STRONGNESS OF COMPANION BASES FOR CLUSTER-TILTED ALGEBRAS OF FINITE TYPE

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Abstract. For every cluster-tilted algebra of simply-laced Dynkin type we provide a companion basis which is strong, i.e. gives the set of dimension vectors of the finitely generated indecomposable modules for the cluster-tilted algebra. This shows in particular that every companion basis of a cluster-tilted algebra of simply-laced Dynkin type is strong. Thus we give a proof of Parsons’s conjecture.

1. Introduction

Cluster algebras were introduced and first investigated by Fomin and Zelevinsky [8] in order to better understand the dual canonical basis of the quantised enveloping algebra of a finite dimensional semisimple Lie algebra. Today cluster algebras are connected to various subjects including representation theory of finite dimensional algebras, Poisson geometry, algebraic geometry, knot theory, Teichmüller theory, dynamical systems, mathematical physics, combinatorics, .... Cluster categories were introduced in [3] (also for type \( A_n \) in [7]) as a categorical model for better understanding of the cluster algebras. The cluster-tilted algebras were introduced by Buan, Marsh and Reiten [4] have a key role in the study of cluster categories. Also an important connection between cluster algebras and cluster-tilted algebras was established in [5] and [6]. More precisely it was proved in [5] and [6] that the quivers of the cluster-tilted algebras of a given simply-laced Dynkin type are precisely the quivers of the exchange matrices of the cluster algebras of that type.

Fomin and Zelevinsky [9] classified the cluster algebras of finite type. Their classification is identical to the Cartan-Killing classification of semisimple Lie algebras. The conditions of their classification is hard to check in general. For solving this problem, Barot, Geiss and Zelevinsky [2] considered the positive quasi-Cartan matrices which are related to the Cartan matrices. The main Theorem of [2] shows that the exchange matrix associated to the quiver of any cluster-tilted algebra of simply-laced Dynkin type have a positive quasi-Cartan companion. More precisely they showed that there exists a Z-basis of roots of the integral root lattice of the corresponding root system of simply-laced Dynkin type such that the matrix of inner products associated to this basis creates the positive quasi-Cartan companion. Parsons in [11] and [12] called such a Z-basis a companion basis. Parsons used companion bases to study indecomposable modules over cluster-tilted algebras. Note that the knowledge of the indecomposable modules over cluster-tilted algebras is not only important in the representation theory of cluster-tilted algebras but also has applications in the corresponding cluster algebra. Nakanishi and Stella [10] showed that the set of all
dimension vectors of the indecomposable modules over a cluster-tilted algebra of finite type, the set of the non-initial $d$-vectors of the corresponding cluster algebra and the set of positive $c$-vectors of the corresponding cluster algebra coincide.

For any companion basis of a cluster-tilted algebra of simply-laced Dynkin type, Parsons defined a collection of positive vectors. Any positive root of the corresponding root system of simply-laced Dynkin type is a linear combination of elements of the companion basis with integer coefficients. That collection is the set of the absolute values of these coefficients. A companion basis of a cluster-tilted algebra of simply-laced Dynkin is called strong [11], [12] if the associated collection of positive vectors are precisely the dimension vectors of the finitely generated indecomposable modules over the given cluster-tilted algebra. Parsons proved that for any cluster-tilted algebra of Dynkin type $A$, any companion basis is strong. He also conjectured that any companion basis of any cluster-tilted algebra of simply-laced Dynkin type is strong (Conjecture 6.3 of [11]).

On the other hand Ringel [13] studied finitely generated indecomposable modules over cluster-concealed algebras (i.e. cluster-tilted algebras where the associated cluster-tilting object corresponds to a preprojective tilting module). Note that cluster-tilted algebras of simply-laced Dynkin type are cluster-concealed algebras, but there are many cluster-concealed algebras which are tame or wild. Ringel [13] used a theorem due to Assem et al. [1] and Zhu [14], which gives an equivalent definition of cluster-tilted algebras. According to that theorem, any cluster-tilted algebra is isomorphic to the relation-extension of some tilted algebra. By using the tilting functor Ringel defined a linear bijection map $g$ between the Grothendieck group of the corresponding hereditary algebra $K_0(H)$ and the Grothendieck group of the tilted algebras $K_0(B)$. He proved that the dimension vectors of indecomposable modules over a cluster-concealed algebra are precisely the vectors $\text{abs}(x)$ of absolute values with $x \in \Phi_B$, where $\Phi_B$ is the image of the root system $\Phi_H$ of $H$ under $g$ and where for each $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$, $\text{abs}(x) = (|x_1|, \cdots, |x_n|)$. He also proved that indecomposable modules over cluster-tilted algebras of simply-laced Dynkin type are uniquely determined by their dimension vectors.

Given a cluster-tilted algebra of simply-laced Dynkin type, according to Ringel’s Theorem there exist $\{x_1, x_2, \cdots, x_n\} \subseteq \Phi_H$ such that $\text{abs}(g(x_i)) = e_i$ for each $i$, where $\{e_1, \cdots, e_n\}$ is the canonical basis of $\mathbb{Z}^n$. The goal of this paper is to show that the set $\{x_1, x_2, \cdots, x_n\}$ is a companion basis for the cluster-tilted algebra and that it is strong. This result not only illustrates the connection between Ringel’s results [13] and Parsons’s results [11], [12], it also proves Parsons’s conjecture (Conjecture 6.3 of [11]). Before proving our results in Section 3 we provide the necessary background in the following section.

2. Preliminaries

Let $\mathbb{F} = \mathbb{Q}(u_1, \cdots, u_n)$ be the field of rational functions in $n$ indeterminates. Let $\mathbf{x} \subseteq \mathbb{F}$ be a transcendence basis over $\mathbb{Q}$, and let $C = (c_{xy})_{x,y \in \mathbf{x}}$ be an $n \times n$ sign-skew-symmetric integer matrix with rows and columns indexed by $\mathbf{x}$. The pair $(\mathbf{x}, C)$ is called a seed. Given such a seed, and an element $z \in \mathbf{x}$, define a new element $z' \in \mathbb{F}$ via the exchange relation:

$$zz' = \prod_{x \in \mathbf{x}, c_{xz} > 0} x^{c_{xz}} + \prod_{x \in \mathbf{x}, c_{xz} < 0} x^{-c_{xz}}.$$
Let \( x' = x \cup \{ z' \} \setminus \{ z \} \), it is a new transcendence basis of \( \mathbb{F} \). Let \( C' \) be the mutation of the matrix \( C \) in direction \( z' \):

\[
c'_{xy} = \begin{cases} 
-c_{xy} & \text{if } x = z \text{ or } y = z, \\
\frac{|c_{xz}|c_{xy} + |c_{xz}|c_{zy}}{2} & \text{otherwise.}
\end{cases}
\]

The row and column labeled \( z \) in \( C \) are relabeled \( z' \) in \( C' \). The pair \((x', C')\) is called the mutation of the seed \((x, C)\) in direction \( z \). Let \( \mathcal{S} \) be the set of seeds obtained by iterated mutation of \((x, C)\). Then the set of cluster variables is, by definition, the union \( \chi \) of all the elements of the transcendence bases appearing in the seeds in \( \mathcal{S} \). These bases are known as clusters, and the cluster algebra \( \mathcal{A}(x, C) \) is the subring of \( \mathbb{F} \) generated by \( \chi \). Up to isomorphism of cluster algebras, it does not depend on the initial choice \( x \) of transcendence basis, so it is just denoted by \( \mathcal{A}_C \). If \( \chi \) is finite, the cluster algebra \( \mathcal{A}_C \) is said to be of finite type.

Let \( H \) be a hereditary finite dimensional \( k \)-algebra, where \( k \) is an algebraically closed field and let \( \mathcal{D} = D^b(\text{mod}H) \) be the bounded derived category of finitely generated left \( H \)-modules with shift functor \([1]\). Also, let \( \tau \) be the \( AR \)-translation in \( \mathcal{D} \). The cluster category is defined as the orbit category \( C_H = \mathcal{D}/F \), where \( F = \tau^{-1}[1] \). The objects of \( C_H \) are the same as the objects of \( \mathcal{D} \), but maps are given by \( \text{Hom}_{C_H}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_D(X, F^iY) \). An object \( \tilde{T} \) in \( C_H \) is called cluster-tilting provided for any object \( X \) of \( C_H \), we have \( \text{Ext}^1_{C_H}(\tilde{T}, X) = 0 \) if and only if \( X \) lies in the additive subcategory \( \text{add}(\tilde{T}) \) of \( C_H \) generated by \( \tilde{T} \). Let \( \tilde{T} \) be a cluster-tilting object in \( C_H \). The cluster-tilted algebra associated to \( \tilde{T} \) is the algebra \( \text{End}^\text{op}_{C_H}(\tilde{T}) \).

Let \( Q \) be a quiver of simply-laced Dynkin type with underlying graph \( \Delta \) and \( n \) vertices. Let \( H = kQ \) and \( \tilde{T} \) be a basic cluster-tilting object in the cluster category \( C_H \). Let \( \Lambda = \text{End}_{C_H}(\tilde{T})^\text{op} \) be the corresponding cluster-tilted algebra. Let \( \mathcal{A} \) be the cluster algebra of type \( \Delta \) and \((x, C)\) be the seed in \( \mathcal{A} \) corresponding to \( \tilde{T} \). According to Theorem 3.1 of \([1]\) (also Section 6 of \([3]\)), \( \Gamma = \Gamma(C) \) is the quiver of \( \Lambda \), where \( \Gamma(C) \) is the quiver with vertices corresponding to the rows and columns of \( C \), and \( c_{ij} \) arrows from the vertex \( i \) to the vertex \( j \) whenever \( c_{ij} > 0 \). Let \( V \) be the Euclidean space with positive definite symmetric bilinear form \((-, -)\) and \( \Phi_H \subseteq V \) be the root system of Dynkin type \( \Delta \).

**Definition 2.1.** (Definitions 4.1, 5.1 and 5.2 of \([1]\))

1. The subset \( \Psi = \{ \gamma_i | 1 \leq i \leq n \} \subseteq \Phi_H \) is called a companion basis for \( \Gamma = \Gamma(C) \) if it satisfies the following properties:
   (i) \( \{ \gamma_i | 1 \leq i \leq n \} \) is a \( \mathbb{Z} \)-basis for \( \mathbb{Z}\Phi_H \),
   (ii) \( |(\gamma_i, \gamma_j)| = |c_{ij}| \) for all \( i \neq j, 1 \leq i, j \leq n \).

2. Let \( \alpha \in \Phi_H \) and suppose that \( \alpha = \sum_{i=1}^{n} c_i \gamma_i \) with \( c_i \in \mathbb{Z} \) for each \( i \). \( d^\Psi_{\alpha} \) is defined to be the vector \( d^\Psi_{\alpha} = (|c_1|, \ldots, |c_n|) \).

3. The companion basis \( \Psi = \{ \gamma_i | 1 \leq i \leq n \} \) of \( \Gamma \) is called strong companion basis if the vectors \( d^\Psi_{\alpha} \) for \( \alpha \in \Phi_H^+ \) are the dimension vectors of the finitely generated indecomposable \( \Lambda \)-modules.

Let \( \Lambda \) be a cluster-tilted algebra of simply-laced Dynkin type and \( \Gamma = \Gamma(C) \) be the quiver of \( \Lambda \). Parsons conjectured in Conjecture 6.3 of \([1]\) that all companion bases for \( \Gamma \) are strong. He proved this conjecture for cluster-tilted algebras of simply-laced type \( A_n \).
By Proposition 6.2 of [11], there exists a tilted algebra of cluster-tilted algebras. They proved that an algebra Λ is cluster-tilted if and only if bases of Γ are strong. This tells us that the existence of a strong companion basis for Γ shows that all companion bases of Γ are strong.

Assem, Brüstle and Schiffler [1] and Zhu [14] independently provided a characterization of cluster-tilted algebras. They proved that an algebra Λ is cluster-tilted if and only if there exists a tilted algebra B such that Λ ∼= Bc, where Bc is trivial extension algebra Bc = B × Ext2 mod(B, B), with D = Hom(−, k) the k-duality. Recall that a k-algebra B is said to be tilted provided B is the endomorphism ring of a tilting H-module T, where H is a finite-dimensional hereditary k-algebra. A tilted algebra B is said to be concealed provided B is the endomorphism ring of a preprojective tilting H-module (i.e. a tilting H-module whose summands are all preprojective). If B is a concealed algebra, then Bc is called a cluster-concealed algebra.

Let B = EndH(T) be concealed algebra and Λ = Bc the corresponding cluster-concealed algebra and K0(H) be the Grothendieck group of H. We identify K0(H) with Zn, where n is the number of isomorphism class of simple H-modules. Let M be a finitely generated H-module. The dimension vector of M is the vector dim M ∈ Zn, whose coefficients are the Jordan-Hölder multiplicities of M. We denote by <−, −>H the bilinear form on K0(H) given by

\[<\text{dim} M, \text{dim} N>_H = \sum_{i \geq 0} (-1)^i \text{dim}_k \text{Ext}_H^i(M, N)\]

for all H-modules M, N. The corresponding symmetric bilinear form (−, −)H is defined as (dim M, dim N)H = <dim M, dim N>_H + <dim N, dim M>_H.

Let T1, · · · , Ts be indecomposable direct summands of T, one from each isomorphism class. Let g : K0(H) → K0(B) given by g(x) = (dim TI, x >H)i. Then for each i, we have g(dim TI) = dim G(Ti), where G = HomH(T, −) : modH → modB is a tilting functor. Since {dim T1, · · · , dim Ts} is a basis of K0(H) and {dim G(T1), · · · , dim G(Ts)} is a basis of K0(B), g is a linear bijection.

Let ΦH be the root system in K0(H) corresponding to the underlying graph of the quiver of H and ΦB = g(ΦH). In Theorem 2 of [13], Ringel proved that the dimension vectors of the indecomposable Λ-modules are precisely the vectors abs(x) with x ∈ ΦB.

Remark 2.2. Let ΦH+ = g(ΦH+), where ΦH+ is the set of positive roots of H. For every x ∈ ΦB there exists a ∈ ΦH+ such that x = g(a). The root a is either positive or negative. If a is negative then −a ∈ ΦH+ and x = −g(−a). Thus −x ∈ ΦB+. Since abs(x) = abs(−x), it is enough to consider positive roots. So according to the Ringel’s Theorem, dim M|M ∈ indΛ = {abs(x)|x ∈ ΦB+}.

In the following example we illustrate the connection between Parsons’s results [11] and Ringel’s results [13].
Example 2.3. Consider the quiver $\Gamma$ given by

![Quiver Diagram]

In fact $\Gamma$ is the mutation of the quiver $Q$ at vertex $3$, where $Q$ is the quiver

![Quiver Diagram]

Thus $\Gamma$ is a quiver of a cluster-tilted algebra $\Lambda$ of type $D_4$. More precisely $\Lambda \cong k\Gamma/I$, where $I$ is an admissible ideal of $k\Gamma$ generated by $\beta\gamma, \gamma\alpha, \eta\gamma, \gamma\delta, \alpha\beta - \delta\eta$. By Theorem 6.1 of [11], $\Psi = \{\alpha_1, \alpha_2 + \alpha_3, \alpha_3, \alpha_3 + \alpha_4\}$ is a companion basis for $\Gamma$, where $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a simple system of $\Phi_H$ and $H = kQ$. Recall that the positive roots $\Phi_H^+$ are $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$. In terms of $\Psi$, their dimension vectors are $\{d^\Psi_\alpha | \alpha \in \Phi_H^+\} = \{(1,0,0,0), (0,1,1,0), (0,0,1,0), (0,1,0,1), (0,1,1,0), (1,0,0,0), (0,0,0,1), (1,1,0,1), (1,1,1,1), (1,0,1,0)\}$. On the other hand $\Lambda \cong B \ltimes \text{Ext}^2_B(DB, B)$, where $B = \text{End}_H(T')$, $T' = P_1 \oplus P_2 \oplus P_3 \oplus P_4$, $P_1, P_2, P_3, P_4$ are indecomposable projective $H$-modules and $P_3$ is the indecomposable $H$-module with the dimension vector $(1,1,1,1)$ (in fact $H \cong \text{End}_H(H)$ and $T'$ is a mutation of $H$ at 3). An easy calculation shows that $\Phi_H^+ = \{(1,0,0,0), (0,1,1,0), (0,0,1,0), (0,1,0,0), (1,0,0,0), (0,1,0,0), (0,0,0,1), (1,1,0,0), (1,0,1,0), (1,1,1,0), (1,1,1,1), (1,0,1,0)\}$ and so by the Ringel’s Theorem the dimension vectors of the indecomposable $\Lambda$-modules are precisely $\{\text{abs}(x) | x \in \Phi_H^+\} = \{(1,0,0,0), (0,1,1,0), (0,0,1,0), (0,1,0,1), (0,1,1,0), (0,1,0,0), (0,0,0,1), (1,1,0,0), (1,0,1,0), (1,1,1,0), (1,1,0,1)\}$. This shows that $\Psi$ is a strong companion basis for $\Gamma$, the vectors of absolute values of the images of the four roots of the companion basis are $\text{abs}(g(\alpha_1)) = (1,0,0,0)$, $\text{abs}(g(\alpha_2 + \alpha_3)) = (0,1,0,0)$, $\text{abs}(g(\alpha_3)) = (0,0,1,0)$ and $\text{abs}(g(\alpha_3 + \alpha_4)) = (0,0,0,1)$. In fact we have a companion basis for $\Gamma$ such that its image under $g$ is the canonical basis of $K_0(B) = \mathbb{Z}^4$. In the next section we prove this fact for arbitrary cluster-tilted algebras of simply-laced Dynkin type.

3. Main Result

Let $\Lambda$ be an arbitrary cluster-tilted algebra of finite type. It is known that there exists $H = kQ$, $Q$ of Dynkin type $\Delta$, and a cluster-tilting object $\widetilde{T}$ of $\mathcal{C}_H$ such that $\Lambda \cong \text{End}_{\mathcal{C}_H}(\widetilde{T})$. Furthermore, it is known that there exists a tilting $H$-module $T$ such that $\Lambda \cong B^c$ where $B = \text{End}_H(T)$, cf. [1] or [14].

Let $\mathcal{A}$ be a cluster algebra of type $\Delta$, and suppose that $(x, C)$ is the seed in $\mathcal{A}$ corresponding to $\widetilde{T}$. Then $\Gamma = \Gamma(C)$ is the quiver of $\Lambda$ ([5], [6]).
Let \( g : K_0(H) \to K_0(B) \) be the linear bijection given by \( g(x) = (\dim T_i, x > H)_i \), and \( S_1, S_2, \ldots, S_n \) be the isomorphism class of simple \( \Lambda \)-modules. By the Ringel’s Theorem there exist \( y_1, \ldots, y_n \in \Phi_B^+ \) such that \( \text{abs}(y_i) = \dim S_i = \epsilon_i \), for each \( 1 \leq i \leq n \). Also for \( 1 \leq i \leq n \) there exists \( x_i \in \Phi_B^+ \) such that \( g(x_i) = y_i \).

**Theorem 3.1.** The subset \( \{x_1, \ldots, x_n\} \subseteq \Phi_B^+ \) is a companion basis for \( \Gamma \).

**Proof.** First we show that \( \{x_1, \ldots, x_n\} \) is a \( \mathbb{Z} \)-basis for \( \mathbb{Z} \Phi_H \). Let \( x \in \mathbb{Z} \Phi_H \), then \( g(x) \in \mathbb{Z} \Phi_B \), and so \( g(x) = d_1e_1 + \cdots + d_ne_n \), where \( d_i \in \mathbb{Z} \) for each \( i \) and \( \{e_1, \ldots, e_n\} \) is the canonical basis of \( K_0(B) \). Since \( e_i = \text{abs}(g(x_i)) \) for each \( i \), \( g(x) = d_1x_1 + \cdots + d_nx_n \), since \( g \) is a linear bijection.

Now we show that for any \( 1 \leq i, j \leq n \), \( i \neq j \), we have \( |(x_i, x_j)_H| = |\epsilon_i, \epsilon_j| \). We have \( |(x_i, x_j)_H| = |(g(x_i), g(x_j))_B| = \epsilon_{e_i, e_j} \), with \( \epsilon_{e_i, e_j} \in \{1, -1\} \). Then \( |(x_i, x_j)_H| = |\epsilon_i, \epsilon_j| = 1 \) if \( e_i > B \) and \( e_j > B \). Thus \( H \) is hereditary, \( \text{gl.dim} B \leq 2 \) and so \( \text{dim} E_{ij} \) \( \Leftrightarrow \dim k \text{Hom}_B(S_i, S_j) - \dim k \text{Ext}_B^1(S_i, S_j) - \dim k \text{Ext}_B^2(S_i, S_j) \). Thus \( |(x_i, x_j)_H| = |\dim k \text{Hom}_B(S_i, S_j) - \dim k \text{Ext}_B^1(S_i, S_j) - \dim k \text{Ext}_B^2(S_i, S_j)| \). It is known that the quivers of cluster-tilted algebras contain no 2-cycles (i.e. oriented cycles of length two). Therefore, \( |c_{ij}| \) is equal to the number of arrows in \( \Gamma = \Gamma(C) \) from \( i \) to \( j \) plus the number of arrows in \( \Gamma = \Gamma(C) \) from \( j \) to \( i \). But \( \Gamma \) is the quiver of \( B^c \) and according to Theorem 2.6 of [11] (see also the proof of this theorem), the number of arrows in the quiver of \( B^c \), \( Q_{B^c} \) from \( i \) to \( j \) is equal to the number of arrows in the quiver of \( B, Q_B \) from \( i \) to \( j \) plus \( \dim k \text{Ext}_B(S_i, S_j) \) additional arrows. Also it is known that the number of arrows in \( Q_B \) from \( i \) to \( j \) equals \( \dim k \text{Ext}_B^1(S_i, S_j) \). Hence \( |c_{ij}| = \dim k \text{Ext}_B^1(S_i, S_j) + \dim k \text{Ext}_B^2(S_i, S_j) + \dim k \text{Ext}_B^3(S_i, S_j) \). We claim that \( \dim k \text{Ext}_B^1(S_i, S_j) + \dim k \text{Ext}_B^2(S_i, S_j) + \dim k \text{Ext}_B^3(S_i, S_j) \) equals the \( \text{dim} k \text{Ext}_B^1(S_i, S_j) \) for any \( 1 \leq i, j \leq n \), \( i \neq j \) and our result follows.

Now we are ready to prove our main Theorem.

**Theorem 3.2.** Let \( \Lambda = \text{End}_{c_H}(\widehat{T}) \) be a cluster-tilted algebra of simply laced-Dynkin type. Suppose that \( (x, C) \) is the seed corresponding to \( \widehat{T} \). Then all companion bases for \( \Gamma = \Gamma(C) \) are strong.

**Proof.** By Proposition 6.2 of [11], it is enough to show that there exits a strong companion basis for \( \Gamma \). According to Theorem 3.1 \( \Psi = \{x_1, \ldots, x_n\} \subseteq \Phi_B^+ \) is a companion basis for \( \Gamma \). We show that \( \Psi \) is a strong companion basis for \( \Gamma \). Let \( \alpha \in \Phi_B^+ \), \( \alpha = c_1x_1 + \cdots + c_nx_n \), with \( c_i \in \mathbb{Z} \) for each \( i \). Then \( g(\alpha) = c_1g(x_1) + \cdots + c_ng(x_n) = c_1 \epsilon_1e_1 + \cdots + c_ne_n \).
\[\cdots + c_n \varepsilon_n e_n \text{ with } \varepsilon_i \in \{1, -1\}, \text{ for each } i. \text{ Also } g(\alpha) \in \Phi_B^+ \text{ and so by Theorem 2 of } [13], \]
\[\text{abs}(g(\alpha)) = (|c_1|, \cdots, |c_n|) \text{ is a dimension vector of some indecomposable } \Lambda\text{-module. But} \]
\[d^\Psi_\alpha = (|c_1|, \cdots, |c_n|) \text{ by definition and hence } \Psi \text{ is a strong companion basis for } \Gamma. \]

It is known that different tilted algebras \(B\) and \(B'\) may correspond to the same cluster-tilted algebra \(B \ltimes \text{Ext}_B^2(DB, B)\). In that case, the linear bijection \(g: K_0(H) \rightarrow K_0(B)\) depends on the choice of the tilted algebra \(B\). In the following example we show that different tilted algebras give a different companion basis.

**Example 3.3.** Consider the cluster-tilted algebra \(\Lambda = B \ltimes \text{Ext}_B^2(DB, B)\) of Example 2.3. Let \(B'\) be the tilted algebra given by the quiver

\[
\begin{array}{ccc}
4 & \beta \\
\gamma & 3 & \alpha \\
2 & & 1
\end{array}
\]

bound by \(\alpha \beta = 0\) and \(\alpha \gamma = 0\). Then it is easy to see that \(B \ltimes \text{Ext}_B^2(DB, B) \cong B' \ltimes \text{Ext}_B^2(DB', B')\) and \(B' = \text{End}_{H'}(T''), \text{ where } T'' = (0, 0, 0, 1) \oplus (1, 0, 0, 0) \oplus (1, 1, 1, 1) \oplus (0, 1, 0, 0) \text{ and } H' = kQ', \text{ where } Q' \text{ is the quiver}
\[
\begin{array}{ccc}
2 \\
1 & 3 & 4
\end{array}
\]

Let \(g': K_0(H') \rightarrow K_0(B')\) be the linear bijection given by \(g'(x) = (\dim T''_i, x > H')_i. \text{ An easy calculation shows that } \text{abs}(g'(1, 0, 0, 0)) = (0, 1, 0, 0), \text{ abs}(g'(0, 1, 0, 0)) = (0, 0, 0, 1), \text{ abs}(g'(0, 0, 1, 0)) = (1, 0, 0, 0) \text{ and } \text{abs}(g'(0, 0, 1, 1)) = (0, 0, 1, 0). \text{ Then by Theorems 3.1 and 3.2 the set } \Psi' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_3 + \alpha_4\} \text{ is a strong companion basis for } \Lambda. \]

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