Nakajima’s remark on Henn’s proof

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Abstract

We fill up a gap in Henn’s proof concerning large automorphism groups of function fields of degree 1 over an algebraically closed field of positive characteristic.

1 Introduction

In 1973, Stichtenoth [4] showed that the Hermitian function fields are the unique function fields $K$ of transcendency degree 1 over an algebraically closed ground field $\Omega$ of characteristic $p$ whose automorphism group $G = \text{Aut}(K/\Omega)$ has order at least $16g^4$ where $g \geq 2$ is the genus of $K$.

In 1978, Henn [1] gave a complete classification of such function fields under the weaker hypothesis $|G| \geq 8g^3$. Later Nakajima [3] improved Henn’s result for ordinary curves. In a footnote of his paper, Nakajima claimed: “Stichtenoth’s result was improved by Henn. But his proof contains a gap (last paragraph of pg. 104). I do not know if the gap can be covered.” The gap appears to be in the penultimate line on pg. 104 when Henn claims “und ersichtlich $z \cong \zeta(z)$ gilt, folgt hieraus $E \leq 2$”. The purpose of the present note is to fill up this gap. Actually, we are going to show the missing details in Henn’s proof.

We keep notation and terminology from [1]. In the last paragraph on pg. 104, the following case is investigated: $K$ has two places $\mathfrak{B}$ and $\mathfrak{B}_2$ with $G_0(\mathfrak{B}) = G_0(\mathfrak{B}_2)$ but none of the hypotheses in Proposition 1 holds.

For this case, Henn explicitly proves the following claims:

(i) $\nu_1 = 2$;

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(ii) the genus $g_2$ of $K^G_2(\mathcal{B})$ is equal to $q_1 - 1$ with $q_1 = |G_1(\mathcal{B})/G_2(\mathcal{B})|$. 

(iii) the smallest non-gap at $\tilde{\mathcal{B}}$ is $q_1$.

Then he observes that $q_1 + 1$ must also be a pole number at $\tilde{\mathcal{B}}$, and chooses an element $z \in K$ such that

$$z \sim \frac{\tilde{\mathcal{B}}^a}{\mathcal{B}^{q_1+1}}, \quad a > 1, \quad \tilde{\mathcal{B}}_2 \nmid a.$$

For $\sigma \in \text{Gal}(K^G_2(\mathcal{B})/K^G_1(\mathcal{B}))$, he shows that $\sigma(z) = z + \alpha$ with $\alpha \in \Omega$ and $\alpha \neq 0$. From this and Lemma 2, he deduces that

$$2 = \nu_1 = 1 - a + q_1,$$

whence $a = q_1 - 1$ and $\deg(a) = 2$ follow. At this point Henn takes an element $\zeta$ of order $|\zeta| = E$ from $G_0(\mathcal{B})$. To end the proof it is sufficient to show that $\zeta$ has order at most 2, as the hypothesis $1 < e < E$ will give then a contradiction. Henn’s idea is to show first that

$$\zeta(z) = cz \text{ with } c \in \Omega \setminus \{0\}, \quad (1)$$

and then to deduce $E = 2$ from it. He claims that $(1)$ follows from the fact that “jeder Punkt $\neq \mathcal{B}, \mathcal{B}_2$ unter $\zeta$ genau $E$ Konjugierte hat.”

We are going to show that $z$ may be chosen such a way that $(1)$ holds indeed.

Since $q_1$ and $q_1 + 1$ are coprime, we have $K^G_2(\mathcal{B}) = \Omega(y, z)$ where $y$ as in Henn’s paper has the property:

$$y \sim \frac{\tilde{\mathcal{B}}^{q_1}}{\mathcal{B}^{q_1+1}}.$$

Let $f \in \Omega[Z, Y]$ be an irreducible polynomial over $\Omega$ such that $f(z, y) = 0$. It may be noted that the plane irreducible curve $C$ with affine equation $f(Z, Y) = 0$ is in its Weierstrass normal form:

$$f(Z, Y) = Y^{q_1+1} + \gamma Z^{q_1} + U_1(Z)Y^{q_1} + \ldots + U_{q_1}(Z)Y + U_{q_1+1}(Z),$$

where $\gamma \in \Omega \setminus \{0\}$ and $\deg U_i(Z) \leq iq_1/(q_1 + 1)$ for $i = 1, \ldots, q_1$ and $\deg U_{q_1+1}(Z) < q_1$. 

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In particular, $C$ has only one point at infinity, namely the infinity point $Z_{\infty}$ of the $Z$-axis which is the center of the place $\tilde{\mathfrak{B}}$. Furthermore, the origin $O$ is the center of $\tilde{\mathfrak{B}}_2$, and no other place of $K^{G_2}(\mathfrak{B})$ is centered either at $Z_{\infty}$ or $O$.

Since $\zeta$ belongs to the normalizer of $G_2(\mathfrak{B})$, $\zeta$ may be viewed as an $\Omega$-automorphism of $K^{G_2}(\mathfrak{B})$ of order $E$. Moreover, $\zeta$ is a linear collineation that preserves $C$. Since $\zeta$ fixes both $\tilde{\mathfrak{B}}$ and $\tilde{\mathfrak{B}}_2$, there exist $\alpha, \beta, \gamma \in \Omega$ with $\alpha, \gamma \neq 0$, such that

$$\zeta(y) = \alpha y;$$
$$\zeta(z) = \beta y + \gamma z.$$

If $\beta = 0$, then (1) holds. Now assume that $\beta \neq 0$. Then $\alpha \neq \gamma$. In fact, if $\alpha = \gamma$ then $\zeta^j(y) = \alpha^j(y)$ and $\zeta^j(z) = j\alpha^{j-1}\beta y + \alpha^j z$ for every positive integer $j$. Now, for $j = E$ this implies that $\alpha^E = 1$ and hence that $E \equiv 0 \pmod{p}$, a contradiction.

Let $u = \beta/(\alpha - \gamma)$, and $z' = z - uy$. Then $\zeta(z') = \zeta(z) - u\zeta(y) = \gamma z + \beta y - u\alpha y = \gamma(z - uy)$. Replacing $z$ by $z'$ we obtain

$$\zeta(z') = \gamma z'.$$

Actually $z$ may be replaced by $z'$ from the very beginning of the argument, therefore Henn’s claim (1) may be assumed to be true.

Henn’s also claims without proof that (1) implies $E \leq 2$. This can be shown as follows. From $\text{div} \zeta(z) = \text{div} z$ it follows that $\zeta$ must preserve the above divisor $a$. Since $\deg a = 2$, this implies that $\zeta^2$ fixes each place in the support of $a$. Therefore, $\zeta^2$ as an $\Omega$-automorphism of the rational function field $K^{G_1}(\mathfrak{B})$ fixes at least three distinct places, namely, $\mathfrak{B}', \mathfrak{B}_2'$ and each place lying under those in the support of $a$ in the covering $K^{G_2}(\mathfrak{B}) \to K^{G_1}(\mathfrak{B})$. But then $\zeta^2$ is the identity, and hence $E = 2$.

**Remark**

A revised proof of Henn’s classification is found in [2, Chapter 11.12].

**Acknowledgements**

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References

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