Evolution kernels of skewed parton distributions: method and two-loop results.

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Abstract

We present a formalism and explicit results for two-loop flavor singlet evolution kernels of skewed parton distributions in the minimal subtraction scheme. This approach avoids explicit multiloop calculations in QCD and is based on the known pattern of conformal symmetry breaking in this scheme as well as constraints arising from the graded algebra of the \( \mathcal{N} = 1 \) super Yang-Mills theory. The conformal symmetry breaking part of the kernels is deduced from commutator relations between scale and special conformal anomalies while the symmetric piece is recovered from the next-to-leading order splitting functions and \( \mathcal{N} = 1 \) supersymmetry relations.

Keywords: evolution equation, two-loop exclusive kernels, conformal and supersymmetric constraints

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1 Introduction

A main source of information about the structure of matter is gained from reactions involving hadrons, with the theoretical description being most reliable for processes with one or more hard scales as for example the photon momentum transfer square, $Q^2$, in $ep$ reactions. The lowest approximation for a process, when only the Fock components with the minimal number of constituents in the hadron wave function at tree level are accounted for, corresponds to a simple picture of Feynman’s parton model, which neglects QCD dynamics. Once this is taken into account, it changes (sometimes drastically) the results. The specific predictions of QCD are given by scaling violations phenomena which arise due to two sources which play a complementary role depending on the magnitude of the hard momentum transfer. For moderately large $Q^2$ it is enough to include higher order perturbative corrections which scale as $\log Q^2$. On the other hand approaching the low $Q^2 \sim \text{few GeV}^2$ domain one has to take into account contributions relatively suppressed by powers of the momentum transfer $1/Q^{\tau-2}$, with $\tau$ being the twist of the operators contributing to the amplitude of the process. Obviously, in this region this latter $Q^2$ behavior overwhelms the weaker $\log Q^2$ due to perturbative evolution. However, even at rather small $Q^2$ the perturbative dependence on $Q^2$ cannot be discarded: being small at large $Q^2$ it becomes prominent at small $Q^2$ and modifies in a significant way the shape of the leading parton configuration of the hadronic wave function. Therefore, the knowledge of perturbative evolution is indispensable for the construction of the leading twist component from the genuinely non-perturbative distribution at low scales. Although this evolution can be weaker than the real non-perturbative evolution — which has to be used at very low $Q^2 \sim m^2_{\text{hadron}}$ — it gives the right direction of change and can be seen as a builder of the rough features of hadronic distributions or amplitudes. The same applies for higher twists, but these contributions are usually discarded.

In the present study we concentrate on the first source of scaling violation, namely, focus on the description of the perturbative evolution in the first two orders of the perturbation series in the QCD coupling at leading twist level, for the so-called skewed parton distributions (SPD) which arise in a number of processes such as deeply virtual Compton scattering [1, 2, 5], electroproduction of mesons [4, 5] etc. The arguments in [3, 4, 6, 7] convince us that these processes indeed factorize at this level into a perturbatively calculable hard scattering amplitude and a SPD. At twist two level, the dominant parton configuration in a hadronic wave function is that of two quarks or gluons. The main difference in the partonic description of these processes as compared to conventional deep inelastic scattering (DIS) is due to the fact that the longitudinal momentum fractions of particles propagating in the $t$-channel are different from one another and this difference is called skewedness $\eta$.

Unfortunately, a direct extraction of the SPDs from experimental data will hardly be possible
in general, since the processes are predicted as a convolution even at leading order (LO). Therefore, to deduce them one has to confront data with different models which are usually given at a much lower scale than factorization is expected to work and data are taken in experiments. Moreover, one of the central issues in the future should be to test the scaling violation phenomena in the cross section via evolution, similar to DIS. Thus, the main task is to describe evolution effects of SPDs as precise as possible. This includes the understanding of perturbative corrections at LO and beyond.

The SPD is defined as a Fourier transform to the momentum fraction space of a light-ray operator constructed from $\varphi$-parton fields and sandwiched between hadronic states non-diagonal in momenta, schematically given by

$$\phi(x, \eta, Q) = \frac{1}{2\pi} \int dz_\varepsilon e^{ixz_\varepsilon} \langle h(p')|O(z_\varepsilon, Q)|h(p)\rangle \quad \text{with} \quad O(z_\varepsilon, Q) = \varphi^\dagger(0)\varphi(z_\varepsilon)\big|_Q.$$  \hspace{1cm} (1)

Such a light-cone operator, $O(z_\varepsilon, Q)$, is a formal resummation of the usual local ones with definite twist. Its scale dependence is governed by a renormalization group equation (RGE), but with anomalous dimensions replaced by integral kernels depending on the positions on the light cone. Therefore, a SPD obeys an evolution equation with kernels given by integral transformations of the kernel in the light-cone position representation. However, the generalized skewed kinematics for the perturbative kernels can unambiguously be restored \cite{8} from the conventional exclusive one, known as Efremov-Radyushkin-Brodsky-Lepage (ER-BL) region $\eta = 1$. Throughout the paper we formally deal with ER-BL type equations \cite{9, 10}

$$\frac{d}{d\ln Q^2} \phi(x, Q) = \int_0^1 dy V(x, y|\alpha_s(Q)) \phi(y, Q),$$  \hspace{1cm} (2)

where $\phi = (\alpha, \varphi)$ is the two-dimensional vector of the quark and gluon distribution amplitudes which mix under renormalization and $V(x, y|\alpha_s)$ for vector or axial-vector distribution amplitudes is a $2 \times 2$-matrix of evolution kernels given by a series in the coupling:

$$V(x, y|\alpha_s) = \sum_{\ell=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^\ell V^{(\ell)}(x, y).$$ \hspace{1cm} (3)

Until recently, only one-loop exclusive evolution kernels were available \cite{3, 11, 12, 13, 14} and apart from the nonsinglet two-loop result \cite{15, 16, 17} nothing has been known about higher order kernels. In our previous studies we have been able to calculate the two-loop anomalous dimensions matrices of the so-called conformal operators which are in one-to-one correspondence with the given distributions and finally obtain the singlet kernels in the form of exclusive convolutions. In the present study we conclude with giving, in great detail, the formalism we have used as well as presenting explicit results for the next-to-leading (NLO) kernels. As has been said above, a
simple continuation \cite{8} allows one to easily deduce the generalized non-forward evolution kernels from the ER-BL ones.

Different methods have been offered so far to solve the nonforward evolution equation. Two of them rely on direct numerical integration \cite{18, 19} of integro-differential equations. However, the first of them was designed for the treatment of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) region only, while the second one does not resum the $\ln Q^2$ terms. Next, one can reconstruct a SPD from its expansion w.r.t. an appropriate basis of polynomials \cite{20, 21}, with the expansion coefficients expressed in terms of conformal moments, which do not mix under renormalization in the one-loop approximation. This latter method can be employed to describe the evolution for general kinematics, however, it becomes inefficient in the small $\eta$ region and in the $t \sim \eta$ domain where SPDs rapidly change their shape. Note that the accuracy of this method is under control, however, the summation of a sufficient number of polynomials is time consuming. Another possibility is to use the conformal covariance in order to map the SPD to a forward distribution \cite{24}, which has in general no physical meaning, and then solve the corresponding DGLAP evolution equations with one of the standard methods. Yet another way is to solve the evolution equation in configuration space in terms of non-local conformal operators \cite{25, 26}, however, in this case, to evolve a given function explicitly the authors have used the same method of orthogonal polynomial reconstruction as used by us in previous studies.

Beyond LO all methods except for the first two can be used in a straightforward manner only in an unrealistic conformal limit of QCD when the Gell-Man–Low $\beta$-function is set equal to zero and by making use of the conformal subtraction (CS) scheme \cite{27, 28} which removes the special conformal symmetry breaking anomaly appearing in the minimal subtraction scheme, or they can be approximately applied for small skewedness, when the conformal non-covariant piece will die out. Thus, in order to be able to describe a realistic situation, we are left with a direct numerical integration of the evolution equations or with the orthogonal polynomial reconstruction method. Since all anomalous dimensions in NLO of conformal operators have been available for some time \cite{29, 30, 31}, we studied the evolution of several models of SPDs and have gained a first insight into the magnitude of evolution effects beyond LO for both the flavor non-singlet \cite{32} and singlet sector \cite{33}. However, due to complications as mentioned above, one should develop a more efficient numerical treatment which can be achieved by direct numerical integration routines. Therefore, the corresponding evolution kernels, whose Gegenbauer moments define the anomalous dimensions \cite{29, 30, 31} mentioned earlier, are needed in two-loop approximation.

The analytical structure of these kernels is expected to be more complex than for the DGLAP ones. Therefore, a direct diagrammatic calculation in NLO is envisaged to be very cumbersome. Although, there is experience in the calculation of the ER-BL kernel in the flavor non-singlet
sector at two-loop order [15, 16, 17], no appropriate technology has been developed so far for the remaining calculations. A technical analysis of the mathematical structure, which appears to be completely analogous in all channels, is given in [17, 34]. Some particularly simple parts, i.e. renormalon chain contributions, of the non-singlet kernels were analyzed anew in Ref. [35].

The goal of this paper is to present the method we employ for the reconstruction of the evolution kernels in NLO. It is based on the analysis of general properties of the kernels and known Gegenbauer moments. Assuming the CS scheme, which implies that the (local) conformal twist-two operators do not mix under renormalization\(^1\), the anomalous dimensions are the same (up to a normalization factor) as in the forward case for DIS. This relation implies a one-to-one mapping between the evolution kernels of SPDs and the DGLAP-kernels. In the conformal limit of this special scheme, the reconstruction of the ER-BL kernels is reduced to an integral transformation from the known DGLAP-kernels [29]. The realistic situation is not as simple as that and beyond LO conformal covariance is broken in the conventional scheme, i.e. using dimensional regularization and (modified) minimal subtraction (\(\overline{\text{MS}}\)), but it presents only a slight complication as compared to the restoration of the conformal covariant part of the kernels from the splitting functions.

The outline of the paper is the following. We collect the properties of the ER-BL kernels in section 2: their support as well as relations to the DGLAP kernels, consequences of conformal constraints in QCD and conformal symmetry in \(\mathcal{N} = 1\) supersymmetric Yang-Mills (SYM) theory. We analyze the implications of these results for the structure of the flavor non-singlet ER-BL kernel in the two-loop approximation and extend it to all other channels. In section 3 we reconstruct first the whole crossed-ladder diagram contributions by means of conformal and supersymmetric constraints from the known non-singlet results, then we restore the remaining diagonal pieces making use of the known splitting functions and present the results for the evolution kernels in the chiral even sector as convolutions of simple kernels having a one-loop structure. In section 4 we perform the convolutions and present our results in an explicit form for the ER-BL type representation and extract from them the so-called skewed DGLAP kernels, which are needed for the numerical treatment of evolution in the DGLAP region. In section 5 we give a summary and conclusions. Different appendices are attached to give technical details.

\(^1\)If we neglect non-covariant terms proportional to the Gell-Mann–Low \(\beta\)-function which are induced by the trace anomaly, the conformal covariance can always be ensured by appropriate normalization conditions for the renormalized operators.
2 Structure of evolution kernels.

In the following four subsections we give a survey of general properties of the twist-two singlet evolution kernels. Their support properties are pointed out in the next subsection 2.1, which contains a generalization of the results obtained by Geyer, Robaschik, and collaborators in the past for the flavor non-singlet case \cite{1, 8, 36, 37}. Let us now give a few results which will be used later on: (i) The non-forward evolution kernels of SPDs are uniquely defined by an extension procedure from the ER-BL kernels. (ii) A simple limiting procedure reduces these generalized ER-BL kernels to the DGLAP kernels. In subsection 2.2 we demonstrate that if conformal symmetry holds true in perturbative QCD, the ER-BL kernels can be deduced from the DGLAP kernels by an integral transformation with a well-defined resolvent. We then discuss the breaking of conformal symmetry in the \( \overline{\text{MS}} \) scheme and explain the third important issue, namely, (iii) how one part of the total evolution kernel is induced by special conformal anomalies. It is shown how to evaluate these anomalies in the ER-BL representation. In subsection 2.3, we explain the use of the \( \mathcal{N} = 1 \) SYM theory as a meaningful tool for the construction of the ER-BL kernels. Based on the results in the previous three subsections, we give a method for the construction of the two-loop ER-BL kernels in subsection 2.4 and demonstrate it in the flavor non-singlet case. The main result of this section is a representation of the NLO kernels in terms of convolutions of simple kernels possessing a one-loop structure, which turns out to be valid in all channels and allows one to restore the missing information from the known DGLAP kernels.

2.1 Support properties of evolution kernels.

In this subsection we recall the support properties of evolution kernels for SPDs. Let us start with the definition of these distributions at leading twist-two level in terms of light-ray operators that are defined for the vector (V) and axial-vector (A) case by:

\[
\begin{align*}
\left\{ \mathcal{Q}^{O V}_1, \mathcal{Q}^{O A}_1 \right\}(\kappa_1, \kappa_2) &= \bar{\psi}_i(\kappa_2 n) \begin{pmatrix} \gamma_+ \\ \gamma_+ \gamma_5 \end{pmatrix} \psi_i(\kappa_1 n), \\
\left\{ \mathcal{Q}^{O V}_2, \mathcal{Q}^{O A}_2 \right\}(\kappa_1, \kappa_2) &= G^a_{\mu+}(\kappa_2 n) \begin{pmatrix} g_{\mu\nu} \\ i\epsilon_{\mu\nu}^{\mu+} \end{pmatrix} G^a_{\nu+}(\kappa_1 n),
\end{align*}
\]

where for brevity we have omitted a path ordered link factor that ensures gauge invariance. Here \( n \) and \( n^\ast \) are light-like vectors with \( nn^\ast = 1 \) and the plus and minus components of a four-vector are obtained by contraction with \( n \) and \( n^\ast \), respectively. Expansion of Eqs. (4,5) into a series of local operators immediately guarantees the contribution from the leading twist-two only. They have even chirality and even (odd) parity for the (axial-)vector case.

The SPDs are Fourier transforms w.r.t. the light-like distance between the fields of the light-ray
operators (4,5), which are sandwiched between off-diagonal hadronic states:

\[ \left\{ \frac{Q_i^\Gamma}{c_{q_i^\Gamma}} \right\}(t, \eta; \mu) = \left\{ \frac{1}{P_+} \right\} \int \frac{d\kappa}{2\pi} e^{i t P_+ (P_2 S_2)} \left\{ \frac{Q_i^\Gamma}{c_{q_i^\Gamma}} \right\}(\kappa, -\kappa)|P_1 S_1\rangle \mu \]

(6)
given at a renormalization point \( \mu \). It can be shown that the support of the SPDs is \(|t| \leq 1\). Here the conjugate variable \( t \) plays the role of a momentum fraction and the skewedness parameter \( \eta = \Delta_+/P_+ \) is defined as the + component of the momentum transfer \( \Delta = P_1 - P_2 \) normalized to the + component of \( P_1 + P_2 \), i.e. \( P_+ = n.(P_1 + P_2) \). More precisely, the momentum fractions of the incoming and outgoing partons are \( \frac{t+\eta}{1+\eta} P_+ \) and \( \frac{t-\eta}{1+\eta} P_+ \), respectively. Note that the SPDs also depend on the momentum transfer square \( \Delta^2 \), which, however, is irrelevant for the evolution and thus will be suppressed in what follows. A form factor decomposition would give us the functions \( H, E \) and \( \tilde{H}, \tilde{E} \) for the parity even and odd sectors, respectively, as introduced in ref. [2], which are governed by the same evolution equations as the distributions (6).

However, we choose the normalization\(^2\) in such a way that in the forward limit \( A_{q_i^\Gamma}(t = z) \) they coincide for \( z \geq 0 \) with the parton densities \( q_i^\Gamma(z) \) and \( z g_i^\Gamma(z) \) for \( A = Q, G \), respectively, while \( \mp Q_i^\Gamma(-z) \) with \( z \geq 0 \) are interpreted as antiquark distributions.

The evolution equations for the SPDs (6) arise from the renormalization group equations (RGEs) of the light-ray operators (4) and (5). In the following we will discuss the flavor singlet case, where the quark and gluon operators mix with each other. For brevity we introduce the two dimensional vector

\[ \mathcal{O}^\Gamma \equiv \frac{1}{2} \sum_{i=u,d,s,...} \left( \frac{Q_i^\Gamma(\kappa_1, \kappa_2) \mp Q_i^\Gamma(\kappa_2, \kappa_1)}{2 N_f G_i^\Gamma(\kappa_1, \kappa_2)} \right) \quad \text{with} \quad \mp \quad \text{for} \quad \{ \Gamma = V \Gamma = A \}
\]

(7)

and \( N_f \) is the number of active quark flavors. Note that due to Bose symmetry, the gluon operator also has definite symmetry with respect to the interchange of \( \kappa_1 \leftrightarrow \kappa_2 \), i.e. it is (anti)symmetric in the case of (axial-)vector couplings. The properties of (the non-local version of) the anomalous dimensions were intensively studied in the past on general grounds. For instance, the scaling and translation properties of the operators tell us that the general form of the RGE reads \([37]\):

\[ \mu \frac{d}{d\mu} \mathcal{O}(\kappa_1, \kappa_2) = - \int dy \int dz \gamma(y, z; \kappa_2 - \kappa_1) \mathcal{O}(\kappa_1[1 - y] + \kappa_2 y, \kappa_2[1 - z] + \kappa_1 z), \]

(8)

where the \( 2 \times 2 \) matrix valued kernel \( \gamma \) explicitly depends on the light-cone position \( \kappa_2 - \kappa_1 \) in the mixed channels:

\[ \gamma(y, z; \kappa_2 - \kappa_1) = \begin{pmatrix} Q_i^\Gamma(\kappa_2, \kappa_1) & \frac{\kappa_2 - \kappa_1}{\kappa_2 - \kappa_1} Q_i^\Gamma(\kappa_2, \kappa_1) \\ \frac{\kappa_2 - \kappa_1}{\kappa_2 - \kappa_1} G_i^\Gamma(\kappa_2, \kappa_1) & G_i^\Gamma(\kappa_2, \kappa_1) \end{pmatrix}. \]

(9)

\(^2\)For the gluon distribution we have included an extra pre-factor of 2 in comparison with the definition given in Ref. [3].
The first important issue to understand is the anatomy of the evolution kernels for SPDs, which arise from the support property of the kernels $\gamma$. This problem can be solved by means of the $\alpha$ or Feynman-parameter representation of Green functions with a non-local operator insertion. It is sufficient to work in light-cone gauge and to formally generalize the $\alpha$-representation for the gluon propagator [1]. From these studies one can deduce the support of the kernels shown in Fig. 1 (a):

$$\gamma(y, z; \kappa_2 - \kappa_1) \neq 0, \text{ for } 0 \leq y, z \leq 1; \ 0, \text{ otherwise.}$$

(10)

Invariance under charge conjugation implies the following symmetry relation

$$\gamma(y, z; \kappa_2 - \kappa_1) = \gamma(z, y; \kappa_2 - \kappa_1).$$

(11)

It is also worth noting that the symmetry properties of the flavor singlet operators w.r.t. the interchange of their light cone arguments, i.e. $\kappa_1 \leftrightarrow \kappa_2$, can be used to map the region $y + z \geq 1$ into $1 \geq y + z$ by the substitution $y \to 1 - z$ and $z \to 1 - y$. Here the region $1 \geq y + z$ corresponds in the forward case to quark-quark mixing as it is the case in LO, while $y + z \geq 1$ appears due to a quark-antiquark interaction.

From the definition of a SPD (6) and the RGE (8) we easily derive the evolution equation:

$$\mu \frac{d}{d\mu} q(t, \eta; \mu) = - \int dt' \gamma(t, t'; \eta; \alpha_s(\mu)) q(t', \eta; \mu).$$

(12)

Here we use a definition of the singlet distributions, which is motivated by the one of normal parton densities,

$$q^\Gamma(t, \eta; \mu) \equiv \sum_{i=u,d,s,\ldots} \left\{ \frac{Q_i^\Gamma(t, \eta; \mu) \mp Q_i^\Gamma(-t, \eta; \mu)}{N_f G} \right\} \text{ with } \mp \text{ for } \begin{cases} \Gamma = V, \\ \Gamma = A, \end{cases}$$

(13)

where the quark sector contains the sum of “parton” and “anti-parton” distributions. It is a simple exercise to see that the kernels for the SPDs are obtained by the integration

$$\gamma(t, t'; \eta) = \int_0^1 dy \int_0^1 dz \left( \frac{Q Q_i^\gamma(y, z)}{G Q_i^\gamma(y, z)} \frac{Q Q_i^\gamma(y, z)}{G Q_i^\gamma(y, z)} d_i \right) \delta(t - t'(1 - y - z) - (y - z)\eta),$$

(14)

where $d_i \equiv d/dt$ and $d_i^{-1} \equiv f^t dt$. Here the indefinite integration limits in the $GQ$ channel induces an ambiguity which affects the unphysical moments only and has to be fixed by hand, e.g. by comparison of moments calculated in both representations. Note that this ambiguity implicitly appears also in the diagrammatic calculation of Feynman diagrams in the light-cone fraction representation and is responsible for different results given in the literature. It is worth mentioning that the representation [14] implies a simple scaling relation:

$$\gamma(t, t'; \eta) = \frac{1}{|\eta|} \left( \frac{Q Q_i^\gamma(t', \eta)}{G Q_i^\gamma(t', \eta)} \eta^{-1} \frac{Q Q_i^\gamma(t, \eta)}{G Q_i^\gamma(t, \eta)} \right),$$

(15)
so that the entries are in fact two-variable functions of ratios.

The invariance under charge conjugation implies now the symmetry for diagonal $A^A\gamma(t,t') = A^A\gamma(-t,-t')$ and off-diagonal $A \neq B$, $A^B\gamma(t,t') = -A^B\gamma(-t,-t')$ elements, while the connection between the parton-parton and parton-antiparton regions makes itself apparent in the substitution $t' \rightarrow -t'$. As explained, for the flavor singlet case it is enough to consider the region $0 \leq 1 - y - z$ of the support $0 \leq y, z \leq 1$. Then the integral representation (14) implies the support shown in Fig. 1 (b), or formally

$$
\gamma(t,t') = \Theta(t,t') f(t,t') \pm \{ t \rightarrow -t, t' \rightarrow -t' \},
$$

where $(-)^+$ stands for (off-) diagonal entries. The entries of the matrix $f(t,t')$ are given by

$$
A^A f(t,t') = \int_0^{\frac{1+t}{1+t'}} dw A^A \gamma \left( y = \frac{1+t-(1+t')w}{2}, z = \frac{1-t-(1-t')w}{2} \right),
$$

$$
A^B f(t,t') = \left\{ \frac{d_t}{d_t^{-1}} \right\} \int_0^{\frac{1+t}{1+t'}} dw A^B \gamma \left( y = \frac{1+t-(1+t')w}{2}, z = \frac{1-t-(1-t')w}{2} \right),
$$

with $d_t (d_t^{-1})$ corresponding to $QG$ $(GQ)$.

Figure 1: Support of the singlet anomalous dimensions $\gamma$ in light-cone position (a) and fraction (b) representation. In (b) we only show the support which arises from $1 - y - z \geq 0$ in the light-cone position representation. A second contribution that comes from the region $1 - y - z \leq 0$ can be formally obtained by $t' \rightarrow -t'$, however, in the flavor singlet case it can be reduced to the first one by means of symmetry. Here the entries of $f_{\pm \pm} = f(\pm t, \pm t')$ are defined by Eqs. (17) and (18).

A first glance at Fig. illustrates the impression that the whole kernel can be obtained from the region $|t|, |t'| \leq 1$. Indeed, it was proved in [1] that the continuation is unique (see
appendix A). For practical purposes it is sufficient to replace the $\theta$ structure:

$$\theta(t-t')|_{|t|,|t'|\leq 1} \rightarrow \Theta(t, t').$$  \hfill (19)

Thus, the evolution kernel for the corresponding SPD can be considered as a generalized ER-BL kernel and its restoration from a given ER-BL kernel is simple.

There are further consequences arising from the evolution equation (12). If we replace in the definition (6) for SPDs the off-diagonal hadronic states by diagonal ones, we immediately obtain the definition of the usual forward parton distributions (up to an additional $1/z$ for the gluon density) with $t = z$ and $\eta = 0$. Comparing the evolution equations, the DGLAP kernel appears as a limit of the generalized ER-BL kernel:

$$P(z) = -\lim_{\eta \to 0} \frac{1}{|\eta|} \left( \frac{QQ_y^\gamma}{z} \frac{QG_y^\gamma}{1} \right)^{ext} \left( \frac{z}{\eta}, \frac{1}{\eta} \right).$$  \hfill (20)

An alternative derivation of this limit using moments is presented in appendix B.

As we have just shown, it is sufficient for the following considerations to work in the exclusive kinematics. Thus, from now on we will deal only with the momentum fraction $x = (1 + t)/2$ and the common definition of ER-BL type kernels:

$$V(x, y) = -\gamma(x - \bar{x}, y - \bar{y})|_{0 \leq x, y \leq 1},$$  \hfill (21)

here and everywhere else we use $\bar{x} \equiv 1 - x$. The general structure of the entries is

$$^{AB}v(x, y) = \theta(y - x)^{AB} f(x, y) + \theta(\bar{y} - x)^{AB} g(x, y) \pm \left\{ \begin{array}{l} x \to \bar{x} \\ y \to \bar{y} \end{array} \right. \text{ for } \left\{ \begin{array}{l} A = B \\ A \neq B \end{array} \right..$$  \hfill (22)

As discussed above, the second $\theta$-structure, i.e. $\theta(\bar{y} - x)$, in the singlet case, can be removed by the (anti-) symmetry of the parton distributions w.r.t. $y \to \bar{y}$. Again, the extension of the kernel in the whole region is done by a simple replacement of $\theta$ functions, e.g.

$$\theta(y - x) \rightarrow \theta \left( 1 - \frac{x}{y} \right) \theta \left( \frac{x}{y} \right) \text{sign}(y).$$  \hfill (23)

### 2.2 Conformal properties of evolution kernels.

To understand the classification of different contributions to the evolution kernel $V$ with respect to the conformal transformation most clearly, we deal in this subsection with the so-called conformal operators $^{t}O_{jl}^{\gamma}$ and their anomalous dimension matrix $\gamma$. These operators build an infinite dimensional, irreducible representation of the collinear conformal algebra $so(2, 1)$ in the space spanned by the bilinear field operators. This algebra arises from the full conformal algebra $so(4, 2)$ by projection onto the light cone and consists of the generators of dilatation $D$, special conformal transformation $K_\pm$, boost along the light-cone $M_{\pm}$, and translation $P_\pm$. Conformal operators

\footnote{The special conformal transformation is given by the product $R\mathcal{P}_c R$, where the inversion $R$ acts as $Rx_\mu = x_\mu/x^2$ and $\mathcal{P}_c$ is a translation with a vector $c$.}
are generated from the light-ray operators (31) and (32) by differentiation w.r.t. $\kappa_1, \kappa_2$:

$$A^{\Gamma}_{jj} = \left( i \partial_{\kappa_1} + i \partial_{\kappa_2} \right)^{j+\nu(A)-3/2} C_j^{\nu(A)} \left( \frac{\partial_{\kappa_1} - \partial_{\kappa_2}}{\partial_{\kappa_1} + \partial_{\kappa_2}} \right) A^{\Gamma}(\kappa_1, \kappa_2)|_{\kappa_1=\kappa_2=0},$$  

where $\nu(Q) = 3/2$ and $\nu(G) = 5/2$. The index of the Gegenbauer polynomial $C^\nu_j$ is determined by group theory: $2\nu(A) = 2d(A) + 2s(A) - 1$, where $d(A)$ and $s(A)$ are the canonical dimension and the spin of the field of species $A$. The operators in Eq. (24) are the highest weight vectors $K_- A^{\Gamma}_{jj} = 0$ and they carry the conformal spin $j + 1$ and the angular momentum $l + 1$. Acting with the generator of translation, i.e. applying $P^{(j-l)}_+$ on $A^{\Gamma}_{jj}$, we generate the whole conformal tower of operators. In LO all members of a tower do not mix under renormalization, i.e. they have the same anomalous dimension $\gamma_j$. This is a consequence of classical conformal symmetry and arises from the commutator constraints between the generators of dilatation $D$ and special conformal transformation $K_-$: $[D, K_-] = iK_-$, where as we have established above $K_-$ acts in the conformal tower as a step down operator.

Scale symmetry is known to be broken by the trace anomaly in the energy momentum tensor [38, 39, 40] and is proportional to the Gell-Mann–Low $\beta$-function. For the Green functions involving only local field operators at different space-time points there is a one-to-one correspondence between breaking of special conformal and scale symmetries [41]. However, as we will discuss in detail below, the renormalization of the Green functions with composite operator insertions there is an extra source of breaking of the special conformal covariance besides terms proportional to $\beta$. Poincaré invariance implies that the mixing matrix is triangular, thus, the RGE has the general form

$$\mu \frac{d}{d\mu} \mathcal{O}_{jl} = - \sum_{k=0}^{j} \gamma_{jk} \mathcal{O}_{kl}, \quad \text{with} \quad \mathcal{O}_{jl} = \sum_{i=u,d,s,...} \left\{ \frac{\mathcal{O}_{ij}^{\Gamma}}{N_f} \mathcal{O}_{jl} \right\}.$$  

(25)

To make contact with the (generalized) ER-BL kernel, we use the fact that the conformal moments of the SPDs (3) are given by the expectation values of conformal operators (24). For the flavor singlet case [see Eq. (13)] we have

$$\eta^{j+\nu(A)-3/2} \int_{-1}^{1} dt C^{\nu(A)}_{j+\nu(A)-3/2} \left( \frac{t}{\eta} \right) A^{\Gamma}(t, \eta; \mu) = \left( \frac{2}{P^+} \right)^{j+1} \langle P_2, S_2 | A^{\Gamma}_{jj} | P_1, S_1 \rangle_{\mu}.$$  

(26)

Moreover, since the Gegenbauer polynomials form a complete basis in the region $[-1, 1]$, we can represent the kernel in the ER-BL region by the following sum

$$A^{B}_{ij}(x, y) = \sum_{j=\nu(A)-3/2}^{\infty} \sum_{k=\nu(B)-3/2}^{j} \frac{w(x|\nu(A))}{N_j(\nu(A))} C^{\nu(A)}_{j+3/2-\nu(A)}(x - \bar{x}) A^{B}_{jk} C^{\nu(B)}_{k+3/2-\nu(B)}(y - \bar{y}),$$  

(27)

where $w(x|\nu) = (x\bar{x})^{\nu-1/2}$ is the weight function and $N_j(\nu) = 2^{-4\nu+1} \Gamma^{2}(\nu) \Gamma(2\nu+j) \Gamma(\nu+j+\frac{1}{2})$ is the normalization coefficient. Up to an overall normalization, the conformal moments are identical with the
anomalous dimensions, i.e. \( AB_{jk}^i = -AB_{jk}^i / 2 \). Note that invariance under charge conjugation implies \( AB_{jk}^i = 0 \) for odd \( j - k \).

Fortunately, the additional special conformal symmetry breaking alluded to above is scheme dependent and as mentioned in the introduction it can be avoided in a special CS scheme. In such a CS scheme and in the formal conformal limit of QCD, when we set the Gell-Mann–Low function to zero, conformal operators do not mix with each other. However, once the \( \beta \)-function is kept non-zero the RGE to all orders of perturbation theory is now

\[
\frac{d}{d\mu} \Delta_{ij} = -\gamma_{ij} \Delta_{ij} - \frac{\beta}{g} \sum_{k=0}^{j-2} \Delta_{jk} \Delta_{kl}.
\]

Thus, in the CS scheme and up to a term proportional to the \( \beta \)-function the ER-BL kernel is diagonal w.r.t. Gegenbauer polynomials:

\[
AB_{jk}^i(x, y) = \sum_{j=0}^{\infty} \frac{w(x|\nu(A))}{N_j(\nu(A))} C_{j+3/2-\nu(A)}^{\nu(A)}(x - \bar{x}) AB_{jk}^i C_{j+3/2-\nu(B)}^{\nu(B)}(y - \bar{y}),
\]

where the sum starts at \( j = 0 \) for \( AB = QQ \) in the parity even case and \( j = 1 \) otherwise. The eigenvalues \( AB_{jk}^i \) are given by the diagonal entries, i.e. \( AB_{j} = AB_{j,j}^i \). From the representation \( (29) \) it necessarily follows that the diagonal entries have the symmetry properties

\[
(y\bar{y})^{\nu(A)-1/2} AA_{ij}^D(x, y) = (x\bar{x})^{\nu(A)-1/2} AA_{ij}^D(y, x).
\]

Taking into account the equalities among Gegenbauer polynomials with shifted index \( \nu \),

\[
\frac{d}{dx} C_{j}^{5/2}(x - \bar{x}) = 6C_{j-1}^{5/2}(x - \bar{x}), \quad \frac{d}{dx} C_{j-1}^{5/2}(x - \bar{x}) = 6C_{j}^{5/2}(x - \bar{x}),
\]

we can easily derive necessary conditions for the kernels in the mixed channels, too:

\[
y\bar{y} \frac{\partial}{\partial x} QG_{ij}^{D}(x, y) = x\bar{x} \frac{\partial}{\partial y} QG_{ij}^{D}(y, x), \quad \frac{\partial}{\partial x} QG_{ij}^{D}(x, y) = \frac{\partial}{\partial y} QG_{ij}^{D}(y, x).
\]

Since the matrix of conformal moments \( \mathbf{v}_j \) are given by the forward anomalous dimensions known from DIS up to NLO,

\[
\mathbf{v}_j = -\frac{1}{2} \begin{pmatrix} QQ \gamma_j & GQ \gamma_j \\ GQ \gamma_j & GG \gamma_j \end{pmatrix},
\]

we may reconstruct the kernels from these information with the help of Eq. \( (31) \). This representation as an infinite sum can be converted into an integral by means of the generating function

\[
G(x, y; z|\nu) = \sum_{j=0}^{\infty} \frac{w(x|\nu)}{N_j(\nu)} C_{j}^{\nu}(x - \bar{x})z^j C_{j}^{\nu}(y - \bar{y})
\]

\[
= \frac{\Gamma(\nu)\Gamma(\nu + 1)}{\Gamma^2(\nu + 1)} \frac{2^{4\nu-1}(x\bar{x})^{\nu-1/2}(1 - z^2)}{[1 - 2((x - \bar{x})(y - \bar{y}) - 4\sqrt{x\bar{x}y\bar{y}})z + z^2]^\nu+1} \times _2F_1\left(\nu + 1, \nu \bigg| \frac{16\sqrt{x\bar{x}y\bar{y}}}{1 - 2((x - \bar{x})(y - \bar{y}) - 4\sqrt{x\bar{x}y\bar{y}})z + z^2}\right).
\]
which was obtained by Gegenbauer’s summation theorem \[42\]. For completeness we give here a simple reduction formula that is obtained from $G(x, y; z|3/2)$ together with Eq. (31):

$$V_{D}(x, y) = \int_{0}^{1} dz \left( -\frac{1}{6} \int_{-1}^{x} dx QG V \frac{\partial}{\partial y} QG V - \frac{1}{36} \int_{-1}^{x} dx \frac{\partial}{\partial y} GG V \right)(z) G(x, y; z|3/2),$$  \tag{35}

where $ABV(z)$ are related to the DGLAP kernels, however, defined as a Mellin transformation of conformal moments:

$$V(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dj v_j z^{-j+1}.$$  \tag{36}

Note that in the $GQ$ and $GG$ channel the first term of the expansion w.r.t. Gegenbauer polynomials does not exist and must be removed by hand. A different version of the reduction formula avoiding this problem has been applied at LO for the restoration of the ER-BL kernels from the DGLAP ones \[29\]. The evaluation of the integrals has been reduced by the Cauchy theorem to the evaluation of residues of simple poles, therefore, it was essential that the one-loop DGLAP kernels do not generate cuts in the complex plane. Beyond LO this property is lost and the derived reduction formula (35) is extremely hard to handle.

To calculate the complete ER-BL kernel in any scheme, e.g. in the \text{MS} scheme with dimensional regularization, we have to take into account the breaking of conformal covariance induced by the conformal anomaly in the action\[4\]. There are different ways to account for this breaking:

- An explicit calculation of the anomalous dimensions or kernels. However, depending on the representation, the calculation and/or extraction of the conformal non-covariant piece is extremely difficult or at least not straightforward.

- Taking normalization conditions that ensure the conformal covariance of the renormalized operator w.r.t. the special conformal transformation for vanishing Gell-Man–Low function. For instance, doing so in LO, the anomalous dimensions are diagonal in NLO and the result in the \text{MS} scheme can be obtained by a finite renormalization extracted in LO.

- One can analyze the conformal symmetry breaking with the help of conformal Ward identities (CWI) and derive constraints for the appearing anomalies which in turn will fix the piece we are interested in. This approach provides us also information on terms proportional to the Gell-Man–Low function.

\[4\]Beside the trace anomaly there also appear equation of motion terms and exact BRST operator insertions. In $4 - 2\epsilon$ dimensions the trace anomaly is proportional to $\beta_\epsilon(\epsilon, g) = -g\epsilon + \beta(g)$ and we have to renormalize the product of trace anomaly and composite operators. Thus, this procedure together with the $\epsilon$ dependent term causes anomalous contributions, e.g. anomalous dimensions in the case of dilatation.
All of the approaches sketched above are equivalent in the formal conformal limit. In order to reduce the effort in obtaining the desired result we prefer to work with the third one. Moreover, it enables us to get a deeper insight into the structure of conformal symmetry breaking counterterms. It offers as well the technical tools to get the off-diagonal anomalous dimensions and the corresponding kernels in \( n^{\text{th}} \) order by calculating Feynman graphs at \((n - 1)^{\text{th}} \) order. For the reader’s convenience we quickly outline this approach. The exact technical steps are published in great detail elsewhere [28, 29, 30, 31, 43].

The major steps consist of:

(i) Derivation of conformal Ward identities for the Green functions with conformal operator insertion

\[
G_{jl}(x_1, x_2, \ldots, x_N) = \langle O_{jl}(0)\phi(x_1)\phi(x_2)\ldots\phi(x_N) \rangle
\]  

(37)

by means of the path integral using dimensional regularization. The renormalization of the Ward identities provides us with a prescription for the calculation of the dilatation and the special conformal anomalies. Here we present these Ward identities in a simplified form required just to demonstrate the way conformal anomalies appear

\[
\sum_{j=1}^{N} \left( -x_{\mu_j} \frac{\partial}{\partial x_{\mu_j}} - d_{\phi_j} \right) G_{jl}(x_1, x_2, \ldots, x_N) = \sum_{k=0}^{j} \{ (l + 3) \mathbf{1} + \gamma \}_{jk} G_{kl}(x_1, x_2, \ldots, x_N) + \ldots, 
\]

(38)

\[
\sum_{j=1}^{N} \eta_{\mu_j}^* \left( -2x_{\mu_j}x_{\nu_j} \frac{\partial}{\partial x_{\nu_j}} + x_{\nu_j}^2 \frac{\partial}{\partial x_{\mu_j}} - 2(d_{\phi_j} + s_{\phi_j})x_{\mu_j} \right) G_{jl}(x_1, x_2, \ldots, x_N) = \sum_{k=0}^{j} \{ a(j, l) \mathbf{1} + \gamma^c(l) \}_{jk} G_{kl}(x_1, x_2, \ldots, x_N) + \ldots, 
\]

(39)

where the anomalous dimensions matrix \( \gamma_{jk} \) and the so-called special conformal anomaly matrix \( \gamma^c_{jk} \) are induced by quantum fluctuations. We introduced the conventions \( a(j, l) = 2(j - l)(j + l + 3) \) for the coefficient which appears at tree level in the transformation of a conformal operator under special conformal transformations. The dots in Eqs. (38,39) stand for terms involving Green functions with the renormalized product of a conformal operator and a conformal variation of the action \( \langle [O_{jl}\delta S]\phi(x_1)\phi(x_2)\ldots\phi(x_N) \rangle \).

(ii) As a next step we derive matrix constraints for the conformal anomalies and find their solution. It turns out that there are two commutator relations stemming from the algebra of the collinear conformal group which provide non-trivial relations between the conformal anomalies. For vanishing Gell-Mann–Low function (complete constraints) we have

\[
[D, K_-]_\_ = iK_- \quad \Rightarrow \quad [a(l) + \gamma^c(l), \gamma]_\_ = 0, 
\]

(40)

\[
[P_+, K_-]_\_ = 2i(D + \mathcal{M}_{++}) \quad \Rightarrow \quad \gamma^c(l + 1) - \gamma^c(l) = -2\gamma. 
\]

(41)
Since the matrix $a$ is diagonal, $a_{jk} = a(j, l)\delta_{jk}$, the constraint (11) tells us that the off-diagonal matrix elements of the anomalous dimension $\gamma$ are indeed induced by the special conformal anomaly matrix $\gamma^c(l)$, while the spin dependence of $\gamma^c(l)$, induced by the breaking of Poincaré invariance due to the special conformal transformation, is governed by Eq. (11). Note that the diagonal form of the special conformal anomaly $\gamma^c(l)$ is necessary and sufficient for the vanishing of the off-diagonal part of the anomalous dimension matrix $\gamma$. This ensures the existence of the CS scheme, in which the conformal covariance of the operators holds true in the conformal limit, $\beta = 0$, at any order of perturbation theory [27].

The extension of these constraints (11) to nonvanishing $\beta$-function is straightforward, however, they require additional algebra [29, 30, 31], which results in the following change of Eq. (11)

$$\left[a(l) + \gamma^c(l) + 2\frac{\beta}{g}b(l), \gamma\right] = 0,$$

(42)

where $b_{jk}(l) = \theta_{jk} \left\{ 2(l + k + 3)\delta_{jk} - \left[ 1 + (-1)^{j-k} \right](2k + 3) \right\} 1$. Decomposing the anomalous dimension matrix in its diagonal (D) and off-diagonal (ND) part, the solution of this constraint can be constructed by successive approximations:

$$\gamma^{ND}(g) = -\frac{g}{1+g} \gamma^{D}(g) = -g \gamma^{D}(g) + \cdots,$$

with $g A_{jk} = \left[ \gamma^c(l) + 2\frac{\beta}{g}b(l), A \right]_{jk}$. (43)

(iii) The last step consists of the explicit evaluation of the anomalies. The LO anomalous dimensions of conformal operators have been known for a long time [12, 22, 23]:

$$QQ\gamma_j^{(0)} = -C_F \left( 3 + \frac{2}{(j+1)(j+2)} - 4\psi(j+2) + 4\psi(1) \right)$$

(44)

$$QG\gamma_j^{(0)} = \frac{-24N_f T_F}{j(j+1)(j+2)(j+3)} \times \left\{ \begin{array}{c} j^2 + 3j + 4, \\
j(j+3), \end{array} \right\}$$

(45)

$$GQ\gamma_j^{(0)} = \frac{-C_F}{3(j+1)(j+2)} \times \left\{ \begin{array}{c} j^2 + 3j + 4, \\
j(j+3), \end{array} \right\}$$

(46)

$$GG\gamma_j^{(0)} = -C_A \left( -4\psi(j+2) + 4\psi(1) - \frac{\beta_0}{C_A} \right)$$

$$- \frac{8C_A}{j(j+1)(j+2)(j+3)} \times \left\{ \begin{array}{c} j^2 + 3j + 3, \\
j(j+3), \end{array} \right\},$$

(47)

where the upper (lower) row corresponds to even (odd) parity and $\beta_0 = \frac{4}{3}T_f N_F - \frac{11}{3}C_A$. For the special conformal anomalies we found at one-loop level [29, 30, 31]:

$$\gamma^{c(0)} = -b\gamma^{(0)} + w.$$  \hspace{1cm} (48)

\footnote{We use the following definition of the step-function: $\theta_{jk} = \{1, \text{ for } j - k \geq 0; 0, \text{ for } j - k < 0\}$.}
The first term on the r.h.s. appears as a result of the one-loop renormalization of a conformal operator in a subgraph and we observe that it is induced by the breaking of scale invariance. For the Green functions constructed from local field operators this exhausts the sources of symmetry breaking, however, for the case at hand it is no longer true and we have an addendum \(w\). The \(w\) matrix contains new information from the renormalization of the operator product of conformal operators and the conformal variation of the action in \(4 - 2\epsilon\) dimensions. These renormalization constants are calculated by means of modified Feynman rules. It turns out that the \(w\) matrix is universal for vector and axial-vector operators and its matrix elements read

\[
\begin{align*}
Q^Q_{ij} & = -2C_F \left[ 1 + (-1)^{j-k} \right] \theta_{j-2,k}(3 + 2k) \\
& \times \left\{ 2A_{ij} + (A_{ij} - \psi(j + 2) + \psi(1)) \left( \frac{j - k}{k + 1}(k + 3) \right) \right\}, \\
Q^G_{ij} & = 0, \\
G^Q_{ij} & = -2C_F \left[ 1 + (-1)^{j-k} \right] \theta_{j-2,k}(3 + 2k) \left( \frac{1}{6} \frac{j - k}{k + 1}(k + 3) \right), \\
G^G_{ij} & = -2C_A \left[ 1 + (-1)^{j-k} \right] \theta_{j-2,k}(3 + 2k) \\
& \times \left\{ 2A_{ij} + (A_{ij} - \psi(j + 2) + \psi(1)) \left[ \frac{\Gamma(j + 4)\Gamma(k)}{\Gamma(j)\Gamma(k + 4)} - 1 \right] + 2(j - k)(j + k + 3) \frac{\Gamma(k)}{\Gamma(k + 4)} \right\}.
\end{align*}
\]

The elements of the matrix \(A\) are rather complicated

\[
A_{ij} = \psi \left( \frac{j + k + 4}{2} \right) - \psi \left( \frac{j - k}{2} \right) + 2\psi(j - k) - \psi(j + 2) - \psi(1).
\]

It is instructive to note that the special conformal anomalies \(\alpha_{c(0)}\) obey certain constraints which originate from the anomalous superconformal WI and a commutator of the generators of the special conformal transformation and restricted supersymmetry, and imply that one can restore all entries in Eqs. (49)-(52) from the knowledge of e.g. \(Q^Q_{ij}(0)\) and superconformal anomalies \([44]\).

The form of the special conformal anomaly \(\alpha_{c(0)}\) implies the following structure for the off-diagonal part of the anomalous dimension matrix

\[
\gamma^{\text{ND}(0)} = \left[ \gamma^{(0)}, \left( \beta_0 \mathbf{1} - \gamma^{(0)} \right) \mathbf{d} + \mathbf{g} \right],
\]

where we defined the matrices \(\mathbf{d}_{ij} = b_{ij}/a(j, k)\) and \(\mathbf{g}_{ij} = w_{ij}/a(j, k)\). This is the final result of the conformal approach which give together with the known anomalous dimensions from DIS, see for instance \([45, 46]\), the complete anomalous dimension matrix to NLO in the \(\overline{\text{MS}}\) scheme relevant to non-forward processes.

As was explained in the introduction it is paramount to know the ER-BL kernels. To obtain them, one can transform, in a straightforward way, the conformal anomalies into the momentum
fraction representation. For instance, for the $QQ$ channel this kernel reads

$$\frac{QQw(x,y)}{\gamma_{\nu}} = \left[QQw(x,y) - \delta(x-y) \int_{0}^{1} dz \theta(z-y) + d \frac{dz}{dz} \delta(x-y) \int_{0}^{1} dz (z-y)QQw(z,y), \right.$$  
$$\left. QQw(x,y) = -C_{F} \theta(y-x) \left( \frac{2}{y(x-y)^{2}} \right) + \left\{ x \rightarrow 1-x, \hspace{0.5cm} y \rightarrow 1-y \right\}. \right.$$  
(55)

In the next step we construct the $QQg(x,y)$ kernel with its conformal moments satisfying the equality $2[(j+1)(j+2) - (k+1)(k+2)]QQg_{jk} = QQw_{jk}$. Using the eigenvalue equation for Gegenbauer polynomials,

$$\frac{d^{2}}{dx^{2}} \left[ x^{\frac{3}{2}} C_{j}^{3/2}(x-x) \right] = -(j+1)(j+2)C_{j}^{3/2}(x-x),$$  
(56)

and the representation (27), we can easily write down a second order differential equation for the $QQg(x,y)$ kernel

$$x \frac{\partial^{2}}{\partial x^{2}} QQg(x,y) - \frac{\partial^{2}}{\partial y^{2}} \left[ y^{2} QQg(x,y) \right] = -\frac{1}{2} QQw(x,y), \hspace{0.5cm} \text{for} \hspace{0.5cm} x \neq y. \hspace{0.5cm} (57)$$

The solution of the homogeneous equation is purely diagonal w.r.t. Gegenbauer polynomials, thus, we find from Eq. (57) the $QQg$ kernel, which contains beside the desired off-diagonal also (arbitrary) diagonal conformal moments. Similarly one constructs the other channels. Our results for the whole singlet sector reads

$$g(x,y) = \theta(y-x) \left( -C_{F} \left[ \frac{\ln(1-x)}{y-x} \right] + \frac{0}{C_{F} x/y} \right) \pm \left\{ x \rightarrow \bar{x}, \hspace{0.5cm} y \rightarrow \bar{y} \right\}, \hspace{0.5cm} (58)$$

with the $(-)+$ sign corresponding to (non-) diagonal entries and the “+-”-prescription defined as $[V(x,y)]_{+} = V(x,y) - \delta(x-y) \int_{0}^{1} dz V(z,y) + \text{const} \cdot \delta(x-y)$, where the constant term is fixed in appendix B.

Now we come to the dotted kernels $\hat{v}(x,y)$ which posses the conformal moments $[\gamma^{(0)}, d]_{-}$. The matrix $d_{jk}$ can be generated by a derivative w.r.t. the index of Gegenbauer polynomials

$$\frac{d}{d\nu} C_{j}^{\nu}(t)_{|\nu=3/2} = -2 \sum_{k=0}^{j} d_{jk} C_{k}^{3/2}(t), \hspace{0.5cm} \frac{d}{d\nu} C_{j-1}^{\nu}(t)_{|\nu=5/2} = -2 \sum_{k=1}^{j} d_{jk} C_{k-1}^{5/2}(t), \hspace{0.5cm} (59)$$

and analogous relations, but with a different sign and range of summation on the r.h.s., follow immediately from the orthogonality relation, for the function $w(x|\nu) C_{j}^{\nu}(x-x) / N_{j}(\nu)$. Thus, it is obvious that the dotted kernels can be generated from

$$AB_{\nu}^{i}(x,y) = \sum_{j=0,1}^{\infty} \frac{w(x|\nu A) + \nu}{N_{j}(\nu A) + \nu} C_{j+3/2-\nu A}(x-x) \frac{AB_{\nu}^{i}(x-x) + \nu}{N_{j+3/2-\nu B}(y-y) + \nu} \hspace{0.5cm} (60)$$
by differentiation w.r.t. the parameter $\epsilon$ at $\epsilon = 0$. Of course, for diagonal entries the generating kernel $AA^i(x, y | 0)$ is symmetric w.r.t. the weight function $AAw(x | \nu)$ [see relation (30)]. Thus Eq. (31) provides essentially a logarithmic modification of the LO kernel

$$AB^i(x, y) = \theta(y - x)ABf^i(x, y) \pm \left\{ x \rightarrow \bar{x} \atop y \rightarrow \bar{y} \right\} \quad \text{for} \quad \begin{cases} A = B \\ A \neq B. \end{cases} \quad (61)$$

However, using the symmetry relation (32) to get the mixed channels one has to fix the integration constant appropriately. Thus, the generic form of the dotted kernel is

$$AB^i(x, y) = \theta(y - x)ABf^i(x, y) \ln \frac{x}{y} + \Delta^{AB^i}(x, y) \pm \left\{ x \rightarrow \bar{x} \atop y \rightarrow \bar{y} \right\} \quad \text{for} \quad \begin{cases} A = B \\ A \neq B. \end{cases} \quad (62)$$

To obtain the addendum $\Delta^{AB^i}$, it is necessary to have a closer look at the structure of LO kernels, which will be given in the next subsection.

The factorized structure of $NLO$ off-diagonal anomalous dimensions is transferred to the momentum fraction kernels and can be constructed out of the conformal anomalies we have just found. In convolution form they read

$$V^{ND(1)}(x, y) = - (I - D) \left\{ \hat{V} \otimes \left( V^{(0)} + \frac{\beta_0}{2} I \right) + \left[ g \otimes V^{(0)} \right] \right\} (x, y), \quad (63)$$

where $\tau_1 \otimes \tau_2(x, y) \equiv \int_0^1 dz \tau_1(x, z)\tau_2(z, y)$ defines the exclusive convolution and $(I - D)$ projects out the diagonal part.

### 2.3 $\mathcal{N} = 1$ supersymmetric constraints.

Since the two-loop off-diagonal kernels (63) are available we are left with the construction of their diagonal counterparts. Unfortunately, the direct use of the integral transformation (35) with the generating function given in Eq. (34) is difficult due to the complicated analytical structure of the integrand in the complex plane: cuts appear on top of poles. Therefore, we have to look for complimentary sources of restrictions on the form of the NLO kernels. Sufficient information comes from the constraints deduced from the graded commutator algebra of $\mathcal{N} = 1$ SYM theory [14, 50].

There is a one-to-one correspondence between QCD and SYM Lagrangians provided one identifies quarks with gluinos and put the former in the adjoint representation of the colour group. We can map thus the QCD result to SYM theory by equating the colour factors: $C_F = 2T_F = C_A \equiv N_c$.

The main use of the previously mentioned constraints will be in the reconstruction of the crossed ladder diagram contributions which have the most complicated analytical structure. For this type of Feynman graphs the restoration of colour factors is unique.

Since the conformal symmetry breaking part has been previously fixed, we can legitimately assume in what follows that conformal covariance holds for the anomalous dimensions. In addition
to this, we have to use the renormalization procedure preserving supersymmetry. In reality none of these statements hold true in the $\overline{\text{MS}}$ scheme beyond LO. From the commutator of the dilatation and restricted supersymmetry generators $[Q,D]_- = \frac{i}{2} Q$ applied to the Green function (B7) one finds six constraints for eight anomalous dimensions in the chiral even case

$$QQ_{\gamma j}^i + \frac{6}{j} GQ_{\gamma j}^i = \frac{j}{6} QQ_{\gamma j}^i + GG_{\gamma j}^i, \quad i = V, A \tag{64}$$

$$QQ_{\gamma j}^V + \frac{6}{j+1} GQ_{\gamma j}^V = QQ_{\gamma j}^A - \frac{j}{6} QQ_{\gamma j}^A, \quad \text{and} \quad V \leftrightarrow A, \tag{65}$$

$$\frac{6}{j} GQ_{\gamma j}^V - \frac{j+3}{6} QQ_{\gamma j}^V = 0, \quad \frac{6}{j} GQ_{\gamma j}^A - \frac{j+3}{6} QQ_{\gamma j}^A = 0, \tag{66}$$

which we call type-I (or Dokshitzer relation [49]), II and III constraints, respectively. It is worth to mention that the first two constraints are valid to all orders of perturbation theory in the supersymmetry preserving regularization and renormalization scheme while the last one gets modified beyond one-loop order by conformal non-covariant, off-diagonal elements of the anomalous dimension matrix [50].

As an example of the power of these constraints we demonstrate their use at LO. We show that all entries in the singlet sector can be reconstructed from the $QQ$ channels by solving the constraints (64)-(66). At first we consider the anomalous dimensions, and eliminate with the help of the type-III relation the $QG$ entries in the type-II relation. By elimination of the $GQ$ entry either in the vector or axial-vector channel we obtain a recurrence relation for the remaining mixed channel. For instance, the solution in the parity odd sector reads

$$GQ_{\gamma j}^A = \frac{2 GQ_{\gamma j}^A}{(j+1)(j+2)} \tag{67}$$

$$+ \sum_{i=2,4,...} \frac{i(i+1)}{6(j+1)(j+2)} \left[ (i+2) \left( QQ_{\gamma i}^V - QQ_{\gamma i}^A \right) + (i-1) \left( QQ_{\gamma i-1}^V - QQ_{\gamma i-1}^A \right) \right],$$

where in addition the (analytic continuation to the non-physical region of the) lowest moment has to be known, i.e. $GQ_{\gamma 0}^A$. The remaining five entries are obviously linear combinations of the known ones: The type-II relation gives the $QG$ entry for even parity, the type-III constraints provide the remaining two mixed channel entries, and the $GG$ channel follows from the Dokshitzer relation. In this way we obtain the whole anomalous dimension matrix for SYM theory in LO. Let us add for completeness that the restoration of colour factors is unique up to the $GG$ channels, where the self-energy contribution provides a constant term proportional to $N_f$. This constant can be fixed either by explicit calculation of self-energy insertions in QCD or by the fact that the energy-momentum tensor is conserved and its anomalous dimension vanishes.

The remaining goal is to derive and solve constraints for the ER-BL kernels by means of the representation (29). Unfortunately, we can only write down the type-II constraints in terms of
integral transformations such as given in Eq. \((35)\). Therefore, we expect a cumbersome solution of these constraints, which is maybe useless for practical purposes. Fortunately, the knowledge of the eigenvalues give us a hint to overcome this problem as described below. The representation \((23)\) together with the type-III relation \((66)\) provides us with the relation between the kernels in the mixed channels

\[
GQ_{\nu^i}(x, y) = (\bar{x}x)^2 \bar{y}y QG_{\nu^i}(y, x). \quad (68)
\]

Next we can eliminate the \(GQ\) entry in the Dokshitzer relation and take into account the relations \((31)\) between the Gegenbauer polynomials, to find a differential equation for the \(GG\) kernels:

\[
\frac{\partial}{\partial y} QQ_{\nu^i}(x, y) + \frac{\partial}{\partial x} GG_{\nu^i}(x, y) = -3QG_{\nu^i}(x, y) \quad \text{for} \quad i = V, A. \quad (69)
\]

Thus, it remains to find equations which determine the mixed channel contribution through the knowledge of the quark one. A closer look at the anomalous dimensions \((44)-(47)\) shows that the \(QG\) entry for parity even and the difference between parity even and odd in the \(GQ\) channel, i.e. \(GQ_{\gamma^\delta} = GQ_{\gamma^V} - GQ_{\gamma^A}\), are proportional \(1/(j+1)(j+2)\), which coincide with the eigenvalues of the scalar \(\phi^3_{(D=6)}\) theory kernel \(QQ_{\nu^a}(x, y) = \theta(y-x)x/y + \theta(x-y)\bar{x}/\bar{y}\), which appears as a part of the whole \(QQ\) kernel. Thus, we can write down for the kernels \(QG_{\nu^A}\) as well as for \(QG_{\nu^\delta} = QG_{\nu^V} - QG_{\nu^A}\) the following differential equations

\[
QG_{\nu^A}(x, y) = \frac{\partial}{\partial y} QQ_{\nu^a}(x, y), \quad (70)
\]

\[
\frac{\partial}{\partial x} QG_{\nu^\delta}(x, y) = -4QQ_{\nu^a}(x, y) + \text{const} \cdot x\bar{x} \quad \text{and} \quad \frac{\partial}{\partial y} QG_{\nu^\delta}(x, y) = 2GG_{\nu^\delta}(x, y), \quad (71)
\]

where the kernel \(QG_{\nu^a}(x, y)\) has the eigenvalues \(2/(j+1)(j+2)\) and can be constructed from the \(QQ\) ones by means of the differential equation \((39)\) with the r.h.s. set equal to zero. In the first equation of the set \((71)\) we included a term proportional to \(x\bar{x}\), which reflect the fact that the conformal expansion of \(QG_{\nu^A}(x, y)\) starts with \(j = 1\) [compare with Eq. \((29)\)]. Consequently, the lowest conformal moment of the r.h.s. has to vanish. Thus, in our case this constant is \(\text{const} = 4QQ_{\nu^a}/N_0(3/2) = 12\).

Let us now discuss the construction of the LO kernels from the knowledge of the \(QQ\) entry

\[
C_F \left[ QQ_{\nu}(x, y) \right]_+ \quad \text{with} \quad QQ_{\nu} \equiv QQ_{\nu^a} + QQ_{\nu^b}, \quad QQ_{\nu^b}(x, y) = \theta(y-x)\frac{1}{y y - x} + \left\{ x \rightarrow \bar{x}, y \rightarrow \bar{y} \right\}. \quad (72)
\]

In the parity odd sector Eq. \((70)\) give us the \(QG\) entry and Eq. \((68)\) the \(GQ\) one. The integration of the constraint \((63)\) provides us then with the missing \(GG\) kernel. The integration constant \(c(y)\) as a function of \(y\) is almost determined by the necessary condition \((30)\) for the kernels to have
a diagonal form. The remaining degree of freedom, i.e. \((yy')^2c'(y) = (xx')^2c'(x) = \text{const}\), can be easily fixed by the requirement that \(GGV_{j1} = 0\) for \(j > 0\). Now we consider the difference between parity even and odd cases. The set of differential equations (71) gives us the corresponding \(GQ\) entry, while the remaining integration constant is fixed from the requirement that the conformal moments \(GQV_{j0}\) vanish for \(j > 0\). Then the difference in the \(QG\) channel follows from the symmetry relation (78). Finally, the integration of Eq. (79) gives us the contribution in the \(GG\) channel.

The results obtained, with a minimal calculation of Feynman diagrams, has the advantage that the kernels are diagonal for all conformal moments. A direct calculation suffers in the parity even case from subtleties, which generate in the unphysical sector off-diagonal conformal moments. Now we present the improved kernels in such a way, that the underlying symmetries are explicitly manifest. For \(I = \{A, V\}\) the kernels read

\[
V^{(0)}_{(I)}(x, y) = \left( \frac{C_F(QQ\epsilon(x, y) + 2T_{(I)}^{QQ}QV_{(I)}(x, y))}{C_F^{QQ}V_{(I)}(x, y) + C_A^{QQ}V_{(I)}(x, y) - \delta(x - y)} \right),
\]

where the entries for parity odd are given by only two types of kernels and the difference to the even case arise from a third one:

\[
QQV \equiv QQV^a + QQV^b, \quad QQV^A \equiv -QQV^a, \quad QQV^A \equiv QQV^a, \quad QQV^A \equiv \left[ 2QQV^a + QQV^b \right]^+, \quad (74)
\]

\[
QQV \equiv QQV^A - 2QQV^c, \quad GBV \equiv GBV^A + 2GBV^c \quad \text{for} \quad B = \{Q, G\}. \quad (75)
\]

The functions \(^{AB}v^i\) are defined by Eq. (71) with

\[
\left\{ \begin{array}{l}
^{AB}f^a \\
^{AB}f^b
\end{array} \right. = \frac{x^{(A)-1/2}}{y^{(B)-1/2}} \begin{cases} 1 \\ \frac{1}{y-x} \end{cases} \quad \text{for} \quad \left\{ \begin{array}{l} A, B = \{Q, G\} \\
A = B
\end{array} \right., \quad (76)
\]

\[
AAf^c = \frac{x^{(A)-1/2}}{y^{(A)-1/2}} \left\{ \begin{array}{l}
2\bar{xy} \left[ \frac{4 - \ln(\bar{xy})}{2\bar{xy}} \right] + y - x \\
2\bar{xy} + y - x
\end{array} \right. \quad \text{for} \quad A = \left\{ \begin{array}{l} Q \\ G
\end{array} \right., \quad (77)
\]

\[
ABf^c = \frac{x^{(b)-1/2}}{y^{(b)-1/2}} \left\{ \begin{array}{l}
2\bar{xy} - x \\
2\bar{xy} - \bar{y}
\end{array} \right. \quad \text{for} \quad A = \left\{ \begin{array}{l} Q \\ G
\end{array} \right\} \neq B. \quad (78)
\]

The index \(\nu(A)\) coincides again with the index of the Gegenbauer polynomials. It is important to note that we have introduced different "+"-definitions in the \(QQ\) and \(GG\) channels in order to have a one-to-one correspondence between exclusive and inclusive kernels which require a regularization. For the exclusive \(QQ\) kernels we use the conventional prescription

\[
[V(x, y)]_+ = V(x, y) - \delta(x - y) \int_0^1 dz V(z, y),
\]

while for the \(GG\) kernel an extra term const(V)\(\delta(x - y)\) is subtracted, with the constant expressed in terms of an integral of the kernel itself [see Eq. (200)].
We also introduced in Eq. (77) the $QQ^v$ kernel, which does not show up in the LO kernel, but is of importance beyond one-loop approximation. It satisfies the differential equation
\[ \frac{\partial}{\partial y} QQ^v(x,y) = -GG^v(x,y), \] (80)
and is diagonal w.r.t. Gegenbauer polynomials. The given explicit representation of this kernel will be essential for the analysis of the ER-BL kernel in NLO.

It is worth to mention that the eigenvalues of the same $v^i$-kernel in different channels are related to each other (here $v_{ij} \equiv v_j$) by the relations
\[ QQ^v_{ij} = \frac{1}{6} QQ^v = \frac{6}{j(j+3)} GG^v_{ij} = \frac{1}{2} \frac{1}{(j+1)(j+2)}, \] (81)
\[ QQ^v_{ij} = -2\psi(j+2) + 2\psi(1) + 2, \] (82)
\[ QQ^v_{ij} = \frac{1}{6} QQ^v = \frac{6}{j(j+3)} GG^v_{ij} = \frac{1}{3} \frac{2}{j(j+1)(j+2)(j+3)}, \] (83)
where for the $AA^v_{ij}$ kernel the proper "+"-prescription has been taken into account.

Even though we do not consider the chiral odd sector in our presentation we just want to mention that the corresponding constraint can be easily implemented for the kernels
\[ \frac{\partial}{\partial y} QQ^T(x,y) + \frac{\partial}{\partial x} GG^T(x,y) = 0. \] (84)

It allows us to find the $GG$ kernel in a unique way from the $QQ$ entry as discussed above:
\[ AAV^{(0)T}(x,y) = \left\{ \begin{array}{ll}
C_F & \left[ AA^v(x,y) \right] + \frac{1}{2} \delta(x-y) \left\{ \frac{C_F}{4C_A + \beta_0} \right\} \\
\end{array} \right. \] for $A = \{ Q, G \}$. (85)

### 2.4 Method of reconstruction of two-loop kernels.

In the previous sections we have established the complete structure of the ER-BL kernels in LO and also partially in NLO. For the MS scheme the piece containing the off-diagonal part to NLO is known as a convolution (63), however, we have no explicit representation of the projection operator $(I - D)$ at hand which makes the following analysis more complicated. Let us drop it and represent the whole contribution to NLO as
\[ V^{(1)}(x,y) = -\hat{V}^e \otimes \left( V^{(0)} + \frac{\beta_0}{2} \mathbb{1} \right)(x,y) - \left[ g^e \otimes V^{(0)} \right]_(x,y) + \mathcal{D}(x,y), \] (86)
where $\mathcal{D}(x,y)$ is a pure diagonal part. We adopt the following strategy for our considerations. To restore this part one goes first to the forward limit (20) in which all non-diagonal terms die out and compares the result with the known two-loop DGLAP kernels $P^{(1)}(z)$ in order to get the DGLAP representation of $\mathcal{D}(x,y)$:
\[ \mathcal{D}(z) = \text{LIM} \mathcal{D}(x,y) \]
\[ = P^{(1)}(z) - \text{LIM} \left\{ -\hat{V}^e \otimes \left( V^{(0)} + \frac{\beta_0}{2} \mathbb{1} \right)(x,y) - \left[ g^e \otimes V^{(0)} \right]_-(x,y) \right\}. \] (87)
Afterwards one may try to use the reduction formula (35) to restore the ER-BL representation. Unfortunately, as we mentioned several times, this last step is too cumbersome to be performed in an analytical manner. Thus, we are forced to apply supersymmetry to get a better understanding of the structure and in order to reconstruct the missing diagonal pieces. If one looks to the DGLAP kernels, one immediately sees that the most complicated terms are Spence functions multiplied by the kernels appearing in LO

$$\pm 2^{AB}p(-z)S_2(-z) + ABp(z) \left[ \ln^2 z - 2\zeta(2) \right], \quad \text{where} \quad S_2(z) = \int_{z/(1+z)}^{1/(1+z)} \frac{dx}{x} \ln \frac{1-x}{x}. \quad (88)$$

Such contributions arise from the crossed ladder Feynman diagrams. Since they have no ultraviolet divergent subgraphs and, thus, require no subtraction, it appears just as in LO only as a $1/\epsilon$ divergence. Therefore, conformal covariance and supersymmetric constraints hold true for these contributions. Of course, it is questionable if these six constraints relate all contributions of this set of Feynman graphs in a given gauge, e.g. light-cone gauge. Fortunately, the answer is irrelevant for the reconstruction of the most complicated part containing the Spence functions. Since the ER-BL kernel is known in the non-singlet case [15, 16, 17], we have the one-to-one correspondence of this piece in the DGLAP and ER-BL representation for the $QQ$ channel at hand [4] which helps in the reconstruction of the other channels.

It is convenient to introduce the following notation for the NLO result,

$$V^{(0)}(x,y) = C_F [v(x,y)]_+, \quad v(x,y) = \theta(y-x)f(x,y) + \left\{ x \to \bar{x}, \quad y \to \bar{y} \right\}, \quad f \equiv QQf^a + QQf^b,$$

$$V^{(1)}(x,y) = C_F \left[ C_F V_F(x,y) - \frac{\beta_0}{2} V_\beta(x,y) - \left( C_F - C_A \right) \frac{1}{2} V_G(x,y) \right]_+, \quad (89)$$

where the functions $QQf^i$ with $i = \{a,b\}$ are given in Eq. (76). The piece which mainly originates from the crossed ladder diagram is proportional to $(C_F - C_A/2)$,

$$V_G(x,y) = 2v^a(x,y) + \frac{4}{3} v(x,y) + \left( G(x,y) + \left\{ x \to \bar{x}, \quad y \to \bar{y} \right\} \right), \quad (90)$$

and is diagonal w.r.t. the Gegenbauer polynomials. This is obvious for the first two terms appearing in Eq. (90). The function $G(x,y)$ contains both $\theta$-structures

$$G(x,y) = \theta(y-x)H(x,y) + \theta(y-\bar{x})\overline{H}(x,y), \quad (91)$$

where the functions $H(x,y)$ and $\overline{H}(x,y)$ are build from the LO function $f(x,y)$ and $\overline{f}(x,y) = f(\bar{x},\bar{y})$ in combination with the Spence function $\text{Li}_2$ and double logs:

$$H(x,y) = 2 \left[ \overline{f} \left( \text{Li}_2(\bar{x}) + \ln y \ln \bar{x} \right) - f \text{Li}_2(\bar{y}) \right], \quad (92)$$

$$\overline{H}(x,y) = 2 \left[ (f - \overline{f}) \left( \text{Li}_2 \left( 1 - \frac{x}{y} \right) + \frac{1}{2} \ln^2 y \right) + f \left( \text{Li}_2(\bar{y}) - \text{Li}_2(\bar{x}) - \ln y \ln x \right) \right], \quad (93)$$

We have slightly changed the original definition given in [13, 16] by $G(x,y) + 2\theta(y-x)\overline{f} \ln y \ln \bar{x} \to G(x,y)$.\end{footnote}
where \( \text{Li}_2(y) = -\int_0^y dx \ln(1-x)/x \). It can be easily checked that the \( G \)-contribution is symmetric w.r.t. the weight \( x\bar{x} \). Note that the two different \( \theta \)-structures do not obey separately this symmetry and contain therefore off-diagonal conformal moments. At this stage it is not so important to redefine \( H \) and \( \overline{H} \) so that they are separately symmetric and diagonal. This will be done for the presentation of the explicit NLO result in section 4. Performing the limit (20) [of course, only the \( QQ \) channel is relevant] we obtain the following correspondence with the non-singlet DGLAP kernel [1]:

\[
G(z) \equiv \text{LIM} G(x, y) = \theta(z)\theta(1-z)H(z) + \theta(-z)\theta(1+z)\overline{H}(z),
\]

where

\[
H(z) \equiv \text{LIM} H(x, y) = p(z) \left( \ln^2 z - 2\zeta(2) \right) + T(z),
\]

\[
\overline{H}(z) \equiv \text{LIM} \overline{H}(x, y) = 2p(z)S_2(-z) + T(-z), \quad T(z) = 2(1+z)\ln z + 4(1-z).
\]

Up to the simpler term \( T(z) \), we recover by mapping of \(-1 \leq z \leq 0\) into the \( 0 \leq z \leq 1 \) region the desired expression (88).

As demonstrated in subsection 2.3 for the LO kernels, we can now apply six supersymmetric constraints to find the \( G \) kernels in all other channels from the kernel (91) in the \( QQ \) channel. This task will be achieved in the next sections.

Now we study the structure of the remaining diagonal piece \( D(x, y) \) in the non-singlet sector. Knowing the correspondence of the \( G \) kernels in different representations, the remaining diagonal piece of \( V_G(x, y) \) can be easily restored from the DGLAP kernel [51]

\[
P_G(z) = 2p^\alpha(z) + \frac{4}{3}p(z) + G(z),
\]

where \( p^\alpha(z) \equiv \text{LIM} v^\alpha(x, y) = 1 - z \) and \( p(z) \equiv \text{LIM} v(x, y) = (1 + z^2)/(1 - z) \) by substitution

\[
p^\alpha(z) \rightarrow v^\alpha(x, y), \quad p(z) \rightarrow v(x, y).
\]

Next, the Feynman diagrams containing vertex and self-energy corrections provide \( V_\beta \) proportional to \( \beta_0 \). Its off-diagonal part is induced by the renormalization of the coupling and is contained in the dotted kernel:

\[
V_\beta(x, y) = \dot{v}(x, y) + D_\beta(x, y).
\]

This dotted kernel is given in Eq. (62), where \( \Delta f \) is zero in the \( QQ \) sector:

\[
\dot{v}(x, y) = \theta(y - x)f(x, y)\ln \frac{x}{y} + \left\{ \begin{array}{l} x \rightarrow \bar{x} \\ y \rightarrow \bar{y} \end{array} \right\}.
\]
This can be proved easily by forming conformal moments and using the triangular form of this matrix. The remaining diagonal piece, \( D_\beta \), is deduced from the known NLO DGLAP kernel \[^{[51]}\]

\[
P_\beta(z) = \frac{5}{3} p(z) + p^\alpha(z) + \dot{p}(z), \quad \dot{p}(z) \equiv \text{LIM} \dot{v}(x, y) = p(z) \ln z + 1 - z, \tag{101}
\]

by going to the forward kinematics and restoring the missing contributions from it by the substitutions \[^{[58]}:\]

\[
D_\beta(x, y) = \frac{5}{3} v(x, y) + v^\alpha(x, y). \tag{102}
\]

Indeed, the final result coincides with \[^{[15, 16, 17]}\].

Making use of the known non-diagonal part in the flavor non-singlet sector [which is the same as appearing in the \( QQ \) channel of Eq. \(^{[58]}\)], \( V_F \) can be represented up to a pure diagonal term, denoted as \( D_F(x, y) \), by the convolution \[^{[55]}\]

\[
V_F(x, y) = - \left( \dot{v} \otimes v + g \otimes v - v \otimes g \right)(x, y) + D_F(x, y), \tag{103}
\]

where the \( g \) kernel is given by the \( QQ \) entry of the matrix \(^{[58]}\). To find an appropriate representation of this missing diagonal element we first take the forward limit. Since the forward limit of the convolution is

\[
\text{LIM} \left\{ [\dot{v}]_+ \otimes [v]_+ \right\} = \{\text{LIM} [\dot{v}]_+\} \otimes \{\text{LIM} [v]_+\}, \tag{104}
\]

where we have introduced the inclusive convolution

\[
P_1(z) \otimes P_2(z) \equiv \int_0^1 dx \int_0^1 dy \delta(z - xy) P_1(x) P_2(y),
\]

with the commutator \( g \otimes V^{(0)} - V^{(0)} \otimes g \) dropping out in the forward limit, we obtain

\[
\text{LIM} V_F(x, y) = - \dot{p} \otimes p + \text{LIM} D_F(x, y). \tag{105}
\]

The comparison of \( \text{LIM} V_F(x, y) \) with the corresponding part of the DGLAP kernel \[^{[51]}\]

\[
P_F(z) = \left\{ \frac{4}{3} - 2 \zeta(2) - \frac{3}{2} \ln z + \ln^2 z - 2 \ln z \ln(1 - z) \right\} p(z)
\]

\[
+ \quad 1 - z + \frac{1 - 3z}{2} \ln z - \frac{1 + z}{2} \ln^2 z, \tag{106}
\]

yields the result in which all double log terms are contained in the convolution

\[
\dot{p} \otimes p = \left[ \frac{-27}{12} + 2 \zeta(2) + \frac{3}{2} \ln z - \ln^2 z + 2 \ln z \ln(1 - z) \right] p(z)
\]

\[
+ \quad \frac{z}{1 - z} + 2(1 - z) \ln \frac{1 - z}{z} + \frac{1 + z}{2} \ln^2 z \tag{107}
\]

\[^{7}\] We remind the reader that all flavor non-singlet kernels are supplemented by a “+”-prescription. Therefore, we use for simplicity in this section the convention that \( C = A \otimes B \) is indeed defined by the convolution of \( \left[C\right]_+ = [A]_+ \otimes [B]_+ \).
and, therefore, only single logs survive in $D_F(z) = \lim D_F(x, y)$:

$$
D_F(z) = P_F + \hat{p} \hat{p}^T \hat{p} = -\frac{1}{2} p^a(-z) \ln z - p^a(z) \left\{ \ln z - 2 \ln(1-z) + \frac{1}{2} \right\} - \frac{5}{12} p(z). \quad (108)
$$

Here we have introduced for convenience the kernel $p^a(z) = 1 - z$. The next important point is that the remaining log terms can be represented as convolutions of $p^a$ and $p$, i.e.

$$
\begin{align*}
[p^a \otimes p^a]_+ &= - [3p^a(z) + p^a(-z) \ln z]_+, \quad (109) \\
[p^a \otimes p]_+ &= - \left\{ \frac{1}{2} p^a(z) + p^a(z) \left( \frac{1}{2} + \ln z - 2 \ln(1-z) \right) \right\}_+. \quad (110)
\end{align*}
$$

Thus, we finally have

$$
D_F(z) = \frac{1}{2} p^a \otimes \{ 2p + p^a \} (z) + \frac{1}{12} p(z) + \frac{5}{2} p^a(z). \quad (111)
$$

Since $D_F(x, y)$ is by definition diagonal, the extension of $D_F(z)$ towards the ER-BL kinematics is trivial:

$$
D_F(z) \rightarrow D_F(x, y) = \frac{1}{2} v^a \otimes (2v + v^a) (x, y) + \frac{1}{12} v(x, y) + \frac{5}{2} v^a(x, y). \quad (112)
$$

Evaluating the convolutions one can establish the equivalence of this equation with the explicit calculated expression in Ref. [15, 16, 17]:

$$
V_F(x, y) = \theta(y-x) \left\{ \left( \frac{4}{3} - 2\zeta(2) \right) f + 3\frac{x}{y} - \left( \frac{3}{2} f - \frac{x}{2y} \right) \ln \frac{x}{y} - (f - f) \ln \frac{x}{y} \ln \left( 1 - \frac{x}{y} \right) \right\} + \left( f + \frac{x}{2y} \right) \ln^2 \frac{x}{y} - \frac{x}{2y} \ln x (1 + \ln x - 2 \ln \bar{x}) + \left\{ \begin{array}{c} x \rightarrow \bar{x} \\ y \rightarrow \bar{y} \end{array} \right\}. \quad (113)
$$

Recapitulating our results presented in this section, the ER-BL kernel to NLO has a rather simple structure governed by conformal anomalies, the crossed ladder contribution $G(x, y)$, as well as a remaining diagonal piece $D(x, y)$:

$$
V^{(1)I}(x, y) = - \hat{V}^I \otimes \left( V^{(0)I} + \frac{\beta_0}{2} \mathbb{1} \right) (x, y) - \left[ g \otimes V^{(0)I} \right]_- (x, y) + G^I(x, y) + D^I(x, y). \quad (114)
$$

The first two terms on the r.h.s. are known, the kernel $G(x, y)$ can be restored from the $QQ$ entry by means of supersymmetric constraints, and the missing diagonal piece can be extracted in the forward case from the DGLAP kernels $P^{(1)I}(z)$:

$$
D^I(z) = P^{(1)I}(z) - \lim \left\{ - \hat{V}^I \otimes \left( V^{(0)I} + \frac{\beta_0}{2} \mathbb{1} \right) - \left[ g \otimes V^{(0)} \right]_- + G^I \right\}. \quad (115)
$$

In the non-singlet case we saw that this piece does not contain double logs or Spence functions. Moreover, it can be represented in a rather straightforward manner by simple kernels, known
from the LO result, and their convolutions. This immediately allows one to obtain the ER-BL representation. Since the established structure is deeply related to the topology of Feynman graphs and the renormalization of the subgraphs, we expect that the missing entry $D^I(x,y)$ in all other cases can be build by means of known kernels, too.

### 3 Reconstruction of evolution kernels in NLO.

In this section we present, in detail, the reconstruction of all eight evolution kernels appearing in the vector and axial-vector case. In the following two subsections we complete the construction of $\hat{V}^I$ and of $G^I$ kernels, respectively. Afterwards we extract the remaining diagonal part and represent the ER–BL type kernels as convolution of kernels, appearing in LO or dotted kernels, and the $G$ kernels.

#### 3.1 Construction of $\hat{V}$.

In this subsection we present a systematic construction of the dotted kernels $QQ\hat{v}^i(x,y)$, which possess the off-diagonal conformal moments $(ABv^i_j - ABv^i_k)_{jk}$. Their generic form is given in Eq. (62) and the remaining goal is to determine the addenda $\Delta^{AB}\hat{f}^i(x,y)$. As we already saw in the previous sections, in the $QQ$ channel there is no such extra term for the $a$ and $b$ kernels. From the generating kernel (59) for the $QQ$ channel we can derive all the other ones. Since the eigenvalues of the $a$-kernel in the $QG$ channel are $\frac{1}{6}QGv^a_{ij} = -QQv^a_{ij}$, we can obtain the dotted kernel by differentiation w.r.t. $y$:

$$QQv^a(x,y) = -\frac{\partial}{\partial y}QQv^a(x,y) + \cdots,$$

where the ellipsis denotes possible diagonal terms. One can easily check that this equation is satisfied by Eq. (62) with $\Delta^{QG}\hat{f}^a(x,y) = 0$. To find the dotted $a$-kernel in the $GQ$ entry, we use a symmetry relation for the generating kernels, which arise from $-QQv^a_{ij}/6 = 6^{GQv^a_{ij}}/j(j+3)$:

$$GQv^a(x,y) = \frac{d}{d\epsilon}GQv^a(x,y|\epsilon)_{\epsilon=0} = -\frac{d}{d\epsilon} \left[ \frac{(x\bar{x})^{2+\epsilon}}{(yy)^{1+\epsilon}}QQv^a(y,x|\epsilon) \right]_{\epsilon=0} + \cdots,$$

$$= -\frac{(x\bar{x})^2}{yy}QQv^a(y,x) + GQv^a(x,y) \ln \frac{x\bar{x}}{yy} + \cdots. (117)$$

From this equation one can easily deduce that $\Delta^{GQ}\hat{f}^a(x,y) = 0$. The $a$ and $b$-kernels in the $GG$ channel may be obtained form the $QQ$ ones by means of the differential equation

$$\frac{\partial}{\partial x}GQv^i(x,y) = -2\frac{\partial}{\partial y}QQv^i(x,y), \quad \text{for} \quad i = a, b,$$  

(118)
which immediately provides the generic form (62) as a solution with $\Delta^{GG}\bar{f}_i(x,y) = \Delta^{GG}\bar{f}_i(y)$. Since the lowest moment, i.e. $\int_0^1 dx \, GQ_j(x,y)$, is a constant, $\Delta^{GG}\bar{f}_i(y)$ with $i = a, b$ itself has to be a constant, which turns out to be zero.

Now we come to the dotted $c$-kernels. Since the eigenvalues of the LO kernel in the $GQ$ channel are given by the $a$-kernels in the $AA$ channels, i.e. $GQ_{\dot{c}} = \frac{GQ_c}{3} = \frac{GQ_c}{6}$ [see Eq. (83)], we can derive the following two differential equations

$$\frac{\partial}{\partial y} GQ_c(x,y) = GQ_a(x,y) + GQ_a(x,y), \quad \frac{\partial}{\partial x} GQ_c(x,y) = -2 GQ_a(x,y) - GQ_a(x,y),$$

(119)

where the added diagonal terms on the r.h.s are fixed in the way that the equations are solvable. Taking the generic form (60), we obtain two differential equations for the addendum:

$$\frac{\partial}{\partial y} \Delta^{GQ}\bar{f}_c(x,y) = -\frac{x^2}{y} (2x - 3), \quad \frac{\partial}{\partial x} \Delta^{GQ}\bar{f}_c(x,y) = x (2x - 3) - 6x \bar{x} \ln \frac{x}{y}.$$  

(120)

The solution $\Delta^{GQ}\bar{f}_c = x^2 (2x - 3) \ln \frac{x}{y}$ is fixed up to a constant, which by comparison with the lowest conformal moments turns out to be zero.

Since the $c$-kernels in the mixed channels are related by supersymmetry, we may use Eq. (117) to find the dotted $c$-kernel in the $QG$ channel from the $GQ$ one. The $GG$ entry follows then by integration from

$$QG_{\dot{c}}(x,y) = \frac{1}{3} \frac{\partial}{\partial x} GQ_c(x,y).$$

(121)

Since $\Delta^{GQ}\bar{f}_c(x,y) = \frac{2x^2}{y} (y - x)$ has been obtained by a direct calculation of a graph with a quark bubble insertion [see conformal predictions (63)] in Ref. 31, we can use Eq. (121) to get

$$\Delta^{GQ}\bar{f}_c = -\frac{x}{3y^2} (4x - 5y + 2xy).$$

Now we are ready to present all dotted kernels. For odd parity we introduce the dotted matrix

$$\hat{V}^I(x,y) = \left( \begin{array}{cc} C_F \left[ QG_{\dot{c}}(x,y) \right] & 2 T_F N_f QG_{\dot{c}}(x,y) \\ C_F GQ_{\dot{c}}(x,y) & C_A GQ_{\dot{c}}(x,y) \end{array} \right) \quad \text{for} \quad I = \{A, V\}.$$  

(122)

Here we use the decomposition as for the LO kernels (74) and (75),

$$QG_{\dot{c}} \equiv QG_{\dot{c}}^a + QG_{\dot{c}}^b, \quad QG_{\dot{c}}^a \equiv -QG_{\dot{c}}^a, \quad QG_{\dot{c}}^A \equiv GQ_{\dot{c}}, \quad GQ_{\dot{c}}^A \equiv 2 GQ_{\dot{c}} + GQ_{\dot{c}}^b,$$

(123)

$$QG_{\dot{c}}^V \equiv QG_{\dot{c}}^A - 2 GQ_{\dot{c}}^c, \quad GB_{\dot{c}}^V \equiv GB_{\dot{c}}^A + 2 GB_{\dot{c}}^c \quad \text{for} \quad B = \{Q, G\},$$

(124)

and included the same “+”-prescription, although the dotted kernels are regular at the point $x = y$. The general structure of $AB_{\dot{c}}$ are given in Eq. (83), where the addenda are

$$\Delta^{AB}\bar{f}_a = \Delta^{AA}\bar{b}_b \equiv 0, \quad \Delta^{GG}\bar{f}_c(x,y) = 2 \frac{x^2}{y^2} (y - x)$$

(125)

$$\Delta^{GQ}\bar{f}_c = -\frac{x}{3y^2} (4x - 5y + 2xy), \quad \Delta^{GQ}\bar{f}_c = x^2 (2x - 3) \ln \frac{x}{y}.$$  

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3.2 Construction of $G$ kernels.

The construction of the diagonal $G(x, y)$ kernel, related to the crossed ladder diagrams, goes in the same manner as demonstrated in subsection 2.3 for the reconstruction of the LO kernels from the $QQ$ one by means of conformal and supersymmetric constraints. Since not all steps are quite obvious, we give here the construction in detail.

The colour structure of the entries in the $G$ kernel is

$$G^I(x, y) = -\frac{1}{2} \left( 2C_F \left( C_F - C_A^2 \right) Q^I(x, y) + 2C_A T_F N_f Q^I(x, y) \right),$$  

(126)

for $I = \{A, V\}$. As explained in subsection 2.4, from the result (91)-(93) in the flavor non-singlet sector [13, 16, 17] and its correspondence to the forward case given in Eqs. (94)-(96), we conclude that all entries in this matrix have the generic form

$$A^B G^I(x, y) = \theta(y - x) \left( A^B H^I + \Delta A^B H^I \right)(x, y) + \theta(y - \bar{x}) \left( A^B \overline{H^I} + \Delta A^B \overline{H^I} \right)(x, y),$$  

(127)

with the following expressions for $H$ and $\overline{H}$

$$A^B H^I(x, y) = 2 \left[ \pm A^B \overline{H} \left( \text{Li}_2(\bar{x}) + \ln y \ln \bar{x} \right) - A^B f^I \text{Li}_2(\bar{y}) \right],$$  

(128)

$$A^B \overline{H}^I(x, y) = 2 \left[ \left( A^B f^I \mp A^B \overline{H}^I \right) \left( \text{Li}_2 \left( 1 - \frac{x}{y} \right) + \frac{1}{2} \ln^2 y \right) + A^B f^I \left( \text{Li}_2(\bar{y}) - \text{Li}_2(x) - \ln y \ln x \right) \right].$$  

(129)

Here the upper (lower) sign corresponds to the $A = B$ ($A \neq B$) channels. In the $QQ$ sector we have $\Delta QQH = \Delta QQ\overline{H} = 0$. However, in general it turns out that the non-vanishing addenda are needed to ensure the diagonal form of the kernels. The forward limit of these $A^B G^I$ kernels has as a generalization of Eqs. (104)-(106) the following form:

$$A^B G^I(z) \equiv \left[ \text{LIM} A^B G^I(x, y) \right]_{z \leq 0 \Rightarrow z \geq 0} = \theta(z) \theta(1 - z) \left[ A^B H^I(z) \pm A^B \overline{H}^I(-z) \right] \quad \text{for} \quad \{I = A, I = V \},$$  

(130)

where we employed the (anti) symmetry of the singlet parton densities to map the mixing between partons and anti-partons, given for $-1 \leq z \leq 0$ into the region $0 \leq z \leq 1$. The $A^B H^I$ and $A^B \overline{H}^I$ functions are defined as

$$A^B H^I(z) \equiv \text{LIM} \left( A^B H^I + \Delta A^B H^I \right)(x, y) = A^B p^I(z) \left( \ln^2 z - 2\zeta(2) \right) + A^B T^I(z),$$  

(131)

$$A^B \overline{H}^I(z) \equiv \text{LIM} \left( A^B \overline{H}^I + \Delta A^B \overline{H}^I \right)(x, y) = 2 A^B p^I(z) S_2(z) + A^B \overline{T}^I(-z),$$  

(132)

where we have to require that $A^B T^I$ and $A^B \overline{T}^I$ are rational functions and/or terms containing single logs of momentum fractions. Note that the mapping, which took place in Eq. (130), can be also performed in the ER-BL representation by the substitution $y \rightarrow \bar{y}$ and is discussed in section 4.
First we reconstruct the missing three entries of $\Delta^{GH^A}$ and $\Delta^{GH^A}$ in the axial-vector case. Since the $ABG^A$ are essentially determined by the LO functions, it is convenient to decompose them in the same way as the LO kernels in Eqs. (72) and (74):

$$QQG^G \equiv QQ^a_G + QQ^b_G, \quad QQG^G \equiv -QQ^a_G, \quad GGG^A \equiv GG^A, \quad GG^A \equiv 2GG^a + GG^b.$$  \hspace{1cm} (133)

As for the kernels $ABG^D(x,y)$ to LO, we can obtain the $QQ$ entry from the $a$-kernel in the $QQ$ channel by a derivative w.r.t. $y$ as in Eq. (70). Since the $QQG^G$ kernel is defined in Eqs. (127)-(129) with $\Delta^{GQH^A} = \Delta^{GQH^A} = 0$, we find the addenda in the $QQ$ channel from

$$\Delta^{GQH^A}(x,y) = -QQH^A(x,y) + \frac{\partial}{\partial y}QQH^a(x,y), \quad \Delta^{GQH^A}(x,y) = -QQH^A(x,y) + \frac{\partial}{\partial y}QQH^a(x,y).$$

Since conformal covariance and supersymmetry connect these functions in a simple way with the $GQ$ ones [see Eq. (68)],

$$GQG^A(x,y) = \frac{(\bar{x}x)^2}{\bar{y}y}QQG^A(y,x) \Rightarrow \begin{cases} \Delta^{GQH^A}(x,y) = -\frac{(\bar{x}x)^2}{\bar{y}y}QQH^A(\bar{y}, \bar{x}) \\ \Delta^{GQH^A}(x,y) = \frac{(\bar{x}x)^2}{\bar{y}y}QQH^A(y,x) \end{cases},$$

we can write our findings in the following symmetric manner

$$\Delta^{GQH^A}(x,y) = \Delta^{GQH^A}(\bar{x}, \bar{y}), \quad \Delta^{GQH^A}(x,y) = \frac{x\bar{x}}{(\bar{y}y)^2} \Delta^{GQH^A}(y,x), \hspace{1cm} (134)$$

$$\Delta^{GQH^A}(x,y) = -\Delta^{GQH^A}(\bar{x}, \bar{y}), \quad \Delta^{GQH^A}(x,y) = -2\frac{x\bar{x}}{y} \ln x + 2\frac{\bar{x}\bar{y}}{\bar{y}} \ln y. \hspace{1cm} (135)$$

To find the remaining $GG$ entries we employ the differential equation (69), i.e.

$$\Delta^{GGH^A}(x,y) = -GGH^A(x,y) - \int^x dx \left[ \frac{\partial}{\partial y}QQH^A(x', y) + 3QQH^A(x', y) + 3\Delta^{GQH^A}(x', y) \right]$$

and an analogous equation for $\Delta^{GGH^A}(x,y)$. Again the remaining freedom can be fixed by the necessary condition (60) for diagonality w.r.t. Gegenbauer polynomials and from the requirement that the moments $GG_{1j}^A$ vanishes for $j > 1$. To simplify the result, we remove a symmetric function $f$ (w.r.t. the simultaneous interchange $x \to \bar{x}$ and $y \to \bar{y}$) which enters in both $\Delta^{GQH^A}$ and $\Delta^{GQH^A}$ kernels, however, with different overall signs and, therefore, disappears from $GG^A$: \hspace{1cm} (136)

$$\Delta^{GGH^A}(x,y) = -\Delta^{GQH^A}(\bar{x}, \bar{y}), \quad \Delta^{GGH^A}(x,y) = -\frac{\bar{x}^2}{\bar{y}y} - 2\frac{x\bar{x}}{yy} - 2\frac{x(\bar{y} + 3\bar{y} - 3\bar{y}y)}{y^2y} \ln x - 2\frac{x(\bar{y} + \bar{y} - 3\bar{y}x)}{yy^2} \ln y.$$

Now we come to the parity even sector. Instead of dealing with the whole sector, we can consider only the difference between vector and axial-vector kernels

$$ABG^V = ABG^A + ABG^b \quad \text{with} \quad QQG^\delta = -2QQG^c, \quad GB^G = 2GB^c \quad \text{for} \quad B = \{Q, G\}. \hspace{1cm} (137)$$

\[8\] Here and below for the vector case this functions slightly differ from that one in Ref. [52].
and the same notation for \(ABH, \overline{ABH}\) and their addenda.

In LO we were able to write down the set (71) of differential equations that determine the entry in the \(GQ\) channel depending on the \(a\)-kernels. Unfortunately, the analogous equations for \(ABG^c\) provide us a solution, which does not preserve the generic form of \(GQG^c\) in the forward limit given in Eq. (130) - (132). However, it turns out that we can restore the generic form by adding convolutions of \(c\)-kernels of the diagonal channels given in Eqs. (74) and (77):

\[
\begin{align*}
\frac{\partial}{\partial x} GQG^\delta(x, y) &= -4 \left[QQG^a(x, y) + 9 QQv^c \otimes QQv^c(x, y) - 9 QQv^c \otimes QQv^c(\bar{x}, y) \right], \quad (138) \\
\frac{\partial}{\partial y} GQG^\delta(x, y) &= 2 \left(GG^a + 2 GGv^c \otimes GGv^c(x, y) + 2 GGv^c \otimes GGv^c(\bar{x}, y) \right), \quad (139)
\end{align*}
\]

where the kernel \(GGG^a\) are part of the whole parity odd functions derived in the fashion already explained above. Note that all of these sets of differential equations represents after separation of the \(\theta\)-structures in fact two sets, one for \(\Delta^{QG}H^\delta(x, y)\) and the other one for \(\Delta^{QG}H^\delta(x, y)\). The two integration constants can be easily determined from the vanishing of the conformal moments \(GQG^c\times j0 = 0\) for \(j > 0\). Finally, we simplify the solution by adding pure diagonal pieces containing \(a\) and \(c\) kernels and their convolution as well as by removing symmetric terms that die out in \(GQG^c\).

Using again the supersymmetric relation (68), we write our findings in the mixed channels in the following way:

\[
\begin{align*}
\Delta^{QG}H^\delta(x, y) &= -\frac{x\bar{x}}{(y\bar{y})^2} \Delta^{QG}H^\delta(\bar{y}, \bar{x}), \quad \Delta^{QG}H^\delta(x, y) = \frac{x\bar{x}}{(y\bar{y})^2} \Delta^{QG}H^\delta(y, x), \\
\Delta^{QG}H^\delta(x, y) &= \Delta^{QG}H^\delta(\bar{x}, y) + 20 \frac{x(x - \bar{x})}{3y} - 4 \frac{\bar{x}(3 + 2\bar{x})}{3y} \ln \bar{x} + 4 \frac{x(3 + 2x)}{3\bar{y}} \ln y \\
\Delta^{QG}H^\delta(x, y) &= -\frac{61}{9} \bar{x} + 2x\bar{x} \left(1 + \frac{25}{18} (x - \bar{x}) - 3(10 \bar{x}) \ln y + (3 - 10x) \ln x \right) \\
&\quad + \frac{x(6 - 19x + 6x^2)}{3y} - \frac{\bar{x}(y + x(\bar{x} - x))}{\bar{y}} \ln y + 2 \frac{x(\bar{y} + x(\bar{x} - x))}{\bar{y}} \ln x,
\end{align*}
\]

The last task is to construct the \(GG\) kernel by means of the constraint (89) where \(QQG^c \equiv 0\). As already described, the solution is obtained in a straightforward manner and reads:

\[
\begin{align*}
\Delta^{GG}H^\delta(x, y) &= \Delta^{QG}H^\delta(\bar{x}, y) + \frac{x(3 - 13\bar{x})}{y^2} - 2 \frac{x(2 + 3x)}{y\bar{y}} - \frac{2\bar{x}}{y} \left(2 \frac{x - x}{y} + 2 \frac{3x}{y} \right) \ln \bar{x} \\
&\quad - \frac{2x}{y} \left(2 \frac{x - \bar{x}}{y} + 2 \frac{3x}{\bar{y}} \right) \ln y, \\
\Delta^{GG}H^\delta(x, y) &= \frac{(1 - x^2)(1 - 21x)}{3y^2} - \frac{x(19(1 + x) - 36x^2)}{3y} + \frac{2\bar{x}^3}{3\bar{y}} \\
&\quad + 2 \left(\frac{x^3}{3y^2} - \frac{x^2(21 - 20x)}{3y} - \frac{2x^2\bar{x}}{\bar{y}} \right) \ln x + 2 \left(\frac{x^3}{3\bar{y}^2} - \frac{x^2(21 - 20\bar{x})}{3\bar{y}} - \frac{2\bar{x}^2y}{y} \right) \ln y.
\end{align*}
\]

This completes our construction of the complete \(G\) matrices relevant for parity odd and even sector.
3.3 Restoration of remaining diagonal terms.

Now we have to find the remaining diagonal pieces of the ER-BL kernels, which are expected to have a simple representation in terms of the kernels known from Eqs. (74)-(78). The DGLAP representations of the missing terms are obtained with the help of Eq. (113), where we can interchange the exclusive convolution with the forward limit procedure:

\[
D^I(z) = P^{(3)I}(z) - \left\{ - \hat{P}^I \otimes \left( P^{(0)I} + \frac{\beta_0}{2} \mathbb{1} \right) - \left[ \frac{1}{g} \otimes P^{(0)I} \right] - G^I \right\}(z). \tag{143}
\]

Here we have introduced the analogous notation for the DGLAP representations as used in the non-forward kinematics. The DGLAP kernels to LO read

\[
P^{(0)I} \equiv \text{LIM} V^{(0)I} = \left( \begin{array}{cc}
C_F & \frac{2 T_F N_f}{Q G p^I(z) + 2 T_F N_f} \\
C_F & C_A [G G p^I(z)] + \frac{3}{2} \beta_a (1 - z) \\
\end{array} \right) \quad \text{for } I = \{ A, V \}, \tag{144}
\]

where the entries are decomposed in the same manner as in Eqs. (74) and (75), i.e.

\[
\begin{align*}
Q Q_p & \equiv Q Q p^a + Q Q p^b, \quad Q G p^A \equiv -Q G p^a, \quad G G p^A \equiv G G p^a, \quad G G p^A \equiv \left[ 2 G G p^a + G G p^b \right]_+ , \\
Q Q_V & \equiv Q Q^A - 2 Q Q p^c, \quad G B V \equiv G B p^A + 2 G B p^c \quad \text{for } B = \{ Q, G \},
\end{align*}
\]

and the functions \( A B p^i \) are defined in the following way

\[
\begin{align*}
Q Q p^a & = \frac{1}{2} G G p^a = 1 - z, \quad Q Q p^b = G G p^b = \frac{2 z}{1 - z}, \quad Q Q p^c = \frac{1}{3} G G p^c = \frac{(1 - z)^3}{3 z}, \\
Q Q p^a & = 1 - 2 z, \quad Q Q p^c = -(1 - z)^2, \quad G G p^a = 2 - z, \quad G G p^c = \frac{(1 - z)^2}{z}. 
\end{align*}
\]

The “+”-prescription is now uniquely defined in the \( QQ \) and \( GG \) channel as the conventional one:

\[
[ A A p(z) ]_+ = \frac{A A p(z) - \delta(1 - z) \int_0^1 dy \ A A p(y)}{1}. 
\]

The limit of the dotted kernels defined in Eqs. (122)-(125) give

\[
\hat{P}^I \equiv \text{LIM} \tilde{V}^I = \left( \begin{array}{cc}
C_F Q Q p(z) & 2 T_F N_f Q Q p^I(z) \\
C_F Q Q p^I(z) & C_A G G p^I(z) \\
\end{array} \right), \tag{148}
\]

with

\[
\begin{align*}
A A p^A & = \left[ A A p^A \ln z + A A p^A \right]_+, \quad A B p^A = A B p^A \ln z \mp Q Q p^a \quad \text{for } \left\{ \begin{array}{c} A B = Q G \\
A B = G Q \end{array} \right. , \\
Q Q p^V & = \frac{Q Q p^V}{3} \left( \ln z + \frac{13}{6} \right) - \frac{2}{3} G G p^c + \frac{13}{6} Q Q p^a - Q Q p^s, \quad G G p^V = -G Q p^A, \\
G G p^V & = G G p^A + 2 G G p^c \ln z + \frac{11}{3} G G p^c.
\end{align*}
\]

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The limit of the $g$ matrix defined in Eq. (58) is

$$
g(z) \equiv \text{LIM} g(x,y) = 2 \begin{pmatrix} -C_F \left[ \frac{\ln(1-z)}{1-z} \right] + \frac{0}{C_F} \\ C_F -CA \left[ \frac{\ln(1-z)}{z(1-z)} \right] + \end{pmatrix}. \tag{151}$$

Finally, we need the complete forward limit of the $G$ kernel given in Eqs. (130) - (132), i.e. we have to determine $ABT^I$ and $ABT^I$. From the result of the $G$ matrix given in Eqs. (126)-(129) with the addenda (134)-(136) and (140)-(136), we obtain the following expressions for the sum or differences in the parity odd and even sector, respectively:

$$AA^T(z) + AA^T(-z) = 4Aa^a(-z) \ln z + 8Aa^a(z), \tag{152}$$

$$AB^T(z) + AB^T(-z) = \mp 4Ba^a(-z) \ln z \mp 12Qp^a(z) \text{ for } \begin{cases} AB = QG \\ AB = GQ \end{cases},$$

$$QQ^T(z) - QQ^T(-z) = 0, \quad GG^T(z) - GG^T(-z) = -\frac{8}{3}Qp^c(-z) \ln z, \tag{153}$$

$$AB^T(z) - AB^T(-z) = \frac{8}{3}Qp^c(-z) (2 \pm 3 \ln z) + \frac{16}{3}Qp^a(z) \ln z \text{ for } \begin{cases} AB = QG \\ AB = GQ \end{cases}.$$

With these results we are now able to find $D^I(z)$ from the known DGLAP kernels in NLO by means of Eq. (143). It remains a simple exercise to express these findings in terms of convolutions of $a,b$ and $c$-type kernels defined in Eqs. (147). In the following two subsubsections we treat the parity odd and even sector separately. We start with the parity odd sector responsible for the evolution of axial-vector distribution amplitudes, since the structure is simpler than the one in the parity even sector.

### 3.3.1 Parity odd sector.

We already have explicit expressions for all kernels entering on the r.h.s. of Eq. (143). It remains to convolute the $\dot{P}$ and $g$ kernels with the LO ones. Since cancellations between these separate terms drastically simplify the final result, we present it as entries of the matrix

$$\mathcal{A}^I = -\dot{P}^I \otimes \left( P^{(0)I} + \frac{\beta_{0}^I}{2} \mathbb{1} \right) - \left[ g \otimes P^{(0)I} \right]. \tag{154}$$

In other words this matrix contains the diagonal part that we could not separate from the conformal anomalies. The calculation is straightforward and results in

$$QQ^A = 2C_F T_F N_f \left\{ 4(1-z) -(1-3z) \ln z -(1+z) \ln^2 z \right\} + \text{NS} \mathcal{A}, \tag{155}$$

where the convolution in the non-singlet part $\text{NS} \mathcal{A} = -C_F^2 Q^a \otimes Q^a - C_F \beta_{0}^a Qp^2 / 2$ has been worked out in Eq. (107). The remaining channels read:

$$QQ^A = QG^A = 2C_F T_F N_f \left\{ (1-2z) \left[ \frac{7}{4} + 2\zeta(2) - \frac{1}{2} \ln^2 z + 2 \ln z \ln(1-z) - \ln^2(1-z) \right] \right\}$$

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\[(1-z)\left[-4-3 \ln z + 4 \ln(1-z)\right]\right] \\
+2C_A T_F N_f \left\{(1-2z) \left[2\zeta(2) + \ln^2(1-z)\right] - 2(1-z) \left[1 + 2 \ln(1-z)\right] \right\} \\
-6 \ln z - 2(1+z) \ln^2 z,\]

\[G^A A^A = C_F^2 \left\{ \frac{1-7z}{2} + \frac{4+3z}{2} \ln z - 4 \ln(1-z) + (2-z) \left[\frac{1}{2} \ln^2 z - \ln^2(1-z)\right] \right\} \\
-C_F \frac{\beta_0}{2} \left\{ 3-z + (2-z) \ln z \right\} + C_F C_A \left\{ -5(1-z) + (4-9z) \ln z \right\} \\
+3 - 2z + \frac{3z^2}{\ln(1-z) + (4+z) \ln^2 z + (2-z) \ln(1-z) \ln(1-z) - 2 \ln z \right\},\]

\[G^A A^A = C_A^2 \left\{ \frac{2-3z+2z^2}{1-z} \left\{ -2\zeta(2) + \ln^2 z - 2 \ln z \ln(1-z) \right\} + 4(1-z) \left[4 - \ln(1-z)\right] + 12(2-z) \ln z + 4(1+z) \ln^2 z \right\} \\
+2C_F T_F N_f \left\{ -10(1-z) - (7+z) \ln z - (1+z) \ln^2 z \right\}.\]

Now we are ready to extract the remaining diagonal terms $^{ABC}D^A$ from

\[D^A(z) = P^{(1)} A^A(z) - \left\{ A^A + G^A \right\}(z).\] (159)

As observed in the flavor non-singlet case, all double logs and Spence functions appearing in the DGLAP kernel in NLO \cite{16,17} are contained in the sum of $A^A$ given in Eqs. (155)-(158) and $G^A$ defined in Eqs. (130)-(132) together with Eq. (152). The remaining single log terms give us a hint to write the entries of $D^A(z)$ as convolution of the simple kernels \cite{14c}, which allows us to restore the ER-BL representation. In the following we present this issue in detail.

In the $QQ$ channel the only new information arises from the pure singlet term. Employing formula (159), we immediately find that this term is

\[-6 C_F T_F N_f Q^a \Rightarrow -6 C_F T_F N_f Q^a/Q^a.\]

Adding the result of the flavor non-singlet sector, the $QQ$ entry reads

\[Q^A Q^A = NS D - 6 C_F T_F N_f Q^a/Q^a,\] (160)

\[NS D = C_F^2 [D_F]_+ - C_F \frac{\beta_0}{2} [D_\beta]_+ - C_F \left( C_F - \frac{C_A}{2} \right) \left[ \frac{4}{3} Q^a Q^a + 2 Q^a Q^a \right]_+,\] (161)

where $D_F, D_\beta$ are given by Eqs. (112) and (113), respectively.

In the $QG$ channel we find in an analogous way

\[Q^A G^A(z) = 3 C_F T_F N_f \left\{ -(1+2z) \ln z - 3(1-z) - \frac{1}{2}(1-2z) \right\} \\
-2 C_A T_F N_f \left\{ -3(1+2z) \ln z - 9(1-z) + [1+2\zeta(2)](1-2z) \right\}\]
by a convolution of $a$-kernels,

$$QQp^a \otimes Q^G p^a = -(1 + 2z) \ln z - 3(1 - z),$$

together with $Q^G p^a = 1 - 2z$ the ER–BL-representation:

$$Q^G D^A = 3 C_F T_F N_f \left\{ QQ v^a \otimes Q^G v^a - \frac{1}{2} Q^G v^a \right\}$$

$$-2 C_A T_F N_f \left\{ 3 QQ v^a \otimes Q^G v^a + [1 + 2 \zeta(2)] Q^G v^a \right\}.$$ (162)

A more complicated expression arises in the $GQ$ channel:

$$GQ D^A(z) = C_F^2 \left\{ (2 - z) \ln(1 - z) - (4 + z) \ln z - 6(1 - z) - \frac{3}{2}(2 - z) \right\}$$

$$- C_F \beta_0 \frac{1}{2} \left\{ 2(2 - z) \ln(1 - z) - (2 - z) \ln z + \frac{2}{3} z - \frac{1}{6} (2 - z) \right\},$$

$$- C_F C_A \left\{ \frac{3 + 2z - z^2}{z} \ln(1 - z) - 2 \ln z - 6(1 - z) - \left[ \frac{7}{3} - 2 \zeta(2) \right] (2 - z) \right\},$$

which is characterized by new terms containing $(2 - z) \ln(1 - z)$ and $[\ln(1 - z)]/z$. Such contributions can be generated by convolutions of $b$-kernels with $a$- and $c$-kernels, respectively\footnote{Note that we can eliminate one of them e.g. $QQp^a \otimes [QQp]_+ = [GGp^A]_+ \otimes QQp^a + 3G^G p^a \otimes QQp^a/2 + 3G^G p^a/2.$}

$$GQ p^a \otimes [QQp]_+ = 2(2 - z) \ln(1 - z) - (2 - z) \ln z + 3/2z,$$

$$GQ p^A \otimes Q^G p^a = 2(2 - z) \ln(1 - z) - 2(4 + z) \ln z - 12(1 - z),$$

$$GQ p^A \otimes Q^G p^c = 2 (\frac{1 - z}{z}) \ln(1 - z) + 2(4 + z) \ln z + 10(1 - z),$$

$$GQ p^a \otimes Q^G p^a - 2(2 + z) \ln z - 6(1 - z), \quad GQ p^a = 2 - z.$$  

The result can be written as

$$GQ D^A = C_F^2 \left\{ \frac{1}{2} G^G v^A \otimes Q^G v^a - \frac{3}{2} G^G v^a \right\} - C_F \beta_0 \frac{1}{2} \left\{ G^G v^a \otimes [QQv]_+ - \frac{1}{6} G^G v^a \right\},$$ (163)

$$- C_F C_A \left\{ \frac{3}{2} G^G v^A \otimes Q^G v^c + G^G v^A - \frac{1}{2} G^G v^a \right\} \otimes Q^G v^a - \left[ \frac{7}{3} - 2 \zeta(2) \right] G^G v^a.$$ 

The remaining entry reads in the DGLAP representation

$$G^G D^A(z) = C_A^2 \left\{ 4(1 - z) \ln(1 - z) - 2(5 + 3z) \ln z + \frac{4}{3} \left[ \frac{2 - 3z + 2z^2}{1 - z} \right]_+ - \frac{41}{2} (1 - z) \right\}$$

$$- 2 \delta(1 - z) \} \right\} - C_A \beta_0 \frac{1}{2} \left\{ 2(1 + z) \ln z + \frac{10}{3} \left[ \frac{2 - 3z + 2z^2}{1 - z} \right]_+ + 6(1 - z) + 2 \delta(1 - z) \right\}$$

$$- C_F T_F N_f \left\{ -4(1 + z) \ln z - 10(1 - z) + 6(1 - z) \right\}.$$
and can be expressed by convolutions of $a$- and $b$-type kernels,

$$GG^A p_i \otimes GG^a p^a = 4(1 - z) \ln(1 - z) - 4(2 + z) \ln z - 16(1 - z)$$

$$GG^A p_i \otimes GG^a p^a = -4(1 + z) \ln z - 8(1 - z), \quad GG^A p_i = 2\frac{2 - 3z + 2z^2}{1 - z}, \quad GG^a p^a = 2(1 - z),$$

from which the desired kernel follows:

$$GG D^A = C^2_A \left\{ \left[ GG v^A \frac{1}{2} GG v^a \right] e \otimes GG v^a + \frac{2}{3} GG v^A \right\}$$

$$- C_A \beta_0 \left\{ \frac{1}{2} GG v^a \otimes GG v^a + 2 \frac{3}{5} GG v^A + 5 \delta(x - y) \right\}$$

$$- C_F T_F N_f \left\{ GG v^a \otimes GG v^a - GG v^a + \delta(x - y) \right\}.$$ 

These results provide us the missing information to obtain the whole kernel from Eq. (114) in the flavor singlet sector for odd parity.

### 3.3.2 Parity even sector.

In analogous way we treat now the parity even sector, starting with the calculation of the matrix $A^V$. For brevity we present here only the results for the difference between vector and axial-vector case, i.e. $A^d = A^V - A^A$ defined in Eq. (154). This difference is induced by the $c$-kernels. Thus, it is not surprising that $QG A^d$ consists of terms $1/z \ln i z \ln j (1 - z)$ and $z^2 \ln i z \ln j (1 - z)$ with $0 \leq i + j \leq 2$:

$$QQ A^d = -2 C_F T_F N_f \left\{ \frac{1}{9z} (1 - z) (31 + 142z - 5z^2) + \frac{4}{3z} (1 + 7z + 7z^2 - z^3) \ln z \right\},$$

$$QG A^d = 2 C_F T_F N_f \left\{ (1 - z) (10 - 7z) + (1 - z)^2 \right\} \left[ -2 \zeta(2) + 3 \ln \frac{z}{1 - z} + \ln^2 \frac{z}{1 - z} \right]$$

$$- (1 - 4z + (1 - 2z) \ln z) \ln z \right\} + 2 C_A T_F N_f \left\{ - \frac{1 - z}{18z} (88 + 175z + 367z^2) \right\}$$

$$- \frac{1}{3z} (4 + 13z + 88z^2) \ln z + (1 - 2z) \ln^2 z$$

$$- (1 - z)^2 \left[ 4 \zeta(2) + \frac{4}{3} \ln (1 - z) + 2 \ln^2 (1 - z) \right],$$

$$GG A^d = C^2_F \left\{ -3 (1 - z) - [3z + (2 - z) \ln z] \ln z - 2 \frac{(1 - z)^2}{z} \ln^2 (1 - z) \right\}$$

$$+ C_F C_A \left\{ \frac{1}{z} (17 + 3z - 21z^2 + z^3) + \frac{1}{3z} (31 + 24z + 57z^2 - 4z^3) \ln z \right\}$$

$$+ \frac{1}{z} (2 - 2z + z^2) \ln^2 z + \frac{2(1 - z)^2}{z} \left[ -3 - 2 \ln z + \ln (1 - z) \right] \ln (1 - z) \right\}$$

$$+ C_F \beta_0 \left\{ 1 - z + (2 - z) \ln z \right\},$$

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\[ \mathcal{G} \mathcal{A}^\delta = C_2^2 \left\{ \frac{2(1 - z)}{9z} (112 - 2z + 112z^2) + \frac{2}{3z} (22 - 18z + 81z^2 - 11z^3) \ln z \\
- \frac{2(1 - z)^3}{3z} \left[ 6\zeta(2) + 11 \ln(1 - z) - 3 \ln^2 z + 6 \ln z \ln(1 - z) \right] \right\} \]  
\[ + 2C_F T_F N_f \left\{ 4(2 - z)(1 - z) + 2(3 - z) \ln z \right\} \]  
\[ (168) \]

Taking these results and the corresponding ones for the axial-vector case in Eqs. (155)-(158) as well as the definitions (130)-(132) together with (153) for the \( G \) kernels, we obtain from the NLO kernel \([48]\) the diagonal terms
\[ D^V(z) = P^{(1)V}(z) - \left\{ \mathcal{A}^A + \mathcal{A}^\delta + \mathcal{G}^V \right\} (z). \]  
\[ (169) \]

Again we observe a cancellation of all double log terms. As to be expected, the convolution formulae from the previous subsection are not able to express all terms in \( D^V \) as convolutions. What is missing are convolutions of \( c \)-type kernels with \( a, b \) and \( c \) ones. We show in the following, that this is enough to restore \( D^V \) for non-forward kinematics.

For the pure singlet part in the \( QQ \) channel we have
\[ 2C_F T_F N_f \left\{ \frac{2}{3} Q_Q^{a} \otimes Q_Q^{a} - 2Q_Q^{a} + (1 - z) \frac{17 + 46z + 17z^2}{3z} + 4(1 + z) \frac{1 + 8z + z^2}{3z} \ln z \right\}, \]

where the convolution of \( a \) kernels is \( Q_Q^{a} \otimes Q_Q^{a} = -2Q_Q^{a}(z) - Q_Q^{a}(-z) \ln z \). Making use of
\[ Q_Q^{c} \otimes Q_Q^{c} = -(1 - z) \frac{11 + 38z + 11z^2}{27z} - (1 + z) \frac{1 + 8z + z^2}{9z} \ln z, \quad Q_Q^{c} = \frac{(1 - z)^3}{3z}. \]

we can restore the ER-BL representation
\[ Q_Q D^V = \text{NSD} + 4C_F T_F N_f \left\{ \frac{1}{3} Q_Q^{a} \otimes Q_Q^{a} - 6Q_Q^{c} \otimes Q_Q^{c} - Q_Q^{a} + \frac{7}{6} Q_Q^{c} \right\}, \]  
\[ (170) \]

For the \( QG \) entry we find
\[ Q_G D^V(z) = 2C_F T_F N_f \left\{ -\frac{1}{2} Q_Q^{a} \otimes Q_G^{a} - \frac{3}{4} Q_Q^{a} - 3Q_Q^{c} - (1 - z) - (1 - 2z + 2z^2) \ln z \\
+ 2(1 - z)^2 \ln(1 - z) \right\} + 2C_A T_F N_f \left\{ \frac{130}{3} Q_Q^{a} \otimes Q_G^{a} + \left( \frac{55}{9} - 2\zeta(2) \right) Q_G^{a} \\
- \left( \frac{301}{18} + 4\zeta(2) \right) Q_G^{c} + 8(1 - z) \frac{7 + 130z + 70z^2}{9z} + 8 \frac{3(20 + 44z + 5z^2)}{3} \ln z \\
+ \frac{16}{3} (1 - z)^2 \ln(1 - z) \right\}. \]

With the help of
\[ [Q_Q^{a} \otimes Q_Q^{c}] = 1 - z + (1 - 2z + 2z^2) \ln z - 2(1 - z)^2 \ln(1 - z), \]
\[ Q_Q^{c} \otimes Q_G^{c} = -(1 - z) \frac{1 + 19z + 10z^2}{9z} - \frac{1}{3}(3 + 6z + z^2) \ln z, \]

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we can express the remaining terms as convolutions and obtain so the ER-BL representation:

$$\begin{align*}
QGD^V &= -C_F T_F N_f \left\{ 2 \left[ QQ_v \right] + \frac{6}{2} QG_v \otimes QG_v + QG_v \otimes QQ_v + \frac{3}{2} QQ_v + 6 QG_v \right\} \\
&+ 2 C_A T_F N_f \left\{ - \left[ \frac{8}{3} QQ_v \right] + 56 QG_v \otimes QG_v + \frac{130}{3} QG_v \otimes QG_v \right\} \\
&+ \left[ \frac{55}{9} - 2 \zeta(2) \right] QG_v - \left[ \frac{301}{18} + 4 \zeta(2) \right] QG_v \right\}. \\
\end{align*}$$

A similar situation appears in the $GQ$ and $GG$ channel. Beside the known convolutions we need only the following ones

$$\begin{align*}
GG_p^A \otimes GQ_p^c &= 5(1-z) + (4+z) \ln z + \frac{(1-z^2)}{z} \ln(1-z), \\
GG_p^c \otimes GQ_p^c &= -(1-z) \frac{10 + 19z + z^2}{3z} - \frac{1 + 6z + 3z^2}{z} \ln z, \\
GG_p^A \otimes GG_p^c &= 10(1-z) + (6 + 3z + z^2) \ln z + \frac{(1-z)^3}{z} \ln(1-z), \\
GG_p^c \otimes GG_p^c &= -(1-z) \frac{11 + 38z + 11z^2}{3z} - (1+z) \frac{1 + 8z + z^2}{z} \ln z,
\end{align*}$$

and thus find the following diagonal kernels

$$\begin{align*}
GQD^V &= C_F^2 \left\{ -GG_v^A \otimes \left[ \frac{1}{2} GG_v^a + 3 GQ_v \right] - 5 GQ_v \otimes GG_v^a - 3 GQ_v^a \right\} \\
&- C_F \beta_0 \left\{ GG_v^A \otimes \left[ \frac{1}{2} GG_v^a + GG_v \right] + \frac{3}{4} GG_v \otimes GG_v^a + \frac{5}{3} GG_v^a \right\} \\
&+ C_F C_A \left\{ - GG_v^A \otimes \left[ GG_v^a - \frac{3}{2} GQ_v \right] - \frac{25}{6} GG_v \otimes GG_v^a + GQ_v \otimes GG_v^c \right. \\
&\quad \left. - \frac{43}{9} + 2 \zeta(2) \right) GG_v^a + \left( \frac{8}{9} - 4 \zeta(2) \right) GQ_v^c \right\}, \\
GGD^V &= C_A^2 \left\{ GG_v^A \otimes \left[ GG_v^a + \frac{11}{3} GG_v \right] - 14 GG_v \otimes GG_v^a + 12 GQ_v \otimes GQ_v \right\} \\
&+ \frac{2}{3} GG_v^A - \frac{131}{12} GG_v^a + \frac{91}{18} GG_v^c - 2 \delta(x-y) \right\} \\
&- C_A \beta_0 \left\{ \frac{-1}{2} GG_v \otimes GG_v^a + \frac{5}{3} GG_v^a + 3 GG_v + \frac{13}{3} GQ_v^c + 2 \delta(x-y)\right\} \\
&+ C_F T_F N_f \left\{ GG_v^a \otimes GG_v + \frac{4}{3} GQ_v \otimes GG_v - \delta(x-y)\right\}.
\end{align*}$$

It is worth mentioning that our result for the evolution kernels in the parity even singlet sector possesses the correct conformal moments in both the physical and unphysical sectors. This is to be contrasted with an explicit momentum fraction space calculation at LO and quark bubble insertions in NLO kernels for the mixed channels \[23\] where the improved kernels do not appear.
4 Explicit representations of the kernels.

In this section we give the explicit form of the ER-BL type kernels. The analytical continuation to the whole \((x,y)\)-plane allows to extend our results to the skewed kinematics and extract from them the so-called skewed DGLAP kernels, which are responsible for the evolution of SPDs in the DGLAP region. The exclusive convolutions we need are straightforward to handle and details are collected in appendix C. To represent the final result in the shortest possible manner, we expanded the output of the convolutions in terms of powers of logs. Then we rewrite their arguments in terms of the ratio \(x/y\) where the remaining ones possess only an \(x\)-dependence. It turns out that these terms for the (non-)diagonal entries are (anti)symmetric with respect to the interchanges \(x \to \bar{x}, \ y \to \bar{y}\), so they have the support \(0 \leq x, y \leq 1\).

Now we should comment on the \(G\)-kernels constructed in section 3.2. Although they are diagonal, this property is lost for the unphysical moments when we map the second \(\theta\)-structure to the first one by means of symmetry. A slight change in the definition of \(ABH^I\) and \(AB\overline{H}^I\) in Eqs. (128) and (129), by subtracting a symmetric function, i.e.

\[
2 \left[ \theta(y-x) \pm \theta(x-y) \right] \left[ ABf^I \ln x \ln y - 2 ABf^I [\ln x + \ln y] \right],
\]

from the common \(\theta\)-structure and adding them to the second one, ensures that both parts are separately diagonal. Thus, in the following we use instead of the definitions (128) and (129) the definitions:

\[
ABH^I = \pm 2 AB\overline{H}^I \ln x \ln y - 2 ABf^I [\ln x + \ln y],
\]

(174)

\[
AB\overline{H}^I = \left( + AB\overline{f}^I + ABf^I \right) \left[ 2 \text{Li}_2 \left( 1 - \frac{x}{y} \right) + \ln^2 y \right] + 2 ABf^I [\ln y - \ln x \ln y] \pm 2 AB\overline{f}^I \text{Li}_2(x),
\]

where again the upper (lower) sign corresponds to the (non)diagonal entries. Together with the addenda given in eqs. (134)-(136) as well as eqs. (140)-(142) the kernels are separately diagonal w.r.t. the Gegenbauer polynomials.

The LO kernels are defined in Eqs. (73)-(78) and the NLO kernels we write for convenience as

\[
\mathbb{V}^{(1)I}(x,y) = \left( \begin{array}{cc} QQV^{(1)I}(x,y) & \text{QG\mathbb{V}^{(1)I}(x,y)} \\ \text{GQ\mathbb{V}^{(1)I}(x,y)} & \text{GG\mathbb{V}^{(1)I}(x,y)} \end{array} \right) + \left( \begin{array}{cc} 0 & 0 \\ 0 & GG\mathbb{V}^{(1)I}_{11} \end{array} \right) \delta(x-y),
\]

(175)

where contrary to our previous definitions \(]\text{AA\mathbb{V}^{(1)I}}\) now denotes the conventional regularization (73) applied also in the \(GG\) channel. The contributions concentrated in \(x = y\) are simply fixed by the lowest conformal moment. We have in the \(GG\) channel:

\[
\text{GG}\mathbb{V}^{(V)I}_{11} = - \frac{N_f}{108} (35C_A + 74C_F),
\]

(176)

\[
\text{GG}\mathbb{V}^{(A)I}_{11} = C_A^2 \left( \frac{95}{27} - \frac{14}{3} \zeta(2) + 2\zeta(3) \right) + \frac{N_f}{54} (29C_A - 28C_F).
\]
The $QQ$ channel contains a contribution coming from the non-singlet kernel and the so-called pure singlet part. Analogous to the forward case we define the $\pm$ kernels in the flavor non-singlet sector to be

$$Q_{QQ}^{(1)\pm} = Q_{QQ}^{(1)\pm} = Q_{QQ}^{(1)A\pm} =$$

$$\left[C_F \left\{ \theta(y - x) \left[ \left( \frac{4}{3} - 2\zeta(2) \right) QQ_f - \frac{3x}{y} - \left( \frac{3}{2} \frac{QQ_f - x}{2y} \right) \ln \frac{x}{y} - (QQ_f - QQ_T) \ln \frac{x}{y} \right] \right. \right.$$

$$\left. \times \ln \left( 1 - \frac{x}{y} \right) + \left( \frac{QQ_f + x}{2y} \right) \ln \frac{x}{y} - \frac{x}{2y} \ln x \left( 1 + \ln x - 2 \ln \bar{x} \right) \right\} - C_F \frac{\beta_0}{2} \theta(y - x) \right] (177)$$

$$\times \left\{ \frac{5}{3} QQ_f + \frac{x}{y} + QQ_f \ln \frac{x}{y} \right\} - C_F \left( \frac{C_F - C_A}{2} \right) \theta(y - x) \left\{ \frac{4}{3} QQ_f + 2 \frac{x}{y} + QQ_h(x, y) \right\}$$

$$\mp QQ_T(x, y) + \left\{ \frac{x}{y} \right\} \right\} + \frac{1 \pm C_F}{2} \left( C_F - C_A \right) \left[ \frac{13}{2} - 6\zeta(2) + 4\zeta(3) \right] \delta(x - y).$$

The kernels $Q_{QQ}^{(1)\pm}$ are responsible for the evolution of the sum (difference) and difference (sum) of quark and anti-quark distributions in the parity even (odd) sector. Our results for the entries of the parity odd sector read:

$$Q_{QQ}^{(1)A} =$$

$$Q_{QQ}^{(1)A} = -2C_F T_F N_f \left\{ \theta(y - x) \left[ \frac{1}{y^2} \ln \frac{x}{y} - \frac{x}{y} \ln x \ln \left( 1 - \ln x \right) \right] + \left\{ \frac{x}{y} \right\} \right\},$$

$$Q_{QQ}^{(1)A} =$$

$$\left[C_F \left\{ \theta(y - x) \left[ \frac{2}{y^2} \ln \frac{x}{y} - \frac{x}{y} \ln x \left( 1 - \ln x \right) \right] + \left\{ \frac{x}{y} \right\} \right. \right.$$

$$\left. \times \ln \left( 1 - \frac{x}{y} \right) + \left( \frac{QQ_f - \frac{3x}{y^2}}{2y^2} \right) \ln \frac{x}{y} - 2 \left( QQ_f - QQ_T \right) \ln \frac{x}{y} \right\} - \frac{1}{y^2} \ln x \ln \left( 2 \ln \bar{x} \right)$$

$$+ \left\{ \frac{x}{y} \right\},$$

$$Q_{QQ}^{(1)A} =$$

$$\frac{C_F^2}{2} \left\{ \theta(y - x) \left[ - \frac{3}{y} (1 \pm x) \ln \frac{x}{y} - \frac{x^2 y}{y^2} \ln \frac{x}{y} - \frac{x(\bar{x} + \bar{y})}{y^2} \ln \left( 1 - \frac{x}{y} \right) + \frac{QQ_f}{2} \right] \right.$$

$$\times \ln \frac{x}{y} - \left( QQ_f + QQ_T \right) \ln \left( 1 - \frac{x}{y} \right) \right\} - \frac{x^2 y}{y^2} \ln x - \frac{x(\bar{x} + \bar{y})}{y^2} \ln x \left( 1 - \frac{x}{y} \right) \right\} - C_F \frac{\beta_0}{2}$$

$$\times \left\{ \theta(y - x) \left[ - \frac{2}{3} QQ_f + \frac{2}{3} \left( QQ_f + QQ_T \right) \ln \left( 1 - \frac{x}{y} \right) \right] + \frac{QQ_f}{2} \ln x \right\} + C_F C_A.$$
which die out in the forward limit, we recover in the parity even sector a similar structure as

\[ \left\{ x \to \bar{x}, y \to \bar{y} \right\}, \]

\[ G_{QV}^{(1)A} = \]

\[ C_A^2 \left\{ \theta(y - x) \left[ \frac{2}{3} - 2\zeta(2) \right] G_{f^A} - \frac{15 x^2}{4 y^2} - 6 \frac{\bar{x} \bar{y}}{y^2 \bar{y}} - \left( \frac{2 x \bar{y} - 6 \bar{x} \bar{y} + x^2}{2 y^2} \right) \right\} \]

\[ \times \ln \left( \frac{1 - \frac{x}{y}}{1 - \frac{1}{y}} \right) - \frac{1}{2} \left( G_{f^A} + \frac{3 x^2}{y} \right) \ln^2 \frac{x}{y} + \frac{1}{2} \left( G_{f^A} - \bar{G}_{\bar{T}} \right) \ln^2 \left( \frac{y}{x} - 1 \right) - \frac{1}{2} G_{\bar{h}^A} \]

\[ + \frac{1}{2} G_{\bar{h}^A} \ln \frac{x}{y} + \left( \frac{5 \bar{x} \bar{y}}{y} + \frac{x}{y} + G_{f^A} \left( -2 + \frac{1}{2} \ln x - \ln \bar{x} \right) + \frac{3 x^2}{2 y} \ln x \right) \ln \left\{ x \to \bar{x}, y \to \bar{y} \right\}, \]

\[ \left\{ x \to \bar{x}, y \to \bar{y} \right\}. \]

Except of the pure singlet part, where the convolution of two c-kernels generates Spence functions which die out in the forward limit, we recover in the parity even sector a similar structure as observed in the previous cases:

\[ Q_{QV}^{(1)V} = \]

\[ Q_{QV}^{(1)+} + 2 C_{f^F} T_{f^F} N_f \left\{ \theta(y - x) \left[ \frac{x(3 - 8x \bar{y})}{y} + \frac{x(5 - 8x)}{\bar{y}} \ln \frac{x}{y} + \left( \frac{x}{\bar{y}} - 4x \bar{x} \right) \right] \right\} \]

\[ \times \ln \frac{x}{y} + 8x \bar{x} \left( \mathrm{Li}_2(\bar{x}) - \mathrm{Li}_2(\bar{y}) + \ln \bar{x} \ln y \right) - \frac{290}{9} x \bar{x} - \left( \frac{x(5 - 8x)}{\bar{y}} + \frac{2x(9 - 19\bar{x})}{3} \right) \]

\[ \times \ln x - \left( \frac{x}{y} - 4x \bar{x} \right) \ln^2 x + 4x \bar{x} \ln x \ln \bar{x} \right\} \right\} \right\} \right\} \]

\[ Q_{QV}^{(1)V} = \]

\[ N_{f^F} T_{f^F} C_A \left\{ \theta(y - x) \left[ 2(5 - 2 \zeta(2)) G_{f^V} - 4 G_{f^A} + \frac{x}{y \bar{y}} + 2 \left( G_{f^V} + \bar{G}_{\bar{T}} + \frac{2x - 2y + xy}{2y \bar{y}^2} \right) \right] \right\} \]

\[ \times \ln \frac{x}{y} - 2 \left( G_{f^V} + \bar{G}_{\bar{T}} + \frac{1}{y \bar{y}} \right) \ln \left( 1 - \frac{x}{y} \right) + \bar{G}_{\bar{T}} \ln^2 \frac{x}{y} + \left( G_{f^V} + \bar{G}_{\bar{T}} \right) \ln^2 \left( \frac{y}{x} - 1 \right) \]

\[ + 2 \left( G_{f^V} + \bar{G}_{\bar{T}} + \frac{x(4y^2)}{2y \bar{y}^2} \right) \ln x - 2 G_{f^V} \ln \bar{x} \ln x + \left( G_{f^V} - \bar{G}_{\bar{T}} \right) \ln^2 \left( \frac{y}{x} - 1 \right) \]

\[ \times \ln^2 x \right\} + N_{f^F} T_{f^F} C_A \left\{ \theta(y - x) \left[ 6x(3 - 4x) \frac{2x(11 - 16x)}{\bar{y}} + 4x(1 - 3x) \right] \right\} \]

\[ - 2 \left( G_{f^V} - 5 \bar{G}_{\bar{T}} + \frac{5 - 7x}{y \bar{y}} + 2x(3 - 2x - 12 \bar{y}) \right) \ln \frac{x}{y} + 2 \left( G_{f^V} + \bar{G}_{\bar{T}} + \frac{1}{y \bar{y}} \right) \]
Now we will derive the NLO skewed DGLAP kernels which govern the evolution of SPDs in...
the kinematic range of $\zeta < z < 1$ and $-1 + \zeta < z < 0$ with $z = \frac{i1+y}{1+y}$ and $\zeta = \frac{2n}{1+y}$. We chose this set of variables such that we have the closest resemblance to the usual DGLAP kinematics. In fact for DVCS and vector meson production $\zeta = x_{Bj}$.

After the continuation of the ER-BL kernels to the whole $x - y$ plane by the replacement of the $\theta$-function structure, we derive the kernels in the following way which is suggested by the non-zero support of the $\theta$-functions after the replacement $x \to z/\zeta, \bar{x} \to 1 - z/\zeta \equiv \frac{\bar{z}}{\zeta}, y \to 1/\zeta$ and $\bar{y} \to 1 - 1/\zeta \equiv \frac{1}{\zeta}$ for the above mentioned kinematical regime:

$$P^{(1)}_{I}(z, \zeta) = \left( \frac{1}{\zeta} \left( QQf^{(1)}_{I} \left( \frac{\zeta}{x}, \frac{1}{\zeta} \right) - QQf^{(1)}_{I} \left( \frac{\zeta}{\bar{x}}, \frac{1}{\zeta} \right) \right) + \frac{1}{\zeta} \left( GQf^{(1)}_{I} \left( \frac{\zeta}{x}, \frac{1}{\zeta} \right) + GQf^{(1)}_{I} \left( \frac{\zeta}{\bar{x}}, \frac{1}{\zeta} \right) \right) \right),$$

where the functions $ABf^{(1)}_{I}(x, y)$ are the coefficients of the $\theta(y - x)$-term appearing in the ER-BL kernels. Note that we omitted the additional factors of $1/z$ in the $GG$ and $GQ$ kernels so as to take into account the fact that in the forward limit, the skewed gluon distribution turns into $zG(z, Q^2)$ rather than $G(z, Q^2)$. Thus, we can take the ”+-”-prescription from the ER-BL kernels, where contributions concentrated in $x = y$ turns into end-point concentrated ones, i.e. $\delta(x - y) \to \delta(1 - z)$.

Following our afore mentioned prescription, the result for the parity odd and even skewed DGLAP kernels are up to end-point concentrated terms, which can be easily restored, given by:

$$QQ P^{\pm}_{NS} =$$

$$\frac{C_F}{2} \left[ \frac{1}{2} \ln^2 \left( 1 - \frac{\bar{z}}{\zeta} \right) \left[ \frac{1}{\zeta} \left( 1 - \frac{z\bar{z}}{\zeta^2} \right) + \frac{z}{\bar{z}} \left( 1 - \frac{\bar{z}}{\zeta} \right) \right] + \frac{1}{2} \ln^2(z) \left[ \frac{2\bar{z}}{z} \left( 1 - \frac{\bar{z}}{\zeta} \right) + \frac{z}{\bar{z}} \right] \right]$$

$$-QQ \zeta \left[ \ln \left( 1 - \frac{\bar{z}}{\zeta} \right) \ln \left( \frac{\bar{z}}{\zeta} \right) + \ln(z) \ln(z) \right] - 2 \ln \left( 1 - \frac{\bar{z}}{\zeta} \right) \left( 1 - \frac{\bar{z}}{\zeta} \right) \left[ \frac{1 - \bar{z} - \bar{z} - \frac{z\bar{z}}{\zeta^2}}{\zeta^2} \right]$$

$$- \frac{1}{2} \ln(z) \left[ \frac{4z}{\zeta} + \frac{3z}{\bar{z}} \left( 1 - \frac{\bar{z}}{\zeta} \right) \right] + \frac{3\bar{z}}{\zeta} + \left[ \frac{4 - \pi^2}{3} \right] QQ \zeta$$

$$- \frac{C_F \beta_0}{2} \left[ \frac{5}{3} QQ \zeta + \frac{\bar{z}}{\zeta} + \ln \left( 1 - \frac{\bar{z}}{\zeta} \right) \left[ \frac{1 - \frac{z\bar{z}}{\zeta^2}}{\zeta} \right] + \ln(z) \left[ \frac{\bar{z}}{z} \left( 1 - \frac{\bar{z}}{\zeta} \right) + \frac{z}{\zeta} \right] \right]$$

$$- \frac{C_F}{2} \left( C_F - C_A \right) \left[ \frac{2\bar{z}}{\zeta} + \frac{4}{3} QQ \zeta + QQ h + QQ \tilde{h} \right].$$

(187)

$$QQ P^A_{S} =$$

$$+ 2C_F T_F N_F \left\{ \left( 1 - \frac{\bar{z}}{\zeta} \right) \ln \left( 1 - \frac{\bar{z}}{\zeta} \right) \left[ \ln \left( 1 - \frac{\bar{z}}{\zeta} \right) - 1 \right] + \frac{z}{\zeta} \ln(z) \left[ \ln(z) - 1 \right] \right\}$$

$$+ QQ P^{-}_{NS},$$

(188)

$$QG P^A_{S} =$$
\[
+ C_A N_F T_F \left\{ \left( \frac{3}{2} Q^G p^A + \frac{4}{\zeta^2} \left( 1 - \frac{z}{\zeta} \right) + 3 \right) \ln^2 \left( 1 - \frac{\bar{z}}{\zeta} \right) - \frac{1}{2} Q^G p^A \left[ \ln^2 \left( \frac{\bar{z}}{\zeta} \right) + \ln^2 (\bar{z}) \right] - \ln^2 (z) \left[ 1 + \frac{2}{\zeta^2} \right] + 2 \ln (1 - \frac{\bar{z}}{\zeta}) \left[ Q^G p^A + 2 + \frac{3}{\zeta^2} \left( 1 - \frac{z}{\zeta} \right) \right] + 2 \ln (z) \left[ 1 + \frac{2}{\zeta^2} \right] \right\} \\
+ 2 \left[ \ln \left( \frac{\bar{z}}{\zeta} \right) + \ln (\bar{z}) \right] \left( \frac{1}{2} Q^G p^A - \frac{1}{\zeta} \right) - 2 Q^G p^A + \frac{6}{\zeta} - \frac{1}{2} Q^G h^A - \frac{1}{2} Q^G h^A \right\} \\
+ C_F N_F T_F \left\{ \frac{z}{\zeta} \left[ \ln^2 \left( 1 - \frac{\bar{z}}{\zeta} \right) - \ln^2 (z) \right] + \frac{1}{2} Q^G p^A \left[ \ln^2 \left( \frac{\bar{z}}{\zeta} - 1 \right) + \ln^2 \left( \frac{\bar{z}}{\zeta} \right) - \ln^2 (z) \right] + \ln (z - \zeta) \left( 1 - \frac{\bar{z}}{\zeta} \right) \right\} + 3 \left( 1 - \frac{z}{\zeta} \right) - 2 z^{-\frac{1 + \bar{z}}{\zeta}} + 4 \ln (\bar{z}) \left[ 1 + \frac{\bar{z}}{\zeta^2} \right] \\
- \ln (z) \left[ \frac{z}{\zeta} \left( 2 + \frac{5}{\zeta} \right) + 3 \frac{z}{\zeta^2} \right] + \left( 5 - \frac{\pi^2}{3} \right) Q^G p^A - \frac{5}{\zeta} \right\},
\]

\[
G_G P^A_s =
\]

\[
+ C_F \left\{ \frac{1}{2} \ln^2 \left( 1 - \frac{\bar{z}}{\zeta} \right) \left[ G^G p^A - \frac{z^2}{\zeta} \right] - \frac{1}{2} G^G p^A \left[ \ln^2 \left( \frac{\bar{z}}{\zeta} \right) + \ln^2 (\bar{z}) \right] + \frac{z^2}{2 \zeta} \ln^2 (z) \right\} \\
- \frac{1}{2} \ln (z - \zeta) \left( 1 - \frac{\bar{z}}{\zeta} \right) \left[ z - \zeta + 3 - \frac{5 z}{\zeta} \right] + \frac{1}{2} \ln (\bar{z}) \left[ \left( 1 - \frac{\bar{z}}{\zeta} \right) \left( 1 - \frac{z}{\zeta} \right) \left( 4 - \zeta \right) + z \left( 2 - \frac{z}{\zeta} \right) \right] \\
\frac{1}{2} \ln (z) \left[ \frac{z}{\zeta} \left( 1 - \frac{\bar{z}}{\zeta} \right) \left( 1 - \frac{z}{\zeta} \right) + z \left( 2 + \frac{1}{\zeta} \right) \right] - \ln (\bar{z}) \left[ \left( 1 - \frac{\bar{z}}{\zeta} \right) \left( 2 + z - \zeta + \zeta \right) \right] \\
- \frac{3}{2} G^G p^A - \frac{3}{2} z - \frac{2}{3} \left( 1 - \frac{\bar{z}}{\zeta} \right) \right\} - \frac{2}{3} C_F N_F T_F \left\{ G^G p^A \left[ \ln \left( \frac{\bar{z}}{\zeta} \right) + \ln (\bar{z}) - \frac{2}{3} \right] + 2 z + 2 \left( 1 - \frac{\bar{z}}{\zeta} \right) \right\} \\
+ C_F C_A \left\{ \frac{1}{2} \ln^2 \left( 1 - \frac{\bar{z}}{\zeta} \right) \left[ 3 G^G p^A \zeta - \frac{z^2}{\zeta} \left( 1 + 3 \bar{z} \right) \right] + \frac{1}{2} \ln^2 (z) \left[ \frac{z^2}{\zeta} (4 - \zeta) - G^G p^A \right] \right\} \\
+ \frac{1}{3} G^G p^A \left[ \ln^2 \left( \frac{\bar{z}}{\zeta} - 1 \right) + \ln^2 \left( \frac{\bar{z}}{\zeta} \right) \right] + \ln \left( 1 - \frac{\bar{z}}{\zeta} \right) \left( 1 - \frac{z}{\zeta} \right) \left[ 4 - 5 z + 3 \zeta \left( 1 - \frac{z}{\zeta} \right) \right] \\
- \frac{1}{6} \ln \left( \frac{\bar{z}}{\zeta} \right) + \ln (\bar{z}) \right\} \left[ 5 z + \left( 1 - \frac{\bar{z}}{\zeta} \right) \left( 5 + z \right) \right] - \ln (z) \left[ \frac{4 z}{\zeta} \left( 1 - \frac{\bar{z}}{\zeta} \right) + z \left( 1 - \frac{z}{\zeta} \right) \right] \\
\left( \frac{28}{9} - \frac{\pi^2}{3} \right) G^G p^A + \frac{8}{3} \left( 1 + z - \frac{\bar{z}}{\zeta} \right) + \frac{1}{2} G^G h^A - \frac{1}{2} G^G h^A \right\}.
\]

\[
G_G P^A_s =
\]

\[
- \frac{2}{3} C_A N_F T_F \left\{ \ln \left( 1 - \frac{\bar{z}}{\zeta} \right) \zeta \left( 1 - \frac{z}{\zeta} \right)^2 - \frac{z^2}{\zeta^2} \ln (z) - \frac{8}{3} G^G p^A + \frac{z}{\zeta} \ln (1 - \frac{\bar{z}}{\zeta}) \left( 1 - \frac{\bar{z}}{\zeta} \right) + \right\} \\
- \frac{3}{2} G^G p^A + \frac{2 z}{\zeta^2} \left( 1 - z \zeta \right) \right\} \\
+ C_F N_F T_F \left\{ \ln^2 \left( 1 - \frac{\bar{z}}{\zeta} \right) \zeta \left( 1 - \frac{z}{\zeta} \right) - \frac{z^2}{\zeta^2} \ln^2 (z) - 2 \ln \left( 1 - \frac{\bar{z}}{\zeta} \right) \left( 1 - \frac{\bar{z}}{\zeta} \right) - \zeta \left( 1 - \frac{z}{\zeta} \right) \right\}
\]

\[-2 \ln(z) \frac{1}{\zeta} \left[ z \left( 1 - \frac{z}{\zeta} \right) + \frac{z(2 - z)}{\zeta} \right] - 8 \frac{z\zeta}{\zeta^2} + 4 \frac{\zeta}{\zeta^2} (1 - z^2) - 2 \left( \frac{\zeta}{\zeta} \right)^2 \]
\[+ C_A^2 \left\{ \frac{1}{2} \ln^2 \left( 1 - \frac{z}{\zeta} \right) \left[ 1 + 9z - 4\zeta - \frac{2}{\zeta} (4 - 3z^2) - \frac{8}{\zeta} \left( 1 - \frac{z}{\zeta} \right)^2 \left( \frac{z}{\zeta} \right)^2 (7 - 6z) \right] \right\} \]
\[+ \ln^2(z) \left( \frac{z}{\zeta} \right)^2 \left[ \frac{z}{\zeta} + 2\zeta + \frac{\zeta^2}{\zeta} \right] + \frac{1}{2} \frac{GG}{P} \left[ \ln^2 \left( \frac{z}{\zeta} - 1 \right) - \ln^2 \left( \frac{z}{\zeta} \right) - 2 \ln(z) \ln(z) \right] \]
\[-\ln \left( 1 - \frac{z}{\zeta} \right) \left( 1 - \frac{z}{\zeta} \right) \left[ \frac{2\zeta}{\zeta} + \frac{31}{6} z + \frac{5}{6} \zeta \right] + \ln(z) \frac{z}{\zeta^2} \left[ 6 + 2\zeta - \frac{31}{6} z \right] \]
\[+ \left( \frac{67}{18} - \frac{\pi^2}{3} \right) \frac{GG}{P} + \frac{5}{6} \left( \frac{\zeta}{\zeta} \right) \frac{z}{\zeta} \frac{z}{\zeta} - \frac{11 \zeta}{3} \frac{z}{\zeta^2} - \frac{11 \zeta}{6} \frac{z}{\zeta^2} (1 - z^2) + \frac{1}{2} \frac{GG}{h} + \frac{1}{2} \frac{GG}{h} \right\}, \hspace{1cm} (191) \]

\[QQ P^V_{PS} = \]
\[-2 C_F T_F N_f \left\{ -2 \ln^2 \left( 1 - \frac{z}{\zeta} \right) \left[ \frac{QG}{P} A + \frac{3}{2} + \frac{1}{\zeta^2} \left( 1 - \frac{z}{\zeta} \right) \left( 1 + 6 \frac{z}{\zeta} - 4 \frac{z}{\zeta^2} \right) \right] \right\} \]
\[-\frac{1}{2} \frac{QG}{P} \left[ \ln^2 \left( 1 - \frac{z}{\zeta} \right) + \ln^2 \left( \frac{z}{\zeta} \right) + \ln^2(z) - \ln^2(z) \right] + \ln^2(z) \frac{2}{\zeta^2} \left[ \left( 1 - \frac{z}{\zeta} \right) \left( 1 + 6 \frac{z}{\zeta} - 4 \frac{z}{\zeta^2} \right) \right] \]
\[-\frac{3}{2} + 2 \ln \left( 1 - \frac{z}{\zeta} \right) \left[ \frac{6 QG}{P} - 7 QG \right] + \frac{1}{\zeta^2} \left[ 1 - \frac{z}{\zeta} \right] \left( 3 - 2 \zeta + 20 z - 8 \frac{z}{\zeta} \right) \]
\[+ \ln \left( \frac{z}{\zeta} \right) + \ln \left( \frac{z}{\zeta} \right) \left[ \frac{QG}{P^V} - \frac{2}{\zeta} \right] + \ln(z) \frac{2}{\zeta^2} \left[ 1 + \zeta - \left( 1 - \frac{z}{\zeta} \right) \left( 1 + 4 \zeta + 8 \frac{z}{\zeta} \right) + 2 \frac{z}{\zeta} \right] \]
\[-2 \left( \frac{QG}{P^V} + \frac{QG}{P^V} \right) - 32 \frac{z}{\zeta} \left( 1 - \frac{z}{\zeta} \right) + \frac{2}{\zeta} \left( 3 + 8 \frac{z}{\zeta} \right) + \frac{1}{2} \frac{QG}{h^V} - \frac{1}{2} \frac{QG}{h^V} \right\} \]
\[+ C_F N_f T_F \left\{ - \ln^2 \left( 1 - \frac{z}{\zeta} \right) \frac{z}{\zeta} + \frac{1}{2} \frac{QG}{P} \left[ \ln^2 \left( \frac{z}{\zeta} - 1 \right) + \ln^2 \left( \frac{z}{\zeta} \right) \right] - \ln^2(z) \frac{1}{\zeta^2} \left( 1 - \frac{z}{\zeta} \right) \right\} \]
\[+ \ln(z - \zeta) \frac{1}{\zeta} \left[ 4z - \zeta - \frac{1}{\zeta} \right] - \ln(\bar{z}) \left[ 1 - \frac{z}{\zeta} + 2 \frac{z}{\zeta} \right] + \ln(z) \left[ \frac{z}{\zeta} + 3 \frac{z}{\zeta} \left( 1 - \frac{z}{\zeta} \right) \right] \]
\[+ \left( \frac{z}{\zeta} \right)^2 \right\] + 4 \ln(\bar{z}) \frac{z}{\zeta^2} (2z - \zeta) - \frac{\pi^2}{3} \frac{QG}{P} - 20 \frac{z\zeta}{\zeta^2} + 8 \frac{z}{\zeta^2} + 6 \frac{z}{\zeta^2} + 5 \frac{z}{\zeta} \right\}, \hspace{1cm} (193) \]
\[ + C_F \left\{ + \frac{1}{2} \ln^2 \left( 1 - \frac{\bar{z}}{\zeta} \right) \left[ G_F p^V - 2 + \frac{z^2}{\zeta} \right] - \frac{1}{2} \ln^2 (z) - \frac{1}{2} G_F p^V \left[ \ln^2 \left( \frac{\bar{z}}{\zeta} \right) + \ln^2 (\bar{z}) \right] \right. \\
- \frac{1}{2} \ln \left( 1 - \frac{\bar{z}}{\zeta} \right) \left( 1 - \frac{z}{\zeta} \right) \left[ 3 + \frac{z}{\zeta} \left( 3 - \frac{1}{\zeta} \right) \right] - 2 \ln (\bar{z}) \left[ z \left( 1 - \frac{\bar{z}}{\zeta} \right) + \frac{3}{2} \left( 1 + \frac{z^2}{\zeta} \right) \right] \\
+ \frac{1}{2} \ln \left( \bar{\zeta} \right) \left[ 5 \left( 1 - \frac{\bar{z}}{\zeta} \right) + 3 \bar{z} \left( 1 + \bar{\zeta} \right) \right] + \ln (z) \left[ 1 + \frac{z}{\zeta} \left( 1 + \frac{1}{2\zeta} \right) \right] - \frac{3}{2} \left( 1 - \frac{\bar{z}}{\zeta} \right) (1 + 3z) \\
\left. \right\} - \frac{2}{3} C_F N_F T_F \left\{ G_F p^V \left[ \ln \left( \frac{\bar{z}}{\zeta} \right) + \ln (\bar{z}) \right] + \frac{10}{3} G_F p^V - 2 G_F p^A + 2z + 2 \left( 1 - \frac{\bar{z}}{\zeta} \right) \right\} \\
+ C_F C_A \left\{ + \frac{1}{2} \ln^2 \left( 1 - \frac{\bar{z}}{\zeta} \right) \left[ \left( 1 - \frac{z}{\zeta} \right)^2 \left( 4 - 3\zeta + 8 \frac{\bar{z}}{\zeta} \right) - 2 + \frac{z^2}{\zeta} \right] + \frac{1}{2} G_F p^V \left[ \ln^2 \left( \frac{\bar{z}}{\zeta} \right) - 1 \right] \right. \\
+ \ln^2 \left( \frac{\bar{z}}{z} \right) + \frac{1}{2} \ln^2 (z) \left[ \frac{3}{\zeta} + \bar{z} + 8 \left( 1 - \frac{z}{\zeta} \right) - \left( \frac{\zeta}{z} \right)^2 G_F p^V \right] + \ln \left( \frac{\bar{z}}{z} \right) + \ln \left( \frac{\bar{z}}{z} \right) \\
\times \left[ \left( \frac{\zeta}{z} \right)^2 + \ln (z) \left[ \frac{11}{6} - \frac{11 z^2}{3 \zeta} + \frac{11 - 18z - 11 z^2}{6 \zeta} - \frac{1}{\zeta} \left( 1 - \frac{z}{\zeta} \right) \left( \frac{4 + 8 \zeta - 8 \bar{z}}{\zeta} \right) \right] \\
\right. \\
- \left( \frac{\zeta}{z} \right)^2 + \ln (z) \left[ \frac{11 \bar{z}^2}{\zeta^2} - \frac{2 \bar{z}^2}{\zeta^2} \right] + 2 \ln \left( 1 - \frac{\bar{z}}{\zeta} \right) \left[ (\bar{z} - 2z) \left( 1 - \frac{z}{\zeta} \right) - \bar{z} \left( 1 - \frac{z}{\zeta} \right)^2 \right] \\
- 20 \frac{G_F p^c}{3} - 12 G_F p^a - 4 G_F p^A + 8 \frac{1}{\zeta} \left( 1 - \frac{\bar{z}}{\zeta} \right) - 4 \bar{z} \left( \frac{\zeta}{z} \right)^2 \right\} \\
+ C_A^2 \left\{ \ln^2 \left( 1 - \frac{\bar{z}}{\zeta} \right) \left[ 1 - \frac{1}{\bar{z}} - 2\zeta + 5z - 2 \left( \frac{\zeta}{z} \right)^2 \left[ 2 + \zeta \left( 1 - \frac{z}{\zeta} \right) + 1 + \zeta \right] \right] \\
+ \frac{G_F p^V}{\zeta} \left[ \ln \left( 1 - \frac{\bar{z}}{\zeta} \right) \ln \left( \frac{\bar{z}}{z} - 1 \right) - \ln (z) \ln (\bar{z}) \right] \\
+ \ln^2 (z) \left[ G_F p^V - \frac{1}{\bar{z}} - \frac{1 - 3z}{\zeta^2} + \frac{2}{\zeta} \left( \frac{\zeta}{z} \right)^2 \left[ \left( 3 - \frac{1}{\zeta} \right) \left( 1 - \frac{z}{\zeta} \right) + \bar{z} \right] \right] \right\} \\
\] (194)
where the $AB^{I}$ and $AB^{I}$ are given by

$$
AB^{I} = +2 p^{I} (z, \zeta) \left[ \log(\zeta) \log \left( 1 - \frac{z}{\zeta} \right) + \text{Li}_2 \left( 1 - z \right) - \text{Li}_2 \left( 1 - \frac{z}{\zeta} \right) - \frac{\pi^2}{6} \right]
$$

$$
AB_{\bar{I}}^{I} = +2 p^{I} (\zeta - z, \zeta) \left[ \log(\zeta) \log \left( 1 - \frac{z}{\zeta} \right) + \frac{1}{2} \log(\zeta) \log \left( \frac{z}{\zeta} \right) + \log(z) - \log \left( \frac{z}{\zeta} \right) \right] + \text{Li}_2 \left( 1 - \frac{z}{\zeta} \right)
$$

$$
- \text{Li}_2 \left( 1 - \frac{z}{\zeta} \right) - \frac{\pi^2}{6} - \log(z) \log(1 - z) - \text{Li}_2(\zeta - z) - \text{Li}_2 \left( \frac{z}{\zeta} \right)
$$

$$
+2 k^{I} (z, \zeta) \left[ \log \left( 1 - \frac{z}{\zeta} \right) \log \left( \frac{z}{\zeta} \right) + 2 \text{Li}_2 \left( 1 - \frac{z}{\zeta} \right) - \text{Li}_2 \left( 1 - z \right) \right]
$$

The LO skewed kernels appearing in the above formulas are given by:

$$
QQ^{I}_p = 1 + z^2 - \zeta(1 + z), \quad QQ^A_p = 2 z^2 - z \zeta, \quad QQ^V_p = 2 z^2 + \zeta z - \zeta^2,
$$

$$
gG^A_p = \frac{z(2 - z) - \zeta}{\zeta}, \quad gG^V_p = \frac{1 + z^2 - \zeta}{\zeta},
$$

$$
gG^A = \frac{\zeta^2(1 + z^2) + 2 z(\zeta z - \zeta)}{\zeta^2 z^2} + 4 \frac{z \zeta}{\zeta^2} - \frac{2 \zeta}{\zeta^2}(1 - z^2),
$$

$$
GG^V = \frac{1}{\zeta} \left( z^2 + \left( 1 - \frac{z}{\zeta} \right) \right) + 2 \left( \frac{z}{\zeta} \right)^2 + 2 \frac{1 - z^2}{\zeta} \left( 1 - \frac{z}{\zeta} \right)^2, \quad QQ^C_p = 2 \left( \frac{z}{\zeta} \right)^2
$$

$$
gG^A = \frac{2 z \zeta}{\zeta^2} - \frac{z}{\zeta^2}(1 - z^2), \quad gG^C_p = \frac{z^3}{\zeta^2}.
$$

(197)

where in comparison to our previous conventions we included here a factor 2 in the LO kernels of the QG-channel. The $AB^{I}_{k_i}$ were found to be:

$$
QQ_{k_1} = \frac{z}{\zeta} + \frac{z}{\bar{z}}, \quad QQ_{k_2} = \frac{z}{\zeta} \left( 1 - \frac{\zeta}{\zeta + z} \right), \quad QQ_{k_1}^A = -2 \frac{z}{\zeta}, \quad QQ_{k_2}^A = -2 \frac{z}{\zeta \zeta^2},
$$

46
\[GQ_k^1 = \frac{z^2}{\zeta}, \quad GQ_k^2 = -\frac{z^2}{\zeta}, \quad GG_k^1 = 2\frac{z^2}{\zeta} + \frac{z^2}{\bar{z}}, \quad GG_k^2 = -\frac{z^2}{\zeta^2} \left(2 - \frac{\zeta}{\zeta + z}\right),\]

\[QG_k^1 = -2 \left[\frac{z}{\zeta} - 2 \left(\frac{z}{\zeta}\right)^2 + 4 \left(1 - \frac{z}{\zeta}\right)\frac{z}{\zeta^2}\right], \quad QG_k^2 = 2\frac{z}{\zeta^2} \left[1 + \left(1 - \frac{z}{\zeta}\right)\left(\frac{4}{\zeta} - 6\right)\right],\]

\[GQ_k^1 = -\frac{z^2}{\zeta} \left[1 - \frac{1}{\zeta}\left(6 - 4\frac{z}{\zeta}\right)\right], \quad GG_k^2 = \frac{z^2}{\zeta^2} \left[1 + \frac{z}{\zeta}\left(6 - 4\frac{z}{\zeta}\right)\right],\]

\[GG_k^1 = \frac{2}{\bar{z}} + 2 \left(\frac{z}{\zeta}\right)^2 + 2 \left(1 - \frac{z}{\zeta}\right)\frac{z^2}{\zeta} \left(1 + \frac{2}{\zeta}\right),\]

\[GG_k^2 = \left(\frac{z}{\zeta}\right)^2 \left[10 + 6z - 9\zeta + \frac{1}{\zeta + z}\left(\zeta(1+z) \left(1 - 4\frac{z}{\zeta^2}\right) - 4\bar{\zeta}\right)\right].\]  

The set of explicit formulae given in this section represents the main result of the present paper.

### 5 Conclusions.

In this paper we have given a detailed description of the formalism for construction of the two-loop exclusive evolution kernels in the flavor singlet sectors and presented explicit results. Our formalism allowed us to avoid complicated NLO calculations and was based on three main ingredients: (i) the known form of conformal symmetry breaking counterterms for renormalization of conformal operators in NLO transformed into the language of ER-BL kernels; (ii) supersymmetric constraints which allowed us to construct the contribution of cross ladder diagrams; (iii) reduction formulae and known two-loop splitting functions which completely constrained the diagonal part of NLO ER-BL kernels. The kernels predicted here are numerically checked by direct comparison of their Gegenbauer moments to the anomalous dimensions of conformal operators whose correctness was shown by supersymmetric [50] and superconformal [44] constraints. Moreover, the predicted $\beta_0$ terms have been checked by a direct calculation of diagrams with quark bubble insertions. Note also that the predictions of the conformal operator product expansion rotated to the $\overline{\text{MS}}$ scheme coincide with the NLO coefficient functions. This proofs again our results of conformal symmetry breaking in $QQ$ and $QG$ channels.

As a byproduct, our understanding of the general structure of ER-BL type kernels implies the discovery of a simple structure for the DGLAP kernels, which results from the topology of Feynman graphs. Indeed, the only new functions appearing in NLO is $^{ABG}(z)$ arising from the crossed ladder diagrams. Everything else can be represented as convolutions of LO kernels and the ones obtained in the forward limit of the conformal anomalies. This is interesting, but unfortunately not very restrictive, since the conformal algebra does not constrain these kernels.

With the ER-BL kernels we have calculated, we open up a new way to effectively perform the evolution of singlet distribution amplitudes and skewed parton distributions by direct numerical
integration of two-loop evolution equations. The numerical algorithms are still to be developed but a priori it is clear that they will provide a superior alternative to the ones which rely on the orthogonal reconstruction approach.

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A Uniqueness of extension.

In Eq. (23) we gave a simple recipe for the extension of the ER-BL kernels into the whole region. Based on the holomorphic properties of the Fourier transform of \( \gamma(t, t', \eta) \) we give here a more complicated method which proves the uniqueness of the procedure [4]. Due to the scaling relation (15) we can restrict ourselves to \( \eta = 1 \).

First we perform a Fourier transform of the anomalous dimension \( \gamma(t, t', 1) \) with respect to \( t \), restricted to the ER-BL region \( |t, t'| \leq 1 \). Due to the known support properties (see fig. 1b) it is sufficient to restrict only the variable \( t' \), i.e. \( |t'| \leq 1 \) (the anomalous dimension matrix vanishes then for \( |t| > 1 \)). Taking into account the support restrictions arising from the connection (14) with the anomalous dimension of light-ray operators, we observe that

\[
\tilde{\gamma}(\lambda, t')_{|t'| \leq 1} = \int dt e^{i\lambda t} \gamma(t, t')_{|t'| \leq 1}
\]

\[
= \int_0^1 dy \int_0^1 dz \left( \begin{array}{cc}
QQ\gamma(y, z) & i\lambda QG\gamma(y, z) \\
GQ\gamma(y, z) & GG\gamma(y, z)
\end{array} \right) e^{i\lambda(t' (1-y-z)+y-z)}
\]

is an entire function in \( \lambda \) and \( t' \) for \( |t'| \leq 1 \). Thus we can perform the analytical continuation which coincide with the Fourier transform of \( \gamma(t, t', 1) \). This proves the uniqueness of the extension procedure.

B Forward limit.

We give now a more heuristic derivation of the forward limit procedure (20) of the generalized ER-BL kernels. The moments of the kernels are related to the anomalous dimensions of local twist-2 operators by

\[
\int_0^1 dx \left( \begin{array}{cc}
 QQV & QQV \\
 GQV & GQV
\end{array} \right) (x, y) = -\frac{1}{2} \sum_{k=0}^j \left( \begin{array}{cc}
 QQ\gamma & QQ\gamma \\
 GQ\gamma & GQ\gamma
\end{array} \right) \left( \begin{array}{c}
y_k \\
y_{k-1}
\end{array} \right).
\]

On the other hand the diagonal entries of the anomalous dimensions are given by the moments of the DGLAP kernels

\[
\int dz z^j \left( \begin{array}{cc}
 QQP & QQP \\
 GQP & GQP
\end{array} \right) (z) = -\frac{1}{2} \left( \begin{array}{cc}
 QQ\gamma & QQ\gamma \\
 GQ\gamma & GQ\gamma
\end{array} \right) \left( \begin{array}{c}
y_k \\
y_{k-1}
\end{array} \right).
\]
To extract the diagonal ones from Eq. (200), we substitute \( y \) by \( 1/\eta \) and multiply each entry on both sides with a sufficient power of \( \eta \). After extension of the kernels, we can rescale the integration variable and find in the limit \( \eta \to 0 \):

\[
\lim_{\eta \to 0} \int_{[\eta]} dx^j \left( \frac{Q^Q V}{\eta} \frac{1}{Q^Q V} \frac{1}{x} \frac{Q^G V}{x} \frac{1}{G^G V} \right) \text{ext} \left( \frac{z}{\eta}, \frac{1}{\eta} \right) = -\frac{1}{2} \left( \frac{Q^Q}{Q^G}, \frac{Q^G}{G^G} \right)_{jj}.
\]  

(202)

Interchanging integration and limit and comparison with Eq. (201) provides us with the desired formula:

\[
P(z) = \lim V(x,y) \equiv \lim_{\eta \to 0} \frac{1}{|\eta|} \left( \frac{Q^Q V}{\eta} \frac{1}{Q^Q V} \frac{1}{x} \frac{Q^G V}{x} \frac{1}{G^G V} \right) \text{ext} \left( \frac{z}{\eta}, \frac{1}{\eta} \right).
\]

(203)

Let us look more closely at the forward limit of the \(+\)-prescription. In the QQ-channel the formal prescriptions for exclusive and inclusive channels are simple related to each other:

\[
\lim \left[ Q^Q V(x,y) \right]_+ = \lim Q^Q V(x,y) - \delta(1-z) \int dz \lim Q^Q V(x,y) = \left[ Q^Q P(z) \right]_+.
\]

(204)

However, in the GG-channel an extra \( z \) factor appears which induce a finite term concentrated at \( z = 1 \):

\[
\lim \left[ G^G V(x,y) - \delta(x-y) \int dz G^G V(z,y) \right] = z^{G^G P(z)} - \delta(1-z) \int dz z^{G^G P(z)}
\]

(205)

In order to have the same simple correspondence between the \(+\)-prescriptions for exclusive and inclusive kernels as given in Eq. (204) for the QQ channel, we redefine, as implicitly done in our previous papers \cite{52}, the standard \(+\)-definition by a finite part:

\[
\left[ G^G V(x,y) \right]_+ = G^G V(x,y) - \delta(x-y) \int dz G^G V(z,y) + \int dz(1-z) G^G P(z).
\]

(206)

If the kernel \( v(x,y) \) contains only the usual \( \theta \)-structure, i.e. \( \theta(y-x) \), the limit yields

\[
\lim v(x,y) = \theta(1-z) \theta(z) \lim F(x,y),
\]

\[
\lim F(x,y) \equiv \lim_{\eta \to 0} \frac{1}{|\eta|} \left( \frac{Q^Q F}{\eta} \frac{1}{Q^Q F} \frac{1}{x} \frac{Q^G F}{x} \frac{1}{G^G F} \right) \left( \frac{z}{\eta}, \frac{1}{\eta} \right).
\]

In the case of \( \theta(y-\bar{x}) \)-structure, we have to replace in the above equation \( \theta(1-z) \theta(z) \) by \( \theta(1+z) \theta(-z) \).

### C Exclusive convolutions.

In this appendix we present the exclusive convolution formula. Due to the two different regions \( y > x \) and \( y < x \) appearing in the kernels, the convolution looks more complicated than for the
inclusive case. Fortunately, it is sufficient to consider one of the regions and for the convolution of two regular kernels,

\[
ABv_i(x, y) = \theta(y - x)^{AB}f(x, y) - (-1)^{\nu(A) + \nu(B)} \left\{ \begin{array}{c} x \to \bar{x} \\ y \to \bar{y} \end{array} \right\} \quad \text{with} \quad i = \{1, 2\},
\]

we use their symmetry and define the convolution of the \(f_i(x, y)\) functions by

\[
ABv_1 \otimes^{BC} v_2(x, y) = \theta(y - x)^{AB}f_1 \otimes^{BC} f_2(x, y) - (-1)^{\nu(A) + \nu(C)} \left\{ \begin{array}{c} x \to \bar{x} \\ y \to \bar{y} \end{array} \right\},
\]

where

\[
ABf_1 \otimes^{BC} f_2(x, y) \equiv \int_x^y dz \ ABf_1(x, z)^{BC} f_2(z, y) - (-1)^{\nu(B) + \nu(C)} \int_y^1 dz \ ABf_1(x, z)^{BC} f_2(z, y) - (-1)^{\nu(A) + \nu(B)} \int_1^x dz \ f_1(x, z)^{BC} f_2(z, y).
\]

In the case that the kernels need a regularization, e.g. the conventional one

\[
ABf_+(x, y) = ABf(x, y) - \delta(x - y) \int_0^y dz ABf(z, y),
\]

we have to arrange the integrals in such a way that each of them is defined. Since the regularization only appears in the diagonal channels, we have to treat the following two cases for convolutions in the mixed channels:

\[
ABf_1 \otimes^{BB} f_2(x, y) = \int_x^y dz \ f_1(x, z) f_2(z, y) + \int_y^1 dz \ f_1(x, z) f_2(z, y) - \int_0^x dz \ f_1(\bar{x}, \bar{z}) f_2(z, y),
\]

\[
f_{1,+}^{AA} \otimes f_{2}^{AB} = \int_x^y dz \ f_1(x, z) f_2(z, y) + \left\{ f_1(x, z) - f_1(\bar{z}, \bar{x}) \right\} f_2(x, y)
\]

\[
- \int_y^1 dz \ f_1(x, z) f_2(z, y) + f_1(\bar{z}, \bar{x}) f_2(x, y)
\]

\[
+ \int_0^x dz \ f_1(\bar{x}, \bar{z}) f_2(z, y) + \left\{ f_1(\bar{x}, \bar{z}) - f_1(z, x) \right\} f_2(x, y),
\]

where \( A \neq B \) and \( f_i \equiv \left\{ f_i^{AA}, f_i^{AB} \right\} \) and each term is separately integrable. The corresponding equations for the convolution of the diagonal kernels follows by changing the sign in the third or second integral, respectively. For the convolution of two kernels with "+"-prescription we write

\[
\left[ AA v_1 \right]_+ \otimes \left[ AA v_2 \right]_+ (x, y) = \left[ \theta(y - x)^{AA}f_1 \otimes^{AA} f_2(x, y) + \left\{ \begin{array}{c} x \to \bar{x} \\ y \to \bar{y} \end{array} \right\} \right]_+,
\]

where the convolution on the r.h.s. is given by

\[
f_{1}^{AA} \otimes f_2^{AA} = \int_x^y dz \left\{ [f_1(x, z) - f_1(x, y)] [f_2(z, y) - f_2(x, y)] + [f_1(x, z) - f_1(\bar{z}, \bar{x})] f_2(x, y) \right\}
\]

\[
+ \int_y^1 dz \left\{ [f_1(x, z) - f_1(x, y)] f_2(\bar{z}, \bar{y}) - [f_1(\bar{z}, \bar{x}) - f_1(x, y)] f_2(x, y) \right\}
\]

\[
+ \int_0^x dz \left\{ [f_1(\bar{x}, \bar{z}) - f_1(x, y)] [f_2(z, y) - f_2(x, y)] + [f_1(\bar{x}, \bar{z}) - f_1(z, x)] f_2(x, y) \right\}.
\]
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