The Loewner framework for nonlinear identification and reduction of Hammerstein cascaded dynamical systems

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We present an algorithm for data-driven identification and reduction of nonlinear cascaded systems with Hammerstein structure. The proposed algorithm relies on the Loewner framework (LF) which constitutes a non-intrusive algorithm for identification and reduction of dynamical systems based on interpolation. We address the following problem: the actuator (control input) enters a static nonlinear block. Then, this processed signal is used as an input for a linear time-invariant system (LTI). Additionally, it is considered that the orders of the linear transfer function and of the static nonlinearity are not a priori known.

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1 Introduction

In some engineering applications that deal with the study of dynamical control systems, the control input enters the differential equations in a nonlinear fashion [5]. It is of interest to identify the hidden nonlinearity while at the same time reduction is needed for robust simulations and control design [1]. The LF [2–4] constitutes a non-intrusive method that uses only input-output data. The matrix pencil composed of two Loewner matrices reveals the minimality (in terms of McMillan degree) of the LTI system. By means of a singular value decomposition (SVD), one can find left and right projection matrices that are used to construct a low order model.

The Hammerstein system is characterized by two blocks connected in series, where the static nonlinear (memoryless) block is followed by a linear time-invariant system (LTI) as in Fig. 1. The scalar control input $u(t)$ is used as an argument to the static nonlinearity $F$ and then the signal $F(u(t))$ passes through a linear time-invariant (LTI) system. The static polynomial map approximates other non-polynomial maps (Taylor series expansion) s.a. $\tanh(\cdot)$, $\exp(\cdot)$, etc. The aim is to identify the cascaded system by estimating the coefficients of the polynomial map $k_i$, $i = 1, 2, \ldots, n$ and the hidden LTI system by using only input-output data $(u(t), y(t))$, $t \geq 0$.

Fig. 1: The input-output scheme of a cascaded system with a static nonlinear (polynomial) map of $n^\text{th}$ order followed by an LTI. The connection describes a Hammerstein nonlinear model.

The steady state output solution can be computed explicitly with the convolution integral\(^1\), the impulse response $h(t)$, $t \geq 0$ and the linear transfer function $H(j\omega)$, $j\omega \in \mathbb{C}$ of the LTI as:

$$
g(t) = (k_1 u(t) + k_2 u^2(t) + \ldots + k_n u^n(t)) * h(t) = k_1(u*h)(t) + k_2(u^2*h)(t) + \ldots + k_n(u^n*h)(t) = k_1 \int_{-\infty}^{\infty} h(\tau) u(t-\tau)d\tau + \ldots + k_n \int_{-\infty}^{\infty} h(\tau) u^n(t-\tau)d\tau = \sum_{i=1}^{n} k_i \int_{-\infty}^{\infty} h(\tau) u^i(t-\tau)d\tau. \quad (1)
$$

Let the singleton real input be defined as $u(t) = A \cos(\omega t) = \alpha e^{j\omega t} + \alpha e^{-j\omega t}$ with the amplitude $\alpha = A/2$, the imaginary unit $j$, the driving frequency $\omega > 0$ and time $t \geq 0$. By substituting the above input in Eq. (1) and by making use of the binomial theorem, we conclude that:

$$
y(t) = \sum_{i=1}^{n} k_i \int_{-\infty}^{\infty} h(\tau) \left(\alpha e^{j\omega(t-\tau)} + \alpha e^{-j\omega(t-\tau)}\right)^i d\tau = \sum_{i=1}^{n} \sum_{m=0}^{i} \binom{i}{m} k_i \alpha^i \frac{i!}{(i-m)!m!} \int_{-\infty}^{\infty} H(j\omega(2m-i)) e^{j\omega(2m-i)} d\omega. \quad (2)
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\(^1\) $(f \ast g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$

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At frequency $\omega$ the $\ell$th harmonic is computed by applying the single-sided Fourier transform in Eq. (2) as:

$$Y_{\omega,\ell}(j \ell \omega) = H(j \ell \omega)\delta(j \ell \omega) \sum_{\ell \leq \ell_0} k_i \phi_{i,\ell}, \; \ell = 0, \ldots, n,$$

$$\phi_{i,\ell} = \begin{cases} 2 \alpha^i \cdot \frac{1}{\ell \omega}\delta(i+\ell)/2, & (i \geq \ell) \text{ and } (i + \ell \text{ even}) \text{,} \\ 0, & (\ell \text{ odd}) \text{,} \end{cases}$$ \hspace{1cm} \text{where } nC_m = \frac{n!}{(n-m)!m!} \tag{3}$$

### 2 The Loewner-Hammerstein identification method

As we have computed the total output of the Hammerstein cascaded system, we proceed with the method of determining the unknowns from input-output data. The symmetry in Eq. (3) allows the cancellation of the unknown contribution of the transfer function. Thus, we first determine the unknown coefficients $k_i$, and afterwards, we fit the LTI system by means of the LF. For this purpose, it is important to define the following invariant frequency quantities $\lambda_{p,q}$.

**Definition 2.1 (Frequency invariant quantities)**

The $Y_{p,q}$ denotes the $q$th harmonic at $p$ frequency.

$$\lambda_{p,q} = \frac{Y_{p,q}}{\sum_{i=q}^{\infty} k_i \phi_{i,q}}$$ \hspace{1cm} \text{for } p \neq q. \tag{4}$$

The entries $\lambda_{p,q}$ are independent of $\omega$.

The above harmonic map allows the construction of the following linear system. Due to the mixing linearity (i.e. $k_1 u(t)$ and $(u \ast h)(t)$), we can fix $k_1$ to an arbitrary value. For $p = 1$ and $q = 2, \ldots, n$, results:

$$\begin{bmatrix} \phi_{21} - \lambda_{12}\phi_{22} & \phi_{31} - \lambda_{13}\phi_{32} & \cdots & \phi_{n1} - \lambda_{1n}\phi_{n2} \\ \phi_{21} & \phi_{31} & \cdots & \phi_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{21} & \phi_{31} & \cdots & \phi_{n1} - \lambda_{1n}\phi_{n3} \end{bmatrix} \begin{bmatrix} k_2 \\ k_3 \\ \vdots \\ k_n \end{bmatrix} = -k_1 \phi_{11} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \forall k_1 \in \mathbb{R} \setminus \{0\}. \tag{5}$$

Finally, as we have identify the scaled $(k_1)$ arbitrary coefficient vector $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, we can transform the above harmonic map into a measurement map for the linear transfer function as $H(j \ell \omega) = Y_{\omega,\ell}/\sum_{\ell \leq \ell_0} k_i \phi_{i,\ell}$. The identification and reduction of the LTI system is done by applying the LF [2–4].

**Algorithm 1: Hammerstein identification with the Loewner framework**

**Input:** Apply signals $u(t) = \alpha \cos(\omega_1 t)$ with driving frequencies $\omega_1, \alpha = 1, \ldots, n$ where $n$ is the maximum nonzero harmonic index.

**Output:** An identified Hammerstein system.

1. Apply FT and measure $U(j \omega_1), Y_1(j \omega_1), Y_2(j \omega_2), \ldots, Y_{n}(j \omega_n)$ from the power spectrum.
2. Fix $k_1$ to an arbitrary value and determine the scaled coefficient vector $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ by solving the system in Eq. (5).
3. Estimate the measurements of the linear transfer function from $H(j \ell \omega) = Y_{\omega,\ell}/\sum_{\ell \leq \ell_0} k_i \phi_{i,\ell}$.
4. Apply the linear Loewner framework for identification and reduction of the LTI.

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**Fig. 2:** The singular value decay of the Loewner matrices. $\sigma_3/\sigma_1 \sim 1e-10$.  
**Fig. 3:** The identified linear transfer function with $\| H - H_r \|_\infty \sim 1e-7$. Comparison with the original one. $\| y - y_r \|_\infty \sim 1e-7$.  
**Fig. 4:** The simulated identified Hammerstein system in Loewner matrices.
domain simulation in Fig. 4 is independent of the choice of $k_1$. The large input as $u(t) = 2\text{sawtooth}(0.1 \cdot 2\pi t)e^{-0.01t}\cos(0.1 \cdot 2\pi t)$ certifies that the method is able to perform well under large inputs for nonlinear Hammerstein systems.

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