TWIN TQFTs AND FROBENIUS ALGEBRAS

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Abstract. We introduce the category of singular 2-dimensional cobordisms and show that it admits a completely algebraic description as the free symmetric monoidal category on a twin Frobenius algebra, by providing a description of this category in terms of generators and relations. A twin Frobenius algebra \((C, W, z, z^*)\) consists of a commutative Frobenius algebra \(C\), a symmetric Frobenius algebra \(W\), and an algebra isomorphism \(z: C \to W\) with dual \(z^*: W \to C\) satisfying some extra conditions. We also introduce a special type of extended 2D Topological Quantum Field Theory defined on singular 2-dimensional cobordisms and show that it is equivalent to a twin Frobenius algebra in a symmetric monoidal category.

1. Introduction

A 2-dimensional Topological Quantum Field Theory (TQFT) is a symmetric monoidal functor from the category \(\mathbf{2Cob}\) of 2-dimensional cobordisms to the category \(\mathbf{Vect}_k\) of vector spaces over a field \(k\). The objects in \(\mathbf{2Cob}\) are smooth compact 1-manifolds without boundary, and the morphisms are the equivalence classes of smooth compact oriented cobordisms between them, modulo diffeomorphisms that restrict to the identity on the boundary. The category \(\mathbf{2Cob}\) of 2-cobordisms and that of 2D TQFTs are well understood, and it is known that 2D TQFTs are characterized by commutative Frobenius algebras, in the sense that the category of 2D TQFTs is equivalent as a symmetric monoidal category to the category of commutative Frobenius algebras. For the classic results involving these concepts, we refer to [1, 6, 12] and the book [10].

Lauda and Pfeiffer studied in [11] a special type of extended TQFTs defined on open-closed cobordisms. These cobordisms are certain smooth oriented 2-manifolds with corners that can be viewed as cobordisms between compact 1-manifolds with boundary, that is, between disjoint unions of circles \(S^1\) and unit intervals \(I = [0, 1]\). An open-closed TQFT is a symmetric monoidal functor \(Z: \mathbf{2Cob}^{ext} \to \mathbf{Vect}_k\), where \(\mathbf{2Cob}^{ext}\) denotes the category of open-closed cobordisms. Lauda and Pfeiffer showed that open-closed TQFTs are characterized by what they call knowledgeable Frobenius algebras \((A, C, \iota, \iota^*)\), where the vector space \(C := Z(S^1)\) associated with the circle has the structure of a commutative Frobenius algebra, the vector space \(A := Z(I)\) associated with the interval has the structure of a symmetric Frobenius algebra, and there are

2000 Mathematics Subject Classification. 57R56, 57M99; 81T40; 19D23.

Key words and phrases. Cobordisms, Frobenius algebras, symmetric monoidal categories, TQFTs.

arXiv:0901.2979v2 [math.GT] 10 Jan 2010
linear maps \( \iota: C \to A \) and \( \iota^*: A \to C \) satisfying certain conditions. This result was obtained by providing a description of the category of open-closed cobordisms in terms of generators and the Moore-Segal relations. They defined a normal form for such cobordisms, characterized by topological invariants, and then proved the sufficiency of the relations by constructing a sequence of moves which transforms the given cobordism into the normal form. They also showed that the category \( \mathbf{2Cob}^{\text{ext}} \) of open-closed cobordisms is equivalent to the symmetric monoidal category freely generated by a knowledgeable Frobenius algebra. We remark that the entire construction in [11] was given for an arbitrary symmetric monoidal category, and not only for \( \text{Vect}_k \).

In [4] the author constructed the universal \( \mathfrak{sl}(2) \) link cohomology that uses dotted foams modulo local relations. This theory corresponds to a certain Frobenius algebra structure defined on \( \mathbb{Z}[i, a, h, X]/(X^2 - hX - a) \), and for the case of \( a = h = 0 \) it gives rise to an isomorphic version of the \( \mathfrak{sl}(2) \) Khovanov homology [2] [8]. One of the features of the construction in [4] is its functoriality property with respect to link cobordisms relative to boundaries, with no sign indeterminacy. In particular, it brings a properly functorial method for computing the Khovanov homology groups. In [5] the author described a method that computes fast and efficient the foam cohomology groups, and also provided a purely topological version of the foam theory in which no dots are required on cobordisms.

We give below a short review of the universal \( \mathfrak{sl}(2) \) foam link cohomology. Given a link diagram \( L \), one associates to it a formal complex \( [L] \) whose objects are formally graded resolutions of \( L \), called webs, and whose morphisms are formal linear combinations of singular cobordisms, also called foams, whose source and target boundaries are resolutions. These singular cobordisms are considered modulo certain local relations.

Each crossing of \( L \) is resolved in one of the following ways:

\[
\begin{bmatrix}
& \rightarrow \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
& \\
\end{bmatrix}
\]

A diagram obtained by resolving all crossings of \( L \) is a disjoint union of webs. A web is a planar oriented graph with \( 2k, k \geq 0 \) bivalent vertices near which the two incident edges are oriented either towards the vertex or away from it. The simplest closed webs are depicted in (3.1).

A foam is an abstract cobordisms between such webs—regarded up to boundary-preserving isotopy—and has singular arcs (and/or singular circles) where orientations disagree, and near which the facets incident with a certain singular arc are compatibly oriented, inducing an orientation on that arc. The presence of these singular arcs (circles) is the reason for the second name of foams, namely singular cobordisms. Examples of such cobordisms are given in (3.2). The category of foams is denoted by \( \mathbf{Foams} \).

The construction of the universal \( \mathfrak{sl}(2) \) foam link cohomology is in the spirit of Bar-Natan’s approach [3] to local Khovanov homology, with the difference that it uses webs and foams modulo a finite set of relations \( \ell \). From these relations, one obtains certain isomorphisms of webs in \( \mathbf{Foams}_/\ell \) (see [4] Corollary 1) allowing the removal of
adjacent bivalent vertices satisfying some extra condition, and replacing each web by an isomorphic web containing exactly two vertices or, better, no vertices at all. Hence vertices are just some artifacts whose presence yields a homology theory for links which is functorial under link cobordisms with no sign ambiguity.

The original Khovanov homology relies on a 2D TQFT, but the \( \mathfrak{sl}(2) \) foam link cohomology uses a “tautological functor”, as the 1-manifolds in the underlying theory are not oriented circles but piecewise oriented circles. Therefore, some type of extended 2D TQFT defined on foams would be quite desirable to have at hand for the foam cohomology theory for links/knots.

In this paper we make the first steps in achieving this goal. The cobordisms considered here are a particular case of those used in [4 5], in the sense that the 1-manifolds are disjoint unions of oriented circles and bi-webs, which are webs with exactly two vertices. The corresponding cobordisms between such 1-manifolds are called singular 2-cobordisms. We introduce the category \( \text{Sing-2Cob} \) of singular 2-cobordisms and show that it is equivalent as a symmetric monoidal category to the category freely generated by what we call a twin Frobenius algebra. A twin Frobenius algebra and knowledgeable Frobenius algebra resemble each other, in the sense that almost all properties of the second are satisfied by the first, except for the “Cardy relation”, which is replaced by the so-called “isomorphism condition” for a twin Frobenius algebra. The definition of twin Frobenius algebras and their category is given in Section 2.

A question arises now: how can one address the more general case in which webs are allowed to have an arbitrary even number of bivalent vertices as in [4 5]? One way to do this is by imposing—in \text{Foams}—the isomorphisms of webs mentioned earlier, in particular, by imposing relations (CI) and (SR) obtained in [4]. Then the category \( \text{Sing-2Cob} \) can be regarded as a quotient of the category \( \text{Foams} \).

We present in Section 3 a normal form for an arbitrary singular 2-cobordism and characterize the category \( \text{Sing-2Cob} \) in terms of generators and relations. We obtain that this category is equivalent, as a symmetric monoidal category, to the category freely generated by a twin Frobenius algebra. In Section 4 we define 2-dimensional twin TQFTs in \( \mathcal{C} \) as symmetric monoidal functors \( \text{Sing-2Cob} \to \mathcal{C} \), where \( \mathcal{C} \) is a symmetric monoidal category, and prove that the category of twin TQFTs in \( \mathcal{C} \) is equivalent as a symmetric monoidal category to the category of twin Frobenius algebras in \( \mathcal{C} \). For our purpose, the category \( \mathcal{C} \) will be either \( \text{Vect}_k \) or \( \text{R-Mod} \), where \( k \) is a field and \( R \) a commutative ring, both containing the primitive fourth root of unity \( i \).

In Section 5 we provide the example of twin Frobenius algebra and its corresponding 2-dimensional twin TQFT that, we believe, can be used in describing the \( \mathfrak{sl}(2) \) foam link cohomology theory with no dots.

2. Twin Frobenius Algebras

Consider a symmetric monoidal (tensor) category \((\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho, \tau)\) with unit object \( 1 \in \mathcal{C} \), associativity law \( \alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z) \), left-unit and right-unit laws \( \lambda_X : 1 \otimes X \to X \) and \( \rho_X : X \otimes 1 \to X \), and with symmetric braiding \( \tau_{X,Y} : X \otimes Y \to Y \otimes X \), for \( X, Y \) and \( Z \) objects in \( \mathcal{C} \).
As examples for the category $\mathcal{C}$, we are interested in $\text{Vect}_k$, the category of vector spaces over a field $k$ and $k$-linear maps, and $\text{R-Mod}$, the category of modules over a commutative ring $R$ and module homomorphisms. The ground field $k$ or ring $R$ is required to contain $i$, where $i^2 = -1$. Note that the unit object $1 \in \mathcal{C}$ is then $k$ or $R$.

For reader’s convenience, we recall first a few definitions.

An algebra object $(C, m, \iota)$ in $\mathcal{C}$ consists of an object $C$ and morphisms $m: C \otimes C \to C$ and $\iota: 1 \to C$ in $\mathcal{C}$ such that:

$$m \circ (\text{id}_C \otimes m) \circ \alpha_{C,C,C} = m \circ (m \otimes \text{id}_C)$$

$$m \circ (\text{id}_C \otimes \iota) = \rho_C \quad \text{and} \quad m \circ (\iota \otimes \text{id}_C) = \lambda_C.$$

A coalgebra object $(C, \Delta, \epsilon)$ in $\mathcal{C}$ is an object $C$ and morphisms $\Delta: C \to C \otimes C$ and $\epsilon: C \to 1$ such that:

$$(\text{id}_C \otimes \Delta) \circ \Delta = \alpha_{C,C,C} \circ (\Delta \otimes \text{id}_C) \circ \Delta$$

$$(\text{id}_C \otimes \epsilon) \circ \Delta = \rho_C^{-1} \quad \text{and} \quad (\epsilon \otimes \text{id}_C) \circ \Delta = \lambda_C^{-1}.$$

A homomorphism of algebras $f: C \to C'$ between two algebra objects $(C, m, \iota)$ and $(C', m', \iota')$ in $\mathcal{C}$ is a morphism $f$ of $C$ such that:

$$f \circ m = m' \circ (f \otimes f) \quad \text{and} \quad f \circ \iota = \iota'.$$

A homomorphism of coalgebras $f: C \to C'$ between two coalgebra objects $(C, \Delta, \epsilon)$ and $(C', \Delta', \epsilon')$ in $\mathcal{C}$ is a morphism $f$ of $C$ such that:

$$(f \otimes f) \circ \Delta = \Delta' \circ f \quad \text{and} \quad \epsilon' \circ f = \epsilon.$$

A Frobenius algebra object $(C, m, \iota, \Delta, \epsilon)$ in $\mathcal{C}$ consists of an object $C$ together with morphisms $m, \iota, \Delta, \epsilon$ such that:

- $(C, m, \iota)$ is an algebra object and $(C, \Delta, \epsilon)$ is a coalgebra object in $\mathcal{C},$
- $(m \otimes \text{id}_C) \circ \alpha_{C,C,C} \circ (\text{id}_C \otimes \Delta) = \Delta \circ m = (\text{id}_C \otimes m) \circ \alpha_{C,C,C} \circ (\Delta \otimes \text{id}_C).$

A Frobenius object $(C, m, \iota, \Delta, \epsilon)$ in $\mathcal{C}$ is called commutative if $m \circ \tau = m,$ and is called symmetric if $\epsilon \circ m = \epsilon \circ m \circ \tau.$ Given two Frobenius algebra objects $(C, m, \iota, \Delta, \epsilon)$ and $(C', m', \iota', \Delta', \epsilon'),$ a homomorphism of Frobenius algebras $f: C \to C'$ is a morphism $f$ in $\mathcal{C}$ which is both a homomorphism of algebra and coalgebra objects.

**Definition 1.** A twin Frobenius algebra $T := (C, W, z, z^*)$ in $\mathcal{C}$ consists of

- a commutative Frobenius algebra $C = (C, m_C, \iota_C, \Delta_C, \epsilon_C),$
- a symmetric Frobenius algebra $W = (W, m_W, \iota_W, \Delta_W, \epsilon_W),$
- two morphisms $z: C \to W$ and $z^*: W \to C,$

such that $z$ is a homomorphism of algebra objects in $\mathcal{C}$ and

$$\epsilon_C \circ m_C \circ (\text{id}_C \otimes z^*) = \epsilon_W \circ m_W \circ (z \otimes \text{id}_W),$$

**(duality condition)**

$$m_W \circ (\text{id}_W \otimes z) = m_W \circ \tau_{W,W} \circ (\text{id}_W \otimes z),$$

**(centrality condition)**

$$iz^* \circ z = \text{id}_C, \quad z \circ (iz^*) = \text{id}_W,$$

**(isomorphism condition)**
The first equality says that \( z^* \) is the morphism dual to \( z \), which implies that \( z^* \) is a homomorphism of coalgebra objects in \( C \). The second equality says that \( z(C) \) is contained in the center of the algebra \( W \), and the last condition states that \( z \) and \( iz^* \) are mutually inverse isomorphisms in \( C \). Hence \( z \) (respectively \( z^* \)) is an isomorphism of algebra (respectively coalgebra) objects.

**Definition 2.** A homomorphism of twin Frobenius algebras

\[
(C_1, W_1, z_1, z_1^*) \rightarrow (C_2, W_2, z_2, z_2^*)
\]

in a symmetric monoidal category \( C \) consists of a pair \((f, g)\) of Frobenius algebra homomorphisms \( f: C_1 \rightarrow C_2 \) and \( g: W_1 \rightarrow W_2 \) such that \( z_2 \circ f = g \circ z_1 \) and \( z_2^* \circ g = f \circ z_1^* \).

**Definition 3.** We denote by \( \text{T-Frob}(C) \) the category whose objects are twin Frobenius algebras in \( C \) and whose morphisms are twin Frobenius algebra homomorphisms.

**Proposition 1.** The category \( \text{T-Frob}(C) \) forms a symmetric monoidal category in the following sense:

- The tensor product of two twin Frobenius algebra objects \( \mathbf{T}_1 = (C_1, W_1, z_1, z_1^*) \) and \( \mathbf{T}_2 = (C_2, W_2, z_2, z_2^*) \) is defined as\[
\mathbf{T}_1 \otimes \mathbf{T}_2 := (C_1 \otimes C_2, W_1 \otimes W_2, z_1 \otimes z_2, z_1^* \otimes z_2^*)
\];
- The unit object is given by \( \mathbf{1} := (1, 1, \text{id}_1, \text{id}_1) \);
- The associativity and unit laws and the symmetric braiding are induced by those of \( C \);
- The tensor product of two homomorphisms \( f = (f_1, g_1) \) and \( g = (f_2, g_2) \) of twin Frobenius algebras is defined as \( f \otimes g := (f_1 \otimes f_2, g_1 \otimes g_2) \).

3. SINGULAR COBORDISMS AND THE CATEGORY \( \text{Sing-2Cob} \)

In this section we define the category of a singular 2-cobordisms and give a presentation of it in terms of generators and relations.

3.1. Description and topological invariants.

**Definition 4.** A singular 2-cobordism is an abstract piecewise oriented smooth 2-dimensional manifold \( \Sigma \) with boundary \( \partial \Sigma = \partial^- \Sigma \cup \partial^+ \Sigma \), where \( \partial^- \Sigma \) is \( \partial^- \Sigma \) with opposite orientation. Both \( \partial^- \Sigma \) and \( \partial^+ \Sigma \) are embedded closed 1-manifolds and we call them the source and target boundary, respectively. A singular 2-cobordism has singular arcs and/or singular circles where orientations disagree. There are exactly two compatibly oriented 2-cells of the underlying 2-dimensional CW-complex \( \Sigma \) that meet at a singular arc/circle, and orientations of two neighboring 2-cells induce an orientation on the singular arc/circle that they share.

Two singular 2-cobordisms \( \Sigma_1 \) and \( \Sigma_2 \) are considered equivalent, and we write \( \Sigma_1 \cong \Sigma_2 \), if there exists an orientation-preserving diffeomorphism \( \Sigma_1 \rightarrow \Sigma_2 \) which restricts to the identity on the boundary.
The objects in the category of singular 2-cobordisms are diffeomorphism classes of compact 1-manifolds. Specifically, such a manifold is diffeomorphic to a disjoint union of clockwise oriented circles and bi-webs; a bi-web is a closed graph with two bivalent vertices, as depicted below. The order of these manifolds is important, and there is a one-to-one correspondence between the objects of \( \text{Sing-2Cob} \) and sequences of zeros and ones.

\[
0 = \begin{array}{c}
\text{circle}
\end{array} \\
1 = \begin{array}{c}
\text{bi-web}
\end{array}
\]

**Definition 5.** An object in the category \( \text{Sing-2Cob} \) consists of a finite sequence \( n = (n_1, n_2, \ldots, n_k) \), where \( n_j \in \{0, 1\} \). The length of the sequence, denoted by \( |n| = k \), can be any nonnegative integer, and equals the number of disjoint connected components of the corresponding object. Each such tuple stands for a 1-dimensional submanifold \( C_n = \bigcup_{j=1}^k C(n_j) \) of \( \mathbb{R}^2 \), where \( C(n_j) \) is a clockwise oriented circle if \( n_j = 0 \) and a bi-web if \( n_j = 1 \).

A morphism \( \Sigma: n \to m \) in \( \text{Sing-2Cob} \) is an equivalence class \( [\Sigma] \) (induced by \( \cong \)) of singular 2-cobordisms with source boundary \( n \) and target boundary \( m \). Given \( \Sigma_1 \) a morphism from \( n \) to \( m \) and \( \Sigma_2 \) a morphism from \( m \) to \( k \), their composition \( \Sigma_2 \circ \Sigma_1 \) is the singular 2-cobordism obtained by placing \( \Sigma_1 \) on top \( \Sigma_2 \) (by using rigid shifts along the \( z \)-axis). All our surfaces are assumed to have collars so that composition yields a smooth surface.

Examples of singular 2-cobordisms are given below. The source of our cobordisms is at the top and the target at the bottom of drawings, in other words, we read morphisms as cobordisms from top to bottom, by convention (note that this is the opposite convention of that used in [4, 5]).

\[
(3.2) \quad (0, 0) \rightarrow (0) \\
(1) \rightarrow (1, 1) \\
(1) \rightarrow (0)
\]

The concatenation \( n \amalg m := (n_1, n_2, \ldots, n_{|n|}, m_1, m_2, \ldots, m_{|m|}) \) of sequences together with the free union of singular 2-cobordisms, which we denote also by \( \amalg \), endows the category \( \text{Sing-2Cob} \) with the structure of a symmetric monoidal category.

For each \( k \in \mathbb{N} \), there is an action of the symmetric group \( S_k \) on the subset of objects \( n \) in \( \text{Sing-2Cob} \) for which \( |n| = k \), defined by

\[
\sigma \ast n := (n_{\sigma^{-1}(1)}, n_{\sigma^{-1}(2)}, \ldots, n_{\sigma^{-1}(k)}).
\]

Given any object \( n \) in \( \text{Sing-2Cob} \) and any permutation \( \sigma \in S_{|n|} \), there is an obvious induced cobordism

\[
\sigma^n : n \rightarrow \sigma \ast n.
\]

For example, if \( n = (0, 1, 1, 0, 1) \) and \( \sigma = (12)(354) \in S_5 \), the corresponding morphism \( \sigma^n \) is the singular cobordism given in (3.3).
We remark that as morphisms of Sing-2Cob, these cobordisms satisfy \( \tau^{n} \circ \sigma^{n} = (\tau \circ \sigma)^{n} \), for any object \( n \) and \( \sigma, \tau \in S_{|n|} \).

**Definition 6.** Let \( \Sigma : n \to m \) be a morphism in Sing-2Cob and let \( l \) be the number of its boundary components that are diffeomorphic to the bi-web. In other words, \( l \) is the number of 1 entries of \( n \ll m \). Number these components by 1, 2, \ldots, \( l \). The orientation of \( \Sigma \) induces an orientation on all singular arcs of \( \Sigma \) and defines a permutation \( \sigma(\Sigma) \in S_{l} \), called the *singular boundary permutation* of \( \Sigma \).

For exemplification, we consider the morphism \( \Sigma \) depicted in (3.4) and we number the bi-webs in its boundary by 1, 2, 3 and 4 from left to right, starting with those in the source and followed by those in the target. The singular boundary permutation of this morphism is \( \sigma(\Sigma) = (1)(234) = (234) \in S_{4} \).

![Diagram](3.4)

We remark that two equivalent singular 2-cobordisms are required to have the same singular boundary permutation (but two singular cobordisms having the same boundary permutation might not be equivalent). With this in mind, we need to refine the definition of the morphisms in Sing-2Cob.

**Definition 7.** A morphism \( \Sigma : n \to m \) is a pair \( \Sigma = ([\Sigma], \sigma) \) consisting of an equivalence class \( [\Sigma] \) of singular 2-cobordisms with source boundary \( n \) and target boundary \( m \), and with singular boundary permutation \( \sigma(\Sigma) = \sigma \).

### 3.2 Generators.

We use Morse theory to provide a generators and relations description of the category Sing-2Cob. Specifically, we decompose each singular 2-cobordism into cobordisms each of which contains exactly one critical point, and the components of such a decomposition are the *generators* for the morphisms.

Every singular 2-cobordism \( \Sigma \in \text{Sing-2Cob} \) admits a Morse function \( f : \Sigma \to \mathbb{R} \). The set of critical points of \( f \) is finite, and they are all non-degenerate and isolated.

**Proposition 2.** Let \( \Sigma \in \text{Sing-2Cob} \) be a connected singular 2-cobordism and \( f : \Sigma \to \mathbb{R} \) a Morse function such that \( f \) has precisely one critical point. Then \( \Sigma \) is equivalent to one of the following singular 2-cobordisms:

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.15\textwidth]{delta_c} \\
= \Delta_{C}
\end{array} & \quad \begin{array}{c}
\includegraphics[width=0.15\textwidth]{m_c} \\
= m_{C}
\end{array} & \quad \begin{array}{c}
\includegraphics[width=0.15\textwidth]{i_c} \\
= i_{C}
\end{array} & \quad \begin{array}{c}
\includegraphics[width=0.15\textwidth]{epsilon_c} \\
= \epsilon_{C}
\end{array} \\
\begin{array}{c}
\includegraphics[width=0.15\textwidth]{delta_w} \\
= \Delta_{W}
\end{array} & \quad \begin{array}{c}
\includegraphics[width=0.15\textwidth]{m_w} \\
= m_{W}
\end{array} & \quad \begin{array}{c}
\includegraphics[width=0.15\textwidth]{i_w} \\
= i_{W}
\end{array} & \quad \begin{array}{c}
\includegraphics[width=0.15\textwidth]{epsilon_w} \\
= \epsilon_{W}
\end{array}
\end{align*}
\]
or to one of the following compositions of singular 2-cobordisms:

(3.7) \[ \text{Diagram} \]

(3.8) \[ \text{Diagram} \]

(3.9) \[ \text{Diagram} \]

(3.10) \[ \text{Diagram} \]

(3.11) \[ \text{Diagram} \]

(3.12) \[ \text{Diagram} \]

These singular cobordisms are embedded in $\mathbb{R}^3$ and have the source at the top and the target at the bottom of the diagram. The vertical axis of the drawing plane is $-f$.

**Proof.** It is well known that the critical point $p$ is characterized by its index $i(p)$, the number of negative eigenvalues of the Hessian of $f$ at $p$.

1. If $i(p) = 2$, the Morse function $f$ has a local maximum. If $p$ does not lie on a singular arc or singular circle, then $\Sigma$ is diffeomorphic to $\epsilon_C$ or to the first composition depicted in (3.7). If $p$ lies on a singular arc or singular circle, then $\Sigma$ is diffeomorphic to $\epsilon_W$ or to the second composition in (3.7).

2. If $i(p) = 1$, then we have a saddle point. If $p$ does not lie on a singular arc, then $\Sigma$ is of the form $m_C$ or $\Delta_C$, or is equivalent to one of the compositions given in (3.9) or (3.10). If the saddle point $p$ lies on a singular arc, then $\Sigma$ is equivalent to $m_W$ or $\Delta_W$, or to one of the compositions depicted in (3.11) or (3.12).

3. If $i(p) = 0$, then $f$ has a local minimum. If $p$ does not belong to a singular arc or singular circle, then $\Sigma$ is of the form $\iota_C$ or is equivalent to the first composition displayed in (3.8). If $p$ does belong to a singular arc or singular circle, then $\Sigma$ is diffeomorphic to $\iota_W$ or to the second composition given in (3.8).
Corollary 1. Every connected singular 2-cobordism can be obtained by gluing the cobordisms depicted in (3.5), (3.6) and those given in (3.13) below:

\[(3.13)\]

\[\begin{array}{c}
\text{z} = id_C \\
\text{id} \\
\text{z}^* = id_W
\end{array}\]

3.3. Non-connected singular 2-cobordisms. We treat the case of non-connected cobordisms via disjoint unions and permutations of the factors of disjoint unions, following Kock’s work [10] for the case of ordinary 2-cobordisms. Since every permutation can be written as a product of transpositions, the following singular 2-cobordisms are sufficient to do this:

\[(3.14)\]

\[\begin{array}{c}
\text{X} \\
\text{X} \\
\text{X} \\
\text{X}
\end{array}\]

There is no need to talk about crossing over or under, since our cobordisms are abstract manifolds, thus not embedded anywhere.

Without loss of generality, we assume that \(\Sigma: n \to m\) has two connected components, \(\Sigma_1\) and \(\Sigma_2\), and that \(n = (n_1, n_2, \ldots, n_{|n|})\). The source boundary of \(\Sigma_1\) is a tuple \(p\) whose components form a subset of \(\{n_1, n_2, \ldots, n_{|n|}\}\), and the source boundary of \(\Sigma_2\) is the tuple \(q\), which is the complement of \(p\) in \(\{n_1, n_2, \ldots, n_{|n|}\}\).

We can permute the components of \(n\) by applying a diffeomorphism \(n \to n\), so that the components of \(p\) come before those of \(q\). This diffeomorphism induces a cobordism \(S\), and we can consider the singular 2-cobordism \(S\Sigma\). Applying the same method to the target boundary of \(\Sigma\), which is also the target boundary of \(S\Sigma\), there is a permutation singular 2-cobordism \(T: m \to m\) so that \(\Sigma' = S\Sigma T\) is a singular 2-cobordism which is the disjoint union (as a cobordism) of \(\Sigma_1\) and \(\Sigma_2\). Then \(\Sigma \cong S^{-1}\Sigma T^{-1}\), where \(S^{-1}\) and \(T^{-1}\) are the permutation cobordisms which are the inverses of \(S\) and \(T\), respectively. For example, \(S^{-1}\) is the diffeomorphism that permutes the components of \(n\) such that the components of \(p\) come after those of \(q\).

As an example, we consider the following singular cobordism:

\[\Sigma\]

For the given cobordism we don’t need to permute the source boundary of \(\Sigma\), thus \(S\) is the disjoint union (as a cobordism) of two cylinders, but we do permute the target boundary of \(\Sigma\) by composing with a cobordism \(T\). The composed cobordism \(S\Sigma T\) is the disjoint union of its connected components \((S\Sigma T)_1\) and \((S\Sigma T)_2\):
We have proved the following:

**Lemma 1.** Every singular 2-cobordism is equivalent to a composition of a permutation cobordism with a disjoint union of connected cobordisms, followed by a permutation cobordism.

Putting together the results of this subsection, we obtain:

**Proposition 3.** The symmetric monoidal category \( \text{Sing-2Cob} \) is generated under composition and disjoint union by the following singular 2-cobordisms:

\[
\begin{align*}
\begin{array}{cccc}
\text{S} & \Sigma & \text{T} & \Sigma_{T_1} & \Sigma_{T_2}
\end{array}
\end{align*}
\]

3.4. **Relations.** In this section we give a list of relations satisfied by the generators of \( \text{Sing-2Cob} \). In Section 3.5 we define a normal form for singular 2-cobordisms with a given topological structure, namely the genus and singular boundary permutation. In Section 3.6 we prove that the relations given in Proposition 4 below are sufficient to completely describe the category \( \text{Sing-2Cob} \); the techniques used are similar in spirit to those in [11].

*Imposed local relations.* We allow formal linear combinations of morphisms, with coefficients in the ground field/ring, and we extend the composition maps in the natural bilinear way. Then we impose the following local relations on the set of morphisms:

\[
(3.15) \quad \begin{array}{c}
\begin{array}{c}
\text{Manifold}
\end{array}
\end{array} = -i \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
\text{Manifold}
\end{array}
\end{array} = -i
\]

We remark that the singular 2-cobordisms in the left equality above do not have the same singular boundary permutation.
3.4.1. **Sufficient relations.**

**Proposition 4.** The following relations hold in the symmetric monoidal category Sing-2Cob:

1. The object $n = (0)$ forms a commutative Frobenius algebra object.

   \[
   \begin{align*}
   (3.16) & \quad \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array}
   \\
   (3.17) & \quad \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array}
   \\
   (3.18) & \quad \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array}
   \\
   (3.19) & \quad \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array}
   
   \end{align*}
   \]

2. The object $n = (1)$ forms a symmetric Frobenius algebra object.

   \[
   \begin{align*}
   (3.20) & \quad \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array}
   \\
   (3.21) & \quad \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array}
   \\
   (3.22) & \quad \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array}
   \\
   (3.23) & \quad \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array}
   
   \end{align*}
   \]

3. The zipper $\begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array}$ forms an algebra homomorphism.

   \[
   \begin{align*}
   (3.24) & \quad \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array} \cong \begin{array}{c}
   \begin{tikzpicture}
   \end{tikzpicture}
   \end{array}
   \end{align*}
   \]
(4) The cozipper \( \begin{array}{c}
\end{array} \) is dual to the zipper.

(3.25)

(5) Centrality relation.

(3.26)

(6) Isomorphism relations: cobordisms \( \begin{array}{c}
\end{array} \) and \( \begin{array}{c}
\end{array} \) are mutually inverse isomorphisms (this is exactly what the local relations (3.15) state).

3.4.2. Consequences of relations. We provide now additional relations that are implied by those described in Proposition 4, and which will be useful for the remaining of the paper.

**Proposition 5.** The cozipper \( \begin{array}{c}
\end{array} \) is a coalgebra homomorphism, that is, the following singular cobordisms are equivalent:

(3.27)

It will be useful to define the following singular cobordisms called *singular pairing* and *singular copairing*:

\[ \begin{array}{c}
\end{array} \]  \( := \) \[ \begin{array}{c}
\end{array} \]

\[ \begin{array}{c}
\end{array} \]  \( := \) \[ \begin{array}{c}
\end{array} \]

Similarly, we define the cobordisms which we call the *ordinary pairing* and *ordinary copairing*:

\[ \begin{array}{c}
\end{array} \]  \( := \) \[ \begin{array}{c}
\end{array} \]

\[ \begin{array}{c}
\end{array} \]  \( := \) \[ \begin{array}{c}
\end{array} \]

These cobordisms satisfy the zig-zag identities:

(3.28)

(3.29)
It follows from Equations (3.23) and (3.20) that the singular pairing is invariant and symmetric:

\[(3.30) \]

Similarly, it follows from Equations (3.19) and (3.16) that the ordinary pairing is invariant and symmetric:

\[(3.31) \]

It is easy to see that similar results hold for both singular and ordinary copairings.

**Proposition 6.** The following singular cobordisms are equivalent:

\[(3.32) \]

**Proof.** The first equivalence of cobordisms in Equation (3.32) is the same as the axiom in Equation (3.25). The proof of the second diffeomorphism in Equation (3.32) is given below, where by “Nat” we denote the diffeomorphisms which express the natural behavior of the symmetric braiding:

\[(3.33) \]

The proof of Equation (3.33) is given below:
Proposition 7. The following singular cobordisms are equivalent:

\[(3.34) \cong \cong \cong \cong \]

\[(3.35) \cong \cong \cong \cong \]

Proof. The diffeomorphisms given in Equation (3.34) follow from the following sequences of diffeomorphisms:

\[\cong \quad (3.20) \quad \cong \quad (3.22) \quad \cong\]

\[\cong \quad (3.20) \quad \cong \quad (3.22) \quad \cong\]

\[\cong \quad (3.28) \quad \cong \quad (3.30) \quad \cong\]

The proof of Equation (3.35) is done similarly by replacing the bi-web with an oriented circle, thus replacing the singular cobordisms above by their ordinary cobordisms counterparts.

Proposition 8. The following singular cobordisms are equivalent:

\[(3.36) \cong \cong \cong \]

\[(3.37) \cong \cong \cong \]

Proof. The proof of these equivalences is done in a similar manner as in the previous Proposition.

Proposition 9. The following singular cobordisms are equivalent:

\[(3.38) \cong \cong \cong \]

\[(3.39) \cong \cong \cong \]
Proposition 10. The singular genus-one operator $\begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0.5) .. (0.5,0); \draw (0.5,0) .. controls (0,0.5) and (0,-0.5) .. (-0.5,0); \end{tikzpicture}$ can be moved around freely in any diagram. Specifically, the following cobordisms are equivalent:

$\begin{align*}
(3.39) \quad & \begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0.5) .. (0.5,0); \draw (0.5,0) .. controls (0,0.5) and (0,-0.5) .. (-0.5,0); \end{tikzpicture} \cong \begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0) .. (0.5,0); \draw (0.5,0) .. controls (0,0) and (0,0.5) .. (-0.5,0); \end{tikzpicture} \\
(3.40) \quad & \begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0) .. (0.5,0); \draw (0.5,0) .. controls (0,0) and (0,0.5) .. (-0.5,0); \end{tikzpicture} \cong \begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0.5) .. (0.5,0); \draw (0.5,0) .. controls (0,0.5) and (0,-0.5) .. (-0.5,0); \end{tikzpicture}
\end{align*}$

Proposition 11. The genus-one operator $\begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0.5) .. (0.5,0); \draw (0.5,0) .. controls (0,0.5) and (0,-0.5) .. (-0.5,0); \end{tikzpicture}$ can be moved around freely in any diagram. Specifically, the following cobordisms are equivalent:

$\begin{align*}
(3.41) \quad & \begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0) .. (0.5,0); \draw (0.5,0) .. controls (0,0) and (0,0.5) .. (-0.5,0); \end{tikzpicture} \cong \begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0.5) .. (0.5,0); \draw (0.5,0) .. controls (0,0.5) and (0,-0.5) .. (-0.5,0); \end{tikzpicture} \\
(3.42) \quad & \begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0) .. (0.5,0); \draw (0.5,0) .. controls (0,0) and (0,0.5) .. (-0.5,0); \end{tikzpicture} \cong \begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0.5) .. (0.5,0); \draw (0.5,0) .. controls (0,0.5) and (0,-0.5) .. (-0.5,0); \end{tikzpicture}
\end{align*}$

Proposition 12. The following relations containing cobordisms with the same singular boundary permutation hold in the category $\text{Sing-2Cob}$:

$\begin{align*}
(3.43) \quad & \begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0) .. (0.5,0); \draw (0.5,0) .. controls (0,0) and (0,0.5) .. (-0.5,0); \end{tikzpicture} = - \begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0) .. (0.5,0); \draw (0.5,0) .. controls (0,0) and (0,0.5) .. (-0.5,0); \end{tikzpicture}
\end{align*}$

$\begin{align*}
(3.44) \quad & \begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0) .. (0.5,0); \draw (0.5,0) .. controls (0,0) and (0,0.5) .. (-0.5,0); \end{tikzpicture} = - \begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0) .. (0.5,0); \draw (0.5,0) .. controls (0,0) and (0,0.5) .. (-0.5,0); \end{tikzpicture} = - \begin{tikzpicture}[baseline=0,thick, scale=0.5] \draw (-0.5,0) .. controls (0,-0.5) and (0,0) .. (0.5,0); \draw (0.5,0) .. controls (0,0) and (0,0.5) .. (-0.5,0); \end{tikzpicture}
\end{align*}$

Proof. The statement follows easily from the first imposed local relation in (3.15) and other results of this subsection.

3.5. The normal form of a singular 2-cobordism. We give in this section the normal form of an arbitrary connected singular 2-cobordism $\Sigma$. The normal form is characterized by the singular boundary permutation $\sigma(\Sigma)$ and genus $g(\Sigma)$. 

3.5.1. Particular case. We first describe the normal form of a connected singular cobordism whose source consists entirely of bi-webs and whose target consists entirely of circles. Thus, we consider singular cobordisms $\Sigma : \mathbf{n} \rightarrow \mathbf{m}$ for which $\mathbf{n} = (1,1,\ldots,1)$ and $\mathbf{m} = (0,0,\ldots,0)$, and denote the set of all such cobordisms by $\textbf{Sing-2Cob}_{W \rightarrow C}(\mathbf{n}, \mathbf{m})$. Then we give the normal form for an arbitrary connected singular cobordism by using the zig-zag identities (3.28) and (3.29).

Notice that relations of the form $(\Sigma' \coprod \text{id}_m) \circ (\text{id}_n' \coprod \Sigma) = \Sigma' \coprod \Sigma$ hold in $\textbf{Sing-2Cob}$ for any $\Sigma : \mathbf{n} \rightarrow \mathbf{m}$ and $\Sigma' : \mathbf{n}' \rightarrow \mathbf{m}'$, and we will make use of them in order to have small heights for diagrams.

**Definition 8.** Let $\Sigma \in \textbf{Sing-2Cob}_{W \rightarrow C}(\mathbf{n}, \mathbf{m})$ be a connected cobordism with singular boundary permutation $\sigma(\Sigma)$ and genus $g(\Sigma)$, and write the singular boundary permutation as a product of disjoint cycles $\sigma(\Sigma) = \sigma_1 \sigma_2 \ldots \sigma_r$, $r \in \mathbb{N} \cup \{0\}$, where $\sigma_k$ has length $q_k \in \mathbb{N}$, $1 \leq k \leq r$. The normal form of $\Sigma$ is the composition

$$\text{NF}_{W \rightarrow C}(\Sigma) = (i)^l D_{|\mathbf{n}|} \circ C_{g(\Sigma)} \circ B_r \circ (\coprod_{k=1}^r A(q_k)) \circ \Sigma_{\sigma(\Sigma)}$$

for some $l \in \mathbb{N} \cup \{0\}$, of the following singular 2-cobordisms:

1. For each cycle $\sigma_k$, the singular cobordism $A(q_k)$ consists of $q_k - 1$ singular multiplications followed by a cozipper, as depicted below:

   $$A(q_k) := \cdots$$

   The normal form contains the free union of such cobordisms for each cycle $\sigma_k, 1 \leq k \leq r$. If $q_k = 1$ then $A(q_k)$ is a cozipper, and if $|\mathbf{n}| = 0$ then $r = 0$, and the free union $\coprod_{k=1}^r A(q_k)$ is replaced by the empty set.

2. If $r \geq 1$, then the singular cobordism $B_r$ consists of $r - 1$ multiplications

   $$B_r := \cdots$$

   If $r = 0$ then $B_0 := \underbrace{\circ \cdots \circ}_0$.

3. If $g(\Sigma) \geq 1$, the singular cobordism $C_{g(\Sigma)}$ is the composite

   $$C_{g(\Sigma)} := G \circ G \circ \cdots \circ G$$

   where $G := \underbrace{\circ \cdots \circ}_0$. If $g(\Sigma) = 0$ then $C_{g(\Sigma)} = \emptyset$. 

4. If \(|m| \geq 1\), then the singular cobordism \(D_{|m|}\) consists of \(|m| - 1\) comultiplications, as depicted below:

\[
D_{|m|} := \quad \vdots \quad \vdots
\]

If \(|m| = 0\) then \(D_0 := \) 

5. \(\Sigma_{\sigma(\Sigma)}\) represents the singular cobordisms induced by the permutation \(\sigma(\Sigma)\) given below. Denote by \(\tau(\sigma)\) the singular boundary permutation of the cobordism \(3.46\)

\[
\sigma(\Sigma) = \sigma(\Sigma)^{-1} \cdot \tau(\Sigma) \cdot \sigma(\Sigma).
\]

In Figure 1 we show a cobordism of the form \(3.46\), that is, the normal form of a cobordism in \(\text{Sing-2Cob}_{W \to C}(n, m)\), without precomposition with \(\Sigma_{\sigma(\Sigma)}\).

The following two results say that a cobordism given in its normal form is invariant, up to equivalence, under composition with certain permutation morphisms.

**Proposition 13.** Let \([\Sigma] \in \text{Sing-2Cob}_{W \to C}(n, m)\). Then

\[
[\sigma^m \circ \text{NF}_{W \to C}(\Sigma)] = [\text{NF}_{W \to C}(\sigma) \circ \sigma^m]
\]

for any \(\sigma \in S_{|m|}\) and for all cycles \(\sigma_k \in S_{|n|}, 1 \leq k \leq r\), that appear in the decomposition of \(\sigma(\Sigma) = \sigma_1 \sigma_2 \ldots \sigma_r\) into disjoint cycles.

3.5.2. **General case.** We use the normal form for a connected singular cobordism in \(\text{Sing-2Cob}_{W \to C}(n, m)\) and the duality property for the bi-web and circle to obtain the normal form of a generic connected morphism \([\Sigma] \in \text{Sing-2Cob}(n, m)\).

Let \(\Sigma\) be a representative of the equivalence class \([\Sigma]\) and let \(n_0 \coprod n_1\) be the permutation of \(n\) such that \(n_0 = (0, 0, \ldots, 0)\) and \(n_1 = (1, 1, \ldots, 1)\). Similarly, let \(m_0 \coprod m_1\) be the permutation of \(m\) such that \(m_0 = (0, 0, \ldots, 0)\) and \(m_1 = (1, 1, \ldots, 1)\). In order to use the normal form described above, we need to associate to \([\Sigma]\) a singular cobordism whose source contains only bi-webs and whose target contains only circles. We define the map

\[
f : \text{Sing-2Cob}(n, m) \to \text{Sing-2Cob}_{W \to C}(m_1 \coprod n_1, m_0 \coprod n_0)
\]

where the singular cobordism \(f([\Sigma])\) is defined as follows. Let \(\sigma_1\) be the permutation corresponding to \(\sigma_1 \in S_{|n|}\) that sends \(n_0\) to \(n_1 \coprod n_0\). Similarly, denote by \(\sigma_2\) the permutation corresponding to the permutation \(\sigma_2 \in S_{|m|}\) that sends \(m_0\) to \(m_1 \coprod m_0\).
We define $f([\Sigma])$ as the singular cobordism obtained from $[\Sigma]$ by precomposing with $\sigma_1^{-1}$, postcomposing with $\sigma_2$ and gluing copairings on every circle that $n_0$ contains, and singular pairings on every bi-web that $m_1$ contains.

For exemplification, consider $n = (0,1,1), m = (0,1,0)$ and $[\Sigma]$ an arbitrary singular cobordism from $n$ to $m$ (note that $|n|$ and $|m|$ do not have to be equal). The corresponding permutation cobordisms $\sigma_1, \sigma_2$ and the image of $[\Sigma]$ under $f$ are given in Figure 2.

Notice that the mapping $f$ is well-defined, namely $f([\Sigma])$ is a morphism in the category $\text{Sing-2Cob}_{W \rightarrow C}(m_1 \coprod n_1, m_0 \coprod n_0)$, and $[f([\Sigma])] = [f([\Sigma'])]$ whenever $[\Sigma] = [\Sigma']$. Therefore it makes sense to consider the normal form $\text{NF}_{W \rightarrow C}(f([\Sigma])).$

We also remark that $f([\Sigma])$ has a certain structure, in the sense that its source $n'$ and target $m'$ can be decomposed into free unions $n' = n'_t \coprod n'_s$ and $m' = m'_t \coprod m'_s$, such that the bi-webs in $n'_t$ (or $n'_s$) and the circles in $m'_t$ (or $m'_s$) correspond to the bi-webs
σ₁ = \begin{align*}
\sigma₂ &= \begin{array}{c}
\text{Figure 2. The image of } [Σ] \text{ under the map } f \\
\end{array}
\end{align*}

and circles coming from the target (or source) of \(σ₂ \circ Σ \circ σ₂^{-1}\). The permutation \(σ₁\) is an element of \(S_{n'+|m'|}\) while \(σ₂\) is an element of \(S_{m'|+|n'|}\).

We define an inverse mapping \(f^{-1}\) that associates to \([Φ] ∈ \text{Sing-}2\text{Cob}_{W→C}(n', m')\) the singular cobordism \(f^{-1}([Φ]) ∈ \text{Sing-}2\text{Cob}_{W→C}(σ₂(-[m'], m'), σ₁(n', [m']))\). The cobordism \(f^{-1}([Φ])\) is obtained by gluing singular copairings to the bi-webs in \(n'_i\) and ordinary pairings to the circles in \(m'_i\), and then by precomposing the resulting cobordism with the cobordism corresponding to \(σ₁\) and by postcomposing it with the cobordism corresponding to \(σ₂\).

This map is well-defined as well, and defines a bijection between the morphisms in \(\text{Sing-}2\text{Cob}(n, m)\) and those in \(\text{Sing-}2\text{Cob}_{W→C}(m_1 \coprod n_1, m_0 \coprod n_0)\).

Going back to the example above, we give in (3.47) the singular cobordism \([f^{-1}(f([Σ]))]) \cong [Σ].

\begin{align*}
(3.47) & \quad \begin{array}{c}
\end{array}
\end{align*}

Definition 9. Let \([Σ] ∈ \text{Sing-}2\text{Cob}(n, m)\) where Σ is a connected cobordism. We define the normal form of \([Σ]\) by

\begin{align*}
(3.48) & \quad [\text{NF}(Σ)] := f^{-1}([\text{NF}_{W→C}(f([Σ]))]).
\end{align*}
3.6. **Sufficiency of the relations.** We show now that the relations described in Proposition 4 are sufficient in order to relate any connected singular 2-cobordism $[\Sigma] \in \text{Sing-2Cob}_W \rightarrow C(n, m)$ to its normal form $NF_W \rightarrow C(\Sigma)$.

Recall that any connected singular 2-cobordism can be decomposed into the cobordisms (generators) given in (3.5), (3.6) and (3.13) (note that we ignore identity morphisms).

We use the notation $X$ (or $X^{-1}$) for an arbitrary singular 2-cobordism $X$ whose target (or source) is not glued to any other cobordism in the decomposition of $\Sigma$.

The following terminology is borrowed from [11, Definition 3.21].

**Definition 10.** Let $[\Sigma] \in \text{Sing-2Cob}(n, m)$ be connected. The height of a generator $G$ in the decomposition of $\Sigma$ is the following number defined inductively:

\[
\begin{align*}
    h(\emptyset) &= h(\otimes) = h(\begin{array}{c}X \end{array}) := 0, \\
    h(\begin{array}{c}G \end{array}) &= 1 + h(X), \\
    h(\begin{array}{c}G \end{array}) &= 1 + h(X) + h(Y),
\end{align*}
\]

where $X$ and $Y$ are arbitrary cobordisms in the decomposition of $\Sigma$.

**Theorem 1.** Let $[\Sigma] \in \text{Sing-2Cob}_W \rightarrow C(n, m)$ be a connected singular 2-cobordism. Then $\Sigma$ is equivalent to its normal form, namely we have $[\Sigma] = [NF_W \rightarrow C(\Sigma)]$.

**Proof.** The proof is similar in spirit to that of [11, Theorem 3.22], with the difference that it uses our cobordisms and relations. We consider $\Sigma$ be given in an arbitrary decomposition and construct a step by step diffeomorphism (relative to boundary) from this decomposition to the normal form $NF_W \rightarrow C(\Sigma)$.

**I.** The decomposition of $\Sigma$ is equivalent to one without singular cups $\otimes$ and singular caps $\emptyset$, by applying the following diffeomorphism:

\[
\begin{align*}
    \text{(3.27)} & \quad \rightarrow \quad \text{(3.24)}
\end{align*}
\]

**II.** The decomposition of $\Sigma$ is equivalent to one in which every singular comultiplication $\otimes$ appears in one of the following situations:

\[
\begin{align*}
    \text{(3.49)} & \quad \rightarrow \quad \text{(3.50)}
\end{align*}
\]

This is done by considering every possible situation in which the singular comultiplication may appear.
a) We can exclude the cases \( \includegraphics{case1} \) and \( \includegraphics{case2} \) by step (I).

b) Apply the diffeomorphism \( \includegraphics{step1} \) whenever possible.

c) The following diffeomorphisms reduce the height of the singular comultiplication:

i) \( \includegraphics{step2} \)

ii) \( \includegraphics{step3} \)

d) The following eliminates the singular comultiplication:

i) \( \includegraphics{step4} \)

ii) \( \includegraphics{step5} \)

and

\( \includegraphics{step6} \)

\( \includegraphics{step7} \)

\( \includegraphics{step8} \)

\( \includegraphics{step9} \)

e) Iterate steps (IIa)-(IId). Since each step either removes the singular comultiplication or reduces its height, and since the target of \( \Sigma \) does not contain biwebs, this process terminates with every singular comultiplication in one of the situations described above.

III. We look now at the possible cases in which the source and target of the singular multiplication \( \includegraphics{singular} \) may appear.

a) The decomposition of \( \Sigma \) is equivalent to one in which the source of the singular multiplication appears in one of the following situations:
or those depicted in the statement of step (II). One can see that this claim holds by considering all possible situations of the singular comultiplication and then applying the following diffeomorphisms. Each iteration of the following steps either removes the singular multiplication or increases its height. As a result, each singular multiplication ends up into one of the situations above.

i) \[ (3.20) \]

ii) \[ (3.24) \]

iii) \[ (3.40) \]

iv) The diffeomorphisms of (IIc)i and (IIId)i.

b) The decomposition of \( \Sigma \) is equivalent to one in which the target of the singular multiplication appears in one of the following situations:

i) We first show that the source of every cozipper \( \) can be put so that it appears in one of the following situations:

\[ \]

This claim is proved by applying the steps (I), (IIId)ii and the following:

\[ (3.44) \]

whenever possible.

ii) The singular genus-one operator \( \) can be removed by iterating the step (IIIa)iii, equation \[ (3.44) \] and the following diffeomorphism:

\[ (3.39) \]
This process either reduces the height of the singular genus-one operator or removes it. Since the height of the operator cannot be zero, the process guarantees to remove the singular genus-one operator.

IV. In this step we show that there exists a sequence of diffeomorphisms that removes all singular comultiplications. Consider the set of all such comultiplications that appear in the decomposition of $\Sigma$ and choose one of minimal height.

We can exclude the case since the singular comultiplication is of minimal height. From steps (II) and (IIIb)ii we know that the remaining situations to consider are:

where “?” may be any singular cobordism that contains no singular comultiplication. Since the above cobordisms are symmetric, it is enough to consider only one case, say the first one. Using step IIIb and the assumption that the singular comultiplication is of minimal height, there are exactly two possible situations for the first generator in the decomposition of “?” , namely:

Iteratively applying the diffeomorphism and considering again the next two possible situations in the decomposition of “?” , we see that after all, there are the following two possible cases:

a) In the first case, the singular comultiplication is eliminated by applying the following sequence of diffeomorphisms:
b) To remove the singular comultiplication in the second case above, we apply the following sequence of diffeomorphisms:

V. After the first four steps of the proof, all singular cups, caps and comultiplications have been eliminated from the decomposition of $\Sigma$, and the resulting decomposition has the following properties:

a) Every singular multiplication has its source in one of the following situations

and its target in one of the following situations

b) Every cozipper $\square$ appears in one of the situations explained in (IIIb)i.

c) Every singular genus-one operator $\bigcirc$ has been eliminated from the decomposition.
VI. We show now that the zipper can be eliminated from the decomposition of $\Sigma$.

The following situations:

![Diagram](image1)

are excluded by steps I, IV, (IIIb)ii, and the second equality in (3.15), respectively. It remains to consider the following cases:

![Diagram](image2)

The second case can be reduced to the first one by applying the following sequence of diffeomorphisms:

![Diagram](image3)

Then, by taking into account the second part of step (Va), we need to consider either:

- ![Diagram](image4)

  a) For the first possibility we apply the diffeomorphism which reduces the height of the zipper.

  b) For the second possibility we apply a sequence of diffeomorphisms that removes the zipper.

![Diagram](image5)

By repeating these steps if necessary and applying $\rightarrow -i$, we guarantee that the zipper has been eliminated from the decomposition of $\Sigma$.

VII. The (resulting) decomposition of $\Sigma$ is equivalent to one in which the ordinary multiplication $\otimes$ has its source in one of the following situations:

![Diagram](image6)
a) We can exclude the cases

\[
\begin{array}{ccc}
\includegraphics[width=0.1\textwidth]{case1} & \includegraphics[width=0.1\textwidth]{case2} & \includegraphics[width=0.1\textwidth]{case3}
\end{array}
\]

since we assume that the source of $\Sigma$ is a free union of bi-webs. To prove the claim we iterate the following diffeomorphisms whenever possible:

b) \[
\begin{array}{ccc}
\includegraphics[width=0.1\textwidth]{diffeomorphism1} & \includegraphics[width=0.1\textwidth]{diffeomorphism2} & \includegraphics[width=0.1\textwidth]{diffeomorphism3}
\end{array}
\]

c) \[
\begin{array}{ccc}
\includegraphics[width=0.1\textwidth]{diffeomorphism4}
\end{array}
\]

d) \[
\begin{array}{ccc}
\includegraphics[width=0.1\textwidth]{diffeomorphism5} & \includegraphics[width=0.1\textwidth]{diffeomorphism6}
\end{array}
\]

e) \[
\begin{array}{ccc}
\includegraphics[width=0.1\textwidth]{diffeomorphism7} & \includegraphics[width=0.1\textwidth]{diffeomorphism8}
\end{array}
\]

Each of the above diffeomorphisms either removes the ordinary multiplication or increases its height, therefore applying these moves whenever possible assures that the process ends with each ordinary multiplication in the decomposition of $\Sigma$ in one of the claimed situations.

VIII. The resulting decomposition of $\Sigma$ is equivalent to one in which each ordinary comultiplication $\includegraphics[width=0.1\textwidth]{comultiplication}$ is in one of the following situations:

\[
\begin{array}{ccc}
\includegraphics[width=0.1\textwidth]{situation1} & \includegraphics[width=0.1\textwidth]{situation2} & \includegraphics[width=0.1\textwidth]{situation3}
\end{array}
\]

a) Employing step VII, we can exclude the cases:

\[
\begin{array}{ccc}
\includegraphics[width=0.1\textwidth]{excluded1} & \includegraphics[width=0.1\textwidth]{excluded2} & \includegraphics[width=0.1\textwidth]{excluded3}
\end{array}
\]

Moreover, every zipper has been eliminated at step VI, thus we can also exclude:

\[
\begin{array}{ccc}
\includegraphics[width=0.1\textwidth]{excluded4} & \includegraphics[width=0.1\textwidth]{excluded5} & \includegraphics[width=0.1\textwidth]{excluded6}
\end{array}
\]

The claim follows by iterating whenever possible the following diffeomorphisms:

b) \[
\begin{array}{ccc}
\includegraphics[width=0.1\textwidth]{diffeomorphism9} & \includegraphics[width=0.1\textwidth]{diffeomorphism10} & \includegraphics[width=0.1\textwidth]{diffeomorphism11}
\end{array}
\]
Notice that each of the above diffeomorphisms either decreases the height of the ordinary comultiplication or removes it, thus the process must end after a finite number of iterations.

IX. We claim that the resulting decomposition of $\Sigma$ is now in the normal form. This follows from steps Va, VI, VII, and VIII, and the following two remarks.

a) Whenever an ordinary cap $\cap$ appears in the resulting decomposition of $\Sigma$, then $\cap$ has its target in one of the following situations:

The other situations are excluded by steps VI and VIIb.

b) Whenever an ordinary cup $\cup$ appears in the resulting decomposition of $\Sigma$, then the source of $\cup$ is in one of the following situations:

The other situations are excluded by step VIIIb. This completes the proof.

Corollary 2. Let $[\Sigma] \in \text{Sing-2Cob}(n,m)$ be connected. Then $[\Sigma] = [\text{NF}(\Sigma)]$.

Proof. $[\text{NF}(\Sigma)] = [f^{-1}([\text{NF}_{W \rightarrow C}(f([\Sigma]))])] = [f^{-1}([f([\Sigma])])] = [\Sigma]$. \qed

Corollary 3. If $[\Sigma], [\Sigma'] \in \text{Sing-2Cob}_{W \rightarrow C}(n,m)$ are connected singular 2-cobordisms with the same genus and singular boundary permutation, then $[\Sigma] = [\Sigma']$.

Proof. This follows at once from the fact that the normal form of a singular 2-cobordism is characterized by the singular boundary permutation and genus of the cobordism. \qed

Proposition 14. The category $\text{Sing-2Cob}$ of singular 2-dimensional cobordisms is equivalent to the symmetric monoidal category freely generated by a twin Frobenius algebra.
4. Twin TQFTs

In this section we define the notion of 2-dimensional twin TQFTs and show that the category of twin TQFTs is equivalent to the category of twin Frobenius algebras.

**Definition 11.** Let $\mathcal{C}$ be a symmetric monoidal category as before. A 2-dimensional twin Topological Quantum Field Theory in $\mathcal{C}$ is a symmetric monoidal functor $\text{Sing-2Cob} \to \mathcal{C}$. A homomorphism of twin TQFTs is a monoidal natural transformation of such functors. We denote by $\text{T-TQFT}(\mathcal{C})$ the category of twin TQFTs in $\mathcal{C}$ and their homomorphisms.

**Theorem 2.** There is a canonical equivalence of categories

\[ \text{T-TQFT}(\mathcal{C}) \simeq \text{T-Frob}(\mathcal{C}) \]

**Proof.** Let $\Lambda : \text{Sing-2Cob} \to \mathcal{C}$ be a 2-dimensional twin TQFT. In general, a monoidal functor is determined completely by its values on the generators of the source category, so let $C$ and $W$ be objects in $\mathcal{C}$ such that:

\[ \emptyset \xrightarrow{\Lambda} 1, \quad \xrightarrow{\Lambda} C, \quad \xrightarrow{\Lambda} W. \]

Since $\Lambda$ is monoidal, it implies that given a general object $n = (n_1, n_2, n_3, \cdots, n_{|n|})$ in $\text{Sing-2Cob}$, the functor $\Lambda$ associates the tensor product of copies of $C$ and $W$, with all parenthesis to the left. That is, $\Lambda(n) = (((\Lambda(n_1) \otimes \Lambda(n_2)) \otimes \Lambda(n_3)) \cdots \Lambda(n_{|n|}))$ with $n_i \in \{0, 1\}$ and $\Lambda(0) := C$ and $\Lambda(1) := W$. Moreover, the fact that $\Lambda$ is a symmetric monoidal functor also imply the following:

\[ \xrightarrow{\iota_C : 1 \to C} \quad [1_C : C \to C] \quad \xrightarrow{\tau_{C,C} : C \otimes C \to C \otimes C} \quad [\tau_{C,C} : W \otimes W \to W \otimes W] \]

Let the images of the generating morphisms in $\text{Sing-2Cob}$ be denoted like below:

\[ \xrightarrow{\iota_C : 1 \to C} \quad [1_C : C \to C] \quad \xrightarrow{\tau_{C,C} : C \otimes C \to C \otimes C} \quad [\tau_{C,C} : W \otimes W \to W \otimes W] \]

\[ \xrightarrow{\iota_W : 1 \to W} \quad [\iota_W : 1 \to W] \quad \xrightarrow{\epsilon_W : W \to 1} \quad [\epsilon_W : W \to 1] \quad \xrightarrow{\tau_{W,C} : W \otimes C \to C \otimes W} \quad [\tau_{W,C} : W \otimes C \to C \otimes W]. \]
The relations that hold in $\text{Sing-2Cob}$ translate into relations among these maps and imply that $(C, W, z, z^*)$ is a twin Frobenius algebra in $C$.

Conversely, let $(C, W, z, z^*)$ be a twin Frobenius algebra in $C$. Thus $(C, m_C, \iota_C, \Delta_C, \epsilon_C)$ is a commutative Frobenius algebra and $(W, m_W, \iota_W, \Delta_W, \epsilon_W)$ is a symmetric Frobenius algebra. We can construct then a 2-dimensional twin TQFT $\Lambda$ by using the above description as definition. Since the relations in $\text{Sing-2Cob}$ correspond precisely to the axioms for a twin Frobenius algebra, the symmetric monoidal functor $\Lambda$ is well-defined.

It is clear that these two constructions are inverse to each other. So we have established a one-to-one correspondence between 2-dimensional twin TQFTs and twin Frobenius algebras.

This correspondence also works for arrows (homomorphisms). An arrow in $T\text{-TQFT}(C)$ is a monoidal natural transformation. Given two twin TQFTs $\Lambda_1$ and $\Lambda_2$, a natural transformation $\eta$ between them consists of maps $\Lambda_1(n) \to \Lambda_2(n)$ for each $n = (n_1, n_2, n_3, \ldots, n|n|)$ in $\text{Sing-2Cob}$. Since $\eta$ is monoidal, the map $\Lambda_1(n) \to \Lambda_2(n)$ is the tensor product of maps $\Lambda_1(n_l) \to \Lambda_2(n_l)$, for $1 \leq l \leq |n|$, and the naturality of $\eta$ means that all these maps are compatible with morphisms in $\text{Sing-2Cob}$. Every such morphism is a composition of generators, so naturality translates into a commutative diagram for each generator. If we denote the twin Frobenius algebras corresponding to $\Lambda_1$ and $\Lambda_2$ by $T_1 = (C_1, W_1, z_1, z_1^*)$ and $T_2 = (C_2, W_2, z_2, z_2^*)$, respectively, then the diagrams corresponding to $\eta$ are

\[
\begin{array}{ccc}
W_1 \otimes W_1 & \xrightarrow{g \otimes g} & W_2 \otimes W_2 \\
m_{W_1} & & m_{W_2} \\
W_1 & \xrightarrow{g} & W_2
\end{array}
\quad
\begin{array}{ccc}
W_1 & \xrightarrow{g} & W_2 \\
\iota_{W_1} & & \iota_{W_2} \\
1 & = & 1
\end{array}
\]

and they imply that $g: W_1 \to W_2$ is a homomorphism of algebra objects. Similarly, the commutative diagrams corresponding to $\Delta$ and $\epsilon$ are

\[
\begin{array}{ccc}
W_1 \otimes W_1 & \xrightarrow{\Delta \otimes \Delta} & W_2 \otimes W_2 \\
\Delta_{W_1} & & \Delta_{W_2} \\
W_1 & \xrightarrow{\Delta} & W_2
\end{array}
\quad
\begin{array}{ccc}
W_1 & \xrightarrow{\epsilon} & 1 \\
\epsilon_{W_1} & & \epsilon_{W_2} \\
1 & = & 1
\end{array}
\]

which states that $g: W_1 \to W_2$ is a homomorphism of coalgebra objects. Hence, $g: W_1 \to W_2$ is a Frobenius algebra homomorphism. There are similar commutative diagrams for the generators $\otimes$, $\iota$, $\Delta$ and $\epsilon$, which amounts to the statement that $f: C_1 \to C_2$ is also a Frobenius algebra homomorphism. Finally, the diagrams
corresponding to the zipper $\z_1$ and cozipper $\z_2$ look like

\[
\begin{array}{c}
C_1 \xrightarrow{f} C_2 \\
W_1 \xrightarrow{g} W_2 \\
\downarrow \quad \downarrow \\
\z_1 \quad \z_2 \\
\end{array}
\quad
\begin{array}{c}
C_1 \xrightarrow{f} C_2 \\
W_1 \xrightarrow{g} W_2 \\
\uparrow \quad \uparrow \\
\z_1^* \quad \z_2^* \\
\end{array}
\]

which are precisely the remaining conditions which $f$ and $g$ must satisfy in order for $(f,g)$ to be a homomorphism of twin Frobenius algebras.

Conversely, given a homomorphism of twin Frobenius algebras $T_1 \rightarrow T_2$, we can use the above arguments backwards to construct a monoidal natural transformation between the twin TQFTs corresponding to $T_1$ and $T_2$. □

5. Examples of twin Frobenius algebras

Example 1. Let $\mathcal{C} = R\text{-Mod}$ be the category of $R$-modules and module homomorphisms, where $R = \mathbb{Z}[i][a,h]$ is the ring of polynomials in indeterminates $a$ and $h$ and with Gaussian integer coefficients. Consider also the ring $\mathcal{A} = R[X]/(X^2 - hX - a) = \langle 1, X \rangle_R$ with inclusion map $\iota: R \rightarrow \mathcal{A}$, $\iota(1) = 1$. We remark that we consider these two rings for our main example, because they play an important role in [4, 5].

Consider $\mathcal{A}_C = (\mathcal{A}, m_C, \iota_C, \Delta_C, \epsilon_C), \mathcal{A}_W = (\mathcal{A}, m_W, \iota_W, \Delta_W, \epsilon_W)$, with $\iota_{C,W} = \iota$, and

\[
\begin{align*}
\epsilon_C(1) &= 0, & \epsilon_W(1) &= 0, \\
\epsilon_C(X) &= 1, & \epsilon_W(X) &= -i, \\
\Delta_C(1) &= 1 \otimes X + X \otimes 1 - h1 \otimes 1, & \Delta_W(1) &= i(1 \otimes X + X \otimes 1 - h1 \otimes 1), \\
\Delta_C(X) &= X \otimes X + a1 \otimes 1, & \Delta_W(X) &= i(X \otimes X + a1 \otimes 1).
\end{align*}
\]

Both $\mathcal{A}_C$ and $\mathcal{A}_W$ are commutative (thus symmetric) Frobenius algebras in $R\text{-Mod}$. It is clear that the multiplication maps $m_{C,W}: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ are defined by the rules:

\[
\begin{align*}
m_{C,W}(1 \otimes X) &= m_{C,W}(X \otimes 1) = X, \\
m_{C,W}(1 \otimes 1) &= 1, m_{C,W}(X \otimes X) = hX + a.
\end{align*}
\]

Note that $\mathcal{A}_W$ is a twisting of $\mathcal{A}_C$; that is, the comultiplication $\Delta_W$ and counit $\epsilon_W$ are obtained from $\Delta_C$ and $\epsilon_C$ by “twisting” them with invertible element $-i \in \mathcal{A}$:

$\epsilon_W(x) = \epsilon_C(-ix), \quad \Delta_W(x) = \Delta_C((-i)^{-1}x) = \Delta_C(ix)$, for all $x \in \mathcal{A}$.

The fact that the above Frobenius structures differ by a twist is not surprising. Kadison showed that twisting by invertible elements of $\mathcal{A}$ is the only way to modify the counit and comultiplication in Frobenius systems (see [7, Theorem 1.6] and [9]). Note that a Frobenius system is the term used in the literature for a Frobenius algebra in $R\text{-Mod}$, where $R$ is some ring.
We define the following homomorphisms:
\[
z: \mathcal{A}_C \to \mathcal{A}_W, \begin{cases} z(1) = 1 \\
z(X) = X, \end{cases} \quad z^*: \mathcal{A}_W \to \mathcal{A}_C, \begin{cases} z^*(1) = -i \\
z^*(X) = -iX. \end{cases}
\]

A straightforward computation shows that \((\mathcal{A}_C, \mathcal{A}_W, z, z^*)\) satisfies the axioms of a twin Frobenius algebra in \(\text{R-Mod}\).

We denote by \(\mathcal{T}: \text{Sing-2Cob} \to \text{R-Mod}\) the 2-dimensional twin TQFT corresponding to \((\mathcal{A}_C, \mathcal{A}_W, z, z^*)\), which assigns the ground ring \(R\) to the empty 1-manifold and assigns \(\mathcal{A}^\otimes_k\) to a generic object \(n = (n_1, n_2, \ldots, n_{|n|})\) in \(\text{Sing-2Cob}\). The \(i\)-th factor of \(\mathcal{A}^\otimes_k\) is endowed with the structure \(\mathcal{A}_C\) if \(n_i = 0\), and with the structure \(\mathcal{A}_W\) if \(n_i = 1\).

On the generating morphisms of the category \(\text{Sing-2Cob}\), the functor \(\mathcal{T}\) is defined as follows:
\[
\begin{align*}
\mathcal{T}: & \quad \triangleleft \to \Delta_C & \quad \mathcal{T}: & \quad \bigcirc \to m_C & \quad \mathcal{T}: & \quad \bigcirc \to \iota_C \\
\mathcal{T}: & \quad \bigcirc \to \Delta_W & \quad \mathcal{T}: & \quad m_W \to \iota_W & \quad \mathcal{T}: & \quad \iota_W \to \epsilon_W \\
\mathcal{T}: & \quad z \to z & \quad \mathcal{T}: & \quad \bigcirc \to \text{id}_{\mathcal{A}_C} & \quad \mathcal{T}: & \quad \bigcirc \to \text{id}_{\mathcal{A}_W}.
\end{align*}
\]

It is well worth noting that this twin TQFT satisfies the local relations for the dot-free version of the universal \(\mathfrak{sl}(2)\) foam cohomology for links (see [5, Section 4]). To be precise, the following identities hold:
\[
\begin{align*}
2\mathcal{T}(\bigcirc) &= \mathcal{T}(\bigcirc) + \mathcal{T}(\bigcirc), & \mathcal{T}(\bigcirc) &= 0 \\
\mathcal{T}(\bigcirc) &= (h^2 + 4a)\mathcal{T}(\bigcirc), & \mathcal{T}(\bigcirc) &= 2 \\
\mathcal{T}(\bigcirc) &= 0, & \mathcal{T}(\bigcirc) &= -2i.
\end{align*}
\]

The last two identities are the ‘UFO’ local relations used in [5] and depicted below:
\[
\bigcirc = 0, \quad \bigcirc = -2i.
\]

Motivated by the above remarks, we believe that the twin Frobenius algebra and its associated twin TQFT considered in this example will play the key role in describing the universal dot-free \(\mathfrak{sl}(2)\) foam link cohomology using twin TQFTs. We will consider this problem in a subsequent paper.
Example 2. Let $S$ be any commutative ring (or field) such that $i \in R$, and consider the truncated polynomial algebras $C = S[x]/(x^n)$ and $W = S[y]/(y^n)$, where $n \geq 2$. Both $C$ and $W$ admit commutative (thus symmetric) Frobenius structures $(C, m_C, \Delta_C, \epsilon_C)$ and $(W, m_W, \Delta_W, \epsilon_W)$ with counit maps given by

$$\epsilon_C(x^{n-1}) = 1, \quad \epsilon_C(x^k) = 0, \quad \text{for all } 0 \leq k \leq n - 2,$$

$$\epsilon_W(y^{n-1}) = -i, \quad \epsilon_W(y^k) = 0, \quad \text{for all } 0 \leq k \leq n - 2,$$

and comultiplication maps defined by

$$\Delta_C(x^k) = \sum_{j=0}^{n-1-k} x^{j+k} \otimes x^{n-1-j}, \quad \text{for all } 0 \leq k \leq n - 1,$$

$$\Delta_W(y^k) = i \sum_{j=0}^{n-1-k} y^{j+k} \otimes y^{n-1-j}, \quad \text{for all } 0 \leq k \leq n - 1.$$

Then $(C, W, z, z^*)$ is a twin Frobenius algebra in $\textbf{S-Mod}$ (or in $\textbf{Vect}_S$, if $S$ is a field), where $z(x^k) = y^k$ and $z^*(y^k) = -ix^k$, for every $0 \leq k \leq n - 1$.

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