Control and synchronization of fractional order complex valued chaotic Chen systems

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Abstract. In this paper, the control and synchronization of the fractional order complex valued chaotic Chen system are investigated. Based on the synchronization stability theorem of fractional order nonlinear systems, a feedback control method is designed to realize the controlling of the system. Meanwhile, a new approach for designing a synchronization controller, based on a special matrix, is proposed. Numerical simulations are given to demonstrate the effectiveness and clarity of the approach.

1. Introduction

Chaos occurs in the system of stochastic irregular motion, which is sensitive to initial conditions; its behaviour is uncertainty, non-repeatable and unpredictable. In 1990, Pecora and Carroll proposed the concept of chaotic synchronization, and successfully realized the synchronization of two coupled chaotic systems in the laboratory [1]. In 1999, Chen Guanrong found a new chaotic system Chen system [2-4], due to its great potential applications such as mathematical science, secure communication, etc. a variety of chaotic synchronization methods have been proposed in previous studies, such as lag synchronization [5], adaptive synchronization [6], complete synchronization [7], impulsive synchronization [8], etc.

For more than 300 years, the fractional calculus remained a purely mathematical subject with little understanding of its physical applications [9]. However, in recent years, with the further development of cross disciplinary application, the fractional derivative has becoming a hot topic in the interdisciplinary application of fractional calculus [10]. A lot of researchers have begun to pay more attention to fractional calculus from the point of view of applications, and it has been found that fractional derivatives can well describe the systems in multidisciplinary fields [11].

However, for a long period time, many researches about fractional-order chaotic systems only concentrated on the state variables in real space, not in complex space. But in many real cases, it is found that the fractional order complex valued system has more research significance [12]. The complex valued systems can be widely used to describe physical phenomena, such as the amplitude of the electromagnetic field [13]. In recent years, the fractional order complex chaotic system has received more and more attention [14]. However, due to multiple factors, the synchronization and control of the system become more difficult. So it is important to study the synchronization and control of fractional order complex valued systems.

This paper is organized as follows: in Sections 2, the basic knowledge and the chaotic attractor of the fractional order Chen system are introduced. In Section 3, by separating the real part and imaginary part of each variable of the system, we design a controller to stabilize the fractional order...
complex chaotic Chen system. In Section 4, a linear feedback controller is designed by using the matrix collocation method to realize the synchronization of fractional order complex chaotic Chen systems. In section 5, some numerical examples are presented to demonstrate the effectiveness of the controllers. Finally, a conclusion is given in Section 6.

2. Preliminaries and model description

2.1. Fractional calculus

Definition [15]: The Caputo fractional derivative of order \( \alpha \) for a function \( f \in C^{n+1} ([0, \infty), R) \) is defined as

\[
D^\alpha_t f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{n-\alpha}} d\tau,
\]

where \( t > t_0 \) and \( n \) is a positive integer such that \( n-1 < \alpha < n \in \mathbb{Z}^+ \).

Lemma 1 [16]: For fractional order systems, if the fractional order \( 0 < q < 1 \), and there exist positive definite matrix \( P \) and semi-positive definite matrix \( Q \), for any state variable \( x(t) \in R^N \), satisfy:

\[
x^T(t)PD^q_t x(t) + x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau) \leq 0,
\]

then the fractional-order delay system is Lyapunov stable.

Remark: When the time delay \( \tau = 0 \), the fractional order \( 0 < q < 1 \), if there exists a positive definite matrix \( P \), for any state variable \( x(t) \in R^N \), such that: \( x^T(t)P_D^q x(t) \leq 0 \), then the system is Lyapunov stable.

Lemma 2 [17]: For nonlinear fractional order systems \( D^q_t = A(x)x \), where \( 0 < q < 1 \), \( x = (x_1, x_2, x_3, \cdots, x_n) \) are the state vectors of the system, \( A(x) \in R^{nm} \) is the coefficient matrix. If \( A(x) = A_1(x) + A_2 \), and \( A_1^T(x) = -A_1(x) \), \( A_2 = \text{diag}(a_i) \), \( a_i < 0 \), \( i = 1,2,\cdots,n \), then the system is asymptotically stable.

2.2. Fractional-order complex-valued Chen system

The mathematical model of the fractional order complex valued Chen system is as follows:

\[
\begin{align*}
D^q_t x(t) &= a_1(y - x), \\
D^q_t y(t) &= (a_2 - a_1)x - xz + a_3y, \\
D^q_t z(t) &= -a_2z + 1/2(\overline{x}y - \overline{y}x),
\end{align*}
\]

where \( x_1 + jx_2, y = x_3 + jx_4 \) are complex state variables for the system (3), \( j = \sqrt{-1} \) and \( z = x_5 \), is a real state variable. The overbar \( \overline{x} (\overline{y}) \) stands for the complex conjugate of \( x(y) \), \((a_1, a_2, a_3)^T\) is the real parameter vector.

First, we rewrite the system (3) by separating the real and imaginary parts of the variable in the form of

\[
\begin{align*}
D^q_t x_1(t) &= a_1(x_3(t) - x_1(t)), \\
D^q_t x_2(t) &= a_1(x_4(t) - x_2(t)), \\
D^q_t x_3(t) &= (a_2 - a_1)x_1(t) - x_1(t)x_4(t) + a_3x_2(t), \\
D^q_t x_4(t) &= (a_2 - a_1)x_2(t) - x_2(t)x_3(t) + a_3x_4(t), \\
D^q_t x_5(t) &= -a_2x_3(t) + x_5(t)x_5(t) + x_3(t)x_4(t),
\end{align*}
\]
where $0 < q < 1$, when $q = 0.96, a_1 = 35, a_2 = 3, a_3 = 28, a_4 - a_1 = -7$, the initial states of the system (4) are chosen to be $x_1(0) = 7, x_2(0) = 8, x_3(0) = 4, x_4(0) = 6, x_5(0) = 11$, the chaotic attractor of fractional order complex valued Chen system are shown in Figure 1. From this it can be seen that the system is chaotic.

![Figure 1. Chaotic attractor in phase space.](image)

3. The control of fractional-order complex-valued Chen system

In this section, by using feedback control method, we will establish the sufficient conditions for the stability of fractional order complex valued Chen system. Moreover, the asymptotic stability conditions are also derived.

**Theorem 1:** If the parameters $k_1, k_2, k_3, k_4, k_5$ satisfy: $k_1 \geq -21, k_2 \geq -21, k_3 \geq 42, k_4 \geq 42, k_5 \geq -3$, then the following Chen system (5) is asymptotically stable.

\[
\begin{align*}
D_q^\alpha x_1(t) &= a_1(x_1(t) - x_2(t)) - k_1 x_1(t), \\
D_q^\alpha x_2(t) &= a_2(x_3(t) - x_2(t)) - k_2 x_2(t), \\
D_q^\alpha x_3(t) &= (a_3 - a_4) x_1(t) - x_4(t) x_5(t) + a_4 x_1(t) - k_3 x_3(t), \\
D_q^\alpha x_4(t) &= (a_1 - a_5) x_1(t) - x_2(t) x_3(t) + a_5 x_1(t) - k_4 x_4(t), \\
D_q^\alpha x_5(t) &= -a_2 x_3(t) + x_1(t) x_3(t) + x_2(t) x_4(t) - k_5 x_5(t).
\end{align*}
\]

**Proof:** According to Lemma 1, take the matrix $P$ as the unit matrix, and construct a function $f(x(t))$:

\[
f(x(t)) = x_1(t) D_q^\alpha x_1(t) + x_2(t) D_q^\alpha x_2(t) + x_3(t) D_q^\alpha x_3(t) + x_4(t) D_q^\alpha x_4(t) + x_5(t) D_q^\alpha x_5(t)
\]

\[
= a_1 x_1(t) x_1(t) - a_2 x_1(t) x_2(t) - a_3 x_1(t) x_4(t) - a_4 x_1(t) x_5(t) - k_1 x_1^2(t) - k_2 x_2^2(t) - k_3 x_3^2(t) - k_4 x_4^2(t) - k_5 x_5^2(t)
\]

\[
+ a_5 x_1(t) x_1(t) + a_6 x_1(t) x_2(t) + a_7 x_1(t) x_3(t) + a_8 x_1(t) x_4(t) + a_9 x_1(t) x_5(t)
\]

\[
-x_1(t) x_1(t) x_5(t) + (a_1 - a_5) x_1(t) x_5(t) - k_4 x_4^2(t) + x_1(t) x_3(t) x_5(t)
\]

\[
+ x_1(t) x_4(t) x_5(t)
\]

\[
= -(a_1 + k_1) x_1^2(t) - (a_2 + k_2) x_2^2(t) - (a_3 + k_3) x_3^2(t) - (a_4 + k_4) x_4^2(t) - (a_5 + k_5) x_5^2(t)
\]

\[
- (a_1 - a_5) x_1(t) x_5(t) + x_1(t) x_3(t) x_5(t)
\]

\[
\leq -(k_1 + \frac{2a_1 - a_5}{2}) x_1^2(t) - (k_2 + \frac{2a_2 - a_2}{2}) x_2^2(t) - (k_3 + \frac{3a_3}{2}) x_3^2(t) - (k_4 + \frac{3a_4}{2}) x_4^2(t) - (k_5 + \frac{3a_5}{2}) x_5^2(t)
\]

\[
- (k_1 + a_2)x_2^2(t),
\]
when $a_1 = 35, a_2 = 3, a_3 = 28,$ and $k_1 \geq -21, k_2 \geq -21, k_3 \geq 42, k_4 \geq 42, k_5 \geq -3$; $f(x(t)) \leq 0$, the system (5) is stable by Lemma 1.

4. Synchronization of fractional order complex valued Chen system

Based on the synchronization stability theorem of fractional-order nonlinear systems, a controller is designed to realize the chaotic synchronization of fractional order complex Chen system.

We design the controller $n_i(t)$ ($i = 1,2,3,4,5$), the response system is given as the follows:

\[
\begin{align*}
D^\alpha y_1(t) &= a_1(y_2(t) - y_1(t)) + n_1(t), \\
D^\alpha y_2(t) &= a_1(y_3(t) - y_2(t)) + n_2(t), \\
D^\alpha y_3(t) &= (a_3 - a_1)y_1(t) - y_2(t)y_2(t) + a_2y_3(t) + n_3(t), \\
D^\alpha y_4(t) &= (a_3 - a_1)y_1(t) - y_2(t)y_2(t) + a_2y_3(t) + n_4(t), \\
D^\alpha y_5(t) &= -a_2y_3(t) + y_1(t)y_2(t) + y_2(t)y_3(t) + n_5(t).
\end{align*}
\]  

(6)

Let $D^\alpha e_i(t) = D^\alpha y_i(t) - D^\alpha x_i(t)$, then the error system is

\[
\begin{align*}
D^\alpha e_1(t) &= a_1(e_1(t) - e_1(t)) + n_1(t), \\
D^\alpha e_2(t) &= a_1(e_1(t) - e_2(t)) + n_2(t), \\
D^\alpha e_3(t) &= (a_3 - a_1)e_1(t) - x_1(t)e_3(t) + a_2e_3(t) + n_3(t), \\
D^\alpha e_4(t) &= (a_3 - a_1)e_1(t) - x_1(t)e_3(t) + a_2e_3(t) + n_4(t), \\
D^\alpha e_5(t) &= y_1(t)e_1(t) + y_2(t)e_2(t) + x_1(t)e_3(t) + x_2(t)e_4(t) - a_2e_3(t) + n_5(t).
\end{align*}
\]  

(7)

**Theorem 2:** For the error system (7), if the designed controllers are

\[
\begin{align*}
n_1(t) &= (y_1(t) - a_1)e_3(t) + y_2(t)e_3(t) = (y_1(t) - 28)e_3(t) - y_1(t)e_1(t), \\
n_2(t) &= (y_4(t) - a_1)e_4(t) + y_5(t)e_4(t) = (y_4(t) - 28)e_4(t) - y_4(t)e_4(t), \\
n_3(t) &= -(a_3 + 3)e_4(t) = -31e_4(t), \\
n_4(t) &= -(a_3 + 3)e_4(t) = -31e_4(t), \\
n_5(t) &= 0.
\end{align*}
\]

then the error system (7) is asymptotically stable.

**Proof:** Rewrite the error system in the form:

\[
\begin{align*}
D^\alpha e_1(t) &= -a_1 & 0 & a_1 & 0 & 0 & e_1(t) & n_1(t) \\
D^\alpha e_2(t) &= 0 & -a_1 & 0 & a_1 & 0 & e_2(t) & n_2(t) \\
D^\alpha e_3(t) &= a_3 - a_1 - y_5(t) & 0 & a_3 & 0 & -x_1(t) & e_3(t) & n_3(t) \\
D^\alpha e_4(t) &= 0 & a_3 - a_1 - y_5(t) & 0 & a_3 & -x_2(t) & e_4(t) & n_4(t) \\
D^\alpha e_5(t) &= y_1(t) & y_2(t) & x_1(t) & x_2(t) & -a_2 & e_5(t) & n_5(t)
\end{align*}
\]  

(8)

According to Lemma 2, the designed error system becomes:

\[
\begin{align*}
D^\alpha e_1(t) &= -a_1 & y_1(t) - a_2 & a_1 & 0 & -y_1(t) & e_1(t) \\
D^\alpha e_2(t) &= 0 & -a_1 & y_1(t) - a_2 & a_1 & 0 & -y_1(t) & e_2(t) \\
D^\alpha e_3(t) &= a_3 - a_1 - y_5(t) & 0 & -3 & 0 & -x_1(t) & e_3(t) \\
D^\alpha e_4(t) &= 0 & a_3 - a_1 - y_5(t) & 0 & -3 & 0 & -x_2(t) & e_4(t) \\
D^\alpha e_5(t) &= y_1(t) & y_2(t) & x_1(t) & x_2(t) & -a_2 & e_5(t) \\
&= a_3 - a_1 & y_1(t) & -a_2 & 0 & y_1(t) & e_1(t) \\
&= a_3 - a_1 - a_1 & 0 & 0 & y_1(t) & -a_2 & e_1(t) \\
&= a_3 - a_1 - y_5(t) & 0 & 0 & 0 & y_1(t) & e_1(t) \\
&= a_3 - a_1 - y_5(t) & 0 & 0 & 0 & -x_1(t) & e_2(t) \\
&= a_3 - a_1 - y_5(t) & 0 & 0 & 0 & -x_2(t) & e_2(t) \\
&= y_1(t) & y_2(t) & x_1(t) & x_2(t) & 0 & e_5(t)
\end{align*}
\]
5. Numerical examples

For section 3, we present a numerical example to demonstrate the effectiveness of the theoretical result. Where $a_1 = 35$, $a_2 = 3$, $a_3 = 28$, the unknown parameters and initial states of the system are chosen to be $k_1 = 10$, $k_2 = 10$, $k_3 = 45$, $k_4 = 45$, $k_5 = 1$, $x_1(0) = 1$, $x_2(0) = 1$, $x_3(0) = 2$, $x_4(0) = 2$, $x_5(0) = 3$. The simulation results of state variables are shown in figure 2. As expected, all the state variables converge to zero and system (5) asymptotic stability and the designed controller is valid.

For section 4, in numerical simulations, $a_1 = 35$, $a_2 = 3$, $a_3 = 28$, the initial states of the drive system and response system are $x_1(0) = (1,1,2,2,3)$, $y_1(0) = (-1,-1,-4,-2,6)$, the simulation results of state variables are shown in figure 3. As expected, the synchronizing error system (7) is stable to zero and the designed controller is valid.
6. Conclusions
In this paper, the feedback control method has been designed to realize the controlling of the system. Based on a special matrix structure, a new approach is used for designing a controller and the synchronization is realized. Numerical simulations are demonstrated the effectiveness and clarity of the approach.

Acknowledgement
This work was supported by the National Natural Science Foundation of China (61573008, 61473178), and Post-Doctoral Applied Research Projects of Qingdao (No. 20161115).

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