Exact analytic two-loop expressions for some QCD observables in the time-like region

D.S. Kourashev

Bogoliubov Lab. Theor. Phys., JINR, Dubna, 141980, Russia;
Moscow State University, Physical Department, Moscow, Vorobyevy Gory, 119899, Russia

kourashev@mtu-net.ru

Abstract

The author of this work have got explicit expressions for the timelike region at Next-to-Leading-Order (NLO) as for the coupling function so for QCD observables. These expressions were compared with approximate ones obtained in terms of “double logarithms”. Next-Next-to-Leading-Order (NNLO) Padé approximation was also discussed.

In this paper some particular role of NLO was emphasised. It is related with the possibility to express any multi-loop expressions in terms of the two-loop ones. Thus we can use NLO functions as a basis of some functional expansion, where higher loop terms incorporation affects the expansion coefficients only.

1 Introduction

Renorm-group equation (RGE) for QCD coupling function $\tilde{\alpha}_s$ could be written as

$$ x \frac{\partial \tilde{\alpha}_s}{\partial x} = - (\beta_0 \tilde{\alpha}_s^2 + \beta_1 \tilde{\alpha}_s^3 + \beta_2 \tilde{\alpha}_s^4 + ...) $$

with $\beta_n$ corresponding to $(n+1)$-loop contribution. Its solution has a very simple form in the one-loop case: $\tilde{\alpha}_s^{(1)}(x) = \frac{1}{\beta_0 \ln x}$. At NLO (two-loop approximation) it can be expressed through the Lambert function $[1, 2, 3, 4]$

$$ \tilde{\alpha}_s^{(2)}(x) = - \frac{\beta_0}{\beta_1} \frac{1}{1 + W(z)} , \quad z \equiv - \frac{\beta_2}{\beta_1 x} e^{-\frac{\beta_3}{\beta_1 x}}. $$

defined by a transcendental equation $W(z)e^{W(z)} = z$. Here and further $x = \frac{Q^2}{\Lambda^2}$, $\Lambda$ is the scale parameter defining unphysical singularities positions.

It should be mentioned that all RGE solutions obtained perturbatively have unphysical singularities such as “ghost poles”. In the two- and three-loop cases the cut lying in infrared region emerges. Hence, the coupling behaviour in this region can’t be described this way. Analytic Approach suggested four years ago by D.V. Shirkov and I.L. Solovtsov $[5, 6, 7]$ allowed to obtain analytic expressions for the running coupling with the same ultraviolet asymptotic that ordinary RGE solutions have but with a stable infrared behaviour. Afterwards, this procedure was slightly corrected and generalised by D.V. Shirkov $[8, 9]$. In the Sec.2 some basic facts from this approach were stated and also some two-loop calculations results were presented.

In the Sec.3 the analytic Approach was applied for the two- and three-loop cases. It was proved in this work that we can obtain exact expressions for the two-loop case and Padé approximated third order $\beta$-function. These results were compared with the ones obtained using ”double logarithms” approximation.

As follows from the publication $[13]$ it’s possible to express the coupling constant for any-loop case as a power series of the two-loop coupling. It’s also possible to
apply Analyticization Procedure to this expansion. The opportunity to express some arbitrary coupling function or observable in terms of two-loop functions is discussed in the Sec.4.

2 Modified Analyticization Procedure

Standard Analytic Approach allows us to get rid of unphysical singularities and make it possible to research the coupling (and observables also) behaviour in the low energy region. Moreover it reduces the scheme dependence considerably. However it is also necessary to incorporate quark thresholds and to construct the procedure allowing to transfer the coupling (and observables also) to the time-like region. These problems were solved by D.V. Shirkov \cite{8,9} in his recent works.

2.1 $Q^2$-channel

As it had been proposed earlier by the Analyticization procedure, it is possible to make the coupling analytic in a whole $Q^2$ complex plane except the cut on the real negative axis. However, this approach does not take into account the thresholds problem. Analyticization procedure is based on Källen-Lehmann spectral representation, it leads to the modified coupling function that is devoid of unphysical singularities. Besides, it has the correct asymptotic behaviour consistent with the perturbative result. The spectral representation has the form

$$\{a(x)\}_an = \frac{1}{\pi} \int_0^\infty \frac{\rho(\sigma)}{\sigma + x} d\sigma,$$

the spectral density can be taken as $\rho(\sigma) = 3\alpha_s^{f=3}(-\sigma)$ that is natural for the low energy processes analysis when there are three active quarks ($f = 3$). Here, the new coupling function is defined through an imaginary part of usual, RG invariant, effective coupling $\bar{\alpha}_s$ continued on the physical cut.

However, it is also possible to involve the threshold matching procedure. It is well known that one can relate the couplings for the regions with a different flavours numbers. One should match $\bar{\alpha}_s^{f=n-1}(M_n^2)$ and $\bar{\alpha}_s^{f=n}(M_n^2)$ by the proper scale parameter redefinition. As a result, spectral density is discontinuous \cite{8} and can be presented as

$$\rho(\sigma) = \rho^{f=3}(\sigma) + \sum_{f>3} \theta(\sigma - M_f^2)(\rho^f(\sigma) - \rho^{f=3}(\sigma)).$$

Of course, the coupling obtained from this density is continuous.

2.2 $s$-channel

Some interesting experiments data correspond to the timelike region ($s$-channel) that leads to the necessity of the relevant redefinition of coupling function. Some approximate estimates, such as $\mathfrak{A}(s) = \bar{\alpha}_s(s)$ or $\mathfrak{A}(s) = |\bar{\alpha}_s(-s)|$, were used earlier for this purpose. These estimates are suitable for qualitative analysis only in the region higher than $5 GeV$, and they can not reflect the correct analytical properties. The difference between these estimates and exact expressions is within several per cents in this region and this precision can be accurate enough for some calculations.
However, it is better to use the “dipole representation” for the Adler function in terms of an observable $R(s)$ in the timelike region

$$D(Q^2) = Q^2 \int_0^\infty \frac{ds}{(s + Q^2)^2} R(s)$$  \hspace{1cm} (5)

and an inverse expression

$$R(s) = \frac{i}{2\pi} \int_{-s-\i}^{-s+\i} \frac{D(x)}{x} dx.$$  \hspace{1cm} (6)

This transformation leads to the coupling function definition for $s$-channel. It can be implemented as

$$\mathcal{A} = \frac{i}{2\pi} \int_{-s-\i}^{-s+\i} \frac{\alpha_s(x)}{x} dx.$$  \hspace{1cm} (7)

It can also be expressed in terms of the spectral density \[10\]

$$\mathcal{A}(s) = \frac{1}{\pi} \int_{s}^{\infty} \frac{d\sigma}{\sigma} \rho(\sigma).$$  \hspace{1cm} (8)

These formulae have been applied to one-loop case by K. Milton and O. Solovtsova \[11\]. Corresponding expression for $A(s)$ is

$$\mathcal{A} = \frac{1}{2\beta_0} - \frac{1}{\pi\beta_0} \arctan \frac{\sqrt{s}}{\Lambda}.$$  \hspace{1cm} (9)

Higher orders was also obtained in this paper.

### 2.3 Thresholds

As follows from the work by D.V. Shirkov \[8\], the effective $s$-channel coupling $\mathcal{A}(s)$ differs from $A^f$ by a shift constant

$$\mathcal{A}(s) = A^f + c^f \text{ at } M_f^2 \leq s \leq M_{f+1}^2,$$  \hspace{1cm} (10)

the shift constants can be easily calculated

$$c^{f-1} = c^f - A^f(M_f^2) - A^{f-1}(M_f^2).$$  \hspace{1cm} (11)

These constants values correspond to 4th, 5th and 6th quarks contributions. The shift constants provide the continuity of the analytic coupling function. Indeed, in spite of discontinuous spectral density we should get the continuous expressions from the spectral representation.

Here are some typical values calculated at NLO:

| $\Lambda_i, MeV$ | $\Lambda_3, MeV$ | $\Lambda_4, MeV$ | $\Lambda_5, MeV$ | $\Lambda_6, MeV$ | $c^3$ | $c^4$ | $c^5$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-------|-------|-------|
| 300             | 251             | 174             | 72              | 62              | 0.0107| 0.0028| 0.0003|
| 400             | 346             | 249             | 108             | 92              | 0.0149| 0.0035| 0.0003|

Note that these contributions gather about 5 per cent to the effective coupling in the low energy region and can considerably influence the scale parameter value.

Quite analogously, second shift constant and third shift constant are defined

$$\mathcal{A}_i(s) = A^f_i + c^f_i \text{ at } M_f^2 \leq s \leq M_{f+1}^2,$$  \hspace{1cm} (12)
\[ c_i^{f-1} = c_i^f + \mathcal{A}_i^f(M_f^2) - \mathcal{A}_i^{f-1}(M_f^2), \quad i=2,3. \]  \tag{13}

Some typical values of \( c_2 \) and \( c_3 \) are presented below:

| \( \Lambda_4, \text{MeV} \) | \( c_2^3 \) | \( c_2^4 \) | \( c_2^5 \) | \( c_3^2 \) | \( c_3^3 \) | \( c_3^4 \) |
|-------------------------|--------|--------|--------|--------|--------|--------|
| 300                     | 0.0021 | -0.0024 | -0.0008 | 0.0026 | 0.0005 | 0.00002 |
| 500                     | 0.0007 | -0.0024 | -0.0004 | 0.0047 | 0.0009 | 0.00003 |

3 Two-loop and three-loop applications

3.1 Exact NLO results

At NLO we can even obtain the analytical expressions for the s-channel Analyticized coupling function. As it was mentioned the two-loop renormgroup equation solution can be expressed in terms of the Lambert function \( W \). This representation also allowed us to obtain \( \mathcal{A}_i(s) \) functions exactly

\[ \mathcal{A}(s) = -\frac{\beta_0}{\beta_1} + \frac{1}{\pi} \Im \frac{1}{\beta_1 \bar{\alpha}_s(-s)}. \]  \tag{14}

\[ \mathcal{A}_2(s) = \frac{1}{\pi \beta_1} \Im \ln(1 + \frac{\beta_1}{\beta_0} \bar{\alpha}_s(-s)). \]  \tag{15}

\[ \mathcal{A}_3(s) = -\frac{\beta_0}{\beta_1} \frac{1}{\pi \beta_1} \Im \left\{ \ln(1 + \frac{\beta_1}{\beta_0} \bar{\alpha}_s(-s)) - \frac{\beta_1}{\beta_0} \bar{\alpha}_s(-s) \right\}. \]  \tag{16}

\[ \mathcal{A}_4(s) = \left( -\frac{\beta_0}{\beta_1} \right)^2 \frac{1}{\pi \beta_1} \Im \left\{ \ln(1 + \frac{\beta_1}{\beta_0} \bar{\alpha}_s(-s)) - \frac{\beta_1}{\beta_0} \bar{\alpha}_s(-s) + \frac{\beta^2_2}{2 \beta^2_0} \bar{\alpha}_s^2(-s) \right\}, \text{etc.} \]  \tag{17}

These expressions are obtained by using the contour integral \( \int_{s^{-i\epsilon}}^{s+i\epsilon} \frac{\alpha_n^2(x)}{x} dx \) in the generalized form

\[ \mathcal{A}_n = \int_{s^{-i\epsilon}}^{s+i\epsilon} \frac{\alpha_n^2(x)}{x} dx. \]  \tag{18}

The integration can be implemented with rather simple variables substitutes. It’s interesting that \( \mathcal{A}_n(s) \) can be presented as \( n - 2 \) residual terms of the Taylor expansion of \( \frac{1}{\pi \beta_1} \Im \ln(1 + \frac{\beta_1}{\beta_0} \bar{\alpha}_s(-s)) \) over the powers of \( \bar{\alpha}_s(-s) \) multiplied by \( -\frac{\beta_0}{\beta_1} \).

3.2 Three-loop Padé approximation

The renorm-group equation at NNLO can not be solved explicitly. However the Padé-approximated \( \beta \)-function

\[ \beta_{\text{Padé}} = -\beta_0 \bar{\alpha}_s^2 \left( 1 + \frac{\beta_1 \bar{\alpha}_s}{\beta_0 - \frac{\beta_0 \beta_2}{\beta_1}} a \right) = \beta_0 \bar{\alpha}_s^2 + \beta_1 \bar{\alpha}_s^3 + \beta_2 \bar{\alpha}_s^4 + \frac{\beta_0 \beta_2}{\beta_1} \bar{\alpha}_s^5 \ldots \]  \tag{19}

can be used in this case and the solution can be also obtained through Lambert function \( W \)

\[ a_{\text{Padé}}^{(3)}(x) = -\frac{\beta_0}{\beta_1} \frac{1}{1 - c + W(z)}, \quad z = -\frac{1}{eb} e^{\frac{1}{c}} x^{-\frac{1}{b}}, \quad b = \frac{\beta_1}{\beta_0}, \quad c = \frac{\beta_0 \beta_2}{\beta_1}. \]  \tag{20}
Without Padé-approximation even in the three-loop case we can’t get analytically
the RGE solution for both coupling and observables. So it is natural to use the the
expansion through two-loop functions to analyse multi-loop functions.

One can obtain the s-channel coupling in this case exactly like in the two-loop case
\[
\mathfrak{A}(s) = -\frac{1}{\pi\beta_0} \left\{ \ln \left( -\frac{1}{c} \left( 1 + W(z) \right) + 1 \right) \right\}, \quad z = -\frac{1}{e\beta_0} e^{-c(-s)^{-\frac{1}{c}}}. \quad (21)
\]

3.3 Using power expansion over the two-loop functions
for NNLO analysis

We can also get the three-loop functions without Padé approximation using the method
suggested in the work [15]. The expansion (31) allows us to obtain the following
expressions
\[
\mathfrak{A}^{NNLO}(s) = k_1 \mathfrak{A}^{(2)}(s) + k_2 \mathfrak{A}^{(2)}(s)^2 + k_3 \mathfrak{A}^{(2)}(s)^3 + k_4 \mathfrak{A}^{(2)}(s)^4 + k_5 \mathfrak{A}^{(2)}(s)^5 + \ldots, \quad (22)
\]
for NNLO case \(k_1 = 1, k_2 = 0, k_3 = \frac{\beta_2}{\beta_0}, k_4 = 0, k_5 = \frac{5\beta_2^2}{2\beta_0^2}\). This case differs from the
case with Padé-transformed \(\beta\)-function as the latter originates some “trail” of higher
order terms. However this difference is quite small (when comparing new shift constants
the difference between NLO and NNLO is about 15 per cents, while it is only 1 per
cent between NNLO and “Padé” case).

3.4 Comparing exact two-loop functions with approximate ones

Two-loop RGE solution (3) is explicit and as it is argued in the present work the
Lambert function is also very convenient for the timelike region research. In spite of
these facts as a rule one takes some approximate expressions like
\[
\alpha^{(2)}_{sapp}(x) = \frac{1}{\beta_0 \ln(x)} \left\{ 1 - \frac{\beta_1}{\beta_0^2} \ln(\ln(x)) + \frac{\beta_2^2}{\beta_0^4} \ln^2(\ln(x)) \left( \ln(\ln(x)) - \frac{1}{2} \right)^2 - \frac{5}{4} \right\}. \quad (23)
\]
As it was proved we can express this function in terms of (2)
\[
\alpha^{(2)}_{sapp}(x) = \alpha^{(2)}_s(x) + k(x) \left( \alpha^{(2)}_s(x) \right)^4 + O \left( \alpha^{(5)}_s(x) \right) \quad (24)
\]
Here \(k(x)\) is some limited function. Some typical values of this function are\[1\]
\[
\begin{array}{ccc}
Q, \text{MeV} & k \left( \frac{\alpha^2}{\alpha_s} \right) \\
600 & -0.7 \\
900 & -0.78 \\
1200 & -0.83 \\
1500 & -0.85 \\
2100 & -0.91
\end{array}
\]
This terms proportional to the fourth power of coupling function contributes for about
4 per cent difference in 1GeV region. We have to note that this can not be corrected
by the scale parameter choice as it would originate nonzero coefficients before 2nd and
3rd powers of \(\alpha^{(2)}_s\) in the expansion (24).

\[1\]We’ve taken \(\Lambda_3 = 300 \text{MeV}\)
4 Obtaining the multi-loop functions in terms of two-loop ones

4.1 Expansion

Suppose some observable can be expressed as a power series of the coupling constant

\[ F(Q^2) = F_1 \bar{\alpha}_s(Q^2) + F_2 \bar{\alpha}_s^2(Q^2) + F_3 \bar{\alpha}_s^3(Q^2) \ldots \]  

(25)

However we face unphysical singularities, wrong analytic behaviour in the infrared (IR) region. We can not consider early attempts to express this observable in s-channel like

\[ F_s(s) = F_1' \bar{\alpha}_s(s) + F_2' \bar{\alpha}_s^2(s) + F_3' \bar{\alpha}_s^3(s) \ldots \]  

(26)

to be very successful. Indeed, it involves the so called \( \pi^2 \)-terms and leads to the expansion coefficients augmenting [12, 13, 14], this way also preserves unphysical singularities. As a result of applying the Analyticization procedure the analytic coupling function and the functional expansion of observables both in \( Q^2 \)- and s- channels can be obtained

\[ F(Q^2) = F_1 A(Q^2) + F_2 A_2(Q^2) + F_3 A_3(Q^2) \ldots \]  

(27)

\[ F_s(s) = F_1 A(s) + F_2 A_2(s) + F_3 A_3(s) \ldots \]  

(28)

After all, the Analytic approach leads to the scheme dependence reduction that is very important physical consequence.

4.2 The expansion over two-loop functions

As follows from the paper [15] we can obtain the expressions for multi-loop functions in terms of two-loop functions. Without analyticization we have the an expansion over the coupling function powers like

\[ a^{\text{multi-loop}} = a^{(2)} + k_2 a^{(2)2} + k_3 a^{(2)3} + \ldots \]  

(29)

Here we can easily apply the analytic approach. It leads to some modifications when we’ve got the functional expansion instead of the power one

\[ A^{\text{multi-loop}} = A^{(2)} + k_2 A_2^{(2)} + k_3 A_3^{(2)} + \ldots \]  

(30)

Thus two-loop functions can be considered as a minimal basis of any orders perturbation expansions. So any observable \( F \) (devoid of anomalous dimensions) can be presented over these functions in arbitrary loop order

\[ F = F_1 A^{(2)} + F_2 A_2^{(2)} + F_3 A_3^{(2)} + \ldots \]  

(31)

Some similar result (observables expansion as a power expansion over two-loop functions) was obtained independently by C. Maxwell [16].
5 Conclusions

In this paper the author has got an exact expressions for analytic running coupling at the two-loop order. These expressions differ from approximate ones by a term proportional to the fourth power of the coupling function. Timelike functions were also obtained just in the terms of Lambert function. As it was shown higher orders loop approximations can be obtained as an expansion over two-loop functions. We need to be only provided with $\beta$-coefficients.

This approach allows us to get obtain accurate theoretical predictions of the coupling function consistent with its analyticity properties. As it was argued the difference between exact expressions and approximate ones in the infrared region is high enough to be considered.

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