Disorder-Induced Long-Ranged Correlations in Scalar Active Matter

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We study the impact of a random quenched potential on scalar active matter. Microscopic simulations reveal that motility-induced phase separation is replaced in two-dimensions by an asymptotically homogeneous phase with anomalous long-ranged correlations and non-vanishing steady-state currents. Using a combination of phenomenological models and a field-theoretical treatment, we show the existence of a lower-critical dimension, \(d_c = 4\), below which phase separation is only observed for systems smaller than an Imry-Ma length-scale. We identify a weak-disorder regime in which the structure factor scales as \(S(q) \sim 1/q^2\) which accounts for our numerics. In \(d = 2\) we predict that, at larger scales, the behaviour should cross over to a strong-disorder regime.

In this Letter, we address this question by studying passive systems in the presence of a quenched random potential using a combination of analytical and numerical approaches. We show that MIPS is destroyed in dimensions \(d \leq d_c\) with \(d_c = 4\): The system only shows the existence of a lower-critical dimension \(d_c = 4\) [12–18]. This makes such active systems more robust to disorder than equilibrium ones with a continuous symmetry, for which \(d_c = 4\) [19–25].

While a lot of effort has been devoted to polar aligning active matter, comparatively less is known on the influence of disorder on the collective properties of scalar active matter, where the sole hydrodynamic mode is the density field. There, it is known that the combination of self-propulsion and kinetic hindrance leads to motility-induced phase separation (MIPS), even in the absence of attractive interactions, in dimensions \(d \geq 2\) [26–41]. Despite important differences, MIPS shares many features of an equilibrium liquid-gas phase separation. The latter is stable to disorder above a lower-critical dimension \(d_c = 2\), and it is natural to ask whether the same holds for MIPS.

In this Letter, we address this question by studying scalar active matter in the presence of a quenched random potential using a combination of analytical and numerical approaches. We show that MIPS is destroyed in dimensions \(d \leq d_c\) with \(d_c = 4\): The system only looks phase separated below an Imry-Ma length scale. Instead, disorder always leads to asymptotically homogeneous systems with persistent steady-state currents. For dimensions \(d > 2\), the system is either in a weak-disorder regime or in a strong-disorder one depending on the strength of the random potential. In the weak-disorder regime, the system is shown to exhibit self-similar correlations with a structure factor decaying as a power law, \(S(q) \sim q^{-2}\), at small wave numbers \(q\). This behavior is very different from that of an equilibrium scalar system, where correlations are known to be short-ranged with a structure factor behaving as a Lorentzian squared [23,42]. In \(d = 2\), we instead predict a crossover between weak- and strong-disorder regimes at a length scale that we identify. In our two-dimensional numerics we only observe the weak-disorder regime, in which we measure a pair-correlation function that decays logarithmically, in agreement with our analytical predictions. Interestingly our results show that, contrary to what was reported for the transition to collective motion [13,14], scalar active systems are more fragile to disorder than passive ones.

**Numerical Simulations.** We start by presenting results from numerical simulations of \(N\) run-and-tumble particles (RTPs) with excluded volume interactions on a two-dimensional lattice of size \(L \times L\) and periodic boundary conditions. Each particle has an attractive interaction with a structure factor behaving as a Lorentzian squared [23,42]. In \(d = 2\), we instead predict a crossover between weak- and strong-disorder regimes at a length scale that we identify. In our two-dimensional numerics we only observe the weak-disorder regime, in which we measure a pair-correlation function that decays logarithmically, in agreement with our analytical predictions. Interestingly our results show that, contrary to what was reported for the transition to collective motion [13,14], scalar active systems are more fragile to disorder than passive ones.

![FIG. 1. Snapshots of simulations without (a) and with (b) disorder. Color encodes density, obtained by averaging occupancies over 4 neighbouring sites. (c) The pair correlation functions are shown in linear scale with (black) and without (red) disorder. (d) Pair correlation with disorder using log-linear scale. The dashed lines correspond to linear (red) and logarithmic (black) decays. Parameters: \(L = 300\) (a-b), \(\Delta V = 7.5\) (b), \(v = 13\), \(\alpha = 1\), \(n_{\text{ATT}} = 2\), \(\rho_0 = N/L^2 = 1\).](image)
orientation \( \hat{e}_\theta = (\cos \theta, \sin \theta) \) with \( \theta \in [0, 2\pi) \), and reorients to a new random direction with rate \( \alpha \). In the absence of disorder, activity is accounted for by hops from the initial position of a particle \( \vec{i} \) to one of the neighboring sites \( \vec{j} = \vec{i} + \vec{u} \) with rate \( W_{\vec{i}, \vec{j}} = \max [v \hat{u} \cdot \hat{e}_\theta, 0] \), where \( v \) controls the propulsion speed. Steric repulsion between the particles is accounted for by modifying the hopping rates according to \( W_{\vec{i}, \vec{j}} = W_{\vec{i}, \vec{j}}(1 - n_j/n_M) \) with \( n_j \) the number of particles at \( \vec{j} \) and \( n_M \) the maximal number of allowed particles on a site. For large enough \( v/\alpha \) and densities, as shown in Fig. (Ia), the system displays MIPS \( [43] \). To introduce a quenched disorder we use \( W_{\vec{i}, \vec{j}} = \max [v \hat{u} \cdot \hat{e}_\theta - (V_j - V_i)/2, 0] \) with \( V_i \) a random potential whose values are drawn from a bounded uniform distribution, \( V_i \in [-\Delta V, \Delta V] \). Here, the lattice spacing and the mobility are set to one. Surprisingly, Fig. (Ib) suggests that the phase separation is washed out by the random potential. The resulting disordered phase displays, however, a non-trivial structure, suggestive of interesting correlations. We quantify the latter using the pair correlation function \( g(r) = \langle n_j n_{j+r} \rangle \) where \( r = |r| \), the brackets represent an average over lattice sites in the steady state, and the overline an average over disorder realizations. In the absence of disorder, phase separation translates into a linear decay of \( g(r) \), as shown in Fig. (Ic). On the contrary, in the presence of disorder, the correlations are found to decay logarithmically, \( g(r) \sim \log(L/r) \), as shown in Fig. (Id). This corresponds to a structure factor \( S(q) \sim q^{-2} \) for small values of \( q \).

To explain the disappearance of phase separation and the emergence of non-trivial correlations, we first introduce a phenomenological model which captures the essential underlying physics. To provide a more quantitative description, we then build on this model to introduce a field-theoretic perspective which predicts the existence of weak- and strong-disorder regimes. In addition, this allows us to characterize the disorder-induced persistent currents that flow in the system, identify the lower critical dimension as \( d_{\ast} = 4 \), and estimate the Imry-Ma length scale.

**Phenomenological model for a dilute system.** Asymmetric potentials in active media lead, through ratchet effects, to long-range density gradients and currents \([45-48]\). When a single localized asymmetric potential, centered around \( r_0 \), is placed in an active fluid of non-interacting run-and-tumble particles, the stationary density profile \( \langle \rho(r) \rangle \) in the far field of the potential is \([47]\)

\[
\langle \rho(r) \rangle = \rho_0 + \frac{\beta_{\text{eff}} (r - r_0) \cdot \vec{p}}{S_d |r - r_0|^d} + O(|r - r_0|^{-d}). \tag{1}
\]

Here, \( S_d = 2\pi^{d/2}/\Gamma(d/2) \), \( \rho_0 \) is the density of the active fluid, \( \beta_{\text{eff}} \equiv 2\alpha/v^2 \), the mobility of the active particles is again set to one, and the angular brackets denote a steady-state average. The vector \( \vec{p} \) is given by the average force exerted by the potential on the active fluid and is thus proportional to the overall density. Given the analogy between Eq. (1) and electrostatics, we follow Ref. [47] and refer to the force \( \vec{p} \) as a dipole.

With Eq. (1) in mind, we consider a phenomenological model in which the bounded random potential is modeled as a superposition of random independent dipoles. Each dipole exerts a force on the active particles in a direction dictated by the local realization of the potential, as sketched in Figs. (a) and (b). To test this random dipole picture numerically, we show in Fig. (c) the measurement of the force \( f(A) \) exerted along an arbitrary direction by the random potential on the particles inside an area \( A \). Consistent with our phenomenological model, \( f(A) \) grows linearly with \( \sqrt{A} \). The figure also shows that the force scales as \( \Delta V^3 \) in this dilute regime. Despite the relative large values of \( \Delta V \) used here, this is consistent with a perturbative result which predicts \( |\vec{p}| \sim \Delta V^3 \) as \( \Delta V \to 0 \) \([47]\). Recall that for an equilibrium system in a random bounded potential \( V(r) \), the force density is \( \propto \beta^{-1} \rho_0 \nabla \exp(-\beta V) \). Integrating over an area \( A \) thus leads to a contribution proportional to \( \int_{\partial A} \exp(-\beta V) \vec{n} \vec{d}l \) from the boundary. Only the fluctuations of \( V(r) \) contribute to the sum so that \( f(A) \) in this case is expected to scale as \( A^{1/4} \). Figure (c) thus highlights the non-equilibrium origin of the dipolar force field: the central effect of the random potential is to generate a local current, through a ratchet effect, and a random force field without long-range correlations.
We now use the phenomenological model to predict the structure factor based on the random dipole picture. The dipole density field $\mathbf{P}(\mathbf{r})$ is randomly drawn from a distribution such that the spatial components of $\mathbf{P}$ satisfy $P_i(\mathbf{r}) = 0$ and $\overline{P_i(\mathbf{r})P_j(\mathbf{r}')} = \chi^2 \delta_{ij} \delta^d(\mathbf{r} - \mathbf{r}')$, notably lacking spatial correlations in $\mathbf{P}(\mathbf{r})$. Denoting $\langle \phi(\mathbf{r}) \rangle \equiv \langle \rho(\mathbf{r}) \rangle - \rho_0$, a direct computation, detailed in SI [49], leads from Eq. (1) to the disorder-averaged structure factor:

$$S(\mathbf{q}) = \frac{\langle \phi(\mathbf{q})\phi(-\mathbf{q}) \rangle}{\langle \phi \rangle^2} = \frac{\beta_{\text{eff}}^2 \chi^2}{q^2}, \quad (2)$$

with $q \equiv |\mathbf{q}|$. Note that, in the dilute (noninteracting) regime, the computation simplifies thanks to $\langle \phi(\mathbf{q}) \rangle \langle \phi(-\mathbf{q}) \rangle = \langle \phi(\mathbf{q}) \rangle^2$, including interactions between the particles is possible at the level of Eq. (1) [48], which would only change the prefactor of $q^{-2}$ in Eq. (2).

Remarkably, the functional form of $S(\mathbf{q})$ predicted by the phenomenological model agrees well with our numerical simulations. Namely, the structure factor in Eq. (2) predicts in $d = 2$ the logarithmic decay of the correlation function reported in Fig. 1(d). Building on this success, we now propose a field-theoretical description of scalar active matter [50] to the case of a quenched random-force acting on the active fluid. We have again set the mobility to one and due to the noise and interactions only enter at subleading order. The color code is the steady-state current normalized by the maximum value measured. (b) The variance of the sum of current $J(C)$ along a contour $C$ as a function of the its perimeter $c$. Parameters: $v = 13$, $\alpha = 1$, $\Delta V = 6.5$ in (a). This behavior is contrasted with the classical Lorentzian squared behavior of structure factor in an equilibrium system subject to random potentials [22]– which implies short-range correlations. Note that the small $q$ behavior of the structure factor reproduces the functional form $S(q) \propto q^{-2}$ predicted by the phenomenological model and observed in the numerics of Fig. 1. In fact, comparing Eqs. (2) and (6) shows that $\sigma/u$, in the dilute regime, is proportional to the inverse effective temperature through $\sigma/u \propto \chi_{\text{eff}}$ [53]. Interestingly, fluctuations due to the noise and interactions only enter at subleading level.

To understand these results further, we decompose the delta-correlated random force using a Helmholtz-Hodge decomposition: $\mathbf{f}(\mathbf{r}) = -\nabla U(\mathbf{r}) + \mathbf{\xi}(\mathbf{r})$. Here $U(\mathbf{r})$ is an effective potential reconstructed from the random force. Its statistical properties, as we show below, are very different from those of the potential $V(\mathbf{r})$ which is short-range correlated. The reconstructed vector field $\mathbf{\xi}(\mathbf{r})$ satisfies $\nabla \cdot \mathbf{\xi}(\mathbf{r}) = 0$, so that it impacts the current $\mathbf{j}$ but does not influence the dynamics of the density field. To enforce the delta correlations of $\mathbf{f}(\mathbf{r})$ together with its statistics, we set $\overline{U(\mathbf{q})U(\mathbf{q}')^*} = \sigma^2 q^{-2} \delta_{q^2, q'^2} - \mathbf{q}^2$, $\mathbf{\xi}(\mathbf{q})\mathbf{\xi}(\mathbf{q}') = \sigma^2 \delta_{\mathbf{q}^2, \mathbf{q}'^2}$, and $\overline{U(\mathbf{q})\mathbf{\xi}(\mathbf{q}')^*} = 0$. Inserting the decomposition into Eqs. (3) and (4) shows that the density fluctuations of active particles in the disordered setting behave as those of passive particles in an effective potential $U(\mathbf{r})$. The statistics of $U(\mathbf{r})$ are those of a Gaussian surface [54]– a self-affine fractal with deep wells that lead both to clustering and long-range correlations. This explains the dense static structures observed in numerical simulations of our microscopic models (see Fig. 1(b) and Supplementary Movie 1); namely, the particles are distributed as if trapped in localized deep potential wells.

**Persistent currents.** The random force due to the disorder is a nonconservative vector field. This manifests itself in the divergence-free part of the Helmholtz-Hodge
decomposition. While this term does not influence the distribution of the density, it does induce currents in the system. To quantify these currents, we consider a closed arbitrary contour $\mathcal{C}$. Taking the curl of the current defined in Eq. (4) and averaging over noise, one finds $(\nabla \times \mathbf{j}) = \nabla \times \mathbf{f}$. Integrating this relation over a domain enclosed by $\mathcal{C}$, we obtain, using Stokes theorem, that the circulation of $(\mathbf{j})$ is entirely controlled by $\mathbf{f}$: $J(\mathcal{C}) \equiv \oint_C \mathbf{f} \cdot d\mathbf{r} = \oint_C \mathbf{j} \cdot d\mathbf{l}$. $J(\mathcal{C})$ is thus a sum of uncorrelated random numbers and we predict its variance to scale with the perimeter $c$ of the contour $\mathcal{C}$. Numerical simulations reported in Fig. 3(b) agree well with this prediction. We stress that, for a given realization of a random potential, disorder induces a current whose time average is non-zero as exemplified in Fig. 3(a). This should be distinguished from the equilibrium case in which currents average out in the steady-state.

Strong-disorder regime. The linear theory used in the previous section is valid as long as density fluctuations are small compared to the mean density. To detect a possible departure from this scenario, we measure the density fluctuations across a length $\ell$ through $\langle \delta \rho^2(\ell) \rangle = 2 [g(a) - g(\ell)]$, with $a$ a short-distance cutoff. The fluctuations have to remain small compared to the natural scales of the density field: $\langle \delta \rho^2(\ell) \rangle \ll \rho_0^2$ with $\rho_0 \equiv \min(\rho_0, \rho_M - \rho_0)$. Here $\rho_0$ and $\rho_M$ are the average and maximal particle densities. Using Eq. (6), we find for large $\ell$

$$\frac{\langle \delta \rho^2(\ell) \rangle}{\rho_0^2} = \begin{cases} \frac{\sigma^2 \ln(\ell/a)}{\pi \sigma^2 \rho_0^2} & \text{for } d = 2 \\ \frac{\sigma^2 \ell^{d-2}}{(d-2)S_d u^2 \rho_0^2} & \text{for } d > 2 \end{cases}.$$  

(7)

For $d > 2$, the linear theory thus holds if $\sigma \ll u \rho_0 \sqrt{(d-2)S_d u^2 \rho_0^2}$, namely, whenever the disorder is weak enough. For strong enough disorder, the breakdown of the linear approximations indicates the possibility of a different behavior for $S(q)$. For $d = 2$, the criterion is valid only for length scales satisfying $\ell \ll \ell^*$ with $\ell^* \equiv a \exp(\pi u^2 \rho_0^2/\sigma^2)$. Note that this length scale is exponential in the square of the ratio between the effective temperature and the disorder strength (encoded in $\sigma/u$, which as argued above is proportional to $\beta_{\text{eff}} \chi$). This suggests a very large length scale, which our numerical simulations could never reach. We suggest in the SI an alternative numerical approach to study the strong-disorder regime in $d = 2$ using passive particles in a Gaussian surface. The resulting correlation function shows clear deviations from the logarithmic behavior on large length-scales.

Lower critical dimension. Our linear theory offers an avenue to test the stability of phase separation against weak-disorder. To do so, we note that the Helmholtz-Hodge decomposition implies that the statistics of $\phi(\mathbf{r})$ is similar to an equilibrium density subject to a correlated random potential $U(\mathbf{r})$. This observation allows us to employ an Imry-Ma argument [19, 20] in order to obtain the lower critical dimension $\ell_c$ below which no phase separation takes place at large scales. To do so, we consider a domain of linear size $\ell$. The surface energy of the domain is given by $\gamma \ell^{d-1}$ with $\gamma$ the surface tension. On the other hand, to leading order, the contribution of the disorder to the energy of the domain is $E(\ell) = \int_\mathcal{D} d^d \mathbf{r}' \rho_0 U(\mathbf{r}')$. The typical energy of a domain of size $\ell$ is thus given by $\sqrt{E(\ell)^2} = \sigma \rho_0 (\ell^{d+2})/2$. Comparing the two energy scales shows the lower critical dimension to be $\ell_c = 4$. In lower dimensions the contribution of the surface energy is negligible on large enough length scales and a system of size $L$ does not phase separate if $L \gg \ell_M$, where

$$\ell_M \approx [\gamma/(\sigma \rho_0)]^{1/2},$$

(8)

which we term the Imry-Ma length scale. Numerically, we indeed confirm that the coarsening to a single macroscopic domain is only observed for small system sizes. Correspondingly, a transition from linearly decaying to logarithmically decaying pair-correlation functions with increasing $L$ is reported in SI [19].

Note that the Imry-Ma argument rules out the existence of a macroscopic ordered/dense phase. Alternatively, the absence of MIPS could stem from the suppression of the feedback loop between a slowdown of particles at high density and their tendency to accumulate wherever they move slower. Reformulated as a mean-field theory, this feedback loop translates into an instability criteria for a homogeneous system of density $\rho$ whenever $\rho v'(\rho) < -v(\rho)$ [26, 32], where $v(\rho)$ is an effective propulsion speed in a system of density $\rho$. We report in Fig. 4 the measurement of $v(\rho)$ for our system, defined as the mean hoping rate of particles along their orientation, with and without the random potential. Both systems show a similar decay which, at mean-field level, would predict the occurrence of MIPS. It is thus the non-trivial correlations induced by disorder that make MIPS disappear at large scales, despite an underlying instability (at mean-field level). The disorder-induced disappearance of MIPS thus has a very different origin than its arrest by diffusiophoretic [55, 56] or hydrodynamic [57] interactions that directly prevent a kinetic hindrance at the microscopic scale.

Conclusion. In this Letter we have shown how a random quenched potential leads to a non-trivial phase in scalar active matter with anomalous correlations that prevent phase separation. Interestingly, while the transition to collective motion is more robust to disorder than the corresponding ferromagnetic transition in equilibrium [13, 14], the converse holds for scalar phase separation: the lower critical dimension is larger in the active case ($\ell_c = 4$) than in the passive one ($\ell_c = 2$). We also note a strong difference between the one-dimensional active case, in which disorder promotes clustering [10], and
that the kink observed for \( v = 13 \) and \( \alpha = 1 \). Both systems exhibit a similar decay which, in the absence of disorder, would lead to MIPS. Note that the kink observed for \( v(\rho) \) in the left panel stems from the occurrence of phase separation.

the two-dimensional one, in which MIPS is destroyed by disorder at large scale. Our results call for a more general study of the influence of disorder-induced long-range correlations on other active matter systems, for instance on active nematics [18]. It would also be interesting to study the possible emergence of disorder-induced long-range forces between passive inclusions.

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Supplemental Information

I. PHENOMENOLOGICAL MODEL

Here we evaluate the structure factor based on the random dipole picture of the phenomenological model. The density $\phi(r)$ in the presence of the random dipole field $P(r)$ reads

$$\langle \phi(r) \rangle = \beta_{\text{eff}} \int \text{d}r' P(r') \cdot \nabla_r G(r-r') ,$$  \hspace{1cm} (S1)

where $G(r-r')$ is the Green function of the Poisson equation. Note that if we have a delta-distributed dipole $P(r) = p \delta_d (r-r_0)$ in a $d$-dimensional free space, Eq. (S1) reads

$$\langle \phi(r) \rangle = \frac{\beta_{\text{eff}}}{S_d} \frac{(r-r_0) \cdot p}{|r-r_0|^d} ,$$ \hspace{1cm} (S2)

which recovers Eq. (1) of the main text. In what follows we use the Fourier convention:

$$f(q) = \frac{1}{\sqrt{V}} \int \text{d}^d r \ e^{-i q \cdot r} f(r) \quad \text{and} \quad f(r) = \frac{1}{\sqrt{V}} \sum_q e^{i q \cdot r} f(q) ,$$

with $V$ denoting the volume of the system. In the Fourier representation, the convolution in Eq. (S1) is written as a product

$$\langle \phi(q) \phi(-q) \rangle = \beta_{\text{eff}} V^{1/2} q \cdot P(q) G(q) .$$

Here $G(q) = -V^{-1/2} q^{-2}$ with $q = |q|$, and the first two moments of $P(q)$ reads

$$\overline{P_i(q)} = 0 , \quad \overline{P_i(q) P_j(q')} = \chi^2 \delta_{ij} \delta_{q,-q'} .$$

Using these relations, the structure factor is calculated in the dilute regime where $\langle \phi(q) \phi(-q) \rangle = \langle \phi(q) \rangle \langle \phi(-q) \rangle$ as

$$\langle \phi(q) \phi(-q) \rangle = \frac{\beta_{\text{eff}}^2}{q^2} \sum_{i,j=1}^d q_i q_j \overline{P_i(q) P_j(-q)} = \frac{\beta_{\text{eff}}^2 \chi^2}{q^2} ,$$ \hspace{1cm} (S3)

which is Eq. (2) of the main text.

II. FIELD-THEORETIC TREATMENT

Here we calculate the structure factor within the linear field-theory presented in the main text. Namely, the structure factor corresponding to

$$\frac{\partial}{\partial t} \phi(r,t) = - \nabla \cdot j(r,t) ,$$ \hspace{1cm} (S4)

$$j(r,t) = - \nabla \mu[\phi] + f(r) + \eta(r,t) .$$ \hspace{1cm} (S5)

with

$$\mu[\phi(r,t)] = u \phi(r,t) + K \nabla^2 \phi(r,t) .$$

Here $j_i(r) = 0$ and $j_i(r) j_j(r') = \sigma^2 \delta_{ij} \delta^d(r-r')$. The Gaussian white noise $\eta(r,t)$ is characterized by zero mean and

$$\langle \eta_i(r,t) \eta_j(r',t') \rangle = 2D \delta_{ij} \delta^d(r-r') \delta(t-t') .$$
Writing Eq. (S4) in Fourier space, we find
\[ \frac{\partial}{\partial t} \phi(q, t) = -q^2 (u + K q^2) \phi(q, t) - i q \cdot f(q) - i q \cdot \eta(r, t) . \]
Solving this equation we obtain
\[ \phi(q, t) = \phi(q, 0) e^{-\kappa(q) t} - \frac{i q \cdot f(q)}{\kappa(q)} \left( 1 - e^{-\kappa(q) t} \right) - \int_0^t dt' e^{-\kappa(q)(t-t')} i q \cdot \eta(r, t) . \]
For \( q^2 (u + K q^2) \). Using this solution, we calculate the structure factor in the stationary state
\[ S(q) = \lim_{t \to \infty} \langle \phi(q, t)\phi(-q, t) \rangle . \]
This leads to
\[ S(q) = \frac{q^2 \sigma^2}{\kappa(q) \kappa(-q)} + \frac{2 D q^2}{\kappa(q) + \kappa(-q)} = \frac{\sigma^2}{q^2 (u + K q^2)^2} + \frac{D}{(u + K q^2)} , \]
where we used the fact that the second moments of the random variables in the Fourier representation read
\[ f_i(q) f_j(q') = \sigma^2 \delta_{ij} \delta_{q, -q'} \] and \( \langle \eta_i(q, t) \eta_j(q', t') \rangle = 2 D \delta_{ij} \delta_{q, -q} \delta(t - t') . \)

III. STRONG-DISORDER REGIME

The strong-disorder regime proved beyond the numerical reach of our lattice-based model. Nevertheless, we can use the field-theoretical model to propose a heuristic approach to study the pair correlation function. Note that
this approach will thus ignore any non-equilibrium non-linear contributions that the field theory does not capture. Since the effective potential exhibit deep wells, we expect these to play a less significant role and to leave the results unaltered at a qualitative level.

To proceed further, we generate a random effective potential $U(r)$ that satisfies the statistics of the Gaussian surface. Then, we introduce particles interacting with excluded volume interaction. Given the diverging depth of the potential well, temperature is not expected to play a large role and we consider the system at $T = 0$. To do so, we simply fill up the system from the bottom of its deepest minima. This could be thought of as filling the Fermi-sea of a 2d Fermionic system in a Gaussian surface potential. Finally, we measure the pair correlation function and average over disorder realizations.

In Fig. S1(a) and S1(b), we show configurations obtained from the process described above. Note that the particle distribution is superficially similar to the configurations obtained from the simulations of run-and-tumble particles presented in Fig. 1(b) of the main text. In Fig. S1(c) and (d), we present the pair correlation functions, which we rescale as \( \tilde{g}(r) \equiv \Delta g(r)/\Delta g(a) \) where \( a \) is the lattice spacing \( a \) and \( \Delta g(r) \equiv g(r) - g(L/2) \). As predicted, when \( r/L \) is small, \( \tilde{g}(r) \) shows a logarithmic decay as a function of \( r/L \). For large \( r/L \), however, the pair correlation function deviates from this logarithmic behaviour, consistently with the prediction of a strong-disorder regime. In Fig. S1(d), we show the crossover to be independent of system size, as predicted by our theory.

### IV. IMRY-MA LENGTH SCALE

In the following, we explore how the system transitions from phase-separated to homogeneous at large scales as the system size crosses the Imry-Ma length scale. To do so, we measure the pair-correlation function for different system sizes in the presence of disorder. We show an example of such a measurement in Fig. S2(a) for a disorder strength $\Delta V = 2.5$. As shown in this Figure, there are two distinct regimes which depend on the system size. The data are correspondingly marked in blue and red. The crossover length between the two behaviors provides an approximate measurement of $\ell_{IM}$. Indeed, for $L \ll \ell_{IM}$: $g(r)$ decays linearly with $r/L$ as expected for a phase separated system (see blue curves and black dashed line). In contrast, for $L \gg \ell_{IM}$, $g(r)$ decays logarithmically with $r/L$ as predicted in the main text and confirmed in Fig. S2(b) (see red curves and red dashed line). Sample configurations in each of the regimes are also shown as insets in Fig. S2(a).

![Fig. S2: The density pair correlation function measured for $\Delta V = 2.5$ and $v/\alpha = 10$ for different system sizes. The data in blue corresponds to small system sizes and in red to larger systems. Two different behaviors are observed consistent with the existence of an Imry-Ma length scale.](image)