Compact Riemannian Manifolds with Homogeneous Geodesics

Dmitrii V. ALEKSEEVSKY† and Yuri G. NIKONOROV‡

† School of Mathematics and Maxwell Institute for Mathematical Studies, Edinburgh University, Edinburgh EH9 3JZ, United Kingdom
E-mail: D.Aleksee@ed.ac.uk

‡ Volgodonsk Institute of Service (branch) of South Russian State University of Economics and Service, 16 Mira Ave., Volgodonsk, Rostov region, 347386, Russia
E-mail: nikonorov2006@mail.ru

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Abstract. A homogeneous Riemannian space \((M = G/H, g)\) is called a geodesic orbit space (shortly, GO-space) if any geodesic is an orbit of one-parameter subgroup of the isometry group \(G\). We study the structure of compact GO-spaces and give some sufficient conditions for existence and non-existence of an invariant metric \(g\) with homogeneous geodesics on a homogeneous space of a compact Lie group \(G\). We give a classification of compact simply connected GO-spaces \((M = G/H, g)\) of positive Euler characteristic. If the group \(G\) is simple and the metric \(g\) does not come from a bi-invariant metric of \(G\), then \(M\) is one of the flag manifolds \(M_1 = SO(2n + 1)/U(n)\) or \(M_2 = Sp(n)/U(1) \cdot Sp(n - 1)\) and \(g\) is any invariant metric on \(M\) which depends on two real parameters. In both cases, there exists unique (up to a scaling) symmetric metric \(g_0\) such that \((M, g_0)\) is the symmetric space \(M = SO(2n + 2)/U(n + 1)\) or, respectively, \(\mathbb{C}P^{2n-1}\). The manifolds \(M_1, M_2\) are weakly symmetric spaces.

Key words: homogeneous spaces, weakly symmetric spaces, homogeneous spaces of positive Euler characteristic, geodesic orbit spaces, normal homogeneous Riemannian manifolds, geodesics

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1 Introduction

A Riemannian manifold \((M, g)\) is called a manifold with homogeneous geodesics or geodesic orbit manifold (shortly, GO-manifold) if all its geodesic are orbits of one-parameter groups of isometries of \((M, g)\). Such manifold is a homogeneous manifold and can be identified with a coset space \(M = G/H\) of a transitive Lie group \(G\) of isometries. A Riemannian homogeneous space \((M = G/H, g^M)\) of a group \(G\) is called a space with homogeneous geodesics (or geodesic orbit space, shortly, GO-space) if any geodesic is an orbit of a one-parameter subgroup of the group \(G\). This terminology was introduced by O. Kowalski and L. Vanhecke in [20], who initiated a systematic study of such spaces.

Recall that homogeneous geodesics correspond to “relative equilibria” of the geodesic flow, considered as a hamiltononian system on the cotangent bundle. Due to this, GO-manifolds can be characterized as Riemannian manifolds such that all integral curves of the geodesic flow are relative equilibria.

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GO-spaces may be considered as a natural generalization of symmetric spaces, classified by É. Cartan [10]. Indeed, a simply connected symmetric space can be defined as a Riemannian manifold \((M,g)\) such that any geodesic \(\gamma \subset M\) is an orbit of one-parameter group \(g_t\) of translations, that is one-parameter group of isometries which preserves \(\gamma\) and induces the parallel transport along \(\gamma\). If we remove the assumption that \(g_t\) induces the parallel transport, we get the notion of a GO-space.

The class of GO-spaces is much larger then the class of symmetric spaces. Any homogeneous space \(M = G/H\) of a compact Lie group \(G\) admits a metric \(g_M\) such that \((M, g_M)\) is a GO-space. It is sufficient to take the metric \(g_M\) which is induced with a bi-invariant Riemannian metric \(g\) on the Lie group \(G\) such that \((G, g) \rightarrow (M = G/H, g^M)\) is a Riemannian submersion with totally geodesic fibres. Such GO-space \((M = G/H, g^M)\) is called a \textbf{normal homogeneous space}.

More generally, any naturally reductive manifold is a geodesic orbit manifold. Recall that a Riemannian manifold \((M, g^M)\) is called \textbf{naturally reductive} if it admits a transitive Lie group \(G\) of isometries with a bi-invariant pseudo-Riemannian metric \(g\), which induces the metric \(g^M\) on \(M = G/H\), see [18, 8]. The first example of non naturally reductive GO-manifold had been constructed by A. Kaplan [16]. An important class of GO-spaces consists of weakly symmetric spaces, introduced by A. Selberg [22]. A homogeneous Riemannian space \((M = G/H, g^M)\) is a \textbf{weakly symmetric space} if any two points \(p, q \in M\) can be interchanged by an isometry \(a \in G\). This property does not depend on the particular invariant metric \(g^M\). Weakly symmetric spaces \(M = G/H\) have many interesting properties (for example, the algebra of \(G\)-invariant differential operators on \(M\) is commutative, the representation of \(G\) in the space \(L^2(M)\) of function is multiplicity free, the algebra of \(G\)-invariant Hamiltonians on \(T^*M\) with respect to Poisson bracket is commutative) and are closely related with spherical spaces, commutative spaces and Gelfand pairs etc., see the book by J.A. Wolf [26]. The classification of weakly symmetric reductive homogeneous spaces was given by O.S. Yakimova [28], see also [29].

In [29], O. Kowalski and L. Vanhecke classified all GO-spaces of dimension \(\leq 6\). C. Gordon [14] reduced the classification of GO-spaces to the classification of GO-metrics on nilmanifolds, compact GO-spaces and non-compact GO-spaces of non-compact semisimple Lie group. She described GO-metrics on nilmanifolds. They exist only on two-step nilpotent nilmanifolds. She also presented some constructions of GO-metrics on homogeneous compact manifolds and non compact manifolds of a semisimple group.

Many interesting results about GO-spaces one can find in [7, 12, 27, 28, 24], where there are also extensive references.

Natural generalizations of normal homogeneous Riemannian manifolds are \(\delta\)-homogeneous Riemannian manifolds, studied in [3, 1, 4]. Note that the class of \(\delta\)-homogeneous Riemannian manifolds is a proper subclass of the class of geodesic orbit spaces with non-negative sectional curvature (see the quoted papers for further properties of \(\delta\)-homogeneous Riemannian manifolds).

In [1], a classification of non-normal invariant GO-metrics on flag manifolds \(M = G/H\) was given. The problem reduces to the case when the (compact) group \(G\) is simple. There exist only two series of flag manifolds of a simple group which admit such metric, namely weakly symmetric spaces \(M_1 = SO(2n+1)/U(n)\) and \(M_2 = Sp(n)/U(1) \cdot Sp(n-1)\), equipped with any (non-normal) invariant metric (which depends on two real parameters). Moreover, there exists unique (up to a scaling) invariant metric \(g_0\), such that the Riemannian manifolds \((M_i, g_0)\) are isometric to the symmetric spaces \(SO(2n+2)/U(n+1)\) and \(CP^{2n-1} = SU(2n)/U(2n-1)\), respectively.

The main goal of this paper is a generalization of this result to the case of compact homogeneous manifolds of positive Euler characteristic. We prove that the weakly symmetric manifolds \(M_1, M_2\) exhaust all simply connected compact irreducible Riemannian non-normal GO-manifolds of positive Euler characteristic.
We indicate now the idea of the proof. Let \((M = G/H, g^M)\) be a compact irreducible non-normal GO-space of positive Euler characteristic. Then the stability subgroup \(H\) has maximal rank, which implies that \(G\) is simple. We prove that there is rank 2 regular simple subgroup \(G'\) of \(G\) (associated with a rank 2 subsystem \(R'\) of the root system \(R\) of the Lie algebra \(g = \text{Lie}(G)\)) such that the orbit \(M' = G'o = G'/H'\) of the point \(o = eH \in M\) (with the induced metric) is a non-normal GO-manifold. Using [1, 3], we prove that the only such manifold \(M'\) is \(SU(5)/U(2)\). This implies that the root system \(R\) is not simply-laced and admits a “special” decomposition \(R = R_0 \cup R_1 \cup R_2\) into a disjoint union of three subsets, which satisfies some properties. We determine all such special decompositions of irreducible root systems and show that only root systems of type \(B_n\) and \(C_n\) admit special decomposition and associated homogeneous manifolds are \(M_1\) and \(M_2\).

The structure of the paper is the following. We fix notations and recall basic definitions in Section 2. Some standard facts about totally geodesic submanifolds of a homogeneous Riemannian spaces are collected in Section 3. We discuss some properties of compact GO-spaces in Section 4. These results are used in Section 5 to derive sufficient conditions for existence and non-existence of a non-normal GO-metric on a homogeneous manifold of a compact group. Section 6 is devoted to classification of compact GO-spaces with positive Euler characteristic.

2 Preliminaries and notations

Let \(M = G/H\) be a homogeneous space of a compact connected Lie group \(G\). We will denote by \(b = \langle \cdot, \cdot \rangle\) a fixed \(\text{Ad}_G\)-invariant Euclidean metric on the Lie algebra \(g\) of \(G\) (for example, the minus Killing form if \(G\) is semisimple) and by

\[
g = \mathfrak{h} + \mathfrak{m}
\]

the associated \(b\)-orthogonal reductive decomposition, where \(\mathfrak{h} = \text{Lie}(H)\). An invariant Riemannian metric \(g^M\) on \(M\) is determined by an \(\text{Ad}_H\)-invariant Euclidean metric \(g = \langle \cdot, \cdot \rangle\) on the space \(\mathfrak{m}\) which is identified with the tangent space \(T_oM\) at the initial point \(o = eH\).

If \(\mathfrak{p}\) is a subspace of \(\mathfrak{m}\), we will denote by \(X_p\) the \(b\)-orthogonal projection of a vector \(X \in g\) onto \(\mathfrak{p}\), by \(b_p\) the restriction of the symmetric bilinear form to \(\mathfrak{p}\) and by \(A^p = \text{pr}_p \circ A \circ \text{pr}_p\) the projection of an endomorphism \(A\) to \(\mathfrak{p}\). If \(g\) is a \(\text{Ad}_H\)-invariant metric, the quotient

\[
A = b^{-1}_m \circ g
\]

is an \(\text{Ad}_H\)-equivariant symmetric positively defined endomorphism on \(\mathfrak{m}\), which we call the metric endomorphism. Conversely, any such equivariant positively defined endomorphism \(A\) of \(\mathfrak{m}\) defines an invariant metric \(g = b \circ A = b(A,\cdot)\) on \(\mathfrak{m}\), hence an invariant Riemannian metric \(g^M\) on \(M\).

**Lemma 1.** Let \((M = G/H, g^M)\) be a compact homogeneous Riemannian space with metric endomorphism \(A\) and

\[
m = m_1 \oplus m_2 \oplus \cdots \oplus m_k,
\]

the \(A\)-eigenspace decomposition such that \(A|_{m_i} = \lambda_i \cdot 1_{m_i}\). Then

\[
\langle m_i, m_j \rangle = 0
\]

and \(\text{Ad}_H\)-modules \(m_i\) satisfy \([m_i, m_j] \subset m\) for \(i \neq j\).
Proof. Since $A$ commute with $\text{Ad}_H$, eigenspaces $m_i$ are $\text{Ad}_H$-invariants and for $X \in m_i$, $Y \in m_j$, $i \neq j$, we get

$$\lambda_i\langle X, Y \rangle = \langle AX, Y \rangle = \langle X, AY \rangle = \lambda_j\langle X, Y \rangle.$$ 

This implies [3]. The inclusion $[m_i, m_j] \subset m$ follows from the fact that $m_j$ is $\text{Ad}_H$-invariant and $\langle [m_i, m_j], h \rangle = \langle [m_i, [m_j, h]] \rangle = 0$. \hfill 

For any subspace $p \subset m$ we will denote by $p^\perp$ its orthogonal complement with respect to the metric $g$ and by $1_p$ the identity operator on $p$.

Recall that $\text{Ad}_H$-submodules $p$, $q$ are called disjoint if they have no non-zero equivalent submodules. If $\text{Ad}_H$-module $m$ is decomposed into a direct sum

$$m = m_1 + \cdots + m_k$$

of disjoint submodules, then any $\text{Ad}_H$-invariant metric $g$ and associated metric endomorphism $A$ have the form

$$g = g_{m_1} \oplus \cdots \oplus g_{m_k}, \quad A = A^{m_1} \oplus \cdots \oplus A^{m_k}.$$ 

Let $(M = G/H, g^M)$ be a compact homogeneous Riemannian space with the reductive decomposition [1] and metric endomorphism $A \in \text{End}(m)$.

We identify elements $X, Y \in g$ with Killing vector fields on $M$. Then the covariant derivative $\nabla_X Y$ at the point $o = eH$ is given by

$$\nabla_X Y(o) = -\frac{1}{2}[X, Y]_m + U(X_m, Y_m),$$

where the bilinear symmetric map $U : m \times m \to m$ is given by

$$2(U(X, Y), Z) = \langle \text{ad}_Z^m X, Y \rangle + \langle X, \text{ad}_Z^m Y \rangle$$

for any $X, Y, Z \in m$ and $X_m$ is the $m$-part of a vector $X \in g$ [8].

Definition 1. A homogeneous Riemannian space $(M = G/H, g^M)$ is called a space with homogeneous geodesics shortly, GO-space if any geodesic $\gamma$ of $M$ is an orbit of 1-parameter subgroup of $G$. The invariant metric $g^M$ is called GO-metric.

If $G$ is the full isometry group, then GO-space is called a manifold with homogeneous geodesics or GO-manifold.

Definition 2. A GO-space $(M = G/H, g^M)$ of a simple compact Lie group $G$ is called a proper GO-space if the metric $g^M$ is not $G$-normal, i.e. the metric endomorphism $A$ is not a scalar operator.

Lemma 2 [1]. A compact homogeneous Riemannian space $(M = G/H, g^M)$ with the reductive decomposition [1] and metric endomorphism $A$ is GO-space if and only if for any $X \in m$ there is $H_X \in h$ such that one of the following equivalent conditions holds:

i) $[H_X + X, A(X)] \in h$;

ii) $([H_X + X, Y]_m, X) = 0$ for all $Y \in m$.

This lemma shows that the property to be GO-space depends only on the reductive decomposition [1] and the Euclidean metric $g$ on $m$. In other words, if $(M = G/H, g^M)$ is a GO-space, then any locally isomorphic homogeneous Riemannian space $(M' = G'/H', g^M')$ is a GO-space. Also a direct product of Riemannian manifolds is a manifold with homogeneous geodesics if and only if each factor is a manifold with homogeneous geodesics.
3 Totally geodesic orbits in a homogeneous Riemannian space

In this section we deal with totally geodesic submanifolds of compact homogeneous Riemannian spaces. This is a useful tool for study of GO-spaces due to the following

**Proposition 1** ([3, Theorem 11]). Every closed totally geodesic submanifold of a Riemannian manifold with homogeneous geodesics is a manifold with homogeneous geodesics.

Let \((M = G/H, g^M)\) be a compact Riemannian homogeneous space with the reductive decomposition (1).

**Definition 3.** A subspace \(p \subset m\) is called totally geodesic if it is the tangent space at \(o\) of a totally geodesic orbit \(Ko \subset G/H = M\) of a subgroup \(K \subset G\).

**Proposition 2.** A subspace \(p \subset m\) is totally geodesic if and only if the following two conditions hold:

a) \(p\) generates a subalgebra of the form \(\mathfrak{k} = \mathfrak{h}' + p\), where \(\mathfrak{h}'\) is a subalgebra of \(\mathfrak{h}\);

b) the endomorphism \(\text{ad}^p_Z \in \text{End}(p)\) for \(Z \in p^\perp\) is \(g\)-skew-symmetric or, equivalently,

\[U(p, p) \subset p.\]

**Proof.** If \(p\) is the tangent space of the orbit \(Ko = K/H'\), then \(\text{Lie}(K) = \mathfrak{k} = \mathfrak{h}' + p\), where \(\mathfrak{h}' = \text{Lie}(H')\) is a subalgebra of \(\mathfrak{h}\). Moreover, the formulas (1) and (5) imply \(U(p, p) \subset p\). Conversely, the conditions a) and b) imply that \(p\) is the tangent space of the totally geodesic orbit \(Ko\) of the subgroup \(K\) generated by the subalgebra \(\mathfrak{k}\).

**Corollary 1.**

i) A subspace \(p \subset m\) is totally geodesic if a) holds and \(Ap = p\).

ii) If a totally geodesic subspace \(p\) is \(\text{ad}_h\)-invariant and \(A\)-invariant, then

\[[h + p, p^\perp] \subset p^\perp.\]

**Proof.** i) Assume that \(Ap = p\). Then \(Ap^\perp = p^\perp\) and \(\langle p, p^\perp \rangle = 0\). From i) and \(Ap = p\) we get \(\langle Z, [X, AX] \rangle = 0\) for any \(X \in p\) and \(Z \in p^\perp\). This implies

\[0 = \langle [Z, X], AX \rangle = \langle [Z, X]_m, AX \rangle = \langle [Z, X]_m, X \rangle = \langle [Z, X]_p, X \rangle = (U(X, X), Z).\]

ii) follows from the fact that the endomorphisms \(\text{ad}_{h+p}^p\) are \(b\)-skew-symmetric and preserves the subspace \(p\). Hence, they preserve its \(b\)-orthogonal complement \(p^\perp\).

**Corollary 2.** Let \((M = G/H, g)\) be a compact Riemannian homogeneous space and \(K\) a connected subgroup of \(G\). The orbit \(P = Ko = K/H'\) is a totally geodesic submanifold if and only if the Lie algebra \(\mathfrak{k}\) is consistent with the reductive decomposition (1) (that is \(\mathfrak{k} = \mathfrak{k} \cap \mathfrak{h} + \mathfrak{k} \cap \mathfrak{m} = \mathfrak{h}' + p\)) and

\[U(p, p) \subset p\]

or, equivalently, the endomorphisms \(\text{ad}_Z^p \in \text{End}(p)\), \(Z \in p^\perp\) are \(g\)-skew-symmetric.
4 Properties of GO-spaces

Lemma 3. Let \((M = G/H, g^M)\) be a GO-space with the reductive decomposition and \(m = p + q\) a g-orthogonal \(\text{Ad}_H\)-invariant decomposition. Then

\[ U(p, p) \subset p, \quad U(q, q) \subset q \]

and the endomorphisms \(\text{ad}^q_p\), \(\text{ad}^p_q\) are skew-symmetric.

Proof. For \(X \in p, Y \in q\) we have

\[ 0 = ([Y + H_Y, X]_m, Y) = -(\text{ad}_X Y, Y) = -(U(Y, Y), X), \]

where \(H_Y\) is as in Lemma 2. This shows that \(\text{ad}^q_X\) is skew-symmetric and \(U(q, q) \subset q\). ■

Lemma 3 together with Proposition 2 implies Proposition 3. Let \((M = G/H, g^M)\) be a GO-space with the reductive decomposition. Then any connected subgroup \(K \subset G\) which contains \(H\) has the totally geodesic orbit \(P = Ko = K/H\) which is GO-space (with respect to the induced metric). Moreover, if the space \(p := \mathfrak{k} \cap m\) is \(A\)-invariant, then

\[ [\mathfrak{k}, m^\perp] \subset m^\perp \]

and the metric \(\bar{g} := g|_{p^\perp}\) is \(\text{Ad}_K\)-invariant and defines an invariant GO-metric \(g^N\) on the homogeneous manifolds \(N = G/K\). The projection \(\pi : G/H \to G/K\) is a Riemannian submersion with totally geodesic fibers such that the fibers and the base are GO-spaces.

Proof. The first claim follows from Lemma 3, Lemma 2 and Proposition 2. If \(A p = p\), then \(m = p + p^\perp\) is a b-orthogonal decomposition and since the metric \(b\) is \(\text{Ad}_G\)-invariant, \(\text{Ad}_K p^\perp = p^\perp\). Then Lemma 3 shows that the metric \(g|_{p^\perp}\) is \(\text{Ad}_K\)-invariant and defines an invariant metric \(g^N\) on \(N = G/K\) such that \(N\) becomes GO-space. ■

Note that a subgroup \(K \supset H\) is compatible with any invariant metric on \(G/H\) if \(\text{Ad}_H\)-modules \(p\) and \(m/p\) are strictly disjoint. This remark implies

Proposition 4. Let \((M = G/H, g)\) be a compact homogeneous Riemannian space. Then the connected normalizer \(N_0(Z)\) of a central subgroup \(Z\) of \(H\) and the connected normalizer \(N_0(H)\) are subgroups consistent with any invariant metric on \(M\).

Proposition 5. Let \((M = G/H, g^M)\) be a compact GO-space with metric endomorphism \(A\).

i) Let \(X, Y \in m\) be eigenvectors of the metric endomorphism \(A\) with different eigenvalues \(\lambda, \mu\). Then

\[ [X, Y] = \frac{\lambda}{\lambda - \mu}[H, X] + \frac{\mu}{\lambda - \mu}[H, Y] \]

for some \(H \in \mathfrak{h}\).

ii) Assume that the vectors \(X, Y\) belong to the \(\lambda\)-eigenspace \(m_\lambda\) of \(A\) and \(X\) is g-orthogonal to the subspace \([\mathfrak{h}, Y]\). Then

\[ [X, Y] \in \mathfrak{h} + m_\lambda. \]
Proof. i) Let $X, Y \in m$ be eigenvectors of $A$ with different eigenvalues $\lambda, \mu$ and $H = H_{X+Y} \in \mathfrak{h}$ the element defined in Lemma 2. Then

$$[H + X + Y, A(X + Y)] = [H + X + Y, \lambda X + \mu Y] = \lambda[H, X] + \mu[H, Y] + (\mu - \lambda)[X, Y] \in \mathfrak{h}.$$  

By Lemma 1, $[H, X]$, $[H, Y]$, $[X, Y] \in m$ and the right hand side is zero.

ii) Assume now that $X, Y \in m$ satisfy conditions ii) and $Z$ is an eigenvector of $A$ with an eigenvalue $\mu \neq \lambda$. Then we have

$$([X, Y]_m, Z) = \mu([X, Y], Z) = \mu(X, [Y, Z]_m) = \frac{\mu}{\lambda}(X, [Y, Z]_m)$$

$$= \frac{\mu}{\lambda} \left( X, \frac{\lambda}{\lambda - \mu}[H, Y] + \frac{\mu}{\lambda - \mu}[H, Z] \right) = 0.$$  

This shows that $[X, Y] \in \mathfrak{h} + m$. 

Corollary 3. Let $(M = G/H, g^M)$ be a compact GO-space with the reductive decomposition (1) and metric endomorphism $A$ and

$$m = m_1 + \cdots + m_k$$  

the $A$-eigenspace decomposition such that $A|m_i = \lambda_i I_{m_i}$. Then for any $\text{Ad}_H$-submodules $p_i \subset m_i$, $p_j \subset m_j$, $i \neq j$, we have

$$[p_i, p_j] \subset p_i + p_j.$$  

Moreover, if $p$, $p'$ are $g$-orthogonal $\text{Ad}_H$-submodules of $m_i$ then

$$[p, p'] \subset \mathfrak{h} + m_i.$$  

5 Some applications

5.1 A sufficient condition for non-existence of GO-metric

Here we consider some applications of results of the previous section.

Definition 4. Let $(M = G/H, g^M)$ be a compact homogeneous Riemannian space. A connected closed Lie subgroup $K \subset G$ which contains $H$ is called compatible with the metric $g^M$ if the subspace $p = \mathfrak{k} \cap m$ of $m$ is invariant under the metric endomorphism $A$.

Let $K, K'$ be two subgroups of $G$ which are compatible with the metric of a homogeneous Riemannian space $(M = G/H, g^M)$. Then we can decompose the space $m$ into a $g$-orthogonal sum of $A$-invariant $\text{Ad}_H$-modules

$$m = q + p_1 + p_2 + n$$  

where

$$q = p \cap p', \quad p = \mathfrak{k} \cap m = q + p_1, \quad p' = \mathfrak{k}' \cap m = q + p_2$$  

and $n$ is the orthogonal complement to

$$p + p' = q + p_1 + p_2$$  

in $m$. 

Proposition 6. Let \((M = G/H, g^M)\) be a homogeneous Riemannian space, \(K, K'\) two subgroups of \(G\) which are compatible with \(g^M\) and \((\ref{eq:decomposition})\) the associated decomposition as above. Then \(p_1, p_2, n\) are \(\text{Ad}_{\tilde{H}}\)-modules, where \(\tilde{H} = K \cap K'\) is the Lie group with the Lie algebra \(\mathfrak{h} = \mathfrak{h} + \mathfrak{q}\), and
\[
[p_1, p_2] \subset n.
\]
Moreover, if \((M = G/H, g^M)\) is a GO-space, then the restriction \(A_{\tilde{m}}\) of the metric endomorphism to \(\tilde{m} = p_1 + p_2 + n\) commutes with \(\text{Ad}_{\tilde{H}}|_{\tilde{m}}\) and for any \(\tilde{H}\)-irreducible submodules \(p'_1 \subset p_1\), and \(p'_2 \subset p_2\) such that
\[
[p'_1, p'_2] \neq 0,
\]
the metric endomorphism \(A\) is a scalar on the space
\[
p'_1 + p'_2 + [p'_1, p'_2].
\]

Proof. Since the decomposition \((\ref{eq:decomposition})\) is \(b\)-orthogonal, we conclude that it is \(\text{Ad}_{\tilde{H}}\)-invariant and
\[
[p, p^\perp] = [p, p_2 + n] \subset p_2 + n,
\]
\[
[p', (p')^\perp] = [p', p_1 + n] \subset p_1 + n,
\]
by Proposition \(2\). This implies
\[
[p_1, p_2] \subset n.
\]
If \((M, g^M)\) is a GO-space, then by Proposition \(3\) the metric endomorphism \(A_{\tilde{m}}\) is \(\tilde{H}\)-invariant. If modules \(p'_1, p'_2\) belong to \(A\)-eigenspaces with different eigenvalues, then by Corollary \(5\)
\[
[p'_1, p'_2] \subset p_1 + p_2.
\]
Together with the previous inclusion, it implies \([p'_1, p'_2] = 0\). If these modules belong to the same eigenspace \(m_\lambda\), then by Corollary \(2\)
\[
[p'_1, p'_2] \subset m_\lambda.
\]

As a corollary, we get the following sufficient condition that a homogeneous manifold \(M = G/H\) does not admit a proper GO-metric.

Proposition 7. Let \(M = G/H\) be a homogeneous space of a compact group \(G\) with the reductive decomposition \(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}\). Assume that the Lie algebra \(\mathfrak{g}\) has two subalgebras \(\mathfrak{t} = \mathfrak{h} + \mathfrak{p}\), \(\mathfrak{t}' = \mathfrak{h} + \mathfrak{p}'\) which contain \(\mathfrak{h}\) and generate \(\mathfrak{g}\). Let
\[
m = \mathfrak{q} + p_1 + p_2 + n, \quad \mathfrak{q} = \mathfrak{p} \cap \mathfrak{p}'
\]
be the associated \(b\)\-orthogonal decomposition. Assume that there is no commuting \(\text{ad}_{\mathfrak{h} + \mathfrak{q}}\) submodules of \(p_1\) and \(p_2\). Then for any GO-metric, defined by an operator \(A\) which preserves this decomposition, \(A\) is a scalar operator on \(p_1 + p_2 + n\). In particular, if \(\mathfrak{q}\) is trivial and \(\text{Ad}_{\tilde{H}}\)-modules \(p_1, p_2, n\) are strictly non-equivalent, then the only GO-metric on \(M\) is the normal metric.

Proof. Let \(A\) be an operator on \(m\) which preserves the decomposition \((\ref{eq:decomposition})\) and defines a GO-metric. Then by Proposition \(3\)
\[
A|_{p_1 + p_2 + [p_1, p_2]} = \lambda \cdot 1
\]
for some \(\lambda\). Now \(p_1\) and \([p_1, p_2] \subset n\) are two \(g\)-orthogonal submodules of the \(A\)-eigenspace \(m_\lambda\). Applying Corollary \(5\) we conclude that
\[
[p_1, [p_1, p_2]] \subset m_\lambda.
\]
Iterating this process, we prove that
\[
n = [p_1, p_2] + [p_1, [p_1, p_2]] + [p_2, [p_1, p_2]] + \cdots \subset m_\lambda
\]
and \(A = \lambda \cdot 1\) on \(p_1 + p_2 + n\).
5.2 A sufficient condition for existence of GO-metric

**Lemma 4.** Let $M = G/H$ be a homogeneous space of a compact Lie group with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Assume that $\text{Ad}_H$-module $\mathfrak{m}$ has a decomposition

$$\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_k$$

into invariant submodules, such that for any $i < j$

$$[\mathfrak{m}_i, \mathfrak{m}_j] = 0$$

or this condition valid with one exception $(i, j) = (1, 2)$ and in this case

$$[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_2$$

and for any $X \in \mathfrak{m}_1$, $Y \in \mathfrak{m}_2$ there is $H \in \mathfrak{h}$ such that $\text{ad}_H Y = \text{ad}_X Y$ and

$$\text{ad}_H (\mathfrak{m}_1 + \mathfrak{m}_3 + \cdots + \mathfrak{m}_k) = 0.$$ 

Then any metric endomorphism of the form $A = \sum x_i \cdot 1_{\mathfrak{m}_i}$ defines a GO-metric on $M$.

**Proof.** Under the assumptions of lemma, for $H \in \mathfrak{h}$ and $X_i \in \mathfrak{m}_i$ we have

$$\left[ H + \sum X_i, \sum x_i X_i \right] = \sum_{i < j} (x_j - x_i) [X_i, X_j] + \sum x_i \text{ad}_H X_i$$

$$= (x_2 - x_1) \text{ad}_{X_1} X_2 + x_2 \text{ad}_H X_2 + \text{ad}_H \left( x_1 X_1 + \sum_{k \geq 3} x_k X_k \right).$$

The right-hand side is zero if $H$ is chosen as in the lemma (where $Y = x_2 X_2$ and $X = (x_1 - x_2) X_1$). Now, it suffices to apply Lemma 2.

**Example 1.** The homogeneous space $M = SU_{p+q}/SU_p \times SU_q$ is a GO-space with respect to any invariant metric.

We have the reductive decomposition

$$\mathfrak{su}_{p+q} = \mathfrak{h} + \mathfrak{m} = (\mathfrak{su}_p + \mathfrak{su}_q) + (\mathbb{R}a + p),$$

where $p \simeq \mathbb{C}^p \otimes \mathbb{C}^q$ and $\text{ad}_a |_{p} = i \cdot 1_{p}$. Any metric endomorphism $A = \lambda \cdot 1_{\mathbb{R}a} + \mu \cdot 1_{p}$ defines a GO-metric by above lemma since for any $X \in p$ there is $H \in \mathfrak{h}$ such that $\text{ad}_H X = i X$. Note that for $p \neq q$ these manifolds are weakly symmetric spaces [26].

5.3 GO-metrics on a compact group $G$

**Proposition 8.** A compact Lie group $G$ with a left-invariant metric $g$ is a GO-space if and only if the corresponding Euclidean metric $(\cdot, \cdot)$ on the Lie algebra $\mathfrak{g}$ is bi-invariant.

**Proof.** The condition that $(G, g)$ is a GO-space can be written as

$$0 = (X, [X, Y]) = -(\text{ad}_Y X, X) = 0.$$

This shows that the metric $(\cdot, \cdot)$ is bi-invariant.

Note that a compact Lie group $G$ can admit a non-bi-invariant left-invariant metrics $g$ with homogeneous geodesics. But the corresponding GO-space will have the form $L/H$ where the group $L$ will contain $G$ as a proper subgroup. See [11] for details.
6 Homogeneous GO-spaces with positive Euler characteristic

6.1 Basic facts about homogeneous manifolds of positive Euler characteristic

Here we recall some properties of homogeneous spaces with positive Euler characteristic (see, for example, [21] or [4]). A homogeneous space $M = G/H$ of a compact connected Lie group $G$ has positive Euler characteristic $\chi(M) > 0$ if and only if the stabilizer $H$ has maximal rank ($\text{rk}(H) = \text{rk}(G)$).

If the group $G$ acts on $M$ almost effectively, then it is semisimple and the universal covering $\tilde{M} = \tilde{G}/\tilde{H}$ is a direct product $\tilde{M} = G_1/H_1 \times \cdots \times G_k/H_k$, where $\tilde{G} = G_1 \times G_2 \times \cdots \times G_k$ is the decomposition of the group $\tilde{G}$ (which is a covering of $G$) into a direct product of simple factors and $H_i = \tilde{H} \cap G_i$.

Any invariant metric $g^M$ on $M$ defines an invariant metric $g^{\tilde{M}}$ on $\tilde{M}$ and the homogeneous Riemannian space $(\tilde{M} = \tilde{G}/\tilde{H}, g^{\tilde{M}})$ is a direct product of homogeneous Riemannian spaces $(M_i = G_i/H_i, g^{M_i})$, $i = 1, \ldots, k$, of simple compact Lie groups $G_i$, see [19]. We have

**Proposition 9** ([19]). A compact almost effective homogeneous Riemannian space $(M = G/H, g^M)$ of positive Euler characteristic is irreducible if and only if the group $G$ is simple. If the group $G$ acts effectively on $M$, it has trivial center.

This proposition shows that a simply connected compact GO-space $(M = G/H, g^M)$ of positive Euler characteristic is a direct product of simply connected GO-spaces $(M_i = G_i/H_i, g^{M_i})$ of simple Lie groups with positive Euler characteristic. So it is sufficient to classify simply connected GO-spaces of a simple compact Lie group with positive Euler characteristic.

A description of homogeneous spaces $G/H$ of positive Euler characteristic reduces to description of connected subgroups $H$ of maximal rank of $G$ or equivalently, subalgebras of maximal rank of a simple compact Lie algebra $\mathfrak{g}$, see [9] and also Section 8.10 in [25]. An important subclass of compact homogeneous spaces of positive Euler characteristic consists of flag manifolds. They are described as adjoint orbits $M = \text{Ad}_G x$ of a compact connected semisimple Lie group $G$ or, in other terms as quotients $M = G/H$ of $G$ by the centerizer $H = Z_G(T)$ of a non-trivial torus $T \subset G$.

Note that every compact naturally reductive homogeneous Riemannian space of positive Euler characteristic is necessarily normal homogeneous with respect to some transitive semisimple isometry group [4].

6.2 The main theorem

Let $G$ be a simple compact connected Lie group, $H \subset K \subset G$ its closed connected subgroups. We denote by $b = \langle \cdot, \cdot \rangle$ the minus Killing form on the Lie algebra $\mathfrak{g}$ and consider the following $b$-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

where

$$\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_2$$

is the Lie algebra of the group $K$. Obviously, $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_1$. Let $g^M = g_{x_1,x_2}$ be a $G$-invariant Riemannian metric on $M = G/H$, generated by the Euclidean metric $g = \langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ of the form

$$g = x_1 \cdot b_{\mathfrak{m}_1} + x_2 \cdot b_{\mathfrak{m}_2},$$

(9)
where $x_1$ and $x_2$ are positive numbers, or, equivalently, by the metric endomorphism

$$A = x_1 \cdot 1_{m_1} + x_2 \cdot 1_{m_2}. \quad (10)$$

We consider two examples of such homogeneous Riemannian spaces \((M = G/H, g_{x_1,x_2})\):

a) \((G, K, H) = (SO(2n + 1), U(n), SO(2n)), n \geq 2\). The group \(G = SO(2n + 1)\) acts transitively on the symmetric space \(\text{Com}(\mathbb{R}^{2n+2}) = SO(2n + 2)/U(n)\) of complex structures in \(\mathbb{R}^{2n+2}\) with stabilizer \(H = U(n)\), see [15]. So we can identify \(M = G/H\) with this symmetric space, but the metric \(g_{x_1,x_2}\) is not \(SO(2n + 2)\)-invariant if \(x_2 \neq 2x_1\) [17].

b) \((G, K, H) = (Sp(n), Sp(1) \cdot Sp(n - 1), U(1) \cdot Sp(n - 1)), n \geq 2\). The group \(G = Sp(n)\) acts transitively on the projective space \(\mathbb{C}P^{2n-1} = SU(2n + 2)/U(2n + 1)\) with stabilizer \(H = U(1) \cdot Sp(n - 1)\). So we can identify \(M = G/H\) with \(\mathbb{C}P^{2n-1}\), but the metric \(g_{x_1,x_2}\) is not \(SU(2n + 2)\)-invariant if \(x_2 \neq 2x_1\), see [15, 17].

Now we can state the main theorem about compact GO-spaces of positive Euler characteristic.

**Theorem 1.** Let \((M = G/H, g^M)\) is a simply connected proper GO-space with positive Euler characteristic and simple compact Lie group \(G\). Then \(M = G/H = SO(2n + 1)/U(n), n \geq 2\), or \(G/H = Sp(n)/U(1) \times Sp(n - 1), n \geq 2\), and \(g^M = g_{x_1,x_2}\) is any \(G\)-invariant metric which is not \(G\)-normal homogeneous. The metric \(g^M\) is \(G\)-normal homogeneous (respectively, symmetric) when \(x_2 = x_1\) (respectively, \(x_2 = 2x_1\)). Moreover, these homogeneous spaces are weakly symmetric flag manifolds.

The non-symmetric metrics \(g_{x_1,x_2}\) have \(G\) as the full connected isometry group of the considered GO-spaces \((M = G/H, g_{x_1,x_2})\), see discussion in [17, 21]. The claim that all these homogeneous Riemannian spaces are weakly symmetric spaces was proved in [27]. Note also that Theorem 1 allows to simplify some arguments in the paper [3].

### 6.3 Proof of the main theorem

Using results from [1] and [3], we reduce the proof to a description of some special decompositions of the root system of the Lie algebra \(\mathfrak{g}\) of the isometry group \(G\).

Let \(M = G/H\) be a homogeneous space of a compact simple Lie group of positive characteristic and

\[
\mathfrak{g} = \mathfrak{h} + \mathfrak{m}
\]

associated reductive decomposition. The subgroup \(H\) contains a maximal torus \(T\) of \(G\). We consider the root space decomposition

\[
\mathfrak{g}^C = \mathfrak{t}^C + \sum_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha
\]

of the complexification \(\mathfrak{g}^C\) of the Lie algebra \(\mathfrak{g}\), where \(\mathfrak{t}^C\) is the Cartan subalgebra associated with \(T\) and \(\mathcal{R}\) is the root system.

For any subset \(P \subset \mathcal{R}\) we denote by

\[
\mathfrak{g}(P) = \sum_{\alpha \in P} \mathfrak{g}_\alpha
\]

the subspace spanned by corresponding root space \(\mathfrak{g}_\alpha\). Then \(H\)-module \(\mathfrak{m}^C\) is decomposed into a direct sum

\[
\mathfrak{m}^C = \mathfrak{g}(R_1) + \cdots + \mathfrak{g}(R_k)
\]
of disjoint submodules, where \( R = R_1 \cup \cdots \cup R_k \) is a disjoint decomposition of \( R \) and subsets \( R_i \) are symmetric, i.e. \(-R_i = R_i\). Moreover, real \( H\)-modules \( g \cap g(R_i) \) are irreducible. Any invariant metric on \( M \) is defined by the metric endomorphism \( A \) on \( \mathfrak{m} \) whose extension to \( \mathfrak{m}^\mathbb{C} \) has the form

\[
A = \text{diag}(x_1 \cdot 1_{p_1}, \ldots, x_\ell \cdot 1_{p_\ell}),
\]

where \( x_i \) are arbitrary positive numbers, \( x_i \neq x_j \) and \( p_i \) is a direct sum of modules \( g(R_m) \).

We will assume that \( A \) is not a scalar operator (i.e. \( \ell > 1 \)) and it defines an invariant metric with homogeneous geodesics. We say that a root \( \alpha \) corresponds to eigenvalue \( x_i \) of \( A \) if \( g_\alpha \subset p_i \).

**Lemma 5.** There are two roots \( \alpha, \beta \) which correspond to different eigenvalues of \( A \) such that \( \alpha + \beta \) is a root.

**Proof.** If it is not the case, \([p_1, p_i] = 0 \) for \( i \neq 1 \) and \( g_1 = p_1 + [p_1, p_1] \) would be a proper ideal of a simple Lie algebra \( g \).

Now, consider the roots \( \alpha \) and \( \beta \) as in the previous lemma. Since \( R(\alpha, \beta) := R \cap \text{span}\{\alpha, \beta\} \) is a rank 2 root system, we can always choose roots \( \alpha, \beta \in R \) which form a basis of the root system \( R(\alpha, \beta) \). Then the subalgebra

\[
\mathfrak{g}_{\alpha, \beta} := t^\mathbb{C} + \sum_{\gamma \in R(\alpha, \beta)} \mathfrak{g}_\gamma
\]

of \( \mathfrak{g}^\mathbb{C} \) is the centralizer of the subalgebra \( t' = \ker \alpha \cap \ker \beta \subset t^\mathbb{C} \).

Then the orbit \( G_{\alpha, \beta} o \subset M \) of the corresponding subgroup \( G_{\alpha, \beta} = T' \cdot G_{\alpha, \beta} \subset G \) is a totally geodesic submanifold (see Corollary 2), hence a proper GO-space with the effective action of the rank two simple group \( G_{\alpha, \beta} \) associated with the root system \( R(\alpha, \beta) \) (see Proposition 1).

Note that it has positive Euler characteristic since the stabilizer of the point \( o \) contains the two-dimensional torus generated by vectors \( H_\alpha, H_\beta \in t^\mathbb{C} \) associated with roots \( \alpha, \beta \). Recall that \( H_\alpha = \frac{2}{(\alpha, \alpha)} b^{-1}. \alpha \).

**Proposition 10.** Every proper GO-space \( (M = G/H, g^M) \) with positive Euler characteristic of a simple group \( G \) of rank 2 is locally isometric to the manifold \( M = \text{SO}(5)/U(2) \) with the metric defined by the metric endomorphism

\[
A = x_1 \cdot 1_{g(R^s)} + x_2 \cdot 1_{g(R^l)}, \quad x_1 \neq x_2 > 0
\]

where

\[
R^s = \{\pm \epsilon_1, \pm \epsilon_2\}, \quad R^l = \{\pm \epsilon_1 \pm \epsilon_2\},
\]

are the sets of short and, respectively, long roots of the Lie algebra \( \mathfrak{so}(5) \). We may assume also that

\[
\mathfrak{m}^\mathbb{C} = g(R^s \cup \{\epsilon_1 + \epsilon_2\}) \quad \text{and} \quad \mathfrak{h}^\mathbb{C} = t^\mathbb{C} + g_{t_1} - \epsilon_2.
\]

**Proof.** Proof of this proposition follows from results of the papers [1] and [3]. Indeed, the group \( G \) has the Lie algebra \( g \) isomorphic to \( \text{su}(3) = A_2, \text{so}(5) = \text{sp}(2) = B_2 = C_2 \) or \( g_2 \). Since the universal Riemannian covering of a GO-space is a GO-space (Lemma 2), we may assume without loss of generality that \( G/H \) is simply connected.

If \( g = \text{su}(3) \), then \( G/H = \text{SU}(3)/S(U(2) \times U(1)) \) (a symmetric space) or \( G/H = \text{SU}(3)/T^2 \), where \( T^2 \) is a maximal torus in \( \text{SU}(3) \). Both these spaces are flag manifolds, and results of [1] show that any GO-metric on these spaces is \( SU(3) \)-normal homogeneous.
If \( g = \text{so}(5) = \text{sp}(2) \), then \((g, h) = (\text{so}(5), \mathbb{R}^2), (g, h) = (\text{so}(5), \mathbb{R} \oplus \text{su}(2)_l), (g, h) = (\text{so}(5), \mathbb{R} \oplus \text{su}(2)_s)\), or \((g, h) = (\text{so}(5), \text{su}(2)_l \oplus \text{su}(2)_l)\), where \(\text{su}(2)_l\) (respectively, \(\text{su}(2)_s\)) stands for a three-dimensional subalgebras generated by all long (respectively, short) roots of \(g\). The last pair corresponds to the irreducible symmetric space \(\text{SO}(5)/\text{SO}(4)\), which admits no non-normal invariant metric. All other spaces are flag manifolds. Results of [11] implies that the only possible pair is \((g, h) = (\text{so}(5), \mathbb{R} \oplus \text{su}(2)_l)\), which corresponds to the space \(\text{SO}(5)/U(2) = \text{Sp}(2)/U(1) \cdot \text{Sp}(1)\).

For \(g = g_2\) the statement of proposition is proved in [3, Proposition 23].

**Corollary 4.** Let \(G\) be a simple compact Lie group and \(M = G/H\) a proper GO-space with positive Euler characteristic. Then the root system \(R\) of the complex Lie algebra \(g^C\) admits a disjoint decomposition

\[ R = R_0 \cup R_1 \cup R_2, \]

where \(R_0\) is the root system of the complexified stability subalgebra \(h^C\), with the following properties:

i) If \(\alpha \in R_1\), \(\beta \in R_2\) and \(\alpha + \beta \in R\) then \(\alpha - \beta \in R\) and the rank 2 root system \(R(\alpha, \beta)\) has type \(B_2 = C_2\).

ii) Moreover, if \(\alpha, \beta\) is a basis of \(R(\alpha, \beta)\) (that is \(\langle \alpha, \beta \rangle < 0\)), then one of the roots \(\alpha, \beta\) is short and the other is long and one of the long roots \(\alpha \pm \beta\) belongs to \(R_0\) and second one belongs to \(R_1 \cup R_2\).

iii) If both roots \(\alpha, \beta\) are short, then one of the long roots \(\alpha \pm \beta\) belongs to \(R_0\) and the other belongs to \(R_1 \cup R_2\).

iv) If \(\alpha \in R_1\) and \(\beta \in R_2\) are long roots, then \(\alpha \pm \beta \notin R\).

We will call a decomposition with the above properties a **special decomposition**. Corollary [3] implies

**Corollary 5.** There is no proper GO-spaces of positive Euler characteristic with simple isometry group \(G = \text{SU}(n), \text{SO}(2n), E_6, E_7, E_8\) (these are all simple Lie algebras with all roots of the same length (simply-laced root system)).

**Corollary 6 ([3, Proposition 23]).** Any GO-space \((G/H, \mu)\) of positive Euler characteristic with \(G = G_2\) is normal homogeneous.

Now, we describe all **special decompositions** of the root systems of types \(B_n, C_n, F_4\). We will use notation from [13] for root systems and simple roots.

**Lemma 6.** The root system

\[ R(F_4) = \{ \pm \epsilon_i, 1/2(\pm \epsilon_1 \mp \epsilon_2 \pm \epsilon_3 \pm \epsilon_4, \pm \epsilon_i \pm \epsilon_j) \mid i, j = 1, 2, 3, 4, \ i \neq j \} \]

does not admit a special decomposition.

**Proof.** Assume that such a decomposition exists. Then we can choose roots \(\alpha \in R_1, \beta \in R_2\) such that \(\alpha \pm \beta\) is a root. Then \(\alpha, \beta\) has different length and we may assume that \(|\alpha| < |\beta|\) and \(\langle \alpha, \beta \rangle < 0\). Then we can include \(\alpha, \beta\) into a system of simple roots \(\delta, \alpha, \beta, \gamma\), see [13]. Since all such systems are conjugated, we may assume that \(\alpha = \epsilon_4, \beta = -\epsilon_4 + \epsilon_3\), see [13]. Then we get contradiction, since \(\alpha - \beta\) is not a root.
Now we describe two special decompositions for the root systems

\[ R(B_n) = \{ \pm \epsilon_i, \pm \epsilon_i \pm \epsilon_j, \ i, j = 1, \ldots, n \} \]

and

\[ R(C_n) = \{ \pm 2\epsilon_i, \pm \epsilon_i \pm \epsilon_j, \ i, j = 1, \ldots, n \} \]

of types \( B_n \) and \( C_n \). Note that in both cases \( R_A = \{ \pm (\epsilon_i - \epsilon_j) \} \) is a closed subsystem. We set \( R_A^\pm = \{ \pm (\epsilon_i + \epsilon_j) \} \).

We denote by \( R^+ \) the standard subsystem of positive roots of a root system \( R \) and by \( R^s \) and \( R^l \) the subset of short and, respectively, long roots of \( R \). Then there is a special decomposition \( R = R_0 \cup R_1 \cup R_2 \) of the systems \( R(B_n) \), \( R(C_n) \) which we call the standard decomposition:

\[
R(B_n) = R_A \cup R^s \cup R_A^+,
R(C_n) = R_A \cup R^l \cup R_A^+.
\]

These decompositions define the following reductive decompositions of the homogeneous spaces \( SO(2n + 1)/U(n) \) and \( Sp(n)/U(n) \):

\[
\mathfrak{so}(2n + 1) = \mathfrak{h} + (\mathfrak{m}_1 + \mathfrak{m}_2) = \mathfrak{g}(R_A) + (\mathfrak{g}(R^s) + \mathfrak{g}(R_A^+)),
\]

\[
\mathfrak{sp}(n) = \mathfrak{h} + (\mathfrak{m}_1 + \mathfrak{m}_2) = \mathfrak{g}(R_A) + (\mathfrak{g}(R^l) + \mathfrak{g}(R_A^+)),
\]

where \( \mathfrak{m}_1, \mathfrak{m}_2 \) are irreducible submodules of \( \mathfrak{m} \). It is known [1] that any metric endomorphism \( A = \text{diag}(x_1 \cdot 1_{\mathfrak{m}_1}, x_2 \cdot 1_{\mathfrak{m}_2}) \) defines a metric with homogeneous geodesics on the corresponding manifold \( M = G/H \) (see a discussion before the statement of Theorem [1]). Now, the proof of Theorem [1] follows from the following proposition.

Proposition 11. Any special decomposition of the root systems \( R_B \), \( R_C \) is conjugated to the standard one.

Proof. We give a proof of this proposition for \( R(B_n) \). The proof for \( R(C_n) \) is similar.

Let

\[ R(B_n) = R_0 \cup R_1 \cup R_2 \]

be a special decomposition of \( R(B_n) \). We may assume that there are roots \( \alpha \in R_1 \) and \( \beta \in R_2 \) with \( \langle \alpha, \beta \rangle < 0 \) and \( |\alpha| < |\beta| \). Then we can include \( \alpha, \beta \) into a system of simple roots, which, without loss of generality, can be written as

\[ \alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-2} - \epsilon_{n-1}, \epsilon_{n-1} + \epsilon_n = \beta, -\epsilon_n = \alpha. \]

Then \( (\epsilon_{n-1} - \epsilon_n) \in R_0 \). We need the following lemma.

Lemma 7. Let \( R(B_n) = R_0 \cup R_1 \cup R_2 \) be a special decomposition as above, \( V' = \epsilon_n^+ \) the orthogonal complement of the vector \( \epsilon_n \) and \( R(B_{n-1}) = R' := R \cap V' \) the root system induced in the hyperspace \( V' \). Then the induced decomposition \( R' = R_0' \cup R_1' \cup R_2' \), where \( R_i' := R_i \cap V' \), is a special decomposition.

Proof. It is sufficient to check that subsets \( R_1', R_2' \) are not empty.

We say that two roots \( \gamma, \delta \) are \( R_0 \)-equivalent (\( \gamma \sim \delta \)) if their difference belongs to \( R_0 \). The equivalent roots belong to the same component \( R_i \). The root \( \epsilon_{n-1} = \epsilon_n - (\epsilon_{n-1} - \epsilon_n) \) is \( R_0 \)-equivalent to \( \alpha = \epsilon_n \). Hence it belongs to \( R_1 \).
We say that a pair of roots \( \gamma, \delta \) with \( \langle \gamma, \delta \rangle < 0 \) is **special** if one of the roots belongs to \( R_1 \) and another to \( R_2 \). Then they have different length (say, \( |\gamma| < |\delta| \)). Moreover, the root \( \gamma + \delta \) is short and it belongs to the same part \( R_i, i = 1, 2 \) as the short root \( \delta \) and the root \( 2\gamma + \delta \) is long and it belongs to \( R_0 \).

Consider the roots \( \sigma_{\pm} = \pm \epsilon_{n-2} + \epsilon_{n-1} \). They have negative scalar product with \( \epsilon_{n-1} \in R_1 \) and \( \beta = \epsilon_{n-1} + \epsilon_n \in R_2 \). They can not belong to \( R_1 \) since then we get a special pair \( \delta_{\pm}, \beta \) which consists of long roots. They both can not belong to \( R_0 \) since otherwise the root \( \epsilon_{n-2} \sim \epsilon_{n-1} \in R_1 \) and \( \pm \epsilon_{n-2} + \epsilon_n \sim \epsilon_{n-1} + \epsilon_n \in R_2 \) and we get a special pair

\[
\gamma = \epsilon_{n-2} \in R_1, \quad \delta = -\epsilon_{n-2} + \epsilon_n \in R_2,
\]

such that \( 2\gamma + \delta \in R_0 \), which is impossible. We conclude that one of the roots \( \sigma_{\pm} = \pm \epsilon_{n-2} + \epsilon_{n-1} \in R' \) must belongs to \( R_2 \). Since the root \( \epsilon_{n-1} \in R' \) belongs to \( R_1 \), the lemma is proved. \( \blacksquare \)

Now we prove the proposition by induction on \( n \). The claim is true for \( n = 2 \) by Proposition 10. Assume that it is true for \( R(B(n - 1)) \) and let \( R(B_n) = R_0 \cup R_1 \cup R_2 \) be a special decomposition as above. By lemma, the decomposition \( R' = R'_0 \cup R'_1 \cup R'_2 \), indexed in the hyperplane \( V' = e_n \perp \), is a special decomposition. By inductive hypothesis we may assume that it has the standard form:

\[
R_0 = \{ \pm(\epsilon_i - \epsilon_j) \}, \quad R_1 = \{ \pm \epsilon_i \}, \quad R_2 = \{ \pm(\epsilon_i + \epsilon_j) \}, \quad i, j = 1, \ldots, n - 1. \]

This implies that the initial decomposition is also standard. \( \blacksquare \)

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