Alternance Theorems and Chebyshev Splines
Approximation

Jean-Pierre Crouzeix * Nadezda Sukhorukova †
Julien Ugon ‡

January 23, 2018

Abstract

One of the purposes in this paper is to provide a better understanding of the alternance property which occurs in Chebyshev polynomial approximation and piecewise polynomial approximation problems. In the first part of this paper, we propose an original approach to obtain new proofs of the well known necessary and sufficient optimality conditions. There are two main advantages of this approach. First of all, the proofs are much simpler and easier to understand than the existing proofs. Second, these proofs are constructive and therefore they lead to alternative-based algorithms that can be considered as Remez-type approximation algorithms. In the second part of this paper, we develop new local optimality conditions for free knot polynomial spline

*Université Clermont Auvergne, Clermont Ferrand, France, jp.crouzeix@isima.fr, 24 Avenue des Landais, 63170 AubiÃlée, France, Phone: +33 4 73 40 63 63
†Corresponding author, Swinburne University of Technology, Melbourne, Australia and Federation University Australia, nsukhorukova@swin.edu.au, Swinburne University of Technology PO Box 218 Hawthorn, Victoria, 3122 Australia, Phone: +61 3 9214 8455 Fax: +61 3 9214 8264
‡Deakin University, Melbourne, Australia and Federation University Australia, Ballarat, Australia, julien.ugon@deakin.edu.au, 221 Burwood Highway, Burwood VIC 3125 Australia, Phone +61 3 9244 6100
approximation. The proofs for free knot approximation are relying on the techniques developed in the first part of this paper.

**Keywords:** Chebyshev Approximation, Polynomial Splines, Fixed and Free knots.

**AMS classification:** 49J52, 90C26, 41A15, 41A50.

## 1 Introduction

For a given continuous function \( f : [a, b] \to \mathbb{R} \), the Chebyshev polynomial approximation problem is

\[
\min_{\pi \in \Pi_n} \left[ \| \pi - f \| = \sup_{t \in [a,b]} | \pi(t) - f(t) | \right],
\]

where \( \Pi_n \) denotes the set of polynomial functions of degree less or equal to \( n \). It is known that this problem has a unique optimal solution which is characterized by the existence of \( \varepsilon \in \{-1, 1\} \) and \( n + 2 \) points \( t_i \) such that

\[
a \leq t_0 < t_1 < \cdots t_n < t_{n+1} \leq b, \quad \| \pi - f \| = \varepsilon(-1)^i[\pi(t_i) - f(t_i)] \quad \forall i.
\]

It can be solved by the celebrated Remez algorithm (1959) [5].

This problem can be also formulated as

\[
\min_{(a,\lambda) \in \mathbb{R}^{n+2}} \left[ \lambda : f(t) - \sum_{i=0}^{n} a_i t^i \leq \lambda, \quad \sum_{i=0}^{n} a_i t^i - f(t) \leq \lambda \ \forall t \in [a, b] \right] (lsp)
\]

and therefore it belongs to the class of linear semi-infinite programs. One observes that the number of alternating points \( t_i \) corresponds to the dimension of the space of primal variables in \((lsp)\). The exchange rules in the Remez algorithm roughly correspond to the leaving/entering rules in the simplex algorithm running over the corresponding dual problem. However, the linear semi-infinite programming theory does not explain the alternance property which is due, as shown in this paper, both to the continuity of \( f - \pi \) and the structure of \( \Pi_n \).
Alternance conditions also appear in Chebyshev spline approximation where \( f \) is approximated by a continuous piecewise polynomial function \([4, 6, 10]\). When the knots (points that connect polynomial pieces) are fixed, the problem is no more linear but convex, when the knots are not known, the problem is no more convex and therefore very hard to solve. There have been several attempts to extend the results to the case of free knots polynomial spline approximation \([4, 7, 11]\). The most advanced results have been obtained in \([11]\), where the most accurate necessary optimality conditions are obtained. Theses results are equivalent to Demyanov-Rubinov stationarity \([2, 3]\) and characterise local optimality.

In this paper, we consider exactly the same problems as the ones recently investigated by Sukhorukova and Ugon \([11]\) and Crouzeix et al. \([1]\). However, our approach and techniques are very different in their essence. The main advantages of our approach is that the proofs are easier to understand and, most importantly, the proofs are constructive. The goal of this study is also to enhance the comprehension of the alternance property in Chebyshev piecewise polynomial approximation problems.

In section 2, we introduce a very general alternance result which holds for any continuous function. In the same section we also introduce a generalisation of the notion of alternating (\(\beta\)-alternating) which is more suitable for computational purposes. This result is applied in section 3 to the Chebyshev (uniform) polynomial approximation and provides an alternative proof of the existence, unicity and characterization of the optimal solution. The proof is very simple, constructive and therefore it gives rise to alternative algorithms with the celebrated Remez algorithm, developed for polynomial approximation. Next, the results of sections 2 and 3 is also successfully applied in section 4 to the fixed knots spline approximation problem.

Finally, section 5 treats of the free knots spline problems. The problem is no more convex so that only local optimality conditions can be obtained.
2 Some general results

Through the paper, given \( a, b \in \mathbb{R} \) with \( a < b \), the norm of \( f : [a, b] \to \mathbb{R} \) is defined by

\[
\|f\| = \max_{t \in [a, b]} |f(t)|.
\]

**Theorem 2.1.** Let \( f : [a, b] \to \mathbb{R} \) be continuous and such that \( M \) := \( \max \{ f(t) : a \leq t \leq b \} = -\min \{ f(t) : a \leq t \leq b \} > 0 \).

1. There exist \( k \geq 1 \) integer, \( \varepsilon \in \{-1, 1\} \), and \( \{(t^-_i, t^+_i)\}_{i=0}^{k} \subset [a, b]^2 \) such that

\[
a \leq t^-_0 \leq t^+_0 < t^-_1 \leq t^+_1 < t^-_2 \leq t^+_2 < \cdots < t^-_k \leq t^+_k \leq b,
\]

\[
\varepsilon M = (-1)^i f(t^-_i) = (-1)^i f(t^+_i), \quad i = 0, 1, \cdots, k,
\]

\[
-M < \varepsilon (-1)^i f(t) \leq M \text{ if } t \in [t^-_i, t^+_i] \text{ for some } i,
\]

\[
|f(t)| < M \text{ for all other } t \in [a, b].
\]

The quantities \( k, \varepsilon, t^-_i, t^+_i \) are uniquely defined.

2. \( \|f - p\| \geq \|f\| \) for any polynomial function \( p \) of degree at most \( k - 1 \).

3. Let us consider the polynomial function of degree \( k \)

\[
\gamma(t) = \varepsilon \prod_{i=0}^{k-1} (\xi_i - t),
\]

where, for \( i = 0, 1, \cdots, k-1 \), \( \xi_i \) is arbitrarily chosen in \( (t^+_i, t^-_{i+1}) \). Then, for \( \lambda > 0 \) small enough, it holds \( \|f - \lambda \gamma\| < \|f\| \).

Below we will give a proof for a more general version of this theorem, which is better adapted for numerical purposes. Indeed, except for some particular functions, it is quite unrealistic to consider that the maximum of \( |f(x)| \) can be reached in more than three points.
Theorem 2.2. Let \( f : [a, b] \to \mathbb{R} \) be continuous and such that

\[
M := \max \{ f(t) : a \leq t \leq b \} = -\min \{ f(t) : a \leq t \leq b \} > 0.
\]

Let \( \beta \in (0, 1] \).

1. There exist \( k(\beta) \geq 1 \) integer, \( \varepsilon(\beta) \in \{-1, 1\} \), \( t_i^- (\beta), t_i^+ (\beta) \in \mathbb{R} \) for \( i = 0, \cdots, k(\beta) \) such that

\[
a \leq t_0^- (\beta) \leq t_0^+ (\beta), \quad t_{k(\beta)}^- (\beta) \leq t_{k(\beta)}^+ (\beta) \leq b,
\]

\[
t_i^- (\beta) \leq t_i^+ (\beta) < t_{i+1}^- (\beta) \leq t_{i+1}^+ (\beta), \quad i = 0, 1, \cdots, k(\beta) - 1,
\]

\[
-\beta M < \varepsilon(\beta)(-1)^i f(t) \leq M \quad \forall t \in [t_i^- (\beta), t_i^+ (\beta)], \quad \forall i,
\]

\[
\varepsilon(\beta)(-1)^i f(t_i^+ (\beta)) = \varepsilon(\beta)(-1)^{i+1} f(t_{i+1}^- (\beta)) = \beta M, \quad i = 0, \cdots, k(\beta) - 1,
\]

\[
\varepsilon(\beta) f(t_0^- (\beta)) \geq \beta M, \quad \varepsilon(\beta)(-1)^{k(\beta)} f(t_{k(\beta)}^+ (\beta)) \geq \beta M,
\]

\[
|f(t)| < \beta M \quad \text{for all other } t \in [a, b].
\]

The quantities \( \varepsilon(\beta), k(\beta), t_i^- (\beta), t_i^+ (\beta) \) are uniquely defined.

2. \( \| f - p \| \geq \beta \| f \| \) for any polynomial function \( p \) of degree at most \( k(\beta) - 1 \).

3. Let us consider the function

\[
\gamma_\beta(t) = \varepsilon(\beta) \prod_{i=0}^{k(\beta)-1} (\xi_i(\beta) - t),
\]

where, for \( i = 0, 1, \cdots, k(\beta) - 1 \), \( \xi_i \) is arbitrarily chosen in \( (t_i^+ (\beta), t_{i+1}^- (\beta)) \).

Then, \( \| f - \lambda \gamma_\beta \| < \| f \| \) for a suitably chosen \( \lambda > 0 \).
Proof. 1) We construct the sequence \( t_0^{-}(\beta), \ldots, t_{k(\beta)}^{+}(\beta) \) using Algorithm 1.

**Algorithm 1:** Construction of the alternating sequence

**Step 0:** Initialisation
Define

\[
t_0^{-}(\beta) = \min_{t \in [a,b]} \{ t : |f(t)| \geq \beta M \} \quad \text{and} \quad \varepsilon(\beta) = \frac{f(t_0^{-}(\beta))}{|f(t_0^{-}(\beta))|}.
\]

By construction,

\[
|f(t)| < \beta M \leq \varepsilon(\beta)f(t_0^{-}(\beta)) \quad \text{if} \ a \leq t < t_0^{-}(\beta).
\]

Set \( i := 1. \)

while \( \{ t : (-1)^{i-1}\varepsilon(\beta)f(t) \leq -\beta M \} \neq \emptyset \) do

**Step i:**
Let

\[
\begin{align*}
t_i^{-}(\beta) &= \min_{t \in [t_{i-1}^{-}(\beta),b]} \{ t : (-1)^{i-1}\varepsilon(\beta)f(t) \leq -\beta M \} \quad \text{and} \\
t_i^{+}(\beta) &= \max_{t \in [t_{i-1}^{-}(\beta),t_i^{-}(\beta)]} \{ t : (-1)^{i-1}\varepsilon(\beta)f(t) \geq \beta M \}.
\end{align*}
\]

Such \( t_{i-1}^{+}(\beta), t_i^{-}(\beta) \) exist and \( t_{i-1}^{-}(\beta) \leq t_{i-1}^{+}(\beta) < t_i^{-}(\beta) \leq b. \) By construction,

\[
\begin{align*}
\beta M &= (-1)^{i}\varepsilon(\beta)f(t_i^{-}(\beta)) = (-1)^{i-1}\varepsilon(\beta)f(t_{i-1}^{+}(\beta)), \\
-\beta M &< \varepsilon(\beta)(-1)^{i-1}f(t) \leq M \quad \text{if} \ t_{i-1}^{-}(\beta) \leq t \leq t_{i-1}^{+}(\beta), \\
|f(t)| &< \beta M \quad \text{if} \ t_{i-1}^{+}(\beta) < t < t_i^{-}(\beta).
\end{align*}
\]

**Step k:** Set \( k(\beta) = i. \)

Let \( t_{k(\beta)}^{+}(\beta) = \max_{t \in [t_{k(\beta)-1}^{-}(\beta),b]} \{ t : (-1)^{k(\beta)}\varepsilon(\beta)f(t) \geq \beta M \}. \)

By construction,

\[
\begin{align*}
\beta M &= (-1)^{k(\beta)}\varepsilon(\beta)f(t_{k(\beta)}^{-}(\beta)) \leq (-1)^{k(\beta)}\varepsilon(\beta)f(t_{k(\beta)}^{+}(\beta)), \\
-\beta M &< \varepsilon(\beta)(-1)^{k(\beta)}f(t) \leq M \quad \text{if} \ t_{k(\beta)-1}^{-}(\beta) \leq t \leq t_{k(\beta)}^{+}(\beta), \\
|f(t)| &< \beta M \quad \text{if} \ t_{k(\beta)}^{+}(\beta) < t \leq b.
\end{align*}
\]

It remains to prove that the construction ends after a finite number of steps. We proceed by contradiction. If not, we have built a strictly increasing
sequence \( \{ t_i^- (\beta) \}_i \subset [a, b] \). This sequence converges to some \( \bar{t} \in [a, b] \). Since \( f \) is continuous, \( f(t_i^- (\beta)) \) converges to \( f(\bar{t}) \). But \( f(t_i^- (\beta)) \) equals either \( \beta M \) or \( -\beta M \) depending on the parity of \( i \). This is not possible.

2) Assume that \( p \in \Pi_{k(\beta)-1} \) exists such that \( \| f - p \| < \beta \| f \| = \beta M \).

Then, for each \( i = 0, \ldots, k(\beta) \)

\[
\beta M > \varepsilon(\beta)(-1)^i[f(t_i^- (\beta) - p(t_i^- (\beta))] \geq \beta M - \varepsilon(\beta)(-1)^i p(t_i^- (\beta)).
\]

Hence, \( p(t_i^- (\beta))p(t_{i+1}^- (\beta)) < 0 \) for \( i = 0, \ldots, k(\beta) - 1 \) which is not possible because the degree of the polynomial function \( p \).

3) It remains to choose \( \lambda \) in such a way that \( \| f - \lambda \gamma \| < \| f \| \). The stronger decrease corresponds to the optimal solution \( \lambda_{opt} \) of the minimisation problem

\[
\mu_{opt} = \min_{\lambda \geq 0} \max_{t \in [a, b]} |f(t) - \lambda \gamma(t)| = \min_{\lambda, \mu} \left[ \begin{array}{c} f(t) - \lambda \gamma(t) \leq \mu, \\ \mu : -f(t) + \lambda \gamma(t) \leq \mu, \\ \forall t \in [a, b]. \end{array} \right] . \tag{1}
\]

The second formulation in problem (1) is a linear programming problem with only two variables but with an infinite number of constraints, hence \( \lambda_{opt} \) and \( \mu_{opt} \) cannot be easily obtained. In order to obtain upper-bounds of \( \mu_{opt} \), let us introduce

\[
m_- (\beta) = \min_t \left| \gamma(t) \right| : t \in [t_i^- (\beta), t_i^+ (\beta)] \text{ for some } i \right], \]

\[
m_+ (\beta) = \max_t \left[ \gamma(t) \right] : t \in [a, b] , \quad \rho(\beta) = m_- (\beta)(m_+ (\beta))^{-1}.
\]

It follows from the construction of the function \( \gamma \) that \( 0 < \rho(\beta) \leq 1 \) and \( m_- (\beta) = \min_{t} \left[ \gamma(t) \right] : t \in [a, b] \) for some \( i \). Then \( \varepsilon(\beta)(-1)^i \gamma(t) > 0 \) and therefore, for all \( \lambda > 0 \),

\[
-\beta \| f \| - \lambda m_+ (\beta) < \varepsilon(\beta)(-1)^i (f - \lambda \gamma)(t) \leq \| f \| - \lambda m_- (\beta). \tag{2}
\]

Next, for the other \( t \in [a, b] \), one has

\[
-\beta \| f \| - \lambda m_- (\beta) < (f - \lambda \gamma)(t) < \beta \| f \| + \lambda m_+ (\beta). \tag{3}
\]
It follows immediately from the inequalities (2)-(3) that for \( \lambda > 0 \) small enough, \( \| f - \lambda \gamma \| < \| f \| \). Furthermore, for \( \beta \in (0, 1) \) and \( \lambda > 0 \)

\[
\| f - \lambda \gamma \| \leq \max \{ \| f \| - \lambda m_-(\beta), \beta \| f \| + \lambda m_+ (\beta) \}.
\] (4)

In particular, for \( \bar{\lambda} = \| f \| (1 - \beta)(m_-(\beta) + m_+(\beta))^{-1} \),

\[
\mu_{\text{opt}} \leq \| f - \bar{\lambda} \gamma \| \leq \frac{1 + \beta \rho(\beta)}{1 + \rho(\beta)} \| f \| < \| f \|.
\] (5)

These bounds are very rough: \( \bar{\lambda} \) is not optimal and the inequalities (2), (3) and (4) correspond to the worst possible cases.

**Definition 2.1.** Consider a sequence of points

\[ a \leq t_0 < t_1 < \cdots < t_k \leq b, \quad k > 0. \]

We call this sequence of points a \( \beta \)-alternating sequence if there exists \( \varepsilon = \{1, -1\} \), such that

\[ \varepsilon f(t_0) \geq \beta M, \quad (-1)^i \varepsilon f(t_i) \geq \beta M, \quad i = 1, \ldots, k. \]

The next proposition analyses the behaviour of \( k(\beta) \) and the points \( t_i^- (\beta) \) and \( t_i^+ (\beta) \) when \( \beta \to 1 \).

**Proposition 2.1.** a) \( k(\beta_1) \geq k(\beta_2) \) when \( 0 < \beta_1 < \beta_2 \leq 1 \).

b) There is \( \bar{\beta} \in (0, 1) \) such that \( k(\bar{\beta}) = k(1) \) for all \( \beta \in (\bar{\beta}, 1] \).

c) For all \( i \), \( t_i^- (\beta) \to t_i^- \) and \( t_i^+ (\beta) \to t_i^+ \) when \( \beta \to 1 \).

**Proof.** a) It is clear that \( t_0^- (\beta_1) \leq t_0^- (\beta_2) \). Next, \( t_1^- (\beta_1) \leq t_1^- (\beta_2) \), \( t_i^- (\beta_1) \leq t_i^- (\beta_2) \) for \( i \geq 2 \).

b) Denote by \( r_i^- \) and \( r_i^+ \) respectively the smallest and the greatest \( t \in [t_i^-, t_{i+1}^-] \) such that \( f(t) = 0 \). Then, \( t_i^- < r_i^- \leq r_i^+ < t_{i+1}^- \). Next, set

\[
\alpha_{-1} = \max_t \left\{ -\varepsilon f(t) : a \leq t \leq t_0^- \right\}.
\]
\[ \alpha_i = \max_t \left[ |f(t)| : r_i^- \leq t \leq r_i^+ \right], \quad i = 0, \ldots, k(1) - 1, \]
\[ \alpha_{k(1)} = \max_t \left[ -\varepsilon(-1)^{k(1)}f(t) : t_{k(1)}^+ \leq t \leq b \right]. \]

Next, let \( \alpha = \max_i \left[ \alpha_i \right] \). By construction, \( 0 < \alpha < M \). Take \( \hat{\beta} = \alpha M^{-1} \). Let any \( \beta \in (\hat{\beta}, 1) \).

1. \( t_0^- (\beta) \in [a, t_0^-] \) and \( t_1^- (\beta) \notin [a, r_0^+] \) since \( -\beta M < \varepsilon f(t) \) for all \( t \in [a, r_0^+] \) and, in case where \( a \neq t_0^- \), \( \varepsilon f(t_0^-) = M \).

Next, \( t_1^- (\beta) \in (r_0^+, t_1^-) \) because \( f(r_0^+) = 0 \), \( \varepsilon f(t_1^-) = -M \) and \( 0 > \varepsilon f(t) \) for all \( t \in (r_0^+, t_1^-) \).

Finally, \( t_0^+ (\beta) \in (t_0^+, r_0^-) \) because \( \varepsilon f(t_0^+) = M \), \( f(r_0^-) = 0 \) and \( |f(t)| < \beta M \) for all \( t \in [r_0^-, t_1^-) \).

2. \( t_2^- (\beta) \notin [t_1^- (\beta), r_1^+] \) since \( -\beta M < (-1)^1 \varepsilon f(t) \) for all \( t \in [t_1^-, r_1^+] \).

Next, \( t_2^- (\beta) \in (r_1^+, t_2^-) \) because \( f(r_1^+) = 0 \), \( (-1)^2 \varepsilon f(t_2^-) = M \) and \( 0 > (-1)^1 \varepsilon f(t) \) for all \( t \in (r_1^+, t_2^-) \).

Finally, \( t_1^+ (\beta) \in (t_1^+, r_2^-) \) because \( (-1)^1 \varepsilon f(t_1^+) = M \), \( f(r_2^-) = 0 \) and \( |f(t)| < \beta M \) for all \( t \in [r_2^-, t_2^-) \).

3. Proceed similarly for other \( i \).

It follows that \( k(\beta) = k(1) \) for \( \beta \in (\hat{\beta}, 1) \).

c) Assume that \( \hat{\beta} < \beta_1 < \beta_2 < 1 \). Then, for all \( i \),
\[ r_{i-1}^+ < t_i^- (\beta_1) < t_i^- (\beta_2) < t_i^- < t_i^+ (\beta_2) < t_i^+ (\beta_2) < r_i^- . \]

Assume, for contradiction, that \( t_i^- (\beta) \) converges to some \( \bar{t} < t_i^- \). The definition of \( t_i^- \) implies \( |f(\bar{t})| < M \) in contradiction with \( f \) continuous and \( f(t_i^- (\beta)) = \beta M \) and \( \beta \to 1 \). The other convergencies are treated similarly. \( \square \)

Next, let us bring our attention on the reduction rate in (5) in case where we take for \( \xi_i \) the middle of the interval \([t_i^+(\beta), t_{i+1}^- (\beta)] \). The quantity \( m_+ (\beta) \) is bounded from above by \([b - a]^{k(\beta)} \). Hence, roughly speaking, a small value
of $\rho(\beta)$ corresponds for some $i$ to a small value of $m_-(\beta)$ which corresponds to a small value of some $\xi_i(\beta) - t_i^-(\beta) \text{ or } t_{i+1}^+(\beta) - \xi_i(\beta)$ and/or a small value of some $\xi_{i+1}(\beta) - \xi_i(\beta)$. Let us explicit that in terms of continuity of the function $f$.

$f$ being continuous on the compact set $[a, b]$ is uniformly continuous. Hence, for all $\delta > 0$, there is $\mu > 0$ such that $|f(t) - f(s)| < \delta$ when $|t - s| < \mu$. This motivates the introduction of the following function

$$
\mu_f(\delta) = \sup_{s, t, \mu} [\mu > 0 : s, t \in [a, b] \text{ and } |t - s| < \mu \implies |f(t) - f(s)| < \delta],
$$

$$
\mu(\delta, f) = \mu_f(\delta) = \inf_{s, t} [\inf \{ |t - s| : s, t \in [a, b], |f(t) - f(s)| \geq \delta \}].
$$

This function is in some way an inverse modulus of continuity of $f$. It is clear that

$$
0 < \mu_f(\delta_1) \leq \mu_f(\delta_2) \text{ whenever } 0 < \delta_1 < \delta_2.
$$

In case where $f$ is Lipschitz, i.e., if there exists $L$ such that

$$
|f(t) - f(s)| \geq L|t - s|
$$

for all $s, t$, one has $\mu_f(\delta) \leq L^{-1}\delta$.

Let us return to our problem. By construction of the points $t_i^-(\beta), t_i^+(\beta)$, one has $|f(t_i^-_{i+1}(\beta)) - f(t_i^+(\beta))| = 2\beta \|f\|$ and therefore for all $i$

$$
t_{i+1}^-(\beta) - t_i^+(\beta) \geq \mu_f(2\beta \|f\|) / 2, \quad t_{i+1}^-(\beta) - \xi_i(\beta) \geq \mu_f(2\beta \|f\|) / 2,
$$

$$
|\xi_i(\beta) - t_i^-(\beta)| \geq \mu_f(2\beta \|f\|) / 2, \quad |\xi_{i+1}(\beta) - \xi_i(\beta)| \geq \mu_f(2\beta \|f\|),
$$

$$
(t_i^+(\beta), t_{i+1}^-(\beta)) \supset (\xi_i(\beta) - \mu_f(2\beta \|f\|) / 2, \xi_i(\beta) + \mu_f(2\beta \|f\|) / 2).
$$

Furthermore,

$$
a \leq t_0^+(\beta) \leq \xi_0(\beta) - \mu_f(2\beta \|f\|) / 2), \quad \xi_{k-1} + \mu_f(2\beta \|f\|) / 2) \leq t_k^- (\beta) \leq b.
$$

Let us observe that

$$
\bigcup_i [t_i^- (\beta), t_i^+(\beta)] \subset \bigcup_i (t_i^+(\beta), t_{i+1}^-(\beta)) \subset \cdots
$$
where $c$ stands for complementary set.

Hence,

$$m_-(\beta) \geq \inf_t \left[ \prod_{i=0}^{k(\beta)-1} |t - \xi_i(\beta)| : t \in [a, b] \cap \mathcal{T}_f(\alpha) \right].$$

(6)

It remains to obtain a lower bound of the product. Let us introduce the following function $\Gamma_k$ which does not depend on $f$ and $\beta$.

$$\Gamma_k(r) = \inf_{t, \xi_i} \left[ \prod_{i=0}^{k-1} |t - \xi_i| : a \leq t \leq b, \xi_0 + r \leq \xi_1 \leq \cdots \leq \xi_{k-1} + r \leq b, \xi_{i+1} - \xi_i \geq 2r, |t - \xi_i| \geq r \forall i. \right], r > 0.$$

By construction $0 < \Gamma_k(r_1) \leq \Gamma_k(r_2)$ whenever $0 < r_1 < r_2$. One has necessarily $2kr \leq b - a$.

$\Gamma_k(r)$ can be explicitly determined. Indeed,

- $\Gamma_2(r) = r^2$ is reached for $\xi_0 = a + r, \xi_1 = \xi_0 + 2r, t = \frac{\xi_0 + \xi_1}{2}$.
- $\Gamma_3(r) = 3r^3$ is reached for $\xi_0 = a + r, \xi_1 = \xi_0 + 2r, \xi_2 = \xi_1 + 2r$ and $t = \frac{\xi_0 + \xi_1 + \xi_2}{2}$.
- $\Gamma_4(r) = 3^2r^4$ reached for $\xi_0 = a + r, \xi_i + 1 = \xi_i + 2r, i = 0, 1, 2$ and $t = \frac{\xi_1 + \xi_2}{2}$.
- $\Gamma_5(r) = 3^25r^5$ reached for $\xi_0 = a + r, \xi_i + 1 = \xi_i + 2r, i = 0, 1, 2, 3$ and $t = \frac{\xi_2 + \xi_3}{2}$.

More generally, $\Gamma_k(r) = c_k r^k$ where

$$c_{2q} = 1^23^25^5 \cdots (2q-1)^2, \quad c_{2q+1} = \frac{1^23^25^5 \cdots (2q+1)^2}{2q+1}.$$ 

Since the logarithmic function is increasing

$$\int_1^{2q-1} \ln(t) dt \leq 2[\ln(3) + \ln(5) \cdots + \ln(2q-1)];$$

$$(2q-1) \ln\left(\frac{2q-1}{e}\right) + 1 \leq \ln[3^25^5 \cdots (2q-1)^2].$$
It follows that
\[
\Gamma_{2q}(r) \geq \frac{e^2}{k-1} \left[ \frac{(k-1)r}{e} \right]^k \geq e \left[ \frac{(k-1)r}{e} \right]^k \text{ if } k = 2q;
\]
\[
\Gamma_{2q+1}(r) \geq \frac{e}{k} \left[ \frac{kr}{e} \right]^k \geq e \left[ \frac{(k-1)r}{e} \right]^k \text{ if } k = 2q+1.
\]

Going back to (6) we obtain
\[
m_-(\beta) \geq \Gamma_{k(\beta)} \left( \frac{\mu_f(2\beta \| f \|)}{2} \right) \geq \frac{e}{k(\beta)} \left[ \frac{(k(\beta) - 1)\mu_f(2\beta \| f \|)}{2e} \right]^{k(\beta)}.
\]

Finally,
\[
\rho(\beta) = \frac{m_-(\beta)}{m_+(\beta)} \geq \frac{e}{k(\beta)} \left[ \frac{(k(\beta) - 1)\mu_f(2\beta \| f \|)}{2e(b-a)} \right]^{k(\beta)}.
\]

Next, since the function \( \rho \rightarrow (1 + \beta \rho)(1 + \rho)^{-1} \) decreases on \([0, \infty)\), we obtain the following theorem which provides an upper bound of the reduction rate in terms of the degree of continuity of \( f \) and the parameter \( \beta \).

**Theorem 2.3.** In case where \( \xi_i = \frac{1}{2}(t_i^+(\beta) + t_i^-(\beta)) \) for all \( i \), for a suitably chosen \( \lambda \).

\[
\| f - \lambda \gamma(\beta) \| \leq \left[ 1 - \frac{(1 - \beta)\tau}{1 + \tau} \right] \| f \|,
\]

where \( \tau = \frac{e}{k(\beta)} \left[ \frac{(k(\beta) - 1)\mu_f(2\beta \| f \|)}{2e(b-a)} \right]^{k(\beta)} \).

This theorem will be the clue for the convergence of algorithms in the next sections.

## 3 The Chebyshev alternance theorem

Let us denote by \( \Pi_n \) the set of polynomial functions with degree less than or equal to \( n \).

The problem consists to solve the convex optimization problem

\[
\min_{\pi \in \Pi_n} [ \| \pi - f \| ].
\]
Based on Theorem 2.1, we present a very short and original proof of the celebrated result of Chebyshev on polynomial approximation. The intriguing $n + 2$ alternate points condition appears as the conjunction of the alternance propriety on continuous functions with the dimension of the linear space $\Pi_n$.

**Theorem 3.1** (Chebyshev theorem). Assume that $f : [a, b] \to \mathbb{R}$ is continuous. Then (7) has one and only one optimal solution. Furthermore, $\pi \in \Pi_n$ is the optimal solution if and only if there exist $\varepsilon \in \{-1, 1\}$, $k \geq n + 1$ and $t_0, t_1, \cdots, t_k$ such that

$$a \leq t_0 < t_1 < \cdots < t_n < t_k \leq b$$

and $f(t_i) - \pi(t_i) = \varepsilon (-1)^i \| f - \pi \|$ for all $i$.

**Proof.**

1. **Existence** The function $\gamma$ defined by $\gamma(\pi) = \| \pi - f \|$ is convex and continuous on $\Pi_n$. To prove the existence of one optimal solution it is enough to prove that the set $\mathcal{A} := \{ \pi \in \Pi_n : \| \pi - f \| \leq \| f \| \}$ is bounded. Given $n + 1$ arbitrary distinct points $t_1, t_2, \cdots, t_{n+1} \in [a, b]$, $\pi \in \Pi_n$ is uniquely defined by the data of $n + 1$ values $\alpha_i$ via the formula

$$\pi(t) = \sum_{i=1}^{n+1} \alpha_i \prod_{j \neq i} \frac{t - t_j}{t_i - t_j}.$$ 

If $\pi \in \mathcal{A}$, then for all $i$,

$$|\alpha_i - f(t_i)| = |\pi(t_i) - f(t_i)| \leq \| \pi - f \| \leq \| f \|.$$ 

Hence, $|\alpha_i| \leq 2 \| f \|$ for all $i$. Thus, $\mathcal{A}$ is bounded.

2. **Necessity** Let

$$M = \max \{ \pi(t) - f(t) : a \leq t \leq b \}, \quad m = \min \{ \pi(t) - f(t) : a \leq t \leq b \}.$$ 

Let us consider the case where $M + m \neq 0$. Set $\delta(t) = \frac{1}{2}(M + m)$ for all $t$. Then, $\pi - \delta \in \Pi_n$ and $\| \pi - \delta - f \| \leq \| \pi - f \| - \frac{1}{2}(M - m)$. Hence, $\pi$ is not an optimal solution.
We are left with the case $M + m = 0$. Apply Theorem 2.1 to the function $\pi - f$. Take $t_i = t_i^-$ for $i = 0, 1, \ldots, k$. Assume that $k < n + 1$. Consider the function $\gamma$ defined in part 2 of the theorem. Then, $\gamma \in \Pi_n$. We have seen that, for $\lambda > 0$ small enough, $\| \pi - \lambda \gamma - f \| < \| \pi - f \|$ and therefore $\pi$ is not optimal.

3. **Sufficiency and uniqueness** Next, assume that $k \geq n + 1$. Let $\hat{\pi} \in \Pi_n$ be an optimal solution. We have seen that such an optimal solution exists. We must prove that $\pi = \hat{\pi}$. We have

$$\varepsilon (-1)^i [\hat{\pi}(t_i) - f(t_i)] \leq \| \pi - f \| = \varepsilon (-1)^i [\pi(t_i) - f(t_i)], \quad i = 0, 1, \ldots, k.$$  

Hence, $\varepsilon (-1)^i [\hat{\pi} - \pi](t_i) \leq 0$ for $i = 0, 1, \ldots, k$. Since $\hat{\pi} - \pi \in \Pi_n$ and $k \geq n + 1$, this is possible only if $\pi = \hat{\pi}$.

Theorem 3.1 and its proof are constructive in the sense they allow to determine if some candidate $\pi \in \Pi_n$ to optimality is optimal and in the opposite case to give a better candidate, but they cannot directly used for designing algorithms building sequences converging to the optimal solution. Recall that that the problem consists in the minimisation of $\| \sigma - f \|$ subject to $\sigma \in \Pi_n$, $f$ being a fixed continuous function on $[a, b]$. Because it is numerically improbable that an arbitrary function reaches exactly its absolute maximum at more than two or three points, we shall construct algorithms converging to a solution satisfying an approximate optimality condition.

Given $\beta \in (0, 1)$, let

$$M(\sigma) = \max_{t \in [a, b]} [\sigma(t) - f(t)], \quad m(\sigma) = \min_{t \in [a, b]} [\sigma(t) - f(t)],$$

$$\hat{\sigma} = \sigma - \frac{1}{2} [M(\sigma) + m(\sigma)], \quad \theta(\sigma) = \| \hat{\sigma} - f \| = \frac{1}{2} [M(\sigma) - m(\sigma)].$$

The functions $M, m$ and $\theta$ are convex and defined on the whole space $\Pi_n$.

Next, given $\sigma \in \Pi_n$, let us denote by $k(\beta, \sigma)$ the integer corresponding to $k(\beta)$ in Theorem 2.2 applied to $\hat{\sigma} - f$ in place of $f$. 

14
Proposition 3.1. Let $\pi \in \Pi_n$ and $\beta, \beta'$ such that $0 < \beta' < \beta < 1$. There exists a neighbourhood $V$ of $\pi$ in $\Pi_n$ such that $k(\beta', \sigma) \geq k(\beta, \pi)$ for all $\sigma \in V$.

Proof. There exist $\xi_i \in [a, b]$ and $\varepsilon \in \{-1, 1\}$ such that

$$a \leq \xi_0 < \xi_1 < \cdots < \xi_{k(\beta, \pi)} \leq b \quad \text{and}$$

$$\varepsilon(-1)^i(\pi - f)(\xi_i) \geq \beta M(\pi) > \beta' M(\pi) \quad \forall i.$$ 

Due to the continuity of the functions $M$ and $m$, there is a neighbourhood $V$ of $\pi$ in $\Pi_n$ such that for all $\sigma \in V$

$$\varepsilon(-1)^i(\sigma - f)(\xi_i) > \beta' M(\sigma) \quad \forall i = 0, 1, \ldots, k(\beta, \pi).$$

It follows $k(\beta', \sigma) \geq k(\beta, \pi)$. 

Given $\sigma \in \Pi_n$ and $\beta \in (0, 1)$, we say that $\sigma \in \Pi_n$ fulfills the $\beta$-alternance optimality condition for the problem (7) if $k(\beta, \sigma) \geq n + 1$.

Proposition 3.2. a) If $\bar{\pi} \in \Pi_n$ is the optimal solution to problem (7) there is $\bar{\beta} \in (0, 1)$ such that $k(\beta', \bar{\pi}) \geq n + 1$ for all $\beta \in [\bar{\beta}, 1]$. Furthermore, for any $\beta \in (\bar{\beta}, 1]$, there exists a neighbourhood $V_\beta$ of $\bar{\pi}$ such that $k(\beta', \sigma) \geq n + 1$ for all $\sigma \in V_\beta$ and $\beta' \leq \bar{\beta}$.

b) If $\bar{\pi} \in \Pi_n$ is not the optimal solution of problem (7), there is $\beta \in (0, 1)$ such that $k(\beta', \bar{\pi}) < n + 1$ for all $\beta' \in [\beta, 1]$.

Proof. The proposition is a consequence of Theorem 3.1 Proposition 2.1 and Proposition 3.1. 

Based on Proposition 3.2 we introduce the following approximate optimality condition. Given $\sigma \in \Pi_n$ and $\beta \in (0, 1)$, we say that $\sigma \in \Pi_n$ fulfills the $\beta$-alternance optimality condition for the problem (7) if $k(\beta, \sigma) \geq n + 1$.

Now, we are ready to propose an algorithm converging to some $\sigma \in \Pi_n$ fulfilling this optimality condition. It supposes that we have at our disposi-
tion an auxiliary algorithm giving a rather good estimation of the maximum of a continuous function on the closed interval $[a, b]$. 

**Algorithm 2: 2 knots**

**Input:** Together with $\beta^+, \beta^-$ with $0 < \beta^- < \beta^+ < 1$ two fixed parameters $\gamma^-, \gamma^+$ such that $0 < \gamma^- \leq 1 < \gamma^+$ are given.

**Initialisation**
1. Start with $\sigma$ defined by $\sigma(t) = \frac{t-a}{b-a} f(b) + \frac{t-b}{a-b} f(a)$.
2. Start with $\beta \in [\beta^-, \beta^+]$.

**Main Step**
1. Compute $M = \max_{t \in [a, b]} [\sigma(t) - f(t)]$, $m = \min_{t \in [a, b]} [\sigma(t) - f(t)]$.
2. Update $\sigma$: take $\sigma(t) = \sigma(t) - M + m$ for all $t$.
3. Apply Theorem 2.2 to $\sigma - f$.
4. If $k(\beta) \geq n + 1$ and $\beta = \beta^+$: STOP, we have found a $\beta^+$-approximation of the solution.
5. If $k(\beta) \geq n + 1$ and $\beta < \beta^+$: take $\beta = \min [\gamma^+ \beta, \beta^+]$ and return to 3.
6. If $k(\beta) \leq n$: set $\xi_i = t_i^{(\beta)} + t_{i+1}^{(\beta)}$, $i = 0, 1, \cdots, k(\beta) - 1$.
   Set $\gamma(t) = \varepsilon(\beta) \prod_{i=0}^{k(\beta)-1} (\xi_i - t)$.
   Choose $\lambda > 0$ such that $\|\sigma - f - \lambda \gamma\| < \frac{1 + \beta \rho(\beta)}{1 + \rho(\beta)} \|\sigma - f\|$, such $\lambda$ exist in view of Theorem 2.2.
   Do $\sigma = \sigma - \lambda \gamma$ and $\beta = \max [\beta^-, \gamma^- \beta]$.
   Go to main step.

**Theorem 3.2.** The algorithm converges in a finite number of steps to some $\hat{\sigma} \in \Pi_n$ such that $k(\beta^+, \hat{\sigma}) \geq n + 1$. Furthermore, $\beta^+ \|\hat{\sigma} - f\| \leq \|\pi_n - f\|$.

**Proof.** The second part of the theorem follows from the first, and part 2) of Theorem 2.2 Assume for contradiction that the algorithm does not stop in a finite number of iterations. Let us denote by $\pi_n$ the unique optimal solution of problem (7) and by $\sigma^l$ the polynomial function at the $l$-th iteration after substep 6. of main step. By construction,

$$0 \leq \|\pi_n - f\| \leq \|f - \sigma^{l+1}\| < \|f - \sigma^l\| \quad \forall l.$$ 

Set $\alpha = \lim_l \|f - \sigma^l\|$. 

16
The function $\sigma \to \|\sigma - f\|$ is convex and continuous, it reaches its minimum on $\Pi_n$ at one unique point. It follows the compactness of the set

$$\Sigma = \{ \sigma \in \Pi_n : \|f - \sigma\| \leq \|f - \sigma^0\| \}.$$  

Let us define for $\delta > 0$

$$\bar{\mu}(\delta) = \inf_{\sigma} [\mu(\delta, f - \sigma) : \sigma \in \Sigma].$$

More explicitly,

$$\bar{\mu}(\delta) = \inf_{t,s,\sigma} \|t - s\| : s, t \in [a, b], \sigma \in \Sigma, |f(t) - \sigma(t) - f(s) + \sigma(s)| \geq \delta].$$

Due to the compactness of the sets $[a, b]$ and $\Sigma$, the infimum is reached at some $(\bar{s}, \bar{t}) \in [a, b]^2$ and $\bar{\sigma} \in \Sigma$. Hence, $\bar{\mu}(\delta) > 0$.

i) Firstly, consider the case where $\alpha > 0$. Set

$$\bar{\tau} = \min_{k \leq n} \left\{ \frac{1}{k} \left( \frac{k \mu_f(2\beta - \alpha)}{2c(b - a)} \right)^k \right\}.$$ 

Then $\bar{\tau} > 0$. Since $\beta - \alpha \leq \beta^l \|f - \sigma^l\|$, Theorem 2.3 implies

$$0 < \alpha < \|f - \sigma^{l+1}\| \leq \left[ 1 - \frac{(1 - \beta^+) \bar{\tau}}{1 + \bar{\tau}} \right] \|f - \sigma^l\| \quad \forall l$$

which is not possible.

ii) It remains to consider the case where $\alpha = 0$. Then $f = \pi_n$ and the whole sequence $\{\sigma^l\}$ converges to the optimal solution $\bar{\pi} = \pi_n$ of problem (7).

Let $\hat{\beta}$ as in a) of Proposition 3.2. Set $\hat{\beta} = \max \{ \hat{\beta}, \beta^+ \}$ and let $V$ neighbourhood of $\pi_n$ such that $k(\beta', \sigma) \geq n + 1$ for all $\sigma \in V$ and $\beta' \leq \hat{\beta}$. Since $\sigma^l$ goes to $\pi_n$ when $l$ goes to $+\infty$, there is a finite integer $l_0$ such that $\sigma^{l_0} \in V$ and thereby $k(\beta^{l_0}, \sigma^{l_0}) \geq n + 1$. After a finite number of iterations of item 5 where $\sigma$ remains unchanged, we are in situation 4. The algorithm stops.
4 Spline approximation with $p + 2$ fixed knots

In this section, we are given $p + 1$ integers $n_i \geq 1$, $i = 0, \ldots, p$ and $p + 2$ points $x_i \in [a, b]$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_p < x_{p+1} = b.$$ 

We define $\Sigma$ as the set of functions $\sigma$ on $[a, b]$ such that for $i = 0, \ldots, p$ there exist $\sigma_i \in \Pi_{n_i}$ such that

$$\sigma(t) = \sigma_i(t) \quad \forall t \in I_i := [x_i, x_{i+1}], \quad \sigma_i(x_{i+1}) = \sigma_{i+1}(x_{i+1}), \quad i = 0, \ldots, p.$$ 

The functions $\sigma \in \Sigma$ are called splines, the points $x_i$ are called knots. $\Sigma$ is a linear space with dimension $n_0 + n_1 + \cdots + n_p$. We use the following notation $\sigma = (\sigma_0, \sigma_1, \cdots, \sigma_p)$.

We are concerned with the convex optimization problem

$$\min_{\sigma \in \Sigma} \|\sigma - f\|,$$ \hspace{1cm} (8)

where $f$ is continuous on $[a, b]$.

Using a similar argument as in section 2, it is not difficult to see that the set $A = \{\sigma \in \Sigma : \|\sigma - f\| \leq \|f\|\}$ is bounded. Hence, the problem has at least one optimal solution.

Set $\|\sigma - f\|_k = \max_{t \in I_k} |\sigma(t) - f(t)| = \max_{t \in I_k} |\sigma_k(t) - f(t)|$. Without loss of coherence, we write $\|\sigma - f\|_k = \|\sigma_k - f\|_k$. Then, $\|\sigma - f\| = \max_k \|\sigma - f\|_k$.

Let $\sigma \in \Sigma$. Set

$$M := \max_{t \in [a, b]} [\sigma(t) - f(t)], \quad m := \min_{t \in [a, b]} [\sigma(t) - f(t)].$$

It is clear that $M + m = 0$ is a necessary condition for $\sigma$ to be an optimal spline. To see that, take $\delta(t) = \frac{1}{2}(M + m)$ for all $t$. Then, $\sigma - \delta \in \Sigma$ and $\|\sigma - \delta - f\| \leq \|\sigma - f\| - \frac{1}{2}(M - m)$.

4.1 A sufficient condition for optimality

Assume that $M + m = 0$ and $\sigma \in \Sigma$ is not optimal. Then, there is some $\delta \in \Sigma$ such that $\|\sigma + \delta - f\| < \|\sigma - f\|$. Then $\varepsilon(-1)^j \delta(t_j) < 0$ and
\( \varepsilon(-1)^j \delta(t_j^+) < 0 \), for all \( j = 0, 1, \cdots, k \) where \( k \) \( \parallel \sigma \) obtained via Theorem 2.1 applied to the continuous function \( \sigma - f \).

Without loss of generality, \( \delta \) can be taken so that \( \delta(x_i) \neq 0 \) for all \( i \). If not, add to \( \delta \) a constant function \( \iota \) such that \( (\delta + \iota)(x_i) \neq 0 \) for all \( i \) and small enough to have \( \| \sigma + \delta + \iota - f \| < \| \sigma - f \| \).

The number of roots of the equation \( \delta(t) = 0 \) in the interval \( [t_j^-, t_j^+] \) is a strictly positive odd number. The number of roots contained in the interval \( [t_j^-, t_j^+] \) is even, possibly 0. More generally, the number of roots of the equation \( \delta(t) = 0 \) in the interval \( [t_j^-, t_j^+] \) is of the form \( l + 2m \) with \( m \geq 0 \) integer.

Given \( i_1, i_2 \) with \( 0 \leq i_1 < i_2 \leq p \), set

\[
J(i_1, i_2) = \{ j : t_j^+ \text{ and/or } t_j^- \in [x_{i_1}^+, x_{i_2}^-] \}.
\]

It follows that the number of roots of the equation \( \delta(t) = 0 \) in the interval \( [x_{i_1}, x_{i_2}] \) is at least \( \text{card}(J(i_1, i_2)) - 1 \).

Recall that \( \delta(x_i) \neq 0 \) for all \( i \) and, on each interval \( [x_i, x_{i+1}] \), \( \delta \) is a polynomial function \( \delta_i \) with degree less or equal to \( n_i \). Therefore, the total number of roots of the equation \( \delta(t) = 0 \) in the interval \( [x_{i_1}, x_{i_2}] \) is at most \( n_{i_1} + n_{i_1+1} + \cdots + n_{i_2-1} \). Hence, for the existence of \( \delta \), it is necessary that for all \( i_1, i_2 \) such that \( i_1 < i_2 \) the following holds

\[
\text{card}(J(i_1, i_2)) - 1 \leq n_{i_1} + n_{i_1+1} + \cdots + n_{i_2-1}.
\]

We have proved the following two propositions.

**Proposition 4.1.** Assume \( M + m = 0 \). A sufficient condition for \( \sigma \) to be optimal is

\[
\exists i_1, i_2, \ i_1 < i_2, \ such \ that \ \text{card}(J(i_1, i_2)) \geq n_{i_1} + n_{i_1+1} + \cdots + n_{i_2-1} + 2. \quad (CS)
\]

**Proposition 4.2.** Assume that we are given \( q \) points \( \xi_i \) such that

\[
a \leq \xi_1 < \xi_2 < \cdots < \xi_{q-1} < \xi_q \leq b. \ If \ q \geq n_0 + n_1 + \cdots + n_p + 2, \ there \ is \ no \ \delta \in \Sigma \ such \ that \ \delta(\xi_i)\delta(\xi_{i+1}) < 0 \ for \ i = 1, \cdots, q - 1.
\]

19
4.2 A sufficient condition for nonoptimality

**Proposition 4.3.** Assume \( M + m = 0 \). Let the integer \( k, \varepsilon \) and the points \( t_j^- \) and \( t_j^+ \) obtained from Theorem 2.1 applied to the function \( \sigma - f \). Assume that for each \( j \) there is \( \xi_j \in (t_j^+, t_{j+1}^-) \) such that \( \xi_j \neq x_i \) for all \( i, j \) and the number \( r_i \) of \( \xi_j \) belonging to the interval \([x_i, x_{i+1}]\) is less than or equal to \( n_i \). Then \( \sigma \) is not an optimal spline.

**Proof.** We build a function \( \gamma : [a, b] \rightarrow \mathbb{R} \) and functions \( \gamma_i : [x_i, x_{i+1}] \rightarrow \mathbb{R} \) as follows:

\[
\gamma(t) = \varepsilon \prod_{j=0}^{k-1} (\xi_j - t), \quad \gamma_i(t) = \varepsilon_i \prod_{j \in J(i)} (\xi_j - t), \quad i = 0, \ldots, p, \tag{9}
\]

where \( J(i) = \{ j \in (x_i, x_{i+1}) \} \), and, for each \( i, \varepsilon_i \in \{-1, 1\} \) is chosen so that the signs of \( \gamma \) and \( \gamma_i \) are the same on the interval \( I_i = [x_i, x_{i+1}] \). In case where \( r_i = \text{card}(J(i)) = 0 \), the product is taken equal to 1.

Set \( \delta_0 = -\gamma_0 \) and, for \( i = 1, \ldots, p \), \( \delta_i = \alpha_i \gamma_i \) where \( \alpha_i > 0 \) is taken so that \( \delta_i(x_i) = \delta_{i-1}(x_i) \). By construction, \( \delta = (\delta_0, \delta_1, \ldots, \delta_p) \in \Sigma \) and

\[
\varepsilon(-1)^j \delta(t) > 0 \quad \text{if} \quad \xi_j < t < \xi_{j+1}, \quad \forall \ j = 0, 1, \ldots, k,
\]

and, in particular,

\[
\varepsilon(-1)^j \delta(t) < 0 \quad \text{if} \quad t_j^- \leq t \leq t_j^+, \quad \forall \ j = 0, 1, \ldots, k.
\]

We shall prove that, for \( \lambda > 0 \) small enough,

\[
|\sigma(t) + \lambda \delta(t) - f(t)| < M \quad \forall \ t \in [a, b]. \tag{10}
\]

Assume, for contradiction, that for each positive positive integer \( m \) there is some \( t_m \in [a, b] \) such that

\[
|\sigma(t_m) + \frac{1}{m} \delta(t_m) - f(t_m)| \geq M. \tag{11}
\]

Let \( \bar{t} \) be a cluster point of the sequence \( \{t_m\} \). Then, \( |\sigma(\bar{t}) - f(\bar{t})| = M \) and thereby there is some \( j \) such that \( \bar{t} \in [t_j^-, t_j^+] \). For \( t \) in a neighbourhood of \( \bar{t} \)
one has $-M < \varepsilon(-1)^j(\sigma(t) - f(t)) \leq M$ and $\varepsilon(-1)^j\delta(t) < 0$. For $m$ large enough, $t_m$ belongs to the neighbourhood and therefore

$$-M < \varepsilon(-1)^j(\sigma(t) + \frac{1}{m}\delta(t) - f(t)) < M,$$

in contradiction with (11).

Since (10) holds for small $\lambda$, $\sigma$ is not an optimal spline.

4.3 Condition (CS) is necessary and sufficient for optimality

Now, we present an algorithm which, in case where condition (CS) does not hold, builds points $\xi_j$ which fulfill the conditions of Proposition 4.3.

Firstly, observe that, in case where (CS) does not hold, necessarily for all $i$ one has $\text{card}(J(i, i+1)) \leq n_i + 1$ and, in case where $\text{card}(J(i, i+1)) = n_i + 1$, this
Algorithm 3: Construction of intermediary points $\xi_j$

Initialisation
Set $r_i = 0$ and $J(i) = \emptyset$ for all $i$.
Set $j = 0$.

Step $j$
Let $i$ be such that $t_j^+ \in [x_i, x_{i+1})$. We consider four cases:

First case: $x_i \leq t_j^+ < t_{j+1}^- \leq x_{i+1}$ and $r_i < n_i$.
- Do $\xi_j = \frac{1}{2}[t_j^+ + t_{j+1}^-]$. By construction $\xi_j \in (x_i, x_{i+1})$.
- Do $r_i = r_i + 1$ and $J(i) = J(i) \cup \{j\}$.
- If $j = k$ go to End, otherwise do $j = j + 1$ and go to Step $j$.

Second case: $x_i \leq t_j^+ < x_{i+1} < t_{j+1}^- < t^− \leq x_{i+1}$ and $r_i < n_i$.
- Do $\xi_j = \frac{1}{2}[t_j^+ + x_{i+1}]$. By construction, $\xi_j \in (x_i, x_{i+1})$.
- Do $r_i = r_i + 1$ and $J(i) = J(i) \cup \{j\}$.
- If $j = k$ go to End, otherwise do $j = j + 1$ and go to Step $j$.

Third case: $x_i \leq t_j^+ < x_{i+1} < t_{j+1}^- < t^− \leq x_{i+1}$ and $r_i = n_i$.
- Do $\xi_j = \frac{1}{2}[x_{i+1} + \min(x_{i+2}, t_{j+1}^-)]$. By construction, $\xi_j \in (x_{i+1}, x_{i+2})$.
- Do $i = i + 1$, $r_i = 1$ and $J(i) = \{\xi_j\}$.
- If $j = k$ go to End, otherwise do $j = j + 1$ and go to Step $j$.

Fourth case: $x_i \leq t_j^+ < t_{j+1}^- \leq x_{i+1}$ and $r_i = n_i$.
- Stop: (CS) holds, $\sigma$ is one optimal spline.

End

$\xi_0, \xi_1, \cdots, \xi_k$ fulfilling the requirements of Proposition 4.3 have been obtained. Hence, $\sigma$ is not one optimal spline.

Proof. It is enough to prove that condition (CS) holds as soon as we encounter the fourth case. We are faced with $i$ and $j$ such that $r_l = n_l$ for all $l < i$, $r_i = n_i$ and $x_i \leq t_j^+ < t_{j+1}^- \leq x_{i+1}$. Then,

$$x_i < \xi_{j-n_i} < t_{j+1-n_i}^- \leq t_{j+1-n_i}^+ < \xi_{j+1-n_i} < \cdots < \xi_{j-1} < t_j^- \leq t_j^+ < t_{j+1}^- \leq x_{i+1}.$$  

There are two possibilities:
i) \( x_i \leq t^\pm_{j-n_i} < \xi_{j-n_i} < t^\pm_{j+1-n_i} < \cdots < t^\pm_{j} < t^\pm_{j+1} \leq x_{i+1} \):

Then \( \text{card}(J(i, i+1)) \geq n_i + 2 \). Hence, condition (CS) holds.

ii) \( t^\pm_{j-n_i} < x_i < \xi_{j-n_i} < t^\pm_{j+1-n_i} < \cdots < t^\pm_{j} < t^\pm_{j+1} \leq x_{i+1} \):

Report to the determination of \( \xi_{j-n_i} \), according to the rules given in the second and third cases. Necessarily, \( n_{i-1} = r_{i-1} \). For simplicity, set \( l = j - n_i \), one has

\[
x_{i-1} < \xi_{l-n_{i-1}} < t^-_{j+1-n_i} < \cdots < \xi_{l-1} < t^-_l \leq x_i.
\]

There are two possibilities:

a) \( x_{i-1} \leq t^+_l-n_{i-1} < \xi_{l-n_{i-1}} < t^-_{j+1-n_i} < \cdots < t^-_l \leq x_i \):

Then \( \text{card}(J(i-1, i+1)) \geq n_{i-1} + n_i + 2 \). Hence, condition (CS) holds.

b) \( t^+_l-n_{i-1} < x_{i-1} < \xi_{l-n_{i-1}} < \cdots < t^-_l \leq x_i \):

Report to the obtention of \( \xi_{l-n_{i-1}} \). Necessarily, \( n_{i-2} = r_{i-2} \). For simplicity, set \( m = l - n_{i-1} \), one has

\[
x_{i-2} < \xi_{m-n_{i-2}} < \cdots < \xi_{m-1} < t^-_m \leq x_{i-1}.
\]

There are two possibilities, etc, repeat the process as long as necessary. \( \square \)

The following theorem is a consequence of the construction.

**Theorem 4.1.** Assume that \( f : [a, b] \to \mathbb{R} \) is continuous and \( \sigma \in \Sigma \) is such that \( M := \max \{ \sigma(t) - f(t) : a \leq t \leq b \} = -\min \{ \sigma(t) - f(t) : a \leq t \leq b \} > 0 \). Then \( \sigma \in \Sigma \) is an optimal solution of (3) if and only if (CS) holds.

We are ready for describing a prototype algorithm. As in Algorithm 1.
we seek splines satisfying a $\beta$-optimal condition with $\beta \in (0, 1)$ close to 1.

**Algorithm 4: $p + 2$ fixed knots**

**Initialisation**

Start with $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_p)$ defined by

$$\sigma_i(t) = \frac{t-x_i}{x_{i+1}-x_i}f(x_{i+1}) + \frac{t-x_{i+1}}{x_i-x_{i+1}}f(x_i), \quad i = 0, 1, \ldots, p.$$ 

**Main Step**

1. Find $M = \max_{t \in [a,b]}[\sigma(t) - f(t)]$, $m = \min_{t \in [a,b]}[\sigma(t) - f(t)]$.
2. Update $\sigma$: take $\sigma(t) = \sigma(t) - (M + m)/2$ for all $t$.
3. Apply Theorem 2.2 to $\sigma - f$. One obtains the quantities $\varepsilon(\beta), k(\beta), t_j^-(\beta), t_j^+(\beta)$.
4. Apply Algorithm 2 with $\varepsilon(\beta), k(\beta), t_j^-(\beta), t_j^+(\beta)$ in place of $\varepsilon, k, t_j^-, t_j^+$ to obtain intermediary values $\xi_j \in (t_j^+(\beta), t_{j+1}^-(\beta))$.
5. If Algorithm 2 stops at the fourth case of step $j$ with $k(\beta) < j$: STOP, we have found a $\beta$-optimal spline.
6. If Algorithm 2 stops at step $j$ with $k(\beta) = j$, construct functions, $\gamma, \gamma_i, \delta_i$ and $\delta$ as in Proposition 4.3. Next choose $\lambda > 0$ such that $\|\sigma + \lambda\gamma - f\| < \|\sigma - f\|$.
7. Do $\sigma = \sigma + \lambda\gamma$ and go to main step.

5 The free knots spline approximation problem

5.1 Local properties of the objective function

In this section, we are given $p + 1$ integers $n_i \geq 1, i = 0, 1, \ldots, p$ and

$$X = \{x \in [a,b]^{n+2} : a = x_0 < x_1 < x_2 < \cdots < x_p < x_{p+1} = b\}.$$ 

Given $x \in X$, $\Sigma(x)$ is the set of functions $\sigma$ on $[a,b]$ such that for each $i = 0, 1, \ldots, p - 1$ there exists $\sigma_i \in \Pi_{n_i}$ such that

$$\sigma(t) = \sigma_i(t) \quad \forall t \in I_i(x) := [x_i, x_{i+1}], \quad i = 0, \ldots, p$$

24
and
\[ \sigma_i(x_{i+1}) = \sigma_{i+1}(x_{i+1}), \quad i = 0, \ldots, p - 1. \]

Here again, we use the following notation \( \sigma = (\sigma_0, \sigma_1, \cdots, \sigma_p) \).

The free knots problem is the minimisation problem
\[
\min_{x,\sigma} \| \sigma - f \| : x \in X, \sigma \in \Sigma(x),
\]
where \( f \) is a given continuous function on \([a, b]\).

Set
\[
\theta(x) = \min_{\sigma \in \Sigma(x)} \| \sigma - f \|.
\]

(13)

Then \( \theta(x) \leq \| f \| \) for any \( x \) since the null function belongs to \( \Sigma(x) \). The next proposition is concerned with the continuity properties of the function \( \theta \).

**Proposition 5.1.** \( \theta \) is locally Lipschitz on \( X \).

**Proof.** i) Let \( \bar{x} \in X \). Set for \( i = 0, 1, \cdots, p \) and \( k = 1, 2, \cdots, n_i + 1 \)
\[
t^k_i = \bar{x}_i + \frac{k}{n_i + 2} (\bar{x}_{i+1} - \bar{x}_i).
\]

We consider a neighbourhood \( V \) of \( \bar{x} \) such that for all \( x \in V \)
\[
x_i < t^1_i < t^{n_i+1}_i < x_{i+1} \quad i = 0, 1, \cdots, p.
\]

Let \( x \in V \) and \( \sigma = (\sigma_0, \sigma_1, \cdots, \sigma_p) \in \Sigma(x) \) be an optimal solution of (13). We know by Section 4 that such a \( \sigma \) exists. On the interval \([x_i, x_{i+1}]\), \( \sigma \) is expressed in a unique way under the form
\[
\sigma(t) = \sigma_i(t) = \sum_{k=1}^{n_i+1} \alpha^k_i \prod_{l \neq k} \frac{t - t^l_i}{t^k_i - t^l_i}.
\]

Since \( \| \sigma - f \| \leq \| f \| \) one has \( |\alpha^k_i| \leq 2 \| f \| \) for all \( i, k \). One deduces that there exists \( K \), not depending on \( x \in V \) and \( i \), such that \( |\sigma_i(t) - \sigma_i(t')| \leq K |t - t'| \) for all \( t, t' \in [x_i, x_{i+1}] \).
ii) Let again \( x \in V \) and \( \sigma = (\sigma_0, \ldots, \sigma_p) \in \Sigma(x) \) be an optimal solution of (13). Let \( y \in V \). We build \( \tau = (\tau_0, \ldots, \tau_p) \in \Sigma(y) \) as follows

\[
\begin{align*}
\tau_0(t) &= \sigma_0(t) \quad \forall t \in [y_0, y_1], \\
\tau_1(t) &= \sigma_1(t) + \tau_0(y_1) - \sigma_1(y_1) \quad \forall t \in [y_1, y_2], \\
\tau_2(t) &= \sigma_2(t) + \tau_1(y_2) - \sigma_2(y_2) \quad \forall t \in [y_2, y_3], \\
&\ldots
\end{align*}
\]

From what we deduce that for all \( t \in [y_i, y_{i+1}] \) one has

\[
\tau_i(t) - \sigma_i(t) = \sum_{k=0}^{i-1} \sigma_k(y_{k+1}) - \sigma_{k+1}(y_{k+1}).
\]

Recall that \( \sigma_k(x_{k+1}) - \sigma_{k+1}(x_{k+1}) = 0 \) for all \( k \). Then, it follows from part i), that there exists \( L \) such that \(|(\tau_i - \sigma_i)(t)| \leq L |y_i - y_{i+1}| \) for all \( t \in [y_i, y_{i+1}] \). Combining with other intervals one obtains for all \( t \in [a, b] \)

\[
|(|\tau - f)(t)| \leq |(|\sigma - f)(t) + |(|\tau - \sigma)(t)| \leq |(|\sigma - f)(t)| + L \|x - y\|_1,
\]

where, for \( v \in \mathbb{R}^{p+2}, \|v\|_1 = \sum_{i=0}^{p+1} |u_i| \). Hence,

\[
\theta(y) \leq \|\tau - f\| \leq \|\sigma - f\| + L \|y - x\|_1 = \theta(x) + L \|y - x\|_1.
\]

Finally, permuting the roles playing by \( x \) and \( y \),

\[
|\theta(y) - \theta(x)| \leq L \|y - x\|_1 \quad \forall x, y \in V.
\]

Thus, \( \theta \) is Lipschitz on \( V \).

\[\square\]

It is clear that if \( (x, \sigma) \) is a (global) local optimal solution of (12), then \( \theta(x) = \|\sigma - f\| \). Since the set \( \{(x, \sigma) : x \in X, \sigma \in \Sigma(x)\} \) is not convex, the problem (12) is non convex. This leads to investigate local optimality.
5.2 Local \( w \)-minimality

Through this subsection, \( x \in X \) and \( \sigma \in \Sigma(x) \) is such that \( \theta(x) = \|\sigma - f\| \). A point \( t \) for which \( \theta(x) = |\sigma(t) - f(t)| \) is said to be extreme.

We are interested in the existence of local moves around \( x \) which potentially induce a decrease of \( \theta \).

Apply Theorem 2.1 to the function \( \sigma - f \). Let \( \varepsilon, k, t_j^-, t_j^+ \) be as in the theorem. Apply the algorithm for the construction of points \( \xi_j \) given in the last section.

Since \( \sigma \) is optimal, the algorithm stops in the fourth case with \( i^0 \geq 0 \) and \( j^0 \) such that

\[
 r_{i^0} = n_{i^0}, \quad x_{i^0} \leq t_{j^0}^+ < t_{j^0+1}^- \leq x_{i^0+1}, \quad r_l \leq n_l \quad \forall \ l = 0, 1, \ldots, i^0.
\]

Let \( i^- \) be the smallest \( i \leq i^0 \) such that \( r_k = n_k \) for all \( k \in [i, i^0] \).

Next, let \( i^+ \) be the greatest \( i \geq i^0 \) such that

\[
 \text{card}(J(l, l + 1)) \geq n_l \quad \forall \ l = i^0, i^0 + 1, \ldots, i^+.
\]

Then,

\[
 i^- \leq i^0 \leq i^+, \quad \text{card}(J(l, l + 1)) \geq n_l \quad \forall \ l = i^-, i^- + 1, \ldots, i^+,
\]

\[
 \text{card}(J(i^-, i^+ + 1)) \geq 2 + n_{i^-} + n_{i^-+1} + \cdots + n_{i^+}.
\]

Let \( j^- \) be the smallest \( j \) such that \( x_{i^-} \leq t_{j^+}^- \).

Let \( j^+ \) be the greatest \( j \) such that \( t_{j^-}^+ \leq x_{i^+1} \). Then,

\[
 x_{i^-} \leq t_{j^-}^+ < t_{j^+}^- \leq x_{i^+1}, \quad j^+ \geq j^- + 1 + n_{i^-} + n_{i^-+1} + \cdots + n_{i^+}.
\]

**Proposition 5.2.** Assume that \( y \in X \) is such that

\[
 x_{i^-} \leq y_{i^-} \leq t_{j^-}^+, \quad t_{j^+}^- \leq y_{i^+1} \leq x_{i^+1}
\]

and \( y_i = x_i \) for all \( i \) such that \( t_{j^-}^+ \leq x_i \leq t_{j^+}^- \). Then, \( \theta(y) \geq \theta(x) \).
Proof. Let us consider the fixed knots spline approximation problem on the interval $[t_{j-}^+, t_{j+}^-]$ with knots $t_{j-}^+, x_{i-1}+1, x_{i-1}+2, \ldots, x_i^+, t_{j+}^-$ and respective degrees $n_{i-}, n_{i-1}+1, \ldots, n_i$. Then, due to Proposition 4.1, the restriction of $\sigma$ to the interval is one optimal spline and therefore for any $y \in X$ and $\tau \in \Sigma(y)$

$$\sup_{t_{j-}^+ \leq t \leq t_{j+}^-} |\tau(t) - f(t)| \geq \sup_{t \in [t_{j-}^+, t_{j+}^-]} |\sigma(t) - f(t)| = \theta(x).$$

It follows that $\theta(y) \geq \theta(x)$. \hfill $\square$

With regard to this proposition, we focus on the knots $x_i$ which belong to the interval $[t_{j-}^+, t_{j+}^-]$. Our strategy consists in seeking if a small move of a given knot $x_i$ can produce a decrease of $\text{card}(J(i^-, i^+ + 1))$. Only knots $x_i$ such that $|\sigma(x_i) - f(x_i)| = \theta(x)$ are relevant in this purpose. We shall describe two cases where such a move does work.

1) Move on the right.

Let $s \in \{-1, 1\}$ be such that $\sigma(x_i) - f(x_i) = s \theta(x)$. Assume that there exist $\bar{\lambda} > 0$, $\delta_0 \in \Pi_0, \delta_1 \in \Pi_1, \ldots, \delta_{i-1} \in \Pi_{i-1}$ such that

1. $s \delta_{i-1}(x_i) > 0$,

and for $k = 0, 1, \ldots, i - 1$,

2. $\theta(x) \geq |\sigma_k(t) - \lambda \delta_k(t) - f(t)|$ for all $t \in [x_k, x_k+1]$, for all $\lambda \in [0, \bar{\lambda}]$,

3. $\delta_k(x_{k+1}) = \delta_{k+1}(x_{k+1})$.

Such a situation occurs for $i \leq i_0$. Indeed, before stopping at $i_0$, Algorithm 2 has built intermediary points $\xi_j \in [a, x_{i_0}]$. Report to the construction of the functions $\delta_i$ in the proof of Proposition 4.3. The functions $s\delta_0, s\delta_1, \ldots, s\delta_{i-1}$ fulfill conditions 1, 2 and 3. In the next subsection, we shall study in detail the existence of such functions in the general case.

Let $y \in X$ be such that $y_k = x_k$ for all $k \neq i$. Let $y_i \in (x_i, x_{i+1})$ be close to $x_i$. Let $\lambda \in (0, \bar{\lambda}]$. Set

$$\tau_k = \begin{cases} 
\sigma_k - \lambda \delta_k & \text{if } k \leq i - 1, \\
\sigma_k & \text{if } k \geq i.
\end{cases}$$

28
The idea is to take, when this is possible, $y_i \in (x_i, x_{i+1})$ so that the function
\[ \tau = (\tau_0, \ldots, \tau_p) \] belongs to $\Sigma(y)$. This is the case if and only if
\[ (\sigma_{i-1} - \delta_{i-1})(y_i) = \sigma_i(y_i) \]
Let us introduce
\[ H(t, \lambda) = (\sigma_{i-1} - \sigma_i)(x_i + t) - \lambda \delta_{i-1}(x_i + t). \]
Then, $H(0, 0) = 0$. Assume that $(\sigma_{i-1} - \sigma_i)'(x_i) \neq 0$. The implicit function theorem says that there exists a differentiable function $t(.)$ such that in a
neighbourhood of 0,
\[ 0 = H(t(\lambda), \lambda), \quad t'(\lambda) = \frac{\delta_{i-1}(x_i + t(\lambda))}{(\sigma_{i-1} - \sigma_i - \lambda \delta_{i-1})'(x_i + t(\lambda))}, \]
\[ t(0) = 0, \quad t'(0) = \frac{\delta_{i-1}(x_i)}{(\sigma_{i-1} - \sigma_i)'(x_i)}. \]
Set $y_i = x_i + t(\lambda)$. Then, $(\sigma_{i-1} - \delta_{i-1})(y_i) = \sigma_i(y_i)$ and $\tau \in \Sigma(y)$.
A move on the right of $x_i$ means $t(\lambda) > 0$ which is obtained for small
values of $\lambda \in (0, \bar{\lambda}]$ under the condition $t'(0) > 0$, i.e.,
\[ 4. \quad s(\sigma_{i-1} - \sigma_i)'(x_i) > 0. \]
Then, by construction, $x_i < y_i$ and
\[ |\tau(t) - f(t)| \leq |\sigma(t) - f(t)| \leq \theta(x) \quad \forall t \in [a, b]. \]
Moreover, in view of 1, there exists $x'_i < x_i$ such that for $\lambda > 0$ small enough
\[ |\tau(t) - f(t)| < \theta(x) \quad \forall t \in [x'_i, y_i]. \]
Summarizing,
\[ \{ t \in [a, b] : |\tau(t) - f(t)| \leq \theta(x) \} \subset \{ t \in [a, b] : |\sigma(t) - f(t)| \leq \theta(x) \}, \]
the inclusion being strict. Therefore, $\theta(y) \leq \|\tau - f\| \leq \theta(x)$. Unlike $x_i$, the
new knot $y_i$ is not an extreme point. The sequence of alternating extreme
points is modified with a possible consequence that \( \theta(y) < \theta(x) \). Anyway, in the case where \( \theta(y) = \theta(x) \), the number of knots which are extreme has decreased.

In line with this result, we introduce the following definition of weak local optimal: we say that \( \theta \) has not a local w-minimum at \( x \in X \) if for any neighbourhood \( V \) of \( x \) there exists \( y \in V \cap X \) such that either \( \theta(y) < \theta(x) \) or \( \theta(y) = \theta(x) \) with a smaller number of extreme knots.

Therefore the definition of of a local w-minimum is as follows.

**Definition 5.1.** \( \theta \) has a local w-minimum at \( x \in X \) if for any neighbourhood \( V \) of \( x \) there exists no \( y \in V \cap X \) such that either \( \theta(y) < \theta(x) \) and \( \theta(y) = \theta(x) \) with a smaller number of extreme knots.

### ii) Move on the left.

Here again, let \( s \in \{-1, 1\} \) be such that \( \sigma(x_i) - f(x_i) = s \theta(x) \). Assume that there exist \( \bar{\lambda} > 0 \), \( \delta_i \in \Pi_{n_i}, \delta_{i+1} \in \Pi_{n_{i+1}}, \ldots, \delta_p \in \Pi_{n_p} \) such that

5. \( s \delta_i(x_i) > 0 \),

and for \( k = i, i + 1, \ldots, p \)

6. \( \theta(x) \geq |\sigma_k(t) - \lambda \delta_k(t) - f(t)| \) for all \( t \in [x_k, x_{k+1}] \), for all \( \lambda \in [0, \bar{\lambda}] \),

7. \( \delta_k(x_{k+1}) = \delta_{k+1}(x_{k+1}) \).

Let \( y \in X \) be such that \( y_k = x_k \) for all \( k \neq i \). Let \( y_i \in (x_{i-1}, x_i) \) be close to \( x_i \). Let \( \lambda \in (0, \bar{\lambda}] \). Set

\[
\tau_k = \begin{cases} 
\sigma_k - \lambda \delta_k & \text{if } k \geq i, \\
\sigma_k & \text{if } k < i.
\end{cases}
\]

In order that \( \tau \) belongs to \( \Sigma(y) \), one requires

\[
(\sigma_i - \lambda \delta_i)(y_i) = \sigma_{i-1}(y_i).
\]

Let us introduce

\[
H(t, \lambda) = (\sigma_i - \sigma_{i-1})(x_i + t) - \lambda \delta_i(x_i + t).
\]
In case where \((\sigma_{i-1} - \sigma_i)'(x_i) \neq 0\), the implicit function theorem says that there exists a differentiable function \(t(.)\) such that in a neighbourhood of 0,

\[
0 = H(t(\lambda), \lambda), \quad t'(\lambda) = \frac{\delta_i(x_i + t(\lambda))}{(\sigma_i - \sigma_{i-1} - \lambda \delta_i)'(x_i + t(\lambda))},
\]

\(t(0) = 0, \quad t'(0) = \frac{\delta_i(x_i)}{(\sigma_i - \sigma_{i-1})'(x_i)}\).

Set \(y_i = x_i + t(\lambda)\). Then, \(\tau \in \Sigma(y)\). The condition \(t'(0) < 0\) is necessary for a move on the left \((y_i < x_i)\). This is the case when

\(s(\sigma_{i-1} - \sigma_i)'(x_i) > 0\).

Then, for \(\lambda > 0\) small enough, one obtains \(y_i < x_i\) and \(\theta(y) \leq \|\tau - f\| \leq \theta(x)\). Moreover, there exists \(x_i' > x_i\) such that \(|\tau(t) - f(t)| < \theta(x)\) for all \(t \in [y_i, x_i']\).

Unlike \(x_i\), the new knot \(y_i\) is not an extreme point. Same consequence as for the left move, \(\theta\) has not a local \(w\)-minimum at \(x \in X\).

We are ready to establish the main result of this subsection.

**Theorem 5.1** (Necessary condition for optimality). Let \(x \in X\) and \(\sigma \in \Sigma(x)\) be such that \(\theta(x) = \|\sigma - f\|\). A necessary condition for local \(w\)-minimality of \(\theta\) at \(x\) is that at each knot \(x_i\) such that \(\theta(x_i) = s(\sigma(x_i) - f(x_i))\) with \(s \in \{-1, 1\}\) and \(s(\sigma_{i-1}'(x_i) - \sigma_i'(x_i)) > 0\) the two following conditions hold:

i) there are no functions \(\delta_k, k = 0, 1, \cdots, i - 1\) fulfilling conditions 1,2,3.

ii) there are no functions \(\delta_k, k = i, i + 1, \cdots, p\) fulfilling conditions 5,6,7.

We are left with the question of the existence of such functions.

### 5.3 Existence and constructions of functions \(\delta_k\)

Let the knot \(x_i\) be such that \(\theta(x) = s(\sigma - f)(x_i)\) with \(s \in \{-1, 1\}\) and \(s(\sigma_{i-1} - \sigma_i)'(x_i) > 0\). Then, there is some \(j\) such that \(x_i \in [t_j^-, t_j^+]\).

We shall describe a process which concludes to the existence or the no existence of functions \(\delta_k\) such are in the theorem. It is in the same spirit as Algorithm 2.
i) We start with the existence of functions $\delta_k$, $k \geq i$.

**Algorithm 5:** Step i

Let $m_i = \text{card}(J(i, i + 1))$. Then $t_{j+m_i-1}^- \leq x_{i+1} < t_{j+m_i}^-$. 

**Case i = p.** Then $x_{i+1} = x_{p+1} = b$.

1. if $m_p \geq n_p + 2$ then 
   | No existence.

2. if $m_p \leq n_p + 1$ then 
   | Choose $\xi_l \in (t_{j+m_p}^+, t_{j+i+1}^-)$, $l = 0, \ldots, m_i - 1$.
   | Take $\delta_p(t) = s \prod_{\xi_l=0}^{m_i-1} (\xi_l - t)$. 
   | The function $\delta_p$ responds to the question: Existence.

**Case i < p**

1. if $m_i \geq n_i + 2$ then 
   | No existence.

2. if $m_i \leq n_i$ or ($m_i = n_i + 1$ and $x_{i+1}$ is an extreme point) then 
   | Choose $\xi_l \in (t_{j+m_i}^+, t_{j+i+1}^-)$, $l = 0, \ldots, m_i - 2$.
   | Take $\xi_{m_i-1} = x_{i+1}$.
   | Take $\delta_i(t) = s \prod_{l=0}^{m_i-1} (\xi_l - t)$ and $\delta_k(t) = 0$ for $k \geq i + 1$.
   | The functions $\delta_k$ respond to the question: Existence.

3. In all other cases go to step i + 1.

**Algorithm 6:** Step i+1

One has $m_i = n_i + 1$. Let $m_{i+1} = \text{card}(J(i, i + 2))$. Then $t_{j+1+m_{i+1}}^- \leq x_{i+2} < t_{j+m_{i+1}}^-$. 

**Case i = p - 1.** Then $x_{i+2} = x_{p+1} = b$.

1. if $m_{i+1} \geq 2 + n_i + n_{i+1}$ then 
   | No existence.

2. if $m_{i+1} \leq 1 + n_i + n_{i+1}$ then 
   | Choose $\xi_l \in (t_{j+m_{i+1}}^+, t_{j+i+1}^-) \cap (x_i, x_{i+1})$, $l = 0, \ldots, n_i - 1$, 
   | Choose $\xi_l \in (t_{j+m_{i+1}}^+, t_{j+i+1}^-) \cap (x_{i+1}, x_{i+2})$, $l = n_i, \ldots, m_{i+1} - 1$.
   | Such choices are possible. Next, take $\delta_{p-1}(t) = s \prod_{l=0}^{n_i-1} (\xi_l - t)$, 
   | take $\delta_p(t) = \lambda \prod_{l=n_i}^{m_{i+1}-1} (\xi_l - t)$,
   | with $\lambda$ taken so that $\delta_{p-1}(x_p) = \delta_p(x_p)$,
   | The functions $\delta_{p-1}, \delta_p$ respond to the question: Existence.

**Case i < p - 1**

1. if $m_{i+1} \geq n_i + n_{i+1} + 2$ then 
   | No existence.

2. if $m_{i+1} \leq n_i + n_{i+1}$ or ($m_{i+1} = n_i + n_{i+1} + 1$ and $x_{i+2}$ is an extreme point). then 
   | Choose $\xi_l \in (t_{j+m_{i+1}}^+, t_{j+i+1}^-) \cap (x_i, x_{i+1})$, $l = 0, \ldots, n_i - 1$, 
   | Choose $\xi_l \in (t_{j+m_{i+1}}^+, t_{j+i+1}^-) \cap (x_{i+1}, x_{i+2})$, $l = n_i, \ldots, m_{i+1} - 2$.
   | Take $\xi_{m_{i+1}-1} = x_{i+2}$
   | The functions $\delta_{p-1}, \delta_p$ respond to the question: Existence.
ii) A symmetric process allows to conclude to the existence or non existence of functions $\delta_k$ fulfilling conditions 1 to 3.

5.4 Another necessary condition for local optimality and connection with existing results

Combining the last two subsections, we obtain a reformulation of Theorem 5.1.

**Theorem 5.2.** Let $x \in X$ and $x \in \Sigma(x)$ be such that $\theta(x) = \|\sigma - f\|$. A necessary condition for local w-minimality of $\theta$ at $x$ is that if there is at least one knot $x_j$, $i \in I$, such that $\theta(x) = s(\sigma(x_j) - f(x_j))$, $s \in \{1, 1\}$ and $s(\sigma_{i-1}^j(x_j) - \sigma_i^j(x_j)) > 0$ then there exists at least one index $i \in I$, such that the following two conditions hold.

1. There exists $k > i$ such that $\text{card}(J(i, k)) \geq 2 + n_i + \cdots + n_k$ and, for all $l$ such that $i < l < k$, $\text{card}(J(i, l)) \geq 1 + n_i + \cdots + n_l$.

2. There exists $k < i$ such that $\text{card}(J(k, i)) \geq 2 + n_k + \cdots + n_{i-1}$ and, for all $l$ such that $k < l < i$, $\text{card}(J(l, i)) \geq 1 + n_l + \cdots + n_{i-1}$.

Let us place this theorem among the recent results on the question. In [10, 11], a knot $x_i$ such that $\theta(x) = s(\sigma(x_i) - f(x_i)), s \in \{1, 1\}$ is called unstable if $s(\sigma_{i-1} - \sigma_i)'(x_i) > 0$, neutral if $s(\sigma_{i-1} - \sigma_i)'(x_i) = 0$ and stable if $s(\sigma_{i-1} - \sigma_i)'(x_i) < 0$. The formulations of the main theorems in [10, 1, 11] and the present paper are very close except that local optimality is considered in the sense of inf-stationarity in the sense of Demyanov and Rubinov [2, 3] and in the local w-minimality in this paper. These two notions are not equivalent, but both aim at describing necessary conditions for optimality. Remark that the situation is a little more general in [11] since the value of the spline may be fixed at some knots.

The approach in [10, 1, 11] is analytic since based on a minimality criterion at $x \in X$ using quasidifferentiability in the sense of Demyanov [2, 3]. The approach in this paper is of a constructive type, it is based on the construction of better local candidates at minimality. However, the two different
approaches together participates to a better understanding of this difficult problem.

It is valuable to note that Theorem 5.1 and Theorem 5.2 do not contain any reference to neutral knots (that is, \( s(\sigma_{i-1} - \sigma_i)'(x_i) = 0 \)). At the same time, these knots play an essential role for detecting inf-stationarity [II]. Therefore, one of our future research directions is to investigate the connection between neutral knots and the reduction of the number of extreme deviation knots.

References

[1] Crouzeix J.-P., Ugon J., Sukhorukova N. *Characterization theorem for best polynomial spline approximation with free knots, variable degree and fixed tails*, Journal of Optimization Theory and Applications 172(3), 950–964 (2017).

[2] Demyanov V., Rubinov A. *Constructive Nonsmooth Analysis*, Peter Lang, Frankfurt am Main (1995).

[3] Demyanov V., Rubinov A. (editors) *Quasidifferentiability and Related Topics*, Non-convex Optimization and Its Applications, vol. 43. Kluwer Academic, Dordrecht/Boston/London (2000).

[4] Nurnberger G. *Approximation by Spline functions*, Springer-Verlag (1989).

[5] Remez E. *General computational methods of Chebyshev approximation*, Atomic Energy Translation 4491 (1957).

[6] Schumaker L. *Uniform approximation by Chebyshev spline functions. II: free knots*, SIAM Journal of Numerical Analysis 5, 647–656 (1968).

[7] Sukhorukova N. A generalisation of Remez algorithm to the case of polynomial splines, Ph.D. thesis, St-Petersburg State University (2006), 134 pp. (in Russian).
[8] Sukhorukova N. Vallée theorem and Remez algorithm in the case of generalized degree polynomial spline approximation, Pacific Journal of Optimization 6(1), 103–114 (2010).

[9] Sukhorukova N. Uniform approximation by the highest defect continuous polynomial splines: necessary and sufficient optimality conditions and their generalisations, Journal of Optimization Theory and Applications 147 (2), 378–394 (2010).

[10] Sukhorukova N., Ugon J. Characterization theorem for best polynomial spline approximation with free knots, Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications and Algorithms 5(5) (2010).

[11] Sukhorukova N., Ugon J. Characterisation theorem for best polynomial spline approximation with free knots, Transactions of the American Mathematical Society 369 (9), 6389–6405 (2017).