A STUDY ON TOPOLOGICAL INTEGER ADDITIVE SET-LABELING OF GRAPHS

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Abstract

A set-labeling of a graph $G$ is an injective function $f : V(G) \to \mathcal{P}(X)$, where $X$ is a finite set and a set-indexer of $G$ is a set-labeling such that the induced function $f^\oplus : E(G) \to \mathcal{P}(X) - \{\emptyset\}$ defined by $f^\oplus(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective. Let $G$ be a graph and let $X$ be a non-empty set. A set-indexer $f : V(G) \to \mathcal{P}(X)$ is called a topological set-labeling of $G$ if $f(V(G))$ is a topology of $X$. An integer additive set-labeling is an injective function $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$, whose associated function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ is defined by $f^+(uv) = f(u) + f(v)$, $uv \in E(G)$, where $\mathbb{N}_0$ is the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ is its power set. An integer additive set-indexer is an integer additive set-labeling such that the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective. In this paper, we extend the concepts of topological set-labeling of graphs to topological integer additive set-labeling of graphs.

Key words: Set-labeling of graphs, Integer additive set-labeling of graphs, topological set-labeling of graphs, topological integer additive set-labeling of graphs.

AMS Subject Classification : 05C78

1 Introduction

For all terms and definitions of graphs and graph classes, not defined specifically in this paper, we refer to [10], [28] and [8]. For more about graph labeling, we refer
to [12] and for terms and definitions in topology, we refer to [11], [18] and [20]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

If any one of $A$ and $B$ is countably infinite, the their sum set $A + B$ will also be a countably infinite set. Hence, all sets mentioned in this paper are finite sets of non-negative integers. We denote the cardinality of a set $A$ by $|A|$. We denote, by $X$, the finite ground set of non-negative integers that is used for set-labeling the elements of $G$.

The research in graph labeling commenced with the introduction of $\beta$-valuations of graphs in [22]. Analogous to the number valuations of graphs, the concepts of set-labelings and set-indexers of graphs are introduced in [1] as follows.

Let $G$ be a $(p, q)$-graph. Let $X$, $Y$ and $Z$ be non-empty sets and $\mathcal{P}(X)$, $\mathcal{P}(Y)$ and $\mathcal{P}(Z)$ be their power sets. Then, the functions $f : V(G) \rightarrow \mathcal{P}(X)$, $f : E(G) \rightarrow \mathcal{P}(Y)$ and $f : V(G) \cup E(G) \rightarrow \mathcal{P}(Z)$ are called the set-assignments of vertices, edges and elements of $G$ respectively. By a set-assignment of a graph, we mean any one of them.

A set-assignment is called a set-labeling or a set-valuation if it is injective. A graph with a set-labeling $f$ is denoted by $(G, f)$ and is referred to as a set-labeled graph.

For a $(p, q)$-graph $G(V, E)$ and a non-empty set $X$ of cardinality $n$, a set-indexer of $G$ is defined as an injective set-valued function $f : V(G) \rightarrow \mathcal{P}(X)$ such that the function $f^\oplus : E(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$ defined by $f^\oplus(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where $\mathcal{P}(X)$ is the set of all subsets of $X$ and $\oplus$ is the symmetric difference of sets. A graph that admits a set-indexer is called a set-indexed graph.

**Theorem 1.1.** [1] Every graph has a set-indexer.

The concept of topological set-labeling of a graph is defined in [4] as follows.

Let $G$ be a graph and let $X$ be a non-empty set. A set-indexer $f : V(G) \rightarrow \mathcal{P}(X)$ is called a topological set-labeling of $G$ if $f(V(G))$ is a topology of $X$. A graph $G$ which admits a topological set-labeling is called a topologically set-labeled graph.

Let $\mathbb{N}_0$ be the set of all non-negative integers. An integer additive set-labeling (IASL, in short) is an injective function $f : V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ such that the associated function $f^+ : E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ is defined by $f^+(uv) = f(u) + f(v)$ for any two adjacent vertices $u$ and $v$ of $G$. A graph $G$ which admits an IASL is called an IASL graph.

An integer additive set-labeling $f$ is an integer additive set-indexer (IASI, in short) if the induced function $f^+ : E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is injective. A graph $G$ which admits an IASI is called an IASI graph.

The cardinality of the set-label of an element (vertex or edge) of a graph $G$ is called the set-indexing number of that element. An IASL (or an IASI) is said to be a $k$-uniform IASL (or $k$-uniform IASI) if $|f^+(e)| = k \forall e \in E(G)$. The vertex set $V(G)$ is called $l$-uniformly set-indexed, if all the vertices of $G$ have the set-indexing number $l$.

In this paper, we extend the concepts of topological set-labelings to integer additive set-labels of a given graph $G$ and establish some results on them.
2 Topological IASI Graphs

Note that under an integer additive set-labeling, no vertex of a given graph $G$ can have $\emptyset$ as its set-labeling. Hence, we need to consider only non-empty subsets of $X$ for set-labeling the elements of $G$.

Analogous to topological set-labels of graphs, we introduce the following notion of topological integer additive set-labels.

**Definition 2.1.** Let $G$ be a graph and let $X$ be a non-empty set of non-negative integers. An integer additive set-indexer $f : V(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$ is called a **topological integer additive set-labeling** (Top-IASL, in short) of $G$ if $f(V(G)) \cup \{\emptyset\}$ is a topology of $X$. A graph $G$ which admits a topological integer additive set-labeling is called a **topological integer additive set-labeled graph** (in short, Top-IASL graph).

**Definition 2.2.** A topological integer additive set-labeling $f$ is called a **topological integer additive set-indexer** (Top-IASI, in short) if the associated function $f^+ : E(G) \rightarrow \mathcal{P}(X)$ defined by $f^+(uv) = f(u) + f(v)$, $u, v \in V(G)$, is also injective. A graph $G$ which admits an integer additive set-graceful indexer is called an **topological integer additive set-indexed graph** (Top-IASI graph, in short).

**Remark 2.3.** Let $X$ be a finite set of non-negative integers and let $f : V(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$ be an integer additive set-labeling on a graph $G$. Note that, here the induced function $f^+ : E(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$ defined by $f^+(uv) = f(u) + f(v)$, is the sum set of the sets $f(u)$ and $f(v)$. Hence, $\{0\}$ can not be the set-label of any edge of $G$.

**Remark 2.4.** Let $X$ be a finite set of non-negative integers and let $f : V(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$ be a topological integer additive set-labeling on a graph $G$. Then, $X \in f(V(G))$ and by the definition of IASL, since $f^+(uv) = f(u) + f(v) \in \mathcal{P}(X)$, the set $\{0\}$ will be the set-label of a vertex that is adjacent to the vertex whose set-label is $X$ in $G$.

What are the structural characteristics of the Top-IASL graphs? We now proceed to find out the answers to the questions related to the structural properties of Top-IASL graphs.

**Proposition 2.5.** If $f : V(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$ is a Top-IASL of a graph $G$, then $G$ must have at least one pendant vertex.

**Proof.** Let $f$ be a Top-IASL defined on a graph $G$. Then, clearly $X \in f(V)$. That is, for some vertex $v \in V(G)$, $f(v) = X$. Then, by Remark 2.4, $v$ is adjacent to a vertex whose set-label is $\{0\}$. Now we claim that, the vertex $v$ can be adjacent to only one vertex that has the set-label $\{0\}$. This can be proved as follows.

Let $u$ be a vertex that is adjacent to the vertex $v$ and let $b$ be a non-zero element of $X$. Also, let $l$ be the maximal element of $X$. If $b \in f(u)$, the element $b + l$ is greater than $l$ and hence it will not be an element of $f^+(uv) = f(u) + f(v)$, contradicting the fact that $f$ is an IASL of $G$. Therefore, the vertex that has the set-label $X$ can be adjacent to a unique vertex that has the set-label $\{0\}$. That is, $G$ has at least one pendant vertex and hence at least one pendant edges. \qed
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Figure 1 depicts the Top-IASL, say $f$, of a graph $G$, with respect to a ground set $X = \{0, 1, 2, 3\}$ and a topology $T = \{\emptyset, X, \{0\}, \{1\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$ of $X$, where $f(V(G)) = T - \{\emptyset\}$ is the collection of the set-labels of the vertices in $G$.

The number of elements in the ground set $X$ is very important in all the studies of set-labeling of graphs. Keeping this in mind, we define

**Definition 2.6.** The minimum cardinality of the ground set $X$ required for a given graph to admit a topological IASL is known as the *topological set-indexing number* (Top-set-indexing number) of that graph.

**Proposition 2.7.** Let $f : V(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$ is a Top-IASL of a graph $G$. Then, the vertices whose set-labels containing the maximal element of the ground set $X$ are pendant vertices which are adjacent to the vertex having the set-label $\{0\}$.

**Proof.** For given ground set $X$ of non-negative integers, let $f : V(G) \rightarrow \mathcal{P}(X) - \emptyset$ be a Top-IASL of $G$. Let $l$ be the maximal element of the ground set $X$. Let $v$ be a vertex of $G$ whose set-label contains the element $l \in X$. Let $u$ be an adjacent vertex of $v$ whose set-label contains a non-zero element $b \in X$. Then, $b + l \not\in X$, contradicting the fact that $f$ is an IASL of $G$. If $l \in f(v)$ for $v \in V(G)$, then its adjacent vertices can have a set-label $\{0\}$. That is, all the vertices whose set-label contains the maximal element of the ground set $X$ must be adjacent to a unique vertex whose set-label is $\{0\}$. \qed

Invoking Proposition 2.5 and Proposition 2.7, we have

**Proposition 2.8.** Let $X$ be the ground set and $T$ be the topology of $X$ which are used for set-labeling the vertices of a Top-IASL graph $G$. Then, an element $x_r$ in $X$ can be an element of the set-label $f(v)$ of a vertex $v$ of $G$ if and only if $x_r + x_s \leq l$, where $x_s$ is any element of the set-label of another vertex $u$ which is adjacent to $v$ in $G$ and $l$ is the maximal element in $X$.

The following result is an immediate consequence of the above results.
Proposition 2.9. If $G$ has only one pendant vertex and if $G$ admits a Top-IASL, then $X$ is the only set-label of the vertices of $G$ containing the maximal element of $X$.

Is the converse of Proposition 2.5 true? The answer to this question depends upon the choice of $X$. First we verify the existence of Top-IASL for certain graphs having some pendant edges by choosing a ground set $X$ suitably. For this consider the following graphs.

Let $G$ be a graph on $m$ vertices and let $P_n$ be a path that has no common vertex with $G$. We call the graph obtained by identifying one vertex of $G$ and one end vertex of $P_n$ an $(m,n)$-ladle.

If $G$ is a cycle $C_m$, then this ladle graph is called an $(m,n)$-tadpole graph or a dragon graph. If $n = 1$ in a tadpole graph, then $G$ is called an $m$-pan.

If $G$ is a complete graph on $m$ vertices, then the corresponding $(m,n)$-ladle graph is called an $(m,n)$-shovel.

Now, we proceed to discuss the admissibility of Top-IASL by these types of graphs. The following result establishes the admissibility of Top-IASL by a pan graph.

Proposition 2.10. A pan graph admits a Top-IASL.

Proof. Let $G$ be an $m$-pan graph. Let $v_1, v_2, \ldots, v_m$ be the vertices of $C_n$. Without loss of generality, let $v_1$ be the unique vertex adjacent to $v$ in $G$. Choose a ground set $X$, containing the element 0, with sufficient cardinality. Label the vertex $v$ of $G$ by the set $X$ itself. Then, by proposition 2.5, $v_1$ must have the set-label $\{0\}$. Now, choose a collection of subsets $\{A_i\}; 1 \leq i \leq m$ of $X$, such that the set-label the vertex $v_i$ by a subset $A_i$ of $X$ in such a way that the sum of the maximal elements of the set-labels of any two adjacent vertices of $C_m$ is less than or equal to the maximal element of $X$ and for two set-labels $A_i$ and $A_j$ of two vertices, $A_i \cap A_j$ is also a set-label of a vertex in $G$. Then, the set $f(V(G)) \cup \{\emptyset\}$ is a topology of $X$. That is, $G$ admits a Top-IASL.

We now proceed to verify the admissibility of Top-IASL by the general tadpole graphs.

Proposition 2.11. A tadpole graph admits a Top-IASL.

Proof. Let $G$ be an $(m,n)$-tadpole graph. Let $\{v_1, v_2, v_3, \ldots, v_m\}$ be the vertex set of $C_m$ and let $\{u_1, u_2, u_3, \ldots, u_n\}$ be the vertex set of $P_n$. Without loss of generality, let $u_n$ be the pendant vertex of $P_n$ in $G$. Choose a ground set $X$, containing the element 0, with sufficient cardinality. Label the vertex $u_n$ of $G$ by the set $X$ itself and the vertex $u_{n-1}$, that is adjacent to $u_n$, by the set $\{0\}$. Now, choose a collection of subsets $\mathcal{A}$, such that the set-label other vertices $u_i$ of $P_n$ and the vertices $C_m$ by the sets in $\mathcal{A}$ in such a way that the sum of the maximal elements of the set-labels of any two adjacent vertices of $C_m$ is less than or equal to the maximal element of $X$ and for two set-labels $A_i$ and $A_j$ of two vertices, $A_i \cap A_j$ is also a set-label of a vertex in $G$. Then, the set $f(V(G)) \cup \{\emptyset\}$ is a topology of $X$. That is, $G$ admits a Top-IASL.
Figure 2a illustrates the admissibility of Top-IASL by the 5-pan and Figure 2a illustrates the admissibility of Top-IASL by the (5, 3)-tadpole graph.

(a) 5-pan with a Top-IASL.  
(b) (5, 3)-tadpole with a Top-IASL.

Figure 2

We can extend the above results to shovel graphs also. The following result establishes the admissibility of Top-IASL by shovel graphs by properly choosing the ground set $X$.

**Proposition 2.12.** The $(m, n)$-shovel graph admits a Top-IASL.

**Proof.** Let $G$ be an $(m, n)$-shovel graph. Let $\{v_1, v_2, v_3, \ldots, v_m\}$ be the vertex set of $K_m$ and let $\{u_1, u_2, u_3, \ldots, u_n\}$ be the vertex set of $P_n$. Without loss of generality, let $u_n$ be the pendant vertex of $P_n$ in $G$. Choose a ground set $X$, containing the element 0, with sufficient cardinality. Label the vertex $u_n$ of $G$ by the set $X$ itself and the vertex $u_{n-1}$, that is adjacent to $u_n$, by the set $\{0\}$. Now, choose a collection of subsets $A$, such that the vertices $K_m$ are labeled by the sets in $A$ in such a way that the maximal elements of the set-labels of any vertex of $K_m$ is less than or equal to half of the maximal element of $X$ and the other vertices $u_i$ of $P_n$ are labeled by the sets in $A$ such that the sum of the maximal elements of the set labels of two adjacent vertices in $P_n$ is less than or equal to the maximal element of $X$ and for two set-labels $A_i$ and $A_j$ of two vertices in $G$, $A_i \cap A_j$ is also a set-label of a vertex in $G$. Then, the set $f(V(G)) \cup \{\emptyset\}$ is a topology of $X$. That is, $G$ admits a Top-IASL.

Figure 3a depicts the admissibility of Top-IASL by the graph (5, 1)-shovel graph PTIASI2 and Figure 3b depicts the admissibility of Top-IASL by the graph (5, 3)-shovel graph.

The above propositions arise the question whether the existence of a pendant vertex in a given graph $G$ results in the admissibility of Top-IALSL by it. The choice of $X$ in all the above results played a major role in establishing a Top-IASL for $G$. The following is a necessary and sufficient condition for a given graph with at least one pendant vertex to admit a Top-IALSL.

**Theorem 2.13.** A graph $G$ admits a Top-IASL if and only if $G$ has at least one pendant vertex.
Proof. Let $G$ be a graph on $n$ vertices with at least one pendant vertex. The necessary part of the theorem follows from Proposition 2.5. Conversely, assume that $G$ has a pendant vertex, say $v$. Choose a ground set $X$ with sufficiently large cardinality so that we can choose a collection $\mathcal{A}$ of those proper subsets of $X$ with the following properties.

1. $\mathcal{A}$ has $n$ elements including the two sets $\{0\}$ and $X$ in it.
2. The sum of the maximal elements of any two sets in $\mathcal{A}$ is less than or equal to the maximal element of $X$.
3. For any two sets $A_i$ and $A_j$ in $\mathcal{A}$, their union $A_i \cup A_j$ is also in $\mathcal{A}$.
4. For any two non-singleton sets $A_r$ and $A_s$ in $\mathcal{A}$, their intersection $A_r \cap A_s$ is also in $\mathcal{A}$.

Let $v$ be labeled by the ground set $X$ and the vertex $u$ adjacent to $v$ in $G$ be labeled by the set $\{0\}$. Now, label the other vertices of $G$ by those subsets of $X$ in $\mathcal{A}$. Then, $f(V) \cup \{\emptyset\}$ will be a topology of $X$ and hence, $f$ is a Top-IASL of $G$.

Theorem 2.13 gives rise to the following result.

**Theorem 2.14.** Let $G$ be a graph with a pendant vertex $v$ which admits a Top-IASL, say $f$, with respect to a ground set $X$. Let $f_1$ be the restriction of $f$ to the graph $G - v$. Then, there exists a collection $\mathcal{B}$ of proper subsets of $X$ which together with $\{\emptyset\}$ form a topology of the union of all elements of $\mathcal{B}$.

**Proof.** Let $G$ be a graph with one pendant vertex, say $v$ and $X$ be the ground set for labeling the vertices of $G$. Choose the collection $\mathcal{B}$ of proper subsets of $X$ which contains the set $\{0\}$ and has the cardinality $n - 1$ such that the sum of the maximal elements of any two sets in it is less than or equal to the maximal element of $X$ and the union of any two sets and the intersection of any two non-singleton sets in $\mathcal{B}$ are also in $\mathcal{B}$. Then, by Theorem 2.13, the set-labeling $f$ under which the pendant vertex $v$ is labeled by the set $X$ and other vertices of $G$ by the elements of $\mathcal{B}$ is a Top-IASL of $G$. 

![Diagram of graphs](image_url)
Let \( f_1 \) be the restriction of \( f \) to the graph \( G - v \). Therefore, \( B = f_1(V(G - v)) \).

Now let \( B = \bigcup_{B_i \in \mathcal{B}} B_i \) and let \( \mathcal{T'} = \mathcal{B} \cup \{\emptyset\} \). Since \( G \) has only one end vertex, by Proposition 2.9, no element of \( \mathcal{A} \) contains the maximal element of \( X \). Therefore, \( B \) also does not contain the maximal element of \( X \). Since the union of any number of sets in \( \mathcal{B} \) is also in \( \mathcal{B} \) and \( \emptyset \in \mathcal{T'} \), the finite intersection of elements in \( \mathcal{T'} \) is also in \( \mathcal{T'} \). The set \( \mathcal{T'} = \mathcal{B} \cup \{\emptyset\} \) is a topology of the maximal set \( B \) in \( \mathcal{B} \).

**Remark 2.15.** If \( v \) is the only pendant vertex of a given graph \( G \), then the collection \( B = f(V(G - v)) \), chosen as explained in Theorem 2.14 does not induce a topological IASL on the graph \( G - v \), since \( f^+(uw) \neq f(u) + f(w) \), for some edge \( uw \in E(G - v) \).

### 3 Top-IASLs with respect to Certain Topologies

In this section, we discuss the existence and admissibility of topological IASLs with respect to some standard topologies like indiscrete topologies and discrete topologies.

A topology \( \mathcal{T} \) is said to be an indiscrete topology of \( X \) if \( \mathcal{T} = \{\emptyset, X\} \). Hence the following result is immediate.

**Theorem 3.1.** A graph \( G \) admits a Top-IASL with respect to the indiscrete topology \( \mathcal{T} \) if and only if \( G \cong K_1 \).

**Proof.** Let \( v \) be the single vertex of the graph \( G = K_1 \). Let \( X \) be the ground set for set-labeling \( G \). Let \( f(v) = X \). Then \( f(V) = \{X\} \) and \( f(V) \cup \{\emptyset\} = \{\emptyset, X\} \), which is the indiscrete topology on \( X \). Conversely, assume that \( G \) admits a Top-IASL with respect to the indiscrete topology \( \mathcal{T} \) of the ground set \( X \). Then, \( f(V(G)) = \mathcal{T} - \{\emptyset\} = \{X\} \), a singleton set. Therefore, \( G \) can have only a single vertex. That is, \( G \cong K_1 \).

From Proposition 3.1, we have the following result.

**Proposition 3.2.** The Top-set-indexing number of \( K_1 \) is 1.

Another basic topology of a set \( X \) is the sierpinski’s topology. If \( X \) is a two point set, say \( X = \{0, 1\} \), then the topology \( \mathcal{T} = \{\emptyset, \{0\}, \{1\}, X\} \) is called the Sierpinski’s topology. The following result establishes the conditions required for a graph to admit a Top-IASL with respect to the Sierpinski’s topology.

**Theorem 3.3.** A graph \( G \) admits a Top-IASL with respect to the Sierpinski’s topology if and only if \( G \cong K_2 \).

**Proof.** Let \( G \) be the given graph, with vertex set \( V \), which admits a Top-IASL with respect to the Sierpinski’s topology. Let a two point set \( X = \{0, 1\} \) be the ground set used for set-labeling the graph \( G \). Then, \( f(V) = \{\{0\}, X\} \). Therefore, \( G \) can have exactly two vertices. That is, \( G \cong K_2 \).
Conversely, assume that $G \cong K_2$. Let $u$ and $v$ be the two vertices of $G$. Choose a two point set $X$ as the ground set to set-label the vertices of $G$. Label the vertex $u$ by $X$. Then by Proposition 2.5, $v$ must have the set-label $\{0\}$. Then $f(V(G)) = \{\{0\}, X\}$. Then, $f(v(G)) \cup \{\emptyset\}$ is a topology on $X$, which is a Sierpenski’s topology of $X$. Therefore, $G \cong K_2$ admits a Top-IASL with respect to the Sierpenski’s topology.

In view of Proposition 3.3, we claim that for any ground set $X$ containing two or more elements, one of which is $0$, induces a Top-IASL on $K_2$. Therefore, the following result is immediate.

**Proposition 3.4.** The Top-set-indexing number of $K_2$ is $2$.

The following results are the immediate consequences of 2.13.

**Proposition 3.5.** For $n \geq 3$, no complete graph $K_n$ admits a Top-IASL.

*Proof.* The proof follows from Theorem 2.13 and from the fact that a complete graph on more than two vertices does not have any pendant vertex. \( \square \)

**Proposition 3.6.** For $m,n \geq 2$, no complete bipartite graph $K_{m,n}$ admits a Top-IASL.

*Proof.* The proof is immediate from the fact that a complete bipartite graph has no pendant vertices. \( \square \)

**Corollary 3.7.** A path graph $P_m$ admits a Top-IASL.

*Proof.* Every path graph $P_m$ has two pendant vertices and hence satisfy the condition mentioned in Theorem 2.13. Hence $P_m$ admits a Top-IASL. \( \square \)

**Proposition 3.8.** Every tree admits a Top-IASL.

*Proof.* Since every tree $G$ has at least two pendant vertices, by Theorem 2.5, $G$ admits a Top-IASL. \( \square \)

**Proposition 3.9.** No cycle graph $C_n$ admits a Top-IASL.

*Proof.* A cycle does not have any pendant vertex. Then, the proof follows immediately by Theorem 2.13. \( \square \)

In view of the above results, we arrive at the following inference.

**Proposition 3.10.** For $k \geq 2$, no $k$-connected graph admits a Top-IASL with respect to a ground set $X$.

*Proof.* No biconnected graph $G$ can have pendant vertices. Hence, by Theorem 3.11, $G$ can not admit a Top-IASL. \( \square \)
We have already discussed the admissibility of a Top-IASL by a graph with respect to the indiscrete topology of the ground set $X$. In this context, it is natural to ask whether a given graph admits the Top-IASL with respect to the discrete topology of a given set $X$. The following theorem establishes the condition required for $G$ to admit a Top-IASL with respect to the discrete topology of $X$.

**Theorem 3.11.** A graph $G$, on $n$ vertices, admits a Top-IASL with respect to the discrete topology of the ground set $X$ if and only if $G$ has at least $2^{|X|-1}$ pendant vertices which are adjacent to a single vertex of $G$.

**Proof.** Let $|X| = m$. Let the graph $G$ admits a Top-IASL $f$ with respect to the discrete topology $T$ of $X$. Therefore, $f(V(G)) = P(X) - \{\emptyset\}$. Then, $|f(V(G))| = 2^{|X|} - 1$. Now, let $l$ be the maximal element in $X$. The number of subsets of $X$ containing $l$ is $2^{|X|} - 1$. Since $f$ is a Top-IASL with respect to the discrete topology, all these sets containing $l$ must also be the set-labels of some vertices of $G$. By Proposition 2.7, all these vertices must be adjacent to the vertex whose set-label is $\{0\}$. By Proposition 2.8, no two vertices whose set-labels contain $l$ can be adjacent among themselves or to any other vertex which has a set-label with non-zero elements. Therefore, $G$ has $2^m - 1$ pendant vertices which are adjacent to a single vertex whose set-label is $\{0\}$.

Conversely, let $G$ be a graph with $n = 2^{|X|} - 1$ vertices such that at least $2^{|X|-1}$ of them are pendant vertices incident on a single vertex of $G$. Label these pendant vertices by the $2^{|X|-1}$ subsets of $X$ containing the maximal element $l$ of $X$. Label remaining vertices of $G$ by the remaining $2^{|X|-1} - 1$ subsets of $X$ which do not contain the element $l$, in such a way that the sum of the maximal elements of the set-labels of two adjacent vertices is less than or equal to $l$. This labeling is clearly a Top-IASL on $G$. That is, $G$ admits a Top-IASL with respect to the discrete topology of $X$.

This completes the proof.

Figure 4 depicts the existence of a Top-IASL with respect to the discrete topology of the ground set $X$ for a graph $G$.

Since the necessary and sufficient condition for a graph to admit a Top-IASL with respect to the discrete topology of ground set $X$ is that $G$ has at least $2^{|X|-1}$ pendant edges that incident at a single vertex of $G$, no paths $P_n$; $n \geq 3$, cycles, complete graphs and complete bipartite graphs can have a Top-IASL with respect to discrete topology of $X$.

**Theorem 2.7** gives rise to the following results also.

**Corollary 3.12.** A graph on even number of vertices does not admit a Top-IASL with respect to the discrete topology of the ground set $X$.

**Proof.** If a graph on $n$ vertices admits a Top-IASL with respect to the discrete topology of the ground set $X$, then by Theorem 3.11, $n = 2^{|X|-1}$, which can never be an even integer. Therefore, $G$ on even number of vertices does not admit a Top-IASL with respect to the discrete topology of $X$. 

\[\Box\]
Corollary 3.13. A star graph $K_{1,r}$ admits a Top-IASL with respect to the discrete topology of the ground set $X$, if and only if $r = 2^{|X|} - 2$.

Proof. First assume that the star graph $G = K_{1,r}$ admits a Top-IASL $f$ with respect to the discrete topology of the ground set $X$. Then, $f(V(G)) = \mathcal{P}(X) - \{\emptyset\}$. That is, $|f(V(G))| = 2^{|X|} - 1$. Hence, $G$ must have $2^{|X|} - 1$ vertices. That is, $r + 1 = 2^{|X|} - 1$. Therefore, $r = 2^{|X|} - 2$.

Conversely, consider a star graph $G = K_{1,r}$, where $r = 2^n - 2$ for some positive integer $n$. Choose a set $X$ with cardinality $n$, which consists of the element 0. Note that the number of non-empty subsets of $X$ is $2^n - 1$. Define a set-labeling $f$ of $G$ which assigns $\{0\}$ to the central vertex of $G$ and the other non-empty subsets of $X$ to the pendant vertices of $G$. Clearly, this labeling is an IASL of $G$. Also, $f(V(G)) = \mathcal{P}(X) - \{\emptyset\}$. Therefore, $f$ is a Top-IASL of $G$ with respect to the discrete topology of $X$.

Figure 5 illustrates the existence of a Top-IASL with respect to the discrete topology of the ground set $X$ for a star graph.

4 Conclusion

In this paper, we have discussed the concepts and properties of topological integer additive set-indexed graphs analogous to those of topological IASI graphs and have done a characterisation based on this labeling.

We note that the admissibility of topological integer additive set-indexers by the given graphs depends also upon the number and nature of the elements in $X$ and the topology $T$ of $X$ concerned. Hence, choosing a ground set $X$ is very important in the process of checking whether a given graph admits a Top-IASL graph.

Certain problems in this area are still open. Some of the areas which seem to be promising for further studies are listed below.
Problem 4.1. Characterise different graph classes which admit topological integer additive set-labelings.

Problem 4.2. Estimate the Top-set-indexing number of different graphs and graph classes which admit topological integer additive set-labelings.

Problem 4.3. Verify the existence of topological integer additive set-labelings for different graph operations and graph products.

Problem 4.4. Establish the necessary and sufficient condition for a graph to admit topological integer additive set-indexer.

Problem 4.5. Characterise the graphs and graph classes which admit Top-IASI.

The integer additive set-indexers under which the vertices of a given graph are labeled by different standard sequences of non negative integers, are also worth studying. All these facts highlight a wide scope for further studies in this area.

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Figure 5:

Figure 6: A Star graph with a Top-IASL with respect to the discrete topology of $X$. 
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