ON INFINITENESS OF INTEGRAL OVERCONVERGENT DE RHAM–WITT COHOMOLOGY MODULO TORSION

VERONIKA ERTL AND ATSUSHI SHIHO

Abstract. In this article, we give examples of smooth varieties of positive characteristic whose first integral overconvergent de Rham–Witt cohomology modulo torsion is not finitely generated over the Witt ring of the base field.

Key Words: infiniteness, overconvergent de Rham–Witt cohomology, $C_{ab}$-curves

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INTRODUCTION

Let $k$ be a perfect field of characteristic $p > 0$, let $W := W(k)$ be the ring of $p$-typical Witt vectors of $k$ and let $K$ be the fraction field of $W$. For a smooth variety $X$ over $k$, its crystalline cohomology $H^i_{	ext{crys}}(X/W)$ is defined by Berthelot [1] and it is shown that, when $X$ is proper, it is finitely generated over $W$. (See [4] for example.) Also, Illusie [11] introduced the notion of de Rham–Witt complex $WΩ^•_X$ which is a complex of étale sheaves on $X$ and proved that its cohomology $H^i(X, WΩ^•_X)$ (called the de Rham–Witt cohomology) is isomorphic to the crystalline cohomology $H^i_{	ext{crys}}(X/W)$. Thus the de Rham–Witt cohomology is finitely generated over $W$ when $X$ is proper smooth over $k$.

When $X$ is smooth but not proper, its crystalline cohomology (hence its de Rham–Witt cohomology also) is not necessarily finitely generated. To remedy this infiniteness, Berthelot [2], [3] introduced the notion of rigid cohomology $H^i_{	ext{rig}}(X/K)$ as a corrected variant of crystalline cohomology tensored with $Q$ using $p$-adic analytic geometry, and proved that it is finite dimensional over $K$. However, rigid cohomology does not a priori have a canonical $W$-lattice. So it would be an interesting problem to construct a finitely generated $W$-lattice of rigid cohomology which has nice properties.

For a smooth variety $X$ over $k$, Davis–Langer–Zink [6] introduced the overconvergent de Rham–Witt complex $W^iΩ^•_X$ as a certain subcomplex of $WΩ^•_X$ and proved that, when $X$ is quasi-projective, its rational cohomology $H^i(X, W^iΩ^•_X) \otimes Q$ is isomorphic to the rigid cohomology $H^i_{	ext{rig}}(X/K)$. Although it is well-known that the integral overconvergent de Rham–Witt cohomology $H^i(X, W^iΩ^•_X)$ can have infinitely generated torsions (e.g. in the case $X = A^1_k$ and $i = 1$), one may naively expect that the image

$$\overline{\mathcal{I}}(X, W^iΩ^•_X) := \text{Im}(H^i(X, W^iΩ^•_X) \to H^i(X, W^iΩ^•_X) \otimes Q),$$

which we will call the integral overconvergent de Rham–Witt cohomology modulo torsion, might give a finitely generated $W$-lattice of the rigid cohomology.

However, various problems seem to arise when one tries to adapt the proofs for finiteness of rigid cohomology in [3], [15], [19], [20], and [13] to the case of $\overline{\mathcal{I}}(X, W^iΩ^•_X)$, because all of these proofs use homological algebra at some instance which is rather delicate when dividing by torsion. In this article, we give a negative answer to the above expectation, by giving counterexamples. In particular, we prove the following result.

**Theorem 0.1** (= Corollary 3.2). For any prime number $p$ and any perfect field $k$ of characteristic $p$, there exists an affine smooth curve $X$ over $k$ such that the first integral overconvergent de Rham–Witt cohomology modulo torsion $\overline{\mathcal{I}}^1(X, W^1Ω^•_X)$ is not finitely generated over $W$. 

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Conventions. Throughout the paper, $p$ will be a fixed prime number. $k$ will be a perfect field of positive characteristic $p > 0$, $W := W(k)$ its ring of $p$-typical Witt vectors and $K = \text{Frac}(W(k))$ will be the fraction field of $W$. By a variety over $k$ we always mean a separated and integral scheme of finite type over $k$. Let moreover $\nu_p$ denote the $p$-adic valuation.

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1. Overconvergent de Rham–Witt cohomology and Monsky–Washnitzer cohomology

As we recalled in the introduction, the integral overconvergent de Rham–Witt cohomology $H^i(X, W^iΩ^\bullet_X)$ is defined for any smooth variety $X$ over $k$, but it is not so easy to compute it directly for general $X$. When $X$ is affine and smooth, there exists a simpler construction of the cohomology $H^i_{MW}(X/W)$ (called the integral Monsky–Washnitzer cohomology), which is due to Monsky and Washnitzer [17]. In this section, we briefly recall the definition of the integral Monsky–Washnitzer cohomology and recall the comparison theorem between the integral overconvergent de Rham–Witt cohomology and the integral Monsky–Washnitzer cohomology.

Let $X = \text{Spec}(A)$ be an affine smooth variety over $k$ and take a lift $X = \text{Spec}(A)$ of $X$ to an affine smooth scheme over $W$. (The existence of such a lift is due to Elkik [9].) Let $A^{\dagger}$ be the weak completion of $A$ (defined by Monsky–Washnitzer), and let $Ω^\bullet_{A^{\dagger}}$ be the de Rham complex of continuous differentials of $A^{\dagger}$ over $W$. We define the integral Monsky–Washnitzer cohomology $H^i_{MW}(X/W)$ of $X$ by

$$H^i_{MW}(X/W) := \text{H}^i(Ω^\bullet_{A^{\dagger}}) .$$

It is known that this definition is independent of the choice of the lift $X = \text{Spec}(A)$ [10]. Then we have the following comparison theorem:

**Theorem 1.1** (Davis–Langer–Zink [6], Davis–Zureick-Brown [7], Ertl–Sprang [10]). Let $X$ be an affine smooth variety over $k$. Then there exists a canonical isomorphism $H^i_{MW}(X/W) \cong H^i(X, W^iΩ^\bullet_X)$.

As in the case of overconvergent de Rham–Witt cohomology, we define the integral Monsky–Washnitzer cohomology modulo torsion $\overline{H}^i_{MW}(X/W)$ by

$$\overline{H}^i_{MW}(X/W) := \text{Im}(H^i_{MW}(X/W) \to H^i_{MW}(X/W) \otimes \mathbb{Q}) .$$

Then, Theorem 1.1 implies the isomorphism

$$\overline{H}^i_{MW}(X/W) \cong \overline{H}^i(X, W^iΩ^\bullet_X).$$

We next recall a relation between Monsky–Washnitzer cohomology and algebraic de Rham cohomology. Let $X = \text{Spec}(A)$, $X = \text{Spec}(A)$ as before Theorem 1.1 and let $Ω^\bullet_A$ be the de Rham complex of algebraic differentials of $A$ over $W$. We define the integral algebraic de Rham cohomology $H^i_{\text{dR}}(X/W)$ by

$$H^i_{\text{dR}}(X/W) := \text{H}^i(Ω^\bullet_A) ,$$

and the integral algebraic de Rham cohomology modulo torsion by

$$\overline{H}^i_{\text{dR}}(X/W) := \text{Im}(H^i_{\text{dR}}(X/W) \to H^i_{\text{dR}}(X/W) \otimes \mathbb{Q}) .$$

The canonical map of weak completion $ι : A \to A^{\dagger}$ induces a morphism

$$ι_* : H^i_{\text{dR}}(X/W) \to H^i_{MW}(X/W) ,$$
hence the commutative diagram with injective vertical arrows:

$$\begin{array}{ccc}
\mathcal{T}_{\text{dR}}(X/W) & \xrightarrow{\tau_*} & \mathcal{T}_{\text{MW}}(X/W) \\
H'_{\text{dR}}(X/W) \otimes \mathbb{Q} & \xrightarrow{\iota_* \mathbb{Q}} & H'_{\text{MW}}(X/W) \otimes \mathbb{Q}.
\end{array}$$

In general, it is not necessarily true that $\iota_* \mathbb{Q}$ is an isomorphism.

2. $C_{ab}$-CURVES

In this section, we give a review of the result of Denef–Vercauteren [8 §3] on the computation of rational algebraic de Rham cohomology and rational Monsky–Washnitzer cohomology of $C_{ab}$-curves. (This is a generalization of the computation by Kedlaya [12] in the case of hyperelliptic curves.)

First we recall the definition of $C_{ab}$-curves.

**Definition 2.1.** Let $a, b$ be coprime positive integers and let $L$ be a field of characteristic prime to $ab$. A $C_{ab}$-curve over $L$ is an affine smooth plane curve $X$ over $L$ defined by an equation of the form

$$\overline{f}(x, y) := y^a + \sum_{j=1}^{a-1} f_j(x) y^j + f_0(x) = 0,$$

where $f_j(x) \in L[x]$ for $0 \leq j \leq a - 1$ with $\deg f_0 = b$ and $a \deg f_j + bj < ab$ for $1 \leq j \leq a - 1$.

**Remark 2.2.**

(i) In some references, the smooth compactification of $X$ is called a $C_{ab}$-curve. However, we adopt the above definition because we will not use the compactification so much.

(ii) When $f_j(x) = 0$ for all $1 \leq j \leq a - 1$, the curve $X$ is called a superelliptic curve (minus one point). When we assume moreover that $a = 2$, the curve $X$ is called a hyperelliptic curve (minus one point of characteristic prime to $2b$). When we assume moreover that $b = 3$, the curve $X$ is called an elliptic curve (minus one point of characteristic prime to 6).

(iii) The smoothness assumption on $X$ is nothing but the Jacobian criterion associated to the equation (3). In the case of superelliptic curves, $X$ is smooth if and only if $f_0(x)$ does not have multiple roots in the algebraic closure of $L$.

Then the following facts are known:

**Fact 2.3.** [8 p.81], [14]

(i) There exists a unique $L$-rational point at infinity (the point in the smooth compactification of $X$ which is not in $X$), which we denote by $P_{\infty}$.

(ii) $\text{ord}_{P_{\infty}}(x) = -a$, $\text{ord}_{P_{\infty}}(y) = -b$.

(iii) The genus of the compactification of $X$ is equal to $(a - 1)(b - 1)/2$.

Now let $a, b$ be coprime positive integers prime to $p$ and consider a $C_{ab}$-curve $X$ over $k$ defined by the equation (3) (with $L$ replaced by $k$).

For each $0 \leq j \leq a$, we take a lift $f_j(x) \in W[x]$ of $\overline{f}_j(x)$ with $\deg f_j = \deg \overline{f}_j$. By definition, we can write explicitly that

$$f_0(x) = \sum_{i=0}^{b} c_{i0} x^i, \quad f_j(x) = \sum_{ai+bj<ab} c_{ij} x^i \quad (1 \leq j \leq a - 1)$$

with $c_{i0}, c_{ij} \in W$. Then we have the following:

**Lemma 2.4.** The equation

$$f(x, y) := y^a + \sum_{j=1}^{a-1} f_j(x) y^j + f_0(x) = 0$$

defines a smooth lift $X$ of $X$ over $W$. 
Proof. This is implicitly proven in [8] §3, but we sketch the argument for the convenience of the reader. Let $F_0 = f(X^a, Y^b)$, $F_1 = \frac{\partial f}{\partial x}(X^a, Y^b)$, $F_2 = \frac{\partial f}{\partial y}(X^a, Y^b)$ and let $F_i^h(X, Y, Z) (i = 0, 1, 2)$ be the homogenization of $F_i$.

Let $\mathbb{F}_i (i = 0, 1, 2)$ be the mod $p$ reduction of $F_i^h$. Then the equation $\mathbb{F}_0 = \mathbb{F}_1 = \mathbb{F}_2 = 0$ has no solution in the projective space $\mathbb{P}^2(k^{alg})$. Indeed, when $Z \neq 0$, this follows from the smoothness of the curve $X$ over $k$ and when $Z = 0$, the above equality becomes

$$Y^{ab} + X^{ab} = bX^{a(b-1)} = aY^{(a-1)b} = 0,$$

which has no nontrivial solution.

Then, [8] Theorem 2 implies that there exist $G_0, G_1, G_2 \in W[X, Y]$ with $\sum_{i=0}^2 G_iF_i = 1$. We may assume that $G_0, G_1, G_2$ are linear combinations of the monomials $X^aiY^bj (i, j \in \mathbb{N})$ because the equality $\sum_{i=0}^2 G_iF_i = 1$ remains true when we discard all the other monomials from $G_0, G_1, G_2$. Then, if we set $g_i(x, y) \in W[x, y] (i = 0, 1, 2)$ so that $g_i(X^a, Y^b) = G_i(X, Y)$, we see the equality $g_0f + g_1 \frac{\partial f}{\partial x} + g_2 \frac{\partial f}{\partial y} = 1$. So $X$ is smooth over $W$ by the Jacobian criterion.

Let $X_K$ be the generic fiber of $X/W$, which is a $C_{ab}$-curve over $K$. Let $P_\infty$ be the point at infinity of $X_K$. (See Fact 2.3(i).)

We have $X = \text{Spec}(A)$, $X = \text{Spec}(A)$ with

$$\overline{A} = k[x, y]/(\overline{f}(x, y)), \quad A = W[x, y]/(f(x, y)).$$

Let $A^1$ be the weak completion of $A$. By definition given in the previous section, we have the first cohomologies

$$H^1_{\text{dR}}(X/W) = H^1(\Omega^*_A), \quad H^1_{\text{MW}}(X/W) = H^1(\Omega^*_{A^1})$$

which induce the diagram (2) (with $i = 1$).

Denef–Vercauteren first compute a basis of the first rational algebraic de Rham cohomology

$$(6) 
H^1_{\text{dR}}(X/W) \otimes \mathbb{Q} = H^1(\Omega^*_{A^1} \otimes \mathbb{Q}).$$

To explain their result and its proof, for $\omega \in \Omega^*_{A^1} \otimes \mathbb{Q}$, we denote its cohomology class in the groups (6) by $[\omega]$.

**Proposition 2.5 ([8] p.89).** The elements $[x^iy^jdx] (0 \leq i \leq b - 2, 1 \leq j \leq a - 1)$ form a basis of $H^1_{\text{dR}}(X/W) \otimes \mathbb{Q} = H^1(\Omega^*_{A^1} \otimes \mathbb{Q})$ over $K$.

We give a sketch of the proof because it is important for us.

**Proof.** The proof is done in several steps.

**Step 1.** By definition, the group $H^1(\Omega^*_{A^1} \otimes \mathbb{Q})$ is generated by $[x^iy^jdx], [x^iy^jd] (i, j \in \mathbb{N})$ over $K$. Using the defining equation (5), we see that the group $H^1(\Omega^*_{A^1} \otimes \mathbb{Q})$ is generated by $[x^iy^jdx], [x^iy^jd] (i \in \mathbb{N}, 0 \leq j \leq a - 1)$ over $K$.

**Step 2.** Next, using the equality

$$0 = [d(x^iy^{j+1})] = (j + 1)[x^iy^jd] + i[x^i-1y^{j+1}dx]$$

and the defining equation (5) again if necessary, we see that the group $H^1(\Omega^*_{A^1} \otimes \mathbb{Q})$ is generated by $[x^iy^jd] (i \in \mathbb{N}, 0 \leq j \leq a - 1)$ over $K$.

**Step 3.** For each $j, l \in \mathbb{N}$, Denef–Vercauteren prove the following equality [8] (18) in p.89:

$$(7) 
[x^l \left( \sum_{k=1}^{a-1} \frac{j}{k+j} f_k(x)y^k + f_0(x) \right) y^jdx - lx^{l-1} \left( \frac{a}{a+j} y^a + \sum_{k=1}^{a-1} \frac{j}{k+j} f_k(x)y^k \right) y^jdx] = 0.$$

We compute the order at $P_\infty$ of terms in the equality (7), noting that $\text{ord}_{P_\infty}(x) = -a, \text{ord}_{P_\infty}(y) = -b, \text{ord}_{P_\infty}(dx) = -(a + 1)$. (See Fact 2.3(ii).) The terms with lowest order are $\omega_1 = x^lf_0(x)y^jdx$ and
\[\omega_2 = -lx^{l-1}\frac{a}{a+j}y^{a+j}dx\] and the order is \(-(a(l) + j b + 1)\). Because
\[\omega_1 = b \omega_0 x^{l+b-1}y^{l}dx + (\text{forms of higher order}), \quad \omega_2 = \frac{la}{a+j} c_{0} x^{l+b-1}y^jdx + (\text{forms of higher order})\]

(we used (3) for \(\omega_2\)), we conclude that the differential form inside the bracket \([\ ]\) in (4) has the form
\[\left(b + \frac{la}{a+j}\right) c_{0} x^{l+b-1}y^jdx + (\text{forms of higher order}).\]

Because the coefficient \((b + \frac{la}{a+j}) c_{0}\) is nonzero, the order of the differential form inside the bracket \([\ ]\) in (4) is equal to \(- (a(l) + j b + 1)\).

Now we go back to consider \([x^iy^jdx]\) \((i \in \mathbb{N}, 0 \leq j \leq a - 1)\). The order at \(P_\infty\) of \(x^iy^jdx\) is \(-(a(i) + j b + 1)\).

So, if \(i \geq b - 1\), we can use (3) and (4) to rewrite \([x^iy^jdx]\) as a linear combination over \(K\) of the elements \([x^i'y^jdx]\) \((i' \in \mathbb{N}, 0 \leq j' \leq a - 1)\) such that the order at \(P_\infty\) of \(x^i'y^jdx\) is strictly larger than that of \(x^iy^jdx\). If \(i' \geq b - 1\) for some \([x^i'y^jdx]\) appearing in the linear combination, we use again (3) and (4) to rewrite this term. We repeat this process as long as there is a term \([x^i'y^jdx]\) with \(i' \geq b - 1\) in the linear combination. Because the order at \(P_\infty\) is always an integer, this process stops at some point and so we conclude that, for \(i \geq b - 1\) and \(0 \leq j \leq a - 1\), \([x^iy^jdx]\) is written as a linear combination of the elements \([x^i'y^jdx]\) with \(0 \leq i' \leq b - 2, 0 \leq j' \leq a - 1\). Hence the group \(H^1(W^1, \Omega^*_A \otimes Q)\) is generated by \([x^i'y^jdx]\) \((0 \leq i \leq b - 2, 0 \leq j \leq a - 1)\) over \(K\). Because the genus of the compactification of \(X\) is \((a - 1)(b - 1)/2\) (see Fact (2.3(iii))), we see that the elements \([x^i'y^jdx]\) \((0 \leq i \leq b - 2, 0 \leq j \leq a - 1)\) form a basis of \(H^1(W^1, \Omega^*_A \otimes Q)\).

**Remark 2.6.** Let \(i,j \in \mathbb{N}\). By Proposition 2.7, \([x^i'y^jdx]\) is written uniquely in the form
\[\sum_{0 \leq i' \leq b - 2 \atop 0 \leq j' \leq a - 1} g^{i,j}_{i',j'}[x^{i'}y^{j'}dx]\]
with \(g^{i,j}_{i',j'} \in K\). By looking at the proof of Proposition 2.7, we see that each \(g^{i,j}_{i',j'}\) is a polynomial function in the coefficients \(c_{st}(s,t \geq 0, as+bt < ab\ or\ \langle s,t \rangle = (b,0))\) appearing in the defining equation (4) of \(X\), divided by some power of \(c_{0}\). Namely, there exist \(G^{i,j}_{i',j'} \in \mathbb{Q}[z_{st}]_{s,t \geq 0, as+bt < ab\ or\ \langle s,t \rangle = (b,0)}\) for any \(i,j,i',j' \in \mathbb{N}, 0 \leq i' \leq b - 2, 1 \leq j' \leq a - 1\) such that \(g^{i,j}_{i',j'}\) is the value of \(G^{i,j}_{i',j'}\) at \(z_{st} = c_{st}\).

Next Denef–Vercauteren compute a basis of the first rational Monsky–Washnitzer cohomology
\[H^1(W^1)(X/W) \otimes Q = H^1(W^1, \Omega^*_A \otimes Q)\].

Following the previous notation, for \(\omega \in \Omega^*_A \otimes Q\), we denote its cohomology class in the groups (9) by \([\omega]\).

**Proposition 2.7 (\([\ref{2.7}]\) p.90–93).** The elements \([x^i'y^jdx]\) \((0 \leq i \leq b - 2, 1 \leq j \leq a - 1)\) form a basis of \(H^1(W^1, \Omega^*_A \otimes Q)\) over \(K\).

We omit the proof of Proposition 2.7 because it is not necessary for us.

**Corollary 2.8.** For a \(C_{ab}\)-curve \(X\), the map \(\iota_*\) in the diagram (2) is an isomorphism.

**Proof.** Because the map \(\iota_*\) sends \([x^i'y^jdx]\) to \([x^{i'}y^{j'}dx]\), the claim follows from Propositions 2.6 and 2.7.

### 3. Infiniteness

In this section, we give examples of affine smooth varieties \(X\) over \(k\) such that the first integral overconvergent de Rham–Witt cohomology modulo torsion \(\overline{H}^1(X, W^1\Omega^*_X)\) is not finitely generated over \(W\). Our basic example is the following one:

**Theorem 3.1.** Let \(a, b\) be coprime positive integers prime to \(p\), let \(\overline{\sigma} \in k^\times\) and let \(X\) be the superelliptic curve \(y^a + x^b + \overline{\sigma} = 0\) (the affine smooth plane curve defined by this equation). Then \(\overline{H}^1(X, W^1\Omega^*_X)\) is not finitely generated over \(W\).
Proof. Note that $X$ is a special case (the case $f_0(x) = x^b + \alpha, f_j(x) = 0$ ($1 \leq j \leq a-1$)) of the $C_{ab}$-curve $X$ in the previous section. Take a lift $\alpha \in W$ of $\bar{\alpha}$ and let $\mathcal{X}$ be the smooth lift of $X$ defined by the equation $y^a + x^b + \alpha = 0$. This is a special case (the case $f_0(x) = x^b + \alpha, f_j(x) = 0$ ($1 \leq j \leq a-1$)) of the lift $\mathcal{X}$ in the previous section. Let $A = W[y]/(y^a + x^b + \alpha)$ be as in the previous section.

We consider the algebraic de Rham cohomology $H^j_{dR}(\mathcal{X}/W) \otimes \mathbb{Q} = H^j(\Omega^*_\mathcal{X} \otimes \mathbb{Q})$. The equality (7) in the cohomology is written as

$$[b x^{l+b-1} y^j dx - \frac{la}{a+j} x^{l-1} y^{a+j} dx] = 0$$

in the case at hand. Using the defining equation $y^a + x^b + \alpha = 0$, it is rewritten as

$$[b x^{l+b-1} y^j dx + \frac{la}{a+j} x^{l-1} (x^b + \alpha) y^j dx] = 0,$$

which is equivalent to the equality

$$(10) \quad [x^{l+b-1} y^j dx] = \left[ -\frac{la\alpha}{la+jb+ab} x^{l-1} y^j dx \right].$$

Recall that $1 \leq j \leq a-1$. Let $1 \leq r \leq b-1$, $N \in \mathbb{N}$ and consider the element $[x^{(N+1)b+(r-1)} y^j dx]$. By using the equality (10) with $l = Nb + r, (N-1)b + r, \ldots, r$, we obtain the equality

$$(11) \quad [x^{(N+1)b+(r-1)} y^j dx] = \prod_{n=0}^{N} \left( -\frac{(nb+r)\alpha}{(nb+r)a+jb+ab} \right) [x^{r-1} y^j dx]$$

$$= \text{(unit)} \prod_{n=0}^{N} \frac{nb+r}{nab+(ra+jb+ab)}. [x^{r-1} y^j dx],$$

where (unit) means an element in $W^\times$.

Now we fix $j, r$ so that $p \equiv jb \text{ (mod a)}, p \equiv ra \text{ (mod b)}$. This is possible because $a, b$ are coprime and $p$ does not divide $ab$. Then we have $p \equiv ra+jb+ab \text{ (mod ab)}$. Hence the set

$$\mathcal{M} := \{ M \in \mathbb{N} \mid p^M \geq ra+jb+ab, p^M \equiv ra+jb+ab \text{ (mod ab)} \}$$

is infinite. Take any $M \in \mathcal{M}$ and put $N := (p^M - (ra+jb+ab))/ab$. For such $N$, we compute the $p$-adic valuation $\nu := \nu_p \left( \prod_{n=0}^{N} \frac{nb+r}{nab+(ra+jb+ab)} \right)$ of $\prod_{n=0}^{N} \frac{nb+r}{nab+(ra+jb+ab)}$.

For $M' \in \mathbb{N}$, define the sets $P_{M'}, Q_{M'}$ by

$$P_{M'} := \{ n \mid 0 \leq n \leq N, p^M | nb+r \}, \quad Q_{M'} := \{ n \mid 0 \leq n \leq N, p^M | nab+(ra+jb+ab) \}.$$

Then

$$(12) \quad \nu = \sum_{M'=1}^{\infty} M'((|P_{M'}| - |P_{M'+1}|) - (|Q_{M'}| - |Q_{M'+1}|))$$

$$= \sum_{M'=1}^{\infty} (|P_{M'}| - |Q_{M'}|) = \sum_{M'=1}^{M} (|P_{M'}| - |Q_{M'}|).$$

(For the last equality, note that $Nb+r \leq Nab+(ra+jb+ab) = p^M$.) So we estimate the terms $|P_{M'}| - |Q_{M'}|$ for $1 \leq M' \leq M$.

In general, for $n_0 \in P_{M'}$ and $0 \leq n \leq N$, we have the equivalence

$$n \in P_{M'} \iff n - n_0 \in p^{M'} \mathbb{Z}$$

because $b$ is prime to $p$, and the same property holds for the set $Q_{M'}$. So, if we denote the maximal element of $P_{M'}$ by $n_0$, we have the equality

$$P_{M'} = \{ n_0, n_0 - p^{M'}, n_0 - 2p^{M'}, \ldots \} \cap \mathcal{N}.$$
Also, since $N \in Q_{M'}$ (because $p^{M'}|p^M = Nab + (ra + jb + ab)$), we have the equality
\[ Q_{M'} = \{ N, N - p^{M'}, N - 2p^{M'}, \ldots \} \cap N. \]

Since $n_0 \leq N$ by definition, we see that $|P_{M'}| \leq |Q_{M'}|$. So $|P_{M'}| - |Q_{M'}| \leq 0$ for any $1 \leq M' \leq M$.

Next we give a stronger estimate of $|P_{M'}| - |Q_{M'}|$ for $1 \leq M' \leq M$ when $M' \in M$. If we define the sets $\tilde{P}_{M'}, \tilde{Q}_{M'}$ by
\[ \tilde{P}_{M'} := \{ a(nb + r) : n \in P_{M'} \}, \quad \tilde{Q}_{M'} := \{ nab + (ra + jb + ab) : n \in Q_{M'} \}, \]
we have the inequality
\[ |P_{M'}| - |Q_{M'}| = |\tilde{P}_{M'}| - |\tilde{Q}_{M'}|. \]
If we denote the maximal (resp. minimal) element of $P_{M'}$ by $n_0$ (resp. $n_1$) and put $\tilde{n}_0 := a(n_0 b + r)$ (resp. $\tilde{n}_1 := a(n_1 b + r)$), we have the equality
\[ \tilde{P}_{M'} = \{ \tilde{n}_0, \tilde{n}_0 - abp^{M'}, \tilde{n}_0 - 2abp^{M'}, \ldots, \tilde{n}_1 \}. \]

On the other hand, $N$ is the maximal element of $Q_{M'}$, and since $M' \in M$, there exists $0 \leq N_1 \leq N$ with $N_1 ab + (ra + jb + ab) = p^{M'}$, which is the minimal element of $Q_{M'}$. Then
\[ Q_{M'} = \{ N, N - p^{M'}, N - 2p^{M'}, \ldots, N_1 \}, \]
and so we see the equality
\[ \tilde{Q}_{M'} = \{ p^M, p^M - abp^{M'}, p^M - 2abp^{M'}, \ldots, p^{M'} \}. \]

Now, noting the inequalities
\[ \tilde{n}_1 = a(n_1 b + r) > n_1 b + r \geq p^{M'}, \quad \tilde{n}_0 = a(n_0 b + r) \leq a(Nb + r) < Nab + (ra + jb + ab) = p^M, \]
we see that $|\tilde{P}_{M'}| < |\tilde{Q}_{M'}|$. So $|P_{M'}| - |Q_{M'}| \leq -1$ for any $1 \leq M' \leq M$ with $M' \in M$.

By combining the inequalities proved in the previous two paragraphs and the equality (13), we see that, if $M$ is the $d$-th element of the set $M, \nu \leq -d$. Then, by putting it into the equality (11), we see that, for some fixed $1 \leq j \leq a - 1$ and $1 \leq r \leq b - 1$, there exists a sequence of natural numbers $\{N_d\}_{d=0}^\infty$ such that
\[ (13) \quad [x^{(N_d+1)b+(r-1)y}] \equiv C_d[x^{r-1}y^d] \quad \nu_p(C_d) \leq -d. \]

Because $[x^{(N_d+1)b+(r-1)y}]$ is the cohomology class coming from the integral algebraic de Rham cohomology $H^1_{\text{dR}}(\mathcal{X}/W)$, we see from (13) that $\overline{H}^1(\mathcal{X}/W)$ contains $K[x^{r-1}y^d] \equiv K$. (The last isomorphism follows from Proposition 2.3.) Then, by the diagram (2) and the fact that $\iota_\ast Q$ is an isomorphism, we conclude that $\overline{H}^1_{\text{MW}}(\mathcal{X}/W)$ also contains $K[x^{r-1}y^d] \equiv K$. Since the last isomorphism group is isomorphic to $\overline{H}^1(\mathcal{X}, W^1\Omega^\natural_{\mathcal{X}})$, we conclude that $\overline{H}^1(\mathcal{X}, W^1\Omega^\natural_{\mathcal{X}})$ is not finitely generated over $W$. \qed

**Corollary 3.2.** For any prime number $p$ and any perfect field $k$ of characteristic $p$, there exists an affine smooth curve $X$ over $k$ such that $\overline{H}^1(\mathcal{X}, W^1\Omega^\natural_{\mathcal{X}})$ is not finitely generated over $W$.

**Proof.** If we take coprime positive integers $a, b$ prime to $p$ and take as $X$ the superelliptic curve $y^a + x^b + 1 = 0$, $\overline{H}^1(\mathcal{X}, W^1\Omega^\natural_{\mathcal{X}})$ is not finitely generated over $W$. \qed

It would be possible to provide more examples of infiniteness using the following:

**Proposition 3.3.** If $X_1 \to X_2$ be a generically étale morphism of affine smooth curves. Then, if $\overline{H}^1(X_2, W^1\Omega^\natural_{\mathcal{X}})$ is not finitely generated over $W$, $\overline{H}^1(X_1, W^1\Omega^\natural_{\mathcal{X}})$ is not finitely generated over $W$ either.

**Proof.** One can take open subschemes $X_1' \subset X_1, X_2' \subset X_2$ such that $X_1' \to X_2'$ is finite étale. Since we have the commutative diagram
\[
\begin{array}{cccc}
\overline{H}^1(X_2, W^1\Omega^\natural_{\mathcal{X}}) & \longrightarrow & H^1_{\text{rig}}(X_2/K) & \longrightarrow & H^1_{\text{rig}}(X_2'/K) \\
\downarrow & & \downarrow & & \downarrow \\
\overline{H}^1(X_1, W^1\Omega^\natural_{\mathcal{X}}) & \longrightarrow & H^1_{\text{rig}}(X_1/K) & \longrightarrow & H^1_{\text{rig}}(X_1'/K)
\end{array}
\]
with horizontal arrows and the right vertical arrow injective, we see that the morphism \( \overline{T}^i(X_2, W^i \Omega^*_X) \to \overline{T}^i(X_1, W^i \Omega^*_X) \) is injective as well, and hence the claim follows.

The following corollary is a stronger version of Corollary 3.2.

**Corollary 3.4.** For any prime number \( p \) and any perfect field \( k \) of characteristic \( p \), there exist infinitely many affine smooth curves \( X_i \) (\( i \in \mathbb{N} \)) over \( k \) whose compactifications are all non-isomorphic such that \( \overline{T}^i(X_1, W^i \Omega^*_X) \) (\( i \in \mathbb{N} \)) are not finitely generated over \( W \).

**Proof.** In the proof of Corollary 3.2, we may take the affine curve \( X \) so that the genus \( g(X^{cpt}) = (a-1)(b-1)/2 \) of the compactification \( X^{cpt} \) of \( X \) is \( \geq 2 \). Then, by taking a family \( f_i : X_i^{cpt} \to X^{cpt} \) (\( i \in \mathbb{N} \)) of finite étale coverings with \( g(X_i^{cpt}) > g(X_i^{cpt}) \) and putting \( X_i := f_i^{-1}(X) \), we obtain the required family \( X_i \) (\( i \in \mathbb{N} \)) thanks to Proposition 3.3.

Next we consider the case of ‘general’ \( C_{ab} \)-curves. Let \( p \) be a fixed prime, let \( a, b \) be coprime positive integers prime to \( p \) and let \( k_0 \) be a field of characteristic \( p \). Let \( S \) be the set of pairs \((i, j) \in \mathbb{N}^2\) with \( ai + bj < ab \) or \((i, j) = (b, 0)\), and let \( \text{Spec}(k_0)^{\circ}((i, j) \in S) = A_{k_0}^{[S]} \) be the affine space with respect to the indeterminates \( \{\tau_{ij}\}_{(i, j) \in S} \). Let \( g : X \to A_{k_0}^{[S]} \) be the relative plane curve defined by the equation \((\text{with})\)

\[
\overline{T}_0(x) = \sum_{i=0}^{b} \tau_{0i} x^i, \quad \overline{T}_j(x) = \sum_{ai+jb<ab} \tau_{ij} x^i (1 \leq j \leq a - 1).
\]

For a point \( \mathbf{c} \) in \( A_{k_0}^{[S]} \), let \( \mathcal{X}_c \) be the fiber of the map \( g \) at \( \mathbf{c} \). Then we have the following proposition:

**Proposition 3.5.** There exists an open dense subscheme \( U \) of \( A_{k_0}^{[S]} \) such that a point \( \mathbf{c} \in A_{k_0}^{[S]} \) belongs to \( U \) if and only if \( \mathcal{X}_c \) is a (smooth) \( C_{ab} \)-curve.

**Proof.** The following proof is inspired by the proof of [3, Proposition 1]. Let \( Z \) be the closed subscheme of \( A_{k_0}^{[S]} \times (A_{k_0} \setminus \{0\})^2 \) (with the coordinates \( \tau_{ij} \) \((i, j) \in S\), \( x, y \)) defined by the equation \( \overline{T} = \overline{T}_0 = \overline{T}_y = 0 \), regarded as the equation in \(|S| + 2 \) variables \( \tau_{ij} \) \((i, j) \in S\), \( x, y \).

First we prove that, for each \( \alpha, \beta \in k_0^{alg} \setminus \{0\} \), the pullback \( Z_{\alpha, \beta} \) of \( Z \) to \( A_{k_0}^{[S]} \times \{\langle \alpha, \beta \rangle \} \) is a closed subscheme of codimension \( \geq 3 \) in \( A_{k_0}^{[S]} \times \{\langle \alpha, \beta \rangle \} \): Note that we have

\[
\overline{T} = y^a + \sum_{(i,j) \in S} \tau_{ij} x^i y^j, \quad \frac{\partial \overline{T}}{\partial x} = \sum_{(i,j) \in S} \tau_{ij} i x^i - y^j, \quad \frac{\partial \overline{T}}{\partial y} = ay^a + \sum_{(i,j) \in S} \tau_{ij} j x^i y^j - 1.
\]

Hence, if we set \( \overline{S} := S \cup \{0, a\} \), \( Z_{\alpha, \beta} \) is isomorphic to the closed subscheme

\[
\left\{ (\tau_{ij})_{i,j \in \overline{S}} \mid \sum_{(i,j) \in \overline{S}} \tau_{ij} x^i y^j = \sum_{(i,j) \in \overline{S}} \tau_{ij} i a^{i-1} y^j = \sum_{(i,j) \in \overline{S}} \tau_{ij} j a_i \beta^{j-1} = 0 \right\} \cap \{ \tau_{00} = 1 \}
\]

in \( A_{k_0}^{[S]} \times \{\tau_{00} = 1\} \) via the natural isomorphism

\[
A_{k_0}^{[S]} \times \{\tau_{00} = 1\} \cong A_{k_0}^{[S]} \cong A_{k_0}^{[S]} \times \{\langle \alpha, \beta \rangle \}.
\]

Since the vectors \( (a^i \beta^j, i a^{i-1} \beta^j, j a_i \beta^{j-1}) \) for \((i, j) = (b, 0), (0, a), (0, 0)\) are linearly independent, we see that the former set of \((14)\) is a linear subscheme of codimension \( 3 \) in \( A_{k_0}^{[S]} \), and by taking intersection with the latter set in \((14)\), we see that \( Z_{\alpha, \beta} \) is a closed subscheme of codimension \( \geq 3 \) in \( A_{k_0}^{[S]} \times \{\langle \alpha, \beta \rangle \} \), as required.

Since \( Z_{\alpha, \beta} \) is a closed subscheme of codimension \( \geq 3 \) in \( A_{k_0}^{[S]} \times \{\langle \alpha, \beta \rangle \} \) for any \( \langle \alpha, \beta \rangle \), \( Z \) is a closed subscheme of codimension \( \geq 3 \) in \( A_{k_0}^{[S]} \times (A_{k_0} \setminus \{0\})^2 \). If we define \( Y_1 \) to be the image of \( Z \) by the
projection $A^{[S]}_{A_0} \times (A_{A_0} \setminus \{0\})^2 \to A^{[S]}_{A_0}$, it is a closed subscheme of codimension $\geq 1$, and this is the set of points $c$ in $A^{[S]}_{A_0}$ such that the fiber $X_c$ is not smooth at some point $(x, y)$ with $x \neq 0, y \neq 0$.

On the other hand, let $Y_2$ be the closed subscheme of codimension 1 in $A^{[S]}_{A_0}$ defined by $\{a_{10} = 0\}$, and let $Y_3$ be the closed subscheme of codimension 1 in $A^{[S]}_{A_0} \setminus Y_2$ by $\{\text{disc}(f(x, 0)) = 0 \} \cup \{\text{disc}(f(0, y)) = 0\}$. $Y_2$ is the set of points $c$ in $A^{[S]}_{A_0} \setminus Y_2$ such that the defining equation of $X_c$ has degree $< b$ in $x$, and $Y_3$ is the set of points $c$ in $A^{[S]}_{A_0} \setminus Y_2$ such that $X_c$ is not smooth at some point $(x, y)$ with $x = 0$ or $y = 0$. If we define $Y_3$ to be the union of $Y_2$ and the closure of $Y_3$, it is of codimension 1 in $A^{[S]}_{A_0}$.

Now let $U := A^{[S]}_{A_0} \setminus (Y_1 \cup Y_2)$. Then $U$ is dense open in $A^{[S]}_{A_0}$ and a point $c \in A^{[S]}_{A_0}$ belongs to $U$ if and only if $X_c$ is a (smooth) $C_{ab}$-curve. So the proof of the proposition is finished. □

We denote the pullback of the map $g : X \to A^{[S]}_{A_0}$ to $U$ by $g_U : X_U \to U$. This is a family of $C_{ab}$-curves.

Then we can prove the following infiniteness result using Theorem 3.1.

**Theorem 3.6.** Let the notations be as above. Then, there exists a sequence of closed subschemes $Z_d (d \in \mathbb{N})$ of codimension $\geq 1$ in $U$ satisfying the following: For any perfect field $k$ containing $k_0$ and for any morphism $h : \text{Spec}(k) \to U$ whose image is not contained in $\bigcup_{d \in \mathbb{N}} (\bigcap_{d' = d} Z_{d'})$, if we denote the pullback of $g_U$ with respect to $h$ by $X \to \text{Spec}(k)$, then $\overline{\mathcal{H}}(X, W^1 \Omega^1_X)$ is not finitely generated over $W$.

Theorem 3.6 is applicable when the image of $h$ is the generic point of $U$. Also, when $k_0$ is uncountable, the set $U \setminus \bigcup_{d \in \mathbb{N}} (\bigcap_{d' = d} Z_{d'})$ contains uncountably many closed points. In these senses, $\overline{\mathcal{H}}(X, W^1 \Omega^1_X)$ is not finitely generated over $W$ for a ‘general’ $C_{ab}$-curve $X$.

**Proof.** As before, let $S$ be the set of pairs $(i, j) \in \mathbb{N}^2$ with $a_i + b_j < a$ or $(i, j) = (b, 0)$. Take a perfect field $k$ containing $k_0$ and a morphism $h : \text{Spec}(k) \to U$ such that its image is not contained in $\bigcup_{d \in \mathbb{N}} (\bigcap_{d' = d} Z_{d'})$. Let $\tau_{ij} ((i, j) \in S)$ be the image of $\tau_{ij}$ by $h^* : k_0 (\tau_{ij})_{(i, j) \in S} \to k$ and let $\sigma_{ij} \in W = W(k)$ be a lift of $\tau_{ij}$.

Consider the relative curve $X$ over $W$ defined by (14). By Lemma 2.4, $X$ is smooth over $W$ and the generic fiber $X_K$ of $X/W$ is a $C_{ab}$-curve. Take $1 \leq r \leq b - 1, 1 \leq j \leq a - 1$ and the sequence of natural numbers $\{N_d\}_{d=0}^\infty$ as in the proof of Theorem 3.1. Let $i_d := (N_d + 1)b + (r - 1)$. Then, by Remark 2.6 we can write

$$[x^{i_d} y^j dx] = \sum_{1 \leq i' \leq b - 1 \leq j} G_{i', j'} (c_{st}) [x^{i'} y^j dx]$$

for some $G_{i', j'} \in \mathbb{Q}[z_{st} | (i, j) \in S \setminus z_{00}^{-1}]$. (Note that $G_{i', j'}$ is independent of the choice of $h$ and the choice of lifts of $\{c_{st} | (s, t) \in S\}$.) By the proof of Theorem 3.1 the equality (13) is the one we obtain from (14) by specializing $z_{st} ((s, t) \in S)$ as

$$z_{st} \mapsto 0 \quad ((s, t) \neq (0, 0), (b, 0)), \quad z_{00} \mapsto a \in W^\times, \quad z_{00} \mapsto 1.$$

Define $l_d$ to be the least integer such that $p^{l_d} G_{r, j} \in \mathbb{Z}_p [z_{st} | (i, j) \in S \setminus z_{00}^{-1}]$. Then, by the specialization (10), $p^{l_d} G_{r, j}$ is sent to $p^{l_d} C_d$, where $C_d$ is as in (13). Because this specialization takes value in $\mathbb{Z}_p$, we see that $p^{l_d} C_d \in \mathbb{Z}_p$, hence $l_d - d \geq 0 \geq l_d$. Thus $l_d \geq d$.

Let $\overline{G}_{r, j} \neq 0$ be the image of $p^{l_d} G_{r, j} \in \mathbb{Z}_p [z_{st} | (i, j) \in S \setminus z_{00}^{-1}]$ by the reduction map $\mathbb{Z}_p [z_{st} | (i, j) \in S \setminus z_{00}^{-1}] \to \mathbb{F}_p [z_{st} | (i, j) \in S \setminus z_{00}^{-1}]$, and let $Z_d$ be the zero locus of $\overline{G}_{r, j}$ in $U$. Note that $\tau_{00}$ is invertible in $U$ and so $Z_d$ is well-defined as a closed subscheme of $U$, and it is of codimension $\geq 1$ because $G_{r, j}$ is nonzero. Also, if the image of $h$ is not contained in $Z_d$, then $\overline{G}_{r, j}$ is nonzero in $k$ and so $\nu_p (G_{r, j} (c_{st})) = -l_d \leq -d$.

Now suppose that the image of $h$ is not contained in $\bigcup_{d \in \mathbb{N}} (\bigcap_{d' = d} Z_{d'})$. Because $[x^{i_d} y^j dx]$ is a cohomology class coming from the integral algebraic de Rham cohomology $H^1 (X/W)$, we see from (14) and the calculation in the previous paragraph that $\overline{\mathcal{H}}_{\text{dR}} (X/W)$ is not contained in any of $p^{-d} (\bigoplus_{1 \leq i \leq b - 2} \bigoplus_{1 \leq j \leq a - 1} [x^{i_d} y^j dx]) (d \in \mathbb{N})$ and so $\overline{\mathcal{H}}_{\text{dR}} (X/W)$ is not contained in any of $p^{-d} (\bigoplus_{1 \leq i \leq b - 2} \bigoplus_{1 \leq j \leq a - 1} [x^{i_d} y^j dx]) (d \in \mathbb{N})$. Since the
last cohomology group is isomorphic to $\overline{H}^i(X, W^1\Omega^*_X)$, we conclude that $\overline{H}^i(X, W^1\Omega^*_X)$ is not finitely generated over $W$. □

Remark 3.7. If infinitely many $Z_d$’s ($d \in \mathbb{N}$) intersect properly, the set $\bigcup_{d \in \mathbb{N}}(\bigcap_{d' \equiv d} Z_{d'})$ in Theorem 3.6 is empty. Thus we expect that $\overline{H}^i(X, W^1\Omega^*_X)$ would not be finitely generated over $W$ for any $C_{ab}$-curve $X$.

Our results suggest that, for most affine smooth curves $X$, the group $\overline{H}^i(X_1, W^1\Omega^*_X)$ are not finitely generated over $W$. On the other hand, we have the following proposition.

Proposition 3.8. For an affine smooth curve $X$ over $k$ whose smooth compactification has genus 0, $\overline{H}^i(X, W^1\Omega^*_X)$ is finitely generated over $W$.

Proof. For a finite extension $k'$ of $k$, we have the base change property $H^1(X, W^1\Omega^*_X) \otimes_W W(k') \cong H^1(X \otimes_k k', W^1\Omega^*_{X\otimes_k k'})$ and so $\overline{H}^i(X, W^1\Omega^*_X) \otimes_W W(k') \cong \overline{H}^i(X \otimes_k k', W^1\Omega^*_{X\otimes_k k'})$. Hence, if $\overline{H}^i(X \otimes_k k', W^1\Omega^*_{X\otimes_k k'})$ is finitely generated over $W(k')$, $\overline{H}^i(X, W^1\Omega^*_X)$ is finitely generated over $W$. Thus we may replace $k$ by a finite extension of it to prove the proposition. Thus we may assume that $X = A^1_k \setminus \{\bar{\tau}_1, \ldots, \bar{\tau}_r\}$ for some distinct $\bar{\tau}_i \in \bar{k}$ ($1 \leq i \leq r$).

We prove the finiteness of $\overline{H}^i(X, W^1\Omega^*_X)$ by induction on $r$. If $r = 0$, $X = A^1_k$. In this case, $\overline{H}^i(X, W^1\Omega^*_X) \subset H^1_{rig}(X/K) = \{0\}$ and so the claim is true.

If $r = 1$, we may assume that $X = A^1_k \setminus \{0\}$. Then $\overline{H}^i(X, W^1\Omega^*_X) \cong \overline{H}^i_{MW}(X/W) = \overline{H}^i(\Omega^*_A)$, where $A^1 = \{\sum a_nx^n \mid a_n \in W \text{ and } \exists \varepsilon > 0, \exists C \in \mathbb{R}, \forall n \in \mathbb{Z}, \nu_p(a_n) \geq \varepsilon|n| + C\}$ is the weak completion of $W[x, x^{-1}]$. Then any element of $\overline{H}^i(\Omega^*_A)$ is written in the form $[(\sum_{n \in \mathbb{Z}} a_n x^n)dx]$ with $\sum_{n \in \mathbb{Z}} a_n x^n \in A^1$, and it can be rewritten as

$$a_{-1}[x^{-1}dx] + \left[\sum_{n \neq 1} a_n x^n dx\right] = a_{-1}[x^{-1}dx] + d\left(\sum_{n \neq 1} \frac{a_n}{n+1} x^{n+1}\right) = a_{-1}[x^{-1}dx] \in W[x^{-1}dx],$$

because $\sum_{n \neq 1} \frac{a_n}{n+1} x^{n+1} \in A^1 \otimes \mathbb{Q}$. Thus $\overline{H}^i(\Omega^*_A) = W[x^{-1}dx]$ and it is finitely generated over $W$, as required.

If $r \geq 2$, we set $X_1 := A^1_k \setminus \{\bar{\tau}_1, \ldots, \bar{\tau}_{r-1}\}$, $X_2 := A^1_k \setminus \{\bar{\tau}_r\}$. Take lifts $\alpha_i \in W$ of $\bar{\tau}_i$ for $1 \leq i \leq r$. Then, for $i = 1, 2$, $\overline{H}^i(X_i, W^1\Omega^*_X) \cong \overline{H}^i_{MW}(X_i/W) = \overline{H}^i(\Omega^*_A)$, where $A^1_i$ is the weak completion of $W[x, (x - \alpha_i)^{-1}, \ldots, (x - \alpha_{i-1})^{-1}]$ and $A^1_2$ is the weak completion of $W[x, (x - \alpha_r)^{-1}]$. Also, $\overline{H}^i(X, W^1\Omega^*_X) \cong \overline{H}^i_{MW}(X/W) = \overline{H}^i(\Omega^*_A)$, where $A^1$ is the weak completion of $W[x, (x - \alpha_1)^{-1}, \ldots, (x - \alpha_r)^{-1}]$. Then, as a special case of [16] Lemma 7, the canonical map $A^1_1 \oplus A^1_2 \to A^1$ is surjective, and so the map $\Omega^1_A \oplus \Omega^1_A \to \Omega^1_A$ is also surjective. Thus we see that the map $\overline{H}^i(\Omega^*_A_1) \oplus \overline{H}^i(\Omega^*_A_2) \to \overline{H}^i(\Omega^*_A)$ is surjective. Because $\overline{H}^i(\Omega^*_A_1) \oplus \overline{H}^i(\Omega^*_A_2)$ is finitely generated over $W$ by induction hypothesis, so is $\overline{H}^i(\Omega^*_A)$, as required. □

Consequently, on the finiteness of $\overline{H}^i(X_1, W^1\Omega^*_X)$ for an affine smooth curve $X$, we conjecture the following.

Conjecture 3.9. Let $X$ be an affine smooth curve over $k$. Then $\overline{H}^i(X_1, W^1\Omega^*_X)$ is finitely generated over $W$ if and only if its smooth compactification has genus 0.

For higher dimensional varieties, we have the following as a simple consequence of Theorem 3.6.

Theorem 3.10. Let $X$ be a projective smooth variety over $k$ and let $a$ be a positive integer prime to $p$. Then there exists a generically étale morphism $f : Y \to X$ of degree $a$ with $Y$ smooth such that $\overline{H}^i(Y, W^1\Omega^*_Y)$ is not finitely generated over $W$. 
Proof. Take a closed embedding $X \subset P^N_k$ into a projective space and take two transversal hyperplanes $H_1, H_2$ in $P^N_k$ which meet $X$ smoothly and transversally. Let $A := H_1 \cap H_2 \cap X$ and let $b : \tilde{X} \to X$ be the blow-up of $X$ along $A$. Then we have canonically a pencil structure $g : \tilde{X} \to P^1_k$.

Let $h : C \to A^1_k \hookrightarrow P^1_k$ be the superelliptic curve $y^p = x^b + 1$ (where $b$ is a positive integer coprime to $p$ and $x$ is the coordinate of $A^1_k$), and let $C' \subset C$ be the open subscheme on which $h$ is étale. Now let $g' : Y \to C'$ be the pullback of $g : \tilde{X} \to P^1_k$ by the composite $C' \to C \xrightarrow{h} P^1_k$. Then we have the canonical map $Y \to X$ which is generically étale of degree $a$, and $Y$ is smooth.

Take a finite extension $k'$ of $k$ such that $A$ admits a $k'$-rational point $t$. In the following, for a scheme or a morphism of schemes $\phi$, we denote $\phi \otimes_k k'$ simply by $\phi_{k'}$. Then $b_{k'}^{-1}(t) \cong P^1_{k'}$ and so it defines a section $s : P^1_{k'} \to \tilde{X}_{k'}$ of $g_{k'}$. Then we have canonically a pencil structure $s : C'_{k'} \to Y_{k'}$ of the map $g'_{k'} : Y_{k'} \to C'_{k'}$.

Thus we have maps between cohomology groups modulo torsion

$$\overline{\Phi}^i\left(C'_{k'}, W^i\Omega^*_{C'_{k'}}\right) \xrightarrow{g'_{k'}^*} \overline{\Phi}^i\left(Y_{k'}, W^i\Omega^*_{Y_{k'}}\right) \xrightarrow{s^*} \overline{\Phi}^i\left(C'_{k'}, W^i\Omega^*_{C'_{k'}}\right)$$

whose composition is the identity. So the map $g'_{k'}^*$ is injective. By Theorem 3.1 and Proposition 3.3, $\overline{\Phi}^i\left(C'_{k'}, W^i\Omega^*_{C'_{k'}}\right)$ is not finitely generated over $W(k')$. Hence $\overline{\Phi}^i\left(Y_{k'}, W^i\Omega^*_{Y_{k'}}\right)$ is not finitely generated over $W(k')$ either. Then, since $\overline{\Phi}^i\left(Y_{k'}, W^i\Omega^*_{Y_{k'}}\right) \cong \overline{\Phi}^i\left(Y, W^i\Omega^*_{Y}\right) \otimes_{W} W(k')$,

we conclude that $\overline{\Phi}^i\left(Y, W^i\Omega^*_{Y}\right)$ is not finitely generated over $W$. So the proof is finished. □

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