Sombor index of directed graphs

Roberto Cruz, Juan Monsalve, Juan Rada *

Instituto de Matemáticas, Universidad de Antioquia, Medellín, Colombia

A R T I C L E   I N F O

Keywords:
Sombor index
Digraphs
Connected digraphs
Strongly connected digraphs
Orientations

A B S T R A C T

Let $D$ be a digraph with set of arcs $A$. The Sombor index of $D$ is defined as

$$\text{SO}(D) = \frac{1}{2} \sum_{\{a \in A\}} \sqrt{d^+_a + d^-_a},$$

where $d^+_a$ and $d^-_a$ are the out-degree and in-degree of the vertices $u$ and $v$ of $D$. When $D$ is a graph, we recover the Sombor index of graphs, a molecular descriptor recently introduced with a good predictive potential and a great research activity this year. In this paper we initiate the study of the Sombor index of digraphs. Specifically, we find sharp upper and lower bounds for $\text{SO}$ over the class $D_n$ of digraphs with $n$ non-isolated vertices, the classes $C_n$ and $S_n$ of connected and strongly connected digraphs on $n$ vertices, respectively, the class of oriented trees $OT(n)$ with $n$ vertices, and the class $\mathcal{O}(G)$ of orientations of a fixed graph $G$.

1. Introduction

A digraph $D$ is an ordered tuple $(V, A)$, where $V$ is a nonempty set, called the set of vertices of $D$, and $A$ is a subset of the set of ordered pairs of different vertices of $D$. The elements of $A$ are called arcs. If $a = (u, v)$ is an arc of $D$ then we simply write $a = uv$ and say that $u$ is adjacent to $v$ or $v$ is adjacent from $u$. The outdegree $d^+_a$ of a vertex $u$ of $D$ is the number of vertices of $D$ that are adjacent from $u$. The indegree $d^-_a$ of $u$ is the number of vertices of $D$ adjacent to $u$. If $u$ is a vertex of $D$ such that $d^+_u = 0$, then $u$ is called a sink vertex of $D$ whereas if $d^-_u = 0$, then $u$ is called a source vertex of $D$. We denote by $p = p(D)$ the number of sink vertices and $q = q(D)$ the number of source vertices of $D$. If $d^+_u = d^-_u = 0$, then $u$ is an isolated vertex. The set of digraphs with $n$ non-isolated vertices is denoted by $D_n$.

A digraph $D$ is symmetric if $uv \in A$ implies $vu \in A$, where $u, v \in V$. There is a natural one-to-one correspondence between the symmetric digraphs and the set of graphs. Thus, the set $G_n$ of graphs with $n$ non-isolated vertices satisfies $G_n \subseteq D_n$. On the other side, a digraph $D$ is called an asymmetric digraph or an oriented graph if whenever $uv \in A$ then $vu \notin A$. Consequently, an oriented graph $G$ can be obtained from a graph $G$ by assigning a direction to each edge of $G$. $D$ is called an orientation of $G$. For instance, in Fig. 1 we show the directed path $P_n$ and the directed cycle $C_n$, which are orientations of the path $P_n$ and the cycle $C_n$, respectively. If $D$ is an orientation of $G$ such that each vertex of $D$ is a sink vertex or a source vertex, then $D$ is called a sink-source orientation of $G$. The digraphs $\overline{K}_{n-1}$ and $\overline{C}_{n-1}$ in Fig. 2 are sink-source orientations of the star $S_n$.

Let $D \in D_n$. Given $1 \leq i, j \leq n - 1$, consider the set

$$A_{ij} = \{uv \in A : d^+_u = i \text{ and } d^-_v = j\}.$$

If $uv \in A_{ij}$ then we say $uv$ is a $(i,j)$-arc. Let $a_{ij}$ be the cardinality of $A_{ij}$ and define

---

* Corresponding author.

E-mail address: pablo.rada@udea.edu.co (J. Rada).

https://doi.org/10.1016/j.heliyon.2022.e09035
Received 21 November 2021; Received in revised form 13 January 2022; Accepted 25 February 2022

2405-8440/© 2022 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).
\( p_{ij} = a_{ij} + a_{ji} \),
for all \( 1 \leq i, j \leq n - 1 \), and
\( p_{ii} = a_{ii} \),
for all \( i = 1, \ldots, n - 1 \). The Sombor index of \( D \) is defined as \([12]\)
\[
SO(D) = \frac{1}{2} \sum_{(i,j) \in K} p_{ij} \sqrt{i^2 + j^2},
\]
(1)
where
\( K = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq j \leq n-1 \} \).
Equivalently,
\[
SO(D) = \frac{1}{2} \sum_{v \in V} \sqrt{(d^+_v)^2 + (d^-_v)^2}.
\]
(2)
The formulas (1) and (2) are particular cases of the general theory of vertex-degree-based topological indices over digraphs \([12, 13]\). When \( D \) is a graph, we recover the recently introduced Sombor index of graphs [7], a molecular descriptor with a good predictive potential [16] and with a great research activity this year \([1, 2, 3, 5, 8, 9, 10, 11, 15, 17]\).

In this paper we initiate the study of the Sombor index of digraphs. Specifically, we find sharp upper and lower bounds for \( SO \) over the class \( D_n \) of digraphs with \( n \) non-isolated vertices, the classes \( C_n \) and \( S_n \) of connected and strongly connected digraphs on \( n \) vertices, respectively, the class of oriented trees \( O\mathcal{T}(n) \) with \( n \) vertices, and the class \( \mathcal{O}(G) \) of orientations of a fixed graph \( G \) with non-isolated vertices.

2. Upper and lower bounds of the Sombor index over significant classes of digraphs

We begin with the study of \( SO \) over the set \( D_n \) of digraphs with \( n \) non-isolated vertices. Let \( D_1 = (V_1, A_1) \) and \( D_2 = (V_2, A_2) \) be digraphs with no common vertices. The digraph \( D = (V_1 \cup V_2, A_1 \cup A_2) \) is called the direct sum of \( D_1 \) and \( D_2 \) and denoted by \( D_1 \oplus D_2 \). In general, \( \bigoplus_{i=1}^k D_i \) denotes the direct sum of the digraphs \( D_1 = (V_1, A_1), \ldots, D_k = (V_k, A_k) \) and \( kD \) denotes the direct sum of \( k \) copies of digraph \( D \).

**Theorem 2.1.** Let \( D \in D_n \) with \( p \) sink vertices and \( q \) source vertices. Then
\[
\frac{\sqrt{2}}{4} (2n - p - q) \leq SO(D) \leq \frac{\sqrt{2}}{4} (n-1)^2 (2n - p - q).
\]
Equality on the left occurs if and only if
\[
D = \bigoplus_{i=1}^{k_1} P_{n_i} \oplus \bigoplus_{j=1}^{k_2} \overline{C}_{n_j},
\]
for some nonnegative integers \( k_1 \) and \( k_2 \). Equality on the right occurs if and only if \( D = K_n \), the complete graph on \( n \) vertices.

**Proof.** Consider the function \( f(x, y) = \frac{\sqrt{x^2 + y^2}}{x + y} \) defined over the compact set
\( \hat{K} = \{(x,y) \in \mathbb{R} \times \mathbb{R} : 1 \leq x \leq y \leq n-1 \} \).
Since
\[
\frac{\partial f(x, y)}{\partial x} = \frac{y (x^3 + 2x^2 + y^3)}{\sqrt{x^2 + y^2} (x + y)^2} > 0
\]
and
\[
\frac{\partial f(x, y)}{\partial y} = \frac{x (x^3 + 2x^2 + y^3)}{\sqrt{x^2 + y^2} (x + y)^2} > 0
\]
for all \((x, y) \in \hat{K}\), we conclude that
\[
\max_{(x,y) \in \hat{K}} f(x, y) = f(n-1, n-1) = \frac{\sqrt{2}}{2} (n-1)^2,
\]
and
\[
\min_{(x,y) \in \hat{K}} f(x, y) = f(1, 1) = \frac{\sqrt{2}}{2}.
\]
It follows from \([13, \text{Theorem 1}]\) that
\[
\frac{2n - p - q}{2\sqrt{2}} \leq SO(D) \leq \frac{(n-1)^2 (2n - p - q)}{2\sqrt{2}}
\]
(3)
Moreover, equality on the left of (3) occurs if and only if \( p_{ij} = 0 \) for all \((1, i) \neq (j, i) \in K\). Since the only connected digraphs such that every arc is a \((1,1)\)-arc are the directed paths and the directed cycles, this is equivalent to
\[
D = \bigoplus_{i=1}^{k_1} P_{n_i} \oplus \bigoplus_{j=1}^{k_2} \overline{C}_{n_j},
\]
for some nonnegative integers \( k_1 \) and \( k_2 \). The equality on the right of (3) occurs if and only if \( p_i = 0 \) for all \((n-1, n-1) \neq (i, j) \in K\), or equivalently, \( D = K_n \).

**Corollary 2.2.** Let \( D \in D_n \). Then
\[
\frac{\sqrt{2}}{4} n \leq SO(D) \leq \frac{\sqrt{2}}{4} (n-1)^2 n.
\]
Equality on the left occurs if and only if \( n \) is even and \( D = \frac{n}{2} \overline{P}_2 \). Equality on the right occurs if and only if \( D = K_n \).

**Proof.** By Theorem 2.1, bearing in mind that \( 0 \leq p + q \leq n \),
\[
\frac{\sqrt{2}}{4} n \leq \frac{\sqrt{2}}{4} (2n - p - q) \leq SO(D)
\]
and
\[
SO(D) \leq \frac{\sqrt{2}}{4} (n-1)^2 (2n - p - q)
\]
\[
\leq \frac{\sqrt{2}}{4} (n-1)^2 n.
\]
(5)
If \( SO(D) = \frac{\sqrt{2}}{4} (n-1)^2 n \) then, by (5), \( p + q = 0 \) and again, by Theorem 2.1, \( D = K_n \). Conversely,
\[
SO(K_n) = \frac{1}{2} n (n-1) \sqrt{(n-1)^2 + (n-1)^2}
\]
\[
= \frac{\sqrt{2}}{2} n (n-1)^2.
\]
On the other side, assume that \( SO(D) = \frac{\sqrt{2}}{4} n \). Then \( p + q = n \) by (4) and by Theorem 2.1, \( n \) is even and \( D = \frac{n}{2} \overline{P}_2 \), since every vertex of \( D \) is a sink vertex or a source vertex. Conversely, if \( n \) is even then
\[
SO \left( \frac{n}{2} \overline{P}_2 \right) = \frac{n}{2} SO \left( \overline{P}_2 \right) = \frac{n}{4} \sqrt{2}.
\]
**Corollary 2.3.** Let \( n \) be odd and \( D \in D_n \). Then
\[
\frac{\sqrt{2}}{4} (n+1) \leq SO(D).
\]
Equality holds if and only if \( D = \frac{n+3}{2} \overline{P}_2 \oplus \overline{P}_3 \).

**Proof.** Let \( p \) and \( q \) be the number of sink and source vertices of \( D \), respectively and assume first that \( p+q \leq n-1 \). Then, by Theorem 2.1,
\[
SO(D) \geq \frac{\sqrt{2}}{4} (2n - p - q) \geq \frac{\sqrt{2}}{4} (n+1).
\]
(6)
Now assume that \( p + q = n \). Let \( P \) be the set of sink vertices of \( D \) and \( Q \) the set of source vertices of \( D \). Then \( V = P \cup Q \) and

\[
2a = \sum_{v \in P} d_v^s + \sum_{v \in Q} d_v^s \geq p + q = n.
\]

Hence \( a \geq \frac{n+1}{2} \) since \( n \) is odd. Now, clearly \( (d_v^s)^2 + (d_v^s)^2 \geq 2 \) for all \( uv \in A \), consequently,

\[
SO(D) = \frac{1}{2} \sum_{u,v \in A} \sqrt{(d_u^s)^2 + (d_v^s)^2} \geq \frac{1}{2} \sqrt{2a} \geq \frac{\sqrt{2}}{4} (n+1). \tag{7}
\]

For the second part, assume that \( \sqrt{\frac{2}{n}} (n+1) = SO(D) \). If \( p + q \leq n-1 \) then, by (6) and Theorem 2.1, \( p_j = 0 \) for all \((i,j) \neq (i,j) \in K \) and \( p + q = n-1 \). It follows from [13, Lemma 2 (6)] that \( D = \frac{n-1}{2} \overline{P}_2 \oplus \overline{P}_1 \).

If \( p + q = n \) then, by (7), \( d_v^s = d_v^s = 1 \) for all \( uv \in A \). It follows that \( D = \bigoplus_{j=1}^n \overline{P}_2 \oplus \bigoplus_{j=1}^n \overline{C}_n \). But then \( D = \frac{n}{2} \overline{P}_2 \), since every vertex of \( D \) is sink or source, and this is a contradiction because \( n \) is odd.

Conversely,

\[
SO \left( \frac{n-3}{2} \overline{P}_2 \oplus \overline{P}_1 \right) = \frac{1}{2} \left( \frac{n-3}{2} \sqrt{\frac{2}{n}} + \frac{2 \sqrt{2}}{n} \right)
\]

\[
= \frac{\sqrt{2}}{4} (n+1). \tag{8}
\]

Let us consider other interesting classes of digraphs. Recall that a digraph is connected if its underlying graph is connected. Moreover, a digraph is strongly connected if there is a directed path between any pair of vertices. We denote by \( C_n \) the set of connected digraphs with \( n \) vertices and \( S_n \) the set of strongly connected digraphs with \( n \) vertices. Clearly, \( S_n \subseteq C_n \subseteq D_n \).

**Theorem 2.4.** Let \( D \in C_n \). Then

\[
\sqrt{\frac{2}{n}} (n-1) \leq SO(D) \leq \sqrt{\frac{2}{n}} (n-1)^2 n. \tag{9}
\]

Equality in the left occurs if and only if \( D = \overline{P}_n \). Equality on the right occurs if and only if \( D = K_n \).

**Proof.** The right part of (9) is a direct application of Corollary 2.2, because \( C_n \subseteq D_n \) and the complete graph \( K_n \in C_n \).

Consider the function \( g(x,y) = \sqrt{x^2 + y^2} \) defined over the compact set

\[
\tilde{K} = \{(x,y) \in \mathbb{R} \times \mathbb{R} : 1 \leq x \leq y \leq n-1 \}
\]

Since

\[
\frac{\partial g(x,y)}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} > 0,
\]

and

\[
\frac{\partial g(x,y)}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} > 0,
\]

for all \((x,y) \in \tilde{K}\), we conclude that

\[
\min_{(x,y) \in \tilde{K}} g(x,y) = g(1,1) = \sqrt{2}.
\]

Let \( a \) be the number of arcs of \( D \). Since \( D \) is connected, \( a \geq n-1 \). It follows from [13, Theorem 2] that

\[
SO(D) \geq \frac{1}{2} a \min_{(x,y) \in K} g(x,y)
\]

\[
= \frac{\sqrt{2}}{2} a \geq \frac{\sqrt{2}}{2} (n-1). \tag{10}
\]

By (9) and [13, Theorem 2], \( SO(D) = \frac{\sqrt{2}}{2} (n-1) \) if and only if \( a = n-1 \) and \( a_{ij} = 0 \) for all \((i,j) \neq (i,j) \in K \) such that \((i,j) \neq (1,1) \). This is equivalent to \( D = \overline{P}_n \).

**Theorem 2.5.** Let \( D \in S_n \). Then

\[
\sqrt{\frac{2}{n}} n \leq SO(D) \leq \sqrt{\frac{2}{n}} (n-1)^2 n.
\]

Equality in the left occurs if and only if \( D = \overline{P}_n \). Equality on the right occurs if and only if \( D = K_n \).

**Proof.** The right part of (10) is a direct application of Corollary 2.2, because \( S_n \subseteq D_n \) and the complete graph \( K_n \in S_n \).

Since \( D \) is strongly connected, \( d_v^s \geq 1 \) for all vertex \( u \) of \( D \). Consequently,

\[
a = \sum_{u \in V} d_v^s \geq n. \tag{11}
\]

Now by [13, Theorem 2] and (11),

\[
SO(D) \geq \frac{1}{2} a \min_{(x,y) \in K} g(x,y) = \frac{\sqrt{2}}{2} a \geq \frac{\sqrt{2}}{2} n.
\]

Moreover, by (12) and [13, Theorem 2], \( SO(D) = \frac{\sqrt{2}}{2} n \) if and only if \( a = n \) and \( a_{ij} = 0 \) for all \((i,j) \in K \) such that \((i,j) \neq (1,1) \). This is equivalent to \( D = \overline{P}_n \).

Let us denote by \( \Theta(G) \) the set of all orientations of the graph \( G \). One natural problem is to determine the extremal values of the Sombor index in \( \Theta(G) \). Let us consider the partial order over \( K \) defined as follows: if \((i,j),(k,l) \in K \), then

\[
(i,j) \leq (k,l) \iff i \leq k \text{ and } j \leq l.
\]

Note that if \((i,j) \leq (k,l) \), then

\[
\sqrt{i^2 + j^2} \leq \sqrt{k^2 + l^2}.
\]

Moreover, if \((i,j) \leq (k,l) \) and \( \sqrt{i^2 + j^2} = \sqrt{k^2 + l^2} \), then \((i,j) = (k,l) \). In other words, \( \Theta \) is a strictly nondecreasing VDB topological index as defined in [13, Definition 2].

**Theorem 2.6.** Let \( G \) be a graph and \( D \) any orientation of \( G \). Then

\[
SO(D) \leq \frac{1}{2} \Theta(G).
\]

Equality holds if and only if \( D \) is a sink-source orientation of \( G \).

**Proof.** This is a direct application of [15, Theorem 5] bearing in mind that \( \Theta \) is a strictly nondecreasing VDB topological index.

Note that a graph \( G \) has a sink-source orientation if and only if \( G \) is a bipartite graph [14, Proposition 2.2]. A consequence of Theorem 2.6 is that among all orientations of a fixed bipartite graph \( G \), the orientation with maximal Sombor index is a sink-source orientation of \( G \). Indeed, let \( E \) be a sink-source orientation of \( G \) and \( D \) any orientation of \( G \). Then

\[
SO(D) \leq \frac{1}{2} \Theta(G) = \Theta(E).
\]

What can we say about the minimal value of \( SO \) in \( \Theta(G) \)? For a general bipartite graph \( G \) this is apparently a difficult problem. However, in our next results we will analyze the case when \( G \) is the path \( P_n \) or the cycle \( C_n \) on \( n \) vertices.

**Theorem 2.7.** Let \( D \) be any orientation of \( P_n \). Then

\[
SO(\overline{P}_n) \leq SO(D) \leq SO(\overline{P}_n),
\]

where \( \overline{P}_n \) is any of the sink-source orientations of \( P_n \) shown in Fig. 3.
Proof. This is a consequence of Theorem 2.6 and Theorem 2.4. □

In order to study the orientations of the cycle $C_n$, we say that $D$ is a quasi-sink-source orientation of $C_n$ if all vertices are sink or source vertices, except for one. In this case, $n$ is odd, $(p_{11}, p_{12}, p_{22}) = (0, 2, n - 2)$, and
\[
SO(D) = \frac{1}{2} \left(2\sqrt{5} + (n - 2)\sqrt{2}\right) = \sqrt{5} + (n - 2)\sqrt{2}.
\]

Theorem 2.8. Let $C_n$ be the cycle on $n$ vertices. Let $D$ be any orientation of $C_n$. Then
\[
\frac{\sqrt{2}}{2} n \leq SO(D) \leq \begin{cases} 
\sqrt{2}n & \text{if } n \text{ is even} \\
\sqrt{3} + (n - 2)\sqrt{2} & \text{if } n \text{ is odd}.
\end{cases}
\]

Equality in the left occurs if and only if $D = \overline{C}_n$. Equality in the right occurs if and only if $D$ is a sink-source (resp. quasi sink-source) orientation when $n$ is even (resp. when $n$ is odd).

Proof. Let $g(x, y) = \sqrt{x^2 + y^2}$ defined over $K$. Then $\min_{(x, y) \in K} g(x, y) = \sqrt{2}$ (see proof of Theorem 2.4). By [13, Theorem 2],
\[
\frac{\sqrt{2}}{2} n \leq SO(D).
\]

Moreover, $\sqrt{2}n = SO(D)$ if and only if all $a_{ij} = 0$ for all $(i, j) \in K$ such that $(i, j) \neq (1, 1)$, which is equivalent to $D = \overline{C}_n$.

On the other hand, assume first that $n$ is even. Then $C_n$ is a bipartite graph and so by Theorem 2.6, if $E$ is a sink-source orientation of $C_n$ (all arcs of $E$ are $(2, 2)$-arcs), then
\[
SO(D) \leq SO(E) = \frac{n}{2} \sqrt{8} = \sqrt{2}n.
\]

Moreover, by (13), $SO(D) = \frac{\sqrt{2}}{2} n$ if and only if $D$ is a sink-source orientation of $C_n$.

Now assume that $n$ is odd. We affirm that for every orientation $D$ of $C_n$,
\[
\frac{\sqrt{2}}{2} p_{11} + \left(\sqrt{2} - \frac{\sqrt{2}}{2}\right) p_{12} \geq \sqrt{2} - \sqrt{5}.
\]

In fact, (14) only fails when $p_{11} = p_{12} = 0$ or when $p_{11} = 0$ and $p_{12} = 1$. In the first case, since
\[
p_{11} + p_{12} + p_{22} = a = n,
\]

we deduce that $(p_{11}, p_{12}, p_{22}) = (0, 0, n)$ and so by [12, Lemma 5.2], $D$ is a sink-source orientation of $C_n$. But this is not possible because $n$ is odd. In the second case we have $(p_{11}, p_{12}, p_{22}) = (1, 0, n - 1)$, which is not possible by [12, Lemma 5.2]. It follows from (14) that
\[
SO(D) = \frac{1}{2} \left(p_{11}\sqrt{2} + p_{12}\sqrt{5} + 2\sqrt{2}p_{22}\right) \leq \sqrt{5} + (p_{11} + p_{12} + p_{22} - 2)\sqrt{2} = \sqrt{5} + (n - 2)\sqrt{2}.
\]

If $SO(D) = \sqrt{5} + (n - 2)\sqrt{2}$, then by (15) and (14) we deduce
\[
\sqrt{2} p_{11} + \left(\sqrt{2} - \frac{\sqrt{2}}{2}\right) p_{12} = 2\sqrt{2} - \sqrt{5}.
\]

Equivalently,
\[
\sqrt{2} (p_{11} + 2p_{12} - 4) = \sqrt{5} (p_{11} - 2).
\]

Hence $p_{11} - 2 = 0$ and $p_{11} + 2p_{12} - 4 = 0$ which implies $p_{11} = 2$ and $p_{12} = 0$. In other words, $(p_{11}, p_{12}, p_{22}) = (0, 2, n - 2)$. It follows from [12, Lemma 5.2] that $D$ is a quasi-sink-source orientation. The converse is clear. □

In Theorems 2.7 and 2.8 the extremal value problem of $SO$ over the orientations of the path $P_n$ and the cycle $C_n$ were solved. Another interesting problem would be to find the extremal values of $SO$ over the set of orientations of the complete graph $K_n$.

Problem 2.9. Find the extremal values of $SO$ among all tournaments of $n$ vertices.

Let $\mathcal{O}T(n)$ denote the set of all oriented trees with $n$ vertices.

Theorem 2.10. Let $T \in \mathcal{O}T(n)$. Then
\[
\frac{\sqrt{2}}{2} (n - 1) \leq SO(T) \leq \frac{1}{2} \sqrt{(n - 1)^2 - 2n + 2}.
\]

Equality on the left occurs if and only if $T = \overline{P}_n$. Equality on the right occurs if and only if $T = K_{n-1,1}$ or $T = \overline{K}_{1,n-1}$.

Proof. The left part of (16) is a direct application of Theorem 2.4, because $\mathcal{O}T(n) \subseteq \overline{C}_n$ and $\overline{P}_n \in \mathcal{O}T(n)$.

Let $T \in \mathcal{O}T(n)$. Assume that $T$ is an orientation of a tree $G$ order $n$. Then by [7, Theorem 3] and Theorem 2.6,
\[
SO(T) \leq \frac{1}{2} SO(G) \leq \frac{1}{2} SO(S_n) = \frac{1}{2} (n - 1) \sqrt{n^2 - 2n + 2}.
\]

By [7, Theorem 3], Theorem 2.6, and (17), equality occurs if and only if $G = S_n$ and $T$ is a sink-source orientation of $G$, and this is equivalent to $T = \overline{K}_{n-1,1}$ or $T = \overline{K}_{1,n-1}$.

3. Conclusion

In 2021, Ivan Gutman proposed a new topological index on graphs called the Sombor index [7], which gave rise to numerous publications on its mathematical properties and chemical applications [8], a clear indicator of the quality of the index. On the other hand, motivated by the important role of the directed graphs in real-world situations, for instance, chemical and biological networks, transportation networks, social networks, just to mention a few [4, 6], the concept of vertex-degree-based topological indices was recently introduced in [12]. Bearing in mind these ideas, we initiated the study of the Sombor index of directed graphs. Concretely, we determined extremal values of the Sombor index over significant classes of digraphs, and characterized the digraphs were these extremal values are attained.
Declarations

Author contribution statement

Roberto Cruz; Juan Monsalve; Juan Rada: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data availability statement

No data was used for the research described in the article.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

References

[1] S. Alikhani, N. Ghanbari, Sombor index of polymers, MATCH Commun. Math. Comput. Chem. 86 (2021) 715–728.
[2] R. Cruz, I. Gutman, J. Rada, Sombor index of chemical graphs, Appl. Math. Comput. 399 (2021) 126018.
[3] K.C. Das, A.S. Cevik, I.N. Cangul, On Sombor index, Symmetry 13 (2021) 140.
[4] M. Dehmer, Z. Chen, F. Emmert-Streib, S. Tripathi, A. Mowshowitz, A. Levitski, L. Feng, Y. Shi, J. Tao, Measuring the complexity of directed graphs: a polynomial-based approach, PLoS ONE 14 (2019) e0223745.
[5] D. Deng, Z. Tang, R. Wu, Molecular trees with extremal values of Sombor indices, Int. J. Quant. Chem. 121 (2021) e26622.
[6] E. Estrada, N. Hatano, Returnability in complex directed networks (digraphs), Linear Algebra Appl. 430 (2009) 1886–1896.
[7] I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, MATCH Commun. Math. Comput. Chem. 86 (2021) 11–16.
[8] I. Gutman, Sombor index - one year later, Bull. Cl. Sci. Math. Nat. Sci. Math. 45 (2020) 43–55.
[9] B. Horoldagava, C. Xu, On Sombor index of graphs, MATCH Commun. Math. Comput. Chem. 86 (2021) 703–713.
[10] H. Liu, L. You, Z. Tang, J.B. Liu, On the reduced Sombor index and its applications, MATCH Commun. Math. Comput. Chem. 86 (2021) 729–733.
[11] H. Liu, H. Chen, Q. Xiao, X. Fang, Z. Tang, More on Sombor indices of chemical graphs and their applications to the boiling point of benzenoid hydrocarbons, Int. J. Quant. Chem. 121 (2021) e26689.
[12] J. Monsalve, J. Rada, Vertex-degree-based topological indices of digraphs, Discrete Appl. Math. 295 (2021) 13–24.
[13] J. Monsalve, J. Rada, Sharp upper and lower bounds of VDB topological indices of digraphs, Symmetry 13 (2021) 1093.
[14] J. Monsalve, J. Rada, Oriented bipartite graphs with minimal trace norm, Linear Multilinear Algebra 67 (2019) 1121–1131.
[15] J. Rada, J.M. Rodríguez, J.M. Sigarreta, General properties on Sombor indices, Discrete Appl. Math. 299 (2021) 87–97.
[16] I. Redžepović, Chemical applicability of Sombor indices, J. Serb. Chem. Soc. 86 (2021) 445–457.
[17] Z. Wang, Y. Mao, Y. Li, B. Furtula, On relations between Sombor and other degree-based indices, J. Appl. Math. Comput. (2021).