Effect of system energy on quantum signatures of chaos in the two-photon Dicke model

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Abstract

We have studied entanglement entropy and Husimi $Q$ distribution as a tool to explore chaos in the quantum two-photon Dicke model. With the increase of the energy of system, the linear entanglement entropy of coherent state prepared in the classical chaotic and regular regions become more distinguishable, and the correspondence relationship between the distribution of time-averaged entanglement entropy and the classical Poincaré section has been improved obviously. Moreover, Husimi $Q$ distribution for the initial states corresponded to the points in the chaotic region in the higher energy system disperses more quickly than that in the lower energy system. Our result imply that higher system energy has contributed to distinguish the chaotic and regular behavior in the quantum two-photon Dicke model.

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I. INTRODUCTION

It is well known that classical chaos is a kind of very important and complex motion with the high sensitivity to initial conditions, which appears in the nonlinearly dynamical systems and can be usually detected by methods including the Poincaré surfaces of section and the Lyapunov characteristic exponents. However, the chaos in the quantum mechanics becomes more intriguing and challenging since there is no general quantum counterpart of classical phase space trajectories due to uncertainty principle \[1–4\]. Moreover, the scalar product between two nearby states with different initial conditions remains a constant for all time rather than the exponential divergence because of the unitary time-evolution operator. Therefore, it is very important how to explore and explain signatures of chaos in quantum system, which could help us to understand further the quantum dynamics itself and the correspondence principle between classical and quantum mechanics.

With the Random Matrix Theory, Wigner \[5, 6\] et al analysed statistical properties at different energy levels in a quantum chaos state, and found that the distribution of spacings between adjacent energy levels for quantum chaos states obeys Wigner distributions governed by the Gaussian ensemble of matrices rather than usual Poisson distribution. Moreover, the study of entanglement entropy indicates that in general the chaotic systems tend to own larger entanglement entropy than the regular systems \[7–12\]. However, the correspondence relationship between entanglement entropy and chaos is not always hold since there exist certain cases in which the entanglement entropy for the initial state prepared in the regular region is higher than that of in chaotic region \[13–15\]. The intrinsic physics is still uncertain and needs to be further investigated. Recently, Ruebeck \[15\] et al studied the entangling quantum kicked top and divided the infinite-time-averaged entanglement entropy \(S_Q\) into two parts: \(I_Q\) and \(R_Q\), which come respectively from the “diagonal” and “off-diagonal” matrix elements of the angular momentum operators obtained by the Floquet eigenstates of the system. They found that the entanglement entropy related to classical chaos is the quantity \(I_Q\) rather than the whole entanglement entropy \(S_Q\) since \(R_Q\) has no possible classical equivalent. In the quantum kicked top model, Piga \[16\] et al also found that with increasing the number of qubits the entanglement entropy of the initial states in the classical regular and chaotic regions become more distinguishable, which leads to a more clear correspondence between the entanglement entropy and the classical features of the phase space. Moreover, they also found that there exists certain similar behaviors between the low entanglement entropy tori and Kolmogorov-Arnol’d-Noser (KAM) tori, which implies that entanglement could play an important role in quantum KAM theory. The efforts have been devoted to developing a quantum analogue of KAM theory in
Refs. [17–20]. Other quantum sources, such as, spin squeezing [21], quantum discord [22], out-of-time ordered correlator [23, 24] have been applied as signatures to explore chaos in quantum systems.

In this paper, we will focus on the two-photonDicke model where $N$ identical two-level atoms couple to a bosonic mode through two-photon interaction. Such a kind of two-photon interaction have been commonly applied to describe the second-order process in physical devices including quantum dots [26, 27], trapped ions [28, 29] and Rydberg atoms in microwave superconducting cavities [30, 31]. Comparing with the standard Dicke model [32], the presence of two-photon interaction results in some new properties appeared in this quantum system. For example, the discrete system spectrum collapses into a continuous band for a specific value of the coupling strength [33–35]. In the transition from the strong to the ultrastrong coupling regime, a continuous symmetry breaks down into a four-folded discrete symmetry described by a generalized-parity operator [36]. Moreover, a super-radiant phase transition [37] also occurs in the two-photon Dicke model due to coherent radiations of the atoms. The behavior of finite-size scaling functions [38] in two-photon Dicke model indicates that the super-radiant phase transition has the same scaling features as in the standard Dicke model. Since two-photon coupling is a nonlinear interaction, chaos phenomenon would appear in such dynamical systems. However, quantum signatures of chaos and the correspondence between entanglement and classical chaos are still open in the two-photon Dicke model. The main motivation in this paper is to study entanglement entropy and Husimi $Q$ distribution and probe further the relationship between entanglement and classical chaos. We find that with higher system energy the values of linear entanglement entropy of the points in these chaotic and regular regions become more distinguishable for the two-photon Dicke model. Meanwhile, the higher system energy improves obviously the correspondence relationship between the distribution of time-averaged entanglement entropy and the classical Poincaré section in this model.

The paper is organized as follows. In Sec.II, we introduces briefly the two-photon Dicke model and its properties. In Sec.III, we study the entanglement entropy and Husimi $Q$ distribution and probe further the effects of system energy on quantum signatures of chaos in the two-photon Dicke model. Finally, we present results and a brief summary.

II. THE TWO-PHOTON DICKE MODEL

Let us now briefly introduce the two-photon Dicke model in which $N$ two-level identical atoms interact with a single bosonic mode by a two-photon interaction. The system Hamiltonian can be expressed as [32]

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{J}_z + \frac{g}{N} (\hat{J}_+ + \hat{J}_-)(\hat{a}^2 + \hat{a}^\dagger 2),$$

(1)
where \( \hat{a} \) and \( \hat{a}^\dagger \), respectively, are the annihilation and creation operators of the single-mode cavity with frequency \( \omega \). \( \omega_0 \) is the atomic transition frequency and \( g \) is the collective coupling strength of the two-photon interaction. \( \hat{J}_z = \sum_{n=1}^{N} \hat{\sigma}_z^{(i)}/2 \) is the operator of two-level atomic inversion, \( \hat{J}_+ = \sum_{n=1}^{N} \hat{\sigma}_+^{(i)}/2 \) and \( \hat{J}_- = \sum_{n=1}^{N} \hat{\sigma}_-^{(i)}/2 \) are the collective atomic raising and lowering operators. The operators \( \hat{J}_z, \hat{J}_+ \) and \( \hat{J}_- \) form the \( SU(2) \) Lie algebra and obey commutation relations

\[
[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z, \quad [\hat{J}_z, \hat{J}_\pm] = \pm \hat{J}_\pm.
\]

Comparing with the usual standard Dicke model, the Hamiltonian in the two-photon Dicke model owns a generalized \( Z_4 \) parity operator \( \prod = (-1)^N \otimes \hat{\sigma}_z^n e^{i\pi \hat{a}^\dagger \hat{a}/2} \) with four eigenvalues \( \pm 1 \) and \( \pm i \) rather than the \( Z_2 \) parity in the standard Dicke model \([39, 40]\).

In order to study the relationship between entanglement and classical chaos in the two-photon-Dicke model, we here take the initial states to be coherent states since they correspond to the minimum uncertainty wave packets centered in the classical phase space. As in Refs. \([8, 39–41]\), we choose the initial quantum states as

\[
|\psi(0)\rangle = |\tau\rangle \otimes |\beta\rangle \equiv |\tau\beta\rangle,
\]

where \( |\tau\rangle \) and \( |\beta\rangle \), respectively, are the atomic and bosonic coherent states with the forms

\[
|\tau\rangle = (1 - \tau\tau^*)^{-j/2} e^{\tau J_+} |J, -J\rangle,
|\beta\rangle = e^{-\beta J^2/2} e^{\beta J^\dagger} |0\rangle.
\]

Here

\[
\tau = \frac{q_1 + ip_1}{\sqrt{4j - q_1^2 - p_1^2}}, \quad \beta = \frac{1}{\sqrt{2}}(q_2 + ip_2).
\]

\( |J, -J\rangle \) denotes the state with spin \( J \) and \( \hat{J}_z = -J \), and \( |0\rangle \) is the bosonic field ground state. Here \( j = N/2 \) and the quantities \( q_1, p_1, q_2, p_2 \) describe the phase space of the system. The indices 1 and 2 denote the atomic and field subsystem, respectively. With the standard procedure \([41]\), one can obtain the classical Hamiltonian corresponding to Eq. \( (1) \)

\[
H_{cl} \equiv \langle \tau\beta | \hat{H} | \tau\beta \rangle = \frac{\omega_0}{2}(q_1^2 + p_1^2 - 2j) + \frac{\omega}{2}(q_2^2 + p_2^2) + \frac{q_1 \sqrt{4j - q_1^2 - p_1^2}(q_2^2 - p_2^2)g}{2j}.
\]
And then, the corresponding equations of motion are given by

\[
\begin{align*}
\dot{q}_1 &= \omega_0 p_1 + g q_1 (p_2^2 - q_2^2) / 2j \sqrt{4j - q_1^2 - p_1^2}, \\
\dot{q}_2 &= \omega p_2 - g q_1 (2q_2^2 - p_2^2) / j, \\
\dot{p}_1 &= -\omega_0 q_1 + g (4j - 2q_1^2 - p_1^2)(p_2^2 - q_2^2) / 2j \sqrt{4j - q_1^2 - p_1^2}, \\
\dot{p}_2 &= -\omega q_2 - g q_1 \sqrt{4j - q_1^2 - p_1^2} / j.
\end{align*}
\]  

(7)

For the two-photon Dicke model, there exists a spectral collapse at \( g_{\text{collapse}} = \frac{\omega}{2} \), which yields that the energy levels of the system collapse into a continuum as \( g \geq \frac{\omega}{2} \) and then the ground state of Hamiltonian (1) is no longer defined in this regime. Therefore, we will focus on the strong-coupling regime \( g < \frac{\omega}{2} \). For the sake of simplicity, we limit our consider on the resonant case \( \omega_0 = 2\omega \) in which the transition between two energy levels occurs only if the atom absorbs (or emits) two photons. The presence of the two-photon interaction yields that the equation of motion (7) in the classical correspondence are not be variable-separable, which means that the corresponding motion could be chaotic. In Fig. (a)-(c), we present the Poincaré section for certain parameters and initial values, which show that there exist chaos for system described by the classical Hamiltonian (6) corresponding to the two-photon Dicke model (1).

### III. EFFECTS OF ENERGY OF SYSTEM ON QUANTUM SIGNATURES OF CHAOS IN THE TWO-PHOTON DICKE MODEL

In this section, we will investigate how the system energy improve quantum signatures of chaos in the two-photon Dicke model. The linear entanglement entropy and Husimi \( Q \) distribution are two common tools to explore chaos in quantum systems including Dicke model [8] and Kicked top model [12]. The linear entanglement entropy is defined as

\[
S(t) = 1 - Tr_1 \rho_1(t)^2,
\]

(8)

with the reduced-density matrix

\[
\rho_1(t) = Tr_2 |\psi(t)\rangle \langle \psi(t)|,
\]

(9)

where \( Tr_i \) is a trace over the \( i \)th subsystem \( (i = 1, 2) \), and the vector \( |\psi(t)\rangle \) is the quantum state of the full system evolved in time under the action of Hamiltonian (1). The quantity \( S(t) \) describes the degree of purity of the subsystems and the degree of decoherence. In Figs (d)-(i), we show that the evolution of linear
entanglement entropy with time $t$ for the different initial states which correspond to different points in the classical phase spaces. Points $A_1$, $A_3$, $B_2$, $B_3$, $C_2$ and $C_3$ are in classical chaotic regions, and points $A_2$, $A_4$, $B_1$ and $C_1$ are in classical regular regions. As the energy of system $E = 1$, from Figs. (d) and (g), it seems that the linear entanglement entropy increases more rapidly for the initial states corresponded to these points in the classical chaotic region. However, the values of linear entanglement entropy for the points $A_1$-$A_4$ are very close so that they are overlapped at sometime, which means that actually it is difficult to distinguish classical chaotic and regular behaviors by using of the linear entanglement entropy in this case. With the increase of
the energy of system $E$, one can obtain that the linear entanglement entropies of the points in these chaotic and regular regions become more distinguishable. As $E = 10$, from Figs. (f) and (i), we find that with the time the linear entanglement entropy tends to different limit values for different initial states. The limit value of linear entanglement entropy for the states corresponded to the points ($C_2$, $C_3$) in chaotic regions is much higher than that of the points ($C_1$) in regular regions. This means that higher energy of system can enhance the availability of entanglement entropy as a tool to explore quantum chaos in the two-photon Dicke model.

Now, we adopt the time-averaged entanglement entropy to investigation the correlation between classical dynamics and quantum entanglement the two-photon Dicke model for different system energies. The time-averaged entanglement entropy is defined by

$$S_m = \frac{1}{T} \int_0^T S(t) dt, \quad (10)$$

where $T$ is the total time of evolution. In Fig. 2, (a)-(c), we present the classical phase space for the two-photon Dicke model with the system energy $E = 1$ for fixed coupling parameter $g = 0.1, 0.25$ and 0.4, respectively.
FIG. 3: Figures (a)-(c) correspond to the classical phase spaces for the two-photon Dicke model with the coupling parameter \( g = 0.1, 0.25, 0.4 \), respectively. Figures (d)-(f) denotes the time-average entanglement entropy for the two-photon Dicke model with the coupling parameter \( g = 0.1, 0.25, 0.4 \), respectively. Here, we set \( \omega = \omega_0/2 = 1 \), \( j = 5 \) and \( E = 10 \).

The corresponding time-averaged entanglement entropy \( S_m \) with \( T = 30 \) are plotted in Fig. 2 (d)-(f) for the initial states related to the points in the whole classical phase space. In Fig. 2 comparing figure (a) (or (c)) with figure (d) (or (f)), it seems that there exist a correspondence between the classical phase space and the distribution of time-averaged entanglement entropy (i.e., the initial state located in the chaotic region in classical phase space owns the high time-averaged entanglement entropy and the initial state in the regular region has the lower entanglement entropy). However, we must point out that the time-averaged entanglement entropy for certain points lied in the regular region is higher than that of in chaotic region. Moreover, from figures 2(b) and 2(e), one can find directly that the correspondence relationship between the classical phase space and the time-averaged entanglement entropy does not hold as the system energy is set to \( E = 1 \). In Fig. 3, we present the classical phase space and the time-averaged entanglement entropy for the two-photon Dicke model with the same parameters as in Fig. 2 except the system energy \( E = 10 \). Comparing Figs. 2 and 3, it is obvious that higher system energy improves the correspondence between the classical phase space and
the distribution of time-averaged entanglement entropy.

Husimi $Q$ distribution is a quasiprobability distribution, which can provide a visualization of high-dimensional quantum states and demonstrates the dynamical evolution of the quantum state with time. It is shown that Husimi $Q$ distribution displays a rapid dispersion over the phase space as the initial coherent state is in the classically chaotic region. Thus, with Husimi $Q$ distribution, one can diagnose chaotic behavior in quantum system \[42\]. For a coherent state, Husimi $Q$ function is defined as

$$Q(q_1, p_1) = \frac{1}{\pi} \langle q_1, p_1 | \hat{\rho}_1 | q_1, p_1 \rangle,$$

where $|q_1, p_1\rangle$ is a coherent state and $\rho_1$ is the reduced-density matrix of the 1st subsystem. In Figs.4-5, we present the change of Husimi $Q$ distribution in phase space for the quantum system with energy $E = 1$ and $E = 10$, respectively. In Fig.4 subfigures (a)-(f) and (g)-(l) demonstrate the dynamical evolution of the coherent state with the initial state corresponding to the point $A_4$ in the regular region and the point $A_1$ in the chaotic region in Fig.1(a), respectively. Husimi $Q$ function owns almost the same dispersion rate in the phase space for these two different states, which indicates again that it is difficult to distinguish classical chaotic and regular behaviors in the two-photon Dicke model with the system energy $E = 1$. However, for the case with the system energy $E = 10$ as shown in Fig.5, one can find Husimi $Q$ function for the initial state corresponded to point $C_3$ in the chaotic region disperses more quickly than that of the corresponding point $C_1$ in the regular region. Thus, the change of Husimi $Q$ distribution also supports that higher system energy has contributed to distinguish the chaotic and regular behavior in the quantum two-photon Dicke model.

IV. SUMMARY

We have studied entanglement entropy and Husimi $Q$ distribution in the two-photon Dicke model. It is shown that in the cases with higher system energy the increasing rate of linear entanglement entropy becomes more rapidly for the initial states corresponded to these points in the classical chaotic region. With the increase of the energy of system, the values of linear entanglement entropy of the points in these chaotic and regular regions become more distinguishable. Moreover, there is an obvious improvement in the correspondence relationship between the distribution of time-averaged entanglement entropy and the classical Poincaré section in the cases with higher system energy. Finally, we also present Husimi $Q$ distribution for a coherent state in the two-photon Dicke model with different system energies, and find that Husimi $Q$ distribution for the initial state corresponded to point in the chaotic region in the higher energy system disperses more quickly than
FIG. 4: Change of Husimi $Q$ distribution with time for fixed coupling parameter $g = 0.3$ and system energy $E = 1$. Figs.(a)-(f) denote the case where the initial coherent state corresponds to the point $A_4$ in the regular region in Fig. 1 (a). Figs.(g)-(l) denote the case where the initial coherent state corresponds to the point $A_1$ in the chaotic region in Fig. 1 (a).

that in the lower energy system. Our result imply that higher system energy has contributed to distinguish the chaotic and regular behavior in the quantum two-photon Dicke model.

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FIG. 5: Change of Husimi $Q$ distribution with time for fixed coupling parameter $g = 0.3$ and system energy $E = 10$. Figs.(a)-(f) denote the case where the initial coherent state corresponds to the point $C_1$ in the regular region in Fig.1 (c). Figs.(g)-(l) denote the case where the initial coherent state corresponds to the point $C_3$ in the chaotic region in Fig.1 (c).

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