DELIGNE PAIRING AND DETERMINANT BUNDLE

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Abstract. Let $X \rightarrow S$ be a smooth projective surjective morphism, where $X$ and $S$ are integral schemes over $\mathbb{C}$. Let $L_0, L_1, \cdots, L_{n-1}, L_n$ be line bundles over $X$. There is a natural isomorphism of the Deligne pairing $\langle L_0, \cdots, L_n \rangle$ with the determinant line bundle $\text{Det}(\otimes_{i=0}^n (L_i - \mathcal{O}_X))$.

1. Introduction

Let $X \rightarrow S$ be a smooth family of complex projective curves parameterized by an integral scheme $S/\mathbb{C}$. Let $L_0$ and $L_1$ be line bundles over $X$. In [2], P. Deligne associated to this data a line bundle $\langle L_0, L_1 \rangle$ over the parameter space $S$. This construction is now known as the Deligne pairing. S. Zhang extended the Deligne pairing to arbitrary relative dimension [13]. Let $S$ and $X$ be integral schemes over $\mathbb{C}$, and let

$$f : X \rightarrow S$$

be a smooth projective surjective morphism. Let $n$ be the dimension of the fibers of $f$. Take algebraic line bundles $L_0, L_1, \cdots, L_{n-1}, L_n$ over $X$. The Deligne pairing, [13], is a line bundle

$$\langle L_0, \cdots, L_n \rangle \rightarrow S$$

(the construction is briefly recalled in Section [2]). The map

$$\text{Pic}(X)^{n+1} \rightarrow \text{Pic}(S)$$

defined by $(L_0, \cdots, L_n) \mapsto \langle L_0, \cdots, L_n \rangle$ is symmetric, and it is bilinear with respect to the group structure defined by the tensor product of line bundles and dualization; it is also compatible with base change.

The Deligne pairing has turned out to be very useful; see [11], [10], [5], [3], [4].

Given a locally free coherent sheaf $F$ on $X$, we have a line bundle $\text{Det}(F)$ on $S$ (see [7], [1]). This extends to a homomorphism to $\text{Pic}(S)$ from the Grothendieck group of locally free coherent sheaves on $X$.

The aim of this note is to announce the following:

Theorem. There is a canonical isomorphism $\langle L_0, \cdots, L_n \rangle \rightarrow \text{Det}(\otimes_{i=0}^n (L_i - \mathcal{O}_X))$. 

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A theorem connecting Deligne pairing to determinant line bundle is proved in [11] (see [11] p. 476, Theorem 1(1)).

If each \( L_i, i \in [0, n] \), is equipped with a \( C^\infty \) hermitian structure \( h_i \), then \( \langle L_0, \cdots, L_n \rangle \) inherits a hermitian structure [2], [13]. On the other hand, the hermitian structures \( h_1, \cdots, h_n \), the trivial hermitian structure on the trivial line bundle \( \mathcal{O}_X \), and a relative Kähler structure on \( X \) together define a hermitian structure on the determinant bundle \( \text{Det}(\otimes_{i=0}^n (L_i - \mathcal{O}_X)) \) according to [1], [12]. The curvatures of \( \langle L_0, \cdots, L_n \rangle \) and \( \text{Det}(\otimes_{i=0}^n (L_i - \mathcal{O}_X)) \) coincide (see Proposition 5). Finally we observe that the Weil–Petersson metric for families of canonically polarized varieties can be interpreted as the curvature form of a certain Deligne pairing.

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2. A Canonical Isomorphism

We continue with the notation of the introduction. Let \( L_0, L_1, \cdots, L_{n-1}, L_n \) be line bundles over \( X \). A local trivialization of \( \langle L_0, \cdots, L_n \rangle \) over some Zariski open subset \( U \) of \( S \) is given by fixing a rational section \( l_i \) of \( L_i \) for each \( i \in [0, n] \), such that the intersection \( (\bigcap_{i=0}^n \text{div}(l_i)) \bigcap f^{-1}(U) \) is empty. The generator for \( \langle L_0, \cdots, L_n \rangle |_U \) corresponding to \( \{l_i\}_{i=0}^n \) is denoted by \( \langle l_0, \ldots, l_n \rangle \).

To describe the line bundle \( \langle L_0, \cdots, L_n \rangle \) in terms of these trivializations, we need to give the transition functions for ordered pairs of such trivializations. Let \( g \) be a rational function on \( X \), and \( i \in [1, n] \). Assume that \( \bigcap_{j \neq i} \text{div}(l_j) = \sum_k n_k Y_k \) is finite over \( S \), and that it has empty intersection with \( \text{div}(g) \) over an open subset \( U' \subset U \) (this subset is the intersection of two open subsets of the above type). Then \( \langle l_0, \ldots, g l_i, \ldots, l_n \rangle \) is another generator. The transition function is given by the following equation:

\[
\langle l_0, \ldots, g l_i, \ldots, l_n \rangle = \prod_k \text{Norm}_{Y_k/S}(g)^{n_k} \langle l_0, \ldots, l_n \rangle.
\]

It is sufficient to describe this type of transition functions, because a general transition function is a product of such functions.

Given a coherent sheaf \( V \) on \( S \), we have a line bundle \( \det(V) \) on \( S \) (see [8] Ch. V, § 6)). For a coherent sheaf \( F \) on \( X \), we have a line bundle

\[
\text{Det}(F) := \det f^* F = \otimes_{i=0}^n \det(R^i f^* F)^{(-1)^i} \longrightarrow S
\]

[7], [1]. For coherent sheaf \( F_1 \) and \( F_2 \) on \( X \), define

\[
\text{Det}(F_1 + F_2) := \text{Det}(F_1) \otimes \text{Det}(F_2) \quad \text{and} \quad \text{Det}(F_1 - F_2) := \text{Det}(F_1) \otimes \text{Det}(F_2)^*.
\]

**Theorem 1.** Let \( L_0, L_1, \cdots, L_n \) be line bundles over \( X \). Then there is a canonical isomorphism

\[
\varphi : \langle L_0, \cdots, L_n \rangle \longrightarrow \text{Det}(\otimes_{i=0}^n (L_i - \mathcal{O}_X)).
\]
We begin with the following observation:

**Lemma 2.** Let $D'$ and $D''$ be effective divisors on $X$ such that the intersection $Y := D' \cap D''$ does not contain any divisor. Then the following equality of elements of $K(X)$ holds:

$$
\mathcal{O}_X(D' - D'') - \mathcal{O}_X = \mathcal{O}_{D'}(D') - \mathcal{O}_{D''} - \mathcal{F},
$$

where $\mathcal{F}$ is supported on $Y$.

**Proof.** This follows immediately from the identity

$$
\mathcal{O}_X(D' - D'') - \mathcal{O}_X = (\mathcal{O}_X(D') - \mathcal{O}_X) \otimes (\mathcal{O}_X(-D'') - \mathcal{O}_X) + (\mathcal{O}_X(D') - \mathcal{O}_X) + (\mathcal{O}_X(-D'') - \mathcal{O}_X)
$$

with $\mathcal{F} = (\mathcal{O}_X(D') - \mathcal{O}_X) \otimes (\mathcal{O}_X(-D'') - \mathcal{O}_X)$.

We will describe the determinant line bundle $\text{Det}(\otimes_{i=0}^n (L_i - \mathcal{O}_X))$ in terms of local trivializations and transition functions. For that purpose, we construct a covering of the base $S$ by Zariski open subsets $S'$ over which each line bundle $L_i$, $i \in [0, n]$, is given by a divisor $D_i^+ - D_i^-$, where both $D_i^+$ and $D_i^-$ are effective, such that the following conditions hold: any intersection of $n$ hypersurfaces in the union of all these divisors is reduced, the intersection is of the expected codimension, and it is finite over the base.

We note that if $Z$ is an intersection of $n$ hypersurfaces in the union these divisors, then $f(Z)$ is contained in a divisor on $S$ because $Z$ is of expected codimension. This choice of divisors for the line bundles gives a trivialization of $\text{Det}(\otimes_{i=0}^n (L_i - \mathcal{O}_X))$ over the complement of the union of all $f(Z_I)$, where $I$ runs over the set of $n$ hypersurfaces in the union the divisors. This open subset, which will be denoted by $S_0$, is nonempty because each $f(Z_I)$ is of codimension at least one. There is a trivialization

$$
\lambda_1 \in H^0(S_0, \text{Det}(\otimes_{i=0}^n (L_i - \mathcal{O}_X))|_{S_0}).
$$

We now pick a rational function

$$
g \in H^0(X', \mathcal{O}_{X'})
$$

on $X$. We assume that the divisor $(g) := \text{div}(g)$ can be included in the above system of divisors so that the above properties continue to hold for the enlarged system. The line bundle $L_0$ is given by the divisor $D_0^+ - D_0^- + (g)$. Let

$$
\lambda_2 \in H^0(S_0', \text{Det}(\otimes_{i=0}^n (L_i - \mathcal{O}_X))|_{S_0})
$$

be the trivialization obtained by replacing $D_0^+ - D_0^-$ with $D_0^+ - D_0^- + (g)$ in the construction of the trivialization $\lambda_1$ in (2.4). Let

$$
t := \lambda_2 \otimes \lambda_1^* \in H^0(S_0 \cap S_0', \mathcal{O}_{S_0 \cap S_0'}^*)
$$

be the transition function. For convenience, $S_0 \cap S_0'$ will be denoted by $S'$. 
Lemma 3. The transition function \( t \) in (2.7) has the following expression:
\[
t = \prod_{\sigma \in \{+,-\}^n} \text{Norm}_{Y_\sigma/S}(g)^{n_\sigma},
\]
where \( Y_\sigma \) and \( n_\sigma \) are defined above, and \( g \) is the function in (2.5).

Theorem 1 is proved using Lemma 3. The details will appear elsewhere.

3. Some applications of Theorem 1

Let \( f : X \to S \) be as before. Take \( n + 2 \) line bundles \( L_0, \ldots, L_{n+1} \) on \( X \), where \( n = \dim X - \dim S \).

Corollary 4. The line bundle \( \text{Det}(\otimes_{i=0}^{n+1}(L_i - \mathcal{O}_X)) \) on \( S \) has a canonical trivialization.

Proof. We know that
\[
\langle L_0, L_2, L_3, \ldots, L_{n+1} \rangle \otimes \langle L_1, \ldots, L_{n+1} \rangle = \langle L_0 \otimes L_1, L_2, \ldots, L_{n+1} \rangle
\]

Therefore, from Theorem 1
\[
\text{Det}((L_0 \otimes L_1 - \mathcal{O}_X) \otimes_{i=2}^{n+1}(L_i - \mathcal{O}_X)) = \text{Det}((L_0 - \mathcal{O}_X) \otimes_{i=2}^{n+1}(L_i - \mathcal{O}_X)) \otimes \text{Det}(\otimes_{i=0}^{n+1}(L_i - \mathcal{O}_X)).
\]

As \( (L_0 \otimes L_1 - \mathcal{O}_X) - (L_0 - \mathcal{O}_X) - (L_0 - \mathcal{O}_X) = L_0 \otimes L_1 - L_0 - L_1 + \mathcal{O}_X \), this isomorphism gives a trivialization of
\[
\text{Det}((L_0 \otimes L_1 - L_0 - L_1 + \mathcal{O}_X) \otimes_{i=2}^{n+1}(L_i - \mathcal{O}_X)).
\]

Since
\[
\text{Det}((L_0 \otimes L_1 - L_0 - L_1 + \mathcal{O}_X) \otimes_{i=2}^{n+1}(L_i - \mathcal{O}_X)) = \text{Det}(\otimes_{i=0}^{n+1}(L_i - \mathcal{O}_X))
\]
we get a trivialization of \( \text{Det}(\otimes_{i=0}^{n+1}(L_i - \mathcal{O}_X)) \).

Fix a relative Kähler form \( \omega_{X/S} \) on the fibration \( f \) in (1.1). By definition, for some open covering \( \{U_i\} \) of \( S \), there exist Kähler forms \( \omega_{f^{-1}U_i} \) on \( f^{-1}U_i \) which induce the relative real \((1,1)\)-form \( \omega_{X/S} \) on \( f^{-1}U_i \). If \( S \) is singular, we require that a Kähler form possesses locally a \( \partial \overline{\partial} \)-potential on some smooth ambient space.

If \( F \) is a vector bundle on \( X \) equipped with a hermitian structure \( h_F \), then there is a natural hermitian structure on the line bundle \( \text{Det}(F) \to S \) which is constructed using \( h_F \) and \( \omega_{X/S} \) [12, 11]; it is known as the Quillen metric. If \( F_1 \) and \( F_2 \) are vector bundles equipped with hermitian structure, then the hermitian structures on \( \text{Det}(F_1) \) and \( \text{Det}(F_2)^* \) together induce a hermitian structure on \( \text{Det}(F_1 - F_2) = \text{Det}(F_1) \otimes \text{Det}(F_2)^* \).

For each \( l \in [0, n] \), fix a hermitian metric \( h_j \) on the line bundle \( L_j \) over \( X \). These \( h_j \) produce a hermitian metric on \( \langle L_0, \ldots, L_n \rangle \) [2, 13 § 1.2]. Therefore, both the line bundles \( \text{Det}(\otimes_{i=0}^{n}(L_i - \mathcal{O}_X)) \) and \( \langle L_0, \ldots, L_n \rangle \) are equipped with a hermitian metric.

Proposition 5. The curvature of the hermitian metric on \( \langle L_0, \ldots, L_n \rangle \) coincides with the curvature of the Quillen metric on \( \text{Det}(\otimes_{i=0}^{n}(L_i - \mathcal{O}_X)) \).
Proof. The Chern form of the metric on $\langle L_0, \ldots, L_n \rangle$ equals the fiber integral

$$\int_{X/S} c_1(L_0, h_0) \wedge \ldots \wedge c_1(L_n, h_n)$$

(see [13]). On the other hand, a theorem of Bismut, Gillet and Soulé [1] says that the Chern form of the determinant line bundle is the degree two component of the Riemann–Roch fiber integral

$$c_1(\text{Det}(\otimes_{i=0}^n (L_i - O_X)), h^Q) = \left( \int_{X/S} \text{ch}(L_0 - O_X, h_0) \cdot \ldots \cdot \text{ch}(L_n - O_X, h_n) t_d(X/S) \right)_{(2)},$$

where $h^Q$ is the Quillen metric on $\text{Det}(\otimes_{i=0}^n (L_i - O_X))$ (this theorem of [1] was extended to (smooth) Kähler fibrations over singular base spaces in [6, §12]).

Note that

$$\text{ch}(L - O_X) = c_1(L, h) + \text{higher order terms}.$$

Hence the only contribution of $t_d(X/S)$ in (3.2) is the constant one, and also the higher order terms in $\text{ch}(L - O_X)$ do not contribute. Consequently, (3.2) coincides with (3.1). □

Let $X \rightarrow S$ be a projective family of canonically polarized varieties. Equip the relative canonical bundle $K_{X/S}$ with the hermitian metric that is induced by the fiberwise Kähler-Einstein metrics. It was shown in [6] that the generalized Weil-Petersson form is equal, up to a numerical factor, to the fiber integral

$$\omega_{WP} \simeq \int_{X/S} c_1(K_{X/S}, h)^{n+1}.$$

Therefore, we have the following:

**Proposition 6.** Let $X \rightarrow S$ be a projective family of canonically polarized varieties. The curvature of the metric on the Deligne pairing $\langle K_{X/S}, \ldots, K_{X/S} \rangle$ given by the fiberwise Kähler-Einstein metric coincides with the generalized Weil-Petersson form $\omega_{WP}$ on $S$.

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