1/4 BPS Dyonic Calorons

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ABSTRACT

We explore the 1/4 BPS configurations of the supersymmetric gauge theories on $R^{1+3} \times S^1$. The BPS bound for energy and the BPS equations are obtained and the characteristics of the BPS solutions are studied. These BPS configurations describe electrically charged calorons, which are constituted of dyons and carry linear momentum along the compact direction. We carry out various approaches to the single caloron case in the theory of $SU(2)$ gauge group.

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I. INTRODUCTION

It has been known for sometime that instantons on $R^3 \times S^1$, or so-called calorons, can be considered to be made of magnetic monopoles when there is a nontrivial Wilson loop which breaks the gauge symmetry to its abelian subgroups \cite{1–3}. For the $SU(N)$ gauge group, there are $N$ different kinds of fundamental monopoles, each corresponding to the roots in the extended Dynkin diagram. The relation between instantons and magnetic monopoles can also be understood by exploring the five dimensional Yang-Mills theories which appears as the low energy Lagrangian on parallel D4 branes and its T-dual version. In their $N = 4$ supersymmetric version on $R^{1+3} \times S^1$, instantons appear as 1/2 BPS objects. The low energy dynamics of calorons or instantons are given by the metric on the moduli space of caloron solutions.

There has been some works done some time ago on the 1/4 BPS dyons on $N = 4$ supersymmetric Yang-Mills theories on $R^{1+3}$, which can arise when several Higgs fields takes expectation values \cite{4,5}. In the five dimensional Yang-Mills theories, there are five Higgs fields and they can take nontrivial expectation values, besides the nontrivial Wilson loop along the compact circle. In those theories also the 1/4 BPS and non BPS configurations are possible. In this paper we explore these 1/4 BPS configurations. Especially we work out a single 1/4 BPS dyonic caloron case in the $SU(2)$ gauge theory.

More recently there has been some work on dyonic or electrically charged instantons in five dimensional field theory \cite{6–8}. As in the four dimensional theory, these dyonic instantons are 1/4 BPS instead of 1/2 BPS in the Yang-Mills theories with sixteen supersymmetries. These dyonic instantons also carry nontrivial angular momentum. They would become caloron when the space is compactified. As we will show in this paper, BPS calorons come with richer characteristics.

Usually we consider a BPS configuration to be at rest. However, they remain BPS when the configuration is Lorentz boosted. As our space is compactified along a circle, a BPS configuration can carry nonzero linear momentum along the circle. However the linear momentum is not topological and cannot be expressed as the boundary term in general.

The 1/4 BPS dyons can be understood as a planar web of fundamental strings and D-strings connecting parallel D3 branes in type IIB theory \cite{9}. Similarly the 1/4 BPS dyonic calorons have the string web picture. We explore this in our simple model.

The low energy dynamics of magnetic monopoles can be approached by the moduli space dynamics. When an additional scalar field is turned on, its effect can be incorporated as a potential term. It was shown in \cite{10} that the BPS configuration of this low energy dynamics corresponds to the 1/4 BPS dyonic configurations. From this correspondence one can read the electric charge of dyons for a given set of moduli parameters. This result can be also found directly from the field theory analysis. We consider the low energy dynamics of 1/4 BPS dyonic caloron and work out in detail these results in the $SU(2)$ case.

The plan of this paper is as follows. In Sec.2, we find the BPS bound on the energy functional. In Sec.3, we find the BPS equations which are satisfied by the configurations saturating the BPS bound. In Sec.4, we find the BPS caloron configurations, which can be regarded to be composed of monopoles and dyons. In Sec.5, we study the $SU(2)$ gauge group case in detail. Especially, we relate our 1/4 BPS configuration with the string web picture. In Sec.6, we...
II. THE BPS BOUND

The underlying spacetime is chosen to be five dimensional, with one of the space being compactified to a circle. The coordinates, $x^M$, where $M = 0, ..., 4$, are split to the time coordinate $x^0$ and space coordinates $x^\mu$ with $\mu = 1, 2, 3, 4$. The compactified coordinate $x^4$ has the finite range

$$0 \leq x^4 < \beta. \quad (1)$$

We consider only the periodic gauge and scalar fields. The allowed gauge transforms are those which leave the gauge fields periodic: small gauge transformations whose gauge functions are periodic, and large gauge transformations whose gauge functions are multi-valued. While our consideration can be easily generalized to arbitrary semi-simple gauge group, we focus on the $SU(N)$ gauge group for simplicity. We consider the Hermitian generators $T^a$ in $N$ dimensional fundamental representation with normalization $\text{tr} T^a T^b = \delta^{ab}/2$. The gauge field is then $A^a_M T^a$.

The Lagrangian we start with is

$$\mathcal{L} = \frac{1}{e^2} \text{tr} \left( -\frac{1}{2} F_{MN} F^{MN} + D_M \phi_I D^M \phi_I - \sum_{I<J} (-i[\phi_I, \phi_J])^2 \right), \quad (2)$$

where $D_M \phi_I = \partial_M \phi_I - i [A_M, \phi_I]$ and $e^2$ is the five dimensional coupling constant of length dimension. We decompose the Higgs field into one component and the rest,

$$\phi_I = a_I \phi + \zeta_I, \quad (3)$$

where $a_I$ is a unit vector in five dimension and $\zeta_I$ is orthogonal to $a_I$. The Gauss law is

$$D_i E_i + D_4 F_{04} - i[\phi, D_0 \phi] - i[\zeta, D_0 \zeta] = 0, \quad (4)$$

where $E_i = F_{0i}$ with $i = 1, 2, 3$.

The energy density is given by

$$\mathcal{E} = \frac{1}{e^2} \text{tr} \left( E_i^2 + B_i^2 + F_{04}^2 + F_{4i}^2 + D_0 \phi^2 + D_i \phi^2 + D_4 \phi^2 \right) + \mathcal{E}_\zeta, \quad (5)$$

where $B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}$ and

$$e^2 \mathcal{E}_\zeta = \text{tr} \left( (D_0 \zeta_I)^2 + (D_\mu \zeta_I)^2 - \sum_I [\phi, \zeta_I]^2 - \sum_{I<J} [\zeta_I, \zeta_J]^2 \right). \quad (6)$$

The energy density can be written as

$$\mathcal{E} = \frac{1}{e^2} \text{tr} \left\{ (E_i + F_{4i} \sin \alpha - D_i \phi \cos \alpha)^2 + (B_i - F_{4i} \cos \alpha - D_i \phi \sin \alpha)^2 \right. \right.$$

$$\left. + (F_{04} - D_4 \phi \cos \alpha)^2 + (D_0 \phi + D_4 \phi \sin \alpha)^2 \right\} + 2 \cos \alpha \left( \text{tr} B_i F_{4i} + \partial_i \text{tr} (E_i \phi) \right)$$

$$+ 2 \sin \alpha \left\{ \partial_i \text{tr} (B_i \phi) - \text{tr} (E_i F_{4i} + D_0 \phi D_4 \phi + D_0 \zeta_I D_4 \zeta_I) \right\} + \mathcal{E}_\zeta, \quad (7)$$

conclude with some remarks.
where
\[ e^2 \hat{\xi} = \text{tr} \left( (D_0 \xi - i[\phi, \xi] \cos \alpha + D_4 \xi \sin \alpha)^2 + \text{tr} \left( D_4 \xi \cos \alpha + i[\phi, \xi] \sin \alpha \right)^2 + (D_i \xi)^2 - \sum_{i<j} [\xi_i, \xi_j]^2. \] (8)

In the above we have used the Gauss law and the single-valuedness of the fields in \( x_4 \).

We introduce four conserved charges;
\[ Q^E = \frac{2}{e^2} \int d^4 x \partial_i \text{tr} (E_i \phi), \] (9)
\[ Q^M = \frac{2}{e^2} \int d^4 x \partial_i \text{tr} (B_i \phi), \] (10)
\[ P^4 = -\frac{2}{e^2} \int d^4 x \text{tr} (E_i F_{4i} + D_0 \phi D_4 \phi + D_0 \xi_i D_4 \xi_i), \] (11)
\[ T = \frac{8\pi^2}{e^2} \nu_P, \] (12)

where \( d^4 x \) is the volume element of the four dimensional space. The linear momentum along the circle \( P^4 \) is conserved but is not topological. The rest of them are topological. Especially \( T \) is the related to the Pontriyagin index by
\[ \nu_P = \frac{1}{8\pi^2} \int d^4 x 2 \text{tr} (B_i F_{4i}). \] (13)

A bound on the energy functional \( H = \int d^4 x E \) is then
\[ H \geq (T + Q^E) \cos \alpha + (Q^M + P^4) \sin \alpha \] (14)
for any angle \( \alpha \). The maximum possible value of the right side is obtained when
\[ \cos \alpha = (T + Q^E)/\sqrt{(T + Q^E)^2 + (Q^M + P^4)^2}, \] (15)
\[ \sin \alpha = (Q^M + P_4)/\sqrt{(T + Q^E)^2 + (Q^M + P^4)^2}. \] (16)

Then the strictest energy bound is then
\[ H \geq Z_\pm = \sqrt{(T \pm Q^E)^2 + (Q^M \pm P^4)^2}, \] (17)
which is the so-called BPS energy bound. \( Z_- \) is obtained by changing the sign of \( \phi \) field.

### III. BPS EQUATIONS

This energy bound is saturated, say, \( H = Z_+ \), by the configurations which satisfy the following BPS equations,
\[ B_i = F_{4i} \cos \alpha + D_i \phi \sin \alpha, \] (18)
\[ E_i = -F_{4i} \sin \alpha + D_i \phi \cos \alpha, \] (19)
\[ F_{04} = D_4 \phi \cos \alpha, \] (20)
\[ D_0 \phi = -D_4 \phi \sin \alpha. \] (21)
and the Gauss law (4). In addition, there are conditions for the \( \zeta_I \) field,

\[
D_0 \zeta_I - i [\phi, \zeta_I] \cos \alpha + D_4 \zeta_I \sin \alpha = 0, \tag{22}
\]

\[
D_4 \zeta_I \cos \alpha + i [\phi, \zeta_I] \sin \alpha = 0, \tag{23}
\]

\[
D_i \zeta_I = 0, \tag{24}
\]

\[
[\zeta_I, \zeta_J] = 0. \tag{25}
\]

The above BPS equations seem to be complicated to solve. However, a considerable simplification can be made by noticing Eq. (18) can be written as a self-dual equation. Let us introduce a new coordinate

\[
\tilde{x}^4 = \frac{1}{\cos \alpha} x^4, \tag{26}
\]

and a new fourth component gauge field

\[
\tilde{A}_4 = A_4 \cos \alpha - \phi \sin \alpha. \tag{27}
\]

Then the equation (18) becomes

\[
B_i = \tilde{D}_4 A_i - D_i \tilde{A}_4 = \tilde{F}_{4i}. \tag{28}
\]

This is the self-dual equation for calorons on \( R^3 \times S^1 \), where \( \tilde{x}^4 \in [0, \beta/\cos \alpha] \). Since the fields are periodic under \( x^4 \), they are period under \( \tilde{x}^4 \). If \( \alpha = \pi/2 \), the above method fails. However, the BPS equations in this case become those of 1/2 BPS monopoles with a single scalar field \( \phi \) and studied extensively. (Not all 1/4 BPS configurations of the theory is not described by the above BPS equation. For those configurations, two Higgs fields are involved and \( A_4 = 0 \), and so the compactified direction does not play any role.)

Introducing

\[
\tilde{D}_4 = \tilde{\partial}_4 - i \tilde{A}_4, \tag{29}
\]

Eq. (20) becomes \( F_{04} = \tilde{D}_4 \phi \) and Eq. (21) becomes \( D_0 \phi = -\tilde{D}_4 \phi \tan \alpha \). Taking a covariant divergence of Eq. (18), we get

\[
D_i F_{4i} \cos \alpha + D_4^2 \phi \sin \alpha = 0 \tag{30}
\]

Using the above relations with Eqs. (22), (23), Eq. (13) and the Gauss law (4) can be put in a single equation,

\[
D_i^2 \phi + \tilde{D}_4^2 \phi - [\zeta_I, [\zeta_I, \phi]] = 0. \tag{31}
\]

Eq. (23) can be put in a form \( \tilde{D}_4 \zeta_I = 0 \). Since we are interested in the 1/4 BPS configuration such that the vacuum expectation values of \( \zeta_I \) vanish (\( \langle \zeta_I \rangle = 0 \)) and \( D_i \zeta_I = \tilde{D}_4 \zeta_I = 0, \zeta_I \) should vanish. Hence, we can drop the \( \zeta_I \) fields from the further discussion. Thus, Eq. (31) becomes

\[
D_i^2 \phi + \tilde{D}_4^2 \phi = 0. \tag{32}
\]

The topological charge of the new variables
\[ \hat{T} = \frac{2}{c^2} \int d^4 \tilde{x} \text{tr} B_i \tilde{F}_i = T - Q^M \tan \alpha. \]  

Note that \( \hat{T} \) or \( T \) need not be integer number for the generic configuration. Another topological quantity appearing naturally here is

\[ \hat{Q}^E = \frac{2}{c^2} \int d^4 \tilde{x} \tilde{\partial}_\mu \text{tr} \tilde{D}_\mu \phi = Q^E + Q^M \tan \alpha \]  

for BPS configurations.

A. The Case where \( \alpha = 0 \)

What is the reason behind this simplification of BPS equation? Let us first consider the \( \alpha = 0 \) case. The BPS equations comes quite simplified:

\[ B_i = F_{4i}, \]  

\[ E_i = D_i \phi, \]  

\[ F_{04} = D_4 \phi, \]  

\[ D_0 \phi = 0. \]

In the gauge \( A_0 = -\phi \), the field configurations become time independent. Especially Eqs. (37) and (38) are automatically satisfied. The fields \( A_\mu \) satisfy the self-dual equation and the Gauss law constraint can be put in a simple form

\[ D^2 \phi = 0. \]

From the above simplified BPS equations, we can see \( P^4 = -Q_M \) from Eqs. (11,10), which is consistent with \( \alpha = 0 \) picture in BPS bound (17).

B. Lorentz boost along \( x^4 \)

Let us start with the \( \alpha = 0 \) case. We call its spacetime coordinates \( \tilde{x}_M \) and its BPS field configurations \( \tilde{A}_M \) and \( \tilde{\phi} \). We can Lorentz boost this coordinate to get a new coordinate \((x^4, t)\) such that

\[ \tilde{t} = \frac{t}{\cos \alpha} - x^4 \tan \alpha \]  

\[ \tilde{x}^4 = -t \tan \alpha + \frac{x^4}{\cos \alpha}. \]

Note that \( 1/(\cos \alpha)^2 - (\tan \alpha)^2 = 1 \) so that it is a Lorentz boost. The spatial coordinates \( x^i \) remain unchanged. We start with the compact radius \( 0 < \tilde{x}^4 < \beta / \cos \alpha \), and so the compact radius of \( x^4 \) becomes \( \beta \).

The \( A_0 \) and \( A_4 \) fields transform as

\[ \tilde{A}_0 = \frac{A_0}{\cos \alpha} + A_4 \tan \alpha, \]  

\[ \tilde{A}_4 = A_0 \tan \alpha + \frac{A_4}{\cos \alpha}. \]
and the $\phi$ and $A_i$ fields are invariant. We work in the gauge $\tilde{A}_0 = \tilde{\phi} = \phi$. The BPS equations (35$\sim$38) of $\alpha = 0$ case becomes the old BPS equations (18$\sim$21) in terms of new variables.

What is interesting about this Lorentz transformation is that $T + Q_E$ is like the rest mass and $P^4 + Q_M$ is like the four momentum of the Lorentz boost. (The Lorentz boost interpretation of Eqs. (33) and (34) shows that while $T$ and $Q_E$ transform nontrivially $T + Q_E$ remains invariant as the rest mass.) Thus when $\alpha = 0$, the momentum $P^4 + Q_M = 0$ as noted before. Of course our Lorentz boost is not an exact symmetry as the interval of space changes and is a kind of a combination of Lorentz boost and rescaling of the $x^4$ coordinate. However, this shows that the BPS configuration with nonzero $\alpha$ can be obtained from the BPS configurations with $\alpha = 0$. This is exactly what we have seen in Eq. (28) and (32). While the BPS configuration of the $\alpha = 0$ case can be chosen to be time independent, this is not true for those of the $\alpha \neq 0$.

IV. MONOPOLES AND CALORONS

The solutions of the primary BPS equation (28) is identical to the self-dual equations for the caloron. Since we are interested in the case where $\alpha \neq \pi/2$ and so it can be obtained from the Lorentz boost from the case $\alpha = 0$, we focus mainly on the $\alpha = 0$ case. The first one (35) is the self-dual equations of $A_\mu$ on $R^3 \times S^1$. The second BPS equation is the zero eigenvalue equation (39) for the $\phi$, around the solution of the first BPS equation.

Let us first consider the solution of the primary BPS equation. This is the BPS equation for the 1/2 BPS configurations. First of all we need the boundary condition for $A_4$. The vacuum expectation value of $A_4$ is single-valued and takes the form

$$< A_4 > = \text{diag}(h_1, h_2, ..., h_N) = h \cdot H,$$

(44)

where $\sum a h_a = 0$ and by gauge choice

$$h_1 < h_2 < ... < h_N < h_1 + \frac{2\pi}{\beta}$$

(45)

This leads to a nontrivial Wilson-loop $P \exp(i \int dx^4 A_4)$ and the symmetry is spontaneously broken to $U(1)^{N-1}$. If any of two $h_a$'s are coincide, the gauge symmetry will have unbroken nonabelian symmetry. While this possibility is quite interesting by itself, we will not pursue this direction in the present paper.

As shown in Ref. [1], the general solutions of the self-dual equation (35) describe the superpositions of $N$ fundamental monopoles corresponding to the simple roots $\beta_i$, $i = 1, ..., N - 1$ and the lowest negative root $\beta_0$. These roots form the extended Dynkin diagram. The topological charge $\nu_{P}$ of each type of monopole is fractional and takes the fractional value,

$$\mu_r = \frac{\beta}{2\pi} (h_{r+1} - h_r), \quad r = 1, ..., N - 1$$

$$\mu_0 = 1 - \frac{\beta}{2\pi} (h_N - h_1)$$

(46)

While all monopoles are on the equal footing, the magnetic monopoles of $\beta_0$ are called Kaluza-Klein monopoles as it has intrinsic $x^4$ dependence in the gauge (45).
Thus the general solution of the primary BPS equation is characterized by the \( N \) nonnegative integers \( n_r \), each of which is the number of \( \beta_r \) monopoles. The total topological charge is then
\[
\nu_p = \sum_{0}^{N-1} n_r \mu_r ,
\]
and the magnetic charge obtained from asymptotic of \( B_i = (r_i/r^4)g \cdot H \) is given by
\[
g = 4\pi \sum_{0}^{N-1} n_r \beta_r ,
\]
The topological charge and the magnetic charge are \( N \) charges together and so determine the monopole numbers \( n_r \) \( r = 0, ..., N \) uniquely. The total number of the partons or monopoles is
\[
N_m = \sum_{0}^{N-1} n_r ,
\]
and the total number of zero modes of the first BPS equation is \( 4N_m \). A single caloron or instanton can be regarded as composite of \( N \) distinct fundamental monopoles, whose topological charge is one and magnetic charge is equal to zero. The number of zero mode of a single instanton is then \( 4N \), as expected from the index theorem. For a given set \( \{n_r\} \) of monopoles, the solution of the first BPS equation is uniquely determined by the moduli parameters. The dimension of the moduli space of these configurations is \( 4N_m \). The general method to solve the first BPS equation is the ADHMN construction, as detailed in Ref. [3]. Thus, the solutions of the primary BPS equations are identical to 1/2 BPS configurations.

The secondary BPS equation can be regarded as the gauge zero modes of the primary BPS equation. The linear fluctuations \( \delta A_\mu \) should satisfy the linearized BPS equation and the gauge fixing condition,
\[
\epsilon_{ijk}D_j \delta A_k = D_4 \delta A_i - D_i \delta A_4 \, ,
\]
\[
D_\mu \delta A_\mu = 0 .
\]
When the linear fluctuation is due to the gauge zero mode, \( \delta A_\mu = D_\mu \Lambda \), the linearized BPS equation is automatically satisfied and the gauge fixing condition becomes
\[
D_\mu^2 \Lambda = 0 ,
\]
which is identical to the secondary BPS equation (39). Since there is \( N - 1 \) unbroken U(1) symmetries, there will be \( N - 1 \) linearly independent solutions of the second BPS equation. The general method for solving the second BPS equation is given in the appendix of Ref. [4].

The BPS equation (39) gives the electric field in terms of the solution \( \phi \). The solution of the second BPS equation is again uniquely determined for a given monopole background and the asymptotic value,
\[
\langle \phi \rangle = (a_1, a_2, ..., a_N) = a \cdot H , \quad \sum_{i=1}^{N} a_i = 0 .
\]
Thus, the moduli of the 1/2 BPS configurations determine the solution of the secondary BPS equation uniquely. From the asymptotics of the field \( \phi \)
\[ \phi = \langle \phi \rangle + \frac{1}{4\pi r} \mathbf{q} \cdot \mathbf{H} + \mathcal{O}\left( \frac{1}{r^2} \right), \]  

we can read the electric field and so the electric charge \( Q^E \).

However, the story is more complicated. There are \( N \) distinct monopoles, each of which can carry its own electric fields. Thus for a group of \( \beta_r \) monopoles, there will be total \( q_r \beta_r \) electric charge and the electric charge would be determined by the asymptotics of \( \phi \) field is given by

\[ \mathbf{q} = \sum_{0}^{N-1} q_r \beta_r. \]  

Similarly to the magnetic charge, the asymptotics alone cannot decide each \( q_r \) since \( \beta_r \) is not independent although it determines the relative electric charges. The electric charge \( \mathbf{q} \) is determined by the moduli parameters of the magnetic background and the asymptotic value \( \mathbf{a} \).

Besides the \( N - 1 \) global \( U(1) \) abelian symmetries of the gauge group \( SU(N) \), there is an additional \( U(1) \) related the translation along \( x^4 \) direction. Thus there are \( N \) global \( U(1) \) charges which consist of the \( N - 1 \) electric charge and the linear momentum \( P^4 \), which in turn determines the \( N \) parameters \( q_r \). The linear momentum \( P^4 \) is not topological and so it is hard to see its relation to \( q_r \) explicitly. When constituent monopoles are well-separated from each other, one may be able to assign electric charge and linear momentum to each monopole. Especially when \( \alpha = 0 \), the linear momentum (11) for very isolated \( \beta_r \) monopoles of electric charge \( q_r \) carries linear momentum \( q_r \mu_r \). Then the total linear momentum becomes \( \sum_r q_r \mu_r \), which should be \( -Q^M \) due to the equation (16). This leads an additional relation between \( q_r \)’s. The configuration with nonzero \( \alpha \) can be obtained from the Lorentz boost of the above case.

V. LOW ENERGY DYNAMICS OF CALORONS

From the consideration in the previous sections, one can see that 1/4 BPS dyonic calorons can be constructed of fundamental dyons. The low energy dynamics of 1/4 BPS configurations has been explored in recent years [10,11]. It was shown that the low energy dynamics is possible for 1/2 BPS configurations. The kinetic energy is given by the moduli space metric of the 1/2 BPS configurations. The additional Higgs field appearing in the 1/4 BPS configuration contributes to the potential to the low energy Lagrangian. The form of the potential is given by the sum of the square norm of Killing vectors. For the low energy dynamics of fundamental objects one has to specify what class of 1/2 BPS configurations one starts with. For example one can start from the case where \( \langle A_4 \rangle \neq 0 \) and \( \langle \phi \rangle = 0 \). In this case one starts with the constituent fundamental monopoles of calorons. Then we consider the low energy dynamics of monopoles with small but nonzero \( \phi \) expectation value. This is the case we will focus in this work.

There are other cases which we will not explore here. With \( \langle A_4 \rangle = 0 \) and \( \langle \phi \rangle \neq 0 \), one start with the the 1/2 BPS monopoles without any Kaluza-Klein monopoles, which would make calorons. It would be interesting to consider the low energy dynamics of these monopoles with small but nonzero \( \langle A_4 \rangle \). One could consider more complicated 1/2 BPS configurations with \( \langle A_4 \rangle \propto \langle \phi \rangle \) but without Kaluza-Klein monopoles.

The moduli space dynamics of dyonic instantons on \( R^4 \) has been studied before in Refs. [12,13]. It is somewhat simpler than our case as there is no symmetry breaking due to the \( A_4 \) expectation value.
A. Caloron Moduli Space

We start with the caloron case with \( \langle \phi \rangle = 0 \). Its low energy dynamics would be described by the caloron moduli space metric. The kinetic energy due to the spatial motion and electric charge would be much smaller than the rest mass of monopoles. We consider the modification of the low energy dynamics when \( \langle \phi \rangle \neq 0 \) but very small. Thus the angle \( \alpha \) would be very small. The \( \alpha \neq 0 \) case can be obtained by the Lorentz transformation and rescaling of the \( x^4 \) coordinate. As \( \alpha \) is very small, it would become an infinitesimal Galilean transformation along \( x^4 \) direction. Under this transformation the electric field transforms more complicatedly as one can see from Eq. (19). The moduli space metric gets split to that for the center of mass motion and that for the relative motion. Thus we just focus on the case \( \alpha = 0 \).

In this case the solution of the primary BPS equation is identical to the 1/2 BPS caloron solutions. We assume that all \( n_r \) are positive and so all constituent monopoles are interacting. In this case, the 1/2 BPS configuration has intrinsic \( x^4 \) dependence which cannot be gauged away.

The dimension of the moduli space is \( 4N_m \) and the moduli space coordinates are \( z^M \) with \( M = 1, \ldots, 4N_m \). The moduli space metric can be obtained from the study of the linear fluctuations around the 1/2 BPS configurations \( A_\mu(x,z) \),

\[
\delta_M A_\mu = \frac{\partial A_\mu}{\partial z^M} - D_\mu \epsilon_M
\]  

(55)

which satisfies Eq. (50). The moduli space metric is

\[
g_{MN}(z) = 2 \int d^4 x \, \text{tr} (\delta_M A_\mu \delta_N A_\mu).
\]  

(56)

For instanton or caloron solutions of \( SU(N) \) gauge theory, the moduli space is \( 4N \) dimensional and has orbifold singularity at the point where all monopoles come together and the caloron collapses. This moduli space metric was obtained using the constituent monopole picture and the nature of singularity is identified as \( R^{4N}/Z_N \). It was also known that the singularity is resolved if we turn on the non-commutativity [12].

The effective Lagrangian for the low energy dynamics can be genericly written as

\[
L_K = \frac{1}{2e^2} g_{MN} z^M \dot{z}^N.
\]  

(57)

There is a natural \( N = 4 \) supersymmetric generalization of the above Lagrangian. Since there are \( N \) fundamental monopoles in calorons, instead of \( N - 1 \) for \( SU(N) \) gauge theory, there are \( N \) conserved \( U(1) \) symmetries, each leading to the \( q_r \beta_r \) electric charges on \( \beta_r \) monopoles. This matches with the field theoretic symmetries, \( N - 1 \) of them are made of unbroken abelian subgroups of \( SU(N) \) and one is for the translation along \( x^4 \) direction.

To understand the dynamics better, let us split the center of mass motion and the relative motion. The Lagrangian (57) would become the sum of the Lagrangian \( L_K_{cm} \) for the center of mass motion and \( L_K_{rel} \) for the relative motion.

As the position of individual monopole is not well defined when two identical monopoles are coming together, it is hard to express the center of mass position in general. However one can argue that the charge for the central \( U(1) \) should be
\[ q_{cm} = \frac{1}{\nu P} \sum_{r=0}^{n} q_r \mu_r \]  

(58)

It is true in large separation. (See for example Ref. [13].) Also \( q_{cm} \) should be a linear combination of \( q_r \) and is conserved, and so the coefficient should be independent of the monopole positions. We would like to identify \( P_4 = \frac{2\pi}{a^2} \nu_{P} q_{cm} \) in large separation. As we consider the \( \alpha = 0 \) case, the center of mass charge would be constrained to be \(-Q_M = -\frac{1}{e^2} a \cdot g\). The nonzero \( \alpha \) case is obtained by the infinitesimal Galilean transformation along the \( x^4 \) direction. Eq. (16) implies \( P_4 + Q_M = \frac{8\pi^2}{e^2} \nu_{P} \alpha \) with \( \nu_{P} \) in Eq. (47). The Killing vector on the moduli space corresponding to the \( x^4 \) translation would be \( K_{cm} = \partial / \partial \psi_{cm} \) with the center of mass angle variable \( \psi_{cm} \).

The relative charges arise as the Noether charges for the \( N - 1 \) Killing vectors \( K^M_r \partial M \) with \( r = 1, 2, ..., N - 1 \) on the moduli space, which correspond to the \( N - 1 \) unbroken generators of \( SU(N) \). For each of this Killing vector, there would be a cyclic coordinate \( \psi_r \).

We can separate the moduli space coordinates \( z^M \) to \( (r_{cm}, \psi_{cm}) \) for the center of mass motion and \( (y^i, \psi^r) \) with \( r = 1, 2, ..., N - 1 \) for the relative motion. The kinetic energy (57) also get split to \( L_{cm}(r_{cm}, \psi_{cm}) \) for the center of mass motion and \( L_{rel}(y^i, \psi^r) \) for the relative motion. The center of mass motion is free and trivial. The relative part \( L_{rel} \) is independent of cyclic coordinates \( \psi_r \) and can be expressed as

\[ L_{rel} = \frac{1}{2e^2} h_{ij}(y) \dot{y}^i \dot{y}^j + \frac{1}{2e^2} L_{rs}(y)(\dot{\psi}^r + w^r_i(y)\dot{y}^i)(\dot{\psi}^s + w^s_j(y)\dot{y}^j) \]  

(59)

The \( N - 1 \) conserved abelian charges of the above Lagrangian is

\[ t_r = \frac{1}{e^2} L_{rs}(y)(\dot{\psi}^s + w^s_i(y)\dot{y}^i) \]  

(60)

**B. The Potential**

For a given 1/2 BPS caloron background specified by its moduli parameters, we introduce the scalar field \( \phi \) which takes nonzero but very small expectation value. The scalar field would take the lowest possible energy when it satisfy \( D^2_\mu \phi = 0 \). This scalar field in turn modifies the caloron dynamics in the second order in \( \phi \). Especially its asymptotic behaviour would be given by Eqs. (52) and (53). Especially \( \mathbf{q} = \sum_r (q_r - q_0) \mathbf{\beta}_r = \sum_{r=1}^{N-1} q_r^{rel} \mathbf{\beta}_r \) would depend on \( \mathbf{a} \) and the caloron moduli parameters \( z^M \). At this moment \( \mathbf{q} \cdot \mathbf{H} \) are not electric charge but simply the \( 1/r \) piece of the asymptotic \( \phi \).

The solution of \( D^2_\mu \phi = 0 \) would depend on the caloron moduli parameters. As \( \phi \) satisfies the same equation (51) as the global gauge transformation parameters, we can interpret

\[ D_\mu \phi = \mathbf{a} \cdot \mathbf{K}^M \delta_M A_\mu \]  

(61)

where \( \mathbf{K}^m \partial_M = \sum_{r=1}^{N-1} \mathbf{\beta}_r \mathbf{K}^M_r \partial_M \) are \( N - 1 \) Killing vectors for the \( N - 1 \) \( U(1) \) gauge generators. As the global gauge transformations change the cyclic coordinates \( \psi^r \) of the relative motion, we choose these cyclic coordinates such that

\[ \mathbf{K}^M_r \frac{\partial}{\partial z^M} = \frac{e^2}{\beta} \frac{\partial}{\partial \psi^r} \]  

(62)
Then the part of the relative moduli space metric becomes \( L_{rs}(y) = (e^4/\beta^2)g_{MN}K^r_MK^s_N \). The induced potential energy [14,10] is then given by

\[
U = \frac{1}{e^2} \int d^4x \text{tr} (D_\mu \phi)^2
= \frac{\beta}{2e^2} q \cdot a
= \frac{1}{2e^2} g_{MN}a \cdot K^M a \cdot K^N
= \frac{\beta^2}{2e^6} L_{rs}a^r a^s
\]

(63)

where \( a = \sum r a^r \lambda_r \) with the fundamental weights satisfying \( \lambda_r \cdot \beta = \delta_{rs} \). The Tong formula [14] fixes the asymptotic value \( q \) explicitly in terms of the moduli space metric and the asymptotic value

\[
\beta q = g_{MN}a \cdot K^M K^N
= \frac{\beta^2}{e^4} L_{rs}a^r a^s
\]

(64)

(65)

The Hamiltonian, \( H_{rel} = L_{rel} + U \) has a BPS bound because

\[
H = \frac{1}{2e^2} h_{ij}(y) \dot{y}^i \dot{y}^j + \frac{e^2}{2} L_{rs}(y)(t_r + \frac{\beta}{e^2} L_{rs}a^r)(t_s + \frac{\beta}{e^4} L_{rs}a^s) \pm \frac{\beta}{e^2} q_r a^r
\]

(66)

which is saturated when \( \dot{y}^i = 0 \) and

\[
t_r = q_r^\text{rel} \equiv \frac{\beta}{e^4} L_{rs}a^s
\]

(67)

The energy for the BPS configuration is then

\[
\frac{\beta}{e^2} a^r q_r^\text{rel} = \frac{\beta^2}{e^6} L_{rs}(y)a^r a^s.
\]

(68)

The above results for the electric charge and energy for the BPS configuration match exactly that from the field theoretic one (54) with \( \alpha = 0 \), and also the above BPS energy plus the rest mass \( T \) becomes the field theoretic BPS energy (17).

VI. THE SU(2) CASE

We explore more in detail the SU(2) case. This shows the above description of 1/4 BPS configurations in a more concrete form. The first BPS equation describes the self-dual calorons, and can be solved by the ADHMN method. The calorons in the SU(2) case has been studied in detail before [3,2]. Using the small and large gauge transformations, we choose the expectation value

\[
\langle A_4 \rangle = -\frac{v}{2} \sigma^3 = \text{diag}(h_1, h_2)
\]

(69)

with \( 0 < v \leq \pi/\beta \). There are two fundamental monopoles of opposite magnetic charge. One is ordinary BPS monopole of topological charge \( \mu_1 = \beta v/2\pi \). Another is the KK monopole of opposite magnetic charge and topological index \( \mu_0 = 1 - \mu_1 \). There is no force between these distinct monopoles as the magnetic attraction is cancelled by the Higgs
repulsion. A single caloron of a unit Pontriyagin index is made of one of these monopoles, so that the net magnetic charge is equal to zero. A single caloron configuration has been obtained explicitly. This complicated configuration describes nonlinear superposition of two distinct monopoles.

Consider the \( \alpha = 0 \) case. The solution of the first BPS equation is characterized by the two nonnegative integers \((n_0, n_1)\). The lowest root is \( \beta_0 = -\beta_1 \) and so the magnetic charge is

\[
g = 4\pi(-n_0 + n_1)\beta_1
\]  

and the total topological charge is

\[
\nu_R = n_0\mu_0 + n_1\mu_1
\]

For a single caloron \( n_0 = n_1 = 1 \), the solution of the primary BPS equation has been found by the ADHMN method and was shown to be superposition of two these monopoles. Furthermore, the moduli space of the 1/2 BPS configurations was shown to be 8 dimensional such that the center of mass part is flat space and the relative part is Taub-NUT space.

**A. Second BPS equation**

The secondary BPS equation around the first BPS equation can be found also explicitly by the method summarized in the appendix of Ref. \[4\]. Here we briefly outline the solution following the reference. The Nahm data for \( SU(2) \) one caloron case is given by \[3,2\]

\[
T_0 = -a_0, \quad T_1 = -a_1,
\]

where \( a_{0,1} \) are constant vectors representing the positions of constituent monopoles. We can put \( a_{0,1} \) by spatial rotation and translation at the \( z \)-axis such that \( a_0^i = -\frac{R}{2}\delta^i_3, a_1^i = \frac{R}{2}\delta^i_3 \). In ADHMN formalism the covariant Laplacian for adjoint scalar field \( \phi \) is replaced by the equation for function \( p(t) \)

\[
\ddot{p}(t) - W(t)p(t) + \Lambda(t) = 0,
\]

where \( W(t) \) and \( \Lambda(t) \) are given by

\[
W(t) \equiv \text{tr}_2 \sum_{a=1}^{2} \delta(t - h_a) w_a^i w_a^i, \quad \Lambda = \text{tr}_2 \sum_{a=1}^{2} \delta(t - h_a) w_a^i \langle \phi \rangle w_a^i.
\]

For the given Nahm data, \( w_{1,2} \) are

\[
w_1 = (\sqrt{2R}, 0), \quad w_2 = (0, \sqrt{2R}),
\]

which is determined by the jumping condition

\[
T(h_a^+) - T(h_a^-) = \frac{1}{2} \text{tr}_2 (\sigma^i w_a^i w_a^i).
\]

The solution of Eq. \[73\] for these data and the boundary value of the adjoint scalar field \( \phi \)
\( \langle \phi \rangle = -\frac{\eta}{2} \sigma_3 = a \cdot H , \quad a \equiv \eta \beta_1 \) \hspace{1cm} (77)

is given by

\[
p(t) = \begin{cases} 
p_0(t + \frac{\pi}{2}), & t \in [-\frac{\pi}{2}, -\frac{v}{2}] \\
p_1 t, & t \in [-\frac{v}{2}, \frac{\pi}{2}] \\
p_0(t - \frac{\pi}{2}), & t \in [\frac{\pi}{2}, \frac{v}{2}] \end{cases} ,
\]

where

\[
p_0 = -\eta \frac{\beta}{2\pi v R} \frac{1}{1 + (1 - \frac{v}{2\pi})v R} ,
\]

\[
p_1 = \eta \frac{(1 - \frac{v}{2\pi}R)}{1 + (1 - \frac{v}{2\pi})v R} .
\]

Then, we can construct the adjoint scalar field by

\[
\phi = N^{-1/2} \langle \phi \rangle N^{-1/2} + N^{-1/2} \int dt u(t) u N^{-1/2} ,
\]

where \( N \) and \( u(t) \) are the functions given in the reference [3]. It is straightforward to obtain the \( \phi \) field in terms of the functions given in the reference [3] and extract the asymptotic behaviour of \( \phi \).

From the asymptotic behaviour, we get

\[
\phi = -\frac{\eta}{2} \sigma_3 \left( 1 - \frac{R}{r} \frac{\beta}{\beta + 2\pi \mu_0 \mu_1 R} \right) + \mathcal{O}\left( \frac{1}{r^2} \right) ,
\]

and thus the electric charge is given by

\[
Q^E = \frac{4\pi}{e^2} \frac{\eta^2 \beta^2 R}{\beta + 2\pi \mu_0 \mu_1 R} .
\]

### B. Moduli space metric and Tong’s method

The moduli space of \( SU(2) \) one caloron is 8 dimensional and this 8 dimensional moduli space can be split into the center of motion space and the relative motion space and their metrics are given by [1, 3, 16]

\[
ds^2_{cm} = 8\pi^2 \left\{ (d\chi)^2 + (d\chi_{cm})^2 \right\} ,
\]

\[
ds^2_{rel} = 8\pi^2 \mu_0 \mu_1 \left\{ (1 + \frac{r}{r})(dr)^2 + r^2 (1 + \frac{r}{r})^{-1}(d\psi + \mathbf{w} \cdot d\mathbf{r})^2 \right\} ,
\]

where \( r = \frac{\beta}{2\pi \mu_0 \mu_1} \). The relative metric \( ds^2_{rel} \) is the one for the well-known Taub-NUT space. For this case

\[
\frac{\beta}{e^2} q = \frac{1}{e^2} g_{\psi \psi}(a \cdot K)^\psi .
\]

Since \( q_\psi = q_1 - q_0, q = q_\psi \beta_1 \) and \( K^\psi = \beta_1 \), one can see that

\[
\frac{\beta}{e^2} q_\psi = \frac{1}{e^2} g_{\psi \psi}(a \cdot K)^\psi = \frac{\eta}{e^2} g_{\psi \psi}
\]

\[
= \frac{4\pi}{e^2} \frac{\eta \beta^2 R}{\beta + 2\pi \mu_0 \mu_1 R} .
\]

This gives us the electric charge \( Q^E \)
\[ Q^E = \frac{\beta}{e^2} a \cdot q = \frac{4\pi}{e^2} \frac{\eta^2 \beta^2 R}{\beta + 2\pi \mu_0 \mu_1 R}, \] (89)

which is the identical expression with one obtained in the previous section.

The low energy dynamics of this configuration is described by the free c.m. part and the relative Taub-NUT metric with potential term \( U = \frac{1}{2}Q^E \). Note that the field theory BPS energy bound \[ (17) \] is saturated by \( T = \frac{\pi}{2\mu_0 \mu_1} \) and the above \( Q^E \).

C. String Picture

A dyon in maximally supersymmetric four-dimensional Yang-Mills theory appears as fundamental and D-string composition connecting D3-branes. The positions of D3-branes are specified by the adjoint scalar vacuum expectation values and the total energy of strings connecting D3-branes matches with the field theoretic energy of dyons \([3]\). In caloron case the string picture appears after the T-dual is taken and then the vacuum expectation value of \( A_4 \) specify the position of D3-branes.

With the tension for a single fundamental string \( T = 1/2\pi \alpha' \), the tension for the D-string is \( gT \) with \( g = 4\pi \beta/e^2 \) in terms of the field theory parameters. The tension of \( s = \beta q/e^2 \) fundamental strings on a single D-string is \( \sqrt{g^2 + s^2 T} \), where \( q \) is the field theory electric charge.

A single dyonic caloron in the \( SU(2) \) gauge group can be represented in a the string picture by the T-dual transformation. Both \( A_4 \) and \( \phi \) expectation values denote the position of the D3 branes in the compact dual space and transverse \( x^5 \) direction given by the \( \phi \) field. The detail of the string picture of this configuration is given below.

![String web picture of SU(2) single caloron case](image)

**Figure 1:** String web picture of SU(2) single caloron case

The above string configuration is 1/4 BPS when there is a tension balance at the string junctions \([15]\), which implies that

\[
\sin \theta_1 = \frac{g}{\sqrt{g^2 + s_1^2}} \]
(90)
\[
\sin \theta_0 = \frac{g}{\sqrt{g^2 + s_0^2}} \]
(91)

where \( s_i = \frac{\beta q_i}{e^2} \).
Let us consider the total energy of the above string configuration, which is the sum of the energy, that is the tension times the length, of the individual string segments. The total string energy is then

\[ E = (s_1 - s_0)T(D - L) + \sqrt{g^2 + s_1^2 T \frac{L}{\cos \theta_1}} + \sqrt{g^2 + s_0^2 T \frac{L}{\cos \theta_0}}. \] (92)

This can be written as

\[ E = gTL(\tan \theta_1 + \tan \theta_0) + (s_1 - s_0)TD \] (93)

which is what we expect from the BPS energy from field theory since we can identify the string theory parameters with the field theory ones as \( T\beta = \eta, \beta TL \tan \theta_i = 2\pi \mu_i, \) \( i = 0, 1 \). The critical charge appears when \( L = D \) or,

\[ \Delta \tilde{q}_c \equiv (s_1 - s_0) = \frac{\beta T}{2\pi \mu_1 \mu_0} gD \] (94)

which is identical with \( \lim_{R \to \infty} (q_1 - q_0) \frac{\beta}{\sigma} \) from the field theory result (89).

**VII. CONCLUSION**

In this paper we have investigated 1/4 BPS configuration on \( R^{3+1} \times S^1 \) in \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory. The generic BPS bound and BPS equations are obtained when \( P^4, Q^E, Q^M \) and \( T \) are turned on. The BPS equations are complicated than 1/4 BPS configuration on \( R^{3+1} \). Nevertheless, BPS equations can be simplified by suitable transformation, resulting in two equations. Then, the static solution for 1/4 BPS configuration can be obtained by solving first primary equation which is much studied 1/2 BPS caloron equation on \( R^3 \times S^1 \) and next solving the secondary equation on this 1/2 BPS caloron background. This is quite similar to the approach for 1/4 BPS dyonic state in the four dimensional field theory [12].

New feature in the 1/4 BPS dyonic caloron case compared to 1/4 BPS dyon is the appearance of topological charge \( T \) and four momentum \( P^4 \) in BPS bound. In the theory of the \( SU(2) \) gauge group, there is no 1/4 BPS dyons in four dimensions but there are 1/4 BPS caloron configurations in five dimensions. In the infinite \( \beta \) limit these dyonic calorons become the dyonic instantons which were studied before [6, 7].

We have also considered the low energy dynamics of this 1/4 BPS configuration and it turns out that it is described by the same type non-linear \( \sigma \)-model Lagrangian with 1/4 BPS dyons. But the potential term is more complicated by the existence of \( P^4 \) and \( Q^M \).

There are several directions needed for further study. First, it would be interesting to understand the \( \alpha \neq 0 \) case from the low energy dynamics view point. We have argued that it is related to the center of mass motion of low energy Lagrangian.

Second, it would be interesting to study the quantum spectrum of this configuration. After the quantization the translational generator along \( x^4 \) direction takes discrete integer values. However the momentum \( P^4 \) could take fractional value. Thus, it would be interesting to elucidate this point.
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