Modeling of vibration for functionally graded beams

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Abstract: In this study, a vibration problem of Euler-Bernoulli beam manufactured with Functionally Graded Material (FGM), which is modelled by fourth-order partial differential equations with variable coefficients, is examined by using the Adomian Decomposition Method (ADM). The method is one of the useful and powerful methods which can be easily applied to linear and nonlinear initial and boundary value problems. As to functionally graded materials, they are composites mixed by two or more materials at a certain rate. This mixture at a certain rate is expressed with an exponential function in order to try to minimize singularities from transition between different surfaces of materials as much as possible. According to the structure of the ADM in terms of initial conditions of the problem, a Fourier series expansion method is used along with the ADM for the solution of simply supported functionally graded Euler-Bernoulli beams. Finally, by choosing an appropriate mixture rate for the material, the results are shown in figures and compared with those of a standard (homogeneous) Euler-Bernoulli beam.

Keywords: Adomian decomposition method, Functionally graded beam, Fourier analysis, Orthogonality

MSC: 35L35, 74E05, 74K10

1 Introduction

In recent years, the ADM has been applied to solve many problems in science and engineering due to the rapid convergence and reduced computational cost. The advantage of this method is that it solves the problems directly without linearization, perturbation or other transforms [1, 2]. In the solution of linear as well as nonlinear deterministic and stochastic equations, this method introduces a rapidly convergent series solution [3, 4]. The method is modified by researchers to solve many problems, which include linear and nonlinear ordinary and partial differential equations, integral equations and their systems. Having exact solutions in some types of problems as a sum of infinite series has encouraged scientists to conduct research by applying the ADM to a wide range of problems and functional equations. Many well known problems in science and engineering are examined by the ADM; the results, which are similar or exactly the same, are compared with either analytical or numerical solutions of these problems [5].

The important part of the ADM is representing the nonlinear terms in the equation by Adomian polynomials built on different techniques. The most widely used Adomian polynomials are generated by George Adomian based on a Taylor series expansion [1, 2]. Using a Taylor series expansion around initial iteration point, the nonlinear terms in the equation can be expressed as a power series of unknown functions. Then, setting the same degrees of unknown values, the Adomian polynomials can be obtained. Another way to calculate Adomian polynomials is built...
on a Neumann series expansion, where the nonlinear terms can be represented by a Neumann series depending on a parameter, such as \( \lambda \), and the Adomian polynomials can be represented as the coefficients of \( \lambda \) in the series. Finally, the parametrization method is also suitable while working with nonlinear cases. Basically, the Adomian polynomials can be obtained by comparison with the expansion of a Neumann series and the expansion of nonlinear terms.

Although the method gives rapidly convergent results compared with other semi-analytic methods, there are some studies about the treatment of the method for convergence and computing time issues for the calculation of Adomian polynomials [6, 7]. The most important modification on the ADM, called Modified ADM, is to eliminate the repeating process in the recurrence formulation by choosing the initial term as two parts. Thus, the acceleration in the convergence of the solution can be increased and the number of iterations is minimized to approach the solution more rapidly [8].

Applications of the ADM to engineering problems have been carried out by researchers. The simple solution of the Cauchy problem for the wave equation is examined by Lesnic [9] using the ADM. As a conclusion, it is shown that the ADM can be applied to similar problems with various types of boundary conditions. Since the method can be easily applied to linear as well as nonlinear problems, many engineering problems can be solved with it. The study of bending vibration of beams is very important in a wide variety of areas such as bridges, tall buildings, trains, aircraft, the dynamics and control of rockets and producing machine tools. The Euler-Bernoulli beam equation, which is governed by the fourth-order differential equation, was solved by both numerical and analytical methods by many researchers. The vibration of beams was examined numerically by Reddy [10] using the finite element method. Also, in Haddadpour [11], the vibration of an Euler-Bernoulli beam was studied using the ADM and an exact closed-form solution was given for a simply supported beam. These results were compared with the solution of the same problem by the modal analysis method. Many researchers have examined the vibration of composite materials made of Functionally Graded Materials (FGMs) which are essentially two-phase particulate composites synthesized in a way such that the volume fractions of the constituents vary constantly from one point to another. The concept of FGMs could provide great flexibility in material design by controlling both the composition profile and the microstructure [12–14]. As an application of the ADM on FGMs, the one dimensional diffusion equation with variable coefficients is examined by Sahin and Karatay [15]. In the problem, the material composition, which is named as a nonhomogeneity parameter, is modeled as both an exponential function and a power function. Finally, results were introduced in terms of a varying nonhomogeneity parameter of the composition profile. For more details about the beams and iterative methods, one can see [16–21].

In this study, the clamped simply supported composite Euler-Bernoulli beam that is governed by the fourth-order differential equation with variable coefficients was solved using the ADM. The composition profile was modeled with an exponential function of the spatial coordinate with a parameter named as the nonhomogeneity parameter. The results in terms of the natural frequencies of the beam were compared with closed-form solutions for clamped simply supported Euler-Bernoulli beam made of homogeneous material, in which the nonhomogeneity parameter was assumed as zero. These results were also introduced in figures for different values of the nonhomogeneity parameter for a composite beam. In the Euler-Bernoulli beam theory, any point on the beam can only move through the vertical direction and this movement is called deflection by \( \omega(x,t) \) of that point. The transverse deflection of the composite beam is governed by the fourth-order partial differential equation with variable coefficients:

\[
\mu(x) \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} \left[ EI(x) \frac{\partial^2 w}{\partial x^2} \right] = q(x,t),
\]

where \( \mu(x) \) is the mass per unit length, \( EI(x) = E(x)I(x) \), \( E(x) \) is the elastic modulus, \( I(x) \) is the moment of inertia and \( q(x,t) \) is the load per unit length. The composition profile for a functionally graded Euler-Bernoulli beam can be represented mathematically by an exponential function such that the engineering parameters are given as

\[
EI(x) = E_0 I_0 e^{\alpha x}, \quad \mu(x) = \mu_0 e^{\alpha x}, \quad \alpha > 0,
\]

where the parameter \( \alpha \) is named as a nonhomogeneity parameter of FGM and the values \( E_0 I_0 \) and \( \mu_0 \) are constants. The ADM will be applied to solve the problem in which the corresponding eigenfunctions of the eigenvalues will be held in terms of generalized Fourier series.
2 Adomian Decomposition Method (ADM)

In this section, we examine a differential equation of the form

\[ Lu + Ru + Nu = g, \]  

where \( L \) is an easily invertible operator, \( R \) is the remainder of the linear operator, \( N \) represents the nonlinear operator and the function \( g \) represents the nonhomogeneous part of a differential equation. Since \( L \) is an easily invertible \( n \)-th order differential operator, its inverse \( L^{-1} \) to both sides of (3) along with the initial conditions, we obtain

\[ u(x, t) = f(x, t) + L^{-1} g - L^{-1} Ru - L^{-1} Nu, \]  

where

\[ f(x, t) = u(x, 0) + tu'(x, 0) + \frac{t^2}{2} u''(x, 0) + \ldots + \frac{t^{n-1}}{(n-1)!} u^{n-1}(x, 0). \]

The ADM defines a solution with an infinite series of the same form as in [22],

\[ u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \]  

and the nonlinear term \( Nu \) can be decomposed by an infinite series of polynomials as

\[ Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \ldots, u_n), \]  

where \( A_0, A_1, A_2, \ldots, A_n \) are Adomian polynomials of \( u_0, u_1, u_2, \ldots, u_n \) given by

\[ A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{k=0}^{\infty} \lambda^k u_k \right) \right]_{\lambda=0} \quad n \geq 0, \]  

and explicitly formulated by

\[ A_0 = f(u_0), \quad A_1 = u_1 f'(u_0), \quad A_2 = u_2 f''(u_0) + \frac{1}{2} u_1^2 f'''(u_0), \]

\[ A_3 = u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{1}{3!} u_1^3 f'''(u_0), \]

\[ A_4 = u_4 f'(u_0) + \left( \frac{1}{2!} u_1^2 + u_1 u_3 \right) f''(u_0) + \frac{1}{2!} u_1^2 u_2 f'''(u_0) + \frac{1}{4!} u_1^4 u_2 f^{(4)}(u_0), \]

\[ \vdots \]

Substituting (5) along with (6) into (4), we obtain

\[ \sum_{n=0}^{\infty} u_n = f(x, t) + L^{-1} g(x, t) - L^{-1} R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n, \]  

where the recurrence relation is given as follows:

\[ u_0 = f(x, t), \]

\[ u_1 = -L^{-1} (Ru_0) - L^{-1} (A_0), \]

\[ \vdots \]

\[ u_{n+1} = -L^{-1} (Ru_n) - L^{-1} (A_n), \quad n \geq 0. \]  

If the series converges in a suitable way, the approximated general solution of a differential equation can be expressed as

\[ u(x) = \lim_{n \to \infty} \sum_{n=0}^{\infty} u_k(x). \]
3 Vibration of functionally graded beam

The Equation (1) can be rewritten in an operator form, with definitions given in (2), as

\[ L_t w + L_x w = Q(x, t), \]  

(11)

where

\[ L_t = \frac{\partial^2}{\partial t^2}, \quad Q(x, t) = \frac{q(x, t)}{\mu_0 e^{-\alpha x}} \]

and

\[ L_x = \frac{1}{\mu_0 e^{\alpha x}} \frac{\partial^2}{\partial x^2} \left[ E_0 I_0 e^{\alpha x} \frac{\partial^2}{\partial x^2} \right] = \alpha^2 \epsilon \frac{\partial^2}{\partial x^2} + 2 \alpha \epsilon \frac{\partial^3}{\partial x^3} + \epsilon \frac{\partial^4}{\partial x^4}, \quad \epsilon = \frac{E_0 I_0}{\mu_0} \]

(12)

subject to the initial conditions

\[ w(x, 0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t} w(x, 0) = g(x) \]

which also satisfy the boundary conditions

\[ w(0, t) = \frac{\partial}{\partial x} w(0, t) = 0 \quad \text{and} \quad w(l, t) = \frac{\partial^2}{\partial x^2} w(l, t) = 0 \]

(13)

related to the clamped simply-supported beam like.

In the solution process, the problem will be separated into two parts:

\[ w(x, t) = u(x, t) + v(x, t) \]

(14)

where \( u(x, t) \) and \( v(x, t) \) are the respective solutions of the homogeneous and nonhomogeneous equations,

\[ L_t u + L_x u = 0, \quad \text{and} \quad L_t v + L_x v = Q(x, t). \]

(15)

(16)

3.1 Solution of homogeneous part

Applying the inverse operator, \( L_t^{-1} = \int_0^t \int_0^t dt \) to both sides of (15), the solution of the homogeneous equation subject to initial conditions can be obtained as

\[ u(x, t) = f(x) + t g(x) - L_t^{-1}[L_x u]. \]

(17)

Finally, using the ADM along with Equation (5), the approximated solution of homogeneous part (15) can be determined by the following recurrence relation:

\[ u_0(x, t) = f(x) + t g(x), \]
\[ u_1(x, t) = -L_x \left( f(x) \frac{t^2}{2} + g(x) \frac{t^3}{6} \right), \]
\[ u_2(x, t) = (-1)^2 L_x^2 \left( f(x) \frac{t^4}{4!} + g(x) \frac{t^5}{5!} \right), \]
\[ u_3(x, t) = (-1)^3 L_x^3 \left( f(x) \frac{t^6}{6!} + g(x) \frac{t^7}{7!} \right), \]

\[ \vdots \]

Consequently, the solution is given by

\[ u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = (-1)^n L_x^n \left( f(x) \frac{t^{(2i)!}}{(2i)!} + g(x) \frac{t^{(2i+1)!}}{(2i+1)!} \right) \]

(18)
where \( L^j_x = l_x L^{-1}_x \). It is clear that a deficient solution may be obtained due to the iteration process, along with the solution \( u_j(x,t) \) that is found by applying the differential operator \( L^j_x \). In the solution process, it is easy to obtain a zero solution or a solution which does not satisfy the boundary conditions of the problem. From now on, to accomplish the deficiency, both functions \( f(x) \) and \( g(x) \) will be expressed in the form of a generalized Fourier series in terms of the Sturm-Liouville boundary value problem. Thus, the coefficients of the Fourier series will be chosen such that these functions satisfy the boundary conditions of the problem. Assume that \( f(x) \) and \( g(x) \) are given functions in a continuous function space \( C_p(a,b) \). When an orthonormal set of functions \( \phi_j(x) \), \( j = 1, 2, \ldots \) in \( C_p(a,b) \) is specified, it may be possible to represent \( f(x) \) and \( g(x) \) by a linear combination of those functions. This may be generalized to an infinite series that converges to \( f(x) \) and \( g(x) \) at all but possibly a finite number of points on the fundamental interval \( 0 < x < l \) as follows:

\[
f(x) = \sum_{j=1}^{\infty} a_j \phi_j(x), \quad g(x) = \sum_{j=1}^{\infty} b_j \phi_j(x),
\]

and, due to orthogonality it can be shown that

\[
a_j = \int_0^l f(x) \phi_j(x) \, dx, \quad b_j = \int_0^l g(x) \phi_j(x) \, dx.
\]

The appropriate boundary value problem to find suitable eigenvalues and eigenfunctions for the graded beam problem can be expressed as

\[
L^j_x \phi_j(x) = \lambda^j \phi_j(x),
\]

where \( \lambda^j \) and \( \phi_j(x) \) represent eigenvalues and eigenfunctions, respectively. Then, the equation (21) can be written explicitly as the fourth order differential equation

\[
\frac{\partial^4 \phi}{\partial x^4} + 2\alpha \frac{\partial^3 \phi}{\partial x^3} + \alpha^2 \frac{\partial^2 \phi}{\partial x^2} - \frac{\lambda}{\epsilon} \phi = 0
\]

and the solution is obtained as

\[
\phi(x) = e^{\frac{-\beta x}{2}} \left( c_1 \cosh \left( \frac{\sqrt{\alpha^2 + 4\beta^2}}{2} x \right) + c_2 \sinh \left( \frac{\sqrt{\alpha^2 + 4\beta^2}}{2} x \right) \right. \\
\left. + c_3 \cos \left( \frac{\sqrt{4\beta^2 - \alpha^2}}{2} x \right) + c_4 \sin \left( \frac{\sqrt{4\beta^2 - \alpha^2}}{2} x \right) \right).
\]

The unknowns \( (c_1, c_2, c_3, c_4) \) are determined by the end conditions of the beam such as

\[
\phi(0) = \phi'(0) = 0, \quad \phi(l) = \phi''(l) = 0.
\]

The application of boundary conditions gives us the zero solution for \( \phi(x) \) that is not acceptable for general vibrating systems. The characteristic equation of the determinant obtained by the end conditions for fixed values of the nonhomogeneity parameter \( \alpha \) infinitely many solutions. For fixed values of \( l = 1 \), \( \epsilon = 1 \) and \( \alpha = 1 \), the first four roots of the characteristic equation which are named as natural frequencies of the vibration can be found as

\[
\beta_1 = 3.7939, \quad \beta_2 = 7.0076, \quad \beta_3 = 10.1697, \quad \beta_4 = 13.3215.
\]

For each value of \( \beta_j \), it is easy to obtain all unknown coefficients assuming that \( c_1 = 1 \) and then, the normalized shape functions (eigenfunctions) \( \phi_j(x) \) in Equation (23) can be easily found. On the other hand, the solution of the homogeneous equation in (15) can be obtained by substituting initial conditions, along with Equation (20), into (18) as

\[
u(x,t) = \sum_{j=1}^{\infty} a_j \left[ \sum_{i=0}^{\infty} (-1)^i \frac{(t \sqrt{\lambda_j})^{2i}}{(2i)!} \right] \phi_j(x) + \sum_{j=1}^{\infty} b_j \left[ \sum_{i=0}^{\infty} (-1)^i \frac{(t \sqrt{\lambda_j})^{2i+1}}{(2i+1)!} \right] \phi_j(x).
\]
Finally, using the identities for series expansions of sine and cosine functions, the solution of the homogeneous equation can be simplified to

\[
    u(x, t) = \sum_{j=1}^{\infty} \left( a_j \cos(\sqrt{\lambda_j} t) + \frac{b_j}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j} t) \right) \phi_j(x)
\]  

(27)

where the normalized shape functions for the first four eigenvalues can be obtained as

\[
    \phi_1(x) = 1.349e^{-0.5x} \times (\cosh(3.824x) - 1.010\sinh(3.824x) - \cos(3.758x) + 1.028\sin(3.758x)) \]

(28)

\[
    \phi_2(x) = 1.281e^{-0.5x} \times (\cosh(7.025x) - 1.010\sinh(7.025x) - \cos(6.989x) + 1.028\sin(6.989x)) \]

(29)

\[
    \phi_3(x) = 1.2647e^{-0.5x} \times (\cosh(10.1820x) - 1.000\sinh(10.182x) - \cos(10.157x) + 1.028\sin(10.157x)) \]

(30)

\[
    \phi_4(x) = 1.264e^{-0.5x} \times (\cosh(13.330x) - 0.999\sinh(13.330x) - \cos(13.312x) + 1.001\sin(13.312x)) \]

(31)

and, for initial conditions associated with (17) as

\[
    w(x, 0) = f(x) = x^2, \quad w_t(x, 0) = g(x) = 0.
\]  

(32)

So, the corresponding solutions are given as follows:

\[
    u_1(x, t) = 0.393e^{-0.5x} \cos(14.378t) \times (\cosh(3.824x) - 1.010\sinh(3.824x) - \cos(3.758x) + 1.028\sin(3.758x)),
\]

(33)

\[
    u_2(x, t) = -0.191e^{-0.5x} \cos(49.107t) \times (\cosh(7.025x) - 1.010\sinh(7.025x) - \cos(6.989x) + 1.028\sin(6.989x)),
\]

(34)

\[
    u_3(x, t) = 0.130e^{-0.5x} \cos(103.42t) \times (\cosh(10.182x) - 1.006\sinh(10.182x) - \cos(10.157x) + 1.028\sin(10.157x)),
\]

(35)

\[
    u_4(x, t) = -0.099e^{-0.5x} \cos(117.46t) \times (\cosh(13.330x) - 0.999\sinh(13.330x) - \cos(13.312x) + 1.001\sin(13.312x)).
\]  

(36)

3.2 Solution of nonhomogeneous part

Now, let us consider the nonhomogeneous problem given by (16). Expressing the function \(v(x, t)\) as an infinite series, the problem can be rewritten using the ADM as

\[
    \sum_{i=0}^{\infty} v_j(x, t) = -L_t^{-1}L_x \sum_{i=0}^{\infty} v_j(x, t) - L_t^{-1}Q(x, t).
\]

(37)

The components \(v_0(x, t)\) are identified as

\[
    v_0(x, t) = L_t^{-1}Q(x, t),
\]

and, using the following recursive scheme, it is possible to obtain:

\[
    v_1(x, t) = -L_t^{-1}L_x v_0 = -L_t^{-1}\left( L_x L_t^{-1}Q(x, t) \right),
\]

\[
    v_2(x, t) = -L_t^{-1}L_x v_1 = -L_t^{-1}\left( L_x L_t^{-1}\left[ -L_t^{-1}\left( L_x L_t^{-1}Q(x, t) \right) \right] \right)
\]

\[
    \vdots
\]

\[
    v_i(x, t) = (-1)^i L_t^{-1}\left( L_t^{-1}\right)^{i+1} L_x Q(x, t).
\]

(38)

Using the eigenfunction expansion method, the function \(Q(x, t)\) can be expressed as a multiplication of unknown functions \(h_j(t)\) and orthogonal functions \(\phi_j(x)\), namely,

\[
    Q(x, t) = \sum_{j=1}^{\infty} h_j(t) \phi_j(x)
\]

(39)
where the unknown functions \( h_j(t) \) can be represented using orthogonality as
\[
h_j(t) = \int_0^L Q(x,t) \phi_j(x) \, dx. \tag{40}
\]

Using (39) along with (21), the equation (38) can be written as
\[
v_i(x,t) = (-1)^i \left( L_T^{-1} \right)^{(i+1)} L_x^i \sum_{j=1}^\infty h_j(t) \phi_j(x),
\]
\[
v_i(x,t) = \sum_{j=1}^\infty (-1)^i \lambda_j^i \phi_j(x) \left( L_T^{-1} \right)^{(i+1)} \{ h_j(t) \},
\]
\[
v_i(x,t) = \sum_{j=1}^\infty (-1)^i \lambda_j^i \phi_j(x) \int_0^t h_j(\tau) \frac{(t-\tau)^{2i+1}}{(2i+1)!} \, d\tau. \tag{41}
\]

Finally, using the identities for series expansion of sine functions, the solution of the nonhomogeneous equation in (16) can be simplified as
\[
v(x,t) = \sum_{i=1}^\infty \sum_{j=1}^\infty (-1)^i \lambda_j^i \phi_j(x) \int_0^t h_j(\tau) \frac{(t-\tau)^{2i+1}}{(2i+1)!} \, d\tau,
\]
\[
v(x,t) = \sum_{j=1}^\infty \frac{\phi_j(x)}{\sqrt{\lambda_j}} \int_0^t h_j(\tau) \sum_{i=0}^\infty (-1)^i \left[ \frac{(t-\tau)^{2i+1}}{(2i+1)!} \right] \, d\tau,
\]
\[
v(x,t) = \sum_{j=1}^\infty \frac{\phi_j(x)}{\sqrt{\lambda_j}} \int_0^t h_j(\tau) \sin \left( \sqrt{\lambda_j}(t-\tau) \right) \, d\tau. \tag{42}
\]

Let us choose the source term in (1) as
\[
q(x,t) = x^2 e^{8x-t}, \tag{43}
\]
and then, for fixed values of the nonhomogeneity parameter \( \alpha = 1 \) and \( \mu_0 = 1 \), it can be obtained that
\[
Q(x,t) = \frac{q(x,t)}{\mu_0} e^{-\alpha x} = x^2 e^{7x-t}. \tag{44}
\]

Thus, the first four values of the function \( h_j(t) \) along with the shape functions in (28)-(30) can be evaluated by the integral given in (40) as follows:
\[
h_1(t) = 57.419 e^{-t}, \quad h_2(t) = -68.227 e^{-t}, \quad h_3(t) = 68.440 e^{-t}, \quad h_4(t) = -62.921 e^{-t}. \tag{45}
\]

Using these values in (42), the solutions for the nonhomogeneous equation are obtained as
\[
v_1(x,t) = -6527 e^{-t-0.5x} (0.574 \cos(14.378t) - 0.574) \times (\cosh(3.824x) - 1.010 \sinh(3.824x) - \cos(3.759x) + 1.028 \sin(3.759x)),
\]
\[
v_2(x,t) = 0.159 e^{-t-0.5x} (0.227 \cos(49.106t) - 0.227) \times (\cosh(7.025x) - 0.999 \sinh(7.025x) - \cos(6.989x) + 1.005 \sin(6.989x)),
\]
\[
v_3(x,t) = 0.130 e^{-t-0.5x} (0.136 \cos(103.42t) - 0.136) \times (\cosh(10.182x) - 1.000 \sinh(10.182x) - \cos(10.157x) + 1.002 \sin(10.157x)),
\]
\[
v_4(x,t) = 0.802 e^{-t-0.5x} (0.314 \cos(177.46t) - 0.314) \times (\cosh(13.330x) - 1.000 \sinh(13.330x) - \cos(13.330x) + 1.002 \sin(13.330x)).
\]
Finally, the general solution of the functionally graded clamped simply-supported beam problem can be written as a superposition of homogeneous and nonhomogeneous solutions as

\[
w(x, t) = \sum_{j=1}^{\infty} \left( b_j \cos(\sqrt{\lambda_j}x) + \frac{c_j}{\omega_j} \cos(\sqrt{\lambda_j}t) + \frac{1}{\sqrt{\lambda_j}} \int_0^t h_j(\tau) \sin(\sqrt{\lambda_j}(t - \tau)d\tau)\phi_j(x) \right).
\]

4 Results and discussion

This study considered a specific functionally graded composite beam which has a clamped end and a simply supported end as shown in Figure 1. The solution of the problem is obtained as an expansion of convergent series by the Adomian decomposition method. The deflection \(w(x, t)\) was represented by the first six terms of the series along with the corresponding eigenvalues and eigenfunctions. Natural frequencies for different nonhomogeneity parameters are shown in Table 1. Results for sufficiently small nonhomogeneity parameters (like \(\alpha = 0.001\)) are compared with the exact solution of the homogeneous beam problem subject to the same boundary conditions given in Appendix A. It is shown that eigenvalues of corresponding eigenfunctions agreed with the exact solution for the beam made of homogeneous material. The figures show how the deflection is changing on the composite beam in terms of the effect of the nonhomogeneity parameter \(\alpha\). For different values of nonhomogeneity parameters – deflections \(u(x, t)\) and \(v(x, t)\) – solutions of the homogeneous and nonhomogeneous parts of the beam problem are shown in Figure 2 and Figure 3, respectively. The resultant of the deflections \(w(x, t)\) is given in Figure 4. In both cases, as the nonhomogeneity parameter \(\alpha\) increases, the deflection decreases through the simply supported end for fixed values of time.

Table 1. The first four natural frequencies for different values of nonhomogeneity parameter

| Eigenvalues | Exact Solution | \(\alpha = 0.001\) | \(\alpha = 0.5\) | \(\alpha = 1.0\) | \(\alpha = 2.0\) |
|-------------|----------------|---------------------|-----------------|-----------------|-----------------|
| \(\beta_1\) | 3.9266         | 3.9265              | 3.8594          | 3.7919          | 3.6529          |
| \(\beta_2\) | 7.0685         | 7.0685              | 7.0358          | 7.0076          | 6.9662          |
| \(\beta_3\) | 10.2102        | 10.2101             | 10.1878         | 10.1697         | 10.1463         |
| \(\beta_4\) | 13.3517        | 13.3517             | 13.3348         | 13.3215         | 13.3057         |
5 Conclusion

The aim of the present work is to examine the behavior of a functionally graded composite beam. Using the Adomian decomposition method, deflections of the composite beam for various values of the nonhomogeneity parameter $\alpha$ are obtained and the results for small values closely match the results for a homogeneous beam.
Appendix: An exact solution for a uniform beam

\[ \mu(x) \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 w}{\partial x^2} \right] = q(x, t). \]  \hspace{2cm} (46)

\[ w(0, t) = \frac{\partial}{\partial x} w(0, t) = 0, \quad w(l, t) = \frac{\partial^2}{\partial x^2} w(l, t) = 0, \]

\[ w(x, 0) = f(x) = x^2 \quad w_t(x, 0) = g(x) = 0, \quad q(x, t) = x^2 e^{8x-t}. \]  \hspace{2cm} (47)

An exact solution of the vibration problem can be obtained by using the eigenfunction expansion method. In the solution process, the problem will be separated into two parts:

\[ w(x, t) = u(x, t) + v(x, t). \]

where \( u(x, t) \) and \( v(x, t) \) are the solutions of homogeneous and nonhomogeneous equations, respectively. Setting \( q(x, t) = 0 \) in the solution of (46), the function \( u(x, t) \) can be defined as

\[ \mu(x) \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 u}{\partial x^2} \right] = 0. \]  \hspace{2cm} (48)

For a uniform beam, \( \mu(x) \) and \( EI(x) \) can be expressed as constants. So, equation (48) becomes

\[ \frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0. \]  \hspace{2cm} (49)

To solve this PDE, we use separation of variables. This method is based on the assumption that the solution \( u(x, t) \) can be expressed as the product of \( \phi(x) \) and \( \psi(t) \) such that

\[ u(x, t) = \phi(x)\psi(t). \]  \hspace{2cm} (50)

Substituting (50) into (46),

\[ \psi''(t)\phi(x) + \psi(t)\phi^4(x) = 0, \]

\[ \frac{1}{\psi(x)} \frac{d^4 \phi(x)}{dx^4} = -\frac{1}{\psi(t)} \frac{d^2 \psi(t)}{dt^2} = \omega^2. \]

Eigenfunctions related to the homogeneous problem are given by

\[ \frac{d^4 \phi(x)}{dx^4} - \omega^2 \phi(x) = 0, \]  \hspace{2cm} (51)

where \( \omega^2 \) is the separation constant of the system and \( \lambda \) is defined by

\[ \lambda^4 = \frac{\mu(x)}{E_0 I_0} \omega^2. \]  \hspace{2cm} (52)

The solution of equation (51) will provide \( \phi(x) \) of the beam. The solution of (51) can be found as

\[ \phi(x) = C_1 \cosh(\lambda x) + C_2 \sinh(\lambda x) + C_3 \cos(\lambda x) + C_4 \sin(\lambda x), \]  \hspace{2cm} (53)

and the unknown values of \( C_1, C_2, C_3, C_4 \) can be determined by the end conditions of the beam. An infinite number of nontrivial solutions can be found, corresponding to an infinite number of natural frequencies \( \lambda_i, i = 1, 2, 3, ..., \) such that

\[ \phi_i(x) = C_1 \left( \sin(\lambda_i x) - \sinh(\lambda_i x) \right) + C_2 \left( \cos(\lambda_i x) - \cosh(\lambda_i x) \right), \]  \hspace{2cm} (54)

where the first four roots of the equation are given as,

\[ \lambda_1 = 3.9266, \quad \lambda_2 = 7.0685, \quad \lambda_3 = 10.2102, \quad \lambda_4 = 13.3517. \]  \hspace{2cm} (55)
Assuming $C_1 = 1$ and using natural frequencies $\lambda_i$, the corresponding nodal shapes $\phi_i(x)$ can be expressed. Let us apply the method of eigenfunction expansion by defining $\psi_n(t)$ as an unknown function such as

$$v(x,t) = \sum_{n=1}^{\infty} \psi_n(t) \phi_n(x).$$  \tag{56}

Substituting (56) into (46),

$$\sum_{n=1}^{\infty} \psi''(t) \phi(x) + \sum_{n=1}^{\infty} \psi(t) \phi^4(x) = q(x,t)$$  \tag{57}

Using (51) we write,

$$\frac{d^4 \phi(x)}{dx^4} = \omega^2 \phi(x) = 0,$$

$$\sum_{n=1}^{\infty} \psi_n(t) \phi_n(x) + \omega^2 \sum_{n=1}^{\infty} \psi_n(t) \phi_n(x) = q(x,t).$$

Using orthogonality properties of eigenfunctions,

$$\int_0^l \psi_n(t) \phi_n(x) \phi_m(x) dx + \omega^2 \int_0^l \psi_n(t) \phi_n(x) \phi_m(x) dx = \int_0^l q(x,t) \phi_m(x) dx.$$  \tag{58}

Thus, we obtain the differential equation

$$\psi''(t) + \omega^2 \psi_n(t) = F_n(t),$$  \tag{59}

where

$$F_n(t) = \frac{\int_0^l q(x,t) \phi_n(x) dx}{\int_0^l \phi_n^2(x) dx}.$$

The solution of this ODE, given in (59), can be written as

$$\psi_n(t) = C_n \cos \omega t + D_n \sin \omega t + A.$$

So the general solution is

$$w(x,t) = \sum_{n=1}^{\infty} \phi_n(x) [C_n \cos \omega t + D_n \sin \omega t + A].$$

Now, we should determine the initial conditions to find $C_n$ and $D_n$. Using initial conditions, we have

$$\psi_n(0) = \frac{\int f(x) \phi_n(x) dx}{\int \phi_n^2(x) dx}, \quad \psi'_n(0) = \frac{\int g(x) \phi_n(x) dx}{\int \phi_n^2(x) dx} = 0.$$  \tag{60}

Finally, the nonhomogeneous PDE can be determined as

$$v(x,t) = \sum_{n=1}^{\infty} \psi_n(t) \phi_n(x).$$

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