Predicting shock response in uncertain structures using the Hybrid method

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Abstract. Using the Hybrid method (FE + SEA) it is possible to estimate the frequency response of an uncertain structure. The current work develops the Hybrid method to allow for time domain analysis of the shock response of a structure. Problems to be overcome when taking Hybrid method results into the time domain are a) the Hybrid method frequency response has no phase information, and b) the Hybrid method frequency response is smoothed in frequency and shows no modal peaks. In this paper the first problem has been overcome, using minimum phase reconstruction. Explanation of minimum phase reconstruction and its limitations are described, and application to shock problems described.

1. Introduction
Structural vibration problems are conventionally divided into two sets: low frequency problems, where the wavelengths of vibration are long compared with the structural dimensions; and high frequency problems, where the wavelengths are short. For these problems there are well developed tools for simulating and predicting vibrations: Finite Element Analysis (FEA)\textsuperscript{[1]} for low frequencies, and Statistical Energy Analysis (SEA)\textsuperscript{[2]} for high frequencies.

Another class of problem exists, when a structure contains different elements that show both low and high frequency behaviour simultaneously. This is a ‘mid-frequency’ problem, and can be tackled using the Hybrid method\textsuperscript{[3]}. The Hybrid method couples together both FEA and SEA modelled subsystems into one structural vibration model. The Hybrid method gives a frequency response result which is averaged over an ensemble of statistically similar structures, and so includes any uncertainty about the nature of a structure.

To predict the shock response of a structure a time domain result is typically sought. The nature of shock is to excite a structure at a large range of frequencies (both low and high) and so for structures with significant resonances extending over this range neither FEA nor SEA is suitable. The Hybrid method is a more suitable tool for such a problem.

Hybrid method results are frequency domain based, and must be transformed into the time domain for shock analysis. Two problems are encountered when transforming to the time domain, a) Hybrid method results lack phase information, and b) ensemble averaging has the effect of smoothing the frequency response in frequency and so removing modal peaks in the spectrum. Both of these frequency domain features have a huge effect on the time domain response.

The work contained in this paper seeks to extend the functionality of the Hybrid method to predict time domain shock responses. The first problem of phase retrieval is tackled using...
minimum phase reconstruction. The meaning of minimum phase is explained in Section 2, then a discussion of how minimum phase reconstruction can be carried out in practise is given in Section 3. Initial experiences of using minimum phase reconstruction for the shock response of uncertain structures are given in Section 4, and concluding remarks given in Section 5.

2. Explanation of minimum phase
For a system $F(u)$ the phase of the system is defined as $\theta(u)$, where $F(u) = |F(u)|e^{i\theta(u)}$. In the case of Hybrid method produced frequency responses, the magnitude $|F(\omega)|$ is known but the phase is not. Phase information is needed for accurate time domain realisation. In some cases phase retrieval is possible, but this is usually ambiguous. It has been shown that, except for certain special cases, there exists an infinity of functions $f_i(t)$ whose Fourier transforms $F_i(\omega)$ have the same modulus $|F(\omega)|$ on the real frequency axis [4].

2.1. Defining minimum phase
One special case where a unique relationship between the magnitude and phase of a system exists is when a system is known to be minimum phase. A system is defined as minimum phase if the system and its inverse are causal and stable. In terms of the poles and zeros, for a frequency response this means that all poles and zeros must lie in the positive imaginary half of the complex frequency plane. A minimum phase system can be also be thought of as the ‘fastest decaying’ system [5]. Formally, for a causal signal $f_i(n)$ that has a given magnitude spectrum $|F(\omega)|$, the minimum phase signal $f_{mp}(n)$ is the one that maximises initial energy:

$$\sum_{n=0}^{K} |f_{mp}(n)|^2 \geq \sum_{n=0}^{K} |f_i(n)|^2, \quad \text{for } K = 0, 1, 2, \ldots, N - 1.$$  

Minimum phase systems occur frequently when studying structural vibration frequency responses. The driving point response of any physical system must be minimum phase, with one minimum phase zero between every pole. To see this, the driving point response is described by mobility — the velocity at a point resulting from a prescribed force input at the same point. This is causal and stable for all real systems. The inverse of the driving point mobility is the driving point impedance — the force that results from applying a prescribed velocity input. This is also causal and stable for all real systems. Thus, the drive point response function and its inverse are causal and stable and so both must be minimum phase functions. For non-drive point responses, the inverse function is not a simple physical function like drive point impedance, and so it cannot be assumed that the inverse is stable. Thus, the non-drive point response cannot be assumed to be minimum phase.

2.2. Phase retrieval using Hilbert Transform
As shown in Section 2, for minimum phase systems, there is a unique mapping between the magnitude and phase of a system’s frequency response [6]. This uses a function known as the Hilbert Transform, $\mathcal{H}\{f(x)\}$, defined as

$$\mathcal{H}\{f(x)\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{x - \tau} d\tau.$$  

The Hilbert Transform relates the transfer function phase and magnitude of a minimum phase system uniquely:

$$\text{arg}[F(\omega)] = -\mathcal{H}\{\log(|F(\omega)|)\},$$  

or inversely:

$$\log(|F(\omega)|) = \log(|F(\infty)|) + \mathcal{H}\{\text{arg}[F(\omega)]\}.$$
3. Using minimum phase reconstruction

For a system with magnitude frequency response $|F(\omega)|$ the phase cannot be retrieved uniquely. By assuming that the system is minimum phase the phase can be found using the Hilbert transform, this is called minimum phase reconstruction (MPR). This section details artefacts encountered when implementing an MPR algorithm, and the errors that result in the time domain when taking a minimum phase assumption for a non-minimum phase system.

3.1. Approximating the Hilbert Transform

This unique mapping given by the Hilbert transform, from phase to magnitude in minimum phase systems, is for continuous functions only. Such a function is not realisable on a digital computer, and it is to be expected that artefacts will occur as a result. For discrete systems there exists the discrete Hilbert transform (DHT) [7]. For a signal $f(k)$ with discrete Fourier transform $F(k)$, the DHT $g(k)$ is given by the inverse discrete Fourier transform of $G(k)$, where

$$G(k) = H(k) \cdot F(k),$$

and

$$H(k) = \begin{cases} -j, & k = 1, 2, \ldots, \frac{N}{2} - 1 \\ 0, & k = 0, \frac{N}{2} \\ +j, & k = \frac{N}{2} + 1, \frac{N}{2} + 2, \ldots, N - 1. \end{cases}$$

For large $N$ the DHT approximates the continuous time Hilbert transform well. Errors in the DHT approximation of the Hilbert transform occur due to the using the DFT approximation of the Fourier transform. By using an FFT length that is long enough to give a good approximation of the Fourier transform, errors in the DHT can be reduced.

An MPR algorithm has been written in MatLab that reconstructs the minimum phase realisation of a magnitude spectrum $|F(\omega)|$, using Eq. (3) and the DHT in the frequency domain. To test the accuracy of the MPR algorithm, the MPR was validated using the magnitude spectrum of a driving point impulse response — a minimum phase signal. The spectrum $H_D(\omega)$ for the driving point response of a random system was calculated using a modal summation,

$$H_D(\omega) = \sum_{n=1}^{N} \frac{\phi_n^2}{\omega_n^2 + j\eta\omega_n\omega - \omega^2},$$

where modal amplitudes $\phi_n$ had a Gaussian distribution, resonant frequencies $\omega_n$ had a Rayleigh distribution, and damping $\eta$ was assumed constant. The DHT algorithm was validated by reconstructing the phase of $H_D(\omega)$ to give $\hat{H}_D(\omega)$.

Figure 1 shows the bode plot for a driving point response and the MPR using the DHT. It can be seen that the MPR does not reproduce the phase of original signal exactly. The magnitudes of $H_D(\omega)$ and $\hat{H}_D(\omega)$ are equal, but the phases are not quite. The MPR, $\hat{H}_D(\omega)$, shows an approximately linear phase term that increases the phase by $2\pi$ over the frequency range calculated. This error in calculated phase is due to the FFT approximation of a Fourier transform used in the calculation of the DHT. Though the phase error at high frequencies (i.e. above the Nyquist frequency) look to be large, these are all actually $2\pi - \theta_e$, where $\theta_e$ is the small phase error at the symmetric point below the Nyquist frequency.

Phase errors are worst at the Nyquist frequency, there being a $\pi$ rads difference between the phase of $H_D(\omega)$ and $\hat{H}_D(\omega)$ here. As long as the magnitude of the response is low here, these errors around the Nyquist frequency do not give a significant error in the time domain. The corresponding time domain signals $h_D(t)$ and $\hat{h}_D(t)$ show only a very slight error, and only in low
magnitude high frequency components which decay very rapidly. Quantitatively, the normalised L2-norm error of $h_D(t)$ and $\hat{h}_D(t)$ is given by

$$\epsilon_{L_2} = \frac{\sum_n (h_D(nT) - \hat{h}_D(nT))^2}{\sqrt{(\sum_n h_D(nT)^2)(\sum_n \hat{h}_D(nT)^2)}}$$

$n = 0, 1, \ldots, N - 1,$

and has a very low value of $\epsilon_{L_2} = 2.77 \times 10^{-4}$. This error can be reduced still further by increasing the frequency window of the spectrum.

Errors can also occur for MPR when a system of low damping is used. The main effect of these errors is an increase in the decay rate of the MPR signal $\hat{h}_D(t)$. These errors are not found for spectra produced from systems with higher damping. It has been found that this error comes from the discrete approximation of $H_D(\omega)$ used in computer calculations. By increasing the frequency resolution of the discrete approximation of $H_D(\omega)$ the error seen in $\hat{h}_D(t)$ is reduced.

A rule of thumb has been found for satisfactory reproduction of $\hat{h}_D(t)$: by ensuring that there are 5 frequency data points covering the half power bandwidth of every resonance of $H_D(\omega)$, no noticeable error in the time domain is observed.

Though the Hilbert transform provides a unique mapping between phase and magnitude for minimum phase signals, for work on a computer the DHT must be used, and this has been shown to produce errors. The solution has been to increase the frequency range and resolution of the spectrum being analysed. Put simply, the only way to reduce the errors that result from using the discrete Hilbert transform is to increase the size and resolution of the sampled space to better approximate the continuous space. This is a limitation of the algorithm, and cannot be avoided.

It should be noted that the use of the continuous variable $\omega$ to describe spectra here is not strictly correct for discrete spectra modelled using computers. However, effort has been made to make data lengths long enough to reduce the artefacts that occur from approximating a continuous function by a finite sampled function. By reducing these artefacts, the discrete and continuous functions become equivalent, and so the term $\omega$ (and $t$ for time) is used where formally a discrete variable would be more appropriate.
3.2. Errors using MPR for non-minimum phase systems

Though the driving point response of a system is minimum phase, non-driving point responses typically are not. As explained in Section 2 the phase of a non-minimum phase response \( H_{NMP}(\omega) \) cannot be determined uniquely from its magnitude. By assuming that the system is minimum phase, the MPR algorithm can be used to (incorrectly) reconstruct the phase of \( H_{NMP}(\omega) \).

Fig. 2 shows the sonogram of an impulse response of a non-minimum phase system. The system analysed is an idealised model of a periodic drill string, and the response shown is the end to end impulse response. Fig. 2(a) shows the dispersive nature of the system, and shows echoes occurring after the initial impulse.

![Figure 2: Sonogram of end to end impulse response: (a) original and (b) MPR](image)

The MPR shown in Fig. 2(b) shows three main features that are of interest. Firstly, any initial delay in the impulse response is removed in the MPR. For dispersive systems it is found that different frequency components experience a different amount of time delay removal, such that all frequency components have a large response at \( t = 0 \). The removal of delay terms means that waves are changed in phase with one another and so response pulse shapes are changed in the time domain. Secondly, the decay of the response is not noticeably affected by MPR. And thirdly, echoes are preserved by MPR.

The lack of initial delay in MPR responses can be explained by remembering the ‘fastest decay’ definition of minimum phase systems given in Section 3. Clearly, any initial delay must be removed if Eq. (1) is to be satisfied.

The features of MPR seen in Fig. 2 have been observed for a range of non-minimum phase systems, including model strings, beams, acoustics ducts, and random spectra.

4. Application: peak shock prediction in uncertain structures

An uncertain structure was simulated by using a random spectrum. The MPR was found of the spectrum, and a half cosine pulse applied to both the original and MPR spectra. The time domain shock responses were obtained using the IFFT, and the magnitude of the peak response found. In this section a comparison of results given by the original non-minimum phase system and its MPR are presented.
4.1. Modelling an uncertain structure

The inertance of the structure is of interest for shock analysis, and the inertance transfer function was modelled using the modal summation

\[ H_D(\omega) = \frac{a(x_2)}{F(x_1)} = \sum_{n=1}^{N} \frac{\omega_n^2 \phi_n(x_1) \phi_n(x_2)}{\omega_n^2 + j\eta \omega_n - \omega^2}. \]  

(9)

where \( x_1 \) defines the position of the input force, \( x_2 \) defines the listening position for the acceleration output, and loss factor \( \eta \) is assumed constant. Modal amplitudes \( \phi_n \) are randomly assigned with Gaussian distribution, and resonant frequencies \( \omega_n \) have a Rayleigh distribution.

The transfer function described in Eq. (9) does not decay at high frequencies, but instead has a constant trend line. For a true impulse response an infinite frequency range would be needed. To avoid this, the response for a non-idealised impulse was used.

The impulse was assumed to be a half cosine pulse, centred on \( t = 0 \) and with area = 1. By widening an idealised impulse in this way the high frequency components of the shock are lost. Also, this is a much more realistic model of a shock input that may occur in practise. By taking the Fourier transform of the cosine pulse the decay at high frequencies is seen:

\[ H_{\text{shock}}(\omega) = \pi^2 \cos \left( \frac{\pi b}{2} \right) \left( \frac{\pi^2}{b^2} - \omega^2 \right) \left\{ \frac{\pi}{2b} \cos \left( \frac{\pi b}{2} \right) \right\}, \]

(10)

where \( b \) defines the pulse width. The power spectrum \( H_{\text{shock}} \) has a ‘sinc’ type structure, with rapidly decaying lobes that decrease with frequency. After the second side lobe the amplitude of the spectrum is at most 2% of the DC amplitude, and so it is assumed that after this point the effect of higher frequencies is negligible to the overall response. Thus, the rectangular windowing that must be used to create a finite length spectrum will not cause large artefacts.

Multiplying the spectra of the cosine pulse and the random structure gives the shock response of the structure. Figure 3 shows the shock response for the original system, its MPR, and for comparison the zero-phase system with the same magnitude as the original. It can be seen that the MPR response is much closer to the original than the zero-phase system. The MPR response over-predicts the value of the peak shock response, and this is expected due to the ‘fastest decay’ property of minimum phase systems.

4.2. Ensemble average results

Due to the random nature of the structure modelled above, the exact nature of the shock response varies depending on the random spectrum used. Using an ensemble of statistically similar structures, the peak shock response was found for original, MPR, and zero-phase spectra.

Table 1 shows the results found from an ensemble of 100 random spectra. The sampling frequency was set to 44100Hz, and an impulse duration of 160\( \mu \)s. Frequency resolution was varied in accordance with the rule of thumb, at least five data points covering every spectral peak. There were 100 modes in the frequency range 0-22050Hz (ie the Nyquist frequency), and a loss factor of \( \eta = 0.01 \) was used. ‘Mean Max’ refers to the ensemble mean value of the peak, ‘Relative Variance Max’ to the relative variance of peak response results across the ensemble. These results show that using the MPR gives a prediction of peak shock response that is comparable with the true result, and much closer in value than results predicted using a zero phase spectrum. However, the minimum phase filter over-predicts the response, by 27% in this case.

The over-prediction in the MPR peak shock response value is found for a wide range of ensembles, with different damping and modal density values. This over-prediction is typically within 50% of the original response; an acceptable error when compared with safety factors of
Figures 3: Impulse responses for random structure

Table 1: Peak shock response results for an ensemble of 100 random filters

|                  | Original | Minimum Phase | Zero Phase |
|------------------|----------|---------------|------------|
| Mean Max         | 3.07E-04 | 3.90E-04      | 8.39E-04   |
| Relative Variance Max | 0.24     | 0.25          | 0.18       |

300-400% typically used to design against shock failure. Due to being an over prediction, the MPR result provides a conservative estimate for the mean peak shock response. The variance of results does not change noticeably for MPR predicted responses, and so can be used to give likelihood estimates for shock responses of uncertain structures.

4.3. Changing the correlation of modal amplitudes

The random spectra used in Sections 4.1 and 4.2 have modal amplitudes $\phi_n(x_1)$ and $\phi_n(x_2)$ that are uncorrelated, produced using the ‘randu’ command in MatLab. Modal amplitudes are only truly uncorrelated in a structure when the listening point $x_2$ is in the far field, i.e. $r = |x_1 - x_2| \simeq \infty$. For random structures the level of correlation between modal amplitudes at different points depends on the distance between these points, $r$. When $r = 0$ (driving point response), the correlation $R(r) = 1$ for all systems.

For a 2-D system the diffuse field correlation function $R(r)$ is given by:

$$R(r) = J_0(kr)$$  \hspace{1cm} (11)

where $J_0$ is a zeroth order Bessel function of the first kind, and $k$ is the wave number of the system.

To allow shock responses to be calculated at a variety of listen positions, modal amplitudes $\phi_n(x_1)$ and $\phi_n(x_2)$ were chosen such that they were correlated according to Eq. (11). Again
Table 2: Peak shock response results for an ensemble of 100 random filters

|                  | Original | Minimum Phase | Zero Phase |
|------------------|----------|---------------|------------|
| Mean Max         | 5.13E-04 | 5.53E-04      | 9.16E-04   |
| Relative Variance Max | 0.34      | 0.30          | 0.32       |

an ensemble of similar spectra was used to find the peak response values for original and MPR shock responses. Results are given in Table 2 for a system where the distance \( r \) was low, meaning that the correlation between modal amplitudes was high.

When the distance between input and listening points is low it can be seen that the MPR still over-predicts the mean shock response of the ensemble, but is closer to the true result than for the uncorrelated far field case. When \( r = 0 \), MPR gives a very accurate prediction of the mean, and this is to be expected as at \( r = 0 \) a driving point response is seen.

5. Conclusions
The Hybrid method is currently not suited to shock response predictions: frequency domain results produced by the Hybrid method lack phase information and are smoothed in frequency. Using an assumption of minimum phase, the phase can be reconstructed using the Hilbert transform. A minimum phase reconstruction algorithm has been implemented, and it has been found that for accurate MPR the frequency range and resolution of the spectra being studied must both be high. Applying MPR to non-minimum phase systems produces errors in the time domain, but these are limited to a removal of any delay in waveforms. Using MPR to predict the shock response of uncertain (non-minimum phase) structures has proved very acceptable — MPR gives a mean ensemble shock response that over-predicts by less than 50% and provides a conservative estimate. MPR gives much more accurate results than using a zero phase assumption. The problem of Hybrid method frequency smoothing has not been tackled, future work must concentrate on this area to allow Hybrid method models to be used for time domain based shock prediction.

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