TOPOLOGICAL CLASSIFICATION OF ZERO-DIMENSIONAL $\mathcal{M}_\omega$-GROUPS

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Abstract. A topological group $G$ is called an $\mathcal{M}_\omega$-group if it admits a countable cover $\mathcal{K}$ by closed metrizable subspaces of $G$ such that a subset $U$ of $G$ is open in $G$ if and only if $U \cap K$ is open in $K$ for every $K \in \mathcal{K}$.

It is shown that any two non-metrizable uncountable separable zero-dimensional $\mathcal{M}_\omega$-groups are homeomorphic. Together with Zelenyuk’s classification of countable $k_\omega$-groups this implies that the topology of a non-metrizable zero-dimensional $\mathcal{M}_\omega$-group $G$ is completely determined by its density and the compact scatteredness rank $r(G)$ which, by definition, is equal to the least upper bound of scatteredness indices of compact scattered subspaces of $G$.

In [Ze] (see also [PZ, §4.3]) E.Zelenyuk has proven that the topology of a countable topological $k_\omega$-group $G$ is completely determined by its compact scatteredness rank $r(G)$ which, by definition, is equal to the least upper bound of scatteredness indices of compact scattered subsets of $G$. In this note we extend this Zelenyuk’s classification result onto the class of punctiform $\mathcal{M}_\omega$-groups.

Let us recall that a topological space $X$ is scattered if every non-empty subset of $X$ has an isolated point. For a scattered space $X$ its scatteredness index $i(X)$ is defined as the smallest ordinal $\alpha$ such that the $\alpha$-th derived set $X^{(\alpha)}$ of $X$ is finite. Derived sets $X^{(\beta)}$ of $X$ are defined by transfinite induction: $X^{(0)} = X$, $X^{(1)}$ is the set of all non-isolated points of $X$; $X^{(\beta+1)} = (X^{(\beta)})^{(1)}$ and $X^{(\beta)} = \bigcap_{\gamma < \beta} X^{(\gamma)}$ if $\beta$ is a limit ordinal. It can be easily shown that $i(X) < \omega_1$ if $X$ is a hereditarily Lindelöf scattered topological space (in particular, a countable compactum). For a topological space $X$ let

$$r(X) = \sup \{i(K) : K \text{ is a compact scattered subset of } X\}$$

be the compact scattered rank of $X$.

A topological space $X$ is defined to be a $k_\omega$-space (resp. an $\mathcal{M}_\omega$-space) if $X$ admits a countable cover $\mathcal{K}$ by compact Hausdorff subspaces (resp. by closed metrizable subspaces) of $X$ such that a subset $U$ of $X$ is open in $X$ if and only if $U \cap K$ is open in $K$ for every $K \in \mathcal{K}$. A space $X$ is called an $\mathcal{MK}_\omega$-space if $X$ is both

1991 Mathematics Subject Classification. 54H11, 22A05, 54G12, 54F45.
Research supported in part by grant INTAS-96-0753.
a $k_\omega$-space and an $\mathcal{M}_\omega$-space. A topological group $G$ is called a $k_\omega$-group (resp. $\mathcal{MK}_\omega$-group, $\mathcal{M}_\omega$-group) if its underlying topological space is $k_\omega$-space (resp. an $\mathcal{MK}_\omega$-space, an $\mathcal{M}_\omega$-space). Since each countable compactum is metrizable, we conclude that each countable $k_\omega$-space is an $\mathcal{MK}_\omega$-space. On the other hand, according to Theorem 4 of [Ba], every non-metrizable $\mathcal{M}_\omega$-group is homeomorphic to the product $H \times D$, where $H$ is an open $\mathcal{MK}_\omega$-subgroup in $G$ and $D$ is a discrete space.

Following [En$_2$, 1.4.3], we say that a topological space $X$ is punctiform if it contains no connected compact subspace containing more than one point. Each punctiform $\sigma$-compact space is zero-dimensional [En$_2$, §1.4]. On the other hand, there exist strongly infinite-dimensional separable complete-metrizable punctiform spaces [En$_2$, 6.2.4]. Given a topological space $X$ by $d(X)$ its density is denoted.

**Main Theorem.** The topology of a non-metrizable punctiform $\mathcal{M}_\omega$-group is completely determined by its density and its compact scatteredness rank. In other words, two non-metrizable punctiform $\mathcal{M}_\omega$-groups $G$, $H$ are homeomorphic if and only if $d(G) = d(H)$ and $r(G) = r(H)$.

To prove this theorem we need to make first some preliminary work. We say that a topological space $X$ carries the direct limit topology with respect to a tower $X_1 \subset X_2 \subset X_3 \subset \ldots$ of subsets of $X$ (this is denoted by $X = \varinjlim X_n$) if $X = \bigcup_{n=1}^{\infty} X_n$ and a subset $U \subset X$ is open if and only if $U \cap X_n$ is open in $X_n$ for every $n \in \mathbb{N}$.

Since the union of any two compact (resp. closed metrizable) subspaces in a topological space is compact (resp. closed and metrizable, see [En$_1$, 4.4.19]), we get the following

**Lemma 1.** A topological space $X$ is an $\mathcal{M}_\omega$-space (an $\mathcal{MK}_\omega$-space) if and only if $X$ carries the direct limit topology with respect to a tower $X_1 \subset X_2 \subset \ldots$ of closed metrizable (compact) subsets of $X$.

Under a Cantor set we understand a zero-dimensional metrizable compactum without isolated points.

**Lemma 2** [Ke, 6.5]. Each uncountable metrizable compactum contains a Cantor set.

According to a classical theorem of Brouwer [Ke, 7.4], each Cantor set is homeomorphic to the Cantor cube $2^\omega = \{0, 1\}^\omega$. It is well known that the Cantor cube is universal for the class of metrizable zero-dimensional compacta. In fact, it is universal is a stronger sense, see [vE], [Po].

**Lemma 3.** Suppose $A$ is a closed subset of a zero-dimensional metrizable compactum $B$. Every embedding $f : A \to 2^\omega$ such that $f(A)$ is nowhere dense in $2^\omega$ extends to an embedding $\bar{f} : B \to 2^\omega$.

Given a cardinal $\tau$ denote by $(2^\tau)^\infty = \varinjlim (2^\tau)^n$ the direct limit of the tower

$$2^\tau \subset (2^\tau)^2 \subset (2^\tau)^3 \subset \ldots$$
consisting of finite powers of the Cantor discontinuum $2^\tau$ (here $(2^\tau)^n$ is identified with the subspace $(2^\tau)^n \times \{\ast\}$ of $(2^\tau)^{n+1}$, where $\ast$ is any fixed point of $2^\tau$).

Using Lemma 3 by standard “back-and-forth” arguments (see [Sa]) one may prove

**Lemma 4.** A space $X$ is homeomorphic to $(2^\omega)^\infty$ if and only if $X$ is a zero-dimensional $\mathcal{M}_\omega$-space satisfying the following property:

$(SU)$ every embedding $f : B \to X$ of a closed subspace $B$ of a zero-dimensional metrizable compactum $A$ may be extended to an embedding $\bar{f} : A \to X$.

Now we are able to prove a “separable” version of Main Theorem.

**Theorem.** Every non-metrizable uncountable separable punctiform $\mathcal{M}_\omega$-group is homeomorphic to $(2^\omega)^\infty$.

**Proof.** Suppose $G$ is a non-metrizable uncountable separable punctiform $\mathcal{M}_\omega$-group. It follows from Theorem 4 of [Ba] that $G$ is an $\mathcal{M}_\omega$-group. Then $G$, being $\sigma$-compact and punctiform, is zero-dimensional, see [En2, §1.4]. According to Lemma 4, to show that $G$ is homeomorphic to $(2^\omega)^\infty$ it remains to verify the property $(SU)$ for the group $G$.

Fix any embedding $f : B \to G$ of a closed subspace of a metrizable zero-dimensional compactum $A$. By the continuity of the multiplication $\ast$ on $G$, the set $f(B)^{-1} \ast f(B) = \{f(b)^{-1} \ast f(b') : b, b' \in B\} \subset G$ is compact. It follows from Theorem 4 of [Ba] that there exists a sequence $(x_n)_{n=1}^{\infty} \subset G$ converging to the neutral element $e$ of $G$ and such that $x_n \notin f(B)^{-1} \ast f(B)$ for every $n \in \mathbb{N}$. This implies that $f(B)$ is a nowhere dense subset in the compactum $f(B) \ast S_0$, where $S_0 = \{e\} \cup \{x_n : n \in \mathbb{N}\}$. Next, since the $\mathcal{M}_\omega$-group $G$ is uncountable and $\sigma$-compact, it contains an uncountable metrizable compactum which in its turn, contains a Cantor set $C \subset G$ according to Lemma 2. Without loss of generality, $C \ni e$. It can be easily shown that the compactum $f(B) \ast S_0 \ast C$ has no isolated point and contains $f(B)$ as a nowhere dense subset. Since $f(B) \ast S_0 \ast C$ is a zero-dimensional metrizable compactum without isolated points, it is homeomorphic to the Cantor cube $2^\omega$, which allows us to apply Lemma 3 to produce an embedding $\bar{f} : A \to f(B) \ast S_0 \ast C \subset G$ extending the embedding $f$. Thus the space $G$ satisfies the condition $(SU)$ and $G$ is homeomorphic to $(2^\omega)^\infty$. □

**Lemma 5.** If $G$ is a non-metrizable $\mathcal{M}_\omega$-group, then $r(G) \leq \omega_1$. Moreover, $r(G) = \omega_1$ if and only if $G$ contains a Cantor set.

**Proof.** Suppose $G$ is a non-metrizable $\mathcal{M}_\omega$-group. Write $G = \lim_{\leftarrow} M_i$, where $M_1 \subset M_2 \subset \ldots$ of a tower of closed metrizable subspaces of $G$ with $G = \bigcup_{i=1}^{\infty} M_i$. It follows that each scattered compactum $K \subset G$ is contained in some $M_i$ and being metrizable and scattered, is countable, see Lemma 2. Consequently, $r(K) < \omega_1$ for every such $K \subset G$. Hence $r(G) \leq \omega_1$.

If $G$ contains a Cantor set $C$, then $r(G) \geq r(C) \geq \omega_1$ because $C$, being universal in the class of zero-dimensional metrizable compacta, contains copies of all
countable compacta (whose scatteredness indices run over all countable ordinals, see [Ke, 6.13]).

Assume finally that \( r(G) = \omega_1 \). According to Theorem 4 of [Ba], \( G \) is homeomorphic to the product \( H \times D \) of an \( \mathcal{KM}_{\omega} \)-group \( H \subset G \) and a discrete space \( D \). Clearly, \( \omega_1 = r(G) = r(H \times D) = r(H) \). Write \( H = \lim_{\to} K_i \), where \( K_1 \subset K_2 \subset \ldots \) is a tower of metrizable compacta in \( H \). One of these compacta is uncountable (otherwise we would get \( r(H) = \sup \{ r(K_i) : i \in \mathbb{N} \} < \omega_1 \), a contradiction with \( r(H) = \omega_1 \)). Consequently, the group \( H \) contains a Cantor set \( C \), see Lemma 2. □

Proof of Main Theorem. Suppose \( G_1, G_2 \) are two non-metrizable \( \mathcal{M}_{\omega} \)-groups with \( r(G_1) = r(G_2) \) and \( d(G_1) = d(G_2) \). By Theorem 4 of [Ba], for every \( i = 1, 2 \) the space \( G_i \) is homeomorphic to the product \( H_i \times D_i \), where \( H_i \subset G_i \) is an \( \mathcal{KM}_{\omega} \)-group and \( D_i \) is a discrete space. Since \( d(G_1) = d(G_2) \) and the spaces \( H_1, H_2 \) are separable, we may assume that \( |D_1| = |D_2| \) (if \( d(G_1) = d(G_2) \) is countable, then replacing \( H_i \) by \( G_i \), we may assume that \( |D_1| = |D_2| = 1 \)). Thus to prove that the groups \( G_1 \) and \( G_2 \) are homeomorphic, it suffices to verify that the groups \( H_1 \) and \( H_2 \) are homeomorphic. Observe that \( r(G_i) = r(H_i \times D_i) = r(H_i) \) for \( i = 1, 2 \) and hence \( r(H_1) = r(H_2) \).

If \( r(H_1) = r(H_2) < \omega_1 \), then by Lemmas 2 and 6, the \( \mathcal{KM}_{\omega} \)-groups \( H_1 \) and \( H_2 \) are countable and by Zelenyuk’s theorem [Ze], they are homeomorphic. If \( r(H_1) = r(H_2) = \omega_1 \), then we may apply Theorem and Lemmas 2, 5 to conclude that both groups \( H_1 \) and \( H_2 \) are homeomorphic to \( (2^\omega)^\omega \). □

A topological space \( X \) is defined to be an AE(0)-space if every continuous map \( f : B \to X \) from a closed subset of a zero-dimensional compact Hausdorff space \( A \) can be extended to a continuous map \( \bar{f} : A \to X \).

Conjecture. An uncountable zero-dimensional \( k_{\omega} \)-group \( G \) is homeomorphic to \( (2^\omega)^\omega \times 2^\kappa \) for some cardinals \( \tau \leq \kappa \) if and only if \( G \) is an AE(0)-space.

References

[Ba] T. Banakh, On topological groups containing a Fréchet-Urysohn fan, Matem. Studii 9:2 (1998), 149–154.
[vE] F. van Engelen, Homogeneous zero-dimensional absolute Borel sets (CWI Tracts), North-Holland, Amsterdam, 1986.
[En1] R. Engelking, General topology, PWN, Warsaw, 1977.
[En2] R. Engelking, Theory of dimensions, finite and infinite, Heldermann Verlag, Lemgo, 1995.
[Ke] A.S. Kechris, Classical descriptive set theory, Springer-Verlag, 1995.
[Po] J. Pollard, On extending homeomorphisms on zero-dimensional spaces, Fund. Math. 67 (1970), 39–48.
[PZ] I. Protasov, E.Zelenyuk, Matem. Studii. Monograph Series (1999), VNTL, Lviv.
[Sa] K. Sakai, On \( R^\infty \)-manifolds and \( Q^\infty \)-manifolds, Topol. Appl. 18 (1984), 69–79.
[Ze] E. Zelenyuk, Group topologies determined by compacta(in Russian), Mat. Stud. 5 (1995), 5–16.

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