ON THE SITUATION OF NODES OF PLANE CURVES

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1. Introduction. We consider complex plane algebraic curves with nodes (i.e., ordinary double points). Such a curve is said to be nodal if it has only nodes as singularities. Salmon proposed the following problem: Describe the situation of nodes of an irreducible nodal curve ([4, Art. 45], [2, pp. 389–393]). Let \( n \) denote the degree of a nodal curve and \( d \) the number of nodes. The problem is trivial if \( n \leq 6 \) and \( d \leq 8 \). The first nontrivial case, \( (n, d) = (6, 9) \), has been analyzed by Halphen (cf. [2, p. 390]). The case

\[
d \leq \min\{\frac{n(n + 3)}{6}, \frac{(n - 1)(n - 2)}{2}\} \quad \text{and} \quad (n, d) \neq (6, 9)
\]

was investigated by Arbarello and Cornalba [1, Theorem 3.2]; we give another proof (see Proposition 3(i)). We consider the remaining cases, which are particularly important as they have applications to the moduli variety of curves.

Let \( V_{n,d} \) be the variety of irreducible nodal curves of degree \( n \) with \( d \) nodes. For \( n(n + 3)/6 \leq d \leq (n - 1)(n - 2)/2 \) and \( (n, d) \neq (6, 9) \), we prove that the map \( p_d: V_{n,d} \to \text{Sym}^d(P^2) \), which sends a curve to the set of its nodes, is a birational morphism onto its image (Theorem, Part (i)) and give a rough description of the image (Corollary) and of the generic curve of \( V_{n,d} \). In fact, we prove our results for the subvariety \( V'_{n,d} \subseteq V_{n,d} \) of those nodal curves which can be degenerated into a sum of \( n \) lines in general position \( (V'_{n,d} \) is irreducible by [5, §11]). We then apply a recent result of Harris to the effect that \( V_{n,d} = V'_{n,d} \).

2. Zero-dimensional schemes. Let \( \text{Hilb}^e \) be the Hilbert scheme of zero-dimensional subschemes of degree \( e \) in \( P^2 \). One can stratify \( \text{Hilb}^e: Y, Z \in \text{Hilb}^e \) belong to the same stratum iff \( h^0(P^2, I_Y(l)) = h^0(P^2, I_Z(l)) \) for all \( l \). Let \( D^e \) denote the dense stratum. It is easy to show that \( D^e \) consists of \( m \)-regular (in the sense of Castelnuovo) schemes not lying on curves of degree \( m - 2 \), where \( m = \min\{i \in \mathbb{Z}|e \leq i(i + 1)/2\} \). We denote by \( \tilde{D}^e \) the subset of \( D^e \) consisting of schemes of the form \( \sum_{i=1}^c P_i \), where \( P_i \neq P_j \) for \( i \neq j \) and \( \sum_{k=1}^d P_{i_k} \in D^d \) for every \( \{i_1, \ldots, i_d\} \subseteq \{1, \ldots, e\} \).

3. Main results. We need four propositions; they are of independent interest.

**Proposition 1.** Let \( d \leq (n - 1)(n - 2)/2 \). If a reduced curve of degree \( n \) with \( d \) assigned singular points \( P_1, \ldots, P_d \) is not a specialization of an irreducible curve with \( d \) assigned nodes, then \( \sum_{i=1}^d P_i \notin D^d \).

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To prove Proposition 1, we suppose \( \sum_{i=1}^{d} P_i \in D^d \) and make a reduction to a curve consisting of two smooth components. We then derive a contradiction by the Cayley-Bacharach theorem.

Let now \( L = L_1 + \cdots + L_n \in \mathbb{P}^2 \) be a sum of \( n \) general lines. Set \( \{P_1\} = L_1 \cap L_2, \{P_2, P_3\} = (L_1 + L_2) \cap L_3, \{P_4, P_5, P_6\} = (L_1 + L_2 + L_3) \cap L_4 \), etc. Then \( \{P_1, \ldots, P_d\} \in D^d \) for \( d \leq n(n-1)/2 \). By Severi [5, §11], for \( d \leq (n-1)(n-2)/2 \), \( L \) with the assigned nodes \( P_1, \ldots, P_d \) is a specialization of a curve of \( V_{n,d} \), and we can prove

PROPOSITION 2. The scheme consisting of \( d \) nodes of a general curve of \( V_{n,d} \) is a point of \( \hat{D}^d \).

In Proposition 3 below, we estimate the dimensions of some families of nonreduced curves. Let \( f(x, y, z) = \sum a_{ijk} x^i y^j z^k \) be the homogeneous polynomial of degree \( n \) with generic coefficients. We consider the following system of \( 3d \) equations in \( a \)'s,

\[
\begin{align*}
&f'_x(x_1, y_1, z_1) = 0, \\
&f'_y(x_1, y_1, z_1) = 0, \\
&\ldots, \\
&f'_z(x_d, y_d, z_d) = 0,
\end{align*}
\]

where \( (x_1 : y_1 : z_1 : \ldots : x_d : y_d : z_d) \in (\mathbb{P}^2)^d \). Let \( M^d \subset (\mathbb{P}^2)^d \) be the closed subscheme where the system has nontrivial solutions. We have two natural maps \( \text{Hilb}^d \to \text{Sym}^d(\mathbb{P}^2) \overset{\sigma_d}{\to} (\mathbb{P}^2)^d \).

PROPOSITION 3. We assume \([n(n + 3)/6] \leq d \leq (n-1)(n-2)/2 \) and \((n, d) \neq (6, 9)\).

(i) Let \( K \subset M^d \) be an irreducible component and \( (Q_1; \ldots; Q_d) \in K \) a general point. If \( d \geq n(n+3)/6 \) and \( \sigma_d(K) \cap \phi_d(\hat{D}^d) \neq \emptyset \), then a curve of degree \( n \) having singularities at \( Q_1, \ldots, Q_d \) is an irreducible nodal curve with \( d \) nodes and \( \dim K = \dim V_{n,d} \). If \( d = [n(n + 3)/6] \), then there exists an irreducible nodal curve with \( d \) nodes in general position.

(ii) Let \( C \in V_{n,d} \) be a general curve and \( P_1, \ldots, P_d \) its nodes. If \( l \) is the degree of a nonreduced curve of minimal degree having singularities at \( P_1, \ldots, P_d \), then \( l > n \) unless \( (n, d) = (8, 14) \).

We also need a generalization of a theorem of Arbarello-Cornalba and Zariski (cf. [6, Theorem 2]).

PROPOSITION 4. Let \( \mathcal{A} \) be an irreducible analytic family of curves of degree \( n \) with \( d \) assigned singular points whose general curve, say \( B \), is reduced and has \( q \) singular points \( P_1, \ldots, P_d, \ldots, P_q \) \( (e \leq d \leq q) \). We assume \( P_1, \ldots, P_d \) are the assigned singularities, \( P_1, \ldots, P_e \) are nodes, and \( P_{e+1}, \ldots, P_d \) are not nodes. We also assume:

(i) there exists a curve \( C \) of degree \( n \) with singularities at \( P_1, \ldots, P_d \), and \( C \) and \( B \) have no common components,

(ii) \( \dim \mathcal{A} \geq \dim V_{n,d} - \min\{d - e, n + 1\} \). Then \( \dim \mathcal{A} = \dim V_{n,d} - d + e, \) \( q = d, \) and \( P_{e+1}, \ldots, P_d \) are cusps. Furthermore, if \( B \) is irreducible, we can drop condition (i) and replace (ii) by the condition:

\[
\dim \mathcal{A} \geq \dim V_{n,d} - \min\{d - e, 3(n - 1)\}.
\]
THEOREM. (i) If \( n(n + 3)/6 \leq d \leq (n - 1)(n - 2)/2 \) and \((n,d) \neq (6,9)\),
then \( p_d: \text{Sym}^d(\mathbb{P}^2) \to \mathbb{P}^2 \) is a birational morphism of \( V'_{n,d} \) onto its image.

(ii) If \( d \leq \min\{n(n + 3)/6, (n - 1)(n - 2)/2\} \) and \((n,d) \neq (6,9)\), then
for general \( P_1, \ldots, P_d \in \mathbb{P}^2 \), there exists a curve in \( V'_{n,d} \) having nodes at
\( P_1, \ldots, P_d \).

We prove both assertions simultaneously, first assuming that \( n(n + 3)/6 \leq d \leq (n - 1)(n - 2)/2 \) and \( n \geq 7 \). Choose an irreducible component \( K \subset M^d \)
such that \( \sigma_1^{-1}(p_d(V'_{n,d})) \subseteq K \) and \( \dim K = \min\{\dim V'_{n,d}, 2d\} \). For this \( K \),
one can find a complete irreducible system \( W \) of nodal curves of degree \( n \) with \( d \) nodes such that
\( p_d(W) = \sigma_d(K) \). We then show that \( \dim W \cap V'_{n,d} \geq \dim V'_{n,d} - n - 1 \), and this allows us to deduce from Proposition 4 that \( V'_{n,d} = W \).

The theorem then follows.

We observe that Part (ii) of Theorem also follows from [1, Theorem 3.2] together with [3]. From now on we assume \( n(n + 3)/6 \leq d \leq (n - 1)(n - 2)/2 \) and \((n,d) \neq (6,9)\). Combining our results with the theorem of Harris [3]
\((V_{n,d} = V'_{n,d})\), we obtain

**COROLLARY.** \( p_d(V_{n,d}) = \sigma_{d}(M^d) \cap \phi_d(D^d) \), and for \( n(n + 3)/6 < t < d \), any \( t \) nodes of a general curve \( C \in V_{n,d} \) determine the location of the remaining nodes of \( C \).

**REMARK.** We can solve (*) on the open subset of \( M^d \cap \sigma_1^{-1}(\phi_1(D^d)) \) where
the system has unique solutions, and we obtain the equation of the generic curve of \( V_{n,d} \).

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