Quantum Droplets in Imbalanced Atomic Mixtures

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Quantum droplets are a quantum analogue to classical fluid droplets in that they are self-bound and display liquid-like properties — such as incompressibility and surface tension — though their stability is the result of quantum fluctuations. One of the major systems for observing quantum droplets is two-component Bose gases. To remain self-bound in free space, two-component droplets must retain a fixed ratio between the densities of the two fluids. The work presented here goes beyond this fixed density ratio by investigating spherical droplets in imbalanced mixtures. This work finds that these systems give rise to a droplet immersed in a background gas. By exploring the breathing modes of these imbalanced mixtures, it is found that the droplet can radiate particles into the surrounding gas, yielding an intricate superposition of simultaneously decaying collective oscillations.

I. INTRODUCTION

Quantum gases are highly controllable systems due in part to their diluteness, yielding a highly versatile platform for exploring quantum many-body physics [1]. For a system of bosons, a mean-field model for quantum gases is Gross-Pitaevskii (GP) theory [2, 3], which captures much of the physics of these systems, whilst neglecting beyond-mean-field effects such as quantum fluctuations [4, 5]. For a single-component homogeneous Bose gas stability is governed by the atomic interactions, with the gas becoming unstable to collapse for attractive interactions. This has been demonstrated experimentally by using a Feshbach resonance to tune the interactions from repulsive to attractive [6, 7]. For a two-component system, the stability depends on both the inter- and intra-species contact interactions, which can likewise be tuned via Feshbach resonances [8–10]. For repulsive intra-species interactions — to ensure stability of each individual component — and attractive inter-species interactions, mean-field theory once more predicts an unstable gas. However, this attractive collapse can be stabilised at high densities by quantum fluctuations [11] forming a self-bound quantum droplet; this has inspired work in so called Lee-Huang-Yang (LHY) fluids [12, 13], named for the correction describing quantum fluctuations to first-order. By tuning mean-field interactions between the components to vanish, the interactions of LHY fluids are described by quantum fluctuations alone [12, 13].

In free space quantum droplets exist in equilibrium with the vacuum and form a seemingly counter-intuitive dilute liquid-like state [11, 14]. This adds to the variety of properties that quantum gases can be used to investigate, as the majority of experiments are inherently within the gas phase — a property widely exploited in time-of-flight imaging [15] — whereas quantum droplets open up the field to explore the properties of quantum liquids, such as surface tension [11, 16, 17] and incompressibility [18, 19], within controllable experimental conditions.

Quantum droplets have been experimentally observed in two systems: (i) dipolar gases of Dy [20–22] and Er [23]; (ii) homonuclear 39K [19, 24–26] and heteronuclear, 41K-87Rb and 23Na-87Rb [27, 28], two-component mixtures. The underlying mechanism for droplet formation is the result of a vanishing mean-field interspecies attraction tending to cause instability, that is countered by the repulsive quantum fluctuations which become significant with increasing density [11]. However, the mean-field interactions within a dipolar Bose gas include both the two-body, short-ranged interactions of a non-dipolar gas, and anisotropic, long-ranged interactions resulting from strong atomic magnetic dipoles [29]. This anisotropy results in elongated droplet profiles [30–34], which differentiates dipolar droplets from non-dipolar, two-component droplets.

A defining property of three-dimensional (3D) two-component quantum droplets is self-evaporation [11]. In certain regimes of the droplets’ phase diagram, excitations will cause the droplet to shed atoms in order to relax to a system of lower energy. This occurs when the energies of these excitations exceed $-\mu$, the particle emission threshold, where $\mu$ is the chemical potential of the droplet. Furthermore, the greatest evaporative region is of the lowest energy monopole mode — the breathing mode — which decays for a large proportion of the droplets phase diagram [11, 35]. Self-evaporation is a non-intuitive and remarkable property that is not exhibited by dipolar quantum droplets [18], or for the breathing mode of a 1D two-component droplet [36], and has not been experimentally observed, but it is crucial in understanding the dynamics of these objects.

The formation of a two-component quantum droplet also has an interesting property, density balancing. A key result from the pioneering work of Petrov is the energetic favourability for the two component densities to maintain a fixed ratio $n_2/n_1 = \text{const.}$ [11] where $n_i$ is the

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number density of the \(i\)th component. By pairing this assumption with negligible spin modes [11] — i.e., assuming only in-phase density oscillations — the mixture can be modelled via a single macroscopic wavefunction. The majority of the literature has focused on such balanced droplets, with theoretical studies of imbalanced systems limited to dipolar mixtures [37, 38] and low-dimensions [39, 40]. This work is a systematic study of the ground states and breathing modes of 3D spherical imbalanced quantum droplets in homonuclear mixtures. Imbalancing the mixture yields a lower energy configuration than a balanced droplet, with a liquid-gas interface between the self-bound droplet and the unbound atoms [40].

This work begins with a discussion of the underlying theory in Section II. This theory is then applied in Sections III and IV to first isolate imbalanced droplet ground states and then to propagate these states in time, subject to an initial perturbation, to analyse the droplet breathing modes. Finally the main results and potential future research avenues will be discussed.

II. THE MODEL

A zero-temperature mixture of two weakly-interacting, dilute Bose gases can be described by the energy functional [11, 41]

\[
E = \int \left[ \frac{\hbar^2}{2m_1} |\nabla \Psi_1|^2 + \frac{\hbar^2}{2m_2} |\nabla \Psi_2|^2 + \frac{g_{11}}{2} |\Psi_1|^4 + \frac{g_{22}}{2} |\Psi_2|^4 + g_{12} |\Psi_1|^2 |\Psi_2|^2 + \frac{E_{\text{LHY}}}{V} \right] d^3r,
\]

in which \(m_i\) are the atomic masses of the \(i\)-th component, \(g_{ii}\) and \(g_{12}\) are the effective intra- and inter-species interaction strengths respectively, and are related to the intra- and inter-species scattering lengths by \(g_{ii} = 4\pi\hbar^2a_{ii}/m_i\) and \(g_{12} = 2\pi\hbar^2a_{12}(1/m_1 + 1/m_2)\). The first two terms of Equation (1) are the kinetic energy contributions, the next three terms describe the two-body interactions within and between the components. The final term is the LHY correction which, to first-order, describes the effects of quantum fluctuations on the condensate [42]. For a Bose-Bose mixture, the LHY correction takes the form [11]

\[
\frac{E_{\text{LHY}}}{V} = \frac{8}{15\pi^2} \left( \frac{m_1}{\hbar^2} \right)^{3/2} g_{11}^{1/2} g_{22} g_{12} f(z, u, x),
\]

where \(f\) is a function, defined in [16, 43], with arguments \(z = m_1/m_2, u = g_{12}^2/g_{11}g_{22}\) and \(x = g_{22}n_2/g_{11}n_1\). The function \(f(z, u, x)\) means that the LHY term can be computed analytically for homonuclear mixtures \((m_2 = m_1)\) [11], but it must be computed numerically for heteronuclear mixtures \((m_2 \neq m_1)\) [11, 16]. The function argument, \(u\), reduces to one by assuming that the mixtures lie at the critical point of attractive stability, i.e., \(g_{12}^2 = g_{11}g_{22} \implies u = 1\), removing the issue of complex contributions resulting from an unstable phonon mode [11, 44, 45]. Crucial to the formation of quantum droplets, quantum fluctuations stabilise the attractive mixture against collapse, and furthermore are ubiquitous in nature though often play a limited role in the physics of many quantum gas experiments as discussed in Section I.

The two component densities can each be related to a macroscopic wavefunction or order parameter, \(n_i = |\Psi_i|^2 = \Psi_i^* \Psi_i\). A variational approach, \(i\hbar \partial \Psi_i/\partial t = \delta E/\delta \Psi_i^*\) [4], can then be used to derive the coupled extended GP equations,

\[
i\hbar \frac{\partial \Psi_1}{\partial t} = \left[ -\frac{\hbar^2}{2m_1} \nabla^2 + \frac{4\pi\hbar^2a_{11}}{m} |\Psi_1|^2 + \frac{4\pi\hbar^2a_{12}}{m} |\Psi_2|^2 + \frac{128\sqrt{\pi}\hbar^2a_{11}}{3m} (a_{11} |\Psi_1|^2 + a_{22} |\Psi_2|^2)^{3/2} \right] \Psi_1,
\]

\[
i\hbar \frac{\partial \Psi_2}{\partial t} = \left[ -\frac{\hbar^2}{2m_2} \nabla^2 + \frac{4\pi\hbar^2a_{22}}{m} |\Psi_2|^2 + \frac{4\pi\hbar^2a_{12}}{m} |\Psi_1|^2 + \frac{128\sqrt{\pi}\hbar^2a_{22}}{3m} (a_{11} |\Psi_1|^2 + a_{22} |\Psi_2|^2)^{3/2} \right] \Psi_2.
\]

Finally, the dimensional scalings \(\bar{r} = \xi \tilde{r}, \tau = \tau \tilde{t}\) and \(\Psi_1 = \rho^{1/2} \tilde{\Psi}_1\) result in the dimensionless, equal-mass coupled extended GP equations,

\[
i\frac{\partial \Psi_1}{\partial \tau} = \left[ -\frac{1}{2} \nabla^2 + |\Psi_1|^2 + \eta |\Psi_2|^2 \right] \Psi_1 + \alpha (|\Psi_1|^2 + |\Psi_2|^2)^{3/2} \Psi_1,
\]

\[
i\frac{\partial \Psi_2}{\partial \tau} = \left[ -\frac{1}{2} \nabla^2 + |\Psi_2|^2 + \eta |\Psi_1|^2 \right] \Psi_2 + \alpha \beta^2 (|\Psi_1|^2 + |\Psi_2|^2)^{3/2} \Psi_2,
\]

in which the dimensionless parameters are

\[
\eta = \frac{a_{12}}{\sqrt{a_{11}a_{22}}},
\]

\(\alpha\) and \(\beta\) being defined in [84, 85].
\[ \alpha = \frac{32}{3} \left[ \frac{2}{3\pi \sqrt{a_{11} + \sqrt{a_{11}a_{22}}}} \right]^{1/2}, \quad \beta = \left( \frac{a_{22}}{a_{11}} \right)^{1/2}, \]

with dimensional parameters

\[ \xi = \sqrt{\frac{3}{8\pi}} \frac{\delta a}{\sqrt{a_{11}n_1^{(0)}}}, \quad \tau = \frac{3n}{8\pi \hbar} \frac{\delta a}{\sqrt{a_{11}n_1^{(0)}}}, \]

\[ \rho_1 = \frac{2}{3} \frac{\delta a}{\sqrt{a_{11}(\sqrt{a_{11}} + \sqrt{a_{22}})}}, \quad \rho_2 = \frac{2}{3} \frac{\delta a}{\sqrt{a_{22}(\sqrt{a_{11}} + \sqrt{a_{22}})}}, \]

where \( \delta a = a_{12} + \sqrt{a_{11}a_{22}} \) and \( n_1^{(0)} \) is the equilibrium density of component-1 for the balanced mixture. The expression of the equilibrium density is calculated in a homogeneous infinite system under the criterion of a vanishing pressure — i.e., the droplet in equilibrium with the vacuum — and takes the form [11]

\[ n_1^{(0)} = \frac{25\pi}{1024} \frac{a_{12} + \sqrt{a_{11}a_{22}}^2}{a_{11}^{3/2}a_{22}(\sqrt{a_{11}} + \sqrt{a_{22}})^5}. \]

By assuming a constant density ratio, \( n_2/n_1 = \sqrt{g_{11}/g_{22}} \), the two component wavefunctions can be expressed in terms of a single wavefunction, \( \Psi_i = \sqrt{N_i}\phi \), neglecting any out-of-phase motion between the components [11, 46, 47]. Equations (2) reduce to a single equation,

\[ i\frac{\partial \phi}{\partial t} = \left[ -\frac{1}{2} \nabla^2 - 3|\phi|^2 + \frac{5}{2}|\phi|^3 \right] \phi, \]

with the system described by a single parameter, an effective atom number, \( \tilde{N} \), given by [11]

\[ \tilde{N} = \left( \frac{\sqrt{g_{22}}}{n_1^{(0)}(\sqrt{g_{11}} + \sqrt{g_{22}})} \right) N \frac{3}{5}, \]

in which \( N \) is the total atom number \( N = N_1 + N_2 \). Thus for a given set of interaction strengths, \( g_{ii} \) and \( g_{ij} \), the atom numbers must maintain a fixed ratio, \( N_2/N_1 = \sqrt{g_{11}/g_{22}} \), imposing that the atoms can only reside bound to the droplet. By breaking the assumption of density-locking, excess atoms may be added to one of the components resulting in modified droplets surrounded by a non-zero density of unbound particles [39, 40]. The effect of the unbound atoms on the 3D spherical droplet structure can be seen in the ground state profiles of the spherical droplets considered here.

### III. GROUND STATES

This work considers spherically-symmetric droplets. Density is hence assumed to be a function of radius only, reducing the computational problem to an effective 1D system with the kinetic term becoming \( \nabla^2\Phi_i \rightarrow (\partial^2\Psi_i/\partial r^2)/r \). Further assumptions within this work include: a homonuclear mixture and balanced intra-species scattering lengths \( a_{11} = a_{22} \Rightarrow \beta = 1 \), to simplify the problem. Thus, the only differences between the components can arise from an imposed atom number imbalance of \( N_1 \neq N_2 \).

To find ground state solutions, the coupled GP equations are propagated numerically in imaginary time until the energy of the mixture is deemed adequately converged. The numerical scheme is a 4th-order Runge-Kutta method, with centred finite-difference methods for the kinetic energy derivatives. Finally, there are Neumann boundary conditions (\( \partial\Psi_i/\partial r = 0 \)) at the edges of the computational box as the density at the centre of the droplet and the far edge is approximately constant.

Figure 1(a) presents an example of an imbalanced droplet with the corresponding balanced solution. A
small computational box has been used to highlight the changes but the key point is that the central densities of the imbalanced components separate from the balanced densities. Notably, the minority component density is depleted within the droplet, whilst the majority component density profile has a non-zero density tail [at the right of the box in Figure 1(a)], and an increased density in the droplet core. This suggests that the majority component displaces some of the minority component from the centre towards the surface of the droplet, i.e., the region in which the balanced droplet density decreases to zero. The difference between the two densities is not a finite size effect. The inset of Figure 1(b) shows that the system chemical potential, \( \Gamma \), of the minority component, similar to submerged droplets [11]. Recalling that the droplet-gas mixture.

The fixed density ratio of a balanced droplet results from energy minimisation, but Figure 1(b) demonstrates that the energy of the mixture can be reduced by imbal-ancing the atom numbers. For increasing imbalance, Figure 1(b) shows that the system energy is reduced, until reaching the plateau discussed above. It is hence possible to lower the energy of the balanced mixture by adding atoms to one of the components, with the system reacting to this imbalance by splitting the central densities of the droplet and expelling a certain proportion of majority component atoms into a surrounding homogeneous gas.

A crude approximation can be given for this energy reduction, by assuming a balanced mixture and adding a constant density equivalent to the excess atoms, to one of the components. Thus, the total density of the majority component in this model takes the form \( n_{1}^{\text{mod}} = n_{1}^{\text{bal}} + \delta n_{1} \) where \( \delta n_{1} = \delta N_{1} / V \) for the spherical volume \( V = \frac{4}{3} \pi L_{r}^{3} \), and \( \delta N_{1} \) corresponding to the excess atoms \( \delta N_{1} = N_{1} - N_{2} \) for \( N_{1} \neq N_{2} \). This rudimentary argument is here applied to a 10% imbalance in a small box of \( L_{r} = 16 \), giving a dimensionless energy of \( E[n_{1}^{\text{bal}}, n_{2}] = -1.57 \times 10^{5} \) for the model of a balanced droplet plus a constant density. When compared to the balanced droplet, of energy \( E[n_{1}^{\text{bal}}, n_{2}] = -1.56 \times 10^{5} \), there is a slight reduction in energy though it is not as low as the calculated imbalanced ground states, \( E[n_{1}^{\text{imbal}}, n_{2}] = -1.59 \times 10^{5} \). There are numerous possibilities for why this model cannot recreate the energy of the truly imbalanced mixtures such as the different structures these ground states exhibit at, for example, the droplet boundary or potentially due to a greater subtlety in the proportion of expelled majority component density, as again seems to be the arguments in Ref [39].

Quantum droplets experiments are largely conducted in a vacuum [19, 25, 27, 28] and hence the unbound gas would be lost to the surrounding free space. To address this issue, absorbing boundary conditions can be applied to the far edge of the box \( (r = L_{r}) \) to remove the non-zero gas density. Through this process, the non-zero density tail can be gradually depleted leading to the convergence of the central densities \( (\Delta n \to 0) \), recovering the balanced droplet. Though stability in free space is a defining property of quantum droplets, imbalanced droplets are nonetheless experimentally observable. The unbound atoms of an imbalanced mixture would be retained by the trapping potentials. Early experimental evidence of this can be seen in homonuclear mixtures [26], alongside recent experimental observations of imbalanced heteronuclear droplets [48]. To accompany experimental research in this area, it is important to systematically investigate the influence that the unbound cloud has upon the droplet. For example the ground states, as shown in Figure 1, depend on both the droplet size (i.e. the value of \( N \)) and the magnitude of the imbalance, up to the plateau region. This indicates that experimentally there will be subtleties in exploring imbalanced droplets, as the size of the imbalance is a defining property of the droplet-gas mixture.

These ground states indicate a modified droplet surrounded by a gas of unbound atoms, that is of lower energy than the balanced analogue, but the next natural question is one of stability. Section IV will explore the dynamical stability of these modified droplet-gas mixtures.

IV. BREATHING MODE

Self-evaporation is a key property of 3D two-component quantum droplets [11]. Recalling that the density-locked droplet can be described by a single parameter, \( \tilde{N} \), there are three main regimes of interest for the droplet collective modes: 1) all modes exceed the particle emission threshold and hence are evaporated \( (\tilde{N} < 94.2) \); 2) the monopole mode evaporates but other non-zero angular momentum modes are stable \( (94.2 < \tilde{N} < 933.7) \); 3) the monopole is stabilised \( (\tilde{N} > 933.7) \) [11, 35]. This first phase is the principal
argument behind the self-evaporative property of a two-component droplet, since a finite-temperature droplet is inherently excited by contributions from, for example, the non-condensate components. Thus, these excitations are evaporated and the droplet relaxes to a lower energy configuration, akin to self-cooling [11].

In this work, the system is restricted to spherically symmetric droplets, meaning that the only observable mode is the breathing mode. The restriction of spherical symmetry is applied to reduce computational cost. This is due to particle shedding of self-evaporative droplets resulting in reflections from the computational box boundaries which become a substantial issue for long-time dynamics. To avoid this issue, very large box sizes — approximately 500 times the droplet sizes considered here — are used to observe the dynamics of the droplets without interference from reflected particles. If simulating a general 3D droplet, these box sizes would quickly become infeasible. By excluding any non-zero angular momentum modes, the spherical symmetry yields two regimes: a decaying and a stable breathing mode, thus this work analyses the stability of breathing modes in the presence of an imbalance.

The breathing mode frequencies of the balanced, density-locked droplet are a function of the single parameter, \( N \) [11]. However, for an imbalanced system, a further degree of freedom is introduced, the size of the imbalance. As shown in the previous section, the central density differences, chemical potentials and energies, as functions of percentage imbalance, reach a plateau, and any further imbalance has no significant effect on the structure of the droplet. Note again that analysis of the breathing mode frequencies here will only be considered up to this saturated limit. In order to observe these collective modes, the ground state solutions found via imaginary time propagation, shown in Section III, are then evolved in real time via Equation (3). To trigger a breathing mode in the droplet, a harmonic phase is imprinted on the ground state wavefunction, i.e., \( e^{i\epsilon r^2} \) where \( \epsilon \) takes a small value (here \( \epsilon = 10^{-5} \)) [4, 49]. This phase is always imprinted onto the minority component due to a mismatch between the non-zero density of the majority component at the boundaries, and the harmonic potential of the imprinted phase. However, for smaller imbalances, these boundary issues are not present due to the small non-zero density tails at the box edges. In these mixtures, no difference was found in triggering the mode in either component.

### A. Self-Evaporative Regime

The first modes to consider are within the self-evaporative regime (\( \tilde{N} < 933.7 \)). Breathing modes of spherical, balanced droplets have been quite extensively investigated [11, 35, 50]. Dynamically, within this regime, the droplet begins to oscillate at a frequency exceeding the particle emission threshold, \(-\mu\), and rapidly decays with the decay rate asymptotically tending to zero and the oscillation frequency tending to the particle emission threshold [50]. The initial rapid decay is due to an initial high dissipation of energy through particle emission, though in the long-time limit this corresponds to limited particle emissions at the energy of the chemical potential. This asymptotically decaying behaviour is likewise recovered in the imbalanced case through with
three modes instead of one due to the imbalance yielding two chemical potentials. Figure 2(a) describes the three super-imposed modes: the early-time, rapidly decaying mode — see purple dashed lines in Figure 2(a) denoting the extracted equal frequencies of the two components — that varies marginally with the percentage imbalance; at late times there are two further modes replacing the early-time mode. The late-time modes arise from the splitting of the chemical potentials which diverge with increasing percentage imbalance before again achieving the plateau [see orange lines in Figure 2(a)].

To visualise the single early mode and two late modes, Figure 2(b) and (c) represent a measure of the droplet central density, $\bar{n}_i(t) = n_i(r = 0, t) - \langle n_i(r = 0) \rangle_t$, with insets showing the associated power spectra $|\mathcal{F}[\bar{n}_i]|^2$ in which $\mathcal{F}[\cdot]$ denotes the power spectrum rescaled by the mean, and all negative frequencies set to zero purely for data visualisation. The early-time mode is a high-amplitude, rapidly decaying mode that is given in Figure 2(b), with an inset of the associated power spectrum highlighting the frequency (vertical dashed line) corresponding to the purple dashed lines in Figure 2(a). As this mode decays, it is then replaced by the two late-time modes corresponding to the split chemical potentials. The late-time dynamics is thus a superposition of a higher and lower frequency mode, as can be seen in Figure 2(c), once more with the associated power spectra showing peaks at the two chemical potentials (given by the vertical dashed lines). It should be noted that evidence of these late-time modes can even be seen at early times, such as the shorter peak in the inset of Figure 2(b), which roughly corresponds to the chemical potential of the majority component.

In summary, at early times the droplet oscillates with an unstable, high-amplitude mode that decays rapidly due to energy dissipation from particle emission. In the long-time limit however, the particle emission is considerably reduced, with particles only emitted at energies of the two chemical potentials. Hence, these late-time modes decay at much lower rates than the initial mode. This is analogous to the density-locked mixture [50], in which the mode frequency asymptotically converges to the chemical potential of each component given by the vertical dashed lines. Thus, for a sufficiently small imbalance, it is possible to have an imbalanced droplet with a stable breathing mode.

Figure 3(a) shows that, as in the self-evaporative case, the chemical potentials diverge with the imbalance until reaching the plateau. The dynamics are again dominated by a high-amplitude mode in both components — given by the central purple dashed lines of Figure 3(a) — that is relatively constant with the changing imbalance. In the balanced case, the stability of the breathing mode for droplets of this size is due to the particle emission threshold exceeding the mode frequency. With increasing imbalance however, as the chemical potentials split, the majority component chemical potential [i.e., the lower chemical potential branch in Figure 3(a)] eventually crosses over the frequency of the stable high-amplitude mode, at which point this mode will begin to decay. This mode crossing implies that the imbalanced non-self-evaporative regime can instead be split into two regions: 1) a stable breathing mode; 2) a decaying breathing mode. This behaviour is highlighted in the upper inset of Figure 3(a), focusing on smaller imbalances. Decay of the high-amplitude mode occurs when the frequency exceeds the negative of the majority component chemical potential. The lower left inset of Figure 3(a) shows that the mode is stable, for an imbalance of 0.2%, as the frequency lies beneath both chemical potentials. However, the lower right inset shows that this mode becomes unstable, for an imbalance of 0.8%, and decays as the frequency exceeds one of the chemical potential branches. The critical imbalance between stability and instability of this mode is $\approx 0.55\%$ here. Thus for a sufficiently small imbalance, it is possible to have an imbalanced droplet with a stable breathing mode.

Figures 3(b) and (c) focus on the early and late times of an unstable breathing mode in this regime. At early times, the initial high-amplitude mode dominates the system, and oscillates at approximately the frequency of the balanced case. This is highlighted in the power spectrum shown in the inset of Figure 3(b) with the frequency given by the vertical dashed line. As found in the self-evaporative regime, the energy of the droplet is dissipated through atom shedding which then leads to the evaporation of the initial mode. The oscillations at late times — shown in Figure 3(c) — are instead dominated by two other modes, with frequencies corresponding to the chemical potentials of each component given by the two outer vertical dashed lines. There are still residual oscillations from the decaying initial mode which hence explains the interference seen in the oscillations of Figure 3(c) and the central peak shown in the power spectrum. Thus, in both the self-evaporative and non-self-evaporative regimes, providing there is decay of the initial high-amplitude mode, there are two main regions of the dynamics: early times — where the dynamics is dominated principally by a high-amplitude mode that is re-

B. Non-Self-Evaporative Regime

The second set of breathing modes, are within the non-self-evaporative regime ($N > 933.7$). For a balanced droplet the breathing mode frequency is lower than the particle emission threshold resulting in a stable oscillation [11, 50]. Whereas in the self-evaporative regime, the dynamics of the droplet is highly reminiscent of the balanced case but with further modes corresponding to the split chemical potentials, the non-self-evaporative region of the droplet phase has some greater subtleties.
lated to the balanced droplet; late times — where there is a superposition of two modes corresponding to each chemical potential. All of these modes are decaying but over different timescales due to the rate of atom shedding. The initial mode decays relatively rapidly due to a high dissipation of energy from the particle emission. However, at late times the particle emission is considerably reduced, with slower emission of particles. This slower emission of particles, at energies of the two chemical potentials, is negligible relative to the droplet kinetic energy and hence the dynamics decay asymptotically [50].

V. DISCUSSION AND CONCLUSIONS

By solving the spherically symmetric, coupled generalised GP equations, this work has systematically investigated the spherical ground states and breathing modes of imbalanced droplets across both the size of the droplet and the imbalance. The ground states of these systems are not simply the associated balanced droplets with all excess atoms unbound in the surrounding gas. Instead the excess atoms modify the droplet core, by splitting the two central component densities, yielding a quantitatively different state that is dynamically stable and exhibits a superposition of breathing mode oscillations.

One of the main experimental probes to justify the observation of a quantum droplet is measuring a constant width of the atomic cloud, after switching off all traps, i.e., the object is self-bound in free space [25, 27, 28]. In this scenario any unbound gas would be lost and, as presented in the discussion of absorbing boundary conditions in Section III, the droplet recovers the fixed density ratio profiles. However, within these experimental scenarios, residual weak trapping is retained [25]. In weak trapping, the breathing mode frequencies and ground state profiles are relatively unaffected, and thus do not differ far from a droplet in free space [51]. Hence, with weak trapping and an imbalance, the excess atoms of an imbalanced mixture would be retained. This could be achieved in homonuclear mixtures, for which the number of atoms in each component can be tuned by radio-frequency pulses on the cloud [26], and there have been recent experimental observations of imbalanced heteronuclear mixtures [48]. However these system will likewise be affected by three- and higher-body losses.

Ideas for further theoretical works include studying extreme imbalanced cases, akin to the 1D work in Ref [40] in which the central density of the minority component becomes comparable to the density tails of the majority component. This would then go beyond the plateau region as the majority component chemical potential becomes positive. Another avenue is to investigate the spherically symmetric ground states and dynamics of a heteronuclear mixture, as the differing kinetic energy contributions of the two components may lead to novel dynamics. This is computationally challenging however, due to the two-component LHY correction for a heteronu-
clear mixture being no longer analytic [11, 16]. There are approximations for this term, though they are least accurate at the droplet centres [43]. This presents an issue for accurate analysis of imbalanced ground state solutions and dynamics, especially for the superimposed, time-dependent collective excitations shown here. As quantum droplet experiments continue to develop—for example through the realisation of new mixtures [52]—it will become increasingly important to understand how these mixtures can minimise their energy, from self-evaporating through the realisation of new mixtures [52]—it will continue to develop— for example.

The data presented in this paper are available [53].

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