USING INDICES OF POINTS ON AN ELLIPTIC CURVE TO CONSTRUCT A
DIOPHANTINE MODEL OF \( \mathbb{Z} \) AND DEFINE \( \mathbb{Z} \) USING ONE UNIVERSAL QUANTIFIER
IN VERY LARGE SUBRINGS OF NUMBER FIELDS, INCLUDING \( \mathbb{Q} \)

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ABSTRACT. Let \( K \) be a number field such that there exists an elliptic curve \( E \) of rank one
over \( K \). For a set \( \mathcal{W}_K \) of primes of \( K \), let \( O_{K,\mathcal{W}_K} = \{ x \in K : \text{ord}_p x \geq 0, \forall p \notin \mathcal{W}_K \} \). Let
\( P \in E(K) \) be a generator of \( E(K) \) modulo the torsion subgroup. Let \( (x_n(P), y_n(P)) \) be the
affine coordinates of \( \lbrack n \rceil P \) with respect to a fixed Weierstrass equation of \( E \). We show that
there exists a set \( \mathcal{W}_K \) of primes of \( K \) of natural density one such that in \( O_{K,\mathcal{W}_K} \) multiplication
of indices (with respect to some fixed multiple of \( P \)) is existentially definable and therefore
these indices can be used to construct a Diophantine model of \( \mathbb{Z} \). We also show that \( \mathbb{Z} \)
is definable over \( O_{K,\mathcal{W}_K} \) using just one universal quantifier. Both, the construction of a
Diophantine model using the indices and the first-order definition of \( \mathbb{Z} \) can be lifted to the
integral closure of \( O_{K,\mathcal{W}_K} \) in any infinite extension \( K_\infty \) of \( K \) as long as \( E(K_\infty) \) is finitely
generated and of rank one.

1. Introduction

The interest in constructing Diophantine models of \( \mathbb{Z} \) over various rings and related is-
SUES of Diophantine decidability and definability over rings goes back to a question that
was posed by Hilbert: given an arbitrary polynomial equation in several variables over
\( \mathbb{Z} \), is there a uniform algorithm to determine whether such an equation has solutions in
\( \mathbb{Z} \)? This question, otherwise known as Hilbert’s Tenth Problem, has been answered neg-
atively in the work of M. Davis, H. Putnam, J. Robinson and Yu. Matijasevich. (See [5],
[6] and [14].) Since the time when this result was obtained, similar questions have been
raised for other fields and rings. In other words, if \( R \) is a recursive ring, then, given an
arbitrary polynomial equation in several variables over \( R \), is there a uniform algorithm to
determine whether such an equation has solutions in \( R \)? One way to resolve the ques-
tion of Diophantine decidability negatively over a ring of characteristic \( 0 \) is to construct a
Diophantine definition of \( \mathbb{Z} \) over such a ring. This notion is defined below.

Definition 1.1. Let \( R \) be a ring and let \( A \subset R^k, k \in \mathbb{Z}_{>0} \). Then we say that \( A \) has a
Diophantine definition over \( R \) if there exists a polynomial

\[ f(t_1, \ldots, t_k, x_1, \ldots, x_n) \in R[t_1, \ldots, t_k, x_1, \ldots, x_n] \]
such that for any $\bar{t} \in R^k$,
\[ \exists x_1, \ldots, x_n \in R, f(t_1, \ldots, t_k, x_1, \ldots, x_n) = 0 \iff \bar{t} \in A. \]

If the quotient field of $R$ is not algebraically closed, we can allow a Diophantine definition to consist of several polynomials without changing the nature of the relation. (See [6] for more details.)

The usefulness of Diophantine definitions stems from the following easy lemma.

**Lemma 1.2.** Let $R_1 \subset R_2$ be two recursive rings such that the quotient field of $R_2$ is not algebraically closed. Assume that Hilbert’s Tenth Problem (abbreviated as “HTP” in the future) is undecidable over $R_1$, and $R_1$ has a Diophantine definition over $R_2$. Then HTP is undecidable over $R_2$.

Using norm equations, Diophantine definitions have been obtained for $\mathbb{Z}$ over the rings of algebraic integers of some number fields. Jan Denef has constructed a Diophantine definition of $\mathbb{Z}$ for the finite degree totally real extensions of $\mathbb{Q}$. Jan Denef and Leonard Lipshitz extended Denef’s results to all the extensions of degree 2 of the finite degree totally real fields. Thanases Pheidas and the author of this paper have independently constructed Diophantine definitions of $\mathbb{Z}$ for number fields with exactly one pair of non-real embeddings. Finally Harold N. Shapiro and the author of this paper showed that the subfields of all the fields mentioned above “inherited” the Diophantine definitions of $\mathbb{Z}$. (These subfields include all the abelian extensions.) The proofs of the results listed above can be found in [7], [9], [8], [19], [27], and [29].

The author modified the norm method to obtain Diophantine definitions of $\mathbb{Z}$ for “large” subrings of totally real number fields (not equal to $\mathbb{Q}$) and their extensions of degree 2. (See [31], [32], [34], [36].) Further, again using norm equations, the author also showed that in some totally real infinite algebraic extensions of $\mathbb{Q}$ and extensions of degree 2 of such fields one can give a Diophantine definition of $\mathbb{Z}$ over integral closures of “small” and “large” rings, though not over the rings of algebraic integers. (The terms “large” and “small” rings will be explained below in Definition 1.4.)

Using elliptic curves Bjorn Poonen has shown the following in [23].

**Theorem 1.3.** Let $M/K$ be a number field extension with an elliptic curve $E$ defined over $K$, of rank one over $K$, such that the rank of $E$ over $M$ is also one. Then $O_K$ (the ring of integers of $K$) is Diophantine over $O_M$.

Cornelissen, Pheidas and Zahidi weakened somewhat assumptions of Poonen’s theorem. Instead of requiring a rank 1 curve retaining its rank in the extension, they require existence of a rank 1 elliptic curve over the bigger field and an abelian variety over the smaller field retaining its positive rank in the extension (see [1]). Further, Poonen and the author have independently shown that the conditions of Theorem 1.3 can be weakened to remove the assumption that the rank is one and require only that the rank in the extension is positive and the same as the rank over the ground field (see [37] and [22]). Following Denef in [9], the author also considered the situations where elliptic curves had finite rank in infinite extensions and showed that when this happens in a totally real field one can
existentially define \( \mathbb{Z} \) over the ring of integers of this field and the ring of integers of any extension of degree 2 of such a field (see [38]).

Recently, in [18], Mazur and Rubin showed that if Shafarevich-Tate conjecture held over a number field \( K \), then for any cyclic extension \( M \) of \( K \), there existed an elliptic curve of rank one over \( K \), keeping its rank over \( M \). Combined with Theorem 1.3 this new result showed that Shafarevich-Tate conjecture implied HTP is undecidable over the rings of integers of any number field. Similar consequences can be derived for big rings in any number field.

Perhaps the most prominent open question in the subject is the Diophantine status of \( \mathbb{Q} \). As indicated above, one way to show unsolvability of HTP over \( \mathbb{Q} \) would be to construct a Diophantine definition of \( \mathbb{Z} \) over \( \mathbb{Q} \). A Diophantine definition is a type of a Diophantine model. Given two recursive rings \( R_1 \) and \( R_2 \) we say that \( R_2 \) has a Diophantine model of \( R_1 \) if there exists an injective and recursive map \( \phi : R_1 \rightarrow R_2 \) sending Diophantine sets to Diophantine sets. If \( R_1 \) has undecidable Diophantine sets, then so does \( R_2 \). Therefore, any recursive ring with a Diophantine model of \( \mathbb{Z} \) has undecidable Diophantine sets and thus HTP is unsolvable over this ring.

It is also not hard to show that given an injection \( \phi \) of \( \mathbb{Z} \) into a recursive ring \( R \), it is enough to show that the images of the graphs of addition and multiplication are Diophantine over \( R \), in order to conclude that \( \phi \) is a Diophantine model. An old plan for constructing a Diophantine model of \( \mathbb{Z} \) over \( \mathbb{Q} \) involved elliptic curves of rank one (see [21]). More specifically let \( E \) be an elliptic curve defined and of rank one over \( \mathbb{Q} \). Fix an affine Weierstrass equation for \( E \), as well as a generator \( Q \). Let \( r \) be the size of the torsion group and let \( P = [r]Q \). Let \( (x_n(P), y_n(P)) \) be the coordinates of \([n]P \) derived from our fixed affine Weierstrass equation. Now for \( n \neq 0 \) send \( n \) to \( y_n \). It is easy to see that the graph of addition is Diophantine over \( \mathbb{Q} \), but it is not clear what happens to the graph of multiplication. This plan has another potentially fatal complication: Mazur’s conjectures (see [15], [16], [17]). As was shown in [3], if Mazur’s conjecture on topology of rational points holds, there is no Diophantine model of \( \mathbb{Z} \) over \( \mathbb{Q} \). It is precisely these difficulties preventing the resolution of the problem over \( \mathbb{Q} \) that motivated the investigation of Diophantine definability and decidability over “large” or “big” rings. These rings can be found in any number field and we define them below.

**Definition 1.4.** Let \( K \) be a number field and let \( \mathcal{W}_K \) be a set of primes of \( K \). Define \( O_{K,\mathcal{W}_K} \) to be the following ring:

\[
O_{K,\mathcal{W}_K} := \left\{ x \in K : \text{ord}_{p} x \geq 0, \forall p \notin \mathcal{W}_K \right\}.
\]

If \( \mathcal{W}_K \) is infinite we will call these rings “big” or “large”. If \( \mathcal{W}_K \) is finite we refer to the corresponding rings as “small”. Such rings are also known as the rings of \( \mathcal{W} \)-integers.

Perhaps the most significant result concerning big rings was obtained by Poonen in [24]. In this paper he showed that there exists a big ring inside \( \mathbb{Q} \) where the set of primes allowed in the denominator is of natural density one and the ring possesses a Diophantine model of \( \mathbb{Z} \). To carry out his construction, Poonen modeled integers by approximation. More specifically in [24] he proved the following. Let \( E \) be a curve of rank one over \( \mathbb{Q} \) without complex multiplication and with only one connected component. Let \( P \) be a generator of \( E(\mathbb{Q}) \). Then for some set \( \mathcal{W}_Q \) of rational primes of natural density one, we have that \( E(O_{K,\mathcal{W}_Q}) = \{(x_t, y_t), i \in \mathbb{Z}_{>0} \} \cup \{ \text{finite set} \} \), where \((x_n, y_n)\) are the coordinates of...
Let \( R \) be a number field. Let \( E \) be an elliptic curve defined and of rank one over \( K \). Let \( P \) be a generator of \( E(K) \) modulo the torsion subgroup, and fix an affine Weierstrass equation for \( E \) of the form \( y^2 = x^3 + ax + b \), with \( a, b \in O_K \), where \( O_K \) is the ring of integers of \( K \). Let \( (x_n, y_n) \) be the coordinates of \( [n]P \) derived from this Weierstrass equation. Then there exists a set of \( K \)-primes \( \mathcal{W}_K \) of natural density one, and a positive integer \( m_0 \) such that the following set \( \Pi \subset O_K^2, \mathcal{W}_K \) is Diophantine over \( O_K, \mathcal{W}_K \).

\[
(U_1, U_2, U_3, X_1, X_2, X_3, V_1, V_2, V_3, Y_1, Y_2, Y_3) \in \Pi \iff \\
\exists \text{ unique } k_1, k_2, k_3 \in \mathbb{Z}_{\neq 0} \text{ such that } \left( \frac{U_i}{V_i}, \frac{X_i}{Y_i} \right) = (x_{m_0k_i}, y_{m_0k_i}), \text{ for } i = 1, 2, 3, \text{ and } k_3 = k_1k_2.
\]

We can use this result to construct yet another variation of a Diophantine model of \( \mathbb{Z} \).

**Definition 1.6.** Let \( R \) be a countable recursive ring, let \( D \subset R^k, k \in \mathbb{Z}_{>0}, \) be a Diophantine subset, and let \( \approx \) be a (Diophantine) equivalence relation on \( D \), i.e assume that the set \( \{(\bar{x}, \bar{y}) : \bar{x}, \bar{y} \in D, \bar{x} \approx \bar{y}\} \) is a Diophantine subset of \( R^{2k} \). Let \( D = \bigcup_{i \in \mathbb{Z}} D_i \), where \( D_i \) is an equivalence class of \( \approx \), and let \( \phi : \mathbb{Z} \to \{D_i, i \in \mathbb{Z}\} \) be defined by \( \phi(i) = D_i \). Finally assume that the sets

\[
\text{Plus} = \{(\bar{x}, \bar{y}, \bar{z}) : \bar{x} \in D_i, \bar{y} \in D_j, \bar{z} \in D_{i+j}\}
\]

and

\[
\text{Times} = \{(\bar{x}, \bar{y}, \bar{z}) : \bar{x} \in D_i, \bar{y} \in D_j, \bar{z} \in D_{ij}\}
\]

are Diophantine over \( R \).

Then we will say that \( R \) has a class Diophantine model of \( \mathbb{Z} \).

It is clear that if \( R \) does have a class Diophantine model of \( \mathbb{Z} \) then HTP is not solvable over \( R \). Such a model of \( \mathbb{Z} \) has been used already to show Diophantine undecidability of function fields of positive characteristic (see \([10], [12], [20], [28], [30], [33]\)).

As a corollary of Theorem 1.5, we immediately obtain the following statement.

**Corollary 1.7.** In the notation above, for \( n \neq 0 \) let \( \phi(n) = [(U_{m_0n}, X_{m_0n}, V_{m_0n}, Y_{m_0n})] \), the equivalence class of \((U_{m_0n}, X_{m_0n}, V_{m_0n}, Y_{m_0n})\) under the equivalence relation described below,

where \( U_{m_0n}, X_{m_0n}, V_{m_0n}, Y_{m_0n} \in O_K, \mathcal{W}_K \), \( V_{m_0n} Y_{m_0n} \neq 0 \), and \( (x_{m_0n}: y_{m_0n}) = \left( \frac{U_{m_0n}}{V_{m_0n}}, \frac{X_{m_0n}}{Y_{m_0n}} \right) \).

Let \( \phi(0) = \{(0, 0, 0, 0)\} \). Then \( \phi \) is a class Diophantine model of \( \mathbb{Z} \). (Here if \( VVYY \neq 0 \) we set \((U, X, V, Y) \approx (\bar{U}, \bar{X}, \bar{V}, \bar{Y}) \) if and only if \( \frac{U}{V} = \bar{U} \) and \( \frac{X}{Y} = \bar{X} \).)
Using Theorem 1.5 we also prove the following.

**Theorem 1.8.** Let $K$ be a number field. Let $E$ be an elliptic curve defined and of rank one over $K$. Then there exists a set $\mathcal{W}_K$ of primes of $K$ of natural density one such that $\mathbb{Z}$ is first-order definable over $O_{K,\mathcal{W}_K}$ using just one universal quantifier.

This result is an improvement of the first-order definability results for big rings in [2] and [25], where the first-order definition of $\mathbb{Z}$ was given using just one universal quantifier over big rings contained in $\mathbb{Q}$ in [25] and in some number fields in [2] with the natural density of the inverted primes arbitrarily close but not equal to one. (We should also note here that the main result of [25] is defining $\mathbb{Z}$ over $\mathbb{Q}$ using two universal quantifiers.) The result of this paper is also a natural complement to the results of [4] where it was shown that a model of $\mathbb{Z}$ can be defined over $\mathbb{Q}$ using just one universal quantifier provided a certain conjecture on elliptic curves is true.

Finally, Theorem 1.5 allows us to simplify some results concerning infinite extensions from [38]. The result of Theorem 1.5 holds for any algebraic extension of $\mathbb{Q}$ with a rank 1 finitely generated elliptic curve. No additional assumptions are required. In the past we needed some way to define integrality at a prime in an infinite extension to use this kind of elliptic curve technique.

We finish this section with a notation set to be used in the rest of the paper.

**Notation 1.9.**

- Let $\mathcal{P}_\mathbb{Q} = \{2, 3, 5, \ldots\}$ denote the set of rational primes.
- Let $K$ be a number field.
- Let $\mathcal{P}_K$ be the set of all finite primes of $K$.
- Given $x \in K$, let $n(x) = \prod_p p^{\text{ord}_p x}$, where the product is taken over all $p \in \mathcal{P}_K$ such that $\text{ord}_p x > 0$. Let $\mathfrak{d}(x) = n(x^{-1})$.
- Let $\mathcal{W}_K \subset \mathcal{P}_K$ (we will make $\mathcal{W}_K$ more specific in the next section).
- Let $A, B \in O_{K,\mathcal{W}_K}$. Then we will say that $(A, B)_{\mathcal{W}_K} = 1$ if for all $p \in \mathcal{P}_K \setminus \mathcal{W}_K$ we have that either $\text{ord}_p A = 0$ or $\text{ord}_p B = 0$.
- Let $A, B \in O_{K,\mathcal{W}_K}$. Then we will say that $A \big|_{\mathcal{W}_K} B$ if for all $p \notin \mathcal{W}_K$ we have that $\text{ord}_p B \geq \text{ord}_p A$ or in other words $A$ divides $B$ in the ring $O_{K,\mathcal{W}_K}$.
- Let $h_K$ be the class number of $K$. (See [11], Chapter I, §4 for the definition of a class number.)
- Let $\mathfrak{A}, \mathfrak{B}$ be two integral divisors of $K$. Then we will say that $\mathfrak{A} \big| \mathfrak{B}$ to mean that for all $p \in \mathcal{P}(K)$ we have that $\text{ord}_p \mathfrak{A} \leq \text{ord}_p \mathfrak{B}$.
- Suppose $\mathfrak{A}, \mathfrak{B}$ are two divisors of $K$ with $\mathfrak{B} = \mathfrak{A}^j$. Then we set $\sqrt[\mathfrak{B}]{} = \mathfrak{A}$.

2. An Outline of the Proof of Theorem 1.5

Let $K$ be a number field with an elliptic curve of rank 1. The key to the proof of Theorem 1.5, that is the key to the construction of a big subring of $K$ where the theorem holds, is the choice of $K$-primes to invert in the ring. In [24] and [26] the inverted primes were chosen so that only a specific sequence of the elliptic curve points had its coordinates in the ring. (We remind the reader that an element of our number field is in the ring if and only if all the primes occurring in the denominator of its divisor are inverted in the ring.) In our case, almost no point of the elliptic curve will have its coordinates in the ring and
we will have to represent each coordinate by a pair consisting of a “numerator” and the corresponding “denominator”. This is the reason for having a class Diophantine model at the end instead of a regular Diophantine model: every coordinate of an elliptic curve point will be represented by an equivalence class of pairs of “numerator” and “denominator”, as in a standard construction of the fraction field of a ring.

To explain the main ideas of the proof we for the moment simplify the situation assuming that $K = \mathbb{Q}$, there are no torsion points, and every non-trivial multiple of the generator $P$ has a primitive divisor. In other words we assume that for every $n > 0$, there exists a prime dividing the reduced denominators of the affine coordinates of $[n]P$ such that this prime does not divide the reduced denominators of the coordinates of any $[m]P$ with $0 < m < n$. (In general this will be true for sufficiently large $n$ only. See Proposition 4.4.) We will also assume that the coordinates of $P$ itself are non-zero integers. (In “real life” we will invert the primes which appear in the denominator of the coordinates of $P$. Also the primitive divisor requirement and the chosen form of the Weierstrass equation will force all the non-trivial multiples of $P$ to have non-zero coordinates.) Under our assumptions we can represent $[n]P$ for a non-zero integer $n$, as a pair \( \left( \frac{U_n}{V_n}, \frac{X_n}{Y_n} \right) \), where \( U_n \neq 0, V_n > 0, X_n \neq 0, Y_n > 0 \) are integers and \( (U_n, V_n) = 1, (X_n, Y_n) = 1 \). Later we will not be able to assume that $U_n, V_n, X_n, Y_n$ are integers but only that these are elements of our big ring. However, we will able to treat the variables ranging over the big rings almost in the same way as if they were integers.

From 24 (see Proposition 4.3, Lemma 4.7, and Lemma 4.9 in this paper) we know that

\[(2.1) \quad \text{if } m, n \in \mathbb{Z}_{\neq 0} \text{ with } m|n, \text{ then } V_m|V_n \text{ in } \mathbb{Z}, \text{ and conversely if } V_m|V_n \text{ in } \mathbb{Z} \text{ then } m|n, \text{ and} \]

\[(2.2) \quad \text{if } k, m \in \mathbb{Z}_{>0}, \text{ then all the primes occurring in } (V_k, V_m) \text{ occur in } V_{(k,m)}, \]

where \( (V_k, V_m) = \gcd(V_k, V_m) \) and \( (k, m) = \gcd(k, m) \) in \( \mathbb{Z} \). (Since we assumed that every non-trivial multiple of $P$ has a primitive divisor, we do not have to worry about $k$ and $m$ being large enough.) Given our assumptions on the coordinates of $P$, we have that $V_1 = 1$, and if \( (k, m) = 1, \text{ then } (V_k, V_m) = 1 \). Thus, if $k$ and $m$ are non-zero relatively prime integers, then $V_kV_m|V_{km}$. Unfortunately, in general $V_{km}$ does not divide $V_kV_m$. In particular, $V_{km}$ is divisible by some prime powers which do not occur in $V_k$ and $V_m$. So the main idea behind the proof is to invert these extra primes to force $V_{km}$ to divide $V_kV_m$ in the resulting ring. Of course we have to leave enough primes uninverted so that (2.1) still holds in the ring.

We now describe the primes we do not invert. For each rational prime $p$ and any positive integer $\ell$ we keep uninveted the largest primitive divisor of $[p^\ell]P$. We call these primes indicator primes. (The idea that the indicator primes are enough to identify uniquely positive multiples of a generator was first investigated in 2.) We invert all the other primes and denote by $R$ the resulting subring of $\mathbb{Q}$. Observe that for $m = \prod p_i^{\ell_i}$, we have that $V_m$ is divisible by the indicator prime of each $[p_i^{\ell_i}]P$ for all $i$ and all $\ell = 1, \ldots, \ell_i$, and, because of (2.2), by no other indicator primes. Indeed, first suppose $q$ is an indicator prime $V_{p^r}$, where $p \neq p_i$ for any $i$. In this case by (2.2), $q$ divide $V_{(p^r,m)} = V_1 = 1$ and we have a contradiction. Next assume that $q$ is an indicator prime for some $V_{p_i^r}$, where $r > \ell_i$. By definition of an indicator prime we have that $q|V_{p_i^{\ell_i}}$ but $q$ does not divide $V_{p_i^{\ell_i}}$ for
any \( j \in \{1, \ldots, r - 1\} \). Applying (2.2) again we obtain \( q | V_{(p', m)} = V_{p'^{j}} \), contradicting our assumptions in this case also.

So now we are in a situation where for \( k, m \in \mathbb{Z}_{\neq 0} \) and relatively prime, \( V_k V_m \) and \( V_{km} \) are divisible by the same uninvited primes. Unfortunately, there is one more point to take care of. The indicator primes do not necessarily appear to the same power in \( V_k, V_m \) and \( V_{km} \) (see Lemma 4.5 in this paper). However in the case of \( \mathbb{Q} \) the difference in powers is at most 2 (in the general case one can bound this difference by twice the degree of the field over \( \mathbb{Q} \), but we will avoid all the cases where the power can go up by more than 2.) Let \( q \neq p \) be rational primes.

(2.3) If \( \text{ord}_q V_m > 0 \), then \( \text{ord}_q V_{pm} = \text{ord}_q V_m \), and \( \text{ord}_q V_{qm} = 2 + \text{ord}_q V_m \).

Observe that for \( t = p \) or \( t = q \) we have that \( \text{ord}_q V_{tm} \leq 3 \text{ord}_q V_m \) since

\[
\text{ord}_q V_{tm} \leq 2 + \text{ord}_q V_m \leq 3 \text{ord}_q V_m,
\]

as by assumption \( \text{ord}_q V_m \geq 1 \). Now we can conclude that if \( (V_k, V_m)_R = 1 \) it is the case that \( V_{km} | R V_k V_m^3 \) in our ring. (Here for \( A, B \in R \) we write \( \langle A, B \rangle_R = 1 \) to mean that the reduced numerators of \( A \) and \( B \) are not simultaneously divisible by any non-inverted prime, and we write \( \langle A | R B \rangle \) to indicate the divisibility in the ring, i.e. the fact that \( \frac{B}{A} \in R \).)

To summarize the discussion above we can now say

\[
(k, m) = 1 \implies \forall n : (V_n | R V_m^3 V_k^3 V_m V_k | R V_n \iff |n| = |km|).
\]

(See Proposition 4.15 and Lemma 4.17)

If \( (k, m) = 1 \), we say that the indices \( k \) and \( m \) can be “multiplied directly”. Note also that for any triple of non-zero indices \( k, m \) and \( n \) we have that

(2.4) \( V_n | R V_m^3 V_k^3 \) and \( V_m V_k | R V_n \) implies \( |n| = |km| \).

(As above, the divisibility bar with a subscript \( R \) here refers to the divisibility in our ring.) Further, as a general matter, for any ring of characteristic not equal to 2, to define multiplication, it is enough to define squaring: \( xy = \frac{1}{2}((x + y)^2 - x^2 - y^2) \). Observe also that for any \( k \neq 0 \) we can always multiply \( k \) and \( k + 1 \) directly. We use this fact to define squaring of an index: given an index \( k \in \mathbb{Z}_{\neq 0} \), we require that for some \( s \in \mathbb{Z}_{\neq 0} \) it is the case that

(2.5) \( V_k V_{k+1} | R V_k V_s | R V_k^3 V_{k+1} \)

or, in other words,

(2.6) \(|(k + 1)k| = |s| \)

If not for absolute values in (2.6), we would be done, since we would be able to define a square of \( k \) by subtracting \( k \) from \( s \). We deal with absolute values via considering all possible cases and using (2.1) in Lemma 5.10.

Over \( \mathbb{Z} \), given integers \( U > 0, V > 0, X \neq 0, Y \neq 0 \) such that \( \frac{U}{V} \) and \( \frac{X}{Y} \) satisfy the chosen Weierstrass equation and such that \( (U, V) = 1, (X, Y) = 1 \), we can conclude that \( (U, V, X, Y) = (U_n, V_n, X_n, Y_n) \) for some unique \( n \in \mathbb{Z}_{\neq 0} \). Unfortunately, if we now assume that \( (U > 0, V > 0, X \neq 0, Y \neq 0) \in R^4 \), where \( R \) is, as above, our ring with infinitely many primes inverted, and \( \frac{U}{V} \) and \( \frac{X}{Y} \) satisfy the chosen Weierstrass equation with \( (U, V)_R = 1, (X, Y)_R = 1 \), then we will be able to conclude only that \( U = \tilde{U}_n = U_n \tilde{U}_n, V = \tilde{V}_n = V_n \tilde{V}_n \).
$V_n V_n$, where $\bar{U}_n, \bar{V}_n$ are rational numbers whose reduced numerators and denominators are divisible by the inverted primes only. (A similar conclusion will apply to $(X, Y)$.) However, since we are only interested in the divisibility by the non-inverted indicator primes, the “bar” parts do not matter or in other words, for any $k, m \in \mathbb{Z} \neq 0$ we still have that

$$(k, m) = 1 \iff \forall n : (\bar{V}_n |_{R} \bar{V}_n^3 \bar{V}_k \land \bar{V}_m \bar{V}_k |_{R} \bar{V}_n \iff |n| = |km|).$$

This is so, because $(k, m) = 1 \iff (\bar{V}_k, \bar{V}_m)_{R} = 1$ and $\bar{V}_n |_{R} \bar{V}_n^3 \bar{V}_k \iff \bar{V}_n |_{R} \bar{V}_n^3 \bar{V}_k$, etc. The fact that we can express the condition of being relatively prime in our ring in polynomial terms is demonstrated in Lemma 5.2. Unfortunately, when the underlying field has a class number greater than one, there are other technical complications requiring raising variables to the power divisible by a class number to obtain relatively prime numerators and denominators. (See Notation 5.3, Item 3 and Remark 5.4.)

The last point that needs to be explained is the density of the inverted and the non-inverted prime sets. In [24] and [26], it was shown that the natural density of the indicator primes corresponding to the prime multiples of any infinite order point is 0. So the only remaining question is the density of the indicator primes corresponding to prime power multiples of such a point, when the power is at least 2. This density is also 0 and the corresponding calculation is much easier. It was first carried out in [2] and is reproduced in the appendix of this paper for the convenience of the reader.

3. An Outline of the Proof of Theorem 1.8.

In this section we keep for the moment the simplifying assumptions and notation of the preceding section, i.e. we assume that we are dealing with a rank one elliptic curve over $\mathbb{Q}$ with a trivial torsion group, and a Weierstrass equation as above, and that every non-trivial multiple of a generator has a primitive divisor. We also assume that Theorem 1.5 holds or in other words in a big subring $R$ of $\mathbb{Q}$ described above we have defined existentially multiplication of indices.

If $x \in \mathbb{Q}$ and $x = \frac{A}{B}$, where $A, B \in R$ with $AB \neq 0$, then we say that $A$ and $B$ are a reduced numerator and a denominator respectively, if $(A, B)_{R} = 1$. In other words, neither $A$, nor $B$ are divisible by “extra” non-inverted primes. If $R = \mathbb{Z}$, this definition is the same as the usual one. We now need the following results from [23] (Lemmas 4.8 and 4.18 of this paper):

(3.7) "For any sufficiently large $l \in \mathbb{Z}_{>0}$, for any $k \in \mathbb{Z}_{>0}$ we have that

the reduced denominator of $x_l$ divides the reduced numerator of \( \left( \frac{x_l}{x_{kl}} - k^2 \right)^2 \) in $\mathbb{Z}$",

and

(3.8) "For any $n \in \mathbb{Z}_{>0}$ there exists $l \in \mathbb{Z}_{>0}$ such that

$n$ divides the reduced denominator of $x_l$ in $\mathbb{Z}$".

Now let $z$ be an arbitrary element of our big ring with the following property: there exists a non-zero integer $k$, such that for all rational numbers $b$ in our ring, there exist non-zero integers $i$ and $j$ satisfying the equations (3.9)–(3.11) below.

(3.9) $b^2$ divides the reduced denominator of $x_i$ in our ring.
(3.10) \[ j = ik \]

The reduced denominator of \( x_i \) divides the reduced numerator of \((z - \frac{x_i}{x_j})^2\) in our ring.

(3.11) (Here, as above, \( x_k, x_i, x_j \) are the \( x \)-coordinates of \([k]P, [i]P \) and \([j]P \) respectively.) Then \( z \in \mathbb{Z} \).

Conversely, if \( z \) above is a square of a non-zero integer, then we can find a \( k \in \mathbb{Z}_{\neq 0} \) such that for every \( b \) in our big ring there exist \( i \) and \( j \) so that (3.9) – (3.11) are satisfied.

First assume that \( z \), a rational number in our ring, is fixed. Let \( k \) be the corresponding non-zero integer, \( b \) an arbitrary element of the ring and assume that \( i, j \in \mathbb{Z}_{\neq 0} \) are such that the equations above are satisfied. From (3.7) and (3.9) we conclude that \( b \) divides the reduced numerator of \( (x_i - k^2) \) as well as the reduced numerator of \( (z - \frac{x_i}{x_j}) = (z - \frac{x_i}{x_{ik}}) \) in our ring. Thus, \( b \) divides the reduced numerator of \( z - k^2 \) in our ring. If \( z = \frac{z_1}{z_2} \), where \( z_1, z_2 \in \mathbb{Z}_{\neq 0} \), then \( b \) divides \( z_1 - z_2k^2 \) in our ring. If we pick \( b \) to be divisible by \( q^m \), where \( q \) is a prime which is not inverted in our ring and \( m \) is a positive integer large enough so that \( q^m > |z_1 - z_2k^2| \), then \( q^m \) divides \( z_1 - z_2k^2 \) in \( \mathbb{Z} \) and the only way the divisibility condition can hold is for \( z_1 = z_2k \). Without loss of generality we can assume that \( z_1 \) and \( z_2 \) were picked to be relatively prime in \( \mathbb{Z} \), and since \( k \) is a non-zero integer, we must conclude that \( z_2 = 1 \), and \( z = z_1 = k^2 \).

Assume now that \( z = k^2 \) where \( k \in \mathbb{Z}_{\neq 0} \). Let \( b \) be any rational number in our ring. Let \( i > 0 \) be such that \( b^2 \) divides the reduced denominator of \( x_i \) and \( i \) is sufficiently large so that (3.7) holds for \( l = ik \). Such an \( i \) exists by (3.8). Finally let \( j = ik \) and observe that (3.11) now holds by (3.7).

4. Elliptic curves

We now proceed with the detailed description of the proof. In this section we lay down the elliptic curve foundations of our results. Many of the technical details in this section are taken from [21], [23], [24] and [26]. Below we indicate which technical results have been taken from other papers.

**Notation and Assumptions 4.1.** We add the following notation and assumptions to the list above.

- Let \( E \) be an elliptic curve of rank 1 defined over \( K \) (in particular, we assume such an \( E \) exists).
- We fix a Weierstrass equation \( W : y^2 = x^3 + ax + b \) for \( E \) with all the coefficients in the ring of integers of \( K \).
- Let \( E(K)_{\text{tors}} \) be the torsion subgroup of \( E(K) \).
- Let \( t \) be a multiple of \( \#E(K)_{\text{tors}} \).
- Let \( Q \in E(K) \) be such that \( Q \) generates \( E(K)/E(K)_{\text{tors}} \).
- Let \( P := [t]Q \).
• Let $\mathcal{S}_{\text{bad}} = \mathcal{S}_{\text{bad}}(W, P, K) \subseteq P_K$ consist of the primes that ramify in $K/\mathbb{Q}$, the primes for which the reduction of the chosen Weierstrass model is singular (this includes all primes above 2), and the primes at which the coordinates of $P$ are not integral.
• For $n \in \mathbb{Z}_{\geq 0}$ write $[n]P = (x_n, y_n) = (x_n(P), y_n(P))$ where $x_n, y_n \in K$.
• For $n \in \mathbb{Z}_{\neq 0}$, let the divisor of $x_n(P)$ be of the form

$$\frac{a_n}{\mathfrak{d}_n}b_n = \frac{a_n(P)}{\mathfrak{d}_n(P)}b_n(P)$$

where

- $\mathfrak{d}_n = \prod q^{-a_n}$, where the product is taken over all primes $q$ of $K$ not in $\mathcal{S}_{\text{bad}}$ such that $a_q = \text{ord}_q x_n < 0$.
- $a_n = \prod q^{a_q}$, where the product is taken over all primes $q$ of $K$ not in $\mathcal{S}_{\text{bad}}$ such that $a_q = \text{ord}_q x_n > 0$.
- $b_n = \prod q^{a_q}$, where the product is taken over all primes $q \in \mathcal{S}_{\text{bad}}$ and $a_q = \text{ord}_q x_n$.
• For $n$ as above, let $\mathcal{S}_n = \mathcal{S}_n(P) = \{ p \in P_K : p|\mathfrak{d}_n \}$. By definition of $\mathcal{S}_{\text{bad}}$ and $\mathfrak{d}_n$, we have $\mathcal{S}_1 = \emptyset$.
• For $\ell \in P_{\mathbb{Q}}$, define $a_\ell$ to be the smallest positive integer such that for any $j \geq a_\ell$ we have that $\mathcal{S}_\ell \setminus \mathcal{S}_{\ell-1} \neq \emptyset$. By Proposition [4.4] below, for all but finitely many primes $\ell$ we have that $a_\ell = 1$.
• For $j \in \mathbb{Z}_{\geq 1}$, let $p_{\ell^j}(P) = p_{\ell^j}$ be a prime of the largest norm in $\mathcal{S}_\ell \setminus \mathcal{S}_{\ell-1}$, if such a prime exists. (This prime will be called the indicator prime for $[p_{\ell^j}](P)$.)
• Let $m_0 = \prod_{\ell \geq 1} \ell^{a_\ell-1}$. (Note that $m_0$ is well defined since, as we have observed above, for all but finitely many primes $\ell$ we have that $a_\ell = 1$.)
• For all $j \in \mathbb{Z}_{\geq 1}$ let $q_{\ell^j} = p_{\ell^{j+\text{ord}_\ell m_0}}$.
• Let $T = [m_0]P$.
• Let $\mathcal{Y}_K = \mathcal{Y}_K(P) = \{ p_\ell : \ell \in P_{\mathbb{Q}}, j \in \mathbb{Z}_{\geq 0} \}$.
• Let $\mathcal{Y}_K = (P_K \setminus \mathcal{Y}_K) \cup \mathcal{S}_{m_0}$. (\(\mathcal{Y}_K\) will be the set of the inverted primes.)
• Let $\mathcal{C}_n = (\mathcal{S}_n \cap \mathcal{Y}_K) \setminus \mathcal{S}_{m_0}$. Note that $\mathcal{C}_{m_0} = \emptyset$. (\(\mathcal{C}_n\) will be the collection of the prime factors of the "$n$"th denominator which are not inverted.)
• Let $\mathcal{X}_n = \mathcal{S}_{m_0n}$. (\(\mathcal{X}_n\) will be the set of the "not-bad denominator primes" for $[n]T$.)
• Let $\mathcal{B}_n = \mathcal{C}_{m_0n}$ and observe that $\mathcal{B}_1$ is empty. (\(\mathcal{B}_n\) will be the set of the non-inverted "denominator" primes for $[n]T$.)
• Let $c_n = \prod q^{-a_q}$, where the product is taken over all primes $q$ of $K$ not in $\mathcal{Y}_K$ such that $a_q = \text{ord}_q x_n < 0$. (The divisor $c_n$ will be the non-inverted part of the "$n$"th denominator.)
• Let $f_n = c_{m_0n}$.
• For $x \in K$, let $\mathfrak{d}(x) = \prod q^{-a_q}$, where the product is taken over all primes $q$ of $K$ such that $a_q = \text{ord}_q x < 0$. Let $n(x) = \mathfrak{d}(x^{-1})$.
• For $x \in K$, let $\mathfrak{d}_{\mathcal{Y}_K}(x) = \prod q^{-a_q}$, where the product is taken over all primes $q$ of $K$ not in $\mathcal{Y}_K$ such that $a_q = \text{ord}_q x < 0$. Let $n_{\mathcal{Y}_K}(x) = \mathfrak{d}_{\mathcal{Y}_K}(x^{-1})$.

Below we combine ideas from [23], [26] and [2] to show that it is enough to have one non-inverted indicator prime for every prime power of the index to identify the index of a point uniquely (up to a sign). At the same time, if we don’t invert only the indicator
primes of the index prime powers, we will have “almost” arranged for the multiplication of indices.

As pointed out above, denominator prime sets are not enough to establish a sign of an index. This is demonstrated by the lemma below.

**Lemma 4.2.** For any \( n \in \mathbb{Z}_{\neq 0} \) we have that \( \mathcal{I}_n = \mathcal{I}_{-n}, \mathcal{C}_n = \mathcal{C}_{-n}, \mathcal{X}_n = \mathcal{X}_{-n}, \mathcal{Y}_n = \mathcal{Y}_{-n}, \) and \( \mathcal{I}_n = \mathcal{I}_{-n}. \)

**Proof.** Given the choice of our Weierstrass equation, we have that \( x_{-n} = x_n. \) \( \square \)

Our next step is to establish several important properties of the primes which appear in the denominators in Propositions 4.3–4.14. Fortunately for us, most of the technical work has already been done elsewhere.

**Proposition 4.3** (Lemma 3.1 of [26]). Let \( \mathfrak{A} \) be an integral divisor of \( K. \) Then
\[
\{ n \in \mathbb{Z} \setminus \{0\} : \mathfrak{A} \mid \mathfrak{d}_n(P) \} \cup \{0\}
\]
is a subgroup of \( \mathbb{Z}. \)

**Proposition 4.4** (Proposition 3.5 of [26]). There exists \( C > 0 \) such that for all \( \ell, m \in \mathcal{P}_Q \) with \( \max(\ell, m) > C \) we have that \( \mathcal{I}_{\ell m} \setminus (\mathcal{I}_\ell \cup \mathcal{I}_m) \neq \emptyset. \)

**Lemma 4.5.** Let \( n \in \mathbb{Z}_{\geq 1}. \) Suppose that \( t \in \mathcal{P}_K \) divides \( \mathfrak{d}_n, \) and \( p \geq 2 \) is a rational prime.

1. If \( t \mid p, \) then \( \operatorname{ord}_t \mathfrak{d}_n p = 2 + \operatorname{ord}_t \mathfrak{d}_n. \)
2. If \( t \nmid p, \) then \( \operatorname{ord}_t \mathfrak{d}_n p = \operatorname{ord}_t \mathfrak{d}_n. \)

**Proof.** The proof of the lemma is almost identical to the proof of Lemma 3.3 of [26] except for the fact that we allow \( p = 2. \) We also remind the reader that any \( t \) dividing \( \mathfrak{d}_n \) is automatically not in \( \mathcal{S}_{\text{bad}} \) and therefore is not dyadic, ramified over \( \mathbb{Q} \) or is among primes at which our Weierstrass model has a bad reduction.

**Remark 4.6.** As was explained in Section 2, in either case, \( \operatorname{ord}_t \mathfrak{d}_n p \leq 3 \operatorname{ord}_t \mathfrak{d}_n. \) \( \square \)

**Corollary 4.7.** Let \( n \in \mathbb{Z}_{\geq 1}. \) Suppose that \( t \in \mathcal{P}_K \) divides \( c_n \) (or \( \mathfrak{f}_n \)), and \( p \geq 2 \) is a rational prime.

1. If \( t \mid p, \) then \( \operatorname{ord}_t c_n p = 2 + \operatorname{ord}_t c_n \) (or \( \operatorname{ord}_t \mathfrak{f}_n p = 2 + \operatorname{ord}_t \mathfrak{f}_n \)).
2. If \( t \nmid p, \) then \( \operatorname{ord}_t c_n p = \operatorname{ord}_t c_n \) (or \( \operatorname{ord}_t \mathfrak{f}_n p = \operatorname{ord}_t \mathfrak{f}_n \)).

**Proof.** The corollary follows immediately from the lemma above if we note that we obtain \( c_n \) from \( \mathfrak{d}_n \) by removing factors of \( \mathfrak{d}_n \) which are in \( \mathfrak{W}_K, \) and \( \mathfrak{f}_n = c_{\text{mon}}. \) \( \square \)

**Lemma 4.8** (Lemma 10 of [23]). Let \( \mathfrak{A} \) be any integral divisor of \( K. \) Then there exists \( k \in \mathbb{Z} > 0 \) such that \( \mathfrak{A} \mid \mathfrak{d}(x_k). \)

**Lemma 4.9.** Let \( m, n \in \mathbb{Z} \setminus \{0\}, \) and let \((m, n)\) be their GCD. Then
\[
\mathcal{I}_m \cap \mathcal{I}_n = \mathcal{I}_{(m,n)}, \quad \mathcal{X}_m \cap \mathcal{X}_n = \mathcal{X}_{(m,n)}, \quad \mathcal{C}_m \cap \mathcal{C}_n = \mathcal{C}_{(m,n)},
\]
and

\[ Y_m \cap Y_n = Y_{(m,n)}. \]

In particular, if \((m, n) = 1\), then

\[ S_m \cap S_n = \emptyset, \]
\[ X_m \cap X_n = X_1 = \mathcal{I}_{m_0}, \]
\[ C_m \cap C_n = C_1 = \emptyset, \]

and

\[ Y_m \cap Y_n = C_{m_0} = \emptyset. \]

**Proof.** The assertion \( \mathcal{I}_m \cap \mathcal{I}_n = \mathcal{I}_{(m,n)} \) is exactly Lemma 3.2 of [26]. Therefore if \((m, n) = 1\) we have that \( \mathcal{I}_{(m,n)} = \mathcal{I}_1 = \emptyset \) by definition of \( \mathcal{I}_n \). Further, by definition,

\[ X_n = \mathcal{I}_{m_0n}, X_m = \mathcal{I}_{m_0n} \]

and therefore,

\[ X_m \cap X_n = \mathcal{I}_{m_0n} \cap \mathcal{I}_{m_0m} = \mathcal{I}_{m_0(m,n)} = \mathcal{I}_{(m,n)}. \]

Thus, if \((m, n) = 1\) we have

\[ X_m \cap X_n = \emptyset = X_1 = S_{m_0}. \]

Also by definition,

\[ C_n = (\mathcal{I}_n \cap V_K) \setminus \mathcal{I}_{m_0}, C_m = (\mathcal{I}_m \cap V_K) \setminus \mathcal{I}_{m_0} \]

and therefore,

\[ C_m \cap C_n = (\mathcal{I}_m \cap \mathcal{I}_n \cap V_K) \setminus \mathcal{I}_{m_0} = (\mathcal{I}_{(m,n)} \cap V_K) \setminus \mathcal{I}_{m_0} = \mathcal{C}_{(m,n)}. \]

Consequently, if \((m, n) = 1\) we have that

\[ C_m \cap C_n = C_1 = (\mathcal{I}_1 \cap V_K) \setminus \mathcal{I}_{m_0} = \emptyset. \]

Finally, again by definition,

\[ Y_n = C_{m_0n}, Y_m = C_{m_0m} \]

and therefore,

\[ Y_m \cap Y_n = C_{m_0n} \cap C_{m_0m} = C_{m_0(m,n)} = \mathcal{Y}_{(m,n)}. \]

Consequently, if \((m, n) = 1\) we have that

\[ Y_m \cap Y_n = Y_1 = C_{m_0} = (\mathcal{I}_{m_0} \cap V_K) \setminus \mathcal{I}_{m_0} = \emptyset. \]

\[ \square \]

**Corollary 4.10.** For any \( \ell \in \mathcal{P}(\mathbb{Q}) \) and any \( j \in \mathbb{Z}_{>0} \) we have that \( q_{\ell j} \) exists, and \( q_{\ell j} \in \mathcal{Q}_k = \mathcal{Q}_{km_0} \) if and only if \( \ell \) divides \( k \). (We remind the reader that by definition, \( q_{\ell j} = p_{\ell j + \ord_{m_0}(m_0)} \) is the indicator prime of \( [\ell_{j + \ord_{m_0}(m_0)}] \mathcal{P}. \))

**Proof.** By definition of \( q_{\ell j} \), to establish its existence it is enough to show that

\[ \mathcal{I}_{\ell m_0} \setminus \mathcal{I}_{\ell j + \ord_{m_0}(m_0)} \neq \emptyset. \]

At the same time, from the definitions of \( m_0 \) and \( a_\ell \) we have that

\[ \mathcal{I}_{\ell a_\ell m_0} \setminus \mathcal{I}_{\ell j + \ord_{m_0}(m_0)} = \mathcal{I}_{\ell a_\ell j} \setminus \mathcal{I}_{\ell j + a_\ell - 1} \neq \emptyset, \]

and therefore \( q_{\ell j} \) exists.
Now suppose \( j > 0 \) and \( q_{\ell j} \in \mathcal{I}_k = \mathcal{I}_{km_0} \). Then by definition of \( q_{\ell j} \), we have that
\[
p_{\ell j + \text{ord}_\ell m_0} \in \mathcal{I}_{km_0} \cap \mathcal{I}_{\ell j + \text{ord}_\ell m_0} = \mathcal{I}_{\text{GCD}(km_0, \ell j + \text{ord}_\ell m_0)} \subseteq \mathcal{I}_{\text{ord}_\ell (km_0)}
\]
by Lemma 4.9. But by the same lemma \( p_{\ell j + \text{ord}_\ell m_0} \in \mathcal{I}_{\text{ord}_\ell (km_0)} \) if and only if \( j \leq \text{ord}_\ell k \).
Conversely, suppose \( j > 0 \) and \( j \leq \text{ord}_\ell k \). Then \( p_{\ell j + \text{ord}_\ell m_0} \in \mathcal{I}_{\ell j + \text{ord}_\ell m_0} \subseteq \mathcal{I}_{km_0} \) by Lemma 4.9 once again and \( q_{\ell j} \in \mathcal{I}_k \).

**Corollary 4.11.** \( 1 \) For any \( k \in \mathbb{Z}_{>1} \) we have that
\[\mathcal{Y}_k = \{ q_{\ell j} : \ell \in \mathcal{P}_Q, 0 < j \leq \text{ord}_\ell k \}.\]
\( 2 \) For \( k, n \in \mathbb{Z}_{>1} \) we have that \( \mathcal{Y}_k \subseteq \mathcal{Y}_n \) if and only if \( k \mid n \).
\( 3 \) For \( k, n \in \mathbb{Z}_{>1} \) we have that \( f_k \mid f_n \) if and only if \( k \mid n \).
\( 4 \) For \( k, n \in \mathbb{Z}_{>1} \) we have that \( (k, n) = 1 \) if and only if \( (f_k, f_n) = (1) \), where \( (1) \) is a trivial divisor.

**Proof.** \( 1 \) First we observe that by definition of \( \mathcal{Y}_k = \mathcal{E}_{mk} = \mathcal{I}_{mk} \setminus \mathcal{H}_k = \mathcal{I}_k \setminus \mathcal{H}_k \), these prime sets contain only the primes of the form \( p_{\ell j} \) for some \( \ell \in \mathcal{P}_Q \) and some \( j \in \mathbb{Z}_{>0} \). Secondly, by Corollary 4.10, we also have that \( q_{\ell j} \in \mathcal{I}_k \) if and only if \( 0 < j \leq \text{ord}_\ell k \).

\( 2 \) If we assume that \( k \mid n \), then \( \mathcal{I}_k \subseteq \mathcal{I}_n \) by Lemma 4.9 and consequently, \( \mathcal{Y}_k \subseteq \mathcal{Y}_n \). Conversely, if we suppose that \( \mathcal{Y}_k \subseteq \mathcal{Y}_n \), then for every rational prime \( \ell \) we have that \( q_{\ell \text{ord}_\ell (k)} \in \mathcal{Y}_n \) by Part 1 of this corollary. Thus, by Part 1 again, for every rational prime \( \ell \) we have that \( \ell \mid \text{ord}_\ell (k) \) if and only if \( \ell \mid \text{ord}_\ell (k) \) divides \( n \). Consequently \( k \) divides \( n \).

\( 3 \) If we first assume that \( f_k \mid f_n \), then \( \mathcal{Y}_k \subseteq \mathcal{Y}_n \) and \( k \mid n \) by Part 2 of this corollary. Next if we suppose \( k \mid n \), then \( \mathcal{Y}_k \subseteq \mathcal{Y}_n \) by Part 2 of this corollary again, and consequently \( f_k \mid f_n \) by Corollary 4.7.

\( 4 \) Suppose \( (k, n) = 1 \), then \( \mathcal{Y}_k \cap \mathcal{Y}_n = \emptyset \) by Corollary 4.9. Since all the prime divisors of \( f_k \) are in \( \mathcal{Y}_k \), and all the prime divisors of \( f_n \) are in \( \mathcal{Y}_n \), we must conclude that \( (f_k, f_n) = (1) \). Conversely, if \( (f_k, f_n) = (1) \), then \( \mathcal{Y}_k \cap \mathcal{Y}_n = \emptyset = \mathcal{Y}_{(k,n)} \), where the last equality holds by Corollary 4.9. But from Corollary 4.10 we conclude that \( (k, n) = 1 \) since \( \mathcal{Y}_1 \) is the only \( \mathcal{Y}_m \) with \( m > 0 \) which is an empty set.

The next corollary is the first step towards the existential definition of multiplication of indices.

**Corollary 4.12.** Let \( m, k \in \mathbb{Z}_{\neq 0} \) with \( (m, k) = 1 \). Then \( \mathcal{Y}_{mk} = \mathcal{Y}_m \cup \mathcal{Y}_k \)

**Proof.** Since \( (m, k) = 1 \) the assertion follows from the Part 1 of Corollary 4.11 since for any \( j \in \mathbb{Z}_{>0} \) and \( \ell \in \mathcal{P}_Q \) we have that \( 0 < j \leq \text{ord}_\ell mk \) if and only if either \( 0 < j \leq \text{ord}_\ell m \) or \( 0 < j \leq \text{ord}_\ell k \).

While we established already that the denominator prime sets cannot distinguish between positive and negative indices, the result below tells us that the indicator primes identify the absolute value of the index for a multiple of \( T \) uniquely.
Corollary 4.13. Let \( n_1, n_2 \in \mathbb{Z}_{>0} \) be such that \( \mathcal{Y}_{n_1} = \mathcal{Y}_{n_2} \). Then \( n_1 = n_2 \).

Proof. By Corollary 4.11 we have that \( n_1 \) divides \( n_2 \) and \( n_2 \) divides \( n_1 \). Thus, \( n_1 = n_2 \). □

From Corollary 4.13 we immediately obtain the proposition below.

Corollary 4.14. Let \( n_1, n_2 \in \mathbb{Z}_{>0} \) be such that \( f_{n_1} = f_{n_2} \). Then \( n_1 = n_2 \).

Proof. The equality \( f_{n_1} = f_{n_2} \) implies \( \mathcal{Y}_{n_1} = \mathcal{Y}_{n_2} \) and we are done by Corollary 4.13. □

We are now ready to conclude that under our definitions and under certain relative primality assumptions, the denominator of the product is “almost” equal to the product of the denominators.

Proposition 4.15. If \( m, k \in \mathbb{Z}_{>0} \) with \( (m, k) = 1 \), then \( f_{mk} | f_{km}^3 f_{m}^3 \).

Proof. Let \( p \in \mathcal{P}_K \) be such that \( \text{ord}_p f_{mk} > 0 \). Then by Corollary 4.12 either \( \text{ord}_p f_m > 0 \) or \( \text{ord}_p f_k > 0 \), but both inequalities cannot hold at the same time since \( (k, m) = 1 \). (See Lemma 4.9.) Without loss of generality, assume the first alternative holds, and therefore by Corollary 4.7 and Remark 4.6 we have that \( \text{ord}_p f_{mk} \leq 3 \text{ord}_p f_m \). □

Definition 4.16. If \( m, k \in \mathbb{Z}_{>0} \) are such that \( (m, k) = 1 \), then we will say that \( m \) and \( k \) can be multiplied directly.

The next lemma is a converse of sorts to the Proposition 4.15.

Lemma 4.17. Let \( m, k, n \in \mathbb{Z}_{>0} \). \( (f_k, f_m) = (1) \), \( f_n | f_{km}^3 f_{m}^3 \), and \( f_m | f_n \). Then \( (k, m) = 1 \), and \( n = mk \).

Proof. First we show that \( (k, m) = 1 \). Suppose not. Let \( \ell \) divide \( (m, k) \). Then \( q_\ell \in \mathcal{Y}_m \cap \mathcal{Y}_k \) and \( (f_k, f_m) \neq (1) \). Thus \( (k, m) = 1 \) and by assumption and Corollary 4.12 we now have that \( \mathcal{Y}_n = \mathcal{Y}_k \cup \mathcal{Y}_m = \mathcal{Y}_{mk} \). By Corollary 4.13 we conclude that \( n = mk \). □

The remaining Propositions 4.18 – 4.22 of this section will be necessary for defining integers using just one universal quantifier. We start with a lemma which allows us to generate integers.

Lemma 4.18 (Lemma 11 of [23]). There exists a positive integer \( m_1 \) such that for any positive integers \( l, k \),

\[
(4.1) \quad \mathcal{d}(x_{l m_1}) \bigg| n \left( \frac{x_{l m_1}}{x_{k l m_1}} - k^2 \right)^2
\]

in the integral divisor semigroup of \( K \).

Remark 4.19. If we restrict our attention to the non-inverted primes only, we can rewrite (4.1) as

\[
(4.2) \quad \mathcal{d}_{\mathcal{Y}_K}(x_{l m_1}) \bigg| n_{\mathcal{Y}_K} \left( \frac{x_{l m_1}}{x_{k l m_1}} - k^2 \right)^2
\]

Lemma 4.20. With \( m_1 \) as in Lemma 4.18 \( (\mathcal{d}_{\mathcal{Y}_K}(x_{l m_1}), n_{\mathcal{Y}_K}(x_{k l m_1})) = (1) \) in the integral divisor semigroup of \( K \).

Proof. From Lemma 4.5 and Lemma 4.9 it follows that \( \mathcal{d}_{\mathcal{Y}_K}(x_{l m_1}) \) divides \( \mathcal{d}_{\mathcal{Y}_K}(x_{k l m_1}) \) and by definition \( (\mathcal{d}_{\mathcal{Y}_K}(x_{k l m_1}), n_{\mathcal{Y}_K}(x_{k l m_1})) = (1) \). □
From Lemma 4.18 and Lemma 4.20 we also deduce the following corollary.

**Corollary 4.21.**

\[ \mathfrak{d}_{\omega_K}(x_{lm_1}) \mid n_{\omega_K} \left( \frac{x_{h_K}^{x_{lm_1}}}{x_{klm_1}^{h_K}} - k^{2h_K} \right)^2 \]

**Proof.** From an elementary algebra calculation we have

\[
\frac{x_{h_K}^{x_{lm_1}}}{x_{klm_1}^{h_K}} - k^{2h_K} = \left( \frac{x_{lm_1}}{x_{klm_1}} - k^2 \right)^{h_K-1} \sum_{r=0}^{h_K-1} \left( \frac{x_{lm_1}}{x_{klm_1}} \right)^{h_K-1-r} k^{2r},
\]

and therefore

\[ n_{\omega_K} \left( \frac{x_{lm_1}}{x_{klm_1}} - k^2 \right) \mid n_{\omega_K} \left( \frac{x_{h_K}^{x_{lm_1}}}{x_{klm_1}^{h_K}} - k^{2h_K} \right) \mathfrak{d}_{\omega_K} \left( \sum_{r=0}^{h_K-1} \left( \frac{x_{lm_1}}{x_{klm_1}} \right)^{h_K-1-r} k^{2r} \right). \]

However, the only primes which can appear in

\[ \mathfrak{d}_{\omega_K} \left( \sum_{r=0}^{h_K-1} \left( \frac{x_{lm_1}}{x_{klm_1}} \right)^{h_K-1-r} k^{2r} \right) \]

are the primes occurring in

\[ \mathfrak{d}_{\omega_K} \left( \frac{x_{lm_1}}{x_{klm_1}} \right). \]

The non-inverted part of the divisor of \( \frac{x_{lm_1}}{x_{klm_1}} \) is equal to \( \frac{n_{\omega_K}(x_{lm_1}) \mathfrak{d}_{\omega_K}(x_{klm_1})}{n_{\omega_K}(x_{klm_1}) \mathfrak{d}_{\omega_K}(x_{lm_1})} \), where \( \frac{\mathfrak{d}_{\omega_K}(x_{klm_1})}{\mathfrak{d}_{\omega_K}(x_{lm_1})} \) is an integral divisor by Lemma 4.7 and Lemma 4.5. This leaves only primes from \( n_{\omega_K}(x_{klm_1}) \) in the denominator. Since none of these primes is present in \( \mathfrak{d}_{\omega_K}(x_{lm_1}) \) due to Lemma 4.20, we have that

\[ \mathfrak{d}_{\omega_K}(x_{lm_1}) \mid n_{\omega_K} \left( \frac{x_{h_K}^{x_{lm_1}}}{x_{klm_1}^{h_K}} - k^{2h_K} \right)^2 \Leftrightarrow \]

\[ \mathfrak{d}_{\omega_K}(x_{lm_1}) \mid n_{\omega_K} \left[ \left( \frac{x_{lm_1}}{x_{klm_1}} - k^2 \right)^2 \left( \sum_{r=0}^{h_K-1} \left( \frac{x_{lm_1}}{x_{klm_1}} \right)^{h_K-1-r} k^{2r} \right)^2 \right] \Leftrightarrow \mathfrak{d}_{\omega_K}(x_{lm_1}) \mid n_{\omega_K} \left( \frac{x_{lm_1}}{x_{klm_1}} - k^2 \right)^2 \]

**Lemma 4.22.** For any \( k \in \mathbb{Z}_{>0} \) we have that \( \mathfrak{d}(x_k), \mathfrak{d}_k \) are squares of some integral divisors of \( K \).

**Proof.** From the Weierstrass equation \( y^2 = x^3 + ax + b \) we have that for any prime \( p \) of \( K \), if \( \text{ord}_p x < 0 \), then \( \text{ord}_p (x^3 + ax + b) = \text{ord}_p x^3 < 0 \) and \( \text{ord}_p y < 0 \) implying that \( \text{ord}_p x \equiv 0 \mod 2 \). \( \square \)
5. DIOPHANTINE DEFINITION OF MULTIPLICATION ON INDICES

We start with a basic fact and some easy lemmas.

**Lemma 5.1.** The set \( \{ x \in O_{K, \mathbb{W}_K} : x \neq 0 \} \) is Diophantine over \( O_{K, \mathbb{W}_K} \). (See Definition 2.2.3 and Proposition 2.2.4 of [35].)

We now use the fact that we can define the set of non-zero integers of our ring to define relative primality over the ring.

**Lemma 5.2.** The set \( R = \{(A, B) \in O_{K, \mathbb{W}_K}^2 : AB \neq 0 \land (A, B)_{\mathbb{W}_K} = 1 \} \) is Diophantine over \( O_{K, \mathbb{W}_K} \).

**Proof.** It is easy to see with the help of the Strong Approximation Theorem that for \((A, B) \in O_{K, \mathbb{W}_K}^2 \) with \( AB \neq 0 \) the following statements are equivalent

1. \((A, B)_{\mathbb{W}_K} = 1\)
2. \( \exists X, Y \in O_{K, \mathbb{W}_K} : XA + YB = 1 \)

\[ \square \]

**Notation 5.3.** We define three sets: one to represent the points on our elliptic curve, one to represent the elliptic curve addition, and one to represent the divisors of the denominators:

1. Let \( E = \{(U, V, X, Y) \in O_{K, \mathbb{W}_K}^4 \mid \exists k \in \mathbb{Z}_{\neq 0} : \frac{U}{V} = x_{m_0}, \frac{X}{Y} = y_{m_0} \} \).

   For each quadruple \((U, V, X, Y)\) the index \( k = k(U, V, X, Y) \) will be unique (since the size of the torsion group divides \( m_0 \)) and will be called the corresponding (to \((U, V, X, Y)\)) index.

2. Let \( Plus = \{(U_1, V_1, X_1, Y_1), (U_2, V_2, X_2, Y_2), (U_3, V_3, X_3, Y_3)\} \subset E^3 \)

   consist of triples of quadruples possessing corresponding indices \( k_1, k_2, k_3 \) satisfying \( k_1 + k_2 = k_3 \).

3. Given \((U, V, X, Y) \in E\), let \( d(U, V, X, Y) = \{(A, B) \in O_{K, \mathbb{W}_K}^2 : \left( \frac{U}{V} \right)^{h_K} = \frac{A}{B}, (A, B)_{\mathbb{W}_K} = 1 \} \).

**Remark 5.4.** The reason for defining the set \( d(U, V, X, Y) \) is that over an arbitrary number field \( K \) we cannot make sure that the numerators and denominators are relatively prime in our ring. Thus a denominator can have “too many” primes in it and the divisibility conditions from Proposition 4.15 can fail if we replace the divisors by the denominators. At the same time, by the definition of the class number, if we raise the \( x \)-coordinate to the power equal to the class number, we can obtain a relatively prime numerator and denominator.

Given Lemma 5.2, the following assertion is obvious.

**Lemma 5.5.** \( E, Plus, \) and \( d(U, V, X, Y) \) for fixed values of \( U, V, X, Y \), are Diophantine over \( O_{K, \mathbb{W}_K} \).
The next lemma and its corollary establish a connection between \( d(U, V, X, Y) \) and the divisor \( f_k \) of the corresponding point on the elliptic curve.

**Lemma 5.6.** If \((U, V, X, Y) \in E, (A, B) \in d(U, V, X, Y), \) and \( k \) is the corresponding index, then for all \( p \not\in \mathcal{W}_K \) we have that \( h_K \cdot \operatorname{ord}_p f_k = -\operatorname{ord}_p n_{\mathcal{W}_K}(B) \) (Here we remind the reader that \( n_{\mathcal{W}_K}(B) \) is the non-inverted part of the numerator of the divisor of \( B \)).

**Proof.** By definition of \( E \) and \( d(U, V, X, Y) \) we have that \( \frac{A}{B} = x^{h_K}_{m_{a_k}} \) for the corresponding to \((U, V, X, Y)\) index \( k \in \mathbb{Z}_{\neq 0} \). Without loss of generality we can assume that \( k > 0 \). ("\(-k\" gives the same \( B \) and the same \( f_k \) by Lemma 4.2) Let \( p \not\in \mathcal{W}_K \) be such that \( \operatorname{ord}_p x_{m_{a_k}} < 0 \). Then either \( \operatorname{ord}_p A < 0 \) or \( \operatorname{ord}_p B > 0 \). The first alternative is impossible because \( A \in O_K, \mathcal{W}_K \) and \( p \not\in \mathcal{W}_K \). Hence we conclude that \( \operatorname{ord}_p B > 0 \). Further we also have that \( \operatorname{ord}_p A = 0 \) because otherwise the relative primeness conditions requiring that \( A \) and \( B \) are not simultaneously divisible by any prime outside \( \mathcal{W}_K \) are violated. Now we see that

\[
h_K \cdot \operatorname{ord}_p x_{m_{a_k}} = \operatorname{ord}_p A - \operatorname{ord}_p B = -\operatorname{ord}_p B.
\]

Suppose now that for some \( p \not\in \mathcal{W}_K \) it is the case that \( \operatorname{ord}_p x_{m_{a_k}} \geq 0 \) and \( \operatorname{ord}_p B > 0 \). In this case we also must have that \( \operatorname{ord}_p A > 0 \) which again is impossible since \((A,B)\mathcal{W}_K = 1. \)

Given the lemma above we immediately conclude the following.

**Corollary 5.7.** If \( I, I_1 \subset I, I_2 \subset I \) are finite subsets of non-zero integers, \((U_i, V_i, X_i, Y_i) \in E, (A_i, B_i) \in d(U_i, V_i, X_i, Y_i), i \in I \) with \( k_i \) being the corresponding indices, then

\[
\left( \prod_{i \in I_1} B_i \right) \left| \mathcal{W}_K \left( \prod_{i \in I_2} B_i \right) \right\Leftarrow \iff \left( \prod_{i \in I_1} f_{k_i} \right) \left| \left( \prod_{i \in I_2} f_{k_i} \right) \right.\]

Next we show that divisibility of indices is Diophantine in our ring.

**Lemma 5.8.** If \( \text{Divide} = \{(U_1, V_1, X_1, Y_1), (U_2, V_2, X_2, Y_2)\} \subset E^3 \) consists of pairs of quadruples with the corresponding indices \( k_1 \) and \( k_2 \) such that \( k_1|k_2 \), then \( \text{Divide} \) is Diophantine over \( O_K, \mathcal{W}_K \).

**Proof.** If \((A_i, B_i) \in d(U_i, V_i, X_i, Y_i), i = 1, 2, \) then by Corollary 5.7 we have that \( B_1 \left|_{\mathcal{W}_K} B_2 \right. \) if and only if \( f_{k_1}|f_{k_2} \). At the same time by Corollary 4.11, Part 3 and Lemma 4.2 we have that \( f_{k_1}|f_{k_2} \) if and only if \( k_1|k_2 \).

We can now define multiplication on the absolute values of indices.

**Lemma 5.9.** If \((U_i, V_i, X_i, Y_i) \in E, (A_i, B_i) \in d(X_i, Y_i, U_i, V_i), i = 1, 2, 3 \) with

\[
(B_1, B_2)_{\mathcal{W}_K} = 1
\]

and

\[
B_1B_2 \left|_{\mathcal{W}_K} B_3 \wedge B_3 \left|_{\mathcal{W}_K} B_1^3B_2^3, \right.
\]

then for the corresponding indices \( k_1, k_2, k_3 \in \mathbb{Z}_{\neq 0} \) we have that \(|k_1||k_2| = |k_3| \).
Proof. If \( k_i \) is the index corresponding to \((U_i, V_i, X_i, Y_i)\), then from (5.1), Lemma 4.2, Corollary 4.11 Part 4, and Lemma 5.6 we conclude that \( f_{k_1}f_{k_2} = f_{k_3} \). Now from Corollary 5.7 and (5.2) it follows that \( f_{k_1}f_{k_2} \mid f_{k_3} \) while \( f_{k_3}^{-1}f_{k_2}^3 \), and the assertion of the lemma is true by Lemma 4.17.

Our final step in this section is to define a square of an index. This is all we need to define multiplication.

Lemma 5.10. For any \((U_1, V_1, X_1, Y_1), (U_2, V_2, X_2, Y_2), (U_3, V_3, X_3, Y_3), (U_4, V_4, X_4, Y_4) \in E\) with the corresponding indices \( k_1, k_2, k_3, \) and \( k_4 \) respectively such that

- (1) \( |k_1| > 4 \)
- (2) \( k_2 = k_1 + 1 \)
- (3) \( k_3 \neq 0 \)
- (4) \( k_4 = k_3 - k_1 \)
- (5) \( (k_1 - 1)(k_4 - 1) \)

while (5.3) and (5.4) are satisfied, we have that \( k_4 = k_1^2 \).

Conversely, for any \((U_1, V_1, X_1, Y_1), (U_2, V_2, X_2, Y_2), (U_3, V_3, X_3, Y_3), (U_4, V_4, X_4, Y_4) \in E\) with the corresponding indices \( |k_1| > 4, k_2 = k_1 + 1, k_3 = k_1^2 + k_1, k_4 = k_1^2, (5.3) \) and (5.4) will be satisfied.

\[ (A_i, B_i) \in d(U_i, V_i, X_i, Y_i), i = 1, 2, 3, 4 \]

\[ B_1B_2 \mid B_3 \quad \text{and} \quad B_3 \mid B_1^3B_2^3, \]

Proof. First assume that \((U_i, V_i, X_i, Z_i) \in E, i = 1, \ldots, 4\) with the corresponding indices \( k_1, \ldots, k_4 \) as above are given and (5.3), (5.4) are satisfied. In this case observe that \( (k_1, k_2) = 1 \) and therefore by Corollary 4.11 we have that \( f_{k_1}f_{k_2} = 1 \). Thus, by Corollary 5.7 we conclude that \( (B_1, B_2) \mid B_3 \). Further, from Lemma 5.9 we can now deduce that \( |k_3| = |k_1(k_1 + 1)| \). Consequently, \( k_3 = \pm(k_1(k_1 + 1)) \). Thus, \( k_4 = k_1^2 \) or \( k_4 = -k_1^2 - 2k_1 \). If the second alternative holds, then \( (k_1 - 1)(-k_1^2 - 2k_1) \) and \( (k_1 - 1)|3 \) which is impossible since \( |k_1| > 4 \). Thus we must have \( k_2 = k_1^2 \).

Suppose now that \( k_2 = k_1 + 1, k_3 = k_1^2 + k_1, \) and \( k_4 = k_1^2 - 1 \). By Corollary 4.11 we have that \( f_{k_1}f_{k_2} \mid f_{k_3} \) and \( f_{k_3} \mid f_{k_2}^3 \). Hence by Corollary 5.7 we can conclude that (5.4) is satisfied by some \( B_1, B_2, B_3, B_4 \in O_K \mid \) which also satisfy (5.3).

Lemma 5.8 together with Lemma 5.10 complete the proof of Theorem 1.5 and Corollary 1.7. (The density computation is in the Appendix.)

We finish this section with a new notation to be used below.

Notation 5.11. Given \((U_i, V_i, X_i, Y_i) \in E, i = 1, 2, 3\) we will say that \(((U_1, V_1, X_1, Y_1), (U_2, V_2, X_2, Y_2), (U_3, V_3, X_3, Y_3)) \in \Pi\)

\[ \text{to mean that the corresponding indices } k_1, k_2, k_3 \text{ satisfy } k_3 = k_1k_2. \]
• Let
\[ E_1 = \{(U, V, X, Y) \in O_K^4 \mid \exists \text{unique } k \in \mathbb{Z}_{\neq 0} : \frac{U}{V} = x_{m_1m_0^k}, \frac{X}{Y} = y_{m_1m_0^k}\}. \]

The positive integer \( m_1 \) is defined in Lemma 4.18.

6. DEFINING \( Z \) OVER \( O_K, \mathbb{W}_K \) USING ONE UNIVERSAL QUANTIFIER

In this section we use the existential definition of multiplication on indices to give a first-order definition of \( Z \) over \( O_K, \mathbb{W}_K \) using just one universal quantifier. We start with a technical lemma.

**Lemma 6.1.** If \( z \in O_K, \mathbb{W}_K \) has the following property:

\[ \exists U_1, V_1, X_1, Y_1, \forall b, \exists U_2, V_2, X_2, Y_2, U_3, V_3, X_3, Y_3, A_1, A_2, A_3, B_1, B_2, B_3, C \]

(with all the variables ranging over \( O_K, \mathbb{W}_K \)) such that

\[
\begin{align*}
(6.1) & \quad (U_1, V_1, X_1, Y_1), (U_3, V_3, X_3, Y_3) \in E_1, (U_2, V_2, X_2, Y_2) \in E_1, \\
(6.2) & \quad ((U_1, V_1, X_1, Y_1), (U_2, V_2, X_2, Y_2), (U_3, V_3, X_3, Y_3)) \in \Pi, \\
(6.3) & \quad (A_i, B_i) \in d(U_i, V_i, X_i, Y_i), i = 1, 2, 3, \\
(6.4) & \quad b^{2h_K} \mid \mathbb{W}_K B_2, \\
(6.5) & \quad (A_3B_2z - B_3A_2)^{2h_K} = B_2^{2h_K + 1}C, \end{align*}
\]

then \( z \in Z \).

Conversely, if \( z_0 \in \mathbb{Z}_{\neq 0} \) and \( z = z_0^{2h_K} \), then Equations (6.1)–(6.5) can be satisfied with variables as above ranging over \( O_K, \mathbb{W}_K \).

**Proof.** From (6.1)–(6.3), we conclude that if \( k_1, k_2, k_3 \) are the indices corresponding to \( (U_1, V_1, X_1, Y_1) \), \( (U_2, V_2, X_2, Y_2) \), and \( (U_3, V_3, X_3, Y_3) \) respectively, then \( k_3 = k_1k_2, k_2 \equiv 0 \mod m_1 \) and \( n_{\mathbb{W}_K}(B_i) = 1^{h_K} \). Further for the discussion below \( k_1 \) is fixed. From equation (6.5), we obtain that

\[ n_{\mathbb{W}_K}(B_2^{2h_K + 1}) \mid n_{\mathbb{W}_K}(A_3B_2z - A_2B_3)^{2h_K} \]

and therefore

\[ n_{\mathbb{W}_K}(B_2) \mid n_{\mathbb{W}_K}(A_3z - \frac{A_2B_3}{B_2})^{2h_K} \]

Further, since \( (B_2, A_3)_{\mathbb{W}_K} = 1 \) by Lemma 4.20, we have that

\[ n_{\mathbb{W}_K}(B_2) \mid n_{\mathbb{W}_K}(z - \frac{A_2B_3}{A_3B_2})^{2h_K}. \]

Thus, since \( n_{\mathbb{W}_K}(B_2) \) is a \( 2h_K \)-th power of another divisor in \( K \) by Lemma 4.22 and by the definition of \( B_2 \) we have that

\[ 2h_K \mid n_{\mathbb{W}_K}(B_2) \mid n_{\mathbb{W}_K}(z - \frac{A_2B_3}{A_3B_2}). \]
From Corollary 4.21 and Lemma 4.22, since \( k_3 = k_1k_2 \), and the indices \( k_2 \) and \( k_3 \) are divisible by \( m_1 \), we conclude that

\[
\sqrt{\mathfrak{d}_K(x_{k_m})} \mid_{\|}_K \mathbf{n}_K \left( \frac{x_{k_m}^{h_K}}{x_{k_m}^{h_{K_0}} - k_1^{h_K}} \right),
\]

and therefore, using the definition of \( B_2 \), we have

\[
2h_K \sqrt{\mathfrak{d}_K(B_2)} \mid_{\|}_K \mathbf{n}_K \left( \frac{A_2B_3}{A_3B_2} - k_1^{h_K} \right).
\]

Substituting \( \frac{A_2B_3}{A_3B_2} \) for \( \frac{x_{k_m}^{h_K}}{x_{k_m}^{h_{K_0}}} \) we obtain

(6.8)

\[
2h_K \sqrt{\mathfrak{d}_K(B_2)} \mid_{\|}_K \mathbf{n}_K \left( \frac{A_2B_3}{A_3B_2} - k_1^{h_K} \right).
\]

Combining (6.7) and (6.8), we obtain

\[
2h_K \sqrt{\mathfrak{d}_K(B_2)} \mid_{\|}_K \mathbf{n}_K (z - k_1^{2h_K}),
\]

and

\[
\mathbf{n}_K (b) \mid_{\|}_K (z - k_1^{2h_K}).
\]

Since the last divisibility condition has to hold for all \( b \), we must conclude that \( z = k_1^{2h_K} \).

Conversely, suppose \( z = z_0^{2h_K} \) for \( z_0 \in \mathbb{Z}_{\neq 0} \). Let \( (U_1, V_1, X_1, Y_1) \in E \) with the corresponding index \( k_1 = z_0 \). Let \( b \in O_{K,\|_K} \) be given. Let \( k_2 \equiv 0 \text{ mod } m_1 \) be such that \( b^2 \mid_{\|_K} \mathfrak{d}_K(x_{k_m}) \). Such an index \( k_2 \) exists by Lemma 4.8. Let \( (U_2, V_2, X_2, Y_2) \in E \) correspond to \( k_2 \). Let \( k_3 = k_1k_2 \) and let \( (U_3, V_3, X_3, Z_3) \in E \) correspond to the index \( k_3 \). Observe that conditions (6.1) and (6.2) are now satisfied. Further note that equation (6.6) holds by Corollary 4.21 and therefore equation (6.5) holds also. \( \square \)

To deal with the case of an arbitrary non-zero integer we add the following corollary.

**Corollary 6.2.** If \( z_0 \in O_{K,\|_K} \) has the following property:

\[
\exists z_1, \ldots, z_{2h_K},
\]

\[
\exists U_{1,0}, \ldots, U_{1,2h_K}, V_{1,0}, \ldots, V_{1,2h_K},
\]

\[
\exists X_{1,0}, \ldots, X_{1,2h_K}, Y_{1,0}, \ldots, Y_{1,2h_K},
\]

\[
\forall b,
\]

\[
\exists U_2, V_2, X_2, Y_2,
\]

\[
\exists U_{3,0} \ldots U_{3,2h_K}, V_{3,0}, \ldots, V_{3,2h_K},
\]

\[
\exists X_{3,0}, \ldots, X_{3,2h_K}, Y_{3,0}, \ldots, Y_{3,2h_K},
\]

\[
\exists A_{1,0}, B_{1,0}, \ldots, A_{1,2h_K}, B_{1,2h_K}, A_2, B_2, A_{3,0}, B_{3,0}, \ldots, A_{3,2h_K}, B_{3,2h_K}, C_0, \ldots, C_{2h_K},
\]

(with all the variables ranging over \( O_{K,\|_K} \)) such that

(6.9)

\[
z_j = (z_0 + j)^{2h_K}, j = 0, \ldots, 2h_K,
\]

(6.10)

\[
(U_{i,j}, V_{i,j}, X_{i,j}, Y_{i,j}) \in E, i = 1, 3, j = 0, \ldots, 2h_K,
\]
(6.11) \[(U_2, V_2, X_2, Y_2) \in E_1,\]

(6.12) \[
[(U_{1,j}, V_{1,j}, X_{1,j}, Y_{1,j}), (U_2, V_2, X_2, Y_2), (U_{3,j}, V_{3,j}, X_{3,j}, Y_{3,j})] \in \Pi, j = 0, \ldots, 2h_K,
\]

(6.13) \[
(A_{i,j}, B_{i,j}) \in d(U_{i,j}, V_{i,j}, X_{i,j}, Y_{i,j}), i = 1, 3, j = 0, \ldots, 2h_K,
\]

(6.14) \[
(A_2, B_2) \in d(U_2, V_2, X_2, Y_2),
\]

(6.15) \[
b^{2h_K} \mid \mathcal{W}_K B_2,
\]

(6.16) \[
(A_{3,j} B_{2j} - B_{3,j} A_2)^{2h_K} = B_2^{2h_K+1} C_j, j = 0, \ldots, 2h_K,
\]

then \(z_0 \in \mathbb{Z}\).

Conversely, if \(z_0 \in \mathbb{Z}_{\neq 0}\), then Equations (6.9) – (6.16) can be satisfied with variables as described above ranging over \(O_{K, \mathcal{W}_K}\).

**Proof.** If the assumptions of the corollary are true, then by Lemma 6.1 we have that

\[
z_0^{2h_K}, \ldots, (z_0 + 2h_K)^{2h_K} \in \mathbb{Z},
\]

and by Corollary B.10.10 of [35], we have that \(z_0 \in \mathbb{Q}\). At the same time, since \(z_0^{2h_K} \in \mathbb{Z}\) we have that \(z_0\) is an algebraic integer, and hence in \(\mathbb{Z}\). The rest of the proof is analogous to the proof of the second part of Lemma 6.1.

The last proposition concludes the proof of Theorem 1.8.

### 7. Infinite Extensions

**Notation and Assumptions 7.1.** We add the following to our assumption list.

- Let \(K_\infty\) be a possibly infinite algebraic extension of \(K\).
- Assume \(E(K_\infty) = E(K)\).
- Let \(O_{K_\infty, \mathcal{W}_{K_\infty}}\) be the integral closure of \(O_{K, \mathcal{W}_K}\) in \(K_\infty\).

Given the assumptions on our elliptic curve, it is easy to see that the results of the previous section will carry over, and therefore we have the following theorem:

**Theorem 7.2.** (1) Let \(K\) be a number field. Let \(E\) be an elliptic curve defined and of rank one over \(K\). Let \(P\) be a generator of \(E(K)\) modulo the torsion subgroup, and fix an affine Weierstrass equation for \(E\) of the form \(y^2 = x^3 + ax + b\), with \(a, b \in O_K\), where \(O_K\) is the ring of integers of \(K\). Let \((x_n, y_n)\) be the coordinates of \([n]P\) derived from this Weierstrass equation. Then there exists a set of \(K\)-primes \(\mathcal{W}_K\) of natural density one, and a positive integer \(m_0\) such that the following set \(\Pi_\infty \subset O_{K_\infty, \mathcal{W}_{K_\infty}}^{12}\) is Diophantine over \(O_{K_\infty, \mathcal{W}_{K_\infty}}\):

\[
(U_1, U_2, U_3, X_1, X_2, X_3, V_1, V_2, V_3, Y_1, Y_2, Y_3) \in \Pi_\infty \iff
\]

\[
\exists \text{ unique } k_1, k_2, k_3 \in \mathbb{Z}_{\neq 0} \text{ such that } \left(\frac{U_i}{V_i}, \frac{X_i}{Y_i}\right) = (x_{m_{k_1}k}, y_{m_{k_1}k}) \text{ and } k_3 = k_1k_2.
\]
(2) For \( n \neq 0 \) let \( \phi_\infty(n) = [(U_n, V_n, X_n, Y_n)] \), the class of \( (U_n, V_n, X_n, Y_n) \) under the equivalence relation described below, where \( U_n, V_n, X_n, Y_n \in O_{K,\infty}^{\#K,\infty}, Y_nV_n \neq 0 \), and 

\[
(x_{mn}, y_{mn}) = \left( \frac{U_n}{V_n}, \frac{X_n}{Y_n} \right). \]

Let \( \phi_\infty(0) = \{[0, 0, 0, 0]\} \). Then \( \phi_\infty \) is a class Diophantine model of \( \mathbb{Z} \). (Here if \( YV \neq 0 \) we have that \( (U, V, X, Y) \approx (\tilde{U}, \tilde{V}, \tilde{X}, \tilde{Y}) \) if and only if 

\[
\left( \frac{\tilde{U}}{\tilde{V}}, \frac{\tilde{X}}{\tilde{Y}} \right) = \left( \frac{U}{V}, \frac{X}{Y} \right). \]

(3) \( \mathbb{Z} \) is definable over \( O_{K,\infty}^{\#K,\infty} \) using one universal quantifier.

8. Appendix

In this Appendix we calculate the natural density of \( \mathcal{V}_K \). This calculation is similar to the one carried in [2]. We use Notation 4.1 and a new notation: for a prime \( p \) of a number field \( K \) we let \( Np \) denote the size of the residue field of \( p \).

Lemma 8.1. Let \( \ell \in \mathcal{P}(\mathbb{Q}) \) and suppose \( p \in \mathcal{I}_{\ell^{n+1}} \setminus \mathcal{I}_{\ell^n} \) for some \( n \in \mathbb{Z}_{\geq 0} \). (Such a \( p \) exists, if \( n \geq a_\ell \)) Then \( \ell^{n+1} < 3Np \).

Proof. If \( p \in \mathcal{I}_{\ell^{n+1}} \setminus \mathcal{I}_{\ell^n} \), then \( p \) does not divide the discriminant of our Weierstrass equation and \( \tilde{E} \), the reduction of \( E \mod p \) is non-singular. Further, \( x_{\ell^n}, y_{\ell^n} \) are integral at \( p \), while \( \text{ord}_p x_{\ell^{n+1}} < 0, \text{ord}_p y_{\ell^{n+1}} < 0 \). Therefore, under the reduction mod \( p \), the image of \([\ell^n]P\) is not \( \tilde{O} \) – the image of \( O \mod p \), while \([\ell^{n+1}]P = \tilde{O} \). Thus we must conclude that \( E(F_p) \) has an element of order \( \ell^{n+1} \) and therefore \( \ell^{n+1} \#E(F_p) \). Let \( \#E(F_p) = Np = q \). From a theorem of Hasse we know that \( \#E(F_p) \leq q + 1 + 2\sqrt{q} \leq 3q \) (see [39], Chapter V, Section 1, Theorem 1.1). \( \square \)

Lemma 8.2. The natural density of the set \( \mathcal{A} = \{p_{\ell^k} : \ell \in \mathcal{P}(\mathbb{Q}), k \in \mathbb{Z}_{\geq 1} \land k \geq a_\ell \} \) is zero.

Proof. For \( p = p_{\ell^k} \in \mathcal{A} \), the preceding lemma says that \( 3Np_{\ell^k} > \ell^k \). Thus, since each \( p \in \mathcal{A} \) corresponds to a distinct pair \( (\ell, k) \) with \( \ell \in \mathcal{P}(\mathbb{Q}) \) and \( k \in \mathbb{Z}_{\geq 2} \) with \( 3Np > \ell^k \), we have the following inequality:

\[
\#\{p \in \mathcal{A} : Np \leq X\} \leq \#\{ (\ell, k) \in \mathcal{P}(\mathbb{Q}) \times \mathbb{Z}_{\geq 1} : \ell \leq \sqrt[3]{3X} \}
\]

Clearly if \( \sqrt[3]{3X} < 2 \), there will be no prime \( \ell \) with \( \ell \leq \sqrt[3]{3X} \). Thus, we can limit ourselves to positive integers \( k \) such that \( k \leq \log_2(3X) \).

By the Prime Number Theorem (see [13], Theorem 4, Section 5, Chapter XV), for some positive constant \( C \) we have that \( \#\{\ell \in \mathcal{P}(\mathbb{Q}) : \ell \leq X\} \leq CX/\log X \) for all \( X \in \mathbb{Z}_{\geq 0} \). From the discussion above we now have the following sequence of inequalities:

\[
\{p \in \mathcal{A} : Np \leq X\} \leq \sum_{k=2}^{[\log_2(3X)]} \#\{\ell \in \mathcal{P}(\mathbb{Q}) : \ell \leq \sqrt[3]{3X} \}
\leq \sum_{k=2}^{[\log_2(3X)]} \#\{\ell \in \mathcal{P}(\mathbb{Q}) : \ell \leq \sqrt[3]{3X} \}
\leq \log_2(3X)[C \frac{\sqrt[3]{3X}}{\log \sqrt[3]{3X}}] = \tilde{C} \sqrt[3]{X}
\]
for some positive constant $\tilde{C}$. At the same time by the Prime Number Theorem again we also know that for some positive constant $\bar{C}$ we have $\#\{p \in \mathcal{P}_K : Np \leq X\} \geq \bar{C}X / \log X$. Thus the upper density of $\mathcal{A}$ is
\[
\limsup_{X \to \infty} \frac{\#\{p \in \mathcal{A} : Np \leq X\}}{\#\{p \in \mathcal{P}_K : Np \leq X\}} \leq \limsup_{X \to \infty} \frac{\tilde{C} \sqrt{X} \log X}{\bar{C}X} = 0.
\]
Hence $\mathcal{A}$ has a natural density, and it is zero. $\square$

**Proposition 8.3.** The set $\mathcal{V}_K(P)$ has natural density zero.

**Proof.** We first observe that it was proven in [23] and [26] that the set
\[
\mathcal{B} = \{p_\ell : \ell \in \mathcal{P}_Q \land a_\ell = 1\}
\]
has a natural density that is zero. Finally we note that $\mathcal{B} \cup \mathcal{A} = \mathcal{V}_K(P)$. $\square$

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