Bohr’s Theorem for Monogenic Power Series

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Abstract

The main goal of this paper is to generalize Bohr’s phenomenon from complex one-dimensional analysis to higher dimensions in the framework of Quaternionic Analysis.

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1 Introduction

In 1914 Bohr discovered that there exists a radius $r \in (0, 1)$ such that if a power series of a holomorphic function converges in the unit disk and its sum has a modulus less than 1, then for $|z| < r$ the sum of the absolute values of its terms is again less than 1. This radius does not depend on the function.

Theorem 1.1 (Bohr, 1914) Let $f$ be a bounded analytic function in the open unit disk, with Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ convergent in the unit disk and with modulus less than 1. Then $\sum_{n=0}^{\infty} |a_n| r^n < 1$ for $0 \leq r < \frac{1}{3}$. This inequality known as Bohr’s inequality is true for $0 \leq r < \frac{1}{3}$ and the constant $\frac{1}{3}$ cannot be improved.

Originally, this theorem was proved for $0 \leq r < \frac{1}{6}$ but soon improved to the sharp result. In Bohr’s paper [20] his own proof was published as well as a proof by Wiener based on function theory methods. Later, S. Sidon gave a different proof (see [22]).

Recently, several papers were published, generalizing Bohr’s theorem to functions of $n$ complex variables (see [8], [21], [1]). Using the standard multi-index notations $\alpha := (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = \alpha_1 + ... + \alpha_n$, $z := (z_1, ..., z_n)$, $z_j \in \mathbb{C}$, $z_\alpha := z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n}$, it is shown in [21] that if a power series $\sum_{\alpha} c_\alpha z_\alpha$ has a modulus less than 1 in the unit polydisc $\{(z_1, ..., z_n) : \max_{1 \leq j \leq n} |z_j| < 1\}$, then the sum of the moduli of the terms is less than 1 in the polydisc of radius $\frac{1}{\sqrt[n]{3}}$.

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In [15], the result shows the possibility to obtain a Bohr type theorem for monogenic functions in the ball in the Euclidean space $\mathbb{R}^3$ with the additionally condition $f(0) = 0$. It is shown that for $r < 0.047$, the inequality is satisfied. The main purpose of this paper is to check if this theorem can be extended to all monogenic functions with $|f(x)| < 1$ in $B_1(0)$.

Having in mind the analogy to the one-dimensional complex function theory we want to know if the result can be proved for a ball in the Euclidean space and not for a polydisc. It is not the goal here to find a sharp estimate for the most general class of functions.

2 Preliminaries

Let $\{e_0, e_1, e_2, e_3\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$. The vector $e_0$ is the scalar unit while the generalized imaginary units $e_1, e_2, e_3$ satisfy the following multiplication rules

$$
e_i e_j + e_j e_i = -2 \delta_{i,j} e_0, \quad i, j = 1, 2, 3$$

$$e_0 e_i = e_i e_0 = e_i, \quad i = 0, 1, 2, 3.$$

This non-commutative product generates the algebra of real quaternions denoted by $\mathbb{H}$. The real vector space $\mathbb{R}^4$ will be embedded in $\mathbb{H}$ by identifying $a := (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$ with the element

$$a = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{H},$$

where $a_i$ ($i = 0, 1, 2, 3$) are real numbers. Remark that $e_0 = (1, 0, 0, 0)^T$ is the multiplicative unit element of $\mathbb{H}$ and by identifying $e_0$ with 1, it will therefore neglected in the following notation.

The real number $\text{Sc} a := a_0$ is called the scalar part of $a$ and $\text{Vec} a := a_1 e_1 + a_2 e_2 + a_3 e_3$ is the vector part of $a$. Analogously to the complex case, the conjugate of $a$ is the quaternion $\overline{a} := a_0 - a_1 e_1 - a_2 e_2 - a_3 e_3$. The norm of $a$ is given by $|a| = \sqrt{(a_0^2 + a_1^2 + a_2^2 + a_3^2)}$ and coincides with the corresponding Euclidean norm of $a$, as a vector in $\mathbb{R}^4$. Considering the subset

$$\mathcal{A} := \text{span}_{\mathbb{R}} \{1, e_1, e_2\}$$

of $\mathbb{H}$, the real vector space $\mathbb{R}^3$ can be embedded in $\mathcal{A}$ by the identification of each element $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ with the reduced quaternion

$$x = x_0 + x_1 e_1 + x_2 e_2 \in \mathcal{A}.$$

As a consequence, we will often use the same symbol $x$ to represent a point in $\mathbb{R}^3$ as well as to represent the corresponding reduced quaternion. Note that the set $\mathcal{A}$ is only a real vector space but not a sub-algebra of $\mathbb{H}$.

Let us consider an open set $\Omega \subset \mathbb{R}^3$ with a piecewise smooth boundary. An $\mathbb{H}$-valued function is a mapping $f : \Omega \rightarrow \mathbb{H}$ such that

$$f(x) = f_0(x) + f_1(x)e_1 + f_2(x)e_2 + f_3(x)e_3,$$

where the coordinates $f_i$ are real-valued functions defined in $\Omega$. For continuously real-differentiable functions $f : \Omega \rightarrow \mathbb{H}$, the operator

$$D = \partial_{x_0} + e_1 \partial_{x_1} + e_2 \partial_{x_2}$$

(1)
is called the generalized Cauchy-Riemann operator. We define the conjugate
generalized Cauchy-Riemann operator by
\[
\mathcal{D} = \partial_{x_0} - e_1 \partial_{x_1} - e_2 \partial_{x_2}.
\]  
(2)
A function \(f : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{H}\) is called left (resp. right) monogenic in \(\Omega\) if
\[Df = 0 \quad \text{in} \quad \Omega \quad \text{(resp.}, \quad fD = 0 \quad \text{in} \quad \Omega).\]
From now on we only use left monogenic functions. For simplicity, we will
call them monogenic. The generalized Cauchy-Riemann operator \(D\) and its
conjugate \(\mathcal{D}\) factorize the Laplace operator in \(\mathbb{R}^3\). In fact, it holds
\[\Delta_3 = D\mathcal{D} = \mathcal{D}D\]
and implies that any monogenic function is also a harmonic function.

From now on, we will consider the following notations: \(B := B_1(0)\) is the unit
ball in \(\mathbb{R}^3\) centered at the origin, \(S = \partial B\) its boundary and \(d\sigma\) is the Lebesgue
measure on \(S\). In what follows, we will denote by \(L_2(S; A; F)\) (resp. \(L_2(B; A; F)\))
the \(F\)-linear Hilbert space of square integrable functions on \(S\) (resp. \(B\)) with
values in \(X\) ( \(X = \mathbb{R}\) or \(A\) or \(\mathbb{H}\)), where \(F = \mathbb{H}\) or \(\mathbb{R}\). For any \(f, g \in L_2(S; A; \mathbb{R})\)
the real-valued inner product is given by
\[\langle f, g \rangle_{L_2(S)} = \int_S \mathbf{Sc}(g)d\sigma.\]  
(3)
Each homogeneous harmonic polynomial \(P_n\) of order \(n\) can be written in
spherical coordinates as
\[P_n(x) = r^n P_n(\omega), \quad \omega \in S,\]
its restriction, \(P_n(\omega)\), to the boundary of the unit ball is called spherical harmonic
of degree \(n\). From \(4\), it is clear that a homogeneous polynomial is determined by its restriction to \(S\). Denoting by \(\mathcal{H}_n(S)\) the space of real-valued spherical harmonics of degree \(n\) in \(S\), it is well-known (see \[3\] and \[10\]) that
\[\dim \mathcal{H}_n(S) = 2n + 1.\]
It is also known (see \[3\] and \[10\]) that if \(n \neq m\), the spaces \(\mathcal{H}_n(S)\) and \(\mathcal{H}_m(S)\)
are orthogonal in \(L_2(S; \mathbb{R}; \mathbb{R})\).

Homogeneous monogenic polynomial of degree \(n\) will be denoted in general
by \(H_n\). In an analogously way to the spherical harmonics, the restriction of \(H_n\)
to the boundary of the unit ball is called spherical monogenic of degree \(n\). We
denote by \(\mathcal{M}_n(\mathbb{H}; F)\) the subspace of \(L_2(B; \mathbb{H}; F) \cap \ker D(B)\) of all homogeneous
monogenic polynomials of degree \(n\). Sudbery proved in \[17\] that the dimension
of \(\mathcal{M}_n(\mathbb{H}; \mathbb{H})\) is \(n + 1\). In \[5\], it is proved that the dimension of \(\mathcal{M}_n(\mathbb{H}; \mathbb{R})\) is
\[4n + 4.\]
Consider, for each \(n \in \mathbb{N}_0\), a basis \(\{H_n^{\nu} : \nu = 1, \ldots, \dim \mathcal{M}_n(\mathbb{H}; F)\}\) of
\(\mathcal{M}_n(\mathbb{H}; F)\), \(F = \mathbb{H}\) or \(F = \mathbb{R}\). Taking into account that the coordinates of \(H_n^{\nu}\)
are harmonic, for arbitrary \(n, k = 0, 1, \ldots,\) we have
\[\langle H_n^{\nu}, H_k^{\mu} \rangle_{L_2(B; \mathbb{H}; F)} = \frac{1}{n + k + 3} \langle H_n^{\nu}, H_k^{\mu} \rangle_{L_2(S; \mathbb{H}; F)},\]  
(5)
3 Homogeneous Monogenic Polynomials

Based on the Fueter variables \( z_1 = x_1 - e_1 x_0 \) and \( z_2 = x_2 - e_2 x_0 \), several systems of homogeneous monogenic polynomials are constructed and used for different purposes (see, e.g., [4, 9, 7, 10, 12, 17]). Following [12], being \( \gamma = (\gamma_1, \gamma_2) \) a multi-index with \( \gamma_1 + \gamma_2 = n \), the generalized powers (or also Fueter polynomials) of degree \( n \) are defined by

\[
z_1^{\gamma_1} \times z_2^{\gamma_2} = \frac{1}{n!} \sum_{\pi(i_1, \ldots, i_n)} z_{i_1} \cdots z_{i_n},
\]

where the sum is taken over all permutations \( \pi(i_1, \ldots, i_n) \) of \((1, \ldots, 1, 2, \ldots, 2)\).

The general form of the Taylor series of a monogenic function \( f : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{H} \) in the neighborhood of the origin (see, e.g., [4, 12]) is given by

\[
f = \sum_{n=0}^{\infty} \sum_{|\gamma|=n} \left( z_1^{\gamma_1} \times z_2^{\gamma_2} \right) c_\gamma, \tag{6}
\]

where \( c_\gamma = \frac{1}{\gamma_1! \gamma_2!} \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} f(x) \bigg|_{x=0} \in \mathbb{H} \) are the Taylor coefficients.

In order to prove a collection of inequalities related to Bohr’s inequality, we need also the Fourier expansion of monogenic functions.

In ([5] and [6]) \( \mathbb{R} \)-linear and \( \mathbb{H} \)-linear complete orthonormal systems of \( \mathbb{H} \)-valued homogeneous monogenic polynomials in the unit ball of \( \mathbb{R}^3 \) are constructed. The main idea of these constructions is based on the factorization of the Laplace operator. We take a system of real-valued homogeneous harmonic polynomials and apply the \( \overline{D} \) operator to get systems of \( \mathbb{H} \)-valued homogeneous monogenic polynomials. To be precise, we introduce the spherical coordinates,

\[
x_0 = r \cos \theta, \ x_1 = r \sin \theta \cos \varphi, \ x_2 = r \sin \theta \sin \varphi,
\]

where \( 0 < r < \infty, 0 < \theta \leq \pi, 0 < \varphi \leq 2\pi \). Each point \( x = (x_0, x_1, x_2) \in \mathbb{R}^3 \setminus \{0\} \) admits a unique representation \( x = rw \), where for each \( i = 0, 1, 2 \) \( w_i = \frac{x_i}{r} \) and \( |w| = 1 \). Now, we apply for each \( n \in \mathbb{N}_0 \), the operator \( \frac{1}{2} \overline{D} \) to the homogeneous harmonic polynomials,

\[
\{ r^{n+1} U_{n+1}^0, r^{n+1} U_{n+1}^m, r^{n+1} V_{n+1}^m, m = 1, \ldots, n + 1 \} \}_{n \in \mathbb{N}_0} \tag{7}
\]

formed by the extensions in the ball of the spherical harmonics (considered, e.g., in [18]),

\[
U_{n+1}^0(\theta, \varphi) = P_{n+1}(\cos \theta), \quad U_{n+1}^m(\theta, \varphi) = P_{n+1}(\cos \theta) \cos m \varphi, \quad V_{n+1}^m(\theta, \varphi) = P_{n+1}(\cos \theta) \sin m \varphi, m = 1, \ldots, n + 1. \tag{8}
\]
Here, \( P_{n+1} \) stands for the Legendre polynomial of degree \( n + 1 \), given by

\[
\begin{cases}
P_{n+1}(t) = \sum_{k=0}^{[s]} a_{n+1,k} t^{n+1-2k} \\
P_0(t) = 1, \quad t \in (-1, 1),
\end{cases}
\]

with

\[
a_{n+1,k} = (-1)^k \frac{1}{2^{n+1}} \frac{(2n + 2 - 2k)!}{k!(n + 1 - k)!(n + 1 - 2k)!},
\]

where \([s]\) denotes the integer part of \( s \in \mathbb{R} \). Also, we stipulate this sum to be zero whenever the upper index is less than the lower one.

The functions \( P_{n+1}^m \) are called the associated Legendre functions,

\[
P_{n+1}^m(t) := (1 - t^2)^{m/2} \frac{d^m}{dt^m} P_{n+1}(t), \quad m = 1, \ldots, n + 1.
\]  

(9)

We remark that if \( m = 0 \), the corresponding associated Legendre function \( P_{n+1}^0 \) coincides with the Legendre polynomial \( P_{n+1} \).

Notice that the Legendre polynomials together with the associated Legendre functions satisfy several recurrence formulae. We point out only some of them, which will be used several times in the next section. Following [2], Legendre polynomials and associated Legendre functions are solutions of a second order differential equation, called \( \text{Legendre differential equation} \), given by

\[
(1 - t^2)(P_{n+1}^m(t))'' - 2t(P_{n+1}^m(t))' + \left( (n + 1)(n + 2) - m^2 \frac{1}{1 - t^2} \right) P_{n+1}^m(t) = 0,
\]

\( m = 0, \ldots, n + 1 \). They also satisfy the recurrence formula

\[
(1 - t^2)(P_{n+1}^m(t))' = (n + m + 1)P_{n}^m(t) - (n + 1)tP_{n+1}^m(t),
\]

(10)

\( m = 0, \ldots, n + 1 \). An additional and useful identity is given by

\[
P_{n}^m(t) = (2m - 1)!!(1 - t^2)^{m/2},
\]

(11)

\( m = 1, \ldots, n + 1 \).

These functions are mutually orthogonal in \( L_2([-1, 1]) \),

\[
\int_{-1}^{1} P_{n+1}^m(t) P_{k+1}^m(t) \, dt = 0, \quad n \neq k
\]

and their norms are

\[
\int_{-1}^{1} (P_{n+1}^m(t))^2 \, dt = \frac{2}{2n + 3}(n + 1 + m)! (n + 1 - m)!, \quad m = 0, \ldots, n + 1.
\]

(11)

For a detailed study of Legendre polynomials and associated Legendre functions we refer, for example, [2] and [18].
Restricting the functions of the set \([7]\) to the sphere, we obtain the spherical monogenics

\[
X_n^0 := \left. \frac{1}{2^{n+1}} \right|_{r=1} (r^{n+1} U_n^0) \\
X_n^m := \left. \frac{1}{2^{n+1}} \right|_{r=1} (r^{n+1} U_n^m) \\
Y_n^m := \left. \frac{1}{2^{n+1}} \right|_{r=1} (r^{n+1} V_n^m), \quad m = 1, ..., n + 1.
\]

For each \(n \in \mathbb{N}_0\), taking the monogenic extensions of the spherical monogenics into the ball, we obtain the set of homogeneous monogenic polynomials

\[
\{ r^n X_n^0, \ r^n X_n^m, \ r^n Y_n^m : m = 1, ..., n + 1 \}. \quad (13)
\]

We need norm estimates of our functions in terms of its Taylor and Fourier expansion are needed. In this way, we begin now to write the homogeneous monogenic polynomials in Cartesian coordinates. In parts, these results were already obtained in [13] and [14], without proof.

**Lemma 3.1** The homogeneous monogenic polynomials \(r^n X_n^l(l = 0, 1, ..., n + 1)\) in terms of Cartesian coordinates can be written as:

\[
r^n X_n^l(x) = [r^n X_n^l(x)]_0 + [r^n X_n^l(x)]_1 e_1 + [r^n X_n^l(x)]_2 e_2,
\]

where

\[
[r^n X_n^l(x)]_0 = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{n+1,l,k} (n + 1 - 2k - l) x_0^{n-2k-l} r^{2k} \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^j \left( \frac{l}{2j} \right) x_1^{l-2j} x_2^{2j} \\
+ \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \beta_{n+1,l,k} (2k) x_0^{n+2-2k-l} r^{2(k-1)} \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^j \left( \frac{l}{2j} \right) x_1^{l-2j} x_2^{2j}
\]

\[
[r^n X_n^l(x)]_1 = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \beta_{n+1,l,k} (2k) x_0^{n+1-2k-l} r^{2(k-1)} \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^j+1 \left( \frac{l}{2j} \right) x_1^{l-2j+1} x_2^{2j} \\
+ \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \beta_{n+1,l,k} x_0^{n+1-2k-l} r^{2k} \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^j+1 \left( \frac{l}{2j} \right) (l-2j) x_1^{l-2j-1} x_2^{2j}
\]

\[
[r^n X_n^l(x)]_2 = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \beta_{n+1,l,k} (2k) x_0^{n+1-2k-l} r^{2(k-1)} \sum_{j=0}^{\lfloor \frac{j}{2} \rfloor} (-1)^j+1 \left( \frac{l}{2j} \right) x_1^{l-2j} x_2^{2j+1} \\
+ \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \beta_{n+1,l,k} x_0^{n+1-2k-l} r^{2k} \sum_{j=1}^{\lfloor \frac{j}{2} \rfloor} (-1)^j+1 \left( \frac{l}{2j} \right) (2j) x_1^{l-2j} x_2^{2j-1},
\]

being

\[
\beta_{n+1,l,k} = (-1)^k \frac{1}{2^{n+2}} \binom{2n+2-2k}{n+1-k} \binom{n+1-k}{k} (n+1-2k)_{l-1}
\]

and \((n+1-2k)_{l-1}\) stands for the Pochhammer symbol.
Proof. Let us consider the spherical monogenics given by (12), explicitly described in (8). By the definition of the Legendre polynomials we have

\[ P_{n+1}(t) = \frac{d}{dt} \left[ \sum_{k=0}^{n+1} a_{n+1,k} t^{n+1-2k} \right] = \sum_{k=0}^{n+1} a_{n+1,k}(n + 1 - 2k) t^{(n+1-2k)-1}. \]

Now, derivating recursively in order to \( t \) \((l-1)\) times,

\[ \partial_t^l P_{n+1}(t) = P_l^{(l)}(t) = \sum_{k=0}^{(n+1)} a_{n+1,k}(n + 1 - 2k)(n + 1 - 2k - 1) \cdots (n + 1 - 2k - (l-1)) t^{(n+1-2k)-l}. \]

By simplicity, we set

\[ \beta_{n+1,l,k} = 2(n + 1 - 2k)(n + 1 - 2k - 1) \cdots (n + 1 - 2k - (l-1)), \]

so that, finally we get for (9) the expression

\[ P_l^{(l)}(\cos \theta) = \left[ \frac{n+1-i}{(n+1-l)} \right] \sum_{k=0}^{(n+1)} 2\beta_{n+1,l,k} (\sin \theta)^l (\cos \theta)^{n+1-2k-l}. \]

In order to express the set \( \{ X^l_n : l = 0, 1, ..., n+1 \} \) in cartesian coordinates, we consider the coordinate’s relation:

\[
\begin{align*}
\cos \theta &= \frac{x_0}{r} & \cos \varphi &= \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \\
\sin \theta &= \frac{\sqrt{x_1^2 + x_2^2}}{r} & \sin \varphi &= \frac{x_2}{\sqrt{x_1^2 + x_2^2}}.
\end{align*}
\]

Now, using

\[ \cos(m \varphi) = \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^j \left( \frac{l}{2j} \right) (\cos \varphi)^{(l-2j)} (\sin \varphi)^{2j} \]

and substituting in (12) we obtain

\[ r^{n+1}U_{n+1}^l(x) = 2 \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{l}{2} \rfloor} \beta_{n+1,l,k} x_0^{n+1-2k-l} r^{2k} (-1)^j \left( \frac{l}{2j} \right) x_1^{l-2j} x_2^{2j}. \]

Applying the hypercomplex derivative \( \frac{1}{2}D^l \) to this expression carries our polynomials in Cartesian coordinates, respectively.

Similar results holds for \( r^n Y^m_n \) \( m = 1, ..., n + 1 \). Let us consider now the following function:

**Definition 3.1** Let \( i, j \in \mathbb{N}_0 \). The function \( g_{i,j} \) is given by

\[
g_{i,j} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ have the same parity} \\ 0, & \text{if } i \text{ and } j \text{ have different parity} \end{cases}.
\]
Proposition 3.1 The Taylor coefficients of the homogeneous monogenic polynomials $r^nX^l_n$ ($l = 0, 1, \ldots, n + 1$) are given by

$$[a^l_{2x}]_0 = g_{l, n} g_{a_1, l} g_{a_2, 0} \beta_{n+1, l, \frac{m-1}{2}} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^j \left( \begin{array}{c} l \\ 2j \end{array} \right) \left( \frac{m-1}{2} + j \right)$$

$$[a^l_{2x}]_1 = g_{l-1, n} g_{l-1, a_1} g_{a_2, 0}$$

$$\left[ \begin{array}{c} \frac{m-1}{2} \\ 2p \end{array} \right] \sum_{p=1}^{\lfloor \frac{m-1}{2} \rfloor} \beta_{n+1, l, p} (2p) \left( \frac{p-1}{2} + p \right) \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^{j+1} \left( \begin{array}{c} l \\ 2j \end{array} \right) \left( \frac{m-1}{2} + j \right)$$

$$+ \left[ \begin{array}{c} \frac{m-1}{2} \\ 2p \end{array} \right] \sum_{p=0}^{\lfloor \frac{m-1}{2} \rfloor} \beta_{n+1, l, p} \left( \frac{p-1}{2} + p \right) \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} (-1)^{j+1} \left( \begin{array}{c} l \\ 2j \end{array} \right) (l-2j) \left( \frac{m-1}{2} + j \right)$$

$$[a^l_{2x}]_2 = g_{l-1, n} g_{l-1, a_1} g_{a_2, 1}$$

$$\left[ \begin{array}{c} \frac{m-1}{2} \\ 2p \end{array} \right] \sum_{p=1}^{\lfloor \frac{m-1}{2} \rfloor} \beta_{n+1, l, p} (2p) \left( \frac{p-1}{2} + p \right) \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^{j+1} \left( \begin{array}{c} l \\ 2j \end{array} \right) \left( \frac{m-1}{2} + j \right)$$

$$+ \left[ \begin{array}{c} \frac{m-1}{2} \\ 2p \end{array} \right] \sum_{p=0}^{\lfloor \frac{m-1}{2} \rfloor} \beta_{n+1, l, p} \left( \frac{p-1}{2} + p \right) \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} (-1)^{j+1} \left( \begin{array}{c} l \\ 2j \end{array} \right) (2j) \left( \frac{m-1}{2} + j \right)$$

Proof. The proof follows directly from Lemma [5.1] by applying the partial derivatives $\partial x^i \partial x^j$.

Again, we obtain analogous formulae for the Taylor coefficients of $r^nY^m_n$ ($m = 1, \ldots, n + 1$).

Corollary 3.1 Let $\gamma = (\gamma_1, \gamma_2)$ be a multi-index with $|\gamma| = n$. The Taylor coefficients $a^n_{\gamma_1}, a^n_{\gamma_2}$ and $b^n_{\gamma_2}$ of the homogeneous monogenic polynomials $r^nX^0_n$, $r^nX^m_n$ and $r^nY^m_n$ satisfy the following inequalities:

$$|a^n_{\gamma_1}| \leq \frac{1}{2^n} (n+1)! \sqrt{\frac{\pi(n+1)}{2n+3}}$$

$$|a^n_{\gamma_2}| \leq \frac{1}{2^n} \frac{(n+1)! \sqrt{\pi(n+1)(n+1+m)!}}{2(2n+3)(n+1-m)!}$$

$$|b^n_{\gamma_2}| \leq \frac{1}{2^n} \frac{(n+1)! \sqrt{\pi(n+1)(n+1+m)!}}{2(2n+3)(n+1-m)!}$$

$m = 1, \ldots, n + 1$.

Proof. Let $B_r(x) \subset \mathbb{R}^3$ be a ball of radius $r$ centered at $x$. From [11] we know the Cauchy integral formula for the ball $B_1(x)$,

$$f(x) = \frac{1}{4\pi} \int_S \frac{x - y}{|x - y|^3} n(y)f(y)dS_y,$$
where \( n \) stands for the outward pointing normal unit vector to \( S \) at \( y \). For simplicity we just present the proof for the homogeneous monogenic polynomials \( r^n X_n^m \) \((m = 1, \ldots, n + 1)\). Applying the Cauchy integral formula to these polynomials in the ball \( B \) and taking partial derivatives with respect to \( x_1 \) and \( x_2 \), we get

\[
a_{x_1}^{m,*} = \frac{1}{\pi^2} \int_B \frac{1}{4\pi} \frac{1}{|x-y|^2} \left| \frac{x-y}{|x-y|^2} \right|_{x=0} n(y) X_n^m(y) dS_y
\]

taking the modulus and applying the Schwarz inequality we finally obtain

\[
|a_{x_1}^{m,*}| \leq \frac{1}{n!} \frac{(n+1)!}{(n+1-m)!} \sqrt{\frac{\pi}{2}} 
\]

where \( a_{x_1}^{m,*} \) denotes the Taylor coefficients associated to the functions \( X_n^m \). The previous inequality is based on [5] where the following relation is proved

\[
\| X_n^m \|_{L^2(S)} = \| Y_n^m \|_{L^2(S)} = \frac{\sqrt{\frac{\pi}{2}}}{|y|^n+2} \] \( m = 1, \ldots, n + 1 \)

and on the paper [26] where it was obtained that

\[
\left\| \frac{\partial^{\gamma_1} \partial^{\gamma_2}}{|x-y|^3} \right\|_{L^2(S)} \leq \frac{(n+1)!}{|y|^{n+2}}.
\]

Using the relation [5] we get the Taylor coefficients associated to the homogeneous monogenic polynomials \( r^n X_n^m \). The case \( m = 0 \) is proved analogously. \( \blacksquare \)

Besides norm estimates we also need pointwise estimates of our basis polynomials.

**Proposition 3.2** For \( n \in \mathbb{N} \) the homogeneous monogenic polynomials satisfy the following inequalities:

\[
|r^n X_n^0(x)| \leq r^n(n+1)2^n \sqrt{\frac{\pi(n+1)}{2n+3}}
\]

\[
|r^n X_n^m(x)| \leq r^n(n+1)2^n \sqrt{\frac{\pi}{2}} \frac{(n+1)(n+1+m)!}{(n+1-m)!} \]

\[
|r^n Y_n^m(x)| \leq r^n(n+1)2^n \sqrt{\frac{\pi}{2}} \frac{(n+1)(n+1+m)!}{(n+1-m)!} \]

\( m = 1, \ldots, n + 1 \).

**Proof.** Again, we prove only the case of the polynomials \( r^n X_n^m \) \((m = 1, \ldots, n + 1)\), the proof for \( r^n Y_n^m \) being similar. We write these polynomials as a Taylor expansion \( 6 \)

\[ r^n X_n^m(x) = \sum_{|\gamma| = n} (z_1^{\gamma_1} \times z_2^{\gamma_2}) a_{x_1}^{m,*} \]

Consequently, modulus of \( r^n X_n^m \) leads to

\[
|r^n X_n^m(x)| \leq r^n(n+1)! \sqrt{\frac{\pi}{2}} \frac{(n+1)(n+1+m)!}{(n+1-m)!} 2^n \sqrt{\frac{\pi}{2}} \frac{(n+1)(n+1+m)!}{(n+1-m)!} n!.
\]
Having in mind \([12]\) we have
\[
\left| z_1^\gamma_1 \times z_2^\gamma_2 \right| \leq r^n
\]
for every multi-index \(\gamma = (\gamma_1, \gamma_2)\) with \(|\gamma| = n\).

For future use in this paper we will need estimates for the real part of the spherical monogenics described in \([12]\).

**Theorem 3.1**
Given a fixed \(n \in \mathbb{N}_0\), the spherical harmonics
\[
\{ \text{Sc}(X^0_n), \text{Sc}(X^m_n), \text{Sc}(Y^m_n) : m = 1, ..., n \}
\]
are orthogonal to each other with respect to the inner product \([19]\).

**Proposition 3.3**
Given a fixed \(n \in \mathbb{N}_0\), the moduli of the spherical harmonics \(\text{Sc}(X^0_n), \text{Sc}(X^m_n)\) and \(\text{Sc}(Y^m_n)\) satisfy the following inequalities
\[
|\text{Sc}(X^l_n)| \leq \frac{1}{2} \frac{(n + 1 + l)!}{n!}, \quad l = 0, ..., n
\]
\[
|\text{Sc}(Y^m_n)| \leq \frac{1}{2} \frac{(n + 1 + m)!}{n!}, \quad m = 1, ..., n.
\]

**Proof.**
According to the results from \([5]\), the real parts of the spherical monogenics are given by
\[
\begin{align*}
\text{Sc}(X^0_n) &= A^{0,n}(\theta) \\
\text{Sc}(X^m_n) &= A^{m,n}(\theta) \cos(m\varphi) \\
\text{Sc}(Y^m_n) &= A^{m,n}(\theta) \sin(m\varphi),
\end{align*}
\]
where
\[
A^{l,n}(\theta) = \frac{1}{2} \left( \sin^2 \theta \frac{d}{dt} \left[ P^l_{n+1}(t) \right]_{t = \cos \theta} + (n + 1) \cos \theta P^l_{n+1}(\cos \theta) \right), \quad l = 0, ..., n.
\]

For simplicity sake we only present the proof for the spherical harmonics \(\text{Sc}(X^m_n)\) \((m = 1, ..., n + 1)\). Making the change of variable \(t = \cos \theta\) and using the recurrence formula \([10]\), it follows that
\[
\text{Sc}(X^m_n) = \frac{1}{2} (n + 1 + m) P^m_n(t).
\]

Applying the modulus in the previous expression and using the inequality proved in \([19]\)
\[
|P^m_n(t)| \leq \frac{(n + m)!}{n!},
\]
for \(-1 \leq t \leq 1\) and \(n \geq m\), we finally obtain the estimate
\[
|\text{Sc}(X^m_n)| \leq \frac{1}{2} \frac{(n + 1 + m)!}{n!}.
\]

Some of the basis polynomials described in \([13]\) play a special role. Applying results from \([5]\, Proposition 3.4.3) we get:
Proposition 3.4 For \( n \in \mathbb{N}_0 \), the spherical monogenics \( X_{n+1}^n \) and \( Y_{n+1}^n \) are given by
\[
X_{n+1}^n = -C_{n+1,n}^n \cos n\varphi e_1 + C_{n+1,n}^n \sin n\varphi e_2
\]
\[
Y_{n+1}^n = -C_{n+1,n}^n \sin n\varphi e_1 - C_{n+1,n}^n \cos n\varphi e_2
\]
where
\[
C_{n+1,n}^n = \frac{n+1}{2} \frac{1}{\sin \theta} P_{n+1}^n(\cos \theta),
\]
and their monogenic extensions into the ball belong to \( \text{ker} \mathcal{D}(B) \cap \text{ker} \mathcal{D}(B) \).

Remark 3.1 The spherical monogenics \( X_{n+1}^n \) and \( Y_{n+1}^n \) are monogenic constants, i.e., monogenic functions which depend only on \( x_1 \) and \( x_2 \). Moreover, they play the role of constants with respect to the hypercomplex differentiation \( \frac{1}{2} \mathcal{D} \).

Proposition 3.5 Given a fixed \( n \in \mathbb{N}_0 \), the spherical harmonics \( \text{Sc}(X_{n+1}^n e_1) \) and \( \text{Sc}(Y_{n+1}^n e_1) \) are orthogonal to each other with respect to the inner product \( \langle \rangle \) and their moduli satisfy the following inequalities
\[
|\text{Sc}\{X_{n+1}^n e_1\}| \leq \frac{1}{2} \frac{(n+1)(2n+1)!}{2^n n!}
\]
\[
|\text{Sc}\{Y_{n+1}^n e_1\}| \leq \frac{1}{2} \frac{(n+1)(2n+1)!}{2^n n!}.
\]

Proof. Again, we present the proof for the spherical harmonics \( \text{Sc}\{X_{n+1}^n e_1\} \), the one for \( \text{Sc}\{Y_{n+1}^n e_1\} \) being similar. According to (15), the real part of the spherical harmonic \( X_{n+1}^n e_1 \) is given by
\[
\text{Sc}\{X_{n+1}^n e_1\} = C_{n+1,n}^n \cos n\varphi.
\]
Making the change of variable \( t = \cos \theta \) and applying the modulus in the previous expression, we get
\[
|\text{Sc}\{X_{n+1}^n e_1\}| = \frac{n+1}{2} \left| \frac{1}{\sqrt{1-t^2}} P_{n+1}^n(t) \right|
\]
and due to the recurrence formula (11) we finally obtain
\[
|\text{Sc}\{X_{n+1}^n e_1\}| = \frac{n+1}{2} \left| \frac{1}{\sqrt{1-t^2}} (2n+1)!!(1-t^2)^{\frac{n+1}{2}} \right| \leq \frac{1}{2} (n+1)(2n+1)!!.
\]

Proposition 3.6 Given a fixed \( n \in \mathbb{N}_0 \), the norms of the spherical harmonics \( \text{Sc}(X_n^0) \), \( \text{Sc}(X_n^m) \) and \( \text{Sc}(Y_n^m) \) are given by
\[
\|\text{Sc}(X_n^0)\|_{L_2(S)} = (n+1) \sqrt{\frac{\pi}{2n+1}}
\]
and
\[
\|\text{Sc}(X_n^m)\|_{L_2(S)} = \|\text{Sc}(Y_n^m)\|_{L_2(S)} = \sqrt{\frac{\pi}{2}} \left( \frac{(n+1+m)(n+1+m)!}{(2n+1)(n-m)!} \right), \quad m = 1, \ldots, n.
\]
Proposition 3.7 Given a fixed \( n \in \mathbb{N}_0 \), the spherical harmonics \( \mathbf{Sc}(X_n^{n+1}e_1) \) and \( \mathbf{Sc}(Y_n^{n+1}e_1) \) are orthogonal to each other with respect to the inner product \( < \), and their norms are given by
\[
|\mathbf{Sc}(X_n^{n+1}e_1)||_{L^2(S)} = |\mathbf{Sc}(Y_n^{n+1}e_1)||_{L^2(S)} = \frac{1}{2} \sqrt{\pi(n+1)(2n+2)!}.
\]

4 Bohr’s Theorem

We will denote by \( X_n^{0,*}, \ldots \) the normalized basis functions in \( L_2(S; \mathbb{H}; \mathbb{H}) \).

Theorem 4.1 (see \textsuperscript{[5]}) Let \( M_n(\mathbb{R}^3; A) \) be the space of \( A \)-valued homogeneous monogenic polynomials of degree \( n \) in \( \mathbb{R}^3 \). For each \( n \), the set of \( 2n+3 \) homogeneous monogenic polynomials
\[
\{ \sqrt{2n+3} \alpha^n X_n^{0,*}, \sqrt{2n+3} \alpha^n X_n^{m,*}, \sqrt{2n+3} \alpha^n Y_n^{m,*}, \ m = 1, \ldots, n+1 \}
\]  
forms an orthonormal basis in \( M_n(\mathbb{R}^3; A) \).

In \textsuperscript{15}, a first version of a quaternionic Bohr’s theorem was considered, therein we restricted ourselves to the case of functions with \( f(0) = 0 \) and we obtained an estimate in terms of a radius of \( r = 0.047 \).

Here, we extend our result to all monogenic functions with \( |f(x)| < 1 \) in \( B \), estimating a value for the radius.

Theorem 4.2 Let \( f \) be a square integrable \( A \)-valued monogenic function with \( |f(x)| < 1 \) in \( B \), \( \mathbf{Sc}\{f\} \) be positive and let
\[
\sum_{n=0}^{\infty} \sqrt{2n+3} \ r^n \left( X_n^{0,*} \alpha_n^0 + \sum_{m=1}^{n+1} [X_n^{m,*} \alpha_n^m + Y_n^{m,*} \beta_n^m] \right)
\]
be its Fourier expansion. Then
\[
\sum_{n=0}^{\infty} \sqrt{2n+3} \ r^n \left| \left( X_n^{0,*} \alpha_n^0 + \sum_{m=1}^{n+1} [X_n^{m,*} \alpha_n^m + Y_n^{m,*} \beta_n^m] \right) \right| < 1
\]
holds in the ball of radius \( r \), with \( 0 \leq r < 0.05 \).

Proof. According to Theorem 4.1 a monogenic \( L_2 \)-function \( f : \Omega \subset \mathbb{R}^3 \rightarrow A \) can be written as Fourier series
\[
f = \sum_{n=0}^{\infty} \sqrt{2n+3} \ r^n \left( X_n^{0,*} \alpha_n^0 + \sum_{m=1}^{n+1} [X_n^{m,*} \alpha_n^m + Y_n^{m,*} \beta_n^m] \right),
\]
where \( \alpha_n^0, \alpha_n^m \) and \( \beta_n^m (m=1, \ldots, n+1) \) are the associated Fourier coefficients.

Let us denote by \( \mathbf{Sc}\{f\} \) the real part of \( f \). Then,
\[
\mathbf{Sc}\{f\} = \frac{f + \overline{f}}{2} = \sum_{n=0}^{\infty} \sqrt{2n+3} \ r^n \left( \mathbf{Sc}\{X_n^{0,*}\} \alpha_n^0 + \sum_{m=1}^{n} [\mathbf{Sc}\{X_n^{m,*}\} \alpha_n^m + \mathbf{Sc}\{Y_n^{m,*}\} \beta_n^m] \right).
\]
Due to Remark 2.1 we split the function $f$ in the following way
\[
f = \sqrt{3} \alpha_0^0 X_0^0 + \sqrt{3} \alpha_0^1 X_0^1 + \sqrt{3} \beta_0^1 Y_0^1 + \sum_{n=1}^{\infty} \sqrt{2n+3} r^n \left\{ X_n^0 \alpha_0^0 + \sum_{m=1}^{n} [X_n^m \alpha_m^0 + Y_n^m \beta_m^1] \right\} + \sum_{n=1}^{\infty} \sqrt{2n+3} r^n [X_n^{n+1} \alpha_{n+1}^0 + Y_n^{n+1} \beta_{n+1}^1].
\]

Based in this splitting, we introduce
\[
f_1 = \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 + \sum_{n=1}^{\infty} \sqrt{2n+3} r^n \left\{ X_n^0 \alpha_0^0 + \sum_{m=1}^{n} [X_n^m \alpha_m^0 + Y_n^m \beta_m^1] \right\}
\]
\[
f_2 = -\frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 e_1 - \frac{1}{2} \sqrt{\frac{3}{\pi}} \beta_0^1 e_2.
\]

Let us assume that there exists $0 < \delta < 1$ such that $|f_1| < \delta$ and $|f_2| < 1 - \delta$. In this way, the modulus of $f$ is preserved. We start now to study the function $f_1$. The main idea is to compare each Fourier coefficient with the coefficient $\alpha_0^0$. In fact, multiplying both sides of the expression
\[
\text{Sc}\{\delta - f_1\} = \delta - \text{Sc}\{f_1\}
\]
by each real part of the homogeneous monogenic polynomials described in (13) and integrating over the sphere, we get these relations. For simplicity we just present the idea applied to the coefficients of $X_k^0$, i.e., $\alpha_0^0$. Multiplying both sides of the expression (17) by $\text{Sc}\{X_k^0\}$ and integrating, we obtain
\[
-\sqrt{2k+3} \alpha_0^0 = \int_S \text{Sc}\{\delta - f_1\} \text{Sc}\{X_k^0\} d\sigma
\]
with $0 < \delta < 1$. Now, applying the modulus we obtain finally
\[
|\alpha_0^0| \sqrt{2k+3} \leq 2 \sqrt{\pi} \frac{\|\text{Sc}\{X_k^0\}\|_{L^2(S)}}{\|\text{Sc}\{X_k^0\}\|_{L^2(S)}} \left( \delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right).
\]

In an analogous way, we can state the following results:
\[
|\alpha_k^p| \sqrt{2k+3} \leq 2 \sqrt{\pi} \frac{\|\text{Sc}\{X_k^p\}\|_{L^2(S)}}{\|\text{Sc}\{X_k^p\}\|_{L^2(S)}} \left( \delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right), \quad p = 1, \ldots, k.
\]
Finally, the previous expressions can be rewritten

\[
\frac{|\mathcal{S}_c(X_k^p)|}{\|\mathcal{S}_c(X_k^p)\|_{L_2(S)}^2} \leq \frac{1}{2\pi} \frac{(2k+1)}{k+1}
\]

\[
\frac{|\mathcal{S}_c(Y_k^p)|}{\|\mathcal{S}_c(Y_k^p)\|_{L_2(S)}^2} \leq \frac{1}{\pi} \frac{(2k+1)(k-p)!}{(k+1+p)k!}, \quad p = 1, \ldots, k.
\]

Finally, the previous expressions can be rewritten

\[
|\alpha_k^p| \sqrt{2k+3} \leq \frac{1}{\sqrt{\pi}} \frac{(2k+1)}{k+1} \left( \delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right)
\]

\[
|\alpha_k^p| \sqrt{2k+3} \leq \frac{2}{\sqrt{\pi}} \frac{(2k+1)(k-p)!}{(k+1+p)k!} \left( \delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right)
\]

\[
|\beta_k^p| \sqrt{2k+3} \leq \frac{2}{\sqrt{\pi}} \frac{(2k+1)(k-p)!}{(k+1+p)k!} \left( \delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right).
\]

Consequently, we can state the following inequalities:

\[
|X_{k,n}^0||\alpha_{k,0}^0| \sqrt{2k+3} \leq \frac{1}{\sqrt{\pi}} (2r)^k (2k+1) \left( \delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right)
\]

\[
\sum_{p=1}^{k} |X_{k,n}^p||\alpha_{k,0}^p| \sqrt{2k+3} \leq \frac{2}{\sqrt{\pi}} (2r)^k (2k+1) \left( \delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right)
\]

\[
\sum_{p=1}^{k} |Y_{k,n}^p||\beta_{k,0}^p| \sqrt{2k+3} \leq \frac{2}{\sqrt{\pi}} (2r)^k (2k+1) \left( \delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right).
\]

Now, using the previous inequalities we end with

\[
|f_1| \leq \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 + \sum_{n=1}^{\infty} \sqrt{2n+3} \, r^n \left[ |X_{n,n}^0||\alpha_{n,0}^0| + \sum_{m=1}^{n} (|X_{n,m}^m||\alpha_{n,m}^m| + |Y_{n,m}^m||\beta_{n,m}^m|) \right]
\]

\[
\leq \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 + \frac{5}{\sqrt{\pi}} \left( \delta - \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^0 \right) \sum_{n=1}^{\infty} (2r)^n (2n+1).
\]

Thus, we have that

\[
|f_1| \leq \delta \implies \frac{5}{\sqrt{\pi}} \sum_{n=1}^{\infty} (2r)^n (2n+1) \leq 1,
\]

and, the last series is convergent for $r < 0.05$. In the same way, we can study the function $f_2$. Let

\[
f_2 = \sqrt{3} \alpha_0^1 + X_0^1 + \sqrt{3} \beta_0^1 Y_0^1 + \sum_{n=1}^{\infty} \sqrt{2n+3} \, r^n \left[ X_{n+1}^n \alpha_{n+1}^n + Y_{n+1}^n \beta_{n+1}^n \right].
\]
Multiplying $f_2$ in the right side by $e_1$ we get
\[ \tilde{f}_2 := f_2 e_1 = \sqrt{\frac{3}{\alpha_0^3}} (r_0 X_0^{1,*} e_1) + \sqrt{\frac{3}{\beta_0^3}} (r_0 Y_0^{1,*} e_1) + \sum_{n=1}^{\infty} \sqrt{\frac{2}{2n+3}} r^n \left[ (X_n^{n+1,*} e_1) \alpha_n^{n+1} + (Y_n^{n+1,*} e_1) \beta_n^{n+1} \right]. \]

We want to apply the same idea previously used for $f_1$. Taking in consideration that $f$ is an $\mathcal{A}$-valued function, we obtain an estimate for the coefficient $\alpha_0^3$. In a similar way, we obtain an estimate for $\beta_0^3$ if we multiply $f_2$ at right by $e_2$. This leads to the inequalities
\[
|\alpha_k^{k+1}| \sqrt{2k+3} \leq 2 \sqrt{\frac{\pi}{3}} \left\| \frac{\text{Sc} \{X_k^{k+1} e_1\}}{\text{Sc} \{X_k^{k+1} e_1\}} \right\|_2 \left(1 - \frac{\delta}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 \right),
\]
\[
|\beta_k^{k+1}| \sqrt{2k+3} \leq 2 \sqrt{\frac{\pi}{3}} \left\| \frac{\text{Sc} \{Y_k^{k+1} e_1\}}{\text{Sc} \{Y_k^{k+1} e_1\}} \right\|_2 \left(1 - \frac{\delta}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 \right).
\]

being
\[
\frac{|\text{Sc} \{X_k^{k+1} e_1\}|}{\left\| \frac{\text{Sc} \{X_k^{k+1} e_1\}}{\text{Sc} \{X_k^{k+1} e_1\}} \right\|_2} \leq \frac{2}{\pi^2} \frac{1}{2^{n+1}}.
\]

Consequently, we have proved:
\[
|X_k^{k+1,*} e_1| \alpha_k^{k+1} \sqrt{2k+3} \leq \frac{2 \sqrt{\frac{\pi}{3}}}{k!} \left(1 - \frac{\delta}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 \right),
\]
\[
|Y_k^{k+1,*} e_1| \beta_k^{k+1} \sqrt{2k+3} \leq \frac{2 \sqrt{\frac{\pi}{3}}}{k!} \left(1 - \frac{\delta}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 \right).
\]

With the previous inequalities we get
\[
|\tilde{f}_2| = |f_2| \leq \frac{1}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 + \frac{4}{\sqrt{3} \pi} \left(1 - \frac{\delta}{2} \sqrt{\frac{3}{\pi}} \alpha_0^1 \right) \sum_{n=1}^{\infty} \frac{r^n}{n!}.
\]

Finally, we end with
\[
|f_2| \leq 1 - \delta \implies \frac{4}{\sqrt{3} \pi} \sum_{n=1}^{\infty} \frac{r^n}{n!} \leq 1,
\]

and, the last series is convergent for $r < 0.56$. Finally,
\[
\sum_{n=0}^{\infty} \sqrt{\frac{2}{2n+3}} r^n \left[ \left( X_n^{0,*} \alpha_n^0 + \sum_{m=1}^{n+1} X_n^{m,*} \alpha_n^m + Y_n^{m,*} \beta_n^m \right) \right] < 1
\]
converges for $0 \leq r < 0.05$. ■
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