A scalar field governed cosmological model from noncompact Kaluza-Klein theory

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Abstract

This paper is a review of a recently introduced cosmological model from a noncompact Kaluza-Klein theory for a single scalar field minimally coupled to gravity. We obtain that the 4D scalar potential has a geometrical origin and assume different representations in different frames. It should be responsible for the expansion of the universe. In this framework we explain the (neutral scalar field governed) evolution of the universe from an initially inflationary expansion that has a change of phase towards a decelerated expansion and thereafter evolves towards the present day observed accelerated (quintessential) expansion. Finally, using the Hamilton-Jacobi formalism, we study extra force and extra mass from this 5D cosmological model.

I. INTRODUCTION

The idea that the Universe may have more than 4 dimensions is due to Kaluza (1921), who with a brilliant insight realized that a 5D manifold could be used to unify Einstein’s theory of general relativity with Maxwell’s theory of electromagnetism. After some delay, Einstein endorsed the idea, but a major impetus was provided by Klein (1926). He made the connection to quantum theory by assuming that the extra dimension was microscopically small, with a size in fact connected via the Planck’s constant to the magnitude of the electron charge. The development of particle physics, quantum field theory and the strings theory
led to a resurgence of interest in higher dimensional field theories as a means of unifying the long range and short range interactions of physics. Thus Kaluza-Klein 5D theory lay the foundation for modern developments such as 10D superstrings and 11D supergravity. There are several versions of this theory such as noncompactified induced matter or space-time-matter theory. The Kaluza-Klein theory is essentially general relativity in 5D, and physically have the motivation of explaining why we perceive 4 dimensions of the space-time and (apparently) do not see the fifth dimension. It is constrained by two conditions. (1) The so called “cylinder condition” was introduced by Kaluza, and consists in setting all partial derivatives with respect to the fifth coordinate to zero. (2) The condition of compactification was introduced by Klein, and consists in the assumption that the fifth dimension is not small in size but has a closed topology (a circle if we are only considering one extra dimension). It is a constraint that may be applied retroactively to a solution. This condition introduces periodicity and allows one to use Fourier and other decompositions of the theory. The field equations would logically be expected to be $G_{AB} = kT_{AB}$ (where $A, B = 0, 1, 2, 3, 4$) with some appropriate coupling constant $k$ and a 5D energy momentum tensor. From the time of Kaluza-Klein onward much work has been done with the “apparent vacuum” or “empty” form of the field equations $G_{AB} = 0$. In the practice is very difficult determine that relations without some starting assumption about $g_{AB}$. This is usually connected with the physical situation being investigated. In gravitational theory, an assumption about $g_{AB} = g_{AB}(x^c)$ is commonly called a choice of coordinates, while in particle physics it is commonly called a choice of gauge. The traditional Kaluza-Klein theory has been worked by many people, including Jordan [1,2], Bergmann [3], Lessner [4], Thiry [5], and Liu and Wesson [6]. In this theory the coordinates are chosen so as to write the 5D metric tensor in the form

$$g_{AB} = \left( g_{\alpha\beta} - k^2 \Phi^2 A_{\alpha} A_{\beta} - k \Phi^2 A_{\alpha} - k \Phi^2 A_{\beta} - \Phi^2 \right),$$  \hspace{1cm} (1)

where $k$ is a coupling constant. Then the Einstein’s field equations in the vacuum reduce to:

$$G_{\mu\nu} = \frac{k^2 \Phi^2}{2} T_{\mu\nu} - \frac{1}{\Phi} \left( \nabla_\mu \nabla_\nu \Phi - g_{\mu\nu} \Box \Phi \right),$$  \hspace{1cm} (2)

$$\nabla^\mu F_{\mu\nu} = -3 \frac{\nabla^\mu \Phi}{\Phi} F_{\mu\nu},$$  \hspace{1cm} (3)

$$\Box \Phi = -\frac{k^2 \Phi^3}{4} F_{\mu\nu} F^{\mu\nu},$$  \hspace{1cm} (4)

where $\mu, \nu = 0, 1, 2, 3$. In these equations $G_{\mu\nu}$ is the Einstein’s tensor, $F_{\mu\nu}$ is the Maxwell’s tensor and $T_{\mu\nu}$ is the energy-momentum tensor for an electromagnetic field given by $T_{\mu\nu} = \frac{1}{2}(g_{\alpha\beta} F_{\alpha\beta} F^{\alpha\beta} - F^\gamma_{\mu} F_{\nu\gamma})$. Also $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the wave operator, and summation convention is in effect. The equations (3) are the 4 equations of electromagnetism modified by a function, which by (4) can be thought of as depending on a wave-like scalar field. The right side of (2) in some sense represents an energy-momentum tensor that is effectively derived from the fifth dimension. In short the traditional Kaluza-Klein theory is
in general a unified account of gravity, electromagnetism and scalar field. In the language of particle physics, the field equations $G_{AB} = 0$ of Kaluza-Klein theory describe a spin-2 graviton, a spin-1 photon and a spin-0 boson which is connected with how particles acquire mass.

II. COSMOLOGICAL MODEL FROM INDUCED MATTER THEORY OR SPACE-TIME MATTER THEORY

Einstein introduce the idea that the physical quantities should be given a geometrical interpretation, as envisaged by many people through time. An early attempt at this was made by Kaluza and Klein, who extended general relativity from 4 to 5 dimensions, but also applied severe restrictions to the geometry (the condition of cylindricity and compactification). In the 90’s Paul Wesson and Ponce de Leon showed that it is possible to interpret most properties of matter as the result of 5D Riemannian geometry, where however the latter allows dependence on the fifth coordinate and does not make assumptions about the topology of the fifth dimension. This theory is called the induced matter theory. The induced matter theory has seen most work in 3 areas: the case of uniform cosmological models, the soliton case and the case of neutral matter. The first case is easiest to treat because of the high degree of symmetry involved. The second case is more complicated, but important because 5D solitons are the analogs of isolated 4D masses, and the 5D class of soliton solution contains the unique 4D Schwarzschild solution. The last case can be treated quite generally, and lays the foundation for many applications where electromagnetic effects are not involved. We are going to give a brief review of the main features considering only the cosmological case. The other cases go beyond the scope of this work.

In the cosmological context the extra dimension is already known to be of great importance for cosmology. There is a class of 5D cosmological models which reduce to the usual four dimensional ones, on hypersuperfaces defined by setting the value of the extra coordinate constant. In these models the matter is explained as the consequence of geometry in five dimensions. The physics of this follows from a mathematical results. The basic idea of this models is explained below. The 5D Einstein’s field equations for apparent vacuum are:

$$G_{AB} = 0,$$

(5)

where the 5D Einstein’s tensor is $G_{AB} = R_{AB} - \frac{1}{2}Rg_{AB}$, with $R_{AB}$ the 5D Ricci’s tensor and $g_{AB}$ the 5D metric. The central thesis of induced matter theory is that from equations (5) we obtain the 4D field equations with matter given by:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}.$$  

(6)

In other words, the equations (6) are a subset of (5) with an effective or induced 4D energy-momentum tensor $T_{\mu\nu}$ which contains the classical properties of matter. This idea can be explained as a consequence of the Campell’s theorem. It says that any analytic N-dimensional Riemannian manifold can be locally embedded in an (N+1)-dimensional Ricci flat Riemannian manifold [7]. This is of great importance for establishing the generality of the proposal that 4D field equations with sources can be locally embedded in 5D field equations without sources. Besides, it can be used to study lower dimensional gravity
It can be employed to find new classes of 5D solutions [9]. Some of the latter have the remarkable property that they are 5D flat but contain 4D subspaces that are curved and correspond to known physical situations [10,11]. However, the principle is clear: curved 4D physics can be embedded in curved or flat 5D geometry.

In this theory an exact solution of (5) is of cosmological type if resembles that of Friedmann-Robertson-Walker (FRW), and the dynamics is governed by equations like those of Friedmann. Paul Wesson, Ponce de Leon and co-workers found several classes of exact cosmological solutions of (5) whose metrics are separable and reduce to the standard 4D FRW ones on the hypersurfaces with the fifth coordinate constant.

Following the idea suggested by Wesson and co-workers and to illustrate the transition from 5D field equations (5) for apparent vacuum to the 4D equations (6) with matter, it is convenient to start considering a 3D spatially, isotropic and flat spherically symmetric 5D line element:

\[ ds^2 = -e^{\alpha(t)} dt^2 + e^{\beta(t)} dr^2 + e^{\gamma(t)} d\psi^2, \] (7)

where \( dr^2 = dx^2 + dy^2 + dz^2 \) and \( \psi \) is the fifth coordinate. We assume that \( e^\alpha, e^\beta \) and \( e^\gamma \) are separable functions of the variables \( \psi \) and \( t \). The equations for the relevant Einstein’s elements are:

\[ G^0_0 = -e^{-\alpha} \left[ \frac{3 \beta^2}{4} + \frac{3 \beta \dot{\gamma}}{4} \right] + e^{-\gamma} \left[ \frac{3 \dot{\beta}^2}{2} + \frac{3 \dot{\beta} \dot{\gamma}}{2} \right], \] (8)

\[ G^0_4 = e^{-\alpha} \left[ \frac{3 \dot{\beta}^2}{4} + \frac{3 \dot{\beta} \dot{\gamma}}{4} - \frac{3 \dot{\beta} \ddot{\alpha}}{4} - \frac{3 \dot{\gamma} \ddot{\alpha}}{4} \right], \] (9)

\[ G^i_i = -e^{-\alpha} \left[ \dddot{\beta} + \frac{3 \dot{\beta}^2}{4} + \ddot{\gamma} + \frac{\dot{\gamma}^2}{4} + \frac{\dot{\beta} \dot{\gamma}}{2} - \frac{\dot{\alpha} \ddot{\beta}}{2} - \frac{\dot{\alpha} \ddot{\gamma}}{2} \right] + e^{-\gamma} \left[ \dddot{\beta} + \frac{3 \dot{\beta}^2}{4} + \ddot{\alpha} + \frac{\dot{\alpha}^2}{4} + \frac{\dot{\beta} \dot{\alpha}}{2} - \frac{\dot{\gamma} \ddot{\alpha}}{2} - \frac{\ddot{\alpha} \ddot{\gamma}}{4} \right], \] (10)

\[ G^4_4 = -e^{-\alpha} \left[ \frac{3 \dddot{\beta}}{2} + \frac{3 \dot{\beta}^2}{2} - \frac{3 \dot{\alpha} \ddot{\beta}}{4} \right] + e^{-\gamma} \left[ \frac{3 \dot{\beta}^2}{4} + \frac{3 \dot{\beta} \dot{\alpha}}{4} \right], \] (11)

where the overstar and the overdot denote, respectively, \( \partial/\partial \psi \) and \( \partial/\partial t \), and \( i = 1, 2, 3 \). Following the convention \((-,-,+,-,+\)) for the 4D metric, we define \( T^0_0 = -\rho_t \) and \( T^i_i = P \) (we are considering a 3D isotropic and homogeneous universe), where \( \rho_t \) is the total energy density and \( P \) is the pressure. The 5D vacuum conditions (5) are given by [17]:

\[ 8\pi G \rho_t = \frac{3}{4} e^{-\alpha} \dot{\beta}^2, \] (12)
\[ 8\pi GP = e^{-\alpha} \left[ \frac{\dot{\alpha} \dot{\beta}}{2} - \ddot{\beta} - \frac{3\dot{\beta}^2}{4} \right], \quad (13) \]

\[ e^\alpha \left[ \frac{3 \dot{\beta}^2}{4} + \frac{3 \beta^* \dot{\alpha}^*}{4} \right] = e^\gamma \left[ \frac{\ddot{\beta}}{2} + \frac{3\dot{\beta}^2}{2} - \frac{\dot{\alpha} \dot{\beta}}{4} \right], \quad (14) \]

where \( G \) is the gravitational constant. Hence, from the equations (12) and (13) and taking \( \dot{\alpha} = 0 \), we obtain the equation of state for the induced matter:

\[ P = -\left( \frac{4}{3} \frac{\ddot{\beta}}{\dot{\beta}^2} + 1 \right) \rho_t. \quad (15) \]

Notice that for \( \ddot{\beta}/\dot{\beta}^2 \leq 0 \) and \( |\ddot{\beta}/\dot{\beta}^2| \ll 1 \) (or zero), this equation describes an inflationary universe. The equality \( \ddot{\beta}/\dot{\beta}^2 = 0 \) corresponds with a 4D de Sitter expansion for the universe. This theory is gauge depending because for different choice of coordinates, one have several metrics all of them solutions of (5). In 1988 Ponce de Leon obtained one of the classes of solutions to (5) which are solutions of cosmological and astrophysical importance. With those line elements it is possible to develop models that reduce to the standard FRW ones with flat 3D space sections on hypersurfaces \( \psi = \text{const.} \). This is one of the most interesting aspects of this theory because one can ensure that the 5D models reduce to the 4D on hypersurfaces \( x^4 = \text{const.} \). Taking the solution \( e^\alpha = \psi^2 \), \( e^\beta = t^\frac{2}{\alpha} \psi \), \( e^\gamma = \alpha^2 (1 - \alpha)^{-2} t^2 \), the basic line element (7) can be written:

\[ ds^2 = -\psi^2 dt^2 + t^{2/\alpha} \psi^{2(1-\alpha)}[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\theta^2)] + \alpha^2 (1 - \alpha)^{-2} t^2 d\psi^2, \quad (16) \]

where \( \alpha \) is a constant related in the Space-Time-Matter (STM) theory to the properties of matter. This constant is determined by induced energy momentum tensor related at the theory. From the Einstein’s equations and eq. (16) the equations of state are:

\[ 8\pi \rho_t = \frac{3}{\alpha^2 \psi^2 t^2}, \quad 8\pi P = \frac{(2\alpha - 3)}{\alpha^2 \psi^2 t^2}, \quad P = \left( \frac{2\alpha}{3} - 1 \right) \rho_t. \quad (17) \]

The choice \( \alpha = 2 \) gives \( P = \frac{4}{3} \rho_t \) which is typical of radiation, and a scale factor \( a(t) \sim t^{1/2} \). The choice \( \alpha = 3/2 \) gives \( P = 0 \) which is typical of dust, and a scale factor that grows as \( t^{2/3} \). Thus on hypersurfaces \( \psi = \text{const.} \) the standard models for the early and late universe are recovered.

The coordinates in (16) are spatially comoving as in the usual presentation of the 4D models. That is, \( u^i = \frac{dx^i}{ds} = 0 \). The other components can be found by solving the 5D geodesic equation to be:

\[ u^0 = \pm \frac{\alpha}{\sqrt{2\alpha - 1} \psi}, \quad u^4 = \pm \frac{(1 - \alpha)^2}{\alpha \sqrt{2\alpha - 1} t}. \quad (18) \]

If we now change coordinates to:

\[ T = t\psi, \quad R = t^{1/\alpha}, \quad \Psi = At^{zA} \psi, \quad (19) \]
we find $u^2 = u^3 = u^4 = 0$ and:

$$u^0 = \mp \frac{\sqrt{2\alpha - 1}}{\alpha}, \quad u^1 = \mp \frac{1}{\sqrt{2\alpha - 1}} \frac{R}{T},$$

where $g^{00} = (2\alpha - 1)/\alpha^2 = \text{const.}$ The density and pressure (17) change to:

$$8\pi \rho_t = \frac{3}{\alpha^2 T^2}, \quad 8\pi P = \frac{2\alpha - 3}{\alpha^2 T^2}.$$ (21)

Energy density and pressure are identical to their 4D values for radiation and dust, without $\psi$ factor. The presence or absence of the latter, and the question of whether $u^4$ is zero or not, clearly depends on the choice of coordinates. So the functional form of $\rho_t$ and $P$ can change depending on the choice of the fifth coordinate. This feature means that a 5D model may take different 4D guises depending on the coordinate frame. A particularly interesting consequence of 5D covariance may be derived by considering the simple coordinate transformation:

$$T = \left(\frac{\alpha}{2}\right) t^{1/\alpha} \psi^{1/(1-\alpha)} \left(1 + \frac{r^2}{\alpha^2}\right) - \frac{\alpha}{2(1-2\alpha)} \left[t^{-1} \psi^{\alpha/(1-\alpha)}\right]^{(1-2\alpha)/\alpha},$$

$$R = r t^{1/\alpha} \psi^{1/(1-\alpha)},$$

$$\Psi = \left(\frac{\alpha}{2}\right) t^{1/\alpha} \psi^{1/(1-\alpha)} \left(1 - \frac{r^2}{\alpha^2}\right) + \frac{\alpha}{2(1-2\alpha)} \left[t^{-1} \psi^{\alpha/(1-\alpha)}\right]^{(1-2\alpha)/\alpha}.$$ (22)

Then the line element (16) becomes:

$$ds^2 = -dT^2 + dR^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) - d\Psi^2,$$ (23)

which is manifestly flat. This surprising result may be verified by computer, which shows that all of the components of the Riemann-Christoffel tensor for the 5D metric (16) are zero. Despite this, the model’s 4D part is not flat, since the 4D Ricci scalar may be calculated to be $6(\alpha - 2)/(\alpha^2 t^2 \psi^2)$. We see that while the universe may be curved in 4D, it is flat in 5D. If we have a conclusion in the space time matter theory is that one extra dimension is enough to explain the phenomenological properties of classical matter. For a complete treatment you must to see [12].

**III. THE EVOLUTION OF THE UNIVERSE FROM NONCOMPACT KALUZA KLEIN THEORY.**

In a cosmological context, the energy density of scalar fields has been reconized to contribute to the expansion of the universe [13], and has been proposed to explain inflation [14], as well as the presently observed accelerated expansion [15]. The observed isotropy and homogeneity of the universe do not allow for the existence of long-range electric and magnetic fields, but neutral scalar fields can have non-trivial dynamics in an expanding FRW-type universe. An attempt to confront the data with the predictions for a minimally coupled scalar field with an a priori unknown potential was made recently [16].
A very important question in theoretical physics consists to provide a good geometrical description of matter using only one extra coordinate (say $\psi$). The explanation of this issue in the framework of the early universe, in particular for inflationary theory [14], should be of great importance in cosmology. In this section, we are aimed to study this topic using the Kaluza-Klein formalism where the fifth coordinate is noncompact. In this framework should be interesting to explain the origin of an effective four dimensional (4D) scalar potential $V(\varphi)$ which could be originated from a 5D apparent vacuum. For example, an attempt to understand inflation [which is governed by the neutral scalar (inflaton) field], from a 5D flat Riemannian manifold was made in [17]. During inflation, the scale factor of the universe accelerates and this acceleration is driven by the potential energy related with the self-interactions of a scalar field. However, Campell’s theorem implies that all inflationary solutions can be generated, at least in principle, from a 5D vacuum Einstein gravity [12]. But, could be possible to develop a formalism to describe all the evolution of the universe? The other aim of this section consists to develop a 5D mechanism inspired in the Campbell’s theorem, to explain the (neutral scalar field governed) evolution of the universe from a initially inflationary (superluminal) expansion that has a change of phase towards a decelerated (radiation and later matter dominated) expansion and thereinafter evolves towards the present day observed accelerated expansion (quintessence) [18]. Although Campell’s theorem relates $N$-dimensional theories to vacuum $(N+1)$-dimensional theories, it does not establish a strict equivalence between them. It is therefore important to determinate when such theories are equivalent. Two notions of equivalence that could be considered are dynamical equivalence and geodesic equivalence. Dynamical equivalence would imply that the dynamics of vacuum $N$-dimensional theories is included in a vacuum $(N+1)$-dimensional theories. Alternatively, one may consider geodesic equivalence, in the sense on Mashhoon et al. [19]. In this case the $(3 + 1)$ geodesic equation induces a $(2 + 1)$ geodesic equation plus a force (per unity of mass) term $F_C^C$:

$$\frac{dU_C}{dS} + \Gamma^C_{AB} U^A U^B = F_C^C.$$ 

For the geodesic equivalence approach one would therefore require $F_C^C = 0$, that describes the trajectories of free-falling observers. In this work we shall use the geodesic equivalence approach.

In the last years extra force and extra mass has been subject of study [20]. It should be an observable effect from extra dimensions on the 4D spacetime. The final aim of this section is to extend the Hamilton-Jacobi formalism developed by Ponce de Leon [21] to cosmological models where the expansion of the universe is governed by a single inflaton field. This interpretation has the advantage of being free of the complications and ambiguities of the geodesic approach.

A. Formalism

To make it, we consider the 5D metric introduced by Ledesma and Bellini [22]

$$dS^2 = \epsilon \left( \psi^2 dN^2 - \psi^2 e^{2N} dr^2 - d\psi^2 \right).$$

(24)
Where $dr^2 = dx^2 + dy^2 + dz^2$. Here, the coordinates $(N,r)$ are dimensionless, the fifth coordinate $\psi$ has spatial unities and $\epsilon$ is a dimensionless parameter that can take the values $\epsilon = 1, -1$. The metric (24) describes a flat 5D manifold in apparent vacuum ($G_{AB} = 0$). We consider a diagonal metric because we are dealing only with gravitational effects, which are the important ones in the global evolution for the universe. To describe neutral matter in a 5D geometrical vacuum (24) we can consider the Lagrangian

$$\mathcal{L}^{(5)}(\varphi, \varphi, A) = -\sqrt{|^{(5)}g|^{(5)}g_0}^{(5)} \mathcal{L}(\varphi, \varphi, A),$$

(25)

where $|^{(5)}g| = \psi^8 e^{6N}$ is the absolute value of the determinant for the 5D metric tensor with components $g_{AB}$ and $|^{(5)}g_0| = \psi^8 e^{6N_0}$ is a constant of dimensionalization determined by $|^{(5)}g|$ evaluated at $\psi = \psi_0$ and $N = N_0$. In this work we shall consider $N_0 = 0$, so that $^{(5)}g_0 = \psi^8$. Here, the index "0" denotes the values at the end of inflation. Furthermore, we shall consider an action

$$I = -\int d^4x \ d\psi \sqrt{|^{(5)}g|^{(5)}g_0}^{(5)} \left[ \frac{^{(5)}R}{16\pi G} +^{(5)} \mathcal{L}(\varphi, \varphi, A) \right],$$

(26)

for a scalar field $\varphi$, which is minimally coupled to gravity. Here, $^{(5)}R$ is the 5D Ricci scalar, which of course, is zero for the 5D flat metric (24) and $G$ is the gravitational constant. Since the 5D metric (24) describes a manifold in apparent vacuum, the density Lagrangian $\mathcal{L}$ in (25) is

$$(^{(5)}g_0)^{\frac{1}{2}} g^{AB} \varphi_A \varphi_B,$$

(27)

which represents a free scalar field. In other words, we define the vacuum as a purely kinetic 5D-lagrangian on a globally 5D-flat metric [in our case, the metric (24)]. Taking into account the metric (24) and the Lagrangian (25), we obtain the equation of motion for $\varphi$

$$\left( 2\psi \frac{\partial \psi}{\partial N} + 3\psi^2 \right) \frac{\partial \varphi}{\partial N} + \psi^2 \frac{\partial^2 \varphi}{\partial N^2} \varphi - \psi^2 e^{-2N} \nabla^2 \varphi - 4\psi^3 \frac{\partial \varphi}{\partial \psi} \frac{\partial^2 \varphi}{\partial \psi^2} = 0,$$

(28)

where $\frac{\partial N}{\partial \psi}$ is zero because the coordinates $(N, \vec{r}, \psi)$ are independents.

In this work we shall consider the case where $N = N(t)$. The relevant Christoffel symbols for the geodesic of the 5D metric (24) in a 3D comoving frame $U^r = 0$, are

$$\Gamma^N_{\psi\psi} = 0, \quad \Gamma^N_{\psi N} = 1/\psi, \quad \Gamma^N_N = \psi, \quad \Gamma^\psi_{N\psi} = 0,$$

(29)

so that the geodesic dynamics $\frac{dU^C}{ds} = \Gamma^C_{AB} U^A U^B$ is described by the following equations of motion for the velocities $U^A$

$$\frac{dU^N}{ds} = -\frac{2}{\psi} U^N U^\psi,$$

(30)

$$\frac{dU^\psi}{ds} = -\psi U^N U^N,$$

(31)

$$\psi^2 U^N U^N - U^\psi U^\psi = 1,$$

(32)
where the eq. (32) describes the condition $g_{AB}U^A U^B = 1$. From the general solution $\psi^N = \cosh[S(N)], \ U^\psi = -\sinh[S(N)]$, we obtain the equation that describes the geodesic evolution for $\psi$

$$ \frac{d\psi}{dN} = \frac{U^\psi}{U^N} = -\psi \tanh[S(N)]. $$

(33)

If we take $\tanh[S(N)] = -1/u(N)$, we obtain the velocities $U^A$:

$$ U^\psi = -\frac{1}{\sqrt{u^2(N) - 1}}, \quad U^r = 0, \quad U^N = \frac{u(N)}{\psi \sqrt{u^2(N) - 1}}, $$

(34)

which are satisfied for $S(N) = \pm |N|$. In this work we shall consider the case $S(N) = |N|$. In this representation $\frac{d\psi}{dN} = \psi/u(N)$. Thus, the fifth coordinate evolves as

$$ \psi(N) = \psi_0 e^{\int dN/u(N)}. $$

(35)

From the mathematical point of view, we are taking a foliation of the 5D metric (24) with $r$ constant. Hence, to describe the metric in physical coordinates we must to make the following transformations:

$$ t = \int \psi(N) dN, \quad R = r\psi, \quad L = \psi(N) e^{-\int dN/u(N)}, $$

(36)

such that for $\psi(t) = 1/h(t)$, we obtain the 5D metric

$$ dS^2 = \epsilon \left( dt^2 - e^{2\int h(t) dt} dR^2 - dL^2 \right), $$

(37)

where $L = \psi_0$ is a constant and $h(t) = \dot{b}/b$ is the effective Hubble parameter defined from the effective scale factor of the universe $b$. The metric (37) describes a 5D generalized FRW metric, which is 3D spatially flat [i.e., it is flat in terms of $\vec{R} = (X,Y,Z)$], isotropic and homogeneous. In the representation $(\vec{R}, t, L)$, the velocities $U^A = \frac{\partial \hat{\psi}^A}{\partial \hat{\psi}^B} U^B$, are

$$ U^t = \frac{2u(t)}{\sqrt{u^2(t) - 1}}, \quad U^R = -\frac{2r}{\sqrt{u^2(t) - 1}}, \quad U^L = 0, $$

(38)

where the old velocities $U^B$ are $U^N, U^r = 0$ and $U^\psi$ and the velocities $\dot{U}^B$ are constrained by the condition

$$ \hat{g}_{AB} \dot{U}^A \dot{U}^B = 1. $$

(39)

Furthermore, the function $u$ can be written as a function of time $u(t) = -\frac{b^2}{\dot{b}}$, where the overdot represents the derivative with respect to the time. The solution $N = \arctanh[1/u(t)]$ corresponds to a time dependent power-law expanding universe $h(t) = p(t)t^{-1}$, such that the effective scale factor go as $b \sim e^{\int p(t) dt}$. When $u^2(t) > 1$, the velocities $U^t$ and $U^R$ are real, so the condition (39) implies that $\epsilon = 1$. [Note that the function $u(t)$ can be related to the deceleration parameter $q(t) = -\ddot{b}/b^2$: $u(t) = 1/[1 + q(t)]$]. In such that case the expansion of the universe is accelerated ($\dot{b} > 0$). However, when $u^2 < 1$ the velocities $U^t$
and \( U^R \) are imaginary and the condition (39) holds for \( \epsilon = -1 \). In this case the expansion of the universe is decelerated because \( \ddot{b} < 0 \). So, the parameter \( \epsilon \) is introduced in the metric (37) to preserve the hyperbolic condition (39). Moreover, the coordinates \((\vec{R}, t, L)\) has physical meaning, because \( t \) is the cosmic time and \((\vec{R}, L)\) are spatial variables. Since the line element is a function of time \( t \) (i.e., \( S \equiv S(t) \)), the new coordinate \( R \) give us the physical distance between galaxies separated by cosmological distances: \( R(t) = r(t)/h(t) \). Note that for \( r > 1 \) \((r < 1)\), the 3D spatial distance \( R(t) \) is defined on super (sub) Hubble scales. Furthermore \( b(t) \) is the effective scale factor of the universe and describes its effective 3D euclidean (spatial) volume. Hence, the effective 4D metric is a spatially (3D) flat FRW one

\[
dS^2 \to ds^2 = \epsilon \left( dt^2 - e^2 \int h(t) dt dR^2 \right),
\]

and has a effective 4D scalar curvature \( ^{(4)}\mathcal{R} = 6(h + 2h^2) \). The metric (40) has a metric tensor with components \( g_{\mu \nu} \). The absolute value of the determinant for this tensor is \( |^{(4)}g| = (b/b_0)^6 \). Now we can make the same treatment to the density Lagrangian (27) and the differential equation (28). Using the transformations (36) we obtain

\[
^{(4)}\mathcal{L}[\phi(\vec{R}, t), \varphi, \mu(\vec{R}, t)] = \frac{1}{2} g^{\mu \nu} \varphi, \mu \varphi, \nu - \frac{1}{2} \left[ (Rh)^2 - \frac{b_0^2}{b^2} \right] \left( \nabla_R \phi \right)^2, \tag{41}
\]

and the equation of motion for \( \varphi \) yields

\[
\ddot{\varphi} + 3h \dot{\varphi} - \frac{b_0^2}{b^2} \nabla_R^2 \varphi + \left[ \left( 4h^3 \right) - \left( 3h \right) - \left( 3h^2 \right) \right] \dot{\varphi} + \left( \frac{b_0^2}{b^2} - h^2 R^2 \right) \nabla_R^2 \varphi \right] = 0. \tag{42}
\]

From eqs. (41) and (42), we obtain respectively the effective scalar 4D potential \( V(\varphi) \) and its derivative with respect to \( \varphi(\vec{R}, t) \) are

\[
V(\varphi) \equiv \frac{1}{2} \left[ (Rh)^2 - \left( \frac{b_0}{b} \right) \right] \left( \nabla_R \varphi \right)^2, \tag{43}
\]

\[
V'(\varphi) \equiv \left[ 4h^3 \right] - \left[ 3h \right] - \left[ 3h^2 \right] \dot{\varphi} + \left( \frac{b_0^2}{b^2} - h^2 R^2 \right) \nabla_R^2 \varphi, \tag{44}
\]

where the prime denotes the derivative with respect to \( \varphi \). The equations (41) and (42) describe the dynamics of the inflaton field \( \varphi(\vec{R}, t) \) in a metric (40) with a Lagrangian

\[
^{(4)}\mathcal{L}[\varphi(\vec{R}, t), \varphi, \mu(\vec{R}, t)] = - \sqrt{\left| ^{(4)}g \right|} \left[ \frac{1}{2} g^{\mu \nu} \varphi, \mu \varphi, \nu + V(\varphi) \right], \tag{45}
\]

where \( \left| ^{(4)}g_0 \right| = 1 \).

In this frame, the 4D energy density \( \rho_t \) and the pressure \( P \) are [22]

\[
8\pi G \rho_t = 3h^2, \tag{46}
\]

\[
8\pi G P = -(3h^2 + 2h). \tag{47}
\]

10
From the condition (39) we can differentiate some different stages of the universe. If \( u^2(t) = \frac{4r^2(b/b_0)^2 - 1}{3} > 1 \), we obtain that \( r \) can take the values \( r > 1 \) (\( r < 1 \)) for \( b/b_0 < 1 \) (\( b/b_0 > 1 \)), respectively. In this case \( q < 0 \), so that the expansion is accelerated. On the other hand if \( u^2(t) = \frac{4r^2(b/b_0)^2 - 1}{3} < 1 \), \( r \) can take the values \( r < 1 \) (\( r > 1 \)) for \( b/b_0 > 1 \) (\( b/b_0 < 1 \)), respectively. In this stage \( q > 0 \) and the expansion of the universe is decelerated, so that the function \( u(t) \) take the values \( 0 < u(t) < 1 \) and the velocities (38) become imaginary. Thus, the metric (40) shifts its signature from \((+, -, -, -)\) to \((-+, +, +, +)\). When \( u(t) = 1 \) the deceleration parameter becomes zero because \( \dot{b} = 0 \). At this moment the velocities (38) rotates synchronically in the complex plane and \( r \) take the values \( 1 < r < 1 \) or \( b/b_0 = 1 \), respectively.

On the other hand, \( V(\varphi) \) and \( V'(\varphi) \) can be written as a function of the old coordinates \((N, r, \psi)\) in the comoving frame \( U^r = 0 \)

\[
\begin{align*}
V(\varphi) & \equiv \frac{1}{2} \left[ r^2 - e^{-2N} \right] \frac{1}{r^2} \left( \frac{1}{\psi} \frac{\dot{\varphi}}{\dot{\psi}} \right)^2 , \\
V'(\varphi) & \equiv \left( 3 \frac{\dot{\psi}}{\dot{\psi}^3} - \frac{4}{\dot{\psi}} \frac{\dot{\varphi}}{\dot{\psi}} - 3 \frac{\dot{\varphi}}{\dot{\psi}} \right) + \left[ \left( \frac{a_0}{a} \right)^2 - 1 \right] \frac{\partial^2 \varphi}{\partial \psi^2} .
\end{align*}
\] (48)

Here, the overstar denotes the derivative with respect to \( N \). Note that \( \Delta N \) is the number of e-folds of the universe. To inflation solves the horizon/flatness problems it is required that \( \Delta N \geq 60 \) at the end of inflation.

At this point we can introduce the 4D Hamiltonian \( \mathcal{H} = \pi^0 \dot{\varphi} - (^{(4)}L) \), where the 4D Lagrangian is \( (^{(4)}L)(\varphi, \varphi_\mu) = \sqrt{\frac{(^{(4)}g)}{|^{(4)}g_0|}} ^{(4)}L(\varphi, \varphi_\mu) \) [see eq. (45)]:

\[
\mathcal{H} = \frac{1}{2} a_0^{-1} \left[ \frac{\dot{\varphi}^2}{a_0^2} + \frac{a_0^2}{a^2} (\nabla \varphi)^2 + 2V(\varphi) \right] .
\] (50)

Hence, we can define the effective 4D energy density operator \( \rho_t \) such that

\[
\rho_t = \frac{1}{2} \left[ \frac{\dot{\varphi}^2}{a_0^2} + \frac{b_0^2}{b^2} (\nabla \varphi)^2 + 2V(\varphi) \right] .
\] (51)

Hence, the 4D expectation value of the Einstein equation \( \langle H^2 \rangle = \frac{8\pi G}{3} \langle \rho_t \rangle \) on the 4D FRW metric (40), will be

\[
\langle H^2 \rangle = \frac{4\pi G}{3} \left( \dot{\varphi}^2 + \frac{b_0^2}{b^2} (\nabla \varphi)^2 + 2V(\varphi) \right) ,
\] (52)

where \( G \) is the gravitational constant and \( \langle H^2 \rangle \equiv \hbar^2 = \dot{b}^2 / b^2 \). Now we can make a semiclassical treatment [17] for the effective 4D quantum field \( \varphi(\vec{R}, t) \), such that \( \langle \varphi \rangle = \phi_c(t) \):

\[
\varphi(\vec{R}, t) = \phi_c(t) + \phi(\vec{R}, t) ,
\] (53)

For consistence we take \( \langle \phi \rangle = 0 \) and \( \langle \dot{\phi} \rangle = 0 \). With this approach the classical dynamics on the background 4D FRW metric (40) is well described by the equations
\[ \ddot{\phi}_c + 3 \frac{\dot{b}}{b} \dot{\phi}_c + V'(\phi_c) = 0, \quad (54) \]

\[ H_c^2 = \frac{8\pi G}{3} \left( \frac{\dot{\phi}_c^2}{2} + V(\phi_c) \right), \quad (55) \]

where \( H_c^2 = \dot{a}^2 / a^2 \) and the prime denotes derivative with respect to the field. In other words the scale factor \( a \) only takes into account the expansion due to the classical Hubble parameter, but the effective scale factor \( b \) takes into account both, classical and quantum contributions in the energy density: \( \frac{\dot{b}^2}{b^2} = \frac{8\pi G}{3} \langle \rho \rangle \). Since \( \dot{\phi}_c = -\frac{H_c'}{4\pi G} \), from eq. (55) we obtain the classical scalar potential \( V(\phi_c) \) as a function of the classical Hubble parameter \( H_c \)

\[ V(\phi_c) = \frac{3M_p^2}{8\pi} \left[ H_c^2 - \frac{M_p^2}{12\pi} (H_c')^2 \right], \]

where \( M_p = G^{-1/2} \) is the Planckian mass. The quantum dynamics is described by

\[ \langle H^2 \rangle = H_c^2 + \frac{8\pi G}{3} \left\langle \frac{\dot{\phi}^2}{2} + \frac{b_0^2}{2b^2} (\nabla \phi)^2 + \sum_{n=1}^{\infty} \frac{1}{n!} V^{(n)}(\phi_c) \phi^n \right\rangle, \quad (56) \]

\[ \ddot{\phi} + 3 \frac{\dot{b}}{b} \phi - \frac{b_0^2}{b^2} \nabla^2 \phi + \sum_{n=1}^{\infty} \frac{1}{n!} V^{(n+1)}(\phi_c) \phi^n = 0, \quad (57) \]

In what follows we shall make the following identification:

\[ \Lambda(t) = 8\pi G \left\langle \frac{\dot{\phi}^2}{2} + \frac{b_0^2}{2b^2} (\nabla \phi)^2 + \sum_{n=1}^{\infty} \frac{1}{n!} V^{(n)}(\phi_c) \phi^n \right\rangle, \quad (58) \]

such that

\[ \frac{\dot{b}^2}{b^2} = \frac{\dot{a}^2}{a^2} + \frac{\Lambda}{3}. \quad (59) \]

On cosmological scales, the fluctuations \( \phi \) are small, so that it is sufficient to make a linear approximation \((n = 1)\) for the fluctuations. Thus, the second term in (59) is negligible on such scales. However, the second term in (59) could be important in the ultraviolet spectrum and more exactly at Planckian scales. At these scales the modes for \( \phi \) should be coherent and the matter inside these regions can be considered as dark. Hence, the significative contribution for the function \( \Lambda(t) \) is given by

\[ \Lambda(t) \approx 8\pi G \left\langle \frac{\dot{\phi}^2}{2} + \frac{b_0^2}{2b^2} (\nabla \phi)^2 + \sum_{n=1}^{\infty} \frac{1}{n!} V^{(n)}(\phi_c) \phi^n \right\rangle_{\text{Planck}}, \quad (60) \]

In this sense, we could make the identification for \( \Lambda \) as a cosmological parameter which only takes into account the “coherent quantum modes” (or dark matter) contribution for the expectation value of energy density: \( \langle \rho \Lambda \rangle = \Lambda / (8\pi G) \).

Once done the linear approximation \((n = 1)\) for the semiclassical treatment (53), we can make the identification of the squared mass for the inflaton field \( m^2 = V''(\phi_c) \). Hence, after make a linear expansion for \( V'(\varphi) \) in eq. (44), we obtain
\[ V'(\phi_c) \equiv \left( 4 \frac{\dot{h}^3}{h} - 3 \frac{\dot{h}}{h} - 3 \frac{h^5}{h^2} \right) \dot{\phi}_c, \]  
(61) 

\[ m^2 \phi \equiv \left( 4 \frac{\dot{h}^3}{h} - 3 \frac{\dot{h}}{h} - 3 \frac{h^5}{h^2} \right) \frac{\partial \phi}{\partial t} + \left( b_0^2 - h^2 R^2 \right) \nabla_R^2 \phi. \]  
(62)

Taking into account the expressions (54) with (61) and (57) with (62), we obtain the dynamics for \( \phi_c \) and \( \phi \). Hence, the equations  
\[ \ddot{\phi}_c + 3h \dot{\phi} + V'(\phi_c) = 0 \]  
and  
\[ \ddot{\phi} + 3h \dot{\phi} - \frac{(b/b_0)^2}{h^2} \nabla_R^2 \phi + V''(\phi_c) \phi = 0 \]  
now take the form [24]

\[ \ddot{\phi}_c + [3h + f(t)] \dot{\phi}_c = 0, \]  
(63) 

\[ \ddot{\phi} + [3h(t) + f(t)] \dot{\phi} - h^2 R^2 \nabla_R^2 \phi = 0, \]  
(64)

where

\[ f(t) = \left( 4 \frac{\dot{h}^3}{h} - 3 \frac{\dot{h}}{h} - 3 \frac{h^5}{h^2} \right). \]  
(65)

**B. Examples**

To illustrate the formalism we shall consider two examples. The first one is an application of this formalism to construct a simple inflationary model, and the second one is a proposal cosmological model in which we consider the cosmological constant, including the inflationary era.

1. **Inflation with \( \Lambda = 0 \)**

On cosmological scales and during inflation, the quantum fluctuations are small, so that the linear approximation in eq. (57) is sufficient to make a realistic description for the evolution of \( \phi \). Furthermore, the cosmological parameter \( \Lambda(t) \) is negligible during inflation when \( \phi \) is considered 3D spatially homogeneous. However, such term could be important in other times of the evolution of the universe. Taking this into account in eq. (59) the effective scale factor \( b(t) \) is equals to the classical scale factor \( a(t) \), and the same is for the effective hubble parameter \( h(t) \) and the classical hubble parameter \( H_c(t) \). During the inflationary epoch, the slow-roll condition \( \gamma(t) = -\dot{H}_c/H_c^2 \ll 1 \) holds [23]. Since \( u(t) = 1/\gamma(t) \), we obtain that \( u \gg 1 \). This assures that all the velocities in \( U^A \) in (34) and \( \dot{U}^A \) in (38) to be real, and imposes the condition \( r \gg 1 \) [24]. Furthermore the equation of state can be written in terms of the function \( u(t) \)

\[ \langle P \rangle = -\left[ 1 - \frac{2}{3u(t)} \right] \langle \rho_t \rangle, \]

which, since \( u \gg 1 \) during inflation, complies with the required condition for this stage: \( \langle P \rangle \simeq -\langle \rho_t \rangle \). Moreover, speaking in terms of the effective 4D FRW metric (40), the geodesic evolution of the fifth coordinate give us the Hubble horizon \( \psi(t) = 1/H(t) \) and the resulting
fifth (constant) coordinate \( L = \psi_0 \) is given by the Hubble horizon at the end of inflation: \( L = 1/H_c(t_0) \).

We can define the redefined quantum fluctuations \( \chi(\vec{R}, t) = e^{1/2 \int \left[ 3H_c(t) + f(t) \right] dt} \phi \), so that the equation of motion for \( \chi \) yields
\[
\ddot{\chi} - \left[ H_c^2 R^2 \nabla^2_R + \frac{1}{4} (3H_c + f(t))^2 + \frac{1}{2} \left( 3H_c + f(t) \right) \right] \chi = 0,
\]
so that the modes \( \chi_k(t) \) of the field \( \chi \) complies the differential equation
\[
\ddot{\chi}_k + H_c^2 R^2 \left( k^2 - k_0^2(t) \right) \chi_k = 0,
\]
with
\[
k_0^2(t) = \frac{1}{R^2 H_c^2} \left[ \frac{1}{4} (3H_c + f(t))^2 + \frac{1}{2} \left( 3H_c + f(t) \right) \right],
\]
where \( f(t) \) is a function of the classical Hubble parameter [see eq. (65)]. Hence, all the dynamics of the quantum fluctuations being described only by the classical Hubble parameter \( H_c = \dot{a}/a \).

Now we are going to study an example where \( \psi(N) = -1/(\alpha N) \), so that \( H_c(N) = -\alpha N \). This implies that the classical Hubble parameter (written as a function of time) is given by \( H_c(t) = H_0 e^\alpha \Delta t \). At the end of inflation \( \alpha \Delta t \ll 1 \), so that \( H_c(t) \approx H_0(1 + \alpha \Delta t) \) and \( 3H_c(t) + f(t) \approx 3H_0(1 + \alpha \Delta t) + 3\alpha - (4H_0^2/\alpha)(1 + 2\alpha \Delta t) - (3H_0^3/\alpha^2)(1 + 3\alpha \Delta t) \), where \( \Delta t = t_0 - t \) and \( t_0 \) is the time for which inflation ends. At the end of inflation it is sufficient to make a \( \Delta t \)-first order expansion for \( k_0^2 \), so that it can be approximated to
\[
k_0^2(t) = \frac{1}{r^2} (A_1 - A_2 t).
\]
With this approximation, the general solution for the modes \( \chi_k(t) \) is
\[
\chi_k(t) = C_1 \text{Ai} [x(t)] + C_2 \text{Bi} [x(t)],
\]
where \( \text{Ai} [x(t)] \) and \( \text{Bi} [x(t)] \) are the Airy functions with argument \( x(t) \). Furthermore, \((C_1,C_2)\) are some constants and
\[
A_1 = \frac{1}{4} \left[ 3H_0 - 3 \frac{H_0^3}{\alpha^2} + \alpha - 3 \frac{H_0^2}{\alpha} \right]^2 + \frac{1}{2} \left( 8H_0^2 - 9 \frac{H_0^3}{\alpha} - \alpha H_0 \right)\]
\[
- \frac{1}{2} \left( 3H_0 - 3 \frac{H_0^3}{\alpha^2} + 3\alpha - 8 \frac{H_0^2}{\alpha} \right) \left( 8H_0^2 + 9 \frac{H_0^3}{\alpha} - 3H_0\alpha \right) t_0,
\]
\[
A_2 = \frac{1}{2} \left( 3H_0 - 3 \frac{H_0^3}{\alpha^2} + 3\alpha - 8 \frac{H_0^2}{\alpha} \right) \left( 3H_0\alpha - 8H_0^2 - 9 \frac{H_0^3}{\alpha} \right),
\]
\[
x(t) = \frac{[A_1 - k^2 - A_2 t]}{A_2} \left( \frac{A_2}{r^2} \right)^{1/3}.
\]
Note that in this example \( H_0 \) denotes the value of the Hubble parameter at the end of inflation. On cosmological scales (i.e., for \( k^2 \ll A_1 - A_2 t \)), the solution for \( \chi_k \) is unstable.
However in the UV sector (i.e., for $k^2 \gg A_1 - A_2 t$), the modes oscillate. This behavior is well described by the function $\text{Bi}[x(t)]$, so that we shall take $C_1 = 0$. Hence, at the end of inflation the modes $\chi_k$ will be

$$\chi_k(t) = C_2 \text{Bi}[x(t)].$$

(74)

Since the modes of the quantum fluctuations $\phi$ are $\phi_k = e^{-1/2 \int [3H_c + f(t)] dt} \chi_k$, the squared fluctuations are given by

$$\langle \phi^2 \rangle \simeq \frac{1}{2\pi^2} e^{-\left[3(H_0 + \alpha) - \frac{H_c^2}{2} - \frac{3H_0^2}{2}\right] t} \int dk \ k^2 \left| \chi_k \right|^2,$$

(75)

where the modes $\chi_k$ are given by eq. (74). Furthermore the density fluctuations at the end of inflation can be estimated by the expression

$$\frac{\delta \rho_t}{\rho_t} \sim \frac{H_c^2}{\phi_c} \sim 2\pi^{1/2} \frac{H_0^{3/2}}{M_p \alpha^{1/2}},$$

(76)

which are of the order of $10^{-5}$ for $H_0 \sim 10^{-5} M_p$ and $\alpha \sim 10^{-5} M_p$. In our case, the spectral index $n_s$ being given by $n_s - 1 = -\frac{6}{u(t)}$. During inflation $u \gg 1$, so that $|n_s - 1| \ll 1$. Hence, during inflation the spectrum approaches very well with a Harrison - Zeldovich one.

2. A more general cosmological model with $\Lambda \neq 0$

As a second example we propose a cosmological model without the above consideration about the cosmological parameter $\Lambda$, because in this model we are considering $\Lambda$ as a constant. That implies the effective Hubble parameter is different to the classical Hubble parameter ($h \neq H_c$). Taking this into account we consider a time dependent power expansion $p(t) = 2/3 - Bt^{-1} + At^{-2}$, such that the classical Hubble parameter is given by $H_c(t) = p(t)/t$ and $(A,B)$ are constants. The effective power $p_1(t)$ for the effective Hubble parameter $h(t)$ will be $p_1(t) = \sqrt{[2/3 + At^{-2} - Bt^{-1}]^2 + \Lambda/3t^2}$, because $h^2 = H_c^2 + \Lambda/3$. This implies that the total density parameter will be $\Omega_T = \Omega_r + \Omega_m + \Omega_\Lambda = 1$, for a critical energy density given by $\rho_t = \frac{3}{8\pi G} h^2$, such that

$$\Omega_r + \Omega_m = \frac{H_c^2}{h^2}, \quad \Omega_\Lambda = \frac{\Lambda}{3h^2},$$

(77)

where $\Omega_r$, $\Omega_m$ and $\Omega_\Lambda$ are respectively the contributions for radiation, matter and $\Lambda$. In our case, because we consider $\Omega_T = 1$, this implies that

$$p_1^2(t) = p^2(t) + \frac{1}{3} \Lambda t^2,$$

(78)

where $t > 0$ is the cosmic time. We define $b/b_0 = e^N$, such that $b_0 \equiv b(t = t_0)$, where $t_0$ is the time when inflation ends (i.e., the time for which $\dot{b} = 0$). Thus $N$ will be greater than zero only for times larger than $t_0$, but negative for $t < t_0$ (i.e., during the previous inflationary phase). This means that the parameter $N$ give us the number of e-folds with respect to the
The interesting here is that the velocities (38) becomes purely imaginary and the signa-
tion and later by matter. The function $u(t)$ evolves from an accelerated to a decelerated expansion and $r = 1$, because $b(t = t_0) = b_0$.

During the second phase (i.e., decelerated expansion) the universe is governed by radiation and later by matter. The function $u^2(t)$ is smaller than the unity (but $u^2 > 0$), so that $r$ take values $\frac{1}{2}b^2 e^{-N} = \frac{1}{2}b_0/b < r < 1$, for $N > 0$. This means that, during this phase, the metric (40) describes the universe on scales smaller than the Hubble radius: $r/h < 1/h$. The interesting here is that the velocities (38) becomes purely imaginary and the signa-
ture of the 4-D effective metric (40) changes synchronically (with re-
spect to the signature during the inflationary phase): $(+, -, -, -) \rightarrow (-, +, +, +)$; that is, $\epsilon$ jumps from 1 to $-1$ to preserve the global geometry in (39). In this sense we can say that the 4-D effective metric (40) is “dynamical”. The fig. (1) shows the evolution of the powers $p_1[x(t)]$ (dashed line) and $p[x(t)]$ (continuous line) as a function of $x(t) = \log_{10}(t)$ for $A = 1.5 \times 10^{30}$ G$^1$ and $B = 10^{15}$ G$^{1/2}$. Numerical calculations give us the time for which $\dot{b} = q = 0$ at the end of inflation: $x(t_0) \approx 14.778$. At this moment $N(t_0) = 0$, but after it becomes positive. Note that for $x(t) < 60.22$ both curves are very similar, but for $x(t) > x(t_0)$ (with $x(t_0) \approx 60.22$), $p_1$ increases very rapidly but not $p$, which remains almost constant with a value close to $p \approx 2/3$. The difference between both curves is due to the presence of a nonzero “cosmological constant” ($\Lambda$), which was valued as: $\Lambda = 1.5 \times 10^{-121}$ G$^{-1}$. In other words, at $t_0 \approx 1.66 \times 10^{60}$ G$^{1/2}$ the deceleration parameter becomes zero and later negative. At this mo-
time, the universe changes from a decelerated to an accelerated phase and $\epsilon$ jumps from $-1$ to 1 because $u(t)$ evolves from $u(t < t_0) < 1$ (decelerated expansion) to $u(t > t_0) > 1$ (acceler-
ated expansion). It should be when the universe was nearly 0.4 $10^{10}$ years old. The present day age of the universe was considered as $x(t) = 60.653$ G$^{1/2}$ (i.e., 1.5 $10^{10}$ light years). Note that $\Omega_r + \Omega_m$ decreases for late times [see figure (2)], so that its present day value should be $(\Omega_r + \Omega_m)[x(t = 60.653$ G$^{1/2})] \approx 0.32$. Thus, the present day value for the vacuum density parameter $\Omega_\Lambda = 1 - (\Omega_r + \Omega_m)$ should be $\Omega_\Lambda[x(t = 60.653$ G$^{1/2})] \approx 0.68$. With these parameter values we obtain the present day deceleration parameter: $q[x(t = 60.653$ G$^{1/2})] \approx -0.747$, so that the present day cosmological parameter should be: $\omega[x(t = 60.653$ G$^{1/2})] \approx -0.831$. Note that all these results are in very good agreement with observation [28,27].

Evolution of $p_1[x(t)]$ (dashed line) and $p[x(t)]$ (continuous line) as a function of $x(t) =

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1 At the moment the consensus has emerged about the experimental value of the cosmological constant [25,26]. It is on the order of magnitude of the matter energy density: $\rho_\Lambda \sim (2 - 3)\rho_m$. The Wilkinson Microwave Anisotropy Probe (WMAP) data suggest that the universe is very nearly spatially flat, with a density parameter $\Omega_r = 1.02 \pm 0.02$ [27].
log_{10}(t), for \(A = 1.5 \times 10^{30} \text{G}^{1}, B = 10^{15} \text{G}^{1/2}\).

Evolution of \((\Omega_{m} + \Omega_{r})[x(t)]\) as a function of \(x(t) = \log_{10}(t)\), for \(A = 1.5 \times 10^{30} \text{G}^{1}, B = 10^{15} \text{G}^{1/2}\).

IV. EXTRA FORCE AND EXTRA MASS

As we saw in the previous section, it is possible to consider a cosmological model governed by a neutral scalar field that initially suffers an inflationary expansion that has a change of phase towards a decelerated (radiation and later matter dominated) expansion that thereinafter evolves towards the observed present day (quintessential) expansion. In this section we shall study the possibility to have fifth force and fifth mass making an extension of the formalism developed by Ponce de Leon to cosmological models. In particular we shall extend this formalism to the cosmological model developed in the previous section considering two frames. This way to obtain equations of practical use, we can introduce the action \(S(x^{A})\) as a function of the generalyzed coordinates \(x^{A}\). Hence, since the momentum \(\mathcal{P}^{A} = -\frac{\partial S}{\partial x^{A}}\), for a diagonal tensor metric \(g^{AB}\) we obtain the Hamilton-Jacobi equation

\[
g^{AB} \frac{\partial S}{\partial x^{A}} \frac{\partial S}{\partial x^{B}} = M_{(5)}^{2},
\]

where \(M_{(5)}\) is the invariant 5D gravitational mass of the object under study (in our case, the mass of the inflaton field). In the particular frame (34), with the Lagrangian (25) and (27), \(M_{(5)}\) describes the 5D mass of the scalar field \(\varphi\). In this case the tensor metric is symmetric (and diagonal), and the Hamilton-Jacobi equation (79) adopts the particular form

\[
g^{\varphi N} \left( \frac{\partial S}{\partial \varphi, N} \right)^{2} + g^{\psi \psi} \left( \frac{\partial S}{\partial \varphi, \psi} \right)^{2} = M_{(5)}^{2},
\]

On the other hand, in general, the line element (24) can be written as:

\[
dS^{2} = ds^{2} + dS_{(4)}^{2},
\]

where \(ds^{2}\) describes the 4D line element and \(dS_{(4)}^{2}\) only the line element related with the fifth coordinate. We shall define the extra force
\[ F_{\text{ext}} = \frac{d\mathcal{P}^{x^4}}{ds}, \]  

(82)

as the force on the sub manifold \( ds^2 \) due to the motion of the fifth coordinate. In general, the momentum \( \mathcal{P}^{x^4} \) is defined as

\[ \mathcal{P}^{x^4} = \frac{\partial(5) L}{\partial \psi, x^4}, \]  

(83)

In the frame (34) \( \mathcal{P}^{x^4} \equiv \mathcal{P}^{\psi} \), and is given by

\[ \mathcal{P}^{\psi} = -\frac{\psi^4 e^{3B}}{\psi_0} \left( g^{\psi \psi} \right)^2 \psi, \]  

(84)

Hence, the extra force holds

\[ F_{\text{ext}} = \frac{\psi^3 e^{3N}}{\psi_0^2} \left( 3^{*} \psi, \psi + 3 \psi, \psi + \psi, \psi^* \right), \]  

(85)

where the overstar denotes the derivative with respect to \( N \).

On the other hand, from the equation \( g_{AB} U^A U^B = 1 \), we obtain the invariant 5D mass \( M_{(5)} \)

\[ g_{AB} \mathcal{P}^A \mathcal{P}^B = M_{(5)}^2, \]  

(86)

where \( \mathcal{P}^A = M_{(5)} U^A \). For example, in the frame (34) the 4D mass \( m_0 \) and the 5D invariant mass \( M_{(5)} \) are given respectively by

\[ M_{(5)}^2 = g_{NN} \left( \frac{\partial S}{\partial \phi, N} \right)^2 + g^{\psi \psi} \left( \frac{\partial S}{\partial \psi, \psi} \right)^2, \quad m_0^2 = g_{NN} \left( \frac{\partial S}{\partial \phi, N} \right)^2, \]  

(87)

so that its difference

\[ m_0^2 - M_{(5)}^2 = -g^{\psi \psi} \left( \frac{\partial S}{\partial \psi, \psi} \right)^2, \]  

(88)

is nonzero. The interesting here is that \( m_0^2 > M_{(5)}^2 \). In other words, in the frame (34) the motion of the fifth coordinate has an antigravitational effect on the field \( \phi \) in the submanifold (or bulk) \( ds^2 \). However, this frame is not very instructive because \( N \) and \( r \) are not physical coordinates. Next we shall study some examples which could be relevant in cosmological models. The first one is the case of the cosmological model developed previously seen from the frame \( (t, R, L) \) defined by the speeds (38). It is easy to see that in this frame the 5D momentum \( \mathcal{P}^L \) is null: \( \mathcal{P}^L = 0 \). This implies that the extra force will be

\[ F_{\text{ext}} = 0. \]  

(89)

It also can be viewed from the point of view of the extra mass. In this frame \( m_0^2 = M_{(5)}^2 \) where
\[
\left( \frac{\partial S}{\partial \varphi, t} \right)^2 - e^2 \int h(t) dt \left( \frac{\partial S}{\partial \varphi, R} \right)^2 = M_{(5)}^2. \quad (90)
\]

Hence, the inertial 4D mass \( m_0 \) is the same than the invariant 5D mass \( M_{(5)} \), so that there is not extra force on the effective 4D frame. This can be justified from the fact that the fifth coordinate \( L \) do not varies in this frame. In other words the 4D bulk \( ds^2 \) is the same that the 5D manifold \( dS^2 \), because \( dS_{(4)} = 0 \) for an observer that “expands with the universe” in an inertial frame.

Other interesting frame it is that whose fifth coordinate is variable. This can be described by means of the transformation \( t = \int \psi(N) dN, \ R = r\psi \) and \( \xi = \psi(N)e^{\int H(N)/H(N)dN} \), so that the 5D velocities are

\[
\begin{align*}
U^t &= \frac{2u(t)}{\sqrt{u^2(t) - 1}}, \\
U^R &= -\frac{2r}{\sqrt{u^2(t) - 1}}, \\
U^\xi &= \frac{u(t)}{\sqrt{u^2(t) - 1}} \left( \frac{\dot{H}}{hH} - \frac{\dot{h}}{h^2} \right) \frac{H}{H_0}.
\end{align*}
\]

In this frame the 5D line element is given by [29]

\[
dS^2 = \epsilon \left[ dt^2 - e^2 \int h(t) dt dR^2 - \left( \frac{H_0}{H} \right)^2 d\xi^2 \right],
\]

where the 4D line element (or “bulk”) \( ds^2 \) is given by the first two terms in (94)

\[
ds^2 = \epsilon \left( dt^2 - e^2 \int h(t) dt dR^2 \right),
\]

and \( h^2(t) = H^2(t) + \frac{C}{3} \) for a given constant \( C \). Hence, the extra force on the 4D bulk will be \( F^\text{ext} = \frac{dP^\xi}{dt} \). Note that extra force becomes from the motion of the fifth coordinate in the effective 4D bulk. In other words, an observer in the 4D bulk (95), will move under the influence of an extra force that, in the example here studied, takes the form

\[
F^\text{ext} = \left( 1 - \frac{r^2 \dot{h}^2}{h^4 e^2 \int h dt} \right)^{-1/2} \frac{dP^\xi}{dt}, \quad (96)
\]

which is invariant under changes of signature (i.e., \( \epsilon = 1 \to \epsilon = -1 \)). The 5D Lagrangian in this frame takes the form

\[
^{(5)}L(\varphi, \varphi, A) = - \left( \frac{b}{b_0} \right)^3 \frac{H_0}{H} \left( \frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi, \partial_\beta + V(\varphi) \right), \quad (97)
\]

so that the momentum \( P^\xi \) is

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In this representation the potential \( V(\varphi) \) assumes the form

\[
V(\varphi) = \frac{1}{2} \left( (R_{\varphi})^2 - \left( \frac{b_0}{b} \right)^2 \right) (\nabla_R \varphi)^2 - \frac{1}{2} \left( \frac{H}{H_0} \right)^2 \varphi^2 - \frac{H}{H_0} (R_{\varphi}) \varphi \xi \nabla_R \varphi, \tag{99}
\]

so that the momentum \( P^\xi \) is

\[
P^\xi = \left( \frac{b}{b_0} \right)^3 \left[ \left( \frac{H}{H_0} \right) \varphi^\xi + (R_{\varphi}) \nabla_R \varphi \right]. \tag{100}
\]

Note that the effective kinetic component in the 5D Lagrangian (97) is 4D, but the potential (99) is evaluated in the 5D frame (91),(92),(93). From eqs. (96) and (100), we obtain the extra force for this frame

\[
F^{\text{ext}} = \left( \frac{b}{b_0} \right)^3 \left[ 1 - \left( \frac{R_{\varphi}}{H} \right)^2 \left( \frac{b}{b_0} \right)^2 \right]^{-1/2} \left[ \left( \frac{\dot{b} H}{b H_0} + \frac{\dot{H}}{H} \right) \varphi^\xi + \left( 3 \frac{\dot{b}}{b} (R_{\varphi}) + (\dot{R}_{\varphi} + \dot{R} \varphi) \right) \nabla_R \varphi \nabla_R \varphi + H \left[ \frac{d}{dt} (\varphi^\xi) + (R_{\varphi}) \frac{d}{dt} (\nabla_R \varphi) \right], \tag{101}
\]

where \( \left( \frac{b}{b_0} \right)^2 = e^{2 \int h dt} \). The extra force is originated in the last two terms of the 5D potential (99), which depends on the fifth coordinate \( \xi \).

On the other hand the 4D squared mass of the inflation field \( \varphi \) on the 4D bulk (95), is given by

\[
m_0^2 = \left( \frac{\partial S}{\partial \varphi, t} \right)^2 - e^{-2 \int h dt} \left( \frac{\partial S}{\partial \varphi, R} \right)^2, \tag{102}
\]

so that one obtains

\[
m_0^2 - M^2_{(5)} = \left( \frac{H}{H_0} \right)^2 \left( \frac{\partial S}{\partial \varphi^\xi} \right)^2, \tag{103}
\]

which gives \( m_0^2 \geq M^2_{(5)} \) because the right hand of the equation (103) is positive (for \( C > 0 \)). This is an important result which shows that the motion of the fifth coordinate has an antigravitational effect on an observer in a 4D bulk in which the inflaton field has a 4D mass \( m_0 \). This fact should be responsible for the extra force (101) because the observer “is placed” in a non inertial frame (or 4D bulk). In this framework the motion of the fifth coordinate is viewed on the bulk as an extra force. Note that it becomes zero as \( C \to 0 \), because in this limit \( U^{\xi} \to 0 \) and \( V(\varphi) \to \frac{1}{2} \left[ (R_{\varphi})^2 - \left( \frac{b_0}{b} \right)^2 \right] (\nabla_R \varphi)^2 \). On the other hand, \( U^{\xi} \to 0 \) as \( t \to \infty \), because \( \dot{H} < 0 \) (and \( \dot{h} < 0 \)) along all the history of the universe, such that \( \frac{H}{H_0} \to 0 \). Hence, for very late times the external force (101) on the bulk becomes negligible. However,
this force should be very important in the early universe when \( H/H_0 \gg 1 \) (note that \( H_0 \) is the value of the Hubble parameter at the end of inflation).

To illustrate the formalism we can consider the case where \( h(t) = t^{-1} p_1(t) \) and \( H(t) = t^{-1} p(t) \), where

\[
\begin{align*}
p_1(t) &= \sqrt{(2/3 + At^{-2} - Bt^{-1})^2 + \frac{C}{3} t^2}, \\
p(t) &= \sqrt{2/3 - Bt^{-1} + At^{-2}}
\end{align*}
\]

Here \( A = 1.5 \times 10^{30} \text{G}^4 \), \( B = 10^{15} \text{G}^{1/2} \) and we take the special case where the constant \( C \) is the cosmological constant \( \Lambda \): \( \Lambda = 1.5 \times 10^{-121} \text{G}^{-1} \). Furthermore, \( G = M_p^{-2} \) is the gravitational constant and \( M_p = 1.2 \times 10^{19} \text{GeV} \) is the Planckian mass. Numerical calculations give us the time for which \( \ddot{b} = \dot{q} = 0 \) at the end of inflation: \( x(t_0) \simeq 14.778 \) [we take \( x(t) = \log_{10}(t) \)]. At this moment \( N(t_0) = 0 \), but after it becomes positive. Furthermore, for \( x(t) > x(t_*) \) [with \( x(t_*) \simeq 60.22 \)], \( p_1 \) increases from the value \( p_1 \simeq 2/3 \) and the 4D bulk universe is accelerated.

\section*{V. CONCLUSIONS}

In this work we have studied a model for the evolution of the universe which is globally described by a single scalar field from 5D apparent vacuum. Such vacuum is described by a flat 5D metric with coordinates \((N,r,\psi)\) and a Lagrangian for a free and minimally coupled to gravity scalar field. The interesting is that the scalar potential \( V(\varphi) \) appears in the 3D comoving frame characterized by \( U_r = 0 \) [see eq. (48)]. A further transformation to physical coordinates \( t = \int \psi(N)dN, \quad R = r\psi \) and \( L = \psi e^{-\int dN/u(N)} \) give us the possibility to describe the system in an effective 4D (but 3D spatially flat) FRW metric. Such that metric is viewed as a particular frame (characterized with \( U^L = 0 \)), where the potential \( V(\varphi) \) is represented as the differential operator (43). In other words, the potential, which assume different representations in different frames, has a geometrical origin. Moreover, the mass of the inflaton field appears in the frame \( U^L = 0 \) as a differential operator applied to the quantum fluctuations \( \phi(\vec{R},t) \). Hence, for the semiclassical treatment here developed, \( m^2\phi(\vec{R},t) \) is a local operator with nonzero expectation value. At this point we must to exalt this result, because a particular frame in physics is intrinsically related to an observer (or experimental result).

This 5D formalism could be extended to other particular frames or quantum fields. Moreover, the evolution of the universe could be examined taking into account also electromagnetism by introducing off-diagonal terms in the metric (24), which should be relevant to study 3D spatial anisotropies in the universe on astrophysical scales. However, all these issues go beyond the scope of this work. Another important aspect that we have studied in this work is the possible origin of extra force and extra mass from a noncompact Kaluza-Klein formalism by using the Hamilton-Jacobi formalism in the framework of cosmological models. We have examined the inertial 4D mass \( m_0 \) of the inflaton field on a 4D FRW bulk in two examples. In the first one there is not motion of the fifth coordinate with respect to the 4D FRW bulk, so that the inertial mass \( m_0 \) is the same than the 5D gravitational mass \( M_5 \) of the inflaton field. As consequence of this fact there is not extra force on the 4D bulk
$ds^2$ because $dS^2 = ds^2$. However, in the second example antigravitational effects on a non inertial 4D bulk should be a consequence of the motion of the fifth coordinate with respect to this bulk, because $dS^2 \neq ds^2$ so that $m_0^2 > M_{(5)}^2$. This disagreement between the 4D inertial and 5D gravitational masses is viewed on the 4D bulk as an extra force. The important here is that $m_0$ has a geometrical origin and depends on the frame of the observer. However, $M_{(5)}$ is a 5D invariant gravitational mass and do not depends on the frame of the observer. This is the same situation as in the Randall-Sundrum brane-world scenario [30,31] and other non-compact Kaluza-Klein theories, where the motion of test particles is higher-dimensional in nature. In other words, all test particles travel on five-dimensional geodesics but observers, who are bounded to spacetime, have access only to the 4D part of the trajectory. Finally, in the cosmological model here studied, we find that both, the discrepancy between $m_0$ and $M_{(5)}$ and extra force, are bigger in the early universe [i.e., during inflation ($x(t) < 14.778$)], but become negligible for large (present day) times.
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