MIXING PROPERTIES OF GENERALIZED $T, T^{-1}$ TRANSFORMATIONS

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Abstract. We study mixing properties of generalized $T, T^{-1}$ transformation. We discuss two mixing mechanisms. In the case the fiber dynamics is mixing, it is sufficient that the driving cocycle is small with small probability. In the case the fiber dynamics is only assumed to be ergodic, one needs to use the shearing properties of the cocycle. Applications include the Central Limit Theorem for sufficiently fast mixing systems and the estimates on deviations of ergodic averages.

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1. Introduction

An important discovery made in the last century is that deterministic systems can exhibit chaotic behavior. Currently there are many examples of systems exhibiting a full array of chaotic properties including Bernoulli property, exponential decay of correlations and central limit theorem (see e.g. [9, 10, 14, 64]). Systems which satisfy only some of the above properties are less understood. In fact, it is desirable to have more examples of such systems in order to understand the full range of possible behaviors of partially chaotic systems.

Generalized $T, T^{-1}$ transformations are a rich source of examples in probability and ergodic theory. In fact, they were used to exhibit examples of systems with unusual limit laws [47, 16], central limit theorem with non standard normalization [8], K but non Bernoulli systems in abstract [43] and smooth setting in various dimensions [45, 58, 44], very weak Bernoulli but not weak Bernoulli partitions [18], slowly mixing systems [19, 49], systems with multiple Gibbs measures [31, 53].

A comprehensive survey of probabilistic version of $T, T^{-1}$ transformations, which is a random walk in random scenery is contained in [20]. On the other hand there are no works addressing how statistical properties of $T, T^{-1}$ transformations depend on the properties of the base and the fiber dynamics. Our paper provides a first step in this direction by investigating mixing properties of $T, T^{-1}$ transformations.

Let us explain what we mean by smooth $T, T^{-1}$ transformations. Let $X, Y$ be compact manifolds, $f : X \to X$ be a smooth map preserving a measure $\mu$ and $G_t : Y \to Y$ be a $d$ parameter flow on $Y$ preserving a measure $\nu$. Let $\tau : X \to \mathbb{R}^d$ be a smooth map. We study the following map $F : (X \times Y) \to (X \times Y)$

$$F(x, y) = (f(x), G_{\tau(x)}y).$$

Note that $F$ preserves the measure $\zeta = \mu \times \nu$ and that

$$F^N(x, y) = (f^N x, G_{\tau_N(x)}y) \quad \text{where} \quad \tau_N(x) = \sum_{n=0}^{N-1} \tau(f^n x).$$

Clearly both mixing of $f$ and ergodicity of $G$ are necessary for $F$ to be mixing. Under these assumptions there are two mechanisms for $F$ to be mixing.

1. If $G$ itself is mixing then it is enough to ensure that $\tau_N$ does not take small values with large probability (cf. [19, 49]).

2. On the other hand if we only assume that $G$ is ergodic then we need to rely on shearing properties of $\tau$ to ensure that $\tau_N$ is uniformly distributed in boxes of size 1. This can be done by assuming various extension of the Central Limit Theorem (cf. [11, 26]).

Abstract results detailing sufficient conditions for each of the two mechanisms described above are presented in Section 2. Estimates on the rates of mixing of $F$ under the assumption that $G$ is mixing are given in Section 4. In Section 5 we prove the Central Limit Theorem in case $F$ mixes sufficiently quickly. Section 6 contains mixing estimates in the case $G$ is only assumed to be ergodic (however, we need much stronger assumptions on the base map $f$). The results presented in Sections 3–6 rely on preliminary facts contained in Section 3. In Section 7 we discuss several examples which
require a combination of ideas from Sections 4 and 6. Section 8 presents application of our mixing results to deviations of ergodic averages and also contains a survey of examples of systems satisfying various assumptions required in our results. We will have some strong assumptions that are sometimes non-trivial to check. In the appendices, we include two examples where we check some of our conditions, namely the anticoncentration large deviation bounds for subshifts of finite type and an estimate on the deviation of ergodic averages for area preserving flows on surfaces of genus $\geq 2$. The results of the appendices may be interesting outside of the scope of the present work.

We also mention that in a forthcoming paper we provide a description of further statistical properties of the generalized $T, T^{-1}$ transformation, using the mixing bounds obtained in the present paper.

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2. Sufficient conditions for mixing

2.1. Local Limit Theorem and Mixing. For a function $A \in L^1(X, \mu)$ we denote $\mu(A(\cdot)) := \int_X A(x) \, d\mu$.

Definition 2.1. $\tau$ satisfies mixing LLT if there exist sequences $(L_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, $(D_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ and a bounded probability density $p$ on $\mathbb{R}^d$ such that for any sequence $(\delta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ with $\lim_{n \to \infty} \delta_n = 0$, $(z_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ such that $|\frac{z_n}{\delta_n} - z| < \delta_n$ for any cube $C \subset \mathbb{R}^d$ and any continuous functions $A_0, A_1 : X \to \mathbb{R}$,

$$
\lim_{n \to \infty} L_n^d \mu \left( A_0(\cdot)A_1(f^n(\cdot))\mathbb{1}_C(\tau_n - D_n - z_n) \right) = p(z) \mu(A_0) \mu(A_1) \text{Vol}(C),
$$

and the convergence is uniform once $(\delta_n)_{n \in \mathbb{N}}$ is fixed and $A_0, A_1, z$ range over compact subsets of $C(X), C(X)$ and $\mathbb{R}^d$ respectively.

Definition 2.2. We say that, $\tau$ satisfies mixing multiple LLT if for each $m \in \mathbb{N}$, any sequence $(\delta_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ with $\lim_{n \to \infty} \delta_n = 0$, and any family of sequences $(z_n^{(1)}, \ldots, z_n^{(m)})_{n \in \mathbb{N}}$ with $|\frac{z_n^{(j)}}{\delta_n} - z^{(j)}| < \delta_n$, any cubes $\{C_j\}_{j \leq m} \subset \mathbb{R}^d$ and continuous functions $A_0, \ldots, A_m : X \to \mathbb{R}$, for any sequences $n_k^{(1)}, \ldots, n_k^{(m)} \in \mathbb{N}$ such that $n_k^{(j)} - n_k^{(j-1)} \geq \delta_k^{-1}$ (with $n_k^{(0)} = 0$),

$$
\lim_{k \to \infty} \left( \prod_{j=1}^{m} I_{n_k^{(j)} - n_k^{(j-1)}} \right) \mu \left( \prod_{j=0}^{m} A_j \left( f^{n_k^{(j)}}(\cdot) \right) \prod_{j=1}^{m} \mathbb{1}_{C_j} \left( \tau_{n_k^{(j)}} - D_{n_k^{(j)}} - z^{(j)}_{n_k^{(j)}} \right) \right) = \prod_{j=0}^{m} \mu(A_j) \prod_{j=1}^{m} p \left( z^{(j)} - z^{(j-1)} \right) \prod_{j=1}^{m} \text{Vol}(C_j)
$$

where $z^{(0)} = 0$. Moreover, the convergence is uniform once $(\delta_n)_{n \in \mathbb{N}}$ is fixed, $A_0, \ldots, A_m$ range over compact subsets of $C(X)$ and $z^{(j)}$ range over a compact subset of $\mathbb{R}^d$ for every $j \leq m$. 

Theorem 2.5. Let \( \hat{\mathcal{M}} \) be centered at 0 with radius \( \delta_n \) and denote by \( A_n(y) := A_0(y) B_i(y) \) be a continuous function on \( X \times Y \).

(a) For \( i = 1, 2 \), let \( \Phi_i(x, y) = A_i(x) B_i(y) \) be a continuous function on \( X \times Y \). Since linear combinations of products as above are dense in \( L^2(\mu \times \nu) \), it suffices to show that for every \( \epsilon > 0 \) there exists \( N_0 \in \mathbb{N} \) such that for every \( N \geq N_0 \), we have

\[
\lim_{n \to \infty} L_n^d \mu \left( A_0(\cdot) A_1(f^n(\cdot)) \phi(\tau_n - D_n - z_n) \right) = p(z) \mu(A_0) \mu(A_1) \int_{\mathbb{R}^d} \phi(t) dt
\]

and the convergence is uniform if \( A_0, A_1 \) range over compact subsets of \( C(X) \) and \( z \) ranges over a compact subset of \( \mathbb{R}^d \). A similar remark applies to the multiple mixing LLT.

Theorem 2.5. Suppose that \( (G_t) \) is ergodic.

(a) If \( \tau \) satisfies the mixing LLT then \( F \) is mixing.

(b) If \( \tau \) satisfies the mixing multiple LLT then \( F \) is multiple mixing.

Proof. (a) For \( i = 1, 2 \), let \( \Phi_i(x, y) = A_i(x) B_i(y) \) be a continuous function on \( X \times Y \). Since linear combinations of products as above are dense in \( L^2(\mu \times \nu) \), it suffices to show that for every \( \epsilon > 0 \) there exists \( N_0 \in \mathbb{N} \) such that for every \( N \geq N_0 \), we have

\[
\bigg| \int_{X \times Y} \Phi_1(x, y) \Phi_2(F^N(x, y)) d(\mu \times \nu) - \mu(A_1) \mu(A_2) \nu(B_1) \nu(B_2) \bigg| < \epsilon.
\]

Let \( \rho(t) := \int_Y B_1(y) B_2(G_t y) d\nu(y) \). Note that

\[
\int_{X \times Y} \Phi_1(x, y) \Phi_2(F^N(x, y)) d(\mu \times \nu) = \int_X A_1(x) A_2(f^N(x)) \cdot \rho(\tau_N(x)) d\mu(x).
\]

Let \( \delta = \delta(\epsilon) > 0 \) be small with respect to \( \epsilon \), and \( I_0 \subset \mathbb{R}^d \) be a cube of volume \( \delta^d \), centered at 0. Consider a (disjoint) cover of \( \mathbb{R}^d \) by a union of small cubes \( \{ I_j \} \), where \( I_j \) is a translation of \( I_0 \) and let \( t_j \) denote the center of \( I_j \). Now let \( B \subset \mathbb{R}^d \) be a ball centered at 0 with radius \( \ell \), and denote \( S_\ell := \{ j : I_j \cap B \neq \emptyset \} \). By the mixing LLT (with \( A_0 = A_1 = 1 \)) it follows that there exists \( K = K(\epsilon) \) and \( N'_0 \in \mathbb{N} \) such that for every \( N \geq N'_0 \),

\[
\mu \left( \{ x \in X : |\tau_N - D_N| > K L N / 2 \} \right) < \epsilon / 2.
\]

Let \( \hat{\mathcal{S}}_1 := S_{KL N} \). Therefore (see 2.2 and 2.3) it is enough to show that

\[
\left| \sum_{j \in \hat{\mathcal{S}}_1} \int_{X \times Y} A_1(x) A_2(f^N(x)) \cdot \rho(\tau_N(x)) 1_{I_j + D_N}(\tau_N(x)) d\mu(x) - \mu(A_1) \mu(A_2) \nu(B_1) \nu(B_2) \right| < \epsilon / 2.
\]

If \( \delta \) is small enough (using continuity of \( (G_t) \)), the above sum is, up to an error less than \( \epsilon / 16 \), equal to

\[
\sum_{j \in \hat{\mathcal{S}}_1} \rho(D_N + t_j) \mu \left( A_1(\cdot) A_2(f^N(\cdot)) 1_{I_j}(\tau_N(\cdot) - D_N) \right).
\]
By the definition of mixing LLT (with $A_1, A_2, C = I_0$ and $z = t_j$), and since the number of $j$'s such that $j \in \hat{S}_1$ is bounded above by $C(\delta, \epsilon)L_N^d$ there exists $N_1 = N_1(\epsilon, \delta) \in \mathbb{N}$ such that for every $N \geq N_1$, the above expression is, up to an error less than $\epsilon/16$, equal to

$$
(2.6) \quad \sum_{j \in \hat{S}_1} \frac{1}{L_N^d} \text{Vol}(I_0) p \left( \frac{t_j}{L_N} \right) \mu(A_1) \mu(A_2) \rho(D_N + t_j).
$$

Enlarging $K$ and $N$, if necessary, we can guarantee that

$$
(2.7) \quad \left| \sum_{j \in \hat{S}_1} \frac{1}{L_N^d} \text{Vol}(I_j) p \left( \frac{t_j}{L_N} \right) - 1 \right| < \frac{\epsilon}{16}.
$$

Now, fix $R > 0$ and for $c \in B_R$, let

$$
\alpha(c) := \sum_{j \in \hat{S}_1} \frac{1}{L_N^d} \text{Vol}(I_0) p \left( \frac{t_j}{L_N} \right) \rho(D_N + t_j + c).
$$

We claim that there exists $N_2 = N_2(R)$ such that for $N \geq N_2$, we have

$$
|\alpha(c) - \alpha(0)| < \epsilon/16.
$$

Indeed, let $k$ be such that $c \in I_k$, then $|t_k| \leq R + 1$ and $|t_k - c| \leq \delta$, by choosing $\delta \ll \epsilon$ small enough, and $N_2$ large so that $\frac{R+1}{L_N} \leq \delta$, we have

$$
|\alpha(c) - \alpha(0)| \leq |\alpha(c) - \alpha(t_k)| + |\alpha(t_k) - \alpha(0)|
\leq \frac{\text{Vol}(I_0)}{L_N^d} \sum_{j \in \hat{S}_1} p \left( \frac{t_j}{L_N} \right) |\rho(D_N + t_j + c) - \rho(D_N + t_j + t_k)| +
\frac{\text{Vol}(I_0)}{L_N^d} \sum_{j \in \hat{S}_1} \left| p \left( \frac{t_j}{L_N} \right) - p \left( \frac{t_j - t_k}{L_N} \right) \right| |\rho(D_N + t_j)| +
\frac{\text{Vol}(I_0)}{L_N^d} \sum_{j \in \hat{S}_1; |t_j - t_k| \geq KL_N} p \left( \frac{t_j}{L_N} \right) |\rho(D_N + t_j + t_k)|
\leq C_1(p, \rho) |t_k - c| + C_2(p, \rho, K) R/L_N + K^d C(\rho) R/L_N
\leq \epsilon/64 + \epsilon/64 + \epsilon/64 < \epsilon/16,
$$

where for the inequality $(\star)$, the first term is due to the fact that $\rho$ is continuous on $t$ and (2.7), the second term is due to continuity of $p$ and the choice of $N_2$ (that is, $\frac{R+1}{L_N} \leq \delta$), and the last term contains a sum of $K^d R L_N^{d-1}$ many terms and hence $\leq K^d C(\rho) R/L_N$.

Therefore

$$
(2.8) \quad \left| \alpha(0) - \frac{1}{\text{Vol}(B_R)} \int_{c \in B_R} \alpha(c) dc \right| < \epsilon/16.
$$

Now by the ergodicity of $G$ and the mean ergodic theorem for the $G$-action, there exist a subset $Y_0 \subset Y$ with $\nu(Y_0) \geq 1 - \frac{\epsilon}{32} \nu_0^3$ and $R_0 > 0$, such that for any $y \in Y_0$ and
\[ R \geq R_0, \]
\[
\left| \frac{1}{\text{Vol}(B_R)} \int_{t \in B_R} B_2(G_{t}y)dt - \nu(B_2) \right| < \frac{\epsilon}{32C_3}.
\]

Here the constant \(C_3 := 10\max_{y \in \{\|B_1(y)\|, \|B_2(y)\|\}}\). Hence for any \(t\), if \(R \geq R_0\),

\[
(2.9) \quad \left| \frac{1}{\text{Vol}(B_R)} \int_{t \in B_R} \rho(t + c)dc - \nu(B_1)\nu(B_2) \right|
\leq \int_{G^{-1}(Y_0)} B_1(y) \left( \frac{1}{\text{Vol}(B_R)} \int_{t \in B_R} B_2(G_{t+c}y)dc - \nu(B_2) \right) d\nu(y)
\]
\[
+ \int_{Y \setminus G^{-1}(Y_0)} B_1(y) \left( \frac{1}{\text{Vol}(B_R)} \int_{t \in B_R} B_2(G_{t+c}y)dc - \nu(B_2) \right) d\nu(y)
\]
\[
\leq \max\{\|B_1\|\} \frac{\epsilon}{32C_3} + \max\{\|B_1\|\} \max\{\|B_2\|\}2(1 - \nu(Y_0)) \leq \frac{\epsilon}{16}.
\]

Note that (2.6) is equal to \(\mu(A_1)\mu(A_2)\alpha(0)\). By (2.8) and (2.9), \(\mu(A_1)\mu(A_2)\alpha(0)\) is, up to an error less than \(\epsilon/8\), equal to

\[
\mu(A_1)\mu(A_2)\nu(B_1)\nu(B_2) \left[ \sum_{j \in S_i} \frac{1}{L_N^4} \text{Vol}(I_j) \mathbb{P} \left( \frac{t_j}{L_N} \right) \right].
\]

Combining the estimates (2.7), (2.5) and (2.6) we obtain (2.4) (and consequently (2.2)), completing the proof.

(b) The proof is essentially the same as that for (a), therefore we leave it to the reader. \(\square\)

2.2. Associated skew product. Given \(F\) as above consider the map \(T\) of \(X \times \mathbb{R}^d\) defined by

\[ T(x, t) = (f(x, t + \tau(x))). \]

Note that \(T\) preserves a measure \(\lambda = \mu \times \text{Lebesgue} \).

**Definition 2.6.** \(T\) is **KS (Krengel–Sucheston) mixing** if for any sets \(\Omega_1, \Omega_2\) of finite measure

\[ \lim_{n \to \infty} \lambda(\Omega_1 \cap T^{-n}\Omega_2) = 0. \]

**Proposition 2.7.** If \(f\) and \(G\) are mixing and \(T\) is KS-mixing, then \(F\) is mixing.

**Proof.** Let \(\Phi_i(x, y) = A_i(x)B_i(y)\). Since \(f\) is mixing we may assume (after subtracting, if necessary the mean of \(B_i\)) that \(\nu(B_i) = 0\). Then

\[
\left| \int \Phi_1(x, y)\Phi_2(F^n(x, y))d\mu(x)d\nu(y) \right| = \left| \int A_1(x)A_2(f^n x)\rho_{\tau_n(x)}(B_1, B_2)d\mu(x) \right|
\]
\[
\leq \|A_1\|_{\infty} \|A_2\|_{\infty} \int |\rho_{\tau_n(x)}(B_1, B_2)| d\mu(x).
\]
We claim that the last integral tends to 0. Indeed, given \( \varepsilon > 0 \) we can find \( R \) such that \( |\rho_t(B_1, B_2)| < \frac{\varepsilon}{2} \) if \( |t| \geq R \). Then
\[
\int |\rho_{\tau_n(x)}(B_1, B_2)| \, d\mu(x) \leq \frac{\varepsilon}{2} + \mu(|\tau_n| < R) \cdot \max_{|t| \leq R} \rho_t(B_1, B_2).
\]
The second term tends to 0 as \( n \to \infty \) due to KS-mixing, so it is smaller than \( \frac{\varepsilon}{2} \) for large \( n \).

It is not known whether the converse of Proposition 2.7 holds or not, so we pose

**Question 2.8.** Suppose that \( f, G \) and \( F \) are mixing. Is \( T \) KS-mixing?

Let \( \mathcal{C}_R \) be the set of \((x, t) \in X \times \mathbb{R}^d\) such that \( t \) is inside a cube centered at the origin with side \( R \). Consider the space \( \mathbb{G}_O \) of functions \( \Phi : X \times \mathbb{R}^d \to \mathbb{R} \) such that \( \Phi \) is equicontinuous, and the limit
\[
\bar{\Phi} := \lim_{R \to \infty} \frac{1}{\lambda(\mathcal{C}_R)} \int_{\mathcal{C}_R} \Phi(x, t) \, d\lambda
\]
even exists.

**Definition 2.9.** \( T \) is **local global mixing** with respect to \( \mathbb{G}_O \) if for each \( \phi \in L^1(\lambda) \), \( \Phi \in \mathbb{G}_O \)
\[
\lim_{n \to \infty} \int \phi(x, t) \Phi(T^n(x, t)) \, d\lambda = \bar{\Phi} \int \phi \, d\lambda.
\]

**Proposition 2.10.** If \( G \) is ergodic and \( T \) is local global mixing with respect to \( \mathbb{G}_O \), then \( F \) is mixing.

**Proof.** Let \( \mathcal{B} \) be a compact set in \( \mathbb{R}^d \) of unit measure. Since \( G \) is measure preserving we have
\[
\iint \Phi_1(x, y) \Phi_2(f^n x, G_{\tau_n(x)} y) \, d\zeta(x, y) = \iint \iint 1_{\mathcal{B}}(s) \Phi_1(x, G_s y) \Phi_2(f^n x, G_{\tau_n(x)+s} y) \, dsd\zeta(x, y)
\]
\[
= \int_Y \left( \iint \psi_y(x, s) \Psi_y(T^n(x, s)) \, d\lambda(x, s) \right) \, d\nu(y)
\]
where \( \psi_y(x, s) = 1_{\mathcal{B}}(s) \Phi_1(x, G_s y) \) \( \Psi_y(x, s) = \Phi_2(x, G_s y) \). Since the inner integral is bounded, it suffices to compute the pointwise limit of this integral for a.e. \( y \).

Applying ergodicity of \( G \) to the ergodic averages of \( \bar{\Psi}(y) := \int \Phi_2(x, y) \, d\mu(x) \), we obtain that for a.e. \( y \) the function \( \Psi_y(x, s) \) belongs to \( \mathbb{G}_O \) with average value \( \zeta(\Phi_2) \). Since \( T \) is local global mixing, the inner integral converges almost surely to
\[
\iint 1_{\mathcal{B}}(s) \Phi_1(x, G_s y) \, d\mu(x) \, ds \, \zeta(\Phi_2)
\]
Integrating with respect to \( y \), we obtain
\[
\lim_{n \to \infty} \iint \Phi_1(x, y) \Phi_2(f^n x, G_{\tau_n(x)} y) \, d\zeta(x, y) =
\]
\[
\iint 1_{\mathcal{B}}(s) \Phi_1(x, G_s y) \, d\mu(x) \, d\nu(y) \, ds \cdot \zeta(\Phi_2) = \text{Vol}(\mathcal{B}) \zeta(\Phi_1) \zeta(\Phi_2) = \zeta(\Phi_1) \zeta(\Phi_2)
\]
since \( G \) preserves \( \nu \) and \( \text{Vol}(\mathcal{B}) = 1 \). \( \square \)
Remark 2.11. According to [29] if $\tau$ satisfies mixing LLT with zero drift, then $T$ is local global mixing with respect to $G_0$. So in case of zero drift, Theorem 2.5(a) follows from Proposition 2.10.

3. Background

Definition 3.1. We say that $G$ is mixing with rate $\psi(t)$ on a space $\mathbb{B}$ if

$$
(3.1) \quad \left| \int B_1(y)B_2(G_t y) d\nu(y) - \nu(B_1)\nu(B_2) \right| \leq C \psi(t)\|B_1\|_{\mathbb{B}}\|B_2\|_{\mathbb{B}}.
$$

We call $G$ exponentially mixing if $(3.1)$ holds with $\mathbb{B} = C^r$ for some $r > 0$ and $\psi(t) = e^{-\delta|t|}$ for some $\delta > 0$.

We call $G$ polynomially mixing if $(3.1)$ holds with $\mathbb{B} = C^r$ for some $r > 0$ and $\psi(t) = \|t\|^{-\delta}$ for some $\delta > 0$.

We call $G$ rapidly mixing if for each $m$ there exists $r$ such that $(3.1)$ holds with $\mathbb{B} = C^r$ and $\psi(t) = \|t\|^{-m}$.

These definitions extend to maps (such as to $f$ and $F$) in the natural way.

Definition 3.2. $\tau$ satisfies exponential large deviation bounds, if for each $\varepsilon > 0$ there exist $C$ and $\delta > 0$ such that for any $N \in \mathbb{N}$,

$$
(3.2) \quad \mu \left( \left| \frac{T_N}{N} - \mu(\tau) \right| \geq \varepsilon \right) \leq C e^{-\delta N}.
$$

$\tau$ satisfies polynomial large deviation bounds, if for each $\varepsilon > 0$ there exist $C$ and $\delta > 0$ such that for any $N \in \mathbb{N}$,

$$
\mu \left( \left| \frac{T_N}{N} - \mu(\tau) \right| \geq \varepsilon \right) \leq C N^{-\delta}.
$$

$\tau$ satisfies superpolynomial large deviation bounds, if for each $w > 0$, $\varepsilon > 0$ there exist $C = C(\varepsilon, w)$ such that for any $N \in \mathbb{N}$,

$$
\mu \left( \left| \frac{T_N}{N} - \mu(\tau) \right| \geq \varepsilon \right) \leq C N^{-w}.
$$

We will often use the following standard fact.

Lemma 3.3. For each $r$, there is $w = w(r)$ such that functions $\Phi \in C^r(X \times Y)$ admit a decomposition $\Phi(x, y) = \sum_{k=1}^{\infty} A_k(x)B_k(y)$, where $A_k \in C^r(X)$, $B_k \in C^r(Y)$ and

$$
(3.3) \quad \sum_k \|A_k\|_{C^r(X)}\|B_k\|_{C^r(Y)} \leq C(r, w)\|\Phi\|_{C^w(X \times Y)}.
$$

Corollary 3.4. Suppose that there are positive constants $K$ and $r$, such that

$$
\left| \int \int A'(x)B'(y)A''(f^n x)B''(G_{\tau_n} y) d\mu(x) d\nu(y) - \mu(A')\nu(B')\mu(A'')\nu(B'') \right|
$$

$$
(3.4) \quad \leq K \|A'\|_{C^r(X)}\|B'\|_{C^r(Y)}\|A''\|_{C^r(X)}\|B''\|_{C^r(Y)} \psi(n).
$$

Then $F$ is mixing with rate $\psi$. 
Proof. Let
\[ \tilde{\rho}_n(\Phi', \Phi'') := \zeta(\Phi'(\Phi'' \circ F^n)) - \zeta(\Phi') \zeta(\Phi''). \]
Decomposing \( \Phi', \Phi'' \in C^w \) as in (3.3), we get
\[ |\tilde{\rho}_n(\Phi', \Phi'')| \leq K \psi(n) \sum_{j,k} (\|A'_j\|_r \|B'_j\|_r \|A''_k\|_r \|B''_k\|_r) \]
\[ \leq K \psi(n) \sum_j (\|A'_j\|_r \|B'_j\|_r) \sum_k (\|A''_k\|_r \|B''_k\|_r) \leq K \psi(n) C^2(r, w) \|\Phi'\|_w \|\Phi''\|_w. \]
\[ \square \]

4. Mixing rates for mixing fibers

4.1. Double mixing.

**Theorem 4.1.** Suppose that \( \mu(\tau) \neq 0 \).
(a) If \( \tau \) satisfies exponential large deviation bounds and \( f \) and \( G \) are exponentially mixing, then \( F \) is exponentially mixing.
(b) If \( \tau \) satisfies polynomial large deviation bounds and \( f \) and \( G \) are polynomially mixing, then \( F \) is polynomially mixing.
(c) If \( \tau \) satisfies superpolynomial large deviation bounds and \( f \) and \( G \) are rapidly mixing, then \( F \) is rapidly mixing.

**Proof.** (a) For \( i = 1, 2 \), let \( \Phi_i(x, y) = A_i(x)B_i(y) \) be a \( C^r \) function on \( X \times Y \). Let \( \rho(t) := \int Y B_1(y)B_2(G_1y)d\nu(y) \). Since \( G \) is exponentially mixing, there exist constants \( C_1 > 0 \) and \( \kappa > 0 \) such that
\[ |\rho(t) - \nu(B_1)\nu(B_2)| \leq C_1 \|\rho\|_{L^1} \|B_1\|_{C^r} \|B_2\|_{C^r} e^{-\kappa\|t\|}. \]
Taking \( \varepsilon = \|\mu(\tau)\|/2 \) in the definition of exponential large deviation bounds, we find that there exist \( C_0 > 0 \) and \( \delta > 0 \) such that \( \mu(T_N) \leq C_0 e^{-\delta N} \), where
\[ T_N := \{x \in X : \|\tau_N(x) - N\mu(\tau)\| \geq N\|\mu(\tau)\|/2\}. \]

Now note that
\[ \int_{x \times Y} \Phi_1(x, y)\Phi_2(F^N(x, y))d(\mu \times \nu) = \int_X A_1(x)A_2(f^N(x))(\rho(\tau_N(x)))d\mu(x). \]
We rewrite the last integral as the sum of two integrals \( I_1 + I_2 \), where
\[ I_1 = \int_{T_N} A_1(x)A_2(f^N(x))(\rho(\tau_N(x)))d\mu(x), \]
and
\[ I_2 = \int_{X \setminus T_N} A_1(x)A_2(f^N(x))(\rho(\tau_N(x)))d\mu(x). \]
By exponential large deviation bounds, \( |I_1| \leq C_2 \mu(T_N) \leq C_3 e^{-\delta N} \). For \( I_2 \), since \( f \) is exponentially mixing, it is enough to show that
\[ \Delta := |I_2 - (\nu(B_1)\nu(B_2)) \int_{X \setminus T_N} A_1(x)A_2(f^N(x))d\mu(x)| \]
is exponentially small. Indeed, by (4.1)
\[ \Delta \leq \left| \int_{X \setminus T_N} |A_1(x)||A_2(f^N(x))||\rho(\tau_N(x)) - \nu(B_1)\nu(B_2)|d\mu(x) \right| \]
\[ \leq C_4 \|A_1\|_0 \|A_2\|_0 \|B_1\|_r \|B_2\|_r \cdot e^{-\kappa_1 N} \leq C_4 \|A_1 \times B_1\|_r \|A_2 \times B_2\|_r \cdot e^{-\kappa_1 N} \]
with \( \kappa_1 = \kappa/2 \). This finishes the proof. The proofs of parts (b) and (c) are analogous to part (a). We will omit them. \( \square \)

**Remark 4.2.** In part (b) above, if \( \tau \) satisfies polynomial large deviation bounds with rate \( N^{-\delta_1} \), and \( f, G \) are polynomially mixing with rate \( N^{-\delta_2} \) and \( N^{-\delta_3} \) respectively, then \( F \) is polynomially mixing with rate \( N^{-\min\{\delta_1, \delta_2, \delta_3\}} \).

**Remark 4.3.** Observe that the LLT was not needed in Theorem 4.1 and so the theorem remains valid if \( \mathbb{R}^d \) is replaced by an arbitrary Lie group, in which case \( \tau_N \) means the product \( \tau_N(x) = \tau(f^{N-1}x) \ldots \tau(fx)\tau(x) \).

**Definition 4.4.** Assume that a cocycle \( \tau \) is such that \( \tau_n - D_n \) converges as \( n \to \infty \) to a non atomic distribution. We say that \( \tau \) satisfies the anticoncentration inequality if for every unit cube \( C \subset \mathbb{R}^d \),
\[ \mu\left( \{ x \in X : \tau_N(x) \in C \} \right) \leq C L_N^{-d}, \]
for some global constant \( C > 0 \).

**Remark 4.5.** Note that by assumption there is a constant \( R \) such that \( \mu(\|\tau_n\| \leq RL_n) \geq 0.5 \) so the power of \( L_N \) in the anticoncentration inequality is optimal.

**Theorem 4.6.** Assume that for some \( r \in \mathbb{N} \), \( f \) is mixing with rate \( \psi_f(N) = L_N^{-\alpha} \), for some \( \alpha > 0 \) on \( C^r \), \( \tau \) satisfies the anticoncentration inequality and \( G \) is mixing with rate \( \psi_G(\cdot) \) on \( C^r \), where
\begin{equation}
\tag{4.3}
\int_{\mathbb{R}^d} \psi_G(t)dt < +\infty.
\end{equation}
Then \( F \) is mixing with rate \( \psi_F(N) := L_N^{-\min\{d, \alpha\}} \) on \( C^w \) for some \( w = w(r) \in \mathbb{N} \).

**Theorem 4.7.** Assume that for some \( r \in \mathbb{N} \), \( f \) is mixing with rate \( \psi_f(N) = L_N^{-\alpha} \), for some \( \alpha > 0 \) on \( C^r \), \( G \) is mixing with rate \( \psi_G(\cdot) \) on \( C^r \), \( \tau \) satisfies the mixing LLT with zero drift.

(a) Suppose \( \tau \) satisfies the anticoncentration inequality. If \( \psi_G(\cdot) \) satisfies (4.3) and
\begin{equation}
\tag{4.4}
\int \Phi_1(x,y)d\nu(y) \equiv 0,
\end{equation}
then
\begin{equation}
\tag{4.5}
\int \Phi_1(z)\Phi_2(F^Nz)d\zeta(z) =
\end{equation}
\[ p(0)L_N^{-d} \int \int \int \Phi_1(x, y) \Phi_2(\bar{x}, G_t y) d\mu(x) d\nu(y) d\mu(\bar{x}) dt + o(L_N^{-d}). \]

(b) If \( \psi_G(t) = ||t||^{-\beta} \), for \( \beta < d \), then \( F \) is mixing with rate \( \psi_F(N) := L_N^{-\min\{\beta, d\}} \) on \( C^w \) for some \( w = w(r) \in \mathbb{N} \).

(c) If \( \min\{\alpha, d\} > \beta \) and for zero mean functions we have
\[
\int B_1(y) B_2(G_t y) d\nu = q(B_1, B_2) \Psi(t) + o(||t||^{-\beta})
\]
where \( q \) is a bounded bilinear form on \( C^r(Y) \) and \( \Psi \) is a homogeneous function of degree \( -\beta \), then
\[
\Phi_1(z) \Phi_2(F^N z) d\zeta(z) = L_N^{-\beta} Q(\Phi_1, \Phi_2) \int_{\mathbb{R}^d} p(t) \Psi(t) dt + o(L_N^{-\beta})
\]
where
\[ Q(\Phi_1, \Phi_2) = \int q(\Phi(x_1, \cdot), \Phi_2(x_2, \cdot)) d\mu(x_1) d\mu(x_2). \]

**Remark 4.8.** In the case \( d = 1 \), (4.5) is proven in [49] under a slightly more restrictive condition.

**Remark 4.9.** We note that the integral in (4.6) converges. In fact, convergence near 0 follows because \( p \) is bounded and \( d > \beta \), while convergence near infinity follows since \( \Psi \) is bounded outside of the unit sphere. We also observe that for \( \Phi_j(x, y) = A_j(x) B_j(y) \)
\[
Q(\Phi_1, \Phi_2) = \mu(A_1) \mu(A_2) q(B_1, B_2).
\]

**Proof of Theorem 4.6.** For \( i = 1, 2 \), let \( \Phi_i(x, y) = A_i(x) \tilde{B}_i(y) \), where \( A_i \in C^r(X) \) and \( \tilde{B}_i \in C^r(Y) \). Let \( B_i = \tilde{B}_i - \nu(\tilde{B}_i) \). Let \( \rho(t) := \int_{Y} B_1(y) B_2(G_t y) d\nu(y) \). Note that
\[
\int_{X \times Y} \Phi_1(x, y) \Phi_2(F^N (x, y)) d(\mu \times \nu) = \int_X A_1(x) A_2(f^N(x)) \cdot \rho(\tau_N(x)) d\mu(x) + \nu(\tilde{B}_1) \nu(\tilde{B}_2) \int_X A_1(x) A_2(f^N(x)) d\mu(x).
\]

Since \( f \) is mixing with rate \( L_N^{-\alpha} \) on \( C^r \), the second summand is equal to \( \mu(A_1) \mu(A_2) \) up to an error less than \( C \|A_1\|_r \|A_2\|_r L_N^{-\alpha} \). It remains to estimate the first summand.

Let \( \{C_i\}_{i=1}^{\infty} \) be a countable disjoint family of unit cubes in \( \mathbb{R}^d \) such that \( \mathbb{R}^d = \bigcup_i C_i \).

Below we assume without the loss of generality that the function \( \psi \) from (4.3) satisfies
\[
\sup_{C_i} \psi(t) \leq K \inf_{C_i} \psi(t).
\]

Indeed, given \( t, \tilde{t} \in C_i \) we have
\[
\nu(B_1 \cdot B_2 \circ G_t) = \nu(B_1 \cdot \hat{B}_2 \circ G_{\tilde{t}})
\]
where \( \hat{B}_2 = B_2 \circ G_{t-\tilde{t}} \). The last integral is smaller in absolute value than
\[
\psi(\tilde{t}) \|B_1\| C^r \|\hat{B}_2\| C^r \leq K \psi(\tilde{t}) \|B_1\| C^r \|B_2\| C^r.
\]

Thus decreasing \( \psi \) if necessary we may assume that (4.9) holds.
Note first that since $\tau$ is bounded, we have
\begin{equation}
\int_X A_1(x)A_2(f^N(x))\cdot \rho(\tau_N(x))d\mu(x) = \sum_{i=1}^{\infty} \int_X A_1(x)A_2(f^N(x))\cdot \rho(\tau_N(x))\mathbb{1}_{C_i}(\tau_N(x))d\mu(x).
\end{equation}

Using that $G$ is mixing with rate $\psi_G$ on $C''$, \((4.10)\) shows that
\[ C\|A_1\|_0\|A_2\|_0\|B_1\|_r\|B_2\|_r \sum_{i=1}^{\infty} \sup_{t \in C_i} \psi_G(t) |\{ x \in X : \tau_N(x) \in C_i \} |. \]

Together with the anticoncentration inequality, we have
\begin{equation}
\int_X A_1(x)A_2(f^N(x))\cdot \rho(\tau_N(x))d\mu(x) \leq C \cdot \|A_1\|_0\|A_2\|_0\|B_1\|_r\|B_2\|_r \sum_{i=1}^{\infty} \sup_{t \in C_i} \psi_G(t).
\end{equation}

Now by \((4.9)\)
\begin{equation}
\sum_{i=1}^{\infty} \sup_{t \in C_i} \psi_G(t) \leq C' \int_{\mathbb{R}^d} \psi_G(t)dt < C''.
\end{equation}

Summarizing, we get
\[ \int_X A_1(x)A_2(f^N(x))\cdot \rho(\tau_N(x))d\mu(x) \leq C''\|A_1\|_0\|A_2\|_0\|B_1\|_r\|B_2\|_r L^{-d}_N \sum_{i=1}^{\infty} \sup_{t \in C_i} \psi_G(t) \]
showing that $F$ is mixing with rate $L^{-\min\{d,\alpha\}}_N$. \hfill \Box

\textbf{Proof of Theorem 4.7} By the same argument in the proof of Theorem 4.6 we just need to estimate
\[ \int_X A_1(x)A_2(f^N(x))\cdot \rho(\tau_N(x))d\mu(x). \]

To prove part (a) note that due to \((2.1)\) for each fixed $i$,
\[ \lim_{N \to \infty} L^d_N \int_X A_1(x)A_2(f^N(x))\cdot \rho(\tau_N(x))\mathbb{1}_{C_i}(\tau_N(x))d\mu(x) = p(0) \int_{C_i} \rho(t)dt \mu(A_1)\mu(A_2). \]

This together with the Dominated Convergence Theorem (note that in part (a) we assume the conditions of Theorem 4.6 whence \((4.11)\) and \((4.12)\) apply) shows that
\[ \lim_{N \to \infty} L^d_N \int_X A_1(x)A_2(f^N(x))\cdot \rho(\tau_N(x))d\mu(x) = p(0)\mu(A_1)\mu(A_2) \int_{\mathbb{R}^d} \rho(t)dt \]
proving \((4.5)\).

To prove part (b), split $\int_X A_1(x)A_2(f^N(x))\cdot \rho(\tau_N(x))d\mu(x) = S_1 + S_2$, where
\[ S_1 := \int_X A_1(x)A_2(f^N(x))\cdot \rho(\tau_N(x))\mathbb{1}_{[-L_N,L_N]^d}(\tau_N(x))d\mu(x), \]
and
\[ S_2 := \int_X A_1(x)A_2(f^N(x)) \cdot \rho(\tau_N(x))\mathbb{1}_{\mathbb{R}^d \setminus [-L_N, L_N]^d}(\tau_N(x))d\mu(x). \]

To estimate \( S_2 \), notice that for \( x \) as in \( S_2 \),
\[ \rho(\tau_N(x)) \leq C\|B_1\|_r\|B_2\|_r\psi(\tau_N(x)) \leq C_0\|B_1\|_r\|B_2\|_r\psi(L_N) \leq C_0\|B_1\|_r\|B_2\|_r L_N^{-\beta}. \]

Therefore \( S_2 \leq C_0\|A_1\|_r\|A_2\|_r\|B_1\|_r\|B_2\|_r L_N^{-\beta}. \)

It remains to estimate \( S_1 \). We trivially have
\[ |S_1| = \left| \int_X A_1(x)A_2(f^N(x)) \cdot \rho(\tau_N(x))\mathbb{1}_{[-L_N, L_N]^d}(\tau_N(x))d\mu(x) \right| \]
\[ \leq \|A_1\|_r\|A_2\|_r \int_X |\rho(\tau_N(x))|\mathbb{1}_{[-L_N, L_N]^d}(\tau_N(x))d\mu(x). \]
\[ \tag{4.13} \]

Cover \([-L_N, L_N]^d\) with (at most) \((|L_N| + 1)^d\) disjoint cubes \(I_j\) of size 1 centered at \(t_j\), so that \(I_j\)'s are translates of the cube \(I_0\). By the mixing LLT for \(z_n = t_j\) (notice that \(\|t_j\| \leq dL_N\) and so \(t_j/L_N\) belongs to a compact set), and \(A_0 = A_1 = 1\), we get (for sufficiently large \(N\)),
\[ L_N^d\mu(\{x \in X : \tau_N(x) \in I_j\}) < 2p^*\text{Vol}(I_0) = 2p^* \]
where \(p^* = \sup_i p_i\). Therefore,
\[ \int_X |\rho(\tau_N(x))|\mathbb{1}_{[-L_N, L_N]^d}(\tau_N(x))d\mu(x) = \sum_j \int_X |\rho(\tau_N(x))|\mathbb{1}_{I_j}(\tau_N(x))d\mu(x) \]
\[ \leq 2p^*L_N^{-d} \sum_j \sup_{t \in I_j} |\rho(t)| \leq CL_N^{-d} \int_{[-L_N, L_N]^d} \rho(t)dt \leq CL_N^{-d}L_N^{-\beta} = CL_N^{-\beta}, \]
completing the proof of (b).

To prove part (c), fix a small \(\delta\) and split
\[ \int_X A_1(x)A_2(f^N(x)) \cdot \rho(\tau_N(x))d\mu(x) = S_1 + S_2 + S_3 \]
where the integrand in \(S_1\) is multiplied by \(\mathbb{1}_{[-\delta L_N, \delta L_N]^d}(\tau_N(x))\), the integrand in \(S_2\) is multiplied by
\[ \mathbb{1}_{[-\delta L_N, \delta L_N]^d \setminus [-\delta L_N, \delta L_N]^d}(\tau_N(x)) \]
and the integrand in \(S_3\) is multiplied by \(\mathbb{1}_{\mathbb{R}^d \setminus [-\delta L_N, \delta L_N]^d}(\tau_N(x))\).
Arguing as in the proof of part (b) we obtain that \(S_3 = O\left(\left(\frac{\delta}{L_N}\right)^\beta\right)\). Since the integrand is bounded, we have \(S_1 = O\left(\left(\frac{\delta}{L_N}\right)^d\right) = O\left(\left(\frac{\delta}{L_N}\right)^\beta\right)\). To handle \(S_2\) we divide the domain of integration into unit cubes \(I_j\). Let \(t_j\) to be the center of \(I_j\). Using the homogenuity of \(\Psi\) we conclude from the mixing LLT that
\[ \int A_1(x)A_2(f^N x)\rho(\tau_N(x))\mathbb{1}_{I_j}(\tau_N(x))d\mu(x) \]
\[ L^{-d+\beta} \mu(A_1)\mu(A_2)q(B_1, B_2) \left( \frac{t_j}{L_N} \right) \Psi \left( \frac{t_j}{L_N} \right) + o \left( L^{-d+\beta}_N \right). \]

Summing over \( j \) and using (4.7) we obtain

\[ S_2 = L^{-\beta}Q(\Phi_1, \Phi_2) \int_{\mathcal{T}_\delta} p(t)\Psi(t)dt + o \left( L^{-\beta}_N \right) \]

where the domain of integration is \( \mathcal{T}_\delta = \left[ -\frac{1}{\delta}, \frac{1}{\delta} \right] \setminus [-\delta, \delta]^d \). Combing our estimates for \( S_1, S_2 \) and \( S_3 \) we obtain

\[ \int \Phi_1(z)\Phi_2(F^nz)\zeta(z) = L^{-\beta}Q(\Phi_1, \Phi_2) \int_{\mathcal{T}_\delta} p(t)\Psi(t)dt + o \left( L^{-\beta}_N \right) + O \left( \left( \frac{\delta}{L_N} \right)^\beta \right) \]

Letting \( \delta \to 0 \) we obtain (4.6) for product observables, which by Lemma 3.3 is sufficient to conclude the general case.

Remark 4.10. Note that the fact that \( B = C^r \) was only used to decompose any \( \Phi \in C^\omega(X \times Y) \) as

(4.14) \( \Phi(x, y) = \sum_n A_n(x)B_n(y) \), where \( \sum_n \|A_n\|_{C^r} \|B_n\|_{C^r} < \infty. \)

Therefore the conclusions of Theorems 4.6, 4.7 remain valid if (3.1) holds on arbitrary space \( \mathcal{B} \) provided that \( \Phi_1, \Phi_2 \) admit decomposition (4.14).

Remark 4.11. The results of this section apply (with obvious modifications) to continuous time \( T, T^{-1} \) systems of the form

(4.15) \( F^t(x, y) = (\phi^t(x), G_{\tau_t(x)}y) \)

where \( \phi \) is a flow on \( X \) and

(4.16) \( \tau_1(x) = \int_0^t \tau(\phi^s(x))ds. \)

Note that due to the fact that \( \zeta(H_1(H_2 \circ F^{n+\delta})) = \zeta((H_1 \circ (H_2 \circ F^\delta) \circ F^n)) \) it is sufficient to control the correlation at integer times. Next \( F^1 \) is \( T, T^{-1} \)-transformation corresponding to \( f = \phi^1, \tau = \tau_1 \). We note however, that in several case for time one maps of the flow the LLT is unknown (or false) unless the observable is the time integral given by (4.16). We refer the reader to [28] for the discussion of mixing LLT for continuous time systems.

Example 4.12. (a) Let \( g \) be an exponentially mixing Anosov flow on some manifold \( M \). Consider a continuous \( T, T^{-1} \) system \( F^t \) with \( X = Y = M \) and \( \phi^t = G_t = g^t \). Then Theorem 4.7(a) shows that for smooth zero mean observables

\[ \lim_{t \to \infty} \sqrt{t} \zeta(H_1(H_2 \circ F^t)) = Q_1(H_1, H_2) \]

where \( Q_1 \) is given by (4.5). Indeed, the condition (4.4) can be relaxed and the conclusion of Theorem 4.7(a) holds for all zero mean smooth observables assuming that \( \alpha > d \) (in this example, \( \alpha \) is arbitrarily large and \( d = 1 \)).
(b) For any positive integer \(k\), define inducively a continuous \(T, T^{-1}\) system \(F_t^k\) with \(X = M, Y = M^k, \phi^t = g^t\) and \(G_t = F^k_t\), where \(F^t_1\) is the flow from the part (a). Then Theorem \([4.7] (c)\) shows that for smooth zero mean observables

\[
\lim_{t \to \infty} t^{2-k} \zeta(H_1(H_2 \circ F^t)) = Q_k(H_1, H_2)
\]

where \(Q_k\) is given in terms of \(Q_{k-1}\) by \([4.6]\).

4.2. Multiple mixing.

**Definition 4.13.** \(G_t\) is mixing of order \(s\) with rate \(\psi\) on a space \(\mathbb{B}\) if

\[
\left| \nu \left( \prod_{j=1}^{s} B_j(G_t y) \right) - \prod_{j=1}^{s} \nu(B_j) \right| \leq C\psi(\delta(t_1, \ldots, t_s)) \prod_{j=1}^{s} \|B_j\|_{\mathbb{B}}
\]

where

\[
\delta(t_1, \ldots, t_s) = \min_{i \neq j} \|t_i - t_j\|
\]

This definition extends to maps (such as to \(f\) and \(F\)) in the natural way.

**Theorem 4.14.** If \(\tau\) satisfies mixing LLT with zero drift and \(f\) and \(G\) are mixing of order \(s\) with rate \(t^{-\alpha}\) with \(\alpha > d\), then \(F\) is mixing of order \(s\) with rate \(\psi_F(N) = L_N^{-d}\).

**Proof.** For \(i = 1, \ldots, s\), let \(\Phi_i(x, y) = A_i(x) B_i(y)\), where \(A_i \in C^\nu(X)\) and \(B_i \in C^\nu(Y)\). Let \(\rho(t_1, t_2, \ldots, t_s) := \int_Y \prod_{i=1}^{s} B_i(G_{t_i} y) d\nu(y)\) (with \(t_1 = 0\)). We have

\[
\int_X \prod_{i=1}^{s} A_i(f^{N_i} x) d\mu(x) = \int_X \prod_{i=1}^{s} A_i(f^{N_i} x) \cdot \rho(\tau_{N_1}(x), \ldots, \tau_{N_s}(x)) d\mu(x)
\]

\[
= \int_X \prod_{i=1}^{s} A_i(f^{N_i} x) \cdot \left( \rho(\tau_{N_1}(x), \ldots, \tau_{N_s}(x)) - \prod_{i=1}^{s} \nu(B_i) \right) d\mu(x)
\]

\[
+ \sum_{i=1}^{s} \nu(B_i) \int_X \prod_{i=1}^{s} A_i(f^{N_i} x) d\mu(x).
\]

Note that since \(f\) is mixing of order \(s\) with rate \(N^{-\alpha}\), the last term above is equal to \(\prod_{i=1}^{s} \mu(A_i) \nu(B_i)\) up to an error of size at most \(O(\prod_{i=1}^{s} \|A_i\|, \min_{i \neq j} |N_i - N_j|^{-\alpha})\). It is therefore enough to bound the first term. Notice moreover that since \(\tau\) is bounded and satisfies mixing LLT with zero drift, we have \(L_N \leq C^n N\) (see Definition \([2.1]\)).

Denote \(N := \min_{i \neq j} |N_i - N_j|\).

Let \(Z \subset X\) be defined by setting: \(x \in Z\) iff \(\min_{i \neq j} \|\tau_{N_i}(x) - \tau_{N_j}(x)\| \geq L_{\bar{N}}\). Using that \(G\) is mixing of order \(s\) with rate \(\|t\|^{-\alpha}\), we get

\[
\int_Z \prod_{i=1}^{s} A_i(f^{N_i} x) \cdot \left( \rho(\tau_{N_1}(x), \ldots, \tau_{N_s}(x)) - \prod_{i=1}^{s} \nu(B_i) \right) d\mu(x) \leq C \prod_{i=1}^{s} \|A_i\|_0 \prod_{i=1}^{s} \|B_i\|_r L_N^{-\alpha}.
\]

So it remains to estimate the above integral on \(Z^c\). By definition, for every \(x \in Z^c\), there exists \(i_x \neq j_x\) such that

\[
\|\tau_{N_{i_x}}(x) - \tau_{N_{j_x}}(x)\| = \min_{i \neq j} \|\tau_{N_i}(x) - \tau_{N_j}(x)\| \leq L_{\bar{N}}.
\]
Let $Z_{ij} := \{x \in Z^c : (i_x, j_x) = (i, j)\}$ (if there are several pairs satisfying (4.19) we take the smallest with respect to the lexicographic order). Let $\{C_k\}_{k=1}^M$ be a finite family of unit cubes centered at $\{e_k\}_{k=1}^M$ in $\mathbb{R}^d$ such that $[-L_N, L_N]^d = \bigcup_k C_k$. Then

(4.20) \[
\left| \int_{Z_{ij}} \prod_{l=1}^s A_l(f^{N_i}x) \cdot \left( \rho(\tau_{N_i}(x), \ldots, \tau_{N_s}(x)) - \prod_{i=1}^s \nu(B_i) \right) d\mu(x) \right| =
\]

\[
\sum_{k=1}^M \int_{Z_{ij}} \prod_{l=1}^s A_l(f^{N_i}x) \cdot \left( \rho(\tau_{N_i}(x), \ldots, \tau_{N_s}(x)) - \prod_{i=1}^s \nu(B_i) \right) 1_{C_k}(\tau_{N_i}(x) - \tau_{N_j}(x)) d\mu(x) .
\]

Using that $G$ is mixing of order $s$ with rate $\|t\|^{-\alpha}$, and

$$\min\{\sup_{C_k} \|t\|^{-\alpha}, 1\} \leq C \inf_{C_k} \|t\|^{-\alpha}$$

we get that LHS of (4.20) is bounded above by

(4.21) \[
C' \prod_{l=1}^s (\|A_l\|_0 \|B_l\|_r) \sum_{k=1}^M \left( \int_{C_k} \min \{ \|t\|^{-\alpha}, 1 \} dt \right) \mu(\{x \in X : \tau_{N_i}(x) - \tau_{N_j}(x) \in C_k\}).
\]

Note that $\tau_{N_i}(x) - \tau_{N_j}(x) = \tau_{N_i - N_j}(f^{N_i}x)$. Hence, by the mixing LLT with $A_0 = A_1 = 1$, $D_n \equiv 0$, we get (by preservation of measure)

$$\mu(\{x \in X : \tau_{N_i}(x) - \tau_{N_j}(x) \in C_j\}) \leq 2L_{N_i - N_j}^{-d} p(c_i/L_N) < CL_N^{-d} .$$

Therefore, (4.21) (and hence also (4.20)) is bounded above by (recall that $\alpha > d$)

$$C'' \prod_{l=1}^s (\|A_l\|_0 \|B_l\|_r) L_N^{-d} .$$

Summing over all $i, j$ and using (4.18), we get that the LHS of (4.17) is bounded by

$$C''' \prod_{l=1}^s (\|A_l\|_0 \|B_l\|_r) L_N^{-d} .$$

This finishes the proof. \qedh

**Theorem 4.15.** If $\tau$ has non zero drift and satisfies exponential large deviation bounds, and $f$ and $G$ are exponentially mixing of order $s$ then $F$ is exponentially mixing of order $s$.

**Proof.** For $i = 1, 2, \ldots, s$, let $\Phi_i(x, y) = A_i(x)B_i(y)$ be a $C''$ function on $X \times Y$. Let $\rho(t_1, \ldots, t_s) := \int Y \prod_{i=1}^s B_i(G_{t_i}y) dv(y)$ (with $t_1 = 0$). Since $G$ is exponentially mixing, there exist a constant $C_1 > 0$ and $\kappa > 0$ such that

(4.22) \[
|\rho(t_1, \ldots, t_s) - \prod_{i=1}^s \nu(B_i)| \leq C_1 \|B_1\|_{C^r} \|B_2\|_{C^r} e^{-\kappa\|t_1, \ldots, t_s\|} .
\]

Fix $0 = N_1 \leq N_2 \leq \ldots \leq N_s$. We again use the decomposition (4.17). By exponential mixing of order $s$ of $f$, the second term in (4.17) is exponentially close to $\prod_{i=1}^s \nu(B_i) \prod_{i=1}^s \mu(A_i)$, and hence we only need to estimate the first term.
Let $T_{ij} := \{ x \in X : \| \tau_{N_i}(x) - \tau_{N_j}(x) - (N_i - N_j)\mu(\tau) \| \geq (N_i - N_j)\|\mu(\tau)\|/2 \}$. Let $\bar{T} = \bigcup_{i \neq j} T_{ij}$. By exponential large deviation bounds (and preservation of measure),

$$
\mu(\bar{T}) \leq s^2 \max_{ij} \mu(T_{ij}) \leq Ce^{-\delta N}.
$$

Therefore it is enough to bound the integral of the first term in the RHS on $X \setminus \bar{T}$. By exponential mixing of $G$,

$$
\int_{X \setminus \bar{T}} \prod_{i=1}^{s} A_i(f^{N_i}x) \cdot \left( \rho(\tau_{N_1}(x), \ldots, \tau_{N_s}(x)) - \prod_{i=1}^{s} \nu(B_i) \right) d\mu(x)
$$

$$
\leq C \prod_{i=1}^{s} \| A_i \| \prod_{i=1}^{s} \| B_i \| \cdot \min_{x \notin \bar{T}} e^{-\kappa \delta(\tau_{N_1}(x), \ldots, \tau_{N_s}(x))}.
$$

By the definition of $\bar{T}$, $\delta(\tau_{N_1}(x), \ldots, \tau_{N_s}(x)) \geq \frac{\|\mu(\tau)\|}{2} \min_{i \neq j} |N_i - N_j|$ completing the proof. \hfill \qed

Let $n_1 \leq n_2 \leq \cdots \leq n_s$ be a $s$ tuple. A partition $\mathfrak{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_k$ of a set $\{n_1, n_2, \ldots, n_s\}$ (where an item may be listed more than once) is called social if for each $j \in \{1, \ldots, k\}$, Card($\mathcal{P}_j$) > 1. An element $n_j$ is called forward free (backward free) for partition $\mathfrak{P}$ if it is the smallest (respectively, the largest) in its atom. We call $n_j$ forward (or backward) fixed if it is not forward (backward) free. We let $F^\pm$ be the set of all forward (or backward) fixed elements. Let

$$
\kappa^{\pm}(\mathfrak{P}) = \prod_{n_j \in F^\pm} L_{n_j - n_{j-1}}.
$$

For $\mathfrak{P} = (P_1, \ldots, P_k)$, let $(n_{i_\ell})_{\ell=1}^{k}$ be the collection of forward free elements, i.e. $n_{i_\ell}$ is the smallest element of $P_\ell$. Analogously we define $(n_{j_\ell})_{\ell=1}^{k}$ to be the collection of backward free elements. Notice that we have the following formula for $\kappa^{\pm}(\mathfrak{P})$:

\begin{equation}
(4.23)
\kappa^{+}(\mathfrak{P}) = \left( \prod_{j=1}^{s} L_{n_j - n_{j-1}} \right) \cdot \left( \prod_{\ell=1}^{k} L_{n_{i_\ell} - n_{i_\ell-1}} \right)^{-1},
\end{equation}

with $n_0 = 0$ and analogously

\begin{equation}
(4.24)
\kappa^{-}(\mathfrak{P}) = \left( \prod_{j=1}^{s} L_{n_j - n_{j-1}} \right) \cdot \left( \prod_{\ell=1}^{k} L_{n_{j_\ell+1} - n_{j_\ell}} \right)^{-1},
\end{equation}

with $n_{k+1} := n_1 + n_s$.

We have the following

**Definition 4.16.** $\tau$ satisfies anticoncentration large deviation bound of order $s$ if there exist a constant $K$ and a decreasing function $\Theta$ such that $\int_{r=1}^{\infty} \Theta(r)r^d < \infty$, and for any unit cubes $C_1, C_2, \ldots, C_s$ centered at $c_1, c_2, \ldots, c_s$

$$
\mu \left( x : \tau_{n_j} \in C_j \text{ for } j = 1, \ldots, s \right) \leq K \left( \prod_{j=1}^{s} L_{n_j - n_{j-1}}^{-d} \right) \Theta \left( \max_{j} \frac{\|C_j - C_{j-1}\|}{L_{n_j - n_{j-1}}} \right)
$$
Remark 4.17. For $s = 2$ anticoncentration large deviation bounds were considered in [27].

Theorem 4.18. If $\tau$ satisfies anticoncentration large deviation bounds of order $s$ and $f$ and $G$ are exponentially mixing of order $s$, then

$$
\int \left( \prod_{j=1}^{s} H_j(F^{n_j}z) \right) d\zeta(z) - \int \prod_{j=1}^{s} \zeta(H_j) \right) \leq C \prod_{j=1}^{s} \|H_j\|_{C^r} \left( \min_{\mathcal{P}} \kappa(\mathcal{P}) \right)^{-d},
$$

where

$$
\kappa(\mathcal{P}) = \max \{ \kappa^+(\mathcal{P}), \kappa^-(\mathcal{P}) \}.
$$

and the minimum in (4.25) is taken over all social partitions of $\{n_1, \ldots, n_s\}$.

We first recall the following result, which simplifies our analysis.

Lemma 4.19. ([27]) If $G$ is exponentially mixing of order $s$, then for some $\eta > 0$

$$
\nu \left( \prod_{j=1}^{s} B_j(G_t, y) \right) - \prod_{j=1}^{s} \nu(B_j) \leq C e^{-\eta \Delta(t_1, \ldots, t_s)} \prod_{j=1}^{s} \|B_j\|_{\mathcal{B}},
$$

where

$$
\Delta(t_1, \ldots, t_s) = \max \min_{j \neq i} \|t_i - t_j\|.
$$

With the above lemma, we prove Theorem 4.18.

Proof of Theorem 4.18. By Lemma 3.3 it is enough to show the statement for $H_j = A_j \times B_j \in C^r(M)$. Let

$$
\rho(t_1, \ldots, t_s) := \nu \left( \prod_{j=1}^{s} B_j(G_t, y) \right) - \prod_{j=1}^{s} \nu(B_j).
$$

Then

$$
\int \left( \prod_{j=1}^{s} H_j(F^{n_j}z) \right) d\zeta(z) =
$$

$$
\int \left( \prod_{j=1}^{s} A_j(f^{n_j}x) \right) \rho(\tau_{n_1}(x), \ldots, \tau_{n_s}(x)) d\mu(x) + \left( \prod_{j=1}^{s} \nu(B_j) \right) \mu \left( \prod_{j=1}^{s} A_j(f^{n_j}x) \right).
$$

Since $f$ is exponentially mixing of order $s$,

$$
\mu \left( \prod_{j=1}^{s} A_j(f^{n_j}x) \right) - \prod_{j=1}^{s} \mu(A_j) \leq C \prod_{j=1}^{s} \|A_j\|_{\mathcal{B}} e^{-\eta \Delta},
$$

where $\Delta = \Delta(n_1, \ldots, n_s)$.

Let $\mathcal{P}$ be the following partition of $n_1 < \ldots < n_s$. Let $i_1 \in \{2, \ldots, s-1\}$ be the smallest index $i$ such that $|n_i - n_{i-1}| > \Delta$. Then the first atom of $\mathcal{P}$ is $\{n_0, \ldots, n_{i-1}\}$. Notice that $|n_{i_1} - n_{i_1+1}| \leq \Delta$ by the definition of $\Delta$. Now recursively, let $i_{k+1} \in \{i_k + 1, \ldots, s\}$ be the smallest index $i$ such that $|n_i - n_{i-1}| > \Delta$. Then the $(k + 1)$-th atom of $\mathcal{P}$ is $\{n_{i_k}, \ldots, n_{i_{k+1}-1}\}$. We continue until we partition all of $n_1 < \ldots n_s$. Then by the
definition of $\Delta$, every atom of $\mathcal{P}$ has at least two elements, and so $\mathcal{P}$ is social. Moreover, all elements in one atom are at distance at most $s\Delta$ (since the number of elements is $\leq s$). Using that $\tau$ is bounded (and so $|L_n| < Cn$) together with (4.23) and (4.24), we conclude

$$
\min\{\kappa^+ (\mathcal{P})^{-d}, \kappa^+ (\mathcal{P})^{-d}\} \geq \left( \prod_{j=1}^{s} L_{{n_j}-n_{j-1}} \right)^{-d} \geq [s\Delta]^{-sd} \geq C\Delta^{-sd} \geq C e^{-\eta \Delta}.
$$

Combining this estimate with (4.28) we find that the second term in (4.27) equals $\prod_{j=1}^{s} \zeta (H_j)$ up to an error which is bounded by the RHS of (4.25). It remains to show that

$$
\left| \int \left( \prod_{j=1}^{s} A_j (f^{n_j} x) \right) \rho (\tau_{n_1} (x), \ldots, \tau_{n_s} (x)) d\mu (x) \right| \leq C \prod_{j=1}^{s} \| A_j \times B_j \|_{C^r} \left( \min \frac{\kappa (\mathcal{P})}{\mathcal{Q}} \right)^{-d},
$$

which will follow by showing that

$$
\int \left| \rho (\tau_{n_1} (x), \ldots, \tau_{n_s} (x)) \right| d\mu (x) \leq C \prod_{j=1}^{s} \| B_j \|_{C^r} \left( \min \frac{\kappa (\mathcal{P})}{\mathcal{Q}} \right)^{-d}.
$$

Let $C_0 := \prod_{j=1}^{s} \| B_j \|_{C^r}$ and let $D_m := \left\{ x : |\rho (\tau_{n_1} (x), \ldots, \tau_{n_s} (x))| \in [C_0 2^{-m}, C_0 2^{-m+1}] \right\}$. Then

$$
\text{(4.29)}
\int \left| \rho (\tau_{n_1} (x), \ldots, \tau_{n_s} (x)) \right| d\mu (x) \leq 2C_0 \sum_{m \geq 0} \frac{1}{2^m} \mu (D_m).
$$

We will estimate the measure of $D_m$. Note that by Lemma 4.19 for some $C_\eta \in \mathbb{N}$,

$$
D_m \subset A_m := \left\{ x : \Delta (\tau_{n_1} (x), \ldots, \tau_{n_s} (x)) \leq C_\eta m \right\}.
$$

We will therefore give an upper bound on the measure of $A_m$. By the definition of $\Delta$ it follows that there exists a social partition $\mathcal{Q} = (P_1, \ldots, P_k)$ of $n_1 < n_2 < \ldots < n_s$ such that for any atom of $\mathcal{Q}$ and any two $n_i, n_j$ in the same atom we have

$$
\text{(4.30)}
|\tau_{n_i} (x) - \tau_{n_j} (x)| < C_\eta s m.
$$

Let $A_m, \mathcal{Q} \subset A_m$ be the set of $x$ for which $\mathcal{Q}$ is social partition of $n_1 < n_2 < \ldots < n_s$ satisfying (4.30). Then

$$
A_m = \bigcup_{\mathcal{Q} \text{ social}} A_m, \mathcal{Q},
$$

and so we will estimate the measure of $A_m, \mathcal{Q}$.

Let $\{ \tilde{C}_{j} \}$ be a disjoint cover of $\mathbb{R}^d$ by cubes of side length $C_\eta s \cdot m$ centered and $\tilde{c}_j$. Note that by the anticoncentration large deviation bounds of order $s$ (decomposing $\tilde{C}_j$ into unit cubes),

$$
\mu \left( x : \tau_{n_j} (x) \in \tilde{C}_j \text{ for } j = 1, \ldots, s \right) \leq K'(sm)^{sd} \left( \prod_{j=1}^{s} L_{{n_j}-n_{j-1}} \right) \Theta \left( \max_j \frac{\| \tilde{c}_j - \tilde{c}_{j-1} \|}{m \cdot L_{{n_j}-n_{j-1}}} \right).
$$

}
It follows by the definition of $\mathfrak{P}$ and \cite{4.30} that all the $\{\tau_{n_j}(x)\}_{n_j \in P_f}$ belong to one cube $\tilde{C}_{r \ell}$. Below, we use the notation $\tau_{P_f}(x) \in C_{r \ell}$ which means that for every $n_j \in P_f$, $\tau_{n_j}(x) \in \tilde{C}_{r \ell}$. Therefore, we have

$$
\mu(A_m, \mathfrak{P}) \leq \sum_{r_1, \ldots, r_k} \mu(\{x : \tau_{P_f}(x) \in \tilde{C}_{r \ell}, \ell \leq k\}).
$$

Let $n_i$ (and $n_j$) be the smallest (the largest) element of $P_f$, $\ell \leq k$. Below we will argue with $(n_i)$ (analogous reasoning can be done for $(n_j)$). Let $u(\ell)$ be such that $n_i - 1 \leq u(\ell)$. By \cite{1.31}, monotonicity of $\Theta$ and the above discussion (using that $n_i$ and $n_i - 1$ are in different atoms), we obtain

$$
\mu(\{x : \tau_{P_f}(x) \in C_{r \ell}, \ell \leq k\} \leq K'm^s d^d \left( \prod_{j=1}^{s} L_{n_j - n_{j-1}}^{-d} \right) \Theta \left( \max_{\ell \leq k} \frac{||\tilde{c}_{r \ell} - \tilde{c}_{u(\ell)}||}{m \cdot L_{n_i - n_{i-1}}} \right).
$$

Therefore

$$
\mu(A_m, \mathfrak{P}) \leq K'm^s d^d \left( \prod_{j=1}^{s} L_{n_j - n_{j-1}}^{-d} \right) \sum_{r_1, \ldots, r_k} \Theta \left( \max_{\ell \leq k} \frac{||\tilde{c}_{r \ell} - \tilde{c}_{u(\ell)}||}{m \cdot L_{n_i - n_{i-1}}} \right).
$$

Note that

$$
\sum_{r_1, \ldots, r_k} \Theta \left( \max_{\ell \leq k} \frac{||\tilde{c}_{r \ell} - \tilde{c}_{u(\ell)}||}{m \cdot L_{n_i - n_{i-1}}} \right) \leq \sum_{\ell} \Theta(\ell) \cdot \left\{ (r_1, \ldots, r_k) : ||\tilde{c}_{r \ell} - \tilde{c}_{u(\ell)}|| \leq \ell \cdot m \cdot L_{n_i - n_{i-1}} \text{ for every } \ell \leq k \right\} \leq \sum_{\ell} \Theta(\ell) \ell^d \cdot m^d \cdot \left( \prod_{\ell \leq k} L_{n_i - n_{i-1}} \right)^d.
$$

Therefore, by the decay assumptions on $\Theta$ and \cite{4.23},

$$
\mu(A_m, \mathfrak{P}) \leq K'm^s d^d + \left( \prod_{j=1}^{s} L_{n_j - n_{j-1}}^{-d} \right) \cdot \left( \prod_{\ell \leq k} L_{n_i - n_{i-1}} \right)^d = K'm^{s+d} \kappa^+(\mathfrak{P})^{-d}.
$$

Analogously we have that

$$
\mu(A_m, \mathfrak{P}) \leq K'm^{s+d} \kappa^+(\mathfrak{P})^{-d}.
$$

Therefore,

$$
\mu(A_m, \mathfrak{P}) \leq K'm^{s+d} \kappa(\mathfrak{P})^{-d}.
$$

Using that $A_m = \bigcup_\mathfrak{P} A_m, \mathfrak{P}$, we get

$$
\mu(A_m) \leq K'C_s m^{s+d} (\min_\mathfrak{P} \kappa(\mathfrak{P}))^{-d},
$$

for some constant $C_s > 0$. Summarizing, by \cite{4.29} (since $D_m \subset A_m$), we get

$$
\int \left| p(\tau_{n_1}(x), \ldots, \tau_{n_s}(x)) \right| d\mu(x) \leq 2K'C_s \prod_{j=1}^{s} \|B_j\|_{C^r(\min_\mathfrak{P} \kappa(\mathfrak{P}))^{-d}} \sum_{m \geq 0} 2^{-m} m^{s+d} \leq
$$
\[ C_{s,d} \prod_{j=1}^{s} \|B_j\| C^r(\min_{\Psi} \kappa(\Psi))^{-d}. \]

This finishes the proof. \(\square\)

5. Central Limit Theorem

Let \(H(x, y)\) be a \(C^r\) function not cohomologous to a constant function. Let \(\Sigma_N(H) := \sum_{n=0}^{N-1} H(F^n(x, y))\). Assume that \(\zeta(H) = 0\). Let \(Z = X \times Y\).

**Theorem 5.1.** Suppose that \(F\) satisfies (4.25) and \(\sum_{n=1}^{\infty} L_n^{-d}\) converges. Then \(\frac{\Sigma_N(H)}{\sqrt{N}}\) converges as \(N \to \infty\) to the normal distribution with zero mean and variance \(\sigma^2\) given by formula (5.1) below.

**Corollary 5.2.** If \(F\) satisfies either the assumptions of Theorem 4.15 or the assumptions of Theorem 4.18 with \(L_N \geq c\sqrt{N}\) and \(d \geq 3\), then \(F\) satisfies the CLT.

**Proof.** In the case of Theorem 4.15 this follows from the CLT for exponentially mixing systems ([13, 7]). In the case of Theorem 4.18 the result follows from Theorem 5.1. \(\square\)

**Proof of Theorem 5.1.** By (4.25) with \(n_1 = 0, n_2 = n\)

\[ \sigma^2 := \sum_{n=-\infty}^{\infty} \zeta(H(H \circ F^n)) \]

exists and is finite. Hence

\[ \zeta\left(\frac{\Sigma_N^2(H)}{N}\right) = \frac{1}{N} \sum_{1 \leq i,j \leq N} \zeta((H \circ F^i)(H \circ F^j)) \]

\[ = \sum_{k=-N+1}^{N-1} \frac{N-|k|}{N} \zeta(H(H \circ F^k)) \to \sum_{n=-\infty}^{\infty} \zeta(H(H \circ F^n)). \]

To finish our proof, we need to estimate the asymptotics of moments \(\zeta(\Sigma_N^m(H))\), for any \(m \geq 3\). Denote

\[ \Omega(k_1, \ldots, k_m) = \int_Z \left( \prod_{i=1}^{m} H(F^{k_i}z) \right) d\zeta(z) \]

so that

\[ \zeta(\Sigma_N^m(H)) = \sum_{k_1, \ldots, k_m=1}^N \Omega(k_1, \ldots, k_m). \]

For the vector \((k_1, \ldots, k_m)\) we associate another vector \((n_1, \ldots, n_m)\) which is the permutation of the elements of \((k_1, \ldots, k_m)\) in increasing order, that is \(n_1 \leq n_2 \leq \ldots \leq n_m\). Noting that \(\Omega\) is symmetric, we have \(\Omega(k_1, \ldots, k_m) = \Omega(n_1, \ldots, n_m)\). We rewrite the
above sum into two terms as $I_1 + I_2$, where $I_1$ is the sum of terms, whose social partition minimizing the RHS of (4.25) is not pairing (i.e. at least one atom contains more than two elements), and $I_2$ is the sum of terms, whose corresponding social partition is pairing. (If there are more than one partition minimizing $\kappa$ at least one of which is not pairing then we put the corresponding term into $I_1$.)

We need two auxiliary estimates. Let $Q = \{Q_1, ..., Q_r\}$ be a fixed social partition of the set $\{1, 2, ..., m\}$. We say that $Q(n_1, ..., n_m) = Q$ if the partition $\mathcal{Q}$ minimizing the RHS of (4.25) for the given numbers $n_1, ..., n_m$ is of the form $\mathcal{Q} = \{P_1, ..., P_r\}$ with $\{i : n_i \in P_k\} = Q_k$ for all $k = 1, ..., r$. Next we write

$$I_Q = \sum_{k_1, ..., k_m : \mathcal{Q}(n_1, ..., n_m) = Q} \Omega(n_1, ..., n_m).$$

**Lemma 5.3.** (a) $I_Q = O(N^r)$.

(b) If $Q = Q_1 \cup \cdots \cup Q_r$ is not pairing, then the sum $I_Q = O\left(N^{(m-1)/2}\right)$.

**Proof.** Since $1/\kappa_{Q}(n_1, ..., n_m) \leq 1/\kappa_{\mathcal{Q}}(n_1, ..., n_m)$, by (4.25) it suffices to estimate

$$\sum_{n_1, ..., n_m} \frac{1}{(\kappa_{\mathcal{Q}}(n_1, ..., n_m))^d}. \tag{5.3}$$

Let $n'_1 < n'_2 < \cdots < n'_r$ be the forward free elements among $\{n_1, ..., n_m\}$ and $n''_1, ..., n''_{m-r}$ be the forward fixed elements. For each fixed element $n''_j$, let $\bar{n}_j$ be the previous element in $\{n_1, ..., n_m\}$. Rewrite (5.3) as

$$\sum_{n'_1, ..., n'_r} \left[ \sum_{n''_1, ..., n''_{m-r}} \left( \frac{1}{\prod_{j=1}^{m-r} L_{n''_j - \bar{n}_j}} \right)^d \right]. \tag{5.4}$$

Since $L_n^{-d}$ is summable, the inner sum is uniformly bounded, so that (5.4) is bounded by $N^r$. This proves (a).

(b) follows from (a) because if $Q$ is not pairing, then $r < \lfloor m/2 \rfloor$. \qed

Since there are finitely many partitions of $\{1, ..., m\}$, Lemma 5.3 implies that $|I_1|$ is bounded above by $O\left(N^{(m-1)/2}\right)$. In particular, for odd $m$, $\zeta(\sum^m_N(H)) = O\left(N^{(m-1)/2}\right)$.

Now let $m$ be even, and $Q$ be a pairing, that is $Q = \{Q_1, ..., Q_{m/2}\}$ with all atoms $Q_k$ containing exactly two numbers. By forward (backward) step we mean $n_j - n_{j-1}$ where $n_j$ is forward (backward) fixed in the partition $Q(n_1, ..., n_m)$. Let $\Gamma_Q(n_1, ..., n_m)$ be a largest among all forward and backward steps in the partition $Q$ and let $\Gamma(n_1, ..., n_m) = \Gamma_Q(n_1, ..., n_m)(n_1, ..., n_m)$.

**Lemma 5.4.** For any $\epsilon > 0$, there exists $M > 0$, such that

$$\left| \sum_{k_1, ..., k_m : \Gamma(n_1, ..., n_m) > M} \Omega(n_1, ..., n_m) \right| \leq N^{m/2}/\epsilon.$$

**Proof.** It is enough to prove the lemma for $\Gamma$ replaced by $\Gamma^+$ and also for $\Gamma$ replaced by $\Gamma^-$, where $\Gamma^+$ is a largest among all forward steps and $\Gamma^-$ is a largest among all
forward steps. We only consider $\Gamma^+$ as $\Gamma^-$ is similar. The proof for $\Gamma^+$ proceed in the same way as the proof of Lemma 5.3 except we estimate the inner sum in (5.4) by

$$
C \left( \sum_{n=1}^{\infty} L_n^{-d} \right)^{m-r-1} \left( \sum_{n=M}^{\infty} L_n^{-d} \right).
$$

Indeed there are $m - r$ factors in the inner sum in (5.4), and by our assumptions one of them should be greater than $M$ As the second factor can be made as small as we wish by taking $M$ large and since $r = m/2$, the result follows.

**Lemma 5.5.** Let $Q$ be a pairing which is different from

$$
\bar{Q} := [(12), (34), \ldots, ((m - 1) m)].
$$

Then the number of $m$-tuples $(k_1, \ldots, k_m)$ with $\Gamma_Q(n_1, n_2, \ldots, n_m) < L$ is $O(N(m/2)^{-1})$, where the implicit constant depends on $L$.

**Proof.** We claim that if $Q \neq \bar{Q}$ then the sets of forward fixed and backward fixed edges are different. If follows that if both $\Gamma_Q^+(n_1, \ldots, n_m) < M$ and $\Gamma_Q^-(n_1, \ldots, n_m) < M$, then there are at least $m/2 + 1$ edges which are shorter that $M$. The number of such tuples is $O(N(m/2)^{-1})$ and the result follows.

It remains to prove the claim. That is, we show that if the sets of forward fixed and backward fixed edges are the same, then $Q = \bar{Q}$. We proceed by induction. If $m = 0$ or 2 then there are no pairings different from $\bar{Q}$. Suppose $m > 2$. Then $(n_{m-1}, n_m)$ is forward fixed, so it should be backward fixed but this is only possible if $(m - 1)$ is paired to $m$. Likewise $(n_1, n_2)$ is backward fixed, hence it is forward fixed. But this is only possible if 1 is paired to 2. Removing 1, 2, $(m - 1)$ and $m$ from $Q$ we obtain a partition of $m - 4$ elements for which the set of forward fixed and backward fixed edges coincide. By induction 3 is paired to 4, 5 to 6, \ldots, $(m - 3)$ to $(m - 2)$. The proof is complete.

By the above lemmas, it suffices to consider indeces $k_1, \ldots, k_m$ so that

$$
\forall i = 1, \ldots, m/2 : M_i := n_{2i} - n_{2i-1} \leq M \text{ and } \forall i = 1, \ldots, m/2 - 1 : n_{2i+1} - n_{2i} > L
$$

for some large $M$ and $L = L(M)$. Indeed, by choosing $M = M(\varepsilon)$ and $N > N_0$, $N_0 = N_0(L)$, the above lemmas give that the contribution of other terms is less than $\varepsilon N^{m/2}$. Now we choose $L$ so that for any fixed $M_1, \ldots, M_{m/2}$ (finitely many choices), the RHS of (4.23) with $s = m/2$ and $H_j = H(H \circ T^{M_j})$ is less than $\varepsilon$. We conclude

$$
\left| \zeta(\Sigma_N^m) - \sum_{k_1, \ldots, k_m \text{ satisfying } (5.7)} \prod_{i=1}^{m/2} \left( \int_Z (H(H \circ T^{M_i})) d\zeta(z) \right) \right| \leq 2\varepsilon N^{m/2}
$$

Let us write $A_\ell = \int_Z (H(H \circ T^\ell)) d\zeta(z)$. Now we claim

$$
\sum_{k_1, \ldots, k_m \text{ satisfying } (5.7)} \prod_{i=1}^{m/2} A_{M_i} = (m - 1)!! N^{m/2} (1 + o(1)) \left[ \sum_{\ell=0}^M (A_\ell(1 + \frac{1}{\ell+1})) \right]^{m/2}.
$$
To prove the claim, first note that
\[ \sum_{M} M_{1} \ldots M_{m/2} = 0 \]
\[ A_{M_{1}} \ldots A_{M_{m/2}} = \left( \sum_{\ell=0}^{M} A_{\ell} \right)^{m/2}. \]

Now it remains to count the number of tuples \((k_{1}, \ldots, k_{m})\) corresponding to the values \(M_{1}, \ldots, M_{m/2}\). Assume for example that \(M_{i} > 0\) for all \(i\). To count the number of possibilities, we first fix a pairing of indeces \(1, \ldots, m\) which can be done in \((m - 1)!!\) different ways. Then we have \(\approx N^{m/2}\) choices to prescribe exactly one element of each pair. Let us say these values are \(s_{1} < s_{2} < \ldots < s_{m/2}\). Except for a \(o(N^{m/2})\) of these choices, we have \(s_{i} - s_{i-1} > 2M + L\) and so for each remaining index \(k_{j}\) we have two choices: if it is paired to \(s_{i}\), then either \(k_{j} = s_{i} - M_{i}\) or \(k_{j} = s_{i} + M_{i}\). Thus the total number of choices is \((m - 1)!!2^{m/2}N^{m/2}(1 + o(1))\) which verifies the claim for the case \(M_{i} > 0\) for all \(i\). If \(M_{i} = 0\) for some \(i\), then we only have one choice for the corresponding \(k_{j}\) and so we lose a factor of 2. The claim follows.

To finish the proof, notice that
\[ \sum_{\ell=0}^{M} A_{\ell}(1 + 1_{\ell>0}) = \sum_{\ell=-M}^{M} \zeta(H(H \circ F^{\ell})) \rightarrow \sigma^{2} \text{ as } M \rightarrow \infty. \]

Thus we have verified
\[ \zeta(\Sigma_{N}(H)) = \begin{cases} o(N^{m/2}), & m \text{ is odd}, \\ (m - 1)!!N^{m/2}\sigma^{m} + o(N^{m/2}), & m \text{ is even}. \end{cases} \]

completing the proof of the theorem. \(\square\)

**Remark 5.6.** The asymptotic variance given by (5.1) is typically non-zero. In particular, if either the drift is non zero, or \(d \geq 5\), then a direct calculation shows that
\[ \lim_{N \rightarrow \infty} \zeta(\Sigma_{N}^{2}) - N\sigma^{2} = - \sum_{n=-\infty}^{\infty} n\zeta(H(H \circ F^{n})) \]

(the convergence of the right hand side follows from the assumptions imposed above). Thus if \(\sigma^{2} = 0\) then \(\zeta(\Sigma_{N}^{2})\) is bounded so by \(L^{2}\)-Gotshalk-Hedlund Theorem \(H\) is an \(L_{2}\) coboundary. It is an open question if the same conclusion holds if \(\mu(\tau) = 0\) and \(d\) is 3 or 4. However, by assumption, \(f\) is exponentially mixing, so if \(H\) does not depend on \(y\) then \(\sigma^{2} > 0\) unless \(H\) is an \(L^{2}\) coboundary. Thus in many (possibly all) cases \(\sigma^{2}\) is a positive semidefinite quadratic form which is not identically equal to zero, and so its null set is a a linear subspace of positive (or infinite) codimension.

### 6. Mixing Rates for Ergodic Fibers

#### 6.1. Results.

**Definition 6.1.** We say that \((f, \tau)\) satisfies a mixing averaged Edgeworth expansion of order \(r\) if there are constants \(k_{1}, k_{2}\) and a sequence \(\delta_{N} \rightarrow 0\) so that for any function \(\bar{\phi} = \phi_{N} \in C^{k_{2}}(\mathbb{R}^{d}, \mathbb{R})\) supported on the box \(J = J_{N}\), the expression

\[ \bar{\phi} \]
\[ \mathcal{I}_{A_1,A_2,\phi}(N) := \mu(A_1(x)A_2(f^N x)\phi(\tau_N(x))) \]
satisfies
\[
\left| \mathcal{I}_{A_1,A_2,\phi}(N) - N^{-d/2} \int_{s \in \mathbb{R}^d} \phi(s) \mathcal{E}_r^{A_1,A_2}(s/\sqrt{N}) ds \right| \leq \|A_1\|_{C^{k_1}} \|A_2\|_{C^{k_1}} \|\phi\|_{C^{k_2}} \text{Vol}(J) \delta_N N^{-(d+r)/2}
\]
where
\[ \mathcal{E}_r(s) = \mathcal{E}_r^{A_1,A_2}(s) = g(s) \sum_{p=0}^r \frac{P_{A_1,A_2}(s)}{N^{p/2}}. \]

\( g(\cdot) \) is a centered Gaussian density with positive definite covariance matrix and \( P_p(s) \) are polynomials in \( s \) whose coefficients are bounded bilinear forms in \( (A_1, A_2) \), in absolute value are bounded by \( C \|A_1\|_{C^{k_1}} \|A_2\|_{C^{k_1}} \), and \( P_0^{A_1,A_2}(s) = \mu(A_1)\mu(A_2) \).

**Definition 6.2.** We say that \((f, \tau)\) satisfies a mixing averaged double Edgeworth expansion of order \( r \) if there are constants \( k_1, k_2 \) and a sequence \( \delta_N \to 0 \) so that for any functions \( \phi_i = \phi_i(N_i) \in C^{k_2}(\mathbb{R}) \) supported on the interval \( J_i = J_i(N_i) \) \( (i = 1, 2) \), the expression
\[ \mathcal{I}_{A_1,A_2,A_3,\phi_1,\phi_2}(N_1, N_2) := \mu(A_1(x)A_2(f^{N_1}(x))A_3(f^{N_2}(x))\phi_1(\tau_{N_1}(x))\phi_2(\tau_{N_2}(x))) \]
satisfies
\[
- \iint \phi_1(s_1) g \left( \frac{s_1}{\sqrt{N_1}} \right) \phi_2(s_2) g \left( \frac{s_2 - s_1}{\sqrt{N_2 - N_1}} \right) N_1^{-d/2} N_2^{-d/2} \sum_{p_1,p_2=0}^r \frac{P_{p_1,p_2}^{A_1,A_2,A_3}(s_1/\sqrt{N_1}, (s_2 - s_1)/\sqrt{N_2 - N_1})}{N_1^{p_1} (N_2 - N_1)^{p_2}} ds_1 ds_2 \leq \left( \prod_{j=1}^3 \|A_j\|_{C^{k_1}} \right) \left( \prod_{i=1,2} \|\phi_i\|_{C^{k_2}} \text{Vol}(J_i) \right) ~ \delta_{\min\{N_1,N_2-N_1\}}^{-d/2} \left( \min\{N_1,N_2-N_1\} \right)^{-1-(d+r)/2}
\]
where \( P_{p_1,p_2}^{A_1,A_2,A_3}(s_1, s_2) \) are polynomials in \( s_1, s_2 \) whose coefficients are bounded trilinear forms in \( (A_1, A_2, A_3) \), bounded in absolute value by \( C \prod_{j=1}^3 \|A_j\|_{C^{k_1}} \), and
\[ P_{0,0}^{A_1,A_2,A_3}(s) = \mu(A_1)\mu(A_2)\mu(A_3). \]

We will use the following hypotheses.

(A1) \( \tau_N \) satisfies a mixing averaged Edgeworth expansion of order \( r_1 \);

(A1') \( \tau_N \) satisfies a mixing averaged double Edgeworth expansion of order \( r_1 \);

(A2) For each \( \delta > 0 \), we have \( \mu(|\tau_N| > N^{1/2+\delta}) = O_\delta(N^{-r_2}) \).
Assume (A1') with (A2), (A3'), (A4) with \( H \) appearing in \([26]\), section 4.

The exponents and the ideas of the proofs are similar to those for any \( r \) noting that \( H \) by (A1).

By Corollary 3.4 and (A4), it suffices to consider the case (6.2)

\[
H_j(x, y) = A_j(x)B_j(y) \quad \text{where} \quad \nu(B_j) = 0.
\]

with \( A_j, B_j \in C^{k_3} \). Without loss of generality we can assume \( k_3 \geq k_2 \), where \( k_2 \) is given by (A1).

The proof of the above results use integrations by parts combined with various versions of (A1) and (A3). The exponents and the ideas of the proofs are similar to those appearing in [26], section 4.

6.2. Proof of Theorem 6.3. Case of \( d = 1 \). Let \( \psi \) be a \( C^\infty \) function such that \( 0 \leq \psi(s) \leq 1 \), \( \psi(0) = 0 \) and \( \psi(1) = 1 \). Given \( L > 0 \), let

\[
\psi_L(s) = \begin{cases} 
\psi(s + L + 1) & \text{if } s \in [-L - 1, -L] \\
1 & \text{if } s \in (-L, L) \\
1 - \psi(s - L) & \text{if } s \in [L, L + 1] \\
0 & \text{otherwise}.
\end{cases}
\]

By Corollary 3.4 and (A4), it suffices to consider the case (6.5)
Let $L = N^{1/2+\delta}$. Then

$$
\rho_{H_1,H_2}(N) = \iint A_1(x)A_2(f^N x)B_1(y)B_2(G_{\tau_N(x)}y)\,d\mu(x)d\nu(y)
$$

(6.6)

$$
= \iint A_1(x)A_2(f^N x)B_1(y)B_2(G_{\tau_N(x)}y)\psi_L(\tau_N(x))\,d\mu(x)d\nu(y)
$$

$$
+ \iint A_1(x)A_2(f^N x)B_1(y)B_2(G_{\tau_N(x)}y)(1 - \psi_L(\tau_N(x)))\,d\mu(x)d\nu(y).
$$

The integrand in the last line is zero unless $|\tau_N(x)| \geq L$, so by (A2) the last line is

$$
O(|H_1|_{C^0}|H_2|_{C^0}N^{-r_2})
$$

and so we need only to bound (6.6). First, observe that we can restrict the integral to $\bar{Y}$, the set of points where

$$
|S^B_{1,2}(y)| < L^{d_\beta} = L^{\beta} \text{ for } t \in [-L, L].
$$

Indeed, by (A3), the integral over $Y \setminus \bar{Y}$ is in

(6.7)

$$
O(|H_1|_{C^0}|H_2|_{C^0}L^{-r_3})
$$

and so is negligible. Next observe that (6.6), restricted to $\bar{Y}$ is of the form

$$
\int_{\bar{Y}} T_{A_1,A_2,\phi_y}(N)\,d\nu(y) \quad \text{with} \quad \phi_y(s) = B_1(y)B_2(G_s y)\psi_L(s).
$$

Now by (6.2), $r_1 \geq 1$ and so by (A1), the above expression can be replaced by

$$
N^{-1/2} \int_{\bar{Y}} \left( \int_{-L}^L \phi_y(s)\mathcal{E}_1(s/\sqrt{N})\,ds \right) \,d\nu(y)
$$

with error

(6.8)

$$
o \left( |A_1|_{C^{k_1}}|B_1|_{C^{k_0}}|A_2|_{C^{k_1}}|B_2|_{C^{k_2}} L N^{-1} \right) = o(N^{\frac{\beta - 1}{2} + \delta})
$$

where $\bar{L} = L + 1$. Integrating by parts, we obtain

$$
\int_{\bar{Y}} \left( \int_{-L}^L \phi_y(s)\mathcal{E}_1(s/\sqrt{N}) \frac{ds}{\sqrt{N}} \right) \,d\nu(y)
$$

$$
- \int_{\bar{Y}} \left( \int_{-L}^L \mathcal{E}_1(s/\sqrt{N}) \tilde{S}_y(s) \frac{ds}{N} \right) \,d\nu(y)
$$

$$
= \int_{\bar{Y}} \mathcal{E}_1(s/\sqrt{N}) \tilde{S}_y(s) \frac{ds}{N} \,d\nu(y) + O \left( |H_1|_{C^0}|H_2|_{C^0}Lg(L/\sqrt{N}) \right)
$$

where $\tilde{S}_y(s) = B_1(y) \int_0^s \psi_L(u)B_2(G_u y)\,du$. Since

$$
\tilde{S}_y(1)_{|s| \leq L} = B_1(y)S^{B_2}_{1,s}(y)_{|s| \leq L}
$$

it follows from the definition of $\bar{Y}$ that the last integral is

$$
O \left( |A_1|_{C^{k_1}}|B_1|_{C^0}|A_2|_{C^{k_1}}|B_2|_{C^{k_3}} \frac{L^{1+\beta}}{N} \right).
$$

This completes the proof of the theorem.
6.3. **Proof of Theorem 6.3**  Case of $d \geq 2$. We follow the approach of the one dimensional case. Let us assume (6.5) (the general case follows from Corollary 3.4). Now $\tau \in \mathbb{R}^d$ and so we define

$$\psi_L(s) = \prod_{j=1}^d \psi_L(s_j) \text{ for } s = (s_1, \ldots, s_d)$$

Let $\bar{Y}$ be defined as

$$\bar{Y} = \{ y : |S_t^{B_2}(y)| < L^{d\beta} \text{ for } t \in [-L, L] \}.$$ 

Next we claim

$$\rho_{H_1, H_2}(N) \approx N^{-d/2} \int_{\bar{Y}} \left( \int_{s \in [-L, L]^d} \phi_y(s) \mathcal{E}_{\tau_1}(s/\sqrt{N}) ds \right) d\nu(y)$$

where $a_N \approx b_N$ means $|a_n - b_N| = o(\|H_1\|_{C^{k_1}} \|H_2\|_{C^{k_2}} N^{d\beta-1+\epsilon})$. Indeed, repeating the argument for $d = 1$, the error term (6.7) remains valid and the error term corresponding to (6.8) is $O(\bar{L}^{d} N^{-(d+r_1)/2})$ which is in $o(N^{d(\beta-1)/2+\delta})$ by the assumption (6.2).

Performing $d$ integrations by parts, one in each coordinate direction, we conclude

$$\rho_{H_1, H_2}(N) \approx -N^{-d} \int_{\bar{Y}} \left( \int_{s \in [-L, L]^d} \tilde{S}_s(y) \frac{\partial^d}{\partial s_1 \ldots \partial s_d} \mathcal{E}_{\tau_1}(s/\sqrt{N}) ds \right) d\nu(y).$$

Now by the definition of $\bar{Y}$,

$$\rho_{H_1, H_2}(N) = O(\|H_1\|_{C^{k_1}} \|H_2\|_{C^{k_2}} N^{-d} L^{d(1+\beta)}),$$

and the theorem follows.

6.4. **Proof of Theorem 6.4**  Case of $d = 1$. Assume (6.5) (the general case follows from Corollary 3.4)

For fixed $y$, let us write

$$\sigma_N = \sigma_N(y) = \int H_1(F^N(x, y)) H_2(F^{2N}(x, y)) d\mu(x)$$

so that

$$\rho_{H_1, H_2}(N) = \zeta(H_1 (H_2 \circ F^N)) = \int \sigma_N(y) d\nu(y).$$

We will prove that for any $\delta > 0$ and for any $y \in \bar{Y}$,

$$\sigma_N = o(N^{\beta-1+\delta})$$

where $\bar{Y}$ (to be defined later) satisfies

$$\nu(\bar{Y}) > 1 - N^{-100}$$

(and so the contribution of its complement is negligible). As in the case of Theorem 6.3 the constant in the convergence in (6.9) can be bounded above by

$$C_{\delta} \|A_1\|_{C^{k_1}} \|A_2\|_{C^{k_1}} \|B_1\|_{C^{k_2}} \|B_2\|_{C^{k_2}}.$$ 

To simplify formulas, we do not indicate this dependence in the sequel.

Denote

$$Y_{L, \eta} = \{ y \in Y : \exists t \in \mathbb{R} : |t| \in [L^\eta, L] : |S_t^B| > t^{\beta+\eta} \}.$$
Next we claim that for any \( \eta > 0 \) and for any \( M \) there is some \( C \) so that \( \nu(Y_{L,\eta}) < CL^{-M} \). To prove this claim, observe that for \( y \in Y_{L,\eta} \) there is some \( t_\ast = t_\ast(y) \) with \( |t_\ast| \in [L^\eta, L] \) and \( |S_{t_\ast}(y)| > t_\ast^{\beta + \eta} \). Then \( |S_{t_\ast}(y)| > \frac{1}{2} |t_\ast|^{\beta + \eta} \) and so

\[
Y_{L,\eta} \subset \bigcup_{k=0}^{[L]} Y_{L,\eta,k}, \text{ where } Y_{L,\eta,k} = \left\{ y \in Y : |S_{k}(y)| > \frac{1}{2} k^{\beta + \eta} \text{ or } |S_{-k}(y)| > \frac{1}{2} k^{\beta + \eta} \right\}.
\]

Now we apply (A3'), with \( M \) replaced by \( (M + 1)/\eta \) to conclude

\[
\nu(Y_{L,\eta,k}) < 2Ck^{-(M + 1)/\eta} < CL^{-M-1}
\]

for all \( k \geq [L^\eta] \). The claim follows.

Next, define

\[
\bar{Y} = Y \setminus \bigcup_{l=0}^{[N]} G_{l}^{-1}(Y_{N^{1/2+\varepsilon}, \delta/4})
\]

with a small \( \varepsilon = \varepsilon(\delta) \). By the previous claim, \( \bar{Y} \) satisfies (6.10).

Denote \( L_1 = N^{1/2+\varepsilon}, L_2 = 2N^{1/2+\varepsilon} \) and \( \bar{L} = L_i + 1 \). We start by computing

\[
\sigma_N = e_1 + \int A_1(f^N(x))A_2(f^{2N}(x))B_1(G_{T_{N}}(y))B_2(G_{T_{2N}}(y))\psi_{L_1}(\tau_N)\psi_{L_2}(\tau_{2N})d\mu(x)
\]

\[
= e_1 + I_{1,1}, A_1, A_2, \phi_{y_1}, \phi_{y_2}(N, 2N)
\]

where

\[
\phi_{y,i}(s) = B_i(G_{s}(y))\psi_{L_i}(s),
\]

and the error term \( e_1 \) satisfies

(6.11) \[
|e_1| = O(N^{-r_1}) = o(N^{\beta-1})
\]

by (A2).

Now using (A1'), we derive

\[
\sigma_N = e_1 + e_2 + \sum_{p_1, p_2 = 0}^{r_1} \frac{1}{2} N^{-p_1 + p_2 + 1} J,
\]

where

\[
J = \int_{-L_1}^{L_1} \phi_{y_1}(s_1) \phi_{y_2}(s_2) ds_1 ds_2,
\]

and where by the error term in (A1') and by (6.3), \( e_2 \) satisfies

(6.12) \[
|e_2| = O(\bar{L}_1 \bar{L}_2 N^{-1/2} N^{-(1+r_1)/2}) = O(N^{2\varepsilon - r_1/2}) = o(N^{\beta-1+\delta}).
\]

Next, we write the integral w.r.t. \( s_2 \) in \( J \) as

\[
J_1 + J_2 = \int_{s_1 - N^{1/2+\varepsilon}}^{s_1 + N^{1/2+\varepsilon}} \ldots ds_2 + \int_{s_2 \in [-\bar{L}_2, \bar{L}_2] \setminus [s_1 - N^{1/2+\varepsilon}, s_1 + N^{1/2+\varepsilon}]} \ldots ds_2.
\]
The integrand in $J_2$ is bounded by a polynomial term times $g(N^\varepsilon)$ and so $J_2$ is negligible. Now let us write

$$
\partial_2(Pg)(x,y) = \frac{\partial}{\partial y}(P(x,y)g(y)).
$$

Then using integration by parts in $J_1$ we conclude that

$$
(6.13) \quad \sigma_N \approx - \sum_{p_1,p_2=0}^{r_1} \frac{1}{N^{p_1+p_2+\frac{1}{2}}} \int_{-L_1}^{L_1} \phi_{y,1}(s_1) g \left( \frac{s_1}{\sqrt{N}} \right) K_{p_1,p_2}(s_1) ds_1,
$$

where

$$
K(s_1) = K_{p_1,p_2}(s_1) := \int_{s_1-N^{1/2+\varepsilon}}^{s_1+N^{1/2+\varepsilon}} S_{s_2-s_1}^{B_2}(G_{s_1},y) \left[ \partial_2 \left( P^{1,A_1,A_2}_{p_1,p_2} g \right) \left( \frac{s_1}{\sqrt{N}}, \frac{s_2-s_1}{\sqrt{N}} \right) \right] ds_2
$$

$$
= \int_{-N^{1/2+\varepsilon}}^{N^{1/2+\varepsilon}} S_u^{B_2}(G_{s_1},y) \left[ \partial_2 \left( P^{1,A_1,A_2}_{p_1,p_2} g \right) \left( \frac{s_1}{\sqrt{N}}, \frac{u}{\sqrt{N}} \right) \right] du
$$

and $\approx$ means that the difference between the two sides is in $o(N^{\beta-1+\delta})$.

Using the fact that $y \in \bar{Y}$ and assuming that $\varepsilon = \varepsilon(\delta)$ is small enough, we have

$$
(6.14) \quad K_{p_1,p_2}(s_1) = O(N^{1+\delta/2})
$$

for any $p_1, p_2$. If $p_1 + p_2 \geq 1$, then by (6.14), the term corresponding to $p_1, p_2$ in (6.13) is

$$
O(N^{-2} N^{1/2+\varepsilon} N^{1/2+\delta/2}) = o(N^{\beta-1+\delta}).
$$

Next, we claim

$$
(6.15) \quad K_{0,0}'(s_1) = O \left( N^{\frac{\beta}{2}+\delta/2} \right).
$$

Note that by (A1'), $P_{0,0}^{1,A_1,A_2}(x,y) = \mu(A_1)\mu(A_2)$ and so

$$
K_{0,0}'(s_1) = \mu(A_1)\mu(A_2) \int_{-N^{1/2+\varepsilon}}^{N^{1/2+\varepsilon}} \left[ \frac{\partial}{\partial s_1} S_u^{B_2}(G_{s_1},y) \right] g' \left( \frac{u}{\sqrt{N}} \right) du
$$

$$
= \mu(A_1)\mu(A_2) \int_{-N^{1/2+\varepsilon}}^{N^{1/2+\varepsilon}} B_2(G_{s_1+u},y) g' \left( \frac{u}{\sqrt{N}} \right) du
$$

$$
- \mu(A_1)\mu(A_2) \int_{-N^{1/2+\varepsilon}}^{N^{1/2+\varepsilon}} B_2(G_{s_1},y) g' \left( \frac{u}{\sqrt{N}} \right) du.
$$

The integral in the penultimate line is $O \left( N^{\frac{\beta}{2}+\delta/2} \right)$ since we can perform one more integration by parts with respect to $u$. The integral in the last line is equal to

$$
\sqrt{N} B_2(G_{s_1},y) \left[ g(N^\varepsilon) - g(-N^\varepsilon) \right],
$$

which decays rapidly (i.e. faster than any polynomial) in $N$ and so is negligible. Thus we have verified (6.15).
Now we use (6.15) and an integration by parts with respect to \( s_1 \) to conclude that the term corresponding to \( p_1 = p_2 = 0 \) in (6.13) is

\[
\approx N^{-3/2} \int_{-L_1}^{L_1} S_{s_1}^{B_1}(y) \frac{\partial}{\partial s_1} \left( g \left( \frac{s_1}{\sqrt{N}} \right) \right) \mathcal{K}_{0,0}(s_1) \, ds_1.
\]

Now the definition of \( \bar{Y} \) together with (6.14) and (6.15) imply that the last expression is \( O(N^{\beta-1+\delta}) \) which completes the proof of (6.9).

We remark that the bound (6.15) can be derived in case \( p_1 + p_2 \geq 1 \) as well. This was not needed in case \( d = 1 \) but will be needed in case \( d \geq 2 \) which we discuss next.

### 6.5. Proof of Theorem 6.4. Case of \( d \geq 2 \)

Assume (6.5) (the general case follows from Corollary 3.4).

We proceed as in the case of \( d = 1 \). That is, we need to show that

\[
(6.16) \quad \sigma_N = o(N^{d(\beta-1)+\delta})
\]

for \( y \in \bar{Y} \) where \( \bar{Y} \) satisfies

\[
(6.17) \quad \nu(\bar{Y}) > 1 - N^{-100d}.
\]

First, we obtain \( |e_1| = O(N^{-r_2}) = o(N^{d(\beta-1)}) \) as in (6.11). Similarly, (6.12) reads as

\[
|e_2| = O(L_1^4 L_2^d N^{-d/2} N^{-(d+r_1)/2}) = O(N^{d-\beta/2}) = o(N^{d(\beta-1)+\delta})
\]

by (6.3) and by assuming that \( \varepsilon = \varepsilon(\delta, d) \) is small. Next, we write

\[
\bar{\partial}_2(P \mathcal{g})(x, y) = \frac{\partial}{\partial y_1 \ldots \partial y_d} (P(x, y) \mathcal{g}(y)).
\]

Then as in (6.13), we derive

\[
(6.18) \quad \sigma_N \approx - \sum_{p_1, p_2=0}^{r_1} N^{-p_1+p_2+3d} J_{p_1, p_2},
\]

where \( \approx \) means that the difference between the two sides is in \( o(N^{d(\beta-1)+\delta}) \) and

\[
J_{p_1, p_2} = \int_{s_1 \in [-L_1, L_1]^d} \phi_y(s_1) g \left( \frac{s_1}{\sqrt{N}} \right) \mathcal{K}_{p_1, p_2}(s_1) \, ds_1,
\]

where

\[
\mathcal{K}_{p_1, p_2}(s_1) = \frac{1}{L_{1/2+\varepsilon}} \int_{u \in [-N^{1/2+\varepsilon}, N^{1/2+\varepsilon}]^d} \left[ \partial_2 \left( P_{p_1, p_2} \mathcal{g} \right) \left( \frac{s_1}{\sqrt{N}}, \frac{u}{\sqrt{N}} \right) \right] \, du,
\]

and for \( u \in \mathbb{R}^d \),

\[
S_u^{B_2} (\bar{y}) = \int_{0 \leq u_i \leq |u_i|} B(G_{v_1 \cdots u_d}(s_1, \ldots, v_d sgn(u_d)(\bar{y}))) \, dv_1 \ldots dv_d
\]

where \( sgn \) is the sign function \( sgn(w) = -1 \) if \( w < 0 \) and \( sgn(w) = 1 \) if \( w > 0 \). For \( I = \{i_1, \ldots, i_{|I|}\} \subset \{1, 2, \ldots, d\} \), let us write

\[
\partial^I = \frac{\partial}{\partial s_{1,i_1} \ldots \partial s_{1,i_{|I|}}}, \quad \bar{\partial} = \partial^{(1,\ldots,d)}.
\]
We use $d$ integrations by parts with respect to the variables $s_{i1}, ..., s_{id}$ to write

\begin{equation}
\mathcal{J}_{p_1,p_2} = \int_{s_1 \in [\bar{L}_1, L_1]^d} S_{s_1}^{B_1}(y) \bar{\partial} \left[ \mathfrak{g} \left( \frac{s_1}{\sqrt{N}} \right) \mathcal{K}_{p_1,p_2}(s_1) \right] ds_1.
\end{equation}

We will show that for any $I \subset \{1, ..., d\}$ and for any $p_1, p_2$,

\begin{equation}
|\partial^I \mathcal{K}_{p_1,p_2}| \lesssim N^{d/2} N^{d/2} N^{d/2},
\end{equation}

where $a_N \lesssim b_N$ means that $a_N < b_N N^{\delta/2}$ (assuming that $\varepsilon = \varepsilon(\delta)$ is small enough). Assume first that (6.20) hold. Then observe that

\begin{equation}
\partial^I \mathcal{K}_{p_1,p_2} \lesssim N^{d/2} N^{d/2} N^{d/2}.
\end{equation}

Substituting this estimate to (6.19), we obtain

\begin{equation}
|\mathcal{J}_{p_1,p_2}| \lesssim N^{d/2} N^{d/2} N^{d/2},
\end{equation}

which, implies (6.16). Thus it remains to prove (6.20).

Assume that $\mathfrak{g}$ is the standard Gaussian density (if this is not the case, we can compute all integrals on a parallelepiped of side length $cN^{1/2+\varepsilon}$, then apply a linear change of variables to reduce to the case of standard Gaussian). To prove (6.20) we write

\begin{equation}
h = \bar{\partial}_2 \left( P_{p_1,p_2}^{1,A_1,A_2} \mathfrak{g} \right).
\end{equation}

Recall that $I = \{i_1, ..., 1|_I\}$, the set of indices $i$ such that we are differentiating with respect to $s_{i,1}$, is given. We need to differentiate the integrand in $\mathcal{K}$, which is a product. Let $I' = \{i'_1, ..., i'_{|I'|}\} \subset I$ denote the set of indices $i'$ so that we differentiate the term $S_u^{B_2}(G_{s_1}(y))$ with respect to $s_{1,i'}$. For $i \in I \setminus I'$, we differentiate $h$ with respect to $s_{i,1}$. We also write $J = \{1, ..., d\} \setminus I$ and $J' = \{1, ..., d\} \setminus I'$. Performing the differentiation, we find

\begin{equation}
\partial^I \mathcal{K}_{p_1,p_2} = \sum_{|I'| \leq I} \int_{u \in [-N^{1/2+\varepsilon}, N^{1/2+\varepsilon}]^d} \int_{w \in [0, |u_{J'}|]} \sum_{\delta_{i'} \in \{0, 1\} \text{ for } i' \in I'} (-1)^{|I'|-|\delta_{i'}|} B_2(G_{i';s_{1,i'}+\delta_{i'}u_{i'},j';s_{1,j'}+w_{j'}\text{sgn}(u_{j'})}(y)) \left[ \partial^{I\setminus I'} h \left( \frac{s_1}{\sqrt{N}}, \frac{u}{\sqrt{N}} \right) \right] dw_{J'} du,
\end{equation}

where in the subscript of $G$ the notation $(i' : a_{i'}; j' : b_{j'})$ means that for coordinates $i' \in I'$ we use $a_{i'}$ and for $j' \in J'$, we use $b_{j'}$. Note that

\begin{equation}
\partial^{I\setminus I'} h \left( \frac{s_1}{\sqrt{N}}, \frac{u}{\sqrt{N}} \right) = N^{\frac{||I|-|I'||}{2}} \bar{h} \left( \frac{s_1}{\sqrt{N}}, \frac{u}{\sqrt{N}} \right),
\end{equation}

where

\begin{equation}
\bar{h}(x, y) = \frac{\partial^{I\setminus I'}}{\partial x_{i_1} \cdots \partial x_{i_{|I'|}}} \frac{\partial}{\partial y_1 \cdots \partial y_d} (P(x, y) \mathfrak{g}(y)).
\end{equation}
Now assume there is some $i'$ so that $\delta_{i'} = 0$. Then $B_2(...) = 0$, and so performing the integral with respect to $u_{i'}$, first, we obtain

\begin{equation}
\int_{u_{i'} \in [-N^{1/2+\varepsilon}, N^{1/2+\varepsilon}]} \tilde{h} \left( \frac{s_1}{\sqrt{N}}, \frac{u}{\sqrt{N}} \right) du_{i'} = \sqrt{N} \sum_{a=1,2} (-1)^a \tilde{h}_i \left( \frac{s_1}{\sqrt{N}}, \left( \frac{u_1}{\sqrt{N}}, \ldots, \frac{u_{i'-1}}{\sqrt{N}}, (1)^a N^\varepsilon, \frac{u_{i'+1}}{\sqrt{N}}, \ldots, \frac{u_d}{\sqrt{N}} \right) \right),
\end{equation}

where

$$
\tilde{h}_i(x, y) = \frac{\partial^{1|-i'|}}{\partial x_{i'} \ldots \partial x_{i'}} \frac{\partial^{d-1}}{\partial y_{i'} \ldots \partial y_{i'+1}} (P(x, y) g(y)).
$$

Recalling that $g(y) = \frac{1}{(2\pi)^{d/2}} \exp \left( -\sum_{j=1}^{d} y_j^2/2 \right)$, we see that $\tilde{h}_i(x, y)$ decays rapidly as $y_i \to \infty$ (i.e. faster than any polynomial). Since we have $|y_i| = N^\varepsilon$, (6.23) decays rapidly as $N \to \infty$. Thus this term, even when integrated with respect to all other variables, decays rapidly and consequently we can neglect all terms in (6.21) where there is some $i'$ so that $\delta_{i'} = 0$.

It remains to study the case when $\delta_{i'} = 1$ for all $i' \in I'$. Then we perform the integrals in (6.21) with respect to $w_{i'}, j' \in J'$ and we integrate by parts with respect to $u_{i'}, i' \in I'$ to obtain that

$$
|\partial^I K_{0,0} - \mathcal{I}|
$$

decays rapidly as $N \to \infty$, where

$$
\mathcal{I} = \int_{u_{i'}' \in I'} \int_{u_{i',i'}' \in I'} S_{b i'}^B \left( y \right) \left[ \partial^I h \left( \frac{s_1}{\sqrt{N}}, \frac{u}{\sqrt{N}} \right) \right] du_{i'} du_{i'}
$$

and

$$
b = (i' : N^{1/2+\varepsilon}, j' : u_{i'}').
$$

As in (6.22), we have

\begin{equation}
\partial^I h \left( \frac{s_1}{\sqrt{N}}, \frac{u}{\sqrt{N}} \right) = N^{-|u|} \hat{h} \left( \frac{s_1}{\sqrt{N}}, \frac{u}{\sqrt{N}} \right)
\end{equation}

where

$$
\hat{h}(x, y) = \frac{\partial^{1|I|}}{\partial x_{i_1} \ldots \partial x_{i_{|I|}}} \frac{\partial^d}{\partial y_{i_1} \ldots \partial y_{d}} (P(x, y) g(y)).
$$

Note that we can assume $|S_{b i'}^B| \lesssim N^{d\beta/2}$. Indeed, we can subdivide the rectangular box with opposite corners 0 and $b$ into small cubes of side length $N^\varepsilon$ and we can assume that the integral of $G_s(y)$ over all of the boxes is smaller than $N^{d\varepsilon}$ for $y \in \bar{Y}$ by (A3') ($\bar{Y}$ satisfies (6.17) similarly to the case $d = 1$). Combining this observation with (6.24), we conclude

$$
|\mathcal{I}| \leq N^{\frac{d-|I|}{2}} \int_{u \in [-N^{1/2+\varepsilon}, N^{1/2+\varepsilon}]} \|\hat{h}\|_\infty du \leq C N^{d\frac{d+1-|I|}{2} + \delta/2}
$$

if $\varepsilon(\delta)$ is small enough. This completes the proof of (6.20) and so the theorem follows.
7. Toral translations and related systems

7.1. Rapid mixing. Let $f$ be an Axiom A diffeomorphism, and $\mu$ be a Gibbs measure with Hölder potential. Let $Y = \mathbb{T}^m$ and $G_t$ be a $d$-parameter flow: $G_{(t_1, \ldots, t_d)}(y) = y + \sum_{j=1}^{d} \alpha_j t_j$ for some $\alpha_1, \ldots, \alpha_d \in \mathbb{R}^m$. Note that $G_t$ has discrete spectrum, so it is far from being mixing. However, according to [23] the mixing properties of the corresponding skew products are typically much better than the results obtained in Section 4 for the case of the mixing fibers. Namely, let $\Pi$ be the linear subspace generated by $\alpha_1, \ldots, \alpha_d$. We say that $\Pi$ is Diophantine if there exist numbers $K, s$ such that for any unit vector $v \in \Pi$ for any $k \in \mathbb{Z}^m$ we have $|\langle v, k \rangle| \geq K|k|^{-s}$.

**Proposition 7.1.** ([23]) If $\Pi$ is Diophantine, then $F$ is rapidly mixing except for the set $\tau : X \rightarrow \Pi$ lying in an infinite codimension submanifold.

Below we describe two applications of this result.

7.2. Heisenberg fibers. Let $f$ be as in §7.1 $Y$ be a Heisenberg nilmanifold, that is $Y = H/\Gamma$ where $H$ is a Heisenberg group, i.e. the set of triples $(p, q, z)$ with $p, q \in \mathbb{R}^m$, $z \in \mathbb{R}$ with multiplication law

$$(p_1, q_1, z_1) (p_2, q_2, z_2) = (p_1 + p_2, q_1 + q_2, z_1 + z_2 + \langle p_1, q_2 \rangle - \langle q_1, p_2 \rangle)$$

and $\Gamma$ is a cocompact discrete subgroup of $H$. Let $G_t$ be a $d$-parameter group of translations. Note that $Y$ fibers over a $2m$ dimensional torus $T$ and $F$ is a skew product whose base map $\mathcal{F}$ is a $T, T^{-1}$ transformation on $X \times T$. We assume that $\mathcal{F}$ is rapidly mixing (this assumption is generic in view of Proposition 7.1). Let $\Phi_1, \Phi_2$ be functions on $X \times Y$. Split $\Phi_j = \tilde{\Phi}_j + \Phi_j$ where $\tilde{\Phi}_j$ is invariant with respect to the center action on $Y$.

**Proposition 7.2.** (a) If $\mu(\tau) = 0$ and $\tau$ is not a coboundary, then

\[ \lim_{n \to \infty} \sqrt{n} \rho_{\Phi_1, \Phi_2}(n) = \frac{1}{\sqrt{2\pi \sigma^2}} \int \int \int \int \tilde{\Phi}_1(x_1, y) \tilde{\Phi}_2(x_2, G_t y) d\mu(x_1) d\mu(x_2) d\nu(y) dt. \]

where $\sigma^2 = \sum_{k=-\infty}^{\infty} \mu(\tau \circ f^k)$.

(b) If $\mu(\tau) \neq 0$ and $\tau$ is not cohomologous to a constant, then $F$ is rapidly mixing.

**Proof.** (a) $\rho_{\Phi_1, \Phi_2}(n) = \rho_{\Phi_1, \Phi_2}(n) + \rho_{\tilde{\Phi}_1, \Phi_2}(n)$. The first term decays faster than any polynomial by our assumption while the second term is asymptotic to the RHS of (7.1) due to Remark 4.10. This completes the proof of (a). The proof of (b) is similar. □

7.3. Constant suspensions in the fiber. Again we take $f$ as in §7.1 but now we consider constant suspensions acting in the fiber. That is let $\mathcal{G}^n$ be a $\mathbb{Z}^d$ an exponentially mixing action on a manifold $\mathcal{Y}$ preserving a measure $\nu$, let $Y = \mathcal{Y} \times \mathbb{R}^d/\sim$ where $\sim$ is the identification $\tilde{y}, z + n \sim \tilde{y}^n, z$. Let $G_t$ be the action $(\tilde{y}, z) \rightarrow (\tilde{y}, z + t)$. It preserves measure $d\nu = d\tilde{\nu} dz$.

Given a $T, T^{-1}$ map as above, consider an associated action $\mathcal{F}$ on $X \times T^d$ given by $\mathcal{F}(x, \theta) = (fx, \theta + \tau(x))$. 
Proposition 7.3. Suppose that $F$ is rapidly mixing. Then (4.5) holds.

Proof. The proof is similar to the proof of Proposition 7.2. Namely we split $H = \tilde{H} + \tilde{\tilde{H}}$ where $\tilde{H}(x, z) = \int H(x, \tilde{y}, z) d\tilde{\nu}(\tilde{y})$. Note that $G_t$ and hence $F$ preserves this splitting and that $\tilde{H}$ is $\mathbb{Z}^d$ invariant, because $G^n$ preserves $\tilde{\nu}$ and

$$\int H(x, \tilde{y}, z + n) d\tilde{\nu}(\tilde{y}) = \int H(x, G^n\tilde{y}, z) d\tilde{\nu}(\tilde{y}) = \tilde{H}(x, z).$$

It follows that

$$\rho_{H_1, H_2}(n) = \rho_{H_1, H_2}(n) + \rho_{\tilde{H}_1, \tilde{\tilde{H}}_2}(n).$$

The first term decays faster than any polynomial, because $F$ is rapidly mixing and the second term is $O(n^{-d/2})$ due to Remark 4.10. However to apply the remark, we need to check that $G_t$ is exponentially mixing on the space $B$ of $CL$ functions such that

$$\int H(x, \tilde{y}, z) d\tilde{\nu}(\tilde{y}) = 0 \text{ for all } (x, z).$$

To check mixing, we write $t = n + \hat{t}$, where $n \in \mathbb{Z}^d$ and $\hat{t}$ belongs to the unit cube. Then

$$\int H_1(x_1, (\tilde{y}, z)) H_2(x_2, G_t(\tilde{y}, z)) d\nu = \int \int H(x_1, (\tilde{y}, z - \hat{t})) H_2(x_2, (G^n\tilde{y}, z)) d\tilde{\nu}(\tilde{y}) dz.$$

Integrating first with respect to $\tilde{y}$, we see that the RHS decays exponentially as needed.

\[ \Box \]

8. Deviations of ergodic averages

8.1. Mixing and deviations. Here we recall some results about the relations of mixing and deviations of ergodic averages.

Lemma 8.1. Let $X_1, X_2, ...$ be a stationary sequence of random variables on a probability space $(\Omega, P)$ and $S_N = \sum_{k=1}^{N} X_k$. Assume that there are constants $C$ and $\rho$ such that for every $n$

$$E(S_n^2) < Cn^{2\rho}.$$ 

Then $S_n/n^{\max\{\rho, \frac{1}{2}\}+\varepsilon}$ converges to zero almost surely for all $\varepsilon > 0$.

Proof. Let us assume $\rho > 1/2$ (the case $\rho \leq 1/2$ is a simple consequence). For a positive integer $m$, let $D_m$ denote the collection of intervals of the form $I_{i,j} = [j2^i + 1, (j + 1)2^i]$ for all non-negative integers $i, j$ so that $(j + 1)2^i \leq 2^m$. By the stationarity assumption,

$$E \left( \sum_{I_j \in D_m} \left( \sum_{k \in I_j} X_n \right)^2 \right) \leq \sum_{i=0}^{m} 2^{m-i} E(S_{2^i}^2) \leq C \sum_{i=0}^{m} 2^{m-i} 2^{2i\rho} \leq C2^{2m\rho}.$$

Now for given positive integer $n$, let $m$ be so that $2^{m-1} < n \leq 2^m$. Then the interval $[1, n]$ can be written as a disjoint union of at most $2m$ intervals from the family $D_m$. Let
us denote this collection of intervals by \( D(n) \). Then by the Cauchy Schwartz inequality,

\[
S_n^2 = \left( \sum_{I \in D(n)} \left( \sum_{k \in I} X_k \right) \right)^2 \leq 2m \sum_{I \in D(n)} \left( \sum_{k \in I} X_k \right)^2 \leq 2m \sum_{I \in D_m(n)} \left( \sum_{k \in I} X_k \right)^2
\]

Thus we have

\[
P(\exists n = 2^{m-1}+1, ..., 2^{m} : S_n^2 > \eta n^{2\rho+\varepsilon})
\]

\[
\leq P(2m \sum_{I \in D_m(n)} \left( \sum_{k \in I} X_k \right)^2 > \eta 2^{(m-1)(2\rho+\varepsilon)})
\]

\[
\leq 2m \eta^{-1} 2^{-2(m-1)(\rho+\varepsilon)} \eta^{-1} 2^{-m\varepsilon}
\]

Using the Borel-Cantelli lemma and the fact that \( \eta > 0 \) is arbitrary, Lemma 8.1 follows. □

Lemma 8.2. Under the assumptions of Lemma 8.1 suppose that \(|E(X_iX_j)| \leq C|i-j|^{-\beta}\) then (8.1) is satisfied with

\[
\rho = \begin{cases} 
\frac{1}{2}, & \text{if } \beta > 1, \\
1 - \frac{\beta}{2}, & \text{if } \beta < 1.
\end{cases}
\]

Proof. (8.1) follows since \( E(S_N^2) = NE(X_0^2) + 2 \sum_{n=0}^{N-1} (N-n) E(X_0X_n) \). □

8.2. Examples and open questions. Here we describe several classes of systems satisfying our assumptions on the base and the fiber dynamics made in previous sections. We also present several open questions pertaining to establishing those properties in several new cases.

Mixing of the base system is required in all our results. In addition the results of Sections 4 require mixing in the fiber, so we begin with reviewing known results for mixing.

Exponential mixing is known in the following cases: uniformly hyperbolic diffeomorphisms with Gibbs measures (10, 55); nonuniformly hyperbolic systems admitting Young towers with exponential tails (64); partially hyperbolic translations on homogeneous spaces (18, 6); contact Anosov flows (52) as well as Anosov flows with suitable assumptions on Lyapunov spectrum (1, 61); some singular hyperbolic flows (2); ergodic automorphisms of tori (46) and of nilmanifolds (39). In all the examples of \( \mathbb{R} \) or \( \mathbb{Z} \) actions listed above, we also have multiple exponential mixing (see e.g. (24)) while in higher rank the multiple exponential mixing is only known for partially hyperbolic translations on homogeneous spaces (6), (partial results for some \( \mathbb{Z}^d \) actions are obtained in 40).

Rapid mixing is known for generic Axiom A flows with Gibbs measures (21, 22, 33), hyperbolic flows having Young towers with exponential tails (5), some singular hyperbolic flows (3), and general compact group extensions of uniformly hyperbolic systems (23).
Polynomial mixing is known for nonuniformly hyperbolic diffeomorphisms and flows having Young towers with polynomial tails ([59], [41], [5]), unipotent actions ([48], [6]), time changes of nilflows ([38]), and some flows on surfaces with degenerate singularities ([32]).

Additional assumptions imposed on base dynamics in various results include large deviations, anticoncentration, and LLT. An easiest way to get large deviation is to have unique ergodicity since in that case the set in LHS of (3.2) is empty. A relative version of unique ergodicity is so called unique ergodicity (see [24] for a definition), which holds for partially hyperbolic systems with unique measure absolutely continuous with respect to the unstable foliation. In this case (3.2) holds due to [24]. Exponential large deviations also hold for non-uniformly hyperbolic systems admitting Young towers with exponential tails for return times [57], while in case the tail is polynomial, polynomial large deviations hold [42].

Anticoncentration inequality is established for systems admitting Young towers provided that the return time tail has second moment [56].

The LLT is known for Axiom A diffeomorphisms with Gibbs measures ([55]), the systems admitting Young tower under the assumptions that the tails admit the second moment ([60]) as well as flows which can be represented as suspensions of flows admitting nice symbolic dynamics [28] including Axiom A flows and certain Lorenz type attractors. The results of [28] can be applied to continuous time $T, T^{-1}$ systems given by (4.15).

For fiber dynamics we require control on ergodic averages. For mixing systems such control can be obtain using moment estimates (cf. Lemma 8.1).

Systems satisfying assumption (A3) (or (A3')) for $d = 1$ include exponentially mixing systems described above, as well as flows which can be represented as suspensions of flows admitting nice symbolic dynamics [28] including Axiom A flows and certain Lorenz type attractors. The results of [28] can be applied to continuous time $T, T^{-1}$ systems given by (4.15).

The results of this paper motivate the study of the statistical properties discussed above for a wider class of dynamical systems. In particular, it is of interest to

(a) construct example of systems satisfying mixing multiple Edgeworth expansion;
(b) prove mixing LLTs for partially hyperbolic systems;
(c) investigate mixing LLTs and anticoncentration bounds for parabolic systems.

### 8.3. Deviations of ergodic averages for generalized $T, T^{-1}$ transformations.

Here we illustrate the information the results obtained in this paper provide about the growth of ergodic sums in several special cases. In the examples below we assume that the base dynamics $f$ is given by an Anosov diffeomorphism equipped with a Gibbs measure and for each fiber flow (1–12) we give an exponent $\alpha$ such that with probability one the ergodic sums of the corresponding generalized $T, T^{-1}$ transformation grow slower than $N^{\alpha+\varepsilon}$ for every $\varepsilon > 0$. This is going to be a simple consequence of Lemmas 8.1 and 8.2. For each example we list the result that implies the assumption of Lemma 8.2.
with a suitable $\beta$. In case we use the results of Section 6, we also assume that $f$ satisfies mixing double averaged Edgeworth expansion of any order. Currently no examples of such systems is known but we expect this property to hold for large class of map (cf. e.g. the computations in [30]).

(1) Anosov diffeomorphisms. In this case we have exponential mixing ([10] [55]);
(a) zero drift : $\alpha = 3/4$ (Thm 4.7);
(b) positive drift : $\alpha = 1/2$ (Thm 4.1).
(2) Diophantine toral translations–here ($A3'$) holds for any $\beta > 0$ and so $\alpha = 1/2$ by Thm 6.4 (cf. also Prop 7.1).
(3) Product of Anosov diffeomorphisms and toral translation: $\alpha = 3/4$ (Thm 6.4).
(4) horocycle flows (see [34]): Thm 6.4 gives
(a) no small eigenvalues of $\Delta$, zero drift–($A3$) holds for any $\beta > 1/2$, so $\alpha = \rho_1(\beta) = 3/4$;
(b) smallest eigenvalue of $\Delta$ is $\lambda \in (0, 1/4)$–($A3$) holds for any $\beta > \frac{1+\sqrt{1-4\lambda}}{2}$, so $\alpha = \rho_1(\beta) = \frac{1+\sqrt{1-4\lambda}}{2}$.
(5) translations flows–($A3'$) holds for any $\beta > \lambda_2$ ([36]) where $\lambda_2$ is the second exponent of Kontsevich-Zorich cocycle. So $\alpha = \rho_1(\beta) = \frac{\lambda_2+1}{\lambda_2}$ (Thm 6.4).
(6) area preserving flows on surfaces: $\alpha = \frac{7+\lambda_2}{8}$ (Thm 6.4 and Pr B.1).
(7) partially hyperbolic translations on homogenous spaces. In this case we have exponential mixing ([48] [6]);
(a) zero drift: $\alpha = 3/4$ (Thm 4.7);
(b) positive drift: $\alpha = 1/2$ (Thm 4.1).
(8) multidimensional Cartan actions on homogenous spaces: $\alpha = 1/2$ (Thms 4.4 and 4.1).
(9) constant suspensions of Cartan actions on tori: $\alpha = 1/2$ (Pr 7.3).
(10) Heisenberg translations : Pr 7.2 gives
(a) zero drift: $\alpha = 3/4$;
(b) positive drift: $\alpha = 1/2$.
(11) continuous time $T, T^{-1}$ system given by (4.15) with both base flow $\phi^t$ and fiber flow $G_t$ given by geodesic flow on a unit tangent bundle over a negatively curve manifold: $\alpha = 7/8$ by Example 4.12(b) with $k = 2$. In fact, Example 4.12(b) shows that for all positive integers $k$, we can obtain a system with $\alpha = 1 - 2^{-k-1}$.
(12) generic higher rank abelian action on Heisenberg nilmanifolds : $\alpha = 1/2$ (17) and Thm 6.4).

Appendix A. Anticoncentration large deviation bounds for subshifts of finite type

We follow the argument in [27].

Let $(\Sigma, \sigma)$ be a subshift of finite type, $\mu$ be a Gibbs measure and $\tau : \Sigma \to \mathbb{R}^d$ be a Hölder function of zero mean. We assume that for each $a \in \mathbb{R}^d \setminus \{0\}$ the function $\langle a, \tau \rangle$ is not a coboundary.

Lemma A.1. ([55]) There are constants $c_1, \delta_0$ such that for $|\xi| < \delta_0$
\begin{equation}
\mu \left( e^{\langle \xi, \tau N \rangle} \right) \leq e^{c_1 N \xi^2}.
\end{equation}
\[ |\Phi_N(\xi)| \leq e^{-c_1N\xi^2}, \quad \text{where} \quad \Phi_N(\xi) = \mu(e^{i\langle \xi, \tau_N \rangle}). \]

**Corollary A.2.** There are constants \( C_2, c_2 \) such that
\[ \mu(|\tau_N| > L) \leq C_2 e^{-c_2L^2/N} \]
and for each unit cube \( Q \)
\[ \mu(\tau_N \in Q) \leq \frac{C_2}{N^{d/2}}. \]

**Proof.** To prove the first inequality we may assume without the loss of generality that \( d = 1 \) and that \( \sqrt{N} \leq L \leq 2c_1\delta_0N \) (we obtain the general result by increasing \( C_2 \) and decreasing \( c_2 \)). We estimate \( \mu(\tau_N > L) \), the bound for \( \mu(\tau_N < -L) \) being similar. We have that for each \( \xi \in (0,\delta_0) \)
\[ \mu(\tau_N > L) = \mu(e^{\xi\tau_N} > e^{\xi L}) \leq e^{-\xi L} \mu(e^{\xi\tau_N}) \leq e^{-\xi L + c_1N\xi^2} \]
Taking \( \xi = \frac{L}{2c_1N} \) we obtain the result.

It is enough to prove (A.4) for cubes of any fixed size \( \rho \) since the unit cube can be covered by a finite number of cubes of size \( \rho \). Let
\[ g(x) = \prod_{l=1}^{d} \left( 1 - \cos(\hat{\delta}x_l) \right) \]
where \( \hat{\delta} = \delta_0/d \) and \( \delta_0 \) is the constant from Lemma A.1. Then
\[ \hat{g}(\xi) = (\pi \hat{\delta})^d \prod_{l=1}^{d} \left( \left( 1 - \frac{|\xi|}{\hat{\delta}} \right) I_{|\xi| \leq \hat{\delta}} \right). \]
Hence for each \( a \)
\[ \mathbb{E}(g(\tau_N - a)) = \int_{\mathbb{R}^d} \hat{g}(-\xi) e^{i\xi a} \Phi_N(\xi)d\xi \leq \int_{|s| < \delta_0} \hat{g}(s)|\Phi_N(s)|ds \]
since \( \hat{g} \) is real, positive, and supported inside the ball of radius \( \delta_0 \). Thus (A.2) implies that there is a constant \( \hat{D} \) such that
\[ \mathbb{E}(g(\tau_N - a)) \leq \frac{\hat{D}}{N^{d/2}}. \]
On the other hand \( g(0) = \frac{1}{2^d} \) so there is a constant \( \rho \) such that \( g(x) > \frac{1}{4^d} \) on the cube of size \( \rho \) centered at 0. Hence if \( Q \) is a cube of size \( \rho \) centered at \( a \) then
\[ \mathbb{E}(g(\tau_N - a)) \geq \frac{\mathbb{P}(S_N \in Q)}{4^d}. \]
Combining the last two displays we obtain the result. \( \square \)

We now prove the anticoncentration large deviation estimate with \( \Theta(r) = e^{-c_4r^2} \).

**Lemma A.3.** If \( Q \) is a unit cube centered at \( z \), then
\[ \mu(\tau_N \in Q) \leq \frac{C_3}{N^{d/2}} e^{-c_4z^2/N}. \]
Proof. There is a constant $R$ such that
\[
\mu(\tau_N \in Q) \leq \mu(\tau_N \in Q, |\tau_{N/2}| > |z/2 - R|) + \mu(\tau_N \in Q, |\tau_N - \tau_{N/2}| > |z/2 - R|).
\]
We will estimate the first term, the estimate of the second is obtained by replacing $\sigma$ by $\sigma^{-1}$. We have
\[
\mu(\tau_N \in Q, |\tau_{N/2}| > |z/2 - R|) \leq \sum_{C',C''} \mu(C'C''),
\]
where the sum is over all pairs of cylinders $(C',C'')$ such that
1. length$(C') = length(C'') = N/2$,
2. there exists $\omega' \in C'$ such that $|\tau_{N/2}(\omega')| > |z/2 - R|$,
3. there exists $\omega'' \in C''$ such that $|\tau_{N/2}(\omega') + \tau_{N/2}(\omega'') - z| < 2R$.

By Gibbs property
\[
\sum_{C',C''} \mu(C'C'') \leq K \sum_{C',C''} \mu(C') \mu(C'').
\]
By (A.3) for each $C'$ the sum of $\mu(C'')$ over the cylinders $C''$ satisfying (iii) is smaller than $(2R)^d C_2$. Summing over $C'$ satisfying (ii) and using (A.3), we obtain the result. \(\square\)

Lemma A.4. Let $Q_1, \ldots Q_s$ be unit cubes centered at $z_1, \ldots z_s$. Then with the notation
\[
z_0 = 0 \in \mathbb{R}^d, n_0 = 0,
\]
\[
\mu(\tau_{n_j} \in Q_j \text{ for } j = 1, \ldots s) \leq \prod_{j=1}^s \left[ \left( \frac{C_4}{(n_j - n_{j-1})^{d/2}} \right) e^{-c_4 |z_j - z_{j-1}|^2} \right].
\]

Proof. The LHS can be bounded by $\sum (\mu(C_1C_2 \ldots C_s))$ where the sum is over all tuples of cylinders such that
1. length$(C_j) = n_j - n_{j-1}$ and
2. On $C_j$, $\tau_{n_j-n_{j-1}}$ is contained in a cube of size $R$ centered at $z_j - z_{j-1}$.

Using Gibbs property the last can be bounded by
\[
K \prod_{j=1}^s \left[ \sum_{C_j:(\text{i}) \text{ and } (\text{ii}) \text{ hold}} \mu(C_j) \right].
\]
Now the result follows by Lemma A.3. \(\square\)

Appendix B. Deviation of ergodic averages for area preserving flows on surfaces

In this section, we will show that typical smooth area preserving flows on surfaces (with non-degenerate singularities) satisfy condition $(A3')$. Let $(\varphi_t)$ be a multivalued Hamiltonian flow on a closed surface of genus $g \geq 2$ such that $(\varphi_t)$ has only simple saddles and no saddle loops homologous to zero (the last assumption implies minimality, [54]). It is classical (see e.g. [4, 62]) that $(\varphi_t)$ can be represented as a special flow over
an interval exchange transformation (IET) $T$ and under a roof function with symmetric logarithmic singularities. More precisely, let $T : [0, 1) \to [0, 1)$ be a $d$-IET given by a permutation $\pi$ and a length vector $\lambda = (\lambda_1, \ldots, \lambda_d)$. Let $(\beta_i)_{i=0}^d$, $\beta_0 = 0$, $\beta_d = 1$ denote the discontinuities of $T$ (along with the endpoints of $[0, 1)$). Let $f : [0, 1) \to \mathbb{R}_+$ be given by

$$f(x) = \sum_{i=0}^{d-1} C_i^+ \log(x - \beta_i) + \sum_{j=1}^d C_j^- \log(\beta_j - x) + g(x),$$

where $g \in C^2([0, 1))$ and

$$\sum_{i=0}^{d-1} C_i^+ = \sum_{j=1}^d C_j^-$$

(in this case we say $f$ has symmetric logarithmic singularities). We will also assume without the loss of generality that $\int_{[0, 1)} f \, d\text{Leb} = 1$. Then, $(\phi_t)$ is conjugated to a flow $(T^f_t)$ acting on $X^f := \{(x, s) : x \in [0, 1), 0 \leq s < f(x)\}$, by

$$T^f_t(x, s) = (T^{n(x,s,t)} x, s + t - S_{n(x,s,t)}(f)(x)),$$

where $n(x, s, t) \in \mathbb{Z}$ is the unique number such that

$$0 \leq s + t - S_{n(x,s,t)}(f)(x) < f(T^{n(x,s,t)} x).$$

Here $S_n(f)(x) = \sum_{k=0}^{n-1} f(T^k(x))$ for $n \geq 0$ and $S_n(f)(x) = S_{-n}(f)(T^n x)$ for $n < 0$. Let $\mu^f$ denote the measure $\text{Leb}_{[0, 1)} \times \text{Leb}_{\mathbb{R}_+}$ restricted to $X^f$. We will phrase the result on deviation of ergodic averages in terms of the special representation of $(\varphi_t)$. Recall that for $B \in L^1(X^f, \mu^f)$,

$$S^B_R(x, s) := \int_0^R B(T^f_t(x, s)) dt$$

Let $\lambda_2 \in \mathbb{R}$ denote the second Lyapunov exponent of the Kontsevich-Zorich cocycle. By a result of Veech, \[63\] (see also Forni, \[36\]), we have $0 \leq \lambda_2 < 1$.

Our main result is:

**Proposition B.1.** Assume that $f$ has symmetric logarithmic singularities. Then for every $d \geq 2$ there exists $C_d > 0$ such that for almost every\[d\] d-IET $T$, we have

$$\mu^f\left(\{(x, s) \in X^f : |S^B_R(x, s)| \geq R^\beta\}\right) < C_B e^{-R^{1/4}},$$

for every $\beta > (3 + \lambda_2)/4$, and every $B \in C^1(X^f)$ of zero-mean.

\[1\] Almost every refers to every irreducible permutation $\pi$ and Lebesgue almost every $\lambda \in \Sigma_d := \{v = (v_1, \ldots, v_d) \in \mathbb{R}_+^d : \sum v_i = 1\}$. 
Notice that the above proposition immediately implies that \((A3')\) is verified for almost every \((T_t^f)\) and therefore also almost every \((\varphi_t)\). Before we prove Proposition 3.1, let us recall some results on deviation of ergodic averages for translation flows. We now fix \(d \geq 2\) and skip it in the notation. We also take \(\varepsilon > 0\) so that \(\beta > (3 + \lambda_2 + 4\varepsilon)/4\).

The first result of Forni, \cite{36} (see also a related result by Zorich, \cite{66} or Bufetov, \cite{12}) is as follows:

**Lemma B.2.** For almost every \(T\), every Lipschitz function \(\phi\), every \(y \in [0, 1)\) and every \(M \in \mathbb{N}\),

\[
\left| \sum_{j=0}^{M-1} \phi(T^j y) - M \int \phi \, d\text{Leb} \right| = O\left( M^{\lambda_2 + \varepsilon} \|\phi\|_{\text{Lip}} \right).
\]

Let \(\eta = \frac{1 - \lambda_2 - \varepsilon}{2}\). Let \([a, b] \subset [0, 1)\) be an interval and let \(\phi^+\) be a Lipschitz function such that \(\phi^+ = 1\) on \([a, b]\) and \(\phi^+ = 0\) outside \([a - M^{-\eta}, b + M^{-\eta}]\) and \(\|\phi^+\|_{\text{Lip}} = O(M^\eta)\). Then

\[
\sum_{j=0}^{M} \mathbb{1}_{[a,b]}(T^j y) \leq \sum_{j=0}^{M} \phi^+(T^j y) \leq M \int \phi^+ \, d\text{Leb} + O\left( M^{\lambda_2 + \varepsilon} \cdot M^\eta \right) \leq M(b - a) + O\left( M^{\lambda_2 + \varepsilon + \eta} + M^{1-\eta} \right) = M(b - a) + O\left( M^{(1+\lambda_2 + \varepsilon)/2} \right),
\]

Analogously, letting \(\phi^- = 1\) on \([a + M^{-\eta}, b - M^{-\eta}]\) and \(\phi^- = 0\) outside \([a, b]\), we get

\[
\sum_{j=0}^{M} \mathbb{1}_{[a,b]}(T^j y) \geq M(b - a) + O\left( M^{(1+\lambda_2 + \varepsilon)/2} \right).
\]

Therefore,

\[
(B.2) \quad \left| \sum_{j=0}^{M} \mathbb{1}_{[a,b]}(T^j y) - M(a - b) \right| = O\left( M^{(1+\lambda_2 + \varepsilon)/2} \right).
\]

Let \(Y_t := \left\{ (y, r) \in X^f : T_s^f(y, r) \in \bigcup_{i=1}^{d} [e^{-t_1/3} + \beta_i, e^{-t_1/3} + \beta_i] \times \mathbb{R} \text{ for some } s \leq t \right\}\).

We have the following lemma:

**Lemma B.3.** For almost every IET \(T\), we have

\[
|n(x, s, R) - R| = O(R^{\max\{\lambda_2,1/3\} + \varepsilon}),
\]

uniformly for \((x, s) \notin Y_t\). Moreover, for every \(i \leq d\) and every \((x, s) \in X^f\), the set

\[
(B.4) \quad \{T_s^f(x, s)\}_{t \leq R} \cap [\beta_i - R^{-2}, \beta_i + R^{-2}] \times \{0\}
\]

is either empty or a singleton for sufficiently large \(R > 0\).
Proof. The proof uses the arguments of Proposition 4.2. in [62]. We only give a sketch. By the definition of special flow,

\[ S_{n(x,s,R)}(f)(x) \leq s + R < S_{n(x,s,R)+1}(f)(x) \]

Since \((x, s) \notin \bar{Y}_R\), it follows that \(|s| = O(f(e^{-R^{1/3}})) = O(R^{1/3})\) and analogously,

\[ f(T^{n(x,s,R)}x) = O(f(e^{-R^{1/3}})) = O(R^{1/3}). \]

Therefore \(|S_{n(x,s,R)}(f)(x) - R| = O(R^{1/3})\). So it is enough to show that

\[ \left| n(x, s, R) - S_{n(x,s,R)}(f)(x) \right| = O(R^{\lambda_2+\epsilon}). \] \hspace{1cm} (B.5)

For this, we will require that the IET \(T\) is as in Proposition 4.2. in [62] (notice that this is a full measure set of IET’s). The strategy of proving (B.5) is analogous to the proof of Proposition 4.2. In particular, after \(n_k\) steps of the (accelerated) Rauzy-Veech induction, the interval \([0, 1]\) can be decomposed as a union of \(d\) towers of heights \((h^k_j)_{j \leq d}\) whose bases are intervals \((I^k_j)_{j \leq d}\). Then for any \(j \leq d\), for every \(0 \leq w < h^k_j\) and every \(z \in I^k_j\), we have

\[ |w - S_w(f)(z)| = O(\left| \log(z_{min,w}) \right|), \] \hspace{1cm} (B.6)

where \(z_{min,w}\) is the minimal distance of the orbit \(\{T^jz\}_{j < w}\) to the set of singularities. In fact, in [62, formula (38)] the estimates are done for \(f'\) rather than \(f\), but one can get (B.6) by repeating the proof of Proposition 4.2. with \(f\) instead \(f'\). Now to get (B.5), one decomposes the orbit \(\{T^jz\}_{j < n(x,s,R)}\) into orbits of length \(h^k_j\), analogously to the proof of Proposition 4.2.

For (B.4) notice that this follows by an analogous statement with \(\{T^kx\}_{k \leq n(x,s,R)}\) instead of \(\{T^k_i(x,s)\}_{i \leq R}\). This in turn is also a consequence of the diophantine assumption, which gives a control on the spacing of the points \(\{T^kx\}_{k \leq n(x,s,R)}\). More precisely, let \(d^\pm_j(x)\) be the ordered distances from the orbit of \(\{T^v x\}_{v < r_k}\) and \(j\)-th discontinuity (+ and − stand for approaches from the right and from the left respectively). Corollary 4.1. in [62] gives a quantitative control on the deviations of the special orbit \(d^\pm_j(x)\) from an arithmetic progression. One then puts \(n(x, s, R)\) between two heights \(r_k \leq n(x, s, R) \leq r_{k+1}\) and uses the fact that \(r_{k+1} = O(r_k^{1+\eta})\) for some \(\eta < 1\). \(\Box\)

With the above lemmas, we can prove Proposition B.4.

Proof of Proposition B.4. Fix \(R > 0\). It is enough to estimate the \(S^R_{R}(x,s)\) for \((x,s) \in X^f\) satisfying \(T^t_i(x,s) \notin \bar{Y}_R\) for \(t < R\) (see (B.3)) because

\[ \mu \left( \{(x,s) : \exists t < R : T^t_i(x,s) \in \bar{Y}_R \} \right) = O(R \cdot e^{-R^{1/3}} \leq e^{-R^{1/4}} \]

for \(R\) large enough.

Let \(\gamma > 0\). We partition \([0,1]\) into intervals \(\{I_i\}_{i=1}^{m_i}\) of equal length \(\sim R^{-\gamma}\) so that all discontinuities of \(T\) are centers of some of the \(I_i\). For each \(i\), we partition the rectangle \(L_i := I_i \times [0, \text{min}_{I_i} f]\) into rectangles \(C_{ij} = I_i \times J_j\), where \(\{J_j\}_{j=1}^{m_j}\) are intervals of length \(\sim R^{-\gamma}\). Let

\[ Z_R(x,s) = \left\{ t < R : T^t_i(x,s) \in \bigcup_i L_i \right\}. \]
Then

\[ S_R^B(x, s) = \int_0^R B(T_t^f(x, s))dt = \int_{Z_R(x,s)} B(T_t^f(x, s))dt + \int_{Z_R^c(x,s)} B(T_t^f(x, s))dt. \]

We will estimate the integrals separately. Since \( B \in C^1(X^f) \) it follows that for \( t \in Z_R \), we can approximate

\[(B.7) \quad \left| B(T_t^f(x, s)) - \sum_{i,j} a_{ij} \mathbb{1}_{C_{ij}}(T_t^f(x, s)) \right| = O(R^{-\gamma}). \]

Therefore,

\[ \int_{Z_R} B(T_t^f(x, s))dt = \sum_{i,j} a_{ij} \int_0^R \mathbb{1}_{C_{ij}}(T_t^f(x, s))dt + O(R \cdot R^{-\gamma}). \]

Moreover,

\[ \int_0^R \mathbb{1}_{C_{ij}}(T_t^f(x, s))dt = \left| \{ m \leq n(x, s, R) : T^m(x) \in I_i \} \right| \cdot |J_j| + O(R^{-\gamma}), \]

the error coming from coming from \( n = 0 \) and \( n = n(x, s, R) \). By Lemma B.3

\[ \left| \{ m \leq n(x, s, R) : T^m(x) \in I_i \} \right| = \left| \{ m \leq R : T^m(x) \in I_i \} \right| + O(R^{\max\{\lambda_2, 1/3\}+\epsilon}). \]

So by (B.7),

\[(B.8) \quad \left| \{ m \leq n(x, s, R) : T^m(x) \in I_i \} \right| = R I_i + O(R^{(1+\lambda_2+\epsilon)/2}). \]

Therefore,

\[ \int_0^R \mathbb{1}_{C_{ij}}(T_t^f(x, s))dt = R \cdot |I_i| \cdot |J_j| + O(R^{(1+\lambda_2+\epsilon)/2}|J_j|) + O(R^{-\gamma}). \]

This implies that

\[ \int_{Z_R} B(T_t^f(x, s))dt = \]

\[ R \cdot \sum_{i,j} a_{ij} |I_i| \cdot |J_j| + O(R^{(1+\lambda_2+\epsilon)/2} \cdot m_I) + O(R^{-\gamma} \cdot m_I \cdot m_J) + O(R^{1-\gamma}). \]

Note that since the length of each \( \{I_i\} \) is of order \( R^{-\gamma} \), we have \( m_I = O(R^\gamma) \). Moreover, since \( f \) has symmetric logarithmic singularities it follows that for every \( i \leq m_I \),

\[ |\min_{I_i} f| = O(\log(R^{-\gamma})) = O(\log R). \]

This implies that \( m_J = O(R^\gamma \log R) \). Putting the two estimates together, we get

\[ \int_{Z_R} B(T_t^f(x, s))dt = R \cdot \sum_{i,j} a_{ij} |I_i| \cdot |J_j| + O(R^{\gamma+(1+\lambda_2+\epsilon)/2}) + O(R^\gamma \log R) + O(R^{1-\gamma}). \]
Then by mean value theorem, for good $i$, we have the following:

$$\left| \int_{Z_R} B(T_i^f(x, s)) dt \right| = O(R^R).$$

It remains to estimate the integral on $Z_{R}^c$. Notice that

$$Z_{R}^c(x, s) = \left\{ t < R : \exists i, T_i^f(x, s) \in I_i \times [\min f, \infty] \right\}.$$

We say $i$ is good if the center of $I_i$ is NOT a discontinuity. We will split the integral over $Z_{R}^c$ according to whether $i$ is good or not. Let

$$Z_{R}^g := \left\{ t < R : \exists i\text{-good } T_i^f(x, s) \in I_i \times [\min f, \max f] \right\}.$$

Then by mean value theorem, for good $i$,

$$\left| \{ t < R : T_i^f(x, s) \in I_i \times [\min f, \max f] \} \right| \leq \left| \{ j \leq n(x, s, R) : T_j(x) \in I_i \} \right| \cdot |I_i| \cdot \max_{I_i} |f'|.$$

We have the following:

**Lemma B.4.** $\sum_{i \text{ good}} \max_{I_i} |f'| = O(R^g \log R)$.

Before we prove the claim let us show how it implies the proposition. By (B.8), using that the $\{I_i\}$ have length $\sim R^{-\gamma}$,

$$\left| \int_{Z_{R}^g} B(T_i^f(x, s)) dt \right| \ll R \cdot R^{-2\gamma} \cdot \sum_{i \text{ good}} \max_{I_i} |f'| \ll R^{1-2\gamma} \cdot R^\gamma \log R \leq R^\beta,$$

by the choice of $\gamma$ and $\beta$. Hence it remains to estimate the integral on $Z_{R}^c \setminus Z_{R}^g$. Let $i$ be not good (the number of such intervals is at most $d$) and assume without the loss of generality that the center of $I_i$ is $\beta_i$. The set $\{ t < R : T_i^f(x, s) \in I_i \times [\min I_i f, \infty] \}$ is a finite union of intervals $\{K_i^r\}$. Notice that for every $r$,

$$|K_i^r| = O(f(x_i^r)),$$

where $x_i^r$ is the distance to $\beta_i$ of the projection of $\{T_i(x, s)\}_{t \in K_i}$ onto $[0, 1)$. Since $T_i(x, s) \notin Y_R$, for all $r$, we have $f(x_i^r) = O(f(e^{-R^{1/2}})) = O(R^{1/2})$. Moreover, for all but at most one $r$ by (B.4), we have $f(x_i^r)O(f(R^2)) = O(\log R)$. Hence,

$$R^{1/2} + \left| \{ t < R : T_i^f(x, s) \in I_i \times [\min f, \infty) \} \right| \ll R^{1/2} + \frac{1}{d} R^\beta.$$

Since the number of not good $i$ is bounded above by $d$, we get that for $(x, s)$ satisfying $T_i(x, s) \notin Y_R$ for $t \leq R$, we have

$$|S_R^B(x, s)| \leq R^\beta.$$
This finishes the proof of the proposition modulo the proof of Lemma [B.4] given below.

□

Proof of Lemma [B.4] By (B.1) it follows that

\[ f'(x) = \sum_{i=0}^{d-1} C_i^+(x - \beta_i)^{-1} + \sum_{j=1}^{d} C_j^-(\beta_j - x)^{-1} + g'(x), \]

Note that the number of intervals satisfies \( m_I = O(R^\gamma) \). Therefore,

\[ \sum_{i \text{ good}} \max_{I_i} |w| = O(R^\gamma) \]

It remains to notice that the functions \( v_j(x) = (x - \beta_j)^{-1} \) and \( u(x) = (\beta_j - x)^{-1} \) are monotone and hence for \( w = v_j, u_j, \sum_{i \text{ good}} \max_{I_i} |w| \) is just a sum over the endpoints of \( I_i \). But all \( \{I_i\} \) have equal length and for good \( i \), \( I_i \) is \( \sim R^{-\gamma} \) separated from all the discontinuities. Therefore, \( \sum_{i \text{ good}} \max_{I_i} |w| \) is just a sum over an arithmetic progression:

\[ \sum_{i \text{ good}} \max_{I_i} |w| = O \left( m \sum_{k=1}^{m} \frac{1}{k \cdot R^{-\gamma}} \right) = O(R^\gamma \log R). \]

This finishes the proof of the lemma.

□

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