Classifying Spaces of Subgroups of Profinite Groups

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Abstract

The set of all closed subgroups of a profinite carries a natural profinite topology. This space of subgroups can be classified up to homeomorphism in many cases, and tight bounds placed on its complexity as expressed by its scattered height.

1 Introduction

In this paper we continue the task of topologically classifying the space of closed subgroups of a profinite group. By definition, a profinite group $G$ is one that can be represented as a projective limit of finite groups, $\lim\leftarrow G_{\lambda}$. Writing $\mathcal{S}(G)$ for the set of closed subgroups of a profinite group, $G = \lim\leftarrow G_{\lambda}$, one sees that $\mathcal{S}(G) = \lim\leftarrow \mathcal{S}(G_{\lambda})$. Giving the finite set $\mathcal{S}(G_{\lambda})$ the discrete topology for each $\lambda$, we see that the projective limit $\mathcal{S}(G)$ picks up a natural topology. This topology is profinite (compact, Hausdorff and zero-dimensional). An alternative description of the topology on $\mathcal{S}(G)$ is that it is the subspace topology inherited by $\mathcal{S}(G)$ from the space of all compact subsets of $G$ with the Vietoris topology, and so the topology is independent of the particular projective representation of $G$.

In previous work [2] we have shown how to use the space of subgroups to count the closed (and closed normal) subgroups of a profinite group, and
gave a partial classification (up to homeomorphism) in certain cases. From that work we know to concentrate on countably based (i.e. second countable) profinite groups.

We know that the space of subgroups \( S(G) \) is countable if and only if \( G \) is a central extension of \( \bigoplus_{i=1}^{k} \mathbb{Z}_{p_i} \) where the \( p_i \)'s are distinct primes, and \( \mathbb{Z}_{p_i} \) is the \( p_i \)-adic integers. In this situation, \( S(G) \cong \omega^k.n + 1 \).

Recall that a profinite space is homeomorphic to the Cantor set if and only if it is countably based and has no isolated points. Hence \( S(G) \) is homeomorphic to the Cantor space if and only if \( S(G) \) has no isolated points; and we proved that \( S(G) \) has isolated points if and only if \( G \) is finitely generated virtually nilpotent and only finitely many primes divide the order of \( G \).

What remains is the case when we have a countably based profinite group \( G \), whose subgroup space is not countable but does have isolated points. We tackle the problem of classifying these subgroup spaces by analyzing the structure of the isolated points via the Cantor-Bendixson process.

Details of this process, and relevant topological results, are given in Section 2. In summary, for any space \( X \) let \( i(X) \) be the set of isolated points in \( X \), and call \( X' = X \setminus i(X) \) the derived set of \( X \). We can then take the derived set of the derived set of \( X \), and so on (potentially transfinitely). For a countably based profinite space this procedure will terminate at some least countable ordinal \( \alpha \), either with an empty derived set – in which case the space \( X \) is countable – or with a derived set homeomorphic to the Cantor set. The scattered height of \( X \) is the ordinal \( \alpha \).

We show (Section 3) that if \( S(G) \) has any isolated points then in fact all (the infinitely many) open subgroups are isolated and they are dense in \( S(G) \). Next we investigate the properties of solitary subgroups – those which are isolated in \( S(G)' \). From this we prove a very tight restriction on the scattered height of a subgroup space (which is potentially any one of the uncountable family of countable ordinals), namely that if \( G \) is profinite and \( S(G) \) has any isolated points, then \( ht(S(G)) \leq k + 1 \) where \( k \) is the finite number of primes \( p \) such that \( p^\infty \) divides the order of \( G \). In particular, if \( G \) is pro-\( p \) then \( ht(S(G)) \leq 2 \). We show in what follows that all scattered heights of \( S(G) \) permitted by these results are attained.

Since a profinite group \( G \) with isolated subgroups has an open normal subgroup isomorphic to \( N = \bigoplus_{i=1}^{k} G_{p_i} \) where each \( G_{p_i} \) is a pro-\( p_i \) group, its space of subgroups has a clopen subspace homeomorphic to \( S(N) = \prod_{i=1}^{k} S(G_{p_i}) \). Thus we next focus on classifying subgroup spaces of pro-\( p \)
groups.

Now suppose $G$ is a pro-$p$ group with isolated subgroups but an uncountable subgroup space. A complete topological classification of $S(G)$ seems well out of reach. Instead our results (in Section 4) show the diversity of possibilities.

Up to homeomorphism, there is a unique countably based profinite space of scattered height 1 with a dense set of isolated points. This space is called the Pelczyński space, $P$. We start, then, by investigating when subgroup spaces are homeomorphic to $P$ – in other words those groups with no solitary subgroups. A wide variety of pro-$p$ groups $G$ have $S(G) \cong P$ including: $\mathbb{Z}_p^n$ and the free pro-$p$ group on $n$ generators ($n > 1$), insoluble just-infinite pro-$p$ groups, and finitely generated nilpotent pro-$p$ groups with $h(G) > 1$.

Next we consider the possibilities for $S(G)$ to have scattered height 2 – in other words has solitary subgroups. One case is that there are (countably) infinitely many solitary subgroups. This occurs if a non-abelian just-infinite pro-$p$ group is multiplied by $\mathbb{Z}_p$. There are even poly-$\mathbb{Z}_p$ with $h(G) > 1$ for which this is true. We note that there are uncountably many pair-wise non-homeomorphic countably based uncountable profinite spaces of scattered height 2.

The second case is that there are just finitely many solitary subgroups. Unlike the previous case, there is just a unique (up to homeomorphism) countably based profinite space $X_n$ such that the isolated points are dense, and there are exactly $n$ isolated points in $X_n'$. This situation arises when the pro-$p$ group $G$ is virtually $\mathbb{Z}_p$. In fact if $G$ is profinite with an open non-central subgroup isomorphic to $\mathbb{Z}_p$, then $S(G) \cong P \oplus (\omega \cdot n + 1)$ (so there are $n$ solitary subgroups). While if $G$ is a non-abelian just-infinite pro-$p$ group then $S(G) \cong P \oplus (\omega + 1)$ (with exactly one solitary subgroup).

This leaves a natural open question: can an infinite finitely generated pro-$p$ group which is not virtually $\mathbb{Z}_p$ have a finite (non-zero) number of solitary subgroups? A natural conjecture leading to a negative solution to this problem is refuted, and a potential example exhibited, but not proved to have just finitely many solitary subgroups.

We conclude (Section 5) with some observations on the case of general profinite groups. In particular we present examples showing that all permitted scattered heights are attained.
2 Background Material

The Cantor–Bendixon Process

Definition 2.1 Let $X$ be a topological space. For $Y \subseteq X$, let $Y'$ denote the set of all limit points of $Y$, that is $Y \setminus Y'$ is the set of points of $Y$ which are isolated in $Y$. We define the following transfinite sequence.

\[
X^{(0)} = X,
X^{(\alpha + 1)} = (X^{(\alpha)})', \text{ for } \alpha \text{ an ordinal},
X^{(\lambda)} = \bigcap_{\mu < \lambda} X^{(\mu)}, \text{ for } \lambda \text{ a limit ordinal}.
\]

A space is perfect if it has no isolated points.

Lemma 2.2 Let $X$ be a Hausdorff space.

(i) $X^{(\alpha)}$ is closed in $X$ for every ordinal $\alpha$.

(ii) If $\alpha \leq \beta$ then $X^{(\alpha)} \supseteq X^{(\beta)}$.

(iii) $X^{(\alpha)} \setminus X^{(\alpha+1)}$ is the set of isolated points of $X^{(\alpha)}$ for every ordinal $\alpha$, and is countable if $X$ is countably based.

(iv) $(X^{(\alpha)})^{(\beta)} = X^{(\alpha + \beta)}$ for all ordinals $\alpha$ and $\beta$.

(v) If $Y \subseteq Z \subseteq X$ then $Y^{(\alpha)} \subseteq Z^{(\alpha)}$ for every ordinal $\alpha$.

(vi) There is a least ordinal $\lambda$ such that $X^{(\lambda)} = X^{(\lambda+1)}$ and if $\alpha \geq \lambda$ then $X^{(\alpha)} = X^{(\lambda)}$. $X^{(\lambda)}$ is perfect in itself. If $X$ is countably based then $\lambda$ is countable.

(vii) If $X$ is compact, and $X^{(\alpha)} = \emptyset$ for some ordinal $\alpha$, then there is a successor ordinal $\lambda$ such that $X^{(\lambda)} = \emptyset$ and $X^{(\lambda-1)}$ is a finite non-empty discrete space.

(ix) If $Y$ is open in $X$ then $Y^{(\alpha)} = Y \cap X^{(\alpha)}$ for every ordinal $\alpha$.

It follows from Lemma 2.2 that every countably based Hausdorff space can be written as the disjoint union of a countable set (the scattered part of $X$) and a set which is perfect in itself (the perfect hull of $X$); this is the Cantor–Bendixon theorem.
Definitions 2.3 Let $X$ be a Hausdorff space. The scattered height of $X$, $ht(X)$ is the least ordinal $\lambda$ such that $X^{(\lambda)} = X^{(\lambda+1)}$.

Let $x \in X \setminus X^{(ht(X))}$. $ht(x, X)$ is defined to be the least ordinal $\alpha$ such that $x \not\in X^{(\alpha+1)}$.

Of course to say that $x \in X \setminus X^{(ht(X))}$ has $ht(X) = \alpha$ is precisely the same as saying that $x \in X^{(\alpha)}$ and that $x$ is isolated in $X^{(\alpha)}$. The next lemma follows immediately from the definitions.

Lemma 2.4 Let $X$ be a Hausdorff space, $Y$ an open set in $X$ and $x \in Y$. Then $ht(Y) \leq ht(X)$ and $ht(x, X) = ht(x, Y)$.

In relation to Lemma 2.2(vii), a compact Hausdorff space $X$ for which $X^{(ht(X))} = \emptyset$ is sometimes known as a scattered space. It is clear that a countably based profinite space is scattered if and only if it is countable. On the other hand it is clear from Lemma 2.2 and Proposition 2.7 that if $X$ is an uncountable countably based profinite space then $X^{(ht(X))}$ is homeomorphic to the Cantor set. In this sense, in order to understand the uncountable countably based profinite spaces we need to understand how to ‘glue’ a countable set onto a Cantor set.

Characterizations of Profinite Spaces We now describe some countably based profinite spaces and some well-known characterisation theorems concerning them. These characterisation theorems are all consequences of a celebrated theorem of Vaught; see, for example Section 27 of [7].

Every ordinal is a topological space when considered with the order topology. Successor ordinals, $\alpha + 1$, are profinite, and countably based if and only if $\alpha$ is countable. Note that $\omega + 1$ is homeomorphic to a convergent sequence.

Lemma 2.5 Let $\alpha$ be a non-zero countable ordinal, and $n$ be a positive integer. Then $\omega^{\alpha}n + 1$ is a countably infinite profinite space. Moreover $(\omega^{\alpha}n + 1)^{(\alpha)} = \{\omega^{\alpha}, \omega^{\alpha}2, \ldots, \omega^{\alpha}n\}$. In particular $ht(\omega^{\alpha}n + 1) = \alpha + 1$ and $|(\omega^{\alpha}n + 1)^{(\alpha)}| = n$.

Proposition 2.6 Let $X$ be a countable profinite space. Then $X$ is homeomorphic to $\omega^{ht(X)-1}|X^{(ht(X)-1)}| + 1$.

Proposition 2.7 A space is homeomorphic to the Cantor set if and only if it a countably based profinite space with no isolated points.
Definition 2.8 Let $F_0 = P_0 = [0, 1]$, the closed unit interval.

Let $F_1 = F_0 \setminus (1/3, 2/3)$ and let $P_1 = F_1 \cup \{1/2\}$. That is to form $F_1$ we have removed the open middle third interval of $F_0$, and to form $P_1$ we have reinserted the mid-point of the deleted interval.

Continuing we remove the middle third intervals of the remaining segments in $F_1$ to form $F_2$, that is $F_2 = F_1 \setminus ((1/9, 2/9) \cup (7/9, 8/9))$. We let $P_2 = F_2 \cup \{1/6, 5/6\}$.

We continue, forming $F_i$ by removing the middle third intervals of the remaining segments in $F_{i-1}$ and forming $P_i$ by reinserting the mid-points of the deleted intervals in $F_i$.

We let $F = \bigcap_{i=0}^{\infty} F_i$ and $P = \bigcap_{i=0}^{\infty} P_i$. $P$ is called Pełczyński space.

Clearly $F$ is the standard construction of the middle third Cantor set, $P$ is a profinite space and it is clear that $P \setminus F$ is the set of isolated points in $P$ and that this is a countable dense subset of $P$. Since $P' = F$, $ht(P) = 1$. These facts characterise Pełczyński space as a countably based profinite space.

Proposition 2.9 Let $X$ be an infinite countably based profinite space. Then $X$ is homeomorphic to Pełczyński space, $P$, if and only if $ht(X) = 1$ and $X \setminus X'$ is dense in $X$.

Pełczyński space will arise frequently in the sequel as the space of closed subgroups of a profinite group. Now Pełczyński space has scattered height 1. We now briefly consider uncountable countably based profinite spaces $X$ of scattered height 2 and with $X \setminus X'$ dense in $X$. Now if $X' \setminus X^{(2)}$ is countably infinite then there are a lot of possibilities for $X$ up to homeomorphism. But if $X' \setminus X^{(2)}$ is finite of size $n$ then $n$ determines the space up to homeomorphism.

Proposition 2.10 Let $X$ be an uncountable countably based profinite space and let $n$ be a positive integer. Then $X$ is homeomorphic to $P \oplus (\omega n + 1)$ if and only if $ht(X) = 2$, $X \setminus X'$ is dense in $X$ and $|X' \setminus X^{(2)}| = n$.

Dimension, and Pro-$p$ Groups of Finite Rank In Section 4 we make frequent use of the dimension function of pro-$p$ groups of finite rank. A profinite group $G$ has finite rank if there is an integer $n$ such that every closed subgroup of $G$ can be topologically generated by $\leq n$ elements. It turns out that pro-$p$ groups of finite rank are precisely the class of $p$-adic analytic pro-$p$ groups. Hence $G$ has a dimension, $\dim(G)$, as a $p$-adic
analytic manifold. A more algebraic definition of \( \dim(G) \) is as the minimal size of a (topological) generating set of any powerful open subgroup. A pro-

\( p \)-group \( G \) is powerful if \( p \) is odd and \( G/G^p \) is abelian or \( p = 2 \) and \( G/G^4 \) is abelian. The properties of \( \dim \) of use here are as follows, see [3]:

**Theorem 2.11** Let \( G \) be a pro-

\( p \)-group of finite rank, \( H \leq C_G \) and \( K \triangleleft C_G \).

Then (i) \( \dim(H) \leq \dim(G) \), (ii) \( \dim(G) = \dim(K) + \dim(G/K) \), (iii) \( H \) is finite if and only if \( \dim(H) = 0 \), and (iv) \( H \leq O_G \) if and only if \( \dim(H) = \dim(G) \).

**Subgroups of Products** We frequently need to analyse subgroups of direct products. For this we use a profinite version of a well-known result for abstract groups. The abstract version has been variously attributed to Goursat, Remak, and to Klein and Fricke. It is clear from the proof of the result for abstract groups (see Theorem 1.61 of [6] and Theorem 8.19 and p. 185 of [3]), that the same argument works for profinite groups.

**Proposition 2.12** Let \( G = G_1 \times G_2 \) be a profinite group and let \( \pi \) and \( \rho \) be the corresponding projections onto \( G_1 \) and \( G_2 \) respectively. Let \( H \leq C_G \).

Then \( \pi(H) = HG_2 \cap G_1 \) and \( H \cap G_1 \triangleleft C \) \( HG_2 \cap G_1 \). Similarly \( \rho(H) = HG_1 \cap G_2 \) and \( H \cap G_2 \triangleleft C \) \( HG_1 \cap G_2 \). Also \( (H \cap G_1) \times (H \cap G_2) \triangleleft C \) \( H \). We have the following topological isomorphisms:

\[
\frac{HG_2 \cap G_1}{H \cap G_1} \cong H G_1 \cap G_2 \quad \frac{HG_2 \cap G_1}{H \cap G_2} \cong \frac{H}{(H \cap G_1) \times (H \cap G_2)}.
\]

**The Space of Closed Subgroups** We can concretely describe canonical basic open neighbourhoods of a subgroup in \( S(G) \) for a profinite group \( G \) as follows.

**Definition 2.13** Let \( G \) be a profinite group. For \( H \leq C_G \) and \( N \triangleleft O_G \), we define \( B(H, N) = \{ K \leq C_G \mid KN = HN \} \).

**Lemma 2.14** Let \( G \) be a profinite group, and \( H \leq C_G \). Suppose that \( (N_\lambda)_{\lambda \in \Lambda} \) is a family of open normal subgroups of \( G \), forming a base for the open neighbourhoods of \( 1 \) in \( G \). Then \( (B(H, N_\lambda))_{\lambda \in \Lambda} \) forms a base for the open neighbourhoods of \( H \) in \( S(G) \).

**Corollary 2.15** Let \( G \) be a profinite group. Then \( S(G) \) is countably based if and only if \( G \) is countably based.
A frequently used result, helping to reduce problems about the space of subgroups of general profinite groups to the space of subgroups of pro-$p$ groups, follows from Proposition 2.12.

**Proposition 2.16** If a profinite group $G$ is topologically isomorphic to $\prod_{i=1}^{n} G_{p_i}$ where $p_1, \ldots, p_n$ are distinct primes, and $G_{p_i}$ is a pro-$p_i$ group, then $S(G) \cong \prod_{i=1}^{n} S(G_{p_i})$.

## 3 Isolated and Solitary Subgroups, Scattered Height

### Isolated Subgroups

In [2] we proved:

**Theorem 3.1 (Gartside and Smith)** Let $G$ be a profinite group. Then the following are equivalent.

(i) $S(G)$ is not perfect.

(ii) $G$ has an isolated open subgroup.

(iii) Every open subgroup of $G$ is isolated.

(iv) The Frattini subgroup, $\Phi(G)$, the intersection of the maximal proper open subgroups, is an open normal subgroup of $G$.

(v) $G$ is finitely generated, virtually pronilpotent and only finitely many primes divide the order of $G$.

From Proposition 3.1 we see that if $G$ is a profinite group with $S(G)$ not perfect then the set of isolated subgroups of $G$ coincides with the set of open subgroups of $G$. We make the following observation about the set of open subgroups.

**Lemma 3.2** Let $G$ be a profinite group. Then the set of open subgroups of $G$ is dense in $S(G)$.

**Proof.** If $H \leq_G G$ and $N \triangleleft_O G$ then $HN \leq_O G$ and $HN \in B(H, N)$. Thus by Lemma 2.14 the set of open subgroups is dense in $S(G)$. $lacksquare$

We are thus led to the following description of $S(G)$ when $G$ is a profinite group and $S(G)$ is not perfect.
Proposition 3.3  Let $G$ be a profinite group. If $S(G)$ is not perfect then the set of open subgroups of $G$ is a countable open dense discrete subspace of $S(G)$ and the set of closed non-open subgroups is a perfect set in $S(G)$.

Proof. If $G$ is a finitely generated profinite group then $G$ has only countably many open subgroups. The result now follows immediately from Theorem 3.1 and Lemma 3.2. □

Proposition 3.4  Let $G$ be a countably based profinite group. Then $S(G)$ is not perfect if and only if the set of open subgroups of $G$ is open in $S(G)$.

Proof. By Theorem 3.1, if $S(G)$ is not perfect then the set of open subgroups of $G$ is open in $S(G)$. Now suppose that the set of open subgroups of $G$, $O(G)$, is open in $S(G)$. Then $O(G)$ is a Baire space. $O(G) = \bigcup_{H \leq O G} \{H\}$, and each $\{H\}$ is closed in $O(G)$. Since $G$ is countably based, $O(G)$ is countable and since $O(G)$ is a Baire space there exists an open subgroup $H$ of $G$ such that $\{H\}$ is open in $O(G)$. But then $\{H\}$ is open in $S(G)$; that is, $H$ is an isolated open subgroup of $G$. Hence $S(G)$ is not perfect. □

Solitary subgroups  In this section we further analyse the space of closed subgroups of profinite groups with isolated subgroups.

Definition 3.5  Let $G$ be a profinite group with $S(G)$ not perfect and $H \leq C G$. $H$ is said to be a solitary subgroup of $G$ if $H \in S(G)'$ and $H$ is isolated in $S(G)'$.

Lemma 3.6  Let $G$ be a profinite group with $S(G)$ not perfect, and let $H \in S(G)'$.

(i) $H$ is solitary in $G$ if and only if there exists an open normal subgroup $N$ of $G$ such that if $K \leq C G$ and $KN = H N$ then either $K = H$ or $K \leq O G$.

(ii) If $K \leq O G$ with $H \leq K$ then $H$ is solitary in $G$ if and only if $H$ is solitary in $K$.

(iii) If $K \vartriangleleft C G$ with $K \leq H$ and $H$ is solitary in $G$ then $H/K$ is solitary in $G/K$. 

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(iv) Let \( g \in G \). Then conjugation by \( g \) induces a homeomorphism from \( S(G)' \) to \( S(G)' \). In particular if \( H \) is solitary in \( G \) then \( H^g \) is solitary in \( G \). Moreover if \( N \) is an open normal subgroup of \( G \) such that \( B(H, N) \cap S(G)' = \{H\} \) then \( B(H^g, N) \cap S(G)' = \{H^g\} \).

(v) If \( H \) is solitary in \( G \) then \( \Phi(H) \not\leq_O H \).

(vi) If \( H \) is solitary in \( G \) then there exists \( K \leq_O G \) with \( H \not\leq_C K \) such that \( K/H \cong \mathbb{Z}_p \) for some prime \( p \).

(vii) \( 1 \) is solitary in \( G \) if and only if \( G \) is virtually \( \mathbb{Z}_p \) for some prime \( p \).

**Proof.** By Lemma 2.13 \( \{B(H, N) \cap S(G)' \mid N \not\leq_O G\} \) is a base for the open neighbourhoods of \( H \) in \( S(G)' \). From this, (i) follows immediately.

For (ii), let \( K \leq_O G \). Then \( S(K) \) is open in \( S(G) \). Since \( S(K)' = S(K) \cap S(G)' \), \( S(K)' \) is open in \( S(G)' \). The claim now follows from the fact that a point isolated in an open subspace is isolated.

For (iii), let \( K \not\leq_C G \) with \( K \leq H \) and suppose that \( H \) is solitary in \( G \). Then there exists an open neighbourhood \( B \) of \( H \) in \( S(G) \) such that \( B \cap S(G)' = \{H\} \). Let \( \pi \colon G \to G/K \) be the natural map. Then \( \pi \) is an open continuous map and so by Proposition 5 of \( \mathbf{[I]} \) \( S(\pi) \) is an open continuous map. Thus \( B^* = S(\pi)(B) \) is open in \( S(G/K) \). Clearly \( H/K \in B^* \cap S(G/K)' \) and so \( \{H/K\} \subseteq B^* \cap S(G/K)' \). Let \( L/K \in B^* \cap S(G/K)' \). Then \( L \subseteq B \cap S(G)' = \{H\} \). Thus \( L = H \) and so \( B^* \cap S(G/K)' = \{H/K\} \) is open in \( S(G/K)' \) as required.

For (iv), let \( g \in G \). Then conjugation by \( g \) is a topological automorphism of \( G \), and so conjugation by \( g \) is a homeomorphism from \( S(G) \) to \( S(G) \). Clearly if \( K \leq_C G \) then \( K \in S(G)' \) if and only if \( K^g \in S(G)' \). Also by Proposition 3.3 \( S(G)' \) is closed in \( S(G) \). Thus conjugation by \( g \) is a homeomorphism from \( S(G)' \) to \( S(G)' \). It is now clear that if \( H \) is solitary in \( G \) then \( H^g \) is solitary in \( G \). Now suppose that \( N \) is an open normal subgroup of \( G \) such that \( B(H, N) \cap S(G)' = \{H\} \). Let \( L \subseteq B(H^g, N) \cap S(G)' \) and let \( K = L^{g^{-1}} \). Then \( K \in S(G)' \) and \( K^g N = L N = H^g N \). So \( K N = H N \) and hence \( K = H \). Thus \( L = H^g \) as required.

For (v), suppose that \( \Phi(H) \not\leq_O H \). Let \( N \not\leq_O G \). Then \( N \cap H \not\leq_O H \) and so \( N \cap H \not\leq \Phi(H) \). So there is a maximal open subgroup \( M \) of \( H \) such that \( N \cap H \not\leq M \). Now \( (N \cap H) M = H \) and so \( M N \cap H = H \). Thus \( M N = H N \). But \( M \in S(G)' \) and \( M \neq H \). So by (i), \( H \) is not solitary in \( G \).
For (vi) suppose that $H$ is solitary in $G$. Then by (i) there is an open normal subgroup $N$ of $G$ such that if $L \leq C G$ and $LN = HN$ then either $L = H$ or $L \leq O G$. Let $K = HN$. Clearly $K \leq O G$. If $k \in K$ then $k = hn$ for some $h \in H$ and $n \in N$. Then $H^kN = H^{hn}N = H^nN = HN$. So $H^k \in B(H, N)$. But $H^k \in S(G)'$ and so $H^k = H$. Thus $H \triangleright C K$. Let $L/H \leq C K/H$ with $L \neq H$. Clearly $L \in B(H, N)$. If $L/H \not\leq O K/H$ then $L \not\leq O K$ and so $L \in S(G)'$. But then $L = H$, a contradiction. Hence $L/H \leq O K/H$. But now $K/H \cong \mathbb{Z}_p$ for some prime $p$, because the $p$-adic integers are characterised among all profinite groups as those whose only non-open subgroup is the trivial subgroup.

Finally for (vii) suppose there is an open normal subgroup $N$ of $G$ with $N \cong \mathbb{Z}_p$ for some prime $p$. Then all subgroups of the form $p^n\mathbb{Z}_p$ are isolated in $S(\mathbb{Z}_p)$, with $1$ as their unique limit. So $1$ is solitary in $\mathbb{Z}_p$ (and in $N$). Then by (ii), $1$ is solitary in $G$. Conversely suppose that $1$ is solitary in $G$. Then by (vi) there exists $K \leq O G$ with $K \cong \mathbb{Z}_p$ for some prime $p$. Clearly if $N = K^G$ then $N \triangleleft G$ and $N \cong \mathbb{Z}_p$. ■

Note that by Lemma 3.6(vi), if $G$ is an infinite pro-$p$ group of finite rank and $H$ is a solitary subgroup of $G$ then $\dim(H) = \dim(G) - 1$.

**Bounding The Scattered Height**

**Lemma 3.7** Let $G$ be a profinite group, $g \in G$, $H \leq C G$ and $k$ be a non-negative integer. If $ht(H, S(G)) = k$ then $ht(H^g, S(G)) = k$.

**Proof.** As in the proof of Lemma 3.6(iv) conjugation by $g$ is a homeomorphism from $S(G)$ to $S(G)$. We now proceed by induction on $k$. Clearly if $k = 0$ then $H \leq O G$ and so $H^g \leq O G$ and $ht(H^g) = 0$. So suppose the result holds for all closed subgroups $K$ with $ht(K, S(G)) = k$. Then if $K \leq C G$ then $K \in S(G)^{(k+1)}$ if and only if $K^g \in S(G)^{(k+1)}$. Also by Lemma 2.2, $S(G)^{(k+1)}$ is closed in $S(G)$. Thus conjugation by $g$ is a homeomorphism from $S(G)^{(k+1)}$ to $S(G)^{(k+1)}$. Now let $H \leq C G$ with $ht(H, S(G)) = k + 1$. Then $H \in S(G)^{(k+1)}$ and $H$ is isolated in $S(G)^{(k+1)}$. So clearly using the above homeomorphism, $H^g$ is isolated in $S(G)^{(k+1)}$. That is $ht(H, S(G)) = k + 1$ as required. ■

**Theorem 3.8** Let $G$ be a profinite group with $S(G)$ not perfect. Let $k$ equal the number of primes $p$ such that $p^\infty \mid o(G)$. Then $ht(S(G)) \leq k + 1$. In particular if $G$ is any pro-$p$ group then $ht(S(G)) \leq 2$. 

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Proof. Note that by Theorem 3.1, \( n \) is finite. Suppose for a contradiction that there is a profinite group \( G \) with \( S(G) \) not perfect and \( ht(S(G)) > k + 1 \). Then there exists \( H \leq C \) with \( ht(H, S(G)) = k + 1 \). So \( H \in S(G)^{(k+1)} \) and \( H \) is isolated in \( S(G)^{(k+1)} \). Hence by Lemma 2.14 there exists an open normal subgroup \( N \) of \( G \) such that \( B(H, N) \cap S(G)^{(k+1)} = \{H\} \). Let \( K = HN \). Then as in the proof of Lemma 3.6(vi) if \( k' \in K \) then \( H^{k'} \in B(H, N) \). But by Lemma 3.7 \( H^{k'} \in S(G)^{(k+1)} \). So \( H^{k'} = H \). Hence \( H \triangleleft C \).

We now consider \( S(K/H) \). Now \( H \) is isolated in \( S(K)^{(k+1)} \) and so the trivial subgroup of \( K/H \) is isolated in \( S(K/H)^{(k+1)} \), that is \( ht(1, S(K, H)) = k + 1 \). Let \( L/H \leq C \) with \( L \neq H \). Then \( ht(L/H, S(K, H)) \leq k \). Thus \( S(K/H) \) is countable and \( ht(S(K/H)) = k + 2 \). But by hypothesis the number of primes \( p \) such that \( p \nmid o(K/H) \) is at most \( k \). So by of [2], \( ht(S(K/H)) \leq k + 1 \), a contradiction. ■

4 Subgroup Spaces of pro-\( p \) Groups

When is \( S(G) \) Homeomorphic to Pełczyński’s Space? Lemma 3.6 provides us with the techniques to show that many classes of pro-\( p \) groups \( G \) have \( S(G) \) homeomorphic to Pełczyński space. Of course we have a useful characterisation of such groups using Proposition 3.1, Proposition 3.3 and Proposition 2.9.

Proposition 4.1 Let \( G \) be a pro-\( p \) group. Then \( S(G) \) is homeomorphic to Pełczyński space if and only if \( G \) is infinite, finitely generated and has no solitary subgroups.

Examples 4.2 Let \( p \) be a prime and let \( n \) be an integer \( > 1 \).

(i) \( S(\mathbb{Z}_p^n) \) is homeomorphic to Pełczyński space.

(ii) If \( G \) is a free pro-\( p \) group of rank \( n \) then \( S(G) \) is homeomorphic to Pełczyński space.

Proof. For (i), firstly note that 1 is not solitary in \( \mathbb{Z}_p^n \) by Lemma 3.6(vii). Now let \( H \in S(\mathbb{Z}_p^n)^{\prime} \) with \( H \neq 1 \) and let \( N \leq C \mathbb{Z}_p^n \). Since \( H \) is not isolated, it is not open, so \( \dim(H) < \dim(\mathbb{Z}_p^n) \), and there is a positive integer \( m < n \) such that \( H \) can be generated (topologically) by \( m \) elements. Also \( HN \neq H \). Hence there exist \( h_1, \ldots, h_m \in H \) and \( n_1, \ldots, n_m \in N \) with \( n_1 \not\in H \) such
that \( \langle h_1n_1, \ldots, h_mn_m \rangle N/N = HN/N \). Let \( K = \langle h_1n_1, \ldots, h_mn_m \rangle \). Now \( KN = HN \), and as \( \dim(K) < \dim(\mathbb{Z}_p^n) \), \( K \) is not open, so not isolated, and \( K \in S(\mathbb{Z}_p^n) \). But \( K \neq H \). So by Lemma 3.6(i), \( H \) is not solitary in \( \mathbb{Z}_p^n \). Hence by Proposition 4.1, \( S(\mathbb{Z}_p^n) \) is homeomorphic to Pelczyński space.

Now for (ii) let \( G \) be a free pro-\( p \) group of rank \( n \). For the definition and basic properties of free pro-\( p \) groups see Chapter 5 of [8]. Again 1 is not solitary in \( G \), and so let \( H \in S(G)' \) with \( H \neq 1 \). As before by Proposition 14 it suffices to show that \( H \) is not solitary in \( G \). So suppose for a contradiction that \( H \) is solitary in \( G \). Then by Lemma 3.6 vii there exists \( K \subseteq G \) with \( H \trianglelefteq G \) and \( K/H \cong \mathbb{Z}_p^m \). Also by Lemma 3.6 ii \( H \) is solitary in \( K \). By Theorem 5.4.4 of [8] \( K \) is a free pro-\( p \) group of rank \( m \) where \( m \) is an integer \( > 1 \). Since \( K/H \) is abelian, \( H \triangleright K' \). By Proposition 5.1.5 and Proposition 5.1.3' of [8], \( K/K' \cong \mathbb{Z}_p^m \). But by Lemma 3.6 iii \( H/K' \) is solitary in \( K/K' \). This contradicts (i). Thus \( H \) is not solitary in \( G \) as required.

**Proposition 4.3** Let \( G \) be an infinite finitely generated pro-\( p \) group, and \( H \) be a solitary subgroup of \( G \). Then \( G/H \) has an open normal subgroup, topologically isomorphic to \( \mathbb{Z}_p^n \) for some positive integer \( n \) where \( n \) is at most the (necessarily finite) number of conjugates of \( H \) in \( G \).

**Proof.** By Lemma 3.6 vi) \( N_G(H) \subseteq G \) and so \( H \) has only finitely many conjugates in \( G \). Let \( H_1, \ldots, H_k \) be the conjugates of \( H \) in \( G \). Now by Lemma 3.6 iv and the proof of Lemma 3.6 vi there exists \( N \trianglelefteq G \) such that for every \( i \), \( H_i \trianglelefteq H \) and \( H_iN/H_i \cong \mathbb{Z}_p^m \). Clearly \( H_G \trianglelefteq (HN)_G \), \( (HN)_G \trianglelefteq G \) and for every \( i \), \( (N_i)_G = (HN)_G \). Also for every \( i \), \( (HN)_G/(H_i \cap (HN)_G) \cong ((HN)_G/H_i)/H_i \) which is a non-trivial closed subgroup of \( H_iN/H_i \). So for every \( i \), \( (HN)_G/(H_i \cap (HN)_G) \cong \mathbb{Z}_p^m \). Now

\[
\frac{(HN)_G}{H_G} = \frac{(HN)_G}{\bigcap_{i=1}^k (H_i \cap (HN)_G)} \text{ which embeds in } \bigoplus_{i=1}^k \frac{(HN)_G}{H_i \cap (HN)_G} \cong \mathbb{Z}_p^k.
\]

Thus \( (HN)_G/H_G \cong \mathbb{Z}_p^n \) for some positive integer \( n \) with \( n \leq k \), as required.

**Corollary 4.4** Let \( G \) be an insoluble just-infinite pro-\( p \) group. Then \( S(G) \) is homeomorphic to Pelczyński space.

**Proof.** Suppose for a contradiction that \( S(G) \) is not homeomorphic to Pelczyński space. Then by Proposition 14 \( G \) has a solitary subgroup \( H \).
Now as $G$ is just-infinite and $H \not\subset_{O} G$, $H_{G} = 1$. But by Proposition 4.3, $G$ is then soluble, a contradiction.

A profinite group $G$ is \textbf{poly-procyclic} if it has a finite series of closed subgroups $1 = G_{0} \leq_{C} G_{1} \leq_{C} \cdots \leq_{C} G_{n} = G$ such that $G_{i-1} \vartriangleleft_{C} G_{i}$ and $G_{i}/G_{i-1}$ is procyclic (topologically generated by one element).

Let $G$ be a poly-procyclic group. For each prime $p$, the $\mathbb{Z}_{p}$-\textbf{length} of $G$, denoted $h_{p}(G)$, is the number of factors having Sylow $p$-subgroups isomorphic to $\mathbb{Z}_{p}$ in a series in $G$ with procyclic factors. If $G$ is pro-$p$ then we write $h(G)$ for $h_{p}(G)$, and refer to the $\mathbb{Z}_{p}$-length as the \textbf{Hirsch length}. It turns out that for a poly-procyclic pro-$p$ group $G$, $h(G) = \dim(G)$.

\textbf{Proposition 4.5} Let $G$ be a finitely generated nilpotent pro-$p$ group with $h(G) > 1$. Then $S(G)$ is homeomorphic to Pełczyński space.

\textbf{Proof.} We first show that without loss of generality we may assume that $G$ is torsion-free. Suppose for a contradiction that $S(G)$ is not homeomorphic to Pełczyński space but that the result holds for finitely generated torsion-free nilpotent pro-$p$ groups. Again by Proposition 4.1, $G$ has a solitary subgroup $H$. By Lemma 3.6(vi) there exists $K \leq_{O} G$ with $H \vartriangleleft_{C} K$ and $K/H \cong \mathbb{Z}_{p}$. Let $T = \text{Tor}(H)$. Then $T$ is finite. (This is the pro-$p$ analogue of the well known fact that every finitely generated nilpotent abstract group has a finite torsion subgroup. The proof is by induction on the Hirsch length). Also $T \vartriangleleft_{C} K$. Clearly $K/T$ is a finitely generated torsion-free nilpotent pro-$p$ group. Since $K$ is open in $G$ and $T$ finite, $h(K/T) = \dim(K/T) = \dim(K) = \dim(G) = h(G) > 1$. But by Lemma 3.6(iii) $H/T$ is solitary in $K/T$, a contradiction.

So we now assume that $G$ is torsion-free. We proceed by induction on $h(G)$. Again assume for a contradiction that $G$ has a solitary subgroup $H$. Then, again there exists $K \leq_{O} G$ with $H \vartriangleleft_{C} K$ and $K/H \cong \mathbb{Z}_{p}$. Since $K$ is torsion-free and nilpotent $HZ(K)/Z(K)$ is torsion-free (see 5.2.19 of $[4]$ for example). Thus $H/(H \cap Z(K)) \cong HZ(K)/Z(K)$ is torsion-free.

Clearly $0 \leq h(H/(H \cap Z(K))) \leq h(G) - 1$. We now exclude the end cases. If $h(H/(H \cap Z(K))) = 0$ then $H/(H \cap Z(K))$ is finite, and $H \cap Z(K) \leq_{O} H$, so $H$ is central in $K$. Thus $K = H \times K/H$. But $H$ is abelian and so $K \cong \mathbb{Z}_{p}^{h(G)}$. By Example 4.2(i) $H$ is not solitary in $K$ and so by Lemma 3.6(ii) $H$ is not solitary in $G$, a contradiction. Hence $h(H/(H \cap Z(K))) > 0$. If $h(H/(H \cap Z(K))) = h(G) - 1$ then $h(H \cap Z(K)) = 0$ and so $H \cap Z(K) = 1$, a contradiction as $K$ is nilpotent (see 5.2.1 of $[4]$ for example). So $h(H/(H \cap Z(K))) \leq h(G) - 2$, and $h(G) > 2$. 

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Hence there exists $L \leq C K$ with $H \cap Z(K) \leq L$ such that $K/(H \cap Z(K)) = H/(H \cap Z(K)) \rtimes L/(H \cap Z(K))$ and $L/(H \cap Z(K)) \cong K/H \cong \mathbb{Z}_p$. Thus $1 < h(K/(H \cap Z(K))) < h(G)$ and $K/(H \cap Z(K))$ is a finitely generated torsion-free nilpotent pro-$p$ group. Hence by the inductive hypothesis $S(K/(H \cap Z(K)))$ is homeomorphic to Pełczyński space. But by Lemma 3.6(ii,iii) $H/(H \cap Z(K))$ is solitary in $K/(H \cap Z(K))$, a contradiction. Hence $H$ is not solitary in $G$, a contradiction. □

In this section we have been attempting to solve:

**Problem 1** Characterise algebraically, the pro-$p$ groups $G$ with $S(G)$ homeomorphic to Pełczyński space.

From the above, we know the class of groups we hope to characterise here is very diverse; it includes uncomplicated groups like $\mathbb{Z}_p^2$, and also very complicated groups like the Nottingham group.

Perhaps this question is more tractable for pro-$p$ groups of finite rank. In particular we ask:

**Problem 2** If $G$ is a uniform pro-$p$ group with $\dim(G) > 1$, is $S(G)$ homeomorphic to Pełczyński space?

A positive answer to this question seems reasonable, given that the group operation in a uniform pro-$p$ group of dimension $n$ can be ‘smoothed out’, to give a new group which is topologically isomorphic to $\mathbb{Z}_p^n$ (see §4.3 of [3]). Also a positive answer to this question would imply that if $G$ is a pro-$p$ of finite rank and dimension $> 1$ then $G$ has an open normal subgroup $N$ such that $S(N)$ is homeomorphic to Pełczyński space.

An Almost Complete Classification for Nilpotent pro-$p$ Groups We now have done enough to describe $S(G)$ for $G$ any nilpotent pro-$p$ group with $w(G) \leq \aleph_1$ where $w(G)$, the weight of $G$, is the cardinality of a minimal sized base of $G$. We can say which spaces can occur and characterise when they occur. First an easy lemma.

**Lemma 4.6** Let $G$ be a pronilpotent group which is virtually $\mathbb{Z}_p$ for some prime $p$. Then $S(G)$ is countably infinite if and only if $G$ is nilpotent.

**Proof.** Let $N \trianglelefteq G$ with $N \cong \mathbb{Z}_p$. Suppose that $S(G)$ is countably infinite. Then by Theorem 3.7 of [2] we know that $N$ is central in $G$. But
$G/Z(G)$ is then nilpotent and so $G$ is nilpotent. Conversely suppose that $G$ is nilpotent. Then $N \cap Z(G)$ is non-trivial (see 5.2.1 of [4] for example). So $N \cap Z(G) \leq O N$ and thus $N \cap Z(G) \cong \mathbb{Z}_p$. Hence $G$ has a central open subgroup topologically isomorphic to $\mathbb{Z}_p$. Thus $S(G)$ is countably infinite by Theorem 3.7 of [2].

**Theorem 4.7** Let $G$ be a nilpotent pro-$p$ group with $w(G) \leq \aleph_1$. Then $S(G)$ lies in precisely one of the following four classes. We give specific characterisations of when $S(G)$ is in each class in terms of properties of $G$.

(i) $S(G)$ is a finite discrete space precisely when $G$ is finite.

(ii) $S(G)$ is homeomorphic to $\omega n + 1$ for some positive integer $n$ precisely when $G$ is virtually $\mathbb{Z}_p$.

(iii) $S(G)$ is homeomorphic to Pełczyński space precisely when $G$ is finitely generated and $h(G) > 1$.

(iv) $S(G)$ is homeomorphic to $\{0, 1\}^{w(G)}$ precisely when $G$ is not finitely generated.

**Proof.** The result follows immediately from Lemma 4.6, Proposition 3.1, and Proposition 4.5, and Corollary 4.3, Corollary 5.7 and Theorem 6.8 from our earlier paper on subgroup spaces of profinite groups, [2].

**Scattered Height = 2, Infinitely Many Solitary Subgroups** We now show that there are pro-$p$ groups with subgroup spaces which have scattered height 2 and infinitely many isolated points in the derived set. There are even such examples amongst poly-procyclic pro-$p$ groups. In order to see this we need to analyse subgroups of direct products.

**Lemma 4.8** Let $H$ be a finitely generated pro-$p$ group, $Z \cong \mathbb{Z}_p$ and $G = H \times Z$. Then $H$ is solitary in $G$ if and only if $H' \leq O H$.

**Proof.** Firstly suppose that $H$ is solitary in $G$. Then by Lemma 3.6(iii) $H/H'$ is solitary in $G/H'$. Now $G/H'$ is a finitely generated abelian pro-$p$ group. If $H' \not\leq O H$ then $h(G/H') > 1$ contradicting Proposition 4.5. Thus $H' \leq O H$. It is not hard to give a direct argument avoiding the use of Proposition 4.5 by considering the torsion subgroup of $H/H'$. 

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Now suppose that $H' \leq_o H$. Let $N = \Phi(H) \times Z$. Then $N \trianglelefteq_o G$. Also $HN = H(\Phi(H) \times Z) = G$. Suppose for $K \leq_C G$ that $KN = G$. Now $(KZ \cap H)\Phi(H) = H \cap (\Phi(H)KZ) = H \cap G = H$. So by standard properties of the Frattini subgroup, $KZ \cap H = H$. Thus by Proposition 2.12 $H/(K \cap H) \cong (KH \cap Z)/(K \cap Z)$. But the latter group is clearly abelian and so $K \cap H \succeq H'$. As $H' \leq_o H$, $K \cap H \leq_o H$ and by the above isomorphism $K \cap Z \leq_o KH \cap Z$. Suppose that $K \neq H$. Then since $K \cap Z \leq_o Z$ and $K \cap Z \leq_o H$ and $K \leq_o G$ so $K \cap Z \leq_o Z$ and $K \leq_o G$. Thus $H$ is solitary in $G$ by Lemma 3.6(i).

**Corollary 4.9** Let $G$ be an infinite finitely generated pro-$p$ group which is not virtually $\mathbb{Z}_p$. Let $H \in S(G)'$. If $H$ is central in $G$ then $H$ is not solitary in $G$.

**Proof.** Suppose that $H$ is central in $G$ but that $H$ is solitary in $G$. Then by Lemma 3.6(vi) there exists $K \leq_o G$ with $H \trianglelefteq_C K$ and $K/H \cong \mathbb{Z}_p$. Thus there exists $Z \leq_C G$ with $K = H \times Z$ and $Z \cong \mathbb{Z}_p$. Now by Lemma 3.6(ii) and Lemma 4.8 $H' \leq_o H$. But $H$ is abelian and so $H$ is finite. Hence $G$ is virtually $\mathbb{Z}_p$, a contradiction. ■

**Corollary 4.10** Let $H$ be a non-abelian just-infinite pro-$p$ group, $Z \cong \mathbb{Z}_p$ and $G = H \times Z$. Then $G$ has precisely countably infinitely many solitary subgroups.

**Proof.** By Lemma 2.2(iii), $G$ has at most countably infinitely many solitary subgroups. Let $N \trianglelefteq_o H$. Then $N' \trianglelefteq_o H$ since $H$ is non-abelian and just-infinite. By Lemma 4.8 $N$ is solitary in $N \times Z$. But $N \times Z \leq_o G$ and so by Lemma 3.6(ii), $N$ is solitary in $G$. Since $H$ has precisely countably infinitely many open normal subgroups, we are done. ■

By taking $H$ to be a non-abelian soluble just-infinite pro-$p$ group in Corollary 4.10 we obtain examples of poly-procyclic pro-$p$ groups with countably infinitely many solitary subgroups. Even poly-$\mathbb{Z}_p$ groups $G$ with $h(G) > 1$ may not have $S(G)$ homeomorphic to Pełczyński space.

**Proposition 4.11** Let $G$ be a pro-$p$ group. Suppose $G$ has a closed normal subgroup $H$ with $H \cong \mathbb{Z}_p^n$, for $n$ an integer $> 1$, and $G/H \cong \mathbb{Z}_p$. Suppose further that $H$ is a rationally irreducible $L$-module for every $L \leq_o G$, with $H \leq L$. (This means that if $K \trianglelefteq_C L$ with $K \leq H$ then either $K = 1$ or $K \trianglelefteq_o H$.) Then $H$ is solitary in $G$. 17
and as cyclic, a contradiction as $H \leq G$ we must specify a group of the form $\mathbb{Q}$.

There are poly-

Example 4.12 There are poly-$\mathbb{Z}_p$ groups $G$ with scattered height 2 and infinitely many solitary subgroups.

Proof. Every extension by $\mathbb{Z}_3$ splits (since $\mathbb{Z}_3$ is a free pro-3 group). So we must specify a group of the form $G = H \times T$ where $H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $T = \langle \alpha \rangle \cong \mathbb{Z}_3$. Consequently we must specify a continuous homomorphism from $T$ to $\text{Aut}(H)$, and since $T$ is a free pro-3 group and $\text{Aut}(H) \cong GL_2(\mathbb{Z}_3)$, it suffices to specify a matrix in a pro-3 subgroup of $GL_2(\mathbb{Z}_3)$ to which $t$ is mapped. Let $\alpha = 1 + \sqrt{5}$. Consider the field extension $\mathbb{Q}_3(\alpha) : \mathbb{Q}_3$ where $\mathbb{Q}_3$ is the field of 3-adic numbers; that is, the field of fractions of $\mathbb{Z}_3$. We consider $\mathbb{Q}_3(\alpha)$ as a vector space over $\mathbb{Q}_3$. Multiplication (on the right) by $\alpha$ is an vector space endomorphism of $\mathbb{Q}_3(\alpha)$. Let $A = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$. Then $A$ is the matrix corresponding to this endomorphism with respect to the basis $\{1, \alpha\}$. Let $K_1 = \{g \in GL_2(\mathbb{Z}_3) \mid g - 1 \in 3M_2(\mathbb{Z}_3)\}$. This is the first congruence subgroup of $GL_2(\mathbb{Z}_3)$, and is an open normal pro-3 subgroup of $GL_2(\mathbb{Z}_3)$; see, for example, §8.5 of [8]. We choose a power $A_1$ of $A$ lying in $K_1$ and define $G$ as above by mapping $t$ to $A_1$.
Now for $n$ a positive integer, $\alpha^n = a_n + b_n\alpha$ for some $a_n, b_n \in \mathbb{Q}_3$. Clearly if $n = 1$ or 2, then $b_n > 0$. It is straightforward to show that if $n > 2$ then $b_n = 2b_{n-1} + 4b_{n-2}$. So by induction, $b_n > 0$ for every positive integer $n$. Thus $\alpha^n \notin \mathbb{Q}_3$ for every $n$. It also follows that $(1 - \sqrt{5})^n \notin \mathbb{Q}_3$ for every $n$. The eigenvalues of $A$ are $1 \pm \sqrt{5}$. For each $n$, the eigenvalues of $A^n$ are the $n$th power of the eigenvalues of $A$, namely $(1 \pm \sqrt{5})^n$. Hence $A^n$ has no eigenvalues in $\mathbb{Q}_3$ for each positive integer $n$. Consequently $A^n_1$ also has no eigenvalues in $\mathbb{Q}_3$ for each positive integer $n$.

We want to show that every open subgroup of $T$ acts rationally irreducibly on $H$. Suppose for a contradiction that there exists $L \leq O T$ and $K \triangleleft H$ with $K \neq 1$ and $K \triangleleft O H$. Now $L = T^{3r}$ for some non-negative integer $r$. $L$ acts on $H$ by mapping $t^{3r}$ to $A_1^{3r}$. Then $K \otimes_{\mathbb{Z}_3} \mathbb{Q}_3$ is a 1-dimensional $\mathbb{Q}_3$-subspace of $H \otimes_{\mathbb{Z}_3} \mathbb{Q}_3$. So $t^{3r}$ acts as scalar multiplication on $K \otimes_{\mathbb{Z}_3} \mathbb{Q}_3$ and thus the matrix $A_1^{3r}$ has an eigenvalue in $\mathbb{Q}_3$, a contradiction. ■

Scattered Height Equal to 2, and Finitely Many Isolated Subgroups

- $G$ virtually $\mathbb{Z}_p$. In Corollary 4.10 we gave examples of pro-$p$ groups with infinitely many solitary subgroups. There also exist profinite groups $G$ with $S(G)$ not homeomorphic to Pelczyński space but with only finitely many solitary subgroups. For example if $G$ is a pro-$p$ group with $S(G)$ countably infinite, then (see [2]) $S(G)'$ is a finite discrete space.

**Lemma 4.13** Let $G$ be a profinite group. Then $G$ is virtually $\mathbb{Z}_p$ for some prime $p$ if and only if $G$ has a finite solitary subgroup. Also if $G$ is virtually $\mathbb{Z}_p$ for some prime $p$ then every $H \in S(G)'$ is finite.

**Proof.** By Lemma 3.6(vii) if $G$ is virtually $\mathbb{Z}_p$ for some prime $p$ then 1 is solitary in $G$. Conversely suppose that $G$ has a finite and solitary subgroup $H$. By Lemma 3.6(vi) there exists $K \leq O G$ with $H \triangleleft C K$ and $K/H \cong \mathbb{Z}_p$ for some prime $p$. So there exists $Z \leq C G$ with $HZ = K$, $H \cap K = 1$ and $Z \cong \mathbb{Z}_p$. But $|K : Z| = |HZ : Z| = |H|$ which is finite by hypothesis. Thus $Z \leq O G$ and $G$ is virtually $\mathbb{Z}_p$.

Now suppose that $G$ is virtually $\mathbb{Z}_p$ for some prime $p$. Let $Z \triangleleft O G$ with $Z \cong \mathbb{Z}_p$. Let $H \in S(G)'$. If $H \cap Z$ is non-trivial then $H \cap Z \leq O Z$ and so $H \triangleleft O G$, a contradiction. So $H \cap Z = 1$. Hence $|H| = |H : H \cap Z| = |HZ : Z|$ which is finite. ■

**Proposition 4.14** Let $G$ be a profinite group with an open normal subgroup $Z$, topologically isomorphic to $\mathbb{Z}_p$ for some prime $p$. Let $H \in S(G)'$. Then $H$
is solitary in $G$ if and only if $H \leq C_G(Z)$. Moreover there are only finitely many such subgroups.

**Proof.** Clearly $Z \leq Z(C_G(Z))$. So $C_G(Z)$ has an open central subgroup topologically isomorphic to $\mathbb{Z}_p$. Hence by Theorem 4.2 of [2], $S(C_G(Z))'$ is finite. Now if $H \leq C_G(Z)$ then $H \in S(C_G(Z))'$ and so $H$ must be solitary in $C_G(Z)$. Also $C_G(Z) \trianglelefteq G$. So by Lemma 3.6(ii) $H$ is solitary in $G$.

Now suppose that $H$ is solitary in $G$. We know by Lemma 4.13 that $H$ is finite and by the proof of Lemma 4.13 that $H \cap Z = 1$. For each $h, k \in H$ the maps $g \mapsto h^g$ and $g \mapsto k$ are continuous maps from $N_G(H)$ to $G$. Thus for each $h, k \in H$, $\{g \in N_G(H) \mid h^g = k\}$ is closed in $N_G(H)$. Hence for each $h \in H$ we can write $N_G(H) = \bigcup_{k \in H} \{g \in N_G(H) \mid h^g = k\}$ a finite union of closed sets. So by Baire’s category theorem for each $h \in H$ there exists $k_h \in H$ such that $\{g \in N_G(H) \mid h^g = k_h\}$ has non-empty interior in $N_G(H)$. But by Lemma 3.6(vi) $N_G(H) \leq_O G$ and so this set has non-empty interior in $G$. Consequently (as the cosets of open normal subgroups form a base for the topology on $G$) for each $h \in H$ there exists $k_h \in H, g_h \in G$ and $N_h \trianglelefteq G$ such that $\{g \in N_G(H) \mid h^g = k_h\} \supseteq N_h g_h$. Let $h \in H$ and $n \in N_h$. Then $h^{n g_h} = k_h$ and so $h^n = k_h^{g_h^{-1}}$ which is independent of $n$. Consequently, $h^n = h$ for every $n \in N_h$.

Now let $N = Z \cap \bigcap_{h \in H} N_h$. Then since $H$ is finite, $N \trianglelefteq O G$. Also as $H \cap Z = 1$, $N \cap Z = 1$. Let $h \in H$ and $n \in N$. Then as $n \in N_h$, $[h, n] = 1$. Thus $H$ acts trivially on $N$ by conjugation. Also $H$ acts on $Z$. $Z$ has a natural ring structure, and $Aut(Z) \cong U(Z)$, the group of units of $Z$. Let $h \in H$. Then there exists $u \in U(Z)$ such that $u \cdot z = z^h$ for every $z \in Z$ where $\cdot$ is the ring multiplication in $Z$. But $H$ acts trivially on $N$ and so $u \cdot n = n$ for every $n \in N$. If we choose a non-trivial (i.e. non-zero in the ring) $n \in N$ then we may cancel it since $N$ is an integral domain and conclude that $u = 1$. Thus $z^h = z$ for every $h \in H$ and $z \in Z$. Hence $H \leq C_G(Z)$ as required. ■

**Corollary 4.15** Let $G$ be a profinite group which is virtually $\mathbb{Z}_p$ for some prime $p$. Suppose that $S(G)$ is uncountable. Then $S(G)$ is homeomorphic to $P \oplus (\omega n + 1)$ where $n = |S(C_G(Z))'|$ (and $P$ is the Pelczyński space).

**Proof.** The result follows immediately from Proposition 4.14 and Proposition 2.10 ■

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Corollary 4.16 Let $G$ be a just-infinite non-abelian pro-$p$ group which is virtually $\mathbb{Z}_p$. Then $\mathcal{S}(G)$ is homeomorphic to $P \oplus (\omega + 1)$.

Proof. Suppose $G$ has non-trivial centre. Then by Proposition 3.6(i) of [2], $G'$ is finite. Thus as $G$ is just-infinite, $G'$ is trivial; that is, $G$ is abelian, a contradiction. Hence $G$ has trivial centre and so by Theorem 3.7 of [2], $\mathcal{S}(G)$ is uncountable. Let $Z$ be a maximal normal abelian subgroup of $G$. Then $Z \vartriangleleft C_G(Z)$ and since $G$ is just-infinite, $Z \vartriangleleft O_G$. Since $G$ is virtually $\mathbb{Z}_p$ it has solitary subgroups, and so definitely isolated subgroups. So $G$ and its open subgroup $Z$ are finitely generated. Also as $G$ is just-infinite, $\text{Tor}(Z) = 1$. Thus as $G$ is virtually $\mathbb{Z}_p$, $Z$ must be topologically isomorphic to $\mathbb{Z}_p$. Now $G/Z$ is nilpotent and so by a standard argument (see for example the proof of 5.2.3 of [4]), $Z = C_G(Z)$. The result now follows from Corollary 4.15. ■

Scattered Height Equal to 2, and Finitely Many Solitary Subgroups
– $G$ Not Virtually $\mathbb{Z}_p$. It is natural to ask whether an infinite finitely generated pro-$p$ group which is not virtually $\mathbb{Z}_p$ can have solitary subgroups but have only finitely many solitary subgroups. It is clear from the proof of Corollary 4.10 that it might not be possible to find such examples if it is the case that if an infinite finitely generated pro-$p$ group has an open commutator subgroup then it has a proper open subgroup with an open commutator subgroup. This though does not happen.

Example 4.17 (Wilson) Let $G$ be the pro-$2$ group with presentation $\langle x_1, x_2 \mid (x_2)^{x_2} = x_1^{-2}, (x_2^{x_1})^{x_2} = x_2^{-2}, ((x_1x_2)^{x_1})^{x_2} = (x_1x_2)^{-2} \rangle$. Then $G$ is a poly-procyclic pro-$2$ group. $G' \leq O_G$ but for every proper open subgroup $H$ of $G$, $H' \not\leq O_H$.

Proof. We sketch a proof that $G$ has these properties. For the definition of pro-$2$ presentations, see §12.1 of [8]. In particular $G$ is the pro-$2$ completion of the corresponding abstract group with this presentation.

Let $a_1 = x_1^2, a_2 = x_2^2, a_3 = (x_1x_2)^2$, and $A = \langle a_1, a_2, a_3 \rangle$. Since $a_3 = a_3^{a_1}, a_3^2 = (a_3^{-1})^{x_2}, a_3^{x_2} = a_3^{-1}$. Thus $A \vartriangleleft C_G$. Also it is easy to check that $[a_1, a_2] = [a_1, a_3] = [a_2, a_3] = 1$. So $A$ is abelian. Now each $a_i$ is of infinite order. This can be seen, for example, by embedding $G$ into the direct product of three copies of the pro-$2$ analogue of the infinite dihedral group. Consequently the $a_i$’s are distinct and $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Clearly $G/A$ is a Klein 4-group. So $G$ is a poly-procyclic pro-$2$ group of Hirsch length 3.
Now \([x_2, a_1] = a_1^2\), \([x_1, a_2] = a_2^2\), and \([x_1, a_3] = a_3^2\). So \((a_1^2, a_2^2, a_3^2) \leq G'\). Thus \(G' \leq O\ G\). Since \(G^2 = A\), and \(G' \leq C\ A\ (G/A\ abelian)\), we see that \(\Phi(G) = A\) (for any pro-\(p\) group \(G\) we have \(\Phi(G) = \overline{G/G^p}\), see Proposition 2.5.2 [8]). Thus the maximal open subgroups of \(G\) are \(\langle x_1, A \rangle\), \(\langle x_2, A \rangle\), and \(\langle x_1x_2, A \rangle\). It is now easy to check that \(\langle x_1, A \rangle' = \langle a_2^2, a_3^2 \rangle\), \(\langle x_2, A \rangle' = \langle a_1^2, a_3^2 \rangle\), and \(\langle x_1x_2, A \rangle' = \langle a_1^2, a_2^2 \rangle\). Now each of these groups is topologically isomorphic to \(\mathbb{Z}_2 \times \mathbb{Z}_2\); in particular they each have Hirsch length 2. Let \(H\) be a proper open subgroup of \(G\). Then \(H\) is contained in a maximal open subgroup \(M\) of \(G\). Now \(H' \leq M'\) and so \(h(H') \leq 2 < 3 = h(G)\). Hence \(H' \not\leq_O H\).

**Proposition 4.18** Let \(W\) be an infinite pro-\(p\) group of finite rank, \(Z \cong \mathbb{Z}_p\) and \(G = W \times Z\). Let \(H \in S(G)\).

(i) Suppose that \(H\) is solitary in \(G\). Then either \(H \leq_O W\) or \(H \cap Z \neq 1\). If \(H \cap Z \neq 1\) then \(H' \leq_O H\). In either case \((H \cap W) \times (H \cap Z) \leq_O H \leq_O (HZ \cap W) \times (HW \cap Z)\).

(ii) Suppose \(H\) factorises, i.e. \(H = (H \cap W) \times (H \cap Z)\). Then \(H\) is solitary in \(G\) if and only if \(H \leq_O W\) and \(H' \leq_O H\).

**Proof.** For (i), suppose that \(H\) is solitary in \(G\). Then by Lemma 3.6(vi), for some open subgroup \(K\) of \(G\), \(H \triangleleft_O K\) and \(K/N \cong \mathbb{Z}_p\). Hence \(\dim(H) = \dim(K) - \dim(\mathbb{Z}_p) = \dim(G) - 1 = \dim(W)\).

Firstly suppose that \(H \leq C\ W\). Since \(\dim(H) = \dim(W)\), \(H \leq_O W\). Clearly \((H \cap W) \times (H \cap Z) \leq_O H \leq_O (HZ \cap W) \times (HW \cap Z)\). Now suppose that \(H \not\leq W\), that is that \(HW \cap Z\) is non-trivial. Suppose for a contradiction that \(H \cap Z\) is trivial. Then \((HW \cap Z)/(H \cap Z) \cong \mathbb{Z}_p\). Thus by Proposition 2.12

\[
\frac{(HZ \cap W) \times (HW \cap Z)}{(H \cap W) \times (H \cap Z)} \cong \mathbb{Z}_p \times \mathbb{Z}_p.
\]

If \((HZ \cap W) \times (HW \cap Z) \leq_O G\) then by Lemma 3.6(ii,iii),

\[
\frac{H}{(H \cap W) \times (H \cap Z)}\]

is isolated in \(S\left(\frac{(HZ \cap W) \times (HW \cap Z)}{(H \cap W) \times (H \cap Z)}\right)\), contradicting Examples 1.2(i). Thus \((HZ \cap W) \times (HW \cap Z)\) is not open \(G\), so \(\dim((HZ \cap W) \times (HW \cap Z)) \leq \dim(W) = \dim(G) - 1\). From the above isomorphism \(\dim(H \cap W) \leq \dim(W) - 2\). But by Proposition 2.12...
Suppose that $H / ((H \cap W) \times (H \cap Z)) \cong \mathbb{Z}_p$. So $\dim(H \cap W) = \dim(H) - \dim(\mathbb{Z}_p) = \dim(W) - 1$, a contradiction. Thus $H \cap Z$ is non-trivial.

Now from Proposition 2.12 $(H \cap W) \times (H \cap Z) \leq_o H \leq_o (HZ \cap W) \times (HW \cap Z)$. Note that since $HW \cap Z \cong \mathbb{Z}_p$, $HZ \cap W \nleq_o (HZ \cap W) \times (HW \cap Z)$.

Suppose that $H' \leq_o H$. Then $H' \leq (HZ \cap W) \times (HW \cap Z)$. Hence $((HZ \cap W) \times (HW \cap Z))' \leq_o (HZ \cap W) \times (HW \cap Z)$. But $((HZ \cap W) \times (HW \cap Z))' = (HZ \cap W)'$. So $HZ \cap W \leq_o (HZ \cap W) \times (HW \cap Z)$, a contradiction. Hence $H' \nleq_o H$.

For (ii), suppose that $H$ factorises and that $H$ is solitary in $G$. Suppose for a contradiction that $H \nleq_o W$. By Lemma 3.6(vi) there exists $K \leq_o G$ with $H \triangleleft C K$ and $K / H \cong \mathbb{Z}_p$. Now

$$\frac{K \cap W}{H \cap W} \cong \frac{H(K \cap W)}{H} \leq \frac{K}{H}.$$  

If $H = H(K \cap W)$ then $K \cap W \leq H$ and so $H \cap W = K \cap W$. But $\dim(H \cap W) = \dim(W) - 1$ where as $\dim(K \cap W) = \dim(W)$, a contradiction. Thus $H \neq H(K \cap W)$ and so $(K \cap W)/(H \cap W) \cong \mathbb{Z}_p$. Now $(K \cap W) \times Z \leq_o G$ and $H = (H \cap W) \times (H \cap Z) \leq_o (K \cap W) \times Z$. So by Lemma 3.6(ii,iii),

$$\frac{H}{H \cap W}$$  

is isolated in $S\left(\frac{K \cap W}{H \cap W} \times Z\right)'$.

But by Examples 4.2(i) this is a contradiction since $\frac{K \cap W}{H \cap W} \times Z \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Thus $H \nleq_o W$.

Now by Lemma 3.6(ii) $H$ is solitary in $H \times Z$. So by Lemma 4.8, $H' \leq_o H$.

Conversely suppose that $H$ factorises, $H \leq_o W$ and $H' \leq_o H$. By Lemma 4.8 $H$ is solitary in $H \times Z$ and so by Lemma 3.6(ii), $H$ is solitary in $G$. □

**Corollary 4.19** Let $W$ be the group of Example 4.17, $Z \cong \mathbb{Z}_2$, and $G = W \times Z$. Let $H \in S(G)'$. If $H$ is solitary in $G$ then either $H = W$ or $H \cap Z \neq 1$ and $H' \nleq_o H$. If $H$ factorises then $H$ is solitary in $G$ if and only if $H = W$.

**Proof.** Suppose that $H$ is solitary in $G$. If $H \triangleleft C W$ then by Proposition 4.18(i), $H \leq_o W$. So by Lemma 3.6(ii) and Lemma 4.8 $H' \leq_o H$. But then by the properties of $W$, $H = W$. If $H \nleq W$ then by Proposition 4.18(i) $H \cap Z \neq 1$ and $H' \nleq_o H$.

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Now suppose that $H$ factorises and is solitary in $G$. Then by Proposition 4.18(ii) and the above, clearly $H = W$. Conversely if $H = W$ then again by Proposition 4.18(ii) $H$ is solitary in $G$ since $W' \leq O W$. ■

Problem 3 Can an infinite finitely generated pro-$p$ group which is not virtually $\mathbb{Z}_p$, have solitary subgroups, but have only finitely many solitary subgroups?

Does the group given in Example 4.17 above, provide a positive answer to this problem?

5 General Profinite Groups Attain All Permitted Scattered Heights

We have seen above that pro-$p$ groups can have subgroup spaces of all heights permitted by Theorem 3.8 – namely scattered height 1 or 2. Now we show that all scattered heights permitted by Theorem 3.8 for a general profinite group can also be attained.

Theorem 5.1 For each $n \leq k + 1$, there is a profinite group $G(n, k)$ satisfying:

the number of primes $p$ such that $p^\infty$ divides $o(G(n, k))$ equals $k$ and $ht(S(G)) = n$.

Proof. Fix $k$. We define appropriate profinite groups $G(n, k)$ for $n = 1$, $1 < n \leq k$ and $n = k + 1$. We choose $G(n, k)$ equal to $\bigoplus_{i=1}^{k} G_{p_i}$, where the $p_i$s are distinct primes. In this case $S(G(n, k)) \cong \prod_{i=1}^{k} S(G_{p_i})$.

Case $n = 1$: Let $G_{p_i} = \mathbb{Z}_{p_i}^2$. Then $S(G_{p_i}) = P$ (the Pelczyński space), and $S(G(n, k)) \cong P^k \cong P$, which has scattered height 1 ( 1).

Case $1 < n \leq k$: Let $G_{p_i} = \mathbb{Z}_{p_i}$ for $i = 1, \ldots, n$ and $G_{p_i} = \mathbb{Z}_{p_{n+1}}^{k+1}$ for $i = n+1, \ldots, k+1$. Then $S(G(n, k)) \cong (\omega + 1)^n \times P^{k+1-n}$, which has scattered height $n$.

Case $n = k + 1$: Let $G_{p_i} = \mathbb{Z}_{p_i}$ for $i = 1, \ldots, k$ and $G_{p_{k+1}}$ be any pro-$p_{k+1}$ group with uncountable subgroup space $X$ with scattered height 2 – for example a non-abelian just-infinite pro-$p_{k+1}$ direct producted with $\mathbb{Z}_{p_{k+1}}$. Then $S(G(n, k)) \cong (\omega + 1)^k \times X$, which has scattered height $k + 1$ ( 1). ■
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