ENDOMORPHISMS AND AUTOMORPHISMS OF MINIMAL SYMBOLIC SYSTEMS WITH SUBLINEAR COMPLEXITY

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Abstract. We show that if the complexity difference function \( p(n+1) - p(n) \) of an infinite minimal shift is bounded, then the automorphism group of the one-sided shift is finite, and the automorphism group of the corresponding two-sided shift “modulo the shift” is finite. For the minimal Sturmian and minimal substitution shifts, the bounds can be explicitly computed, and for linearly recurrent shifts, the bound can be expressed as a function of the linear recurrence constant. We also show that any endomorphism of a linearly recurrent shift is a root of a power of the shift map.

1. Introduction

In this note, we use a result by Cassaigne [Cas96] to obtain a simple proof of the fact that any automorphism \( \Phi \) of a one- or two-sided minimal shift \((X, \sigma)\) with sublinear complexity satisfies \( \Phi^k = \sigma^n \) for some \( k \geq 1 \) and \( n \in \mathbb{Z} \). If \( X \) is one-sided, we have \( n = 0 \). (All terms are defined in Section 2.) In Theorem 3.1 we give bounds on the values that \( k \) can assume and in some cases describe how to compute these bounds explicitly. We also describe endomorphisms of a one-sided linearly recurrent shift in Theorem 4.3.

Earlier results for families of “small” shifts include [Cov72], where the endomorphism monoid of \((X, \sigma)\) is explicitly described for any nontrivial constant length substitution shift on two letters, and [HP89], where measurable factor maps between constant length substitution shifts which do not have a purely discrete spectrum are characterized. More recently, any endomorphism of a Sturmian shift is shown to be a power of the shift in [Oll13], the set of factor maps between pairs of constant length or Pisot substitution shifts with the same Perron value are shown in [ST14] to be generated by a finite set, and by the result of [CK14b], we see that modulo the subgroup generated by the shift, every automorphism of a transitive shift with subquadratic complexity has finite index. All these results are for two-sided shifts.

We recently became aware of [DDMP14] and [CK14a], which contain similar results for the two-sided case. Though there seem to be common threads in all three approaches, the directions taken are different.

2. Notation

Let \( \mathcal{A} \) be a finite alphabet, with the discrete topology, and let \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \). We endow \( \mathcal{A}^{\mathbb{N}_0} \) and \( \mathcal{A}^\mathbb{Z} \) with the product topology, and let \( \sigma : \mathcal{A}^{\mathbb{N}_0} \to \mathcal{A}^{\mathbb{N}_0} \) (or \( \sigma : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z} \)) denote the shift map. We consider only minimal and infinite \((X, \sigma)\), which can be either one or two-sided. An endomorphism of \((X, \sigma)\) is a map \( \Phi : X \to X \) which is continuous, onto.

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and commutes with \( \sigma \); if in addition \( \Phi \) is one-to-one, then \( \Phi \) is called an automorphism. We say \( \Phi \) has finite order \( m \) if \( m \geq 1 \) is the least number such that \( \Phi^m \) is the identity, and that \( \Phi \) is a \( k \)-th root of a power of the shift if there exists an \( n \) such that \( \Phi^k = \sigma^n \). Let \( \text{Aut}(X, \sigma) \) denote the automorphism group of \((X, \sigma)\). If \( X \) is two-sided, then \( \text{Aut}(X, \sigma) \) contains the (normal) subgroup generated by the shift, which we denote by \( \{ \sigma^n \} \).

The language of a minimal shift \((X, \sigma)\), denoted \( L_X \), is the set of all finite words that we see in points in \( X \). We denote by \( p(n) \), \( n \geq 1 \), the complexity function of \((X, \sigma)\); \( p(n) \) is the number of words in \( L_X \) of length \( n \). A symbolic system \((X, \sigma)\) has sublinear complexity if its complexity function is bounded by a linear function.

If \((X, \sigma)\) is a one-sided minimal shift, let \((\bar{X}, \bar{\sigma})\) denote the two-sided version of \((X, \sigma)\) i.e. \( \bar{X} \) consists of all bi-infinite sequences such that \( L_X = L_{\bar{X}} \). Given a two-sided shift \((\bar{X}, \bar{\sigma})\) we define similarly its one-sided version. If \((X, \sigma)\) is minimal, then the one-sided version of \((\bar{X}, \bar{\sigma})\) is \((X, \sigma)\) itself. Henceforth \((X, \sigma)\) refers to a one-sided shift and \((\bar{X}, \bar{\sigma})\) refers to a two-sided shift. We say that \( x \in X \) is a branch point of order \( k > 1 \) if \( |\sigma^{-1}(x)| = k \). We say that \( \bar{x} = \ldots \bar{x}_{-1} \bar{x}_0 \bar{x}_1 \ldots \) and \( \bar{y} = \ldots \bar{y}_{-1} \bar{y}_0 \bar{y}_1 \ldots \) in \( \bar{X} \) are right asymptotic if there is some \( N \in \mathbb{Z} \) such that \( \bar{x}_n = \bar{y}_n \) for \( n \geq N \).

Two orbits \( O_x = \{ \bar{\sigma}^n(x) : n \in \mathbb{Z} \} \) and \( O_y = \{ \bar{\sigma}^n(y) : n \in \mathbb{Z} \} \) in \( \bar{X} \) are right asymptotic if there exist \( \bar{x}' \in O_x \) and \( \bar{y}' \in O_y \) that are right asymptotic. A single orbit \( O_x \) is right asymptotic if there is a point \( \bar{y} \) with \( O_x \cap O_y = \emptyset \), and \( O_x \) and \( O_y \) are right asymptotic.

Define the equivalence relation \( \sim \) on the set of asymptotic orbits of \((\bar{X}, \bar{\sigma})\) by \( O_x \sim O_y \) if \( O_x \) and \( O_y \) are right asymptotic.

3. Results

Our main result is

**Theorem 3.1.** Let \((X, \sigma)\) be minimal and infinite with sublinear complexity. Then

1. if \( X \) has \( M_k \) branch points of order \( k \) in \( X \), then \( \text{Aut}(X, \sigma) \) has at most \( M := \min_k M_k \) elements. As a consequence any \( \Phi \in \text{Aut}(X, \sigma) \) has order at most \( M \).
2. if for each \( k \geq 2 \), \( X \) has \( M_k \) \(~\)-equivalence classes of size \( k \), and \( M := \min_k M_k \), then \( \text{Aut}(\bar{X}, \bar{\sigma})/\{ \bar{\sigma}^n \} \) has at most \( M \) elements. As a consequence any \( \Phi \in \text{Aut}(\bar{X}, \bar{\sigma}) \) is at most an \( M \)-root of a power of the shift.

**Proof.** We shall use Lemmas 3.2 and 3.3 below in our proof.

Using Lemma 3.2 the set \( E \) of branch points in \( X \) and the collection \( \mathcal{E} \) of right-asymptotic orbits in \( \bar{X} \) are finite. The minimality of our systems implies that an automorphism is determined by its action on any one point, and the fact that our systems are infinite implies that \( X \), resp. \( \bar{X} \), has at least one branch point, resp. right asymptotic orbit.

An automorphism of a one-sided system must map a branch point of order \( k \) to a branch point of order \( k \), so we apply Lemma 3.3(1) to the set of branch points of order \( K \), where \( M_K \) is a minimum of \( \{ M_k : k \geq 2 \} \).

An automorphism \( \bar{\Phi} \) of a two-sided system must map a \(~\)-equivalence class of size \( k \) to a \(~\)-equivalence class of size \( k \), so we apply Lemma 3.3(2) to the set \( \mathcal{E} \) of orbits belonging to any \(~\)-equivalence class of size \( K \), where \( M_K \) is a minimum of \( \{ M_k : k \geq 2 \} \).

Note that if \( \bar{\Phi} \) sends an equivalence class to itself, then \( \bar{\Phi} \) must be a power of the shift. For if \( \bar{x}_i = \bar{y}_i \) for \( i \geq 0 \) and \( \bar{\Phi} \) maps the \( \bar{\sigma} \)-orbit of \( \bar{x} \) to the \( \bar{\sigma} \)-orbit of \( \bar{y} \), then for some \( n \) it
maps \( \tilde{x}_{[0, \infty)} = \tilde{y}_{[0, \infty)} \) to \( \tilde{y}_{[n, \infty)} \). Minimality now implies that \( \tilde{\Phi} = \tilde{\sigma}^n \). This argument also implies that if \( \Phi_1(x) = y \) and \( \Phi_2(x) = y' \) where \( y \sim y' \), then \( \Phi_1 = \Phi_2 \circ \sigma^n \) for some integer \( n \). Thus there are at most \( M_K \) automorphisms in \( \text{Aut}(\tilde{X}, \tilde{\sigma})/\{\tilde{\sigma}^n\} \).

□

As \( \text{Aut}(X, \sigma) \) and \( \text{Aut}(\tilde{X}, \tilde{\sigma})/\{\sigma^n\} \) are both (finite) groups, then the order of any element of \( \text{Aut}(X, \sigma) \) divides \( |\text{Aut}(X, \sigma)| \), and any element of \( \text{Aut}(\tilde{X}, \tilde{\sigma}) \) is a \( k \)-th root of the shift, where \( k \) divides \( |\text{Aut}(\tilde{X}, \tilde{\sigma})/\{\sigma^n\}| \).

Let \( s(n) := p(n+1) - p(n) \) be the complexity difference function.

**Lemma 3.2.** Let \( (X, \sigma) \) be minimal and infinite. Then (1) is equivalent to (2), (2) implies (3), and (3) is equivalent to (4).

1. \( (X, \sigma) \) has sublinear complexity.
2. There exists a constant \( L \) such that \( s(n) \leq L \) for all \( n \geq 1 \).
3. \( (X, \sigma) \) has finitely branch points.
4. \( (\tilde{X}, \tilde{\sigma}) \) has finitely many asymptotic orbits.

**Proof.** The only difficult implication is that sublinear complexity implies that the difference function \( s(n) \) is bounded, and this is proved in [Cas96]. Conversely if \( s(n) \leq L \) for all \( n \), then \( p(n) \leq Kn \) for all \( n \), where \( K := \max\{L, p(1)\} \).

To see that (2) implies (3), first note that \( L \) is an upper bound for the number of words of length \( n \) which can be extended to the left in at least two ways. If \( x \) is a branch point in \( X \), then for each \( n \), the prefix of \( x \) of length \( n \) can be extended in at least two ways. If there were more than \( L \) branch points, choose \( n \) large enough so that the prefixes of these branch points are distinct, a contradiction.

Finally note that the sums of the orders of all the branch points in \( (X, \sigma) \) is an upper bound (possibly strict) for the number of right asymptotic orbits in \( (\tilde{X}, \tilde{\sigma}) \). This shows that (3) is equivalent to (4).

The proof of the following lemma is straightforward. It follows directly from the minimality of our systems, and will be our main tool. Here our systems need not be symbolic, nor invertible in (1), and we will not use bars to indicate invertibility.

**Lemma 3.3.** Let \( (X, S) \) and \( (Y, T) \) be infinite minimal systems.

1. If there exist finite sets \( E \subset X \) and \( F \subset Y \) such that \( \Phi(E) \subseteq F \) for any isomorphism \( \Phi : (X, S) \to (Y, T) \), then there are at most \( |E| \) isomorphisms from \( (X, S) \) to \( (Y, T) \).
2. Let \( S \) and \( T \) be invertible. If there exist finite collections \( E \) of \( S \)-orbits and \( F \) of \( T \)-orbits such that any isomorphism \( \Phi : (X, S) \to (Y, T) \) satisfies \( \Phi(E) \subseteq F \), then there are at most \( |E| \) isomorphisms \( \{\Phi_i : i \in I\} \) such that any isomorphism from \( (X, S) \) to \( (Y, T) \) is of the form \( \Phi = \Phi_i \circ S^n \) for some \( n \in \mathbb{Z} \) and some \( i \).

Note that we can also use Lemma 3.3 to bound the size of the set of isomorphisms \( \Phi : (X, S) \to (Y, T) \) between two minimal systems “modulo powers of \( S \”).

4. EXAMPLES AND BOUNDS

We now describe some families of shifts which satisfy the conditions of Theorem 3.1.
- **Sturmian shifts.** All infinite minimal Sturmian shifts have complexity function $p(n) = n + 1$. This result is implicit in [MH40], although [MH40] does not ever mention blocks. For details of why $p(n) = n + 1$, see [CH73]. Since one-sided, resp. two-sided Sturmian shifts have only one branch point, resp. pair of positively asymptotic orbits, from Theorem 3.1 we get

**Corollary 4.1.** For infinite minimal Sturmian systems, both $\text{Aut}(X, \sigma)$ and $\text{Aut}(\bar{X}, \bar{\sigma})/\{\bar{\sigma}^n\}$ consist of only the identity map.

This has been proved by J. Olli [Oll13] using different methods.

- **Linearly recurrent shifts.** The notions below are the same for one-sided and two-sided shifts, so we will state them only for one-sided shifts, thus avoiding the bars.

  If $u, w \in L_X$, we say that $w$ is a return word to $u$ if (a) $u$ is a prefix of $w$, (b) $wu \in L_X$, and (c) there are exactly two occurrences of $u$ in $wu$. Letting $\ell(u)$ denote the length of $u$, a minimal one-sided shift $(X, \sigma)$ is linearly recurrent if there exists a constant $K$ such that for any word $u \in L_X$ and any return word (to $u$) $w$, $\ell(w) \leq K\ell(u)$.

  Such a $K$ is called a recurrence constant. By [DHS99, Theorem 23], if $(X, \sigma)$ has linear recurrence constant $K$, then its complexity is bounded above by $Kn$ for large $n$. Cassaigne [Cas96] shows that if $p(n) \leq Kn + 1$, then $s(n) \leq 2K(2K+1)^2$ for all large $n$. It then follows from Theorem 3.1 that

**Corollary 4.2.** Let $(X, \sigma)$ be a linearly recurrent shift with recurrence constant $K$. Then $|\text{Aut}(X, \sigma)| \leq 2(2(K+1)(2K+3)^2$ and $|\text{Aut}(\bar{X}, \bar{\sigma})/\{\bar{\sigma}^n\}| \leq 2(2(K+1)(2K+3)^2$.

Note that for linearly recurrent two-sided shifts, any endomorphism is an automorphism ([Dur00, Corollary 18]). The appropriate one-sided version of coalescence - when any endomorphism is an automorphism - is then

**Theorem 4.3.** Let $(X, \sigma)$ be a one-sided linearly recurrent shift with recurrence constant $K$. Then every endomorphism of $(X, \sigma)$ is a $k$-th root of a power of the shift for some positive $k \leq 2(K+1)(2K+3)^2$.

**Proof.** Any endomorphism $\Phi$ of $(X, \sigma)$ defines an endomorphism $\tilde{\Phi}$ of $(\bar{X}, \bar{\sigma})$, which must be an automorphism by [Dur00, Corollary 18]. By Corollary 4.2, $\Phi^k = \bar{\sigma}^n$ for some positive $k \leq 2(K+1)(2K+3)^2$ and some $n \in \mathbb{Z}$, so $\Phi^k = \sigma^n$. $\square$

This last result cannot be improved: Example 4.6 tells us that $\Phi$ is not necessarily a power of the shift.

- **Substitution shifts.** A more tractable subclass of the linearly recurrent shifts are the primitive substitution shifts: see [BDH03] for definitions, where a primitive substitution on $k$ letters is shown to have at most $k^2$ right asymptotic orbits. We can then deduce

**Corollary 4.4.** Let $(\bar{X}, \bar{\sigma})$ be the minimal shift generated by a primitive substitution on $k$ letters. Then any automorphism of $(\bar{X}, \bar{\sigma})$ is an at most $k^2$-th root of the shift.

A substitution $\theta$ is (one-sided) recognizable [Mos92] if any one sided point can, except for possibly some initial segment, be uniquely de-substituted. There exist algorithms for finding branch points of recognizable primitive substitutions, and hence
right asymptotic orbits of primitive substitution systems. For these substitutions \( \theta \), branch points can arise in one of two ways. A branch point can be a \( \theta \)-periodic point satisfying \( \theta^n(y) = y \), where there is more than one word \( a \theta y \in \mathcal{L}_X \) such that \( a \) is a suffix of \( \theta^n(a) \). The only other way a branch point \( y \) can arise is when we see maximal proper common suffixes of at least two substitution words \( \theta^n(a) \); in this case \( y \) satisfies \( w \theta^n(y) = y \) for some \( w \in \mathcal{L}_X \). In both cases such branch points can be listed.

**Example 4.5.** Take the substitution \( \theta \) on the alphabet \( \{a, b, c\} \) defined by \( \theta(a) = acb \), \( \theta(b) = aba \) and \( \theta(c) = acc \). Then the right \( \theta \)-fixed point \( u \) is a 2-branch point. Note that \( \theta(b) \) and \( \theta(c) \) both have \( a \) as a maximal proper common suffix, and \( \theta^n(b) \) and \( \theta^n(c) \) both have \( a \theta(a) \ldots \theta^{n-1}(a) \) as a maximal common suffix. Letting \( n \to \infty \), we see that \( y = a \theta(a) \theta^2(a) \ldots \) is a 2-branch point and that it satisfies the equation \( a \theta(y) = y \). There are no other branch points. According to our bounds, \( |\text{Aut}(X, \sigma)| \leq 2 \).

With a little extra work, we can see that \( \text{Aut}(X, \sigma) \) is trivial. Note that \( u \in \theta(X) \) and, since \( a \theta(y) = y \), then \( y \in \sigma^2(\theta(X)) \). The recognizability of \( \theta \) tells us that each point \( x \in X \) belongs to exactly one of \( \sigma^i \theta(X) \), \( i = 0, 1, 2 \). If \( \Phi(u) = y \), \( \Phi(\theta(X)) = \sigma^2(\theta(X)) \) and \( \Phi(\sigma^2(\theta(X))) = \sigma(\theta(X)) \), which implies that \( \Phi(y) \neq u \), a contradiction.

The following example is a modification of the example in [Hed69, page 372] and shows that not every endomorphism of a one sided shift is a power of the shift.

**Example 4.6.** Let \( B = 1001 \) and \( C = 1101 \) and let \( X_0 \) be the set of all concatenations of \( B \) and \( C \), where \( B \) and \( C \) occur starting at multiples of 4, that “mirror” points in the Morse-Thue Minimal System (in the sense of [CKLJS]). \( (X_0, \sigma^4) \) is isomorphic to the one-sided Morse-Thue system. Define \( X = X_0 \cup \sigma(X_0) \cup \sigma^2(X_0) \cup \sigma^3(X_0) \). Let \( \Phi \) be the endomorphism of \( (X, \sigma) \) whose right radius 3 local rule \( \phi \) is

\[
\phi(xyzw) = \begin{cases} 
1 & \text{if } xzw = 1001, \\
0 & \text{if } xzw = 1101, \\
y & \text{otherwise.} 
\end{cases}
\]

Then the local rule of \( \Phi \) “depends on the first variable”, is not a power of \( \sigma \), but is \( \sigma \) followed by interchanging all occurrences of \( B \) and \( C \). Therefore \( \Phi^2 = \sigma^2 \).

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