THE C-(SYMMETRIC) QUADRILATERAL LATTICE, ITS TRANSFORMATIONS AND THE ALGEBRO-GEOMETRIC CONSTRUCTION

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Abstract. The C-quadrilateral lattice (CQL), called also the symmetric lattice, provides geometric interpretation of the discrete CKP equation within the quadrilateral lattice (QL) theory. We discuss affine-geometric properties of the lattice emphasizing the role of the Gallucci theorem in the multidimensional consistency of the CQL. Then we give the algebro-geometric construction of the lattice. We also present the reduction of the vectorial fundamental transformation of the QL to the CQL case. In the Appendix we show a relation between the QL and the so called Darboux maps.

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References

1. Introduction

The difference equations play an increasing role in science. We have learned to appreciate the inherent discreteness of physical phenomena at the atomic and subatomic level. Also in modern theoretical physics the assumption of a space-time continuum is often being abandoned. Apart from physics difference equations have numerous applications, e.g., in numerical analysis, computer science, mathematical biology and economics.

The domain of discrete (difference) systems forms nowadays one of the focal points in integrable systems research. Also the connection between geometry and integrability, well known in the case of integrable differential equations \([19, 34, 31, 24]\), has been transferred to the discrete level (see \([19]\) and references therein). A successful general approach towards description of this relation is provided by the

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theory of multidimensional quadrilateral lattices (QLs) [17]. These are just maps \( x : \mathbb{Z}^N \to \mathbb{P}^M \) (\( 3 \leq N \leq M \)) of multidimensional integer lattice into projective space, with planar elementary quadrilaterals. The integrable partial difference equation counterpart of the QLs are the discrete Darboux equations (see Section 2 for details), being found first [6] as the most general difference system integrable by the non-local \( \partial \) dressing method. It turns out that integrability of the discrete Darboux system is encoded in a very simple geometric statement (see Fig. 1).

**Lemma 1.1** (The geometric integrability scheme). Consider points \( x_0, x_1, x_2 \) and \( x_3 \) in general position in \( \mathbb{P}^M \), \( M \geq 3 \). On the plane \( \langle x_0, x_i, x_j \rangle \), \( 1 \leq i < j \leq 3 \) choose a point \( x_{ij} \) not on the lines \( \langle x_0, x_i \rangle \), \( \langle x_0, x_j \rangle \) and \( \langle x_i, x_j \rangle \). Then there exists the unique point \( x_{123} \) which belongs simultaneously to the three planes \( \langle x_3, x_{13}, x_{23} \rangle \), \( \langle x_2, x_{12}, x_{23} \rangle \) and \( \langle x_1, x_{12}, x_{13} \rangle \).

Integrable reductions of the quadrilateral lattice (and thus of the discrete Darboux equations) arise from additional constraints which are compatible with geometric integrability scheme (see, for example [14, 18, 16]). Because application of several integrable reductions preserves integrability of the lattice, it is important to isolate the basic ones. Integrable constraints of the quadrilateral lattice may have local or global nature. The global constraints are related with existence of an additional geometric structure in the projective space which allows to impose some restrictions on the lattice. The reduction considered in this paper can be described within the affine geometry approach.

The differential Darboux equations, which have appeared first in projective differential geometry of multidimensional conjugate nets [9], play an important role in the multicomponent Kadomtsev–Petviashvili (KP) hierarchy, which is commonly considered as the fundamental system of equations in integrability theory. One of the most important reductions of the KP hierarchy of nonlinear equations is the so called CKP hierarchy [11] (here “C” appears in the context of the classification theory of simple Lie algebras). In [2] it was shown that the differential Darboux equations of the multicomponent KP hierarchy should be in this case supplemented by certain symmetry condition. In fact such equations were considered first within the differential geometry context in [9, 5]. Their discrete counterpart was studied in [18] under the name of the symmetric quadrilateral lattice. In [18] the symmetric Darboux equations have been solved by the non-local \( \partial \) dressing method. The corresponding reduction of the fundamental transformation [20] of QL was given on the algebraic level in [29].

Within the context of the so called Darboux maps such a reduction of the discrete Darboux equations was studied in [33], where the system was reformulated in a convenient scalar form (see also [12]) and related with the superposition formulas for the Darboux transformations of the CKP hierarchy. In [20] such special Darboux maps were characterized geometrically using the classical Pascal’s theorem for hexagons inscribed in conics. Also the Bäcklund transformation for such maps was investigated geometrically in [20]. In fact, the attempt to understand results of [33, 20] in the quadrilateral lattice approach was my motivation to undertake again, after [18], studies of the symmetric reduction of the Darboux equations.

**Figure 1.** The geometric integrability scheme

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An important reduction of the quadrilateral lattice with the symmetric rotation coefficients is the Egorov lattice [32]. It was solved in [18] using the inverse spectral transform. In [1] the algebro-geometric techniques were used to construct large class of such lattices. In 1998 Peter Grinevich isolated from [1] these of the algebro-geometric conditions which give rise the symmetric quadrilateral lattices [23].

The main goal of the paper is to give new pure geometric characterization of the symmetric quadrilateral lattice and of the corresponding reduction of the fundamental transformation. Because of the intimate relation of the lattice with the discrete CKP equation we will call it also the C-quadrilateral lattice. We study the geometric and algebraic properties of the CQL reduction of the fundamental transformation. In particular, we show that the reduced transformation satisfies the Bianchi permutability principle. To make the theory of such lattices complete we apply the algebro-geometric techniques to construct large classes of the lattices together with corresponding solutions of the Darboux equations. In doing that we introduce the backward and the dual Baker–Akhiezer functions of the quadrilateral lattice.

The paper is organized as follows. In Section 2 we summarize relevant facts from the quadrilateral lattice theory and we introduce the notation. In Section 3 we introduce new geometric definition of the C-quadrilateral lattice and we study its integrability (multidimensional consistency) using geometric means. We also present its algebraic description in terms of symmetric Darboux equations. In Section 4 we are devoted to presentation of the algebro-geometric construction of the C-quadrilateral (symmetric) lattices. In Section 5 we introduce geometrically the C-reduction of the fundamental transformation of the quadrilateral lattice [20] and we link it with the earlier algebraic results of [29]. We also prove the corresponding permutability theorem for this transformation. Finally in the Appendix we present briefly the relation between the Darboux maps [33] and the quadrilateral lattices.

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2. The multidimensional quadrilateral lattice (affine description)

2.1. The discrete Laplace and Darboux equations. Consider a multidimensional quadrilateral lattice (MQL); i.e., a mapping \( x : \mathbb{Z}^N \to \mathbb{P}^M(\mathbb{F}) \), \( 3 \leq N \leq M \), with all the elementary quadrilaterals planar [17]; here \( \mathbb{Z}^N \) is the \( N \)-dimensional integer lattice, and \( \mathbb{P}^M(\mathbb{F}) \) is \( M \)-dimensional projective space over the field \( \mathbb{F} \). In the affine gauge (which will be extremely useful in the paper) the lattice is represented by a mapping \( x : \mathbb{Z}^N \to \mathbb{F}^M \) and the planarity condition can be formulated in terms of the Laplace equations

\[
\Delta_i \Delta_j x = (T_i A_{ij}) \Delta_j x + (T_j A_{ji}) \Delta_i x, \quad i \neq j, \quad i, j = 1, \ldots, N,
\]

where \( T_i \) is the translation operator in the \( i \)-th direction, and \( \Delta_i = T_i - 1 \) is the corresponding partial difference operator.

Due to compatibility of the system (2.1) the coefficients \( A_{ij} \) satisfy the MQL (or discrete Darboux) system of equations

\[
\Delta_k A_{ij} + (T_k A_{ij}) A_{ik} = (T_j A_{jk}) A_{ij} + (T_k A_{kj}) A_{ik}, \quad i, j, k \text{ distinct}.
\]

The \( j \leftrightarrow k \) symmetry of RHS of (2.2) implies existence of the potentials \( H_i, i = 1, \ldots, N \), (called the Lamé coefficients) such that

\[
A_{ij} = \frac{\Delta_j H_i}{H_i}, \quad i \neq j.
\]

If we introduce the suitably scaled tangent vectors \( X_i, i = 1, \ldots, N \),

\[
\Delta_i x = (T_i H_i) X_i,
\]

then equations (2.1) can be rewritten as a first order system expressing the fact that \( j \)-th variation of \( X_i \) is proportional to \( X_j \) only

\[
\Delta_j X_i = (T_j Q_{ij}) X_j, \quad i \neq j.
\]
The proportionality factors $Q_{ij}$, called the rotation coefficients, can be found from the linear equations

\begin{equation}
\Delta_i H_j = (T_i H_i) Q_{ij}, \quad i \neq j,
\end{equation}

adjoint to (2.5). The compatibility condition for the system (2.5) (or its adjoint) gives the following new form of the discrete Darboux equations

\begin{equation}
\Delta_k Q_{ij} = (T_k Q_{ik}) Q_{kj}, \quad i \neq j \neq k \neq i.
\end{equation}

2.2. The backward data, the connection factors and the $\tau$-function. The backward tangent vectors $\tilde{X}_i$, $i = 1, \ldots, N$, are defined similarly to the forward tangent vectors $X_i$ but with the help of the backward shifts $T_i^{-1}$ and of the backward difference operator $\tilde{\Delta}_i := 1 - T_i^{-1}$:

\begin{equation}
\tilde{\Delta}_i \tilde{X}_j = (T_i^{-1} \tilde{Q}_{ij}) \tilde{X}_i, \quad \text{or} \quad \Delta_i \tilde{X}_j = (T_i \tilde{X}_i) \tilde{Q}_{ij}, \quad i \neq j;
\end{equation}

we define also the backward tangent rotation coefficients $\tilde{Q}_{ij}$ as the corresponding proportionality factors. Notice that the backward tangent vectors $\tilde{X}_i$ satisfy the adjoint linear system (2.6).

The backward Lamé coefficients $\tilde{H}_i$, $i = 1, \ldots, N$, are defined by

\begin{equation}
\tilde{\Delta}_i \tilde{H}_i = (T_i^{-1} \tilde{H}_i) \tilde{Q}_{ij}, \quad \text{or} \quad \Delta_i \tilde{H}_i = (T_i \tilde{H}_i) \tilde{Q}_{ij}, \quad i \neq j.
\end{equation}

As a consequence of equations (2.8) or equations (2.10) the functions $\tilde{Q}_{ij}$ satisfy the MQL equations (2.7).

Define backward Laplace coefficients

\begin{equation}
\tilde{A}_{ij} = \frac{\tilde{\Delta}_i \tilde{H}_i}{\tilde{H}_i},
\end{equation}

then $x$ satisfies the backward Laplace equation

\begin{equation}
\tilde{\Delta}_i \tilde{A}_{ij} x = (T_i^{-1} \tilde{A}_{ij}) \tilde{\Delta}_i x + (T_i^{-1} \tilde{A}_{ij}) \tilde{\Delta}_j x, \quad i \neq j.
\end{equation}

The connection factors $\rho_i : \mathbb{Z}^N \to \mathbb{F}$ are the proportionality coefficients between $X_i$ and $T_i \tilde{X}_i$ (both vectors are proportional to $\Delta_i x$):

\begin{equation}
X_i = \rho_i (T_i \tilde{X}_i), \quad i = 1, \ldots, N.
\end{equation}

Other forward and backward data of the lattice $x$ are related through the following formulas

\begin{align}
\tilde{H}_i &= \rho_i T_i H_i, \\
\rho_j T_j \tilde{Q}_{ij} &= \rho_i T_i Q_{ji}.
\end{align}
Moreover, the factors $\rho_i$ are satisfy equations
\begin{equation}
\frac{T_i\rho_i}{\rho_i} = 1 - (T_i(Q_{ji})(T_jQ_{ij}), \quad i \neq j,
\end{equation}
which imply existence of yet another potential (the $\tau$-function of the quadrilateral lattice) such that
\begin{equation}
\rho_i = \frac{T_i\tau}{\tau}.
\end{equation}

Given a quadrilateral lattice $x$, the forward data $\{X_i, H_i, Q_{ij}\}$ and the backward data $\{\tilde{X}_i, \tilde{H}_i, \tilde{Q}_{ij}\}$ are defined up to rescaling by functions $a_i(m_i)$, $b_i(m_i)$ of single variables
\begin{align}
X_i &\rightarrow a_iX_i, & T_iH_i &\rightarrow \frac{1}{a_i}T_iH_i, & T_jQ_{ij} &\rightarrow \frac{a_i}{a_j}T_jQ_{ij}, \\
\tilde{X}_i &\rightarrow \frac{1}{b_i}T_i\tilde{X}_i, & \tilde{H}_i &\rightarrow b_i\tilde{H}_i, & \tilde{T}_j\tilde{Q}_{ij} &\rightarrow \frac{b_i}{b_j}T_j\tilde{Q}_{ij},
\end{align}
then also
\begin{equation}
\rho_i \rightarrow a_ib_i\rho_i.
\end{equation}

Remark. Since $Q_{ij}$ and $\tilde{Q}_{ij}$ are both solutions of the discrete Darboux equations (2.7), then equations (2.13)-(2.16) describe a special symmetry transformations of (2.7), first found in [27] without any associated geometric meaning.

3. The C-quadrilateral lattice

3.1. Geometric definition of the C-quadrilateral lattice. The geometric arena, where the reduction of the quadrilateral lattice studied in this paper lives, is the affine space. Recall that the affine transformations are these projective transformations which leave invariant a fixed hyperplane $H_\infty \subset \mathbb{P}^M$, called the hyperplane at infinity (see, for example [8]). Two lines of $\mathbb{A}^M = \mathbb{P}^M \setminus H_\infty$ (identified with $FF^M$, up to fixing the origin) are called parallel if they intersect in a point of $H_\infty$.

Definition 3.1. A hexahedron with planar faces in the affine space $\mathbb{A}^M$ is called a C-hexahedron if the three points obtained by intersection of the common lines of the pairs of planes of its opposite faces with the hyperplane $H_\infty$ at infinity are collinear (see Figure 3).

Remark. We exclude for a time being from our considerations the degenerate case when a pair of opposite faces of the hexahedron is parallel, i.e. their common line belongs to $H_\infty$.

Corollary 3.1. Equivalently the C-hexahedron is characterized by coplanarity of suitable parallel shifts of the intersection lines of the opposite face planes.

Definition 3.2. A quadrilateral lattice $x: \mathbb{Z}^N \rightarrow \mathbb{A}^M$ is called a C-quadrilateral lattice (CQL) if all its hexahedra are C-hexahedra.

The following Proposition gives an analytic characterization of C-quadrilateral lattices in terms of their rotation coefficients.

Proposition 3.2. A quadrilateral lattice is subject to C-reduction if and only if its rotation coefficients satisfy the constraint
\begin{equation}
(T_jQ_{ij})(T_kQ_{jk})(T_iQ_{ki}) = (T_jQ_{kj})(T_kQ_{ik})(T_iQ_{ji}), \quad i,j,k \text{ distinct}.
\end{equation}

Proof. Denote by $t^i_{ij}$ ($i,j,k$ are distinct) the direction vector of the common line of the plane $\langle x, T_i x, T_j x \rangle$ and its $k$-opposite $\langle T_k x, T_i x, T_j x \rangle$. It must be therefore decomposed in the basis $\{X_i, X_j\}$ and in the basis $\{T_iX_i, T_kX_k\}$. Assuming its decomposition in the second basis we get
\begin{equation}
t^i_{ij} = aT_kX_i + bT_kX_j = aX_i + bX_j + (aT_kQ_{ik} + bT_kQ_{jk})X_k,
\end{equation}
where we have used the linear problem (2.5). Because the coefficient in front of $X_k$ must vanish, the vector can be therefore chosen as
\begin{equation}
t^k_{ij} = T_kQ_{jk}X_i - T_kQ_{ik}X_j.
\end{equation}
Figure 3. The C-hexahedron

Notice that the condition in Definition 3.2 is equivalent to coplanarity of the vectors \( t_{ij} \), \( t_{ik} \), and \( t_{jk} \), and the statement follows from equation

\[
t_{ij} \wedge t_{ik} \wedge t_{jk} = (T_j Q_{ij})(T_k Q_{jk})(T_i Q_{ki}) - (T_j Q_{kj})(T_k Q_{ik})(T_i Q_{ji}) X_i \wedge X_j \wedge X_k, \quad i, j, k \text{ distinct.}
\]

□

Remark. In the degenerate case, when a pair of opposite faces of the hexahedron is parallel, the corresponding rotation coefficients vanish. Because they appear on different sides of equation (3.1), then the constraint is automatically satisfied.

As it was shown in [18], the constraint (3.1) allows to rescale the forward and backward data, using possibility given by equations (2.18)-(2.19), to the form such that

\[
Q_{ij} = \tilde{Q}_{ij}, \quad \text{or} \quad \rho_i T_i Q_{ji} = \rho_j T_j Q_{ij}, \quad i \neq j,
\]

i.e., the rotation coefficients of CQL are, in a sense, symmetric with respect to interchanging of their indices. Under such a constraint equations (2.16) and (2.17) allow to express the rotation coefficients in terms of the \( \tau \)-function as follows [33]

\[
(T_j Q_{ij})^2 = \frac{T_i \tau}{T_j \tau} \left( 1 - \frac{(T_i T_j \tau)(T_i \tau)}{(T_i \tau)(T_j \tau)} \right).
\]
Then the discrete Darboux equations equations \((3.7)\) can be rewritten in the following quartic form
\[
(T_{i} T_j T_k τ − T_j T_k T_i τ − T_k T_i T_j τ − T_i T_k T_j τ)^2 = 0
\]
\[(3.5)\]
\[
4(T_i T_j T_k T_l + T_i T_k T_l T_j + T_j T_k T_l T_i − T_j T_k T_l T_i)
\]
called in \([33]\) the discrete CKP equation.

Finally, we present a result which we will use in Section \([5]\). Its formal algebraic proof can be found in \([29]\) but, essentially, it uses the facts that (i) the functions \(ρ_i\) connect solutions of the forward and backward linear problems, (ii) a solution of the adjoint linear problem \((2.10)\) is connected in this correspondence with solution of the linear problem \((2.5)\) but with backward rotation coefficients, and (iii) backward and forward rotation coefficients in the CQL reduction coincide.

**Lemma 3.3** (\([29]\)). Given solution \(Y_i^* : Z^N \to (P^K)^*\) of the adjoint linear problem \((2.6)\) for “symmetric” rotation coefficients \((3.3)\) then \(ρ_i(T_i Y_i^*)^T : Z^N \to P^K\) satisfies the corresponding linear problem \((2.5)\).

### 3.2. Multidimensional consistency of the C-constraint

As it was shown in \([17]\) the planarity condition, which allows to construct the point \(x_{123}\) as in Lemma \([1.1]\) does not lead to any further restrictions if we increase dimension of the lattice. Because in the CQL case the constraint is imposed on the 3D (elementary hexahedra) level, then to assure its multidimensional consistency we have to check the four dimensional consistency. The multidimensional consistency of the C-reduction with the geometric integrability scheme would be the immediate consequence of its 4D consistency.

In fact, the consistency of the constraint has been proved algebraically in \([18]\) starting from its algebraic form \((3.3)\). However, in that proof the main difficulty was shifted to the proof of existence (in multidimensions) of the special choice of the \(τ\)-function. In this paper we present pure geometric proof of the 4D consistency of the CQL. We first recall the relevant result on four dimensional consistency of the QL, which is the consequence of the of the following geometric observation.

**Lemma 3.4** (The 4D consistency of the geometric integrability scheme). Consider points \(x_0, x_1, x_2, x_3\) and \(x_4\) in general position in \(P^M\), \(M ≥ 4\). Choose generic points \(x_{ij} \in \langle x_0, x_i, x_j \rangle\), \(1 ≤ i < j ≤ 4\), on the corresponding planes, and using the planarity condition construct the points \(x_{ijk} \in \langle x_0, x_i, x_j, x_k \rangle\), \(1 ≤ i < j < k ≤ 4\) — the remaining vertices of the four (combinatorial) cubes. Then the intersection point \(x_{1234}\) of the three planes
\[
\langle x_{12}, x_{123}, x_{124} \rangle, \langle x_{13}, x_{123}, x_{134} \rangle, \langle x_{14}, x_{124}, x_{134} \rangle \quad \text{in} \quad \langle x_1, x_{12}, x_{13}, x_{14} \rangle,
\]
coincides with the intersection point of the three planes
\[
\langle x_{12}, x_{123}, x_{124} \rangle, \langle x_{23}, x_{123}, x_{234} \rangle, \langle x_{24}, x_{124}, x_{234} \rangle \quad \text{in} \quad \langle x_2, x_{12}, x_{23}, x_{24} \rangle,
\]
which is the same as the intersection point of the three planes
\[
\langle x_{13}, x_{123}, x_{134} \rangle, \langle x_{23}, x_{123}, x_{234} \rangle, \langle x_{34}, x_{134}, x_{234} \rangle \quad \text{in} \quad \langle x_3, x_{13}, x_{23}, x_{34} \rangle,
\]
and the intersection point of the three planes
\[
\langle x_{14}, x_{124}, x_{134} \rangle, \langle x_{24}, x_{124}, x_{234} \rangle, \langle x_{34}, x_{134}, x_{234} \rangle \quad \text{in} \quad \langle x_4, x_{14}, x_{24}, x_{34} \rangle.
\]

**Remark.** In fact, the point \(x_{1234}\) is the unique intersection point of the four three dimensional subspaces \(\langle x_1, x_{12}, x_{13}, x_{14} \rangle, \langle x_2, x_{12}, x_{23}, x_{24} \rangle, \langle x_3, x_{13}, x_{23}, x_{34} \rangle, \langle x_4, x_{14}, x_{24}, x_{34} \rangle\) of the four dimensional subspace \(\langle x_0, x_1, x_2, x_3, x_4 \rangle\). This observation generalizes naturally to the case of more dimensional hypercube with the planar facets.

It turns out that the geometric core of the integrability of CQL is provided by the Gallucci’s theorem on eight skew lines (see, for example \([8]\)).

**Theorem 3.5** (The Gallucci theorem). If three skew lines all meet three other skew lines, any transversal to the first set of three meets any transversal to the second set.

Motivated by the corresponding results of \([26]\) we first prove a useful Lemma, where the Gallucci theorem is used.

**Lemma 3.6.** Consider four adjacent hexahedra with planar faces which share a vertex of a 4D hypercube. If three of them are C-hexahedra then the same holds also for the forth one.
Proof. We start from generic 4D hexahedron with planar faces. Denote by \(\pi_{i,j,k}\) the three dimensional subspace of \(H_\infty\) being its intersection with the 4D subspace \(\langle x_0, x_1, x_2, x_3, x_4 \rangle\) of the hexahedron. By \(\pi_{i,j,k}\) denote the intersection line of the 3D subspace \(\langle x_0, x_1, x_j, x_k \rangle\) of one of the four 3D hexahedra with \(H_\infty\).

Denote by \(h_{i,j}^k\) the intersection points of the opposite planes \(\langle x_0, x_i, x_j, x_k \rangle\), \(\langle x_k, x_i, x_j, x_0 \rangle\) of the four 3D hexahedra with \(H_\infty\) (all indices are distinct and range from 1 to 4, eventually they should be reordered); these are the points entering into the definition of the C-hexahedra. By \(h_{i,j}^k\) denote the intersection lines of the three dimensional subspaces \(\langle x_0, x_i, x_j, x_k \rangle\), \(\langle x_0, x_i, x_j, x_0 \rangle\) and \(H_\infty\). There are four lines of this type (the \(h\)-family). Notice that point \(h_{i,j}^k\) is the intersection point of \(h_{i,j}^k\) with the plane \(\pi_{i,j,k}\). On the plane \(\pi_{i,j,k}\) we have therefore apart from three collinear point \(h_{i,j}^k\), \(h_{i,k}^j\) and \(h_{j,k}^i\) (they belong to \(h_{i,j}^k\)) also three other points \(h_{i,j}^k\), \(h_{i,k}^j\) and \(h_{j,k}^i\).

The C-reduction condition of the hexahedron with with vertices \(x_0, x_i, x_j\) and \(x_k\) means collinearity of the points points \(h_{i,j}^k\), \(h_{i,k}^j\) and \(h_{j,k}^i\). If such a line exists we denote it by \(g_{i,j,k}\). It intersects not only lines \(h_{i,j}^k\), \(h_{i,k}^j\) and \(h_{j,k}^i\) (in points \(h_{i,j}^k\), \(h_{i,k}^j\) and \(h_{j,k}^i\), correspondingly), but also the fourth line \(h_{i,j}^k\) of the \(h\)-family, as both lines belong to the plane \(\pi_{i,j,k}\).

Let us assume that the C-reduction condition is satisfied for three hexahedra, i.e., three such \(g\)-lines exist, all transversal to four \(h\)-lines. Assume that \(g_{i,j,k}\), \(g_{j,i,k}\) and \(g_{j,k,i}\) exist, define the line \(G_{j,k,l}\) as the unique line passing through the points \(h_{i,j}^k\) and \(h_{i,j}^l\), thus transversal to the lines \(h_{i,k}^j\) and \(h_{j,k}^i\). Because \(G_{j,k,l}\) is contained in \(\pi_{j,k,l}\) then it must intersect also the line \(h_{i,j}^k\) — the third line of the \(h\)-family. By Gallucci’s theorem it intersects therefore the forth line \(h_{i,j}^k\).

That intersection point belongs to the plane \(\pi_{j,k,l}\) (containing the points \(h_{i,j}^k\) and \(h_{i,j}^l\) which define \(G_{j,k,l}\) therefore it must be \(h_{i,j}^k = h_{i,j}^k \cap \pi_{j,k,l}\). We have therefore shown that also the three points \(h_{i,j}^k\), \(h_{i,k}^j\) and \(h_{j,k}^i\) are collinear, i.e., \(G_{j,k,l} = g_{j,k,l}\).

\[\text{Proposition 3.7.} \quad \text{Under hypotheses of Lemma 3.6 assume that the C-reduction condition is satisfied for 3D hexahedra meeting in one vertex of a 4D hypercube. Then the condition holds for all hexahedra of the hypercube.}\]

\[\text{Proof.} \quad \text{Without loss of generality assume that the common vertex of the four hexahedra (we know that it is enough to assume the C-reduction condition for three of them) is } x_0. \text{ Then three of them share the vertex } x_i, \text{ which implies the constraint for the forth one. All remaining four hexahedra of the 4D hypercube which do not share the point } x_0 \text{ are of that type.}\]

On the level of the multidimensional quadrilateral lattice there exists simple alternative algebraic proof of an analogue of Lemma 3.6 which combined with ideas behind the proof of Proposition 3.2 would give the algebraic proof of the Lemma.

\[\text{Corollary 3.8.} \quad \text{Consider a 4D hexahedron with vertices } x, T_jx, T_jx, T_kx, T_kx \text{ of the quadrilateral lattice } x : \mathbb{Z}^N \to \mathbb{Z}^M, 4 \leq N \leq M. \text{ If three of the four 3D hexahedra meeting in the vertex } x \text{ are the C-hexahedra then the same holds also for the forth one.}\]

\[\text{Proof.} \quad \text{Assume that the fourth one is the hexahedron with basic vertices } x, T_jx, T_kx \text{ and } T_kx. \text{ By Proposition 3.2 the C-reduction condition for the remaining three reads}\]

\[\text{(3.6) } (T_jQ_{ij})(T_kQ_{jk})(T_kQ_{ki}) = (T_jQ_{ij})(T_kQ_{ik})(T_jQ_{ji}), \]

\[\text{(3.7) } (T_kQ_{ik})(T_jQ_{ij})(T_kQ_{il}) = (T_jQ_{ij})(T_jQ_{jl})(T_kQ_{il}), \]

\[\text{(3.8) } (T_kQ_{ik})(T_jQ_{ij})(T_kQ_{li}) = (T_kQ_{ki})(T_kQ_{kl})(T_kQ_{li}). \]

Multiplying the equations we obtain (assuming that the rotation coefficients do not vanish)

\[\text{(3.9) } (T_kQ_{jk})(T_jQ_{ij})(T_kQ_{lk}) = (T_jQ_{kj})(T_kQ_{jl})(T_kQ_{lk}). \]

\[\square\]
4. Algebro-geometric construction of the C-quadrilateral lattice

In this Section we apply the algebro-geometric approach, well known in the theory of integrable systems \([3]\), to the symmetric Darboux equations and to the C-quadrilateral lattice. Similar restrictions on the algebro-geometric data appeared in \([1]\) in construction of quasi periodic solutions of the CKP hierarchy. For discrete Laplace equations the algebro-geometric techniques were first applied in \([28]\). In \([1]\) the Egorov reduction of the symmetric lattice \([22]\) was studied by means of the algebro-geometric methods. In 1998 Peter Grinevich isolated from \([1]\) these of the algebro-geometric conditions which give rise the symmetric quadrilateral lattices \([23]\). Below we present that result on algebro-geometric description of the C-(symmetric) quadrilateral lattice within broader context of various discrete Baker–Akhiezer functions.

In this Section we work over the field of complex numbers, i.e., \(F = C\), and we discuss the corresponding reality conditions. We mention that many of the results given below can be transferred to the finite-field case, as in \([3, 22]\).

4.1. The Baker–Akhiezer function of the quadrilateral lattice. Let us consider a compact non-degenerate Riemann surface \(\mathcal{R}\) of genus \(g\), a non-special divisor \(D = P_1 + \cdots + P_n\) on \(\mathcal{R}\), the \(N\) pairs of points \(Q_i^\pm \in \mathcal{R}\), and the normalization point \(Q_\infty \in \mathcal{R}\). The system \(\{\mathcal{R}, D, Q_i^\pm, Q_\infty\}\) is called the algebro-geometric data used in the construction of the quadrilateral lattice. For simplicity we assume that all the points above (including the points of \(D\)) are distinct.

For \(m \in \mathbb{Z}^N\) we define the meromorphic function \(\psi(m) : \mathcal{R} \to \mathbb{CP}\) by prescribing its analytical properties:

(i) as a function on \(\mathcal{R} \setminus \cup_{i=1}^N \{Q_i^\pm\}\) it may have as singularities only simple poles in points of the divisor \(D\);

(ii) in points \(Q_i^+\) (in points \(Q_i^-\)) it has poles (correspondingly, zeros) of the order \(m_i\), where by the pole of the negative order we mean zero of the corresponding order;

(iii) in the point \(Q_\infty\), the function \(\psi\) is normalized to 1.

By the standard (see \([1, 3]\)) application of the Riemann–Roch theorem, such a function exists and is unique. When \(z_i^\pm(P)\) is a local coordinate centered at \(Q_i^\pm\) then \(\psi\) in a neighbourhood of the point \(Q_i^\pm\) is of the form

\[
\psi(m|P) = (z_i^\pm(P))^{-m_i} \left( \sum_{s=0}^\infty \xi_s^i(m) (z_i^\pm(P))^s \right).
\]

By the standard reasoning in the finite-gap theory one can prove the following result \([1]\) which connects the function \(\psi\) with the quadrilateral lattice theory.

**Theorem 4.1** \((1)\). For arbitrary point \(P \in \mathcal{R}\) the Baker–Akhiezer function \(\psi\) satisfies in the variable \(m \in \mathbb{Z}^N\) the following system of linear equations

\[
\Delta_i \Delta_j \psi(m|P) = \left( T_i^0 \frac{\xi_j^i(m)}{\xi_j^i(0)} \right) \Delta_j \psi(m|P) + \left( T_j^0 \frac{\xi_j^i(m)}{\xi_j^i(0)} \right) \Delta_j \psi(m|P), \quad i \neq j.
\]

**Corollary 4.2.** Let \(d\mu(P) = (d\mu_1(P), \ldots, d\mu_M(P))^t\) be a vector valued measure on \(\mathcal{R}\), then the function

\[
\mathbf{x}(m) = \int_\mathcal{R} \psi(m|P) d\mu(P),
\]

defines a quadrilateral lattice (in the complex affine space) with the functions \(\xi_0^i\) as the Lamé coefficients \(H_i\).

**Remark.** Similarly one shows that \(\mathbf{x}\) satisfies the backward Laplace equations \([2, 12]\) with the backward Lamé coefficients (proportional to) \(\xi_0^{-i}\); see also Proposition \([4, 6]\).

In order to have real quadrilateral lattices we must impose on the algebro-geometric data certain reality restrictions.

**Corollary 4.3.** Assume that the Riemann surface \(\mathcal{R}\) allows for an anti-holomorphic involution \(i\). If

\[
i(D) = D, \quad i(Q_i^\pm) = Q_i^\mp, \quad i(Q_\infty) = Q_\infty, \quad d\mu(i(P)) = d\mu(P),
\]

then...
then the lattice \( x \), given by equation (4.3), is real. Moreover, if the local coordinate systems \( z_i^\pm \) are compatible with \( i \), i.e.,

\[
z_i^\pm(i(P)) = z_i^\pm(P),
\]

then the Lamé coefficients, as defined above, are real functions.

4.2. Other Baker–Akhiezer functions. In this Section we define auxiliary Baker-Akhiezer functions which play the role of the forward and backward normalized tangent vectors. Then we define the dual (adjoint) Baker-Akhiezer function of the quadrilateral lattice.

4.2.1. The forward Baker–Akhiezer functions \( \psi_i \). Below we consider new functions \( \psi_i \) whose relation to \( \psi \) is analogous to that between \( X_i \) to \( x \). Given \( m \in \mathbb{Z}^N \) define \([\Pi]\) the functions \( \psi_i(m) \) as meromorphic functions on \( \mathcal{R} \) having the following analytic properties:

(i) as a function on \( \mathcal{R} \setminus \bigcup_{i=1}^N \{ Q_i^\pm \} \) it may have as singularities only simple poles in points of the divisor \( D \);

(ii) in points \( Q_j^+ \) (in points \( Q_j^- \)) it has poles of the order \( m_j + \delta_{ij} \) (correspondingly, zeros of the order \( m_j \));

(iii) in the point \( Q_\infty \) the function \( \psi \) is equal to 0.

By the Riemann–Roch theorem the space of such functions is one dimensional. By choosing local coordinates \( \psi_j^+ \) near \( Q_j^+ \) the function \( \psi_i \) can be made unique by fixing its lowest order term at \( Q_j^+ \) to one. Then near \( Q_j^+ \) we have the following local expansions

\[
\psi_i(m|P) = \left( z_j^+(P) \right)^{m_j} \left( \frac{\delta_{ij} \zeta_i^+ \left( \frac{\psi_i(m)}{z_j^+(P)} \right) + \sum_{s=0}^{\infty} \zeta_{i,s}^+(m) \left( \frac{\psi_i(m)}{z_j^+(P)} \right)^s}{\zeta_j^+(P)} \right).
\]

**Theorem 4.4 ([\Pi]).** The functions \( \psi_i \) satisfy the equations

\[
\Delta_i \psi_i(m|P) = (T_i \xi_{i,0}^+(m)) \psi_i(m|P),
\]

\[
\Delta_j \psi_i(m|P) = (T_j \zeta_{i,j}^+(m)) \psi_i(m|P), \quad j \neq i,
\]

whose expansion at \( Q_k^+ \), gives

\[
\Delta_i \xi_{k,0}^+ = (T_i \xi_{i,0}^+(m)) \xi_{i,k}^+, \quad k \neq i,
\]

\[
\Delta_j \zeta_{i,j}^+ = (T_j \zeta_{i,j}^+(m)) \zeta_{i,j}^+, \quad k \neq j,
\]

and allows for the identification

\[
Q_{ij}(m) = \zeta_{i,j}^+(m).
\]

**Corollary 4.5.** In notation of Corollary 4.2 we have

\[
X_i(m) = \int_\mathcal{R} \psi_i(m|P) d\mu(P).
\]

**Remark.** Notice that different choices of local coordinates \( z_i^+ \) correspond to rescaling of the forward data in agreement with equation (2.18).

4.2.2. The backward Baker–Akhiezer functions \( \tilde{\psi}_i \). We define the corresponding algebro-geometric analog of the backward normalized tangent vectors \( \tilde{X}_i \). Given \( m \in \mathbb{Z}^N \) define the functions \( \tilde{\psi}_i(m) \) as meromorphic functions on \( \mathcal{R} \) having the following analytic properties:

(i) as a function on \( \mathcal{R} \setminus \bigcup_{i=1}^N \{ Q_i^\pm \} \) it may have as singularities only simple poles in points of the divisor \( D \);

(ii) in points \( Q_j^+ \) (in points \( Q_j^- \)) it has poles of the order \( m_j \) (correspondingly, zeros of the order \( m_j - \delta_{ij} \));

(iii) in the point \( Q_\infty \) the function \( \psi \) is equal to 0.

By the Riemann–Roch theorem the space of such functions is one dimensional. By choosing local coordinates \( \psi_j^- \) near \( Q_j^- \) the function \( \psi_i \) can be made unique by fixing its lowest order term at \( Q_j^- \) to one. Then near \( Q_j^- \) we have the following local expansions

\[
\tilde{\psi}_i(m|P) = \left( z_j^-(P) \right)^{m_j} \left( \frac{\delta_{ij} \zeta_i^- \left( \frac{\psi_i(m)}{z_j^-(P)} \right) + \sum_{s=0}^{\infty} \zeta_{i,s}^- (m) \left( \frac{\psi_i(m)}{z_j^-(P)} \right)^s}{\zeta_j^- (P)} \right).
\]
By the standard methods [3] one can prove the following analog of Theorem [4.3].

**Proposition 4.6.** The functions $\psi$ and $\tilde{\psi}_i$ are connected by the formulas

\begin{align}
\Delta_i \psi(m|P) &= - \left( T_i \tilde{\psi}_i(m|P) \right) \xi_{i0}^-(m), \\
\Delta_j \tilde{\psi}_i(m|P) &= - \left( T_j \tilde{\psi}_j(m|P) \right) \tilde{\xi}_{i0}^-(m), \quad j \neq i,
\end{align}

whose expansion at $Q_k^\pm$, gives

\begin{align}
\Delta_i \xi_{0}^{k,-} &= - \left( T_{i} \tilde{\xi}_{i0}^{k,-} \right) \xi_{0}^{k,-}, \quad k \neq i, \\
\Delta_j \tilde{\xi}_{i0}^{k,-} &= \left( T_{j} \tilde{\xi}_{j0}^{k,-} \right) \tilde{\xi}_{i0}^{k,-}, \quad k \neq j,
\end{align}

and allows for the identification

\begin{align}
\tilde{H}_i(m) &= - \xi_{i0}^-(m), \quad \tilde{Q}_{ij}(m) = - \tilde{\xi}_{j0}^-(m).
\end{align}

**Corollary 4.7.** In notation of Corollary [4.3] we have

\begin{align}
\tilde{X}_i(m) &= \int_{R} \tilde{\psi}_i(m|P) d\mu(P).
\end{align}

Moreover, by comparing the analytical properties the functions $\psi_i$ and $\tilde{\psi}_i$ we obtain

\begin{align}
\psi_i(m|P) &= \left( T_i \tilde{\psi}_i(m|P) \right) \xi_{i0}^+(m), \\
T_i \tilde{\psi}_i(m|P) &= \left( T_i \tilde{\xi}_{i0}^{i,+} \right) \psi_i(m|P),
\end{align}

which allows for the identification

\begin{align}
\rho_i(m) &= \xi_{i0}^{i,+} \left( m \right) = \frac{1}{T_i \xi_{i0}^{i,+}}.
\end{align}

**Remark.** Notice that different choices of local coordinates $z_i^-$ correspond to rescaling of the backward data given by equation (2.19).

**Remark.** We are not concerned here about explicit theta-function formulas for the Baker–Akhiezer functions and related potentials, see however [1]. In particular, the $\tau$-function of the quadrilateral lattice is, essentially [15], the Riemann theta function.

### 4.2.3. The dual Baker–Akhiezer functions.

In definition of the dual (adjoint) Baker–Akhiezer function of the quadrilateral lattice we use the idea applied in [10] to construction of the adjoint Baker–Akhiezer function of the KP hierarchy.

Denote by $\omega_\infty$ the meromorphic differential with the only singularity being the second order pole at $Q_\infty$, and whose holomorphic part is normalized by vanishing of $\omega_\infty$ at points of the divisor $D$

\begin{align}
\omega_\infty(P_i) &= 0, \quad i = 1, \ldots, g.
\end{align}

**Remark.** By choosing a coordinate system $z_\infty(P)$ centered at $Q_\infty$ the differential $\omega_\infty$ can be made unique by fixing its singular part in $z_\infty(P)$ as

\begin{align}
\omega_\infty(P) &= \left( \frac{1}{z_\infty(P)^2} + O(1) \right) dz_\infty(P),
\end{align}

but we will not use that in the sequel.

Denote by $D^*$ the divisor of other $g$ zeros of $\omega_\infty$, and use it to define the dual Baker–Akhiezer function $\psi^*$ exchanging also the role of the points $Q_\infty^+$ and $Q_\infty^-;

(i) as a function on $\mathcal{R} \setminus \cup_{i=1}^{n} \{ Q_i^+ \}$ it may have as singularities only simple poles in points of the divisor $D^*$;

(ii) in points $Q_i^+$ (in points $Q_i^-$) it has zeros (correspondingly, poles) of the order $m_i$;

(iii) in the point $Q_\infty$ the function $\psi^*$ is normalized to 1.
Using the Riemann–Roch theorem one can show that such function \( \psi^*(m|P) \) exists and is unique. In a neighbourhood of the point \( Q^+_i \) it is of the form

\[
(4.25) \quad \psi^*(m|P) = \left( z^\pm_i(P) \right)^{\pm m_i} \left( \sum_{s=0}^{\infty} \xi^{{*i},\pm}_s(m) \left( z^\pm_i(P) \right)^s \right).
\]

Using the similar procedure like in the previous section it can be shown that the dual function \( \psi^* \) satisfies the Laplace equations with Lamé coefficients \( \xi^{{*i},-}_0 \), and it satisfies the backward Laplace equations with the backward Lamé coefficients \( \xi^{{*i},+}_0 \).

**Remark.** The meromorphic differential form

\[
(4.26) \quad \omega = \psi^* \omega_\infty
\]

is singular only at \( Q_\infty \) with the singularity being the second order pole. By the residue theorem the integral of \( \omega \) around a closed contour around \( Q_\infty \) vanishes, which is the quadrilateral lattice counterpart of the celebrated bilinear identity [10] on the algebro-geometric level.

In analogy to the Baker–Akhiezer functions \( \psi \) and \( \tilde{\psi}_i \) we may define the corresponding dual Baker–Akhiezer functions. In the sequel we will need the analog of \( \tilde{\psi}_i \), which is defined as follows. Given \( m \in \mathbb{Z}^N \) define the functions \( \psi^*_i(m) \) as meromorphic functions on \( \mathcal{R} \) having the following analytic properties:

(i) as a function on \( \mathbb{R} \setminus \psi_\circ \mathcal{R} \); \( \sigma \)

\[
(4.31) \quad (\psi_\circ \psi^*_i(m))_{\mathcal{R} \setminus \psi_\circ \mathcal{R}} = (\psi^*_i(m))_{\mathcal{R} \setminus \psi_\circ \mathcal{R}}.
\]

**Proposition 4.9.** Assume that the type of restrictions appeared in [11] in construction of quasi-periodic solutions of the CKP hierarchy.

\[
(4.29) \quad a^\pm_{i,0}(m) + a^\pm_{j,0}(m) = 0, \quad i \neq j.
\]

**4.3. The algebro-geometric C-quadrilateral lattices.** Finally, we show that under certain restrictions on the algebro-geometric data the finite-gap construction gives C-reduced quadrilateral lattice. This type of restrictions appeared in [11] in construction of quasi-periodic solutions of the CKP hierarchy.

**Proposition 4.9.** Assume that \( \mathcal{R} \) is equipped with the holomorphic involution \( \sigma : \mathcal{R} \to \mathcal{R} \) such that

\[
(4.30) \quad \sigma(D^*) = D, \quad \sigma(Q^+_i) = Q^+_i, \quad \sigma(Q^+_\infty) = Q^+_\infty,
\]

then

\[
(4.31) \quad \psi \circ \sigma = \psi^*,
\]

\[
(4.32) \quad \tilde{\psi}_i \circ \sigma = c_i \psi^*_i, \quad c_i \in \mathbb{C}.
\]

**Proof.** In the standard way we compare analytic properties of both sides of each equation. The function \( \psi \circ \sigma \) has the following analytic properties:

(i) as a function on \( \mathcal{R} \setminus \cup_{i=1}^N \{ Q^+_i \} \) it may have as singularities only simple poles in points of the divisor \( D^* \);

(ii) in points \( Q^+_i \) (in points \( Q^+_i \)) it has zeros (correspondingly, poles) of the order \( m_i \);
(iii) in the point $Q_\infty$ the function $\psi \circ \sigma$ is normalized to 1.

Comparison with the analytic properties of $\psi^*$ and the Riemann–Roch theorem gives equation (4.31).

Let us describe the analytic properties of the superposition $\tilde{\psi}_i \circ \sigma$:

(i) as a function on $R \setminus \cup_{j=1}^N \{Q_j^+\}$ it may have as singularities only simple poles in points of the divisor $D^*$;

(ii) in points $Q_j^+$ (in points $Q_j^-$) it has zeros of the order $m_j - \delta_{ij}$ (correspondingly, poles of the order $m_j$);

(iii) in the point $Q_\infty$ the function $\tilde{\psi}_i \circ \sigma$ is equal to 0.

Therefore the function $\tilde{\psi}_i \circ \sigma$ must be proportional to $\psi^*_i$.

**Corollary 4.10.** Notice that when the local coordinates, which fix normalization of the functions, are chosen in agreement with the involution $\sigma$

\begin{equation}
\tag{4.33}
z_i^-(\sigma(P)) = z_i^+(P),
\end{equation}

then the proportionality in equation (4.32) becomes equality (i.e., $c_i = 1$). Moreover, under such conditions the expansions (4.13) and (4.27) give

\begin{equation}
\tag{4.34}
c_{i,s}^{j,+}(m) = c_{i,s}^{j,-}(m).
\end{equation}

**Theorem 4.11 (23).** Under assumptions of Proposition 4.9, the quadrilateral lattice constructed according to Corollary 4.2 is subject to the C-(symmetric) reduction.

**Proof.** Assume for a time being that the local coordinates $z_i^+$ are chosen in such a way that the first coefficients $a_i^+$ of the expansion of $\omega_i$ near points $Q_i^+$ are equal (see Proposition 4.8), and the local coordinates $z_i^-$ are chosen according to equation (4.33) we may think of this special choice as using the allowed freedom (2.18)-(2.19) in definition of the backward and forward data. Then equations (4.29) and (4.34) imply that

\begin{equation}
\tag{4.35}
c_{i,0}^{j,\pm}(m) = -\tilde{c}_{i,0}^{j,\pm}(m), \quad \text{or} \quad Q_{ij} = \tilde{Q}_{ij}.
\end{equation}

\[\square\]

## 5. Transformation of the C-quadrilateral lattice

We introduce geometrically the C-reduction of the fundamental transformation of the quadrilateral lattice. We also connect this definition with earlier algebraic results of [29]. Then we prove the corresponding permutability theorem for this transformation.

### 5.1. The vectorial fundamental transformation of the quadrilateral lattice

Let us first recall some basic facts concerning the vectorial fundamental transformation of the quadrilateral lattice. Geometrically, the (scalar) fundamental transformation is the relation between two quadrilateral lattices $x$ and $x'$ such that for each direction $i$ the points $x$, $x'_i$, and $x'_i$ are coplanar.

We present below the algebraic description of its vectorial extension (see [30] [20] [29] for details) in the affine formalism. Given the solution $Y_i : \mathbb{Z}^N \rightarrow \mathbb{R}^K$, of the linear system (2.4), and given the solution $Y_i^* : \mathbb{Z}^N \rightarrow (\mathbb{R}^K)^*$, of the linear system (2.0). These allow to construct the linear operator valued potential $\Omega(Y, Y^*) : \mathbb{Z}^N \rightarrow M_K^K(\mathbb{R})$, defined by

\begin{equation}
\tag{5.1}
\Delta_i \Omega(Y, Y^*) = Y_i \otimes T_i Y_i^*, \quad i = 1, \ldots, N;
\end{equation}

similarly, one defines $\Omega(X, Y^*) : \mathbb{Z}^N \rightarrow M_K^K(\mathbb{R})$ and $\Omega(Y, H) : \mathbb{Z}^N \rightarrow \mathbb{R}^K$ by

\begin{align}
\Delta_i \Omega(X, Y^*) &= X_i \otimes T_i Y_i^*, \\
\Delta_i \Omega(Y, H) &= Y_i \otimes T_i H_i.
\end{align}

**Proposition 5.1.** If $\Omega(Y, Y^*)$ is invertible then the vector function $x' : \mathbb{Z}^N \rightarrow \mathbb{R}^M$ given by

\begin{equation}
\tag{5.4}
x' = x - \Omega(X, Y^*) \Omega(Y, Y^*)^{-1} \Omega(Y, H),
\end{equation}

\[\square\]
represents a quadrilateral lattice (the fundamental transform of \( x \)), whose Lamé coefficients \( H'_i \), normalized tangent vectors \( X'_i \) and rotation coefficients \( Q'_{ij} \) are given by

\[
(5.5) \quad H'_i = H_i - Y'_i \Omega(Y, Y'^{-1}) Y_i, \\
(5.6) \quad X'_i = X_i - \Omega(X, Y'^{-1}) Y_i, \\
(5.7) \quad Q'_{ij} = Q_{ij} - Y'_i \Omega(Y, Y'^{-1}) Y_i.
\]

Moreover \cite{20}, the connection coefficients \( \rho_i \) and the \( \tau \)-function transform according to

\[
(5.8) \quad \rho'_i = \rho_i (1 + T Y'_i \Omega(Y, Y'^{-1}) Y_i), \\
(5.9) \quad \tau' = \tau \det \Omega(Y, Y'^{-1}).
\]

The vectorial fundamental transformation can be considered as superposition of \( K \) (scalar) fundamental transformations; on intermediate stages the rest of the transformation data should be suitably transformed as well. Such a description contains already the principle of permutability of such transformations, which follows from the following observation \cite{20}.

**Proposition 5.2.** Assume the following splitting of the data of the vectorial fundamental transformation

\[
(5.10) \quad Y_i = \left( \begin{array}{c} Y^a_i \\ Y^b_i \end{array} \right), \quad Y'^*_i = \left( \begin{array}{cc} Y^*_{ai} & Y^*_{bi} \end{array} \right),
\]

associated with the partition \( \mathbb{F}^K = \mathbb{F}^{K_a} \oplus \mathbb{F}^{K_b} \), which implies the following splitting of the potentials

\[
(5.11) \quad \Omega(Y, H) = \left( \begin{array}{c} \Omega(Y^a, H) \\ \Omega(Y^b, H) \end{array} \right), \quad \Omega(Y, Y'^*) = \left( \begin{array}{cc} \Omega(Y^a, Y'^*_a) & \Omega(Y^a, Y'^*_b) \\ \Omega(Y^b, Y'^*_a) & \Omega(Y^b, Y'^*_b) \end{array} \right),
\]

\[
(5.12) \quad \Omega(X, Y'^*) = \left( \begin{array}{cc} \Omega(X, Y'^*_a) & \Omega(X, Y'^*_b) \end{array} \right).
\]

Then the vectorial fundamental transformation is equivalent to the following superposition of vectorial fundamental transformations:

1) Transformation \( x \to x^{[a]} \) with the data \( Y^a_i, Y'^*_i \) and the corresponding potentials \( \Omega(Y^a, H), \Omega(Y^a, Y'^*_a), \Omega(X, Y'^*_a) \)

\[
(5.13) \quad x^{[a]} = x - \Omega(X, Y'^*_a) \Omega(Y^a, Y'^*_a)^{-1} \Omega(Y^a, H), \\
(5.14) \quad X'_i^{[a]} = X_i - \Omega(X, Y'^*_a) \Omega(Y^a, Y'^*_a)^{-1} Y'_i, \\
(5.15) \quad H'_i^{[a]} = H_i - Y'^*_a \Omega(Y^a, Y'^*_a)^{-1} \Omega(Y^a, H).
\]

2) Application on the result the vectorial fundamental transformation with the transformed data

\[
(5.16) \quad Y_i^{b(a)} = Y_i^b - \Omega(Y^b, Y'^*_a) \Omega(Y^a, Y'^*_a)^{-1} Y_i^a, \\
(5.17) \quad Y^{*a}_i = Y^{*a}_i - Y'^*_a \Omega(Y^a, Y'^*_a)^{-1} \Omega(Y^a, Y'^*_b),
\]

and potentials

\[
(5.18) \quad \Omega(Y^b, H)^{[a]} = \Omega(Y^b, H) - \Omega(Y^b, Y'^*_a) \Omega(Y^a, Y'^*_a)^{-1} \Omega(Y^a, H) = \Omega(Y^{b(a)}, H^{[a]}), \\
(5.19) \quad \Omega(Y^b, Y'^*_b)^{[a]} = \Omega(Y^b, Y'^*_b) - \Omega(Y^b, Y'^*_a) \Omega(Y^a, Y'^*_a)^{-1} \Omega(Y^a, Y'^*_b) = \Omega(Y^{b(a)}, Y'^*_b^{[a]}), \\
(5.20) \quad \Omega(X, Y'^*_b)^{[a]} = \Omega(X, Y'^*_b) - \Omega(X, Y'^*_a) \Omega(Y^a, Y'^*_a)^{-1} \Omega(Y^a, Y'^*_b) = \Omega(X^{[a]}, Y'^*_b^{[a]}),
\]

i.e.,

\[
(5.21) \quad x' = x^{(a,b)} = x^{[a]} - \Omega(X, Y'^*_b)^{[a]} \Omega(Y^b, Y'^*_b)^{[a]}^{-1} \Omega(Y^b, H)^{[a]},
\]

*Remark.* The same result \( x' = x^{(a,b)} = x^{(b,a)} \) is obtained exchanging the order of transformations, exchanging also the indices \( a \) and \( b \) in formulas \((5.13)-(5.21)\).
5.2. The CQL (symmetric) reduction of the fundamental transformation. In this section we describe restrictions on the data of the fundamental transformation in order to preserve the reduction from QL to CQL. As usually (see, for example [20], [13], [16]) a reduction of the fundamental transformation for a special quadrilateral lattice mimics the geometric properties of the lattice. Because the basic geometric property of the (scalar) fundamental transformation can be interpreted as construction of a ”new level” of the quadrilateral lattice, then it is natural to define the reduced transformation in a similar spirit. Our definition of the CQL reduction of the fundamental transformation is therefore based on the following observation.

Lemma 5.3. Given quadrilateral lattice \( x : \mathbb{Z}^{N} \rightarrow \mathbb{H}^{M} \) and its fundamental transform \( x' \) constructed under additional assumption that for any point \( x \) of the lattice and any pair \( i, j \) of different directions, the hexahedra with basic vertices \( x, T_{i}x, T_{j}x \) and \( x' \) satisfy the C-reduction condition. Then both the starting lattice \( x : \mathbb{Z}^{N} \rightarrow \mathbb{H}^{M} \) and its transform \( x' : \mathbb{Z}^{N} \rightarrow \mathbb{H}^{M} \) are C-quadrilateral lattices.

Proof. As \( N \geq 3 \), by Lemma 3.6 we have that also the hexahedra with basic vertices \( x, T_{i}x, T_{j}x \) and \( T_{k}x \), with \( i, j, k \) distinct, satisfy the C-reduction condition. The similar statement for the transformed lattice is a consequence of the 4-dimensional consistency of the CQL lattice. \( \square \)

Definition 5.1. The fundamental transform \( x' \) of a C-quadrilateral lattice \( x : \mathbb{Z}^{N} \rightarrow \mathbb{H}^{M} \) constructed under additional assumption that for any point \( x \) of the lattice and any pair \( i, j \) of different directions, the hexahedra with basic vertices \( x, T_{i}x, T_{j}x \) and \( x' \) satisfy the C-reduction condition, is called the CQL reduction of the fundamental transformation.

The following result gives the corresponding restriction of the data of the (scalar) fundamental transformation.

Proposition 5.4. Let \( x \) be a C-quadrilateral lattice with rotation coefficients satisfying constraint (5.3), and \( x' \) its C-reduced fundamental transform. Then there exists a constant \( c \) such that the data \( Y_{i} : \mathbb{Z}^{N} \rightarrow \mathbb{F} \) and \( Y'_{i} : \mathbb{Z}^{N} \rightarrow \mathbb{F} \) of the transformation are connected by relation

\[
Y_{i} = c \rho_{i} T_{i} Y'_{i}, \quad i = 1, \ldots, N, \quad c \in \mathbb{F}.
\]

Proof. We start from considerations similar to that of proof of Proposition 3.2. The idea is to interpret the fundamental transformation as construction of a new level of the quadrilateral lattice. The potential \( \Omega(X,Y) \), called also the Combescure vector of the transformation, serves as the normalized tangent vector \( x' \), which we denote by \( X'_{i} \), in the transformation direction “\( x' \).”

Denote by \( t_{ij} \) (\( i, j \) are distinct) the direction vector of the common line of the planes \( \langle x, T_{i}x, T_{j}x \rangle \) and \( \langle x', T_{i}x', T_{j}x' \rangle \). It must be therefore decomposed in the basis \( \{X_{i}, X_{j}\} \) and in the basis \( \{X'_{i}, X'_{j}\} \). Assuming its decomposition in the second basis we get

\[
t_{ij} = aX'_{i} + bX'_{j} = aX_{i} + bX_{j} - X_{i}(aY_{i} + bY_{j}) \frac{1}{\Omega(X,Y)}.
\]

where we have used the transformation equation (5.6). Because the coefficient in front of \( X'_{i} \) must vanish, the vector can be therefore chosen as

\[
t_{ij} = Y_{j}X_{i} - Y_{i}X_{j}.
\]

Similarly, denote by \( t_{ii} \) the direction vector of the intersection line of the plane \( \langle x, x', T_{i}x \rangle \) with \( \langle T_{j}x, T_{i}x', T_{j}x \rangle \). It must be therefore decomposed in the basis \( \{X_{i}, X_{i}\} \) and in the basis \( \{T_{j}X_{i}, T_{j}X_{i}\} \). By using equations (5.2) and (2.5) we can choose the vector as

\[
t'_{ij} = (T_{j}Q_{ij})T_{j}X_{i} - (T_{j}Y_{i}^{*})T_{j}X_{i} = (T_{j}Q_{ij})X_{i} - (T_{j}Y_{i}^{*})X_{i}.
\]

Because

\[
t'_{ij} \wedge t'_{ij} \wedge t'_{ij} = (T_{j}Q_{ij})Y_{j}(T_{j}Y_{i}^{*}) - (T_{j}Q_{ij})Y_{i}(T_{j}Y_{j}^{*}) \wedge X_{i} \wedge X_{j} \quad i \neq j,
\]

then the C-reduction condition of the \( i, j \) hexahedron takes the form of equation (3.1)

\[
(T_{j}Q_{ij})Y_{j}(T_{j}Y_{i}^{*}) = (T_{j}Q_{ij})Y_{i}(T_{j}Y_{j}^{*}).
\]

Making use of condition (3.3) we obtain that

\[
\rho_{i}(T_{j}Y_{j}^{*})Y_{i} = \rho_{i}(T_{i}Y_{i}^{*})Y_{j}.
\]
Then we use Lemma 3.3, which states that \( \rho_i T_i Y_i^* \) satisfies the same linear problem (2.5) as \( Y_i \) does. Finally, application of the following Lemma concludes the proof. □

**Lemma 5.5.** Any two scalar solutions \( Y_i \) and \( \hat{Y}_i \) of the linear problem (2.5), which satisfy the constraint
\[
\hat{Y}_j Y_i = \hat{Y}_i Y_j,
\]
must be proportional.

**Proof of the Lemma.** Assume that none of the solutions is trivial (then the proportionality constant would be zero) and define
\[
r_i = \frac{\hat{Y}_i}{Y_i}.
\]
By equation (2.5) we find
\[
\Delta_j r_i = \frac{(T_j Q_{ij})(\hat{Y}_j Y_i - \hat{Y}_i Y_j)}{Y_i (T_i Y_i)}, \quad j \neq i, \tag{5.29}
\]
which vanishes because of the assumption (5.27), therefore \( r_i \) may depend on the variable \( m_i \) only. Inserting then \( \hat{Y}_i = r_i Y_i \) into equation (5.27) we obtain \( r_i = r_j \), which implies that all the \( r \)'s are equal to the same constant. □

**Remark.** In the non-degenerate situation, i.e. \( c \neq 0 \), which we assume in the sequel, we can put \( c = 1 \), because (up to initial value) \( \Omega(cY, Y^*) = c\Omega(Y, Y^*) \), and \( \Omega(cY, H) = c\Omega(Y, H) \), and the final result is independent of \( c \).

### 5.3. Permutability theorem for the CQL reduction of the fundamental transformation.

In this section we study restrictions of the data of the vectorial fundamental transformation, which are compatible with the CQL reduction. In that part we follow the corresponding results of [29] (see also Proposition 4.9 of [18]). Then we show the corresponding permutability property of the transformation.

**Proposition 5.6 (29).** Given a solution \( Y_i^* \) of the adjoint linear problem (2.6) for the C-quadrilateral lattice whose rotation coefficients satisfy the CQL constraint (3.3) then
\[
Y_i = \rho_i (T_i Y_i^*)^t \tag{5.30}
\]
provides a vectorial solution of the linear problem (2.5), and the corresponding potential \( \Omega(Y, Y^*) \) allows for the following constraint
\[
\Omega(Y, Y^*)^t = \Omega(Y, Y^*). \tag{5.31}
\]
With such a data the transformed lattice \( x' \) given by (6.1) is C-quadrilateral lattice as well.

**Remark.** In [29], instead of relation (5.30) it was used more general relation
\[
\hat{Y}_i = A \rho_i (T_i Y_i^*)^t, \tag{5.32}
\]
where \( A \) is an arbitrary linear operator. Then also the constraint (5.31) had to be replaced by
\[
A\Omega(\hat{Y}, Y^*)^t = \Omega(\hat{Y}, Y^*)A^t, \tag{5.33}
\]
which is however equivalent, up to initial data, to (5.31) due to
\[
A\Omega(AY, Y^*)^t - \Omega(AY, Y^*)A^t = A(\Omega(Y, Y^*)^t - \Omega(Y, Y^*)) A^t. \tag{5.34}
\]
Moreover, because (up to initial value) \( \Omega(AY, Y^*) = A\Omega(Y, Y^*) \), and \( \Omega(AY, H) = A\Omega(Y, H) \) the final result (5.3) is independent of (non-degenerate) \( A \).

**Proposition 5.7.** The fundamental vectorial transform given by (5.4) with the data restricted by conditions (5.30) and (5.31) can be considered as the superposition of \( K \) (scalar) discrete CQL reduced fundamental transforms.
regarded as a set of $N_j \rho$ maps if the four images of the edges of any face of the $Z^v$ lateral lattice theory. Denote by $x$ the lattice the hexahedra with basic vertices thus showing the fundamental role of the quadrilateral lattice in integrable discrete geometry.

Two different faces of a more fundamental result concerning the so called quadrangular sets of points. Equations was a consequence of the Gallucci theorem. However, it turns out [8] that both theorems are of the Möbius theorem on mutually inscribed tetrahedrons, and the integrability of the discrete CKP integrable geometry. We remark that the integrability of the discrete BKP equations was a consequence of the M"{o}bius theorem on mutually inscribed tetrahedrons, and the integrability of the discrete CKP equations was a consequence of the Gallucci theorem. However, it turns out [8] that both theorems are two different faces of a more fundamental result concerning the so called quadrangular sets of points.

Results of the paper show once again the fundamental role of the incidence geometry structures in the format similar to [16], where we presented novel geometric interpretation of the discrete BKP equation.

**Proposition 5.4** (with $c = 1)$. For $K = 1$ we obtain the CQL reduction of the fundamental transformation in the setting of Proposition 5.1 (with $c = 1$). For $K > 1$ the statement follows from the standard reasoning applied to superposition of two reduced vectorial fundamental transformations (compare with [20, 14, 16]).

Assume the splitting $F^K = F^K_+ \oplus F^K_-$ and the induced splitting

$$Y^*_i = \left( Y^*_a, Y^*_b \right),$$

of the basic data $Y^*_i$ of the transformation. Then we have also

$$Y_i = \left( Y^a_i, Y^b_i \right) = \left( \rho_i(T_iY^*_a), \rho_i(T_iY^*_b) \right),$$

and (in the shorthand notation, compare equations (5.11)-(5.12))

$$\Omega(Y, Y^*) = \left( \Omega^a_a, \Omega^a_b, \Omega^b_a, \Omega^b_b \right),$$

while the constraint (5.31) reads

$$\Omega^a_i = \Omega^a_i, \quad \Omega^b_i = \Omega^b_i,$$

By straightforward algebra, using equations (5.38), one checks that the transformed data satisfy the CQL constraints (5.30) and (5.31) as well, i.e.,

$$Y^{b[a]}_i = \rho_i(T_iY^{b[a]}_a), \quad (\Omega^b_i)^t = \Omega^b_i,$$

which concludes the proof. \( \square \)

**Remark.** In the case with matrix $A$ as in previous Remark, the scalar components of the vectorial transformation do not satisfy (unless $A$ is diagonal) the CQL reduction condition of Proposition 5.4.

**Remark.** Because the CQL-reduced fundamental transformation can be considered as construction of new levels of the C-quadrilateral lattice, then if we denote by $x^{(1,2)}$ the C-quadrilateral lattice obtained by superposition of two (scalar) such transforms from $x$ to $x^{(1)}$ and $x^{(2)}$, then for each direction $i$ of the lattice the hexahedra with basic vertices $x_i$, $T_i x$, $x^{(1)}$ and $x^{(2)}$ are $C$-hexahedra. Similarly, if we consider superpositions of three (scalar) transforms of the C-quadrilateral lattice $x$ then the hexahedra with basic vertices $x_i$, $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$ are $C$-hexahedra.

**6. Conclusion and remarks**

We presented new geometric interpretation of the discrete CKP equation within the theory of quadrilateral lattices. The paper should be considered as supplementary to [16]. It has been also written in a format similar to [12], where we presented novel geometric interpretation of the discrete BKP equation. Results of the paper show once again the fundamental role of the incidence geometry structures in the integrable geometry. We remark that the integrability of the discrete BKP equations was a consequence of the Möbius theorem on mutually inscribed tetrahedrons, and the integrability of the discrete CKP equations was a consequence of the Gallucci theorem. However, it turns out [8] that both theorems are two different faces of a more fundamental result concerning the so called quadrangular sets of points.

In the Appendix we present the theory of the Darboux maps within the quadrilateral lattice theory thus showing the fundamental role of the quadrilateral lattice in integrable discrete geometry.

**Appendix A. The Darboux maps within the quadrilateral lattice theory**

We would like to present an interpretation of the so called Darboux maps [33, 24] within the quadrilateral lattice theory. Denote by $E = E(Z^N)$ the set of edges of the $Z^N$ lattice. Consider a map

$$v : E \rightarrow \mathbb{R}^M,$$

regarded as a set of $N$ maps $v^i : \mathbb{Z}^N \rightarrow \mathbb{R}^M$ of edges in ith direction. It is termed a discrete Darboux map if the four images of the edges of any face of the $Z^N$ lattice are collinear, i.e., there exist functions $\rho_j$, $i \neq j$, such that

$$\Delta_j v^i = (T_i \rho_j^j)(T_i v^j - T_j v^i), \quad i \neq j.$$
Compatibility of equations (A.2) implies [33] that the functions $\rho^{ij}$ satisfy the discrete Darboux equations (2.2).

Remark. In order to use the Darboux equations in the form (2.2) the definition of $\rho^{ij}$ in this paper is shifted with respect to that used in [33, 26].

We will briefly demonstrate that the Darboux maps can be interpreted as suitably rescaled normalized backward tangent vectors $\tilde{X}_i$; compare Figure 2 with Figure 4, where also the geometric construction of the Darboux map is given.

Proposition A.1. Consider the quadrilateral lattice $x : \mathbb{Z}^N \to \mathbb{R}^M$ together with its backward tangent vectors $\tilde{X}_i$ and the corresponding backward rotation coefficients $\tilde{Q}_{ij}$. Let $\tilde{v}_i$ be a scalar solution of the backward linear problem (2.8)

(A.3) $\tilde{\Delta}_i \tilde{v}_j = (T_i^{-1} \tilde{Q}_{ij}) \tilde{v}_i$, \hspace{1cm} or \hspace{1cm} $\Delta_i \tilde{v}_j = (T_i \tilde{v}_i) \tilde{Q}_{ij}$, \hspace{1cm} $i \neq j$,

define the maps $y_i : \mathbb{Z}^N \to \mathbb{R}^M$, $i = 1, \ldots, N$,

(A.4) $y_i = T_i \left( \frac{1}{\tilde{v}_i} \tilde{X}_i \right)$.

Then the maps $y_i$ satisfy the Darboux map equations

(A.5) $\Delta_i y_j = (T_j B_{ji}) (T_j y_i - T_i y_j)$,

with the coefficients

(A.6) $B_{ij} = \frac{\Delta_j \tilde{v}_i}{\tilde{v}_i}$, \hspace{1cm} $i \neq j$, \hspace{1cm} $i, j = 1, \ldots, N$.

Proof. By direct verification using the fact that both $\tilde{X}_i$ and $\tilde{v}_i$ satisfy the same linear system (2.8). Geometrically, by results of Appendix A1 of [21], it means that $y_i$ and $y_j$ represent mutual Laplace transforms [13] in the affine gauge, i.e. $y_i$, $T_j y_i$, $y_j$ and $T_i y_j$ are collinear and satisfy equation of the form of (A.5). \hfill \Box

Corollary A.2. The above result can be reversed, i.e., any Darboux map gives rise via equations (A.4) and (A.6) to a system of normalized backward tangent vectors of a quadrilateral lattice. Thus the correspondence between Darboux maps and quadrilateral lattices occurs on the geometric linear level.
Remark. Notice that the functions \( \tilde{v} \) satisfy the forward adjoint linear problem \( (2.3) \) with the rotation coefficients \( Q_j \) which satisfy the MQL equations \( (2.7) \). Then without any calculation we infer that the coefficients \( B_{ij} \) are solutions of the discrete Darboux (MQL) equations \( (2.2) \).

Finally, we mention that to the vectors \( y_i \), it can be given geometric meaning as non-homogeneous coordinates in \( H_\infty \) of the intersections points \( (x,T_x) \cap H_\infty \) of the tangent lines to the quadrilateral lattice \( x: Z^N \to \mathbb{P}^M \) with the hyperplane at infinity \( H_\infty \). (see Section 5.2). Using the Pascal hexagon theorem it can be shown that within this interpretation the “conic condition” of \( (2.9) \) is equivalent to Definition \( (3.1) \) of the C-hexahedron.

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