On a $p$-adic Cubic Generalized Logistic Dynamical System

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Abstract. Applications of $p$-adic numbers mathematical physics, quantum mechanics stimulated increasing interest in the study of $p$-adic dynamical system. One of the interesting investigations is $p$-adic logistics map. In this paper, we consider a new generalization, namely we study a dynamical system of the form $f_a(x) = ax(1-x^2)$. The paper is devoted to the investigation of a trajectory of the given system. We investigate the generalized logistic dynamical system with respect to parameter $a$ and we restrict ourselves for the investigation of the case $|a|_p < 1$. We study the existence of the fixed points and their behavior. Moreover, we describe their size of attractors and Siegel discs since the structure of the orbits of the system is related to the geometry of the $p$-adic Siegel discs.

1. Introduction

Over the last century, $p$-adic numbers and $p$-adic analysis have come to play major role in the number theory. $P$-adic numbers were first introduced by K. Hensel. During a century after their discovery they were considered mainly objects of pure mathematics. Starting from 1980’s various models described in the language of $p$-adic analysis have been actively studied. $P$-adic numbers have been widely used in many applications mostly in $p$-adic mathematical physics, quantum mechanics and others which rises the interest in the study of $p$-adic dynamical systems (see for example, [6, 18, 42, 44, 45]).

On the other hand, the study of $p$-adic dynamical systems arises in Diophantine geometry in the constructions of canonical heights, used for counting rational points on algebraic varieties over a number field, as in [12]. In [24] the $p$-adic field have arisen in physics in the theory of superstrings, promoting questions about their dynamics. Also some applications of $p$-adic dynamical systems to some biological, physical systems were proposed in [8, 3, 4, 15, 25]. In [9],[27] dynamical systems (not only monomial) over finite field extensions of the $p$-adic numbers were considered. Other studies of non-Archimedean dynamics in the neighborhood of a periodic and of the counting of periodic points over global fields using local fields appeared in [26, 20, 29, 28, 36]. Certain rational $p$-adic dynamical systems were investigated in [22],[31],[33], which appear from problems of $p$-adic Gibbs measures [23, 32, 34, 35]. Note that in [37, 10, 11] a general theory of $p$-adic rational dynamical systems over complex $p$-adic filed $\mathbb{C}_p$ has been developed.

The most studied discrete $p$-adic dynamical systems (iterations of maps) are so-called monomial systems. In [5],[25] the behavior of a $p$-adic dynamical system $f(x) = x^n$ in the fields of $p$-adic numbers $\mathbb{Q}_p$ and $\mathbb{C}_p$ was investigated. In [25] perturbated monomial dynamical systems defined by functions $f_q(x) = x^n + q(x)$, where the perturbation $q(x)$ is a polynomial whose coefficients have small $p$-adic absolute value, have been studied. It was investigated the connection between monomial and perturbated monomial systems. These investigations show that the study of perturbated dynamical systems is important (see [16]). Note that for a quadratic function $f(x) = x^2 + c$, $c \in \mathbb{Q}_p$ its chaotic...
behavior is complicated (see [41, 4, 40, 46]). In [40, 14] the Fatou and Julia sets of logistic $p$-adic dynamical system, i.e. $f(x) = rx(1-x)$, have been studied. Note that the logistic map and generalized logistic maps are well known in the literature and it is of great importance in the study of dynamical systems (see [7, 13, 21]). Certain ergodic and mixing properties of monomial and perturbated dynamical systems have been considered in [1, 17, 19].

In [30] the asymptotic behavior of a nonlinear $p$-adic dynamical system, especially a generalized $p$-adic logistic map $G(x) = (ax)^2(x + 1)$ has been investigated. On the other hand, such a cubic dynamical system is also a perturbated cubic dynamical system, since it can be reduced to the form $f(x) = x^3 + ax^2$. In the present paper we are going to consider an other kind of cubic logistic mapping, i.e. $f_α(x) = ax(1 - x^2)$. We will show that the defined two cubic dynamical systems are not almost conjugate. In this paper, we restrict ourselves to the case $|a|_p < 1$. Other values will be studied elsewhere. So, for such a mapping we will investigate the basin of attraction of such a dynamical system. Note that globally attracting sets play an important role in dynamics, restricting the asymptotic behavior to certain regions of the phase space. However, descriptions of the global attractor can be difficult as it may contain complicated chaotic dynamics. Moreover, we also describe the Siegel discs of the system, since the structure of the orbits of the system is related to the geometry of the $p$-adic Siegel discs (see [2]).

2. Preliminaries

2.1. $p$-adic numbers

Let $\mathbb{Q}$ be the field of rational numbers. Throughout the paper $p$ will be a fixed prime number. Every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{r}{n}$, where $r, n \in \mathbb{Z}$, $m$ is a positive integer and $p, n, m$ are relatively prime. The $p$-adic norm of $x$ is given by $|x|_p = p^{-r}$ and $|0|_p = 0$. This norm satisfies so called the strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

From this inequality one can infer that

$$\text{if } |x|_p \neq |y|_p, \text{ then } |x - y|_p = \max\{|x|_p, |y|_p\} \quad (1)$$

$$\text{if } |x|_p = |y|_p, \text{ then } |x - y|_p \leq |2x|_p. \quad (2)$$

This is a ultrametricity of the norm. The completion of $\mathbb{Q}$ with respect to the $p$-adic norm defines the $p$-adic field which is denoted by $\mathbb{Q}_p$. Note that any $p$-adic number $x \neq 0$ can be uniquely represented in the canonical series:

$$x = p^{γ(x)}(x_0 + x_1p + x_2p^2 + ...), \quad (3)$$

where $γ = γ(x) \in \mathbb{Z}$ and $x_j$ are integers, $0 \leq x_j \leq p - 1$, $x_0 > 0$, $j = 0, 1, 2, ...$ (see more detail [?, 38, 39]). Observe that in this case $|x|_p = p^{-γ(x)}$.

Denote

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}, \quad \mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x|_p = 1\}.$$

We recall that an integer $a \in \mathbb{Z}$ is called a quadratic residue modulo $p$ if the equation $x^2 \equiv a (\text{mod} \ p)$ has a solution $x \in \mathbb{Z}$.

**Definition 2.1** Let $p$ be an odd prime i.e. $p > 2$ and $a$ be an integer not divisible by $p$, i.e. $(a, p) = 1$. The Legendre Symbol

$$\left( \frac{a}{b} \right) = \begin{cases} 1, & \text{If } a \text{ is a quadratic residue of } p; \\ -1, & \text{If } a \text{ is not a quadratic residue of } p. \end{cases}$$

One has the following useful
Theorem 2.2 (Euler’s criterion) Let $p > 2$ be prime number and $a$ be an integer not divisible by $p$. Then

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$$

Now using the last theorem we infer the following

Lemma 2.3 [38] Let $a \in \mathbb{Q}_p$, $a \neq 0$ and $p \geq 3$ such that

$$a = \frac{a^*}{|a|_p}, a^* \in \mathbb{Z}_p^*$$

Then, $\sqrt{a}$ exists if and only if it satisfies these two conditions:

(i) $\log_p |a|_p$ is even

(ii) $(a^*)^{\frac{p-1}{2}} \equiv 1 \pmod{p}$

However, at $p = 2$, $\sqrt{a}$ exists if and only if it satisfies these two conditions:

(i) $\log_2 |a|_2$ is even

(ii) $(a^*) \equiv 1 \pmod{2^3}$

In particular case, let $a^* \in \mathbb{Z}_p^* (p > 2)$, then $\sqrt{a}$ exists if and only if

$$(a^*)^{\frac{p-1}{2}} \equiv 1 \pmod{p}$$

Theorem 2.4 Let $p$ be prime number and $p > 2$. Then we have

(i) The $p$-adic number $-1$ is a square in $\mathbb{Q}_p$ if and only if $p \equiv 1 \pmod{4}$ and the $p$-adic number $-1$ is not a square in $\mathbb{Q}_p$ if and only if $p \equiv 3 \pmod{4}$

(ii) The $p$-adic number 2 is a square in $\mathbb{Q}_p$ if and only if $p \equiv \pm 1 \pmod{8}$ and the $p$-adic number 2 is not a square in $\mathbb{Q}_p$ if and only if $p \equiv \pm 3 \pmod{8}$

(iii) The $p$-adic number $-2$ is a square in $\mathbb{Q}_p$ if and only if $p \equiv 1$ or $3 \pmod{8}$ and the $p$-adic number $-2$ is not a square in $\mathbb{Q}_p$ if and only if $p \equiv -1$ or $-3 \pmod{8}$

(iv) Let $p > 3$ then the $p$-adic number 3 is a square in $\mathbb{Q}_p$ if and only if $p \equiv \pm 1 \pmod{12}$ and the $p$-adic number 3 is not a square in $\mathbb{Q}_p$ if and only if $p \equiv \pm 5 \pmod{12}$

(v) Let $p > 3$ then the $p$-adic number $-3$ is a square in $\mathbb{Q}_p$ if and only if $p \equiv 1$ or $-5 \pmod{12}$ and the $p$-adic number $-3$ is not a square in $\mathbb{Q}_p$ if and only if $p \equiv -1$ or $5 \pmod{12}$

2.2. Dynamical Systems in $\mathbb{Q}_p$

For any $a \in \mathbb{Q}_p$ and $r > 0$ denote

$$\bar{B}_r(a) = \{ x \in \mathbb{Q}_p : |x - a|_p \leq r \}, \quad B_r(a) = \{ x \in \mathbb{Q}_p : |x - a|_p < r \},$$

$$S_r(a) = \{ x \in \mathbb{Q}_p : |x - a|_p = r \}.$$ 

A function $f : B_r(a) \to \mathbb{Q}_p$ is said to be analytic if it can be represented by

$$f(x) = \sum_{n=0}^{\infty} f_n (x-a)^n, \quad f_n \in \mathbb{Q}_p,$$

which converges uniformly on the ball $B_r(a)$. 
Let \( \tau \) be a fixed point of an analytic function \( f \) (contained in \( A \)) such that for all points \( x \in U(x^{(0)}) \) it holds \( \lim_{n \to \infty} y^{(n)} = x^{(0)} \), where \( y^{(n)} = f^n(y) \). If \( x^{(0)} \) is an attractor then its basin of attraction is \( A(x^{(0)}) = \{ y \in \mathbb{Q}_p : y^{(n)} \to x^{(0)}, n \to \infty \} \).

A fixed point \( x^{(0)} \) is called repeller if there exists a neighborhood \( U(x^{(0)}) \) of \( x^{(0)} \) such that \( |f(x) - x^{(0)}|_p > |x - x^{(0)}|_p \) for \( x \in U(x^{(0)}) \). For a fixed point \( x^{(0)} \) of a function \( f(x) \) a ball \( B_r(x^{(0)}) \) is called a Siegel disc if each sphere \( S_p(x^{(0)}) \), \( p < r \) is an invariant sphere of \( f(x) \), i.e. if \( x \in S_p(x^{(0)}) \) then all iterated points \( x^{(n)} \in S_p(x^{(0)}) \) for all \( n = 1, 2, \ldots \). The union of all Siegel discs with the center at \( x^{(0)} \) is said to a maximum Siegel disc and is denoted by \( SI(x^{(0)}) \).

**Remark 2.5** In non-Archimedean geometry, a center of a disc is nothing but a point which belongs to the disc, therefore, in principle, different fixed points may have the same Siegel disc.

Let \( x^{(0)} \) be a fixed point of an analytic function \( f(x) \). Set

\[
\lambda = \frac{d}{dx}f(x^{(0)}).
\]

The point \( x^{(0)} \) is called attractive if \( 0 \leq |\lambda|_p < 1 \), indifferent if \( |\lambda|_p = 1 \), and repelling if \( |\lambda|_p > 1 \).

**Theorem 2.6** [25] Let \( a \) be a fixed point of the analytic function \( f : B \to K \). Then,

i. If \( r > 0 \) satisfies the inequality

\[
\max_{1 \leq n \leq \infty} \left| \frac{1}{n!} \cdot \frac{d^n f}{dx^n}(a) \right| r^{n-1} < 1
\]

and \( B_r(a) \subset B \), then \( B_r(a) \subset A(a) \).

ii. If \( a \) is a neutral point of \( f \) then it is the center of a Siegel disc. If \( r > 0 \) satisfies the inequality

\[
S = \max_{2 \leq n \leq \infty} \left| \frac{1}{n!} \cdot \frac{d^n f}{dx^n}(a) \right| r^{n-1} < |f'(a)|_k
\]

and \( B_r(a) \subset B \), then \( B_r(a) \subset SI(a) \).

iii. If \( a \) is a repelling point of \( f \), then \( a \) is a repeller of the dynamical system.

### 3. A generalized cubic \( p \)-adic logistic map

In this section we are going to study the dynamical system given by

\[
f_a(x) = ax(1 - x^2), x \in \mathbb{Q}_p
\]

where \( a \in \mathbb{Q}_p \). Such a mapping is call a generalized cubic logistic mapping, and the rest of the paper is devoted to the investigation of such a dynamical system. As mentioned before, another kind of generalize logistic map given by

\[
G_b(x) = x^3 + bx^2, x \in \mathbb{Q}_p
\]

where \( b \in \mathbb{Q}_p \), has been studied in [30]. We will show for a few certain values of parameter \( a \) and \( b \), they are conjugate but for the rest they are not. Here by conjugacy we mean a one-to-one mapping \( \tau : \mathbb{Q}_p \to \mathbb{Q}_p \) such that \( \tau \circ f = g \circ \tau \).
Proposition 3.1 The mappings $f_a$ and $G_b$ are conjugate via $\tau(x) = \alpha x + \beta$ iff one has

\[
\alpha = \sqrt{-\frac{3}{2}}, \quad \beta = \sqrt{-\frac{1}{2}}, \quad a = \frac{3}{2}, \quad b = -3\sqrt{-\frac{1}{2}}.
\]

Now using Lemma 2.3 one can establish the following

Proposition 3.2 The numbers $-\frac{3}{2}$ and $-\frac{1}{2}$ are the square of some numbers in $\mathbb{Q}_p$ (the same time) if and only if

\[
\left(\frac{p-3}{2}\right)^{\frac{p-1}{2}} \equiv 1 \pmod{p}
\]

and

\[
\left(\frac{p-1}{2}\right)^{\frac{p-1}{2}} \equiv 1 \pmod{p}
\]

Remark 3.3 From the Propositions 3.1, 3.2, we conclude that if $f_a$ and $G_b$ are conjugate, then $|a|_p = |b|_p = 1$. Moreover, such a case in [30] is not fully investigated. For the other values of $a$, this two mappings are not conjugate. Therefore, it is reasonable to investigate behavior of the dynamical system $f_a(x)$.

3.1. Description and existence of the fixed points

In this section we will investigate the dynamics of generalized cubic logistic function by finding their fixed points, and their existence for the case $|a|_p < 1$.

Let us find its fixed points which means we need to solve the equation $f_a(x) = x$. One can find three formal fixed points:

\[
x_1 = 0 \quad \text{and} \quad x_{2,3} = \pm \sqrt{\frac{a-1}{a}}.
\]

We can denote the fixed points as

\[
Fix = \left\{ x_1 = 0, x_2 = \sqrt{\frac{a-1}{a}}, x_3 = -\sqrt{\frac{a-1}{a}} \right\}
\]

We have done finding the formal solutions for the fixed points. Note that not all these fixed points do exist. Therefore, let us move on to the existence of the fixed points. Note that Lemma 2.3 is useful in most parts of our investigation in establishing the existence of the fixed points.

Take into account that the fixed point $x_1 = 0$ is a trivial case and it always exist for any prime number. But the fixed points $x_{2,3} = \pm \sqrt{\frac{a-1}{a}}$ are not always exist for any prime number. Let us check the existence of the fixed points $x_{2,3}$. Here, we restrict ourselves for the case $|a|_p < 1$.

As before, we have $|a|_p < 1$, then from the canonical representation of $a$, one finds

\[
a = p^k (a_0 + a_1 p + \cdots), \quad k \geq 1, \quad a_0 \neq 0.
\]

Proposition 3.4 Let $|a|_p < 1$, then the following assertions hold true:

(i) Let $p \geq 3$, then $\sqrt{\frac{a-1}{a}}$ exists if and only if $\log_p |a|_p$ is even and $(-a_0)^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ has solution.

(ii) Let $p = 2$, then $\sqrt{\frac{a-1}{a}}$ exists if and only if $\log_2 |a|_p$ is even number which is less than $-3$ and $a_0 \equiv 1 \pmod{2^3}$.
Proof. From (9) we have
\[ 1 - a = 1 - p^k (a_0 + a_1 p + \cdots) \quad (10) \]
If \( p \geq 3 \), then according to Lemma 2.3, the equation
\[ x^2 \equiv 1 \pmod{p} \]
has solution in \( \mathbb{Q} \) (i.e. \( x = p - 1 \)). Therefore, \( \sqrt{1 - a} \) exists. If \( p = 2 \), then from (10) and again Lemma 2.3 implies that \( \sqrt{1 - a} \) exists if and only if \( k \geq 3 \). Hence from \( \sqrt{\frac{a - 1}{a}} = \sqrt{\frac{1 - \frac{1}{a}}{a}} \), we conclude that \( \sqrt{\frac{a - 1}{a}} \) exists if and only if \( \sqrt{-a} \) exists. Now using Lemma 2.3 we get the desired assertion. This completes the proof.

3.2. Description Of the Behavior
In this section, we will describe the behavior of the function (5) with respect to the parameter \( a \) \((|a|_p < 1)\) whether the fixed points \( x_1, x_2, x_3 \) is attracting, neutral or repelling. Form (5), we have
\[ f'(x) = a - 3ax^2 \quad (11) \]
We already know those fixed points exist for some values of prime number. Therefore in what follows, we assume the existence ones. Now, let us check their behavior and study the value of
\[ |f'(x^{(0)})|_p = |a - 3a(x^{(0)})^2|_p, \quad (12) \]
where \( x^{(0)} \) is a fixed point. We substitute the fixed points into (12), one gets
\[ |f'(x_1)|_p = |a|_p \quad \text{and} \quad |f'(x_{2/3})|_p = |a - 3a \left( \frac{a - 1}{a} \right)|_p = |3 - 2a|_p \quad (13) \]
We have used the equality \( x_{2/3}^2 = \frac{a - 1}{a} \).

Let us start with the first fixed point \( x_1 = 0 \). From (13) immediately find that \( x_1 \) is an attracting fixed point.

Next, let us turn to fixed point \( x_{2/3} \). If we consider \( p = 3 \), then from (13) we find
\[ |3 - 2a|_3 < 1 \quad (14) \]
This means \( x_{2/3} \) are also attracting fixed points. However, if we let \( p \neq 3 \), from the strong triangle inequality we obtain
\[ |f'(x_{2/3})|_p = |3 - 2a| = 1 \]
Hence, the fixed points \( x_{2/3} \) are neutral or indifferent fixed point. So taking into account Proposition 3.4 we can formulate the following:

Proposition 3.5 Let \( 0 < |a|_p < 1 \), and \( a = \frac{a^*}{|a|_p} \) with \( a^* = a_0 + a_1 p + a_2 p^2 + \cdots \), where \( a^* \in \mathbb{Z}_p^* \). Then the following statements hold true:

(I) Let \( p > 3 \),
(a) If \( \log_p |a|_p \) is even and \( (-a_0)^{p-1} \equiv 1 \pmod{p} \), then \( x_1 = 0 \) is an attracting and \( x_{2/3} \) are indifferent fixed points.
(b) In the rest cases, the dynamical system has unique attracting fixed point \( x_1 = 0 \).

(II) Let \( p = 3 \),
(a) If \( \log_3 |a|_3 \) is even and \( a_0 = 2 \), then all three fixed points are attracting;
(b) In the rest cases, the dynamical system has unique attracting fixed point \( x_1 = 0 \).

(III) Let \( p = 2 \),
(a) If \( \log_2 |a|_2 \leq 4 \) and is even with \( a^* \equiv -1( \mod 8) \). Then the fixed point \( x_1 = 0 \) is an attracting and \( x_{2/3} \) are indifferent fixed points.
(b) In the rest cases, the dynamical system has unique attracting fixed point \( x_1 = 0 \).

4. Attractors and Siegel Discs of Fixed Points
In the previous section, we have established the behavior of the fixed points of the dynamical system. Using those results, in this section we are going to describe the size of attractors and Siegel discs of the system. Before going to details, let us formulate certain auxiliary facts. Let us assume that \( x^{(0)} \) is a fixed point of \( f \). Now taking into account Theorem 2.6 one gets

**Proposition 4.1** Let \( x^{(0)} \) be a fixed point of \( f \) given by \( f_a(x) = ax(1-x^2) \) then the following assertions hold true:

(i) If \( x^{(0)} \) is an attractive point of \( f \), then it is an attractor of the dynamical system. If \( r > 0 \) satisfies the inequality
\[
\max\{|3ax^{(0)}| \cdot r; |a|_p \cdot r^2\} < 1
\]  
then \( B_r(x^{(0)}) \subset A(x^{(0)}) \).
(ii) If \( x^{(0)} \) is a neutral point of \( f \), then it is the center of a Siegel disc. If \( r > 0 \) satisfies the inequality
\[
(15) \quad \max\{|3ax^{(0)}| \cdot r; |a|_p \cdot r^2\} < 1
\]
then \( B_r(x^{(0)}) \subset SI(x^{(0)}) \).

Let us study the fixed point \( x_1 \), which is attractive. One can see that
\[
\max\{|3ax^{(0)}| \cdot r; |a|_p \cdot r^2\} < 1 \quad \text{if and only if} \quad r < \frac{1}{\sqrt{|a|_p}}.
\]

Therefore due to Proposition 4.1 we have
\[
B_r(x_1) \subset A(x_1), \quad r < \frac{1}{\sqrt{|a|_p}}.
\]

Now assume that \( |x|_p \geq \frac{1}{\sqrt{|a|_p}} \). From \( |a|_p < 1 \) one gets \( |x|_p > 1 \). By means of the strong triangle inequality we get
\[
|f_a(x)|_p = |a|_p |x|_p |1 - x^2|_p = |a|_p |x|_p^3 \geq |a|_p \cdot \frac{1}{|a|_p \sqrt{|a|_p}} = \frac{1}{\sqrt{|a|_p}} > 1
\]

Therefore
\[
A(x_1) \subset B_{\frac{1}{\sqrt{|a|_p}}}(x_1) \quad \text{and} \quad B_r(x_1) \subset A(x_1) \quad \text{for any} \quad r < \frac{1}{\sqrt{|a|_p}}.
\]

Therefore, one may formulate the following:

**Proposition 4.2** Let \( |a|_p < 1 \), then \( A(x_1) = B_{\frac{1}{\sqrt{|a|_p}}}(x_1) \).

Now, let us turn to the fixed points \( x_{2/3} = \sqrt{\frac{a-1}{a}} \). Due to Proposition 3.5, we should consider two cases (i) \( p \neq 3 \) and (ii) \( p = 3 \).
**Proposition 4.3** Let $|a|_p < 1$ and $p = 3$, then $x_{2/3}$ is attractive, and one has

$$A(x_{2/3}) = B_{\frac{1}{\sqrt{|a|_p}}} (x_{2/3}).$$

**Proof.** Due to Proposition 4.1 we have $r < \frac{1}{\sqrt{|a|_p}}$, so $\overline{B_r(x_{2/3})} \subset A(x_{2/3})$.

Let $|x - x_{2/3}|_p \geq \frac{1}{\sqrt{|a|_p}}$, then one has

$$|f_a(x) - x_2|_p = |a|_p |x - x_2|_p x_2^2 + x \cdot x_2 + x^2 - 1|_p$$

By letting $x - x_2 = \gamma$, where $|\gamma|_p \geq \frac{1}{\sqrt{|a|_p}}$, one gets

$$|f_a(x) - x_2|_p = |\gamma|_p a \gamma^2 + 3a \gamma x_2 + 2a - 3|_p.$$  \hspace{1cm} (16)

By putting $P(\gamma) = a \gamma^2 + 3a \gamma x_2 + 2a - 3$, we find

$$|P(\gamma)|_p = |\gamma|_p a \gamma^2 + 3a \gamma x_2 + 2a - 3|_p = |a \gamma + 3x_2| + 2a - 3|_p.$$  \hspace{1cm} (17)

Now taking into account $|2a - 3|_p < 1$, $|\gamma + 3x_2| = |\gamma|_p$, $|\gamma|_p \geq \frac{1}{\sqrt{|a|_p}}$ and $|3x_2| = \frac{1}{\sqrt{|a|_p}}$ with the strong triangle inequality we obtain $|P(\gamma)|_p = |a \gamma|^2$. This with (16) yields

$$|f_a(x) - x_2|_p = |a \gamma|^3 \geq \frac{1}{\sqrt{|a|_p}}.$$  

Therefore,

$$B_{\frac{1}{\sqrt{|a|_p}}} (x_{2/3}) \supset A(x_{2/3})$$

which gives us that

$$\overline{B_r(x_{2/3})} \subset A(x_{2/3})$$

for any $r < \frac{1}{\sqrt{|a|_p}}$

this completes the proof.

**Proposition 4.4** Let $|a|_p < 1$, $p \neq 3$ and $\sqrt{-a}$ exists. Then the following statements hold true:

i. If $|P(\gamma)|_p = 1$ for all $\gamma$ with $|\gamma| = \frac{1}{\sqrt{|a|_p}}$, then

$$SI(x_{2/3}) = B_{\frac{1}{\sqrt{|a|_p}}} (x_{2/3}).$$

ii. If $|P(\gamma)|_p < 1$ for some $\gamma_0$ with $|\gamma_0|_p = \frac{1}{\sqrt{|a|_p}}$, then

$$SI(x_{2/3}) = B_{\frac{1}{\sqrt{|a|_p}}} (x_{2/3}).$$

(iii) If $p \neq 2$, then $SI(x_2) \cap SI(x_3) = \emptyset$. If $p = 2$, then $SI(x_2) = SI(x_3)$.

**Proof.** The items (i) and (ii) can be proven by the same argument as above. To establish (iii) it is enough to find a relation between two the balls $B_{\frac{1}{\sqrt{|a|_p}}} (x_2)$ and $B_{\frac{1}{\sqrt{|a|_p}}} (x_3)$ whether they are coincide to each other or not. First, let us consider

$$|x_2 - x_3|_p = \left| \sqrt{\frac{a - 1}{a}} + \sqrt{\frac{a - 1}{a}} \right|_p = \left| 2 \sqrt{\frac{a - 1}{a}} \right|_p$$

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Now, suppose \( p \neq 2 \), then we obtain
\[
|x_2 - x_3|_p = \frac{1}{\sqrt{|a|_p}}
\]
Hence, at \( p \neq 2 \) we have
\[
x_2 \notin B_{\frac{1}{\sqrt{|a|_p}}}(x_3),
\]
this means
\[
B_{\frac{1}{\sqrt{|a|_p}}}(x_2) \cap B_{\frac{1}{\sqrt{|a|_p}}}(x_3) = \varnothing.
\]
Next, let us consider the case at \( p = 2 \). Again by distance we have
\[
|x_2 - x_3|_p = \frac{1}{2} \cdot \frac{1}{\sqrt{|a|_p}} < \frac{1}{\sqrt{|a|_p}}
\]
This means
\[
x_3 \in B_{\frac{1}{\sqrt{|a|_p}}}(x_2) \Rightarrow B_{\frac{1}{\sqrt{|a|_p}}}(x_2) = B_{\frac{1}{\sqrt{|a|_p}}}(x_3)
\]
This completes the proof.

Now we want to find some sufficient conditions for the equality \( SI(x_{2/3}) = B_{\frac{1}{\sqrt{|a|_p}}}(x_{2/3}) \).

**Corollary 4.5** Let \( |a|_p < 1, p \neq 2, 3 \) and \( \sqrt{-a} \) exist. If \( \sqrt{-3} \) exists, then
\[
SI(x_{2/3}) = B_{\frac{1}{\sqrt{|a|_p}}}(x_{2/3})
\]

**Proof.** To establish \( SI(x_{2/3}) = B_{\frac{1}{\sqrt{|a|_p}}}(x_{2/3}) \) it is enough to solve the equation \( P(\gamma) = 0 \). One can see the discriminant of the last equation is
\[
D = a^2 + 3a.
\]
Let \( a = p^k \cdot \varepsilon \), then one gets
\[
3a + a^2 = 3p^k \cdot \varepsilon + (p^{2k} \cdot \varepsilon^2) = p^k(3\varepsilon + p^k \cdot \varepsilon^2),
\]
therefore \( \sqrt{a^2 + 3a} \) exists if and only if \( k \) is even and
\[
x^2 \equiv 3\varepsilon \pmod{p}
\]
has a solution at \( p \neq 3 \) (see Lemma 2.3). But in the last conclusion equivalent to the existence of \( \sqrt{3a} \). We know that \( \sqrt{-a} \) exists, therefore, \( \sqrt{3a} \) exists iff \( \sqrt{-3} \) exists. Hence, we conclude \( \sqrt{a^2 + 3a} \) exists if and only if \( \sqrt{-3} \) exists. Now assume that \( \sqrt{a^2 + 3a} \) exists. Then one can see that the solutions of (17) are the following:
\[
\gamma_{1/2} = \frac{-3ax_2 \pm \sqrt{D}}{2a}.
\]
From \( p \neq 2 \) and Vietta’s theorem, we can directly find that
\[
|\gamma_1 + \gamma_2|_p = | -3x_2 |_p = \frac{1}{\sqrt{|a|_p}} \tag{19}
\]
\[
|\gamma_1 \cdot \gamma_2| = \frac{|2a - 3|_p}{|a|_p} = \frac{1}{|a|_p}. \tag{20}
\]
Now, we are interested in finding the explicit value of $|\gamma_1|_p$. Assume $|\gamma_1|_p \neq |\gamma_2|_p$ or more explicitly $|\gamma_1|_p > |\gamma_2|_p$. Then, from the strong triangle inequality, we have

$$|\gamma_1 + \gamma_2|_p = |\gamma_1| = \frac{1}{\sqrt{|a|_p}}$$

On the other hand,

$$|\gamma_1 \cdot \gamma_2|_p = |\gamma_1|_p \cdot |\gamma_2|_p = \frac{1}{\sqrt{|a|_p}} \cdot |\gamma_2|_p = \frac{1}{|a|_p}.$$  

Correspondingly, $|\gamma|_2 = \frac{1}{\sqrt{|a|_p}}$, but this contradicts to our assumption. Hence, $|\gamma_1|_p = |\gamma_2|_p$. So from (19), we got

$$|\gamma_1|_p = |\gamma_2|_p = \frac{1}{\sqrt{|a|_p}}$$

This with (16) implies

$$|f_a(\tilde{x}_0) - x_2|_p = 0;$$

where $\tilde{x}_0 = x_2 + \gamma_1$, which means

$$S \frac{1}{\sqrt{|a|_p}} \neq SI(x_2).$$

So, we have

$$SI(x_2) = B \frac{1}{\sqrt{|a|_p}}(x_2)$$

This completes the proof.

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