ON THE RELATIVE WEAK ASYMPTOTIC HOMOMORPHISM PROPERITY FOR TRIPLES OF GROUP VON NEUMANN ALGEBRAS

PAUL JOLISSAINT

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Abstract. A triple of finite von Neumann algebras \( B \subset N \subset M \) is said to have the relative weak asymptotic homomorphism property if there exists a net of unitaries \( (u_i)_{i \in I} \subset U(B) \) such that

\[
\lim_{i \in I} \| E_B(xu_iy) - E_B(E_N(x)u_iE_N(y)) \|_2 = 0
\]

for all \( x, y \in M \). Recently, J. Fang, M. Gao and R. Smith proved that the triple \( B \subset N \subset M \) has the relative weak asymptotic homomorphism property if and only if \( N \) contains the set of all \( x \in M \) such that \( Bx \subset \bigoplus_{i=1}^n x_iB \) for finitely many elements \( x_1, \ldots, x_n \in M \). Furthermore, if \( H < G \) is a pair of groups, they get a purely algebraic characterization of the weak asymptotic homomorphism property for the pair of von Neumann algebras \( L(H) \subset L(G) \), but their proof requires a result which is very general and whose proof is rather long. We extend the result to the case of a triple of groups \( H < K < G \), we present a direct and elementary proof of the above-mentioned characterization, and we introduce three more equivalent conditions on the triple \( H < K < G \), one of them stating that the subspace of \( H \)-compact vectors of the quasi-regular representation of \( H \) on \( \ell^2(G/H) \) is contained in \( \ell^2(K/H) \).

1. Introduction

Let \( 1 \in B \subset N \subset M \) be a triple of finite von Neumann algebras endowed with a fixed, normal, finite, faithful and normalized trace \( \tau \). Then \( E_N \) (resp. \( E_B \)) denotes the \( \tau \)-preserving conditional expectation from \( M \) onto \( N \) (resp. \( B \)); we also set \( M \cap N = \{ x \in M : E_N(x) = 0 \} \).

Following [3], we say that the triple \( B \subset N \subset M \) has the relative weak asymptotic homomorphism property if there exists a net of unitaries \( (u_i)_{i \in I} \subset U(B) \) such that, for all \( x, y \in M \),

\[
\lim_{i \in I} \| E_B(xu_iy) - E_B(E_N(x)u_iE_N(y)) \|_2 = 0.
\]

Using the identity

\[
E_B(xu y) - E_B(E_N(x)uE_N(y)) = E_B([x - E_N(x)]u[y - E_N(y)]),
\]

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1393
which holds for every $u \in U(B)$ and all $x, y \in M$, it is readily seen that the triple $B \subset N \subset M$ has the relative weak asymptotic homomorphism property if and only if one can find a net $(u_i)_{i \in I} \subset U(B)$ such that

$$\lim_{i \in I} \|E_B(xu_i,y)\|_2 = 0$$

for all $x, y \in M \ominus N$.

The one-sided quasi-normalizer of $B$ in $M$ is the set of elements $x \in M$ for which there exist finitely many elements $x_1, \ldots, x_n \in M$ such that $Bx \subset \sum_{i=1}^n x_i B$. It is denoted by $qN^{(1)}_M(B)$.

Inspired by [2], the authors of [3] prove in Theorem 3.1 that the triple $B \subset N \subset M$ has the relative weak asymptotic homomorphism property if and only if $qN^{(1)}_M(B) \subset N$. Furthermore, they also study the case of group algebras that we recall now.

Let $G$ be a discrete group and let $H$ be a subgroup of $G$. Then there is a natural analogue of the one-sided quasi-normalizer for such a pair of groups: we denote by $qN^{(1)}_G(H)$ the set of elements $g \in G$ for which there exist finitely many elements $g_1, \ldots, g_n \in G$ such that $Hg \subset \bigcup_{i=1}^n g_i H$.

Thus, if $H < K < G$ is a triple of groups and if $B = L(H) \subset N = L(K) \subset M = L(G)$ denotes the triple of von Neumann algebras associated to $H < K < G$, then it is reasonable to ask whether $B \subset N \subset M$ has the relative weak asymptotic homomorphism property if and only if $qN^{(1)}_G(H) \subset K$. Corollary 5.4 in [3] states that this is indeed true when $K = H$, but the proof presented there relies heavily on the main theorem of the article. It is thus natural to look for a more direct and elementary proof of the above-mentioned result, and the aim of the present paper is to provide such a proof and to add three more equivalent conditions.

2. The main result

Before stating our result, let us fix some additional notation. For each element $g \in G$ we denote by $\lambda_g$ the unitary operator acting by left translation on $\ell^2(G)$, i.e. $(\lambda_g \xi)(g') = \xi(g^{-1}g')$ for every $\xi \in \ell^2(G)$ and every $g' \in G$. We denote also by $L_f(G)$ the subalgebra of all elements of $L(G)$ with finite support; i.e., $L_f(G)$ is the linear span of $\lambda(G)$ in $B(\ell^2(G))$.

We fix a triple of groups $H < K < G$ for the rest of the article.

Let $\pi$ denote the quasi-regular representation of $G$ on $\ell^2(G/H)$; we denote by $[g]$ the equivalence class $[g] = gH$, so that $\pi([g])\xi([g']) = \xi([g^{-1}g'])$ for all $g, g' \in G$ and $\xi \in \ell^2(G/H)$. Following [1], we say that a vector $\xi \in \ell^2(G/H)$ is $H$-compact if the norm closure of its $H$-orbit $\{\pi(h)\xi : h \in H\}$ is a compact subset of $\ell^2(G/H)$.

The set of all $H$-compact vectors is a closed subspace of $\ell^2(G/H)$ that we denote by $\ell^2(G/H)_{c,H}$. We also set $\ell^2(G/H)^H = \{\xi \in \ell^2(G/H) : \pi(h)\xi = \xi \forall h \in H\}$, which is the subspace of all $H$-invariant vectors of $\ell^2(G/H)$. It is contained in $\ell^2(G/H)_{c,H}$.

**Theorem 2.1.** Let $H < K < G$ and $B = L(H) \subset N = L(K) \subset M = L(G)$ be as above. Then the following conditions are equivalent:

1. There exists a net $(h_i)_{i \in I} \subset H$ such that, for all $x, y \in M \ominus N$, one has
   $$\lim_{i \in I} \|E_B(xh_i,y)\|_2 = 0;$$
i.e., the net of unitaries in the relative weak asymptotic homomorphism property may be chosen in the subgroup $\lambda(H)$ of $U(B)$. 
(2) The triple $B \subset N \subset M$ has the relative weak asymptotic homomorphism property.
(3) If $g \in G$ and $F \subset G$ finite are such that $Hg \subset FH$, then $g \in K$, i.e. $qN_G^{(1)}(H) \subset K$.
(4) The subspace of $H$-compact vectors $\ell^2(G/H)_{c,H}$ is contained in $\ell^2(K/H)$.
(5) The subspace of $H$-fixed vectors $\ell^2(G/H)^H$ is contained in $\ell^2(K/H)$.
(6) For every nonempty finite set $F \subset G \setminus K$, there exists $h \in H$ such that $FhF \cap H = \emptyset$.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Observe that condition (3) is equivalent to the following statement (since, if $g \notin K$, then $HgH \cap K = \emptyset$):

For every $g \in G \setminus K$, and for every nonempty finite set $F \subset G \setminus K$, there exists $h \in H$ such that $Fhg \cap H = \emptyset$.

Thus, let us assume that condition (3) does not hold. There exists $g \in G \setminus K$ and a nonempty finite set $F \subset G \setminus K$ such that $Fhg \cap H \neq \emptyset$ for every $h \in H$. Then let $u \in U(B)$. One has:

$$\sum_{g' \in F} \|\mathbb{E}_B(\lambda_{g'}u\lambda_g)\|^2 = \sum_{g' \in F} \left( \sum_{h \in H, g'h \in H} |u(h)|^2 \right)$$

$$= \sum_{h \in H} \left( \sum_{g' \in F, g'h \in H} |u(h)|^2 \right)$$

$$\geq \sum_{h \in H} |u(h)|^2 = \|u\|^2 = 1$$

since, for every $h \in H$, one can find $g'(h) \in F$ such that $g'(h)h \in H$. Hence there cannot exist a net $(u_t)_{t \in T} \subset U(B)$ as above, and the triple $B \subset N \subset M$ does not have the relative weak asymptotic homomorphism property.

(3) $\Rightarrow$ (4). We choose a set of representatives $T \ni e$ of left classes so that $G = \bigsqcup_{t \in T} tH$, and let $\xi \neq 0$ be an $H$-compact vector.

Let $s \in T$ be such that $\epsilon := |\xi([s])| > 0$. There exist then finitely many vectors $\xi_1, \ldots, \xi_n \in \ell^2(G/H)$ such that, for every $h \in H$, there exists $1 \leq j \leq n$ such that $\|\pi(h)\xi - \xi_j\| \leq \epsilon/2$. Set

$$F = \bigcup_{j=1}^n \{ t \in T : |\xi_j([t])| \geq \epsilon/2 \},$$

which is a finite set. Then we claim that $Hs \subset FH$. Indeed, if $h \in H$, let $t \in T$ be such that $[hs] = [t]$, and let $j$ be such that $\|\pi(h)\xi - \xi_j\| \leq \epsilon/2$. Then

$$\epsilon - |\xi_j([t])| = |\xi([s])| - |\xi_j([t])| \leq |\xi([s]) - \xi_j([hs])| \leq \|\pi(h)\xi - \xi_j\| \leq \epsilon/2;$$

hence $\epsilon/2 \leq |\xi_j([t])|$ and $t \in F$. Thus $Hs \subset FH$, and condition (3) implies that $s \in K$. This proves that $\xi \in \ell^2(K/H)$.

(4) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (6). Let us assume that the triple $H < K < G$ satisfies condition (5) but not (6). Then there exists a finite set $F = F^{-1} \subset G \setminus K$ such that $FhF \cap H \neq \emptyset$.
for every $h \in H$. Set

$$\xi = \sum_{g \in F} \delta_{[g]}.$$ 

Then $\xi \perp \ell^2(K/H)$, and one has for every $h \in H$:

$$\langle \pi(h)\xi, \xi \rangle = \sum_{g,g' \in F} \langle \delta_{[hg]}, \delta_{[g']} \rangle \geq 1$$

since the condition on $F$ implies that for every $h \in H$, there exist $g,g' \in F$ such that $hgH = g'H$. Let $C$ be the closed convex hull of $\{\pi(h)\xi : h \in H\}$. Then it is easy to see that $\langle \zeta, \xi \rangle \geq 1$ for every $\zeta \in C$. Let $\eta \in C$ be the vector with minimal norm. By its uniqueness, it is $H$-invariant and nonzero by the above observation. Thus, $\eta$ is supported in $K/H$ and orthogonal to $\ell^2(K/H)$ since $\xi$ is. This is the expected contradiction.

$(6) \Rightarrow (1)$. Let $I = \{F \subset G \setminus K : F \neq \emptyset, \text{ finite}\}$ be the directed set of all nonempty finite subsets of $G \setminus K$. Condition $(6)$ states that, for every $F \in I$, there exists $h_F \in H$ such that $Fh_F \cap H = \emptyset$. Let $x$ and $y$ in $L_f(G)$ satisfy $E_N(x) = E_N(y) = 0$. Then let $F_0 \in I$ be chosen so that the supports of $x$ and $y$ are contained in $F_0$. Then $x\lambda_{h_F}y = \sum_{g,g' \in F_0} x(g)y(g')\lambda_{gh_Fg'}$ for every $F \supset F_0$; thus $E_B(x\lambda_{h_F}y) = 0$ for every $F \supset F_0$. This proves that the triple $B \subset N \subset M$ satisfies condition $(1)$ by the density of $L_f(G)$ in $L(G)$. □

Remark 2.2. In the case of a pair of groups $H < G$, which corresponds to $H = K$, condition $(4)$ means that all $H$-invariant vectors in $\ell^2(G/H)$ are multiples of $\delta_{[e]}$, and this means that the unitary representation $\rho$ of $H$ on the subspace $\ell^2(G/H) \ominus \mathbb{C}\delta_{[e]}$ is ergodic in the sense of [1].

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Université de Neuchâtel, Institut de Mathématiques, Emile-Argand 11, 2000 Neuchâtel, Switzerland

E-mail address: paul.jolissaint@unine.ch