Abstract. For Fano homogeneous toric bundles, we derive a formula for the greatest lower bound on Ricci curvature. We also give criteria for the ampleness of a kind of line bundles over general homogeneous toric bundles.

1. Introduction

The existence of canonical metrics on Kähler manifolds has been a central problem in Kähler geometry. For Fano toric manifolds, Wang and Zhu [WZ] showed that there always exists Kähler-Ricci soliton. This implies that it admits Kähler-Einstein metric if and only if the Futaki invariant vanishes. This is also equivalent to that the barycenter of the associated polytope coincides with the origin.

A natural generalization of toric manifolds is the homogeneous toric bundles. Given a generalized flag manifold $G_C/P$ and a toric manifold $F$ with an action by complex torus $T_{m}^C$, we can construct a fiber bundle via a homomorphism $\tau : P \rightarrow T_{m}^C$. 

$$M \triangleq G_C \times_{P,\tau} F \rightarrow G_C/P.$$

In [PS-1, PS-2] Podestà and Spiro determined when $M$ will be Fano, and in that case showed the existence of Kähler-Ricci solitons.

To find Kähler-Einstein metrics on a general Fano manifold $X$, we usually consider the following equation depending on parameter $t \in [0, 1]$, i.e. continuation method along Aubin’s path,

$$Ric(\omega_t) = t\omega_t + (1-t)\omega_0.$$  

It is solvable when $t$ is close to zero and solutions of the case $t = 1$ give K-E metrics.

When the K-E metrics does not exist, consider

$$\sup\{T > 0 \mid \text{L.1 is solvable for } t \in [0, T]\}.$$  

In [Sz], Székelyhidi showed that this supremum is independent on $\omega_0$ and equals to the greatest lower bounds on Ricci curvature

$$R(X) = \sup\{t \geq 0 \mid \exists \omega \in c_1(X) \text{ s.t. } Ric(\omega) > t\omega\}.$$  

For toric manifolds, Li [Li-1] obtained an explicit formula for $R(X)$ in terms of the associated polytope. In addition, for the bi-equivariant Fano compactifications of complex reductive groups, including toric manifolds, Delcroix [De] obtained a similar formula.

One aim of this paper is to derive a similar formula for Fano homogeneous toric bundles. Comparing to the toric case, it turns out the main difference is that the real-reduced equation of [L1] has an additional term. But since this term is
uniformly bounded, we can obtain the key estimates by following the way of \[WZ\] without essential modifications. With these estimates at hand, the rest of the proof is similar to \[Li-1\].

Now we state the result, see Section 2 for the notations. Let

\[
\triangle_F = \{ y \in t^* \mid \langle p_i, y \rangle + 1 \geq 0 \text{ for } i = 1, \cdots, N \}
\]

be the polytope associated to Fano toric manifold \(F\). Using the dual of \(d\tau|_{Z(t)} : Z(t) \rightarrow t\), we denote

\[
\triangle_M = (d\tau)^* \left( \frac{1}{2\pi} \triangle_F \right) + \Gamma_V^\vee \subset Z(t)^*,
\]

where \(\Gamma_V\) is defined by \(2.3\). With respect to the \(-B\)-orthogonal decomposition \(H = Z(t) \oplus Z(t)^\perp\). Let \(iH_\alpha\mid_Z\) be the \(Z(t)\)-component of \(iH_\alpha\).

**Theorem 1.** Let \(M\) be a Fano homogeneous toric bundle given by data \((G^C, P, F, \tau)\). Let

\[
P \triangleq \int_{\triangle_M} \vec{x} \cdot \prod_{\alpha \in R_m^+} (iH_\alpha)|_Z \, d\mu \div \int_{\triangle_M} \prod_{\alpha \in R_m^+} (iH_\alpha)|_Z \, d\mu \in \triangle_M
\]

where \((iH_\alpha)|_Z\) are treated as linear functions on \(Z(t)^*\), \(d\mu\) is the Lebesgue measure on the affine subspace spanned by \(\triangle_M\). Then

\[
R(M) = \sup \{ 0 \leq t < 1 \mid \frac{-t}{1-t}P + \frac{1}{1-t}\Gamma_V^\vee \in \triangle_M \}.
\]

Another aim is to determine the ample line bundles over \(M\). In \([PS-1]\) they gave a criteria for anti-canonical bundle. We only consider line bundles in the following form

\[
\mathcal{L} = \left( G^C \times_{P, \tau} L_F \right) \otimes \pi^*L_\chi,
\]

where \(L_F\) is a line bundle over \(F\) with a lifted torus action, \(L_\chi\) is a line bundle over \(G^C/P\) given by a character \(\chi : P \rightarrow \mathbb{C}^*\). Let \(\Delta_{L_F}\) be the weight polytope associated to \(L_F\), namely the convex hull of the weights of action on fibers over the fixed points.

**Theorem 2.** The line bundle \(\mathcal{L}\) is ample if and only if \(L_F\) is ample and

\[
(d\tau)^* \left( \frac{1}{2\pi} \Delta_{L_F} \right) + \frac{i}{2\pi} d\chi|_{Z(t)} \subset C^\vee
\]

where \(C^\vee \subset Z(t)^*\) is the \(-B\)-dual cone of the Weyl chamber \(C\).

The main difficult to prove this is how to compute the curvature of \(\mathcal{L}\), we use a Koszul type formula \(3.5\).

### 2. Toric Bundles over Flag Manifolds

In this section we follow \([PS-1]\). The only new observation is Proposition \(3\).
2.1. Generalized flag manifolds. Let \( G \) be a compact semi-simple Lie group, \( S \) be a subtorus. Take \( T \) is a maximal torus containing \( S \). Let \( K = C_G(S) \), then \( K \supset T \). Denote \( \mathfrak{g}, \mathfrak{t}, \mathfrak{h} \) the Lie algebras of \( G, K, T \). Let \( \mathfrak{m} \) be the orthogonal complement of \( \mathfrak{t} \) with respect to the Killing form \( B \), which is a symmetric and negative definite bilinear form on \( \mathfrak{g} \). Then we have \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \) and \( [\mathfrak{t}, \mathfrak{m}] \subset \mathfrak{m} \).

Denote \( G^C \) the complexification of \( G \). Since \( T^C \) is also a maximal torus of \( G^C \), we have the root decomposition

\[
\mathfrak{g}^C = \mathfrak{h}^C \oplus \sum_{\alpha \in R} \mathbb{C}E_\alpha,
\]

where \( R \subset i\mathfrak{h}^* \) is the root system and \( \{E_\alpha\} \) are root vectors such that \( H_\alpha = [E_\alpha, E_{-\alpha}] \) is the \( \mathcal{B} \)-dual of \( \alpha \). Let

\[
F_\alpha = \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha}), \ G_\alpha = \frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha}),
\]

then \( \{F_\alpha, G_\alpha\} \mathbb{R} \)-spans \( (\mathbb{C}E_\alpha \oplus \mathbb{C}E_{-\alpha}) \cap \mathfrak{g} \) and \( [F_\alpha, G_\alpha] = iH_\alpha \).

\( K \) corresponds a partition of roots \( R = R_t \sqcup R_m \) such that

\[
\mathfrak{t}^C = \mathfrak{h}^C \oplus \sum_{\alpha \in R_t} \mathbb{C}E_\alpha, \ \mathfrak{m}^C = \sum_{\alpha \in R_m} \mathbb{C}E_\alpha.
\]

Denote \( Z(\mathfrak{t}) \subset \mathfrak{h} \) the center of \( \mathfrak{t} \). Its orthogonal complement \( Z(\mathfrak{t})^\perp \) in \( \mathfrak{h} \) is spanned by \( \{iH_\alpha\}_{\alpha \in R_t} \).

From these data we have a generalized flag manifold \( V = G/K \). There is a 1-to-1 correspondence between \( G \)-invariant complex structures \( J_V \) and the partitions \( R_m = R_{m}^+ \sqcup R_{m}^- \) satisfying

1. \( R_{m}^+ = -R_{m}^- \),
2. If \( \alpha \in R_t \sqcup R_{m}^+ \) and \( \beta \in R_{m}^+ \), then \( \alpha + \beta \in R_{m}^+ \).

Identify \( T_{eK}V \) with \( \mathfrak{m} \), then the decomposition of \( T_{eK}V \otimes \mathbb{C} \) induced by \( J_V \) is

\[
\mathfrak{m}^{(1,0)} = \sum_{\alpha \in R_m^+} \mathbb{C}E_\alpha, \ \mathfrak{m}^{(0,1)} = \sum_{\alpha \in R_m^-} \mathbb{C}E_\alpha.
\]

It also gives a parabolic subgroup \( P \subset G^C \) with Lie algebra \( \mathfrak{p} = \mathfrak{t}^C + \mathfrak{m}^{(0,1)} \). Then we have a complex model \( V = G^C/P \). Now \( V \) is a complex manifold with a holomorphic \( G^C \)-action.

Let \( \omega \) be a \( G \)-invariant Kähler metric on \( V \), there exists a unique \( I_\omega \in Z(\mathfrak{t}) \) such that

\[
\omega(\hat{X}, \hat{Y})|_{eK} = B(I_\omega, [X, Y])
\]

for all \( X, Y \in \mathfrak{g} \), where \( \hat{X} \) is the induced vector field by \( G \)-action. In order that \( \omega \) is positive, we need \( I_\omega \) belongs to Weyl chamber

\[
\mathcal{C} = \{I \in Z(\mathfrak{t}) \mid i\alpha(I) > 0 \text{ for all } \alpha \in R_m^+\}.
\]

If we take

\[
I_\omega = I_V \triangleq -\frac{i}{2\pi} \sum_{\alpha \in R_m^+} H_\alpha,
\]

then \( \omega \) is the \( G \)-invariant Kähler-Einstein metric.
2.2. Toric bundles. Let $F$ be a toric manifold with an action by complex torus $T^m$. Denote $T^m$ the real torus and $\mathfrak{t}$ its Lie algebra.

Let $\phi: P \rightarrow T^m$ be a surjective homomorphism which takes $K$ into $T^m$. Since $d\phi([X,Y]) = 0$ for all $X, Y \in \mathfrak{p}$, we see that $d\phi|_{Z(\mathfrak{t})}: Z(\mathfrak{t}) \rightarrow \mathfrak{t}$ is surjective. Then choose a $-\mathcal{B}$-orthonormal basis $\{Z_i\}_{i=1}^m$ for $(\ker d\phi|_{Z(\mathfrak{t})})^\perp$.

The toric bundles have the following two models,

$$M \cong G \times K \times F \cong G^C \times F_{\mathfrak{t}},$$

where $G \times K \times F = G \times F/\{(g, x) \sim (gk^{-1}, \phi(k).x) \mid k \in K\}$.

Denote $\tau$ the natural projection $M \rightarrow V$ and $F_0 \subset M$ the fiber over $eK$, which can be identified with $F$ via $x \mapsto [e, x]$ for $x \in F$. By the second model $M$ is a complex manifold. Note that $TM|_{F_0} = TF_0 \oplus \mathfrak{m}$, then the complex structure $J$ is the direct sum of the ones on $TF$ and $\mathfrak{m}$. Namely, $JA = J\mathfrak{m}$ for all $A \in \mathfrak{m}$.

Moreover, $M$ has a holomorphic $G^C \times T^m$-action defined by

$$(h, z).([g, x]) = [hg, z.x]$$

where $g, h \in G^C$, $z \in T^m$ and $x \in F$. In the following we identify subgroups $G^C \times \{e\}, \{e\} \times T^m$ with $G^C, T^m$.

Note that $F_0$ is stabilized by $P$ and $T^m$. Actually, these two actions on $F_0$ are equivalent with each other through $\tau$. Since

\begin{equation}
(p, e).[e, x] = [p, x] = [e, \tau(p).x] = (e, \tau(p)).[e, x],
\end{equation}

for all $p \in P$, $x \in F$.

2.3. Algebraic representations. Algebraic representation is a way to describe the invariant forms on $G$-manifolds.

Let $\phi$ be a $G$-invariant closed two-form on $M$. According to [PS-T], there exists a map $Z_{\phi}: M \rightarrow \mathfrak{g}$, called the algebraic representation of $\phi$, such that

1. $Z_{\phi}(g.x) = \text{Ad}(g).Z_{\phi}(x)$ for all $g \in G, x \in M$.
2. For all $X, Y \in \mathfrak{g}$,

\begin{equation}
\phi(\hat{X}, \hat{Y}) = \mathcal{B}(Z_{\phi}, [X, Y]).
\end{equation}

Moreover, if $\phi$ is non-degenerate, i.e. a symplectic form, then $Z_{\phi}^\vee = -\mathcal{B}(\cdot, \cdot)$ is the moment map for the $G$-action.

The following Proposition tells us when $\phi$ is also invariant under $T^m$-action.

Proposition 3. Let $\phi$ be a $G$-invariant and $J$-invariant closed two-form on $M$, then $\phi$ is invariant under $T^m$-action if and only if $Z_{\phi}|_{F_0}$ takes value in $Z(\mathfrak{t})$.

Proof. Suppose that $\phi$ is $T^m$-invariant. Since $G$-action commutes with $T^m$-action, we have $t_*\hat{X} = \hat{X}$ for all $t \in T^m$ and $X \in \mathfrak{g}$. Then by the property of $Z_{\phi}$,

$$\mathcal{B}(Z_{\phi}(x), [X, Y]) = \phi(x)(\hat{X}, \hat{Y}) = \phi(t.x)(t_*\hat{X}, t_*\hat{Y}) = \mathcal{B}(Z_{\phi}(t.x), [X, Y])$$

for all $X, Y \in \mathfrak{g}$, $t \in T^m$ and $x \in M$. Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, it follows that $Z_{\phi}(x) = Z_{\phi}(t.x)$.

In the following we assume $x \in F_0$. Take $k \in K$ such that $\tau(k) = t$. By the equivalence of actions on $F_0$ (2.4),

$$Z_{\phi}(t.x) = Z_{\phi}(k.x) = \text{Ad}(k).Z_{\phi}(x).$$
Thus $\text{Ad}(k), Z_\phi(x) = Z_\phi(x)$ for all $k \in K$. It turns out $[\mathfrak{t}, Z_\phi(x)] = 0$. From the fact $[H, E_\alpha] = \alpha(H)E_\alpha$ for all $H \in \mathfrak{h}$, we can deduce that $Z_\phi(x) \in \mathfrak{t}$. Thus $Z_\phi(x) \in Z(\mathfrak{t})$ as desired.

Conversely, tracing back the above arguments, we know that
\begin{equation}
\phi|_x(\hat{X}, \hat{Y}) = \phi|_{\mathfrak{t}, x}(t_x \hat{X}, t_x \hat{Y})
\end{equation}
for all $X, Y \in \mathfrak{g}$, $t \in T^m$ and $x \in F_o$. On the other hand, since $[\mathfrak{t}, \mathfrak{m}] \subset \mathfrak{m}$ we have
$$
\phi|_{\mathfrak{F}_c}(\hat{Z}_j, \hat{m}) = \mathcal{B}(Z_\phi|_{\mathfrak{F}_c}, [Z_j, \hat{m}]) = 0.
$$
Moreover, since $J \hat{m} = \hat{m}$ on $F_o$ and $\phi$ is $J$-invariant, we also have
$$
\phi|_{\mathfrak{F}_c}(J \hat{Z}_j, \hat{m}) = -\phi|_{\mathfrak{F}_c}(\hat{Z}_j, J \hat{m}) = 0.
$$
Thus we have
\begin{equation}
\phi|_{\mathfrak{F}_c}(T F_o, \hat{m}) = 0,
\end{equation}
since that $T F_o$ is spanned by $\{\hat{Z}_j, J \hat{Z}_j\}_{j=1}^m$. Then together with (2.6), it implies that $t^* (\phi|_{\mathfrak{t}, x}) = \phi|_x$ for all $t \in T^m, x \in F_o$. By using the $G$-action, we know this holds for all $x \in M$. \hfill \Box

3. Ampleness of Line Bundles

In this section we discuss the ampleness of line bundles over $M$.

Let $L_F$ be an ample line bundle over $F$ with a lifted torus action. Take a $T^m$-invariant metric $h_F$ on $L_F$, its curvature is denoted by $0 < \omega_F \in c_1(L_F)$. Then $h_F$ together with the lifted action induce a moment map. In fact, let $U \cong T_{c1}^m$ be the open orbit and $s(z)$ be an equivariant section on $U$. Let $\varphi = -\log \|s\|^2_{h_F}$ which is $T^m$-invariant. The basis $\{d\tau(Z_k)\}_{k=1}^m$ of $t$ gives logarithmic coordinates $\{x_k + i\theta_k\}$ on $U$. Then the moment map is given by
\begin{equation}
\mu = \sum_k \mu_k d\tau(Z_k)^* : F \to \mathfrak{t}^*, \quad \mu_k = -\frac{1}{4\pi} \frac{\partial \varphi}{\partial x_k}
\end{equation}
where $\{d\tau(Z_k)^*\}$ is the dual basis. Denote $\frac{1}{2\pi i} \Delta_{L_F}$ the image of $\mu$, which does not depend on $h_F$. Actually, $\Delta_{L_F}$ is the convex hull of the weights of action on fibers over the fixed points.

Let $\chi : P \to \mathbb{C}^*$ be a character. It gives a line bundle
$$
L_{\chi} \cong G^C \times_{P, \chi} \mathbb{C} \cong G \times_{K, \chi} \mathbb{C}
$$
over $V$. Let $h_{\chi}$ be a $G$-invariant metric on $L_{\chi}$ defined by
$$
\|g, \lambda\|^2_{h_{\chi}} = |\lambda|^2
$$
where $g \in G$. Denote $\omega_{\chi}$ the induced curvature form. Let $I_{\chi} \in Z(\mathfrak{t})$ such that
$$
I_{\chi}^\chi = -\mathcal{B}(I_{\chi}, \cdot) = \frac{i}{2\pi} d\chi |_{Z(\mathfrak{t})},
$$
then we can check that $\omega_{\chi}(\hat{X}, \hat{Y})|_{\mathfrak{t}K} = \mathcal{B}(I_{\chi}, [X, Y])$ for all $X, Y \in \mathfrak{g}$.

Consider the following holomorphic line bundle over $M$,
\begin{equation}
\mathcal{L} = (G^C \times_{P, \tau} L_F) \otimes \pi^* L_{\chi}.
\end{equation}
Note that its isomorphism class depends on the lifted torus action on $L_F$.

It is easy to see that $\mathcal{L}$ is isomorphic to $G^C \times_{P, \tau, t} L_F$, where $P$ acts on $L_F$ by
$$
p.s = \chi(p) (\tau(p) s).
$$
Use this model we define a $G^C \times T^{m}_C$-action on $L$ by $(h, z)\cdot [g, s] = [hg, z, s]$, where $g, h \in G^C$, $s \in L_F$.

Now let us first consider $L_1 \cong G^C \times_{P, \tau} L_F \cong G \times_{K, \tau} L_F$. There is a natural metric $h_1$ on it defined by

$$\| [g, s]\|_{h_1}^2 = |s|^2_{h_F}.$$  

There is also a $G^C \times T^{m}_C$-action on $L_1$ and this metric is invariant under $G \times T^m$-action. Let $\omega_1$ be the induced curvature form. Since the invariance, we only consider its restriction on $TM|_{F_0}$. By (2.7) we have

$$\omega_1|_{TF_0} = \omega_F, \quad \omega_1(TF_0, \hat{m}) = 0.$$  

In order to compute $\omega_1|_{\hat{m}}$ we need an open dense subset with a product structure. Denote $N$ the kernel of $\tau : K \to T^m$. Recall that $U \subset F$ is the open orbit.

**Lemma 4.** The open dense subset $G \times_{K, \tau} U \subset M$ is diffeomorphism to $G/N \times \mathbb{R}^m_+$. And on this open set $L_1$ can be identified with $G/N \times \mathbb{R}^m_+ \times \mathbb{C}$.

**Proof.** We denote $r_k = \exp x_k$ the coordinates of $\mathbb{R}^m_+$. Define $\phi : G/N \times \mathbb{R}^m_+ \to G \times_{K, \tau} U$ by

$$(gN, r) \mapsto [g, r].$$

Identify $L_F|_U$ with $U \times \mathbb{C}$ via $\lambda s(z) \mapsto (z, \lambda)$, then we define $\Phi : G/N \times \mathbb{R}^m_+ \times \mathbb{C} \to G \times_{K, \tau} (L_F|_U)$ by

$$(gN, r, \lambda) \mapsto [g, (r, \lambda)].$$

Then $\phi, \Phi$ give the desired identifications.

We only verify that $\Phi$ is surjective. For any $z \in T^m$, let $z = rt$ where $r \in \mathbb{R}^m_+$, $t \in T^m$. Take $k \in K$ such that $\tau(k) = t$. Since

$$[g, (z, \lambda)] = [g, \tau(k). (r, \lambda)] = [gk, (r, \lambda)],$$

thus $\Phi(gkN, r, \lambda) = [g, (z, \lambda)]$. Note that $gkN$ dose not depend on the choice of $k$. \hfill \Box

Under this identification, the $G \times T^m$-action on $L_1$ becomes $(h, t).(gN, r, \lambda) = (h^gkN, r, \lambda)$, where $k \in K$ is any element such that $\tau(k) = t$. And the metric (3.3) becomes

$$\|(gN, r, \lambda)\|_{h}^2 = |\lambda|^2 e^{-\varphi(t)}.$$  

The following result can be seen as a Koszul type formula. See [Be] (2.134) for the origin one.

**Proposition 5.** Let $(L, h) \to M$ be a holomorphic Hermitian line bundle with curvature $\omega \in c_1(L)$. Suppose that there is a compact Lie group $G$ acts holomorphically on $L \to M$ and preserves $h$. For $A \in \mathfrak{g}$, denote $A$ the induced vector field on $M$ and $\hat{A}$ the induced vector field on $L$. The complex structure on $L$ and $M$ both are denoted by $J$. Let $H$ be a function on $L$ defined by $H(s) = |s|^2_{h}$. Then for any $A, B \in \mathfrak{g}$, $x \in M$ we have

$$\omega(\hat{A}, \hat{B})|_x = \frac{1}{4\pi} J[\hat{A}, \hat{B}] \log H,$$  

where the derivative is taken at any $\bar{x} \in L_{\bar{x}} \setminus \{0\}$. 

Proof. Take an open subset $U$ including $x$ and a local frame $s$ of $L$ on it. Identify $L|_U$ with $U \times \mathbb{C}$ by $\lambda s(p) \mapsto (p, \lambda)$. Let $\phi = -\log |s|^2$, then $2\pi \omega = i\partial \bar{\partial} \phi$.

For $p \in U$ and $X \in T_pM$, the horizontal lifting of $X$ w.r.t. Chern connection is

$$\tilde{X}|_{(p, \lambda)} = (X, \lambda \partial \phi(X)) \in T_pM \oplus \mathbb{C}.$$ 

For $A \in \mathfrak{g}$, the induced vector field $\tilde{A}$ on $L$ has the form

$$(3.6) \quad \tilde{A}|_{(p, \lambda)} = (\hat{A}, \lambda \theta_A),$$

where $\theta_A$ is a complex-value function on $U$. Since the action is holomorphic, $\theta_A$ is holomorphic. Moreover, by the preservation of $h$ we can deduce that

$$(3.7) \quad \text{Re} \theta_A = \frac{1}{2} \hat{A} \phi.$$ 

It follows that

$$\hat{A} - \tilde{A}|_{(p, \lambda)} = \left(0, \lambda(\theta_A - \partial \phi(\hat{A}))\right).$$

Denote $i f_A = \theta_A - \partial \phi(\hat{A})$, since $(3.7)$ $f_A = \text{Im} \theta_A + \frac{1}{2} J \hat{A} \cdot \phi$ is real. We claim

$$df_A = 2\pi i \hat{A} \omega.$$ 

Actually, by $\hat{A}$ is holomorphic and Cartan's formula,

$$\partial(\hat{A} \phi) = L_{\hat{A}} \partial \phi = d \left(\partial \phi(\hat{A})\right) - \iota_{\hat{A}} \partial \bar{\partial} \phi.$$ 

And $(3.7)$ implies $\theta_A - \hat{A} \phi = -\bar{\theta}_A$, then we have

$$idf_A = \partial \theta_A - d \left(\partial \phi(\hat{A})\right) = \partial(\theta_A - \hat{A} \phi) - \iota_{\hat{A}} \partial \bar{\partial} \phi = -\iota_{\hat{A}} \partial \bar{\partial} \phi$$

as desired.

Now for $A, B \in \mathfrak{g}$, by $(3.6)$ we have

$$[A, B]|_{(p, \lambda)} = \left([\hat{A}, \hat{B}], \lambda(\hat{A} \theta_B - \hat{B} \theta_A)\right).$$

Since $\log H(p, \lambda) = \log |\lambda|^2 - \phi(p)$, then

$$(3.8) \quad \frac{1}{2} J[\hat{A}, \hat{B}] \cdot \log H = -\hat{A} \text{Im} \theta_B + \hat{B} \text{Im} \theta_A - \frac{1}{2} J[\hat{A}, \hat{B}] \cdot \phi.$$ 

The last term

$$J[\hat{A}, \hat{B}] \cdot \phi = [J \hat{A}, \hat{B}] \cdot \phi = J\hat{A}(\hat{B} \cdot \phi) - \hat{B}(J \hat{A} \cdot \phi).$$

On the other hand, by $(3.8)$,

$$2\pi \omega(\hat{A}, \hat{B}) = \hat{B} \cdot f_A = \hat{B} \cdot \text{Im} \theta_A + \frac{1}{2} \hat{B}(J \hat{A} \cdot \phi).$$

Thus

$$\frac{1}{2} J[\hat{A}, \hat{B}] \cdot \log H = -\hat{A} \text{Im} \theta_B - \frac{1}{2} J\hat{A}(\hat{B} \cdot \phi) + 2\pi \omega(\hat{A}, \hat{B})$$

$$= -2 \text{Im} \left(\bar{\partial} \theta_B(\hat{A})\right) + 2\pi \omega(\hat{A}, \hat{B})$$

$$= 2\pi \omega(\hat{A}, \hat{B}),$$

where we used $\hat{B} \cdot \phi = 2\text{Re} \theta_B$. □
Now we use (3.3) to compute the other component of \( \omega_1 \). Denote \( \Omega \) the curvature of the metric \( h_1 \otimes \pi^* h_\chi \) on \( L \). Obviously, \( \Omega = \omega_1 + \pi^* \omega_\chi \).

**Theorem 6.** For \( A, B \in \mathfrak{m} \) we have

\[
\omega_1(\hat{A}, \hat{B})|_{F_0} = B \left( \sum_i \mu_i Z_i, [A, B] \right),
\]

where the \( \{ \mu_i \} \) are given by (3.11). And the algebraic representation of \( \Omega \) restricted on \( F_0 \) is equal to \( \sum_i \mu_i Z_i + I_\chi \).

**Proof.** Restrict on the open set \( G \times K, r U \), under the identification in Lemma 4, the induced vector field on \( \mathcal{L}_1 = G/N \times \mathbb{R}_+^m \times \mathbb{C} \) is

\[
\hat{A}|_{(gN, r, \lambda)} = g_*(\text{Ad}(g^{-1})A|_{(eN, r, \lambda)}) = g_*(\text{Ad}(g^{-1})A, 0, 0).
\]

Where \( \text{Ad}(g)^{-1}A \) is considered that belongs to \( T_{eN} G/N \approx \mathfrak{g}/\mathfrak{n} = \mathfrak{m} \oplus \sum_i \mathbb{R} \cdot Z_i \).

Thus

\[
\mathcal{J}[\hat{A}, \hat{B}]|_{(gN, r, \lambda)} = -\mathcal{J}[A, B]|_{(gN, r, \lambda)} = -g_*(\text{Ad}(g^{-1})[A, B], 0, 0),
\]

where \( \mathcal{J} \) is the complex structure on \( \mathcal{L}_1 \).

Note that on

\[
T\mathcal{L}_1|_{(eN, r, \lambda)} = \mathfrak{m} \oplus \sum_i \mathbb{R} \cdot Z_i \oplus \sum_i \mathbb{R} \cdot r_i \frac{\partial}{\partial r_i} \oplus \mathbb{C},
\]

the complex structure satisfies \( \mathcal{J} Z_i = r_i \frac{\partial}{\partial r_i} = -\frac{\partial}{\partial x_i} \). And

\[
\log H(gN, r, \lambda) = \log |\lambda|^2 - \varphi(r).
\]

It turns out that to compute \( \mathcal{J}[A, B] \log H \) we only need the \( Z_i \)-components of \( \text{Ad}(g)^{-1}[A, B] \).

Next we further restrict on \( (G \times K, r U) \cap F_0 = K/N \times \mathbb{R}_+^m, \) so \( g \in K \).

For \( k \in K \), we have

\[
\text{Ad}(k)^{-1}[A, B] = -\sum_i B(\text{Ad}(k)^{-1}[A, B], Z_i) Z_i + C
\]

\[
= -\sum_i B([A, B], Z_i) Z_i + C
\]

in \( \mathfrak{g}/\mathfrak{n} \), where \( C \in \mathfrak{m} \) and we used \( Z_i \in Z(\mathfrak{k}) \). Thus

\[
\mathcal{J}[\hat{A}, \hat{B}]|_{(gN, r, \lambda)} = \left( -g_* J Y, C, \sum_i B([A, B], Z_i) \frac{\partial}{\partial x_i}, 0 \right).
\]

Then by (3.3)

\[
\omega_1(\hat{A}, \hat{B}) = \frac{1}{4\pi} \mathcal{J}[\hat{A}, \hat{B}] \log H = -\frac{1}{4\pi} \sum_i B([A, B], Z_i) \frac{\partial \varphi}{\partial x_i},
\]

\[
= B \left( \sum_i \mu_i Z_i, [A, B] \right).
\]

Combine this with (5.34), it is easy to check that

\[
(\omega_1 + \pi^* \omega_\chi)(\hat{X}, \hat{Y}) = B \left( \sum_i \mu_i Z_i + I_\chi, [X, Y] \right)
\]
Once we have the algebraic representation of $\Omega$, the proof of Theorem 2 will be same as [PS-1].

Proof of Theorem 2. “Sufficiency” Take a metric $h_F$ on $L_F$ with positive curvature. It gives a metric $h_1 \otimes \pi^* h_\chi$ on $\cal L$. We show that its curvature form $\Omega$ is positive. By the $G$-invariance, we only check this on $F_o$. Since $TM|_{F_o} = TF_o \oplus \tilde{m}$, (4.1) and the facts $[E_\alpha, E_\beta] \in \bar{C} : E_{\alpha+\beta}$, it turns out that we only need to check $\Omega(F_\alpha, J\hat{F}_\alpha) > 0$ for all $\alpha \in R_m^+$.

Thus there is a $\mu$-moment map $\Lambda \subset \bar{C}$ with positive curvature. We can assume that this metric is $G$-invariant. Thus there is a $G$-invariant function $\phi$ such that $\Omega + i\partial\bar{\partial}\phi > 0$. By Lemma 3.1(b) in [PS-1], restrict on $F_o$, the algebraic representation of $\partial\bar{\partial}\phi$ is $-\sum_i \theta_x Z_i$. Hence the algebraic representation of $\Omega + i\partial\bar{\partial}\phi$ is $\sum_i (\mu_i - \phi_x(x)) Z_i + I_x$. Then by (3.10) the positivity of $\Omega + i\partial\bar{\partial}\phi$ implies

$$\sqrt{-1} \alpha \left( \sum_i (\mu_i - \phi_x(x)) Z_i + I_x \right) > 0$$

on $F_o$, for all $\alpha \in R_m^+$. Since the additional $\phi_x$-terms dose not change the image of moment map $\mu$, it implies (4.1).

4. Proof of the Theorem 1

Now we assume that $M$ is Fano. From the short exact sequence

$$0 \rightarrow T_u M \rightarrow TM \xrightarrow{\pi} \pi^* TV \rightarrow 0,$$

we have

$$K_M^{-1} = \left( G^\mathbb{C} \times_{P,\tau} K_F^{-1} \right) \otimes \pi^* K_V^{-1},$$

where the torus acts on $K_F^{-1}$ in the canonical way. In particular, we have $K_M^{-1}|_{F_o} = K_{F_o}^{-1}$. Thus $F$ is also Fano.

Let $\triangle_F$ be the polytope associated to $F$, it has the form (1.2). Take $h_0^F$ be the pullback of the Fubini-Study metric via the map given by global sections of $K_F^{-1}$. Then it induces a moment map $\mu_0 = \sum_i \mu_0 d\tau(Z_i)^*$ with image $\frac{1}{2\pi} \triangle F_L$, where

$$\mu_0 = -\frac{1}{2\pi} \partial_\pi \delta_{\alpha},$$

and

$$u_0 = \log \sum_{\lambda \in \triangle F \cap \Lambda} e^{(-\lambda)},$$

$\Lambda \subset \mathfrak{t}^*$ is the weight lattice.
Follow the way in the last section, \( h_0^0 \) induces a \( G \times T^m \)-invariant Kähler metric \( \omega_0 \) in \( c_1(M) \) with algebraic representation

\[
Z_{\omega_0} = \sum_i \mu_0 i Z_i + I_V.
\]

Now consider the equation

\[
(4.3) \quad \text{Ric}(\omega_{\varphi_t}) = t\omega_{\varphi_t} + (1 - t)\omega_0
\]

where \( \omega_{\varphi_t} = \omega_0 + \frac{1}{4\pi} \partial \bar{\partial} \varphi_t \). It is solvable for \( t \in [0, R(M)) \).

It follows from the uniqueness of the twisted Kähler-Einstein metrics that \( \{\varphi_t\} \) are \( G \times T^m \)-invariant. Moreover, \( \omega_{\varphi_t} \) can be seen as the induced Kähler metric by \( e^{-\varphi_t(F_0)} \cdot h_0^0 \). Thus

\[
Z_{\omega_{\varphi_t}} = \sum_i \mu_t i Z_i + I_V, \quad \mu_t i = \mu_0 i - \frac{1}{4\pi} \frac{\partial \varphi_t}{\partial x_i}.
\]

By [PS-I] Proposition 3.2,

\[
Z_{\text{Ric}(\omega_{\varphi_t})} = \sum_i \frac{1}{4\pi} \frac{\partial \log D}{\partial x_i} Z_i + I_V, \quad D = \det \left( -\frac{\partial \mu_t}{\partial x_j} \right)_{i,j} \cdot \prod_{\alpha \in R^+ m} \sqrt{-1} \alpha(Z_{\omega_{\varphi_t}}).
\]

Combine these equalities with (4.3), it follows that

\[
\frac{1}{4\pi} \frac{\partial \log D}{\partial x_i} = \mu_0 i + \frac{1}{4\pi} \frac{\partial \varphi_t}{\partial x_i}.
\]

Thus \( \log D + u_0 + t \varphi_t = C_t \) for some constant \( C_t \). Adjust \( \varphi_t \) by adding some constant, we have

\[
\prod_{\alpha \in R^+ m} \sqrt{-1} \alpha(Z_{\omega_{\varphi_t}}) \cdot \det(u_{t,ij}) = e^{-u_0 - t \varphi_t},
\]

where \( u_t = u_0 + \varphi_t \). Denote \( \Gamma(x, t) = \prod_{\alpha \in R^+ m} \sqrt{-1} \alpha(Z_{\omega_{\varphi_t}}) \) then

\[
\Gamma(x, t) \cdot \det(u_{t,ij}) = e^{-(1-t)u_0 - t u_t}.
\]

Since \( K^{-1}_M \) is ample, by Theorem 2, (4.11) and (3.10), we see that \( Z_{\omega_{\varphi_t}}(F_0) \subset C \).

Since the image \( Z_{\omega_{\varphi_t}}(F_0) \) does not depend on \( t \), there exists constant \( C > 0 \) such that

\[
c \leq \Gamma(x, t) \leq C
\]

for all \( x \in \mathbb{R}^m \) and \( t < R(M) \).

With this property, follow [WZ] we can obtain the following key estimates without essential modifications.

Denote \( w_t = (1 - t)u_0 + t u_t, \quad m_t = \min\{w_t(x) \mid x \in \mathbb{R}^m\} \), and suppose that the minimum is attained at \( x_t \).

**Proposition 7.** [WZ] There exists a time-independent constant \( a, C \) and \( C' \), such that

\[
|m_t| < C, \quad w_t(x) \geq a |x - x_t| - C'
\]

for all \( t < R(M) \) and \( x \in \mathbb{R}^m \).
By passing to a subsequence, we can assume
\[ \nabla \{ \text{is solvable}. \] Thus there exists a sequence
\[ k \] is independent of
\[ \Delta \] the image \( \nabla u_t(\mathbb{R}^m) \) which dose not depend on \( t \), and
\[ \rho_{DH}(y) = \prod_{\alpha \in R^+_m} \sqrt{-1} \alpha \left( \sum_i -\frac{y_i}{4\pi} Z_i + I_i \right). \]

By the divergence theorem, \( \int_{\mathbb{R}^m} \nabla w_t \cdot e^{-w_t} \, dx = 0. \) Then it follows that
\[ \frac{1}{|\Delta|} \int_{\mathbb{R}^m} \nabla u_0 \cdot e^{-w_t} \, dx = \frac{-t}{1-t} P_{\triangle} \triangleq \frac{-t}{1-t} \int_{\Delta} y \cdot \rho_{DH}(y) \, dy, \]
where \( |\Delta| = \int_{\Delta} \rho_{DH}(y) \, dy = \int_{\mathbb{R}^m} e^{-w_t} \, dx. \)

Note that \( 0 \in \triangle \), we define
\[ R_{\triangle} = \sup\{0 < t < 1 \mid \frac{-t}{1-t} P_{\triangle} \in \triangle \}. \]

Since \( \nabla u_0(x) \in \triangle \), it follows from \( (4.3) \) that \( \frac{-t}{1-t} P_{\triangle} \in \triangle \) for \( t < R(M) \). Thus
\[ R(M) \leq R_{\triangle}. \]

To prove \( R(M) = R_{\triangle} \), the following arguments are due to \cite{Li-1}.

If \( R(M) = 1 \) we are done. In the following we assume \( R(M) < 1 \).

It is shown in \cite{WZ} that if \( |x_i| \) is uniformly bounded for \( t \in [0, t_0] \), then \( R(M) \) is solvable. Thus there exists a sequence \( \{t_k\}, t_k \to R(M)^- \), such that \( |x_{tk}| \to \infty \).

By passing to a subsequence, we can assume \( \nabla u_0(x_{tk}) \to \partial \triangle \). We take an affine function \( l(y) \) such that \( l(\triangle) \geq 0 \) and \( \lim_k l(\nabla u_0(x_{tk})) = 0. \)

Denote \( dx = \frac{1}{|\Delta|} \, dx \). For any \( \epsilon > 0 \), by Proposition \( 7 \) there exists \( R_\epsilon > 0 \) which is independent of \( k \) such that
\[ \int_{\mathbb{R}^m \setminus B_{R_\epsilon}(x_{tk})} l(\nabla u_0) \cdot e^{-w_{tk}} \, dx < C \cdot \int_{\mathbb{R}^m \setminus B_{R_\epsilon}(x_{tk})} e^{-\alpha |x-x_{i(t)}|} \, dx < \epsilon. \]

On the other hand, use the explicit formula of \( u_0 \) \( (4.2) \), it is shown in \cite{Li-1} that there exists \( C > 0 \) which only depends on \( \triangle \) such that
\[ l(\nabla u_0(x)) \leq e^{CR_\epsilon} l(\nabla u_0(x_{tk})) \]
for all \( x \in B_{R_\epsilon}(x_{tk}) \).

Then by \( (4.4) \)
\[ l\left( \frac{-t_k}{1-t_k} P_{\triangle} \right) = \int_{\mathbb{R}^m} l(\nabla u_0) \cdot e^{-w_{tk}} \, dx \]
\[ = \int_{\mathbb{R}^m \setminus B_{R_\epsilon}(x_{tk})} l(\nabla u_0) \cdot e^{-w_{tk}} \, dx + \int_{B_{R_\epsilon}(x_{tk})} l(\nabla u_0) \cdot e^{-w_{tk}} \, dx \]
\[ < \epsilon + e^{CR_\epsilon} l(\nabla u_0(x_{tk})), \]
then let \( k > N \) such that \( e^{CR_\epsilon} l(\nabla u_0(x_{tk})) < \epsilon. \)

Hence \( l\left( \frac{-t_k}{1-t_k} P_{\triangle} \right) \to 0 \), this implies \( \frac{-t_k}{1-t_k} P_{\triangle} \to \partial \triangle \). Thus we have \( R(M) = R_{\triangle} \) as desired.
Finally, it is easy to see that $R_\Delta$ is same to the right hand side of (1.3).

5. Example

Take $G = SU(2)$, $K = \{\text{diag}(e^{i\theta}, e^{-i\theta}) \mid \theta \in \mathbb{R}\}$, $G^C = SL(2, \mathbb{C})$, $P = SL(2, \mathbb{C}) \cap \{\text{upper triangle matrices}\}$. Let

$$H_\alpha = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad E_\alpha = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{-\alpha} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

then $\mathfrak{h} = \mathfrak{k} = \mathbb{R} \cdot \sqrt{-1}H_\alpha$, where the root $\sqrt{-1}\alpha \in \mathfrak{h}^*$ such that $\alpha(H_\alpha) = 2$. The Killing form such that $\mathcal{B}(H_\alpha, H_\alpha) = 2$. The root decomposition is

$$\mathfrak{g}^C = \mathfrak{h}^C \oplus \mathbb{C} \cdot E_\alpha \oplus \mathbb{C} \cdot E_{-\alpha},$$

where $R = R_\mathfrak{m} = \{\pm \alpha\}$, $R_+ = \{-\alpha\}$. Thus by (2.3), $I_\mathfrak{V} = \frac{1}{2\pi}H_\alpha$. $G/K = \mathbb{C}P^1$ and $I_\mathfrak{V}$ gives the Fubini-Study metric.

Take $F = \mathbb{C}P^1$, the associated polytope $\Delta_F = [-1, 1] \subset \mathfrak{t}^*$. And $\tau : P \to \mathbb{C}^*$ be that $\tau \left( \begin{bmatrix} z & w \\ 0 & z^{-1} \end{bmatrix} \right) = z$. The toric bundle is the $\mathbb{C}P^2$ with 1-point blowup.

Identify $\mathfrak{h}^*$ with $\mathbb{R}$ via the basis $\{\sqrt{-1}H_\alpha\}$. Then $I_\mathfrak{V} = \frac{1}{2\pi}$ and $(d\tau)^* \left( \frac{1}{4\pi} \Delta_F \right) = \left[ -\frac{1}{4\pi}, \frac{1}{4\pi} \right]$, so $\Delta_M = (d\tau)^* \left( \frac{1}{4\pi} \Delta_F \right) + I_\mathfrak{V}' = \left[ \frac{1}{4\pi}, \frac{3}{4\pi} \right], \quad P = \frac{\int_{\Delta_M} x \cdot x \ dx}{\int_{\Delta_M} x \ dx} = \frac{13}{24\pi}.$

Thus $R(M) = \sup \{0 \leq t < 1 \mid \frac{1}{1-t}P + \frac{1}{1-t}I_\mathfrak{V}' \in \Delta_M \} = \frac{6}{7}.$

It coincides with the previous results in [Sz] and [Li-1].

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