrak{H}$ such that $AH^x = H^xA$ for all $H \in \mathfrak{H}$ and all $x \in G$. Then:

1. If $A$ is a $\sigma_i$-group, then $O^{\sigma_i}(G) \leq N_G(A)$.
2. $A/A_G$ is $\sigma$-nilpotent.
3. If $B$ is a $\sigma_i$-group and $O^{\sigma_i}(G) \leq N_G(B)$, then $G$ possesses a complete Hall $\sigma$-set $\mathcal{L}$ such that $BL^x = L^xB$ for all $L \in \mathcal{L}$ and all $x \in G$.

**Proposition 2.5.** Let $A$ be a $\sigma$-nilpotent $\sigma$-subnormal subgroup of $G$ and $V$ a characteristic subgroup of $A$. Let $H$ be a Hall $\sigma_i$-subgroup of $G$. If $AH = HA$, then $VH = HV$.

**Proof.** Assume that this proposition is false and let $G$ be a counterexample with $|G|+|V|+|A|$ minimal.

By hypothesis, $A = A_1 \times \cdots \times A_t$, where $\{A_1, \ldots, A_t\}$ is a complete Hall $\sigma$-set of $A$. Hence $V = (A_1 \cap V) \times \cdots \times (A_t \cap V)$, where $\{A_1 \cap V, \ldots, A_t \cap V\}$ is a complete Hall $\sigma$-set of $V$. We can assume without loss of generality that $A_k$ is a $\sigma_k$-subgroup of $A$ for all $k = 1, \ldots, t$.

It is clear that $A_i \cap V$ is characteristic in $A$ for all $i = 1, \ldots, t$. Therefore, if $A_i \cap V < V$, then $(A_i \cap V)H = H(A_i \cap V)$ by the choice of $G$ and so for some $j, j = 1$ say, we have $A_1 \cap V = V$ since otherwise we have
\[VH = ((A_1 \cap V) \times \cdots \times (A_t \cap V))H = H((A_1 \cap V) \times \cdots \times (A_t \cap V)) = HV.
\]
Thus $V \leq A_1$. It is clear that $A_1$ is a $\sigma$-subnormal subgroup of $G$, so in the case when $i = 1$ we have $V \leq A_1 \leq H$ by Lemma 2.3(5). But then $VH = H = HV$, a contradiction. Thus $i > 1$.

Now we show that $A_1H = HA_1$. Indeed, it is clear that $A = A_1 \times V \times A_i$ and $A_i$ is $\sigma$-subnormal in $G$. Thus $A_i \leq H$ by Lemma 2.3(5). Therefore
\[AH = HA = (A_1 \times V \times A_i)H = (A_1 \times V)H = (A_1 \times V)H,
\]
where $A_1 \times V$ is a $\sigma$-subnormal $\sigma_i$-subgroup of $G$. Then $A_1 \times V$ is $\sigma$-subnormal in $(A_1 \times V)H$ by Lemma 2.3(1). Hence $H \leq N_G(A_1 \times V)$ by Lemma 2.3(2). Since $A_1$ is a characteristic subgroup of $A_1 \times V$, we have $H \leq N_G(A_1)$ and so $A_1H = HA_1$. Therefore $H \leq N_G(A_1)$ by Lemma 2.3(2). But $V$
is a characteristic subgroup of $A_1$ since $V$ is characteristic in $A$ by hypothesis and $A = A_1 \times \cdots \times A_t$. Therefore $H \leq N_G(V)$ and so $VH = HV$, a contradiction. The proposition is proved.

**Corollary 2.6.** Let $A$ be a $\sigma$-nilpotent subgroup of a $\sigma$-full group $G$. Then the following statements are equivalent:

(i) $A$ is $\sigma$-permutable in $G$.

(ii) Each Hall $\sigma_i$-subgroup of $A$ is $\sigma$-permutable in $G$ for all $i$.

(iii) Each characteristic subgroup of $A$ is $\sigma$-permutable in $G$.

**Proof.** By hypothesis, $A = A_1 \times \cdots \times A_t$, where $\{A_1, \ldots, A_t\}$ is a complete Hall $\sigma$-set of $A$. Then $A_i$ is characteristic in $A$ for all $i = 1, \ldots, t$. Therefore (ii), (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii), (iii) This follows from Proposition 2.5.

The corollary is proved.

**Proof of Theorem C.** This directly follows from Corollary 2.6.

3 Proof of Theorem A

**Proposition 3.1.** Let $A$ be a subgroup of $G$. If $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ such that $AL^x = L^xA$ for all $L \in \mathcal{H}$ and all $x \in G$, then $A$ is $\sigma$-permutable in $G$.

**Proof.** Assume that this proposition is false and let $G$ be a counterexample with $|G| + |A|$ minimal. Then for some $i$ and some Hall $\sigma_i$-subgroup $H$ of $G$ we have $AH \neq HA$. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$. We can assume without loss of generality that $H_k$ is a $\sigma_k$-group for all $k = 1, \ldots, t$. Let $V = H_i$.

First we show that $A_G = 1$. Indeed, assume that $R = A_G \neq 1$. Then $\mathcal{H}_0 = \{H_1R/R, \ldots, H_tR/R\}$ is a complete Hall $\sigma$-set of $G/R$ such that

$$AL^x/R = (A/R)(LR/R)^xR = (LR/R)^xR(A/R) = L^xA/R$$

for all $LR/R \in \mathcal{H}_0$ and all $xR \in G/R$. On the other hand, $HR/R$ is Hall $\sigma_i$-subgroup of $G/R$. Hence the choice of $G$ implies that

$$AH/R = (A/R)(HR/R) = (HR/R)(A/R) = HA/R$$

and so $AH = HA$, a contradiction. Therefore $A_G = 1$, hence $A = A_1 \times \cdots \times A_t$, where $\{A_1, \ldots, A_t\}$ is a complete Hall $\sigma$-set of $A$ by Lemma 2.4(2). Moreover, Lemma 2.2 implies that $A$ is $\sigma$-subnormal in $G$.

First assume that $A = A_1$ is a $\sigma_j$-group. If $j = i$, then $A \cap H = A$ by Lemma 2.3(5) and so $AH = H = HA$. Hence $j \neq i$. By hypothesis, $AV^x = V^xA$ for each $x \in G$. Then $V^x \leq N_G(A)$ for all $x \in G$ by Lemma 2.3(1)(2). Hence $V^G \leq N_G(A)$. But then $H \leq V^G \leq N_G(A)$, which implies that $AH = HA$. This contradiction shows that $t > 1$. 

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The subgroups $A_1, \ldots, A_t$ are characteristic in $A$, so $A_1 L^x = L^x A_1$ for all $L \in \mathcal{H}$ and all $x \in G$ by Proposition 2.5. Therefore the minimality of $|G| + |A|$ implies that $A_i H = H A_i$ for all $i = 1, \ldots, t$, so $A H = H A$. This contradiction completes the proof of the result.

**Lemma 3.2.** Let $R \leq V$ and $H$ be subgroups of a $\sigma$-full group $G$. Suppose that $H$ is $\sigma$-permutable in $G$ and $R$ is normal in $G$. Then:

(1) If $V/R$ is a $\sigma$-permutable subgroup of $G/R$, then $V$ is a $\sigma$-permutable subgroup of $G$.

(2) The subgroup $HR/R$ is $\sigma$-permutable in $G/R$.

**Proof.** (1) Let $i \in I$ and $H$ be a Hall $\sigma_i$-subgroup of $G$. Then $HR/R$ is a Hall $\sigma_i$-subgroup of $G/R$, so

$$VH/R = (V/R)(HR/R) = (HR/R)(V/R) = HV/R$$

by hypothesis and hence $VH = HV$.

(2) By hypothesis, $G$ possesses a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ and $HL^x = L^x H$ for all $L \in \mathcal{H}$ and all $x \in G$. Then $\mathcal{H}_0 = \{H_1 R/R, \ldots, H_t R/R\}$ is a complete Hall $\sigma$-set in $G/R$ and

$$(HR/R)(LR/R)^xR = HL^xR/R = L^x HR/R = (LR/R)^xR (HR/R)$$

for all $LR/R \in \mathcal{H}_0$ and $xR \in G/R$. Therefore $HR/R$ is $\sigma$-permutable in $G/R$ by Proposition 3.1.

The lemma is proved.

**Lemma 3.3** (See Lemma 5.2 in [11]). Let $\mathcal{L}$ be a modular sublattice of the lattice $\mathcal{L}(G)$, and $U, V, N \in \mathcal{L}$ with $N \leq \langle U, V \rangle$. If $U$ permutes both with $V \cap UN$ and $VN$, then $U$ permutes with $V$.

**Proposition 3.4.** Let $G$ be $\sigma$-full and $\mathcal{L} = \mathcal{L}_{\sigma}(G)$. Then: (i) $\mathcal{L}$ is a sublattice of $\mathcal{L}_{\sigma_{per}}(G)$, and (ii) If $\mathcal{L}$ is distributive, then $AB = BA$ for all $A, B \in \mathcal{L}$.

**Proof.** (i) Let $A, B \in \mathcal{L}$. The subgroups $A$ and $B$ are $\sigma$-subnormal in $G$ by Lemma 2.2, so $A, B \leq O_{\sigma_i}(G)$ by Lemma 2.3(4). Thus $\langle A, B \rangle$ is a $\sigma_i$-subgroup of $G$ and this subgroup is $\sigma$-permutable in $G$ by Lemma 2.1. Finally, $A \cap B$ is also a $\sigma_i$-subgroup of $G$ and this subgroup is $\sigma$-permutable in $G$ by Theorem B. Thus we have (i).

(ii) Suppose that this assertion is false and let $G$ be a counterexample with $|G| + |A| + |B|$ minimal. Thus $AB \neq BA$ but $A_1 B_1 = B_1 A_1$ for all $A_1, B_1 \in \mathcal{L}$ such that $A_1 \leq A, B_1 \leq B$ and either $A_1 \neq A$ or $B_1 \neq B$. Let $V = \langle A, B \rangle O^{\sigma}(G)$ and $R = \langle A, B \rangle \cap O^{\sigma_i}(G)$. Then $V$ is $\sigma$-subnormal in $G$.

(1) $\mathcal{L}_{\sigma_i}(V)$ is a sublattice of $\mathcal{L}$.

Indeed, let $H \in \mathcal{L}_{\sigma_i}(V)$. Then $H$ is $\sigma$-subnormal in $V$ by Lemma 2.2 and so, because of Lemma 2.3(4), $H \leq O_{\sigma_i}(V) \leq O_{\sigma_i}(G)$. Therefore $H$ permutes with each Hall $\sigma_{i}$-subgroup of $G$. On the other hand, each Hall $\sigma_{j}$-subgroup $W$ of $G$, where $j \neq i$, is contained in $V$ by Lemma 2.3(5) since $|W|$ divides $|V|$, so $HW = WH$. Hence $H \in \mathcal{L}_{\sigma_i}(G)$, which implies (1).

(2) $V = G$, so $\langle A, B \rangle \leq G$. 

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Claim (1) implies that the hypothesis holds for $\mathcal{L}_{\sigma_i}(V)$ and so in the case when $V \neq G$ the choice of $G$ implies that $AB = BA$. Thus $G = \langle A, B \rangle O^{\sigma_i}(G)$. Therefore, since $O^{\sigma_i}(G) \leq N_G(\langle A, B \rangle)$ by Lemma 2.4(1), $\langle A, B \rangle$ is normal in $G$.

(3) $R = 1$.

Assume that $R = \langle A, B \rangle \cap O^{\sigma_i}(G) \neq 1$. First we show that $BRA = \langle A, B \rangle R$. Indeed, let $H/R$ be a $\sigma_i$-subgroup of $G/R$. Then $H$ is a $\sigma_i$-group since $\langle A, B \rangle \leq O_{\sigma_i}(G)$. Moreover, Lemma 3.2(1)(2) implies that $H/R$ is $\sigma$-permutable in $G/R$ if and only if $H$ is $\sigma$-permutable in $G$. Therefore the lattice $\mathcal{L}_{\sigma_i}(G/R)$ is isomorphic to the interval $[G/R]$ in the distributive lattice $\mathcal{L}$. Therefore, by the minimality of $G$, $(AR/R)(BR/R) = (BR/R)(AR/R)$ by Lemma 3.2(2) and so $BRA = \langle A, B \rangle R$.

Now we show that $BRA = BR$. Assume that this is false. Then $A \cap BR < A$. But Theorem B implies that $A \cap BR$ is $\sigma$-permutable in $G$, so the minimality of $|G| + |A| + |B|$ implies that $B$ permutes with $A \cap BR$. Also, $B$ permutes with $RA$ since $B(RA) = \langle A, B \rangle R$, so $AB = BA$ by Lemma 3.3, Part (i) and Theorem B. This contradiction shows that $A \leq BR$, so $BRA = BR$. But $R \leq O^{\sigma_i}(G) \leq N_G(B)$ by Lemma 2.4(1), hence $B$ is normal in $BR$ and since $A \leq BR$ it follows that $AB = BA$. This contradiction shows that we have (3).

**Final contradiction.** Claims (2) and (3) imply that

$$G = \langle A, B \rangle O^{\sigma_i}(G) = \langle A, B \rangle \times O^{\sigma_i}(G),$$

so every subgroup $H$ of $\langle A, B \rangle$ is $O^{\sigma_i}(G)$-invariant. It follows that every subgroup of $\langle A, B \rangle$ is $\sigma$-permutable in $G$ by Lemma 2.4(3) and Proposition 3.1. Hence $\mathcal{L}(\langle A, B \rangle)$ is a sublattice of the distributive lattice $\mathcal{L}$. Thus $\langle A, B \rangle$ is cyclic by the Ore theorem by [2] Theorem 1.2.3], so $AB = BA$, a contradiction. The proposition is proved.

**Corollary 3.5.** If the lattice $\mathcal{L} = \mathcal{L}_{\sigma}(G)$ is distributive, then every two members $A$ and $B$ of $\mathcal{L}$ are permutable.

**Proof.** Suppose that this corollary is false and let $G$ be a counterexample with $|G| + |A| + |B|$ minimal.

Let $R$ be a minimal normal subgroup of $G$. Lemma 3.2 implies that $\mathcal{L}_{\sigma}(G/R)$ is isomorphic to the interval $[G/R]$ in the modular lattice $\mathcal{L}_{\sigma_{\text{per}}}(G)$. Therefore, Lemma 3.2(2) and the minimality of $G$ imply that $(AR/R)(BR/R) = (BR/R)(AR/R)$. It follows that $RAB = \langle A, B \rangle R$ is a subgroup of $G$, so $A_G = 1$ and $B_G = 1$. Hence, because of Lemma 2.4(2), $A$ and $B$ are $\sigma$-nilpotent. The minimality of $|G| + |A| + |B|$ implies that for some $i$ we have $A, B \leq O_{\sigma_i}(G)$ and so $A, B \in \mathcal{L}_{\sigma_i}(G)$. But $\mathcal{L}_{\sigma_i}(G)$ is a sublattice of the distributive lattice $\mathcal{L}_{\sigma}(G)$ by Proposition 3.4(i). Therefore $AB = BA$ by Proposition 3.4(ii), a contradiction. The corollary is proved.

**Lemma 3.6** (See [12] p. 59]). A modular lattice $\mathcal{L}$ is distributive if and only if $\mathcal{L}$ has no distinct elements $a, b$ and $c$ such that $a \vee b = a \vee c = b \vee c$ and $a \wedge b = a \wedge c = b \wedge c$.

**Lemma 3.7** (See [2] Theorem 1.6.2]). Let $G = A \times B$, $f : A \to B$ is an isomorphism and $C = \{aa^f \mid a \in A\}$. Then $G = AC = BC$ and $A \cap C = 1 = B \cap C$. 

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Lemma 3.8 (See Corollary 2.4 and Lemma 2.5 in [3]). The class of all \( \sigma \)-nilpotent groups \( \mathfrak{N}_\sigma \) is closed under taking products of normal subgroups, homomorphic images and subgroups.

In view of Proposition 2.2.8 in [13], we get from Lemma 3.8 the following

**Lemma 3.9.** If \( N \) is a normal subgroup of \( G \), then \( (G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma} N/N \).

**Proof of Theorem A.** Necessity. First note that every two members of \( L \) are permutable by Theorem B. Moreover, since the lattice \( L_n(G) \) is a sublattice of the lattice \( L \), it is distributive. Since \( G/D = G/G^{\mathfrak{N}_\sigma} \) is \( \sigma \)-nilpotent by Lemmas 3.8 and 3.9, every subgroup \( E \) of \( G \) satisfying \( D \leq E \leq G \) is \( \sigma \)-permutable in \( G \) by Lemma 3.2(1) and Remark 1.2(ii). Hence \( L(G/D) = L_{\sigma\text{-perm}}(G/D) \) is distributive and so \( G/D \) is cyclic by the Ore theorem [2, Theorem 1.2.3]. It is clear also that \( D \) is a meet-distributive element of \( L \). Thus Conditions (i)-(iii) hold on \( G \).

Now we show that Condition (iv) holds on \( G \). First note that since, in view of Lemma 3.2, the lattice \( L_{\sigma\text{-perm}}(G/R) \) is isomorphic to the interval \([G/R]\) in the distributive lattice \( L \), it is enough to consider the case when \( \bar{G} = G \) and \( \bar{K} = K, \bar{H} = H, \bar{L} = L \in L \).

Suppose that \( H \neq L \). Then \( H \neq K \). Let \( K < H_0 \leq H \), where \( H_0 \) covers \( K \) in \( L \), and let \( L_0/K = (H_0/K)^f \), where \( f : H/K \rightarrow L/K \) is a \( O^{\sigma_i}(G) \)-isomorphism. For \( g \in O^{\sigma_i}(G) \) and \( l_0K = (hK)^f \in L_0/K, \) where \( h \in H \), we have

\[
(l_0K)^g = ((hK)^f)^g = ((hK)^g)^f = (h^gK)^f = (h_0K)^f,
\]

where \( h_0 \in H_0 \) since \( H_0 \) is \( O^{\sigma_i}(G) \)-invariant by Lemma 2.4(1). Hence \((l_0K)^g \in L_0/K \). It follows that \( L_0 \) is \( O^{\sigma_i}(G) \)-invariant and so \( L_0 \) covers \( K \) in \( L \) since the inverse map \( f^{-1} : L/K \rightarrow H/K \) is a \( O^{\sigma_i}(G) \)-isomorphism too.

First assume that \( H_0 \neq L_0 \) and let \( E_0/K = \{ hK(hK)^f \mid hK \in H_0/K \} \). Then

\[
(H_0/K)(L_0/K) = (H_0/K) \times (L_0/K).
\]

Indeed, if \( H_0^x \neq H_0 \) for some \( x \in L_0 \), then (i) and the fact that \( H_0 \) and \( L_0 \) cover \( K \) in \( L \) would imply that \( \{ K; H_0; H_0^x; L_0; H_0L_0 \} \) would be a diamond in the distributive lattice \( L \), contradicting Lemma 3.6. Hence, by Lemma 3.7, \( E_0/K \) is a subgroup of \( (H_0/K) \times (L_0/K) \) and we have

\[
(H_0/K) \times (L_0/K) = (H_0/K) \times (E_0/K) = (L_0/K) \times (E_0/K).
\]

Note that if \( g \in O^{\sigma_i}(G) \) and \( hK(hK)^f \in E_0/K \), then

\[
(hK(hK)^f)^g = (hK)^g((hK)^f)^g = (h^gK)(h^gK)^f \in E_0/K
\]
since \( f_{H_0/K} \) is a \( O^{\sigma_i}(G) \)-isomorphism from \( H_0/K \) onto \( L_0/K = (H_0/K)^f \). Hence \( E_0/K \) is \( O^{\sigma_i}(G) \)-invariant, so \( O^{\sigma_i}(G) \leq N_G(E_0) \). Therefore \( H_0, L_0 \) and \( E_0 \) are distinct elements of \( L \) such that \( H_0 \cap L_0 = H_0 \cap E_0 = L_0 \cap E_0 = K \) and \( H_0L_0 = H_0E_0 = L_0E_0 \), which is impossible by Lemma
3.6 since \( H_0L_0 \) is a \( \sigma \)-permutable subgroup of \( G \). Therefore \( H_0 = L_0 \). Now \( f \) induces a \( O^{\sigma_i}(G) \)-isomorphism \( f' : H/H_0 \to L/H_0 \) and an obvious induction yields that \( H = L \). Hence we have (iv).

**Sufficiency.** This follows from the following

**Proposition 3.10.** Let \( D = G^{\sigma_0} \) and \( \mathcal{L} = \mathcal{L}_{\sigma_{\text{per}}}(G) \). Suppose that the following conditions hold:

(i) Every two members of \( \mathcal{L} \) are permutable.

(ii) The lattice \( \mathcal{L}_n(G) \) of all normal subgroups of \( G \) is distributive.

(iii) \( G/D \) is cyclic and \( D \) is a meet-distributive element of \( \mathcal{L} \).

(iv) In every factor group \( \bar{G} = G/R \), any two \( O^{\sigma_i}(\bar{G}) \)-isomorphic sections \( \bar{H}/\bar{K} \) and \( \bar{L}/\bar{K} \), where \( \bar{K}, \bar{H}, \bar{L} \in \mathcal{L}_{\sigma_i}(\bar{G}) \) (for some \( i \)) and the subgroups \( \bar{H} \) and \( \bar{L} \) cover \( \bar{K} \) in \( \mathcal{L}_{\sigma_{\text{per}}}(\bar{G}) \), coincide.

Then \( \mathcal{L} \) is distributive.

**Proof.** Suppose that this is false and let \( G \) be a counterexample of minimal order.

First note that if \( A, B, C \in \mathcal{L}_{\sigma_{\text{per}}}(G) \) and \( A \leq C \), then

\[
C \cap \langle A, B \rangle = C \cap AB = A(C \cap B) = \langle A, C \cap B \rangle
\]

by Condition (i), so the lattice \( \mathcal{L}_\sigma(G) \) is modular. Hence, by Lemma 3.6, there are distinct \( \sigma \)-permutable subgroups \( A, B \) and \( C \) of \( G \) such that for some \( \sigma \)-permutable subgroups \( E \) and \( T \) of \( G \) we have \( E = A \cap B = A \cap C = B \cap C \) and \( T = AB = AC = BC \).

(1) The lattice \( \mathcal{L}_{\sigma_{\text{per}}}(G/R) \) is distributive for each non-identity normal subgroup \( R \) of \( G \).

In view of the choice of \( G \), it is enough to show that Conditions (i), (ii), (iii) and (iv) hold for \( G/R \).

Let \( K/R, H/R \in \mathcal{L}_{\sigma_{\text{per}}}(G/R) \). Then \( K, H \in \mathcal{L}(G) \) by Lemma 3.2(1) and so \( KH = HK \) by Condition (i), which implies that \( (K/R)(H/R) = (H/R)(K/R) \). It is clear also that the lattice \( \mathcal{L}_n(G/R) \) is isomorphic to some sublattice of the lattice \( \mathcal{L}_n(G) \), so \( \mathcal{L}_n(G/R) \) is distributive. Thus Conditions (i) and (ii) hold on \( G/R \).

By Lemma 3.9 we have

\[
(G/R)^{\mathcal{L}_n} = G^{\mathcal{L}_n} R/R = DR/R.
\]

Thus

\[
(G/R)/(G/R)^{\mathcal{L}_n} = (G/R)/(DR/R) \simeq G/DR \simeq (G/D)/(GR/D)
\]

is cyclic by Condition (iii). Conditions (i) and (iii) imply that

\[
D \cap \langle K, H \rangle = D \cap \langle D \cap K, D \cap H \rangle = (D \cap K)(D \cap H)
\]

since \( D \cap K \) and \( D \cap H \) are \( \sigma \)-permutable in \( G \) by Theorem B, so

\[
(G/R)^{\mathcal{L}_n} \cap \langle (K/R), (H/R) \rangle = (DR/R) \cap (K/R)(H/R) = (DR \cap KH)/R = R(D \cap KH)/R =
\]
\[ R(D \cap K)(D \cap H)/R = ((D \cap K)R/R)((D \cap H)R/R) = \]
\[ = ((DR/R) \cap (K/R))((DR/R) \cap (H/R)) = ((G/R)^{3T} \cap (K/R)), (G/R)^{3T} \cap (H/R)). \]

Hence \((G/R)^{3T}\) is a meet-distributive element of \(L_{oper}(G/R)\). Thus Condition (iii) hold on \(G/R\). Finally, Condition (iv), evidently, hold on \(G/R\). Thus we have (1).

(2) \(E_G = 1\) (In view of Lemma 3.2, this follows from Claim (1), Lemma 3.6 and the choice of \(G\)).

(3) \(A_G B_G \cap A_G C_G \cap B_G C_G = 1\).

Since \(A \cap B = E\), we have \(B_G \cap A_G \leq E_G = 1\) by Claim (2). Similarly, \(B_G \cap C_G = 1\) and \(A_G \cap C_G = 1\). Therefore

\[ (A_G B_G \cap A_G C_G) \cap B_G C_G = A_G(B_G \cap A_G C_G) \cap B_G C_G = \]
\[ = A_G(B_G \cap A_G)(B_G \cap C_G) \cap B_G C_G = \]
\[ = A_G \cap B_G C_G = (A_G \cap B_G)(A_G \cap C_G) = 1 \]

by Claim (ii).

(4) The subgroup \(T\) is \(\sigma\)-nilpotent.

Note that

\[ T/A_G B_G = AB/A_G B_G = (AAG_B G/A_G B_G)(BAG_B G/A_G B_G), \]

where

\[ AA_G B_G/A_G B_G \simeq A/A \cap A_G B_G = \]
\[ = A/A_G(A \cap B_G) \simeq (A/A_G)/((A_G(A \cap B_G)/A_G) \]

and

\[ BAG_B G/A_G B_G \simeq (B/B_G)/(B_G(B \cap A_G)/B_G) \]

are \(\sigma\)-nilpotent by Lemma 2.4(2). The subgroups \(AA_G B_G/A_G B_G\) and \(BAG_B G/A_G B_G\) are \(\sigma\)-subnormal in \(G/A_G B_G\) by Lemmas 2.2 and 3.2(2). Hence \(T/A_G B_G\) is \(\sigma\)-nilpotent by Lemma 2.3(6). Similarly, \(T/A_G C_G\) and \(T/C_G B_G\) are \(\sigma\)-nilpotent. Hence from Claim (3) it follows that \(T \simeq T/(A_G B_G \cap A_G C_G \cap B_G C_G)\) is \(\sigma\)-nilpotent by Lemma 3.8.

(5) For some prime \(i\), there are distinct \(\sigma_i\)-subgroups \(A_i, B_i, C_i \in \mathcal{L}\) such that \(H_i = A_i B_i = A_i C_i = B_i C_i\) and \(K_i = A_i \cap B_i = A_i \cap C_i = B_i \cap C_i\) are \(\sigma\)-permutably subnormal subgroups of \(G\).

Let \(\sigma_i \in \sigma(T)\), that is, \(\sigma_i \cap \pi(T) \neq \emptyset\). Then, by Claim (4), \(H_i = O_{\sigma_i}(T)\) is the Hall \(\sigma_i\)-subgroup of \(T\) and \(A_i = O_{\sigma_i}(A)\), \(B_i = O_{\sigma_i}(B)\) and \(C_i = O_{\sigma_i}(C)\) are the Hall \(\sigma_i\)-subgroups of \(A\), \(B\) and \(C\), respectively. Hence \(H_i = A_i B_i = A_i C_i = B_i C_i\). Moreover, \(A_i, B_i\) and \(C_i\) are \(\sigma\)-permutably subnormal in \(G\) by Theorem C. It is clear also that \(K_i = A_i \cap B_i = A_i \cap C_i = B_i \cap C_i = O_{\sigma_i}(E)\).

Suppose that \(A_i = B_i\). Then \(H_i = A_i B_i = A_i = B_i = K_i \leq C_i \leq H_i\). Hence \(A_i = B_i = C_i\). Therefore, since \(A \neq B \neq C\) and \(A \neq C\), there is \(\sigma_i \in \sigma(T)\) such that \(A_i \neq B_i \neq C_i\) and \(A_i \neq C_i\). Finally, \(H_i\) and \(K_i\) are \(\sigma\)-permutably subnormal subgroups of \(G\) by Condition (i), so we have (5).
There are distinct $\sigma_i$-subgroups $A_0, B_0, C_0 \in \mathcal{L}$ such that $H_0 = A_0B_0 = A_0C_0 = B_0C_0$ and $K_0 = A_0 \cap B_0 = A_0 \cap C_0 = B_0 \cap C_0$ are $\sigma$-permutable subgroups of $G$ and $A_0, B_0, C_0$ are normal subgroups of $O^{\sigma_i}(G)$.

Let $A_0 = A_i \cap D$, $B_0 = B_i \cap D$ and $C_0 = C_i \cap D$. Then $A_0$, $B_0$ and $C_0$ are $\sigma$-permutable $\sigma_i$-subgroups of $G$ by Claim (5) and Theorem B. Moreover, Claim (5) implies that

$$K_0 = A_0 \cap B_0 = A_i \cap B_i \cap D = A_i \cap C_i \cap D = A_0 \cap C_0 = B_i \cap C_i \cap D = B_0 \cap C_0.$$ Since $D$ is a meet-distributive element of $\mathcal{L}$ by Condition (iii),

$$H_0 = D \cap A_iB_i = (D \cap A_i)(D \cap B_i) = A_0B_0 = A_0C_0 = D \cap A_iC_i = D \cap B_iC_i = B_0C_0.$$ Now we show that $A_0, B_0, C_0$ are distinct elements of $\mathcal{L}$. First note that

$$|H_i : K_i| = |A_i : K_i||B_i : K_i| = |A_i : K_i||C_i : K_i| = |B_i : K_i||C_i : K_i|,$$

so $|A_i : K_i| = |B_i : K_i| = |C_i : K_i|$. Hence $|A_i| = |B_i| = |C_i|$. Suppose that $A_0 = B_0$. Then

$$D \cap H_i = D \cap A_iB_i = (D \cap A_i)(D \cap B_i) = A_0B_0 = A_0 = B_0 = D \cap K_i.$$ Hence $K_iD \cap H_i = K_i(D \cap H_i) = K_i$ is normal in $H_i$ and

$$DH_i/DK_i \simeq H_i/(H_i \cap K_iD) = H_i/K_i(H_i \cap D) = H_i/K_i$$

is cyclic since $G/D$ is cyclic by Condition (iii). On the other hand, $H_i/K_i = (A_i/K_i)(B_i/K_i)$, where $|A_i/K_i| = |B_i/K_i|$, so $A_i/K_i = B_i/K_i = 1$, which implies that $A_i = B_i$. This contradiction shows that $A_0 \neq B_0$. Similarly, one can show that $A_0 \neq C_0$ and $B_0 \neq C_0$. Finally, $A_0, B_0, C_0$ are normal subgroups of $O^{\sigma_i}(G)$ by Lemma 2.4(1), and Claim (5) and Theorem B imply that $K_0$ and $H_0$ are $\sigma$-permutable in $G$.

(7) $A_0/K_0$ and $B_0/K_0$ are $O^{\sigma_i}(G)$-isomorphic.

From Claim (6) we get that

$$H_0/K_0 = (A_0/K_0) \times (B_0/K_0) = (A_0/K_0) \times (C_0/K_0) = (B_0/K_0) \times (C_0/K_0).$$

Therefore

$$A_0/K_0 \simeq ((A_0/K_0) \times (C_0/K_0))/(C_0/K_0) = (H_0/K_0)/(C_0/K_0)$$

and

$$B_0/K_0 \simeq ((B_0/K_0) \times (C_0/K_0))/(C_0/K_0) = (H_0/K_0)/(C_0/K_0)$$

are $O^{\sigma}(G)$-isomorphisms by Lemma 2.4(1). Hence we have (7).

Final contradiction. Let $f : A_0/K_0 \to B_0/K_0$ be a $O^{\sigma_i}(G)$-isomorphism. Let $K_0 < X \leq A_0$, where $X$ covers $K_0$ in $\mathcal{L}$. Then $X/K_0$ is a chief factor of $O^{\sigma_i}(G)$ by Lemma 2.4(3) and Proposition 3.1, so $L/K_0 = f(A_0/K_0)$ is also a chief factor of $O^{\sigma}(G)$. Hence $L$ covers $K_0$ in $\mathcal{L}$ by Lemma 2.4(1). Now $f$ induces a $O^{\sigma}(G)$-isomorphism from $X/K_0$ onto $L/K_0$ and so $L = T$ by Condition (iv). Hence $K_0 < A_0 \cap B_0$, contrary to (6).

The proposition is proved.
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