Generalized Yang-Mills Theory under Rotor Model

B.T.T.Wong

Abstract

This paper follows the previous work on generalized abelian gauge field theory of higher-order derivatives under rotor model and extends the study to the most generalized non-abelian case. We find that the rotor mechanism from the abelian case applies nicely to the non-abelian case under the Lorentz gauge condition. Under the rotor mechanism, the gauge field transforms as $T_n^\mu \to \Box^n T_n^\mu$. When the order of field derivative is $n = 0$, this restores back to the original Yang-Mills action. Our work gives an extensive generalization of the Yang-Mills theory with higher-order field derivatives. Finally, we compute the equation of motion and Noether’s current of the generalized non-abelian gauge field theory.

1 Introduction

Quantum field theories with high-order derivatives are of appealing interest because they have the possibilities to eliminate ultraviolet divergences in the calculation of scattering amplitudes [1, 2, 3, 4, 5, 6, 7, 8]. Vast amount of studies in higher order derivative field theories including both scalar fields and gauge fields have been performed [9, 10, 11, 12]. However, there are difficulties in establishing the formalism as these theories are often non-renormalizable and dynamically unstable with unbounded Hamiltonian [13, 14, 15, 16, 17]. Yet, higher derivative field theories give insight to the study of quantum gravity and modified theories of gravity and hence it is worth to establish new formalisms on high-order theories [18, 19, 20, 21, 22].

In our previous paper [23], we have established the formalism of generalized abelian gauge field theory under rotor model with higher-order derivatives. We have shown the following theorem:

\[
S = -\frac{1}{4} \int d^Dx G_{n,\mu\nu} G_n^{\mu\nu} = \frac{1}{4^n} \int d^Dx (\Box^n T^\mu) \tilde{R}_{\mu\nu}(\Box^n T^\nu) = -\frac{1}{4^{n+1}} \int d^Dx \Box^n G_{\mu\nu} \Box^n G^{\mu\nu},
\]

where $\tilde{R}_{\mu\nu}$ is the projection tensor $\tilde{R}_{\mu\nu} = \frac{1}{2} (\gamma_{\mu\nu} - \partial_\mu \partial_\nu)$ and $G_{n,\mu\nu} = \partial_\mu T_{n,\nu} - \partial_\nu T_{n,\mu}$ is the field strength of the $n$-th order rotor model. We have the $n$-th order gauge field strength identified as [23]

\[
G_{n,\mu\nu} \equiv \frac{1}{2^n} \Box^n G_{\mu\nu}.
\]

The projection is regarded as the second-order rotation [23]. Therefore, the $n$-th order rotation of gauge field $T_{n,\mu}$ is attained by successive second-order rotation given by

\*CERN, u3500478@connect.hku.hk
also define the Maxwell-like gauge field strength as ˜ \[ \Lambda \] in such a way that we can use all the results we have in our previous work [23]. We case. For the non-abelian case, we have the Yang-Mills action in general mechanism applies due to the existence of extra terms in the action for the non-abelian (Yang-Mills theory) under the rotor mechanism. There is no guarantee that the old

\[ T_{\mu \nu} = \hat{R}_{\mu_1 \mu_2 \mu_3} \hat{R}^{\mu_1 \mu_2 \mu_3} \hat{R}_{\mu_1 \mu_2 \mu_3} T^{\mu_1 \mu_2 \mu_3} \]

for which

\[ P_{\mu j}^{\mu j-1} = \Box \delta_{\mu j}^{\mu j-1} \]

is called the propagator. This is known as the rotor transformation which generates high-order derivative gauge fields. In the generalized theory, the action changes by the transformation of gauge field as \( T^\mu \rightarrow \Box^n T^\mu \). The working dimension \( D \) for renormalizability is \( 4n + 4 \) for unity gauge field dimension [23]. When \( n = 0 \) this reduces back to the conventional Maxwell action

\[ S = -\frac{1}{4} \int d^4 x G_{\mu \nu} G^{\mu \nu} = -\frac{1}{4} \int d^4 x T_{\mu \nu} T^{\mu \nu} . \]

In this article, we aim at constructing a generalized non-abelian gauge field theory (Yang-Mills theory) under the rotor mechanism. There is no guarantee that the old mechanism applies due to the existence of extra terms in the action for the non-abelian case. For the non-abelian case, we have the Yang-Mills action in general \( D \) dimensional spacetime as [24],

\[ S_{\text{YM}} = -\frac{1}{2} \int d^D x \text{Tr} G_{\mu \nu} G^{\mu \nu} = -\frac{1}{4} \int d^D x G^a_{\mu \nu} G^{\mu \nu a} \]

where

\[ G_{\mu \nu} = \partial_\mu T_\nu - \partial_\nu T_\mu - i g [T_\mu, T_\nu] , \]

for which \( F_{\mu \nu} = F^a_{\mu \nu} t^a \) and \( T_\mu = T^a_\mu t^a \) are matrices with \( t^a \) the generators of \( \text{SU}(N) \) Lie group. We have to sum over repeated generator indices \( a \). Using the Lie algebra \([t^a, t^b] = i f^{abc} t^c\), this gives the gauge field strength as,

\[ G^a_{\mu \nu} = \partial_\mu T^a_\nu - \partial_\nu T^a_\mu + g f^{abc} T^b_\mu T^c_\nu . \]

Using integration by parts on the kinetic term, equation(6) gives

\[ S_{\text{YM}} = \int d^D x \left( T_{\mu \nu} \hat{R}_{\rho \sigma} T^{\rho \sigma} + 2 g f^{abc} (\partial_\mu T^{\nu a} - \partial_\nu T^{\mu a}) T^{b c}_{\mu \nu} + g^2 f^{abc} f^{def} T^b_\mu T^c_\nu T^d_\rho T^{ef}_\rho \right) , \]

where the first term is Maxwell-like, and we have the second term and the third as the extra self-coupling terms compared to the abelian case. Let’s define the Maxwell-like action as

\[ S_{\text{Maxwell}} = \int d^D x (T_{\mu \nu} \hat{R}_{\rho \sigma} T^{\rho \sigma}) , \]

in such a way that we can use all the results we have in our previous work [23]. We also define the Maxwell-like gauge field strength as \( \tilde{G}_{\mu \nu} = \partial_\mu T_\nu - \partial_\nu T_\mu \), so that

\[ G_{\mu \nu} = \tilde{G}_{\mu \nu} - i g [T_\mu, T_\nu] , \]

and

\[ G^a_{\mu \nu} \rightarrow \tilde{G}^a_{\mu \nu} + g f^{abc} T^b_\mu T^c_\nu . \]

Under the rotor mechanism, the non-abelian gauge field transforms as

\[ T_{\mu \nu a} \rightarrow T_{\mu \nu a}^{\prime} = (\hat{R}_{\mu_1 \mu_2 \mu_3} \hat{R}^{\mu_1 \mu_2 \mu_3} \hat{R}_{\mu_1 \mu_2 \mu_3}) T_{\mu \nu a} . \]
2 The generalized non-abelian gauge field theorem under rotor model

In this paper, in analogy to the abelian case of our previous work in [23], we aim at proving the following theorem for the non-abelian case,

\[
S_{\text{YM}}^{(n)} = \frac{1}{4} \int d^Dx \, G_{n \mu \nu} G_{n a}^{\mu \nu a} = \int d^Dx \left( \frac{1}{4^n} \Box^n T^{a \mu} \hat{R}_{\mu \nu} \Box^n T^{a \nu} + \frac{2}{8^n} g f^{abc} (\partial^{\mu} \Box^n T^{a \nu} - \partial^{\nu} \Box^n T^{a \mu}) \Box^n T^{b \mu} \Box^n T^{c \nu} + \frac{g^2}{16^n} f^{abc} f^{def} \Box^n T^{b \mu} \Box^n T^{c \mu} \Box^n T^{d \nu} \Box^n T^{e \nu} \right) \quad (14)
\]

We will see that indeed the rotor mechanism in the abelian case applies to the non-abelian case as well without modification. The only necessary condition is the Lorentz gauge condition \( \partial_{\mu} T^{a \mu} = 0 \). In such way the rotor mechanism functions the gauge field to transform as

\[
T^{a \mu} \rightarrow \Box^n T^{a \mu}. \quad (15)
\]

And the gauge field strength transforms as

\[
G^{a \mu \nu} \rightarrow \partial_{\mu} \Box^n T^{a \nu} - \partial_{\nu} \Box^n T^{a \mu} - ig[\Box^n T^{a \mu}, \Box^n T^{a \nu}] = \Box^n \tilde{G}^{a \mu \nu} - ig(\Box^n T^{a \mu} \Box^n T^{a \nu} - \Box^n T^{a \nu} \Box^n T^{a \mu}). \quad (16)
\]

or explicitly

\[
G^{a \mu \nu} \rightarrow \partial_{\mu} \Box^n T^{a \nu} - \partial_{\nu} \Box^n T^{a \mu} + g f^{abc} \Box^n T^{b \mu} \Box^n T^{c \nu} = \Box^n \tilde{G}^{a \mu \nu} + g f^{abc} \Box^n T^{b \mu} \Box^n T^{c \nu}. \quad (17)
\]

When \( n = 0 \), then general form of equation (14) will reduce back to (9). We will first prove the theorem for \( n = 1 \), followed by \( n = 2 \) case and finally the general \( n \) case as we do in [23].

2.1 The \( n = 1 \) case

For the first-ordered rotation, the first-ordered rotated field is

\[
L^{a \mu} = \hat{R}_{\mu \nu} T^{a \nu}. \quad (18)
\]

The new gauge field strength is

\[
H^{a \mu \nu} = \partial_{\mu} L^{a \nu} - \partial_{\nu} L^{a \mu} - g f^{abc} L^{b \mu} L^{c \nu}. \quad (19)
\]

The new action becomes

\[
S_{\text{YM}}^{(1)} = \int d^Dx \left( L^{a \mu} \hat{R}_{\mu \nu} L^{a \nu} + \frac{2}{3} g f^{abc} (\partial^{\mu} L^{a \nu} - \partial^{\nu} L^{a \mu}) L^{b \mu} L^{c \nu} + \frac{g^2}{4} f^{abc} f^{def} L^{b \mu} L^{c \mu} L^{d \nu} L^{e \nu} \right) \quad (20)
\]

For the first Maxwell-like term, we can apply the result we obtained in our previous work [23] (equation 27), in which we have

\[
\int d^Dx L^{a \mu} \hat{R}_{\mu \nu} L^{a \nu} = \frac{1}{4} \int d^Dx \hat{R}_{a \mu \nu} \hat{R}^{a \mu \nu}. \quad (21)
\]
Then we have

\[
S^{(1)}_{YM} = \int d^D x \left( \frac{1}{4} \square T^{\mu a} \hat{R}_{\mu \nu} \square T^{\nu a} + 2 g f^{abc} (\partial^\mu L^{\nu a} - \partial^\nu L^{\mu a}) L^{b}_\mu L^{c}_\nu + g^2 f^{abc} f^{ade} L^{b}_\mu L^{c}_\nu L^{d}_\rho L^{e}_\sigma \right)
\]

\[
= \int d^D x \left( \frac{1}{4} \square T^{\mu a} \hat{R}_{\mu \nu} \square T^{\nu a} + 2 g f^{abc} (\partial^\mu \hat{R}^{\nu \delta T^a_\delta} - \partial^\nu \hat{R}^{\mu \lambda T^a_\lambda}) \hat{R}_{\mu \alpha} T^{\alpha b} \hat{R}_{\nu \beta} T^{\beta c} + g^2 f^{abc} f^{ade} \hat{R}_{\mu a} T^{\alpha b} \hat{R}_{\nu \beta} T^{\beta c} R^{\mu \gamma T^c_\gamma} R^{\nu \delta T^e_\delta} \right).
\]

(22)

Now we investigate the second term and the third term of the last line. For the second term, we have

\[
2 g f^{abc} (\partial^\mu \hat{R}^{\nu \delta T^a_\delta} - \partial^\nu \hat{R}^{\mu \lambda T^a_\lambda}) \hat{R}_{\mu \alpha} T^{\alpha b} \hat{R}_{\nu \beta} T^{\beta c}
\]

\[
= \frac{2}{8} g f^{abc} \left( (\partial^\mu (\square \eta^\nu - \partial^\nu \hat{R}^{\mu \lambda T^a_\lambda}) - \partial^\nu (\square \eta^\mu - \partial^\mu \hat{R}^{\nu \lambda T^a_\lambda}) \right) \left( \square \eta^{\alpha \alpha} - \partial^\alpha \partial^\alpha \right) T^{\alpha b}
\]

\[
\times (\square \eta^{\nu \beta} - \partial^\nu \partial^\beta) T^{\beta c}
\]

\[
= \frac{2}{8} g f^{abc} \left( (\partial^\mu (\square \eta^\nu - \partial^\nu \hat{R}^{\mu \lambda T^a_\lambda}) - \partial^\nu (\square \eta^\mu - \partial^\mu \hat{R}^{\nu \lambda T^a_\lambda}) \right) \left( \square \eta^{\alpha \alpha} - \partial^\alpha \partial^\alpha \right) T^{\alpha b}
\]

\[
\times (\square \eta^{\nu \beta} - \partial^\nu \partial^\beta) T^{\beta c}
\]

\[
= \frac{2}{8} g f^{abc} \left( (\partial^\mu (\square \eta^\nu - \partial^\nu \hat{R}^{\mu \lambda T^a_\lambda}) - \partial^\nu (\square \eta^\mu - \partial^\mu \hat{R}^{\nu \lambda T^a_\lambda}) \right) \left( \square \eta^{\alpha \alpha} - \partial^\alpha \partial^\alpha \right) T^{\alpha b}
\]

\[
\times (\square \eta^{\nu \beta} - \partial^\nu \partial^\beta) T^{\beta c}
\]

\[
= \frac{2}{8} g f^{abc} \left( (\partial^\mu (\square \eta^\nu - \partial^\nu \hat{R}^{\mu \lambda T^a_\lambda}) - \partial^\nu (\square \eta^\mu - \partial^\mu \hat{R}^{\nu \lambda T^a_\lambda}) \right) \left( \square \eta^{\alpha \alpha} - \partial^\alpha \partial^\alpha \right) T^{\alpha b}
\]

\[
\times (\square \eta^{\nu \beta} - \partial^\nu \partial^\beta) T^{\beta c}
\]

\[
= \frac{2}{8} g f^{abc} (\partial^\mu \square T^{\nu a} - \partial^\nu \square T^{\mu a}) \square T^{\beta}_{\mu} \square T^{\nu c}.
\]

(23)

Now we impose the Lorentz gauge condition, then the last three terms vanish. Therefore we have

\[
2 g f^{abc} (\partial^\mu \hat{R}^{\nu \delta T^a_\delta} - \partial^\nu \hat{R}^{\mu \lambda T^a_\lambda}) \hat{R}_{\mu \alpha} T^{\alpha b} \hat{R}_{\nu \beta} T^{\beta c}
\]

\[
= \frac{2}{8} g f^{abc} (\partial^\mu \square T^{\nu a} - \partial^\nu \square T^{\mu a}) \square T^{\beta}_{\mu} \square T^{\nu c}.
\]

(24)

For the third term, we would also impose Lorentz gauge condition,

\[
\frac{g^2}{16} f^{abc} f^{ade} \hat{R}_{\mu a} T^{\alpha b} \hat{R}_{\nu \beta} T^{\beta c} R^{\mu \gamma T^c_\gamma} R^{\nu \delta T^e_\delta}
\]

\[
= \frac{g^2}{16} f^{abc} f^{ade} \left( \square \eta^{\alpha \beta} T^{\gamma}_{\gamma} - \partial^\alpha \partial^\beta T^{\gamma}_{\gamma} \right) \left( \square \eta^{\nu \delta} T^{\epsilon}_{\epsilon} - \partial^\nu \partial^\delta T^{\epsilon}_{\epsilon} \right)
\]

\[
= \frac{g^2}{16} f^{abc} f^{ade} \left( \square \eta^{\alpha \beta} T^{\gamma}_{\gamma} - \partial^\alpha \partial^\beta T^{\gamma}_{\gamma} \right) \left( \square \eta^{\nu \delta} T^{\epsilon}_{\epsilon} - \partial^\nu \partial^\delta T^{\epsilon}_{\epsilon} \right)
\]

\[
= \frac{g^2}{16} f^{abc} f^{ade} \square T^{\beta}_{\mu} \square T^{\gamma}_{\nu} \square T^{\epsilon}_{\rho} \square T^{\epsilon}_{\sigma}.
\]

(25)

Therefore, for the \( n = 1 \) case, we obtain

\[
S^{(4)}_{YM} = \int d^D x \left( \frac{1}{4} \square T^{\mu a} \hat{R}_{\mu \nu} \square T^{\nu a} + \frac{2}{8} g f^{abc} (\partial^\mu \square T^{\nu a} - \partial^\nu \square T^{\mu a}) \square T^{\beta}_{\mu} \square T^{\nu c} + \frac{g^2}{16} f^{abc} f^{ade} \square T^{\beta}_{\mu} \square T^{\gamma}_{\nu} \square T^{\epsilon}_{\rho} \square T^{\epsilon}_{\sigma} \right)
\]

(26)

Thus the proof is completed.

\[2.2 \quad \text{The } n = 2 \text{ case}\]

For the second-ordered rotation, the second-ordered rotation field is

\[
J^{\mu a} = R^{\mu a} \hat{R}_{\mu \nu} T^{\nu a}.
\]

(27)
Applying Lorentz gauge condition, we obtain
\[ J^{\mu a} = \frac{1}{4} (\Box^2 \delta^{\mu}_{\alpha} - \Box \partial^{\mu} \partial_{\alpha}) T^{\alpha a}. \] (28)

Also we have
\[ J^{\alpha}_{\sigma} = \eta_{\sigma \rho} \hat{R}^{\rho a}_{\mu \nu} T^{\nu a} = \frac{1}{4} (\Box^2 \eta_{\sigma \alpha} - \Box \partial_{\sigma} \partial_{\alpha}) T^{\alpha a}. \] (29)

The new gauge field strength is
\[ K^{\mu}_{\nu} = \partial_{\mu} J^{\nu} - \partial_{\nu} J^{\mu} - g f^{abc} J^{b}_{\mu} J^{c}_{\nu}. \] (30)

The new action becomes
\[ S^{(2)}_{YM} = \int d^D x \left( J^{\mu a} \hat{R}^{\nu a}_{\mu \rho} - 2g f^{abc} (\partial^{\mu} \hat{R}^{\nu a}_{\mu \rho} - \partial^{\nu} L^{\mu a}), J^{b}_{\mu} J^{c}_{\nu} + g^2 f^{abc} f^{ade} J^{b}_{\mu} J^{c}_{\nu} J^{d}_{\rho} J^{e}_{\sigma} \right), \] (31)

For the first Maxwell-like term, we can apply the result we obtained in our previous paper \cite{23} (equation (43)), in which we have,
\[ \int d^D x J^{\mu a} \hat{R}^{\nu a}_{\mu \rho} = \frac{1}{16} \int d^D x \Box^2 T^{\mu a} \hat{R}^{\nu a}_{\mu \rho} \Box^2 T^{\nu a}. \] (32)

Explicitly, this is
\[ S^{(2)}_{YM} = \int d^D x \left( \frac{1}{16} \Box^2 T^{\mu a} \hat{R}^{\nu a}_{\mu \rho} \Box^2 T^{\nu a} + \frac{2}{64} g f^{abc} \left( (\partial^{\mu} \Box^2 \delta^{\nu}_{\alpha} - \Box \partial^{\nu} \partial_{\alpha}) T^{\alpha a} - \partial^{\nu} (\Box^2 \delta^{\mu}_{\alpha} - \Box \partial^{\mu} \partial_{\alpha}) T^{\alpha a} \right) \right. \]
\[ \times \left. \eta_{\mu \rho} \eta_{\nu \alpha} \hat{R}^{\sigma \mu \alpha \rho \nu} \hat{R}^{\sigma \nu \rho \mu \alpha} T^{\beta \epsilon \gamma \delta} \right. \]
\[ + \left. \frac{g^2}{256} (\Box^2 \eta_{\mu \rho} T^{\alpha \beta} - \Box \partial_{\mu} \partial_{\rho} T^{\alpha \beta}) (\Box^2 \eta_{\nu \beta} T^{\gamma \delta} - \Box \partial_{\nu} \partial_{\beta} T^{\gamma \delta}) (\Box^2 \delta^{\epsilon}_{\gamma} T^{\rho \delta} - \Box \partial^{\epsilon} \partial_{\rho} T^{\delta \gamma}) \right) \times \left( \Box^2 \delta^{\gamma}_{\delta} T^{\epsilon \lambda} - \Box \partial^{\gamma} \partial_{\epsilon} T^{\delta \lambda} \right) \right). \] (33)

Then we have
\[ S^{(2)}_{YM} = \int d^D x \left( \frac{1}{16} \Box^2 T^{\mu a} \hat{R}^{\nu a}_{\mu \rho} \Box^2 T^{\nu a} + \frac{2}{64} g f^{abc} \left( (\partial^{\mu} \Box^2 \delta^{\nu}_{\alpha} - \Box \partial^{\nu} \partial_{\alpha}) T^{\alpha a} - \partial^{\nu} (\Box^2 \delta^{\mu}_{\alpha} - \Box \partial^{\mu} \partial_{\alpha}) T^{\alpha a} \right) \right. \]
\[ \times \left. \eta_{\mu \rho} \eta_{\nu \alpha} \hat{R}^{\sigma \mu \alpha \rho \nu} \hat{R}^{\sigma \nu \rho \mu \alpha} T^{\beta \epsilon \gamma \delta} \right. \]
\[ + \left. \frac{g^2}{256} (\Box^2 \eta_{\mu \rho} T^{\alpha \beta} - \Box \partial_{\mu} \partial_{\rho} T^{\alpha \beta}) (\Box^2 \eta_{\nu \beta} T^{\gamma \delta} - \Box \partial_{\nu} \partial_{\beta} T^{\gamma \delta}) (\Box^2 \delta^{\epsilon}_{\gamma} T^{\rho \delta} - \Box \partial^{\epsilon} \partial_{\rho} T^{\delta \gamma}) \right) \times \left( \Box^2 \delta^{\gamma}_{\delta} T^{\epsilon \lambda} - \Box \partial^{\gamma} \partial_{\epsilon} T^{\delta \lambda} \right) \right). \] (34)

Applying Lorentz gauge condition, we obtain
\[ S^{(2)}_{YM} = \int d^D x \left( \frac{1}{16} \Box^2 T^{\mu a} \hat{R}^{\nu a}_{\mu \rho} \Box^2 T^{\nu a} + \frac{2}{64} g f^{abc} \left( (\partial^{\mu} \Box^2 \delta^{\nu}_{\alpha} - \Box \partial^{\nu} \partial_{\alpha}) T^{\alpha a} - \partial^{\nu} (\Box^2 \delta^{\mu}_{\alpha} - \Box \partial^{\mu} \partial_{\alpha}) T^{\alpha a} \right) \right. \]
\[ \times \left. \eta_{\mu \rho} \eta_{\nu \alpha} \hat{R}^{\sigma \mu \alpha \rho \nu} \hat{R}^{\sigma \nu \rho \mu \alpha} T^{\beta \epsilon \gamma \delta} \right. \]
\[ + \left. \frac{g^2}{256} (\Box^2 \eta_{\mu \rho} T^{\alpha \beta} - \Box \partial_{\mu} \partial_{\rho} T^{\alpha \beta}) (\Box^2 \eta_{\nu \beta} T^{\gamma \delta} - \Box \partial_{\nu} \partial_{\beta} T^{\gamma \delta}) (\Box^2 \delta^{\epsilon}_{\gamma} T^{\rho \delta} - \Box \partial^{\epsilon} \partial_{\rho} T^{\delta \gamma}) \right) \times \left( \Box^2 \delta^{\gamma}_{\delta} T^{\epsilon \lambda} - \Box \partial^{\gamma} \partial_{\epsilon} T^{\delta \lambda} \right) \right) \]
\[ + \frac{g^2}{256} (\Box^2 \hat{R}^{\mu a}_{\nu b} \Box^2 \hat{R}^{\nu b}_{\mu a} - \Box \partial^{\mu} \partial^{\nu} T^{a b} - \Box \partial^{\mu} \partial^{\nu} T^{a b}) \right), \] (35)

which completes the proof for \( n = 2 \) case.
3 The general \( n \) case

To prove the general \( n \) case, we use the results in our previous paper \[23\] for \( n \)-th order rotation and apply to the non-abelian case. Equation (3) and equation give (4)

\[
T_{n \mu \nu} = (\hat{R}_{\mu \nu} - 1 \hat{R}_{\mu - 1 \nu - 2} \ldots \hat{R}_{\mu \nu}) T_{\mu \nu} = \frac{1}{2n-1} \left( \square^{n-1} \delta_{\mu}^{1} \right) \hat{R}_{\mu \nu} T^{\mu \nu},
\]

(36)

The new gauge field transforms as

\[
G_{n \mu \nu} = \partial \nu T_{n \mu \nu} + g f^{abc} T_{n \mu} \nabla^c.
\]

(37)

The new action becomes

\[
S^{(2)}_{\text{YM}} = \int d^4 x \left( T_{n \mu \nu} \hat{R}_{\mu \nu} - 2 g f^{abc} \left( \partial \mu T_{n \nu} - \partial \nu T_{n \mu} \right) T_{a \mu \nu} + g^2 f^{abc} T_{n \mu} \nabla^c \right),
\]

(38)

For the first Maxwell-like term, we can apply the result we obtained in our previous paper \[23\] (equation (75)), in which we have,

\[
\int d^4 x \hat{R}_{\mu \nu} T_{n \mu \nu} = \frac{1}{4n} \int d^4 x \nabla^{\mu} T_{n \mu} T_{n \nu}.
\]

(39)

Therefore the action is

\[
S^{(2)}_{\text{YM}} = \int d^4 x \left( \frac{1}{4n} \nabla^{\mu} T_{n \mu} \nabla^{\nu} T_{n \nu} + \frac{2}{2n-2} g f^{abc} \left( \partial \mu \nabla^{\nu} \eta^{\mu \nu \mu} \hat{R}_{\mu \nu} T_{\mu \nu} \right) \right.
\]

\[
- \partial \nu \nabla^{\mu} \eta^{\mu \nu \mu} \hat{R}_{\mu \nu} T_{\mu \nu} \right)
\]

\[
\left. - \delta \nu \nabla^{\mu} \eta^{\mu \nu \mu} \hat{R}_{\mu \nu} T_{\mu \nu} \right)
\]

\[
\left. + \frac{g^2}{24n} f^{abc} \left( \nabla^{\mu} \nabla^{\nu} T_{n \mu} T_{n \nu} + g^2 f^{abc} T_{n \mu} \nabla^c \right) \right),
\]

(40)

But since under the Lorentz gauge, we immediately have

\[
T_{n \mu \nu} = \frac{1}{2n-1} \left( \square^{n-1} \delta_{\mu}^{1} \right) \cdot \frac{1}{2} \left( \square \eta_{\mu \nu} - \partial \mu \partial \nu \right) T_{\mu \nu} = \frac{1}{2n} \nabla^{\mu} T_{\mu \nu}.
\]

(41)

Therefore we have

\[
S^{(2)}_{\text{YM}} = -\frac{1}{4} \int d^4 x F_{n \mu \nu} \nabla^{\mu} \nabla^{\nu}
\]

\[
= \int d^4 x \left( \frac{1}{4n} \nabla^{\mu} T_{n \mu} \nabla^{\nu} T_{n \nu} + \frac{2}{2n-2} g f^{abc} \left( \partial \mu \nabla^{\nu} T_{n \nu} - \partial \nu \nabla^{\mu} T_{n \mu} \right) \right)
\]

\[
+ \frac{g^2}{24n} f^{abc} \left( \nabla^{\mu} \nabla^{\nu} T_{n \mu} T_{n \nu} + g^2 f^{abc} T_{n \mu} \nabla^c \right)
\]

(42)

which completes the proof of the general \( n \) case. Therefore, under the rotor mechanism, the non-abelian gauge field transforms as \( T_{\mu}^{n} \rightarrow \nabla^{n} T_{\mu}^{n} \) as we can compare to the original action in equation (3). When \( n = 0 \), this would give us back the original Yang-Mills action. The renormalization dimension that requires to keep unity dimension of \( T_{\mu}^{n} \) field would be same as the abelian case, which is is \( D = 4n + 4 \).
4 Equation of motion and Noether’s current of the generalized abelian gauge field theory

The equation of motion of the original Yang-Mill’s theory is given by \[24, 25\],
\[
\partial_{\mu}F^{\mu\nu} + gf^{abc}T^{b}_{\mu}G^{\mu\nu c} = 0. \tag{43}
\]

The covariant derivative is defined by
\[
D_{\mu} = \partial_{\mu} - igT^{c}_{\mu}t^{c}. \tag{44}
\]

In tensor form,
\[
D_{\mu}^{ab} = \partial_{\mu}\delta^{ab} - igT^{c}_{\mu}(t^{c})^{ab}. \tag{45}
\]

Using the adjoint representation \((t^{k})^{ij} = -if^{kij}\), we would have
\[
D_{\mu}^{ab} = \partial_{\mu}\delta^{ab} - gf^{abc}T^{c}_{\mu} = \partial_{\mu}\delta^{ab} + gf^{acbt}t^{c}_{\mu}. \tag{46}
\]

Therefore, the equation of motion in \((43)\) can be neatly written by the covariant derivative acting on the non-abelian gauge field strength,
\[
D_{\mu}G^{\mu\nu a} = 0. \tag{47}
\]

Expanding \((43)\) or \((47)\) explicitly gives
\[
2\hat{R}^{\mu\nu}T^{a}_{\mu} + g f^{abc}(2T^{a}_{\mu}\partial_{\mu}T^{\nu c} + (\partial_{\mu}T^{ab})T^{c\nu} + T^{b}_{\mu}\partial^{c}T^{\nu k} + gf^{cde}T^{a}_{\mu}\partial^{d}T^{\nu c}) = 0. \tag{48}
\]

Under Lorentz gauge, the equation of motion simplifies to
\[
2\Box T^{a}_{\mu} + g f^{abc}(2T^{a}_{\mu}\partial_{\mu}T^{\nu c} + T^{b}_{\mu}\partial^{c}T^{\nu k} + gf^{cde}T^{a}_{\mu}\partial^{d}T^{\nu c}) = 0. \tag{49}
\]

When we compare it to the equation of motion of the abelian case \(2\hat{R}^{\mu\nu}T^{a}_{\mu} = 0\), we see that the non-abelian case contains extra non-linear terms.

To find the equation of motion for our generalized rotor model for non-abelian case, we need to vary the action as follow,
\[
\delta S_{YM}^{(n)} = -\frac{1}{2} \int d^{D}x G^{\mu\nu a}_{n} \delta G^{a}_{n\mu\nu}. \tag{50}
\]

The remaining work is to compute \(\delta G^{a}_{n\mu\nu}\). Using the transformation of field under the Lorentz gauge as in \((42)\),
\[
T^{a}_{\mu\nu} = \frac{1}{2n}\Box^{n}T^{a}_{\mu}, \tag{51}
\]

first we have,
\[
G^{a}_{n\mu\nu} = \frac{1}{2n}\partial_{\mu}\Box^{n}T^{a}_{\nu} - \frac{1}{2n}\partial_{\nu}\Box^{n}T^{a}_{\mu} + \frac{1}{4n}gf^{abc}\Box^{n}T^{b}_{\mu}\Box^{n}T^{c}_{\nu}. \tag{52}
\]

Then we have the variation as
\[
\delta G^{a}_{n\mu\nu} = \frac{1}{2n}\partial_{\mu}\delta\Box^{n}T^{a}_{\nu} - \frac{1}{2n}\partial_{\nu}\delta\Box^{n}T^{a}_{\mu} + \frac{1}{4n}gf^{abc}\delta\Box^{n}T^{b}_{\mu}\delta\Box^{n}T^{c}_{\nu} + \delta\Box^{n}T^{a}_{\mu}\delta\Box^{n}T^{\nu c} + \delta\Box^{n}T^{b}_{\mu}\delta\Box^{n}T^{c}_{\nu}
\]
\[
= \frac{1}{2n}\partial_{\mu}\delta\Box^{n}T^{a}_{\nu} - \frac{1}{2n}\partial_{\nu}\delta\Box^{n}T^{a}_{\mu} + \frac{1}{4n}gf^{abc}\delta\Box^{n}T^{b}_{\mu}\delta\Box^{n}T^{c}_{\nu} + \frac{1}{4n}gf^{acbt}\delta\Box^{n}T^{b}_{\mu}\delta\Box^{n}T^{c}_{\nu}
\]
\[
= \frac{1}{2n}\partial_{\mu}\delta\Box^{n}T^{a}_{\nu} - \frac{1}{2n}\partial_{\nu}\delta\Box^{n}T^{a}_{\mu} + \frac{1}{4n}gf^{abc}\delta\Box^{n}T^{b}_{\mu}\delta\Box^{n}T^{c}_{\nu} - \frac{1}{4n}gf^{abc}\delta\Box^{n}T^{b}_{\mu}\delta\Box^{n}T^{c}_{\nu}
\]
\[
= \left(\frac{1}{2n}\partial_{\mu}\delta\Box^{n}T^{a}_{\nu} + \frac{1}{4n}gf^{abc}\delta\Box^{n}T^{b}_{\mu}\delta\Box^{n}T^{c}_{\nu}\right) - \left(\frac{1}{2n}\partial_{\nu}\delta\Box^{n}T^{a}_{\mu} + \frac{1}{4n}gf^{abc}\delta\Box^{n}T^{b}_{\mu}\delta\Box^{n}T^{c}_{\nu}\right)
\]
\[
:= \frac{1}{2n}\left(\partial_{\mu}\delta\Box^{n}T^{a}_{\nu} - \partial_{\nu}\delta\Box^{n}T^{a}_{\mu}\right), \tag{53}
\]
where we define the covariant derivative of the rotor model as
\[ D_{n \mu} = \partial_\mu - \frac{i}{2n} g \Box^n T_\mu^c, \]  
(54)
or in tensor form
\[ D_{ab}^{n \mu} = \partial_\mu \delta_{ab} - \frac{i}{2n} g f^{cab} \Box^n T_\mu^c \]  
(55)
and using the adjoint representation \((t^c)_{ab} = -if_{cab}\), then we have
\[ D_{ab}^{n \mu} = \partial_\mu \delta_{ab} - \frac{1}{2n} g f^{abc} \Box^n T_\mu^c \]  
(56)
Therefore the covariant derivative in adjoint representation is
\[ D_{ab}^{n \mu} = \partial_\mu \delta_{ab} + \frac{1}{2n} g f^{acb} \Box^n T_\mu^c. \]  
(57)
Therefore we have
\[ \frac{1}{2n} D_{ab}^{n \mu} \delta \Box^n T_\nu^b = \frac{1}{2n} D_{n \mu} \delta \Box^n T_\nu^a = \frac{1}{2n} \left( \partial_\mu \delta_{ab} + \frac{1}{2n} g f^{acb} \Box^n T_\mu^c \right) \delta \Box^n T_\nu^b \]  
(58)
as expected. Similarly,
\[ \frac{1}{2n} D_{ab}^{n \nu} \delta \Box^n T_\mu^b = \frac{1}{2n} \partial_\nu \delta \Box^n T_\mu^a + \frac{1}{4n} g f^{abc} \Box^n T_\mu^b \delta \Box^n T_\nu^c \]  
(59)
as expected. Now the variation of action becomes
\[ \delta S_{YM}^{(n)} = -\frac{1}{2} \int d^D x G_\mu^n \omega_a \delta G_{n \mu}^a \]  
\[ = -\frac{1}{2} \int d^D x G_\mu^{n a} (D_{n \mu} \delta \Box^n T_\nu^a - D_{n \nu} \delta \Box^n T_\mu^a) \]  
(60)\[ = \int d^D x G_\mu^{n a} D_{n \mu} \delta \Box T_\nu^a \]  
\[ = -\int d^D x (D_{n \mu} G_\mu^{n a} \delta \Box T_\nu^a), \]
where from the third line to the forth line we have performed integration by parts. The action is extremeized when
\[ \frac{\delta S_{YM}^{(n)}}{\delta \Box^n T_\nu^a} = 0. \]  
(61)
Thus we obtain the equation of motion for the generalized non-abelian gauge field theory as follow
\[ D_{n \mu} G_\mu^{n a} = 0, \]  
(62)
or using equation (57), acting on \(G_\mu^{n a}\)
\[ \partial_\mu G_\mu^{n a} + \frac{1}{2n} g f^{abc} \Box^n T_\mu^b G_\mu^{n c} = 0. \]  
(63)
Explicitly expanding equation (63), the original equation of motion becomes (48) under
the rotor mechanism, which is as follow,
\[
2 \hat{\mathcal{R}}^{\mu \nu} T^a_{\mu} + g f^{abc} \left( \frac{1}{2^n} \left( 2 \Box^n T^a_{\mu} \partial^\mu \Box^n T^{\nu c} + (\partial^\mu \Box^n T^{\nu b}) \Box^n T^{\nu c} + \Box^n T^{\nu b} \partial^\nu \Box^n T^{\mu k} \right) + \frac{1}{8^n} g f^{\alpha \beta \gamma} \Box^n T^a_{\mu} \Box^n T^{\alpha \beta} \right) = 0 .
\] (64)

Under the Lorentz gauge condition, the equation of motion simplifies to
\[
2 \Box^n T^{\nu a} + g f^{abc} \left( \frac{1}{2^n} \left( 2 \Box^n T^a_{\mu} \partial^\mu \Box^n T^{\nu c} + \Box^n T^{\nu b} \partial^\nu \Box^n T^{\mu k} \right) + \frac{1}{8^n} g f^{\alpha \beta \gamma} \Box^n T^a_{\mu} \Box^n T^{\alpha \beta} \right) = 0 .
\] (65)
The general covariant derivative of the rotor model in (54) in matrix form satisfies the
Jacobi identity,
\[
[D_{n \mu}, [D_{n \mu}, D_{n \nu}]] + [D_{n \mu}, [D_{n \nu}, D_{n \rho}]] + [D_{n \nu}, [D_{n \rho}, D_{n \mu}]] = 0 .
\] (66)

And since the commutator of covariant derivative is promoted to
\[
[D_{n \mu}, D_{n \nu}] = -i g F_{\mu \nu} ,
\] (67)
therefore we obtain the Bianchi identity of high-order covariant derivative as
\[
D_{n \mu} F^{a}_{\mu \nu} + D_{n \mu} F^{a}_{\nu \rho} + D_{n \nu} F^{a}_{\rho \mu} = 0 .
\] (68)

Finally, we will compute the Noether’s current and the associated Noether’s charge
of our high-order non-abelian gauge field theory under rotor model. First we know
that by the transformation of field due to the rotor mechanism \( T^a_{\mu} \rightarrow \Box^n T^a_{\mu} \), the Euler-Lagrangian equation becomes,
\[
\frac{\partial \mathcal{L}_{YM}^{(n)}}{\partial \Box^n T^a_{\mu}} = \partial^\mu \frac{\partial \mathcal{L}_{YM}^{(n)}}{\partial (\partial^\mu \Box^n T^a_{\mu})} .
\] (69)

Using the Euler-Lagrangian equation, it can be shown that by the same derivation as
in [23](equation 103), that the Noether current is identified by
\[
J^\mu = \frac{\partial \mathcal{L}_{YM}^{(n)}}{\partial (\partial^\mu \Box^n T^a_{\mu})} \delta \Box^n T^a_{\mu} .
\] (70)

The Lagrangian can be broken down into three separate parts: the Maxwell part, the
mixing part and the self-coupling part.
\[
\mathcal{L}_{YM}^{(n)} = \mathcal{L}_{\text{Maxwell}}^{(n)} + \mathcal{L}_{\text{mixing}}^{(n)} + \mathcal{L}_{\text{self-coupling}}^{(n)} .
\] (71)

The Maxwell part is in 2n-order derivatives,
\[
\mathcal{L}_{\text{Maxwell}}^{(n)} = \frac{1}{4^n} \Box^n T^{\mu a} \hat{R}_{\mu \nu} \Box^n T^{\nu a} = - \frac{1}{4^{n+1}} \Box^n \hat{G}^{\mu a} \Box^n \hat{G}^{\nu a} .
\] (72)

The mixing part is in n-order derivatives,
\[
\mathcal{L}_{\text{mixing}}^{(n)} = \frac{2}{2^n} g f^{abc} (\partial^\mu \Box^n T^{\nu a} - \partial^\nu \Box^n T^{\mu a}) \Box^n T^{\mu b} \Box^n T^{\nu c} .
\] (73)
The self-coupling part is in 0-order derivatives,
\[ L_{self-coupling}^{(n)} = \frac{g^2}{24n} f^{abc} f^{ade} \square^n T^b_\mu \square^n T^c_\nu \square^n T^d_\rho \square^n T^{\mu} . \]  

(74)

Hence the Noether current is
\[ J^\mu = \left( \frac{\partial L_{Maxwell}^{(n)}}{\partial (\partial_\mu \square^n T^a_\nu)} + \frac{\partial L_{mixing}^{(n)}}{\partial (\partial_\mu \square^n T^a_\nu)} + \frac{\partial L_{self-coupling}^{(n)}}{\partial (\partial_\mu \square^n T^a_\nu)} \right) \delta \square^n T^a_\nu . \]  

(75)

The first term is the standard Maxwell case, which is evaluated as
\[ \frac{\partial L_{Maxwell}^{(n)}}{\partial (\partial_\alpha \square^n T^a_\beta)} = - \frac{1}{4^n} \square^n G^{\alpha \beta} . \]  

(76)

The second term is evaluated to be
\[ \frac{\partial L_{mixing}^{(n)}}{\partial (\partial_\alpha \square^n T^a_\beta)} = \frac{2}{24n} g f^{abc} \left( \frac{\partial}{\partial (\partial_\mu \square^n T^a_\nu)} - \frac{\partial}{\partial (\partial_\alpha \square^n T^a_\nu)} \right) \square^n T^b_\mu \square^n T^c_\nu \]  

(77)

\[ = \frac{2}{24n} g f^{abc} \left( \delta^{\mu a} \delta^{\nu b} \delta^k - \delta^{\nu a} \delta^{\mu b} \delta^k \right) \square^n T^b_\mu \square^n T^c_\nu \]  

\[ = \frac{2}{24n} g f^{kbc} \left( \square^n T^{a b} \square^n T^{b c} - \square^n T^{a c} \square^n T^{b b} \right) \]  

\[ = \frac{2}{24n} g f^{kbc} \square^n T^{a [b \square^n T^{c] \beta}} . \]

The third term is zero as there are no derivatives of the rotor field,
\[ \frac{\partial L_{self-coupling}^{(n)}}{\partial (\partial_\alpha \square^n T^a_\beta)} = 0 . \]  

(78)

Therefore, the Noether’s current is
\[ J^\alpha = \left( - \frac{1}{4^n} \square^n G^{\alpha \beta} + \frac{2}{8^n} g f^{kbc} \left( \square^n T^{a b} \square^n T^{b c} - \square^n T^{a c} \square^n T^{b b} \right) \right) \delta \square^n T^b_\beta . \]  

(79)

The associated Noether’s charge is given by
\[ Q = \int d^{D-1} x J^0 = \int d^{D-1} x \left( - \frac{1}{4^n} \square^n G^{0 \beta} + \frac{2}{8^n} g f^{kbc} \left( \square^n T^{0 b} \square^n T^{b c} - \square^n T^{0 c} \square^n T^{b b} \right) \right) \delta \square^n T^b_\beta . \]  

(80)

5 Conclusion

In this paper, we have established the generalized non-abelian gauge field theorem under the rotor mechanism. Under the Lorentz gauge condition, the rotor transformation of gauge field for the general non-abelian case is same as the abelian case. The gauge field transforms as \( T^a_\mu \rightarrow \square^n T^a_\mu \) under the rotor mechanism. When \( n = 0 \) this restores back to the original Yang-Mills theory. We also compute the equation of motion and Noether’s current for our theory. In the future, this theory can help to develop the generalized field theory of higher order derivatives for the standard model of particles.
References

[1] B. Podolsky. A Generalized Electrodynamics Part I—Non-Quantum. *Phys. Rev.* 62, 68. 1942.

[2] B. Podolsky and C. Kikuchi. A Generalized Electrodynamics Part II-Quantum. *Phys. Rev.* 65, 228. 1944.

[3] B. Podolsky and C. Kikuchi. Auxiliary Conditions and Electrostatic Interaction in Generalized Quantum Electrodynamics. *Phys. Rev.* 67, 184. 1945.

[4] B. Podolsky and P. Schwed. Review of a Generalized Electrodynamics. *Rev. Mod. Phys.* 20, 40. 1948.

[5] D. J. Montgomery. Relativistic Interaction of Electrons on Podolsky’s Generalized Electrodynamics. *Phys. Rev.* 69, 117. 1946.

[6] T. D. Lee and G. C. Wick *Nucl.Phys.B* 9, 209. 1969.

[7] T. D. Lee and G. C. Wick. *Phys.Rev.D* 2, 1033. 1970.

[8] B. Grinstein, D. O’Connell, M. B. Wise. The Lee-Wick Standard Model. *Phys.Rev.D* 77:025012. 2008.

[9] A. Pais and G. E. Uhlenbeck. On Field Theories with Non-Localized Action. *Phys.Rev.* 79, 14. 1950.

[10] G.W. Gibbons, C.N. Pope and Sergey Solodukhin. Higher Derivative Scalar Quantum Field Theory in Curved Spacetime. *Phys.Rev.D* 100. 2019.

[11] J. Dai. Stability in the higher derivative Abelian gauge field theory. *Nuclear Physics B.* Vol 961. 2020.

[12] D.S. Kaparulin. A stable higher-derivative theory with the Yang-Mills gauge symmetry. [arXiv:2011.12928 [hep-th]]

[13] M. Ostrogradsky. Mem. Ac. St. Petersbourg VI 4 (1850) 385.

[14] R. P. Woodard. The Theorem of Ostrogradsky. [arXiv:1506.02210 [hep-th]].

[15] V.V. Nesterenko. On the instability of classical dynamics in theories with higher derivatives, *Phys.Rev.D* 75. 2007.

[16] N.G. Stephen. On the Ostrogradski instability for higher-order derivative theories and pseudo-mechanical energy. *J. Sound. Vib* 310(3): 729-739. 2008.

[17] H. Motohashi and T. Suyama. Third order equations of motion and the Ostrogradsky instability, *Phys.Rev.D* 91. 2015.

[18] K. S. Stelle. Renormalization of higher-derivative quantum gravity. *Phys.Rev.D* 16, 953. 1977.

[19] E. S. Fradkin and A.A. Tseytlin. Renormalizable asymptotically free quantum theory of gravity. *Nucl. Phys.B* 201. 469.
[20] S. Nojiri and S. D. Odintso. Introduction to Modified Gravity and Gravitational Alternative for Dark Energy. *Int. J. Geom. Meth. Mod. Phys.* **4**, 2007.

[21] T. P. Sotiriou. $f(R)$ Theories of Gravity. *Rev. Mod. Phys.* **82**, 451-497, 2010.

[22] S. Nojiri and S. D. Odintsov. *Phys. Rept.* **505**, 2011, 59.

[23] B. T. T. Wong. Generalized abelian gauge field theory under rotor model. *Mod. Phys. Lett. A.* Vol. 36, No. 27, 2150194, 2021. [arXiv:2104.14472](https://arxiv.org/abs/2104.14472).

[24] Yang, C. N.; Mills, R. Conservation of Isotopic Spin and Isotopic Gauge Invariance. *Physical Review* **96** (1): 191–195, 1954.

[25] M. E. Peskin and D. V. Schroeder. An introduction to quantum field theory. *ABP.* 1995.