Non-commutative oscillator with Kepler-type dynamical symmetry

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Abstract

A 3-dimensional non-commutative oscillator with no mass term but with a certain momentum-dependent potential admits a conserved Runge-Lenz vector, derived from the dual description in momentum space. The latter corresponds to a Dirac monopole with a fine-tuned inverse-square plus Newtonian potential, introduced by McIntosh, Cisneros, and by Zwanziger some time ago. The trajectories are (arcs of) ellipses, which, in the commutative limit, reduce to the circular hodographs of the Kepler problem. The dynamical symmetry allows for an algebraic determination of the bound-state spectrum and actually extends to the conformal algebra \( o(4,2) \).

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The “Berry phase” induced in the semiclassical analysis of a Bloch electron in a 3-dimensional crystal makes the coordinates non-commutative \[1\],
\[
\{x_i, x_j\} = \epsilon_{ijk} \Theta_k, \quad \{x_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0,
\]
where \(\Theta_i = \Theta_i(\vec{p})\). Then the Jacobi identity requires the consistency condition \(\vec{\nabla}_{\vec{p}} \cdot \vec{\Theta} = 0\) \[1\]. Choosing, for example, the non-commutative vector aligned in the third direction, \(\Theta_i = \theta \delta_{i3}\), \(\theta = \text{const.}\), the 3D theory reduces to the planar mechanics based on “exotic” Galilean symmetry \[2\]. For the Kepler potential \(V \propto r^{-1}\), for example, interesting results, including perihelion precession, can be derived \[3\]. Other applications concern the Quantum Hall Effect \[2, 4\].

Such a choice only allows for axial symmetry, though. Full rotational symmetry can, however, be restored by choosing instead \(\vec{\Theta}\) to be a “monopole in \(\vec{p}\)-space” \[5\],
\[
\Theta_i = \theta \frac{p_i}{p^3}, \quad \theta = \text{const.},
\]
where \(p = |\vec{p}|\). We record for further use that the associated symplectic structure has an extra, “monopole” term, \(\Omega = dp_i \wedge dx_i + (\theta/2p^3) \epsilon_{ijk} p_i dp_j \wedge dp_k\), that of a mass-zero, spin-\(\theta\) orbit of the Poincaré [and indeed of the o(4, 2) conformal] group \[6\].

In this Letter we study the 3D mechanics with non-commutativity \[2\], augmented with a Hamiltonian of the form \(H = (p^2/2 + V(\vec{x}, \vec{p}))\). Note that the potential may also depend on the momentum, \(\vec{p}\). The equations of motion read
\[
\dot{x}_i = p_i + \frac{\partial V}{\partial p_i} + \theta \epsilon_{ijk} \frac{p_k}{p^3} \frac{\partial V}{\partial x_j}, \quad \dot{p}_i = -\frac{\partial V}{\partial x_i}.
\]
Note, in the first relation, the “anomalous velocity terms” due to our assumptions.

We are particularly interested in finding conserved quantities. This task is conveniently achieved by using van Holten’s covariant framework \[7\], which amounts to searching for an expansion in integer powers of the momentum, \(Q = C_0(\vec{x}) + C_1(\vec{x}) p_i + \frac{1}{2} C_{ij}(\vec{x}) p_i p_j + \frac{1}{3} C_{ijk}(\vec{x}) p_i p_j p_k + \ldots\). Requiring \(Q\) to Poisson-commute with the Hamiltonian yields an infinite series of constraints. The expansion can be truncated at a finite order \(n\), however, provided the Killing equation is satisfied, \(D_{(i_1} C_{i_2 \ldots i_n)} = 0\), when we can set \(C_{i_1 \ldots i_n+1 \ldots} = 0\).

\[1\] For a more general theory which also includes magnetic fields, see, e.g., \[1, 2\]. For simplicity, the mass has been chosen unity.
Let us assume that the potential has the form \( V(r, p) \), and try to find the conserved angular momentum, associated with the Killing vector \( \vec{C} = \vec{n} \times \vec{x} \), which represents a space rotation around \( \vec{n} \). An easy calculation shows, however, that the procedure fails to work, owing to the “monopole” term, which is not an integer power of \( \vec{p} \). We propose therefore to work instead in “dual” (i.e. momentum) space, and search for conserved quantities expanded rather into powers of the position,

\[
Q = C_0(\vec{p}) + C_i(\vec{p})x_i + \frac{1}{2!}C_{ij}(\vec{p})x_ix_j + \frac{1}{3!}C_{ijk}(\vec{p})x_ix_jx_k \ldots
\]

Then van Holten’s algorithm is replaced by

\[
C_i \left( p_i + \frac{\partial V}{\partial p_i} \right) = 0 \quad o(0)
\]

\[
\frac{1}{r} \frac{\partial V}{\partial r} \left( \theta \epsilon_{ijk} \frac{p_k}{p^3} C_i - \frac{\partial C_i}{\partial p_j} \right) + C_{ij} \left( p_i + \frac{\partial V}{\partial p_i} \right) = 0 \quad o(1)
\]

\[
\frac{1}{r} \frac{\partial V}{\partial r} \left( \theta \frac{p_m}{p^3} (\epsilon_{ijm} C_{ik} + \epsilon_{ikm} C_{ij}) - \left( \frac{\partial C_k}{\partial p_j} + \frac{\partial C_j}{\partial p_k} \right) \right) + C_{ijk} \left( p_i + \frac{\partial V}{\partial p_i} \right) = 0 \quad o(2)
\]

\[
\frac{1}{r} \frac{\partial V}{\partial r} \left( \frac{p_m}{p^3} (\epsilon_{lm} C_{ij} + \epsilon_{ilm} C_{ij} + \epsilon_{kim} C_{ij}) - \left( \frac{\partial C_{ij}}{\partial p_k} + \frac{\partial C_{jk}}{\partial p_i} + \frac{\partial C_{ki}}{\partial p_j} \right) \right) + C_{lijk} \left( p_l + \frac{\partial V}{\partial p_l} \right) = 0 \quad o(3)
\]

where \( r = |\vec{x}| \). Then, for the dual Killing vector \( \vec{C} = \vec{n} \times \vec{p} \), the algorithm provides us with the conserved angular momentum,

\[
\vec{J} = \vec{L} - \theta \vec{p} = \vec{x} \times \vec{p} - \theta \vec{p} = \vec{p}/p,
\]

which is what one would expect, due to the “monopole in p-space”, whereas the NC parameter, \( \theta \), behaves as the “monopole charge” [8].

The next step is to inquire about second order conserved quantities. The usual Runge-Lenz vector is generated by the Killing tensor \( C_{ij} = 2 \delta_{ij} \vec{n} \cdot \vec{x} - n_i x_j - n_j x_i \), where \( \vec{n} \) is some fixed unit vector [7]. Not surprisingly, the original procedure fails once again. The dual procedure [4] works, though. The two-tensor

\[
C_{ij} = 2 \delta_{ij} \vec{n} \cdot \vec{p} - n_i p_j - n_j p_i,
\]

verifies the dual Killing equation of order 3. Then the order-2 equation yields \( C_k = \theta (\vec{n} \times \vec{p})_k/p \).

Inserting into the first-order constraint and assuming \( \partial_r V \neq 0 \), the constraint is satisfied
with \( C = \alpha \vec{n} \cdot \hat{p}, \) provided the potential and the Hamiltonian are
\[
V = \frac{x^2}{2} - \frac{p^2}{2} + \frac{\theta^2}{2p^2} + \frac{\alpha}{p}, \quad H = \frac{x^2}{2} + \frac{\theta^2}{2p^2} + \frac{\alpha}{p},
\]
respectively, where \( \alpha \) is an arbitrary constant. Then the dual algorithm provides us with the Runge-Lenz-type vector
\[
\vec{K} = \vec{x} \times \vec{J} - \alpha \hat{p}. \tag{8}
\]
Its conservation can also be checked by a direct calculation, using the equations of motion,
\[
\dot{x} = -\left(\frac{\theta^2}{p^4} + \frac{\alpha}{p^3}\right) \vec{p} + \theta \frac{\vec{p} \times \vec{x}}{p^3}, \quad \dot{p} = -\vec{x}. \tag{9}
\]

Note that the \(-p^2/2\) term in the potential cancels the usual kinetic term, and our system describes a non-relativistic, non-commutative particle with no mass term in an oscillator field, plus some momentum-dependent interaction. Writing the Hamiltonian as
\[
H = \frac{x^2}{2} + \frac{\theta^2}{2p^2} - \alpha^2/2\theta^2
\]
shows, moreover, that \( H \geq -\alpha^2/2\theta^2 \) with equality only attained when \( p = -\theta^2/\alpha \), which plainly requires \( \alpha < 0 \).

It is easy to understand the reason why our modified algorithm did work: calling \( \vec{p} \) “position” and \(-\vec{x}\) “momentum”, the system can also be interpreted as an “ordinary” (i.e. massive and commutative) non-relativistic charged particle in the field of a Dirac monopole of strength \( \theta \), augmented with an inverse-square plus a Newtonian potential. This is the well-known “McIntosh-Cisneros – Zwanziger” (MICZ) system [6, 9], for which the fine-tuned inverse-square potential is known to cancel the effect of the monopole, allowing for a Kepler-type dynamical symmetry [6, 9]. The angular momentum, (5), and the Runge-Lenz vector, (8), are, in particular, that of the MICZ problem [9] in “dual” [momentum] space.

The conserved quantities provide us with valuable information on the motion. Mimicking what is done in the MICZ case, we note that \( \vec{J} \cdot \hat{p} = -\theta \) implies that the vector \( \vec{p} \) moves on a cone of opening angle \( \arccos(-\theta/J) \). On the other hand, for \( \vec{N} = \alpha \vec{J} - \theta \vec{K} \) we have \( \vec{N} \cdot \vec{p} = \theta(J^2 - \theta^2) = \theta L^2 \) a constant, so that the \( \vec{p} \)-motion is in the plane perpendicular to \( \vec{N} \). The trajectory in \( p \)-space belongs therefore to a conic section.

For the MICZ problem, this is the main result, but for us here our main interest lies in finding the real space trajectories, \( \vec{x}(t) \). By [9], this amounts to finding the [momentum-] “hodograph” of the MICZ problem. Curiously, while the hodograph of the Kepler problem is well-known [it is a circle or a circular arc], we could not find the corresponding result in
the (vast) literature of MICZ. Returning to our notations, we note that due to $\vec{N} \cdot \vec{x} = 0$, $\vec{x}(t)$ also belongs to the same oblique plane, whose normal is $\vec{N}$.

The problem is conveniently studied in an adapted coordinate system. One proves indeed that

$$\{\hat{i}, \hat{j}, \hat{k}\} = \left\{ \frac{1}{|\alpha L|} \vec{K} \times \vec{J}, \frac{1}{|\lambda\epsilon|} (2\theta H \vec{J} + \alpha \vec{K}), \frac{1}{|\lambda\epsilon|} (\alpha \vec{J} - \theta \vec{K}) \right\}$$

(10)

[where $\lambda^2 = \alpha^2 + 2H\theta^2$, $\epsilon^2 = \alpha^2 + 2HJ^2$ and $L^2 = J^2 - \theta^2$] is an orthonormal basis. (In $\vec{k}$ we recognize, in particular, $\vec{N}/N$.)

Firstly, projecting onto these axis, $p_z = \vec{p} \cdot \hat{k} = \theta L/|\lambda| = \text{const}$, while for $p_x = \vec{p} \cdot \hat{i}$ and $p_y = \vec{p} \cdot \hat{j}$ we find

$$\left(\frac{p_y + \frac{|\epsilon| |\alpha|}{2|\lambda|H}}{\lambda^2/4H^2} \right)^2 - \frac{p_x^2}{L^2/2H} = 1,$$

(11)

which is the equation of a hyperbola or of an ellipse, depending on the sign of $H$. (For vanishing $H$ one gets a parabola). This confirms what is known for the MICZ problem [9].

Next, for

$$X = \vec{x} \cdot \hat{i} = -\frac{2|L|}{|\epsilon|} (H - \frac{\alpha}{2p}), \quad Y = \vec{x} \cdot \hat{j} = -\frac{|\lambda| \vec{x} \cdot \vec{p}}{|\epsilon| p}$$

(12)

an easy calculation yields

$$(X + \frac{|\epsilon| |L|}{J^2})^2 + \frac{\alpha^2 L^2}{\lambda^2 J^2} Y^2 = \frac{L^2 \alpha^2}{J^4}.$$  

(13)

[completed with $Z = \vec{x} \cdot \hat{k} = 0$] which is an ellipse, since $\lambda^2 = \alpha^2 + 2H\theta^2 \geq 0$. The center has been shifted to $-|\epsilon|L/J^2$ along the axis $\hat{i}$. The major axis is directed along $\hat{j}$.

Note that, unlike as in $\vec{p}$-space, the $\vec{x}$-trajectories are always bounded. When the energy is negative, $H < 0$ [which is only possible when the Newtonian potential is attractive, $\alpha < 0$], the whole ellipse is described. When $H > 0$, [which is the only possibility in the repulsive case $\alpha > 0$] it is only the arc between the tangents drawn from the origin which is obtained. When the non-commutativity is turned off, $\theta \to 0$, the known circular hodographs of the Kepler problem are recovered. As $\alpha \to 0$, the trajectory becomes unbounded, and follows the $y$-axis.

So far, we only discussed classical mechanics. Quantization is now straightforward using the known group theoretical properties of the MICZ problem in dual space. The non-commutativity [alias monopole charge] $\theta$ has to be an integer or half integer – this is indeed the first indication about the quantization of the NC parameter – ; wave functions should be
FIG. 1: When the Newtonian potential is attractive, $\alpha < 0$, one can have $H < 0$, and the trajectories are full ellipses. The origin is inside the ellipse. When $H > 0$, which can arise either in the high-energy attractive or in the repulsive ($\alpha > 0$) case, the origin is outside our ellipse. The particle is confined to an arc [denoted with the heavy line] of the ellipse, determined by the tangents drawn from the origin. For $H = 0$, the origin lies on the ellipse, and “motion” reduces to this single point.

chosen in the momentum representation, $\psi(\vec{p})$. The angular momentum, $\vec{J}$, and the rescaled Runge-Lenz vector, $\vec{K}/\sqrt{2\vert H\vert}$, close into $o(3,1)/o(4)$ depending on the sign of the energy. In the last case, the representation theory provides us with the discrete energy spectrum [in units $\hbar = 1$]

$$E_n = -\frac{\alpha^2}{2n^2}, \quad n = n_r + \frac{1}{2} + (l + \frac{1}{2})\sqrt{1 + \frac{4\theta^2}{(2l + 1)^2}}, \quad (14)$$

where $n = 0, 1, \ldots$, $l = 0, 1, \ldots$, with degeneracy $n^2 - \theta^2$. The same result can plainly be derived directly by solving the Schrödinger equation in $\vec{p}$-space [9].

Moreover, the symmetry extends to the conformal $o(4,2)$ symmetry, due to the fact that the massless Poincaré orbits with helicity $\theta$ are in fact orbits of the conformal group, cf. [6].

In conclusion, we observe that in most approaches one studies the properties (like trajectories, symmetries, etc.) of some given physical system. Here we followed in the reverse direction: after positing the fundamental commutation relations, we were looking for potentials with nice properties. This leads us to the momentum-dependent potentials [7], which is indeed a McIntosh-Cisneros — Zwanziger system [9] in dual space. Unlike as in a constant electric field [10], the motions lie in an (oblique) plane. The particle is confined to bounded trajectories, namely to (arcs of) ellipses.

The best way to figure our motions is to think of them as analogs of the circular
hodographs of the Kepler problem to which they indeed reduce when the non-commutativity is turned off. For $H < 0$, for example, the dual motions are bound, and the velocity turns around the whole ellipse; for $H > 0$ instead, our motion along a finite arc, starting from one extreme point and tending to the other one at the end of the arc, corresponds to the variation of the velocity in the course of a hyperbolic motion (of a comet, or in Rutherford scattering) in dual space.

There is also some analogy with particles with torsion [11].

Our system, with monopole-type non-commutativity [2], has some remarkable properties. Momentum-dependent potentials are rather unusual in high-energy physics. They are, however, widely used in nuclear physics, namely in the study of heavy ion collisions, where they correspond to non-local interactions [12]. Remarkably, in NC field theory, a $1/p^2$ contribution to the propagator emerges from UV-IR mixing. See the middle equation in equation (22) page 11 of Ref. [13].

The absence of a mass term should not be thought of as the system being massless; it is rather reminiscent of “Chern-Simons dynamics” [4].

One can be puzzled how the system would look like in configuration space. Trying to eliminate the momentum from the phase-space equations (9) in the usual way, which amounts to deriving $\dot{x}$ w.r.t. time and using the equations for $\dot{\vec{p}}$, fails, however, owing to the presence of underived $\vec{p}$ in the resulting equation. This reflects the non-local character of the system.

One can, instead, eliminate $\vec{x}$ using the same procedure, but in dual space. This yields in fact the equations of motion of MICZ in momentum space,

\[ \ddot{\vec{p}} = \frac{J^2}{p^3} + \frac{\alpha}{p^2}, \quad \ddot{\vec{p}} = \frac{\alpha}{p^3} \vec{p} - \frac{\theta}{p^3} \vec{J}. \] (15)

Are these equations related to a theory with higher-order derivatives of the type [14]? The answer is yes and no. The clue is that time is not a “good” parameter for Kepler-type problems, owing to the impossibility of expressing it from the Kepler equation [15]. This is also the reason for which we describe the shape of the trajectories, but we do not integrate the equations of motion. A “better” parameter can be found along the lines indicated by Souriau [16, 17] and then, deriving w.r.t. the new parameter, transforms (15) into a fourth-order linear matrix differential equation, which can be solved.

It is, however, not clear at all if these equations derive from some higher-order Lagrangian, and if they happen to do, what would be the physical meaning of the latter.
The fourth-order equations do certainly not come from one of the type in Ref. [14]: the latter lives in fact in two space dimensions and has constant scalar non-commutativity $\theta$ — while our system is 3-dimensional and has a momentum-dependent vector $\vec{\Theta}(\vec{p})$, given in [2].

It is tempting to ask if the relation to the “closest physical theory” with a momentum-dependent potential, namely nuclear physics, can be further developed and if similar (super)symmetries can be found also in nuclear physics. Once again, the answer seems to be negative, though: while dynamical symmetries do play a role in nuclear physics [18], those used so far do not seem to be of a momentum-dependent Keplerian types.

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