RECURRENCE RELATION FOR JONES POLYNOMIALS

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ABSTRACT. Using a simple recurrence relation we give a new method to compute Jones polynomials of closed braids: we find a general expansion formula and a rational generating function for Jones polynomials. The method is used to estimate degree of Jones polynomials for some families of braids and to obtain general qualitative results.

1. Introduction

The Jones polynomial $V_L(q)$ of an oriented link $L$ is a Laurent polynomial in the variable $\sqrt{q}$ satisfying the skein relation

$$q^{-1}V_{L_+} - qV_{L_-} = (q^{1/2} - q^{-1/2})V_{L_0},$$

and such that the value of the unknot is 1 (see [10], [12], [14]). The relation holds for any oriented links having diagrams which are identical except near one crossing where they differ as in the figure below:

Any link $L$ can be obtained as a closure of a braid $\beta \in B_n$ (for some $n$), $L = \hat{\beta}$. We will use classical Artin presentation of braids ([1], [5]) with generators $x_1, \ldots, x_{n-1}$, where $x_i$ is:

$$x_i \begin{array}{cccccc}
1 & i-1 & i & i+1 & i+2 & n \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

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Fixing the sequence \((x_{i_1}, \ldots, x_{i_k})\) of generators, \(i_h \in \{1, 2, \ldots, n-1\}\), and all the exponents \(a_1, \ldots, a_k\) (\(a_h \in \mathbb{Z}\)), but the \(j\)-th exponent is variable, we have the braid 
\(\beta(e) = x_{i_1}^{a_1} \cdots x_{i_j}^{e} \cdots x_{i_k}^{a_k} \in B_n\) and its Jones polynomial \(V_n(e) = V(\beta(e))\); in general 
\(V_n(\beta)\) stands for \(V(\beta)\), where \(\beta \in B_n\). We will freely use Artin braid relations and Markov moves in proofs and some computations.

We also change the variable \(s = q^{-1/2}\) in order to obtain, for large \(e\), polynomials in \(s\) (and not Laurent polynomials in \(\sqrt{q}\)). Our first result is:

**Theorem 1.1** (Recurrence relation). For any \(e \in \mathbb{Z}\) we have

\[ V_n(e + 2) = (s^3 - s)V_n(e + 1) + s^4V_n(e). \]

This formula shows that in computations with Jones polynomial of braids the exponents can be reduced to 0 and 1 and is nothing new here: this comes from quadratic relations in Hecke algebras and V.F.R. Jones and A. Ocneanu traces \([11], [18]\). See also \([15]\) for applications of these ideas to computations.

Systematic and elementary algebraic consequences of quadratic reduction gives us a general expansion formula and the generating function for Jones polynomials. Let us introduce two basic polynomials \(P_0^{[a]}(s) = s^{3a} + (-1)^as^{a+2}\) and \(P_1^{[a]}(s) = s^{3a-1} + (-1)^{a+1}s^{a-1}\) (if \(a\) is not positive, these are Laurent polynomials).

**Theorem 1.2** (Expansion formula). The following formula holds for braids in \(B_n\) 
\((a_1, \ldots, a_k \in \mathbb{Z}\) and \(J_* = (j_1, \ldots, j_k)\)):

\[ V_n(x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k})(s) = \frac{1}{(s^2 + 1)^k} \sum_{J_* \in \{0,1\}^k} P_{j_1}^{[a_1]}(s) \cdots P_{j_k}^{[a_k]}(s) V_n(x_1^{j_1} \cdots x_k^{j_k})(s). \]

We define the generating function for Jones polynomials corresponding to the braids \(\beta^{A_*} = x_1^{a_1} \cdots x_k^{a_k} \in B_n\) with a fixed sequence \(I_* = (i_1, \ldots, i_k) \in \{1, \ldots, n-1\}^k\), as a formal series in \(t_1, \ldots, t_k\)

\[ V_{n, I_*}(t_1, \ldots, t_k) = \sum_{A_* \in \mathbb{Z}^k} V_n(x_1^{a_1} \cdots x_k^{a_k}) t_1^{a_1} \cdots t_k^{a_k}. \]

In the next formula we use new polynomials in the formal variable \(t\): \(q(t) = (1 - s^3t)(1 + st), Q_0(t) = 1 - (s^3 - s)t, \) and \(Q_1(t) = t\).

**Theorem 1.3.** The generating function of Jones polynomials of type \((n, I_*)\) is a rational function in \(t_1, \ldots, t_k\) given by

\[ V_{n, I_*}(t_1, \ldots, t_k) = \frac{1}{q(t_1) \cdots q(t_k)} \sum_{J_* \in \{0,1\}^k} Q_{j_1}(t_1) \cdots Q_{j_k}(t_k) V_n(x_1^{j_1} \cdots x_k^{j_k}). \]

In Section 2 we give proofs for Theorems 1, 2 and 3. The necessary algebraic background is given in Appendix (for details see \([17]\)).
In Section 3 we will use recurrence relation to evaluate the degree of Jones polynomial. One result is the following:

**Proposition 1.4.** If \( e \gg 0 \), then \( V_n(e) \) is a polynomial in \( s \) and

\[
\lim_{e \to \infty} \deg V_n(e) = +\infty.
\]

There are still open problems relating Jones invariant with closed 3-braids, see J. Birman paper [6]. In Section 4 we compute Jones polynomials of 2-braids and some families of 3-braids and powers of Garside braid \( \Delta_3 = x_1x_2x_1 \) (see [9], [5]) and we establish the following results:

**Proposition 1.5.** For any \( k \geq 0 \), we have

\[
\begin{align*}
V_3(\Delta_3^{2k}) &= 2s^{12k} + s^{6k+2} + s^{6k-2}, \\
V_3(\Delta_3^{2k+1}) &= -s^{6k+5} - s^{6k+1}.
\end{align*}
\]

**Theorem 1.6.**

a) For exponents \( a_i \geq 1 \) \((i = 1, \ldots, 2L)\), Jones polynomial of the 3-braid \( \beta^A = x_1^{a_1}x_2^{a_2} \cdots x_1^{a_{2L-1}}x_2^{a_{2L}} \) of total degree \( D = \sum a_i \) satisfies:

\[
\deg V_3(x_1^{a_1}x_2^{a_2} \cdots x_1^{a_{2L-1}}x_2^{a_{2L}}) \leq 3D - 2L.
\]

b) For exponents \( a_i \geq 2 \) \((i = 1, \ldots, 2L)\) and \( k \leq 4 \) the equality holds:

\[
\deg V_3(x_1^{a_1}x_2^{a_2} \cdots x_1^{a_{2L-1}}x_2^{a_{2L}}) = 3D - 2L
\]

and the leading coefficient of \( V_3(\beta^A) \) is 1.

The degree, coefficients and breadth of Jones polynomial are well understood for special classes of links ([16], [20]). See also [19] for a study of this subject. Research in this area ([4], [8], [7]) is motivated by a natural question: are there nontrivial solutions for the equation \( V(\hat{\beta}) = 1 \) ? In Section 5 we find that:

**Theorem 1.7.** The sequence \( (V_n(e))_{e \in \mathbb{Z}} \) could contain at most two polynomials \( V_n(a) \) and \( V_n(b) \) equal to 1, and in this case \( |a - b| = 2 \).

2. Proofs of the main results

In this section we fix \( n \geq 2 \), generators \( x_{i_1}, \ldots, x_{i_k} \in B_n \), the index \( j \), and the exponents \( (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k) \). First we translate the skein relation for Jones polynomials into a recurrence relation of Fibonacci type:

\[
V_n(e + 2) = (s^3 - s)V_n(e + 1) + s^4V_n(e).
\]

**Proof of Theorem 1.7.** Let \( \gamma = \alpha x_{i_j}^e \beta \), where \( \alpha = x_1^{a_1} \cdots x_{i_{j-1}}^{a_{j-1}} \) and \( \beta = x_{i_{j+1}}^{a_{j+1}} \cdots x_k^{a_k} \) be fixed. Let us take \( e \) positive. Since the geometrical change appears only at the \( j \)-th position we draw a local picture (for the \( j \)-th factor only):
Now it is clear from the figure that the relation $q^{-1}V_{L_+} - qV_{L_-} = (q^{1/2} - q^{-1/2})V_{L_0}$ becomes, after the changes $q \to s^{-2}$ and $L_- \to \beta(e + 2)$, $L_0 \to \beta(e + 1)$ and $L_+ \to \beta(e)$, simply $V_n(e + 2) = (s^3 - s)V_n(e + 1) + s^4V_n(e)$. The negative case ($e < 0$) can be reduced to the positive case using the transformation $V_n(x_{i_1}^{a_1} \ldots x_{i_j}^{e} \ldots x_{i_k}^{a_k}) = V_n(x_{i_1}^{a_1} \ldots x_{i_j}^{m + e} \ldots x_{i_k}^{a_k})$ with $m$ big enough.

Remark 2.1. We can find a similar recurrence relation for Alexander-Conway polynomial and, in general, for HOMFLY polynomial. It is interesting to remark that in the classical cases, Jones and Alexander, the roots of the characteristic equation are rational functions: $r_1 = -s$, $r_2 = s^3$ for Jones polynomial and $r_1 = s$, $r_2 = -s^{-1}$ for Alexander polynomial. Jones recurrence is nicer because the roots are polynomials. For Alexander-Conway polynomial and more results on the roots of the characteristic equation for HOMFLY polynomial, see [2].

In Appendix (or see [17] for full details), multiple Fibonacci sequences are introduced. Our main example is the multiple Fibonacci sequence given by Jones polynomials of closures of braids $V_n,(x_{i_1}^{a_1}x_{i_2}^{a_2} \ldots x_{i_k}^{a_k})$, where $a_1, \ldots, a_k \in \mathbb{Z}$; we fix $n$ (all braids are in $B_n$), $k$, and also $I_*= (i_1, \ldots, i_k)$ with indices $i_h \in \{1, \ldots, n-1\}$. Applying Theorem 6.1 we obtain:

Proof of Theorem 1.2. From the basic recurrence relation, we have $r^2 = (s^2 - s)r + s^4$, with the roots $r_1 = -s$ and $r_2 = s^3$. Hence $D = s^3 + s = s(s^2 + 1)$, $S_0^{[n]} = s^{3n+1} + (-1)^ns^{n+3} = s[s^{3n} + (-1)^ns^{n+2}]$, $S_1^{[n]} = s^{3n} + (-1)^{n+1}s^n = s[s^{3n-1} + (-1)^{n+1}s^{n-1}]$.

Now, let us introduce two new sequences of polynomials (if $n \geq 1$), $P_0^{[n]}(s) = s^{3n} + (-1)^n s^{n+2}$ and $P_1^{[n]}(s) = s^{3n-1} + (-1)^{n+1}s^{n-1}$. Using Theorem 6.1 we get

$$V_n(x_{i_1}^{a_1}x_{i_2}^{a_2} \ldots x_{i_k}^{a_k}) = \frac{1}{s^k(s^2 + 1)^k} \sum_{J_+ \in \{0,1\}^k} (sP_{j_1}^{[a_1]})(sP_{j_2}^{[a_2]})(sP_{j_k}^{[a_k]})V_n(x_{j_1}^{i_1} \ldots x_{j_k}^{i_k})$$
\[ \frac{1}{(s^2 + 1)^k} \sum_{J_r \in \{0, 1\}^k} P_{j_1}^{[a_1]} \ldots P_{j_k}^{[a_k]} V_n(x_{i_1}^{j_1} \ldots x_{i_k}^{j_k}). \]

**Proof of Theorem 1.3.** In the second part of Theorem 6.1 take \( r_1 = -s \) and \( r_2 = s^3 \).

**Remark 2.2.**

a) The expansion formula has \( 2^k \) terms, where \( k \) is the number of factors of the braid \( \beta = x_{i_1}^{a_1} \ldots x_{i_k}^{a_k} \), less than \( 2|a_1| + \cdots + |a_k| \), the number of terms in Kauffman expansion (see [13], [15]).

b) This formula reduces the computation of Jones polynomial of \( \beta \) to the computation of Jones polynomial of (simpler) positive braids \( \beta_J = x_{i_{i_1}}^{j_1} \ldots x_{i_k}^{j_k} \), with \( J_r \in \{0, 1\}^k \). If these braids contain exponents \( \geq 2 \) (after possible concatenations the number of factors is less than \( k \)), another application of expansion formula will reduce degrees and the number of factors. Therefore, iterated application of the expansion formula reduces computation to Jones polynomials of Markov square-free braids (see [3] for definition and proofs) \( \beta_I = x_{i_1} x_{i_2} \ldots x_{i_k} \), where \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n - 1 \), and in this case \( V_n(\beta_I) = (-s - s^{-1})^{n-k-1} \).

3. Degree of Jones Polynomial

First we study the behavior of the function \( e \mapsto \deg V_n(\beta(e)) = \deg V(e) \) for an \( n \)-braid \( \beta(e) = x_{i_1}^{a_1} \ldots x_{i_j}^{a_j} \ldots x_{i_k}^{a_k} \). More precise results are given for some families of 3-braids in Section 4. If the Laurent polynomial \( f = a_q s^q + a_{q+1} s^{q+1} + \cdots + a_p s^p \) has coefficients \( a_q, a_p \neq 0 \), we denote the degree \( \deg(f) = p \), the order \( \text{ord}(f) = q \), and the leading coefficient \( \text{coeff}(f) = a_p \); we also use the convention \( \deg(0) \leq 0 \).

**Definition 3.1.**

a) \( \beta(e), \beta(e + 1) \) is a **stable pair** if
\[ \deg V(e + 1) > 1 + \deg V(e). \]

b) \( \beta(e), \beta(e + 1) \) is a **semistable pair** if
\[ \deg V(e + 1) \leq \deg V(e). \]

c) \( \beta(e), \beta(e + 1) \) is a **critical pair** if
\[ \deg V(e + 1) = 1 + \deg V(e). \]

**Proposition 3.2 (stable case).** If \( \beta(e), \beta(e + 1) \) is a stable pair then for all \( m \in \mathbb{N} \) the pair \( \beta(e + m), \beta(e + m + 1) \) is also stable, and for \( m \geq 1 \)
\[ \deg V(e + m) = \deg V(e + 1) + 3(m - 1) \]
and \( \beta(e + 1) \) and \( \beta(e + m) \) have the same leading coefficient.
Proof. We prove the statement by induction on \( m \). Hypothesis gives the case \( m = 0 \). If \( \beta(e + m - 1) \), \( \beta(e + m) \) is a stable pair, then Theorem \[ \text{(3.1)} \] implies

\[
V(e + m + 1) = (s^3 - s)V(e + m) + s^4V(e + m - 1)
\]
and \( \deg V(e+m) > 1 + \deg V(e+m-1) \) implies \( \deg V(e+m+1) = \deg V(e+m)+3 = \deg V(e+1) + 3m \) (hence \( \beta(e + m) \), \( \beta(e + m + 1) \) is stable) and \( \text{coeff} V(e + m + 1) = \text{coeff} V(e + m) \).

**Proposition 3.3** (semistable case). If \( \beta(e) \), \( \beta(e+1) \) is a semistable pair then for all \( m \geq 1 \) the pair \( \beta(e+m) \), \( \beta(e + m + 1) \) is stable, and for \( m \geq 2 \),

\[
\deg V(e + m) = \deg V(e) + 3m - 2
\]
and also \( \beta(e) \) and \( \beta(e + m) \) have the same leading coefficient.

**Proof.** The pair \( \beta(e) \), \( \beta(e+1) \) is semistable and the recurrence

\[
V(e + 2) = (s^3 - s)V(e + 1) + s^4V(e)
\]
implies \( \deg V(e + 2) = \deg V(e) + 4 > \deg V(e + 1) + 1 \), hence the pair \( \beta(e + 1) \), \( \beta(e + 2) \) is stable and we can apply Proposition \[ \text{3.2} \] below.

In the last (critical) case we cannot obtain complete results; see the Case 2c) below.

**Proposition 3.4** (critical case). Let \( \beta(e) \), \( \beta(e+1) \) be a critical pair and \( C \) the sum of leading coefficients of \( \beta(e) \) and \( \beta(e+1) \).

**Case 1) If** \( C \neq 0 \), **then for all** \( m \geq 1 \) **the pair** \( \beta(e+m) \), \( \beta(e + m + 1) \) **is stable, and**

\[
\deg V(e + m) = \deg V(e + 1) + 3(m - 1),
\]
and also the leading coefficient of \( V(e + m) \) is \( C \).

**Case 2) If** \( C = 0 \) **and**

2a) \( \deg V(e + 2) = 2 + \deg V(e + 1) \), **then for all** \( m \geq 1 \) **the pair** \( \beta(e + m) \), \( \beta(e + m + 1) \) **is stable and for all** \( m \geq 2 \) **the following holds**

\[
\deg V(e + m) = \deg V(e + 1) + 3m - 2.
\]

2b) \( \deg V(e + 2) \leq \deg V(e + 1) \), **then** \( \beta(e + 2) \), \( \beta(e + 3) \) **is a stable pair and for all** \( m \geq 2 \) **the pair** \( \beta(e + m) \), \( \beta(e + m + 1) \) **is stable. For all** \( m \geq 3 \) **we have the relation**

\[
\deg V(e + m) = \deg V(e + 1) + 3m - 5.
\]

2c) \( \deg V(e + 2) = 1 + \deg V(e + 1) \), **then the pair** \( \beta(e + 1) \), \( \beta(e + 2) \) **is critical.**
Proof. Case 1) We prove it by induction on $m$. For $m = 1$, Theorem 1.1 implies

$$V(e + 2) = (s^3 - s)V(e + 1) + s^4V(e),$$

and condition $C \neq 0$ implies that $V(e + 2)$ has degree $\deg V(e + 1) + 3$ and leading coefficient $C$. We can apply Proposition 3.2 for the stable pair $\beta(e + 1), \beta(e + 2)$.

Case 2) The hypotheses imply that $\beta(e + 1), \beta(e + 2)$ is a stable, b) semistable, and c) critical, respectively. □

Remark 3.5. In the critical case the degree of $V(e + 2)$ can be arbitrarily small compared to degree of $V(e)$; see, in Section 4, the family $V_3(a_1, 1, 3, 1)$ and also the degrees in the sequence $V_\Delta(6k + 1), V_\Delta(6k + 2), V_\Delta(6k + 3)$.

The order of $V(e)$ is increasing for large value of $e$.

Proposition 3.6. With the same notations we have:

a) $\ord V(e + 2) \geq 1 + \min(\ord V(e), \ord V(e + 1))$.

b) For any $m \geq 2$, $\ord V(e + m) \geq \min(\ord V(e), \ord V(e + 1)) + (m - 1)$.

In particular, if $e \gg 0$, then $V(e)$ is a polynomial in $s$.

Proof. a) is a direct consequence of Theorem 1.1 and b) comes from a) by induction. □

Proof of Proposition 1.4. First choose $e_0 \geq e$ such that for any $m \geq e_0$, $V(m)$ is a polynomial in $s$, as in the previous proposition. Let us suppose that in the sequence $\beta(e_0), \beta(e_0 + 1), \ldots$ there is a stable (or semistable) pair, say $\beta(e_1), \beta(e_1 + 1)$. Then according to Proposition 3.2 (or Proposition 3.3), all the consecutive pairs in the sequence $\beta(e_1 + 1), \beta(e_1 + 2), \ldots$ are stable. Therefore $\deg V(e_1 + m) \to \infty$ as $m \to \infty$. If any pair of the sequence $\beta(e_0), \beta(e_0 + 1), \ldots$ is critical, then the sequence $\deg \beta(e_0), \deg \beta(e_0 + 1), \ldots$ is arithmetic and again we get the desired result.

Second Proof. We can use Proposition 3.6 along with the fact that Jones polynomial cannot be 0. □

Similar results can be proved for negative exponents.

Proposition 3.7. With $e < 0$ we have:

a) $\deg V(e - 2) \leq \max(\deg V(e - 1), \deg V(e)) - 1$.

b) For any $m \geq 2$, $\deg V(e - m) \leq \max(\deg V(e - 1), \deg V(e)) - (m - 1)$.

In particular, for $e \ll 0$, $V(e)$ is a polynomial in $s^{-1}$ and

$$\lim_{e \to -\infty} \deg V(e) = -\infty.$$

Proof. Use the recurrence relation in the form $V(e - 2) = (s^{-3} - s^{-1})V(e - 1) + s^{-4}V(e)$. □
4. Examples

Three types of examples are given: arbitrary braids in $B_2$, braids $x_1^{a_1} x_2^{a_2} \ldots x_2^{a_{2L}}$ in $B_3$ (with complete results for $L \leq 2$) and powers of Garside braid $\Delta_3$. Some of the results are well known, especially those which are connected with torus links, some seem to be new, and all of them show how to use the recurrence relation.

**Proposition 4.1.** Let $x_1^a$, $a \in \mathbb{Z}$, be a braid in $B_2$, and $\hat{x}_1^a$ the corresponding link. Its Jones polynomial is given by: for $a \leq -2$

$$V_2(a) = -s^{3a+1} + s^{3a+3} - s^{3a+5} + \ldots - (-1)^a s^{-5} + (-1)^{a+1} s^{-3} + (-1)^{a+1} s^{a+1},$$

for the next three values

$$V_2(-1) = 1, V_2(0) = -s - s^{-1}, V_2(1) = 1,$$

and for $a \geq 2$

$$V_2(a) = -s^{3a-1} + s^{3a-3} - s^{3a-5} + \ldots - (-1)^a s^{a+5} + (-1)^{a+1} s^{a+3} + (-1)^{a+1} s^{a-1}.$$

**Proof.** The Jones polynomials of the trivial two-component link and trivial knot are given by

$$V_2(0) = -s - s^{-1}, V_2(\pm 1) = 1. \quad (4.1)$$

From the basic recurrence relation the general term is

$$V_2(a) = \left(\frac{-s}{1 + s^2}\right) (s^3)^a + \left(\frac{-1 - s^2 - s^4}{s + s^3}\right) (-s)^a. \quad (4.2)$$

Elementary computations give the desired result.

For $a \leq -2$, the coefficients $\frac{-s}{1 + s^2}$ and $\frac{-1 - s^2 - s^4}{s + s^3}$ are invariant under $s \rightarrow s^{-1}$, therefore

$$V_2(a)(s) = \left(\frac{-s}{1 + s^2}\right) (s^3)^a + \left(\frac{-1 - s^2 - s^4}{s + s^3}\right) (-s)^a = V_2(-a)(s^{-1}), \quad (4.3)$$

and this proves the formula for negative exponents. \qed

**Proposition 4.2.** Let $\alpha, \beta \in B_n$, $\gamma = \alpha \beta$, and $\tilde{\gamma}_k = \alpha x_n^k \beta \in B_{n+1}$ ($k \in \mathbb{Z}$), then

$$V_{n+1}(\tilde{\gamma}_k) = V_n(\gamma) V_2(x_1^k).$$

**Proof.** Define $f(k) = V_{n+1}(\tilde{\gamma}_k)$ and $g(k) = V_n(\gamma) V_2(x_1^k)$. These coincide for $k = 0$: $g(0) = V_n(\gamma) V_2(1) = V_n(\gamma)(-s - s^{-1})$ and $f(0) = V_{n+1}(\tilde{\gamma}_0) = V_{n+1}(\gamma) = V_n(\gamma)(-s - s^{-1})$, and $k = 1$: $g(1) = V_n(\gamma) V_2(x_1) = V_n(\gamma)$ and $f(1) = V_{n+1}(\alpha x_n \beta) = V_{n+1}(\beta x_n \alpha) = V_n(\beta \alpha) = V_n(\alpha \beta) = V_n(\gamma)$; also $f(k)$ and $g(k)$ satisfy the same recurrence relation. \qed

**Corollary 4.3.** $V_3(x_1^{a_1} x_2^{a_2}) = V_2(x_1^{a_1}) V_2(x_1^{a_2}).$
Now we compute the degree and the leading coefficient for Jones polynomial $V_3(a_1, a_2, a_3, a_4) = V_3(x_1^{a_1}x_2^{a_2}x_1^{a_3}x_2^{a_4})$, where $a_i \geq 1$. We denote by $D$ the total degree $a_1 + a_2 + a_3 + a_4$.

**Proposition 4.4** (generic case). If $a_1, a_2, a_3, a_4 \geq 2$, then the leading term of Jones polynomial $V_3(a_1, a_2, a_3, a_4)$ is $+s^{3D-4}$.

**Proof.** Using the general expansion formula we find

$$V_3(a_1, a_2, a_3, a_4) = \frac{1}{(s^2 + 1)^4} \sum_{j_4 \in \{0, 1\}^4} P_{j_1}^{[a_1]} P_{j_2}^{[a_2]} P_{j_3}^{[a_3]} P_{j_4}^{[a_4]} V_3(j_1, j_2, j_3, j_4).$$

The degree and leading coefficient of $V_3(j_1, j_2, j_3, j_4)$ are in Table 2 (proof of Theorem 1.6 contains a user guide for the table). In the above formula maximal degree is obtained from $V_3(1, 0, 1, 0) = V_3(0, 1, 0, 1)$ (coefficient $1 + 1$) and from $V_3(1, 1, 1, 1)$ (coefficient $-1$).

**Proposition 4.5.** The leading terms of Jones polynomials $V_3(a_1, a_2, a_3, a_4)$ for positive exponents are given by the next table:

| $(a_1, a_2, a_3, a_4)$ | Critical cases | Stable cases |
|-----------------------|----------------|--------------|
| $(1, 1, a_3, a_4)$   | $-s^{4D-4} + \ldots$ | $a_3, a_4 \geq 1 : -s^{4D-4} + \ldots$ |
| $(a_1, 2, 1)$        | $a_1 = 2 : 2s^{3D-6} + \ldots$ | $a_1 \geq 3 : s^{3D-6} + \ldots$ |
| $(a_1, 1, 3, 1)$     | $a_1 = 3 : -s^{3D-8} + \ldots$ | $a_1 \geq 5 : s^{3D-6} + \ldots$ |
| $(a_1, 1, a_3, 1)$   | $a_1 = 4 : -s^{3D-16} + \ldots$ | $a_1, a_3 \geq 4 : s^{3D-8} + \ldots$ |
| $(a_1, 1, 3, 2)$     | $-s^{3D-16} + \ldots$ | $a_1 \geq 3 : -s^{3D-6} + \ldots$ |
| $(a_1, 1, a_3, a_4)$ | $a_1, a_3 \geq 3 : -s^{3D-6} + \ldots$ | $a_1, a_2, a_3, a_4 \geq 2 : s^{3D-4} + \ldots$ |

**Proof.** For the first line we can use $x_1x_2x_1^{a_1}x_2^{a_2} = x_2x_1^{a_1}x_2^{a_4+1} \sim x_1^{a_1+a_4+1}x_2$. In the second line we start the recurrence with $V_3(1, 1, 2, 1)$ and $V_3(2, 1, 2, 1) = V_3(6)$, a critical case with $C = 1$. For the family $V_3(a_1, 1, 3, 1)$ the recurrence starts with $V_3(1, 1, 3, 1)$ and $V_3(2, 1, 3, 1)$ and we obtain (a critical case) $V_3(3, 1, 3, 1) = -s^{16} + s^{10} + s^6$ and $V_3(4, 1, 3, 1) = -s^{11} - s^7$ and this is semistable. In the generic case $V_3(a_1, 1, a_3, 1)$, $a_1, a_3 \geq 4$, the expansion formula has only 4 $J_i$-blocks:

$$P_0^{[a_1]} P_0^{[a_3]} (s^6 + \ldots) + \left( P_1^{[a_1]} P_0^{[a_3]} + P_0^{[a_1]} P_1^{[a_3]} \right) (-s - s^{-1}) + P_1^{[a_1]} P_1^{[a_3]} (-s^8 + \ldots).$$

The case $V_3(a_1, 1, 3, 2)$ starts with the semistable pair $V_3(1, 1, 3, 2) = -s^{17} + \ldots$ and $V_3(2, 1, 3, 2) = -s^{16} + \ldots$. The line $(a_1, 1, a_3, a_4)$ is given by the expansion formula with 8 $J_i$-blocks. □
The missing cases \((2, 1, a_3, a_4)\) and \((a_1, 1, 2, a_4)\) can be reduced to the previous list: \(x_1^2 x_2 x_1^{a_3} x_2^{a_4} = x_1 x_2^{a_3} x_1^2 = x_2^{a_4} x_2^{a_3+1} = x_2 x_1 x_2^2 x_1^{a_4} = x_1 x_2 x_1^{a_3+1} x_2 = x_1 x_2^{a_4+1} x_1 x_2^{a_3} \sim x_1 x_2^{a_4+1} x_2^2 \). If \(a_1 \geq 3, a_3 \geq 4\), then \((a_1, 1, 3, 2)\) can be reduced, too: \(x_1^{a_1} x_2 x_1^{a_2} x_2 = x_2 x_1 x_2^{a_1} x_1 x_2^{a_2} \sim x_1 x_2 x_1^{a_1} x_2^{a_2-1} x_2^2 \sim x_3 x_2 x_1^{a_1} x_2^{a_2-1} \).

Now our purpose is to compute Jones polynomial of the braid \(\alpha(n) = x_1 x_2 x_1 x_2 \ldots (n \text{ factors})\); this sequence contains the powers of \(\Delta = \Delta_3 = \Delta_k\). We will use the next table where \(X\) is the canonical form of \(\alpha(n)\) (i.e. the smallest word in the length-lexicographic order with \(x_1 < x_2\)) and \(Y\) is a conjugate of \(X\), suitable for computations. The number of factors of the six \(Y\)’s is \(2k + 2\).

| \(\alpha(n)\) | \(X\) | \(Y\) |
|---|---|---|
| \(\Delta^{2k}\) | \(x_1^{2k} x_2 x_1^{i_2} x_2 \ldots x_1^{i_2} x_2\) | \(x_1^{2k} x_2 x_1^{i_2} x_2 \ldots x_1^{i_2} x_2\) |
| \(\Delta^{2k} x_1\) | \(x_1^{2k+1} x_2 x_1^{i_2} x_2 \ldots x_1^{i_2} x_2\) | \(x_1^{2k+1} x_2 x_1^{i_2} x_2 \ldots x_1^{i_2} x_2\) |
| \(\Delta^{2k} x_2\) | \(x_1^{2k+1} x_2 x_1^{i_2} x_2 \ldots x_1^{i_2} x_2\) | \(x_1^{2k+1} x_2 x_1^{i_2} x_2 \ldots x_1^{i_2} x_2\) |
| \(\Delta^{2k+1}\) | \(x_1^{2k+2} x_2 x_1^{i_2} x_2 \ldots x_1^{i_2} x_2\) | \(x_1^{2k+2} x_2 x_1^{i_2} x_2 \ldots x_1^{i_2} x_2\) |
| \(\Delta^{2k+1} x_2\) | \(x_1^{2k+2} x_2 x_1^{i_2} x_2 \ldots x_1^{i_2} x_2\) | \(x_1^{2k+2} x_2 x_1^{i_2} x_2 \ldots x_1^{i_2} x_2\) |
| \(\Delta^{2k+1} x_2 x_1\) | \(x_1^{2k+2} x_2 x_1^{i_2} x_2 \ldots x_1^{i_2} x_2\) | \(x_1^{2k+2} x_2 x_1^{i_2} x_2 \ldots x_1^{i_2} x_2\) |

In order to simplify the notation we denote by \(V_\Delta(n) = V_3(\alpha(n))\) Jones polynomial of the closure of 3-braid \(x_1 x_2 x_1 x_2 \ldots (n \text{ factors})\).

**Proposition 4.6.** The Jones polynomials \(V_\Delta(n)\) satisfy the following recurrence relations:

\[
\begin{align*}
V_\Delta(2k + 1) &= (s^3 - s)V_\Delta(2k) + s^4 V_\Delta(2k - 1) \\
V_\Delta(6k + 4) &= (s^3 - s)V_\Delta(6k + 3) + s^4 V_\Delta(6k + 2) \\
V_\Delta(6k + 2) &= (s^3 - s)V_\Delta(6k + 1) + (s^7 - s^5)V_\Delta(6k - 1) + s^8 V_\Delta(6k - 2) \\
V_\Delta(6k) &= (s^3 - s)[V_\Delta(6k - 1) + s^4 V_\Delta(6k - 3) + s^8 V_\Delta(6k - 5)] + \\
&\quad + s^{12} V_\Delta(6k - 6).
\end{align*}
\]

**Proof.** The proof is by induction. For the first relation, the case \(6k + 5\) is given by the basic recurrence relation and the table. We give a general proof, using a different idea: \(\alpha(2k - 1) = x_1 x_2 \ldots x_1 (2k - 1 \text{ factors}), \alpha(2k) = x_1 x_2 \ldots x_2 (2k \text{ factors}), \alpha(2k + 1) = x_1 x_2 \ldots x_1 x_2 x_1 \sim x_2 x_1 \ldots x_1 x_2 \sim x_1 x_2 \ldots x_1 x_2^2\) and now we can apply the basic recurrence relation.

\(V_\Delta(6k + 4)\) is also given by the basic recurrence relation with \(V_3(x_1^{2k+3} x_2 x_1 \ldots), V_3(x_1^{2k+2} x_2 x_1 \ldots)\) and \(V_3(x_1^{2k+1} x_2 x_1 \ldots)\). For the last two relation we have to apply Theorem 4.1 two or three times.

To compute \(V_\Delta(6k + 2)\) we use the basic recurrence relation twice: first, the recurrence relation among \(\alpha(6k + 2) = x_1^{2k+1} x_2 x_1 \ldots x_2 x_2^2, \alpha(6k + 1) = x_1^{2k+1} x_2 x_1 \ldots x_2 x_2^2\) and \(x_1^{2k+1} x_2 x_1 \ldots x_2 x_2^2\) which is conjugate to \(x_1^{2k+3} x_2 x_1 \ldots x_2^2 = \beta\). The new braid
\(\beta, \alpha(6k - 1) \sim x_1^{2k+2}x_2^2x_1^{2} \ldots x_2^{2} \) and \(\alpha(6k - 2) \sim x_1^{2k+1}x_2^2x_1^{2} \ldots x_2^{2}\) are related by the basic recurrence relation.

Finally, to verify the last formula we start with \(\alpha(6k) \sim x_1^{2k}x_2^2x_1^{2} \ldots x_2^{3}\). Changing the under- and over-crossing status in the second last crossing we get the braid \(x_1^{2k}x_2^2x_1^{2} \ldots x_1^{2}x_2^{1} \sim x_1^{2k+2}x_2^2x_1^{2} \ldots x_1^{2}x_2^{2}\) whose Jones polynomial is \(V_\Delta(6k - 1)\). Destroying the same crossing we obtain the braid word \(x_1^{2k}x_2^2x_1^{2} \ldots x_1^{2}x_2^{2}\) with \(2k\) letters. We denote it by \(\gamma\) and we obtain \(V_\Delta(6k) = (s^3 - s)V_\Delta(6k - 1) + s^4V_3(\gamma)\). To compute \(V_3(\gamma)\), we interchange the over- and under-crossing status of the last crossing of \(\gamma\) and obtain \(x_1^{2k}x_2^2x_1^{2} \ldots x_1^{2}x_2^{2}\) whose Jones polynomial is \(V_\Delta(6k - 3)\). The elimination of last crossing of \(\gamma\) gives \(x_1^{2k}x_2^2x_1^{2} \ldots x_1^{2}x_2^{1}\) which we denote by \(\delta\). Hence \(V_3(\gamma) = (s^3 - s)V_\Delta(6k - 3) + s^4V_3(\delta)\). The smoothing of first crossing of \(\delta\) ultimately gives the relation \(V_3(\delta) = (s^3 - s)V_\Delta(6k - 5) + s^4V_\Delta(6k - 6)\) and the result follows. \(\square\)

**Lemma 4.7.** The first Jones polynomials are given by:

\[
\begin{align*}
V_\Delta(0) & = s^2 + 2 + s^{-2} \\
V_\Delta(1) & = -s - s^{-1} \\
V_\Delta(2) & = 1 \\
V_\Delta(3) & = -s^5 - s \\
V_\Delta(4) & = -s^8 + s^6 + s^2 \\
V_\Delta(5) & = -s^{11} + s^9 - s^7 - s^3.
\end{align*}
\]

**Proof.** First three are obvious; next use Proposition 4.6 for \(V_\Delta(3), V_\Delta(4)\) and \(V_\Delta(5)\). \(\square\)

**Remark 4.8.** The recurrence relations of Proposition 4.6 give the following recurrence matrix:

\[
\begin{pmatrix}
V_\Delta(6k) \\
V_\Delta(6k + 1) \\
\vdots \\
V_\Delta(6k + 5)
\end{pmatrix}
= A(k)
\begin{pmatrix}
V_\Delta(6k - 6) \\
V_\Delta(6k - 5) \\
\vdots \\
V_\Delta(6k - 1)
\end{pmatrix},
\]

where the Jordan normal form of \(A(k)\) has a nice structure:

\[
A(k) \sim
\begin{pmatrix}
J_3 & 0 & 0 & 0 \\
0 & s^6 & 0 & 0 \\
0 & 0 & s^{12} & 0 \\
0 & 0 & 0 & s^{18}
\end{pmatrix},
\]

\[
J_3 =
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
\]

This fact is reflected in the following very simple general formulae of Jones polynomials \(V_\Delta(n)\).
Proposition 4.9. For any $k \geq 0$ Jones polynomials of $\alpha(n) = x_1 x_2 x_1 \ldots (n\text{-times})$ are given by:

$$
\begin{align*}
V_\Delta(6k) &= 2s^{12k} + s^{6k+2} + s^{6k-2} \\
V_\Delta(6k + 1) &= s^{12k+3} - s^{12k+1} - s^{6k+3} - s^{6k-1} \\
V_\Delta(6k + 2) &= -s^{12k+4} + s^{6k+4} + s^{6k} \\
V_\Delta(6k + 3) &= -s^{6k+5} - s^{6k+1} \\
V_\Delta(6k + 4) &= -s^{12k+8} + s^{6k+6} + s^{6k+2} \\
V_\Delta(6k + 5) &= -s^{12k+11} + s^{6k+9} - s^{6k+7} - s^{6k+3}.
\end{align*}
$$

Proof. The proof is by induction: the case $k = 0$ is covered by Lemma 4.7 and induction step can be checked with Proposition 4.6. □

Now we analyze Jones polynomial of general positive 3-braids $\beta^{A_*} = x_1^{a_1} x_2^{a_2} \ldots x_2^{a_{2L}}$, where $A_* = (a_1, ..., a_{2L})$, $a_i \geq 0$; we use the short notations $D = \sum_{i=1}^{2L} a_i$, the total degree of $\beta^{A_*}$, $Z = \text{card}\{i|1 \leq i \leq 2L, a_i = 0\}$. As an example, $J_*^{1,0} = (1, 0, \ldots, 1, 0)$ and $J_*^{0,1} = (0, 1, \ldots, 0, 1)$ have the same total sum and number of zeros $J_*^{1,0} = J_*^{1,0} = L = Z$. For a positive sequence $A_*$ and $J_* \in \{0, 1\}^{2L}$, the $J_*$-block is the Laurent polynomial in $s$ $P^{A_*}_{J_*} V_{J_*} = P^{[a_1]}_{J_1} P^{[a_2]}_{J_2} \ldots P^{[a_{2L}]}_{J_{2L}} V_3(\beta^{J_*})$.

Lemma 4.10. If all $a_i$ are positive, the inequality holds:

$$\deg P^{A_*}_{J_*} \leq 3D - J,$$

with equality if and only if $j_i = 0$ implies $a_i \geq 2$.

Proof. By definition, $\deg P^{[1]}_0 \leq 0$, $\deg P^{[a]}_0 = 3a$ if $a \geq 2$, and $\deg P^{[a]}_1 = 3a - 1$ if $a \geq 1$. □

We start the proof of Theorem 1.6 with a more general result:

Theorem 4.11. If the positive 3-braid $\beta^{A_*} = x_1^{a_1} x_2^{a_2} \ldots x_2^{a_{2L-1}} x_1^{a_{2L}}$ has degree $D$ and number of zeros $Z$, then the inequality holds:

$$\deg V_3(\beta^{A_*}) \leq 3D - 2L + 2Z.$$

Lemma 4.12. The inequality in Theorem 4.11 for $L - 1$ factors and the inequality in Theorem 1.6 for $L$ factors imply the inequality in Theorem 4.11 for $L$ factors.

Proof. If the braid $\beta^{A_*} = x_1^{a_1} x_2^{a_2} \ldots x_2^{a_{2L}}$ has all exponents $\geq 1$, then Theorem 1.6 $(L)$ gives the result. If at least one exponent is zero, then $\beta^{A_*} = \beta^{A'_*}$, where $A'_*$ is a sequence of length $2(L - 1)$ obtained from $A_*$ by deletion of a zero exponent and concatenation of its neighbors: as an example, if $A_* = (2, 1, 0, 3, 4, 0)$, then $A'_*$ can be $(2, 4, 4, 0)$ or $(6, 1, 0, 3)$ (any choice gives the braid $x_1^6 x_2^4$). The new ingredients of
\[ A' \text{ are } D' = D, \ L' = L - 1, \text{ and } Z' = Z - 1 \text{ or } Z - 2 \text{ (if we delete one zero and concatenate two others), therefore}
\]
\[ \deg V_3(\beta^{A'}) = \deg V_3(\beta^{A'}) \leq 3D' - 2L' + 2Z' \leq 3D - 2L + 2Z. \]

\[ \square \]

**Proof of Theorem 1.6.**  

a) We start induction on \( L \geq 1 \). If \( L = 1 \), \( V_3(x_1^{a_1}x_2^{a_2}) = V_2(x_1^{a_1})V_2(x_1^{a_2}) \), and we have, up to a symmetry, the next cases:

| \((a_1, a_2)\) | \(\deg V_3(x_1^{a_1}x_2^{a_2})\) | \(3D - 2L + 2Z\) |
|-----------------|-------------------|------------------|
| \((0, 0)\)      | \(2\)              | \(2\)            |
| \((1, 0)\)      | \(1\)              | \(3\)            |
| \((1, 1)\)      | \(\leq 0\)         | \(4\)            |
| \((\geq 2, 0)\) | \(3a_1\)           | \(3a_1\)         |
| \((\geq 2, 1)\) | \(3a_1 - 1\)       | \(3a_1 + 1\)     |
| \((\geq 2, 2)\) | \(3a_1 + 3a_2 - 2\) | \(3a_1 + 3a_2 - 2\) |

Suppose that \( L \geq 2 \). According to Lemma 8 it is enough to show that any \( J_* \)-block of the expansion formula for \( V_3(\beta^{A_*}) \) \((a_i \geq 1)\)

\[ (s^2 + 1)^{2L}V_3(\beta^{A_*}) = \sum_{J_* \in \{0, 1\}^{2L}} P_{J_*}^{A_*}V_{J_*} \]

has \( \deg \leq 3D + 2L \); because there is no factor \( P_{J_*}^{[0]} \), we have \( \deg P_{J_*}^{A_*} \leq 3D - J \), and it is enough to show that \( \deg V_{J_*} \leq J + 2L \). We begin with terms having a small contribution:

- **case 1:** \( J \leq L - 1 \). In this case the 0,1 sequence \( J_* \) contains at least two zeros which are neighbors \((j_{2L} \text{ and } j_1 \text{ are neighbors})\), we delete both and obtain a new sequence \( J'_* \) of length \( 2(L - 1) \) with \( J' = J \) and number of zeros \( Z' = 2(L - 1) - J \). Theorem 4.11 \((L - 1)\) gives \( \deg V_{J_*} \leq 3J' - 2L' + 2Z' = J + 2L - Z \).

- **case 2:** \( J = L \) but \( J_* \) is different from \( J_*^{1,0} \) and \( J_*^{0,1} \). This is similar with case 1 because one can find two zero neighbors.

Now we are looking for the main contributors:

- **case 3:** \( J_*^{1,0} \) and \( J_*^{0,1} \). Their Jones polynomials coincide with \( V_3(x_1^L) \) of degree \( 3L = J + 2L \).

- **case 4:** \( L + 1 \leq J \leq 2L - 1 \). Consider the new sequence \( J'_* \) obtained after deletion of all zeros and concatenation of (nonzero) neighbors, if necessary. For this new sequence we have \( J' = J \), \( Z' = 0 \) and deletion of \( i \) consecutive zeros reduces the length of \( J_* \) by \( i^* = 2(i - \left\lfloor \frac{i}{2} \right\rfloor) = i \) (for \( i \) even) and \( i + 1 \) (for \( i \) odd). Denote by \( Z_i \) the number of sequences of \( i \) consecutive zeros; for example, if \( J_* = (1, 0, 1, 0, 0, 1, 1, 1, 1, 0) \), then \( J'_* = (3, 1, 1, 1) \) and \( Z_1 = 2, Z_2 = 1, Z_{\geq 3} = 0 \).
Therefore the total number of zeros in $J_*$ is $Z = \sum_{i \geq 1} iZ_i = 2L - J$ and the length of $J'$ is $2L' = 2L - \sum_{i \geq 1} i^+Z_i$, so we can evaluate the degrees:

$$\deg V_{J_*} = \deg V_{J'_*} \leq 3J' - 2L' = 3J - 2L + \sum_{i \geq 1} i^+Z_i = 3J - 2L + 2\sum_{i \geq 1} iZ_i - 2\sum_{i \geq 2} \left\lfloor \frac{i}{2} \right\rfloor Z_i \leq 3J - 2L + 2(2L - J) = J + 2L.$$

**case 5:** $J_* = (1, 1, \ldots, 1)$. This is the example studied in Proposition 4.9 with $2L$ factors and we found the general formula

$$\deg V_{\Delta(2L)} = 4L = J + 2L.$$

**Proof of Theorem 1.6 b) The case $L = 1$ is a consequence of Proposition 4.1 and Corollary 4.3.** The next cases $L = 2, 3, 4$ are given by three tables containing: $\delta = \deg(P_{J_*)} - \deg(P_{111\ldots1})$ (we assume $a_i \geq 2$), $J_* = (j_1, \ldots, j_{2L})$, $w$ is a word conjugate to $x_1^{j_1}x_2^{j_2}x_1^{j_{2k-1}}x_2^{j_{2k}}$, $N$ is the number of positive braids in the same conjugacy class with $w$, $T$ is the leading term of $V_3(w)$, and $\deg = \delta + \deg(T)$. In the last column the top degrees are in bold characters.

**Table 2.** $L = 2, V_{J_*} = V(x_1^{a_1}x_2^{a_2}x_1^{a_3}x_2^{a_4})$

| $\delta$ | $J_* = (j_1, j_2, j_3, j_4)$ | $w$ | $N$ | $T$ | $\deg$ |
|----------|------------------|-----|-----|-----|-------|
| 4        | 0000             | 1   | 1   | $s^2$ | 6     |
| 3        | 1000             | $x_1$ | 4   | $-s$ | 4     |
| 2        | 1100             | $x_1x_2$ | 4   | 1    | 2     |
| 2        | 1010             | $x_2^2$ | 2   | $s^6$ | 8     |
| 1        | 1110             | $x_2^3x_2$ | 4   | $-s^5$ | 6     |
| 0        | 1111             | $x_1^3x_2$ | 1   | $-s^8$ | 8     |
Table 3. \( L = 3, V_{J*} = V(x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5}x_6^{a_6}) \)

| \( \delta \) | \( J* = (j_1, j_2, j_3, j_4, j_5, j_6) \) | \( w \) | \( N \) | \( T \) | \( \text{deg} \) |
|---|---|---|---|---|---|
| 6 | 000000 | 1 | 1 | \( s^2 \) | 8 |
| 5 | 100000 | \( x_1 \) | 6 | \(-s\) | 6 |
| 4 | 110000 | \( x_1x_2 \) | 9 | 1 | 4 |
| 4 | 101000 | \( x_1^2 \) | 6 | \( s^6 \) | 10 |
| 3 | 110000 | \( x_1^2x_2 \) | 18 | \(-s^5\) | 8 |
| 3 | 101010 | \( x_1^3 \) | 2 | \( s^9 \) | 12 |
| 2 | 111100 | \( x_1^3x_2 \) | 12 | \(-s^8\) | 10 |
| 2 | 110110 | \( x_1^2x_2^2 \) | 3 | \( s^{10} \) | 12 |
| 1 | 111110 | \( x_1x_2 \) | 6 | \(-s^{11}\) | 12 |
| 0 | 111111 | \( x_1^2x_2x_1^2x_2 \) | 1 | \( 2s^{12} \) | 12 |

Table 4. \( L = 4, V_{J*} = V(x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5}x_6^{a_6}x_1^{a_7}x_2^{a_8}) \)

| \( \delta \) | \( J* = (j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8) \) | \( w \) | \( N \) | \( T \) | \( \text{deg} \) |
|---|---|---|---|---|---|
| 8 | 00000000 | 1 | 1 | \( s^2 \) | 10 |
| 7 | 10000000 | \( x_1 \) | 8 | \(-s\) | 6 |
| 6 | 11000000 | \( x_1x_2 \) | 16 | 1 | 6 |
| 6 | 10100000 | \( x_1^2 \) | 12 | \( s^6 \) | 12 |
| 5 | 11100000 | \( x_1^2x_2 \) | 48 | \(-s^5\) | 10 |
| 5 | 10101000 | \( x_1^3 \) | 8 | \( s^9 \) | 14 |
| 4 | 11110000 | \( x_1^3x_2 \) | 55 | \(-s^8\) | 12 |
| 4 | 11011000 | \( x_1^2x_2^2 \) | 13 | \( s^{10} \) | 14 |
| 4 | 10101010 | \( x_1^4 \) | 2 | \( s^{12} \) | 16 |
| 3 | 11111000 | \( x_1^4x_2 \) | 48 | \(-s^{11}\) | 14 |
| 3 | 10101101 | \( x_1^3x_2^2 \) | 8 | \( s^{13} \) | 16 |
| 2 | 11111100 | \( x_1^2x_2x_1^2x_2 \) | 12 | \( 2s^{12} \) | 14 |
| 2 | 11110110 | \( x_1^2x_2^2 \) | 16 | \(-s^{14}\) | 16 |
| 1 | 11111110 | \( x_1^2x_2x_1^2x_2 \) | 8 | \( s^{15} \) | 16 |
| 0 | 11111111 | \( x_1^2x_2x_1^2x_2 \) | 1 | \(-s^{16}\) | 16 |

From these tables, one can compute the degree and the leading term of \( P_{J*}^AV_{J*} \) using the formula

\[ \delta + \text{deg}(T) + \sum_{i=1}^{2L} (3a_i - 1) = \text{deg} + 3D - 2L. \]

Next, using the top degree from the table with the corresponding coefficient and subtracting the degree of the denominator \((s^2 + 1)^{2L}\) we get the result. For instance,
using the Table 4, the leading coefficient of $V_3(x_1^{a_1}x_2^{a_2}x_1^{a_3}x_2^{a_4}x_1^{a_5}x_2^{a_6}x_1^{a_7}x_2^{a_8})$, $a_i \geq 2$, is

$$2 + 8 - 16 + 8 - 1 = 1$$

and its degree is

$$16 + 3D - 2 \cdot 4 - 2 \cdot 8 = 3D - 8.$$  

\[ \square \]

As we can see from the tables, there are braids $\beta^{A*}$ with positive exponents having maximal degree $\deg V_3(\beta^{A*}) = 3D - 2L$, but not all their exponents are $\geq 2$; other examples with maximal degree are given by the families $V_3(a_1, 1, \ldots, 1)$ and $V_3(a_1, a_2, 1, \ldots, 1)$, $a_1, a_2 \geq 2$, $2L$ indices, with leading terms $s^{3D-2L}$ if $L \equiv 0(\mod 3)$ and $-s^{3D-2L}$ if $L \equiv 1(\mod 3)$. These examples shows that combinatorics of maximal degree in the expansion formula is not so simple.

**Conjecture 4.13.** If $a_i \geq 2$, the leading term of $V_3(x_1^{a_1}x_2^{a_2} \ldots x_1^{a_{2L-1}}x_2^{a_{2L}})$ is $s^{3D-2L}$.

5. **There are few unit polynomials in a row**

Now we apply the recurrence relation to evaluate the number of solutions of the equation $V(e) = 1$, where $V(e) = V_n(x_1^{a_1} \ldots x_i^{e} \ldots x_k^{a_k})$, with the same conventions: the sequence $x_1, \ldots, x_k$ of generators of $B_n$ is fixed, the exponents $a_1, \ldots, a_j, \ldots a_k$ are fixed, and $e$ is an arbitrary integer.

**Proposition 5.1.** a) If $V(e)$ and $V(e + 1)$ are polynomials in $s$ and $k \geq 2$, then $V(e + k)$ is a polynomial different from 1.

b) If $V(e)$ and $V(e - 1)$ are polynomials in $s^{-1}$ and $k \geq 2$, then $V(e - k)$ is a polynomial in $s^{-1}$ different from 1.

**Proof.** a) From the recurrence relation $V(e + k)$ is a polynomial in $s$ for $k \geq 2$. If $V(e + k) = 1$ for some $k \geq 2$, then $1 = (s^3 - s)V(e + k - 1) + s^4V(e + k - 2)$, and this is impossible with polynomials. Similarly for part b).  

An obvious consequence of Propositions 3.6 and 3.7 is the fact that in any sequence $(V(e))_{e \in \mathbb{Z}}$ there are only finitely many terms equal to 1. Now we prove a stronger statement: in such a sequence there are at most two polynomials equal to 1, and in this case they are very close:

**Lemma 5.2.** The sequence $(V(e))_{e \in \mathbb{Z}}$ could contain at most two terms equal to 1. If $V(a) = V(b) = 1, a \neq b$, then $|a - b| = 1$ or 2.

**Proof.** Suppose $V(a) = 1$ and $V(b) = 1$ ($a < b$). We can take $Q_n = V(n + a)$, where $n \in \mathbb{Z}$. Hence $Q_0 = 1$. From the recurrence relation we have

$$Q_n = \frac{1}{s^3 + s}[(Q_1 + s)s^{3n} + (s^3 - Q_1)(-s)^n].$$
Let us suppose that $Q_n = 1$ for a positive $n$; in our case $b - a$ is such an example. If $n$ is even, we have $s^3 + s = (s^{3n} - s^n)Q_1 + s^{3n+1} - s^{n+3}$. This implies $Q_1 = \frac{1}{s^n(s^{3n+1})}[s^3 + s - s^{3n+1} - s^{n+3}] = - s - \frac{1}{s^n(s^{3n+1})}(s^2 + 1)$. This gives Laurent polynomials only for $n = 0$ and $n = 2$. In the last case, $Q_1 = - s - \frac{1}{7}$, hence only $Q_0$ and $Q_2$ are equal to 1. If $n$ is odd, we have $s^3 + s = (s^{3n} + s^n)Q_1 + s^{3n+1} - s^{n+3}$. This implies $Q_1 = \frac{1}{s^n(s^{3n+1})}[s^3 + s - s^{3n+1} + s^{n+3}] = - s + \frac{1}{s^n(s^{3n+1})} \times s^2 + 1)(s^n + 1)$. Laurent polynomials are obtained only for $n = 0$ and $n = 1$. In this last case, $Q_0 = Q_1 = 1$ and $Q_2 \neq 1$.

**Example 5.3.** The closures of the braids $x_1^{-1}$, $x_0^0$, and $x_1^1$ in $B_2$ give the links in the following figure:

![Diagram of links](image)

and the corresponding Jones polynomials are: $V_2(-1) = 1$, $V_2(0) = - s - s^{-1}$, and $V_2(1) = 1$. This shows that the case $V(a) = V(a + 2) = 1$ is possible.

**Lemma 5.4.** Let $\beta(e) = ax_i^e \gamma$ and $\beta(e + 1) = ax_i^{e+1} \gamma$ be two braids in $B_n$. Then $\beta(e)$ and $\beta(e + 1)$ cannot be simultaneously knots. In particular, $V(e)$ and $V(e + 1)$ evaluated at 1 cannot be 1 at the same time.

**Proof.** If $\beta(e)$ is a knot, then the associated permutation $\pi(\beta(e))$ is an $n$-cycle. The permutation associated with $\beta(e + 1)$ is $\pi(\alpha x_i^{e+1} \gamma) = \pi(\alpha x_i \gamma) \pi(\gamma^{-1} x_i \gamma)$, and this cannot be an $n$-cycle because the signature of the last factor is $-1$.

**Proof of Theorem 1.7** This is a consequence of lemmas 5.2 and 5.4.

6. **APPENDIX**

In [17] multiple Fibonacci sequences (and multiple Fibonacci modules) are introduced. A multiple sequence $(x_{n_1, \ldots, n_p})_{n_1, \ldots, n_p \in \mathbb{Z}}$ of elements in a ring $\mathcal{R}$ is called a multiple Fibonacci sequence of type $(\beta, \gamma) \in \mathcal{R}^2$ if for any $i \in \{1, \ldots, p\}$ and any $k \in \mathbb{Z}$ we have

$$a_{n_1, \ldots, n_{i-1}, k+2, n_{i+1}, \ldots, n_p} = \beta a_{n_1, \ldots, n_{i-1}, k+1, n_{i+1}, \ldots, n_p} + \gamma a_{n_1, \ldots, n_{i-1}, k, n_{i+1}, \ldots, n_p}.$$ 

The $\mathcal{R}$-module of such multiple sequences is denoted by $\mathcal{F}[p](\beta, \gamma)$ and it is isomorphic with the tensor product $\mathcal{F}(\beta, \gamma)^{\otimes p}$. 
Theorem 6.1. Let \((x_{n_1},...x_{n_p}) \geq 0\) be an element in \(\mathcal{F}[p](r_1 + r_2, -r_1 r_2)\).

a) The general term is given by

\[ x_{n_1,\ldots,n_p} \equiv D^{-r} \sum_{0 \leq j_1,\ldots,j_p \leq 1} S_{j_1}^{[n_1]}(r_1, r_2) \ldots S_{j_p}^{[n_p]}(r_1, r_2) x_{j_1,\ldots,j_p}, \]

where \(D = r_2 - r_1\), \(S_0^{[n]}(r_1, r_2) = r_1^n r_2 - r_1 r_2^n\), \(S_1^{[n]}(r_1, r_2) = r_2^n - r_1^n\).

b) The generating function of \((x_{n_1},...x_{n_p})\) is given by

\[ G(t_1,\ldots,t_p) = q(t_1)^{-1} \ldots q(t_p)^{-1} \sum_{0 \leq j_1,\ldots,j_p \leq 1} Q_{j_1}(t_1) \ldots Q_{j_p}(t_p)x_{j_1,\ldots,j_p}, \]

where \(q(t) = (1 - r_1 t)(1 - r_2 t)\), \(Q_0(t) = 1 - (r_1 + r_2)t\) and \(Q_1(t) = t\).

Proof. These require only elementary computations; one can find all the details in [17].

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