Structure of magnetic fields in non-convective stars

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ABSTRACT

We develop a theoretical framework to construct axisymmetric magnetic equilibria in stars, consisting of both poloidal and toroidal magnetic field components. In a stationary axisymmetric configuration, the poloidal current is a function of the poloidal magnetic flux only, and thus should vanish on field lines extending outside of the star. Non-zero poloidal current (and the corresponding non-zero toroidal magnetic field) is limited to a set of toroid-shape flux surfaces fully enclosed inside the star. If we demand that there are no current sheets then on the separatrix delineating the regions of zero and finite toroidal magnetic field both the poloidal flux function (related to the toroidal component of the magnetic field) and its derivative (related to the poloidal component) should match. Thus, for a given magnetic field in the bulk of the star, the elliptical Grad–Shafranov equation that describes magnetic field structure inside the toroid is an ill-posed problem, with both Dirichlet and Newman boundary conditions and a priori unknown distribution of toroidal and poloidal electric currents. We discuss a procedure which allows to solve this ill-posed problem by adjusting the unknown current functions. We illustrate the method by constructing a number of semi-analytical equilibria connecting to outside dipole and having various poloidal current distribution on the flux surfaces closing inside the star. In particular, we find a poloidal current-carrying solution that leaves the shape of the flux function and, correspondingly, the toroidal component of the electric current, the same as in the case of no poloidal current. The equilibria discussed in this paper may have arbitrary large toroidal magnetic field, and may include a set of stable equilibria. The method developed here can also be applied to magnetic structure of differentially rotating stars, as well as to calculate velocity field in incompressible isolated fluid vortex with a swirl.

Key words: MHD – stars: magnetic fields.

1 INTRODUCTION

Structure of magnetic fields in non-convective stars is a long-standing issue in astrophysics (Prendergast 1956). In fluid non-convective stars, very general arguments prove that both purely poloidal and purely toroidal magnetic fields are unstable (Tayler 1973; Wright 1973; Markey & Tayler 1973; Flowers & Ruderman 1977). As a result, magnetic fields inside the star should consist of a combination of both (Prendergast 1956). An example of such stable (on resistive times-scales) equilibrium were found numerically by Braithwaite & Spruit (2004). No acceptable analytical solution is known.

In this paper, we discuss a semi-analytical procedure to construct magnetic structures of fluid stars. We are interested in developing a procedure how to construct poloidal–toroidal equilibria of stars, a non-trivial mathematical problem on its own. We do not discuss the stability of the resulting configuration. Magnetohydrodynamic (MHD) simulations (e.g. Braithwaite & Nordlund 2006) indicate that stable configurations correspond to stably stratified stars with toroidal magnetic field reaching inside the star the local value of the poloidal magnetic field. Thus, we are looking for equilibria where the toroidal and poloidal magnetic field are, generally, of the same order somewhere inside the star.

Recent attempts to construct analytical and semi-analytical poloidal–toroidal equilibria of a spheromak-type (Broderick & Narayan 2008; Duez & Mathis 2009) suffer from a common drawback: they require unphysical surface currents. Spheromaks are stable force-free equilibria with the poloidal current being a linear function of the magnetic flux. As is well known, linear force-free magnetic field configurations minimize magnetic energy given a total helicity (Woltjer 1958). Applications of such constructions to real stars is not justified: linear force-free fields cannot avoid unphysical surface currents and corresponding infinite Lorentz forces in principle, since poloidal current is non-zero everywhere inside the star by the nature of the spheromak-type solutions (see the discussion after equation 7); plus any current sheet will be resistive and strongly diffusive. In addition, as is well known, magnetic field
cannot be force-free everywhere in space. The main goal of this work is to find equilibria with no current sheets inside the star or on the surface.

2 MAGNETIC EQUILIBRIA: THE GRAD–SHAFFRANOV EQUATION

In MHD equilibria, the Lorentz forces are balanced by gradients of pressure and gravitational forces,

\[ \mathbf{J} \times \mathbf{B} = \nabla p + \rho \nabla \Phi, \tag{1} \]

where \( \Phi \) is gravitational potential. Dividing equation (1) by \( \rho \), taking a curl, and assuming barotropic fluid, \( p = p(\rho) \) (see Appendix A for discussion of non-barotropic fluid), equation (1) reduces to

\[ \nabla \times \left( \frac{\mathbf{J} \times \mathbf{B}}{\rho} \right) = 0 \tag{2} \]

Axially symmetric magnetic fields can be written as

\[ \mathbf{B} = \frac{\nabla P \times \mathbf{e}_\phi + 2I \mathbf{e}_\phi}{\sigma}, \tag{3} \]

where \( 2\pi P(\sigma, z) \) is the poloidal magnetic flux and \( I(\sigma, z) \) is the poloidal current enclosed by an axially symmetric loop located at cylindrical coordinates \( \sigma, z \) (e.g. Shafranov 1966); we set the speed of light to unity. Axial symmetry and the fluid approximation (so the pressure is derivable from a potential and thus cannot have an azimuthal component) give \( \nabla \times \mathbf{J}_\phi \propto (\nabla \times \nabla P) \cdot \mathbf{e}_\phi = 0 \), implying that poloidal current is a function of \( P \) only \( I = I(P) \). The force balance equation (2) gives

\[ \nabla P \times \nabla \left( \frac{\Delta^* \sigma P + 4I'I}{\sigma^2 \rho} \cdot \mathbf{e}_\phi \right) = 0. \tag{4} \]

Taking into account that \( I = I(P) \), this reduces to the Grad–Shafranov equation, which in cylindrical coordinates become (e.g. Shafranov 1966)

\[ \Delta^* P = -\rho \sigma^2 F(P) - G(P), \tag{5} \]

where

\[ \Delta^* = \sigma \partial_\sigma \left( \frac{\partial_\sigma}{\partial_\sigma} \right) + \frac{\partial^2}{\partial z^2} \tag{6} \]

is the Grad–Shafranov operator, \( G(P) = 4II' \), \( I = I(P) \) and \( F = F(P) \) are functions of the flux function only. Also note, that toroidal current depends exclusively on the shape of the flux functions:

\[ J_\phi = -\frac{\Delta^* P}{\sigma}. \tag{7} \]

As a crucial physical constraint, we will assume that the current cannot leave the star. One can then identify three regions of space, with different requirements on the functions \( I(P) \) and \( \rho F(P) \) (e.g. Monaghan 1976, Fig. 1). First, there is the outside vacuum, where \( I = 0 \) and \( \rho = 0 \). Secondly, there is a part of the stellar interior threaded by field lines connecting to the outside. Since on these field lines the condition \( I(P) = 0 \) was imposed outside the star, it must still be true inside; non-zero density implies that the term containing \( F \) is non-zero inside the star. Finally, if a flux surface is completely closed within the star, on it \( I \) and \( \rho F \) can both be non-zero.

Since we are interested in general properties of magnetic equilibria, we make a simplifying assumption of a constant density,
\[ \rho = \text{constant}. \]

Redefining the function \( F \) to include the assumed constant density, the Grad–Shafranov equation becomes

\[ \Delta^* P = -\frac{15}{2} F(P) \sigma^2 + G(P). \tag{8} \]

The coefficient -15/2 is introduced for numerical convenience (see a comment after equation 9). Function \( F \) is assumed to be zero outside of the star (since the new definition of \( F \) includes density, which is zero outside). Below, we refer to function \( F \) as the pressure function (it is related to gradient of presser with respect to flux function Shafranov 1966) and the function \( G \) as the poloidal current function.

### 2.1 The mathematical problem

Stable equilibria of fluid stars must have considerable toroidal magnetic field component. As discussed above, the toroidal magnetic field cannot be non-zero everywhere inside the star, it must be limited to a smaller region of toroidal shape (in case of axial symmetry). On the separatrix between the regions of non-zero toroidal magnetic field and the bulk (which has zero toroidal magnetic field), both the poloidal and the toroidal components of the magnetic field must connect smoothly, without a surface current. For axisymmetric configurations, the toroidal components of the magnetic field is related to flux function, while the poloidal component of the magnetic field is related to the gradient of the flux function (see equation 3) with \( I = I(P) \). In order to avoid current sheets, on the separatrix both the flux function \( P \) and its normal derivative \( \partial_n P \) should be continuous.

Thus, we are faced with an unusual mathematical problem: we need to find solutions of an elliptical equation (8) while matching on a given boundary both the flux function (equivalently, the toroidal B-field) and the flux function normal derivative (equivalently, the poloidal B-field). This makes an elliptical equation (8) is formally overconstrained, as it should satisfy both Newman and Dirichlet boundary conditions simultaneously. In addition, the functions \( F \) and \( G \) are unknown a priori and should be found as a part of the solution. Thus, we are trying to solve an overconstrained equation with two unknown functions, which themselves are functions of the solution only.

For a given magnetic field structure in the bulk (with no poloidal current), we are looking for magnetic field structure inside a poloidal current-carrying toroid of a given shape. In principle, the shape of this toroid also is not known a priori. But, this does not create any additional mathematical complications; for any shape of the boundary, the solutions inside and outside of the toroid should be matched satisfying both Newman and Dirichlet boundary conditions simultaneously. (As a side note, a one-dimensional problem with both Newman and Dirichlet boundary conditions can be solved if there exist several eigenvalues for the solution of either Newman or Dirichlet problem.) The four corresponding boundary conditions can be satisfied by choosing two integration constants and two eigenvalues. This method does not work for the two-dimensional domain, since the shapes of the flux functions corresponding to different eigenvalue problems are different.) In this paper, we devise a procedure that allows to built both the functions \( F \) and \( G \) and the flux function \( P \) inside the toroid for given boundary conditions at the boundary of the toroid.

### 3 MATCHING THE BULK AND THE TOROID SOLUTIONS

#### 3.1 Solution in the bulk

As we discussed in Section 2.1, we are faced with an unusual mathematic problem of continuously matching the solution of the elliptical Grad–Shafranov equation (8) with zero poloidal current \( (I = 0) \) and some distribution of the toroidal current \( (F \neq 0) \) in the bulk and unknown non-zero poloidal current \( (I \neq 0) \) and some toroidal current inside a toroid. As a basic solution in the bulk, we take a well-known dipolar, purely poloidal solution connecting to outside dipole (Ferraro 1954; Shafranov 1966) (the procedure developed below can be repeated for any shape of the enclosed toroid)

\[ P_0 = \frac{(5R^2 - 3(\sigma^2 - z^2))\sigma^2}{4R^2} B_0, \]

\[ B = \frac{B_0}{2\pi} \left[ 3\sigma z \epsilon + (5R^2 - 6\sigma^2 - 3z^2) \epsilon \right]. \tag{9} \]

This solution corresponds to \( I = 0 \) and \( F = 1 \). It has a set of flux surfaces closing inside a star (see Fig. 1). Under the fluid approximation, this solution is unstable to non-axisymmetric perturbations (Taylor 1973; Flowers & Ruderman 1977).

The flux surfaces are given by solution \( z(\sigma) \) of (9) for \( 0 < P < (25/48) \). (Below, for conciseness we set \( R = 1, B_0 = 1 \).) For a given meridional plane, they satisfy

\[ z^2 + \sigma^2 = \frac{5}{3} \frac{P}{\sigma^2}. \tag{10} \]

There is a set of nested toroids enclosing within a star and centred on the apex point \( \sigma = \sqrt{5/6} \) and \( z = 0 \). The ellipticity of the boundary is 1/2 (the flux surfaces are not ellipses, though). The first closed flux surface (separatrix) corresponds to \( P_s = 1/2 \) and equation for it can be written as

\[ (z)^2 = \frac{5}{3} - \frac{2}{3 \sqrt{5}} - \left( \hat{x} + 2/\sqrt{5} \right)^2, \tag{11} \]

where \( \hat{x} = \sigma R - 2/\sqrt{5} \) are coordinates centred on the centre line of the enclosed toroids, corresponding to \( \sigma = (2/3)^{1/5}, z = 0 \) (since the flux surfaces are not ellipses, the symmetry axis of the separatrix does not intersect the apex). The separatrix intersects the equatorial plane at points \( \sigma = \sqrt{2/3} \) and \( \sigma = 1 \). The maximum value of \( P \) equals \( P_{\text{max}} = (25/48) \). This value is higher than the maximum value for vacuum dipolar field at a given radius \( P_s = 1/2 \); this allows a smooth sewing of the vacuum dipole potential, which is a decreasing function of radius, with increasing potential at small radii of the internal solution.

#### 3.2 The algorithm to calculate magnetic field structure inside the toroid

The solution (9) has a set of flux surfaces closing inside a star. Toroidal magnetic field should be confined to this region. On the separatrix between the regions of zero and finite magnetic field, both the flux function \( P \) and its derivative should be continuous. In this section, we discuss how the unknown functions \( F \) and \( G \) can be adjusted to satisfy the overconstrained boundary conditions on the separatrix \( \partial \).
We expand the unknown functions \( F(P) \) and \( G(P) \) in terms of their argument \( P \) near the separatrix \( \delta \), corresponding to \( P = P_0 \) (subscript 0 corresponds to quantities evaluated on the separatrix)

\[
\left( \frac{F}{G} \right) = \sum_{j=0}^{\infty} \left( \frac{P - P_0|_0}{j!} \right) \left( \frac{F^{(j)}(P_0|_0)}{G^{(j)}(P_0|_0)} \right)
\]

Here, \( F^{(j)}(P_0|_0) \) and \( G^{(j)}(P_0|_0) \) are values of the derivatives on the separatrix. These are numerical coefficients to be determined. The upper bound in the sum is determined by the number of initial starting points and the imposed smoothness order of the solutions at the points of intersection. The first terms in the series (12) are known: at the separatrix \( F = 1, G = 0 \).

At each point inside the separatrix, the function \( P \) can be expanded in Taylor series in a following manner. On the separatrix, the flux function is constant \( P = P_0|_0 \), while its derivative is normal to the separatrix and is a known function, \( \delta_n P_0|_0 \). This allows us to express from equation (8) the second normal derivative at the boundary \( \delta \) as some function \( \mathcal{F} \) of \( P_0|_0, \delta_n P_0|_0, F(P_0|_0), G(P_0|_0) \):

\[
\delta_n^2 P|_0 = \mathcal{F} \left[ P_0|_0, \delta_n P_0|_0, F(P_0|_0), G(P_0|_0) \right]
\]

From the procedure described above is implemented starting from three particular points on the toroid corresponding to the symmetry axes of the separatrix: at the points where the separatrix intersects the magnetic equator \( z = 0, r = 5/3R e_x \) and \( r = R e_y \), as well as at the point where magnetic field on the separatrix is parallel to the equator, \( r = (2/3)^{1/4} R e_x + (5/2 - 2/3) e_y \). Thus, the point where the integration curves intersect is \( \sigma = (2/3)^{1/4} \) and \( z = 0 \). Note that the point where magnetic field on the separatrix is parallel to the equator has different cylindrical radius than the apex point of the enclosed cylinders; enclosed flux surfaces are not ellipses. This is true both for the solution (9) and for the numerical solutions found here.

Since the procedure described above is not unique, we expect various internal structures of the enclosed toroid. We have two unknown functions, \( F \) and \( G \): choosing one of them determines the
other. This will create a set of various equilibria. Below, we describe two procedures: first, expansion of functions \( F \) and \( G \) in terms of \( P \) up to a given order and secondly specifying \( F \) and solving for \( G \). As is shown in Section 3.3, simultaneous expansion of functions \( F \) and \( G \) in terms of \( P \) (as opposed to arbitrary specifying one of them) selects a particular class of solutions, when the toroidal current and the form of the flux functions remain the same as in the case of zero poloidal current. (Recall that the toroidal current depends only on the shape of the flux function, equation 7.)

### 3.3 Expansion of \( F(P) \) and \( G(P) \) in terms of \( P - P_{0} \)

For three starting points on the separatrix, there are five matching conditions: equality of flux functions (two conditions); equalities of first and second radial derivative and zero derivative along the direction normal to the equatorial plane. We expand the functions \( F \) and \( G \) up to the third order in \( P - P_{0} \); boundary conditions require \( G = 0 \) and \( F = 1 \) on the boundary, then five expansion parameters are determined from the matching conditions leaving one free parameter. The free parameter normalizes the overall strength of the poloidal current inside the toroid.

Results of calculations are presented in Figs 2 and 3. Somewhat surprisingly, in case of non-zero poloidal current \( G \neq 0 \) and non-constant pressure function \( F \neq 1 \), the algorithm leaves the shape of the flux functions, the same as for zero poloidal current, \( P(r, z) \approx P_{0}(r, z) \) (see equation 9 and Fig. 2). At the same time, the functions \( F \) and \( G \) on the right-hand side of equation (8) are adjusted to match the zero poloidal current case, \(-(15/2) F(P) \sigma^{2} + G(P) \sim -(15/2) \sigma^{2}(Fig. 3)\).

![Figure 3](image-url)  
**Figure 3.** Structure of current and pressure functions for the case when functions \( G \) and \( F \) are expanded in terms of the flux function at the separatrix. Left-hand panel: function \( G = 4I/17 \); Centre panel: current function \( I(P) \); right-hand panel: function \( F(P) - 1 \). This figure illustrates the value of the right-hand side terms in equation (8) for different values of the amplitude of the poloidal current.

![Figure 4](image-url)  
**Figure 4.** Shapes of the flux function \( P \) for given \( F = \xi \). Upper-left panel: analytical solution equation (9). Upper-right panel: numerical solution for \( \xi = 1 \). In this case, the numerical solution closely matches the analytical solution equation (9); this vindicates the fitting procedure. Lower-left panel: \( \xi = 10 \). Lower-right panel: \( \xi = -10 \).
Thus, the toroidal current and the shape of the poloidal magnetic field remain the same. In Fig. 3, the current-carrying flux functions closely resemble the current-free one, with precision of $\sim 10^{-4}$. We remind that this calculation is based on only three extrapolation points. In comparison, when the pressure function $F$ is imposed (Section 3.4) the distortion of the form of the flux function is of the order of unity.

### 3.4 Other solutions

Next, we find the flux function $P$ and coefficients $G^m(P_0)$ assuming that inside the toroid $F$ is a given function of $P$.

#### 3.4.1 Constant $F \neq 1$

Let us find the flux function $P$ and coefficients $G^m(P_0)$ assuming that inside the toroid $F = \xi \neq 1$. We expand solutions to fifth-order derivative near the separatrix in order to match expansion at the equatorial plane to the second order (three coefficients), match it to expansion along $z$ axis, and condition of vanishing radial magnetic field at the equator. We investigate solutions as functions of the parameter $\xi$.

First, for $\xi = 1$ the procedure described above closely reproduces the solution (9) (see Figs 4 and 5). Thus, if we keep the function $F$ the same in the bulk and inside the toroid, the procedure described above converges to the solution $G = 0$.

For $\xi \neq 1$, there is non-zero poloidal current and, correspondingly non-zero toroidal magnetic field (Fig. 4). For large positive or large negative $\xi$, the topology of the magnetic field inside the toroid changes (see Figs 4–6). For $\xi > 1.99$, there forms a new set of flux surfaces centred close to the axis of the toroid where the poloidal field reverse. For $-4.1 \leq \xi \leq -3.4$, there is an off-centred set of flux surfaces where the poloidal field reverse. For $\xi < -4.1$, additional off-centred set of flux surfaces forms. For large absolute values of $|\xi|$, the sizes of the internal toroids increase, while no other change of topology occur.

The plot of $I(P)$ indicates that when solutions extend to values of the flux function large than approximately 0.6 (in dimensionless units $R = B_0 = 1$) or smaller than 0.4 for the different sign of $F$, the toroidal field, $\propto 2I/r$, approaches poloidal field ($=1/2$ at equator). (In case of $I = 0$, the maximum value of $P$ is 25/48.) The maximum value of $P$ $\sim 0.6$ or minimum $P$ $\sim 0.4$ is reached (approximately) for $|\xi| \geq 10$. Thus, we expect that for $|\xi| \geq 10$ the resulting configuration is stable.

Some of the found configurations have poloidal magnetic field reversing inside the submerged toroid. Though such configurations have not so far been seen in simulations, we hypothesize that this may be related to initial conditions used in simulations.

#### 3.4.2 $F = 1 + \alpha(P - 1/2)$

Solutions corresponding to $F = \xi \neq 1$ has a disadvantage that the toroidal current density experiences a jump on the separatrix. If

![Figure 5](https://academic.oup.com/mnras/article-abstract/402/1/345/1032294)

Figure 5. Current as function of flux function $P$, $P$ in the equatorial plane, $P$ along the line $\varpi = (2/3)^{1/4}$ as function of $z$, $B_z$ in the equatorial plane for different values of parameter $\xi = 1, \pm 10$ [except in the $I(P)$ plot where $\xi = 1, \pm 5, \pm 10$]. The middle line in the plots (b–d) is both the analytical solution (9) and numerical solution for $\xi = 1$ (the difference between them is tiny on this scale).
we impose a smooth variation of toroidal current at the separatrix, e.g. in the form $F = 1 + \alpha(P - 1/2)$, the behaviour of the solutions qualitatively remains the same (Fig. 7).

### 4 CONCLUSION

In this paper, we develop a procedure to construct poloidal–toroidal equilibria of fluid non-convective stars. This involves a mathematically unusual procedure to find the distribution of unknown poloidal and toroidal currents, which depend only on the shape of the magnetic flux function, and the shape of the magnetic flux function itself; all these quantities must match the shape of the separatrix and specified Dirichlet and Newman boundary conditions on it. The procedure we propose uses expansion of the current functions in powers of the flux function to a specified order, which depends on the number of points used to extrapolate the flux function. Generally, this procedure is not unique; for example, given some pressure function $F$ it will determine the poloidal current function $G$, or vice versa. This is qualitatively consistent with results of numerical simulations (Braithwaite & Nordlund 2006) which show that depending on the initial conditions a variety of final states are achieved.

One particular configuration stands out among the all possibilities. It corresponds to simultaneous expansion of two functions $F$ and $G$ in terms of the value of the flux function $P$ on the separatrix. In this particular case, the shape of the poloidal current-carrying flux surfaces remains the same as in the case of no poloidal current: newly found functions $F$ and $G$ add up to produce the toroidal current of the poloidal current-free configuration.

An important feature of our solutions is that there no current sheets, neither on the surface of the enclosed toroid nor on the surface of the star. On the other hand, we do not test for the stability of the resulting magnetic structures. This could be done, in principle, by minimizing magnetic energy at a given helicity for magnetic field structure inside the toroid. Results of numerical simulations (Braithwaite & Nordlund 2006) indicate that stable magnetic field configurations require both stable stratification and typically have toroidal magnetic field of the order of the poloidal magnetic field. Our procedure, in principle, allows arbitrary ratios of toroidal and poloidal magnetic fields, and we expect that some of the found magnetic field configurations to be stable.

One of the main limitations of our approach is that we assume barotropic equation of state. In contrast, in stably stratified stellar interiors buoyancy forces play an important role, dominating typically over magnetic forces. Still, even in non-barotropic fluid the poloidal current is still a function of $P$ (Appendix A), so that the toroidal magnetic field is still limited to a set of fully submerged flux surfaces. On the separatrix of the regions with vanishing and finite toroidal magnetic field, again, both the flux function and its derivative should match. We leave this problem to future considerations.

The method developed here can also be applied to rotating stars when the magnetic axis and the rotation axis are aligned. In this case, the Grad–Shafranov equation has a form similar to (8) with $F \rightarrow F + m^2 \Omega^2 \xi$, where $\Omega(P)$ is the angular velocity of rotation, which is constant on a given flux surface (Chandrasekhar 1956). The new function $\Omega(P)$ can be expanded near the separatrix in a same way as the pressure $F$ and the current function $G$.

There is a hydrodynamical analogue to the solutions presented here, describing an isolated fluid vortex with a swirl. Stationary magnetic configurations are related to velocity field of a time-independent flow of an incompressible fluid, with the substitution $v \rightarrow B$ (Shafranov 1966). In case of a steady, axially symmetric fluid motion, the velocity potential satisfies the same equation as (5), the so-called Bragg–Hawthorne and/or Squire-Long equation (see e.g. Lamb 1975, equation 165.13). Our method then describes the swirling velocity of a core of an overall non-swirling vortex (see also Moffatt 1969).

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Figure 7. Form of the flux functions and magnetic field for a given $F = 1 + \alpha(P - 1/2)$. Qualitatively, solutions remain the same as in the case $F = \xi \neq 1$. Left-hand panel: values of $P$ at the equatorial plane. Centre panel: values of magnetic field at the equatorial plane. Right-hand panel: values of $P$ at $\sigma = (2/3)^{1/4}$. Qualitatively, the structure of the solutions remain the same as in Figs 5–6

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APPENDIX A: NON-BAROTROPIC EOS

In non-barotropic fluid equation (2) becomes

$$\nabla \times \left( J \times B \right)_\rho = \nabla p \times \nabla \rho \rho^2 \rho^2. \quad (A1)$$

Axial symmetry and the fluid approximation still imply $I = I(P)$. The force balance equation (A1) gives

$$\nabla P \times \nabla \tilde{\sigma}^2 \rho F + 4I' \tilde{\sigma}^2 = \frac{p \times \nabla \rho}{\rho^2}, \quad (A2)$$

writing $p = p(s, \rho)$, where $s$ is entropy and assuming

$$\tilde{\sigma}^2 \rho F + 4I' \tilde{\sigma}^2 = F(s, P), \quad (A3)$$

we find

$$\nabla P \times \nabla F = \frac{\partial F}{\partial s} \nabla P \times \nabla s = \left( \frac{\partial p}{\partial s} \right)_\rho \nabla s \times \nabla \rho \rho^2. \quad (A4)$$

If we assume that distribution of density is spherically symmetric, \( \rho = \rho(r) \), we find

$$F = F(P) - \int \left( \frac{\partial p}{\partial s} \right)_\rho \frac{\rho}{\rho^2} ds \rho^2. \quad (A5)$$

This form of $F$ should be used in the Grad–Shafranov equation (8), which makes it an integro-differential equation for $P$ (cf. Villata & Ferrari 1994, equation 20). Thus, in case of non-barotropic fluid the poloidal current is still function of $P$ only, $I = I(P)$, plus there will be an extra term in the Grad–Shafranov equation (equation A5). Thus, one still encounters the same problem that on the boundary of the enclosed toroid with toroidal magnetic field both flux function and its derivative should match those in the bulk.

APPENDIX B: EXPLICIT EXAMPLE OF EXPANSION OF $P$ NEAR THE POINT $\sigma = 1, Z = 0$

Let us illustrate the procedure of finding the flux function inside the toroid starting from a particular point on the separatrix, $\sigma = 1, z = 0$. In this case, the normal coincides with the radial direction. We know the value of $P|_0 = 1/2$ and its derivative $\partial_s P|_0 = -1/2$ at this point. In addition, the flux function is zero on the separatrix, $G(1/2) = 0$, while the pressure function is unity, $F(1/2) = 1$. From (8), we find $\partial^3_\sigma P$ :

$$\partial^3_\sigma P = -\partial^2_\sigma P \frac{\partial s_\sigma}{\partial \sigma} - \frac{15}{2} \sigma^2 F(P) + G(P). \quad (B1)$$

On the boundary, it is equal to

$$\partial^3_\sigma P|_0 = \frac{1}{2\sigma^2} - \frac{15}{2} \sigma^2 F(1/2) - G(1/2) \bigg|_0 = -\frac{13}{2}. \quad (B2)$$

Taking a derivative of equation (B1) with respect to $\sigma$, we find

$$\partial^3_\sigma P|_0 = -18 + \frac{15}{4} F'(1/2) + \frac{1}{2} G'(1/2). \quad (B3)$$

Thus, expansion of the flux function near the point $\sigma = 1, z = 0$ reads

$$P = \frac{1}{2} - \frac{1}{2} (\sigma - 1) - \frac{13}{2} (\sigma - 1)^2 + \left( -18 + \frac{15}{4} F'(1/2) + \frac{1}{2} G'(1/2) \right) \frac{(\sigma - 1)^3}{6} + \cdots. \quad (B4)$$

It can be continued to arbitrary high order. Matching of the expansions of the flux functions from different points then gives the coefficients $F^{(3)}(1/2)$ and $G^{(3)}(1/2)$, thus determining simultaneously both the flux function $P$ and functions $F$ and $G$.

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