New Results in the Physics of Neutrino Oscillations

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1. Introduction

Neutrino mixing and oscillations [1] are among the most interesting topics of modern Particle Physics: at theoretical level, the understanding of the origin of the small masses of neutrinos and of the mixing among generations is still a puzzling problem [2]. At experimental level, recent results [3] seem finally to confirm the existence of neutrino oscillations and (consequently) of non vanishing masses for these particles.

However, since the pioneering work of Pontecorvo [4], who first pointed out the possibility of flavor oscillations for mixed massive neutrinos, a careful analysis of the structure of the Hilbert space for mixed particles was not carried out successfully [5,6].

This was achieved only recently [7] and in the present paper I report about these results. I show that a study of the mixing transformations in Quantum Field Theory (QFT), reveals a rich non-perturbative structure of the vacuum for the mixed fields. This fact has phenomenological consequences on neutrino oscillations: the oscillation formula turns out to have an additional oscillating piece and energy dependent amplitudes, in contrast with the usual (quantum mechanical) Pontecorvo formula, which is however recovered in the relativistic limit. I also show how the concept of a topological (Berry) phase naturally enters the physics of neutrino oscillations [8].

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2. Mixing of Fermion Fields

In order to discuss neutrino oscillations in QFT, I consider the following Lagrangian for two Dirac fields $\nu_e$ and $\nu_\mu$ (omitting space-time dependence for brevity)

$$\mathcal{L} = \bar{\nu}_e (i \gamma^\mu m_e) \nu_e + \bar{\nu}_\mu (i \gamma^\mu m_\mu) \nu_\mu - m_{e\mu} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e) ,$$

which is sufficient to describe the single particle evolution of a mixed fermion\(^1\). Mixing arises when the above Lagrangian is diagonalized by means of the transformations

$$\begin{align*}
\nu_e(x) &= \nu_1(x) \cos \theta + \nu_2(x) \sin \theta \\
\nu_\mu(x) &= -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta ,
\end{align*}$$

where $\theta$ is the mixing angle. $\nu_e$ and $\nu_\mu$ are neutrino fields with definite flavors. $\nu_1$ while $\nu_2$ are (free) neutrino fields with definite masses $m_1$ and $m_2$, respectively. In terms of $\nu_1$ and $\nu_2$, the above Lagrangian reads

$$\mathcal{L} = \bar{\nu}_1 (i \gamma^\mu m_1 - m_e) \nu_1 + \bar{\nu}_2 (i \gamma^\mu m_2 - m_\mu) \nu_2 .$$

with $m_e = m_1 \cos^2 \theta + m_2 \sin^2 \theta$, $m_\mu = m_1 \sin^2 \theta + m_2 \cos^2 \theta$, $m_{e\mu} = (m_2 - m_1) \sin \theta \cos \theta$.

The free fields $\nu_1(x)$ and $\nu_2(x)$ are written as

$$\nu_i(x) = \frac{1}{\sqrt{\lambda}} \sum_{k,r} \left[ u_{k,i}^r e^{-i\omega_{k,i} t} \alpha_{k,i}^r + v_{-k,i}^r e^{i\omega_{k,i} t} \beta_{-k,i}^r \right] e^{ikx} , \quad i = 1, 2 .$$

(4)

with $\omega_{k,i} = \sqrt{k^2 + m_i^2}$. The $\alpha_i$ and the $\beta_i$ ($i = 1, 2$), are defined with respect to the vacuum state $|0\rangle_{1,2}$: $\alpha_i |0\rangle_{1,2} = \beta_i |0\rangle_{1,2} = 0$. The anticommutation relations are the usual ones. The orthonormality and completeness relations are:

$$u_{k,i}^r u_{k,i}^{r\dagger} = v_{k,i}^{r\dagger} v_{k,i}^r = \delta_{rs} , \quad u_{k,i}^r v_{-k,i}^s = v_{-k,i}^s u_{k,i}^r = 0 , \quad \sum_r (u_{k,i}^r u_{k,i}^{r\dagger} + v_{-k,i}^r v_{-k,i}^{r\dagger}) = \mathbb{I} .$$

(5)

Our first step is the study of the generator of eqs.(2) and of the underlying group theoretical structure.

Eqs.(3) can be put in the form\(^1\):

$$\begin{align*}
\nu_e^\alpha(x) &= \mathcal{G}^{-1}_\theta(t) \nu_1^\alpha(x) \mathcal{G}_\theta(t) = \frac{1}{\sqrt{\lambda}} \sum_{k,r} \left[ u_{k,1}^{r\alpha} e^{-i\omega_{k,1} t} \alpha_{k,e}^r (t) + v_{-k,1}^{r\alpha} e^{i\omega_{k,1} t} \beta_{-k,e}^r (t) \right] e^{ikx} , \\
\nu_\mu^\alpha(x) &= \mathcal{G}^{-1}_\theta(t) \nu_2^\alpha(x) \mathcal{G}_\theta(t) = \frac{1}{\sqrt{\lambda}} \sum_{k,r} \left[ u_{k,2}^{r\alpha} e^{-i\omega_{k,2} t} \alpha_{k,\mu}^r (t) + v_{-k,2}^{r\alpha} e^{i\omega_{k,2} t} \beta_{-k,\mu}^r (t) \right] e^{ikx} ,
\end{align*}$$

(6)

(7)

where $\alpha = 1, \ldots, 4$. The annihilation operators for the flavor fields are defined as (indices are suppressed): $\alpha_{e,\mu}(t) \equiv \mathcal{G}^{-1}_\theta(t) \alpha_{1,2} \mathcal{G}_\theta(t)$ and $\beta_{e,\mu}(t) \equiv \mathcal{G}^{-1}_\theta(t) \beta_{1,2} \mathcal{G}_\theta(t)$.

\(^1\)The inclusion of interaction terms in the Lagrangian eq.(1) does not alter the following discussion which is about the transformations eq.(2). For a discussion of three flavor mixing, see ref.[I].
The generator $G_\theta(t)$ is given by

$$G_\theta(t) = \exp \left[ \theta \left( S_+(t) - S_-(t) \right) \right].$$  

\hspace{1cm} (8)

$$S_+(t) \equiv \int d^3x \nu_1^\dagger(x)\nu_2(x), \quad S_-(t) \equiv \int d^3x \nu_2^\dagger(x)\nu_1(x),$$

\hspace{1cm} (9)

and is (at finite volume) an unitary operator: $G_\theta^{-1}(t) = G_{-\theta}(t) = G_\theta^\dagger(t)$, preserving the canonical anticommutation relations.

Eqs. (3), (7) follow from $d^2_{\mu \nu}v_\mu^\alpha = -v_\nu^\alpha$, $d^2_{\mu \nu}v_\mu^\alpha = -v_\nu^\alpha$ with the initial conditions $\nu_\mu^{1,2}|_{\theta=0} = \nu_1^{1,2}$, $d_\theta v_\mu^{1,2}|_{\theta=0} = \nu_2^{1,2}$ and $\nu_\mu^{1,2}|_{\theta=0} = \nu_2^{1,2}$, $d_\theta v_\mu^{1,2}|_{\theta=0} = -\nu_1^{1,2}$.

Note that $G_\theta$ is an element of $SU(2)$. Indeed, if one introduces

$$S_3 \equiv \frac{1}{2} \int d^3x \left( \nu_1^\dagger(x)\nu_1(x) - \nu_2^\dagger(x)\nu_2(x) \right),$$

\hspace{1cm} (10)

and the total charge (Casimir operator),

$$S_0 \equiv \frac{1}{2} \int d^3x \left( \nu_1^\dagger(x)\nu_1(x) + \nu_2^\dagger(x)\nu_2(x) \right),$$

\hspace{1cm} (11)

then the $su(2)$ algebra is closed:

$$[S_+, S_-] = 2S_3, \quad [S_3, S_\pm] = \pm S_\pm, \quad [S_0, S_3] = [S_0, S_\pm] = 0.$$  

\hspace{1cm} (12)

The crucial point about the above generator is that it does not leave invariant the vacuum $|0\rangle_{1,2}$. Its action on it results in a new state,

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t) |0\rangle_{1,2},$$

\hspace{1cm} (13)

which is the flavor vacuum, i.e. the vacuum for the flavor fields. Let us define $|0\rangle_{e,\mu} \equiv |0(0)\rangle_{e,\mu}$ and compute $1,2(0)|0\rangle_{e,\mu}$. By writing $|0\rangle_{e,\mu} = \prod_k |0\rangle_{e,\mu}^k$, we obtain [4]:

$$1,2(0)|0\rangle_{e,\mu} = \prod_k \left( 1 - \sin^2 \theta |V_k|^2 \right)^2 = e^{\sum_k \ln \left( 1 - \sin^2 \theta |V_k|^2 \right)^2},$$

\hspace{1cm} (14)

where the function $V_k$ is defined in eq. (18). Note that $|V_k|^2$ depends on $k$ only through its modulus, takes values in the interval $[0, \frac{1}{2}]$ and $|V_k|^2 \to 0$ when $|k| \to \infty$ (see Fig.1).

By using the customary continuous limit relation $\sum_k \to \frac{V}{(2\pi)} \int d^3k$, in the infinite volume limit we obtain

$$\lim_{V \to \infty} 1,2(0)|0\rangle_{e,\mu} = \lim_{V \to \infty} e^{\frac{V}{(2\pi)} \int d^3k \ln \left( 1 - \sin^2 \theta |V_k|^2 \right)^2} = 0$$

\hspace{1cm} (15)

for any value of the parameters $\theta$, $m_1$ and $m_2$.

Eq. (13) expresses the unitary inequivalence of the flavor and the mass representations and shows the non-trivial nature of the mixing transformations [2]. In other words, the mixing transformations induce a structure in the flavor vacuum which indeed turns out to be an $SU(2)$ generalized coherent state [1] (cf. eqs. (13) and (15)).

Of course, the orthogonality between $|0\rangle_{e,\mu}$ and $|0\rangle_{1,2}$ disappears when $\theta = 0$ and/or $m_1 = m_2$ (in this case $V_k = 0$ for any $k$ and also no mixing occurs in the Lagrangian [2]).
Let us now look at the explicit form for the flavor annihilation operators. Without loss of generality, we can choose the reference frame such that \( k = (0, 0, |k|) \). In this case the annihilation operators have the simple form:

\[
\alpha_{k,e}^r(t) = \cos \theta \alpha_{k,1}^r + \sin \theta \left( U_k^r(t) \alpha_{k,2}^r + e^r V_k(t) \beta_{-k,2}^r \right)
\]

\[
\alpha_{k,\mu}^r(t) = \cos \theta \alpha_{k,\mu}^r - \sin \theta \left( U_k^r(t) \alpha_{k,\mu}^r - e^r V_k(t) \beta_{-k,\mu}^r \right)
\]

\[
\beta_{-k,e}^r(t) = \cos \theta \beta_{-k,1}^r + \sin \theta \left( U_k^r(t) \beta_{-k,2}^r - e^r V_k(t) \alpha_{k,2}^r \right)
\]

\[
\beta_{-k,\mu}^r(t) = \cos \theta \beta_{-k,\mu}^r - \sin \theta \left( U_k^r(t) \beta_{-k,\mu}^r + e^r V_k(t) \alpha_{k,\mu}^r \right)
\]

where \( e^r = (-1)^r \). Also, \( V_k(t) = |V_k| e^{i(\omega_{k,2} + \omega_{k,1}) t} \) and \( U_k(t) = |U_k| e^{i(\omega_{k,2} - \omega_{k,1}) t} \), with

\[
|U_k| \equiv u_{k,1}^r u_{k,1}^t = \frac{|k|^2}{\frac{1}{(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)}}
\]

\[
|V_k| \equiv e^r u_{k,1}^r v_{-k,2}^r = -e^r u_{-k,2}^r v_{-k,1}^r.
\]

Explicitly,

\[
|U_k| = \left( \frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left( \frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left( 1 + \frac{|k|^2}{(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)} \right)
\]

\[
|V_k| = \left( \frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left( \frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left( \frac{k}{(\omega_{k,2} + m_2)} - \frac{k}{(\omega_{k,1} + m_1)} \right)
\]

\[
|U_k|^2 + |V_k|^2 = 1
\]

We thus see that, at the level of annihilation operators, the structure of the mixing transformation is that of a Bogoliubov transformation nested into a rotation. The functions \( U_k \) and \( V_k \) play indeed the role of Bogoliubov coefficients.

It is also possible to exhibit the full explicit expression of \( |0\rangle_{e,\mu}^k \) (at time \( t = 0 \) and for \( k = (0, 0, |k|) \)):

\[
|0\rangle_{e,\mu}^k = \prod_r \left[ (1 - \sin^2 \theta |V_k|^2) - e^r \sin \theta \cos \theta |V_k| \left( \alpha_{k,1}^{r\dagger} \beta_{-k,2}^{r\dagger} + \alpha_{k,2}^{r\dagger} \beta_{-k,1}^{r\dagger} \right) + \right.
\]

\[
+ e^r \sin^2 \theta |V_k| \left( U_k \left( \alpha_{k,1}^{r\dagger} \beta_{-k,1}^{r\dagger} - \alpha_{k,2}^{r\dagger} \beta_{-k,2}^{r\dagger} \right) + \sin^2 \theta |V_k|^2 \right) \left[ \alpha_{k,1}^{r\dagger} \alpha_{k,2}^{r\dagger} \beta_{-k,1}^{r\dagger} \beta_{-k,2}^{r\dagger} \right] |0\rangle_{1,2}
\]

We see that the expression of the flavor vacuum \( |0\rangle_{e,\mu} \) involves four different particle-antiparticle “couples”. It is interesting to compare \( |0\rangle_{e,\mu} \) with the BCS superconducting ground state \( |\Gamma\rangle \), which involves only one kind of couple and is generated by a Bogoliubov transformation.

Finally, the condensation density is given by

\[
e_{\mu}(0|\alpha_{k,i}^{r\dagger} \alpha_{k,j}^{r} |0\rangle_{e,\mu} = e_{\mu}(0|\beta_{k,i}^{r\dagger} \beta_{k,j}^{r} |0\rangle_{e,\mu} = \sin^2 \theta |V_k|^2, \quad i = 1, 2.
\]

The function \( |V_k|^2 \) has a maximum at \( |k| = \sqrt{m_1 m_2} \) (see Fig.1).

\[\text{2} \text{The flavor operators of eq.} (19) \text{ do annihilate the flavor vacuum. For example: } \alpha_{k,e}^r(t)|0(t)\rangle_{e,\mu} = G_{\theta}^{-1}(t) \alpha_{k,1}^r G_{\theta}(t) G_{\theta}^{-1}(t)|0\rangle_{1,2} = 0.\]
3. Neutrino Oscillations: the Exact Formula

Equipped with the results of the previous Section, we can now study the time evolution of a flavor state, i.e. neutrino oscillations. We know from eq.(13) that we have to work in the Hilbert space built on $|0\rangle_{e,\mu}$, since this is the space for the flavor fields.

At time $t = 0$, the vacuum state is $|0\rangle_{e,\mu}$ and the one electron neutrino state is (for $k = (0, 0, |k|)$):

$$|\nu_e\rangle \equiv \alpha^{r\dagger}_{k,e}|0\rangle_{e,\mu} = \begin{bmatrix} \cos \theta \alpha^{r\dagger}_{k,1} + |U_k| \sin \theta \alpha^{r\dagger}_{k,2} + \epsilon^r |V_k| \sin \theta \alpha^{r\dagger}_{k,2} \alpha^{r\dagger}_{k,1} \beta^{r\dagger}_{-k,1} \end{bmatrix} |0\rangle_{1,2}. \quad (22)$$

In this state a multiparticle component is present, disappearing in the relativistic limit $|k| \gg \sqrt{m_1m_2}$: in this limit the (quantum-mechanical) Pontecorvo state is recovered.

If we now assume that the neutrino state at time $t$ is given by $|\nu_e(t)\rangle = e^{-iHt}|\nu_e\rangle$, we see that it is not possible to compare directly this state with the one at time $t = 0$ given in eq.(22). The reason is that the flavor vacuum $|0\rangle_{e,\mu}$ is not eigenstate of the free Hamiltonian $H$ and it “rotates” under the action of the time evolution generator: one indeed finds

$$\lim_{V \to \infty} e,\mu \langle 0 | 0(t)\rangle_{e,\mu} = 0. \quad \text{Thus at different times we have unitarily inequivalent flavor vacua and this implies that we cannot directly compare flavor states at different times.}$$

A way to circumvent this problem is to study the propagators for the mixed fields $\nu_e, \nu_\mu$ [11]

The crucial point is about how to compute these propagators: if one (naively) uses the vacuum $|0\rangle_{1,2}$, one gets an inconsistent result (cf. eq.(29)). Let us show this by considering the Feynman propagator for electron neutrinos,

$$S_{ee}(x, y) = \langle 1, 2 | T[\nu_e(x)\bar{\nu}_e(y)] |0\rangle_{1,2} \quad (23)$$

where $T$ denotes time ordering. Use of (2) gives $S_{ee}$ in momentum representation as
\[
S_{ee}(k_0, k) = \cos^2 \theta \frac{k + m_1}{k^2 - m_1^2 + i\delta} + \sin^2 \theta \frac{k + m_2}{k^2 - m_2^2 + i\delta} ,
\]

which is just the weighted sum of the two propagators for the free fields \(\nu_1\) and \(\nu_2\). It coincides with the Feynman propagator obtained by resumming (to all orders) the perturbative series

\[
S_{ee} = S_e \left(1 + m_{e\mu} S\mu S_e + m_{e\mu} S\mu S_e S_{ee} S_e + \ldots\right) = S_e \left(1 - m_{e\mu} S\mu S_e\right)^{-1} ,
\]

where the “bare” propagators are defined as \(S_{e/\mu} = (\not{k} - m_{e/\mu} + i\delta)^{-1}\).

The transition amplitude for an electron neutrino created by \(\alpha_{k,e}^r\) at time \(t = 0\) going into the same particle at time \(t\), is given by

\[
\mathcal{P}_{ee}^r(k, t) = i u_{k,1}^r e^{i\omega_{k,1}^r t} S_{ee}^r(k, t) \gamma^0 u_{k,1}^r ,
\]

where the spinors \(u_1\) and \(v_1\) form the basis in which the field \(\nu_e\) is expanded (cf. eq. (3)). Here, \(S_{ee}^r\) denotes the unordered Green’s function (or Wightman function): \(i S_{ee}^r(t, x; 0, y) = \sum_{1,2} \langle 0 | \nu_e(t, x) \bar{\nu}_e(0, y) | 0 \rangle_{1,2} \). The explicit expression for \(S_{ee}^r(k, t)\) is

\[
S_{ee}^r(k, t) = -i \sum_r \left( \cos^2 \theta e^{-i\omega_{k,1}^r t} u_{k,1}^r \bar{u}_{k,1}^r + \sin^2 \theta e^{-i\omega_{k,2}^r t} u_{k,2}^r \bar{u}_{k,2}^r \right) .
\]

The amplitude eq. (26) is independent of the spin orientation and given by

\[
\mathcal{P}_{ee}(k, t) = \cos^2 \theta + \sin^2 \theta |U_k|^2 e^{-i(\omega_{k,2} - \omega_{k,1}) t} .
\]

For different masses and for \(k \neq 0\), \(|U_k|^2\) is always < 1 (cf. eq. (13)). Notice also that \(|U_k|^2 \rightarrow 1\) in the relativistic limit \(|k| \gg \sqrt{m_1 m_2}\); only in this limit the squared modulus of \(\mathcal{P}_{ee}(k, t)\) does reproduce the Pontecorvo oscillation formula.

Of course, it should be \(\lim_{t \rightarrow 0} \mathcal{P}_{ee}(t) = 1\). Instead, one obtains the unacceptable result

\[
\mathcal{P}_{ee}(k, 0^+) = \cos^2 \theta + \sin^2 \theta |U_k|^2 < 1 .
\]

This means that the choice of the state \(|0\rangle_{1,2}\) in eq. (23) and in the computation of the Wightman function is \emph{not} the correct one. The reason is that, as previously shown, \(|0\rangle_{1,2}\) is \emph{not} the vacuum state for the flavor fields \(\bar{\nu}_e\).

We are then led to define the propagators on the flavor vacuum \(|0\rangle_{e,\mu}\). Considering again the propagator for the electron neutrinos, we have

\[
G_{ee}(x, y) \equiv e,\mu \langle 0(y_0) | T [\nu_e(x) \bar{\nu}_e(y)] | 0(y_0) \rangle_{e,\mu} .
\]

Notice that here the time argument \(y_0\) (or, equally well, \(x_0\)) of the flavor ground state, is chosen to be equal on both sides of the expectation value. This is necessary since, as already observed, transition matrix elements of the type \(e,\mu \langle 0 | \alpha_e \exp [-i H t ] \alpha^\dagger_e | 0 \rangle_{e,\mu}\), do not represent physical transition amplitudes: they actually vanish (in the infinite volume limit) due to the unitary inequivalence of flavor vacua at different times \(1\). Therefore the comparison of states at different times necessitates a \emph{parallel transport} of these states to a common point of reference. The definition \(21\) includes this concept of parallel transport, which is a sort of “gauge fixing”: a geometric structure associated with the simple dynamical system of eq.(11) is thus uncovered.
In mixed \((k, t)\) representation, we have (for \(k = (0, 0, |k|)\)):

\[
G_{ee}(k_0, k) = S_{ee}(k_0, k) + 2\pi i \sin^2 \theta \left[ |V_k|^2 (k + m_2) \delta(k^2 - m_2^2) - |U_k||V_k| \sum_r \left( \epsilon^r u_{k,2}^r \bar{v}_{k,2}^r \delta(k_0 - \omega_2) + \epsilon^r v_{k,2}^r \bar{u}_{k,2}^r \delta(k_0 + \omega_2) \right) \right], \tag{31}
\]

Comparison of eq.(31) with eq.(24) shows that the difference between the full and the perturbative propagators is in the imaginary part.

I define the Wightman functions for an electron neutrino as 
\[iG_{ee}^>(t, x; 0, y) = e_{\mu} \langle 0 | \nu_e(t, x) \bar{\nu}_e(0, y) | 0 \rangle_{e, \mu} \],
and 
\[iG_{\mu e}^>(t, x; 0, y) = e_{\mu} \langle 0 | \nu_\mu(t, x) \bar{\nu}_e(0, y) | 0 \rangle_{e, \mu} \].
These are conveniently expressed in terms of anticommutators at different times as

\[
iG_{ee}^>(k, t) = \sum_r \left[ u_{k,1}^r \bar{u}_{k,1}^r \left\{ \alpha_{k,e}^r(t), \alpha_{k,e}^r \right\} e^{-i\omega_{k,1}t} + v_{k,1}^r \bar{u}_{k,1}^r \left\{ \beta_{k,e}^r(t), \alpha_{k,e}^r \right\} e^{i\omega_{k,1}t} \right], \tag{32}
\]

\[
iG_{\mu e}^>(k, t) = \sum_r \left[ u_{k,2}^r \bar{u}_{k,1}^r \left\{ \alpha_{k,\mu}^r(t), \alpha_{k,e}^r \right\} e^{-i\omega_{k,2}t} + v_{k,2}^r \bar{u}_{k,1}^r \left\{ \beta_{k,\mu}^r(t), \alpha_{k,e}^r \right\} e^{i\omega_{k,2}t} \right]. \tag{33}
\]

Here and in the following \(\alpha_{k,e}^r\) stands for \(\alpha_{k,e}^r(0)\). We now have four distinct transition amplitudes, given by anticommutators of flavor operators at different times:

\[
P_{ee}^r(k, t) \equiv i \left| u_{k,1}^r \right|^2 e^{i\omega_{k,1}t} G_{ee}^>(k, t) \gamma^0 u_{k,1} = \left\{ \alpha_{k,e}^r(t), \alpha_{k,e}^r \right\} \cos^2 \theta + \sin^2 \theta \left[ |U_k|^2 e^{-i(\omega_{k,2} - \omega_{k,1})t} + |V_k|^2 e^{i(\omega_{k,2} + \omega_{k,1})t} \right], \tag{34}
\]

\[
P_{ee}^r(k, t) \equiv i \left| v_{k,1}^r \right|^2 e^{i\omega_{k,1}t} G_{ee}^>(k, t) \gamma^0 u_{k,1} = \left\{ \beta_{k,e}^r(t), \alpha_{k,e}^r \right\} \cos^2 \theta \sin^2 \theta \left[ e^{i(\omega_{k,2} - \omega_{k,1})t} - e^{-i(\omega_{k,2} + \omega_{k,1})t} \right], \tag{35}
\]

\[
P_{\mu e}^r(k, t) \equiv i \left| u_{k,2}^r \right|^2 e^{i\omega_{k,2}t} G_{\mu e}^>(k, t) \gamma^0 u_{k,1} = \left\{ \alpha_{k,\mu}^r(t), \alpha_{k,e}^r \right\} \cos \theta \sin \theta \left[ 1 - e^{i(\omega_{k,2} - \omega_{k,1})t} \right], \tag{36}
\]

\[
P_{\mu e}^r(k, t) \equiv i \left| v_{k,2}^r \right|^2 e^{i\omega_{k,2}t} G_{\mu e}^>(k, t) \gamma^0 u_{k,1} = \left\{ \beta_{k,\mu}^r(t), \alpha_{k,e}^r \right\} \cos \theta \sin \theta \left[ 1 - e^{-i(\omega_{k,2} + \omega_{k,1})t} \right]. \tag{37}
\]

All other anticommutators with \(\alpha_{k,e}^r\) vanish. The probability amplitude is now correctly normalized: \(\lim_{t \to 0^+} P_{ee}(k, t) = 1\), and \(P_{ee}, P_{\mu e}, P_{\mu \mu}\) go to zero in the same limit \(t \to 0^+\). Moreover,

\[
|P_{ee}^r(k, t)|^2 + |P_{ee}^r(k, t)|^2 + |P_{\mu e}^r(k, t)|^2 + |P_{\mu e}^r(k, t)|^2 = 1, \tag{38}
\]

as the conservation of the total probability requires. We also note that these transition probabilities are independent of the spin orientation.

In order to understand the above transition amplitudes, consider the flavor charge operators, defined as

\[
Q_{e/\mu} \equiv \sum_{k, \mu} \left( \alpha_{k,e/\mu}^r \alpha_{k,e/\mu}^r + \beta_{k,e/\mu}^r \beta_{k,e/\mu}^r \right). \]

We then have:
The state \( \nu(0) \) at time \( t \), we get:

\[
\langle \nu\nu \rangle \equiv \langle \nu(0) \nu(0) \rangle = \langle \nu(0) \nu(0) \rangle = 0 , \tag{39}
\]

\[
\langle \nu\nu \rangle \equiv \langle \nu\nu \rangle = \left| \alpha_{\nu}^{r} (t), \alpha_{\nu}^{l} (t) \right|^2 + \left| \beta_{\nu}^{r} (t), \alpha_{\nu}^{l} (t) \right|^2 , \tag{40}
\]

\[
\langle \nu\nu \rangle \equiv \langle \nu\nu \rangle = \left| \alpha_{\nu}^{r} (t), \alpha_{\nu}^{l} (t) \right|^2 + \left| \beta_{\nu}^{r} (t), \alpha_{\nu}^{l} (t) \right|^2 . \tag{41}
\]

Charge conservation is ensured at any time: \( \langle \nu\nu \rangle (Q_{e} + Q_{\mu} \nu(0)) = 1 \) and the oscillation formula readily follows as

\[
P_{\nu \nu \nu} (k, t) = \left| \alpha_{\nu}^{r} (t), \alpha_{\nu}^{l} (t) \right|^2 + \left| \beta_{\nu}^{r} (t), \alpha_{\nu}^{l} (t) \right|^2 \tag{42}
\]

\[
P_{\nu \nu \nu} (k, t) = \left| \alpha_{\nu}^{r} (t), \alpha_{\nu}^{l} (t) \right|^2 + \left| \beta_{\nu}^{r} (t), \alpha_{\nu}^{l} (t) \right|^2 \tag{43}
\]

This result is exact. There are two differences with respect to the usual formula for neutrino oscillations: the amplitudes are energy dependent, and there is an additional oscillating term. For \( |k| \gg \sqrt{m_{1} m_{2}} \), \( U_{k} \) \( 1 \) and \( \nu(0) \) \( 0 \) and the traditional oscillation formula is recovered. However, also in this case we have that the neutrino state remains a coherent state, thus phenomenological implications of our analysis are possible also for relativistic neutrinos. Further work in this direction is in progress.

4. Berry Phase for Oscillating Neutrinos

Here I report on preliminary results \([8] \) about the existence of a topological (Berry) phase in the evolution of a mixed state, more specifically in neutrino oscillations. Let us consider an electron neutrino state in the usual (Pontecorvo) approximation \([1] \); at time \( t \) it is given by

\[
| \nu_{e}(0) \rangle \equiv e^{-iHt} | \nu_{e}(0) \rangle = e^{-i \omega_{1} t} \left( \cos \theta | \nu_{1} \rangle + e^{-i (\omega_{2} - \omega_{1}) t} \sin \theta | \nu_{2} \rangle \right) . \tag{44}
\]

The state \( | \nu_{e}(0) \rangle \), apart from a phase factor, returns to the initial state \( | \nu_{e}(0) \rangle \) after a period \( T = \frac{2 \pi}{\omega_{2} - \omega_{1}} \):

\[
| \nu_{e}(T) \rangle \equiv e^{i \phi} | \nu_{e}(0) \rangle , \quad \phi = \frac{2 \pi \omega_{1}}{\omega_{2} - \omega_{1}} . \tag{45}
\]

It has been shown \([13] \) that any state which has a cyclic quantum evolution, can acquire a geometrical (Berry) phase factor after a cycle. This means that the phase factor \( \phi \) of eq. (15) contains in general two contributions, a dynamical one and a geometrical one. Thus the task is that of separate these two contributions. Following the procedure stated in ref. \([13] \), we get:

\[
\beta = \phi + \frac{2 \pi}{\omega_{2} - \omega_{1}} \left( \omega_{1} \cos^{2} \theta + \omega_{2} \sin^{2} \theta \right) = 2 \pi \sin^{2} \theta . \tag{46}
\]
We thus see that there is indeed a non-zero geometrical phase $\beta$, related to the mixing angle $\theta$, and that it is independent from the neutrino energies $\omega_1, \omega_2$ and masses $m_1, m_2$.

An alternative way for calculating the geometrical phase is the following. Define the state

$$|\tilde{\nu}_e(t)\rangle \equiv e^{-if(t)}|\nu_e(t)\rangle,$$

with $f(t) = -\omega_1 t$ such that $f(T) - f(0) = \phi$. Then the Berry phase is defined as:

$$\beta = \int_0^T \langle \tilde{\nu}_e(t)|i\frac{d}{dt}|\tilde{\nu}_e(t)\rangle \, dt = 2\pi \sin^2 \theta,$$

which coincides with the result (46).

The topological phase factor of eq. (46) acts then as a “counter” of oscillations: after each period (oscillation length), the neutrino state gets an additional $\beta = 2\pi \sin^2 \theta$ in its phase. In principle, it is possible to think to (interference) experiments in which one could measure this phase, in analogy to what is done in other situations (see ref. [14] for example): this would give a direct measurement of the mixing angle.

5. Discussion and Conclusions

I have shown how a simple dynamical system, such as the one describing neutrino oscillations (eq.(4)), can exhibit many interesting features if a proper analysis is carried out in the context of Quantum Field Theory (QFT).

This is a crucial point: indeed it is well known that in QFT there exist many inequivalent representations of the field algebra [15] (many vacua), and this makes the difference with Quantum Mechanics, where only one Hilbert space is admitted. This considerations are far to be academic: I have shown in eqs. (42) and (43) that the condensate structure of the flavor vacuum has physical consequences on the neutrino oscillation formula.

It is also important to stress the generality of the above analysis: a similar situation (with the due changes) holds for the case of mixing of boson fields [16] and in this respect the work is in progress.

This is true also for the topological phase associated to “flavor” oscillations, which is not peculiar of fermion systems but is a general feature of mixed states of the form (44), so the result (46) is valid for a boson system as well. The work on the Berry phase is in progress [8], in particular in the direction of extending the above result to the three flavors case and to the full QFT neutrino state (cf. eq. (22)).

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