SO(4) Landau Models and Matrix Geometry

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Abstract

We develop an in-depth analysis of the SO(4) Landau models on $S^3$ in the SU(2) monopole background and their associated matrix geometry. The Schwinger and Dirac gauges for the SU(2) monopole are introduced to provide a concrete coordinate representation of SO(4) operators and wavefunctions. The gauge fixing enables us to demonstrate algebraic relations of the operators and the SO(4) covariance of the eigenfunctions. With the spin connection of $S^3$, we construct an SO(4) invariant Weyl-Landau operator and analyze its eigenvalue problem with explicit form of the eigenstates. The obtained results include the known formulae of the free Weyl operator eigenstates in the free field limit. Other eigenvalue problems of variant relativistic Landau models, such as massive Dirac-Landau and supersymmetric Landau models, are investigated too. With the developed SO(4) technologies, we derive the three-dimensional matrix geometry in the Landau models. By applying the level projection method to the Landau models, we identify the matrix elements of the $S^3$ coordinates as the fuzzy three-sphere. For the non-relativistic model, it is shown that the fuzzy three-sphere geometry emerges in each of the Landau levels and only in the degenerate lowest energy sub-bands. We also point out that Dirac-Landau operator accommodates two fuzzy three-spheres in each Landau level and the mass term induces interaction between them.
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1 Introduction

The Landau models are physical models that manifest the non-commutative geometry in a most obvious way. It is well known [1, 2] that the fuzzy two-sphere geometry [3, 4, 5] is realized in the SO(3) Landau model [6, 7] that provides a set-up of the 2D quantum Hall effect [8]. Similarly the set-up of the SO(5) Landau model [9, 10] is used for the construction of the 4D quantum Hall effect [11] whose underlying geometry is the fuzzy four-sphere [12, 13, 14]. The correspondence was further explored on $S^{2k}$ [15, 16] and the SO(2$k$ + 1) Landau model was shown to realize the geometry of fuzzy 2$k$-sphere [17, 18]. Besides spheres, there are many manifolds that incorporate non-commutative geometry, and Landau models have been constructed on various manifolds, i.e. $\mathbb{C}P^n$, supermanifolds, hyperboloids, etc. [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. The works have brought deeper understanding of the Landau physics and the associated fuzzy geometry as well. The magnetic field is the vital for the realization of the non-commutative geometry in the Landau model, and for spheres, the magnetic field is brought by the monopole at the center of the spheres. Since the monopole charge mathematically corresponds to the Chern number that is defined on even dimensional manifold, all of the manifolds used in the above works are even dimensional. Also in the viewpoint of the non-commutative geometry, adoption of the even dimensional manifolds is quite reasonable, because the geometric quantization is performed by replacing the Poisson bracket with the commutator, and even dimensional symplectic manifold generally accommodates non-commutative structure by such a quantization procedure.

From above point of view, the Landau model on $S^3$ which Nair and Randjbar-Daemi first proposed [30] was rather exotic, though the model nicely fits in between the SO(3) and SO(5) Landau models (see Table 1). In the model, the quantization of the SU(2) monopole charge was assumed, but there is no reason to justify the assumption: The Chern number is not defined in odd dimensions, and so the monopole charge quantization is not guaranteed. Also for odd dimensional manifolds, the symplectic structure cannot be embedded and then the geometric quantization procedure mentioned above is useless. Even if we adopt the quantum Nambu three-bracket instead of the usual commutator [31], we encounter other problems, such as the violation of the Jacobi identity [32]. It thus seemed to exist fundamental difficulties for Landau models and non-commutative geometry in odd dimensional space. In Refs. [33, 34, 35] however, it was pointed out that the usage of the odd dimensional bracket can be circumvented by treating the odd dimensional bracket as a sub-bracket of the one-dimension higher even bracket, which indicates that the odd dimensional non-commutative space is not apparently consistent by itself but consistent as a subspace of one-dimension higher even dimensional space. Inspired by this observation, we proposed a resolution for the difficulty of odd dimensional Landau model. We showed that the SO(4) Landau model is naturally embedded in the SO(5) Landau model, and the monopole charge quantization is accounted for by that on one-dimension higher space $S^4$ [36]. We also demonstrated that similar relation holds for arbitrary odd and even dimensional Landau models [37] and the dimensional relation has its origin in differential topology; the dimensional ladder of anomaly or the spectral flow of Atiyah-Patodi-Singer. Though the foundation of

| / | 2D | 3D | 4D |
|---|---|---|---|
| Base-manifold | $S^2$ | $S^3$ | $S^4$ |
| Holonomy | $SO(2) \simeq U(1)$ | $SO(3) \simeq SU(2)/Z_2$ | $SO(4) \simeq SU(2) \otimes SU(2)/Z_2$ |
| Monopole gauge group | $U(1)$ | $SU(2)$ | $SU(2)$ |
| Landau model | SO(3) Landau model | SO(4) Landau model | SO(5) Landau model |
| Quantum Hall effect | 2D QHE | 3D QHE | 4D QHE |

Table 1: Landau models on low dimensional spheres and associated monopoles
the odd dimensional Landau models was thus established, there are merely a handful of works about them up to the present [30, 36, 37, 38]. (See also [39, 40, 41] for odd dimensional topological insulator Landau models based on the Dirac oscillator.)

In this paper, we revisit the $SO(4)$ Landau model – the minimal model of the odd dimensional Landau models. Through a full investigation of the $SO(4)$ Landau model, we learn properties specific to the odd dimensional Landau model and associated non-commutative geometry whose analyses are technically difficult in higher dimensions. The main achievements are as follows: (i) We introduce the Schwinger gauge and the Dirac gauge for $S^3$ and solve the Landau problem with explicit form of the wavefunction and the operators. The gauge fixing enables us to demonstrate important algebraic relations, such as the $SO(4)$ invariance of the Dirac-Landau operator and covariance of the $SO(4)$ Landau level eigenstates. (ii) We analyze relativistic Landau operators on $S^3$ with spin connection. Besides the eigenvalues, we derive a concrete coordinate representation of the eigenstates. It is shown that the obtained results indeed include the known formulae of the (free) relativistic operator [45, 46, 47] in the free background limit. (iii) The matrix elements of the arbitrary Landau levels of the $SO(4)$ Landau models are derived explicitly. We demonstrate that the obtained matrix geometry is identical to that of the fuzzy three-sphere. This is the first derivation of the odd dimensional matrix geometry in the context of the Landau model. Especially, we point out that the mass parameter of the Dirac-Landau model induces interaction between two fuzzy three-spheres realized in each of the relativistic Landau levels.

This paper is organized as follows. In Sec.2, we introduce the Schwinger and Dirac gauges for geometric quantities of three-sphere. Sec.3 discusses the non-relativistic Landau model in the Dirac gauge. We analyze the eigenvalue problem of the spinor Landau model with synthesized connection in Sec.4. Subsequently, the eigenvalue problem of relativistic Landau models is solved for Weyl-type, Dirac-type and supersymmetric-type in Sec.5. The matrix geometries of the Landau models are identified as fuzzy three-sphere in Sec.6. Sec.7 is devoted to summary and discussions.

2 Geometric Quantities of Three-sphere

Here, we summarize basic geometric quantities of $S^3$ based on the exterior derivative method [18, 19]. (For the component method, see Appendix A.)

We first parameterize the coordinates on $S^3$ as

$$x_1 = \sin \chi \sin \theta \cos \phi, \quad x_2 = \sin \chi \sin \theta \sin \phi, \quad x_3 = \sin \chi \cos \theta, \quad x_4 = \cos \chi,$$

with the ranges

$$0 \leq \chi \leq \pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi.$$  

(1)

The world-line on $S^3$ is given by

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = d\chi^2 + \sin^2 \chi \ d\theta^2 + \sin^2 \chi \sin^2 \theta \ d\phi^2,$$

and the area of $S^3$ is

$$A(S^3) = \int_{S^3} d\Omega_3 = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \int_0^\pi d\chi \sin^2 \chi = 2\pi^2.$$  

(3)

From the formula

$$ds^2 = \delta_{ab} e^a e^b,$$

(5)

The previous work [30] adopted the Dirac-Landau operator without the spin connection, and so the role of the spin connection has not been taken into account. We then investigate the eigenvalue problem of the Dirac-Landau operator with the spin connection from the beginning.
we can read off the dreibein as
\[ e^1_S = d\chi, \quad e^2_S = \sin \chi \ d\theta, \quad e^3_S = \sin \chi \sin \theta \ d\phi, \] (6)
where \( a, b = 1, 2, 3 \) denote locally flat space coordinate indices. Since \( \delta_{ab} \) is the \( SO(3) \) invariant tensor, the choice of the dreibein is not unique due to the degrees of freedom of \( SO(3) \) rotation. The choice (6) is the simplest one, and we refer to it as the Schwinger gauge. Another useful choice is the Dirac gauge
\[ e^1_D = \cos \theta d\chi - \sin \chi \sin \theta d\theta, \quad e^2_D = \sin \theta \cos \phi d\chi + \sin \chi \cos \phi d\theta - \sin \chi \sin \theta \phi d\phi, \]
\[ e^3_D = \sin \theta \sin \phi d\chi + \sin \chi \cos \theta \sin \phi d\theta + \sin \chi \sin \theta \cos \phi d\phi, \] (7)
which is related to the Schwinger gauge by the following \( SO(3) \) rotation
\[ \begin{pmatrix} e^1_D \\ e^2_D \\ e^3_D \end{pmatrix} = O(\theta, \phi) \begin{pmatrix} e^1_S \\ e^2_S \\ e^3_S \end{pmatrix}, \] (8)
where
\[ O(\theta, \phi) = e^{-i\phi t_1} e^{-i\theta t_3} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta \cos \phi & \cos \theta \cos \phi & \sin \theta \sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \end{pmatrix}, \] (9)
with \((t_a)_{bc} = -i\epsilon_{abc}\) being the \( SO(3) \) generators of the adjoint representation. The spin connection can be found from the torsion free condition
\[ de^a = -\omega^a{}_{b} e^b. \] (10)
For the Schwinger gauge (6), we have\[^3\] \[^3\]
\[ \omega^1{}_{2}(= -\omega^2{}_{1}) = -\cos \chi \ d\theta, \quad \omega^3{}_{1}(= -\omega^1{}_{3}) = \cos \chi \sin \theta \ d\phi, \]
\[ \omega^2{}_{3}(= -\omega^3{}_{2}) = -\cos \theta \ d\phi. \] (13)
The matrix form of the spin connection is then constructed as
\[ \omega = \sum_{a<b} \omega_{ab} \sigma^{ab}, \] (14)
where \( \sigma^{ab} \) \((a, b = 1, 2, 3)\) are the \( SO(3) \) generators in the spinor representation:
\[ \sigma_{ab} = -i \frac{1}{4} [\gamma_a, \gamma_b] \quad \sigma_{31} = \frac{1}{2} \epsilon_{abc} \gamma_c. \] (15)
\( \gamma_a \) \((a = 1, 2, 3)\) denote the \( SO(3) \) gamma matrices, which throughout the paper we will take
\[ \gamma_1 = \sigma_3, \quad \gamma_2 = \sigma_1, \quad \gamma_3 = \sigma_2. \] (16)
\[^3\]Since the present dreibein \( \Phi \) is not the Maurer-Cartan one-form, they do not satisfy \( \omega_{ab} = \epsilon_{abc} e^c \) (Appendix B of \[1\]). Non-zero components of the Riemann curvature 2 form, \( R^a{}_{b} = d\omega^a{}_{b} + \omega^a{}_{c} \omega^c{}_{b} \), are derived as
\[ R_{12} = -R_{21} = \sin \chi \ d\chi \wedge d\theta = e_1 \wedge e_2, \quad R_{31} = -R_{13} = -\sin \chi \sin \theta \ d\chi \wedge d\phi = e_3 \wedge e_1, \]
\[ R_{23} = -R_{32} = \sin^2 \chi \sin \theta \ d\theta \wedge d\phi = e_2 \wedge e_3. \] (11)
For spheres, a special relation, \( R_{ab} = e_a \wedge e_b \), holds in arbitrary dimensions (p.378 in \[18\]). Reading off the Riemann curvature \( R^n{}_{bcd} \) from \( R^a{}_{b} = \frac{1}{4} R^a{}_{bcd} e^c \wedge e_d \), we obtain \( R_{1212} = R_{2323} = R_{3131} = 1 \) (other non-zero components are determined by the symmetry of \( R_{abcd} \)) to construct the scalar curvature
\[ R = R^n{}_{bab} = 6. \] (12)
The spin connection is given by
\[
\omega_S = \frac{1}{2} \begin{pmatrix}
-\cos \theta \, d\phi & i \cos \chi \, d\theta + \cos \chi \sin \theta \, d\phi \\
-i \cos \chi \, d\theta + \cos \chi \sin \theta \, d\phi & \cos \theta \, d\phi
\end{pmatrix}.
\] (17)

Notice that the holonomy of \( S^3 \simeq SO(4)/SO(3) \) is \( SO(3) \simeq SU(2) \), and the spin connection (17) is formally equivalent to the \( SU(2) \) monopole gauge field with minimal charge (see Sec. 3). In the Dirac gauge (7), the spin connection is given by
\[
\omega_D = \frac{1}{2} (1 - \cos \chi) \begin{pmatrix}
\sin^2 \theta d\phi & -(i d\theta + \sin \theta \cos \theta d\phi)e^{-i\phi} \\
(i d\theta - \sin \theta \cos \theta d\phi)e^{i\phi} & -\sin^2 \theta d\phi
\end{pmatrix}.
\] (18)

(17) and (18) are related by the \( SU(2) \) gauge transformation:
\[
\omega_S = g^{(1/2)\dagger} \omega_D g^{(1/2)} - ig^{(1/2)\dagger} dg^{(1/2)},
\] (19)

where
\[
g^{(1/2)}(\theta, \phi) = e^{-i\frac{1}{2} \phi \sigma_3} e^{-i\frac{1}{2} \theta \sigma_2} = \begin{pmatrix}
e^{-i\frac{1}{2} \phi} \cos \frac{\theta}{2} & -e^{-i\frac{1}{2} \phi} \sin \frac{\theta}{2} \\
e^{i\frac{1}{2} \phi} \sin \frac{\theta}{2} & e^{i\frac{1}{2} \phi} \cos \frac{\theta}{2}
\end{pmatrix}.
\] (20)

\( g(\theta, \phi) \) is the \( SU(2) \) group element corresponding to the \( SO(3) \) element (9)
\[
g^{(1/2)\dagger}(\theta, \phi) \gamma_a g^{(1/2)}(\theta, \phi) = O(\theta, \phi)_{ab} \gamma_b.
\] (23)

For the local polar coordinates on \( S^3 \), the gauge field takes a simple form in the Schwinger gauge (17), while in the Dirac gauge the representation is rather clumsy (18). On the other hand, in the target space Cartesian coordinates, the Schwinger gauge representation (17) becomes lengthy
\[
\omega_S = -\frac{1}{2(1 - x_4^2)\sqrt{x_1^2 + x_2^2}} \times \left( x_4 \sqrt{1 - x_4^2} (x_2 dx_1 - x_1 dx_2) \sigma_1 + x_4 (x_1 x_3 dx_1 + x_2 x_3 dx_2 + (x_1^2 + x_2^2) dx_3) \sigma_2 - \frac{x_3}{\sqrt{x_1^2 + x_2^2}} (x_2 dx_1 - x_1 dx_2) \sigma_3 \right),
\] (24)

while the Dirac gauge representation (18) is much concise
\[
\omega_D = -\frac{1}{2(1 + x_4)} \epsilon_{abc} x_b \sigma_c dx_a.
\] (25)

Thus, the Schwinger gauge is an appropriate gauge in the usage of the local polar coordinates on \( S^3 \), and the Dirac gauge in the target space Cartesian coordinates. The spin connection has the singularity both at the north pole \( x_4 = 1 \) and the south pole \( x_4 = -1 \), and hence the name, the Schwinger gauge [50, 1]. Meanwhile, the singularity of (25) is only at the south pole \( x_4 = -1 \), and the name the Dirac gauge.

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4 In other words,
\[
g^{(1/2)}(\theta, \phi) \gamma_a g^{(1/2)\dagger}(\theta, \phi) = \gamma_b O(\theta, \phi)_{ab}.
\] (21)

or
\[
O(\theta, \phi)_{ab} = \frac{1}{2} \text{tr}(g^{(1/2)\dagger}(\theta, \phi) \gamma_a g^{(1/2)}(\theta, \phi) \gamma_b) = \frac{1}{2} \text{tr}(g^{(1/2)}(\theta, \phi) \gamma_b g^{(1/2)\dagger}(\theta, \phi) \gamma_a).
\] (22)
3 Non-relativistic Landau Model

In this section, we perform a thorough investigation of the eigenvalue problem of the SO(4) Landau Hamiltonian. The obtained results are utilized throughout the paper, and we provide a detail explanation for readers not to stumble against any logical gap or technical difficulty. We first present an expanded discussion of [30, 36] about the mathematical background of the SO(4) Landau model (Sec.3.1), the SO(4) operators and their algebraic relations (Sec.3.2) and the SO(4) Landau problem (Sec.3.3). Next, in Sec.3.4, we fix the gauge and provide new results about the basic properties of the SO(4) monopole harmonics (Sec.3.4.2) and the SO(4) covariance (Sec.3.4.3) in which we give a verification of the SO(4) monopole harmonics to be the eigenstates of the SO(4) Landau Hamiltonian. In Sec.3.5, we check that the derived SO(4) monopole harmonics indeed reduce to the known SO(4) spherical harmonics in the free background limit.

For notational brevity, we adopt the following abbreviation of the angular coordinates in (1):

\[ \chi = (\chi, \theta, \phi) \]  

(26)

and

\[ -\chi = (-\chi, \theta, \phi). \]

(27)

The sign flip of \( \chi \) represents the parity transformation on \( S^3 \):

\[ (x_1, x_2, x_3, x_4) \rightarrow (-x_1, -x_2, -x_3, x_4), \]

(28)

which interchanges the left-handed and right-handed coordinate systems, and so we call it the LR transformation.

3.1 The chiral Hopf map and the SU(2) monopole gauge field

The underlying geometry of the SO(4) Landau model is the chiral Hopf map [36],

\[ S^3_L \otimes S^3_R \xrightarrow{\text{diag}} S^3. \]

(29)

Here, the projected space \( S^3 \) denotes the base-manifold, and \( S^3_L \simeq SU(2) \) corresponds to the \( SU(2) \) fiber, and the map gives a set-up of the SO(4) Landau model on \( S^3 \) with \( SU(2) \) monopole at the center. We represent the coordinates on \( S_L \) and \( S_R \) as two two-component chiral Hopf spinors, \( \psi_L = (\psi_{L1} \psi_{L2}) \) and \( \psi_R = (\psi_{R1} \psi_{R2}) \), subject to

\[ \psi_L \psi_L^\dagger = \psi_R \psi_R^\dagger = \frac{1}{2}. \]

(30)

(29) can be expressed as

\[ \psi_L, \psi_R \rightarrow x_\mu = \psi_L \bar{q}_\mu \psi_R^\dagger + \psi_R q_\mu \psi_L^\dagger \]  

(31)

(\( \mu = 1, 2, 3, 4 \)),

where \( q_\mu \) and \( \bar{q}_\mu \) are the quaternions and conjugate-quaternions,

\[ q_\mu = (q_\mu, 1) = (-i\sigma_i, 1), \quad \bar{q}_\mu = (-q_\mu, 1) = (i\sigma_i, 1). \]

(32)

From (31), we have

\[ \sum_{\mu=1}^{4} x_\mu x_\mu = 4(\psi_L \psi_L^\dagger)(\psi_R \psi_R^\dagger) = 1. \]

(33)
$x_\mu$ are invariant under the simultaneous $SU(2)$ transformation of $\psi_L$ and $\psi_R$, $\psi_{L/R} \to e^{\alpha_\mu \nu^\mu} \psi_{L/R}$, and we denote such diagonal $SU(2)$ rotation as $SU(2)_D$. The chiral Hopf spinors, $\psi_L$ and $\psi_R$, are represented as

$$\psi_L(x) = \phi \, \Psi_L^{(1/2)}(x), \quad \psi_R(x) = \phi \, \Psi_R^{(1/2)}(x),$$

where $\phi = (\phi_1, \phi_2)$ is a normalized two-component spinor $\phi \phi^\dagger = 1/2$ representing the $S^3$-fibre. $\Psi_L^{(1/2)}$ and $\Psi_R^{(1/2)}$ are given by

$$\Psi_L^{(1/2)}(x) = \frac{1}{\sqrt{2(1 + x_4)}} (1 + x_\mu q_\mu), \hspace{1cm} \Psi_R^{(1/2)}(x) = \frac{1}{\sqrt{2(1 + x_4)}} (1 + x_\mu \bar{q}_\mu) = \Psi_L^{(1/2)}(x)^\dagger,$$

each of which is an $SU(2)$ group element and their squares yield $\Psi_L^{(1/2)^2} = (\Psi_R^{(1/2)^\dagger})^2 = x_\mu q_\mu$. Using $\Psi_L^{(1/2)}$ and $\Psi_R^{(1/2)}$, we can derive the $SU(2)$ monopole connection as

$$A^{(1/2)} = -\frac{1}{2} (\Psi_L^{(1/2)^\dagger} d\Psi_L^{(1/2)} + \Psi_R^{(1/2)^\dagger} d\Psi_R^{(1/2)}) = -\frac{1}{2(1 + x_4)} \epsilon_{ijk} x_j \sigma_k dx_i. \quad (36)$$

Note that the gauge connection $\Psi_L^{(1/2)}$ is formally identical to the spin connection $\Psi_R^{(1/2)}$. With the angular coordinates, $\Psi_L$ and $\Psi_R$ can be expressed as

$$\Psi_L^{(1/2)}(x) = e^{-i\chi \hat{x} \cdot \sigma}, \quad \Psi_R^{(1/2)}(x) = e^{i\chi \hat{x} \cdot \sigma}, \quad (38)$$

where $\hat{x}$ signifies a position on the $(S^2)$-equator of $S^3$:

$$\hat{x} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \quad (39)$$

Promoting the Pauli matrices in (38) to $SU(2)$ arbitrary matrices with spin magnitude $I/2$, we introduce

$$\Psi_D^{(1/2)}(\chi) \equiv e^{-i\chi \hat{x} \cdot S(I/2)}, \quad (40)$$

and

$$\Psi_L^{(1/2)}(x) = \Psi_D^{(1/2)}(\chi), \quad \Psi_R^{(1/2)}(x) = \Psi_D^{(1/2)}(-\chi). \quad (41)$$

$\Psi_L$ and $\Psi_R$ are interchanged by the LR transformation (28). They satisfy

$$\Psi_D^{(1/2)}(\chi)^{-1} = \Psi_D^{(1/2)}(\chi^\dagger) = \Psi_D^{(1/2)}(-\chi). \quad (42)$$

In a same manner to (36), we obtain the $SU(2)$ monopole gauge field in the Dirac gauge

$$A^{(1/2)} = -\frac{1}{2} (\Psi_L^{(1/2)^\dagger} d\Psi_L^{(1/2)} + \Psi_R^{(1/2)^\dagger} d\Psi_R^{(1/2)}) = -\frac{1}{2(1 + x_4)} \epsilon_{ijk} x_j S_k^{(1/2)} dx_i, \quad (43)$$

or

$$A_i^{(1/2)} = -\frac{1}{2 + x_4} \epsilon_{ijk} x_j S_k^{(1/2)} \quad (i = 1, 2, 3), \quad A_4^{(1/2)} = 0. \quad (44)$$

The winding number associated with the $SU(2)$ gauge field (43) is

$$\nu = \frac{1}{24\pi^2} \int_{S^3} \text{tr}(-i g^i dg)^3, \quad (45)$$

\footnote{Indeed,}

$$\Psi_L^{(1/2)} = \frac{1}{\sqrt{2(1 + x_4)}} \left[ \begin{array}{cc} 1 + x_4 - i x_3 & -x_2 - i x_1 \\ x_2 + i x_1 & 1 + x_4 + i x_3 \end{array} \right], \quad \cos \frac{\chi}{2} - i \sin \frac{\chi}{2} \cos \theta, \quad -i \sin \frac{\chi}{2} \sin \theta e^{-i\phi}, \quad \cos \frac{\chi}{2} + i \sin \frac{\chi}{2} \cos \theta, \quad (37)$$

$$\Psi_R^{(1/2)} = \frac{1}{\sqrt{2(1 + x_4)}} \left[ \begin{array}{cc} 1 + x_4 + i x_3 & x_2 + i x_1 \\ -x_2 - i x_1 & 1 + x_4 - i x_3 \end{array} \right], \quad \cos \frac{\chi}{2} + i \sin \frac{\chi}{2} \cos \theta, \quad i \sin \frac{\chi}{2} \sin \theta e^{-i\phi}, \quad \cos \frac{\chi}{2} + i \sin \frac{\chi}{2} \cos \theta. \quad (38)$$
where
\[ g = \Psi_D^{(1/2)}(\chi)^2 = e^{-2i\chi \hat{x} \cdot \sigma^{(1/2)}}, \] (46)
and (45) is evaluated as
\[ \nu = \frac{1}{6} I(I + 1)(I + 2). \] (47)
It should be mentioned that the chiral Hopf map and the present \( SU(2) \) monopole are naturally understood by embedding \( S^3 \) in one-dimension higher \( S^4 \). [36, 37]

### 3.2 \( SO(4) \) operators

The \( SU(2) \) magnetic field is perpendicular to \( S^3 \) surface, and so the present system respects the \( SO(4) \) rotational symmetry. We construct total angular momentum operators that generate the simultaneous \( SO(4) \) rotations of the base-manifold and the gauge space. We clarify analogies and differences to the \( U(1) \) monopole system on \( S^2 \). [1]

#### 3.2.1 \( SO(4) \) angular momentum operators

For \( A_\mu (\mu = 1, 2, 3, 4) \), let us introduce the covariant derivative
\[ D_\mu = \partial_\mu + iA_\mu, \] (48)
and the field strength is given by
\[ F_{\mu\nu} = -i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \] (49)
with
\[ F_{ij} = -x_i A_j + x_j A_i + \epsilon_{ijk} S^{(1/2)}_k, \quad F_{i4} = (1 + x_4) A_j = -\epsilon_{ijk} x_j S^{(1/2)}_k. \] (50)
With the dreibein \( \mathbf{e} \), (50) is concisely represented as
\[ F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} \epsilon_{ijk} e^i_4 \wedge e^j_4 S^{(1/2)}_k. \] (51)
The \( SO(4) \) covariant angular momentum is introduced as
\[ \Lambda_{\mu\nu} = -ix_\mu D_\nu + ix_\nu D_\mu \quad (\mu, \nu = 1, 2, 3, 4), \] (52)
and the conserved \( SO(4) \) angular momentum operator consists of particle angular momentum and the field angular momentum of the monopole:
\[ L_{\mu\nu} = \Lambda_{\mu\nu} + F_{\mu\nu}. \] (53)
In the Dirac gauge, \( L_{\mu\nu} \) are expressed as
\[ L_{ij} = l_{ij} + \epsilon_{ijk} S^{(1/2)}_k, \quad L_{i4} = l_{i4} - \frac{1}{1 + x_4} \epsilon_{ijk} x_j S^{(1/2)}_k, \] (54)
where \( l_{\mu\nu} \) denote the free \( SO(4) \) angular momentum operators
\[ l_{\mu\nu} = -ix_\mu \partial_\nu + ix_\nu \partial_\mu. \] (55)
With the explicit coordinate representations, it is straightforward to check that \( T_{\mu\nu} = L_{\mu\nu}, \Lambda_{\mu\nu} \) and \( F_{\mu\nu} \) transform as two-rank tensors under the \( SO(4) \) transformations generated by \( L_{\mu\nu} \):
\[ [L_{\mu\nu}, T_{\rho\sigma}] = i\delta_{\rho\mu} T_{\nu\sigma} - i\delta_{\mu\rho} T_{\nu\sigma} + i\delta_{\nu\sigma} T_{\mu\rho} - i\delta_{\nu\rho} T_{\mu\sigma}. \] (56)
From the orthogonality between $\Lambda_{\mu\nu}$ and $F_{\mu\nu}$

$$\sum_{\mu<\nu=1}^{4} F_{\mu\nu} A_{\mu\nu} = \sum_{\mu<\nu=1}^{4} \Lambda_{\mu\nu} F_{\mu\nu} = 0,$$

(57)

the SO(4) Casimir is given by

$$\sum_{\mu<\nu=1}^{4} L_{\mu\nu}^2 = \sum_{\mu<\nu=1}^{4} \Lambda_{\mu\nu}^2 + \sum_{\mu<\nu=1}^{4} F_{\mu\nu}^2,$$

(58)

which can be rewritten as

$$\sum_{\mu<\nu=1}^{4} L_{\mu\nu}^2 = \sum_{\mu<\nu}^{4} l_{\mu\nu}^2 + \frac{2}{1 + x_4} I \cdot S^{(l/2)} + \frac{1}{2(1 + x_4)} I(I + 2) - \frac{1}{(1 + x_4)^2} (\mathbf{x} \cdot S^{(l/2)})^2,$$

(59)

with $I$ the free SO(3) angular momentum operator:

$$l_i = \frac{1}{2} \epsilon_{ijk} l_{jk} = -i \epsilon_{ijk} x_j \partial_k.$$

(60)

The first term is the SO(4) free angular momentum Casimir, and the second term is formally equivalent to the spin-orbit coupling in three-dimension. Meanwhile, the SO(3) Casimir of the SO(3) Landau model on $S^2$ is represented as

$$\sum_{i=1}^{3} \hat{l}_i^2 = \sum_{i=1}^{3} l_i^2 + \frac{I}{1 + x_3} l_3^2 + \frac{1}{2(1 + x_3)} I^2,$$

(61)

where

$$\hat{L}_i = -i \epsilon_{ijk} x_j (\partial_k + i \hat{A}_k) + \hat{F}_i.$$

(62)

$\hat{A}_i = -I \frac{1}{2(1 + x_3)} \epsilon_{ijk} x_j$ is the $U(1)$ monopole gauge field and $\hat{F}_i = \epsilon_{ijk} \partial_j \hat{A}_k = \frac{1}{2} x_i$. Comparison between (59) and (61) shows the the last term of (59), $-\frac{1}{(1 + x_4)^2} (\mathbf{x} \cdot S^{(l/2)})^2$, is specific to the SO(4) Landau model with the non-Abelian gauge field.

### 3.2.2 $SU(2)_L \otimes SU(2)_R$ group generators

Since $SO(4) \simeq SU(2)_L \otimes SU(2)_R$, we can construct $su(2)_L \oplus su(2)_R$ generators from the $so(4)$ generators:

$$L_i = \frac{1}{4} \eta_{\mu\nu}^i L_{\mu\nu}, \quad \bar{L}_i = \frac{1}{4} \eta_{\mu\nu}^i \bar{L}_{\mu\nu},$$

(63)

which satisfy

$$[L_i, L_j] = i \epsilon_{ijk} L_k, \quad [\bar{L}_i, \bar{L}_j] = i \epsilon_{ijk} \bar{L}_k, \quad [L_i, \bar{L}_j] = 0.$$

(64)

Here, $\eta_{\mu\nu}^i$ and $\eta_{\mu\nu}^i$ are the 't Hooft symbols:

$$\eta_{\mu\nu}^i = \epsilon_{\mu
u4} + \delta_{\mu4} \delta_{\nu4} - \delta_{\mu4} \delta_{\nu4}, \quad \eta_{\mu\nu}^i = \epsilon_{\mu\nu4} - \delta_{\mu4} \delta_{\nu4} + \delta_{\mu4} \delta_{\nu4}.\quad (65)$$

The two independent $su(2)$ algebras (54) can be verified by the $so(4)$ algebra and properties of the 't Hooft symbols. The SO(4) Casimir is also given by a simple sum of the two $SU(2)$ Casimirs:

$$\sum_{\mu<\nu=1}^{4} L_{\mu\nu}^2 = 2 (L_i^2 + \bar{L}_i^2)$$

(67)

---

6When $I = 1$, this term is reduced to $-\frac{1}{4} \frac{1 - x_4}{1 + x_4}$.

7The following properties will be useful:

$$\eta_{\mu\nu} \eta_{\rho\sigma} = \delta_{\mu\rho} \delta_{\nu\sigma} + \epsilon_{ij} \eta_{\nu\sigma}^k, \quad \eta_{\mu\nu} \eta_{\rho\sigma} = \delta_{\mu\rho} \delta_{\nu\sigma} + \epsilon_{ij} \eta_{\nu\sigma}^k.$$
whose eigenvalues are readily obtained as
\[ \sum_{\mu<\nu} L_{\mu\nu}^2 = 2l_L(l_L + 1) + 2l_R(l_R + 1). \] (68)

Here, \( l_L \) and \( l_R \) denote the SU(2)\(_L\) and SU(2)\(_R\) Casimir indices,
\[ L^2 = l_L(l_L + 1), \quad \bar{L}^2 = l_R(l_R + 1), \] (69)
and their sum is bounded below by the monopole charge
\[ l_L + l_R = n + \frac{I}{2}. \quad (n = 0, 1, 2, \cdots) \] (70)

\( n \) comes from the particle angular momentum, while \( I/2 \) comes from the field angular momentum of the monopole (recall (63)). In the case \( I = 0 \), the SO(4) irreducible representation is reduced to the SO(4) spherical harmonics with the SO(4) indices \((l_L,l_R) = (n/2, n/2)\) (Appendix B.1), and so each of \( l_L \) and \( l_R \) is bounded below by \( n/2 \):
\[ l_L, l_R \geq \frac{n}{2}. \] (71)

Consequently, for given \( n \), \((l_L,l_R)\) can take the following \((I + 1)\) distinct values
\[ (l_L, l_R) = \left( \frac{n}{2} + \frac{I}{4} + \frac{s}{2}, \frac{n}{2} + \frac{I}{4} - \frac{s}{2} \right), \quad \left( \frac{n}{2} + \frac{I}{4} + \frac{s}{2} + 1, \frac{n}{2} + \frac{I}{4} - \frac{s}{2} + 1 \right), \cdots, \left( \frac{n}{2} + \frac{I}{4} - \frac{s}{2} + 1, \frac{n}{2} + \frac{I}{4} + \frac{s}{2} + 1 \right). \] (72)

Introducing the chirality parameter \( s \)
\[ s \equiv l_L - l_R = \frac{I}{2} + \frac{1}{2} - \frac{1}{2} \cdot \frac{I}{2} - \cdots - \frac{I}{2}, \] (73)
we specify \((l_L,l_R)\) as
\[ (l_L, l_R) = \left( \frac{n}{2} + \frac{I}{4} + \frac{s}{2}, \frac{n}{2} + \frac{I}{4} - \frac{s}{2} \right). \] (74)

The SO(4) Casimir eigenvalue (68) is now represented as
\[ \sum_{\mu<\nu} L_{\mu\nu}^2 = (n + \frac{I}{2})(n + \frac{I}{2} + 2) + s^2. \] (75)

### 3.2.3 SU(2)\(_{\text{diag}}\) generators and “boost” generators

In the Dirac gauge, (63) are explicitly represented as
\[ L_i = -i\frac{1}{2} \epsilon_{ijk} x_j \partial_k - i\frac{1}{2} x_i \partial_4 + i\frac{1}{2} x_4 \partial_i + \frac{1}{2} \epsilon_i^{(1/2)} - \frac{1}{2(1 + x_4)} \epsilon_{ijk} x_j S_k^{(1/2)}, \]
\[ \bar{L}_i = -i\frac{1}{2} \epsilon_{ijk} x_j \partial_k + i\frac{1}{2} x_i \partial_4 - i\frac{1}{2} x_4 \partial_i + \frac{1}{2} \epsilon_i^{(1/2)} + \frac{1}{2(1 + x_4)} \epsilon_{ijk} x_j S_k^{(1/2)}. \] (76)

\( L_i \) and \( \bar{L}_i \) are interchanged by the LR transformation [28]. For the comparison to the SO(3) Landau model [1], it is useful to introduce the SU(2) diagonal group generators and “boost” generators. From the two sets of SU(2) operators, we define the SU(2) diagonal group generators,
\[ L_i^{\text{diag}} \equiv L_i + \bar{L}_i = -i \epsilon_{ijk} x_j \partial_k + S_i^{(1/2)} \] (77)
where
\[ l_i = -i \epsilon_{ijk} x_j \partial_k. \]  

(78)

\( L^D_i \) do not depend on \( x_4 \) and are formally identical to the angular momentum operators of a free particle with higher spin \( I/2 \) in three dimensions that obviously satisfy the \( su(2) \) algebra. In the \( SO(3) \) Landau model on \( S^2 \), the \( SO(2) \) operator \( \hat{L}_z \) consists of the free angular momentum and the \( U(1) \) gauge generator, \( \hat{L}_z = -i \frac{\partial}{\partial \phi} - \frac{I}{2} \), and then \( (77) \) may be regarded as its \( SU(2) \) generalization. From analogy to the Lorentz group, we refer to the remaining \( SO(4) \) operators as the “boost” operators:

\[ K_i = L_i - \bar{L}_i = -i x_i \partial_4 + i x_4 \partial_i - \frac{1}{1 + x_4^2} \epsilon_{ijk} x_j S^{(I/2)}_k. \]  

(79)

It is easy to see that \( L^\text{diag} \) and \( K \) satisfy

\[ [L^\text{diag}_i, L^\text{diag}_j] = i \epsilon_{ijk} L^\text{diag}_k, \quad [L^\text{diag}_i, K_j] = i \epsilon_{ijk} K_k, \quad [K_i, K_j] = i \epsilon_{ijk} L^\text{diag}_k. \]  

(80)

\( K_i \) thus transform as the \( SU(2)^\text{diag} \) vector. Though \( K_2 \) (and \( L^\text{diag}_2 \)) is not invariant under general \( SO(4) \) transformations

\[ [K_i, K^2] = -[K_i, L^\text{diag}^2] = i \epsilon_{ijk} (K_j L^\text{diag}_k - L^\text{diag}_j K_k) \neq 0, \]  

(81)

\( K^2 \) is invariant under the \( SU(2)_D \) transformations

\[ [L^\text{diag}_i, K^2] = 0. \]  

(82)

While \( L^\text{diag}^2 \) represents \( SU(2) \) Casimir, the square of the boost operators yields

\[ K^2 = (\hat{L} - \bar{L})^2 = 2(\hat{L}^2 + \bar{L}^2) - L^{\text{diag}^2} = \sum_{\mu < \nu} L_{\mu \nu}^2 - L^{\text{diag}^2}. \]  

(83)

In the \( SO(3) \) Landau model \cite{1}, the boost generators \( K_i \) correspond to \( \hat{L}_x \) and \( \hat{L}_y \) \cite{2}, and the sum of their squares gives

\[ \hat{L}_x^2 + \hat{L}_y^2 = \hat{L}^2 - \hat{L}_z^2. \]  

(84)

The \( SO(3) \) Landau Hamiltonian can be obtained from \cite{3} by replacing the eigenvalues of \( \hat{L}_z \) with the \( U(1) \) gauge generators \( I/2 \)

\[ \hat{L}_x^2 + \hat{L}_y^2 \rightarrow \hat{L}^2 - (I/2)^2. \]  

(85)

We heuristically apply this procedure to the present case and replace \( L^D \) in \cite{4} with the gauge group generator \( S^{(I/2)} \) to obtain

\[ K^2 \rightarrow \sum_{\mu < \nu} L_{\mu \nu}^2 - S^{(I/2)^2}, \]  

(86)

which may give \( SO(4) \) Landau Hamiltonian. We shall confirm it in Sec.3.3.

3.3 \( SO(4) \) Landau problem

3.3.1 \( SO(4) \) Landau levels and subbands

From the Landau Hamiltonian on \( \mathbb{R}^4 \)

\[ H = \frac{1}{2M} \sum_{\mu=1}^{4} D_{\mu}^2 = -\frac{1}{2M} \frac{\partial^2}{\partial r^2} - \frac{3}{2Mr} \frac{\partial}{\partial r} + \frac{1}{2Mr^2} \sum_{\mu < \nu=1}^{4} \Lambda_{\mu \nu}^2, \]  

(87)

\[ \text{Recall that in the beginning} \quad S_1^{(I/2)} = \text{the SU(2) gauge group generators and the particle was a spinless particle, but here we reinterpret} \quad S_1^{(I/2)} \text{ as the intrinsic spin of particle. This interpretation is similar to that of Wilczek} \cite{5} \.]
the $SO(4)$ Landau Hamiltonian on $S^3$ is obtained as

$$H = -\frac{1}{2M} \sum_{\mu} D_\mu^2 |_{r=1} = \frac{1}{2M} \sum_{\mu<\nu} \Lambda_{\mu\nu}^2,$$

(88)

which is invariant under the $SO(4)$ global rotations generated by $L_{\mu\nu}$. Using the relation (58), we can express (88) as

$$H = \frac{1}{2M} \sum_{\mu<\nu} L_{\mu\nu}^2 - \frac{1}{2M} \sum_{\mu<\nu} F_{\mu\nu}^2,$$

(89)

where the second term on the right-hand side is equal to the $SO(3)$ Casimir

$$\sum_{\mu<\nu} F_{\mu\nu}^2 = \frac{3}{I(I+1)} \sum_{i=1}^3 S_i^{(I/2)^2} = \frac{1}{4} I(I+2),$$

(90)

and then

$$H = \frac{1}{2M} \left( \sum_{\mu<\nu} L_{\mu\nu}^2 - S^{(I/2)^2} \right) = \frac{1}{M} \sum_{i=1}^3 (L_i^2 + \bar{L}_i^2) - \frac{1}{2M} S^{(I/2)^2}.$$

(91)

We thus verified that (86) is equal to the $SO(4)$ Landau Hamiltonian up to the unimportant proportional factor. $H$ is obviously invariant under the LR transformation (28), i.e., $L_i \leftrightarrow \bar{L}_i$, and respects the LR symmetry. From (68) or (75), we can readily derive the energy of the $SO(4)$ Landau Hamiltonian (91):

$$E_n(s) = \frac{1}{M} (l_L(l_L+1) + l_R(l_R+1)) - \frac{1}{8M} I(I+2)$$

$$= \frac{1}{2M} (n(n+2) + \frac{I}{2}(2n+1) + s^2).$$

(92)

Notice that the energy eigenvalues depend both on $n$ and $s$. While $n$ denotes the Landau level index, the chirality parameter $s$ corresponds to subband of each Landau level. Notice that the energy eigenvalue (92) is invariant under the sign flip of the chirality parameter:

$$s \leftrightarrow -s,$$

(93)

which is a direct consequence of the LR symmetry, since the $SU(2)_L$ and $SU(2)_R$ quantum numbers are interchanged by the LR transformation.

The dimension of the $SO(4)$ irreducible representation specified by $n$ and $s$ is

$$d_n(s) = (2l_L+1)(2l_R+1) = (n+\frac{I}{2}+1+s)(n+\frac{I}{2}+1-s).$$

(94)

Since $E_n(s)$ depends on $s^2$, the $SO(4)$ eigenstates with $(n, s)$ and $(n, -s)$ are degenerate. The degeneracy of the subband of the Landau level $E_n(\{|s|\})$ for $s \neq 0$ is given by

$$2d_n(|s|) = 2((n+\frac{I}{2}+1)^2 - |s|^2).$$

(95)

When $I$ is odd, $s$ can take $s = 0$ and the degeneracy is

$$d_n(s = 0) = (n+\frac{I}{2}+1)^2.$$

(96)

A schematic picture is given by Fig. 1.

---

9The energy interval between adjacent subbands, $\Delta E_n(s) = s/(MR^2)$, collapses in the thermodynamic limit $I, R \rightarrow \infty$ with $I/(R^2)$ fixed. The subbands (for $s << I$) thus become degenerate in the thermodynamic limit, and hence $n$ is referred to as the Landau level and $s$ subband. The origin of subbands of Landau levels may be accounted for by the curvature of the sphere.
3.3.2 Landau level eigenstates

In [36], we constructed the lowest Landau level ($n = 0$) basis states by taking the symmetric product of the chiral Hopf spinors. Here, we provide a precise meaning of the construction. For $n = 0$, the $SO(4) \cong SU(2)_L \otimes SU(2)_R$ indices are given by

$$l_L = \frac{1}{2}(I_2 + s), \quad l_R = \frac{1}{2}(I_2 - s).$$ (97)

For each of $SU(2)$, we take the fully symmetric product of the chiral Hopf spinors (34):

$$\Psi_{l,L,m_L}^L = \frac{1}{\sqrt{(l_L + m_L)!(l_L - m_L)!}}\psi_{L_1}^{l_L+m_L}\psi_{L_2}^{l_L-m_L},$$

$$\Psi_{l,R,m_R}^R = \frac{1}{\sqrt{(l_R + m_R)!(l_R - m_R)!}}\psi_{R_1}^{l_R+m_R}\psi_{R_2}^{l_R-m_R},$$ (98)

with $m_L = l_L, l_L - 1, \cdots, -l_L$ and $m_R = l_R, l_R - 1, \cdots, -l_R$, and the lowest Landau level basis states are constructed as

$$\Phi_{m_L, m_R}^{0,s,I/2} = \frac{1}{\sqrt{I!}} \Psi_{l,L,m_L}^L \otimes \Psi_{l,R,m_R}^R.$$ (99)

Recall that the $S^3_D$-fibre $\phi$ is common to $\psi_L$ and $\psi_R$ (34). As the expansion basis of $\Phi_{m_L, m_R}^{0,s,I/2}$, we adopt the $SU(2)_D$ higher spin representation made of $\phi$:

$$e_A^{(I/2)} = \frac{1}{\sqrt{(I_2 + A)!(I_2 - A)!}}\phi_{1+\frac{I}{2}}^{l_L+A}\phi_{2-\frac{I}{2}}^{l_L-A}. \quad (A = I/2, I/2 - 1, \cdots, -I/2)$$ (100)
Using the basis \( \Phi_{0,s,1/2} \), we expand \( \Phi_{m_L,m_R}^{(0,s,1/2)} \) as
\[
\Phi_{m_L,m_R}^{(0,s,1/2)}(x) = \sum_{A=-1/2}^{1/2} \Phi_{m_L,m_R}^{(0,s,1/2)}(x)_A e^{i A/2} \tag{101}
\]
to define the expansion coefficients carrying the internal \( SU(2) \) gauge index \( A \). In particular for \( s = 1/2 \) and \( s = -1/2 \), the coefficients are obtained as
\[
\Phi_{m_L,0}^{(0,1/2,1/2)}(x)_A = \Psi_L(x)_A m_L = \Psi(\chi)_{A,m_L}, \quad (m_L = \frac{1}{2}, \frac{3}{2}, \ldots, -1, \ldots, - \frac{3}{2})
\]
\[
\Phi_{0,m_R}^{(0,-1/2,1/2)}(x)_A = \Psi_R(x)_A m_R = \Psi(-\chi)_{A,m_R}, \quad (m_R = \frac{1}{2}, \frac{3}{2}, \ldots, -1, \ldots, - \frac{3}{2}) \tag{102}
\]
where \( \Psi(\chi) \) is the \( SU(2) \) group element in the Dirac gauge \( [10] \). Also from other exercises, such as \( \Phi_{1/2,-1/2}^{(0,0,1)}(x)_0 = \frac{i}{\sqrt{2}}(\Psi_{L11} \Psi_{R22} + \Psi_{L12} \Psi_{R21}) \), we can deduce a general formula for the lowest Landau level eigenstate:
\[
\Phi_{m_L,m_R}^{(n,s,1/2)}(x)_A = \sum_{l_L} \sum_{l_R} |l_L, m_L; l_R, m_R\rangle \langle l_L, m_L; l_R, m_R| \Psi^{(l_L)}(\chi)_{m'_L,m_L} \Psi^{(l_R)}(-\chi)_{m'_R,m_R}, \tag{103}
\]
where \( l_L \) and \( l_R \) are given by \([11] \) and \( |l_L, m_L; l_R, m_R\rangle \) denotes the Clebsch-Gordan coefficient. Replacing the relation \([67] \) of the lowest Landau level with that of the higher Landau level \([41] \), we may expect that \( \Phi_{103}^{(0,1/2)} \) realizes the higher Landau level basis states. (This expectation turns out to be true as we shall see below.) We will refer to \( \Phi_{103}^{(0,1/2)} \) as the \( SO(5) \) monopole harmonics \([11] \).

Nair and Randjbar-Daemi gave the first derivation of the Landau level eigenstates \([51] \), in which harmonic expansion on coset space \([52] \) was applied (see \([53, 54] \) also). On the coset
\[
S^3 \simeq SU(2)_L \otimes SU(2)_R / SU(2)_{\text{diag}}, \tag{104}
\]
the following matrix element was considered
\[
\Phi_{m_L,m_R}^{(l_L,l_R,1/2)}(x)_A = \langle l_L, A | g_L \otimes g_R | l_L, m_L; l_R, m_R \rangle. \tag{105}
\]
Inserting the complete basis relation
\[
1 = \sum_{l_L} \sum_{l_R} |l_L, m_L; l_R, m_R\rangle \langle l_L, m_L; l_R, m_R| \tag{106}
\]
to \( \Phi_{105}^{(l_L,l_R,1/2)} \), we have\([11] \]
\[
\Phi_{m_L,m_R}^{(n,s,1/2)}(x)_A = \sum_{m'_L=-l_L}^{l_L} \sum_{m'_R=-l_R}^{l_R} \langle l_L, A | g_L \otimes g_R | l_L, m'_L; l_R, m'_R \rangle D^{(l_L)}(\chi)_{m'_L,m_L} D^{(l_R)}(-\chi)_{m'_R,m_R}, \tag{107}
\]
where \( D \) is the Wigner’s \( D \) function
\[
D^{(l_L)}(\chi)_{m,n} = \langle l, m | g_L | l, n \rangle, \quad D^{(l_R)}(-\chi)_{m,n} = \langle l, m | g_R | l, n \rangle. \tag{108}
\]

\[\text{In the terminology, the usual monopole harmonics} \ [7] \ \text{are called the} \ SO(3) \ \text{monopole harmonics, and the Yang’s} \ SU(2) \ \text{monopole harmonics} \ [10] \ \text{will be the} \ SO(5) \ \text{monopole harmonics.}\]

\[\text{In} \ [53, 54, 107] \ \text{is referred to as the spin(-weighted) spherical harmonics on} \ S^3.\]
\( \Psi^{(l)}(\chi) \) corresponds to \( D^{(l)}(\chi) \) in the Dirac gauge (Appendix D), and so we find (107) is equivalent to (107). By binding the left magnetic quantum numbers \( (m_L', m_R') \) of two \( D \) functions with the Clebsch-Gordan coefficients, we can construct the \( \text{SO}(4) \) Landau level basis states with internal magnetic quantum number \( (A) \) as in (107). The condition

\[
\int_{S^3} d\Omega_3 \Phi_{m_L, m_R}^{(n, s, I/2)}(x) = \sqrt{\frac{(2l_L + 1)(2l_R + 1)}{2\pi^2}} \sum_{m_L' = -l_L}^{l_L} \sum_{m_R' = -l_R}^{l_R} \langle I/2, A|l_L, l_R, m_L', m_R' \rangle D^{(l_L)}(\chi)_{m_L', m_L} D^{(l_R)}(-\chi)_{m_R', m_R}.
\]

Using (107), we can construct a vector-like notation of the \( \text{SO}(4) \) monopole harmonics:

\[
\Phi_{m_L, m_R}^{(n, s, I/2)}(x) = \frac{1}{\sqrt{I+1}} \begin{pmatrix} \Phi_{m_L, m_R}^{(n, s, I/2)}(x)_{I/2} \\ \Phi_{m_L, m_R}^{(n, s, I/2)}(x)_{I/2-1} \\ \vdots \\ \Phi_{m_L, m_R}^{(n, s, I/2)}(x)_{I/2} \end{pmatrix},
\]

which satisfies

\[
\int_{S^3} d\Omega_3 \Phi_{m_L, m_R}^{(n, s, I/2)}(x) \Phi_{m_L', m_R'}^{(n', s', I'/2)}(x) = \delta_{m_L, m_L'} \delta_F m_R, \delta_{s, s'}, \delta_{I, I'}.
\]

Since the \( D \)-functions depend on the \( \text{SO}(4) \) Casimir indices determined only through \( n + \frac{l}{2} \) (and \( s \)), different \( n \) and \( I/2 \) can give rise to same \( D \) functions if \( n + \frac{\frac{1}{2}}{2} \) is fixed. The Clebsch-Gordan coefficients account for their difference.

### 3.4 Gauge fixing analysis

With gauge fixing, we further pursue the properties of the \( \text{SO}(4) \) Landau Hamiltonian eigenstates.

#### 3.4.1 Dirac gauge and Schwinger gauge

We first establish relations between the Dirac and the Schwinger gauges. In the Schwinger gauge, the \( D \)-function is given by

\[
\Psi_{S}^{(l/2)}(\chi) = D^{(l/2)}(\chi, -\theta, -\phi) \equiv e^{-i\chi S_z^{(l/2)}} e^{i\theta S_y^{(l/2)}} e^{i\phi S_x^{(l/2)}},
\]

which is related to \( \Psi_{D}^{(l)}(\chi) \) (10) by the \( \text{SU}(2) \) gauge transformation (see Appendix D for details)

\[
\Psi_{D}^{(l)}(\chi) = g^{(l)}(\theta, \phi) \Psi_{S}^{(l/2)}(\chi),
\]

where

\[
g^{(l)}(\theta, \phi) = e^{-i\phi S_z^{(l)}} e^{-i\theta S_y^{(l)}},
\]

As in the case of the Dirac gauge (113), the \( \text{SU}(2) \) gauge field in the Schwinger gauge can be expressed as

\[
A_S = -\frac{1}{2}(\Psi_L d\Psi_L^\dagger + \Psi_R d\Psi_R^\dagger) = S_z^{(l/2)} \cos \chi \sin \theta d\phi - S_y^{(l/2)} \cos \chi d\theta - S_x^{(l/2)} \cos \theta d\phi,
\]

\[
A_S = -\frac{1}{2}(\Psi_L d\Psi_L^\dagger + \Psi_R d\Psi_R^\dagger) = S_z^{(l/2)} \cos \chi \sin \theta d\phi - S_y^{(l/2)} \cos \chi d\theta - S_x^{(l/2)} \cos \theta d\phi.
\]
where
\[ \Psi_L(x) = \Psi_S^{(1/2)}(\chi), \quad \Psi_R(x) = \Psi_S^{(1/2)}(-\chi). \] (117)
We can read off the components of the gauge field from (116) as
\[ A_\chi = 0, \quad A_\theta = -\cos \chi S_y^{(1/2)}, \quad A_\phi = \cos \chi \sin \theta S_z^{(1/2)} - \cos \theta S_z^{(1/2)}. \] (118)
For \( I = 1 \), (118) is reduced to the \( SU(2) \) spin connection (17). From (114), one may find
\[ A_S = g(1/2)^\dagger A_D g(1/2) - ig(1/2)^\dagger dg(1/2). \] (119)
Actually this is a generalization of (13) for arbitrary spin magnitude. With (117), in the Schwinger gauge the \( SO(4) \) monopole harmonics are constructed as \[ \Phi_{m_L,m_R}^{(n,s,I/2)}(\chi)_A = \sqrt{\frac{(2l_L + 1)(2l_R + 1)}{2\pi^2}} \times \sum_{m_L'=-l_L}^{l_L} \sum_{m_R'=-l_R}^{l_R} \langle I/2, A|l_L, m'_L; l_R, m'_R \rangle \Psi_S^{(l)}(\chi)_{m_L,m_R} \Psi_S^{(l)}(-\chi)_{m_L',m_R'}, \] (120)
For instance, for \( (n,I/2,s) = (1,1/2,1/2) \), the \( SU(2)_L \otimes SU(2)_R \) indices are given by \( (l_L, l_R) = (1,1/2) \) and the dimension of the multiplet is \( (2l_L + 1)(2l_R + 1) = 6 \). As the gauge field undergoes the gauge transformation (119), the \( SO(4) \) monopole harmonics in the Dirac and the Schwinger gauges should be related as
\[ \Phi_{m_L,m_R}^{(n,s,I/2)}(\chi)_D = g(1/2)(\theta, \phi) \Phi_{m_L,m_R}^{(n,s,I/2)}(\chi)_S. \] (122)
It is easy to verify (122) and the property of the Clebsch-Gordan coefficient
\[ \langle I/2, A|l_L, m'_L; l_R, m'_R \rangle g^{(l)}(\theta, \phi)_{m'_L,m_L} g^{(l)}(\theta, \phi)_{m'_R,m_R} = g^{(l)}(\theta, \phi)_{AB} \langle I/2, B|l_L, m_L; l_R, m_R \rangle. \] (123)

### 3.4.2 Properties of the \( SO(4) \) monopole harmonics

From the properties of the \( D \) function and the Clebsch-Gordan coefficients
\[ D^{(l)}(\chi, \theta, \phi)_{m,n} = (-1)^{m-n} D^{(l)}(\chi, \theta, \phi)_{-m,-n}, \quad C_{l,-m;l',-m'} = (-1)^{l+l'+M} C^{l,M}_{l,-m;l',-m'}, \] (124)
the complex conjugate of the \( SO(4) \) monopole harmonics is given by
\[ \Phi_{m_L,m_R}^{(n,s,I/2)}(\chi)_A^* = (-1)^{l_L+l_R+\frac{1}{2}-m_L-m_R-A} \Phi_{-m_L,-m_R}^{(n,s,I/2)}(\chi) - A|l_L=\frac{1}{2}(n+s), l_R=\frac{1}{2}(n-s). \] (125)

\[ ^{12} \text{Unlike the Dirac gauge } \Psi_S^{(1)}(-\chi) = \Psi_S^{(1)}(\chi)^{-1}, \text{ in the Schwinger gauge } \Psi_S^{(1)}(-\chi) \neq \Psi_S^{(1)}(\chi)^{-1}. \]
\[ ^{13} \text{In this case, the } SO(4) \text{ monopole harmonics } \Phi^{(1)} \text{ are explicitly given by} \]
\[ \Phi^{(1,1/2,1/2)}_{1,1/2} = \frac{1}{\pi} e^{-\frac{i}{2} \phi} \sin \chi \sin \theta \left( e^{-\frac{i}{2} \chi \cos \theta} e^{\frac{i}{2} \chi \sin \theta} \right), \quad \Phi^{(1,1/2,1/2)}_{1,-1/2} = \frac{1}{\pi} e^{\frac{i}{2} \phi} \left( e^{\frac{i}{2} \chi \cos \theta} e^{-\frac{i}{2} \chi \sin \theta} \right), \]
\[ \Phi^{(1,1/2,1/2)}_{0,1/2} = \frac{1}{\sqrt{2\pi}} e^{\frac{i}{2} \phi} \left( -e^{-\frac{1}{2} \chi \cos \theta} e^{\frac{1}{2} \chi \sin \theta} \right), \quad \Phi^{(1,1/2,1/2)}_{0,-1/2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{i}{2} \phi} \left( e^{\frac{1}{2} \chi \cos \theta} e^{-\frac{1}{2} \chi \sin \theta} \right), \]
\[ \Phi^{(1,1/2,1/2)}_{-1,1/2} = -\frac{1}{\pi} e^{-\frac{i}{2} \phi} \left( e^{\frac{1}{2} \chi \cos \theta} e^{\frac{i}{2} \chi \sin \theta} \right), \quad \Phi^{(1,1/2,1/2)}_{-1,-1/2} = -\frac{1}{\pi} e^{-\frac{i}{2} \phi} \left( e^{\frac{1}{2} \chi \cos \theta} e^{\frac{i}{2} \chi \sin \theta} \right). \] (121)
Notice that the complex conjugate flips both of the \( SO(4) \) magnetic quantum numbers, \( m_L \) and \( m_R \). Integration of the product of three \( SO(4) \) monopole harmonics is given by (see Appendix C)

\[
\frac{1}{I+1} \int_{S^3} d\Omega_3 \left( \sum_{\Delta = -\frac{I}{2}}^{\frac{I}{2}} \Phi^{[l_L,l_R,\frac{I}{2}]}_{m_L,m_R}(\chi)^*_A \cdot \Phi^{[\frac{I}{2},0]}_{m'_L,m'_R}(\chi) \cdot \Phi^{[l_L,l_R,\frac{I}{2}]}_{m''_L,m''_R}(\chi)_A \right)
\]

\[
= \sqrt{(p+1)(2l_L+1)(2l_R+1)} \frac{(-1)^{(l_L+l_R+\frac{I}{2})}}{2\pi^2} \left\{ \frac{l_L}{l_R} \frac{l_R}{l_L} \right\}^\frac{I}{2} C^{l_L,m_L}_{l_R,m_R} C^{l_R,m_R}_{l_L,m_L} (126)
\]

and

\[
\frac{1}{I+1} \int_{S^3} d\Omega_3 \left( \sum_{\Delta = -\frac{I}{2}}^{\frac{I}{2}} \Phi^{[l_L,l_R,\frac{I}{2}]}_{m_L,m_R}(\chi)^*_A \cdot \Phi^{[\frac{I}{2},0]}_{m'_L,m'_R}(\chi) \cdot \Phi^{[l_L,l_R,\frac{I}{2}]}_{m''_L,m''_R}(\chi)_A \right)
\]

\[
= \sqrt{(p+1)(2l_L+1)(2l_R+1)} \frac{(-1)^{2l_L+\frac{I}{2}+\frac{I}{2}p}}{2\pi^2} \left\{ \frac{l_L}{l_R} \frac{l_R}{l_L} \right\}^\frac{I}{2} C^{l_L,m_L}_{l_R,m_R} C^{l_R,m_R}_{l_L,m_L} (127)
\]

where

\[
l_L = \frac{1}{2} (n + \frac{I}{2}) + \frac{1}{2} s, \quad l_R = \frac{1}{2} (n - \frac{I}{2}) - \frac{1}{2} s, (128)
\]

and \{ \cdots \} on the right-hand side is the 6-j symbol:

\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} = \sum_{\text{all } m} (-1)^{\Sigma (j_i - m_i) + 1} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ m_4 & m_2 & -m_6 \end{pmatrix} \begin{pmatrix} j_4 & j_2 & j_6 \\ -m_4 & -m_5 & m_3 \end{pmatrix}, (129)
\]

The integration formula (126) will be crucial to derive the matrix geometry of the \( SO(4) \) Landau level in Sec 6.

### 3.4.3 \( SO(4) \) covariance

The \( SO(4) \) covariance of the \( SO(4) \) monopole harmonics is essential for the monopole harmonics to be the eigenstates of the \( SO(4) \) Landau Hamiltonian. The gauge fixing indeed allows us to demonstrate the covariance of the \( SO(4) \) monopole harmonics.

In the Dirac gauge, the angular momentum operators are represented as (76) and the \( SO(4) \) monopole
harmonics are \( \text{I} \) with \( \text{III} \). Using these, we can explicitly show \( 1 \)

\[
L_i \Phi_{m_L, m_R}^{(n, s, \frac{j}{2})} = \sum_{m_L=-l_L}^{l_L} \Phi_{m_L, m_R}^{(n, s, \frac{j}{2})} (S_i^{(L)})_{m_L} m_L, \quad \bar{L}_i \Phi_{m_L, m_R}^{(n, s, \frac{j}{2})} = \sum_{m_R=-l_R}^{l_R} \Phi_{m_L, m_R}^{(n, s, \frac{j}{2})} (S_i^{(R)})_{m_R} m_R, \tag{133}
\]

or more concisely,

\[
L_i \Phi^{(n, s, \frac{j}{2})} = (S_i^{(L)})^t \Phi^{(n, s, \frac{j}{2})}, \quad \bar{L}_i \Phi^{(n, s, \frac{j}{2})} = \Phi^{(n, s, \frac{j}{2})} (S_i^{(R)}), \tag{134}
\]

which manifests that the \( SO(4) \) monopole harmonics are the irreducible representation of the \( SU(2)_L \) and \( SU(2)_R \). From

\[
L_{\mu \nu} = \eta_{\mu \nu} L_i + \bar{\eta}_{\mu \nu} \bar{L}_i, \tag{135}
\]

we have \( 15 \)

\[
L_{\mu \nu} \Phi^{(n, s, \frac{j}{2})} = \eta_{\mu \nu} (S_i^{(L)})^t \Phi^{(n, s, \frac{j}{2})} + \bar{\eta}_{\mu \nu} \Phi^{(n, s, \frac{j}{2})} (S_i^{(R)}), \tag{139}
\]

and then

\[
\sum_{\mu < \nu} L_{\mu \nu}^2 \Phi_{m_L, m_R}^{(n, s, \frac{j}{2})} = 2 \Phi_{m_L, m_R}^{(n, s, \frac{j}{2})} (S_i^{(L)})^2 (m_L m_L) + 2 \Phi_{m_L, m_R}^{(n, s, \frac{j}{2})} (S_i^{(R)})^2 (m_R m_R) = 2 (l(L + 1) + l(R + 1)) \Phi_{m_L, m_R}^{(n, s, \frac{j}{2})}. \tag{140}
\]

Obviously, \( \Phi_{m_L, m_R}^{(n, s, \frac{j}{2})} \) are the Landau level eigenstates with the \( SO(4) \) index \( 128 \).

### 3.5 Reduction to the \( SO(4) \) spherical harmonics

We can check that the \( SO(4) \) monopole harmonics are reduced to the \( SO(4) \) spherical harmonics in the free \( SU(2) \) background limit. In literature \( 42, 43, 44 \), the \( SO(4) \) spherical harmonics is usually given by

\[\Phi_{m_L, m_R}^{(n, s, \frac{j}{2})} \equiv \Phi_{m_L, m_R}^{(n, s, \frac{j}{2})} \quad (136)\]

the \( SO(4) \) transformation \( 139 \) can be concisely expressed as

\[
L_{\mu \nu} \Phi_{m_L, m_R}^{(n, s, \frac{j}{2})} = \sum_{N=1}^{d(n,s)} \Phi_{N}^{(n, s, \frac{j}{2})} (\Sigma_{\mu \nu})_{NM}, \tag{137}
\]

where \( \Sigma_{\mu \nu} \) are the \( SO(4) \) generators of the \( (l_L, l_R) \) representation:

\[
\Sigma_{\mu \nu} = \eta_{\mu \nu} S_i^{(L)} \otimes 1 + \bar{\eta}_{\mu \nu} 1 \otimes S_i^{(R)}. \tag{138}
\]
(see Appendix [B.1])

\[
Y_{nlm}(\chi) = 2^l l! \sqrt{\frac{2(n+1)(n-l)!}{\pi(n+l+1)!}} \sin^l(\chi) C_{n-l}^{l+1} \cos(\chi) \cdot Y_{lm}(\theta, \phi), \quad (l = 0, 1, 2, \ldots, n \text{ and } m = -l, -l+1, \ldots, l)
\]

where \(Y_{lm}\) are the SO(3) spherical harmonics and \(C_{n-l}^{l+1}\) are the Gegenbauer polynomials:

\[
C_{n}^{\alpha}(x) = \frac{(-2)^n \Gamma(n+\alpha)\Gamma(n+2\alpha)}{n! \Gamma(\alpha)\Gamma(2n+2\alpha)} (1-x^2)^{-\alpha+\frac{1}{2}} \frac{d^n}{dx^n}[(1-x^2)^{n+\alpha}].
\]

The SO(4) spherical harmonics \([141]\) carry the \(SU(2)_L \otimes SU(2)_R\) index

\[
(l_L, l_R) = \left(\frac{n}{2}, \frac{n}{2}\right).
\]

Meanwhile, the SO(4) monopole harmonics in the Dirac gauge \([110]\) yield \((n/2, n/2)\) representation as\(^{16}\)

\[
\Phi_{m_L, m_R}^{(n, 0, 0)}(x) = \frac{n+1}{\sqrt{2\pi^2}} \sum_{m'=-n/2}^{-n/2} \langle 0, 0 \mid \frac{n}{2}, m'_{L}; \frac{n}{2}, m'_{R} \rangle \Psi^{(n/2)}(\chi)_{m'_{L}, m'_{R}} \Psi^{(n/2)}(-\chi)_{m_{R}, m_{R}},
\]

where we used

\[
\langle 0, 0 \mid \frac{n}{2}, m'_{L}; \frac{n}{2}, m'_{R} \rangle = (-i)^n \frac{1}{\sqrt{n+1}} e^{2m'_{L} \delta_{m'_{L}, 0}}.
\]

For instance,

\[
\Phi_{m_L, m_R}^{(n=1, 0, 0)}(\chi) = \frac{1}{\pi} \left( \begin{array}{cc} \cos \chi - i \sin \chi \cos \theta & -i \sin \chi \sin \theta \\ -\cos \chi & \cos \chi - i \sin \chi \cos \theta \end{array} \right)_{m_{L}, m_{R}},
\]

The Clebsch-Gordan coefficients simply relate the two superficially different expressions of the SO(4) spherical harmonics, \([141]\) and \([147]\), as

\[
Y_{nlm}(x) = i^l \sum_{m_{L}, m_{R} = -n/2}^{n/2} \langle l, m_{L}; \frac{n}{2}, m_{R} \rangle \Phi_{m_{L}, m_{R}}^{(n, 0, 0)}(x).
\]

The SO(4) monopole harmonics are thus reduced to the SO(4) spherical harmonics in the free background limit.

\(^{16}\) In the Schwinger gauge \([120]\), we have

\[
\Phi_{m_{L}, m_{R}}^{(n, 0, 0)}(S)_{m_{L}, m_{R}}(x) = (-i)^n \sqrt{\frac{n+1}{4\pi^2}} \sum_{m = -n/2}^{n/2} i^{2m} \psi^{(n/2)}(S)_{m_{L}, m_{R}} \psi^{(n/2)}(-\chi)_{-m_{L}, m_{R}, R},
\]

which is equal to \([147]\):

\[
\Phi_{m_{L}, m_{R}}^{(n, 0, 0)}(S)_{m_{L}, m_{R}}(x) = \Phi_{m_{L}, m_{R}}^{(n, 0, 0)}(x).
\]

The SO(4) free angular momentum operator acts to \(\Phi_{m_{L}, m_{R}}^{(n, 0, 0)}(\chi)\) as

\[
l_{\mu \nu} \Phi_{m_{L}, m_{R}}^{(n, 0, 0)}(x) = \eta_{\mu \nu} \sum_{m'_{L} = -n/2}^{n/2} \Phi_{m_{L}, m_{R}}^{(n, 0, 0)}(x) \langle S_{m'_{L}}^{(n/2)} \rangle_{m_{L}, m_{L}} + \eta_{\mu \nu} \sum_{m'_{R} = -n/2}^{n/2} \Phi_{m_{L}, m_{R}}^{(n, 0, 0)}(x) \langle S_{m'_{R}}^{(n/2)} \rangle_{m_{R}, m_{R}}.
\]
4 Spinor Landau Model

Before going to the analysis of relativistic Landau models, we investigate the spinor $SO(4)$ Landau model whose Hamiltonian is the square of a relativistic Landau operator. The analysis of the spinor Landau model is a preliminary step to more complicated relativistic Landau models, but the spinor Landau model has importance in its own right. The spinor Landau model includes a synthesized connection of the spin and the gauge connections just like the relativistic Landau models. In Sec.4.1 we discuss special properties of the synthesized connection to present a basic idea to solve the eigenvalue problem. Based on the observation, we introduce the $SO(4)$ angular momentum operators in Sec.4.2 and explicitly solve the eigenvalue problem in Sec.4.3. We discuss the physics described by the spinor $SO(4)$ Landau model in Sec.4.4.

4.1 Synthesized connection

As mentioned in Sec.3, the components of the spin connection and the $SU(2)$ gauge field are exactly equivalent and their difference is just the representation of the $SU(2)$ generators:

$$\omega_\mu = \omega_\mu^i \frac{1}{2} \sigma_i, \quad A_\mu = \omega_\mu^i S_i^{(1/2)}. \quad (151)$$

The synthesized connection of the spin connection and the gauge field is

$$A_\mu = \omega_\mu \otimes 1_{I+1} + 1_2 \otimes A_\mu^{(1/2)} = \omega_\mu^i \left( \frac{1}{2} \sigma_i \otimes 1_{I+1} + 1_2 \otimes S_i^{(1/2)} \right). \quad (\mu = 1, 2, 3, 4) \quad (153)$$

Notice that in the present model, the common factor $\omega_\mu^i$ can be extracted in front of the synthesized $SU(2)$ generators. We can then decompose the $SU(2)$ representations to two direct sum of the two irreducible representations. The generators of the synthesized $SU(2)$ gauge group are

$$\frac{1}{2} \sigma_i \otimes 1_{I+1} + 1_2 \otimes S_i^{(1/2)}, \quad (154)$$

which are irreducibly decomposed as

$$\frac{1}{2} \otimes \frac{I}{2} = J^+ (\equiv \frac{I}{2} + \frac{1}{2}) \oplus J^- (\equiv \frac{I}{2} - \frac{1}{2}) \quad (155)$$

or more explicitly

$$\frac{1}{2} \sigma_i \otimes 1_{I+1} + 1_2 \otimes S_i^{(1/2)} \Rightarrow U^\dagger \left( \frac{1}{2} \sigma_i \otimes 1_{I+1} + 1_2 \otimes S_i^{(1/2)} \right) U = \begin{pmatrix} S_i^{(J^+)} & 0 \\ 0 & S_i^{(J^-)} \end{pmatrix}. \quad (156)$$

$U$ denotes $2(I + 1) \times 2(I + 1)$ unitary matrix constructed by the Clebsch-Gordan coefficients:

$$U = \left( C^+ \quad C^- \right) = \left( C^{J^+, A^+}_{\frac{1}{2}, \frac{1}{2} \sigma} ; \frac{1}{2}, m \quad C^{J^-, A^-}_{\frac{1}{2}, \frac{1}{2} \sigma} ; \frac{1}{2}, m \right). \quad (157)$$

The column is specified by the index $(\sigma, m)$ $(\sigma = +, -, m = I/2, I/2 - 1, \cdots, -I/2)$ and the row by $A^+ (= J^+, J^+, \cdots, -J^+), A^- (= J^-, J^-, \cdots, -J^-) \quad (158)$ $C^+$ and $C^-$ represent $2(I + 1) \times (I + 2)$

\[\text{In the local coordinates on } S^3, \text{ the synthesized connection is given by}\]

$$A_\alpha = \omega_\alpha \otimes 1_{I+1} + 1_2 \otimes A_\alpha^{(1/2)} = \omega_\alpha^i \left( \frac{1}{2} \sigma_i \otimes 1_{I+1} + 1_2 \otimes S_i^{(1/2)} \right). \quad (\alpha = \chi, \theta, \phi) \quad (152)$$

\[\text{For instance } I/2 = 1/2, \text{ we have}\]

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (158)$$

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and $2(I + 1) \times I$ rectangular matrices, respectively. The unitary transformation decomposes the synthesized $SU(2)$ connection space to the direct sum of the two $SU(2)$ spaces of spin $J^+$ and $J^-$, such as

$$U^\dagger A_\mu U = \begin{pmatrix} A_\mu^{(J^+)} & 0 \\ 0 & A_\mu^{(J^-)} \end{pmatrix}. \quad (159)$$

As usual, we construct the covariant derivatives for the synthesized connection:

$$D_\mu = \partial_\mu + iA_\mu \equiv \partial_\mu + i\omega_\mu \otimes 1 + i1 \otimes A_\mu, \quad (160)$$

and the field strength:

$$F_{\mu\nu} = -i[D_\mu, D_\nu] = f_{\mu\nu} \otimes 1 + 1 \otimes F_{\mu\nu} = f_{\mu\nu}^I \sigma_i \otimes 1_{I+1} + 1_2 \otimes S_i^{(1/2)}, \quad (162)$$

where

$$f_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + i[\omega_\mu, \omega_\nu]. \quad (163)$$

They are also block diagonalized as

$$U^\dagger D_\mu U = \begin{pmatrix} D_\mu^{(J^+)} & 0 \\ 0 & D_\mu^{(J^-)} \end{pmatrix}, \quad U^\dagger F U = \begin{pmatrix} F^{(J^+)} & 0 \\ 0 & F^{(J^-)} \end{pmatrix}. \quad (164)$$

The spinor Landau model is thus decomposed to the direct sum of two $SO(4)$ Landau models of $SU(2)$ gauge fields with $SU(2)$ index $J^+$ and $J^-$, and the eigenvalue problem is boiled down to those of the $SO(4)$ Landau models with $J^+$ and $J^-$ sectors. Therefore, to solve the eigenvalue problem, what we need to do is just to apply the method of Sec.3 to each of the sectors.

### 4.2 $SO(4)$ synthesized angular momentum

As the gauge connection is simply replaced with the synthesized connection, we introduce the synthesized $SO(4)$ angular momentum operators as

$$A_{\mu\nu} = -ix_\mu D_\nu + ix_\nu D_\mu, \quad L_{\mu\nu} = A_{\mu\nu} + F_{\mu\nu}. \quad (165)$$

By the unitary transformation, they are transformed to the block-diagonalized forms:

$$U^\dagger A_{\mu\nu} U = \begin{pmatrix} A_{\mu\nu}^{(J^+)} & 0 \\ 0 & A_{\mu\nu}^{(J^-)} \end{pmatrix}, \quad U^\dagger L_{\mu\nu} U = \begin{pmatrix} L_{\mu\nu}^{(J^+)} & 0 \\ 0 & L_{\mu\nu}^{(J^-)} \end{pmatrix}. \quad (166)$$

It is obvious that they satisfy the similar relations in Sec.3.2

$$\sum_{\mu<\nu} A_{\mu\nu} F_{\mu\nu} = \sum_{\mu<\nu} F_{\mu\nu} A_{\mu\nu} = 0, \quad \sum_{\mu<\nu} L_{\mu\nu}^2 = \sum_{\mu<\nu} A_{\mu\nu}^2 + \sum_{\mu<\nu} F_{\mu\nu}^2. \quad (167)$$

For

$$T_{\mu\nu} = L_{\mu\nu}, \quad A_{\mu\nu}, \quad F_{\mu\nu}, \quad (168)$$

the $SO(4)$ covariance is realized as

$$[L_{\mu\nu}, T_{\rho\sigma}] = i\delta_{\mu\rho} T_{\nu\sigma} - i\delta_{\mu\sigma} T_{\nu\rho} + i\delta_{\nu\rho} T_{\mu\sigma} - i\delta_{\nu\sigma} T_{\mu\rho}. \quad (169)$$

More concisely,

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} \epsilon_{ijk} e^i \wedge e^j (\frac{1}{2} \sigma_k \otimes 1_{I+1} + 1_2 \otimes S_i^{(1/2)}). \quad (161)$$

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The SO(4) Casimir

\[ \sum_{\mu < \nu} \mathcal{L}_{\mu\nu}^2 = U \left( \begin{pmatrix} \sum_{\mu < \nu} I_{\mu\nu}^{(J^+)} & 0 \\ 0 & \sum_{\mu < \nu} I_{\mu\nu}^{(J^-)} \end{pmatrix} \right) U^\dagger, \]

is diagonalized as

\[ \sum_{\mu < \nu} \mathcal{L}_{\mu\nu}^2 \Rightarrow U^\dagger \sum_{\mu < \nu} \mathcal{L}_{\mu\nu}^2 U = \begin{pmatrix} C^+(n^+, s^+) \mathbf{1}_{2J^++1} & 0 \\ 0 & C^-(n^-, s^-) \mathbf{1}_{2J^-+1} \end{pmatrix}, \]

where

\[ C^+(n^+, s^+) = (n^+ + J^+) (n^+ + J^+ + 2) + s^+^2, \]
\[ C^-(n^-, s^-) = (n^- + J^-) (n^- + J^- + 2) + s^-^2, \]

(172a, 172b)

with \( n^+, n^- = 0, 1, 2, 3, \cdots \), \( s^+ = -J^+, -J^++1, \cdots, J^+ \) and \( s^- = -J^-, -J^- + 1, \cdots, J^- \). There exist degeneracies, \( C^+(n, s) = C^+(n, -s) = C^-(n+1, s) = C^-(n+1, -s) \). The corresponding eigenstates are given by the following 2\((I + 1)\) component states:

\[ \Psi_{M_L^+, M_R^+}^{(n^+, \pm s^+, J^+)}(x) \equiv U \begin{pmatrix} \Phi_{M_L^+, M_R^+}^{(n^+, \pm s^+, J^+)} \\ 0 \end{pmatrix}, \]

(173a)
\[ \Psi_{M_L^-, M_R^-}^{(n^-, \pm s^-, J^-)}(x) \equiv U \begin{pmatrix} 0 \\ \Phi_{M_L^-, M_R^-}^{(n^-, \pm s^-, J^-)} \end{pmatrix}, \]

(173b)

where \( \Phi_{M_L^+, M_R^+}^{(n^+, \pm s^+, J^+)} \) and \( \Phi_{M_L^-, M_R^-}^{(n^-, \pm s^-, J^-)} \) denote the SO(4) monopole harmonics with \((I + 2)\) and \( I \) components, respectively.

### 4.3 Eigenvalue Problem

With the synthesized connection, we adopt the following as the spinor Landau Hamiltonian

\[ H = \sum_{\mu < \nu} A_{\mu\nu}^2 + \frac{1}{2} ((I + 3) C^+ C^+ \dagger - (I - 1) C^- C^- \dagger) \]
\[ = \sum_{\mu < \nu} A_{\mu\nu}^2 + \frac{1}{2} U \begin{pmatrix} (I + 3) \mathbf{1}_{I+2} & 0 \\ 0 & -(I - 1) \mathbf{1}_I \end{pmatrix} U, \]

(176)

Using the explicit form of \( C^+ \) and \( C^- \), the components of \( \Psi_{M_L^+, M_R^+}^{(n^+, \pm s^+, J^+)} \) can be expressed as

\[ \Psi_{M_L^+, M_R^+}^{(n^+, \pm s^+, J^+)}(x) = \sum_{M_L^-=L^+} L^+ \sum_{R^+=L^-} R^- (\frac{1}{2} \pm \frac{1}{2}) m \lvert P_{J^+} \lvert L^+, M_L^+; R^+, M_R^+ \rvert \Psi^{(L^+)}(x) M_L^+, M_L^+ \Psi^{(R^+)}(-x) M_R^+, M_R^+; \]
\[ \Psi_{M_L^-, M_R^-}^{(n^-, \pm s^-, J^-)}(x) = \sum_{M_L^-=L^-} L^- \sum_{R^-=L^+} R^+ (\frac{1}{2} \pm \frac{1}{2}) m \lvert P_{J^-} \lvert L^-, M_L^-; R^-, M_R^- \rvert \Psi^{(L^-)}(x) M_L^-, M_L^- \Psi^{(R^-)}(-x) M_R^-, M_R^-; \]

(174)

where \( \sigma = +1, -1, m = I/2, I/2 - 1, \cdots, -I/2 \), and \( P_{J^\pm} \) is the projection operator to the Hilbert space of the \( SU(2) \) index \( J^\pm \):

\[ P_{J^\pm} \equiv \sum_{A = -J^\pm}^{J^\pm} \lvert J^\pm, A \rangle \langle J^\pm, A \rvert. \]

(175)
The constant matrix term on the right-hand side was added for the spinor Landau Hamiltonian to be the precise square of the Weyl-Landau operator (see Sec.<sup>167</sup>). From (167) and
\[
\sum_{\mu<\nu} F_{\mu\nu}^2 = U \left( \begin{array}{cc} S^{(J^+)^2} & 0 \\ 0 & S^{(J^-)^2} \end{array} \right) U^\dagger = U \left( \begin{array}{cc} \frac{1}{4}(I+1)(I+3)1_{I+2} & 0 \\ 0 & \frac{1}{4}(I-1)(I+1)1_{I} \end{array} \right) U^\dagger, \tag{177} \]
we find that (176) is essentially the SO(4) Casimir made of \( L_{\mu\nu} \):
\[
H = \sum_{\mu<\nu} L_{\mu\nu}^2 - \frac{1}{4}(I-1)(I+3). \tag{178} \]

The spinor Landau Hamiltonian is invariant under the SO(4) rotations. Recalling the results in Sec.<sup>172</sup> we can derive the energy eigenvalues, \( E^+_n(s) \) for \( J^+ \)-sector and \( E^-_n(s) \) for \( J^- \)-sector. For \( E^+_n(s) \), the range of the indices are \( n = 0, 1, 2, \ldots \), \( s = -J^+, -J^+ + 1, \ldots, J^+ \), and hence we have \( E_n(s) = n(I + n + 1) + s^2 \) with \( n = 0, 1, 2, \ldots \) and \( s = -J^+, -J^+ + 1, \ldots, J^+ \). In detail,
\[
|s| = J^+ : E_n(J^+) \equiv E_{n-1}^+(\pm J^+) = (n + \frac{I-1}{2})^2, \tag{179a} \\
|s| \leq J^- : E_n(s) \equiv E_{n-1}^-(\pm s) = E_{n-1}^+(\pm s) = n(I + n + 1) + s^2, \tag{179b} \\
n = 0 : E_0^-(s) = E_0^-(s) = s^2 \quad (|s| \leq J^-), \tag{179c} \]
where
\[
n = 1, 2, 3, \ldots, \quad s = -\frac{I}{2}, -\frac{I}{2} + 1, \ldots, \frac{3I}{2}, \ldots, \frac{I}{2}, \ldots, \frac{I}{2} - 1. \tag{180} \]

The schematic picture of the spectrum is given by Fig.2. Note that \( J^- = \frac{I}{2} - \frac{1}{2} \geq 0 \) is not well defined for \( I = 0 \), and then the energy levels (179a) and (179b) vanish in the free background limit. (See Appendix B.3 also.) The degeneracies are
\[
2(2L^+(n - 1, J^+) + 1)(2R^+(n - 1, J^+) + 1) = 2n(n + I + 1), \tag{181a} \\
4(2L^-(n, s) + 1)(2R^-(n, s) + 1)(1 - \delta_{I,0}) = (2n + I + 2s + 1)(2n + I - 2s + 1)(1 - \delta_{I,0}) \tag{181b} \\
(2L^- (0, s) + 1)(2R^- (0, s) + 1)(2 - \delta_{s,0})(1 - \delta_{I,0}) = (2 - \delta_{s,0})(\frac{I + 1}{2} + s)(\frac{I + 1}{2} - s)(1 - \delta_{I,0}). \tag{181c} \]

In (181c), \( s = 0 \) is special:
\[
(\frac{I + 1}{2})^2 = 1, 2^2, 3^2, \ldots, \tag{182} \]
which is equal to the degeneracy \( (2j + 1)^2 \) \( (j = 0, 1/2, 1, 3/2, \ldots) \) of two particles with identical spin \( j \). The explicit forms of the eigenstates are given by
\[
\Psi_{M_L,M_R}^{(n-1,\pm j^+ J^+)}(J^+ \>
\Psi_{M_L,M_R}^{(n,\pm s,J^-)}(J^- \>
\Psi_{M_L,M_R}^{(n,\pm s,J')}(J^- \>
\Psi_{M_L,M_R}^{(0,\pm s,J^-)}(J^- \>. \tag{183c} \]

The lowest Landau level of the SO(4) spinor Landau model (170a) comes only from that of the \( J^- \)-sector, and (179a) from \( n \)th Landau level in the \( J^+ \)-sector and \( (n - 1) \)th Landau level in the \( J^- \)-sector, because of \( n + J^+ = (n + 1) + J^- \). These features are similar to those of the SO(3) spinor Landau model on \( S^2 \). However, (179a) comes only from the \( |s| = J^+ \) subband of \( n \)th Landau level, which is a new feature not observed in the SO(3) spinor Landau model.
Figure 2: Energy spectra of the spinor Landau model
4.4 Interactions in the spinor Landau Hamiltonian

We discuss interactions that the spinor Landau model describes. In \(109\), we saw that the non-relativistic \(SO(4)\) Landau Hamiltonian includes the (gauge) spin-orbit interaction. Since the synthesized connection consists of two kinds of spins coming from the holonomy and gauge groups, the spinor Landau Hamiltonian is expected to represent their interactions also.

We first decompose the \(SO(4)\) synthesized angular momentum operator as
\[
L_{\mu \nu} = L_{\mu \nu}^0 + s_{\mu \nu} \tag{184}
\]
where \(L_{\mu \nu}^0\) contain the gauge spin and the holonomy spin \(s_{\mu \nu}\) :
\[
L_{\mu \nu}^0 = -ix_\mu D_\nu + ix_\nu D_\mu + 1 \otimes F_{\mu \nu}, \tag{185a}
\]
\[
s_{\mu \nu} = (x_\mu \omega_\nu - x_\nu \omega_\mu + f_{\mu \nu}) \otimes 1. \tag{185b}
\]
The square of \(SO(4)\) total angular momentum is derived as
\[
L_{\mu \nu}^2 = L_{\mu \nu}^2 + 2s_{\mu \nu} L_{\mu \nu} + s_{\mu \nu}^2, \tag{186}
\]
where we used
\[
l_{\mu \nu}s_{\mu \nu} = -2ix_\mu \partial_\nu s_{\mu \nu} = 0. \tag{187}
\]
\(L_{\mu \nu}^2\) is given by \(109\). The second term on the right-hand side of \(186\) contains the holonomy spin-orbit interaction and the holonomy gauge spin-spin interaction. Indeed in the Dirac gauge
\[
\omega_i = -\frac{1}{2(1 + x_4)} \epsilon_{ijk} x_j \sigma_k, \quad \omega_4 = 0, \quad f_{ij} = -x_i \omega_j + x_j \omega_i + \frac{1}{2} \epsilon_{ijk} \sigma_k, \quad f_{i4} = (1 + x_4) \omega_i, \tag{188}
\]
the holonomy part \(185b\) is simply represented as
\[
s_{ij} = \frac{1}{2} \epsilon_{ijk} \sigma_k, \quad s_{i4} = \omega_i, \tag{189}
\]
and the second term of \(186\) becomes
\[
\sum_{\mu < \nu} s_{\mu \nu} L_{\mu \nu} = \sigma_i \otimes L_i^D + 2\omega_i \otimes K_i = \frac{1}{1 + x_4} \sigma \cdot l + \frac{2}{1 + x_4} \left( \sigma \otimes S^{(1/2)} - \frac{1}{2(1 + x_4)} (x \cdot \sigma) \otimes (x \cdot S^{(1/2)}) \right), \tag{190}
\]
where both \(L_i^D\) \(177\) and \(K_i\) \(79\) contain the gauge spin \(S_i^{(1/2)}\). The last term on the right-hand side of \(186\) can be expressed as
\[
\sum_{\mu < \nu} s_{\mu \nu}^2 = \omega_{\mu}^2 + \sum_{\mu < \nu} f_{\mu \nu}^2 = \frac{1}{1 + x_4} + \frac{1}{4}. \tag{191}
\]
Consequently, the spinor Landau Hamiltonian \(178\) is represented as
\[
H = \sum_{\mu < \nu} l_{\mu \nu}^2 + \frac{2}{1 + x_4} S^{(1/2)} \cdot l + \frac{1}{1 + x_4} \sigma \cdot l + \frac{2}{1 + x_4} \left( \sigma \otimes S^{(1/2)} - \frac{1}{2(1 + x_4)} (x \cdot \sigma) \otimes (x \cdot S^{(1/2)}) \right) - \frac{1}{(1 + x_4)^2} (x \cdot S^{(1/2)})^2 + \frac{1 - x_4}{4(1 + x_4)} I(I + 2) + \frac{2 + x_4}{1 + x_4}. \tag{192}
\]
The second, third, and fourth terms respectively stand for the gauge spin-orbit, holonomy spin-orbit, and holonomy gauge spin-spin interactions. With special combination of such interactions, the spinor Landau Hamiltonian \(192\) respects the \(SO(4)\) symmetry.
5 Relativistic Landau Models

Behind the SO(4) Landau models, we implicitly assumed (3 + 1) space-time (whose spacial manifold is $S^3$), and we refer to $2 \times 2$ Dirac operator as the Weyl operator and $4 \times 4$ Dirac operator as the Dirac operator simply. The previous results of the spinor Landau models are applied to solve the eigenvalue problems of the relativistic Landau models. In Sec.5.1, we utilize the spin connection to construct the Weyl-Landau Hamiltonian on $S^3$ and analyze the eigenvalue problem, in which verification of the SO(4) invariance of the Dirac-Landau operator and an explicit form of the eigenstates are derived. We also account for the existence of the zero-modes from the non-commutative geometry point of view. It is shown that the obtained results are reduced to the known formulae of the free Weyl operator in the free background limit. In Sec.5.2, we make use of the results of the Weyl-Landau model to solve the eigenvalue problem of the massive Dirac-Landau model. The supersymmetric Landau operator made of the square of the Dirac-Landau operator is also analyzed.

A convenient gauge to express the relativistic operators on $S^3$ is the Schwinger gauge, which we will adopt in this section.

5.1 SO(4) Weyl-Landau model

Let us begin with the construction of the Weyl-Landau operator

$$-i\mathcal{P} = -ie_\alpha \gamma^\alpha \mathcal{D}_\alpha = -ie_\alpha \gamma^\alpha (\partial_\alpha + i\omega_\alpha \otimes 1_{t+1} + i1_2 \otimes A^{(1/2)}_\alpha).$$

(193)

It is a $(2 \cdot (I + 1)) \times (2 \cdot (I + 1))$ matrix-valued differential operator. From (13), the covariant derivatives

$$-i\mathcal{D}_\alpha = -i\partial_\alpha + A_\alpha$$

(194)

are given by

$$-i\mathcal{D}_\chi = -i\partial_\chi,$$

$$-i\mathcal{D}_\theta = -i\partial_\theta - \cos \chi (\frac{1}{2} \sigma_y \otimes 1 + 1 \otimes S_y^{(1/2)}),$$

$$-i\mathcal{D}_\phi = -i\partial_\phi + \sin \theta \cos \chi (\frac{1}{2} \sigma_x \otimes 1 + 1 \otimes S_x^{(1/2)}) - \cos \theta (\frac{1}{2} \sigma_z \otimes 1 + 1 \otimes S_z^{(1/2)}),$$

(195)

and the Weyl-Landau operator is expressed as

$$-i\mathcal{P} = -i\gamma^1 \mathcal{D}_\chi - i\frac{1}{\sin \chi} \gamma^2 \mathcal{D}_\theta - i\frac{1}{\sin \chi \sin \theta} \gamma^3 \mathcal{D}_\phi$$

$$= -i\sigma_z \hat{D}_\chi - i\frac{1}{\sin \chi} \sigma_x \hat{D}_\theta - i\frac{1}{\sin \chi \sin \theta} \sigma_y \hat{D}_\phi,$$

(196)

or

$$-i\mathcal{P} = \left( -i\frac{1}{\sin \chi} \hat{D}_\theta + \frac{1}{\sin \chi \sin \theta} \hat{D}_\phi, -i\frac{1}{\sin \chi} \hat{D}_\theta - \frac{1}{\sin \chi \sin \theta} \hat{D}_\phi \right),$$

(197)

where

$$\hat{D}_\chi \equiv \partial_\chi + \cot \chi = D_\chi + \cot \chi,$$

$$\hat{D}_\theta \equiv \partial_\theta - i\cos \chi S_y^{(1/2)} + \frac{1}{2} \cot \theta = D_\theta + \frac{1}{2} \cot \theta,$$

$$\hat{D}_\phi \equiv \partial_\phi - i\cos \theta S_z^{(1/2)} + i\cos \chi \sin \theta S_x^{(1/2)} = D_\phi.$$
The last terms of \( \hat{D}_\chi \) and \( \hat{D}_\theta \) are non-hermitian terms that come from the spin connection \(^{[19]}\) (as in the case of the Dirac-Landau operator on \( S^3 \)) \(^{[1]}\): \( \cot \chi \) in \( \hat{D}_\chi \) is from \( \omega^1 \) and \( \omega^3 \), and \( \frac{1}{2} \cot \theta \) in \( \hat{D}_\theta \) from \( \omega^2 \) \(^{[2]}\). In the previous study of the \( SO(3) \) Landau model \(^{[1]}\), we showed that the Dirac-Landau operator is invariant under the \( SO(3) \) rotation. In the present case, the Weyl-Landau operator \(^{[19]}\) is invariant under the transformations generated by the \( SO(4) \) synthesized angular momentum operators. To see this, we use explicit coordinate representation of the Weyl-Dirac operator and \( SO(4) \) angular momentum operators. Having established the gauge transformation (Sec.3.4.1), it is not difficult to work either in the Dirac gauge or in the Schwinger gauge to demonstrate

\[ [-i\mathcal{P}, \mathcal{L}_{\mu \nu}] = 0. \]  

As the Weyl-Landau operator is \( SO(4) \) singlet, the Weyl-Landau operator eigenvalues should have degeneracies due to the existence of the \( SO(4) \) symmetry. In other words, the Weyl-Landau operator eigenstates consist of the \( SO(4) \) Landau level basis states of the spinor Landau model. Let us first derive the eigenvalues of the Weyl-Landau operator. The square of the Weyl-Landau operator can also be derived from a general formula \(^{[55, 37, 38]}\)

\[ (-i\mathcal{P})^2 = \sum_{\mu<\nu} \mathcal{L}^2 - S^{(I/2)}^2 + \frac{3}{4} = \sum_{\mu<\nu} \mathcal{L}^2 - \frac{1}{4}(I + 3)(I - 1). \]  

The first equation of \( \ref{201} \) can be checked rigorously from the Weyl-Landau operator \( \ref{197} \) and the \( SO(4) \) angular momentum operators \(^{[2]}\). With the results in Sec.4.3, we can readily obtain the Weyl-Landau operator eigenvalues as \( \pm \lambda_n(s) = \pm \sqrt{n(2n + 1) + s^2} \) with \( n = 0, 1, 2, \ldots \) and \( s = -J^+, -J^+ + 1, \ldots, J^+ \), or

\[
\begin{align*}
|s| = J^+ : & + \lambda(n, \frac{I + 1}{2}) = \lambda(n, \frac{I + 1}{2}), & - \lambda(n, \frac{I + 1}{2}) = -(n + \frac{I + 1}{2}) \quad \text{(204a)} \\
|s| \leq J^- : & + \lambda(n, s) = +\sqrt{n(I + 1 + 1) + s^2}, & - \lambda(n, s) = -\sqrt{n(I + 1 + 1) + s^2}, \quad \text{(204b)} \\
n = 0 : & + \lambda(0, s) = s \quad (s \geq 0), & - \lambda(0, s) = s \quad (s \leq 0), \quad \text{(204c)}
\end{align*}
\]

where

\[ n = 1, 2, 3, \ldots, \quad s = -\frac{I}{2} + \frac{1}{2}, -\frac{I}{2} + \frac{3}{2}, \ldots, -\frac{I}{2}. \]  

(In the free background limit \( I = 0 \), there do not exist the eigenstates for \( \ref{204b} \) and \( \ref{204c} \), as in the case of the spinor Landau model.) Notice that the zeroth Landau level \( \ref{204d} \) does not explicitly depend on the monopole charge and remains in low energy region even in \( I \to \infty \) limit. The schematic picture of the Weyl-Landau operator is given by Fig.3. The degeneracies for \( \ref{204} \) are respectively given by

\(^{21}\) The spin connection \( \varphi = \epsilon_a \gamma^a \omega_a \) is given by

\[
\begin{align*}
\text{Schwinger gauge : } \varphi_S &= -i \cot \chi \sigma_z - i \frac{1}{2 \sin \chi} \cot \theta \sigma_x, \\
\text{Dirac gauge : } \varphi_D &= i \tan \left( \frac{\chi}{2} \right) \left( \sin \theta \cos \phi \sigma_x + \sin \theta \sin \phi \sigma_y + \cos \theta \sigma_z \right) = i \tan \left( \frac{\chi}{2} \right) \hat{x} \cdot \sigma.
\end{align*}
\]

\(^{22}\) \( \ref{201} \) can also be derived from a general formula \(^{[55, 37, 38]}\)

\[ (-i\mathcal{P})^2 = C_{SO(4)} - C_{SO(3)} + \frac{1}{8} R_{S^3} \]  

with

\[ R_{S^3} = d(d - 1)|_{d=3} = 6. \]
Figure 3: The Weyl-Landau operator spectra. The spectral patterns are different with respect to the parity of $I$. In particular for odd $I$, there appear zero-modes ($s = 0$).

\begin{align}
(n + I + 1)n, \\
2(n + I + 1)(n + I + 1 - s) & \times (n + I + 1 - 2s), \\
\frac{(I + 1)}{2} + s & \times (1 - \delta_{I,0}).
\end{align}

The explicit forms of the eigenstates are

\begin{align}
\alpha_s \Psi_{ML,MR}^{(n-1,s,J^+)} + \beta_s \Psi_{ML,MR}^{(n,s,J^-)}, \\
\beta_s \Psi_{ML,MR}^{(n-1,s,J^+)} - \alpha_s \Psi_{ML,MR}^{(n-1,s,J^-)}, \\
\Psi_{ML,MR}^{(0,s,J^-)} (s \geq 0), \\
\Psi_{ML,MR}^{(0,s,J^-)} (s \leq 0),
\end{align}

where $\alpha_s$ and $\beta_s$ are the coefficients (subject to $\alpha_s^2 + \beta_s^2 = 1$) that are determined so that \ref{eq:207b} be the eigenstates of the Weyl-Landau operator with the eigenvalues \ref{eq:204b}. In Appendix E we explicitly derive $\alpha_s$ and $\beta_s$ and construct the Weyl-Landau operator eigenstates for several cases. As mentioned above, the Weyl-Landau operator eigenstates \ref{eq:207} are the $SO(4) \simeq SU(2)_L \otimes SU(2)_R$ irreducible representations with the indices $(L, R)$:

\begin{align}
\left(\frac{1}{2}(n - 1), \frac{1}{2}(n + I)\right), \\
\left(\frac{1}{2}(n + I + 1), \frac{1}{2}(n + I)\right), \\
\left(\frac{1}{2}(n + I + 1 - 2s), \frac{1}{2}(n + I + 1 - s)\right), \\
\left(\frac{1}{2}(n + I + 1 - s), \frac{1}{2}(n + I + 1 - s)\right), \\
\left(\frac{1}{2}(n + I + 1), \frac{1}{2}(n + I)\right), \\
\left(\frac{1}{2}(n + I + 1), \frac{1}{2}(n + I)\right).
\end{align}
For \( s = 0 \), two representations of \( SU(4) \) coincide to be \( (L, R) = (\frac{1}{4}(I - 1), \frac{1}{4}(I - 1)) \). (207c) consists of both the \((n - 1)\)th Landau level basis states in \( J^+ \)-sector and the \( n \)th Landau level in \( J^- \)-sector, because \( n - 1 + J^+ = n + J^- \). This feature is similar to that of the \( SO(3) \) Dirac-Landau model on \( S^2 \). Meanwhile, (207a) comes only from the non-relativistic \( n \)th Landau level with replacement of \( I/2 \) with \( J^+ \), and this is a new feature in the \( SO(4) \) model.

Notice that the zeroth Landau level (204c) comes only from the lowest Landau level of \( J^+ \)-sector, and the relativistic zero Landau level is exactly equal to that of the non-relativistic lowest Landau level with replacement of \( I/2 \) with \( J^- = (I - 1)/2 \). This property is also observed in the \( SO(3) \) model on \( S^2 \). For the zeroth Landau level, the spectrum of the subbands is exactly equal to the corresponding chirality parameter:

\[
- i \mathcal{P}_{n=0} = s.
\]

Therefore the reflection symmetry between the positive and negative eigenvalues in the zeroth Landau level is identical to the LR symmetry:

\[
s \leftrightarrow - s.
\]

Thus in the relativistic Landau model, the left-right symmetry of the non-relativistic Landau model is realized as the chiral symmetry (of the zeroth Landau level).

5.1.1 The lowest Landau level and the zero-modes

In [37], we showed that the total degeneracy of the \( SO(4) \) zeroth Landau level is equal to the 2nd Chern number of the \( SO(5) \) Landau model:

\[
\nu_{3D}^{\text{total}} = \sum_{s=0}^{J^- - \frac{1}{2}} (\frac{J^- + 1}{2} + s)(\frac{J^- + 1}{2} - s) = \frac{1}{6} I(I + 1)(I + 2) = c_2,
\]

as a manifestation of the dimensional ladder of anomaly. The total degeneracy of the zeroth Landau level thus finds its topological origin in one dimension higher space.

In even dimensions, the existence of Dirac-Landau operator zero-modes is accounted for by the Atiyah-Singer index theorem [55][15]. Though constructed on three-sphere, the \( SO(4) \) relativistic Landau model also accommodates the zero-modes for odd \( I \). Here we discuss the origin of such zero-modes. For odd \( I = 2q - 1 \) \((q = 1, 2, 3, \cdots)\), the explicit form of the zero-modes is given by

\[
\Psi^{(n=0,s=0,J^-)}_{L,M_L,M_R} \propto \psi_{L1}^{L^-+M_L} \psi_{L2}^{L^-+M_L} \otimes \psi_{R1}^{L^-+M_R} \psi_{R2}^{L^-+M_R}, \quad (\frac{J^-}{2} \leq M_L, M_R \leq \frac{J^-}{2})
\]

with degeneracy

\[
\nu_{3D}^{\text{zero}} \equiv (J^- + 1)^2 = q^2.
\]

Each of the \( SU(2) \) representations, \( \psi_{L1}^{L^-+M_L} \psi_{L2}^{L^-+M_L} \) and \( \psi_{R1}^{L^-+M_R} \psi_{R2}^{L^-+M_R} \), is the zero-modes of the \( SO(3) \) relativistic models with \( U(1) \) monopole charge \( q/2 = (I + 1)/4 \). Due to the Atiyah-Singer index theorem, the degeneracy of such zero-modes on two-sphere is equal to the 1st Chern number, \( c_1 = \frac{1}{2\pi} \int_{S^2} \tilde{F} = q \). Therefore, the zero-mode degeneracy of the Weyl-Landau operator can be expressed by the 1st Chern-number:

\[
\nu_{3D}^{\text{zero}} = c_1^2
\]

In this sense, the zero-modes on \( S^3 \) originates from the topological quantity in one dimension lower 2D space. From the viewpoint of the non-commutative geometry, the fuzzy three-sphere is represented as [35][30][23]

\[
S^3_F \cong (SU(2)_L \otimes SU(2)_R)/(U(1)_L \otimes U(1)_R) \cong S^2_L \otimes S^2_R,
\]

\( 23 \) (217) is naturally induced from the chiral Hopf map [30]:

\[
S^2_L \otimes S^2_R \xrightarrow{S^3_L \otimes S^3_R} S^2_L \otimes S^2_R.
\]

(215)
meaning that the fuzzy three-sphere is essentially the product of two independent fuzzy two-spheres. As the
zero-mode degeneracy is equal to the dimension of fuzzy sphere \[15\], it may be natural that the zero-modes
on \(S^3\) is given by the product of the zero-modes on two \(S^2\)s.

### 5.1.2 Reduction to the free Weyl model

In the free background limit \(I \to 0\), the synthesized gauge field becomes the spin connection, and
\(L_{\mu\nu}|_{I/2=0} = -ix_{\mu}(\partial_{\nu} + i\omega_{\nu}) + ix_{\nu}(\partial_{\mu} + i\omega_{\mu})\) is formally equivalent to the non-relativistic angular momentum,
\(L_{\mu\nu} = -ix_{\mu}(\partial_{\nu} + iA_{\nu}^{(1/2)}) + ix_{\nu}(\partial_{\mu} + iA_{\mu}^{(1/2)})\) with minimal monopole charge \(I/2 = 1/2\) (see Appendices \[13.2\] and \[13.3\]). Therefore, the eigenstates of \(L_{\mu\nu}|^{(I/2=0)}\) are given by the \(SO(4)\) monopole harmonics with \(I/2 = 1/2\).

Also in the relativistic Landau model for \(I = 0\), the Weyl-Landau operator is reduced to the free Weyl
operator, \(-i\bar{\psi} = -ie_a \gamma^a(\partial_\alpha + i\omega_\alpha)\), and the chirality parameter \(n_{\alpha}\) becomes
\[
s = \frac{1}{2} - \frac{1}{2},
\]
and only \((204a)\) survives in \((204)\) to give the eigenvalue
\[
- i\bar{\psi} = \pm (n + \frac{3}{2}) \quad (n = 0, 1, 2, \cdots),
\]
with degeneracy \((206a)\)
\[
(n + 2)(n + 1).
\]
(In the convention \(219\), \(n\) starts from 0 not 1 unlike \(205\).) These indeed coincide with the known results
of the free Weyl operator (Appendix \[13.2\]). The free Weyl operator spectra are given by Fig.4. When \(I = 0\),
\(J^-\) no longer exists, and only \((204a)\) with \(J^+ = 1/2\) becomes the eigenstates:
\[
- i\bar{\psi} = \pm (\frac{3}{2} + n) : \Phi_{M_L^+, M_R^+\,}^{(n, -1/2, 1/2)}(x) = \Phi_{M_L^+, M_R^+\,}^{(n, -1/2, 1/2)}(x),
- i\bar{\psi} = \mp (\frac{3}{2} + n) : \Phi_{M_L^+, M_R^+\,}^{(n, +1/2, 1/2)}(x) = \Phi_{M_L^+, M_R^+\,}^{(n, +1/2, 1/2)}(x),
\]
where \(\Phi_{M_L^+, M_R^+\,}^{(n, \pm1/2)}\) denote the \(SO(4)\) monopole harmonics \([131]\). We can also show that \((221)\) is transformed
to the known expression of the free Weyl operator eigenstates \(\psi_{n,l,m,\sigma}^{(\pm)}\) \([315]\) as
\[
\psi_{n,l,m,\sigma}(\chi, \theta, \phi) = i^{l} \sigma^m \sum_{M_L = -L}^{L} \sum_{M_R = -R}^{R} \langle l + \frac{1}{2}, -\sigma(m + \frac{1}{2})|L, M_L; R, M_R\rangle \Phi_{M_L, M_R}^{(n, -1/2, 1/2)}(x),
\]
\[
\psi_{n,l,m,\sigma}(\chi, \theta, \phi) = i^{l} \sigma^m \sum_{M_L = -L}^{L} \sum_{M_R = -R}^{R} \langle l + \frac{1}{2}, -\sigma(m + \frac{1}{2})|L, M_L; R, M_R\rangle \Phi_{M_L, M_R}^{(n, +1/2, 1/2)}(x),
\]
with \(\sigma = +, -\). We thus established the relations between the obtained results of the \(SO(4)\) Weyl-Landau
model and the free Weyl model \([45, 46, 47]\).

In arbitrary odd dimension, \((216)\) is generalized as \((218)\)
\[
S_{F}^{2k-1} \simeq SO(2k)/(U(1) \times U(k - 1)).
\]
Figure 4: The free Weyl operator spectrum. The special subbands with $s = \pm J^+$ (denoted by blue in Fig.3) survive in the free limit.

5.2 $SO(4)$ Dirac-Landau model

Next we analyze the Weyl-Landau operator to the Dirac-Landau operator. From $SO(4)$ gamma matrices

$$\Gamma^a = \begin{pmatrix} \gamma^a & 0 \\ 0 & -\gamma^a \end{pmatrix} \quad (a = 1, 2, 3), \quad \Gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (223)$$

with $\gamma^a$, the massive Dirac-Landau Hamiltonian is constructed as

$$-i\not{D} + M\Gamma^4 = -i\epsilon^a_{\alpha} \Gamma^a (\partial_\alpha + iA_\alpha) + M\Gamma^4 = \begin{pmatrix} -i\not{D} & M \\ M & i\not{D} \end{pmatrix}. \quad (224)$$

The mass term appears as the off-diagonal blocks. The $SO(4)$ angular momentum operators are similarly given by

$$L_{\mu\nu} = \begin{pmatrix} L_{\mu\nu} & 0 \\ 0 & L_{\mu\nu} \end{pmatrix}, \quad (225)$$

with $L_{\mu\nu}$ on the right-hand side. The chirality matrix is

$$\Gamma^5 = -\Gamma^1\Gamma^2\Gamma^3\Gamma^4 = \begin{pmatrix} 0 & -i1_2 \\ i1_2 & 0 \end{pmatrix}. \quad (226)$$

The massive Dirac Hamiltonian respects both of the $SO(4)$ symmetry

$$[-i\not{D} + M\Gamma^4, L_{\mu\nu}] = 0, \quad (227)$$

and the chiral symmetry

$$\{-i\not{D} + M\Gamma^4, \Gamma^5\} = 0 \quad (228)$$

or

$$\Gamma^5(-i\not{D} + M\Gamma^4)\Gamma^5 = i\not{D} - M\Gamma^4. \quad (229)$$
The chiral symmetry guarantees the reflection symmetry of the positive and negative energy levels with respect to the zero-energy. The Dirac-Landau operator can then be regarded as a Hamiltonian of chiral topological insulator. (229) suggests that the chiral transformation is equivalent to the sign flips of the Dirac-Landau operator and the mass parameter:

\[-i\mathcal{P} \rightarrow +i\mathcal{P},\]  
\[M \rightarrow -M.\]  

(230a)  
(230b)

In particular for the zeroth Landau level, the chiral transformation corresponds to

\[s \rightarrow -s.\]  

(231)

Thus, the left-right symmetry of the non-relativistic \(SO(4)\) Landau model can be translated as the chiral symmetry in the Dirac-Landau model.

Having solved the eigenvalue problem of the Weyl-Landau operator, we can readily analyze the massive Dirac-Landau problem

\[(-i\mathcal{D} + M\Gamma^4)\mathcal{E}_{\pm\lambda_n(s)} = \pm\lambda_n(s)\mathcal{E}_{\pm\lambda_n(s)},\]  

(232)

Since \((-i\mathcal{D} + M\Gamma^4)^2 = (-i\mathcal{D})^2 + M^2 = \lambda^2 + M^2\), the eigenvalues are derived as

\[\pm\lambda_n(s) = \pm\sqrt{\lambda_n(s)^2 + M^2},\]  

(233)

or

\[\lambda_n(I + \frac{3}{2}) = \pm\sqrt{(I + \frac{3}{2})^2 + n^2 + M^2}, \quad -\Lambda_n(I + \frac{1}{2}) = -\sqrt{(I + \frac{1}{2})^2 + n^2 + M^2},\]  
\[\lambda_n(s) = \pm\sqrt{n(I + 1 + n) + s^2 + M^2}, \quad -\Lambda_n(s) = -\sqrt{n(I + 1 + n) + s^2 + M^2},\]  
\[\lambda_0(s) = \pm\sqrt{s^2 + M^2}, \quad -\Lambda_0(s) = -\sqrt{s^2 + M^2},\]  

(234a)  
(234b)  
(234c)

where \(n = 1, 2, \cdots\) and \(s = \frac{I + 1}{2}, \frac{I - 1}{2}, -\frac{I - 1}{2}, \cdots\). Notice the gap between the positive and negative subbands of the zeroth Landau level is \(\sim M\), and the gap does not close even in the thermodynamic limit. Each of the eigenvalues has the following degeneracy:

\[2(n + I + 1)n,\]  
\[(2n + I + 2s + 1)(2n + I - 2s + 1) \times (1 - \delta_{I,0}),\]  
\[(I + \frac{1}{2} + s)((I + \frac{1}{2}) - s) \times (2 - \delta_{s,0}(1 - \delta_{M,0})(1 - \delta_{I,0}).\]  

(235a)  
(235b)  
(235c)

For even \(I\), zero-modes do not appear, while for odd \(I\), the states with energies \(M\) and \(-M\) \((s = 0)\) coincide to become degenerate zero-modes in the massless limit. For \((234c)\), the degenerate eigenstates are given by

\[\mathcal{E}^{(1)}_{\pm\lambda_n(s)} = \frac{\lambda_n(s) + \lambda_0(s)}{2\lambda_n(s)} \left(\frac{\Psi_{\lambda_n(s)} + \lambda_0(s)}{\lambda_n(s) + \lambda_0(s)} \Psi_{\lambda_n(s)}\right), \quad \mathcal{E}^{(2)}_{\pm\lambda_n(s)} = \frac{\lambda_n(s) + \lambda_0(s)}{2\lambda_n(s)} \left(\frac{\Psi_{\lambda_n(s)} + \lambda_0(s)}{\lambda_n(s) + \lambda_0(s)} \Psi_{\lambda_n(s)}\right),\]  
\[\mathcal{E}^{(1)}_{-\lambda_n(s)} = \frac{\lambda_n(s) + \lambda_0(s)}{2\lambda_n(s)} \left(-\frac{\Psi_{\lambda_n(s)} + \lambda_0(s)}{\lambda_n(s) + \lambda_0(s)} \Psi_{\lambda_n(s)}\right), \quad \mathcal{E}^{(2)}_{-\lambda_n(s)} = \frac{\lambda_n(s) + \lambda_0(s)}{2\lambda_n(s)} \left(-\frac{\Psi_{\lambda_n(s)} + \lambda_0(s)}{\lambda_n(s) + \lambda_0(s)} \Psi_{\lambda_n(s)}\right),\]  

(236a)  
(236b)
Figure 5: Energy spectra of the massive Dirac-Landau model.

where $\Psi_{\pm \lambda_n(s)}$ represents the Weyl-Landau operator eigenstates with eigenvalues $\pm \lambda_n(s)$. $\Xi_{\pm \lambda_n(s)}$ transforms as a representation of the $SO(4)$ group generated by $L_{\mu\nu}$. The chiral symmetry relates the eigenstates as

$$
\Xi^{(1)} + \Lambda_n(s) = +i \Gamma^5 \Xi^{(2)} - \Lambda_n(s),
\Xi^{(2)} + \Lambda_n(s) = -i \Gamma^5 \Xi^{(1)} - \Lambda_n(s).
$$

Similarly for (234a) and (234c), the degenerate eigenstates are respectively given by

$$
\Xi^{(1)} \Lambda_0(s) (s \geq 0), \Xi^{(2)} \Lambda_0(s) (s \leq 0), \Xi^{(1)} (-\Lambda_0(s)) (s \geq 0), \Xi^{(2)} (-\Lambda_0(s)) (s \leq 0).
$$

In the zeroth Landau level of $s = 0$, $\Xi^{(1)}$ and $\Xi^{(2)}$ are not independent:

$$
\Xi^{(1)} \Lambda_0(0) = M = \Xi^{(2)} \Lambda_0(0) = M = \frac{1}{\sqrt{2}} \left( \Psi_{\lambda=0} \Psi_{\lambda=0} \right), \quad \Xi^{(2)} (-\Lambda_0(0)) = -M = -\Xi^{(1)} (-\Lambda_0(0)) = -M = \frac{1}{\sqrt{2}} \left( -\Psi_{\lambda=0} \Psi_{\lambda=0} \right).
$$

### 5.2.1 Supersymmetric Landau model

The square of the Dirac-Landau operator yields a supersymmetric quantum mechanical Hamiltonian:

$$
H_{\text{SUSY}} = (-i \mathcal{D})^2 = \begin{pmatrix} (-i \mathcal{D})^2 & 0 \\ 0 & (+i \mathcal{D})^2 \end{pmatrix} = \{Q, Q^\dagger\},
$$

where

$$
Q = \begin{pmatrix} 0 & -i \mathcal{D} \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ -i \mathcal{D} & 0 \end{pmatrix}.
$$
Q and \( Q^\dagger \) are supercharges that satisfy

\[
Q^2 = 0, \quad Q^\dagger 2 = 0, \quad [H_{\text{SUSY}}, Q] = [H_{\text{SUSY}}, Q^\dagger] = 0.
\] (242)

The SUSY Hamiltonian \( H_{\text{SUSY}} \) consists of two identical spinor Landau Hamiltonians, and its eigenvalues are \( \lambda_n(s)^2 = n(n + I + 1) + s^2 \) with the degenerate eigenstates

\[
\Xi^{(1)}_{\lambda_n(s)} = \begin{pmatrix} \Psi_{\lambda_n(s)} \\ 0 \end{pmatrix}, \quad \Xi^{(2)}_{\lambda_n(s)} = \begin{pmatrix} 0 \\ \Psi_{-\lambda_n(s)} \end{pmatrix}, \quad \Xi^{(1)}_{-\lambda_n(s)} = \begin{pmatrix} 0 \\ \Psi_{\lambda_n(s)} \end{pmatrix}, \quad \Xi^{(2)}_{-\lambda_n(s)} = \begin{pmatrix} \Psi_{-\lambda_n(s)} \\ 0 \end{pmatrix}.
\] (243)

The Witten parity

\[
W = \frac{[Q, Q^\dagger]}{\{Q, Q^\dagger\}}
\] (244)
is given by

\[
W = \frac{1}{\lambda^2} \begin{pmatrix} (-i\mathbb{P})^2 & 0 \\ 0 & -(i\mathbb{P})^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (245)

\( \Psi_{\pm\lambda} \) belongs to the Witten parity + ("fermionic") sector, while \( \Psi_{\pm\lambda} \) the Witten parity − ("bosonic") sector. They are the superpartner related by the SUSY transformation:

\[
\begin{pmatrix} \Psi_{\pm\lambda} \\ 0 \end{pmatrix} = \pm \frac{1}{\lambda} Q \begin{pmatrix} 0 \\ \Psi_{\pm\lambda} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \Psi_{\pm\lambda} \end{pmatrix} = \pm \frac{1}{\lambda} Q^\dagger \begin{pmatrix} \Psi_{\pm\lambda} \\ 0 \end{pmatrix}, \quad (\lambda \neq 0)
\] (246)

For odd \( I \), the ground-state of \( H_{\text{SUSY}} \) is given by SUSY invariant zero-energy state (good SUSY), while for even \( I \), the ground-state is not zero-energy state and does not respect the SUSY (broken SUSY). We can also consider the square of the massive Dirac-Landau Hamiltonian

\[
H_M = (-i\mathbb{D} + \Gamma^4)^2 = (-i\mathbb{D})^2 + M^2 = H_{\text{SUSY}} + M^2.
\] (247)

The mass term just shifts the zero-energy of the spectrum of \( H_{\text{SUSY}} \).

6 Matrix Geometry

In the above, we introduced the various \( SO(4) \) Landau models and solved their eigenvalue problem. With the developed technologies and results, we are now ready to evaluate the matrix geometry of the \( SO(4) \) Landau levels. We concretely derive the matrix elements of \( S^3 \) coordinates in each of the Landau levels based on the level projection method to appreciate the emergent non-commutative geometry. We first derive the fuzzy three-sphere geometry of the \( SO(4) \) non-relativistic Landau model in Sec.6.1, and next we explore fuzzy geometry of the spinor Landau model and relativistic Landau models in Sec.6.2 and Sec.6.3. We point out that the massive Dirac-Landau model accommodates the two fuzzy three-spheres whose interaction is induced by the mass term (Sec.6.3).

We switch the notation from \( \Phi_{m_{\text{L}},m_{\text{R}},(n,s,I/2)} \) to \( \Phi_{m_{\text{L}},m_{\text{R}},(l_{\text{L}},l_{\text{R}},I/2)} \) in this section.

6.1 Landau level projection

We apply the level projection method to the \( SO(4) \) non-relativistic Landau model. The basis states of the Landau level subband \( E_n(s) \) are the \( SO(4) \) monopole harmonics carrying the indices

\[
(n, I/2, s), \quad (n, I/2, -s),
\] (248)
or in the SO(4) \( \simeq SU(2) \otimes SU(2) \) notation,
\[
(l_L, l_R) = \left( \frac{1}{2}(n + \frac{I}{2} + s), \frac{1}{2}(n + \frac{I}{2} - s) \right), \quad (l'_L, l'_R) = (l_R, l_L).
\]

The matrix elements of \( x_\mu \) in the Landau level subband are then given by
\[
X_\mu(n) = \begin{pmatrix}
\langle l_L, l'_R | x_\mu | l_L, l'_R \rangle & \langle l_L, l_R | x_\mu | l'_L, l'_R \rangle \\
\langle l'_L, l'_R | x_\mu | l'_L, l'_R \rangle & \langle l'_L, l_R | x_\mu | l'_L, l'_R \rangle
\end{pmatrix}.
\]

Each of the blocks in (250) denotes a square matrix of \( d(n, s) \times d(n, s) \) with
\[
d(n, s) = d(n, -s) = (2l + 1)(2l + 1) = (n + \frac{I}{2} + s + 1)(n + \frac{I}{2} - s + 1).
\]

To evaluate (250), from (149) we represent \( x_\mu \) as
\[
x_1 = -\frac{\pi}{2}(\Phi_{1/2,1/2}(x) - \Phi_{1/2,-1/2}(x)),
\]
\[
x_2 = -\frac{\pi}{2}(\Phi_{1/2,1/2}(x) + \Phi_{1/2,-1/2}(x)),
\]
\[
x_3 = \frac{\pi}{2}(\Phi_{1/2,1/2}(x) + \Phi_{1/2,-1/2}(x)),
\]
\[
x_4 = \frac{\pi}{2}(\Phi_{1/2,1/2}(x) - \Phi_{1/2,-1/2}(x)),
\]
and derive the matrix elements, such as \( \langle l_L, l'_R | \Phi_{1/2,1/2} | l'_L, l'_R \rangle \). The SO(4) vector \( x_\mu \) carries the SU(2) \( L \) \( \otimes \) SU(2) \( R \) index, \( (l_L, l_R) = (1/2, 1/2) \), and so the matrix elements of (250) take finite values only for the cases that each SU(2) index of SU(2) \( L \) \( \otimes \) SU(2) \( R \) between the ket and bra differs by \( \pm 1/2 \), i.e. \( |l_L - l_R| = 1/2 \) or \( |s| = 1/2 \) (recall (249)). Explicit evaluation of (250) is as follows. From (120), we have
\[
\langle l_L, l'_R | \Phi_{1/2,1/2} | l'_L, l_R \rangle = (1 + 1)\langle l_L | l'_L \rangle \langle l'_R | l_R \rangle \rho_{l_L, l'_R, l'_L, l_R}.
\]

where we used \( C_{1/2,1/2}^{l_L, l'_R, l'_L} = C_{1/2,1/2}^{l'_L, l'_R, l_L} = 0 \). Therefore, the diagonal blocks of (250) always vanish:
\[
\langle l_L, l_R | x_\mu | l_L, l_R \rangle = \langle l_R, l_L | x_\mu | l'_L, l'_R \rangle = 0 \quad (2l + 1)(2l + 1),
\]
and then \( X_\mu(n) \) take the form of
\[
X_\mu(n) = \begin{pmatrix}
\langle l_L, l'_R | x_\mu | l_L, l'_R \rangle & \langle l_L, l_R | x_\mu | l'_L, l'_R \rangle \\
\langle l'_L, l'_R | x_\mu | l'_L, l'_R \rangle & \langle l'_L, l_R | x_\mu | l'_L, l'_R \rangle
\end{pmatrix} = \begin{pmatrix}
\rho_{d(n, s)} & Y_\mu^{(I)}(n) \\
Y_\mu^{(I)}(n)^\dagger & \rho_{d(n, s)}
\end{pmatrix}.
\]

The matrix elements of \( Y_\mu^{(I)}(n) \) can be derived as follows. Also from (120), we have
\[
\langle l_L, l'_R | \Phi_{1/2,1/2} | l'_L, l_R \rangle = \int d\Omega_3 \Phi_{m_L, m_R}^{[l_L, l'_R]} \Phi_{m_L, m_R}^{[l'_L, l_R]} \Pi_{m_L, m_R}^{[l_L, l'_R]}
\]
\[
\rangle = \frac{(-1)^n + 1}{\sqrt{(2l + 1)(2l + 1)}} \frac{1}{\pi} \delta_{m_L, m_R} \delta_{m_L, m_R} \rho_{d(n, s)} \rho_{d(n, s)}
\]
\[
= \frac{1}{\pi} \delta_{m_L, m_R} \delta_{m_L, m_R} \times \sqrt{(l_L - \tau n_L)(l_L + 1 + \sigma n_R)} \langle l_L | l'_R | l'_L | l_R \rangle,
\]
\[
\rho_{d(n, s)} = \frac{1}{\pi} \delta_{m_L, m_R} \delta_{m_L, m_R} \times \sqrt{(l_L - \tau n_L)(l_L + 1 + \sigma n_R)} \langle l_L | l'_R | l'_L | l_R \rangle,
\]
where we used (for \( l_L = l_R + \frac{1}{2} \))

\[
C_{1/2, \sigma, 2}^{l_L, m_L, l_R, n_R} = \sqrt{\frac{l_R + 1 + \sigma n_R}{2l_R + 1}} \delta_{m_L, n_R + \frac{\sigma}{2}}, \quad C_{1/2, \sigma, 2}^{l_L, m_L, l_R, n_R} = \tau \sqrt{\frac{l_L - \tau n_L}{2l_L + 1}} \delta_{m_R, n_R + \frac{\tau}{2}}.
\]

(257)

For even \( I \),

\[
\begin{bmatrix}
l_L \\
l_R \\
l_R \\
l_L \\
\end{bmatrix} = 0, \quad (I = 0, 2, 4, \ldots)
\]

(258)

and then \( Y_\mu = 0 \). Meanwhile for odd \( I \),

\[
\begin{bmatrix}
l_L \\
l_R \\
l_R \\
l_L \\
\end{bmatrix} = (-1)^{n+1} \frac{2(I + 1)}{(2n + I + 1)(2n + I + 3)} \delta_{|s|, \frac{1}{2}},
\]

(259)

and, as mentioned above, non-zero matrix elements appear only for

\[
|s| = 1/2,
\]

(260)

and from (258) we obtain

\[
Y^{(l)}_\mu (n) = \frac{I + 1}{(2n + I)(2n + I + 3)} Y^{(2(l + 1) = 2n + 1)}_\mu,
\]

(261)

where

\[
\hat{Y}^{(I)}_1 = \frac{i}{2}(l_L - n_L)(l_R + 1 + n_R) + \frac{i}{2}(l_L + n_L)(l_R + 1 - n_R) \right|_{(l_L, l_R) = (\frac{1}{2} + \frac{1}{2} + \frac{1}{2})},
\]

\[
\hat{Y}^{(I)}_2 = \frac{i}{2}(l_L - n_L)(l_R + 1 + n_R) - \frac{i}{2}(l_L + n_L)(l_R + 1 - n_R) \right|_{(l_L, l_R) = (\frac{1}{2} + \frac{1}{2} - \frac{1}{2})},
\]

\[
\hat{Y}^{(I)}_3 = \frac{i}{2}(l_L + n_L)(l_R + 1 + n_R) + \frac{i}{2}(l_L + n_L)(l_R + 1 - n_R) \right|_{(l_L, l_R) = (\frac{1}{2} + \frac{1}{2} - \frac{1}{2})},
\]

\[
\hat{Y}^{(I)}_4 = \frac{i}{2}(l_L + n_L)(l_R + 1 + n_R) - \frac{i}{2}(l_L + n_L)(l_R + 1 - n_R) \right|_{(l_L, l_R) = (\frac{1}{2} + \frac{1}{2} - \frac{1}{2})}.
\]

(262)

\( \hat{Y}^{(l)}_\mu \) are the off-diagonal blocks of the \( SO(4) \) gamma matrices in the symmetric representation\(^{24}\)

\[
\Gamma^{(l)}_\mu = \begin{pmatrix} 0 & \hat{Y}^{(l)}_\mu \\ \hat{Y}^{(l)}_\mu & 0 \end{pmatrix}, \quad (I = 1, 3, 5, \ldots)
\]

(264)

which satisfy \(^{33, 63}\)

\[
\sum_{\mu=1}^{4} \Gamma^{(l)}_\mu^2 = \frac{1}{2} (I + 1)(I + 3) 1_{(I+3)(I+1)} \cdot (I = 1, 3, 5, \ldots)
\]

(265)

---

\(^{24}\)For instance,

\[
\Gamma^{(l=1)}_\mu = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 12 \\ 12 & 0 \end{pmatrix}.
\]

(263)

37
is now obtained as
\[ X_\mu(n) = \frac{I + 1}{(2n + I + 1)(2n + I + 3)} \Gamma^{(2n+1)}_\mu. \] (266)

To summarize, the non-trivial matrix geometry appears only in the subband \(|s| = 1/2\) for odd \(I\):
\[ X_\mu(n) = \begin{pmatrix} 0_{d(n,1/2)} & Y_\mu^{(I)}(n) \\ Y_\mu^{(I)}(n) & 0_{d(n,1/2)} \end{pmatrix} = \frac{I + 1}{(2n + I + 1)(2n + I + 3)} \Gamma^{(2n+1)}_\mu. \] (267)

In particular, for the lowest Landau level \((n = 0)\),
\[ X_\mu(n = 0) = \frac{1}{I + 3} \Gamma^{(I)}_\mu. \] (I = 1, 3, 5, ⋯.) (268)

\[ X_\mu \] (267) satisfy the relation
\[ \sum_{\mu=0}^{4} X_\mu(n)X_\mu(n) = R_n^{(I)} 1_{2(2n+I+1)(2n+I+3)} \] (269)

with
\[ R_n^{(I)} = \frac{I + 1}{\sqrt{2(2n + I + 1)(2n + I + 3)}}. \] (270)

The relation (269) is invariant under the \(SO(4)\) rotations and is a non-commutative counterpart of the definition of three-sphere. We thus find that \(X_\mu\) denote the matrix coordinates of fuzzy three-sphere. The radius decreases as the Landau level increases. A similar behavior is observed in the fuzzy two-sphere of the \(SO(3)\) Landau model [11]. In each Landau level, only the \(|s| = 1/2\) subband realizes the fuzzy three-sphere geometry and each Landau level accommodates just one fuzzy three-sphere.

The fuzzy three-sphere geometry of the \(SO(4)\) Landau model is naturally understood as a subspace embedded in the one dimension higher fuzzy four-sphere of the \(SO(5)\) Landau model [36, 33]. If we regard the chirality parameter \(s\) as an extra fifth coordinate \(x_5\), \(X_\mu\) can be interpreted as the coordinates of two \(S^3\)-latitudes with \(x_5 = \pm s\) on the virtual fuzzy \(S^4\) (Fig.6), as suggested by the equation
\[ \sum_{\frac{I}{2}}^{\frac{I}{2}} d_{n=0}(s) = \frac{1}{6} (I + 1)(I + 2)(I + 3) = d_{n=0}^{SO(5)}, \] (271)

where \(d_{n=0}(s)\) [24] and \(d_{n=0}^{SO(5)}\) respectively denote the lowest Landau level degeneracy of the \(SO(4)\) and \(SO(5)\) Landau models.

### 6.2 Spinor Landau model

As discussed in Sec 4, the spinor Landau model consists of two independent Landau models:

\[
\text{Spinor Landau model with monopole charge } \frac{I}{2} = \text{Landau models with } J^+ = \frac{I + 1}{2} \oplus J^- = \frac{I - 1}{2}. \] (272)
$s = \pm 1/2$ can be realized for even $I$. From the upper figure of Fig. 6 we find that the non-trivial cases occur in

\begin{align}
I = 0, \ n = 1, 2, \cdots \quad &\rightarrow \ S_F^3(n, J^+ = \frac{I-1}{2}, |s| = \frac{1}{2}), \quad (273a) \\
I = 2, 4, \cdots, n = 1, 2, \cdots \quad &\rightarrow \ S_F^3(n, J^- = \frac{I+1}{2}, |s| = \frac{1}{2}) \oplus S_F^3(n-1, J^+ = \frac{I-1}{2}, |s| = \frac{1}{2}), \quad (273b) \\
I = 2, 4, \cdots, n = 0 \quad &\rightarrow \ S_F^3(n = 0, J^- = \frac{I-1}{2}, |s| = \frac{1}{2}). \quad (273c)
\end{align}

The radii of the corresponding fuzzy three-spheres are respectively given by

\begin{align}
R_n^{(1)} &= \frac{1}{\sqrt{2(n+1)(n+2)}}, \quad (274a) \\
R_n^{(I-1)} &= \frac{I}{\sqrt{2(2n+I)(2n+I+2)}}, \quad R_n^{(I+1)} = \frac{I+2}{\sqrt{2(2n+I)(2n+I+2)}}, \quad (274b) \\
R_{n=0}^{(I-1)} &= \frac{I}{2(I+2)}. \quad (274c)
\end{align}

and the matrix sizes of $X_\mu$ are

\begin{align}
2(n+1)n, \quad (275a) \\
\frac{1}{2}(2n+I+2)(2n+I), \quad \frac{1}{2}(2n+I+2)(2n+I), \quad (275b) \\
\frac{1}{2}I(I+2). \quad (275c)
\end{align}

### 6.3 Relativistic Landau models

The Weyl-Landau model is a “square root” of the spinor Landau model and the non-trivial matrix geometry can occur for even $I$. In the spectrum of the Weyl-Landau model, the degeneracies of $s = 1/2$ and $-1/2$ corresponding to (273a) and (273c) are completely lifted (see the left-figure of Fig. 8), and only the half of the degeneracies in (273b) survives in the Weyl-Landau model to generate the fuzzy three-sphere. From the left figure of Fig. 8 we can see

\begin{align}
I = 2, 4, \cdots, \quad \pm n = \pm 1, \pm 2, \cdots \quad &\rightarrow \ S_F^3. \quad (276)
\end{align}

\[\text{Even if there was not external magnetic field, the SU(2) connection of the holonomy would act as a fictitious magnetic field to generate fuzzy three-sphere geometry.}\]
In the Weyl-Landau level, \(-i\mathcal{D} = +\lambda_n(s)\) for \(n = 1, 2, \cdots\)\(^{26}\), the degenerate eigenstates with \(s = \pm 1/2\) are given by (277a):

\[
\Psi_{s=+1/2} = U \left( \begin{array}{c} \alpha_+ \Phi^{(n-1,s=1/2,J^+)}_i \\ \beta_+ \Phi^{(n,s=1/2,J^-)}_i \end{array} \right), \quad (\alpha_+^2 + \beta_+^2 = 1)
\]

(277a)

\[
\Psi_{s=-1/2} = U \left( \begin{array}{c} \alpha_- \Phi^{(n-1,s=-1/2,J^+)}_i \\ \beta_- \Phi^{(n,s=-1/2,J^-)}_i \end{array} \right), \quad (\alpha_-^2 + \beta_-^2 = 1)
\]

(277b)

and the matrix elements

\[
X_\mu(n, I/2) = \frac{1}{2n+I} \left( \alpha_+ + \frac{I}{2n+I} - \frac{I}{2n+I+2} + \beta_+ \right) \Gamma^{(2n+I-1)}_{\mu}
\]

(278)

are evaluated as

\[
X_\mu(n, I/2) = \alpha_+ \alpha_- X_\mu(n-1, J^+) + \beta_+ \beta_- X_\mu(n, J^-),
\]

(279)

where \(X_\mu(n, J)\) denote (281) with the replacement of \(I\) with \(2J\):

\[
X_\mu(n, J) = \frac{2J+1}{(2n+2J)(2n+2J+3)} \Gamma^{(2n+2J)}_{\mu}
\]

(280)

Consequently, \(X_\mu\) (279) are given by

\[
X_\mu(n, I/2) = \frac{1}{2n+I} \left( \alpha_+ + \frac{I}{2n+I} - \frac{I}{2n+I+2} + \beta_+ \right) \Gamma^{(2n+I-1)}_{\mu}
\]

(281)

whose radius becomes

\[
R_n^{(I)} = \sqrt{\frac{2n+I+2}{2(2n+I)} \left( \alpha_+ + \frac{I}{2n+I} - \frac{I}{2n+I+2} + \beta_+ \right)}.
\]

(282)

Next, we investigate the case of the Dirac-Landau model. In the massless case, the Dirac-Landau operator becomes a simple direct sum of the two Weyl-Landau operators, and there exist two identical fuzzy three-spheres, each of which originates from each Weyl-Landau operator. Since \(\Gamma^4\) is an off-diagonal block matrix (223), the mass term brings “interaction” between such two Weyl-Landau sectors. For \(-i\mathcal{D} + M \Gamma^4 = \Lambda_n(s = 1/2)\), we have degenerate eigenstates, \(\Xi^{(1)}_{\Lambda_n(s = 1/2)}\), \(\Xi^{(1)}_{\Lambda_n(s = -1/2)}\) and \(\Xi^{(2)}_{\Lambda_n(s = 1/2)}\), \(\Xi^{(2)}_{\Lambda_n(s = -1/2)}\) (274). Taking the matrix elements of \(x_\mu\) between them, we obtain

\[
X_\mu(M) = \begin{pmatrix}
\frac{1}{2n+I/2} & 0 \\
0 & \frac{1}{2n-I/2}
\end{pmatrix} + \frac{M}{\Lambda_n} \begin{pmatrix}
0 & 0 & Z_\mu^{(+)} \\
0 & 0 & Z_\mu^{(-)} \\
Z_\mu^{(+)*} & Z_\mu^{(-)*} & 0
\end{pmatrix},
\]

(283)

where \(\Lambda_n \equiv \Lambda_n(s = 1/2)\), \(\lambda_n \equiv \lambda_n(s = 1/2)\), and

\[
W^{(+)}_\mu = \begin{pmatrix}
0 & Z_\mu^{(+)} \\
Z_\mu^{(+)*} & 0
\end{pmatrix}, \quad W^{(-)}_\mu = \begin{pmatrix}
0 & Z_\mu^{(-)} \\
Z_\mu^{(-)*} & 0
\end{pmatrix}
\]

\[
\equiv \frac{I \alpha_+ \beta_+ - (I + 2) \alpha_- \beta_-}{(2n+I-1)(2n+I+2)} \Gamma^{(2n+I-1)}_{\mu},
\]

(284)

\[
\equiv \frac{I \alpha_- \beta_+ - (I + 2) \alpha_+ \beta_-}{(2n+I-1)(2n+I+2)} \Gamma^{(2n+I-1)}_{\mu}.
\]

\(^{26}\)In the following, we discuss the positive relativistic Landau levels. The extension to the negative Landau level is obvious.
The square of \( X_\mu \) is derived as

\[
\begin{align*}
X_\mu(M)^2 &= \left( \begin{array}{cc}
X_\mu^2 & 0 \\
0 & X_\mu^2
\end{array} \right) + \left( \frac{M}{\Lambda} \right)^2 \left( \begin{array}{cc}
W_\mu^{(+)} & W_\mu^{(+)} \\
W_\mu^{(-)} & W_\mu^{(-)}
\end{array} \right) \\
&+ \frac{M}{\Lambda} \left( \begin{array}{cc}
0 & W_\mu^{(+)} \\
W_\mu^{(-)} & X_\mu + X_\mu W_\mu^{(-)}
\end{array} \right),
\end{align*}
\]

where we interchanged the second and fourth columns and rows. In the massless limit, the off-diagonal blocks of \( X_\mu(M) \) vanish to yield \( X_\mu(M) \to \left( \begin{array}{cc}
X_\mu & 0 \\
0 & X_\mu
\end{array} \right) \) that actually represents the two identical non-interacting fuzzy three-spheres. When the mass term is turned on, the off-diagonal block matrices appear to bring interactions between the two fuzzy three-spheres. \( M/\Lambda \) can be interpreted as the coupling of the interaction. Meanwhile, for each of the cases of \((285)\), the degenerate subbands with \( s = \pm 1/2 \) appear for even \( I \), and the fuzzy three-spheres are respectively realized as

\[
\begin{align*}
X_\mu(M) &= \frac{1}{\Lambda n} X_\mu(n, I/2)_{| n, I/2 = 0, s = 1/2} = \frac{M}{\sqrt{(n + 1/2)^2 + M^2}} X_\mu(n, I = 1/2), \quad (n = 1, 2, \cdots) \quad (286a) \\
X_\mu(M) &= \frac{M}{\Lambda n} X_\mu(n, I/2)_{| n = 0, I/2, s = 1/2} = \frac{M}{\sqrt{M^2 + 1/4}} X_\mu(n = 0, I = 1/2). \quad (286b)
\end{align*}
\]

In the massless limit, the fuzzy geometries collapse in either case.

In the supersymmetric Landau model, the Hamiltonian \( (285) \) consists of two independent spinor Landau Hamiltonians and accommodates twice the number of fuzzy three-spheres of the spinor Landau model. The mass term in the SUSY Landau model does not bring any particular effect to fuzzy geometry, as the mass term just shifts the energy levels uniformly \( (287) \).

### 7 Summary and Discussions

We thoroughly investigated the \( SO(4) \) Landau models based on the Dirac and the Schwinger gauges. The gauge fixing enabled us to elaborate the previous works about the \( SO(4) \) Landau models and bring the new observations, such as the properties of the \( SO(4) \) monopole harmonics and the \( SO(4) \) symmetry of the relativistic operators. In the present analysis, we took into account the spin connection of three-sphere to construct the relativistic Landau operators. With the synthesized connection of the spin connection and gauge field, we solved the eigenvalue problem of the spinor Landau operator and of the Weyl-Landau operator also. The obtained results are confirmed to reproduce the known results of the \( SO(4) \) spherical harmonics and the free Dirac operator in the free background limit. The eigenvalue problems of the massive Dirac-Landau and the supersymmetric Landau model are analyzed too. We applied the level projection method to the \( SO(4) \) Landau models and derived the odd dimensional matrix geometry for the first time. It was shown that, for each of the (non-relativistic) Landau levels, the fuzzy three-sphere geometry appears only in the lowest energy \(|s| = 1/2 \) subband, and the size of fuzzy three-sphere depends on the Landau level. We also clarified realizations of the fuzzy three-sphere geometry in the relativistic Landau models. In particular, we designated that the mass term of the Dirac-Landau model induces interaction between two fuzzy spheres realized in the relativistic Landau level.

As the \( SO(3) \) Landau model has a wide range of applications, we expect that the \( SO(4) \) Landau model may also find its playing fields in many branches of physics. Even if limited to condensed matter physics, one may conceive its possible applications to Weyl/Dirac semi-metal, three-dimensional quantum Hall effect and chiral topological insulator. We have clarified the emergent fuzzy three-sphere geometry in the Landau...
physics. It is interesting to see that such an exotic geometry realizes “inside” the physical models, and the dynamics of the fuzzy spaces can be controlled by a physical parameter which in principle can be controlled by experiment.

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A Geometric quantities of three-sphere: component method

A.1 Metric and curvature

The metric on $S^3$ is given by (3):

$$g_{\alpha\beta} = \begin{pmatrix}
  g_{\chi\chi} & g_{\chi\theta} & g_{\chi\phi} \\
  g_{\theta\chi} & g_{\theta\theta} & g_{\theta\phi} \\
  g_{\phi\chi} & g_{\phi\theta} & g_{\phi\phi}
\end{pmatrix} = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & \sin^2 \chi & 0 \\
  0 & 0 & \sin^2 \chi \sin^2 \theta
\end{pmatrix}.$$  \hspace{1cm} (287)

From (287), the non-zero components of Christoffel symbol, $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = \frac{1}{2} g^{\alpha\delta} (\partial_\delta g_{\gamma\beta} + \partial_{\gamma} g_{\beta\delta} - \partial_{\delta} g_{\beta\gamma})$, are derived as

$$\Gamma^\chi_{\theta\theta} = -\sin \chi \cos \chi, \quad \Gamma^\chi_{\phi\phi} = -\sin \chi \cos \chi \sin^2 \theta,$$
$$\Gamma^\theta_{\chi\theta} = \Gamma^\theta_{\theta\chi} = \cot \chi, \quad \Gamma^\theta_{\phi\phi} = -\sin \theta \cos \theta,$$
$$\Gamma^\phi_{\chi\phi} = \Gamma^\phi_{\phi\chi} = \cot \chi, \quad \Gamma^\phi_{\theta\theta} = \Gamma^\phi_{\phi\theta} = \cot \theta,$$ \hspace{1cm} (288)

and the non-zero components of curvature, $R^\alpha_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\gamma\epsilon} \Gamma^\epsilon_{\beta\delta} - \Gamma^\alpha_{\delta\epsilon} \Gamma^\epsilon_{\beta\gamma}$, are derived as

$$R^\chi_{\theta\chi\theta} = \sin^2 \chi, \quad R^\chi_{\phi\chi\phi} = \sin^2 \chi \sin^2 \theta,$$
$$R^\theta_{\chi\chi\theta} = -1, \quad R^\theta_{\phi\phi\theta} = \sin^2 \chi \sin^2 \theta,$$
$$R^\phi_{\chi\chi\phi} = -1, \quad R^\phi_{\theta\theta\phi} = -\sin^2 \chi.$$ \hspace{1cm} (289)

The non-zero components of Ricci tensor, $R_{\mu\nu} \equiv R^\rho_{\mu\rho\nu}$, are given by

$$R_{\chi\chi} = 2, \quad R_{\theta\theta} = 2 \sin^2 \chi, \quad R_{\phi\phi} = 2 \sin^2 \chi \sin^2 \theta,$$ \hspace{1cm} (290)

and the Ricci scalar, $R = R^\mu_{\mu}$, is

$$R = 2 \times 3 = 6.$$ \hspace{1cm} (291)

A.2 Spin connection

From $e^\alpha = e^\alpha_a dx^a$, we can read off the components of the dreibein (9):

$$e^\alpha_a = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & \sin \chi & 0 \\
  0 & 0 & \sin \chi \sin \theta
\end{pmatrix}.$$  \hspace{1cm} (292)

The inverse is

$$e_a^\alpha = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & \sin^{-1} \chi & 0 \\
  0 & 0 & \sin^{-1} \chi \sin^{-1} \theta
\end{pmatrix}.$$ \hspace{1cm} (293)
The components of the spin-connection are obtained by the formula

$$\omega_{ab\alpha} = - e_b a \nabla_a e^\beta = - e_b a (\partial_a e^\beta + \Gamma^\beta_{a\gamma} e^\gamma),$$

(294)
as

$$\omega_{12\alpha} (= - \omega_{21\alpha}) = \{\omega_{12\chi}, \omega_{12\theta}, \omega_{12\phi}\} = \{0, - \cos \chi, 0\},$$

$$\omega_{31\alpha} (= - \omega_{13\alpha}) = \{\omega_{31\chi}, \omega_{31\theta}, \omega_{31\phi}\} = \{0, 0, \sin \theta \cos \chi\},$$

$$\omega_{23\alpha} (= - \omega_{32\alpha}) = \{\omega_{23\chi}, \omega_{23\theta}, \omega_{23\phi}\} = \{0, 0, - \cos \theta\},$$

(295)
which are consistent with [13]. The matrix form of the spin connection is given by

$$\omega_{\alpha} = \sum_{a<b=1,2,3} \omega_{ab\alpha} \sigma^{ab},$$

(296)
with

$$\omega_\chi = 0, \quad \omega_\theta = - \cos \chi \sigma^{12}, \quad \omega_\phi = \sin \theta \cos \chi \sigma^{31} - \cos \theta \sigma^{23},$$

(297)
or

$$\omega = \omega_\alpha dx^\alpha = - \frac{1}{2} (\gamma_1 \cos \theta - \gamma_2 \cos \chi \sin \theta) d\phi - \frac{1}{2} \gamma_3 \cos \chi d\theta,$$

(298)
where we used

$$\sigma^{ab} = - \frac{i}{4} [\gamma^a, \gamma^b] = \frac{1}{2} e^{abc} \gamma^c.$$

(299)
The curvature is obtained as

$$f = d\omega + i\omega^2$$

$$= \frac{1}{2} \gamma_1 \sin^2 \chi \sin \theta \ d\theta \wedge d\phi + \frac{1}{2} \gamma_2 \sin \chi \sin \theta \ d\phi \wedge d\chi + \frac{1}{2} \gamma_3 \sin \chi \ d\chi \wedge d\theta$$

$$= \frac{1}{2} \epsilon_{abc} \gamma^b \wedge e^c \frac{1}{2} \gamma_a.$$  

(300)

**B** The *SO*(4) spherical harmonics and free Dirac operator

**B.1** The *SO*(4) spherical harmonics

In the polar coordinates, the *SO*(4) free angular momentum operators

$$l_{\mu\nu} = -ix_\mu \partial_\nu + ix_\nu \partial_\mu$$

(301)
are given by

$$l_{12} = -i \partial_\phi,$$

$$l_{13} = i (\cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi),$$

$$l_{14} = i (\sin \theta \cos \phi \partial_\chi + \cot \chi \cos \theta \cos \phi \partial_\theta - \cot \chi \frac{1}{\sin \theta} \sin \phi \partial_\phi),$$

$$l_{23} = i (\sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi),$$

$$l_{24} = i (\sin \theta \sin \phi \partial_\chi + \cot \chi \cos \theta \sin \phi \partial_\theta + \cot \chi \frac{1}{\sin \theta} \cos \phi \partial_\phi),$$

$$l_{34} = i (\cos \theta \partial_\chi - \cot \chi \sin \theta \partial_\phi),$$

(302)
and the $SO(4)$ Casimir is derived as

$$
\sum_{\mu<\nu}^{4} l_{\mu\nu}^2 = -\frac{1}{\sin^2 \chi} \partial_{\chi}(\sin^2 \chi \partial_{\chi}) - \frac{1}{\sin^2 \chi \sin \theta} \left( \partial_{\theta}(\sin \theta \partial_{\theta}) + \frac{1}{\sin \theta} \partial_{\phi}^2 \right) = -\Delta_{S^3},
$$

(303)

where

$$
\Delta_{S^3} = \frac{1}{\sqrt{g}} \partial_{\mu}(\sqrt{g} g^{\mu\nu} \partial_{\nu}) = -\frac{1}{\sin^2 \chi} \partial_{\chi}(\sin^2 \chi \partial_{\chi}) - \frac{1}{\sin^2 \chi \sin^2 \theta} \partial_{\phi}^2.
$$

(304)

The $SO(4)$ spherical harmonics that satisfy

$$
\sum_{\mu<\nu}^{4} l_{\mu\nu}^2 \Phi^{(n,0,0)}_{m_L,m_R}(x) = n(n+2)\Phi^{(n,0,0)}_{m_L,m_R}(x), \quad (n = 0, 1, 2, \cdots)
$$

(306)

are usually denoted as \[42, 43, 44\]

$$
Y_{nlm}(x) = \frac{2}{\sqrt{2}} \left( \frac{n+1}{n+2} \right)^{\frac{1}{2}} \Gamma(n+2) \cdot Y_{lm}(\theta, \phi).
$$

(307)

Here $Y_{lm}$ are the $SO(3)$ spherical harmonics, and $C^{l+1}_{n-l}$ are the Gegenbauer polynomials:

$$
C^\alpha_n(x) = \frac{(2\alpha)_n}{(\alpha+\frac{1}{2})_n} \frac{P_n^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2})}(x)}{\Gamma(\alpha+\frac{1}{2})} = \frac{(\alpha)_n}{n!} \frac{\Gamma(n+\alpha)}{\Gamma(2n+\alpha)} (1-x^2)^{-\alpha+\frac{1}{4}} d^n/(1-x^2)^n d^n, \quad (308)
$$

with $P_n^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2})}(x)$ being Jacobi polynomials and $(\alpha)_n \equiv \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$. The degeneracy is

$$
\sum_{l=0}^{n} (2l+1) = (n+1)^2.
$$

(309)

### B.2 The free Weyl operator and eigenstates

The eigenvalue problems of free Dirac operators on arbitrary dimensional spheres are generally solved in \[45, 46, 47\]. Here, we apply the results to the $S^3$ case. For spinor particle, the covariant derivatives on $S^3$ are given by

$$
- i\nabla_{\chi} = -i\partial_{\chi},
- i\nabla_{\theta} = -i\partial_{\theta} - \frac{1}{2} \gamma_3 \cos \chi,
- i\nabla_{\phi} = -i\partial_{\phi} - \frac{1}{2} \gamma_1 \cos \theta + \frac{1}{2} \gamma_2 \cos \chi \sin \theta,
$$

(310)

\[From the general formula, we have

$$
\sum_{\mu<\nu}^{4} l_{\mu\nu}^{(0)}^2 = 2(L(L+1) + R(R+1))_{L=R=\frac{3}{2}} = n(n+2).
$$

(305)
where we used the spin-connection \( \mathbf{S} \). (310) is formally equivalent to the covariant derivatives of spinless particle in the \( SU(2) \) monopole background with 1/2 = 1/2. We construct the free Weyl operator as

\[
-i \nabla_{S^3} = -i e_a \gamma^a \nabla_a = -i \gamma_1 \nabla_\chi - i \frac{1}{\sin \chi} \gamma_2 \nabla_\theta - i \frac{1}{\sin \chi \sin \theta} \gamma_3 \nabla_\phi \\
= -i \gamma_1 (\partial_\chi + \cot \chi) - i \frac{1}{\sin \chi} (\partial_\theta + \frac{1}{2} \cot \theta) - i \gamma_3 \frac{1}{\sin \chi \sin \theta} \partial_\phi,
\]

(314)

or

\[
-i \nabla_{S^3} = -i \sigma_3 (\partial_\chi + \cot \chi) - i \sigma_1 \frac{1}{\sin \chi} (\partial_\theta + \frac{1}{2} \cot \theta) - i \sigma_2 \frac{1}{\sin \chi \sin \theta} \partial_\phi
\]

\[
= \left( \begin{array}{cc} -i (\partial_\chi + \cot \chi) & \frac{1}{\sin \chi} \left( -i \partial_\theta - \frac{1}{\sin \theta} (\partial_\phi + i \frac{3}{2} \cos \theta) \right) \\ \frac{1}{\sin \chi} \left( -i \partial_\theta + \frac{1}{\sin \theta} (\partial_\phi - i \frac{3}{2} \cos \theta) \right) & i (\partial_\chi + \cot \chi) \end{array} \right). 
\]

(315)

The eigenvalue problem of the Weyl operator \( \nabla_{S^3} \) is expressed as

\[
- i \nabla_{S^3} \psi_{n,l,m}^{(\pm)}(x) = \pm \left( \frac{3}{2} + n \right) \psi_{n,l,m}^{(\pm)}(x),
\]

(316)

with

\[
n = 0, 1, 2, \ldots, \quad l = 0, 1, 2, \ldots, n, \quad m = 0, 1, 2, \ldots, l.
\]

The corresponding eigenstates are

\[
- i \nabla_{S^3} \psi_{n,l,m}^{(\pm)}(x) = \pm \left( \frac{3}{2} + n \right) \psi_{n,l,m}^{(\pm)}(x),
\]

(317)

\[
\begin{align*}
\psi_{n,l,m}^{(+)}(x) &= \frac{1}{A} \left( \cos \frac{\chi}{2} \right)^l \left( \cos \frac{\theta}{2} \right)^m \\
&\quad \cdot \left( \cos \frac{\gamma}{2} P_{n-l}^{m+\frac{1}{2},l+\frac{1}{2}}(\cos \chi) + i \sin \frac{\gamma}{2} P_{n-l}^{m+\frac{1}{2},l+\frac{1}{2}}(\cos \chi) \right) \\
&\quad \cdot \left( \pm \left( \cos \frac{\gamma}{2} P_{n-l}^{m+\frac{1}{2},l+\frac{1}{2}}(\cos \chi) \right) \right) \\
&\quad \cdot \left( \cos \frac{\theta}{2} P_{l-m}^{m+\frac{1}{2},l+\frac{1}{2}}(\cos \theta) \right) \right) e^{\pm i(m+\frac{1}{2})\phi},
\end{align*}
\]

\[
- i \nabla_{S^3} = -\left( \frac{3}{2} + n \right) :
\]

\[
\begin{align*}
\psi_{n,l,m}^{(-)}(x) &= \frac{1}{A} \left( \cos \frac{\chi}{2} \right)^l \left( \cos \frac{\theta}{2} \right)^m \\
&\quad \cdot \left( \cos \frac{\gamma}{2} P_{n-l}^{m+\frac{1}{2},l+\frac{1}{2}}(\cos \chi) - i \sin \frac{\gamma}{2} P_{n-l}^{m+\frac{1}{2},l+\frac{1}{2}}(\cos \chi) \right) \\
&\quad \cdot \left( \pm \left( \cos \frac{\gamma}{2} P_{n-l}^{m+\frac{1}{2},l+\frac{1}{2}}(\cos \chi) \right) \right) \\
&\quad \cdot \left( \cos \frac{\theta}{2} P_{l-m}^{m+\frac{1}{2},l+\frac{1}{2}}(\cos \theta) \right) \right) e^{\pm i(m+\frac{1}{2})\phi},
\end{align*}
\]

(318)

\[\text{can be written as}
\]

\[
- i \nabla_{S^3} = -i \gamma_1 (\partial_\chi + \cot \chi) - i \frac{1}{\sin \chi} \nabla_{S^2},
\]

(311)

where

\[
\nabla_{S^2} \equiv \gamma^2 (\partial_\theta + \frac{1}{2} \cot \theta) + \gamma_3 \frac{1}{\sin \theta} \partial_\phi.
\]

(312)

\[\text{is consistent with the general relation in Sec.3.1 of 15:}
\]

\[
- i \nabla_{S^3} = -i \gamma_1 (\partial_\chi + \frac{N-1}{2} (N-3) \cot \chi) - i \frac{1}{\sin \chi} \nabla_{S^2}.
\]

(313)
The number of each of the eigenstates \(318\) is

\[
\sum_{l=0}^{n} (l + 1) = \frac{1}{2} (n + 1)(n + 2).
\]

(321)

Hence for \(-i\nabla = +\left(\frac{3}{2} + n\right)\), the degeneracy is

\[
2 \times \frac{1}{2} (n + 1)(n + 2) = (n + 1)(n + 2),
\]

(322)

and for \(-i\nabla = -\left(\frac{3}{2} + n\right)\),

\[
2 \times \frac{1}{2} (n + 1)(n + 2) = (n + 1)(n + 2).
\]

(323)

From the covariant derivatives \(310\), we construct the \(SO(4)\) angular momentum operators as

\[
J_{\mu\nu} = -ix_\mu \nabla_\nu + ix_\nu \nabla_\mu + f_{\mu\nu},
\]

(324)

where \(\nabla_\mu = \partial_\mu + i\omega_\mu\) and \(f_{\mu\nu} = -i[\nabla_\mu, \nabla_\nu]\) \(300\). \(J_{\mu\nu} \] is formally equivalent to the non-relativistic angular momentum with minimal monopole charge \(I/2 = 1/2\). The Weyl operator \(-i\nabla\) is invariant under the \(SO(4)\) transformation generated by \(J_{\mu\nu}\):

\[
[-i\nabla, J_{\mu\nu}] = 0.
\]

(325)

The Weyl operator eigenstates \(318\) correspond to the \(SO(4) \simeq SU(2) \otimes SU(2)\) irreducible representations:

\[
-i\nabla = +\left(\frac{3}{2} + n\right) \leftrightarrow (L, R) = \left(\frac{n}{2}, \frac{n}{2}, + \frac{1}{2}\right),
\]

\[
-i\nabla = -\left(\frac{3}{2} + n\right) \leftrightarrow (L, R) = \left(\frac{n}{2}, \frac{n}{2}, + \frac{1}{2}\right).
\]

(326)

In particular for \(n = 0\), \(318\) becomes

\[
-i\nabla = +\frac{3}{2} : \psi_{n=0, l=0, m=0, +}^{(+)} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix}
\sin \frac{\theta}{2} e^{i\chi/(\gamma-\phi)} \\
\cos \frac{\theta}{2} e^{-i\chi/(\gamma+\phi)}
\end{pmatrix}, \quad \psi_{0,0,0,-}^{(+)} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix}
\cos \frac{\theta}{2} e^{i\chi/(\gamma+\phi)} \\
-\sin \frac{\theta}{2} e^{-i\chi/(\gamma-\phi)}
\end{pmatrix},
\]

\[
-i\nabla = -\frac{3}{2} : \psi_{n=0, l=0, m=0, +}^{(-)} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix}
\sin \frac{\theta}{2} e^{-i\chi/(\gamma-\phi)} \\
\cos \frac{\theta}{2} e^{i\chi/(\gamma+\phi)}
\end{pmatrix}, \quad \psi_{0,0,0,-}^{(-)} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix}
\cos \frac{\theta}{2} e^{-i\chi/(\gamma+\phi)} \\
-\sin \frac{\theta}{2} e^{i\chi/(\gamma-\phi)}
\end{pmatrix}.
\]

(327)

Meanwhile for the \(SO(4)\) monopole harmonics \(110\), we have

\[
s = +\frac{1}{2} : \Phi_{0,1/2,0}^{(0,1/2,1/2)} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix}
\cos \frac{\theta}{2} e^{-i\chi/(\gamma-\phi)} \\
-\sin \frac{\theta}{2} e^{i\chi/(\gamma+\phi)}
\end{pmatrix}, \quad \Phi_{1/2,0}^{(-1/2,1/2)} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix}
\sin \frac{\theta}{2} e^{i\chi/(\gamma-\phi)} \\
\cos \frac{\theta}{2} e^{-i\chi/(\gamma+\phi)}
\end{pmatrix},
\]

\[
s = -\frac{1}{2} : \Phi_{0,1/2}^{(0,1/2,1/2)} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix}
\cos \frac{\theta}{2} e^{i\chi/(\gamma+\phi)} \\
-\sin \frac{\theta}{2} e^{-i\chi/(\gamma-\phi)}
\end{pmatrix}, \quad \Phi_{1/2,0}^{(-1/2,1/2)} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix}
\sin \frac{\theta}{2} e^{-i\chi/(\gamma+\phi)} \\
\cos \frac{\theta}{2} e^{i\chi/(\gamma-\phi)}
\end{pmatrix}.
\]

(328)

\[29\] The normalization constant \(A\) is determined so as to satisfy

\[
\int_0^\pi d\chi \sin^2 \chi \int_0^\pi d\vartheta \sin \vartheta \int_0^{2\pi} d\varphi \psi_{n,l,m,l,m}^{(+)}(x)^\dagger \psi_{n,l,m,l,m}^{(-)}(x) = 1.
\]

(319)

46
They are related as
\[
\psi_{0,0,0,+}^{(+)} = \Phi_{0,-1/2,1/2}^{(0,-1/2,1/2)}, \quad \psi_{0,0,0,-}^{(+)} = \Phi_{0,1/2,1/2}^{(0,-1/2,1/2)},
\]
\[
\psi_{0,0,0,+}^{(-)} = \Phi_{-1/2,0}^{(0,1/2,1/2)}, \quad \psi_{0,0,0,-}^{(-)} = \Phi_{1/2,0}^{(0,1/2,1/2)}.
\]

(329) is a special case of (222).

### B.3 Square of the free Weyl operator

The square of the free Weyl operator on \(S^3\) is derived as
\[
(-i\nabla_{S^3})^2 = -\left(\partial_\chi + \cot \chi\right)^2 + \frac{1}{\sin^2 \chi} (-i\nabla_{S^2})^2 + \frac{\cot \chi}{\sin \chi} \gamma^1 \nabla_{S^2},
\]

or
\[
(-i\nabla_{S^3})^2 = -\left(\partial_\chi + \cot \chi\right)^2 - \frac{1}{\sin^2 \chi} (\partial_\theta + \frac{1}{2} \cot \theta)^2 - \frac{1}{\sin^2 \chi \sin^2 \theta} \partial_\phi^2
\]
\[
+ i\gamma^1 \frac{\cot \theta}{\sin^2 \chi \sin \theta} \partial_\phi - i\gamma^2 \frac{\cot \chi}{\sin \chi \sin \theta} \partial_\phi + i\gamma^3 \frac{\cot \chi}{\sin \chi} (\partial_\theta + \frac{1}{2} \cot \theta).
\]

Using (310), we can show
\[
(-i\nabla_{S^3})^2 = \Delta_{S^3}^{\text{SO}(4)} = -\frac{R}{4}|_{R=6}
\]
\[
= \frac{4}{\mu<\nu} J_{\mu\nu}^2 + \frac{R}{8}|_{R=6},
\]
where
\[
\Delta_{S^3} = \frac{1}{\sqrt{g}} \nabla_\alpha (\sqrt{g} g^{\alpha\beta} \nabla_\beta) = \frac{1}{\sin^2 \chi} \nabla_\chi (\sin^2 \chi \nabla_\chi) + \frac{1}{\sin^2 \chi} \nabla_\theta (\sin \theta \nabla_\theta) + \frac{1}{\sin \chi} \nabla_\phi^2 + \frac{3}{2},
\]

(333a)
\[
\sum_{\mu<\nu} J_{\mu\nu}^2 = -\frac{1}{\sin^2 \chi} \nabla_\chi (\sin^2 \chi \nabla_\chi) - \frac{1}{\sin^2 \chi} \nabla_\theta (\sin \theta \nabla_\theta) - \frac{1}{\sin \chi} \nabla_\phi^2 + \frac{3}{4}.
\]

(333b)

(333a) and (333b) are simply related as

\[
\Delta_{S^3} = -\sum_{\mu<\nu} J_{\mu\nu}^2 + \frac{3}{4}.
\]

(335)

The eigenvalues of the \(SO(4)\) Casimir (333b) can be obtained from the non-relativistic result (75) for \(I/2 = |s| = 1/2\):
\[
\sum_{\mu<\nu} J_{\mu\nu}^2 = n^2 + 3n + \frac{3}{2}.
\]

(336)

\[\text{Also from the general formula in \cite{55, 57}, we have}
\]
\[
\Delta_{S^3} = -\sum_{\mu<\nu} J_{\mu\nu}^2 + \frac{R}{8}|_{R=6}.
\]

(334)
and then
\[ (-i\nabla_{S^3})^2 = \left(\frac{3}{2} + n\right)^2, \]  
(337)

or
\[ -i\nabla_{S^3} = \pm\left(\frac{3}{2} + n\right). \]  
(338)

C Integral of the product of three SO(4) monopole harmonics

From the explicit form of the SO(4) monopole harmonics (110), we can evaluate the integral of the product of three SO(4) monopole harmonics as

\[
\frac{1}{I+1} \int_{S^3} d\Omega_3 \left( \sum_{A=-\frac{1}{2}}^{\frac{1}{2}} \Phi^{[l_L,l_R;\frac{1}{2}]}_{m_L,m_R} (\chi)^A \cdot \Phi^{[\frac{3}{2},m_R;0]}_{m_L,m_R} (\chi) \cdot \Phi^{[l_L,l_R;\frac{1}{2}]}_{m_L',m_R'} (\chi) \right)_A
\]

\[
= (p+1) \sqrt{\frac{(I+1)(2L+1)(2R+1)}{2\pi^2}} \left\{ \begin{array}{c} \frac{l_L}{2} \\ \frac{l_R}{2} \\ \frac{l}{2} \end{array} \right\} C^{l_L,m_L}_{\frac{3}{2},m_L'} C^{l_R,m_R}_{\frac{1}{2},m_R'} ; l_L,m_L \end{array} \right\} C^{l_R,m_R}_{\frac{3}{2},m_R'} ; l_R,m_R' \right\} C^{l_L,m_L}_{\frac{1}{2},m_R} ; l_L,m_L \end{array} \right\} C^{l_R,m_R}_{\frac{3}{2},m_R'} ; l_R,m_R' \right\} C^{l_L,m_L}_{\frac{1}{2},m_R} ; l_L,m_L \}
(339)

This is a special formula of Eq.(3.11) in (111). For \( \Phi^{[\frac{3}{2},m_R;0]}_{m_L,m_R} |_{n_{L}=m_{L}=m_{R}=0} = \frac{1}{\sqrt{2\pi^2}}, \) (339) becomes

\[
\frac{1}{\sqrt{2\pi^2}} \int_{S^3} d\Omega_3 \sum_{A} \Phi^{[l_L,l_R;\frac{1}{2}]}_{m_L,m_R} (\chi)^A \cdot \Phi^{[l_L,l_R;\frac{1}{2}]}_{m_L',m_R'} (\chi) \right\} \]

\[
= \sqrt{\frac{(2L+1)(2R+1)}{2\pi^2}} \left\{ \begin{array}{c} l_L \\ l_R \\ \frac{l}{2} \end{array} \right\} C^{l_L,m_L}_{0,0} ; l_L,m_L \end{array} \right\} C^{l_R,m_R}_{0,0} ; l_R,m_R' \right\} C^{l_L,m_L}_{\frac{3}{2},m_R} ; l_L,m_L \}
(340)

where, in the third equation, we used

\[
\left\{ \begin{array}{c} l_L \\ l_R \\ \frac{l}{2} \end{array} \right\} = \frac{1}{\sqrt{(2L+1)(2R+1)}} \left\{ \begin{array}{c} \frac{l_l}{2} \end{array} \right\} C^{l',m'}_{0,0} ; l_m = \delta_{l,l'} \delta_{m,m'}. \]
(341)

(340) is consistent with (112).
Similarly, we have

\[
\frac{1}{l+1} \int_{S^3} d\Omega_3 \left( \sum_{A=-\frac{1}{2}}^{\frac{1}{2}} \Phi^{[l_L,l_R,\frac{1}{2}]}_{m_L,m_R} (\chi)_A \cdot \Phi^{[\frac{3}{2},0]}_{m'_L,m'_R}(\chi) \cdot \Phi^{[l_R,l_L,\frac{1}{2}]}_{m'_R,m'_L}(\chi)_A \right)
\]

\[
= (p+1) \sqrt{\frac{(I+1)(2l_L+1)(2l_R+1)}{2\pi^2}} \left( l_L \frac{l}{2} l_R \frac{l}{2} \right) C^{l_L,m_L}_{\frac{3}{2},m'_L} : l_R,m'_R : l_L,m''_L
\]

\[
= (p+1) \sqrt{\frac{(p+1)(2l_L+1)(2l_R+1)}{2\pi^2}} \left( (-1)^{2l_L+2l_R+I+p} l_L \frac{l}{2} l_R \frac{l}{2} \right) C^{l_L,m_L}_{\frac{3}{2},m'_L} : l_R,m'_R : l_L,m''_L
\]

\[
= \sqrt{\frac{(p+1)(2l_L+1)(2l_R+1)}{2\pi^2}} \left( (-1)^{2l_L+4l_R+I+2p} l_L \frac{l}{2} l_R \frac{l}{2} \right) C^{l_L,m_L}_{\frac{3}{2},m'_L} : l_R,m''_L : l_L,m''_L
\]

\[
= \sqrt{\frac{(p+1)(2l_L+1)(2l_R+1)}{2\pi^2}} \left( (-1)^{2l_L+2l_R+I+p} l_L \frac{l}{2} l_R \frac{l}{2} \right) C^{l_L,m_L}_{\frac{3}{2},m'_L} : l_R,m'_R : l_L,m''_L
\]

**D SU(2) transformation between Dirac and Schwinger gauges**

**D.1 Curvature for the spin connection**

With respect to the spin connections in the Schwiger gauge \[17\] and the Dirac gauge \[18\], the curvatures

\[
f = d\omega + i\omega^2,
\]

are respectively derived as

\[
f_S = \frac{1}{2} \gamma_1 \sin^2 \chi \sin \theta \, d\theta \wedge d\phi + \frac{1}{2} \gamma_2 \sin \chi \sin \theta \, d\phi \wedge d\chi + \frac{1}{2} \gamma_3 \sin \chi \, d\chi \wedge d\theta,
\]

\[
f_D = \frac{1}{2} \gamma_1 \sin \chi \sin \theta (\cos \theta \, d\theta \wedge d\phi - \sin \theta \, d \phi \wedge d \chi)
\]

\[
- \frac{1}{2} \gamma_2 \sin \chi \sin \phi (d\chi \wedge d\theta - \sin \theta \cos \theta \cot \phi \, d\phi \wedge d\chi - \sin \chi \sin^2 \theta \cot \phi \, d\theta \wedge d\phi)
\]

\[
+ \frac{1}{2} \gamma_3 \sin \chi \cos \phi (d\chi \wedge d\theta + \sin \theta \cos \theta \tan \phi \, d\phi \wedge d\chi + \sin \chi \sin^2 \theta \tan \phi \, d\theta \wedge d\phi). \tag{344}
\]

Using the dreibeins, \[13\] and \[17\], they are concisely represented as

\[
f_S = \frac{1}{4} \epsilon_{abc} e_8^a \gamma_c, \quad f_D = \frac{1}{4} \epsilon_{abc} e_D^a \gamma_c. \tag{345}
\]

\(f_S\) and \(f_D\) are related by the \(SU(2)\) transformation:

\[
f_D = \frac{1}{4} \epsilon_{abc} e_D^a \wedge e_8^b \gamma_c = \frac{1}{4} \epsilon_{abc} O_{ad} O_{be} e_S^d \wedge e_8^a \gamma_c = \frac{1}{2} \epsilon_{cde} O_{cf} e_S^b \wedge e_8^d \gamma_c = g f_S g^\dagger, \tag{346}
\]

where in the third equation we used the \(SO(3)\) invariance of the Levi-Civita tensor

\[
\epsilon_{ade} O_{db} O_{ec} = \epsilon_{bca} O_{ad}, \tag{347}
\]

and in the last equation \[28\].
D.2 \( D \) function and gauge transformation

We introduce two parameterizations of the \( SU(2) \) group elements:

\[
\Psi_D^{(l)}(\chi) \equiv e^{-i\chi S_D^{(l)}},
\]
\[
\Psi_S^{(l)}(\chi) \equiv e^{-i\chi S_S^{(l)}} e^{i\phi S_S^{(l)}}.
\]

(348a)

(348b)

In particular for \( l = 1/2 \),

\[
\Psi_D^{(1/2)}(\chi) = \begin{pmatrix}
\cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \cos \phi & -i \sin \frac{\theta}{2} \sin \theta e^{-i\phi} \\
-i \sin \frac{\theta}{2} \sin \theta e^{i\phi} & \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \cos \phi
\end{pmatrix},
\]
\[
\Psi_S^{(1/2)}(\chi) = \begin{pmatrix}
\cos \frac{\theta}{2} e^{-i\frac{\pi}{4}(\chi - \phi)} & \sin \frac{\theta}{2} e^{-i\frac{\pi}{4}(\chi + \phi)} \\
-\sin \frac{\theta}{2} e^{i\frac{\pi}{4}(\chi + \phi)} & \cos \frac{\theta}{2} e^{i\frac{\pi}{4}(\chi - \phi)}
\end{pmatrix}.
\]

(349a)

(349b)

With the \( D \)-function

\[
D^{(l)}(\chi, \theta, \phi) \equiv e^{-i\chi S_D^{(l)}} e^{-i\theta S_S^{(l)}} e^{-i\phi S_S^{(l)}},
\]

(350)

is given by

\[
\Psi_D^{(l)}(\chi) = D^{(l)}(\phi, \theta, 0) D^{(l)}(\chi, -\theta, -\phi),
\]
\[
\Psi_S^{(l)}(\chi) = D^{(l)}(\chi, -\theta, -\phi).
\]

(351a)

(351b)

For the \( SO(4) \) representations, \((l_L, l_R) = (l, 0)\) and \((0, l)\), we have

\[
(l_L, l_R) = (l, 0) : \Psi_L^{(D)} \equiv \Psi_D(\chi), \quad (l_L, l_R) = (0, l) : \Psi_R^{(D)} \equiv \Psi_D(-\chi),
\]
\[
\Psi_L^{(S)} \equiv \Psi_S^{(l)}(\chi), \quad \Psi_R^{(S)} \equiv \Psi_S^{(l)}(-\chi).
\]

(352a)

(352b)

They satisfy

\[
\Psi_{L(D)}^{(l)} = \Psi_{R(D)}^{(l)}, \quad \Psi_{L(S)}^{(l)} = g^{(l)} \Psi_{R(S)}^{(l)} g^{(l)},
\]

(353)

with

\[
g^{(l)}(\theta, \phi) \equiv D^{(l)}(\phi, \theta, 0) = e^{-i\phi S_D^{(l)}} e^{-i\theta S_S^{(l)}}.
\]

(354)

For \( l = 1/2 \),

\[
g^{(1/2)}(\theta, \phi) = \begin{pmatrix}
\cos \frac{\theta}{2} e^{-i\frac{\pi}{4}} & -\sin \frac{\theta}{2} e^{-i\frac{\pi}{4}} \\
\sin \frac{\theta}{2} e^{i\frac{\pi}{4}} & \cos \frac{\theta}{2} e^{i\frac{\pi}{4}}
\end{pmatrix}.
\]

(355)

\[
\text{[354]} \text{ also acts as the gauge function between the Dirac and Schwinger gauges}
\]
\[
\Psi_L^{(D)} = g^{(l)} \Psi_L^{(S)}, \quad \Psi_R^{(D)} = g^{(l)} \Psi_R^{(S)}.
\]

(356)

Using \([353]\), we can rewrite \([356]\) as

\[
\Psi_L^{(D)} = \Psi_R^{(S)} g, \quad \Psi_R^{(D)} = \Psi_R^{(S)} g\dagger.
\]

(357)

\[
\text{[357]} \text{ implies that the } SU(2) \text{ gauge fields in the two gauges}
\]
\[
A^{(l)}_D = -\frac{i}{2} (\Psi_L^{(D)} g - d\Psi_L^{(D)} + \Psi_R^{(D)} d\Psi_R^{(D)}),
\]
\[
A^{(l)}_S = -\frac{i}{2} (\Psi_L^{(S)} g + d\Psi_L^{(S)} + \Psi_R^{(S)} d\Psi_R^{(S)}),
\]

(358a)

(358b)
are related as
\[ A_D^{(l)} = g^{(l)} A_S^{(l)} g^{(l)\dagger} - ig^{(l)} d g^{(l)\dagger}. \]  
(359)

From (358a), we have
\[ A_D^{(1/2)} = \sum_{i=1}^{3} A_D^{i} S_{i}^{(1/2)}, \]  
(360)
with
\[ A_D^{1} = -(1 - \cos \chi)(\sin \phi d \theta + \sin \theta \cos \phi d \phi), \]
\[ A_D^{2} = (1 - \cos \chi)(\cos \phi d \theta - \sin \theta \sin \phi d \phi), \]
\[ A_D^{3} = (1 - \cos \chi)(\sin \theta d \phi). \]  
(361)

In the Cartesian coordinate, they are given by
\[ A_D^{i} = -\frac{1}{1 + x_4} \epsilon_{ijk} x_j S_k^{(1/2)}, \quad A_D^{4} = 0. \]  
(362)

Also from (358b), we obtain
\[ A_S^{(1/2)} = \sum_{i=1}^{3} A_S^{i} S_{i}^{(1/2)}, \]  
(363)
with
\[ A_S^{1} = \cos \chi \sin \theta d \phi, \quad A_S^{2} = -\cos \chi d \theta, \quad A_S^{3} = -\sin \theta d \phi. \]  
(364)

With the explicit expressions (361) and (364), it is not difficult to verify (359).

Using the gauge field (364), we can construct the covariant derivative as
\[ D_{\alpha} = \partial_{\alpha} + i A_{\alpha} S_{i}^{(1/2)} \quad (\alpha = \chi, \theta, \phi), \]  
(365)
and the covariant angular momentum operators as
\[ \Lambda_{\mu \nu} = -ix_{\mu} D_{\nu} + ix_{\nu} D_{\mu}. \]  
(366)

are expressed as
\[ \Lambda_{12} = -iD_{\phi}, \]
\[ \Lambda_{13} = i(\cos \phi D_{\theta} - \cot \theta \sin \phi D_{\phi}), \]
\[ \Lambda_{14} = i(\sin \theta \cos \phi D_{\chi} + \cot \chi \cos \theta \cos \phi D_{\theta} - \cot \chi \frac{1}{\sin \theta} \sin \phi D_{\phi}), \]
\[ \Lambda_{23} = i(\sin \phi D_{\theta} + \cot \theta \cos \phi D_{\phi}), \]
\[ \Lambda_{24} = i(\sin \theta \sin \phi D_{\chi} + \cot \chi \cos \theta \sin \phi D_{\theta} + \cot \chi \frac{1}{\sin \theta} \cos \phi D_{\phi}), \]
\[ \Lambda_{34} = i(\cos \theta D_{\chi} - \cot \chi \sin \theta D_{\theta}). \]  
(367)

The sum of their squares gives
\[ \sum_{\mu < \nu}^{4} \Lambda_{\mu \nu}^2 = -\frac{1}{\sin^2 \chi} D_{\chi}(\sin^2 \chi D_{\chi}) - \frac{1}{\sin^2 \chi} \frac{1}{\sin \theta} \left( D_{\theta}(\sin \theta D_{\theta}) + \frac{1}{\sin \theta} D_{\phi}^2 \right), \]  
(368)
which is equal to (303) with the replacement of \( \partial_{\alpha} \) with \( D_{\alpha} \). One can confirm that \( \Phi_{m_L, m_R}^{(n,s,1/2)} \) are the eigenstates of \( \sum_{\mu < \nu}^{4} \Lambda_{\mu \nu}^2 \) by using their explicit coordinate representation.

\[ ^{32} \text{One can alternatively adopt (361) in the Dirac gauge, but the concrete calculations are rather laborious in the polar coordinates.} \]
D.3 Gauge covariance of the Weyl-Landau operator

From the relations $e_a^a = O_{ab} e_b^b$ and $g \gamma_a g^\dagger = \gamma_b O_{ab}$ (see Sec. 2), we have

$$g^{(1/2)} (e_S^a \gamma_a) g^{(1/2)\dagger} = e_D^a \gamma_a.$$  

(369)

Meanwhile, the covariant derivatives, $\nabla \alpha = \partial \alpha + i \omega \alpha$, in the Dirac and Schwinger gauges are related as

$$g^{(1/2)} \nabla^S \alpha g^{(1/2)\dagger} = \nabla^D \alpha,$$  

(370)

with

$$g^{(1/2)} \omega^S_\alpha g^{(1/2)\dagger} - ig^{(1/2)} \partial_\alpha g^{(1/2)\dagger} = \omega^D_\alpha.$$  

(371)

Consequently, the free Weyl operators

$$\nabla^S = \gamma^a e_{Sa} ^{\alpha} \nabla^S_\alpha, \quad \nabla^D = \gamma^a e_{Da} ^{\alpha} \nabla^D_\alpha,$$  

(372)

are related as

$$g^{(1/2)} \nabla^S g^{(1/2)\dagger} = \nabla^D.$$  

(373)

We can verify similar relations for the Weyl-Landau operator, $\mathcal{P} = \gamma^a e_{a} ^{\alpha} D_\alpha$. The synthesized gauge function

$$G(\theta, \phi) = g^{(1/2)}(\theta, \phi) \otimes g^{(1/2)}(\theta, \phi),$$  

(374)

generates the gauge transformation for $A = \omega \otimes 1 + 1 \otimes A$,

$$G A^S_\alpha G^\dagger - i G \partial_\alpha G^\dagger = A^D_\alpha,$$  

(375)

and

$$G (e_S^a \gamma_a \otimes 1) G^\dagger = (g^{(1/2)} (e_S^a \gamma_a) g^{(1/2)\dagger}) \otimes (g^{(1/2)} g^{(1/2)\dagger}) = e_D^a \gamma_a \otimes 1.$$  

(376)

From (375) and (376), we easily see that the Weyl-Landau operator satisfies

$$G \mathcal{P}^S G^\dagger = \mathcal{P}^D,$$  

(377)

and then

$$G \Psi^S = \Psi^D.$$  

(378)

E Examples of the Weyl-Landau operator eigenstates

For several Weyl-Landau operators, we explicitly derive a coordinate representation of the eigenstates based on the general procedure presented in Sec. 5.1.

E.1 $I/2 = 1/2$

The synthesized spin magnitudes are given by

$$J^+ = I + 1/2 = 1, \quad J^- = I - 1/2 = 0,$$  

(379)

and the Weyl-Landau operator spectrum becomes

$$\pm \lambda(n, s) = \pm \sqrt{n(n + 2) + s^2},$$  

(380)
Figure 7: The Weyl-Landau spectrum for $\ell/2 = 1/2$.

with

\[ n = 0 : \; s = 0, \quad n \neq 0 : \; s = 1, 0, -1. \]  \hspace{1cm} (381)

For instance (Fig. 7),

\[ \lambda(0, 0) = 0, \quad \lambda(1, 0) = \sqrt{3}, \quad \lambda(1, 1) = 2, \quad \lambda(2, 0) = 2\sqrt{2}, \quad \lambda(2, 1) = 3, \cdots. \]  \hspace{1cm} (382)

In the following, we use the notation:

\[ \Phi^{(n, s, -)}_{ML, MR} \equiv \Phi^{(n, s, J^- = 0)}_{ML, MR}, \quad \Phi^{(n, s, +)}_{ML, MR} \equiv \Phi^{(n, J^+ = 1)}_{ML, MR}, \]  \hspace{1cm} (383)

where $\Phi^{(-, n, s)}$ and $\Phi^{(+, n, s)}$ respectively denote the one-component ($S = 0$) and three component ($S = 1$) irreducible representation of the $SU(2)$ gauge group. The unitary matrix for the irreducible decomposition, $1/2 \otimes 1/2 = 1 \oplus 0$, is given by

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}. \]  \hspace{1cm} (384)

It is straightforward to construct the eigenstates of the spinor Landau model by acting (384) to (383).

Derivation of the Weyl-Landau operator eigenstates is not difficult. We act the Weyl-Landau operator to linear combination of the eigenstates and determine the coefficients of the linear combination ($\alpha$ and $\beta$ in (207b)) so that the linear combination to be the eigenstate of the Weyl-Landau operator. The results are as follows.

- $\lambda = 0 : \; (n^-, s^-) = (0, 0)$

\[ \Psi_{\lambda=0} = U \begin{pmatrix} 0 \\ 0 \\ 0 \Phi^{(0, 0, -)} \end{pmatrix} = U \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \]  \hspace{1cm} (385)
\[ \pm \lambda = \pm \sqrt{3} : (n^-, s^-) = (1, 0), \quad (n^+, s^+) = (0, 0) \]

\[
\Psi_{\pm \lambda = \pm \sqrt{3}, M_L, M_R} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_{M_L, M_R}^+ + \psi_{M_L, M_R}^- \end{pmatrix} = \frac{1}{\sqrt{2}} U \begin{pmatrix} \Phi_{M_L, M_R}^{(1, 0, +)} \\
\Phi_{M_L, M_R}^{(0, 0, -)} \end{pmatrix}, \quad (M_L, M_R = 1/2, -1/2)
\]

\[
\Psi_{-\lambda = -\sqrt{3}, M_L, M_R} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_{M_L, M_R}^+ - \psi_{M_L, M_R}^- \end{pmatrix} = \frac{1}{\sqrt{2}} U \begin{pmatrix} \Phi_{M_L, M_R}^{(1, 0, +)} \\
-\Phi_{M_L, M_R}^{(0, 0, -)} \end{pmatrix}, \quad (M_L, M_R = 1/2, -1/2) \tag{386}
\]

where

\[
\begin{pmatrix}
0 \\
0 \\
0 \\
\Phi_{M_L, M_R}^{(0, 0, -)}
\end{pmatrix}, \quad \begin{pmatrix}
\psi_{M_L, M_R}^+ \\
\Phi_{M_L, M_R}^{(1, 0, +)} \\
0
\end{pmatrix} \tag{387}
\]

\[ \pm \lambda = \pm 2 : (n^+, s^+) = (0, \pm 1) \]

\[
\psi_{\lambda = 2, M_L = 0, M_R} = U \begin{pmatrix} \Phi_{M_L = 0, M_R}^{(0, 1, +)} \\
0 \end{pmatrix}, \quad (M_R = +1, 0, -1)
\]

\[
\psi_{-\lambda = -2, M_L, M_R = 0} = U \begin{pmatrix} \Phi_{M_L, M_R = 0}^{(0, 1, +)} \\
0 \end{pmatrix}. \quad (M_L = +1, 0, -1) \tag{388}
\]

\[ \pm \lambda = \pm 2 \sqrt{2} : (n^-, s^-) = (2, 0), \quad (n^+, s^+) = (1, 0) \]

\[
\psi_{\lambda = 2 \sqrt{2}, M_L, M_R} = \frac{1}{\sqrt{2}} U \begin{pmatrix} \Phi_{M_L, M_R}^{(2, 0, +)} \\
\Phi_{M_L, M_R}^{(1, 0, -)} \end{pmatrix}, \quad (M_L, M_R = 1, 0, -1)
\]

\[
\psi_{-\lambda = -2 \sqrt{2}, M_L, M_R} = \frac{1}{\sqrt{2}} U \begin{pmatrix} \Phi_{M_L, M_R}^{(2, 0, +)} \\
-\Phi_{M_L, M_R}^{(1, 0, -)} \end{pmatrix}. \quad (M_L, M_R = 1, 0, -1) \tag{389}
\]

\[ \pm \lambda = \pm 3 : (n^-, s^-) = (2, \pm 1), \quad (n^+, s^+) = (1, \pm 1) \]

\[
\psi_{\lambda = 3, M_L, M_R} = U \begin{pmatrix} \Phi_{M_L, M_R}^{(1, -1, +)} \\
0 \end{pmatrix}, \quad (M_L = 1/2, -1/2, \quad M_R = 3/2, 1/2, -1/2, -3/2)
\]

\[
\psi_{-\lambda = -3, M_L, M_R} = U \begin{pmatrix} \Phi_{M_L, M_R}^{(1, 1, +)} \\
0 \end{pmatrix}. \quad (M_L = 3/2, 1/2, -1/2, -3/2, \quad M_R = 1/2, -1/2) \tag{390}
\]

### E.2 \( I/2 = 1 \)

The synthesized spins are

\[
J^+ = \frac{I}{2} + \frac{1}{2} = \frac{3}{2}, \quad J^- = \frac{I}{2} - \frac{1}{2} = \frac{1}{2}, \tag{391}
\]

and the Weyl-Landau operator spectrum is

\[
\pm \lambda(n, s) = \pm \sqrt{n(n+3) + s^2}, \tag{392}
\]
with
\[ n = 0 : \quad s = \frac{1}{2}, -\frac{1}{2}, \quad n \neq 0 : \quad s = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}. \]

For instance (Fig. [3]),
\[ \lambda(0, 1/2) = 1/2, \quad \lambda(1, 1/2) = \sqrt{17}/2, \quad \lambda(1, 3/2) = 5/2, \quad \lambda(2, 1/2) = \sqrt{41}/2, \quad \lambda(2, 3/2) = 7/2, \quad \cdots. \]

\[ \text{Figure 8: The Weyl-Landau spectrum for } I/2 = 1. \]

We adopt the notation:
\[ \Phi^{(n, s, -)}_{M_L, M_R} \equiv \Phi^{(n, s, J^- = 1/2)}_{M_L, M_R}, \quad \Phi^{(n, s, +)}_{M_L, M_R} \equiv \Phi^{(n, s, J^+ = 3/2)}_{M_L, M_R}, \]

where \( \Phi^{(-, n, s)} \) and \( \Phi^{(+, n, s)} \) respectively denote the two-component \( (S = 1/2) \) and four-component \( (S = 3/2) \) irreducible representations of the \( SU(2) \) gauge group. The unitary matrix for the decomposition \( 1/2 \otimes 1 = 3/2 \oplus 1/2 \) is given by
\[ U = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \]

- \( \pm \lambda = \pm 1/2 : \quad (n^-, s^-) = (0, \pm 1/2) \)
\[ \Psi_{\lambda=+1/2,M_L,M_R=0} = U \begin{pmatrix} 0 \\ \Phi_{(0,1/2,-)}^{(0,1/2,-)} \\ \Phi_{M_L,M_R=0}^{(1,1/2,-)} \end{pmatrix}, \quad (M_L = 1/2, -1/2) \]

\[ \Psi_{-\lambda=-1/2,M_L=0,M_R} = U \begin{pmatrix} 0 \\ \Phi_{M_L,M_R=0}^{(0,-1/2,-)} \end{pmatrix}. \quad (M_R = 1/2, -1/2) \] (397)

- \( \pm \lambda = \pm \sqrt{17}/2 \) : \((n^-, s^-) = (1, \pm 1/2), \quad (n^+, s^+) = (0, \pm 1/2)\)

For \( s = +1/2 \) sector,

\[ \Psi_{\lambda=+\sqrt{17}/2,M_L,M_R} = \frac{1}{\sqrt{6(51 + 5\sqrt{17})}} U \begin{pmatrix} 8\sqrt{2}\Phi_{(0,1/2,+)}^{(0,1/2,+)} \\ -(5 + 3\sqrt{17})\Phi_{M_L,M_R}^{(1,1/2,-)} \end{pmatrix}, \quad (M_L = 1, 0, -1, \quad M_R = 1/2, -1/2) \]

\[ \Psi_{-\lambda=-\sqrt{17}/2,M_L,M_R} = \frac{1}{\sqrt{6(51 + 5\sqrt{17})}} U \begin{pmatrix} (5 + 3\sqrt{17})\sqrt{2}\Phi_{M_L,M_R}^{(0,1/2,+)} \\ 8\sqrt{2}\Phi_{M_L,M_R}^{(1,1/2,-)} \end{pmatrix}. \quad (M_L = 1, 0, -1, \quad M_R = 1/2, -1/2) \] (398)

For \( s = -1/2 \) sector,

\[ \Psi_{\lambda=+\sqrt{17}/2,M_L,M_R} = \frac{1}{\sqrt{6(51 - 5\sqrt{17})}} U \begin{pmatrix} 8\sqrt{2}\Phi_{(0,-1/2,+)}^{(0,-1/2,+)} \\ -(5 - 3\sqrt{17})\Phi_{M_L,M_R}^{(1,-1/2,-)} \end{pmatrix}, \quad (M_L = 1/2, -1/2, \quad M_R = 1, 0, -1) \]

\[ \Psi_{-\lambda=-\sqrt{17}/2,M_L,M_R} = \frac{1}{\sqrt{6(51 - 5\sqrt{17})}} U \begin{pmatrix} (5 - 3\sqrt{17})\sqrt{2}\Phi_{M_L,M_R}^{(0,-1/2,+)} \\ -8\sqrt{2}\Phi_{M_L,M_R}^{(1,-1/2,-)} \end{pmatrix}. \quad (M_L = 1/2, -1/2, \quad M_R = 1, 0, -1) \] (399)

- \( \pm \lambda = \pm 5/2 \) : \((n^+, s^+) = (0, \pm 3/2)\)

\[ \Psi_{\lambda=5/2,M_L=0,M_R} = U \Phi_{M_L=0,M_R=0}^{(0,-3/2,+)} \left( \Phi_{M_L=0,M_R=0}^{(0,3/2,+)} \right), \quad (M_R = 3/2, 1/2, -1/2, -3/2) \]

\[ \Psi_{-\lambda=-5/2,M_L,M_R=0} = U \Phi_{M_L,M_R=0}^{(0,3/2,+)} \left( \Phi_{M_L,M_R=0}^{(0,-3/2,+)} \right). \quad (M_L = 3/2, 1/2, -1/2, -3/2) \] (400)

References

[1] Kazuki Hasebe, “Relativistic Landau Models and Generation of Fuzzy Spheres”, Int.J.Mod.Phys.A 31 (2016) 1650117; [arXiv:1511.04681].

[2] Machiko Hatsuda, Satoshi Iso, Hiroshi Umetsu, “Noncommutative superspace, supermatrix and lowest Landau level”, Nucl.Phys. B671 (2003) 217-242; [hep-th/0306251].

[3] F.A. Berezin, “General Concept of Quantization”, Commun.Math. Phys. 40 (1975) 153-174.

[4] Jens Hoppe, “Quantum Theory of a Massless Relativistic Surface and a Two-dimensional Bound State Problem”, MIT PhD Thesis (1982). “Membranes and integrable systems”, Phys.Lett. B 250 (1990) 44-48.
[5] J. Madore, “The Fuzzy Sphere”, Class. Quant. Grav. 9 (1992) 69.

[6] P.A.M. Dirac, “Quantized singularities in the electromagnetic field”, Proc. Royal Soc. London, A133 (1931) 60-72.

[7] T.T. Wu, C.N. Yang, “Dirac Monopoles without Strings: Monopole Harmonics”, Nucl.Phys. B107 (1976) 1030-1033.

[8] F.D.M. Haldane, “Fractional quantization of the Hall effect: a hierarchy of incompressible quantum fluid states”, Phys. Rev. Lett. 51 (1983) 605-608.

[9] Chen Ning Yang, “Generalization of Dirac’s Monopole to SU2 Gauge Fields”, J.Math.Phys. 19 (1978) 320.

[10] Chen Ning Yang, “SU2 monopole harmonics”, J.Math.Phys. 19 (1978) 2622.

[11] S. C. Zhang and J. P. Hu, “A Four Dimensional Generalization of the Quantum Hall Effect”, Science 294 (2001) 823; cond-mat/0110572.

[12] H. Grosse, C. Klimcik, P. Presnajder, “On Finite 4D Quantum Field Theory in Non-Commutative Geometry”, Commun.Math.Phys. 180 (1996) 429-438; hep-th/9602115.

[13] Judith Castelino, Sangmin Lee, Washington Taylor, “Longitudinal 5-branes as 4-spheres in Matrix theory”, Nucl.Phys.B526 (1998) 334-350; hep-th/9712105.

[14] Yusuke Kimura, “Noncommutative gauge theory on fuzzy four-sphere and matrix model”, Nucl.Phys.B 637 (2002) 177; hep-th/0204256.

[15] Kazuki Hasebe, “Higher Dimensional Quantum Hall Effect as A-Class Topological Insulator”, Nucl.Phys. B 886 (2014) 952-1002; arXiv:1403.5066.

[16] K. Hasebe and Y. Kimura, “Dimensional Hierarchy in Quantum Hall Effects on Fuzzy Spheres”, Phys.Lett. B 602 (2004) 255; hep-th/0310274.

[17] P. M. Ho and S. Ramgoolam, “Higher dimensional geometries from matrix brane constructions”, Nucl.Phys.B 627 (2002) 266; hep-th/0111278.

[18] Yusuke Kimura, “On higher dimensional fuzzy spherical branes”, Nucl.Phys.B 664 (2003) 512; hep-th/0301055.

[19] Dimitra Karabali, V.P. Nair, “Quantum Hall Effect in Higher Dimensions”, Nucl.Phys. B641 (2002) 533-546; hep-th/0203264.

[20] B.A. Bernevig, J.P. Hu, N. Toumbas, S.C. Zhang, “The Eight Dimensional Quantum Hall Effect and the Octonions”, Phys.Rev.Lett. 91 (2003) 236803; cond-mat/0306045.

[21] Kazuki Hasebe, “Supersymmetric Quantum Hall Effect on Fuzzy Supersphere”, Phys.Rev.Lett. 94 (2005) 206802; hep-th/0411137.

[22] Ahmed Jellal, “Quantum Hall Effect on Higher Dimensional Spaces”, Nucl.Phys. B725 (2005) 554-576; hep-th/0505095.

[23] Kazuki Hasebe, “Hyperbolic supersymmetric quantum Hall effect”, Phys.Rev.D78 (2008) 125024; arXiv:0809.4885.
[24] Mohammed Daoud, Ahmed Jellal, “Quantum Hall Effect on the Flag Manifold $F_2$”, Int.J.Mod.Phys.A23 (2008) 3129-3154; hep-th/0610157.

[25] Kazuki Hasebe, “Split-Quaternionic Hopf Map, Quantum Hall Effect, and Twistor Theory”, Phys.Rev.D81 (2010) 041702; arXiv:0902.2523.

[26] F. Balli, A. Behtash, S. Kurkcuoglu, G. Unal, “Quantum Hall Effect on the Grassmannians $Gr_2(C^N)$”, Phys. Rev. D 89 (2014) 105031; arXiv:1403.3823.

[27] Dimitra Karabali, V.P. Nair, “The Geometry of Quantum Hall Effect: An Effective Action for all Dimensions”, Phys. Rev. D 94 (2016) 024022, arXiv:1604.00722.

[28] Matthew F. Lapa, Chao-Ming Jian, Peng Ye, Taylor L. Hughes, “Topological Electromagnetic Responses of Bosonic Quantum Hall, Topological Insulator, and Chiral Semi-Metal phases in All Dimensions”, Phys. Rev. B 95 (2017) 035149, arXiv:1611.03504.

[29] Jonathan J. Heckman, Luigi Tizzano, “6D Fractional Quantum Hall Effect”, arXiv:1708.02250.

[30] V.P. Nair, S. Randjbar-Daemi, “Quantum Hall effect on $S^3$, edge states and fuzzy $S^3/Z_2$”, Nucl.Phys. B679 (2004) 447-463; hep-th/0309212.

[31] Yoichiro Nambu, “Generalized Hamiltonian Dynamics”, Phys.Rev.D7 (1973) 2405-2412.

[32] Thomas Curtright, Cosmas Zachos, “Classical and Quantum Nambu Mechanics”, Phys.Rev.D68 (2003) 085001; hep-th/0212267.

[33] M. M. Sheikh-Jabbari, M. Torabian, “Classification of All 1/2 BPS Solutions of the Tiny Graviton Matrix Theory”, JHEP 0504 (2005) 001; hep-th/0501001.

[34] M. M. Sheikh-Jabbari, “Tiny Graviton Matrix Theory: DLCQ of IIB Plane-Wave String Theory, A Conjecture”, JHEP 0409 (2004) 017; hep-th/0406214.

[35] Sanjaye Ramgoolam, “Higher dimensional geometries related to fuzzy odd-dimensional spheres”, JHEP 0210 (2002) 064; hep-th/0207111.

[36] Kazuki Hasebe, “Chiral topological insulator on Nambu 3-algebraic geometry”, Nucl.Phys. B 886 (2014) 681-690; arXiv:1403.7810.

[37] Kazuki Hasebe, “Higher (Odd) Dimensional Quantum Hall Effect and Extended Dimensional Hierarchy”, Nucl.Phys. B 920 (2017) 475-520; arXiv:1612.05853.

[38] U.H. Coskun, S. Kurkcuoglu, G.C.Toga, “Quantum Hall Effect on Odd Spheres”, Phys. Rev. D 95 (2017) 065021; arXiv:1612.03855.

[39] Yi Li, Congjun Wu, “High-Dimensional Topological Insulators with Quaternionic Analytic Landau Levels”, Phys. Rev. Lett. 110 (2013) 216802; arXiv:1103.5422.

[40] Yi Li, Kenneth Intriligator, Yue Yu, Congjun Wu, “Isotropic Landau levels of Dirac fermions in high dimensions”, Phys. Rev. B 85 (2012) 085132; arXiv:1108.5650.

[41] Yi Li, Shou-Cheng Zhang, Congjun Wu, “Topological insulators with SU(2) Landau levels”, Phys. Rev. Lett. 111 (2013) 186803, arXiv:1208.1562.

[42] L. C. Biedenharn, “Wigner Coefficients for the $R_4$ Group and Some Applications”, J. Math. Phys. 2 (1961) 433.
[43] G. Domokos, “Four-Dimensional Symmetry”, Phys. Rev. 159 (1967) 1387.

[44] Harry Hochstadt, “The functions of Mathematical Physics”, Dover 2012.

[45] R. Camporesi, A. Higuchi, “On the eigenfunctions of the Dirac operator on spheres and real hyperbolic spaces”, J.Geom.Phys. 20 (1996) 1-18; [gr-qc/9505009]

[46] A. Trautman, “The Dirac operator on hypersurfaces”, Acta Phys.Pol.B26 (1995) 1283-1310; [hep-th/9810018]

[47] A. Trautman, “Spin structures on hypersurfaces and the spectrum of the Dirac operator on spheres”, in Spinors, Twistors, Clifford Algebras and Quantum Deformations (Kluwer Academic Publishers, 1993).

[48] Tohru Eguchi, Peter B. Gilkey, Andrew J. Hanson “Gravitation, Gauge Theories and Differential Geometry”, Phys.Rep. 66 (1980) 213-390.

[49] David G. Boulware, “Spin-$\frac{3}{2}$ field theory in Schwarzschild space”, Phys.Rev. D 12 (1975) 350.

[50] Bjoern Felsager, “Geometry, Particles, and Fields” (Graduate Texts in Contemporary Physics), Springer (1998).

[51] F. Wilczek, “Magnetic Flux, Angular Momentum, and Statistics”, Phys. Rev. Lett. 48 (1982) 1144-1146.

[52] Abdus Salam, J. Strathdee, “On Kaluza-Klein Theory”, Ann. Phys. 141 (1982) 316-352.

[53] Goro Ishiki, Yastoshi Takayama, Asato Tsuchiya, “N=4 SYM on $R \times S^3$ and Theories with 16 Supercharges”, JHEP10 (2006) 007; [hep-th/0605163]

[54] Goro Ishiki, Shinji Shimasaki, Yastoshi Takayama, Asato Tsuchiya, “Embedding of theories with SU(2|4) symmetry into the plane wave matrix model”, JHEP11 (2006) 089; [hep-th/0610038]

[55] Brian P. Dolan, “The Spectrum of the Dirac Operator on Coset Spaces with Homogeneous Gauge Fields”, JHEP 0305 (2003) 018; [hep-th/0302122]