Polaron to molecule transition in a strongly imbalanced Fermi gas

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A single down spin Fermion with an attractive, zero range interaction with a Fermi sea of up-spin Fermions forms a polaronic quasiparticle. The associated quasiparticle weight vanishes beyond a critical strength of the attractive interaction, where a many-body bound state is formed. From a variational wavefunction in the molecular limit, we determine the critical value for the polaron to molecule transition. The value agrees well with the diagrammatic Monte Carlo results of Prokof’ev and Svistunov and is consistent with recent rf-spectroscopy measurements of the quasiparticle weight by Schirotzek et al. In addition, we calculate the contact coefficient of the strongly imbalanced gas, using the adiabatic theorem of Tan and discuss the implications of the polaron to molecule transition for the phase diagram of the attractive Fermi gas at finite imbalance.

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I. INTRODUCTION

The physics of single particles immersed in an environment is ubiquitous in physics. It appears, for example, in the large polaron problem where a single electron is dressed by its interaction with phonons [1] or in models for dissipation and decoherence in quantum mechanics [2, 3]. In recent years, new directions for exploring quantum many-body problems have been opened through ultracold atoms [4]. In particular, for degenerate Fermi gases, the interaction strength can be tuned over a wide range using Feshbach resonances. This allows to study impurity problems in a fermionic environment. A specific example is a gas of fermionic 6Li, where the two lowest hyperfine states are populated in a highly imbalanced situation. For this system, recent experiments have shown that the minority atoms (‘down spins’) apparently form a liquid of quasiparticles [5]. Due to the strong attractive interaction to the up-spin Fermi sea, the associated quasiparticle weight, as determined from a sharp peak in the rf-spectrum, is found to vanish beyond a critical interaction strength. This transition may be interpreted as one, in which a single ↓-Fermion immersed in sea of ↑-Fermions can no longer propagate as a quasiparticle but forms a many-body bound state with the Fermi sea. The existence of such a transition has been predicted by Prokof’ev and Svistunov [6, 7]. Using a novel diagrammatic Monte Carlo method, they have shown that for strong attractive interactions, a molecular state is energetically favored compared to one in which the single down-spin forms a polaronic quasiparticle in the up-spin Fermi sea. In the present work, we analyze this problem by a simple variational wavefunction. It provides an analytically tractable model for the physics on the molecular side, thus complementing the wavefunction. It supports efficiently the results of the diagrammatic Monte Carlo method. Moreover, it describes correctly the three-body physics of repulsive atom-dimer interactions in the deep molecular limit and has zero residue for the down-spin Green function. The variational wave function is also used to determine the saturation field $\hbar_s$, beyond which a two-component Fermi gas is fully polarized and the behavior of the so-called contact coefficient introduced by Tan [9] in the limit of strong imbalance.

II. THE FERMI POLARON AND ITS QUASIPARTICLE WEIGHT

A simple variational wavefunction for the (N+1)-particle problem of a single down-spin Fermion immersed in a sea of spin-up Fermions has been introduced by Chevy [8]. It is based on an expansion up to single particle-hole excitations around the unperturbed Fermi sea

$$|\psi_0\rangle = (\phi_0 d_0^\dagger + \sum_{k,q} \phi_{kq} d_{q-k}^\dagger u_k^q u_q^d) |FS_N^F\rangle. \quad (2.1)$$

Here and in the following sums on $k$ and $q$ with a prime are restricted to $k > k_F$ and $q < k_F$, respectively. Moreover, $|FS_N^F\rangle$ is the N-particle Fermi sea and the creation operators of up- and down-Fermions with momentum $k$ are denoted by $u_k^u$ and $d_k^d$. Despite the restriction to single particle-hole excitations, which is difficult to justify for the relevant case of zero range interactions that can create particle-hole pairs at arbitrary momentum, Monte Carlo calculations show that the ansatz (2.1) gives a ground state energy that is very accurate, in particular at unitarity, where the scattering length $a$ is infinite [6, 7]. The reason why the leading term in an expansion in the number of particle-hole excitations gives very good results for the ground state energy can be traced back to the decoupling of higher order terms for vanishing hole momenta $q = 0$ [10], i.e. contributions with more than one particle hole excitation interfere destructively.

The wavefunction (2.1) describes the added down-spin as a quasiparticle dressed by its interaction with the up-spin Fermi sea. The virtual cloud of particle-hole excitations leads to a quasiparticle energy

$$E(p) = A\varepsilon_F + \frac{p^2}{2m^*} + \ldots \quad (2.2)$$

at low momenta $|p| \ll k_F$ that contains a ‘binding energy’
$A \varepsilon_F < 0$ of a single down-spin to the Fermi sea and an effective mass $m^* [11]$. Here, the Fermi energy is defined by $\varepsilon_F = k_F^2/(2m)$ (we use $\hbar = 1$ throughout the paper) with a Fermi momentum $k_F$ that is related to the up-spin density by the standard relation $n_1 = k_F^2/(6\pi^2)$ for a single component Fermi gas. Since we are interested in the limit of vanishing down-spin density $n_\downarrow \to 0$, these are the relevant energy and momentum scales. The dimensionless coefficient $A$ and the effective mass $m^*$ have been determined from variational Monte Carlo calculations at the unitarity point [11] and from a T-matrix approximation at arbitrary values of the dimensionless interaction strength $v = 1/(k_Fa)$ [12]. Very recently they have also been measured experimentally, giving $A \approx -0.64(7)$ [5] and $m^*/m = 1.17(10)$ [13] at unitarity, in rather good agreement with the theoretical predictions.

From a many-body point of view, the criterion that a single added down-spin is indeed a proper quasiparticle can be expressed by defining the quasiparticle residue $Z_\downarrow$ from the long-time limit

$$Z_\downarrow = \lim_{t \to \infty} |G_\downarrow(p = 0, t)| \neq 0 \quad (2.3)$$

of the down-spin Green-function at zero momentum. Within the variational wavefunction (2.1) this residue is simply given by the probability $Z_\downarrow = |\phi_0|^2$ that an added down-spin at momentum $p = 0$ is not mixed with plane waves at nonzero momenta $q - k \neq 0$ through particle-hole excitations. The fact that the coefficient $|\phi_0|^2$ of the Chevy wavefunction coincides with the quasiparticle weight can be derived formally by noting that the ansatz (2.1) is equivalent to a non-selfconsistent T-matrix approach for the down-spin Green function, which sums the particle-particle ladder for the vertex part $\Gamma(k, \omega)$ [12]. It is then straightforward to see that $|\phi_0|^2 = |1 - \partial_\omega \Sigma|_{\omega = 0}^{-1}$ coincides with the standard definition of the quasiparticle weight via the energy derivative of the down-spin self energy $\Sigma(p, \omega)$ at zero frequency and momentum. The numerical value of $Z_\downarrow$ at unitarity $v = 0$ is $Z_\downarrow(v = 0) \approx 0.78$ within the Chevy ansatz. This is much larger than the experimentally observed value $Z_\downarrow = 0.39(9)$ which is likely to be a lower bound, however [5]. Smaller values $Z_\downarrow(v = 0) = 0.47$ of the quasiparticle weight at unitarity are found from a 1/N-expansion of the attractive fermion problem at strong imbalance, which is equivalent to a non-selfconsistent T-matrix approximation with the bare chemical potential [14].

In Fig. 1 we show the quasiparticle residue $Z_\downarrow$ for the minority Fermion as a function of $v = 1/(k_Fa)$ within the ansatz (2.1) in comparison with the recent experimental results [5]. Apparently, the expansion up to single particle-hole excitations considerably overestimates the quasiparticle residue even though it gives reliable results for the ground state energy. A much more basic shortcoming of the ansatz (2.1), however, appears if one considers the BEC-limit $v \gg 1$. Indeed, the ansatz predicts a finite value of $Z_\downarrow$ at arbitrary interaction strengths, even in the deep molecular limit. In this limit, however, an added down-spin will form a bound state with one of the up-spin fermions and can no longer propagate as a coherent quasiparticle. One thus expects that $Z_\downarrow$ vanishes identically beyond a critical strength $v_M > 0$ of the interaction, consistent with the experimental findings [5]. It is important to note, that the formation of the bound state is a genuine many-body effect at any finite density of the up-spin Fermi sea. Indeed, the binding is of a two-body nature only in the trivial limit $v \gg 1$ (that is, effectively, for $k_F \to 0$), where the bound state is formed with a single up-spin Fermion. By contrast, just beyond the critical value $v_M$, the down-spin is effectively compensated by forming a singlet with many up-spin Fermions, somewhat similar to the physics of a localized Kondo-spin interacting antiferromagnetically with a sea of conduction electrons at temperatures much below the Kondo-temperature [15]. Note, however, that in the Kondo problem the impurity spin is not fixed and the transition from an uncompensated spin to an effective singlet state appears as a continuous crossover from high to low temperatures. In the present problem, instead, there is a discontinuous transition in the ground state as a function of the attractive coupling $v$.

An indication that the variational wavefunction (2.1) is not applicable for strong attractive interactions is provided by considering the Thouless criterion for a superfluid instability in which up-and down-spins are paired in an s-wave superfluid [16]. Evaluating the relevant vertex function within the T-matrix approximation for arbitrary values of the down-spin chemical potential $\mu_\downarrow$, it is found that the Thouless criterion $\Gamma^{-1}(k = 0, \omega = 0) = 0$ leads to a critical value for $\mu_\downarrow$ that is below the value $\mu_\downarrow = E(p = 0)$ obtained from the ground state energy of the variational state (2.1) provided that $v \geq 1.27$. A second argument that indicates the breakdown of the ansatz (2.1) in the regime $v \gg 1$ is the behavior of the ground state energy. Indeed, in a systematic expansion in powers of the scattering length $a \to 0^+$ the ground state energy of a single added down-spin relative to the free up-spin Fermi sea is expected to be of the form

$$E^a \sim E_\text{b} - \varepsilon_F + g_\text{ad}n_\downarrow + O(a^2). \quad (2.4)$$
Its leading contribution is just the molecular binding energy $E_b = -1/(3ma^2) < 0$. The contribution $\varepsilon_F$ of order $a^0$ accounts for the removal of one ↑-Fermion from the Fermi-sea that is required for the formation of the molecule. The last term, of order $a$ is the mean field repulsion between the molecule and the Fermi-sea. Its interaction strength $g_{ad} = 3\pi a a_d/m$ is related to the exact atom-dimer scattering length $a_{ad} = 1.18 a$ that has first been calculated in connection with neutron-deuteron scattering [17] (for a recent derivation in the cold gas context see Petrov et. al. [18]). It turns out, however, that the variational ansatz Eq. (2.1) leads to $E \to E_b - \varepsilon_F/2 + O(a)$ which is too high by $\varepsilon_F/2$ compared to the exact asymptotics (2.4). The reason for this discrepancy can be seen easily from the structure of Chevys’ wavefunction. On the BEC side, the dominant contribution comes from the $q = 0$ terms, i.e.

$$\sum_k \phi_k |d_{k}u_{|q}\rangle |FS_k^N|$$

(2.5)

describing the molecule formation of an up- and a down-spin with opposite momenta. This contribution is not optimal however, since it creates a hole in the center of the ↑-Fermi-sphere. Energetically, it would be favorable to replace $u_0|FS_0^N|$ with a $(N-1)$-particle Fermi sea $|FS_{N-1}^N\rangle$, leading to a ground state energy that is lower by $\varepsilon_F$. Within the ansatz (2.1), this would require terms with an arbitrary number of particle-hole excitations in order to reshuffle the Fermi-sea in such a way that the hole vanishes.

III. VARIATIONAL ANSATZ IN THE MOLECULAR REGIME

In order to describe the physics of bound state formation in the regime $v \gg 1$, we propose a variational ansatz for the $(N+1)$-body problem that complements the ansatz (2.1) describing a Fermi polaron with a finite quasiparticle residue. Our ansatz gives the exact behavior (2.4) of the ground state energy in the BEC-limit up to linear order in $a$. The associated variational wavefunction

$$|\psi_0\rangle = \left( \sum_k \varepsilon_k d_{k}u_{k}\right) |FS_k^N|$$

(3.1)

is a natural generalization of the Chevys ansatz and is constructed by adding a $(↑, ↑)$-pair to a $(N-1)$-particle Fermi sea of ↑-Fermions, together with the leading order term in an expansion in particle-hole excitations. Again, sums on $k, k’, q$ and $q'$ are restricted to $k, k’ > k_F$ and $q < k_F$, respectively. The first term accounts for the formation of the molecule in the presence of the ↑-Fermi sea and gives the correct next-to-leading-order ground state energy in the BEC-limit, avoiding the problem of creating a hole in the ↑-Fermi sea. The single particle-hole excitation in the second term describes the leading order contribution to the interaction of the dimer with the Fermi sea apart from Pauli-blocking effects, that are already accounted for in the first term. An important feature brought about by the inclusion of the second term in Eq. (3.1) is that it amounts to an exact treatment of the three-particle problem. Indeed, as is shown in detail in the Appendix, the set of coupled equations (3.6)-(3.9) that determine the coefficients of the variational many-body wavefunction reduce, in the three-particle limit, precisely to the integral equation for the exact solution of the three-body problem by Skorniakov and Tom-martirosian [17]. As a result, the exact atom-dimer scattering length $a_{ad} = 1.18 a$ appears in the asymptotic behavior of the ground state energy (2.4), giving rise to the correct next-to-next-to-leading order behavior of the ground state energy in the BEC-limit.

Obviously, the ansatz (3.1) is not capable of describing the whole range of scattering lengths correctly. In particular, it does not capture the weak coupling limit $a \to 0^−$. Indeed, the ↑-Fermion in the first term is always added at momenta $k > k_F$, leading to a ground state energy that is too high by $\varepsilon_F$ in the weak coupling limit. Our ansatz (3.1) is therefore complementary to the Chevy wavefunction (2.1), which correctly describes the situation at weak coupling up to and slightly beyond the unitarity limit.

From a physical point of view, the two variational wavefunctions (2.1) and (3.1) characterize very different ground states. Chevys’ ansatz describes a Fermi polaron with a finite quasiparticle residue, which allows to build a normal Fermi liquid at a finite concentration of the down-spin Fermions, provided that interactions between the quasiparticles have no attractive channels (see section IV. below). By contrast, the wavefunction (3.1) describes a bosonic molecule interacting with a Fermi sea. At a finite concentration $n_1 \neq 0$, the resulting ground state is expected to be a superfluid, coexisting with unpaired up-spin Fermions. The critical coupling $v_M$, where the ground state energies of the two variational wavefunctions intersect is thus expected to separate a normal fluid from a superfluid ground state of the attractive Fermi gas in the limit of very strong imbalance.

The variational ansatz (3.1) is based on a single channel model that describes the attractive interactions between both spin states. For computational purposes, however, it turns out to be easier to start from the more general two-channel model, which is defined by the Hamiltonian

$$H = \sum_p \left( \frac{\varepsilon_p}{2} + \nu_0 \right) b_p^\dagger b_p + \sum_{p, \sigma} \varepsilon_p c_p^\dagger c_{p, \sigma} + \frac{g_0}{\sqrt{V}} \sum_{p' p} \left( b_p^\dagger c_{p', \uparrow} c_{p', \downarrow} + h.c. \right).$$

(3.2)

Here, $b_p^\dagger$ denotes the bosonic creation operator of a molecule with momentum $p$ and $c_p^\dagger_{\sigma, \sigma}$ are the fermionic creation operators for the two species $\sigma = \uparrow, \downarrow$. The free particle dispersion is denoted by $\varepsilon_p = p^2/(2m)$. In the first term accounts for the factor two in the molecule to single Fermion mass ratio. The bare values of the detuning $\nu_0$ and the Feshbach coupling strength $g_0$ can be related to the physical s-wave scattering length $a$ and the interaction range $r_0$ via
\[ \frac{\nu_0}{g_0} = -\frac{m}{4\pi a} + \frac{1}{V} \sum_k \frac{1}{2\varepsilon_k}, \quad (3.3) \]
\[ r_0 = -\frac{8\pi}{g_0m^2}. \quad (3.4) \]

The two-channel Hamiltonian (3.2) is equivalent to a single channel model in the interesting limit of zero-range interactions $r_0 \to 0$ (i.e. for broad Feshbach resonances), as can be seen easily by integrating out the bosonic degrees of freedom.

The corresponding variational ansatz to (3.1) in the two-channel model has two additional terms ($\sim \eta_0, \eta_{kq}$) where the closed-channel state is occupied

\[ |\psi_0\rangle = \left( \eta_0 b_0^\dagger + \sum_k \xi_k d_k^\dagger u_k + \sum_{k,q} \eta_{kq} b_{kq}^\dagger u_k u_q \right. \]
\[ \left. + \sum_{k',k,q} \xi_{k'kq} d_{k'q}^\dagger u_{k'} u_k u_q \right) |FS_1^{N-1}\rangle. \quad (3.5) \]

Calculating the expectation value $\langle \psi_0 | \hat{H} - E | \psi_0 \rangle$, taking the derivatives with respect to the infinite set of variational parameters $\eta_0, \xi_k, \eta_{kq}, \xi_{k'kq}$ and setting them equal to zero leads to the following set of coupled equations

\[ (E + \varepsilon_F - \nu_0) \eta_0 = -\frac{g_0}{\sqrt{V}} \sum_k \xi_k \quad (3.6) \]
\[ (E + \varepsilon_F - 2\varepsilon_k) \xi_k = -\frac{g_0}{\sqrt{V}} \eta_0 + \frac{g_0}{\sqrt{V}} \sum_{k'} \eta_{k'k} \quad (3.7) \]
\[ (E + \varepsilon_F - \nu_0 - \frac{\varepsilon_q - \varepsilon_k}{2} - \varepsilon_k + \varepsilon_q) \eta_{kq} = \frac{g_0}{\sqrt{V}} \xi_k - \frac{2g_0}{\sqrt{V}} \sum_{k'} \xi_{k'kq} \quad (3.8) \]
\[ (E + \varepsilon_F - \varepsilon_{k-q-k'} - \varepsilon_{k'} - \varepsilon_k + \varepsilon_q) \xi_{k'kq} = -\frac{g_0}{2\sqrt{V}} (\eta_{kq} - \eta_{k'q}) \quad (3.9) \]

Note that the ground state energy $E$ is measured with respect to the N-particle Fermi sea, which explains the occurrence of the $\varepsilon_F$ terms in the above equations. Moreover, using the N-particle Fermi sea as reference scale, the ground state energy $E$ is equivalent to the chemical potential $\mu_1 \equiv E$ of the single down-spin.

### A. No particle-hole excitation

Neglecting for a moment the contribution of particle-hole excitations in (3.5), i.e. setting $\eta_{kq} = \xi_{k'kq} = 0$, the ground-state energy is determined by the equations (3.6) and (3.7) alone. Performing the integrations and taking the zero-range limit $r_0 \to 0$, they reduce to a simple transcendental equation

\[ \frac{\pi}{2k_F a} = 1 + \sqrt{\frac{E + \varepsilon_F}{2\varepsilon_F}} \arctan \left( \sqrt{\frac{E + \varepsilon_F}{2\varepsilon_F}} \right). \quad (3.10) \]

In the BEC-limit $a \to 0^+$, Eq. (3.10) gives rise to a ground state energy of the form (2.4). The associated atom-dimer scattering length, however, is given by its value $a_{\text{ad}}^{\text{Born}} = (8/3) a$ in the Born approximation. More generally, it turns out that Eq. (3.10) is exactly equivalent to the Thouless-criterion $\Gamma^{-1}(k = 0, \omega = 0) = 0$ if the vertex function is calculated within a non-self-consistent T-matrix approach where only the particle-particle ladder is summed, as discussed in section II. The resulting ground state energy is below that of the Fermi polaron if $v \geq 1.27$.

### B. Full variational treatment

In the general case $\eta_{kq} \neq 0, \xi_{k'kq} \neq 0$, the equations (3.6)-(3.9) can be reduced to a single homogeneous Fredholm equation of the second kind for the variational parameters $\eta_{kq}$ in the thermodynamic limit (again, the zero range limit has been taken already)

\[ \frac{1}{V^2} \sum_{k',q'} K(E; k, q; k', q') \eta_{k'q'} = 0. \quad (3.11) \]

The associated Kernel $K(E; k, q; k', q')$ is given by

\[ K = \frac{V \delta_{k,k'}}{E_k} - \frac{1}{\gamma E_k E_{k'}} - \frac{V \delta_{q,q'}}{E_{k'kq}} - \alpha_{kq} V^2 \delta_{k,k'} \delta_{q,q'} \quad (3.12) \]

with

\[ E_k \triangleq E + \varepsilon_F - 2\varepsilon_k \quad (3.13) \]
\[ E_{k'kq} \triangleq E + \varepsilon_F - \varepsilon_{k-q-k'} - \varepsilon_{k'} + \varepsilon_q \quad (3.14) \]
\[ \alpha_{kq} \triangleq -\frac{\nu_0}{g_0} - \frac{1}{V} \sum_{k'} \frac{1}{E_{k'kq}} \quad (3.15) \]
\[ \gamma \triangleq \frac{\nu_0}{g_0} + \frac{1}{V} \sum_k \frac{1}{E_k} \quad (3.16) \]

Due to the isotropy of the system, the variational parameters $\eta_{kq} \equiv \eta(k,q,\cos \theta_{kq})$ depend only on the magnitudes of the
two momenta k and q and the angle between them. This allows Eq. (3.11) to be reduced to a three dimensional integral equation.

The ground state energy \( E \) is now simply obtained by the condition that the Fredholm determinant of the kernel \( K \) vanishes. We evaluate the Fredholm determinant numerically by discretizing the integral equation using a Gauss-Legendre quadrature and calculating the determinant of the corresponding linear equation system. The order of the quadrature for the \( k, q \) and \( \cos b_{kq} \) integral were chosen as 11, 11 and 4, leading to an error of \( \sim 10^{-4} \) of the ground state energy at the unitarity point \( a \to \infty \), where the convergence is slowest.

The ground state energy as function of \( (k_F a)^{-1} \) is shown in Fig. 2. Apparently, our ansatz (3.1) leads to a ground state energy that is below that of Chevy’s ansatz for interaction strengths larger than \( (k_F a)^{-1} = 0.84 \). This value is in good agreement with the diagrammatic Monte-Carlo results by Prokof’ev and Svistunov [6, 7], who obtain \( (k_F a)^{-1} = 0.90 \). In fact, the small discrepancy is entirely due to the fact that we intersect our molecular ground state energy with that obtained using the Chevy wavefunction, which is not precise near \( v_M \). The Monte Carlo results in turn give better values for the polaron energy, shifting the intersection slightly towards the BEC regime, as can be seen in Fig. 2. Yet, as far as the molecular ground state energy is concerned, our results agree perfectly with the Monte Carlo data, down to the smallest coupling \( v \approx 0.6 \) where they have been calculated.

It is interesting to note, that the approximation \( q = 0 \) in the wavefunction (3.5) (i.e. pinning the hole-wavevector at zero momentum), leads to a ground state energy that differs from the calculation with the full wavefunction by at most 3% in the regime \( (k_F a)^{-1} > 0.84 \), where the ansatz is valid. The situation thus appears similar to that in the polaron case, where Combescot et al. [10] have shown that an expansion in hole-wavevectors works very well for Chevy’s ansatz at unitarity.

C. Quasiparticle Residue

We now show that the quasiparticle residue \( Z_1 \) of the ↓-Fermion, which can be thought of as a kind of order parameter of the transition from the polaron to the molecular state, vanishes identically in the thermodynamic limit for the variational wavefunction (3.5), that gives a lower ground state energy on the molecular side of the critical coupling \( v_M \).

Since the variational ground state wavefunction does not allow to calculate the full down-spin Green Function, the definition (2.3) of the quasiparticle residue is not applicable. Instead, we use the standard connection between \( Z_1 \) and the jump in the momentum distribution at the Fermi momentum \( k_F \), and the latter is zero in the limit of a single down-spin. The momentum distribution of the ↓-Fermion within the variational ansatz (3.5) is given by

\[
n^{↓}_p = |\xi_p|^2 + 2 \sum_{k'q} |\xi'_{k'q}|^2 \delta_{p,q-k'k} \tag{3.17}
\]

and is normalized via

\[
1 = \sum_p n^{↓}_p = \sum_k |\xi_k|^2 + 2 \sum_{k'q} |\xi'_{k'q}|^2 . \tag{3.18}
\]

The normalization condition requires the coefficients to scale with the system volume as \( |\xi_k| \sim 1/\sqrt{V} \) and \( |\xi_{k'q}| \sim 1/V^{3/2} \). Since an upper bound to the quasiparticle residue \( Z^{\downarrow} \) is given by the momentum distribution at \( p = 0 \) and \( |\xi_p| \approx 0 \) for \( p < 0 \), we find that

\[
Z^{\downarrow} \leq n^{↓}_{p=0} = 2 \sum_{k'q} |\xi'_{k'q}|^2 \delta_{0,k'-k} \sim \frac{1}{V} . \tag{3.19}
\]

As a result, the quasiparticle residue \( Z^{\downarrow} \) of the molecular wavefunction scales inversely with the volume of the system and thus vanishes in the thermodynamic limit. This is in contrast to Chevy’s wavefunction, where \( Z_1 = |\phi_0|^2 \) is always finite. The two wavefunctions (2.1) and (3.1) therefore indeed describe qualitatively different ground states. In particular, no sharp peak is expected in the minority rf-spectrum at coupling strengths \( v > v_M \), consistent with the experimental observation [5].

In the \( q = 0 \) approximation, which captures the essential properties of the variational ansatz (3.1), the quasiparticle residue \( Z^{\downarrow} \) in fact vanishes identically. Indeed,

\[
Z^{\downarrow} \leq 2 \sum_k |\xi_{-kko}|^2 = 0 , \tag{3.20}
\]

since, as can be seen from Eq. (3.9), the coefficients \( \xi_{-kko} \propto \eta_{ko} - \eta_{-kko} \) vanish because \( \eta_{ko} \) only depends on the length of \( k \).
IV. CONTACT COEFFICIENT AND PHASE DIAGRAM

The analysis of the polaron to molecule transition in the previous section leaves two important questions open: what is the nature of the transition and what are its implications for the phase diagram of the strongly imbalanced gas? Now for the case of a single down-spin in an up-spin Fermi sea, the transition from a polaronic to a molecular state is a first order transition, where the quasiparticle residue $Z_1$ exhibits a discontinuous jump from a finite value to zero at the critical coupling $v_M \approx 0.9$. This is a result of the fact that the energies of the two ground states, which have different quantum numbers, cross with a finite slope at $v_M$ (see Fig. 2). It is important to note that this crossing is not an artefact of extending the different variational states beyond their domain of validity. Indeed, as shown by Prokof’ev and Svistunov [6, 7], both the polaronic and the molecular state exist as stable excitations for $v > v_M$ or $v < v_M$ respectively because the phase space for decay vanishes linearly with the magnitude of the energy difference. Both states are thus reachable as metastable configurations coming from the weak coupling or the molecular side, as expected for a first order transition.

A different perspective on the first order nature of the polaron to molecule transition is provided by considering the so-called contact coefficient $C$. As shown by Tan [9], the momentum distribution of Fermi gases with zero range interactions generically decays with a power law $n_{\downarrow}(k) \rightarrow C/k^{4}$ for large momenta. The associated coefficient $C$ is identical for both spin components $\sigma = \uparrow, \downarrow$ [20] and is a measure of the probability that two Fermions with opposite spin are close to each other [21]. Using the adiabatic theorem derived by Tan [22], the contact density can be determined from the derivative

$$\frac{\partial u}{\partial (1/a)} = - \frac{\hbar^2}{4\pi m} C$$

(4.1)

of the ground state energy density $u = E/V$ with respect to the inverse scattering length. Now the definition of the down-spin chemical potential $\mu_\downarrow$ implies that the energy $u$ of the strongly imbalanced Fermi gas $n_\downarrow \ll n_\uparrow$ to linear order in the minority density $n_\downarrow$ is of the form

$$u = \frac{3}{5} \varepsilon_{F\downarrow} n_\downarrow + \mu_\downarrow n_\downarrow + \ldots$$

(4.2)

where the first term is simply the energy of a non-interacting gas of spin-up Fermions. The dimensionless contact coefficient $s$ defined by $C = s \cdot k_F^3$ for a strongly imbalanced Fermi gas can thus be obtained from the derivative

$$s = \frac{1}{3\pi} \frac{\partial (-\mu_\downarrow/\varepsilon_F)}{\partial v}$$

(4.3)

of the negative down-spin chemical potential in units of the Fermi energy with respect to the coupling constant $v$. Since $\mu_\downarrow$ is precisely the energy $E$ associated with adding a single down-spin, our result for the ground state energy of the $(N+1)$-particle problem immediately gives the contact density of an almost fully polarized attractive Fermi gas (note that this applies even on the molecular side $v > v_M$, where the single added down-spin is not a propagating quasiparticle). The associated dimensionless constant $s$ is shown in Fig. 3. It increases monotonically from weak coupling to unitarity and up to the critical coupling $v_M$. At this point, there is a discontinuous jump upwards, that reflects the transition to a molecular state. Note, that the proportionality $C \sim k_F^3 \sim n_\downarrow$ of the contact to the down-spin density makes $C$ vanish in the limit of full polarization. This is expected, because the fully polarized gas at zero temperature is an ideal Fermi gas, with no tails in the momentum distribution. Apart from the jump at $v_M$, the behavior of the dimensionless contact coefficient $s$ is rather close to that obtained for the contact coefficient $C = s \cdot k_F^3$ of the balanced superfluid along the BCS-BEC crossover [23] (note that the Fermi momentum $k_F$ of the balanced gas is related to that of the up-spin component used here by $k_F^2 = k_F^2 (1+\sigma)$, where $\sigma = (n_\uparrow - n_\downarrow)/(n_\uparrow + n_\downarrow)$ is the degree of polarization at a fixed total number of particles). Indeed, in weak coupling one obtains $s_{\text{BCS}} = (2/3\pi^2)^2$ from the mean field attraction of the polaron to the up-spin Fermi sea, while $s \approx 0.08$ at unitarity and $s_{\text{BEC}} = 4v/3\pi$ in the molecular limit, very similar to the behavior that is found for $s$ in the balanced superfluid [23].

The solution of the $(N+1)$-body problem for arbitrary coupling strengths $v$ has also implications for the phase diagram of the imbalanced Fermi gas in the regime near complete polarization. For a discussion of this issue, it is convenient to introduce an effective magnetic field $h$ that couples to the two different spin-states $\sigma = \uparrow, \downarrow$ in the standard form

$$\hat{H}' = -h \left( \hat{N}_\uparrow - \hat{N}_\downarrow \right)$$

(4.4)

of a ‘Zeeman’ field that favors a finite population imbalance $\sigma = (n_\uparrow - n_\downarrow)/(n_\uparrow + n_\downarrow) > 0$. At a fixed total density $n$, the ground state energy $u$ per volume is then a function of $n$ and

![FIG. 3: (Color online) Dimensionless contact coefficient $s$ as function of $(k_Fa)^{-1}$, calculated from the two variational wavefunctions (2.1) and (3.1). Within this approach $s$ is discontinuous at the critical coupling $v_M$. The black dashed line marks the asymptotics in the molecular limit, where $s = 4v/3\pi$ is fixed by the two-particle bound state wavefunction in momentum space.](image-url)
It determines the chemical potentials of the majority and minority species from $\mu_{\uparrow,\downarrow} = \mu \pm h$ where $\mu = \partial u(n,h)/\partial n$ is the average chemical potential. In addition, it also fixes the imbalance from $n_\uparrow - n_\downarrow = -\partial u(n,h)/\partial h$. The choice of an ensemble with fixed values of $n$ and $h$ is convenient for a discussion of the ground state phase diagram of the attractive Fermi gas at arbitrary coupling $v$, both in the homogeneous case and in the presence of a harmonic trap [24]. Indeed, there are two critical fields $h_{\downarrow}(v)$ and $h_{\uparrow}(v)$ that separate two simple limiting phases from a regime, in which nontrivial ground states are expected: The lower critical field $h_{\downarrow}$ is defined by $\sigma(h) = 0$ for $h < h_{\downarrow}$ and determines the boundary of the balanced superfluid phase (denoted by $SF_0$ in Fig. 4, following the notation used by Pilati and Giorgini [25]). The upper critical (or ‘saturation’) field $h_{\uparrow}$, in turn, is defined by the condition of complete polarization $\sigma(h) = 1$ for $h > h_{\uparrow}$. Since a single component Fermi system has vanishing interactions in the ultracold limit, this regime is just an ideal Fermi gas, i.e. it is a normal, fully polarized state. The qualitative structure of the zero temperature phase diagram as a function of the interaction parameter $v = 1/(k_F a)$ and the effective magnetic field $h$ in units of the bare Fermi energy $\varepsilon_F$ of the fully polarized gas is shown in Fig. 4.

In this diagram, the upper line $h_{\uparrow}(v)$ is completely fixed by our calculation above of the energy $\mu_{\uparrow}$ associated with adding a single down-spin to an up-spin Fermi gas. Indeed, since $\mu_{\uparrow} \equiv \varepsilon_F$ along this line, we have $h_{\uparrow} = (\varepsilon_F - \mu_{\uparrow})/2$. In terms of the constant $A(v)$ introduced in Eq. (2.2), this leads to $h_{\uparrow}/\varepsilon_F = (1 - A)/2$, giving $h_{\uparrow} = 0.81 \varepsilon_F$ at unitarity from the precise numerical value of the polaron energy [7]. On the molecular side, Eq. (2.4) gives

$$h_{\downarrow}/\varepsilon_F = v^2 + 1 - \frac{a_{ad}/a}{2\pi v} + \ldots,$$

which is very accurate even at $v = v_M$. The point $M$ along this line separates a regime where a single down-spin is a well defined fermionic quasiparticle from one, in which it is bound to the up-spin Fermi sea. The first order nature of the transition shows up as a discontinuity of the slope in $h_{\downarrow}(v)$ at $M$ which is, however, hardly visible in Fig. 4. For a finite density of down-spins, the point $M$ appears as an endpoint of a line that separates a phase with a finite Fermi surface volume $\Omega_\downarrow \neq 0$ to its left from one with $\Omega_\downarrow = 0$ [26]. Using the generalized Luttinger theorem derived by Sachdev and Yang [27], the expected polarized superfluid ($SF_p$) phase on the molecular side has a condensate of ‘dimers’ plus an up-spin Fermi sea, whose volume $\Omega_\uparrow = (2\pi)^3(n_{\uparrow} - n_\downarrow)$ is set by the imbalance. This is consistent with the naive picture that the density of unpaired up-spins is simply $n_{\uparrow} - n_\downarrow$ even though the ‘dimers’ in the vicinity of the transition are far from local ($\uparrow\downarrow$)-pairs. In principle, this simple picture of the $SF_p$-phase as a BEC coexisting with a sharp, single Fermi surface of unpaired up-spins is unstable with respect to p-wave pairing due to the induced interactions between the unpaired fermions through the superfluid [28]. In practice, the nontrivial superfluid phase of the unpaired up-spins is exponentially suppressed for strong imbalance. Moreover, quantitative results for the p-wave instability can be derived only in second order in $1/v \ll 1$, where the resulting energy scales are exponentially small compared with $\varepsilon_F$. In practice, therefore, the phase with p-wave pairing among the unpaired up-spins seems hardly accessible experimentally.

A nontrivial issue that has been neglected in the discussion so far is the question whether a gas of polarons or bound molecules is indeed stable at low but finite densities $n_\downarrow$. On the weak coupling side, there is again an induced attractive interaction in the p-wave channel among both the up-spins and the down-spins, mediated by the other species. The ground state is thus expected to be a two-component p-wave superfluid and not a normal Fermi liquid state [28]. Similar to the situation in the BEC-limit, however the energy scale for this instability is exponentially small in the regime where the calculation can be controlled. More importantly, as has been shown recently by Nishida [29], the effective interaction between two heavy down-spin fermions immersed in an up-spin Fermi sea is attractive in the p-wave channel only for weak coupling. Approaching unitarity, the p-wave interaction becomes repulsive. Assuming that this result carries over to the relevant case of equal masses of the up- and down-spin Fermions, a finite density gas of down-spins will indeed form a normal Fermi liquid at unitarity, as was implicitly assumed in the calculations of the equation of state and density profiles of the unitary gas beyond the critical imbalance $\sigma_c \approx 0.4$, where the balanced superfluid is no longer stable [11, 30]. On the molecular side, the phase immediately below the saturation field line $h_{\downarrow}(v)$ is expected to be a superfluid of $\langle \uparrow\downarrow\rangle$-pairs at a very low density $n_{\downarrow} \rightarrow 0$ immersed in an up-spin Fermi sea. The fact that the atom-dimer repulsion
\(a_{ad} = 1.18 a\) is much larger than the dimer-dimer repulsion \(a_{dd} = 0.6 a\) [18], however indicates that a low density gas of molecules tends to phase separate from the up-spin Fermi gas. This phase separation has indeed been found from an extended BCS-description of the BCS-BEC crossover in an imbalanced gas [31, 32, 33, 34]. It has recently been seen also in the variational Monte Carlo calculations by Pilati and Giorgini [25]. Their results indicate that a section between \(v_N \approx 0.73\) and a triple point at \(v_F \approx 1.7\) along the \(h_s\)-line is actually a first order line, where the polarized superfluid disappears with a finite jump in density as the effective field \(h\) increases through \(h_s\). As shown above, the point \(M\) lies in the interval between \(v_N\) and \(v_F\) and thus the polaron to molecule transition would not be accessible at any finite minority density, at least not in an equilibrium situation. Clearly, our variational calculation for the single down-spin problem cannot address the question of phase separation. An unexpected feature of the \(h_s\)-line in the presence of phase separation is the fact that the transition across \(h_s\) is predicted to be continuous up to \(v_N\), first order between \(v_N\) and \(v_F\) and continuous again for \(v > v_F\). The rather large value \(v_F \approx 1.7\) up to which phase separation is predicted also appears surprising. Indeed, in this regime a mean field theory describing a Fermi gas coexisting with a BEC of free molecules gives for the energy per volume as a functional of the density difference \(\delta n = n_1 - n_1\) and the dimensionless field \(\eta = h / \varepsilon_F\) the simple form

\[
\frac{u(n, \delta n, \eta)}{\varepsilon_F} = \frac{3}{\varepsilon} (\frac{\delta n}{n})^{2/3} \delta n - u^2(n - \delta n) + \frac{\tilde{a}_{ad}}{2\pi \varepsilon} \frac{(n - \delta n)}{n} + \frac{\tilde{a}_{dd}}{12 \pi \varepsilon} \frac{(n - \delta n)}{n} - \frac{1}{2} \frac{\eta}{h} \delta n \quad (4.6)
\]

Here, \(\tilde{a}_{ad} = \frac{a_{ad}}{4}\) and \(\tilde{a}_{dd} = \frac{a_{dd}}{4}\) are the atom-dimer and dimer-dimer scattering length measured in units of the atom-atom scattering length and \(v = (k_F a)^{-1}\). The true ground state energy density \(u(n, \eta)\) as function of the total density \(n\) and the effective magnetic field \(\eta\) is determined by the minimum of the Landau energy (4.6) with respect to the density imbalance

\[
\frac{\partial (u / \varepsilon_F)}{\partial \delta n} = \frac{3}{\varepsilon} \left( \frac{\delta n}{n} \right)^{2/3} + \frac{\tilde{a}_{ad}}{2 \pi \varepsilon} (1 - 2 \frac{\delta n}{n}) + \frac{\tilde{a}_{dd}}{6 \pi \varepsilon} (\frac{\delta n}{n} - 1) - \frac{\eta}{h} \delta n \quad (4.7)
\]

This equation determines the imbalance \(\sigma = \delta n / n\) as a function of the field \(\eta\) and indeed, it correctly describes the exact asymptotic results for both the saturation field \(h_s\) and the lower critical field \(h_c\) in the limit \(\nu \gg 1\) (see Eqs. (4.5) and (4.9)). In this simple model, phase separation between a polarized superfluid phase and a fully polarized Fermi gas appears for coupling constants \(\nu < v_{c,PS}\), below which the energy density (4.7) has a second minimum at full polarization \(\sigma = 1\). This occurs at

\[
v_{c,PS} = \frac{3}{2 \pi} \left( \frac{\tilde{a}_{aa} - \tilde{a}_{dd}}{6} \right) . \quad (4.8)
\]

With the exact values \(\tilde{a}_{ad} = 1.18\) and \(\tilde{a}_{dd} = 0.6\) one obtains \(v_{c,PS} = 0.516\), where the simple expansion (4.6), however, is no longer valid. From the calculation above, the triple point \(v_F\) beyond which phase separation appears in an almost fully polarized gas lies at a much smaller value of the coupling strengths than found previously [25, 34]. At finite temperatures, phase separation is suppressed by the presence of a mixing entropy, which may explain that it is not observed in the experiments, where \(T \approx 0.15 T_F\).

Concerning the lower critical field \(h_c(v)\), its weak-coupling limit is determined by the well known Chandrasekhar-Clogston result \(h_c = \Delta / \sqrt{2}\), beyond which the balanced BCS-pairstate is unstable [35, 36]. In the molecular limit \(v \gg 1\), the critical field

\[
h_c(\varepsilon_F) = v^2 + \frac{1}{2 \pi \varepsilon} (a_{ad} - a_{dd} / 6) + \ldots \quad (4.9)
\]

follows essentially the two-particle binding energy with corrections due to the atom-dimer and dimer-dimer scattering lengths \(a_{ad}\) and \(a_{dd}\), respectively. At unitarity, \(h_c = 0.25 \varepsilon_F\) is a universal constant times the bare up-spin Fermi energy \(\varepsilon_F\) [11]. As a result, there is a wide range \(h_s / h_c = 3.12\) between the balanced superfluid and the fully polarized gas, much larger than that found in an \(N = \infty\) theory of the imbalanced attractive Fermi gas, where \(h_s / h_c = 1.24 \approx [37, 38, 39\). For \(h > h_c\) the balanced superfluid is destroyed by the onset of a finite polarization \(\sigma \neq 0\), which leads to a mismatch of the Fermi energies. An effective field theory due to Son and Stephanov [38] indicates, that the phase beyond the balanced superfluid exhibits a spatially varying superfluid order of the FFLO type as also found at weak coupling. The transition is first order with a jump both in polarization \(\sigma\) an total density, a situation that is also found in the case of a direct transition between a balanced superfluid and a partially polarized normal phase [11, 30]. In an ensemble with a given density that is used here, the \(h_c\)-line would then split into two distinct lines \(h_{c1}\) and \(h_{c2}\) as noted by Sheehy and Radzihovsky [32]. In our diagram in Fig. 4, \(h_c\) denotes the boundary of the balanced superfluid at given density, which is well defined without specifying the state that is reached at nonzero polarization. The first order nature of the transition is found only up to a splitting point \(S\), beyond which the fermionic excitations have their minimum at \(p = 0\). When this is the case, additional up-spin Fermions can be added by filling up a Fermi surface whose volume \(\Omega_f \sim \sigma \sim (h - h_c)^{3/2}\) increases continuously from zero. The transition from the balanced superfluid to a polarized superfluid with unpaired excess Fermions is therefore continuous and preserves superfluidity. The precise location of the splitting point \(S\) has been determined recently from a calculation of the fermionic excitation spectrum along the BCS-BEC crossover of the balanced gas [23]. It is located at \(\nu_S \approx 0.63\) and \(h_c(\nu_S) = \Delta \approx 0.6 \varepsilon_F\), at considerably larger coupling strengths than predicted by mean-field theory where the splitting point coincides with the zero crossing of the chemical potential (note the factor \(2^{1/3}\)difference with the result in Ref. [23], which is due to the fact that the up-spin Fermi wave vector and not that of the balanced case appears in our present coupling constant \(\nu\)). This is in agreement with the calculation of the splitting point within an \(\varepsilon - 4 - d\)-expansion by Nishida and Son [39] but is larger than the value \(\nu_S \approx 0.5\)
found for the splitting point in the Monte Carlo calculations of Pilati and Giorgini [25]. Note that the possibility of extracting the critical coupling \( v_S \) of the splitting point from a calculation of the balanced gas relies on the fact that the ground state energy is independent of the field \( h \) in the whole regime \( h \leq h_c \), where the polarization \( \sigma = -n \partial u(n, h)/\partial h \) vanishes. The nature of the phase diagram near the splitting point has been discussed by Son and Stephanov [38] using an effective field theory. In particular, the phase immediately beyond \( h_c \) is expected to be of the FFLO-type, with a spatially oscillating superfluid order parameter that appears also in weak coupling beyond the Chandrasekhar-Clogston limit [40]. It is an open question, how this nontrivial superfluid evolves into a normal phase in which the two spin components each form a Fermi liquid. In fact it is this latter phase, which describes the experimentally observed density profiles [41] at unitary extremely well [11, 30]. It is also an open issue, how to separate in detail the regime between the lower critical field and the saturation field into a regime where an imbalanced Fermi liquid or a polarized superfluid phase appear as ground states. In the phase diagram of Pilati and Giorgini, the first order line that bounds the balanced superfluid up to the splitting point \( S \) extends as a first order line up to \( h_s \) at the coupling \( v_N \) and then continues along \( h_s \) up to the tricritical point \( v_T \).

V. CONCLUSIONS

From a variational wavefunction that describes the \((N + 1)\)-particle problem of a single down-spin interacting strongly with an up-spin Fermi sea, we have discussed the physics of the strongly imbalanced Fermi gas. In particular, we have focused our attention on the quasiparticle residue and the contact coefficient \( C \). The latter exhibits a discontinuous jump at the polaron to molecule transition which might be detected by measurements of the closed channel fraction similar to the analysis of the experiments by Partridge et.al. [42] due to Werner, Tarruell and Castin [43]. A motivation for this work were the recent experiments of Schirotzek et al. [5], who have observed a transition from a Fermi liquid phase of polaronic quasiparticles near unitarity to a phase in which the quasiparticle residue vanishes. As shown above, this transition is expected to be a discontinuous one in the single down-spin limit. Apparently, however, \( Z_1 \) vanishes in a continuous manner in the experiment (see Fig. 1). Apart from the uncertainty in extracting \( Z_1 \) from the sharp structure in the minority rf-spectrum, this discrepancy is probably due to the fact that the Chevy wavefunction (2.1) strongly overestimates the quasiparticle residue \( Z_1 \) near the polaron to molecule transition. A reliable quantitative calculation of the rf-spectra for both finite concentrations and at finite temperatures is, unfortunately, not available. The existence of a stable finite density gas of polarons in the regime up to \( v \simeq 0.9 \) however indicates that the interaction between them is repulsive, so that they indeed form a Landau Fermi liquid below the critical coupling \( v_M \). As discussed in section IV, the detailed structure of how this phase connects to the nontrivial superfluid phases expected near the splitting point \( S \) and on the BEC-side is a major and still open problem.

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Note added: The variational wavefunction (3.1) for the \((N + 1)\)-particle problem in the molecular regime has been found independently by C. Mora and F. Chevy [44]. For the calculation of the ground state energy they have assumed a vanishing hole wavevector \( q = 0 \), which reduces the resulting integral equation to a one-dimensional problem. Furthermore, we are grateful to R. Combescot for pointing out their closely related work [45].

APPENDIX: THREE PARTICLE LIMIT

In the following we briefly show, how the exact solution of the three Fermion problem can be obtained from the integral equation (3.11) for the full variational wavefunction (3.5). If the \((N-1)\)-particle Fermi sea is reduced to a single \( \uparrow \)-Fermion, only the \( q = 0 \) terms remain in the variational wavefunction (3.5). Thus, starting from the integral equation (3.11) in the thermodynamic limit, taking the limit \( k_F \rightarrow 0 \) and setting \( q = 0 \), one arrives at the simplified equation

\[
\alpha_{k_0} \eta_{k_0} = -\int \frac{d^3 k'}{(2\pi)^3} \frac{\eta_{k'0}}{E_{k'k_0}} .
\]  

(A.1)

Inserting the coefficients \( \alpha_{k_0} \) and \( E_{k'k_0} \) from Eqs. (3.14) and (3.15) explicitly, the integral equation (A.1) takes the form

\[
\left( \frac{1}{a} - \sqrt{\frac{3k^2}{4} - m E} \right) \eta_k = \int \frac{d^3 k'}{(2\pi)^3} \frac{4\pi \eta_{k'}}{k'^2 + k'^2 + k \cdot k' - m E} .
\]  

(A.2)

Note that \( \eta_k \equiv \eta_{k0} \) corresponds to the Fourier transform of the relative wavefunction between the \( (\uparrow, \downarrow) \)-molecule and the additional \( \uparrow \)-Fermion. The integral equation (A.2) is exactly the same as the one obtained by Skorniakov and Tom-Tartosirosian [17] for the three Fermion problem. In particular, it is equivalent to Eqn. (29) in [17], which corresponds to three-nucleon scattering with total isospin \( T = 1/2 \) and total spin \( S = 3/2 \) (note that the spin part only contributes an unimportant prefactor to the wavefunction in this case).
Within a local density approximation, the different phases that appear in a trap with a spatially dependent coupling constant $\nu(x)$ due to the decrease of the local Fermi wavevector $k_F(x)$ from the center of the trap to its edge simply follow from a parabolic (vertical at unitarity) line in Fig. 4:

$$\frac{\hbar}{\varepsilon_F(x)} = \frac{2\hbar}{|E_h|} \cdot v^2(x),$$  \hspace{1cm} (A.3)

where $|E_h| = 1/(ma^2)$ for both positive or negative scattering lengths $a$. Its curvature $2\hbar/|E_h|$ is fixed by the global imbalance, while the initial point $\nu(x = 0)$ is determined by the local Fermi wavevector at the trap center $x = 0$. Note that in a trap different phases are spatially separated for both continuous and first order transition lines. By contrast, genuine phase separation is associated with a first order transition and appears only in the homogeneous system.

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