AMENABILITY OF GROUPS ACTING ON TREES

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Abstract. This note describes the first example of a group that is amenable, but cannot be obtained by subgroups, quotients, extensions and direct limits from the class of groups locally of subexponential growth. It has a balanced presentation

\[ \Delta = \langle b, t \mid [b, t^{2}b^{-1}, [[[b, t^{−1}], b], b]] \rangle. \]

In the proof, I show that \( \Delta \) acts transitively on a 3-regular tree, and that \( \Gamma = \langle b, b^{t^{−1}} \rangle \) stabilizes a vertex and acts by restriction on a binary rooted tree. \( \Gamma \) is a “fractal group”, generated by a 3-state automaton, and is a discrete analogue of the monodromy action of iterates of \( f(z) = z^{2} − 1 \) on associated coverings of the Riemann sphere. \( \Delta \) shares many properties with the Thompson group \( F \).

I prove briefly some algebraic properties of \( \Gamma \), and in particular the convergence of quotient Cayley graphs of \( \Gamma \) (aka “Schreier graphs”) to the Julia set of \( f \).

Whenever convenient, the results are expressed in the framework of weakly branch groups, with extra hypotheses such as contraction.

1. Introduction

The purpose of this note is twofold: it hints at the connection between groups acting on trees (à la Bass-Serre) and groups acting on rooted trees (à la Grigorchuk); and it gives a criterion for amenability and intermediate growth of the latter (and sometimes the former).

This paper was written in least possible generality that makes the proofs non-artificial. Many generalizations are possible, and in particular to the class of “monomial groups” defined below.

As a concrete byproduct, the group

\[ \Delta = \langle b, t \mid [b, t^{2}b^{-1}, [[[b, t^{−1}], b], b]] \rangle \]

is the first example of a group that is amenable, but cannot be obtained by subgroups, quotients, extensions and direct limits from groups locally of subexponential growth (see Theorem 2.2); and it furthermore has a balanced presentation and acts vertex-transitively on a 3-regular tree (see Theorem 2.5).

1.1. Groups of intermediate growth. Let \( G = \langle S \rangle \) be a finitely generated group. Its growth function is \( \gamma(n) = \# \{g \in G : g \in S^{n}\} \). Define a preorder on growth functions by \( \gamma \lesssim \delta \) if \( \gamma(n) \leq \delta(Cn) \) for some \( C \in \mathbb{N} \) and all \( n \in \mathbb{N} \), and denote its symmetric closure by \( \sim \). The \( \sim \)-equivalence class of \( \gamma \) is independent of the choice of \( S \). If \( \gamma \sim 2^{n} \), then \( G \) has subexponential growth. If furthermore \( \gamma \lesssim n^{D} \)

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for all $D$, then $G$ has intermediate growth. John Milnor asked in 1968 [Mil68] whether such groups existed, and the first example was constructed in the 1980’s by Grigorchuk [Gri83]; see Equation (7).

1.2. Amenability. A group is amenable [vN29] if it admits a finitely additive invariant measure. Examples are finite groups and abelian groups. Amenability is preserved by taking subgroups, quotients, extensions, and direct limits. The class $EG$ of elementary amenable groups are those obtained by these constructions, starting from finite and abelian groups. Groups of subexponential growth are also amenable; the elementary amenable groups of subexponential growth are all of polynomial growth [Cho80].

On the other hand, non-abelian free groups are not amenable, and hence we have a tower $\{\text{elementary amenable groups}\} \subseteq \{\text{amenable groups}\} \subseteq \{\text{groups with no free subgroup}\}$. Mahlon Day asked in [Day57] whether these inclusions are strict. The first one is, since Grigorchuk’s group of intermediate growth is not elementary amenable. The second one is also strict [Ol’80]; for example, the free Burnside group of exponent $n$ odd at least 665 is not amenable [Ady82].

Even in the class of finitely presented groups, both inclusions are strict: the Grigorchuk group can be embedded in a finitely presented amenable group[Gri98], and Alexander Ol’shanskii and Mark Sapir constructed in [OS01], for all sufficiently large odd $n$, a non-amenable finitely presented group satisfying the identity $[X,Y]^n$.

Following Pierre de la Harpe, Rostislav Grigorchuk and Tullio Ceccherini-Silberstein [CGH99, §14], we denote by $BG$ the smallest class of groups containing all groups locally of subexponential growth\footnote{i.e. whose finitely-generated subgroups have subexponential growth.}, and closed under taking subgroups, quotients, extensions and direct limits. I show in this note that $\Delta$ is amenable, but does not belong to $BG$.

1.3. Groups acting on trees. Although $\Delta$ is given by a finite presentation, it may also be defined by an action on the 3-regular tree. Let $U$ be the binary rooted tree. Among the many ways the 3-regular tree $T$ can be represented, we choose the following two:

- an infinite horizontal line, called the axis, with an edge hanging down at each integer coordinate, and a copy of $U$ attached to that edge’s other extremity;
- a copy of $U$, in which the root vertex has been removed and its two adjoining edges have been fused together.

The advantage of the first model is that it contains a natural hyperbolic element, namely the translation one step to the left along the axis. For any $n \in \mathbb{N}$, the set of vertices connecting to the axis at coordinate $\leq n$ span a subtree $T_n$ isomorphic to $U$.

We start by describing the action of $\Delta$ in the first model. Let $t$ act on $T$ by shifting one step to the left along the axis, and define the tree isometry $b$ as follows: first, its restriction $b_0$ to the rooted binary tree $T_0$ switches the downward and leftward branches at $-1 - 2n$ for every $n \in \mathbb{N}$, starting from $-\infty$ and moving towards 0. Next, identify each of the binary trees below $n > 0$ with $T_0$ in a translation-invariant way. Then $b$ fixes the half-axis $\mathbb{N}$ and the subtrees below $2n$ for every $n \in \mathbb{N}$, and acts on the binary tree below $1 + 2n$ like $b_0^n$ acts on $T_0$; see Figure 1.
Figure 1. The action of the generators $b$ and $t$ of $\Delta$ on the 3-regular tree $T$.

For convenience, in the sequel, we will always write $b$ for $b_0$ and $\tilde{b}$ for $b$. The action in the second model will be described in Subsection 2.4. Let me just remark that in that picture $t$ acts as one of the standard generators of the Thompson group [CFP96].

The stabilizer of 0 contains $\Gamma = \langle b, b_t^{-1} \rangle$, and $\Delta$ is an ascending HNN extension of $\Gamma$ by $t$. Indeed writing $a = b_t^{-1}$ we have $a^t = b$ and $b^t = a^2$ in $\Delta$.

The action of $\Gamma$ restricts to a faithful action on $U$, whose vertices can be naturally labelled by words over $\{1, 2\}$, with 1 corresponding to left and horizontal edges and 2 corresponding to right and vertical ones. The action can then be described by

$$
\begin{align*}
(1w)^a &= 2w^b, & (2w)^a &= 1w, & (1w)^b &= 1w^a, & (2w)^b &= 2w.
\end{align*}
$$

This is an example of a group generated by a finite-state automaton. A transducer is a tuple $A = (Q, X, \lambda, \tau)$ with $Q, X$ finite sets called states and letters, $\lambda : Q \times X \to X$ an output function and $\tau : Q \times X \to Q$ a transition function. A choice of initial state $q \in Q$ defines an action of $A_q$ on the tree $X^*$, by

$$
(jA_q)^a = (), \quad (xwA_q)^a = \lambda(q, x)wA_q^{(a, x)}.
$$

If each of these transformations is invertible, the group of $A$ is defined as the group $G(A)$ generated by $\{A_q\}_{q \in Q}$.

Automata can be described as graphs, with states as vertices, and an edge from $q$ to $\tau(q, x)$ labelled $x/\lambda(q, x)$ for all $q \in Q$ and $x \in X$. Figure 2 (top left) gives an automaton generating $\Gamma$, and Figure 2 (bottom left) gives an automaton generating the Grigorchuk group mentioned above and in Equation (7).

1.4. Automata groups. Automata groups are mainly studied using their decomposition map: given $g \in G$ acting on the rooted tree $X^*$, its action may be decomposed in $\#X$ actions on the subtrees connected to the root, followed by a permutation of the branches at the root. This induces a group homomorphism $\psi : G \to G \wr \mathfrak{S}_X$, written $\psi(g)(x) = \langle \langle g_x : x \in X \rangle \rangle \pi_g$, into a wreath product.

$^2\mathfrak{S}_X$ denotes the symmetric group on $X$; the wreath product is $G^X \rtimes \mathfrak{S}_X$. 
Two favourable situations may occur: first, the definition of an automaton is dual in that $X, Q$ and $\lambda, \tau$ may be switched simultaneously. If the dual automaton $A^*$ generates invertible transformations of $Q^*$, then the group

$$\Pi = \langle Q \cup X \mid xq = \tau(g,x)\lambda(q,x) \text{ for all } q \in Q, x \in X \rangle$$

naturally acts on the product of trees $F_X \times F_Q$, and the original group $G$ can be recovered as the quotient $\langle Q \rangle / \langle Q \rangle^{X^*}$.

An important example is the automaton $A$ in Figure 2 (bottom right); it is conjectured that the group it generates is free on $A$'s states, though the “proof” in [Ale83] appears to be incomplete.

Another favourable situation is the existence of a word metric $| \cdot |$ on $G$ such that $|gx| \leq \eta |g| + C$ for some $\eta < 1$ and all $g \in G$; this property, called contraction, opens the road to inductive proofs on word length.
If a group is contracting, then the projection map $g \mapsto g_x$ is not injective, so $x$, as a state of the dual automaton, cannot be invertible. It is in that sense that contraction and invertibility-of-dual are opposites.

The following notion is due to Volodymyr Nekrashevych [BGN02]. For an automaton group with states $Q$ and and alphabet $X$, construct the following graph $\Sigma(A)$ on the vertex set $X^*$; for all $w \in X^*$, $x \in X$, $s \in S$ it has an edge (of the \textit{first} kind) from $w$ to $w^s$, and an edge (of the \textit{second} kind) from $w$ to $xw$. The edges of the first kind span the tree $X^*$, and the edges of the second kind span the disjoint union of the Schreier graphs on $X^n$, for all $n \in \mathbb{N}$.

If $A$ has an invertible dual, then $\Sigma(A)$ is a quotient of the subset of $\Pi$’s Cayley graph spanned by $X^*$. On the other hand, if $G(A)$ is contracting, then $\Sigma(A)$ is Gromov-hyperbolic.

The \textit{limit space} of $G$ is then the hyperbolic boundary of $\Sigma(A)$. It can be defined as the equivalence classes of infinite rays in $\Sigma(A)$ mutually at bounded distance from each other.

An even stronger property than contraction is that $\sum_{x \in X} |g_x| \leq \eta|g| + C$ again for some $\eta < 1$ and all $g \in G$. Such a property implies that $G$ has subexponential growth (see Lemma 2.1).

A weaker, probabilistic version of this strong contraction property implies that $G$ is amenable. Namely, if given a uniformly distributed random group element of length $n$ the distribution of $\sum_{x \in X} |g_x|$ has mean less than $\eta n + C$ for some $\eta < 1$. There are groups (for instance $\Gamma$) that satisfy this probabilistic strong contraction property while have exponential growth; this occurs because even though there is strong contraction on average, the words in a geodesic normal form are very far from “average”.

1.5. \textbf{Notation.} For $a, b \in G$ and $x, y \in G \cup \mathbb{Z}$ we write

$$a^b = b^{-1}ab; \quad a^{x+y} = a^xa^y; \quad a^{xy} = (a^x)^y; \quad [a, b] = a^{-1+b} - b^{-a+1}. $$

2. Definitions and Statement of Results

In this section, $G$ denotes an arbitrary group, and $\Gamma$ denotes the specific example (1) generated by the automaton in Figure 2 (top left).

2.1. \textbf{Actions on rooted trees.} Fix a finite alphabet $X = \{1, \ldots, d\}$. The free monoid $X^*$ naturally has the structure of regular rooted tree, rooted at the empty word $\emptyset$, with an edge connecting $w$ to $wx$ for all $w \in X^*$ and $x \in X$. By $wX^*$ we mean the subtree isomorphic to $X^*$ and attached to the root $\emptyset$ of $X^*$ at its vertex $w$.

Let $W$ denote the automorphism group of $X^*$. Every $g \in W$ induces a permutation $\pi_g$ of $X$ by restriction, and $g\pi_g^{-1}$ fixes $X$, so induces for each $x \in X$ an automorphism $g_x$ of $xX^* \cong X^*$. We therefore have a wreath product decomposition, written

$$\psi : W \rightarrow W \wr \mathfrak{S}_X, \quad g \mapsto \langle g_1, \ldots, g_d \rangle \pi_g.$$ 

We will sometimes avoid $\psi$ from the notation for greater clarity. We also fix a $d$-cycle $(1, 2, \ldots, d) \in \mathfrak{S}_X$.

Let $S$ be a finite subset of $W$; assume that each $s \in S$ appears exactly once among the $s_x$ for $s \in S, x \in X$; that all other $s_x$ are trivial; and that $\pi_s$ is a power of the fixed $d$-cycle. We then call the group $G = \langle S \rangle$ a \textit{monomial group}. 


As examples on $X = \{1, 2\}$, we have:

- $S = \{\lambda^{\pm 1}, \mu^{\pm 1}\}$, with
  \[
  \lambda^{\psi} = \ll \lambda, 1 \gg (1, 2), \quad \mu^{\psi} = \ll \mu^{-1}, 1 \gg (1, 2).
  \]
  This is the “Brunner-Sidki-Vieira group” [BSV99], abbreviated BSV in the sequel. It is generated by the automaton in Figure 2 (top right).
- $S = \{a, b\}$, with
  \[
  a^{\psi} = \ll b, 1 \gg (1, 2), \quad b^{\psi} = \ll a, 1 \gg.
  \]
  This group was discovered by Richard Pink in connection with the Galois group of the iterates of the polynomial $z^2 - 1$. It will be called $\Gamma$ in the sequel.
- More generally, $S = \{a_1, \ldots, a_n\}$ with
  \[
  a_1^{\psi} = \ll a_n, 1 \gg (1, 2), \quad a_i^{\psi} = \ll a_{i-1}, 1 \gg \text{ or } \ll a_{i-1}, 1 \gg.
  \]
  These groups are the “iterated monodromy groups” of polynomials $z^2 + c$, with $c$ a periodic point in the Mandelbrot set — see Subsection 2.2.

A group $G$ acting on a rooted tree $X^*$ is \emph{fractal} if for every $w \in X^*$ the stabilizer of $w$ in $G$ maps to $G$ by restriction to and identification of $wX^*$ with $wX^*$ pointwise.

Assume $G$ is finitely generated, and let $| \cdot |$ denote a word metric on $G$. Then $G$ is \emph{contracting} if there are $\eta < 1$ and $C$ such that $|g_x| \leq \eta |g| + C$ holds for all $g \in G, x \in X$, and is \emph{strongly contracting} if there are $\eta < 1$ and $C$ such that $\sum_{x \in X} |g_x| \leq \eta |g| + C$ holds for all $g \in G$.

\begin{lemma} \textup{(Bar98).} \label{lem:contracting_growth}
Let $G$ be strongly contracting, with contraction constant $\eta$. Then $G$ has intermediate growth, and its growth function $\gamma$ satisfies

$$\gamma(n) \preceq e^{\alpha n}, \quad \text{with } \alpha = \frac{\log \# X}{\log(\# X/\eta)}.$$ 

\end{lemma}

By $H^X$ we denote the direct product of $\# X$ copies of $H < W$, acting independently on the subtrees $xX^*$ for all $x \in X$. Let $G$ be a fractal group. If it contains a non-trivial subgroup $K$ such that $K^\psi$ contains $K^X$ in its base group, then $G$ is \emph{weakly branch over} $K$; this implies that $G$ is weakly branch.

The main result of this note is the following:

\begin{theorem} \label{thm:main}
Let $G$ be a monomial group. Then $G$ is fractal. If $G^\psi$ maps to a transitive subgroup of $S_X$, and $\# S \geq 2$ with $S$ not of the form $\{a, a^{-1}\}$, then $G$ is weakly branch over $G'$. If $G'/G^\psi X$ is amenable, then $G$ is amenable.
\end{theorem}

I do not know whether all monomial groups have exponential growth; the last two examples do, and this question is open for the BSV group (4).

\section{Groups and covering maps}

The last example above (6) is a special case of a construction due to Volodymyr Nekrashevych [BGN02]. It was inspired by research by Richard Pink on Galois groups of iterated polynomials — see Point (8) in Theorem 2.3.
Let $f$ be a branched self-covering of a Riemann surface $\mathcal{S}$. A point $z \in \mathcal{S}$ is critical if $f'(z) = 0$, and is a ramification point if it is the $f$-image of a critical point. The postcritical set of $f$ is $\{f^n(z) : n \geq 1, z \text{ critical}\}$.

Assume $P$ is finite, and write $\mathcal{M} = \mathcal{S} \setminus P$. Then $f$ induces by restriction an étale map of $\mathcal{M}$.

Let $*$ be a generic point in $\mathcal{M}$, i.e. be such that the iterated inverse $f$-images of $*$ are all distinct. If $f$ has degree $d$, then for all $n \in \mathbb{N}$ there are $d^n$ points in $f^{-n}(*)$, and all these points naturally form a $d$-regular tree $U$ with root $*$ and an edge between $z$ and $f(z)$ for all $z \in U \setminus \{*, 0\}$. Denote by $X = f^{-1}(*)$ the first level of $U$.

Let $\gamma$ be a loop at $* \in \mathcal{M}$. Then for every $v \in U$ at level $n$ there is a unique lift $\gamma_v$ of $\gamma$ starting at $v$ such that $f^n(\gamma_v) = \gamma$; and furthermore the endpoint $v^\gamma$ of $\gamma_v$ also belongs to the $n$th level of $U$.

For any such $\gamma$ the map $v \mapsto v^\gamma$ is a tree automorphism of $U$, and depends only on the homotopy class of $\gamma$ in $\pi_1(\mathcal{M}, *)$. We therefore define the \textit{iterated monodromy group} $G_U(f)$ of $f$ as the subgroup of $\text{Aut}(U)$ generated by all maps $v \to v^\gamma$, as $\gamma$ ranges over $\pi_1(\mathcal{M}, *)$.

This definition is actually independent of the choice of $*$: if $*'$ is another generic basepoint, with tree $U'$, then choose a path $p$ from $*$ to $*'$ and then $\phi : U \to U'$ such that

$$\begin{array}{c}
\pi_1(\mathcal{M}, *) \xrightarrow{p^*} \pi_1(\mathcal{M}, *') \\
\downarrow \text{act} \quad \downarrow \text{act} \\
G_U(f) \xrightarrow{\phi_*} G_{U'}(f)
\end{array}$$

commutes; we write $G(f)$ for $G_U(f)$, defined up to conjugation in $\text{Aut}(U)$. Abstractly, $G(f)$ is a presented as a quotient of $\pi_1(\mathcal{M}, *)$.

We now identify $U$ with the standard tree $X^*$. Enumerate $X = \{v_1, v_2, \ldots, v_d\}$, and choose for each $v \in X$ a path $\ell_v$ from $*$ to $v$ in $\mathcal{M}$. Consider a loop $\gamma$ at $*$; then it induces the permutation $v \mapsto v^\gamma$ of $X$, and for each $v \in X$ its lift $\gamma_v$ at $v$ yields a loop $\ell_v \gamma_v \ell_v^{-1}$ at $*$, which depends only on the class of $\gamma$ in $G(f)$.

We therefore have a natural wreath product decomposition (3)

$$\phi : G(f) \to G(f) \wr S_X, \quad g \mapsto \langle g_1, \ldots, g_d \rangle \pi_g,$$

where, if $g$ is represented by a loop $\gamma$ and $v = v_i \in X$, we have $v^\gamma = v^{g_i}$ and $g_i$ is the class of $\ell_v \gamma_v \ell_v^{-1}$ in $\pi_1(\mathcal{M}, *)$.

Consider a polynomial self-mapping of the Riemann sphere $f(z) = z^N + c \in \mathbb{C}[z]$ such that $f^N(0) = 0$ for some $N \in \mathbb{N}$. Then $G(f)$ is a monomial group, acting on $X^*$ where $\#X = d$. The only example I consider here is $f(z) = z^2 - 1$; it gives the group $G(f) = \Gamma$.

The postcritical set $P$ is $\{0, 1, \infty\}$, so $\mathcal{M}$ is a thrice-punctured sphere and $G$ is $2$-generated.

For convenience, pick as base point $*$ a point close, but not equal, to $(1 - \sqrt{5})/2$; then $X = \{x, y\}$ with $x$ close to $*$ and $y$ close to $-*$.

Consider the following representatives of $\pi_1(\mathcal{M}, *)$'s generators: $a$ is a straight path approaching $-1$, turning a small loop in the positive orientation around $-1$, and returning to $*$. similarly, $b$ is a straight path approaching $0$, turning around $0$ in the positive orientation, and returning to $*$.
Let $\ell_x$ be a short arc from $*$ to $x$, and let $\ell_y$ be a half-circle above the origin from $*$ to $y$.

Let us compute first $f^{-1}(a)$, i.e. the path traced by $\pm \sqrt{z + 1}$ as $z$ moves along $a$. Its lift at $x$ moves towards 0, passes below it, and continues towards $y$. Its lift at $y$ moves towards 0, passes above it, and continues towards $x$. We have $a_x = b$ and $a_y = 1$, so the wreath decomposition of $a$ is $\phi(a) = \langle b, 1 \rangle \langle 1, 2 \rangle$.

Consider next $f^{-1}(b)$. Its lift at $x$ moves towards $-1$, loops around $-1$, and returns to $x$. Its lift at $y$ moves towards 1, loops and returns to $y$. We have $b_x = a$ and $b_y = 1$, so the wreath decomposition of $b$ is $\phi(b) = \langle a, 1 \rangle$.

These paths are presented in Figure 3.

2.3. $\Gamma$ and $\Delta$. The next claims concern only the specific example $\Gamma = \langle a, b \rangle$. For clarity, its action (5) on $\{1, 2\}^*$ is given by

$(1w)^a = 2w^b$, $(2w)^a = 1w$, $(1w)^b = 1w^a$, $(2w)^b = 2w$.

**Theorem 2.3.** The group $\Gamma$ is

1. fractal, contracting, and weakly branch over $\Gamma'$;
2. torsion free;
3. of exponential growth, containing $\{a, b\}^*$ as a free submonoid;
4. has as quotients along its lower central series $\gamma_1/\gamma_2 = \mathbb{Z}^2$, $\gamma_2/\gamma_3 = \mathbb{Z}$, and $\gamma_3/\gamma_4 = \mathbb{Z}/4$. Therefore all successive quotients except the first two in the lower central series are finite.

In the lower 2-central series\(^3\) defined by $\Gamma_1 = \Gamma$ and $\Gamma_{n+1} = [\Gamma, \Gamma_n]\Gamma_{[n/2]}^2$, we have

$$\dim_{\mathbb{F}_2} \Gamma_n/\Gamma_{n+1} = \begin{cases} i + 2 & \text{if } n = 2^i \text{ for some } i; \\ \max \{i + 1 \mid 2^i \text{ divides } n\} & \text{otherwise.} \end{cases}$$

5. right-orderable, but not bi-orderable\(^4\);
6. not solvable; however, every proper quotient of $\Gamma$ is nilpotent-by-(finite 2-group), and every non-trivial normal subgroup of $G$ has a subgroup mapping onto $\Gamma$;
7. has solvable word problem, and is recursively presented as

$$\Gamma = \langle a, b \mid [a^p, b^p], [b^p, a^{2p}], a^{2p} \rangle \text{ for all } p \text{ a power of } 2;$$

its Schur multiplier is $H_2(G, \mathbb{Z}) = \mathbb{Z}^\infty$.

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\(^3\)aka “Zassenhaus series”, “Jennings series”, “Lazard series”, or “dimension series”

\(^4\)i.e., there is a total order $\leq$ on $\Gamma$ with $x \leq y \Rightarrow xz \leq yz$ for all $x, y, z \in \Gamma$, but there is no order satisfying $x \leq y \Rightarrow wxz \leq wyz$ for all $w, x, y, z \in \Gamma$. 
(8) Set \( f(z) = z^2 - 1 \). Then the closure of \( \Gamma \) in the profinite group \( W \) is the Galois group of \( \Lambda / \mathbb{C}(z) \), where \( \Lambda = \bigcup_{n \geq 0} \Lambda_n \) and \( \Lambda_n \) is the splitting field of \( f^n(t) - z \) over \( \mathbb{C}(z) \). It has Hausdorff dimension \( 2/3 \), and contains the BSV group;

(9) \( \Gamma \) has as limit space the Julia set \( J \) of \( z^2 - 1 \); the Schreier graphs of the action of \( \Gamma \) on \( X^* \) are planar, and can be metrized so as to converge to \( J \) in the Gromov-Hausdorff metric;

(10) The spectrum of the Hecke-type operator\(^5 \) \( \frac{1}{2}(a + a^{-1} + b + b^{-1}) \) on \( L^2(X^*, \mu) \) is a Cantor set of null measure, while its spectrum on \( \ell^2(\Gamma) \) is the interval \([-1, 1]\).

Consider the endomorphism \( \sigma : \Gamma \to \Gamma \) given by \( a^* = b, b^* = a^2 \), and form the HNN extension \( \Delta = \langle \Gamma, t | a^t = b, b^t = a^2 \rangle \).

**Theorem 2.4.** \( \Gamma \) and \( \Delta \) are amenable, but do not belong to \( \text{BG} \).

\( \Gamma \) is infinitely presented, and \( \Delta \) has a balanced, finite presentation

\[
\Delta = \langle b, t | b^{t^2 - 2}, [[[b, t^{-1}], b], b] \rangle.
\]

2.4. **Transitive actions on trees.** We now consider extension of actions on rooted trees to transitive actions on regular trees containing the original rooted tree.

**Theorem 2.5.** Let \( G \) act on \( X^* \) and be weakly branch over \( K \). Assume that the map \( K \to K \times 1 \times \cdots \times 1 \) given by \( k \mapsto \ll k, 1, \ldots, 1 \gg \) lifts to an endomorphism \( \sigma : g \mapsto \ll g, *, \ldots, * \gg \) of \( G \). Then the HNN extension \( \tilde{G} = \langle G, \sigma \rangle \) acts transitively on a \((\#X + 1)\)-regular tree; \( \sigma \) is a hyperbolic translation, and \( G \) is a split quotient of the stabilizer of a vertex \(*\) on \( \sigma \)'s axis. Deleting from \( * \) the edge on \( \sigma \)'s axis and keeping the connected component of \( * \) gives a \( \#X \)-regular rooted tree carrying \( G \)'s original action.

If furthermore \( G \) is contracting, and \( G/K \) and \( K/K^X \) are both finitely presented, then \( \tilde{G} \) is finitely presented\(^6 \).

This result applies to the Pink group \( \Gamma \), to the BSV group (see Equation (4) or Figure 2 (top right)), and to the Grigorchuk group; we start with \( G = \Gamma \).

Consider a 3-regular tree \( T \); it can be viewed as a rooted binary tree \( U = \{1, 2\}^* \), in which the root vertex \( \emptyset \) was removed, and its two adjacent edges were replaced by a new edge \( e \) joining their extremities \( 1, 2 \); conversely, a binary tree isomorphic to \( U \) is obtained by inserting a root vertex in the middle of an edge. Consider the

\(^5\) The automorphism group \( W \) of \( X^* \) is a profinite group, a basis of neighbourhoods of the identity being given by the family of pointwise fixators \( W_n \) of \( X^n \). For a subgroup \( G \) of \( W \), its Hausdorff dimension \([\text{BS97}]\) is defined by

\[
\dim(G) = \lim_{n \to \infty} \frac{|GW_n/W_n|}{|W/W_n|}.
\]

\(^6\) i.e. the operator defined as the averaged sum of the generators of a group in a unitary representation. Here \( \ell^2(\Gamma) \) denotes the left-regular representation of \( \Gamma \) by left-multiplication on the space of square-summable functions on \( \Gamma \), and \( L^2(X^*, \mu) \) denotes the “natural” representation of \( \Gamma \) by permutation on the space of square-integrable functions on the boundary of the tree \( X^* \), with \( \mu \) the Bernoulli measure.

\(^7\) There is a standard presentation, due to Bass and Serre \([\text{Ser80}]\), for a group acting transitively on a tree. This result should be understood in that spirit.
automorphisms $c, d \in W = \text{Aut} U$ given by
\[c^\psi = \langle \langle b, d^2 \rangle \rangle, \quad d^\psi = \langle 1, c \rangle.\]
The pointwise fixator of $e$ is $W \times W$ acting disjointly on $1U$ and $2U$; we still write $\langle g_1, g_2 \rangle$ its elements. Extend the action of $\Gamma$ to isometries of $T$ fixing $e$ by letting $a$ act as $\tilde{a} = \langle \langle a, c \rangle \rangle$ and letting $b$ act as $\tilde{b} = \langle \langle b, d \rangle \rangle$. Note that $[c, d] = 1$ so the subgroup of $\text{Aut} T$ generated by $\{\tilde{a}, \tilde{b}\}$ is still $\Gamma$. Let $t$ act by shifting toward the root in $2U$ along $2^\infty$, crossing $e$, and shifting away from the root in $1U$ along $1^\infty$. In symbols, we have
\[(22w)^t = 2w, \quad (21w)^t = 12w, \quad 2^t = 1, \quad (1w)^t = 11w,\]
and conjugation by $t$ is given in $\Delta$ by $\langle \langle x, \langle x, y, z \rangle > \rangle^t = \langle \langle x, y, z \rangle \rangle$. It is then easy to check that $\langle \langle b, t \rangle \rangle = \Delta$ in this action, described in Figure 1. The actions in this setting are given in Figure 4.

The Thompson group $F$ is the group of piecewise linear orientation-preserving self-homeomorphisms of $[0, 1] \cap \mathbb{Z}[\frac{1}{2}]$; see [GS87, CFP96]. It has a finite, balanced presentation
\[F = \langle t, u | [tu^{-1}, u^t], [tu^{-1}, u^{t^2}] \rangle.\]
It is known that $F$ is torsion-free, not in the class $\text{EG}$, and does not contain any non-abelian free subgroup; however, it is open whether $F$ is amenable.

$[0, 1] \cap \mathbb{Z}[\frac{1}{2}]$ can be identified with $U = \{1, 2\}^*$ by mapping the dyadic number $0.x_1 \ldots x_n$ to $x_1 \ldots x_n$. In this way $F$ acts by homeomorphisms on the boundary of $U$. This action is described in Figure 4; note that the generator $t$ of $F$ acts in the same way as the generator $t$ of $\Delta$, and $u = \langle \langle t, 1 \rangle \rangle$ in our notation — but beware that $u$ is not an isometry of $T$. The arguments in [Röv99] show that $\langle \langle b, t, u \rangle \rangle$ is a finitely presented simple group. I do not know whether it is amenable, though this question is probably harder than the corresponding one for $F$.

Consider next the Grigorchuk group $G$ from Figure 2 (bottom left). It may be defined as $G = \langle a, b, c, d \rangle$ acting on $\{1, 2\}^*$, with
\[(7) \quad a^\psi = \langle \langle 1, 1 \rangle \rangle(1, 2), \quad b^\psi = \langle \langle a, c \rangle \rangle, \quad c^\psi = \langle \langle a, d \rangle \rangle, \quad d^\psi = \langle \langle 1, b \rangle \rangle.\]
This group is contracting, and even strongly contracting [Bar98], with $|g_1| + |g_2| \leq 7 \eta(|g| + 1)$ for $\eta \approx 0.811$ the real root of $X^3 + X^2 + X - 2$. It is therefore of intermediate growth, of rate at most $e^{0.768}$, and hence is amenable. It embeds...
in the finitely presented group $\tilde{G} = \langle G, t \rangle$, with $t$ acting by conjugation as the endomorphism $\sigma : G \to G$ given by

$$a^\sigma = c^a, \quad b^\sigma = d, \quad c^\sigma = b, \quad d^\sigma = c.$$ 

Consider the following isometries of the 3-regular tree $T$ described above:

$$\tilde{a} = \langle a, d, \langle a^d, d \rangle \rangle, \quad \tilde{b} = \langle b, d \rangle, \quad \tilde{c} = \langle c, c \rangle, \quad \tilde{d} = \langle d, b \rangle.$$ 

Then $G \cong \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle$, and $\tilde{G}$ is generated by $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ and a hyperbolic element $t$ moving toward the root along $2^\infty$ in $\mathbf{2}U$ and away from the root along $2^\infty$ in $\mathbf{1}U$; we have $\langle \langle x, \langle y, z \rangle \rangle \rangle^t = \langle \langle x, \langle y \rangle \rangle \rangle$. The action of $\tilde{G}$ is described in Figure 5.

A presentation of $\tilde{G}$ with 2 generators and 4 relators, obtained using $c = a^t$, $b = a^{tat}$, $d = a^{tat^2}$, is

$$\tilde{G} = \langle a, t | a^2, a^{tat^2+tat+t}, a^{(1+ta)8}, a^{(1+ta+2)(1+ta)4} \rangle.$$ 

2.5. Reddite Caesare. Some of the results in Theorem 2.3 were obtained independently by Rostislav Grigorchuk and Andrzej Żuk, whom the author thanks for their communication. The proof technique follows ideas appearing in the original works of Rostislav Grigorchuk, Said Sidki [BSV99] and Edmeia da Silva [Sil01]. The author is also extremely grateful to Professors de la Harpe, Grigorchuk and Nekrashevych for their generous sharing of knowledge and ideas.

3. Proofs

We use $S$ as a natural generating set of $G$, and write $|w|$ the length of a word, and $|g|$ the minimal length of a group element. Most of the proofs follow by induction on $|g|$.

**Proof of Theorem 2.2.** Write $K = G'$. For $s, t \in S$, pick $s', t' \in S$ such that $s' = s, t' = t$, and let $n$ be the order of $\pi_n$. Then $[s', (t')^n]^v$ will have precisely one non-trivial coordinate, containing $[s, t]$. By conjugating, $K$ contains $K^X$. Finally $K \neq 1$ by our assumption.

Consider next the set $F_n$ of freely reduced words of length $n$ over $S$, and the subset $N_n$ of words evaluating to 1 in $G$. $F = \bigcup_{n \geq 0} F_n$ is the free group on $S$, and $N = \bigcup_{n \geq 0} N_n$ is the kernel of the natural map $F \to G$.

**Lemma 3.1** (Kesten [Kes59]; Grigorchuk [Gri80]). $G$ is amenable if and only if $\#N_n/\#F_n > \rho^n$ for all $\rho < 1$ and all $n$ even and large enough.

The decomposition map $\psi : G \to G \setminus \langle (1, \ldots, d) \rangle$ induces a map $F \to F \setminus \langle (1, \ldots, d) \rangle$ on freely reduced words, again written $w^\psi = \langle \langle w_1, \ldots, w_d \rangle \rangle \pi_w$. By construction, we have $|w_1| + \cdots + |w_d| \leq |w|$; and usually the inequality is strict: since $G$ is weakly branch, there are non-trivial reduced words $u, v$ with $u_x = 1$ for all $x \neq 1$ and $v_1 = 1$; then $w = [u, v]$ has positive length but $w_1 = \cdots = w_d = 1$.

The cancellation that occurs in the $w_d$ is determined by the local rules specifying the decomposition of generators. Therefore, if $w$ is chosen uniformly at random in $F_n$ with $n$ large, then $w_x$ will again be uniformly distributed within $F_{|w_x|}$, and the length of each $w_x$ will follow a binomial distribution; hence $|w_1| + \cdots + |w_d|$ will also follow a binomial distribution.
Figure 5. The action of the generators $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, t$ of $\tilde{G}$ on the 3-regular tree $T$.

Assume that the mean of this distribution is $\mu n$ and its variance is $\mu(1 - \mu)n/\eta$. This means that the probability that $w$ of length $n$ yields via $\psi$ freely reduced words $w_1, \ldots, w_d$ of total length $m$ is

\[(8) \quad C_{m,n}^{\mu,\eta} = \eta^{\eta m}(1 - \mu)^{\eta(n-m)}.\]

By the above argument we have $\mu < 1$, although the precise value is unimportant for the present.
For a subgroup $A < G$ we write $p_A(n)$ the probability that $w$ chosen uniformly at random in $F_n$ evaluates to $1 \in G$, conditionally on knowing that $w$ evaluates to an element of $A$. For $A < B < G$ we write $p_{A/B}(n) = p_A(n)/p_B(n)$.

Assume now for contradiction that $p(n)$ decays exponentially at rate $\rho$, say $(\rho - \epsilon)^n < p(n) < (\rho + \epsilon)^n$ for any $\epsilon > 0$, provided $n$ is large enough. Since $G/K$ is abelian, $p_{G/K}(n)$ decays subexponentially, so we also have $(\rho - \epsilon)^n < p_K(n) < (\rho + \epsilon)^n$. Then for large $n$

$$(\rho + \epsilon)^n \geq p(n) \geq p_{G/K}(n)p_K(n),$$

and $p_K(n) \geq p_{K/K^*}(n) \sum_{0 \leq m \leq n} C_{n,m}^\rho p_K(i_1) \cdots p_K(i_d)$;

Writing $E(n)$ a function that decays subexponentially, and takes into account both the $\approx \binom{n}{d}$ ways of partitioning $m$ in $d$ parts, and $p_{K/K^*}(n)$, of subexponential decay since $K/K^*$ is assumed to be amenable,

$$p_K(n) \geq E(n) \sum_{m=0}^n \binom{\eta m}{\eta m} \mu^m (1 - \mu)^{n-m} (\rho - \epsilon)^n$$

$$\approx E(n) \sum_{m'=0}^{\eta m} \binom{\eta m'}{\eta m'} (1 - \mu)^{n-m'} (\rho - \epsilon)^n$$

$$= E(n) (1 - \mu)^n (1 - \mu + \sqrt[]{\rho - \epsilon})^n.$$

Letting $n$ tend to $\infty$ and taking $n$th roots, we get $\sqrt[]{\rho - \epsilon} \geq (1 - \mu) + \sqrt[]{\rho - \epsilon}$ and hence $\rho \geq 1$, since $\epsilon > 0$ is arbitrary, $\mu < 1$, and $\eta > 0$.

We note that the parameters $\mu, \eta$ were experimentally found to be $\mu \approx 0.699$ and $\eta \approx 0.326$ for the Pink group $\Gamma$, and $\mu \approx 0.781$ and $\eta \approx 0.282$ for the BSV group. These values were obtained by a Monte-Carlo simulation using $10 000 000$ words of length $50 000$.

I now proceed with the proof of Theorem 2.3 describing algebraic properties of $\Gamma$. Alternate proofs of some of the points were found independently by Grigorchuk and Žuk, and appear in [GZ].

For convenience, we write $c = [a, b]$, $d = [c, a]$ and $e = [d, a]$ in $\Gamma$.

**Point (1) of Theorem 2.3.** $\Gamma$ is fractal and weakly branch by Theorem 2.2. By letting $a$ have length $1$ and $b$ have length $\sqrt[]{2}$, we have $|g_\alpha| \leq (|g| + 1)/\sqrt[]{2}$ for all $g \in G$, $x \in X$; hence $\Gamma$ is contracting.

Next, we prove by induction on length of words that we have $\Gamma/\Gamma' \approx \mathbb{Z}^2$ generated by the images of $a, b$ and $\Gamma'/\Gamma' \times \Gamma' \approx \mathbb{Z}$, generated by the image of $c$.

Assume for contradiction that $a^mb^n \in \Gamma'$ with $|m| + |n|$ minimal. Then clearly $m$ is even, say $m = 2p$. We have, for some $k \in \mathbb{Z}$ with $|k| \leq |m| + |n|$,

$$a^mb^n = \langle b^p a^n, b^p \rangle = \langle g, h \rangle c^k = \langle g a^k, h a^{-bk} \rangle$$

and therefore $a^{-nk}b^p$ and $a^kb^p$ both belong to $\Gamma'$. This contradicts our assumption on minimality.

Assume next that $c^k \in \Gamma' \times \Gamma'$ with $|k|$ minimal. Then $a^k \in \Gamma'$ which contradicts the second claim. \qed
**Point (2).** Since \( \Gamma \) acts on the binary tree, it is residually a 2-group, and its only torsion must be 2-torsion. Assume for contraction that \( \Gamma \) contains an element \( g \) of order 2, of minimal norm.

By the previous point, \( a \) and \( b \) are of infinite order. We may therefore assume \( |g| \geq 2 \). If \( g \) fixes \( 1 \), then its restrictions \( g_x, x \in X \) are shorter, and at least one of them has order 2, contradicting \( |g|'s \) minimality.

If \( g \) does not fix \( x \), then we may write \( g = \langle g_1, g_2 \rangle a \) for some \( g_1, g_2 \in \Gamma \). We then have \( h = g^2 = \langle g_1 b g_2, g_2 g_1 b \rangle = 1 \) and therefore \( g_2 g_1 b = 1 \). Now for any element \( h \) fixing \( 1 \) we have \( h_1 h_2 \in \langle a, b^2, \Gamma' \rangle \); this last subgroup does not contain \( b \) by the previous Point, so we have a contradiction. □

**Point (3).** Consider two words \( u, v \) in \( \{a, b\}^* \) that are equal in \( \Gamma \), and assume \( |u| + |v| \) is minimal. We have \( u_1 = v_1 \) and \( u_2 = v_2 \) in \( \Gamma \), which are shorter relations, so we may assume these words are equal by induction.

Now if \( u_1 \) and \( v_1 \) start with the same letter \( a \) or \( b \), this implies that \( u \) and \( v \) also start with the same letter \( b \) or \( a \) respectively, and cancelling these letters would give a shorter pair of words \( u, v \) equal in \( \Gamma \).

It follows that \( \{a, b\}^* \) is a free submonoid, and hence that \( \Gamma \) has exponential growth. □

**Point (4).** This follows from writing generators for \( \gamma_i \), and using induction on length. Writing \( c = [a, b], d = [c, a] \) and \( e = [d, a] \),

\[
\begin{align*}
\gamma_1 &= (a, b); \\
\gamma_2 &= (c = [a, b] = (a, a^{-b}), c^{-1} - a = \langle c, 1 \rangle, c^{-a^{-1}} - 1 = \langle 1, c \rangle); \\
\gamma_3 &= (d = [c, a], e = [d, a], [e^{-1}, b] = \langle d, 1 \rangle, \langle e, 1 \rangle, \langle 1, d \rangle, \langle 1, e \rangle); \\
\gamma_4 &= (d^4, e, \langle d, 1 \rangle, \langle e, 1 \rangle, \langle 1, d \rangle, \langle 1, e \rangle).
\end{align*}
\]

Only \( d^4 \in \gamma_4 \) deserves some justification; writing \( \equiv \) for congruence modulo \( \gamma_4 \), we have

\[
\begin{align*}
d^2 &\equiv d^2 e = b^{-1} a^{-1} b a^{-2} b^{-1} a b a^2 \\
&= b^{-1} a b a^{-2} b^{-1} a^{-1} b a^2 \text{ using the relation } [a^{2b}, a^2] = 1 \\
&= (d^2 e)^{-a-b} \equiv d^{-2}.
\end{align*}
\]

For the 2-central series see [Bar02a], where the same answer is proven for the BSV group. □

**Point (5).** We consider for all \( n \in \mathbb{N} \) the subgroups \( \Gamma_n = (\Gamma')^\times_n \) of \( \Gamma \). Then \( \Gamma / \Gamma_0 \cong \mathbb{Z}^2 \) and \( \Gamma_n / \Gamma_{n+1} \cong \mathbb{Z}^2 \) are both right-orderable, and \( \cap_{n \geq 0} \Gamma_n = 1 \). Define a right order on \( \Gamma \) by

\[
x \leq y \iff x = y \text{ or } xy^{-1} < 1 \text{ in } \Gamma_n / \Gamma_{n+1}, \text{ where } n \text{ is maximal with } xy^{-1} \in \Gamma_n.
\]

Note that this is not a bi-ordering, since \( \Gamma_{n+1} \) is not central in \( \Gamma_n \). That no bi-ordering exists follows from \( d^2 e \) being conjugate to \( (d^2 e)^{-1} \), see Point (4).

**Point (6).** The calculations in Point (4) show that \( \Gamma'' \) is the normal closure of \( [c, \langle c, 1 \rangle] = \langle d, 1 \rangle \); therefore \( \Gamma'' = \gamma_3 \times \gamma_3 \), and so \( G'' > G'' \times G'' \); hence \( G^{(n)} > G^{(n-1)} \times G^{(n-1)} \) for all \( n \). Assume for contradiction that \( G \) is solvable; this means \( G^{(n)} = 1 \) for some minimal \( n \), a contradiction with the above statement.
Now consider a non-trivial normal subgroup \( N \) of \( G \). By [Gri00, Theorem 4], we have \((\gamma_3)^{X^n} < N \) for some \( n \). Since \( G \) is an abelian-by-(finite 2) extension of \((\gamma_3)^{X^n}\), we conclude that \( G/N \) is nilpotent-by-(finite 2).

To show that a subgroup of \( N \) maps onto \( \Gamma \), since \( N \) contains \((\gamma_3)^{X^n}\), it is sufficient to show that \( \gamma_3 \) maps onto \( \Gamma \). Now \( \gamma_3 \) is the normal closure of \( d = [[a, b], a] \) in \( \Gamma \). We have \( d^\psi = \langle a^{-1-b}, a^{2b} \rangle \), and \( a^{(1-b)\psi} = \langle b^{-1}a, b^{-1}a^{-1} \rangle \) and \( a^{2b\psi} = \langle b^a, b \rangle \); therefore projection twice on the first factor maps \( \gamma_3 \) to \( \Gamma \).

**Point (7).** Let \( F \) be the free group on \( \{a, b\} \), and write \( \Gamma = F/R \). Then (5) defines a homomorphism \( F \to F/G \). Letting \( \sigma \) denote the \( F \)-endomorphism \( a \mapsto b, b \mapsto a^2 \) we have a diagram

\[
\begin{array}{c}
\langle a^2, b, b^a \rangle \\
\downarrow R^n \downarrow \downarrow R \times R \downarrow \downarrow R \times 1
\end{array}
\]

Set \( R_0 = 1 < F \) and inductively \( R_{n+1} = (R_n \times R_n)^{\pi^{-1}} \). Then \( R = \bigcup_{n \geq 0} R_n \) because \( \Gamma \) is contracting, and we have \( R_{n+1} = (R_1 R_n^n)^F \). Since \( R_1 = \langle [b, b^a] : \text{odd } i \rangle F \), we have \( R = \langle [b^{2^i}, b^{\sigma^i}] : n \in \mathbb{N}, \text{ odd } i \rangle F \). Now \( b^{2i} = [b^{-1}, a^{-2}]^{a^{2i}} b^{a^{2i-2}} \) using the relation \([a^{2^i}, a^2] = [b^a, b]^\sigma\); therefore \( b^{2i}, b \) follows from \([a^{2^i}, b] \) and \([b^{2i-2}, a^{-2}] b, [b^{-1}, a^{-2}]^{a^2}, b \) which in turn is a consequence of \([b^a, b] \); the presentation of \( \Gamma \) follows.

The Schur multiplier of \( G \) is \((R \cap [F, F])/(R, F)\), by Hopf’s formula. Writing \( R = \langle [b, b^a] \rangle R^\sigma \), we get

\[
R/[R, F] = \langle [b, b^a] \rangle \oplus R^\sigma/[R, F]^\sigma,
\]

so \( H^2(G, \mathbb{Z}) \cong \mathbb{Z} \) with \( \sigma \) acting on it as a one-sided shift.

**Point (8).** Write \( Q_n \) the quotient \( GW_n/W_n \). Induction shows that \( a \) has order \( 2^{[n/2]} \) in \( Q_n \) and \( b \) and \( [a, b] \) have order \( 2^{[n/2]} \) in \( Q_n \); hence \( Q_n' \) has index \( 2^n \) in \( Q_n \) and \( Q_{n-1}' \times Q_{n-1}' \) has index \( 2^{[n/2]} \) in \( Q_n' \). Since \( |Q_0'| = 1 \), we get

\[
|Q_n| = 2^{2^{n-1}}, \quad \text{since } |W/W_n| = 2^{2^{n-1}}, \quad \text{we have } \dim(G) = \frac{2}{3}.
\]

The generator \( \mu \) of the BSV group is \( b^{-1}a \in \Gamma \), since \( b^{-1}a = \langle a^{-1}b, 1 \rangle \) satisfies \( \mu \)'s recursion (4). We do not have \( \tau \in \Gamma \); but defining \( c_n \in \Gamma \) by the recursion \( c_0 = 1 \) and \( c_{n+1} = [a, b] c_n \), we have \( bac_{n+1} = \langle ab[b, a] c_n, 1 \rangle \) (1, 2); hence setting \( c = \lim_{n \to \infty} c_n \in \Gamma \), we have \( babc = \langle abc, 1 \rangle \) (1, 2) = \( \tau \in \Gamma \).

**Point (9).** Consider next the Schreier graphs \( \mathfrak{S}_n \) with \( X^n \) as the vertex set. \( \mathfrak{S}_n \) is constructed as follows: it is built of two parts \( A_n, B_n \) connected at a distinguished vertex. Each of these parts is 4-regular, except at the connection vertex where each is 2-regular, and \( A_n \) contains only the \( a^{\pm 1} \)-edges while \( B_n \) contains only the \( b^{\pm 1} \)-edges.

\( A_0 \) and \( B_0 \) are the graphs on 1 vertex with a single loop of the appropriate label.
If \( n = 2k \) is even, then \( B_{2k+1} = B_{2k} \), and \( A_{2k+1} \) is obtained by taking an \( a \)-labelled \( 2^{k+1} \)-gon \( v_0, \ldots, v_{2k+1} \), and attaching to each \( v_i \) with \( i \neq 0 \) a copy of \( B_{2j} \) where \( 2! \| i \). Its distinguished vertex is \( v_0 \).

If \( n = 2k - 1 \) is odd, then \( A_{2k} = A_{2k-1} \), and \( B_{2k} \) is obtained by taking a \( b \)-labelled \( 2^k \)-gon \( v_0, \ldots, v_{2k-1} \), and attaching to each \( v_i \) with \( i \neq 0 \) a copy of \( A_{2j+1} \) where \( 2! \| i \). Its distinguished vertex is \( v_0 \).

The first Schreier graphs \( \Theta_n \) of \( G \) are drawn in Figure 6. Compare with the Julia set in Figure 7.

**Point (10).** Consider first the spectrum on \( \mathcal{H} = L^2(X^\omega, \mu) \). Since \( X^\omega = 1 \times X^\omega \sqcup 2 \times X^\omega \), we may decompose \( a, b \) and write them as \( 2 \times 2 \)-matrices over \( B(\mathcal{H}) \). We have \( a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \); finite approximations \( a_n, b_n \) can be obtained by expanding to \( 2^n \times 2^n \)-matrices and replacing all \( a \)'s and \( b \)'s by 1; we have

\[
a_0 = b_0 = (1), \quad a_{n+1} = \begin{pmatrix} 0 & b_n \\ 1 & 0 \end{pmatrix}, \quad b_{n+1} = \begin{pmatrix} a_n & 0 \\ 0 & 1 \end{pmatrix}.
\]

Introduce for \( n \geq 0 \) the following homogeneous polynomials of degree \( 2^n \):

\[
Q_n(\lambda, \mu, \nu) = \det \left( \lambda + \mu (a_n + a_n^{-1}) + \nu (b_n + b_n^{-1}) \right).
\]

Then the solution of \( Q_n(\lambda, -\frac{1}{4}, -\frac{1}{4}) = 0 \) is the spectrum of the Hecke-type operator \( \frac{1}{4} \left( a_n + a_n^{-1} + b_n + b_n^{-1} \right) \) of \( \Gamma \)'s action on \( \mathbb{C}X^n \); this is also the spectrum of the Schreier graph \( \Theta_n \).

Define the homogeneous polynomial mapping \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) by

\[(\lambda, \mu, \nu) \mapsto (\lambda^2 + 2\lambda\nu - 2\mu^2, \lambda\nu + 2\nu^2, -\mu^2).
\]

Then \( Q_n \) is given, for \( n \geq 1 \), by

\[
Q_0(\lambda, \mu, \nu) = \lambda + 2\mu + 2\nu;
\]

\[
Q_1(\lambda, \mu, \nu) = Q_0(\lambda, \mu, \nu) \cdot (\lambda - 2\mu + 2\nu);
\]

\[
Q_{n+1}(\lambda, \mu, \nu) = \det \begin{pmatrix} \lambda + \nu (a_n + a_n^{-1}) & \mu (1 + b_n) \\ \mu (b_n^{-1} + 1) & \lambda + 2\nu \end{pmatrix}
\]

\[
= \det (\lambda + \nu (a_n + a_n^{-1})) (\lambda + 2\nu) - \mu^2 (1 + b_n) (b_n^{-1} + 1)
\]

\[
= Q_n(\lambda^2 + 2\lambda\nu - 2\mu^2, \lambda\nu + 2\nu^2, -\mu^2)
\]

\[
= Q_n(F(\lambda, \mu, \nu)).
\]

Define \( K \) as the closure of the set of all backwards \( F \)-iterates of \( \{Q_1 = 0\} \). Then the spectrum of \( \pi \) is the intersection of the line \( \{\mu = \nu = -\frac{1}{4}\} \) with \( K \), and is easily seen to be a Cantor set — see Figure 8.

By contrast, the spectrum of \( \ell^2(\Gamma) \) is the interval \([-1, 1]\) by [HK07], since \( \Gamma \) is amenable and torsion-free (and hence satisfies the Baum-Connes conjecture).

My proof that \( \Gamma \) is not in \( \mathcal{BG} \) is inspired by [GZ]. Define for ordinals \( \alpha \) the following subclasses of \( \mathcal{BG} \): first, \( \mathcal{BG}_0 \) is the class of groups locally of subexponential growth. Let \( \mathcal{BG}_{\alpha+1} \) be the class of subgroups, quotients, extensions and direct limits of groups in \( \mathcal{BG}_\alpha \), and for a limit ordinal \( \beta \) set \( \mathcal{BG}_\beta = \bigcup_{\alpha < \beta} \mathcal{BG}_\alpha \). Note that it is actually not necessary to consider subgroups and quotients in the inductive construction of \( \mathcal{BG}_{\alpha+1} \).

**Proof of Theorem 2.4.** \( \Gamma \) is amenable by Theorem 2.2. Since \( \Delta \) is an ascending extension of \( \Gamma \), it is also amenable.
Figure 6. The Schreier graphs $\mathcal{G}_n$ for $1 \leq n \leq 6$. The solid lines represent $a$'s action on $X^n$, and the dotted lines represent $b$'s action. All vertices have degree 4; the $b$ loops are represented only for $n \leq 3$.

Figure 7. The Julia set of the polynomial $z^2 - 1$.

Figure 8. The spectrum of $\pi$, in its level-6 approximation.
Assume $\Gamma \in \mathcal{BG}$ for contradiction. Then $\Gamma \in \mathcal{BG}_\alpha$ for some minimal ordinal $\alpha$, which of course is not a limit ordinal. Since $\Gamma$ has exponential growth, we have $\alpha > 0$.

By minimality of $\alpha$, $\Gamma$ cannot be a subgroup or quotient of a group in $\mathcal{BG}_{\alpha-1}$. It cannot be a direct limit, since it is finitely generated. Therefore there are $N, Q \in \mathcal{BG}_{\alpha-1}$ with $G/N = Q$. Now by Theorem 2.3, Point (6), $N$ has admits a subgroup mapping onto $\Gamma$, so $\Gamma \in \mathcal{BG}_{\alpha-1}$, a contradiction.

The presentation of $\Delta$ is obtained as the HNN extension of $\Gamma$ identifying $\Gamma$ and $\Gamma^\sigma$. To $\Gamma$’s presentation we add a generator $t$ and relations $a^t = b, b^t = a^2$; and note then that of the relations of $\Gamma$ all can be removed but the first, and $a$ can be removed from the generating set and replaced by $b^{-1}$ in $[b^a, b]$ and $b^{-1}a^{-2}$.

Proof of Theorem 2.5. Consider the sequence of trees with basepoint $T_n = (X^*, 1^n)$, where $1^n$ is the leftmost vertex at level $n$ in $X^*$. The direct limit $\varinjlim T_n$ is a $(\#X + 1)$-regular tree with a distinguished vertex $*$. The tree injection $T_n \to T_{n+1}$ given by $w \mapsto 1w$ extends to an invertible hyperbolic isometry $t$ of $T$.

Let $g \in G$ act on $T_n$ as $g^\sigma$ acts on $X^*$; this action extends to the limit $T$, and we have $g^\sigma = g^t$, so $\langle G, t \rangle$ is an HNN extension.

Let $U$ denote the connected component of $T \setminus \{\text{the edge } 1 \text{ at } *\}$. Then $U$ is naturally isomorphic to $X^*$ and carries the original action of $G$. Therefore restriction to $U$ gives a split epimorphism from the stabilizer of $U$ to $G$.

Any $v \in T$ can be mapped to a vertex in $U$ by a sufficiently large power of $t$; then since $G$ is transitive on $X^n$ for all $n$, it further can be mapped to some vertex $1^n$; and mapped to $*$ by $t^{-n}$; therefore $\tilde{G}$ acts transitively on $T$.

Finally, if $G/K$ and $K/K^X$ are finitely presented and $G$ is contracting, then $\tilde{G}$ is finitely presented, by the argument in [Bar02b].

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