ON SOLUTIONS OF LINEAR EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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Abstract. We show that a linear functional equation with polynomial coefficients need not admit an arc-analytic solution even if it admits a continuous semialgebraic one. We also show that such an equation need not admit a Nash regulous solution even if it admits an arc-analytic one.

1. Introduction

The present note is concerned with existence of solutions to linear equations with polynomial coefficients in various classes of semialgebraic functions in $\mathbb{R}^n$. Recall that a set $X$ in $\mathbb{R}^n$ is called semialgebraic if it can be written as a finite union of sets of the form $\{x \in \mathbb{R}^n : p(x) = 0, q_1(x) > 0, \ldots, q_r(x) > 0\}$, where $r \in \mathbb{N}$ and $p, q_1, \ldots, q_r$ are polynomial functions. Given $X \subset \mathbb{R}^n$, a semialgebraic function $f : X \to \mathbb{R}$ is one whose graph is a semialgebraic subset of $\mathbb{R}^{n+1}$.

A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be regulous if there exist polynomial functions $p$ and $q$ such that the zero locus of $q$ is nowhere dense in $\mathbb{R}^n$ and $f(x) = p(x)/q(x)$ whenever $q(x) \neq 0$. A real analytic semialgebraic function on $\mathbb{R}^n$ is called Nash. A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be Nash regulous if there exist Nash functions $g$ and $h$ such that the zero locus of $h$ is nowhere dense in $\mathbb{R}^n$ and $f(x) = g(x)/h(x)$ whenever $h(x) \neq 0$. Finally, recall that a function $f : X \to \mathbb{R}$ is called arc-analytic if it is analytic along every arc, that is, $f \circ \gamma$ is analytic for every real analytic $\gamma : (-1, 1) \to X$. We shall denote the regulous, Nash regulous, and arc-analytic semialgebraic functions on $\mathbb{R}^n$ by $\mathcal{R}^0(\mathbb{R}^n)$, $\mathcal{N}^0(\mathbb{R}^n)$ and $\mathcal{A}_a(\mathbb{R}^n)$, respectively. We have

$$\mathcal{R}^0(\mathbb{R}^n) \subset \mathcal{N}^0(\mathbb{R}^n) \subset \mathcal{A}_a(\mathbb{R}^n).$$

The first inclusion is trivial and the second one follows from [1, Prop. 3.1]. Both inclusions are strict.

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The above classes of semialgebraic functions have been extensively studied recently (see, e.g., [1, 2, 6, 8] and the references therein), in particular, in the context of the following problem of Fefferman and Kollár [5].

Consider a linear equation

\[(1.2) \quad f_1 \varphi_1 + \cdots + f_r \varphi_r = g,\]

where \(g\) and the \(f_j\) are continuous (real-valued) functions on \(\mathbb{R}^n\). Fefferman-Kollár asked whether assuming that \(g\) and the \(f_j\) have some regularity properties, one could find a solution \((\varphi_1, \ldots, \varphi_r)\) to \((1.2)\) with similar regularity properties.

This is a difficult problem, even when the coefficients of \((1.2)\) are polynomial. One line of attack is to instead consider a somewhat easier question:

**Problem 1.1.** Suppose that \((1.2)\) admits a solution \((\varphi_1, \ldots, \varphi_r)\) within some class of functions. Does there exist then a solution to \((1.2)\) within a strictly smaller class?

In the semialgebraic setting, the most general positive answer to this problem is given by [5, Cor. 29(1)]: If \(f_1, \ldots, f_r\) are polynomial, \(g\) is semialgebraic and \((1.2)\) admits a continuous solution, then it admits a continuous semialgebraic solution. In a similar vein, Kucharz and Kurdyka showed that, in case \(n = 2\), if \(f_1, \ldots, f_r, g\) are regulous then \((1.2)\) admits a continuous solution if and only if it admits a regulous solution (cf. [9, Cor. 1.7]).

On the other hand, the above is known to fail for \(n \geq 3\). Namely, by [7, Ex.6], there exist \(f_1, f_2, g \in \mathbb{R}[x, y, z]\) such that \(f_1 \varphi_1 + f_2 \varphi_2 = g\) admits a continuous solution, but no regulous one. Nonetheless, the solution from [7, Ex.6] is Nash regulous, and in [8] Kucharz conjectured that existence of a continuous solution to \((1.2)\) should imply the existence of a Nash regulous one, for any \(n \geq 1\), provided \(f_1, \ldots, f_r, g\) are polynomial.

The main goal of this note is to show that the latter is not the case. In Example 3.1 we show that there exists a linear equation with polynomial coefficients which admits a continuous solution, but no arc-analytic one. By \((1.1)\), it follows that there is no Nash regulous solution either. Perhaps even more interestingly, in Example 3.2 we show a linear equation with polynomial coefficients that does admit an arc-analytic solution and has no Nash regulous solution nonetheless. Both our examples are modifications of [7, Ex.6].

2. Toolbox

The following facts will be needed in Examples 3.1 and 3.2.
Proposition 2.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a semialgebraic function. Then, $f$ is arc-analytic if and only if there exists a mapping $\pi : \tilde{R} \to \mathbb{R}^n$ which is a finite sequence of blowings-up with smooth algebraic centers, such that the composite $f \circ \pi$ is a Nash function.

Proof. This is a special case of [3, Thm. 1.4].

Functions satisfying the conclusion of Proposition 2.1 are called blow-Nash.

Remark 2.2. A function $f : \mathbb{R} \to \mathbb{R}$ is arc-analytic if and only if it is real analytic. This follows directly from the definition of arc-analytic functions.

Recall that a Nash set (i.e., the zero set of a Nash function) in $\mathbb{R}^n$ is said to be Nash irreducible if it cannot be realized as a union of two proper Nash subsets. A set is called Nash constructible if it belongs to the Boolean algebra generated by the Nash subsets in $\mathbb{R}^n$.

Remark 2.3 (cf. [10, Ex. 2.3]). The graph $\Gamma_f$ of $f(x, y) = \sqrt{x^4 + y^4}$ is not Nash constructible in $\mathbb{R}^3$.

Indeed, let $X := \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^4 + y^4\}$. We claim that $X$ is Nash irreducible. First, note that $z^2 - x^4 - y^4$ is an irreducible element in the ring of convergent power series over $\mathbb{C}$. This implies that the set $\{z^2 - x^4 - y^4 = 0\} \subset \mathbb{C}^3$ has an irreducible (complex analytic) germ at the origin, of (complex) dimension 2. On the other hand, the (real analytic) germ of $X$ at the origin is of (real) dimension 2. Hence, its complexification has to be given by precisely $\{z^2 - x^4 - y^4 = 0\}$. It follows that the germ $X_0$ is irreducible, and there is thus no way to decompose $X$ into proper analytic subsets. (See [4] for details on real analytic germs and their complexifications.)

The irreducibility of $X$ implies that $X$ is the smallest Nash set in $\mathbb{R}^3$ containing $\Gamma_f$. Therefore, by [8, Prop. 2.1], if $\Gamma_f$ were Nash constructible then it would need to contain the smooth locus of $X$. This is not the case, however, because $X$ contains also the graph of $g(x, y) = -\sqrt{x^4 + y^4}$.

The following result is new, though it follows easily from [8].

Lemma 2.4. Let $n \geq 1$ and let $f, g \in \mathcal{A}_a(\mathbb{R}^n)$. If the zero locus of $g$ is nowhere-dense in $\mathbb{R}^n$ and the function $f/g$ extends continuously to $\mathbb{R}^n$, then this extension is in $\mathcal{A}_a(\mathbb{R}^n)$.

Proof. By Proposition 2.1 above, there is a finite sequence $\pi : \tilde{R} \to \mathbb{R}^n$ of blowings-up with smooth algebraic centers such that $f \circ \pi$ and $g \circ \pi$ are Nash functions on the Nash manifold $\tilde{R}$. Continuity of $f/g$ implies that
(f \circ \pi)/(g \circ \pi) : \tilde{R} \to \mathbb{R} is a Nash regulous function. By [8, Prop. 3.1], Nash regulous functions are arc-analytic, and hence there is a finite sequence \( \sigma : \tilde{R} \to \tilde{R} \) of blowings-up with smooth algebraic centers such that \((f/g) \circ \pi \circ \sigma = f \circ \pi = g \circ \pi \circ \sigma : \tilde{R} \to \mathbb{R} is Nash, by Proposition 2.1 again. Therefore, \( f/g \) is arc-analytic. \(\square\)

3. Examples

Example 3.1. Consider the equation

\[
(3.1) \quad x^3 y \varphi_1 + (x^3 - y^3 z) \varphi_2 = x^4.
\]

We claim that

\[
\varphi_1(x, y, z) = z^{1/3}, \quad \varphi_2(x, y, z) = \frac{x^3}{x^2 + xyz^{1/3} + y^2 z^{2/3}}
\]

is a continuous solution to (3.1), but no semialgebraic arc-analytic solution exists. The function \( \varphi_1 \) is clearly continuous. To see that \( \varphi_2 \) is continuous, first note that the set

\[
\{(x, y, z) \in \mathbb{R}^3 : x^2 + xyz^{1/3} + y^2 z^{2/3} = 0\}
\]

is the union of the \( y \)-axis and the \( z \)-axis. Therefore, \( x \to 0 \) whenever \( (x, y, z) \) approaches the locus of indeterminacy of \( \varphi_2 \). On the other hand, we have

\[
x^2 + xyz^{1/3} + y^2 z^{2/3} \geq \frac{1}{2} (x^2 + y^2 z^{2/3}),
\]

which shows that \( \frac{x^2}{x^2 + xyz^{1/3} + y^2 z^{2/3}} \) is bounded. Hence, \( \varphi_2 \) can be continuously extended by zero to \( \mathbb{R}^3 \).

Suppose now that (3.1) has an arc-analytic solution \( (\psi_1, \psi_2) \). Set

\[
S := \{(x, y, z) \in \mathbb{R}^3 : x^3 = y^3 z\},
\]

and note that \( y \) vanishes on \( S \) only when \( x \) does so. Therefore, \( x/y \) is a well defined function on \( S \setminus \{x = 0\} \), and thus, by (3.1), we obtain that

\[
\psi_1|_{S \setminus \{x = 0\}} = \frac{x}{y}|_{S \setminus \{x = 0\}}.
\]

Note that every point \((0, 0, c)\) of the \( z \)-axis can be approached within \( S \setminus \{x = 0\} \), even by an analytic arc. Indeed, for instance, by the arc \((\sqrt[3]{c}t, t, c)\) for \( c \neq 0 \) and the arc \((t^2, t, t^3)\) for \( c = 0 \). This allows us to write

\[
\lim_{(x,y,z) \to (0,0,c)} \psi_1(x, y, z) = \lim_{(x,y,z) \to (0,0,c)} \frac{x}{y}|_{S \setminus \{x = 0\}} = c^{1/3}.
\]

Therefore, \( \psi_1|_{z-axis} = z^{1/3} \), by continuity. This contradicts the arc-analyticity of \( \psi_1 \), by Remark 2.2. \(\square\)
Example 3.2. Consider now the equation

\[(3.2) \quad x^4y^2 \varphi_1 + (x^4 - y^4(z^4 + w^4)) \varphi_2 = x^6. \]

We claim that

\[ \varphi_1 = \sqrt{z^4 + w^4}, \quad \varphi_2 = \frac{x^4}{x^2 + y^2 \sqrt{z^4 + w^4}} \]

is an arc-analytic solution to (3.2), but no Nash regulous solution exists. It is easy to see that the function \(\sqrt{z^4 + w^4}\) is blow-Nash, and hence arc-analytic, by Proposition 2.1. Thus, by Lemma 2.4, to see that \(\varphi_2\) is arc-analytic, it suffices to show that it extends continuously to \(\mathbb{R}^4\). First, note that the set

\[ \{ (x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 \sqrt{z^4 + w^4} = 0 \} \]

is the union of the \(y\)-axis and the \((z, w)\)-plane. Therefore, \(x \to 0\) whenever \((x, y, z, w)\) approaches the locus of indeterminacy of \(\varphi_2\). On the other hand, the function \(\frac{x^2}{x^2 + y^2 \sqrt{z^4 + w^4}}\) is clearly bounded. Hence, \(\varphi_2\) can be continuously extended by zero to \(\mathbb{R}^4\).

Suppose now that (3.2) has a Nash regulous solution \((\psi_1, \psi_2)\). Set

\[ S := \{ (x, y, z, w) \in \mathbb{R}^4 : x^4 = y^4(z^4 + w^4) \}, \]

and note that \(y\) vanishes on \(S\) only when \(x\) does so. Therefore, \((x/y)^2\) is a well defined function on \(S \setminus \{ x = 0 \}\), and thus, by (3.2), we obtain that

\[ \psi_1|_{S \setminus \{ x = 0 \}} = \frac{x^2}{y^2}|_{S \setminus \{ x = 0 \}}. \]

Note that the \((z, w)\)-plane is contained in \(S\), and every point \((0, 0, c, d)\) of the \((z, w)\)-plane can be approached within \(S \setminus \{ x = 0 \}\), even by an analytic arc. Indeed, for instance, by the arc \((\sqrt{c^4 + d^4}t, t, c, d)\) for \(c^4 + d^4 \neq 0\) and the arc \((\sqrt{2t^2}, t, t, t)\) for \(c^4 + d^4 = 0\). This allows us to write

\[ \lim_{(x, \ldots, w) \to (0, 0, c, d)} \psi_1(x, y, z, w) = \lim_{(x, \ldots, w) \to (0, 0, c, d)} \frac{x^2}{y^2}|_{S \setminus \{ x = 0 \}} = \sqrt{c^4 + d^4}. \]

Therefore, \(\psi_1|_{(z, w)\text{-plane}} = \sqrt{z^4 + w^4}\), by continuity. This is impossible for a Nash regulous function though, because by [8, Cor. 3.2] the graph of a Nash regulous function (and hence its intersection with any coordinate plane) is a closed Nash constructible set. However, the graph of \(f(z, w) = \sqrt{z^4 + w^4}\) is not Nash constructible, by Remark 2.3.

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