A Minimal Model for Dilatonic Gravity

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We study a new minimal scalar-tensor model of gravity with Brans-Dicke factor \( \omega(\Phi) \equiv 0 \) and cosmological factor \( \Pi(\Phi) \). The constraints on \( \Pi(\Phi) \) from known gravitational experiments are derived. We show that almost any time evolution of the scale factor in a homogeneous isotropic Universe can be obtained via a properly chosen \( \Pi(\Phi) \) and discuss the general properties of models of this type.

I. INTRODUCTION

At present general relativity (GR) is the most successful theory of gravity in describing gravitational phenomena at laboratory–, earth-surface–, solar-system– and star-systems scales [1]. Although it gives quite a good description of these phenomena at galaxies scales and at the scales of the whole Universe, there exist well known problems such as the rotation of galaxies, the initial singularity problem, the early Universe, the recent discovery of the accelerated expansion of the Universe, etc. It is likely, that the indicated problems call for an extension of the GR framework.

The most promising modern theories which incorporate naturally GR like supergravity and (super)string theories, unfortunately introduce, at least at their present stage of development, a large number of new fields without any real physical basis and are far from being experimentally testable. Therefore, it seems sensible to explore some other models which are (i) compatible with the established gravitational experiments, (ii) obtained through a minimal extension of GR, (iii) promising candidates for overcoming the above problems, and which (iv) may be considered as a part of some more general modern theories. In the present article we outline the general properties of one such model which is built upon an additional scalar field \( \Phi \) and that differs from the known inflationary and quintessential models.

We call minimal dilatonic gravity (MDG) the scalar-tensor model of gravity [1], [2] with an action

\[
A_{G, \Lambda} = -\frac{c}{2\kappa} \int d^4x \sqrt{|g|} \Phi \left( R + 2\Lambda \Pi(\Phi) \right).
\]

Here, the well known Brans-Dicke factor has been denoted by \( \omega(\Phi) \equiv 0 \), \( \Lambda \) stays for the cosmological constant, while \( \Pi(\Phi) \) is the dimensionless cosmological factor. The matter action, \( A_M \), and the matter equations of motion will have the usual GR form and do not depend directly on the dilaton field \( \Phi \).

The field equations for the metric \( g_{\alpha\beta} \) and the dilaton field \( \Phi \) are given by:

\[
\Phi \left( G_{\alpha\beta} - \Lambda \Pi(\Phi) g_{\alpha\beta} \right) - \left( \nabla_\alpha \nabla_\beta - g_{\alpha\beta} \Box \right) \Phi = \frac{c}{2\kappa} T_{\alpha\beta},
\]

\[
\Box \Phi + \Lambda \frac{dV}{d\Phi}(\Phi) = \frac{c}{3\kappa} T
\]

and yield the energy-momentum conservation law, \( \nabla_\alpha T^\alpha_{\beta} = 0 \).

The variation of the action [1] with respect to the metric \( g_{\alpha\beta} \) leads to the first one of the equations [2]. The variation of [2] with respect to the dilaton field \( \Phi \) produces the following local algebraic relation between this field and the scalar curvature \( R \):

\[
R + 2\Lambda \left( \Phi \frac{dV}{d\Phi} + \Pi(\Phi) \right) = 0.
\]

In combining the latter relation with the trace of the first of the equations [2], the field equation for the dilaton \( \Phi \) is obtained. Because of the absence of the standard kinetic term \( \sim \omega(\Phi)(\nabla\Phi)^2 \) for the dilaton in action [1], its dependence on nonzero space-time curvature \( (R \neq 0) \) is essential for the existence of this equation. There, \( \frac{dV}{d\Phi} = \frac{d}{d\Phi} \left( \Phi \frac{dV}{d\Phi} - \Pi \right) \) is the derivative of the dilatonic potential \( V(\Phi) \).

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The presence of a potential term for the scalar fields in the action of the theory, was considered many times from different points of view. For example, the extended inflation models and the quintessential models contain such a term, see [3] and the references therein.

In Brans-Dicke’s theory, $\Pi(\Phi) \equiv 0$, and one has to make a choice for the function $\omega(\Phi)$ only, in order to fix the model. However, recent astrophysical data from type Ia supernovae at high red-shifts, on the one hand, as well as from cosmic microwave background, on the other hand, lead to a strong support to the absence of the Brans-Dicke kinetic term $\omega(\Phi)$ in the action of the theory [4].

In more general scalar-tensor theories of gravity, two unknown functions, $\omega(\Phi)$ and $\Pi(\Phi)$, are to be fixed. The old attempts to extract from the known gravitational experiments simultaneously the dilaton mass $m_\Phi \neq 0$ also a constant parameter $\omega = const \neq 0$, turned out to be less successful, living great uncertainties for the corresponding values of the above parameters, see [3]. A new promising method to determine $\omega(\Phi)$ and $\Pi(\Phi)$ (or some equivalent functions) using astrophysical observations, was developed in the recent work [3]. Unfortunately, at present, we don’t have sufficiently reliable data, as required for applying this method, so that, at present, the form of both functions still remains an open problem.

As a scalar-tensor theory with only one unknown function $\Pi(\Phi)$, the MDG is much better defined. Its investigation was pioneered by O’Hanlon in connection with Fujii’s theory of massive dilatons as early as in [7], however, without any connection with the cosmological constant problem.

In general, such a minimal model may be related with the so called nonlinear theories of gravity (see [3] and the references therein). Indeed, the variation of the action (1) with respect to the dilaton field $\Phi$ gives the local algebraic relation (3) between scalar curvature $R$ and dilaton $\Phi$ in place of the differential equation, which is the absence of the Brans-Dicke kinetic term $\sim \omega(\Phi)(\nabla \Phi)^2$ in the action for $\omega \equiv 0$. If $\omega(\Phi)$ is not constant, one can solve the relation (3) with respect to the field $\Phi$ and it becomes a local function of the scalar curvature: $\Phi = \Phi(R)$. (For $\omega \neq 0$ the last relation will have non local integral form.) The substitution of this function back into the action (1) transforms it to the action of nonlinear theories of gravity $A_{NG} = \int d^4x \sqrt{|g|} f(R)$ with $f(R) = R\Phi(R) + 2\Lambda \Phi(R)\Pi(\Phi(R))$. Hence, the action (1) is of the Helmholtz type for nonlinear gravity [3].

MDG may be considered also as a part of more general theories of gravity like the (4+1)-dimensional Kaluza-Klein model described by Fujii in [3], or, the model of gravity with violated local conformal symmetry in affine-connected spaces, which probably may be related to string theory [10].

In this special scalar-tensor model of gravity the cosmological factor $\Pi(\Phi)$ is the only function which has to be chosen to comply with gravitational experiments and astrophysical observations. In the present article, we shall demonstrate that this results in much better defined predictions for the dilaton mass, which is now derived from known gravitational experiments. It turns out that the results of these experiments on Earth surface and in solar, or star systems depend mainly on the mass term in the dilaton potential. A numerical analysis of the boson stars in massive dilatonic gravity based on MDG was performed very recently [11]. It shows that the boson star structure is sensitive, typically within a few percent accuracy only, to the dilaton mass rather than to the exact form of the potential. Hence, we need some other source of information about the form of the cosmological factor.

In the present article we shall show how one can solve the inverse cosmological problem in MDG. By inverse cosmological problem we mean the determination of the cosmological factor $\Pi(\Phi)$ which yields a given time evolution of the scale parameter $A(t)$ in the Robertson-Walker (RW) model of the Universe. Such an approach for the determination of the cosmological factor may be considered as a further development of the general idea to find the dilaton potential from astrophysical observations as discussed first in the articles [2]. There, the relation between the dilaton potential and the luminosity distance was obtained from observations of type Ia supernovae explosions at high red-shifts. Similar consideration based on real observational data may take place in the model of MDG, too. Our approach, in being based upon the evolution of the cosmological factor, gives another possibility to extract the dilaton potential from data and seems to be more profound.

An essential new element of our MDG is the nonzero constant $\Lambda$ which we identify with standard cosmological constant. The new astrophysical data $\Omega_{\Lambda} = 0.65 \pm 0.13$, $H_0 = (65 \pm 5) km s^{-1} Mpc^{-1}$ [4] give observed value $\Lambda_{obs} = 3\Omega_{\Lambda} H_0^2 c^2 = (0.85 \pm 0.28) \times 10^{-56} cm^{-2}$. Though the present confidence level of $\Omega_{\Lambda}$ is not as high, we accept its value as a basic quantity which defines the natural units for all cosmological quantities: the cosmological length $A_c = 1/\sqrt{\Lambda obs} = (1.02 \pm 0.18) \times 10^{28} cm$, the
cosmological time \( T_c := A_c/c = (3.4 \pm 0.6) \times 10^{17} \) s = (10.8 \pm 1.9) Gyr, the cosmological energy density \( \varepsilon_c := \frac{\Delta \varepsilon}{c} = (1.16 \pm 0.41) \times 10^{-5} \text{g cm}^{-1} \text{s}^{-2}, \) the cosmological energy \( E_c := 3 \alpha_1^3 \varepsilon_c = 3 \alpha_1^{-1/2} c^2 \kappa^{-1} = (3.7 \pm .7) \times 10^{77} \text{erg} \) the cosmological momentum \( P_c := 3c/(\kappa \Lambda \Omega_{\text{obs}}) = (1.2 \pm .2) \times 10^{67} \text{g cm s}^{-1} \) and new the cosmological unit for the action \( A_c := 3c/(\kappa \Lambda \Omega_{\text{obs}}) = (1.2 \pm .4) \times 10^{122} \text{h}, \) \( \kappa \) being Einstein constant. Further, we use dimensionless variables like: \( \tau := t/T_c, \ a := A/A_c, \ h := H T_c \) (\( H := A^{-1} \text{d}A/\text{dt} \) being Hubble parameter), \( \varepsilon_c = \varepsilon_c/|\varepsilon_c| = \pm 1, \ \varepsilon := \varepsilon/|\varepsilon_c| \)-matter energy density, etc.

II. SOLAR SYSTEM AND EARTH-SURFACE GRAVITATIONAL EXPERIMENTS

We use the well studied scalar-tensor theories of gravity [1], [2], [5], [7] to derive the properties of cosmological factor \( \Pi(\Phi) \) and predictions for gravitational experiments.

A. General Consideration

1. The MDG with \( \Lambda = 0 \) is a Brans-Dicke theory with \( \omega = 0 \) and contradicts the data from the solar system gravitational experiments, because the value \( \gamma \) of the post Newtonian parameter \( \gamma = \frac{\Delta \varepsilon}{\varepsilon_c} \) is far from the experimental constraint \( \gamma = 1 \pm .001 \) [1]. We must introduce cosmological term \( \Lambda \Pi(\Phi) \neq 0 \) in action (1) to overcome this problem.
2. In contrast to O’Hanlon’s model [1] we wish MDG to reproduce GR with \( \Lambda \neq 0 \) for \( \Phi = \hat{\Phi} = \text{const} \neq 0 \). Then from action (1), we obtain normalization condition \( \Pi(\Phi) = 1 \) and Einstein constant \( \kappa = \frac{\kappa}{\kappa} \).
3. In vacuum, far from matter MDG have to allow weak field approximation \( \Phi = \bar{\Phi}(1+\zeta), |\zeta| \ll 1 \) which we consider in harmonic gauge. Then the second of equations (2), written in linearized form (Einstein constant. Further, we use dimensionless variables like: \( \tau := t/T_c, \ a := A/A_c, \ h := H T_c \) (\( H := A^{-1} \text{d}A/\text{dt} \) being Hubble parameter), \( \varepsilon_c = \varepsilon_c/|\varepsilon_c| = \pm 1, \ \varepsilon := \varepsilon/|\varepsilon_c| \)-matter energy density, etc.

4. For few point particles of masses \( m_\alpha \) being source of metric and dilaton fields in equations (3) we obtain Newtonian approximation \( (g_{\alpha \beta} = h_{\alpha \beta} + h_{\alpha \beta}, |h_{\alpha \beta}| \ll 1) \) for the gravitational potential \( \varphi(\mathbf{r}) \) and the dilaton field \( \Phi(\mathbf{r}) \):

\[
\varphi(\mathbf{r})/c^2 = -\frac{G}{c^2} \sum_{\alpha} \frac{m_\alpha}{r_\alpha} \left( 1 + \alpha(p) e^{-|\mathbf{r} - \mathbf{r}_\alpha|/l_\Phi} \right)
- \frac{1}{2} \beta^2 \sum_{\alpha} \frac{m_\alpha}{r_\alpha^3} (|\mathbf{r} - \mathbf{r}_\alpha|/l_\Phi)^2,
\]

\[
\Phi(\mathbf{r})/\bar{\Phi} = 1 + \frac{2}{3} \frac{G}{c^2} \sum_{\alpha} \frac{m_\alpha}{r_\alpha} e^{-|\mathbf{r} - \mathbf{r}_\alpha|/l_\Phi}. \tag{5}
\]

\( G = \frac{c^2}{8\pi} (1 - \frac{4}{3} \beta^2) \) is Newton’s constant, \( M = \sum_\alpha m_\alpha \). The term

\[-\frac{1}{2} \beta^2 \sum_\alpha \frac{m_\alpha}{r_\alpha^3} (|\mathbf{r} - \mathbf{r}_\alpha|/l_\Phi)^2 = -\frac{1}{2} \lambda |\mathbf{r} - \sum_\alpha \frac{m_\alpha}{r_\alpha^3} r_\alpha| ^2 + \text{const}
\]

in \( \varphi(\mathbf{r}) \) is known from GR with \( \Lambda \neq 0 \). It represents an universal anty-gravitational interaction of test mass \( m \) with any mass \( m_\alpha \) via repellant elastic force

\[
\mathbf{F}_{\lambda \alpha} = \frac{1}{2} \lambda mc^2 \frac{m_\alpha}{M}(\mathbf{r} - \mathbf{r}_\alpha). \tag{6}
\]
For solar system distances \( l \leq 1000\text{AU} \) the whole second term in \( \varphi \) may be neglected [13], being of order \( \leq 10^{-24} \). Then we arrive to the known form of gravitational potential \( \varphi \) [5], but with specific for MDG coefficient

\[
\alpha(p) = \frac{1+4p^2}{3-4p^2}
\]

and the comparison of both possibilities: \( \alpha \geq \frac{1}{3} \), or \( \alpha \leq -1 \) with Cavendish type experiments yields the experimental constraint \( l_{\Phi} \leq 1.6 \text{ [mm]} \) (\( p < 2 \times 10^{-29} \)), see the articles by De Rújula and by Fischbach and Talmadge in [5] and the references therein. Hence, in the solar system phenomena the factor \( e^{-l/l_{\Phi}} \) has fantastically small values (\( < \exp(-10^{11}) \) for the Earth-Sun distances \( l \), or \( < \exp(-3 \times 10^{10}) \) for the Earth-Moon distances \( l \)) and there is no hope to find some differences between MDG and GR in this domain.

The constraint \( m_{\Phi}c^2 \geq 10^{-4} \text{[eV]} \) does not exclude a small value (with respect to the elementary particles scales) for the rest energy of the hypothetical \( \Phi \)-particle.

5. The parameterized-post-Newtonian (PPN) solution of equation (2) is complicated, but because of the constraint \( p < 10^{-28} \) we can neglect the second term in \( \varphi \), set \( \alpha = \frac{1}{3} \) and use with great precision Helbig’s PPN formalism which differs essentially from the standard one [5] for zero mass dilaton fields. We obtain much more definite predictions then general relations between \( \alpha \) and the length \( l_{\Phi} \) given in the articles by De Rújula and by Helbig [5], because of the condition \( \omega \equiv 0 \).

**B. The Basic Gravitational Effects in MDG**

A brief treatment of basic gravitational effects gives:

1. **Nordtvedt Effect**

In MDG a body with significant gravitational self-energy \( E_G = \sum_{b \neq c} \frac{m_b m_c}{|r_b - r_c|} \) will not move along geodesics due to additional universal anty-gravitational force:

\[
F_N = -\frac{2}{3}E_G \nabla \Phi.
\]

For usual bodies it is too small even at distances \( |r - r_a| \leq l_{\Phi} \), because of the small factor \( E_G \). Hence, in MDG we have no strict strong equivalence principle. Nevertheless, the week equivalence principle is not violated.

The experimental data for Nordtvedt effect caused by the Sun are formulated as a constraint \( \eta = 0 \pm .0015 \) [1] on the parameter \( \eta \) which in MDG becomes a function of the distance \( l \) to the source:

\[
\eta(l) = -\frac{1}{2} (1 + l/l_{\Phi}) e^{-l/l_{\Phi}}.
\]

Taking into account the value of the Astronomical Unit (AU) \( l_{AU} \approx 1.5 \times 10^{11}[m] \) we obtain from the experimental value of \( \eta \) the constraint \( l_{\Phi} \leq 2 \times 10^{10}[m] \).

2. **Time Delay of Electromagnetic Waves**

The standard action for electromagnetic field and the Maxwell equations do not depend directly on the field \( \Phi \). Therefore the influence of this field on the electromagnetic phenomena like motion of electromagnetic waves in vacuum is possible only indirectly – via its influence on the space-time metric. The solar system measurements of the time delay of the electromagnetic pulses give the above used value \( \gamma = 1 \pm .001 \) of this post Newtonian parameter [6]. In MDG this yields the relation \( (1 \pm .001)g(l_{AU}) = 1 \) and gives once more the constraint \( l_{\Phi} \leq 2 \times 10^{10}[m] \). Here

\[
g(l) := 1 + \frac{1}{2}(1 + l/l_{\Phi}) e^{-l/l_{\Phi}}.
\]
3. Perihelion Shift

Helbig’s results [3] applied in MDG give the following formula for the perihelion shift of a planet orbiting around the Sun (with mass $M$

$$\delta \varphi = \frac{k(l_\varphi)}{g(l_\varphi)} \delta \varphi_{GR}.$$ 

Here $l_\varphi$ is the semi-major axis of the orbit of planet and

$$k(l_\varphi) \approx 1 + \frac{1}{12} \left( 4 + \frac{l_\varphi^2}{l_p^2} \right) e^{-l_\varphi/l_p} - \frac{1}{27} e^{-2l_\varphi/l_p}$$

is obtained neglecting its eccentricity. The observed value of perihelion shift of Mercury gives the constraint $l_\varphi \leq 10^9\,m$ (see the article by De Rújula [3]).

The above week restrictions on $l_\varphi$ derived in MDG from gravitational experiments show that in presence of dilaton field $\Phi$ are impossible essential deviations from GR in solar system. Formulae [3] show that observable deviations from Newton law of gravity can not be expected at distances greater then few cm, too.

III. APPLICATION OF MDG IN COSMOLOGY

A. General Equations of MDG for RW Universe

The design of realistic model of the Universe lies beyond the scope of the present article. Here we wish only to outline some general features of the MDG, applied to cosmological problems and to show that in general it is able to correct the well known shortcomings of GR in this domain. We derive the equations of the inverse cosmological problem in MDG and demonstrate some simple solutions, which give indications for unexpected new physics.

Consider RW adiabatic homogeneous isotropic Universe with $ds^2_{RW} = c^2 dt^2 - A^2 dl^2$, $t = Tc\tau$, $A(t) = A(a(\tau))$ and dimensionless $dl^2 = \frac{dt^2}{A^2} + l_\lambda^2, l_\lambda^2 = \frac{\sum}{a^2} + l^2(d\theta^2 + \sin^2\theta) d\varphi^2$ ($k = -1, 0, 1$) in presence of matter with energy-density $\varepsilon = \varepsilon_\varphi(a)/\Phi$ and pressure $p = \varepsilon_\varphi(a)/\Phi$.

From equations (2) and (3) we obtain the following basic dynamical equations of MDG for RW Universe (see for details the Appendix):

\begin{align}
\frac{1}{a^2} \frac{da}{d\tau} + \frac{1}{a^2} \left( \frac{d\tau}{a} \right)^2 + \frac{1}{a^2} = \frac{1}{2} \left( \Phi \frac{d\tau}{a} (\Phi) + \Pi(\Phi) \right), \\
\frac{1}{a^2} \frac{d\tau}{d\tau} + \frac{1}{a^2} \left( \frac{d\tau}{a^2} \right)^2 + \frac{1}{a^2} = \frac{1}{2} \left( \Phi \Pi(\Phi) + \varepsilon(a) \right). \quad (8)
\end{align}

Using Hubble parameter $h(a) = \frac{1}{a^2} (\frac{d\tau}{a})$ ($\tau(a)$ is the inverse function of $a(\tau)$), new variable $\lambda = \ln a$ and denoting by prime the differentiation with respect to $\lambda$ we write down the equations [6] as a second order non-autonomous system for the functions $\Phi(\lambda)$ and $h^2(\lambda)$:

\begin{align}
\frac{1}{2} (h^2)' + 2h^2 + ke^{-2\lambda} = \frac{1}{2} \left( \Phi \frac{d\lambda}{a} (\Phi) + \Pi(\Phi) \right), \\
h^2\Phi' + (h^2 + ke^{-2\lambda}) \Phi = \frac{1}{2} \left( \Phi \Pi(\Phi) + \varepsilon(e^\lambda) \right). \quad (9)
\end{align}

For given function $\Pi(\Phi)$ the solutions of system [6] determine the functions $\Phi(a)$ and $h(a)$ and the time dependence of the scale parameter $a(\tau)$ is given implicitly by the relation

$$\tau(a) = \int_{a_{in}}^{a} \frac{da}{a(h(a)) + \tau_{in}}. \quad (10)$$

Instead, we can exclude the cosmological factor $\Pi(\Phi)$ arriving to linear differential equation of second order for the function $\Phi(\lambda)$:

$$\Phi'' + \left( \frac{k}{h} - 1 \right) \Phi' + 2 \left( \frac{k}{h} - kh^{-2}e^{-2\lambda} \right) \Phi = \frac{1}{2} \varepsilon'. \quad (11)$$

In terms of the function $\psi(a) = \sqrt{|h(a)|/a} \Phi(a)$ it reads:

$$\psi'' + n^2 \psi = \delta. \quad (12)$$
where we have introduced the new functions

\[
-n^2 = \frac{1}{2} h'' - \frac{1}{4} (\frac{\lambda'}{\frac{\lambda}{3}})^2 - \frac{\delta}{2} \frac{\delta'}{\frac{\delta}{3} e^{-2\lambda}},
\]

\[
\delta = \frac{1}{2} \sqrt{a/h^3 h''/\frac{h''}{h'}}.
\]

For the important coefficient \( n \) one can obtain the representation

\[
-n^2 = 3 + \frac{2k}{a^2h} + \frac{s}{a^2} + \frac{q}{a^2},
\]

where \(-q = \frac{1}{a^2h^2} \frac{d^2a}{da^2} = 1 + \frac{h'}{h} \) is decelerating parameter and \( S[a(\tau)] = \frac{d^2a}{da^2} - \frac{3}{2} \left( \frac{d^2a}{da^2} \right)^2 \) is Schwartz derivative of the scale factor \( a(\tau) \).

### B. The Inverse Cosmological Problem

Now we are ready to consider the inverse cosmological problem: to find the corresponding cosmological factor \( \Pi(\Phi) \) and dilatonic potential \( V(\Phi) \) which leads to a given evolution of the Universe, determined by function \( a(\tau) \). It is remarkable that in MDG an unique solution of this problem exist for already three times differentiable function \( a(\tau) \). Indeed, for given \( a(\tau) \) we may construct a function \( h(\lambda) \) and find the general solution \( \Phi(\lambda, C_1, C_2) \) of the linear second order differential equation (11). It depends on two arbitrary constants \( C_{1,2} \) which have to be determined from the additional conditions \( \Pi(\Phi) = 1, \frac{d\Pi}{d\Phi}(\Phi) = \Phi^{-1}, \frac{d^2\Pi}{d\Phi^2}(\Phi) = \frac{4}{3} p^{-2} \Phi^{-2} \). They ensure the self-consistence of the MDG model yielding initial conditions for equation (11) at point \( \lambda \) which is real solution of the algebraic equation

\[
r(\lambda) = -4.
\]

This is a new form of relation (3) and \( r(\lambda) = -6 \left( \frac{1}{2} (h^2)' + 2h^2 + ke^{-2\lambda} \right) \) is the dimensionless scalar curvature \( r = R/\Lambda \). Then:

\[
\bar{\Phi} = \frac{-4\bar{\epsilon} (1 + \frac{3}{4} p^2)}{\left( j_{00}(1 + \frac{3}{4} p^2) + 4p^2 h^2 r' \right)},
\]

\[
\bar{\Phi}'/\bar{\Phi} = \frac{-\frac{3}{4} p^2 r' / (1 + \frac{3}{4} p^2)}.
\]

Here and later on, the bar sign means that the corresponding quantities are calculated at the point \( \lambda \), and \( j_{00} = G_{00}/\Lambda = 3 \left( h^2 + ke^{-2\lambda} \right) \) is the dimensionless 00-component of Einstein tensor. Hence, the values of all “bar” quantities (including \( \bar{\epsilon} \) in action (3)) may be determined from the time evolution \( a(\tau) \) of the Universe via the solution \( \lambda = \ln a \) of the equation (3).

In their turn the quantities \( \bar{\Phi} \) and \( \bar{\Phi}' \) determine the values of the constants \( C_{1,2} \) and an unique solution \( \Phi(\lambda) \) of the equation (11). Indeed, let \( \Phi_1(\lambda) \) and \( \Phi_2(\lambda) \) are fundamental system of solutions of the homogeneous equation associated with non-homogeneous one (11). Then \( \Delta(\lambda) := \Phi_1 \Phi_2 - \Phi_2 \Phi_1 = (\Delta \bar{h}/\bar{a}) e^\lambda h(\lambda) \neq 0 \) and

\[
\Phi(\lambda) = C_1 \Phi_1(\lambda) + C_2 \Phi_2(\lambda) + \Phi_3(\lambda)
\]

is the general solution of equation (11) which depends on two constants

\[
C_1 = (\bar{\Phi}_1 \bar{\Phi} - \bar{\Phi}_2 \bar{\Phi}')/\Delta, \quad C_2 = (\bar{\Phi}_1 \bar{\Phi}' - \bar{\Phi}_2 \bar{\Phi})/\Delta
\]

and on contribution of matter described by the term:

\[
\Phi_3 = \bar{a}/(3\bar{h}\Delta) \left( \Phi_2 \int_\lambda^\lambda \frac{de}{a} / (ah) - \Phi_1 \int_\lambda^\lambda \frac{de}{a} / (ah) \right).
\]

The cosmological factor \( \Pi \) and the potential \( V \) as functions of the variable \( \lambda \) are determined by the equations

\[
\Pi(\lambda) = j_{00} + 3h^2 \Phi'/\Phi - \epsilon/\Phi, \quad V(\lambda) = \frac{4}{3} \int \Phi (\Phi\Pi - \Phi'\Pi) d\lambda
\]

which define the functions \( \Pi(\Phi) \) and \( V(\Phi) \) implicitly, as well.
C. Simple Examples

Let us demonstrate the above procedure for solution of the inverse cosmological problem on several simple examples:

1. Evolution law $a(\tau) = (\omega \tau)^{1/\nu}$, (where $\omega$ is a free parameter) gives

$$h(\lambda) = \omega^{\nu} e^{-\nu \lambda}, \quad q = \nu - 1, \quad -n^2(\lambda) = \frac{1 + 10 \nu + \lambda^2}{4} + 2k e^{2(\nu - 1)\lambda}$$

and the equation

$$\bar{a}^2 + \frac{3(\nu - 2)\omega^2}{2(\nu - 1)} \bar{a}^{2(1 - \nu)} = k$$

for $\bar{a}$. Then:

i) For $\nu \geq 2$ we have real solution $\bar{a}$ only if $k = +1$:

$$\Phi_1(a) = a^{\frac{1 + \mu}{2}} I_{\mu/2} (ba^{\nu - 1}) \quad \text{and} \quad \Phi_2(a) = a^{\frac{1 + \mu}{2}} K_{\mu/2} (ba^{\nu - 1}),$$

$\mu := \sqrt{\frac{1 + 10 \nu^2 + \lambda^2}{2(\nu - 1)}}$ being the order of Bessel functions $I_{\mu}, J_{\mu}, K_{\mu}, Y_{\mu}$ and $b := \sqrt{\frac{2 \nu^2}{(\nu - 1)^2}}$. For GR-like law $a \sim \sqrt{\tau}$ ($\nu = 2$) in MDG we obtain positive value ($k = +1$) for the three-space curvature, $\lambda = 0$, $\mu = \frac{1}{2}$ and Bessel functions are reduced to elementary functions.

ii) When $\nu < 2$ all values $k = -1, 0, +1$ are admissible:

- for $k = +1$ the solutions $\Phi_{1,2}$ are the same as above;
- for $k = 0$ we have

$$\Phi_{1,2} = a^{\frac{1 + \mu}{2}(\nu - 1)};$$

- for $k = -1$ the solutions are:

$$\Phi_1 = a^{\frac{1 + \mu}{2} I_{\mu/2} (ba^{\nu - 1})}, \quad \Phi_2 = a^{\frac{1 + \mu}{2} Y_{\mu/2} (ba^{\nu - 1})}.$$ 

In the special case of linear evolution $a \sim \tau$ ($\nu = 1$) $-n^2(\lambda) = 3 + \frac{2k}{\omega^2}$,

$$\Phi_{1,2}(a) = a^{1 + \sqrt{-n^2}}$$

and the root $\bar{\lambda} = \frac{1}{2} \ln \frac{3}{2}(k + \omega^2)$ is real for all values of $\omega^2 > 0$, if $k = 0, +1$. For $k = -1$ the root $\bar{\lambda}$ will be real if $|\omega| > 1$.

2. Evolution law $a(\tau) = e^{\omega \tau}$ gives

$$h(\lambda) = \omega, \quad q = -1, \quad -n^2(\lambda) = \frac{1}{4} + \frac{2k}{\omega^2} e^{-2\lambda}$$

and the equation $\frac{2}{3} \bar{a}^2 (1 - 3 \omega^2) = k$ whith root $\bar{a} = \sqrt{\frac{3}{2(1 - 3 \omega^2)}}$. Now we have the following solutions:

i) For $|\omega| < \frac{\sqrt{2}}{3}$ and $k = +1$:

$$\Phi_1 = a \cosh \left( \frac{\sqrt{2}}{\omega |\omega|} \right), \quad \Phi_2 = a \sinh \left( \frac{\sqrt{2}}{\omega |\omega|} \right);$$

ii) For $|\omega| > \frac{\sqrt{2}}{3}$ and $k = -1$:

$$\Phi_1 = a \cos \left( \frac{\sqrt{2}}{\omega |\omega|} \right), \quad \Phi_2 = a \sin \left( \frac{\sqrt{2}}{\omega |\omega|} \right).$$

Conditions (14) and (17) exclude exact exponential expansion of spatially flat Universe ($k = 0$).

For inflationary scenario in this case one may use a scale factor $a(\tau) = e^{\omega \tau} + \text{const}$ which turns out to be possible if $\text{const} \neq 0$. 

7
IV. SOME SPECIFIC PROPERTIES AND POSSIBLE FURTHER DEVELOPMENTS OF THE MDG MODEL

In the following concluding remarks we stress some basic properties of MDG models and outline some possible important further developments to be considered elsewhere. First of all we have to stress that actually here we consider the general properties of certain class of such models. Depending on the form of cosmological factor $\Pi(\Phi)$ the concrete representatives of this class of models of extended gravity will have different physical properties and consequences, having in the same time some general features.

As we see, the MDG models of the RW Universe have the following specific properties:

1) In domains where $n > 0$ the dilatonic field $\Phi(a)$ oscillates; if $n < 0$ such oscillations do not exist (see equation (12)). To judge whether the oscillating solutions of MDG are physically acceptable, one has to find their physical consequences and possible manifestations in the real world.

2) The dilatonic field $\Phi$ can change its sign, i.e. in general phase transitions of the Universe from gravity ($\Phi > 0$) to anty-gravity ($\Phi < 0$) and vice-versa are possible in MDG. In general, the hyper-surfaces $\Phi(x) = 0$ in space-time are dangerous singular Cauchy surfaces: as seen from first of equations (2), initial conditions on them do not determine second time-derivatives of metrics and evolution. This problem needs careful investigation, but in the case of the RW Universe considered here it has a simple positive solution: the second of equations (2) and equation (3) instead of the first of equations (2) determine the evolution even for initial condition $\Phi(\tau_0) = 0$.

Depending on the form of cosmological factor $\Pi(\Phi)$ in some of MDG models the above transition from gravity to anty-gravity are possible, but such new phenomena do not exist in models with proper cosmological factor. To decide which model describes better the real Universe we first have to study theoretically the properties of this rich class of models and then to look for observational evidences in favor of some of them.

3) In the spirit of Max principle Newton’s constant depends on the presence of matter: $G \sim 1/\Phi \sim 1/\epsilon$, see equation (13).

4) For simple functions $\epsilon(\tau)$ the cosmological factor $\Pi(\Phi)$ and the dilatonic potential $V(\Phi)$ may show unexpected catastrophic behavior including terms $\sim (\Delta \Phi)^{3/2} (\Delta \Phi = \Phi - \Phi(\lambda^*))$ in vicinity of the critical points $\lambda^*$: $\Pi'(\lambda^*) = 0$ of the projection of analytical curve $\{\Pi(\lambda), \Phi(\lambda), \lambda\}$ on the plane $\{\Pi, \Phi\}$. The conditions $\Pi'(\lambda^*) = 0, \Pi''(\lambda^*) = 0, \Pi''(\lambda^*) \neq 0; \Pi'(\lambda^*) = 0, \Pi''(\lambda^*) \neq 0$ ensure a behavior

\[
\Pi(\Phi) = e^{\sigma(\lambda)} \Delta \Phi \left( \Pi^* + O_{3/2}(\Delta \Phi) + O_2(\Delta \Phi) \right),
\]

where $\sigma(\lambda) := (3(1 + \frac{\lambda}{\lambda_0})h^2 - \epsilon(\Phi))/ (j_{\lambda 0} - \epsilon(\Phi))$.

The potential $V(\Phi)$ has the same critical points $\Phi^*$ as the factor $\Pi(\Phi)$ and in numerical study of above examples shows the following behavior: a local minima around the point $\Phi^*$ and critical points at one, or at both ends of some interval $[\Phi_{left}, \Phi_{right}] \ni \Phi$. An interesting open problem is to find the general conditions on the scale factor $\epsilon(\tau)$ which make impossible these catastrophes, yielding an everywhere analytical cosmological factor $\Pi(\Phi)$ and potential $V(\Phi)$ with desired properties. The existence of such class of scale factors $\epsilon(\tau)$ is clear: in the direct cosmological problem we can choose a desired functions $\Pi(\Phi)$ and $V(\Phi)$ and then we will obtain the corresponding function $\epsilon(\tau)$.

5) Clearly one can construct MDG model of Universe without the typical for GR initial singularities: $\epsilon(\tau_0) = 0$ and with any desired kind of inflation.

6) Because dilaton field $\Phi$ is quite massive, in it will be stored significant amount of energy. In RW model this energy has a homogeneous and isotropic distribution in the Universe. An interesting open question is: may the field $\Phi$ play the role of a dark matter, or of a dark energy in the Universe?

7) A very important problem is to reconstruct the cosmological factor $\Pi(\Phi)$ of the real Universe using a proper experimental data and the astrophysical observations (see 12, where such problem was considered first in the framework of more complicated models with two unknown functions).

8) As a further steeps for a justification of the parameters of the MDG model and its relevance to the physical reality one should study the possible consequences of the existence of the dilaton $\Phi$ for binary pulsars (which give precise test of the extended theories of gravity) and for the curves of the rotation of galaxies which are not explained by Newtonian theory of gravity and GR.

It’s clear that MDG offers new curious possibilities which deserve further careful investigation.

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V. APPENDIX

Here we give a more extended explanation of the derivation of the basic dynamical equations (8) of the MDG for RW Universe. The first of these equations is a direct representation of the equation (19) written for the case of RW metric. The second one represents the (0,0) component of the first equation of the system (2). (Note that in the case of RW metric one obtains the same equations for i = 1, 2, 3.) For the RW Universe this equation for the field Φ = Φ(τ) acquires the form:

\[ \ddot{\Phi} + 3\frac{\dot{a}\dot{\Phi}}{a} + \frac{d\Phi}{d\tau} = \frac{1}{3}(\epsilon - 3p_c). \]  

(18)

From the conservation law \( \nabla_a T^a_\beta = 0 \), applied for the RW metric, one obtains the same relation as in GR: \( \frac{d}{da}(a^2\epsilon) = -3p_c a^3 \). It makes possible to exclude the pressure \( p_c \) from equation (18). Now, just as in GR, it becomes possible to prove that the equation (18) follows from the basic system of dynamical equations (8): differentiating the second one with respect to the time \( \tau \), using the first one and once more the second one, after some algebra we derive the equation (18). This means that any solution of the system (8) satisfies the equation (18) and there is no need to take it into account, solving dynamical problems for the RW Universe.

One has to stress that the second order dynamical equations for the RW Universe in MDG: the first of the equations (8) and the equation (18), are equivalent to the Euler-Lagrange equations for action

\[ \int \alpha_{nw} \left( -\Phi a\dot{a}^2 - a^2\dot{\Phi} + ka\Phi - \frac{1}{3}a^3(\Phi \Pi(\Phi) + \epsilon(a)) \right) d\tau. \]

(19)

It is obtained from the original action (11) (per unit volume) substituting in it the RW metric and dropping out the corresponding common factor with dimension of action (i.e. the cosmological unit for action \( A_c \), introduced in Sec.1). The second of the equations (8), being a first order differential equation, represents the corresponding energy integral of this Euler-Lagrange system:

\[ \epsilon_{total} = \dot{a}p_a + \dot{\Phi}p_\Phi - \left( -\Phi a\dot{a}^2 - a^2\dot{\Phi} + ka\Phi - \frac{1}{3}a^3(\Phi \Pi(\Phi) + \epsilon(a)) \right) = -a^3 \left( \dot{\Phi} \left( \frac{da}{d\tau} \right)^2 + \frac{d\Phi}{d\tau} \right) - \dot{\Phi} \left( \Phi \Pi(\Phi) + \epsilon(a) \right) = 0. \]

(20)

Just as in GR, it must be zero, because of the invariance of MDG under general coordinate transformations. In the equation (21) \( p_a = -2\Phi a\dot{a} - a^2\dot{\Phi} \) and \( p_\Phi = -a^2\dot{\Phi} \) are the corresponding (dimensionless) canonical momenta for the action (14).

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