Stefan Wewers

**Fiercely ramified cyclic extensions of \( p \)-adic fields with imperfect residue field**

Received: 29 April 2011 / Revised: 10 April 2013
Published online: 9 May 2013

**Abstract.** We study the ramification of fierce cyclic Galois extensions of a local field \( K \) of characteristic zero with a one-dimensional residue field of characteristic \( p > 0 \). Using Kato’s theory of the refined Swan conductor, we associate to such an extension a *ramification datum*, consisting of a sequence of pairs \((\delta_i, \omega_i)\), where \( \delta_i \) is a positive rational number and \( \omega_i \) a differential form on the residue field of \( K \). Our main result gives necessary and sufficient conditions on such sequences to occur as a ramification datum of a fierce cyclic extension of \( K \).

1. **Introduction**

1.1. *A classical result on wildly ramified cyclic extensions*

Let \( K \) be a complete discrete valued field with residue field \( \bar{K} \) of characteristic \( p > 0 \). Let \( L/K \) be a finite Galois extension with Galois group \( G \). Let us assume, for the moment, that the residue field \( \bar{K} \) is perfect. Then the classical theory of ramification groups gives rise to a descending filtration \((G^t)_{t \in \mathbb{Q}, t \geq -1}\) on \( G \) (the *upper numbering filtration*), which has the following nice properties (see [14]):

- The definition of the filtration is compatible with taking quotients of \( G \).
- The subgroup \( G^0 \) is the inertia subgroup of \( G \). Moreover, the graded pieces \( \text{Gr}^t(G) \) for \( t > 0 \) are elementary abelian and killed by \( p \).
- If \( G \) is abelian then all the breaks of the filtration are integers (Theorem of Hasse and Arf).

Now suppose in addition that the field \( K \) has characteristic \( p \), \( G \cong \mathbb{Z}/p^n\mathbb{Z} \) is cyclic of \( p \)-power order and that \( L/K \) is totally ramified. It is clear from the above that the sequence of breaks of the filtration \((G^t)_t\) is now of the form

\[
0 < u_1 < u_2 < \cdots < u_n,
\]

where each \( u_i \) is a positive integer such that \( |G^{u_i}| = p^{n-i+1} \). In this situation we have the following (more or less well known) result.

---

S. Wewers (✉): Institut für Reine Mathematik, Universität Ulm, Helmholtzstr. 18, 89081 Ulm, Germany. e-mail: stefan.wewers@uni-ulm.de

*Mathematics Subject Classification (2000):* 11S15, 11S31, 14F05, 19F05

DOI: 10.1007/s00229-013-0630-1
Theorem 1.1. Suppose that char(K) = p. Let 0 < u_1 < u_2 \ldots < u_n be a strictly increasing sequence of integers. Then (u_i) is the sequence of ramification breaks associated to a totally ramified Galois extension L/K with cyclic Galois group of order p^n if and only if the following holds:

(i) p \nmid u_1,
(ii) for i > 1 we either have u_i = pu_{i-1} or u_i > pu_{i-1} and p \nmid u_i.

A more general version of this result is proved in [11]. As stated above, the theorem is an immediate consequence of the classical formula of Schmid [13, Satz 3] which computes the breaks u_i in terms of the Artin–Schreier–Witt representation of L/K. See also [15] and [4].

The goal of the present paper is to prove results that are analogous to Theorem 1.1 in the case when K has characteristic zero and the residue field \overline{K} is ‘one-dimensional’ in a suitable sense (in particular: not perfect!). These results are used in joint work with Obus [12] on the problem of ‘lifting’ a cyclic Galois extension as in Theorem 1.1 to characteristic zero (this problem is called the local lifting problem). In the context of these applications, the results of the present paper can be seen as a ‘deformation’ of Theorem 1.1.

1.2. Fierce extensions

If the residue field of K is not perfect, the theory of ramification groups becomes more delicate. A general construction of a filtration (G^i_t) for the Galois group of an extension of the field K has been given by Abbes and Saito in [1]. In subsequent papers, the same authors have also shown that their filtration has all the expected properties (e.g. the Hasse–Arf–Theorem holds).

From the point of view of ramification theory, the situation considered in the present paper is rather special. We therefore get by with using the much simpler construction of ramification groups explored in [10], which is similar to the classical construction. We do use in an essential way the beautiful results of Kato which give rise to the refined Swan conductor and characterizes it in terms of higher class field theory [6,9,10].

Our assumptions on K are as follows. We assume that K has characteristic zero and is complete with respect to a discrete valuation v with residue field \overline{K}. We normalize v such that v(p) = 1. The crucial assumption is that the residue field \overline{K} has a p-basis of length one (i.e. [\overline{K} : \overline{K}^p] = p) and that \(H^1(\overline{K}, \mathbb{Z}/p\mathbb{Z}) \neq 0\). For instance, this assumption holds if \overline{K} is itself a local field with perfect residue field.

A finite separable extension \(L/K\) is called fierce if the extension of residue fields \(\overline{L}/\overline{K}\) is purely inseparable and \([L : K] = [\overline{L} : \overline{K}]\).\(^1\) Note that a fierce extension is weakly unramified, i.e. that K contains a prime element of L. Moreover, our assumption on \(\overline{K}\) implies that any finite fierce extension \(L/K\) is monogenic (which means that the corresponding extension of valuation rings is generated by one element). So if \(L/K\) is fierce and Galois with Galois group G, then one can define

---

\(^1\) Some authors call such an extension ferociously ramified.
a filtration of higher ramification groups \( (G^t) \), on \( G \) essentially as in the classical situation \([10]\). The point is that for the kind of applications we have in mind (related to reduction and lifting of covers of curves) it is very natural to replace, if necessary, the field \( K \) by a suitable constant extension (see Sect. 2.2, Definition 2.3). Using a result of Epp \([3]\) we may therefore assume from the start that our Galois extension \( L/K \) is fierce.

1.3. Ramification data for fierce cyclic extensions

Let us now assume that \( L/K \) is a fierce Galois extension with cyclic Galois group \( G \cong \mathbb{Z}/p^n\mathbb{Z} \). Using the results of Kato mentioned before, we associate to \( L/K \) a so called ramification datum \((\delta_i, \omega_i)_{i=1,\ldots,n}\). Here \( \delta_1, \ldots, \delta_n \) are positive rational numbers and \( \omega_1, \ldots, \omega_n \) are differential forms on the residue field \( \overline{K} \). For fixed \( i \), the pair \((\delta_i, \omega_i)\) represents the refined Swan conductor of a character \( \chi_i \) of \( G \) of order \( p^i \), in the sense of \([9]\). The rational numbers \( \delta_1 < \delta_2 < \cdots < \delta_n \) are simply the breaks of the filtration \( (G^t) \); the differentials \( \omega_1, \ldots, \omega_n \) contain finer information on the ramification of \( L/K \). For instance, if \( v_1: \overline{K}^\times \to \mathbb{Z} \) is a normalized discrete valuation on the residue field of \( K \) and \( \bar{v}: \overline{K} \to \mathbb{Q} \times \mathbb{Z} \) is the rank-2-valuation obtained from the pair \((v, v_1)\), then the breaks of the ramification filtration on \( G \) with respect to the valuation \( \bar{v} \) are

\[
(\delta_i, v_1(\omega_i) + 1), \quad i = 1, \ldots, n.
\]

This follows from \([10]\), Corollary 4.6.

Our main result (Theorems 4.3 and 4.6) gives a necessary and sufficient condition on a tuple \((\delta_i, \omega_i)_{i=1,\ldots,n}\) to occur as the ramification datum of a fierce cyclic extension \( L/K \). The full statement of this condition is a bit lengthy (see Theorem 4.3). The following theorem states a partial result which is analogous to Condition (ii) in Theorem 1.1 (the letter \( C \) stands for the Cartier operator).

**Theorem 1.2.** Let \( L/K \) be a fierce Galois extension with cyclic Galois group \( G \cong \mathbb{Z}/p^n\mathbb{Z} \). Let \((\delta_i, \omega_i)_{i=1,\ldots,n}\) be the ramification datum associated to \( L/K \). Suppose \( \delta_{i-1} < 1/(p-1) \) for some \( i > 1 \). Then either

\[
\delta_i = p \delta_{i-1}, \quad C(\omega_i) = \omega_{i-1},
\]

or

\[
\delta_i > p \delta_{i-1}.
\]

In the latter case, we have \( C(\omega_i) = \omega_i \) if and only if \( \delta_i = p/(p-1) \) and \( C(\omega_i) = 0 \) otherwise.

The inequality \( \delta_i \geq p \delta_{i-1} \) resulting from this theorem (which holds if \( \delta_{i-1} \leq 1/(p-1) \)) is not an entirely new result. For instance, it follows from the results of Hyodo \([5]\). However, Hyodo’s results are valid in much greater generality than ours and are therefore weaker. In particular the statements in Theorem 1.2 concerning the differential forms \( \omega_i \) are false if we omit the assumption \( [\overline{K}:\overline{K}^p] = p \). It is precisely these relations between the differentials \( \omega_i \) which are crucial for the applications of our results in \([12]\).
2. Preliminaries

We introduce the basic notation and assumptions concerning the field $K$ we will be working with. We also discuss (almost) constant and fierce extensions of $K$, following [16].

2.1. The main assumption

Let $K$ be a complete discrete valuation field of characteristic zero, with residue field $\bar{K}$ of characteristic $p > 0$. The valuation on $K$ is denoted by $v$, and it is normalized such that $v(p) = 1$. Let $e \in \mathbb{N}$ be the absolute ramification index of $K$, so that $v(K^\times) = \mathbb{Z}e^{-1}$. For $t \in \mathbb{Z}e^{-1}$ we set

$$p_K^t := \{ x \in K \mid v(x) \geq t \}$$

and

$$U_K^t := \{ x \in K \mid v(x - 1) \geq t \} \quad \text{(for } t \geq 0).$$

Note that $p_K^1 = (p)$ and that $p_K^{1/e}$ is the maximal ideal of $\mathcal{O}_K$.

Our crucial assumption on $K$ is:

**Assumption 2.1.** (a) $[\bar{K} : \bar{K}^p] = p$.
(b) $H^1(\bar{K}, \mathbb{Z}/p\mathbb{Z}) \neq 0$.

Assumption 2.1 is satisfied, for instance, if $\bar{K}$ is a function field of transcendence degree one over a perfect field, or if $\bar{K}$ is complete with respect to a discrete valuation and has a perfect residue field (i.e. if $K$ is a 2-local field).

2.2. Constant extensions

We let

$$F := \bigcap_i \bar{K}^{p_i}$$

denote the maximal perfect subfield of $\bar{K}$. Since $K$ is complete, the inclusion $F \hookrightarrow \bar{K}$ lifts to a unique embedding of the ring of Witt vectors $W(F)$ into $\mathcal{O}_K$, see [14], Proposition II.10 and Theorem II.5. Let $k_0$ denote the fraction field of $W(F)$; we consider $k_0$ as a subfield of $K$. Let $k$ denote the algebraic closure of $k_0$ in $K$. We call $k$ the field of constants of $K$. Elements of $k$ are called constants.

**Proposition 2.2.**

(i) The extension $k/k_0$ is finite and totally ramified. In particular, $k$ is complete with respect to $v$ and has residue field $F$.

(ii) The field $k$ is the largest subfield of $K$ with perfect residue field.

(iii) Let $l/k$ be an algebraic extension. Then $l$ is the field of constants of $L := Kl$. 
Proof. The residue field $\bar{k}$ of $k$ is contained in $\bar{K}$ and is an algebraic extension of $F$. Since $F$ is the maximal perfect subfield of $\bar{K}$, we have $F = \bar{k}$, i.e. the extension $k/k_0$ has trivial residue field extension. Now it follows from [14], Theorem II.4 that $k/k_0$ is finite and totally ramified. So (i) is proved.

Let $M \subset K$ be a subfield with perfect residue field $\bar{M}$. Let $M_0$ denote the fraction field of $W(\bar{M})$. Since $\bar{M}$ is a perfect subfield of $\bar{K}$, we have $M \subset F$ and hence $M_0 \subset k_0$. But $M/M_0$ is finite (use again [14, Theorem II.4]). Therefore, $M \subset k$, which proves (ii).

Let $l/k$ be an algebraic extension and set $L := Kl$. We note that $l$ is algebraically closed in $L$. Let $l'$ denote the field of constants of $L$. Then $l \subset l'$, by (ii). The residue field extension $\bar{L}/\bar{K}$ is algebraic. An easy argument shows that the maximal perfect subfield $\bar{F} := \cap_i \bar{l}_i^{p^i}$ is algebraic over the maximal perfect subfield $F$ of $\bar{K}$. But $F'$ is the residue field of $l'$ and $F$ is contained in the residue field of $l$. It follows that $l'$ is algebraic over $l$. We conclude $l = l'$. This completes the proof of the proposition. \( \square \)

**Definition 2.3.** An extension $L/K$ is called **constant** if $L = Kl$ for some algebraic extension $l/k$. An extension $L/K$ is called **almost constant** if it lies in the composition of a constant and an unramified extension.

A second important assumption on $K$ that we will make throughout is that the extension $K/k$ is **weakly unramified**, i.e. we have

$$v(k^\times) = v(K^\times).$$

By a theorem of Epp (see [3]) this assumption always holds after replacing $K$ by a finite constant extension. So in contrast to Assumption 2.1, this is no real restriction of generality.

### 2.3. Fierce extensions

An extension $L/K$ is called **fierce** if the extension of residue fields $[\bar{L} : \bar{K}]$ is purely inseparable, and moreover

$$[\bar{L} : \bar{K}] = [L : K].$$

Note that if $L/K$ is fierce and $K'/K$ is almost constant, then $L' := LK'/K'$ is again fierce, and we have $[L' : K'] = [L : K]$. Together with Epp’s theorem, this shows the following.

**Proposition 2.4.** Let $L/K$ be a finite extension. Then there exists a finite, almost constant extension $K'/K$ such that the extension $L' := LK'/K'$ is fierce. Furthermore, the degree $[L' : K']$ does not depend on the choice of $K'/K$.

In this paper we study the ramification of finite cyclic extensions $L/K$ ‘up to almost constant extensions’. By the above proposition, it is no loss of generality to assume from the start that $L/K$ is fierce.
2.4. The choice of a prime element

After replacing $K$ by a finite, constant extension, we may assume that $K$ contains a primitive $p$th root of unity $\zeta_p \in \mu_p(K)$. This assumption will be made throughout the paper. We set $\lambda := \zeta_p - 1$. Note that $v(\lambda) = 1/(p - 1)$ and

$$\frac{\lambda^{p-1}}{p} \equiv -1 \pmod{p^{1/e}}. \quad (1)$$

It is clear that $\lambda$ is contained in the field of constants $k$ of $K$.

Let $\pi$ be a prime element of $k$ (which is also a prime element of $K$!). Then $v(\pi) = 1/e$, where $e$ is the absolute ramification index of $k$. Note that $e$ is a multiple of $p - 1$ since $\lambda \in k$. Let $\tilde{c} \in \bar{k}$ denote the residue class of the unit $\pi^{e/(p-1)}/\lambda$. After replacing $k$ by a finite unramified extension we may assume that there exists $\tilde{d} \in \bar{k}$ such that $\tilde{d}^{e/(p-1)}\tilde{c} = 1$. Let $d \in \mathcal{O}_k^\times$ be a lift of $\tilde{d}$. We set

$$\pi_{1/e} := d \cdot \pi.$$

By definition $\pi_{1/e}$ is an element of $k$ with $v(\pi_{1/e}) = 1/e$ and

$$\pi_{1/e}^{e/(p-1)} \equiv \lambda \pmod{p^{1/(p-1)+1/e}}.$$

Since the absolute ramification index of $k$ has not changed by passing to an unramified extension, $\pi_{1/e}$ is a prime element of $k$. For any $t \in v(K^\times) = e^{-1} \cdot \mathbb{Z}$ we set

$$\pi_t := \pi_{1/e}^{et}.$$

Now the following holds:

Remark 2.5. (i) $v(\pi_t) = t$, for all $t \in v(K^\times)$.
(ii) $\pi_t^k = \pi_{kt}$, for all $k \in \mathbb{Z}$.
(iii) $\pi_{1/(p-1)} \equiv \lambda \pmod{p^{1/(p-1)+1/e}}$.
(iv) $\pi_1 \equiv -p \pmod{p^{1+1/e}}$.

Remark 2.6. We will often have to replace $K$ by some finite constant extension $K'/K$. Then it may not be possible to extend the definition of $\pi_t$ to $K'$ in a compatible way. However, if we allow to further replace $K'$ by an unramified constant extension, then it is possible to define $\pi'_t$ for all $t \in v((K')^\times)$ in such a way that Remark 2.5 holds and $\pi'_t/\pi_t \equiv 1$ for all $t \in v(K^\times)$.

As a result, if we replace $K$ by a constant extension then we need to choose $\pi_t$ once again, but this will never cause any problems.
3. Ramification and Swan conductors

3.1. The ramification filtration

Let $L/K$ be a fierce Galois extension. Then the extension of valuation rings $O_L/O_K$ is monogenic. In fact, the purely inseparable extension of residue fields $\bar{L}/\bar{K}$ can be generated by one element $\bar{x} \in \bar{L}$ by Assumptions 2.1, and then any element $x \in O_L$ lifting $\bar{x}$ generates $O_L$ over $O_K$.

It follows that we have the usual theory of ramifications groups available, see e.g. [10], Sect. 2–Sect. 3. Let $G = \text{Gal}(L/K)$ denote the Galois group of $L/K$. There are two filtrations by subgroups of $G$,

\[(G^t)_{t \geq 0}, \quad (G_t)_{t \geq 0},\]

with index set $\mathbb{R}_{\geq 0}$ (upper and lower numbering), defined as follows.

For $\sigma \in G$, let

\[i_G(\sigma) := \min \{ v(\sigma(x) - x) \mid x \in O_L \}.\]

Then the lower numbering filtration is defined by

\[G_t := \{ \sigma \in G \mid i_G(\sigma) \geq t \}.\]

Note that this definition differs from the standard definition given e.g. in [14], but it agrees with the definition used in [10] (put $\epsilon := 0$ in [10, Corollary 3.3]).

Set

\[\phi_{L/K}(t) := \int_0^t |G_s| ds.\]

Then $\phi_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is piecewise linear and strictly increasing, hence bijective. Let $\psi_{L/K} := \phi_{L/K}^{-1}$ be the inverse map. The upper numbering filtration is now defined by

\[G^t := G_{\psi_{L/K}(t)}.\]

One shows that

\[\psi_{L/K}(t) = \int_0^t |G^s|^{-1} ds.\]

Remark 3.1. The filtrations $(G^t)_t$ and $(G_t)_t$ have the following properties:

(i) If $s \leq t$, then $G^s \supseteq G^t$ and $G_s \supseteq G_t$.

(ii) $G = G^0 = G_0$.

(iii) For each $\sigma \in G \setminus \{1\}$, the sets $\{ t \geq 0 \mid \sigma \in G^t \}$ and $\{ s \geq 0 \mid \sigma \in G_s \}$ have a maximum.

(iv) Let $H \triangleleft G$ be a normal subgroup and $M := L^H$ the corresponding subfield. Then
\((G/H)^t = G^t / (G^t \cap H), \quad G^t \cap H = H^{\Psi_{M/K}(t)}\),

where the filtration \((H^t)_t\) is induced from the isomorphism \(H \cong \text{Gal}(L/M)\).

See e.g. [10, Lemma 2.9].

It will be important for us that the ramification filtrations are invariant under constant extensions of \(K\).

**Lemma 3.2.** Let \(K'/K\) be an almost constant extension. Set \(L' := LK'\) and \(G' := \text{Gal}(L'/K')\). Then the natural isomorphism \(G' \sim G\) is compatible with the upper and lower filtrations, i.e.

\[(G')_t \sim G_t, \quad (G')^t \sim G^t,\]

for all \(t \geq 0\).

**Proof.** Let \(x \in \mathcal{O}_L\) be an element whose residue class generates the extension \(\bar{L}/\bar{K}\). Then \(\mathcal{O}_L = \mathcal{O}_K[x]\) and \(\mathcal{O}_{L'} = \mathcal{O}_{K'}[x]\). We conclude that

\[i_G(\sigma) = v(\sigma(x) - x) = i_{G'}(\sigma),\]

for all \(\sigma \in G\). The statement of the lemma follows immediately. \(\square\)

### 3.2. The refined Swan conductor

By a **character** on \(K\) we shall always mean a continous group homomorphism \(\chi : \text{Gal}(K^{\text{sep}}/K) \to \mathbb{Q}_p/\mathbb{Z}_p\). Let \(L/K\) be the unique finite Galois extension \(L/K\) such that \(\chi\) factors over \(G = \text{Gal}(L/K)\) and induces an injective homomorphism \(G \hookrightarrow \mathbb{Q}_p/\mathbb{Z}_p\). Then \(G\) is cyclic of order \(p^n\), for some \(n \geq 0\). We call \(p^n\) the **order** of \(\chi\). We call \(\chi\) **weakly unramified** (resp. **fierce**) if the extension \(L/K\) is weakly unramified (resp. fierce).

We are interested in ramification of characters, but only up to an almost constant extension. By Proposition 2.4 we may therefore assume from the start that a given character on \(K\) is fierce.

**Definition 3.3.** Let \(\chi\) be a fierce character on \(K\). We identify \(\chi\) with a group homomorphism \(\chi : G \to \mathbb{Q}_p/\mathbb{Z}_p\), where \(G = \text{Gal}(L/K)\) is the Galois group of a suitable finite fierce Galois extension \(L/K\) (but we do not assume that \(\chi\) is injective). The (usual) **Swan conductor** of \(\chi\) is the rational number

\[\text{sw}(\chi) := \max \{ t \geq 0 \mid \chi|_{G^t} \neq 1 \} .\]

Note that \(\text{sw}(\chi)\) is independent of the chosen extension \(L/K\) and does not change if we restrict \(\chi\) to an almost constant extension of \(K\).

Following [9], we can also define a **refined Swan conductor** of \(\chi\), as follows. We define for \(q, n \geq 1\)
Fiercely ramified cyclic extensions

\[ H^q_{p^n}(K) := H^q(K, \mathbb{Z}/p^n\mathbb{Z}(q-1)) \]

and

\[ H^q(K) := \lim_{\rightarrow n} H^q_{p^n}(K). \]

An element of \( H^1(K) \) is simply a character on \( K \) as defined above, and

\[ H^2(K) = \text{Br}(K)[p^\infty]. \]

Let

\[ (\cdot, \cdot)_{K} : H^1(K) \times K^\times \to H^2(K) \]

be the pairing defined in [14], XIV, §1. We also need the two morphisms

\[ i_1 : \bar{K} \to H^1_p(K), \quad i_2 : \Omega^1_{\bar{K}} \to H^2_p(K) \]

defined as follows. The morphism \( i_1 \) is the composition of the Artin-Schreier map

\[ \bar{K} \to H^1_p(\bar{K}) := H^1(\bar{K}, \mathbb{Z}/p\mathbb{Z}) \]

with the restriction map

\[ H^1_p(\bar{K}) \to H^1_p(K). \]

Note that an element \( \bar{a} \in \bar{K} \) lies in the kernel of \( i_1 \) if and only if \( \bar{a} = \bar{b}p - \bar{b} \) for some \( \bar{b} \in \bar{K} \).

In order to define \( i_2 \) we choose an element \( x \in O_K \) with \( \bar{x} \notin \bar{K}^p \). By Assumption 2.1 (a), every differential \( \omega \in \Omega^1_{\bar{K}} \) can be written as

\[ \omega = \bar{a} \frac{d\bar{x}}{\bar{x}}, \]

for a unique element \( \bar{a} \in \bar{K} \). We set

\[ i_2(\omega) := (i_1(\bar{a}), x)_K. \]

We can also write \( \bar{a} = \bar{b}_0^p + \bar{b}_1^p \bar{x} + \cdots + \bar{b}_{p-1}^p \bar{x}^{p-1} \). The Cartier operator \( C : \Omega_{\bar{K}} \to \Omega_{\bar{K}} \) is the map defined by

\[ C(\omega) := \bar{b}_0 \frac{d\bar{x}}{\bar{x}}, \]

see [2]. It is shown in [8] that \( i_2(\omega) \) lies in \( H^2_p(K) \) and does not depend on the choice of \( x \). Moreover, \( i_2(\omega) = 0 \) if and only if \( \omega = C(\eta) - \eta \) for some \( \eta \in \Omega^1_{\bar{K}} \).

Using the above definition and Assumption 2.1 (b) one easily proves:

**Remark 3.4.**  (i) The map

\[ \Omega^1_{\bar{K}} \to H^2_p(K), \quad \omega \mapsto (\bar{z} \mapsto i_2(\bar{z}\omega)) \]

is \( \bar{K} \)-linear and injective.
(ii) We have
\[ i_2(\tilde{z}^p \omega) = i_2(\tilde{z} C(\omega)), \]
for all \( \omega \in \Omega^1_{\bar{K}} \) and \( \bar{z} \in \bar{K} \).

**Theorem 3.5** (Kato). Let \( \chi \in H^1(K) \) be a fierce character on \( K \). Then the following holds.

(i) The Swan conductor \( sw(\chi) \) is an element of \( v(K^\times) = \mathbb{Z} \cdot \frac{1}{\ell'} \).
(ii) For \( s \geq 0 \) the restriction of the morphism
\[ K^\times \to H^2(K), \quad b \mapsto (\chi, b)_K \]
to the subgroup
\[ U^s_K := \{ x \in K^\times \mid v(x - 1) \geq s \} \]
is trivial if and only if \( s > sw(\chi) \).
(iii) Let \( \pi \in K \) be an element with \( v(\pi) = sw(\chi) \) (which exists by (i)). Then there exists a unique nonzero differential \( \omega \in \Omega^1_{\bar{K}} \) such that
\[ (\chi, 1 - \pi z)_K = i_2(\tilde{z} \omega), \]
for all \( z \in \mathcal{O}_K \).

**Proof.** See [9, Theorem 3.6]. \( \Box \)

**Corollary 3.6.** Let \( \chi \in H^1(K) \) be a fierce character on \( K \) of order \( > 1 \). Then \( sw(\chi^p) < sw(\chi) \).

**Definition 3.7.** Let \( \chi \in H^1(K) \) be a fierce character on \( K \), and set \( \delta := sw(\chi) \). Let \( \pi \in K \) be an element with \( v(\pi) = \delta \) and \( \omega \in \Omega^1_{\bar{K}} \) as in Theorem 3.5. The refined Swan conductor of \( \chi \) is the element
\[ rsw(\chi) := \pi^{-1} \otimes \omega \in p^{-\delta}_{K} \otimes_{\mathcal{O}_K} \Omega^1_{\bar{K}} \]
(which does not depend on the choice of \( \pi \)).

**Lemma 3.8.** Let \( K'/K \) be an almost constant extension. Then \( sw(\chi|_{K'}) = sw(\chi) \). Moreover, \( rsw(\chi|_{K'}) \) is equal to the image of \( rsw(\chi) \) under the (injective!) morphism
\[ p^{-\delta}_{K} \otimes_{\mathcal{O}_K} \Omega^1_{\bar{K}} \to p^{-\delta}_{K'} \otimes_{\mathcal{O}_{K'}} \Omega^1_{\bar{K'}}. \]

**Proof.** The equality \( sw(\chi|_{K'}) = sw(\chi) \) follows immediately from Lemma 3.2. The second claim then follows from Theorem 3.5 and the formula
\[ (\chi|_{K'}, a)_{K'} = \text{Res}^{K'}_K (\chi, a)_K, \]
for \( a \in K^\times \) (see [14, XIV, Sect.1]). \( \Box \)
The refined Swan conductor $rsw(\chi)$, as it is defined above, is intrinsic and encodes the usual Swan conductor $sw(\chi)$. For our purposes, the following definition, which depends on the choice of the elements $\pi_t$ in §2.4, will be more convenient.

**Definition 3.9.** Let $\chi$ be a fierce character on $K$ of order $> 1$ and set $\delta := sw(\chi)$. The **differential Swan conductor** of $\chi$ is the unique element $\omega = dsw(\chi) \in \Omega^1_{\bar{K}}$ such that

$$rsw(\chi) = \pi^{-1}_\delta \otimes \omega$$

(see Sect. 2.4 for the definition of $\pi_\delta$). By definition, we have

$$(\chi, 1 - \pi_\delta \cdot z)_K = i_2(\bar{z}\omega),$$

for all $z \in \mathcal{O}_K$.

**Remark 3.10.** The definition of $\omega = dsw(\chi)$ depends on the choice of $\pi_t$ made in Sect. 2.4, but only up to a constant in $\bar{k}^\times$. Moreover, the definition of $\omega = dsw(\chi)$ is invariant under replacing $K$ by a finite constant extension, by Lemma 3.8 and Remark 2.6.

The following lemma shows how the refined Swan conductor behaves with respect to addition of characters.

**Lemma 3.11.** Let $\chi_i, i = 1, 2, 3$, be fierce characters on $K$ satisfying the relation $\chi_3 = \chi_1 + \chi_2$. Set $\delta_i := sw(\chi_i)$ and $\omega_i := dsw(\chi_i)$, for $i = 1, 2, 3$. Then the following holds.

(i) If $\delta_1 \neq \delta_2$ then $\delta_3 = \max\{\delta_1, \delta_2\}$. Furthermore, we have $\omega_3 = \omega_1$ if $\delta_1 > \delta_2$ and $\omega_3 = \omega_2$ otherwise.

(ii) If $\delta_1 = \delta_2$ and $\omega_1 + \omega_2 \neq 0$ then $\delta_1 = \delta_2 = \delta_3$ and $\omega_3 = \omega_1 + \omega_2$.

(iii) If $\delta_1 = \delta_2$ and $\omega_1 + \omega_2 = 0$ then $\delta_3 < \delta_1$.

**Proof.** Follows from Theorem 3.5 and Definition 3.9. Details are left to the reader. □

4. Cyclic extensions and ramification data

Before we can state our main results, we need to define the notion of a ramification datum.

**Definition 4.1.** (i) Let $n \geq 1$. A **ramification datum** is a tupel $(\delta_i, \omega_i)_{i=1,...,n}$, where $\delta_i \in \mathbb{Q}_{\geq 0}$ is a nonnegative rational number and $\omega_i \in \Omega^1_{\bar{K}} \setminus \{0\}$ is a nonzero differential form.

(ii) Let $\chi \in H^1(K)$ be a fierce character on $K$ of order $p^n$, with $n \geq 1$. For $i = 1, \ldots, n$ we set

$$\chi_i := p^{n-i} \cdot \chi, \quad \delta_i := sw(\chi_i), \quad \omega_i := dsw(\chi_i).$$

The tupel $(\delta_i, \omega_i)_{i=1,...,n}$ is called the **ramification datum** associated to $\chi$. 

Remark 4.2. Let $L/K$ be the cyclic extension of degree $p^n$ corresponding to the character $\chi$, and $G = \text{Gal}(L/K)$. Let $(\delta_i, \omega_i)_i$ be the ramification datum associated to $\chi$. The following statements follow from Definition 4.1 and Corollary 3.6.

(i) The numbers $\delta_i$ are precisely the breaks for the upper numbering filtration $(G^t)_t$. For $i = 1, \ldots, n$ we have

$$|G^{\delta_i}| = p^{n-i+1}.$$ 

It follows that

$$0 < \delta_1 < \delta_2 < \cdots < \delta_n.$$ 

(ii) It follows from (i) that the tuple $(\delta_i)$ only depends on the extension $L/K$. This is not quite true for the differentials $\omega_i$. If we replace $\chi$ by $\chi^a$, where $a \in \mathbb{Z}$ is prime to $p$, then $\omega_i$ gets replaced by $\bar{a} \omega_i$ (and where $\bar{a}$ denotes the residue of $a$ modulo $p$).

Here is our first main theorem.

Theorem 4.3. Let $\chi \in H^1(K)$ be a fierce character of order $p^n > 1$ and let $(\delta_i, \omega_i)_i$ denote the ramification datum associated to $\chi$. Then for all $i = 1, \ldots, n$ the following holds.

(i) $0 < \delta_1 \leq p/(p-1)$. Moreover, we have

(a) $\delta_1 = p/(p-1) \iff C(\omega_1) = \omega_1$,

(b) $\delta_1 < p/(p-1) \iff C(\omega_1) = 0$.

(ii) Suppose $i > 1$. If $\delta_{i-1} > 1/(p-1)$ then we have

$$\delta_i = \delta_{i-1} + 1, \quad \omega_i = -\omega_{i-1}.$$ 

(iii) Suppose $i > 1$. If $\delta_{i-1} \leq 1/(p-1)$ then

$$p \delta_{i-1} \leq \delta_i \leq \frac{p}{p-1}.$$ 

Moreover, we have

(a) $\delta_i = p \delta_{i-1} < p/(p-1) \Rightarrow C(\omega_i) = \omega_{i-1}$,

(b) $p \delta_{i-1} < \delta_i < p/(p-1) \Rightarrow C(\omega_i) = 0$,

(c) $p \delta_{i-1} < \delta_i = p/(p-1) \Rightarrow C(\omega_i) = \omega_i$,

(d) $p \delta_{i-1} = \delta_i = p/(p-1) \Rightarrow C(\omega_i) = \omega_i + \omega_{i-1}$.

Remark 4.4. In [5] Hyodo studies the ramification of cyclic Galois extensions $L/K$, where $K$ is a complete discretely valued field of mixed characteristic and imperfect residue field. His results imply some parts of Theorem 4.3. For simplicity we give details only in the case $n = 2$. Namely, in the situation of Theorem 4.3, [5. Lemma 4.1] implies the following inequalities:

(a) If $\delta_1 \geq 1/(p-1)$ then

$$\delta_1 + 1 \leq \delta_2 \leq \frac{p}{p-1} + \delta_1 \cdot \frac{p-1}{p}.$$
(b) If \( \delta_1 \leq 1/(p - 1) \) then

\[
p \delta_1 \leq \delta_2 \leq \frac{p}{p - 1} + \delta_1 \cdot \frac{p - 1}{p}.
\]

The statement of Theorem 4.3 is stronger than (a) and (b) above. In fact, Theorem 4.3 is false without Assumption 2.1, whereas Hyodo’s results are valid in much greater generality.

Remark 4.5. In the statement of Theorem 4.3 (iii) the converse implications hold in Case (a) and (c), but not in Case (b) and (d). This can be seen as follows.

Let us first consider (a) and assume that \( C(\omega_i) = \omega_i - 1 \). Since \( \omega_j \neq 0 \) for all \( j \) by definition, we conclude that \( C(\omega_i) \neq 0 \), \( \omega_i + \omega_{i-1} \). This rules out Case (b) and (d). So suppose that (c) holds and therefore \( \omega_i = C(\omega_i) = \omega_{i-1} \). Using \( \delta_j \leq p \delta_i < p/(p - 1) \) for \( j < i \) and the forward direction of (a) one shows inductively that \( \omega_i = \omega_{i-1} = \cdots = \omega_1 \). But then \( \delta_1 < p/(p - 1) \) and \( C(\omega_1) = \omega_1 \). This contradicts Part (i) of Theorem 4.3 and proves that the converse implication holds in Case (a). The proof in Case (c) is similar.

On the other hand, it is possible that \( C(\omega_i) = \omega_i + \omega_{i-1} = 0 \) which shows that the converse implication can’t hold in Case (b) and (d).

Our second main result states that every ramification datum satisfying the conclusion of Theorem 4.3 is realized by some fierce character. More precisely:

**Theorem 4.6.** Let \( (\delta_i, \omega_i)_i \) be a ramification datum satisfying Conditions (i)–(iii) in Theorem 4.3. Then there exists a finite constant extension \( K'/K \) and a fierce character \( \chi \) of order \( p^n \) on \( K' \), such that \( (\delta_i, \omega_i) \) is the ramification datum associated to \( \chi \).

The proof of Theorems 4.3 and 4.6 occupies the rest of this paper. Both theorems are proved by induction on \( n \). The case \( n = 1 \) is proved in Sect. 5, and most of the induction step is done in Sect. 6. Finally, a crucial case occuring in the induction step for Theorem 4.6 is dealt with in Sect. 7.

5. Cyclic extensions of order \( p \)

In this section we prove Theorems 4.3 and 4.6 for fierce characters of order \( p \). The proof relies on an explicit description of the refined Swan conductor \( \text{rsw}(\chi) \) of such characters in terms of Kummer theory. This description will also be useful for the study of characters of higher order.

Recall that we assume \( \zeta_p \in K \). Therefore, Kummer theory defines an isomorphism

\[
K^\times/(K^\times)^p \xrightarrow{\sim} H^1_p(K).
\]

We write \( \chi = \chi_u \in H^1_p(K) \) for the character corresponding to an element \( u \in K^\times \). Explicitly, \( \chi \) is given as follows. If \( u \in K^p \), then \( \chi \) is the trivial character. Otherwise,
let \( v \in K^{\text{alg}} \) be a \( p \)th root of \( u \) and \( L := K[v] \). Clearly, \( L/K \) is a Galois extension, and the map
\[
G = \text{Gal}(L/K) \to \mu_p(K), \quad \sigma \mapsto \sigma(v)/v,
\]
is an isomorphism of cyclic groups of order \( p \). The value \( \chi(\sigma) \in \mathbb{Z}/p\mathbb{Z} \) is determined by the identity
\[
\sigma(v)/v = \zeta_p^{\chi(\sigma)}.
\]
Note that \( \chi \) depends on the choice of \( \zeta_p \) but not on the choice of \( v \).

**Definition 5.1.** An element \( u \in K^\times \) is said to be **reduced** if one of the following two conditions hold.

(a) \( u \in O_K^\times \) and \( \bar{u} \notin \bar{K}^p \).
(b) \( u = 1 + \pi^p w \), with \( \pi \in k \), \( 0 < v(\pi) < 1/(p - 1) \), \( w \in O_K \) and \( \bar{w} \notin \bar{K}^p \).

We say that \( u \) is **reducible** if there exists \( a \in K^\times \) such that \( ua^p \) is reduced.

**Proposition 5.2.** The character \( \chi = \chi_u \) is fierce if and only if \( u \) is reducible.

*Proof.* This follows from [5], Lemma 2.16. For convenience of the reader we sketch a proof of the ‘if’-direction. By definition, the character \( \chi \) corresponds to the Galois extension \( L := K[v]/K \), were \( v^p = u \). Assume that \( u \) is reducible. In order to show that \( L/K \) is fierce we may even assume that \( u \) is reduced. In Case (a), the residue \( \bar{v} \in \bar{L} \) generates an inseparable extension of \( \bar{K} \) of degree \( p \). It follows that \( [L : K] = [\bar{L} : \bar{K}] \), i.e. \( L/K \) is fierce. Note also that \( O_L = O_K[v] \) in this case.

In Case (b) we set \( z := (v - 1)/\pi \in L \). Then
\[
\frac{(\pi z + 1)^p - 1}{\pi^p} = z^p + p\pi^{-1}z^{p-1} + \cdots + p\pi^{1-p}z = w.
\]

By the assumption on \( \pi \), the valuation of the coefficient of \( z^i \) in (3) is
\[
v\left(\binom{p}{i}\pi^{i-p}\right) = 1 - (p - i)v(\pi) > 0, \quad \text{for} \quad i = 1, \ldots, p - 1.
\]
It follows that \( z \in O_L \) is integral and that its residue satisfies the irreducible equation
\[
\bar{z}^p = \bar{w}.
\]
As in Case (a), we conclude that \( L/K \) is fierce and that \( O_L = O_K[z] \).

**Proposition 5.3.** Let \( \chi = \chi_u \in H^1_p(K) \), where \( u \in K^\times \) is reduced. Then the refined Swan conductor \( rsw(\chi) \) is given as follows.

(i) If \( \bar{u} \notin \bar{K}^p \) [Case (a) in Definition 5.1] then
\[
rsw(\chi) = \lambda^{-p} \otimes \frac{d\bar{u}}{\bar{u}}.
\]
(ii) If \( u = 1 + \pi p w \) [Case (b) in Definition 5.1] then
\[
\text{rsw}(\chi) = (\pi^p \lambda^{-p}) \otimes d\tilde{w}.
\]

**Proof.** We use the notation of the proof of Proposition 5.2. Let \( \sigma \in G = \text{Gal}(L/K) \) be the element with \( \chi(\sigma) = 1/p \), i.e. \( \sigma(v) = \zeta_p v \). Fix an element \( x \in \mathcal{O}_L^\times \) with \( \bar{L} = \bar{K}[\bar{x}] \). Write \( y := N_{L/K}(x) \in \mathcal{O}_K \). Then \( \bar{y} = \bar{x}^p \in \bar{K}\setminus \bar{K}^p \). Set \( a := \sigma(x)/x - 1, b := N_{L/K}(a) \). By [7, Sect. 3.3, Lemma 15], we have
\[
(\chi, 1 - bc)_K = (i_1(\bar{c}), y)_K = i_2\left(\frac{\bar{c}}{\bar{y}}\right),
\]
for all \( c \in \mathcal{O}_K \). It follows that
\[
\text{rsw}(\chi) = b^{-1} \otimes \frac{d\bar{y}}{y}.
\]

In Case (a), we set \( x := v \). Then \( a = \zeta_p - 1 = \lambda, b = \lambda^p \) and \( y = u \). Hence \( \text{rsw}(\chi) = \lambda^{-p} \otimes d\bar{u}/\bar{u} \) by (5), and (i) is proved.

In Case (b), we set \( x := z \). Then \( a = \lambda \pi z^{-1} + \lambda \) and \( y = w \) (if \( p = 2 \) then \( y = -w \)). The assumption on \( \pi \) implies \( 0 < v(a) = 1/(p-1) - v(\pi) < v(\lambda) \) and
\[
a \equiv \lambda \pi z^{-1} \pmod{p^{v(a)+\epsilon}}.
\]
It follows that
\[
b \equiv \lambda^p \pi^{-p} w^{-1} \pmod{p^{v(a)+\epsilon}}.
\]
(Note that the case \( p = 2 \) is no exception since \( -1 = 1 \) in \( \bar{K} \).) As in Case (a) we conclude that
\[
\text{rsw}(\chi) = b^{-1} \otimes d\bar{w}/\bar{w} = (\pi^p \lambda^{-p}) \otimes d\tilde{w}.
\]
\( \square \)

**Corollary 5.4.** (i) Let \((\delta, \omega)\) be the ramification datum associated to a fierce character \( \chi \) of order \( p \). Then we have either
\[
\delta = \frac{p}{p-1}, \quad \omega = \frac{d\bar{y}}{\bar{y}},
\]
or
\[
0 < \delta < \frac{p}{p-1}, \quad \omega = d\bar{y},
\]
for some element \( \bar{y} \in \bar{K}\setminus \bar{K}^p \).

(ii) Theorems 4.3 and 4.6 hold true for \( n = 1 \).
Proof. Claim (i) follows immediately from (1) and Proposition 5.3. For the proof of (ii), recall the well known fact that for a differential form $\omega \in \Omega^1$ we have $C(\omega) = \omega$ if and only if $\omega = d\bar{y}/\bar{y}$, for some element $\bar{y} \in \bar{K}$. Similarly, $C(\omega) = 0$ if and only if $\omega = d\bar{y}$. In both cases, $\omega \neq 0$ if and only if $\bar{y} \notin \bar{K}^p$. It is now immediate that Theorem 4.3 follows from (i) for $n = 1$.

Conversely, let $(\delta, \omega)$ be a ramification datum of length $n = 1$ which satisfies Condition (i) of Theorem 4.3. In Case (a), we can write $\omega = d\bar{u}/\bar{u}$ for some $\bar{u} \in \bar{K} \setminus \bar{K}^p$. Then for any lift $u \in \mathcal{O}_K$ of $\bar{u}$, the ramification datum associated to the character $\chi_u$ is equal to $(\delta, \omega)$. Case (b) is similar, the only difference being that we need that the rational number $\delta$ is contained in $v(K^\times)$, which holds after a finite constant extension of $K$. Claim (ii) is now proved. 

6. Higher order

The case $n = 1$ of Theorems 4.3 and 4.6 having been proved, we continue by induction with the case $n > 1$. Theorem 4.3 follows immediately, by induction, from the case $n = 1$ and the following proposition.

Proposition 6.1. Let $\chi$ be a fierce character of order $p^n$, $n > 1$. Set $\bar{\chi} := p \cdot \chi$, $\delta := \text{sw}(\chi)$, $\omega := \text{ds}(\chi)$, $\bar{\delta} := \text{sw}(\bar{\chi})$, $\bar{\omega} := \text{ds}(\bar{\chi})$.

Then the following holds. 

(i) $\delta \leq \max\{ \bar{\delta} + 1, p/(p - 1) \}$.

(ii) Suppose $\bar{\delta} > 1/(p - 1)$. Then

$$\delta = \bar{\delta} + 1, \quad \omega = -\bar{\omega}.$$ 

(iii) Suppose $\bar{\delta} \leq 1/(p - 1)$. Then

$$p \bar{\delta} \leq \delta \leq \frac{p}{p - 1}.$$ 

Moreover,

(a) $\delta = p \bar{\delta} < p/(p - 1) \Rightarrow C(\omega) = \bar{\omega}$.

(b) $p \bar{\delta} < \delta < p/(p - 1) \Rightarrow C(\omega) = 0$.

(c) $p \bar{\delta} < \delta = p/(p - 1) \Rightarrow C(\omega) = \omega$.

(d) $p \bar{\delta} = \delta = p/(p - 1) \Rightarrow C(\omega) = \omega + \bar{\omega}$.

Proof. The relation $p \cdot \chi = \bar{\chi}$ and the bilinearity of the symbol $(\cdot, \cdot)_K$ imply the relation

$$(\bar{\chi}, a)_K = (\chi, a^p)_K,$$

for all $a \in K^\times$. For an element of the form $a = 1 - \pi_t z$, with $t \in v(p_K)$ and $z \in \mathcal{O}_K$, we get, using Remark 2.5:

$$a^p = \begin{cases} 
1 - \pi_{pt} \cdot z^p + \ldots, & t < 1/(p - 1),
1 - \pi_{pt} \cdot (z^p - z) + \ldots, & t = 1/(p - 1),
1 + \pi_{t+1} \cdot z + \ldots, & t > 1/(p - 1).
\end{cases}$$

(7)

Here the dots indicate terms of higher valuation than the preceding term.
In order to prove (i), we assume that $\delta > \bar{\delta} + 1$ and $\delta > p/(p - 1)$. Then $\delta - 1 > \bar{\delta}$, so Theorem 3.5 (ii) shows that

$$(\bar{\chi}, 1 - \pi_{\bar{\delta} - 1} \cdot z)_K = 0,$$

for all $z \in \mathcal{O}_K$. Using (6), (7), the inequality $\delta - 1 > 1/(p - 1)$ and Theorem 3.5 (ii)-(iii) we deduce

$$0 = (\chi, 1 - \pi_{\bar{\delta} - 1} \cdot z)_K$$

$$= (\chi, 1 + \pi_{\bar{\delta}} \cdot z + \ldots)_K$$

$$= (\chi, 1 + \pi_{\bar{\delta}} \cdot z)_K,$$

for all $z \in \mathcal{O}_K$. But this is a contradiction to Theorem 3.5 (ii). We have proved (i).

We proceed to prove (ii). By (i) and the assumption we have $\delta \leq \bar{\delta} + 1$. By the definition of $\bar{\delta}$ and $\bar{\omega}$ we have

$$(\bar{\chi}, 1 - \pi_{\bar{\delta}} \cdot z)_K = i_2(\bar{z}\bar{\omega}) \neq 0,$$

for some $z \in \mathcal{O}_K$. Using (6) and (7) we deduce

$$(\chi, 1 - \pi_{\bar{\delta}} \cdot z)_K = (\chi, 1 + \pi_{\bar{\delta} + 1} \cdot z + \ldots)_K \neq 0.$$  (8)

Now the definition of $\delta$ and Theorem 3.5 (ii) shows that $\delta \geq \bar{\delta} + 1$. Hence $\delta = \bar{\delta} + 1$, and the above calculation shows that

$$i_2(\bar{z}\bar{\omega}) = (\chi, 1 + \pi_{\bar{\delta}} \cdot z)_K = i_2(-\bar{z}\omega),$$

for all $z \in \bar{K}$. Hence $\omega = -\bar{\omega}$, and (ii) is proved.

The proof of (iii) follows the same line of argument. First of all, (i) and the assumption imply $\delta \leq p/(p - 1)$. We have

$$i_2(\bar{z}\bar{\omega}) = (\chi, 1 - \pi_{\bar{\delta}} \cdot z)_K$$

$$= \begin{cases} (\chi, 1 - \pi_{\bar{p}\bar{\delta}} \cdot z^p + \ldots)_K, & \bar{\delta} < 1/(p - 1), \\ (\chi, 1 - \pi_{\bar{p}\bar{\delta}} \cdot (z^p - z) + \ldots)_K, & \bar{\delta} = 1/(p - 1). \end{cases}$$  (9)

Since $i_2(\bar{z}\bar{\omega}) \neq 0$ for some $z$, we conclude that $\delta \geq p\bar{\delta}$.

Suppose that $\delta = p\bar{\delta} < p/(p - 1)$. Then from (9) and Remark 3.4 we get

$$i_2(\bar{z}\bar{\omega}) = i_2(\bar{z}\bar{C}(\omega)) = i_2(\bar{z}\bar{C}(\omega)),$$

for all $\bar{z} \in \bar{K}$. This shows that $\bar{C}(\omega) = \bar{\omega}$ [Case (a)]. Similarly, if $p\bar{\delta} = \delta = p/(p - 1)$, we get

$$i_2(\bar{z}\bar{\omega}) = i_2((\bar{z}^p - \bar{z})\omega) = i_2(\bar{z}(\bar{C}(\omega) - \omega)),$$

which shows $\bar{C}(\omega) = \omega + \bar{\omega}$ [Case (d)].

Finally, suppose $p\bar{\delta} < \delta \leq p/(p - 1)$. To prove the remaining cases (b) and (c), we may assume that $\delta/p \in v(K^\times)$. Then

$$0 = (\bar{\chi}, 1 - \pi_{\delta/p} \cdot z)_K$$

$$= (\chi, 1 - \pi_{\delta/p} \cdot z)_K$$

$$= (\chi, 1 - \pi_{\delta} \cdot g(z) + \ldots)_K = i_2(g(\bar{z})\omega),$$
with \( g(z) := z^p \) if \( \delta < p/(p-1) \) and \( g(z) = z^p - z \) if \( \delta = p/(p-1) \). In the first case [Case (b)] we get
\[
i_2(\bar{z} C(\omega)) = i_2(\bar{z}^p \omega) = 0
\]
and conclude \( C(\omega) = 0 \). In the second case [Case (c)], we get
\[
i_2(\bar{z} (\bar{C}(\omega) - \omega)) = i_2((\bar{z}^p - \bar{z}) \omega) = 0
\]
and we conclude \( C(\omega) = \omega \). Now the proposition and Theorem 4.3 are proved. \( \Box \)

We now start the proof of Theorem 4.6. We fix a character \( \tilde{\chi} \) on \( K \) of order \( p^{n-1} \) and we assume that \( \tilde{\chi} \) is fierce. We set
\[
\tilde{\delta} := \text{sw}(\tilde{\chi}), \quad \tilde{\omega} := \text{dsw}(\tilde{\chi}).
\]

**Definition 6.2.** By a lift of \( \tilde{\chi} \) we mean a character \( \chi \), defined (and fierce) over some finite constant extension \( K'/K \), with \( p \cdot \chi = \tilde{\chi}|_{K'} \).

Theorem 4.6 follows by induction from the case \( n = 1 \) and the following proposition.

**Proposition 6.3.** Given \( \delta \in \mathbb{Q}_{>0} \) and \( \omega \in \Omega_1 \setminus \{0\} \) satisfying Condition (i)–(iii) of Proposition 6.1, there exist a lift \( \chi \) of \( \tilde{\chi} \) with \( \delta = \text{sw}(\chi) \) and \( \omega = \text{dsw}(\chi) \).

**Proof.** After a finite constant extension we may assume that \( K \) contains a \( p^n \)th root of unity. Then Kummer theory shows that the following sequence is exact:
\[
0 \rightarrow H^1_p(K) \longrightarrow H^1_{p^n}(K) \overset{\cdot p}{\longrightarrow} H^1_{p^{n-1}}(K) \rightarrow 0.
\]
Therefore, the set of all lifts \( \chi \) of \( \tilde{\chi} \) (defined over \( K \)) is a principal homogenous spaces under the natural action of \( H^1_p(K) \). In particular, it is nonempty. Note that, given an individual lift \( \chi \), we can make it fierce by a finite constant extension of \( K \). However, we can’t do that for all lifts \( \chi \) at a time.

Suppose first that \( \tilde{\delta} > 1/(p-1) \). Then for any (fierce) lift \( \chi \) we have \( \text{sw}(\chi) = \tilde{\delta} + 1 \) and \( \text{dsw}(\chi) = -\tilde{\omega} \) by Proposition 6.1 (ii). Since at least one lift exists, which becomes fierce after a finite constant extension of \( K \), there is nothing more to show.

Next suppose that \( \tilde{\delta} = 1/(p-1) \), and let \( \chi_0 \) be some fierce lift of \( \tilde{\chi} \). Set \( \delta_0 := \text{sw}(\chi_0) \) and \( \omega_0 := \text{dsw}(\chi_0) \). Then we have \( \delta_0 = p/(p-1) \) and \( \bar{C}(\omega_0) = \omega_0 + \tilde{\omega} \) by Proposition 6.1 (iii.d). Our assumption on \( \delta \) and \( \omega \) mean that \( \delta = p/(p-1) \) and \( \bar{C}(\omega) = \omega + \tilde{\omega} \). Set \( \eta := \omega - \omega_0 \). Then \( \bar{C}(\eta) = \eta \). By the Case \( n = 1 \) of Theorem 4.3 (see Corollary 5.4 (ii)) there exists a character \( \psi \) of order \( p \) with \( \text{sw}(\psi) = p/(p-1) \) and \( \text{dsw}(\psi) = \eta \). Set \( \chi := \chi_0 + \psi \). This is a character of order \( p^n \). After a constant extension of \( K \) it is fierce, and then Lemma 3.11 (ii) shows that
\[
\text{sw}(\chi) = \delta, \quad \text{dsw}(\chi) = \omega_0 + \eta = \omega.
\]
So in this case the proposition is proved, too.

Finally we deal with the case \( \tilde{\delta} < 1/(p-1) \). The hard part is to show the following
Lemma 6.4. Let $\tilde{\chi} \in H^1_{p^{n-1}}(K)$ be a fierce character of order $p^{n-1}$, with $\bar{\delta} = \text{sw}(\tilde{\chi}) < 1/(p-1)$. Then there exists a lift $\chi_{\text{min}}$ of $\tilde{\chi}$ (defined over a finite constant extension $K'/K$) such that $\text{sw}(\chi_{\text{min}}) = p\bar{\delta}$.

The proof of Lemma 6.4 will be given in the next section. Let $\chi_{\text{min}}$ be a lift of $\tilde{\chi}$ with $\text{sw}(\chi_{\text{min}}) = p\bar{\delta}$. Set $\omega_{\text{min}} := \text{dsw}(\chi_{\text{min}})$. We have $C(\omega_{\text{min}}) = \bar{\omega}$ by Proposition 6.1 (iii.a). If $\delta = p\bar{\delta}$, then we set $\eta := \omega - \omega_{\text{min}}$. Since $C(\eta) = \bar{\omega} - \bar{\omega} = 0$ and $\delta < p/(p-1)$, there exists a constant extension $K'/K$ and $\psi \in H^1_p(K')$ with $\text{sw}(\psi) = \delta$ and $\text{dsw}(\psi) = \eta$ (Corollary 5.4 (ii)). Set $\chi := \chi_{\text{min}} + \psi$. It follows from Lemma 3.11 that $\text{sw}(\chi) = \delta$ and $\text{dsw}(\chi) = \omega$, as required.

The case $p\bar{\delta} < \delta$ is handled in a similar way. We are in Case (a) or (c) of Proposition 6.1. In Case (a) we have $\delta < p/(p-1)$ and $C(\omega) = 0$, in Case (c) we have $\delta = p/(p-1)$ and $C(\omega) = \omega$. In both cases, Corollary 5.4 (ii) shows that there exists a constant extension $K'/K$ and $\psi \in H^1_p(K')$ with $\text{sw}(\psi) = \delta$ and $\text{dsw}(\psi) = \omega$. Set $\chi := \chi_{\text{min}} + \psi$. Using again Lemma 3.11 we see that $\text{sw}(\chi) = \delta$ and $\text{dsw}(\chi) = \omega$. This completes the proof of Proposition 6.3, under the condition that Lemma 6.4 holds. \qed

7. Construction of a minimal lift

The goal of this section is to give a proof of Lemma 6.4 and thus complete the proof of Theorem 4.6. Since the proof is quite technical, we start by explaining the main idea.

Let $\tilde{\chi}$ be a character of order $p^{n-1}$, with $\bar{\delta} := \text{sw}(\tilde{\chi}) < 1/(p-1)$. Let $\chi_0$ be an arbitrary lift of $\tilde{\chi}$. Then $\delta_0 := \text{sw}(\chi_0) \geq p\bar{\delta}$, by Proposition 6.1. If $\delta_0 = p\bar{\delta}$ then we are done. Otherwise, we can use Lemma 3.11 to find a lift $\chi_1$ of $\tilde{\chi}$ with $\delta_1 := \text{sw}(\chi_1) < \delta_0$. This argument shows the following. If the set

$$\Delta := \{ \text{sw}(\chi) \mid \chi \text{ is a lift of } \tilde{\chi}\}$$

has a minimum, then this minimum is equal to $p\bar{\delta}$, and we are done. However, it is not obvious that a minimum exists: since lifts are defined over finite but arbitrarily large constant extensions of $K$, $\Delta$ is not contained in any discrete subgroup of $\mathbb{R}$.

Our solution to this problem is to consider a certain subset of the set of all lifts of $\tilde{\chi}$. Let us call lifts that lie in this subset moderate (in the actual proof this terminology is used in a slightly different way, see Definition 7.5). We then show that the corresponding subset $\Delta_{\text{mod}} \subset \Delta$ has the following two properties: (a) if $\Delta_{\text{mod}}$ has a minimum, then it is equal to $p\bar{\delta}$, (b) if $\delta_0 > \delta_1 > \ldots$ is a strictly decreasing sequence in $\Delta_{\text{mod}}$, then $\lim \delta_i < p\bar{\delta}$ (again, in the actual proof this is formulated differently). Combining (a) and (b) shows that there exists a minimal lift $\chi$ with $\text{sw}(\chi) = p\bar{\delta}$, proving Lemma 6.4.

7.1. Approximation by $p^t$th powers

We fix a fierce character $\tilde{\chi} \in H^1_{p^{n-1}}(K)$ of order $p^{n-1}$, with $n \geq 2$. It gives rise to a cyclic Galois extension $M/K$, with Galois group $\tilde{G} := \text{Gal}(M/K) \cong \mathbb{Z}/p^{n-1}$. Let $\delta_1, \ldots, \delta_{n-1}$ be the breaks of the upper numbering filtration $(\tilde{G}^t)_t$. Then...
\[ |\tilde{G}'| = \begin{cases} 
    p^{n-1}, & t \leq \delta_1, \\
    p^{n-i}, & \delta_{i-1} < t \leq \delta_i, \\
    1, & t > \delta_{n-1}. 
\end{cases} \]

By definition, the Swan conductor \( \bar{\delta} := \text{sw}(\bar{\chi}) \) of \( \bar{\chi} \) is equal to \( \delta_{n-1} \). We assume that \( \bar{\delta} < 1/(p-1) \).

Let \( \psi_{M/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) be the inverse Herbrand function associated to \( M/K \), see Sect. 3.1. Then for \( t \geq \bar{\delta} \) we have

\[
\psi_{M/K}(t) = \delta_1 p^{-n+1} + (\delta_2 - \delta_1) p^{-n+2} + \cdots + (\delta_{n-1} - \delta_{n-2}) p^{-1} + (t - \bar{\delta})
\]

with

\[
\epsilon := \delta_1 \cdot \frac{p-1}{p^{n-1}} + \delta_2 \cdot \frac{p-1}{p^{n-2}} + \cdots + \delta_{n-1} \cdot \frac{p-1}{p} > 0.
\]

We may assume that \( \epsilon \in v(K^\times) \). It is obvious from (10) that

\[
\epsilon \leq \bar{\delta} < 1/(p-1) \leq 1.
\]

It follows that

\[ \tilde{O}_M^{(\epsilon)} := O_M^p + \pi \epsilon \cdot O_M \]

is a subring of \( O_M \).

**Lemma 7.1.** Let \( a \in O_K \) be given. After replacing \( K \) by a finite constant extension (which may depend on \( a \)), we have \( a \in \tilde{O}_M^{(\epsilon)} \).

**Proof.** Set \( K' := K(a_{1/p}) \), \( M' = MK' = M[a_{1/p}] \). We may assume that, after a finite constant extension of \( K \), \( M'/M \) is fierce of degree \( p \) (otherwise, \( a \) becomes a \( p \)th power in \( O_M \) and we are done). But then \( M'/K \) is a fierce Galois extension such that

\[ \text{Gal}(M'/K) = \text{Gal}(M'/K') \times \text{Gal}(M'/M) \cong \mathbb{Z}/p^{n-1} \mathbb{Z} \times \mathbb{Z}/p \mathbb{Z}. \]

Let \( \delta_{K'/K} \) (resp. \( \delta_{M'/M} \)) denote the Swan conductor of a character of order \( p \) giving rise to the extension \( K'/K \) (resp. \( M'/M \)). Using Remark 3.1 and Definition 3.3 it is easy to see that

\[
\delta_{M'/M} = \psi_{M/K}(\delta_{K'/K}).
\]

By Proposition 5.2 we can write

\[ a = b^p + \pi t \cdot c, \]

with \( b, c \in O_K \), \( \tilde{c} \notin \tilde{K}^p \) and \( t \in v(O_K) \). Then

\[ \delta_{K'/K} = p/(p-1) - t \]
by Proposition 5.3. If $t \geq \epsilon$ then we are done. We may therefore assume

$$\delta_{K'/K} > p/(p - 1) - \epsilon > 1/(p - 1).$$

But then (10) and (13) show that

$$\delta_{M'/M} = \delta_{K'/K} - \epsilon = p/(p - 1) - t',$$

with $t' := t + \epsilon \geq \epsilon$. Using again Propositions 5.2 and 5.3, we see that we can write

$$a = (b')^p + \pi t' \cdot c',$$

with $b', c' \in \mathcal{O}_M$. This proves the lemma. 

\[\square\]

**Corollary 7.2.** After replacing $K$ by a finite constant extension, the following holds. For all $\bar{a} \in \bar{K}$ there exists a lift $a \in \mathcal{O}_K$ which can be written in the form

$$a = b^p + \pi \epsilon \cdot c,$$

with $b, c \in \mathcal{O}_M$.

**Proof.** Choose $x \in \mathcal{O}_K$ with $\bar{x} \not\in \bar{K}^p$. Then

$$\bar{K} = \bar{K}^p[\bar{x}],$$

by Assumption 2.1 (a). By Lemma 7.1 we may assume that $x \in \mathcal{O}_M^{(\epsilon)}$. It follows that $\bar{K} = \bar{K}^p[\bar{x}]$ is contained in the image of the ring homomorphism

$$\mathcal{O}_M^{(\epsilon)} \cap \mathcal{O}_K \to \tilde{M},$$

proving the claim. 

\[\square\]

After replacing $K$ by a finite constant extension once, we may and will assume from now on that the conclusion of Corollary 7.2 holds. We recall that we also assume $\zeta_{p^n} \in K$.

### 7.2. Moderate elements, I

Let $\Lambda := v(K^\times)$ denote the value group of $K$. Then $\Lambda = \mathbb{Z} \cdot \frac{1}{e}$, where $e$ is the absolute ramification index of $K$. For $k \geq 1$ we set

$$v_k := 1 + \frac{1}{p} + \cdots + \frac{1}{p^{k-1}}.$$

Note that $v_k \to p/(p - 1)$ for $k \to \infty$.

**Definition 7.3.** The subset $\Lambda_{\epsilon} \subset \mathbb{Q}$ is defined as follows. A rational number $t \in \mathbb{Q}$ is contained in $\Lambda_{\epsilon}$ if and only if $t \geq 0$ and the implication

$$t < \left(1 - \epsilon/p\right) \cdot v_k \implies p^k t \in \Lambda$$

holds, for all $k \geq 1$. Here $\epsilon \in \Lambda$ is defined by (11).

It is clear that $\Lambda \cap \mathbb{Q}_{\geq 0} \subset \Lambda_{\epsilon}$. Furthermore:
Lemma 7.4. (i) \((\Lambda_\epsilon, +)\) is a submonoid of \((\mathbb{Q}, +)\).

(ii) Suppose \(t \in \Lambda_\epsilon\) and \(t \geq \epsilon\). Then \(t'_i := 1 + i(t - \epsilon)/p\) and \(t_i := 1 + it/p\), for \(i = 1, 2, \ldots\), lie in \(\Lambda_\epsilon\) as well.

(iii) Let \(t_1, t_2, \ldots \in \Lambda_\epsilon\) be a strictly increasing sequence. Then

\[
\limsup_{i} t_i \geq \frac{p - \epsilon}{p - 1}.
\]

(iv) Suppose \(s, t \in \Lambda_\epsilon\), with \(s, t \geq \epsilon\). Then \(s + t - \epsilon \in \Lambda_\epsilon\).

Proof. Part (i) of the lemma follows directly from the definition of \(\Lambda_\epsilon\). To prove (ii) we first note that

\[
p \cdot \left(\left(1 - \frac{\epsilon}{p}\right) v_k - 1\right) + \epsilon = \left(1 - \frac{\epsilon}{p}\right) v_{k-1}. \tag{14}
\]

Assume that

\[
t'_i = 1 + \frac{i(t - \epsilon)}{p} < \left(1 - \frac{\epsilon}{p}\right) v_{k}.
\]

Since \(t'_i \leq t'_i\) this inequality holds in particular for \(i = 1\). Using \((14)\) and a direct computation one shows that

\[
t < \left(1 - \frac{\epsilon}{p}\right) v_{k-1}.
\]

Since \(t \in \Lambda_\epsilon\) by assumption, we conclude that \(p^{k-1}t \in \Lambda\). But then

\[
p^k t'_i = p^k + ip^{k-1}t - ip^{k-1}\epsilon \in \Lambda,
\]

which shows that \(t'_i \in \Lambda_\epsilon\). If \(t_i < (1 - \epsilon/p)v_k\) then we also have \(t'_i < (1 - \epsilon/p)v_k\), and the previous argument shows that \(p^{k-1}t \in \Lambda\). As before, we can conclude that

\[
p^k t_i = p^k + ip^{k-1}t \in \Lambda
\]

and hence \(t_i \in \Lambda_\epsilon\). Now (ii) is proved.

Given a strictly increasing sequence \(t_1 < t_2 < \ldots \in \Lambda_\epsilon\), we distinguish two cases. In the first case, there exists \(k_0\) such that \(t_i \in p^{-k_0} \Lambda\), for all \(i\). Then the sequence is unbounded, and the claim in (iii) is correct. In the other case, there exists a strictly increasing sequence of indices \(i_1 < i_2 < \ldots\) such that \(p^k t_{i_k} \notin \Lambda\). Since we assume \(t_{i_k} \in \Lambda_\epsilon\) this means that

\[
t_{i_k} \geq \left(1 - \frac{\epsilon}{p}\right) v_k \longrightarrow_{k \to \infty} \frac{p - \epsilon}{p - 1}.
\]

Now (iii) follows immediately. Finally, the proof of (iv) is straightforward. \(\square\)
7.3. Moderate elements, II

For the rest of this paper we shall keep the field $K$ fixed. Recall that we have chosen certain elements $\pi_t \in k$ in the field of constants of $K$ with $v(\pi_t) = t$, for all $t \in \Lambda = v(K^\times)$, see §2.4. It is clear that we can extend the definition of $\pi_t$ for all $t \in \mathbb{Q}$, where $\pi_t$ lies in a fixed algebraic closure of $k$ and such that Remark 2.5 holds. Let $N = p^n$ be a power of $p$ and set $\Lambda' := \frac{1}{N}\Lambda$. Then the field

$$K' := K[\pi_t \mid t \in \Lambda']$$

is a totally ramified constant extension of $K$ such that $v((K')^\times) = \Lambda'$. Many of the following statements depend on the choice of $K'$ (i.e. on $m$) and are true only for $K'/K$ (i.e. $m$) sufficiently large.

We set $M' := MK'$. Since $K'/K$ is constant and totally ramified and $M/K$ is fierce, $M'/M$ is again constant and totally ramified, with $[M' : M] = [K' : K]$.

It is easy to see that an element $a \in \mathcal{O}_{M'}$ can be uniquely written in the form

$$a = a_1 \cdot \pi_{t_1} + \cdots + a_r \cdot \pi_{t_r},$$

with $a_i \in \mathcal{O}_{M}^\times, t_i \in \Lambda', 0 \leq t_1 < \cdots < t_r$ and such that $t_i - t_j \notin \Lambda$, for $i \neq j$. We call (15) the canonical form and the numbers $t_1, \ldots, t_r$ the exponents of $a$.

**Definition 7.5.** An element $a \in \mathcal{O}_{M'}$ is called moderate if its exponents $t_1, \ldots, t_r$ are contained in $\Lambda_-$. It is important for us that the condition of being moderate is compatible with enlarging the extension $K'/K$ (i.e. increasing $m$). Given two elements $u, u' \in (M')^\times$, we write $u \sim_{M'} u'$ if $u^{-1} u' \in (M')^P$.

**Lemma 7.6.** (i) Suppose $a \in \mathcal{O}_{M'}$ can be written as

$$a = a_1 \cdot \pi_{t_1} + \cdots + a_r \cdot \pi_{t_r},$$

with $a_i \in \mathcal{O}_{M}$ and $t_i \in \Lambda', t_i \geq 0$. If $t_i \in \Lambda_-$ for all $i$ then $a$ is moderate.

(ii) If $a_1, a_2 \in \mathcal{O}_{M'}$ is moderate, then so is $a_1 \pm a_2$ and $a_1 a_2$. In other words: the set of moderate elements is a subring of $\mathcal{O}_{M'}$.

(iii) Let $u \in \mathcal{O}_{M'}^\times$ be a moderate principal unit. Then

$$v(u - 1) \in \Lambda_-.$$

Furthermore, $u^{-1}$ is again moderate.

(iv) Let $u \in \mathcal{O}_{M'}^\times$ be a moderate principal unit. Let $\psi_u \in H^1_p(M')$ denote the Kummer character associated to $u$. Assume that

$$sw(\psi_u) > \frac{\epsilon}{p - 1}.$$ 

Then (after enlarging the constant extension $K'/K$) $u \sim_{M'} u'$, where $u' \in \mathcal{O}_{M'}^\times$ is moderate and reduced (in the sense of Definition 5.1).
For \( t \in \Lambda_\varepsilon \), with \( t \geq \varepsilon \), and \( a \in \mathcal{O}_K \cap \tilde{\mathcal{O}}_M^{(\varepsilon)} \) (see Sect. 7.1 for notation), we set

\[
u := 1 + \pi_{t-\varepsilon} \cdot a \in \mathcal{O}_M^{\times}.
\]

Then (after enlarging the constant extension \( K'/K \)) we have \( u \sim_{M'} u' \), for a moderate principal unit \( u' \in \mathcal{O}_M^{\times} \).

**Proof.** Let \( a = \sum_i a_i \pi_{t_i} \) be as in (i). We may assume that \( a_i \neq 0 \) for all \( i \). Let \( I \subset \{1, \ldots, r\} \) be the subset of indices \( i \) such that \( t_i \leq t_j \) for all \( j \) with \( t_i - t_j \in \Lambda_\varepsilon \).

Then

\[
a = \sum_{i \in I} \left( \sum_{t_j - t_i \in \Lambda} a_j \pi_{t_j - t_i} \right) \in \mathcal{O}_M.
\]

If \( a'_i = 0 \) then we eliminate \( i \) from the set \( I \). Otherwise, we can write \( a'_i = a''_i \pi_{s_i} \)

with \( a''_i \in \mathcal{O}_M^{\times} \) and \( s_i \in \Lambda_\varepsilon \). Then

\[
a = \sum_{i \in I} a''_i \pi_{t_i + s_i},
\]

is, up to reordering the terms of the sum, the canonical form of \( a \). Since \( t_i \in \Lambda_\varepsilon \) by assumption, \( t_i + s_i \in \Lambda_\varepsilon \) as well. Hence \( a \) is moderate, proving (i). Assertion (ii) follows easily from (i) and Lemma 7.4 (i).

Let \( u = 1 + \pi_t a \) be moderate, with \( a \in \mathcal{O}_M^{\times} \) and \( t > 0 \). We have to show that \( t \in \Lambda_\varepsilon \). Let \( a = a_0 + a_1 \pi_{t_1} + \cdots + a_r \pi_{t_r} \) be the canonical form. Then \( t_i \notin \Lambda_\varepsilon \), for all \( i \). Applying the reasoning of the proof of (i) to

\[
u = 1 + a_0 \pi_t + a_1 \pi_{t_1 + t} + \cdots + a_r \pi_{t_r + t}
\]

we see that \( t \) is either an exponent of \( u \) or \( t \in \Lambda_\varepsilon \). In both cases, \( t \in \Lambda_\varepsilon \), proving the first part of (iii). It remains to show that \( u^{-1} \) is also moderate. We write

\[
u^{-1} = 1 - \pi_t a + \pi_{2t} a^2 - \cdots = 1 + \sum_{i=1}^r a_i \pi_{t_i},
\]

where the last expression is the canonical form of \( u^{-1} \). Arguing again with the proof of (i) we see that for all \( i \) there exists an integer \( k \geq 0 \) such that \( t_i = kt + s \), with \( s \in \Lambda_\varepsilon, s \geq 0 \). Hence \( t_i \in \Lambda_\varepsilon \) and \( u^{-1} \) is moderate.

For the proof of (iv) we continue with the same notation. After enlarging the extension \( K'/K \) we may assume that \( t/p \in \Lambda_\varepsilon \). If \( \bar{a} \notin M^p \) then \( u \) is reduced and we are done. Otherwise, we can find \( b \in \mathcal{O}_M^{\times} \) with \( \bar{b}^p = \bar{a} \) (see Definition 5.1). Set \( v := 1 + \pi_{t/p} \cdot b \in \mathcal{O}_M^{\times} \).

Then

\[
v^p = 1 + \pi_t \cdot b^p + \sum_{i=1}^{p-1} \pi_{1+it/p} \cdot b_i,
\]
with \( b_i \in \mathcal{O}_M \). Since \( t \in \Lambda_\epsilon \), (i) and Remark 7.4 (ii) show that \( v^p \) is moderate. Write

\[
u_1 := uv^{-p} = 1 + \pi t_1 \cdot a_1,
\]

with \( a_1 \in \mathcal{O}_M^\times \). Then \( t_1 > t \), and it follows from (i) and (iii) that \( u_1 \) is moderate. If \( u_1 \) is reduced, we are done. Otherwise, we apply the same procedure again to \( u_1 \). Continuing this way, we obtain a sequence of moderate principal units \( u_1, u_2, \ldots \) such that

\[
t < t_1 < t_2 < \ldots
\]

(16)

We have to show that this process stops after a finite number of steps with a reduced principal unit \( u_k \), for some \( k \in \mathbb{N} \). We argue by contradiction and assume that we obtain an infinite sequence of \( u_i \). Since \( u_i \) is moderate, \( t_i \in \Lambda_\epsilon \) for all \( i \), by (iii). Therefore, Lemma 7.4 (iii) shows that

\[
\limsup_i t_i \geq \frac{p - \epsilon}{p - 1}.
\]

(17)

On the other hand we have

\[
t_i \leq \frac{p}{p - 1} - sw(\psi_u) < \frac{p - \epsilon}{p - 1}
\]

(18)

by Proposition 5.3 and the assumption. But (17) and (18) contradict each other. This completes the proof of (iv).

It remains to prove (v). By assumption we can write \( a = b^p + \pi \epsilon \cdot c \), with \( b, c \in \mathcal{O}_M \), and hence

\[
u = 1 + \pi t - \epsilon \cdot b^p + \pi t \cdot c = u_0 + u_1,
\]

with \( u_0 := 1 + \pi t - \epsilon \cdot b^p \) and \( u_1 := \pi t \cdot c \). After enlarging \( K'/K \) we may assume that \( (t - \epsilon)/p \in \Lambda' \). We set \( w := 1 + \pi (t - \epsilon)/p \cdot b \in \mathcal{O}_{M'} \). Then

\[
w^p = 1 + \pi t - \epsilon \cdot b^p + \sum_{i=1}^{p-1} \pi 1+i(t-\epsilon)/p \cdot b_i = u_0 + w_1,
\]

with \( b_i \in \mathcal{O}_M \) and \( w_1 \in \mathcal{O}_{M'} \). We have \( v(w_1) > 1 > \epsilon \) and, moreover, \( w_1 \) is moderate by Lemma 7.4 (ii). Using (ii) and Lemma 7.4 (iv) one shows that

\[
u_0^{-1} w_1 = (1 - \pi t - \epsilon b^p + \pi 2(t - \epsilon)b^2p - \ldots)w_1
\]

and hence

\[
w_2 := (1 + u_0^{-1} w_1)^{-1}
\]

is moderate. A similar argument shows that \( u_0^{-1} u_1 \) is moderate. Using once more (ii) we conclude that

\[
u' := uw^{-p} = (1 + u_1 u_0^{-1})w_2
\]

is moderate. This finishes the proof of the lemma. \( \square \)
7.4. The proof of Lemma 6.4

We can now give a proof of Lemma 6.4. We argue by contradiction, i.e. we assume that \( sw(\chi) > p\delta \) for all lifts \( \chi \) of \( \tilde{\chi} \). Using this assumption, we shall construct inductively a sequence of lifts \( \chi_0, \chi_1, \chi_2, \ldots \) of \( \tilde{\chi} \) whose Swan conductors \( \delta_i := sw(\chi_i) \) form a strictly decreasing sequence. Afterwards we will show that \( \lim_i \delta_i < p\delta \), which gives the desired contradiction. For a fixed \( i \), the lifts \( \chi_0, \ldots, \chi_i \) will be defined and fierce over a finite constant extension \( K'/K \) as described above. During the induction process, we will have to keep increasing \( K'/K \), but this is not a problem.

We start by choosing an arbitrary lift \( \chi_0 \in H^1_{pr}(K) \) of \( \tilde{\chi} \) defined over \( K \). We may assume that \( \chi_0 \) is fierce over \( K' \). By assumption we have \( \delta_0 := sw(\chi_0) > p\delta \).

By induction, suppose that \( \chi_i \) has been constructed and is defined and fierce over the constant extension \( K'/K \). Set \( \delta_i := sw(\chi_i) \) and \( \omega_i := dsw(\chi_i) \). We have \( \delta < 1/(p - 1) \) and \( p\delta < \delta_i \) by assumption. Therefore, Proposition 6.1 says that \( \delta_i \leq p/(p - 1) \) and that \( C(\omega_i) = 0 \) if \( \delta < p/(p - 1) \) and \( C(\omega_i) = \omega_i \) otherwise.

Note that the second case, \( \delta_i = p/(p - 1) \), can occur only once, for \( i = 0 \). It follows that \( \omega_i = d\tilde{a}_i \) in the first and \( \omega_i = d\tilde{a}_i/\tilde{a}_i \) in the second case, for an element \( \tilde{a}_i \in \tilde{K} \). Using Corollary 7.2 we can choose \( a_i \in \hat{O}_K \cap \tilde{O}_M^{(\epsilon)} \) lifting \( \tilde{a}_i \). Set \( \mu_i := p/(p - 1) - \delta_i \) and \( z_i := 1 - \pi_{\mu_i} \cdot a_i \in O_K^{(\epsilon)} \),

if \( \delta_i < p/(p - 1) \) (and \( z_i := a_i^{-1} \) otherwise). Here we assume that \( \mu_i \in \Lambda' \). Then, as an element of \( O_K^{(\epsilon)} \), \( z_i \) is reduced (since \( \tilde{a}_i \notin \tilde{K}^p \)). Let \( \psi_i \in H^1_{pr}(K') \) be the character associated to \( z_i \) by Kummer theory. By Proposition 5.2 and Proposition 5.3 \( \psi_i \) is fierce over \( K' \) and we have

\[
sw(\psi_i) = p/(p - 1) - \mu_i = \delta_i, \quad dsw(\psi_i) = -\omega_i.
\]

We define \( \chi_{i+1} := \chi_i + \psi_i \in H^1_{pr}(K') \). This is again a lift of \( \tilde{\chi} \), and Lemma 3.11 shows that \( \delta_{i+1} := sw(\chi_{i+1}) < \delta_i \). This completes the construction of the sequence \( \chi_0, \chi_1, \ldots \).

The character \( \chi_0 \) corresponds, via Kummer theory, to the class of an element \( u \in K^\times \) in \( K^\times/(K^\times)^p \). It follows that \( M = K[y] \), where \( y^p - u = 0 \). Moreover, the restriction \( \chi_0|M \in H^1_{pr}(M) \) corresponds to the class of \( y \) in \( M^\times/(M^\times)^p \).

Let \( \chi \in H^1_{pr}(K') \) be an arbitrary lift of \( \tilde{\chi} \) defined over \( K' \). Then \( \chi|M' \) corresponds to the class of \( yz \) in \( M'^\times/(M'^\times)^p \), for some \( z \in K'^\times \). Note that \( \chi \) is fierce if and only if \( \chi|M' \) is, and this is the case if and only if \( yz \in M'^\times \)-equivalent to a principal unit of the form \( y' = 1 + \pi_\lambda \cdot w \), with \( w \in O_M^{(\epsilon)}, \tilde{w} \notin M^p \) and \( \lambda/p \in \Lambda' \). Assume that this is the case. Then \( \delta := sw(\chi|M') = p/(p - 1) - \lambda \). Using (10) we obtain

\[
\delta = \frac{p}{p - 1} + \epsilon - \lambda. \tag{19}
\]

For the sequence of lifts \( \chi_0, \chi_1, \ldots \) constructed above, this means the following. The restriction \( \chi_i|M' \in H^1_{pr}(M') \) corresponds to the class of \( y_i \in M'^\times \), defined by the recursive formula \( y_0 := y, y_{i+1} := y_i z_i \).
Lemma 7.7. We have $y_i \sim_{M'} y'_i$, where $y'_i \in \mathcal{O}_M^{\times}$, is reduced and moderate.

Proof. We prove the lemma by induction on $i$, using Lemma 7.6 repeatedly. For $i = 0$ there is nothing to show. By induction, we may assume that $y_i \sim_{M'} y'_i$, with $y'_i$ reduced and moderate. Write

$$y'_i = 1 + \pi\lambda_i \cdot \nu_i,$$

with $\nu_i \notin M^p$. Then (19) says that $\delta_i = sw(\chi_i) = p/(p - 1) + \epsilon - \lambda_i$. Therefore, $\mu_i = p/(p - 1) - \delta_i = \lambda_i - \epsilon$.

Since $\lambda_i \in \Lambda_{\epsilon}$ by Assertion (iii) of Lemma 7.6, Assertion (v) of that lemma shows that $z_i$ is equivalent to a moderate element (here we use that $a_i \in \hat{O}_M^{(c)}$). Hence $y'_i z_i$ is equivalent to a moderate element by Assertion (ii). The Kummer class of $y'_i z_i$ is the character $\chi_{i+1}|_{M'}$. We have

$$sw(\chi_{i+1}|_{M'}) - \frac{\epsilon}{p - 1} = \delta_{i+1} - \frac{p}{p - 1} \cdot \epsilon$$

$$> p\bar{\delta} - \left(\delta'_{n-1} + \frac{1}{p}\delta'_{n-2} + \cdots + \frac{1}{p^{n-2}}\delta_1\right)$$

$$\geq p\bar{\delta} - \bar{\delta} \left(1 + \frac{1}{p^2} + \cdots + \frac{1}{p^{2n-4}}\right)$$

$$> \bar{\delta} \cdot \left(p - \frac{p^2}{p^2 - 1}\right) > 0.$$

(Here $\delta'_1, \ldots, \delta'_{n-1} = \bar{\delta}$ are the ramification invariants of the extension $M'/K'$, and we use Proposition 6.1 to see that $\delta'_k < \bar{\delta}/p^{n-k-1}$.) So Assertion (iv) of Lemma 7.6 shows that $y_{i+1} \sim_{M'} y'_i z_i \sim_{M'} y'_{i+1}$, with $y'_{i+1}$ reduced and moderate. This completes the proof of Lemma 7.7.

Since $y'_i$ is reduced by Lemma 7.7, (19) shows that

$$\lambda_i = v(y'_i - 1) = p/(p - 1) + \epsilon - \delta_i$$

It follows that the sequence $\lambda_0 < \lambda_1 < \ldots$ is strictly increasing. On the other hand, $y'_i$ is also moderate, so $\lambda_i \in \Lambda_{\epsilon}$ by Lemma 7.6 (iii). Therefore, Lemma 7.4 (iii) implies

$$\limsup_i \lambda_i \geq \frac{p - \epsilon}{p - 1}.$$

We conclude that

$$\liminf_i \delta_i \leq \frac{p}{p - 1} + \epsilon - \frac{p - \epsilon}{p - 1} = \frac{p}{p - 1} \cdot \epsilon.$$

(20)

By the calculation already used we have

$$\frac{p}{p - 1} \cdot \epsilon < \bar{\delta} \cdot \frac{p^2}{p^2 - 1} < p\bar{\delta}.$$

(21)

Combining (20) and (21) shows that $\delta_i < p\bar{\delta}$ for $i$ sufficiently large. But this contradicts Proposition 6.1, and Lemma 6.4 follows. \qed
Acknowledgments I am very grateful for the referee for a careful reading of a previous version of this paper and for pointing out several inaccuracies.

References

[1] Abbes, A., Saito, T.: Ramification of local fields with imperfect residue field. Am. J. Math. 124, 879–920 (2002)
[2] Cartier, P.: Une nouvelles opération sur les formes différentielles. C. R. Acad. Sci. Paris 244, 426–428 (1957)
[3] Epp, H.P.: Eliminating wild ramification. Invent. Math. 19, 235–249 (1973)
[4] Garuti, M.A.: Linear systems attached to cyclic inertia. In: Arithmetic Fundamental Groups and Noncommutative Algebra (Berkeley, CA, 1999), volume 70 of Proceedings of Symposium in Pure Mathematics, pp. 377–386. American Mathematical Society (2002)
[5] Hyodo, O.: Wild ramification in the imperfect residue field case. In: Galois Representations and Arithmetic Algebraic Geometry, Number 12 in Advanced Studies in Pure Mathematics, pp. 287–314 (1987)
[6] Kato, K.: A generalization of local class field theory by using $K$-groups, I. J. Fac. Sci. Univ. Tokyo Sec. IA 26, 303–376 (1979)
[7] Kato, K.: A generalization of local class field theory by using $K$-groups, II. J. Fac. Sci. Univ. Tokyo Sec. IA 27, 603–683 (1980)
[8] Kato, K.: Galois cohomology of complete discrete valuation fields. In: Algebraic $K$-theory II, Number 967 in LNM, pp. 215–236. Springer (1982)
[9] Kato, K.: Swan conductors with differential values. In: Galois Representations and Arithmetic Algebraic Geometry, Number 12 in Advanced Studies in Pure Mathematics, pp. 315–342 (1987)
[10] Kato, K.: Vanishing cycles, ramification of valuations, and class field theory. Duke Math. J. 55(3), 629–659 (1987)
[11] Obus, A., Pries, R.: Wild tame-by-cyclic extensions. J. Pure Appl. Algebra 214(5), 565–573 (2010)
[12] Obus, A., Wewers, S.: Cyclic extensions and the local lifting problem. Ann. Math. (to appear) arXiv:1203.5057 (2012)
[13] Schmid, H.L.: Zur Arithmetik der zyklischen $p$-Körper. J. Reine Angew. Math. 176, 161–167 (1937)
[14] Serre, J.-P.: Corps locaux. Hermann, Paris (1968)
[15] Thomas, L.: Ramification groups in Artin–Schreier–Witt extensions. J. Théorie Des Nombres Bordeaux 17, 689–720 (2005)
[16] Zhukov, I.: On ramification theory in the imperfect residue field case. Sbornik Math. 194(12), 1747–1774 (2003)