The $\alpha$-particle in nuclear matter

M. Beyer$^1$, S.A. Sofianos$^2$, C. Kührts$^1$, G. Röpke$^1$, and P. Schuck$^3$

$^1$Fachbereich Physik, Universität Rostock, 18051 Rostock, Germany
$^2$Dept. of Physics, University of South Africa, Pretoria 0003, South Africa
$^3$Institut de Science Nucléaires, Université Joseph Fourier, CNRS-IN2P3 53, Avenue des Martyrs, F-38026 Grenoble Cedex, France

Abstract

Among the light nuclear clusters the $\alpha$-particle is by far the strongest bound system and therefore expected to play a significant role in the dynamics of nuclei and the phases of nuclear matter. To systematically study the properties of the $\alpha$-particle we have derived an effective four-body equation of the Alt-Grassberger-Sandhas (AGS) type that includes the dominant medium effects, i.e. self energy corrections and Pauli-blocking in a consistent way. The equation is solved utilizing the energy dependent pole expansion for the subsystem amplitudes. We find that the Mott transition of an $\alpha$-particle at rest differs from that expected from perturbation theory and occurs at approximately 1/10 of nuclear matter densities.

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I. INTRODUCTION

The modification of few-body properties such as the binding energy and the wave function of a bound state due to a medium of finite temperature and density is an important subject of many-particle theory. As an example we consider symmetric nuclear matter consisting of nucleons (equal number of protons and neutrons) at density $\rho$ and temperature $T$. The modification of single-nucleon properties can be obtained from a Dyson equation in terms of a self-energy. In a specific approximation the quasiparticle picture can be derived. A more rigorous description leads to the nucleon spectral function. Similarly we can consider the two-nucleon system where the medium modification are obtained from a Bethe-Goldstone equation [1]. In addition to the self-energy shift, also Pauli-blocking has to be taken into account that is of the same order of magnitude. It has been shown [2-3] that, as a consequence, deuterons in nuclear matter become unbound if the density exceeds a certain value, the Mott density.

Of course, the same mechanisms are also responsible for the modification of higher clusters embedded in nuclear matter. However, the solution of the few-body in-medium equa-
tions where the effects of the medium are accounted for by a density and temperature dependent contribution to the Hamiltonian has only been done within perturbation theory, see [4]. Therefore the results achieved for the energy shifts and the Mott densities are only approximations.

Recently, rigorous methods have been used to find solutions for the three-body problem in nuclear matter [5–9]. The Faddeev equations are extended to include the effects of the medium, and the corresponding Alt-Grassberger-Sandhas (AGS) equations have been solved [10]. Different properties of the three-nucleon system in the medium such as the modification of the binding energy of the three-nucleon bound state [8] and the medium modification of the nucleon-deuteron break-up cross section [5,6] have been calculated.

In the present letter we give first results of the solution of the in-medium four-particle equation describing the modification of the binding energy of the α-particle in symmetric nuclear matter. An AGS-type equation has been solved and the results will be compared with those of perturbation theory.

Note that the four-particle correlations in low-density nuclear matter are very important because of the large binding energy of the α-particle. They have to be accounted for not only in equilibrium when considering the nuclear matter equation of state or the contributions of correlations to the single-nucleon spectral function, but also in nonequilibrium such as the light cluster formation in heavy ion collisions.

II. IN-MEDIUM FEW-BODY EQUATIONS

The few-nucleon problem in nuclear matter can be treated using Green function approaches. Within the cluster-mean field expansion [4], a self-consistent system of equations can be derived describing a $n$-nucleon cluster moving in a mean field produced by the equilibrium mixture of clusters with arbitrary nucleon number $m$. A Dyson equation approach to describe clusters at finite temperatures and densities has been given in [11]. However, the self-consistent determination of the composition of the medium is a very challenging task that is not solved until now. We will perform the approximation where the correlations in the medium are neglected so that the embedding nuclear matter is described by the equilibrium distribution of quasiparticles (see also [3] for the two-particle problem, or [5–8] for the three-particle problem).

The extension of this formalism to describe $n$-nucleon correlations in nuclear matter will be given elsewhere. Here, we will give some of the basic relations, which are direct generalizations of the three-particle case.

Let the Hamiltonian of the system be given by

$$H = \sum_{1} \frac{k_{1}^{2}}{2m} a_{1}^{\dagger}a_{1} + \frac{1}{2} \sum_{121'2'} V_{2}(12, 1'2') a_{1}^{\dagger}a_{2}^{\dagger}a_{2}a_{1}$$

where $a_{1}$ etc. denotes the Heisenberg operator of the particle that includes quantum numbers such as spin $s_{1}$ and momentum $k_{1}$. The free resolvent $G_{0}$ for an $n$-particle cluster is given in Matsubara-Fourier representation by

$$G_{0}(z) = (z - H_{0})^{-1} N \equiv R_{0}(z) N,$$

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$$G_{0}(z) = (z - H_{0})^{-1} N \equiv R_{0}(z) N,$$
where \( G_0, H_0, \) and \( N \) is a compact notation for matrices in the space of \( n \) particles with respect to the particle indices given below. Here \( z \) denotes the Matsubara frequencies \( z_\lambda = \frac{\pi \lambda}{(-i\beta)} + \mu \) with \( \lambda = 0, \pm 2, \pm 4, \ldots \) for bosons and \( \lambda = \pm 1, \pm 3, \ldots \) in the case of fermions. To simplify the notation we have further dropped the index \( n \) on the matrices, however use it if explicitly needed. The effective in-medium Hamiltonian \( H_0 \) for noninteracting quasi-particles is given by

\[
H_0 = \sum_{i=1}^{n} \frac{k_i^2}{2m} + \Sigma_i \equiv \sum_{i=1}^{n} \varepsilon_i,
\]

where the energy shift \( \Sigma_1 \) and the Fermi function \( f_1 \) are

\[
\Sigma_1 = \sum_2 V_2(12, \tilde{1}\tilde{2})f_2, \quad \quad \quad \quad \quad f_1 \equiv f(\varepsilon_1) = \frac{1}{e^{(\varepsilon_1 - \mu)k_B T} + 1}.
\]

The notation \( \tilde{1}\tilde{2} \) means antisymmetrisation. The factor \( N \) in Eq. (2) resembles the Pauli blocking or normalization of the Green functions. This factor is different for the different clusters considered depending on the number of particles \( n \). It is given by

\[
N = \bar{f}_1\bar{f}_2 \ldots \bar{f}_n \pm f_1f_2 \ldots f_n
\]

where \( \bar{f} = 1 - f \). The upper sign is for an odd number of fermions (Fermi type) and the lower for an even number of fermions (Bose type). Note that \( NR_0 = R_0N \).

The full resolvent after Matsubara-Fourier transformation may be written in the following way

\[
G(z) = (z - H_0 - V)^{-1}N \equiv R(z)N,
\]

where the potential \( V \) is a sum of two-body interactions between pairs \( \alpha \), i.e.

\[
V = \sum_\alpha V_\alpha = \sum_\alpha N_2^\alpha V_2^\alpha,
\]

and \( V_2^\alpha \) is the two-body potential given in Eq. (8). The sum runs over all unique pairs in the cluster. Note, that as a consequence of Eq. (8) \( V^\dagger \neq V \), also \( R(z)N \neq NR(z) \) that later on leads to right and left eigen–vectors.

To be more specific: If the interaction is between particle 1 and 2 (in the pair \( \alpha = (12) \) of a cluster of \( n \) particles) the effective potential of Eq. (8) reads

\[
\langle 12\rangle N_2^{(12)} V_2^{(12)} |1'2'\rangle = (\bar{f}_1\bar{f}_2 - f_1f_2)V_2(12, 1'2').
\]

A useful notion is the channel resolvent \( G_\alpha(z) \) for an \( n \) particle cluster, where only the pair interaction in channel \( \alpha \) is considered. This may be written as

\[
G_\alpha(z) = (z - H_0 - V_\alpha)^{-1}N = (z - H_0 - N_2^\alpha V_2^\alpha)^{-1}N \equiv R_\alpha(z)N.
\]
Using $R_0^{-1}(z)$, $R_{\alpha}^{-1}(z)$, and $R^{-1}(z)$ it is possible to formally derive the resolvent equations in the standard way. To keep the formal equivalence to the isolated case, the $n$-particle channel $t$-matrix $T_{\alpha}$ is defined by

$$R_{\alpha}(z) = R_0(z) + R_0(z)T_{\alpha}(z)R_0(z).$$  \hspace{1cm} (11)$$

With the use of $T_{\alpha}(z) = N_2^\alpha T_2^\alpha(z)$ Eq. (11) leads to the well-known Bethe-Goldstone equation [1]

$$T_2^\alpha(z) = V_2^\alpha + V_2^\alpha R_0(z)N_2^\alpha T_2^\alpha(z) = V_2^\alpha + T_2(z)R_0(z)N_2^\alpha V_2^\alpha.$$  \hspace{1cm} (12)$$

We remark that similar equations have been written down by various authors previously [12,13].

Note that the above equations are also valid for the two-particle subsystem embedded in a larger cluster (three, four, or more particles). As for the isolated equations the effects of the other particles appear only in the Matsubara frequencies $z$ (energies) of the other particles in the cluster. No additional blocking factors $N$ related to the larger cluster arise. Also note, that the changes due to the Pauli blocking are in the resolvent $G_0$ not in the potential $V_2$. However, it is possible to rewrite this equation and introduce an effective potential as seen in Eq. (12) and use unchanged resolvents instead. Making use of the more intuitive picture of a blocking in the propagation of the particles (related to the resolvents) we find by Eq. (12) the correct expression for the $t$-matrix that enters into the Boltzmann collision integral (see, for example, [14]).

The derivation of the three-body equation is straightforward and has been given elsewhere [6–9]. The AGS operator $U_{\beta\alpha}(z)$ [10] for the three particle system is defined by

$$R(z) = \delta_{\beta\alpha} R_{\alpha}(z) + R_{\beta}(z)U_{\beta\alpha}(z)R_{\alpha}(z).$$  \hspace{1cm} (13)$$

Inserting Eqs. (11) and (12) in the above identity we result with the AGS-type equation

$$U_{\beta\alpha}(z) = \delta_{\beta\alpha} R_0(z)^{-1} + \sum_\gamma \delta_{\beta\gamma} N_2^\gamma T_2^\gamma(z)R_0(z)U_{\gamma\alpha}(z),$$  \hspace{1cm} (14)$$

that includes now medium effects as Pauli blocking and self energy shifts. We used the notation $\delta_{\alpha\beta} = 1 - \delta_{\alpha\beta}$. This equation solves the three-body transition operator for a three-particle cluster as well as for a three-particle cluster embedded in a more-particle cluster, i.e. the effect of the other particles in the cluster is again only in the Matsubara frequency (energy) $z$. The definition of the transition operator given by Eq. (13) was chosen so that no additional factor $N$ appears in the final equation. This guarantees that the cluster equations are valid also if they are part of a larger cluster. Thus, the two-body subsystem $t$-matrix entering in Eq. (14) is the same as the one given in Eq. (12). Therefore, it is possible to use all results of the few-body 'algebra', in particular those based on cluster decomposition.

The in-medium bound state equation for an $n$-particle cluster follows from the homogeneous Lippmann-Schwinger equation and is given by

$$|\psi_B\rangle = R_0(E_B)V|\psi_B\rangle = R_0(E_B) \sum_\gamma N_2^\gamma V_2^\gamma|\psi_B\rangle.$$  \hspace{1cm} (15)$$
where the sum is over all unique pairs in the cluster. As shown in Ref. [8] for the three-body bound state it is convenient to introduce form factors

$$|F_\beta\rangle = \sum_{\gamma=1}^3 \delta_{\beta\gamma} N_2^\gamma V_2^\gamma |\psi_B\rangle$$

(16)

that leads to the homogeneous in-medium AGS-type equation

$$|F_\alpha\rangle = \sum_{\beta=1}^3 \delta_{\alpha\beta} N_2^\beta T_2^\beta R_0(B_\beta)|F_\beta\rangle.$$  

(17)

We may generalize the AGS method given in Refs. [13, 17] to the in-medium four-body case [18]

$$|\psi_\beta\rangle = R_0(B_4)N_2^\beta V_2^\beta |\psi_B\rangle.$$  

(18)

where

$$|\psi_\beta\rangle = R_0(B_4)N_2^\beta V_2^\beta |\psi_B\rangle.$$  

(19)

Introducing the 3 + 1 and 2 + 2 cluster decomposition of the four-body system, denoted by $\tau, \sigma, ...$, the sum on the right hand side of Eq. (18) may be rearranged by introducing four-body form factors

$$|F_\sigma^\beta\rangle = \sum_{\tau} \delta_{\sigma\tau} \sum_{\alpha} \delta_{\tau\alpha} R_0^{-1}(B_\alpha) |\psi_\alpha\rangle,$$

(20)

with $\beta \subset \sigma$, $\delta_{\beta\alpha} = \delta_{\beta\alpha}$, if $\beta, \alpha \subset \tau$ and $\delta_{\beta\alpha} = 0$ otherwise. The homogeneous in-medium AGS-type equation for the four-body form factors is then written

$$|F_\tau^\beta\rangle = \sum_{\tau\gamma} \delta_{\sigma\tau} U_{\beta\gamma}(B_4) R_0(B_4) N_2^{\gamma} T_2^{\gamma}(B_4) R_0(B_4) |F_\gamma^\tau\rangle, \quad \beta \subset \sigma, \gamma \subset \tau.$$  

(21)

The driving kernel consists of the in-medium two-body $t$-matrix defined by the Bethe-Goldstone equation and the in-medium AGS-type transition operator defined in Eq. (14). Note also that an additional Pauli blocking factor $N_2^\gamma$ occurs.

The equations for the three-body scattering and bound state problem have been solved numerically in Refs. [3, 8]. An exploratory calculation to study a possible $\alpha$-like condensate (quartetting) has been carried out using a variational ansatz for the (2+2) channel and by neglecting the (3+1) channel [19].

Because of the medium dependence of the equations the calculation time increases drastically. This is due to the fact that the positions of the deuteron pole as well as the three-nucleon pole vary with the intrinsic momentum and are not fixed at the usual binding energy because of the phase space occupation through other particles. Presently a sufficiently fast and accurate method to solve the three- and four-body equations relies on the separability of the subamplitudes that appear in the AGS equations. To solve the four-body bound
states we utilize the energy dependent pole expansion (EDPE) [18] that needs to be adjusted to the in-medium case, because of different right and left expansion functions due to the nonsymmetric effective potential.

To be more specific, we assume the following expansions for the amplitudes of the respective sub-systems embedded in the four-body equation. For the two-body sub-system we have

$$T_{\gamma}(z) \simeq \sum_n |\tilde{\Gamma}_{\gamma n}(z) t_{\gamma n}(z)\rangle \langle \Gamma_{\gamma n}(z)| = \sum_n N_{\gamma n}^2 |g_{\gamma n}(z)\rangle \langle g_{\gamma n}|. \quad (22)$$

The last equation of the right hand side is used in the present calculation and reflects a simple Yamaguchi ansatz for the form factors [20]. The Pauli blocking factor then appears explicitly. This has been used in Refs. [5–7,9] and a comparison to the Paris potential is given in Ref. [8]. For the present purpose this approximation seems sufficient. For the three-body subamplitudes we use the EDPE expansion

$$\langle g_{\beta m}(z)|R_0(z)U^*_m(z)R_0(z)|\tilde{g}_{\gamma n}(z)\rangle \simeq \sum_{t,\mu \nu} |\tilde{\Gamma}^{\tau t,\mu}_{\beta m}(z) t_{\mu \nu}^{\tau t}(z)\rangle \langle \Gamma^{\tau t,\nu}_{\gamma n}(z)| \quad (23)$$

with

$$|\tilde{\Gamma}^{\tau t,\mu}_{\beta m}(z)\rangle = \langle g_{\alpha n}|R_0(z)|\tilde{g}_{\beta m}(z)\rangle t_{\beta m}(B_3)|\tilde{\Gamma}^{\tau t,\mu}_{\beta m}\rangle. \quad (24)$$

The Sturmian functions corresponding to the fixed energy $B_3$ are given by

$$\eta_{t,\mu} |\tilde{\Gamma}^{\tau t,\mu}_{\alpha n}\rangle = \sum_{\beta m} \langle g_{\alpha n}|R_0(z)|\tilde{g}_{\beta m}(z)\rangle t_{\beta m}(B_3)|\tilde{\Gamma}^{\tau t,\mu}_{\beta m}\rangle \quad (25)$$

$$\eta_{t,\mu} |\Gamma^{\tau t,\mu}_{\alpha n}\rangle = \sum_{\beta m} \langle \tilde{g}_{\alpha n}|R_0(z)|g_{\beta m}(z)\rangle t_{\beta m}(B_3)|\Gamma^{\tau t,\mu}_{\beta m}\rangle \quad (26)$$

Introducing the form factors

$$|F^{s}_{\mu}\rangle = \sum_{\beta m} \langle \Gamma^{s}_{\beta m,\nu}(B_4)|t_{\beta m}(B_4)|g_{\beta m}(B_4)|R_0(B_4)|F^{s}_{\beta}\rangle \quad (27)$$

we obtain the following homogeneous system of integral equations

$$|F^{s}_{\mu}\rangle = \sum_{\tau t} \sum_{\nu \kappa} \sum_{\gamma n} \delta_{\tau \nu} \langle \Gamma^{s,\nu}_{\gamma n}(B_4)|t_{\gamma n}(B_4)|\tilde{\Gamma}^{\tau t,\mu}_{\gamma n}(B_4)\rangle t_{\mu \kappa}^{\tau t}(B_4) |F^{s}_{\kappa}\rangle. \quad (28)$$

Formally these equations resemble the structure of the isolated four-body equations. However, the dominant features of the influence of the medium, i.e. the self-energy correction and the Pauli blocking, are systematically taken into account.

Inclusion of spin-isospin degrees of freedom and symmetrization is a challenging task for the four-body problem and done as for the isolated case. To this end we have introduced angle averaged Pauli factors as explained e.g. in Ref. [4] and fit the self-energy by use of effective masses.
III. RESULTS AND CONCLUSION

To solve the four-body equation numerically we use a Yamaguchi type rank one potential for the $^3S_1$ and $^1S_0$ channels. The parameters are taken from an early work of Gibson and Lehman [21]. We renormalized the calculated binding energy of the α-particle so that it coincides for the isolated particle with the experimental one. Presently, instead of using a more elaborated approach to the isolated four-nucleon problem we merely study the change of the binding energy due to the density and temperature of the surrounding nuclear matter. From our recent results for the three-body systems [8], we argue that the change is not very sensitive to the particular form of the potential. We shall, therefore, leave the study of model dependences for a future communication.

We calculated the binding energy of an α-like cluster with zero center of mass momentum in symmetric nuclear matter at temperature $T = 10$ MeV as a function of the nucleon density. The results are shown as a solid line in Fig. 1. The Mott transition occurs at a single particle density of $\rho_{\text{Mott}} = 0.0207$ fm$^{-3}$. For comparison we have given a perturbative calculation shown as dashed line. This calculation is based on a simple Gaussian wave function for the α-particle with the width fitted to the electric rms radius. Also the binding energy has been renormalized to the experimental value. The Mott density in this case is at 0.0305 fm$^{-3}$ that strongly differs from the value gained from the solution of Eq. (28).

The corresponding curves for the triton and the deuteron are shown as dotted and dashed dotted lines respectively. Note, that these binding energies are for clusters at rest in the medium. Although it is an interesting case, when the sub-clusters embedded in the larger cluster vanishes as a bound state the sub-clusters in this case have a dynamical binding energy that depends on the c.m. momentum. Nevertheless, the question of Boromenian states and the Efimov effect needs further investigation.

Unlike for the triton the α-particle still exists at densities where the Pauli blocking factor, see Eq. (8), becomes negative. The usual procedure of symmetrizing the effective potential by proper square root factors fails. Therefore when solving the four-body equation we have to keep track of the right and left eigenvectors in the subsystem.

In conclusion, we derived and solved for the first time an effective in-medium four-particle equation of the AGS type. Applying it to symmetric nuclear matter, we found that the binding energy of the α-particle decreases with increasing density due to Pauli blocking and disappears at a critical value of the density (about 1/10 of the nuclear matter density for $T = 10$ MeV). The dependence of the results on temperature and center of mass momentum will be the subject of an extended work.

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FIGURE CAPTIONS

Fig. 1. Binding energy of an $\alpha$-like cluster with zero center of mass momentum embedded in symmetric nuclear matter at a temperature of $T = 10$ MeV as a function of nucleon density. Solid line: Yamaguchi potential, renormalized to experimental binding energy at zero density. Dashed line: perturbation approach. For comparison, the medium dependent binding energies of the deuteron (dashed-dotted) and triton (dotted) are also shown.
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