Nonlinear constitutive relations for anisotropic elastic materials

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Abstract. A general approach to constructing of nonlinear variants of connection between stresses and strains in anisotropic materials with different types of symmetry of properties is considered. This approach is based on the concept of elastic proper subspaces of anisotropic materials introduced in the mechanics of solids by J. Rychlewski and on the particular postulate of isotropy proposed by A. A. Il'yushin.

The generalization of the particular postulate on the case of nonlinear anisotropic materials is formulated. Systems of invariants of deformations as lengths of projections of the strain vector into proper subspaces are developed.

Some variants of nonlinear constitutive relations for anisotropic materials are offered. The analysis of these relations from the point of view of their satisfaction to general and limit forms of generalization of partial isotropy postulate on anisotropic materials is performed. The relations for particular cases of anisotropy are written.

1. Introduction
Anisotropic materials characterized by nonlinear behavior in the process of deformation are increasingly becoming the subject of the mechanics of deformable solids. Such materials can include various kinds of tissues of various types, blood vessels and bones of living organisms as well as composite materials widely used in engineering. Models of elastic and inelastic behavior of such materials were represented in the works [1, 2, 3, 4, 5, 6, 7, 8]. Finite deformations of anisotropic materials were considered in the issues [2, 8, 9, 10, 11].

The main problem of constructing of relations which determine the behavior of anisotropic material is the choice of a system of invariants of a strain tensor which characterize material symmetry properties. For a linearly elastic material this problem is solved in works by J. E. Green [12], C. Truesdell [13], A. J. M. Spencer [14], V. V. Novozhilov [15] and many other authors. Systems of invariants used in constitutive relations are also given in the papers [2, 11, 16].

In the present article we use the system of invariants connected with the concept of elastic proper subspaces of materials introduced by J. Rychlewski in [17]. For different materials the number of invariants is different: for the isotropic material the system includes two invariants, for the transversally-isotropic (hexagonal) — four and for orthotropic (rhombic) — six invariants. Functions which determine the behavior of the anisotropic material can depend on all invariants forming the system for the material under consideration. This fact significantly complicates the model of the material. For this reason it is appropriate to introduce additional hypotheses.
concerning the behavior of the nonlinear anisotropic material. In our investigation we use A. A. Il’yushin’s particular postulate as such hypothesis [18].

We also propose variants of nonlinear constitutive relations for anisotropic materials constructed within the frameworks of the generalization of the A. A. Il’yushin’s particular postulate to the case of anisotropic materials [8, 19, 20]. Adopting the particular postulate as a hypothesis reduces the number of arguments of material functions which are included into nonlinear constitutive relations and consequently to reduce the number of experiments which are necessary for concretization of these relations.

2. Elastic proper subspaces

Let us establish the compliance between the process of deforming $\varepsilon(t)$ and its image in the six-dimensional space which represents the trajectory of deforming i.e. the hodograph of the six-dimensional vector $\vec{c}(t)$ with the stress vector $\vec{\sigma}(t)$ built at each point. The relation between the coordinates of six-dimensional vectors $\vec{c}$, $\vec{\sigma}$ and the components of the strain tensor $\varepsilon$ and the stress tensor $S$ is given in the work [20] and is used further:

$$\vec{c} = e_0 \vec{t}_0 + e_1 \vec{t}_1 + e_2 \vec{t}_2 + e_3 \vec{t}_3 + e_4 \vec{t}_4 + e_5 \vec{t}_5,$$

where $\vec{t}_\alpha$, $\alpha = 0, \ldots, 5$ are basis vectors of the six-dimensional space of deformations,

$$e_0 = \frac{1}{\sqrt{3}} (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}), \quad e_1 = \frac{1}{\sqrt{6}} (2\varepsilon_{33} - \varepsilon_{11} - \varepsilon_{22}),$$

$$e_2 = \frac{1}{\sqrt{2}} (\varepsilon_{11} - \varepsilon_{22}), \quad e_3 = \frac{1}{\sqrt{2}} (\varepsilon_{12} + \varepsilon_{21}),$$

$$e_4 = \frac{1}{\sqrt{2}} (\varepsilon_{23} + \varepsilon_{32}), \quad e_5 = \frac{1}{\sqrt{2}} (\varepsilon_{31} + \varepsilon_{13}).$$

The inverse relations are of the form

$$\varepsilon_{11} = \frac{1}{\sqrt{3}} e_0 - \frac{1}{\sqrt{6}} e_1 + \frac{1}{\sqrt{2}} e_2, \quad \varepsilon_{22} = \frac{1}{\sqrt{3}} e_0 - \frac{1}{\sqrt{6}} e_1 - \frac{1}{\sqrt{2}} e_2,$$

$$\varepsilon_3 = \frac{1}{\sqrt{3}} e_0 + \sqrt{\frac{2}{3}} e_1, \quad \varepsilon_{12} = \frac{e_3}{\sqrt{2}}, \quad \varepsilon_{23} = \frac{e_4}{\sqrt{2}}, \quad \varepsilon_{31} = \frac{e_5}{\sqrt{2}}.$$

The requirement that components of the strain tensor $\varepsilon_{ij}$ and the stress tensor $S_{ij}$ must be determined in the Cartesian coordinate system with axes directed along main axes of anisotropy is important in the case of anisotropic materials.

Let us write the Hooke’s law in the six-dimensional space in the form

$$\vec{\sigma} = n \cdot \vec{c},$$

where $n$ is the image of the elasticity tensor of the material in the six-dimensional space. We define eigenvalues $\lambda_\alpha$ and eigenvectors $\vec{\omega}_\alpha$ of the tensor $n$:

$$n \cdot \vec{\omega}_\alpha = \lambda_\alpha \vec{\omega}_\alpha.$$

For a second-order tensor in the six-dimensional space the maximum number of different eigenvalues is 6. It is possible to put into compliance a one-dimensional proper subspace with
the basis tensor $\Omega_\alpha = \vec{\omega}_\alpha \bar{\omega}_\alpha$ and each single root $\lambda_\alpha$. We can also correlate a $k$-dimensional proper subspace with the basis tensor

$$\Omega_\alpha = \sum_{\beta=\alpha}^{\beta=\alpha+k-1} \vec{\omega}_\beta \bar{\omega}_\beta$$

with each root $\lambda_\alpha$ of the multiplicity $k$.

In view of the invariance of eigenvalues and eigenvectors, the basis tensors $\Omega_\alpha$ are invariant with respect to the group of eigenvector transformations.

Elastic proper subspaces of a material in the conception of J. Rychlewski [17] are strain tensors $\varepsilon_\alpha$, which correspond to six-dimensional vectors $\vec{\omega}_\alpha$. Elastic proper subspaces for materials belonging to different crystallographic systems are defined by N. I. Ostrosablin in [21]. In the issues [8, 19, 20] this problem is solved in the six-dimensional space under consideration. In particular it is determined [8, 19, 20] that multidimensional proper subspaces are inherent not in all anisotropic materials, but only in uniaxial crystals (tetragonal, trigonal and hexagonal) as well as in cubic and isotropic media. The elastic properties of a transversally-isotropic material coincide with those of a hexagonal crystal.

We present the expressions for the basis tensors $\Omega_\alpha$ which characterize anisotropic materials of different types. We assign vectors and tensors of the six-dimensional space to the basis $\vec{\iota}_\alpha$.

Isotropic materials have their two proper subspaces: the one-dimensional subspace with the basis tensor $\Omega_1 = \vec{\iota}_0 \vec{\iota}_0$ and the five-dimensional one with the basis tensor $\Omega_2 = \vec{\iota}_1 \vec{\iota}_1 + \vec{\iota}_2 \vec{\iota}_2 + \vec{\iota}_3 \vec{\iota}_3 + \vec{\iota}_4 \vec{\iota}_4 + \vec{\iota}_5 \vec{\iota}_5$. The Elasticity tensor in the six-dimensional space is represented as

$$n^{(is)} = \lambda_1 \Omega_1 + \lambda_2 \Omega_2,$$

where $\lambda_1 = 3K$, $\lambda_2 = 2G$ are the elastic modules.

Cubic materials have three proper subspaces: the one-dimensional subspace with the basis tensor $\Omega_1 = \vec{\iota}_0 \vec{\iota}_0$, the two-dimensional subspace with the basis tensor $\Omega_2 = \vec{\iota}_1 \vec{\iota}_1 + \vec{\iota}_2 \vec{\iota}_2$ and the three-dimensional one with the basis tensor $\Omega_3 = \vec{\iota}_1 \vec{\iota}_1 + \vec{\iota}_2 \vec{\iota}_2 + \vec{\iota}_3 \vec{\iota}_3 + \vec{\iota}_4 \vec{\iota}_4 + \vec{\iota}_5 \vec{\iota}_5 + \vec{\iota}_6 \vec{\iota}_6$.

Hexagonal and trigonal materials have four proper subspaces: two two-dimensional subspaces with the basis tensors $\Omega_1 = \vec{\omega}_1 \vec{\omega}_1$ and $\Omega_2 = \vec{\omega}_2 \vec{\omega}_2$, where the eigenvectors are associated with the basic vectors by the relations

$$\vec{\omega}_1 = \vec{\iota}_0 \cos \varphi + \vec{\iota}_1 \sin \varphi, \quad \vec{\omega}_2 = -\vec{\iota}_0 \sin \varphi + \vec{\iota}_1 \cos \varphi,$$

and the angle $\varphi$ is expressed in terms of components $n_{\alpha\beta} = \vec{\iota}_\alpha \cdot n \cdot \vec{\iota}_\beta$:

$$\tan \varphi = \frac{2n_{01}}{n_{00} - n_{11} + \sqrt{(n_{00} - n_{11})^2 + 4n_{01}^2}}$$

and two two-dimensional proper subspaces with the basis tensors $\Omega_3 = \vec{\omega}_3 \vec{\omega}_3 + \vec{\omega}_4 \vec{\omega}_4 + \vec{\omega}_5 \vec{\omega}_5$, $\Omega_4 = \vec{\omega}_4 \vec{\omega}_4 + \vec{\omega}_6 \vec{\omega}_6$ for hexagonal materials and with basis tensors $\Omega_3 = \vec{\omega}_3 \vec{\omega}_3 + \vec{\omega}_5 \vec{\omega}_5$, $\Omega_4 = \vec{\omega}_4 \vec{\omega}_4 + \vec{\omega}_6 \vec{\omega}_6$ for trigonal materials. In the latter case the orientation of the eigenvectors with respect to the basis $\vec{\iota}_\alpha$ is defined by the following relations:

$$\vec{\omega}_3 = \vec{\iota}_2 \cos \psi + \vec{\iota}_4 \sin \psi, \quad \vec{\omega}_4 = -\vec{\iota}_2 \sin \psi + \vec{\iota}_4 \cos \psi,$$

$$\vec{\omega}_5 = \vec{\iota}_3 \cos \psi + \vec{\iota}_5 \sin \psi, \quad \vec{\omega}_6 = -\vec{\iota}_3 \sin \psi + \vec{\iota}_5 \cos \psi,$$

and the angle $\psi$ is expressed in terms of components $n_{\alpha\beta}$:

$$\tan \psi = \frac{2n_{24}}{n_{22} - n_{44} + \sqrt{(n_{22} - n_{44})^2 + 4n_{24}^2}}.$$
Tetragonal materials have five proper subspaces: four one-dimensional subspaces with basis tensors \( \Omega_1 = \vec{ω}_1 \vec{i}_1 \), \( \Omega_2 = \vec{ω}_2 \vec{i}_2 \), \( \Omega_3 = \vec{ω}_3 \vec{i}_3 \), \( \Omega_4 = \vec{ω}_4 \vec{i}_4 \), where the eigenvectors \( \vec{ω}_1 \) and \( \vec{ω}_2 \) are related to the basis vectors by the same relations as in the case of hexagonal or trigonal materials, and only one two-dimensional proper subspace with the basis tensor \( \Omega_5 = \vec{ω}_5 \vec{i}_5 \).

From the expressions for eigen basis tensors we obtain the invariance of the operators \( n \) relative to the eigen transformations. For example, the operator of the isotropic material (3) is invariant with respect to rotations and reflections of the six-dimensional basis with respect to the vector \( \vec{i}_0 \). This transformation can be interpreted as the rotation of the basis \( \vec{ω}_a (\alpha = 0, ..., 5) \) around the axis \( \vec{i}_0 \), orthogonal to the deviatoric plane \( \vec{i}_1, \vec{i}_2 \). Herewith the angle of the type of the deformed state, which is proportional to the angle of rotation, changes, but the change of this angle does not affect the linear relationship between stress and strain, described by the Hooke’s law.

Thus, the operator \( n \) for the linearly elastic materials satisfies the invariance conditions for the subgroup of eigen orthogonal transformations. This condition is a consequence of the linearity of the relationship between stress and strain tensors and is determined by the uniqueness of this relationship.

3. Variants of constitutive relations

According to A. A. Il’yushin’s particular postulate, the six-dimensional image of the deformation process of the initially isotropic body is invariant not only in relation to the group of orthogonal transformations, associated with the choice of the initial frame of reference, but also with respect to arbitrary transformations of rotation and reflection in the five-dimensional deviatoric subspace. Since under such transformations there are changes in the third invariants of strain and stress tensors, the particular postulate requires that the invariants of this type do not appear explicitly in the number of functions for which the scalar functionals, defining the properties of materials, are given. The particular isotropy postulate is a hypothesis that allows to formulate quasi-linear constitutive equations, with the linear relations of the Hooke’s law satisfying the particular postulate.

The particular postulate of isotropy has found numerous experimental confirmation. In the works of V. S. Lenskiy and I. D. Mashkov, R. A. Vasin and R. I. Shirov, Dao Zuy Bik, L. S. Andreev, R. G. Terekhov and Yu. N. Shevchenko, A. M. Zhukov, V. P. Degtyarev, V. G. Zubchaninov, S. A. Elsuf’ev programs of experiments were developed for the implementation of complex loading for small strains and the results of experiments that confirm the validity of the postulate of isotropy for most materials at small strains were presented. The review and the analysis of these articles is performed in [20].

In his book A. A. Il’yushin [18] mentioned the possibility of generalization of the isotropy postulate for initially anisotropic materials. The idea of this generalization is that it is formulated in the space of generalized stresses, for example, obtained by multiplying the stresses with anisotropic basic tensors. In this space it is required to define a class of invariant transformations preserving the form of the connection between stresses and strains.

We proceed from the fact that for linearly elastic anisotropic materials the stress and strain vectors in each proper subspace are collinear. This relationship is preserved under any orthogonal transformation of the deformation path in each proper space, which is not one-dimensional. In particular, for an isotropic material this condition is satisfied in the five-dimensional deviatoric subspace. In the case of the cubic crystal system the condition of isotropy of images is fulfilled in the plane of the vectors \( \vec{ω}_2, \vec{ω}_3 \) and in a three-dimensional subspace with the basis \( \vec{ω}_4, \vec{ω}_5, \vec{ω}_6 \), in the case of the hexagonal (trigonal) material — in planes \( \vec{ω}_3, \vec{ω}_4 (\vec{ω}_3, \vec{ω}_7) \) and \( \vec{ω}_5, \vec{ω}_6 (\vec{ω}_4, \vec{ω}_6) \) and in the case of the tetragonal material — in the plane \( \vec{ω}_5, \vec{ω}_6 \). We extend this property to all stages of the deformation process.

As a result, we obtain the following generalization of the particular postulate for the non-
linear anisotropic materials: the image of the thermomechanical process with a deformation path disposed in the proper subspace of the tensor of initial elasticity is invariant under the group of intrinsic orthogonal transformations.

From this generalization it follows that the process of loading in each multidimensional proper subspace is defined only by the inner deformation path in this subspace and is not dependent on the orientation of the trajectory relative to the basis vectors of this subspace. This formulation allows the deviation of the stress vector from the strain vector located in the proper subspace. Moreover, this deviation is the effect of the second order with respect to deformation since asymptotically the strain and stress vectors are collinear. According to the postulate the deviation should not change at intrinsic orthogonal transformations. An example of this behavior is the appearance of a hydrostatic stress in shear deformation of an isotropic material.

The limit form of the generalization of the particular postulate is the assumption that the image of the stress process and the strain trajectory lie in the same proper subspace. In view of this assumption the variant of nonlinear relations for anisotropic materials is written [19, 20] in the form

$$\tilde{\sigma} = \left(\sum_{i=k}^{i=p} A_i^{(\alpha)} \tilde{r}_i^{(\alpha)}\right) \sigma_{(i)} = \left(\sum_{i=p}^{i=q} A_j^{(\alpha)} \tilde{r}_j^{(\alpha)}\right)$$

where $A_i^{(\alpha)} \left[ s_i(\tau), \chi_{(\alpha)}^1(\tau), \ldots, \chi_{(\alpha)}^{m-1}(\tau) \right]_{\tau=t}^{\tau=t_0}$ is the functional of the deformation process $\tilde{e}_{(\alpha)}(t)$; $\tilde{r}_i^{(\alpha)}$ is the basis of a $k$-dimensional proper subspace. This basis is covariant with respect to eigen orthogonal transformations. $s_{(\alpha)}$ and $\chi_{(\alpha)}^i$ are the arc length and curvatures of the eigen trajectory respectively. The Frenet basis as the basis $\tilde{r}_i^{(\alpha)}$ is convenient to accept as connected with the trajectory $\tilde{e}_{(\alpha)}(t)$ of the process in the case of smooth eigen trajectories.

If an anisotropic material has $p$ single eigenvalues and $q$ eigenvalues with multiplicities $k_1, k_2, \ldots, k_q$ respectively the connection between stress and strain vectors has the form

$$\tilde{\sigma} = \sum_{i=1}^{i=p} \sigma_{(i)} \tilde{\omega}_i + \sum_{i=p+1}^{i=q} \sum_{j=1}^{j=k} A_j^{(\alpha)} \tilde{r}_j^{(\alpha)}, \quad (4)$$

where $\sigma_{(i)} = \sigma_{(i)} \left[ s_i(\tau) \right]_{\tau=t_0}$ is the functional of the deformation process in the one-dimensional proper subspace with the basis vector $\tilde{\omega}_i$.

The relations (4) are significantly simplified if the deformation process is simple and the strain trajectory is a ray:

$$\tilde{\sigma} = \sum_{i=1}^{i=p} \sigma_{(i)} e_{(i)} \tilde{\omega}_i + \sum_{i=p+1}^{i=q+1} \tilde{A}_{(\alpha)}(s_{(\alpha)}) \tilde{e}_{(\alpha)} \tilde{e}^T_{(\alpha)}, \quad (5)$$

where $e_{(i)} = \tilde{e} \cdot \tilde{\omega}_i$ are the linear invariants of deformations; $\tilde{e}_{(\alpha)} = \tilde{e} \cdot \Omega_\alpha$ are the strain eigenvectors, i.e. the projections of the strain vector into multidimensional proper subspace with basis tensors $\Omega_\alpha$; $s_{(\alpha)} = \sqrt{\tilde{e}_{(\alpha)} \cdot \tilde{e}^{(\alpha)}}$ are the quadratic invariants of deformations.

For anisotropic materials of different types and for the isotropic material linear and quadratic invariants of deformations are defined in the table 1. The invariants given in the table 1 for different anisotropic materials are based on the invariant bases. They coincide with algebraic invariants which were brought in the article of K. F. Chernykh [16].

The orthogonal system of eigenvectors $\tilde{\omega}_i$, $i = 1, \ldots, p$ which match the single eigenvalues is connected with the basis vectors $\tilde{r}_0, \tilde{r}_1, \ldots, \tilde{r}_{p-1}$ of the six-dimensional space by the rotation
Table 1. Invariants of the six-dimensional vector \( \vec{e} \) and of the symmetrical second-order tensor \( \varepsilon \).

| Linear invariants of the vector \( \vec{e} \) | Linear invariants of the tensor \( \varepsilon \) | Quadratic invariants of the vector \( \vec{e} \) | Quadratic invariants of the tensor \( \varepsilon \) |
|---|---|---|---|
| \( e_0, e_1, e_2 \) | \( \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33} \) | | |
| \( e_3, e_4, e_5 \) | \( \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31} \) | | |
| \( \varepsilon_{0}, e_1, e_2, e_3 \) | \( e_{11}, e_{22}, e_{33}, e_{12} \) | \( \varepsilon_{23}, \varepsilon_{31}, \varepsilon_{23} \varepsilon_{31} \) | |
| \( \varepsilon_{0}, e_1, e_2 \) | \( e_{11}, e_{22}, e_{33} \) | \( e_{12}, e_{23}, e_{31} \) | |
| \( \varepsilon_{0}, e_1 \) | \( e_{11} + e_{22}, e_{33} \) | \( (\varepsilon_{11} - e_{22})^2, e_{12}, e_{23} + e_{31}^2, e_{23}^2 + e_{31} \) | |
| \( \varepsilon_{0}, e_1 \) | \( e_{11} + e_{22}, e_{33} \) | \( e_{12}^2 + e_{23}^2 + e_{31}^2, e_{11} - e_{22}^2 + 4e_{12}^2 + e_{23}^2 + e_{31}^2, e_{23}^2 + e_{31} \) | |
| \( \varepsilon_{0} \) | \( e_{11} + e_{22} + e_{33} \) | \( e_{11}^2 + e_{22}^2 + e_{33}^2 + e_{12}^2 + e_{23}^2 + e_{31}^2 \) | |

Let us represent this term in the form

\[
\sum_{i=1}^{i=p} \sigma(i) (e(i)) \vec{\omega} = \sum_{\beta=0}^{\beta=p-1} \sigma_\beta (e_\beta) \vec{t}_\beta,
\]

where \( \sigma_\beta = \vec{\sigma} \cdot \vec{t}_\beta \), \( e_\beta = \vec{e} \cdot \vec{t}_\beta \) are the projections of the stress and strain vectors on coordinate axes.

Let us represent this term in the form

\[
\sum_{\beta=0}^{\beta=p-1} \sigma_\beta (e_\beta) \vec{t}_\beta = \sum_{\alpha=0}^{\alpha=p-1} \sum_{\beta=0}^{\beta=p-1} N_{\alpha \beta} (e_0, e_1, ..., e_{p-1}) e_\alpha \vec{t}_\beta,
\]

where \( N = N_{\alpha \beta} \vec{t}_\alpha \vec{t}_\beta \) is the symmetric second-order tensor in the subspace with the basis \( \vec{t}_0, \vec{t}_1, ..., \vec{t}_{p-1} \). Then the constitutive relations (5) take the form

\[
\vec{\sigma} = \sum_{\alpha=0}^{\alpha=p-1} \sum_{\beta=0}^{\beta=p-1} N_{\alpha \beta} (e_0, e_1, ..., e_{p-1}) e_\alpha \vec{t}_\beta + \sum_{\alpha=p+1}^{\alpha=q+p+1} \vec{A}_\alpha (s_\alpha) \vec{e}_\alpha.
\]
(6) becomes an expression for the Hooke’s law

$$\bar{\sigma} = \sum_{\alpha=0}^{\alpha=q+p} \sum_{\beta=0}^{\beta=p-1} N_{\alpha\beta} e_{\alpha} \vec{e}_{\bar{\beta}} + \sum_{\alpha=p}^{\alpha=q+p} G_{\alpha} \vec{e}_{(\alpha)},$$

where $N_{\alpha\beta}$, $G_{\alpha}$ are the components of the elasticity tensor of a material.

Nonlinear constitutive relations (6) satisfy the limit form of the particular postulate for anisotropic materials, since the functions $N_{\alpha\beta}(e_0, e_1, ..., e_{p-1})$ depend only on the linear invariants of the strain tensor, and the functions $\tilde{A}_{(\alpha)}(s_{(\alpha)})$ depend only on the quadratic invariant of the same name.

A variant of relations taking into account the mutual influence of processes in different proper subspaces and satisfying the generalization of the particular postulate on anisotropic materials can be written in the form

$$\bar{\sigma} = \sum_{i=p}^{i=p} \sigma(i)(e(i), s(p+1), s(p+2), ..., s(q+p+1)) \vec{e}_1 +$$

$$+ \sum_{\alpha=p+1}^{\alpha=q+p+1} \tilde{A}_{(\alpha)}(e(1), e(2), ..., e(p), s(\alpha)) \vec{e}_{(\alpha)}, \quad (7)$$

where the material functions $\sigma(i)$, $\tilde{A}_{(\alpha)}$ depend on linear and quadratic invariants of the strain tensor simultaneously.

If one transforms the first term in the same manner as in the relations (5) a variant of relations will be obtained from (7) in the form

$$\bar{\sigma} = \sum_{\alpha=0}^{\alpha=p-1} \sum_{\beta=0}^{\beta=p-1} N_{\alpha\beta}(e_0, e_1, ..., e_{p-1}, s(p+1), s(p+2), ..., s(q+p+1)) e_{\alpha} \vec{e}_{\bar{\beta}} +$$

$$+ \sum_{\alpha=p+1}^{\alpha=q+p+1} \tilde{A}_{(\alpha)}(e(1), e(2), ..., e(p), s(\alpha)) \vec{e}_{(\alpha)}. \quad (8)$$

In the relations (8) the influence of processes in different proper subspaces affects only on scalar material properties. If some process lies entirely in a multidimensional proper subspace i.e. $\vec{e} = \vec{e}_{(\alpha)}$ the corresponding stress vector is also located in this subspace i.e. $\bar{\sigma} = \bar{\sigma}_{(\alpha)}$.

Let us represent functions $\sigma(i)$ in the relations (7) in the form

$$\sigma(i) = K_i(e(i)) e(i) + \sum_{\alpha=p+1}^{\alpha=q+p+1} G_{\alpha i}(s_{(\alpha)}) s_{(\alpha)}^2,$$

then these relations will be written as

$$\bar{\sigma} = \sum_{i=1}^{i=p} \left( K_i(e(i)) e(i) + \sum_{\alpha=p+1}^{\alpha=q+p+1} G_{\alpha i}(s_{(\alpha)}) s_{(\alpha)}^2 \right) \vec{e}_i +$$

$$+ \sum_{\alpha=p+1}^{\alpha=q+p+1} \left( \tilde{A}_{(\alpha)}(e(1), e(2), ..., e(p), s(\alpha)) + \sum_{i=1}^{i=p} 2G_{\alpha i}(s_{(\alpha)}) s_{(\alpha)} \right) \vec{e}_{(\alpha)}. \quad (9)$$
The relations (9) also satisfy the generalization of the particular postulate on anisotropic materials but allow to describe a deviation of the stress vector from the strain eigen vector: when \( \bar{\varepsilon} = \tilde{\varepsilon}_0 \) the stress vector

\[
\bar{\sigma} = \sum_{i=1}^{i=p} G_{\alpha i} \varepsilon_{(\alpha)}^2 \bar{\omega}_i + \left( \tilde{A}_{(\alpha)} + \sum_{i=1}^{i=p} 2G_{\alpha i} \varepsilon_{(\alpha)} \right) \bar{\varepsilon}_{(\alpha)}.
\]

Besides, this deviation has the second order of infinitesimals. Therefore in the relations (9) the influence of processes in different proper subspaces affects both scalar and vector material properties.

4. Nonlinear relations for the hexagonal material

The hexagonal material has two one-dimensional proper subspaces and two two-dimensional subspaces with basis tensors given above. The linear and quadratic invariants of deformations for such material are brought in the table 1.

The spectral decomposition of the strain vector in this case has the form

\[
\bar{\varepsilon} = \varepsilon_{(1)} \bar{\omega}_{(1)} + \varepsilon_{(2)} \bar{\omega}_{(2)} + \varepsilon_{(3)} \bar{\varepsilon}_{(3)} + \varepsilon_{(4)} \bar{\varepsilon}_{(4)},
\]

where \( \varepsilon_{(1)} = \bar{\varepsilon} \cdot \bar{\omega}_1 \), \( \varepsilon_{(2)} = \bar{\varepsilon} \cdot \bar{\omega}_2 \) are the linear invariants of deformations; \( \varepsilon_{(3)} \), \( \varepsilon_{(4)} \) are the projections of the strain vector into multidimensional proper subspaces.

The relations (9) also satisfy the generalization of the particular postulate on anisotropic materials but allow to describe a deviation of the stress vector from the strain eigen vector: when \( \bar{\varepsilon} = \tilde{\varepsilon}_0 \) the stress vector

\[
\bar{\sigma} = \sum_{i=1}^{i=p} G_{\alpha i} \varepsilon_{(\alpha)}^2 \bar{\omega}_i + \left( \tilde{A}_{(\alpha)} + \sum_{i=1}^{i=p} 2G_{\alpha i} \varepsilon_{(\alpha)} \right) \bar{\varepsilon}_{(\alpha)}.
\]

The relations (9) also satisfy the generalization of the particular postulate on anisotropic materials but allow to describe a deviation of the stress vector from the strain eigen vector: when \( \bar{\varepsilon} = \tilde{\varepsilon}_0 \) the stress vector

\[
\bar{\sigma} = \sum_{i=1}^{i=p} G_{\alpha i} \varepsilon_{(\alpha)}^2 \bar{\omega}_i + \left( \tilde{A}_{(\alpha)} + \sum_{i=1}^{i=p} 2G_{\alpha i} \varepsilon_{(\alpha)} \right) \bar{\varepsilon}_{(\alpha)}.
\]
where \( \mathbf{n} = n_{\alpha \beta} \vec{i}_\alpha \vec{i}_\beta \) is the elasticity tensor.

The relation (8) for the hexagonal material has the form

\[
\sigma = (N_{00}(e_0, e_1, s_{(3)}, s_{(4)}) e_0 + N_{01}(e_0, e_1, s_{(3)}, s_{(4)}) e_1) \vec{i}_0 + \\
+ (N_{01}(e_0, e_1, s_{(3)}, s_{(4)}) e_0 + N_{11}(e_0, e_1, s_{(3)}, s_{(4)}) e_1) \vec{i}_1 + \\
+ \tilde{A}_{(3)}(e_0, e_1, s_{(3)}) \vec{c}_{(3)} + \tilde{A}_{(4)}(e_0, e_1, s_{(4)}) \vec{c}_{(4)}
\]

(14)

and requires to define five material functions for its concretization.

5. Nonlinear relations for the cubic and isotropic materials

The cubic material has one one-dimensional proper subspace, one two-dimensional subspace and one three-dimensional one. The basis tensors for each of three subspaces are mentioned above in the Section 2. The linear and quadratic invariants of deformations for such material are given in the table 1.

The spectral decomposition of the strain vector in the case of the cubic material has the following form:

\[
\vec{e} = e_{(1)} \vec{\omega}_{(1)} + e_{(2)} \vec{\omega}_{(2)} + e_{(3)} \vec{\omega}_{(3)},
\]

(15)

where \( e_{(1)} = \vec{e} \cdot \vec{\omega}_1 \) is the linear invariant of deformations; \( e_{(2)}, e_{(3)} \) are the projections of the strain vector into multidimensional proper subspaces. Since \( \vec{\omega}_1 = \vec{i}_0 \) for the cubic material its linear invariant \( e_{(1)} \) takes the form \( e_{(1)} = \vec{e} \cdot \vec{i}_0 = e_0 \).

For the cubic material the eigenvectors \( \vec{\omega}_\beta \) coincide with the basis vectors \( \vec{i}_\alpha \) in the multidimensional proper subspaces, namely: \( \vec{\omega}_2 = \vec{i}_1, \vec{\omega}_3 = \vec{i}_2 \) in the two-dimensional subspace, \( \vec{\omega}_4 = \vec{i}_3, \vec{\omega}_5 = \vec{i}_4, \vec{\omega}_6 = \vec{i}_5 \) in the three-dimensional subspace.

Therefore the basis tensors in the multidimensional proper subspaces of the cubic material can be determined as \( \Omega_2 = \vec{\omega}_2 \vec{\omega}_2 + \vec{\omega}_3 \vec{\omega}_3 = \vec{i}_1 \vec{i}_1 + \vec{i}_2 \vec{i}_2 \) and \( \Omega_3 = \vec{\omega}_4 \vec{\omega}_4 + \vec{\omega}_5 \vec{\omega}_5 + \vec{\omega}_6 \vec{\omega}_6 = \vec{i}_3 \vec{i}_3 + \vec{i}_4 \vec{i}_4 + \vec{i}_5 \vec{i}_5 \).

Let us write the suggested variants of the constitutive relations for the cubic material. The relations (6) take the following form:

\[
\vec{\sigma} = N_{00}(e_0) e_0 \vec{i}_0 + \tilde{A}_{(2)}(s_{(2)}) \vec{c}_{(2)} + \tilde{A}_{(3)}(s_{(3)}) \vec{c}_{(3)}.
\]

(16)

If the material functions in the relations (16) are assumed to be constants: \( N_{00}(e_0) = n_{00} \), \( \tilde{A}_{(2)}(s_{(2)}) = n_{11} \), \( \tilde{A}_{(3)}(s_{(3)}) = n_{33} \) the expression of the Hooke’s law for the cubic material is obtained

\[
\vec{\sigma} = n_{00} e_0 \vec{i}_0 + n_{11} \left( e_1 \vec{i}_1 + e_2 \vec{i}_2 \right) + n_{33} \left( e_3 \vec{i}_3 + e_4 \vec{i}_4 + e_5 \vec{i}_5 \right),
\]

where \( \mathbf{n} = n_{\alpha \beta} \vec{i}_\alpha \vec{i}_\beta \) is the elasticity tensor.

The relation (8) for the cubic material has the form

\[
\vec{\sigma} = N_{00}(e_0, s_{(2)}, s_{(3)}) e_0 \vec{i}_0 + \tilde{A}_{(2)}(e_0, s_{(2)}) \vec{c}_{(2)} + \tilde{A}_{(3)}(e_0, s_{(3)}) \vec{c}_{(3)}.
\]

(17)

It is necessary to define three material functions for the concretization of the last relation.

On the basis of results given in the sections 2 and 3 the relations (6) for the isotropic material take the form

\[
\vec{\sigma} = N_{00}(e_0) e_0 \vec{i}_0 + \tilde{A}_{(1)}(s_{(1)}) \vec{c}_{(1)},
\]

(18)

where \( e_0 = \vec{e} \cdot \vec{i}_0, \) \( s_{(1)} = \sqrt{e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2} \) are the invariants of deformations for the isotropic material which characterize the change of material volume and the intensity of forming respectively in the case of small deformations.

A more general variant of relations follows from (8):

\[
\vec{\sigma} = N_{00}(e_0, s_{(1)}) e_0 \vec{i}_0 + \tilde{A}_{(1)}(e_0, s_{(1)}) \vec{c}_{(1)}.
\]

(19)

The mutual influence of processes in different proper subspaces of the isotropic material is taken into account in this relation.
6. Results and discussion

The obtained constitutive relations (6) are concretized for materials with different types of symmetry of properties (13), (16) and (18). All these relations satisfy the limit form of the generalization of A. A. Il’yushin’s postulate on anisotropic materials. In the particular case of the isotropic material (18) this hypothesis means that the volume change $e_0$ does not depend on the forming $s_{(1)}$ and the process of forming defined by the vector $\vec{e}_{(1)}$ is independent of the volume change.

The relations (8) and their particular variants for the hexagonal (14), cubic (17) and isotropic (19) materials are constructed on the basis of the general form of generalization of the particular postulate on anisotropic materials. In the case of the isotropic material it means that the material functions $N_{00}(e_0, s_{(1)})$ and $\tilde{A}_{(1)}(e_0, s_{(1)})$ depend on both the material volume change and forming. This fact significantly complicates the process of experimental concretization of the material functions.

7. Conclusion

Systems of invariants of deformations are defined as lengths of projections of the strain vector into elastic proper subspaces. The systems are written for anisotropic materials of different types. In the article several variants of nonlinear constitutive relations for anisotropic materials are offered. For the hexagonal, cubic and isotropic materials the analysis of satisfaction of the constitutive relations to the general and limit forms of the generalization of A. A. Il’yushin’s particular postulate is fulfilled.

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