Pedal curves, orthotomics and catacaustics of frontals in the hyperbolic 2-space

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Abstract

In this paper, firstly the definition of the pedal curves of spacelike frontals is presented. The parametric representation of pedal curves of spacelike frontals is given by using the hyperbolic Legendrian moving frames along these frontals. We mainly deal with the classification and recognition problems of singularities of hyperbolic pedal curves of spacelike frontals constructed by non-singular and singular dual curve germs in hyperbolic 2-space. We show that for non-singular dual curve germs one can determine singularity types of such pedal curves based on singularities of the first spacelike hyperbolic Legendrian curvature germs and locations of pedal points. On the other hand, for singular dual curve germs, singularity types of pedal curves depend upon singularities of both of the spacelike hyperbolic Legendrian curvature germs and also locations of pedal points. Then, after introducing orthotomics of spacelike frontals we present some relationships of such curves with hyperbolic pedal curves. Finally, we present an application to light patterns generated by reflected rays in hyperbolic plane. The theory investigated in the paper is supported by illustrated examples.

1 Introduction

The differential geometry of curves in the Euclidean and Minkowski spaces are widely studied. Obtaining new curves based on some rules from a given curve is one of widely studied problems. Among these curves, the so-called pedal curves have an importance. A pedal curve of a regular curve in the Euclidean plane is the locus of the feets of the
perpendiculars from a fixed point (which is called the pedal point) to the tangent lines along the curve \([12, 31]\). This definition yields pedal coordinates of a point on the curve with respect to the curve and the pedal point. Based on these coordinates, it is reasonable to get the pedal equation of a given curve. Furthermore, pedal coordinates are practical for solving specific force problems in classical and celestial mechanics. In \([3]\), the author stated that the trajectory of a particle under central and Lorentz-like forces can be converted to pedal coordinates at once without need of solving any differential equation. Then, he applied the methods developed in the paper to solve dark Kepler problem.

As a more general definition, we can say that a pedal curve of a regular curve is defined as the locus of the nearest point in the geodesic, which is tangent to the curve at a point, from a given point. This definition yields parametric representation for pedal curves by using the orthogonal projection and the Frenet frame along the curve. But, if the curve is not regular at some points, then the pedal curve can not be defined as above because the Frenet frame is not well-defined along the curve. In \([7]\) Fukunaga and Takahashi examined Legendre curves in the unit tangent bundle of the Euclidean plane and introduced a moving frame which is called Legendrian Frenet frame along the curve. This frame is well-defined even at singular points of the curve. By this means, Fukunaga and Takahashi \([8, 9, 10]\) presented the definitions of the evolute and involute of a curve, which could possess singular points, by utilizing the Legendrian Frenet frame. After this study, spherical fronts in the Euclidean 2-sphere were defined and then the evolutes of spherical fronts were examined by Yu et al. \([29]\). Furthermore, Li and Pei \([19]\) considered pedal curves of spherical fronts by using the definition of pedal curves in \(S^2\) given by \([23]\). Moreover, in \([6]\) Chen and Takahashi presented the definitions of frontals in hyperbolic and de Sitter 2-spaces and found moving frames along spacelike and timelike frontals. Thanks to these frames, they introduced evolutes and parallels of timelike and spacelike frontals. Recently, many studies on frontals and framed curves (in 3D setting) have been proposed. (Some of them \([13, 14, 20, 21, 27]\).

Similar to the above general definition of pedal curves, an orthotomic of a curve \(r\) relative to a point \(Q\) can be defined as the set of the reflections of \(Q\) about the planes in which the geodesics tangent to the \(r\) lie for all \(s \in I\). The evolute of the orthotomic is called the catacaustic. From point of the view of optics, the catacaustic represents the light patterns generated by reflected rays. See for more information to \([1, 5, 11]\).

This paper is organized as follows. In Section 2, we present some preliminaries on Minkowski geometry and spacelike frontals in hyperbolic 2-space. We also define pedal curves of regular curves in hyperbolic 2-space. This definition is given by considering the orthogonal projections in hyperbolic 2-space \([15, 16]\) and a similar way to that of \([23]\). In Section 3, the definitions of pedal curves of spacelike frontals in hyperbolic 2-space are given and some results are presented. In Section 4 we give a complete classification of singularities of these curves when the dual curve germs are non-singular or singular. We are mainly inspired by the papers \([23, 24, 25]\). However, we relatively extend the results given in those mentioned papers to the frontals and hyperbolic 2-space. Then, we realize that such an extension is possible only if we take account of both of the hyperbolic curvatures of the frontal. In Section 5 we introduce hyperbolic orthotomics of spacelike frontals and
examine properties of such curves. Finally, in Section 6 we give an application to optics.

2 Preliminaries

The Minkowski 3-space $\mathbb{R}^3_1$ is the real vector space with a pseudo scalar product given as

$$\langle u, w \rangle = -u_1w_1 + u_2w_2 + u_3w_3,$$

where $u = (u_1, u_2, u_3), w = (w_1, w_2, w_3) \in \mathbb{R}^3$.

The vectors in $\mathbb{R}^3_1$ are classified by the above pseudo scalar product. Consider a non-zero vector $u = (u_1, u_2, u_3) \in \mathbb{R}^3_1$. The vector $u$ is called a spacelike, a timelike or a lightlike (null) vector if $\langle u, u \rangle > 0, \langle u, u \rangle < 0$ or $\langle u, u \rangle = 0$, respectively. The pseudo-norm of the vector $u$ is given by $\| u \| = \sqrt{|\langle u, u \rangle|}$.

For two arbitrary curves $u = (u_1, u_2, u_3)$ and $w = (w_1, w_2, w_3)$, the pseudo vector product is defined by

$$u \wedge v = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (-u_2w_3 + u_3w_2, u_3w_1 - u_1w_3, -u_2w_1 + u_1w_2),$$

where the set $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3_1$.

In Minkowski 3-space, curves are classified depending on their tangent vectors. A curve is said to be spacelike, timelike or lightlike (null) if the tangent vector of the curve is always a spacelike, a timelike or a lightlike (null) vector, respectively.

Given a vector $u$ and a real number $c$, the plane with pseudo-normal $u$ is defined by

$$P(u, c) = \{ x \in \mathbb{R}^3_1 | \langle x, u \rangle_L = c \}.$$

The plane $P(u, c)$ is classified depending on its pseudo-normal $u$. If $u$ is a spacelike, a timelike or a lightlike vector, then the plane $P(u, c)$ is said to be a timelike, a spacelike or a lightlike plane, respectively. Now, we remind pseudo 2-spheres in $\mathbb{R}^3_1$. Hyperbolic 2-space, de Sitter 2-space and lightlike cone at the origin are respectively defined by

$$H^2 = \{ u \in \mathbb{R}^3_1 | \langle u, u \rangle = -1 \},$$

$$dS^2 = \{ u \in \mathbb{R}^3_1 | \langle u, u \rangle = 1 \},$$

$$LC^* = \{ u \in \mathbb{R}^3_1 \backslash \{0\} | \langle u, u \rangle = 0 \}.$$

Let us take a curve obtained by the intersection of $H^2$ (or, $dS^2$) with the plane $P(u, c)$:

$$H^2 \cap P(u, c) \text{ (or, } dS^2 \cap P(u, c) \text{)}.$$

Then, if $u$ is a timelike, a spacelike, or a lightlike curve, then the intersection curve is called hyperbolic (or de Sitter) ellipse, hyperbolic (or de Sitter) parabola or hyperbolic (or de Sitter) hyperbola, respectively. For more details about Minkowski space, please see [26].

Now, we introduce the pedal curves of regular curves in hyperbolic 2-space.
2.1 Pedal curves of regular curves in hyperbolic 2-space

Let \( r_h : I \to \mathcal{H}^2 \) be a regular curve, that is \( \| \dot{r}_h(s) \| \neq 0 \) for all \( s \in I \), where \( \dot{r}_h(s) \) denotes the derivative of \( r_h \) with respect to the arbitrary parameter \( s \). Because \( r_h \) is a regular curve, the unit spacelike tangent vector \( T_h(s) = \dot{r}_h(s)/\| \dot{r}_h(s) \| \) to the curve is well-defined. Hence, taking a unit spacelike vector \( r_h \), we call \( \nu_h \) the dual curve of \( r_h \) in hyperbolic 2-space (resp. in \( \Delta_1 \)).

Hence, taking a unit spacelike vector \( N_h = r_h \wedge T_h \), we obtain a pseudo orthonormal frame \( \{ r_h, T_h, N_h = r_h \wedge T_h \} \) along \( r_h \) called hyperbolic Frenet frame. Then, the hyperbolic Frenet-Serret type formulas of the frame can be given as

\[
\begin{pmatrix}
\dot{r}_h(s) \\
T_h(s) \\
N_h(s)
\end{pmatrix} = \| \dot{r}_h(s) \| \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & \kappa_h(s) \\ 0 & -\kappa_h(s) & 0 \end{pmatrix} \begin{pmatrix} r_h(s) \\ T_h(s) \\ N_h(s) \end{pmatrix},
\]

where \( \kappa_h(s) = \frac{\det(r_h(s), \dot{r}_h(s), \ddot{r}_h(s))}{\| r_h(s) \|^3} \) is called the hyperbolic geodesic curvature.

By using the method in [23, 24] and the orthogonal projections in [16], we give a parametrization for a pedal curve of \( r_h \) relative to \( Q \in \mathcal{H}^2 \) as

\[
Ped_Q(r_h)(s) = \frac{1}{\sqrt{1 + \langle Q, N_h(s) \rangle^2}} (Q - \langle Q, N_h(s) \rangle N_h(s)),
\]

Then, it is easy to see that \( Ped_Q(r_h)(s) \) is located in \( \mathcal{H}^2 \).

2.2 Spacelike frontals in the hyperbolic plane

Suppose that \( r_h : I \to \mathcal{H}^2 \) is a spacelike curve at regular points. If there exists a smooth map \( v_h : I \to dS^2 \) such that \( (r_h, v_h) : I \to \Delta_1 \) satisfies \( (r_h(s), v_h(s))^* \theta = 0 \) for any \( s \in I \), then the curve \( r_h \) (resp. the pair \( (r_h, v_h) \)) is called a spacelike frontal (resp. a spacelike Legendre curve) in hyperbolic 2-space (resp. in \( \Delta_1 \)), where

\[
\Delta_1 = \{(u, w) \mid \langle u, w \rangle = 0 \} \subset \mathcal{H}^2 \times dS^2
\]

is a 3-dimensional manifold and \( \theta \) is a canonical contact 1-form on \( \Delta_1 \) [17, 18]. The condition \( (r_h(s), v_h(s))^* \theta = 0 \) means \( \langle \dot{r}_h(s), v_h(s) \rangle = 0 \) for all \( s \in I \). In addition, if \( (r_h, v_h) \) is an immersion, then it is called a spacelike Legendrian immersion in \( \Delta_1 \) and \( r_h \) is called a spacelike front in hyperbolic 2-space. In this case, the set \( \{ r_h(s), v_h(s), \mu_h(s) = r_h(s) \wedge v_h(s) \} \) is a pseudo orthonormal frame called the hyperbolic Legendrian Frenet frame in \( \mathbb{R}^3 \). Moreover, this frame is well-defined even at a singular point of \( r_h \). Then, the hyperbolic Legendrian Frenet-Serret type formulas of the frame is given by

\[
\begin{pmatrix}
\dot{r}_h(s) \\
\dot{v}_h(s) \\
\dot{\mu}_h(s)
\end{pmatrix} = \begin{pmatrix} 0 & 0 & \ell_h(s) \\ 0 & 0 & m_h(s) \\ \ell_h(s) & -m_h(s) & 0 \end{pmatrix} \begin{pmatrix} r_h(s) \\ v_h(s) \\ \mu_h(s) \end{pmatrix},
\]

where, the pair \( (\ell_h(s) = \langle \dot{r}_h(s), \mu_h(s) \rangle, m_h(s) = \langle \dot{v}_h(s), \mu_h(s) \rangle) \) is called the spacelike hyperbolic Legendrian curvature of spacelike Legendrian curve \( (r_h, v_h) \) (cf. [3]). Furthermore, we call \( \nu_h \) the dual curve of \( r_h \).
3 Pedal curves of spacelike frontals in the hyperbolic plane

Consider a spacelike Legendre curve \((r_h, v_h)\) with spacelike hyperbolic Legendre curvature \((\ell_h, m_h)\). Similar to the regular case, we say that the pedal curve of \(r_h\) with respect to the pedal point \(Q \in \mathcal{H}^2\) is the locus of the nearest point in the geodesic \(G_{v_h(s)}\) from \(Q\), which is tangent to the vector \(\mu_h\) at a point (See Fig. 1).

![Figure 1: Geometry of a pedal curve in hyperbolic 2-space](image)

Due to the above definition, \(Q\) and \(r_h\) must lie on the same part of the hyperbolic 2-space. A pedal curve \(Ped_Q(r_h)(s) : I \to \mathcal{H}^2\) of the spacelike frontal \(r_h(s)\) relative to a point \(Q \in \mathcal{H}^2\) is given by

\[
    Ped_Q(r_h)(s) = \frac{1}{\sqrt{1 + \langle Q, v_h(s) \rangle^2}} (Q - \langle Q, v_h(s) \rangle v_h(s)),
\]

(4)

**Remark 3.1.** It seems like that if \(Q = v_h(s_0)\), then the equation of the pedal curve defined above vanishes at that point. But, we cannot consider this case since \(Q \in \mathcal{H}^2\) while \(v_h(s_0) \in dS^2\).

**Proposition 3.2.** Let \(r_h : I \to \mathcal{H}^2\) be a regular curve and \(Q\) be any point in \(\mathcal{H}^2\). Then, the pedal curve of the regular spacelike curve coincides with one of the spacelike front.

**Proof.** Assume that \(r_h : I \to \mathcal{H}^2\) is a regular curve and \(Q \in \mathcal{H}^2\) is a point. Without loss of generality, by taking \(v_h(s) = N_h(s)\) we obtain a spacelike Legendre immersion \((r_h, N_h)\) with the spacelike hyperbolic Legendre curvature \((-\|\dot{r}_h\|, \|\dot{r}_h\|\kappa_h)\). From the definition of pedal curve of a spacelike regular curve, we get

\[
    Ped_Q(r_h)(s) = \frac{1}{\sqrt{1 + \langle Q, N_h(s) \rangle^2}} (Q - \langle Q, N_h(s) \rangle N_h(s))
\]

\[
    = \frac{1}{\sqrt{1 + \langle Q, v_h(s) \rangle^2}} (Q - \langle Q, v_h(s) \rangle v_h(s)) = Ped_Q(r_h)(s).
\]
**Proposition 3.3.** Let \((r_h,v_h) : I \to \Delta_1\) be a spacelike Legendre curve with spacelike hyperbolic Legendre curvature \((\ell_h,m_h)\) and \(Q\) be any point in \(\mathbb{H}^2\). Then, the pedal curve \(\mathcal{P}_{ed}Q(r_h)\) of the spacelike frontal \(r_h\) is not dependent of the parametrization of \((r_h,v_h)\).

**Proof.** Suppose that \((r_h,v_h) : I \to \Delta_1 \subset \mathbb{H}^2 \times dS^2\) and \((\hat{r}_h,\hat{v}_h) : \hat{I} \to \Delta_1 \subset \mathbb{H}^2 \times dS^2\) are parametrically equivalent by means of a (positive) change of parameter \(s : \hat{I} \to I\). Then, we have \((\hat{r}_h(\xi),\hat{v}_h(\xi)) = (r_h(s(\xi)),v_h(s(\xi)))\) and thus obtain

\[
\mathcal{P}_{ed}Q(\hat{r}_h)(\xi) = \frac{1}{\sqrt{1 + \langle Q, \hat{v}_h(\xi) \rangle^2}} (Q - \langle Q, \hat{v}_h(\xi) \rangle \hat{v}_h(\xi)) = \mathcal{P}_{ed}Q(r_h)(s(\xi)).
\]

\[\square\]

**Theorem 3.4.** Consider a spacelike Legendre curve \((r_h,v_h)\) with the spacelike hyperbolic Legendre curvature \((\ell_h,m_h)\). The pedal curve \(\mathcal{P}_{ed}Q(r_h)\) of the spacelike front \(r_h\) relative to \(Q \in \mathbb{H}^2\) has a singular point at \(s_0\) if and only if \(m_h(s_0) = 0\) or \(Q = r_h(s_0)\).

**Proof.** If we differentiate Eq. \([4]\), then using hyperbolic Legendrian Frenet-Serret type formulas we obtain

\[
\mathcal{P}_{ed}Q(r_h)(s) = \frac{m_h(s)}{\sqrt{1 + \langle Q,v_h(s) \rangle^2}} \left( \langle Q, \mu_h(s) \rangle v_h(s) + \langle Q,v_h(s) \rangle \mu_h(s) \right) - m_h(s) \frac{\langle Q,v_h(s) \rangle (Q,\mu_h(s))}{1 + \langle Q,v_h(s) \rangle^2} \left( -\langle Q, r_h(s) \rangle r_h(s) + \langle Q, \mu_h(s) \rangle \mu_h(s) \right).
\]

Then, it is easy to see that \(s_0 \in I\) is a singular point of \(\mathcal{P}_{ed}Q(r_h)\) if and only if \(m_h(s_0) = 0\) or \(Q = r_h(s_0)\) since \(\{r_h(s),v_h(s),\mu_h(s)\}\) is a pseudo orthonormal frame. \[\square\]

**Corollary 3.5.** Assume that \((r_h,v_h) : I \to \Delta_1\) is a spacelike Legendre curve with the spacelike hyperbolic Legendre curvature \((\ell_h,m_h)\). If \(s_0\) is a singular point of \(v_h\), then the pedal curve \(\mathcal{P}_{ed}Q(r_h)\) is singular at \(s_0\) as well.

**Proof.** Because \(s_0\) is a singular point of \(v_h\), we have \(m_h(s_0) = 0\). Then, by using \([5]\) we conclude the proof. \[\square\]

**Theorem 3.6.** Let \((r_h,v_h)\) be a spacelike Legendre curve with spacelike hyperbolic Legendre curvature \((\ell_h,m_h)\) and \(Q\) be any point in \(\mathbb{H}^2 - r_h(I)\). Then, the pedal curve \(\mathcal{P}_{ed}Q(r_h)\) of the spacelike frontal with respect to \(Q\) is a spacelike frontal that is, \((\mathcal{P}_{ed}Q(r_h),\mathcal{v}_h)\) is a spacelike
Legendre curve with spacelike hyperbolic Legendre curvature \((\ell_h, m_h)\), where

\[
\begin{align*}
\dot{v}_h &= \frac{(Q, \mu_h)^2 r_h + (Q, r_h)(Q, v_h) - (Q, r_h)(Q, \mu_h)\mu_h}{\sqrt{(Q, \mu_h)^2(1 + (Q, v_h)^2) + (Q, r_h)^2(Q, v_h)^2}}, \\
\dot{\mu}_h &= \frac{(Q, r_h)^2(Q, v_h)\mu_h + (Q, \mu_h)((Q, r_h)^2 + (Q, \mu_h)^2)v_h - (Q, r_h)(Q, v_h)(Q, \mu_h)r_h}{\sqrt{1 + (Q, v_h)^2}} \\
\dot{\ell}_h &= \frac{m_h}{1 + (Q, v_h)^2}\sqrt{(Q, \mu_h)^2(1 + (Q, v_h)^2) + (Q, r_h)^2(Q, v_h)^2}, \\
\ddot{m}_h &= (\dot{v}_h(s), \dot{\mu}_h(s)).
\end{align*}
\]

(6)

**Proof.** It is enough to show that \(Ped_Q(r_h), \dot{v}_h\) satisfies the conditions for being a spacelike Legendre curve by definition.

The first condition is \(\langle Ped_Q(r_h), \dot{v}_h \rangle = 0\). By using Eq. (4) and \(\dot{v}_h\) defined above, we get

\[
\langle Ped_Q(r_h), \dot{v}_h \rangle = \frac{(Q, \mu_h)^2(Q, r_h) + (Q, r_h)(Q, v_h)^2 - (Q, r_h)(Q, \mu_h)^2 - (Q, r_h)(Q, v_h)^2}{\sqrt{1 + (Q, v_h(s))^2}} \\
= 0.
\]

The second one is \(\langle Ped_Q(r_h), \dot{v}_h \rangle = 0\). Using Eq. (5) and \(\dot{v}_h\) yields

\[
\langle Ped_Q(r_h), \dot{v}_h \rangle = A(s)\left(\frac{1}{1 + (Q, v_h(s))^2}(-Q, \mu_h(s))^3(Q, r_h(s))(Q, v_h(s))
+ (Q, r_h(s))(Q, \mu_h(s))^3(Q, v_h(s))
- (Q, \mu_h(s))(Q, r_h(s))(Q, v_h(s)) + (Q, r_h(s))(Q, \mu_h(s))(Q, v_h(s)) \right)
= 0,
\]

where

\[
A(s) = \frac{m_h(s)}{\sqrt{1 + (Q, v_h(s))^2}} \sqrt{(Q, \mu_h)^2(1 + (Q, v_h)^2) + (Q, r_h)^2(Q, v_h)^2}.
\]

Then, the above two conditions shows that \((Ped_Q(r_h), \dot{v}_h)\) is a spacelike Legendre curve. Since the set \(\{r_h, v_h, \mu_h\}\) constructs an orthonormal basis it is reasonable to write \(Q - (Q, v_h) v_h = (Q, r_h)r_h + (Q, \mu_h)\mu_h\). By considering this fact one can calculate the wedge product \(Ped_Q(r_h) \wedge \dot{v}_h\) which yields \(\dot{\mu}_h\). Using the above relations, one can obtain the Legendre curvatures of \(Ped_Q(r_h)\).

**Example 3.7.** Given a hyperbolic astroid \(r_h(s) = (\sqrt{1 + \cos^6 s + \sin^6 s}, \cos^3 s, \sin^3 s)\) in hyperbolic 2-space, we get

\[
\dot{r}_h(s) = \left(\frac{-3 \cos^5 s \sin s + 3 \sin^5 s \cos s}{\sqrt{1 + \cos^6 s + \sin^6 s}}, -3 \cos^2 s \sin s, 3 \sin^2 s \cos s\right).
\]
If we take $v_h : [0, 2\pi) \to dS^2$ as

$$v_h(s) = \frac{1}{\sqrt{1 + \sin^2 s \cos^2 s}} \left( \sin s \cos s \sqrt{1 + \cos^6 s + \sin^6 s}, \sin s(1 + \cos^4 s), \cos s(1 + \sin^4 s) \right),$$

then $(r_h, v_h) : [0, 2\pi) \to \mathcal{H}^2 \times dS^2$ is a spacelike Legendre immersion. Moreover, by the cross product $r_h \wedge v_h$ we immediately get

$$\mu_h(s) = \frac{\sqrt{1 + \cos^4 s - \sin^2 s}}{\sqrt{1 + \cos^2 s \sin^2 s}} \left( \frac{\cos^4 s - \sin^4 s}{\sqrt{1 + \cos^6 s + \sin^6 s}}, \cos s, -\sin s \right).$$

If we choose the pedal point as $Q_1 = (1, 0, 0)$, then the hyperbolic pedal of the spacelike front $r_h$ with respect to $Q_1$ is found as follows (See Fig. 2 (a)):

$$\mathcal{P}_{ed_{Q_1}}(r_h) = \frac{1}{\sqrt{1 + \cos^2 s \sin^2 s}} \left( \sqrt{1 + \cos^2 s \sin^2 s (2 + \cos^6 s + \sin^6 s)} \right) \times \left( 1 + \cos^2 s \sin^2 s (2 + \cos^6 s + \sin^6 s), \cos s (1 + \cos^4 s) \sin^2 s \sqrt{1 + \cos^6 s + \sin^6 s}, \cos^2 s \sin s (1 + \sin^4 s) \sqrt{1 + \cos^6 s + \sin^6 s} \right).$$

Now, let us choose the pedal point as $Q_2 = \frac{1}{2}(\sqrt{5}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = r_h(\pi/4)$. Then similar to the case above the hyperbolic pedal of the spacelike front $r_h$ with respect to $Q_2$ can be obtained. (See Fig. 2 (b)).

(a) $Q = (1, 0, 0)$.

(b) $Q = \frac{1}{2}(\sqrt{5}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Figure 2: The hyperbolic astroid (red), its hyperbolic pedal curve (blue).
4 Singularities of hyperbolic pedal curves in the hyperbolic plane

Now, we present the complete classification and recognition of singularities of hyperbolic pedal curves in the hyperbolic plane. We present singularity types of pedal curves for non-singular and singular dual curve germs, respectively. For the rest of the paper, we assume \( Q, r_h \in \mathcal{H}_2^+ \).

Let \( Q \) be a point of \( \mathcal{H}_2^+ \). Consider the \( C^\infty \) map
\[
\psi_Q : dS^2 \to \mathcal{H}_2^+
\]
\[
x \mapsto \psi_Q(x) = \frac{1}{\sqrt{1 + \langle Q, x \rangle^2}} (Q - \langle Q, x \rangle x).
\]
Define \( \hat{H}_Q = \{ y \in \mathcal{H}_2^+ | \langle Q, y \rangle < 0 \} \). Since \( \langle \psi_Q(x), Q \rangle < 0 \) for all \( x \in dS^2 \) we have \( X_Q := \psi_Q(dS^2) \subset \hat{H}_Q \). Let \( dS^2 := dS^2/\{ \pm Id \} \) be the model space (or projective quotient) of \( dS^2 \). By construction, \( dS^2 \) is a subset of the projective space \( \mathbb{R}P^2 \). Indeed, topologically \( dS^2 \) can be defined as
\[
dS^2 = P\{ x \in \mathbb{R}^3_1 | \langle x, x \rangle > 0 \}.
\]
Consider the canonical projection \( f : dS^2 \to dS^2 \). Since \( \psi_Q(x) = \psi_Q(-x) \), the map \( \psi_Q \) induces \( \tilde{\psi}_Q : dS^2 \to X_Q \). Thus, it follows from (4) that
\[
\mathcal{P}ed_Q(r_h)(s) = \tilde{\psi}_Q \circ f \circ \nu_h(s).
\]
Suppose that \( b : B \to \mathbb{R}^3_2 \) is the blow up of \( \mathbb{R}^3_2 \) centered at the origin, where \( B = \{ (x_1, x_2) \times [y_1 : y_2] \in \mathbb{R}^3_1 \times \mathbb{R}P^1 \mid x_1 y_2 = x_2 y_1 \} \). We give the following lemma.

**Lemma 4.1.** Let \( Q \in \mathcal{H}_2^+ \). Then, there exist \( C^\infty \) diffeomorphisms \( h_s : dS^2 \to B \) and \( h_1 : X_Q \to \mathbb{R}^3_2 \) such that \( h_t \circ \tilde{\psi}_Q \equiv b \circ h_s \).

**Proof.** Without loss of generality, let us assume \( Q = (1, 0, 0) \). For
\[
U_1 = \{ (x_1, x_2) \times [y_1 : y_2] \in \mathbb{R}^3_1 \times \mathbb{R}P^1 \mid x_1 y_2 = x_2 y_1, \ x_1 \neq 0 \},
\]
\[
U_2 = \{ (x_1, x_2) \times [y_1 : y_2] \in \mathbb{R}^3_1 \times \mathbb{R}P^1 \mid x_1 y_2 = x_2 y_1, \ x_2 \neq 0 \},
\]
and
\[
\varphi_1 : U_1 \to \mathbb{R}^3_2; \quad (x_1, x_2) \times [y_1 : y_2] \mapsto (u_1, u_2) = (x_1, \frac{y_2}{y_1}),
\]
\[
\varphi_2 : U_2 \to \mathbb{R}^3_2; \quad (x_1, x_2) \times [y_1 : y_2] \mapsto (u_1', u_2') = (\frac{y_1}{y_2}, x_2),
\]
it is well-known that the set \( \{ (U_1, \varphi_1), (U_2, \varphi_2) \} \) is the standard atlas for \( B \) and we have
\[
b \circ \varphi_1^{-1}(u_1, u_2) = (u_1, u_1 u_2),
\]
\[
b \circ \varphi_2^{-1}(u_1, u_2) = (u_1 u_2, u_2).
\]
Remark 4.2. Based on Lemma 4.1, the map \( \lambda \) where

\[
\varphi_{Q,1} : U_{Q,1} \to \mathbb{R}_+^2; \quad \varphi_{Q,1}(f(x_1, x_2, x_3)) = (\tanh(\lambda)x_2, \frac{x_2}{x_3}),
\]

\[
\varphi_{Q,2} : U_{Q,2} \to \mathbb{R}_+^2; \quad \varphi_{Q,2}(f(x_1, x_2, x_3)) = (\frac{x_2}{x_3}, \tanh(\lambda)x_3),
\]

Define the sets

\[
U_{Q,1} = \{f(x_1, x_2, x_3) | x_2 \neq 0\};
\]

\[
U_{Q,2} = \{f(x_1, x_2, x_3) | x_3 \neq 0\};
\]

and the maps

\[
\varphi_{Q,1} : U_{Q,1} \to \mathbb{R}_+^2; \quad \varphi_{Q,1}(f(x_1, x_2, x_3)) = (\tanh(\lambda)x_2, \frac{x_2}{x_3}),
\]

\[
\varphi_{Q,2} : U_{Q,2} \to \mathbb{R}_+^2; \quad \varphi_{Q,2}(f(x_1, x_2, x_3)) = (\frac{x_2}{x_3}, \tanh(\lambda)x_3),
\]

where \( \lambda = \sinh^{-1}(x_1) \in \mathbb{R} \). Then, we see that the following equality holds:

\[
\varphi_{Q,j} \circ \varphi_{Q,i}^{-1} \equiv \varphi_j \circ \varphi_i^{-1}, \quad i, j \in \{1, 2\}.
\]

Thus, the set \( \{(U_{Q,1}, \varphi_{Q,1}), (U_{Q,2}, \varphi_{Q,2})\} \) is an atlas for \( dS^2 \).

Now, we express the map \( \psi_Q \) by the coordinates \((u_1, u_2, u_3)\). Based on the assumption \( Q = (1, 0, 0) \), for \( x = (\sinh \lambda, x_2, x_3) \) we have

\[
\varphi_{Q,1}(x) = (\cosh \lambda, x_2 \tanh \lambda, x_3 \tanh \lambda).
\]

Then, we have

\[
q \circ \psi_{Q} \circ \varphi_{Q,1}^{-1}(u_1, u_2) = (u_1, u_2 u_1),
\]

\[
q \circ \psi_{Q} \circ \varphi_{Q,2}^{-1}(u_1, u_2) = (u_1 u_2, u_2),
\]

where \( q : \mathbb{R}_+^2 \to \mathbb{R}_+^2; (x, y, z) \mapsto (y, z) \) is the canonical projection. The restriction \( q|_{dS^2} : dS^2 \to q(dS^2) \) is a \( C^\infty \)-diffeomorphism. Then, Lemma 4.1 is proved for \( \psi_{Q}|_{U_{Q,i}} \) and \( b|_{U_i} \). Hence, in order to complete the proof it is sufficient to show that the equality

\[
\varphi_{Q,i}^{-1}(\varphi_{Q,j}(f(x_1, x_2, x_3))) = \varphi_{Q,j}^{-1}(\varphi_{Q,i}(f(x_1, x_2, x_3))), \quad i, j \in \{1, 2\}
\]

holds for \( f(x_1, x_2, x_3) \in U_{Q,i} \cap U_{Q,j} \). This equality is satisfied because we know that the patching relations for \( \{(U_{Q,i}, \varphi_{Q,i})\} \) are the same as those for the standard atlas of \( B \).

**Remark 4.2.** Based on Lemma 4.1, the map \( \psi_Q \) is a map of blow up type.

Now, thanks to the above lemma we obtain concrete normal forms for generic singularities and exact locations of the pedal point \( Q \) for such singularities. At first, we give the following definitions and lemmas.

**Definition.** Let \( f, g : (I, s_0) \to \mathbb{R}^n \) be two curve germs. Then, \( f \) and \( g \) are said to be \( C^r \) \( \mathcal{L} \)-equivalent if there exists a \( C^r \) diffeomorphism germ \( \psi : (\mathbb{R}^n, f(s_0)) \to (\mathbb{R}^n, g(s_0)) \) such that \( g = \psi \circ f \). Moreover, these curve germs are said to be \( C^r \) \( \mathcal{A} \)-equivalent provided that there exist two \( C^r \) diffeomorphism germs \( \phi : (I, s_0) \to (I, s_0) \) and \( \psi : (\mathbb{R}^n, f(s_0)) \to (\mathbb{R}^n, g(s_0)) \) such that \( g \circ \phi = \psi \circ f \).
Definition ([4]). A function \( f : I \rightarrow \mathbb{R} \) is said to have \( A_k \)-type singularity (\( k \geq 0 \)) at \( s_0 \in I \) if \( f(s_0) = f'(s_0) = \cdots = f^{(k)}(s_0) = 0 \) and \( f^{(k+1)}(s_0) \neq 0 \).

Lemma 4.3 ([4, Theorem 3.3]). Assume that \( g : (\mathbb{R}, 0) \rightarrow \mathbb{R} \) is a \( C^\infty \) function-germ. Assume further that \( g \) has \( A_k \)-type singularity at 0. In this case, there exists a \( C^\infty \) diffeomorphism germ \( f : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0) \) such that \( g(f(s)) = \pm s^k \), where we have + or − depending on the sign of \( g^{(k+1)}(0) \).

Lemma 4.4 ([4, Hadamard’s Lemma]). Let \( f : (\mathbb{R}, 0) \rightarrow \mathbb{R} \) be smooth, and suppose \( f^{(p)}(0) = 0 \) for all \( p \) with \( 1 \leq p \leq k \). Then, there is a smooth function \( f_1 : (\mathbb{R}, 0) \rightarrow \mathbb{R} \) such that \( f(s) = f(0) + s^{k+1}f_1(s) \) for all \( s \) in some neighbourhood of 0. Moreover, if \( f^{(k+1)}(0) \neq 0 \) then \( f_1(0) \neq 0 \).

Theorem 4.5. Consider a spacelike Legendre curve \((r_h, \nu_h)\) with a spacelike hyperbolic Legendre curvature \((\ell_h, m_h)\). Let \( s_0 \in I \) such that \( m_h(s_0) \neq 0 \) and \( Q \) be a point of \( \mathcal{H}^2_+ \). Suppose that \( \ell_h \) has an \( A_{k-1} \)-type singularity at \( s_0 \in I \). Then, the following statements are satisfied:

1. Assume that \( Q \in \mathcal{H}^2_+ - \{r_h(s_0)\} \). In this case, the map-germ \( \mathcal{P}ed_Q(r_h) : (I, s_0) \rightarrow (\mathcal{H}^2_+, \mathcal{P}ed_Q(r_h)(s_0)) \) is smooth, which means that it is \( C^\infty \) \( A \)-equivalent to the map-germ \( (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0) \) defined by \( t \mapsto (t, 0) \).

2. Assume that \( Q = r_h(s_0) \). Then, the map-germ \( \mathcal{P}ed_Q(r_h) : (I, s_0) \rightarrow (\mathcal{H}^2_+, \mathcal{P}ed_Q(r_h)(s_0)) \) is \( C^1 \) \( A \)-equivalent to the map-germ \( (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0) \) defined by \( t \mapsto (t^{k+2}, t^{k+3}) \).

Proof. 1. Since \( Q \in \mathcal{H}^2_+ - \{r_h(s_0)\} \) and \( m_h(s_0) \neq 0 \) we have \( \mathcal{P}ed_Q(r_h)(s_0) \neq 0 \) from Theorem 3.4. Then, the map-germ \( \mathcal{P}ed_Q(r_h)(s_0) \) is non-singular.

2. By an appropriate rotation of \( \mathcal{H}^2_+ \), we may assume that \( Q = (1, 0, 0) \in \mathcal{H}^2_+ \) and \( r_h(s_0) = (1, 0, 0), \nu_h(s_0) = (0, 1, 0) \) and \( m_h(s_0) = (0, 0, 1) \). Then, it is easy to see that the component function-germs \( \nu_1, \nu_2 \) and \( \nu_3 \) of the map-germ \( \nu_h = (\nu_1, \nu_2, \nu_3) : (I, s_0) \rightarrow dS^2 \) admit the lowest degree of non-zero terms as \( k+2, 0 \) and 1, respectively. Then, by considering the hyperbolic Legendrian Frenet-Serret type formulas we may take the map-germ \( \nu_h : (I, s_0) \rightarrow (dS^2, \nu_h(s_0)) \) as

\[
\nu_h(s) = \begin{pmatrix}
\frac{1}{(k+2)!} \ell_h^{(k)}(s_0)m_h(s_0)(s - s_0)^{k+2} + C(s - s_0) \\
1 + A(s - s_0) \\
m_h(s_0)(s - s_0) + B(s - s_0)
\end{pmatrix}
\]

where \( A, B \) and \( C \) are some \( C^\infty \) function-germs \((\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)\) such that \( \frac{d^pA}{dt^p}(0) = \frac{d^pB}{dt^p}(0) = 0 \ (p = 0, 1) \) and \( \frac{d^pC}{dt^p}(0) \ (p \leq k + 2) \). From Lemma 4.1, we have

\[
\varphi_Q,I(f(\nu_1, \nu_2, \nu_3)) = (\tanh(\lambda)\nu_2, \frac{\nu_3}{\nu_2}),
\]

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where \( \sinh(\lambda) = \nu_1 \). This directly gives

\[
\varphi_{Q,1}(f(\nu_h(s))) = \left( \frac{\tanh(\lambda)(1 + A(s - s_0))}{m_h(s_0)(s - s_0) + B(s - s_0)} \right).
\]

From [7] we have \( q \circ \tilde{\psi}_Q \circ \varphi_{Q,1}^{-1}(u_1, u_2) = (u_1, u_2u_1) \) which yields that the map-germ \( \tilde{\psi}_Q \circ v_h : (I, s_0) \rightarrow (\mathcal{S}^2, \tilde{\psi}_Q \circ v_h(s_0)) \) is \( C^\infty \mathcal{A} \)-equivalent to

\[
\begin{pmatrix}
\tanh(\lambda)(1 + A(s - s_0)) \\
tanh(\lambda)(m_h(s_0)(s - s_0) + B(s - s_0))
\end{pmatrix}.
\]

Thus, the-map germ \( \mathcal{P}edQ(r_h) : (I, s_0) \rightarrow (\mathcal{H}_+^2, \mathcal{P}edQ(r_h)(s_0)) \) is \( C^\infty \mathcal{A} \)-equivalent to

\[
\begin{pmatrix}
\frac{1}{(k + 2)!} (m_h(s_0))^{k+2} + \hat{C}(s - s_0) \\
\frac{1}{(k + 2)!} (m_h(s_0))^{k+3} + \hat{B}(s - s_0)
\end{pmatrix},
\]

where \( \hat{B} \) and \( \hat{C} \) are certain \( C^\infty \) function-germs \( (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0) \) such that \( \hat{B} \) has at least an \( A_{k+3} \) singularity at 0 and \( \hat{C} \) has at least an \( A_{k+2} \) singularity at 0.

From Lemma 4.3 we conclude that \( \mathcal{P}edQ(r_h) \) is \( C^\infty \mathcal{A} \)-equivalent to

\[
(t^{k+2} + D_1(t), t^{k+3} + D_2(t)),
\]

where \( D_i : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0) \) are some \( C^\infty \) function germs with \( \frac{d^p D_i(0)}{dt^p} = 0 \) for \( p \leq k + i + 1 \).

Let us consider the following cases:

Let \( k + 2 \) be odd. In this case, we need to consider

\[
h(x_1, x_2) = (x_1 + D_1(x_1^{1/2}), x_2 + D_2(x_1^{1/2})).
\]

On the other hand, if \( k + 3 \) is odd, then it is reasonable to consider

\[
h(x_1, x_2) = (x_1 + D_1(x_2^{1/3}), x_2 + D_2(x_2^{1/3})).
\]

Now, all we need to do is to show that \( h \) is a germ of \( C^1 \) diffeomorphism for both cases. One can see that both of the maps \( x_p \mapsto D_i(x_p^{1/p+1}) \) and \( x_p \mapsto \frac{dD_i(x_p^{1/p+1})}{dx_p}(x_p) \) are well-defined and continuous even at 0. On the other hand, it is not hard to see that the Jacobian matrix of \( h \) at \((0, 0)\) is the unit matrix. Thus, \( h \) is a germ of \( C^1 \) diffeomorphism which concludes the proof.

\( \square \)
Remark 4.6. As an immediate consequence of Theorem 4.5, we have that if \( Q = r_h(s_0) \) and \( r_h(s_0) \) is a regular point of \( r_h \), then the map-germ \( \mathcal{P}_{\text{Ped}}(r_h) : (I, s_0) \to (\mathcal{H}_+^2, \mathcal{P}_{\text{Ped}}(r_h)(s_0)) \) is \( C^1 \), \( \mathcal{A} \)-equivalent to the map-germ \( (\mathbb{R}, 0) \to (\mathbb{R}^2, 0); t \mapsto (t^2, t^3) \).

We present an example to Theorem 4.5

Example 4.7. Let \( r_h : I \to \mathcal{H}_+^2 \) be a spacelike curve given by \( r_h(s) = (\sqrt{1 + s^4 + s^6}, s^2, s^3) \). By differentiating \( r_h \) with respect to \( s \), we find

\[
\dot{r}_h(s) = \left( \frac{4s^3 + 6s^5}{2\sqrt{1 + s^4 + s^6}}, 2s, 3s^2 \right).
\]

If we take \( v_h : I \to dS^2 \) as

\[
v_h(s) = \frac{1}{\sqrt{s^6 + 9s^2 + 4}} (s^3\sqrt{1 + s^4 + s^6}, s^5 + 3s, s^6 - 2),
\]

then we obtain a spacelike Legendrian curve \( (r_h, v_h) : I \to \Delta_1 \) with spacelike hyperbolic curvature \( (\ell_h, m_h) \), where

\[
\ell_h(s) = \frac{s\sqrt{s^6 + 9s^2 + 4}}{\sqrt{1 + s^4 + s^6}}, \quad m_h(s) = \frac{s^{10} + 15s^6 + 10s^4 + 6}{(s^6 + 9s^2 + 4)s^s + s^6} \neq 0.
\]

Furthermore, one can easily obtain \( \mu_h : I \to dS^2 \) as

\[
\mu_h(s) = \frac{\sqrt{1 + s^4 + s^6}}{\sqrt{s^6 + 9s^2 + 4}} \left( \frac{2s^2 + 3s^4}{\sqrt{1 + s^4 + s^6}}, 2, 3s \right).
\]

Take a point \( Q_1 = (\sqrt{2}, 1, 0) \in \mathcal{H}_+^2 - \{r_h(0)\} \). Then, by Theorem 4.5 (1) the map-germ \( \mathcal{P}_{\text{Ped}}(r_h) : (I, 0) \to (\mathcal{H}_+^2, (\sqrt{2}, 1, 0)) \) is smooth. The hyperbolic pedal curve of \( r_h \) with respect to \( Q_1 \) is given in Fig. 3.

If we choose \( Q_2 = (\sqrt{3}, 1, 1) = r_h(1) \), where \( r_h(1) \) is a regular point of \( r_h \). In this case, by Theorem 4.5 (2) we have that the map-germ \( \mathcal{P}_{\text{Ped}}(r_h) : (I, 1) \to (\mathcal{H}_+^2, (\sqrt{3}, 1, 1)) \) is \( C^1 \), \( \mathcal{A} \)-equivalent to the map-germ defined by \( t \mapsto (t^2, t^3) \). This means that the pedal curve \( \mathcal{P}_{\text{Ped}}(r_h) \) has a 3/2 cusp at \( s_0 = 1 \). The hyperbolic pedal of \( r_h \) with respect to \( Q_2 \) is given in Fig. 4. Let us take \( Q_3 = (1, 0, 0) = r_h(0) \). It is easy to see that by using the canonical projection \( q \), the map-germ \( r_h : (I, 0) \to (\mathcal{H}_+^2, (1, 0, 0)) \) is \( C^1 \), \( \mathcal{A} \)-equivalent to the map-germ defined by \( s \mapsto (s^2, s^3) \). On the other hand, it can be easily computed that \( \ell_h(0) = 0 \) and \( \ell_h'(0) = 2 \neq 0 \) which means that \( \ell_h \) has an \( A_0 \)-type singularity at \( s_0 = 0 \). Thus, by Theorem 4.5 (2) the map-germ \( \mathcal{P}_{\text{Ped}}(r_h) : (I, 0) \to (\mathcal{H}_+^2, (1, 0, 0)) \) is \( C^1 \), \( \mathcal{A} \)-equivalent to the map-germ \( s \mapsto (s^3, s^4) \). The hyperbolic pedal of \( r_h \) with respect to \( Q_3 \) is given in Fig. 5.

Now, we investigate singularities of pedal curves for points \( s_0 \in I \) such that \( m_h(s_0) = 0 \). Take \( s_0 \in I \). For any \( s \) such that \( s + s_0 \in I \), we define

\[
\gamma_j(s) = (m_h(s + s_0), m_h'(s + s_0), \ldots, m_h^{(j-1)}(s + s_0)), \quad j \geq 1
\]

Then, \( \gamma_j^* \mathcal{M}_j \mathcal{E}_1 \) is an ideal of \( \mathcal{E}_1 \). We take in consideration quotient \( \mathcal{E}_1 \) algebras given as \( \mathcal{E}_1 / (\gamma_j^* \mathcal{M}_j \mathcal{E}_1) \).
Lemma 4.8. Let $\ell_h$ has an $A_{k-1}$-singularity ($k \geq 0$) at $s + s_0$, $s_0 \in I$, where $A_{-1}$ means that $\ell_h(s + s_0) \neq 0$. Then, the followings are satisfied:

1. $\langle \nu_h^{(j+1)}(s + s_0), \nu_h(s + s_0) \rangle \in \gamma_j^* \mathcal{M}_j \mathcal{E}_1$.
2. $\langle \nu_h^{(j+1)}(s + s_0), \mu_h(s + s_0) \rangle + \gamma_j^* \mathcal{M}_j \mathcal{E}_1 = m_h^{(j)}(s + s_0) + \gamma_j^* \mathcal{M}_j \mathcal{E}_1$.
3. $\langle \nu_h^{(j+k+2)}(s + s_0), r_h(s + s_0) \rangle + \gamma_j^* \mathcal{M}_j \mathcal{E}_1 = -(j+k+1) \ell_h^{(k)}(s + s_0) m_h^{(j)}(s + s_0) + \gamma_j^* \mathcal{M}_j \mathcal{E}_1$.

Proof. For simplicity, we use just the notation $f$ meaning $f(s + s_0)$. Let us prove the lemma by induction over $j$.

For $j = 1$, it is enough to show the followings:

$$
\langle \nu'_h, \nu_h \rangle = -m_h^2,
$$

$$
\langle \nu'_h, \mu_h \rangle = m_h',
$$

$$
\langle \nu_h^{k+3}, r_h \rangle = -(k + 2)m_h' \ell_h^{(k)} - m_h \ell_h^{(k+1)}.
$$

Since $\langle \nu_h, \nu_h \rangle = 1$, we have $\langle \nu'_h, \nu_h \rangle = 0$. By taking derivative again, we see that $\langle \nu''_h, \nu_h \rangle + \langle \nu'_h, \nu'_h \rangle$ which yields $\langle \nu''_h, \nu_h \rangle = -m_h^2$. By (3), we know $\langle \nu'_h, \mu_h \rangle = m_h$. Then, we find $\langle \nu''_h, \mu_h \rangle + \langle \nu'_h, \mu'_h \rangle = m_h'$. This shows that $\langle \nu''_h, \mu_h \rangle = m_h'$. After routine computations, one can show that

$$
\nu_h^{(k+3)} = m_h^{(k+2)} \mu_h + \binom{k + 2}{1} m_h^{(k+1)} (\ell_h r_h - m_h \nu_h) + \binom{k + 2}{2} m_h^{(k)} (\ell_h r_h - m_h \nu_h)' + \cdots + m_h (\ell_h r_h - m_h \nu_h)^{(k+1)}.
$$

Hence, by using $A_{k-1}$ singularity of $\ell_h$ we find that $\langle \nu_h^{(k+3)}, r_h \rangle = -(k+2)m_h' \ell_h^{(k)} - m_h \ell_h^{(k+1)}$. 

Figure 3: The spacelike frontal $r_h$ (red), its hyperbolic pedal curve relative to $Q_1 = (\sqrt{2}, 1, 0)$ (blue).
Figure 4: The spacelike frontal $r_h$ (red), its hyperbolic pedal curve relative to $Q_2 = (\sqrt{3}, 1, 1)$ (blue).

Now, let us prove the lemma for $j = i + 1$ by assuming that it is satisfied for $j \leq i$. Taking derivative of $\langle \nu_h^{(i+1)}, \nu_h \rangle \in \gamma_{i+1}^* \mathcal{M}_i \mathcal{E}_1$ yields

$$\langle \nu_h^{(i+2)}, \nu_h \rangle + \langle \nu_h^{(i+1)}, \nu'_h \rangle \in \gamma_{i+1}^* \mathcal{M}_{i+1} \mathcal{E}_1.$$ 

Since $\langle \nu_h^{(i+1)}, \nu'_h \rangle = m_h \langle \nu_h^{(i+1)}, \mu_h \rangle \in \gamma_{i}^* \mathcal{M}_{i+1} \mathcal{E}_1$ and $\gamma_{i}^* \mathcal{M}_{i+1} \mathcal{E}_1 \subset \gamma_{i+1}^* \mathcal{M}_{i+1} \mathcal{E}_1$, we obtain $\langle \nu_h^{(i+2)}, \nu_h \rangle \in \gamma_{i+1}^* \mathcal{M}_{i+1} \mathcal{E}_1$.

By differentiating $\langle \nu_h^{(i+1)}, \mu_h \rangle + \gamma_{i}^* \mathcal{M}_{i+1} \mathcal{E}_1 = m_h^{(i)} + \gamma_{i+1}^* \mathcal{M}_{i+1} \mathcal{E}_1$ and using (3), we deduce

$$\langle \nu_h^{(i+2)}, \mu_h \rangle + \ell_h \langle \nu_h^{(i+1)}, r_h \rangle - m_h \langle \nu_h^{(i+1)}, \nu_h \rangle + \gamma_{i+1}^* \mathcal{M}_{i+1} \mathcal{E}_1 = m_h^{(i+1)} + \gamma_{i+1}^* \mathcal{M}_{i+1} \mathcal{E}_1 \quad (9)$$

From the statement (1), we have $\langle \nu_h^{(i+1)}, \nu_h \rangle \in \gamma_{i}^* \mathcal{M}_i \mathcal{E}_1$. Now, we consider two cases.

Let $\ell_h \neq 0$ i.e. $k = 0$. By taking $j = i - 1 \ (i \geq 1)$ in the statement (3) we find that the statement (2) is satisfied for $j = i + 1$ if $\ell_h \neq 0$.

On the other hand, let $\ell_h = 0$ i.e. $k \neq 0$. By substituting $\ell_h = 0$ to (9) we find $\langle \nu_h^{(i+2)}, \mu_h \rangle + \gamma_{i+1}^* \mathcal{M}_{i+1} \mathcal{E}_1 = m_h^{(i+1)} + \gamma_{i+1}^* \mathcal{M}_{i+1} \mathcal{E}_1$. Hence, we see that the statement (2) holds for $j = i + 1$.

Finally, by differentiating $\langle \nu_h^{(i+k+2)}, r_h \rangle + \gamma_{i+1}^* \mathcal{M}_i \mathcal{E}_1 = -\binom{i+k+1}{i} \ell_h^{(i+k)} m_h^{(i)} + \gamma_{i+1}^* \mathcal{M}_i \mathcal{E}_1$ we find

$$\langle \nu_h^{(i+k+3)}, r_h \rangle + \ell_h \langle \nu_h^{(i+k+2)}, \mu_h \rangle + \gamma_{i+1}^* \mathcal{M}_{i+1} \mathcal{E}_1$$

$$= -\binom{i+k+1}{i} (\ell_h^{(i+k)} m_h^{(i)} + \ell_h^{(k)} + m_h^{(i+1)} + \gamma_{i+1}^* \mathcal{M}_{i+1} \mathcal{E}_1).$$

By considering two cases based on $\ell_h$ again, one can conclude the proof. \qed
Figure 5: The spacelike frontal \( r_h \) (red), its hyperbolic pedal curve relative to \( Q_3 = (1, 0, 0) \) (blue).

**Theorem 4.9.** Let \( (r_h, v_h) \) be a spacelike Legendre curve with a spacelike hyperbolic Legendre curvature \( (\ell_h, m_h) \) and \( Q \in H_+^2 \) be a point. Assume that \( m_h \) has an \( A_j^{-1} \)-type singularity and \( \ell_h \) has an \( A_k^{-1} \)-type singularity at \( s_0 \in I \). Then, the following statements are satisfied:

1. Suppose that \( Q = r_h(s_0) \). In this case, the map-germ \( \text{Ped}_Q(r_h) : (I, s_0) \to (H_+^2, \text{Ped}_Q(r_h)(s_0)) \) is \( C^1 \) \( A \)-equivalent to the map-germ \( (\mathbb{R}, 0) \to (\mathbb{R}^2, 0); t \mapsto (t^{j+k+2}, t^{2j+k+3}) \).

2. If \( Q \in G_{\nu h}(s_0) - \{r_h(s_0)\} \), then the map-germ \( \text{Ped}_Q(r_h) : (I, s_0) \to (H_+^2, \text{Ped}_Q(r_h)(s_0)) \) is \( C^1 \) \( A \)-equivalent to the map-germ \( (\mathbb{R}, 0) \to (\mathbb{R}^2, 0); t \mapsto (t^{j+1}, t^{2j+k+3}) \).

3. If \( Q \in H_+^2 - G_{\nu h}(s_0) \), then the map-germ \( \text{Ped}_Q(r_h) : (I, s_0) \to (H_+^2, \text{Ped}_Q(r_h)(s_0)) \) is \( C^1 \) \( A \)-equivalent to the map-germ \( (\mathbb{R}, 0) \to (\mathbb{R}^2, 0); t \mapsto (t^{j+1}, t^{j+k+2}) \).

**Proof.**

1. By a suitable rotation of \( H_+^2 \), it is reasonable to assume that \( Q = (1, 0, 0) \in H_+^2 \) and \( r_h(s_0) = (1, 0, 0) \), \( \nu_h(s_0) = (0, 1, 0) \) and \( \mu_h = (0, 0, 1) \) since \( Q = r_h(s_0) \). By Lemma 4.8 we may take the map-germ \( v_h : (I, s_0) \to (dS^2, v_h(s_0)) \) as

\[
v_h(s) = \begin{pmatrix} \frac{(j+k+1)}{j} \ell_h^{(k)}(s_0)m_h^{(j)}(s_0)(s - s_0)^{j+k+2} + C(s - s_0) \\ 1 + A(s - s_0) \\ \frac{1}{(j+1)!}m_h^{(j)}(s_0)(s - s_0)^{j+1} + B(s - s_0) \end{pmatrix},
\]
where $A, B$ and $C$ are some $C^\infty$ function-germs $(\mathbb{R}, 0) \to (\mathbb{R}, 0)$. Furthermore, $A$ and $B$ have $A_{j+1}$ singularity at 0 while $C$ has at least an $A_{j+k+1}$ singularity at 0. By Lemma 4.1 we know

$$
\varphi_{Q,1}(f(x_1, x_2, x_3)) = (\tanh(\lambda)x_2, \frac{x_3}{x_2}), \quad \sinh \lambda = x_1
$$

which yields

$$
\varphi_{Q,1}(f(u_h(s))) = \left( \frac{\tanh(\lambda) (1 + A(s - s_0))}{m_h^{(j)}(s_0)(s - s_0)^{j+1} + (j + 1)! B(s - s_0)} \right). \quad (10)
$$

Since $q \circ \tilde{\psi}_Q \circ \varphi_{Q,1}^{-1}(u_1, u_2) = (u_1, u_2 u_1)$, it follows from (10) that the map-germ $\tilde{\psi}_Q \circ v_h : (I, s_0) \to (dS^2, \tilde{\psi}_Q \circ v_h(s_0))$ is $C^\infty A$-equivalent to

$$
\begin{pmatrix}
\tanh(\lambda) (1 + A(s - s_0)) \\
\tanh(\lambda) m_h^{(j)}(s_0)(s - s_0)^{j+1} + (j + 1)! B(s - s_0)
\end{pmatrix}.
$$

Then, since the first component of $v_h$ is $\sinh \lambda$ we see that the map germ $\mathcal{P} ed_Q(r_h) : (I, s_0) \to (\mathcal{H}^2_+, \mathcal{P} ed_Q(r_h)(s_0))$ is $C^\infty A$-equivalent to

$$
\begin{pmatrix}
\frac{(j+k+1)}{2} \ell_h^{(j)}(s_0)m_h^{(j)}(s_0)(s - s_0)^{j+k+2} + \hat{C}(s - s_0) \\
\frac{(j+k+1)}{2} \ell_h^{(j)}(s_0)(m_h^{(j)}(s_0))^2(s - s_0)^{2j+k+3} + \hat{B}(s - s_0)
\end{pmatrix},
$$

where $\hat{B}$ and $\hat{C}$ are certain $C^\infty$ function-germs $(\mathbb{R}, 0) \to (\mathbb{R}, 0)$. Moreover, $\hat{B}$ has at least an $A_{2j+k+3}$ singularity at 0 while $\hat{C}$ has at least an $A_{j+k+2}$ singularity at 0.

Since $\frac{(j+k+1)}{2} \ell_h^{(j)}(s_0)m_h^{(j)}(s_0) \neq 0$ and $\hat{C}(s - s_0)$ has at least an $A_{j+k+2}$-type singularity at 0, by using Lemma 4.3 we find that $\mathcal{P} ed_Q(r_h)$ is $C^\infty A$-equivalent to

$$
(t^{j+k+2}, t^{2j+k+3} + D(t)),
$$

where $D : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is a $C^\infty$ function germ with $\frac{d^p D}{dt^p}(0) = 0$ for $p \leq 2j + k + 3$.

We have two cases:

(a) Let $2j + k + 3$ be odd. Then, we consider

$$
h_2(x_1, x_2) = (x_1, x_2 + D(x_2^{j+k+3})).
$$

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Since $2j+k+3$ is odd, we see that $x_2 \mapsto D(x_2^{\frac{1}{2j+k+3}})$ is well-defined and continuous everywhere. In addition to this, there exists a $C^\infty$ function-germ, by Lemma 4.4 $D : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $D(t) = t^{2j+k+4} \tilde{D}(t)$. Therefore, we obtain the following:

\[
\frac{dD(x_2^{\frac{1}{2j+k+3}})}{dx_2} = \lim_{h \to 0} \frac{D((x_2 + h)^{\frac{1}{2j+k+3}}) - D(x_2^{\frac{1}{2j+k+3}})}{h} \\
= \lim_{h \to 0} \frac{(x_2 + h)^{\frac{2j+k+4}{2j+k+3}} \tilde{D}((x_2 + h)^{\frac{1}{2j+k+3}}) - x_2^{\frac{2j+k+4}{2j+k+3}} \tilde{D}(x_2^{\frac{1}{2j+k+3}})}{h} \\
= \lim_{h \to 0} \frac{x_2^{\frac{2j+k+4}{2j+k+3}} \tilde{D}((x_2 + h)^{\frac{1}{2j+k+3}}) - \tilde{D}(x_2^{\frac{1}{2j+k+3}})}{h} \\
+ \frac{2j + k + 4}{2j + k + 3} x_2^{\frac{1}{2j+k+3}} \tilde{D}(x_2^{\frac{1}{2j+k+3}}) \\
+ \frac{1}{2j + k + 3} \tilde{D}'(x_2^{\frac{1}{2j+k+3}}) \\
\]

Hence, $x_2 \mapsto \frac{dD(x_2^{\frac{1}{2j+k+3}})}{dx_2}$ is well-defined and continuous everywhere. Since we have $\frac{dD(x_2^{\frac{1}{2j+k+3}})}{dx_2}(0) = 0$, the Jacobian matrix of $h_2$ at $(0, 0)$ is the unit matrix which yields that $h_2$ is a germ of $C^1$-diffeomorphism.

(b) Let $2j + k + 3$ be even. Consider

\[
h_3(x_1, x_2) = (x_1, x_2 + \hat{D}(x_2)),
\]

where

\[
\hat{D}(x_2) = \begin{cases} 
D(x_2^{\frac{1}{2j+k+3}}) & ; \ x_2 \geq 0 \\
-D((-x_2)^{\frac{1}{2j+k+3}}) & ; \ x_2 < 0.
\end{cases}
\]

One can see that $x_2 \mapsto \hat{D}(x_2)$ is well-defined and continuous even at $x_2 = 0$. Furthermore, we find that

\[
\hat{D}'(x_2) = \begin{cases} 
\frac{D'(x_2^{\frac{1}{2j+k+3}})}{(2j + k + 3)x_2^{\frac{2j+k+2}{2j+k+3}}} \ ; \ x_2 > 0 \\
\frac{D'((-x_2)^{\frac{1}{2j+k+3}})}{(2j + k + 3)(-x_2)^{\frac{2j+k+2}{2j+k+3}}} \ ; \ x_2 < 0 \\
0 \ ; \ x_2 = 0
\end{cases}
\]
Then, we see that $x_2 \mapsto \frac{d\hat{D}(x_2^{2j+k+3})}{dx_2}$ is well-defined and continuous everywhere. In addition, we see that the Jacobian matrix of $h_3$ at $(0,0)$ is the unit matrix. So, $h_3$ is a germ of $C^1$-diffeomorphism. Thus, the map-germ $\mathcal{P}ed_Q(r_h)$ is $C^1$ $\mathcal{A}$-equivalent to the map-germ; $t \mapsto (t^{j+k+2}, t^{2j+k+3})$.

2. Assume that $Q = (1,0,0)$. By an appropriate rotation of $\mathcal{H}_+^2$, we may also assume that $\nu_h(s_0) = (0,1,0)$. Moreover, we may take $r_h(s_0) = (a,0,b)$ and $\mu(s_0) = (b,0,a)$ such that $a^2 - b^2 = 1$, $a,0 \neq b \notin \mathbb{R}$ since $Q \in G_{\nu_h(s_0)} - \{r_h(s_0)\}$. Thus, we obtain that the map-germ $\nu_h : (I,s_0) \mapsto (dS^2,\nu_h(s_0))$ is $C^\infty$ $\mathcal{A}$-equivalent to the map-germ

$$
\nu_h(s) = \begin{pmatrix}
 a\gamma + b\delta \\
 1 + A(s-s_0) \\
 b\gamma + a\delta
\end{pmatrix},
$$

where

$$
\gamma(s) = \frac{1}{(j+1)!} m_h^{(j)}(s_0)(s-s_0)^{j+1} + B(s-s_0)
$$

and

$$
\delta(s) = \frac{(j+k+1)}{(j+2)!} f_h^{(k)}(s_0)m_h^{(j)}(s_0)(s-s_0)^{j+k+2} + C(s-s_0)
$$

and $A,B$ and $C$ are some $C^\infty$ function-germs $([0,0]) \mapsto ([0,0])$. Furthermore, $A$ and $B$ have at least $A_{j+1}$ singularity at $0$ while $C$ has at least an $A_{j+k+2}$ singularity at $0$. Hence, we find that

$$
\varphi_Q,1(f(\nu_h(s))) = \begin{pmatrix}
 \tanh(\lambda)(1 + A(s-s_0)) \\
 b\gamma + a\delta \\
 (1 + A(s-s_0))
\end{pmatrix}.
$$

where $\sinh \lambda = a\gamma + b\delta$. For a linear isomorphism $h_1 : \mathbb{R}^2 \mapsto \mathbb{R}^2$ given by $h_1(u_1,u_2) = (u_1,u_2 + \frac{b}{a}u_1)$ and a $C^\infty$ diffeomorphism $h_2 : \mathbb{R}^2 \mapsto \mathbb{R}^2$ given by $h_2(U_1,U_2) = (U_1,U_2 + \frac{b}{a}U_2)$, it is easy to show that the following is satisfied [25, Lemma 5.1]:

$$
q \circ \tilde{\psi}_Q \circ \varphi_Q^{-1}_{Q,1} \circ h(u_1,u_2) = h_2 \circ q \circ \tilde{\psi}_Q \circ \varphi_Q^{-1}_{Q,1}(u_1,u_2).
$$

Thus, by taking $u_1 = \tanh(\lambda)(1 + A(s-s_0))$ and $u_2 = \frac{b\gamma + a\delta}{(1 + A(s-s_0))}$ we obtain that $q \circ \tilde{\psi}_Q \circ \varphi_Q^{-1}_{Q,1} \circ h(u_1,u_2)$ is $C^\infty$ $\mathcal{A}$-equivalent to $\mathcal{P}ed_Q(r_h)$ near $s_0$. In this case, by using Taylor expansions $q \circ \tilde{\psi}_Q \circ \varphi_Q^{-1}_{Q,1} \circ h(u_1,u_2)$ can be written as

$$
\begin{pmatrix}
 \frac{1}{(j+1)!} m_h^{(j)}(s_0)(s-s_0)^{j+1} + \hat{C}(s-s_0) \\
 b^2 \frac{(j+k+1)}{(j+2)!(j+1)!} f_h^{(k)}(s_0)(m_h^{(j)}(s_0))^2(s-s_0)^{2j+k+3} + \hat{B}(s-s_0)
\end{pmatrix},
$$

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where \( \hat{B} \) and \( \hat{C} \) are some \( C^\infty \) function-germs \((\mathbb{R}, 0) \to (\mathbb{R}, 0)\). Furthermore, \( \hat{B} \) has at least an \( A_{2j+k+3} \) singularity at 0 while \( \hat{C} \) at least has an \( A_{j+1} \) singularity at 0. By using Lemma 4.3 and Lemma 4.4 we obtain that \( P_{Ed_Q}(\nu_h) \) is \( C^\infty A \)-equivalent to 
\[
(t^{j+1}, t^{2j+k+3} + D(t)),
\]
where \( D : (\mathbb{R}, 0) \to (\mathbb{R}, 0) \) is a \( C^\infty \) function-germ with \( \frac{d^pD}{dt^p}(0) = 0 \) for \( p \leq 2j+k+3 \). Hence, the proof follows from similar arguments to those of (1).

3. Let us assume that \( Q = (1, 0, 0) \). Since \( Q \in H^2_+ - G_{\nu_h(s_0)} \), we have that \( \langle Q, \nu_h(s_0) \rangle \neq 0 \). Then, we can assume that \( \nu_h(s_0) = (1, 0, \sqrt{2}) \), \( r_h(s_0) = (\sqrt{2}, 0, 1) \) and \( \mu_h(s_0) = (0, 1, 0) \). By Lemma 4.4, one can see that the map-germ \( \nu_h : (I, s_0) \to (dS^2, \nu_h(s_0)) \) is \( C^\infty A \)-equivalent to the map-germ defined by
\[
\nu_h(s) = \begin{pmatrix}
1 + A(s-s_0) \\
\frac{1}{(j+1)!}m_h^{(j)}(s-s_0)^{j+1} + B(s-s_0) \\
\sqrt{2} + \frac{1}{(j+2)!}l_h^{(k)}(s-s_0)^{j+k+2} + C(s-s_0)
\end{pmatrix},
\]
where \( A, B \) and \( C \) are some \( C^\infty \) function-germs \((\mathbb{R}, 0) \to (\mathbb{R}, 0)\). Furthermore, \( A \) and \( B \) have at least \( A_{j+1} \) singularity at 0 while \( C \) has at least an \( A_{j+k+1} \) singularity at 0. From Lemma 4.3, and by applying suitable scales and reflections along coordinate axes of \( \mathbb{R}^2_1 \) if necessary, it can be seen that \( P_{Ed_Q}(r_h) \) is \( C^\infty A \)-equivalent to the following form:
\[
(t^{j+1}, t^{j+k+2} + D(t)),
\]
where \( D : (\mathbb{R}, 0) \to (\mathbb{R}, 0) \) is \( C^\infty \) function-germs with \( \frac{d^pD}{dt^p}(0) = 0 \) for \( p \leq j+k+2 \). By considering two cases similar to (1), one can conclude the proof.

\[\square\]

**Example 4.10.** Let us consider the curve \( r_h(s) = (\sqrt{1 + s^6 + s^{14}}, s^3, s^7) \) in \( H^2_+ \). Differentiating the curve with respect to \( s \) yields 
\[
\dot{r}_h(s) = \left( \frac{6s^5 + 14s^{13}}{2\sqrt{1 + s^6 + s^{14}}}, 3s^2, 7s^6 \right).
\]
By taking \( v_h : I \to dS^2 \) as
\[
v_h(s) = \frac{1}{\sqrt{16s^{14} + 49s^8 + 9}}(4s^7\sqrt{1 + s^6 + s^{14}}, 7s^4 + 4s^{10}, 4s^{14} - 3),
\]
we obtain that \( (r_h, v_h) : I \to D_1 \) is a spacelike Legendre curve with 
\[
\ell_h(s) = \frac{s^2\sqrt{16s^{14} + 49s^8 + 9}}{\sqrt{1 + s^6 + s^{14}}} , \quad m_h(s) = \frac{4s^3(16s^{20} + 70s^{14} + 30s^6 + 21)}{(16s^{14} + 49s^8 + 9)\sqrt{1 + s^6 + s^{14}}}.
\]

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After simple computations, we see that $m_h(s)$ has an $A_2$-type singularity at $0 \in I$. Take a point $Q = (1, 0, 0) = r_h(0) \in \mathcal{H}^2_+$. It is easy to see that $\ell_h(0) = 0$, $\ell_h'(0) = 0$ and $\ell_h''(0) \neq 0$ which means that $\ell_h$ has an $A_1$ singularity at $s_0 = 0$. Thus, by Theorem 4.9 (2) we have that the map-germ $\text{Ped}_Q(r_h) : (I, 0) \to (\mathcal{H}^2_+,(1,0,0))$ is $C^1$-equivalent to the map-germ given by $s \mapsto (s^7, s^{11})$. The hyperbolic pedal curve of $r_h$ with respect to $Q$ is given in Fig. 6.

5 Orthotomics of spacelike frontals in the hyperbolic plane

We now present the definition of an hyperbolic orthotomic curve of a spacelike frontal in the hyperbolic 2-space. And then, inspired by the paper [28, 22] we shall investigate the singularities of hyperbolic orthotomics and the relationships of such curves with hyperbolic pedal curves defined in the above sections.

Let us take a spacelike Legendre curve $(r_h,v_h)$ with spacelike hyperbolic Legendre curvature $(\ell_h,m_h)$. Based on the definition of a hyperbolic pedal curve in $\mathcal{H}^2$, an hyperbolic orthotomic of the spacelike frontal $r_h(s)$ relative to a point $Q$ is defined as

$$\text{Ort}_Q(r_h)(s) = Q - 2\langle Q, v_h(s) \rangle v_h(s).$$

(11)

It seems that this is a natural generalization of plane orthotomic curves to hyperbolic 2-space. A similar reason to one for pedal curves makes us realize that $Q$ and $r_h$ must lie on the same part of the hyperbolic 2-space. Then we assume both in $\mathcal{H}^2_+$. Similar results to Proposition 3.2 and 3.3 are simply satisfied for (11). The derivative of $\text{Ort}_Q(r_h)(s)$ can be calculated as

$$\dot{\text{Ort}}_Q(r_h)(s) = -2m_h(s)(\langle Q, \mu_h(s) \rangle v_h(s) + \langle Q, v_h(s) \rangle \mu_h(s)).$$
which means that $\text{Ort}_Q(r_h)$ has a singularity at $s_0$ if and only if $m_h(s_0) = 0$ or $Q = r_h(s_0)$. The definitions of pedal curves and orthotomics obviously suggest the following relationship:

$$\frac{\text{Ort}_Q(r_h)(s) + Q}{2} = -(Q, \text{Ped}_Q(r_h)(s)) \text{Ped}_Q(r_h)(s).$$

Then, it is clear that

$$\text{Ort}_Q(r_h)(s) = -2(Q, \text{Ped}_Q(r_h)(s)) \text{Ped}_Q(r_h)(s) - Q.$$ 

Let $Q$ be a point of $H^2_+$. Consider the $C^\infty$ map

$$\phi_Q : H^2_+ \to H^2_+ \quad \quad x \mapsto \phi_Q(x) = -2(Q, x) - Q.$$ 

Then, we can give the following lemma.

**Lemma 5.1.** Let us take a spacelike Legendre curve $(r_h, v_h)$ with spacelike hyperbolic Legendre curvature $(\ell_h, m_h)$. Let $Q \in H^2_+$ be a point. Then, the following is satisfied:

$$\text{Ort}_Q(r_h)(s) = \phi_Q \circ \text{Ped}_Q(r_h)(s) = \phi_Q \circ \psi_Q \circ v_h(s).$$

By the above lemma, we can say that $\text{Ort}_Q(r_h)$ has a singularity at $s_0$ if and only if $\text{Ped}_Q(r_h)$ has a singularity at $s_0$ or $\text{Ped}_Q(r_h)(s_0)$ is a singular point of $\phi_Q$.

**Theorem 5.2.** Let $(r_h, v_h)$ be a spacelike Legendre curve with a spacelike hyperbolic Legendre curvature $(\ell_h, m_h)$ and $Q \in H^2_+$ be a point. For $s_0 \in I$, the hyperbolic orthotomic curve-germ $\text{Ort}_Q(r_h) : (I, s_0) \to (H^2_+, \text{Ort}_Q(r_h)(s_0))$ is $C^\infty \mathcal{L}$-equivalent to the hyperbolic pedal curve-germ $\text{Ped}_Q(r_h) : (I, s_0) \to (H^2_+, \text{Ped}_Q(r_h)(s_0))$.

**Proof.** We already mentioned in Lemma 5.1 that it is possible to write $\text{Ort}_Q(r_h)(s) = \phi_Q \circ \text{Ped}_Q(r_h)(s)$. We defined the set $\tilde{H}_Q = \{y \in H^2_+ | \langle Q, y \rangle < 0\}$ in Section 4. It is easy to see that any point of $\tilde{H}_Q$ is a regular point of $\phi_Q$. On the other hand, obviously the map $\phi_Q|_{\tilde{H}_Q} : \tilde{H}_Q \to \phi_Q(\tilde{H}_Q)$ is bijective. Thus, we see that the map $\phi_Q|_{\tilde{H}_Q}$ is a $C^\infty$ diffeomorphism. So, the fact that $\text{Ped}_Q(r_h)(I)$ is a subset of $\tilde{H}_Q$ concludes the proof.

**Remark 5.3.** The importance of the above theorem is that it naturally characterizes the singularities of hyperbolic orthotomics of spacelike frontals based on the hyperbolic pedal curves whose singularities (based on $C^1$-equivalence) have already investigated in Section 4.

In our last theorem, we see that an orthotomic of a spacelike frontal $r_h$ relative to the point $Q \in H^2 - r_h(I)$ is a spacelike frontal as well.
Theorem 5.4. Let \((r_h, v_h)\) be a spacelike Legendre curve with spacelike hyperbolic Legendre curvature \((\ell_h, m_h)\) and \(Q\) be any point in \(H^2 - r_h(I)\). The orthotomic \(\text{Ort}_Q(r_h)\) of the spacelike frontal relative to \(Q\) is a spacelike frontal that is, \((\text{Ort}_Q(r_h), \tilde{v}_h)\) is a spacelike Legendre curve with spacelike hyperbolic Legendre curvature \((\tilde{\ell}_h, \tilde{m}_h)\), where

\[
\tilde{v}_h = \frac{(Q, r_h)^2 - 1)r_h + (Q, r_h)(Q, v_h)v_h - (Q, r_h)(Q, \mu_h)\mu_h}{\sqrt{(Q, r_h)^2 - 1}},
\]
\[
\tilde{\mu}_h = Q - (Q, r_h)r_h
\]
\[
\tilde{\ell}_h = -2m_h\sqrt{(Q, r_h)^2 - 1},
\]
\[
\tilde{m}_h = (\tilde{v}_h(s), \tilde{\mu}_h(s)).
\]

Proof. The proof can be handled by a similar idea of the proof of Theorem 3.6. \(\square\)

Example 5.5. Let us consider the hyperbolic astroid given in Example 3.7. Chosing the point \(Q_1 = (1, 0, 0)\), we find the hyperbolic orthotomic of \(r_h\) relative to \(Q_1\) as (See Fig 7 (a)):

\[
\text{Ort}_{Q_1}(r_h) = Q_1 - 2(Q_1, v_h)v_h
\]
\[
= \left( \frac{-95 + 28 \cos(4s) + 3 \cos(8s)}{8(-9 + \cos(4s))}, \frac{2 \cos s \sin^2 s(1 + \cos^4 s)\sqrt{1 + \cos^6 s + \sin^6 s}}{1 + \cos^2 s \sin^2 s}, \right)
\]
\[
\frac{2 \cos^2 s \sin s(1 + \sin^4 s)\sqrt{1 + \cos^6 s + \sin^6 s}}{1 + \cos^2 s \sin^2 s}.
\]

If we choose the point as \(Q_2 = \frac{1}{2}(\sqrt{5}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\), then the hyperbolic orthotomic of \(r_h\) relative to \(Q_2\) can be obtained similarly (See Fig. 7 (b)).

6 An application to optics

Based on the above discussions, it would be better to present an application. In a plane, the caustic by reflection (or catacaustic) of a curve with respect to a light source \(Q\) is the envelope of the rays emitted by \(Q\) after reflection on a mirror with profile. By definition, a catacaustic of a curve relative to a point light source \(Q\) is nothing but the evolute of the orthotomic of the curve relative to \(Q\). This definition was generalized to curves on sphere and was examined in terms of both singularity and optics, [2, 30].

Our purpose is to extend such curves to frontals on hyperbolic 2-space. In Section 5, we have already investigated orthotomics of spacelike frontals. On the other hand, in [6] the authors defined the evolute of a spacelike frontal \(r_h\) in hyperbolic 2-space as

\[
\mathcal{E}v(r_h)(s) = \pm \frac{1}{\sqrt{[m_h^2(s) - \ell_h^2(s)]}}(m_h(s)r_h(s) - \ell_h(s)v_h(s)).
\]

(12)
Figure 7: The hyperbolic astroid (red), its hyperbolic orthotomic curve (green).

We see that $E_v(r_h) \in \mathcal{H}^2$ (resp. $E_v(r_h) \in dS^2$) if $m_h^2 > \ell_h^2$ (resp. $m_h^2 < \ell_h^2$). We shall consider only the case $E_v(r_h) \in \mathcal{H}^2_+$ when $m_h^2 > \ell_h^2$ since we could just take $-E_v(r_h)$ instead of $E_v(r_h)$ otherwise.

Taking the orthotomic of $r_h$ relative to the point $Q \in \mathcal{H}^2_+$ instead of $r_h$ in Eq. (12) one could have the catacaustic of $r_h$ with respect to the light source $Q$, that is $E_v(Ort_Q(r_h))$. But, this is reasonable only if Theorem 5.4 is satisfied. This means that the light source cannot be on the mirror $r_h$. On the other hand, we see that the above discussion about singularities of an orthotomic and the information on singularities of the evolute of a spacelike frontal in [6] naturally characterize the singularities of the catacaustic. Let us give some results that can be proved by using the methods presented in [6].

**Theorem 6.1.** Suppose that $(r_h, v_h)$ is a spacelike Legendre curve with spacelike hyperbolic Legendre curvature $(\ell_h, m_h)$ and $Q$ be any point in $\mathcal{H}^2 - r_h(I)$. Suppose further that $Ort_Q(r_h)$ and $C_Q(r_h)$ denote the orthotomic and the catacaustic of $r_h$ with respect to the point $Q$, respectively. Then, the followings are satisfied.

(i) If $s_0$ is a singular point of $Ort_Q(r_h)$, then $C_Q(r_h)(s_0) = \pm Ort_Q(r_h)(s_0)$.

(ii) If $s_0$ is a singular point of $\bar{v}_h$, then $C_Q(r_h)(s_0) = \pm \bar{v}_h(s_0)$.

(iii) $s_0$ is a regular point of $C_Q(r_h)(s_0)$ if and only if $\hat{\ell}_h(s_0) \neq 0$ or equivalently $m_h(s_0) \neq 0$.

(iv) $s_0$ is a singular point of $C_Q(r_h)(s_0)$ if and only if $\hat{Ort}_Q(s_0) = 0$.

Let us present an example.

**Example 6.2.** Consider the hyperbolic astroid given in Example 3.7. Let us choose a point light source located at $Q_1 = (1, 0, 0)$. We have already found the orthotomic of $r_h$ relative...
to \( Q_1 \) in Example 5.5. So, one can compute the catacaustic of \( r_h \) relative to the light source \( Q_1 \) by using Eq. (12) and a straightforward calculation (See Fig. 8). Notice that we just consider the catacaustic in \( \mathcal{H}^2_+ \) here as an example.

Figure 8: The hyperbolic astroid (red), its caustic (purple) relative to the light source (yellow).

7 Conclusion

In this paper, we have focused on pedal curves of spacelike frontals and classifications of their singularities in hyperbolic 2-space. We have classified singularities of pedal curves constructed by non-singular and singular dual curve germs, respectively. We have supported our results by illustrated examples. Our work suggests several avenues for future research. One of them would be investigate similar problems for de Sitter space. Another direction would be present the complete classification of the singularities of hyperbolic pedal curves in the \( n \)-dimensional hyperbolic space. In order to do that, it is necessary to define frontals in Hyperbolic \( n \)-space and to obtain a Legendrian moving frame along these frontals. We gave some singularity results based on \( C^1-\mathcal{L} \) (or \( \mathcal{A} \)) equivalence. It may be possible to extend those results to \( C^\infty-\mathcal{L} \) (or \( \mathcal{A} \)) equivalence.

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