Schrödinger equations in the constrained space with several initial constraints

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A general system constrained with several initial constraint conditions is quantized based on the Dirac formalism and the Schrödinger equation for this system is obtained. These constraint conditions are now allowed to depend not only on the coordinates but also on the velocities. It is shown that the hermiticity for the observables of the system restricts the geometrical structure of our world.

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1 INTRODUCTION

Almost every physical theory, electromagnetism, gravity and etc., is treated as the theory of a constrained system. The theoretical framework of the constrained system was invented in 1950’s by Dirac with the Hamiltonian formalism [1][2]. The essence of the Dirac method is as follows. Due to the Hamiltonian formalism, we normally use the canonical Poisson brackets in the unconstrained phase space. In the constrained phase space, however, a new type of brackets – Dirac brackets – must be introduced à la Dirac. In passing from the classical to the quantum theory, we replace these Dirac brackets by the commutators (×1/\(\bar{\hbar}\)). To obtain the Schrödinger equation, we have to find the representation of the commutation relations, i.e., we must obtain the representation of the operator variables.

The classical theories of constrained systems are completely described in Hamiltonian formalism with the Dirac method[3]. However, when we pass to the quantum theory, we meet some problems[3][4]. One of such problems is that Falck and Hirshfeld presented a simple example of the Dirac method applied to the dynamics of a point particle They quantized a system with a particle on a sphere in the 3-dimensional Euclidean space. In this case the Dirac brackets of the variables, expressed in the Cartesian coordinates, are not canonical. Here “... canonical” means that the Dirac bracket between a coordinate variable and its conjugate momentum variable is 1 and that other Dirac brackets with respect to the variables vanish. As is clear from this example, the Dirac brackets among the conjugate variables are generally not canonical, and the difficulties arise to find out the representation of the operator variables when we pass to the quantum theory [4].

Homma et al. suggested a new method to overcome these difficulties for a constrained particle system on a sphere and on a general hypersurface [5][6]. A constraint \(f(q^i) = 0\) with \(q^i\), the coordinates of a particle, is evidently equivalent to \(df/dt = 0\) with a vanishing constraint at some instant. They defined the system under the constraint \(f(q^i) = 0\) as system(I) and that under the constraint \(df/dt = 0\) as system(II). The two systems should be equivalent both in the classical mechanics and in the quantum mechanics. The Dirac brackets in system(I) are not canonical, but those in system(II) are canonical, whichever is the reason why they introduced system(II). They, therefore, easily found the representation of the operators in the latter system. Now let us call these constraint conditions introduced initially to make a particle constrained on a specific manifold, by the name “initial constraint conditions”.

Homma et al. dealt with the system constrained only with a single initial constraint condition, i.e., the system constrained on a hypersurface. In other words, they formulated the particle dynamics in the constrained coordinate space whose dimension is only reduced by one from that of the unconstrained coordinate space: in the constrained phase space the dimension is reduced by two from that of the unconstrained phase space. However, the system with several initial constraint conditions has been discarded up to now.

Following their method, we, in this paper, treat several initial constraint conditions to complete the dynamics of a particle on various manifolds. Moreover these conditions are made to depend not only on the coordinates but also on the velocities. Thus if we succeed

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1Its details are explored in Appendix
to construct the theory considering the several initial constraint conditions depending on coordinate and velocity variables, we are to have the general theory of constrained systems with arbitrary initial constraint conditions.

This paper is composed as follows. In Sec. 2, we shall review the method of Homma et al., restricted to an example of a particle constrained on a sphere of arbitrary dimensions. In Sec. 3, we shall deal with several initial constraint conditions depending only on the coordinates, and in Sec. 4, with those depending both on the coordinates and the velocities of a particle. The last section will be devoted to the summary and discussions.

In present paper, the summation convention, according to which summation over the repeated indices is meant, is adopted.

### 2 QUANTIZATION ON A SPHERE

Now, we consider, as system(I), a nonrelativistic particle (mass : \( m \)) constrained on an \((N - 1)\)-dimensional sphere (radius : \( \sqrt{A} \) and \( N \geq 2 \)) in the \( N \)-dimensional Euclidean space. The initial constraint condition in this system is \( f(q) \equiv q^i q_i - A = 0 \), and the Lagrangian is

\[
L_I = \frac{1}{2} m \dot{q}^i \dot{q}_i + \lambda (q^i q_i - A) \quad (i = 1, 2, \ldots, N),
\]

where

\[
\dot{q}^i = \frac{dq^i}{dt},
\]

the \( q^i \) denote the coordinates of the particle and \( \lambda \) is a Lagrange multiplier. The conjugate momenta are defined as

\[
p_i \equiv \frac{\partial L_I}{\partial \dot{q}^i} = m \dot{q}^i, \quad (2)
\]

\[
p_\lambda \equiv \frac{\partial L_I}{\partial \lambda} = 0. \quad (3)
\]

Equation (3) is a primary constraint condition

\[
\phi_1 \equiv p_\lambda \approx 0, \quad (4)
\]

where “\( \approx \)” represents Dirac’s weak equality. Following the Dirac method we obtain the secondary constraint conditions

\[
\phi_2 \equiv q^i q_i - A \approx 0, \quad (5)
\]

\[
\phi_3 \equiv q^i p_i \approx 0, \quad (6)
\]

\[
\phi_4 \equiv p^i p_i + 2m \lambda q^i q_i \approx 0. \quad (7)
\]

By Eq.(7), the variable \( \lambda \) is written in terms of \( q^i \) and \( p_i \):

\[
\lambda = \frac{p^i p_i}{2m q^i q_i}. \quad (8)
\]
We, further, eliminate $p_\lambda$ and $\lambda$ from this system with Eqs. (4) and (8), respectively. Therefore the variables of this system are $q^i$ and $p_i$, and the remaining constraint conditions are $\phi_2 \approx 0$ and $\phi_3 \approx 0$. The Dirac brackets of the variables are calculated as follows:

\[
\{ q^i, q^j \}_D = 0 , \tag{9}
\]
\[
\{ q^i, p_j \}_D = \delta^i_j - \frac{q^i q_j}{A} , \tag{10}
\]
\[
\{ p_i, p_j \}_D = \frac{p^i q_j - p^j q_i}{A} . \tag{11}
\]

We find that the Dirac brackets (9)–(11) are not canonical. After we have introduced the Dirac brackets, the extended Hamiltonian is

\[
H_{IE} = \frac{1}{2m} p^i p_i . \tag{12}
\]

In the canonical quantization method, we replace the Dirac brackets (9)–(11) by the commutators ($\times 1/i\hbar$) and regard the variables as operators:

\[
[\hat{q}^i, \hat{q}^j] = 0 , \tag{13}
\]
\[
[\hat{q}^i, \hat{p}_j] = i\hbar \left( \delta^i_j - \frac{\hat{q}^i \hat{q}_j}{A} \right) , \tag{14}
\]
\[
[\hat{p}_i, \hat{p}_j] = i\hbar \frac{\hat{p}^i \hat{q}_j - \hat{p}^j \hat{q}_i}{A} , \tag{15}
\]

where the symbol with caret “^” designates an operator.

In order to obtain the Schrödinger equation, we must find the coordinate representation of $\hat{p}_i$. In our simple case, it is possible to obtain its representation by the trial-and-error method. However, since the commutation relations of $q^i$ and $p_i$ are complicated in general cases, it is difficult to do that. Therefore we discuss the quantization method à la Homma et al.

Firstly, we consider the constrained system with the initial constraint condition $q^i \dot{q}_i = 0$ and refer it to system(II). This condition is the time derivative of the initial constraint condition in system(I): $\dot{f}(q) = 2q^i \dot{q}_i = 0$. The Lagrangian of system(II) is given by

\[
L_{II} = \frac{1}{2} m \dot{q}^i \dot{q}_i + \lambda (q^i \dot{q}_i) . \tag{16}
\]

The conjugate momenta are

\[
\Pi_i \equiv \frac{\partial L_{II}}{\partial \dot{q}^i} = m \dot{q}_i + \lambda q_i , \tag{17}
\]
\[
\Pi_\lambda \equiv \frac{\partial L_{II}}{\partial \dot{\lambda}} = 0 . \tag{18}
\]

The primary constraint condition is

\[
\chi_1 \equiv \Pi_\lambda \approx 0 , \tag{19}
\]
and the secondary constraint condition is
\[ \chi_2 \equiv q^i \Pi_i - \lambda q^i q_i \approx 0 . \] (20)

In system(II), the Dirac brackets of \( q^i \) and \( \Pi_i \) are
\[ \{ q^i, q^j \}_D = 0 , \] (21)
\[ \{ q^i, \Pi_j \}_D = \delta^i_j , \] (22)
\[ \{ \Pi_i, \Pi_j \}_D = 0 . \] (23)

Equations (21)–(23) show that the Dirac brackets are canonical in system(II). The extended Hamiltonian is
\[ H_{IE} = \frac{1}{2m} \left( \Pi_i - \frac{q^i q_j}{q^2} \Pi_j \right)^2 \]
\[ = \frac{1}{2m} (F^i_j \Pi_j)^2 , \] (24)

where
\[ F^i_j \equiv \delta^i_j - \frac{q^i q_j}{q^2} , \] (25)
\[ q^2 \equiv q^i q_i . \]

From Eq.(25), we find that \( F^i_j \) is the projection matrix which projects an arbitrary vector onto the sphere.

In the above discussion, we treat \((q^i, \Pi_i, \lambda, \Pi_\lambda; i = 1, 2, \ldots, N)\) as the variables of system(II). However we can reduce them to \((q^i, \Pi_i; i = 1, 2, \ldots, N)\) by solving the equations \( \chi_1 \approx 0 \) and \( \chi_2 \approx 0 \) to represent \( \Pi_\lambda \) and \( \lambda \) in the other variables. In this case, since all constraint conditions are disappeared, we can deal with the system through the Poisson brackets constructed with \((q^i, \Pi_i; i = 1, 2, \ldots, N)\). The Poisson brackets of \( q^i \) and \( \Pi_i \) take the same form as in Eqs.(21)–(23).

Secondly, we introduce new momentum variables \( P_i \):
\[ P_i \equiv F^i_j \Pi_j . \] (26)

which are interpreted as projected momentum variables. The Dirac brackets including \( P_i \) are calculated as follows:
\[ \{ q^i, P_j \}_D = \delta^i_j - \frac{q^i q_j}{q^2} = F^i_j , \] (27)
\[ \{ P_i, P_j \}_D = \frac{P_i q_j - P_j q_i}{q^2} . \] (28)

Investigate the time development of \( q^2 \),
\[ \frac{dq^2}{dt} = \{ q^2, H_{IE} \}_D = 0 , \] (29)
and we find that \( q^2 \) is a constant of motion. Putting \( q^2 = A \) at \( t = 0 \) as the initial condition of the differential equation (29), the Dirac brackets (27) and (28) take the same form as in Eqs. (11) and (11). We further obtain, from Eq. (26), the constraint condition whose form is the same as that of Eq. (11):

\[
q^i P_i = q^i F^i_j \Pi_j = 0 .
\] (30)

Finally, we regard \( P_i \) as the momenta \( p_i \) of system(I). In this way the constraint structure of system(II) is equivalent to that of system(I). Therefore system(I) and system(II) have the same content in the classical theory.

In order to quantize system(II), we replace the Dirac brackets (21)–(23) by the commutators \( \times 1/i\hbar \) of the operators:

\[
[q^i, q^j] = 0 , \quad [q^i, \Pi_j] = i\hbar \delta^i_j , \quad [\Pi_i, \Pi_j] = 0 .
\] (31) (32) (33)

From these commutation relations, we immediately have the coordinate representation of \( \Pi_i \):

\[
\Pi_i = -i\hbar \frac{\partial}{\partial q^i} .
\] (34)

Therefore it is easy to obtain the Schrödinger equation of system(II) in the coordinate representation.

Requiring the equivalence between system(I) and system(II) in the quantum theory and making use of Eq. (34), we obtain the Schrödinger equation for system(I).

To sum up, in the quantum case, we adopt \( q^2 = A \) as the initial condition, and then regard \( \hat{P}_i \) as being equal to \( \hat{p}_i \) after symmetrizing the order of operators in Eq. (26):

\[
\hat{p}_i = \frac{1}{2} \left[ \hat{F}^i_j \hat{\Pi}_j + \hat{\Pi}_j \hat{F}^j_i \right] .
\] (35)

We, of course, have all constraint conditions of system(I) based on those of system(II) in quantum level. Substitution of Eq. (33) into the quantum Hamiltonian of system(I)

\[
\hat{H}_{IE} = \frac{1}{2m} \hat{p}_i \hat{p}_i ,
\] (36)

gives us the Hamiltonian of the system(II). The resultant quantum Hamiltonian of system(II) is reduced to, with an appropriate ordering,

\[
\hat{H}_{II} = \frac{1}{8m} \left( \hat{F}^i_j \hat{\Pi}_j + \hat{\Pi}_j \hat{F}^j_i \right)^2 .
\] (37)

System(II) thus becomes equivalent to system(I) also in quantum level. In order to obtain the coordinate representation of the Hamilton operator of system(I), we substitute Eq. (34) into Eq. (37).
3 SEVERAL CONSTRAINT CONDITIONS DEPENDING ONLY ON COORDINATE VARIABLES

In this section, we treat the dynamics of a nonrelativistic particle of mass $m$ in an $N$-dimensional Riemannian space with a Riemannian metric $g_{ij}(q)$. Let a particle be constrained with $M$ initial constraint conditions

$$f_s(q) = 0 \quad (s = 1, 2, \cdots, M; M < N) ,$$

where $s$ is the index to distinguish various initial constraint conditions. Namely, the particle is constrained in an $(N - M)$-dimensional subspace. Here we assume that the constraint conditions depend only on the coordinate variables $q^i$ and that these conditions are differentiable with respect to $q^i$. The Lagrangian $L_1$ in this system is expressed as follows:

$$L_1 = \frac{1}{2} m g^{ij} \dot{q}^i \dot{q}^j + A_i(q) \dot{q}^i - V(q) + \lambda^s f_s(q) \quad (i, j = 1, 2, \cdots, N) ,$$

where

$$\dot{q}^i \equiv \frac{dq^i}{dt} ,$$

and $V(q)$, $A_i(q)$ and $\lambda^s$ are the scalar potential, the vector potentials and the Lagrange multipliers, respectively. The momenta conjugate to $q^i$ and $\lambda^s$ are

$$p_i \equiv \frac{\partial L_1}{\partial \dot{q}^i} = m g^{ij} \dot{q}^j + A_i(q) ,$$

$$p'_s \equiv \frac{\partial L_1}{\partial \lambda^s} = 0 .$$

The primary constraint conditions are

$$\phi_{1s} \equiv p'_s \approx 0 .$$

We thus have the total Hamiltonian

$$H_{TT} = \frac{1}{2m} g^{ij} (p_i - A_i)(p_j - A_j) + V(q) - \lambda^s f_s(q) + v^s p'_s ,$$

where

$$v^s \equiv \dot{\lambda}^s ,$$

and $v^s$ are the multipliers for Eq.(42). For the consistency that a constraint condition should hold also after the time development, we succeedingly have secondary constraint conditions

$$\phi_{2s} \equiv f_s(q) \approx 0 ,$$

$$\phi_{3s} \equiv \frac{1}{m} g^{ij} \frac{\partial f_s}{\partial q^i}(p_j - A_j) \approx 0 ,$$

7
The momenta conjugate to $q$ Hamiltonian is all other Poisson brackets vanish. Hence all $\phi$ conditions. The Dirac brackets among $q$ the term thus made is designated by ($i$ independent. This means that the determinant of ($D$ calculated the Poisson brackets. We exchange the indices $i$ where $q$ rewritten as the functions of $q$. Since the initial constraint conditions $f = 0$ are independent each other, the normals to the surfaces $\partial f_s(q)/\partial q^i$ are linearly independent. This means that the determinant of $(D_s)$ does not vanish, and $\lambda^s$ are rewritten as the functions of $q^i$ and $p_i$ with Eq. (40). Moreover $v^s$ are determined as the functions of $q^i$ and $p_i$ for the consistency of $\phi_{4s}$. Hence all the physical quantities are expressed with $q^i$ and $p_i$. We calculate the Poisson brackets among $\phi_{\alpha s}$ ($\alpha = 1, 2, 3, 4$):

$$\{\phi_{1s}, \phi_{4t}\} = -\frac{1}{m} D_{st} , \quad (47)$$

$$\{\phi_{2s}, \phi_{3t}\} = \frac{1}{m} D_{st} , \quad (48)$$

and all other Poisson brackets vanish. Hence all $\phi_{\alpha s} = 0$ are second class constraint conditions. The Dirac brackets among $q^i$ and $p_i$ are

$$\{q^i, q^j\}_D = 0 , \quad (49)$$

$$\{q^i, p_j\}_D = \delta^i_j - g^{ki} (D^{-1})^{st} \frac{\partial f_s}{\partial q^k} \frac{\partial f_t}{\partial q^j} \bigg|_{f=0} \quad (50)$$

$$\{p_i, p_j\}_D = -\frac{\partial f_s}{\partial q^i} (D^{-1})^{st} \left[ \frac{\partial}{\partial q^j} \left( \frac{g^{kl} \partial f_l}{\partial q^k} \right) p_t - \frac{\partial}{\partial q^j} \left( g^{kl} \partial f_l / \partial q^k A_t \right) \right] \bigg|_{f=0} - (i \leftrightarrow j) , \quad (51)$$

where $|_{f=0}$ denotes that we impose all initial constraint conditions $f_s = 0$ after having calculated the Poisson brackets. We exchange the indices $i$ and $j$ in the first term and the term thus made is designated by $(i \leftrightarrow j)$. With the Dirac brackets, the extended Hamiltonian is

$$H_{IE} = \frac{1}{2m} g^{ij} (p_i - A_i)(p_j - A_j) + V(q) . \quad (52)$$

We now consider system(II) following the procedure in Sec. 2. In system(II), the Lagrangian $L_{II}$ is

$$L_{II} = \frac{1}{2} m g_{ij} \dot{q}^i \dot{q}^j + A_i(q) \dot{q}^i - V(q) + \lambda^s \dot{f}_s(q) \quad (i, j = 1, 2, \cdots, N) . \quad (53)$$

The momenta conjugate to $q^i$ and $\lambda^s$ are

$$\Pi_i \equiv \frac{\partial L_{II}}{\partial \dot{q}^i} = m g_{ij} \dot{q}^j + A_i + \lambda^s \frac{\partial f_s}{\partial q^i} , \quad (54)$$

$$\Pi'_s \equiv \frac{\partial L_{II}}{\partial \lambda^s} = 0 . \quad (55)$$
Therefore the primary constraint conditions are
\[ \chi_1s \equiv \Pi'_s \approx 0. \] (56)

Thus the total Hamiltonian is
\[ H_{\text{IT}} = \frac{1}{2m} g^{ij} \left( \Pi_i - A_i - \lambda^s \frac{\partial f_s}{\partial q^i} \right) \left( \Pi_j - A_j - \lambda^t \frac{\partial f_t}{\partial q^j} \right) + V(q) + u^s \Pi'_s, \] (57)

where
\[ u^s \equiv \dot{\lambda}^s, \]
and \( u^s \) are the multipliers for Eq. (56). The consistency requires the following secondary constraint conditions:
\[ \chi_2s \equiv \frac{1}{m} g^{ij} \frac{\partial f_s}{\partial q^i} \left( \Pi_j - A_j - \lambda^t \frac{\partial f_t}{\partial q^j} \right) \approx 0. \] (58)

Eliminating \( \lambda^s \) and \( u^s \) in the same way as in system(I), all the physical quantities are expressed with \( q^i \) and \( \Pi_i \). We calculate the Poisson brackets among \( \chi_\alpha s \) (\( \alpha = 1, 2 \)):
\[ \{ \chi_1s, \chi_2t \} = \frac{1}{m} D_{st}, \] (59)

and all other Poisson brackets vanish, indicating that all \( \chi_\alpha s = 0 \) are second class constraint conditions. The Dirac brackets among \( q^i \) and \( \Pi_i \) are
\[ \{ q^i, q^j \}_D = 0, \] (60)
\[ \{ q^i, \Pi_j \}_D = \delta^i_j, \] (61)
\[ \{ \Pi_i, \Pi_j \}_D = 0. \] (62)

The extended Hamiltonian is
\[ H_{\text{IE}} = \frac{1}{2m} g^{ij} F^k_i (\Pi_k - A_k) F^l_j (\Pi_l - A_l) + V(q), \] (63)

where
\[ F^k_i \equiv \delta^k_i - g^{kl} (D^{-1})^{st} \frac{\partial f_s}{\partial q^k} \frac{\partial f_t}{\partial q^l} \bigg|_{f=0} \] (64)

Here we define new momentum variables \( P_i \):
\[ P_i - A_i \equiv F^k_i (\Pi_k - A_k). \] (65)

The Dirac brackets including \( P_i \) are calculated as follows:
\[ \{ q^i, P_j \}_D = F^i_j, \] (66)
\[ \{ P_i, P_j \}_D = - \frac{\partial f_s}{\partial q^i} (D^{-1})^{st} \left[ \frac{\partial}{\partial q^j} \left( g^{kl} \frac{\partial f_i}{\partial q^k} \right) P_l - \frac{\partial}{\partial q^j} \left( g^{kl} \frac{\partial f_i}{\partial q^k} A_l \right) \right] \bigg|_{f=0} - (i \leftrightarrow j). \] (67)
Moreover we have the time development of $f_s$
\[
\frac{df_s}{dt} = \{ f_s, H_{\text{HE}} \}_D = 0 ,
\] (68)
which shows that $f_s$ are the constants of motion. When we choose $f_s = 0$ as the initial conditions of the differential equations (68) (i.e., at $t = 0$), the Dirac brackets (69) and (57) take the same forms as in Eqs. (50) and (51). Note that since $(P_i - A_i)$ are the tangential components of $(\Pi_k - A_k)$ to the surfaces and $\partial f_s / \partial q_i$ are the normal components, $(P_i - A_i)$ just satisfy the constraint conditions (45) with $p_i$, replaced by $P_i$. The argument mentioned above means that the constraint structure of system(I) is equivalent to that of system(II). We take $f_s = 0$ as the initial conditions and identify $P_i$ with the momenta $p_i$ in system(I).

We pass to the quantum theory under the requirement of the above equivalence of both systems(I) and (II). For the hermiticity, we need to symmetrize the order of the products of operators expressing observables. Firstly, we symmetrize the momentum operators:
\[
\hat{p}_i - \hat{A}_i = \frac{1}{2} [\hat{F}_i^k(\hat{f} = 0), \hat{\Pi}_k - \hat{A}_k]_+, \tag{69}
\]
where $[ , ]_+$ denotes the anticommutator. The commutation relations including $\hat{\Pi}_i$ are
\[
\left[ \hat{q}^i, \hat{\Pi}_j \right] = i\hbar \delta^i_j , \tag{70}
\]
\[
\left[ \hat{\Pi}_i, \hat{\Pi}_j \right] = 0 . \tag{71}
\]
These are canonical. The coordinate representation of $\hat{\Pi}_i$ is
\[
\hat{\Pi}_i = -i\hbar \frac{\partial}{\partial q^i} - \frac{1}{2} i\hbar \left\{ k, \frac{g_{ki}}{2} \right\}(q) . \tag{72}
\]

Here we use the Christoffel symbol
\[
\left\{ j, \frac{k}{ki} \right\}(q) \equiv \frac{1}{2} g^{in}(\partial_k g_{in} + \partial_i g_{kn} - \partial_n g_{ki}) . \tag{73}
\]

Secondly, we obtain the quantum Hamiltonian by symmetrization. We write down the Hamiltonian, based on Eqs.(52) and (68), as follows [7]:
\[
\hat{H}_{\text{HE}} = \frac{1}{2m} [\hat{F}_i^k(\hat{f} = 0), \hat{\Pi}_k - \hat{A}_k]_+ + \hat{g}^{ij} [\hat{F}_j^l(\hat{f} = 0), \hat{\Pi}_l - \hat{A}_l]_+ + V(\hat{q}) + \hbar^2 \hat{Q} , \tag{74}
\]
where
\[
\hat{Q} \equiv \frac{1}{2m} \left[ F_i^k F_j^l \left( g^{ij} \left\{ a, \frac{a}{k} l \right\} - g^{jb} \left\{ i, \frac{b}{k} l \right\} \right) \partial_l + F_i^l g^{ij} \left( \partial_k F_j^l \right) \partial_i \right.
\]
\[
+ \frac{1}{2} F_i^k \partial_k \left( g^{ij} \left( F_j^l \left\{ a, \frac{a}{l} b \right\} + \partial_l F_j^l \right) \right)
\]
\[
+ \frac{1}{4} \left( F_i^k \left\{ b, \frac{b}{k} l \right\} + \partial_k F_i^k \right) g^{ij} \left( F_j^l \left\{ a, \frac{a}{l} b \right\} + \partial_l F_j^l \right) \right] . \tag{75}
\]
The quantum mechanical potential \( \hat{Q} \), typical of quantum mechanics, is a term which makes the Schrödinger equation be covariant under general coordinate transformations. The hermiticity of the Hamiltonian (74) leads to the vanishing of the antisymmetric parts for those multiplied by the metric \( g^{ij} \). Hence we have the hermiticity conditions

\[
-F^k_i \partial_k F^l_j + (\partial_l F^h_j) F^l_h + F^k_j \partial_k F^l_i - (\partial_j F^h_i) F^l_h = 0 ,
\]

\[
\partial_l F^h_j \partial_k F^h_i - \partial_j F^h_i \partial_k F^h_i = 0 ,
\]

\[
(\delta^k_i - F^k_i) \partial_k \{(\delta^l_j - F^l_j) A_l(q)\} - (i \leftrightarrow j) = 0 ,
\]

\[
F^l_i g^{ij}(\partial_k F^k_j) = 0 ,
\]

Note that Eq.(79) comes from the hermiticity of the quantum mechanical potential. With these conditions, we finally obtain the Schrödinger equation

\[
i \hbar \frac{\partial \Psi}{\partial t} = \hat{H}_E \Psi
\]

\[
= -\frac{\hbar^2}{2m} g^{ij} F^k_i (F^l_j \Psi)_l + \frac{i \hbar}{m} F^k_i A_k g^{ij} F^l_j \Psi_l + \frac{i \hbar}{2m} (F^k_i A_k g^{ij} F^l_j)_l \Psi + \frac{1}{2m} (F^k_i A_k g^{ij} F^l_j A_l) \Psi + V(q) \Psi,
\]

where \( \Psi \) is a wave function, and \( .k \) denotes the covariant derivative with respect to \( q^k \), and its concrete expression for \( T^l_i \) is

\[
T^l_i .k \equiv \nabla_k T^l_i = \partial_k T^l_i + \left\{ \begin{array}{c} j \\ l k \end{array} \right\} T^l_i - \left\{ \begin{array}{c} l \\ i k \end{array} \right\} T^l_i.
\]

4 SEVERAL CONSTRAINT CONDITIONS DEPENDING ON COORDINATES AND VELOCITIES

In this section we consider the system of a particle constrained with several initial constraint conditions, depending not only on the coordinates \( q^i \) but also on the velocities \( \dot{q}^i (i = 1, 2, \cdots, N) \).

We proceed by taking an example, namely by putting concrete forms for the initial constraints \( f_s(\dot{q}, q) \). Other forms for \( f_s(\dot{q}, q) \) are, of course, allowed. For instance, \( f_s(\dot{q}, q) \) may include the terms whose powers of velocities are greater than one: \( \alpha'_{sijk} \cdots (q) \dot{q}^i \dot{q}^j \dot{q}^k \cdots \). But this case leads to a complicated nonlinear equation of motion. We thus start with a simple (and general) example:

\[
f_s(\dot{q}, q) = \alpha_{si}(q) \dot{q}^i + \beta_s(q) = 0 \quad (s = 1, 2, \cdots, M; M < N),
\]

where both \( \alpha_{si}(q) \) and \( \beta_s(q) \) are the functions of the coordinates \( q^i \). The conditions (82) are assumed to be independent with respect to the velocity variables. We thus immediately have \( \det(E_{st}) \neq 0 \) with \( E_{st} = g^{ij} \alpha_{si} \alpha_{tj} \). (See the discussion leading to \( \det(D_{st}) \neq 0 \) in the previous section.)
The Lagrangian of this system is given by

\[ L = \frac{1}{2}m g_{ij} \dot{q}^i \dot{q}^j + A_i(q) \dot{q}^i - V(q) + \lambda^s f_s(\dot{q}, q) \quad (s = 1, 2, \ldots, M) , \]  

with \( \lambda^s \), the Lagrange multipliers. The momenta \( p_i \) conjugate to \( q^i \) and \( \Pi_s \) conjugate to \( \lambda^s \) are

\[ p_i = mg_{ij} \dot{q}^j + A_i(q) + \lambda^s \alpha_{si} , \]  
\[ \Pi_s = 0 . \]

From Eq.(83), the primary constraints are deduced:

\[ \chi_{1s} \equiv \Pi_s \approx 0 \quad (s = 1, 2, \ldots, M) . \]

Therefore the total Hamiltonian of the system is given by

\[ H_T = \frac{1}{2m} g^{ij}(p_i - A_i - \lambda^s \alpha_{si})(p_j - A_j - \lambda^t \alpha_{tj}) - \lambda^s \beta_s + V(q) + v^s \Pi_s , \]

where \( v^s (\equiv \dot{\lambda}^s) \) are the Lagrange multipliers for the primary constraints (86).

The consistency for the primary constraints leads us to the secondary constraints

\[ \chi_{2s} \equiv -\frac{1}{m} E_{st} \lambda^t + \frac{1}{m} g^{ij}(p_i - A_i) \alpha_{sj} + \beta_s \approx 0 . \]

We can write \( \lambda^s \) in terms of \( q^i \) and \( p_i \) as follows:

\[ \lambda^s \approx (E^{-1})_{st} [g^{ij}(p_i - A_i) \alpha_{tj} + m \beta_t] . \]

The multipliers \( v_s \) are determined as the functions of \( q^i \) and \( p_i \) from the consistency for \( \chi_{2s} \). Hereafter we eliminate \( \Pi_s \) and \( \lambda^s \) with Eqs.(84) and (89), and regard \( q^i \) and \( p_i \) as the independent variables of the system. The Poisson brackets of \( \chi_{1s} \) and \( \chi_{2s} \) are

\[ \{\chi_{1s}, \chi_{2t}\} = \frac{1}{m} E_{st} , \]

which shows that all the constraints are second class.

Now the Dirac brackets for the variables become

\[ \{q^i, q^j\}_D = 0 , \]
\[ \{q^i, p_j\}_D = \delta^i_j , \]
\[ \{p_i, p_j\}_D = 0 . \]

The total Hamiltonian (87) is reduced to the extended Hamiltonian

\[ H_E = \frac{1}{2m} g^{ij}[p_i - A_i - (E^{-1})_{st} \alpha_{si} \{m \beta_t + g^{kl}(p_k - A_k) \alpha_{ut}\}] \]
\[ \times [p_j - A_j - (E^{-1})_{st} \alpha_{sj} \{m \beta_t + g^{kl}(p_k - A_k) \alpha_{ut}\}] \]
\[ - \beta_s (E^{-1})_{st} [m \beta_t + g^{kl}(p_k - A_k) \alpha_{ut}] + V(q) . \]
The classical dynamics of the system is described with this Hamiltonian and the Dirac brackets (91)–(93).

In order to quantize the system, we must replace the Dirac brackets (91)–(93) by the commutators \((\times 1/i\hbar)\) and regard variables \(q^i\) and \(p_i\) as operator variables \(\hat{q}^i\) and \(\hat{p}_i\). The results are

\[
\begin{align*}
\left[\hat{q}^i, \hat{q}^j\right] &= 0, \\
\left[\hat{q}^i, \hat{p}_j\right] &= i\hbar \delta^i_j, \\
\left[\hat{p}_i, \hat{p}_j\right] &= 0.
\end{align*}
\] (95) (96) (97)

The commutation relations (95)–(97) are canonical, so that we easily obtain the coordinate representation of the momentum operators. Symmetrizing the order of operators in Eq.(94) to guarantee the hermiticity of the quantized Hamiltonian, we have

\[
\hat{H}_E = \frac{1}{8m}[\hat{F}^k_i, \hat{p}_k - \hat{A}_k] + \hat{g}^{ij} [\hat{F}^j_l, \hat{p}_l - \hat{A}_l] + \\
- \frac{1}{2} [\hat{g}^{ij} \hat{F}^k_i (\hat{E}^{-1})^{uv} \hat{\alpha}_{uj} \hat{\beta}_v, \hat{p}_k - \hat{A}_k] + \\
- \frac{1}{2} [\hat{g}^{ij} (\hat{E}^{-1})^{uv} \hat{\beta}_v \hat{\alpha}_{uj}, \hat{p}_k - \hat{A}_k] + \\
\frac{1}{2} m \hat{g}^{ij} (\hat{E}^{-1})^{uv} \hat{\beta}_v \hat{\alpha}_{ui} (\hat{E}^{-1})^{st} \hat{\beta}_t \hat{\alpha}_{sj} + \\
- m \hat{\alpha}_u \hat{\beta}_v (\hat{E}^{-1})^{uv} + \\
\hbar^2 \hat{Q} + V(\hat{q}),
\] (98)

where \(\hat{F}^j_i\) is defined by

\[
\hat{F}^j_i \equiv \delta^j_i - \hat{g}^{jk} (\hat{E}^{-1})^{st} \hat{\alpha}_{sk} \hat{\alpha}_{ti} \big|_{j=0},
\] (99)

and \(\hat{Q}\) is the quantum mechanical potential mentioned in Sec. 3. Henceforth we express all the operators in the coordinate representation and omit the caret “\(^{\hat{}}\)”.

The requirement of hermiticity further leads us to the following two conditions:

\[
F^k_i F^l_j \left(g^{ij} \left\{ \begin{array}{cc} a & k \\ ak & i \\ \end{array} \right\} - g^{jb} \left\{ \begin{array}{cc} i & \\\ k & b \end{array} \right\} \right) \partial_l + F^l_i g^{ij} (\partial_k F^k_j) \partial_l = 0,
\] (100)

\[
(E^{-1})^{uv} \alpha_{uj} \beta_v g^{ij} (\delta^k_i + F^k_i) \left\{ \begin{array}{cc} l & \\\ l & k \end{array} \right\} = 0.
\] (101)

The Schrödinger equation of this system is written in the coordinate representation of the operators, with the conditions (100) and (101), as

\[
i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}_E \Psi
\]

\[
= \frac{1}{2m} g^{ij} F^k_i (-i\hbar \nabla_k - A_k) F^l_j (-i\hbar \nabla_l - A_l) \Psi \\
- g^{ij} (E^{-1})^{uv} \alpha_{uj} \beta_v F^k_i (-i\hbar \nabla_k - A_k) \Psi
\]
\[- \beta_u (E^{-1})^{uv} \alpha_{kl} g^{kl} (-i \hbar \nabla_k - A_k) \Psi + \frac{1}{2} i \hbar \nabla_k (E^{-1})^{uv} \alpha_{uj} \beta_v g^{ij} (\delta^k_i + F^k_i)] \Psi + V(q) \Psi, \] (102)

\( \Psi \) : a wave function

where \( \nabla_k \) is the covariant derivative with respect to \( q^k \), defined in Eq.(81).

5 SUMMARY AND DISCUSSIONS

We have, in this paper, presented a general theory of a particle constrained with several initial constraint conditions, thus aiming at completion of the dynamics of a particle on a general manifold.

First we put the initial constraint conditions as functions of \( q^i \). The Dirac brackets of \( q^i \) and \( p_i \) are complicated in structure, which leads to some difficulties associated with quantization. Therefore we apply the method of Homma et al. to the system and obtain the Schrödinger equation. In this process, requiring the Hamiltonian to be hermitian, we have the conditions (76)–(79). This is the result that was not obtained by Homma et al. and is due to the existence of several initial constraints in the system.

Next we treated the system of a particle constrained with the initial constraint conditions (82), depending not only on the coordinates but also on the velocities. The Dirac brackets of variables simply become canonical, so that we have the coordinate representation of the momentum operators straightforwardly. We have the conditions (100) and (101) by requiring that the Hamiltonian should be hermitian.

Since \( F^j_i \) depends on \( f_s \), we regard the hermiticity conditions as the conditions for the initial constraints \( f_s \). Hence, in the quantum theory, the structure of the manifold on which the particle exists is to be restricted.

Note that there is a case that the initial constraint conditions depend explicitly on time. In this case we cannot express the time evolution of physical quantities only with the Dirac bracket. We must make, at the same time, an explicit use of the Poisson bracket, therefore we cannot apply the method of the present paper to this system. The formalism of the quantization of this system is a subject for future study.

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