ON DAVID TYPE SIEGEL DISKS OF THE SINE FAMILY

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Abstract. In this paper, we develop a method to verify David’s integrability condition for certain Beltrami differentials without using Petersen puzzles. Using this method and trans-quasiconformal surgery, we prove that for any David type rotation number, the boundary of the Siegel disk of \( f_\theta(z) = e^{2\pi i \theta} \sin(z) \) is a Jordan curve which passes through exactly two critical points \( \pi/2 \) and \( -\pi/2 \).

1. Introduction

Let \( 0 < \theta < 1 \) be an irrational number and \([a_1, \ldots, a_n, \ldots]\) be its continued fraction. We call \( \theta \) of bounded type if \( \sup \{ a_n \} < \infty \), and of David type if \( \log a_n = O(\sqrt{n}) \). It was proved in [11] that when \( \theta \) is of bounded type, the Siegel disk of the entire function \( f_\theta(z) = e^{2\pi i \theta} \sin(z) \) is a quasi-disk with exactly two critical points \( \pi/2 \) and \( -\pi/2 \) on the boundary. The main purpose of this paper is to extend this result to the case that \( \theta \) is of David type. We prove

Main Theorem. Let \( 0 < \theta < 1 \) be an irrational number of David type. Then the boundary of the Siegel disk of \( f_\theta(z) = e^{2\pi i \theta} \sin(z) \) is a Jordan curve which passes through exactly two critical points \( \pi/2 \) and \( -\pi/2 \).

A similar result for David type Siegel disks of quadratic polynomials was previously obtained by Petersen and Zakeri in their seminal work [8]. Our proof goes along the same line as theirs. First we construct a Blaschke fraction \( G_\theta \) which models the map \( f_\theta \). Then we perform a trans-quasiconformal surgery on \( G_\theta \). To make such surgery possible, one needs to prove the integrability of some Beltrami differential \( \mu \), and as in [8], this is the heart of the whole paper. After this, we get an entire function \( T_\theta \) which has a Siegel disk of rotation number \( \theta \) such that the boundary of the Siegel disk is a Jordan curve passing through exactly two critical points \( \pi/2 \) and \( -\pi/2 \). The main theorem then follows by showing that \( f_\theta(z) = T_\theta(z) \).

The most remarkable difference between the proof in this paper and that in [8] is as follows. In quadratic polynomial case, one has a set of puzzle pieces with some very nice geometric and dynamical properties which were

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used in an essential way in [8] to prove the integrability of \( \mu \) (these puzzles were previously constructed by Petersen in his famous article [6], and are usually called Petersen puzzles now). But in our case, there are no external rays and equipotential curves for \( f_\theta \), so such puzzle pieces do not exist any more. Thus the puzzle technique used there do not apply here. To solve this problem, a new method will be developed in §6 of this paper by which one can estimate the area of some dynamically defined sets, and the integrability of \( \mu \) then follows. Due to its flexibility, the method can be applied in more general situations. In particular, it is one of the crucial techniques in [12] where it has been proved that every David type Siegel disk of a polynomial map of any degree must be a Jordan domain with at least one of the critical points on its boundary.

Throughout the following, we use \( \hat{C} \), \( C \), \( \Delta \), and \( T \) to denote the Riemann sphere, the complex plane, the open unit disk, and the unit circle, respectively. The following is the organization of the paper.

In §2, we present the background materials about David homeomorphisms and critical circle mappings.

In §3, we construct an odd Blaschke fraction \( G_\theta \) to serve as the model map for \( f_\theta \). The restriction of \( G_\theta \) on \( T \) is a homeomorphism with rotation number \( \theta \) and two critical points \( 1 \) and \( -1 \). Let \( \Phi : C \to C \) be the square map given by \( z \to z^2 \). Then the map

\[
 g_\theta(z) = \Phi \circ G_\theta \circ \Phi^{-1}(z).
\]

is a meromorphic function with exactly two essential singularities at 0 and \( \infty \), and moreover, the restriction of \( g_\theta \) on \( T \) is a critical circle mapping with rotation number \( \alpha \equiv 2\theta \mod (1) \) (that is, \( \alpha = 2\theta \) if \( 0 < \theta < 1/2 \) and \( \alpha = 2\theta - 1 \) if \( 1/2 < \theta < 1 \)) (Lemma 3.7). By Yoccoz’s linearization theorem [9], there is a circle homeomorphism \( h : T \to T \) such that \( h(1) = 1 \) and

\[
 g_\theta|_T(z) = h^{-1} \circ R_\alpha \circ h(z)
\]

where \( R_\alpha \) is the rigid rotation given by \( \alpha \).

In §4, we prove that \( \alpha \) is also of David type (Lemma 4.1).

In §5, we introduce Yoccoz’s cell construction by which one can extend \( h \) to a David homeomorphism \( H : \Delta \to \Delta \). Let

\[
 \nu_H = \frac{\overline{\partial} H}{\partial H} \frac{d\sigma}{dz}
\]

be the Beltrami differential of \( H \) in \( \Delta \). Define

\[
 \overline{g}_\theta(z) = \begin{cases} 
 g_\theta(z) & \text{for } z \in C - \Delta, \\
 H^{-1} \circ R_\alpha \circ H(z) & \text{for } z \in \Delta.
\end{cases}
\]

It follows that \( \nu_H \) is \( \overline{g}_\theta \)-invariant. Let \( \nu \) denote the Beltrami differential in the whole complex plane which is obtained by the pull back of \( \nu_H \) through the iterations of \( \overline{g}_\theta \).
In §6, we prove that the dilatation of the Beltrami differential \( \nu \) satisfies an exponential growth condition, more precisely, there exist constants \( M > 0, \alpha > 0 \), and \( 0 < \epsilon_0 < 1 \), such that for any \( 0 < \epsilon < \epsilon_0 \), the following inequality holds,

\[
\text{area}\{z \mid |\nu(z)| > 1 - \epsilon\} \leq Me^{-\alpha/\epsilon},
\]

where \( \text{area}(X) \) is used to denote the spherical area of a subset \( X \subset \hat{\mathbb{C}} \).

Let \( \mu \) be the Beltrami differential in the complex plane which is defined by the pull back of \( \nu \) through the square map \( \Phi \). It will be proved that \( \mu \) satisfies the condition (2) also (Lemma 6.1). By David’s theorem [2], \( \mu \) is integrable. That is, there is a homeomorphism \( \phi : \mathbb{C} \to \mathbb{C} \) in \( W_{1,1}^{1,1}(\mathbb{C}) \) such that

\[
\partial \phi = \mu \partial \phi.
\]

Define

\[
\tilde{G}_\theta(z) = \begin{cases} 
G_\theta(z) & \text{for } z \in \mathbb{C} - \Delta, \\
\Phi^{-1} \circ H^{-1} \circ R_\alpha \circ H \circ \Phi(z) & \text{for } z \in \Delta.
\end{cases}
\]

It follows that \( \mu \) is \( \tilde{G}_\theta \)-invariant. Now let \( \phi \) be normalized such that it fixes \( 0 \) and the infinity, and maps \( 1 \) to \( \pi/2 \). Then by the same argument as in the proof of Lemma 5.5 of [3], it follows that the map \( T_\theta(z) = \phi \circ \tilde{G}_\theta \circ \phi^{-1}(z) \) is an entire function (Lemma 7.2). From the construction above, \( T_\theta \) has a Siegel disk centered at the origin with rotation number \( \theta \), and moreover, the boundary of the Siegel disk is a Jordan curve passing through exactly two critical points \( \pi/2 \) and \( -\pi/2 \).

In §7, we will prove that \( f_\theta(z) = T_\theta(z) \). We prove this by using a topological rigidity property of the Sine family (Lemma 1 of [4] or Lemma 7.3). The Main Theorem follows.

2. Preliminaries

2.1. David Homeomorphisms. Let \( \Omega \subset \hat{\mathbb{C}} \) be a domain. A Beltrami differential \( \mu = \mu(z)dz/dz \) in \( \Omega \) is a measurable \((-1, 1)\)-form such that \( |\mu(z)| < 1 \) almost everywhere in \( \Omega \). We say \( \mu \) is integrable if there is a homeomorphism \( \phi : \Omega \to \Omega' \) in \( W_{1,1}^{1,1}(\Omega) \) which solves the Beltrami equation

\[
\partial \phi = \mu \partial \phi.
\]

The map \( \phi \) is called a David homeomorphism. When \( \|\mu\|_\infty < 1 \), the map \( \phi \) is the classical quasiconformal mapping.

Recall that \( \text{area}(X) \) is used to denote the spherical area of a subset \( X \subset \hat{\mathbb{C}} \).

**Theorem 2.1** (David [2]). Let \( \Omega \subset \hat{\mathbb{C}} \) be a domain. Let \( \mu \) be a Beltrami differential in \( \Omega \). Then \( \mu \) is integrable if there exist constants \( M > 0, \alpha > 0 \), and \( 0 < \epsilon_0 < 1 \), such that for any \( 0 < \epsilon < \epsilon_0 \), the following inequality holds,

\[
\text{area}\{z \mid |\mu(z)| > 1 - \epsilon\} \leq Me^{-\alpha/\epsilon}.
\]
Moreover, if $\mu$ is integrable, up to postcomposing a conformal map, there is a unique solution $\phi : \Omega \to \Omega'$ in $W^{1,1}_{\text{loc}}(\Omega)$ which solves the Beltrami equation (4). That is, if $\psi : \Omega \to \Omega''$ is another such solution, then there is a conformal map $\sigma : \Omega' \to \Omega''$ such that $\psi = \sigma \circ \phi$.

2.2. Critical Circle Mappings. For our purpose, we say a homeomorphism $f : \mathbb{T} \to \mathbb{T}$ is a critical circle mapping if it is real analytic and has exactly one critical point at 1.

Suppose $f$ is a critical circle mapping with an irrational rotation number $\theta$. Let $p_n/q_n, n \geq 0$ be the continued fractions of $\theta$. For $i \in \mathbb{Z}$, let $x_i \in \mathbb{T}$ denote the point such that $f^i(x_1) = 1$. Let $I_n = [1, x_{q_n}]$. For $i \geq 0$, let $I^i_n \subset \mathbb{T}$ denote the interval such that $f^i(I_n) = I_n$, that is, $I^i_n = [x_i, x_{q_n+i}]$. Then the collection of the intervals

$I^i_n, 0 \leq i \leq q_{n+1} - 1$, and $I^j_{n+1}, 0 \leq j \leq q_n - 1$,

defines a partition of $\mathbb{T}$ modulo the common end points. We call such a partition a dynamical partition of level $n$. It is not difficult to see that the set of all the end points in this partition is

$$\Pi_n = \{x_i \mid 0 \leq j < q_n + q_{n+1}\}.$$

Theorem 2.2 (Swiatek-Herman, see [3]). Let $f : \mathbb{T} \to \mathbb{T}$ be a real analytic critical circle mapping with an irrational rotation number $\theta$. Let $n \geq 0$. Then there is an asymptotically universal bound such that

$$|[x, f^{-q_n}(x)]| \asymp |[x, f^{q_n}(x)]|$$

holds for any point $x$ in $\mathbb{T}$ and

$$|I| \asymp |J|$$

holds for any two adjacent intervals $I$ and $J$ in the dynamical partition of $\mathbb{T}$ of level $n$.

Now let us consider another partition of $\mathbb{T}$. Let

$$\Xi_n = \{x_i \mid 0 \leq i < q_{n+1}\}.$$

The points in $\Xi_n$ separated $\mathbb{T}$ into disjoint intervals. This partition arises in Yoccoz’s cell construction (see §6 of [8] or §5). Let us call it the cell partition of level $n$. The following lemma describes the relation between these two partitions.

Lemma 2.1. Each interval in $\mathbb{T} \setminus \Xi_n$ is either an interval in $\mathbb{T} \setminus \Pi_n$ or the union of two adjacent intervals in $\mathbb{T} \setminus \Pi_n$.

Remark 2.1. For our use, the definition of the cell partition is a little different from that in [8] where the cell partition of level $n$ is defined by the points $\{x_i \mid 0 \leq i < q_n\}$. Therefore, the cells of level $n$ in this paper correspond to the cells of level $n + 1$ there.
Proof. The direction from $|−|_1$ values 1 and 0 type of construction was pioneered by A. Cheritat (see [1]). Recall that $\Delta$

Let us prove the other direction. Consider the intervals $\mathbb{J}$ and $\mathbb{I}$ by considering the action of $f$ by Theorem 2.2 of [3] for Koebe’s distortion principle). Actually, we have

An interval $[x_q, x_{q+1}]$ is also an interval in the cell partition of level $n$. This proves one direction of the second assertion, it follows that $\mathbb{I} \subset \mathbb{J}$ and $|I| < \delta|J|$. The third assertion is implied by Theorem 2.2.

Note that $f^{\pm 1}(1)$ has exactly two critical points on both sides of it.

Let us prove the other direction. Consider the intervals $J = [x_{q-n}, x_{q-n+1}, x_{q-n+2}]$ and $I = [1, x_{q-n}]$. Note that $J \subset [x_{q-n+2}, x_{q-n}] \subset I$. By Theorem 2.2, $J$ has a definite space around it inside $I$. Since $I$ contains at most two points from $f^{\pm 1}(1), 1 \leq k \leq n$, the direction $[x_{q-n}, x_{q-n+1}] \geq [x_{q-n+2}, x_{q-n}]$ then follows by considering the action of $f^{-q_n}$ on $I$ and Koebe’s distortion principle (see Lemma 2.4 of [3] for Koebe’s distortion principle).

In this section, we construct a Ghys-like model map $G_{\theta}$. The idea of such type of construction was pioneered by A. Cheritat (see [11]). Recall that $\Delta$ and $T$ denote the unit disk and the unit circle, respectively.

Let $T(z) = \sin(z)$. It follows that the map $T(z)$ has exactly two critical values 1 and $-1$. Let $D$ be the component of $T^{-1}(\Delta)$ which contains the origin.
Lemma 3.1. \( D \) is a Jordan domain which is symmetric about the origin and the map \( T|\partial D: \partial D \to \mathbb{T} \) is a homeomorphism. Moreover, \( \partial D \) passes through exactly two critical points \( \pi/2 \) and \( -\pi/2 \).

Proof. Since \( T \) is an entire function with no finite asymptotic value by Lemma 1 of [?], \( \partial D \) is bounded and thus a closed and piecewise smooth curve. In addition, since \( \Delta \) contains no critical value of \( T \), the map \( T: D \to \Delta \) is a holomorphic isomorphism. This implies that \( \partial D \) does not intersect with itself and thus is a Jordan curve. It follows that \( T: \partial D \to \mathbb{T} \) is a homeomorphism. The symmetry of \( D \) follows from the odd property of \( T(z) \). The first assertion of the lemma has been proved.

Note that the inverse branch of \( T \) which maps the origin to itself can be continuously extended to 1 along the segment \([0, 1]\). It follows that \( \pi/2 \in \partial D \). The same argument implies that \( -\pi/2 \in \partial D \). Because \( T: \partial D \to \mathbb{T} \) is a homeomorphism, and because 1 and \(-1\) are the only two critical values of \( T \), \( \pi/2 \) and \(-\pi/2 \) are the only two critical points on \( \partial D \). The proof of the lemma is completed. \( \Box \)

For \( k \in \mathbb{Z} \), let \( D_k = \{ z + k\pi | z \in D \} \). It follows that \( D_0 = D \).

Lemma 3.2. The domains \( D_k, k \in \mathbb{Z} \), are all the components of \( T^{-1}(\Delta) \). For any \( k \in \mathbb{Z} \), \( \partial D_k \cap \partial D_{k+1} = \{ k\pi + \pi/2 \} \), and moreover, \( \partial D_i \cap \partial D_j = \emptyset \) for \( i, j \in \mathbb{Z} \) with \( |i - j| > 1 \).

Proof. Since \( D = D_0 \) is symmetric about the origin, \( T(D_k) = \Delta \) for any \( k \in \mathbb{Z} \). The first assertion then follows from the fact that \( T^{-1}(0) = \{ k\pi | k \in \mathbb{Z} \} \). Note that \( \partial D_i \cap \partial D_j \) must consist of critical points if it is non-empty. The second assertion then follows from the fact that every \( \partial D_k \) contains exactly two critical points \( k\pi + \pi/2 \) and \( k\pi - \pi/2 \). \( \Box \)

Let \( \psi: \hat{\mathbb{C}} - \mathbb{D} \to \hat{\mathbb{C}} - \mathbb{T} \) be the Riemann map such that \( \psi(\infty) = \infty \) and \( \psi(1) = \pi/2 \). Since \( \Delta \) and \( D \) are both symmetric about the origin, we have

Lemma 3.3. \( \psi \) is odd.

For \( z \in \mathbb{C} \), let \( z^* \) denote the symmetric image of \( z \) about the unit circle. Define

\[
G(z) = \begin{cases} 
T \circ \psi(z) & \text{for } z \in \mathbb{C} - \Delta, \\
((T \circ \psi)(z^*))^* & \text{for } z \in \Delta - \{0\}.
\end{cases}
\]

By Lemma 3.3 and the construction of \( G(z) \), we have

Lemma 3.4. \( G(z) \) is holomorphic in \( \mathbb{C} - \{0\} \) and is symmetric about the unit circle. Moreover, \( G(z) \) is odd, and \( G|\mathbb{T}: \mathbb{T} \to \mathbb{T} \) is a real analytic circle homeomorphism which has exactly two critical points at 1 and \(-1\).
Let $0 < \theta < 1$ be the David type irrational number in the Main Theorem. Since $G|T : T \to T$ is a critical circle homeomorphism, by Proposition 11.1.9 of [5], we get

**Lemma 3.5.** There exists a unique $t \in [0, 1)$ such that $e^{2\pi it}G|T : T \to T$ is a critical circle homeomorphism of rotation number $\theta$.

Let $t \in [0, 1)$ be the number given in Lemma 3.5. Let us denote $e^{2\pi it}G(z)$ by $G_\theta(z)$. Since $G(z)$ is odd by Lemma 3.4, we have

**Lemma 3.6.** $G_\theta$ is odd.

Let $\Phi : \mathbb{C} \to \mathbb{C}$ be the square map given by $\Phi(z) = z^2$. Define

$$g_\theta(z) = \Phi \circ G_\theta \circ \Phi^{-1}(z).$$

**Lemma 3.7.** $g_\theta$ is a meromorphic function with exactly two essential singularities at 0 and $\infty$, and the restriction of $g_\theta$ to $T$ is a critical circle homeomorphism with exactly one critical point at 1. Moreover, the rotation number of $g_\theta|T$ is $\alpha \equiv 2\theta \pmod{1}$.

**Proof.** Since $G_\theta$ is odd by Lemma 3.6, $g_\theta$ is well defined and has exactly one critical point 1 on the unit circle. The first assertion follows. Now let us prove the second assertion. Let $I$ denote the anticlockwise arc from 1 to $g_\theta(1) = (G_\theta(1))^2$. Consider the orbit segment $O_n = \{g_\theta^k(1) = (G_\theta^k(1))^2, 0 \leq k \leq n\}$. Let $P_n$ denote the numbers of the points in $O_n$ which are contained in $I$. There are two cases.

In the first case, $0 < \theta < 1/2$. Since $G_\theta$ is odd, it follows that any half of the unit circle contains almost half of the number of the points in $O_n$. Thus $G_\theta(1)$ is contained in the upper half of the unit circle. Let $J$ denote the anticlockwise arc from 1 to $G_\theta(1)$. It follows that $J \subset I$. Let $Q^+_n$ and $Q^-_n$ denote the numbers of the points in $O_n$ which are contained in $J$, and $-J$, respectively (Here $-J$ is the anticlockwise arc from $-1$ to $-G_\theta(1)$). It follows that

$$\lim_{n \to \infty} Q^+_n/n = \lim_{n \to \infty} Q^-_n/n = \theta.$$

Note that $g_\theta^k(1) \in I$ if and only if $G_\theta^k(1) \in J \cup (-J)$ and that $J \cap (-J) = \emptyset$. We thus have

$$\lim_{n \to \infty} P_n/n = \lim_{n \to \infty} [(Q^+_n + Q^-_n)/n] = 2\theta.$$

In the second case, $1/2 < \theta < 1$. Since any half of the unit circle contains almost half of the number of the points in $O_n$, it follows that $G_\theta(1)$ is contained in the lower half of the unit circle. Thus $-G_\theta(1)$ is contained in the upper half of the unit circle. Let $J$ denote the anticlockwise arc from 1 to $-G_\theta(1)$. It follows that $J \subset I$. Again let $Q^+_n$ and $Q^-_n$ denote the numbers of the
points in $O_n$ which are contained in $J$, and $-J$, respectively (Here $-J$ is the anticlockwise arc from $-1$ to $G(1)$). It follows that

$$\lim_{n \to \infty} Q^+_n/n = \lim_{n \to \infty} Q^-_n/n = \theta - 1/2.$$ 

As before, $g^k(1) \in I$ if and only if $G^k(1) \in J \cup (-J)$. Since $J \cap (-J) = \emptyset$, we have

$$\lim_{n \to \infty} P_n/n = \lim_{n \to \infty} [(Q^+_n + Q^-_n)/n] = 2\theta - 1.$$ 

The lemma follows.

4. An Arithmetic Property

**Lemma 4.1.** Let $0 < \theta < 1$ be an irrational number of David type. Let $0 < \alpha < 1$ be the irrational number such that

$$\alpha \equiv 2\theta \mod (1).$$

Then $\alpha$ is also of David type.

**Proof.** Let $[b_1, \ldots, b_n, \ldots]$, $s_n/t_n$, and $[a_1, \ldots, a_n, \ldots]$, $p_n/q_n$, be the continued fractions and convergents of $\theta$ and $\alpha$, respectively. Let $n \geq 4$. We claim that there exists an even integer $L = 2m$ among $t_n - 1$, $t_n$, and $t_n - t_n - 1$ and an integer $N \geq 0$ such that the inequality

$$|2m\theta - N| < |2y\theta - x|$$

holds for all integers $x \geq 0$ and $0 < y < m$.

In fact, if one of $t_{n-1}$ and $t_n$ is even, we can take it to be $L$, and take $N$ to be $s_{n-1}$ or $s_n$. Then the claim is obviously true. Otherwise, both $t_{n-1}$ and $t_n$ are odd integers. Then let $L = t_n - t_{n-1}$ and let $N \geq 0$ be the integer such that the left hand of (6) obtains the minimum. If $t_{n-2} = t_n - t_{n-1}$, the claim is obviously true. Otherwise, $t_n - t_{n-1} > t_{n-1}$. Then the claim also follows since the only possible integers $s$ and $t$ such that $t < t_n - t_{n-1}$ and $|t(t_n - t_{n-1})\theta - N| \geq |t\theta - s|$ are $s_{n-1}$ and $t_{n-1}$. But $t_{n-1}$ is odd, hence (6) also holds in the later case.

From (6) and $\alpha \equiv 2\theta \mod (1)$, it follows that there exists some integer $N \geq 0$ such that

$$|m\alpha - N| < |\alpha y - x|$$

holds for all integers $x \geq 0$ and $0 < y < m$. This implies that $m = q_l$ for some $l \geq 0$. Let $k$ be the largest number such that $q_k < t_{n+1}$. Since $m = L/2 < t_{n+1}$, and since $k$ is the largest integer such that $q_k < t_{n+1}$, it follows that $q_k \geq m$. Since $L \geq t_{n-2}$, we have $m = L/2 > t_{n-4}$. Thus we get

$$q_k > t_{n-4}.$$ 

This implies that for every $n \geq 4$, there is some $q_k$ between $t_{n+1}$ and $t_{n-4}$. 

Now for every \( k \geq 1 \), let \( n \geq 1 \) be the least integer such that \( q_k < t_{n+1} \).

It is clear that \( n \geq 9 \) for all \( k \) large. Since for every \( n \geq 4 \), there is some \( q_l \) between \( t_{n-4} \) and \( t_{n-9} \), it follows that

\[
q_k - 1 > t_{n-9}.
\]

Thus we have

\[
a_k \leq q_k / q_{k-1} < t_{n+1} / t_{n-9}.
\]

All these together implies that

\[
\log a_k < \log(t_{n+1}/t_{n-9}) \leq \sum_{n-8 \leq l \leq n+1} \log(b_l + 1) \leq C \sqrt{n} \leq C' \sqrt{k}
\]

holds for all \( k \geq 1 \) large, where \( C, C' > 0 \) are some uniform constants. The lemma follows.  \( \Box \)

5. **Yoccoz’s Cell Construction**

Recall that \( g_\theta|_\mathbb{T} \) is a critical circle homeomorphism with rotation number \( \alpha \) and exactly one critical point at 1. For \( i \in \mathbb{Z} \), let \( x_i \) be the point in \( \mathbb{T} \) such that \( g_\theta^i(x_i) = 1 \). For \( n \geq 0 \), let \( p_n / q_n \) be the continued fraction of \( \alpha \). Consider the cell partition of level \( n \) introduced in \( \S 2 \),

\[
\Xi_n = \{ x_i \mid 0 \leq i < q_{n+1} \}.
\]

For each \( x_i \in \Xi_n \), let \( y_i \) be the point on the radial segment \([0, x_i]\) such that

\[
|y_i - x_i| = d(x_r, x_l)/2
\]

where \( x_r \) and \( x_l \) denote the two points immediately to the right and left of \( x_i \) in \( \Xi_n \), and \( d(x_r, x_l) \) denotes the Euclidean length of the smaller arc connecting \( x_r \) and \( x_l \). Let us assume that \( n \geq 0 \) is large enough such that \( d(x_i, x_r) < 1 \) holds for any two adjacent points \( x_i \) and \( x_r \) in \( \Xi_n \).

Let \( x_i \) and \( x_r \) be any two adjacent points in \( \Xi_n \). Connect \( y_i \) and \( y_r \) by a straight segment. Then the three straight segments \([x_i, y_i], [y_l, y_r], [x_r, y_r]\), and the arc segment \([x_i, x_r]\) bound a domain, which is called a cell of level \( n \). It follows that the union of all the cells of level \( n \) is an annulus with \( \mathbb{T} \) being the outer boundary component. Let us denote this annulus by \( Y_n \).

Let \( K > 1 \). Two straight segments \( I \) and \( J \) are called \( K \)-commensurable if \( |J| / K < |I| < K |J| \). From the construction of the cells, and Theorem 2.2, Lemma 2.4 and Lemma 2.5, one has the following lemma,

**Lemma 5.1.** The four sides of each cell are \( K \)-commensurable for some \( K > 1 \) dependent only on \( g_\theta \). Furthermore, each cell \( E \) of level \( n + 2 \) is well contained in some cell \( E' \) of level \( n \) in the sense that there is a uniform
0 < \sigma < 1 \text{ such that the ratio of the length of each side of } E \text{ to the length of the corresponding side of } E' \text{ is less than } \sigma.

Let \( h : \mathbb{T} \to \mathbb{T} \) be the homeomorphism such that \( h(1) = 1 \) and \( g_\theta|\mathbb{T}(z) = h^{-1} \circ R_\alpha \circ h(z) \). Then by Yoccoz’s extension theorem (see [10] or Theorem 6.5 of [8]), we have

**Lemma 5.2.** There is a \( C > 0 \) such that the map \( h \) can be extended to a homeomorphism \( H : \Delta \to \Delta \) whose dilatation in \( Y_n \) is at most \( C(1 + (\log a_{n+2})^2) \).

By composing with a quasiconformal homeomorphism of the unit disk to itself which fixes 1, we may assume that \( H(0) = 0 \). Let

\[ \nu_H = \frac{\partial H}{\partial \overline{H}} \frac{d\overline{z}}{dz} \]

be the Beltrami differential of \( H \) in \( \Delta \). Define

\[
\tilde{g}_\theta(z) = \begin{cases} g_\theta(z) & \text{for } z \in \mathbb{C} - \Delta, \\ H^{-1} \circ R_\alpha \circ H(z) & \text{for } z \in \Delta. \end{cases}
\]

It follows that \( \nu_H \) is \( \tilde{g}_\theta \)-invariant. Let \( \nu \) denote the Beltrami differential in the whole complex plane which is obtained by the pull back of \( \nu_H \) through the iterations of \( \tilde{g}_\theta \).

Define

\[
\tilde{G}_\theta(z) = \begin{cases} G_\theta(z) & \text{for } z \in \mathbb{C} - \Delta, \\ \Phi^{-1} \circ H^{-1} \circ R_\alpha \circ H \circ \Phi(z) & \text{for } z \in \Delta. \end{cases}
\]

Here the branch of \( \Phi^{-1} \) is taken to be such that

\[ \Phi^{-1} \circ H^{-1} \circ R_\alpha \circ H \circ \Phi(1) = G_\theta(1). \]

Let \( \mu \) be the Beltrami differential in the complex plane which is obtained by the pull back of \( \nu \) through the square map \( \Phi \). The proof of the following lemma is direct, and we leave it to the reader.

**Lemma 5.3.** The map \( \tilde{G}_\theta \) is odd. The Beltrami differential \( \mu \) is \( \tilde{G}_\theta \)-invariant, and moreover, \( \mu(z) = \mu(-z) \).

### 6. The Integrability of \( \mu \)

The purpose of this section is to prove the integrability of \( \mu \).

**6.1. The Integrability of \( \nu \) Implies the Integrability of \( \mu \).**

**Lemma 6.1.** If \( \nu \) satisfies the condition \( \text{[2]} \), then so does \( \mu \) with the same \( 0 < \epsilon_0 < 1 \) but possibly different constants \( M > 0 \) and \( \alpha > 0 \).
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Proof. Let $\Phi: z \to z^2$ be the square map defined in §3. It is sufficient to prove that there exists a $C > 0$ such that for any measurable set $E \subset \mathbb{C}$, the following inequality holds,

\begin{equation}
\text{area}(\Phi^{-1}(E)) < C\text{area}(E)^{1/2}.
\end{equation}

To show this, let $E_1 = E \cap \Delta$ and $E_2 = E \cap (\mathbb{C} \setminus \Delta)$. It is sufficient to prove (10) holds for both $E_1$ and $E_2$. Since the transform $\zeta = 1/z$ commutes with $\Phi$ and preserves the spherical metric $|dz|/(1+|z|^2)$ and maps $E_2$ to some subset of $\Delta$, we need only to prove (10) for $E_1$. Note that in $\Delta$, the Euclidean area is equivalent to the spherical area. Thus it is sufficient to prove (10) in the case of Euclidean area. Note that

\[ \int_{E_1} dx dy = 2 \int_{\Phi^{-1}(E_1)} (s^2 + t^2) ds dt. \]

It follows that for given $\int_{E_1} dx dy$, $\int_{\Phi^{-1}(E_1)} ds dt$ obtains the maximum when $\Phi^{-1}(E_1)$ is a Euclidean disk centered at the origin. This implies (10) in the case of Euclidean area and the lemma follows. □

Let

\[ X = \{ z \in \mathbb{C} \setminus \Delta | g_k^z(z) \in \Delta \text{ for some integer } k > 0 \}. \]

For each $z \in X$, let $k_z$ be the least integer such that $g_k^z(z) \in \Delta$. Define

\[ X_n = \{ z \in X | g_k^z(z) \in Y_n \}. \]

By Lemma 5.2 and the condition that $\log a_n = O(\sqrt{n})$, we have

\textbf{Lemma 6.2.} If there exist $C > 0$ and $0 < \delta < 1$ such that $\text{area}(X_n) < C\delta^n$ holds for all $n$ large enough, then $\nu$ satisfies the condition (2).

It is clear that Lemma 6.2 can be further reduced to the next lemma.

\textbf{Lemma 6.3.} If there exist $C > 0$, $0 < \epsilon < 1$, and $0 < \delta < 1$ such that

\[ \text{area}(X_{n+2}) \leq Ce^n + \delta \text{area}(X_n), \]

then $\nu$ satisfies the condition (2).

The remaining of the section is devoted to the proof of Lemma 6.3.

\textbf{6.2. A covering lemma.} For $z \in \mathbb{C}$ and $r > 0$, let $B_r(z)$ denote the Euclidean disk with radius $r$ and center at $z$.

\textbf{Lemma 6.4.} Let $K > 1$. Then there is a constant $L > 1$ depending only on $K$ such that for any finite family of pairs of sets $\{(U_i, V_i)\}_{i \in \Lambda}$ in $\mathbb{C}$, if for each $i \in \Lambda$, there exist $x_i \in V_i$ and $r_i > 0$ such that

\[ B_{r_i}(x_i) \subset V_i \subset U_i \subset B_{Kr_i}(x_i), \]
then there is a subfamily $\sigma_0$ of $\Lambda$ such that all $B_{r_j}(x_j), j \in \sigma_0,$ are disjoint, and moreover,

$$\bigcup_{i \in \Lambda} U_i \subset \bigcup_{j \in \sigma_0} B_{Lr_j}(x_j).$$

Proof. Let us simply denote $B_{r_i}(x_i)$ as $B_i$. It is sufficient to prove the worst case, that is, $V_i = B_i$, and $U_i = B_{Kr_i}(x_i)$. By considering the subfamily of $\Lambda$ which consists of all those $i$ such that $B_i$ is maximal (that is, $B_i$ is not contained in any other $B_j$), we may assume that for any $i \neq j$ in $\Lambda$, $B_i$ is not contained in $B_j$. Let $\Sigma$ be the class which consists of all the non-empty subsets of $\Lambda$ such that for every $\sigma \in \Sigma$, the sets $B_i, i \in \sigma$ are disjoint with each other. Clearly any subset of $\Lambda$ which contains exactly one element must belong to $\Sigma$. It follows that $\Sigma$ is finite and non-empty. Let $\sigma_0 \in \Sigma$ be such that

$$m\left(\bigcup_{i \in \sigma_0} B_i\right) = \max_{\sigma \in \Sigma} m\left(\bigcup_{i \in \sigma} B_i\right)$$

where $m$ denotes the Euclidean area. Now let us prove that there is an $L > 1$ depending only on $K$ such that for any $i \in \Lambda$, there is some $j \in \sigma_0$ with $U_i \subset B_{Lr_j}(x_j)$.

In fact, if $i \in \sigma_0$, we can take $L = K$ and $j = i$. We may assume that $i \notin \sigma_0$. By the maximal property of $\sigma_0$, the disk $B_i$ must intersect at least one $B_j$ for some $j \in \sigma_0$. Let

$$\Theta = \{j \in \sigma_0 \mid B_i \cap B_j \neq \emptyset\}.$$ 

It follows from the maximal property of $\sigma_0$ again that

$$m(B_i) \leq m\left(\bigcup_{j \in \Theta} B_j\right).$$

(This is because otherwise, one may use $B_i$ to replace all the disks $B_j, j \in \Theta$, then the total Euclidean area will be increased, and this contradicts with the maximal property of $\sigma_0$)

Since by assumption, every $B_j$ for $j \in \Theta$ is not completely contained in $B_i$, it follows that the boundary circle of $B_j$ intersects the boundary circle of $B_i$. It thus follows that

$$r_i \leq 8 \max_{j \in \Theta} r_j.$$ 

Because otherwise, the union of $B_j$ would be a proper subset of the annulus

$$\{z \mid \frac{3}{4} r_i < |z - x_i| < \frac{5}{4} r_i\},$$

whose Euclidean area is equal to that of $B_i$. This contradicts with (11).
Let $L = 8K + 9$. Let $j \in \Theta$ be such that $r_j$ obtains the maximum of $r_l, l \in \Theta$. It is easy to see that $U_i \subset B_{K r_j}(x_i) \subset B_{L r_j}(x_j)$. The proof of the lemma is completed. □

Recall that we use $\text{area}(X)$ to denote the spherical area of a subset $X \subset \hat{C}$. Let $\Omega = \mathbb{C} \setminus \Delta$. For a subset $E \subset \Omega$, let $\text{diam}_\Omega(E)$ denote the diameter of $E$ with respect to the hyperbolic metric in $\Omega$.

**Corollary 6.1.** Let $\{(U_i, V_i)\}_{i \in \Lambda}$ be a finite family of pairs of sets in $\Omega$ satisfying the condition in Lemma 6.4 for some $1 < K < \infty$. If in addition

$$\text{diam}_\Omega(U_i) < K$$

for each $i \in \Lambda$, then

$$\frac{\text{area}(\bigcup_{i \in \Lambda} V_i)}{\text{area}(\bigcup_{i \in \Lambda} U_i)} \geq \lambda(K)$$

where $0 < \lambda(K) < 1$ is a constant dependent only on $K$.

**Proof.** Let $\sigma_0 \subset \Lambda$ and $L$ be given as in Lemma 6.4. Then for any $i \in \Lambda$, from the proof of Lemma 6.4 there is some $j \in \sigma_0$ such that $U_i \subset B_{L r_j}(x_j)$ and $B_{r_j}(x_j)$ intersects $B_{r_i}(x_i)$. Since $B_{r_i}(x_i) \subset U_i$ and $B_{r_j}(x_j) \subset U_j$, we get $U_i \cap U_j \neq \emptyset$. This, together with (14), implies

$$\text{diam}_\Omega(U_i \cup U_j) < 2K.$$  

By (13), there is some constant $1 < \ell(K) < \infty$ depending only on $K$ such that

$$\sup_{z, \xi \in U_i \cup U_j} \frac{1 + |z|^2}{1 + |\xi|^2} < \ell(K).$$

Since the spherical metric is given by $|dz|/(1 + |z|^2)$, this implies that the distortion of the spherical metric in $U_i \cup U_j$ is bounded by $\ell(K)$. But on the other hand, by $U_i \subset B_{L r_j}(x_j)$ we have

$$m(U_i) \leq L^2 m(B_{r_j}(x_j)),$$

where $m(\cdot)$ denotes the Euclidean area. Since $B_{r_j}(x_j) \subset U_j \subset U_i \cup U_j$, it follows from (14) and (15) that

$$\text{area}(U_i) \leq L^2 \ell(K) \text{area}(B_{r_j}(x_j)).$$

Since $L$ depends only on $K$ and all $B_{r_j}(x_j), j \in \sigma_0$, are disjoint, the lemma then follows by taking $\lambda(K) = L^2 \ell(K)$. □
6.3. Hyperbolic neighborhoods. Let us first introduce some concepts. Let $I \subset T$ be an open segment. Set

$$\Omega_I = \mathbb{C} \setminus \{(0) \cup (T \setminus I)\}.$$ 

Let $d_{\Omega_I}(\cdot, \cdot)$ denote the hyperbolic distance in $\Omega_I$. For $d > 0$, the hyperbolic neighborhood of $I$ is defined to be

$$H_d(I) = \{z \in \Omega_I \mid d_{\Omega_I}(z, I) < d\}.$$ 

For given $d > 0$, when $I$ is small, $H_d(I)$ is like the hyperbolic neighborhood of the slit plane. Thus it is like the domain bounded by two arcs of Euclidean circles which are symmetric about $T$. In the following, we always assume that the arc segment $I$ involved is small, and therefore regard $H_d(I)$ as the domain bounded by two symmetric arc segments of Euclidean circles. Let $\alpha$ be the exterior angle between $\partial H_d(I)$ and $T$. For the convenience of our later discussions, let us use $H_\alpha(I)$ to denote the domain $H_d(I)$.

6.4. The construction of the set $Z_n$. Now take $0 < \beta < \alpha < \pi/3$ and let them be fixed throughout the following sections. Recall that for $i \in \mathbb{Z}$, $x_i$ is the point in $T$ such that $(g_\theta|T)'(x_i) = 1$.

For $n > 0$, Let

$$I_n = [1, x_{q_n}], \quad K_n = [1, x_{-q_n+1}], \quad \text{and} \quad L_n = [x_{q_n}, x_{-q_n+1}].$$

Define

$$A_n = H_\alpha(I_n) \setminus \Xi,$$

$$B_n = H_\alpha(I_{n+1}) \setminus \Xi = A_{n+1},$$

$$C_n = H_\alpha(K_n) \setminus \Xi.$$
and

\[ D_n = H_\beta(L_n) \setminus \Delta. \]

**Note.** In the following, we assume that the integer \( n \) in the discussion is large enough such that \( I_n, K_n, \) and \( L_n \) are all small and hence all the domains \( H_\alpha(I_n), H_\alpha(K_n), \) and \( H_\beta(L_n) \) are simply connected.

For an arc segment \( I \subset \mathbb{T} \), let \( I \) denote the interior of \( I \).

For \( 0 \leq i \leq q_n \), \( g_0^i(1) \notin I_{n+1} \). Let \( B_n^i \) denote the domain which is attached to the segment \([x_i, x_i+q_n+1]\) such that \( g_0^i : B_n^i \to B_n \) is a homeomorphism.

For \( 0 \leq q_n+1 \), \( g_0^i(1) \notin K_n \) let \( C_n^i \) denote the domain which is attached to the segment \([x_i-q_n+1, x_i]\) such that \( g_0^i : C_n^i \to C_n \) is a homeomorphism.

For \( 0 \leq i \leq q_n + 1 \), \( g_0^i(1) \notin L_n \). Let \( D_n^i \) denote the domain which is attached to the segment \([x_i+q_n, x_i-q_n+1]\) such that \( g_0^i : D_n^i \to D_n \) is a homeomorphism.

**Lemma 6.5.** \( B_n^i \subset H_\alpha(I_{n+1}^i), 0 \leq i \leq q_n, C_n^i \subset H_\alpha([x_i, x_i-q_n+1]), 0 \leq i \leq q_n+1 \), and \( D_n^i \subset H_\beta([x_q+1, x_i-q_n+1]), 0 \leq i \leq q_n+1 \).

**Proof.** Let us prove the first assertion and the other two can be proved in the same way. For \( 0 \leq i \leq q_n \), let \( P_i \) denote the set of the critical values of \( g_0^i \). Then

\[ P_i = \{ g_0^i(1) \mid 1 \leq j \leq i \}. \]

Note that \( g_0^i \) has exactly one critical value. It follows that \( P_i \cap \Omega_{I_{n+1}^i} = \emptyset \). Let \( \Psi_i \) denote the inverse branch of \( g_0^i \) which maps \( I_{n+1}^i \) to \( I_{n+1}^i \). Since \( H_\alpha(I_{n+1}^i) \) is simply connected by assumption, \( \Psi_i \) can be holomorphically extended to \( H_\alpha(I_{n+1}^i) \). But since \( \Omega_{I_{n+1}^i} \) is not simply connected, the map \( \Psi_i \) may not be extended to a holomorphic function on \( \Omega_{I_{n+1}^i} \). To avoid this problem, let us consider the holomorphic universal covering map \( \pi : \Delta \to \Omega_{I_{n+1}^i} \). Since \( P_i \cap \Omega_{I_{n+1}^i} = \emptyset \), \( \Psi_i \) can be lifted to a holomorphic function \( \tilde{\Psi}_i : \Delta \to \Omega_{I_{n+1}^i} \) such that

\[ \pi = g_0^i \circ \tilde{\Psi}_i. \]

This, together with the Schwarz Contraction Principle, implies that the map \( \Psi_i \) maps \( H_\alpha(I_{n+1}^i) \) into \( H_\alpha(I_{n+1}^i) \). The first assertion then follows. The other two assertions can be proved in the same way. \( \square \)

Define

\[ Z_n = \bigcup_{0 \leq i \leq q_n} B_n^i \cup \bigcup_{0 \leq i \leq q_n+1} C_n^i \cup \bigcup_{0 \leq i \leq q_n+1-1} D_n^i. \]

See Figure 1 for an illustration of \( A_n, B_n, C_n, \) and \( D_n \). The cone, whose two sides have an angle \( \pi/3 \) with \( \mathbb{T} \), represents part of the pre-image of \( \Delta \).
6.5. The construction of the family $\{(U_i, V_i)\}_{i \in \Lambda}$. Let $\Omega = \mathbb{C} \setminus \Sigma$. Let $\text{diam}_\Omega(\cdot)$ denote the hyperbolic diameter of a subset in $\Omega$. Let $\text{diam}(\cdot)$ and $\text{dist}(\cdot, \cdot)$ denote the diameter and distance with respect to the Euclidean metric. Recall that $T(z) = \sin(z)$. The following is a technical lemma about the distortion of $T^{-1}$ in a bounded set.

For two quantities $x, y > 0$, we write $x \preceq y$ if there is some universal constant $0 < K < \infty$ such that $x \preceq K y$. We write $x \simeq y$ if $y \preceq x$. We write $x \asymp y$ if $x \preceq y$ and $y \preceq x$ both hold.

**Lemma 6.6.** Let $1 < M < \infty$. Then there exists a constant $1 < \tau(M) < \infty$ depending only on $M$ such that for any $r > 0$ and $a \in \mathbb{C}$ with $B_{M r}(a) \subset B_2(0)$, and any component $U$ of $T^{-1}(B_{M r}(a))$ and any component $V$ of $T^{-1}(B_r(a))$ with $V \subset U$, there exist an $r' > 0$ and an $a' \in \mathbb{C}$ such that

$$B_{r'}(a') \subset V \subset U \subset B_{r(M)r'}(a').$$

**Proof.** By using a compact argument, we may assume that $r > 0$ is small and $a$ is contained in a small neighborhood of one of the critical values of $T(z)$, $1$ or $-1$. Without loss of generality, let us assume $a$ is close to $1$.

By a direct calculation, it is not difficult to see that there exists a uniform $1 < L < \infty$ such that for any small Euclidean disk $B_R(w)$ near $1$, if $W$ is a component of $T^{-1}(B_R(w))$, then one can find $z \in \mathbb{C}$ and $R' > 0$ such that

$$(17) \quad B_{R'}(z) \subset W \subset B_{L R'}(z)$$

with $R' \leq \sqrt{R + |w - 1|} - \sqrt{|w - 1|}$.

Now we have two cases. In the first case, $r < |a - 1|/10 M$. By $(17)$, we have

$$\text{diam} U \preceq \sqrt{M r + |a - 1|} - \sqrt{|a - 1|} \preceq M r / \sqrt{|a - 1|}.$$ 

By $(17)$, there is an $a' \in V$ and $r' > 0$ such that $B_{r'}(a') \subset V$ with

$$r' \geq \sqrt{r + |a - 1| - \sqrt{|a - 1|}} \geq (r/\sqrt{|a - 1|}).$$

This proves the lemma in the first case.

In the second case, $r \geq |a - 1|/10 M$. Then

$$\text{diam} U \preceq \sqrt{M r + |a - 1|} \preceq \sqrt{11 M r}.$$ 

By $(17)$, there is an $a' \in V$ and $r' > 0$ such that $B_{r'}(a') \subset V$ with

$$r' \geq \sqrt{r + |a - 1| - \sqrt{|a - 1|}} \geq \sqrt{7(\sqrt{1 + 10 M} - \sqrt{10 M})}.$$ 

In the last inequality we use the fact that $\sqrt{1 + x} - \sqrt{x}$ is decreasing for $x > 0$. This proves the lemma in the second case and Lemma 6.6 follows.

**Definition 6.1.** Let $1 < K < \infty$ and $z \in X_{n+2}$. We say $z$ is associated to a $K$–admissible pair $(U, V)$ if $V \subset U \subset \Omega$ are two topological disks such that

1. $z \in U$,
2. $V \subset X_n \setminus X_{n+2}$,
We have two cases. In the first case, Proof.

and any to some $K$ we can always take a Euclidean disk $B$. Logical disk $A$.

**Lemma 6.7.** There is a uniform $1 < K < \infty$ such that for all $n$ large enough and any $z \in X_{n+2}$, if $\omega = g_\theta(z) \in Y_{n+2}$ and $z \notin A_n \cup B_n$, then $z$ is associated to some $K$-admissible pair $(U, V)$.

**Proof.** We have two cases. In the first case, $d(z, T) \geq \phi$. In the second case, $d(z, T) < \phi$.

Suppose that we are in the first case. By assuming that $n$ is large enough, we can always take a Euclidean disk $B$ in $Y_n \setminus Y_{n+2}$ and a small open topological disk $A$ such that

1. $\omega \in A$,
2. $B \subset A$,
3. $diam(A) \leq diam(B)$.

Note that for all $n$ large enough, we can take $A$ small so that the component of $g_\theta^{-1}(A)$ which contains $z$, say $U$, lies in the outside of $\overline{\Delta}$. That is, $U \subset \Omega$. Let $V$ be one of the components of $g_\theta^{-1}(B)$ such that $V \subset U$. By using the previous notations, we have

$$\omega = g_\theta(z) = \Phi \circ R_t \circ T \circ \psi \circ \Phi^{-1}(z)$$

where $\psi: \hat{C} \setminus \overline{\Delta} \to \hat{C} \setminus \overline{D}$, $T: z \to \sin(z)$, $R_t: z \to e^{2\pi it}z$, and $\Phi: z \to z^2$ are the maps as defined in §3.

Since $A$ is a small open topological disk which intersects $\Delta$, $\Phi^{-1}(A)$, and hence $R_t^{-1} \circ \Phi^{-1}(A)$, are small open topological disks which intersect $\Delta$ also (We take one of the branches of $\Phi^{-1}$). By taking $A$ small, the distortion of $R_t^{-1} \circ \Phi^{-1}$ on $A$ is uniformly bounded, and from the third property above, we can thus find a point $a \in C$, an $r > 0$, and a universal $1 < M < \infty$ such that

$$diam(A) \leq diam(B)$$

(18) $B_r(a) \subset R_t^{-1} \circ \Phi^{-1}(B) \subset R_t^{-1} \circ \Phi^{-1}(A) \subset B_{M \cdot r}(a) \subset B_2(0)$.

Since $T$ is periodic, the diameter of any component of $T^{-1} \circ R_t^{-1} \circ \Phi^{-1}(A)$ has a uniform upper bound. Since $d(z, T) \geq \phi$ and the diameter of $A$ is small, it follows that $d((\psi \circ \Phi^{-1})(z), \partial D) \geq \kappa(\phi)$ where $\kappa(p) > 0$ is some constant depending only on $\phi$. Let $A'$ denote the component of $T^{-1} \circ R_t^{-1} \circ \Phi^{-1}(A)$ which contains $(\psi \circ \Phi^{-1})(z)$. Since $T$ is periodic, by taking $A$ small, we can make $A'$ small and $d(A', \partial D) \geq \kappa(\phi)/2$. So we can always assume that

(19) $diam_{\partial \Omega}(A') < 1$. 
by taking \( A \) small.

Let \( U = (\Phi \circ \psi^{-1})(A') \). Since \( \Phi \circ \psi^{-1} : C \smallsetminus \overline{T} \to C \smallsetminus \overline{A} \) is a holomorphic map, it follows from (19) and Schwarz Contraction Principle that

\[
\text{diam}_{C \smallsetminus \overline{A}}(U) < 1.
\]

This verifies the property (3) in Definition 6.1. The first two properties of Definition 6.1 hold automatically. The last property follows since the distortion caused by each map in the composition

\[
g_{\theta}^{-1} = \Phi \circ \psi^{-1} \circ T^{-1} \circ R_t^{-1} \circ \Phi^{-1}
\]

is bounded by some uniform constant provided that \( A \) is small. In fact, by (18) and Lemma 6.6, it is sufficient to show that the distortion of \( \Phi \circ \psi^{-1} \) on \( A' \) is uniformly bounded. From (19), it follows that \( \text{diam}(A') \leq \text{dist}(A', \partial D) \).

This implies that \( \psi^{-1} \) can be defined in a definitely larger domain containing \( A' \). It follows from Koebe’s distortion theorem that the distortion of \( \psi^{-1} \) on \( A' \) is uniformly bounded. Since \( A' \) is small and since the derivative of \( \psi^{-1} \) is bounded in a neighborhood of the infinity, it follows that \( \text{diam}(\psi^{-1}(A')) < 1 \) provided that \( A \) is small. This then implies that the distortion of \( \Phi \) on \( \psi^{-1}(A') \) is uniformly bounded. The last property in Definition 6.1 then follows.

Now suppose that we are in the second case. That means, \( \text{d}(z, \mathbb{T}) < \varphi \).

Since \( z \notin A_n \cup B_n \), it follows that

\[
\text{dist}(\omega, v) \geq |I_n^q - 1|.
\]

Recall that \( I_n = [1, x_{qn}] \) and \( I_n^q \) is the arc segment on \( \mathbb{T} \) such that \( g_{\theta}^q(I_n^i) = I_n^i \). (21) then comes immediately from the fact that the part of the cone, which is contained in \( A_n \cup B_n \), has size \( \sim I_n \sim |I_n^q - 1| \), see Figure 1.

Note that \( I_n^q - 1 \) is the interval in the dynamical partition of level \( n \) which contains \( v \). Let \( I \subset \mathbb{T} \) be any interval in the cell partition of level \( n \) which contains \( v \) or has \( v \) as one of its end points (In the latter case, there are two such intervals in the cell partition of level \( n \)). The inequality (21), together with Theorem 2.2 and Lemma 2.1, implies that

\[
\text{dist}(\omega, v) \geq |I|.
\]

Let \( J' \subset J \subset \mathbb{T} \) be the corresponding intervals to the cells of level \( n + 2 \) and \( n \) whose closures contain \( \omega \). Since any two adjacent intervals in the cell partition are commensurable(This is implied by Theorem 2.2 and Lemma 2.1), we have

\[
\text{dist}(\omega, v) \geq |I| > |J'|.
\]

In fact, if \( J = I \), then (23) follows from (22). Otherwise, let \( M \) denote the interval in the cell partition of level \( n \) which is between \( I \) and \( J \) and which is adjacent to \( J \). Then \( |M| \approx |J| \) by Theorem 2.2 and Lemma 2.1. If \( M \neq I \), we must have \( \text{dist}(\omega, v) \geq |M| \) and (23) follows. If \( M = I \), then (23) follows again from (22). It now follows from Lemma 5.1 that there is a Euclidean
From (23) and (24), it follows that for such Euclidean disk $B$, there is an open topological disk $A \subset \Delta$ such that:

1. $\omega \in A$,
2. $B \subset A$,
3. $\text{diam}(A) \leq \text{diam}(B) \leq \text{dist}(v, A)$.

See Figure 2 for an illustration of the sets $A$ and $B$. Let $U$ and $V$ be the pullbacks of $A$ and $B$ by $g^\theta$ respectively such that $z \in U$ and $V \subset U$. The first two properties in Definition 6.1 hold automatically. Let us verify the property (3). In fact, since $A \subset \Delta$ by the construction, $U$ is contained in the cone. From $\text{diam}(A) \leq \text{dist}(v, A)$, it follows that

$$\frac{\text{diam}(U)}{\text{dist}(U, \mathbb{I})} < \rho$$

for some uniform $\rho > 0$. This implies that $\text{diam}_0(U) < K$ where $K > 1$ is some constant depending only on $\rho$ and the property (3) follows. Since $\text{diam}(A) \leq \text{dist}(v, A)$, $g^\theta$ can thus be defined in a definitely larger domain $E \supset \overline{A}$ such that $\text{mod}(E \setminus \overline{A})$ has a uniform positive lower bound. The last property then follows from Koebe’s distortion theorem. \flushright{$\square$}

**Lemma 6.8.** There is a uniform $1 < K < \infty$ such that for any $0 \leq i \leq q_n - 1$ and $z \in X_{n+2}$, if $\omega = g^\theta(z) \in B^i_n$ but $z \notin B^{i+1}_n$, then $z$ is associated to some $K$-admissible pair $(U, V)$. 

---

**Figure 2.**
Proof. Again we have two cases. In the first case, \(d(z, T) \geq \wp\). In the second case, \(d(z, T) < \wp\).

Suppose that we are in the first case. Note that by Lemma 6.5, \(\omega \in B_{1n} \subset H_\alpha(I_{in+1})\). With the aid of this fact and Lemmas 2.1, 5.1, and Theorem 2.2, the proof of the first case can be completed by using exactly the same argument as in the proof of the first case of Lemma 6.7. The reader shall easily fill up the details of the proof for this case.

Now suppose that we are in the second case. That is, \(d(z, T) < \wp\). Note that \(I_{in+1} \cap I_{in+1}^q = \emptyset\) and that by the third assertion of Lemma 2.4, \(\nu\) separates the interval \(I_{in+1}^q\) into two \(L\)-commensurable subintervals for some uniform \(1 < L < \infty\). Since \(\omega \in H_\alpha(I_{in+1}^q)\), it follows that

\[
\text{dist}(\omega, \nu) \geq |I_{in+1}^q|.
\]

Let \(J\) be the interval in the cell partition of level \(n\) which contains \(I_{in+1}^q\). In particular, \(\nu \in J\). By Theorem 2.2 and Lemma 2.1 it follows that \(|I_{in+1}^q| \asymp |J|\) and therefore by (25) we have

\[
\text{dist}(\omega, \nu) \asymp \text{dist}(I_{in+1}^q, \nu) \asymp \text{dist}(H_\alpha(I_{in+1}^q), \nu) \geq |J|.
\]

Let \(E\) be the cell of level \(n\) corresponding to \(J\). It follows from (26) and the same argument as in the proof of (24), we have

\[
\text{dist}(\omega, \nu) \geq |J|.
\]

Let \(E\) be the cell of level \(n\) corresponding to \(J\). It follows from (26) and Lemma 5.1 that there is a Euclidean disk \(B \subset E \setminus X_{n+2}\) and a topological disk \(A \subset (\Delta \cup H_\alpha(I_{in+1}^q))\) such that
1. \( \omega \in A \),
2. \( B \Subset A \),
3. \( \text{diam}(A) \leq \text{diam}(B) \leq |J| \leq \text{dist}(v, A) \),

See Figure 3 for an illustration of the sets \( A \) and \( B \). Let \( U \) and \( V \) be the pull backs of \( A \) and \( B \) by \( g_0 \) respectively such that \( z \in U \) and \( V \subset U \). It is clear that the first two properties of Definition 6.1 hold automatically. Since \( \text{diam}(A) \leq \text{dist}(v, A) \), \( g_0^{-1} \) can be defined in a definite larger domain containing \( A \), so the last property follows from Koebe’s distortion theorem.

Now let us prove the property (3). Since \( A \subset (\Delta \cup H_\alpha(I_n^i)) \) and \( \text{diam}(A) \leq \text{dist}(v, A) \), it follows that \( \text{diam}(U)/\text{dist}(U, T) < \rho \) for some uniform \( \rho > 0 \). This implies that \( \text{diam}_\Omega(U) < K \) for some \( K > 1 \) depending only on \( \rho \). This proves the property (3) and Lemma 6.8 follows.

\[ \square \]

**Lemma 6.9.** There is a uniform \( 1 < K < \infty \) such that for any \( 0 \leq i \leq q_{n+1} - 1 \) and any \( z \in X_{n+2} \) with \( \omega = g_0(z) \in C_n^i \), if \( z \notin C_{n+1}^i \) for \( 0 \leq i \leq q_{n+1} - 2 \) and \( z \notin A_n \cup B_n \) for \( i = q_{n+1} - 1 \), then \( z \) is associated to some \( K \)-admissible pair \((U, V)\).

**Proof.** Suppose that \( 0 \leq i \leq q_{n+1} - 2 \). As before, we have two cases. In the first case, \( d(z, T) \geq \varphi \). In the second case, \( d(z, T) < \varphi \). Again, the first case can be proved by the same argument as in the proof of the first case of Lemma 6.7. So let us suppose that we are in the second case. That is, \( d(z, T) < \varphi \). By Lemma 6.5 \( C_n^i \subset H_\alpha([x_{i-q_{n+1}}, x_i]) \subset H_\alpha(I_n^i) \) for all \( 0 \leq i \leq q_{n+1} - 1 \). Note that \( I_n^i \cap I_n^{q_{n+1} - 1} = \emptyset \) and that by the third assertion of Lemma 2.4 \( v \) separates the interval \( I_n^{q_{n+1} - 1} \) into two \( L \)-commensurable subintervals for some uniform \( 1 < L < \infty \). Since \( \omega \in C_n^i \subset H_\alpha(I_n^i) \), it follows that

\[ \text{dist}(\omega, v) \geq |I_n^{q_{n+1} - 1}|. \]

Then the same argument as in the proof of the second case of Lemma 6.8 can be used to construct a \( K \)-admissible pair \((U, V)\) associated to \( z \). The reader shall easily supply the details.

Now suppose that \( i = q_{n+1} - 1 \). Again we have two cases. In the first case, \( d(z, T) \geq \varphi \). In the second case, \( d(z, T) < \varphi \). The first case can still be treated in the same way as in the proof of the first case of Lemma 6.7. So let us assume that \( d(z, T) < \varphi \). Note that there are two components of \( g_0^{-1}(C_n^{q_{n+1} - 1}) \) whose boundaries contain the critical point \( 1 \). It is clear that one of them is contained in \( B_n \). Let \( \Omega \) denote the other one. Then \( \Omega \) is a domain which is attached to one side of the cone from the outside. Let \( \Omega' = \Omega \setminus (A_n \cup B_n) \). Since \( z \notin A_n \cup B_n \), we have \( z \in \Omega' \). Note that \( ||x_{q_n+q_{n+1}-1}, v|| > ||v, x_{q_{n+1}-1}|| \) by the third assertion of Lemma 2.4 and \( C_n^{q_{n+1} - 1} \subset H_\alpha([x_{q_{n+1}-1}, v]) \) by Lemma 6.5 it follows that

\[ \text{diam}(\Omega') \leq \text{dist}(\Omega', T) \asymp |I_n|. \]
On the other hand, by Lemma 2.1, Lemma 2.4, and Lemma 5.1, it follows that there is a Euclidean disk $V \subset X_n \setminus X_{n+2}$ which is contained in the cone such that
\[ \text{diam}(V) \approx \text{dist}(V, T) \approx |I_n|. \]
It follows that one can construct an open topological disk $U$ containing $\Omega'$ and $V$ such that
\[ \text{diam}(U) \approx \text{dist}(U, T) \approx |I_n|. \]
See Figure 4 for an illustration of the sets $\Omega'$ and $V$. The properties of Definition 6.1 are obviously satisfied. The lemma follows. \( \square \)

**Lemma 6.10.** There is a uniform $1 < K < \infty$ such that for any $0 \leq i \leq q_{n+1} - 2$ and $z \in X_{n+2}$ with $\omega = g_\theta(z) \in D_{i+1}^n$, if $z \notin D_{i+1}^n$ for $0 \leq i \leq q_{n+1} - 2$ and $z \notin A_n \cup B_n$ for $i = q_{n+1} - 1$, then $z$ is associated to some $K$-admissible pair $(U, V)$.

**Proof.** The case that $0 \leq i \leq q_{n+1} - 2$ can be proved by the same argument as in the proof of the same case of Lemma 6.9. The reader shall easily supply the details.

Suppose that $i = q_{n+1} - 1$. As before, we have two cases. In the first case, $d(z, T) \geq \varphi$. In the second case, $d(z, T) < \varphi$. Again, the first case can be proved by the same argument as in the proof of the same case of Lemma 6.7. So let us assume that $d(z, T) < \varphi$. By Lemma 6.3, $D_{q_{n+1} - 1}^n \subset H_\beta([x_{q_n+q_{n+1}-1}, v])$. There are exactly two components of $g_\theta^{-1}(D_{q_{n+1} - 1}^n)$ which are attached to 1. Let us use $\Omega_1$ to denote the one which is attached to $[x_{q_n+q_{n+1}}, 1]$, and use $\Omega_2$ to denote the other one which is attached to one
side of the cone from the outside. Let \( \Omega'_{i} = \Omega_{i} \setminus (A_{n} \cup B_{n}) \) for \( i = 1, 2 \). By Lemma 6.5 and the third assertion of Lemma 2.4 it follows that for \( i = 1, 2 \),
\[
\text{diam}(\Omega'_{i}) \leq \text{dist}(\Omega'_{i}, T) \asymp |I_{n}|.
\]

Then by Lemma 2.1, Lemma 2.4, and Lemma 5.1, for \( i = 1, 2 \), one can take a Euclidean disk \( V_{i} \subset X_{n} \setminus X_{n+2} \) which is contained in the cone such that
\[
\text{diam}(V_{i}) \asymp \text{dist}(V_{i}, T) \asymp |I_{n}|,
\]
and a topological disk \( U_{i} \) which contains \( \Omega'_{i} \) and \( V_{i} \) such that
\[
\text{diam}(U_{i}) \asymp \text{dist}(U_{i}, T) \asymp |I_{n}|.
\]

See Figure 5 for an illustration of the sets \( \Omega'_{i} \) and \( V_{i} \), \( i = 1, 2 \). The properties of Definition 6.1 are obviously satisfied. The lemma follows. \( \square \)

**Lemma 6.11.** There is a uniform \( 1 < K < \infty \) such that for any \( z \in X_{n+2} \), if \( z \in A_{n} \setminus (B_{n} \cup C_{n} \cup D_{n}) \), then \( z \) is associated to some \( K \)-admissible pair \((U, V)\).

**Proof.** Let \( W = A_{n} \setminus (B_{n} \cup C_{n} \cup D_{n}) \). Note that \( |x_{q_{n}}, x_{-q_{n+1}}| \gg |x_{-q_{n+1}}, 1| \) by the first assertion of Lemma 2.4. By the definition of \( A_{n}, B_{n}, C_{n}, D_{n} \) and the fact that \( 0 < \beta < \alpha \), it follows that
\[
\text{diam}(W) \leq \text{dist}(W, T) \asymp |I_{n}|.
\]

See Figure 1 for an illustration.

Now by Lemma 6.1 we can construct a Euclidean disk \( V \subset X_{n} \setminus X_{n+2} \) in the cone such that
\[
\text{diam}(V) \asymp \text{dist}(V, T) \asymp |I_{n}|.
\]
It follows that there is an open topological disk $U$ containing $W$ and $V$ such that

$$\text{diam}(U) \leq \text{dist}(U, \mathbb{T}) \asymp |I_n|.$$ 

The properties of Definition 6.1 are obviously satisfied. The lemma follows. \hfill $\Box$

**Lemma 6.12.** For every $1 < K < \infty$, there exists an $1 < L < \infty$ depending only on $K$ such that if a point $z \in X_{n+2}$ is associated to some $K$-admissible pair $(U, V)$, then for any point $\xi \in X_{n+2}$ in the inverse orbit of $z$, $\xi$ is associated to some $L$-admissible pair $(U', V')$.

**Proof.** Suppose $z \in X_{n+2}$ is associated to some $K$-admissible pair $(U, V)$. Let $\xi \in X_{n+2}$ be a point in the inverse orbit of $z$, that is, $g^k_\theta(\xi) = z$ for some integer $k \geq 1$. Let $U' \subset U$ be the pull backs of $V$ and $U$ by $g^k_\theta$ such that $\xi \in U'$. The first two properties of Definition 6.1 hold automatically. Since $\text{diam}_\Omega(U) < K$, the branch of $g^k_\theta$, which maps $z$ to $\xi$, can be defined in a definitely larger domain containing $\mathbb{T}$. By Koebe’s distortion theorem, the last property of Definition 6.1 holds for some constant depending only on $K$. It remains to prove the third property.

Recall that $\Omega = \mathbb{C} \setminus \Delta$. Let $\Sigma = \mathbb{C} \setminus (\Delta \cup g^{-1}_\theta(\Delta))$. It follows that $\Sigma \subset \Omega$. Note that $g_\theta(1) = (g_\theta(1))^2$ is the only critical value of $g_\theta$ in $\mathbb{C}$. This implies that $g_\theta : \Sigma \rightarrow \Omega$ is a holomorphic covering map and that any inverse branch of $g_\theta$ contracts the hyperbolic metric in $\Omega$. Thus we get $\text{diam}_\Omega(U') < K$. This proves the third property of Definition 6.1 and the lemma follows. \hfill $\Box$

Let $Z_n$ be the set defined in (10). The importance of the set $Z_n$ is reflected by the following lemma.

**Lemma 6.13.** There is a uniform $K > 1$ such that for any $z \in X_{n+2}$, either $z \in Z_n$, or $z$ is associated to some $K$-admissible pair $(U, V)$.

**Proof.** Take $z \in X_{n+2}$. Suppose $z \notin Z_n$. Recall that $k_z > 0$ is the least integer such that $g^{k_z}_\theta(z) \in \Delta$. Let $l = k_z$. For $0 \leq k \leq l$, let $z_k = g^{l-k}_\theta(z)$. Then $z = z_0$. We may assume that $z_1 \in A_n \cup B_n$. This is because otherwise, $\omega = g_\theta(z_1) = g^{k_z}_\theta(z) \in Y_{n+2}$ and $z_1 \notin A_n \cup B_n$, then by Lemma 6.7, $z_1$ is associated to a $K$-admissible pair $(U, V)$ for some uniform $1 < K < \infty$. Since $z = z_1$, the Lemma then follows by Lemma 6.12.

We may further assume that $z_1 \in Z_n$. Because otherwise, we will have $z_1 \in (A_n \cup B_n) \setminus Z_n \subset A_n \setminus (B_n \cup C_n \cup D_n)$.

By Lemma 6.11 it follows that $z_1$ is associated to a $K$-admissible pair for some uniform $1 < K < \infty$. The lemma then follows again by Lemma 6.12 since $z$ lies in the inverse orbit of $z_1$.

Now suppose that $k \leq l$ is the largest integer such that $z_i \in Z_n$ for all $1 \leq i \leq k$ and $z_k \in A_n \cup B_n$. By assumption that $z_1 = z \notin Z_n$, it follows
that \( k < l \). Now we may assume that one of the following three possibilities happens: \( z_k \in B_n, z_k \in C_n \), or \( z_k \in D_n \). This is because otherwise, \( z_k \in A_n \setminus (B_n \cup C_n \cup D_n) \). So by Lemma 6.11 it follows that \( z_k \) is associated to a \( K \)-admissible pair for some uniform \( 1 < K < \infty \). The lemma then follows again by Lemma 6.12 since \( z = z_i \) lies in the inverse orbit of \( z_k \).

Suppose that \( z_k \in B_n \). By the assumption that \( z \notin Z_n \) and the choice of \( k \), there is either an \( 0 \leq i \leq q_n - 2 \) such that \( z_k+i \in B_n \) but \( z_{k+i+1} \notin B_n \) and \( z_{k+q_n-1} \in B_n \) but \( z_{k+q_n} \notin D_n \), and hence \( z_k \notin B_n \). Then the lemma follows from Lemma 6.8 and Lemma 6.12.

Suppose that \( z_k \in C_n \). By the assumption that \( z \notin Z_n \) and the choice of \( k \), there is either an \( 0 \leq i \leq q_n+1 \) such that \( z_k+i \in C_n \) but \( z_{k+i+1} \notin C_n+1 \) or \( z_{k+q_n+1} \in C_n+1 \) but \( z_{k+q_n+1} \notin A_n \cup B_n \). Then the lemma follows from Lemma 6.9 and Lemma 6.12.

Suppose that \( z_k \in D_n \). By the assumption that \( z \notin Z_n \) and the choice of \( k \), there is either an \( 0 \leq i \leq q_n-1 \) such that \( z_k+i \in D_n \) but \( z_{k+i+1} \notin D_n+1 \) or \( z_{k+q_n-1} \in D_n+1 \) but \( z_{k+q_n} \notin A_n \cup B_n \). Then the lemma follows from Lemma 6.10 and Lemma 6.12.

The proof of the lemma is finished.

\[ \square \]

### 6.6. Proof of Lemma 6.3

**Proof.** Let \( N \geq 1 \) and \( R > 1 \) be large and be fixed. For \( z \in X_{n+2} \), recall that \( k_z \geq 1 \) is the least positive integer such that \( g_{\theta}^{k_z}(z) \in \Delta \). Define

\[
X_{n+2}^{N,R} = \{ z \in X_{n+2} \mid |z| \leq R \text{ and } k_z \leq N \}.
\]

Note that the inner boundary component of \( Y_{n+2} \) is the union of finitely many straight segments and the outer boundary component of \( Y_{n+2} \) is the unit circle. Let \( X_{n+2}^{N,R} \) denote the closure of \( X_{n+2}^{N,R} \). Let

\[
W_n = \mathbb{T}_R \cup (B_R(0) \cap \bigcup_{0 \leq i \leq N} g_{\theta}^{-k}(\partial Y_{n+2}))
\]

where \( \mathbb{T}_R = \{ z \mid |z| = R \} \). It is clear that \( W_n \) is the union of finitely many piecewise smooth curve segments and moreover, we have

\[
X_{n+2}^{N,R} \setminus X_{n+2}^{N,R} \subset W_n.
\]

Since \( Z_n \) is open, it follows that \( X_{n+2}^{N,R} \setminus Z_n \) is a compact set. Take an arbitrary small positive number \( \eta > 0 \). It is clear that there is a finite open cover of \( W_n \), say \( Q_i, 1 \leq i \leq M \), such that

\[
\sum_{1 \leq i \leq M} \text{area}(Q_i) < \eta.
\]
By Lemma 6.13, any point \( x \) in the compact set \( X_{n+2}^N \setminus Z_n \) is either belongs to some \( Q_i \) or is associated to some \( K \)–admissible pair \((U, V)\) for some uniform \( 1 < K < \infty \). We thus have finitely many pairs \((U_i, V_i)\), \( i \in \Lambda \), such that

1. \( X_{n+2}^N \setminus Z_n \subset \bigcup_{1 \leq i \leq M} Q_i \cup \bigcup_{i \in \Lambda} U_i \),
2. \( V_i \subset X_n \setminus X_{n+2} \) for every \( i \in \Lambda \),
3. there is a uniform \( K > 1 \) such that for any \( i \in \Lambda \), there exist \( x_i \in V_i \) and \( r_i > 0 \), such that \( B_{r_i}(x_i) \subset V_i \subset U_i \subset B_{K^{-1}r_i}(x_i) \).

On the other hand, by Theorem 2.2, it follows that there is a \( 0 < \sigma < 1 \) such that for any interval of the dynamical partition of level \( n \), \( |I| < \sigma^n \).

This, together with Lemma 6.3, implies that there is a uniform \( C > 1 \) and \( 0 < \epsilon < 1 \) such that

\[
\text{(28)} \quad \text{area}(Z_n) < Ce^n.
\]

We now claim that there is a \( 0 < \delta < 1 \) such that

\[
\text{(29)} \quad \text{area}(X_{n+2}) \leq Ce^n + \delta \text{area}(X_n).
\]

In fact, by the first property above, we have

\[
\text{(30)} \quad \text{area}(X_{n+2}^N) \leq \text{area}(Z_n) + \text{area}(\bigcup_{1 \leq i \leq M} Q_i) + \text{area}(\bigcup_{i \in \Lambda} U_i).
\]

By Corollary 6.1, we have

\[
\text{(31)} \quad \text{area}(\bigcup_{i \in \Lambda} U_i) \leq \text{area}(\bigcup_{i \in \Lambda} V_i)/\lambda(K).
\]

From the second property above, we have

\[
\text{area}(\bigcup_{i \in \Lambda} V_i) \leq \text{area}(X_n) - \text{area}(X_{n+2}).
\]

Note that

\[
\text{area}(X_{n+2}) \geq \text{area}(X_{n+2}^N) > \text{area}(X_{n+2}^N_1) - \text{area}(\bigcup_{1 \leq i \leq M} Q_i).
\]

We thus have

\[
\text{(32)} \quad \text{area}(\bigcup_{i \in \Lambda} U_i) \leq (\text{area}(X_n) - \text{area}(X_{n+2}^N) + \text{area}(\bigcup_{1 \leq i \leq M} Q_i))/\lambda(K).
\]

Let \( \delta = 1/(1 + \lambda(K)) \). From (30) and (32), we have

\[
\text{area}(X_{n+2}^N) \leq \text{area}(Z_n) + \text{area}(\bigcup_{1 \leq i \leq M} Q_i) + \delta \text{area}(X_n).
\]

By (28), we have

\[
\text{area}(X_{n+2}^N) \leq Ce^n + \delta \text{area}(X_n) + \eta.
\]
Since $\eta > 0$ can be arbitrarily small, we thus have

$$\text{area}(X_{n+1}^{N,R}) \leq Ce^n + \delta \text{area}(X_n).$$

In particular, we get

$$\text{area}(X_{n+1}^{N,R}) \leq Ce^n + \delta \text{area}(X_n).$$

Since the constants $C, \epsilon$, and $\delta$ do not depend on $N$ and $R$, (29) now follows by letting $N, R \to \infty$. This proves the claim and Lemma 6.3 follows. \hfill \Box

It follows that $\nu$, and thus $\mu$ by Lemma 6.1, satisfy the condition (2). We have proved the integrability of $\mu$.

7. Proof of the Main Theorem

Let $\phi : \mathbb{C} \to \mathbb{C}$ be the David homeomorphism given by $\mu$ which fixes 0 and the infinity, and maps 1 to $\pi/2$.

**Lemma 7.1.** The map $\phi$ is odd.

**Proof.** By Lemma 5.4, $\mu(z) = \mu(-z)$. Consider the map $\hat{\phi}(z) = \phi(-z)$. It follows that $\phi$ and $\hat{\phi}$ has the same Beltrami differential. By Theorem 2.1 it follows that $\phi \circ \hat{\phi}^{-1}$ is a conformal map in the plane. Since it fixes 0 and $\infty$, it follows that $(\hat{\phi} \circ \phi^{-1})(z) = az$ for some $a \neq 0$. That is, $\phi(-z) = a\phi(z)$.

It follows that $\phi(-z) = a\phi(-(z)) = a^2\phi(-z)$ for all $z$. This implies that $a^2 = 1$. Clearly $a \neq 1$ since $\phi$ is a homeomorphism of the plane. It follows that $a = -1$ and thus $\phi(-z) = -\phi(z)$. The lemma has been proved. \hfill \Box

**Lemma 7.2.** $T_0 = \phi \circ \widetilde{G}_0 \circ \phi^{-1}$ is an odd entire function.

The proof uses completely the same argument as in the proof of Lemma 5.5 of [8].

**Proof.** Let $X$ denote the set of the critical points of $\widetilde{G}_0$. It is sufficient to show that the map $\phi \circ \widetilde{G}_0$ belongs to $W^{1,1}_{\text{loc}}(\mathbb{C} \setminus X)$. In fact, if $\phi \circ \widetilde{G}_0$ belongs to $W^{1,1}_{\text{loc}}(\mathbb{C} \setminus X)$, then in any small open neighborhood $U$ of a regular point of $\widetilde{G}_0$, since by Lemma 5.3, the Beltrami differential of $\phi \circ \widetilde{G}_0$ and $\phi$ are both equal to $\mu$, it follows from Theorem 2.1 that $\phi \circ \widetilde{G}_0 = \sigma \circ \phi$ where $\sigma$ is a conformal map defined on $\phi(U)$. This implies that $T_0$ is holomorphic in the complex plane except the points in $\phi(X)$. But it is clear that for any point $z \in \phi(X)$, there is a neighborhood $W$ of $z$ such that $T_0$ is bounded in $W$. It follows that all the points in $\phi(X)$ are removable. So $T_0$ is an entire function.

Now let us show that the map $\phi \circ \widetilde{G}_0$ belongs to $W^{1,1}_{\text{loc}}(\mathbb{C} \setminus X)$. Firstly, $\phi \circ \widetilde{G}_0 \in W^{1,1}_{\text{loc}}(\mathbb{C} \setminus (X \cup \bar{X}))$. This is because $\widetilde{G}_0$ is holomorphic in $\mathbb{C} \setminus (X \cup \bar{X})$ and $\phi \in W^{1,1}_{\text{loc}}(\mathbb{C})$. Secondly, we have $\phi \circ \widetilde{G}_0 \in W^{1,1}_{\text{loc}}(\Delta)$. To see this, write $\phi \circ \widetilde{G}_0 = \phi \circ \Phi^{-1} \circ H^{-1} \circ R_\alpha \circ H \circ \Phi$ in $\Delta$. Note that $\phi \circ \Phi^{-1}$ and $H$ has same Beltrami differential in $\Delta$, it follows from Theorem 2.1 again that $\phi \circ \Phi^{-1} \circ H^{-1}$
and therefore $\phi \circ \Phi^{-1} \circ H^{-1} \circ R_\alpha$ is conformal. Since $\Phi$ is conformal, $H \circ \Phi$ belongs to $W^{1,1}_{loc}(\Delta)$. It follows that

$$\phi \circ \tilde{G}_\theta = (\phi \circ \Phi^{-1} \circ H^{-1} \circ R_\alpha) \circ (H \circ \Phi) \in W^{1,1}_{loc}(\Delta).$$

It remains to prove that for every small open disk $U$ centered at the point in $T \setminus \{1, -1\}$, $\phi \circ \tilde{G}_\theta \in W^{1,1}_{loc}(U)$. Note that $\phi \circ \tilde{G}_\theta$ is almost differentiable in $U$. Therefore

$$(33) \quad \int_U \text{Jac}(\phi \circ \tilde{G}_\theta) \leq \text{area } ((\phi \circ \tilde{G}_\theta)(U)) < \infty.$$ 

This implies that $\text{Jac}(\phi \circ \tilde{G}_\theta) \in L^1(U)$. It follows that the ordinary partial derivatives of $\phi \circ \tilde{G}_\theta$ are almost equal to the distributive ones in any compact set in $U \setminus T$. It is sufficient to prove that $\partial(\phi \circ \tilde{G}_\theta) \in L^1(U)$ (Then the distributive partial derivatives coincide with the ordinary partial derivatives in $U$ and are thus integrable in $U$). But this follows from the following argument. Since $\mu_{\phi \circ \tilde{G}_\theta} = \mu$ almost everywhere in $U$, we have

$$|\partial(\phi \circ \tilde{G}_\theta)|^2 \leq \frac{\text{Jac}(\phi \circ \tilde{G}_\theta)}{1 - |\mu_{\phi \circ \tilde{G}_\theta}|^2} = \frac{\text{Jac}(\phi \circ \tilde{G}_\theta)}{1 - |\mu|}$$

and therefore,

$$|\partial(\phi \circ \tilde{G}_\theta)| \leq \frac{\text{Jac}(\phi \circ \tilde{G}_\theta)^{1/2}}{(1 - |\mu|)^{1/2}}.$$

Since $\mu$ satisfies the exponential growth condition (2), the measurable function $1/(1 - |\mu|)$ is integrable in $U$. This, together with (33) and Cauchy inequality, implies the integrability of $\partial(\phi \circ \tilde{G}_\theta)$ in $U$.

The odd property of $T_\theta$ follows from the odd property of $\tilde{G}_\theta$ (see Lemma 5.3) and Lemma 7.1.

**Definition 7.1.** Two maps $f : \mathbb{C} \rightarrow \mathbb{C}$ and $g : \mathbb{C} \rightarrow \mathbb{C}$ are called topologically equivalent if there exist two homeomorphisms $\theta_1$ and $\theta_2$ of the complex plane such that $f = \theta_2^{-1} \circ g \circ \theta_1$.

**Lemma 7.3** (Lemma 1, [1]). Let $f$ be an entire function. If $f(z)$ is topologically equivalent to $\sin(z)$, then $f(z) = a + b\sin(cz + d)$ where $a, b, c, d \in \mathbb{C}$, and $b, c \neq 0$.

For a proof of Lemma 7.3 see [4].

**Lemma 7.4.** Let $f : \mathbb{C} \rightarrow \mathbb{C}$ and $g : \mathbb{C} \rightarrow \mathbb{C}$ be two continuous maps such that $f = g$ on the outside of the unit disk. If in addition, $f : \overline{\Delta} \rightarrow \overline{\Delta}$ and $g : \overline{\Delta} \rightarrow \overline{\Delta}$ are both homeomorphisms, then $f$ and $g$ are topologically equivalent to each other.
Proof. Define \( \theta_2(z) = z \) for \( z \notin \Delta \) and \( \theta_2(z) = g^{-1} \circ f(z) \) for \( z \in \Delta \). It follows that \( \theta_2 : \mathbb{C} \to \mathbb{C} \) is a homeomorphism. Let \( \theta_1 = id \). Then \( f = \theta_1^{-1} \circ g \circ \theta_2 \). The Lemma follows.

Let \( \psi : \hat{\mathbb{C}} \setminus \bar{\Delta} \to \hat{\mathbb{C}} \setminus \bar{D} \) be map in the definition of \( G(z) \) (see \( \S 3 \)). Let \( \eta : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a homeomorphic extension of \( \psi \). As before let \( T(z) = \sin(z) \). It follows that \( T(z) \) is topologically equivalent to \( T \circ \eta \). Let \( t \in [0, 1) \) be the number in Lemma 3.5. Let \( S(z) = e^{2\pi it}(T \circ \eta)(z) \).

Lemma 7.5. \( S(z) \) is topologically equivalent to \( T(z) \) and \( \tilde{G}_\theta(z) \).

Proof. The first topological equivalence follows from the definition of \( S(z) \). The second one follows from the definition of \( \tilde{G}_\theta \) and Lemma 7.4. \( \square \)

Lemma 7.6. \( T_\theta(z) \) is topologically equivalent to \( T(z) \).

Proof. By the construction of \( T_\theta \), it follows that \( T_\theta(z) \) is topologically equivalent to \( \tilde{G}_\theta \). The Lemma then follows from Lemma 7.5. \( \square \)

Now it is the time to prove the Main Theorem.

Proof. By Lemma 7.3 and Lemma 7.6 it follows that \( T_\theta(z) = a + b \sin(cz + d) \) where \( a, b, c, d \in \mathbb{C} \) and \( b, c \neq 0 \). Since \( T_\theta \) is odd by Lemma 7.2 we get

\[
a + b \sin(cz + d) \equiv -a + b \sin(cz - d).
\]

Now by differentiating both sides of (34), we get

\[
\cos(cz + d) \equiv \cos(cz - d).
\]

It follows that

\[
\sin(d) \sin(cz) \equiv 0.
\]

Since \( c \neq 0 \), it follows that \( d = k\pi \) for some integer \( k \). Therefore, we may assume that \( T_\theta(z) = a + b \sin(cz) \) for some \( b, c \neq 0 \). Since \( T_\theta(0) = 0 \), it follows that \( a = 0 \). This implies that \( T_\theta(z) = b \sin(cz) \).

Since \( T_\theta(\pi/2) = 0 \), it follows that \( c \) is some odd integer. By changing the sign of \( b \), we may assume that \( c \) is positive. Suppose \( c = 2l + 1 \) for some integer \( l \geq 0 \). Let \( \Omega_0 \) be the Siegel disk of \( T_\theta \) centered at the origin. For \( k \in \mathbb{Z} \), let

\[
\Omega_k = \{ z + k\pi \mid z \in \Omega_0 \}.
\]

Since \( T_\theta \) is odd by Lemma 7.2, \( \Omega_0 \) is symmetric about the origin. It follows that \( T_\theta(\Omega_k) = \Omega_0 \). Therefore each \( \Omega_k \) is a component of \( T_\theta^{-1}(\Omega_0) \).

Let \( D_k, k \in \mathbb{Z} \), be the domains in Lemma 3.2. Recall that \( D = D_0 \). Let \( \psi : \hat{\mathbb{C}} \setminus \bar{\Delta} \to \hat{\mathbb{C}} \setminus \bar{D} \) be the map defined immediately after Lemma 3.2. Let \( \tilde{\Omega}_0 = \Omega_0 \) and \( \tilde{\Omega}_k = \phi \circ \psi^{-1}(D_k) \).

By Lemma 3.2 we have
1. $\Omega_k, k \in \mathbb{Z}$, are all the components of $T_{\theta}^{-1}(\Omega_0)$,
2. every $\partial \Omega_k$ contains exactly two critical points of $T_\theta$,
3. $\partial \Omega_k \cap \partial \Omega_j = \emptyset$ if $|k - j| > 1$,
4. any critical point of $T_\theta$ is the intersection point of $\partial \Omega_k$ and $\partial \Omega_{k+1}$ for some $k \in \mathbb{Z}$, and for every $k \in \mathbb{Z}$, $\partial \Omega_k \cap \partial \Omega_{k+1}$ contains exactly one critical point of $T_\theta$.

It is clear that every $\Omega_k$ is equal to some $\tilde{\Omega}_j$. We claim that $\Omega_k = \tilde{\Omega}_k$ for all $k \in \mathbb{Z}$. By definition, $\Omega_0 = \tilde{\Omega}_0$. Since $\partial \Omega_0$ contains the critical point $\pi/2$, and since only $\partial \tilde{\Omega}_0$ and $\partial \tilde{\Omega}_1$ contain $\pi/2$ (This is because $\pi/2 \in \partial D$ and $\phi \circ \psi^{-1}(\pi/2) = \pi/2$), we get $\Omega_1 = \tilde{\Omega}_1$ (Since $\Omega_1 \neq \tilde{\Omega}_0 = \Omega_0$). Since only $\partial \Omega_1$ and $\partial \Omega_2$ intersect $\partial \Omega_0$, and since $\partial \Omega_2$ intersects $\partial \Omega_1$, it follows that $\Omega_2 = \tilde{\Omega}_2$ (Since $\Omega_2 \neq \tilde{\Omega}_0 = \Omega_0$). Since only $\partial \Omega_1$ and $\partial \Omega_3$ intersect $\partial \Omega_2$, and since $\partial \Omega_3$ intersects $\partial \Omega_2 = \Omega_2$, it follows that $\Omega_3 = \tilde{\Omega}_3$ (Since $\Omega_3 \neq \tilde{\Omega}_1 = \Omega_1$).

Repeating this argument, we get $\Omega_k = \tilde{\Omega}_k$ for all $k \geq 0$. The same argument implies $\Omega_k = \tilde{\Omega}_k$ for all $k < 0$. The claim has been proved.

Now it follows that the set of the critical points of $T_\theta$ is equal to
\[
\{\pi/2 + k\pi \mid k \in \mathbb{Z}\}.
\]
This implies that $c = 1$. It follows that $b = e^{2\pi i \theta}$ and therefore $T_\theta(z) = f_\theta(z)$.

This completes the proof of the Main Theorem. $\square$

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