Quantum mechanics in general quantum systems (I): 
exact solution

An Min Wang
Quantum Theory Group, Department of Modern Physics,
University of Science and Technology of China, Hefei, 230026, P.R.China

After proving and using a mathematical identity, we first deduce an expansion formula of operator binomials power. Then, starting from our idea of combining the Feynman path integral spirit and the Dyson series kernel, we find an explicit and general form of time evolution operator that is a $c$-number function and a power series of perturbation including all order approximations in the unperturbed Hamiltonian representation. Based on it, we obtain an exact solution of the Schrödinger equation in general quantum systems independent of time. In special, we write down the concrete form of the exact solution of the Schrödinger equation when the solvable part (unperturbed part) of this Hamiltonian is simply taken as the kinetic energy term. Comparison of our exact solution with the existed perturbation theory makes some features and significance of our exact solution clear. The conclusions expressly indicate that our exact solution is obviously consistent with the usual time-independent perturbation theory at any order approximation, it explicitly calculates out the expanding coefficients of the unperturbed state in the non-perturbation method, and it fully solves the recurrence equation of the expansion coefficients of final state in the unperturbed Hamiltonian representation from a view of time-dependent perturbation theory. At the same time, the exact solution of the von Neumann equation is also given. Our results can be thought of as theoretical developments of quantum dynamics, and are helpful for understanding the dynamical behavior and related subjects of general quantum systems in both theory and application. Our exact solution, together with its sequence studies on perturbation theory [An Min Wang, quant-ph/0611217] and open system dynamics [An Min Wang, quant-ph/0601051] can be used to establish the foundation of theoretical formulism of quantum mechanics in general quantum systems. Further applications of our exact solution to quantum theory can be expected.

PACS numbers: 03.65.-w, 03.65.Ca

I. INTRODUCTION

One of the most important tasks of physics is to obtain the time evolution law and form of a physical system, that is, so-called dynamical equation and its solution. This directly relates to physics foundation. In a quantum system, the dynamical equation is the Schrödinger equation [1 2] for a pure state $|\Psi(t)\rangle$ or the von Neumann equation for a mixed state $\rho(t)$ [3], that is

$$
-\frac{i}{\hbar}\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle,
$$

$$
\dot{\rho}(t) = -i[H, \rho(t)].
$$

where $H$ is the Hamiltonian of a quantum system. However, only a few idealized quantum systems are exactly solvable at present by using the existed theory and methods. Therefore, it is extremely interesting and essential important to find the exact solutions to the Schrödinger equation and the von Neumann equation for Hamiltonians of even moderate complexity.

If the Hamiltonian is assumed independent of time, an arbitrary initial state at $t = 0$ is denoted by $|\Psi(0)\rangle$ for a pure state or $\rho(0)$ for a mixed state, then the final state at a given time $t$ will become

$$
|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle,
$$

$$
\rho(t) = e^{-iHt}\rho(0)e^{iHt},
$$

where $e^{-iHt}$ (set $\hbar = 1$ for simplicity) is called as the time evolution operator. The above equations indicate that the final state can be rewritten as the form that the time evolution operator acts on an initial state. In other
words, they are, respectively, a formal solution of the Schrödinger equation for a pure state and one of the von Neumann equation for a mixed state. However, the formal solution is not the practical solution that can be applied to concrete problems and actual calculations. Ones are interested in the practical solution because it can really produce the physical results and conclusions in the quantum theory. Hence, to solve the Schrödinger equation and the von Neumann equation refers to find the practical solution in most cases. It is clear that this task is equivalently to seek for the explicit expression of the time evolution operator.

Historically, the most famous example to study the expression of the time evolution operator is Feynman path integral formalism. In fact, Dyson series is also an important example. From our point of view, only if the expression of the time evolution operator is a c-number function, can we clearly express the practical solution of the final state using (3) and (4), respectively, for the pure state and the mixed state. Moreover, we realize that for a general quantum system without the exact solution in the usual theory, the expression of the final state in c-number function form can be generally expressed by an infinite series. Hence, only if the expressing series of the final state is a power series of perturbation, can we inherit the advantages of perturbation theory. As is well-known, to find the c-number function form of the time evolution operator is a successful linchpin of Feynman path integral formalism, and to expand the time evolution operator as a power series of perturbation is a powerful headstream of Dyson series (in the interaction picture). Consequently, our physical idea in this paper comes from the combination of Feynman path integral spirit and Dyson series kernel. Starting from this combined idea, we would like to find an explicit and general expression of the time evolution operator that it not only is a c-number function but also is a power series expansion of perturbation including all order approximations in terms of our own methods.

To express the time evolution operator as a c-number function needs a representation of Hilbert space, and further to expand it as a power series of perturbation needs to separate its unperturbed part and perturbing part. Actually, the above problems can be studied by using the similar methods to the usual perturbation theory. In the usual perturbation theory, the key idea to research the time evolution of general quantum systems is to split the system Hamiltonian into two parts, that is

\[ H = H_0 + H_1, \]

where the eigenvalue problem of so-called unperturbed Hamiltonian \( H_0 \) is solvable, and so-called perturbing Hamiltonian \( H_1 \) is the rest part of the Hamiltonian. In other words, this splitting is chosen in such a manner that the solutions of \( H_0 \) are known as

\[ H_0 |\Phi\rangle = E_\gamma |\Phi\rangle, \]

where \( |\Phi\rangle \) is the eigenvector of \( H_0 \) and \( E_\gamma \) is the corresponding eigenvalue. Whole \( |\Phi\rangle \), in which \( \gamma \) takes over all possible values, form a representation of the unperturbed Hamiltonian. It must be pointed out that in the perturbation theory, the above Hamiltonian split principle with the best solvability is not unique in more general cases. It is possible to need to consider such an additional condition that the selected split can remove degeneracies as complete as possible in order to suit to apply the perturbation theory, or to let our improved scheme of perturbation theory work well. If the remained degeneracies are allowed, it requires that the off-diagonal element of the perturbing Hamiltonian matrix between arbitrary two degenerate levels are always vanishing. As an example, it has been discussed in our sequence study. Of course, if the cut-off approximation of perturbation is necessary in our improved scheme of perturbation theory, the off-diagonal elements of \( H_1 \) should be small enough compared with the diagonal elements of \( H = H_0 + H_1 \) in the \( H_0 \) representation.

As soon as we find the explicit and general expression of the time evolution operator that is both a c-number function and a power series of perturbation including all order approximations, we definitely can obtain the exact solutions of the Schrödinger equation and von Neumann equation, and so we can solve a lot of problems in quantum theory, for example, entanglement dynamics and open system dynamics. Of course, our solutions are called “exact ones” in the sense including all order approximations of perturbation. However, our final purpose is not limit to these. We, starting from our exact solutions, try to finally establish the foundations of theoretical formalism of quantum mechanics in general quantum systems. Our serial studies on perturbation theory and open system dynamics indicate that we have partially arrived at our purpose.

In this paper, the key and central task is, in the unperturbed Hamiltonian representation, to find an explicit expression of the time evolution operator \( e^{-iHt} \) that is a c-number function and a power series of perturbation including all order approximations according to our idea combining the Feynman path integral spirit and Dyson series kernel. Our method is first to derive out an expansion formula of operator binomials power and then apply it to the Taylor’s expansion of time evolution operator \( e^{-iHt} = e^{-it(H_0+H_1)t} \). Moreover, after proving and using our identity, we derive out the explicit and general form of representation matrix of \( e^{-iHt} \) in the representation of the unperturbed Hamiltonian \( H_0 \). Consequently, we obtain the exact solutions of the Schrödinger equation and the von Neumann equation in general quantum systems independent of time, in particular, a concrete form of the Schrödinger
equation solution when the solvable part of Hamiltonian is simply taken as the kinetic energy term in order to account for the generality and universality of our exact solutions. Furthermore, by comparing our exact solution of the Schrödinger equation with the usual perturbation theory, we reveal their relations and show what is more and what is different. The conclusions clearly indicate that our exact solution is obviously consistent with the usual time-independent perturbation theory at any order approximation, but also in our exact solution we explicitly calculate out the expanding coefficients of unperturbed state in Lippmann-Schwinger equation for the non-perturbative method, and/or fully solves the recurrence equation of the expansion coefficients of final state in the unperturbed Hamiltonian representation from a view of time-dependent perturbation theory. Based on the above all of reasons and results, our exact solutions can be thought of as the theoretical developments of quantum dynamics in general quantum systems, and they are helpful for understanding the dynamical behavior and related subjects of quantum systems in both theory and application. From our point of view, our exact solutions and their sequence studies on perturbation theory and open system dynamics can be used to establish the foundation of quantum mechanics of general quantum systems. Specially, we think that the features and advantages of our exact solutions can not be fully revealed only by the improved scheme of perturbation theory and open system dynamics. We would like to study the more applications to the formulation of quantum mechanics in general quantum systems in the near future.

This paper is organized as the following: in this section, we give an introduction; in Sec. III we first propose the expansion formula of power of operator binomials; in Sec. III we derive out an explicit and general expression of time evolution operator after proving and using our identity; in Sec. IV we obtain the exact solutions of the Schrödinger equation and the von Neumann equation, and then present a concrete example when the solvable part (unperturbed part) of Hamiltonian is taken as the kinetic energy term; in Sec. V by comparison of our exact solution of the Schrödinger equation with the usual perturbation theory, we prove their consistency and their relations, and explain what is more and what is different.; in Sec. VI we summarize our conclusions and give some discussions. Finally, we write an appendix where the proof of our identity is presented.

II. EXPANSION FORMULA OF OPERATOR BINOMIALS POWER

In order to obtain the explicitly exact solutions of the Schrödinger equation and von Neumann equation in general quantum systems independent of time, we need to deduce the expression of the time evolution operator. According to our physical idea, we should first separate the time evolution operator into two parts, respectively, with perturbation and without perturbation. After to split the Hamiltonian, this problem changes to the derivation of expansion formula of operator binomials power. Without loss of generality, we are always able to write the power of operator binomials as two parts

\[(A + B)^n = A^n + f^n(A, B).\]  

(7)

where \(A\) and \(B\) are two operators and do not commute with each other in general. If \(B\) is taken as a perturbation, the above decomposition just splits the operator binomials power into two parts, respectively, with perturbation and without perturbation. Now, the key matter is how to obtain the form of \(f^n(A, B)\).

It is clear that \(f^n(A, B)\) is a polynomial including at least first power of \(B\) and at most \(n\)th power of \(B\) in every term. Thus, a general term with \(l\)th power of \(B\) has the form \((\prod_{i=1}^{l} A^{k_i}B) A^{n-l-\sum_{i=1}^{l} k_i}\). From the symmetry of power of binomials we conclude that every \(k_i\) take the values from 0 to \((n-l)\), but it must keep \(n-l-\sum_{i=1}^{l} k_i \geq 0\). So we have

\[f^n(A, B) = \sum_{l=1}^{n} \sum_{\sum_{i=1}^{l} k_i + l \leq n} \left(\prod_{i=1}^{l} A^{k_i}B\right) A^{n-l-\sum_{i=1}^{l} k_i}\]

(8)

\[= \sum_{l=1}^{n} \sum_{\sum_{i=1}^{l} k_i = 0} \left(\prod_{i=1}^{l} A^{k_i}B\right) A^{n-l-\sum_{i=1}^{l} k_i} \theta(n - l - \sum_{i=1}^{l} k_i),\]

(9)

where \(\theta(x)\) is a step function, that is, \(\theta(x) = 1\) if \(x \geq 0\), and \(\theta(x) = 0\) if \(x < 0\). Obviously based on above definition, we easily verify

\[f^1(A, B) = B, \quad f^2(A, B) = AB + B(A + B).\]

(10)

They imply that the expression is correct for \(n = 1, 2\).
Now we use the mathematical induction to prove the expression $[8]$ of $f^n(A, B)$, that is, let us assume that it is valid for a given $n$, and then prove that it is also valid for $n + 1$. Denoting $F^{n+1}(A, B)$ with the form of expression $[9]$ where $n$ is replaced by $n + 1$, and we extract the part of $l = 1$ in its finite summation for $l$

$$F^{n+1}(A, B) = \sum_{k_1=0}^{n} A^{k_1} B A^{n-k_1} + \sum_{l=2}^{n+1} \sum_{k_1=0}^{(n+1)-l} \left( \prod_{i=1}^{l} A^{k_i} B \right) A^{(n+1)-l-\sum_{i=1}^{l} k_i} \theta \left( (n+1) - l - \sum_{i=1}^{l} k_i \right). \tag{11}$$

We extract the terms $k_1 = 0$ in the first and second summations, and again replace $k_1$ by $k_1 - 1$ in the summation (the summations for $k_1$ from 1 to $(n+1) - l$ change as one from 0 to $n - l$), the result is

$$F^{n+1}(A, B) = B A^n + \sum_{k_1=0}^{n} A^{k_1} B A^{n-1-k_1} + B \sum_{l=2}^{n+1} \sum_{k_1=0}^{(n+1)-l} \left( \prod_{i=1}^{l} A^{k_i} B \right) A^{(n+1)-l-\sum_{i=1}^{l} k_i} \theta \left( (n+1) - l - \sum_{i=1}^{l} k_i \right)
+ \sum_{l=2}^{n+1} \sum_{k_1=0}^{(n+1)-l} \sum_{k_2, \ldots, k_{l-1}} \left( \prod_{i=1}^{l} A^{k_i} B \right) A^{(n+1)-l-\sum_{i=1}^{l} k_i} \theta \left( (n+1) - l - \sum_{i=1}^{l} k_i \right), \tag{12}$$

furthermore

$$F^{n+1}(A, B) = B A^n + \sum_{k_1=0}^{n-1} A^{k_1} B A^{n-1-k_1} + B \sum_{l=2}^{n+1} \sum_{k_1=0}^{(n+1)-l} \left( \prod_{i=1}^{l} A^{k_i} B \right) A^{(n+1)-l-\sum_{i=1}^{l} k_i} \theta \left( (n+1) - l - \sum_{i=1}^{l} k_i \right)
+ \sum_{l=2}^{n+1} \sum_{k_1=0}^{(n+1)-l} \sum_{k_2, \ldots, k_{l-1}} \left( \prod_{i=1}^{l} A^{k_i} B \right) A^{n-l-\sum_{i=1}^{l} k_i} \theta \left( n - l - \sum_{i=1}^{l} k_i \right). \tag{13}$$

Considering the third term in above expression, we change the dummy index $\{k_2, k_3, \ldots, k_l\}$ into $\{k_1, k_2, \ldots, k_{l-1}\}$, rewrite $(n+1) - l$ as $n - (l - 1)$, and finally replace $l$ by $l - 1$ in the summation (the summation for $l$ from 2 to $n + 1$ changes as one from 1 to $n$), we obtain

$$B \sum_{l=2}^{n+1} \sum_{k_2, \ldots, k_{l-1}=0}^{n-l-1} \left( \prod_{i=1}^{l-1} A^{k_i} B \right) A^{n-(l-1)-\sum_{i=1}^{l-1} k_i} \theta \left( n - (l-1) - \sum_{i=1}^{l-1} k_i \right)
= B \sum_{l=2}^{n+1} \sum_{k_1=0}^{n-l-1} \left( \prod_{i=1}^{l-1} A^{k_i} B \right) A^{n-(l-1)-\sum_{i=1}^{l-1} k_i} \theta \left( n - (l-1) - \sum_{i=1}^{l-1} k_i \right)
= B \sum_{l=2}^{n+1} \sum_{k_1=0}^{n-l-1} \left( \prod_{i=1}^{l} A^{k_i} B \right) A^{n-l-\sum_{i=1}^{l} k_i} \theta \left( n - l - \sum_{i=1}^{l} k_i \right)
= B f^n(A, B), \tag{14}$$

where we have used the expression $[9]$ of $f^n(A, B)$. Because of the step function $\theta \left( n - l - \sum_{i=1}^{l} k_i \right)$ in the fourth term of the expression $[13]$, the upper bound of summation for $l$ is abated to $n$, and the upper bound of summation for $k_2, k_3, \ldots, k_l$ is abated to $n - l$. Then, merging the fourth term and the second term in Eq. $[13]$ gives

$$A \sum_{k_1=0}^{n-1} A^{k_1} B A^{n-1-k_1} + A \sum_{l=2}^{n+1} \sum_{k_1=0}^{n-l-1} \sum_{k_2, \ldots, k_{l-1}} \left( \prod_{i=1}^{l} A^{k_i} B \right) A^{n-l-\sum_{i=1}^{l} k_i} \theta \left( n - l - \sum_{i=1}^{l} k_i \right)
= A \sum_{k_1=0}^{n-1} A^{k_1} B A^{n-1-k_1} + A \sum_{l=2}^{n} \sum_{k_1=0}^{n-l} \sum_{k_2, \ldots, k_{l-1}=0} \left( \prod_{i=1}^{l} A^{k_i} B \right) A^{n-l-\sum_{i=1}^{l} k_i} \theta \left( n - l - \sum_{i=1}^{l} k_i \right)
= A \sum_{l=1}^{n} \sum_{k_1=0}^{n-l} \sum_{k_2, \ldots, k_{l-1}=0} \left( \prod_{i=1}^{l} A^{k_i} B \right) A^{n-l-\sum_{i=1}^{l} k_i} \theta \left( n - l - \sum_{i=1}^{l} k_i \right)
= A f^n(A, B), \tag{15}$$
where we have used again the expression (11) of $f^n(A, B)$.

Substituting (14) and (15) into (13), immediately leads to the following result

$$F^{n+1} = BA^n + (A + B)f^n(A, B).$$  \hspace{1cm} (16)

Note that

$$(A + B)^{n+1} = (A + B)(A + B)^n$$

$$= A^{n+1} + BA^n + (A + B)f^n(A, B),$$  \hspace{1cm} (17)

we have the relation

$$f^{n+1}(A, B) = BA^n + (A + B)f^n(A, B).$$  \hspace{1cm} (18)

Therefore, in terms of Eqs. (16) and (18) we have finished our proof that $f^n(A, B)$ has the expression (8) or (9) for any $n$.

**III. EXPRESSION OF THE TIME EVOLUTION OPERATOR**

Now we investigate the expression of the time evolution operator by means of above expansion formula of power of operator binomials, that is, we write

$$e^{-iHt} = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!}(H_0 + H_1)^n = e^{-iH_0t} + \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} f^n(H_0, H_1).$$  \hspace{1cm} (19)

In above equation, inserting the complete relation $\sum_\gamma |\Phi^{\gamma}\rangle\langle\Phi^{\gamma}| = 1$ before every $H_0^{k_i}$, and using the eigen equation of $H_0$, it is easy to see that

$$f^n(H_0, H_1) = \sum_{l=1}^{n} \sum_{\gamma_1, \cdots, \gamma_{l+1}} \sum_{k_1, \cdots, k_{l}=0}^{n-l} \left[ \prod_{i=1}^{l} E_{\gamma_i}^{k_i} \right] E_{\gamma_{l+1}}^{-\sum_{i=1}^{l} k_i - l} \left[ \prod_{i=1}^{l} H_{1}^{\gamma_i\gamma_{i+1}} \right] |\Phi^{\gamma_l}\rangle \langle\Phi^{\gamma_{l+1}}|$$

$$= \sum_{l=1}^{n} \sum_{\gamma_1, \cdots, \gamma_{l+1}} C_l^n(E[\gamma, l]) \left[ \prod_{i=1}^{l} H_{1}^{\gamma_i\gamma_{i+1}} \right] |\Phi^{\gamma_l}\rangle \langle\Phi^{\gamma_{l+1}}|,$$  \hspace{1cm} (20)

where $H_{1}^{\gamma\gamma+1} = (\Phi^{\gamma}|H_1|\Phi^{\gamma+1})$, $E[\gamma, l]$ is a vector with $l + 1$ components denoted by

$$E[\gamma, l] = \{E_{\gamma_1}, E_{\gamma_2}, \cdots, E_{\gamma_l}, E_{\gamma_{l+1}}\}$$  \hspace{1cm} (21)

and we introduce the definition of $C_l^n(E[\gamma, l])$ ($l \geq 1$) as the following

$$C_l^n(E[\gamma, l]) = \sum_{\sum_{i=1}^{l} k_i \leq n} \left[ \prod_{i=1}^{l} E_{\gamma_i}^{k_i} \right] E_{\gamma_{l+1}}^{-\sum_{i=1}^{l} k_i - l}.$$  \hspace{1cm} (22)

The expression (20) and the expression of the time evolution operator depend on it has incarnated our physical idea, that is, it is both a c-number function and a power series of perturbation. Our further task is to find the explicit expression of $C_l^n(E[\gamma, l])$.

Before to find $C_l^n(E[\gamma, l])$, let us mention the cases when energy level degenerations happen, that is, some same values appear in the eigen spectrum $\mathcal{E} = \{E_\gamma | \gamma = 1, 2, \cdots \}$, the eigen spectrum is re-denoted by $\mathcal{E} = \{E_{\gamma a_a} | \gamma = 1, 2, \cdots, a_a = 1, \cdots, m_\gamma \}$ and the numbers of original $\gamma$ is different from the numbers of new $\gamma$. Here, $m_\gamma$ is called degeneracy for a given $\gamma$, which means the number of the same eigenvalues ($m_\gamma$-fold degeneracy). In special, when $m_\gamma = 1$, the energy level $E_{\gamma a_a}$ is not degenerate. It is clear that the above expression (20) has only a little change:

$$f^n(H_0, H_1) = \sum_{l=1}^{n} \sum_{\gamma_1, \cdots, \gamma_{l+1}} \sum_{\alpha_{\gamma_1}, \cdots, \alpha_{\gamma_{l+1}}} C_l^n(E[\gamma, l]) \left[ \prod_{i=1}^{l} H_{1}^{\gamma_i\alpha_{\gamma_i}\gamma_{i+1}\alpha_{\gamma_{i+1}}} \right] |\Phi^{\gamma_1\alpha_{\gamma_1}}\rangle \langle\Phi^{\gamma_{l+1}\alpha_{\gamma_{l+1}}}|,$$  \hspace{1cm} (23)
which has no obvious influence on our following derivation and proof. For simplicity, we do not consider the degenerate cases in this section since no new idea and skill will be needed here.

In order to derive out an explicit and useful expression of \( C^n_l(E[\gamma, l]) \), we first change the dummy index \( k_l \rightarrow n - l - \sum_{i=1}^{l} k_i \) in Eq. (9). Note that at the same time in spite of \( \theta(n - l - \sum_{i=1}^{l} k_i) \) changes as \( \theta(k_l) \), but a hiding factor \( \theta(k_l) \) becomes \( \theta(n - l - \sum_{i=1}^{l} k_i) \). Thus, we can rewrite

\[
f^n(A, B) = \sum_{l=1}^{n} \sum_{k_1, \ldots, k_l=0}^{n-l} \left( \prod_{i=1}^{l-1} A^{k_i} B \right) A^{n-l-\sum_{i=1}^{l} k_i} BA^k \theta \left( n - l - \sum_{i=1}^{l} k_i \right).
\]

(24)

In fact, this new expression of \( f^n(A, B) \) is a result of the symmetry of power of binomials for its every factor. Similarly, in terms of the above method to obtain the definition of \( C^n_l(E[\gamma, l]) \) (22), we have its new definition (where \( k_l \) is replaced by \( k \))

\[
C^n_l(E[\gamma, l]) = \sum_{k=0}^{n-l} \left\{ \sum_{k_1, \ldots, k_{l-1}=0}^{n-l} \left[ \prod_{i=1}^{l-1} E_{\gamma_1}^{k_i} \right] E_{\gamma_1}^{n-l-\sum_{i=1}^{l-1} k_i} \theta \left( (n - k) - l - \sum_{i=1}^{l-1} k_i \right) \right\} E_{\gamma_{l+1}}^{k}.
\]

(25)

Because \( \theta \left( (n - k) - l - \sum_{i=1}^{l-1} k_i \right) \) abates the upper bound of summation for \( k_i (i = 1, 2, \ldots, l) \) from \( (n - l) \) to \( (n - k - 1) - (l - 1) \), we obtain the recurrence equation

\[
C^n_l(E[\gamma, l]) = \sum_{k=0}^{n-l} \left\{ \sum_{k_1, \ldots, k_{l-1}=0}^{n-l} \left[ \prod_{i=1}^{l-1} E_{\gamma_1}^{k_i} \right] \frac{E_{\gamma_1}^{(n-k-1)-(l-1)-\sum_{i=1}^{l-1} k_i} - \theta \left( (n - k - 1) - (l - 1) - \sum_{i=1}^{l-1} k_i \right)}{E_{\gamma_{l+1}}} \right\} E_{\gamma_{l+1}}^{k}.
\]

(26)

In particular, when \( l = 1 \), from the definition of \( C^n_1(E(\gamma, 1)) \) it follows that

\[
C^n_1(E(\gamma, 1)) = \sum_{k_1=0}^{n-1} E_{\gamma_1}^{k_1} E_{\gamma_2}^{n-1-k_1} = E_{\gamma_1}^{n-1} \sum_{k_1=0}^{n-1} \left( \frac{E_{\gamma_2}}{E_{\gamma_2}} \right)^{k_1}.
\]

(27)

By means of the summation formula of a geometric series, we find that

\[
C^n_1(E(\gamma, 1)) = \frac{E_{\gamma_1}^{n}}{E_{\gamma_1} - E_{\gamma_2}} - \frac{E_{\gamma_2}^{n}}{E_{\gamma_2} - E_{\gamma_3}}.
\]

(28)

Based on the recurrence equation (26), we have

\[
C^n_2(E(\gamma, 2)) = \sum_{k=0}^{n-2} \left( \frac{E_{\gamma_1}^{n-k-1}}{E_{\gamma_1} - E_{\gamma_2}} - \frac{E_{\gamma_2}^{n-k-1}}{E_{\gamma_2} - E_{\gamma_3}} \right) E_{\gamma_3}^{k} = \sum_{k=0}^{n-2} \left( \frac{E_{\gamma_1}^{n-k-1}}{E_{\gamma_1} - E_{\gamma_2}} \right)^{k} \sum_{k=0}^{n-2} \left( \frac{E_{\gamma_2}}{E_{\gamma_1}} \right)^{k}
\]

(29)
In the above calculations, the last step is important. In fact, in order to obtain the concrete expression of \( C_l^n \) (\( l \) and \( n \) are both positive integers), we need our identity

\[
\sum_{i=1}^{l+1} (-1)^{i-1} \frac{E^K_{i}}{d_i(E[\gamma, l])} = \begin{cases} 
0 & \text{(If } 0 \leq K < l) \\
1 & \text{(If } K = l) 
\end{cases}.
\]  

(30)

It is proved in Appendix A in detail. The denominators \( d_i(E[\gamma, l]) \) in above identity are defined by

\[
d_1(E[\gamma, l]) = \prod_{i=1}^{l} (E_{\gamma_i} - E_{\gamma_{i+1}}),
\]

(31)

\[
d_i(E[\gamma, l]) = \prod_{j=1}^{i-1} (E_{\gamma_j} - E_{\gamma_i}) \prod_{k=i+1}^{l} (E_{\gamma_i} - E_{\gamma_k}),
\]

(32)

\[
d_{l+1}(E[\gamma, l]) = \prod_{i=1}^{l} (E_{\gamma_i} - E_{\gamma_{l+1}}),
\]

(33)

where \( 2 \leq i \leq l \). Then, using the recurrence equation (30) and our identity (33), we obtain

\[
C_l^n(E[\gamma, l]) = \sum_{i=1}^{l+1} (-1)^{i-1} \frac{E^n_{i}}{d_i(E[\gamma, l])}.
\]

(34)

Here, the mathematical induction shows its power again. If the expression (34) is correct for a given \( n \), for example \( n = 1, 2 \), then for \( n + 1 \) from the recurrence equation (30), it follows that

\[
C_l^{n+1}(E[\gamma, l]) = \sum_{k=0}^{n+1-l} \left( \sum_{i=1}^{l} (-1)^{i-1} \frac{E^{n-k}_{i}}{d_i(E[\gamma, l-1])} \right) E^k_{\gamma_{l+1}}
\]

\[
= \sum_{i=1}^{l} \frac{(-1)^{i-1}E^n_{i}}{d_i(E[\gamma, l-1])} \sum_{k=0}^{n+1-l} \left( \frac{E^{n-k}_{\gamma_{l+1}}}{E^l_{\gamma_i}} \right)^k
\]

\[
= \sum_{i=1}^{l} \frac{(-1)^{i-1}(E^{n+1}_{\gamma_{l+1}} - E^{l+1}_{\gamma_{l+2}} - E^{n+2}_{\gamma_{l+1}})}{d_i(E[\gamma, l-1])(E_{\gamma_i} - E_{\gamma_{l+1}})}.
\]

(35)

Because that \( d_i(E[\gamma, l-1])(E_{\gamma_i} - E_{\gamma_{l+1}}) = d_i(E[\gamma, l]) \) (\( i \leq l + 1 \)) and our identity (30), it is easy to see

\[
\sum_{i=1}^{l} \frac{(-1)^{i-1}E^{n-k}_{i}}{d_i(E[\gamma, l])} = (-1)^{l-1} \frac{E^{l-1}_{\gamma_{l+1}}}{d_{l+1}(E[\gamma, l])}.
\]

(36)

Substitute it into Eq. (35) yields

\[
C_l^{n+1}(E[\gamma, l]) = \sum_{i=1}^{l+1} (-1)^{i-1} \frac{E^n_{i}}{d_i(E[\gamma, l])}.
\]

(37)

That is, we have proved that the expression (34) of \( C_l^n(E[\gamma, l]) \) is valid for any \( n \).

It must be emphasized that since the summation is over all of values of the energy eigenvalues, there are apparent divergences in the expression of \( C_l^n(E[\gamma, l]) \). Here, “apparent” refers to an untrue thing, that is, the apparent divergences are not real singularities and they can be eliminated by mathematical and/or physical methods. Moreover, the same problem appears if with degeneracy. Therefore, we need to understand above expressions in the sense of limitations. For instance, for \( C_l^n(E[\gamma, 1]) = (E^n_{\gamma_1} - E^n_{\gamma_2}) / (E_{\gamma_1} - E_{\gamma_2}) \), we have the expression

\[
lim_{E_{\gamma_2} \to E_{\gamma_1}} C_l^n(E[\gamma, 1]) = (\delta_{\gamma_1} - \Theta(\gamma_2 - \gamma_1))\delta_{E_{\gamma_2}}E_{\gamma_1}^{-1}E_{\gamma_2},
\]

where the step function \( \Theta(x) = 1 \), if \( x > 0 \), and \( \Theta(x) = 0 \), if \( x \leq 0 \). In other words, the apparent divergences here are not real divergences and can be eliminated by finding the correct limitations. There is no theoretical problem when we formally keep the apparent divergences in the above expression. Actually, in our recent work [6], we successfully eliminate all the apparent divergences.
Because of our identity (30), the summation to \( l \) in Eq. (20) can be extended to \( \infty \). Thus, the expression of the time evolution operator is changed to a summation according to the order (or power) of the perturbing Hamiltonian \( H_1 \) as follows

\[
\langle \Phi | e^{-iHt} | \Phi' \rangle = e^{-iE_{\gamma'}t} \delta_{\gamma\gamma'} + \sum_{l=1}^{\infty} \sum_{\gamma_1, \cdots, \gamma_{l+1}} \left[ \sum_{i=1}^{l+1} (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E_{\gamma_i}, l)} \right] \prod_{j=1}^{l} H_1^{\gamma_j \gamma_{j+1}} \delta_{\gamma_{l+1} \gamma'}.
\]  

(38)

When there is degeneracy, it becomes

\[
\langle \Phi | e^{-iHt} | \Phi' \rangle = e^{-iE_{\gamma'}t} \delta_{\gamma\gamma'} + \sum_{l=1}^{\infty} \sum_{\gamma_1, \cdots, \gamma_{l+1}} a_{\gamma_1, \cdots, \gamma_{l+1}} \sum_{i=1}^{l+1} (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E_{\gamma_i}, l)} \prod_{j=1}^{l} H_1^{\gamma_j \gamma_{j+1}} \delta_{\gamma_{l+1} \gamma'}.
\]  

(39)

that is,

\[
e^{-iH_{\text{tot}}t} = \sum_{l=0}^{\infty} A_l(t) = \sum_{l=0}^{\infty} \sum_{\gamma, \gamma'} A_l^{\gamma \gamma'}(t) |\Phi_{\gamma'}\rangle \langle \Phi_{\gamma'}|,
\]  

(40)

where

\[
A_0^{\gamma \gamma'}(t) = e^{-iE_{\gamma'}t} \delta_{\gamma\gamma'},
\]  

(42)

\[
A_l^{\gamma \gamma'}(t) = \sum_{\gamma_1, \cdots, \gamma_{l+1}} \left[ \sum_{i=1}^{l+1} (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E_{\gamma_i}, l)} \right] \prod_{j=1}^{l} H_1^{\gamma_j \gamma_{j+1}} \delta_{\gamma_{l+1} \gamma'}.
\]  

(43)

It is clear that this propagator has the closed time evolution factors. In special, it is a \( c \)-number function just like the Feynman path integral\ [4], and it is a power series of the perturbing Hamiltonian just like Dyson series in interaction picture\ [2]. Therefore it has, at the same time, the advantages that exist in the Feynman path integral famulism and Feynman diagram (Dyson) expansion formula .

IV. SOLUTION OF THE SCHRÖDINGER EQUATION AND THE VON NEUMANN EQUATION

Substituting the expression of the time evolution operator (38) into Eq. (3), we obtain immediately our explicit form of time evolution of an arbitrary initial state in a general quantum system

\[
|\Psi(t)\rangle = \sum_{l=0}^{\infty} A_l(t) |\Psi(0)\rangle = \sum_{l=0}^{\infty} \sum_{\gamma, \gamma'} A_l^{\gamma \gamma'}(t) \langle \Phi_{\gamma'} |\Psi(0)\rangle |\Phi_{\gamma'}\rangle,
\]  

(44)

In form, this solution is also suitable for the degenerate cases, but more apparent divergences will appear. It must be emphasized that these apparent divergences can be easily eliminated by the limitation calculations. To save space, we delay to discuss the disposal of degenerate cases and the elimination of apparent divergences to our sequence study [6].

Similarly, the exact solution of the von Neumann equation for mixed states can be obtained by using our general and explicit expression of the time evolution operator (11). From the formal solution of the von Neumann equation

\[
\rho(t) = e^{-iHt} \rho(0) e^{iHt},
\]  

(45)

it immediately follows that

\[
\rho(t) = \sum_{k, l=0}^{\infty} A_k(t) \rho(0) A_l(-t) = \sum_{k, l=0}^{\infty} A_k(t) \rho(0) A_l(t)
\]  

(46)

\[
= \sum_{k, l=0}^{\infty} \sum_{\gamma, \gamma'} A_k^{\gamma \beta}(t) \rho^{\beta \gamma'} (0) A_l^{\beta' \gamma'} (-t) |\Phi_{\gamma'}\rangle \langle \Phi_{\gamma'}|,
\]  

(47)
Since we have obtained the exact expression of the time evolution operator, it is direct to extend our conclusions from the pure state to the mixed state. For simplicity, we only study the cases of pure states in the following, that is, our exact solution of the Schrödinger equation.

Note that the linearity of the evolution operator and the completeness of eigenvector set of $H_0$, we can simply set the initial state as an eigenvector of $H_0$. Up to now, everything is exact and no any approximation enters. Therefore, it is an exact solution despite that its form is an infinity series. In other words, it exactly includes all order approximations of $H_1$. Only when $H_1$ is taken as a perturbation, can it be cut off based on the needed precision. From a view of formalized theory, it is explicit and general, but is not compact. Moreover, if the convergence is guaranteed, it is strict since its general term is known. Usually, to a practical purpose, if only including the finite (often low) order approximation of $H_1$, above expression is cut off to the finite terms and becomes a perturbed solution.

Actually, since our solution includes the contributions from all order approximations of the rest part of Hamiltonian except for the solvable part $H_0$, or all order approximations of the perturbative part $H_1$ of Hamiltonian, it is not very important whether $H_1$ is (relatively) large or small when compared with $H_0$. In principle, for a general quantum system with the normal form Hamiltonian, we can always write down

$$H = \frac{\hat{k}^2}{2m} + V.$$  

If we take the solvable kinetic energy term as $H_0 = \frac{\hat{k}^2}{2m}$ and the potential energy part as $H_1 = V$, our method is applicable to such quantum system. It should be pointed out that we study this concrete example in order to account for the generality and universality of our exact solution.

It is clear that

$$\frac{\hat{k}^2}{2m}|k\rangle = \frac{k^2}{2m}|k\rangle = E_k|k\rangle, \quad E_k = \frac{k^2}{2m}. \tag{49}$$

And denoting

$$\langle x|k\rangle = \frac{1}{L^{3/2}} e^{ikx}, \tag{50}$$

$$\psi_k(0) = \langle k|\Psi(0)\rangle, \tag{51}$$

$$\Psi(x,t) = \langle x|\Psi(t)\rangle, \tag{52}$$

we obtain the final state as

$$\Psi(x,t) = \sum_k \psi_k(0) \frac{e^{i(kx - E_k t)}}{L^{3/2}} + \sum_{i=1}^{\infty} \sum_{k,k',i_1,\ldots,i_{l+1}} \psi_{k'}(0) \left[ \prod_{j=1}^{l+1} V^{k_j,k_{j+1}} \right] \delta_{k_i,k} \delta_{k_{i+1}k'} \times \left[ \sum_{l=1}^{l+1} \frac{(-1)^{l-1}}{d_l(E[k,l])} \frac{1}{L^{3/2}} e^{i(kx - E_{k_l} t)} \right] \tag{53}$$

$$= \sum_k \psi_k(0) \frac{e^{i(kx - E_k t)}}{L^{3/2}} + \sum_{i=1}^{\infty} \sum_{k,k',i_1,\ldots,i_{l+1}} \psi_{k'}(0) \left[ \prod_{j=1}^{l+1} V(k_j - k_{j+1}) \right] \delta_{k_i,k} \delta_{k_{i+1}k'} \times \left[ \sum_{l=1}^{l+1} \frac{(-1)^{l-1}}{d_l(E[k,l])} \frac{1}{L^{3/2}} e^{i(kx - E_{k_l} t)} \right]. \tag{54}$$

In the second equal mark, we have used the fact that $V = V(x)$ usually, thus

$$V^{k_j,k_{j+1}} = \langle k_j|V(x)|k_{j+1}\rangle$$

$$= \int_{-\infty}^{\infty} d^3x d^3x_{j+1} \langle k_j|x_j\rangle \langle x_j|V(x)|x_{j+1}\rangle \langle x_{j+1}|k_{j+1}\rangle$$

$$= \frac{1}{L^3} \int_{-\infty}^{\infty} d^3x d^3x_{j+1} e^{-ik_j \cdot x_j} V(x_{j+1}) \delta^3(x_j - x_{j+1}) e^{ik_{j+1} \cdot x_{j+1}}$$

$$= \frac{1}{L^3} \int_{-\infty}^{\infty} d^3x V(x) e^{-ik_j \cdot x}$$

$$= V(k_j - k_{j+1}). \tag{55}$$
It is clear that $\mathcal{V}(k)$ is the Fourier transformation of $V(x)$. Eq. (53) or (54) is a concrete form of our solution (44) of Schrödinger equation when the solvable part of Hamiltonian is taken as the kinetic energy term. Therefore, in principle, if the Fourier transformation of $V(x)$ can be found, the evolution of the arbitrarily initial state with time can be obtained. It must be emphasized that it is often that the solvable part of Hamiltonian will be not only for the practical purposes.

Furthermore, we can derive out the propagator

$$\langle x | e^{-iHt} | x' \rangle = \sum_{k,k'} \langle x | k \rangle \langle k | e^{-iHt} | k' \rangle \langle k' | x' \rangle$$

$$= \sum_k \frac{1}{L^3} e^{i(kx-x')-iE_k t} + \sum_{l=1}^{\infty} \sum_{k,k',k_1,\ldots,k_{l+1}} \left( \prod_{j=1}^{l} V^{k_j,k_{j+1}} \right) \delta_{k_1k} \delta_{k_{l+1}k'}$$

$$\times \left( \sum_{i=1}^{l+1} \frac{(-1)^{i-1}}{d_i (E[k,l])} \frac{1}{L^{3/2}} e^{-iE_k t} \right) \frac{1}{L^3} e^{i(kx-x'x')}.$$

(56)

It is different from the known propagator in the expressive form, but they should be equivalent in theory because we still use the usual quantum mechanics principle. However, what is more and what is new in our exact solution? In the following, and in our serial studies [6, 7], we will see them.

V. COMPARISON WITH THE USUAL PERTURBATION THEORY

In this section, we will compare the usual perturbation theory with our solution, reveal their consistency and relation, and point out what is more in our solution and what is different between them. Furthermore, we expect to reveal the features, significance and possible applications of our solution in theory. We will respectively investigate and discuss the cases comparing with the time-independent, time-dependent perturbation theories and the non-perturbative solution method.

A. Comparison with the time-independent perturbation theory

The usual time-independent (stationary) perturbation theory is mainly to study the stationary wave equation in order to obtain the perturbative energies and perturbative states. However, our solution focus on the development of quantum states, which is a solution of the Schrödinger dynamical equation. Although their main purposes are different at their start points, but they are consistent. Moreover, our exact solution is also able to apply to the calculation of transition probability, perturbed energy and perturbed state, which will be seen clearly in Ref. [6]. Moreover, our exact solution has been successfully applied to open system dynamics [7].

In order to verify that the time-independent perturbation theory is consistent with our solution at any order approximation, we suppose that in the initial state, the system is in the eigen state of the total Hamiltonian, that is

$$H |\Psi_{ET}(0)\rangle = E_T |\Psi_{ET}(0)\rangle.$$

(57)

It is clear that

$$|\Psi_{ET}(t)\rangle = e^{-iE_T t} |\Psi_{ET}(0)\rangle = \sum_{\gamma\gamma'} \left\{ e^{-iE_T t} \delta_{\gamma\gamma'} + \sum_{l=1}^{\infty} A_{l\gamma'}^{\gamma'}(t) \right\} \left[ (\Phi^{\gamma'} |\Psi_{ET}(0)\rangle) |\Phi^{\gamma}\rangle.\right.$$  

(58)

Calculating the $K$th time derivative of this equation and then setting $t = 0$, we obtain

$$E_T^K |\Psi_{ET}(0)\rangle = \sum_{\gamma\gamma'} \left\{ E_T^K \delta_{\gamma\gamma'} + (i)K \sum_{l=1}^{\infty} \frac{d^K A_{\gamma'}^{\gamma'}(t)}{dt^K} \bigg|_{t=0} \right\} a_{\gamma'} |\Phi^{\gamma}\rangle$$

(59)

$$= \sum_{\gamma\gamma'} \left\{ E_T^K \delta_{\gamma\gamma'} + \sum_{l=1}^{\infty} B_l^{\gamma'}(K) \right\} a_{\gamma'} |\Phi^{\gamma}\rangle,$$

(60)
where we have used the fact that

\[ |\Psi_{E_\gamma}(0)\rangle = \sum_\gamma a_\gamma |\Phi\rangle, \quad a_\gamma = \langle \Phi | \Psi_{E_\gamma}(0)\rangle, \tag{61} \]

\[
\left. \frac{d^K A^{\gamma'}_l(t)}{dt^K} \right|_{t=0} = (-i)^K B_l^{\gamma'}(K). \tag{62}
\]

It is easy to obtain that

\[
\left. \frac{d^K A^{\gamma'}_l(t)}{dt^K} \right|_{t=0} = (-i)^K \sum_{\gamma_1, \ldots, \gamma_{l+1}} \left[ \sum_{i=1}^{l+1} (-1)^i \frac{E_{\gamma_i}^K}{d_i[E_\gamma, l]} \right] \prod_{j=1}^{l} H^{\gamma_j+1}_1 \delta_{\gamma_1 \gamma_1 \gamma_{l+1} \gamma'}, \tag{63}
\]

\[ B_l^{\gamma'}(K) = \sum_{\gamma_1, \ldots, \gamma_{l+1}} C^K_l(E[\gamma, l]) \theta(K-l) \prod_{j=1}^{l} H^{\gamma_j+1}_1 \delta_{\gamma_1 \gamma_1 \gamma_{l+1} \gamma'}. \tag{64} \]

where we have used our identity $\langle 30 \rangle$. This means that if $l > K$

\[
\left. \frac{d^K A^{\gamma'}_l(t)}{dt^K} \right|_{t=0} = (-i)^K B_l^{\gamma'}(K > l) = 0. \tag{65}
\]

In special

\[
\left. \frac{dA^{\gamma'}_l(t)}{dt} \right|_{t=0} = -iB_l^{\gamma'}(1) = -iH_1^{\gamma'} \delta_{l1}. \tag{66}
\]

\[
\left. \frac{d^2 A^{\gamma'}_l(t)}{dt^2} \right|_{t=0} = -B_l^{\gamma'}(2) = -(E_\gamma + E_{\gamma'})H_1^{\gamma'} \delta_{l1} - \sum_{\gamma} H_1^{\gamma_1} H_1^{\gamma_1 \gamma'} \delta_{l2}. \tag{67}
\]

In other words, the summation to $l$ in the right side of Eq. $(59)$ is cut off to $K$. Therefore, by left multiplying $|\Phi\rangle$ to Eq. $(59)$ we obtain

\[ E_T^K a_\gamma = E_T^K a_\gamma + \sum_{\gamma'} \sum_{l=1}^{K} B_l^{\gamma'}(K) a_{\gamma'}. \tag{68} \]

When $K = 1$, the above equation backs to the start point of time-independent perturbation theory

\[ E_T a_\gamma = E_T a_\gamma + \sum_{\gamma'} H_1^{\gamma'} a_{\gamma'}. \tag{69} \]

It implies that it is consistent with our solution. In order to certify that our solution is compatible with the time-independent perturbation theory, we have to consider the cases when $K \geq 2$. By using $E_T^K a_\gamma = E_T (E_T^{K-1} a_\gamma)$, substituting Eq. $(65)$ twice, then moving the second term to the right side, we have

\[
E_T^{K-1} E_T a_\gamma = E_T^K a_\gamma + \sum_{\gamma'} \sum_{l=1}^{K} B_l^{\gamma'}(K) a_{\gamma'} - \sum_{\gamma' \gamma''} \sum_{l=1}^{K-1} B_l^{\gamma'}(K-1) \sum_{\gamma''} \left( E_T \delta_{\gamma' \gamma''} + H_1^{\gamma'+1} \right) a_{\gamma''}
\]

\[
= E_T^K a_\gamma + \sum_{\gamma'} \sum_{l=1}^{K-1} \left( B_l^{\gamma'}(K) - E_T B_l^{\gamma'}(K-1) \right) a_{\gamma'}
\]

\[
+ \sum_{\gamma'} \left( \prod_{j=1}^{l} H_1^{\gamma_j+1} \right) \delta_{\gamma_1 \gamma_1 \gamma_{l+1} \gamma'} a_{\gamma'} - \sum_{\gamma' \gamma''} \sum_{l=1}^{K-1} B_l^{\gamma'}(K-1) \sum_{\gamma''} H_1^{\gamma''} a_{\gamma''}. \tag{70} \]
Since when $K \geq 2$

$$C^K_1(E[\gamma, 1]) - E_{\gamma_2} C^{K-1}_1(E[\gamma, 1]) = E_{\gamma_1}^{K-1},$$

(71)

and again $l \geq 2$

$$C^K_l(E[\gamma, l]) - E_{\gamma_{l+1}} C^{K-1}_l(E[\gamma, l]) = C^{K-1}_l(E[\gamma, l-1]),$$

(72)

which has been proved in appendix A. Obviously, when $K = 2$, Eq. (70) becomes

$$E_\gamma E_T a_\gamma = E^2_\gamma a_\gamma + E_\gamma \sum \gamma' H^{\gamma'}_1 a_{\gamma'}.$$  

(73)

It can back to Eq. (69). Similarly, when $K \geq 3$, so do they. In fact, from Eqs. (71, 72), Eq. (70) becomes

$$E_{\gamma}^{K-1} E_T a_\gamma = E^K_\gamma a_\gamma + E_{\gamma}^{K-1} \sum \gamma' H^{\gamma'}_1 a_{\gamma'},$$

(74)

where we have used the fact

$$\sum_{\gamma'}^{K-1} \sum_{l=2}^{K-1} \left[ B^\gamma_1(K) - E_{\gamma'} B^\gamma_1(K - 1) \right] a_{\gamma'} + \sum_{\gamma'} \left( \prod_{j=1}^{K} H^{\gamma_j+1}_1 \right) \delta_{\gamma_1 \delta_{\gamma_{l+1}} a_{\gamma'}}$$

$$- \sum_{\gamma'} \sum_{l=1}^{K-1} B^\gamma_1(K - 1) \sum \gamma'' H^{\gamma''}_1 a_{\gamma''} = 0.$$  

(75)

Its proof is not difficult because we can derive out

$$\sum_{\gamma'} \sum_{l=2}^{K-1} \left[ B^\gamma_1(K) - E_{\gamma'} B^\gamma_1(K - 1) \right] a_{\gamma'},$$

$$= \sum_{\gamma'} \sum_{l=2}^{K-1} \sum_{\gamma_1, \ldots, \gamma_{l+1}} \left[ C^K_l(E[\gamma, l]) - E_{\gamma'} C^{K-1}_l(E[\gamma, l]) \right] \left( \prod_{j=1}^{l} H^{\gamma_j+1}_1 \right) \delta_{\gamma_1 \delta_{\gamma_{l+1}} a_{\gamma'}}$$

$$= \sum_{\gamma'} \sum_{l=2}^{K-1} \sum_{\gamma_1, \ldots, \gamma_{l+1}} C^{K-1}_l(E[\gamma, l-1]) \left( \prod_{j=1}^{l} H^{\gamma_j+1}_1 \right) \delta_{\gamma_1 \delta_{\gamma_{l+1}} a_{\gamma'}}$$

$$= \sum_{\gamma'} \sum_{l=1}^{K-2} \sum_{\gamma_1, \ldots, \gamma_{l+1}} C^{K-1}_l(E[\gamma, l]) \left( \prod_{j=1}^{l} H^{\gamma_j+1}_1 \right) H^{\gamma_{l+1} a_{\gamma'}}$$

$$= \sum_{\gamma', \gamma''} \sum_{l=1}^{K-2} B^\gamma_1(K - 1) H^{\gamma''}_1 a_{\gamma''}.$$  

(76)

The first equality has used the definitions of $B^\gamma_1(K)$, the second equality has used Eq. (72), the third equality sets $l - 1 \rightarrow l$ and sums over $\gamma_{l+2}$, the forth equality sets $\gamma' \rightarrow \gamma''$, inserts $\sum_{\gamma'} \delta_{\gamma_{l+1} a_{\gamma'}}$ and uses the definition of $B^\gamma_1(K)$ again. Substituting it into the left side of Eq. (75) and using $B^\gamma_1(K - 1)$ expression, we can finish the proof of Eq. (75). Actually, the above proof further verifies the correctness of our solution from the usual perturbation theory.

In our point of view, only if the expression of the high order approximation has been obtained, can we consider its contribution in the time-independent perturbation theory since its method is to find a given order approximation. However, this task needs to solve a simultaneous equation system, it will be heavy for the enough high order approximation. In Ref. [6], we will give a method to find the improved forms of perturbed energy and perturbed state absorbing the partial contributions from the high order even all order approximations of perturbation.
B. Comparison with the non-perturbative method

Now, let us see what is more in our solution than the nonperturbative method. Actually, since \( H \) is not explicitly time-dependent, its eigenvectors can be given formally by so-called Lippmann-Schwinger equations as follows:

\[
|\Psi_S^\gamma(\pm)\rangle = |\Phi\gamma\rangle + \frac{1}{E\gamma - H_0 \pm i\eta} H_1 |\Psi_S^\gamma(\pm)\rangle
\]

(77)

\[
= |\Phi\gamma\rangle + G_0^\gamma(\pm) H_1 |\Psi_S^\gamma(\pm)\rangle
\]

(78)

\[
= |\Phi\gamma\rangle + \frac{1}{E\gamma - H \pm i\eta} H_1 |\Phi\gamma\rangle
\]

(79)

\[
= |\Phi\gamma\rangle + G^\gamma(\pm) H_1 |\Phi\gamma\rangle,
\]

(80)

where the complete Green’s function \( G^\gamma(\pm) = 1/(E\gamma - H \pm i\eta) \) and the unperturbed Green’s function \( G_0^\gamma(\pm) = 1/(E\gamma - H_0 \pm i\eta) \) satisfy the Dyson’s equation

\[
G^\gamma(\pm) = G_0^\gamma(\pm) + G_0^\gamma(\pm) H_1 G^\gamma(\pm)
\]

(81)

\[
= \sum_{l=0}^{\infty} (G_0^\gamma(\pm) H_1)^l G_0^\gamma(\pm).
\]

(82)

Here, the subscript “S” means the stationary solution:

\[
H |\Psi_S^\gamma(\pm)\rangle = E\gamma |\Psi_S^\gamma(\pm)\rangle.
\]

(83)

More strictly, we should use the “in” and “out” states to express it \(^3\). In historical literature, this solution is known as so-called non-perturbative one. It has played an important role in the formal scattering theory \(^3\).

Back to our attempt, for a given initial state \(|\Psi(0)\rangle\), we have

\[
|\Psi(0)\rangle = \sum_{\gamma'} \langle \Psi_S^{\gamma'}(\pm)|\Psi(0)\rangle |\Psi_S^{\gamma'}(\pm)\rangle.
\]

(84)

Acting the time evolution operator on it, we immediately obtain

\[
|\Psi(t)\rangle = \sum_{\gamma'} \langle \Psi_S^{\gamma'}(\pm)|\Psi(0)\rangle e^{-iE\gamma' t} |\Psi_S^{\gamma'}(\pm)\rangle
\]

\[
= \sum_{\gamma'} \langle \Phi\gamma'\rangle \sum_{l=0}^{\infty} (H_1 G_0^{\gamma'}(\mp))^l |\Psi(0)\rangle e^{-iE\gamma' t} \sum_{l'=0}^{\infty} (G^{\gamma'}(\pm) H_1)^l' |\Phi\gamma'\rangle
\]

\[
= \sum_{\gamma,\gamma'} e^{-iE\gamma t} \delta_{\gamma'\gamma} \langle \Phi\gamma'\rangle |\Psi(0)\rangle + \sum_{l,l'=0 \atop l+l' \neq 0} \langle \Phi\gamma'\rangle \left( H_1 G_0^{\gamma'}(\mp) \right)^l |\Psi(0)\rangle \langle \Phi\gamma | \left( G_0^{\gamma'}(\pm) H_1 \right)^{l'} |\Phi\gamma'\rangle e^{-iE\gamma' t} |\Phi\gamma\rangle,
\]

(85)

comparing this result with our solution \(^4\), we can say that our solution has finished the calculations of explicit form of the expanding coefficients (matrix elements) in the second term of the above equation using our method and rearrange the resulting terms according with the power of elements of the perturbing Hamiltonian matrix \( H_1^{(\gamma)\gamma+1} \). This implies that our solution can have more physical contents and more explicit significance. Therefore, we think that our solution is a new development of the stationary perturbation theory. Seemingly, its physical results are consistent with the Feynman’s diagram expansion of Dyson’s series, but its general form is explicit, and convenient to calculate the time evolution of states with time. Specially, we can obtain the improved form of perturbed solution that absorbs partial contributions from the high order even all order approximations in our subsequent works \(^4\), in which, it will be seen that our improved scheme of perturbation theory based on our exact solution has higher efficiency and better precision in the calculation.

C. Comparison with the time-dependent perturbation theory

The aim of our solution is similar to the time-dependent perturbation theory. But their methods are different. The usual time-dependent perturbation theory \(^2\) considers a quantum state initially in the eigenvector of the Hamiltonian...
where $v$ and inserting this into (90) give

$$\frac{i}{\partial t}|\Psi(t)\rangle = (H_0 + \lambda v(t))|\Psi(t)\rangle$$

(86)

with the initial state

$$|\Psi(0)\rangle = |\Phi^\alpha\rangle.$$  

(87)

It is often used to study the cases that the system returns to an eigenvector $|\Phi_f\rangle$ of the Hamiltonian $H_0$ when the action of the perturbing potential becomes negligible.

In order to compare it with our solution, let us first recall the time-dependent perturbation method. Note that the state $|\Psi(t)\rangle$ at time $t$ can be expanded by the complete orthogonal system of the eigenvectors $|\Phi^\gamma\rangle$ of the Hamiltonian $H_0$, that is

$$|\Psi(t)\rangle = \sum_\gamma c_\gamma(t)|\Phi^\gamma\rangle$$

(88)

with

$$c_\gamma(t) = \langle \Phi^\gamma | \Psi(t) \rangle,$$

(89)

whereas the evolution equation (86) will become

$$\frac{\partial}{\partial t}c_\gamma(t) = E_\gamma c_\gamma(t) + \lambda \sum_{\gamma'} v^{\gamma\gamma'}(t)c_{\gamma'}(t),$$

(90)

where $v^{\gamma\gamma'}(t) = \langle \Phi^\gamma | v(t) | \Phi^{\gamma'} \rangle$. Now setting

$$c_\gamma(t) = b_\gamma(t)e^{-iE_\gamma t},$$

(91)

and inserting this into (90) give

$$i \frac{\partial}{\partial t}b_\gamma(t) = \lambda \sum_{\gamma'} e^{i(E_{\gamma'} - E_{\gamma})t} v^{\gamma\gamma'}(t)b_{\gamma'}(t).$$

(92)

Then, making a series expansion of $b_\gamma(t)$ according to the power of $\lambda$:

$$b_\gamma(t) = \sum_{l=0} \lambda^l b_\gamma^{(l)}(t),$$

(93)

and setting equal the coefficients of $\lambda^l$ on the both sides of the equation (92), ones find:

$$i \frac{\partial}{\partial t}b_\gamma^{(0)}(t) = 0,$$

(94)

$$i \frac{\partial}{\partial t}b_\gamma^{(l)}(t) = \lambda \sum_{\gamma'} e^{i(E_{\gamma'} - E_{\gamma})t} v^{\gamma\gamma'}(t)b_{\gamma'}^{(l-1)}(t), \quad (l \neq 0).$$

(95)

From the initial condition (87), it follows that $c_\gamma(0) = b_\gamma(0) = \delta_{\gamma\alpha} = b_\gamma^{(0)}(0)$ and $b_\gamma^{(l)}(0) = 0$ if $l \geq 1$. By integration over the variable $t$ this will yield

$$b_\gamma^{(0)}(t) = \delta_{\gamma\alpha},$$

(96)

$$b_\gamma^{(1)}(t) = -i \int_0^t dt_1 e^{i(E_{\gamma'} - E_{\gamma})t_1} v^{\gamma\alpha}(t_1),$$

(97)

$$b_\gamma^{(2)}(t) = -i \int_0^t dt_1 \int_0^{t_1} dt_2 e^{i(E_{\gamma'} - E_{\gamma})t_2} v^{\gamma\gamma'}(t_2) b_{\gamma'}^{(1)}(t_2)$$

$$= -\sum_{\gamma'} \int_0^t dt_2 \int_0^{t_2} dt_1 e^{i(E_{\gamma'} - E_{\gamma})t_2} e^{i(E_{\gamma'} - E_{\gamma})t_1} v^{\gamma\gamma'}(t_2)v^{\gamma\alpha}(t_1).$$

(98)
If we take $v$ independent of time, the above method in the time-dependent perturbation theory is still valid. Thus,

$$b^{(1)}_\gamma(t) = -\frac{1}{E_\gamma - E_\alpha} \left( e^{-iE_\alpha t} - e^{-iE_\gamma t} \right) e^{iE_\gamma t} v^{\gamma_2 \gamma_3}.$$  

(99)

$$b^{(2)}_\gamma(t) = \sum_{\gamma_2} \frac{1}{(E_\gamma - E_\alpha)(E_\gamma - E_{\gamma_2})} \left( e^{-iE_{\gamma_2} t} - e^{-iE_\gamma t} \right) e^{iE_\gamma t} v^{\gamma_2 \gamma_3}.$$  

(100)

This means that

$$c^{(1)}_\gamma(t) = \sum_{\gamma_1, \gamma_2} \sum_{i=1}^{2} (-1)^{i-1} \frac{e^{-iE_{\gamma_i}}}{d_i(E[\gamma, 1])} v^{\gamma_1 \gamma_2} \delta_{\gamma_1} \delta_{\gamma_2}.$$  

(101)

$$c^{(2)}_\gamma(t) = \sum_{\gamma_1, \gamma_2, \gamma_3} \sum_{i=1}^{3} (-1)^{i-1} \frac{1}{d_i(E[\gamma, 2])} e^{-iE_{\gamma_i} t} \delta_{\gamma_1} \delta_{\gamma_2} v^{\gamma_1 \gamma_2} v^{\gamma_2 \gamma_3}.$$  

(102)

Now, we use the mathematical induction to prove

$$c^{(l)}_\gamma(t) = \sum_{\gamma_1, \cdots, \gamma_{l+1}} \sum_{i=1}^{l+1} (-1)^{i-1} \frac{e^{-iE_{\gamma_i} t}}{d_i(E[\gamma, l])} \left( \prod_{j=1}^{l} v^{\gamma_j \gamma_{j+1}} \right) \delta_{\gamma_1} \delta_{\gamma_{l+1}} a.$$  

(103)

Obviously, we have seen that it is valid for $l = 1, 2$. Suppose it is also valid for a given $n$, thus form Eqs. (95) and $b^{(l)}_\gamma(0) = 0, l \geq 1$ it follows that

$$b^{(n+1)}_\beta(t) = -i \int_0^t d\tau \sum_{\gamma} e^{i(E_{\beta} - E_\gamma) \tau} v^{\beta \gamma} b^{(n)}_\gamma(\tau).$$  

(104)

Substitute Eqs. (91) and (103) we obtain

$$b^{(n+1)}_\beta(t) = -i \int_0^t d\tau \sum_{\gamma} e^{iE_{\beta} \tau} v^{\beta \gamma} c^{(n)}_\gamma(\tau)$$

$$= -i \int_0^t d\tau \sum_{\gamma} e^{iE_{\beta} \tau} \sum_{\gamma_1, \cdots, \gamma_{n+2}} \sum_{i=1}^{n+1} (-1)^{i-1} \frac{e^{-iE_{\gamma_i} \tau}}{d_i(E[\gamma, n])} \left( \prod_{j=1}^{n} v^{\gamma_j \gamma_{j+1}} \right) \delta_{\gamma_1} \delta_{\gamma_{n+2} \gamma_{n+2}}$$

$$= \sum_{\gamma, \cdots, \gamma_{n+2}} \sum_{i=1}^{n+1} (-1)^{i-1} \frac{e^{i(E_{\beta} - E_{\gamma_{n+2}}) t}}{d_i(E[\gamma, n])}(E_{\gamma_i} - E_{\gamma_{n+2}}) \left( \prod_{j=1}^{n} v^{\gamma_j \gamma_{j+1}} \right) v^{\gamma_{n+2} \gamma_{n+2}} \delta_{\gamma_{n+2} \beta} \delta_{\gamma_{n+2} \gamma_{n+2}}.$$  

(105)

In the last equality, we have inserted $\sum_{\gamma_{n+2}} \delta_{\gamma_{n+2} \beta}$, summed over $\gamma$ and integral over $\tau$. In terms of the definition of $d_i(E[\gamma, n])$, we know that $d_i(E[\gamma, n])(E_{\gamma_i} - E_{\gamma_{n+2}}) = d_i(E[\gamma, n+1])$. Then based on our identity (30), we have

$$\sum_{i=1}^{n+1} (-1)^{i-1} \frac{1}{d_i(E[\gamma, n + 1])} = (-1)^{(n+2)-1} \frac{1}{d_{n+2}(E[\gamma, n + 1])}.$$  

(106)
Thus Eq. (105) becomes

\[
\beta^{(n+1)}(t) = \sum_{\gamma_1, \ldots, \gamma_{n+2}} \sum_{i=1}^{n+2} (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, n+1])} \left( \prod_{j=1}^{n+1} \nu_{\gamma_j} \gamma_{j+1} \right) \nu_{\gamma_{n+2}} \delta_{\gamma_{n+2} \gamma_{n+1}} e^{iE_{\beta}t}.
\]

Set the index taking turns, that is, \( \gamma_1 \to \gamma_{n+2} \) and \( \gamma_i + 1 \to \gamma_i \,(n+1) \geq i \geq 1 \). Again from the definition of \( d_i(E[\gamma, n]) \), we can verify easily under the above index taking turns,

\[
\sum_{i=1}^{n+2} (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, n+1])} \rightarrow \sum_{i=1}^{n+2} (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, n+1])}.
\]

Therefore we obtain the conclusion

\[
c^{(n+1)}(t) = \beta^{(n+1)}(t)e^{-iE_{\beta}t} = \sum_{\gamma_1, \ldots, \gamma_{n+2}} \sum_{i=1}^{n+2} (-1)^{i-1} \frac{e^{-iE_{\gamma_i}t}}{d_i(E[\gamma, n+1])} \left( \prod_{j=1}^{n+1} \nu_{\gamma_j} \gamma_{j+1} \right) \delta_{\gamma_1 \gamma_{n+2}} \delta_{\gamma_{n+2} \gamma_{n+1}}.
\]

This implies that we finish the proof of expression (108). Substitute it into Eq. (88), we immediately obtain the same expression as our solution (11).

From the statement above, it can be seen that our solution actually finishes the task to solve the recurrence equation of \( \beta^{(n)}(t) \) by using our new method. Although the recurrence equation of \( \beta^{(n)}(t) \) can be solved by integral in principle, it is indeed not easy if one does not know our identity (30) and relevant relations. To our knowledge, the general term form has not ever been obtained up to now. This is an obvious difficulty to attempt absorbing high order approximations. However, our solution has done it, and so it will make some calculations of perturbation theory more convenient and more accurate.

However, we pay for the price to require that \( H \) is not explicitly dependent on time. Obviously our solution is written as a symmetric and whole form. Moreover, our method gives up, at least in the sense of formalization, the requirement that \( V \) is small enough compared with \( H_0 \), and our proof is more general and stricter. In addition, we do not need to consider the cases that the perturbing potential is switched at the initial and final time, but the perturbing potential is not explicitly dependent of time. Since we obtain the general term, our solution must have more applications, it is more efficient and more accurate for the practical applications, because we can selectively absorb the partial contributions from the high order even all order approximations. This can be called as the improved forms of perturbed solutions, which will be given in Ref. [6].

VI. DISCUSSION AND CONCLUSION

First of all, we would like to point out that our general and explicit solution (11) or its particular forms (18) of the Schrödinger equation in a general quantum system independent of time is explicit and exact in form because all order approximations of perturbation not only are completely included but also are clearly expressed, although it is an infinite series. Our exact solution is in c-number function rather than operator form. This means that it can inherit the same advantage as the Feynman path integral expression. Moreover, our exact solution is a power series of the perturbing Hamiltonian like as the Dyson series in the interaction picture. This implies that the cut-off approximation of perturbation can be made based on the needed precision of the problems. It must be emphasized that when applying our exact solution to a concrete quantum system, all we need to do for our exact solution is the calculations of perturbing Hamiltonian matrix and the coefficient function limitations. Here, the “perturbing Hamiltonian matrix” refers to the representation matrix of the perturbing Hamiltonian in the unperturbed Hamiltonian representation. Thus, such calculations are definitely much easier than the calculations of path integral and Dyson series for the most systems. On the other hand, because the general term is explicitly given in our exact solution series, we can more conveniently consider the contributions from the high order even all order approximations in order to improve the precision than doing the same thing via the Dyson series.

Although we have obtained the exact solution in general quantum systems, this does not mean that the perturbation theory is unnecessary because our exact solution is still an infinite power series of perturbation. Our solution is called “exact one” in the sense including the contributions from all order approximations of the perturbing Hamiltonian. In practise, if we do not intend to apply our solution to investigations of the formal theory of quantum mechanics, we need to cut off our exact solution series to some given order approximation in the calculations of concrete problems. Perhaps, one argues that our exact solution will back to the usual perturbation theory, and it is, at most, an explicit form that can bring out the efficiency amelioration. Nevertheless, the case is not so. Such a view, in fact, ignores the
significance of the general term in an infinite series, and forgets the technologies to deal with an infinite series in the present mathematics and physics. From our point of view, since the general term is known, we can systematically and reasonably absorb the partial contributions from some high order even all order approximations just like one has done in quantum field theory via summation over a series of different order but similar feature Feynman figures. In our paper [6], we will fixed this problem via proposing and developing an improved scheme of perturbation theory.

In order to account for what is more in our solution, and reveal the relations and differences between our solution and the existed method, we compare our solution with the usual perturbation theory. We find their consistency. In fact, our solution has finished the task to calculate the expanding coefficients of final state in \( H_0 \) representation and obtain the general term up to any order using our own method. But the usual perturbation theory only carries out this task from some given order approximation to the next order approximation step by step. In a sentence, more explicit and general feature of our exact solution can lead to more physical conclusions and applications. In addition, by comparison with the existed perturbation theory, we actually verify the validity of our exact solution.

It is worth pointing out that our solution, different from the time-dependent perturbation theory, presents the explicit solution of recurrence equation of expanding coefficients of final state in \( H_0 \) representation, but we pay for the price that \( H \) is not explicitly dependent on the time. In short, there is gain and there is lose.

It is worthy noticing that there are the apparent divergences in the expression of our exact solution, in special, when the degeneracy happens. These apparent divergences can be easily eliminated via the limitation calculations. The related discussions are arranged in our paper [6].

Therefore, we can say that our exact solution is more explicit than the usual non-perturbed solution of the Schrödinger equation because the general term is given in the \( c \)-number function form, and we can say our solution is more general than the usual perturbation theory because our deducing methods give up some preconditions used in the usual scheme. However, our exact solution can not be used to the time-dependent system at present. Just as its explicit and general features, our exact solution not only has the mathematical delicateness, but also can contain more physical content, obtain the more efficiency and higher precision and result in new applications, which have been revealed in our serial studies [6, 7].

Our physical idea is a combination of Feynman path integral spirit and Dyson series kernel. Hence, we first find an general and explicit expression of the time evolution operator that not only is a \( c \)-number function but also a power series of perturbation including all order approximations. From our point of view, it is one of the most main results in our method. It must be emphasized that the study on the time evolution operator plays a central role in quantum dynamics and perturbation theory. Because the universal significance of our general and explicit expression of time evolution operator, we wish that it will have more applications in quantum theory. Besides the perturbation theory and open system dynamics, it is more interesting to apply our exact solution to the other formalization study of quantum dynamics in order to further and more powerfully show the advantages of our exact solution.

In the process of deducing our exact solution, we obtain an expansion formula of operator binomials power. At our knowledge, this formula is first proposed and strictly proved. Besides its theoretical value in mathematics, we are sure it is interesting and important for expressing some useful operator formula in quantum physics. In addition, we prove an identity of fraction function. It should be interesting in mathematics. Perhaps, it has other applications to be expected finding.

We can see that in our solution, every expanding coefficient (amplitude) before the basis vector \( |\Phi^\gamma\rangle \) has some closed time evolution factors with exponential form, and includes all order approximations so that we can clearly understand the dynamical behavior of quantum systems. Although our solution is obtained in \( H_0 \) representation, its form in other representations can be given by the representation transformation and the development factors with time \( t \) in the expanding coefficients (amplitude) do not change.

Because we obtain the general term form of time evolution of quantum state, it provides the probability considering the partial contributions from the high order even all of order approximations. For example, we obtain the revised Fermi’s Golden Rule [10]. This can be seen in our recent work [6].

Our general and explicit solution can be successfully applied to open systems to obtain the master equation and exact solution. For example, we obtain the Redfield master equation [11] without using Born-Markov approximation. This can be seen in our recent work [7].

From the features of our solution, we believe that it will have interesting applications in the calculation of entanglement dynamics and decoherence process as well as the other physical quantities dependent on the expanding coefficients. Of course, our solution is an exact one, its advantages and features can not be fully revealed only via the perturbative method.

In summary, our results can be thought of as theoretical developments of quantum dynamics, and are helpful for understanding the dynamical behavior and related subjects of general quantum systems in both theory and application. Together with our perturbation theory [6] and open system dynamics [7] they can finally form the foundation of theoretical formalism of quantum mechanics in general quantum systems. Further study on quantum mechanics of general quantum systems is on progressing.
APPENDIX A: THE PROOF OF OUR IDENTITY

In this appendix, we would like to prove our identity \((A3)\). For simplicity in notation and universality, we replace the variables \(E_{\gamma_i}\) by \(x_i\) as well as \(E[\gamma, l]\) by \(x[l]\). It is clear that the common denominator \(D(x[n])\) in the above expression \((A10)\) (the index \(l\) is replaced by \(n\)) reads

\[
D(x[n]) = \prod_{i=1}^{n} \left[ \prod_{j=1}^{n} (x_i - x_{j+1}) \right],
\]

(A1)

while the \(i\)-th numerator is

\[
n_i(x[n], K) = \frac{D(x[n])}{d_i(x[n])} x_i^K,
\]

(A2)

and the total numerator \(N(x[n], K)\) is

\[
N(x[n], K) = \sum_{i=1}^{n+1} n_i(x[n], K).
\]

(A3)

In order to simplify our notation, we denote \(n_i(x[n]) = n_i(x[n], 0)\). Again, introducing a new vector

\[
x_i^D[n] = \{x_2, x_3, \ldots, x_n, x_{n+1}\},
\]

(A4)

\[
x_i^D[n] = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}\},
\]

(A5)

\[
x_{n+1}^D[n] = \{x_1, x_2, \ldots, x_{n-1}, x_n\}.
\]

(A6)

Obviously, \(x_i^D[n]\) with \(n\) components is obtained by deleting the \(i\)-th component from \(x[n]\). From the definition of \(n_i(x[n])\), it follows that

\[
n_i(x[n]) = D(x_i^D[n]).
\]

(A7)

Without loss of generality, for an arbitrary given \(i\), we always can rewrite \(x_i^D[n] = y[n-1] = \{y_1, y_2, \ldots, y_{n-1}, y_n\}\) and consider the general expression of \(D(y[n-1])\). It is easy to verify that

\[
D(y[1]) = -y_2n_1(y[1]) + y_1n_2(y[1]),
\]

(A8)

\[
D(y[2]) = y_2y_3n_1(y[2]) - y_1y_3n_2(y[2]) + y_1y_2n_3(y[2]).
\]

(A9)

Thus, by mathematical induction, we first assume that for \(n \geq 2\),

\[
D(y[n-1]) = \sum_{i=1}^{n} (-1)^{(i-1)+(n-1)} p_i(y[n-1])n_i(y[n-1]),
\]

(A10)

where we have defined

\[
p_i(y[n-1]) = \prod_{j=1}^{n} y_j.
\]

(A11)

As above, we have verified that the expression \((A10)\) is valid for \(n = 2, 3\). Then, we need to prove that the following expression

\[
D(y[n]) = \sum_{i=1}^{n+1} (-1)^{(i-1)+n} p_i(y[n])n_i(y[n])
\]

(A12)
Thus, the right side of Eq. (A14) becomes

$$
\sum_{i=1}^{n+1} (-1)^{i-1} n_i(y[n]) = 0.
$$

(A13)

According to the relation (A7) and substituting the precondition (A10), we have

$$
\sum_{i=1}^{n+1} (-1)^{i-1} n_i(y[n]) = \sum_{i=1}^{n+1} (-1)^{i-1} D(y_i^D[n]) = \sum_{i=1}^{n+1} (-1)^{i-1} \left[ \sum_{j=1}^{n} (-1)^{(j-1)+(n-1)} p_j(y_i^D[n]) n_j(y_i^D[n]) \right]
$$

$$
= (-1)^{n-1} \sum_{j=1}^{n} \sum_{i=1}^{n+1} (-1)^{i+j} p_j(y_i^D[n]) n_j(y_i^D[n]).
$$

(A14)

Based on the definitions of \( p_i(x[n]) \) and \( n_i(x[n]) \), we find that

$$
p_j(y_i^D[n]) = \begin{cases} 
p_{i-1}(y_j^D[n]) & \text{(If } i > j) \\
p_i(y_{j+1}^D[n]) & \text{(If } i \leq j) 
\end{cases}
$$

(A15)

$$
n_j(y_i^D[n]) = \begin{cases} 
n_{i-1}(y_j^D[n]) & \text{(If } i > j) \\
n_i(y_{j+1}^D[n]) & \text{(If } i \leq j) 
\end{cases}
$$

(A16)

Thus, the right side of Eq. (A14) becomes

$$
(-1)^{n-1} \sum_{j=1}^{n} \sum_{i=1}^{n+1} (-1)^{i+j} p_j(y_i^D[n]) n_j(y_i^D[n])
$$

$$
= (-1)^{n-1} \sum_{j=1}^{n} \sum_{i=1}^{n+1} (-1)^{i+j} p_j(y_i^D[n]) n_j(y_i^D[n])
$$

$$
+ (-1)^{n-1} \sum_{j=1}^{n} \sum_{i=1}^{n+1} (-1)^{i+j} p_j(y_i^D[n]) n_j(y_i^D[n])
$$

$$
= (-1)^{n-1} \sum_{j=1}^{n} \sum_{i=1}^{n+1} (-1)^{i+j} p_{i-1}(y_j^D[n]) n_{i-1}(y_j^D[n])
$$

$$
+ (-1)^{n-1} \sum_{j=1}^{n} \sum_{i=1}^{n+1} (-1)^{i+j} p_i(y_{j+1}^D[n]) n_i(y_{j+1}^D[n]).
$$

(A17)

Setting that \( i-1 \to i \) for the first term and \( j+1 \to j \) for the second term in the right side of the above equation, we obtain

$$
(-1)^{n-1} \sum_{j=1}^{n} \sum_{i=1}^{n+1} (-1)^{i+j} p_j(y_i^D[n]) n_j(y_i^D[n])
$$

$$
= (-1)^{n-1} \sum_{j=1}^{n} \sum_{i=1}^{n} (-1)^{i+j-1} p_i(y_j^D[n]) n_i(y_j^D[n])
$$

$$
+ (-1)^{n-1} \sum_{j=2}^{n+1} \sum_{i=1}^{j-1} (-1)^{i+j-1} p_i(y_j^D[n]) n_i(y_j^D[n])
$$

$$
= (-1)^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} (-1)^{i+j} p_i(y_j^D[n]) n_i(y_j^D[n])
$$

$$
+ (-1)^{n} \sum_{i=1}^{n+1} (-1)^{i+(n+1)} p_i(y_{n+1}^D[n]) n_i(y_{n+1}^D[n]).
$$

(A18)
Again, setting that $i \leftrightarrow j$ in the right side of the above equation gives

\[
(-1)^{n-1} \sum_{j=1}^{n} \sum_{i=1}^{n+1} (-1)^{i+j} p_j(y_i^D[n]) n_j(y_j^D[n])
\]

\[= (-1)^n \sum_{j=1}^{n} \sum_{i=1}^{n+1} (-1)^{i+j} p_j(y_i^D[n]) n_j(y_j^D[n]). \tag{A19}
\]

Thus, it implies that

\[
(-1)^{n-1} \sum_{j=1}^{n} \sum_{i=1}^{n+1} (-1)^{i+j} p_j(y_i^D[n]) n_j(y_j^D[n]) = 0. \tag{A20}
\]

From the relation \([A13]\) it follows that the identity \([A13]\) is valid.

Now, let us back to the proof of the expression \([A12]\). Since the definition of $D(y[n])$, we can rewrite it as

\[
D(y[n]) = D(y[n-1])d_{n+1}(y[n]). \tag{A21}
\]

Substituting our precondition \([A10]\) yields

\[
D(y[n]) = \sum_{i=1}^{n} (-1)^{(i-1)+(n-1)} p_i(y[n-1]) n_i(y[n-1]) d_{n+1}(y[n]). \tag{A22}
\]

Note that

\[
n_i(y[n-1])d_{n+1}(y[n]) = n_i(y[n])(y_i - y_{n+1}), \tag{A23}
\]

\[
p_i(y[n-1])y_{n+1} = p_i(y[n]), \tag{A24}
\]

\[
p_i(y[n-1])y_i = p_{n+1}(y[n]), \tag{A25}
\]

we have that

\[
D(y[n]) = \sum_{i=1}^{n} (-1)^{(i-1)+n} p_i(y[n]) n_i(y[n]) + \sum_{i=1}^{n} (-1)^{(i-1)+(n-1)} p_{n+1}(y[n]) n_i(y[n])
\]

\[= \sum_{i=1}^{n} (-1)^{(i-1)+n} p_i(y[n]) n_i(y[n]) + (-1)^{n-1} p_{n+1}(y[n]) \sum_{i=1}^{n} (-1)^{i-1} n_i(y[n]) \]

\[= \sum_{i=1}^{n} (-1)^{(i-1)+n} p_i(y[n]) n_i(y[n]) + (-1)^{n-1} p_{n+1}(y[n]) \left[ (-1)^n n_{n+1}(y[n]) \right]. \tag{A26}
\]

In the last equality we have used the conclusion \([A13]\) of our precondition, that is

\[
\sum_{i=1}^{n} (-1)^{i-1} n_i(y[n]) = (-1)^n n_{n+1}(y[n]). \tag{A27}
\]

Thus, Eq.\((A26)\) becomes

\[
D(y[n]) = \sum_{i=1}^{n+1} (-1)^{(i-1)+n} p_i(y[n]) n_i(y[n]). \tag{A28}
\]

The needed expression \([A12]\) is proved by mathematical induction. That is, we have proved that for any $n \geq 1$, the expression \([A12]\) is valid.

Since our proof of the conclusion \([A13]\) of our precondition is independent of $n$ ($n \geq 1$), we can, in the same way, prove that the identity \([A13]\) is correct for any $n \geq 1$.

Now, let us prove our identity \([30]\). Obviously, when $K = 0$ we have

\[
\sum_{i=1}^{n+1} (-1)^{i-1} \frac{1}{d_i(x[n])} = \frac{1}{D(x[n])} \left[ \sum_{i=1}^{n+1} (-1)^{i-1} n_i(x[n]) \right] = 0, \tag{A29}
\]
where we have used the fact the identity \(\text{(A13)}\) is valid for any \(n \geq 1\). Furthermore, we extend the definition domain of \(C_n^K(x[n])\) from \(K \geq n\) to \(K \geq 0\), and still write its form as

\[
C_n^K(x[n]) = \sum_{i=1}^{n+1} (-1)^{i-1} \frac{x_i^K}{d_i(x[n])}.
\]  
(A30)

Obviously, Eq.\(\text{(A29)}\) means

\[
C_n^0(x[n]) = 0, \quad (n \geq 1).
\]  
(A31)

In order to consider the cases when \(K \neq 0\), by using of \(d_i(x[n])(x_i - x_{n+2}) = d_i(x[n+1])\) \((i \leq n + 1)\), we obtain

\[
C_n^K(x[n]) = \sum_{i=1}^{n+1} (-1)^{i-1} \frac{x_i^K}{d_i(x[n])} - \sum_{i=1}^{n+1} (-1)^{i-1} \frac{x_i^{K+1}}{d_i(x[n+1])} - \sum_{i=1}^{n+1} (-1)^{i-1} \frac{x_i^{K+2}}{d_{i+2}(x[n+1])} - x_{n+2} \sum_{i=1}^{n+1} (-1)^{i-1} \frac{x_i^K}{d_i(x[n+1])}.
\]  
(A32)

It follows the recurrence equation as the following

\[
C_n^K(x[n]) = C_{n+1}^{K+1}(x[n+1]) - x_{n+2}C_{n+1}^K(x[n+1]).
\]  
(A33)

It implies that since \(C_n^0(x[n]) = 0\) for any \(n \geq 1\), then \(C_{n+1}^1(x[n+1]) = 0\) for any \(n \geq 1\) or \(C_n^1(x[n]) = 0\) for any \(n \geq 2\); since \(C_n^2(x[n]) = 0\) for any \(n \geq 2\), then \(C_{n+1}^2(x[n+1]) = 0\) for any \(n \geq 2\) or \(C_n^2(x[n]) = 0\) for any \(n \geq 3\); \ldots, since \(C_n^K(x[n]) = 0\) for any \(n \geq (k + 1)\) \((k \geq 0)\), then \(C_{n+1}^{k+1}(x[n+1]) = 0\) for any \(n \geq (k + 1)\) or \(C_n^{k+1}(x[n]) = 0\) for any \(n \geq (k + 2)\); \ldots. In fact, the mathematical induction tells us this result. Obviously

\[
C_n^K(x[n]) = 0, \quad (\text{If } 0 \leq K < n).
\]  
(A34)

Taking \(K = n\) in Eq.\(\text{(A33)}\) and using Eq.\(\text{(A34)}\), we have

\[
C_n^n(x[n]) = C_{n+1}^{n+1}(x[n+1]) = 1, \quad (n \geq 1),
\]  
(A35)

where we have used the fact that \(C_1^1(x[1]) = 1\). Therefore, the proof of our identity \(\text{(30)}\) is finally finished.

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