Adjoint Transform of Willmore Surfaces in $n$-sphere

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Abstract

After the surface theory of Möbius geometry, this study concerns a pair of conformally immersed surfaces in $n$-sphere. Two new invariants $\theta$ and $\rho$ associated with them are introduced as well as the notion of touch and co-touch. This approach is helpful in research about transforms of certain surface classes. As an application, we define adjoint transform for any given Willmore surface in $n$-sphere. It always exists locally (yet not unique in general) and generalizes known duality theorems of Willmore surfaces. This theory on surface pairs reaches its high point by a characterization of adjoint Willmore surfaces in terms of harmonic maps.

1 Introduction

One fascinating aspect of surface theory in differential geometry is the construction of various transforms preserving certain surface classes [4, 12, 17]. Many of them are classical and discovered more than one hundred years ago. Today geometers are still interested in such transforms, because they indicate a hidden symmetry for the surface class in concern, which shows the deeper connection with integrable systems [5, 9, 12, 17]. Conversely, if a surface class is associated with some integrable equation or enjoys similar features, one would expect to find such transforms.

In this work we are interested in construction of new transforms for Willmore surfaces in $S^n$. It has been shown that they allow a spectral transform similar to isothermic surfaces [8]. Both surface classes are Möbius invariant and connected by the classical Blaschke’s Problem ([12, Ch.3]). This problem requires to find two surfaces enveloping the same 2-sphere congruence and forming conformal correspondence. The non-trivial solutions in $S^3$ consist of Darboux pair of isothermic surfaces and dual Willmore surfaces. Since there already exists a rich transform theory of isothermic surfaces [5, 12], we are encouraged to find a parallel theory of Willmore surfaces.

This aim was partially achieved in [4]. For a Willmore surface in $\mathbb{H}P^1 \cong S^4$ they introduced three kinds of transforms: the forward and backward 1-step Bäcklund transform, which resemble the Christoffel transform of an
isothermic surface (also known as the dual isothermic surface); the forward and backward 2-step Bäcklund transform; the Darboux transform, which is described by a Riccati equation like the Darboux transform of isothermic surfaces \[13\].

Although the quaternionic setup provides new insight into the spinor representation of surfaces in 3- and 4-space, it does not apply to higher dimensional spaces. For this reason it is not realistic to generalize these transforms to \(S^n\) based on the original algebraic description. Therefore, we follow another line: For a pair of surfaces being certain transform to each other, characterize them by geometric conditions. One of such results is the well-known geometric characterization of Bäcklund transforms for pseudospherical surfaces \[18\, \text{Ch.6}\]. Another example is Blaschke's Problem and its solutions mentioned above. In view of such results, it is reasonable to establish a general theory on surface pairs at the beginning. That should be done in a Möbius invariant way in arbitrary dimensional space. We accomplished this task in the light-cone model, and derived two new invariants \(\theta, \rho\) associated with such a pair of oriented surfaces \[14\].

Another difficulty of establishing the transform theory of Willmore surfaces lies in the following fact: there exists no dual surface for a generic Willmore surface in \(S^n\). Notice that the dual isothermic surface is the basic transform for a given isothermic surface, on which the construction of all other transforms rely \[5, 13\]. So the failure of Bryant's duality theorem \(\[3\), see also \[1\] in higher codimension case is really a disappointment.

Thus it might be a surprise to the reader that there does exist a generalization of dual Willmore surface to \(S^n\), which is introduced in this paper and called the adjoint Willmore surface(s). We used the plural at here, because in general they are not unique. Yet such adjoint transforms always exist locally, including the dual S-Willmore surface as a special case. We show the adjoint transform always produces a Willmore surface \(f\) from a given one \(f : M \to S^n\), and \(f\) is also an adjoint transform of \(\hat{f}\) (Theorem 4.7). This generalizes the known duality theorems of Bryant \[3\], Ejiri \[10\] and that about the forward/backward 2-step Bäcklund transforms \[4\]. Indeed, the definition of the adjoint Willmore surface is inspired by the geometric characterization of the 2-step Bäcklund transform in terms of co-touching condition, the case in which our new invariant \(\theta = 0\).

In another viewpoint, a surface pair is just a (Riemann) surface mapped into the moduli space of point pairs in \(S^n\). This map is conformal exactly when \(\theta = 0\). Furthermore, if we ask what is a conformal harmonic map into this semi-Riemannian symmetric space, we find it must be given by a pair of adjoint Willmore surfaces (Theorem 4.8). This interesting result enjoys a similar flavor as the well-known characterization of Willmore surfaces by the harmonicity of its conformal Gauss map (see Theorem 2.6).

We make two remarks relating our new results to the Darboux transform of isothermic surfaces. First, the condition of touching \((\rho = 0)\), the coun-
terpart of co-touching, was originally introduced in [2], subsequently used to define the generalized Darboux transform for a generic surface in $S^4$, the moduli space of which gives the spectral curve. Next, it is worth mentioning that the symmetric space of point pairs has been considered for $S^3$ [7], $S^4$ [11] and $S^n$ [5, 12, Ch.8]. In these works, Darboux pair of isothermic surfaces is characterized in terms of curved flats in this symmetric space, which is a special kind of integrable system. Again we find some parallel between the theory of Willmore surfaces and isothermic surfaces.

In the following, we will briefly review the surface theory in Möbius differential geometry in Section 2 and some basic facts about Willmore surfaces. We go on to develop the theory on surface pairs in Section 3 with a detailed discussion on the meaning of touch and co-touch (the interpretation by quaternions is left to the Appendix). Finally, the definition of adjoint transform of a Willmore surface as well as its properties are presented in Section 4.

2 Surface theory in Möbius geometry

2.1 The surface theory by moving frames

In this paper we will follow [8] in their treatment of surface theory in Möbius geometry. As usual, let $\mathcal{L}$ denote the light cone in the $n + 2$ dimensional Minkowski space $\mathbb{R}^{n+1,1}$ with quadratic form $\langle y, y \rangle = -y_0^2 + \sum_{i=1}^{n+1} y_i^2$. Then the unit sphere $S^n(n \geq 3)$ in Euclidean space may be identified with our projectivized light cone:

$$S^n \cong \mathbb{P}(\mathcal{L}) : x \leftrightarrow [1 : x].$$

The projective action of the Lorentz group on $\mathbb{P}(\mathcal{L})$ yields a representation of the Möbius group. In this model, points are described by light-like vectors (null lines), and hyperspheres correspond to space-like vectors. Generally, a $k$-sphere $S \subset S^n$ is represented by space-like $(n-k)$-dim subspace $U$ (or the orthogonal complement $U^\perp$ equivalently).

For surface $f : M \to S^n \cong \mathbb{P}(\mathcal{L})$, a (local) lift of $f$ is just a map $F$ from $M$ into the light cone such that the null line spanned by $F(p)$ is $f(p)$. Two different local lifts differ by a scaling, so the metric induced from them are conformal to each other. When the underlying $M$ is a Riemann surface, $f$ is a conformal map iff $\langle F_z, F_{\bar{z}} \rangle = 0$ for any $F$ and any coordinate $z$ on $M$; it is immersion iff $\langle F_z, F_{\bar{z}} \rangle > 0$.

In our study we often associate a 2-sphere (congruence) to $f$ and be interested in the contact relationship. Let space-like $(n-2)$-dim subspace $U$ stand for such a 2-sphere, $F$ a lift of $f$. This sphere passes through $f(p)$ iff $\langle F(p), U \rangle = 0$. Suppose this is satisfied, then the sphere is tangent to $f$ at $p$ iff $\langle dF(p), U \rangle = 0$. Identify the 2-sphere with $U^\perp$, then it is tangent to $f$. 

at \( p \) iff \( F(p) \) (the map itself) and \( dF(p) \) (all tangent vectors) are contained in \( U^\perp \).

Given conformal immersion \( f : M \to \mathbb{S}^n \cong \mathbb{P}(\mathcal{L}) \) of Riemann surface \( M \) with local lift \( F \), there is a Möbius invariant decomposition \( M \times \mathbb{R}^{n+1,1} = V \oplus V^\perp \), where

\[
V = \text{Span}\{F, dF, F_{\bar{z}z}\}
\]

is a rank-4 subbundle defined via local lift \( F \) and complex coordinate \( z \) (one readily checks that \( V \) is independent to such choices, thus well-defined). \( V \) is a Lorentzian subbundle, and \( V^\perp \) is a space-like subbundle, which might be identified with the normal bundle of \( f \) in \( \mathbb{S}^n \). The connection \( D \) on \( V^\perp \) defined by orthogonal projection of the derivative in \( \mathbb{R}^{n+1,1} \) is the usual normal connection in metric geometry, which is already known to be Möbius invariant. On the other hand, \( V \) determines a Möbius invariant 2-sphere \( \mathbb{P}(V \cap \mathcal{L}) \) at every point of this immersed surface, we call it the mean curvature sphere or central sphere congruence. The complexification of \( V \) and \( V^\perp \) are denoted respectively as \( V_C, V^\perp_C \).

**Remark 2.1.** The name mean curvature sphere comes from the remarkable property that it is tangent to the surface and has the same mean curvature vector as the surface at the tangent point, where the ambient space is endowed with a metric of Euclidean space (or any space form).

Fix a local coordinate \( z \). Among various choice of local lifts there is a canonical one into the forward light cone, which is denoted by \( Y \). \( Y \) is Möbius invariant and determined by \( |dY|^2 = |dz|^2 \). We choose a Möbius invariant frame of \( V_C \) as \( \{Y, Y_z, Y_{\bar{z}}, N\} \). The real \( N \in \Gamma(V) \) is chosen so that these frame vectors are orthogonal to each other except \( \langle Y_z, Y_{\bar{z}} \rangle = \frac{1}{2}, \langle Y, N \rangle = -1 \). Such a light-like vector \( N \) is also unique.

Since \( Y_{zz} \) is orthogonal to \( Y, Y_z \) and \( Y_{\bar{z}} \), there must be a complex function \( s \) and a section \( \kappa \in \Gamma(V^\perp_C) \) so that the following Hill’s equation holds:

\[
Y_{zz} + \frac{s^2}{2} Y = \kappa.
\]

This defines two basic invariants \( \kappa \) and \( s \) depending on coordinate \( z \). \( \kappa \) may be identified with the normal valued Hopf differential up to a suitable scaling, meanwhile \( s \) is interpreted as the Schwarzian of immersion \( f \). In \( S^3 \), \( \kappa \) and \( s \) form a complete system of invariants. For more explanation, see [8].

Let \( \psi \in \Gamma(V^\perp) \) denote an arbitrary section of the normal bundle. Now it is easy to derive the structure equations:

\[
\begin{align*}
Y_{zz} &= -\frac{s}{2} Y + \kappa, \\
Y_{z\bar{z}} &= -(\kappa, \bar{\kappa}) Y + \frac{1}{2} N, \\
N_z &= -2(\kappa, \bar{\kappa}) Y_z - s Y_{\bar{z}} + 2D_{\bar{z}} \kappa, \\
\psi_z &= D_z \psi + 2(\psi, D_{\bar{z}} \kappa) Y - 2\langle \psi, \kappa \rangle Y_{\bar{z}}.
\end{align*}
\]

4
The computation is straightforward, hence omitted at here. The conformal Gauss, Codazzi and Ricci equations as integrable conditions are given as below:

\[
\frac{1}{2}s_{\bar{z}} = 3\langle D_{\bar{z}}\bar{\kappa}, \kappa \rangle + \langle \bar{\kappa}, D_{\bar{z}}\kappa \rangle,
\]

\[
\text{Im}\left(D_{\bar{z}}D_{\bar{z}}\kappa + \frac{s}{2}\kappa \right) = 0,
\]

\[
R_{\bar{z}z}^{D}\psi := D_{\bar{z}}D_{\bar{z}}\psi - D_{\bar{z}}D_{\bar{z}}\psi = 2\langle \psi, \kappa \rangle \bar{\kappa} - 2\langle \psi, \bar{\kappa} \rangle \kappa.
\]

### 2.2 Willmore functional and Willmore surfaces

**Definition 2.2.** For a conformally immersed surface \( f : M \to S^n \) with decomposition \( M \times \mathbb{R}^{n+1,1} = V \oplus V^\perp \) as before, we define

\[
G := Y \wedge Y_u \wedge Y_v \wedge N = -2i \cdot Y \wedge Y_{\bar{z}} \wedge Y_{\bar{z}} \wedge N, \quad z = u + iv.
\]

It is a map from \( M \) to the Grassmannian \( G_{3,1}(\mathbb{R}^{n+1,1}) \), called the conformal Gauss map of \( f \). This Grassmannian consists of all 4-dimensional Minkowski subspaces. A basic fact about this map is [10]

**Proposition 2.3.** For conformal immersion \( f : M \to S^n \), \( G \) induces a positive definite metric (by the usual inner product between multivectors)

\[
g = \frac{1}{4}(dG, dG) = \langle \kappa, \bar{\kappa} \rangle|dz|^2
\]

on \( M \) except at umbilic points, which is called the Möbius metric. Especially this is a conformal metric, thus justifies the name of conformal Gauss map.

**Definition 2.4.** The Willmore functional of \( f \) is defined at here as the area of \( M \) with respect to the Möbius metric:

\[
W(f) := \frac{i}{2} \int_M |\kappa|^2 dz \wedge d\bar{z}.
\]

**Definition 2.5.** Let \( M \) be a topological surface. Any immersion \( f : M \to S^n \) automatically induces a conformal structure over \( M \), hence defines the Willmore functional \( W(f) \). If \( f \) is a critical point of \( W \) with respect to any variations of the map and the induced conformal structures, it is called a Willmore surface.

For any conformal map \( G : M \to G_{3,1}(\mathbb{R}^{n+1,1}) \), the energy is \( E(G) := \int_M (dG \wedge *dG) \). The Willmore functional of a surface \( f \) is related to the energy of its conformal Gauss map via \( \tilde{W}(f) = -\frac{1}{8}E(G) \). Moreover, Willmore surfaces are characterized by the harmonicity of its conformal Gauss map.

\[
\text{In case of a surface in } \mathbb{R}^3, \text{ the Willmore functional is usually defined as } \tilde{W}(f) = \int_M (H^2 - K) dM. \text{ It differs from our definition by } \tilde{W}(f) = 4W(f).
\]
Theorem 2.6 ([3] [8] [10]). For a conformally immersed surface $f$ in $S^n$, the following three conditions are equivalent:

(i) $f$ is Willmore.

(ii) The Hopf differential and Schwarzian of $f$ satisfy

\[ D \bar{z} \bar{D}z \kappa + \frac{1}{2} \bar{s} \kappa = 0. \]  
(Willmore condition)

This is a condition stronger than the conformal Codazzi equation (3b).

(iii) The conformal Gauss map $G$ is a harmonic map into the Grassmannian $G_{3,1}(\mathbb{R}^{n+1,1})$.

Corollary 2.7. The integrability conditions for a Willmore surface is

\[
\begin{aligned}
\frac{1}{2} \bar{s} \bar{z} &= 3 \langle D \bar{z} \bar{k}, \kappa \rangle + \langle \bar{k}, D \bar{z} \kappa \rangle, \\
D \bar{z} \bar{D}z \kappa + \bar{s} \kappa &= 0, \\
R_{\bar{z}z}^D \psi &= 2 \langle \psi, \kappa \rangle \bar{k} - 2 \langle \psi, \bar{k} \rangle \kappa.
\end{aligned}
\]

This system admits the symmetry

\[ \kappa_\lambda = \lambda \kappa, \quad s_\lambda = s, \]

for unitary $\lambda \in S^1$, which describes the associated family of Willmore surfaces.

Remark 2.8. The characterization of Willmore surfaces in terms of \[4\] was given in \[3\] without proof. Note the equivalence between conditions (ii) and (iii) of Theorem 2.6 is well-known to experts in this field, from which (i) follows easily. Analogous results in Lie sphere geometry and projective geometry were discussed in \[6\].

3 Pair of conformally immersed surfaces

After the general surface theory, we turn to transforms for certain surfaces. Usually they are obtained from a given surface by some integrable equations. Alternatively, many times such transforms might be characterized by some geometric conditions on the surface pair involved. The second approach motivates us to build a general theory of surface pairs, which seems to be a natural development based on the previous section.
3.1 Basic invariants of surface pair

Let us start with a Riemann surface $M$ and two conformal immersions $f, \hat{f} : M \to S^n$ which are assumed to be always distinct. Given coordinate $z$, set $Y$ to be the canonical lift of $f$, with Schwarzian $s$, Hopf differential $\kappa$, and frame $\{Y, Y_z, Y_{\bar{z}}, N\}$. $\hat{Y}$ is a fixed local lift of $\hat{f}$ so that $\langle Y, \hat{Y} \rangle = -1$. We may express explicitly that $\hat{Y} = \lambda Y + \bar{\mu}Y_z + \mu Y_{\bar{z}} + N + \xi$, where $\lambda$ and $\mu$ are real-valued and complex-valued functions respectively, and the real $\xi \in \Gamma(V^\perp)$. Since $\hat{Y}$ is isotropic, there must be $\lambda = \frac{1}{2}(|\mu|^2 + \langle \xi, \xi \rangle)$. So we have $\hat{Y} = \frac{1}{2}(|\mu|^2 + \langle \xi, \xi \rangle)Y + \bar{\mu}Y_z + \mu Y_{\bar{z}} + N + \xi$, (5)

Take derivative on both sides. By (2) we may find the fundamental equation for such a surface pair after a straightforward computation:

$$\hat{Y}_z = \frac{\mu}{2} \hat{Y} + \theta \left( Y_z + \frac{\mu}{2} Y \right) + \rho \left( Y_z + \frac{\mu}{2} Y \right) + \langle \xi, \xi \rangle Y + \zeta,$$ (6)

where

$$\theta = \mu_z - \frac{1}{2} \mu^2 - s - 2\langle \xi, \kappa \rangle,$$ (7a)

$$\rho = \bar{\mu}_z - 2\langle \kappa, \bar{\kappa} \rangle + \frac{1}{2} \langle \xi, \xi \rangle,$$ (7b)

$$\zeta = D_z \xi - \frac{\mu}{2} \xi + 2 \left( D_z \kappa + \frac{\bar{\mu}}{2} \kappa \right) \in \Gamma(V^\perp_C).$$ (7c)

It is easy to check that $\theta$ and $\rho$ corresponds to a $(2, 0)$ form and a $(1, 1)$ form separately. They may be defined alternatively by the inner product between bivectors:

$$\theta = 2 \langle Y \wedge Y_z, \hat{Y} \wedge \hat{Y}_z \rangle,$$ (8a)

$$\rho = 2 \langle Y \wedge Y_z, \hat{Y} \wedge \hat{Y}_z \rangle.$$ (8b)

Although these expressions involve lifts $Y, \hat{Y}$ and coordinate $z$, both $\theta$ and $\rho$ are independent to such choices, hence well-defined invariants associated with such a pair of immersed surfaces. Note if we interchange between $Y$ and $\hat{Y}$, $\rho$ turns to be $\bar{\rho}$, and $\theta$ keeps invariant.

$Y \wedge Y_z$ and $\hat{Y} \wedge \hat{Y}_z$ may be interpreted as complex contact elements. So $\theta$ and $\rho$ are second order invariants of $(f, \hat{f})$. More concretely, a 2-dim contact element (always assumed to be oriented) is just a 2-dim oriented subspace in $T_pS^n$ for some $p \in S^n$, which corresponds to a 3-dim oriented subspace of signature $(2, 0)$ in $\mathbb{R}^{n+1,1}$. We represent this object by its oriented frame $\{X, X_1, X_2\}$ with scalar product matrix diag$(0, 1, 1)$. This determines (up to multiplication by an unitary complex number) the complex contact element
represented by $X \wedge (X_1 - iX_2)$. Its conjugate corresponds to the real contact element with reversed orientation.

Consider two contact elements $\Sigma = \{Y, Y_1, Y_2\}$ and $\widehat{\Sigma} = \{\widehat{Y}, \widehat{Y}_1, \widehat{Y}_2\}$ at distinct points (so $(Y, \widehat{Y}) \neq 0$). Similarly define

$$\theta = \frac{1}{2} \langle Y \wedge (Y_1 - iY_2), \widehat{Y} \wedge (\widehat{Y}_1 - i\widehat{Y}_2) \rangle, \tag{9a}$$

$$\rho = \frac{1}{2} \langle Y \wedge (Y_1 + iY_2), \widehat{Y} \wedge (\widehat{Y}_1 - i\widehat{Y}_2) \rangle. \tag{9b}$$

Note they are independent to the choice of frames.

When $(Y, \widehat{Y}) = 0$, we have two contact elements at the same point. Intuitively we need only to consider the 2-planes $\text{Span}\{Y_1, Y_2\}$ and $\text{Span}\{\widehat{Y}_1, \widehat{Y}_2\}$.

The following two quantities

$$\theta = \frac{1}{2} \langle Y_1 + iY_2, \widehat{Y}_1 - i\widehat{Y}_2 \rangle, \tag{10a}$$

$$\rho = \frac{1}{2} \langle Y_1 - iY_2, \widehat{Y}_1 - i\widehat{Y}_2 \rangle. \tag{10b}$$

are similarly well-defined, i.e. they are independent to the choice of frames of $\Sigma, \widehat{\Sigma}$. Compared to (9a) (9b), here the $\pm$ sign is reversed in two places. Why this convention will be clear in next subsection.

### 3.2 Touch and co-touch

To better understand the geometric meaning of $\theta$ and $\rho$ (as well as their counterparts $\widehat{\theta}, \widehat{\rho}$), let’s consider the special case when either of them vanishes.

**Definition 3.1.** Two contact elements $\Sigma$ and $\widehat{\Sigma}$ at one point are said to **touch** each other if $\rho = 0$ and **co-touch** each other if $\theta = 0$.

Consider two oriented surfaces immersed in $\mathbb{S}^n$ intersecting at $p$. We say they **touch** (co-touch) each other if the contact elements given by their tangent spaces at $p$ touch (co-touch).

**Example 3.2.** For two surfaces tangent to each other at the same point, it is easy to see that they either touch each other at this point when their orientations are compatible, or co-touch when the orientations are opposite.

**Example 3.3.** Given two complex lines in $\mathbb{C}^n, n \geq 2$. Regard them as real 2-planes with the induced orientation (via the complex structure) in $\mathbb{R}^{2n}$. Then they touch each other. In Appendix we will see that all touching 2-plane pairs are constructed in this way.

These examples show that the touching relation (including touch and co-touch) between two surfaces is a generalization of tangency. Such notions
were first introduced by Pedit and Pinkall in the context of quaternions \( \mathbb{H} \), then used to define Darboux transforms for general surfaces in \( S^4 \cong \mathbb{H} P^1 \), which generalize the classical Darboux transforms of isothermal surfaces [2, Section 7.1]. Simply speaking, for a surface immersed in \( \mathbb{H} \cong \mathbb{R}^4 \) one can define the left and right normal vector \( N, R \). Two surfaces having a common point \( p \) are said to left touch each other if they share the same left normal vector \( N \) at \( p \). Co-touching was defined similarly [15] when they have opposite \( N \) or \( R \). Detailed discussion is left to Appendix.

The key observation in [15] is: the touching relation between 2-planes depends only on the Euclidean geometry of \( \mathbb{R}^4 \), whereby independent to the quaternionic structure. Moreover, it might be defined in arbitrary dimensional space. This seems out of one’s expectation and calls for explanation. Note the usual Gauss map identifies a 2-plane in \( \mathbb{R}^n \) with a null line in \( \mathbb{Q} \subset \mathbb{C} \mathbb{P}^{n-1} \). For two such null lines \( l_1, l_2 \in \mathbb{Q} \), there are two noteworthy cases, i.e. when \( l_1 \) is orthogonal to \( l_2 \), or to \( l_2 \)’s conjugate. Either of these two cases corresponds to touch or co-touch between the 2-planes represented by \( l_1, l_2 \).

To clarify the geometric meaning of \( \theta = 0 \) and \( \rho = 0 \) for a pair of conformal immersions with lifts \( Y, \hat{Y} \), observe that given coordinate \( z = u + iv \), contact element \( \Sigma = \{Y, Y_u, Y_v\} \) at \( Y(p) \), and single point \( \hat{Y}(p) \), there is an unique oriented 2-sphere passing through \( Y(p), \hat{Y}(p) \) and tangent to \( Y \) with compatible orientation. It is given by the 4-dim subspace of signature \( (3,1) \) spanned by \( \{Y, Y_u, Y_v, \hat{Y}\} \), with the orientation fixed by the oriented contact element \( \Sigma = \{Y, Y_u, Y_v\} \) or the complexification \( Y \wedge Y \). Denote it as \( S(p) \). Now we may state

**Proposition 3.4.** Given two conformal immersions \( f, \hat{f} \), the invariant \( \rho(p) = 0 \) iff the 2-sphere \( S(p) \) touches \( \hat{f} \) at \( \hat{Y}(p) \), and \( \theta(p) = 0 \) iff \( S(p) \) co-touches \( \hat{Y} \) at \( \hat{f}(p) \).

**Proof.** We may take the normalized lifts \( Y, \hat{Y} \) as before. By [5],

\[
\hat{Y} = \frac{1}{2} \left( |\mu|^2 + \langle \xi, \xi \rangle \right) Y + \bar{\mu} Y_z + \mu Y_{\bar{z}} + N + \xi
\]

\(^2\)In other words, left-touching means the tangent planes of these surfaces at \( p \) can be transformed to each other by right-multiplication of a unit quaternion. Right touch is understood in the similar way.

\(^3\)For a pair of intersecting lines in \( \mathbb{R}^n \), the intersection angle is the only invariant in Euclidean geometry. More generally, for two oriented \( m \)-dim subspaces \( \Sigma_1, \Sigma_2 \subset \mathbb{R}^n \), the singular values of the inner product matrix between their oriented orthonormal frames are a complete system of invariants under the action of \( \text{SO}(n) \). They are independent to the choice of such frames, hence well-defined. In Möbius geometry we find the complete invariants associated with a pair of oriented contact elements at the same point in this way. When \( m = 2 \), suppose the singular values are \( \lambda_1, \lambda_2 \), then \( \Sigma_1 \) touch (co-touch) \( \Sigma_2 \) iff \( \lambda_1 = \lambda_2 \) (\( \lambda_1 = -\lambda_2 \)).
is orthogonal to $Y_z + \frac{\mu}{2}Y$. Note that under the reflection with respect to $Y - \hat{Y}$, $S(p)$ is invariant with reversed orientation, and the complex contact element $\Sigma = Y \wedge (Y_z + \frac{\mu}{2}Y)$ is mapped to $\hat{Y} \wedge (Y_z + \frac{\mu}{2}Y)$. Thus the complex contact element given by $S(p) = \text{Span}\{Y, Y_u, Y_v, \hat{Y}\}$ at $\hat{Y}(p)$ should be $\Sigma' = \hat{Y} \wedge (Y_z + \frac{\mu}{2}Y)$. On the other hand, the complex contact element given by immersion $\hat{Y}$ at $\hat{Y}(p)$ is $\hat{\Sigma} = \hat{Y} \wedge \hat{Y}_z$. Thus at $\hat{Y}(p)$ the invariants associated with $\Sigma'$ and $\hat{\Sigma}$ are computed by the fundamental equation (6):
\[
\theta = 2\langle Y_z + \frac{\mu}{2}Y, \hat{Y}_z \rangle = \theta, \quad \rho = 2\langle Y_z + \frac{\mu}{2}Y, \hat{Y}_z \rangle = \rho.
\]
The conclusion now follows from the definition of touch and co-touch.

4 Adjoint transforms of Willmore surfaces

4.1 Motivation and definition

After Bryant’s work [3], people are interested in the generalization of the duality theorem for Willmore surfaces in $S^n$. Ejiri pointed out that the duality theorem holds true only for a smaller class of Willmore surfaces, the so-called $S$-Willmore surfaces [10]. Although that, the hope to generalize the construction of dual Willmore surface still exists, according to the observations below:

1. In [14], as an application of our theory on surface pairs, Blaschke’s Problem and its solutions were generalized to $S^n$. Dual S-Willmore surfaces arises as the second class of non-trivial solutions, for which the invariant $\theta = 0$ (in the isothermic case, $\rho = 0$). The vanishing of $\theta$ has a nice geometric interpretation as co-touching in general case.

2. The forward and backward 2-step Bäcklund transforms of a Willmore surface in $S^4$ [14] are generalization of the duality theorem above, which might be called the left and right dual Willmore surface respectively. Like the dual Willmore surface in 3-dim case, the 2-step Bäcklund transform also falls on the mean curvature sphere $S$ of the given Willmore surface. But it only co-touches $S$ and not necessarily to be tangent.

Stimulated by these facts, one naturally attempts to characterize the left and right dual Willmore surface in $S^4$ by the geometric properties listed above. It yields a new class of transforms for any Willmore surface in $S^n$.

**Definition 4.1.** A map $\hat{f} : M \to S^n$ is called the adjoint transform of Willmore surface $f : M \to S^n$ if it is conformal and co-touches the mean curvature sphere of $f$ at corresponding point. Especially, $\hat{f}$ must locate on the corresponding mean curvature sphere of $f$. Note that $\hat{f}$ is allowed to be a degenerate point.
This definition gives the conditions characterizing an adjoint transform. Yet we need a more explicit description. Consider surface pair \( f, \hat{f} \) with adapted lifts \( Y, \hat{Y} \), satisfying \( \langle Y, \hat{Y} \rangle = -1 \). Furthermore suppose \( \hat{f} \) is on the mean curvature sphere of \( f \). Then equations (5)(6) take the form
\[
\hat{Y} = \frac{1}{2} |\mu|^2 Y + \bar{\mu} Y_z + \mu Y_z + N, \tag{11}
\]
\[
\hat{Y}_z = \frac{\mu}{2} \hat{Y} + \theta \left( Y_z + \frac{\bar{\mu}}{2} Y \right) + \rho \left( Y_z + \frac{\mu}{2} Y \right) + 2 \eta. \tag{12}
\]
Here \( \mu \) is a complex connection 1-form determined by \( \mu = 2 \langle \hat{Y}, Y_z \rangle \). It further defines those invariants associated with the pair \( f, \hat{f} \) as in Subsection 3.1:
\[
\theta := \frac{1}{2} \mu^2 - s, \quad \rho := \bar{\mu} z - 2 \langle \kappa, \bar{\kappa} \rangle, \quad \eta := D_z \kappa + \frac{\bar{\mu}}{2} \kappa. \tag{13}
\]
There follows
\[
\langle \hat{Y}_z, \hat{Y}_z \rangle = \langle \hat{Y}_z - \frac{\mu}{2} \hat{Y}, \hat{Y}_z - \frac{\bar{\mu}}{2} \hat{Y} \rangle = 4 \langle \eta, \eta \rangle + \theta \cdot \rho, \tag{14}
\]
\[
\langle \hat{Y}_z, \bar{\hat{Y}}_z \rangle = \langle \hat{Y}_z - \frac{\mu}{2} \hat{Y}, \bar{\hat{Y}}_z - \frac{\bar{\mu}}{2} \bar{\hat{Y}} \rangle = 4 \langle \eta, \bar{\eta} \rangle + \frac{1}{2} |\theta|^2 + \frac{1}{2} |\rho|^2. \tag{15}
\]
That \( f \) is Willmore implies
\[
0 = D_z D_z \kappa + \frac{\bar{s}}{2} \kappa = D_z (\eta - \frac{\bar{\mu}}{2} \kappa) + \frac{\bar{s}}{2} \kappa = D_z \eta - \frac{\bar{\mu}}{2} \eta - \frac{\bar{\theta}}{2} \kappa \tag{16}
\]

**Definition 4.2.** The map into \( S^n \) represented by (11) is an adjoint transform of Willmore surface \( Y \) iff \( \mu \) satisfies the following conditions:
\[
\text{Co-touching:} \quad 0 = \theta = \mu_z - \frac{1}{2} |\mu|^2 - s. \tag{17a}
\]
\[
\text{Conformality:} \quad 0 = \langle \eta, \eta \rangle = \langle D_z \kappa + \frac{\bar{\mu}}{2} \kappa, D_z \kappa + \frac{\mu}{2} \kappa \rangle. \tag{17b}
\]

**Example 4.3.** A Willmore surface \( f \) is a S-Willmore surface if \( D_z \kappa \) linearly depends on \( \kappa \). In such a case there exist a function \( \mu \) locally so that \( D_z \kappa + \frac{\bar{\mu}}{2} \kappa = 0 \) when there is no umbilic points. It is easy to check that (17) holds for this \( \mu \), which gives the dual Willmore surface \( \hat{f} \) via (11).

**Remark 4.4.** It is easy to show that \( \mu \) is a connection 1-form of \( K^{-1} \), where \( K \) denotes the canonical bundle of Riemann surface \( M \). Conversely, given any connection 1-form of \( K^{-1} \), if it satisfies (17) with respect to any local coordinate, then it defines an adjoint transform globally.

**Remark 4.5.** The reader should be aware of the problem of singularities. First, the map underlying \( \hat{Y} \) may not be immersion when \( \sigma = 0 \). Thus \( \hat{Y} \) as well as the underlying map \( \hat{f} : M \to S^n \) might has branch points.
Next, the connection 1-form $\mu dz$ may have poles, which corresponds to the coincidence case of $f$ and $\hat{f}$. In this paper we will concentrate on the local aspect of this construction, and ignore this problem temporarily. But when deal with closed Willmore surfaces, this is an inevitable problem related to both global and local geometry.

In [15], this adjoint transform is applied to the study of Willmore 2-spheres in $S^n$, which yields very strong vanishing results. Despite this success as well as the removable singularity theorem utilised there, these singularities still constitute the final obstruction to a complete classification. If we can have a better understanding of them, the known classification results of Willmore 2-spheres [3, 10, 16] might be generalized to $S^n$ based on [15].

4.2 Existence

Our definition of adjoint transforms leads to the natural problem of existence and uniqueness of solutions to system (17a)(17b). Note that when $\langle \kappa, \kappa \rangle \neq 0$, (17b) is a quadratic equation about $\mu$ and much easier to solve. In such a situation, at every point we have two roots for

$$0 = \langle \eta, \eta \rangle = \langle Dz \kappa + \frac{\mu}{2} \kappa, Dz \kappa + \frac{\mu}{2} \kappa \rangle.$$ 

Fix either of such a root $\mu$ and differentiate this equation. By [16],

$$0 = \langle \eta, \eta \rangle z = 2 \langle Dz \eta, \eta \rangle = 2 \langle \frac{\mu}{2} \eta + \frac{\theta}{2} \kappa, \eta \rangle = \bar{\theta} \langle \kappa, \eta \rangle.$$ 

If $\langle \kappa, \eta \rangle \neq 0$, we have $\theta = 0$ as desired. Otherwise, suppose $\langle \kappa, \eta \rangle = 0$ on an open subset and take derivative, one obtains

$$0 = \langle \kappa, \eta \rangle z = \langle Dz \kappa, \eta \rangle + \langle \kappa, Dz \eta \rangle = \langle \eta - \frac{\mu}{2} \kappa, \eta \rangle + \langle \kappa, \frac{\mu}{2} \eta + \frac{\theta}{2} \kappa \rangle = \bar{\theta} \langle \kappa, \kappa \rangle.$$ 

By assumption, $\langle \kappa, \kappa \rangle \neq 0$, so $\theta = 0$. Hence we see that the Willmore condition [16] guarantees a solution $\mu$ of (17) and the existence of adjoint transforms.

How about the case when $\langle \kappa, \kappa \rangle = 0$ on an open subset? By Willmore condition [11] it follows $0 = \langle Dz \kappa, \kappa \rangle = \langle Dz \kappa, Dz \kappa \rangle$. That means (17b) holds automatically for any $\mu$. So we need only to solve the PDE (17a)

$$\mu z - \frac{1}{2} \mu^2 - s = 0$$

independently. It is a Riccati equation about $\mu$ with respect to the given Schwarzian $s$. In S-Willmore case this is solved in Example 4.3. When
immersion \( f : M \to S^n \) is Willmore but not S-Willmore, the Willmore condition \( (14) \) implies that \( \kappa \) and \( D_\bar{z}\kappa \) span a rank 2 holomorphic subbundle of \( V_\mathbb{C}^+ \). By \( (16) \), it is easy to show that there is a 1-1 correspondence between solution \( \mu \) and holomorphic line subbundle spanned by \( D_\bar{z}\kappa + \bar{\mu} \). So there are infinitely many solutions \( \mu \).

**Remark 4.6.** Generally speaking, \( (17a) \) is an under-determined equation, thus admit (infinitely) many solutions. More concretely, there is a well known correspondence between the solutions of Riccati equation \( \mu_z - \frac{1}{2}\mu^2 - s = 0 \) and the solutions to linear equation

\[
y_{zz} + \frac{s}{2}y = 0. \tag{18}
\]

Suppose of \( (18) \) and they are independent, i.e. there exist no anti-holomorphic function \( h \) so that \( \hat{y} = h \cdot y \). The general solution to \( (18) \) is a combination \( hy + \tilde{h}y \), where \( y, \tilde{y} \) are two independent non-trivial solutions, \( h, \tilde{h} \) are all anti-holomorphic. In case that only one non-trivial solution \( y \) is given, the second solution \( \tilde{y} \) might be found by solving a \( \partial \)-problem for \( \lambda : \lambda_z = 1/y^2 \), then \( \tilde{y} = Ay \). This implies that even for a S-Willmore surface with \( \langle \kappa,\kappa \rangle \equiv 0 \), locally there are still infinitely many adjoint transforms.

### 4.3 Duality theorem

In this subsection, we want to prove that the adjoint transform preserve the Willmore condition. Fix the original Willmore surface with lift \( Y \). Assume there is a \( \mu \) solving \( (17a) \) and \( (17b) \), which defines an adjoint transform \( \hat{f} \). Therefore, \( (12) \) is simplified to

\[
\hat{Y}_z = \frac{\mu}{2}\hat{Y} + \rho \left( Y_z + \frac{\mu}{2}Y \right) + 2\eta. \tag{19}
\]

Note \( \theta = 0 \) also implies

\[
\rho = \bar{\mu}z - 2\langle \kappa,\bar{\kappa} \rangle \hat{z} = \bar{s}z + \bar{\mu}\bar{z} - 2\langle \kappa,\bar{\kappa} \rangle = \bar{\mu} + 4\langle \eta,\bar{\kappa} \rangle \tag{20}
\]

by Gauss equation \( (3a) \), and

\[
D_\bar{z}\eta = \frac{\bar{\mu}}{2}\eta \tag{21}
\]

by \( (13) \). Next consider the canonical lift of the adjoint transform, denoted as \( \tilde{Y} \). Let \( \langle \tilde{Y},Y \rangle = -1/\sigma \) be a real function defined on \( M \). Equivalently speaking, \( \tilde{Y} \) is obtained from \( \hat{Y} \) via \( \tilde{Y} = \frac{1}{\sigma}\hat{Y} \). So \( \frac{1}{2} = \langle \tilde{Y}_z,\tilde{Y} \rangle = \frac{1}{\sigma^2}\langle \hat{Y}_z,\hat{Y} \rangle \). Combined with \( (15) \) and \( \theta = 0 \), we get

\[
\sigma^2 = 2\langle \hat{Y}_z,\hat{Y} \rangle = 8\langle \eta,\bar{\eta} \rangle + |\rho|^2. \tag{22}
\]

\(^4\)This is true at least on the open subset where \( \kappa \wedge D_\bar{z}\kappa \neq 0 \). Then this subbundle extends smoothly to zeros of \( \kappa \wedge D_\bar{z}\kappa \). Note \( V_\mathbb{C}^+ \) is a holomorphic bundle w.r.t. \( D_\bar{z} \).
Thus (19) may be written as
\[ \tilde{Y}_z = -\frac{\tilde{\mu}}{2} \tilde{Y} + \frac{\rho}{\sigma} \left( Y_z + \frac{\mu}{2} Y \right) + \frac{2}{\sigma} \eta, \tag{23} \]
where
\[ \tilde{\mu} := \frac{2\sigma z}{\sigma} - \mu. \tag{24} \]
To find \( \tilde{N} \) we should calculate \( \tilde{Y}_z \bar{z} \). It is easy to find
\[ \eta \bar{z} = 2\langle \eta, \bar{\eta} \rangle Y - 2\langle \eta, \bar{\kappa} \rangle \left( Y_z + \frac{\mu}{2} Y \right) + \frac{\tilde{\mu}}{2} \eta, \tag{25a} \]
\[ \eta_z = -2\langle \eta, \kappa \rangle \left( Y_z + \frac{\mu}{2} Y \right) + D_z \eta. \tag{25b} \]
Differentiate both sides of (23). A straightforward computation yields
\[ \tilde{Y}_z \bar{z} = \frac{1}{2} \left( \rho - \tilde{\mu} \bar{z} \right) \tilde{Y} + \frac{1}{2} \left( -\frac{1}{2} \tilde{\mu}^2 \tilde{Z} - \tilde{\mu} \bar{Y}_z - \tilde{\mu} \bar{Y}_z + \sigma Y \right). \]
Define
\[ \tilde{N} := -\frac{1}{2} |\tilde{\mu}|^2 \bar{Z} - \tilde{\mu} \bar{Y}_z - \tilde{\mu} \bar{Y}_z + \sigma Y. \tag{26} \]
We verify \( \langle \tilde{N}, \tilde{Y} \rangle = 0, \langle \tilde{N}, \tilde{Y} \rangle = -1, \langle \tilde{N}, \tilde{N} \rangle = 0 \). So \( \{ \tilde{Y}, \tilde{Y}_z, \tilde{Y}_\bar{z}, \tilde{N} \} \) is the canonical frame of \( \tilde{Y} \) as desired. Compare the structure equation (of \( \tilde{Y} \))
\[ \tilde{Y}_z \bar{z} = -\langle \kappa, \bar{\kappa} \rangle \tilde{Y} + \frac{1}{2} \tilde{N} \]
with previous result, we may similarly define
\[ \tilde{\rho} := \tilde{\mu}_z - 2\langle \kappa, \bar{\kappa} \rangle. \tag{27} \]
Then there must be
\[ \tilde{\rho} = \tilde{\rho}. \tag{28} \]
How about the corresponding invariants \( \kappa \) and \( s \)? According to structure equations of \( \tilde{Y} \), \( s \) is determined by \( \kappa = \bar{Y}_z + \frac{s}{2} \bar{Y} \in \tilde{V}_z \), where \( \tilde{V} := \text{Span}\{ \tilde{Y}, \tilde{Y}_z, \tilde{Y}_\bar{z}, \tilde{Y}_zz \} = \text{Span}\{ \tilde{Y}, \tilde{Y}_z, \tilde{Y}_\bar{z}, Y \} \) by (26). Since \( \tilde{Y}_zz \) and \( \tilde{Y} \) are always orthogonal to \( \tilde{Y}, \tilde{Y}_z, \tilde{Y}_\bar{z} \), We find \( s \) by solving
\[ 0 = \langle \tilde{Y}_zz + \frac{s}{2} \tilde{Y}, Y \rangle = \langle \tilde{Y}_z, Y \rangle z - \langle \tilde{Y}_z, \bar{Y}_z \rangle - \frac{s}{2\sigma} \left( \tilde{\mu}_z - \frac{1}{2} \tilde{\mu}^2 - \tilde{s} \right). \]
Therefore
\[ \tilde{s} = \tilde{\mu}_z - \frac{1}{2} \tilde{\mu}^2. \tag{29} \]
Denote the normal connection of \( \tilde{Y} \) as \( \bar{D} \). We have structure equation
\[ 2\bar{D}_z \kappa = \tilde{N}_z + 2\langle \kappa, \bar{\kappa} \rangle \tilde{Y}_z + \tilde{s} \tilde{Y}_z. \tag{30} \]
Differentiate (30) and modulo components of $\hat{V}$, which is spanned by $\{\hat{Y}, \hat{Y}_z, \hat{Y}_{zz}, Y\}$, one obtains

$$2(\hat{D}_z \tilde{k})_z + \tilde{s} \tilde{k} \equiv \hat{N}_{zz} + \hat{s} \hat{Y}_{zz} + \tilde{s} \tilde{k}$$

$$\equiv \left(-\frac{1}{2} |\mu|^2 \hat{Y} - \tilde{\mu} \hat{Y}_z - \tilde{\mu} \hat{Y}_{zz} + \sigma Y\right)_{zz} + \hat{s} \hat{Y}_{zz} + \tilde{s} \tilde{k}$$

$$\equiv (\tilde{s} - \tilde{\mu}_{zz}) \hat{Y}_{zz} - \tilde{\mu}_z \hat{Y}_z - \tilde{\mu} \hat{Y}_{zz} - \tilde{\mu} \hat{Y}_{zz} + (\sigma Y)_{zz} + \tilde{s} \tilde{k}$$

$$\equiv (\tilde{s} - \tilde{\mu}_z) \tilde{k} + (\tilde{s} - \tilde{\mu}_z) \tilde{k} - \tilde{\mu} \tilde{k} - \tilde{\mu} \tilde{k} + (\sigma Y)_{zz}$$

$$\equiv -\frac{1}{2} \mu^2 \hat{k} - \frac{1}{2} \hat{\mu}^2 \hat{k} - \tilde{\mu} \hat{D}_z \tilde{k} - \tilde{\mu} \tilde{D}_z \tilde{k} + (\sigma Y)_{zz}$$

$$\equiv -\frac{1}{2} \mu^2 \hat{k} - \frac{1}{2} \hat{\mu}^2 \hat{k} - \tilde{\mu} \hat{N}_z - \tilde{\mu} \tilde{N}_z + (\sigma Y)_{zz}$$

$$\equiv -\frac{1}{2} \mu^2 \hat{k} - \frac{1}{2} \hat{\mu}^2 \hat{k} - \tilde{\mu} \left[ -\tilde{\mu} \hat{Y}_{zz} + (\sigma Y)_{zz} \right]$$

$$\equiv -\frac{\hat{\mu}}{2} (\sigma Y)_z - \frac{\mu}{2} (\sigma Y)_z + (\sigma Y)_{zz}$$

$$\equiv \left(\sigma_z - \frac{\mu}{2} \sigma\right) \hat{Y}_z + \left(\sigma_z - \frac{\mu}{2} \sigma\right) Y_z + \sigma \cdot \frac{1}{2} N$$

$$\equiv \frac{\sigma}{2} (\mu \hat{Y}_z + \tilde{\mu} \hat{Y}_z + \tilde{N})$$

$$\equiv \frac{\sigma}{2} \left( \sigma \hat{Y} - \frac{1}{2} |\mu|^2 \hat{Y} \right)$$

$$\equiv 0.$$ 

Thus we have proved that the Willmore condition (16) is also satisfied for $\hat{Y}$. Furthermore, equation (29) shows that $\hat{Y}$ may be viewed as an adjoint transform of $Y$, because $\hat{\mu}$ satisfies (29), which amounts to say that the similarly defined quantity $\hat{\theta}$ also vanishes, meanwhile we already know the conformality between $\hat{Y}$ and $Y$. This remarkable duality is just what one expected, since such a relationship between a Willmore surface in $\mathbb{S}^4$ and its forward/backward two-step Bäcklund transforms (I) is already known. Sum together, we get

**Theorem 4.7.** An adjoint transform $\hat{Y}$ of a Willmore surface $Y$ is also Willmore, which is called an adjoint Willmore surface of $Y$ or a Willmore surface adjoint to $Y$. Vice versa, $Y$ is also an adjoint transform of $\hat{Y}$. The relationship between their corresponding invariants are given by

$$-\frac{1}{\sigma} = (\tilde{\theta}, Y), \quad \frac{2\sigma_z}{\sigma} = \hat{\mu} + \mu, \quad \hat{\rho} = \rho.$$
4.4 Characterization by conformal harmonic maps

In the last section, we have developed a theory of pairs of conformally immersed surfaces \( f, \hat{f} \) from a Riemann surface \( M \) into \( \mathbb{S}^n \). Another way to look at them is considering the 2-plane spanned by their lifts \( Y, \hat{Y} \). This defines a map

\[
H : M \to G_{1,1}(\mathbb{R}^{n+1,1}),
\]

\[
p \mapsto Y(p) \wedge \hat{Y}(p).
\]

Similar to the description of the conformal Gauss map, here the Grassmannian \( G_{1,1}(\mathbb{R}^{n+1,1}) \) consists of all 2-dim Minkowski subspaces, and we regard it as a submanifold embedded in \( \wedge^2 \mathbb{R}^{n+1,1} \). The bivector is uniquely determined if we put the restriction \( \langle Y, \hat{Y} \rangle = -1 \) (hence \( \langle H, H \rangle = -1 \)). Conversely, such a map corresponds to a pair of surfaces in \( \mathbb{S}^n \).

Associated with \( Y, \hat{Y} \) are invariants \( \theta, \rho \) defined via (7). They appear also as invariants of \( H \). It turns out

\[
\langle H_z, H_z \rangle = \theta, \quad \langle H_z, \bar{H}_z \rangle = \frac{1}{2} (\rho + \bar{\rho}).
\]

\[
\implies \langle dH, dH \rangle = \theta dz^2 + \frac{1}{2} (\rho + \bar{\rho}) (dz d\bar{z} + d\bar{z} dz) + \bar{\theta} d\bar{z}^2. \tag{31}
\]

So the co-touching condition is equivalent to the conformality of \( H \).

On the other hand, (31) gives the energy of \( H \):

\[
E(H) := \int_M \langle dH \wedge *dH \rangle = -i \int_M (\rho + \bar{\rho}) \cdot dz \wedge d\bar{z}
\]

Now comes another natural question: What is the condition that \( H \) being conformal harmonic? Note that \( H \) is similar to the conformal Gauss map in that each of them is into some Grassmannian associated with \( \mathbb{R}^{n+1,1} \). The latter being harmonic iff the original surface is Willmore (Theorem 2.6). By analogy one would expect some similar result for \( H \). Of course we should assume that the underlying maps \( f, \hat{f} \) are also conformal. Surprisingly, these simple conditions give a nice characterization of adjoint Willmore surfaces.

**Theorem 4.8.** Let \( M \) be a Riemann surface. Assume \( Y, \hat{Y} \) are local lifts of immersions \( f, \hat{f} : M \to \mathbb{S}^n \) satisfying \( \langle Y, \hat{Y} \rangle = -1 \), which induce map \( H = Y \wedge \hat{Y} : M \to G_{1,1}(\mathbb{R}^{n+1,1}) \). Then the three conditions below are equivalent:

(i) \( f, \hat{f} \) are two Willmore surfaces adjoint to each other.

(ii) \( f, \hat{f} \) and \( H \) are conformal maps, and \( f, \hat{f} \) locate on the mean curvature sphere of each other.

\(^5\)In the computation, without loss of generality we may assume \( Y \) is the canonical lift of \( f \), and using the formulae in Subsection 3.1.
(iii) \( f, \hat{f} \) are conformal to each other and \( H = Y \wedge \hat{Y} \) is conformal harmonic.

**Proof.** Choose \( Y, \hat{Y} \) as in Subsection 3.1, with
\[
\hat{Y} = \frac{1}{2} \left( |\mu|^2 + \langle \xi, \xi \rangle \right) Y + \hat{\mu} Y_z + \mu Y_{\bar{z}} + N + \xi,
\]
\[
\hat{Y}_z = \frac{\mu}{2} \hat{Y} + \theta \left( Y_z + \frac{\mu}{2} Y \right) + \rho \left( Y_z + \frac{\mu}{2} Y \right) + \langle \xi, \xi \rangle Y + \xi,
\]
where \( \theta, \rho, \xi \) are associated invariants given in (7). Let \( H_t = Y_t \wedge \hat{Y}_t \) be a variation of \( H = H_0 = Y \wedge \hat{Y} \), so that
\[
\langle Y_t, Y_t \rangle = \langle \hat{Y}_t, \hat{Y}_t \rangle = 0, \quad \langle Y_t, \hat{Y}_t \rangle = -1, \quad \Rightarrow \quad \langle H_t, H_t \rangle = -1.
\]
We abbreviate \( \frac{d}{dt} \bigg|_{t=0} \) by a dot. The only restrictions on the variational vector field \( \dot{H} \), or equivalently on \( \dot{Y}, \dot{\hat{Y}} \), are (\( \langle \dot{H}, H \rangle = 0 \))
\[
\langle \dot{Y}, Y \rangle = \langle \dot{\hat{Y}}, \hat{Y} \rangle = \langle \dot{Y}, \hat{Y} \rangle + \langle Y, \dot{\hat{Y}} \rangle = 0. \tag{32}
\]
The first variation of the energy of \( H_t \) is:
\[
\frac{d}{dt} \bigg|_{t=0} E(H_t) = -2 \int_M \langle \dot{H}, d \ast dH \rangle = 4i \int_M \langle \dot{H}, H_{zz} \rangle \cdot dz \wedge d\bar{z}.
\]
So \( H \) is conformal harmonic iff \( \theta = 0 \) and \( \langle \dot{H}, H_{zz} \rangle = 0, \quad \forall \dot{H} \).

First we show (iii) \( \Rightarrow \) (ii). Take special variational vector fields
\[
\dot{Y} = 0, \quad \dot{\hat{Y}} = \langle \xi, \xi \rangle Y + \xi.
\]
It is easy to verify that they satisfy (32) by checking \( \langle \dot{\hat{Y}}, Y \rangle = 0 = \langle \dot{Y}, \hat{Y} \rangle \).

Computation shows
\[
\langle \dot{H}, H_{zz} \rangle = \frac{1}{2} \langle \dot{\hat{Y}}, \mu Y_z + \mu Y_{\bar{z}} + N \rangle = \frac{1}{2} \langle \dot{\hat{Y}}, \hat{Y} - \frac{1}{2} (|\mu|^2 + \langle \xi, \xi \rangle) Y - \xi \rangle = -\frac{1}{2} \langle \xi, \xi \rangle.
\]
Since the restriction of the Minkowski metric on the M"obius normal bundle \( V^\perp \) is positive definite, \( E = 0 \) implies \( \xi = 0 \), i.e. \( \hat{f} \) is on the mean curvature sphere of \( f \). But there is no bias for \( f \) or \( \hat{f} \) in the assumptions, so these two surfaces should be dual to each other. Hence \( f \) is also on the mean curvature sphere of \( \hat{f} \).

Next we prove (ii) \( \Rightarrow \) (i). With \( \xi = 0, \theta = 0 \) we have the simplified formulae below:
\[
\hat{Y} = \frac{1}{2} \left( |\mu|^2 + \langle \xi, \xi \rangle \right) Y + \hat{\mu} Y_z + \mu Y_{\bar{z}} + N,
\]
\[
\hat{Y}_z = \frac{\mu}{2} \hat{Y} + \rho \left( Y_z + \frac{\mu}{2} Y \right) + 2\eta,
\]

\[\begin{array}{l}
\text{\footnotesize \( \ddagger \)}\text{Intuitively, this is because the expression of } \rho \text{ contains the term } \langle \xi, \xi \rangle, \text{ hence } \xi \text{ must vanish if the integral of } \rho + \hat{\rho} \text{ is critical.}
\end{array}\]
where
\[ \rho := \bar{\mu}_z - 2\langle \kappa, \bar{\kappa} \rangle, \quad \eta := D_z \kappa + \frac{\bar{\mu}}{2} \kappa. \]

As in last subsection, \( \theta := \mu_z - \frac{1}{2}\mu^2 - s = 0 \) further implies
\[ \rho \bar{z} = \bar{\mu}_z + 4\langle \eta, \bar{\kappa} \rangle \]
by Gauss equation (3a), and
\[ D_z \eta - \frac{\bar{\mu}}{2} \eta = D_z D_z \kappa + \frac{s}{2} \kappa, \]
which is real-valued by Codazzi equation (3b). Also note
\[ \eta \bar{z} = D_z \eta + 2\langle \eta, \bar{\eta} \rangle Y - 2\langle \eta, \bar{\kappa} \rangle \left( Y_z + \frac{\mu}{2} Y \right) \]
by (2). Now the differentiation of \( \hat{Y}_z \) can be computed out with the outcome
\begin{equation}
\hat{Y}_{zz} = \frac{\mu}{2} \hat{Y}_z + \frac{\mu_z}{2} \hat{Y}_z + \left( \frac{\mu_z}{2} + \frac{\rho}{2} - \frac{|\mu|^2}{4} \right) \hat{Y} + \left( \frac{1}{2} |\rho|^2 + 4\langle \eta, \bar{\eta} \rangle \right) Y + 2 \left( D_z \eta - \frac{\bar{\mu}}{2} \eta \right). \tag{33}
\end{equation}
Since \( \hat{f} \) is also on the mean curvature sphere of \( f \), \( Y \) is a linear combination of \( \{ \hat{Y}, \hat{Y}_z, \hat{Y}_z, \hat{Y}_{zz} \} \).

The \( \hat{Y}_{zz} \)-component of \( Y \) is not zero. (Otherwise \( Y \) is a combination of \( \hat{Y}, \hat{Y}_z, \hat{Y}_{zz} \), hence \( \langle Y, \hat{Y} \rangle = 0 \), a contradiction.) So \( \hat{Y}_{zz} \), as well as \( D_z \eta - \frac{\bar{\mu}}{2} \eta \), is contained in \( \text{Span}\{ \hat{Y}, \hat{Y}_z, \hat{Y}_{zz}, Y \} \). By the expressions of \( \hat{Y}, \hat{Y}_z \), this is true only if \( 0 = D_z \eta - \frac{\bar{\mu}}{2} \eta = D_z D_z \kappa + \frac{s}{2} \kappa \), i.e. \( f \) is Willmore. Again by the duality between \( f \) and \( \hat{f} \) we know \( \hat{f} \) is also Willmore. The assumptions directly imply that they form adjoint transform to each other.

Finally one should verify (i) \( \Rightarrow \) (iii). This case \( \theta = 0, \xi = 0, \ D_z \eta - \frac{\bar{\mu}}{2} \eta = 0 \), and (33) takes the following form:
\[ \hat{Y}_{zz} = \frac{\mu}{2} \hat{Y}_z + \frac{\mu_z}{2} \hat{Y}_z + \left( \frac{\mu_z}{2} + \frac{\rho}{2} - \frac{|\mu|^2}{4} \right) \hat{Y} + \left( \frac{1}{2} |\rho|^2 + 4\langle \eta, \bar{\eta} \rangle \right) Y. \]
We compute \( \langle \hat{H}, H_{zz} \rangle \) for arbitrary \( \hat{H} \), or equivalently, for any variational
vector fields $\dot{Y}, \ddot{Y}$. Invoking the restrictions (32), there follows

\[
\langle \dot{H}, H \bar{z} \rangle = \langle \dot{Y}, \ddot{Y} \bar{z} - \frac{\mu}{2} \dot{Y} \bar{z} - \frac{\bar{\mu}}{2} \dot{Y} \bar{z} + \left( \frac{|\mu|^2}{4} - \langle \kappa, \bar{\kappa} \rangle \right) \dot{Y} \rangle \\
+ \langle \ddot{Y}, \frac{\mu}{2} Y + \frac{\bar{\mu}}{2} Y + \frac{1}{2} N - \left( \langle Y, \ddot{Y} \bar{z} \rangle + \langle \kappa, \bar{\kappa} \rangle Y \right) \rangle \\
= \langle \dot{Y}, \left( \frac{\mu}{2} + \frac{\rho}{2} - \langle \kappa, \bar{\kappa} \rangle \right) \ddot{Y} + \left( \frac{1}{2} |\rho|^2 + 4 \langle \eta, \bar{\eta} \rangle \right) Y \rangle \\
+ \langle \ddot{Y}, \frac{1}{2} \dot{Y} - \frac{|\mu|^2}{4} Y + \left( \frac{\mu}{2} + \frac{\rho}{2} + \frac{|\mu|^2}{4} - \langle \kappa, \bar{\kappa} \rangle \right) Y \rangle \\
= 0.
\]

So $H$ is harmonic. This completes our proof. □

Remark 4.9. Indeed, our discussion above provides another proof to the Duality Theorem 4.7.

Remark 4.10. As Burstall pointed out to the author, one striking feature of the theorem above is that the condition (iii) is a collection of first and second order conditions, yet they force a fourth order equation (condition (i), that $f, \bar{f}$ being Willmore surfaces).

Appendix

In this part we want to discuss the relationship of left/right touching between 2-planes in the setup of quaternions, then point out the connection with the general notion of touch and co-touch. Our starting point is the following lemma.

Lemma A-1 (the Fundamental Lemma in [4]; also Lemma 6 in [2]).

For every oriented real subspace $U \subset \mathbb{H}$ of dimension 2 there are unique vectors $N, R$ satisfying $N^2 = R^2 = -1$ and

\[
U = \{ x \in \mathbb{H} \mid Nx = -xR \}.
\]

Conversely, every pair of vectors $N$ and $R$ satisfying $N^2 = R^2 = -1$ defines, via (34), an oriented 2-plane.

$N$ and $R$ are called the left and right normal vector of $U$ respectively, though in general they are not orthogonal to $U$. For an oriented immersed surface $M$ in $\mathbb{H}$, we can define the pair $\{N, R\}$ similarly, which might be identified with the usual Gauss map of $M$ in $\mathbb{R}^4$.

Definition A-2. Let $U_i$ be oriented 2-plane with $N_i$ and $R_i$ as their left and right normal vectors respectively, $i = 1, 2$. Then
1. \( U_1 \) and \( U_2 \) touch each other from left (right) if \( N_1 = N_2 \) (\( R_1 = R_2 \)).

2. \( U_1 \) and \( U_2 \) co-touch each other from left (right) if \( N_1 = -N_2 \) (\( R_1 = -R_2 \)).

Similarly, we can define (co-)touch of two conformal immersions at their intersection point. When touch is both from left and right, these two immersions are tangent at the intersection point with the same induced orientation. So left/right touch may be viewed as the generalization of tangency.

Given two oriented 2-planes in an oriented 4-space, this definition seems algebraic and depending on the way in which we identify \( \mathbb{R}^4 \simeq \mathbb{H} \). Yet by the following two lemmas, we find they are well-defined geometric notions (depending only on different choices of orientations).

**Lemma A-3.** Every orientation preserving linear isometry of \( \mathbb{H} \) is of the form \( x \in \mathbb{H} \mapsto \mu x \lambda \). Here \( \mu, \lambda \in S^3 \) are unit quaternions.

**Lemma A-4.**

(i) Every orientation preserving linear isometry of \( \mathbb{H} \) leaves the relationship of left (co-)touch and right (co-)touch invariant.

(ii) Every orientation reversing linear isometry of \( \mathbb{H} \) preserves the property of touch and co-touch, but interchanges between left and right.

(iii) Suppose \( U_1 \) touches (co-touches) \( U_2 \) from left/right. Then \( U_1 \) with opposite orientation co-touches (touches) \( U_2 \) from left/right respectively.

The second lemma is easy to obtain as a corollary of the first one, which is a well-known fact and we omit both proofs at here. They tell us that the difference between left and right is due to the orientation induced by the identification \( \mathbb{R}^4 = \mathbb{H} \), hence not essential. The proposition below confirms this observation, and unifies two different definitions of touch and co-touch.

**Proposition A-5.** Let \( U \) and \( \hat{U} \) be a pair of oriented 2-dim subspaces in \( \mathbb{R}^4 \). They (co-)touch each other as contact elements if, and only if, they (co-)touch each other from left or right. (Whether it is from left or right depends on the orientation induced by the identification \( \mathbb{R}^4 = \mathbb{H} \).)

**Proof.** Equipped \( U, \hat{U} \) with oriented orthonormal basis \( \{\alpha, \beta\} \) and \( \{\hat{\alpha}, \hat{\beta}\} \) respectively. Regarding \( U \) as a conformally embedded submanifold of \( \mathbb{R}^4 \subset S^4 \), we fix a lift \( U \subset \mathbb{R}^4 \to \mathbb{R}^5 \) as

\[
\begin{align*}
v \in U \mapsto \left( \frac{1}{2} (1 + |v|^2), \frac{1}{2} (1 - |v|^2), v \right),
\end{align*}
\]

which projects down to \( \mathbb{P}(\mathcal{L}) \). The image of \( 0 \in \mathbb{R}^4 \) is \( Y = (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0) \). The induced contact element at 0 is given by \( \Sigma = \{Y, Y_1, Y_2\} \), where

\[
Y_1 = (0, 0, \alpha), \quad Y_2 = (0, 0, \beta).
\]
We have similar representation $\hat{\Sigma} = \{Y, \hat{Y}_1, \hat{Y}_2\}$ for $\hat{U}$. Now consider the invariant $\rho$ associated with $\Sigma, \hat{\Sigma}$. By definition (10),

$$\rho = \frac{1}{2} \langle \alpha - i\beta, \hat{\alpha} - i\hat{\beta} \rangle = 0 \iff \begin{cases} \langle \alpha, \hat{\alpha} \rangle = \langle \beta, \hat{\beta} \rangle, \\ \langle \alpha, \hat{\beta} \rangle = -\langle \beta, \hat{\alpha} \rangle. \end{cases}$$

We define a complex structure $J$ on $\mathbb{R}^4$ via $J\{\alpha, \beta, \hat{\alpha}, \hat{\beta}\} = \{\beta, -\alpha, \hat{\beta}, -\hat{\alpha}\}$. In this term the condition of touch holds if, and only if, there is $J$ satisfying the formula above and compatible with the Euclidean metric. By Lemma A-3, it is easy to show that such a complex structure must be of the form $\alpha \mapsto N\alpha$ or $\alpha \mapsto \alpha R$, where $N, R \in \mathbb{H}, N^2 = R^2 = -1$. This implies our conclusion on touch. For co-touch the similar argument applies. □

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