THE STRONG AND GRAVITATIONAL COUPLINGS
OF KNOTTED SOLITONS

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Abstract. We extend our earlier study of the electroweak interactions of quantum knots to their gravitational and strong interactions. The knots are defined by appropriate quantum groups and are intended to describe all knotted field structures that conserve mass and spin, charge and hypercharge, as well as color charge and color hypercharge. As sources of the gravitational fields the knots are described as representations of the quantum group $SL_q(2)$ and as sources of the electroweak and strong fields they are described by $SU_q(2)$. When the point sources of the standard theory are replaced by the quantum knots, the interaction terms of the new Lagrangian density acquire knot form factors and the standard local gauge invariance is supplemented by an additional global $U(1) \times U(1)$ invariance of the $SU_q(2)$ algebra.
1 Introduction.

In this paper we are interested in extending our previous discussion of knots as sources of the electroweak interactions$^{1,2,3}$ to their gravitational and strong couplings. In the standard theory the elementary fermions are sources of the weak, strong, and gravitational fields. They are labelled by quantum numbers originating in the corresponding symmetry groups $SU(2) \times U(1)$, $SU(3)$, and Lorentz respectively, and the connections of these groups give rise through the quantum theory of fields to electroweak bosons, gluons, and gravitons, respectively. The fermionic sources are usually understood to be structureless point particles, but this assumption cannot be correct if one believes that some structure is needed to endow the sources with their quantum numbers. We shall hypothesize that the sources are solitons and further that the solitons are quantum knots.

All knots, either oriented or unoriented, may be described by $SL_q(2)$. We shall describe the subset of oriented knots not only by $SL_q(2)$ but also by the subgroup $SU_q(2)$. We shall also assume that all knots, either oriented or unoriented, are gravitationally coupled but that only the oriented knots are coupled to the electroweak and gluon fields. Under this assumption the non-oriented knots are candidates for dark matter. In describing the gravitational interactions we shall assume that the internal state functions of all knots appear as irreducible representations of $SL_q(2)$ while in the case of the electroweak and strong interactions, it is assumed that only the oriented knots appear and they appear with internal state functions that are irreducible representations of $SU_q(2)$. These state functions will be labelled by the quantum numbers of the standard theory. In this way we shall assume that the quantum knot is defined by irreducible representations of its symmetry group $SL_q(2)$.

A similar characterization of a fundamental quantum system by irreducible representations of its symmetry group is provided by the wave function of the spherical top, $D_{mm'}^j$, where $D_{mm'}^j$ is an irreducible representation of $O(3)$ and where $j(j+1)$ are the eigenvalues of $J^2$, while $m$ and $m'$ are the eigenvalues of the $z$ components of $\vec{J}$ in inertial and body fixed frames. Another fundamental example is the wave function of the H atom expressed as the solution of an integral equation on the group space of $O(3)$, the symmetry group of
the hydrogen atom. There the wave function $D^j_{mm'}$ is an element of an irreducible representation of $O(3)$; and $2j + 1$ is the principal quantum number, while $m$ and $m'$ are eigenvalues of the $z$ components of the angular momentum and the Runge-Lenz vector.

In developing the knot model we have begun with the observation that not only are there 4 trefoils and 4 families of elementary fermions, but also that there is a unique correspondence between the 4 families and the 4 trefoils, leading to the labelling of the fermionic solitons as the following irreducible representations of $SU_q(2)$: $D^{3t}_{-3t_3-3t_0}$ where $t$ and $t_3$ are isotopic spin and its third component, while $t_0$ is the hypercharge. We have shown that these quantum numbers are connected with the topological characterization of the oriented knot by

$$
t = \frac{N}{6}, \quad t_3 = -\frac{w}{6}, \quad t_0 = -\frac{r + 1}{6}$$

where $N, w,$ and $r$ are the number of crossings, the writhe, and the rotation of the knot.\(^3\)

We now seek a similar representation of the generic knot as the source of gravitational couplings.

## 2 The Generic Knot and the Lorentz Group.

The quantum algebra, $SL_q(2)$, which describes the symmetry of the generic knot, oriented or not, may be defined as follows. Let

$$L_q^\dagger \epsilon_q L_q = \epsilon_q$$

where

$$\epsilon_q = \begin{pmatrix} 0 & q_1^{1/2} \\ -q_1^{-1/2} & 0 \end{pmatrix}, \quad q_1 = q^{-1}$$

Then $L_q$ is a two-dimensional representation of $SL_q(2)$.

If

$$L_q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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then
\[ ab = qba \quad bd = qdb \quad bc = cb \quad ad - qbc = 1 \]
\[ ac = qca \quad cd = qdc \quad da - q_1bc = 1 \quad (A) \]

There are no finite matrix representations of the elements \((a, b, c, d)\) of the algebra \((A)\) unless \(q\) is a root of unity.

When \(q = 1\) we have

\[ L' \epsilon L = \epsilon \quad (2.3a) \]
\[ \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.3b) \]

Then \((A)\) is satisfied by complex numbers and \((2.3)\) may be rewritten as simply

\[ \text{det } L = 1 \quad (2.4) \]

since

\[ \epsilon_{ij} L_{im} L_{jn} = \epsilon_{mn} \text{det } L \quad (2.5) \]

Hence

\[ \lim_{q \to 1} SL_q(2) = SL(2, C) \quad (2.6) \]

We now recall the familiar argument that \(SL(2, C)\) describes the Lorentz group.

Let

\[ P = \bar{p} \bar{\sigma} + p_o \quad (2.7) \]

Consider

\[ P' = L^+ P L \quad (2.8a) \]

where

\[ \text{det } L = 1 \quad (2.8b) \]

Then

\[ (P')^+ = P' \quad (2.9) \]
\[ \text{det } P' = \text{det } P \quad (2.10) \]
By (2.7) and (2.9), Eq. (2.10) may be rewritten as

\[(p_o^2 - \vec{p}^2)' = p_o^2 - \vec{p}^2 \equiv m_o^2 \tag{2.11}\]

Since \(L\) has complex matrix elements and \(\det L = 1\) it is a six-parameter matrix that induces, by (2.8) and (2.11), a Lorentz transformation on \((p_o, \vec{p})\).

The affine connection of the local Lorentz group leads to the gravitational interactions. We shall now postulate that the gravitational couplings are blind to the orientation of the solitonic knot and that all solitons couple to gravity as states of \(SL_q(2)\) whether or not the soliton is oriented. We shall assume that the action is invariant under global transformation of the \(SL_q(2)\) algebra and in conformity with standard theory, local transformations of the \(SL(2, C)\) algebra.

As the detailed treatment of the gravitational couplings will follow along the same lines as the electroweak couplings, let us next describe the algebra that we have used to describe the oriented knot and the electroweak couplings.

## 3 The Oriented Knot and the Electroweak Couplings.

The quantum algebra \(SU_q(2)\), with which we describe the symmetry of the oriented knot, is the unitary subalgebra of \(SL_q(2)\) obtained by setting

\[d = \bar{a}, \quad c = -q_1\bar{b} \tag{3.1}\]

Then (A) reduces to the following

\[ab = qba, \quad a\bar{a} + b\bar{b} = 1, \quad b\bar{b} = \bar{b}b \]
\[\bar{a}b = q\bar{b}a, \quad \bar{a}a + q_1^2\bar{b}b = 1 \tag{A}'\]

Denote the \(2j + 1\)-dimensional irreducible representations of \(SL_q(2)\) by \(D^j_{mm'}\) and the \(2j + 1\)-dimensional irreducible representations of \(SU_q(2)\) by \(D^j_{mm'}\), where we set in the knot context:

\[j = \frac{N}{2}, \quad m = \frac{w}{2}, \quad m' = \frac{r + 1}{2} \tag{3.3}\]
and \((N, w, r)\) are the number of crossings, the writhe, and the rotation of the knot.

When the elementary particles are identified as quantum knots, the topological structure of the knot is determined by the electric charge, hypercharge, and isotopic spin as stated earlier in Eq. (1.1).³

We again assume that the action is invariant under global transformations of the \(SU_q(2)\) algebra and local transformations of the \(SU(2) \times U(1)\) group. The connection of the local \(SU(2) \times U(1)\) group leads to the weak interactions. The addition of global invariance of the knot symmetries will slightly modify the standard theory but will not give rise to new fields, as would be the case if the new symmetries were local.²

The uniform treatment of the sources of the gravitational and electroweak interactions will now be based on the gauge invariance of the knot algebras \(SL_q(2)\) and \(SU_q(2)\).

### 4 The Gauge Invariance of \(SL_q(2)\) and \(SU_q(2)\).

The \(2j + 1\)-irreducible representations of \(SL_q(2)\) and \(SU_q(2)\) that we take to describe the symmetries of the generic knot and the oriented knot are respectively as follows:

\[
SL_q(2) \quad D^j_{mm'} = \sum_{s,t} A^j_{mm'}(s, t) \delta(s + t, n'_+) a^s b^{n_+ - s} c'^t d^{n_- - t} 
\]

\[
SU_q(2) \quad D^j_{mm'} = \sum_{s,t} A^j_{mm'}(s, t) \delta(s + t, n'_+) (-q_1)^t a^s b^{n_+ - s} b^t a^{n_- - t} 
\]

where

\[
A^j_{mm'}(s, t) = \left[ \frac{\langle n'_+ \rangle_1! \langle n'_- \rangle_1!}{\langle n_+ \rangle_1! \langle n_- \rangle_1!} \right]^{1/2} \left( \begin{array}{c} n_+ \\ s \end{array} \right)_1 \left( \begin{array}{c} n_- \\ t \end{array} \right)_1
\]

and

\[
n_\pm = j \pm m \\
n'_\pm = j \pm m'
\]

The algebra \((A)\) of \(SL_q(2)\) is invariant under the following gauge transformation

\[
a' = e^{i\varphi_a} a \\
b' = e^{i\varphi_b} b \\
d' = e^{-i\varphi_a} d \\
c' = e^{-i\varphi_b} c
\]
Let us examine the resulting transformation of an arbitrary term of (4.1):

\[(a^n b^m c^n d^m)' = e^{i\varphi_a(n_a-n_d)} e^{i\varphi_b(n_b-n_c)} (a^n b^m c^n d^m)\]  
(4.6)

But by (4.1)-(4.4),

\[
\begin{align*}
n_a - n_d &= s + t - n_\pm = n'_+ - n_\pm = m' + m \\
n_b - n_c &= n_+ - s - t = n_+ - n'_+ = m - m' 
\end{align*}
\]
(4.7)

Then

\[
(a^n b^m c^n d^m)' = e^{i\varphi_a(m+m')} e^{i\varphi_b(m-m')} (a^n b^m c^n d^m) 
\]
(4.9)

Then every term of (4.1) is multiplied by the same factor:

\[e^{i\varphi_a(m+m')} e^{i\varphi_b(m-m')} \]
(4.10)

independent of \(j\).

Hence the gauge transformation (4.5) of the \(SL_q(2)\) algebra \((A)\) induces the following gauge transformation on the irreducible representations of \(SL_q(2)\)

\[
\mathcal{D}^j_{mm'} = e^{i(m+m')\varphi_a} e^{i(m-m')\varphi_b} \mathcal{D}^j_{mm'}
\]
(4.11)

and in particular one gets back (4.5) when \(j = 1/2\).

Let us rewrite (4.11) as follows

\[
\mathcal{D}^j_{mm'} = e^{i(\varphi_a + \varphi_b)m} e^{i(\varphi_a - \varphi_b)m'} \mathcal{D}^j_{mm'}
\]
(4.12)

All of the preceding relations still hold when one passes from \(SL_q(2)\) to \(SU_q(2)\) by setting

\[
d = \bar{a} \\
c = -q_1 \bar{b}
\]
(4.13)

Then (4.12) becomes

\[
\mathcal{D}^j_{mm'} = e^{i\varphi(w)Q(w)} e^{i\varphi(r)Q(r)} \mathcal{D}^j_{mm'}
\]
(4.14)

where

\[
Q(w) \equiv km = k\frac{w}{2} \\
Q(r) \equiv km' = k\frac{r+1}{2}
\]
(4.15)
Here we have set $m = w/2$ and $m' = \frac{r+1}{2}$ as in (3.3). In this way the (4.12) phase transformation introduces two knot charges, $Q(w)$ and $Q(r)$, associated with the writhe and rotation respectively. These are directly related by (3.3) and (1.1) to the charges $t_3$ and $t_0$ of the standard model as shown in Ref. 3. These relations may also be expressed as

$$D^{j}_{mm'} = D^{3t}_{3t_3-3t_0}$$

(4.17)

The $t_3$ and $t_0$ charges stem from the $SU(2) \times U(1)$ symmetries of the standard model while the writhe and rotation charges originate in the $SU_q(2)$ symmetry of the knot.

5 Gravitational “Charges”.

The knot and gravitational symmetries are also closely related since the former symmetry is characterized by $SL_q(2)$ and the latter by gauged $SL(2, C)$. Let us next compare the knot and gravitational “charges”.

We shall assume that the $D^{3/2}_{mm'}$ are those internal wave functions of the fermionic soliton that interact with the gravitational field just as the $D^{3/2}_{mm'}$ are the internal wave functions that interact with the electroweak fields. In both cases $m = \frac{w}{2}$ and $m' = \frac{r+1}{2}$, since the gravitational and electroweak fields probe different aspects of the single solitonic structure defined by $w$ and $r$.

The irreducible representations $D^{j}_{mm'}$ of $SL_q(2)$ are shown in (4.1) and the gauge transformations on $D^{j}_{mm'}$ are given by (4.11). To pass from $SL_q(2)$ to $SL(2, C)$ set $q = 1$ as we have done in (2.3). The algebra (A) is then satisfied by complex numbers and $\det L = 1$ as in (2.4). Then the irreducible representations $D^{j}_{mm'}(q = 1)$ of $SL(2, C)$ may be obtained from (4.1) as

$$D^{j}_{mm'}(q = 1) = \sum_{s,t} A^{j}_{mm'}(q = 1)\delta(s+t, j+m')\delta(ad-bc, 1)a^s b^{n+s} c^t d^{n-t}$$

(5.1)

where the $a, b, c, d$ are now complex commuting numbers.

If the complex numbers $a, b, c, d$ are subjected to the gauge transformations (4.5) then the algebra (A) with $q = 1$ is invariant but the $D^{j}_{mm'}(q = 1)$ are gauge transformed since the complete set of equations from (4.6) and (4.12) is unchanged and therefore (4.12) reads
as follows when \( q = 1 \):

\[
D^j_{mm'}(q = 1) = e^{i(\varphi_a + \varphi_b)m}e^{i(\varphi_a - \varphi_b)m'}D^j_{mm'}(q = 1)
\]  

(5.2)

Since we are requiring all interactions to be invariant under the gauge transformations (4.5) that leave the knot algebra invariant, all interactions are therefore also invariant under the transformations (5.2). Then \( m \) and \( m' \) are conserved in every gravitational interaction and define two gravitational "charges", \( Q_1 \) and \( Q_2 \).

The values of the two conserved charges, \( Q_1 \) and \( Q_2 \), mass and spin, associated with the Lorentz group are fixed by specifying their source and hence by the assignment of \( m \) and \( m' \). Therefore \( Q_1 \) and \( Q_2 \) may be regarded as functions of the conserved \( m \) and \( m' \):

\[
Q_1 = f_1(m, m')
\]

(5.3)

\[
Q_2 = f_2(m, m')
\]

(5.4)

where again \( m = \frac{w}{2} \) and \( m' = \frac{r+1}{2} \).

Using the factorization of a generic matrix into the product of a Hermitian and a unitary matrix, we may explicitly display \( Q_1 \) and \( Q_2 \) by writing the two-dimensional representation of the Lorentz group as the product of a boost, \( H \), and a unitary rotation \( U \)

\[
L = HU
\]

(5.5)

where \( H \) is a Hermitian matrix with \( \det H = 1 \) and \( U \) is a \( SU(2) \) matrix. Then we may write

\[
L = \begin{pmatrix}
\pi_0 + \pi_3 & \pi_1 - i\pi_2 \\
\pi_1 + i\pi_2 & \pi_0 - \pi_3
\end{pmatrix} e^{\frac{i}{2}\vec{\sigma}\vec{\theta}}
\]

(5.6a)

where

\[
\pi_k = p_k/m_0
\]

(5.6b)

Here \((\vec{p}, p_0)\) is the four-momentum of the soliton and \( m_0 \) is its rest mass. Then \( m_0H \) is the matrix defined in (2.7), and \( L \) is a two-dimensional representation of the Lorentz group. Here \( U \) depends on the spin \( \vec{\sigma}/2 \) of the fermionic soliton, and \( H \) depends on its rest mass \( m_0 \). The two "charges" \( Q_1 \) and \( Q_2 \) in (5.3) and (5.4) may be identified with the rest mass and any component, say \( s_3 \) of the spin. (If one bases this discussion on the Poincaré group...
instead of the complex Lorentz group, then the spin is defined by the Pauli-Lubansky vector and the mass is again defined by $p_\mu$.)

6 The Gravitational and Electroweak Interactions.

The affine connection of the local Lorentz group leads to the gravitational couplings. Since we have postulated that the gravitational couplings are blind as to whether the knot is oriented, we shall assume that all solitons enter the gravitational interaction as states of $SL_q(2)$. Therefore we shall express the electroweak and gravitational interactions as follows:

$$\bar{\psi} D\psi \Gamma D + \bar{\psi} \Gamma R$$

(6.1)

and

$$\bar{\psi} \bar{D}\bar{\psi} \Gamma \bar{D} + \bar{\psi} \Gamma \bar{D}$$

(6.2)

where $LD$ and $L\bar{D}$ describe left-chiral normal modes with $D$ and $\bar{D}$ representing $SU_q(2)$ and $SL_q(2)$ symmetries respectively, and where $W$ and $\Gamma$ are the electroweak and gravitational connections respectively. Here $R$ represents a right-chiral singlet which is also an internal singlet. An oriented knot is represented in both (6.1) and (6.2) and an unoriented knot would appear in only the gravitational interaction (6.2). Here $\bar{D}$ is the adjoint of $D$ computed according to (4.2) and

$$a \leftrightarrow \bar{a} \quad \bar{x}y = \bar{y}\bar{x}$$

(6.3)

$$b \leftrightarrow \bar{b}$$

Likewise $\bar{D}$ is the adjoint of $D$ computed according to (4.1) and

$$a \leftrightarrow \bar{d} \quad \bar{x}y = \bar{y}\bar{x}$$

(6.4)

$$b \leftrightarrow \bar{c}$$

Under the gauge transformations (4.11) and (4.14) $\bar{D}D$ and $\bar{D}\bar{D}$ are invariant by (4.7) and (4.8). Therefore (6.1) and (6.2) are invariant under gauge transformations (4.11) and (4.14).

The interactions (6.1) and (6.2) arise in the covariant derivative terms of the following form

$$\bar{\psi} \nabla \psi$$

(6.5)
where
\[ \nabla = \partial + W + \Gamma \]  

(6.6)

The term (6.5) is invariant under the local group of the standard theory but also under the
global gauge transformations (4.11) and (4.14).

To evaluate (6.1) and (6.2) we next turn to the spectrum of \( SL_q(2) \).

7 Spectrum of \( SL_q(2) \) Algebra \((A)\).

Since \( b \) and \( c \) commute, they have common eigenstates. Let \( |0\rangle \) be designated as a ground
state and let

\[
\begin{align*}
    b|0\rangle &= \beta|0\rangle \\
    c|0\rangle &= \gamma|0\rangle \\
    bc|0\rangle &= \beta\gamma|0\rangle
\end{align*}
\]

(7.1)  (7.2)  (7.3)

We assume that \( b \) and \( c \) are Hermitian:

\[
\begin{align*}
    b &= \bar{b} \\
    c &= \bar{c}
\end{align*}
\]

(7.4)  (7.5)

Then the eigenvalues \( \beta, \gamma \) are real and the eigenfunctions are orthogonal.

From the algebra we see that

\[ bc|n\rangle = E_n|n\rangle \]  

(7.6)

where

\[
\begin{align*}
    |n\rangle &\sim d^n|0\rangle \\
    E_n &= q^{2n}\beta\gamma
\end{align*}
\]

(7.7)  (7.8)

Here \( d \) and \( a \) are raising and lowering operators respectively.

\[
\begin{align*}
    d|n\rangle &= \lambda_n|n+1\rangle \\
    a|n\rangle &= \mu_n|n-1\rangle
\end{align*}
\]

(7.9)  (7.10)
Then

\[ \begin{align*}
ad |n\rangle &= a\lambda_n |n+1\rangle \\
&= \lambda_n \mu_{n+1} |n\rangle \\
a |n\rangle &= d\mu_n |n-1\rangle \\
&= \mu_n \lambda_{n-1} |n\rangle
\end{align*} \]  
(7.11)

From the algebra (A)

\[ \begin{align*}
(1 + qbc) |n\rangle &= \lambda_n \mu_{n+1} |n\rangle \\
&= \lambda_n \mu_{n+1} |n\rangle \\
(1 + q_1bc) |n\rangle &= \mu_n \lambda_{n-1} |n\rangle \\
&= \mu_n \lambda_{n-1} |n\rangle
\end{align*} \]  
(7.12)

If there is a highest state \( M \), and a lowest state \( M' \), then

\[ \lambda_M = \mu_{M'} = 0 \quad M' < M \]  
(7.15)

By (7.13) and (7.14)

\[ \begin{align*}
(1 + qbc) |M\rangle &= 0 \\
(1 + q_1bc) |M'\rangle &= 0
\end{align*} \]  
(7.16)

Then by (7.8)

\[ q^{2M+1} \beta \gamma = q^{2M'-1} \beta \gamma \]  
(7.17)

or

\[ (q^2)^{M-M'+1} = 1 \]  
(7.18)

As we continue to assume that \( q \) is real,

\[ M' = M + 1 \]  
(7.19)

Since (7.15) and (7.20) are contradictory, there may be either a highest or a lowest state, but not both.

According to the general rules of our model,\(^{1,2,3}\) the individual states of excitation of the soliton represented by \( D^i_{mm'} \) are \( D^i_{mm'} |n\rangle \). Since the empirical evidence appears to restrict the number of states, there must be an externally required physical boundary condition to cut off the otherwise infinite spectrum that is formally required by (7.15) and (7.20).
8 Evaluation of Electroweak and Gravitational Interactions.

To evaluate the interactions (6.1) and (6.2) between left-chiral solitonic states let us first consider the generic electroweak interaction:

\[ \mathcal{F}_3 \mathcal{W}_2 \mathcal{F}_1 \]  

where

\[ \mathcal{F}_1 = F_1(p, s, t) D_{m_1 m_1'}^{3/2} |n_1\rangle \]  

\[ \mathcal{F}_3 = F_3(p, s, t) D_{m_3 m_3'}^{3/2} |n_3\rangle \]  

\[ \mathcal{W}_2 = W_2(p, s, t) D_{m_2 m_2'}^{j} \]  

Here \( F(p, w, t) \) and \( W(p, s, t) \) are the standard left chiral fermionic and electroweak normal modes where \((p, s, t)\) represent momentum, spin and isotopic spin. The \( D_{m m'}^j \) are the internal state factors attached to the standard normal modes of the electroweak connection.

Then (8.1) may be rewritten as

\[ (\mathcal{F}_3 \mathcal{W}_2 \mathcal{F}_1) \langle n_3 | D_{m_3 m_3'}^{3/2} D_{m_2 m_2'}^{j} D_{m_1 m_1'}^{3/2} | n_1 \rangle \]  

The correction to the standard matrix element appears in the second factor:

\[ \langle n_3 | D_{m_3 m_3'}^{3/2} D_{m_2 m_2'}^{j} D_{m_1 m_1'}^{3/2} | n_1 \rangle \]  

We require that this internal factor or form factor be invariant under \( U_a(1) \times U_b(1) \) defined by (4.5) and (4.14). It then follows that

\[ m_1 + m_2 = m_3 \]  

\[ m_1' + m_2' = m_3' \]  

i.e., the topological and isotopic observables whose eigenvalues are \( m \) and \( m' \) are conserved.

The conservation of \( m \) and \( m' \) then make it possible to label the \( D_{m m'}^j \) by either topological or isotopic quantum numbers as follows:

\[ j = 3t \quad m = -3t_3 \quad m' = -3t_0 \]  

\[ j = N/2 \quad m = w/2 \quad m' = r + 1/2 \]
and to follow the same rules for the bosons as for the fermions. These identifications are discussed in Refs. 2 and 3.

We shall assume the same construction for the gravitational matrix element corresponding to (8.6) but also in accordance with (6.2), namely:

$$\langle n_3 | \tilde{D}_{m_3m_3}^{3/2} D_{m_2m_2}^{i} D_{m_1m_1}^{3/2} | n_1 \rangle$$

(8.11)

where the $D_{mm'}$ correspond to $SL_q(2)$ rather than $SU_q(2)$ and where $\tilde{D}_{mm'}^{3/2}$ is the modified adjoint computed according to (4.1) and (6.4).

We now propose to use the same conservation laws (8.7) and (8.8) to assign quantum numbers to the internal boson factor $D_{m_2m_2}^{j}$ in (8.11). For the graviton, however, we choose $j = 0$. Therefore we set

$$D_{m_2m_2}^{j} = D_{00}^{0} = 1$$

(8.12)

Then the graviton is an unknotted clockwise loop. Therefore the correction factor or form factor, in the gravitational case is simply

$$\langle n_3 | \tilde{D}_{m_3m_3}^{3/2} D_{m_3m_3}^{3/2} | n_1 \rangle \delta(n_3, n_1)$$

(8.13)

In this section the distinction that we have made between the electroweak and gravitational form factors lies entirely in the use of $D$ and $\tilde{D}$, while the labelling (quantum numbers) is the same for both $D$ and $\tilde{D}$ since they describe different aspects of the same trefoil.

9 The Quark-Gluon Interaction.

Let us consider the tensor products

$$a' = U_a a \quad b' = U_b b$$

$$d' = U_a^{-1} d \quad c' = U_b^{-1} c$$

(9.1)

where $U_a$ and $U_b$ are commuting elements of a unimodular unitary group that also commute with $(a, b, c, d)$. Then $(a', b', c', d')$ satisfy the same algebra $(A)$ as $(a, b, c, d)$.

Let $D_{nmn'}^{ij}$ be an element of an irreducible unitary representation of the algebra $(A)$. Then by the same argument that leads to (4.11)

$$D_{nmn'}^{ij} = U_a^{m-m'} U_b^{m-m'} D_{nmn'}^{ij}$$

(9.2)
set

\[ U_a = e^{iQ_a \theta_a} \quad \text{Tr } Q_a = 0 \]  
\[ U_b = e^{iQ_b \theta_b} \quad \text{Tr } Q_b = 0 \] 

where \( Q_a \) and \( Q_b \) are both Hermitian and diagonal and hence real. Let us now restrict \( U_a \) and \( U_b \) to the center of the \( SU(3) \) group that describes the strong couplings. These couplings, the gluon-quark interactions in the standard theory, have the following form

\[ \mathcal{L}_{\text{int}}(\text{quark} - \text{gluon}) = -\frac{g_0}{2} \bar{q}(Q_A A + Q_B B) q - \frac{g_0}{\sqrt{2}} [\bar{q}(\tau_{21} X + \tau_{31} Y + \tau_{32} Z) q + \text{c.c.}] \]  

where the generators of \( SU(3) \) in the fundamental representation are

\[ Q_A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Q_B = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix} \]  

and

\[ \tau_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \tau_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \tau_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]  

In the present model the quarks have a knot structure as already described. If a knot structure is also assigned to the gluons, then each term of (9.5) will acquire a knot factor. We shall require that the so amended action be invariant not only under local \( SU(3) \) strong, as required by standard theory, but also under the global transformation (9.1) and therefore (9.2). These global transformations may be re-expressed as

\[ \mathcal{D}^j_{mm'} = (U_a U_b)^m (U_a U_b^{-1})^{m'} \mathcal{D}^j_{mm'} \]
\[ = e^{i(Q_a \theta_a + Q_b \theta_b)m} e^{i(Q_a \theta_a - Q_b \theta_b)m'} \mathcal{D}^j_{mm'} \]
\[ = e^{iQ_A \theta_A} e^{iQ_B \theta_B} \mathcal{D}^j_{mm'} \]  

where

\[ Q_A \theta_A = (Q_a \theta_a + Q_b \theta_b)m \]  
\[ = (Q_a \theta_a + Q_b \theta_b)(-3t_3) \]
where
\[
Q_B \theta_B = (Q_a \theta_a - Q_b \theta_b) m' = (Q_a \theta_a - Q_b \theta_b)(-3t_0)
\] (9.10a) (9.10b)

The same equations (9.8)-(9.10) hold if $D$ is replaced by $D$. We shall now identify $Q_A$ and $Q_B$ as the color charge and the color hypercharge respectively as described by the standard theory. Here $Q_A \sim t_3$ and $Q_B \sim t_0$ according to Eqs. (9.9b) and (9.10b). (Note, however, that $Q_A \sim t_3$ and not $t_3 + t_0$.) The color charge and hypercharge are given by (9.6) and Table 1.

**Table 1.**

| Color | $Q_A$ | $Q_B$ |
|-------|-------|-------|
| red   | 1/2   | $\sqrt{3}/6$ |
| yellow| $-1/2$| $\sqrt{3}/6$ |
| green | 0     | $-\sqrt{3}/3$ |

These are the color charge and color hypercharge of the fermionic soliton with $r = -2$ (or weak hypercharge =1/6). If $r = 2$ (or $t_0 = -1/2$) $Q_A = Q_B = 0$.

In Table 2 one finds for the solitons of the $u$-family and the $d$-family the explicit relations between the weak and the strong charges that are shown in Table 1. There the strong color charge $Q_A$ is related to the weak $t_3$ and the strong color hypercharge $Q_b$ is related to $t_0$ as indicated in (9.9b) and (9.10b).

**Table 2.**

| $t_3$ | $t_0$ | $D^{3/2}_{-3t_3-3t_0}$ | $Q_A$ | $Q_B$ | $Q_B$ |
|-------|-------|------------------------|-------|-------|-------|
| $u$   | $\frac{1}{2}$ | $\frac{1}{6}$ | $D^{3/2}_{-3/2}$ | $-t_3$ | $-t_3$ | $0$ | $\sqrt{3}t_0$ | $\sqrt{3}t_0$ | $-2\sqrt{3}t_0$ |
| $d$   | $-\frac{1}{2}$ | $\frac{1}{6}$ | $D^{3/2}_{3/2}$ | $-1/2$ | $t_3$ | $t_3$ | $0$ | $\sqrt{3}t_0$ | $\sqrt{3}t_0$ | $-2\sqrt{3}t_0$ |

As illustrated in Table 1 there are three varieties of solitons with $r = -2$ that differ in gluon charge and are conventionally labelled $R$, $Y$, and $G$. There is only one variety of soliton with $r = 2$. There are eight gluons including the charged triplet ($X, Y, Z$) and their antigluons as
well as the uncharged $A$ and $B$ displayed in Eq. (9.5). The charges of $(X,Y,Z)$ are fixed by the conservation of gluon charge and hypercharge together with the assignments to $u$ and $d$ of $Q_A$ and $Q_B$ shown in Table 2. The charges of $(X,Y,Z)$ are shown in Table 3 expressed in terms of $t_3$ and $t_0$ for the $u$ soliton.

Table 3.

|   | $Q_A$  | $Q_B$  |
|---|--------|--------|
| $X$ | $-2t_3$ | 0      |
| $Y$ | $-t_3$  | $-3\sqrt{3}t_0$ |
| $Z$ | $t_3$   | $-3\sqrt{3}t_0$ |

The strong charges that are, according to Eqs. (9.9b) and (9.10b) and to Tables 1 and 3, simply related to the weak $t_3$ and $t_0$, are in turn simply related to the topological description of the knot source since $t_3 = -\frac{w}{6}$ and $t_0 = -\frac{r+1}{6}$.

The content of the preceding tables may be simply expressed by the following relations (induced by the transformations (9.1) on the knot algebra).

By (9.9b) and (9.10b)

\[ Q_A = \hat{Q}_A t_3 \quad (9.11) \]
\[ Q_B = \hat{Q}_B t_0 \quad (9.12) \]

where

\[ \text{Tr } \hat{Q}_A = \text{Tr } \hat{Q}_B = 0 \quad (9.13) \]

and

\[ t_3 = \pm \frac{1}{2}, \quad t_0 = \frac{1}{6} \quad (9.14) \]

Since (9.11) and (9.12) must agree with (9.6) of the standard theory, we have

\[ \hat{Q}_A = \begin{pmatrix} \pm 1 \\ \mp 1 \\ 0 \end{pmatrix} \quad (9.15) \]
\[ \hat{Q}_B = \sqrt{3} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad (9.16) \]
where the upper and lower signs correspond to $u$ and $d$ respectively. The linear relation between $(Q_A, Q_B)$ and $(w, r)$ that is implied by (9.11) and (9.12) is completed by (9.15) and (9.16), as well as by $t_3 = \frac{w}{2}$ and $t_0 = \frac{1}{2}(r + 1)$.

We shall now describe a modification of (9.5) that is similar to (8.11) and (8.6) where we have introduced knot form factors for the gravitational and electroweak interactions. For the gluon form factors we assume that the quark knot is oriented and therefore choose

$$\langle n_3 | \bar{D}^{3/2}_{m_3m_3'}(c_3) D^j_{m_2m_2'}(c_2) \bar{D}^{3/2}_{m_1m_1'}(c_1) | n_1 \rangle$$

(9.17)

where $m = w/2$ and $m' = \frac{r+1}{2}$ as in the gravitational and electroweak cases. Here the first and third factors are the internal state functions of the interacting quarks while the second factor is the internal state function of a gluon. Each quark and gluon may acquire one of three colors according to (9.11) and (9.12) but the internal state function does not depend explicitly on the color index $(c)$. The transformation of $D^j_{m_2m_2'}(c)$ under $U_a(1) \times U_b(1)$ does depend on $c$ according to (9.8) holding for $D$. In addition to the conservation of color in the standard theory there is now the conservation of $m$ and $m'$ that stems from invariance under global $U_a(1) \times U_b(1)$ and fixes $m_2$ and $m_2'$ in terms of the $m$ and $m'$ of the initial and final quarks.

The different facets of the knotted soliton required to describe its different couplings may be related by the charges (integrals of the motion) corresponding to these different couplings. First, as a topological object the knotted soliton has two “charges”, the writhe and the rotation. Next, as a source of the electroweak fields it also has two charges, the electric charge and the hypercharge coming from $SU(2) \times U(1)$. Then there are again two integrals of the motion, mass and spin, stemming from the Lorentz group, which, as a local group, leads to the gravitational couplings. Finally, as a source of the gluon field, the quarks have two additional charges, color charge and color hypercharge, eigenvalues of the commuting generators of $SU(3)$. In summary, if one labels the particle by assigning the conserved $w$ and $r$, then, depending on the context, two conserved charges $Q_A$ and $Q_B$ are fixed by

$$Q_A = F_A(w, r)$$

$$Q_B = F_B(w, r)$$

(9.18)
where $Q_A$ and $Q_B$ come from one of the groups $SL(2c)$, $SU(2) \times U(1)$, or $SU(3)$ and correspond to the different mappings of $(w, r)$ onto $(Q_A, Q_B)$ that are denoted by $F_A$ and $F_B$.

In the case of the electroweak and the gluon couplings the relations $F_A$ and $F_B$ are linear. In the Lorentz or gravitational case one may write $s_3 = \pm \frac{w}{6}$ for the spin, but the relation of the mass to $(w, r)$ is not linear and will be considered next.

10 Hamiltonian and Integrals of the Motion of the Quantum Knot.

Let the Hamiltonian of the quantum knot be $H(b, c)$. Then since the generic knot is defined by (4.1), we have

$$H(b, c)D^{j}_{mm'}|n⟩ = H(b, c) \left[ \sum_{s, t} A_{mm'}^j \delta(s + t, n'_+)a^s b^{n_+ - 3} c^t d^{m_+ - t} \right]|n⟩$$

(10.1)

$$= D^{j}_{mm'} H(q_1^{n_a-n_d b}, q_1^{n_a-n_d c})|n⟩$$

(10.2)

where $n_a$ and $n_d$ are the exponents of $a$ and $d$ respectively. Then by relations like (7.8)

$$H(b, c)D^{j}_{mm'}|n⟩ = D^{j}_{mm'} H(q_1^{n_a-n_d b}, q_1^{n_a-n_d c})|n⟩$$

(10.3)

where

$$E^{j}_{mm'}(n) = H(\lambda \beta, \lambda \gamma)$$

(10.4a)

and

$$\lambda = q^{n-(m+m')}$$

(10.4b)

by (4.7). Therefore the $D^{j}_{mm'}$ are eigenstates of $H(b, c)$ and the indices on $D^{j}_{mm'}$ are the eigenvalues of the integrals of the motion. The eigenvalues of $H(b, c)$ are $H(\lambda \beta, \lambda \gamma)$ by (10.4).

The operators that represent integrals of the motion may be expressed in terms of an elementary operator $\omega_x$ that may be defined by its action on every term of $D^{j}_{mm'}$ as follows:

$$\omega_x(\ldots x^{n_x} \ldots) = n_x(\ldots x^{n_x} \ldots) \quad x = (a, b, c, d)$$

(10.5)
i.e., $\omega_x$ acts like $x \frac{\partial}{\partial x}$.

Then define

\[ J = \frac{1}{2}(\omega_a + \omega_b + \omega_c + \omega_d) \quad (10.6) \]
\[ W = \frac{1}{2}(\omega_a - \omega_d + \omega_b - \omega_c) \quad (10.7) \]
\[ R = \frac{1}{2}(\omega_a - \omega_d - \omega_b + \omega_c) \quad (10.8) \]

When $J$, $W$, and $R$ act on $D_{mm'}^j$ one finds by (10.5)

\[ J D_{mm'}^j = j D_{mm'}^j \quad (10.9) \]
\[ W D_{mm'}^j = m D_{mm'}^j \quad (10.10) \]
\[ R D_{mm'}^j = m' D_{mm'}^j \quad (10.11) \]

The operators ($J$, $W$, $R$) are all interpreted as integrals of the motion with different physical identifications of the eigenvalues $m$ and $m'$ for $SL_q(2)$, $SU_q(2)$, and $SU(2)$ in the different examples we have discussed. For $SL_q(2)$ $m$ and $m'$ will be functions of mass and spin; for $SU_q(2)$ $m$ and $m'$ will be functions of charge and hypercharge or alternatively of writhe and rotation. For $SU(2)$ one has the non-solitonic examples of the spherical top and $H$-atom mentioned in the introduction.

We also introduce the inversion operator transforming $D$ into $\tilde{D}$:

\[ \tilde{D} = ID \quad (10.12) \]

$I$ which is defined by (6.4) interchanges the exponents of $a$ and $d$ and of $b$ and $c$ and transforms the algebra $A$ into the same algebra with $q \rightarrow q_1$. Note that

\[ I J = J \]
\[ I W = -W \quad (10.13) \]
\[ I R = -R \]

In this section the state function of the knot has been taken to be $D_{mm'}^j$. All of the statements of this section remain correct for $D_{mm'}^j$ when one passes to the subalgebra $(A)'$. 

20
11 Masses of the Trefoils.

In an atomic or nuclear system the energy levels are calculated with the aid of a Hamiltonian which in turn is determined by the assumed potential energy. If on the other hand the system is defined, as it is here, solely by its symmetry group, the only restriction on the Hamiltonian is provided by the group itself. We have taken the Hamiltonian of the knot to be a function of the operators $b$ and $c$, say, $f(b, c)$. Since the knot is here defined by state functions $D_{\frac{3}{2}, \frac{w}{2} + \frac{r}{2} + 1}$ we see by (10.4) that the eigenvalues of $f(b, c)$ are given by

$$E(w, r, n) = f(\lambda\beta, \lambda\gamma)$$

$$\lambda = q^{n-(w+r+1)/2}$$

where $E(w, r, n)$ will be interpreted as the mass of the $n^{th}$ excited state of the trefoil with writhe $w$ and rotation $r$.

The choice of the function $f(b, c)$ can be fixed by the requirement that $f(b, c)$ agree with the mass term in the action of the standard theory, namely

$$\bar{L}\varphi R + \bar{R}\varphi L$$

In the standard theory $L$ and $R$ are the left- and right-chiral components of the fermion field and $\varphi$ is the Higgs scalar. $L$ and $\varphi$ are $SU(2)$ isotopic doublets and $R$ is an isotopic singlet so that $\bar{L}\varphi R$ is an isotopic invariant.

In the present model, we assume that $R$ is a singlet in the knot algebra as well as in the isotopic group. Then (11.3) reduces to

$$\bar{L}\varphi + \varphi L$$

where we have dropped the external spacetime factors in $L, \varphi$ and $R$ and retained only the internal state factors.

To translate (11.3) into knot language, note that an oriented knot is represented in gravitational interactions by $D_{\frac{3}{2}, \frac{w}{2} + \frac{r}{2} + 1}$ and in electroweak interactions by $D_{\frac{3}{2}, \frac{w}{2} + \frac{r}{2} + 1}$ according to (6.1) and (6.2). In the mass term (11.4) we choose to represent $L$ by the form, $D_{\frac{3}{2}, \frac{w}{2} + \frac{r}{2} + 1}$, appropriate to gravitational interactions, since the gravitational field of a particle measures
the mass of that particle. Since we require that (11.4) be a knot invariant as well as an isotopic invariant, we finally assume that $\varphi$ is also represented by $D_{\frac{3}{2}+1}^3$. 

Then we fix $f(b, c)$, the Hamiltonian of the knot by setting

$$f(b, c) \sim \bar{L}_\varphi + \varphi L$$

(11.5)

or

$$f(b, c) \sim D_{\frac{3}{2}+1}^3 D_{\frac{3}{2}+1}^3$$

(11.6)

By the algebra $(A)$ every expression (11.6) is a function of $b$ and $c$ only. Then the masses are

$$m(w, r, n) \sim \langle n | D_{\frac{3}{2}+1}^3 D_{\frac{3}{2}+1}^3 | n \rangle$$

(11.7)

Since this expression (11.7) is the same as (8.13), it follows from this result that the gravitational couplings are proportional to the mass in this model.

In previous work\(^1,^2\) we have compared the empirical masses of the elementary fermions with

$$m(w, r, n) \sim \langle n | D_{\frac{3}{2}+1}^3 D_{\frac{3}{2}+1}^3 | n \rangle$$

(11.8)

It is possible to fit the observed masses with either (11.7) or (11.8) by adjusting $q$ and $\beta$ but the spectrum predicted by both must be cut off since there appears to be not more than three members in each family of fermions. Here, as in our earlier work, we assume that the three members of each family occupy the three lowest states of energy.

Remarks.

In assuming that the elementary particles are knots of field one assigns a unifying role to the knot algebra. In this way the electroweak, color, and gravitational symmetries may be related to the symmetries of the knot. As a result, charge and hypercharge, color charge and color hypercharge, as well as mass and spin of the elementary fermions may be expressed as functions of the writhe and rotation of the corresponding trefoils. These functions are determined jointly by the gauge invariance of the knot algebra and by the $SU(2) \times U(1)$, and $SU(3)$, and $SU(2c)$ groups. In the field theoretic implementation of this correspondence it is proposed that the state functions of the quantum knots be attached to the normal modes of the standard theory with the result that the interaction terms of the standard theory...
are multiplied by form factors reflecting the trefoil origins of the quantum numbers of the standard theory.

If it is correct that the trefoils play a fundamental role in the structure of the elementary particles, then it is natural to ask if the elementary trefoil can be realized as an elementary boson as well as an elementary fermion, i.e., as a scalar as well as a chiral spinor. (The fact that there are 4 trefoils and 4 classes of fermions as well as a unique correspondence between them, makes it possible to interpret the 3 members of each fermionic class as eigenstates of the corresponding knot.) In comparing the conjectured fermionic trefoils with the physical fermions, one finds that the topology of the trefoil, as defined by the writhe and rotation, correlates uniquely via the knot algebra with the isotopic structure of the physical fermion as defined by its charge and hypercharge. If one conjectures that the topology in fact determines the charge structure, then if the scalar trefoil in fact exists, one may also conjecture that this trefoil, sharing the topology of the spinor trefoil, will also share the same charge and hypercharge structure. This extended model would suggest a search for the charged scalars with the same charge structure as the fermions.

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