Abstract

We inquire into some properties of diagonalizable pseudo-Hermitian operators, showing that their definition can be relaxed and that the pseudo-Hermiticity property is strictly connected with the existence of an antilinear symmetry. This result is then illustrated by considering the particular case of the complex Morse potential.

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1 Introduction

In the last years the study of some non-Hermitian Hamiltonians with a real spectrum, and the conjecture on the connection between the reality of the spectrum and the PT-invariance of these Hamiltonians due to Bender and Boettcher\textsuperscript{1}, have given rise to a growing interest in the literature\textsuperscript{2}. Indeed, the above-mentioned Hamiltonians form a subclass of the class of “pseudo-Hermitian” operators, i.e., those operators which satisfy the equation

\[ A^\dagger = \eta A \eta^{-1} \quad (1) \]

with

\[ \eta = \eta^\dagger. \quad (2) \]

Pseudo-Hermitian operators were introduced in the early 40’s by Dirac\textsuperscript{3} and Pauli\textsuperscript{4} in order to overcome some divergence difficulties in physics by using an indefinite metric associated with \( \eta \), and were later resumed by Lee and Wick\textsuperscript{5} (who firstly, at the best of our knowledge, used the term “pseudo-Hermiticity”). More recently, many interesting properties of such operators have been examined and their spectrum has been suitably characterized\textsuperscript{6,7}.

We aim in the present paper to point out further properties of pseudo-Hermitian operators that are relevant from a physical viewpoint. To this end, we introduce in Sect. II the possibly broader class of \emph{weakly} pseudo-Hermitian operators, i.e., those operators which satisfy Eq. (1) without any constraint on the (linear, invertible) operator \( \eta \), and show that, whenever one considers only diagonalizable operators, this class actually coincides with the class of all the pseudo-Hermitian operators. Hence the condition in Eq. (2) can be dropped when defining (diagonalizable) pseudo-Hermitian operators, which is useful from several viewpoints (in particular, it simplifies checking Eq. (1)). Moreover, we show in Sect. III that a diagonalizable operator \( H \) is (weakly) pseudo-Hermitian if and only if an antilinear involutory operator exists which commutes with it. This result has a number of relevant consequences; in particular, in every theory which admits a time-reversal invariance, or a CPT-invariance, the Hamiltonian must necessarily be a (weakly) pseudo-Hermitian operator. Furthermore, the above result is strictly intertwined with an old theorem\textsuperscript{8} of group representation theory, according to which a set of operators admits an involutory antilinear mapping that commutes with it if and only if all the operators in the set can assume conjointly a real form in a suitable basis. Indeed, by using this theorem together with the above results, we conclude in Sect. IV that for any diagonalizable (weakly) pseudo-Hermitian operator \( H \) a basis exists in which \( H \) has a real form. If this basis coincides with the eigenbasis of \( H \), then \( H \) also has a real spectrum. Finally, we illustrate our results by means of an example, considering the special case of the complex Morse potential\textsuperscript{9} in Sect. V.
2 The spectra of weakly pseudo-Hermitian operators

As we wrote in the Introduction, we introduce here a new class of operators, whose properties will be studied in the following.

**Definition 1.** A linear operator $A$ is said to be weakly pseudo-Hermitian if a linear, invertible operator $\eta$ exists such that

$$\eta A \eta^{-1} = A^\dagger.$$  

(1)

The above definition generalizes the definition of pseudo-Hermitian operators since we do not assume $\eta = \eta^\dagger$ as is required in the standard definition of pseudo-Hermitian operators$^{5,6}$. As in Refs. 6 and 7, we consider here only diagonalizable operators; moreover, for the sake of simplicity, we consider only discrete spectra (see however Sect. V, where a potential with a continuous spectrum is explicitly studied). Whenever $H$ is a diagonalizable operator with a discrete spectrum, a complete biorthonormal eigenbasis $\{|\psi_n, a\rangle, |\phi_n, a\rangle\}$ exists$^{11}$, i.e., a basis such that

$$H |\psi_n, a\rangle = E_n |\psi_n, a\rangle, \quad H^\dagger |\phi_n, a\rangle = E_n^* |\phi_n, a\rangle$$  (3)

$$\langle \phi_m, b |\psi_n, a\rangle = \delta_{mn} \delta_{ab}$$  (4)

$$\sum_n \sum_a d_n |\phi_n, a\rangle \langle \psi_n, a| = \sum_n \sum_a d_n |\psi_n, a\rangle \langle \phi_n, a| = 1$$  (5)

where $d_n$ denotes the degeneracy of $E_n$, and $a$ and $b$ are degeneracy labels.

The operator $H$ can then be written in the form

$$H = \sum_n \sum_a d_n |\psi_n, a\rangle E_n \langle \phi_n, a|. $$

For the sake of brevity we also write the above basis $\{|\psi_m\rangle, |\phi_m\rangle\}$ in the following, with an obvious meaning of symbols. Then $H$ can also be written in the form

$$H = \sum_m |\psi_m\rangle E_m \langle \phi_m|$$

(where it may occur that $E_m = E_{m'}$ even if $m \neq m'$). Furthermore, if $\{u_m\}$ is any complete, orthonormal basis in our space, we put in the following

$$O = \sum_m |\psi_m\rangle \langle u_m|$  (6)
By using Eq. (6), we get

\[ O^{-1} = \sum_m |u_m\rangle \langle \phi_m| \]

and

\[ O^{-1} H O = \sum_m |\phi_m\rangle \langle \sum_{m'} |\psi_{m'}\rangle E_{m'} \langle \phi_{m'}| \sum_{m''} |\psi_{m''}\rangle \langle u_{m''}| = \sum_m |u_m\rangle E_m \langle u_m|. \]

Moreover,

\[ (OO^\dagger) H^\dagger (OO^\dagger)^{-1} = \sum_m |\psi_m\rangle E_m^* \langle \phi_m|. \] (7)

We can now state the following proposition, which embodies some results in Ref. 6 on pseudo-Hermitian operators.

**Proposition 1.** Let \( H \) be a diagonalizable operator with a discrete spectrum. Then, the following conditions are equivalent:

i) \( H \) is weakly pseudo-Hermitian;

ii) the eigenvalues of \( H \) occur in complex conjugate pairs, and for each pair the multiplicities of both the eigenvalues are the same;

iii) \( H \) is pseudo-Hermitian.

**Proof.** The implication iii) \( \Rightarrow \) ii) is proven in Prop. 7 of Ref. 6. By observing that only the invertibility of \( \eta \) is used in this proof, in order to show that \( \eta^{-1} \) maps the eigensubspace of \( H^\dagger \) associated with \( E_n \) to that of \( H \) associated with \( E_n^* \), and both the subspaces have the same dimension, one immediately transforms this proof into a proof of the implication i) \( \Rightarrow \) ii).

The implication ii) \( \Rightarrow \) iii) is also proven in Ref. 6. We provide here, however, a somewhat different proof of it, which produces a useful decomposition of \( \eta \) (see Eq. (11)).

Let us therefore assume that condition ii) holds, and use (whenever it is necessary) the subscript ‘\( q \)’ to denote real eigenvalues and the corresponding eigenvectors, and the subscript ‘\( \pm \)’ to denote the complex eigenvalues with positive or negative imaginary part, respectively, and the corresponding eigenvectors.

Then, let us consider the involutory operator \( T \), defined as follows:

\[ T |\psi_{n_\pm}, a\rangle = |\psi_{n_\pm}, a\rangle \] (hence, \( T |\psi_{n_0}, a\rangle = |\psi_{n_0}, a\rangle \)).

The explicit form of \( T \) is

\[ T = T(\sum_{n_0} \sum_{a=1}^{d_{n_0}} |\psi_{n_0}, a\rangle \langle \phi_{n_0}, a| + \sum_{n_+} \sum_{a=1}^{d_{n_+}} |\psi_{n_+}, a\rangle \langle \phi_{n_+}, a| + \sum_{n_-} \sum_{a=1}^{d_{n_-}} |\psi_{n_-}, a\rangle \langle \phi_{n_-}, a| ) = \sum_{n_0, a} |\psi_{n_0}, a\rangle \langle \phi_{n_0}, a| + \sum_{n_+, n_-} (|\psi_{n_+}, a\rangle \langle \phi_{n_+}, a| + |\psi_{n_-}, a\rangle \langle \phi_{n_-}, a| ). \]
The action of $T$ on the bras $\langle \phi_{n\pm}, a \rangle$ easily follows from the expression above:

$$\langle \phi_{n\pm}, a | T = \langle \phi_{n\pm}, a \rangle$$

(hence, $\langle \phi_{n_0}, a | T = \langle \phi_{n_0}, a \rangle$).

(9)

Then, by simple calculations, one has

$$THT = \sum_{m} |\psi_m \rangle E^*_m \langle \phi_m|,$$  

(10)

and finally, comparing Eqs. (7) and (10), it follows

$$THT = (OO^\dagger)H^\dagger(OO^\dagger)^{-1},$$

hence condition iii) follows at once, with

$$\eta = (OO^\dagger)^{-1}T = \sum_{n_0, a} |\phi_{n_0}, a \rangle \langle \phi_{n_0}, a | + \sum_{n_+, n_-, a} \langle |\phi_{n_+}, a \rangle \langle \phi_{n_-}, a | + \langle |\phi_{n_-}, a \rangle \langle \phi_{n_+}, a | = \eta^\dagger.$$

(11)

Finally, the proof of the Proposition can be completed by observing that the implication iii) $\Rightarrow$ i) is obvious. ■

The introduction of the operator $T$ on the above proof and the decomposition $\eta = (OO^\dagger)^{-1}T$ in Eq. (11) allows one to obtain immediately the characterization of the case of real spectrum. Indeed, noting that $T = 1$ if and only if the spectrum of $H$ is real, the following statement holds (see also the Theorem in Ref. 7).

**Proposition 2.** The spectrum of a diagonalizable weakly pseudo-Hermitian (hence, of a diagonalizable pseudo-Hermitian) operator $H$ is real if and only if an operator $\eta$ exists such that $\eta = (OO^\dagger)^{-1}$.

Furthermore, the existence of an Hermitian operator $\eta$ whenever $H$ is weakly pseudo-Hermitian implies that also in this case one can introduce an Hermitian, indefinite inner product $^4$-$^6$, $^10$ which is invariant under the time translation generated by $H$.

## 3 Weakly pseudo-Hermitian operators and antilinear symmetries

In order to discuss properly the next argument, we state the following definition.

**Definition 2**$^5$. Given the biorthonormal basis $\mathcal{E} = \{ |\psi_m \rangle, |\phi_m \rangle \}$ in a Hilbert space, we call conjugation associated with it the involutory antilinear operator

$$\Theta_E = \sum_{m} |\psi_m \rangle K \langle \phi_m|,$$

(12)

where the operator $K$ acts transforming each complex number on the right into its complex conjugate.
Let us discuss now the connection between the (weak) pseudo-Hermiticity condition and the antilinear commutant\(^8\) of \(H\) \((\text{i.e., the set of the antilinear, invertible operators which commute with it})\). This connection was already acknowledged in Ref. 7, where the author shows that, if \(H\) commutes with an antilinear operator \(A\), then condition \(ii\) of Proposition 1 holds, and that a Hamiltonian with an antilinear symmetry \(A\) has a real spectrum if and only if the symmetry is exact\(^7\)(\text{i.e., its eigenvectors are invariant under the action of } A\). The latter statement can be rephrased, using Definition 2, by saying that the spectrum of \(H\) is real if and only if \([H, \Theta_E] = 0\).

However, the above results enlighten only partially the key role of the antilinear commutant of \(H\). Indeed, the following, more complete proposition holds.

**Proposition 3.** Let \(H\) be a diagonalizable operator with a discrete spectrum. Then, the following conditions are equivalent:

\(i\) an antilinear, invertible operator \(\Omega\) exists such that \([H, \Omega] = 0\);

\(ii\) \(H\) is (weakly) pseudo-Hermitian;

\(iii\) an antilinear, involutory operator \(\hat{\Omega}\) exists such that \([H, \hat{\Omega}] = 0\).

**Proof.** \(i \Rightarrow ii\). Let \(\Omega\) exist. Then, the linear operator

\[ \eta = (OO^\dagger)^{-1} \Theta_E \Omega \]

(where \(E\) is the biorthonormal basis associated with \(H\), and \(O\) and \(\Theta_E\) are defined as in Eqs. (6) and (12), respectively) fulfils the condition stated by Eq. (1), hence \(H\) is (weakly) pseudo-Hermitian. Indeed,

\[ \Theta_E H \Theta_E^{-1} = \Theta_E H \Theta_E = \sum_m \langle \psi_m | E^*_m \langle \phi_m | , \]

so that, recalling Eq. (7),

\[ \eta H \eta^{-1} = (OO^\dagger)^{-1} \Theta_E \Omega H \Omega^{-1} \Theta_E (OO^\dagger) = (OO^\dagger)^{-1} \Theta_E H \Theta_E (OO^\dagger) = H^\dagger. \]

\(ii \Rightarrow iii\). If \(H\) is (weakly) pseudo-Hermitian, the eigenvalues of \(H\) occur in complex conjugate pairs, and for each pair the multiplicities of both the eigenvalues are the same (Proposition 1). Then, one easily sees, recalling the definition of the operator \(T\) provided in the proof of Proposition 1 and Eq. (10), that

\[ \Theta_E H \Theta_E = T H T. \]

Hence the antilinear operator

\[ \hat{\Omega} = \Theta_E T = \sum_{n_0, a} |\psi_{n_0, a}\rangle K \langle \phi_{n_0, a}| + \sum_{n_+, n_-, a} (|\psi_{n_+, a}\rangle K \langle \phi_{n_+, a}| + |\psi_{n_-, a}\rangle K \langle \phi_{n_-, a}|) \]

commutes with \(H\). Finally, \(\hat{\Omega}\) is involutory, (i.e., \(\hat{\Omega}^2 = 1\)) as one can immediately verify by using the explicit expression of \(\hat{\Omega}\) in Eq. (14).
Proposition 3 has an interesting physical interpretation, as we have emphasized in the Introduction. Indeed, whenever $H$ is the Hamiltonian of some physical system, it establishes a link between the properties of $H$ (and of its spectrum) and the symmetries of the physical system described by it. For, the time-reversal symmetry is associated, in complex quantum mechanics, with an antilinear operator. Hence, whenever a physical system admits such a symmetry (or else, more generally, it is invariant under the combined action of the time-reversal operator times a linear one) the antilinear commutant of its Hamiltonian must be non-void, hence $H$ is a (weakly) pseudo-Hermitian operator. Vice versa any (weakly) pseudo-Hermitian Hamiltonian admits an antilinear (involutory) symmetry.

Finally, since in the case of real spectrum the operator $T$ defined in the proof of Proposition 1 is such that $T = 1$, hence $\hat{\Omega} = \Theta_{e}$, one obtains the following proposition.

**Proposition 4.** A diagonalizable, weakly pseudo-Hermitian operator $H$ has a real spectrum if and only if it commutes with the conjugation associated with its eigenbasis.

**Remark.** While we were writing the final version of this paper, some similar results have been obtained by other authors\(^\text{12}\) (in particular, having in mind the equivalence \(i) \Leftrightarrow \text{iii} \)) in our Proposition 1, Theorem 2 of Ref. 12 essentially states the equivalence \(i) \Leftrightarrow \text{ii} \)) of our Proposition 3). Nevertheless, our presentation is rather different and embodies the new condition \(\text{iii}) \) in Proposition 3, which has a number of interesting consequences, that we are going to explore in the next section.

### 4 Real form of the (weakly) pseudo-Hermitian operators

According to Proposition 3, for any (weakly) pseudo-Hermitian operator $H$, at least one involutory antilinear operator exists which commutes with it. Then, it has been proven elsewhere\(^\text{8}\) that any involutory antilinear operator $\hat{\Omega}$ is a conjugation in some suitable basis; moreover, in the basis associated with $\hat{\Omega}$, any operator commuting with $\hat{\Omega}$ has a real form.

The proof of this latter statement can be sketched as follows. If we denote by $S$ the linear part of $\hat{\Omega}$, i.e., $\hat{\Omega} = SK$ (where $K$ is the complex conjugation operator, see Sect. III), then $\hat{\Omega}^2 = 1$ implies $SS^* = 1$ and this is possible if and only if an $U$ exists such that $S = UU^*$. Then $[H, \hat{\Omega}] = 0$ implies $HUU^* = UU^*H^*$, hence

$$U^{-1}HU = (U^{*-1}H^*U^*) = (U^{-1}HU)^*.$$  

Referring to the notation introduced in the present paper, let $\mathcal{F} = \{|v_m\rangle, |w_m\rangle\}$ be the biorthonormal basis associated, in the above sense, to the conjugation $\hat{\Omega}$ which commutes with $H$ (of course, it may be an orthonormal basis which
occurs if and only if, for all \( m \), \( |v_m\rangle = |w_m\rangle \), and let us consider the matrix elements of \( H \) in such basis. It is easy to verify that they are real; indeed, on one hand,

\[
\langle w_i | H | v_k \rangle = \langle w_i | \sum_n \langle \psi_n | E_n \langle \phi_n | v_k \rangle
\]

and, on the other hand

\[
\langle w_i | \hat{\Omega} H \hat{\Omega} | v_k \rangle = \langle w_i | \sum_m \langle v_m | K \langle \psi_n | E_n \langle \phi_n | v_{m'} \rangle K \langle w_{m'} | v_k \rangle =
\]

\[
\sum_{m,m',n} \delta_{im} K \langle w_m | \phi_n \rangle E_n \langle \psi_n | v_{m'} \rangle K \delta_{m',k} =
\]

\[
\langle v_k | \sum_n \phi_n \rangle E_n^* \langle \psi_n | w_i \rangle = \left( \langle w_i | \sum_n \langle \psi_n | E_n \langle \phi_n | v_k \rangle \right)^*.
\]

Since, trivially, every operator which assumes a real form in some basis \( \mathfrak{B} \) commutes with the conjugation associated with \( \mathfrak{B} \), we have thus proven the following proposition.

**Proposition 5.** An operator \( H \) is (weakly) pseudo-Hermitian if and only if a basis exists in which it assumes a real form.

The results in Propositions 1, 3 and 5 can be collected together, obtaining a set of six equivalent conditions which can be useful in order to characterize the Hamiltonians that we are considering. In particular, the statement in Proposition 5 can be used to write a (weakly) pseudo-Hermitian operator in a more manageable form (an example of basis transformation which puts a particular Hamiltonian in real form is in the next Section).

### 5 An example: the complex Morse potential

Let us verify the results obtained in the previous sections in the special case of the complex Morse potential\(^8\). This was extensively studied, for instance, in Ref. 9 and its spectrum was predicted to be real by means of group theoretic techniques\(^13\).

The Morse potential is given by

\[
V(x) = (A + iB)^2 e^{-2x} - (A + iB)(2C + 1)e^{-x} \quad (A, B, C \in \mathbb{R}).
\]  

Putting \( \rho = \sqrt{A^2 + B^2}, \theta = \arctan \frac{2B}{A}, k = 2C + 1 \), we get

\[
V(x) = \rho^2 e^{-2x + i\theta} - k \rho e^{-x + i\theta/2}.
\]
Following Ref. 9, let us introduce the (Hermitian) operator \( e^{-\theta p} \) \((\theta \in \mathbb{R}, \ p = -i\frac{d}{dx})\). Hence, we obtain the following Equation\(^9\)
\[
e^{-\theta p}V(x)e^{\theta p} = V(x + i\theta) = V^*(x).
\] (17)
This equation shows that \( V \) is a pseudo-Hermitian (but non PT-symmetric) operator. Let us put now \( \Omega = SK = e^{\theta p}K \). By using Eq. (17) one gets
\[
\Omega V = V\Omega,
\]
which agrees with Proposition 3. Then, a straightforward calculation shows that \( S^* = e^{-\theta p} = S^{-1} \), hence \( \Omega \) is involutory, which also agrees with Proposition 3. Moreover, one gets by inspection that
\[
S = e^{\theta p/2}(e^{\theta p/2})^{* -1} = UU^* - 1.
\]
Thus, finally,
\[
U^{-1}VU = e^{-\theta p/2}V e^{\theta p/2} = V(x + i\theta/2) = \rho^2 e^{-2x} - k\rho e^{-x} = (U^{-1}VU)^*
\]
which agrees with Proposition 5.

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