Iwasawa Main Conjecture for Non-Ordinary Modular Forms

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Abstract

Let \( p > 2 \) be a prime. Under mild assumptions, we prove the Iwasawa main conjecture of Kato, for modular forms with general weight and conductor prime to \( p \). This generalizes an earlier work of the author on supersingular elliptic curves.

1 Introduction

Let \( p > 2 \) be a prime. The history of Iwasawa theory dates back to the 1950’s, when Iwasawa studied the \( p \)-part of class groups of cyclotomic extensions of \( \mathbb{Q} \) of \( p \)-power degree. It turns out that one can pass to the cyclotomic \( \mathbb{Z}_p \) extension field \( \mathbb{Q}_\infty \) and get a finitely generated torsion module structure over the so called Iwasawa algebra \( \Lambda := \mathbb{Z}_p[[\Gamma]] \), where \( \Gamma \) is defined to be \( \text{Gal}(\mathbb{Q}_\infty / \mathbb{Q}) \). In 1960’s Kobota and Leopoldt discovered an analytic counterpart of Iwasawa’s \( \Lambda \)-module, namely the \( p \)-adic \( L \)-function which packages together the algebraic part of special values of \( L \)-functions of \( \mathbb{Q} \) twisted by finite order cyclotomic characters. An old philosophy, which is explicitly formulated as the Iwasawa main conjecture in this context, is that such special \( L \)-values should give the size of some class groups with corresponding actions of the Galois group \( \Gamma \). Later in the 1970’s, Barry Mazur made a key observation that Iwasawa’s idea can be applied to elliptic curves, or more generally, abelian varieties. This has important application to the Birch and Swinnerton-Dyer conjecture. Later on the formulation has been vastly generalized to motives, including the case of modular forms, by Greenberg and other people.

Towards a proof there are in principal two approaches. One is using Euler system. This idea originated in the work of Thaine in late 1980’s and later on axiomized by Rubin in a handful of different contexts. The Euler system method is especially useful in giving the upper bound of the arithmetic objects (namely Selmer groups). With the help of the class number formula, this is enough to establish the full equality in Iwasawa main conjecture for certain Hecke characters. The other approach is to use congruences between modular forms. Such idea first appeared in a work of Ribet in early 1980’s (called the Ribet’s lemma). This method is used to deduce the lower bound of Selmer groups and is employed by Mazur-Wiles and Wiles to prove the Iwasawa main conjecture for totally real fields. However to study Iwasawa theory for motives of rank larger than one, where one does not have the class number formula, one needs to apply both the Euler system method and the modular form method to give the full equality. This is illustrated by the recent work of Kato and Skinner-Urban in the proof of Iwasawa main conjecture for modular forms ordinary at \( p \). Kato proved the upper bound for Selmer groups by constructing an Euler system using \( K \)-theory of modular curves, while Skinner-Urban used modular form method on the rank four unitary group...
U(2, 2). Now we discuss some details about Kato and Skinner-Urban’s work, which is closely related to the present paper.

**Strict Selmer Groups**

Let

$$f = \sum_{n=1}^{\infty} a_n q^n$$

be a normalized cuspidal eigenform for GL$_2$/\mathbb{Q}$ with even weight $k$, trivial character and prime to $p$ conductor $N$. By work of Shimura, Deligne, Langlands and others, one can associate a two dimensional irreducible Galois representation $\rho_f : G_\mathbb{Q} \to$ GL$_2(\mathcal{O}_L)$. Here $L$ is a finite extension of \mathbb{Q}$_p$ and $\mathcal{O}_L$ is the integer ring of it. We choose the $L$ so that it contains all the coefficients of $\Gamma_0(N)$ cusp forms. This Galois representation is determined by requiring that for all primes $\ell \nmid pN$,

$$\text{tr}\rho_f(\text{Frob}_\ell) = a_\ell.$$ Write $T_f$ for the representation space for $\rho_f$. The $T_f|_{G_\mathbb{Q}_p}$ is crystalline in the sense of Fontaine with Hodge-Tate weights $(0, k-1)$. (We use the convention that the cyclotomic character has Hodge-Tate weight 1.) In this paper we define the Iwasawa algebra $\Lambda = \mathcal{O}_L[[\Gamma]]$. Let $S$ be the set of primes dividing $pN$. In this paper we will write $H^i(\mathbb{Q}^S/\mathbb{Q}, -)$ or $H^i(\mathbb{Q}^S, -)$ for the $i$-th cohomology of the Galois group of $\mathbb{Q}$ unramified outside $S$. We define

$$H^i_{\text{cl, iw}}(\mathbb{Q}^S, T_f(-\frac{k-2}{2})) := \varprojlim_n H^1(\mathbb{Q}_n^S, T_f(-\frac{k-2}{2}))$$

where $\mathbb{Q}_n$ is running over all intermediate field extensions between $\mathbb{Q}_\infty$ and $\mathbb{Q}$. In this paper we will often write $T$ for $T_f(-\frac{k-2}{2})$. Kato proved that it is a torsion-free rank one module over $\Lambda$, and defined a zeta element $z_{\text{Kato}}$ in it. On the arithmetic side, we define the Selmer group

$$\text{Sel}_{\text{str}, \mathbb{Q}_n}(f) := \ker\{H^1(\mathbb{Q}_n^S, T_f(-\frac{k-2}{2}) \otimes_{\mathcal{O}_L} \mathcal{O}_L^*) \to \prod_{v \mid p} H^1(\mathbb{Q}_{n,v}, T_f(-\frac{k-2}{2}) \otimes_{\mathcal{O}_L} \mathcal{O}_L^*) \}$$

$$\times H^1(\mathbb{Q}_{n,p}, T_f(-\frac{k-2}{2}) \otimes_{\mathcal{O}_L} \mathcal{O}_L^*))\}.$$ Here the superscript * means Pontryagin dual. We define

$$H^1_f(\mathbb{Q}_{n,v}, V_f(-\frac{k-2}{2}) := \ker\{H^1(\mathbb{Q}_{n,v}, V_f(-\frac{k-2}{2})) \to H^1(I_{n,v}, V_f(-\frac{k-2}{2}))\}$$

and define $H^1_f(\mathbb{Q}_{n,v}, T_f(-\frac{k-2}{2}) \otimes_{\mathcal{O}_L} \mathcal{O}_L^*)$ as the image of $H^1_f(\mathbb{Q}_{n,v}, V_f(-\frac{k-2}{2}))$.

Define also

$$\text{Sel}_{\text{str}, \mathbb{Q}_\infty}(f) := \varprojlim_n \text{Sel}_{\text{str}, \mathbb{Q}_n}(f)$$

and

$$X_{\text{str}} := \text{Sel}_{\text{str}, \mathbb{Q}_\infty}(f)^*.$$ Kato formulated the Iwasawa main conjecture as

**Conjecture 1.1.**

$$\text{char}_\Lambda X_{\text{str}} = \text{char}(\frac{H^1_{\text{cl, iw}}(\mathbb{Q}^S, T_f(-\frac{k-2}{2}))}{\Lambda z_{\text{Kato}}}).$$
Remark 1.2. In [28] Kato used the $\Lambda$-module $H^2(Q^S, T_f(-\frac{k}{2}) \otimes \Lambda)$, which is isomorphic to $X_{str}$ here (see [33, Page 12]).

Kato proved the following

Theorem 1.3. We have

$$\text{char}_{\Lambda \otimes \mathbb{Q}_p}(X_{str}) \supseteq \text{char}_{\Lambda \otimes \mathbb{Q}_p}(\frac{H^1_{\text{Iw}}(Q^S, T_f(-\frac{k-2}{2}))}{\Lambda_{2 \text{Kato}}}).$$

If moreover $\text{Im}(G_Q)$ contains $\text{SL}_2(\mathbb{Z}_p)$, then the above containment is true as ideals of $\Lambda$.

In the case when $f$ has CM the theorem depends on the result of Karl Rubin. Under some hypothesis Skinner and Urban [48] proved the other side containment in the Iwasawa main conjecture, in the case when the form $f$ is ordinary at $p$ (meaning $a_p$ is a $p$-adic unit). If $f$ is not ordinary the situation is more complicated. Our goal in this paper is to prove the lower bound for Selmer groups for modular forms with $p \nmid N$.

We make the following assumption

(Irred) The residual Galois representation $\bar{T}_f$ is irreducible over $G_{\mathbb{Q}_p}$.

This assumption is made to ensure that the local Iwasawa cohomology group at $p$ is free over the Iwasawa algebra, which simplifies the argument. It seems plausible to relax such assumption by more refined argument. Our main theorem is the following

Theorem 1.4. Assume $2 | k$, $p \nmid N$, (Irred), $f$ has trivial character, and that $\bar{T}_f|_{Q(\zeta_p)}$ is absolutely irreducible. Assume moreover that the $p$-component of the automorphic representation $\pi_f$ is a principal series representation with distinct Satake parameters. If there is an $\ell | N$ such that $\pi_{\ell}$ is the Steinberg representation twisted by $\chi_{\text{ur}}$ for $\chi_{\text{ur}}$ being the unramified character sending $p$ to $(-1)^{\frac{k}{2}}p^{\frac{k}{2}-1}$, then

- We have
  $$\text{char}_{\Lambda|1/p}(X_{str}) = \text{char}_{\Lambda|1/p}(\frac{H^1_{\text{cl, Iw}}(Q^S, T_f(-\frac{k-2}{2}))}{\Lambda_{2 \text{Kato}}}).$$

- If moreover the image of the Galois representation $\rho_f$ contains $\text{SL}_2(\mathbb{Z}_p)$. Then
  $$\text{char}_{\Lambda}(X_{str}) = \text{char}_{\Lambda}(\frac{H^1_{\text{cl, Iw}}(Q^S, T_f(-\frac{k-2}{2}))}{\Lambda_{2 \text{Kato}}}).$$

The assumption on the Satake parameter is conjecturally automatic. The assumption on the $\ell$ on the second part is due to using results of Hsieh in [21] and can probably be weakened if one has a general weight version of [3]. (It is used to ensure the local root numbers of $f$ over some auxiliary quadratic field are $+1$.) A corollary of part two is that if $L(f, \frac{k}{2}) \neq 0$ then the $p$-part of the Tamagawa number conjecture is true (see Corollary 4.35), for all primes $p < k$ (i.e. within the Fontaine-Laffaille range). The reason for the low weight assumption is we need an integral comparison isomorphism between étale cohomology and crystalline cohomology. We note that there is recent progress by Bhargav-Morrow-Scholze [7] on integral $p$-adic Hodge theory (for trivial coefficient local systems). See Remark 4.36 for more details of the relations to our work.
The first general result in the non-ordinary case is the Greenberg-Iwasawa main conjecture. In application we suppose that $p$ splits as $v_0\bar{v}_0$. Let $\Gamma_K$ be the Galois group of $K_\infty/K$ for $K_\infty$ being the $\mathbb{Z}_p$ extension of $K$. Write $\Gamma^\pm$ for the rank one over $\mathbb{Z}_p$ submodules of $\Gamma_K$ on which the action of the complex conjugation $c$ is given by $\pm 1$. Then any form $g$ with complex multiplication by $K$ is ordinary at $p$. Suppose the weight of $g$ is greater than the weight of $f$. Then the Iwasawa theory of the Rankin-Selberg product $f \otimes g$ has the same form as ordinary forms, since it satisfies the Panchishkin’s condition. This makes the corresponding Iwasawa main conjecture more accessible. Moreover this Rankin-Selberg product is closely related to the Iwasawa theory of the original modular form $f$. We first give the precise formulation of the main conjecture. In application we suppose $g$ is the Hida family corresponding to characters of $\Gamma_K$ and we identify the Galois representation of $g$ with the induced representation from $G_K$ to $G_Q$ of the natural character $\Psi$ of $\Gamma_K \hookrightarrow \mathbb{Z}_p[[\Gamma_K]]^\times$. On the arithmetic side we defined

$$\text{Sel}_{K,f}^{\text{Gr}} = \ker \{ H^1(K^S, T_f(-\frac{k-2}{2}) \otimes \Lambda^*(\Psi)) \rightarrow \prod_{v | p} H^1(K_{v}, T_f(-\frac{k-2}{2}) \otimes \Lambda^*(\Psi)) \times H^1(K_{\bar{v}_0}, T_f(-\frac{k-2}{2}) \otimes \Lambda^*(\Psi)) \}. $$

$$X_{K,f}^{\text{Gr}} := (\text{Sel}_{K,f}^{\text{Gr}})^*. $$

On the analytic side, there is a Greenberg $p$-adic $L$-function $L_{f,K}^{\text{Gr}} \in \mathcal{O}_L[[\Gamma_K]]$ with interpolation property given in Proposition 4.1. Here we write $\mathcal{O}_L^{ur}$ for the completion of the maximal unramified extension of $\mathcal{O}_L$. The Greenberg-Iwasawa main conjecture is the following:

**Conjecture 1.6. (Greenberg Main Conjecture)** The $X_{K,f}^{\text{Gr}}$ is a torsion $\mathcal{O}_L[[\Gamma_K]]$-module. Moreover

$$\text{char}_{\mathcal{O}_L^{ur}[[\Gamma_K]]}(X_{K,f}^{\text{Gr}} \otimes_{\mathcal{O}_L} \mathcal{O}_L^{ur}) = (L_{f,K}^{\text{Gr}}).$$
We will call this conjecture (GMC) in this paper. In our case the fact that $\mathcal{L}^{Gr}_{f,K}$ is integral is explained in the text. Under some hypothesis we proved in [51], [52] that if $f$ has weight two, then up to powers of $p$ we have

$$\text{char}_{\mathcal{O}^\text{ur}_L[[\Gamma_K]]}(X_{K,f}^{Gr} \otimes \mathcal{O}_L \mathcal{O}^\text{ur}_L) \subseteq (\mathcal{L}^{Gr}_{f,K}).$$

After this is proved, in [51] the author developed some local $\pm$ theory in the similar style of Kobayashi and B.D.Kim, and used the explicit reciprocity law for Beilinson-Flach element and Poitou-Tate exact sequence to deduce Conjecture 1.1 from Conjecture 1.6.

To prove the main theorem in this paper for general modular forms of any even weight, the difficulty is two-fold. First of all, the author only proved Greenberg’s main conjecture when $f$ has weight two. The obstacle is that if $f$ has higher weight then one needs to compute the Fourier-Jacobi expansion for vector valued Eisenstein series, which seems formidable. Secondly we do not have an explicit local theory as in the $\pm$ case (except in the special case when $a_p = 0$, and for elliptic curves over $\mathbb{Q}$ but $a_p \neq 0$ by Florian Sprung), while the work [52] used such theory in a crucial way.

Our first result is the following theorem on one containment of Conjecture 1.6, which generalizes the result in [52] to forms $f$ of any even weight $k$.

**Theorem 1.7.** Suppose there is at least one rational prime $q$ where the automorphic representation $\pi_f$ associated to $f$ is not an unramified principal series representation. Assume moreover that the $p$-component of the automorphic representation $\pi_f$ is a principal series representation with distinct Satake parameters, and that the residual Galois representation $\bar{\rho}_f$ is irreducible over $G_{K(\zeta_p)}$. Then up to primes which are pullbacks of primes in $\mathcal{O}_L[[\Gamma^+]]$ we have

$$\text{char}_{\mathcal{O}^\text{ur}_L[[\Gamma_K]]}(X_{K,f}^{Gr} \otimes \mathcal{O}_L \mathcal{O}^\text{ur}_L) \subseteq (\mathcal{L}^{Gr}_{f,K}).$$

Moreover if for each $\ell|N$ non-split in $K$, we have $\ell||N$ is ramified in $K$ and $\pi_\ell$ is the Steinberg representation twisted by $\chi_{ur}^{\frac{k}{2}}$ for $\chi_{ur}$ being the unramified character sending $p$ to $(-1)^{\frac{k}{2}} p^{\frac{k}{2}-1}$. Then the whole containment above is true.

The last part is by appealing to the result of Hsieh [21] on the vanishing of anti-cyclotomic $\mu$-invariant of $\mathcal{L}^{Gr}_{f,K}$. To prove this theorem, we use the full strength of our joint work with Eischen [14] on constructing vector-valued Klingen Eisenstein families on $U(3,1)$, from pullbacks of nearly holomorphic Siegel Eisenstein series on $U(3,3)$. We combine our earlier work in [51], [52] on the $p$-adic property for Fourier-Jacobi coefficients with the general theory of Ikeda to show that the Fourier-Jacobi coefficient of nearly holomorphic Siegel Eisenstein series can be expressed as finite sum of products of Eisenstein series and theta functions on the Jacobi group containing $U(2,2)$. We thank Ikeda for showing us the simple argument using lowest weight representations (In the scalar valued case the Siegel Eisenstein series on $U(3,3)$ is holomorphic, and the local Fourier-Jacobi coefficient at the Archimedean place can be explicitly expressed as the product of a Siegel section on $U(2,2)$ and a Schwartz function. In general it is difficult to compute it explicitly.) We fix one Archimedean weight and vary the $p$-adic nebentypus in families. Instead of computing the Archimedean Fourier-Jacobi integral, we can use a conceptual argument to factor out a finite sum of Archimedean integrals for it and prove the factor is non-zero. After this we can apply the techniques we developed in our previous work to prove that certain Fourier-Jacobi coefficient is co-prime to the $p$-adic $L$-function we study. Along the way we determine the constant coming from the local pullback integrals for doubling methods at Archimedean places (which seems hard to
compute directly), by comparing our construction with Hida’s construction using Rankin-Selberg method. This is necessary for the proof. We also remove the square-free conductor assumption for $f$ in our previous works, by doing some local triple product computation allowing some supercuspidal components.

In the second step, to prove the main theorem, we first prove the result after inverting $p$. Our idea is to use the analytic Iwasawa theory of Pottharst and Iwasawa theory for $(\varphi, \Gamma)$-modules (upgraded to a two variable setting), in the context of Nekovar’s Selmer complexes [12]. It turns out that Pottharst’s triangline-ordinary theory works in similar way as the ordinary case when working with the more flexible analytic Iwasawa theory. In the two-variable setting there are subtleties to take care of – for example there is a finite set of height one primes where the regulator map vanishes. Also we need to compare differently constructed analytic $p$-adic $L$-functions – although they agree on all arithmetic points, however these do not uniquely determine the analytic functions themselves.

In the third step, after this, we only need to study powers of $p$. After carefully studying the control theorem, we only need to compute the cardinality (which is $< \infty$) of the Selmer group for $T_f(-\frac{k-2}{2})$ twisted by some finite order cyclotomic character. We use a different idea and work directly with Kato’s zeta element. We avoid the search for nice integral local theory analogous to the $\pm$ theory. Instead we take one “generic” finite order cyclotomic character twist of $T_f$, and consider deformations of it along the one variable family which corresponds to the $\mathbb{Z}_p$-extension of $K$ that is totally ramified at $v_0$ and unramified at $v_0$. In this family we do have a nice integral local theory at $v_0$. The key fact is a uniform boundedness result for Bloch-Kato’s logarithm map for families of unramified twists. We prove this by a careful study of Fontaine’s rings $B_{dR}$ and $A_{cris}$. Fortunately this theory is enough for our purposes. Another key trick is to combine the arithmetic point with its dual points together in the argument, so that we can use Tate duality and deduce Kato’s main conjecture without computing the Tamagawa number at $p$ at generic arithmetic points.

Remark 1.8. In the argument of step three above, we need to use global duality to study some non-trivial pseudo-null Iwasawa modules of certain dual Selmer groups, a phenomenon not seen in classical Iwasawa theory.

In fact our argument combing the last two steps implies a slightly stronger conclusion, that in the case when we do not know $\text{Im}(G_{Q_p}) \supseteq \text{SL}_2(\mathbb{Z}_p)$, we still have the size of the strict Selmer group is bounded below by the index of $z_{Kato}$ in the integral Iwasawa cohomology (i.e. can treat powers of $p$ for one side divisibility) in the case when the zeta element is inside the latter. (If not then this statement is empty).

We managed to make our proof to work as general as possible. In fact a lot of the argument can be applied to the case when $f$ has ramification at $p$ (most interestingly when $\pi_f$ is supercuspidal at $p$). However the essential difficulty in proving the main conjecture in this case is to prove the reciprocity law for Beilinson-Flach element at the arithmetic points corresponding to $L(f, \chi, \frac{s}{2})$ for finite order characters $\chi$ of $\Gamma$. These correspond to Rankin-Selberg products of $f$ with weight one forms $g$, in which geometry does not give the required formula directly. In the crystalline case, Kings-Loeffler-Zerbes achieved this by constructing a big regulator map interpolating the Bloch-Kato $\text{exp}^*$ map and log map, and do some analytic continuation. In the non-crystalline case it seems very hard to work out such a big regulator map and the interpolation formula explicitly. One also needs to understand certain $p$-adic $L$-function for such $f$, which has infinite slope. It seems there are some ongoing work on this using explicit description of $p$-adic local Langlands correspondence, follows an early idea of M.Emerton. We hope experts in such areas can shed some light on such problems. (And this paper provides more motivation for such investigations).
Recently, Olivier Fouquet informed us that in view of the Zariski density of crystalline points in the Hecke algebra for \( p \)-adic modular forms (see Emerton [15, Proposition 5.4.1 and Remark 5.4.3]), it seems plausible to use his method of studying equivariant Tamagawa number conjectures over Hecke algebras to deduce Iwasawa main conjectures for modular forms in all cases (allowing very general ramification at \( p \)), from the result of this paper as a key input. This may enable us to complete the picture for the problem of Kato’s main conjectures.

The paper is organized as follows: in Section 2 we recall and develop some \( p \)-adic local theory needed for the argument. One key result is to study the Iwasawa theory for the \( \mathbb{Z}_p \)-extension of \( K \) which is unramified in \( v_0 \) and totally ramified in \( \bar{v}_0 \). In Section 3 we prove the Greenberg type main conjecture for general weight modular forms \( f \), by studying the local integral at the Archimedean place. In Section 4 we give the proof of the main theorem of this paper.

Acknowledgement We would like to thank Bhargav Bhatt, Tamotsu Ikeda, David Loeffler, Antonio Lei, Ruochuan Liu, Matthew Morrow, Jonathan Pottharst, Christopher Skinner, Richard Taylor, Michael Woodbury and Sarah Zerbes for useful communications. We would also like to thank Manjul Bhargava for his constant interest and encouragement on part of this work. The author is partially supported by the Chinese Academy of Science grant Y729025EE1, NSFC grant 11688101, 11621061 and an NSFC grant associated to the â€œRecruitment Program of Global Expertsâ€. Notations: In this paper we often write \( T = T_f(-\tfrac{k^2}{2}) \) for the Galois representation \( T_f \) associated to \( f \). We write \( \mathbb{Q}_\infty \) as the cyclotomic \( \mathbb{Z}_p \)-extension of \( \mathbb{Q} \). Let \( \mathcal{K} \) be a quadratic imaginary extension of \( \mathbb{Q} \) where \( p \) splits as \( v_0 \bar{v}_0 \). Let \( \mathcal{K}_\infty \) be the \( \mathbb{Z}_p \)-extension of \( \mathcal{K} \). Let \( \mathbb{Q}_{\infty, p}^\text{ur} \) be the unramified \( \mathbb{Z}_p \)-extension of \( \mathbb{Q}_{\infty, p} \). Let \( \Gamma = \Gamma_\mathbb{Q} = \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \), \( \Gamma_\mathcal{K} = \text{Gal}(\mathcal{K}_\infty/\mathcal{K}) \) and \( \Gamma_1 = \text{Gal}(\mathbb{Q}_{\infty, p}/\mathbb{Q}_p) \). Let \( L/\mathbb{Q}_p \) be a finite extension as before with integer ring \( \mathcal{O}_L \). Define \( \Lambda = \mathcal{O}_L[[\Gamma]] \) and \( \Lambda_\mathcal{K} = \mathcal{O}_L[[\Gamma_\mathcal{K}]] \). Let \( \mathcal{K}_v^{v_0} (\mathcal{K}_{\bar{v}_0}^{\bar{v}_0}) \) be the \( \mathbb{Z}_p \)-extension of \( \mathcal{K} \) unramified outside \( v_0 \) (unramified outside \( \bar{v}_0 \), respectively). Define \( \Gamma_v = \text{Gal}(\mathcal{K}_v^{v_0}/\mathcal{K}) \) and \( \Gamma_{\bar{v}_0} = \text{Gal}(\mathcal{K}_{\bar{v}_0}^{\bar{v}_0}/\mathcal{K}) \). We write \( \epsilon \) for the cyclotomic character.

2 \((\varphi, \Gamma)\)-modules and Iwasawa Theory

2.1 Iwasawa Cohomology Groups

Lemma 2.1. We have \( H^1(\mathbb{Q}_p, T/pT) \simeq \mathbb{F}_p^2 \).

Proof. It follows from the local Euler characteristic formula that

\[
\frac{\sharp H^0(\mathbb{Q}_p, T/pT) \cdot \sharp H^2(\mathbb{Q}_p, T/pT)}{\sharp H^1(\mathbb{Q}_p, T/pT)} = p^2.
\]

By (Irred) and Tate local duality we have \( H^0(\mathbb{Q}_p, T/pT) = H^2(\mathbb{Q}_p, T/pT) = 0 \). So \( \sharp H^1(\mathbb{Q}_p, T/pT) = p^2 \). Thus \( H^1(\mathbb{Q}_p, T/pT) \simeq \mathbb{F}_p^2 \). \( \square \)

Lemma 2.2. We have \( H^1(\mathbb{Q}_p, T) \simeq \mathbb{Z}_p^2 \).

Proof. Using the local Euler characteristic formula we have

\[
\sharp H^1(\mathbb{Q}_p, T/p^nT) = p^{2n}.
\]
From the cohomology long exact sequence of

\[ 0 \longrightarrow T \xrightarrow{xP^n} T \longrightarrow T/p^nT \longrightarrow 0 \]

we see that \( H^1(\mathbb{Q}_p, T) \) is a torsion-free \( \mathbb{Z}_p \)-modules and that \( H^1(\mathbb{Q}_p, T)/p^nH^1(\mathbb{Q}_p, T) \simeq H^1(\mathbb{Q}_p, T/p^nT) \). Since \( H^2(\mathbb{Q}_p, T) = 0 \) by (Irred) and Tate local duality. These altogether gives the lemma. \( \square \)

We define the classical Iwasawa cohomology \( H^1_{cl, Iw}(\mathbb{Q}_p, \infty, T) \) to be the inverse limit with respect to the co-restriction map.

\[ \lim_{\longrightarrow} H^1(\mathbb{Q}_{p,n}, T). \]

**Lemma 2.3.** We have

\[ H^1_{cl, Iw}(\mathbb{Q}_p, \infty, T) \simeq \Lambda \]

and

\[ H^1_{cl, Iw}(\mathbb{Q}_p, \infty, T) \simeq \mathbb{Z}_p[[\Gamma_1]]. \]

**Proof.** From (Irred) we know \( H^0(K, T[p]) = 0 \) for any \( K \) Abelian over \( \mathbb{Q}_p \). As above for any \( 0 \neq f \in \Lambda \) it follows from the cohomological long exact sequence of

\[ 0 \longrightarrow T \otimes \Lambda \xrightarrow{xf} T \otimes \Lambda \longrightarrow T \otimes \Lambda/f \Lambda \longrightarrow 0 \]

and (Irred) that \( H^1_{cl, Iw}(\mathbb{Q}_p, \infty, T) \) is a torsion-free \( \Lambda \)-module, and that \( H^1_{cl, Iw}(\mathbb{Q}_p, \infty, T)/fH^1_{cl, Iw}(\mathbb{Q}_p, \infty, T) \simeq H^1(\mathbb{Q}_p, T \otimes \Lambda/f \Lambda) \). So \( H^1_{cl, Iw}(\mathbb{Q}_p, \infty, T)/(T, p)H^1_{cl, Iw}(\mathbb{Q}_p, \infty, T) \simeq H^1(\mathbb{Q}_p, T/pT) \). By Nakayama’s lemma \( H^1_{cl, Iw}(\mathbb{Q}_p, \infty, T) \) is a \( \Lambda \)-module generated by two elements and is thus a quotient of \( \Lambda \oplus \Lambda \). Taking \( \omega_n \) as above, we see that \( H^1(\mathbb{Q}_{p,n}, T) \) is a \( \Lambda_n \)-module generated by two elements. As before using Euler local characteristic formula we can show that the \( \mathbb{Z}_p \)-rank of \( H^1(\mathbb{Q}_{p,n}, T) \) is the same as that of \( \Lambda_n \oplus \Lambda_n \). So the two generators of \( H^1_{cl, Iw}(\mathbb{Q}_p, \infty, T) \) gives an isomorphism \( H^1(\mathbb{Q}_p, \infty, T) \simeq \Lambda_n \oplus \Lambda_n \). Take inverse limit we get \( H^1_{cl, Iw}(\mathbb{Q}_p, \infty, T) \simeq \Lambda \oplus \Lambda \). The second statement follows similarly. \( \square \)

2.2 (\( \varphi, \Gamma \))-modules

We first recall some standard notions of \( p \)-adic Hodge theory. Let \( \hat{\mathbb{Q}}_p \) be the \( p \)-adic completion of \( \mathbb{Q}_p \) and \( \mathcal{O}_{\hat{\mathbb{Q}}_p} \) be the elements whose \( p \)-adic valuations are less than or equal to 1. Fix once for all \( p^n \)-th root of unity \( \zeta_{p^n} \) with \( \zeta_{p^n+1} = \zeta_{p^n} \). Let \( \hat{E}^+ := \lim_{\longrightarrow} \mathcal{O}_{\hat{\mathbb{Q}}_p}/p \) with respect to the \( p \)-th power map as the transition map. We define a valuation \( v \) on \( \hat{E}^+ \) as follows. Suppose \( x = (x_n)_n \) the define \( v(x) = \lim_n p^n v(x_n) \). Here the valuation \( v \) is normalized so that \( v(p) = 1 \) and for \( n \) large we take a lifting of \( x_n \) in \( \mathcal{O}_{\hat{\mathbb{Q}}} \) and use its valuation to define \( v(x_n) \). This valuation on \( \hat{E}^+ \) is easily seen to be well defined. Define \( \hat{E} \) as the fraction field of \( \hat{E}^+ \). Let \( \varepsilon := (\zeta_{p^n})_{n \geq 0} \in \hat{E}^+ \) (\( \bar{x} \) being the image of \( x \)). Let \( \hat{A}^+ := W(\hat{E}^+) \) and \( \hat{A} := W(\hat{E}) \) be the ring of Witt vectors of \( \hat{E}^+ \) and \( \hat{E} \) respectively. Write \([x]\) for the Teichmuller lift of \( x \in \hat{E}^+ \) or \( \hat{E} \). There is a surjective ring homomorphism \( \theta : \hat{A}^+ \rightarrow \mathcal{O}_{\hat{\mathbb{Q}}_p} \) with \( \theta([x_n]) := \lim_{n \rightarrow \infty} x_{p^n} \). Define \( B_{dR}^+ := \lim_{\longrightarrow} \hat{A}^+ [1/p]/(\ker(\theta)[1/p])^n \). Define \( t = \log([\varepsilon]) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} ([\varepsilon] - 1)^n \in B_{dR}^+ \). Then \( B_{dR}^+ \) is a discrete valuation ring with maximal ideal \( (t) \) and residue field \( \mathbb{C}_p \).
For $0 \leq r \leq s < \infty$, $r, s \in \mathbb{Q}$, let $\tilde{A}^{[r,s]}$ be the $p$-adic completion of $\tilde{A}^+[\frac{-1}{p}, \frac{[s-1]p}{p}]$. Let $\tilde{B}^{[r,s]} = \tilde{A}^{[r,s]}[1/p]$. Write $\tilde{B}_{\text{rig}}^{1,r} = \cap_{r \geq s > \infty} \tilde{B}^{[r,s]}$ and $\tilde{B}_{\text{rig}}^1 := \cup_r \tilde{B}_{\text{rig}}^{1,r}$. So there is a natural injection $\tilde{B}^{[\frac{r-1}{p}, \frac{r-1}{p}]} \to \tilde{B}^{1,r}_{\text{rig}}$ and injection

$$
\iota_n : \tilde{B}^{1,r,p^{n-1}(p-1)}_{\text{rig}} \to \tilde{B}^{1,r}_{\text{rig}} \subseteq \tilde{B}^{1,r}_{\text{rig}},
$$

for each $n \geq 0$. For any finite extension $K/\mathbb{Q}_p$ there is a Robba ring $\tilde{B}_{\text{rig},K}^1 \subseteq \tilde{B}_{\text{rig}}^1$ of it. If $K$ is unramified over $\mathbb{Q}_p$ then

$$
\tilde{B}_{\text{rig},K}^1 = \cup_{r > 0} \tilde{B}_{\text{rig},K}^{1,r}
$$

for

$$
\tilde{B}_{\text{rig},K}^{1,r} := \{ f(T) = \sum_{n \in \mathbb{Z}} a_n T^n | a_n \in K, f(T) \text{ convergent in } p^{-1/r} < |T|_p < 1 \}
$$

with $T = [e] - 1$. In this case we write

$$
\tilde{B}_{\text{rig},K}^1 := \{ f(T) = \sum_{n \in \mathbb{Z}} a_n T^n | a_n \in K, f(T) \text{ convergent in } 0 \leq |T|_p < 1 \}.
$$

If $K$ is ramified over $\mathbb{Q}_p$ then the construction of $\tilde{B}_{\text{rig},K}^1$ requires the theory of norm fields. There is also a $\psi$ operator $\tilde{B}_{\text{rig},K}^1 \to \tilde{B}_{\text{rig},K}^1$ defined as follows: we have $\tilde{B}_{\text{rig},K}^1 = \oplus_{i=1}^{p-1} (T + 1)^i \varphi(\tilde{B}_{\text{rig},K}^1)$. For any

$$
x = \sum_{i=1}^{p-1} (1 + T)^i \varphi(x_i)
$$

define $\psi(x) = x_0$. The $\iota_n$ defined in (1) satisfies

$$
\iota_n(\tilde{B}_{\text{rig},K}^{1,p^{n-1}(p-1)}) \to K_n[[t]]
$$

with $K_n := K(\zeta_{p^n})$.

**Definition 2.4.** A $(\varphi, \Gamma_K)$-module $D$ of rank $d$ over $\tilde{B}_{\text{rig},K}^1$ if

- $D$ is a finite free $\tilde{B}_{\text{rig},K}^1$-module of rank $d$;
- $D$ is equipped with a $\varphi$-semilinear map $\varphi : D \to D$ such that
  
  $$
  \varphi^*(D) : \tilde{B}_{\text{rig},K}^1 \otimes \varphi, \tilde{B}_{\text{rig},K}^1 \to D : a \otimes x \to a\varphi(x)
  $$

  is an isomorphism;
- $D$ is equipped with a continuous semilinear action of $\Gamma_K$ which commutes with $\varphi$.

Write $r_n = p^{n-1}(p-1)$. It is well known that any $(\varphi,\Gamma_K)$-module $D$ is overconvergent, i.e. there is $n(D) > 0$ and a unique finite free $\tilde{B}_{\text{rig},K}^{1,r_n(D)}$-submodule $D^{n(D)} \subseteq D$ of rank $d$ with $\tilde{B}_{\text{rig},K}^1 \otimes \tilde{B}_{\text{rig},K}^{1,r_n(D)}$ $D^{n(D)} = D$. 

9
Bloch-Kato exponential maps
Recall the fundamental exact sequence in $p$-adic Hodge theory
\[
0 \to \mathbb{Q}_p \to B_{\mathrm{cris}}^{\varphi=1} \oplus B_{\mathrm{dR}}^+ \to B_{\mathrm{dR}} \to 0.
\] (2)

Tensoring with the Galois representation $V$ and taking the $K$-Galois cohomology long exact sequence
we define the Bloch-Kato’s exponential map to be the coboundary map
\[
\frac{D_{\mathrm{dR}}(V)}{D_{\mathrm{dR}}^+(V) + D_{\mathrm{cris}}^{\varphi=1}(V)} \to H^1(K, V).
\]

In general Nakamura defined the $\exp$ and $\exp^*$ maps for deRham $(\varphi, \Gamma)$-modules.

For any $n > n(D)$ we define
\[
D^+_{\mathrm{dR}}(D) := K_n([t]) \otimes_{\mathbb{Z}_p, B^{\Gamma_K=1}} D^{(n)}
\]
and
\[
D_{\mathrm{dR}}(D) := K_n((t)) \otimes_{\mathbb{Z}_p, B^{\Gamma_K=1}} D^{(n)}.
\]

We also define
\[
D^K_{\mathrm{dR}}(D) = D_{\mathrm{dR}}(D)^{\Gamma_K=1}, D^K_{\mathrm{crys}}(D) = D[1/t]^{\Gamma_K=1}.
\]

The filtration on $D^K_{\mathrm{dR}}(D)$ is given by
\[
\text{Fil}^i D^K_{\mathrm{dR}}(D) = D^K_{\mathrm{dR}}(D) \cap i^* D^+_{\mathrm{dR}}(D), i \in \mathbb{Z}.
\]

We define a $(\varphi, \Gamma)$-module $D$ to be crystalline (deRham) if the rank $D$ is equal to the $\mathbb{Z}_p$-rank
of $D^\mathrm{crys}(D_{\mathrm{dR}})$. If $V$ is a representation over some finite extension $L$ of $\mathbb{Q}_p$, we make all these
definitions by regarding it as a $\mathbb{Q}_p$ representation.

2.3 $\Lambda_\infty$ and Co-admissible modules

We summarize some facts and definitions in [44] for later use.

Definition 2.5. ([44, Definition 3.1]) The analytic Iwasawa algebra $\Lambda_\infty := \lim_{\leftarrow n} \Lambda[\mathfrak{m}^n/p][1/p]$. This is the ring of rigid analytic functions on the open unit disc and is a Bezout domain. The analytic Iwasawa cohomology
\[
H^q_{\mathrm{Iw}}(K, D) := \lim_{\leftarrow n} H^q(K, D \otimes_K \hat{\Lambda}_n^e)
\]
as a $\Lambda_\infty$-module.

Definition 2.6. A co-admissible $\Lambda_\infty$-module $M$ is the module of global sections of coherent analytic sheaves on $W$. That means, there is an inverse system $(M_n)_n$ of finitely generated $\Lambda_n[1/p]$-modules such that the map $M_{n+1} \to M_n$ induces isomorphisms $M_{n+1} \otimes_{\Lambda_{n+1}[1/p]} \Lambda_n[1/p] \simeq M_n$. Then
\[
M = \lim_{\leftarrow n} M_n.
\]

The following proposition is proved by Pottharst [41].
Proposition 2.7. (1) The torsion submodule $M_{\text{tors}}$ of an admissible $\Lambda_\infty$-module $M$ is also co-admissible, and $M/M_{\text{tors}}$ is a finitely generated free $\Lambda_\infty$-module.

(2) The torsion co-admissible $\Lambda_\infty$-modules are those isomorphic to $\prod_{\alpha \in I} \Lambda_\infty p_\alpha^n$ for some collections $\{p_\alpha\}_{\alpha \in I}$ of closed points of $\cup_n \text{Spec} \Lambda_n[1/p]$ ($n_\alpha$ are positive integers) such that for each $n$ there are only finitely many $\alpha$ with $p_\alpha \in \text{Spec} \Lambda[1/p]$.

Definition 2.8. (Pottharst) Let $M$ as above be torsion. We define the divisor for $M$ as the formal sum $\sum n_\alpha p_\alpha$. We define the characteristic ideal $\text{char}_\Lambda(M)$ to be the principal ideal generated by some $f_M \in \Lambda$ such that the divisor of $f_M$ is the same as the divisor for $M$. (Such $f_M$ exists by a well known result of Lazard).

Then as in [13] Page 7

$$H^0_{\text{lw}}(K, V) = H^0_{\text{cl}, \text{lw}}(K, V) \otimes_{\Lambda} \Lambda_\infty.$$  

2.4 Unramified Iwasawa Theory

In this subsection we prove some key facts about uniform boundedness of Bloch-Kato logarithm map along unramified field extensions. Later on we are going to study the $\mathbb{Z}_p$-extension of $K$ which is unramified at $v_0$ but totally ramified at $v_0$, and the result proved here will be of crucial importance. We write in this subsection $r$ the highest Hodge-Tate weight of $T$. Recall $T = [\ell] - 1$. We also define $\gamma_n(x) = \frac{x^n}{n!}$. Let $q' = \varphi^{-1}(q)$ for $q = \sum_{a \in \mathbb{F}_p, \ell} [\ell]^{[\ell]}$.

Lemma 2.9. We have $\theta(q') = 0$, but $\theta(q'/T) \neq 0$.

Proof. It is clear that $\theta(q') = 0$. Take $\varpi = (\varpi^0, \varpi^1, \cdots) \in \hat{E}^+$ with $\varpi^0 = -p$ and $\xi := [\varpi] + p$. Then $\ker(\theta : W(R) \to \mathcal{O}_C)$ is the principal ideal generated by $\xi$ ([13] Proposition 5.12]). If $\theta(q'/T) = 0$ then $q' = \xi^2 \lambda$ for some $\lambda \in W(\hat{E}^+)$. So $\bar{q}' = \varpi^2 \bar{\lambda}$. But $v(\bar{q}') = 1$ and $v(\varpi^2) = 2$, a contradiction. 

Definition 2.10. $a = \varphi(q'/T)$.

Definition 2.11. Let $A^0_{\text{cris}}$ be the divided power envelop of $W(\hat{E}^+)$ with respect to $\ker \theta$, that is, by adding all elements $a^m/m!$ for all $a \in \ker \theta$. Defining the ring $A_{\text{cris}} = \varinjlim_n A_{\text{cris}}^0/p^n A_{\text{cris}}^0$ and $B^+_{\text{cris}} := A_{\text{cris}}[1/p]$.

Now let $\text{Fil}^r A_{\text{cris}} = A_{\text{cris}} \cap \text{Fil}^r B_{\text{dR}}$ and $\text{Fil}^r A_{\text{cris}}^0 = \{x \in \text{Fil}^r A_{\text{cris}} | \varphi x \in p^r A\}$. Then we have the following

Lemma 2.12. For every $x \in \text{Fil}^r A_{\text{cris}}$, $p^a x \in \text{Fil}^r A_{\text{cris}}$ for a the largest integer such that $(p-1)a < r$. Moreover $\text{Fil}^r A_{\text{cris}}$ is the associated sub $W(\hat{E}^+)$-module of $A_{\text{cris}}$ generated by $q^j \gamma_b(p^{-1} t^{p-1})$ for $j + (p-1)b \geq r$.

This is just [13] Proposition 6.24).

Let $T$ be a two dimensional Galois representation of $G_{\mathbb{Q}_p}$ over $\mathcal{O}_L$ with $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ being deRham. Let $v_1, v_2$ be a basis of $T$. Suppose the Hodge-Tate weight of $V$ is $(r, s)$ with $r > 0, s \leq 0$. Let $\omega_V$ be a fixed generator of the one dimensional space $\text{Fil}^0 D_{\text{dR}}(V)$ over $L$. Then $t^r \omega_V \in B^+_{\text{dR}} \otimes T$.

There is an element $z \in W(\hat{E}^+)[\frac{1}{p}] \otimes T$ such that

$$t^r \omega_V - z \in \text{Fil}^{r+1} B_{\text{dR}} \otimes T.$$  

(3)
where
\[
\text{Im}(\text{map}) = (t^{r-1} + (\tilde{\nu} + \tilde{\nu})^{-1}) v_1 + (\tilde{\nu} + \tilde{\nu}^{-1}) v_2 \mod \text{Fil}^{-1} B_{\text{dR}} \otimes T.
\]

Then by (3)
\[
t^r (p^{r-i+m_r(r-i-1)+n} a \cdot \omega_V - (\tilde{b}_{r-1} \cdot \tilde{v}^{r-1} + \tilde{b}_i \cdot \tilde{v}^i) v_1)
\]
is in \(p^{-m_r} \mathcal{O}_{\mathbb{C}_p}\), and that the image under \(\theta\) of the coefficient of \(v_2\) in
\[
(t^i (p^{r-i+m_r(r-i-1)+n} a \cdot \omega_V - (\tilde{c}_{r-1} \cdot \tilde{v}^{r-1} + \tilde{c}_i \cdot \tilde{v}^i) v_1) v_2)
\]
is in \(p^{-m_r} \mathcal{O}_{\mathbb{C}_p}\). So there is a choice of \(\tilde{b}_{r-1}, \ldots, \tilde{b}_i\) and \(\tilde{c}_{r-1}, \ldots, \tilde{c}_i\) such that
\[
p^{r-i+1+m_r(r-i)+n} a \cdot \omega_V - (\tilde{b}_{r-1}(\tilde{v}^{r-1} + \tilde{b}_i \cdot \tilde{v}^i) v_1 - (\tilde{c}_{r-1}(\tilde{v}^{r-1} + \tilde{c}_i \cdot \tilde{v}^i) v_2
\]
is in \(\text{Fil}^{-1} B_{\text{dR}} \otimes T\).

(\text{The } \tilde{b}_{r-1}, \ldots, \tilde{b}_i \text{ and } \tilde{c}_{r-1}, \ldots, \tilde{c}_i \text{ are the previous } \tilde{b}_{r-1}, \ldots, \tilde{b}_i \text{ and } \tilde{c}_{r-1}, \ldots, \tilde{c}_i \text{ multiplied by } p^{1+m_r}.)

Continuing this process we can find \(\tilde{b}_{r-1}, \ldots, \tilde{b}_0\) and \(\tilde{c}_{r-1}, \ldots, \tilde{c}_0\) such that
\[
p^{r+m_r(r)+n} a \cdot \omega_V - (\tilde{b}_{r-1}(\tilde{v}^{r-1} + \tilde{b}_0) v_1 - (\tilde{c}_{r-1}(\tilde{v}^{r-1} + \tilde{c}_0) v_2 \in B_{\text{dR}}^+ \otimes T.
\]

This proves the claim.

Another observation is that \(U_1 \cap \ker(B_{\text{dR}}^{p=1} \to B_{\text{dR}}^+) \subseteq p^{-m_r} \mathbb{Z}_p\). This can be seen by noting that for any element \(C\) in this intersection, \(t^r C \in \text{Fil}^r A_{\text{cris}}\). So \(\theta(C) \in p^{-m_r} \mathcal{O}_{\mathbb{C}_p}\). The following proposition follows immediately.
Proposition 2.13. We consider the co-boundary map
\[
\exp : \frac{(B_{\text{dR}} \otimes V)^{I_{Q_p}}}{(B_{\text{dR}} \otimes V)^{I_{Q_p}} + (B_{\text{cris}} \otimes V)^{I_{Q_p}} \cdot \sigma = 1} \to H^1(I_{Q_p}, V).
\]
Then there is an \( m > 0 \) such that for any \( a \in W(F_{\mathbb{P}_p}) \) we have \( \exp(a \cdot \omega) \in p^{-m}H^1(I_{Q_p}, T) \).

Proof. To see this we just use the diagram
\[
\begin{array}{ccccccc}
0 & \to & \ker & \to & U_r^1 \oplus B_{\text{dR}}^+ & \to & U_r^1 + B_{\text{dR}}^+ & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & . \\
0 & \to & \mathbb{Q}_p & \to & B_{\text{cris}}^{\sigma = 1} \oplus B_{\text{dR}}^+ & \to & B_{\text{dR}} & \to & 0 \\
\end{array}
\]

We refer to Section 4.1 for some discussion of Yager modules and periods for unramified representations, and the definitions of the \( d \) and \( \rho(d) \).

Corollary 2.14. Let \( \rho \) be an unramified \( p \)-adic character of \( G_{\mathbb{Q}_p} \) such that \( \rho(\text{Frob}_p) = 1 + m \in \mathcal{O}_{\mathbb{Q}_p} \) with \( \text{val}_p(m) > 0 \). Then the map
\[
\exp : \frac{B_{\text{dR}}}{B_{\text{dR}}^+} \otimes V(\rho)^{G_{\mathbb{Q}_p}} \to H^1(G_{\mathbb{Q}_p}, V(\rho))
\]
can be constructed as
\[
\exp(\lim_n \sum_{\sigma \in U_n} d_n^\sigma \rho(\sigma) \cdot \omega_f) = \lim_n \sum_{\sigma \in U_n} \rho(\sigma) \exp(d_n^\sigma \cdot \omega_f).
\]

The right hand side is well defined thanks to Proposition 2.13.

To see this, we consider the natural unramified rank one Galois representation of \( U = \text{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}) \) over the Iwasawa algebra \( \mathbb{Z}_p[[U]] \). We consider the map from \( \mathbb{Z}_p[[U]] \) to \( \rho \) mapping \( u \) to \( \rho(\sigma) \). Tensoring this map with \( \mathcal{O}_{\mathbb{Q}_p} \) and taking the long exact sequence of Galois cohomology, we get the required formula.

Corollary 2.15. For some integer \( m \) we consider the inverse limit of the maps for \( \mathbb{Q}_p \subset F_n \subset \mathbb{Q}_p^{ur} \),
\[
\exp \mathcal{O}_{F_n} \cdot \omega_f \to p^{-m}H^1(F_n, T)
\]
and get a map
\[
\exp : (\lim_n \mathcal{O}_{F_n} \cdot \omega_f) \to p^{-m}H^1(\mathbb{Q}_p, T \otimes \mathbb{Z}_p[[U]]).
\]
Then for some choice of such an integer \( m \) we have \( p^m \exp(d \cdot \omega_f) \) generates a \( \mathbb{Z}_p[[U]] \)-direct summand of
\[
H^1(\mathbb{Q}_p, T \otimes \mathbb{Z}_p[[U]]) \simeq \mathbb{Z}_p[[U]] \oplus \mathbb{Z}_p[[U]].
\]

Proof. By Corollary 2.14 the specialization of \( \exp(d \cdot \omega_f) \) to any \( \phi \in \text{Spec} \mathbb{Z}_p[[U]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) with \( u \) mapping to \( 1 + m \) with \( \text{val}_p(m) > 0 \) is non-zero (since the exp map for \( V(\rho) \) is injective on \( B_{\text{dR}}(V(\rho)) \)).
The corollary follows by Weierstrass preparation theorem of the ring \( \mathbb{Z}_p[[U]] \) and observing that \( \mathbb{Z}_p[[U]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) is a Bezout domain.
3 Iwasawa-Greenberg Main Conjecture

The Idea

Our goal in this section is to prove Theorem 1.7. The idea in proving [51, Theorem 5.3] when \( f \) has weight two is roughly summarized as follows. We first construct families of Klingen Eisenstein series \( E_{\text{Kling}} \) on the unitary group \( U(3, 1) \) using [14]. The Hida theory developed in [51, Section 3] for semi-ordinary forms (we refer to loc.cit for precise definition) enables us to construct a family of cusp forms, which is congruent to the Klingen Eisenstein family modulo \( L_{f, \mathbb{K}} \). Then we proved there is a functional (constructed via Fourier-Jacobi expansion map) acting on the space of families of semi-ordinary forms on \( U(3, 1) \), which maps \( E_{\text{Kling}} \) to an element which is a unit of the coefficient ring \( \mathcal{O}^a[[\Gamma_{\mathbb{K}}]] \), up to multiplying by an element in \( \overline{\mathbb{Q}}_p^\times \). (i.e. Proposition 5.2 of loc.cit. This is the hard part of the whole argument). With this in hand, this functional and the cuspidal family we mentioned above gives a map from the cuspidal Hecke algebra to \( \mathcal{O}^a[[\Gamma_{\mathbb{K}}]] \) which, modulo \( L_{f, \mathbb{K}} \), gives the Hecke eigenvalues acting on the space of families of cusp forms, which is congruent to the Klingen Eisenstein family modulo \( \mathcal{O}^a[[\Gamma_{\mathbb{K}}]] \), up to multiplying by an element in \( \overline{\mathbb{Q}}_p^\times \) (a number fixed throughout the whole family). Then we proved the lower bound of the Selmer group. (See the proof of [51, Theorem 5.3] and [52, Section 9.3]).

Now we return to the situation in this paper (i.e. general weight). All ingredients are available, except that we need some new idea in the vector valued case to construct the corresponding functional on semi-ordinary forms on \( U(3, 1) \) using Fourier-Jacobi expansion map, so that its value on the Klingen Eisenstein family is an element in \( \mathcal{O}^a[[\Gamma_{\mathbb{K}}]]^\times \), up to multiplying by a non-zero constant in \( \overline{\mathbb{Q}}_p^\times \). Recall that in [51] and [52], since the Klingen Eisenstein series \( E_{\text{Kling}} \) is realized using pullback formula under

\[
U(3, 1) \times U(2) \hookrightarrow U(3, 3)
\]

for the Siegel Eisenstein series \( E_{\text{sieg}} \) on \( U(3, 3) \), we computed that the \( \beta \)-th Fourier-Jacobi coefficient \( FJ_1 E_{\text{sieg}} = E_{2,2} \cdot \Theta \) for \( \beta = 1 \), where \( E_{2,2} \) is a Siegel Eisenstein series on \( U(2, 2) \) and \( \Theta \) is a theta function on \( N'U(2, 2) \), where the \( N' \) is an unipotent such group in \( U(3, 3) \) defined as the \( N' \) in [52, Introduction]. In this section we will often write \( D \) for the Jacobi group \( N'U(2, 2) \) as in [26]. Let \( N \) be the unipotent subgroup of the Klingen parabolic subgroup \( P \) of \( U(3, 1) \) and \( U(2) \) is embedded into the Levi part of \( P \) (see [52, Section 3] for details). We constructed a theta function \( \theta_1 \) on \( NU(2) \) and defined a functional \( l_\theta \) by pairing with \( \theta_1 \) along \( N \), from forms on \( D(\mathbb{Q}) \backslash D(h) \) to forms on \( U(2) \) (see [52, Section 8.5, Corollary 6.36]). We also constructed auxiliary theta functions \( h \) and \( \theta \) on \( U(2) \) and deformed them in Hida families \( \mathbf{h} \) and \( \mathbf{\theta} \). We used the doubling method for \( h \) under \( U(2) \times U(2) \hookrightarrow U(2, 2) \) to see that \( \langle \ell_\theta, FJ_1(E), h \rangle = \langle e^{ord} l_\theta, FJ_1(E), h \rangle \) where \( e^{ord} \) is Hida’s ordinary projector on \( U(2) \), since the \( h \) we construct is ordinary) is essentially the triple product integral \( \int h(g) \theta^{low}(g) f(g) dg \). Here the superscript \( low \) means translating \( \theta \) by \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). We also construct families of forms \( \tilde{h} \) and \( \tilde{\theta} \) in the dual space of \( h \) and \( \theta \), respectively. The product

\[
\int h(g) \theta^{low}(g) f(g) dg \int \tilde{h}(g) \tilde{\theta}^{low}(g) \tilde{f}(g) dg
\]

can be evaluated using Ichino’s triple product formula. In [51] we have seen that both \( \int h(g) \theta^{low}(g) f(g) dg \) and \( \int \tilde{h}(g) \tilde{\theta}^{low}(g) \tilde{f}(g) dg \) are interpolated by elements in \( \mathcal{O}^a[[\Gamma_{\mathbb{K}}]] \), and the product is in \( (\mathcal{O}^a[[\Gamma_{\mathbb{K}}]])^\times \) up to multiplying by an element in \( \overline{\mathbb{Q}}_p^\times \) (a number fixed throughout the whole family). Then the required property of the Fourier-Jacobi coefficient of \( E_{\text{Kling}} \) follows directly.
In this paper we use the construction in [14] of the Klingen Eisenstein series, using the pullback formula for

\[ U(3, 1) \times U(0, 2) \hookrightarrow U(3, 3) \]

from the nearly holomorphic Siegel Eisenstein series on \( U(3, 1) \).

**Proposition 3.1.** Suppose the unitary automorphic representation \( \pi = \pi_f \) generated by the weight \( k \) form \( f \) is such that \( \pi_p \) is an unramified principal series representation \( \pi(\chi_1, \chi_2) \) with distinct Satake parameters. Let \( \overline{\pi} \) be the dual representation of \( \pi \). Let \( \Sigma \) be a finite set of primes containing all the bad primes

(i) There is an element \( \mathcal{L}_{f, \overline{\pi}}^{\Sigma} \in \Lambda_K, \mathcal{O}_L^\times \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) such that for any character \( \xi_\phi \) of \( \Gamma_K \), which is the avatar of a Hecke character of conductor \( p \), infinite type \( \left( \frac{2}{2} + m_\phi, -\frac{\kappa_\phi}{2} - m_\phi \right) \) with \( \kappa_\phi \) an even integer which is at least \( 6 \), \( m_\phi \geq \frac{k-2}{2} \), we have

\[
\phi(\mathcal{L}_{f, \overline{\pi}}^{\Sigma}) = \frac{L^\Sigma(\overline{\pi}, \xi_\phi, \frac{\kappa_\phi-1}{2})\Omega_{2m_\phi+2\kappa_\phi}}{\Omega_{2m_\phi+2\kappa_\phi}^\Sigma} c'_\phi p^{\kappa_\phi-3} g(\xi_{\phi, 2})^2 \prod_{i=1}^2 (\chi_i^{-1} \xi_{\phi, 2})^2(p)
\]

c'_\phi is a constant coming from an Archimedean integral.

(ii) There is a set of formal \( q \)-expansions \( \mathbf{E}_{f, \xi_0} := \{ \sum_\beta a^t_{[g]}(\beta)q^\beta \}_{[g], t} \) for \( \sum_\beta a^t_{[g]}(\beta)q^\beta \in \Lambda_K, \mathcal{O}_L^\times \otimes_{\mathbb{Z}_p} \mathcal{R}_{[g], \infty} \) where \( \mathcal{R}_{[g], \infty} \) is some ring to be defined later, \( ([g], t) \) are \( p \)-adic cusp labels, such that for a Zariski dense set of arithmetic points \( \phi \in \text{Spec}_{K, \mathcal{O}_L}, \phi(\mathbf{E}_{f, \xi_0}) \) is the Fourier-Jacobi expansion of the highest weight vector of the holomorphic Klingen Eisenstein series constructed by pullback formula which is an eigenvector for \( U_{1+} \) with non-zero eigenvalue. The weight for \( \phi(\mathbf{E}_{f, \xi_0}) \) is \( (m_\phi + \frac{k-2}{2}, m_\phi - \frac{k-2}{2}, 0; \kappa_\phi) \).

(iii) The \( a^t_{[g]}(0) \)'s are divisible by \( \mathcal{L}_{f, \overline{\pi}}^{\Sigma}, \mathcal{L}_{\overline{\pi}}^{\Sigma} \), where \( \mathcal{L}_{\overline{\pi}}^{\Sigma} \) is the \( p \)-adic \( L \)-function of a Dirichlet character as in [13].

We also refer to [14] for the convention of weights of automorphic forms on \( U(2) \) and \( U(3, 1) \). This is just a translation of the main theorem of [11] to the situation here. We can recover the full \( p \)-adic \( L \)-function \( \mathcal{L}_{f, \overline{\pi}} \) by putting back the Euler factors at primes in \( \Sigma \). We again write \( E_{\text{Kling}} \) for the family of Klingen Eisenstein series constructed in the above theorem. We can actually make the constant \( c'_\phi \) precise.

**Lemma 3.2.** The constant \( c'_\phi \) above is given by

\[
\Gamma(\kappa_\phi + m_\phi - \frac{k}{2}) \Gamma(\kappa_\phi + m_\phi + \frac{k}{2} - 1) 2^{-3}\kappa_\phi - 4m_\phi + 1 \pi^{-1} \pi - 2m_\phi 2k - \kappa_\phi - 2m_\phi - 1.
\]

**Proof.** It is not easy to compute the \( c'_\phi \) directly. We prove the lemma by a comparison of the above \( p \)-adic \( L \)-function and Hida’s Rankin-Selberg \( p \)-adic \( L \)-function. Let \( \mathbf{g} \) be the Hida family corresponding to the family of characters of \( \Gamma_K \). We pick an auxiliary Hida family of ordinary forms \( \mathbf{f}' \) and compare

- The product \( \mathcal{L}_{\mathbf{f}', \mathbf{g}}^{\text{Hida}} \cdot \mathcal{L}_{\text{Katz}}^{\Sigma} h_K \), where \( \mathcal{L}_{\text{Katz}}^{\Sigma} h_K \) is the class number \( h_K \) of \( K \) times the Katz \( p \)-adic \( L \)-function, which interpolates the Petersson inner product of specializations of \( \mathbf{g} \) (see [24]). The \( \mathcal{L}_{\mathbf{f}', \mathbf{g}}^{\text{Hida}} \) is the Rankin-Selberg \( p \)-adic \( L \)-function constructed by Hida in [20] interpolating algebraic part of the critical values of Rankin-Selberg \( L \)-functions for specializations of \( \mathbf{f}' \) and \( \mathbf{g} \), where the specializations of \( \mathbf{g} \) has higher weight.
• The $p$-adic $L$-function $L_{f,K}$ constructed using doubling method as above.

We first look at the arithmetic points where the Siegel Eisenstein series are of scalar weight. The computations are essentially done in [53] (although the ramifications in loc.cit is slightly different, however those assumptions are put for constructing the family of Klingen Eisenstein series. The computations in the doubling method construction of the $p$-adic $L$-function carries out in the same way in the situation here). We see that the above two items have the same value at these points. As these arithmetic points are Zariski dense, the two should be identical. Then we look at the arithmetic points considered in the above proposition. Comparing the interpolation formulas here and in [20] Theorem 1, we get the formulas for $c'_\phi$ (note that the critical $L$-value is not zero since it is away from center).

In application we will twist our Klingen Eisenstein series by an anticyclotomic character so that the specializations are of the weight $(\frac{k-2}{2}, -\frac{k-2}{2}, -m_\phi; \kappa_\phi + m_\phi)$. Now we still write $L_{f,\otimes g}^{\text{Hida}}$ for the Rankin-Selberg Hida $p$-adic $L$-function interpolating critical values of the Rankin-Selberg $L$-function for $f$ and specializations of $g$ whose weight is higher than $f$. Since the higher weight form $g$ is ordinary, Hida’s construction works in the same way even though $f$ is not ordinary. We have the following

**Corollary 3.3.**

$$L_{f,\otimes g}^{\text{Hida}} \cdot L_V^{\text{Katz}_{\otimes h_K}} = L_{f,K}^{Gr}.$$

The corollary follows from above lemma and the interpolation formulas on both hand sides. From now on we write $L_{f,K}^{Gr}$ for the $L_{f,K}$ constructed above, since it corresponds to the Greenberg’s main conjecture. We prove the following

**Lemma 3.4.** The $L_{f,K}^{Gr}$ is in $O_{L}^{ur}[[\Gamma_K]]$.

**Proof.** From the construction the denominator of $L_{f,K}^{Gr}$ can only be powers of $p$ times the product of the Euler factors of a finite number of primes of $L_{f,K}^{Gr}$. But by the argument in [51] Proposition 8.3 we know the denominator can at most be powers of $U$ if we take $\mathbb{Z}_p[[U]]$ as the coefficient ring of $g$. These two sets are disjoint. Thus the denominator must be a unit. \hfill $\square$

Now let us return to the proof of Theorem 1.7. The Hida theory for general weight is already developed in [51] Section 3. The main difficulty here is that the explicit local Fourier-Jacobi computation at Archimedean place is very hard to study if the weight is not scalar. Our idea is not to compute such local Fourier-Jacobi integrals at $\infty$. Instead we fix the weight $(k, \kappa_\phi, m_\phi)$ (notation as before) and varying its nebentypus at $p$. Such arithmetic points are Zariski dense in $\text{Spec}O_L[[\Gamma_K]]$. We show that there is a number $C_\infty$ depending only on the Archimedean data (and is thus the same number for all arithmetic points), which can be proved to be non-zero, and an element $L \in O_L^{ur}[[\Gamma_K]] \times \bar{Q}_p^\times$, such that at each of such arithmetic point we have

$$\langle l_{\theta_1} (\text{FJ}_b, (\phi(E))), h_\phi \rangle \cdot \int_{\tilde{h}_\phi} (g) \theta_\phi^{\text{low}} (g) \tilde{f}(g) dg = C_\infty \cdot \phi(L).$$

**Interpolating Inner Products**

We take a basis $\{v_1, v_2, \cdots, v_t\}$ of $V_{(\frac{k-2}{2}, -\frac{k-2}{2})}$ and $\{v_1', v_2', \cdots, v_t'\}$ of $\tilde{V}_{(\frac{k-2}{2}, -\frac{k-2}{2})}$. If

$$f \in A(U(2), V_{(\frac{k-2}{2}, -\frac{k-2}{2})})$$

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(the space of \(V_{\left(\frac{k-2}{2}, -\frac{k-2}{2}\right)}\)-valued automorphic forms on the definite unitary group),

\[
h \in \mathcal{A}(U(2), V_{\left(\frac{k-2}{2}, -\frac{k-2}{2}\right)}).
\]

Writing \(f = f_1 v_1 + \cdots + f_t v_t\) and \(h = h_1 v_1^\vee + \cdots + h_t v_t^\vee\). Then define \(\langle f, h \rangle := \sum_{i=1}^t \langle f_i, h_i \rangle\).

We first develop a vector valued generalization in [52, Section 7.3] of pairing of a family of forms \(f\) of weight \(V_{\left(\frac{k-2}{2}, -\frac{k-2}{2}\right)}\) (but with nebentypus at \(p\) varying) on \(U(2)\) to a Hida family of eigenforms \(g\) of weight \(V_{\left(\frac{k-2}{2}, -\frac{k-2}{2}\right)}^\vee\). We define \(V_{\left(\frac{k-2}{2}, -\frac{k-2}{2}\right)}^\vee\)-valued measure \(d\mu\) as in loc.cit (note that a normalization factor \(p^{k-2}\) needs to be put to keep the integrality of the \(U_p\) operator and to make its eigenvalue on ordinary forms to be \(p\)-adic unit) and use the above pairing to define

\[
\int_{[U(2)]} f d\mu_g
\]

as an element in \(\Lambda\), such that for each arithmetic points \(\phi\),

\[
\phi\left(\int_{[U(2)]} f d\mu_g\right) = p^\phi \cdot \langle f_\phi, g_\phi^{\text{low}} \rangle.
\]

As in loc.cit, this \(\int_{[U(2)]} f d\mu_g\) is written with a better notation

\[
\langle f, g \rangle.
\]

Hsieh made a construction in [23] which is actually the same as ours. He defined in [23] Definition 4.3 a \(\Lambda\)-adic pairing between ordinary Hida families on \(D^\times\). Since when specializing to all arithmetic points \(Q\) of weight 2 (a Zariski dense set), his construction and ours both give

\[
\langle U_p^{-\alpha} f_Q, f_Q \rangle
\]

for \(p^\alpha\) being the conductor of \(Q\), we see the two constructions coincide. Hsieh in fact went one step further in that he proved when specializing to arithmetic points of weight higher than 2, it still gives

\[
\langle U_p^{-\alpha} f_Q, f_Q \rangle.
\]

(Petersson inner product of vector valued forms in loc.cit). Recall we defined Hida families \(h, \tilde{h}, \theta, \tilde{\theta}\) of ordinary forms on \(D^\times\) in [52] Section 8.3], by interpolating constructions via theta correspondence for \(U(1) \times U(2)\) at arithmetic points of weight 2. By standard facts about Hida control theorems, we know the families \(h\) and \(\tilde{h}\) also interpolate the highest weight vectors of CM forms at arithmetic points of weight \(k\). From the computations in [52], we know \(\langle h, \tilde{h} \rangle\) and \(\langle \theta, \tilde{\theta} \rangle\) (Hsieh’s \(\Lambda\)-adic pairing) are interpolated by \(p\)-adic \(L\)-functions \(L_3\) and \(L_4\) defined in [52] Section 8.5].

**Ikeda Theory**

We refer to [51] Section 6.1] for the background of Siegel Eisenstein series and write \(I_n(\tau)\) for the corresponding space of (local or global) Siegel sections defined using a Hecke or local character \(\tau\). Write \(E = E_{\text{Sieg}}\) for the Siegel Eisenstein series on \(U(3, 3)\) that we use in the pullback formula. Now we put ourselves in the context of [26]. We are in the \(m = 1\) and \(n = 2\) of [26] Section 2, case 2] (see the definitions of \(X, Y, Z, V\) there). We also refer to loc.cit for the background of theta functions.
and pairings $\langle , \rangle$, $\langle , \rangle$ between kernel functions and theta functions. Let $\psi$ be an additive character of $K$ and $\omega_\psi$ be the Weil representation there.

We first make some observations. We are thankful to Professor Ikeda for pointing out these to us. For $\beta \in \mathbb{Q}^\times$, let $\pi$ be the representation of the Jacobi group $NU(2, 2)$ generated by the $\beta$-th Fourier-Jacobi coefficient of a nearly holomorphic Siegel Eisenstein series on $U(3, 3)$. Then by [26, Proposition 1.3], there is a map

$$\omega_\psi \otimes I \hookrightarrow \pi$$

with dense image, where $\omega_\psi$ is the Weil representation of the Jacobi group defined there, and $I$ is the representation generated by the integral of loc.cit. By [26, Theorem 3.2] this is the sub-representation of the automorphic representation corresponding to Siegel Eisenstein series on $U(2, 2)$. Let $g$ be the Lie algebra for $U(2, 2)(\mathbb{R})$ and $K$ be a maximal compact subgroup of it. Let $\mathfrak{k}$ be its Lie algebra. Write the Harish-Chandra decomposition of the complex Lie algebra by

$$g^C = \mathfrak{k}^+ \oplus p^+ \oplus p^-.$$

By a lowest weight representation we mean a $(g, K)$-module generated by elements which are killed by some finite power of $p^-$. Then as $E_{\text{Siegel}}$ is nearly holomorphic, we know $\pi$ is a lowest weight representation. Also as $\omega_\psi$ is a Weil representation, it is also a lowest weight representation. By the Leibnitz rule

$$X^-(v_1 \otimes v_2) = X^-v_1 \otimes v_2 + v_1 \otimes X^-v_2$$

for $v_1 \in \omega_\psi$, $v_2 \in I$ and $X^- \in p^-$, we see $I$ is also a lowest weight representation. Note that under the Archimedean theta correspondence of $U(1)(\mathbb{R}) \times U(2, 2)(\mathbb{R})$, any component of the Weil representation of $U(2, 2)(\mathbb{R})$ with given central character must be irreducible (since $U(1)(\mathbb{R})$ is compact, see e.g. [11]). We also note that the representation of $U(2, 2)(\mathbb{R})$ admissibly induced from a character of the Siegel parabolic subgroup is of finite length. Now we claim that for a fixed $K$-type $\sigma$ of $\pi$, there are only finitely many $K$-types of $\omega_\psi$ and $I$, whose tensor product contains $\sigma$. This can be seen, for example by noting that in any lowest weight representation generated by a single vector, the $U(1)(\mathbb{R})$-weights are monotonically going up with finite dimensional eigen-spaces for each eigen-character. It follows that the map

$$\omega_\psi \otimes I \hookrightarrow \pi$$

is surjective for $K$-finite vectors, from the density of the image of the above $\hookrightarrow$.

We refer to [26, Theorem 3.2] for the definition of an integral operator $R(f, \phi)$, where $f$ is a Siegel Eisenstein section on $U(3, 3)$ and $\phi$ is a Schwartz function. It also makes sense to talk about local versions of these integrals. From the above discussion there is a finite number of Archimedean Schwartz functions $\phi_{4, i, \infty}$, $\phi_{2, i, \infty}$ depending only on $k_i$ ($i = 1, 2, \cdots$) such that $FJ_{\beta}(f)$ can be written as finite sums of expressions as

$$\Theta_{\phi_{4, i, \infty} \prod \phi_{\infty}}(nh)E(R(f_{\infty}, \phi_{2, \infty}) \cdot \prod_{v \in \infty} f^+ v)$$

for some Siegel section $\prod f^+ v \in I_2(\tau)$ and Schwartz functions $\prod \phi_{\infty}$. Define the local $\beta$-th Fourier-Jacobi integral for $f \in I_3(\tau)$ as

$$FJ_{\beta}(f, \phi) = \int_{\mathcal{Q}_v} f_v \left(\begin{array}{c} 13 \\ -13 \end{array}\right) \left(\begin{array}{ccc} I_3 & S_v & 0 \\ 0 & 0 & 0 \\ 13 & 13 \end{array}\right) (nh)e(-\beta S_v) dS_v.$$
Note also the computations in [52] Section 6] imply for each $v < \infty$ and $f_v$ the Siegel section on $U(3, 3)$ that we use to define the Siegel Eisenstein series $E_{\text{sieg}}$, the local Fourier-Jacobi integral can be written in the form

$$FJ_\beta(f_v) = \sum_{j_v = 1}^{n_v} f_{j_v} \phi_{j_v}$$

for $f_{j_v} \in L_2(\tau_v)$ Siegel sections on $U(2, 2)$ and $\phi_{j_v}$ local Schwartz functions. Then from the computation in [22] Page 628] on

$$\langle FJ_\beta(E), \Theta_\phi \rangle$$

and choosing the test Schwartz function $\phi$ properly, we obtain that by possibly scaling the $\phi_{2,1,\infty}$ (which again only depends on our Archimedean datum $k$ chosen),

$$FJ_\beta(E) = \sum_i \Big( \prod_v \sum_{j_v} E(R(f_{\infty}, \phi_{2,i,\infty}) \cdot \prod_v f_{j_v}, -) \cdot \Theta_{\phi_{4,\infty}} \prod_{v < \infty} \phi_{j_v} \Big),$$

Now we look at the Fourier-Jacobi expansion theory for forms on $GU(3, 1)$. Recall [52] Section 3.6] that for $\beta \in \mathbb{Q}^\times$ there is a line bundle $L(\beta)$ on the boundary component $Z_{[g]}$ ($[g]$ is some cusp label) of the Shimura variety for $GU(3, 1)$. We refer to loc.cit. and [22] Section 3.6] for the theory of Fourier-Jacobi coefficients for forms on $GU(3, 1)$. Some points are worth pointing out when working with vector valued forms. We refer to [33] for the comparison between algebraic and analytic Fourier-Jacobi coefficients. As noted in [22] Section 3.6], the algebraic Fourier-Jacobi coefficient takes values only in the $N^1_H$-coinvariants (see 3.6.2 of loc.cit) of the representation $L_{\mathfrak{k}}$, due to the ambiguity in choosing a basis for the differentials of the Mumford family. Note that the $\beta$-th Fourier-Jacobi expansion takes values in the space of forms on $U(2)$ tensored with the space of global sections of $L(\beta)$. We only look at the $L_{\mathfrak{k}} := L_{\frac{k-2}{2}, -\frac{k-2}{2}}$-components (regarded as a sub-representation of the restriction of the representation $L_{\mathfrak{k}}$ to $GL_2$, which clearly appears with multiplicity one in this restriction). In fact the ambiguity on the choice above does not make any differences when looking at this $L_{\mathfrak{k}}$-component, by the description of the $N^1_H$-coinvariants in the proof of [22] Lemma 3.12] (this corresponds to the $L_{\mathfrak{k}}$-component there). According to the description in [33] Section 5.3] this corresponds to looking at a quotient of the $E_{M,\text{an}}(W)$ in loc.cit. which is the pullback to $Z_{[g]}$ of an automorphic vector bundle of weight $\mathfrak{k}'$ on the Igusa variety for the definite unitary group $U(2)$, tensoring with the line bundle $L(\beta)$. We look at theta functions $\theta_1$, which are exactly the ones considered in [52] and [31], i.e. corresponds to the dual of the space of global sections of $L(\beta)$. We can make the choice of $\phi_{1,\infty}$ as in [52] Definition 6.12] and $\phi_1 = \phi_{1,\infty} \times \prod_{v < \infty} \phi_{1,v}$, and define the functional $l_{\theta_1}$ as in [52] Sections 4.9, 8.5], such that $\theta_1 = \theta_{\phi_1}$ is such that $l_{\theta_1} \in \text{Hom}(H^0(Z_{[g]}, L(\beta)), \mathcal{O}_L)$. We can ensure that the $(\frac{k-2}{2}, -\frac{k-2}{2})$-component of $l_{\theta_1}(FJ_\beta(E_{\text{Kling}}))$ is non-zero for some $\beta \in \mathbb{Q}^\times \cap \mathbb{Z}_p^\times$. If not then the $(\frac{k-2}{2}, -\frac{k-2}{2})$-component of $E_{\text{Kling}}$ is a constant function on the Shimura variety of $GU(3, 1)$. This contradicts the description of the boundary restriction $E_{\text{Kling}}$, namely the $(\frac{k-2}{2}, -\frac{k-2}{2})$-component is non-zero at some cusp (namely at $w_3$, see [52] Section 6.8]) while is zero at other cusps.

Thus there must be a $\beta' \neq 0$ such that $\text{proj}_{(\frac{k-2}{2}, -\frac{k-2}{2})}(FJ_{\beta'}) = 0$. Let $\beta' = p^n \beta''$ for $\beta'' \in \mathbb{Z}_p^\times$ and $n \in \mathbb{Z}$. Let $y$ be an element in $\mathcal{K}^\times$ which is very close to $(p, 1)$ in the $p$-adic topology of $\mathcal{K}_p$. Then $\text{diag}(y, y, 1)^n \in U(3, 1)(\mathbb{Q})$. Set $\beta = \beta'(y) - n \in \mathbb{Z}_p^\times \cap \mathbb{Q}$ then

$$\text{proj}_{(\frac{k-2}{2}, -\frac{k-2}{2})}(FJ_{\beta} \rho(\text{diag}(y, y, 1)_p)E)$$

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is not the zero function. So there must be a choice of \( \theta_1 \) and some weight \( (\frac{k-2}{2}, \frac{k-2}{2}) \) form \( h \) such that the \( l_{\theta_1}(\text{FJ}_\beta(E)) \) above is non-zero. The reason of making sure that \( \beta \in \mathbb{Z}_p^\times \) is that only for those \( \beta \) we did the Fourier-Jacobi coefficient computation at \( p \) for the Klingen Eisenstein series in [52].

Let \( \theta \) be a theta function on \( U(2) \) defined as in [52, Proposition 8.4]. Let \( L_5, L_6 \) be the two Katz \( p \)-adic \( L \)-functions defined in [52, Section 8.5]. By our choices at loc. cit they are units in the Iwasawa algebra. Suppose we are doing our computations at an arithmetic point \( z \) (notation as in [52]) and write \( h \) for the specialization of \( h \) at \( z \). By the doubling method for \( h \) under \( U(2) \times U(2) \hookrightarrow U(2,2) \) with the Siegel Eisenstein series on \( U(2,2) \)

\[
E(R(f_\infty, \phi_{2,i,\infty}) \cdot \prod_v f_{j_v}, -)
\]

above (see [52 Proposition 6.1] for details), we know (see [52 Proposition 8.24]) there is a constant \( C_{i,\infty} \) such that

\[
\langle l_{\theta_1}(\prod_v \sum_{j_v} E(R(f_\infty, \phi_{2,i,\infty}) \otimes_{v<\infty} f_{j_v}))(g) \Theta_{\phi_{2,i,\infty}} \prod_{v<\infty} \phi_{j_v}(ng), h \rangle = z(L_5 L_6) C_{i,\infty} \int_{[U(2)]} h(g) \cdot \theta^{\text{low}}(g) f(g) dg
\]

(4)

\[
= z(L_5 L_6) C_{i,\infty} \int_{[U(2)]} h(g) \theta^{\text{low}}(g) f(g) dg.
\]

(5)

Here the \( \theta \) is the specialization of \( \theta \) at \( z \), and the \( \theta^{\text{low}}(g) := \theta(g \left( \begin{array}{cc} 1 & 1 \\ 0 & p \end{array} \right) ) \) as in [52]. That the “\( \theta \)” part appearing is the eigen-component \( \theta \) of trivial weight comes from considering the central character. We write

\[
C_\infty = \sum_i C_{i,\infty}.
\]

Note that the \( C_\infty \) only depends on our Archimedean datum as the \( C_{i,\infty} \)’s are. We see

\[
C_\infty \neq 0
\]

by our choices.

As in [51] Section 8.5] we consider triple product expression

\[
\int \tilde{h} \tilde{f} d\mu_{\tilde{\theta}_3} = \langle \tilde{h} \tilde{f}, \tilde{\theta}_3 \rangle
\]

(6)

again Hsieh’s \( \Lambda \)-adic pairing \( \langle, \rangle \), recall the specializations of \( \tilde{h}, \tilde{f}, \tilde{\theta}_3 \) are in the dual automorphic representation space for \( h, f \) and \( \theta \), respectively). Here by the product \( \tilde{h} \tilde{f} \) we mean the scalar valued form obtained using the natural pairing between the coefficient rings of \( \tilde{h} \) and \( \tilde{f} \). This expression is interpolated by an element in \( \mathcal{O}_L[[\Gamma_K]] \). As in [52 Sections 8.4, 8.5] we appeal to Ichino’s formula to evaluate the product of the two triple product integrals above (namely the product of (1) and (5)). Note that at the Archimedean place, the representation \( L^{k-2} \) for \( \pi_f \) has dimension \( k-1 \). We
Suppose \( \pi_\ell \) is supercuspidal representation with trivial character and conductor \( p^t, t \geq 2 \) and \( \varphi_\ell \in \pi_\ell \) is a new vector. Let \( \tilde{\pi}_\ell \) be the contragradient representation of \( \pi_\ell \) and \( \tilde{\varphi}_\ell \in \tilde{\pi}_\ell \) be the new vector. Consider the matrix coefficient \( \Phi = \Phi_{\varphi_\ell, \tilde{\varphi}_\ell}(g) := (\pi(g)\varphi_\ell, \tilde{\varphi}_\ell), \) normalized such that \( \Phi_{\varphi_\ell, \tilde{\varphi}_\ell}(1) = 1. \) Then for \( g \in \text{diag}(\ell^n, 1) \left( \begin{smallmatrix} 1 & \ell^{-n}\mathbb{Z}_\ell \\ \ell^{-n}\mathbb{Z}_\ell & 1 \end{smallmatrix} \right) K_\ell, \Phi(g) \neq 0 \) only when \( n = 0. \) In that case \( \Phi(g) = 1. \) For \( g \in \text{diag}(1, \ell^n) \left( \begin{smallmatrix} 1 & \ell^{-n}\mathbb{Z}_\ell \\ \ell^{-n}\mathbb{Z}_\ell & 1 \end{smallmatrix} \right) K_\ell, \Phi(g) \neq 0 \) only when \( n = 0. \) In this case \( \Phi(g) = 1. \)

This is an easy consequence of [25, Proposition 3.1]. The following corollary follows from the above lemma and the computations in [52, Section 8.4].
Corollary 3.8. Let $\pi_\ell$ be a supercuspidal representation of $GL_2(\mathbb{Q}_\ell)$ with trivial character and conductor $\ell^t$. Let $\varphi_\ell \in \pi_\ell$ and $\tilde{\varphi}_\ell \in \tilde{\pi}_\ell$ be as above. Let $\pi_h = \pi(\chi_{h,1}, \chi_{h,2})$, $\pi_\theta(\chi_{\theta,1}, \chi_{\theta,2})$ and $\chi_{h,1}$, $\chi_{h,2}$, $\chi_{\theta,1}$, $\chi_{\theta,2}$ be characters of $\mathbb{Q}_\ell^\times$ with conductors $\ell^{t_1}$ and $t_1 > t$ such that $\chi_{h,1}\chi_{\theta,1}$ and $\chi_{h,2}\chi_{\theta,2}$ are both unramified. Suppose $f_{\chi_\theta}$ and $f_{\chi_h}$ are as in [52, Section 8.4]. Then Ichino's local triple product integral

$$I_\ell(\varphi_\ell \otimes f_{\chi_\theta} \otimes f_{\chi_h}, \tilde{\varphi}_\ell \otimes \tilde{f}_{\chi_\theta} \otimes \tilde{f}_{\chi_h}) = \text{Vol}(K_{t_1}).$$

Now we state the main theorem of this section.

Theorem 3.9. Suppose that

- There is at least one prime $\ell$ such that $\pi_{f,\ell}$ is not a principal series representation.
- The $p$ component of $\pi_f$ is an unramified principal series representation with distinct Satake parameters.
- The residual Galois representation $\bar{\rho}_f$ is absolutely irreducible over $K[\sqrt{(-1)^{\frac{p-1}{2}}}]$.
- For each $\ell | N$ non-split in $K$, the $\ell$ is ramified in $K$ and the representation $\pi_{f,\ell}$ is the Steinberg representation twisted by the quadratic unramified character.

Then we have

$$\text{char}_{\mathcal{O}[[\Gamma]]}(X_{f,K}^{Gr}) \subseteq (L_{f,K}^{Gr}).$$

Proof. The argument goes in the same way as [51, Theorem 5.3]. The assumption on $\bar{\rho}$ is made to apply the modularity lifting result as in loc. cit. (There the weight of $f$ is assumed to be 2 and the assumption is redundant). The last assumption is put to apply [21] on the vanishing of the anticyclotomic $\mu$-invariant of the $p$-adic $L$-function in our theorem and see that it is not contained in any height one prime of $\mathcal{O}_L[[\Gamma^+]]$.

4 Proof of Main Results

4.1 Beilinson-Flach Elements and Yager Modules

Now we reproduce some constructions in [51]. Recall $g$ be the Hida family of normalized CM forms attached to characters of $\Gamma_K$ with the coefficient ring $\Lambda_g := \mathbb{Z}_p[[U]]$ (the trivial character of $\Gamma_K$ is a specialization of this family). We write $L_g$ for the fraction ring of $\Lambda_g$. As in [30] let $M(f)^*$ ($M(g)^*$) be the part of the cohomology of the modular curves which is the Galois representation associated to $f$ ($g$). The corresponding coefficients for $M(f)^*$ and $M(g)^*$ is $Q_p$ and $L_g$. (Note that the Hida family $g$ is not quite a Hida family considered in loc. cit. It plays the role of a branch a there). Note also that $g$ is cuspidal (which is called “generically non-Eisenstein” in an earlier version) in the sense of [32]. We have $M(g)^*$ is a rank two $L_g$ vector space and, there is a short exact sequence of $L_g$ vector spaces with $G_{Q_p}$ action:

$$0 \to \mathfrak{F}_g^+ \to M(g)^* \to \mathfrak{F}_g^- \to 0$$

with $\mathfrak{F}_g^+$ being rank one $L_g$ vector spaces such that the Galois action on $\mathfrak{F}_g^-$ is unramified. Since $g$ is a CM family with $p$ splits in $K$, the above exact sequence in fact splits as $G_{Q_p}$ representation. For an arithmetic specialization $g_\phi$ of $g$ the Galois representation $M(f)^* \otimes M(g_\phi)^*$ is the induced
representation from \(G_K\) to \(G_Q\) of \(M(f)^* \otimes \xi_{g^*}\), where \(\xi_{g^*}\) is the Hecke character corresponding to \(g^*\). This identification will be used implicitly later. We also write \(D_{dR}(f) = (M(f)^* \otimes B_{dR})^{G_Q}\). The transition map is given by co-restriction. For \(f\) let \(D_{dR}(f)\) be the Dieudonne module for \(M(f)^*\) and let \(\eta^\vee_f\) be any basis of \(\text{Fil}^0D_{dR}(f)\). Let \(\omega^\vee_f\) be a basis of \(\frac{D_{dR}(f)}{\text{Fil}^0D_{dR}(f)}\) such that \((\omega^\vee_f, \omega_f) = 1\).

We mainly follow [38] to present the theory of Yager modules. Let \(K/\mathbb{Q}_p\) be a finite unramified extension. For \(x \in \mathcal{O}_K\) we define \(y_K/\mathbb{Q}_p(x) = \sum_{\sigma \in \text{Gal}(K/\mathbb{Q}_p)} x^\sigma[\sigma] \in \mathcal{O}_K[\text{Gal}(K/\mathbb{Q}_p)]\) (note our convention is slightly different from [38]). Let \(\mathbb{Q}_p^{ur}/\mathbb{Q}_p\) be an unramified \(\mathbb{Z}_p\)-extension with Galois group \(U\). Then the above map induces an isomorphism of \(\Lambda_{\mathcal{O}_F}(U)\)-modules

\[
y_{\mathbb{Q}_p^{ur}/\mathbb{Q}_p} : \lim_{\mathbb{Q}_p \subseteq K \subseteq \mathbb{Q}_p^{ur}} \mathcal{O}_F \simeq S_{\mathbb{Q}_p^{ur}/\mathbb{Q}_p} = \{ f \in \hat{\mathbb{Z}}_p^{ur}[[U]] : f^u = [u]f \}
\]

for any \(u \in U\) a topological generator. Here the superscript means \(u\) acting on the coefficient ring while \([u]\) means multiplying by the group-like element \(u^{-1}\). The module \(S_{\mathbb{Q}_p^{ur}/\mathbb{Q}_p}\) is called the Yager module. It is explained in loc.cit that the \(S_{\mathbb{Q}_p^{ur}/\mathbb{Q}_p}\) is a free rank one module over \(\mathbb{Z}_p\). Let \(\mathcal{F}\) be a \(\mathbb{Z}_p\) representation of \(U\) then they defined a map \(\phi : \hat{\mathbb{Z}}_p^{ur}[[U]] \rightarrow \text{Aut}(\mathcal{F} \otimes \hat{\mathbb{Z}}_p^{ur})\) by mapping \(u\) to its action on \(\mathcal{F}\) and extend linearly. As is noted in loc.cit the image of elements in the Yager module is in \((\mathcal{F} \otimes \hat{\mathbb{Z}}_p^{ur})^{G_{Q_p}}\). We define as a generator of the Yager module for \(\mathbb{Q}_p\). Then we can define \(\rho(d)\) and let \(\rho(d)^\vee\) be the element in \(\hat{\mathbb{Z}}_p^{ur}[[U]]\) which is the inverse of \(\lim_{\rho \in \mathbb{U}/\rho \equiv U} \rho d \cdot \sigma^{-1}\).

Now we recall some notations in [33]. Let \(E_{SP}(D_K) := \lim_{\mathcal{U}} H^1(X_1(D_K^{ur}) \otimes \hat{\mathbb{Q}}, \mathbb{Z}_p)\) and \(GES_p(D_K) := \lim_{\mathcal{U}} H^1(Y_1(D_K^{ur}) \otimes \hat{\mathbb{Q}}, \mathbb{Z}_p)\) which are modules equipped with Galois action of \(G_Q\). Here \(Y_1(D_K^{ur})\) and \(Y_1(D_K^{pr})\) are corresponding compact and non-compact modular curves. Recall in loc.cit there is an ordinary idempotent \(e^*\) associated to the covariant Hecke operator \(U_p\). Let \(\mathcal{A}_{\infty}^* = e^*E_{SP}(D_K)^{U_p} = e^*GES_p(D_K)^{U_p}\) (see the Theorem in loc.cit). Let \(\mathcal{B}_{\infty}^*\) be the quotient of \(e^*E_{SP}(D_K)\) over \(\mathcal{A}_{\infty}^*\).

In an earlier version of [32] the authors defined elements \(\omega_g^\vee \in (\mathcal{F}_g^+(\chi_g)^{-1} \otimes \hat{\mathbb{Z}}_p^{ur})^{G_{Q_p}}\) and \(\eta_g^\vee \in (\mathcal{F}_g^-(\chi_g)^{-1} \otimes \hat{\mathbb{Z}}_p^{ur})^{G_{Q_p}}\). Here the \(\chi_g\) is the central character for \(g\). We briefly recall the definitions since they are more convenient for our use (these notions are replaced by their dual in the current version of [32]). In the natural isomorphism

\[
\mathcal{A}_* \otimes \mathbb{Z}_p[[T]] \simeq \text{Hom}_{\mathcal{Z}_p}((\mathcal{S}_{ord}(D_K, \chi_K), \hat{\mathbb{Z}}_p^{ur}[[T]]), \hat{\mathbb{Z}}_p^{ur}[[T]])
\]

(see the proof of [43, Corollary 2.3.6]), the \(\omega_g^\vee\) is corresponds to the functional which maps each normalized eigenform to 1. On the other hand \(\eta_g^\vee\) is defined to be the element in \(\mathcal{B}_{\infty}^*\) which, under the pairing in [43, Theorem 2.3.5], pairs with \(\omega_g^\vee\) to the product of local root numbers at primes to \(p\) places of \(g\). This product moves \(p\)-adic analytically and is a unit.

We take basis \(v^\pm\) of \(\mathcal{F}_g\) with respect to which \(\omega_g^\vee\) and \(\eta_g^\vee\) are \(\rho(d)^\vee v^+\) and \(\rho(d)^- v^-\) (see the discussion for Yager Modules). Let \(\Psi_g\) be the \(G_{\mathbb{Q}}\)-valued Galois character of \(G_K\) corresponding to the Galois representation associated to \(g\) (i.e. \(\text{Ind}_{G_{\mathbb{Q}}}^{G_K}(\Psi_g) = M(g)^*\)). Since \(p\) splits as \(\mathfrak{m}\mathfrak{g}\mathfrak{m}\mathfrak{m}\) in \(\mathfrak{K}\), there is a canonical identification \((\text{Ind}_{G_{\mathbb{Q}}}^{G_K}(\Psi_g))|_{G_{\mathbb{Q}}} \simeq \Psi_{g}|_{G_{\mathbb{Q}}} \oplus \Psi_{g}|_{G_{\mathbb{Q}}}\) and can take a \(\Lambda_g\)-basis of the right side as \(\{v, c \cdot v\}\) where \(c\) is the complex conjugation. (Note that there are two choices for the \(\Psi_g\) and we choose the one so that \(\Psi_g|_{G_{\mathbb{Q}}}\) corresponds to \(\mathcal{F}_g^\pm\).)
Conventions
we use the basis \{v^+, c \cdot v^+\} to identify the Galois representation of \(g\) with the induced representation \(\text{Ind}_{G_q}^K \Psi_g\).

In [37], the authors constructed Beilinson-Flach elements \(BF_\alpha = BF_{f,g} \in H^1_{cl,1w}(Q_\infty, T_f \otimes T_g) \otimes \Lambda_{A\infty}\). (That these classes are in the modular on the right hand side is explained in discussion after [51 Definition 7.2].) Recall that these classes are obtained from writing down classes in \(H^3_{\text{mot}}(Y_1(N)^2,Q_p, T^{[k,k']}_\text{sym}(\mathcal{H}_Q,2-j))\) explicitly for cuspidal eigenforms \(f\) and \(g\) with weights \(k\) and \(k'\) respectively and \(j + 1 < k, k'\), and consider the image under the map

\[
H^3_{\text{mot}}(Y_1(N)^2,Q_p, T^{[k,k']}_\text{sym}(\mathcal{H}_Q,2-j)) \xrightarrow{r^i} H^3_{\text{et}}(Y_1(N)^2,Q_p, T^{[k,k']}_\text{sym}(\mathcal{H}_Q,2-j)) \xrightarrow{\text{AJ}_{f,g}} H^1(Q_p, M_1(f \otimes g)^*(2-j)).
\]

Here \(T^{[k,k']}_\text{sym}\) means the \([k,k']\)-component in the symmetric tensor product of the universal elliptic curve \(\mathcal{H}\) over the modular curves. The \(r^i\) is the etale regulator map and \(\text{AJ}_{f,g}\) is the etale Abelian-Jacobi map, followed by projecting to the \((f \otimes g)\)-component. Deforming \(f\) in a Coleman family \(\mathcal{F}\) and varying the \(k, k'\) and \(j\) in \(p\)-adic families one gets the three-variable Beilinson-Flach class, which specializes to the two-variable class under \(\mathcal{F} \rightarrow f\).

4.2 Control Theorem of Selmer Groups
Let \(P \in \text{Spec}\Lambda\) be a generic arithmetic point (i.e. corresponding to a finite order character of \(\Gamma\)) and \(v \in H^1_{lv}(Q_{p,\infty}, T \otimes \Lambda)\) an element of a \(\Lambda\)-basis of the latter such that the image of \(v\) is an \(\Lambda/P\)-basis of \(H^1_{f}(Q_{p,\infty}, T \otimes \Lambda/P)\). Then it is easy to see that \(v\) satisfies the following

(*) For all but finitely many integers \(m\) and \(x := \gamma - (1 + p)^m\), the map \(H^1(Q_{p,\infty}, T) \xrightarrow{xH^1(Q_{p,\infty}, T)} H^1(Q_{p,\infty}, T)\)

is injective.

Picking up such a \(v\) is important, especially when we prove that certain module of dual Selmer group has non pseudo-null submodules later on. (A pseudo-null submodule over \(\Lambda\) means it has finite cardinality.) We consider the control theorem for \(v\)-Selmer groups. This means the Selmer condition which is the usual one at primes outside \(p\), but is the orthogonal complement of \(\Lambda v\) at \(p\) under Tate local pairing. We look at the following diagram

\[
0 \rightarrow \text{Sel}_{v^\prime}(Q, A[P]) \rightarrow H^1(Q^S/Q, A[P]) \rightarrow \mathcal{P}_{v^\prime}(Q, A[P])
\]

\[
0 \rightarrow \text{Sel}_{v^\prime}(Q_{\infty})^P \rightarrow \lim_{\rightarrow n} H^1(Q^S_n/Q_n, A)^P \rightarrow \mathcal{P}_{v^\prime}(Q_{\infty}, A)
\]

where \(\mathcal{P}_{v^\prime}(Q, A) = \prod_{\ell \neq p} H^1_{\ell}(G_{\ell}, A) \times H^1_{(G_{\ell}, A)^{\vee}}(\Lambda)^{\vee}\) and \(\mathcal{P}_{v^\prime}(Q, A) = \prod_{\ell \neq p} H^1_{\ell}(G_{\ell}, A[P]) \times H^1_{((G_{\ell}, A)[P])}((\Lambda/P)\text{Im}(v))^{\vee}\).

We define the Tamagawa number of \(A[P]\) at \(\ell \neq p\) to be

\[
c_{P,d} = \# \ker \{H^1_{\ell}(G_{\ell}, A[P]) \rightarrow H^1_{\ell}(I_{\ell}, A[P])\}.
\]

Also let \(c'_{P,d} \in O_L \otimes \Lambda/P\) be such that

\[
c_{P,d} = \#(O_L \otimes (\Lambda/P)/c'_{P,d}O_L \otimes (\Lambda/P)).
\]

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We define a number \( c_{P,p} \) at \( p \) as follows (up to a \( p \)-adic unit): let \( V \) be a \( \mathcal{O}_P \)-basis of \( H^1(\mathbb{Q}_p, T \otimes \mathbb{A}/P) \), then
\[
\exp^* V = c_{P,p}^f \omega_f, \quad c_{P,p} := \#(\mathcal{O}_L \otimes (\mathbb{A}/P) \mathcal{O}_L \otimes (\mathbb{A}/P))
\] (7)
Here we identify the \( T_f \) with its realization in the cohomology of the modular curve.

**Remark 4.1.** We discuss a little about the relations between this \( c_{P,p} \) and Tamagawa numbers. We first note that in the Fontaine-Laffaille range \( k < p \), the number is actually a local number. This can be seen using the integral comparison theorem between crystalline and deRham cohomology of modular curves. For details see Section 4.4. Keep this assumption, suppose \( P \) corresponds to the trivial character of \( \Gamma \). Then \( T \otimes \mathbb{A}/P \) is crystalline. Then its Tamagawa number is defined in [35, (5.6)] as
\[
\#(\mathcal{O}_L \otimes (\mathbb{A}/P) \mathcal{O}_L \otimes (\mathbb{A}/P))/\det(1 - \varphi|_{\text{cris}}(V)).
\]
For more backgrounds justifying this definition see [5].

For \( \chi \) a character of \( \text{Gal}(\mathbb{Q}_{p,n}/\mathbb{Q}_p) \) with coefficient ring \( E \), in literature people usually use the convention that
\[
D_{\text{DR}}(V \otimes \chi) = D_{\text{DR}}(V) \otimes D_{\text{DR}}(\chi) = D_{\text{DR}}(V) \otimes (\mathbb{Q}_{p,n} \otimes E)^\chi.
\]
Recall when defining the \( \exp^* \) map for \( T_P \), the pairing on both Galois cohomology and Dieudonné module come from
\[
(T_P \otimes \mathbb{Q}_p) \times (T_P \otimes \mathbb{Q}_p) \to \mathbb{Q}_{p,n}(1) \to \mathbb{Q}_p(1)
\]
where the last is the trace map. By the formula in [38, Lemma B.4], we see for any \( a \in K_n \)
\[
\text{tr}_{K_n/\mathbb{Q}_p}(a(\langle \exp^* x, \log y \rangle - (x,y))) = 0.
\]
Thus \( \langle \exp^* x, \log y \rangle = (x,y)_{K_n} \). We observe that the pairing of cup product
\[
(\cdot, \cdot) : H^1(\mathbb{Q}_p, T \otimes \mathbb{A}/P) \times H^1(\mathbb{Q}_p, T \otimes \mathbb{A}/P^e) \to \mathcal{O}_L
\]
is surjective. Thus for some \( \mathcal{O}_P \)- basis \( \tilde{V} \) of \( H^1(\mathbb{Q}_p, T \otimes \mathbb{A}/P^e) \), we have
\[
\log \tilde{V} = \frac{1}{c_{P,p}} \cdot \omega_f^\gamma.
\] (8)

**Remark 4.2.** In the case when \( f \) is ordinary at \( p \), or \( f \) corresponds to a supersingular elliptic curve with \( a_p = 0 \), it is not hard to explicitly compute the \( c_{p,p} \) using the local theory, e.g. in [51].

The Poitou-Tate exact sequence implies that if \( \#(\text{Sel}_V(\mathbb{Q}, \mathbb{A}[P])) < \infty \) and \( H^1(\mathbb{Q}_S/\mathbb{Q}, T/\mathbb{P}^eT) = 0 \), then
\[
\prod_{\ell} c_{P,\ell} \text{FittSel}_V(\mathbb{Q}, \mathbb{A}[P]) = \text{Fitt}X_V/\text{PX}_V.
\]

**Lemma 4.3.** The cardinality \( \#(H^2(\mathbb{Q}_S/\mathbb{Q}, T))[x] < \infty \) for all but finitely many \( m \)'s and \( x = \gamma - (1+p)^m \).
Proof. The \( H^2(Q^S/Q, T) \) is a finitely generated \( \Lambda \)-module. Then the lemma follows from the well known structure theorem of finitely generated \( \Lambda \)-modules.

Lemma 4.4. The \( \frac{H^1(Q^S/Q, A)}{xH^1(Q^S/Q, A)} \) is a finitely generated \( \Lambda \)-module. Then the lemma follows from the well known structure theorem of finitely generated \( \Lambda \)-modules.

Proof. We have

\[
\frac{H^1(Q^S/Q, A)}{xH^1(Q^S/Q, A)} \hookrightarrow H^2(Q^S/Q, A[x]).
\]

From the Global duality the right side is dual to

\[
\ker \{ H^1(Q^S/Q, T_x) \to \prod_{v \in S} H^1(Q_v, T_x) \}.
\]

We claim this term is 0 for all but finitely many \( m \). Indeed \( H^1(Q^S/Q, T_x) \) is \( p \)-torsion free by (Irred). Moreover we have exact sequence

\[
\frac{H^1(Q^S/Q, T)}{xH^1(Q^S/Q, T)} \hookrightarrow H^1(Q^S/Q, T_x) \to H^2(Q^S/Q, T)[x].
\]

The last term is finite for all but finitely many \( m \) by lemma \( 4.3 \). The \( H^1(Q^S/Q, T) \) is a torsion-free rank one \( \Lambda \)-module such that the localization map \( H^1(Q^S/Q, T) \to H^1(Q_p, T) \) is injective. (Because by \( 46 \) the image of \( z_{Kato} \) under this map is non-zero). Now it is easy to see that

\[
\ker \{ H^1(Q^S/Q, T_x) \to H^1(Q_p, T_x) \}
\]

is 0 for all but finitely many \( m \). The lemma follows readily.

Proposition 4.5. The \( X_{\psi^1} \) has no pseudo-null submodules.

Proof. Let \( x = \gamma - (1 + p)^m \) for some integer \( m \). Then we claim for all but finitely many integers \( m \) we have surjection

\[
H^1(Q^S/Q, A[x]) \to P_{\psi^1}(Q, A[x]).
\]

We first look at the exact sequence

\[
H^1(Q^S/Q, T) \xrightarrow{x} H^1(Q^S/Q, T) \to H^1(Q^S/Q, T_x) \to H^2(Q^S/Q, T)[x].
\]

From Lemma \( 4.3 \) the last term is torsion for all but finitely many \( m \). By property (*) on \( v \) we get

\[
H^1_v(Q^S/Q, T_x) = 0
\]

for these \( x \). From Poitou-Tate exact sequence

\[
H^1_v(Q^S/Q, T_x) \to P_v(Q, T_x) \to H^1(Q^S/Q, A[x])^\vee,
\]

we get the claim.

It is also clear that the map \( H^1(Q^S/Q, A[x]) \to H^1(Q^S/Q, A)[x] \) is an isomorphism, and that the map

\[
P_{\psi^1}(Q, A[x]) \to P_{\psi^1}(Q, A)[x]
\]

is also an isomorphism.
is surjective. These altogether imply
\[ H^1(Q^S/Q, A)[x] \to \mathcal{P}_{v^\vee}(Q, A)[x] \]
is surjective.

Then consider the following diagram
\[
\begin{array}{cccc}
0 & \longrightarrow & \text{Sel}_{v^\vee}(Q, A) & \longrightarrow & H^1(Q^S/Q, A) & \longrightarrow & \mathcal{P}_{v^\vee}(Q, A) \\
\downarrow x & & \downarrow x & & \downarrow x & & \\
0 & \longrightarrow & \text{Sel}_{v^\vee}(Q, A) & \longrightarrow & H^1(Q^S/Q, A) & \longrightarrow & \mathcal{P}_{v^\vee}(Q, A)
\end{array}
\]
By Snake lemma and Lemma 4.4 the \( \text{Sel}_{v^\vee}(Q, A) \) is surjective.

By Nakayama’s lemma, there is no quotient of \( \text{Sel}_{v^\vee}(Q, A) \) of finite cardinality. Thus \( X_{v^\vee} \) has no pseudo-null submodules.

**Definition 4.6.** Write \( F \) for the characteristic polynomial of \( X_{v^\vee} \) and write \( \phi \) for the arithmetic point corresponding to \( P \). We also write \( X_{v^\vee, \phi} \) for the Selmer group for \( A[\phi] \).

Note that the local Selmer condition of it at all primes outside \( p \) is \( \{0\} \). Thus the control theorem as before implies that
\[
\prod_{\ell \mid p} c_{P, \ell}(f)(X_{v^\vee, \phi}) = \#(X_{v^\vee}/PX_{v^\vee}).
\]
(10)

Note for any finitely generated torsion Iwasawa module \( M \) of \( \Lambda \), if \( (x) \) is a prime ideal of \( \Lambda \) with \( \#(M/xM) < \infty \), and \( \mathcal{F} \) is a generator of \( \text{char}_\Lambda(M) \), then
\[
\#(M/xM) \geq \#(\Lambda/(\mathcal{F}, x)).
\]
If \( M \) has no pseudo-null submodule then the above “\( \geq \)” is an “\( = \)”.

So the control theorem argument as before implies that
\[
\prod_{\ell \mid p} c_{P, \ell}(f)(X_{v^\vee, \phi}) = \#(\mathcal{O}_\phi/(\mathcal{F}(\phi))).
\]
(11)

### 4.3 Selmer Complexes and Iwasawa Main Conjecture

Before continuing we explain how we make choice for the quadratic imaginary field \( K \). We choose it to be ramified in the prime \( \ell \) in the assumption of Theorem 1.4 split at \( p \), and is split at all other primes of \( N \). We also ensure that the \( \bar{T} \) is still irreducible over \( G_{K(\zeta_p)} \). Let \( G_{K,S} \) be the Galois group of \( K \) unramified outside \( S \).

We follow [43], [45] to present the analytic Iwasawa theory, in the framework of Nekovar’s Selmer complex. Let \( U \simeq \mathbb{Z}_p \) be the Galois group \( \text{Gal}(Q_{ur}^p/Q_p) \). We define \( \mathcal{A} \) to be the affinoid ring \( \mathcal{O}_L(p^{-r}U) \) for some \( r > 0 \) and \( \Lambda_{A,\infty} = \Lambda_{\infty} \mathcal{O}_{\mathcal{A}} \). We fix a local condition, which means for any \( v \in S \) a bounded complex of finite type \( \Lambda_{A,\infty} \)-modules \( U_v^\bullet \) and a morphism
\[
i_v : U_v^\bullet \to C^\bullet_{cont}(G_v, T \otimes \Lambda_{A,\infty}).
\]
Here we write $C^{ullet}_\text{cont}(G,M)$ for the space of continuous cochains of the $G$-module $M$. We define the Selmer complex for $f$ over $\mathcal{K}$ to be the mapping cone

$$\text{Cone}[C^\bullet_\text{cont}(G_{K,S}, T \otimes Q_L \Lambda_{A,\infty}) \oplus \oplus_{v \in S} U_v \to \oplus_{v \in S} C^\bullet_\text{cont}(G_v, T \otimes \Lambda_{A,\infty})][-1],$$

where the map is given by $\oplus_v (\text{res}_v, -i_v)$. Throughout this paper as in [4], for each $v \in S$ not dividing $p$, we use the unramified local condition by

$$i_v : U_v^\bullet : C^\bullet_\text{cont}(G_v/I_v, (T \otimes \Lambda_{A,\infty})^I_v) \to C^\bullet_\text{cont}(G_v, T \otimes \Lambda_{A,\infty}).$$

We will make several different choices for the local Selmer conditions at $p$. We need some preparations.

**Definition 4.7.** Write $\mathcal{R}$ for the Robba ring $B^+_{\text{rig},Q_p}$ over $Q_p$ and $\mathcal{R}^+$ for $B^+_{\text{rig},Q_p}$. We define a triangulation of a two-dimensional $(\varphi, \Gamma)$-modules $D$ over $\mathcal{R}$ to be a short exact sequence $0 \to F^+D \to D \to F^-D \to 0$ where $F^+D$ are free rank one $(\varphi, \Gamma)$-modules over $\mathcal{R}$. For any finite extension $L$ of $Q_p$, we define $\mathcal{R}_L = \mathcal{R} \otimes_{Q_p} L$ and can talk about triangulations of $(\varphi, \Gamma)$-modules of rank two over $\mathcal{R}_L$.

**Definition 4.8.** If $V$ is a two dimensional crystalline representation of $G_{Q_p}$. A refinement of $V$ is a full $\varphi$-stable filtration of $D_{\text{cris}}(V)$:

$$F_0 = 0 \subseteq F_1 \subseteq F_2 = D_{\text{cris}}(V).$$

This is equivalent to an ordering of $\{\alpha, \beta\}$.

It is summarized in [4, 2.4] that there is a one-to-one correspondence between triangulations of $D(V)$ and refinements of $V$, given by $F^+ = \mathcal{R}[1/t]F_1 \cap D$ and $F_1 = F^+[1/t] \Gamma$. Let the Robba ring over $A$ be $\mathcal{R}_A := \mathcal{R} \hat{\otimes} A$. We also write $\mathcal{R}^+_A = \mathcal{R}^+ \hat{\otimes} A$. There is a nature action $U \hookrightarrow A^\times$. Then we can define a $(\varphi, \Gamma)$-module $D_A$ over $\mathcal{R}_A$ by pulling back the action of $\Gamma$ on $D$ but twisting the action of $\varphi$ on $D$ by the Frobenius action as above. One can define triangulation for families of $(\varphi, \Gamma_K)$-modules over $\mathcal{R}_A$ similarly. We define the analytic Iwasawa cohomology for $D_A$ in the same way as [2.3.4]

**Lemma 4.9.** The $H^1_{\text{Iw}}(Q_p, D_A)$ can be computed using the complex

$$D_A \xrightarrow{\psi^{-1}} D_A$$

concentrated at degrees 1 and 2.

This is just [29, Theorem 4.4.8]. Suppose $D$ has the form $\mathcal{R}(\alpha^{-1})$, then we have

**Proposition 4.10.** There is an exact sequence

$$0 \to \oplus_{m=0}^\infty (t^{m} D_{\text{cris}}(D_A))^{\varphi=1} \to (\mathcal{R}^+_A \otimes D)^{\psi=1} \to (\mathcal{R}^+_A \otimes D)^{\psi=0} \to \bigoplus_{m=0}^N \frac{t^{m} \otimes D_{\text{cris}}(D_A)}{(1 - \varphi)(t^{m} \otimes D_{\text{cris}}(D_A))}$$

where the third arrow is given by $\varphi - 1$ and $N >> 0$. Note that $\alpha, \beta$ are Weil numbers of odd weight $k - 1$. The second term above is easily seen to be 0.
This proposition is the family version of [41, Lemma 3.18]. The analogues in our setting of the results in Lemma 3.17 of loc.cit is proved in [39, Section 2] as well. For the above rank one \((\varphi, \Gamma)\)-module\(D\) we consider a finite set of height one primes \(S(D)\) of \(\Lambda_{K,\infty}\), each generated by an element of the form \((U + 1 - \alpha p^{-m})\) for some non-negative integer \(m\). Note that this is a finite set because for \(m >> 0\), \((U + 1 - \alpha p^{-m})\) is invertible. For any height one prime \(P\) not in \(S(D)\) the localized map at \(P\)

\[
\varphi - 1 : (\mathcal{A}_A \otimes D)_P^{\psi=1} \rightarrow (\mathcal{A}_A \otimes D)_P^{\psi=0}
\]

is an isomorphism. Now suppose \(D\) is the \((\varphi, \Gamma)\)-module of \(V_f = T_f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\). We fix a triangulation of \(D\) by requiring \(\mathcal{F}^- := D/\mathcal{F}^+\) to be the \((\varphi, \Gamma)\)-module \(\mathcal{R}(\alpha^{-1})\). We thus define an induced triangulation of \(D_A\) in the obvious way. We write the corresponding modules as \(\mathcal{F}_A^\pm = \mathcal{F}^\pm(D_A)\). For any \((\varphi, \Gamma)\)-module of the form \(\mathcal{R}_A(\alpha^{-1})\) for some \(\alpha \in A^*\), we define a regulator map as [30, (6.2.1)]

\[
\text{Reg}_{\mathcal{R}_A(\alpha^{-1})} : H^1_{\text{Iw}}(\mathbb{Q}_p, \mathcal{R}_A(\alpha^{-1}) \longrightarrow \mathcal{R}_A(\alpha^{-1})^{\psi=1}
\]

\[
\longrightarrow \mathcal{R}_A(\alpha^{-1})^{\psi=1} 1 - \varphi \longrightarrow \mathcal{R}_A(\alpha^{-1})^{\psi=0} \longrightarrow \mathcal{A} \otimes \Lambda_{\mathbb{Q}, \infty}.
\]

The last is the Mellin transform. As in [30, Section 6] since \(\mathcal{F}_A^{-}\) has the form \(\mathcal{R}_A(\alpha^{-1})\) so we can define the regulator map \(\text{Reg}_{\mathcal{F}^-}\) as above. The \(\mathcal{F}_A^+\) is of the form \(\mathcal{R}_A(\beta^{-1})\) twisted by some character factoring through \(\Gamma\) (in the cyclotomic direction), so as in loc.cit we can still define the regulator map \(\text{Reg}_{\mathcal{F}^+}\) by re-parameterizing the weight space. We have an exact sequence

\[
0 \rightarrow H^1_{\text{Iw}}(\mathbb{Q}_p, \mathcal{F}^+(D_A)) \rightarrow H^1_{\text{Iw}}(\mathbb{Q}_p, D_A) \rightarrow H^1_{\text{Iw}}(\mathbb{Q}_p, \mathcal{F}^-(D_A)) \rightarrow 0.
\]

(As noted in [44, Proof of Proposition 2.9], by the machinery developed in [45] we only need to check that for any specialization of \(\mathcal{F}_A^+\) twisted by a character of \(\Gamma\), the \(H^2\) becomes trivial. This is easily seen by the local duality.)

**Definition 4.11.** We define the \(\alpha\) local Selmer condition \(U_p\) at \(p\) as the second arrow above, using the identification [45, Theorem 2.8] of the derived category of Galois cohomology of Galois representations over an affinoid algebra and the corresponding \((\varphi, \Gamma)\)-module.

By Proposition 4.10 for any \(P \notin S(D)\) the regulator map \(\text{Reg}\) gives isomorphisms

\[
H^1(\mathbb{Q}_p, \mathcal{F}_A^+)_{\alpha, \infty} \simeq \Lambda_{\mathbb{A}, \infty, P},
\]

\[
H^1(\mathbb{Q}_p, \mathcal{F}_A^-)_{\alpha, \infty} \simeq \Lambda_{\mathbb{A}, \infty, P}.
\]

We also have

\[
H^1(\mathbb{Q}_p, D)_{\alpha, \infty} \simeq \Lambda^2_{\mathbb{A}, \infty, P}.
\]

Denote \(\tilde{\Gamma}(G_{K,S}, U^*_v, T \otimes \Lambda_{\mathbb{A}, \infty})\) as the image of the Selmer complex in the derived category of finite type \(\Lambda_{\mathbb{A}, \infty}\)-modules. Let \(\tilde{H}^i(G_{K,S}, T \otimes \Lambda_{\mathbb{A}, \infty})\) be the cohomology groups of the Selmer complex, which are called extended Selmer groups. Replacing \(\Lambda_{\mathbb{A}, \infty}\) by \(\Lambda_{\mathbb{Q}, \infty}\) we similarly define \(\tilde{H}^i(G_{K,S}, T \otimes \Lambda_{\mathbb{Q}, \infty})\). We record here some properties of the Selmer complex.

**Proposition 4.12.** [44, Theorem 4.1]\\

\[
\tilde{\Gamma}_{\alpha}(G_{K,S}, U^*_v, T \otimes \Lambda_{\mathbb{A}, \infty}) \simeq \Lambda_{\mathbb{Q}, \infty}.
\]

In particular

\[
0 \rightarrow \tilde{H}_{\alpha}^1(G_{K,S}, T \otimes \Lambda_{\mathbb{A}, \infty}) \otimes \Lambda_{\mathbb{Q}, \infty} \rightarrow \tilde{H}_{\alpha}^1(G_{K,S}, T \otimes \Lambda_{\mathbb{Q}, \infty}) \rightarrow \text{Tor}^1_{\mathbb{A}, \infty}(\tilde{H}_{\alpha}^1(G_{K,S}, T \otimes \Lambda_{\mathbb{Q}, \infty}), \Lambda_{\mathbb{Q}, \infty}) \rightarrow 0.
\]
Definition 4.13. We define $X_{\alpha\alpha}$ to be $\tilde{H}_2^{rel}(G_{K,S}, T \otimes \Lambda_{A,\infty})$ defined using the $\alpha$-Selmer conditions at both $v_0$ and $\bar{v}_0$. Similarly we define $X_{0,rel}$, $X_{\alpha,rel}$, etc (here $\alpha$ rel stands for “relaxed”).

It follows from the definition of the Selmer complex that

$$0 \to \tilde{H}_{1,rel}^1(G_{K,S}, T \otimes \Lambda_{A,\infty}) \to H^1(G_{K,S}, T \otimes \Lambda_{A,\infty}) \to H^1(G_{v_0}, F^-_A) \to \tilde{H}_{2,rel}^2(G_{K,S}, T \otimes \Lambda_{K,\infty}) \to H^2(G_{K,S}, T \otimes \Lambda_{A,\infty}) \to \oplus_{v|p} H^2(G_{K,v}, T \otimes \Lambda_{A,\infty}). \tag{12}$$

(See [44, Page 18], especially the computation of local Galois cohomology at $v \nmid p$. Note also that

$$0 = H^2_{Iw}(G_p, D_A) = H^2_{Iw}(G_p, F^+_A) = H^2_{Iw}(G_p, F^-_A).$$

Also

$$0 \to \tilde{H}_{0,rel}^1(G_{K,S}, T \otimes \Lambda_{A,\infty}) \to H^1(G_{K,S}, T \otimes \Lambda_{A,\infty}) \to H^1(G_{v_0}, D) \to \tilde{H}_{0,rel}^2(G_{K,S}, T \otimes \Lambda_{A,\infty}) \to H^2(G_{K,S}, T \otimes \Lambda_{A,\infty}) \to \oplus_{v|p} H^2(G_{K,v}, T \otimes \Lambda_{A,\infty}). \tag{13}$$

Denote the third arrows of $\tag{12}$ and $\tag{13}$ as (A) and (B). Define Ker by the exact sequence

$$0 \to \text{Ker} \to \frac{H^1(G_{v_0}, D_A)}{\text{im}(B)} \to \frac{H^1(G_{v_0}, F^-_A)}{\text{im}(A)} \to 0.$$  

Then

$$\text{Ker} = \frac{\text{im}(B) + H^1(G_{v_0}, F^+_A)}{\text{im}(B)} \approx \frac{H^1(G_{v_0}, F^+_A)}{\text{im}(B) \cap H^1(G_{v_0}, F^+_A)} \approx \frac{H^1(G_{v_0}, F^+_A)}{\text{im}(H^2_{Iw,rel}(G_{K,S}, T \otimes \Lambda_{A,\infty})).}$$

Combining this with $\tag{12}$ and $\tag{13}$ we obtain

$$0 \to \frac{H^1(G_{v_0}, F^+_A)}{\text{im}(H^2_{Iw,rel}(G_{K,S}, T \otimes \Lambda_{A,\infty}))) \to \tilde{H}_{0,rel}^2(G_{K,S}, T \otimes \Lambda_{A,\infty}) \to \tilde{H}_{1,rel}^2(G_{K,S}, T \otimes \Lambda_{A,\infty}) \to 0. \tag{14}$$

Similarly we get

$$0 \to \frac{H^1(G_{v_0}, F^-_A)}{\text{im}(H^2_{Iw,rel}(G_{K,S}, T \otimes \Lambda_{A,\infty}))} \to \tilde{H}_{2,rel}^2(G_{K,S}, T \otimes \Lambda_{A,\infty}) \to \tilde{H}_{1,rel}^2(G_{K,S}, T \otimes \Lambda_{A,\infty}) \to 0. \tag{15}$$

Before continuing we need the following

Lemma 4.14. The $H^1_{\alpha,rel}(G_{K,S}, T \otimes \Lambda_{A,\infty})$ has rank one over $\Lambda_{A,\infty}$.

Proof. We only need to know that $H^1_{\alpha\alpha}(G_{K,S}, T \otimes \Lambda_{A,\infty}) = 0$. We need to use the

Lemma 4.15. [3, Proposition 5] Let $\chi$ be a finite order character of $\Gamma$. Then

$$H^1_{\alpha}(\mathbb{Q}_p, T \otimes \chi) = H^1_f(\mathbb{Q}_p, T \otimes \chi).$$

Now if $H^1_{\alpha\alpha}(G_{K,S}, T \otimes \Lambda_{A,\infty}) = 0$ has positive rank, then for all $\phi$ we have the Bloch-Kato Selmer group of $T_\chi$ has positive rank, which contradicts the main theorem of [28].
We have
\[ \text{loc}_{\nu} BF_{\alpha} \in H_{1W}(\mathbb{Q}_p, F^+(D_A)) \]
by [30, Section 7].

**Definition 4.16.** We define the congruence number \( c_f \) of \( f \). Consider the localized Hecke algebra \( T_{m_f} \) acting on the space of \( \mathcal{O}_L \)-valued cusp forms with respect to \( \Gamma_0(N) \), where \( m_f \) is the maximal ideal corresponding to \( f \). Then \( T_{m_f} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq L \oplus B \) for some algebra \( B \), where the \( L \) corresponds to projecting to \( f \)-component. Let \( 1_f \) be the idempotent corresponding to this \( L \). On the other hand we write \( \ell_f \) for the generator of the rank one \( \mathcal{O}_L \)-module \( T_{m_f} \cap L \). Let \( c_f := \frac{\ell_f}{1_f} \).

Now we define several \( p \)-adic \( L \)-functions.

**Definition 4.17.** We define a Rankin-Selberg \( p \)-adic \( L \)-function \( L_{f \otimes g} \) (here we fix one Hecke eigenvalue \( \alpha \) of \( f \) at \( p \)). Notice the difference from the previously defined \( L_{f \otimes g}^{\text{Hida}} \) which interpolates critical Rankin-Selberg \( L \)-values where the specializations of \( g \) has weight higher than \( f \). We multiply the “geometric” \( p \)-adic \( L \)-function for the Rankin-Selberg product \( f \otimes g \) constructed in [30, Appendix], interpolating the critical values of the Rankin-Selberg \( L \)-values of \( f \) and specializations of \( g \) whose weight is less than the weight of \( f \). (In fact there is a gap in the construction of Urban’s Rankin-Selberg \( p \)-adic \( L \)-functions as noted in the appendix of [30]. Loeffler have been able to resolve the problem [30] and proved interpolation formulas of their “geometric” \( p \)-adic \( L \)-functions at all points we are interested in.) Then we multiply it by the congruence number of \( f \) and denote the product as \( L_{f \otimes g} \). In the special case here where \( g \) comes from families of characters of \( \Gamma \) we also denote it as \( L_{\text{aa}} \). (Note that the period for the “geometric” \( p \)-adic \( L \)-function is the Petersson inner product of \( f \) with itself. Such Petersson inner product, divided by the congruence number of \( f \) is the so called canonical period of \( f \).)

We also define the \( p \)-adic \( L \)-function \( L_{\alpha}(f) \) of \( f_\alpha \) over \( \mathbb{Q} \) by requiring that for any finite order character \( \chi \) of \( \Gamma \) with conductor \( p^n \ (n \geq 2) \) and \( r + \frac{k-2}{2} \in [1, k-1] \),
\[
L_{\alpha}(f)(\varepsilon \chi^{-1}) = \frac{(r + \frac{k-2}{2} - 1)!p^{n(r + \frac{k-2}{2})} \alpha^{-n} g(\chi)^{-1} L(p)(f, \chi, r + \frac{k-2}{2})}{(2\pi i)^k \frac{1}{2} + r \Omega_f^{-1} \chi^{(-1)^{r-1}}}. 
\]

The reciprocity law in loc.cit implies that under the convention at the end of Section 14
\[
\langle \text{Reg}_{\nu_0, F^{-}}(BF_{\alpha}), \eta_f \rangle = L_{\text{aa}} / c_f, \tag{16}
\]
\[
\langle \text{Reg}_{\nu_0, F^{+}}(BF_{\alpha}), \omega_f \rangle = L_{f, K}^{G_{F}} \cdot \frac{1}{h_{K} L_{K}^{\text{Katz}}} \cdot \frac{v_{-}}{c \cdot v^{+}}. \tag{17}
\]
We say one word about different conventions about the \( \omega_{g}^{\nu} \) and \( \eta_{g}^{\nu} \): in [30] the \( D(F^{-} \mathcal{M}_g^{*}) \) is identified with the \( (F^{-} \mathcal{M}_g^{*} \otimes \widehat{\mathbb{Z}}_{p}^{ur})^{G_{F}} \) via a choice of the \( \rho(d) \). We need the following

**Lemma 4.18.** If we identify the coefficient ring of \( g \) with \( \mathbb{Z}_p[[U]] \). Then up to some powers of \( U \) we have
\[
\frac{h_{K} L_{K}^{\text{Katz}}}{v^{-}} \cdot \frac{v_{-}}{c \cdot v^{+}} \]
is in \( \mathbb{Z}_p^{ur}[[U]] \).
Proposition 4.19. \( (15) \) we arrived at the following

\[ S \]

In fact one can prove that the set 

\[ v \]

Note that the parameter 

Remark 4.20. \( 0 \)

since the latter is not identically 

for those primes. Observe that when specializing to the cyclotomic line any prime 

in terms of characteristic ideal, we first observe that we only need to prove the equality for ideals after tensoring with \( \Lambda_{Q,n} \) for each \( n \). We replace \( \Lambda_{A,\infty} \) by \( \Lambda_{A,n} \) and define the corresponding Selmer complex for \( T \otimes \Lambda_{A,n} \), using the pullbacks of the local Selmer conditions \( U_v \) under the nature map 

\[ f_n : \Lambda_{A,\infty} \to \Lambda_{A,n} \]. Note that by [15] Theorem 1.6] we have an isomorphism in the derived category

\[ LF_n^* R\Gamma \text{cont}(G, T \otimes \Lambda_{K,\infty}) \simeq R\Gamma \text{cont}(G, T \otimes \Lambda_{K,n}). \]

Note that by an easy argument using global duality (see [42, 5.1.6]), the \( \Lambda_{A,\infty} \)-characteristic ideal of \( \tilde{H}^2(G_{K,S}, T \otimes \Lambda_{A,\infty}) \) is exactly the base change to \( \Lambda_{A,\infty} \) of that for \( X_{f/K}^{\text{cyc}} \). So by [14] and [15] we arrived at the following

**Proposition 4.19.** Let \( P \) be a height one prime of \( \Lambda_{K,\infty} \), which is not in \( S_{v_0}(\mathcal{F}^+(D)) \) (the subscript \( v_0 \) means identifying \( K_{v_0} \) with \( \mathbb{Q}_p \)) or \( S_{\bar{v}_0}(\mathcal{F}^-(D)) \), then

\[ \text{ord}_p \text{char}_{\Lambda_{K,\infty}}(X_{\alpha \alpha}) \geq \text{ord}_p \mathcal{L}_{\alpha \alpha}. \]

Note that the pullback of \( U \) in Lemma [11.18] is a height one prime which does not contain \( \mathcal{L}_{\alpha \alpha} \), since the latter is not identically 0 on the cyclotomic line.

**Remark 4.20.** Note that the parameter \( U \) and \( A \) at \( v_0 \) and \( \bar{v}_0 \) are different as parameters in \( \Gamma_K \). In fact one can prove that the set \( S_{v_0}(\mathcal{F}^+(D)) \) and \( S_{\bar{v}_0}(\mathcal{F}^-(D)) \) are the same. But we will not need it in this paper.

So we have

\[ \text{ord}_p \text{Fitt}(X_{\alpha \alpha}) \geq \text{ord}_p \mathcal{L}_{\alpha \alpha} \]

for those primes. Observe that when specializing to the cyclotomic line any prime \( P \) in \( S_{v_0}(\mathcal{F}^+(D)) \) or \( S_{\bar{v}_0}(\mathcal{F}^-(D)) \) specializes to the unit ideal of \( \Lambda_{\infty} \). We write \( X_{\alpha \alpha} \) also for the one-variable (cyclotomic) dual Selmer group over \( \Lambda \) (defined using the local Selmer condition \( \alpha \alpha \) at \( p \)). So by Proposition 4.12 we have the

**Corollary 4.21.** For any \( n \) we have

\[ \text{char}_{\Lambda_{Q,n}}(X_{\alpha \alpha} \otimes_{\Lambda_{\infty}} \Lambda_{Q,n}) \subseteq (\mathcal{L}_{\alpha \alpha}). \]

Therefore

\[ \text{char}_{\Lambda_{Q,\infty}}(X_{\alpha \alpha}) \subseteq (\mathcal{L}_{\alpha \alpha}). \]

To save notation we also write \( \mathcal{L}_{\alpha \alpha} \) for the specialization of \( \mathcal{L}_{\alpha \alpha} \) to the cyclotomic line. In order to relate this to Kato’s main conjecture we need to study the relations between \( \mathcal{L}_{\alpha \alpha} \) and \( \mathcal{L}_\alpha(f \cdot L(c)) \). Note that although we know they are equal at all arithmetic points, however these interpolation formulas do not determine the element in \( \Lambda_{Q,\infty} \) uniquely. So we need to prove the

**Lemma 4.22.** Up to multiplying by a non-zero constant we have

\[ \mathcal{L}_{\alpha \alpha} = \mathcal{L}_\alpha(f) \cdot \mathcal{L}_\alpha(f^{\text{cyc}}). \]
Proof. There are several ways of proving this and we only give one. The eigencurve machinery implies we can deform \( f_{\alpha} \) into a Coleman family \( F \). Then Bellaiche constructed \([2]\) the corresponding two variable \( p \)-adic \( L \)-functions \( \mathcal{L}_{F, \mathbb{Q}} \) and \( \mathcal{L}_{F \otimes \chi_{\kappa}, \mathbb{Q}} \) which specializes to \( \mathcal{L}_{\alpha}(f) \) and \( \mathcal{L}_{\alpha}(f \otimes \chi_{\kappa}) \). Also there is the Rankin-Selberg three-variable \( p \)-adic \( L \)-function constructed by Loeffler-Zerbes \([30]\) as the “geometric” \( p \)-adic \( L \)-function \( \mathcal{L}(F \otimes g) \) (whose weight space contains the weight space for \( \mathcal{L}(F) \) and \( \mathcal{L}(F \otimes \chi_{\kappa}) \) as closed subspace) which specializes to \( \mathcal{L}_{\alpha \alpha} \). Note that the Loeffler-Zerbes included the interpolation formulas when the specialization of \( g \) has weight one. There is a \( p \)-adically dense set of arithmetic points (crystalline points, see \([30]\)) in the two-variable weight space for \( \mathcal{L}_{\alpha \alpha} \) where the specializations of \( \mathcal{L}(F \otimes g) \) and \( \mathcal{L}(F) \cdot \mathcal{L}(F \otimes \chi_{\kappa}) \) are equal. Here one small subtlety is the different period in each case – the canonical period for each specialization of \( F \), or the product of the \( \pm \) periods. However at least locally the ratio of such periods is interpolated as rigid analytic functions, say by taking the ratio of the two \( p \)-adic \( L \)-functions above. So we multiply one \( p \)-adic \( L \)-function by this ratio and this product should be equal to the second \( p \)-adic \( L \)-function. This specialization to \( f \) of this ratio is the constant mentioned in the lemma (which is just the non-zero number \( \frac{\Omega^\text{can}}{\Omega f f_f} \)). So as rigid analytic functions the two variable \( p \)-adic \( L \)-functions \( \mathcal{L}(F \otimes g) \) and \( \mathcal{L}(F) \cdot \mathcal{L}(F \otimes \chi_{\kappa}) \) must be equal up to multiplying by this constant. Thus we get the corollary. \( \Box \)

It is also easy to see that

\[
\check{\Gamma}_\alpha(G_{\mathbb{Q}, S}, T \otimes \Lambda_{\mathbb{Q}, \infty}, f) \simeq \check{\Gamma}_\alpha(G_{\mathbb{Q}, S}, T \otimes \Lambda_{\mathbb{Q}, \infty}, f) \oplus \check{\Gamma}_\alpha(G_{\mathbb{Q}, S}, T \otimes \Lambda_{\mathbb{Q}, \infty}, f^{\chi_{\kappa}}).
\]

The following theorem is proved by Pottharst in \([44]\) Theorem 5.4, which is essentially a reformulation of Kato’s theorem.

**Theorem 4.23.**

\[
\text{char}_{\Lambda_{\mathbb{Q}, \infty}}(X_{\alpha \alpha}(f, \mathbb{Q})) \supseteq (\mathcal{L}_{\alpha}(f)).
\]

Moreover the “=” is equivalent to Conjecture \([74]\) after inverting \( p \).

Combining what we have proved with Kato’s theorem we have

**Theorem 4.24.**

\[
\text{char}_{\Lambda_{\mathbb{Q}, \infty}}(X_{\alpha \alpha}(f, \mathbb{Q})) = (\mathcal{L}_{\alpha}(f)).
\]

### 4.4 Powers of \( p \)

We briefly discuss the \( \omega_f \) and \( \eta_f \) in \([30]\). We first discuss the case within the Fontaine-Laffaille range as in this case there relates a well formulated Tamagawa number conjecture of Bloch-Kato. We refer to \([12]\) Section 2.1, 2.2 for the background of the motive \( \mathcal{M}_k/O_L \) associated to the space of weight \( k \) modular forms. This comes from the \( k-2 \)-th symmetric power of the universal elliptic curve over the modular curve. Suppose \( k < p \) (i.e. the Fontaine-Laffaille range). Let \( D_{FL} \) be the Fontaine-Laffaille functor. Then by the comparison theorem of Faltings (as noted in loc.cit), we have

\[
D_{FL}(H^1_{dR}(\mathcal{M}_k)) = H^1_{dR}(\mathcal{M}_k).
\]

(See \([37]\) Section 6.10). The Galois representation \( T_f \) is realized as \( H^1_{dR}(\mathcal{M}_k)[\lambda_f] \) (meaning the maximal submodule of \( H^1_{dR}(\mathcal{M}_k) \) on which the Hecke algebra \( T_0(N) \) is acting via its action \( \lambda_f \) on \( f \)). Thus

\[
D_{FL}(T_f) = H^1(\mathcal{M}_k)[\lambda_f].
\]

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On the other hand we have an exact sequence of Hecke modules (localized at the maximal ideal \( m_f \subset \mathbb{T}_0(N) \)):

\[
0 \to H^0(X_0(N), \omega^k/O_L)_{m_f} \to H^1_{dR}(\mathcal{M}_k)_{m_f} \to H^1(X_0(N), \omega^{2-k}/O_L)_{m_f} \to 0.
\]

Here \( \omega^k \) is the weight \( k \) automorphic sheave. As Hecke modules the next to last term is free of rank one over \( \mathbb{T}_0(N)_{m_f} \), and the second term is isomorphic to \( S(X_0(N), \mathcal{O}_L)_{m_f} \) (\( \mathcal{O}_L \)-valued cusp forms). Unravelling the definitions the \( \eta_f \in H^1(X_0(N), \omega^{2-k}/O_L)_{m_f} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) in [30] corresponds to the \( f \)-component projector \( 1_f \) under the identification of \( H^1(X_0(N), \mathbb{Z}_p)_{m_f} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) with \( \mathbb{T}_0(N)_{m_f} \), while

\[
\frac{D_{\mathcal{F}L}(T_f)}{\text{Fil}^0D_{\mathcal{F}L}(T_f)} \simeq H^1(X_0(N), \omega^{2-k})_{m_f}[\lambda_f].
\]

So by definition the ratio of a generator of \( \frac{D_{\mathcal{F}L}(T_f)}{\text{Fil}^0D_{\mathcal{F}L}(T_f)} \) over \( \eta_f \) is the congruence number \( c_f \) of \( f \) (determined up to a \( p \)-adic unit).

**Remark 4.25.** Note that up to multiplying by a \( p \)-adic unit, the canonical period \( \Omega_f^{\text{can}} \) is the Petersson inner product period in the Rankin-Selberg \( p \)-adic \( L \)-function \( \mathcal{L}_{f \otimes \mathbb{R}} \) divided by this \( c_f \).

**Definition 4.26.** We define an arithmetic point \( \tilde{\phi} \in \text{Spec} \mathcal{O}_L[[W]] \) as a \( \bar{\mathbb{Q}}_p \) point mapping \( (1 + W) \) to a \( p \)-power root of unity times \( (1 + p)^{1+j} \) for some \( j \) between 0 and \( k-2 \). The specialization to \( \tilde{\phi} \) of the \( p \)-adic \( L \)-function interpolates the \( L \)-value \( L(f, 1+j, \chi_{\tilde{\phi}}^{-1}) \) for a finite order character \( \chi_{\tilde{\phi}} \), up to normalization factors. We also write \( \tilde{\phi}^{-1} \) for the arithmetic point which is symmetric to \( \tilde{\phi} \) with respect to the central point of the functional equation. At \( \tilde{\phi}^{-1} \), the corresponding \( 1+j \) should be \( k-1-j \).

**Definition 4.27.** We say an arithmetic point \( \tilde{\phi} \in \mathcal{X} \) is generic if \( L(f, \chi_{\tilde{\phi}}, 1+j) \neq 0 \) and \( \mathcal{L}_{f,K}^{\text{Gr}}|_{\mathcal{Y}_{\tilde{\phi}}} \) is not identically 0. It is clear that all but finite many arithmetic points are generic.

We write \((v_1, v_2)\) for an \( \mathcal{O}_{L_n} \)-basis of \( H^1(Q_p, T_{\tilde{\phi}}) \) such that \( v_1 \) is a generator of \( H^1(Q_p, T_{\tilde{\phi}}) \). We sometimes write them as \( v_{1,\mathcal{V}_0},v_{2,\mathcal{V}_0} \). Let \( \tilde{\phi} \) be such that \( \tilde{\phi} \) and \( \tilde{\phi}^{-1} \) are both generic. Write \( \Gamma_{\mathcal{K}} = \Gamma_{\text{cyc}} \times \Gamma_{\mathcal{V}_0} \) and let \( pr: \mathcal{Y} := \text{Spec} \mathbb{Z}_p[[\Gamma_{\text{cyc}} \times \Gamma_{\mathcal{V}_0}]] \to \mathcal{X} := \text{Spec} \mathbb{Z}_p[[\Gamma_{\text{cyc}}]] \) be the natural projection. For \( \tilde{\phi} \in \mathcal{X} \) let

\[
\mathcal{Y}_{\tilde{\phi}} := \mathcal{Y} \otimes_{\mathcal{X}, \tilde{\phi}} A(\tilde{\phi}).
\]

We consider \( \mathcal{Y}_{\tilde{\phi}} \). Define an element \( \mathcal{L}_{\tilde{\phi}}^1 \in \mathcal{O}_L[[\mathcal{V}_0]] \) such that

\[
\text{BF}|_{\mathcal{Y}_{\tilde{\phi}}} \equiv (k-2-j)!(\mathcal{L}_{\tilde{\phi}}^1)(G(\chi_{\tilde{\phi}}^{-1})(\frac{\beta_f}{p^{1+j}})^r)v_{2,\mathcal{V}_0}(\text{mod } v_{1,\mathcal{V}_0}). \tag{18}
\]

Then \( \mathcal{L}_{\tilde{\phi}}^1(0) \neq 0 \). For any \( \tilde{\phi} \) of conductor \( p^r \) which is \( e^{1+j-\frac{k}{2}} \) times a finite order \( \chi_{\tilde{\phi}} \), we have

\[
\alpha_{\tilde{\phi}}^2 \cdot \tilde{\phi}(\mathcal{L}_{f,K}^{\text{Gr}})^2 \mathcal{L}_{\tilde{\phi}}^1(\frac{\beta_f}{p^{1+j}})^r = \frac{(j!)^2 L(1+j, \chi_{\tilde{\phi}}^{-1})}{(2\pi i)^{2j+2}\Omega_f^{\text{can}}}. \tag{19}
\]

Let \( \mathcal{A} \) be the factor appearing in Lemma [4.18]. We also have for any \( \phi \in \mathcal{Y}_{\tilde{\phi}} \)

\[
\log_{\mathcal{V}_0} \phi(\text{BF}) = \phi(\mathcal{A})(k-2-j)!\phi(\mathcal{L}_{f,K}^{\text{Gr}})G(\chi_{\tilde{\phi}}^{-1}) \cdot \frac{\phi(\alpha_{\mathcal{A}}^2)^{\beta_f}}{p^{1+j}} \omega_{\mathcal{Y}}. \tag{19}
\]
Note that $\beta_f, j, r$ only depend on $\tilde{\phi}$, $\alpha_g$ is an element in $\mathbb{Z}_p[[\Gamma_{v_0}]]$ such that $\alpha_g(0) = 1$. Recall also the remark right after (4.3) and (17) for the role played by $\omega_\gamma$ and $\eta_\gamma$. We have

$$\exp_\tilde{\phi}^* \tilde{\phi}(BF) = \frac{1}{\phi_{f,0}} \tilde{\phi}(\mathcal{L}_{f,0}^\gamma \mathcal{L}_{f,0}^{\text{can}}) \cdot G(\chi_{\tilde{\phi}}^{-1}) \cdot \frac{(\alpha_f \beta_g)}{p^{1+j}} \cdot \omega_f$$

where $c_f$ is the congruence number we defined before. So

$$\tilde{\phi}(A) \exp_\tilde{\phi}^* \tilde{\phi}(BF) = \frac{L_K(f, 1 + j, \chi_{\tilde{\phi}}^{-1}) \cdot G(\chi_{\tilde{\phi}}^{-1})}{\phi_{f,0}} \cdot \frac{G^2(\chi_{\tilde{\phi}})(\frac{\alpha_f \beta_g}{p^{1+j}})^r \cdot p^{2r} \cdot \omega_f}{\phi_{f,0}^*}$$

Note here that the factor $\mathcal{E}(f)\mathcal{E}(f^*)$ in the interpolation formula for Rankin-Selberg $p$-adic $L$-function is cancelled by the factor $(1 - \frac{\beta}{\alpha})(1 - \frac{\beta}{p\alpha})$ in [30] Corollary 6.4.3. Thus

$$\exp_\tilde{\phi}^* \tilde{\phi}(BF) = \frac{L_K(f, 1 + j, \chi_{\tilde{\phi}}^{-1})}{\phi_{f,0}} \cdot \frac{G(\chi_{\tilde{\phi}})(\frac{\beta_f}{p^{1+j}})^r \cdot (p^{2j}r^r/p^{(k-1)r}) \cdot \omega_f}{\phi_{f,0}^*}.$$ (20)

Similarly for $\tilde{\phi}$ above we can study everything at arithmetic points $\tilde{\phi}^{-1}, \phi^{-1}$. In particular

$$\log_{\tilde{\phi}}^* \tilde{\phi}^{-1}(BF) = \phi^{-1}(A) \cdot j! \cdot \frac{1}{\phi_{f,0}} \cdot \frac{L_K^{\text{Gr}}(\mathcal{L}_{f,0} \chi_{\tilde{\phi}})^r \cdot \omega_f}{\phi_{f,0}^*}.$$ (21)

and

$$\exp_\tilde{\phi}^* \tilde{\phi}^{-1}(BF) = \frac{L_K(f, k - 1 - j, \chi_{\tilde{\phi}}^{-1})}{(k - 2 - j)!} \cdot \frac{G(\chi_{\tilde{\phi}}^{-1})}{\phi_{f,0}} \cdot \frac{L_K(f, 1 + j, \chi_{\tilde{\phi}}^{-1})}{\phi_{f,0}} \cdot \frac{G(\chi_{\tilde{\phi}}^{-1})}{\phi_{f,0}} \cdot \frac{G(\chi_{\tilde{\phi}}^{-1})}{\phi_{f,0}} \cdot \frac{\phi(v_2, \tilde{\phi})}{\phi(v_2, \tilde{\phi})}.$$ (22)

We have

$$L_{\tilde{\phi}}^1(0) = \frac{j!}{\phi_{f,0}} \cdot \frac{G(\chi_{\tilde{\phi}}^{-1})}{\phi_{f,0}} \cdot \frac{L_K(f, 1 + j, \chi_{\tilde{\phi}}^{-1})}{\phi_{f,0}} \cdot \frac{G(\chi_{\tilde{\phi}}^{-1})}{\phi_{f,0}} \cdot \frac{G(\chi_{\tilde{\phi}}^{-1})}{\phi_{f,0}} \cdot \frac{\phi(v_2, \tilde{\phi})}{\phi(v_2, \tilde{\phi})}.$$ (22)

By the results in Section 2.4 the

$$\lim_{Q_p \subseteq K \subseteq Q^\text{ur}_p} H^1_f(K, T_{\tilde{\phi}})$$

is an $\mathcal{O}_{L_n[[\Gamma_{v_0}]]}$-direct summand of

$$\lim_{Q_p \subseteq K \subseteq Q^\text{ur}_p} H^1_f(K, T_{\tilde{\phi}}) \simeq \mathcal{O}_{L_n[[\Gamma_{v_0}]]}^2.$$

**Definition 4.28.** We define the “unramified” local Selmer condition at $v_0$ (as well as its dual under Tate duality) by this

$$\lim_{Q_p \subseteq K \subseteq Q^\text{ur}_p} H^1_f(K, T_{\tilde{\phi}})$$

along the one-variable family over $\mathcal{O}_{L_n[[\Gamma_{v_0}]]}$. We denote this Selmer condition using the subscript ur.
Now we look at the following exact sequences of $\mathcal{O}_{L_n}[[U]]$-modules
\begin{equation}
0 \to H^1_{\text{ur,rel}}(\mathcal{K}^S, T_\phi \otimes \mathcal{O}_{Y_\phi}) \to H^1_{\text{ur}}(\mathcal{K}_{v_0}, T_\phi \otimes \mathcal{O}_{Y_\phi}) \to X_{\text{rel,str}} \to X_{\text{ur,str}} \to 0 \tag{23}
\end{equation}
\begin{equation}
0 \to H^1_{\text{ur,rel}}(\mathcal{K}^S, T_\phi) \to \frac{H^1(\mathcal{K}_{v_0}, T_\phi)}{\mathcal{O}_{L_n,v_1}} \to X_{\text{ur,v}_1} \to X_{\text{ur,str}} \to 0. \tag{24}
\end{equation}
(In the second exact sequence the Selmer groups are 0-dimensional at the point $\tilde{\phi}$.) From the genericity of $\tilde{\phi}$ it is not hard to see that $H^1_{\text{ur,rel}}(\mathcal{K}^S, T_\phi)$ has rank one over $\mathcal{O}_{L_n}$, by Corollaries $2.14$ and $2.15$ there is an isomorphism
\[ H^1_{\text{ur}}(\mathcal{K}_{v_0}, T \otimes \Lambda_K|_{Y_\phi}) \simeq \mathcal{O}_{L_n}[[U]] \]
which interpolates $c_{\phi,p}$ times the logarithmic map, divided by the specializations of $\rho(d)$ there. Now Theorem $1.7$ and our (8), (19), (20), (21), (22) and (23) implies that
\[ c_f c_{\phi,p}(f) \cdot c'_{\phi^{-1},p}(f) \text{char}_{\mathcal{O}_{L_n}[[U]]}(X_{\text{ur,v}_1,\tilde{\phi}}) \text{char}_{\mathcal{O}_{L_n}[[U]]}(X_{\text{ur,v}_1,\tilde{\phi}^{-1}}) \]
\[ \subseteq \text{char}_{\mathcal{O}_{L_n}[[U]]}
\left( \frac{1}{(k-2-j)!((\chi_{\tilde{\phi}}^{-1})^{(\frac{\beta_f}{p+1})^r})^{O_{L_n}[[U]]}} \right) \cdot \text{char}_{\mathcal{O}_{L_n}[[U]]}
\left( \frac{1}{(k-2-j)!((\chi_{\tilde{\phi}}^{-1})^{(\frac{\beta_f}{p+1})^r})^{O_{L_n}[[U]]}} \right)
\]
up to powers of $U$ (as fractional ideals for the right hand side). However from the genericity of $\tilde{\phi}$ and $\tilde{\phi}^{-1}$ we know the powers of $U$ in these characteristic ideals are less than or equal to 0. So the above containment is actually true. Here in applying Theorem $1.7$ we have used the following argument: the two variable char$(X_{K,f}^\text{Gr}) \subseteq (L_{f,K}^\text{Gr})$ implies two variable Fitt$(X_{K,f}^\text{Gr}) \subseteq (L_{f,K}^\text{Gr})$, which implies one variable Fitt$(X_{K,f}^\text{Gr}) \subseteq (L_{f,K}^\text{Gr})$, which implies the one variable char$(X_{K,f}^\text{Gr}) \subseteq (L_{f,K}^\text{Gr})$ by that $O_L[[\Gamma_K]]$ is a normal domain. Note that the powers of $U$ in Lemma $4.18$ do not have any effect to the argument since the specialization of BF at $\tilde{\phi}$ is not zero. The $c_f$ comes from the discussion in Remark $4.25$. Also pay attention to that the $p^{2j}r + r/p(k-1)r$ and $p^{2j}r + r/p(k-1)r$ at $\tilde{\phi}$ and $\tilde{\phi}^{-1}$ cancel out. We note that any prime $v$ of $\mathcal{K}$ not dividing $p$ is finitely decomposed in the $\mathbb{Z}_p$-extension $\mathcal{K}_{v_0}/\mathcal{K}$, and is completely split in $\mathcal{K}_\infty/\mathcal{K}_{v_0}$. Before doing specializations we need some more preparations.

**Proposition 4.29.** The characteristic ideal of the cokernel of
\[ H^1_{\text{ur,rel}}(\mathcal{K}^S, T_\phi \otimes \mathcal{O}_{L_n}[[U]]) \to H^1_{\text{ur,rel}}(\mathcal{K}^S, T_\phi) \]
as $\mathcal{O}_{L_n}$ modules is the characteristic ideal of $X_{f,\text{str, null}}(T_\phi \otimes \mathcal{O}_{L_n}[[U]])[u-1]$, where the subscript null means the pseudo-null submodule, and $u$ is a topological generator of $U$.

**Proof.** From the long exact sequence of Selmer complexes, we know the cokernel is isomorphic to
\[ \tilde{H}^2_{\text{ur,rel}}(\mathcal{K}^S, T_\phi \otimes \mathcal{O}_{L_n}[[U]])[u-1]. \]
By global duality $\tilde{H}^2_{\text{ur,rel}}(\mathcal{K}^S, T_\phi \otimes \mathcal{O}_{L_n}[[U]])$ is isomorphic to $X_{\text{ur,0}}(T_\phi \otimes \mathcal{O}_{L_n}[[U]])$, which is a torsion module. Moreover it is easy to see that $(u-1)$ is not in the characteristic ideal of it. So the $(u-1)$ torsion part is contained in the pseudo-null submodule. This proves the proposition. \qed
Lemma 4.30. Let $f_{ur,0}$ be the characteristic ideal of $X_{ur,0}(T_{\tilde{\phi}} \otimes O_L \mathbb{[[U]]}^*)$. The characteristic ideal of
\[
\frac{X_{ur,0}(T_{\tilde{\phi}} \otimes O_L \mathbb{[[U]]}^*)}{(u - 1)X_{ur,0}(T_{\tilde{\phi}} \otimes O_L \mathbb{[[U]]}^*)}
\]
is the characteristic ideal of $X_{ur,0,\text{null}}(T_{\tilde{\phi}} \otimes O_L \mathbb{[[U]]}^*)[u - 1]$ times the principal ideal generated by $f_{ur,0}(0)$.

Proof. This follows from the exact sequence
\[
0 \to X_{ur,0,\text{null}} \to X_{ur,0} \to X_{ur,0,\text{tf}} \to 0
\]
where the subscript tf denotes the $\mathbb{Z}_p$-torsion free part. It is well known that
\[
\text{char}(\frac{X_{ur,0,\text{tf}}}{(u - 1)X_{ur,0,\text{tf}}}) = (f_{ur,0}(0)).
\]
Note also that as $X_{ur,0,\text{null}}$ has finite cardinality, the kernel and cokernel of
\[
\times(u - 1) : X_{ur,0,\text{null}} \to X_{ur,0,\text{null}}
\]
have the same cardinality. The lemma is easily seen by Tor-exact sequence noting
\[
\text{Tor}^1(X_{ur,0,\text{tf}}, O_L \mathbb{[[U]]}/(u - 1)) = 0.
\]

\[
\square
\]

With the above proposition and lemma, specializing to $U \to 0$, applying \[24\], we obtain (the $v$ may or may not divide $p$)
\[
\prod_v c'_{K,\tilde{\phi}, v}(f) \prod_v c'_{K,\tilde{\phi}^{-1}, v}(f) \text{char}_{O_L}(X_{v_1, v_1, \phi}) \text{char}_{O_L}(X_{v_1, v_1, \tilde{\phi}^{-1}}) \subseteq \frac{L_K(f, 1 + j, \chi_{\tilde{\phi}^{-1}}^{-1})}{(2\pi i)^{2k - 2 + 2}\Omega_{f\kappa}^\text{can}}(2\pi i)^{2k - 2 - 2}\Omega_{f\kappa}^\text{can} \frac{L_K(f, k - 1 - j, \chi_{\tilde{\phi}^{-1}}^{-1})}{(2\pi i)^{2k - 2 + 2}\Omega_{f\kappa}^\text{can}}.
\]

(25)

Here the subscript $K$ means the local Tamagawa numbers over $K$. It is known that $c_{\kappa, \ell, \phi}(f) = c_{\ell, \phi}(f) \cdot c_{\ell, \phi}(f^{\kappa})$. Note also that as we are only concerned with the lower bound for Selmer groups we do not need results as in Subsection \[4.2\] for the specialization this time. The last preparation is the following

Lemma 4.31. The $\Omega_{f\kappa}^\pm$ is an $O_L$-multiple of $\Omega_f^\pm$.

Proof. We prove it following the first half of \[50\] Lemma 9.6 (note that the running assumption here is slightly different from \[50\]). The $f$ is a new form of level $N$ and $f^{\kappa}$ is a new form whose conductor is some $M$ which is a divisor of $ND^2$. Consider the map
\[
H^1(\Gamma_0(N), L_{/O_L}^k)_{m_f} \to H^1(\Gamma_0(ND^2), L_{/O_L}^k)_{m_f}
\]
constructed in loc.cit. The $\omega_f^\pm$ is mapped to $\omega_{f\kappa}^\pm$ under this map, and $\gamma_f^\pm$ is mapped to some multiple of $\omega_{f\kappa}^\pm$. Note $m_f$ is non-Eisenstein. So the image of $\gamma_f^\pm$ are elements in $H^1(\Gamma_0(M), L_{/O_L}^{k - 2})_{m,f}$. Thus $\text{im}(\gamma_f^\pm)$ are $O_L$-multiples of $\gamma_{f\kappa}^\pm$.

\[
\square
\]

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We can prove the following

**Corollary 4.32.** With the assumptions of Theorem 1.4 part two, we have

\[ \text{char}_\Lambda \left( \frac{H^1_{\text{Iw}}(\mathbb{Q}^S, T)}{\Lambda_{\text{Kato}}} \right) \supseteq \text{char}_\Lambda(X_{\text{str}}). \]

**Proof.** This corollary follows from (25) with the notation as Section 4.2, let \( A \in \Lambda \) be the image of \( z_{\text{Kato}, f} \) in \( H^1_{\text{Iw}}(\mathbb{Q}_p, T) \) upon the choice of a basis of the latter, and \( B \) the image of \( z_{\text{Kato}, f_K} \) in \( H^1_{\text{Iw}}(\mathbb{Q}_p, T) \). Let \( \mathcal{F}_1 = A \cdot B \). Then by Kato’s result and what we proved, \( (\mathcal{F}_1) = (\mathcal{F}) \) as ideals of \( \Lambda \) up to powers of \( p \). On the other hand Kato proved that

\[ \exp^* \tilde{\phi}(z_{\text{Kato}}) = \frac{L_{K}(f, 1 + j, \chi^{-1}_{\phi})}{(2\pi i)^{1+j} \Omega_{f}^{(-1)^{p} \omega_{f}}} . \]

We know that up to multiplying by a \( p \)-adic unit we have

\[ \Omega_{f}^{\text{can}} = \Omega_{f}^{+} \Omega_{f}^{-} \]

by the work in [17]. (One uses the pairing constructed in *loc. cit* and work of Hida, see e.g. [50, Lemma 9.5]. The key is the freeness over the local Hecke ring \( T_{m_f} \) of \( H^1(\Gamma_0(N), L_{/\mathcal{O}_L}^{k-2})_{m_f} \), which is proved in [17] when the weight is in the Fontaine-Laffaille range). Therefore, the discussion at the end of Section 4.2 gives the corollary (Lemma 4.31 is also used here). □

**Remark 4.33.** We do need to argue with the one variable family over \( \mathcal{O}_L[[\Gamma_{\text{Iw}}]] \) instead of only look at the one point \( \tilde{\phi} \), because we cannot prove that \( L_{f,K}^{\text{Gr}} \) is not identically zero along the cyclotomic line.

The following corollary reproves an early result of [49].

**Corollary 4.34.** Assumptions are as part one of Theorem 1.4. If \( L(f, k/2) = 0 \), then \( \text{corankSel}_{f, \mathbb{Q}} \geq 1 \).

**Proof.** Write \( \phi_0 \) for the point in \( \text{Spec}\Lambda \) corresponding to \( \gamma - 1 \mapsto 0 \) (\( \gamma \) being a topological generator of \( \Gamma \)). If the specialization of \( z_{\text{Kato}} \) to \( \phi_0 \) is 0, then \( \gamma - 1 \in \text{char}(\frac{H^1_{\text{Iw}}(\mathbb{Q}, T)}{\Lambda_{\text{Kato}}}) \), which implies \( \gamma - 1 \in \text{char}(X_{\text{str}}) \). By the control theorem of strict Selmer groups (although we do not know if the pseudo-null submodule of the dual Selmer group is zero, it does not have any effect on the rank part), we see \( \text{corankSel}_{E, \mathbb{Q}} \geq 1 \). If the specialization of \( z_{\text{Kato}} \) to \( \phi_0 \) is non-zero, then by our assumption that \( L(f, k/2) = 0 \) it generates a non-zero element in \( H^1_{f}(\mathbb{Q}, V_f(-\frac{k-2}{2})) \). Then it is easily seen that \( \text{corankSel}_{f, \mathbb{Q}} \geq 1 \). □

Finally we prove the Tamagawa number conjecture of Bloch-Kato in the case when the central critical value of \( f \) is non-zero. Recall by our assumption (Irred) the module \( H^1(\mathbb{Q}_p, T) \) is free of rank two over \( \mathcal{O}_L \).

Combining Theorem 1.4 Kato’s result and the control theorem of Selmer groups we get the following

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Corollary 4.35. Assumptions are as in part two of Theorem 1.4. Assume moreover that the weight \( k \) is in the Fontaine-Laffaille range \( k < p \). If \( L(f, \frac{k}{2}) \neq 0 \), then the full Iwasawa main conjecture for \( T \) is true, and up to multiplying by a \( p \)-adic unit we have

\[
\text{Tam}_p(T) \prod_{l|N} c_l(T) \cdot \sharp(\text{Sel}_{p^\infty}(T)) = \sharp\left( \frac{\mathcal{O}_L}{(L(f, \frac{k}{2}))(2\pi i)^{-1/2} \omega_f}\right).
\]

We remark that under the Fontaine-Laffaille assumption \( k < p \), the (Irred) is automatic thanks to a result of Edixhoven \[13\].

**Proof.** Recall that \( \text{Tam}_p(T) \) is defined by

\[
\sharp\left( \frac{\mathcal{O}_L}{c_{p,p} / \det(1 - \varphi|D_{\text{cris}}(V))\mathcal{O}_L} \right)
\]

for \( V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \). On the other hand

\[
\exp^* \phi_0(\mathbf{z}_{\text{Kato}}) = \frac{L([p], \frac{k}{2})}{(2\pi i)^{-1/2} \omega_f} \omega_f
\]

where the \( \{p\} \) means removing the Euler factor at \( p \). We use our argument in the Section on control theorems, taking the \( P \) there to correspond to the trivial character and take some auxiliary \( v \) as in there, and prove the \( X_{v,v} \) has no pseudo-null submodules. These altogether gives the result. \( \square \)

Now we leave the Fontaine-Laffaille range and try to prove Kato’s main conjecture for general (even) weight forms. Let \( \pi : E \to Y(N) \) be the universal elliptic curve over the open modular curve \( Y(N) \) and \( \mathbb{L}_{k-2} \) be \( \text{Sym}^{k-2} \mathbb{R}^1\pi_{*}\mathbb{Z}_p \) which is a rank \( k - 2 \) over \( \mathbb{Z}_p \) local system on \( Y(N) \). Consider the Poincare pairing

\[
H^1_{\text{et}}(Y(N), \mathbb{L}_{k-2})_{m_f} \times H^1_{\text{et,c}}(Y(N), \mathbb{L}_{k-2})_{m_f} \to \mathbb{Z}_p
\]

which is perfect. Note that as \( \tilde{\rho}_f \) is absolutely irreducible, we have

\[
H^1_{\text{et}}(Y(N), \mathbb{L}_{k-2})_{m_f} = H^1_{\text{et,c}}(Y(N), \mathbb{L}_{k-2})_{m_f} = H^1_{\text{et,c}}(Y(N), \mathbb{L}_{k-2})_{m_f}
\]

(the last being the interior cohomology). Such pairing can be slightly generalized if we extend the coefficient from \( \mathbb{Z}_p \) to \( \mathcal{O}_L \). Inverting \( p \) and applying the Faltings comparison map, this Poincare duality induces on the (graded piece of) algebraic deRham cohomology the Serre duality

\[
H^1(X(N), \omega_{-k} \otimes \Omega^1_{X(N)})_{m_f} \times H^0(X, \omega_k)_{m_f} \to \mathbb{Q}_p.
\]

Back to our problem we have to notice one subtlety for the pairing: if we let \( \{x, y\} \) be an \( \mathcal{O}_L \) basis for \( H^1_{\text{et}}(Y, \mathbb{L}_{k-2} \otimes_{\mathbb{Z}_p} \mathcal{O}_L)_{m_f} [\lambda_f] \). Then under the Poincare pairing above, we have

\[
(x, y) = \frac{\langle f, f \rangle_{X(N)}}{\Omega_f^2 \Omega_f} := \mathcal{C}_f
\]

up to multiplying by a \( p \)-adic unit. (See \[11\] Lemma 4.17 to Theorem 4.20] and also \[50\] Lemma 9.5]). This number \( \mathcal{C}_f \) is closely related to the congruence number \( c_f \) as discussed above. Identifying
$T_f$ with $H^1_{\text{et}}(Y(N),\mathcal{L}_{k-2})_{m_f}[\lambda_f]$, the perfect pairing on $T_f \times T_f$ is $\frac{1}{\lambda_f}$ times the one induced from the Poincare pairing above on $X(N)$. Thus the vector which pairs to 1 with $\omega_f$ under the Poincare duality, is paired to $\frac{1}{\lambda_f}$ under the perfect pairing of $T_f$. With this factor into consideration, the previous argument gives the main conjecture for $f$ over $K$ and with $\Omega^+_f \Omega^-_f$ as the period factor. Then we can combine with Kato’s result and argue as before to get the main conjecture for $f$ over $\mathbb{Q}$.

Remark 4.36. Unfortunately beyond the Fontaine-Laffaille case we are unable to get a Tamagawa number conjecture result as in [6], for the reason below. When $k < p - 1$, starting from $T_f$ the Fontaine-Laffaille theory gives a canonical $\mathcal{O}_L$-lattice in the Dieudonne module of $V_f$. The Tamagawa number at $p$ is defined using this lattice and the Bloch-Kato exponential map. When $k$ is large, one needs more general ways to define lattices in rational Dieudonne modules. For example one can use Breuil-Kisin modules as in [7]. If there were nice comparison isomorphisms as the formula on top of page 4 of loc.cit, for the local systems on $X(N)$ considered here, then one can define Tamagawa numbers at $p$ and obtain analogues of Corollary 4.35. One can hope it is the case under some torsion-free assumptions of the crystalline cohomology. (In [2] for trivial local systems, the authors constructed a new $A_{\text{inf}}$-cohomology theory, which takes values in Breuil-Kisin-Fargues modules, and specializes to various known cohomology theories (crystalline, deRham, étale, etc). Under those torsion-free assumptions, it is proved that the $H^i_{A_{\text{inf}}}$‘s are free modules over $A_{\text{inf}}$, and are thus the Breuil-Kisin-Fargues modules associated to $H^i_{\text{et}}$ (together with some other datum). However without such torsion-free assumption there does not seem to be a clear relation between $H^i_{A_{\text{inf}}}$ and $H^i_{\text{et}}$. And it seems these torsion does exist even after localizing at the maximal ideal $m_f$. (see [47])). We expect the subtleties are related to the differences between the congruence number $c_f$ and the $c_f$ defined above.

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