A parabolic problem involving $p(x)$-Laplacian, a power and a singular nonlinearity

Akasmika Panda$^*_{1}$, Debajyoti Choudhuri$^†_{1,1}$ & Kamel Saoudi$^‡_{1,2}$

$^1$Department of Mathematics, National Institute of Technology Rourkela, India

$^2$Basic and Applied Scientific Research Center, Imam Abdulrahman Bin Faisal University,, P.O. Box 1982, 31441, Dammam, Saudi Arabia

Abstract

The purpose of this paper is to study the nonlinear singular parabolic equations with $p(x)$-Laplacian. Precisely, we consider the following problem and discuss the existence of the nonnegative weak solution.

$$\frac{\partial u}{\partial t} - \Delta_{p(x)}u = \lambda u^{q(x)-1} + u^{-\delta(x)} g + f \text{ in } Q_T,$$

$$u = 0 \text{ on } \Sigma_T,$$

$$u(0, \cdot) = u_0(\cdot) \text{ in } \Omega.$$

Here $Q_T = \Omega \times (0, T)$, $\Sigma_T = \partial \Omega \times (0, T)$, $\Omega$ is a bounded domain in $\mathbb{R}^N$ with Lipschitz continuous boundary $\partial \Omega$, $\lambda \in (0, \infty)$, $f, g \in L^1(Q_T)$, $u_0 \in L^r(\Omega)$ with $r \geq 2$, the function $\delta : \overline{\Omega} \to (0, \infty)$ is continuous, $p \in C(\overline{\Omega})$ with $\max_{x \in \overline{\Omega}} p(x) = p^+ < N$, and $q \in C(\overline{\Omega})$.

We distinguish two cases:

- For $f > 0$, we assume $2 - \frac{1}{N+1} < p^- = \min_{x \in \overline{\Omega}} p(x)$, and $\max_{x \in \overline{\Omega}} q(x) = q^+ < p^- + \frac{1}{N+1}$.

- For $f \equiv 0$, we assume $2 < p^- \leq p(x) \leq q(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ for every $x \in \overline{\Omega}$, and $g(x, t) = \tilde{g}(x) \in L^\infty(\Omega)$.

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*akasmika444@gmail.com
†dc.iit12@gmail.com
‡kmsaoudi@iau.edu.sa (Corresponding author)
1 Introduction

The first part of the article is devoted to the study of the following singular parabolic problem with an $L^1$ data given by

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta_{p(x)} u &= \lambda u^{q(x)-1} + u^{-\delta(x)} g + f \text{ in } Q_T, \\
u > 0 \text{ in } Q_T, \\
u = 0 \text{ on } \Sigma_T, \\
u(0, \cdot) &= u_0(\cdot) \text{ in } \Omega,
\end{aligned}
\]

(1.1)

where $u_0 \in L^r(\Omega)$ with $r \geq 2$ is positive, $\lambda > 0$, $f, g \in L^1(Q_T)$ are two positive functions, $\delta : \Omega \to (0, \infty)$ is continuous and $p, q \in C(\Omega)$ satisfy the following hypotheses.

(A1) $2 - \frac{1}{N+1} < \min_{x \in \Omega} p(x) = p^- \leq p^+ = \max_{x \in \Omega} p(x) < N$

(A2) $1 < \min_{x \in \Omega} q(x) = q^- \leq q^+ = \max_{x \in \Omega} q(x) < p^- + \frac{1}{N+1}$

Throughout the article we will consider a bounded domain $\Omega$ in $\mathbb{R}^N$ with a Lipschitz continuous boundary $\partial \Omega$. We denote $Q_T = \Omega \times (0, T)$, $\Sigma_T = \partial \Omega \times (0, T)$ for final time $T > 0$. To the best of our knowledge, parabolic problems of type (1.1) are new in the literature which involves a power nonlinearity, singularity, and an $L^1$ data with the $p(x)$-Laplace operator. The $p(x)$ growth condition is considered as a principal class of nonstandard $(p, q)$ growth condition. In the present work, our first objective is to construct some auxiliary problems where we replace the singular term by $\frac{1}{u + \frac{1}{n}} \delta(\cdot)$ for every $n \in \mathbb{N}$. We point out here that we restrict the class of $L^1$ functions $f$ and the range of $q(\cdot)$, as given in (A2) and (A3) (refer Section 4), to find some required a priori estimates. Then with the help of these estimates, we pass the limit $n \to \infty$ in the auxiliary problem to obtain a SOLA (Solutions Obtained as Limits of Approximations) to (1.1).

The literature pertaining to the elliptic counterpart of the problem and its variant, considered in this article is vast and is impossible to assimilate in here. Therefore, we will only refer the readers to those work which has some connection to the current work. The weak theory for purely singular problems has been developed over many years starting from the works by [9, 15, 33] with linear operators and by [16, 31] with nonlinear operators. Elliptic problems involving a singular nonlinearity, a measure data or an $L^1$ data have been studied in [17, 31, 34] and the references therein. Since problems of type (1.1) is new in the literature, we find a very less number of articles dealing with its stationary case. Further, we cite [23] where the authors have settled the multiplicity result for the stationary problem of (1.1) with $p(\cdot) = p$ (a constant). In the elliptic setting, the literature for the problem as in (1.1) for $f \equiv 0$ with $p(x)$-Laplacian or $p$-Laplacian can be found in [21, 26, 27, 39] and the bibliography therein. More precisely, these seminal papers deal with the existence and multiplicity of the problem both in the subcritical and critical case.

Let us now discuss some of the important parabolic problems which has helped us in the construction of this work. Concerning the case $\lambda = 0$ and $p(\cdot) = p$ of (1.1), the existence result
has been investigated in [10, 11, 12, 32] and the bibliography therein. Bonis & Giachetti in [10], assumed the functions $f$ and $g$ to be in some $L^r(0, T; L^m(\Omega))$ space with $\frac{1}{r} + \frac{N}{pm} < 1$. In the same spirit of [10], in [32] the authors have shown the existence of a weak solution by considering $g$ to be in some $L^1(0, T; L^m(\Omega))$ space with $\frac{1}{r} + \frac{N}{pm} < 1$. In the same spirit of [10], in [32] the authors have shown the existence of a weak solution by considering $g$ to be in $L^1(0, T; L^m(\Omega))$ space with $\frac{1}{r} + \frac{N}{pm} < 1$.

In the same spirit of [10], in [32] the authors have shown the existence of a weak solution by considering $g$ to be in $L^1(0, T; L^m(\Omega))$ space with $\frac{1}{r} + \frac{N}{pm} < 1$. The parabolic problems involving $p(x)$-Laplacian and a measure data or an $L^1$ data (the case $\lambda = 0$ and $g \equiv 0$ of (1.1)) have been analyzed by several authors since the papers [5, 38, 42]. The corresponding constant exponent cases (problems with $p$-Laplacian) are studied by Petitta et al. in [35, 36] and Boccardo et al. in [7, 6], etc. Most of them worked with renormalized solutions and entropy solutions. It is worth mentioning the result in [28] where nonlinear parabolic problems with variable exponent are considered with Neumann-type boundary conditions.

We refer to the works by Badra et al. in [1, 2], Bougherara et al. in [13, 14], and the references therein for model problems as in (1.1) with $f \equiv 0$. More precisely, the paper considered by M. Badra, K. Bal, and J. Giacomoni in [1] is as follows.

$$\frac{\partial u}{\partial t} - \Delta_p u = f(x, u, \nabla u) + u^{-\delta} \text{ in } Q_T,$$

$$u > 0 \text{ in } Q_T,$$

$$u = 0 \text{ on } \Sigma_T,$$

$$u(0, \cdot) = u_0(\cdot) \text{ in } \Omega,$$

where $p \in (1, \infty)$, $\delta < 2 + \frac{1}{p-1}$, $u_0 \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ satisfies a cone condition and $f(x, s, \xi) = f(x, s)$ is a Caratheodory bounded below function which is locally Lipschitz for the second variable $s$ and obeys the following subhomogeneous growth condition.

$$0 \leq \lim_{s \to \infty} \frac{f(x, s)}{s^{p-1}} = \alpha_f < \lambda_1(\Omega).$$

Here $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta_p$ in $\Omega$ with zero Dirichlet boundary condition. The authors have proved the existence and uniqueness of a weak solution to (1.2). Later, in 2015, Bougherara & Giacomoni in [14] generalized the result in [1] for any $p > 1$, $\delta > 0$ and $u_0 \in (C_0(\Omega))^+$. Furthermore, Bougherara et al. in [13] studied the problem (1.2) with $\delta > 0$, $u_0 \in L^r(\Omega)$ and

$$f(x, s, \xi) \leq as^{q-1} + b + c|\xi|^{p-\frac{r}{q}}, \forall x \in \Omega, s \in \mathbb{R}^+, |\xi| \geq M$$

where $a, c, M > 0$, $b \geq 0$, $r \geq q$, $2 \leq p \leq q < \max\{p^*, p(1 + \frac{r}{N})\}$.

Motivated by the former result in [13], we discuss the following singular problem, in the second part of the article, for the variable exponent case.

$$\frac{\partial u}{\partial t} - \Delta_{p(x)} u = \lambda u^{q(x)-1} + u^{-\delta(x)} \tilde{g} \text{ in } Q_T,$$

$$u > 0 \text{ in } Q_T,$$

$$u = 0 \text{ on } \Sigma_T,$$

$$u(0, \cdot) = u_0(\cdot) \text{ in } \Omega$$

(1.3)
where \( u_0 \in L^r(\Omega) \), \( \lambda > 0 \), \( \tilde{g} \in L^{\infty}(\Omega) \) is a positive function, \( \delta : \overline{\Omega} \to (0, \infty) \) is continuous and \( p, q \in C(\overline{\Omega}) \) obey the following restrictions.

\begin{enumerate}[(B1)]
  \item \[ 2 \leq \min_{x \in \overline{\Omega}} p(x) = p^- \leq p^+ = \max_{x \in \overline{\Omega}} p(x) < N \]
  \item \[ p(x) \leq q(x) < p^*(x) = \frac{Np(x)}{N - p(x)} \text{ for all } x \in \overline{\Omega} \]
  \item \[ \max_{x \in \overline{\Omega}} q(x) = q^+ < p^- (1 + \frac{r}{N}) \]
  \item \[ r > \max\{q^+, \delta^+ + 1\} \]
\end{enumerate}

The problems of type (1.1) and (1.3) are related to different models such as turbulent flow of gas in porous media [37], chemical heterogeneous catalyst kinetics [3], thermo-conductivity [19], electromagnetic field [22], boundary layer phenomena for viscous fluids, signal transmission, non-Newtonian flows, etc [20, 30].

We now describe the plan of the paper. In Section 2, we provide the mathematical preliminaries that will be helpful throughout the paper. In addition to that, we define the notion of weak solutions to problems (1.1), (1.3), and also state our two main results. In Section 3, we discuss a singular parabolic problem with an \( L^1 \) data and the approach will be used to prove the existence of a weak solution to (1.1). Section 4 is all about proving the existence result for (1.1). Section 5 is followed by two subsections to prove the existence result for (1.3). In Section 5.1, we present the approximation scheme with the help of a semi-discretization approach in time. Then in Section 5.2, we find some a priori estimates on the sequence of the solutions of the approximating problems.

## 2 Mathematical preliminaries and main results

Consider the domain \( \Omega \subset \mathbb{R}^N \) to be bounded and the function \( p \in C(\overline{\Omega}) \) to be continuous with \( 1 \leq p^- = \min_{x \in \overline{\Omega}} p(x) \leq \max_{x \in \overline{\Omega}} p(x) = p^+ < \infty \). Then the Lebesgue space with variable exponent \( p(\cdot) \) is defined by

\[ L^{p(\cdot)}(\Omega) = \{ u : \Omega \to \mathbb{R} : \int_\Omega |u|^{p(x)} < \infty \}. \]

The space \( L^{p(\cdot)}(\Omega) \) with \( p^- > 1 \) is a reflexive Banach space endowed with the following norm.

\[ \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\{ \mu > 0 : \int_\Omega \frac{u(x)}{\mu} |p(x)| \mu dx < 1 \}. \]

The dual space of \( L^{p(\cdot)} \) is denoted by \( L^{p'(\cdot)} \) where \( p'(\cdot) = \frac{p(\cdot)}{p(\cdot) - 1} \).

If \( u_1 \in L^{p(\cdot)}(\Omega) \) and \( u_2 \in L^{p'(\cdot)}(\Omega) \), then we have the following Hölder type inequality.

\[ \int_\Omega |u_1 u_2| dx \leq \left( \frac{1}{p^-} + \frac{1}{p'^-} \right) \|u_1\|_{L^{p(\cdot)}(\Omega)} \|u_2\|_{L^{p'(\cdot)}(\Omega)}. \]

Let \( q \in C(\overline{\Omega}) \) with \( q(x) \geq p(x) \), for every \( x \in \overline{\Omega} \), then \( L^{q(\cdot)}(\Omega) \) is continuously embedded in \( L^{p(\cdot)}(\Omega) \).
Let us define the modular function as $\rho(u) = \int_{\Omega} |u(x)|^{p(x)} dx$. Then the relations between the modular function and the norm $\| \cdot \|_{L^{p(.)}(\Omega)}$ are as follows.

- $\| u \|_{L^{p(.)}(\Omega)} = \mu \iff \rho(\mu) = 1$.

- $\| u \|_{L^{p(.)}(\Omega)} > 1 \implies \| u \|_{L^{p(.)}(\Omega)}^{p^{-}} \leq \rho(u) \leq \| u \|_{L^{p(.)}(\Omega)}^{p^{+}}$. This inequality reverses if $\| u \|_{L^{p(.)}(\Omega)} < 1$.

- $\lim_{n \to \infty} \| u_n - u \|_{L^{p(.)}(\Omega)} = 0 \iff \lim_{n \to \infty} \rho(u_n - u) = 0$.

The Sobolev space with variable exponent is given by

$$W^{1,p(.)}(\Omega) = \{ u \in L^{p(.)}(\Omega) : \nabla u \in L^{p(.)}(\Omega) \}.$$

The space $W^{1,p(.)}(\Omega)$ with $p^{-} > 1$ is a separable and reflexive Banach space equipped with the following norm.

$$\| u \|_{W^{1,p(.)}(\Omega)} = \| u \|_{L^{p(.)}(\Omega)} + \| \nabla u \|_{L^{p(.)}(\Omega)}.$$

We also define the subspace $W_{0}^{1,p(.)}(\Omega)$ as closure of $C_{c}^{\infty}(\Omega)$ in $W^{1,p(.)}(\Omega)$. The dual of $W_{0}^{1,p(.)}(\Omega)$ is denoted by $W_{0}^{-1,p^{*}(\cdot)}(\Omega)$. For detailed study on these variable exponent spaces one can refer the work of Fan & Zhao [18].

By the Poincaré inequality (see [18]), for every $u \in W_{0}^{1,p(.)}(\Omega)$ we have

$$\| u \|_{L^{p(.)}(\Omega)} \leq C\| \nabla u \|_{L^{p(.)}(\Omega)}$$

where $C = C(\Omega, p) > 0$ and by the Sobolev embedding theorem, the following embedding

$$W^{1,p(.)}(\Omega) \hookrightarrow L^{r(.)}(\Omega)$$

is continuous for any $r \in C(\overline{\Omega})$ with $r(\cdot) \leq p^{*}(\cdot) = \frac{Np(.)}{N-p(.)}$. Moreover, this embedding is compact for any $r \in C(\overline{\Omega})$ with $\inf_{x \in \overline{\Omega}} (p^{*}(x) - r(x)) > 0$.

Let us consider the extended function $p : \overline{Q}_{T} = [0, T] \times \overline{\Omega} \to [1, \infty)$ with $p(t, x) = p(x)$ for every $(t, x) \in \overline{\Omega}$. We now define a generalized Lebesgue space with variable exponent by

$$L^{p(.)}(\mathcal{Q}_{T}) = \{ u : Q_{T} \to \mathbb{R} : \int_{Q_{T}} |u|^{p(x)} dx dt < \infty \}.$$

The properties of $L^{p(.)}(\mathcal{Q}_{T})$ are same as $L^{p(.)}(\Omega)$ when endowed with the norm $\| \cdot \|_{L^{p(.)}(\mathcal{Q}_{T})}$ given by

$$\| u \|_{L^{p(.)}(\mathcal{Q}_{T})} = \inf \{ \mu > 0 : \int_{Q_{T}} \frac{|u(x, t)|^{p(x)}}{\mu} dx dt < 1 \}.$$

If $X$ is a Banach space, then the space $L^{r}(0, T; X)$ with $r \geq 1$ denotes the standard Bochner space and is defined as follows.

$$L^{r}(0, T; X) = \{ u : (0, T) \to X : \| u(t) \|_{X} \in L^{r}(0, T) \}.$$

Furthermore, $C([0, T]; X)$ identifies the space of continuous functions $u : [0, T] \to X$ such that $\| u \|_{C([0, T]; X)} = \max_{t \in [0, T]} \| u(t) \|_{X} < \infty$. 
Remark 2.1. We have the following embedding results. (refer [5])

1. Let \( p(\cdot) \) and \( q(\cdot) \) are two continuous functions with \( p(x) \leq q(x) \) for almost every \( x \in \Omega \). Then the embedding from \( L^{p^-(0,T;W^{1,q(\cdot)}_0(\Omega))} \) to \( L^{p^-(0,T;W^{1,p(\cdot)}_0(\Omega))} \) is continuous.

2. The following inclusions are continuous and dense.

\[
L^{p^+(0,T;L^{p(\cdot)}(\Omega))} \hookrightarrow_d V^{p(\cdot)}(Q_T) \hookrightarrow_d L^{p^-(0,T;L^{p(\cdot)}(\Omega))}.
\]

With the consideration of the above results and remarks, we introduce a natural function space with variable exponent as follows.

\[
V^{p(\cdot)}(Q_T) = \{ u \in L^{p^-(0,T;W^{1,p(\cdot)}_0(\Omega))} : |\nabla u| \in L^{p(\cdot)}(Q_T) \}.
\] (2.4)

The space \( V^{p(\cdot)}(Q_T) \) is a separable and reflexive Banach space endowed with the norm

\[
\|u\|_{V^{p(\cdot)}(Q_T)} = \|\nabla u\|_{L^{p(\cdot)}(Q_T)}.
\]

According to Bendahmane et al. in [5], we have the following continuous dense embeddings.

1. \( L^{p^+(0,T;W^{1,p(\cdot)}_0(\Omega))} \hookrightarrow_d V^{p(\cdot)}(Q_T) \hookrightarrow_d L^{p^-(0,T;W^{1,p(\cdot)}_0(\Omega))} \).

2. \( L^{p^--(0,T;W^{-1,p'(\cdot)}(\Omega))} \hookrightarrow_d V^{p(\cdot)}(Q_T)^* \hookrightarrow_d L^{p^+(0,T;W^{-1,p'(\cdot)}(\Omega))} \), where \( V^{p(\cdot)}(Q_T)^* \) is the dual space of \( V^{p(\cdot)}(Q_T) \).

In this paper, we are also going to deal with local Sobolev spaces with variable exponent defined by

\[
V^{p(\cdot)}_{loc}(Q_T) = \{ u : Q_T \to \mathbb{R} : u \text{ and } \nabla u \in L^{p(\cdot)}((0,T) \times K) \text{ for every compact } K \subset \Omega \}. \quad (2.5)
\]

Thus, we need to define a general sense of trace known as M-boundary trace which is given below.

Definition 2.2. Let \( \{\Omega_m\} \) be a sequence such that \( \bar{\Omega}_m \subset \Omega_{m+1} \subset \Omega \). Then \( \{\Omega_m\} \) is said to be an exhaustion of \( \Omega \) if \( \Omega_m \uparrow \Omega \). If each \( \Omega_m \) is of \( C^2 \) class, then this exhaustion is said to be of class \( C^2 \). Moreover, we say that an exhaustion \( \{\Omega_m\} \) is a uniform \( C^2 \) exhaustion if \( \Omega \) is \( C^2 \) and the sequence \( \{\Omega_m\} \) is uniformly of class \( C^2 \).

Definition 2.3 (M-boundary trace, [24]). Let \( u \in W^{1,p(\cdot)}_{loc}(\Omega) \) for \( p^- > 1 \). Then \( \nu \in M(\partial \Omega) \) is said to be the M-boundary trace of \( u \) on \( \partial \Omega \) if for every \( C^2 \) exhaustion \( \{\Omega_m\} \) and for every \( f \in C(\bar{\Omega}) \)

\[
\int_{\partial \Omega_m} u|_{\partial \Omega_m} f dS \to \int_{\partial \Omega} f d\nu.
\]

Here \( u|_{\partial \Omega_m} \) denotes the Sobolev trace, \( dS = dH^{N-1} \) and \( H^{N-1} \) denotes the \( (N-1) \) dimensional Hausdorff measure. The M-boundary trace \( \nu \) of \( u \) is denoted by \( tr \ u \).

Furthermore, if \( u \in W^{1,p(\cdot)}(\Omega) \), then Sobolev trace of \( u \) is same as M-boundary trace of \( u = tr \ u \).
Before providing the notion of the weak solutions to the problems (1.1) and (1.3), we define the truncation functions, which will be used henceforth very often. For a fixed $k > 0$, the truncation functions $T_k$ and $G_k$ are defined, respectively, as $T_k(s) = \max\{-k, \min\{s, k\}\}$ and $G_k(s) = (|s| - k)^+ \text{sign}(s)$. For $\gamma > 0$ we define

$$T_{k,\gamma}(s) = \int_0^s T_k(\tau)d\tau. \quad (2.6)$$

**Definition 2.4.** Assume $0 < \delta^+ < 1$. Then a weak solution to the problem (1.1) is a function $u \in L^1(0, T; W^{1,1}_0(\Omega))$ such that $\frac{d}{dt} u \in L^1(0, T; L^1_{\text{loc}}(\Omega))$ and

$$-\int_{Q_T} u\varphi_t - \int_{\Omega} u_0\varphi(x, 0) + \int_{Q_T} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla \varphi = \lambda \int_{Q_T} u^{q(x)-1} \varphi + \int_{Q_T} \frac{g\varphi}{u^{\delta(x)}} + \int_{\Omega} f \varphi \quad (2.7)$$

for every $\varphi \in C^1_c(\Omega \times [0, T))$.

Further, assume that $\delta^+ \geq 1$. Then a function $u \in L^1(0, T; W^{1,1}_{\text{loc}}(\Omega))$ is said to be a weak solution to (1.1) if $\frac{d}{dt} u \in L^1(0, T; L^1_{\text{loc}}(\Omega))$, $\text{tr} u(\cdot, t) = 0$ in the sense of Definition 2.3 for every $t \in (0, T)$ and $u$ satisfies (2.7) for every $\varphi \in C^1_c(\Omega \times [0, T))$.

We now state our first main result of the paper in the following theorem.

**Theorem 2.5.** Let $\lambda \leq 1$ and the assumptions (A1)-(A2) hold. Further, assume that $f \in \mathcal{C} \subset L^1(Q_T)$ verifies (A3), provided in Section 4. Then for $\delta^+ < 1$, there exists a non-negative weak solution $u$ to (1.1) in $V^{r(\cdot)}(Q_T)$ (as defined in (2.4)), in the sense of Definition 2.4, for every $1 \leq r(\cdot) < p(\cdot) - \frac{N}{N+1}$. Similarly, for $\delta^+ \geq 1$, problem (1.1) admits a nonnegative weak solution $u \in V^{r(\cdot)}_{\text{loc}}(Q_T)$ (as defined in (2.5)), in the sense of Definition 2.4, for every $1 \leq r(\cdot) < p(\cdot) - \frac{N}{N+1}$.

Next we define the notion weak solution to (1.3) as follows.

**Definition 2.6.** Assume $0 < \delta^+ < 1$. Then a weak solution to the problem (1.3) is a function $u$ such that $u \in V^{p(\cdot)}(Q_T) \cap L^\infty(0, T; L^\infty(\Omega)) \cap L^\infty((\eta, T) \times \Omega)$ for every $\eta \in (0, T)$, $\frac{d}{dt} u \in L^1(0, T; L^1_{\text{loc}}(\Omega))$ and for every $\varphi \in C^1_c(\Omega \times [0, T))$,

$$-\int_{Q_T} u\varphi_t - \int_{\Omega} u_0\varphi(x, 0) + \int_{Q_T} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla \varphi = \lambda \int_{Q_T} u^{q(x)-1} \varphi + \int_{Q_T} \frac{g\varphi}{u^{\delta(x)}}. \quad (2.8)$$

Further, assume that $\delta^+ \geq 1$. Then a function $u$ is said to be a weak solution to (1.3) if $u \in V^{p(\cdot)}_{\text{loc}}(Q_T) \cap L^\infty(0, T; L^\infty(\Omega)) \cap L^\infty((\eta, T) \times \Omega)$ for every $\eta \in (0, T)$, $\text{tr} u(\cdot, t) = 0$ in the sense of Definition 2.3 for every $t \in (0, T)$ and $u$ satisfies (2.8) for every $\varphi \in C^1_c(\Omega \times [0, T))$.

The following theorem is the existence result for (1.3) and is our second main result.

**Theorem 2.7.** Let the assumptions (B1) – (B4) hold. Then there exists $\overline{T} > 0$ such that for any $T < \overline{T}$, the problem (1.3) admits a weak solution $u$, in the sense of Definition 2.6.
We now prove a weak comparison principle which is a very useful tool to establish a comparison between singular parabolic problems with \( p(x) \)-Laplacian.

**Theorem 2.8** (Comparison principle). Let \( u_0, v_0 \in L^2(\Omega) \). Suppose \( u, v \in V^{p(\cdot)}(Q_T) \) such that \( \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \in V^{p(\cdot)}(Q_T)^* \) and

\[
\frac{\partial u}{\partial t} - \Delta_{p(x)}u - \frac{1}{(u + c)^{\delta(x)}} \leq \frac{\partial v}{\partial t} - \Delta_{p(x)}v - \frac{1}{(v + c)^{\delta(x)}} \quad \text{weakly in } Q_T, \ c > 0,
\]

\( u(0, \cdot) = u_0(\cdot) \leq v(0, \cdot) = v_0(\cdot) \) in \( \Omega \), \( u = v = 0 \) on \( \Sigma_T \). Then \( u \leq v \) a.e. in \( Q_T \).

**Proof.** Since \( \frac{\partial u}{\partial t} - \Delta_{p(x)}u - \frac{1}{(u + c)^{\delta(x)}} \leq \frac{\partial v}{\partial t} - \Delta_{p(x)}v - \frac{1}{(v + c)^{\delta(x)}} \) weakly in \( Q_T \) with \( u = v = 0 \) on \( \Sigma_T \), we have

\[
\int_{Q_T} u_t \phi + \int_{Q_T} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla \phi - \int_{Q_T} \frac{\phi}{(u + c)^{\delta(x)}} \leq \int_{Q_T} v_t \phi + \int_{Q_T} |\nabla v|^{p(x)-2}\nabla v \cdot \nabla \phi - \int_{Q_T} \frac{\phi}{(v + c)^{\delta(x)}}
\]

for every \( \phi \in V^{p(\cdot)}(Q_T) \), \( \phi \geq 0 \). We formally choose \( \phi = (u - v)^+ \chi_{(0,t)} \) for any \( t \in (0,T] \) and we get

\[
\int_{Q_t} (u - v)_t (u - v)^+ + \int_{Q_t} (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v) \cdot \nabla (u - v)^+ \leq 0.
\]

Moreover, from the nonnegativity of the second term present in the above inequality, we have

\[
\frac{1}{2} \int_{Q_t} \frac{d}{dt} \left[ (u - v)^+ \right]^2 \leq 0.
\]

Since \( u_0 \leq v_0 \) in \( L^2(\Omega) \), we get \( (u - v)^+ = 0 \) a.e. in \( Q_T \). Thus, \( u \leq v \) a.e. in \( Q_T \).

\[\square\]

### 3 Existence result for a singular problem with an \( L^1 \) data

Prior to proving the existence of solution to (1.1), we consider the following singular problem which will help to prove Theorem 2.5

\[
\frac{\partial v}{\partial t} - \Delta_{p(x)}v = v^{-\delta(x)}g + f \quad \text{in } Q_T,
\]

\[
\begin{align*}
&v > 0 \text{ in } Q_T, \\
&v = 0 \text{ on } \Sigma_T, \\
&v(0, \cdot) = v_0(\cdot) \text{ in } \Omega
\end{align*}
\]

where \( \delta \in C(\overline{\Omega}) \) with \( 0 < \delta^- \leq \delta^+ < \infty \), \( p \in C(\overline{\Omega}) \) with \( 2 - \frac{1}{N+1} < p^- \leq p^+ < N \), \( f, g \in L^1(Q_T) \) and \( v_0 \in L^r(\Omega) \), for \( r \geq 2 \), are three positive functions.
Lemma 3.2. Let $\delta^+ < 1$. Then a weak solution to the problem (3.9) is a function $v \in L^1(0,T;W^{1,1}_0(\Omega))$ such that $v \in L^1(0,T;L^1_{loc}(\Omega))$ and

$$-\int_{Q_T} v\varphi_t - \int_\Omega v_0\varphi(x,0) + \int_{Q_T} |\nabla v|^{p(x)-2} \nabla v \cdot \nabla \varphi = \int_{Q_T} f\varphi + \int_{Q_T} \frac{g\varphi}{v^{\delta(x)}}$$

(3.10)

for every $\varphi \in C^1_c(\Omega \times [0,T))$.

Similarly, if $\delta^- \geq 1$, a function $v \in L^1(0,T;W^{1,1}_0(\Omega))$ is said to be a weak solution to (3.9) if $\frac{v}{v^{\delta(x)}} \in L^1(0,T;L^1_{loc}(\Omega))$, $tr v(\cdot, t) = 0$ in the sense of Definition 2.3 for every $t \in (0,T)$ and $v$ satisfies (3.10) for every $\varphi \in C^1_c(\Omega \times [0,T))$.

We now follow the method of approximation to establish the existence of a weak solution to (3.9). Consider the following scheme of approximation.

$$\frac{\partial v_n}{\partial t} - \Delta_{p(x)} v_n = (v_n + 1/n)^{-\delta(x)} g_n + f_n \text{ in } Q_T,$$

$$v_n > 0 \text{ in } Q_T,$$

$$v_n = 0 \text{ on } \Sigma_T,$$

$$v_n(0,\cdot) = v_{0,n}(\cdot) \text{ in } \Omega.$$  

(3.11)

Here $g_n = T_n(g)$, $v_{0,n} = T_n(v_0)$ and the sequence $\{f_n\} \subset L^\infty(Q_T)$ is such that $f_n$ converges strongly to $f$ in $L^1(Q_T)$. The weak formulation of (3.11) is given by

$$-\int_{Q_T} v_n\varphi_t - \int_\Omega v_{0,n}\varphi(x,0) + \int_{Q_T} |\nabla v_n|^{p(x)-2} \nabla v_n \cdot \nabla \varphi = \int_{Q_T} f_n\varphi + \int_{Q_T} \frac{g_n\varphi}{v_n + 1/n}^{\delta(x)}$$

(3.12)

for every $\varphi \in C^1_c(\Omega \times [0,T))$.

First of all, we will show the existence of a weak solution to (3.11). Then with the help of some a priori estimates we will pass the limit $n \to \infty$ in the weak formulation (3.12) to obtain a weak solution to (3.9).

Lemma 3.2. For a fixed $n \in \mathbb{N}$, the problem (3.11) admits a unique weak solution $v_n \in V^{p(\cdot)}(Q_T) \cap L^\infty(Q_T)$ with $\frac{\partial v_n}{\partial t} \in V^{p(\cdot)}(Q_T)^*$.

Proof. The proof is based on Schauder’s fixed point argument. Let us define a map,

$$G : L^{p(\cdot)}(Q_T) \to L^{p(\cdot)}(Q_T)$$

such that for any $u \in L^{p(\cdot)}(Q_T)$ we obtain a unique positive weak solution $\bar{u} \in V^{p(\cdot)}(Q_T) \cap C(0,T;L^2(\Omega))$ to the following problem.

$$\frac{\partial \bar{u}}{\partial t} - \Delta_{p(x)} \bar{u} = (u + 1/n)^{-\delta(x)} g_n + f_n \text{ in } Q_T,$$

$$\bar{u} = 0 \text{ on } \Sigma_T,$$

$$\bar{u}(0,\cdot) = v_{0,n}(\cdot) \text{ in } \Omega.$$  

(3.13)
Lemma 3.3. For a strongly singular case, and in (3.11) and then integrate over $Q$, the solution space of (3.9) depends on the function $\delta$. Fix $a$

Proof. Bounded in $L^2(Q_T)$, we have
\begin{align*}
\frac{1}{2} \int_{Q_T} \bar{u}^2(x,T) - \frac{1}{2} \int_{\Omega} \bar{u}^2_{0,n} - \int_{Q_T} |\nabla \bar{u}|^{p(x)} &= \int_{Q_T} f_n \bar{u} + \int_{Q_T} \frac{g_n \bar{u}}{(u + 1/n)^{\delta(x)}} \\
&\leq C_1(n,\lambda) \int_{Q_T} \bar{u}.
\end{align*}
This implies,
\begin{equation}
\int_{Q_T} |\nabla \bar{u}|^{p(x)} \leq C_1(n,\lambda) \int_{Q_T} \bar{u} + \frac{1}{2} \int_{\Omega} \bar{u}^2_{0,n}. \tag{3.14}
\end{equation}
On using the Poincaré Inequality, we establish the following.
\begin{align*}
\int_{Q_T} |\bar{u}|^{p(x)} \leq C_2 C_1(n,\lambda) \int_{Q_T} \bar{u} + \frac{C_2}{2} \int_{\Omega} \bar{u}^2_{0,n} \\
&\leq C_3
\end{align*}
where $C_3$ is independent of $u$. Thus, the ball $B_{C_3}(0)$ in $L^{p(\cdot)}(Q_T)$ of radius $C_3$ is invariant for the map $G$. By standard arguments it is easy to prove that the map $G$ is continuous and $G(B_{C_3}(0))$ is relatively compact.
Therefore, on using the Schauder fixed point theorem to $G$, we obtain a fixed point $v_n \in V^{p(\cdot)}(Q_T) \cap L^\infty(Q_T)$ that is also a weak solution to the problem (3.13). Now by the virtue of the weak comparison principle, Theorem 2.8 we conclude that the problem (3.13) admits a unique solution. Hence the proof.

We now prove some required a priori estimates to pass the limit $n \to \infty$ in (3.12). Since the solution space of (3.9) depends on the function $\delta$, the proof is divided into two cases, $\delta^+ < 1$ and $\delta^+ \geq 1$. If $\delta^+ < 1$, we find a global solution to (3.3); otherwise, we find a local solution for a strongly singular case.

**Lemma 3.3.** Let $v_n$ be a positive weak solution to (3.11). Then the sequence $\{v_n\}$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$.

**Proof.** Fix a $t \in (0,T]$ and denote $Q_t = \Omega \times (0,t)$. Let us multiply $T_1^\gamma(v_n)$, for $\gamma = \max\{1,\delta^+\}$, in (3.11) and then integrate over $Q_t$ to establish the following.
\begin{equation}
\int_{Q_t} (v_n)T_1^\gamma(v_n) + \gamma \int_{Q_t} |\nabla T_1^\gamma(v_n)|^{p(x)}T_1^{\gamma-1} = \int_{Q_t} f_n T_1^\gamma(v_n) + \int_{Q_t} \frac{g_n T_1^\gamma(v_n)}{(v_n + 1/n)^{\delta(x)}}. \tag{3.15}
\end{equation}
From the above equation (3.15), we get
\begin{align*}
\int_{Q_t} (T_1^\gamma(v_n))_t &\leq \int_{Q_t} f + \int_{Q_t \cap \{v_n \leq 1\}} v_n^{\gamma-\delta(x)} g_n + \int_{Q_t \cap \{v_n > 1\}} v_n^{-\delta(x)} g_n \\
&\leq C_1 + \|g\|_{L^1(Q_T)}, \tag{3.16}
\end{align*}
where the function $T_{1,\gamma}(\cdot)$ is defined in (2.6). By the definition of $T_{1,\gamma}(\cdot)$ we obtain $T_{1,\gamma}(s) \geq s - 1$. Hence,

$$\int_{\Omega} v_n(x, t) \leq C_1 + \|g\|_{L^1(Q_T)} + |\Omega| + \int_{\Omega} T_{1,\gamma}(v_0).$$

This implies

$$\|v_n\|_{L^\infty(0,T;L^1(\Omega))} \leq C. \quad (3.17)$$

**Lemma 3.4.** Let $\delta^+ < 1$ and $v_n$ be a positive weak solution to (3.11). Then $\{v_n\}$ is bounded in $V^{r(\cdot)}(Q_T)$ for every $1 \leq r(x) < p(x) - \frac{N}{N+1}$ for all $x \in \overline{\Omega}$. Moreover, $\{T_k(v_n)\}$ is bounded in $V^{p(\cdot)}(Q_T)$, for any $k > 0$.

**Proof.** Let us multiply $T_k(v_n)$ in (3.11) and integrate over $Q_T$ to obtain

$$\int_{Q_T} (v_n)_t T_k(v_n) + \int_{Q_T} |\nabla T_k(v_n)|^{p(x)} = \int_{Q_T} f_n T_k(v_n) + \int_{Q_T} \frac{g_n T_k(v_n)}{(n + 1/n)^q(x)} \leq C_2 k.$$

This gives

$$\int_{Q_T} |\nabla T_k(v_n)|^{p(x)} \leq C_3 k,$$

and hence

$$\|T_k(v_n)\|_{V^{p(\cdot)}(Q_T)} \leq C k. \quad (3.18)$$

By proceeding similarly with the test function $T_1(G_k(v_n))$ we get

$$\int_{Q_T \cap \{k < v_n < k + 1\}} |\nabla v_n|^{p(x)} \leq C_4. \quad (3.19)$$

With the consideration of Lemma 3.3 (3.19) and Lemma 2.1 of [5], we conclude that $\{v_n\}$ is bounded in $V^{r(\cdot)}(Q_T)$ for every $1 \leq r(x) < p(x) - \frac{N}{N+1}$ for all $x \in \overline{\Omega}$. \hfill \Box

**Lemma 3.5.** Let $\delta^+ \geq 1$. Then $\{v_n\}$ is bounded in $V^{r(\cdot)}_{loc}(Q_T)$ for every $1 \leq r(x) < p(x) - \frac{N}{N+1}$ for all $x \in \overline{\Omega}$. Moreover, $\{T_k^{p^+}(v_n)\}$ is bounded in $L^{p^-}(0,T;W^1_{0}^{1,p^-}(\Omega))$, for any $k > 0$.

**Proof.** The proof follows the method used in Lemma 3.4. Let $\varphi \in D(\Omega)$ be a nonnegative function and $k > 0$ is a fixed constant. We multiply $(T_1(G_k(v_n)) - 1)\varphi^{p^+}$ in (3.11) and then integrate by parts on $Q_T$ to get

$$\int_{Q_T} (v_n)_t (T_1(G_k(v_n)) - 1)\varphi^{p^+} + \int_{Q_T} |\nabla T_1(G_k(v_n))|^{p(x)}\varphi^{p^+} + p^+ \int_{Q_T} |\nabla v_n|^{p(x) - 2}\nabla v_n \cdot \nabla \varphi^{p^+ - 1}(T_1(G_k(v_n)) - 1) \leq 0.$$  

(3.20)
By the Definition of $T_{k,1}(\cdot)$, we obatin the following.

\[
\left| \int_{Q_T} (v_n)T_1(G_k(v_n)) - 1) \varphi^+ \right| \leq \left| \int_{Q_T} \left( \int_{0}^{v_0,n} T_1(G_k(s)) ds - v_0,n \right) \varphi^+(x) \right| + \left| \int_{Q_T} \left( \int_{0}^{v_n(T)} T_1(G_k(s)) ds - v_n(T) \right) \varphi^+(x) \right| \\
\leq C_1.
\]

Now with the help of Young’s inequality, from (3.20), we have

\[
\int_{Q_T} \left| \nabla T_1(G_k(v_n)) \right| p(x) \varphi^+ \leq C_1 + C_2 \epsilon \varphi^+ \int_{Q_T} \left| \nabla T_1(G_k(v_n)) \right| p(x) \varphi^+ + C_4 \varphi^+ \int_{Q_T} \left| \nabla \varphi \right| p(x).
\]

This implies

\[
\int_{Q_T} \left| \nabla T_1(G_k(v_n)) \right| p(x) \varphi^+ \leq C_3
\]

and

\[
\int_{Q_T \cap \{ k < v_n < k+1 \}} \left| \nabla v_n \right| p(x) \varphi^+ \leq C_4. \tag{3.21}
\]

By Lemma 3.3, 3.21 and Lemma 2.1 of \cite{5}, we conclude that \{ $v_n$ \} is bounded in $V_{loc}^{r(x)}(Q_T)$ for every $1 \leq r(x) < p(x) - \frac{N}{N+1}$ for all $x \in \Omega$.

Let us consider $T_k^{\delta^+}(v_n)$ as the test function in (3.9) and thus we have

\[
\int_{Q_T} (v_n)T_k^{\delta^+}(v_n) + \delta^+ \int_{Q_T} \left| \nabla T_k(v_n) \right| p(x) T_k^{\delta^+ - 1}(v_n) = \int_{Q_T} f_n T_k^{\delta^+}(v_n) + \int_{Q_T} g_n T_k^{\delta^+}(v_n) \left( v_n + 1/n \right)^{\delta(x)} \\
\leq C_4 k^{\delta^+}.
\]

On further simplification we get

\[
\int_{Q_T} (v_n)T_k^{\delta^+}(v_n) + \delta^+ \int_{Q_T} \left| \nabla T_k(v_n) \right| p(x) T_k^{\delta^+ - 1}(v_n) \leq C_4 k^{\delta^+} + \delta^+ \int_{Q_T} T_k^{\delta^+ - 1}(v_n)
\]

and thus

\[
\int_{Q_T} \left| \nabla T_k^{p^-} \right| (v_n) p^- \leq C_5.
\]

Hence, the sequence $\{ T_k^{p^-}(v_n) \}$ is bounded in $L^p^-(0,T; W_0^{1,p^-}(\Omega))$ for every $k > 0$. \hfill \Box

The succeeding theorem is the existence result for (3.9).

**Theorem 3.6.** There exists a nonnegative weak solution $v$ to (3.9) in the sense of Definition 3.1.
Proof. The proof follows the lines used in [32]. According to Lemma 3.4 and Lemma 3.5, the sequence $\{v_n\}$ is bounded in $V^{r(\cdot)}(Q_T)$ if $\delta^+ < 1$ and is bounded in $V^{r(\cdot)}_{loc}(Q_T)$ if $\delta^+ \geq 1$, for every $1 \leq r(\cdot) < p(\cdot) - \frac{N}{N+1}$. Therefore, for $\delta^+ < 1$, there exists a function $v \in V^{r(\cdot)}(Q_T)$ such that, up to a subsequence, $v_n$ converges to $v$ a.e. in $Q_T$, weakly in $V^{r(\cdot)}(Q_T)$. Similarly, for the case $\delta^+ \geq 1$, there exists $v \in V^{r(\cdot)}_{loc}(Q_T)$ such that, up to a subsequence, $v_n$ converges to $v$ a.e. in $Q_T$ and weakly in $V^{r(\cdot)}_{loc}(Q_T)$. Clearly, the right hand side of (3.11) is bounded in $L^1(0,T;L^1_{loc}(\Omega))$. Thus, $\{\frac{\partial (v_n\varphi)}{\partial t}\}$ is a bounded sequence in $L^{s^-}(0,T;W^{-1,s^-}(\Omega)) + L^1(Q_T)$ with $s(\cdot) = \frac{r(\cdot)}{p(\cdot)-1}$ for any $\varphi \in C^1_c(\Omega)$, $\varphi \geq 0$. This allows us to apply Corollary 4 of [40] to guarantee that $v_n$ strongly converges to $v$ in $L^1(0,T;L^1_{loc}(\Omega))$.

By Theorem 4.3 of [8], we deduce that $\nabla T_k(v_n)$ converges strongly to $\nabla T_k(v)$ in $L^p_{loc}(Q_T)$ for every $\widetilde{p}(\cdot) < p(\cdot)$ and for all $k > 0$. Theorem 4.3 of [8] was stated for the constant exponent case, i.e., for problems involving $p$-Laplacian. The same proof also follows in the variable exponent setup. Then in a standard way it follows that

$$\nabla v_n \rightarrow \nabla v \text{ a.e. in } Q_T.$$ 

Thus, by using Vitali’s theorem one can prove that $|\nabla v_n|^{p(\cdot)-1}$ strongly converges to $|\nabla v|^{p(\cdot)-1}$ in $L^1(0,T;L^1_{loc}(\Omega))$.

The weak formulation of (3.11) is given by

$$-\int_{Q_T} v_n \varphi_t - \int_{\Omega} v_0 \varphi(x,0) + \int_{Q_T} |\nabla v_n|^{p(x)-2}\nabla v_n \cdot \nabla \varphi = \int_{Q_T} f_n \varphi + \int_{Q_T} \frac{g_n \varphi}{(v_n + 1/n)^{\delta(x)}} \quad (3.22)$$

for every $\varphi \in C^1_c(\Omega \times [0,T])$. By the above arguments we can pass the limit $n \rightarrow \infty$ in the left hand side terms of (3.22). Since $(v_n + 1/n)^{-\delta(\cdot)} g_n$ is bounded in $L^1(0,T;L^1_{loc}(\Omega))$, using Fatou’s lemma we observe that $v^{-\delta(\cdot)} g \in L^1(0,T;L^1_{loc}(\Omega))$. Thus, by following the work of Oliva & Petitta [Theorem 2.3, [32]], it is clear that for any $\gamma > 0$,

$$\lim_{\gamma \rightarrow 0} \lim_{n \rightarrow \infty} \int_{Q_T \cap \{v_n \leq \gamma\}} \frac{g_n}{(v_n + 1/n)^{\delta(x)}} \varphi = 0. \quad (3.23)$$

This implies,

$$\lim_{\gamma \rightarrow 0} \lim_{n \rightarrow \infty} \int_{Q_T \cap \{v_n > \gamma\}} \frac{g_n}{(v_n + 1/n)^{\delta(x)}} \varphi = \int_{Q_T \cap \{v > 0\}} \frac{g}{v^{\delta(x)}} \varphi = \int_{Q_T} \frac{g}{v^{\delta(x)}} \varphi$$

and

$$\{(x,t) \in Q_T : u(x,t) = 0\} \subset \{(x,t) \in Q_T : g(x,t) = 0\},$$

except for a set of zero measure. Thus, we have

$$\lim_{n \rightarrow \infty} \int_{Q_T} \frac{g_n}{(v_n + 1/n)^{\delta(x)}} \varphi = \int_{Q_T} \frac{g}{v^{\delta(x)}} \varphi, \forall \varphi \in C^1_c(\Omega \times [0,T]).$$

Therefore, after passing the limit we conclude that $v$ is a weak solution to (3.9) in the sense of Definition 3.1 and satisfies the following equation.

$$-\int_{Q_T} v \varphi_t - \int_{\Omega} v_0 \varphi(x,0) + \int_{Q_T} |\nabla v|^{p(x)-2}\nabla v \cdot \nabla \varphi = \int_{Q_T} f \varphi + \int_{Q_T} \frac{g \varphi}{v^{\delta(x)}}. \quad (3.24)$$
According to Lemma 3.3 for $\delta^+ \geq 1$, the sequence $\{T_k^{p-\delta^+-1}(v_n)\}$ is uniformly bounded in $L^p(0, T; W^{1,p}_0(\Omega))$. Thus, $T_k^{p-\delta^+-1}(v(\cdot, t)) \in W^{1,p}_0(\Omega)$ for every $k > 0$ and every $t \in (0, T)$. It is now easy to prove that $\text{tr} \ v(\cdot, t) = 0$ for every $t \in (0, T)$ in the sense of Definition 2.3.

If $\delta^+ < 1$, then $T_k(v(\cdot, t)) \in W^{1,p}(\Omega)$ for every $t \in (0, T)$ and thus the Sobolev trace of $v$ is same as $\text{tr} \ v = 0$.

**Remark 3.7.** The same existence result holds if we replace $f$ by a positive bounded Radon measure $\mu$ in the problem (3.9).

### 4 Proof of Theorem 2.5

In this section, we establish the existence of weak solution to (1.1), i.e. we prove Theorem 2.5 through approximation. We follow the approach used in Section 3 to prove the existence result for (3.9). The preliminary step towards the proof of our main result is to construct the approximating scheme under suitable assumption on the $L^1$ data $f$ so that the problem (1.1) admits at least one solution.

**Step 1:** We assume the following restriction on the the function $f$.

(A3) $f \in L^1(Q_T)$ lies in the following class $\mathcal{C}$.

$$\mathcal{C} = \{ f \in L^1(Q_T) : \exists \{f_n\}_{n \in \mathbb{N}} \subset L^\infty(Q_T) \text{ s.t } f_n \to f \text{ in } L^1(Q_T) \text{ with } f_{n+1} \geq v_n^{q^{(\cdot)}-1} \text{ a.e. in } Q_T \}$$

where the function $q(\cdot)$ satisfies (A2) and $v_n$ is a unique non-negative weak solution to the following problem with $v_0 > 0$.

$$\begin{align*}
\frac{\partial v_n}{\partial t} - \Delta_{p(x)} v_n &= \frac{g_n}{(v_n + 1/n)^{\delta(x)}} + 2f_n \text{ in } Q_T, \\
v_n &= 0 \text{ on } \Sigma_T, \\
v_n(0, \cdot) &= v_{0,n}(\cdot) \text{ in } \Omega.
\end{align*}$$

Here the function $p(\cdot)$ satisfies (A1), $\delta > 0 \in C(\overline{\Omega})$, $v_{0,n} = T_n(v_0)$ with $v_0 \in L^r(\Omega)$ ($r \geq 2$) and $g_n = T_n(g)$, $g \in L^1(Q_T)$.

The existence of a unique $v_n \in V^{p(\cdot)}(Q_T) \cap L^\infty(Q_T)$ is guaranteed from Lemma 3.2. Let $f_n \to f$ in $L^1(Q_T)$, then by Lemma 3.3 and Lemma 3.5 we observe that, for every $r(\cdot) < p(\cdot) - \frac{N}{N+1}$, $\{v_n\}$ is bounded in $V^{r(\cdot)}(Q_T)$, if $\delta^+ < 1$ and is bounded in $V^{r(\cdot)}_{loc}(Q_T)$, if $\delta^+ \geq 1$.

**Step 2:** We construct the approximating problem to (1.1) as follows. Let us define the sequence $\{u_n\}_{n \in \mathbb{N}}$ by

1. $u_0 \leq v_0$ a.e. in $\Omega$.
2. $u_n$ is the unique nonnegative weak solution to the following problem

$$\begin{align*}
\frac{\partial u_n}{\partial t} - \Delta_{p(x)} u_n &= \lambda h_n(u_{n-1}) + \frac{g_n}{(u_n + 1/n)^{\delta(x)}} + f_n \text{ in } Q_T, \\
u_n &= 0 \text{ on } \Sigma_T, \\
u_n(0, \cdot) &= u_{0,n}(\cdot) \text{ in } \Omega.
\end{align*}$$

(4.26)
where \( u_{0,n} = T_n(u_0), \ g_n = T_n(g), \ h_n(u_{n-1}) = T_n(u_{n-1}^{q(-1)}), \) the functions \( p(\cdot), \ q(\cdot) \) satisfy the assumptions (A1)-(A2) and \( f_n \in L^\infty(Q_T) \) such that \( f_n \to f \in C \) in \( L^1(Q_T) \).

The existence of a unique \( u_n \in V^{p(\cdot), q(\cdot)}_0(Q) \) follows from Lemma 3.2.

**Lemma 4.1.** Let \( u_n \) be a unique weak solution to the approximating problem (4.26). Then the sequence \( \{u_n\} \) is uniformly bounded in \( L^{q(-1)}(Q_T) \).

**Proof.** From the construction of the sequence \( \{u_n\} \), we have \( u_0 \leq v_0 \) a.e. in \( \Omega \). Choose \( \lambda \leq 1 \). Thus, by the straight forward application of the weak comparison principle, Theorem 2.8, we get \( u_1 \leq v_1 \) a.e. in \( Q_T \), where \( v_1 \) is the unique weak solution to (4.25) with \( n = 1 \). Let us assume that \( u_{n-1} \leq v_{n-1} \) a.e. in \( Q_T \). Then from (4.26) we have

\[
\frac{\partial u_n}{\partial t} - \Delta_{p(x)} u_n - \frac{g_n}{(u_n + \frac{1}{n})^{\delta(x)}} = \lambda h_n(u_{n-1}) + f_n
\]

\[
\leq u_{n-1}^{q(-1)} + f_n
\]

\[
\leq v_{n-1}^{q(-1)} + f_n
\]

\[
\leq 2f_n
\]

\[
= \frac{\partial v_n}{\partial t} - \Delta_{p(x)} v_n - \frac{g_n}{(v_n + \frac{1}{n})^{\delta(x)}}.
\]

Thus, by the weak comparison principle, we conclude that for every \( n \in \mathbb{N} \),

\[
u_n \leq v_n, \text{ a.e. in } Q_T.
\]

(4.27)

Since \( u_n^{q(-1)} \leq f_{n+1} \) a.e. in \( Q_T \) and \( f_n \to f \in C \subset L^1(Q_T) \), the sequence \( \{u_n\} \) is uniformly bounded in \( L^{q(-1)}(Q_T) \).

**Proof of Theorem 2.5.** Let the functions \( f \in L^1(Q_T), \ p(\cdot) \) and \( q(\cdot) \) satisfy the assumptions (A1) – (A3). Further, let us assume \( \lambda \leq 1, \ g \in L^1(Q_T), \ u_0 \in L^r(\Omega) \) with \( r \geq 2 \) and \( u_n \) is a unique nonnegative weak solution of (4.26). Then from Lemma 4.1, the sequence \( \{u_n\} \) is uniformly bounded in \( L^{q(-1)}(Q_T) \).

On readapting the methods used in Lemma 3.3 and Lemma 3.4, we prove that for \( \delta^+ < 1, \{u_n\} \) is bounded in \( L^\infty(0, T; L^1(\Omega)) \cap V^{p(\cdot)}(Q_T) \) for every \( 1 \leq r(\cdot) < p(\cdot) - \frac{N}{N+1} \). Moreover, for every \( k > 0, \} T_k(u_n) \} \) is bounded in \( V^{p(\cdot)}(Q_T) \). Similarly, by the virtue of Lemma 3.3, we deduce that for \( \delta^+ \geq 1, \} u_n \} \) is bounded in \( L^\infty(0, T; L^1(\Omega)) \cap V^{p(\cdot)}_{loc}(Q_T) \) for every \( 1 \leq r(\cdot) < p(\cdot) - \frac{N}{N+1} \) and \( \{T_k^{p(\cdot) - \frac{1}{p(-1)}}(u_n) \) \) is bounded in \( L^{p(\cdot)}(0, T; W^{1,p(\cdot)}_{loc}(\Omega)) \), for any \( k > 0 \). Therefore, there exists a function \( u \) such that, up to a subsequence, \( u_n \) converges to \( u \) a.e. in \( Q_T \), weakly in \( V^{p(\cdot)}(Q_T) \) if \( \delta^+ < 1 \) and weakly in \( V^{p(\cdot)}_{loc}(Q_T) \) if \( \delta^+ \geq 1 \).

According to Theorem 3.6, \( u_n \to u \) strongly in \( L^1(0, T; L^1_{loc}(\Omega)), u_n^{q(-1)} \to u^{q(-1)} \) strongly in \( L^1(0, T; L^1_{loc}(\Omega)) \) and

\[
\lim_{n \to \infty} \int_{Q_T} \frac{g_n}{(u_n + \frac{1}{n})^{\delta(x)}} \varphi = \int_{Q_T} \frac{g}{u^{\delta(x)}} \varphi, \ \forall \varphi \in C_c^1(\Omega \times [0, T]).
\]
Furthermore, $\nabla u_n$ converges almost everywhere to $\nabla u$ in $Q_T$ and hence $|\nabla u_n|^{p(-1)}$ strongly converges to $|\nabla u|^{p(1)-1}$ in $L^1(0, T; L^1_{loc}(\Omega))$. We are now in the position to pass the limit $n \to \infty$ in the weak formulation of (4.26), i.e. in

$$-\int_{Q_T} u_n \varphi_t - \int_{\Omega} u_{0,n} \varphi(x, 0) + \int_{Q_T} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \varphi = \int_{Q_T} \lambda h_n(u_{n-1}) \varphi + \int_{Q_T} \frac{g_n \varphi}{(u_n + \frac{1}{n})^{p(x)}} + \int_{Q_T} f \varphi$$

for every $\varphi \in C^1_c(\Omega \times [0, T])$. Thus, $u$ is a nonnegative weak solution to (1.1), in the sense of Definition 2.4 such that $u$ satisfies

$$-\int_{Q_T} u \varphi_t - \int_{\Omega} u \varphi(x, 0) + \int_{Q_T} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi = \int_{Q_T} (\lambda u^{q(x)-1} + f) \varphi + \int_{Q_T} \frac{g \varphi}{u^{\delta(x)}}$$

for every $\varphi \in C^1_c(\Omega \times [0, T])$. This completes the proof.

\section{Proof of Theorem 2.7}

In this section, we discuss the problem (1.3) and prove our second main result, i.e. Theorem 2.7. For this purpose, let us consider the following approximating problem.

$$\frac{\partial u_n}{\partial t} - \Delta_{p(x)} u_n = \lambda h_n(u_n) + (u_n + 1/n)^{-\delta(x)} \tilde{g} \text{ in } Q_T,$$

$$u_n > 0 \text{ in } Q_T,$$

$$u_n = 0 \text{ on } \Sigma_T,$$

$$u_n(0, \cdot) = u_{0,n}(\cdot) \text{ in } \Omega$$

(5.28)

Here the functions $p(\cdot)$, $q(\cdot)$ and the constant $r$ satisfy the hypotheses (B1) – (B4), $\lambda > 0$, $\delta : \overline{\Omega} \to (0, \infty)$ is a continuous function, $h_n(u_n) = T_n(u_n^{q(-1)})$, $\tilde{g} \in L^\infty(\Omega)$ and $u_{0,n} = T_n(u_0)$.

\subsection{Existence result for the problem (5.28)}

Let $M \gg 1$, $\eta = \frac{T}{M}$ and for $0 \leq m \leq M$ we define $t_m = m\eta$. Consider the following elliptic problem.

$$\frac{u_n^m - u_n^{m-1}}{\eta} - \Delta_{p(x)} u_n^m = \lambda h_n(u_n^{m-1}) + (u_n^m + 1/n)^{-\delta(x)} \tilde{g} \text{ in } \Omega,$$

$$u_n^m > 0 \text{ in } \Omega,$$

$$u_n^m = 0 \text{ on } \partial \Omega.$$

(5.29)

The iteration starts from the initial condition $u_n^0 = u_{0,n} \in L^\infty(\Omega)$ and $u_n^1$ satisfies

$$\frac{u_n^1 - u_n^0}{\eta} - \Delta_{p(x)} u_n^1 = \lambda h_n(u_n^0) + (u_n^1 + 1/n)^{-\delta(x)} \tilde{g} \text{ in } \Omega,$$

$$u_n^1 > 0 \text{ in } \Omega,$$

$$u_n^1 = 0 \text{ on } \partial \Omega.$$
It is not difficult to show the existence of a weak solution \( u_n \) to (5.29) in \( W_0^{1,p} (\Omega) \) for any \( m \geq 1 \). Before dealing with the problem (5.28), we discuss the corresponding stationary problem (5.29). Then by the semi-discretization approach in time, we obtain a nonnegative weak solution to (5.28).

For \( 0 \leq m \leq M \) and \( t \in [t_{m-1}, t_m) \), motivated by the implicit Euler’s method, we define the functions \( u_{n, \eta} \) and \( \tilde{u}_{n, \eta} \) as follows.

\[
  u_{n, \eta}(t, \cdot) = u_n^m(\cdot)
\]

and

\[
  \tilde{u}_{n, \eta}(t, \cdot) = \frac{u_n^m(\cdot) - u_n^{m-1}(\cdot)}{\eta t}(t - t_{m-1}) + u_n^{m-1}(\cdot), \quad u_{n, \eta}(0, \cdot) = u_n^0(\cdot) = u_{0,n}(\cdot).
\]

Clearly, \( u_{n, \eta} \) and \( \tilde{u}_{n, \eta} \) satisfy

\[
  \frac{\partial \tilde{u}_{n, \eta}}{\partial t} - \Delta_p(x) u_{n, \eta} = \lambda h_n (u_{n, \eta} (\cdot - \eta t)) + (u_{n, \eta} + 1/n)^{-\delta(x)} \tilde{g} \text{ in } (\eta, T) \text{ for any } \eta > \eta_t. \quad (5.31)
\]

We will now follow the proof of Theorem 3.1 of [13] to obtain some uniform estimates for \( u_{n, \eta} \) and \( \tilde{u}_{n, \eta} \) independent of \( \eta_t \).

**Claim 1**: The sequences \( \{ u_{n, \eta} \} \) and \( \{ \tilde{u}_{n, \eta} \} \) are uniformly bounded in \( V^p(Q_T) \) and \( V^p((\eta, T) \times \Omega) \), respectively, for every \( 0 < \eta < T \) independent of \( \eta_t \).

**Proof**: Let us multiply \( \eta_t u_n^m \) in (5.29). Then we integrate over \( \Omega \) and take the sum from \( m = 1 \) to \( m' \leq M \) to obtain

\[
  \sum_{m=1}^{m'} \int_{\Omega} (u_n^m - u_n^{m-1}) u_n^m + \eta_t \sum_{m=1}^{m'} \int_{\Omega} |\nabla u_n^m|^p(x) = \eta_t \sum_{m=1}^{m'} \int_{\Omega} (u_{n, \eta} + 1/n)^{-\delta(x)} \tilde{g} + \lambda \eta_t \sum_{m=1}^{m'} \int_{\Omega} h_n(u_n^{m-1}) u_n^m.
\]

(5.32)

By Young’s inequality we estimate the right hand side terms of (5.32).

\[
  \int_{\Omega} \frac{\tilde{g} u_n^m}{(u_{n, \eta} + 1/n)^{\delta(x)}} \leq \int_{\Omega} (u_n^m)^2 + C(n)
\]

and

\[
  \int_{\Omega} h_n(u_n^{m-1}) u_n^m \leq \int_{\Omega} (u_n^m)^2 + C(n)
\]

where \( C(n) \) is only dependent on \( n \). The first term in the left hand side of (5.32) can be rewritten as follows.

\[
  \sum_{m=1}^{m'} \int_{\Omega} (u_n^m - u_n^{m-1}) u_n^m = \sum_{m=1}^{m'} \int_{\Omega} \frac{1}{2} [(u_n^m - u_n^{m-1})^2 + (u_n^m)^2 - (u_n^{m-1})^2] = \frac{1}{2} \sum_{m=1}^{m'} \int_{\Omega} (u_n^m - u_n^{m-1})^2 + \frac{1}{2} \sum_{m=1}^{m'} \int_{\Omega} (u_n^m)^2 - \frac{1}{2} \sum_{m=1}^{m'} \int_{\Omega} (u_n^0)^2.
\]
Now by substituting the above estimates in (5.32) we get

\[
\frac{1}{2} \sum_{m=1}^{m'} \int_\Omega [(u_n^m - u_{n-1}^m)^2 + \frac{1}{2} \int_\Omega (u_n^m')^2 + \eta \sum_{m=1}^{m'} \int_\Omega |\nabla u_n^m|^p(x) \leq 2\eta \sum_{m=1}^{m'} \int_\Omega (u_n^m)^2 + \tilde{C}(n).
\]

Thus, by [13] it follows that \{u_n,\eta\} and \{\tilde{u}_n,\eta\} are bounded in \(L^\infty(0, T; L^2(\Omega))\) independently of \(\eta\). This proves the claim. \(\square\)

**Claim 2:** \(\{u_n,\eta\}\) and \(\{\tilde{u}_n,\eta\}\) are bounded in \(L^\infty(Q_T)\).

**Proof.** Let us consider the positive function \(v_n\) that satisfies

\[
\frac{\partial v_n}{\partial t} - \Delta_p(x)v_n = \lambda \cdot n + n \delta(x) \tilde{g} \text{ in } Q_T,
\]

\[
v_n = 0 \text{ on } \Sigma_T,
\]

\[
v_n = n \text{ in } \Omega.
\]

Denote \(v_n^m = v_n(t_m)\) for \(0 \leq m \leq M\). Thus, \(v_n^m\) satisfies

\[
\frac{v_n^m - v_{n-1}^m}{\eta} - \Delta_p(x)v_n^m = \lambda \cdot n + n \delta(x) \tilde{g} \text{ in } \Omega,
\]

\[
v_n^m > 0 \text{ in } \Omega,
\]

\[
v_n^m = 0 \text{ on } \partial \Omega.
\]

Therefore, using the weak comparison principle we guarantee that \(v_n^m\) is a weak supersolution to (5.29) and for every \(0 \leq m \leq M\), \(u_n^m \leq v_n^m \leq C(T) < \infty\) independently of \(m\). \(\square\)

**Claim 3:** \(\{u_n,\eta\}\) is bounded in \(L^\infty(\eta, T; W^{1,p}_0(\Omega))\) and \(\{\frac{\partial \tilde{u}_n,\eta}{\partial t}\}\) is bounded in \(L^2((\eta, T) \times \Omega)\) for every \(0 < \eta < T\). Moreover, \(\{\frac{\partial \tilde{u}_n,\eta}{\partial t}\}\) is bounded in \(V^{p,\cdot}(Q_T)^*\).

**Proof.** Let us multiply \(\frac{t_m + t_{m-1}}{2}(u_n^m - u_{n-1}^m)\) in (5.29). Then we integrate over \(\Omega\) and take the sum from \(m = 2\) to \(m' \leq M\) to obtain the following equation.

\[
\frac{\eta}{2} \sum_{m=2}^{m'} (t_m + t_{m-1}) \int_\Omega \left(\frac{u_n^m - u_{n-1}^m}{\eta}\right)^2 + \frac{1}{2} \sum_{m=2}^{m'} (t_m + t_{m-1}) \int_\Omega |\nabla u_n^m|^p(x) - 2u_n^m \cdot \nabla (u_n^m - u_{n-1}^m)
\]

\[
= \frac{1}{2} \sum_{m=2}^{m'} (t_m + t_{m-1}) \int_\Omega \tilde{g}(u_n^m + 1/n)^{-\delta(x)}(u_n^m - u_{n-1}^m) + \frac{1}{2} \sum_{m=2}^{m'} (t_m + t_{m-1}) \int_\Omega \lambda h_n(u_n^m - u_{n-1}^m).
\]

(5.35)
Using the properties of convex functions, we estimate the followings.

\[
\frac{1}{2} \sum_{m=2}^{m'} (t_m + t_{m-1}) \int_{\Omega} |\nabla u_n^m|^{p(x)-2} \nabla u_n^m \cdot (\nabla u_n^m - \nabla u_{n-1}^m) \geq \frac{m'}{2p} \sum_{m=2}^{m'} (t_m + t_{m-1}) \int_{\Omega} (|\nabla u_n^m|^{p(x)} - |\nabla u_{n-1}^m|^{p(x)})
\]

\[
= \frac{t_{m'}}{p} \int_{\Omega} |\nabla u_n^{m'}|^{p(x)} - \frac{\eta_t}{p} \int_{\Omega} |\nabla u_n^1|^{p(x)} - \frac{\eta_t}{2p} \sum_{m=2}^{m'} \int_{\Omega} (|\nabla u_n^m|^{p(x)} + |\nabla u_{n-1}^m|^{p(x)})
\]

\[
\geq \frac{t_{m'}}{p} \int_{\Omega} |\nabla u_n^{m'}|^{p(x)} - \frac{2}{p} \int_0^{t_{m'}} \int_{\Omega} |\nabla u_{n,\eta_t}|^{p(x)}
\]

and

\[
\frac{1}{2} \sum_{m=2}^{m'} (t_m + t_{m-1}) \int_{\Omega} \tilde{g}(u_n^m + 1/n)^{-\delta(x)} (u_n^m - u_{n-1}^m) \leq \frac{1}{2(1-\delta^+)} \sum_{m=2}^{m'} (t_m + t_{m-1}) \int_{\Omega} \tilde{g} ((u_n^m + 1/n)^{1-\delta(x)} - (u_{n-1}^m + 1/n)^{1-\delta(x)})
\]

\[
= \frac{t_{m'}}{2(1-\delta^+)} \int_{\Omega} \tilde{g}(u_n^{m'} + 1/n)^{1-\delta(x)} - \frac{\eta_t}{2(1-\delta^+)} \int_{\Omega} \tilde{g}(u_n^1 + 1/n)^{1-\delta(x)}
\]

\[
- \frac{\eta_t}{(1-\delta^+)} \sum_{m=2}^{m'} \int_{\Omega} (\tilde{g}(u_n^m + 1/n)^{1-\delta(x)} + (u_{n-1}^m + 1/n)^{1-\delta(x)})
\]

\[
\leq \frac{t_{m'}}{2(1-\delta^+)} \int_{\Omega} \tilde{g}(u_n^{m'} + 1/n)^{1-\delta(x)} + C \int_0^{t_{m'}} \int_{\Omega} \tilde{g}(u_{n,\eta_t} + 1/n)^{1-\delta(x)}. \]

The last term of (5.35) can be approximated using Young’s inequality.

\[
\frac{1}{2} \sum_{m=2}^{m'} \lambda h_n(u_{n-1}^m)(u_n^m - u_{n-1}^m) \leq \eta_t \sum_{m=2}^{m'} (t_m + t_{m-1}) \left[ \int_{\Omega} (\lambda h_n(u_{n-1}^m))^2 + \frac{1}{4} \int_{\Omega} \left( \frac{u_n^m - u_{n-1}^m}{\eta_t} \right)^2 \right]
\]

\[
\leq 2T\lambda^2 \int_0^{t_{m'}} \int_{\Omega} (h_n(u_{n,\eta_t}))^2 + \frac{\eta_t}{4} \sum_{m=2}^{m'} (t_m + t_{m-1}) \int_{\Omega} \left( \frac{u_n^m - u_{n-1}^m}{\eta_t} \right)^2.
\]

Thus, by substituting the above bounds in (5.35), we have

\[
\frac{1}{2} \int_0^{t_{m'}} \int_{\Omega} \left| \frac{\partial u_{n,\eta_t}}{\partial t} \right|^2 \leq \frac{t_{m'}}{p} \int_{\Omega} |\nabla u_n^{m'}|^{p(x)} \leq \frac{1}{2} \int_{\Omega} (u_n^1 - u_n^0)^2 + \frac{t_{m'}}{2(1-\delta^+)} \int_{\Omega} \tilde{g}(u_n^{m'} + 1/n)^{1-\delta(x)}
\]

\[
+ C \int_0^{t_{m'}} \int_{\Omega} \tilde{g}(u_{n,\eta_t} + 1/n)^{1-\delta(x)}
\]

\[
+ 2T\lambda^2 \int_0^{t_{m'}} \int_{\Omega} (h_n(u_{n,\eta_t}))^2 + \frac{2}{p} \int_0^{t_{m'}} \int_{\Omega} |\nabla u_{n,\eta_t}|^{p(x)}. \]
On using the estimates obtained from Claim 1 and Claim 2, we obtain
\[
\left\| \frac{\partial \tilde{u}_{n,\eta}}{\partial t} \right\|_{L^2(\eta,T;L^2(\Omega))} \leq C_1. \text{ for any } \eta \in (0, T),
\] (5.36)
and
\[
\sup_{t \in (0,T)} t \int_\Omega |\nabla u_{n,\eta}|^{p(x)} \leq \max_{0 \leq m' \leq M} t_{m'} \int_\Omega |\nabla u_{n,m'}|^{p(x)} \leq C_2
\] (5.37)
where \( C_1, C_2 \) are two positive constants independent of \( \eta \). By considering \( 5.31 \) we deduce that \( \{ \frac{\partial \tilde{u}_{n,\eta}}{\partial t} \} \) is bounded in \( V^{p(\cdot)}(Q_T)^* \).

We have obtained all the required energy estimates. We now state the existence result for \( 5.28 \) in the following theorem.

**Theorem 5.1.** The problem \( 5.28 \) possesses at least one positive weak solution \( u_n \in V^{p(\cdot)}(Q_T) \cap L^\infty(Q_T) \cap C(0, T; L^2(\Omega)) \).

**Proof.** With the consideration of the energy estimates obtained from Claim 1, 2 and 3, there exists \( u_n \) and \( \tilde{u}_n \) such that, up to a subsequence, as \( \eta \to 0^+ \) (i.e. as \( m \to \infty \))
\[
u_{n,\eta} \to u_n \text{ weakly in } V^{p(\cdot)}(Q_T),
\]
\[
\tilde{u}_{n,\eta} \to \tilde{u}_n \text{ weak starly in } L^\infty(Q_T) \cap L^\infty(\eta,T,W^{1,p(\cdot)}(\Omega)), \forall \eta \in (0, T),
\]
\[
\frac{\partial \tilde{u}_{n,\eta}}{\partial t} \to \frac{\partial \tilde{u}_n}{\partial t} \text{ weakly in } V^{p(\cdot)}(Q_T)^* \cap L^2((\eta,T) \times \Omega), \forall \eta \in (0, T).
\]
For \( M' > 1 \), there exists a unique \( M' \) such that \( \eta \in (t_{M'}, t_{M'+1}] \) and
\[
\|u_{n,\eta} - \tilde{u}_{n,\eta}\|_{L^\infty(\eta,T;L^2(\Omega))} \leq 2 \max_{M' \leq m \leq M} \|u_{n} - u_{n,m-1}\|_{L^2(\Omega)} \to 0, \text{ as } \eta \to 0^+.
\] (5.38)

This implies, \( u_n = \tilde{u}_n \) in a.e. \( Q_T \). Since \( u_n \in V^{p(\cdot)}(Q_T)^* \cap V^{p(\cdot)}(Q_T) \), by Theorem 4.2 of \( [4] \), page 167, we deduce that \( u_n \in C(0, T; L^2(\Omega)) \). Further, according to the Aubin-Lions-Simon Lemma (see \( 40 \)), we obtain the following compactness result.
\[
u_{n,\eta} \longrightarrow u_n \text{ in } L^2(Q_T)
\] (5.39)
and by \( 5.38 \)
\[
\tilde{u}_{n,\eta} \longrightarrow u_n \text{ in } L^2((\eta,T) \times \Omega), \forall \eta \in (0, T).
\] (5.40)

The next claim is that \( u_n \) is a weak solution to \( 5.28 \). For this purpose, let us multiply \( (u_{n,\eta} - u_n) \) in \( 5.31 \) and integrate over \( Q_{T,\eta} = (\eta,T) \times \Omega \). Thus, we obtain the following.
\[
\int_{Q_{T,\eta}} \frac{\partial \tilde{u}_{n,\eta}}{\partial t} (u_{n,\eta} - u_n) + \int_{Q_{T,\eta}} |\nabla u_{n,\eta}|^{p(x)} - 2 \nabla u_{n,\eta} \cdot (u_{n,\eta} - u_n) = \lambda \int_{Q_{T,\eta}} h_n(u_{n,\eta}(\cdot - \eta))(u_{n,\eta} - u_n) + \int_{Q_{T,\eta}} g(u_{n,\eta} + 1/n)^{\delta(x)}(u_{n,\eta} - u_n)
\] (5.41)
By using the Dominated convergence theorem, (5.39) and (5.40), we obtain
\[
\int_{Q_{T,\eta}} h_n(u_{n,\eta}(\cdot - \eta t))(u_{n,\eta} - u_n) = o_{\eta}(1)
\]
\[
\int_{Q_{T,\eta}} \tilde{g}(u_{n,\eta} + 1/n)^{-\delta(x)}(u_{n,\eta} - u_n) = o_{\eta}(1)
\]
and thus by convexity argument
\[
\frac{1}{2} \int_\Omega (\tilde{u}_{n,\eta n} - u_n)^2(T) + \frac{1}{p^+} \left( \int_{Q_{T,\eta}} |\nabla u_{n,\eta t}|^{p(x)} - \int_{Q_{T,\eta}} |\nabla u_n|^{p(x)} \right) \leq o_{\eta}(1) + \frac{1}{2} \int_\Omega (\tilde{u}_{n,\eta n} - u_n)^2(\eta).
\]

(5.42)

This implies
\[
\lim_{\eta_t \to 0^+} \int_{Q_{T,\eta}} |\nabla u_{n,\eta t}|^{p(x)} \leq \int_{Q_{T,\eta}} |\nabla u_n|^{p(x)}.
\]

By using the weak convergence of \( u_{n,\eta t} \) to \( u_n \) in \( V^{p(\cdot)}(Q_T) \), we deduce that
\[
\lim_{\eta_t \to 0^+} \int_{Q_{T,\eta}} |\nabla u_{n,\eta t}|^{p(x)} = \int_{Q_{T,\eta}} |\nabla u_n|^{p(x)}.
\]

Thus,
\[
\nabla u_{n,\eta t} \to \nabla u_n \text{ in } L^{p(\cdot)}((\eta, T) \times \Omega), \forall \eta \in (0, T) \text{ as } \eta_t \to 0^+.
\]

Consequently, there exists a sequence \((\eta_t)_m \to 0\) as \( m \to \infty \), such that
\[
\nabla u_{n,\eta_t}_m \to \nabla u_n \text{ a.e. in } Q_T \text{ as } m \to \infty.
\]

(5.43)

Thus, by these compactness results it is easy to show that \( u_n \) is a weak solution to (5.28).

It is left to prove that \( u_n(0) = u_{0,n} \). Let us multiply \( \varphi \in \mathcal{D}(\Omega) \) in (5.31) and integrate over \((t_1, t_2) \times \Omega\) for \(0 < t_1 < t_2\). Thus, we get
\[
\int_\Omega \tilde{u}_{n,\eta_t}(t_2)\varphi - \int_\Omega \tilde{u}_{n,\eta_t}(t_1)\varphi + \int_{t_1}^{t_2} \int_\Omega |\nabla u_{n,\eta_t}|^{p(x)-2} \nabla u_{n,\eta_t} \cdot \nabla \varphi
\]
\[
= \int_{t_1}^{t_2} \int_\Omega \tilde{g}(u_{n,\eta_t} + 1/n)^{-\delta(x)}\varphi + \int_{t_1}^{t_2} \int_\Omega h_n(u_{n,\eta_t}(\cdot - \eta_t))\varphi.
\]

(5.44)

Following (5.40) and passing the limit \( t_1 \to 0 \) in (5.44) we have
\[
\int_\Omega \tilde{u}_{n,\eta_t}(t_2)\varphi - \int_\Omega u_{0,n}\varphi + \int_{0}^{t_2} \int_\Omega |\nabla u_{n,\eta_t}|^{p(x)-2} \nabla u_{n,\eta_t} \cdot \nabla \varphi
\]
\[
= \int_{0}^{t_2} \int_\Omega \tilde{g}(u_{n,\eta_t} + 1/n)^{-\delta(x)}\varphi + \int_{0}^{t_2} \int_\Omega h_n(u_{n,\eta_t}(\cdot - \eta_t))\varphi.
\]

(5.45)
On the interval \((0, \eta)\), we have used \(h_n(u_{n, \eta}(\cdot - \eta_t)) = h_n(u_{0, n})\). By using the Lebesgue theorem and the energy estimates for \(u_{n, \eta}, \tilde{u}_{n, \eta}\), we pass the limit \(\eta_t \to 0\) and \(t_2 \to 0\) in \((5.44)\) to obtain the following.

\[
\lim_{t \to 0} \int_{\Omega} u_n(t) \varphi = \int_{\Omega} u_{0, n} \varphi. \tag{5.46}
\]

Since, \(u_n \in C(0, T; L^2(\Omega))\), we deduce that \(u_n(0, x) = u_{0, n}(x)\) for every \(x \in \Omega\). This concludes the proof. \(\square\)

### 5.2 A priori estimates for \(u_n\)

To obtain a weak solution to the problem \((1.3)\), we have to pass the limit \(n \to \infty\) in the weak formulation of its approximating problem \((5.28)\). Thus, it is essential to prove the following proposition which provides a uniform \(L^\infty\) bound for \(u_n\) independent of \(n\).

**Proposition 5.2.** Let \(u_n\) be a weak solution to \((5.28)\). Then there exists \(\overline{T} > 0\) such that for every \(T < \overline{T}\), \(u_n\) verifies the following properties.

1. There exists \(K_1 > 0\) such that \(\|u_n(t)\|_{L^r(\Omega)} \leq K_1 < \infty\) independent of \(n\), for every \(t \in (0, T)\).

2. For every \(\eta \in (0, T)\), there exists \(K_\eta > 0\) such that \(\|u_n(t)\|_{L^\infty(\Omega)} \leq K_\eta < \infty\) independent of \(n\), for every \(t \in (\eta, T)\).

**Proof.** The proof follows the similar argument as in Proposition 4.2 of \([13]\). For a fixed \(\rho > \max\{\delta - 1, 0\}\), let us multiply \(u_n^{\rho + 1}\) in \((5.28)\) and then integrate by parts on \(\Omega\) to obtain

\[
\frac{1}{2 + \rho \delta} \int_{\Omega} |u_n(t)|^{2 + \rho} dx + (1 + \rho) \int_{\Omega} |\nabla u_n(t)|^\rho |u_n(t)|^\rho = \lambda \int_{\Omega} h_n(u_n(t)) u_n(t)^{1+\rho} + \int_{\Omega} \tilde{g}(u_n + 1/n)^{-\delta(x)} u_n(t)^{1+\rho} \\
\leq \lambda \int_{\Omega} u_n^{q(x)+\rho} + \int_{\Omega} \tilde{g} u_n^{1+\rho-\delta(x)} + \int_{\Omega} \tilde{g} u_n^{1+\rho} \\
\leq \lambda \left( \int_{\Omega} u_n^{q+\rho} + \int_{\Omega} u_n^\rho \right) + \int_{\Omega} \tilde{g} u_n^{1+\rho-\delta} + \int_{\Omega} \tilde{g} u_n^{1+\rho} \\
\leq \lambda \int_{\Omega} u_n^{q+\rho} + C_1 \|u_n\|_{L^{2+\rho}(\Omega)}^{\rho} + C_2 \|u_n\|_{L^{2+\rho}(\Omega)}^{1+\rho-\delta} + C_3 \|u_n\|_{L^{2+\rho}(\Omega)}^{1+\rho}. \tag{5.47}
\]

From \((B3)\) we have \(q^+ < p^- (1 + \frac{r}{N})\), where \(r\) satisfies \((B4)\). Choose \(\rho = r - 2\). Thus, by applying Lemma 5.1 of \([41]\), we establish the following Gagliardo - Nirenberg type estimate.

\[
\lambda \int_{\Omega} u_n^{q+\rho} \leq \frac{1 + \rho}{4} \left( \int_{\Omega} |\nabla u_n|^{p^-} u_n^\rho + C_4 \|u_n\|_{L^{2+\rho}(\Omega)}^{q^++\rho+(q^+-p^-)E(\rho)} \right) \\
\leq \frac{1 + \rho}{4} \left( \int_{\Omega} |\nabla u_n|^{p(x)} u_n^\rho + \int_{\Omega} u_n^\rho \right) + C_4 \|u_n\|_{L^{2+\rho}(\Omega)}^{q^++\rho+(q^+-p^-)E(\rho)} \\
\leq \frac{1 + \rho}{4} \left( \int_{\Omega} |\nabla u_n|^{p(x)} u_n^\rho + C_5 \|u_n\|_{L^{2+\rho}(\Omega)}^{\rho} + C_4 \|u_n\|_{L^{2+\rho}(\Omega)}^{q^++\rho+(q^+-p^-)E(\rho)} \right) \tag{5.48}
\]
where \( E(\rho) = \frac{q^{+2} - 2}{p - (1 + \frac{2}{N})q^{-}} \). The validity of the above inequality can be seen for every \( \bar{\rho} \geq \rho \).

Let us denote \( B(\rho) = \frac{q^{+} + \rho + (q^{-} - p^{-})E(\rho)}{2 + \rho} \geq 1 \). On combining (5.47) and (5.48) we have

\[
\frac{1}{2 + \rho(t)} \frac{d}{dt} \int_{\Omega} |u_n(t)|^{2 + \rho(t)} \, dx = \int_{\Omega} \frac{\partial u_n}{\partial t} \, u_n^{1+\rho(t)} + \frac{\rho'(t)}{2 + \rho(t)} \int_{\Omega} u_n^{2+\rho(t)} \log u_n
\]
\[
= -(1 + \rho(t)) \int_{\Omega} \left| \nabla u_n(t) \right|^{p(x)} |u_n(t)|^\rho + \lambda \int_{\Omega} h_n(u_n(t))u_n(t)^{1+\rho} + \int_{\Omega} \tilde{g}(u_n + 1/n)^{-\delta(x)}u_n(t)^{1+\rho} + \frac{\rho'(t)}{2 + \rho(t)} \int_{\Omega} u_n^{2+\rho} \log u_n.
\]

(5.50)

The above inequality (5.49) is a similar type of differential inequality as given in equation (4.13) of [13]. Thus, with a sub and super solution approach as used in the proof of Proposition 4.2 of [13], we can prove the first property (1).

It is left to prove (2). Let us consider a \( C^1 \) function \( \rho : [0, T) \to [2, \infty) \) with \( \rho'(t) > 0 \) for every \( t \in [0, T) \) and \( \rho(0) = \rho_0 = r - 2 \). By using the result in L. Gross [25], Lemma 1.1, p. 1065], we obtain

\[
\frac{\rho'(t)}{2 + \rho(t)} \int_{\Omega} u_n^{2+\rho(t)} \log u_n \leq \frac{1 + \rho}{4} \int_{\Omega} \left| \nabla u_n \right|^{p^{-}} |u_n|^\rho + C_9 \left( \frac{\sigma \rho'(t)}{(2 + \rho)(1 + \rho)^{2/p^{-}}} \right) \frac{p^{-}}{p^{-} - 2} \| u_n \|_{L^{2+\rho}(\Omega)}
\]
\[
+ \psi_n(\rho)|u_n|^{2+\rho} + \frac{\rho'(t)}{2 + \rho(t)} \| u_n \|_{L^{2+\rho}(\Omega)} \log \| u_n \|_{L^{2+\rho}(\Omega)}
\]
\[
\leq \frac{1 + \rho}{4} \int_{\Omega} \left| \nabla u_n \right|^{p^{-}} |u_n|^\rho + C_{10} \| u_n \|_{L^{2+\rho}(\Omega)}
\]
\[
+ C_9 \left( \frac{\sigma \rho'}{(2 + \rho)(1 + \rho)^{2/p^{-}}} \right) \frac{p^{-}}{p^{-} - 2} \| u_n \|_{L^{2+\rho}(\Omega)}
\]
\[
+ \psi_n(\rho)|u_n|^{2+\rho} + \frac{\rho'(t)}{2 + \rho(t)} \| u_n \|_{L^{2+\rho}(\Omega)} \log \| u_n \|_{L^{2+\rho}(\Omega)}
\]

(5.51)

for \( \sigma > 0 \) with \( \psi_n(\rho) = \frac{N}{2(2+\rho)} \log \left( \frac{16\pi \rho}{2+\rho} \right) \). On using the estimates (5.47) – (5.49) and (5.51)
in (5.50) we establish
\[
\frac{1}{2 + \rho} \frac{d}{dt} \int_{\Omega} |u_n(t)|^{2+\rho} dx \leq C_7 \|u_n\|_{L^{2+\rho}(\Omega)}^{q^+ + \rho(q^+ - p^-)E(\rho)} + C_8 + C_{10} \|u\|_{L^{2+\rho}(\Omega)}^p
\]
\[
+ C_9 \left( \frac{\sigma \rho'}{(2 + \rho)(1 + \rho)^{2/p^-}} \right)^{\frac{p^-}{p^--2}} \|u_n\|_{L^{2+\rho}(\Omega)}^p
\]
\[
+ \psi_n(\rho) \rho' \|u_n\|_{L^{2+\rho}(\Omega)}^{2+\rho} + \frac{\rho'(t)}{2 + \rho(t)} \|u_n\|_{L^{2+\rho}(\Omega)}^{2+\rho} \log \|u_n\|_{L^{2+\rho}(\Omega)}
\]
\[
\leq C_{11} \|u_n\|_{L^{2+\rho}(\Omega)}^{q^+ + \rho(q^+ - p^-)E(\rho)} + C_{12} + C_9 \left( \frac{\sigma \rho'}{(2 + \rho)(1 + \rho)^{2/p^-}} \right)^{\frac{p^-}{p^--2}} \|u_n\|_{L^{2+\rho}(\Omega)}^p
\]
\[
+ \psi_n(\rho) \rho' \|u_n\|_{L^{2+\rho}(\Omega)}^{2+\rho} + \frac{\rho'(t)}{2 + \rho(t)} \|u_n\|_{L^{2+\rho}(\Omega)}^{2+\rho} \log \|u_n\|_{L^{2+\rho}(\Omega)}
\]
for every \( t \in (0, T) \). With the consideration of the above inequality and Proposition 4.2 of [13], we conclude (2), i.e. for every \( \eta \in (0, T) \), there exists \( K_\eta > 0 \) such that \( \|u_n\|_{L^\infty(\Omega)} \leq K_\eta < \infty \) independent of \( n \), for every \( t \in (\eta, T) \). \( \square \)

We now find the uniform Sobolev bounds for \( u_n \).

**Lemma 5.3.** Let \( \delta^+ < 1 \). Then the sequence \( \{u_n\} \) is bounded in \( V^{p(\cdot)}(Q_T) \).

**Proof.** We multiply \( u_n \) in (5.28) and then integrate by parts on \( Q_T \) to get
\[
\frac{1}{2} \left[ \int_{\Omega} u_n^2(T) - \int_{\Omega} u_{0,n}^2 \right] + \int_{Q_T} |\nabla u_n|^{p(x)} = \int_{Q_T} \lambda h_n(u_n)u_n + \int_{Q_T} \tilde{g}u_n \frac{u_n^q}{(u_n + 1/n)^{\delta(x)}} + \int_{Q_T} \tilde{g}u_n^{1-\delta(x)}.
\]
By Proposition 5.2, the sequence \( \{u_n\} \) is bounded in \( L^\infty(0, T; L^r(\Omega)) \) and \( u_{0,n} \leq u_0 \in L^r(\Omega) \). Thus, we get
\[
\int_{Q_T} |\nabla u_n|^{p(x)} \leq C.
\]
This implies \( \{u_n\} \) is uniformly bounded in \( V^{p(\cdot)}(Q_T) \). \( \square \)

**Lemma 5.4.** Let \( \delta^+ \geq 1 \). Then \( \{u_n\} \) is bounded in \( V^{p(\cdot)}_{loc}(Q_T) \). Moreover, \( \{u_n\}_{p^-} \) is bounded in \( L^{p^-}(0, T; W^{1,p^-}_0(\Omega)) \).

**Proof.** Let \( \varphi \in D(\Omega) \) be a nonnegative function and \( k > 0 \). We multiply \( (u_n - 1)\varphi^{p^+} \) in (5.28)
and then integrate by parts on $Q_T$ to get
\[
\frac{1}{2} \int_{\Omega} (u_n(T) - 1)^2 \varphi^+ - \frac{1}{2} \int_{T} (u_{0,n} - 1)^2 \varphi^+ + \int_{Q_T} |\nabla u_n|^{p(x)} \varphi^+ \\
+ p^+ \int_{Q_T} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \varphi \varphi^+ (u_n - 1)
\]
\[
= \int_{Q_T} \lambda h_n(u_n)(u_n - 1)\varphi^+ + \int_{Q_T} \tilde{g}(u_n - 1)\varphi^+
\]
\[
\leq \lambda C\varphi \int_{Q_T} u_n^{q(x)} + \int_{\{u_n \leq 1\}} \tilde{g}(u_n - 1)\varphi^+ + \int_{\{u_n > 1\}} \tilde{g}(u_n - 1)\varphi^+
\]
\[
\leq \lambda C\varphi \int_{Q_T} u_n^{q(x)} + \int_{\{u_n > 1\} \cap \{\delta(x) \leq 1\}} \tilde{g} u_n^{1-\delta(x)}\varphi^+ + \int_{\{u_n > 1\} \cap \{\delta(x) > 1\}} \tilde{g} u_n^{1-\delta(x)}\varphi^+
\]
\[
\leq C\varphi \left[ \lambda \int_{Q_T} u_n^{q(x)} + C_1 \int_{\{u_n > 1\} \cap \{\delta(x) \leq 1\}} u_n^{1-\delta(x)} + C_2 \right] 
\]
\[
\leq \tilde{C}\varphi. \tag{5.53}
\]

We have used the boundedness of $\{u_n\}$ in $L^\infty(0, T; L^r(\Omega))$ to obtain the uniform bound $\tilde{C}\varphi > 0$ independent of $n$. Hence, by using Young’s inequality we have
\[
\int_{Q_T} |\nabla u_n|^{p(x)} \varphi^+ \leq \tilde{C}\varphi + \frac{1}{2} \int_{\Omega} (u_{0,n} - 1)^2 \varphi^+ + p^+ \int_{Q_T} |\nabla u_n|^{p(x)-1} \cdot \nabla \varphi |\varphi^+ u_n - 1| 
\]
\[
\leq C_3 + C_4 p^+ \epsilon \int_{Q_T} |\nabla u_n|^{p(x)} \varphi^+ + p^+ C_\epsilon \int_{Q_T} |\nabla \varphi|^{p(x)} |u_n - 1|^{p(x)}.
\]

This implies
\[
\int_{Q_T} |\nabla u_n|^{p(x)} \varphi^+ \leq C_5 \tag{5.54}
\]

and hence $\{u_n\}$ is uniformly bounded is $V_{loc}^{p(\cdot)}(Q_T)$. With the consideration of the inequalities (5.47) – (5.49) and Proposition 5.2 we have
\[
\frac{1}{r} \int_0^T \frac{d}{dt} \int_{\Omega} |u_n(t)|^r dx dt + \frac{3(r-1)}{4} \int_0^T \int_{\Omega} |\nabla u_n|^{p(x)} |u_n|^{r-2} \leq C_7 \int_0^T \|u_n\|_{L^r(\Omega)}^{rB(r-2)} + TC_8
\]
\[
\leq \tilde{C}. \tag{5.55}
\]

Since $p^- \leq p(x)$ for all $x \in \bar{\Omega}$, the above estimate (5.55) gives
\[
\int_{Q_T} |\nabla u_n|^{p^+-x-2} |p^- \leq C \int_{Q_T} |\nabla u_n|^{p^-} |u_n|^{r-2}
\]
\[
\leq C \int_{Q_T} |\nabla u_n|^{p(x)} |u_n|^{r-2} + C \int_{Q_T} |u_n|^{r-2}
\]
\[
\leq C^* < \infty.
\]

This proves the lemma. \qed
Proof of Theorem 5.7. Let \( u_n \) be a weak solution to the problem (5.28). If \( \delta^+ < 1 \), then according to Proposition 5.2 and Lemma 5.3 there exists \( \tilde{T} > 0 \) such that the sequence \( \{u_n\} \) is bounded in \( V_{p(\cdot)}(Q_T) \cap L^\infty(0, T; L^r(\Omega)) \cap L^\infty((\eta, T) \times \Omega) \) for every \( T < \tilde{T} \), every \( \eta \in (0, T) \). Hence, there exists function \( u \) such that, up to a subsequential level, as \( n \to \infty \)

\[
u_n \to u, \text{ weakly in } V_{p(\cdot)}(Q_T)
\]

and

\[
u_n \to u, \text{ weak starly in } L^\infty(0, T; L^r(\Omega)) \cap L^\infty((\eta, T) \times \Omega).
\]

For the case \( \delta^+ \geq 1 \), by considering Proposition 5.2 and Lemma 5.4 there exists \( \bar{T} > 0 \) and \( u \in V_{p(\cdot)}^l(Q_T) \cap L^\infty(0, T; L^r(\Omega)) \cap L^\infty((\eta, T) \times \Omega) \) such that for every \( T < \bar{T} \), \( u_n \) converges to \( u \) weakly to \( V_{p(\cdot)}^l(Q_T) \) and weak starly in \( L^\infty(0, T; L^r(\Omega)) \cap L^\infty((\eta, T) \times \Omega) \) for all \( \eta \in (0, T) \). The weak formulation of the problem (5.28) is given by

\[
- \int_{Q_T} u_n \varphi_t - \int_{\Omega} u_{0,n} \varphi(x, 0) + \int_{Q_T} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \varphi = \lambda \int_{Q_T} h_n(u_n) \varphi + \int_{Q_T} \frac{\tilde{g}\varphi}{(u_n + 1/n)^{\delta(x)}} \tag{5.56}
\]

for every \( \varphi \in C^1_c([0, T) \times \Omega) \). Following the proof of Theorem 3.6, we can pass the limit \( n \to \infty \) in (5.56) only if we are able show the a.e. convergence of \( \nabla u_n \) towards \( \nabla u \).

Let us consider \( \gamma > 0 \) and \( \phi \in C^1_c(Q_T) \) such that \( 0 \leq \phi \leq 1 \). Let \( u_m \), for \( m \in \mathbb{N} \), be the regularization of \( u \) in time, as in [10]. Then \( u_m \in V_{p(\cdot)}(Q_T) \cap L^\infty((\eta, T) \times \Omega) \) for every \( \eta \in (0, T) \), \( \frac{\partial u_m}{\partial t} \in V_{p(\cdot)}(Q_T)^* \), and \( u_m \to u \) strongly in \( V_{p(\cdot)}(Q_T) \) as \( m \to \infty \). Thus, by multiplying \( (u_n - u_m)\phi^{p^+} \) in (5.28), integrating by parts over \( Q_T \) and using Proposition 5.2, we have

\[
- \frac{1}{2} \int_{Q_T} (u_n - u_m)^2 (\phi^{p^+})_t + \int_{Q_T} (u_m)_t (u_n - u_m) \phi^{p^+} + p^+ \int_{Q_T} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla (u_n - u_m) \phi^{p^+} + \int_{Q_T} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla (u_n - u_m) \phi^{p^+} \leq \lambda \int_{Q_T} u_n^{q(x)-1} |u_n - u_m| \phi^{p^+} + \int_{Q_T} \frac{\tilde{g}(u_n - u_m) \phi^{p^+}}{(u_n + 1/n)^{\delta(x)}} \leq \lambda K^\phi \int_{Q_T} |u_n - u_m| \phi^{p^+} + \int_{Q_T} \frac{\tilde{g}(u_n - u_m) \phi^{p^+}}{(u_n + 1/n)^{\delta(x)}} \leq \lambda K^\phi \int_{Q_T} |u_n - u_m| \phi^{p^+} + K^\phi \int_{Q_T} \frac{\tilde{g} \phi^{p^+}}{(u_n + 1/n)^{\delta(x)}} + C^\gamma^{-\delta^+} \int_{Q_T} |u_n - u_m| \phi^{p^+}. \tag{5.57}
\]

Using Hölder’s inequality, Lemma 5.3 and Lemma 5.4, the third integral in the right hand side of (5.57) can be estimated in the following way:

\[
\int_{Q_T} |\nabla u_n|^{p(x)-1} \phi^{p^+} - 1 |u_n - u_m| \nabla \phi \leq C_1 \left\| \nabla u_n \right\|_{L^\infty(Q_T)}^{p(x)-1} \left\| \phi^{p^+} \right\|_{L^\infty(Q_T)} \left\| u_n - u_m \right\|_{L^\infty(\Omega)} \leq C_2 \left\| u_n - u_m \right\|_{L^p(\Omega)} \phi^{p^+}. \tag{5.58}
\]
On combining (5.57) and (5.58) we obtain
\[
\int_{Q_T} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla (u_n - u_m) \phi^{p^+} \nabla u_n \\
\leq \lambda K_{\phi}^{q^+-1} \int_{Q_T} |u_n - u_m| \phi^{p^+} + K_{\phi} \int_{Q_T \cap \{u_n \leq \gamma\}} (u_n + 1/n)^{q(x)} + C_{\gamma^{-q^+}} \int_{Q_T} |u_n - u_m| \phi^{p^+} \\
+ C_2 ||u_n - u_m| \nabla \phi||_{L^p(\Omega)} + \frac{1}{2} \int_{Q_T} (u_n - u_m)^2 (\phi^{p^+})_t - \int_{Q_T} (u_m)_t (u_n - u_m) \phi^{p^+} \\
= C_{\phi, \omega} (n, m, \gamma). \quad (5.59)
\]

It follows by (5.59) adding and subtracting suitable terms that
\[
\int_{Q_T} (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u) \cdot \nabla (u_n - u) \phi^{p^+} \leq C_{\phi, \omega} (n, k, \gamma). \quad (5.60)
\]

By (3.23), on passing the limit first on \(n\), then on \(m\) (for \(0 < \gamma < 1\) fixed) and finally on \(\gamma\) in (5.60), we establish the following.
\[
\lim_{n \to \infty} \int_{Q_T} (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u) \cdot \nabla (u_n - u) \phi^{p^+} \leq 0,
\]

which implies
\[
\lim_{n \to \infty} \int_{Q_T} |\nabla (u_n - u)|^{p(x)} \phi^{p^+} = 0.
\]

This proves that \(\nabla u_n\) converges to \(\nabla u\) a.e. in \(Q_T\). Thus, we obtain a nonnegative weak solution \(u\) to (1.3), in the sense of Definition 2.6. Moreover, \(u\) satisfies the following equation.
\[
- \int_{Q_T} u \varphi_t - \int_{\Omega} u_0 \varphi(x, 0) + \int_{Q_T} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi = \lambda \int_{Q_T} u^{q(x)-1} \varphi + \int_{Q_T} \frac{\tilde{g} \varphi}{u^\alpha(x)}, \quad (5.61)
\]

for every \(\varphi \in C^1_c((0,T) \times \Omega)\). Thus, we conclude the proof. \(\square\)

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