MINIMAL SURFACES IN THE 3-SPHERE BY STACKING CLIFFORD TORI

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Abstract. Extending work of Kapouleas and Yang, we construct by gluing methods sequences of closed minimal surfaces embedded in the round 3-sphere. Each surface resembles multiple tori, all coaxial with a nearby Clifford torus, that have been connected by many small catenoidal tunnels. The sequences can be indexed by the number N of tori joined and by a rectangular lattice on the corresponding Clifford torus, aligned with its axis circles. The terms of a given sequence themselves are then indexed by the sublattices of this lattice, which determine the locations of the catenoidal tunnels. The elements of O(4) preserving a surface in the sequence constitute a discrete group preserving the corresponding sublattice, and this symmetry group contains isometries exchanging the two sides of the associated Clifford torus precisely when the lattice is square. A given sequence converges to the Clifford torus counted with multiplicity N, while an appropriately blown-up sequence of tunnels at a fixed location converges to a standard Euclidean catenoid. The finite number of tori and rational ratio of lattice spacings may be prescribed at will, but the number of tunnels, or equivalently the surface’s genus, must be taken large in terms of this data in order to complete the construction.

1. Introduction

In [7] Kapouleas and Yang constructed a sequence of embedded minimal surfaces in the round 3-sphere \( S^3 \) converging to the Clifford torus \( \mathcal{T} \) counted with multiplicity 2; each surface approximates two parallel tori coaxial with a nearby Clifford torus (not included in the surface) in between them and joined by many small catenoidal tunnels centered on a square lattice on the central Clifford torus. Accordingly they called their surfaces doublings of the Clifford torus. Kapouleas announced these in [4] as the first examples of a general class of gluing constructions to double given minimal surfaces, subsequently discussed further in [5], and he has recently [6] produced embedded doublings of the equatorial 2-sphere in \( S^3 \).

The surfaces of Kapouleas and Yang are highly symmetric, admitting many horizontal symmetries, which preserve as sets the two sides of the doubled Clifford torus and permute the lattice sites, as well as vertical symmetries, each of which exchanges the two sides of the doubled torus but fixes as a set a catenoidal tunnel. In fact all these symmetries are enforced throughout the construction and exploited to simplify its execution. The present article carries out less vertically symmetric doublings of the torus, with the symmetry broken in two ways. First, we allow the catenoidal tunnels to be arranged on rectangular rather than strictly square lattices. Any isometry of \( S^3 \) exchanging the two sides of the doubled Clifford torus will fail to preserve such a lattice, unless it is square. Second, we interpret doubling in a generalized sense, realizing also triplings, quadruplings, and in fact embedded minimal surfaces resembling an arbitrary finite number of tori coaxial with a nearby Clifford torus and connected to one another by many small catenoidal tunnels. Whenever at least three tori are incorporated, even if these tunnels are centered on square lattices, there will be many tunnels admitting no symmetries that preserve each as a set but exchange its ends, nor will the symmetry group act transitively on the collection of tori.

These new constructions add to the list of known closed minimal embeddings in \( S^3 \), so far comprising those found in [9], [8], [7], [2], [6], and [10]. (Actually in this last reference Marques and Neves apply min-max methods to prove the existence of infinitely many minimally embedded hypersurfaces in every closed manifold having positive Ricci curvature and dimension between 3
A variety of further examples, to be obtained by sweepouts, have been described by Pitts and Rubinstein [11], some of which appear consistent with surfaces obtained elsewhere (among the above references) by alternative techniques (but none look like the surfaces we introduce here). The survey article [1] by Brendle contains an outline of several of the constructions just mentioned. The constructions at hand should be of interest not only as providing new examples of minimal surfaces in \( S^3 \) but also as a basis for further doublings with asymmetric sides. While this work confronts configurations with less vertical symmetry than is present in [7], Kapouleas’ sphere doubling [6] addresses a situation with less horizontal symmetry. We anticipate applications demanding a synthesis of the two refinements. A program toward doubling constructions of increasing generality, including potential applications, is described in [5].

The present work naturally emulates, with a few departures, the approach of [7] and draws extensively from the general gluing technology developed by Kapouleas, much of which can be found summarized in [4] and was itself inspired by techniques from [12]. Although the current article can be read without reference to [7] or any other gluing constructions, for the rest of this introduction we will make use, without detailed explanation, of terminology standardized by Kapouleas, so that the reader already acquainted with it may easily appreciate the principal differences between this construction and [7].

We now outline our procedure in very rough terms. As basic data we take positive integers \( N \geq 2, k \leq \ell, \) and \( m \), where \( N \) specifies the number of copies of \( T \) to be made, while \( k, \ell, \) and \( m \) determine \( N - 1 \) rectangular lattices, with ratio \( k/\ell \) of fundamental edges, on which are centered in total \( k\ell m^2(N - 1) \) catenoidal tunnels. Corresponding to a choice of such data, an initial surface is built as follows. We start with \( N \) tori coaxial with \( T \) that have small constant mean curvature and so lie near \( T \). From each of the two outermost tori we excise \( k\ell m^2 \) discs centered on certain rectangular lattices, and from each of the remaining intermediate tori we excise \( 2k\ell m^2 \) discs. Using local coordinates for \( S^3 \) adapted to the tori, we take standard catenoids, truncated at a pair of large, opposite circles of revolution, shrink them, and glue them to the tori, along boundary circles of the components so as to bridge adjacent tori, bending the ends as necessary so that the resulting connected surface is smooth.

For \( m \) large in terms of the other data, the tori approach \( T \) and the catenoidal tunnels, after rescaling, converge to a standard catenoid in Euclidean space. Thus for large \( m \) each such initial surface is, in a certain sense, approximately minimal, and the construction is completed by perturbing the surface to exact minimality. Two mechanisms of perturbation are applied in tandem. One sort of perturbation is realized by considering graphs of small functions over the initial surface. To select the right function is then to solve the quasilinear elliptic differential equation prescribing zero mean curvature for the corresponding graph. The equation can be studied by comparing the linearization of the operator governing the mean curvature of graphs to certain large-\( m \) limit operators on the limit catenoids and limit torus. In the simplest scenario one could solve the linearized operator on the toral and catenoidal components separately, combine these solutions through an iterative procedure, and finally invoke an inverse function theorem to solve the original nonlinear equation. However, the presence of nontrivial kernel to the limit operators gives rise to approximate kernel that thwarts the approach just described.

The space of admissible perturbing functions is constrained to respect the symmetries enjoyed by the initial surface, and so their imposition has the effect of reducing the dimension of the approximate kernel. Each torus turns out to carry one-dimensional approximate kernel of its own, but in [7] the two tori can be exchanged by reflections through certain great circles, and so together the tori contribute just one dimension to the approximate kernel in [7] versus \( N \) dimensions more generally. Furthermore, in [7] these reflections through circles render trivial the approximate kernel on the catenoidal tunnels. Following the approach of [7] in the absence of these symmetries, each tunnel would feature one-dimensional approximate kernel, but we bypass this kernel altogether by
altering, as compared to \([7]\), the initial data at the tunnel’s waist for the rotationally invariant mode of the solution.

To overcome the obstruction posed by the approximate kernel, \([7]\) introduces substitute kernel, spanned by a single function supported on the tori away from the circles where they attach to the tunnels. By adding multiples of this function to the source term of the linearized equation, the so modified source can be made orthogonal to the approximate kernel, enabling the success of the above scheme, but at the cost of solving the original equation only modulo substitute kernel. For the same purpose the current construction introduces \(N\)-dimensional substitute kernel, spanned by functions each of which is supported on a single torus away from the tunnels. (Actually, in this construction we never explicitly identify the approximate kernel, nor do we invoke the \(h\) metric employed in \([7]\) for its analysis, but our application of substitute kernel is morally identical.)

A further difficulty concerns the vast disparity in scale between the waist radii of the catenoidal tunnels on the one hand and the much greater spacing between the tori on the other. The initial surface’s second fundamental form grows toward the waists of the catenoids from a small value bounded uniformly in \(m\) on the tori to a value diverging with \(m\), and the embeddedness of graphical perturbations is most precarious near the waists. For these reasons as well as to ensure convergence of the iteratively defined global solution, it is necessary to arrange for solutions on the tori to decay toward the catenoidal waists.

All of the catenoids attaching to each of the two outermost tori—the only type of torus appearing in \([7]\)—are equivalent modulo the symmetries, and adjustment of the source term by the substitute kernel suffices to achieve such decay on these catenoids. (Again, our actual approach deviates somewhat from this description, applicable to \([7]\), but just superficially.) However, each of the intermediate tori, \(N - 2\) in number, attaches to catenoids of precisely two classes, and so the appropriate decay of solutions requires the introduction of another \(N - 2\) functions, linear combinations of which are added to the source term to arrange decay, a device originating in \([3]\) but unneeded in \([7]\). In total we arrive at a \((2N - 2)\)-dimensional extended substitute kernel, the sum of the substitute kernel and the span of these functions, modulo which subspace we can, for large \(m\), invert the linearized operator.

Thus an infinite-dimensional problem is reduced to a finite-dimensional one. The resolution of this latter problem requires the second type of perturbation and is best understood in terms of a correspondence, which Kapouleas \([3]\) calls the geometric principle, between the initial geometry and the analytic obstructions that the extended substitute kernel represents. Coarsely put, elements of the extended substitute kernel can be generated, as components of the initial surface’s mean curvature, by certain motions of its building blocks—here catenoids and tori—relative to one another. So motivated, the other type of perturbation is realized by incorporating parameters, one for each dimension of extended substitute kernel, into the definition of the initial surface whose variation repositions the component tori and catenoids. Thus for each choice of \(k, \ell, m,\) and \(N\), we define not one initial surface but a \((2N - 2)\)-parameter family of them.

More specifically, two parameters may be associated with each of the \(N - 1\) classes of catenoids, one for each pair of adjacent tori. One set \(\{\zeta_i\}_{i=1}^{N-1}\) of parameters controls the waist radii, while the other set \(\{\xi_i\}_{i=1}^{N-1}\) adjusts the heights of the centers, that is their signed distance from \(T\). A degree of rigidity, in the form of matching conditions, is maintained to reposition the tori in response to the parameters, and the surface is smoothed using cutoff functions as needed. A single parameter \(\zeta\) works for \([7]\), since there \(N = 2\) and the symmetry between the sides of \(T\) forces \(\xi = 0\).

In the course of the construction it is necessary to solve for the proper selection of parameters along with the perturbing function. The dependence on the parameters of the “extended” components of the extended substitute kernel can be directly estimated with accuracy adequate for our purposes. It turns out that these components are primarily generated by dislocations resulting
from antisymmetric variation in pairs of \( \xi \) parameters associated to catenoids adjoining a common torus.

The parameter dependence of the substitute kernel itself is more conveniently monitored indirectly, as in \cite{7}, via forces. The approximate kernel, and so the substitute kernel, on each torus, may be identified with approximate translations of the torus relative to \( T \). In fact \( S^3 \) admits an exact Killing field which does not generate this variation of the torus but does approximate it locally. The force in the direction of this Killing field through certain neighborhoods of a given torus then serves as an estimate of the projection of the mean curvature onto the approximate kernel and thereby as a proxy for the corresponding component of substitute kernel itself. The balancing equations and the analysis of the parameter dependence of the forces here are substantially more complicated than those of \cite{7} but no different in principle.

Finally, estimates for the initial geometry, the linearized equation, the nonlinear terms, and the parameter dependence of the forces and dislocations are applied in conjunction with the Schauder fixed-point theorem to prove that given \( k, \ell, \) and \( N \), for sufficiently large \( m \) there exists a choice of parameters and a smooth, appropriately symmetric perturbing function such that the resulting surface is minimal and a good approximation to the corresponding initial surface, so in particular embedded.

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2. Initial surfaces

We realize \( S^3 \) as the unit sphere \( \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1 \} \) in \( \mathbb{C}^2 \) and set

\[
T = \left\{ (z_1, z_2) : |z_1| = |z_2| = \frac{1}{\sqrt{2}} \right\},
\]

the Clifford torus whose equidistant axes are simply the coordinate unit circles \( C_1 = \{ z_2 = 0 \} \) and \( C_2 = \{ z_1 = 0 \} \). We define the covering map \( \Phi : \mathbb{R}^3 \rightarrow S^3 \setminus (C_1 \cup C_2) \) by

\[
\Phi(x, y, z) = \left( e^{i\sqrt{2}x} \sin \left( z + \frac{\pi}{4} \right), e^{i\sqrt{2}y} \cos \left( z + \frac{\pi}{4} \right) \right),
\]

which maps (i) horizontal planes to constant-mean-curvature tori having axes \( C_1 \) and \( C_2 \), with \( \Phi(\{ z = 0 \}) = T \) in particular, (ii) vertical lines to quarter great circles orthogonal to \( C_1 \), \( C_2 \), and \( T \), (iii) vertical planes of constant \( x \) to great hemispheres with equator \( C_2 \), (iv) vertical planes of constant \( y \) to great hemispheres with equator \( C_1 \), and (v) vertical planes of constant \( x \pm y \) to half Clifford tori through \( C_1 \) and \( C_2 \), orthogonally intersecting \( T \) along great circles. Writing \( g_S \) for the round metric on \( S^3 \) and \( g_E \) for the flat metric on \( \mathbb{R}^3 \) we find

\[
\Phi^* g_S = g_E + (\sin 2z) \left( dx^2 - dy^2 \right).
\]

The initial surfaces will be built by applying \( \Phi \) to a stack of horizontal planes connected by staggered catenoidal columns.

Half-catenoids bent to planes. Since we shall make frequent use throughout the construction of cutoff functions, we fix now a smooth, nondecreasing \( \Psi : \mathbb{R} \rightarrow [0, 1] \) with \( \Psi \) identically 0 on \( (-\infty, -1] \), identically 1 on \([1, \infty)\), and such that \( \Psi - \frac{1}{2} \) is odd. We then define, for any \( a, b \in \mathbb{R} \), the function \( \psi[a, b] : \mathbb{R} \rightarrow [0, 1] \) by

\[
\psi[a, b] = \Psi \circ L_{a,b},
\]

where \( L_{a,b} \) is a linear map that shifts \( a \) units horizontally and \( b \) units vertically. This function \( \psi[a, b] \) is a smooth, nondecreasing function on \( \mathbb{R} \) that is 1 on \([a, b] \), 0 on \( (-\infty, a) \cup (b, \infty) \), and such that \( \psi - \frac{1}{2} \) is odd.
where \( L_{a,b} : \mathbb{R} \to \mathbb{R} \) is the linear function satisfying \( L(a) = -3 \) and \( L(b) = 3 \). Write \( B[(x, y), r, \mathbb{R}^2, g_E] \) for the open Euclidean disc in \( \mathbb{R}^2 \) with radius \( r \) and center \((x, y)\). Given \( z_K, z_T \in \mathbb{R} \) and \( R, X, Y, \tau > 0 \) with \( \tau < 2R < \min(X, Y) \), set

\[
T_{X,Y,\tau} = \left( [-X, X] \times [-Y, Y] \right) \setminus B[(0, 0), \tau, \mathbb{R}^2, g_E]
\]

and define the function \( \phi[z_K, z_T, R, X, Y, \tau] : T_{X,Y,\tau} \to \mathbb{R} \) by

\[
\phi[z_K, z_T, R, X, Y, \tau](x, y) = z_K + (z_T - z_K)\psi[R, 2R]\left(\sqrt{x^2 + y^2}\right)
+ \text{sgn}(z_T - z_K)\left(\tau \arccosh\frac{\sqrt{x^2 + y^2}}{\tau}\right)\psi[2R, R]\left(\sqrt{x^2 + y^2}\right),
\]

where the sign function \( \text{sgn} : \mathbb{R} \to \mathbb{R} \) takes the value 1 when its argument is nonnegative and the value \(-1\) otherwise. Given further \((x_0, y_0) \in \mathbb{R}^2\), define the embedding \( T_{\text{ext}}(x_0, y_0, z_K, z_T, R, X, Y, \tau) : T_{X,Y,\tau} \to \mathbb{R}^3 \) by

\[
T_{\text{ext}}(x_0, y_0, z_K, z_T, R, X, Y, \tau)(x, y) = (x_0 + x, y_0 + y, \phi[z_K, z_T, R, X, Y, \tau](x, y));
\]

and given also \( z_K' \in \mathbb{R} \) along with \( \tau' > 0 \), now assuming \( \max(\tau, \tau') < 4R < \min(X, Y) \), set

\[
T_{X,Y,\tau,\tau'} = \left( [-X, X] \times [-Y, Y] \right) \setminus \left( B\left(\left(\frac{X}{2}, \frac{Y}{2}\right), \tau, \mathbb{R}^2, g_E\right) \cup B\left(\left(\frac{X}{2}, \frac{Y}{2}\right), \tau', \mathbb{R}^2, g_E\right)\right)
\]
and define the embedding $T_{int}((x_0, y_0), z_K, z'_K, z_T, R, X, Y, \tau, \tau') : T_{X, Y, \tau, \tau'} \rightarrow \mathbb{R}^3$ by

\[
T_{int}(x, y) = \begin{cases} 
(x_0 + x, y_0 + y, \phi [z_K, z_T, R, \frac{X}{2}, \frac{Y}{2}, \tau] (x + \frac{X}{2}, y + \frac{Y}{2})) & \text{for } (x, y) \in [-X, 0] \times [-Y, 0] \\
(x_0 + x, y_0 + y, \phi [z'_K, z_T, R, \frac{X}{2}, \frac{Y}{2}, \tau'] (x - \frac{X}{2}, y - \frac{Y}{2})) & \text{for } (x, y) \in [0, X] \times [0, Y] \\
(x_0 + x, y_0 + y, z_T) & \text{elsewhere.}
\end{cases}
\]

The initial surfaces will be built from various applications of $T_{ext}$, for extreme or outermost tori and adjoining half-catenoids, and of $T_{int}$, for the intermediate tori and pairs of adjoining half-catenoids. The horizontal positions of the catenoids (equivalently the values of $(x_0, y_0)$ in the parametrizations above) are decided by the lattice that the data $k$, $\ell$, and $m$ set. The radii of the annuli of transition (determined by $R$) will be chosen on the order of but sufficiently smaller than the smaller of the two lattice edges. There remain $N - 1$ selections of waist radii ($\tau$ and $\tau'$) and $N - 1$ more of waist heights ($z_K$ and $z'_K$). Balancing conditions studied in the next section allow for the estimation of the values these $2N - 2$ unknowns must assume for the construction to succeed, but their precise specification is made by the $\zeta$ and $\xi$ parameters. Matching conditions then fix the heights ($z_T$) of the tori, by requiring these to agree with the heights of the adjoining catenoids where they meet the transition annuli, in the case of the extreme tori, or to agree with the average of the heights of the upper and lower catenoids at the transition circles they adjoin, in the case of the intermediate tori.

The hierarchy of data. For any positive integers $k$, $\ell$, and $N \geq 2$, the construction produces a sequence, indexed by $m$, of $(2N - 2)$-parameter families of initial surfaces. In order to obtain adequate estimates for (i) the initial mean curvature, (ii) the linearized operator, and (iii) the nonlinear terms, we will routinely make the assumption that $m$ is sufficiently large in terms of $k$, $\ell$, $N$, and all of the parameters. Since we expect the ultimate parameter choices themselves to depend on $m$, it is necessary to assume that they are all bounded in absolute value by a constant $c > 0$ independent of $m$. Of course we do not yet know what range is needed for the parameters, but eventually we will be able to pick $c$ in terms of $k$, $\ell$, and $N$ so that for every sufficiently large $m$ we will be able to find parameters bounded by $c$ so that the corresponding initial surface can be perturbed to minimality.

To continue with the definition of the initial surfaces we fix $k$, $\ell$, and $N \geq 2$ as well as $c > 0$ and parameters $\zeta, \xi \in [-c, c]^{N-1}$. For notational simplicity we assume $k \leq \ell$ and we write $n$ for the greatest integer no greater than $N/2$, so that $N = 2n$ when $N$ is even and $N = 2n + 1$ when $N$ is odd. We acknowledge a certain redundancy in the minimal surfaces ultimately exhibited, one which is easily removed by taking $k$ and $\ell$ relatively prime.

We associate to $k$, $\ell$, and $m$ the rectangular lattice

\[
\left\{ \left( \frac{\sqrt{2\pi}}{km}, \frac{\sqrt{2\pi}}{\ell m}, 0 \right) \in \mathbb{R}^3 : i, j \in \mathbb{Z} \right\},
\]

fixing the horizontal positions of the catenoidal tunnels. The tunnels connecting a given pair of tori have vertical axes either all intersecting this lattice or all intersecting its translate by $\left( \frac{\pi}{\sqrt{2km}}, \frac{\pi}{\sqrt{2\ell m}}, 0 \right)$.

In the next section we will determine a collection $\{\tau_i\}_{i=1}^{N-1}$ of positive real numbers, as functions of $k$, $\ell$, and $m$, to serve as catenoidal waist radii when $\zeta = 0$; the ratio of any two of these waists will be bounded independently of $m$, and we shall find

\[
\tau_1 = \frac{1}{10\ell m} e^{-\frac{km^2}{4\pi}(1-\frac{3}{2})},
\]
where \( c_2 \) is a positive number depending on \( k, \ell, m \), and \( N \) but independently of \( m \) bounded away from 2. For general \( \zeta \) we define the radii

\begin{equation}
\tau_i = \begin{cases} 
\ell m \xi_1 & \text{for } i = 1 \\
\ell m \xi_1 + k^{-1} \ell^{-1} m^{-2} \ell \xi_i & \text{for } 1 < i < N.
\end{cases}
\end{equation}

The \( N \) heights of the tori are

\begin{equation}
z_i = \begin{cases} 
\tau_1 \xi_1 - 2^{N \mod 2} \log_2 \left( \frac{1}{m \tau_n} \right) - 2 \sum_{j=1}^{n-1} \tau_j \log_2 \left( \frac{1}{m \tau_j} \right) & \text{for } i = 1 \\
\frac{\tau_{i-1} \xi_{i-1} + \tau_i \xi_1}{2} - 2^{N \mod 2} \log_2 \left( \frac{1}{m \tau_n} \right) + 2 \left( \sum_{j=1}^{i-1} \sum_{j=1}^{n-1} \tau_j \log_2 \left( \frac{1}{m \tau_j} \right) \right) & \text{for } 1 < i < N \\
\tau_{N-1} \xi_{N-1} - 2^{N \mod 2} \log_2 \left( \frac{1}{m \tau_n} \right) + 2 \sum_{j=n}^{N-1} \tau_j \log_2 \left( \frac{1}{m \tau_j} \right) & \text{for } i = N
\end{cases}
\end{equation}

and the \( N - 1 \) heights of the catenoids’ centers are

\begin{equation}
z_i^K = \tau_i \xi_i + \tau_i \log_2 \left( \frac{1}{m \tau_i} \right) + 2 \sum_{j=1}^{i-1} \sum_{j=1}^{n-1} \tau_j \log_2 \left( \frac{1}{m \tau_j} \right) - 2^{N \mod 2} \log_2 \left( \frac{1}{m \tau_n} \right) \log_2 \left( \frac{1}{m \tau_n} \right) \text{ for } 1 \leq i \leq N - 1.
\end{equation}

These definitions can be understood as implementing the matching conditions mentioned earlier as well as the vertical offsets introduced by the \( \xi \) parameters. Each logarithmic term, ignoring any powers of 2 appearing as prefactors, represents the height achieved by a corresponding catenoid above its waist plane a distance \( \frac{1}{m \tau_n} \) from its axis, where the catenoids are meant to transition to planes (tori under \( \Phi \)). The factor of \( 10 \ell \) is chosen—10 somewhat arbitrarily and \( \ell \) because we assume \( k \leq \ell \)—to ensure the transition is completed on the order \( (m^{-1}) \) of the lattice spacing but well away from neighboring catenoids. Here the logarithm is used, as a matter of convenience, to approximate inverse hyperbolic cosine.
Symmetries. We orient $S^3$ by choosing the standard orientation on $\mathbb{C}^2$, and, given an oriented circle $C$ in $S^3$, we write $R^\theta_C$ for rotation by $\theta$ about $C$. Thus $R^\theta_C(z_1, z_2) = (e^{i\theta}z_1, z_2)$ and $R^\theta_{C_2}(z_1, z_2) = (z_1, e^{i\theta}z_2)$. Note that $\Phi$ intertwines each rotation of these two types with a horizontal translation. Let $X$ denote reflection through the totally geodesic sphere with equator $C_2$ and $(1,0)$ a pole, and let $Y$ denote reflection through the totally geodesic sphere with equator $C_1$ and $(0,1)$ a pole. Then $X(z_1, z_2) = (\overline{z_1}, z_2)$ and $Y(z_1, z_2) = (z_1, \overline{z_2})$; $\Phi$ intertwines each of these with reflection through a vertical coordinate plane.

We next define the subgroup $G$ of $O(4)$ generated as
\begin{equation}
G = \langle R^{2m}_{C_1}, X, ZR^{(N \mod 2)\pi/2m}_{C_1} \rangle.
\end{equation}
Not only does $G$ preserve the lattice on $\mathbb{T}$ defined by $k$, $\ell$, and $m$, but it will also preserve the corresponding initial surface, whatever values the parameters assume, and we will later admit only functions equivariant under a natural action by $G$ as candidate perturbations. Thus all elements of $G$ will be symmetries of the minimal surfaces we construct.

While $\Phi$ intertwines reflections through a plane of constant $x$ or constant $y$ with an isometry of $S^3$ (reflection through the corresponding sphere), it is clear from 2.3 that $(x, y, z) \mapsto (x, y, -z)$ does not define a symmetry of $S^3$ but $(x, y, z) \mapsto (y, x, -z)$ does. Indeed reflection through a diagonal or antidiagonal line on $\{z = 0\}$ corresponds to reflection through a great circle on $\mathbb{T}$. Writing $Z$ for reflection through the great circle $\{z_1 = z_2\}$, so that $Z(z_1, z_2) = (z_2, z_1)$, we see that $\Phi$ intertwines $Z$ with the above reflection through $\{x = y\}$ on $\{z = 0\}$ and further that $Z$ preserves the lattice of the construction precisely when it is square.

Accordingly, when $k = \ell$, we could enforce
\begin{equation}
G_s = \langle R^{2m}_{C_1}, X, ZR^{(N \mod 2)\pi/2m}_{C_1} \rangle.
\end{equation}
as the symmetry group of the construction. Actually, in the square case $G_s$ will automatically preserve the resulting minimal surface provided we constrain the parameters to realize the symmetries of $G_s$ in the initial surfaces, cutting in half the number of free parameters. Specifically we would require that for $N$ even $\zeta_{n-i} = \zeta_{n+i}$ and $\xi_{n-i} = -\xi_{n+i}$, while for $N$ odd $\zeta_{n-i} = \zeta_{n+i+1}$ and $\xi_{n-i} = -\xi_{n+i+1}$, leaving just $N - 1$ independent parameters. The entire construction, including the solution operator $R$, now coming with $(N - 1)$-dimensional extended substitute kernel, will then respect $G_s$ without any modifications. For this reason we will not again explicitly address the square case.

Assembly and basic properties. Assuming the specification of the waist radii given in the next section and setting $R = \frac{1}{16km}$, $X = \frac{\pi}{\sqrt{2km}}$, and $Y = \frac{\pi}{\sqrt{2km}}$, let
\begin{equation}
\Omega_i = \begin{cases} 
\Phi \left(T_{ext}[(0,0), z^K_i, z_1, R, X, Y, \tau_1](T_{X,Y,\tau_1}) \right) & \text{for } i = 1 \\
\Phi \left(T_{int}[(2i-3) \left(\frac{X}{2}, \frac{Y}{2}\right), z^{K-1}_{i-1}, z^K_i, z_i, R, X, Y, \tau_{i-1}, \tau_i] \left(T_{X,Y,\tau_{i-1},\tau_i} \right) \right) & \text{for } 1 < i < N \\
\Phi \left(T_{ext}[(2N-2) \left(\frac{X}{2}, \frac{Y}{2}\right), z^{K-1}_{N-1}, z_N, R, X, Y, \tau_{N-1}] \left(T_{X,Y,\tau_{N-1}} \right) \right) & \text{for } i = N.
\end{cases}
\end{equation}
Then we define the initial surface
\begin{equation}
\Sigma = G \bigcup_{i=1}^{N} \Omega_i.
\end{equation}

Proposition 2.19. Given positive integers $k$, $\ell$, and $N \geq 2$, and a real number $c > 0$, there exists $m_0 > 0$ such that for every integer $m \geq m_0$ and every choice of parameters $\zeta, \xi \in [-c, c]^{N-1}$ the initial surface $\Sigma[N, k, \ell, m, \zeta, \xi]$, defined by 2.18, is a smooth, closed, orientable surface, embedded
in $S^3$, of genus $kℓm^2(N−1)+1$, and invariant as a set under the action of $G[N,k,ℓ,m,ζ,ξ]$, defined by \[2.15\]

Proof. In light of the selection of waist radii in the next section, especially \[3.6\] and \[3.8\] we have

$$
\lim_{m→∞} \tau_i = \lim_{m→∞} \tau_i \ln \frac{1}{10ℓmτ_i} = 0,
$$

and so all of the claims are clear from the construction of $Σ$. □

Of course we have not yet precisely specified $\{τ_i\}$, a gap filled in the next section.

3. Forces and Dislocations

Forces. As mentioned in the introduction, we will eventually discover that on the toral regions the Jacobi operator $L$ of the initial surfaces has approximate kernel, spanned by eigenfunctions with small eigenvalues. Actually we already have reason to anticipate this state of affairs. The Clifford torus $T$ itself has Jacobi operator $Δ_T + 4$, with kernel spanned by $\cos \sqrt{2}x \cos \sqrt{2}y$, $\cos \sqrt{2}x \sin \sqrt{2}y$, $\sin \sqrt{2}x \cos \sqrt{2}y$, and $\sin \sqrt{2}x \sin \sqrt{2}y$. Each of these functions is the normal projection of the restriction to $T$ of a Killing field on $S^3$. (The isometry group $O(4)$ of the sphere has dimension 6, while its subgroup stabilizing the torus has dimension 2.) Evidently there are no exceptional Jacobi fields: all of them originate from families of ambient isometries. Note that while none of these four functions satisfies the symmetries of our construction, the first one almost does, on a single fundamental domain containing $(1,1) ∈ T$.

The Killing field $K$ inducing this Jacobi field is just rotation, toward $C_2$, along the circle preserved by both $X$ and $Y$. The circle has preimage under $Φ$ containing the z-axis, and with respect to the $(x,y,z)$ coordinate system

$$
K = -\frac{1}{\sqrt{2}} \cot \left( z + \frac{π}{4} \right) \sin \sqrt{2}x \cos \sqrt{2}y \partial_x + \frac{1}{\sqrt{2}} \tan \left( z + \frac{π}{4} \right) \cos \sqrt{2}x \sin \sqrt{2}y \partial_y + \cos \sqrt{2}x \cos \sqrt{2}y \partial_z.
$$

Thus $K$ can be regarded as approximating $\partial_z$, which of course generates vertical translations and corresponds to the constant 1 spanning the kernel of the Jacobi operator $Δ$ for horizontal Euclidean planes—the inverse images under $Φ$ of constant-mean-curvature tori parallel to $T$. We will confront this basic eigenfunction, without any explicit mention of Killing fields, in the course of our detailed study of the linearized operator. Right now we intend to calculate the $K$ force (also called flux in the literature) through various regions and to study its dependence on the initial surface. These forces will measure the projection of the surface’s mean curvature onto the approximate kernel, so in the final section we will apply the following computations to complete the proof of the main theorem. More immediately we will impose balancing conditions on the initial surface, such that the $K$ force on various regions vanishes, at least within a margin on the order of the perturbations, by functions and parameters, that we will be making. This balancing will finally determine the waist radii, up to choice of $ζ$, thus completing the construction of the initial surfaces.

Let $F_i = \int_{∂Ω_i} (K, η_i)$, the $K$ force exerted by $Ω_i$, having outward conormal $η_i$. One finds

$$
F_i + O(m^{-2}τ_1) = \begin{cases}
2πτ_1 + \frac{8π^2}{kℓm^2}z_1 & \text{for } i = 1 \\
2π(τ_i - τ_{i-1}) + \frac{8π^2}{kℓm^2}z_i & \text{for } 2 ≤ i ≤ N - 1 \\
-2πτ_N - 1 + \frac{8π^2}{kℓm^2}z_N & \text{for } i = N,
\end{cases}
$$

where the quantity $O(m^{-2}τ_1)$ absorbs terms which, for $m$ sufficiently large—in terms of $k, ℓ, N,$ and $c$—are bounded by $m^{-2}τ_1$ times a constant depending on just $k, ℓ, N$. Actually we are not in a position to justify this bound on the error terms given arbitrary $\{τ_i\}$. Our first application of the forces, however, is merely heuristic, as motivation for the definition of $\{τ_i\}$ via \[3.6\] and \[3.4\] for $\{c_i\}$ bounded independently of $m$. With this attitude one can simply retain the dominant
terms contributing to the force, captured on the right-hand side above, and impose the balancing
described below to obtain \([3.6]\) and \([3.8]\). Armed with these one can return to verify the estimate for
the error terms encapsulated in the expression \(O(m^{-2}\tau_1)\).

The computation of the forces themselves is simple. The term \(2\pi\tau_i\) corresponds to the force
through a waist circle; the error originates from treating the waist as a Euclidean circle and can be
assessed using \([2.3]\). The calculation of the force through a rectangular boundary component is eased
by invoking the first-variation-of-area formula to replace the integral over that component by an
integral over the exact constant-mean-curvature torus \(\{z = z_0\}\) also bounded by it and having the
same conormal there as the initial surface. This torus’ mean curvature is \(2\tan 2z_0\partial_z\), accounting
for the appearance of terms contributing to the force that contain height factors, and its area is
\(2\pi^2\cos 2z_0\). The error here derives from approximating \(\tan z\) by \(z\) and \(\cos 2z_0\) by 1.

Consequently \(\mathcal{F}_{i+1} - \mathcal{F}_i + O(m^{-2}\tau_1) = \)
\[
\begin{cases}
2\pi(\tau_2 - \tau_1) + \frac{8\pi^2}{k\ell m^2} \left(2\tau_1 \ln \frac{1}{10\ell m \tau_1} + \frac{\xi \tau_2 - \xi \tau_1}{2} \right) & \text{for } i = 1 \text{ and } N \geq 3 \\
2\pi(\tau_{i+1} - 2\tau_i + \tau_{i-1}) + \frac{8\pi^2}{k\ell m^2} \left(2\tau_i \ln \frac{1}{10\ell m \tau_i} + \frac{\xi_{i+1} \tau_{i+1} - \xi_{i-1} \tau_{i-1}}{2} \right) & \text{for } 2 \leq i \leq N - 2 \\
2\pi(-2\tau_{N-1} + \tau_{N-2}) + \frac{8\pi^2}{k\ell m^2} \left(2\tau_{N-1} \ln \frac{1}{10\ell m \tau_{N-1}} + \frac{\xi_{N-1} \tau_{N-1} - \xi_{N-2} \tau_{N-2}}{2} \right) & \text{for } i = N - 1 \text{ and } N \geq 3,
\end{cases}
\]
and \(\mathcal{F}_{n+1} + O(m^{-2}\tau_1) = \)
\[
2\pi(\tau_{n+1} - \tau_n) + [(N + 1) \bmod 2] \frac{8\pi^2}{k\ell m^2} \left(\tau_n \ln \frac{1}{10\ell m \tau_n} + \frac{\tau_{n+1} \xi_{n+1} + \tau_n \xi_n}{2} \right),
\]
understanding \(\tau_{n+1} = \xi_{n+1} = 0\) in the last equation when \(N = 2\).

Balancing. The preceding equations will play an indispensable role in selecting the right parameters
at the conclusion of the construction, but at the moment we set \(\zeta = \xi = 0\) and impose balancing
so that all the above forces vanish up to terms of order \(m^{-2}\tau_1\), in order to determine the collection \(\{\tau_i\}_{i=1}^{N-1}\). Starting with the last equation (possibly coupled with the one directly above it)
and then proceeding inductively to compare the force through each \(\Omega_i\) with the force through
its reflection (in the sense of Euclidean \(\mathbb{R}^3\)) through \(\{z = 0\}\), we evidently have
\[
(3.3) \quad \tau_{n-i} = \tau_{n+i+1} \text{ for } N \text{ odd} \\
\tau_{n-i} = \tau_{n+i} \text{ for } N \text{ even.}
\]
Therefore, defining
\[
(3.4) \quad c_i = \tau_i / \tau_1 \text{ for } 1 \leq i \leq N - 1,
\]
we need only solve the \(n\) equations
\[
2\pi(c_2 - 2) = \frac{8\pi^2}{k\ell m^2} 2\ln 10\ell m \tau_1, \text{ provided } n \geq 2
\]
\[
2\pi(c_{i+1} - 2c_i + c_{i-1}) = \frac{8\pi^2}{k\ell m^2} 2c_i \ln 10\ell m \tau_i \text{ for } 2 \leq i \leq n - 1
\]
\[
2\pi(c_{n-1} - c_n) = 2^{N \bmod 2} \frac{8\pi^2}{k\ell m^2} c_n \ln 10\ell m \tau_n \text{ (understanding } c_0 = 0\text{)}
\]
for \( \{c_i\}_{i=2}^n \) and \( \tau_1 \). The first applicable equation directly yields

\[
\tau_1 = \frac{1}{10\ell m} e^{-\frac{k\ell m}{8\pi} (1 - \frac{c}{2})},
\]

where we understand \( c_2 = 0 \) when \( N = 2 \) and we note \( 3.3 \) implies \( c_2 = 1 \) when \( N = 3 \). By subtracting the other equations from multiples of the first we find

\[
c_2 c_2 - c_3 = 1 - \frac{8\pi}{k\ell m} c_2 \ln c_2 \text{ for } n \geq 3
\]

\[
- c_{i-1} + c_2 c_i - c_{i+1} = -\frac{8\pi}{k\ell m} c_i \ln c_i \text{ for } 3 \leq i \leq n - 1
\]

\[
- 2^{(N+1) \mod 2} c_{n-1} + (c_2 - N \mod 2) c_n = -\frac{8\pi}{k\ell m} c_n \ln c_n.
\]

**Lemma 3.8.** Given \( N \geq 4 \) there exist \( d_2 < d_3 < \cdots < d_n \) depending on just \( N \) with \( d_2 \in (1, 2) \) strictly increasing in \( n \), and there exists \( m_0 > 0 \) so that for each \( m > m_0 \) there are \( c_2, c_3, \ldots, c_n \) solving \( 3.7 \) and satisfying \( \lim_{m \to \infty} c_i = d_i \).

**Proof.** Bear in mind that balancing has been accomplished (by \( 3.6 \)) for \( N = 2 \) and \( N = 3 \) and in these cases the system \( 3.7 \) is vacuous. Momentarily ignoring the logarithmic terms, for \( N = 4 \) the system reduces to \( d_2^2 = 2 \), so \( d_2 = \sqrt{2} < 2 \), while for \( N = 5 \) we get \( d_2^2 - d_2 - 1 = 0 \), yielding \( d_2 = \frac{1 + \sqrt{5}}{2} < 2 \). Now the functions \( c_2 \mapsto c_3^2 \) and \( c_2 \mapsto c_3^2 - c_2 - 1 \) have nonzero derivatives at these respective values, so the lemma is established for \( N = 4 \) and \( N = 5 \) by applying the inverse function theorem and taking \( m \) large. Thus we may assume \( n \geq 3 \) and pursue an elaboration of the same strategy.

For real \( \beta \) define the \((n-1) \times (n-1)\) matrices

\[
A_{2n}(\beta) = \begin{pmatrix}
\beta & -1 & 0 & 0 & \cdots & 0 \\
-1 & \beta & -1 & 0 & \cdots & 0 \\
0 & -1 & \beta & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & \beta & -1 \\
0 & 0 & \cdots & 0 & -2 & \beta
\end{pmatrix}
\]

\[
A_{2n+1}(\beta) = \begin{pmatrix}
\beta & -1 & 0 & 0 & \cdots & 0 \\
-1 & \beta & -1 & 0 & \cdots & 0 \\
0 & -1 & \beta & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & \beta & -1 \\
0 & 0 & \cdots & 0 & -1 & \beta - 1
\end{pmatrix}.
\]

Thus, temporarily neglecting the logarithmic terms, we need to solve the system

\[
A_N(\beta) \begin{pmatrix}
d_2 \\
d_3 \\
\vdots \\
d_n
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix} \text{ subject to the constraints } \beta = d_2 \text{ and } d_i > 0 \text{ for } 2 \leq i \leq n.
Using Cramer’s rule and expansion by minors
\[ d_i = \frac{P_{n-i+1}[N \mod 2](\beta)}{P_n[N \mod 2](\beta)}, \]
where
\[ P_i[0](\lambda) = \det A_{2i}(\lambda) \text{ for } i \geq 3, \]
\[ P_i[1](\lambda) = \det A_{2i+1}(\lambda) \text{ for } i \geq 3, \]
\[ P_2[0](\lambda) = \lambda, P_2[1] = \lambda - 1, \]
\[ P_1[0](\lambda) = 2, \text{ and } P_1[1](\lambda) = 1. \]

Further expansion by minors reveals the recursive relations (independent of the parity of \( N \))
\[ P_{i+1}(\lambda) = \lambda P_i(\lambda) - P_{i-1}(\lambda) \text{ for } i \geq 2. \]

On the other hand, the above expression for \( d_2 \), applying the constraint \( d_2 = \beta \), can be rewritten as
\[ \beta P_n(\beta) = P_{n-1}(\beta), \]
whence \( n \geq 3 \) delivers
\[ P_{n+1}(\beta) = 0. \]

We now claim that for each \( n \geq 3 \) (and either parity of \( N \))
- \( P_n \) has a root strictly greater than 1; if \( \gamma_n \) is its largest such root, then
- \( P_{n-1}(x) > 0 \) whenever \( x \geq \gamma_n \), and
- \( P_{n+1}(\gamma_n) < 0. \)

Moreover, \( \gamma_n \) is strictly increasing in \( n \).

These claims can be established by induction on \( n \). The case \( n = 3 \) is easily verified: \( P_2[0](x) = x \), \( P_3[0](x) = x^2 - 2 \), and \( P_4[0](x) = x^3 - 3x \), so \( \gamma_3[0] = \sqrt{2} \), \( P_2[0](x \geq \gamma_3[0]) > 0 \), and \( P_4[0](\gamma_3[0]) = 2\sqrt{2} - 3\sqrt{2} < 0 \), while \( P_2[1](x) = x - 1 \), \( P_3[1](x) = x^2 - x - 1 \), and \( P_4[1](x) = x^3 - x^2 - 2x + 1 \), so \( \gamma_3[1] = \frac{1+\sqrt{5}}{2} \), \( P_2[1](x \geq \gamma_3[1]) \geq \frac{\sqrt{5}-2}{2} > 0 \), and \( P_4[1](\gamma_3[1]) = \frac{1-\sqrt{5}}{2} < 0 \). Now suppose the claims hold for \( n = j \). According to the third claim \( P_{j+1}(\gamma_j) \) is strictly increasing in \( n \), so \( \gamma_{j+1} > \gamma_j \). Then \( P_j(x \geq \gamma_{j+1}) > 0 \) by the maximality of \( \gamma_j \). Finally, using \( 3.13 \) \( P_{j+2}(\gamma_{j+1}) = \gamma_{j+1} P_{j+1}(\gamma_{j+1}) - P_j(\gamma_{j+1}) \),

which is negative, since the first term vanishes and the second has just been established positive.

Thus \( \beta = \gamma_{n+1} \) solves \( 3.15 \). Since \( \gamma_n \) is greater than 1 and strictly increasing in \( n \), we find, using again \( 3.13 \) that
\[ P_j(\gamma_n) - P_{j+1}(\gamma_n) = (1 - \gamma_n)P_j(\gamma_n) + P_{j-1}(\gamma_n) > 0 \]
whenever \( j < n \). It then follows from \( 3.12 \) that \( d_2 < d_3 < d_4 < \cdots < d_n \).

Finally we claim \( d_2 < 2 \). In fact we assert that for each \( n \geq 2 \) (regardless of the parity of \( N \))
- \( P_n(x) - P_{n-1}(x) \geq 0 \) and \( P_n(x) > 0 \) whenever \( x \geq 2 \),
which is proven by induction on \( n \). For \( n = 2 \) and \( x \geq 2 \) we have \( P_2(x) - P_1(x) = x - 2 \geq 0 \) (whatever the parity of \( N \)) and clearly both \( P_2[0](x) = x > 0 \) and \( P_2[1](x) = x - 1 > 0 \). Assuming then that the claim holds for \( n = j \), we get from \( 3.13 \) assuming still \( x \geq 2 \),
\[ P_{j+1}(x) - P_{j}(x) = xP_{j}(x) - P_{j-1}(x) - P_{j}(x) = (x - 1)P_{j}(x) - P_{j-1}(x) \]
\[ \geq P_{j}(x) - P_{j-1}(x) \geq 0 \]
and therefore \( P_j(x) > 0 \) as well. We conclude that for every \( n \geq 2 \) we have \( P_n(x) > 0 \) whenever \( n \geq 2 \), so all roots of \( P_n \) lie to the left of 2, establishing the bound on \( d_2 \).
It remains to reintroduce the logarithmic terms. We define the function $F_n : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$F_N \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = A_N(x_2) \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$$

and calculate its derivative at $(d_i)_{i=2}^n$:

$$dF_N|_{(d_i)_{i=2}^n} = A_N(d_2) + \begin{pmatrix} d_2 & 0 & 0 & \cdots & 0 \\ d_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_n & 0 & 0 & \cdots & 0 \end{pmatrix},$$

whose determinant is

$$\det A_N(d_2) + \det \begin{pmatrix} d_2 & -1 & 0 & 0 & \cdots & 0 \\ d_3 & \beta & -1 & 0 & \cdots & 0 \\ d_4 & -1 & \beta & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ d_{n-1} & 0 & \cdots & -1 & \beta & -1 \\ d_n & 0 & \cdots & 0 & -2(N+1) \mod 2 & \beta - N \mod 2 \end{pmatrix}$$

$$= P_n(d_2) + \sum_{i=2}^n d_i P_{n-i+1}(d_2) > 0.$$

We conclude by invoking the inverse function theorem and taking $m$ large.

\[\square\]

**Parameter dependence.** Henceforth we fix $\{c_i\}_{i=2}^n$ as in the previous subsection, the corresponding $\tau_1$ as defined by (3.6) $\tau_i = c_i \tau_1$ for $2 \leq i \leq n$, and $\tau_i$ for remaining values of the index in accordance with (3.3). The next lemma completes our estimation, at this stage, of the forces. Both the $\xi$ and $\zeta$ parameters influence the forces, which will be analyzed to address the substitute kernel. The $\xi$ parameters also control dislocations $\{D_i\}$, or displacements of a pair of adjacent inequivalent catenoidal regions relative to the toral region they share.

**Lemma 3.21.** Given $c > 0$ and positive integers $k, \ell, m$, and $N \geq 2$, parameters $\zeta, \xi \in [-c, c]^{N-1}$, and $\{\tau_i\}_{i=1}^N$ as defined by (2.12) let

$$D_i = \begin{cases} \frac{1}{2} \tau_i \xi_i - \frac{1}{2} \tau_{i-1} \xi_{i-1} & \text{for } 2 \leq i \leq N - 1 \\ 0 & \text{for } i = 1 \text{ and } i = N. \end{cases}$$

Then for $m$ sufficiently large in terms of $k, \ell, N,$ and $c$,

(i) $\frac{k \omega_m^2}{2 \pi^2} (F_i - F_{i+1}) + \frac{n \pi}{7} (D_i + D_{i+1}) + O(1) = (Z \zeta)i$, where

(ii) the $(N-1) \times (N-1)$ matrix

$$Z = \begin{pmatrix} 8\pi & -c_2 & 0 & 0 & \cdots & 0 & 0 \\ 8\pi c_2 & 1 + c_3 & -c_3 & 0 & \cdots & 0 & 0 \\ 8\pi c_3 & -c_2 & c_2 + c_4 & -c_4 & 0 & \cdots & 0 \\ 8\pi c_4 & 0 & -c_3 & c_3 + c_5 & -c_5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 8\pi c_{N-2} & 0 & \cdots & 0 & -c_{N-3} & c_{N-3} + c_{N-1} & -c_{N-1} \\ 8\pi c_{N-1} & 0 & 0 & \cdots & 0 & -c_{N-2} & c_{N-2} \end{pmatrix}$$
satisfied; that the inverses are bounded as claimed follows from the convergence of $c$ matrices, expansion by minors and a simple inductive argument reveal that the determinants are $E$-structuration, we must first identify certain norms and corresponding spaces of sections. For the most

\[ \xi = \sum_{s,t} \frac{\partial}{\partial s} f_i - \frac{\partial}{\partial t} f_i + O(1); \]

\[ (iii) \frac{k \pi^2}{8 \alpha^2} (F_1 + F_N) = \xi_1 + cN-1 \xi_{N-1} + O(1); \]

\[ (iv) \frac{\partial}{\partial t} D_i = c_i \xi_i - c_{i-1} \xi_{i-1} + O(1); \] and

\[ (v) \text{the} (N-1) \times (N-1) \text{ matrix} \]

\[
\Xi = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & c_{N-1} \\
-1 & c_2 & 0 & \cdots & 0 & 0 & 0 \\
0 & -c_2 & c_3 & \cdots & 0 & 0 & 0 \\
0 & 0 & -c_3 & c_4 & \cdots & 0 & 0 \\
\vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots \\
0 & 0 & \cdots & 0 & -c_{N-3} & c_{N-2} & 0 \\
0 & 0 & \cdots & 0 & 0 & -c_{N-2} & c_{N-1}
\end{pmatrix}
\]

has inverse bounded independently of $m$ and $c$.

**Proof.** The calculations are straightforward, using our previous relations between the forces and taking advantage of the balancing just imposed for $\xi = \xi = 0$. As for the invertibility of the matrices, expansion by minors and a simple inductive argument reveal that the determinants are strictly positive provided merely that each $c_i$ is strictly positive, which condition is obviously satisfied; that the inverses are bounded as claimed follows from the convergence of $c_i$ to $d_i$ with large $m$. \(\square\)

4. Estimates of the initial geometry

**Norms and spaces of sections.** To state the estimates and to carry out the rest of the construction, we must first identify certain norms and corresponding spaces of sections. For the most part our notation is standard and speaks for itself. Given a vector bundle $E \to M$, a nonnegative integer $j$, and an exponent $\alpha \in (0,1)$, we write $\Gamma^j_{\text{loc}}(E)$ and $\Gamma^{j,\alpha}_{\text{loc}}(E)$ for the space of sections of $E$ having component functions of class $C^j_{\text{loc}}$ or $C^{j,\alpha}_{\text{loc}}$ respectively relative to every local chart and trivialization. We set $\Gamma^\infty(E) = \bigcap_{j=0}^\infty \Gamma^j_{\text{loc}}(E)$, and in case $E$ is the trivial bundle $M \times \mathbb{R}$, we denote the spaces here and below by replacing $\Gamma$ with $C$ and $E$ with $M$, as usual.

All of the vector bundles of interest to us are derived from tangent bundles by a combination of duality, tensor product, pullback, and projection; a single Riemannian metric will determine canonical metrics and connections on all these bundles. When there is no danger of confusion we write simply $\|\cdot\|$ for the corresponding pointwise norm and $D$ for the connection. Given a section $u$ of a bundle $E \to M$ so equipped, we define the standard global norms

\[ (4.1) \| u : \Gamma^j(E, g) \| = \| u \|_j = \sum_{i=0}^j \sup_{p \in M} \| D^i u(p) \| \]

as well as the Hölder seminorms

\[ (4.2) [u]_{\alpha} = \sup_{\gamma:[0,1] \to M} \frac{\| u(\gamma(1)) - P^1_{\alpha}[\gamma] u(\gamma(0)) \|}{\| \gamma \|^\alpha} , \]

where the supremum is taken over all piecewise $C^1$ paths and, given such a path $\gamma : [a, b] \to M$ and parameters $s, t \in [a, b]$, $P^1_{\alpha}[\gamma] : E_{\gamma(s)} \to E_{\gamma(t)}$ is parallel transport along $\gamma$ from the fiber over $\gamma(s)$ to the fiber over $\gamma(t)$ and $|\gamma|$ denotes the length of $\gamma$.

Then we can define also the Hölder norms

\[ (4.3) \| u : \Gamma^{j,\alpha}(E, g) \| = \| u \|_{j,\alpha} = \| u \|_j + [D^j u]_{\alpha} . \]
Note that for functions on convex open subsets of Euclidean space these Hölder norms agree with the conventional ones, and on precompact subsets with sufficiently regular boundary the two norms are at least equivalent. Generally, the spaces $\Gamma^{j,\alpha}(E, g)$ and $\Gamma^j(E, g)$ consisting of sections with finite corresponding norm enjoy many of the properties familiar from the Euclidean case; in particular $\Gamma^{j,\beta}(E, g)$ embeds compactly in $\Gamma^{j,\alpha}(E, g)$ whenever $M$ is closed and $0 < \alpha < \beta < 1$.

In the construction we will routinely wish to compare norms of the above type induced by different metrics on a single manifold. The definitions make it easy to see that $\| u : \Gamma^{j,\alpha}(E, h) \| \leq C \| u : \Gamma^{j,\alpha}(E, g) \|$, where $C$ is controlled by the $g$-norms of $h$, its inverse, and finitely many $g$-derivatives of $h$ (the maximum order needed depending in a transparent way on $j$, $\alpha$, and on the bundle $E$).

If $M$ is a two-sided hypersurface immersed in a Riemannian manifold $N$, and $G_M$ is a group of isometries of $N$ preserving $M$ as a set, then $G_M$ acts on a section $u$ of the normal bundle of $M$ by $(g, u)(p) = g_*[u(g^{-1}(p))]$ for each $g \in G_M$; because this bundle is just the trivial $\mathbb{R}$ bundle over $M$, its sections can be identified with functions (for us representing mean curvature or normal perturbations) on which the corresponding action of $G_M$ is given by $(g, f)(p) = (-1)^s f(g^{-1}(p))$, where $(-1)^s$ is 1 if $g$ preserves each of the two sides of $M$ and $-1$ if it exchanges them.

All the elements of the symmetry group $G$ of the construction can be seen to fix each side of $\Sigma$, so this action is trivial, but we remark that $G_s$ contains reflections through great circles which when $N$ is odd exchange the sides. As already explained though, our construction will produce $G_s$-invariant minimal surfaces from $G_s$-invariant initial surfaces despite enforcing just $G$ throughout.

In general we will append the subscript $G$ to a space of functions to denote that subspace consisting of function which are equivariant under the $G_M$ action just described. Sometimes we will append the subscript $c$ to a space of sections to isolate those sections having compact support.

Finally we will often wish to work with weighted versions above the norms. For this construction the following definition suffices:

\[
\| u : \Gamma^{j,\alpha}(E, g, f) \| = \sup_{p \in M} \frac{\| u : \Gamma^{j,\alpha}(E|_{B(p,1,g)}, g) \|}{f(p)},
\]

where $f : M \to (0, \infty)$ is a weight function and $B(p,1,g)$ is the ball of $g$-radius 1 centered at $p$.

We will also make use of weighted $\Gamma^j$ norms, with the obvious definition.

**The $\chi$ metric.** It is the primary task of this section to estimate the intrinsic and extrinsic geometry of the initial surfaces. To fix the extrinsic quantities we pick on each initial surface $\Sigma$ the global unit normal $\nu$ which is directed toward $C_1$ at the points of $\Sigma$ closest to $C_1$. We then define $A$ to be the scalar-valued second fundamental form of $\Sigma$ relative to $\nu$, and we take the scalar-valued mean curvature $H$ to be the trace of the former.

Every initial surface admits by virtue of its construction a natural decomposition into overlapping regions, each of which resembles either a portion of a torus or a truncated catenoid. Modulo the horizontal symmetries, there are $N$ such toral regions, one for each torus incorporated in the construction, and there are $N-1$ catenoidal regions, one for each pair of adjacent tori. Definitions are made in the subsections below. The estimates will then be obtained by treating the catenoidal regions as perturbations of Euclidean catenoids and the toral regions as graphs over the Clifford torus.

Because all these regions shrink with increasing $m$ and because even on a fixed initial surface the characteristic scale $m^{-1}$ of the toral regions dwarfs the characteristic scale $\tau_1$ near the waists, it will be advantageous to uniformize the problem by working with a metric $\chi$ on each initial surface conformal to the natural one $g = \iota^* g_S$ induced by the round spherical metric $g_S$ and by the inclusion $\iota$ of the initial surface in $\mathbb{S}^3$. We will set

\[
\chi = \rho^2 g,
\]
where the conformal factor $\rho^{-1}$ is to be defined as a $G$-equivariant function on $\Sigma$ (i) measuring on each catenoidal region the $\Phi^*g_E$ distance to the axis and (ii) transitioning smoothly to the constant $m^{-1}$ by the edge of the toral regions.

To be precise, if we first define $\tilde{\rho}[z_1, z_2] : \mathbb{R}^3 \to \mathbb{R}$, for given $z_1, z_2 \in \mathbb{R}$, by

$$
(4.6) \quad \tilde{\rho}[z_1, z_2](x, y, z) = \begin{cases} 
\left( \frac{1}{\sqrt{x^2+y^2}} - m \right) \psi \left[ \frac{1}{\sqrt{m}}, \frac{1}{10\ell m} \right] \left( \sqrt{x^2+y^2} \right) & \text{if } z_1 \leq z \leq z_2 \\
0 & \text{otherwise}
\end{cases}
$$

and $\tilde{\rho} : \mathbb{R}^3 \to \mathbb{R}$ by

$$
(4.7) \quad \tilde{\rho}(x, y, z) = m + \sum_{i=1}^{N-1} \sum_{(p, q) \in \mathbb{Z}^2} \tilde{\rho} \left[ z_i^K, z_{i+1}^K \right] \left( x - \frac{(2p + (i + 1) \mod 2)\pi}{\sqrt{2\ell m}}, y - \frac{(2q + (i + 1) \mod 2)\pi}{\sqrt{2\ell m}}, z \right),
$$

taking $z_0^K = -\frac{\pi}{\ell}$ and $z_N^K = \frac{\pi}{\ell}$, then $\rho \in C_G(\Sigma)$ is uniquely defined by

$$
(4.8) \quad \Phi|_{\Phi^{-1}(\Sigma)} \rho = \tilde{\rho} \circ \Phi^{-1}(\Sigma).
$$

Equipped with the $\chi$ metric, each catenoidal region tends with large $m$ to the flat cylinder of radius 1, while each toral region tends, away from the catenoids adjoining it, to a flat $\frac{\sqrt{2\pi}}{k} \times \frac{\sqrt{2\pi}}{t}$ rectangle.

Before proceeding, we briefly mention a couple differences of our approach from [7]. First, our catenoidal and toral regions above correspond to their extended standard regions, but since we never view their standard regions or transition regions in isolation, we omit the modifier extended. Second, our use of the $\chi$ metric follows theirs to study the mean curvature equation on the initial surfaces globally, but whereas Kapouleas and Yang introduce another metric, $h$, conformal to $g$ in order to analyze the approximate kernel, we will apply the $\chi$ metric to this problem as well, in the next section.

**Catenoidal regions.** Given $a > 0$ set $X_a = [-a, a] \times S^1$, and, given further $(x_0, y_0, z_0) \in \mathbb{R}^3$ and $\tau > 0$, define the embedding $\kappa : (x_0, y_0, z_0), \tau, a] : X_a \to \mathbb{R}^3$ by

$$
(4.9) \quad \kappa((x_0, y_0, z_0), \tau, a) = (x_0, y_0, z_0) + \tau(\cosh t \cos \theta, \cosh t \sin \theta, t).
$$

If we endow the cylinder $\mathbb{R} \times S^1$ with its flat metric

$$
(4.10) \quad \hat{\chi} = dt^2 + d\theta^2
$$

and the image of $\kappa$ above with its natural catenoidal metric $\gamma$ induced by the ambient Euclidean metric $g_E$ on the target, we can easily calculate that $\kappa$ is a conformal map with

$$
(4.11) \quad \kappa^*\gamma = (\tau^2 \cosh^2 t) \hat{\chi}.
$$

We also observe that the pointwise inner product of the Killing field $\partial_x$ on $\mathbb{R}^3$ with a global unit normal for the catenoid defines a Jacobi field. Multiplying the Jacobi equation for this field by the conformal factor $\tau^{-2} \sech^2 t$ one can readily solve for the squared norm

$$
(4.12) \quad \|\hat{A}\|_{\gamma}^2 = 2\tau^{-2} \sech^4 t
$$

of the second fundamental form of $\kappa$.

Next, for $1 \leq i \leq N - 1$ set

$$
(4.13) \quad a_i = \text{arcosh} \left( \frac{1}{10\ell m t_i} \right)
$$
and define \( \kappa_i : \mathbb{K}_{a_i} \to S^3 \) by

\[
\kappa_i = \Phi \circ \hat{\kappa} \left( \left( \frac{(i-1)\pi}{\sqrt{2km}}, \frac{(i-1)\pi}{\sqrt{2\ell m}}, z^K_i \right), \tau_i, a_i \right).
\]

Then the image of \( \kappa_i \) is entirely contained in the initial surface and defines the associated catenoidal region

\[
(4.15) \quad \mathcal{K}[i] = \kappa_i(\mathbb{K}_{a_i}).
\]

Since the covering map \( \Phi \) is, on small scales, an approximate isometry, we expect that each catenoidal region will, in a suitably rescaled sense, converge in the large-\( m \) limit to an exact catenoid in Euclidean space. The next proposition quantifies this convergence.

**Proposition 4.16.** Given positive integers \( k, \ell, \) and \( N \geq 2, \) along with \( c > 0, \) there exists \( m_0 > 0 \)—depending on \( k, \ell, N, \) and \( c \)—and there exists \( C > 0 \)—depending on \( k, \ell, \) and \( N, \) but not on \( c \)—such that for each \( \zeta, \xi \in [-c, c]^{N-1}, \) for each \( m > m_0, \) and for \( 1 \leq i \leq N - 1 \)

\[
(i) \quad \| \kappa_i^\ast \chi - \hat{\chi}_K : \Gamma^2(T^\ast \mathbb{K}_{a_i} ; \hat{\chi}_K) \| \leq Cm^2 \tau_1;
\]

\[
(ii) \quad \| \kappa_i^\ast \rho^{-2} \|_{g_2} - 2 \sech^2 t : C^1(\mathbb{K}_{a_i}, \hat{\chi}_K) \| \leq Cm^{-2}; \quad \text{and}
\]

\[
(iii) \quad \| \kappa_i^\ast \rho^{-2} H : C^1(\mathbb{K}_{a_i}, \hat{\chi}_K, m^2 \kappa_i^\ast \rho^{-2} \tau_1 + m^2 \tau_1^2) \| \leq C.
\]

**Proof.** Clearly \( \kappa_i^\ast \rho = \tau_i^{-1} \sech t, \) and from \( 4.14, 4.9 \) and \( 2.3 \) it is simple to calculate

\[
\kappa_i^\ast \chi - \hat{\chi}_K = \sin 2z \left( \tanh^2 t \cos 2\theta dt^2 - 2 \tanh t \sin 2\theta dt^2 - \cos 2\theta d\theta^2 \right),
\]

where \( z = z^K_i + \tau_i t; \) since \( \hat{\chi}_K = dt^2 + d\theta^2, \) since \( dz = \tau_i dt, \) and since the definitions of \( \tau_i, z^K_i, \) and \( a_i \) imply \( |z| \leq Cm^2 \tau_1, \) this establishes (i).

It is fairly efficient to estimate \( \| A \|_{g}^2 \) by the same method outlined above to find \( \| \hat{A} \|_{\hat{g}}^2 \). In this context too, that is relative to \( \Phi^\ast g_S, \partial_x \) is a Killing field, so \( f = \langle \nu, \partial_x \rangle \) defines a Jacobi field on \( \kappa_i \).

We find

\[
\nu = -\frac{N \sech t \cos \theta}{1 + \sin 2z} \partial_x - \frac{N \sech t \sin \theta}{1 - \sin 2z} \partial_y + N \tanh t \partial_z,
\]

with normalization factor

\[
N = \frac{\cos 2z}{\sqrt{1 - \sech^2 t \cos 2\theta \sin 2z - \tanh^2 t \sin^2 2z}},
\]

and so

\[
f(t, \theta) = N \sech t \cos \theta.
\]

Since the Ricci curvature of \( S^3 \) is just \( 2g_S, \) we have Jacobi equation

\[
(4.21) \quad \Delta_g f + \| A \|_{g}^2 f + 2f = 0
\]

on \( \mathcal{K}[i]. \) Multiplying by \( \rho^{-2} \) and rearranging, we obtain

\[
(4.22) \quad \rho^{-2} \| A \|_{g}^2 = -\partial_t^2 f + \partial_y^2 f - \frac{1}{f} \left( \Delta_{\chi} - \Delta_{\hat{\chi}} \right) f - 2\rho^{-2},
\]

which delivers (ii) in light of \( 4.17 \) the expressions just given for \( f \) and \( N, \) the estimates noted above for \( z, \) the inequality \( \rho \geq Cm, \) and the equality \( d\rho^{-1} = \rho \tanh t dt \) on \( \mathcal{K}[i]. \)

Finally, by (a) considering the family of immersions \( \{ \kappa_i[s] = \kappa_i + sv \}_{s \in \mathbb{R}}, \) where \( + \) incidates the obvious application of the Euclidean exponential map transferred via \( \Phi \) to a neighborhood of \( \Sigma \) in \( S^3, \) (b) calculating the rate of change \( \frac{d}{ds} \bigg|_{s=0} \sqrt{|g|} \) at \( s = 0 \) of the volume form of the family
in \((t, \theta)\) coordinates, and (c) appealing to the first-variation-of-area formula, one can without too much labor compute the exact mean curvature

\[
H = \frac{\tanh^3 t \sin 4z \cos 2z + \frac{1}{2} \tau \rho^2 [1 + \tanh^2 t] \cos 2\theta \sin 4z + \mathrm{sech}^2 t \tanh t [4 \sin 2z - (1 + 3 \sin^2 2z) \cos 2\theta]}{(1 - \mathrm{sech}^2 t \cos 2z \sin 2z - \tan^2 t \sin^2 2z)^{3/2}}
\]

proving (iii).

\[
\square
\]

**Graphs over immersions.** The estimates away from the catenoidal regions will be obtained by treating the initial surface there as a graph over the torus, as an application of the following lemma. Its last two items will be used again in the final section to estimate the contribution to the mean curvature of the perturbed surface which are nonlinear in the perturbing function.

**Lemma 4.24.** Let \( \phi : \Sigma \to M \) be a smooth two-sided immersion of a surface \( \Sigma \) into a complete Riemannian 3-manifold \( M \) with smooth metric \( g \), and fix a global unit normal \( \nu \in \phi^* TM \) for \( \phi \). Given \( u \in C^2_{\text{loc}}(\Sigma) \), define the map \( \phi_u : \Sigma \to M \) by \( \phi_u(p) = \exp_{\phi(p)} u(p) \nu(p) \), where \( \exp : TM \to M \) is the exponential map on \( (M, g) \), and define the function \( \omega_u : \Sigma \to \mathbb{R} \) by

\[
\omega_u(p) = \sum_{j=0}^{2} \left( \|D^j A(p)\|_0 + \left\| D^j R|_{B(p, |u(p)|, g)} \right\|_0 \right),
\]

where \( B(p, |u(p)|, g) \) is the open ball in \( M \) of \( g \)-radius \( |u(p)| \) centered at \( p \), \( A \) is the second fundamental form of \( \phi \) relative to \( \nu \), \( R \) is the Riemannian curvature of \( (M, g) \), and each instance of \( D \) is the appropriate connection canonically induced by \( g \) and \( \phi \).

Given \( \alpha \in (0, 1) \), there exist constants \( C, \epsilon > 0 \) with \( C \) depending on \( \alpha \) but both independent of \( M, g, \Sigma, \phi, \) and \( u \) such that if \( \|\omega_u u : C^0(\Sigma)\| + \|du : T^0(T^* \Sigma, \phi^* g)\| < \epsilon \), then \( \phi_u \) is a two-sided immersion satisfying the following estimates, where covariant derivatives, norms, and seminorms are defined relative to \( \phi^* g \), where \( A_u \) and \( H_u \) denote the second fundamental form and mean curvature of \( \phi_u \) relative to the global unit normal \( \nu_u \) distinguished by the condition that \( \nu_u(p) \) have positive inner product with \( \frac{d}{dt}|_{t=0} \exp(\phi(p) tu(p)) \), and where the Jacobi operator \( \mathcal{L} \) is defined by \( \mathcal{L} u = \frac{d}{dt}|_{t=0} H_u \) :

(i) \( \|\phi_u^* g - \phi^* g\| \leq C \left( \omega_u|u| + \|du\|^2 \right) \);

(ii) \( \|D\phi_u^* g\| \leq C \left( \omega_u^2|u| + \omega_u \|du\| + \|du\| \|D^2 u\| \right) \);

(iii) \( \|D\phi_u^* g\|_\alpha \leq C \left( \|\omega_u\|_0 \|du\|_\alpha + \|\omega_u\|^{1+\alpha}_0 \|du\|_0 + \|\omega_u\|^{2+\alpha}_0 \|u\|_0 \right) + \|\omega_u\|^2_0 \|u\|_0 \|du\|_\alpha + \|du\|_0 \|D^2 u\|_\alpha \); and

(iv) \( \|A_u - A\|_\alpha \leq C \left( \|D^2 u\|_0 + \omega_u \|du\| + \omega_u^2 \|u\| \right) \);

(v) \( \|A_u - A\|_\alpha \leq C \left( \|\omega_u\|^{2+\alpha}_0 \|u\|_0 + \|\omega_u\|^{1+\alpha}_0 \|du\|_0 + \|\omega_u\|_0 \|du\|_\alpha + \|D^2 u\|_0 \|du\|_0 \|du\|_\alpha \right) + \|\omega_u\|^2_0 \|du\|^2_0 \|D^2 u\|^2_0 \right) \); and

(vi) \( \|H_u - H_0 - \mathcal{L} u\| \leq C \left( \omega_u^2 \|u\|^2 + \omega_u \|du\|^2 + \omega_u \|u\| \|D^2 u\| + \|du\|^2 \|D^2 u\| \right) \); and

(vii) \( \|H_u - H_0 - \mathcal{L} u\|_\alpha \leq C \left( \|\omega_u\|^{3+\alpha}_0 \|u\|_0 + \|\omega_u\|^{2+\alpha}_0 \|u\|_0 \|du\|_0 + \|\omega_u\|^2_0 \|u\|_0 \|du\|_\alpha \right) + \|\omega_u\|^{1+\alpha}_0 \|du\|^2_0 + \|\omega_u\|_0 \|du\|_0 \|du\|_\alpha + \|du\|_0 \|du\|_\alpha \|D^2 u\|_0 \right) + \|\omega_u\|^{1+\alpha}_0 \|u\|_0 \|D^2 u\|_\alpha + \|\omega_u\|_0 \|u\|_0 \|du\|_\alpha \|D^2 u\|_0 \right) + \|\omega_u\|^2_0 \|u\|_0 \|du\|_\alpha + \|\omega_u\|_0 \|u\|_0 \|D^2 u\|_\alpha \right) \).
We omit the proof, which consists of elementary calculations, making repeated use of the Jacobi equation along geodesics normal to $\Sigma$.

**Toral regions.** The overlap of the toral and catenoidal regions is limited by a collection $\{b_i\}_{i=1}^{N-1}$ of positive constants, bounding below the absolute value of the $t$ parameter on the corresponding catenoidal regions and chosen large a bit later in terms of $k$, $\ell$, and $N$, but bounded independently of $m$ (unlike each $a_i$, which is of order $m^2$). Recalling (2.7) and (2.9) and keeping $X = \frac{\pi}{\sqrt{2km}}$, $Y = \frac{\pi}{\sqrt{2\ell m}}$, and $R = \frac{1}{10\ell m}$, define $T_1 : \mathbb{T}_{X,Y,\tau_1 \cosh b_1} \to S^3$ by

$$ (4.26) \quad T_1(x,y) = \Phi \left( T_{\text{ext}} \left[ (0,0), z^K_1, z_1, R, X, Y, \tau_1 \right] (x,y) \right), $$

define $T_N : \mathbb{T}_{X,Y,\tau_{N-1} \cosh b_{N-1}} \to S^3$ by

$$ (4.27) \quad T_N(x,y) = \Phi \left( T_{\text{ext}} \left[ (N-1)(X,Y), z^K_{N-1}, z_{N-1}, R, X, Y, \tau_{N-1} \right] (x,y) \right), $$

and for $2 \leq i \leq N-1$ define $T_i : \mathbb{T}_{X,Y,\tau_{i-1} \cosh b_{i-1}, \tau_i \cosh b_i} \to S^3$ by

$$ (4.28) \quad T_i(x,y) = \Phi \left( T_{\text{int}} \left[ (2i-3) \left( \frac{X}{2}, \frac{Y}{2} \right), z^K_{i-1}, z^K_i, z_i, R, X, Y, \tau_{i-1}, \tau_i \right] (x,y) \right). $$

The image of $T_i$, entirely contained in the initial surface, gives the toral region

$$ (4.29) \quad \mathcal{T}[i] = \text{Im} T_i. $$

The parametrizing rectangles naturally carry the flat metric $g_E = dx^2 + dy^2$, but we equip each one also with the conformal metric

$$ (4.30) \quad \hat{\chi}_T = (T^* E \rho^2) g_E. $$

In the next section we will define the extended substitute kernel needed to complete the construction, as outlined in the introduction. Then, in the final section, the role of the dislocations will become clear; the dislocation $\mathcal{D}_i$ (recall (3.22)) on the toral region $\mathcal{T}[i]$ will be varied to cancel the “extended” portion of the extended substitute kernel supported there. For this reason it is necessary to isolate the dominant contribution of each dislocation to the mean curvature, and to that end, for $2 \leq i \leq N-2$, we define $v_i \in C^\infty(\mathcal{T}[i])$ by

$$ (4.31) \quad T_i^* v_i(x,y) = \psi \left[ \frac{1}{5\ell m}, \frac{1}{10\ell m} \right] \left( \sqrt{\left( x - \frac{\pi}{2\sqrt{2km}} \right)^2 + \left( y - \frac{\pi}{2\sqrt{2\ell m}} \right)^2} \right), $$

$$ - \psi \left[ \frac{1}{5\ell m}, \frac{1}{10\ell m} \right] \left( \sqrt{\left( x + \frac{\pi}{2\sqrt{2km}} \right)^2 + \left( y + \frac{\pi}{2\sqrt{2\ell m}} \right)^2} \right) $$

and $\overline{w}_i \in C^\infty_c(\mathcal{T}[i])$ by

$$ (4.32) \quad \overline{w}_i = \rho^{-2} \Delta_{T_i^{-1}} \hat{\chi}_T v_i. $$

Restricted to $\mathcal{T}[i]$, the function $v_i$ should be regarded as the section of the normal bundle generating dislocations there, and we will see the function $\overline{w}_i$ then captures the principal effect of dislocation on the mean curvature; later the collection $\{\overline{w}_i\}_{i=1}^{N-1}$ will reappear as the defining basis for the extended part of the extended substitute kernel. For notational convenience, we define $\overline{w}_1 \in C^\infty(\mathcal{T}[1])$ and $\overline{w}_N \in C^\infty(\mathcal{T}[N])$ to be identically 0.

**Proposition 4.33.** Given positive integers $\alpha$, $k$, $\ell$, and $N \geq 2$, as well as $c > 0$, there exists $m_0 > 0$—depending on $k$, $\ell$, $N$, and $c$—and there exists a constant $C > 0$—depending on $k$, $\ell$, and $N$, but not on $c$—such that for each $\zeta, \xi \in [-c, c]^{N-1}$, for each $m > m_0$, and for $1 \leq i \leq N$

(i) $\|T_i^* \chi - \hat{\chi}_T : \Gamma^1,_{\alpha} (T^* T_i^{-1} \mathcal{T}[i] \otimes 2, \hat{\chi}_T) \| \leq C \left( m^2 \tau_1 + \text{sech}^2 \min_j b_j \right)$;
Proof. First observe that if \( T[i] \cap K[j] \neq \emptyset \), then

\[
\kappa_j^{-1} T_i^{-1} \hat{\chi}_T \big|_{T[i] \cap K[j]} = \tanh^2 t^2 dt^2 + d\theta^2,
\]

which with item (i) of 4.16 proves item (i) of the present proposition where the toral and catenoidal regions overlap. Similarly (ii) and (iii) have already been established on these intersections.

We will finish the proof by verifying the estimates on \( \rho < 100m \). For this we apply 4.24 viewing \( T_i \) as a perturbation \( \phi_{u_i} \) of the embedding \( \phi = \varpi \circ T_i \) of \( T_i^{-1} T[i] \) inside the Clifford torus \( T \), with \( \varpi : S^3 \rightarrow T \) indicating orthogonal projection in \( S^3 \) onto \( T \). As the function generating the perturbation (that is playing the role of \( u \) in the statement of 4.24) we have, for \( 2 \leq i \leq N-1 \), \( u_i : \mathbb{T}_X, Y, r_{i-1} \cosh b_{i-1}, r_i \cosh b_i \rightarrow \mathbb{R} \) given by

\[
\begin{align*}
 u_i(x, y) & = z_i + \left( z_{i-1}^{-1} + \tau_{i-1} \operatorname{arcosh} \frac{r_{i-1}}{r_{i-1}} - z_i \right) \psi \left[ \frac{1}{5\ell m}, \frac{1}{10\ell m} \right] (r_{i-1}) \\
 & \quad + \left( z_{i}^{-1} \cosh \frac{r_i}{r_i} - z_i \right) \psi \left[ \frac{1}{5\ell m}, \frac{1}{10\ell m} \right] (r_i),
\end{align*}
\]

where \( r_{i \pm 1} = \sqrt{x^2 + \frac{2}{2\ell m} y^2} \); for \( i = 1 \), \( u_1 : \mathbb{T}_X, Y, r_1 \cosh b_1 \rightarrow \mathbb{R} \) given by

\[
\begin{align*}
 u_1(x, y) & = z_1 + \left( z_{1}^{-1} \cosh \frac{r_0}{r_1} - z_1 \right) \psi \left[ \frac{1}{5\ell m}, \frac{1}{10\ell m} \right] (r_0),
\end{align*}
\]

where \( r_0(x, y) = \sqrt{x^2 + y^2} \); and, for \( i = N \), \( u_N : \mathbb{T}_X, Y, r_{N-1} \cosh b_{N-1} \rightarrow \mathbb{R} \) given by

\[
\begin{align*}
 u_N & = z_N + \left( z_{N-1}^{-1} + \tau_{N-1} \operatorname{arcosh} \frac{r_{N-1}}{r_{N-1}} - z_N \right) \psi \left[ \frac{1}{5\ell m}, \frac{1}{10\ell m} \right] (r_0).
\end{align*}
\]

For \( 1 \leq i \leq N \) and \( m \) sufficiently large in terms of \( k, \ell, N, \) and \( c \), we estimate

\[
\| u_i : C^{2, \alpha} \left( T_i^{-1} (T[i]) \setminus \{ \rho < 100\ell m \} \right), m^2 g_E \| \leq C m^2 \tau_1,
\]

where we emphasize that \( C \) is independent of \( c \) and \( m \). Thus 4.24 yields

\[
\| m^2 T_i^* g_S - m^2 g_E : \Gamma^{1, \alpha} \left( T_i^{-1} (T[i]) \setminus \{ \rho < 100\ell m \} \right), m^2 g_E \| \leq C m^2 \tau_1.
\]

Next observe that \( \frac{T_i}{m} \) and its reciprocal are uniformly bounded on \( \{ \rho < 100\ell m \} \), independently of \( m \) (and \( c \)), as is the \( m^2 g_E \) norm of every \( m^2 g_E \) covariant derivative of \( \frac{\Phi}{m} \). Thus every Hölder norm defined by \( \hat{\chi}_T \) is controlled, uniformly in \( m \) and \( c \), by the corresponding norm defined by \( m^2 g_E \), securing (i). Similarly, since \( T \) has second fundamental form \( (dy^2 - dx^2)\partial_x \) relative to the coordinates induced by \( \Phi \), by appealing to 4.24 and comparing \( \hat{\chi}_T \) to \( m^2 g_E \) on \( \{ \rho < 100m \} \), we obtain (ii).

Finally, the Jacobi operator on \( T \) is just \( \Delta_{g_E} \) + 4, and we find

\[
\| \Delta_{g_E} u_i - D_{\rho^2 \bar{w}_i} : C^{0, \alpha} \left( T_i^{-1} (T[i]) \setminus \{ \rho < 100\ell m \} \right), m^2 g_E \| \leq C m^2 \tau_1.
\]

(Note that without subtracting the dislocation term, this contribution from the Laplacian of \( u_i \) to the mean curvature could not be bounded independently of the parameters.) With this last estimate now in place, the proof is completed by another application of 4.24, the same comparison of \( \hat{\chi}_T \) norms to \( m^2 g_E \) norms, and the equivalence (by constants independent of \( m \) and \( c \)) on every subset of \( T[i] \) of the \( C^{0, \alpha} \) norms induced by \( \hat{\chi}_T \) and \( \chi \), which itself follows from (i).

\[ \square \]
Decay norms and a global estimate of the mean curvature. As mentioned in the introduction, because the characteristic size $\tau_1$ of the catenoidal waists is so much smaller than the characteristic size $m^{-1}$ of the toral regions, we must allow perturbing functions to be much larger on the toral regions than on the core of the catenoidal regions. For this reason we will weight our norms by powers of the factor $m \rho^{-1}$, which takes the value 1 a maximal distance from the catenoidal regions and has order $m \tau_1$ at the waists. Specifically, for each $\alpha \in (0, 1)$, $\gamma \in [0, \infty)$, and nonnegative integer $j$, we define

$$
\|\cdot\|_{j, \alpha, \gamma} = \| : \mathcal{C}^{j, \alpha}_{\rho \gamma} (\Sigma, \chi, \frac{m^\gamma}{\rho^\gamma})
$$

and set

$$
\mathcal{C}^{j, \alpha, \gamma}_G (\Sigma) = \left\{ u \in \mathcal{C}^{j, \alpha}_{\text{loc}, G} (\Sigma) \mid \|u\|_{j, \alpha, \gamma} < \infty \right\},
$$

where, as explained in the first subsection, the subscript $G$ has the effect of admitting only functions invariant under $G$.

In order to secure acceptable decay estimates for solutions to the linearized equation, we will need the following estimate for the initial mean curvature.

**Corollary 4.43.** Given $\alpha \in (0, 1)$ and positive integers $k$, $\ell$, and $N \geq 2$, as well as a real number $c > 0$, there exists a constant $C > 0$—depending on $k$, $\ell$, and $N$ but not on $c$—and there exists a positive integer $m_0$—depending on $k$, $\ell$, $N$, and $c$—such that for each $m > m_0$, each $\zeta, \xi \in [-c, c]^{N-1}$, and each $\gamma \in (0, 1)$

$$
\left\| \rho^{-2} H - \sum_{i=2}^{N-1} D_i \overline{w_i} \right\|_{0, \alpha, \gamma} \leq C \tau_1.
$$

**Proof.** The estimate follows from the definition 2.12 of $\tau_1$, from the definition 4.8 of $\rho$, from item (iii) of 4.16 using also the equivalence of $\chi$ and $\overline{\chi}_K$ norms implied by item (i) of 4.16 and from item (iii) of 4.33.

5. The linearized operator

To review the strategy outlined in the introduction, given an initial surface $\Sigma$, embedded in $\mathbb{S}^3$ by $\iota : \Sigma \to \mathbb{S}^3$, and given a function $u \in \mathcal{C}^{2}_{\text{loc}} (\Sigma)$, we define the perturbation $\iota_u : \Sigma \to \mathbb{S}^3$ by $\iota_u (p) = \exp_{\iota(p)} u(p) \nu(p)$. As observed in 4.24, when $u$ is sufficiently small, $\iota_u$ will be an immersion with well-defined scalar mean curvature relative to the unit normal obtained as a perturbation of $\nu$. For fixed $k$, $\ell$, and $m$, the mean curvature of $\iota_u$ may be considered a functional $\mathcal{H}_{\zeta, \xi} [u]$ of $u$ as well as of the parameters $\zeta$ and $\xi$. We intend to solve the equation $\mathcal{H}_{\zeta, \xi} [u] = 0$, when $m$ is sufficiently large in terms of $k$ and $\ell$, and to obtain estimates for $u$ demonstrating that $\iota_u$ is a smooth embedding close in a precise sense to the original $\iota$.

A major step toward the solution consists in the study of the initial surface’s Jacobi operator, the linearization $\mathcal{L}$ of $\mathcal{H}$ defined by

$$
\mathcal{L} u = \frac{d}{dt} \bigg|_{t=0} \mathcal{H} [\iota u] = (\Delta_g + \|A\|^2 + 2) u.
$$

Actually, because of the uniformity afforded by the $\chi$ metric, it is much more convenient to study instead

$$
\mathcal{L}_\chi = \rho^{-2} \mathcal{L} = \Delta_g + \rho^{-2} \|A\|^2 + 2 \rho^{-2},
$$

which, recalling the estimates of $\rho^{-2} \|A\|^2$ from 4.16 and 4.33, clearly defines a linear map $\mathcal{L}_\chi : \mathcal{C}^{2, \alpha, \gamma}_G (\Sigma) \to \mathcal{C}^{0, \alpha, \gamma}_G (\Sigma)$ which is bounded independently of $m$ and $c$.  


In this section we construct a likewise bounded right inverse $R$ to $L$, modulo extended substitute kernel. We do this by first analyzing $L$ "semilocally", meaning on the toral and catenoidal regions individually, and by observing that on each of these $L$ has a simple limit as $m \to \infty$. We invert these regional limits (modulo extended substitute kernel in the toral cases) and so produce approximate semilocal inverses to $L$, which are applied iteratively, using decay properties of the solutions they yield, to construct $R$.

**Continuity in the parameters.** Because we will have to select the parameter values of the initial surface in parallel with the perturbing function, it will be necessary to compare functions defined on initial surfaces corresponding to different parameter values. Such comparison will be accomplished by pulling back functions through certain diffeomorphisms between the initial surfaces. We define these diffeomorphisms as compromises between natural identifications on the various standard regions. More precisely we set

$$a_i = \text{arcosh} \left( \frac{1}{10\ell m} \right)$$

so that $a_i$ is simply the value taken by $a_i$ when $\zeta = 0$. Then we find that the function $\tilde{\tau}_i : \left[ \tau_i, \infty \right) \to \left[ \tau_i, \infty \right)$ given by

$$\tilde{\tau}_i(r) = \tau_i \cosh \left( \frac{a_i \text{arcosh} \left( \frac{r}{\tau_i} \right)}{\tau_i} \right) \psi \left( \frac{1}{20\ell m}, \frac{1}{30\ell m} \right) (r) + r \psi \left( \frac{1}{30\ell m}, \frac{1}{20\ell m} \right) (r)$$

is the identity on $\left( \frac{1}{20\ell m}, \infty \right)$ and strictly monotonic everywhere for $m$ sufficiently large in terms of $k$, $\ell$, $N$, and $c$. Now we can define the diffeomorphism

$$P_{\zeta,\xi} : \Sigma[N,k,\ell,m,\zeta,\xi] \to \Sigma[N,k,\ell,m,0,0]$$

by demanding that it commute with the action of $G$, that

$$P_{\zeta,\xi} \circ \kappa_i[\zeta,\xi][t,\theta] = k_i[0,0] \left( (\text{sgn} t) \text{arcosh} \tau_i^{-1} \tilde{\tau}_i(\tau_i \cosh t, \theta) \right),$$

and that

$$P_{\zeta,\xi}|_{T[i]}(K[i-1] \cup K[i]) = T_i[0,0] \circ T_i[\zeta,\xi]^{-1},$$

understanding $K[0] = K[N] = \emptyset$. Obviously each diffeomorphism $P_{\zeta,\xi}$ preserves the catenoidal regions, but we arrange also for preservation of the toral regions by decreeing

$$b_i = \frac{a_i b}{a_i}$$

with $b$ a positive constant at our disposal but independent of $m$ and the parameters. Finally we set

$$P_{\zeta,\xi} = P_{\zeta,\xi}^*.$$

**Lemma 5.9.** The map

$$\left( \zeta, \xi \right) \mapsto P_{\zeta,\xi}^{-1} L \chi P_{\zeta,\xi}$$

from the set of parameters $[-c, c]^{N-1} \times [-c, c]^{N-1}$ to the space of bounded linear maps from $C^{2,\alpha,\gamma}(\Sigma[\zeta = \xi = 0])$ to $C^{\alpha,\gamma}(\Sigma[\zeta = \xi = 0])$, equipping this last space of maps with the operator norm, is continuous with bounded image.

**Proof.** In fact it is obvious from the definitions of $L$ and $P_{\zeta,\xi}$ that the coefficients, relative to either $(t, \theta)$ or $(x, y)$ coordinates on $\Sigma[\zeta = \xi = 0]$, of the operator in question are smooth in the parameters and coordinates and are uniformly bounded in $C^1(\Sigma[\zeta = \xi = 0], \chi)$. $\square$
Catenoidal solutions. From items (i) and (ii) of 4.16 we can see
\begin{equation}
\lim_{m \to \infty} \kappa_i^* \mathcal{L} \kappa_i^{*-1} = \tilde{\mathcal{L}}_K := \Delta \tilde{\kappa}_K + 2 \text{sech}^2 t.
\end{equation}
We set
\begin{equation}
\mathcal{G}_{K[i]} = \{ g \in \mathcal{G} \mid gK[i] \subseteq K[i] \}
\end{equation}
and
\begin{equation}
C_{\mathcal{G}}^{j,\alpha,\gamma}(K[i]) = C_{\mathcal{G}_{K[i]}}^{j,\alpha}(K[i], \chi, m^\gamma \rho^{-\gamma})
\end{equation}
and define \( \tilde{\mathcal{L}}_{K[i]} : C_{\mathcal{G}}^{2,\alpha,\gamma}(K[i]) \to C_{\mathcal{G}}^{2,\alpha,\gamma}(K[i]) \) by
\begin{equation}
\tilde{\mathcal{L}}_{K[i]} = \kappa_i^{*-1} \tilde{\mathcal{L}}_K \kappa_i^*
\end{equation}
and in the next proposition construct a suitable inverse.

**Proposition 5.15.** Given \( \alpha, \gamma \in (0, 1) \), \( c > 0 \), and positive integers \( k, \ell \), and \( N \), there exists \( m_0 > 0 \)—depending on the foregoing data—and there exists \( C > 0 \)—independent of \( c \)—such that for \( \zeta, \xi \in [-c, c]^{N-1}, 1 \leq i \leq N - 1 \), and \( m > m_0 \), the map
\begin{equation}
P_{\zeta, \xi}^{-1} \tilde{\mathcal{L}}_{K[i]} P_{\zeta, \xi} : C_{\mathcal{G}}^{2,\alpha,\gamma}(K[i]_{\zeta=\xi=0}) \to C_{\mathcal{G}}^{2,\alpha,\gamma}(K[i]_{\zeta=\xi=0})
\end{equation}
has norm bounded by \( C \), is constant in \( \xi \), and depends continuously (with respect to the operator norm) on \( \zeta \), and moreover there exists a linear map
\begin{equation}
\tilde{\mathcal{R}}_{K[i]} : C_{\mathcal{G},c}^{0,\alpha,\gamma}(K[i]) \to C_{\mathcal{G}}^{2,\alpha,\gamma}(K[i])
\end{equation}
such that if \( f \in C_{\mathcal{G},c}^{0,\alpha,\gamma}(K[i]) \) and \( u = \tilde{\mathcal{R}}_{K[i]} f \), then
\begin{enumerate}
\item \( \tilde{\mathcal{L}}_{K[i]} u = f \);
\item \( \| u \|_{2,\alpha,\gamma} \leq C \| f \|_{0,\alpha,\gamma} \); and
\item the map \( P_{\zeta, \xi}^{-1} \tilde{\mathcal{R}}_{K[i]} P_{\zeta, \xi} : C_{\mathcal{G},c}^{0,\alpha,\gamma}(K[i]_{\zeta=\xi=0}) \to C_{\mathcal{G}}^{2,\alpha,\gamma}(K[i]_{\zeta=\xi=0}) \) has operator norm bounded uniformly in the parameters and \( m \) by \( C \), is constant in \( \xi \), and depends continuously (with respect to the operator norm) on \( \zeta \).
\end{enumerate}

**Proof.** Set \( \mathbb{K} = \mathbb{R} \times S^1 \) and define
\begin{equation}
C_{\mathcal{G}}^{j,\alpha,\gamma}(\mathbb{K}) = \left\{ u \in C_{\mathcal{G}}^{j,\alpha}(\mathbb{K}, \tilde{\kappa}_K, e^\gamma |t|) \mid u(t, \theta) = u(t, -\theta) = u(t, \pi - \theta) \right\}.
\end{equation}
By separation of variables we can reduce the study of \( \tilde{\mathcal{L}}_K : C_{\mathcal{G}}^{2,\alpha,\gamma}(\mathbb{K}) \to C_{\mathcal{G}}^{0,\alpha,\gamma}(\mathbb{K}) \) to the study of the ordinary differential operator
\begin{equation}
L = \partial_t^2 + 2 \text{sech}^2 t.
\end{equation}
The latter’s analysis is greatly facilitated by the observation that
\begin{equation}
(\partial_t - \tanh t)(\partial_t + \tanh t) + 1 = L, \text{ while }
(\partial_t + \tanh t)(\partial_t - \tanh t) + 1 = \partial_t^2,
\end{equation}
which reveals that for each integer \( n \geq 2 \) the kernel of \( L - n^2 \) is spanned by the functions
\begin{equation}
u_{\pm n}(t) = (\partial_t - \tanh t)e^{\pm nt} = (\pm n - \tanh t)e^{\pm nt}.
\end{equation}
The kernel of \( L \) itself is spanned by
\begin{equation}
u_0(t) = -(\partial_t - \tanh t)1 = \tanh t \quad \text{and} \quad \nu_1(t) = (\partial_t - \tanh t)t = 1 - t \tanh t.
\end{equation}
the Jacobi field $u_0$ is the normal projection of the Killing field generating vertical translations, while the Jacobi field $u_0$ is the normal projection of the vector field generating dilations about the origin. Last we record the kernel of $L - 1$, though it plays no role in this construction, spanned by

\[(5.23) \quad u_1(t) = (\partial_t - \tanh t) \sinh t = \text{sech} t \quad \text{and} \quad u_1(t) = (\partial_t - \tanh t) = \sinh t + t \text{sech} t,\]

which, multiplied by linear combinations $\cos \theta$ and $\sin \theta$, respectively generate horizontal translations and rotations about horizontal axes through the origin.

From the above eigenfunctions of $L$, we see that for $n \geq 2$ the operator $L - n^2$ has a bounded Green’s function

\[(5.24) \quad G_n(t, s) = \frac{1}{2n(1 - n^2)} \left\{ e^{n(t-s)(n + \tanh s)(n - \tanh t)} \right. \quad \text{for} \quad t \leq s \]

\[\left. e^{n(s-t)(n - \tanh s)(n + \tanh t)} \right. \quad \text{for} \quad t \geq s.\]

The exceptional case $n = 1$ will not concern us here, but, for $n = 0$, we will take the solution to $Lu = f$ with trivial initial data, given by

\[(5.25) \quad u(t) = \int_0^t G_0^+(t, s)f(s) ds \]

with

\[(5.26) \quad G_0^+(t, s) = \tanh t - \tanh s + (t-s) \tanh t \tanh s.\]

Now suppose $\hat{f} \in C^{0,\beta,\gamma}_G(\mathbb{K})$. Then, setting

\[(5.27) \quad \hat{f}_0(t) = \frac{1}{2\pi} \int_0^{2\pi} \hat{f}(t, \theta) \, d\theta, \]

\[\hat{f}_{0,+(t)} = \frac{2}{\pi} \int_0^{2\pi} \hat{f}(t, \theta) \cos n\theta \, d\theta, \quad \text{and} \]

\[\hat{f}_{0,-(t)} = \frac{2}{\pi} \int_0^{2\pi} \hat{f}(t, \theta) \sin n\theta \, d\theta, \]

the symmetries imply that $\hat{f}_{0,-(t)}$ for arbitrary $n$ and $\hat{f}_{0,+(t)}$ for arbitrary odd $n$ all vanish identically.

We then define

\[(5.28) \quad \hat{u}_{2n}(t) = \begin{cases} \int_0^t G_0^+(t, s)\hat{f}_0(s) ds \quad \text{for} \quad n = 0 \\ \int_{\infty}^t G_2n(t, s)\hat{f}_{2n,+}(s) ds \quad \text{otherwise}. \end{cases} \]

It follows from \ref{5.26} and \ref{5.24} that

\[(5.29) \quad \sup_{t \in \mathbb{R}} e^{-\gamma |t|} |u_{2n}(t)| \leq C \frac{n}{n^2} \|\hat{f}\|_{0,0,\gamma}, \]

so in particular the series $\sum_{n=0}^{\infty} \hat{u}_{2n}(t) \cos 2n\theta$ converges in $C^{0}$; we call the limit $\hat{u}(t)$. On the other hand, the series $\hat{f}_0(t) + \sum_{n=0}^{\infty} \hat{f}_{2n,+}(t) \cos 2n\theta$ converges distributionally to $\hat{f}$. Accordingly $\hat{u}$ solves $\hat{L}_K \hat{u} = \hat{f}$ weakly, but $\hat{f} \in C^{0,\alpha}_G(\mathbb{K})$, so in fact $\hat{u} \in C^{2,\alpha}_G(\mathbb{K})$ is a classical solution to $\hat{L}_K \hat{u} = \hat{f}$. Thus, setting $\hat{R}_K \hat{f} = \hat{u}$, interior Schauder estimates for $\hat{L}_K$ in conjunction with \ref{5.29} confirm that we have constructed a right inverse

\[(5.30) \quad \hat{R}_K : C^{0,\alpha,\gamma}_G(\mathbb{K}) \to C^{2,\alpha,\gamma}_G(\mathbb{K})\]

\[\text{to} \quad \hat{L}_K.\]

In turn, using also interior Schauder estimates for $\hat{L}_{K[i]}$, we see that

\[(5.31) \quad \hat{R}_{K[i]} = \kappa_i^{-1} \hat{R}_K \kappa_i^*\]
satisfies items (i) and (ii) of the proposition.

The independence from $\xi$ of $\mathcal{L}_{K[i]}$ and $\mathcal{R}_{K[i]}$ is already clear from \ref{5.6}. In proving the continuity in $\zeta$, we will emphasize the dependence of certain entities on the parameters by their inclusion, within brackets, in the notation. Of course $\mathcal{L}_K$ and $\mathcal{R}_K$ themselves are independent of the parameters, but motivated by \ref{5.6} and recalling \ref{5.4}, we define

\begin{equation}
(5.32) \quad \lambda_i[\zeta] : \mathbb{K} \to \mathbb{K}
\end{equation}

by

\begin{equation}
(5.33) \quad \lambda_i[\zeta](t, \theta) = (\text{sgn}(t) \arccosh \frac{1}{\zeta}(\tau_i \cosh t), \theta)
\end{equation}

and we introduce

\begin{equation}
(5.34) \quad \mathcal{L}_K[\zeta] = \lambda_i[\zeta]^{*^{-1}} \mathcal{L}_K \lambda_i[\zeta]^* : C^{0,\alpha}_{loc,G}(\mathbb{K}) \to C^{2,\alpha}_{loc,G}(\mathbb{K}),
\end{equation}

so that in particular

\begin{equation}
(5.35) \quad \mathcal{P}_{\zeta,\xi}^{-1} \mathcal{L}_{K[i]}[\zeta, \xi] \mathcal{P}_{\zeta,\xi} = \kappa_i[0, 0]^{*^{-1}} \mathcal{L}_K[\zeta] \kappa_i[0, 0]^*.
\end{equation}

Using \ref{5.33} and comparing \ref{4.13} to \ref{5.3} one sees $\frac{d}{d\zeta} \mathcal{L}_K[\zeta]$ defines a map from $C^{2,\alpha,\gamma}(\mathbb{K})$ to $C^{0,\alpha,\gamma}(\mathbb{K})$ whose norm can be made arbitrarily small, uniformly in $\zeta$, by assuming $m$ sufficiently large in terms of $c$. The relation \ref{5.35} then implies the continuity in $\zeta$ and the uniform boundedness of $\mathcal{L}_{K[i]}$ as asserted at the beginning of the proposition.

Furthermore, we can define $\mathcal{R}_K[\zeta] : C^{0,\alpha,\gamma}(\mathbb{K}) \to C^{2,\alpha,\gamma}(\mathbb{K})$ by the Neumann series

\begin{equation}
(5.36) \quad \mathcal{R}_K[\zeta] = \sum_{n=0}^{\infty} \left[ \mathcal{R}_K \left( \mathcal{L}_K - \mathcal{L}_K[\zeta] \right) \right]^n \mathcal{R}_K,
\end{equation}

whose convergence and boundedness, uniform in $\zeta$, and whose continuity in $\zeta$ can be guaranteed by taking $m$ large in terms of $c$, on account of the previous paragraph. Evidently, for each $\zeta$, the map

\begin{equation}
(5.37) \quad \lambda_i[\zeta]^{*} \mathcal{R}_K[\zeta] \lambda_i[\zeta]^{*^{-1}} : C^{0,\alpha,\gamma}(\mathbb{K}) \to C^{2,\alpha,\gamma}(\mathbb{K})
\end{equation}

is a bounded (by a constant which need not be assumed independent of $\zeta$) right inverse for $\mathcal{L}_K$, imposing trivial data at $t = 0$ for the rotationally invariant mode of the solution. On the other hand, the analysis of $L$ above reveals that there can be just one such right inverse, namely $\mathcal{R}_K$, because all higher modes annihilated by $\mathcal{L}_K$ grow too quickly in one direction or the other. Thus we find

\begin{equation}
(5.38) \quad \mathcal{P}_{\zeta,\xi}^{-1} \mathcal{R}_{K[i]}[\zeta, \xi] \mathcal{P}_{\zeta,\xi} = \kappa_i[0, 0]^{*^{-1}} \mathcal{R}_K[\zeta] \kappa_i[0, 0]^*,
\end{equation}

concluding the proof.

\begin{flushright}
$\Box$
\end{flushright}

**Toral solutions.** Items (i) and (ii) of \ref{4.33} imply

\begin{equation}
(5.39) \quad \lim_{m \to \infty} T_i^* \mathcal{L}_\chi T_i^{*^{-1}} = \Delta_{\chi_T}.
\end{equation}

We set

\begin{equation}
(5.40) \quad C^{2,\alpha,\gamma}_G(\mathcal{T}[i]) = \left\{ u|_{\mathcal{T}[i]} \mid u \in C^{2,\alpha}_G \left( \mathcal{G}\mathcal{T}[i], \chi, \frac{m^2}{\rho^2} \right) \right\}
\end{equation}

and define $\mathcal{L}_{\mathcal{T}[i]}$

\begin{equation}
(5.41) \quad \mathcal{L}_{\mathcal{T}[i]} : C^{2,\alpha,\gamma}_G(\mathcal{T}[i]) \to C^{2,\alpha,\gamma}_G(\mathcal{T}[i])
\end{equation}
by
\begin{equation}
\tilde{\mathcal{L}}_{[i]} = T_i^{-1} \Delta_{T_i} T_i^*.
\end{equation}

To prepare for the next proposition we recall the definition \[4.32\] of \(\tilde{w}_i\), remembering in particular that \(\tilde{w}_1\) and \(\tilde{w}_N\) are both identically 0, and we introduce \(w_i \in \mathcal{C}_0^\infty(M)\), for \(1 \leq i \leq N\), defined by
\begin{equation}
w_i = \psi \left[ \frac{1}{10\ell m}, \frac{1}{5\ell m} \right] \circ \rho.
\end{equation}
Together these functions span the extended substitute kernel, modulo which we can invert \(\tilde{\mathcal{L}}_{[i]}\) as follows.

**Proposition 5.44.** Given \(\alpha, \gamma \in (0, 1)\), \(c > 0\), and positive integers \(k\), \(\ell\), and \(N\), there exists \(m_0 > 0\)—depending on the foregoing data—and there exists \(C > 0\)—independent of \(c\)—such that for \(\zeta, \xi \in [c, c]^{N-1}\), \(1 \leq i \leq N - 1\), and \(m > m_0\), the map
\begin{equation}
P_{\zeta, \xi}^{-1} \tilde{\mathcal{L}}_{[i]} P_{\zeta, \xi} : \mathcal{C}_G^{2, \alpha, \gamma}(T[i], \zeta = \xi = 0) \to \mathcal{C}_G^{0, \alpha, \gamma}(T[i], \zeta = \xi = 0)
\end{equation}
has norm bounded by \(C\), is constant in \(\zeta\), and depends continuously (with respect to the operator norm) on \(\zeta\), and moreover there exists a linear map
\begin{equation}
\tilde{\mathcal{R}}_{[i]} : \left\{ f \in \mathcal{C}_G^{0, \alpha}(T[i], \chi) \mid \text{supp } f \subset \{ \rho < 20\ell m \} \right\} \to \mathcal{C}_G^{2, \alpha, 2}(T[i]) \times \mathbb{R} \times \mathbb{R}
\end{equation}
such that if \(f \in \mathcal{C}_G^{0, \alpha}(T[i], \chi)\) is supported in \(\{ \rho < 20\ell m \}\) and \((u, \mu, \pi) = \tilde{\mathcal{R}}_{[i]} f\), then
(i) \(\tilde{\mathcal{L}}_{[i]} u = f + \mu \tilde{w}_i + \tilde{w}_i\);
(ii) \(\|u\|_{2, \alpha, 2} + |\mu| + |\pi| \leq C \|f\|_{0, \alpha};\)
(iii) the map \(\pi_1 \tilde{\mathcal{R}}_{[i]} P_{\zeta, \xi}\), projecting onto the component \(\mu\) of substitute kernel, is constant in \((\zeta, \xi)\);
(iv) the map
\begin{equation}
P_{\zeta, \xi}^{-1} \pi_1 \tilde{\mathcal{R}}_{[i]} P_{\zeta, \xi} : \left\{ f \in \mathcal{C}_G^{0, \alpha}(T[i], \chi) \mid \text{supp } f \subset \{ \rho < 20\ell m \} \right\} \to \mathcal{C}_G^{2, \alpha, 2}(T[i], \zeta = \xi = 0)
\end{equation}
has operator norm bounded uniformly in the parameters and \(m\) by \(C\), is constant in \(\zeta\), and depends continuously (with respect to the operator norm) on \(\zeta\); and
(v) the map \(\pi_3 \tilde{\mathcal{R}}_{[i]} P_{\zeta, \xi}\) has norm bounded uniformly in the parameters and \(m\) by \(C\), is constant in \(\zeta\), and depends continuously on \(\zeta\).

**Proof.** Suppose \(f \in \mathcal{C}_G^{0, \alpha}(T[i], \chi)\) has support in \(\{ \rho < 20\ell m \}\). Extend its pullback \(T_i^* f\) by 0, without renaming, to a function on the whole rectangle \(T_{X,Y,0}\), and likewise extend \(T_i^* w_i\). Note that, via \(T_i\), the symmetries of \(G\) require that \(T_i^* f\) extend to a function on \(\mathbb{R}^2\) invariant under reflections through the lattices of the construction and so doubly periodic, with periods \(2X\) and \(2Y\). Observe that \(T_i^* w_i\) enjoys these same symmetries, and in particular both functions satisfy periodic boundary conditions on \(T_{X,Y,0}\). Of course, the flat Laplacian \(\Delta_{\mathcal{E}}\) on the torus obtained from \(T_{X,Y,0}\) by identifying opposite edges has one-dimensional kernel consisting of the constant functions, so if we set
\begin{equation}
\mu = \frac{\int_{T_{X,Y,0}} T_i^* (\rho^2 f) \, dx \, dy}{\int_{T_{X,Y,0}} T_i^* (\rho^2 w_i) \, dx \, dy},
\end{equation}
then \(T_i^* [\rho^2 (f + \mu w_i)]\) is orthogonal to this kernel. The support of \(f\), definitions of \(\rho\) and \(w_i\), and dimensions of the rectangle guarantee \(|\mu| \leq C \|f\|_{\mathcal{C}_0}\) and
\begin{equation}
\|T_i^* [\rho^2 (f + \mu w_i)]\|_{L^2(T_{X,Y,0}, \mathcal{E})} \leq C m \|f\|_{\mathcal{C}_0},
\end{equation}
for \(m \geq m_0\).
so a solution $\hat{u}_0 \in H^1(\mathbb{T}_{X,Y,0}, g_E)$ to
\begin{equation}
\Delta_{g_E} \hat{u}_0 = T_i^* \left[ \rho^2 (f + \mu w_i) \right]
\end{equation}
eexists with \( \|d\hat{u}_0\|_{L^2(g_E)} \leq C m^{-1} \|f\|_{C^0} \), which in turn yields
\begin{equation}
\|\hat{u}_0\|_{C^0} \leq C m^{-1} \|d\hat{u}_0\|_{L^1(g_E)} \leq C \|\mu w_i\|_{L^2(g_E)} \left( \|f\|_{C^0} \right).}
\end{equation}

Since \( f \in C^{0,\alpha}(\mathbb{T}[i], \hat{\chi}) \), we have $\hat{u}_0 \in C^{2,\alpha}(\mathbb{T}_{X,Y,0}, g_E)$, and in fact $\hat{u}_0$ is harmonic outside \( \{T_i^* \rho < 20\ell m\} \). This region consists of one disc if $\mathbb{T}[i]$ is extreme and of two discs if $\mathbb{T}[i]$ is intermediate. Of course $T_i^{-1} \mathbb{T}[i]$ does not contain the center(s) of the disc(s), but the rectangle $\mathbb{T}_{X,Y,0}$ does, so we may define $q$ to be the average value of $\hat{u}_0$ on the centers (so simply the value at the single center in case $i \in \{1, N\}$). Then $\hat{u}_1 = \hat{u}_0 - q$ is another solution of (5.50) since $q$ is constant, and also satisfies $\|\hat{u}_1\|_{C^0} \leq \|f\|_{C^0}$.

If $\mathbb{T}[i]$ is extreme, then $\hat{u}_1$ vanishes at the origin, but $\hat{u}_1$ respects all the reflectional symmetries described above, so in particular is invariant under reflections through the origin, so being harmonic on the disc about it of radius $\frac{1}{20\ell m}$, it accordingly decays there like the square of the distance to the center. In the intermediate case, however, we must perform an additional step to arrange the decay. Now, having subtracted $q$, the adjusted solution $\hat{u}_1$ takes opposite values at the centers of the two discs in question. Define $\hat{f}$ to be the value taken at the center where the function $T_i^* v_i$ (4.31) is $-1$. Then $\hat{u} = \hat{u}_1 + \hat{f} T_i^* v_i$ vanishes at both centers and is still harmonic on both of the discs of radius $\frac{1}{20\ell m}$, ensuring the decay, but at the cost of altering the right-hand side of (5.50), so that, recalling (4.32), $\hat{u}$ instead solves
\begin{equation}
\Delta_{g_E} \hat{u} = T_i^* \left[ \rho^2 (f + \mu w_i + \rho w_i) \right]
\end{equation}
and satisfies
\begin{equation}
\|\hat{u} : C^0 (T_i^{-1} \mathbb{T}[i], g_E, \frac{m^2}{\rho^2}) \| \leq C \|f : C^0 (\mathbb{T}[i])\|.
\end{equation}

Now the symmetries of $T_i^* f$, $T_i^* w_i$, $T_i^* \overline{w}_i$, $T_i^* \rho$, and $\Delta_{g_E}$ imply $u = T_i^{-1} \hat{u}$ is $\mathcal{G}$-invariant, and the definitions $\hat{\chi}_T = \rho^2 g_E$ and $\tilde{\mathcal{L}}_{\mathbb{T}[i]} = T_i^{-1} \Delta_{\hat{\chi}_T} T_i^*$ imply
\begin{equation}
\tilde{\mathcal{L}}_{\mathbb{T}[i]} u = f + \mu w_i + \rho w_i,
\end{equation}
so, using also (5.53) coupled with interior Schauder estimates for $\tilde{\mathcal{L}}_{\mathbb{T}[i]}$, we see that by defining $\tilde{\mathcal{R}}_{\mathbb{T}[i]}$ according to $\tilde{\mathcal{R}}_{\mathbb{T}[i]} f = u$ we have established items (i) and (ii).

The continuity and boundedness assertions concerning $\mathcal{P}_{\zeta, \xi}^{-1} \tilde{\mathcal{L}}_{\mathbb{T}[i]} \mathcal{P}_{\zeta, \xi}$, made at the beginning of the proposition, can be verified much like the analogous statements on catenoidal regions, by considering coordinate expressions for the conjugated operators and taking $m$ large in terms of $c$.

Item (iii) is trivial in view of the above construction of $\tilde{\mathcal{R}}_{\mathbb{T}[i]}$ and the observation that
\begin{equation}
T_i[\zeta, \xi]^* \mathcal{P}_{\zeta, \xi} T_i[0, 0]^{-1}
\end{equation}
acts as the identity on functions supported in \( \{\rho < 20\ell m\} \). Off this set each solution produced by $\tilde{\mathcal{R}}_{\mathbb{T}[i]}$ is smooth and in fact harmonic, so in particular we get the estimate
\begin{equation}
\|\pi_1 \tilde{\mathcal{R}}_{\mathbb{T}[i]} f : C^0 \left( \mathbb{T}[i] \cap \{\rho \leq 20\ell m\}, \chi, \frac{m^2}{\rho^2} \right) \| \leq C \|f\|_{0, \alpha},
\end{equation}
but on the other hand, by taking $m$ large in terms of $c$, we can ensure that
\begin{equation}
\mathcal{P}_{\zeta, \xi}^{-1} : C^3 \left( \mathbb{T}[i], \chi, \frac{m^2}{\rho^2} \right) \rightarrow C^{2, \alpha, \gamma} \left( \mathbb{T}[i](0,0) \right)
\end{equation}
is uniformly bounded and depends continuously on $\zeta$, completing the proof.

\[ \square \]

**Global solutions.**

**Proposition 5.58.** Given $\alpha, \gamma \in (0, 1)$ and initial surface data $k, \ell, N$, and parameter range $c > 0$, there exists $C > 0$—independent of $c$—and there exists $m_0 > 0$ such that whenever $\zeta, \xi \in [-c, c]^{N-1}$ and $m > m_0$, there exists a linear map

\begin{equation}
R : C^{0,\alpha,\gamma}_G(\Sigma) \to C^{2,\alpha,\gamma}_G(\Sigma) \times \mathbb{R}^N \times \mathbb{R}^{N-2}
\end{equation}

such that if $f \in C^{0,\alpha,\gamma}_G(\Sigma[\zeta = \xi = 0])$ and $(u, (\mu_1, \cdots, \mu_N), (\bar{\mu}_2, \cdots, \bar{\mu}_{N-1})) = R P_{\zeta,\xi} f$, then

\begin{enumerate}[(i)]
\item $L_\chi u = P_{\zeta,\xi} f + \sum \mu_i w_i + \sum \bar{\mu}_i \bar{w}_i$;
\item $\| P_{\zeta,\xi}^{-1} u \|_{2,\alpha,\gamma} + \sum |\mu_i| + \sum |\bar{\mu}_i| \leq C \| f \|_{0,\alpha,\gamma}$; and
\item \((P_{\zeta,\xi}^{-1} u, (\mu_1, \cdots, \mu_N), (\bar{\mu}_2, \cdots, \bar{\mu}_{N-1}))\) depends continuously on $(\zeta, \xi)$.
\end{enumerate}

**Proof.** We will need the cutoff functions $\Psi_K, \Psi_T \in C^\infty(\Sigma)$ defined by

\begin{equation}
\Psi_K = \psi [20\ell m, 30\ell m] \circ \rho \text{ and }
\end{equation}

\begin{equation}
\Psi_T = \begin{cases} 1 \text{ on } \Sigma \setminus (\mathcal{G} \cup \bigcup_{i=1}^{N-1} K[i]) \\ \psi [\tau_1^{-1} \sech b_1, \tau_1^{-1} \sech 2b_1] \circ \rho \text{ on } K[i], \end{cases}
\end{equation}

and below we will casually identify these functions as well as the constant function 1 with the respective operators on spaces of functions that each defines by multiplication, so that 1 acts as the identity, $\Psi_K$ smoothly cuts off functions to have support contained in the union of the catenoidal regions, and $\Psi_T$ smoothly cuts off functions to have support contained in the union of the toral regions. We can also regard multiplication by these functions as extending to the entire initial surface functions originally defined on a single region. Additionally, we will identify each region, catenoidal or toral, with its image under $\mathcal{G}$ and will accordingly interpret $\tilde{R}_K[i], \tilde{R}_T[i], \tilde{L}_K[i],$ and $\tilde{L}_T[i]$ as acting on these sets by the obvious extension enforcing the symmetries.

Given $f \in C^{0,\alpha,\gamma}_G(\Sigma)$, set

\begin{align}
&u_0 = \sum_{i=1}^{N-1} \Psi_K \tilde{R}_K[i] (\Psi_K f)_{|K[i]}, \\
u_1 = \sum_{i=1}^{N} \Psi_T \pi_1 \tilde{R}_T[i] (f_1 |_{T[i]}),
\end{align}

\begin{align}
&f_1 = \sum_{i=1}^{N-1} [\Psi_K, L_\chi] \tilde{R}_K[i] (\Psi_K f)_{|K[i]} + (1 - \Psi_K^2) f, \tag{5.61}
\end{align}

\begin{align}
\mu_i = \pi_2 \tilde{R}_T[i] (f_1 |_{T[i]}) \text{ for } 1 \leq i \leq N, \text{ and } \\
\bar{\mu}_i = \pi_3 \tilde{R}_T[i] (f_1 |_{T[i]}) \text{ for } 2 \leq i \leq N - 1,
\end{align}

then

\begin{equation}
\tilde{R} f = (u_0 + u_1, (\mu_1, \cdots, \mu_N), (\bar{\mu}_2, \cdots, \bar{\mu}_{N-1}))\ ,
\end{equation}

defines a map $\tilde{R} : C^{0,\alpha,\gamma}_G(\Sigma) \to C^{2,\alpha,\gamma}_G(\Sigma) \times \mathbb{R}^N \times \mathbb{R}^{N-2}$, depending continuously on the parameters (via $P_{\zeta,\xi}$) and bounded uniformly in them, according to the last two propositions, 5.15 and 5.44, and to 5.9.

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Defining also \( L : C^2_{\rho} (\Sigma) \times \mathbb{R}^N \times \mathbb{R}^{N-2} \to C^0_{\rho} (\Sigma) \) by
\[
L((u, (\mu_1, \cdots, \mu_N), (\overline{\mu}_2, \cdots, \overline{\mu}_{N-1})) = \mathcal{L}_u u - \sum_{i=1}^N \mu_i u_i - \sum_{i=2}^{N-1} \overline{\mu}_i \overline{w}_i,
\]
we find
\[
L\tilde{R} f - f = \sum_{i=1}^N [\mathcal{L}_u, \Psi_T] \pi_1 \tilde{R}_{T[i]} (f_1|_{T[i]}) + \sum_{i=1}^N \Psi_T (\mathcal{L}_u - \tilde{L}_{T[i]}) \pi_1 \tilde{R}_{T[i]} (f_1|_{T[i]})
\]
\[+
\sum_{i=1}^{N-1} \Psi_K (\mathcal{L}_u - \tilde{L}_K[i]) \tilde{R}_K[i] (\Psi_K f|_{K[i]}).
\]

The commutator factor in the first term is supported near the catenoidal boundary of the toral region, where by \( 5.44 \) the toral solution to \( f_1 \) has decayed rapidly, and of course the norm of \( f_1 \) is controlled by the norm of \( f \). For the second and third term we can use the first two items of \( 4.33 \) and of \( 4.16 \) respectively to bound the deviation of \( \mathcal{L}_u \) from its regional limits. We get
\[
\| (L\tilde{R} - 1) f \|_{0,\alpha,\gamma} \leq C \left( m^{2-\gamma} r^{2-\gamma} e^{(2-\gamma)b} + m^{-2} + \text{sech}^2 b \right) \| f \|_{0,\alpha,\gamma},
\]
where \( C \) is independent of \( m, c \), the parameters themselves, and \( b \).

Thus by assuming \( b \) large in terms of \( C \) and then \( m \) large in terms of \( b \) and \( C \), we can ensure \( L\tilde{R} \) is invertible on \( C^0_{\rho} (\Sigma) \), with inverse continuous in the parameters (via \( P_{\zeta,\xi} \)) and bounded uniformly in them by 2. Taking \( \mathcal{R} = \tilde{R} \left( L\tilde{R} \right)^{-1} \) concludes the proof.

As an immediate corollary we obtain an estimate for the first correction to the surface as well as a statement of its continuous dependence on the parameters.

**Corollary 5.66.** If
\[
(u_0, (\mu_1, \cdots, \mu_N), (\overline{\mu}_2, \cdots, \overline{\mu}_{N-1})) = -\mathcal{R} \left( \rho^{-2} H - \sum_i D_i \overline{w}_i \right),
\]
then the map
\[
(\zeta, \xi) \mapsto \left( P_{\zeta,\xi}^{-1} u_0, (\mu_1, \cdots, \mu_N), (\overline{\mu}_2, \cdots, \overline{\mu}_{N-1}) \right)
\]
is continuous, using the \( C^{2,\alpha,\gamma} \) norm on the first factor of the target, and there exists \( C > 0 \) depending on just \( k, \ell \), and \( N \) such that for \( m \) sufficiently large in terms of \( k, \ell, N, \) and \( c \)
\[
\| P_{\zeta,\xi}^{-1} u_0 \|_{2,\alpha,\gamma} + \sum_i |\mu_i| + \sum_i |\overline{\mu}_i| \leq C \tau_1.
\]

**Proof.** We note that \( D_i \) is manifestly continuous in the parameters and that the functions \( P_{\zeta,\xi}^{-1} H \), \( P_{\zeta,\xi}^{-1} \rho \), and \( P_{\zeta,\xi}^{-1} \overline{w}_i \) in fact vary smoothly on the initial surface and in the parameters. The continuity then follows from the proposition, as does the estimate, using also \( 4.43 \) (with a larger Hölder exponent). \( \square \)
6. The main theorem

We will need the following estimate for the nonlinear contribution

\[(6.1) \quad Q[u] = \mathcal{H}[u] - \mathcal{H}[0] - Lu\]

that the perturbing function \(u\) makes to the mean curvature.

**Lemma 6.2.** Given \(k, \ell, N, \alpha, \gamma \in (0, 1), \ c > 0, \) and \(C > 0\) independent of \(m\) and \(c\), there exists \(m_0 > 0\) such that whenever \(\zeta, \xi \in [-c, c]^{N-1}, \ m > m_0, \) and \(u : \Sigma[\zeta = \xi = 0] \to \mathbb{R}\) satisfies \(\|u\|_{2, \alpha, \gamma} \leq C \tau_1\), we have

\[(6.3) \quad \left\|P_{\zeta, \xi}^{-1} \rho^{-2} Q[p_{\zeta, \xi} u]\right\|_{0, \alpha, \gamma} \leq \frac{1 + \tau_1}{1 + \tau_1}.\]

Furthermore the map

\[(6.4) \quad P_{\zeta, \xi}: \mathbb{R} \times [-c, c] \to \mathbb{R}\]

is continuous.

**Proof.** If \(t_{\zeta, \xi}\) denotes the defining embedding of the initial surface with parameter values \((\zeta, \xi)\), then \(P_{\zeta, \xi}^{-1} \mathcal{H}[p_{\zeta, \xi} u]\) is simply the mean curvature of the perturbation by \(u\) of the embedding \(t_{\zeta, \xi} \circ P_{\zeta, \xi}^{-1}\). Its value at \(p \in \Sigma[\zeta = \xi = 0]\) thus depends smoothly on \(p\), \((\zeta, \xi)\), and the 2-jet of \(u\) at \(p\), implying the asserted continuity.

Now fix \(p \in \Sigma[\zeta = \xi = 0]\). If we blow up the ambient metric \(g_S\) to \(P_{\zeta, \xi}^{-1}(p)g_S\) and the perturbing function \(u\) to \(P_{\zeta, \xi}^{-1}(p)u\), we find

\[(6.5) \quad P_{\zeta, \xi}Q[p_{\zeta, \xi} u] = Q_{t_{\zeta, \xi} \circ P_{\zeta, \xi}}^{g_S}[u] = \left(P_{\zeta, \xi}^{-1}(p)Q_{t_{\zeta, \xi} \circ P_{\zeta, \xi}}^{g_S}(p)g_S\right) \left(P_{\zeta, \xi}^{-1}(p)u\right),\]

where the latter two instances of \(Q\) indicate the nonlinear part of the mean curvature, relative to the superscripted ambient metric, of the perturbation by the bracketed function (in the normal direction corresponding to \(\nu\)) of the subscripted immersion.

We can apply 4.24 to the blown-up situation to estimate the rightmost nonlinear terms appearing in 6.5 on the ball \(B_p \subset \Sigma[\zeta = \xi = 0]\) with \(\chi\)-radius 1 and center \(p\). Here the metric on \(B_p\) induced by the reparamerized immersion and rescaled ambient metric will be

\[(6.6) \quad \left(P_{\zeta, \xi}^{-1}(p)\right)^2\left(P_{\zeta, \xi}^{-1}(p)\right)^{-1} \chi_{\zeta, \xi},\]

where the subscripts on \(\rho\) and \(\chi\) emphasize that they are defined on the initial surface with parameter values \(\zeta\) and \(\xi\), but then for \(j \in \{0, 2\}\), the \(C_j^\alpha\) norms on \(B_p\) induced by this metric and by \(\chi_{0, 0}\) will be equivalent, with ratio bounded above and below by positive constants independent of \(p, m,\) and \(c\).

Thus, noting that all derivatives of the second fundamental form of \(t_{\zeta, \xi} \circ P_{\zeta, \xi}^{-1}\) relative to the blown-up metric (and of course all derivatives of the curvature of the blown-up metric) are bounded by a constant independent of \(m\) and \(c\), we apply the last two items of 4.24 to obtain from 6.5 the estimate

\[(6.7) \quad \|Q[u] : C_0^\alpha(B_p, \chi_{0, 0})\| \leq C \left(P_{\zeta, \xi}^{-1}(p)\right)^2\left(P_{\zeta, \xi}^{-1}(p)\right)u : C^2_\alpha(B_p, \chi_{0, 0})\|^2 \leq C e^{Cc} \rho_0^3(\rho_0^2 + \rho_0^{-2\gamma} - \rho_0^{-2\gamma}(p) \|u\|_{2, \alpha, \gamma}^2,\]

where \(C\) depends on \(m, \alpha, \gamma, \) and \(c\).
where $C$ may be different from the given one but is still independent of $m$ and $c$. Therefore

\[(6.8) \quad \|\rho^{-2} Q[u]\|_{0,\alpha,\gamma} \leq C e^{Cc} \|m^{\gamma}\rho^{1-\gamma}\|_{0} \tau_{1}^{2} \leq C e^{Cc} m^{\gamma} \tau_{1}^{\gamma-1} \tau_{1}^{2} \leq \frac{1}{1+\gamma},\]

for $m$ large enough, as asserted.

We will need also a refinement of 3.21, after some preliminary definitions. Take $(u_{0},\mu,\rho)$ as in 5.66 and given $v \in C_{G}^{2,\alpha,\gamma}(\Sigma[\zeta = \xi = 0])$, let $\nu = -\pi_{3} R \rho^{-2} Q[u_{0} + \mathcal{P}_{\zeta,\xi} v]$. Now define

\[(6.9) \quad \tilde{D}_{i}[v,\zeta,\xi] = D_{i} + \tilde{m}_{i} + \tilde{v}_{i}\]

for $2 \leq i \leq N - 1$, understanding $\tilde{D}_{1} = \tilde{D}_{N} = 0$, and define

\[(6.10) \quad \tilde{F}_{i}[v,\zeta,\xi]\]

as the force in the direction $K$ on the perturbation by $u_{0} + \mathcal{P}_{\zeta,\xi} v$ of $\Omega_{i}$ (from 2.17).

**Lemma 6.11.** Given $\alpha,\gamma \in (0,1)$ and initial surface data $k,\ell$, and $N$, there exists $\xi > 0$ and there exists $m_{0} > 0$ such that whenever $\zeta,\xi \in [-\zeta,\xi]^{N-1}$, $m > m_{0}$, and $v \in C_{G}^{2}(\Sigma[\zeta = \xi = 0])$ satisfies $\|v\|_{2,\alpha,\gamma} \leq \frac{1}{1+\gamma}$, we have, abbreviating $\tilde{F}_{i}[v,\zeta,\xi]$ as $\tilde{F}_{i}$ and $\tilde{D}_{i}[v,\zeta,\xi]$ as $D_{i}$,

\[(6.12) \quad \left\|\begin{pmatrix} \zeta_{1} \\ \zeta_{2} \\ \vdots \\ \zeta_{N-1} \end{pmatrix} - \frac{k\ell m^{2}}{2\pi r_{1}} Z^{-1} \left( \begin{pmatrix} \tilde{F}_{1} - \tilde{F}_{2} \\ \tilde{F}_{2} - \tilde{F}_{3} \\ \vdots \\ \tilde{F}_{N-1} - \tilde{F}_{N} \end{pmatrix} - \frac{4\pi}{r_{1}} Z^{-1} \begin{pmatrix} \tilde{D}_{1} + \tilde{D}_{2} \\ \tilde{D}_{2} + \tilde{D}_{3} \\ \vdots \\ \tilde{D}_{N-1} + \tilde{D}_{N} \end{pmatrix} \right) \right\| \leq \zeta,\]

recalling $Z : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ from 3.23 and

\[(6.13) \quad \left\|\begin{pmatrix} \xi_{1} \\ \xi_{2} \\ \vdots \\ \xi_{N} \end{pmatrix} - \frac{k\ell m^{2}}{8\pi^{2} r_{1}} Z^{-1} \left( \begin{pmatrix} \tilde{F}_{1} + \tilde{F}_{N} \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \frac{2}{r_{1}} Z^{-1} \begin{pmatrix} \tilde{D}_{1} \\ \tilde{D}_{2} \\ \vdots \\ \tilde{D}_{N-1} \end{pmatrix} \right) \right\| \leq \zeta,\]

recalling $\Xi : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ from 3.24. Furthermore, for $1 \leq i \leq N$, the maps $(v,\zeta,\xi) \mapsto \tilde{F}_{i}$ and $(v,\zeta,\xi) \mapsto \tilde{D}_{i}$ are continuous, using the $C^{2,\alpha}(\Sigma,\chi)$ norm on the first factor of the domain.

**Proof.** The continuity of $\tilde{D}_{i}$ is clear from the definition of $D_{i}$ and from the continuity statements in 5.66, 6.2, and 5.58. The force $\tilde{F}_{i}$ can be expressed as an integral along the image curves of $\partial \Omega_{i} \subset \Sigma[\zeta = \xi = 0]$ under the perturbation by $\mathcal{P}_{\zeta,\xi}^{1} u_{0} + v$ of $\mathcal{P}_{\zeta,\xi} P^{-1}$. Fixing a point $p$ on such a curve, the image point and conormal there can be seen to depend smoothly on the $p$, the parameters, and the 1-jet at $p$ of the perturbing function. Since the Killing field $K$ and the ambient metric $g_{S}$ are themselves smooth, this establishes, in conjunction with the same references just cited, the asserted continuity of $\tilde{F}_{i}$.

To check the estimates, first note that $|\tilde{D}_{i} - D_{i}| \leq C r_{1}$, with $C$ independent of $m$ and the parameters. The proof will be completed by showing that $|\tilde{F}_{i} - F_{i}| \leq C r_{1}$ and then appealing to 3.21. For the force comparison, set $u = u_{0} + \mathcal{P}_{\zeta,\xi} v$, write $\nu_{u}$ for the unit normal obtained from $\nu$ by perturbing the initial surface $\Sigma[\zeta,\xi]$ by $u$, write $\sqrt{|g|}$ for the area density of this perturbed surface, and write $\sqrt{|g|}$ for the area density of the initial surface, as well as for the corresponding
length density on curves therein. Since \( K \) is Killing, we can use the first-variation formula and Green’s second identity to obtain

\[
\begin{aligned}
\mathcal{F}_i - \mathcal{F}_i & = \int_{\Omega_i} \langle \nu_u, K \rangle \mathcal{H}[u] \left( \sqrt{|g_u|} - \sqrt{|g|} \right) + \int_{\Omega_i} \langle \langle \nu_u, K \rangle - \langle \nu, K \rangle \rangle \mathcal{H}[u] \sqrt{|g|} \\
& + \int_{\Omega_i} \langle \nu, K \rangle \mathcal{Q}[u] \sqrt{|g|} + \int_{\partial\Omega_i} \langle \nu, K \rangle \eta u \sqrt{|g|} - \int_{\partial\Omega_i} \eta \langle \nu, K \rangle \sqrt{|g|},
\end{aligned}
\]

(6.14)

recalling that \( \eta \) is the outward conormal on \( \partial\Omega_i \).

From 4.24 (i), \( \| u \|_{1,0, \gamma} \leq C \tau_1 \), and estimates of \( \| A \| \) from 4.16 and 4.33, we find

\[
\left| \sqrt{|g_u|} - \sqrt{|g|} \right| \leq C \tau_1 \sqrt{|g|},
\]

(6.15)

so, using also the obvious boundedness of \( \| K \|_{gs} \), the estimate \( \rho^{-2} \mathcal{H}[u] \|_0 \leq C \tau_1 \), and the upper bound \( C(m^2 + |\zeta|) \) for the \( \chi = \rho^2 g \) area of \( \Omega_i \), we can certainly bound the first term of 6.14 by \( Cm^{-2} \tau_1 \), with \( C \) independent of \( m \) and the parameters, provided the former is chosen large in terms of the latter. Similarly, the difference between \( \nu_u \) and \( \nu \), parallely transported along the geodesics it generates, is controlled by \( (1 + \| A \|_g |u|) \| du \|_g \) and so in turn bounded by \( Cm^{-2} \tau_1 \), and of course \( \| DK \|_{gs} \) is uniformly bounded, so we can bound the second term of 6.14 by \( Cm^{-2} \tau_1 \) as well. To achieve the same bound for the third term we apply 6.2.

On the waist circles of the catenoidal regions we have \( |\eta u| \leq Cm^2 \tau_1 \), while the circles themselves have length of order \( \tau_1 \), ensuring the estimate for the circular component(s) of the boundary in the fourth term of 6.14. On the rectangular component \( R, u \) satisfies periodic boundary conditions, so \( \int_R \eta u \sqrt{|g|} = 0 \), but \( |\langle \nu, K \rangle - 1| \leq Cm^{-2} \), \( |\eta u| \leq Cm \tau_1 \), and \( R \) has length of order \( m^{-2} \), so we have \( Cm^{-2} \tau_1 \) as an upper bound for the entire fourth term. Finally, for the fifth term, on \( R \) we see \( |\eta \langle \nu, K \rangle| \leq Cm^{-1} \tau_1 \), while on the waist circle(s) we use the estimates

\[
|\eta \langle \nu, K \rangle| \leq \| DK \|_{gs} \| A \| \| K \|_{gs} \leq C \tau_1^{-1}
\]

(6.16)

and \( |u| \leq C \tau_1^{1+\gamma} \), concluding the proof.

We can now prove the main theorem.

**Theorem 6.17.** Fix \( \alpha, \gamma \in (0, 1) \). Given positive integers \( k, \ell \), and \( N \geq 2 \), there exist \( \xi, C, m_0 > 0 \) such that for every \( m > m_0 \) there exist parameters \( \zeta, \xi \in [-\xi, \xi]^{N-1} \) and a function \( u \in C^\infty_0(\Sigma[N, k, \ell, m, \zeta, \xi]) \) such that \( \| u \|_{2, \alpha, \gamma} \leq C \tau_1 \) and the normal perturbation \( \nu_u : \Sigma \to S^3 \) by \( u \) of the defining embedding \( \nu : \Sigma \to S^3 \) is a minimal embedding.

**Proof.** Set

\[
B = \left\{ v \in C^2_0(\Sigma[\zeta = \xi = 0], \chi) : \| v \|_{2, \alpha, \gamma} \leq \frac{1+\gamma}{2} \right\} \times [-\xi, \xi]^{N-1} \times [-\xi, \xi]^{N-1},
\]

(6.18)

and define \( \mathcal{J} : B \to B \) by

\[
\mathcal{J}(v, \zeta, \xi) = \left( -\pi \xi \sum \mathcal{D}_i \mathcal{P}_\zeta v + \mathcal{Q}_\xi \xi \left[ -\pi \xi \sum \mathcal{D}_i \mathcal{P}_\zeta v + \mathcal{Q}_\xi \xi \right] - \sum \nabla_i \mathcal{P}_\zeta v \right) + \mathcal{J}(v, \zeta, \xi)
\]

\[
\zeta - \frac{k\mu m^2}{2\pi^2 \tau_1} Z^{-1} \left( \bar{F}_i - \bar{F}_{i+1} \right)_{i=1}^{N-1} - \frac{4\pi}{\tau_1} Z^{-1} \left( \bar{D}_i + \bar{D}_{i+1} \right)_{i=1}^{N-1},
\]

(6.19)

\[
\xi - \frac{k\mu m^2}{8\pi^2 \tau_1} Z^{-1} \left( \bar{F}_i + \bar{F}_N \right) \left( \delta_{i-1} - \frac{2}{\tau_1} Z^{-1} \left( \bar{D}_i \right)_{i=1}^{N-1} \right),
\]

where \( \bar{D} \) and \( \bar{F} \) abbreviate \( \mathcal{D}[\mathcal{P}_\zeta \xi v, \zeta, \xi] \) and \( \mathcal{F}[\mathcal{P}_\zeta \xi v, \zeta, \xi] \) respectively.
By \([5.66] [5.58] [6.2]\) and \([6.11]\) (which provides \(c\)), we are assured that \(J(B) \subseteq B\) as advertised. By the same references \(J\) is continuous, using the \(C^{2,\alpha}(\Sigma, \chi)\) norm on the first factor (and the usual ones on the second two). Moreover, \(B\) is in fact compact under this topology, because of the \(C^{2,\alpha}\) bound on elements of the first factor, and it is obviously convex. Thus Schauder’s fixed-point theorem ensures the existence of a fixed point \(v_1\) to \(J\). Set \(u_0 = -\pi_1 R \left( \rho^{-2} J[0] - \sum \mathcal{D}_i \mathcal{W}_i \right) \), \(v_0 = \mathcal{P}_\chi \xi v_1\) and \(u = u_0 + v_0\).

It follows that \(\tilde{F}_i = \tilde{D}_i = 0\) for every \(i\) and that
\[
\rho^{-2} J[u] = \rho^{-2} J[0] + \mathcal{L}_\chi u_0 + \mathcal{L}_\chi v_0 + \rho^{-2} Q[u_0 + v_0] + \sum \mu_i w_i + \sum \mu_i \mathcal{W}_i \\
= \sum \mathcal{D}_i \mathcal{W}_i + \sum \mu_i w_i + \sum \mu_i \mathcal{W}_i \\
= \sum \mathcal{D}_i \mathcal{W}_i + \sum \mu_i \mathcal{W}_i \\
= \sum \mu_i \mathcal{W}_i,
\]
but then \(F_i = \mu_i \int_{\Omega_i} w_i \langle u, K \rangle\), so, because the integrand has a sign, it is easy to see that the vanishing force condition implies \(\mu_i = 0\) for every \(i\). This shows minimality, whence follows the regularity since \(u \in C^{2,\alpha}(M, \chi)\). The estimate on \(u\) is clear from \([5.66] [6.2]\) and \([5.58]\) and in turn it implies the embeddedness.

\[\square\]

References

[1] S. Brendle, Minimal surfaces in \(S^3\): a survey of recent results, Bulletin of Mathematical Sciences 3 (2013), 133–171.
[2] J. Choe and M. Soret, New minimal surfaces in \(S^3\) desingularizing the Clifford tori, preprint, available at arXiv:1304.3184.
[3] N. Kapouleas, Constant mean curvature surfaces constructed by fusing Wente tori, Inventiones Mathematicae 119 (1995), no. 3, 443–518.
[4] , Constructions of minimal surfaces by gluing minimal immersions, Global Theory of Minimal Surfaces, Clay Mathematics Proceedings, vol. 2, American Mathematical Society, Providence, 2005, pp. 489–524.
[5] , Doubling and desingularization constructions for minimal surfaces, Surveys in Geometric Analysis and Relativity celebrating Richard Schoen’s 60th birthday, Advanced Lectures in Mathematics, vol. 20, Higher Education Press and International Press, Somerville, MA, 2011, pp. 281–325.
[6] , Minimal surfaces in the round three-sphere by doubling the equatorial two-sphere, I, preprint, available at arXiv:1409.0226.
[7] N. Kapouleas and S.D. Yang, Minimal surfaces in the three-sphere by doubling the Clifford torus, American Journal of Mathematics 132 (2010), 257–295.
[8] H. Karcher, U. Pinkall, and I. Sterling, New minimal surfaces in \(S^3\), Journal of Differential Geometry 28 (1988), 169–185.
[9] H.B. Lawson Jr., Complete minimal surfaces in \(S^3\), Annals of Mathematics 92 (1970), 335–374.
[10] F. Marques and A. Neves, Existence of infinitely many minimal hypersurfaces in positive Ricci curvature, preprint, available at arXiv:1311.6501.
[11] J.T. Pitts and J.H. Rubinstein, Equivariant minmax and minimal surfaces in geometric three-manifolds, Bulletin of the American Mathematical Society 19 (1988), no. 1, 303–309.
[12] R.M. Schoen, The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation, Communications on Pure and Applied Mathematics 41 (1988), no. 3, 317–392.

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