Introduction to the study of entropy in Quantum Games

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The present work is an introductory study about entropy its properties and its role in quantum information theory. In a next work, we will use these results to the analysis of a quantum game described by a density operator $\rho$ and with its entropy equal to von Neumann’s.

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I. INTRODUCTION

In a recent work [1] we proposed quantization relationships which would let us describe and solution problems originated by conflicting or cooperative behaviors among the members of a system from the point of view of quantum mechanical interactions. Through these relationships we could described a system through a density operator and its entropy would be given by the von Neumann entropy. The quantum version of the replicator dynamics is the equation of evolution of mixed states from quantum statistical mechanics.

Since Shannon [2], information theory or the mathematical theory of communication changed from an engineering discipline that dealt with communication channels and codes [2] to a physical theory [4] in where the introduction of the concepts of entropy and information were indispensable to our understanding of the physics of measurement. Classical information theory has two primary goals [5]: The first is the development of the fundamental theoretical limits on the achievable performance when communicating a given information source over a given communications channel using coding schemes from within a prescribed class. The second goal is the development of coding schemes that provide performance that is reasonably good in comparison with the optimal performance given by the theory.

Quantum information theory may be defined [6] as the study of the achievable limits to information processing possible within quantum mechanics. Thus, the field of quantum information has two tasks: First, it aims to determine limits on the class of information processing tasks which are possible in quantum mechanics and provide constructive means for achieving information processing tasks. Quantum information theory appears to be the basis for a proper understanding of the emerging fields of quantum computation [7, 8], quantum communication [9, 10], and quantum cryptography [11, 12]. Entropy is the central concept of information theories. The quantum analogue of entropy appeared 21 years before Shannon’s entropy and generalizes Boltzmann’s expression. It was introduced in quantum mechanics by von Neumann [13, 14]. Entropy in quantum information theory plays prominent roles in many contexts, e.g., in studies of the classical capacity of a quantum channel [15, 16] and the compressibility of a quantum source [17, 18].

II. SHANNON, ENTROPY AND CLASSICAL INFORMATION THEORY

Entropy [2, 19] is the central concept of information theory. In classical physics, information processing and communication is best described by Shannon information theory.

The Shannon entropy expresses the average information we expect to gain on performing a probabilistic experiment of a random variable $A$ which takes the value $a_i$ with the respective probability $p_i$. It also can be seen as a measure of uncertainty before we learn the value of $A$. We define the Shannon entropy of a random variable $A$ by

$$ H(A) \equiv H(p_1, \ldots, p_n) \equiv -\sum_{i=1}^n p_i \log_2 p_i. \quad (1) $$

The entropy of a random variable is completely determined by the probabilities of the different possible values that the random variable takes. Due to the fact that $p = (p_1, \ldots, p_n)$ is a probability distribution, it must satisfy $\sum_{i=1}^n p_i = 1$ and $0 \leq p_1, \ldots, p_n \leq 1$. The Shannon entropy of the probability distribution associated with the source gives the minimal number of bits that are needed in order to store the information produced by a source, in the sense that the produced string can later be recovered. Shannon formalized the requirements for an information measure $H(p_1, \ldots, p_n)$ with the following criteria:

1. $H$ should be continuous in the $p_i$.
2. If the $p_i$ are all equal, i.e. $p_i = 1/n$, then $H$ should be a monotonic increasing function of $n$.
3. $H$ should be objective: If a choice be broken down into two successive choices, the original $H$ should be the weighted sum of the individual values of $H$.

$$ H(p_1, \ldots, p_n) = H(p_1 + p_2, p_3, \ldots, p_n) + (p_1 + p_2)H(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}). \quad (2) $$
Suppose $A$ and $B$ are two random variables. The *joint entropy* $H(A, B)$ measures our total uncertainty about the pair $(A, B)$. The joint entropy $H(A, B)$ is defined by

$$H(A, B) = -\sum_{i,j} p_{ij} \log_2 p_{ij}$$

while

$$H(A) = -\sum_{i} p_{i} \log_2 p_{i},$$
$$H(B) = -\sum_{j} p_{j} \log_2 p_{j},$$

where $p_{ij}$ is the joint probability to find $A$ in state $a_i$ and $B$ in state $b_j$.

The *conditional entropy* $H(A \mid B)$ is a measure of how uncertain we are about the value of $A$, given that we know the value of $B$. The entropy of $A$ conditional on knowing that $B$ takes the value $b_j$ is defined by

$$H(A \mid B) = -\sum_{i} p_{ij} \log_2 p_{ij},$$

where $p_{ij} = \frac{p_{ij}}{\sum_j p_{ij}}$ is the conditional probability that $A$ is in state $a_i$ given that $B$ is in state $b_j$.

The *mutual or correlation entropy* $H(A : B)$ of $A$ and $B$ measures how much information $A$ and $B$ have in common. The mutual or correlation entropy $H(A : B)$ is defined by

$$H(A : B) = H(A) + H(B) - H(A, B),$$
$$H(A : B) = -\sum_{i,j} p_{ij} \log_2 p_{ij},$$

where $p_{ij}$ is the mutual probability defined as $p_{ij} = \frac{p_{ij}}{\sum_i \sum_j p_{ij}}$. The mutual or correlation entropy also can be expressed through the conditional entropy via

$$H(A : B) = H(A) - H(A \mid B),$$
$$H(A : B) = H(B) - H(B \mid A).$$

The joint entropy would equal the sum of each of $A$’s and $B$’s entropies only in the case that there are no correlations between $A$’s and $B$’s states. In that case, the mutual entropy or information vanishes and we could not make any predictions about $A$ just from knowing something about $B$.

The *relative entropy* $H(p \parallel q)$ measures the closeness of two probability distributions, $p$ and $q$, defined over the same random variable $A$. We define the relative entropy of $p$ with respect to $q$ by

$$H(p \parallel q) = \sum_i p_i \log_2 \frac{p_i}{q_i},$$
$$H(p \parallel q) = -H(A) - \sum_i p_i \log_2 q_i.$$ 

The relative entropy is non-negative, $H(p \parallel q) \geq 0$, with equality if and only if $p = q$. The classical relative entropy of two probability distributions is related to the probability of distinguishing the two distributions after a large but finite number of independent samples (Sanov’s theorem).

Let’s review some basic properties of entropy $[3]$:

1. $H(A, B) = H(B, A), H(A : B) = H(B : A)$.
2. $H(B \mid A) \geq 0$ and thus $H(A : B) \leq H(B)$, with equality if and only if $B = f(A)$.
3. $H(A) \leq H(A, B)$, with equality if and only if $B = f(A)$.
4. Subadditivity: $H(A, B) \leq H(A) + H(B)$ with equality if and only if $A$ and $B$ are independent random variables.
5. $H(B \mid A) \leq H(B)$ and thus $H(A : B) \geq 0$, with equality in each if and only if $A$ and $B$ are independent random variables.
6. Strong subadditivity: $H(A, B, C) + H(B) \leq H(A, B) + H(B, C)$.

Conditioning reduces entropy

$$H(A \mid B, C) \leq H(A \mid B)$$

and for a set of random variables $A_1, ..., A_n$ and $B$, the chaining for conditional entropies is

$$H(A_1, ..., A_n \mid B) = \sum_{i=1}^n H(A_i \mid B, A_1, ..., A_{i-1}).$$

Suppose $A \rightarrow B \rightarrow C$ is a Markov chain. Then

$$H(A) \geq H(A : B) \geq H(A : C),$$
$$H(C : B) \geq H(C : A).$$

The first inequality is saturated if and only if, given $B$, it is possible to reconstruct $A$. The *data processing inequality* states that the information we have available about a source of information can only decrease with the time: once information has been lost, it is gone forever. If a random variable $A$ is subject to noise, producing $B$, the data processing cannot be used to increase the amount of mutual information between the output of the process and the original information $A$. The *data pipelining inequality* says that any information $C$ shares with $A$ must be information which $C$ also shares with $B$: the information is “pipelined” from $A$ through $B$ to $C$.

### III. Von Neumann, Entropy and Quantun Information Theory

Von Neumann $[13, 14]$ defined the entropy of a quantum state $\rho$ by the formula

$$S(\rho) = -Tr(\rho \ln \rho)$$
which is the quantum analogue of the Shannon entropy $H$ [19]. The entropy $S(\rho)$ is non-negative and takes its maximum value $\ln n$ when $\rho$ is maximally mixed, and its minimum value zero if $\rho$ is pure. If $\lambda_i$ are the eigenvalues of $\rho$ then von Neumann’s definition can be expressed as

$$S(\rho) = -\sum_i \lambda_i \ln \lambda_i. \quad (16)$$

The von Neumann entropy reduces to a Shannon entropy if $\rho$ is a mixed state composed of orthogonal quantum states [20]. If $U$ is a unitary transformation, then

$$S(\rho) = S(U \rho U^\dagger). \quad (17)$$

If a composite system $AB$ is in a pure state, then $S(A) = S(B)$. Suppose $\rho = \sum_i p_i \rho_i$ where $p_i$ are probabilities, and $\rho_i$ are density operators. Then

$$S(\rho) \leq H(p_i) + \sum_i p_i S(\rho_i) \quad (18)$$

with equality if and only if the states $\rho_i$ have support on orthogonal subspaces, i.e. suppose $|i\rangle$ are orthogonal states for a system $A$, and $\rho_i$ is any set of density operators for another system $B$. Then

$$S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i). \quad (19)$$

where $H(p_i)$ is the Shannon entropy of the distribution $p_i$. The entropy is a concave function of its inputs. That is, given real numbers $p_i$, satisfying $p_i \geq 0$, $\sum_i p_i = 1$, and its corresponding density operators $\rho_i$, the entropy satisfies the equation

$$S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i). \quad (20)$$

It means that our uncertainty about this mixture of states should be higher than the average uncertainty of the states $\rho_i$.

The maximum amount of information that we can obtain about the identity of a state is called the accessible information. It is no greater than the von Neumann entropy of the ensemble’s density matrix (Holevo’s theorem) [21] [24] and its greatest lower bound is the subentropy $Q(\rho)$ [25] defined by

$$Q(\rho) = -\sum_{j=1}^n \left( \prod_{k \neq j} \frac{\lambda_j}{\lambda_j - \lambda_k} \right) \lambda_j \ln \lambda_j. \quad (21)$$

The upper bound $\chi = S(\rho) - \sum_i p_i S(\rho_i)$, called Holevo’s, on the mutual information resulting from the measurement of any observable, including POVM’s, which may have more outcomes than the dimensionality of the system being measured is

$$H(A : B) \leq S(\rho) - \sum_i p_i S(\rho_i) \leq H(A), \quad (22)$$

$$H(A : B) \leq S(\rho) \leq \ln n. \quad (23)$$

By analogy with the Shannon entropies it is possible to define conditional, mutual and relative entropies

$$S(A \mid B) \equiv S(A, B) - S(B), \quad (24)$$

$$S(A : B) \equiv S(A) + S(B) - S(A, B), \quad (25)$$

$$S(A : B) = S(A) - S(A|B), \quad (26)$$

$$S(A : B) = S(B) - S(B|A). \quad (27)$$

The negativity of the conditional entropy always indicates that two systems are entangled and indeed, how negative the conditional entropy is provides a lower bound on how entangled the two systems are [6].

The von Neumann entropy is additive, it means that if $\rho_A$ is a state of system $A$ and $\rho_B$ a state of system $B$, then

$$S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) \quad (28)$$

and strongly subadditive, which means that for a tripartite system in the state $\rho_{ABC}$

$$S(\rho_A, \rho_B, \rho_C) + S(\rho_B, \rho_C) \leq S(\rho_A, \rho_B, \rho_C) + S(\rho_B, \rho_C). \quad (29)$$

Suppose distinct quantum systems $A$ and $B$ have a joint state $\rho_{AB}$. The joint entropy for the two systems satisfies the next inequalities

$$S(A, B) \leq S(A) + S(B), \quad (30)$$

$$S(A, B) \geq \ln |\rho_{AB}|. \quad (31)$$

The first inequality is known as subadditivity, and it means that there can be more predictability in the whole than in the sum of the parts. The second inequality is known as triangle inequality.

Let $\rho$ and $\sigma$ be density operators. We define the relative entropy [26] of $\rho$ with respect to $\sigma$ to be

$$S(\rho \parallel \sigma) = Tr(\rho \ln \rho) - Tr(\rho \ln \sigma). \quad (32)$$

This function has a number of useful properties [14]:

1. $S(\rho \parallel \sigma) \geq 0$, with equality if and only if $\rho = \sigma$.
2. $S(\rho \parallel \sigma) < \infty$, if and only if $\text{supp} \rho \subseteq \text{supp} \sigma$. (Here “supp” is the subspace spanned by eigenvectors of $\rho$ with non-zero eigenvalues).
3. The relative entropy is continuous where it is not infinite.
4. The relative entropy is jointly convex in its arguments [27]. That is, if $\rho_1, \rho_2, \sigma_1$ and $\sigma_2$ are density operators, and $p_1$ and $p_2$ are non-negative numbers that sum to unity (i.e., probabilities), then

$$S(\rho \parallel \sigma) \leq p_1 S(\rho_1 \parallel \sigma_1) + p_2 S(\rho_2 \parallel \sigma_2), \quad (33)$$

where $\rho = p_1 \rho_1 + p_2 \rho_2$ and $\sigma = p_1 \sigma_1 + p_2 \sigma_2$. Joint convexity automatically implies convexity in each argument, so that

$$S(\rho \parallel \sigma) \leq p_1 S(\rho_1 \parallel \sigma) + p_2 S(\rho_2 \parallel \sigma). \quad (34)$$
Sanov’s theorem [3] has its quantum analogue [28, 29]. Suppose \( \rho \) and \( \sigma \) are two possible states of the quantum system \( Q \), and suppose we are provided with \( N \) identically prepared copies of \( Q \). A measurement is made to determine whether the prepared state is \( \rho \). The probability \( P_N \) that the state \( \sigma \) is confused with \( \rho \) is
\[
P_N \approx e^{-NS(\rho||\sigma)}.
\] (35)

The relative entropy can be seen as a measure of “distance” or of separation between two density operators. Two states are “close” if they are difficult to distinguish, but “far apart” if the probability of confusing them is small [26].

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