Supertraces on the algebra of observables of the rational Calogero model based on the classical root system

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Abstract

We find a complete set of supertraces on the algebras $H_{W(R)}(\nu)$, the algebra of observables of the rational Calogero model with harmonic interaction based on the classical root systems $R$ of $B_N$, $C_N$ and $D_N$ types. These results extend the results known for the case $A_{N-1}$. It is shown that $H_{W(R)}(\nu)$ admits $q(R)$ independent supertraces where $q(B_N) = q(C_N)$ is a number of partitions of $N$ into a sum of positive integers and $q(D_N)$ is a number of partitions of $N$ into a sum of positive integers with even number of even integers.

1 Introduction

In this paper we continue to investigate some properties of the associative algebras that were shown in [1, 2] to underly the rational Calogero model [3]. We extend the results obtained in [3] to the associative algebras that underly the rational Calogero model associated with the root systems [5] of the classical Lie algebras $B_N$, $C_N$ and $D_N$.

Let $H_G(\nu)$ be the associative algebra generated by $N$ pairs of (deformed) Heisenberg-Weyl oscillators $a_i^\alpha$ which constitute the basises in the pair of $N$-dimensional subspaces $a_i^\alpha \in \mathcal{H}_N$ and group algebra $\mathfrak{G}$ of some finite group $G$ with the following properties:

i) $H_G(\nu)$ possesses the parity $\pi$: $\pi(a_i^\alpha) = 1$, $\pi(g) = 0 \forall g \in G$

ii) $ga_i^\alpha = \sum_j T_{ij}(g^{-1})a_j^\alpha g$, $\forall g \in G$ and for all $\alpha = 0, 1, i = 1, ..., N$, $a_i^\alpha \in \mathcal{H}_N$, where the orthogonal matrices $T_{ij}(g)$ realize $N$-dimensional representation of group $G$.

iii) $[a_i^\alpha, a_j^\beta] = \epsilon^{\alpha\beta} A_{ij}$, where $\epsilon^{\alpha\beta}$ is antisymmetric tensor, $\epsilon^{01} = 1$, and $A_{ij} = A_{ji} \in \mathfrak{G}$ depend on one or more parameters $\nu$.

To formulate the next property let us introduce the subspaces $\mathcal{E}^\alpha(g) \subset \mathcal{H}_N$ as

$$\mathcal{E}^\alpha(g) = \{ h \in \mathcal{H}_N : gh = -hg \},$$

and the grading $E$ on $\mathfrak{G}$,

$$E(g) = \dim \mathcal{E}^\alpha(g).$$

iii) $E(\mathcal{P}([h_0, h_1])g) = E(g) - 1 \forall g \in G, \forall h_\alpha \in \mathcal{E}^\alpha(g)$. Here the notation $\mathcal{P}(g)$ introduced for the projector $\mathfrak{G} \rightarrow \mathfrak{G}$ defined as $\mathcal{P}(\sum_i \alpha_i g_i) = \sum_{i: g_i \neq 1} \alpha_i g_i$ for $g_i \in G$ and any constants $\alpha_i$. 

v) Every element in $H_G(\nu)$ is a polynomial of $a_i^\alpha$.

We define the supertrace as a linear complex-valued function $\text{str}(\cdot)$ on the algebra $H_G(\nu)$ such that

$$\text{str}(fg) = (-1)^{\pi(f)\pi(g)} \text{str}(gf) \quad \forall f, g \in H_G(\nu).$$
On the example of the case $G = S_N$ the following theorem is proved in \cite{4}.

**Theorem 1.** If the superalgebra $H_G(\nu)$ possesses the properties i) - v) then the number of independent supertraces on $H_G(\nu)$ is equal to the number of the supertraces on the algebra $\mathcal{G}$, satisfying the equations

$$\text{str}([h_0, h_1]g) = 0 \quad \forall g \in G \text{ with } E(g) \neq 0 \text{ and } \forall h_\alpha \in \mathcal{E}^\alpha(g).$$

In this paper we apply this theorem to the case where $G$ is Weyl group $W(\mathbb{R})$ of the root system $\mathbb{R}$ of classical Lie algebra, $A_{N-1}$, $B_N$, $C_N$, and $D_N$ types. We will consider three (because $W(B_N) = W(C_N)$) different cases simultaneously. Note that the algebra $SH_N(\nu)$ considered in \cite{6, 4} is the algebra $H_{W(A_{N-1})}(\nu) = H_{S_N}(\nu)$ in our notations.

Consider 3-parametric deformation $H_G(\nu)$ of the associative Heisenberg-Weyl algebra of polynomials of $N$ pairs of oscillators. This algebra is generated by $N$ pairs of generating elements $a^\alpha_i$, $\alpha = 0, 1$, and by the reflections $K_{ij}$ and $R_i$, $i, j = 1, \ldots, N$, satisfying the following relations

\begin{align*}
K_{ij} &= K_{ji}, \quad K_{ij}K_{kl} = K_{kl}K_{ij} \quad \text{for } i \neq j \neq l \neq i, \\
K_{ij}K_{kl} &= K_{kl}K_{ij} \quad \text{if } i, j, k, l \text{ are pairwise different}, (5) \\
R_iR_j &= R_jR_i \quad \forall i, j, \quad R_iR_i = 1, \quad \forall i, \quad (6) \\
K_{ij}R_j &= R_iK_{ij}, \quad K_{ij}R_k = R_kK_{ij} \quad \text{for } i \neq j \neq k \neq i, (7) \\
R_i a^\alpha_i &= -a^\alpha_i R_i, \quad R_i a^\alpha_j = a^\alpha_j R_i \quad \text{for } i \neq j, (8) \\
K_{ij} a^\alpha_j &= a^\alpha_i K_{ij}, \quad K_{ij} a^\alpha_k = a^\alpha_k K_{ij} \quad \text{for } i \neq j \neq k \neq i, (9) \\
\left[a^\alpha_i, a^\beta_j\right] &= \epsilon^\alpha\beta A_{ij}, (10)
\end{align*}

where $\epsilon^\alpha\beta = -\epsilon^\beta\alpha$, $\epsilon^{01} = 1$, and

\begin{align*}
A_{ij} &= \delta_{ij} + \nu_0 \tilde{A}^0_{ij} + \nu_1 \tilde{A}^1_{ij} + \nu_2 \tilde{A}^2_{ij}, \\
\tilde{A}^0_{ij} &= \delta_{ij} \sum_{l=1, l \neq i}^{N} K_{il} - \delta_{i\neq j} K_{ij}, \\
\tilde{A}^1_{ij} &= \delta_{ij} R_i, \\
\tilde{A}^2_{ij} &= \delta_{ij} \sum_{l \neq i} K_{il} R_i R_l + \delta_{i\neq j} K_{ij} R_i R_j. (11)
\end{align*}

The commutation relations defined in such a way are selfconsistent when one of the following conditions takes place:

\footnote{In this paper repeated Latin indices $i, j, k, \ldots$ do not imply summation.}
A) \( \nu_1 = \nu_2 = 0 \), and generating elements \( R_i \) are excluded,
B, C) \( \nu_0 = \nu_2 \),
D) \( \nu_0 = \nu_2, \nu_1 = 0 \), and every monomial in \( H_G(\nu) \) contains even number of reflections \( R_i \).

The reflections \( K_{ij} \) and \( R_i \) determine Weyl group of the root systems \( A_{N-1}, B_N, C_N \) and \( D_N \) correspondingly to these cases and the operators \( a_i^\alpha \) have the presentation
\[
a_i^\alpha = x_i + (-1)^\alpha D_i(x)
\]
where \( D_i(x) \) are Dunkl’s differential-difference operators [7] connected with the corresponding root systems,
\[
D_i = \frac{\partial}{\partial x_i} + \nu_1 \frac{1}{x_i} (1 - R_i) + \sum_{l \neq i}^{N} \left( \nu_0 \frac{1}{x_i - x_l} (1 - K_{il}) + \nu_2 \frac{1}{x_i + x_l} (1 - K_{il} R_l R_i) \right),
\]
and satisfying the condition
\[
[D_i, D_j] = 0 \quad \forall i, j
\]
for values of \( \nu \)-s listed above.

The Hamiltonian of Calogero model associated with corresponding root system [5] is identified with second-order differential operator
\[
H = \frac{1}{2} \sum_{i=1}^{N} \{ a_i^0, a_i^1 \}. 
\]
The operators \( a_i^\alpha \) serve as generalized oscillators underlying the Calogero problem and allow one [2] to construct wave functions via the standard Fock procedure with the Fock vacuum \( |0\rangle \) such that \( a_0^0 |0\rangle = 0 \).

To know the supertraces is useful in various respects. One of the most important is that they define multilinear invariant forms
\[
str(a_1 a_2 ... a_n),
\]
what allows for example to construct the lagrangians for dynamical theories based on these algebras. Another useful property is that since null vectors of any invariant bilinear form span a both-side ideal of the algebra, this gives a powerful device for investigating ideals which decouple from everything under the supertrace operation as it happens in \( SH_2(\nu) \) for half-integer \( \nu \) [8].

An important motivation for the analysis of the supertraces of \( H_{W(R)}(\nu) \) is due to its deep relationship with the analysis of the representations of this algebra, which in its turn gets applications to the analysis of the wave functions of the Calogero models. For example, given representation of \( H_{W(R)}(\nu) \), one can speculate that it induces some supertrace on this algebra as (appropriately regularized) supertrace of (infinite) representation matrices. When the corresponding bilinear form (15) degenerates this would imply that the representation becomes reducible.

For almost all superalgebras considered in this paper the situation is very interesting since almost all of them admit more than one independent supertrace. For finite dimensional algebras the existence of several supertraces means that they have both-side ideals, but for infinite dimensional algebras under consideration the existence of ideals is still an open problem.

Below, in the section 2 we prove the lemma, that ensures the existence of the supertraces on \( H_{W(R)}(\nu) \) and in the section 3 we calculate the number of supertraces using the theorem proved in Appendix.

\[\text{In (13) the reflections } K_{ij} \text{ and } R_i \text{ act on the space of coordinates } x_i\]
2 1-dimensional representations of the elements of the Weyl groups of the classical root systems

In this section we show that the property iii) is satisfied for \( H_G(\nu) \) if the group \( G \) is the Weyl group of the classical root system, and is generated by the reflections \( K_{ij} \) and \( R_i \).

Every element \( g \in G \) for \( G = W(A_{N-1}) \), \( G = W(B_N) \), \( G = W(C_N) \) and \( G = W(D_N) \) can be presented in the form \( g = \sigma \prod_{i \in M_g} R_i \) where the permutation \( \sigma \in S_N \), the symmetric group, and \( M_g \) is some subset of indices \( 1, \ldots, N \). It is well known that every permutation \( \sigma \in S_N \) defines some partition of the set of indices \( 1, \ldots, N \) to the subsets \( C_1, \ldots, C_{t_\sigma} \) in such a way that it can be presented as a product \( \sigma = \prod_{m=1}^{t_\sigma} c_m \) of the commuting cycles \( c_m \) acting on the subsets \( C_m \) correspondingly.

Introduce the elements

\[
r_m = \prod_{i \in M_g \cap C_m} R_i.
\]

Then \( g \) can be presented as

\[
g = \prod_{m=1}^{t_\sigma} \hat{c}_m \quad \text{where} \quad \hat{c}_m = c_m r_m. \tag{17}
\]

We call these elements \( \hat{c}_m \) as cycles.

Given element \( g \in G \), we introduce a new set of basis elements \( \mathcal{B}_g = \{b^\alpha \} \) instead of \( \{a^\alpha \} \) in the following way. For every cycle \( \hat{c}_m \) in the decomposition (17) let us fix some index \( l_m \), that belongs to the subset \( C_m \) associated with the cycle \( \hat{c}_m \). The basis elements \( b^\alpha_{mj}, j = 1, \ldots, |C_m| \), that realize 1-dimensional representations of the commutative cyclic group generated by \( \hat{c}_m \), have the form

\[
b^\alpha_{mj} = \frac{1}{\sqrt{|C_m|}} \sum_{k=1}^{|C_m|} (\lambda_m)^j k \hat{c}_m^{-k} a^\alpha_{lm} \hat{c}_m^k, \tag{18}
\]

where

\[
\lambda_m = \exp \left( \frac{1 + (|\hat{c}_m| + |C_m|) \mod 2}{|C_m|} \pi i \right). \tag{19}
\]

Here the notation \( |C_m| \) denotes the number of elements in the subset \( C_m \), and \( |\hat{c}_m| \) is the length of the cycle \( \hat{c}_m \). The length \( |g| \) of some element \( g \in G \) is the minimal number of the reflections whose product is \( g \).

From the definition (18) it follows that

\[
\hat{c}_m b^\alpha_{mj} = (\lambda_m)^j b^\alpha_{mj} \hat{c}_m, \tag{20}
\]

\[
\hat{c}_m b^\alpha_{nj} = b^\alpha_{nj} \hat{c}_m, \quad \text{for} \quad n \neq m, \tag{21}
\]

and therefore

\[
g b^\alpha_{mj} = (\lambda_m)^j b^\alpha_{mj} g. \tag{22}
\]

\(^3\)In \( [4] \) we used the definition of the length equal to \( |\hat{c}_m| + 1 \).
Now one can easily deduce that for every cycle \( \hat{c}_m \) with even length the matrix \( T_{ij}(\hat{c}_m) \) has no eigenvalues equal to \(-1\) while for the cycle with odd length it has precisely 1 eigenvalue \(-1\). As a consequence the function \( E(g) \) is the number of odd cycles in the decomposition (17) of \( g \).

In what follows, instead of writing \( b^\alpha_{mJ} \) we use the notation \( b^J \) with the label \( I \) accounting for the full information about the index \( \alpha \), the index \( m \) enumerating cycles in (17), and the index \( j \) that enumerates various elements \( b^\alpha_{mJ} \) related to the cycle \( \hat{c}_m \), i.e. \( I \) \((I = 1, \ldots, 2N)\) enumerates all possible triples \{\( \alpha, m, j \)\}. We denote the index \( \alpha \), the cycle, the subset of indices and the eigenvalue in (20) corresponding to some fixed index \( I \) as \( \alpha(I), c(I), C(I) \) and \( \lambda_I = (\lambda_m)^j \), respectively. The notation \( g(I) = g_0 \) implies that \( b^J \in \mathfrak{B}_g \). \( \mathfrak{B}_1 \) is the original basis of the generating elements \( a^\alpha_i \) (here \( 1 \in G \) is the unit element).

Let \( \mathfrak{M}(g) \) be the matrix that maps \( \mathfrak{B}_1 \rightarrow \mathfrak{B}_g \) in accordance with (18),

\[
b^J = \sum_{i,\alpha} \mathfrak{M}_{i\alpha}(g) a^\alpha_i. \tag{23}
\]

Obviously this mapping is invertible. Using the matrix notations one can rewrite (22) as

\[
bg^J - 1 = \sum_{J=1}^{2N} A^J_j(g) b^J, \quad \forall b^J \in \mathfrak{B}_g, \tag{24}
\]

where \( A^J_j(g) = \delta^J_j \lambda_I \).

The commutation relations for the generating elements \( b^J \) follow from (11) and (11)

\[
[b^J, b^K] = f^{IJK} = C^{IJ} + \epsilon^{\alpha(I)\alpha(J)} B^{IJ}, \tag{25}
\]

where \( g(I) = g(J) \),

\[
C^{IJ} = \epsilon^{\alpha(I)\alpha(J)} \delta_{c(I)c(J)} \delta_{\lambda_I \lambda_J}^{-1}, \tag{26}
\]

and

\[
B^{IJ} = \sum_{i,J} \mathfrak{M}^I_{i0}(\sigma) \mathfrak{M}^J_{1}(\sigma) A_{ij}, \quad \epsilon^{\alpha(I)\alpha(J)} B^{IJ} = \mathcal{P} \left( [b^J, b^J] \right). \tag{27}
\]

The indices \( I, J \) are raised and lowered with the aid of the symplectic form \( C^{IJ} \)

\[
b^J = \sum_{J} C^{JL} b^L, \quad b^J = \sum_{J} b^L C^J_L; \quad \sum_{M} C_{LM} C^{MJ} = -\delta^J_I. \tag{28}
\]

Note that the elements \( b^J \) are normalized in (18) in such a way that the \( \nu \)-independent part in (23) has the form (23).

Now we can prove the existence of the supertraces for the cases \( B_N, C_N \) and \( D_N \) using Theorem 1. For this purpose it is sufficient to prove the next lemma (property iii \( ) \).

**Lemma 1.** If \( \lambda_I = \lambda_J = -1 \), where \( g(I) = g(J) = g \) then \( E(B^{IJ} g) = E(g) - 1 \).

The proof is based on the following simple facts from the theory of the symmetric group:

**Proposition 1.** Let \( c_1 \) and \( c_2 \) be two distinct cycles in the decomposition of some permutation from \( S_N \). Let indices \( i_1 \) and \( i_2 \) belong to the subsets of indices associated with the cycles \( c_1 \) and \( c_2 \), respectively. Then the permutation \( c = K_{i_1 i_2} c_1 c_2 \) is a cycle of length \( |c| = |c_1| + |c_2| + 1 \).
Proposition 2. Given cyclic permutation $c \in S_N$, let $i \neq j$ be two indices such that $c^k(i) = j$, where $k$ is some positive integer, $k < |c|$. Then $K_{ij}c = c_1c_2$ where $c_{1,2}$ are some non-coinciding mutually commuting cycles such that $|c_1| = k - 1$ and $|c_2| = |c| - k$.

To prove Lemma 1 let us first consider the case $c(I) = c(J)$. For the definiteness consider the odd cycle

$$c(I) = c(J) = K_{12}K_{23}...K_{(p-1)p}R_{i_1}R_{i_2}...R_{i_l}$$

with $1 \leq i_1 < i_2 < ... < i_l \leq p$ and odd $p - 1 + l$

and

$$b^p = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} (-1)^j (-1)^{\Delta(j)} a_j^{\alpha(p)}, \quad P = I, J,$$

where

$$\Delta(j) = \sum_{k \geq j} \sum_{n=1}^{l} \delta_{k, in}.$$ (31)

Then $B^{IJ}$ can be written in the following form

$$pB^{IJ} = e^{\alpha(I)\alpha(J)} \left( \sum_{i=1, j=1, i \neq j}^{p} \left( \nu_0(1 + (-1)^{i+j+\Delta(i) + \Delta(j)})K_{ij} + \nu_2(1 + (-1)^{i+j+\Delta(i) + \Delta(j)})K_{ij}R_{i}R_{j} \right) + \nu_1 \sum_{i=1}^{p} R_{i} + \sum_{i=1}^{p} \sum_{j=p+1}^{N} (\nu_0K_{ij} + \nu_2K_{ij}R_{i}R_{j}) \right),$$ (32)

and one can easily check using Propositions 1 and 2 that

i) $K_{ij}c(I)$ decomposes in the product of two even cycles when $1 \leq i < j \leq p$ and $i + j + \Delta(i) + \Delta(j)$ is odd;

ii) $K_{ij}R_{i}R_{j}c(I)$ decomposes in the product of two even cycles when $1 \leq i < j \leq p$ and $i + j + \Delta(i) + \Delta(j)$ is even;

iii) $R_{i}c(I)$ is even cycle when $1 \leq i \leq p$ because $c(I)$ is odd cycle;

iv) if $c(K) \neq c(I)$ is some cycle in the decomposition \(|7|\) of $g$ then $K_{ij}c(I)c(K)$ is the cycle with the same parity as $c(K)$ has when $i \in C(I)$ and $j \in C(K)$;

v) if $c(K) \neq c(I)$ is some cycle in the decomposition \(|7|\) of $g$ then $K_{ij}R_{i}R_{j}c(I)c(K)$ is the cycle with the same parity as $c(K)$ has when $i \in C(I)$ and $j \in C(K)$.

The case $c(I) \neq c(J)$ reduces to the subcases iv) and v) considered above what ends the proof of Lemma 1.

3 The supertraces on $G$, Ground Level Conditions and the number of supertraces on $H_{W(R)}(\nu)$.

Due to the $G$-invariance the definition of the supertrace on $G$ is the definition of the central function on $G$ i.e. a function on the conjugacy classes of $G$ and so the number of
the supertraces on $\mathfrak{S}$ is equal to the number of the conjugacy classes in $G$.

Since $\mathfrak{S} \subset H_G(\nu)$ some additional restrictions on these functions follow from (3) and the defining relations (3)-(11) of $H_G(\nu)$. Actually, consider some elements $b^\prime$ such that $\lambda_I = -1$. Then, one finds from (3) and (22) that $\text{str} \left( b^\prime b' g \right) = - \text{str} \left( b' b^\prime g \right) = \text{str} \left( b^\prime b' g \right)$ and therefore $\text{str} \left( \left[ b^\prime, b' \right] g \right) = 0$ or equivalently

$$\delta_{c(I)c(J)} \delta_{-1 \lambda J} \text{str}(g) = -\text{str}(B^{IJ} g).$$

(33)

Since these conditions restrict supertraces of degree-0 polynomials of $a_i^\alpha$ we called them in [4] as ground level conditions (GLC). Due to Lemma 2 proved in Appendix the equations (33) become identities for every supertrace on $\mathfrak{S}$ if $\delta_{c(I)c(J)} \delta_{-1 \lambda J} = 0$. Hence they express the supertrace of elements $g$ with $E(g) = e$ via the supertraces of elements $B^{IJ} g$ with $E(B^{IJ} g) = e - 1$:

$$\text{str}(g) = -\text{str}(B^{IJ} g),$$

(34)

where $B^J$ is $B^{IJ}$ with $J$ defined by relations $c_J = c_I$, $\lambda_I = \lambda_J = -1$, $\alpha(I) + \alpha(J) = 1$. So the number of solutions $Q(G)$ of system (33) does not exceed $O(G)$, the number of conjugacy classes of elements without odd cycles.

In the Appendix the following theorem is proved:

**Theorem 2.** If $G = W(\mathcal{R})$ where $\mathcal{R} = A_{N-1}, B_N, C_N, D_N$, then $Q(G) = O(G)$.

To compute the value $O(G)$ consider some even cycle

$$c = K_{12} K_{23} \ldots K_{(p-1)p} R_{i_1} R_{i_2} \ldots R_{i_l}, \quad (p + l \text{ is odd})$$

(35)

and the sequence of similarity transformations admissible in all cases $\mathcal{R} = B_N, C_N, D_N$:

$$c \rightarrow R_{i_1} R_{i_l-1} c R_{i_l} R_{i_l-1} = K_{12} K_{23} \ldots K_{(p-1)p} R_{i_1} R_{i_2} \ldots R_{i_l-2} \rightarrow \ldots$$

$$\rightarrow \ldots = K_{12} K_{23} \ldots K_{(p-1)p} (R_1)^{l_1} (R_2)^{l_2-l_1}.$$  

(36)

(37)

Here $l_1$ is the number of odd $i_k$ in (35). If $p$ is even then either $l_1$ or $l - l_1$ is odd and (35) is similar to $K_{12} K_{23} \ldots K_{(p-1)p} R_1$. If $p$ is odd then (35) is similar either to $K_{12} K_{23} \ldots K_{(p-1)p} R_1$ or to $K_{12} K_{23} \ldots K_{(p-1)p} R_1 R_2$. The latter expression is similar to $K_{12} K_{23} \ldots K_{(p-1)p} R_1 R_2$ that in its turn is similar to $K_{12} K_{23} \ldots K_{(p-1)p}$ since $p$ is odd.

This consideration shows that the conjugacy class of some even cycle $c(I)$ is determined by two dependent values: the number of indices $p(I) = |C(I)|$ that transformed by $c(I)$ and the parity $\epsilon(I) = (p(I) + 1)_{\mod 2}$ of the number of $R$-s in $c(I)$.

In such a way the conjugacy class with $E = 0$ completely characterized by the set of nonnegative integers $n_1, n_3, \ldots; m_2, m_4, \ldots$, where $n_i$ is the number of cycles $c(I)$ with $p(I) = i$ and $\epsilon(I) = 0$ while $m_i$ is the number of cycles $c(I)$ with $p(I) = i$ and $\epsilon(I) = 1$.

The numbers $n_i$ and $m_i$ have to satisfy the following conditions:

for the case A) $m_i = 0 \forall i$, $\sum_i i n_i = N$;

for the case B, C) $\sum_i i (n_i + m_i) = N$;

for the case D) $\sum_i i (n_i + m_i) = N$, $(\sum_i m_i)_{\mod 2} = 0$.

So the number of supertraces is equal to a number of partitions of $N$ into a sum of odd positive integers for the case $A_{N-1}$, a number of partitions of $N$ into a sum of positive

$^4$R-s are absent in the case $\mathcal{R} = A_{N-1}$. 7
integers for the cases $B_N$ and $C_N$ and $q(D_N)$ is a number of partitions of $N$ into a sum of positive integers with even number of even integers.

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**Appendix**

**The proof of Theorem 2.**

First let us prove the following lemma.

**Lemma 2.** The equations $\text{str}([b^I, b^J]g)$ with $b^I, b^J \in \mathcal{B}_g$, $\lambda_I = -1$ are satisfied equivalently for any supertrace $\text{str}(\cdot)$ on $\mathcal{G}$ if $c(I) \neq c(J)$ or $c(I) = c(J)$ and $\lambda_J \neq -1$.

Indeed, due to $G$-invariance the following identities can be obtained:

$$\text{str}([b^I, b^J]g) = \text{str}(c(I)[b^I, b^J]g(c(I))^{-1})$$

$$\text{str}([c(I)b^I(c(I))^{-1}, c(I)b^J(c(I))^{-1}]c(I)g(c(I))^{-1}) = \text{str}([-b^I, b^J]g)$$

for the case $c(I) \neq c(J)$, and

$$\text{str}([b^I, b^J]g) = \text{str}(c(I)[b^I, b^J]g(c(I))^{-1}) = -\lambda_J \text{str}([b^I, b^J]g)$$

for the case $c(I) = c(J)$.

As a result one can consider only the case $c(I) = c(J)$, $\lambda_I = \lambda_J = -1$, and $\alpha(I) = 1 - \alpha(J)$ to prove Theorem 2.

By induction on a number of odd cycles $e = E(g)$ we show that for $g$ with $E(g) = e \geq 1$ there is only one independent equation on $\text{str}(g)$ provided that all equations (33) with $E(g) = e' < e$ are resolved. The first step of the induction consists of the observation that there are no equations for the case $E(g) = 0$.

Let us consider the case where there are two equations (33) on $\text{str}(g)$ for some $g$. This is only possible if $g = c_1c_2g'$ where $c_1$ and $c_2$ are some odd cycles in the decomposition of $g$ such that $c_1$ is not similar to $c_2$. Note that $E(g') = E(g) - 2 = e - 2$, $E(c_1g') = E(c_2g') = e - 1$.

Without loss of generality let us set

$$c_1 = K_{12}K_{23}...K_{(p-1)p}R_{i_1}R_{i_2}...R_{i_k}$$  \hspace{1cm} (Ap1)

with $1 \leq i_1 < i_2 < ... < i_k \leq p$ and odd $p - 1 + k$, and

$$c_2 = K_{(p+1)(p+2)}...K_{(p+q-1)(p+q)}R_{i_{k+1}}R_{i_{k+2}}...R_{i_{k+l}}$$  \hspace{1cm} (Ap2)
with \( p + 1 \leq i_{k+1} < i_{k+2} < \ldots < i_{k+l} \leq p + q \) and odd \( q - 1 + l \), and introduce the corresponding generating elements

\[
b_1^\alpha = \frac{1}{\sqrt{p}} \sum_{s=1}^{p} (-1)^s (-1)^{\Delta(s)} a_s^\alpha, \tag{Ap3}
\]

\[
b_2^\alpha = \frac{1}{\sqrt{q}} \sum_{s=p+1}^{p+q} (-1)^s (-1)^{\Delta(s)} a_s^\alpha, \tag{Ap4}
\]

where \( \Delta(s) = \sum_{u=s}^{i+1} \delta(u), \) and \( \delta(u) = \sum_{v=1}^{k+l} \delta_{uv} \),

with eigenvalues \( \lambda_1 = \lambda_2 = -1 \).

Let us note that \( c_+ = K_1(p+1)c_1c_2 \) is a cycle such that

\[
c_+ b_+^\alpha = -b_+^\alpha c_+, \quad \text{where} \quad b_+^\alpha = \frac{\sqrt{p}b_1^\alpha + \sqrt{q}b_2^\alpha}{\sqrt{p+q}}, \tag{Ap5}
\]

while \( c_- = K_{p+1} R_1 R_{p+1} c_1c_2 \) is a cycle such that

\[
c_- b_-^\alpha = -b_-^\alpha c_-, \quad \text{where} \quad b_-^\alpha = \frac{\sqrt{p}b_1^\alpha - \sqrt{q}b_2^\alpha}{\sqrt{p+q}}. \tag{Ap6}
\]

Now consider the equation for \( \text{str}(g) \) in the form

\[
\text{str}(g) = -\text{str}([b_1^0, b_1^1] - 1)g =
\]

\[
-\text{str} \left( \left([b_1^0, b_1^1] - 1 - \frac{1}{p} \sum_{i=1}^{p} \sum_{j=p+1}^{p+q} (\nu_0 K_{ij} + \nu_2 K_{ij} R_i R_j) \right) g \right) \tag{Ap7}
\]

\[
-\text{str} \left( \frac{1}{p} \sum_{i=1}^{p} \sum_{j=p+1}^{p+q} (\nu_0 K_{ij} + \nu_2 K_{ij} R_i R_j) g \right). 
\]

Denote

\[
h = \left( \left([b_1^0, b_2^1] - 1 - \frac{1}{p} \sum_{i=1}^{p} \sum_{j=p+1}^{p+q} (\nu_0 K_{ij} + \nu_2 K_{ij} R_i R_j) \right) g \right), \quad h \in \mathfrak{G},
\]

and note that \( E(h) = e - 1 \), and \( h = \sum \gamma_t g_t \) with some \( g_t \in G \) and constants \( \gamma_t \), such that every \( g_t \) contains the cycle \( c_2 \) in its decomposition \( \{7\} \). So due to inductive hypothesis the following identity is true

\[
\text{str}(h) = -\text{str}([b_2^0, b_2^1] - 1)h 
\]

and we obtain

\[
\text{str}(g) = \text{str} \left( \left([b_2^0, b_2^1] - 1 - \frac{1}{p} \sum_{i=1}^{p} \sum_{j=p+1}^{p+q} (\nu_0 K_{ij} + \nu_2 K_{ij} R_i R_j) \right) g \right) 
\]

\[
-\text{str} \left( \frac{1}{p} \sum_{i=1}^{p} \sum_{j=p+1}^{p+q} (\nu_0 K_{ij} + \nu_2 K_{ij} R_i R_j) g \right). \tag{Ap8}
\]
The substitution $1 \leftrightarrow 2$, $p \leftrightarrow q$ gives

$$str(g) = str\left(\left[\frac{[b_0^0, b_1^0]}{p} - \frac{[b_0^1, b_1^1]}{q}\right] - 1 - \frac{1}{p} \sum_{q=1}^{p} \sum_{j=p+1}^{p+q} (\nu_0 K_{ij} + \nu_2 K_{ij} R_i R_j)\right) g$$

$$- str\left(\frac{1}{q} \sum_{i=1}^{p} \sum_{j=p+1}^{p+q} (\nu_0 K_{ij} + \nu_2 K_{ij} R_i R_j)\right) g. \quad (Ap9)$$

Let us show that due to inductive hypothesis the difference between $(Ap8)$ and $(Ap9)$ is vanishing. This difference is equal to

$$X = str\left(\left[\frac{[b_0^0, b_1^0]}{p} - \frac{[b_0^1, b_1^1]}{q}\right] \sum_{i=1}^{p} \sum_{j=p+1}^{p+q} (\nu_0 K_{ij} + \nu_2 K_{ij} R_i R_j)\right), \quad (Ap10)$$

because $str\left([b_0^0, b_1^0], [b_0^1, b_1^1]\right) g = 0$ for every supertrace on $\mathfrak{G}$. Now we can use the following identities for $1 < i \leq p$ and $p + 1 < j \leq p + q$

$$(c_1)^{-1} K_{ij} c_1 = K_{(i-1)j} (R_{i-1} R_j) \delta^{(i-1)},$$

$$(c_2)^{-1} K_{ij} c_2 = K_{ij} (R_{i-1} R_j) \delta^{(i)},$$

$$(c_1)^{-1} K_{ij} R_i R_j c_1 = K_{(i-1)j} (R_{i-1} R_j) \delta^{(i+1)},$$

$$(c_2)^{-1} K_{ij} R_i R_j c_2 = K_{ij} (R_{i-1} R_j) \delta^{(i+1)}$$

to deduce that $X = X_0 + X_2$ for every $G$-invariant supertrace on $\mathfrak{G}$, where

$$X_0 = F_0(\nu_0, \nu_2) str\left(\left[\frac{[b_0^0, b_1^0]}{p} - \frac{[b_0^1, b_1^1]}{q}\right] K_{1(p+1)} g \right)$$

$$X_2 = F_2(\nu_0, \nu_2) str\left(\left[\frac{[b_0^0, b_1^0]}{p} - \frac{[b_0^1, b_1^1]}{q}\right] K_{1(p+1)} R_1 R_{p+1} g \right) \quad (Ap11)$$

and $F_{0,1}(\nu_0, \nu_2)$ are some definite functions. Substituting $b_2^0 = \frac{1}{\sqrt{q}}(\sqrt{p} + q B_+^\alpha - \sqrt{p} b_1^0)$ in $X_0$ and $b_2^0 = \frac{1}{\sqrt{q}}(\sqrt{p} + q B_+^\alpha + \sqrt{p} b_1^0)$ in $X_2$, and using inductive hypothesis and Lemma 2 one obtains $X_0 = X_2 = 0$, what finishes the proof of Theorem 2.

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