Scheduling of Sensor Transmissions Based on Value of Information for Summary Statistics

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Abstract—The optimization of Value of Information (VoI) in sensor networks integrates awareness of the measured process in the communication system. However, most existing scheduling algorithms do not consider the specific needs of monitoring applications, but define VoI as a generic Mean Square Error (MSE) of the whole system state regardless of the relevance of individual components. In this work, we consider different summary statistics, i.e., different functions of the state, which can represent the useful information for a monitoring process, particularly in safety and industrial applications. We propose policies that minimize the estimation error for different summary statistics, showing significant gains by simulation.

Index Terms—Internet of Things, Wireless Sensor Networks, Value of Information, Scheduling policies

I. INTRODUCTION

Over the past few years, the unprecedented development of the Internet of Things (IoT) has made the remote estimation of stochastic processes a central problem in communications and automation [1], where a set of sensors transmit observations to a central Base Station (BS). The possibility to process sensor data either at the BS or in a distributed fashion through in-network processing [2] has led the research community to focus extensively on the scheduling of sensor updates in severely resource-constrained wireless network environments.

For a wide range of remote estimation problems, the freshness of the observations at the BS is a good proxy for the estimation quality. This promotes Age of Information (AoI) [3] as a measure of the time that has passed since the last update from a given sensor. However, if the destination has a model of the observed processes, it is often better to directly minimize the uncertainty of the process estimates instead of the AoI [4]. The problem of scheduling sensors with this goal has been considered for several different policies [5], [6], whose objective is to minimize the Mean Square Error (MSE) of a Kalman filter, considering communication constraints.

More recently, the problem of minimizing the MSE of the process estimates has been referred to as Value of Information (VoI) [4]. A recent work [7] tries to maximize the accuracy of a more complex unscented filter, aiming at optimal sensor selection for maneuvering tasks, and VoI can also be used for data muling applications in underwater or drone networks [8]. Another interesting twist to this is the application of concepts not over time, but in space, placing sensors in the positions that will result in the highest overall accuracy for the estimation of a spatial process [9].

However, there are cases where minimizing the MSE is not the best thing to do: for example, if the application needs to compute a non-linear function of the state, such as the maximum value among all sensors. While minimizing the MSE implicitly gives equal value to all sensors, some might have a larger weight in the non-linear function (e.g., sensors with a higher value for the maximum function). Examples in industrial settings include: (1) triggering a safety warning if the temperature of any of the components in a machine reaches a safety limit, (2) monitoring if the difference in the strain on different parts of a structure is outside the design parameters. Such a scenario is represented in Fig. 1: the remote server sends queries to the BS, which correspond to the non-linear function, and the BS needs to schedule transmission so as to maximize the accuracy. This setup was also used in our previous work [10]. The scheduling in this scenario is driven by the BS which selects the sensor that it believes to have the most useful information at each time slot; the opposite scenario, in which sensors themselves decide whether to transmit or not, is an interesting but different problem, as it requires sensors to maintain an estimate of the system state and a decision algorithm, which consume energy, as well as to coordinate among themselves to avoid collisions. Our scenario is directly applicable to wake-up radio [11], [12] and similar schemes with low-power sensors.

We propose heuristic strategies to schedule sensor updates in a linear dynamic system, which explicitly aim to minimize the error of various summary statistics. We derive the one-step optimal strategies for some well-known function, and give a general Monte Carlo-based algorithm that can deal with different query functions. The simulations show that the proposed strategies can significantly reduce the error on a number of summary statistics, with more significant gains in case of highly non-linear summary statistics.

The rest of this letter is organized as follows. The system model is presented in Sec. II and one-step policies for various
summary statistics are derived in Sec. III. Numerical results are presented in Sec. IV, and finally Sec. V concludes the paper and presents some possible avenues of future work.

II. SYSTEM MODEL

We consider a system with $N$ sensors, which are connected through time-slotted wireless links to a BS equipped with computing and storage resources. Without loss of generality, we assume that the time slots occur at $t = 1, 2, \ldots$ and the sensors are indexed by $n = 1, \ldots, N$. We assume that each sensor observes a value in an $N$-dimensional process, whose state $x(t) = [x_1, \ldots, x_N]^T$ evolves according to

$$x(t) = Ax(t-1) + v(t),$$

where $A \in \mathbb{R}^{N \times N}$ is the transition matrix, $v(t) \sim \mathcal{N}(0, \Sigma_v)$ is the process noise with covariance matrix $\Sigma_v \in \mathbb{R}^{N \times N}$, and $x(0) = 0$. The sensors observe the processes with additive white Gaussian measurement noise $w(t) \sim \mathcal{N}(0, \Sigma_w)$, i.e., $y(t) = x(t) + w(t)$. In general, the covariance matrices $\Sigma_v$ and $\Sigma_w$ are not diagonal. Note that although we assume that the number of sensors is equal to the dimension of the process (to simplify the notation), the analysis can be easily extended to more general observable systems.

We consider a Time Division Multiple Access (TDMA) air interface, in which each time slot, $t$, contains a downlink phase and an uplink phase. The downlink is used by the BS to schedule the sensor, $a(t)$, that transmits its observation $y_{a(t)}(t)$ in the uplink phase. The channel is modeled as a packet erasure channel with error probability $\varepsilon_{a(t)}$, which captures errors both in the transmission of the scheduling decision and the observation. We also assume that the process dynamics are known to the BS, a standard assumption in Kalman filtering, which is practical if the monitored system is well-understood, even if its instantaneous state is hard to measure directly. This condition is common for many Vol applications [13], [14], in which well-known processes are estimated by sensors over wide areas. We also denote the row vector of length $N$ whose only non-zero value is the $n$-th, which is 1, as $1_n$, and the $N \times N$ identity matrix as $I_N$.

A. Kalman Filter Estimation

We assume that the BS maintains a distribution over its belief of the state $p(x(t))$ using a Kalman filter. The Kalman filter is the Minimum Mean Square Error (MMSE) estimator for the model defined in Eq. (1) [15], in which case $p(x(t)) \sim \mathcal{N}(\hat{x}(t), \psi(t))$. The mean vector $\hat{x}(t)$ and the covariance matrix $\psi(t)$ are updated at each timestep $t$ based on the outcome of the scheduling process. The Kalman filter operates in two steps: a prior update, which only depends on the system statistics, and a posterior update, which integrates new observations. The prior update operation is given by:

$$\hat{x}(t) = A \hat{x}(t-1) \triangleq \hat{x}_F(t),$$

$$\psi(t) = A \psi(t-1) A^T + \Sigma_v \triangleq \psi_F(t).$$

If the transmission of the update fails, an event we denote as F, the BS can only rely on the prior update for its estimate. If the update is received, it can be used to improve the estimate. We then compute the Kalman filter gain $k(t)$:

$$k(t) = \psi_F(t) I_{a(t)} \left[ I_{a(t)} (\psi_F(t) + \Sigma_w) I_{a(t)} \right]^{-1}.$$

We finally get the updated estimate in case of a success event:

$$\hat{x}(t) = \hat{x}_F(t) + k(t) (y(t) - \hat{x}_F(t)) \triangleq \hat{x}_{S,a(t)}(t),$$

$$\psi(t) = (I_N - k(t) I_{a(t)}) \psi_F(t) \triangleq \psi_{S,a(t)}(t),$$

Note that the recursive structure of the Kalman filter and the independence of the transmission errors imply that $\hat{x}(t)$ and $\psi(t)$ are sufficient statistics for the state estimate given the full history $\mathcal{H}(t)$ of past actions and observations.

B. Summary Statistics

Unlike the majority of Vol applications, in which the BS aims to minimize the MSE of $\hat{x}(t)$, we consider the case in which an external user requests summary statistics about the state of the system: these correspond to a predefined, fixed function of the system state, e.g., the average value or the number of states with values within a given interval. Formally, we define a summary statistic as a function $z : \mathbb{R}^N \rightarrow \mathbb{R}$ of the true state $x(t)$. However, because $x(t)$ is unknown to the BS, it can only provide an approximate answer to the query based on its state belief $p(x(t))$. We will consider estimators of the summary statistics on the form

$$\hat{z}(t) = \mathbb{E}_{x \sim \mathcal{N}(\hat{x}(t), \psi(t))}[z(x)],$$

which corresponds to the minimum MSE estimator of $z(x(t))$, given the current observation [16], [17]. This is different from minimizing the MSE of $\hat{x}(t)$, particularly when function $z(\cdot)$ is non-linear or the sensors have different weights. We denote the squared error as $\nu_z(t)$:

$$\nu_z(t) = (z(\hat{x}(t)) - \hat{z}(t))^2.$$

III. SCHEDULING STRATEGIES

In our scenario, we seek a scheduling strategy, that is, a function $\pi_z$ from the current Kalman state (which is represented by vector $\hat{x}(t)$ and matrix $\psi(t)$) to an action $a(t)$, that minimizes the expected error for a given summary statistic $z$:

$$\min_{\pi_z \in \Pi} \mathbb{E}_{a(t)}[\nu_z(t) | \pi_z],$$

While we only consider the error in the next time step, the optimal solutions are expected to perform well with respect to the long-term error due to the linearity of the observed process (despite a non-linear summary statistic). Computing $\mathbb{E}_{a(t)}[\nu_z(t) | a(t)]$ is not simple, but it can be expressed in terms of the two possible transmission outcomes:

$$\mathbb{E}_{a(t)}[\nu_z(t) | a(t)] = (1 - \varepsilon_{a(t)}) \mathbb{E}_{a(t)}[\nu_z(t) | \hat{x}_{S,a(t)}(t), x_{S,a(t)}(t)] + \varepsilon_{a(t)} \mathbb{E}_{a(t)}[\nu_z(t) | \hat{x}_F(t), x_{F}(t)],$$

where the expectation is over the state evolution. Since $x_{S,a(t)}(t)$ and $x_{F}(t)$ can be computed using (6) and (5), we can iterate over the possible actions and find the optimal scheduler, as long as we can estimate the MSE for a given
observation. In the following, we derive the optimal schedulers for some well-known summary statistics, along with giving a Monte Carlo-based approximate scheduler that can deal with more complex statistics for which the MSE is hard to express in closed form. Using the result from (10) we can obtain the optimal scheduling decision at time \( t \):

\[
a^*_z(t) = \arg \min_{a(t) \in \{1, \ldots, N \}} \mathbb{E}\left[\nu_z(t) \mid a(t)\right]. \tag{11}
\]

### A. Baseline Scheduler

We start by defining our benchmark scheme, which aims to minimize the MSE between the true state \( x \) and the estimated state \( \hat{x} \). The query is then computed as in (7) based on the state estimate.

The squared state estimation error can be expressed as

\[
E[\nu_MSE(t)] = \mathbb{E}\left[(\hat{x}(t) - \hat{x}(t))^T(x(t) - \hat{x}(t))\right]. \tag{12}
\]

Because \( (x(t) - \hat{x}(t)) \sim \mathcal{N}(0, \psi(t)) \), the expression above is equivalent to the trace of the covariance matrix \( \psi \) [18]:

\[
E[\nu_MSE(t) \mid \psi(t)] = \text{tr}(\psi(t)). \tag{13}
\]

We can then use (11) to compute the minimum MSE schedule.

### B. Sample Mean Scheduling

We now consider the most basic statistic, the sample mean:

\[
\bar{z}_{\text{avg}}(x(t)) = \frac{1}{N} \sum_{n=1}^{N} x_n(t). \tag{14}
\]

The estimation error \( \nu_{\text{avg}}(t) \) is equal to the square of the average difference between the true and the estimated entries of \( x \). Since the sum of all elements in \( x(t) - \hat{x}(t) \) is a Gaussian random variable with zero mean and variance equal to the sum of all elements in \( \psi(t) \), we have:

\[
E[\nu_{\text{avg}}(t) \mid \psi(t)] = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} \psi(i,j)(t)}{N^2}, \tag{15}
\]

where \( \psi(i,j)(t) \) is entry \((i,j)\) of \( \psi(t) \). We can then use the result in (11) to derive the one-step optimal schedule.

### C. Sample Variance Scheduling

Another important summary statistic is the sample variance, quantifying how much the state deviates from the mean:

\[
\bar{z}_{\text{var}}(x(t)) = \frac{1}{N-1} \sum_{n=1}^{N} \left(x_n(t) - \frac{1}{N} \sum_{m=1}^{N} x_m(t)\right)^2. \tag{16}
\]

To derive the scheduling policy, it is convenient to express \( \bar{z}_{\text{var}}(x(t)) \) in quadratic form with matrix \( M = I - 1/N \):

\[
\bar{z}_{\text{var}}(x(t)) = \frac{(Mx(t))^TMx(t)}{N-1} = \frac{x(t)^TMx(t)}{N-1}. \tag{17}
\]

### IV. NUMERICAL EVALUATION

In the following, we show the effects of the sampling strategy on different statistics by simulation, using a Monte Carlo approach: we generate a synthetic process, then try to estimate it at the BS using the different schedulers. The systems below represent two highly asymmetric examples, but

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**Algorithm 1 Monte Carlo scheduling policy**

```plaintext
1. function SCHEDULE(\( \nu(t-1), \psi(t-1), A, \Sigma_u, \Sigma_w, \epsilon, z) 
2. \( u \leftarrow 0 \)
3. for \( n \in \{1, \ldots, N\} \) do
4. \( u \leftarrow 0 \)
5. for \( m \in \{1, \ldots, M\} \) do
6. \( \hat{x}(t-1), \psi(t-1), S \leftarrow \text{PRIOR UPDATE}(\hat{x}, \psi, A, \Sigma_u) \)
7. if RANDOM(0, 1) \( \geq \epsilon_n \) then
8. \( y \leftarrow \text{GAUSSIAN SAMPLE}(z, \hat{x}(t), \psi, \Sigma_w) \)
9. \( x_m \leftarrow \text{GAUSSIAN SAMPLE}(\hat{x}, \psi) \)
10. \( u(n) \leftarrow z(x_m) \) \( \triangleright \) Compute query value
11. \( \nu(n) \leftarrow \text{VAR}(u) \) \( \triangleright \) Sample variance
12. end
13. \( \nu \leftarrow \text{arg min} \nu \)
```

Taking into account the belief \( p(x) \sim \mathcal{N}(\hat{x}(t), \psi(t)) \), the expected value and variance of the sample variance are known from the literature [18]:

\[
\bar{z}_{\text{var}}(t) = \frac{1}{N-1} \left( \text{tr}(M\psi(t)) + \hat{x}(t)^TM\hat{x}(t) \right) \tag{18}
\]

\[
E[\nu_{\text{var}}(t) \mid \psi(t)] = \frac{2\text{tr}(M\psi(t)^2) + 4\hat{x}(t)^TM\psi(t)\hat{x}(t)}{(N-1)^2}. \tag{19}
\]

As for the MSE and sample mean, we can now simply derive the scheduler by using this result in (11).
the strategies we derived are optimal for all observable linear systems. The scenarios are constructed to be stable, i.e., the eigenvalues of the system matrices are all smaller than 1.

A. Scenario and Settings

We evolve the system for 100 episodes of 1000 samples each, and the Monte Carlo scheduler computes a total of $M = 1000$ samples for each state. We consider two systems with $N = 20$ sensors, in which the elements of the update matrix $A$ are known. In the first scenario, the matrix $A_1$ is given by:

$$A_1^{(i,j)} = \begin{cases} \frac{3}{5}, & \text{if } i = j; \\ -\frac{1}{8}, & \text{if } i \neq j, \text{mod}(i - 2j, 7) = 0, \end{cases}$$

where $\text{mod}(m, n)$ is the integer modulo function, and the values are 0 everywhere else. On the other hand, in the second scenario, we have:

$$A_2^{(i,j)} = \begin{cases} \frac{3}{5}, & \text{if } i = j; \\ -\frac{1}{8}, & \text{if } i \neq j, \text{mod}([i - 2.3j], 7) = 0. \end{cases}$$

The other parameters are the same in both scenarios. We also have $\Sigma_w = I$, while the process noise covariance is given by:

$$\Sigma_u^{(i,j)} = \begin{cases} \frac{11 + \text{mod}(i, 10)}{8}, & \text{if } i = j; \\ 1, & \text{if } i \neq j, \text{mod}(i - j, 6) = 0. \end{cases}$$

Sensors with higher indices will have a slightly higher variance. The transmission error probabilities are $\varepsilon_n = 0.02 \left[ \frac{n-1}{10} \right]$. The filter is initialized at step 0 with state $\hat{x}(0) = \mathbf{x}(0) = 0$, and $\psi(0) = I$.

In addition to the baseline scheduler, we also consider the well-known Maximum Age First (MAF) scheduler as a benchmark. If we denote the age of the last received packet from sensor $n$ as $\Delta_n$, the scheduler always picks the sensor with the highest age:

$$a_{\text{MAF}}^*(t) = \arg\max_{n \in \{1, \ldots, N\}} \Delta_n(t).$$

Finally, we consider four different summary statistics, as well as the state $\text{MSE}$ aside from the sample mean and variance, we consider the maximum and count statistics, denoted as $\varepsilon_{\text{max}} = \max_n x_n(t)$ and $\varepsilon_{\text{cnt}}$, which is given by:

$$\varepsilon_{\text{cnt}}(x) = \sum_{n=1}^{N} \mathbb{I}(x_n(t) - a) \mathbb{I}(b - x_n(t)),$$  \hspace{1cm} (24)

where $\mathbb{I}(x)$ is the step function, equal to 1 if $x \geq 0$ and 0 otherwise, and the count interval $[a, b]$. In other words, the count statistic is a simple count of the number of state components that are within $[a, b]$, which we set to $[-5, 5]$.

B. Results

We can first look at the choices of the schemes aimed at each target metric in one of the episodes for each scenario, shown in Fig. 2. As expected, the MAF scheduler selects sensors with a similar frequency in both scenarios: sensors with an index over 10 are selected slightly more frequently, as transmission errors occur more often, but the difference is small. On the other hand, the average scheduler only selects two sensors, 7 and 14, in the first scenario, and only the last, sensor 20, in the second: this holds throughout all episodes, independently from the state of the system. In the first scenario, alternating between these two sensors gives the best estimate of the overall average, as the state of each of these two sensors only depends on the other’s. In the second scenario, no sensor is isolated, but sensor 20 is the one that affects the average the most. The average scheduler then gets the best estimate it can for the other values, concentrating on these sensors and actually getting a better average performance. Naturally, this results in a significantly worse performance when looking at any other summary statistic. We also remark that all other policies excluding MAF never choose sensors 7 and 14 in the first scenario: as errors compound for most of these summary statistics, it does not make sense to choose isolated sensors, as sensors that are more correlated to their neighbors have a better chance to reduce the overall error. We can also note that, in the second scenario, the MSE, count, and sample variance schedulers often make similar choices, while they do not in the first scenario: this similarity is purely due to the specific features of the system, and cannot be relied upon for design.

The Cumulative Distribution Functions (CDFs) of the quadratic estimation errors $\nu_n(t)$ obtained for the various summary statistics are shown in Fig. 3. As expected, optimizing the scheduling for a given summary statistic can reduce the error on it, for all the considered statistics. We did not plot the average statistic, as all policies had a similar performance, although, as expected, the average scheduler performed best for that statistic. However, $\varepsilon_{\text{avg}}^*$ was almost always the worst policy when looking at other summary statistics, often by a

Fig. 2: Sensor selection frequency for the considered policies.
wide margin, in both scenarios, for the reasons we explained above. The maximum scheduler \( \pi^*_{\text{max}} \) was also noticeably worse when looking at the state \( \text{MSE} \) or the interval count, as it tended to pick sensors with a high value, often accepting a larger error on other components of the state. As the maximum value was almost always far over the interval boundaries, the count statistic was also negatively affected. Finally, the similarity in the behavior of the count, sample variance, and the \( \text{MSE} \) scheduler in the second scenario has a similar performance, as Fig. (f) shows.

In general, the average and count statistics tend to be relatively insensitive to the scheduling policy used, with most policies showing similar results: in both cases, estimation errors tend to compensate, and the error is relatively low. On the other hand, the gain from using the appropriate scheduling strategy is clearly noticeable when looking at the \( \text{MSE} \) and at the maximum statistic. In these cases, individual components of the state can have a disproportionate effect, and errors tend to compound rather than compensate each other. In general, while never being the optimum, the \( \text{MAF} \) scheduler is also never the worst, as it is a purely \text{Vol-oriented} approach that does not consider the specific definition of \text{Vol}.

V. CONCLUSION

In this letter, we have considered the optimization of a sensor polling strategy, using different statistics to define the \text{Vol}. The difference between the policies can be important, as errors tend to compensate each other in some cases and compound for other statistics, leading to different choices of sensors. Naturally, one-step optimization is a limited approach, and we plan to consider more complex schemes which can take long-term effects into account, as well as different statistics over the same process. Energy consumption is also another important metric, and we plan to compare \text{Vol}-based strategies to energy-efficient ones and try to find a balance between them.

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