A Non Degenerate Semi-Classical Lagrangian for Dilaton-Gravity in Two Dimensions

Noureddine Mohammedi

International Centre for Theoretical Physics
P. O. Box 586, 34100 Trieste, Italy.

Abstract

An action for two dimensional gravity conformally coupled to two dilaton-type fields is analysed. Classically, the theory has some exact solutions. These include configurations representing black holes. A semi-classical theory is obtained by assuming that these singular solutions are caused by the collapse of some matter fields. The semi-classical equations of motion reveal then that any generic solution must have a flat geometry.
1. Introduction

One of the challenging issues in modern physics is the reconciliation between quantum mechanics and general relativity. This clash followed Hawking’s discovery that matter in an initially pure quantum state can collapse to form a black hole, which then evaporates into a mixed quantum state through Hawking radiation [1,2,3]. Insight into this matter might be gained by analysing the problem in the context of two dimensions. This is mainly because in two dimensions, a closed form expression for the vacuum expectation value of the stress-energy tensor can be computed for a conformally flat space-time metric [4]. This fact enables the calculation of the temperature and energy flux at infinity of an evaporating black hole [5,6,7,8]. In four dimensions, however, the conformal anomaly does not uniquely determine the one-loop effective action, and hence the vacuum expectation value of the energy-momentum tensor [8].

An interesting model for analysing the process of black hole formation under gravitational collapse and Hawking radiation was proposed by Callan, Giddings, Harvey and Strominger in ref.[9]. This consists in coupling two dimensional gravity to a dilaton field and some conformal matter fields. At the classical level, they found a set of exact solutions among which one is a black hole formed by an infalling shock wave. They then, in order to describe the back-reaction on the metric due to Hawking radiation, modified their classical action by including the one-loop conformal anomaly. Furthermore, by allowing for a large number of matter fields, they argued that quantum corrections would prevent the formation of black hole singularities in a gravitational collapse and therefore pure quantum states would not evolve into mixed quantum states. However, it was shown in refs.[10,11] that in this theory gravitational collapse always ends with the formation of a singularity. Hence the puzzles of black hole evaporation are still persistent.

In this paper we present a model for two dimensional gravity in which gravitational collapse does not develop a singularity. It consists of two dimensional gravity conformally coupled to two dilaton fields. The model is

\[ ^{2}\text{Despite this, many interesting papers were devoted to this model [12].} \]
obtained by a dimensional reduction of four dimensional Einstein gravity
conformally coupled to a massless scalar field. We begin in section two by
introducing our general model and describe some classical solutions. These
include the usual dilaton vacuum and the static black hole solutions. We
then include, in section three, the effect of some quantum matter fields and
take into account their one-loop conformal anomaly. A semi-classical treat-
ment reveals then that for any generic solution the metric is always flat. We
also give some exact solutions to the semi-classical equations of motion.

2. The Conformally Invariant Lagrangian

Recently, a model for two dimensional gravity has been put forward [9]. This
is given by the action

\[ S_{\text{flat}} = \int d^2x \sqrt{-g} \left( \omega R + \Lambda \right) , \tag{2.1} \]

where \( \Lambda \) is a cosmological constant and \( \omega(x) \) is a scalar field playing the role
of a dilaton field. The equations of motion resulting from the variation of
\( S_{\text{flat}} \) with respect to \( \omega \) and \( g^{\mu\nu} \) are, respectively, given by

\[ R = 0 \]
\[ g_{\mu\nu} \nabla^2 \omega - \nabla_\mu \nabla_\nu \omega - \frac{1}{2} \Lambda g_{\mu\nu} = 0 \ . \tag{2.2} \]

The first equation shows that the metric is flat (in two dimensions, the van-
ishing of the Ricci scalar is the same as the vanishing of the Riemann tensor).
Taking the trace of the second equation and going to light-cone coordinates,
one finds the following solution for \( \omega \) [14]

\[ \omega(x^+, x^-) = \frac{1}{2} \left[ M - \Lambda (x^+ - x_0^+) (x^- - x_0^-) \right] \]
\[ \eta_{+-} = -1 , \ \eta_{++} = \eta_{--} = 0 , \ x^\pm = \frac{1}{\sqrt{2}} (t \pm r) \ , \tag{2.3} \]

with \( x_0^\pm \) and \( M \) being integration constants.

It is therefore clear that if the "physical metric" is \( g_{\mu\nu} \) then the above
model does not have a black hole solution. However, upon the following

\[ \text{Our conventions for the Riemann and Ricci tensors are those of Misner, Thorne and Wheeler [13].} \]
redefinition of the metric
\[ g_{\mu\nu} = \tilde{g}_{\mu\nu} e^{-2\phi}, \quad \omega \equiv e^{-2\phi}. \] (2.4)

The resulting action is the so-called "string-inspired" model of two dimensional gravity [9]
\[ S_{\text{curved}} = \int d^2x \sqrt{-\tilde{g}} e^{-2\phi} \left( \tilde{R} + 4\tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \Lambda \right), \] (2.5)
where we have used the well-known result
\[ R = e^{2\phi}(\tilde{R} + 2\nabla^2 \phi), \quad \nabla^2 = \frac{1}{\sqrt{-\tilde{g}}} \partial_\mu (\sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\nu). \] (2.6)

It follows that if we take \( \tilde{g}_{\mu\nu} = g_{\mu\nu}/\omega \) as the "physical metric", then the model we started with exhibits a black hole solution. It is crucial to notice that the above transformation on the metric cannot be compensated by a variation of the field \( \omega(x) \) in such a way that the action \( (2.1) \) remains invariant. It is therefore natural to ask what should we take as the physical metric of two dimensional gravity?

In this note we present a model for two dimensional gravity which is not sensitive to any local rescalings of the metric. That is a conformally invariant field theory. Hence, all metrics in this theory are equivalent. The action for such a model is given by
\[ S_{\text{conf}} = \int d^2x \sqrt{-g} \left[ \phi^2 F(\lambda \phi) + A\lambda^2 \partial^2 \phi + B\phi^2 \nabla_\mu \lambda \nabla^\mu \phi + C\phi^2 \nabla_\mu \nabla^\mu \phi + D\phi^2 \right], \] (2.7)
with \( \lambda(x) \) and \( \phi(x) \) being two scalar fields and \( F(\lambda \phi) \) is an arbitrary function of their product. The constant coefficients \( A, B, C \) and \( D \) are determined by requiring conformal invariance of \( (2.7) \). Indeed, our action is invariant under the following conformal transformations
\[ g_{\mu\nu} \to g_{\mu\nu} e^{2\rho(x)} \]
\[ \lambda \to \lambda e^{\rho(x)} \]
\[ \phi \to \phi e^{-\rho(x)} \] (2.8)
provided that
\[ A = -\frac{1}{4}(C - 2D) \]
\[ B = C - D. \] (2.9)
The above action in (2.7) is also attractive from another point of view. In fact, it turns out that if we choose

\[ C = \frac{4}{3}, \quad D = 1, \quad A = \frac{1}{6}, \quad B = \frac{1}{3}, \quad F(\lambda \phi) = \frac{1}{3} \quad (2.10) \]

then our model derives from the action functional for a real massless scalar field \( \phi \) conformally coupled to Einstein gravity in four space-time dimensions upon imposing spherical symmetry [15]. The four dimensional action is written as [8]

\[ S_{(4D)} = \int d^4 x \sqrt{-\hat{g}} \left[ \frac{1}{6} \phi^2 \hat{R} + \hat{g}^{ab} \partial_a \phi \partial_b \phi \right], \quad (2.11) \]

where the indices \( a, b \) range from 1 to 4 and \( \hat{g}_{ab} \) is the four dimensional metric. This action is invariant under the conformal transformations

\[ \hat{g}_{ab}(\hat{x}) \rightarrow \hat{g}_{ab}(\hat{x}) e^{2 \rho(\hat{x})} \]
\[ \phi(\hat{x}) \rightarrow \phi(\hat{x}) e^{-\rho(\hat{x})}. \quad (2.12) \]

Spherical symmetry is imposed through the ansatz:

\[ ds^2 = \hat{g}_{\mu\nu}(x) dx^\mu dx^\nu + \lambda^2(x) S_{ij}(y) dy^i dy^j \]
\[ \phi = \phi(x) \quad, \quad (2.13) \]

where \( g_{\mu\nu}, (\mu, \nu = 1, 2) \), is the metric on a two dimensional manifold in some coordinate patch \( x^\mu \), and \( S_{ij}, (i, j = 3, 4) \), is the metric on the standard two-sphere with coordinates \( y^i \). The dimensionally reduced action is, up to a total derivative and an overall constant volume of the two-sphere, of the form of the model (2.7) with coefficients as given in (2.10).

In what follows and for simplicity, we will be dealing only with the model specified by the coefficients written in (2.10). It is convenient to introduce the new variables \( \theta \) and \( \psi \) as

\[ \phi = e^{-\theta}, \quad \lambda = \psi e^{\theta}. \quad (2.14) \]

The action corresponding to the coefficients (2.10) is then

\[ S_{conf} = \frac{1}{3} \int d^2 x \sqrt{-\hat{g}} \left[ e^{-2\theta} + \frac{1}{2} \psi^2 \hat{R} + \nabla_{\mu} \psi \nabla^\mu \psi - 2 \psi \nabla_{\mu} \theta \nabla^\mu \psi \right]. \quad (2.15) \]
The equations of motion are given by
\[
\begin{align*}
\frac{\delta S}{\delta \theta} &= -\frac{2}{3} e^{-2\theta} + \frac{2}{3} \nabla^\mu (\psi \nabla_\mu \psi) = 0 \\
\frac{\delta S}{\delta \psi} &= \frac{1}{3} \psi R + \frac{2}{3} \psi \nabla^2 \theta - \frac{2}{3} \nabla^2 \psi = 0 \\
\frac{\delta S}{\delta g^\mu_\nu} &= -\frac{1}{6} g^\mu_\nu \left( e^{-2\theta} - \nabla^\alpha \psi \nabla_\alpha \psi - 2\psi \nabla^\alpha \theta \nabla_\alpha \psi - 2\psi \nabla^2 \psi \right) \\
&\quad - \frac{1}{3} (\psi \nabla_\mu \theta \nabla_\nu \psi + \psi \nabla_\nu \theta \nabla_\mu \psi + \psi \nabla_\mu \nabla_\nu \psi) = 0 \quad . \quad (2.16)
\end{align*}
\]

Notice that the equation of motion for \( \theta \) is exactly the trace of the equation of motion for \( g^\mu_\nu \). This means that \( \theta \) is not a dynamical field.

The above equations of motion have the vacuum solution
\[
\begin{align*}
g^\mu_\nu &= \eta^\mu_\nu \\
e^{-\theta} &= \alpha \\
\psi &= \alpha r + \gamma \quad , \quad (2.17)
\end{align*}
\]
where \( \alpha \) and \( \gamma \) are two constants. A static black hole solution is immediately obtained from the dimensionally reduced four dimensional Schwarzschild solution in the presence of a constant scalar field. This is given by
\[
\begin{align*}
ds^2 &= -(1 - \frac{2M}{r}) dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 \\
e^{-\theta} &= \beta \\
\psi &= \beta r \quad , \quad (2.18)
\end{align*}
\]
with \( \beta \) and \( M \) being constants.

In order to describe the metric and the fields \( \theta \) and \( \psi \) due to an infalling shell of matter fields we need to patch in a continuous manner the vacuum solution together with the black hole one across some light-like line. For this, we introduce the null coordinates \((u, v)\) in which the metric takes the form
\[
ds^2 = -2 \left( 1 - \frac{2M}{r} \right) du dv \quad , \quad (2.19)
\]
where \( r \) is now a function of \( u \) and \( v \) and is determined by the equation
\[
\frac{1}{\sqrt{2}} (v - u) = r + 2M \ln \left( \frac{r}{2M} - 1 \right) \quad . \quad (2.20)
\]

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The patching is then carried out across the wordline of the shell \( v = v_0 \), where \( v_0 \) is a constant. However, since equation (2.20) has no solutions in closed form, we were not able to explicitly perform the patching and hence find the matter fields which form the black hole. Nevertheless, we assume in the next section that the black hole is obtained by the collapse of some conformal matter fields.

We can also look for solutions in which the metric is conformally flat

\[
g_{\mu\nu} = \eta_{\mu\nu} e^{2\sigma}.
\]  

(2.21)

In this gauge the equations of motion get modified according to

\[
\begin{align*}
\frac{\delta S}{\delta \theta} &= -\frac{2}{3} e^{-2(\theta-\sigma)} - \frac{2}{3} \partial_+ \partial_- \psi^2 = 0 \\
\frac{\delta S}{\delta \sigma} &= \frac{2}{3} e^{-2(\theta-\sigma)} + \frac{2}{3} \partial_+ \partial_- \psi^2 = 0 \\
\frac{\delta S}{\delta \psi} &= \frac{4}{3} \psi \partial_+ \partial_- \sigma - \frac{4}{3} \psi \partial_+ \partial_- \theta + \frac{4}{3} \partial_+ \partial_- \psi = 0.
\end{align*}
\]  

(2.22)

There are, in addition, two constraints corresponding to the gauge fixing of the metric, \( g_{++} = g_{--} = 0 \),

\[
\begin{align*}
T_{++} &= \frac{2}{3} \psi \partial_+ \psi \partial_+ \sigma - \frac{2}{3} \psi \partial_+ \psi \partial_+ \theta - \frac{1}{3} \psi \partial_+ \partial_+ \psi = 0 \\
T_{--} &= \frac{2}{3} \psi \partial_- \psi \partial_- \sigma - \frac{2}{3} \psi \partial_- \psi \partial_- \theta - \frac{1}{3} \psi \partial_- \partial_- \psi = 0
\end{align*}
\]  

(2.23)

Notice that the equations of motion for \( \theta \) and \( \sigma \) are identical and only the combination

\[
\rho(x) = \sigma(x) - \theta(x)
\]  

(2.24)

appears in the above equations. This is due to the fact that our action (2.15) is conformally invariant.

A general free field solution for the equations of motion and the constraints is found to be given by

\[
\begin{align*}
\psi(x^+, x^-) &= \psi_+(x^+) - \psi_-(x^-) \\
e^{2\rho(x^+, x^-)} &= 2 \frac{d\psi_+}{dx^+} \frac{d\psi_-}{dx^-},
\end{align*}
\]  

(2.25)

where \( \psi_+ \) and \( \psi_- \) are two arbitrary functions. In this solution, however, the conformal factor \( \sigma \) is not determined and can be any expression. Therefore
the curvature can be singular. It is clear that this family of solutions does not include the black hole solution found in (2.18).

In the particular case when $\sigma(x) = \theta(x)$, the dilaton field $\psi$ is linear and is uniquely determined to be

$$\psi(x^+, x^-) = ax^+ - \frac{1}{2a}x^- + b,$$

(2.26)

where $a$ and $b$ are constants of integration. This case, however, is equivalent to getting rid of the $\theta$-dependence in the action (2.15) by redefining $g_{\mu\nu}$. Indeed, by writing

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu}e^{2\theta},$$

(2.27)

the action (2.15) reduces to

$$S = \frac{1}{3} \int d^2x \sqrt{-\bar{g}} \left[ 1 + \frac{1}{2} \psi^2 \bar{R} + \bar{g}^{\mu\nu} \bar{\nabla}_\mu \psi \bar{\nabla}_\nu \psi \right].$$

(2.28)

However, since the action (2.15) is conformally invariant one can easily prevent the above redefinition by shifting at the same time $\theta$ and $\psi$ according to (2.14) and (2.8). As explained earlier, this is not the case in the string inspired model.

3. Including the Back-Reaction

So far we have treated the theory at the classical level only and without any matter fields. A semi-classical effective action might be obtained by including the effect of some quantum matter fields. In this section we are assuming that some of the black hole solutions, especially the one in (2.18), are caused by the collapse of some conformal matter fields. It was shown in [9] that, when there is a number of matter fields, the back-reaction on the geometry due to Hawking radiation can be accounted for by adding the one-loop anomaly term to the classical action. In our case and in the conformal gauge, including the effect of the trace anomaly yields the following effective semi-classical action

$$S = \int d^2x \sqrt{-\eta} \left[ \frac{1}{3} e^{2\sigma} \phi^2 + \frac{1}{6} \lambda^2 \phi^2 R + \frac{2}{3} \phi^2 \lambda \eta^{\mu\nu} \partial_\mu \lambda \partial_\nu \sigma + \frac{2}{3} \phi \lambda^2 \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \sigma 
+ \frac{1}{3} \phi^2 \eta^{\mu\nu} \partial_\mu \lambda \partial_\nu \lambda + \frac{4}{3} \phi \lambda \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \lambda + \lambda^2 \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi 
- \frac{1}{2} \sum_{i=1}^{N} \eta^{\mu\nu} \partial_\mu f^i \partial_\nu f^i + \kappa \eta^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \right],$$

(3.1)
where $f^i$ are $N$ infalling matter fields assumed to form the classical black hole and the term proportional to $\kappa \equiv \frac{\hbar N}{24}$ represents the back-reaction on the metric due to the collapsing $f^i$ fields.\(^4\)

Our effective Lagrangian can be regarded as a non-linear sigma model on a three dimensional target space (the directions along the $f^i$ fields are flat). The metric on this target space is determined by the kinetic term for the fields $\lambda, \phi$ and $\sigma$. This is given by

$$G_{rs} = \begin{pmatrix}
\frac{1}{3} \phi^2 & \frac{2}{3} \lambda \phi & \frac{1}{3} \lambda \phi^2 \\
\frac{2}{3} \lambda \phi & \lambda^2 & \frac{1}{3} \lambda^2 \phi \\
\frac{1}{3} \lambda \phi^2 & \frac{1}{3} \lambda^2 \phi & \kappa
\end{pmatrix}, \quad (3.2)
$$

The determinant of this metric is

$$\det(G_{rs}) = -\frac{1}{9} \kappa \lambda^2 \phi^2 = -\frac{1}{9} \kappa \psi^2. \quad (3.3)$$

Therefore, there is no degeneration of the target space metric and the kinetic term can be always inverted. This allows, in particular, for the carrying out of a weak field perturbation theory in the amplitude of the field $f^i$ in order to solve, for example, the full equations when $\kappa \neq 0$. This is in complete contrast to the string-inspired model written in (2.5). There, in the conformal gauge, the target space metric has a determinant of $4e^{-2\phi}(\kappa - e^{-2\phi})$ which is degenerate when the dilaton takes the value $e^{-2\phi} = \kappa$. This fact was responsible for the singular solutions of the equations of motion of the string-inspired model [10,11].

Now, using the variables $\theta$ and $\psi$, the semi-classical equations of motion are given by

$$\frac{\delta S}{\delta \theta} = -\frac{2}{3} e^{-2(\theta - \sigma)} - \frac{2}{3} \partial_+ \partial_- \psi^2 = 0$$
$$\frac{\delta S}{\delta \sigma} = \frac{2}{3} e^{-2(\theta - \sigma)} + \frac{2}{3} \partial_+ \partial_- \psi^2 + 4\kappa \partial_+ \partial_- \sigma = 0$$
$$\frac{\delta S}{\delta \psi} = \frac{4}{3} \psi \partial_+ \partial_- \sigma - \frac{4}{3} \psi \partial_+ \partial_- \theta + \frac{4}{3} \partial_+ \partial_- \psi = 0$$
$$\frac{\delta S}{\delta f^i} = \partial_+ \partial_- f^i = 0. \quad (3.4)$$

The modified constraints are

$$T_{++} = \frac{2}{3} \psi \partial_+ \psi \partial_+ \sigma - \frac{2}{3} \psi \partial_+ \psi \partial_+ \theta - \frac{1}{3} \psi \partial_+ \partial_+ \psi$$

\(^4\)We are assuming that the Liouville cosmological constant can be set to zero.
\[ T_{--} = \frac{2}{3} \psi \partial_- \psi \partial_- \sigma - \frac{2}{3} \psi \partial_- \psi \partial_- \theta - \frac{1}{3} \psi \partial_- \partial_- \psi + \kappa \left( \partial_- \sigma \partial_- \sigma + t_-(x^-) \right) - \frac{1}{2} \sum_{i=1}^N \partial_- f_i \partial_- f_i = 0. \]  

The functions \( t_+(x^+) \) and \( t_-(x^-) \) are due to the non-locality of the anomaly term added to the classical action and are fixed by the asymptotic physical boundary conditions [9].

The first thing to notice here is that the equations of motion for \( \theta \) and \( \sigma \) imply that

\[ \kappa \partial_+ \partial_- \sigma = 0. \]  

Therefore, the Ricci scalar

\[ R = 4e^{-2\sigma} \partial_+ \partial_- \sigma \]  

must vanish when \( \kappa \neq 0 \). Hence, any generic solution in the conformal gauge is flat. Consequently, perturbing the vacuum solution (2.17) by some infalling matter fields does not produce any singularities.

The solutions for the \( \sigma \) and \( f^i \) fields are straightforward to find

\[ \sigma(x^+, x^-) = \sigma_+(x^+) + \sigma_-(x^-) \]
\[ f^i(x^+, x^-) = f^i_+(x^+) + f^i_-(x^-). \]  

A free field solution for the fields \( \psi \) and \( \theta \) is given by

\[ \psi(x^+, x^-) = h_+(x^+) - h_-(x^-) \]
\[ \theta(x^+, x^-) = \sigma_+(x^+) + \sigma_-(x^-) - \frac{1}{2} \ln \left( \frac{2}{dx^+ dx^-} \right). \]  

The constraints then lead to

\[ \sigma_+(x^+) = \pm \frac{1}{\sqrt{\kappa}} \int_{x^-}^{x^+} dy^+ \left( \frac{1}{2} \sum_{i=1}^N \left[ \frac{df^i_+(y^+)}{dy^+} \right]^2 - \kappa t_+(y^+) \right)^{\frac{1}{4}} \]
\[ \sigma_-(x^-) = \pm \frac{1}{\sqrt{\kappa}} \int_{x^-}^{x^+} dy^- \left( \frac{1}{2} \sum_{i=1}^N \left[ \frac{df^i_-(y^-)}{dy^-} \right]^2 - \kappa t_-(y^-) \right)^{\frac{1}{4}}. \]  

A free field solution for the fields \( \psi \) and \( \theta \) is given by
In conclusion we have presented a model for two dimensional gravity con-
formally coupled to two dilaton fields. At the classical level we found so-
lutions corresponding to black holes. However, we were not able to prove
that these singular solutions are formed by gravitational collapse of a shell
of matter fields. We assumed, nevertheless, that such matter fields do exist
and constructed a semi-classical theory. We then showed that all the singular
configurations of the classical theory are absent in this semi-classical theory.

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