COMBINATORIAL FORMULAS FOR ARITHMETIC DENSITY

Robert Schneider
Department of Mathematics, University of Georgia, Athens, Georgia
robertsc@mtu.edu

Andrew V. Sills
Department of Mathematical Sciences, Georgia Southern University, Statesboro
and Savannah, Georgia
ASills@GeorgiaSouthern.edu

Received: 3/16/22, Accepted: 6/21/22, Published: 7/8/22

Abstract
Let \( d_S \) denote the arithmetic density of a subset \( S \subseteq \mathbb{N} \). We derive a power series in \( q \in \mathbb{C}, |q| < 1 \), with coefficients related to integer partitions and integer compositions, that yields \( 1/d_S \) in the limit as \( q \to 1 \) radially.

1. Introduction and Statement of Results
In recent works [10, 11], Ono, Wagner, and the first author use methods from partition theory to prove \( q \)-series formulas for the arithmetic density \( d_S \) of a subset \( S \) of natural numbers \( \mathbb{N} \):

\[
\frac{d_S}{\ell} := \lim_{N \to \infty} \frac{\#\{n \in S : n \leq N\}}{N}.
\]

(1)

In this paper, we prove a combinatorial limiting formula for the reciprocal value \( 1/d_S \), using ideas related to both partitions and integer compositions.

Let \( P \) denote the set of integer partitions, unordered finite sums of natural numbers (see, e.g., [1]), including the empty partition \( \emptyset \in P \). Let \( S \subseteq \mathbb{N} \), and let \( P_S \) denote the set of partitions whose parts lie in the subset \( S \). For \( \lambda \in P \), let \( |\lambda| \) denote the size of the partition (sum of parts), let \( \ell(\lambda) \) denote the length (number of parts), and let \( m_i(\lambda) \) denote the multiplicity (frequency) of \( i \) as a part of partition \( \lambda \). Note that \( |\emptyset| = \ell(\emptyset) = 0 \).

Define the sum \( C_S(n) \) over partitions \( \lambda \in P_S \) with size at most \( n \), noting \( m_i(\lambda) = 0 \) if \( i \notin S \), by

\[
C_S(n) := \sum_{\lambda \in P_S \atop 0 \leq |\lambda| \leq n} \frac{(-1)^{\ell(\lambda)} \ell(\lambda)!}{m_1! m_2! m_3! \cdots m_n!}.
\]

(2)
Sums over partitions have a history dating back to work of MacMahon [9, p. 61ff.] and Fine [4, §22]. They are important in modern number theory; the $q$-bracket of Bloch–Okounkov [3, 14] is an operator from statistical physics that induces modularity in partition-theoretic $q$-series.

Equation (2) has a natural combinatorial interpretation in terms of integer compositions, which are ordered finite sums of natural numbers (see, e.g., [8, Section IV]). Let $\mathcal{C}$ denote the set of all integer compositions. We will extend the partition terms and notations defined above to compositions, with the same meanings. Let $\mathcal{C}_S$ denote the compositions whose parts lie in $S \subseteq \mathbb{N}$. For each partition $\lambda$ in (2), $\ell(\lambda)!/m_1!m_2!m_3!\cdots$ counts the multiset permutations of the parts of $\lambda$, i.e., all compositions $\gamma \in \mathcal{C}_S$ having the same parts as $\lambda$.

**Proposition 1.** $C_S(n)$ counts the number of compositions $\gamma$ in $\mathcal{C}_S$ of even length, minus those of odd length, having sizes between 0 and $n$ inclusive:

$$C_S(n) = \sum_{\gamma \in \mathcal{C}_S, 0 \leq |\gamma| \leq n} (-1)^{\ell(\gamma)}.$$ 

Let $F_S(q) := \sum_{n \geq 0} C_S(n)q^n$ with domain of convergence depending on $S \subseteq \mathbb{N}$; and define the auxiliary series $f_S(q) := 1 + \sum_{n \in S} q^n, |q| < 1$. Our focus is on the behavior of $F_S(q)$ as $q \to 1$.

**Theorem 1.** Let $S \subseteq \mathbb{N}$ such that $d_S > 0$, and such that $f_S(q) = 1 + \sum_{n \in S} q^n$ is analytic and has no zeros on $\{q \in \mathbb{C} : |q| < 1\}$. Then as $q \to 1$ radially, we have

$$\lim_{q \to 1} F_S(q) = \lim_{q \to 1} \sum_{n=0}^{\infty} C_S(n)q^n = \frac{1}{d_S}.$$ 

Theorem 1 has something of an analytic converse.

**Theorem 2.** Let $S \subseteq \mathbb{N}$ such that $F_S(q) = \sum_{n=0}^{\infty} C_S(n)q^n$ is analytic and has no zeros on $\{q \in \mathbb{C} : |q| < 1\}$, and such that $\lim_{q \to 1} F_S(q) = L$ exists as $q \to 1$ radially. If the limit $L$ is infinite, then $d_S = 0$. If $L \geq 1$ is finite, then $d_S = 1/L$. The case $L < 1$ cannot occur.

We postpone proofs until Section 2. Here is an example that uses Theorem 1.

**Example 1.** For $t, r \in \mathbb{N}$, with $r \leq t$, let $S_{r,t}$ denote the set of positive integers congruent to $r$ modulo $t$. Then as $q \to 1$ radially, we have

$$\lim_{q \to 1} \sum_{n=0}^{\infty} C_{S_{r,t}}(n)q^n = t.$$ 

We confirm that $d_{S_{r,t}} = 1/t > 0$. Moreover, since $f_{S_{r,t}}(q) = 1 + \sum_{n \in S_{r,t}} q^n = 1 + \sum_{n \geq 0} q^{r+nt} = (1 + q^r - q^t)/(1 - q^t)$ is analytic on $|q| < 1$ with no zeros in...
Lemma 1. Our central lemma expresses the coefficients of the reciprocal of a power series as a Cauchy product formula for power series, together with a theorem of Frobenius [5]. We prove the theorems using the multinomial theorem, geometric series and the Cauchy product formula for power series.

2. Proofs of Theorem 1 and Theorem 2

We prove the theorems using the multinomial theorem, geometric series and the Cauchy product formula for power series, together with a theorem of Frobenius [5]. Our central lemma expresses the coefficients of the reciprocal of a power series as a sum over partitions.

Lemma 1. For $a_0 \in \mathbb{C}, a_0 \neq 0$, let $f(q) := \sum_{n \geq 0} a_n q^n$ be analytic on $\{q \in \mathbb{C} : |q| < 1\}$. Then on the domain of analyticity of $\phi(q) := 1/f(q)$ we have

$$\phi(q) = \frac{1}{f(q)} = \sum_{n=0}^{\infty} c_n q^n,$$

where $c_n = \sum_{\lambda \in \mathcal{P}} (-1)^{\ell(\lambda)} \ell(\lambda)! \frac{a_1^{m_1}a_2^{m_2}a_3^{m_3} \cdots a_n^{m_n}}{a_0^{r(\lambda)+1}m_1!m_2!m_3! \cdots m_n!}$. 

Proof. The result is equivalent to [12, Thm. 3.1], which is proved using the Maclaurin expansion $\phi(q) = 1/f(q) = \sum_{n \geq 0} \phi^{(n)}(0)q^n/n! = \sum_{n \geq 0} c_n q^n$. For completeness, we give a self-contained proof of the identity for $c_n$. Begin with the multinomial theorem (see, e.g., [7]), written as a sum over partitions $\lambda$ with largest part $\lg(\lambda)$ at most $k$, and length exactly $r$:

$$(x_1 + x_2 + x_3 + \cdots + x_k)^r = r! \sum_{0 \leq \ell(\lambda) \leq k}^{\ell(\lambda) = r} \frac{x_1^{m_1}x_2^{m_2}x_3^{m_3} \cdots x_k^{m_k}}{m_1!m_2!m_3! \cdots m_k!}. \tag{3}$$

Make the substitution $x_i = \frac{a_i}{a_0} q^i$ with $a_i$ as in Lemma 1. We now let $k$ tend to infinity: if $g(q) := \sum_{0}^{\infty} \frac{a_0}{a_0} q + \frac{a_0}{a_0} q^2 + \frac{a_0}{a_0} q^3 + \cdots = a_0^{-1} f(q) - 1$ converges absolutely on $|q| < 1$, then (3) becomes

$$(g(q))^r = \frac{1}{a_0^r} \sum_{\ell(\lambda) = r} q^{\ell(\lambda)} \frac{a_1^{m_1}a_2^{m_2}a_3^{m_3} \cdots}{m_1!m_2!m_3! \cdots}. \tag{4}$$

For $q \in \mathbb{C}$ such that $|g(q)| < 1$, multiplying both sides of (4) by $(-1)^r$ and summing over $r \geq 0$ gives an infinite geometric series on the left, and a sum over all partitions on the right:

$$\frac{1}{1 + g(q)} = \frac{a_0}{f(q)} = \sum_{\lambda \in \mathcal{P}} q^{\ell(\lambda)} \frac{(-1)^{\ell(\lambda)} \ell(\lambda)! a_1^{m_1}a_2^{m_2}a_3^{m_3} \cdots}{a_0^{r(\lambda)}m_1!m_2!m_3! \cdots}. \tag{5}$$

Dividing through by $a_0$, then collecting coefficients of $q^n$ on the right-hand side to write $1/f(q) = \sum_{n \geq 0} c_n q^n$, proves the identity for $c_n$. While the geometric
series representation of \( a_0/f(q) \) is only valid when \( |q(q)| < 1 \), the formula for \( c_n = \phi^{(n)}(0)/n! \) holds on the domain of analyticity of \( \phi(q) = 1/f(q) \) by uniqueness of the Maclaurin series representation of the analytic function, noting \( f(0) \neq 0 \) so \( 1/f(q) \) can be expressed as a power series centered at \( q = 0 \).

**Lemma 2.** Let \( a_1 \in \mathbb{C}, a_0 \neq 0 \), be such that \( f(q) = \sum_{n \geq 0} a_n q^n \) is analytic and has no zeros on \( \{q \in \mathbb{C} : |q| < 1\} \). Define \( c_n \) as in Lemma 1. Define \( A(n) := \sum_{i=0}^n a_i, C(n) := \sum_{i=0}^n c_i \), and let \( A := \lim_{n \to \infty} A(n)/n \) if the limit exists. Then if \( A \neq 0 \), as \( q \to 1 \) radially we have that

\[
\lim_{q \to 1} \sum_{n=0}^{\infty} C(n) q^n = \frac{1}{A}.
\]

**Proof.** This lemma can be obtained as a special case of the first asymptotic formula in [6]; for completeness, we prove it directly. For the claimed limit to exist as \( q \to 1 \), we need the Maclaurin series for \( \phi(q) = 1/f(q) \) convergent on the unit disk \( |q| < 1 \). Take \( \varepsilon > 0 \) and suppose that \( f(q_0) = 0 \) for some \( q_0 \) with \( |q_0| < 1 - \varepsilon \). Then \( \phi(q_0) = 1/f(q_0) \) represents a pole, so the series \( \phi(q) \) has radius of convergence \( \leq 1 - \varepsilon \) and the limit as \( q \to 1 \) does not exist. Hence the conditions on the domain and zeros are necessary. That this limit is equal to \( 1/A \) follows from [5], which proves when \( q \to 1 \) radially from within the unit disk\(^1\) that

\[
\lim_{q \to 1} (1-q)f(q) = A.
\]

Noting that \( 1/(1-q), \phi(q), \) and thus \( (1-q)^{-1}\phi(q) \) are analytic on \( |q| < 1 \), then Lemma 1 gives

\[
\frac{1}{A} = \lim_{q \to 1} \frac{1}{1-q} \sum_{n \geq 0} c_n q^n = \lim_{q \to 1} \left( \sum_{n \geq 0} q^n \right) \left( \sum_{n \geq 0} c_n q^n \right) = \lim_{q \to 1} \sum_{n \geq 0} C(n) q^n,
\]

with \( C(n) = \sum_{0 \leq i \leq n} c_i = \sum_{0 \leq i \leq n} 1 \cdot c_i \) due to the Cauchy product formula for power series. \( \square \)

**Proof of Theorem 1.** Set \( a_0 = 1 \) and for \( n \geq 1 \), let \( a_n \) be the indicator function of \( S \subseteq \mathbb{N} \), i.e., \( a_n = 1 \) if \( n \in S \), and \( a_n = 0 \) if \( n \notin S \). Since \( 1 + \sum_{n \in S} q^n = \sum_{n \geq 0} a_n q^n \) is analytic on \( |q| < 1 \) by comparison with geometric series, then with the stipulation it has no zeros, \( f_S(q) \) satisfies the analytic conditions of Lemmas 1 and 2. Thus, \( a_1, a_2^2, \ldots a_r^r = 1 \) if \( \lambda \in P_S \) and \( = 0 \) otherwise in the expression for \( c_n \) in Lemma 1, which yields \( C(n) = C_S(n) \) in Lemma 2. Observing that \( A = \lim_{n \to \infty} A(n)/n \) in Lemma 2 gives the theorem. \( \square \)

\(^1\)In fact, this limiting result holds if \( q \to 1 \) through any path in a Stolz sector of the unit disk (see, e.g., [13]), a region with vertex at \( q = 1 \) such that \( \frac{1-q}{1-\eta} \leq M \) for some \( M > 0 \).
Proof of Theorem 2. Since $F_S(q)$ is analytic with no zeros for $|q| < 1$, we have
$\Phi_S(q) := 1/F_S(q)$ is also analytic with no zeros inside the unit disk, and has a
unique power series expansion $\Phi_S(q) = \sum_{n \geq 0} d_n q^n$ around the origin. In Theorem
1, $1/F_S(q)$ has the power series expansion $(1 - q) (1 + \sum_{n \in S} q^n) = (1 - q)f_S(q)$,
analytic for $|q| < 1$. Then by uniqueness of the power series expansion of $1/F_S(q)$ on
its domain of analyticity, $\Phi_S(q) = (1 - q)f_S(q)$ on the unit disk. If $L$ is infinite, then
$\lim_{q \to 1} \Phi_S(q) := \lim_{q \to 1} 1/F_S(q) = 0$ which is equal to $d_S$ by (6). If $L \geq 1$ is finite,
then $1/L = \lim_{q \to 1} 1/F_S(q) = \lim_{q \to 1} \Phi_S(q) = d_S$, also by (6). If the case $L < 1$
were to occur under these hypotheses, it would mean $1/F_S(q) = (1 - q)f_S(q) \to
1/L > 1$ as $q \to 0$. But with all coefficients of $\sum_{n \geq 0} a_n q^n$ being 0 or
1, then by (6), $\lim_{q \to 1} (1 - q)f_S(q) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n \leq 1$, if the limit exists.
Thus $L < 1$ cannot occur. \hfill \Box

3. Further Remarks

Let $c(n) = 2^{n-1}$ denote the number of compositions of size $n \geq 1$ [8, p. 151],
with $c(0) := 1$, and let $c_S(n)$ denote the number of size-$n$ compositions having all
parts from $S \subseteq \mathbb{N}$. Considering the results in Section 1, one wonders about the
"non-alternating" variant of (2):

$$C^+_S(n) := \sum_{\lambda \in \mathcal{P}_S \atop 0 \leq |\lambda| \leq n} \frac{\ell(\lambda)!}{m_1! m_2! \cdots m_n!} = \sum_{0 \leq j \leq n} c_S(j), \quad (8)$$

the number of compositions $\gamma \in \mathcal{C}_S$ having sizes $0 \leq |\gamma| \leq n$. This is a fairly natural
statistic, e.g., for $S = \mathbb{N}$ one has $C^+_S(n) = \sum_{0 \leq j \leq n} c(j) = 1 + (1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}) = 2^n$.

What are the analytic properties of the power series $F^+_S(q) := \sum_{n \geq 0} C^+_S(n) q^n$,
where $|q| < 1$? Let $g_S(q) := f_S(q) - 1$ with $f_S(q)$ analytic on $|q| < 1$. Similar multinomial,
geometric series, Cauchy product and analytic arguments as above, applied
to $\sum_{r \geq 0} (g_S(q))^r$, $|g_S(q)| < 1$, prove the generating function formula $\sum_{n \geq 0} c_S(n) q^n =$
$\sum_{\gamma \in \mathcal{C}_S} q^{|\gamma|} = (1 - f_S(q))^{-1} = (1 - g_S(q))^{-1}$ (see, e.g., [2, Thm. 1.1]), for $q \in \mathbb{C}$ such
that $|f_S(q)| < 2$. Then by (5), together with the Cauchy product for power series
as used in (7) above, we deduce the identity

$$F^+_S(q) = \sum_{n=0}^{\infty} C^+_S(n) q^n = \frac{1}{(1-q)(2-f_S(q))}. \quad (9)$$

However, limiting formulas analogous to Theorem 1 do not result in this case,
as $F^+_S(q)$ is not analytic on the unit disk. To see this, note that if $S$ is a finite
nonempty subset, then $|f_S(q)| \leq 1 + \# S$ when $|q| \leq 1$ since there are $\# S$ terms
of the form \( q^n \), with \( f_S(1) = 1 + \# S \) exactly. If \( S \) is an infinite subset of \( \mathbb{N} \), then \( |f_S(q)| \to \infty \) as \( q \to 1 \). Thus \( |f_S(q)| < 2 \) for all \( |q| < 1 \) if and only if the subset \( S \) has one element. For a subset \( S \) with two or more elements, \( F_S^+ (q) \) converges on a disk strictly smaller than the unit disk, and \( \lim_{q \to 1} F_S^+ (q) \) does not exist.

**Remark 1.** It is possible to find composition-theoretic limiting formulas in a smaller disk, e.g.,

\[
\lim_{q \to 1/2} \frac{1-2q}{1-q} \sum_{n \in S} c(n)q^n = \lim_{q \to 1/2} \frac{\sum_{n \in S} c(n)q^n}{\sum_{n \geq 0} c(n)q^n} = d_S. \tag{10}
\]

This can be deduced from (6):

\[
d_S = \lim_{q \to 1} (1-q) \sum_{n \in S} q^n = \lim_{q \to 1/2} (1-2q) \sum_{n \in S} (2q)^n = \lim_{q \to 1/2} \frac{1-2q}{1-q} \sum_{n \in S} 2^{n-1} q^n,
\]

noting that \( \sum_{n \geq 0} c(n)q^n = 1 + \sum_{n \geq 1} 2^{n-1} q^n = (1-q)/(1-2q) \).

**Acknowledgments** The authors are thankful to Maurice D. Hendon for advice on convergence of complex power series that strengthened our proofs, and for suggesting the special case of Example 1; to George E. Andrews for noting a useful correction in an earlier draft; and to the editor and anonymous referee for carefully reviewing our work.

**References**

[1] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, no. 2, Addison–Wesley, Reading, Massachusetts, 1976. Reissued, Cambridge University Press, 1998.

[2] M. Bicknell, and V. E. Hoggatt, Palindromic compositions, *Fibonacci Quarterly* 13 (1975), 350–356.

[3] S. Bloch and A. Okounkov, *The character of the infinite wedge representation*, Adv. Math. 149 (2000), 1–60.

[4] N. J. Fine, *Basic Hypergeometric Series and Applications*, Mathematical Surveys and Monographs, no. 27, American Mathematical Society, Providence, Rhode Island, 1988.

[5] G. Frobenius, Über die Leibnitzsche Reihe, *J. Reine Angew. Math.* 89 (1880), 262–264.

[6] G. H. Hardy and J. E. Littlewood, Tauberian theorems concerning power series and Dirichlet’s series whose coefficients are positive, *Proc. London Math. Soc. Ser. 2* 13 (1914), 174–191.

[7] K. K. Kataria, A probabilistic proof of the multinomial theorem, *Amer. Math. Monthly* 123 (2016), 94–96.
[8] P. A. MacMahon, *Combinatory Analysis*, vol. I, Cambridge University Press, 1915; Reissued (with volumes I and II bound in one volume), AMS Chelsea, 2001.

[9] P. A. MacMahon, *Combinatory Analysis*, vol. II, Cambridge University Press, 1916; Reissued (with volumes I and II bound in one volume), AMS Chelsea, 2001.

[10] K. Ono, R. Schneider, and I. Wagner, Partition-theoretic formulas for arithmetic densities, *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, Springer Proc. Math. Stat. 221 (2017), 611–624.

[11] K. Ono, R. Schneider, and I. Wagner, Partition-theoretic formulas for arithmetic densities, II, *Hardy–Ramanujan Journal* 43 (2020), 1–16.

[12] A. Salem, Reciprocal of infinite series and partition functions, *Integral Transforms and Special Functions* 22 (2011), 443–452.

[13] L. E. Snyder, Continuous Stolz extensions and boundary functions, *Trans. Amer. Math. Soc.* 119 (1965), 417–427.

[14] D. Zagier, Partitions, quasimodular forms, and the Bloch–Okounkov theorem, *Ramanujan J.* 4 (2016), 345–368.