On the Scattering Phase for $AdS_5 \times S^5$ Strings

NIKLAS BEISERT

Joseph Henry Laboratories
Princeton University
Princeton, NJ 08544, USA
nbeisert@princeton.edu

Abstract

We propose a phase factor of the worldsheet S-matrix for strings on $AdS_5 \times S^5$ apparently solving Janik’s crossing relation exactly.

1 Introduction

The discovery of integrability in planar AdS/CFT \cite{1, 2} has given hope that both participating models, $\mathcal{N} = 4$ gauge theory and string theory on $AdS_5 \times S^5$, can be solved exactly in the planar limit. The spectrum can be obtained, at least to the leading few orders in perturbation theory, by asymptotic Bethe ansätze, see \cite{3} for a review. The leading weak-coupling order in gauge theory was solved in \cite{4}. Reliable Bethe equations presently exist for up to second order (three loops) \cite{5-7}. The spectrum of classical string theory was solved in \cite{8, 9} by means of spectral curves. Based on these results, Bethe equations for quantum strings were proposed \cite{10, 7}. The current state of the art is the expansion to first strong-coupling order \cite{11-13}.

The Bethe equations for gauge theory can be derived by means of an asymptotic Bethe ansatz \cite{14}. This ansatz transforms the spin chain states into a one-dimensional particle model. On the string theory side, one obtains a very similar particle model by an appropriate light cone gauge. After obtaining and diagonalising the S-matrix of the particles one can write down the Bethe equations for periodic states.

The particle model consists of 8 bosons and 8 fermions above the half-BPS vacuum \cite{15}. These particles can be grouped in a $4 \times 4$ matrix. Then the rows transforms under a

$$\mathfrak{h} = \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$$

(1)
superalgebra and the columns under a separate $\mathfrak{h}$ algebra with shared central charges \[16\].
The S-matrix therefore is a product of the S-matrices for rows and for columns

$$S_{12}^{\text{psu}(2,2|4)} = S_{12}^{0,\text{psu}(2,2|4)} S_{12}^{\text{bare}} S_{12}^{\text{bare}}.$$  \(2\)

Consequently, it suffices to restrict to only one row of particles and to find its S-matrix $S_{12}^{\text{bare}}$. Remarkably, it turns out that the flavour structure of this S-matrix is completely determined by its $\mathfrak{h}$ symmetry \[16\]. Moreover, the Yang-Baxter relation is automatically satisfied by this S-matrix. Symmetry alone, however, does not constrain the overall phase factor $S_{12}^{0,\text{psu}(2,2|4)}$.

The properties of the particles were mainly derived from gauge theory, but also in string theory the particles behave similarly \[17, 18\]. It is therefore very reasonable to assume that the flavour structure of the S-matrix is the same for both models. Finding the exact phase factor for $\mathcal{N} = 4$ gauge theory and for string theory on $AdS_5 \times S^5$ remains one of the biggest challenges in this context. If the Bethe equations with the correct phase turn out to apply even at finite coupling, one could compare them and see whether the AdS/CFT prediction of coinciding spectra holds.

## 2 Phase Factor and Crossing

To obtain the phase factor for an integrable model one usually employs crossing symmetry which puts severe restrictions on its form. Furthermore, assuming a minimal set of singularities often fixes the factor uniquely. An equation for crossing symmetry of the phase factor for the $\mathfrak{h}$-symmetric S-matrix $S_{12}^{\mathfrak{h}} = S_{12}^{0,\mathfrak{h}} S_{12}^{\text{bare}}$ was derived by Janik in \[19\]

$$S_{12}^{0,\mathfrak{h}}(1/x_1^+, x_2^+) = \frac{f(x_1^-, x_2^-)}{S_{12}^{0,\mathfrak{h}}(x_1^-, x_2^-)}$$  \(3\)

with the function

$$f(x_1^+, x_2^+) = \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{x_1^+ - 1/x_2^-}{x_1^+ - 1/x_2^+} = \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{1/x_1^- - x_2^-}{1/x_1^+ - x_2^-}.$$  \(4\)

The spectral parameters $x^\pm$ are related to the particle momenta by

$$\exp ip = \frac{x^+}{x^-}.$$  \(5\)

They furthermore obey the equation

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g},$$  \(6\)

where $g$ is the square root of the ’t Hooft coupling constant (up to factors)\[1\]

$$g = \frac{\sqrt{\lambda}}{4\pi}.$$  \(7\)

\[1\]To simplify many expressions, I have chosen a normalisation in this letter which differs from my previous papers, e.g. [3]. The relationship to the old literature is $x^\pm = x^\pm_{\text{old}}/\sqrt{2} g_{\text{old}}$ and $g = g_{\text{old}}/\sqrt{2}$. My excuses for all the previous factors of $1/\sqrt{2}\ldots$
The function $f$ has singularities at
\[ x_1^\pm = x_2^\pm, 1/x_2^\pm \quad \text{and when} \quad x_1^+ = 1/x_2^- \quad \text{or} \quad x_1^- = x_2^+. \quad (8) \]
The latter two singularities are related to two-particle bound states, c.f. \[20\].

In fact, the crossing relation (3) superficially has no solution due to the mismatch of both sides under the antipodal map $x_1^\pm \mapsto 1/x_1^\pm$. However, we have to take into account that the phase factor $S^{0,\mathfrak{h}}$ has branch cuts, e.g. from the opening-up of a two-particle channel. We therefore need to specify how to reach the antipodal point $1/x_1^\pm$ from the point $x_1^\pm$ itself: The equation for the spectral parameters (6) defines a complex torus (19) and there are at least two inequivalent short paths to reach the antipode. Depending on which cuts are crossed by the path, the phase factor $S^{0,\mathfrak{h}}(1/x_1^\pm, x_2^\pm)$ can take different values. One of them should obey (3), the other will obey a related equation.

Going back to $\mathfrak{psu}(2,2|4)$ we can express the crossing relation in terms of the dressing factor $\sigma$ (10) in the conventions of \[7\] ($\sigma$ in \[10\] equals $\sigma^2$ in \[7\]) as
\[ \sigma(1/x_1^\pm, x_2^\pm) = \frac{h(x_1^\pm, x_2^\pm)}{\sigma(x_1^\pm, x_2^\pm)}, \quad \text{where} \quad h(x_1^\pm, x_2^\pm) = \frac{x_2^\pm}{x_2^\pm f(x_1^\pm, x_2^\pm)}. \quad (9) \]

In this letter we shall propose a function obeying the above crossing relation. This also leads to a function obeying crossing (3) for the S-matrix with $\mathfrak{h}$ symmetry via
\[ S^{0,\mathfrak{h}}(x_1^\pm, x_2^\pm) = \sqrt{\frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^-}} \frac{1}{\sigma(x_1^\pm, x_2^\pm)}. \quad (10) \]

### 3 The Proposal

The central proposal of this letter is that a crossing-symmetric phase factor is given by
\[ \sigma(x_1^\pm, x_2^\pm) = \exp i\theta(x_1^\pm, x_2^\pm), \]
\[ \theta(x_1^\pm, x_2^\pm) = \theta_0(x_1^\pm, x_2^\pm) + \theta_1(x_1^\pm, x_2^\pm), \]
\[ \theta_0(x_1^\pm, x_2^\pm) = -\frac{i}{2} \log \left( \sqrt{\frac{x_1^+ x_2^-}{x_1^- x_2^+} \frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^-}} \right), \]
\[ \theta_1(x_1^\pm, x_2^\pm) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s} (q_r(x_1^\pm) q_s(x_2^\pm) - q_s(x_1^\pm) q_r(x_2^\pm)) \quad (11) \]
with the coefficients
\[ c_{r,s} = \frac{(-1)^{r+s} - 1}{\pi} \frac{(r - 1)(s - 1)}{(r + s - 2)(s - r)} \quad (12) \]
and the magnon charges
\[ q_r(x^\pm) = \frac{i}{r - 1} \left( \frac{1}{(x^\pm)^{r-1}} - \frac{1}{(x^-)^{r-1}} \right). \quad (13) \]
The first contribution $\theta_0$ is similar at leading order in strong coupling with classical string theory (8) and the AFS phase (10), but it does not literally agree. It is shown in
a subsequent publication \cite{21} that an additional homogeneous solution of the crossing relation (3) with \( f = 1 \) is required

\[ \delta \theta(x_1^+, x_2^+) = \frac{i}{2} \left( \frac{1}{2} + \frac{ig}{x_2} - \frac{ig}{x_2} \right) \log \frac{x_1^+}{x_1} - \frac{i}{2} \left( \frac{1}{2} + \frac{ig}{x_1} - \frac{ig}{x_1} \right) \log \frac{x_2^+}{x_2}. \] (14)

Together, the two contributions \( \theta_0 + \delta \theta \) yield precisely the phase factor derived from string theory in light cone gauge \cite{22} which is known to be consistent with the classical result \cite{8, 10}. The second contribution \( \theta_1 \) is precisely the one-loop phase proposed by Hernández and López \cite{12}. It was already confirmed that this phase obeys the crossing relations perturbatively up to the first order in the strong-coupling expansion in \cite{23}. The present claim is that the phase (11) obeys the crossing relation exactly even for finite coupling \( g \). In the following I will provide arguments to substantiate the proposal. I will however not give a rigorous proof.

4 Analytic Structure

To motivate why the phase (11,12,13) might receive no further corrections to be able to solve the crossing relation, it is useful to investigate the structure of zeros and poles in the function \( \exp i \theta \). The key insight is that, effectively, an almost analytic function is uniquely defined (up to a constant term) by the positions and multiplicities of its zeros and poles. We would therefore need to show that the structure of poles agrees with the crossing relation. To find zeros and poles, we sum up the expressions for \( \theta_1 \) (11,12,13) in a closed form, see \cite{23}. The phase then has the following general form

\[ \theta_1(x_1^+, x_2^+) = \ldots \pm \frac{1}{2\pi} \log(*) \log(*) + \ldots \pm \frac{1}{2\pi} \text{Li}_2(*) + \ldots \] (15)

with \( * \) representing some combinations of \( x_{1,2}^+ \).

Let us first consider the crossing relation applied twice: It has the form

\[ \sigma(x_1^+, x_2^+) = \frac{f(1/x_1^+, x_2^+)}{f(x_1^+, x_2^+)} \sigma(x_1^+, x_2^+) \quad \text{or} \quad \exp i \theta = \exp i \Delta \theta \exp i \theta. \] (16)

It is understood that the two instances of \( \sigma \) differ by a continuous path once around the imaginary period of the torus. In other words, we are comparing the value of the dressing factor \( \sigma \) on two different Riemann sheets. In detail, the phase shift \( \Delta \theta \) reads

\[ \exp i \Delta \theta = \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-} \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \left( 1 - \frac{1}{x_1^+ x_2^+} \right) \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \left( 1 - \frac{1}{x_1^- x_2^-} \right). \] (17)

Now it is important to know how the functions \( \log \) and \( \text{Li}_2 \) change when crossing a cut

\[ \log z \mapsto \log z \pm 2\pi i, \quad \text{Li}_2 z \mapsto \text{Li}_2 z \pm 2\pi i \log z. \] (18)

Therefore, the expression for the phase shift is of the form

\[ \Delta \theta_1 = \ldots \pm i \log(*) + \ldots \pm 2\pi + \ldots \] (19)
describing zeros and poles in \( \exp i \Delta \theta \). Excitingly, the coefficients in front of the log’s are precisely \( \pm i \) leading to a single zero or pole, in agreement with (17).

The appearance of integer factors in an exponent gives the hint that a non-renormalisation theorem may apply: Further corrections of \( \theta \) would be of the order \( \mathcal{O}(1/g) \) or higher. As the coupling constant is arbitrary, it would be hard to obtain integer coefficients in front of logarithmic singularities. We would most likely introduce zeros or poles with irrational weights which would completely alter the analytic structure and introduce unwanted branch cuts. This is similar to the argument of why some anomalies in a quantum field theory receive corrections at the one-loop level only. In fact, as \( \theta_1 \) is a one-loop contribution, it is conceivable that it represents an anomaly of some sort. A further supporting argument in the form of an explicit example is given in App. A.

Next we should find out where the logarithmic singularities in \( \Delta \theta_1 \) reside. According to (18), the positions are just the zeros and poles of the arguments of log’s and \( \text{Li}_2 \)’s in (15). For the full expression from (11,12,13), the logarithmic singularities happen to be at the positions

\[
x_1 = x_2, 1/x_2, \quad \text{or} \quad x_{1,2} = 0, \infty, \pm 1,
\]

where \( x_{1,2} \) represent any of \( x_{1,2}^\pm \). Among these we clearly identify all the poles and zeros of the function (17). Therefore it is conceivable that (11,12,13) solve the doubled crossing relation (16).

It turns out that \( \theta_1 \) alone solves the doubled crossing relation. The additional term \( \theta_0 \) is required to obey the single crossing relation (9). Using the perturbative machinery of [23] it is straight-forward to verify that the crossing relation is obeyed for several odd powers in \( 1/g \sim \zeta \). In fact, we used this property to construct and guess the phase \( \theta_0 \). Conversely, the even powers in \( 1/g \) are generated only by \( \theta_1 \). One might be able to show that crossing is obeyed perturbatively for even powers in \( 1/g \), but this would be much more involved due to residual logarithms in the expressions.

Should (11,12,13) be the correct physical answer for string theory? Firstly, as shown above, it has about the right set of analytic properties. Furthermore, it has been verified to obey the crossing relation in the leading and sub-leading perturbative orders [23] (by construction, also to second and a few higher even orders). Finally, further corrections of \( \theta \) would be likely change the analytic structure and a non-renormalisation theorem may apply. In conclusion, one of the simplest conceivable solutions of the crossing relation may be (11,12,13). Together with the homogeneous solution (14) [21] it agrees with perturbative string computations [24] at next to leading order [10,25,12,13]. However, it must be noted that the proposed phase does not match with the classical scattering phase for giant magnons derived in [17]. A further homogeneous solution of the crossing relation may be required for agreement with giant magnons and/or higher perturbative orders.

5 Verifications

Of course we need to perform some basic tests of the conjecture. The perturbative test in [23] is a first step. We would like to show at finite coupling \( g \) that the crossing relation (9) is obeyed. The problem is that we need to specify a path when shifting one of the
parameters \( x^\pm \) to its antipode \( 1/x^\pm \) and that the change of phase does depend on it. A full analysis is beyond the scope of the current publication and should be performed in more detail elsewhere.

Here we are moderate and check the validity of the crossing relation for a few random values of \( x_{1,2}^\pm \). We specify a path \( x_1^\pm (t) \) connecting \( x_1^\pm \) to \( 1/x_1^\pm \) and obeying (6). We use a parametrisation where the rapidity \( z \) moves along one imaginary period of the torus which defines the space of solutions \( x^\pm \) to (6), c.f. [19]. In particular, we choose the momentum \( p \), defined via (5), to equal 
\[
p = 2 \text{ am}(z).
\]
Here ‘am’ represents Jacobi’s elliptic amplitude with elliptic modulus \( k = 4 ig \). The imaginary half-period is 
\[
\omega_2 = 2i K(\sqrt{1-k^2}) - 2 K(k).
\]
When moving along the path, we need to be careful about branch cuts. When the path crosses a branch cut, the logarithms have to be replaced according to (18).

Several sets of points \( x_{1,2}^\pm \) were chosen at random and the crossing relation (9) was verified numerically to six-digit precision. Furthermore, the double crossing relation (16) can be verified more explicitly: One simply keeps track of which terms have been added according to (18) when moving along the path. The resulting expression matched (17) (or its inverse, cf. [21]) in all tested cases.

6 Conclusions and Outlook

In this letter I have proposed an overall phase factor (11,12,13) for the worldsheet S-matrix of strings on \( AdS_5 \times S^5 \): Its main property is that it seemingly solves Janik’s crossing relation [19] stated in (9) as the present tests confirm. The S-matrix agrees with string theory at the classical level [8] and the first subleading order in \( 1/\sqrt{\lambda} \) [12] when the homogeneous piece [14, 21] is added. It is useful to write down the overall scattering factor for two excitations in the \( R \times S^3 \) subsector

\[
\sigma^2(x_1^\pm, x_2^\pm) \frac{x_1^\pm - x_2^\pm}{x_1^\pm - x_2^\pm} = \exp\left(2i\theta_1(x_1^\pm, x_2^\pm)\right) \sqrt{\frac{x_1^\pm}{x_2^\pm} \frac{x_1^\pm - x_2^\pm}{x_1^\pm - x_2^\pm}}. \tag{21}
\]

Here \( \theta_1 \) is the Hernández-López scattering phase [12] reproduced in (11). It is interesting that such a simple form of scattering factor remains.

For the complete S-matrix element \( A_{12} \) in the S-matrix with \( h \) symmetry we obtain

\[
A_{12} = S^{0,h}(x_1^\pm, x_2^\pm) \frac{x_2^+ - x_1^-}{x_2^+ - x_1^-} = \exp\left(-i\theta_1(x_1^\pm, x_2^\pm)\right) 4 \sqrt{\frac{x_1^- x_2^+}{x_1^+ x_2^-}} \sqrt{\frac{x_2^- x_1^+}{x_2^+ x_1^-}}. \tag{22}
\]

Intriguingly, the element \( D_{12} \) has just the square root term inverted

\[
D_{12} = -S^{0,h}(x_1^\pm, x_2^\pm) = -\exp\left(-i\theta_1(x_1^\pm, x_2^\pm)\right) 4 \sqrt{\frac{x_1^- x_2^+}{x_1^+ x_2^-}} \sqrt{\frac{x_2^- x_1^+}{x_2^+ x_1^-}}. \tag{23}
\]

The term \( A_{12} \) is precisely the square root of (the inverse of) the above scattering factor for string theory in the \( R \times S^3 \) sector. The full S-matrix for string theory can thus
be written merely as the product of two $h$-symmetric S-matrices without an additional prefactor (the inverse is due to a change of conventions)

$$S_{12}^{\text{psu}(2,2|4)} = S_{12}^h S_{12}^h.$$ (24)

There is a host of further investigations that should be performed: The present conjecture should be completed by a suitable homogeneous solution of the crossing equation to achieve full agreement with perturbative string theory at higher orders. Then it should be derived from string theory along the lines of $^{[23,22,27]}$. It is interesting to see that the one-loop result consists of $\log \cdot \log$ and $\text{Li}_2$ terms. This is what might be expected as the outcome of a one-loop integral in some field theory (albeit a four-dimensional one).

Furthermore, it is very important to study the analytic structure of the phase factor. Where are the zeros, singularities, branch points and how are they connected? What is their meaning? Is there periodicity along the real cycle of the torus? What is the structure of the underlying Riemann surface? Does the function $\theta_1$ of two points on a torus appear in another context? Some of this knowledge should eventually enable one to rigorously prove the crossing relation for the phase factor.

The most pressing question is presumably whether the phase factor interpolates to $\sigma = 1$ at the first few orders of the weak-coupling expansion around $g = 0$ in order to match with gauge theory. This is a crucial test of the AdS/CFT conjecture. Simple agreement with gauge theory would very much count in favour of the exact validity of the correspondence. In the case of disagreement, one may argue that the Bethe ansätze are asymptotic and valid only to the first few orders in perturbation theory (at either strong or weak coupling) $^5$. Therefore the disagreement would be irrelevant and both models would have their own dressing factor $\sigma$. The author’s hope, however, is that the Bethe equations for gauge and string theory are exact and not just asymptotic. No tests of the weak-coupling regime have been performed here. It is only remarked that, according to the conventional logic, the Hernández-López term appears at $O(g^3)$. Perhaps, the correct gauge theory answer ($\sigma = 1$ until at least $O(g^4)$) can be found on a different Riemann sheet? In any case, the appearance of a contribution to anomalous dimensions at two and a half loops (equivalent to $O(g^3)$ in $\sigma$) would seem cumbersome. Results in the BMN limit $^{[15,6]}$, for transcendentality counting $^{[28]}$ and the similarity to the Hubbard spin chain $^{[29,30]}$ suggest that the exact function $\sigma = 1$ is preferable for several reasons. This function however cannot extrapolate to the factor obeying crossing symmetry (unless there are essential singularities).

An important test for the consistency of some Bethe equations is that they reproduce the correct number of states of the underlying model. This is not an easy task but it should be performed for the present model and some tests seems possible. Does the number of states change between strong and weak coupling? A related question was posed in $^{[31]}$: How does the $\mathbb{R} \times S^3$ sector of string theory transform to the (smaller) $\text{su}(2)$ sector in gauge theory?

Finally, it would be remarkable if one could apply sigma model Bethe equations with an additional level of nesting to derive the complete phase. This has already been demonstrated to work in several cases $^{[32]}$ at the leading order in strong coupling (similarly, for gauge theory in $^{[30]}$). Can we also derive the Hernández-López phase in this fashion, perhaps even in more general sigma models?
**Note added.** A previous version of this manuscript posted to arxiv.org claimed full agreement of the phase \(11,12,13\) with perturbative string theory at the leading few orders. It was brought to my attention by J. Maldacena that this is not so for the case of classical giant magnons \[17\] and AFS \[10\] at finite momentum. R. Hernández and E. López noted that it does not even agree with AFS at small momentum, i.e. with classical spinning strings and near-plane wave strings. The correct statement is that it agrees for small momenta after adding the homogeneous piece \(14\) as found in a subsequent article \[21\]. To achieve agreement with classical giant magnons is more subtle, cf. also \[21\]. I am very grateful for their kind remarks and also to S. Frolov and A. Tseytlin for similar statements.

**Acknowledgements.** I thank N. Dorey, S. Frolov, R. Hernández, C. Kristjansen, E. López, J. Maldacena, T. McLoughlin, M. Staudacher and A. Tseytlin for discussions. This work is supported in part by the U.S. National Science Foundation Grant No. PHY02-43680. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

### A Spiral Staircase of Poles

The double crossing relation \[10\] requires that the degree of a pole in \(\sigma\) changes by one when going once around one period of the torus. To gather some experience with functions of this type, let us consider the simple example

\[
f(y) \sim \prod_{n=1}^{\infty} \left( \frac{y-n}{y+n} \right)^n.
\]

This function was constructed to have a zero of degree \(n\) at \(y = n\) for all \(n \in \mathbb{Z}\). It needs regularisation and we can easily evaluate the product by considering a multiple logarithmic derivative and then integrating back. The result is of the form

\[
f(y) \sim \exp S(e^{2\pi iy}) \quad \text{with} \quad S(w) = \frac{\text{Li}_2 w + \log w \log(1-w)}{2\pi i}.
\]

This function is regular at \(w = 1\) and has a logarithmic singularity at \(w = 0\). When moving once around the point \(w = 0\) (and thus past the branch cut) a zero or a pole appears at \(w = 1\). Performing another loop in the same direction will increase the degree of the zero or pole by one. This property led to the conjecture that the Hernández-López term \[12\] might be sufficient to satisfy large parts of the crossing relation: In the summed form by Arutyunov and Frolov \[23\] it consists of only \(\text{Li}_2\) and \(\log \cdot \log\) terms with precisely the right coefficients. Moreover, one can show that \(\theta_1\) is a sum of \(\pm S(w)\)’s with the \(w\)’s some suitable functions of \(x_{1,2}\).

### References

[1] J. A. Minahan and K. Zarembo, JHEP 0303, 013 (2003), [hep-th/0212208](http://arxiv.org/abs/hep-th/0212208)
[2] N. Beisert, C. Kristjansen and M. Staudacher, Nucl. Phys. B664, 131 (2003), hep-th/0303060
I. Beni, J. Polchinski and R. Roiban, Phys. Rev. D69, 046002 (2004), hep-th/0305116
[3] N. Beisert, Phys. Rept. 405, 1 (2004), hep-th/0407277
[4] N. Beisert and M. Staudacher, Nucl. Phys. B670, 439 (2003), hep-th/0307042
[5] D. Serban and M. Staudacher, JHEP 0406, 001 (2004), hep-th/0401057
[6] N. Beisert, V. Dippel and M. Staudacher, JHEP 0407, 075 (2004), hep-th/0405001
[7] N. Beisert and M. Staudacher, Nucl. Phys. B727, 1 (2005), hep-th/0504190
[8] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, JHEP 0405, 024 (2004), hep-th/0402207
[9] N. Beisert, V. Kazakov, K. Sakai and K. Zarembo, Commun. Math. Phys. 263, 659 (2006), hep-th/0502226
[10] G. Arutyunov, S. Frolov and M. Staudacher, JHEP 0410, 016 (2004), hep-th/0406256
[11] N. Beisert and A. A. Tseytlin, Phys. Lett. B629, 102 (2005), hep-th/0509084
S. Schäfer-Nameki and M. Zamaklar, JHEP 0510, 044 (2005), hep-th/0509096
[12] R. Hernández and E. López, JHEP 0607, 004 (2006), hep-th/0603204
[13] L. Freyhult and C. Kristjansen, Phys. Lett. B638, 258 (2006), hep-th/0604069
[14] M. Staudacher, JHEP 0505, 054 (2005), hep-th/0412188
[15] D. Berenstein, J. M. Maldacena and H. Nastase, JHEP 0204, 013 (2002), hep-th/0202021
[16] N. Beisert, hep-th/0511082
[17] D. M. Hofman and J. M. Maldacena, J. Phys. A39, 13119 (2006), hep-th/0604135
[18] G. Arutyunov, S. Frolov and M. Zamaklar, hep-th/0606126
[19] R. A. Janik, Phys. Rev. D73, 086006 (2006), hep-th/0603038
[20] N. Dorey, J. Phys. A39, 13119 (2006), hep-th/0604175
[21] N. Beisert, R. Hernández and E. López, JHEP 0611, 070 (2006), hep-th/0609044
[22] S. Frolov, J. Plefka and M. Zamaklar, J. Phys. A39, 13037 (2006), hep-th/0603008
[23] G. Arutyunov and S. Frolov, Phys. Lett. B639, 378 (2006), hep-th/0604043
[24] C. G. Callan, Jr., T. McLoughlin and I. Swanson, Nucl. Phys. B694, 115 (2004), hep-th/0404007
S. Frolov and A. A. Tseytlin, Phys. Lett. B570, 96 (2003), hep-th/0306143
[25] N. Beisert and L. Freyhult, Phys. Lett. B622, 343 (2005), hep-th/0506243
S. Schäfer-Nameki, M. Zamaklar and K. Zarembo, JHEP 0509, 051 (2005), hep-th/0507189
[26] G. Arutyunov and S. Frolov, JHEP 0601, 055 (2006), hep-th/0510208
[27] T. Klose and K. Zarembo, J. Stat. Mech. 06, P05006 (2006), hep-th/0604039
R. Roiban, A. Tirziu and A. A. Tseytlin, J. Phys. A39, 13129 (2006), hep-th/0604199
[28] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko and V. N. Velizhanin, Phys. Lett. B595, 521 (2004), hep-th/0404092
B. Eden and M. Staudacher, J. Stat. Mech. 06, P11014 (2006), hep-th/0603157
[29] E. H. Lieb and F. Y. Wu, Phys. Rev. Lett. 20, 1445 (1968).
[30] A. Rej, D. Serban and M. Staudacher, JHEP 0603, 018 (2006), hep-th/0512077
[31] J. A. Minahan, Fortsch. Phys. 53, 828 (2005), hep-th/0503143
[32] N. Mann and J. Polchinski, Phys. Rev. D72, 086002 (2005), hep-th/0508232
N. Gromov, V. Kazakov, K. Sakai and P. Vieira, Nucl. Phys. B764, 15 (2007), hep-th/0603043
N. Gromov and V. Kazakov, hep-th/0605028.