Extended Hamiltonian action for arbitrary spin fields in flat and AdS space

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Abstract
Totally symmetric, arbitrary spin massless and massive free fields in flat and AdS space and conformal fields in flat space are studied. An extended gauge-invariant Hamiltonian action for such fields is obtained. The action is constructed out of phase space fields and Lagrangian multipliers which are free of algebraic constraints. Gauge transformations of the phase space fields and Lagrangian multipliers are derived. Use of the Poincaré parametrization of AdS space allows us to treat fields in flat space and AdS space on an equal footing.

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1. Introduction

In view of the aesthetic features of the extended Hamiltonian approach to relativistic field dynamics, an interest in this approach has been periodically renewed (see e.g. [1, 2]). The extended Hamiltonian approach provides a systematic and self-contained way to study many aspects of relativistic field dynamics. Progress in understanding higher-spin field dynamics [3] has lead to an intensive and in-depth study of various aspects of AdS field dynamics. The Lagrangian formulation of higher-spin fields was developed many years ago in [4, 5]. Many interesting approaches to AdS fields are known in the literature. However, we note that an extended Hamiltonian formulation of massless and massive higher-spin fields in the flat and AdS space of arbitrary dimensions and conformal fields in the flat space of arbitrary dimensions, has not yet been worked out1.

The purpose of this paper is to develop a gauge-invariant Hamiltonian approach to totally symmetric arbitrary spin massless and massive fields in flat and AdS space and conformal

1 Discussion of the Hamiltonian formulation of massless fermionic fields in 4d flat and AdS4 space may be found in [6, 7] (for some discussion of massless bosonic fields in AdS4 see [7]). A Discussion of a Hamiltonian formulation of a massive spin 3/2 fermionic field in 4d flat space may be found in [8]. A Hamiltonian formulation of a conformal spin-2 field in 4d space is considered in [9].
fields in flat space. In this paper, we deal with free bosonic fields. Our approach to extended Hamiltonian field dynamics can be summarized as follows.

(i) We start with the Lagrangian formulation of massless and massive fields in flat and AdS space and conformal fields in flat space and use a representation for the Lagrangian in terms of a modified de Donder divergence obtained in [10–14]. We consider fields in $d$-dimensional flat space and $(d + 1)$-dimensional AdS space. We use the Poincaré parametrization of AdS$_{d+1}$ space in which the Lorentz algebra $so(d − 1, 1)$ symmetries are manifestly realized. We use the double-traceless higher-spin fields of the Lorentz algebra $so(d − 1, 1)$. It is the use of such double-traceless fields and the Poincaré parametrization of AdS space that allows us to treat massless and massive fields in flat and AdS space and conformal fields in flat space on an equal footing.

(ii) Our extended Hamiltonian action is formulated in terms of $so(d − 1)$ algebra fields. All the fields appearing in our extended Hamiltonian formulation are free of algebraic constraints. The field content entering our extended Hamiltonian action, involves phase space fields and Lagrangian multipliers. The number of Lagrangian multipliers and half of the phase space fields is equal to the number of gauge fields appearing in the Lagrangian formulation.

Our paper is organized as follows.

In section 2, we review the Lagrangian formulation of massless and massive fields in flat and AdS space and conformal fields in flat space. We discuss the representation for the Lagrangian in terms of the modified de Donder divergence found in [10–14]. Also, we review the realization of the gauge symmetries of the Lagrangian.

Section 3 is devoted to an extended Hamiltonian formulation of massless and massive fields in flat and AdS space and conformal fields in flat space. We start with a description of the field content appearing in our approach. After this we present our results for an extended Hamiltonian action and the corresponding gauge transformations.

In the appendix, we summarize our conventions and the notation.

2. The gauge-invariant Lagrangian via a modified de Donder divergence

In a metric-like approach, the gauge-invariant Lagrangian for free massless fields in flat and AdS$_4$ space was obtained in [4, 5], while the gauge-invariant Lagrangian for free massive fields in flat and AdS$_{d+1}$ space was found in [15]. In [10–14], we noticed that the use of a modified de Donder divergence simplifies the structure of the gauge-invariant Lagrangian considerably. A representation of the gauge-invariant Lagrangian for massive fields in flat space, in terms of a modified de Donder divergence, was obtained in [10]; while a representation of the gauge-invariant Lagrangian for massless and massive fields in AdS space, in terms of a modified de Donder divergence, was found in [11, 12]. A representation of the gauge-invariant Lagrangian for a conformal field in flat space, in terms of a modified de Donder divergence was obtained in [13, 14]. Because the representation of the gauge-invariant Lagrangian via a modified de Donder divergence turns out to be helpful for the derivation of an extended Hamiltonian action, we start with a review of our results in [10–14]. Before we proceed to the review, note that
we use the Cartesian parametrization of Minkowski space and the Poincaré parametrization of AdS$_{d+1}$ space (for the notation, see the appendix),

\[ ds^2 = dx^a dx^b, \quad \text{for flat space}, \]
\[ ds^2 = \frac{1}{z} (dx^a dx^a + dz dz), \quad \text{for AdS space}. \]

The use of such parametrizations allows us, among other things, to treat fields in flat and AdS space on an equal footing. We now discuss the field contents.

**Field content for a massless field in \( R^{d-1,1} \).** The massless spin-\( s \) field in \( d \)-dimensional flat space can be described by the rank-\( s \) totally symmetric tensor field of the Lorentz algebra so\((d-1,1)\) [4]

\[ \phi^{a_1 \cdots a_s}, \] (2.3)

subject to the double-tracelessness constraint, \( \phi^{a_1 b_1 \cdots a_s} = 0 \). To simplify the representation of the gauge-invariant action, we use the oscillators \( \alpha^a \) and introduce the following ket-vector:

\[ |\phi\rangle \equiv \frac{1}{s!} \phi^{a_1 \cdots a_s} |0\rangle. \] (2.4)

**Field content for the massive field in \( R^{d-1,1} \).** As is well known [15], the spin-\( s \) massive field in flat space can be described by the following set of fields:

\[ \phi^{a_1 \cdots a_{s'}}, \quad s' = 0, 1, \ldots, s. \] (2.5)

Fields in (2.5) with \( s' = 0 \), \( s' = 1 \) and \( s' \geq 2 \) are the respective scalar, vector and rank-\( s' \) totally symmetric fields of the Lorentz algebra so\((d-1,1)\). Fields in (2.5) with \( s' \geq 4 \) are double-traceless, \( \phi^{a_1 b_1 \cdots a_{s'}} = 0 \). To streamline the representation of gauge-invariant action, we use the oscillators \( \alpha^a \) and \( \xi \) and introduce the following ket-vector:

\[ |\phi\rangle \equiv \sum_{s'=0}^{s} \xi^{s-s'} \phi^{a_1 \cdots a_{s'}} |0\rangle. \] (2.6)

**Field content for a massless field in AdS$_{d+1}$.** To discuss the gauge-invariant formulation of a spin-\( s \) massless field in AdS$_{d+1}$ we use the following set of fields in [11]:

\[ \phi^{a_1 \cdots a_{s'}}, \quad s' = 0, 1, \ldots, s. \] (2.7)

Fields in (2.7) with \( s' = 0 \), \( s' = 1 \) and \( s' \geq 2 \) are the respective scalar, vector and rank-\( s' \) totally symmetric fields of the Lorentz algebra so\((d-1,1)\). Fields in (2.7) with \( s' \geq 4 \) are double-traceless, \( \phi^{a_1 b_1 \cdots a_{s'}} = 0 \). To discuss the gauge-invariant Lagrangian in an easy-to-use form, we use the oscillators \( \alpha^a \) and \( \xi \) to collect fields (2.7) into the ket-vector

\[ |\phi\rangle \equiv \sum_{s'=0}^{s} \xi^{s-s'} \phi^{a_1 \cdots a_{s'}} |0\rangle. \] (2.8)

**Field content for the massive field in AdS$_{d+1}$.** To discuss the gauge-invariant formulation of the spin-\( s \) massive field in AdS$_{d+1}$ we use the following set of fields in [12]:

\[ \phi^{a_1 \cdots a_{s'}}, \quad s' = 0, 1, \ldots, s-1, s, \quad n \in [s-s'], \] (2.9)

3 In our approach, only so\((d-1,1)\) symmetries are manifestly realized. The so\((d,2)\) symmetries of the fields in AdS$_{d+1}$ could be manifestly realized by using the ambient space approach (see e.g. [42–45]).

4 In [5], a spin-\( s \) massless field in AdS$_{d+1}$ is described by the rank-\( s \) totally symmetric double-traceless tensor field of the Lorentz algebra so\((d,1)\). Note that the so\((d-1,1)\) tensorial components of the tensor field in [5] are not double-traceless. The tensor field in [5] is related to our fields (2.7) by an invertible transformation. This invertible transformation is described in [11].
(for notation, see (A.2) in the appendix). Fields in (2.9) with \( s' = 0 \), \( s' = 1 \) and \( s' \geq 2 \) are the respective scalar, vector and rank-\( s' \) totally symmetric fields of the Lorentz algebra \( so(d-1,1) \). The fields in (2.9) with \( s' \geq 4 \) are double-traceless, \( \phi_{\mu\nu^{a'b'}c'^d'}^{ab} = 0 \). To streamline the representation, we use the oscillators \( \alpha^a, \alpha^c \) and \( \zeta \) and collect fields (2.9) into the ket-vector defined by

\[
|\phi\rangle = \sum_{s' = 0}^{s} \sum_{n \in \{s'-s\}_{1\bar{1}}} \xi^{s'\bar{s}} \alpha^{a_{s'\bar{s}}} \alpha^{a_{s'\bar{s}}} \phi_{\phi_{\mu\nu^{a'b'}c'^d'}^{ab}} |0\rangle.
\]

(2.10)

Field content for a conformal field in flat space. To discuss the ordinary-derivative formulation of a spin-\( s \) conformal field, we use the following set of fields in [13, 14]:

\[
\phi_{\mu\nu^{a'b'}c'^d'}^{ab}, \quad s' = 0, 1, \ldots, s - 1, \quad k' \in [k]_1, \quad k' \equiv s' + d - 6 \nu. \]

(2.11)

Fields in (2.11) with \( s' = 0 \), \( s' = 1 \) and \( s' \geq 2 \) are the respective scalar, vector and rank-\( s' \) totally symmetric fields of the Lorentz algebra \( so(d-1,1) \). The fields in (2.11) with \( s' \geq 4 \) are double-traceless, \( \phi_{\mu\nu^{a'b'}c'^d'}^{ab} = 0 \). To simplify the representation, we use the oscillators \( \alpha^a, \xi, \nu^a \) and \( \nu^c \), and collect fields (2.11) into the ket-vector \( |\phi\rangle \) defined by

\[
|\phi\rangle = \sum_{s' = 0}^{s} \sum_{k' \in [k]_1} \xi^{s'\bar{s}} \nu^{a_{s'\bar{s}}} \phi_{\phi_{\mu\nu^{a'b'}c'^d'}^{ab}} |0\rangle.
\]

(2.12)

Lagrangian. The gauge-invariant action for fields in flat and AdS space is given by

\[
S = \int d^d x \mathcal{L}, \quad \text{for fields in } R^{d-1,1},
\]

\[
S = \int d^d x d\zeta \mathcal{L}, \quad \text{for fields in } AdS_{d+1},
\]

(2.13)

where the Lagrangian we found is given by

\[
\mathcal{L} = -\frac{1}{2} \{ \bar{\partial}^a \phi \mu |\bar{\partial}^a \phi \} - \frac{1}{2} (\phi |\mu \mathcal{M}^2 |\phi \} + \frac{1}{2} (\bar{\mathcal{C}} \phi |\bar{\mathcal{C}} \phi \},
\]

(2.14)

\[
\bar{\mathcal{C}} \equiv \bar{\alpha} \bar{\partial} - \frac{1}{2} \bar{\alpha} \bar{\partial}^2 \bar{\alpha} - \bar{e}_1 \bar{\Pi}^{[\bar{d}]} + \frac{1}{2} \bar{e}_1 \bar{\alpha}^2,
\]

(2.15)

\[
\mathcal{C} \equiv \alpha \partial - \frac{1}{2} \alpha \partial^2 \alpha \partial - e_1 \Pi^{[d]} + \frac{1}{2} e_1 \alpha^2,
\]

(2.16)

\[
\mu \equiv 1 - \frac{1}{4} \alpha^2 \bar{\alpha}^2, \quad \bar{\Pi}^{[\bar{d}]} \equiv 1 - \alpha^2 \bar{\alpha}^2 \frac{1}{2(2N_d + d)} \bar{\alpha}^2,
\]

(2.17)

and expressions like \( \alpha \bar{\partial}, \alpha^2 \bar{\alpha}^2 \) are defined in the appendix (see (A.6) and (A.7)). We note that the Lagrangian for fields in flat and AdS space and conformal fields is distinguished only by the operators \( \mathcal{M}^2, e_1 \) and \( \bar{e}_1 \). To illustrate this, we now present the explicit form of these operators in turn.

Operators \( \mathcal{M}^2, e_1 \) and \( \bar{e}_1 \) for a massless field in flat space:

\[
\mathcal{M}^2 = 0, \quad e_1 = 0, \quad \bar{e}_1 = 0.
\]

(2.18)

Operators \( \mathcal{M}^2, e_1 \) and \( \bar{e}_1 \) for a massive field in \( R^{d-1,1} \) [10]:

\[
\mathcal{M}^2 = m^2, \quad e_1 = m \bar{e}_1, \quad \bar{e}_1 = -m \bar{e}_1 \bar{\zeta},
\]

(2.19)

\[5\] In [15], the spin-\( s \) massive field in \( AdS_{d+1} \) is described by the set of fields involving totally symmetric double-traceless tensor fields of the Lorentz algebra \( so(d,1) \). Note that \( so(d-1,1) \) tensorial components of the tensor fields in [15] are not double-traceless. The fields in [15] are related to our fields (2.9) by an invertible transformation. This invertible transformation is described in [12].
\[ \tilde{e}_1 = \left( \frac{2s + d - 4 - N_\ell}{2s + d - 4 - 2N_\ell} \right)^{1/2} \].

In (2.19), \( m \) stands for the commonly used mass parameter of a massive field.

**Operators \( \mathcal{M}^2 \), \( e_1 \) and \( \tilde{e}_1 \) for a massless field in \( \text{AdS}_{d+1} \) [11]:**

\[
\mathcal{M}^2 = -\tilde{\alpha}_e^2 + \frac{1}{\tilde{z}^2} \left( \nu^2 - \frac{1}{4} \right),
\]

\[
e_1 = \alpha^s T_{v-\frac{1}{2}}, \quad -\tilde{e}_1 = T_{v+\frac{1}{2}} r_\ell \tilde{\alpha},
\]

\[
T_v = \tilde{\alpha}_e + \frac{\nu}{\tilde{z}}, \quad \nu \equiv s + \frac{d - 4}{2} - N_\ell,
\]

\[
r_\ell = \frac{2s + d - 4 - N_\ell}{2s + d - 4 - 2N_\ell}^{1/2}.
\]

**Operators \( \mathcal{M}^2 \), \( e_1 \) and \( \tilde{e}_1 \) for a massive field in \( \text{AdS}_{d+1} \) [12]:**

\[
\mathcal{M}^2 = -\tilde{\alpha}_e^2 + \frac{1}{\tilde{z}^2} \left( \nu^2 - \frac{1}{4} \right),
\]

\[
e_1 = \xi r_\ell T_{v-\frac{1}{2}} + \alpha^s T_{v+\frac{1}{2}}, \quad -\tilde{e}_1 = T_{v+\frac{1}{2}} r_\ell \tilde{\alpha} + T_{v-\frac{1}{2}} r_\ell \tilde{\alpha},
\]

\[
T_v = \tilde{\alpha}_e + \frac{\nu}{\tilde{z}}, \quad \nu \equiv \kappa + N_\ell - N_\ell,
\]

\[
r_\ell = \left( \frac{\left(s + \frac{d-4}{2} - N_\ell\right)\left(\kappa - s - \frac{d-4}{2} + N_\ell\right)\left(\kappa + 1 + N_\ell\right)}{2\left(s + \frac{d-4}{2} - N_\ell - N_\ell\right)\left(\kappa + N_\ell - N_\ell\right)\left(\kappa + N_\ell - N_\ell + 1\right)} \right)^{1/2},
\]

\[
r_\ell = \frac{\left(s + \frac{d-4}{2} - N_\ell\right)\left(\kappa + s + \frac{d-4}{2} - N_\ell\right)\left(\kappa - 1 - N_\ell\right)}{2\left(s + \frac{d-4}{2} - N_\ell - N_\ell\right)\left(\kappa + N_\ell - N_\ell\right)\left(\kappa + N_\ell - N_\ell - 1\right)}^{1/2},
\]

\[
\kappa \equiv \sqrt{m^2 + \left(s + \frac{d-4}{2}\right)^2}.
\]

In (2.30), \( m \) stands for the commonly used mass parameter of the spin-\( s \) massive field in \( \text{AdS}_{d+1} \).

**Operators \( \mathcal{M}^2 \), \( e_1 \) and \( \tilde{e}_1 \) for a conformal field in \( \mathbb{R}^{d-1,1} \) [14]:**

\[
\mathcal{M}^2 = \partial^\rho \tilde{\partial}^\rho, \quad e_1 = \xi \tilde{e}_1 \tilde{\partial}^\rho, \quad \tilde{e}_1 = -\nu^\rho \tilde{e}_1 \tilde{\partial}^\rho,
\]

where \( \tilde{\partial} \) is given in (2.20). Note the following remarks.

(i) It is the quantity \( \tilde{C}(\phi) \) that we refer to as a modified de Donder divergence. This modified de Donder divergence coincides with the standard de Donder divergence only in the case of a massless field in flat space. From (2.14) and (2.15), we see that many complicated terms contributing to the Lagrangian are collected into the modified de Donder divergence. Thus, as outlined above, the use of the modified de Donder divergence allows us to simplify, considerably, the structure of the Lagrangian\(^6\).

\(^6\) Because our modified de Donder gauge leads to a considerably simplified analysis of the AdS field dynamics, we believe that this gauge might also be useful for a better understanding of various aspects of AdS/QCD correspondence which are discussed, for example, in [34]. Interesting applications of the standard de Donder–Feynman gauge to the various problems of higher-spin fields may be found in [35–37].
(iii) The representation for the Lagrangian in (2.14) takes the same form as in [4, 15]. In the case of the massless and massive fields in $\text{AdS}_{d+1}$, in order to cast the Lagrangians in [4, 15] into the form given in (2.14), we use our set of the $so(d-1,1)$ algebra double-traceless gauge fields. We recall that, in [4, 15], the Lagrangians of massless and massive fields in $\text{AdS}_{d+1}$ are formulated in terms of $so(d,1)$ algebra double-traceless gauge fields. Our gauge fields are related to the gauge fields used in [4, 15] by invertible transformations. The invertible transformations are described in [11, 12].

The representation for the Lagrangian in (2.14)–(2.17) is universal and is valid for the arbitrary Poincaré invariant theory. Various Poincaré invariant theories are distinguished by the operators $\mathcal{M}^2$, $e_1$ and $\tilde{e}_1$. Namely, the dependence of the operators $\mathcal{C}$ and $\bar{\mathcal{C}}$ on the oscillators $a^\alpha$ and $\bar{a}^{\bar{\alpha}}$ and the flat derivative $\partial^a$, takes the same form for massless and massive fields in flat and AdS space and conformal fields in flat space. In other words, the operators $\mathcal{C}$ and $\bar{\mathcal{C}}$ for massless and massive fields in flat and AdS space and conformal fields in flat space are distinguished only by the operators $e_1$ and $\tilde{e}_1$.

The representation for the Lagrangian given in (2.14) turns out to be especially helpful for the study of AdS/CFT duality for arbitrary spin massless and massive bulk AdS fields and the corresponding boundary current and shadow fields (see [39–41].)

Gauge symmetries. We now discuss the gauge symmetries of the Lagrangian given in (2.14). We begin with a description of the gauge transformation parameters involved in gauge transformations of gauge fields. We discuss the gauge transformation parameters in turn.

Gauge transformation parameters for a massless field in $R^{d-1,1}$. To discuss the gauge symmetries of a spin-\(s\) massless field in flat space, we use the gauge transformation parameter 

\[
\xi^{\alpha_1 \cdots \alpha_{s-1}},
\]  

which is a rank-(\(s-1\)) totally symmetric tensor field of the Lorentz algebra $so(d - 1, 1)$. For \(s \geq 3\), this parameter is traceless, $\xi^{\alpha_0 \alpha_1 \cdots \alpha_{s-1}} = 0$ (see [4]). To simplify the representation we use the oscillator $a^\alpha$ and introduce the ket-vector

\[
|\xi\rangle \equiv \frac{1}{(s-1)!} a^{\alpha_1} \cdots a^{\alpha_{s-1}} \xi^{\alpha_1 \cdots \alpha_{s-1}} |0\rangle. \tag{2.33}
\]

Gauge transformation parameters for the massive field in $R^{d-1,1}$. The gauge symmetries of the spin-\(s\) massive field in flat space are described by the following set of gauge transformation parameters in [15]:

\[
\xi^{\alpha_1 \cdots \alpha_{s-1}}, \quad s' = 0, 1, \ldots, s - 1. \tag{2.34}
\]

The gauge transformation parameters in (2.34) with \(s' = 0, s' = 1\) and \(s' \geq 2\) are the respective scalar, vector and rank-\(s'\) totally symmetric fields of the Lorentz algebra $so(d - 1, 1)$. The gauge transformation parameters in (2.34) with \(s' \geq 2\) are traceless, $\xi^{\alpha_0 \alpha_1 \cdots \alpha_{s'-1}} = 0$. To streamline the representation, we use the oscillators $a^\alpha$ and $\xi$ and introduce the following ket-vector:

\[
|\xi\rangle \equiv \sum_{s'=0}^{s-1} \frac{1}{s'!} \sqrt{(s - 1 - s')} \xi^{\alpha_1 \cdots \alpha_{s'}} |0\rangle. \tag{2.35}
\]

Gauge transformation parameters for a massless field in $\text{AdS}_{d+1}$. To discuss gauge symmetries of a spin-\(s\) massless field in $\text{AdS}_{d+1}$, we use the following set of gauge transformation parameters in [11]:

\[
\xi^{\alpha_1 \cdots \alpha_{s'}}, \quad s' = 0, 1, \ldots, s - 1. \tag{2.36}
\]
The gauge transformation parameters in (2.36) with \( s' = 0, s' = 1 \) and \( s' \geq 2 \) are the respective scalar, vector and rank-\( s' \) totally symmetric fields of the Lorentz algebra \( so(d - 1, 1) \). The gauge transformation parameters in (2.36) with \( s' \geq 2 \) are traceless. \( \xi^{a_i \cdots a_{s'}}_{a_i \cdots a_{s'}} = 0 \). To simplify the representation, we use the oscillators \( a^d \) and \( \alpha^i \) and collect gauge transformation parameters (2.36) into the ket-vector given by

\[
|\xi\rangle \equiv \sum_{s'=0}^{s-1} \sum_{a_i}^{a_s} \frac{\alpha^{a_i} \cdots \alpha^{a_{s'}}}{s'! (s' - 1)!} \xi^{a_i \cdots a_{s'}}(0).
\]  

(2.37)

Gauge transformation parameters for the massive field in \( \text{AdS}_{d+1} \). To describe the gauge symmetries of the spin-\( s \) massive field in \( \text{AdS}_{d+1} \) we use the following set of gauge transformation parameters in [12]:

\[
\xi_{n}^{a_i \cdots a_{s'}}, \quad s' = 0, 1, \ldots, s - 1, \quad n \in [s - 1 - s' \uparrow].
\]  

(2.38)

The gauge transformation parameters in (2.38) with \( s' = 0, s' = 1 \) and \( s' \geq 2 \) are the respective scalar, vector and rank-\( s' \) totally symmetric fields of the Lorentz algebra \( so(d - 1, 1) \). The gauge transformation parameters in (2.38) with \( s' \geq 2 \) are traceless. \( \xi_{n}^{a_i \cdots a_{s'}} = 0 \). To streamline the representation of the gauge transformations, we use oscillators \( a^d \), \( \alpha^i \) and \( \zeta \) and collect gauge transformation parameters (2.38) into the ket-vector defined by

\[
|\xi\rangle \equiv \sum_{s'=0}^{s-1} \sum_{a_i}^{a_s} \frac{\zeta^{a_i} \cdots \zeta^{a_{s'}}}{s'! (s' - 1)!} \xi^{a_i \cdots a_{s'}}(0).
\]  

(2.39)

Gauge transformation parameters for a conformal field in \( \mathbb{R}^{d-1,1} \). To describe the gauge symmetries of a spin-\( s \) conformal field, we use the following set of gauge transformation parameters in [13, 14]:

\[
\xi_{k'}^{a_i \cdots a_{s'}}, \quad s' = 0, 1, \ldots, s - 1, \quad k' \in [k'_{\pm} + 1]_{\uparrow}.
\]  

(2.40)

The gauge transformation parameters in (2.40) with \( s' = 0, s' = 1 \) and \( s' \geq 2 \) are the respective scalar, vector and rank-\( s' \) totally symmetric fields of the Lorentz algebra \( so(d - 1, 1) \). The gauge transformation parameters in (2.40) with \( s' \geq 2 \) are traceless. \( \xi_{k'}^{a_i \cdots a_{s'}} = 0 \). To simplify the representation, we use the oscillators \( a^d \), \( \zeta \), \( \nu^0 \) and \( \nu^0 \) and collect gauge transformation parameters (2.40) into the ket-vector defined by

\[
|\xi\rangle \equiv \sum_{s'=0}^{s-1} \sum_{a_i}^{a_s} \frac{\zeta^{a_i} \cdots (\nu^0)^{a_{s'}}}{s'! (s' - 1)!} \xi^{a_i \cdots a_{s'}}(0).
\]  

(2.41)

Having represented the field contents and gauge transformation parameters in terms of the ket-vectors \( |\phi\rangle \) and \( |\xi\rangle \), we note that the gauge transformations can be presented, entirely, in terms of these ket-vectors. The representation for the gauge transformations found in [10–14] is given by

\[
\delta|\phi\rangle = G|\xi\rangle, \quad G \equiv a \delta - e \frac{1}{2N_a + d - 2\bar{z}}.
\]  

(2.42)
In the case of massless and massive fields in flat space, the gauge transformations (2.42) coincide with the respective gauge transformations found in [4] and [15]. In the case of the massless and massive fields in AdS_{d+1}, in order to cast the gauge transformations in [5, 15] into the form given in (2.42), we use our set of the so(d − 1, 1) algebra double-traceless gauge fields and the so(d − 1, 1) algebra traceless gauge transformation parameters.

3. Extended gauge-invariant Hamiltonian action

We now discuss the extended gauge-invariant Hamiltonian action for massless and massive fields in flat and AdS space and conformal fields in flat space. We begin our discussion with a description of the field contents. We discuss the field contents in turn.

Field content for a massless field in R^{d−1,1}. To discuss the Hamiltonian action for a spin-s massless field in flat space, we introduce the following set of fields:

\[ \phi^{i_1 \ldots i_s}, \quad \mathcal{P}^{i_1 \ldots i_s}, \quad (3.1) \]

\[ \phi^{i_1 \ldots i_{s-2}}, \quad \mathcal{P}^{i_1 \ldots i_{s-2}}, \quad (3.2) \]

\[ \lambda^{i_1 \ldots i_{s-3}}, \quad \lambda^{i_1 \ldots i_{s-3}}. \quad (3.3) \]

The fields in (3.1)–(3.3) are totally symmetric traceful tensor fields of the so(d − 1) algebra. Thus, we see that all our fields in (3.1)–(3.3) are free from any constraints, i.e. we deal with unconstrained fields. We note that the fields in (3.1) and (3.2) are phase-space variables, while the fields in (3.3) are Lagrangian multipliers. The fields \( \phi^{i_1 \ldots i_s}, \phi^{i_1 \ldots i_{s-2}}, \) and Lagrangian multipliers (3.3) are related to the field in (2.3) by invertible transformation. To simplify the representation, we use the oscillators \( \alpha' \) and collect fields (3.1)–(3.3) into the following ket-vectors:

\[ |\phi_{s}⟩ ≡ \frac{1}{s!} \alpha^{i_1} \cdots \alpha^{i_s} \phi^{i_1 \ldots i_s}|0⟩, \quad (3.4) \]

\[ |\phi_{s-3}⟩ ≡ \frac{1}{(s-3)!} \alpha^{i_1} \cdots \alpha^{i_{s-3}} \phi^{i_1 \ldots i_{s-3}}|0⟩, \]

\[ |\mathcal{P}_{s}⟩ ≡ \frac{1}{s!} \alpha^{i_1} \cdots \alpha^{i_s} \mathcal{P}^{i_1 \ldots i_s}|0⟩, \quad (3.5) \]

\[ |\mathcal{P}_{s-3}⟩ ≡ \frac{1}{(s-3)!} \alpha^{i_1} \cdots \alpha^{i_{s-3}} \mathcal{P}^{i_1 \ldots i_{s-3}}|0⟩, \]

\[ |\lambda_{s-1}⟩ ≡ \frac{1}{(s-1)!} \alpha^{i_1} \cdots \alpha^{i_{s-1}} \lambda^{i_1 \ldots i_{s-1}}|0⟩, \quad (3.6) \]

\[ |\lambda_{s-2}⟩ ≡ \frac{1}{(s-2)!} \alpha^{i_1} \cdots \alpha^{i_{s-2}} \lambda^{i_1 \ldots i_{s-2}}|0⟩. \]

Field content for the massive field in R^{d−1,1}. To develop the Hamiltonian approach to the spin-s massive field in flat space, we introduce the following set of fields:

\[ \phi^{i_1 \ldots i_{s'}}, \quad \mathcal{P}^{i_1 \ldots i_{s'}}, \quad s' = 0, 1, \ldots, s, \quad (3.7) \]

\[ \phi^{i_1 \ldots i_{s-3}}, \quad \mathcal{P}^{i_1 \ldots i_{s-3}}, \quad s' = 0, 1, \ldots, s - 3, \]

\[ \lambda^{i_1 \ldots i_{s-1}}, \quad s' = 0, 1, \ldots, s - 1, \quad (3.8) \]

\[ \lambda^{i_1 \ldots i_{s-2}}, \quad s' = 0, 1, \ldots, s - 2. \]

Fields in (3.7) and (3.8) with \( s' = 0, s' = 1 \) and \( s' \geq 2 \) are the respective scalar, vector and totally symmetric rank-s' traceful tensor fields of the so(d − 1) algebra. To simplify
the representation, we use the oscillators $\alpha'$ and $\zeta$ and collect fields (3.7) and (3.8) into the following ket-vectors:

$$
|\phi_s\rangle = \sum_{r=0}^{s} \frac{\zeta^{-r} \alpha^i_1 \cdots \alpha^i_r}{s'! \sqrt{(s-s')!}} \phi^{i_1 \cdots i_r}_{s'} [0],
$$

(3.9)

$$
|\phi_{s-3}\rangle = \sum_{r=0}^{s-3} \frac{\zeta^{-3-r} \alpha^i_1 \cdots \alpha^i_r}{s'! \sqrt{(s-3-s')!}} \phi^{i_1 \cdots i_r}_{s-3} [0],
$$

$$
|\mathcal{P}_s\rangle = \sum_{r=0}^{s} \frac{\zeta^r \alpha^i_1 \cdots \alpha^i_r}{s'! \sqrt{(s-s')!}} \mathcal{P}^{i_1 \cdots i_r}_{s'} [0],
$$

(3.10)

$$
|\mathcal{P}_{s-3}\rangle = \sum_{r=0}^{s-3} \frac{\zeta^{-3-r} \alpha^i_1 \cdots \alpha^i_r}{s'! \sqrt{(s-3-s')!}} \mathcal{P}^{i_1 \cdots i_r}_{s-3} [0],
$$

$$
|\lambda_{s-1}\rangle = \sum_{r=0}^{s-1} \frac{\zeta^{r-1} \alpha^i_1 \cdots \alpha^i_r}{s'! \sqrt{(s-1-s')!}} \lambda^{i_1 \cdots i_r}_{s-1} [0],
$$

(3.11)

$$
|\lambda_{s-2}\rangle = \sum_{r=0}^{s-2} \frac{\zeta^{r-2} \alpha^i_1 \cdots \alpha^i_r}{s'! \sqrt{(s-2-s')!}} \lambda^{i_1 \cdots i_r}_{s-2} [0].
$$

Field content for a massless field in $\text{AdS}_{d+1}$. To discuss the Hamiltonian action for a spin-$s$ massless field in AdS space, we introduce the following set of fields:

$$
\phi_{s-1}^{i_1 \cdots i_r}, \quad \mathcal{P}_{s-1}^{i_1 \cdots i_r}, \quad s' = 0, 1, \ldots, s,
$$

(3.12)

$$
\phi_{s-3}^{i_1 \cdots i_r}, \quad \mathcal{P}_{s-3}^{i_1 \cdots i_r}, \quad s' = 0, 1, \ldots, s - 3,
$$

$$
\lambda_{s-1}^{i_1 \cdots i_r}, \quad s' = 0, 1, \ldots, s - 1,
$$

(3.13)

$$
\lambda_{s-2}^{i_1 \cdots i_r}, \quad s' = 0, 1, \ldots, s - 2.
$$

We note that the fields in (3.12) and (3.13) with $s' = 0$, $s' = 1$ and $s' \geq 2$ are the respective scalar, vector and totally symmetric rank-$s'$ traceful tensor fields of the $so(d-1)$ algebra. To simplify the representation, we use the oscillators $\alpha'$ and $\zeta$ and collect fields (3.12) and (3.13) into the following ket-vectors:

$$
|\phi_s\rangle = \sum_{r=0}^{s} \frac{\alpha'^{-r} \alpha^i_1 \cdots \alpha^i_r}{s'! \sqrt{(s-s')!}} \phi^{i_1 \cdots i_r}_{s'} [0],
$$

(3.14)

$$
|\phi_{s-3}\rangle = \sum_{r=0}^{s-3} \frac{\alpha'^{-3-r} \alpha^i_1 \cdots \alpha^i_r}{s'! \sqrt{(s-3-s')!}} \phi^{i_1 \cdots i_r}_{s-3} [0],
$$

$$
|\mathcal{P}_s\rangle = \sum_{r=0}^{s} \frac{\alpha'^{r} \alpha^i_1 \cdots \alpha^i_r}{s'! \sqrt{(s-s')!}} \mathcal{P}^{i_1 \cdots i_r}_{s'} [0],
$$

(3.15)

$$
|\mathcal{P}_{s-3}\rangle = \sum_{r=0}^{s-3} \frac{\alpha'^{-3-r} \alpha^i_1 \cdots \alpha^i_r}{s'! \sqrt{(s-3-s')!}} \mathcal{P}^{i_1 \cdots i_r}_{s-3} [0],
$$

$$
|\lambda_{s-1}\rangle = \sum_{r=0}^{s-1} \frac{\alpha'^{r-1} \alpha^i_1 \cdots \alpha^i_r}{s'! \sqrt{(s-1-s')!}} \lambda^{i_1 \cdots i_r}_{s-1} [0],
$$

(3.16)

$$
|\lambda_{s-2}\rangle = \sum_{r=0}^{s-2} \frac{\alpha'^{r-2} \alpha^i_1 \cdots \alpha^i_r}{s'! \sqrt{(s-2-s')!}} \lambda^{i_1 \cdots i_r}_{s-2} [0].
$$
Field content for the massive field in AdS$_{d+1}$. To develop the Hamiltonian approach to the spin-$s$ massive field in AdS space, we introduce the following set of fields:

$$
\phi^{i_1 \ldots i_r}_{s,n}, \quad \mathcal{P}^{i_1 \ldots i_r}_{s,n}, \quad n \in [s - s'], 2], \quad s' = 0, 1, \ldots, s,
\phi^{i_1 \ldots i_r}_{s - 3, n}, \quad \mathcal{P}^{i_1 \ldots i_r}_{s - 3, n}, \quad n \in [s - 3 - s']_2, \quad s' = 0, 1, \ldots, s - 3,
$$

(3.17)

$$
\lambda^{i_1 \ldots i_r}_{s - 1, n}, \quad n \in [s - 1 - s']_2, \quad s' = 0, 1, \ldots, s - 1,
\lambda^{i_1 \ldots i_r}_{s - 2, n}, \quad n \in [s - 2 - s']_2, \quad s' = 0, 1, \ldots, s - 2.
$$

(3.18)

Fields in (3.17) and (3.18) with $s' = 0$, $s' = 1$ and $s' \geq 2$ are the respective scalar, vector and totally symmetric rank-$s'$ tensor fields of the $so(d - 1)$ algebra. To simplify the representation, we use the oscillators $\alpha^i$, $\alpha_\zeta$ and $\zeta$ and collect fields (3.17) and (3.18) into the following ket-vectors:

$$
\langle \phi_s \rangle = \sum_{s' = 0}^{s} \sum_{n \in [s - s']_2} \xi^{i_1 \ldots i_r} \alpha^{i_1 \ldots i_r} \phi^{i_1 \ldots i_r}_{s,n} |0\rangle,
\langle \phi_{s-3} \rangle = \sum_{s' = 0}^{s-3} \sum_{n \in [s - 3 - s']_2} \xi^{i_1 \ldots i_r} \alpha^{i_1 \ldots i_r} \phi^{i_1 \ldots i_r}_{s-3,n} |0\rangle,
$$

(3.19)

$$
\langle \mathcal{P}_s \rangle = \sum_{s' = 0}^{s} \sum_{n \in [s - s']_2} \xi^{i_1 \ldots i_r} \alpha^{i_1 \ldots i_r} \mathcal{P}^{i_1 \ldots i_r}_{s,n} |0\rangle,
\langle \mathcal{P}_{s-3} \rangle = \sum_{s' = 0}^{s-3} \sum_{n \in [s - 3 - s']_2} \xi^{i_1 \ldots i_r} \alpha^{i_1 \ldots i_r} \mathcal{P}^{i_1 \ldots i_r}_{s-3,n} |0\rangle,
$$

(3.20)

$$
\langle \lambda_{s-1} \rangle = \sum_{s' = 0}^{s-1} \sum_{n \in [s - 1 - s']_2} \xi^{i_1 \ldots i_r} \alpha^{i_1 \ldots i_r} \lambda^{i_1 \ldots i_r}_{s-1,n} |0\rangle,
\langle \lambda_{s-2} \rangle = \sum_{s' = 0}^{s-2} \sum_{n \in [s - 2 - s']_2} \xi^{i_1 \ldots i_r} \alpha^{i_1 \ldots i_r} \lambda^{i_1 \ldots i_r}_{s-2,n} |0\rangle.
$$

(3.21)

Field content for a conformal field in $\mathbb{R}^{d-1,1}$. To develop the Hamiltonian approach to a spin-$s$ conformal field in flat space, we introduce the following set of fields:

$$
\phi^{i_1 \ldots i_r}_{s,k}, \quad \mathcal{P}^{i_1 \ldots i_r}_{s,k}, \quad k' \in [k_s]_2, \quad s' = 0, 1, \ldots, s,
\phi^{i_1 \ldots i_r}_{s-3, k}, \quad \mathcal{P}^{i_1 \ldots i_r}_{s-3, k}, \quad k' \in [k_s]_2, \quad s' = 0, 1, \ldots, s - 3,
$$

(3.22)

$$
\lambda^{i_1 \ldots i_r}_{s-1, k}, \quad k' \in [k_s]_2, \quad s' = 0, 1, \ldots, s - 1,
\lambda^{i_1 \ldots i_r}_{s-2, k}, \quad k' \in [k_s]_2, \quad s' = 0, 1, \ldots, s - 2.
$$

(3.23)

Fields in (3.22) and (3.23) with $s' = 0$, $s' = 1$ and $s' \geq 2$ are the respective scalar, vector and totally symmetric rank-$s'$ tensor fields of the $so(d - 1)$ algebra. To simplify the representation, we use the oscillators $\alpha^i$, $\zeta$, $\nu^0$ and $\nu^0$ and collect fields (3.22) and (3.23) into
the following ket-vectors:

\[
|\phi_s\rangle \equiv \sum_{s'=0}^{s} \sum_{k \in \{|k|\}} \xi^{s-s'} (u^0)^{s+k} \sqrt{s'! (s-s')! (k+\frac{1}{2})!} \alpha^h \cdots \alpha^v |\phi_{s',k}\rangle |0\rangle,
\]

(3.24)

\[
|\phi_{s-3}\rangle \equiv \sum_{s'=0}^{s-3} \sum_{k \in \{|k|\}} \xi^{s-3-s'} (u^0)^{s+k} \sqrt{s'! (s-3-s')! (k+\frac{1}{2})!} \alpha^h \cdots \alpha^v |\phi_{s',k}\rangle |0\rangle,
\]

(3.25)

\[
|\mathcal{P}_s\rangle \equiv \sum_{s'=0}^{s} \sum_{k \in \{|k|\}} \xi^{s-s'} (u^0)^{s+k} \sqrt{s'! (s-s')! (k+\frac{1}{2})!} \alpha^h \cdots \alpha^v \mathcal{P}_{s',k} |0\rangle,
\]

\[
|\mathcal{P}_{s-3}\rangle \equiv \sum_{s'=0}^{s-3} \sum_{k \in \{|k|\}} \xi^{s-3-s'} (u^0)^{s+k} \sqrt{s'! (s-3-s')! (k+\frac{1}{2})!} \alpha^h \cdots \alpha^v \mathcal{P}_{s',k} |0\rangle,
\]

(3.26)

\[
|\lambda_{s-1}\rangle \equiv \sum_{s'=0}^{s-1} \sum_{k \in \{|k|\}} \xi^{s-1-s'} (u^0)^{s+k} \sqrt{s'! (s-1-s')! (k+\frac{1}{2})!} \alpha^h \cdots \alpha^v \lambda_{s',k} |0\rangle,
\]

(3.27)

\[
|\lambda_{s-2}\rangle \equiv \sum_{s'=0}^{s-2} \sum_{k \in \{|k|\}} \xi^{s-2-s'} (u^0)^{s+k} \sqrt{s'! (s-2-s')! (k+\frac{1}{2})!} \alpha^h \cdots \alpha^v \lambda_{s',k} |0\rangle.
\]

To summarize, the fields which we use for discussing the extended Hamiltonian approach to massless and massive fields in flat and AdS space, can be collected into the following ket-vectors:

\[
|\phi_s\rangle, \quad |\phi_{s-3}\rangle, \quad |\mathcal{P}_s\rangle, \quad |\mathcal{P}_{s-3}\rangle, \quad |\lambda_{s-1}\rangle, \quad |\lambda_{s-2}\rangle.
\]

(3.28)

We note that fields \(|\phi_s\rangle, |\phi_{s-3}\rangle, |\mathcal{P}_s\rangle, |\mathcal{P}_{s-3}\rangle\) are phase space variables, while the fields \(|\lambda_{s-1}\rangle, |\lambda_{s-2}\rangle\) are Lagrangian multipliers. In order to obtain the gauge-invariant Hamiltonian description in an easy-to-use form, we collect fields (3.27) into the vectors given by

\[
|\phi\rangle = \begin{pmatrix} |\phi_s\rangle \\ |\phi_{s-3}\rangle \end{pmatrix}, \quad |\mathcal{P}\rangle = \begin{pmatrix} |\mathcal{P}_s\rangle \\ |\mathcal{P}_{s-3}\rangle \end{pmatrix}, \quad |\lambda\rangle = \begin{pmatrix} |\lambda_{s-1}\rangle \\ |\lambda_{s-2}\rangle \end{pmatrix}.
\]

(3.29)

Extended Hamiltonian action. Extended gauge-invariant Hamiltonian action for massless and massive fields in flat and AdS space and conformal fields in flat space, takes the form

\[
S = \int dt d^{d-1}x \mathcal{L}, \quad \text{for fields in } R^{d-1,1},
\]

(3.30)

\[
S = \int dt d^{d-1}x dz \mathcal{L}, \quad \text{for fields in } AdS_{d+1},
\]

where \(\mathcal{L}\) is a phase-space Lagrangian. The phase-space Lagrangian we found is given by

\[
\mathcal{L} = \langle \mathcal{P}|\phi\rangle - \frac{1}{2} \langle \mathcal{P}|\mathcal{K}^{-1}|\mathcal{P}\rangle + \langle \mathcal{P}|L|\phi\rangle + \mathcal{L}^* + \langle \lambda|T\rangle,
\]

(3.31)

where \(\mathcal{K}\) is the inverse metric, \(L\) the Lagrangian, \(\lambda\) the Lagrange multiplier, \(T\) the stress-energy tensor, and \(\mathcal{L}^*\) is the conjugate momentum. The operators \(\partial^i\) and \(\partial^r\) are given by

\[
\partial^i \partial^i - \mathcal{M}^2,
\]

(3.32)

\[
\Delta_m \equiv \partial^i \partial^i - \mathcal{M}^2,
\]

(3.33)
where \( 2 \times 2 \) matrices \( \pi_{\pm}, \sigma_{\pm} \) and operators \( C_{mn}, G_{mn}, n_{00}, n_{44}, K_0, K_3 \) are given in the appendix. We note that the operators \( n_{00}, n_{44}, K_0 \) and \( K_3 \) only depend on the spatial oscillators and are independent of the spatial derivative. From (3.30), we see that the fields \( |\phi\rangle \) and \( |\mathcal{P}\rangle \) are realized as phase space variables, while the field \( |\lambda\rangle \) is realized as a Lagrangian multiplier.

**Gauge transformations.** We now discuss the realization of gauge symmetries in the framework of an extended Hamiltonian gauge-invariant approach. We begin our discussion with a description of the gauge transformation parameters to be used for the description of gauge transformations. We discuss the gauge transformation parameters in turn.

**Gauge transformation parameters for a massless field in \( R^{d-1,1} \).** In the framework of our Hamiltonian gauge-invariant approach, the gauge symmetries of a spin-\( s \) massless field in flat space are described by the following two gauge transformation parameters:

\[
\xi^{i_1 i_2 \cdots |i_s}, \quad \eta^{i_1 i_2 \cdots |i_s}.
\]

(3.39)

For the corresponding values of \( s \), the gauge transformation parameters in (3.39) are scalar, vector and traceful tensor fields of the \( \text{so}(d-1) \) algebra. We use the oscillators \( \alpha^i \) to collect the parameters into the two ket-vectors given by

\[
|\xi_{s-1}\rangle \equiv \frac{1}{(s-1)!} \alpha^{i_1} \cdots \alpha^{i_{s-1}} |\xi_{s-1}^{i_1 i_2 \cdots |i_{s-1}}\rangle|0\rangle,
\]

\[
|\xi_{s-2}\rangle \equiv \frac{1}{(s-2)!} \alpha^{i_1} \cdots \alpha^{i_{s-2}} |\xi_{s-2}^{i_1 i_2 \cdots |i_{s-2}}\rangle|0\rangle.
\]

(3.40)

**Gauge transformation parameters for a massive field in \( R^{d-1,1} \).** In the framework of our Hamiltonian gauge-invariant approach, the gauge symmetries of a spin-\( s \) massive field in flat space are described by the following set of gauge transformation parameters:

\[
\xi^{i_1 i_2 \cdots |i_{s'}}, \quad s' = 0, 1, \ldots, s-1,
\]

\[
\eta^{i_1 i_2 \cdots |i_{s'-2}}, \quad s' = 0, 1, \ldots, s-2.
\]

(3.41)

We note that the gauge transformation parameters in (3.41) with \( s' = 0, s' = 1 \) and \( s' \geq 2 \) are the respective scalar, vector and rank-\( s' \) traceful tensor fields of the \( \text{so}(d-1) \) algebra. We use the oscillators \( \alpha^i \) and \( \xi^i \) to collect the gauge transformation parameters into the two ket-vectors given by

\[
|\xi_{s-1}\rangle \equiv \sum_{s' = 0}^{s-1} \frac{s'!}{(s - 1 - s')!} \xi^{i_1 i_2 \cdots |i_{s-1}} |\xi_{s-1}^{i_1 i_2 \cdots |i_{s-1}}\rangle|0\rangle,
\]

\[
|\xi_{s-2}\rangle \equiv \sum_{s' = 0}^{s-2} \frac{s'!}{(s - 2 - s')!} \xi^{i_1 i_2 \cdots |i_{s-2}} |\xi_{s-2}^{i_1 i_2 \cdots |i_{s-2}}\rangle|0\rangle.
\]

(3.42)
Gauge transformation parameters for a massless field in AdS_{d+1}. To discuss gauge symmetries of a spin-$s$ massless AdS field in the framework of a Hamiltonian gauge-invariant approach, we use the following set of gauge transformation parameters:

\[ \xi_{s-1}^{i_{s-1}} \quad s' = 0, 1, \ldots, s - 1, \]
\[ \xi_{s-2}^{i_{s-2}} \quad s' = 0, 1, \ldots, s - 2. \]  

(3.43)

The gauge transformation parameters in (3.43) with $s' = 0, s' = 1$ and $s' \geq 2$ are the respective scalar, vector and rank-$s'$ traceful tensor fields of the $so(d-1)$ algebra. We use the oscillators $\alpha^i$, $\alpha^i$, and $\xi$ to collect the gauge transformation parameters into the two ket-vectors given by

\[ |\xi_{s-1}\rangle = \sum_{r=0}^{s-1} \frac{\alpha_z^{s-1-s'} \alpha^i \ldots \alpha^i}{s'! \sqrt{(s-1-s')!}} \xi_{s-1}^{i_{s-1}} |0\rangle, \]
\[ |\xi_{s-2}\rangle = \sum_{r=0}^{s-2} \frac{\alpha_z^{s-2-s'} \alpha^i \ldots \alpha^i}{s'! \sqrt{(s-2-s')!}} \xi_{s-2}^{i_{s-2}} |0\rangle. \]  

(3.44)

Gauge transformation parameters for a massive field in AdS_{d+1}. To discuss the gauge symmetries of a spin-$s$ massive AdS field in the framework of a Hamiltonian gauge-invariant approach, we use the following set of gauge transformation parameters:

\[ \xi_{s-1, n}^{i_{s-1}} \quad n \in [s - 1 - s', 2], \quad s' = 0, 1, \ldots, s - 1, \]
\[ \xi_{s-2, n}^{i_{s-2}} \quad n \in [s - 2 - s', 2], \quad s' = 0, 1, \ldots, s - 2. \]  

(3.45)

Gauge transformation parameters in (3.45) with $s' = 0, s' = 1$ and $s' \geq 2$ are the respective scalar, vector and rank-$s'$ traceful tensor fields of the $so(d-1)$ algebra. We use the oscillators $\alpha^i$, $\alpha^i$, $\zeta$ and $\rho$ to collect the gauge transformation parameters into the two ket-vectors given by

\[ |\xi_{s-1}\rangle = \sum_{r=0}^{s-1} \sum_{n \in [s-1-s', 2]} \frac{\xi_z^{s-1-s'} \alpha_z^{s-1-s'} \alpha^i \ldots \alpha^i}{s'! \sqrt{(s-1-s')!} (\frac{s-1-s' + n}{2})!} \xi_{s-1, n}^{i_{s-1}} |0\rangle, \]
\[ |\xi_{s-2}\rangle = \sum_{r=0}^{s-2} \sum_{n \in [s-2-s', 2]} \frac{\xi_z^{s-2-s'} \alpha_z^{s-2-s'} \alpha^i \ldots \alpha^i}{s'! \sqrt{(s-2-s')!} (\frac{s-2-s' + n}{2})!} \xi_{s-2, n}^{i_{s-2}} |0\rangle. \]  

(3.46)

(3.47)

Gauge transformation parameters for a conformal field in $R^{d-1,1}$. To discuss the gauge symmetries of a spin-$s$ conformal field in the framework of a Hamiltonian gauge-invariant approach, we use the following set of gauge transformation parameters:

\[ \xi_{s-1, k'}^{i_{s-1}} \quad s' = 0, 1, \ldots, s - 1, \quad k' \in [k_s + 1], \]
\[ \xi_{s-2, k'}^{i_{s-2}} \quad s' = 0, 1, \ldots, s - 2, \quad k' \in [k_s + 1]. \]  

(3.48)

The gauge transformation parameters in (3.48) with $s' = 0, s' = 1$ and $s' \geq 2$ are the respective scalar, vector and rank-$s'$ traceful tensor fields of the $so(d-1)$ algebra. We use the oscillators $\alpha^i$, $\zeta$, $\upsilon^0$ and $\upsilon^0$ to collect the parameters into the two ket-vectors given by

\[ |\xi_{s-1}\rangle = \sum_{r=0}^{s-1} \sum_{k' \in [k_s + 1]} \xi_z^{s-1-s'} (\upsilon^0) \frac{s_z^{s-1-s'} (\upsilon^0)^{k_s+1+k'}}{s'! \sqrt{(s-1-s')!} (\frac{s-1-s' + k_s + 1}{2})!} \alpha^i \ldots \alpha^i \xi_{s-1, k'}^{i_{s-1}} |0\rangle, \]
\[ |\xi_{s-2}\rangle = \sum_{r=0}^{s-2} \sum_{k' \in [k_s + 1]} \xi_z^{s-2-s'} (\upsilon^0) \frac{s_z^{s-2-s'} (\upsilon^0)^{k_s+1+k'}}{s'! \sqrt{(s-2-s')!} (\frac{s-2-s' + k_s + 1}{2})!} \alpha^i \ldots \alpha^i \xi_{s-2, k'}^{i_{s-2}} |0\rangle. \]  

(3.49)
To summarize, we note that, in all the cases considered above, the gauge transformation parameters we use for the description of the gauge symmetries of massless and massive fields in flat and AdS space and conformal fields in flat space, can be collected into two ket-vectors
\[ |\xi_{s-1}\rangle, \quad |\xi_{s-2}\rangle. \] (3.50)

As before, in order to obtain the gauge transformations in an easy-to-use form, we collect the gauge transformation parameters (3.50) into the vector given by
\[ |\xi\rangle = \begin{pmatrix} |\xi_{s-1}\rangle \\ |\xi_{s-2}\rangle \end{pmatrix}. \] (3.51)

We now discuss gauge transformations. The gauge transformations can be presented, entirely, in terms of the ket-vectors discussed above. The gauge transformations we found, take the form
\[ \delta|\phi\rangle = G_\phi|\xi\rangle, \] (3.52)
\[ \delta|\mathcal{P}\rangle = G_P|\xi\rangle, \] (3.53)
\[ \delta|\lambda\rangle = |\dot{\xi}\rangle + G_\lambda|\xi\rangle, \] (3.54)
\[ G_\phi \equiv G_{00}\pi_+ + \bar{G}_{32}\pi_-, \] (3.55)
\[ G_P \equiv G_{02}\sigma_+ + \bar{G}_{31}\sigma_- \] (3.56)
\[ G_\lambda \equiv G_{12}\sigma_+ + \bar{G}_{21}\sigma_- \] (3.57)

where the explicit form of the operators $G_{mn}$ is given in the appendix. The following remarks are relevant.

(i) From (3.54) we see that gauge transformations of the Lagrangian multiplier $|\lambda\rangle$ involve the time derivative of the gauge transformation parameter, as it should be in an extended Hamiltonian approach (see e.g. [2]).

(ii) Introducing the Hamiltonian $H$ and the gauge transformation -generating function $T_\xi$,
\[ H \equiv \int d^{d-1}x \mathcal{H}, \quad \text{for fields in } R^{d-1,1}, \] (3.58)
\[ H \equiv \int d^{d-1}x dz \mathcal{H}, \quad \text{for fields in } AdS_{d+1}, \]
\[ -\mathcal{H} = -\frac{i}{2}\langle\mathcal{P}|K^{-1}|\mathcal{P}\rangle + \langle\mathcal{P}|\mathcal{L}|\phi\rangle + \mathcal{L}^*, \] (3.59)
\[ T_\xi \equiv \int d^{d-1}x |\xi||\delta\rangle, \quad \text{for fields in } R^{d-1,1}, \] (3.60)
\[ T_\xi \equiv \int d^{d-1}x dz |\xi||\delta\rangle, \quad \text{for fields in } AdS_{d+1}, \]

where $|T\rangle$ is given in (3.32), we find that under the gauge transformations (3.52)–(3.54), the constraint $|T\rangle$ and the Hamiltonian $H$ transform as
\[ \delta|T\rangle = 0, \] (3.61)
\[ \delta H = T_{G_\xi}. \] (3.62)

The relation (3.61) shows that the constraint $|T\rangle$ is invariant under the gauge transformations, while from the relation (3.62), we learn that the gauge variation of the Hamiltonian $H$ is proportional to the constraint $|T\rangle$. In other words, $|T\rangle$ is the first-class constraint.
(iii) The Lagrangian (3.30) implies the standard equal-time Poisson bracket,

\[ \{ |x\}, |y\rangle \} = \left\{ |\delta^{d-1}(x - x'), \delta(z - z')\}, \right\}

for fields in \( R^{d-1,1} \),

\[ \{ |x\}, |y\rangle \} = \left\{ |\delta^{d-1}(x - x')\delta(z - z')\}, \right\}

for fields in \( \text{AdS}_{d+1} \).

where \(|\rangle\langle|\) stands for the unit operator on the space of the ket-vectors given in (3.28). Using the Poisson bracket, we check that the gauge transformations of phase space variables \(|\phi\rangle \) and \(|\mathcal{P}\rangle \) given in (3.52) and (3.53) can be represented as

\[ \delta|\phi\rangle = \left\{ |\phi\rangle, T_1\right\}, \]

\[ \delta|\mathcal{P}\rangle = \left\{ |\mathcal{P}\rangle, T_1\right\}. \]

as it should be in the framework of the extended Hamiltonian approach. Also, in terms of the Poisson bracket, the gauge transformations given in (3.61) and (3.62) can be represented as

\[ [T_{\xi}, T_{\xi}] = 0, \]

\[ [T_{\xi}, H] = -T_{G,\xi}. \]

(iv) As an illustration, let us count the physical DOF for a spin-\( s \) massless field in \( d \)-dimensional flat space by using the extended Hamiltonian approach. Using notation \( N_{e,n} \) for the dimension of the totally symmetric rank-\( s \) traceless tensor field of \( so(n) \) algebra,

\[ N_{e,n} = \frac{(s + n - 1)!}{(n - 1)!s!}, \]

we note that the dimensions of the fields \( \phi^{i_1 \cdots i_s} \) and \( \phi^{i_1 \cdots i_s-2} \) are given by \( N_{e,d-1} \) and \( N_{e-3,d-1} \), respectively, while the dimensions of the Lagrangian multipliers \( \lambda^{i_1 \cdots i_s} \) and \( \lambda^{i_1 \cdots i_s-2} \) are given by \( N_{e-1,d-1} \) and \( N_{e-2,d-1} \), respectively. Applying the standard formula for counting physical DOF (see e.g. [2]), we find the relation

\[ N_{e,d-1} + N_{e-3,d-1} - N_{e-1,d-1} - N_{e-2,d-1} = (2s + d - 4) \frac{(s + d - 5)!}{(d - 4)!s!}. \]

The number on the rhs in (3.68) is a dimension of a totally symmetric rank-\( s \) traceless tensor field of \( so(d - 2) \) algebra. This dimension is the number of physical DOF for a spin-\( s \) massless field in \( d \)-dimensional space-time.

To summarize, starting with the Lagrangian formulation of double-traceless higher-spin fields, we obtained an extended Hamiltonian action in terms of fields which are free of algebraic constraints. We believe that the appearance of unconstrained fields in an extended Hamiltonian approach should streamline the application of our approach to the study of various aspects of higher-spin fields\(^9\). Also, we think that the power of the Hamiltonian method will provide new possibilities for analyzing equations of the motion of AdS fields and studying AdS/CFT correspondence\(^10\).

In conclusion, we note a number of potentially interesting generalizations and applications in our approach. This is to say that, although many methods for building interaction vertices for higher-spin fields are known in the literature (see e.g. [54–68]), constructing interaction vertices for concrete field theoretical models of higher-spin fields is still a challenging problem. We believe that the use of the extended Hamiltonian approach will provide interesting new possibilities for studying this important problem. We also think that the extended Hamiltonian approach should streamline the application of our approach to the study of various aspects of higher-spin fields\(^9\). Also, we think that the power of the Hamiltonian method will provide new possibilities for analyzing equations of the motion of AdS fields and studying AdS/CFT correspondence\(^10\).

\(^9\) As a side remark we note that at the Lagrangian level many interesting formulations in terms of unconstrained fields have been developed, recently. That is to say, various Lagrangian formulations of higher-spin field dynamics, in terms of constrained fields, are discussed in [46–50].

\(^10\) A discussion of interesting methods for analyzing equations of the motion of fields in AdS space may be found in [51–53].
approach, discussed in this paper, might be useful for the study of string theory in the AdS background [69–71] and various aspects of AdS/CFT correspondence as in [72–75]. In this paper we considered an extended Hamiltonian action for bosonic totally symmetric fields. Needless to say, a generalization of our approach to the case of fermionic fields [76] and mixed symmetry fields [77–82] could also be of interest.

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Appendix. Notation

The basis of $2 \times 2$ matrices we use is defined as

$$
\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \pi_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
$$

Throughout the paper, the notation $n \in [k]_2$ implies that $n = -k, -k + 2, -k + 4, \ldots, k - 4, k - 2, k$:

$$
n \in [k]_2 \implies n = -k, -k + 2, -k + 4, \ldots, k - 4, k - 2, k.
$$

Notation in the basis of Lorentz algebra so($d-1, 1$). Our conventions are as follows. $x^a$ denotes coordinates in $d$-dimensional flat space-time, while $\partial_a$ denotes derivatives with respect to $x^a$, $\delta_{ab} \equiv \partial \partial x^a$. The vector indices of the Lorentz algebra so($d-1, 1$) take the values $a, b, c, e = 0, 1, \ldots, d - 1$. We use the mostly positive flat metric tensor $\eta^{ab}$. To simplify our expressions we drop $\eta_{ab}$ in scalar products, i.e. we use $X^a Y^b \equiv \eta_{ab} X^a Y^b$. We use a set of creation operators $\alpha^a$, $\alpha^a$, $\xi^a$, $\xi^a$, $\bar{\nu}^a$, $\bar{\nu}^a$ and a respective set of annihilation operators $\bar{\alpha}^a$, $\bar{\alpha}^a$, $\bar{\xi}$, $\bar{\xi}$, $\bar{\nu}$, $\bar{\nu}$. These operators, to be referred to as oscillators, satisfy the commutation relations

$$
[\bar{\alpha}^a, \alpha^b] = \eta^{ab}, \quad [\bar{\alpha}^a, \alpha^b] = 1, \quad [\bar{\xi}, \xi] = 1, \quad [\bar{\nu}, \nu] = 1, \quad [\bar{\nu}, \nu] = 1,
$$

$$
\bar{\alpha}^a |0\rangle = 0, \quad \bar{\alpha}^a |0\rangle = 0, \quad \bar{\xi} |0\rangle = 0, \quad \bar{\nu} |0\rangle = 0, \quad \bar{\bar{\nu}} |0\rangle = 0.
$$

We adopt the following Hermitian conjugation rules for the derivatives and oscillators:

$$
\alpha^{\dagger} = -\bar{\alpha}, \quad \alpha^{\dagger} = \bar{\alpha}, \quad \bar{\xi} = \xi, \quad \bar{\bar{\xi}} = \bar{\xi}, \quad \nu^{\dagger} = \bar{\nu}, \quad \nu^{\dagger} = \bar{\nu}.
$$

We use operators constructed out of the derivatives and oscillators,

$$
\square = \partial^a \partial_a, \quad a \bar{a} = \alpha^a \bar{\alpha}^a, \quad \bar{a} \partial = \bar{\alpha} \partial, \quad (A.6)
$$

$$
\alpha^2 = \alpha^a \alpha^a, \quad \bar{\alpha}^2 = \bar{\alpha}^a \bar{\alpha}^a, \quad N_a = \alpha^a \bar{\alpha}^a, \quad (A.7)
$$

$$
N_z = \alpha^a \bar{\alpha}^a, \quad N_{\xi} = \bar{\xi} \xi.
$$

Notation in the basis of so($d-1$) algebra. In the basis so($d-1$) algebra, we split the space-time coordinates, derivatives and oscillators as follows:

$$
x^a = t, x^i, \quad \partial_a = \partial_t, \partial_i, \quad \partial^a = \partial_t / \partial t, \quad \partial^i = \partial_t / \partial x^i, \quad (A.9)
$$

$$
\alpha^a = \alpha^0, \alpha^i, \quad \bar{\alpha}^a = \bar{\alpha}^0, \bar{\alpha}^i, \quad [\bar{\alpha}^0, \alpha^0] = -1, \quad [\bar{\alpha}^i, \alpha^i] = \delta^{ij}, \quad (A.10)
$$

An extensive study and applications of the oscillator formalism may be found in [83, 84].
Vector indices of the algebra $so(d - 1)$ take the values $i, j = 1, \ldots, d - 1$. We use operators constructed out of the spatial derivative and oscillators,

$$\alpha \partial \equiv \alpha_i \partial^i, \quad \bar{\alpha} \partial \equiv \bar{\alpha}^i \partial_i, \quad \alpha^2 \equiv \alpha^i \alpha^i, \quad \bar{\alpha}^2 \equiv \bar{\alpha}^i \bar{\alpha}_i, \quad N_\alpha \equiv \alpha^i \bar{\alpha}_i,$$

(A.11)

$$n_{00} \equiv \sum_{n=0}^{\infty} \frac{1}{(2n)!} \alpha^{2n} \bar{\alpha}^{2n}, \quad (A.12)$$

$$n_{11} \equiv -\sum_{n=0}^{\infty} \frac{2n + 1}{(2n)!} \alpha^{2n} \bar{\alpha}^{2n}, \quad (A.13)$$

$$n_{22} \equiv \sum_{n=0}^{\infty} \frac{2n + 2}{(2n + 1)!} \alpha^{2n} \bar{\alpha}^{2n}, \quad (A.14)$$

$$n_{33} \equiv -\sum_{n=0}^{\infty} \frac{4(n + 1)^2}{(2n + 3)!} \alpha^{2n} \bar{\alpha}^{2n}. \quad (A.15)$$

$$n_{44} \equiv \sum_{n=0}^{\infty} \frac{1}{(2n + 1)!} \alpha^{2n} \bar{\alpha}^{2n}, \quad (A.16)$$

$$n_{02} \equiv \alpha^2 \sum_{n=0}^{\infty} \frac{1}{(2n + 1)!} \alpha^{2n} \bar{\alpha}^{2n}, \quad (A.17)$$

$$n_{13} \equiv \alpha^2 \sum_{n=0}^{\infty} \frac{1}{(2n + 1)!} \alpha^{2n} \bar{\alpha}^{2n}, \quad (A.18)$$

$$K_0 \equiv \sum_{n=0}^{\infty} \frac{1 - 2n}{(2n)!} \alpha^{2n} \bar{\alpha}^{2n}, \quad (A.19)$$

$$K_3 \equiv \sum_{n=0}^{\infty} \frac{2n + 2}{(2n + 3)!} \alpha^{2n} \bar{\alpha}^{2n}, \quad (A.20)$$

$$K_0 \equiv n_{00} + \alpha^2 n_{44} \bar{\alpha}^2, \quad (A.21)$$

$$K_3 \equiv n_{33} - n_{44}, \quad (A.22)$$

$$G_{01} = \alpha \partial - e_1 - \alpha^2 \frac{1}{2N_\alpha + d - 2} \bar{e}_1, \quad (A.23)$$

$$\bar{G}_{01} = \bar{\alpha} \partial - \bar{e}_1 - e_1 \frac{1}{2N_\alpha + d - 2} \bar{\alpha}^2, \quad (A.24)$$

$$G_{12} = \alpha \partial - e_1 - \alpha^2 \frac{1}{2N_\alpha + d} \bar{\epsilon}_1, \quad (A.25)$$

$$\bar{G}_{12} = \bar{\alpha} \partial - \bar{\epsilon}_1 - e_1 \frac{1}{2N_\alpha + d} \bar{\alpha}^2, \quad (A.26)$$

$$G_{21} = -\alpha \partial + e_1 \frac{2N_\alpha + d}{2N_\alpha + d - 2}, \quad (A.27)$$

$$\bar{G}_{21} = -\bar{\alpha} \partial + \bar{e}_1 \frac{2N_\alpha + d}{2N_\alpha + d - 2}, \quad (A.28)$$
\[ G_{32} = 3\alpha \partial + \alpha^2 \bar{a} \partial - e_1 \left( 3 + \alpha^2 \frac{1}{2N_\alpha + d + 4} \bar{a}^2 \right) - \alpha^2 \bar{e}_1, \quad (A.29) \]

\[ \tilde{G}_{32} = 3\bar{a} \partial + \alpha \partial \bar{a}^2 - \left( 3 + \alpha^2 \frac{1}{2N_\alpha + d + 4} \bar{a}^2 \right) \bar{e}_1 - e_1 \bar{a}^2, \quad (A.30) \]

\[ G_{02} \equiv -\alpha^2 n_{44} \Delta_m - C_{10} n_{00} C_{12}, \quad (A.31) \]

\[ \bar{G}_{02} \equiv n_{44} \bar{a}^2 \Delta_m + \bar{C}_{12} n_{00} \bar{C}_{10}, \quad (A.32) \]

\[ G_{31} \equiv -G_{21} n_{44} C_{23}, \quad (A.33) \]

\[ \tilde{G}_{31} \equiv \tilde{C}_{23} n_{44} \tilde{G}_{21}, \quad (A.34) \]

\[ C_{10} = \alpha \partial - e_1, \quad (A.35) \]

\[ \tilde{C}_{10} = \bar{a} \partial - \bar{e}_1, \quad (A.36) \]

\[ C_{12} = \alpha \partial - e_1 - \alpha^2 \frac{1}{2N_\alpha + d} \bar{e}_1, \quad (A.37) \]

\[ \tilde{C}_{12} = \bar{a} \partial - \bar{e}_1 - \bar{e}_1 \left( \frac{1}{2N_\alpha + d} \bar{a}^2 \right), \quad (A.38) \]

\[ C_{21} = \alpha \partial + \alpha^2 \bar{a} \partial - \left( 1 + \alpha^2 \frac{1}{2N_\alpha + d + 2} \bar{a}^2 \right) e_1 - \alpha^2 \bar{e}_1, \quad (A.39) \]

\[ \tilde{C}_{21} = \bar{a} \partial + \alpha \partial \bar{a}^2 - \left( 1 + \alpha^2 \frac{1}{2N_\alpha + d + 2} \bar{a}^2 \right) \bar{e}_1 - e_1 \bar{a}^2, \quad (A.40) \]

\[ C_{23} = \alpha \partial - e_1 - \alpha^2 \frac{1}{2N_\alpha + d + 2} \bar{e}_1, \quad (A.41) \]

\[ \tilde{C}_{23} = \bar{a} \partial - \bar{e}_1 - \bar{e}_1 \left( \frac{1}{2N_\alpha + d + 2} \bar{a}^2 \right). \quad (A.42) \]

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