Entanglement timescale and mixedness in non-Hermitian quantum systems

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We discuss the short-time perturbative expansion of the linear entropy for finite-dimensional quantum systems whose dynamics can be effectively described by a non-Hermitian Hamiltonian. We derive a timescale for the degree of mixedness for an input state undergoing non-Hermitian dynamics and specialize these results in the case of a driven-dissipative two-level system. Next, we derive a timescale for the growth of entanglement for bipartite quantum systems that depends on the effective non-Hermitian Hamiltonian. In the Hermitian limit, this result recovers the perturbative expansion for coherence loss in Hermitian systems, while it provides an entanglement timescale for initial pure and uncorrelated states. To illustrate these findings, we consider the many-body transverse-field XY Hamiltonian coupled to an imaginary all-to-all Ising model. We find that the non-Hermitian Hamiltonian enhances the short-time dynamics of the linear entropy for the considered input states. Overall, each timescale depends on minimal ingredients such as the probe state and the non-Hermitian Hamiltonian of the system, and its evaluation requires low computational cost. Our results find applications to non-Hermitian quantum sensing, quantum thermodynamics of non-Hermitian systems, and \( \mathcal{PT} \)-symmetric quantum field theory.

I. INTRODUCTION

The study of non-Hermitian systems [1, 2] has paved the way for recent developments across the subjects of quantum sensing [3, 4], \( \mathcal{PT} \)-symmetry and exceptional points [5–12], linear response theory [13], quantum many-body systems [14–22], skin effect [23–25], bulk-edge correspondence [26, 27], phase transitions [28, 29], and the quantum boomerang effect in localized systems [30–33], to cite a few.

Recent theoretical achievements discussed the effects of postselection on the dynamics of open quantum systems, thus reconciling the approaches of effective non-Hermitian Hamiltonians and Liouvillian superoperators [34, 35]. In this setting, several works have addressed the role of non-Hermitian features on legitimate quantum mechanical signatures, e.g., quantum coherence and entanglement [36–39]. For example, it has been shown that quantum coherence can be characterized under the framework of multiple quantum coherences [40–42]. In turn, the dynamics of entanglement in non-Hermitian systems has been widely probed with entropic measures, but their evaluation generally involves the full spectral decomposition of the state driven by the non-Hermitian Hamiltonian [43–46]. This can be a challenging computational task, especially for interacting quantum many-body systems.

To overcome this issue, the onset growth of entanglement at earlier times of the dynamics can be addressed through the so-called linear entropy, a useful information-theoretic quantifier that is related to the second-order Rényi entropy and quantum purity [47, 48]. Remarkably, those quantities have been experimentally probed in optical lattices [49–51], and trapped ion setups [52, 53]. In the Hermitian case, it is known that the short-time perturbative expansion of the linear entropy implies a universal timescale for the entanglement dynamics of interacting bipartite systems [54–56]. It is worth mentioning that this timescale is inversely proportional to the fluctuations of the coupling between subsystems [57, 58]. Importantly, this result also assigns a timescale for the decoherence mechanism within the subsystems of such composite quantum systems [59–61]. We also mention the study of the growth of entanglement through a perturbative expansion of the entanglement negativity [62, 63], and also quantum fidelities [64, 65]. To the best of our knowledge, despite these remarkable achievements in the Hermitian setting, deriving an analogous timescale for non-Hermitian systems remains a gap to be filled.

Here we address timescales for the growth of the linear entropy for finite-dimensional quantum systems described by effective non-Hermitian Hamiltonians. The physical system is initialized in a quantum state which can be chosen as either a pure or mixed one, possibly an entangled state or even an uncorrelated one. We investigate the short-time perturbative expansion of the linear entropy for a given input state driven by a general non-Hermitian Hamiltonian. In this setting, up to the second order in time, the onset growth of mixedness of the evolved state is governed by two competing timescales that are intrinsically related to the anti-Hermitian part of the non-Hermitian Hamiltonian. In particular, we specialize these results in the case of a driven non-Hermitian two-level system and discuss the mixedness of a single-qubit state.

Next, focusing on the reduced dynamics of bipartite systems described by non-Hermitian Hamiltonians, we derive the short-time perturbative expansion of the linear entropy for a given evolved marginal state of the composite system. In the Hermitian limit, these re-
sults recover the perturbative expansion for coherence loss in Hermitian systems [59]. In particular, for initial pure and uncorrelated states, we find the lowest order entanglement timescale for quantum systems described by Hermitian Hamiltonians [54, 55]. To illustrate these findings, we consider a paradigmatic many-body non-Hermitian Hamiltonian, and present analytical calculations and numerical simulations to support our theoretical predictions. We verify that, unlike the Hermitian case, the non-Hermitian Hamiltonian is responsible for an enhancement in the short-time dynamics of the linear entropy for the multipartite states that have been considered.

The paper is organized as follows. In Sec. II, we briefly review useful properties regarding the linear entropy. In Sec. III we investigate the short-time perturbative expansion of the linear entropy for finite-dimensional quantum systems whose dynamics are driven by a non-Hermitian Hamiltonian. In Sec. IIIA, we illustrate our findings by means of the two-level system. In Sec. IV we derive a universal entanglement timescale for bipartite quantum systems evolving under the action of a given non-Hermitian Hamiltonian. In Sec. IVB, we specialize these results to the case of two initially uncorrelated subsystems. In addition, Sec. IVB addresses the case of a many-body system with non-Hermitian Hamiltonian describing the transverse-field XY model perturbed by a fully connected Ising Hamiltonian with imaginary exchange coupling. Finally, in Sec. V we summarize our conclusions.

II. LINEAR ENTROPY

In this section, we review the main properties of linear entropy, i.e., a versatile information-theoretic measure that quantifies the degree of mixedness of a given state [66]. Linear entropy has been used to witness multipartite entanglement [67–69]. Let us consider a quantum system with finite-dimensional Hilbert space $\mathcal{H}$, with $d = \dim \mathcal{H}$. The space of quantum states $S \subset \mathcal{H}$ is a convex set of Hermitian, positive semidefinite, trace-one, $d \times d$ matrices, i.e., $S = \{\rho \in \mathcal{H} \mid \rho^\dagger = \rho, \rho \geq 0, \text{Tr}(\rho) = 1\}$. The normalized linear entropy of the quantum state $\rho$ is defined as [70]

$$S_L(\rho) := \frac{d}{d-1}[1 - f(\rho)], \quad (1)$$

where $f(\rho) = \text{Tr}(\rho^2)$ stands for the quantum purity. The latter quantity is bounded as $1/d \leq f(\rho) \leq 1$, which implies that the linear entropy ranges as $0 \leq S_L(\rho) \leq 1$ for all $\rho \in S$. In addition, given the spectral decomposition $\rho = \sum_j p_j |j\rangle \langle j|$ in terms of the basis of states $\{|j\rangle\}_{j=1}^{d}$, with $0 \leq p_j \leq 1$ and $\sum_j p_j = 1$, one readily concludes that $S_L(\rho) = [d/(d-1)][1 - \sum_j p_j^2]$.

Linear entropy remains invariant under unitary transformations over the input state, i.e., $S_L(V \rho V^\dagger) = S_L(\rho)$, with $V^\dagger V = V^\dagger V = 1$ and for all $\rho \in S$. It is related to the second-order Rényi entropy $S_2(\rho)$ [71, 72], also known as collision entropy [73], and thus becomes $S_L(\rho) = [d/(d-1)][1 - e^{-S_2(\rho)}]$. Furthermore, Eq. (1) is also written as $S_L(\rho) = [d/(d-1)]S_2(\rho)$, with $S_2(\rho)$ being the second-order Tsallis entropy [74]. Interestingly, for a given state $\rho \in S$ and observable $\Lambda \in \mathcal{H}$, it has been shown that the linear entropy satisfies the lower bound $S_L(\rho) \geq [2d/(d-1)][\langle \Lambda^2 \rangle - \langle \Lambda \rangle^2 - (1/4)F(\rho, \Lambda)]/[\lambda_{\text{max}}(\Lambda) - \lambda_{\text{min}}(\Lambda)]^2$, with $F(\rho, \Lambda) = \text{Tr}(\rho \Lambda)$, where $F(\rho, \Lambda)$ is the quantum Fisher information, while $\lambda_{\text{max}}(\Lambda)$ and $\lambda_{\text{min}}(\Lambda)$ are the largest and smallest eigenvalues of $\Lambda$, respectively [75].

Recently, linear entropy has also been discussed for systems described by non-Hermitian Hamiltonians [43, 44, 76]. In this regard, at least two approaches can be highlighted. The first one consists of equipping the Hilbert space with a generalized inner product structure, called metric operator [77, 78]. In the second case, one introduces a renormalized density operator whose dynamics is governed by modified Heisenberg equations of motion [1, 79–81]. Throughout this paper, we will follow this last perspective.

III. MIXEDNESS TIMESCALE FOR NON-HERMITIAN SYSTEMS

We consider a quantum system with finite-dimensional Hilbert space $\mathcal{H}$, with $d = \dim \mathcal{H}$, which is initialized in the state $\rho_0 \in S$ [see Sec. II]. This input state undergoes a nonunitary evolution governed by the time-independent non-Hermitian Hamiltonian $H = H_1 + iH_2$, where $H_1 = (1/2)(H + H^\dagger)$, and $H_2 = (1/2i)(H - H^\dagger)$. In turn, the noncommuting observables $H_1$ and $H_2$ stand for the Hermitian and anti-Hermitian parts of the Hamiltonian $\hat{H}$, respectively. In the remainder of the paper, we set $\hbar = 1$. The dynamics of the normalized time-dependent density matrix $\hat{\rho}_t = \rho_t/\text{Tr}(\rho_t)$ fulfills the equation of motion [79–81]

$$\frac{d\hat{\rho}_t}{dt} = -i[H_1, \hat{\rho}_t] + \{H_2, \hat{\rho}_t\} - 2\text{Tr}(\hat{\rho}_tH_2)\hat{\rho}_t, \quad (2)$$

which in turn represents a completely positive and trace-preserving operation. We point out that the dynamical map in Eq. (2) drives a nonunitary evolution under which the state $\hat{\rho}_t$ exhibits a time-dependent mixedness. In the Hermitian setting, the mixedness stands as a conserved quantity for any quantum state undergoing a unitary evolution generated by a Hermitian Hamiltonian.

Here, we choose the normalized linear entropy $S_L(\hat{\rho}_t) = [d/(d-1)][1 - f(\hat{\rho}_t)]$ as a useful quantum information-theoretic quantifier to probe the mixedness of state $\hat{\rho}_t$, with $f(\hat{\rho}_t) = \text{Tr}(\hat{\rho}_t^2)$. As discussed in Sec. II, for the case of unitary evolutions generated by Hermitian Hamiltonians, both the purity and the linear entropy remain invariant for all $t > 0$. However, for the nonunitary dynamics dictated by non-Hermitian Hamiltonians, the linear entropy becomes a time-dependent quantity, and its evaluation requires the full spectral decomposition of
the evolved state. This task has a high computational cost for many-body quantum systems.

In this section, we are interested in the short-time perturbative expansion of \( S_L(\tilde{\rho}_t) \) to understand the initial growth of the mixedness of the evolved state \( \tilde{\rho}_t \) [see Eq. (2)]. The Taylor expansion of the linear entropy up to second order in \( t \) around \( t = 0 \) yields

\[
S_L(\tilde{\rho}_t) \approx S_L(\rho_0) - \frac{d}{d-1} \left( \frac{t}{T_1^1} + \frac{t^2}{T_2^2} \right) + O(t^3) ,
\]

where we define

\[
T_1^{-1} := 4 \text{cov}_{\rho_0}(\rho_0, H_2) ,
\]

and

\[
T_2^{-2} := -4 f(\rho_0) \text{var}_{\rho_0}(H_2) - 8 (H_2)_{\rho_0} \text{cov}_{\rho_0}(\rho_0, H_2) + 8 \text{cov}_{\rho_0}(H_2, \rho_0 H_2) - 2i \text{cov}_{\rho_0}(\rho_0, [H_2, H_1]) ,
\]

with \((\bullet)_{\rho_0} = \text{Tr}(\rho_0 \bullet)\) being the expectation value at time \( t = 0 \), while \( \text{cov}_{A}(B, C) = (1/2)\text{Tr}(A(\bullet B) - \text{Tr}(AB)\text{Tr}(AC))\) defines the covariance functional. In particular, for \( B = C \), note that the covariance reduces to the variance \( \text{var}_{A}(B) \equiv \text{cov}_{A}(B, B) = \text{Tr}(AB^2) - \text{Tr}(AB)^2 \).

Equations (3), (4) and (5) are the first main result of the paper. The coefficients \(|1/T_1|\) and \(|1/T_2|\) provide timescales for the linear entropy at earlier times of the dynamics, thus predicting the initial growth of the mixedness of the evolved state of the quantum system. Importantly, they can be evaluated once the input state \( \rho_0 \) and the Hamiltonian \( H = H_1 + iH_2 \) of the system have been specified. Note that \( 1/T_1 \) and \( 1/T_2 \) depend on the fluctuations of the observable \( H_2 \) that are captured by its covariance respective to the input state. In particular, choosing \( H_2 \) a zero-valued operator, Eqs. (4) and (5) vanish and one gets that \( S_L(\tilde{\rho}_t) \approx S_L(\rho_0) \). In fact, this is expected since the linear entropy remains invariant for any quantum state undergoing a unitary evolution generated by a Hermitian operator. In addition, for any initial pure state with \( \rho_0^0 = \rho_0 \) and \( f(\rho_0) = \text{Tr}(\rho_0^0 \rho_0) = 1 \), one can verify that Eqs. (4) and (5) imply that \( 1/T_1 = 0 \) and \( 1/T_2 = 0 \), regardless of the operators \( H_1 \) and \( H_2 \). Finally, we note that the last term on the right-hand side of Eq. (5) vanishes for \( [H_1, H_2] = 0 \), i.e., for two commuting operators \( H_1 \) and \( H_2 \).

The proof of Eq. (3) is as follows. We shall begin with the linear entropy of the evolved state \( \tilde{\rho}_t \), and evaluate its Taylor expansion up to second order in \( t \), around \( t = 0 \), which yields

\[
S_L(\tilde{\rho}_t) \approx S_L(\rho_0) - \frac{d}{d-1} t \left[ f^{(1)}(\tilde{\rho}_t) \right]_{t=0} - \frac{d}{d-1} \frac{t^2}{2} \left[ f^{(2)}(\tilde{\rho}_t) \right]_{t=0} + O(t^3) ,
\]

where the \( n \)-th order derivative of the quantum purity becomes

\[
f^{(n)}(\tilde{\rho}_t) = \sum_{k=0}^{n} \frac{n!}{(n-k)! k!} \text{Tr} \left( \frac{d^k \tilde{\rho}_t}{dt^k} \frac{d^{n-k} \tilde{\rho}_t}{dt^{n-k}} \right) .
\]

Therefore, starting from Eqs. (2) and (7), both the first-order and second-order derivatives of the quantum purity at the vicinity of \( t = 0 \) yield

\[
[f^{(1)}(\tilde{\rho}_t)]_{t=0} = 4 \left( \text{Tr}(\rho_0^0 H_2) - \text{Tr}(\rho_0^0 \text{Tr}(\rho_0 H_2)) \right) = 4 \text{cov}_{\rho_0}(\rho_0, H_2) ,
\]

and

\[
(1/2)[f^{(2)}(\tilde{\rho}_t)]_{t=0} = -4 f(\rho_0) \text{var}_{\rho_0}(H_2) - 8 (H_2)_{\rho_0} \text{cov}_{\rho_0}(\rho_0, H_2) + 8 \text{cov}_{\rho_0}(H_2, \rho_0 H_2) - 2i \text{cov}_{\rho_0}(\rho_0, [H_2, H_1]) ,
\]

where we recognize the aforementioned covariance and variance functionals. Finally, bringing together the results in Eqs. (6), (8) and (9), one readily concludes the main result in Eq. (3). We point out that the first-order derivative \( f^{(1)}(\tilde{\rho}_t) \) was previously investigated in the context of gain-loss systems [1], and in the quantum-classical description of non-Hermitian systems [1, 76].

### A. Example: Dissipative two-level system

To illustrate our findings, we consider a driven two-level system described by the Hamiltonian \( H_1 = \Delta|1\rangle\langle 1| + (\Omega/2)|(0\rangle\langle 1| + |1\rangle\langle 0|) \), where the two vectors \(|0\rangle \) and \(|1\rangle \) stand for ground and excited states, respectively, with \( \Delta \) the energy detuning, and \( \Omega \) being their coupling. The system interacts with a zero-temperature thermal reservoir, so that it decays from the excited state \(|1\rangle \) to the ground state \(|0\rangle \) emitting a photon at a rate \( \gamma \). The dynamics is governed by the Markovian master equation \( d\rho_1/dt = -i(H_{\text{eff}}\rho_1 - \rho_1 H_{\text{eff}}^\dagger) + \gamma L\rho_1 L^\dagger \), where \( H_{\text{eff}} = H - i(\gamma/2)L^\dagger L \) is the effective non-Hermitian Hamiltonian, while \( L = |0\rangle\langle 1| \) is the jump operator [82].

In the semiclassical regime, i.e., assuming that the effect of quantum jumps is negligible in the time interval under consideration, an effective description of the master equation can be obtained in terms of the coherent nonunitary dissipation of the system, the latter related to the non-Hermitian Hamiltonian \( H_{\text{eff}} \) [34]. In this case, by discarding the quantum jump term \( \gamma L\rho_1 L^\dagger \), the dynamics of the system is dictated by the equation \( d\rho_1/dt \approx -i(H_{\text{eff}}\rho_1 - \rho_1 H_{\text{eff}}^\dagger) \), which no longer describes a completely positive and trace-preserving evolution.

To overcome this issue, one introduces the normalized time-dependent density matrix \( \tilde{\rho}_t = \rho_t/\text{Tr}(\rho_t) \), which in turn fulfills Eq. (2), with \( H_1 = \Delta|1\rangle\langle 1| + (\Omega/2)|(0\rangle\langle 1| + |1\rangle\langle 0|) \) and \( H_2 = -(\gamma/2)|1\rangle\langle 1| \). The system is initialized in a single-qubit state \( \rho_0 = (1/2)(|0\rangle\langle 0| + i\bar{r} \cdot \vec{\sigma}) \), where \( \bar{r} = \{ r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta \} \) is the Bloch vector, with \( r \in [0, 1], \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \), while \( \vec{\sigma} = \{ \sigma_x, \sigma_y, \sigma_z \} \) is the vector of Pauli matrices, and \( I \) is the 2 \times 2 identity matrix. We will not show the analytical expressions for the exact linear entropy \( S_L(\tilde{\rho}_t) \) of the evolved state as they are cumbersome [see Eq. (1)]. However, it is straightforward to obtain the short-time
series expansion of $S_L(\tilde{\rho}_t)$ applying Eq. (3), with the linear entropy of the input state as $S_L(\rho_0) = 1 - r^2$. Using Eqs. (4) and (5), we obtain the following dimensionless coefficients

$$\frac{1}{\gamma T_1} = (1 - r^2) r \cos \theta ,$$

(10)

and

$$\frac{1}{\gamma^2 T_2^2} = \frac{1}{4} (1 - r^2) \left( 1 - 3r^2 \cos^2 \theta + 2 \frac{\Omega}{\gamma} r \sin \theta \sin \phi \right) ,$$

(11)

respectively. We see that $1/\gamma T_1$ is a function of $r$ and $\theta$, while $1/\gamma^2 T_2^2$ depends on the parameters $r$, $\theta$, $\phi$, and $\Omega/\gamma$. We notice that $S_L(\rho_0)$, $1/\gamma T_1$ and $1/\gamma^2 T_2^2$ approach zero for any initial single-qubit pure state with $r = 1$, thus implying that the linear entropy $S_L(\tilde{\rho}_t)$ is a vanishing quantity in this case.

In Fig. 1 we show the plots of the dimensionless quantities $1/\gamma T_1$ and $1/\gamma^2 T_2^2$, as a function of the mixing parameter $r$ and the azimuthal angle $\theta$. In Fig. 1(a), we see that $1/\gamma T_1 > 0$ for $\theta \in [0, \pi/2)$, while $1/\gamma T_1 < 0$ for $\theta \in (\pi/2, \pi]$, regardless of the mixing parameter $0 \leq r \leq 1$. In addition, it follows that $1/\gamma T_1 = 0$ for any chosen initial state with $\theta = \pi/2$ [see Eq. (10)]. Next, Figs. 1(b)–(d) show the plots of $1/\gamma^2 T_2^2$ in Eq. (11), where we consider the cases $\Omega/\gamma = 0.1$ [see Fig. 1(c)], $\Omega/\gamma = 1$ [see Fig. 1(d)], and $\Omega/\gamma = 10$ [see Fig. 1(b)], also fixing the polar angle $\phi = \pi/4$. On the one hand, for input states with either $\theta = 0$ or $\theta = \pi$, that are all incoherent states respective to the computational basis $\{|0\rangle, |1\rangle\}$, Eq. (11) reduces to $1/\gamma^2 T_2^2 = (1/4)(1 - r^2)(1 - 3r^2)$, which is positive for $0 \leq r \leq 1/\sqrt{3}$ [see Figs. 1(b)–(d)]. On the other hand, for initial states lying in the equatorial $xy$-plane with $\theta = \pi/2$, one gets that $1/\gamma^2 T_2^2 = (1/4)(1 - r^2)[1 + 2r(\Omega/\gamma) \sin \phi]$, which is positive for $0 \leq \phi \leq \pi$ and $0 \leq r \leq 1$. We emphasize that the timescales related to the growth of mixedness can be obtained from the absolute values $|1/\gamma T_1|$ and $|1/\gamma^2 T_2^2|$. Figure 2 shows the plots of the linear entropy $S_L(\tilde{\rho}_t)$, as a function of the dimensionless parameter $\gamma$, for the aforementioned driven two-level system. The blue solid line refer to the exact linear entropy $S_L(\tilde{\rho}_t)$ [see Eq. (1)], while the red dashed line depicts the short time expansion of this quantity in Eq. (3). We set input states with $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$ [see Figs. 2(a)–2(c)], and $\{r, \theta, \phi\} = \{1/4, 3\pi/4, \pi/4\}$ [see Figs. 2(d)–2(f)]. In addition, for a fixed ratio $\Delta/\gamma = 0.5$, we consider the cases $\Omega/\gamma = 0.1$ [see Figs. 2(a) and 2(d)], $\Omega/\gamma = 1$ [see Figs. 2(b) and 2(c)], and $\Omega/\gamma = 10$ [see Figs. 2(c) and 2(f)]. In this setting, Eqs. (10) and (11) show that $1/\gamma T_1$ and $1/\gamma^2 T_2^2$ stand as positive quantities, respectively, regardless of the ratio $\Omega/\gamma$ and the probe states that have been considered. This implies that the linear entropy in Eq. (3) is a concave function.

Overall, Fig. 2 shows that the short-time approximation of $S_L(\tilde{\rho}_t)$ correctly reproduces its growth at earlier
times of the dynamics. Indeed, we find good quantitative agreement for $0 \leq \gamma t \lesssim 0.1$. However, for $\gamma t \gtrsim 0.1$, we have that the latter result is loose and fails to capture the changes in the eigenvalues of the state $\tilde{\rho}_t$ driven by the non-Hermitian Hamiltonian. We emphasize that one should look to higher orders in its Taylor expansion to accurately predict the mixedness degree of the evolved state for later times.

IV. ENTANGLEMENT TIMESCALE FOR NON-HERMITIAN SYSTEMS

In this section, we provide an entanglement timescale for quantum systems whose dynamics can be effectively described by a non-Hermitian Hamiltonian. In detail, using the linear entropy as a useful measure of entanglement, we investigate its short-time expansion up to the second order in $t$ for certain time-dependent marginal states of the composite system.

We consider a bipartite quantum system with a finite-dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ split into the subsystems $\mathcal{H}_A$ and $\mathcal{H}_B$, with $d_{A,B} = \text{dim} \mathcal{H}_{A,B}$. This composite system is initialized in the quantum state $\rho_0^{AB}$, which in turn can be chosen either a pure or mixed state, entangled or uncorrelated one, from which the mixed marginal states $\rho_0^{AB} = Tr_{B,A}(\rho_0^{AB})$ can be obtained. The state $\rho_0^{AB}$ undergoes a nonunitary evolution generated by the time-independent non-Hermitian Hamiltonian $H = H_1 + iH_2$, with $H_1 = (1/2)(H + H^\dagger)$ and $H_2 = -(i/2)(H - H^\dagger)$ being noncommuting observables acting over $\mathcal{H}_A \otimes \mathcal{H}_B$. It is noteworthy that the operators $H_1$ and $H_2$ play the role of the Hamiltonian and anti-Hermitian parts of $H$, respectively. In this setting, it can be proved that the effective dynamics of subsystem $\mathcal{H}_{A,B}$ is governed by the equation of motion \cite{79-81}

$$
\frac{d}{dt} \rho_{t}^{A,B} = -i Tr_{B,A}([H_1, \tilde{\rho}_t^{AB}]) + Tr_{B,A}(\{H_2, \tilde{\rho}_t^{AB}(t)\}) - 2 Tr_{AB}(\tilde{\rho}_t^{AB} H_2) \tilde{\rho}_t^{A,B},
$$

where $\tilde{\rho}_t^{A,B} := Tr_{B,A}(\rho_t^{AB})$ stand for the time-dependent reduced density matrices, while $\tilde{\rho}_t^{AB} := \rho_t^{AB} / Tr_{AB}(\rho_t^{AB})$ is the normalized state of the whole system.

Without loss of generality, hereafter we will address the dynamics of the marginal state $\tilde{\rho}_t^A$, and investigate the short-time behavior of its linear entropy

$$
S_L(\tilde{\rho}_t^A) = \frac{d_A}{(d_A - 1)} (1 - f(\tilde{\rho}_t^A)),
$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{(Color online) Plot of the linear entropy $S_L(\tilde{\rho}_t)$, as a function of the dimensionless parameter $\gamma t$, for the driven two-level system described by the Hamiltonian $H = \Delta |1\rangle \langle 1| + (\Omega/2)(|0\rangle \langle 1| + |1\rangle \langle 0|)$. Here we choose the ratio $\Delta/\gamma = 0.5$, and also set $\Omega/\gamma = 0.1$ [panels (a), (d)], $\Omega/\gamma = 1$ [panels (b), (e)], and $\Omega/\gamma = 10$ [panels (c), (f)]. The system is initialized in the single-qubit state $\rho_0 = (1/2)(I + i \vec{r} \cdot \vec{\sigma})$, with $|r, \theta, \phi\rangle = \{1/4, \pi/4, \pi/4\}$ [panels (a), (b), (c)], and $|r, \theta, \phi\rangle = \{1/4, 3\pi/4, \pi/4\}$ [panels (d), (e), (f)].}
\end{figure}
where \( f(\tilde{\rho}_t^A) = \text{Tr}_A(\tilde{\rho}_t^A)^2 \) is the purity of the aforementioned reduced density matrix. In this case, by performing a Taylor expansion of \( S_L(\tilde{\rho}_t^A) \) up to second order in \( t \), around \( t = 0 \), one gets

\[
S_L(\tilde{\rho}_t^A) \approx S_L(\rho_0^A) - \frac{d_A}{(d_A - 1)} \left( \frac{1}{T_{1,h}} + \frac{1}{T_{1,nh}} \right) t
\]

\[
- \frac{d_A}{(d_A - 1)} \left( \frac{1}{T_{2,h}} + \frac{1}{T_{2,nh}} \right) t^2 + O(t^3) , \tag{14}
\]

with \( T_{1,h} \) and \( T_{1,nh} \) being coefficients related to the first-order derivative of the linear entropy around \( t = 0 \), and defined as

\[
T_{1,h}^{-1} := 2i \langle \text{Tr}_B(\rho_0^A H_1) \rangle_A , \tag{15}
\]

and

\[
T_{1,nh}^{-1} := 2 \langle \text{Tr}_B(\{\rho_0^A, H_2\}) \rangle_A - 4 \langle \rho_0^A \rangle \langle H_2 \rangle_{AB} , \tag{16}
\]

while \( T_{2,h} \) and \( T_{2,nh} \) arise from the second-order derivative of the linear entropy at the vicinity of \( t = 0 \) as follows

\[
T_{2,h}^{-2} := -\langle \text{Tr}_B(\{\rho_0^A, H_1, H_1\}) \rangle_A - \text{Tr}_A(\langle \text{Tr}_B(\rho_0^A H_1) \rangle^2) , \tag{17}
\]

and

\[
T_{2,nh}^{-2} := \langle \text{Tr}_B(\{\rho_0^A, H_2, H_2\}) \rangle_A + \text{Tr}_A(\langle \text{Tr}_B(\{\rho_0^A, H_2\}) \rangle^2) + i\langle \text{Tr}_B(\{\rho_0^A, H_1, H_2\}) \rangle_A + \langle \text{Tr}_B(\{\rho_0^A, H_1\} \rangle_A) + 2f(\rho_0^A) \langle [\langle H_2, H_1 \rangle_{AB} - 2(\langle H_2^2 \rangle_{AB} - 3 \langle H_2 \rangle_{AB}^2) \rangle_{AB} - 2i \text{Tr}_A(\text{Tr}_B(H_1, \rho_0^A)) \text{Tr}_B(\rho_0^A, H_2) \rangle . \tag{18}
\]

Here \( \langle \bullet \rangle_{\mu} := \text{Tr}_\mu(\rho_0^A)^2 \) defines the expectation value at time \( t = 0 \), with \( \mu = \{A, B, AB\} \).

We point out that Eq. (14) [see also Eqs. (15)–(18)] is the second main result of the paper. Overall, we see that the coefficients \( 1/T_{1,h} + 1/T_{1,nh} \) and \( 1/T_{2,h} + 1/T_{2,nh} \) represent first-order and second-order timescales in the initial growth of the entanglement dynamics signaled by linear entropy. On the one hand, both the coefficients \( T_{1,h} \) and \( T_{2,h} \) depend on the initial state of the bipartite system and the Hermitian part \( H_1 \) of the non-Hermitian Hamiltonian. Importantly, Eqs. (15) and (17) recover the two lowest-order terms in the short-time expansion in the so-called idempotency defect for composite systems described by Hermitian Hamiltonians, and constitute a timescale for the entanglement dynamics of subsystems \([54, 55, 59]\). On the other hand, the coefficients \( 1/T_{1,nh} \) and \( 1/T_{2,nh} \) depend on the anti-Hermitian part \( H_2 \) of the effective non-Hermitian Hamiltonian. In particular, note that the result in Eqs. (16) and (18) approach zero in the Hermitian limit \( H_1 = H = H_1 \), i.e., when one sets \( H_2 = 0 \) as a zero-valued observable, regardless of the observable \( H_1 \). This means that \( 1/T_{1,nh} \) and \( 1/T_{2,nh} \) assign first-order and second-order nontrivial corrections to the entanglement timescales that are induced by the effective non-Hermitian Hamiltonian.

The proof of Eq. (14) is as follows. We begin evaluating Taylor’s series of \( S_L(\tilde{\rho}_t^A) \) up to the second order in \( t \), around \( t = 0 \), which yields

\[
S_L(\tilde{\rho}_t^A) \approx S_L(\rho_0^A) - \frac{d_A}{d_A - 1} t [f(1)(\tilde{\rho}_t^A)]_{t=0}
\]

\[
- \frac{d_A}{d_A - 1} \frac{t^2}{2} [f(2)(\tilde{\rho}_t^A)]_{t=0} + O(t^3) . \tag{19}
\]

The first-order and second-order derivatives of the purity of the reduced state around \( t = 0 \) read as

\[
[f(1)(\tilde{\rho}_t^A)]_{t=0} = -2i \langle \text{Tr}_B([H_1, \rho_0^AB]) \rangle_A
\]

\[
+ 2 \langle \text{Tr}_B([\rho_0^A H_1, H_2]) \rangle_A - 4 f_A(0) \langle H_2 \rangle_{AB} , \tag{20}
\]

and

\[
(1/2)[f(2)(\tilde{\rho}_t^A)]_{t=0} =
\]

\[
- \langle [\text{Tr}_B([\rho_0^A H_1, H_1]) A - \text{Tr}_A \langle \text{Tr}_B([\rho_0^A H_1]) \rangle - 8 \langle H_2 \rangle_{AB} \langle \text{Tr}_B([\rho_0^A H_2]) \rangle_A + i \langle \text{Tr}_B([\rho_0^A H_1]) \rangle A + 2f(\rho_0^A) \langle [\langle H_2, H_1 \rangle_{AB} - 2(\langle H_2^2 \rangle_{AB} - 3 \langle H_2 \rangle_{AB}^2) \rangle_{AB} - 2i \text{Tr}_A(\text{Tr}_B(H_1, \rho_0^A)) \text{Tr}_B(\rho_0^A, H_2) \rangle . \tag{21}
\]

Finally, by combining Eqs. (19), (20) and (21), one readily recovers the short-time expansion of the linear entropy in Eq. (14).

To gain insights into understanding the results in Eqs. (14)–(18), in the following, we investigate two cases of interest in view of the nonunitary dynamics of non-Hermitian Hamiltonians. The first case describes bipartite quantum systems with initial uncorrelated states. The second one addresses a multiparticle system whose non-Hermitian Hamiltonian corresponds to the transverse-field XY model with next-nearest neighbor couplings and a perturbing term given by an all-to-all Ising Hamiltonian with an imaginary exchange coupling.

### A. Separable initial pure states

Here we specialize the result in Eq. (14) to the particular case of uncorrelated initial pure state \( \rho_0^AB = \rho_0^A \otimes \rho_0^B \), with \( \rho_0^A \) and \( \rho_0^B \) normalized pure marginal states, i.e.,

\[
\text{Tr}_\mu(\rho_0^A)^2 = \text{Tr}_\mu(\rho_0^B)^2 = 1 \text{ for all } \mu = \{A, B, AB\}.
\]

We consider the non-Hermitian Hamiltonian \( H = H_1 + iH_2 \) with \( H_1 = \sum_A A_n \otimes B_n \) and \( H_2 = \sum_B C_n \otimes D_n \), where \( A_n, C_n \in H_A \) and \( B_n, D_n \in H_B \) represent noncommuting local observables. In this setting, one can prove that both the coefficients \( 1/T_{1,h} = 0 \) [see Eq. (15)] and \( 1/T_{1,nh} = 0 \) [see Eq. (16)] identically vanish, and the linear entropy...
in Eq. (14) becomes

\[ S_L(\rho^t) \approx -\frac{d_A}{(d_A - 1)} \left( \frac{1}{T^2_{2, h}} + \frac{1}{T^2_{2, nh}} \right) t^2 + O(t^3) , \]  

(22)

with the following nonzero coefficients

\[ T^2_{2, h} = -2 \sum_{k,l} \langle (A_k A_l)_A - \langle A_k \rangle_A \langle A_l \rangle_A \rangle \times \langle (B_k B_l)_B - \langle B_k \rangle_B \langle B_l \rangle_B \rangle , \]  

(23)

and

\[ T^2_{2, nh} = -2 \sum_{k,l} \langle (C_k C_l)_A - \langle C_k \rangle_A \langle C_l \rangle_A \rangle \times \langle (D_k D_l)_B - \langle D_k \rangle_B \langle D_l \rangle_B \rangle \]

\[ + 4 \sum_{k,l} \text{Im} \{ \langle A_k C_l \rangle_A - \langle A_k \rangle_A \langle C_l \rangle_A \} \times \langle (B_k D_l)_B - \langle B_k \rangle_B \langle D_l \rangle_B \rangle \} . \]  

(24)

Overall, Eq. (22) implies that the linear entropy varies quadratically at earlier times of the dynamics. We see that \( 1/T_{2, h} \) is proportional to the so-called correlated quantum uncertainty of observables \( A_n \in \mathcal{H}_A \) and \( B_n \in \mathcal{H}_B \), thus being entirely determined by the expectation values of these operators with respect to the initial marginal states \( \rho^t_{A,B} \). It is noteworthy that \( 1/T_{2, h} \) assigns a universal timescale for two initially pure subsystems to become entangled by means of the coupling with a Hermitian Hamiltonian \( H_1 \) \([54, 55, 59]\).

In turn, the coefficient \( 1/T_{2, nh} \) depends on the correlated quantum uncertainty of observables \( C_n \in \mathcal{H}_A \) and \( D_n \in \mathcal{H}_B \), and the imaginary part of cross-correlations of the set of local observables. We see that \( 1/T_{2, nh} \) represents a true signature of the non-Hermitian features of \( H \) in the entanglement dynamics. Thus, in addition to the coefficient \( 1/T_{2, h} \), the effective non-Hermitian Hamiltonian induces the factor \( 1/T_{2, nh} \) on the entanglement timescale for initially separable states. In particular, one verifies that \( 1/T_{2, nh} = 0 \) vanishes in the Hermitian limit \( H = H^\dagger = H_1 \), i.e., when choosing zero valued observables \( C_n \) and \( D_n \).

**B. 1D quantum many-body systems**

We set the non-Hermitian Hamiltonian \( H = H_1 + iH_2 \), where \( H_1 \) describes the transverse-field XY model with open boundary conditions as \([83–87]\)

\[ H_1 = -J \sum_{j=1}^{N-1} (\gamma_+ \sigma^y_j \sigma^y_{j+1} + \gamma_- \sigma^y_j \sigma^y_{j+1}) - h \sum_{j=1}^N \sigma^z_j , \]  

(25)

where \( J \) is the coupling constant, \( h \) represents the external magnetic field along the z-axis, and \( \gamma_\pm = (1 \pm \gamma)/2 \), with \( \gamma \) being the anisotropy parameter. For \( \gamma = 0 \) this Hamiltonian reduces to the isotropic XX model, while for \( \gamma = \pm 1 \) we recover the Ising model. Furthermore, this model exhibits phase transitions at the isotropic line \( \gamma = 0 \) (\( |\gamma| \leq 1 \)), and at the critical magnetic field \( |h| = 1 \).

In turn, \( H_2 \) denotes the many-body fully connected quantum Ising model given by

\[ H_2 = \frac{J_z}{N} \sum_{j<l} \sigma^z_j \sigma^z_l , \]  

(26)

where \( J_z \) is the coupling strength, \( N \) is the number of spins, and \( \{ \sigma^z_{j=1,...,N} \} \) are the Pauli matrices.

We consider a bipartition into first sequential \( k \) sites \((1, \ldots, k)\) as the subsystem \( A \), and its complement of sequential \( N - k \) sites \((k+1, \ldots, N)\) as subsystem \( B \). The system \( A + B \) is initialized in the mixed state

\[ \rho_0^{A,B} = \left( \frac{1 - p}{d^2} \right) \mathbb{I} + p |\text{GHZ}_N \rangle \langle \text{GHZ}_N| , \]  

(27)

with \( d = 2^N \), \( 0 \leq p \leq 1 \), and \( |\text{GHZ}_N \rangle \) is the GHZ state of \( N \) particles defined as

\[ |\text{GHZ}_N \rangle = \frac{1}{\sqrt{2}} (|0^\otimes N \rangle + |1^\otimes N \rangle) , \]  

(28)

and its purity is written as \( f(\rho^{A,B}_0) = (1/2^N)(1 + (2^N - 1)p^2) \).

Furthermore, one can evaluate the averaged values \( \langle H_2 \rangle_{AB} = (J_z/2)(N-1)p \) and \( \langle H_2^2 \rangle_{AB} = J_z^2(N-1)/(4N)(2 + (N-2)(N+1)p) \) of the observable \( H_2 \) respective to the probe state of system \( A + B \). The many-body state \( \rho_0^{A,B} \) undergoes a nonunitary evolution generated by the non-Hermitian Hamiltonian \( H = H_1 + iH_2 \), and the subsystem \( A \) is described by the reduced \( k \)-particle state \( \rho^t_A = \text{Tr}_{N-k}(\rho^t_{A,B}) \) whose dynamics is governed by Eq. (12). The linear entropy \( S_L(\rho^t_A) \) of this marginal state is given in Eq. (13), which in turn reduces to \( S_L(\rho^t_A) = |d_A/(d_A - 1)|[1 - (1/2^k)(1 + (2^{k-1} - 1)p^2)] \) at time \( t = 0 \). The short-time expansion of \( S_L(\rho^t_A) \) is given in Eq. (14). In this setting, it is possible to verify that the coefficient \( 1/T_{1,h} = 0 \) vanishes [see Eq. (15)], while Eq. (16) implies the following nonzero contribution

\[ T^{-2}_{1, nh} = -\frac{J_z p(1 - p)}{2^{k-1}N} (k(k - 1) + N(N - 1)(2^{k-1} - 1)p) . \]  

(29)

Noteworthy, Eq. (29) shows that \( 1/T_{1, nh} \) exhibits a polynomial dependence on the mixing parameter \( p \), thus being a negative quantity for all \( 0 < p < 1 \), and \( k \in \{1, \ldots, N\} \). In particular, it follows that \( 1/T_{1, nh} = 0 \) for the initial pure state \( |\text{GHZ}_N \rangle \langle \text{GHZ}_N| \) \( (p = 1) \) and also for the maximally mixed state \( \mathbb{I}/d \) \( (p = 0) \). We find that \( 1/T_{1, nh} \) is proportional to the coupling strength \( J_z \), and identically vanishes in the Hermitian limit \( (J_z = 0) \).

Next, by using Eq. (17), we obtain

\[ T^2_{2, h} = (\delta_{dB,2} - 1)\gamma^2 J^2 p^2 , \]  

(30)
which depends on the coupling $J$, anisotropy parameter $\gamma$, and vanishes whenever $\mathcal{H}_B$ is a two-dimensional subspace, i.e., one gets $1/T_{2,h} = 0$ for the case $d_B = 2$. Finally, by applying Eq. (18) and performing lengthy calculations, one obtains the result

$$T_{2,nh}^{-2} = \frac{J^2}{2k^3 - N^2} \{3N^2(N - 1)^2(1 - 2k^{-1}) p^4 + N(N - 1)(2k^{-1} - 1)(5N(N - 1) - 2) - 4k(k - 1)\} p^4 + [k(k^2 - 6) + k + 4 + 2Nk(k - 1)3N - 1] - 2N(2k^{-1} - 1)(N^3 - 2N^2 + 1)p^2 - 2k(k - 1) - k(k - 1)[k(k - 5) + 2((N - 1)(N + 2) - 1)] p \} .$$

(31)

We find that $1/T_{2,nh}$ behaves polynomially with the mixing parameter $p$. In particular, it follows that $1/T_{2,nh}^2 = 0$ for $p = 1$, while for $p = 0$ one obtains $1/T_{2,nh}^2 = 2^{-k} J^2 k(k - 1)/N^2$. Hence, for $N \gg k$, the latter case implies $1/T_{2,nh}^2 \sim J^2 / N^2$ for the initial maximally mixed state ($p = 0$), i.e., it scales with the inverse square of the number of particles.

In the following, we will numerically address the short-time dynamics of the linear entropy $S_L(\tilde{\rho}_A^k)$ in Eq. (14). The system $A + B$ is initialized at the GHZ mixed state in Eq. (27), with $H_1$ being the transverse field XY Hamiltonian in Eq. (25), and $H_2$ stand for the all-to-all Ising model in Eq. (26). In each panel, we find an accurate quantitative agreement for $0 \leq Jt \lesssim 0.1$. Nevertheless, for $Jt \gtrsim 0.1$, we have that the result in Eq. (13) becomes loose and it is bounded from above by the exact linear entropy in Eq. (14), failing to predict the entanglement.
dynamics within the subsystem A.

Next, Fig. 4 shows the short-time dynamics of the linear entropy $S_L(\tilde{\rho}_L^k)$ in Eq. (14), for the $k$-particle reduced density matrix $\tilde{\rho}_L^k$, as a function of the dimensionless parameter $Jt$. The nonunitary evolution of subsystem $A$ is governed by Eq. (12), with $H_1$ being the transverse field XY Hamiltonian in Eq. (25), and $H_2$ as the all-to-all Ising model in Eq. (26). The system $A + B$ is initialized in the GHZ mixed state $\rho_{0AB} = ((1-p)/d)I + p|GHZ_N\rangle\langle GHZ_N|$, where $|GHZ_N\rangle = (1/\sqrt{2})(|0_N\rangle\otimes|1_N\rangle + |1_N\rangle\otimes|0_N\rangle)$. Here we set $N = 8$, $\gamma = 0.75$, and the mixing parameter $p = 0.5$. The blue solid lines correspond to the case $J_z/J = 0.5$, and the red dashed lines depict the case $J_z/J = 0$.

In Fig. 5 we display the short-time dynamics of the linear entropy $S_L(\tilde{\rho}_L^k)$ in Eq. (14), as a function of $Jt$, and consider the subsystem $A$ with $k = 5$ sites. We emphasize that each of the blue solid lines corresponds to the case $J_z/J = 0.5$ (non-Hermitian Hamiltonian), and the red ones represent the case $J_z/J = 0$ (Hermitian Hamiltonian). We set the mixing parameters $p = 0.25$ [see Fig. 5(a)], $p = 0.5$ [see Fig. 5(b)], $p = 0.75$ [see Fig. 5(c)], and $p = 1$ [see Fig. 5(d)]. In Figs. 5(a) and 5(b), one finds that the linear entropy is concave whenever $J_z/J \neq 0$, while it turns into a convex function in the Hermitian limit with $J_z/J = 0$. In Fig. 5(c), the linear entropy turn to be a convex function, which is due to the fact that $1/T_{2h} + 1/T_{2,h} < 0$ for $p = 0.75$. Figure 5(d) show that, for $p = 1$, the two linear entropies coincide regardless of the generator $H$. Indeed, we have seen from Eqs. (29) and (31) that $1/T_{1, nh} = 0 = 1/T_{2, nh}$ for $J = 0$ [see Eqs. (29) and (31), respectively], while one readily obtains that $1/T_{1, nh} = 0$ as the subsystem $B$ has dimension $d_B = 2$ [see Eq. (30)]. Hence, bearing in mind that $1/T_{1, nh} = 0$, it follows that the linear entropy $S_L(\tilde{\rho}_L^k) \approx S_L(\rho_0^k)$ is time-independent in the short-time approximation, for $J_z/J = 0$.

As a final remark, Figs. 4 and 5 show that the non-Hermitian Hamiltonian ($J_z/J \neq 0$) enhances the short-time dynamics of $S_L(\tilde{\rho}_L^k)$, which bounds from above the respective linear entropy for the Hermitian Hamiltonian ($J_z/J = 0$). This result suggests that non-Hermitian Hamiltonians can play the role of a resource in the short-time dynamics of subsystem correlations, with potential use in some quantum metrological tasks as shown.
recently in Refs. [3, 4]. Lastly, Figs. 4 and 5 show a crossover behavior between both the non-Hermitian \((\mathcal{J}_z/J \neq 0)\) and Hermitian \((\mathcal{J}_z/J = 0)\) cases, but it should be noted that it occurs in a time window that extrapolates the validity of the short-time approximation. Indeed, Fig. 3 shows that our results find good agreement with the numerical simulation of \(S_{\ell}(\tilde{\rho}_A^k)\) in Eq. (13) for \(0 \leq Jt \lesssim 0.1\).

\section{V. DISCUSSION AND CONCLUSIONS}

In this paper, we discuss the timescales related to the onset growth of linear entropy for finite-dimensional quantum systems described by effective non-Hermitian Hamiltonians. We investigate the short-time perturbative expansion of the linear entropy for a given input state driven by a general non-Hermitian Hamiltonian. We emphasize that our approach takes in account initial quantum states that can be either pure or mixed, possibly entangled or even uncorrelated states.

We address the degree of mixedness of a quantum state that undergoes the nonunitary dynamics generated by an effective non-Hermitian Hamiltonian \(H = H_1 + iH_2\) [see Sec. III]. In this setting, Eq. (3) stands for the short-time expansion of the linear entropy up to second order in time \(t\), around \(t = 0\), which in turn depends on the coefficients \(1/T_1\) and \(1/T_2\) in Eqs. (4) and (5), respectively. Both quantities can be evaluated once the input state and the Hamiltonian \(H\) have been specified. We emphasize that Eqs. (4) and (5) provide two competing timescales in the initial growth of the mixedness of the evolved state at earlier times of the dynamics. In particular, both coefficients vanish whenever the system is initialized in a pure state, regardless of the non-Hermitian part of the Hamiltonian. Moreover, in the Hermitian limit, we have that \(1/T_1 = 0\) and \(1/T_2 = 0\) independently of the initial state of the system. We note that, since the linear entropy defines a conserved quantity for Hermitian quantum systems, it can be proved that any of the coefficients in its perturbative expansion must vanish in this limiting case [see Eq. (3)].

We specialize these results to the case of a dissipative non-Hermitian two-level system initialized in a mixed single-qubit state [see Sec. III A]. We find analytical expressions for the coefficients \(1/T_1\) [see Eq. (10)] and \(1/T_2\) [see Eq. (11)] in terms of the Bloch sphere parameters. In this case, we compare the exact linear entropy \(S_L(\rho_t)\) with its aforementioned short-time expansion around \(t = 0\). We find good quantitative agreement between these two quantities at earlier times of the dynamics. Of course, for later times one should include higher orders in the Taylor expansion to obtain tighter results for the mixedness of the evolved state.

Next, we investigate the reduced dynamics of composite systems described by non-Hermitian Hamiltonians [see Sec. IV]. We derived the short-time perturbative expansion of the linear entropy \(S_{\ell}(\tilde{\rho}_A^k)\) for a given time-dependent marginal state of a bipartite system [see Eq. (14)]. We found that, up to the second order in time \(t\), the growth of the linear entropy is governed by the coefficients \(1/T_{1,h}\) and \(1/T_{1,nh}\) in Eqs. (15) and (16), respectively, and also \(1/T_{2,h}\) and \(1/T_{2,nh}\) in Eqs. (17) and (18), respectively. On the one hand, one gets that \(1/T_{1,h}\) and \(1/T_{2,h}\) depend on \(H_1\) and the input state of the system. On the other hand, we have that \(1/T_{1,nh}\) and \(1/T_{2,nh}\) depend on \(H_2\), thus being intrinsically related to the non-Hermitian features of the Hamiltonian. In the Hermitian limit, i.e., when one sets \(H_2\) being a zero-valued operator, we find \(1/T_{1,nh} = 0\) and \(1/T_{2,nh} = 0\) for any bipartite system. In this case, Eqs. (15) and (17) remain nonzero and recover the perturbative expansion of the idempotency defect measuring the coherence losses for composite systems described by Hermitian Hamiltonians [59].

In particular, specifying an initial pure and uncorrelated state, we find the vanishing coefficients \(1/T_{1,h} = 0\) and \(1/T_{1,nh} = 0\), and the lowest order of the short-time
perturbative expansion of the linear entropy $S_L(\tilde{\rho})$ depends on $1/T_{2,h}^2$ and $1/T_{2, nh}^2$ that are given in Eqs. (23) and (24), respectively [see Sec. IV A]. In the Hermitian limit, the latter result identically vanishes, and $|1/T_{2, h}|$ signals the entanglement timescale for quantum systems described by Hermitian Hamiltonians [54]. It is noteworthy that this result is also related to the timescale that governs the growth of entanglement for Rényi entropies [55].

To illustrate these findings, we investigated the linear entropy of the $k$-particle evolved marginal state for a quantum many-body system described by the transverse-field XY model coupled to the imaginary fully connected Ising Hamiltonian [see Sec. IV B]. We found analytical expressions for $1/T_{1,nh}$ and $1/T_{2, nh}^2$, which in turn scale linearly with the coupling strength of the all-to-all Ising Hamiltonian [see Eqs. (29) and (31), respectively]. In addition, it follows that $1/T_{1,h}$ vanishes, while $1/T_{2,h}^2$ depends on the anisotropy parameter of the XY model [see Eq. (30)]. We compared the short-time expansion of the linear entropy with its exact numerical simulation [see Fig. 3], and discussed its dynamical behavior in both cases of non-Hermitian and Hermitian Hamiltonians [see Figs. 4 and 5]. We find that non-Hermiticity enhances the short-time dynamics of the linear entropy, providing an upper bound for the respective linear entropy for Hermitian Hamiltonian.

Our findings provide insightful qualitative and quantitative information about the initial growth of linear entropy at early times. Importantly, the results require low computational cost and their evaluation involves minimal ingredients as the initial state and the non-Hermitian Hamiltonian that governs the nonunitary dynamics. This might be of interest for higher dimensional systems, where evaluating the linear entropy would require the full spectral decomposition of the evolved system. We point out that one could generalize the present discussion in terms of α-Rényi entropies [55]. Furthermore, one can investigate the interplay of the aforementioned timescales and the quantum speed limit for nonunitary evolutions generated by non-Hermitian Hamiltonians [88, 89]. We hope to address these questions in further investigations. The results in this paper could find applications in the subjects of non-Hermitian quantum sensing [90, 91], quantum thermodynamics of non-Hermitian systems [92], non-Hermitian long-range interacting quantum systems [93], and $\mathcal{PT}$-symmetric quantum field theory [94].

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