Formulae for the Conjugate and the Subdifferential of the Supremum Function

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Abstract The aim of this work is to provide formulae for the subdifferential and the conjugate function of the supremum function over an arbitrary family of functions. The work is principally motivated by the case when data functions are lower semicontinuous proper and convex. Nevertheless, we explore the case when the family of functions is arbitrary, but satisfying that the biconjugate of the supremum functions is equal to the supremum of the biconjugate of the data functions. The study focuses its attention on functions defined in finite-dimensional spaces, in this case the formulae can be simplified under certain qualification conditions. However, we show how to extend these results to arbitrary locally convex spaces without any qualification condition.

Keywords convex analysis · ε-subdifferential · Fenchel conjugate · pointwise supremum function

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1 Introduction

Convex analysis has been one of the most studied topics in nonsmooth analysis; currently, chapters referring to convex analysis can be found in many monographic books related to optimization and variational analysis. The development of convex analysis has brought many tools to establish necessary and sufficient conditions for optimality. A remarkable tool in this scenario is the notion of the subdifferential of a convex function. Everybody interested in
nonsmooth analysis is aware of classical calculus rules and qualification conditions, together with practical algorithms, which involve this mathematical object.

Another important tool in (convex) subdifferential calculus is the so called \textit{approximate subdifferential of convex functions} (see, e.g., [1]). This operator has played a central role in optimization; its impact can be seen in applications such as linear programming, convex and non-convex programming, stochastic programming and semi-infinite programming among many other topics, because this mathematical object has been used to provide calculus rules without qualification conditions (see, e.g., [2,3]), characterize points of sub-optimality (see, e.g., [4]). It has been used in many minimization algorithms (see, e.g., [1]) and also can be found in connection with integration of subdifferentials (see e.g [5,6,7]). Also, the nonemptiness of this operator is guaranteed in arbitrary locally convex spaces under weaker conditions (see [4]) and it represents, in some sense, a continuous multifunction (see, e.g., [8]). All of these properties have motivated some authors to give generalizations of this object to vector optimization (see, e.g., [9,10] and the references therein).

A functional, which appears commonly in applications, is the supremum functional of a family of functions, for this reason many authors have shown several formulæ for the calculus of the subdifferential of the supremum function (see, e.g., [11,12,13,14,15,16] and the reference therein). Moreover in [13] the authors have shown that the supremum function appears to be fundamental in the theory of convex subdifferential calculus, in the sense that the general formula presented in [13, Theorem 4] allows the authors to recover classical and important formulæ available in the literature (see [13, Corollary 16], where the authors recover as a simple corollary of their general formulæ the calculus rules for the sum and composition). However, none of them have studied the \(\varepsilon\)-subdifferential of the supremum function. This observation motivates this work to provide calculus for the subdifferential together with formulæ for the conjugate function of the supremum function.

This work is organized as follows. In Section \ref{sec:prelim} we present some classical notation and definitions of convex analysis which agree with many monograph books (see, e.g., [4,17,18]). In Section \ref{sec:supremum} we give preliminary results concerning the calculus of the conjugate and the subdifferential of the supremum function. Section \ref{sec:conj} is devoted to studying formulæ for the conjugate function of the supremum function of an arbitrary family of (possibly non-convex) functions using the additional assumption that the index set is an ordered set and the functions are epi-pointed; this allows us to give formulæ for the conjugate of the lim sup and lim inf function. Finally, we divide Section \ref{sec:main} where the main results are established, in two subsections. Subsection \ref{sec:finite} is focused on when the functions are defined in a finite-dimensional space; in this scenario we investigate the subdifferential and the conjugate function of the pointwise supremum under the standard assumption that the normal cone of the domain of the supremum function at the point of interest does not contain lines. The main results about the calculus of the subdifferential of the supremum function is Theorem \ref{thm:finite} later under assumptions of compactness of the index set and
some continuity property, which are classical in semi-infinite programming, we derive Theorem 5.2. Posteriorly we get a result that can be understood as the conjugate counterpart of the last two theorems (see Theorem 5.3). Due to the fact that this research is principally motivated by [13], in this section we bypass the convexity of the data function by the weaker assumption that the biconjugate of the supremum function is equal to the supremum of the biconjugate of the data functions. This kind of hypothesis has been recently used in several works (see, e.g., [12,13,14,19,20]). Subsection 5.2 is motivated by the completeness of this work. In this subsection we show how to generalize the mentioned results (given in a finite-dimensional space) to a general locally convex space and without any qualification condition using the family of all finite-dimensional subspaces containing a given point.

2 Notation

Throughout the paper $X$ and $X^*$ will denote two (separated) locally convex spaces (lcs) in duality by the bilinear form $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$, given by $\langle x, x^* \rangle = x^*(x)$. In $X^*$ the weak topology is denoted by $w(X^*, X)$ ($w^*$ for short) and the Mackey topology on $X^*$ is denoted by $\tau(X^*, X)$, the space $X$ will be endowed with a compatible initial topology $\tau$ (i.e., $(X, \tau)^* = X^*$). The family of all closed convex and balanced neighborhoods of zero for the topology $\tau$ will be denoted by $N_0(\tau)$ and we omit the symbol $\tau$ when there is no confusion, similar terminology is adapted for the space $X^*$. We will write $\mathbb{R} := [-\infty, \infty]$ and we adapt the convention $\frac{\alpha}{0} = +\infty$ for all $\alpha > 0$. The symbol $\dim X$ means the dimension of the space $X$.

The closure of $A$ will be denoted by $\text{cl} A$. We denote by $\text{int}(A)$, $\text{conv}(A)$ and $\text{cl conv}(A)$, the interior, the convex hull and the closed convex hull of $A$, respectively. The polar of $A$ is the set $A^\circ := \{ x^* \in X^* : \langle x, x^* \rangle \leq 1, \forall x \in A \}$. When the space $X$ is finite-dimensional, the norm on $X$ and $X^*$ will be denoted by $\| \cdot \|$ and $\| \cdot \|_*$, respectively. Given $x \in X$ (or $x^* \in X^*$) and $r \geq 0$ we denote $B(x, r) := \{ y \in X : \| x - y \| \leq r \}$ ($B(x^*, r) := \{ y^* \in X^* : \| x^* - y^* \|_* \leq r \}$).

For a given function $f : X \to \mathbb{R}$, the (effective) domain and the epigraph of $f$ are

$$\text{dom } f := \{ x \in X : f(x) < +\infty \} \quad \text{and} \quad \text{epi } f := \{ (x, \lambda) \in X \times \mathbb{R} : f(x) \leq \lambda \},$$

respectively. We say that $f$ is proper if $\text{dom } f \neq \emptyset$ and $f > -\infty$, and inf-compact if for every $\lambda \in \mathbb{R}$ the set $\{ x \in X : f(x) \leq \lambda \}$ is compact. We denote $\mathcal{I}_0(X)$ the class of proper lower semicontinuous (lsc) convex functions on $X$. The Fenchel conjugate of $f$ is the function $f^* : X^* \to \mathbb{R}$ defined by

$$f^*(x^*) := \sup_{x \in X} \{ \langle x, x^* \rangle - f(x) \},$$

and the biconjugate of $f$ is $f^{**} := (f^*)^* : X \to \mathbb{R}$. The closed hull, the convex hull and the closed convex hull of $f$ are denoted by $\text{cl } f$, $\text{conv } f$ and $\text{cl conv } f$ respectively, and they are defined as the functions such that $\text{epi } \text{cl } f = \text{cl epigraph } f.$
epi \conv f = \conv epi f, \text{ and } epi \cl \conv f = \cl epi f \text{ respectively. We recall that } f^{**} \leq \cl \conv f \text{ and whenever } f^{**} \text{ is proper one has } f^{**} = \cl \conv f.

For } \varepsilon \geq 0, \text{ the } \varepsilon\text{-subdifferential (or the approximate subdifferential of convex functions) of } f \text{ at a point } x \in X, \text{ where it is finite, is the set } \\
\partial_{\varepsilon}\ f(x) := \{x^* \in X^*: \langle x^*, y - x \rangle \leq f(y) - f(x) + \varepsilon, \ \forall y \in X\};

\text{ if } f(x) \text{ is not finite, we set } \partial_{\varepsilon}\ f(x) := \emptyset. \text{ The special case } \varepsilon = 0 \text{ is the classical convex subdifferential, also called the Moreau-Rockafellar Subdifferential and is denoted by } \partial f(x).

The indicator and the support functions of a set } A (\subset X, X^*) \text{ are, respectively, } \\
\delta_A(x) := \begin{cases} 
0, & x \in A, \\
+\infty, & x \notin A, 
\end{cases} \quad \sigma_A := \delta_A^*.

The asymptotic cone of } A \text{ is defined by } A_\infty := \bigcap_{\varepsilon > 0} \cl ([0, \varepsilon]A) \text{ (see, e.g., } [21]). \text{ When the space } X \text{ is a Banach space the asymptotic cone is commonly defined as } A_\infty = \{d \in X : \exists s_k \searrow 0 \text{ and } x_k \in A \text{ such that } \lim s_k x_k = d\}. \text{ When the set } A \text{ is convex } A_\infty \text{ is also known as the recession cone; in this case it admits the following representation } \\
A_\infty = \{d \in X : x + \lambda d \in A \text{ for some } x \text{ and all } \lambda \geq 0\}.

We set \\
A^- := \{x^* \in X^*: \langle x^*, x \rangle \leq 0, \ \forall x \in A\},

A^\perp := \{x^* \in X^*: \langle x^*, x \rangle = 0, \ \forall x \in A\}

the negative dual cone, and the orthogonal subspace (or annihilator) of } A, \text{ respectively.}

The } \varepsilon\text{-normal set of } A \text{ at a point } x \text{ is defined and denoted by } \\
N^{\varepsilon}_A(x) := \partial_{\varepsilon}\delta_A(x).

Now let us introduce the class of epi-pointed functions. \text{ This class of functions has shown that it shares important properties with the class of convex function, we refer to } [6,7,22,23,24,25,26,27,28] \text{ and the references therein.}

**Definition 2.1** A function } f : X \to \mathbb{R} \text{ is said to be epi-pointed if } f^* \text{ is proper and } \tau(X^*, X)-\text{continuous at some point of its domain.}

As far as we know, this family of functions was introduced in finite dimensions in } [29]. \text{ However, the the above extension was introduced in } [25] \text{ with the name of Mackey epi-pointed functions.}

In a finite-dimensional Banach space } X \text{ the reader can understand the above property in terms of the pointedness of the epigraph of its asymptotic function, which is given by } \\
f^\infty(u) := \liminf_{s \searrow 0} s f(s^{-1} w),
or in terms of a \textit{coercive property}, which is the existence of \( x^* \in X, \; \alpha > 0 \) and \( r \in \mathbb{R} \) such that \( f(x) \geq \langle x^*, x \rangle + \alpha \| x \| + r \) for all \( x \in X \) (see, e.g., [23 Proposition 3.1]).

For a set \( T, \mathcal{P}_f(T) \) denotes the set of all \( F \subseteq T \) such that \( F \) is finite. We define
\[
f(x) := \sup_{t \in T} f_t, \quad \text{for } x \in X
\]
as the pointwise supremum of the family \( \{f_t : t \in T\} \). When \( T \) is a directed set ordered by \( \preceq \) (i.e., \( (T, \preceq) \) is an ordered set and for every \( t_1, t_2 \in T \) there exists \( t_3 \in T \) with \( t_1 \preceq t_3 \) and \( t_2 \preceq t_3 \)) we say that the family of functions is increasing provided that for all \( t_1, t_2 \in T \)
\[
t_1 \preceq t_2 \implies f_{t_1} \leq f_{t_2} \quad \text{(i.e., } f_{t_1}(x) \leq f_{t_2}(x), \; \forall x \in X\).
\]
For convenience we also use the notation \( t_2 \preceq t_2 \) iff \( t_1 \preceq t_2 \).

Following the standard notation (see, e.g., [11, 12, 13]), for a subset \( Z \) of \( \mathbb{R} \) let \( \bigwedge Z \) be the set of all \( (\lambda_t) \in Z^T \) such that \( \lambda_t \neq 0 \) for finitely many \( t \in T \). The support of \( \lambda \) is defined as \( \text{supp} \lambda := \{t \in T:\; \lambda_t \neq 0\} \). The \textit{generalized simplex} on \( T \) is the set \( \bigwedge(T) := \{\lambda \in [0, 1]^T : \sum_{t \in T} \lambda_t = 1\} \), also it will be convenient to denote \( \bigwedge^\varepsilon(T) := \{\lambda \in [0, \varepsilon]^T : \sum_{t \in T} \lambda_t = \varepsilon\} \). For a family of functions \( \{f_t\}_{t \in T} \subseteq \bigwedge \mathbb{R}^T \), a point \( \bar{x} \) and \( \varepsilon \geq 0 \), the set of \( \varepsilon \)-active indexes at \( \bar{x} \) is defined by \( T_\varepsilon(\bar{x}) := \{t \in T : f_t(\bar{x}) \geq f(\bar{x}) - \varepsilon\} \).

3 Preliminary Result

3.1 Basic Properties of the Conjugate and the \( \varepsilon \)-Subdifferential of the Pointwise Supremum

The next lemma shows some basic properties of the the pointwise supremum function.

**Lemma 3.1.** Let \( \{f_t : t \in T\} \) be an arbitrary family of functions and define\( h := \inf_{t \in T} f_t^* \). Then
(a) \( h^* = \sup f_t^* \), consequently if \( h^{**} \neq -\infty \), then \( \text{cl conv } h = h^{**} \). Therefore, if the functions \( f_t \in \Phi_0(X) \) and \( f \) is proper one has
\[
\text{epi } f^* = \text{cl conv } \bigcup_{t \in T} \text{epi } f_t^*.
\]
(b) \( \text{conv } h(x^*) = \text{conv } \{f_t^* : t \in T\}(x^*) \), where
\[
\text{conv } \{f_t^* : t \in T\}(x^*) := \inf \left\{ \sum_{t \in T} \lambda_t f_t(x_t^*) : \lambda \in \bigwedge(T) \text{ and } \sum_{t \in T} \lambda_t x_t^* = x^* \right\}.
\]
Moreover, the infimum can be taken only in \( \lambda \in \bigwedge(T) \) with
\[
\# \text{supp } \lambda \leq \min\{\dim X + 1, \#T\}.
\]
(c) If \( f^* \) is proper and \( f^{**} = \sup_{t \in T} f_t^* \), then \( f^* = \text{cl conv}\{f_t^* : t \in T\} \).

(d) If \( \{f_t : t \in T\} \) is an increasing family of functions, then \( \inf_{t \in T} f_t^* \) is convex.

(e) If \( f_t \leq f_s \) and \( f_t \) is epi-pointed and \( f_s^* \) is proper, then \( f_s \) is epi-pointed.

(f) For every \( t \in T \) we have that \( f_t^* \subseteq \text{dom } f^* \) and \( \text{int dom } f_t^* \subseteq \text{int dom } f^* \).

In addition, if \( f^* \) is proper, \( f^{**} = \sup_{t \in T} f_t^* \) and \( \{f_t : t \in T\} \) is an increasing family of epi-pointed functions one has

\[
\bigcup_{t \in T} \text{int dom } f_t^* = \text{int dom } f^* \quad \text{and} \quad \bigcup_{t \in T} \text{int epi } f_t^* = \text{int epi } f^* .
\]

**Proof**

(a) It is not difficult to see that \( h^* = \sup f_t^{**} \), then \( \text{cl conv } h = h^{**} \) whenever \( h^{**} \) is proper (see, e.g., [1, Theorem 2.3.4]). Moreover, when \( \{f_t\}_{t \in T} \subseteq \mathcal{P}_0(x) \) we have \( f_t^{**} = f_t \) for each \( t \in T \), in particular, \( h^* = f \). Then, using the fact that \( f^* \) is proper we have \( h^{**} \) is also proper and consequently

\[ \text{epi } f^* = \text{cl conv } \bigcup_{t \in T} \text{epi } f_t^* \quad \text{and} \quad f^* = \text{cl conv } \inf_{t \in T} f_t^* . \]

(b) First, \( \text{conv } h \leq \text{conv } \{f_t^* : t \in T\} \) due to the convexity of \( \text{conv } h \). Now, we notice that

\[ \text{epi}_h h = \bigcup_{t \in T} \text{epi}_h f_t^* , \tag{1} \]

where \( \text{epi}_h \) denotes the strict epigraph. It follows that for every \( x^* \in \text{dom conv } h \) and every and \( \epsilon > 0 \), the element \( (x^*, \alpha_\epsilon) \in \text{conv}(\text{epi } h) \), where

\[ \alpha_\epsilon := \begin{cases} \text{conv } h(x^*), & \text{if } h(x^*) \in \mathbb{R}, \\ \epsilon^{-1}, & \text{if } h(x^*) = -\infty . \end{cases} \]

Then, there exists \( \lambda_i \geq 0 \) and \( (x^*_i, \beta_i) \in \text{epi } h \) with \( i = 1, \ldots, p \) and \( \sum_{i=1}^p \lambda_i = 1 \) such that \( (x^*, \alpha_\epsilon) = \sum_{i=1}^p \lambda_i (x^*_i, \beta_i) \). Moreover, by [1] there exists \( t_i \in T \) such that \( (x^*_i, \beta_i + \epsilon) \in \text{epi}_h f_{t_i}^* \). It implies that

\[ \alpha_\epsilon + \epsilon \geq \sum_{i=1}^p \lambda_i f_{t_i}^* (x^*_i) \geq \text{conv } \{f_t^* : t \in T\} . \]

Since \( \alpha_\epsilon + \epsilon \) converges to \( \text{conv } h(x^*) \) as \( \epsilon \searrow 0 \) we conclude the equality. Moreover, when the space is finite-dimensional we can use Carathéodory’s Theorem to consider only \( \lambda \in \Delta(T) \) with \( \#\text{supp } \lambda \leq \dim X + 1 \).

(c) Since \( f^* \) is proper and \( h^{**} \geq f^* \) we have \( h^{**} \neq -\infty \), then by \([a]\) we get \( h^{**} = \text{cl conv } h \). Finally, let us prove that \( \text{cl conv } h \leq f^* \), indeed

\[ h^{**} = \sup_{x \in X} \{\langle . , x \rangle - h^*(x) \} = \sup_{x \in X} \{\langle . , x \rangle - f^{**}(x) \} = f^* . \]

Therefore \( f^* = h^{**} = \text{cl conv } h = \text{cl conv } \{f_t^* : t \in T\} \).
Because the family of functions is increasing, we have $\text{epi } f^*_t \subseteq \text{epi } f^*_s$ for $t \leq s$, therefore $\bigcup_{t \in T} \text{epi } f^*_t$ is convex and then $\inf_{t \in T} f^*_t$ is convex.

If $f_t \leq f_s$ one has $f^*_t \geq f^*_s$ and since $f^*_t$ is bounded in $\text{int}(\text{dom } f^*_t)$, so is $f^*_s$, which implies the continuity of $f^*_s$.

First, one has $\bigcup_{t \in T} \text{dom } f^*_t \subseteq \text{dom } f^*$ and $\bigcup_{t \in T} \text{epi } f^*_t \subseteq \text{epi } f^*$, consequently $\bigcup_{t \in T} \text{int } \text{dom } f^*_t \subseteq \text{int } \text{dom } f^*$ and $\bigcup_{t \in T} \text{int } \text{epi } f^*_t \subseteq \text{int } \text{epi } f^*$.

Now, by (c) and (d) one has

$$\text{epi } f^* = \text{cl}\left( \bigcup_{t \in T} \text{epi } f^*_t \right) = \text{cl} \left( \bigcup_{t \in T} \text{int } \text{epi } f^*_t \right)$$

and then $\text{dom } f^* \subseteq \text{cl} \left( \bigcup_{t \in T} \text{int } \text{dom } f^*_t \right)$. Fix $(u^*_0, \beta_0) \in \bigcup_{t \in T} \text{int } \text{epi } f^*_t$ and take $(x^*_0, \alpha_0) \in \text{int } f^*$; then there exists $\gamma > 0$ such that $(x, \alpha) := (x^*_0, \alpha_0) + \gamma((u^*_0, \beta_0)-(u^*_0, \beta_0)) \in \text{epi } f^*$, because $\bigcup_{t \in T} \text{int } \text{epi } f^*_t$ is a convex open dense subset in $\text{epi } f^*$ (see (2)) one has

$$\frac{1}{1+\gamma}(x, \alpha) + \frac{\gamma}{1+\gamma}(u^*_0, \beta_0) = (x^*_0, \alpha_0) \in \bigcup_{t \in T} \text{int } \text{epi } f^*_t.$$

Using the same argument one also can conclude $\bigcup_{t \in T} \text{int } \text{dom } f^*_t = \text{int } \text{dom } f^*$.

Remark 3.1 Lemma 3.1 (b) appears to be very simple. Nevertheless, it represents a key point in the entire work and it allows us to use a weaker condition than the lsc and convexity of the data function (i.e. $f^*_t \in \Gamma_0(X)$ for all $t \in T$).

In this work we use the hypothesis that $f^{**} = \sup_{t \in T} f_t^{**}$, which has been used recently by some authors (see [13, 29, 30]) and it is weaker than the hypothesis used in [13][14]. More precisely in [13] the authors use the hypothesis that $\{f_t\}_{t \in T}$ are proper convex functions satisfying $\text{cl } f = \sup_{t \in T} \text{cl } f_t$. Posteriorly in [14] the authors use the hypothesis that $\{f_t\}_{t \in T}$ are proper convex functions and $\text{cl } f(x) = \sup_{t \in T} \text{cl } f_t(x)$ for all $x \in \text{cl } (\text{dom } f)$. Recently, in [12] the authors presented an improvement of the last results, in this work the functions $\{f_t\}_{t \in T}$ are assumed to be proper convex functions and $\text{cl } f(x) = \sup \text{cl } f_t(x)$ for all $x \in \text{dom } f$ (see [12, Corollary 6]). It is worth mentioning that to prove Lemma 3.1(c) one can assume the weaker hypothesis that $f^*$ is proper and $f^{**}(x) = \sup_{t \in T} f_t^{**}(x)$ for all $x \in D$, where $D$ is a graphically dense subset of $\text{dom } f^{**}$ (which is satisfied under the assumption of [12, Corollary 6]). For the sake of simplicity, we keep in mind these sophisticated assumptions for a future work.

The next lemma is a slight extension of [23, Lemma 4.5] without the assumption of convexity.

Lemma 3.2 Let $g : X \to \mathbb{R}$ be an epi-pointed function. Consider $\eta > 0$ and $x \in X$ such that $\eta + g^{**}(x) > g(x)$. Then

$$\partial \eta g(x) = \text{cl} \{ \partial \eta g(x) \cap \text{int } \text{dom } g^* \}.$$ (3)
Consequently for every $\eta \geq 0$

$$
\partial_\eta g(x) = \bigcap_{\gamma > 0} \text{cl}^{\ast\ast} \left\{ \partial_{\eta+\gamma} g(x) \cap \text{int dom } g^* \right\}.
$$

(4)

**Proof** First, if $|g^{\ast\ast}(x)| = +\infty$ one has the empty set in both sides of (4). Now consider $g^{\ast\ast}(x) \in \mathbb{R}$ and take $\gamma > 0$ such that $\eta - \gamma + g^{\ast\ast}(x) > g(x)$. Because $g^{\ast\ast}$ is a proper convex and lsc function and $\gamma > 0$, one has $\partial_\gamma g^{\ast\ast}(x)$ is non-empty, and by [33, Lemma 4.5] we get $\partial_\gamma g^{\ast\ast}(x) \cap \text{int dom } g^* \neq \emptyset$.

Then taking $u^* \in \partial_\gamma g^{\ast\ast}(x) \cap \text{int dom } g^*$ one gets $u^* \in \partial_\gamma g(x)$, which implies, $\partial_\gamma g(x) \cap \text{int dom } g^* \neq \emptyset$. Since $\text{int dom } g^*$ is open, convex and dense in the convex set $\text{dom } g^*$, we have that

$$
\text{cl} \left\{ \partial_\gamma g(x) \cap \text{int dom } g^* \right\} = \partial_\gamma g(x) \cap \text{cl int dom } g^* = \partial_\gamma g(x).
$$

Finally, (4) follows noticing that if $\partial_\gamma g(x) \neq \emptyset$ we have $\gamma + \eta + g^{\ast\ast}(x) > g(x)$ and $\partial_\gamma g(x) = \bigcap_{\gamma > 0} \partial_{\eta+\gamma} g(x).$ $\square$

**Lemma 3.3** [3] Lemma 1.1 Let $h$ be an extended-real-valued convex function defined over $X^*$. Then, for all $r \in \mathbb{R}$,

$$
\{ x^* \in X^* : \text{cl } h(x^*) \leq r \} = \bigcap_{\gamma > 0} \text{cl} \{ x^* \in X^* : h(x^*) < r + \gamma \}.
$$

Moreover, if $r > \inf_{X^*} h$, then

$$
\{ x^* \in X^* : \text{cl } h(x^*) \leq r \} = \text{cl} \{ x^* \in X^* : h(x^*) < r \}.
$$

The following results give us a first representation of the $\varepsilon$-subdifferential of $f$, this result corresponds to a slight extension of [33] Theorem 2.6

**Proposition 3.1** Let \{t_1 : t \in T\} be an arbitrary family of functions such that $f^{\ast\ast} = \sup_T f_t^{\ast\ast}$. If $\varepsilon > \inf_{x^* \in X^*} \{ f^*(x^*) + f(x) - \langle x^*, x \rangle \}$, then

$$
\partial_\varepsilon f(x) = \text{cl} \left\{ \sum_{t \in \text{supp } T} \lambda_t \partial_{\eta_t} f_t(x) : \begin{array}{l}
\lambda \in \Delta(T), \eta_t \geq 0, \\
\sum_{t \in T} \lambda_t \cdot \eta_t \in \varepsilon + \text{[0,} \varepsilon \text{]} \text{ and } \\
\sum_{t \in T} \lambda_t \cdot f_t(x) > f(x) + \sum_{t \in T} \lambda_t \cdot \eta_t - \varepsilon
\end{array} \right\}.
$$

(5)

Consequently, for every $\varepsilon \geq 0$

$$
\partial_\varepsilon f(x) = \bigcap_{\gamma > \varepsilon} \text{cl} \left\{ \sum_{t \in \text{supp } T} \lambda_t \partial_{\eta_t} f_t(x) : \begin{array}{l}
\lambda \in \Delta(T), \eta_t \geq 0, \\
\sum_{t \in T} \lambda_t \cdot \eta_t \in \varepsilon + \text{[0,} \varepsilon \text{]} \text{ and } \\
\sum_{t \in T} \lambda_t f_t(x) \geq f(x) + \sum_{t \in T} \lambda_t \cdot \eta_t - \varepsilon
\end{array} \right\}.
$$

(6)

Moreover, when the space $X$ is finite-dimensional the infimum can be taken only $\lambda \in \Delta(T)$ with $\# \text{ supp } \lambda \leq \min \{ \text{dim } X + 1, \# T \}$.\]
Lemma 3.1(c),

Now let \( x^* \in X^* \) satisfying \( \text{conv}\{ f_i^* : t \in T \}\}(x^*) + f(x) < \langle x^*, x \rangle + \varepsilon \).

Since the right side of (5) and (6) are included in \( \partial \varepsilon \cdot x^* \) for \( t \in \supp \lambda \) such that \( \sum_{t \in T} \lambda_t x_t^* = x^* \) and

\[ \text{conv}\{ f_i^* : t \in T \}\}(x^*) + f(x) - \langle x^*, x \rangle \leq \sum_{t \in T} \lambda_t (f_i^*(x_t^*) + f_t(x) - \langle x_t^*, x \rangle) + \sum_{t \in T} \lambda_t (f(x) - f_t(x)) < \varepsilon, \]

then taking \( \eta_t := f_i^*(x_t^*) + f_t(x) - \langle x_t^*, x \rangle \) if \( t \in \supp \lambda \) and \( \eta_t := 0 \) if \( t \notin \supp \lambda \) one gets that \( x_t^* \in \partial \eta_t f_t(x) \) for every \( t \in \supp \lambda \), \( \sum_{t \in T} \lambda_t \cdot \eta_t \in [0, \varepsilon) \) and \( \sum_{t \in T} \lambda_t \cdot f_t(x) > f(x) + \sum_{t \in T} \lambda_t \cdot \eta_t - \varepsilon \), which concludes the proof of (3).

Finally, if \( \varepsilon \geq 0 \) is any real number such that \( \partial \varepsilon f(x) \neq \emptyset \) one has that \( \inf_{x^* \in X^*} \{ f^*(x^*) + f(x) - \langle x^*, x \rangle \} \leq \varepsilon \), consequently using (4) for \( \gamma > \varepsilon \) and the fact that \( \partial \varepsilon f(x) = \bigcap_{\gamma > \varepsilon} \partial \gamma f(x) \) we obtain
\[
\partial \varepsilon f(x) = \bigcap_{\gamma > \varepsilon} \text{cl} \left\{ \sum_{t \in T} \lambda_t \partial \eta_t f_t(x) : \begin{array}{l}
\lambda \in \Delta(T), \ \eta_t \geq 0, \\
\sum_{t \in T} \lambda_t \cdot \eta_t \in [0, \gamma) \text{ and} \\
\sum_{t \in T} \lambda_t \cdot f_t(x) > f(x) + \sum_{t \in T} \lambda_t \cdot \eta_t - \gamma 
\end{array} \right\}.
\]

\[ \square \]

3.2 Characterization of Global \( \varepsilon \)-Minimum of Pointwise Supremum

The intention of this subsection is to introduce the definition of \( \varepsilon \)-Robust infimum, which appears to be promising to characterize \( \varepsilon \)-minimum of the supremum function; we also refer a sufficient condition to guarantee the mentioned property. The introduction of this concept allows us to cover many ideas reflected in max-min theorems that have been broadly studied, this notation is motivated by the concept of Robust infimum or Decoupled Infimum used in subdifferential theory to get fuzzy calculus rules (see, e.g., [32–35]). To be more precise the reader can observe that when \( \varepsilon = 0 \) the definition below corresponds to a max-min equation, that is to say,
\[
\sup_T \inf_X f_t(x) = \inf_X \sup_T f_t(x).
\]

**Definition 3.1 (\( \varepsilon \)-Robust infimum)** We will say that the family of functions \( \{ f_t : t \in T \} \) has an \( \varepsilon \)-Robust infimum on \( B \subseteq X \) at \( \bar{x} \in B \) provided that
\[
f(\bar{x}) \leq \sup_{t \in T} \inf_{x \in B} f_t(x) + \varepsilon. \tag{7}
\]

The special case \( \varepsilon = 0 \) is simply called a Robust minimum.
The min-max problem has been studied in many papers with different states of generality, for this reason it is impossible to recall all the sufficient conditions that establish the interchange between supremum and infimum; we refer to \cite{1,3,5,7,8,9} for some results and further discussions. However, we establish the next lemma which guarantees this class of max-min results and fits perfectly with the framework of our study.

First we need the following result.

**Lemma 3.4** \cite{22} Lemma 4] Let \( X \) be a topological space and let \((f_\alpha)_{\alpha \in D}\) be a net of lsc proper functions defined on \( X \) such that
\[
\alpha, \beta \in D, \ \alpha \leq \beta \Rightarrow f_\alpha \leq f_\beta.
\]

For \((\varepsilon_\alpha)_{\alpha \in D} \searrow 0\), let \((x_\alpha)_{\alpha \in D}\) be a relatively compact net such that
\[
x_\alpha \in \varepsilon_\alpha\text{-argmin} f_\alpha \text{ for each } \alpha.
\]
Then
\[
\inf_{x \in X} \sup_{\alpha \in D} f_\alpha(x) = \sup_{\alpha \in D} \inf_{x \in X} f_\alpha(x),
\]
and every accumulation point of \((x_\alpha)\) is a minimizer of the function \(\sup_{\alpha \in D} f_\alpha\).

**Lemma 3.5** [Sufficient condition for robust local minimum] Let \((X, \tau)\) be a topological space and \(B \subseteq X\). Suppose \((T, \preceq)\) is a directed set and the family of \(\tau\)-lsc functions \(\{f_t : t \in T\}\) is increasing, \(B\) is \(\tau\)-closed and there exists some \(t_0\) such that \(f_{t_0}\) is \(\tau\)-incompact on \(B\). Then the family \(\{f_t : t \in T\}\) has a Robust minimum on \(B\).

**Proof** It is easy to see that \(\inf_{x \in B} \sup_{t \in T} f_t(x) \geq \sup_{t \in T} \inf_{x \in B} f_t(x) =: \eta\). For the opposite inequality let us assume that \(\eta < +\infty\). Consider \(\varepsilon_t \in (0, 1)\) and points \(x_t \in B\) such that \(\varepsilon_t \to 0\) and \(x_t\) belongs to \(\varepsilon_t\text{-argmin}_B \{f_t\}\), then
\[
\inf_{x \in B} f_t(x) \geq \inf_{x \in B} f_{t_0}(x) > -\infty \text{ and } x_t \in \{x : f_{t_0}(x) \leq \eta + 1\} \text{ for all } t \geq t_0,
\]
so the set \((x_t)\) is relatively compact, and by Lemma \ref{3.4} we get the result. \(\Box\)

We finish this section with a very simple result of the \(\varepsilon\)-robust infimum in terms of the \(\varepsilon\)-subdifferential of the initial data.

**Proposition 3.2** Let \(\{f_t : t \in T\}\) be a family of functions which has an \(\varepsilon\)-Robust infimum on \(B \subseteq X\) at \(x\), then
\[
0 \in \bigcap_{\gamma > 0} \bigcup_{t \in T} \partial_{\varepsilon + \gamma} (f_t + \delta_B)(x).
\]
(8)

The condition is also sufficient if \(f(x) = f_t(x)\) for all \(t \in T\).

**Proof** Let \(x\) such that \(f(x) \leq \sup_{t \in T} \inf_{y \in B} f_t(y) + \varepsilon\), then for a given \(\gamma > 0\) one can take \(t \in T\) such that \(f_t(x) \leq f(x) \leq \inf_{y \in B} f_t(y) + \varepsilon + \gamma\), that is, \(0 \in \partial_{\varepsilon + \gamma} (f_t + \delta_B)(x)\). Conversely, if \(f(x) = f_t(x)\) for all \(t \in T\) and \(0\) belongs to \(\bigcap_{\gamma > 0} \bigcup_{t \in T} \partial_{\varepsilon + \gamma} (f_t + \delta_B)(x)\), one has that for every \(\gamma > 0\) there exists some \(t \in T\) such that \(f_t(x) = f(x) \leq f_t(y) + \varepsilon + \gamma\) for all \(y \in B\). Then, \(f(x) \leq \sup_{t \in T} \inf_{y \in B} f_t(y) + \varepsilon + \gamma\), so the arbitrariness of \(\gamma\) gives the result. \(\Box\)
4 Pointwise Supremum Function of an Ordered Family of Epi-pointed Functions

In this section, we investigate the \(\varepsilon\)-subdifferential of the pointwise supremum under the assumption that the data is epi-pointed and the set \(T\) is a directed set. This extra hypothesis allows us to give better estimations for the conjugate function of the supremum function as well as formulae for the \(\limsup\) and \(\liminf\) functions. It is worth mentioning that the property of being a directed set can be satisfied using the family \(P_f(T)\) and the family of functions defined by \(g_A(\cdot) := \max_{x \in A} f_x\), then \(f(x) = \sup_{A \in P_f(T)} g_A(\cdot)\). Moreover the epi-pointed property can be also obtained using an appropriate perturbation of the functions \(f_t\) as will be used in some of the proofs.

**Theorem 4.1** Let \(\{f_t : t \in T\}\) be an increasing family of epi-pointed functions such that \(f^{\ast}\) is proper and \(f^{\ast\ast} = \sup_{t \in T} f^{\ast}\). Then for every \(x^\ast \in \text{int dom } f^{\ast}\) one has

\[
f^{\ast}(x^\ast) = \inf \{ f^{\ast}_t(x^\ast) : t \in T \}. \tag{9}
\]

**Proof** Let \(x_0^\ast \in \text{int dom } f^{\ast}\), by Lemma 3.1 (a) \(f^{\ast}(x_0^\ast) \leq \inf \{ f^{\ast}_t(x_0^\ast) : t \in T \}\). Now, using Lemma 3.1 (c) and (d) we have that \(f^{\ast}\) is the closed hull of the function \(\inf \{ f^{\ast}_t(\cdot) : t \in T \}\), then one can take \(x_0^\ast \to x_0^\ast\) such that

\[
f^{\ast}(x_0^\ast) = \lim_{\alpha} \inf \{ f^{\ast}_t(x_0^\ast) : t \in T \}, \tag{10}
\]

besides by Lemma 3.1 (f) \(x_0^\ast \in \text{int dom } f_{t_0}\) for some \(t_0 \in T\). Because the family of epi-pointed functions \(\{f^{\ast}_s\}_{s \geq t_0}\) is decreasing, they are (uniformly) bounded on \(x_0^\ast + U_{t_0}\) (recall that \(x_0^\ast \in \text{int dom } f^{\ast}_{t_0}\) and \(f^{\ast}\) is continuous at \(x_0^\ast\), then by [26] Theorem 2.2.11) we can find a neighbourhood \(U_{t_0} \in N_0, K_{t_0} > 0\) and a continuous seminorm \(\rho\) such that

\[
|f^{\ast}_s(x^\ast) - f^{\ast}_s(y^\ast)| \leq K_{t_0} \rho(x^\ast - y^\ast) \text{ for all } x, y \in x_0^\ast + U_{t_0}, \text{ for all } s \geq t_0. \tag{11}
\]

Now let \(\gamma_\alpha \to 0\) and \(t_\alpha \in T\) with \(t_\alpha \geq t_0\) such that \(\gamma_\alpha + f^{\ast}(x_0^\ast) \geq f^{\ast}_s(x_0^\ast)\) (recall (11)), then using (11) one yields

\[
\gamma_\alpha + f^{\ast}(x_0^\ast) \geq -K \rho(x_0^\ast - x_0^\ast) + f_{t_\alpha}(x_0^\ast) \geq -K \rho(x_0^\ast - x_0^\ast) + \inf \{ f^{\ast}_t(x^\ast) : t \in T \}. \tag{11}
\]

Then taking the limits we conclude the result. \(\square\)

Now based on Theorem 4.1 together with classical calculus rules for the conjugate of the sum of convex functions we give one answer to the question proposed in [10] Question 3.6.

For this purpose we introduce the set \(\ell^1(\mathbb{N}, X^\ast)\) as the set of all sequences \((x_n^\ast) \in X^\ast\) such that \(\sum_{n \in \mathbb{N}} |\langle x_n^\ast, x \rangle| < +\infty\) for every \(x \in X\) and the linear functional \(\langle x^\ast, \cdot \rangle := \sum_{n \in \mathbb{N}} \langle x_n^\ast, \cdot \rangle\) belongs to \(X^\ast\) (see [28] for more details).
Corollary 4.1 Consider \((x^*_n) \in l^1(N, X^*)\), \((\alpha_n) \in l^1(N, \mathbb{R})\) and a sequence of functions \(f_n \in \Gamma_0(X)\) such that

\[ f_n(x) \geq \langle x^*_n, x \rangle + \alpha_n \ \forall x \in X, \ \forall n \in \mathbb{N}. \]

Consider \(f(x) := \sum_{n \in \mathbb{N}} f_n(x)\). If \(f\) and \(f_n\) are epi-pointed functions, then, for every \(x^* \in \text{int dom } f^*\) one has

\[ f^*(x^*) = \inf \left\{ \sum_{n=1}^{N} f^*_n(x^*_n) : N \in \mathbb{N} \text{ and } \sum_{n=1}^{N} x^*_n = x^* \right\} \quad (12) \]

\[ = \inf \left\{ \sum_{n \in \mathbb{N}} f^*_n(x^*_n) : \sum_{n \in \mathbb{N}} x^*_n = x^* \right\}. \quad (13) \]

Proof We (may) assume \(f_n \geq 0\) for all \(n \in \mathbb{N}\) (otherwise we can consider \(f_n := f_n - x^*_n - \alpha_n\)). Hence, \(f_n^*(0) \leq 0\) for all \(n \in \mathbb{N}\). Thus, \(f = \sup_{n \in \mathbb{N}} g_n\), where \(g_n := \sum_{k \leq n} f_k\). First we recall (see, e.g., [4, Corollary 2.3.5]) that for every \(n \in \mathbb{N}\) one has

\[ (g_n)^*(x^*) = cl \{w^* \triangledown_{k=1}^{n} f_k^*(x^*_k), \ \forall x^* \in X^*, \quad (14) \]

\[ (g_n)^*(x^*) = \{k=1}^{n} f_k^*(x^*_k), \ \forall x^* \in \text{int dom } f_k^*(\cdot), \quad (15) \]

where \(\{k=1}^{n} f_k^*(\cdot) = \inf \{f_1^*(x^*_1) + \ldots + f_n^*(x^*_n) : \sum_{k=1}^{n} x^*_k = 1\}\). Now \((14)\) implies that

\[ \sum_{k=1}^{n} \text{dom } f_k^* \subseteq \text{dom } g_n^* \subseteq \text{cl} \left( \sum_{k=1}^{n} \text{dom } f_k^* \right), \]

whence \(\text{int dom } f_k^*(\cdot) = \text{int dom } g_n^*\).

Now, consider \(x^* \in \text{int dom } f^*\). It is not difficult to prove that

\[ f^*(x^*) \leq \inf \left\{ \sum_{n \in \mathbb{N}} f^*_n(x^*_n) : \sum_{n \in \mathbb{N}} x^*_n = x^* \right\}. \]

For the other inequality we notice that \(\{g_n\}\) form an increasing family of proper lsc convex epi-pointed functions and \(f = \sup_{n \in \mathbb{N}} g_n\), then using Theorem \((11)\) one has that \(f^*(x^*) = \inf \{g_n^*(x^*) : n \in \mathbb{N}\}\), besides by Lemma \((3)\) \(x^* \in \bigcup_{n \in \mathbb{N}} \text{int dom } g_n^*\). Whence \((15)\) yields \(f^*(x^*) = \inf \{\sum_{k=1}^{n} f_k^*(x_k^*) : N \in \mathbb{N} \text{ and } \sum_{k=1}^{n} x_k^* = x^*\}\), which concludes the proof of \((12)\). Finally, in order to prove \((13)\) we recall \(f_n^*(0) \leq 0\), thus \((12)\) implies \(f^*(x^*) \geq \inf \{\sum_{n \in \mathbb{N}} f_n^*(x_n^*) : \sum_{n \in \mathbb{N}} x_n^* = x^*\}\) and that concludes the proof. \(\square\)

A straightforward application of Lemma \((3)\) and Theorem \((4)\) gives us an estimation for the conjugate of \(\text{lim sup}\) and \(\text{lim inf}\) functions.

Corollary 4.2 Consider a directed set \(T\) and an arbitrary family of functions \(\{f_t : t \in T\} \subseteq \Gamma_0(X)\).

(a) If \(h(x) := \text{lim sup}_{t \in T} f_t(x)\) is proper, then

\[ \text{epi } h^* = \bigcap_{t \in T} \text{cl conv} \left\{ \bigcup_{s \geq t} \text{epi } f^*_s \right\}. \quad (16) \]
If \( g(x) := \lim \inf_{t \in T} f_t(x) \) is proper and there exists \( t_0 \in T \) such that 
\[
(\inf_{s \geq t_0} f_s)(x) \text{ is epi-pointed and } g^{**} = \sup g^*_t,
\]
where \( g_t := \inf_{s \geq t} f_s \), then 
\[
g^*(x^*) := \lim \sup f_t^*(x^*) \text{ for all } x^* \in \text{int dom } g^*
\tag{17}
\]
and \( \text{epi } g^* = \bigcup_{t \in T} \bigcap_{s \geq t} \text{epi } f_s^* \).

Proof First define \( h_t := \sup_{s \geq t} f_s \). Then
(a) \( h^* = \sup_{x \in X} \{(x, h(x)) \} = \sup_{t \in T} \sup_{x \in X} \{(x, h_t(x)) \} = \sup_{t \in T} h^*_t \).

Then, we notice that \( h_t \) must be proper for some \( t \in T \), so by Lemma 3.1(c) \( \text{epi } h^*_t = \text{cl conv} \{ \text{epi } f_s^* \} \), and so (17) holds.

(b) \( g = \sup_{t \in T} g_t \) and \( \{ g_t \}_{t \geq t_0} \) is an increasing family of epi-pointed functions satisfying \( g^{**} = \sup_{t \in T} g_t^{**} \). Then, Theorem 4.1 and Lemma 3.1(a) yield (17), also Lemma 3.1(a) implies \( g_t^* = \bigcap_{s \geq t} \text{epi } f_s^* \), and consequently \( \text{epi } g^* = \bigcup_{t \in T} \text{epi } g_t = \bigcup_{t \in T} \bigcap_{s \geq t} \text{epi } f_s^* \).

\( \square \)

**Theorem 4.2** Let \( \{ f_t : t \in T \} \) be an increasing family of epi-pointed functions such that \( f^{**} = \sup_{t \in T} f_t^{**} \), then for all \( x \in X \).

\[
\partial_{\varepsilon} f(x) = \bigcap_{t \in T} \bigg\{ \bigcup_{\gamma > 0} \partial_{\varepsilon + \gamma} f_t(x) \bigg\} \tag{18}
\]

Moreover, the above formulae hold if the functions \( f_t \) are not necessarily epi-pointed, but they belong to \( \Gamma_0(X) \).

Proof First we check that the right side of (18) is included in the \( \varepsilon \)-subdifferential of \( f \) at \( x \). Indeed, let \( \gamma > 0 \) and \( t \in T \) and pick \( x^* \in \bigcap_{\gamma > 0} \partial_{\varepsilon + \gamma} f_t(x) \). Then
\[
(x^*, y - x) \leq f_s(y) - f_s(x) + \varepsilon + \gamma \leq f(y) - f(x) + \varepsilon + \gamma \text{ for all } y \in X \text{ and for all } s \geq s_0, \text{ then } (x^*, y - x) \leq f(y) - f(x) + \varepsilon + \gamma, \text{ which means } x^* \in \partial_{\varepsilon + \gamma} f(x).
\]
Therefore
\[
\bigcap_{\gamma > 0} \bigcap_{t \in T} \bigg\{ \bigcup_{s \geq t, s \geq s_0} \partial_{\varepsilon + \gamma} f_s(x) \bigg\} \subseteq \bigcap_{\gamma > 0} \partial_{\varepsilon + \gamma} f(x) = \partial_{\varepsilon} f(x).
\]

Now, it is easy to see that (18) is included in (19). For the opposite inclusion we notice that for any co-final set \( T' \subseteq T \) we have
\[
\bigcap_{t \in T} \bigg\{ \bigcup_{\gamma > 0} \partial_{\varepsilon + \gamma} f_t(x) \bigg\} = \bigcap_{t \in T} \bigg\{ \bigcup_{\gamma > 0} \partial_{\varepsilon + \gamma} f_s(x) \bigg\}
\]
Then consider $\gamma > 0$ and $t \in T_\gamma(x)$ arbitrary. Pick $s_2 \geq s_1 \geq t$ (it implies that $s_1, s_2 \in T_\gamma(x)$) and $x^* \in \partial_{\epsilon+\gamma} f_{s_1}(x)$, then
\[
\langle x^*, y - x \rangle \leq f_{s_1}(y) - f_{s_1}(x) + \epsilon + \gamma \\
\leq f_{s_2}(y) - f_{s_1}(x) + f_{s_2}(x) - f_{s_2}(x) + \epsilon + \gamma \\
\leq f_{s_2}(y) - f_{s_1}(x) + f(x) - f_{s_2}(x) + \epsilon + \gamma \\
\leq f_{s_2}(y) - f_{s_2}(y) + \epsilon + 2\gamma,
\]
which means $x^* \in \partial_{\epsilon+2\gamma} f_{s_2}(x)$, since it is for every $s_2 \geq s_1 \geq t$ we conclude that
\[
\bigcup_{s \geq t} \partial_{\epsilon+\gamma} f_s(x) \subseteq \bigcup_{s_0 \geq t} \bigcap_{s \geq s_0} \partial_{\epsilon+2\gamma} f_s(x),
\]
and consequently
\[
\bigcap_{t \in T_\gamma} \left\{ \bigcup_{s \geq t} \partial_{\epsilon+\gamma} f_s(x) \right\} \subseteq \bigcap_{t \in T_\gamma} \left\{ \bigcup_{s \geq t} \partial_{\epsilon+\gamma} f_s(x) \right\} \\
\subseteq \bigcap_{t \in T_\gamma} \left\{ \bigcup_{s_0 \geq t} \bigcap_{s \geq s_0} \partial_{\epsilon+\gamma} f_s(x) \right\} \\
\subseteq \bigcap_{t \in T_\gamma} \left\{ \bigcup_{s_0 \geq t} \bigcap_{s \geq s_0} \partial_{\epsilon+\gamma} f_s(x) \right\}.
\]

Now we focus on proving that $\partial_{\epsilon} f(x)$ is included in the right side of (18), so w.l.o.g. we can assume that $\partial_{\epsilon} f(x) \neq \emptyset$, in particular $f^*$ is proper, and by Lemma (e) $f$ is epi-pointed. First we prove this in the case that the functions $f_t$ are epi-pointed. Thanks to Lemma 3.2 it is enough to prove that for every $\gamma > 0$ and $t \in T$
\[
\partial_{\epsilon+\gamma} f(x) \cap \text{int dom } f^* \subseteq \bigcup_{s \geq t} \partial_{\epsilon+2\gamma} f_s(x). \quad (20)
\]

Then, take $x^*$ in the left side of (20), then $f^*(x^*) + f(x) \leq \langle x^*, x \rangle + \epsilon + \gamma$, then using Theorem 4.1 (recall $x^* \in \text{int dom } f^*$) and due to the fact that the family is increasing, we can take $s \geq t$ such that $f^*_s(x) + f_s(x) \leq \langle x^*, x \rangle + \epsilon + 2\gamma$, which implies $x^* \in \partial_{\epsilon+2\gamma} f_s(x)$.

Now when the functions are not necessarily epi-pointed one can use the following argumentation; in order to simplify the notation we assume that $x = 0$. For the sake of understanding the proof first consider that $X$ is a finite-dimensional space. Then, for every $\alpha > 0$ we define the epi-pointed
function \( g^\epsilon_\alpha(\cdot) := f_t(\cdot) + \alpha \| \cdot \| \) (see, e.g., [23] Proposition 3.1) and consider 
\( g^\epsilon = \sup_{t \in T} g^\epsilon_\alpha = f + \alpha \| \cdot \| \). Then, \( \partial \varepsilon f(0) = \bigcap_{\alpha > 0} \partial \varepsilon g^\alpha(0) \), therefore
\[
\partial \varepsilon f(0) = \bigcap_{\alpha > 0} \partial \varepsilon g^\alpha(0) = \bigcap_{\alpha > 0} \bigcap_{t \in T} \cl \left\{ \bigcup_{s \geq t} \partial \varepsilon \gamma g^\alpha_\gamma(0) \right\}
\]
\[
= \bigcap_{t \in T} \bigcap_{\alpha > 0} \bigcup_{s \geq t} \left\{ \partial \varepsilon \gamma f_s(0) + \alpha B(0, 1) \right\}
\]
\[
= \bigcap_{t \in T} \bigcap_{\alpha > 0} \bigcup_{s \geq t} \left\{ \partial \varepsilon \gamma f_s(0) \right\}
\]

Finally, if the space \( X \) is infinite-dimensional we must identify the elements of \( \partial (f + \delta_L)(0) \) with \( \partial f_{\| \cdot \|}(0) \); here \( L \) is a finite-dimensional subspace of \( X \) and \( f_{\| \cdot \|} \) is the restriction of \( f \) to \( L \). Indeed, consider \( t \in T \) and \( \gamma > 0 \), then take \( e_k \in X \) for \( k = 1, \ldots, p \) and define \( V = \{ x^* \in X^* : \| x^* , e_k \| \leq 1 \} \) and \( L \) be the linear subspace generated by \( \{ e_k \}_{k=1}^p \). Now, we are going to prove that there exists \( s \geq t \) such that \( \partial (f + \delta_L)(0) \subseteq \partial \varepsilon \gamma f_s(x) + V \). Let \( x^* \in \partial (f + \delta_L)(0) \) and consider \( P : X \to L \) a continuous projection and \( P^* \) its adjoint operator, then \( x^*_L \in \partial f_{\| \cdot \|}(0) \), then by the last part there exists \( s \geq t \) such that \( x^*_L \in \partial (f_{\| \cdot \|})(0) + W^* \), where \( W^* := \{ x^* \in L^* : \| x^* , e_k \| \leq 1 \} \). Then \( x^* = P^*(x^*_L) + x^* - P^*(x^*_L) \in \partial \varepsilon \gamma (f_s + \delta_L)(0) + V + L^\perp \), besides Hiriart-Urruty and Phelps’s formula (see, e.g., [3] Theorem 3.2) implies
\[
\partial \varepsilon \gamma (f_s + \delta_L)(0) \subseteq \partial \varepsilon \gamma (f_s + \delta_L)(0) + V^* + L^\perp.
\]
Therefore \( x^* \in \partial \varepsilon \gamma f_s(x) + V \).

The following example shows that the closure operator in the above result is necessary even for functions defined in the real line.

**Example 4.1** Consider the functions \( f_n(x) = (1 - 1/n) |x| \), then \( f(x) = |x| \) and \( \partial \varepsilon f_n(0) = [-1 + 1/n, 1 - 1/n] \) and \( \partial \varepsilon f(0) = [-1, 1] \) for all \( \varepsilon \geq 0 \). Whence \( \bigcup_{n \in N} \partial \varepsilon f_n(0) = (-1, 1) \neq [-1, 1] = \partial f(0) \).

The next result corresponds to a general formula for the \( \varepsilon \)-normal set of an intersection of arbitrary closed and convex sets.

**Corollary 4.3** Consider a family of closed convex subsets \( \{ C_t : t \in T \} \) and \( C := \bigcap_{t \in T} C_t \). Then for every \( \varepsilon \geq 0 \) and \( x \in C \)
\[
N_C^\varepsilon(x) = \bigcap_{\eta > 0} \cl \left\{ \sum_{t \in A} N^\eta_C(x) : \eta_t \geq 0 \text{ and } \sum_{t \in A} \eta_t = \varepsilon + \gamma \right\}.
\]
Proof First the right-hand side of (21) is included in \(N^C_\varepsilon(x)\). We focus on the opposite inclusion. Consider the sets \(G_A := \cap_{t \in A} C_t\) for \(A \in \mathcal{P}(T)\). Then 
\[
\delta_C = \sup_{A \in \mathcal{P}(T)} \delta_{G_A},
\]
then by Theorem 4.2 we have that 
\[
N^\varepsilon_\varepsilon(x) = \bigcap_{\gamma > 0} \text{cl}\ \left\{ N^{\varepsilon + \gamma}_{G_A}(x) : A \subseteq T, \ #A < +\infty \right\},
\]
moreover \(N^{\varepsilon + \gamma}_{G_A}(x) = \partial_{\varepsilon + \gamma} \left( \sum_{t \in A} \delta_{C_t} \right)(x)\) and by Hiriart-Urruty and Phelps’s formula
\[
\partial_{\varepsilon + \gamma} \left( \sum_{t \in A} \delta_{C_t} \right)(x) = \text{cl w}^* \left\{ \sum_{t \in A} N^{\varepsilon}_{C_t}(x) : \eta_t \geq 0 \text{ and } \sum_{t \in T} \eta_t = \varepsilon + \gamma \right\}.
\]
Consequently, mixing (22) and (23) we get the result. \(\square\)

5 Calculus for \(\varepsilon\)-Subdifferential and the Fenchel Conjugate of Pointwise Supremum Function

This section is devoted to the study of the \(\varepsilon\)-subdifferential of the pointwise supremum function. In the first part we assume that the space \(X\) is finite-dimensional; this condition allows us to use Carathéodory’s Theorem to bound the cardinal of the support of the elements \(\lambda \in \Delta(T)\) in Proposition 3.1 and with this we obtain more precise calculus rules for the \(\varepsilon\)-subdifferential and the Fenchel conjugate of the pointwise supremum function. In the second part of this section, the finite-dimensional condition over \(X\) is removed using some reduction to finite-dimensional subspaces; for this reason we prefer to use the notation \(X\) in Subsection 5.1 for a finite-dimensional space also instead of using the Euclidean space \(\mathbb{R}^n\), putting emphasis on the fact that \(X\) could be an abstract finite-dimensional space.

5.1 Finite-Dimensional Banach Spaces

In this section we give simplifications of Proposition 3.1 under some classical qualification conditions in a finite-dimensional Banach space \(X\). In this subsection we assume that the functions \(f_t\) satisfy the relation \(f^{**} = \sup_{t \in T} f_t^{**}\).

Let us introduce the following qualification condition at a point \(x \in X\).
\[
N_{\text{dom } f}(x) \text{ does not contain lines.}
\]

Remark 5.1 It is important to recall that for any arbitrary function \(f : X \to \mathbb{R}\) with proper conjugate the following statements are equivalent (i) \(N_{\text{dom } f}(x)\) does not contains lines, (ii) \(\text{cl conv } f\) is continuous at some point and (iii) \(f^*\) is epi-pointed. Moreover, \(\partial_{\varepsilon} f(x) = N_{\text{dom } f}(x)\) for all \(\varepsilon \geq 0\) and all \(x \in X\) such that \(\partial_{\varepsilon} f(x) \neq \emptyset\).
The key tool to establish this exact formulation is the pointedness of the normal cones involved and Carathéodory’s Theorem, which allow us to give a limiting representation of (5) and (6). The next lemma represents the major ideas in the proofs of the main results for this section.

**Lemma 5.1** Consider \( \varepsilon \geq 0 \), \( n := \min\{\#T, \dim X + 1\} \) and \( x^* \in \partial_x f(x) \), then there are sequences \( t_{i,k} \in T \), \( (\lambda_{i,k}) \in \Delta(\{1, \ldots, n\}) \), \( x^*_{i,k} \in \partial_{\eta_{i,k}} f(t_{i,k}) \) and \( \eta_{i,k} \geq 0 \) with \( i \in \{1, \ldots, n\} \), \( k \in \mathbb{N} \) such that \( \sum_{i=1}^{n} \lambda_{i,k} \leq \varepsilon \),

\[
\sum_{i=1}^{n} \lim_{k \to \infty} \lambda_{i,k} (f(x) - f(t_{i,k}(x))) \leq \varepsilon - \sum_{i=1}^{n} \lim_{k \to \infty} \lambda_{i,k} \eta_{i,k},
\]

and \( \sum_{i=1}^{n} \lambda_{i,k} x^*_{i,k} \to x^* \). Moreover, one of the following conditions holds.

(a) There exists \( n_1 \in \mathbb{N} \) with \( n_1 \leq n \) such that \( \lambda_{i,k} \xrightarrow{k \to \infty} \lambda_i \), \( \eta_{i,k} \to \eta_i \) for \( i \leq n_1 \) and \( \lambda_{i,k} \xrightarrow{k \to \infty} 0 \), \( \lambda_{i,k} \cdot x^*_{i,k} \xrightarrow{k \to \infty} x^*_i \) for \( n_1 < i \leq n \),

\[
\sum_{i=1}^{n_1} \lambda_i (f(t_{i,k}(x)) - f(x)) + \sum_{i=n_1+1}^{n} \lim_{k \to \infty} \lambda_{i,k} x^*_i (x) - f(x)) \leq \varepsilon - \sum_{i=1}^{n_1} \lambda_i \lim_{k \to \infty} \eta_{i,k} - \sum_{i=n_1}^{n} \lim_{k \to \infty} \lambda_{i,k} \eta_{i,k},
\]

\( \sum_{i=1}^{n_1} \lambda_i = 1 \) and \( x^* = \sum_{i=1}^{n_1} \lambda_i x^*_i + \sum_{i=n_1+1}^{n} x^*_i \), or

(b) There are \( \nu_k \searrow 0 \) such that \( \nu_k \cdot \lambda_{i,k} \cdot x^*_i \xrightarrow{k \to \infty} x^*_i \) and \( \sum_{i=1}^{n_1} x^*_i = 0 \) with not all \( x^*_i \) equal to zero.

Moreover, if one assumes (vii), then (a) always holds.

**Proof** Consider \( x^* \in \partial_x f(x) \) and \( \gamma_k \to 0 \). Then by Proposition 4.1 there are sequences \( t_{i,k} \in T \), \( (\lambda_{i,k}) \in \Delta(\{1, \ldots, n\}) \), \( \eta_{i,k} \geq 0 \) and \( x_{i,k} \in \partial_{\eta_{i,k}} f(t_{i,k}) \) such that \( \sum_{i \in T} \lambda_{i,k} \cdot \eta_{i,k} \in [\varepsilon, \varepsilon + \gamma_k] \) and \( \sum_{i \in T} \lambda_{i,k} \cdot f(t_{i,k}) \geq f(x) + \sum_{i \in T} \lambda_{i,k} \cdot \eta_{i,k} - \gamma \) and \( u^*_k := \sum_{i=1}^{n} \lambda_{i,k} \cdot x^*_i \to x^* \). We may assume (up to a subsequence) that for every \( i = 1, \ldots, n \) \( (i) \lambda_{i,k} \to \lambda_i \), \( (ii) \lim_{k \to \infty} (\lambda_{i,k} \cdot \eta_{i,k}) \) exists and \( (iii) \lim_{k \to \infty} \lambda_{i,k} \cdot (f(x) - f(t_{i,k}(x))) \) exists. Then, \( \sum_{i=1}^{n} \lim_{k \to \infty} (\lambda_{i,k} \cdot \eta_{i,k}) \leq \varepsilon \) and

\[
\sum_{i=1}^{n} \lim_{k \to \infty} \lambda_{i,k} \cdot (f(x) - f(t_{i,k}(x))) \leq \varepsilon - \sum_{i=1}^{n} \lim_{k \to \infty} \lambda_{i,k} \eta_{i,k}.
\]

Moreover (reordering if it is necessary), we assume that \( \lambda_i > 0 \) for \( i = 1, \ldots, n_1 \) and \( \lambda_i > 0 \) for \( i > n_1 \).
Now, on the one hand, if we assume that \( \sup_{k,i} \| \lambda_{i,k} \cdot x^*_{i,k} \|_* < +\infty \), then up to a subsequence one can assume that \( \lambda_{i,k} \xrightarrow{k \to \infty} \lambda_i > 0 \), \( x_{i,k} \xrightarrow{k \to \infty} x^*_i \), for \( i \leq n_1 \) and \( \lambda_{i,k} \xrightarrow{k \to \infty} 0 \), \( \lambda_{i,k} \cdot x_{i,k} \xrightarrow{k \to \infty} x^*_i \) for \( n_1 < i \leq n_2 \). Hence, for \( k \to \infty \),

\[
\sum_{i=1}^{n_1} \lambda_i (\lim_{k \to \infty} (f_{i,k}(x) - f(x))) + \sum_{i>n_1} \lim_{k \to \infty} (\lambda_{i,k} f_{i,k}(x) - f(x)) \leq \varepsilon - \sum_{i=1}^{n_1} \lambda_i \eta_{i,k} + \sum_{i>n_1} \lim_{k \to \infty} \lambda_{i,k} \eta_{i,k}
\]

and \( x^* = \sum_{i=1}^{n_1} \lambda_i x^*_i + \sum_{i>n_1} x^*_i \).

On the other hand, if \( \sup_{k,i} \| \lambda_{i,k} \cdot x^*_{i,k} \|_* = \sup \{ \max_i \| \lambda_{i,k} \cdot x^*_{i,k} \| \} = +\infty \) up to a subsequence one can assume that \( \nu_k := (\max_i \| \lambda_{i,k} \cdot x^*_{i,k} \|_*)^{-1} \) \( \leq 0 \) and \( \nu_k \lambda_{i,k} \cdot x^*_{i,k} \xrightarrow{k \to \infty} x^*_i \) with not all \( x^*_i \) equal to zero, thus \( \nu_k \lambda_{i,k} \xrightarrow{k \to \infty} \sum_{i=1}^{n_1} x^*_i = 0 \).

Finally, it is not difficult to see that \( (b) \) contradicts \( (a) \).

\( \square \)

Now we present the main result of this section.

**Theorem 5.1** Assume that \( (a) \) holds at \( x \in \dom f \). Then for every \( \varepsilon \geq 0 \) one has

\[
\partial_{\varepsilon} f(x) = \bigcup \left\{ \left. S(x, \varepsilon_1) + N^*_{\dom f}(x) \cap (\varepsilon_1, \varepsilon_2) \in \Delta^e(\{1, 2\}) \right\} \bigcup \left\{ \left. \overline{S}(x, \varepsilon_1) + N^*_{\dom f}(x) \cap (\varepsilon_1, \varepsilon_2) \in \Delta^e(\{1, 2\}) \right\} \right.
\]

where

\[
S(x, \varepsilon_1) := \bigcap_{\gamma \geq 0} \left\{ \left. \sum_{t \in \supp \lambda} \lambda_t \partial_{\varepsilon_1 + \gamma} f_t(x) : \lambda \in \Delta(T), (\varepsilon_t) \in \Delta^e(\{1, 2\}) \right\} f_t(x) + \varepsilon_t/\lambda_t + \gamma \geq f(x) \right\}
\]

\[
\overline{S}(x, \varepsilon_1) := \left\{ \left. \sum_{t \in \supp \lambda} \lambda_t x^*_t : \lambda \in \Delta(T), (\varepsilon_t) \in \Delta^e(\{1, 2\}) \right. x^*_t \in \Delta_{\varepsilon, \lambda}(x, \varepsilon, \lambda) \right\}
\]

and \( \Delta_{\varepsilon, \lambda}(x, \varepsilon, \lambda) := \left\{ x^* \in X^* : \exists t_k \in T, x^*_{i,k} \in \partial_{\varepsilon_1 + \gamma_k} f_{i,k}(x), \gamma_k \geq 0, \text{ such that } x^*_{i,k} \xrightarrow{k \to \infty} x^* \text{ and } \lim f_{i,k}(x) + \varepsilon/\lambda \geq f(x) \right\} \)

**Proof** We focus on the nontrivial inclusions. Let \( x^* \in \partial_{\varepsilon} f(x) \), then consider sequences \( t_{i,k} \in T, (\lambda_{i,k}) \in \Delta(\{1, \ldots, n_1\}), \eta_{i,k} \geq 0 \) and \( x_{i,k} \in \partial_{\eta_{i,k}} f_{i,k}(x) \) as in Lemma 5.1 and by \( (a) \) condition \( (a) \) must hold. Now, using the notation of Lemma 5.1, define \( y^* := \sum_{i=1}^{n_1} \lambda_i x^*_i, z^* := \sum_{i=n_1+1}^{n} x^*_i, \varepsilon_1 := \sum_{i=1}^{n_1} \varepsilon_i \), with \( p_i := \lambda_i (f(x) - \lim f_{i,k}(x) + \eta_i) \) for \( i = 1, \ldots, n_1 \) and

\[
varepsilon_2 := \sum_{i>n_1} \lim_{k \to \infty} \lambda_{i,k} (f(x) - f_{i,k}(x)) + \sum_{i>n_1} \lim_{k \to \infty} \lambda_{i,k} \eta_{i,k}, \quad (25)
\]
so \(x^* = y^* + z^*\) and \(\varepsilon_1 + \varepsilon_2 \leq \varepsilon\). It follows that \(y^* \in \mathcal{S}(x, \varepsilon_1)\). Indeed, define \(\gamma_{i,k} = \frac{\lambda_{i,k}}{\lambda} - (f(x) - f_{t_{i,k}}(x) + \eta_{i,k})\), then one gets \(x^*_{i,k} \in \partial_{\lambda} \lambda_{i,k} f_{t_{i,k}}(x)\) and \(\lim_{k} f_{t_{i,k}}(x) + \eta_{i,k} \geq f(x)\), which means \(x^*_i \in A(x, p_i, \lambda_i)\).

Now we show that \(z^* \in \mathcal{N}_{\text{dom}}^\gamma(x)\) and \(y^* \in \mathcal{S}(x, \varepsilon_1)\). Let \(\gamma\) any number in \((0, \min\{\lambda_i : i = 1, \ldots, n\})\) and define \(M = \sup\{\|x^*_k\| : k \in \mathbb{N}, i = 1, \ldots, n\}\) for some applications it is better to understand this condition in terms of the \(\Delta\) of its equivalent statements (see Remark 5.1), is standard in convex analysis, let \(k \in \mathbb{N}\) be such that \(\|y^* - \sum_{i=1}^{n} \lambda_i x^*_i\| \leq \gamma, (1 - \sum_{i=1}^{n} \lambda_i)M < \gamma\),

\[
\lambda_i f(x) - f_{t_{i,k}}(x) + \eta_{i,k} < p_i + \gamma^2
\]

and \(\varepsilon_1(\lambda_{i,k} - \frac{1}{(1 - \sum_{i=1}^{n} \lambda_{i,k}) + \lambda_{i,k}}) < \gamma\) for all \(i = 1, \ldots, n\). Then, we set \(\lambda \in \Delta(T)\), \((\varepsilon_t) \in \Delta^+(T)\) and \(\eta \in \mathbb{R}^f(T)\) by

\[
\lambda_t = \begin{cases} 
\lambda_{i,k} + (1 - \sum_{i=1}^{n} \lambda_{i,k}), & \text{if } t = t_{i,k}, \\
\lambda_{i,k}, & \text{if } t = t_{i,k} \text{ for } i = 2, \ldots, n, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
\varepsilon_t = \begin{cases} 
p_i, & \text{if } t = t_{i,k}, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
\eta_t = \begin{cases} 
\eta_{i,k}, & \text{if } t = t_{i,k} \text{ for } i = 1, \ldots, n, \\
0, & \text{otherwise,}
\end{cases}
\]

respectively. Then \(y^* \in \sum_{t \in \text{supp} \lambda} \lambda_t \partial_{\eta_t} f_t(x) + \mathbb{B}(0, 2\gamma)\), moreover (recall 26) and \(\gamma < \lambda_{i,k}\) for \(t \neq t_{i,k}\), \(f_t(x) + \varepsilon_t/\lambda_t + \gamma \geq f(x) + \eta_t\), and for \(t = t_{i,k}\), \(f_t(x) + \varepsilon_t/\lambda_{i,k} + \gamma \geq f(x) + \eta_t\), besides

\[
\varepsilon_t(\lambda_{i,k} - \varepsilon_t) + \varepsilon_t \left(\frac{1}{\lambda_{i,k}} - \frac{1}{\lambda_{i,k} + (1 - \sum_{i=1}^{n} \lambda_{i,k})}\right) \leq \varepsilon_t + \gamma.
\]

Therefore \(y^* \in \sum_{t \in \text{supp} \lambda} \lambda_t \partial_{\eta_t} f_t(x) + \mathbb{B}(0, 2\gamma)\).

Finally, for every \(y \in \text{dom} f\), one has

\[
\langle z^*, y - x \rangle = \lim_{k \to \infty} \sum_{i=n+1}^{n} \lambda_{i,k} (x^*_i, y - x) 
\]

\[
\leq \lim_{k \to \infty} \sum_{i=n+1}^{n} \lambda_{i,k} (f_{t_{i,k}}(y) - f_{t_{i,k}}(x) + \eta_{i,k}) 
\]

\[
\leq \lim_{k \to \infty} \sum_{i=n+1}^{n} \lambda_{i,k} (f(y) - f(x)) + \lim_{k \to \infty} \sum_{i=n+1}^{n} \lambda_{i,k} (f(x) - f_{t_{i,k}}(x)) 
\]

\[
\leq \varepsilon_2 \text{ (recall 26).}
\]

Although the assumption that the normal does not contain lines, or some of its equivalent statements (see Remark 5.1), is standard in convex analysis, for some applications it is better to understand this condition in terms of the...
functions $f_t$, for this reason we introduce the next qualification condition in terms of the normal cones of the data function $\{f_t\}_{t \in T}$.

For every $A \subseteq \mathcal{P}_f(T)$ one has

$$x^*_t \in N_{\text{dom} f_t}(x) \text{ for } t \in A \text{ and } \sum_{t \in A} x^*_t = 0 \implies x^*_t = 0 \text{ for all } t \in A. \quad (27)$$

Under the additional assumptions of the compactness of $T$ and some continuity property of the function $t \to f_t(w)$ we can prove that (27) is equivalent to (24).

More precisely we get the following result.

**Theorem 5.2** Assume that $T$ is a compact space, $t \to f_t(w)$ is upper semicontinuous for every $w \in X$ and (24) holds at $x$. Then, for every $\varepsilon \geq 0$

$$N_{\text{epi} f}(x, f(x)) = \bigcup_{t \in \text{supp}(x, t)} \left\{ \sum_{t \in \text{supp} \lambda} N_{\text{epi} f_t}(x, f(x)) : (\varepsilon_t) \in \Delta^\varepsilon(T) \right\} \quad (28)$$

$$\partial_{\varepsilon} f(x) = \bigcup_{t \in \text{supp} \eta} \left\{ S(x, \varepsilon_1, T) + \mathcal{N}(x, \varepsilon_2, T) : (\varepsilon_1, \varepsilon_2) \in \Delta^\varepsilon([1, 2]) \right\} \quad (29)$$

where

$$S(x, \varepsilon_1) := \left\{ \sum_{t \in \text{supp} \lambda} \lambda_{\varepsilon_t} \partial_{\varepsilon_t} f_t(x) : \lambda \in \Delta(T), (\varepsilon_t) \in \Delta^\varepsilon(T) \text{ and } f_t(x) + \varepsilon_t / \lambda_t \geq f(x) \right\},$$

$$\mathcal{N}(x, \varepsilon_2) := \left\{ \sum_{t \in \text{supp} \eta} N_{\text{dom} f_t}(x) : (\eta) \in \Delta^\varepsilon(T) \right\}.$$

**Proof** First we prove (28), the result is trivial if $f^{**} = -\infty$, or $f^{**} = +\infty$. Therefore, we assume that $f^{**}$ is proper. Consider the set $\tilde{T} := \{ t : f^{**}_t \neq -\infty \}$ and the functions $y_t := \delta_{\text{epi} f_t}$ and $g := \sup_{t \in \tilde{T}} y_t$. The relation $f^{**} = \sup f^{**}$ is equivalent to $g^{**} = \sup_{t \in \tilde{T}} g_t^{**}$. Then, we pick $(x^*, \alpha)$ in $N_{\text{epi} f}(x, f(x)) = \partial_{\varepsilon} g(x, f(x))$, so applying Lemma 5.1 (and following its notation) there are sequences $t_{i, k} \in T$, $(\lambda_{i, k}) \in \Delta\{1, \ldots, n\}$, $\eta_{i, k} \geq 0$, and sequences $(x^*_{i, k}, \alpha_{i, k})$ in $\partial_{\varepsilon}\eta_{i, k} y_{i, k} f_t(x, f(x)) = N_{\text{epi} f_t}(x, f(x))$ with $i \in \{1, \ldots, n\}$, $k \in \mathbb{N}$ such that for large enough $k$

$$\sum_{i=1}^n \lim \lambda_{i, k} \cdot \eta_{i, k} \leq \varepsilon, \text{ and } \sum_{i=1}^n \lambda_{i, k} \cdot (x^*_{i, k}, \alpha_{i, k}) \to (x^*, \alpha).$$

We recall that necessarily $\alpha \leq 0$ and $\alpha_{i, k} \leq 0$ for all $i, k$. Since $T$ is compact we may assume (up to a subnet) that $t_{i, k} \to t_i$, let us define $\varepsilon_t = \sum_{i=1}^n \lim \lambda_{i, k} \cdot \eta_{i, k}$ if there exists some $t_i = t$, and $\varepsilon_t = 0$ if $t \neq t_i$ for all $i$.

Now suppose that condition (b) of Lemma 6.1 holds, then the elements $\nu_k \lambda_{i, k} \cdot (x^*_{i, k}, \alpha_{i, k})$ converges to $(x^*, \beta_i)$ and $\sum_{i=1}^n (x^*_{i, k}, \beta_i) = (0, 0)$ for some $\nu_k \neq 0$, and not all $(x^*_{i, k}, \beta_i)$ are equal to zero, then necessarily $\beta_i = 0$, because $\beta_i \leq 0$ for all $i = 1, \ldots, n$. Now we check that $x^*_t \in \text{dom} f_t(x)$, indeed let $y \in \text{dom} f_t$, because $t \to f_t(y)$ is upper semicontinuous (usc) we have that there exists $r \in \mathbb{R}$ such that $(y, r) \in \text{epi} f_{i, k}$ for large enough $k$, consequently

$$\langle x^*_t, y - x \rangle = \lim_k \left( \nu_k \lambda_{i, k} \cdot x^*_{i, k, y - x} + \nu_k \lambda_{i, k} \alpha_{i, k} (r - f(x)) \right)$$

$$= \lim_k \nu_k \lambda_{i, k} \left( \langle x^*_{i, k}, y - x \rangle + \alpha_{i, k} (r - f(x)) \right)$$

$$\leq \lim_k \nu_k \lambda_{i, k} \eta_{i, k} \leq \lim_k \nu_k \varepsilon = 0,$$
the last means \( x^*_i \in N_{\text{dom} f_i} (x) \) and \( \sum_{i=1}^n x^*_i = 0 \) with not all \( x^*_i \) equal to zero, this contradicts (27). Therefore, condition (a) of Lemma 5.1 must hold, then \( \lambda_{i,k} \cdot (x^*_i, \alpha_{i,k}) \rightarrow (x^*, \beta_i) \) and using that \( t \rightarrow f_t(w) \) is usc we get

\[
(u^*_i, \alpha_i) := \sum_{j: t_j = t} (x^*_i, \beta_i) \in N_{\text{epi} f_t}(x, f(x)), \quad \text{if there exists some } t_i = t,
\]

\[
(u^*_i, \alpha_i) := (0, 0) \in N_{\text{epi} f_t}(x, f(x)), \quad \text{otherwise},
\]

and \( \sum_{t \in T} (u^*_i, \alpha_i) = (x^*, \alpha) \). Consequently using (28) (with \( \varepsilon = 0 \)) we get that (27) implies (24). Then, we can apply Theorem 5.1 and following its notation Remark 5.2.

It has not escaped our notice that using (29) one can prove that

\[
\text{dom} f^* \subseteq N \epsilon f^*(x) = \text{epi} f^*(\varepsilon < 0),
\]

\( \lambda \) this contradicts (27). Therefore, condition (a) of Lemma 5.1 must hold, then \( \lambda_i, k, l \cdot (x^*_i, \alpha_{i,k}) \rightarrow (x^*, \beta_i) \) and using that \( t \rightarrow f_t(w) \) is usc we get

\[
(u^*_i, \alpha_i) := \sum_{j: t_j = t} (x^*_i, \beta_i) \in N_{\text{epi} f_t}(x, f(x)), \quad \text{if there exists some } t_i = t,
\]

\[
(u^*_i, \alpha_i) := (0, 0) \in N_{\text{epi} f_t}(x, f(x)), \quad \text{otherwise},
\]

and \( \sum_{t \in T} (u^*_i, \alpha_i) = (x^*, \alpha) \). Consequently using (28) (with \( \varepsilon = 0 \)) we get that (27) implies (24). Then, we can apply Theorem 5.1 and following its notation we use the compactness of \( T \) and the upper semicontinuity of \( t \rightarrow f_t(w) \) (together with similar arguments as in the first part) to prove that \( \Delta(x, \varepsilon_1) \) is contained in \( S(x, \varepsilon_1) \). Moreover, using the fact that \( (x^*, 0) \in N^e_{\text{epi} f_t}(x, f(x)) \) iff \( x^* \in N^e_{\text{dom} f_t}(x) \) together with (28) we derive that \( N^e_{\text{dom} f_t}(x) \) is equal to \( N(x, \varepsilon_2) \). This concludes the proof of (29).

**Remark 5.2** It has not escaped our notice that using (29) one can prove that

\[
\partial_\varepsilon f(x) = \bigcup \left\{ S(x, \varepsilon_1, T_1) + N(x, \varepsilon_2, T_2) : (\varepsilon_1, \varepsilon_2) \in \Delta^e(\{1, 2\}), \right. \\
T_1 \cap T_2 = \emptyset \quad \text{and} \quad T_1 + T_2 \subseteq \text{epi} f(x), \quad \#T_1 + \#T_2 \leq \dim X + 1 \bigg\}.
\]

(30)

where

\[
S(x, \varepsilon_1, T_1) := \left\{ \sum_{t \in \text{supp} \lambda} \lambda_t \partial_\varepsilon f_t(x) : \lambda \in \Delta(T), \quad \varepsilon_t \in \Delta^e(T_t), \right\},
\]

\[
N(x, \varepsilon_2, T_2) := \left\{ \sum_{t \in \text{supp} \eta} N^e_{\text{dom} f_t}(x) : \eta \in \Delta^e(T_2) \right\}.
\]

Indeed, to prove (30) we notice that by the finite dimension of \( X \) every element \( x^* \in \partial_\varepsilon f(x) \) must be expressed as \( x^* = \sum_{t \in T_1} \lambda_t x^*_t + \sum_{s \in T_2} w^*_s \) with some \( x^*_t \in \partial_\varepsilon f_t(x), t \in T_1, w^*_s \in N^e_{\text{dom} f_t} \), with \( \sum_{t \in T_1} \varepsilon_t + \sum_{s \in T_2} \nu_s \leq \varepsilon, T_1, T_2 \subseteq T, \quad \#(T_1 \cup T_2) \leq \dim X + 1 \) and \( f_t(x) + \varepsilon_t/\lambda_t \geq f(x) \) for all \( t \in T_1 \). Then \( \lambda_t \partial_\varepsilon f_t(x) + N^e_{\text{dom} f_t}(x) \subseteq \lambda_t \partial_\varepsilon f_t(x) + N^e_{\text{dom} f_t}(x) \) for \( t \in T_1 \cap T_2 \), consequently \( x^* \in N(x, \varepsilon_1, T_1) + N(x, \varepsilon_2, T_2 \setminus T_1) \), where \( \varepsilon_1 := \sum_{t \in T_1} \varepsilon_t + \sum_{t \in T_1 \cap T_2} \nu_t \) and \( \varepsilon_2 := \sum_{s \in T_2 \setminus T_1} \nu_s \).

The final goal of this section is to give the following characterization of the epigraph of \( f^* \) and consequently an expression for itself. This result can be understood as the conjugate counterpart of Theorems 5.1 and 5.2.

**Theorem 5.3** (a) If \( f^* \) is epi-pointed, one has

\[
\text{epi} f^* = \text{conv} \bigcup \{ \text{epi} f_t : t \in T \} + (\text{epi} f^*)^e.
\]
(b) If $T$ is compact and $t \to f_t(x)$ is upper semicontinuous for every $x \in X$ and for every $B \in \mathcal{P}(T)$ and every $(x^*_t, \alpha_t) \in (\text{epi} f_t^*)_\infty$ with $t \in B$ one has

$$
\sum_{t \in B} (x^*_t, \alpha_t) = (0, 0) \implies (x^*_t, \alpha_t) = (0, 0) \text{ for all } t \in B.
$$

Then

$$
\text{epi } f^* = \text{conv}\{\text{epi } f_t^* : t \in T\} + \text{conv}\{\text{epi } f_t^*_\infty : t \in T\}.
$$

**Proof** Assume that $f^*$ is epi-pointed. Consider $(x^*, \alpha) \in \text{epi } f^*$, then by Lemma 5.1 we have $(x^*, \alpha) = \lim(x^*_n, \alpha_n)$ for some $(x^*_n, \alpha_n) \in \text{conv}\{\bigcup_{t \in T} \text{epi } f_t^*\}$. Moreover, we can write

$$(x^*_n, \alpha_n) = \sum_{k=1}^{N} \lambda_{t_k,n}(y^*_{k,n}, \beta_{k,n})$$

where $N := \text{dim } X + 2$, $(\lambda_{t_k,n}) \in \Delta(T)$ and $(y^*_{k,n}, \beta_{k,n}) \in \text{epi } f_{t_k,n}^*$ for some $t_{k,n} \in T$. Now by a similar argumentation as the one given in Lemma 5.1 up to a subsequence $\lambda_{t_k,n}(y^*_{k,n}, \beta_{k,n})$ must converge to a point $(w^*_k, \beta_k)$ and $\lambda_{k,n} \to \lambda_k$, otherwise there exists $(u^*_k, \mu_k) \in (\text{epi } f^*_\infty = \text{epi}(f^*)_\infty$ not all equal to zero such that $\sum_{k=1}^{N} (u^*_k, \mu_k) = (0, 0)$ which contradicts the fact that $f^*$ is epi-pointed. Now, on the one hand for every $k = 1, ..., N$ with $\lambda_k \neq 0$ one has that

$$(y^*_{k,n}, \beta_{k,n}) \to (y^*_k, \beta_k) \in \text{cl}\{\text{epi } f_t^* : t \in T\}.$$ 

On the other hand for every $k = 1, ..., N$ with $\lambda_k = 0$ one has that

$$\lambda_{k,n}(y^*_k, \beta_k) \to (w^*_k, \gamma_k) \in (\text{epi } f^*_\infty).$$

Therefore,

$$x^* = \sum_{k: \lambda_k > 0} \lambda_k (y^*_k, \beta_k) + (w^*, \beta),$$

where $(y^*_k, \beta_k) \in \text{cl}\{\text{epi } f_t^* : t \in T\}$ and $(w^*, \beta) = \sum_{k: \lambda_k > 0} (w^*_k, \gamma_k) \in (\text{epi } f^*_\infty$.

The proof of (11) follows similar arguments as the previous one, together with the argumentation about the compactness of $T$ and the upper semicontinuity as was given in the proof of Theorem 5.2 so we omit the proof. \(\square\)

**Remark 5.3** A relevant point to consider is that all the convex combinations given in the results of this subsection can be written as convex combinations of no more than $\text{dim } X + 1$ points (or $\text{dim } X + 2$ for formulas related to epigraphs). In the same form, condition (27) and (31) are equivalent to consider $A \subseteq T$ with $\#A \leq \text{dim } X$ and $B \subseteq T$ with $\#B \leq \text{dim } X + 1$ in (27) and (31) respectively.
5.2 Infinite-Dimensional Locally Convex Spaces

For the sake of simplicity we assume that the functions \( f_t \in \Gamma_0(X) \) for all \( t \in T \).

Consider \( x \in X \), by \( \mathcal{F}_x \) we denote the family of all finite-dimensional linear subspaces of \( X \) containing \( x \).

**Theorem 5.4** Consider a family of functions \( \{f_t\}_{t \in T} \subseteq \Gamma_0(X) \). Then, for every \( x \in X \) and \( \varepsilon \geq 0 \) one has

\[
\partial_x f(x) = \bigcap_{L \in \mathcal{F}_x, \gamma \geq 0} \text{cl}^w \left( \{ \mathcal{G}(x, \varepsilon_1, \gamma) + N_{\text{dom} f_t \cap L}(x) \mid (\varepsilon_1, \varepsilon_2) \in \Delta^\varepsilon(\{1,2\}) \} \right),
\]

where

\[
\mathcal{G}(x, \varepsilon_1, \gamma) := \left\{ \sum_{t \in T} \lambda_t \partial_{x_t^* + \gamma} f_t(x) : \lambda \in \Delta(T), (\varepsilon_t) \in \Delta^{\varepsilon_1}(T), \text{and } f_t(x) + \varepsilon_t/\lambda_t + \gamma \geq f(x) \right\}
\]

**Proof** Since the right side of the equation is contained in the \( \varepsilon \)-subdifferential of \( f \) at \( x \), we must focus on the opposite inclusion. Without loss of generality we (may) assume that \( x = 0 \).

Now take \( \gamma > 0 \), \( V^* \in N_0 \) and \( L \in \mathcal{F}_0 \) such that \( L^\perp \subseteq V^* \). Take \( x^* \) in \( \partial_x f(0) \), consider \( F := \text{spn}\{L \cap \text{dom } f\} \) and \( P : X \to F \) a continuous linear projection and let us denote by \( P^* \) its adjoint operator. Then \( x^* = y^* + z^* \), where \( y^* := P^*(x^*_t) \) and \( z^* := x^* - y^* \in F^\perp \), then \( x^*_t \) belongs to \( \partial f_t(0) \). Since the relative interior of \( \text{dom } f_t \) with respect to \( F \) is nonempty, the hypotheses of Theorem 5.1 hold (in \( F \)), then there exists \( \varepsilon_1, \varepsilon_2 \geq 0 \) with \( \varepsilon_1 + \varepsilon_2 = \varepsilon \) together with elements \( u^*, \lambda^* \) in \( S(0, \varepsilon_1) \) and \( N_{\text{dom} f_t \cap L}(0) \) respectively such that \( x^*_t = u^* + \lambda^* \), which implies the existence of \( \lambda \in \Delta(T), (\varepsilon_t) \in \Delta^{\varepsilon_1}(T) \) such that

\[
f_t(x) + \varepsilon_t/\lambda_t + \gamma \geq f(x) \quad \text{and} \quad u^* \in \sum_{t \in \text{supp } \lambda} \partial_{x_t^* + \gamma} f_t(x) + (P^*)^{-1}(V^*).
\]

Whence \( P^*(u^*) \in \sum_{t \in \text{supp } \lambda} \partial_{x_t^* + \gamma} f_t(x) + V^* \) and by [3, Theorem 3.2] we have

\[
\sum_{t \in \text{supp } \lambda} \partial_{x_t^* + \gamma} f_t(x) + (P^*)^{-1}(V^*) \subseteq \partial_{x_t^* + \gamma} f_t(x) + V^*.
\]

Moreover, \( P^*(x^*) \in N_{\text{dom} f_t \cap L}(x) \). Therefore,

\[
x^* \in \mathcal{G}(x, \varepsilon_1, \gamma) + N_{\text{dom} f_t \cap L}(x) + V^* + V^*.
\]

From the arbitrariness of \( L, V^* \) and \( \gamma > 0 \) we get the result. \( \square \)
Remark 5.4 In [30] the author has proposed changing the family $\mathcal{F}_x$ for some family of convex sets $C \subseteq \text{dom } f$ which covers $\text{dom } f$. It is not difficult to see that $N^\eta_{\text{dom } f \cap C \cap L}(x) \subseteq N^\eta_{\text{dom } f \cap C \cap L}(x)$ for every set $C$ containing $x$. Then, in order to get [32] with some special class of sets $C$ one needs a formula in the following form

$$N^\eta_{\text{dom } f \cap C \cap L}(x) = \text{cl}^{\nu^*} \left( N^\eta_{\text{dom } f \cap C}(x) + (L - x)^\perp \right).$$

(33)

or some $\nu$-enlargement of the above expression

$$N^\eta_{\text{dom } f \cap C \cap L}(x) = \bigcap_{\nu > 0} \text{cl}^{\nu^*} \left( N^\eta_{\text{dom } f \cap C}(x) + (L - x)^\perp \right).$$

(34)

Then, in order to obtain a new formula for the $\varepsilon$-subdifferential of $f$ one can consider a family of sets $\{C_\alpha : \alpha \in A\}$ satisfying \((33)\) for large enough $L$ and all $\eta \leq \varepsilon$ which cover $\text{dom } f$, or if one prefers the use of the $\nu$-enlargement, one can consider a family of sets $\{C_\alpha : \alpha \in A\}$ satisfying \((34)\) for large enough $L$ and all $\eta \leq \varepsilon$ which cover $\text{dom } f$. In particular it happens in [30] Theorem 6.2, where the author has used a family of closed convex sets.

A more concrete idea is to understand the $\varepsilon$-normal set of the intersection as the $\varepsilon$-subdifferential of the sum of the indicators of the respective sets, that is,

$$N^\eta_{\text{dom } f \cap C \cap L}(x) = \begin{cases} \partial^\eta_{\text{dom } f \cap C \cap L}(x) = \partial^\eta_{\text{dom } f \cap C + \delta_L}(x), & \text{if } \eta > 0, \\ N_{R_+(\text{dom } f \cap C \cap L)}(x) = \partial(\delta_{R_+(\text{dom } f \cap C \cap L)}(x) + \delta_L)(0), & \text{if } \eta = 0, \end{cases}$$

then one can apply Hiriart-Urruty and Phelps’s formulae. Nevertheless, these formulae need that the sets be closed, or at least satisfied

$$\text{cl}(\text{dom } f \cap C \cap L) = \text{cl}(\text{dom } f \cap C) \cap L$$

(36)

$$\text{cl}(R_+(\text{dom } f \cap C - x) + L) = \text{cl}(R_+(\text{dom } f \cap C - x) + x) \cap L$$

(37)

(see, e.g., [29, 41] Proposition 2), where Hiriart-Urruty and Phelps’s formulae have been imposed with weaker assumptions, in particular under \((33)\) and \((34)\). Then, in order to obtain formulae for the $\varepsilon$-subdifferential (with $\varepsilon > 0$) of $f$ one defines $\mathcal{A}_x^\varepsilon$ as some family of sets $C$ satisfying \((36)\) and \((37)\) for large enough $L$. Similarly, for the Moreau-Rockafellar Subdifferential one can consider $\mathcal{A}_x$ as some family of sets $C$ satisfying \((37)\) for large enough $L$.

Typical examples of sets that satisfy \((36)\) and \((37)\) are sets which satisfy the accessibility lemma, that is there exist some $x_0 \in \text{dom } f \cap C$ such that $[x_0, x] := \{(1 - \lambda)x_0 + \lambda x : \lambda \in [0, 1]\} \subseteq \text{dom } f \cap C$ for every $x \in \text{cl}(\text{dom } f \cap C)$.

This assumption is satisfied under the nonemptiness of some notations of relative interior (see \([12]\)). In this scenario, we can underline \([13] \) Corollary 8 where the authors assume that the relative interior of the domain is not empty, in this case it is enough to use $\mathcal{A}_x = \{\text{dom } f\}$. Another assumption is that $\mathbb{R}_+(\text{dom } - x)$ is closed (see, e.g., \([12] \) Corollary 9), again it is enough to consider $\mathcal{A}_x = \{\text{dom } f\}$. 
As a corollary to our formula we recover the result given in [13] Theorem 4 (see also [11,12,14]).

**Corollary 5.1** In the setting of Theorem 5.4 one has for all \( x \in X \)

\[
\partial_x f(x) = \bigcap_{\gamma > 0} \text{cl} \{ \text{conv} \{ \bigcup_{t \in T(x)} \partial_{\gamma} f_t(x) \} + N_{\text{dom } f \cap L}(x) \}.
\]  

(38)

**Corollary 5.2** In the setting of Theorem 5.4 assume that \( f \) is continuous at some point of its domain, then for \( \varepsilon > 0 \) and all \( x \in X \) one has

\[
\partial_x f(x) = \bigcup_{\gamma > 0} \{ \bigcap_{\gamma > 0} \text{cl} \{ \mathcal{S}(x, \varepsilon_1 + \gamma, \gamma) + N_{\text{dom } f}(x) \} \mid \varepsilon_1 + \varepsilon_2 = \varepsilon \text{ and } \varepsilon_1, \varepsilon_2 \geq 0 \},
\]

Proof Let \( x^* \in \partial_x f(x) \) and \( \gamma > 0 \), then by Theorem 5.4 there exist nets \( y_{\nu,L}^* \in \mathcal{S}(x, \varepsilon_1(\nu, L), \gamma) \) and \( \lambda_{\nu,L}^* \in N_{\text{dom } f}(x) \) such that \( x^* = \lim y_{\nu,L}^* + \lambda_{\nu,L}^* \); because \( f \) is continuous at some point the net \( (y_{\nu,L}^*) \) must be bounded, so (up to a subnet) we may assume \( y_{\nu,L}^* \to y^* \) and \( \lambda_{\nu,L}^* \to \lambda^* \) and \( \varepsilon_1(\nu, L) \to \varepsilon_1 \) and \( \varepsilon_2(\nu, L) \to \varepsilon_2 \). It follows that \( \lambda^* \in N_{\text{dom } f}(x) \). Now it only remains to show that \( y^* \in \text{cl} \mathcal{S}(x, \varepsilon_1 + \gamma, \gamma) \). Indeed, consider \((\nu_0, L_0)\) such that for every \((\nu, L) \succ (\nu_0, L_0), |\varepsilon_1(\nu, L) - \varepsilon_1| \leq \gamma \). Whence \( y_{\nu,L}^* \in \sum_{t \in \text{supp} \lambda} \lambda_t \partial_{\varepsilon_t/\lambda_t + \gamma} f_t(x) \) and \( f_t(x) + \varepsilon_t/\lambda_t + \gamma \geq f(x) \) for some \((\varepsilon_t) \in \Delta^{(\nu,L)}(T)\), then considering the real numbers \( \varepsilon_t := \varepsilon_t - \varepsilon_t/\lambda_t + \gamma \), we get \( y_{\nu,L}^* \in \mathcal{S}(x, \varepsilon_1 + \gamma, \gamma) \) for every \((\nu, L) \succ (\nu_0, L_0)\), that concludes the proof.

Due to the fact that our formulae involve the \( \varepsilon \)-normal set of \( \text{dom } f \cap L \) at a point \( x \), we present the following result, where the set is expressed in terms of the data function \( f_t \). The proof of the following lemma follows [13] Lemma 5.

**Lemma 5.2** In the setting of Theorem 5.4 assume that \( f \) is proper. Then for every \( x \in \text{dom } f \) and every \( \varepsilon \geq 0 \), we have that:

\[
N_{\text{dom } f}(x) = \{ x^* \in X^* : (x^*, \langle x^*, x \rangle + \varepsilon) \in \text{epi}(\sigma_{\text{dom } f}) \} \tag{39}
\]

\[
= \{ x^* \in X^* : (x^*, \langle x^*, x \rangle + \varepsilon) \in \text{epi } f^* \} \tag{40}
\]

\[
= \{ x^* \in X^* : (x^*, \langle x^*, x \rangle + \varepsilon) \in \text{cl conv} \left( \bigcup_{t \in T} \text{epi } f_t^* \right) \} \tag{41}
\]

\[
= \{ x^* \in X^* : (x^*, \langle x^*, x \rangle + \varepsilon) \in \left[ \text{cl conv} \left( \bigcup_{t \in T} \text{gph } f_t^* \right) + \{0\} \times [0, \varepsilon] \right] \} \tag{42}
\]

\[
+ \{0\} \times [0, \varepsilon]
\]
Consequently for every $L \in \mathcal{F}_x$, $N_{\text{dom } f \cap L}(x)$ can be expressed as

\[
\{x^* \in X^* : (x^*, \langle x^*, x \rangle + \varepsilon) \in \left[ \text{cl conv} \left( L^+ \times \{0\} \cup \bigcup_{t \in T} \text{epi } f_t^* \right) \right]_\infty, \text{ or } \quad (43)
\]

\[
\{x^* \in X^* : (x^*, \langle x^*, x \rangle + \varepsilon) \in \left[ \text{cl conv} \left( L^+ \times \{0\} \cup \bigcup_{t \in T} \text{gph } f_t^* \right) \right]_\infty. \quad (44)
\]

Proof The equalities (39) and (40) follow from the definition of $N_{\text{dom } f \cap L}(x)$ and from the fact that $(\text{epi } f^*)_\infty = (\text{epi } \sigma_{\text{dom } f})(\text{see }[13, \text{Lemma 5}])$. The third equation is given by the fact that $\text{epi } f^* = \text{cl conv} \{ f_t^* : t \in T \}$ (see Lemma 3.1). Now we have to prove (42), from the fact that $\text{cl conv} \left( \bigcup_{t \in T} \text{epi } f_t^* \right) \supseteq \text{cl conv} \left( \bigcup_{t \in T} \text{gph } f_t^* \right) + \{0\} \times [0, \varepsilon]$.

The inclusion $\supseteq$ holds in (42), then we only have to prove the opposite inclusion. We have that

\[
\text{cl } w^* \left[ \text{cl conv} \left( \bigcup_{t \in T} \text{gph } f_t^* \right) + \{0\} \times \mathbb{R}_+ \right] = \text{cl conv} \left( \bigcup_{t \in T} \text{epi } f_t^* \right).
\]

Since $f^*$ is proper, we have

\[
\left[ \text{cl conv} \left( \bigcup_{t \in T} \text{gph } f_t^* \right) \right]_\infty - \{0\} \times \mathbb{R}_+ \subseteq \left[ \text{cl conv} \left( \bigcup_{t \in T} \text{epi } f_t^* \right) \right]_\infty - \{0\} \times \mathbb{R}_+ = \{(0, 0)\},
\]

so Dieudonn’s Theorem (see [21, Theorem 2.3.1]) implies that

\[
\text{cl conv} \left( \bigcup_{t \in T} \text{gph } f_t^* \right) + \{0\} \times \mathbb{R}_+\]

is closed, besides $\text{cl conv} \left( \bigcup_{t \in T} \text{epi } f_t^* \right) = \text{cl conv} \left( \bigcup_{t \in T} \text{gph } f_t^* \right) + \{0\} \times \mathbb{R}_+$ and

\[
\left[ \text{cl conv} \left( \bigcup_{t \in T} \text{epi } f_t^* \right) \right]_\infty = \left[ \text{cl conv} \left( \bigcup_{t \in T} \text{gph } f_t^* \right) \right]_\infty + \{0\} \times \mathbb{R}_+. \quad (45)
\]

Then take $x^* \in X^*$ such that $(x^*, \langle x^*, x \rangle + \varepsilon) \in \left[ \text{cl conv} \left( \bigcup_{t \in T} \text{epi } f_t^* \right) \right]_\infty$, by (45) there exist $(y^*, \gamma) \in \left[ \text{cl conv} \left( \bigcup_{t \in T} \text{gph } f_t^* \right) \right]_\infty$ and $\eta \geq 0$ such that $(x^*, \langle x^*, x \rangle + \varepsilon) = (y^*, \gamma + \eta)$, so $x^* = y^*$. Moreover, since

\[
\text{dom } f \times \{-1\} \subseteq \left[ \left( \text{cl conv} \left( \bigcup_{t \in T} f_t^* \right) \right)_\infty \right]^\sim = \left[ \left( \text{cl conv} \left( \bigcup_{t \in T} \text{gph } f_t^* \right) \right)_\infty \right]^\sim \subseteq \left[ \left( \text{cl conv} \left( \bigcup_{t \in T} \text{gph } f_t^* \right) \right)_\infty \right]^\sim,
\]

we get $\langle (x^*, \gamma), (x, -1) \rangle \leq 0$, so $\eta \leq \varepsilon$, and then

$$\langle x^*, \langle x^*, x \rangle + \varepsilon \rangle \in \left[ \text{cl conv} \{ \text{gph} f^*_t : t \in T \} \right]_{\infty} + \{0\} \times [0, \varepsilon].$$

Finally, (43) and (44) follow applying (41) and (42) to the family of functions $\{f_t : t \in T\} \cup \{\delta_L\}$.

$\square$

6 Conclusions

We have derived formulae for the $\varepsilon$-subdifferential and the conjugate functions of the supremum function. We studied two different methods to get these expressions.

The first one corresponds to the case when the index set is a directed set and the family of functions is an increasing family of epi-pointed functions. Nevertheless, these hypotheses can be obtained, as we have shown in Theorem 4.2 and Corollary 4.3, considering the maximum function over all the finite parts of the index set and doing some appropriate perturbation of the function in order to get the epi-pointedness property, in this case the formula obtained does not involve the use of normal cones of the supremum function, which for example, allows us to give a formula for the normal cone of an arbitrary intersection only considering the normal cones of the data, which (at least not directly) cannot be derived from the formulae expressed in Section 5.

The second one corresponds to obtaining general formulae for the conjugate and the $\varepsilon$-subdifferential of $f$ in finite-dimensional spaces. Next, we have shown how to derive these results in general locally convex spaces by a reduction to finite-dimensional subspaces. It is important to say that this reduction to finite-dimensional subspaces allows us to use standard arguments in subdifferential theory to get the results, which in particular extend some of the most general formulae to the Moreau-Rockafellar subdifferential of the supremum function, to the calculus of the $\varepsilon$-subdifferential of the supremum function.

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