THE MACKEY MACHINE FOR CROSSED PRODUCTS BY REGULAR GROUPOIDS. I

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Abstract. We first describe a Rieffel induction system for groupoid crossed products. We then use this induction system to show that, given a regular groupoid $G$ and a dynamical system $(A, G, \alpha)$, every irreducible representation of $A \rtimes G$ is induced from a representation of the group crossed product $A(u) \rtimes S_u$ where $u \in G^{(0)}$, $A(u)$ is a fibre of $A$, and $S_u$ is a stabilizer subgroup of $G$.

Introduction

The term “Mackey Machine” generally refers to a program for extracting information about the representation theory of a group, group action, or dynamical system using the representation theory of certain subsystems as well as some sort of process for inducing representations. The goal of such a program is to, under suitable hypotheses, identify the topology on the primitive ideal space, or spectrum, of an associated universal $C^*$-algebra (e.g. the group $C^*$-algebra). These notions have been applied quite successfully in many different contexts. There is the original work done by Mackey [9, 8] and then later generalized by Fell and Doran [3], as well as Takesaki’s research [18] on group crossed products. In terms of methodology, however, our approach will be closer to that of Rieffel, who recast Mackey’s theory using Hilbert modules and Morita equivalence [16, 17]. This technique was then applied to crossed products by Green [5] and Echterhoff [1]. More recently, work has continued on the Mackey Machine for group crossed products in [2] as well as for groupoid $C^*$-algebras in [7, 6]. In this paper, however, we will develop the Mackey Machine for groupoid crossed products.

In general, when implementing the Mackey Machine for a groupoid dynamical system $(A, G, \alpha)$ one must choose between two natural classes of subgroups. The first is the set of stabilizer subgroups for the action of the groupoid $G$ on its unit space $G^{(0)}$. The second is the set of stabilizer subgroups for the action of the groupoid $G$ on the primitive ideal space of the $C^*$-algebra $A$. This series studies the irreducible representations of $A \rtimes G$ using the former class, and in particular using the stabilizer subgroupoid $S$ of $G$. Our eventual goal will be to use induction to identify the spectrum of $A \rtimes_\alpha G$ as a quotient of the spectrum of $A \rtimes_\alpha S$. This falls under the ideology of the Mackey Machine because, as a set, $(A \rtimes_\alpha S)^\wedge$ is equal to the disjoint union $\coprod A(u) \rtimes S_u$, where the $A(u)$ are the fibres of $A$ and the $S_u$ are the stabilizer subgroups associated to the action of $G$ on its unit space. Our first step towards this goal will be to demonstrate an induction technique for groupoid crossed products. The construction of induced representations contained in this paper is very general and extends “Rieffel”-type induction for groups, groupoids, and group crossed products [16, 7, 5]. After induction, the next step in developing the Mackey

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Machine is to prove that every irreducible representation of the system is induced from a representation of a substructure. Specifically we show that every irreducible representation of $A \rtimes G$ is induced from a representation of a “stabilizer” crossed product $A(u) \rtimes S_u$ for some $u \in G^{(0)}$. This result is similar in spirit to the Gootman-Rosenberg-Sauvageot Theorem or the results of [19] since we make use of a regularity hypothesis. The structure of this paper is roughly as follows. Section 4 covers some basic crossed product theory followed by the induction results in Section 2. Section 3 contains some basic constructions which will be needed in Section 4 where we prove the surjectivity result described above.

Before we begin in earnest we should first make some remarks about our hypotheses. Because we are looking to identify the spectrum of a $C^*$-algebra we will need some sort of “Type I” or “regularity” condition. Specifically, we will need to assume that the groupoid $G$ satisfies the conditions of the Mackey-Glimm Dichotomy [14]. We will refer to such groupoids as regular groupoids. Of course, the assumption that $G$ is regular could be easily replaced by any of the fourteen equivalent conditions in the Mackey-Glimm Dichotomy, including assuming that $G^{(0)}/G$ is $T_0$ or that $G^{(0)}/G$ is almost Hausdorff. Finally, it should be noted that the results of this paper are contained, with more detail and a great deal of background material, in the author’s thesis [3].

1. Groupoid Crossed Products

Throughout the paper we will let $G$ denote a second countable, locally compact Hausdorff groupoid with a Haar system $\Lambda$. Given an element $u \in G^{(0)}$ of the unit space of $G$ we will use $S_u = \{ \gamma \in G : s(\gamma) = r(\gamma) = u \}$ to denote the stabilizer, or isotropy, subgroup of $G$ over $u$. Since groupoids act on fibred objects, we must consider the theory of $C_0(X)$-algebras and upper-semicontinuous $C^*$-bundles (usc-bundles) if we are to work with groupoid dynamical systems. It is assumed that the reader is familiar with $C_0(X)$-algebras and their associated bundles, although either [20] Appendix C or [4] Section 3.1 will serve as a reference. Throughout the rest of the paper we will let $A$ denote a separable $C_0(G^{(0)})$-algebra and will let $A$ be its associated usc-bundle. We use $\Gamma_0(G^{(0)}, A)$ to denote the space of sections of $A$ which vanish at infinity and identify this space with $A$ in the usual fashion.

Remark 1.1. Because it will be fundamental to what follows, let us recall the “usual” method of identifying a $C_0(X)$-algebra $A$ with $\Gamma_0(X, A)$. Given $x \in X$ we let

$$I_x = \text{span}\{ \phi \cdot a : \phi \in C_0(X), a \in A, \phi(x) = 0 \}$$

and define $A(x) := A/I_x$. We denote the image of $a$ in $A(x)$ by $a(x)$. Then [20] Theorem C.26] implies that $A = \prod_{x \in X} A(x)$ has an usc-bundle structure and that the map sending $a$ to $x \mapsto a(x)$ is an isomorphism of $A$ onto $\Gamma_0(X, A)$.

We take this opportunity to establish some basic facts about $C_0(X)$-algebras which we shall need in the next section. Given a $C_0(X)$-algebra $A$, its associated usc-bundle $p : A \to X$ and $Y$ a locally compact Hausdorff subset of $X$, we define $A|_Y := p^{-1}(Y)$. Furthermore, we use $A(Y)$ to denote $\Gamma_0(Y, A|_Y)$, which we shall identify with $\Gamma_0(Y, A)$. Next, if $\tau : Y \to X$ is a continuous map then we let $\tau^*A$ denote the pullback of $A$ by $\tau$. In other words $\tau^*A = \{(y, a) \in Y \times A : \tau(y) = p(a)\}$. In a slight break from the standard notation, we will use $\tau^*A$ to denote the
we can view recall that there is an induced action of $A$, $G$ be the corresponding transformation groupoid.

Given a locally compact Hausdorff group $H$ by the range map, given the convolution and involution operations $c$ of compactly supported sections $\Gamma_f$ is a simple matter to cut down the neighborhood of $\text{supp } A$, $G$, $\alpha$ dynamical system ($\gamma$ and that $K$ limit topology if it converges to $f$.

One of the important facts about pull backs is the following

**Proposition 1.3.** Suppose $X$ and $Y$ are locally compact Hausdorff spaces, $A$ is a $C_0(X)$-algebra, and $\tau : Y \to X$ is a continuous map. Given $f \in C_c(Y)$ and $a \in A$ define $f \otimes a(y) = f(y)a(\tau(y))$ for all $y \in Y$. Then $f \otimes a \in \Gamma_c(Y, \tau^* A)$ and

$$C_c(Y) \circ A := \text{span}\{ f \otimes a : f \in C_c(Y), a \in A \}$$

is dense in $\tau^* A$. What’s more, $C_c(Y) \circ A$ is dense in $\Gamma_c(Y, \tau^* A)$ with respect to the inductive limit topology.

**Proof.** The statements about $f \otimes a$ are trivial, and it follows in a straightforward manner from [20 Proposition C.24] that $C_c(Y) \circ A$ is dense in $\tau^* A$. Given $f \in \Gamma_c(Y, \tau^* A)$ recall that a sequence $f_i$ converges to $f$ with respect to the inductive limit topology if it converges to $f$ uniformly and the $\text{supp } f_i$ are all eventually contained in some compact set $K$. We know that there exists $f_i \in C_c(Y) \circ A$ such that $f_i \to f$ uniformly. Since $C_c(Y) \circ A$ is closed under the action of $C_0(Y)$, it is a simple matter to cut down the $f_i$ so that they are supported on a compact neighborhood of $\text{supp } f$.

Now, given $A$ and $G$ as above we let $\alpha$ denote an action of $G$ on $A$ as defined in [11 Definition 4.1]. Recall that this means $\alpha$ is given by a collection of isomorphisms $\{\alpha_\gamma\}_{\gamma \in G}$ such that $\alpha_\gamma : A(s(\gamma)) \to A(r(\gamma))$, $\alpha_\gamma \circ \alpha_\eta = \alpha_{\gamma \eta}$ whenever $s(\gamma) = r(\eta)$, and that $\gamma \cdot a \mapsto \alpha_\gamma(a)$ is a continuous action of $G$ on $A$. Given a groupoid dynamical system $(A, G, \alpha)$ consisting of $A$, $G$, and $\alpha$ as above, we construct the groupoid crossed product $A \rtimes_\alpha G$ as in [11 Section 4] or [4 Chapter 3]. Without going into details we will mention that $A \rtimes G$ is a universal completion of the algebra of compactly supported sections $\Gamma_c(G, \tau^* A)$, where $\tau^* A$ denotes the pull back of $A$ by the range map, given the convolution and involution operations

$$f * g(\gamma) = \int_G f(\eta) \alpha_\gamma(g(\eta^{-1}\gamma)) d\lambda(\gamma)(\eta) \quad \text{and} \quad f^*(\gamma) = \alpha_\gamma(g(\gamma^{-1})^*)$$

What is, perhaps, more motivating than the details of the construction of the crossed product is the following example.

**Example 1.4.** Suppose $A$ is a $C^*$-algebra with Hausdorff spectrum $\hat{A}$ and recall that we can view $A$ as a $C_0(\hat{A})$-algebra with fibres $A / \ker \pi$. Let $\mathcal{A}$ be the corresponding usc-bundle. Given a locally compact Hausdorff group $H$ and an action $\alpha$ of $H$ on $A$ recall that there is an induced action of $H$ on $\hat{A}$ given by $s \cdot \pi = \pi \circ \alpha_s^{-1}$. Let $G$ be the corresponding transformation groupoid $G = H \times \hat{A}$. Now, given $(s, \pi) \in G$ define $\beta_{(s, \pi)} : A(s^{-1} \cdot \pi) \to A(\pi)$ by

$$\beta_{(s, \pi)}(a(s^{-1} \cdot \pi)) = \alpha_s(a)(\pi).$$
A straightforward computation shows that $\beta_{(s,\pi)}$ is a well defined isomorphism. It is similarly easy to prove that $\beta_{(s,\pi)(t,\rho)} = \beta_{(s,\pi)} \circ \beta_{(t,\rho)}$ whenever $s^{-1} \cdot \pi = \rho$. The last thing we need to do to show that $\beta$ is a groupoid action is to prove it is continuous. This is, again, straightforward as long as you apply [20, Proposition C.20]. Since $(A,G,\beta)$ is a groupoid dynamical system, we can form the crossed product $A \rtimes_\beta G$. We claim that this is naturally isomorphic to the group crossed product $A \rtimes_\alpha H$. First, recall that $A \rtimes_\alpha H$ is a universal completion of the function algebra $C_c(H, A)$. Now, given $f \in \Gamma_c(G, r^*A)$ define $\Phi(f)(s)(\pi) = f(s, \pi)$ for $s \in H$ and $\pi \in A$. It follows quickly that $\Phi(f)(s)$ defines an element of $A$ and that $\Phi(f)$ is a continuous compactly supported function on $H$. Thus $\Phi : \Gamma_c(G, r^*A) \to C_c(H, A)$ and simple calculations show that $\Phi$ is a *-homomorphism which is continuous with respect to the inductive limit topology. It is now an immediate result of Renault’s Disintegration Theorem [11, Theorem 7.8,7.12] that $\Phi$ extends to a homomorphism from $A \rtimes G$ into $A \rtimes H$. The goal is to show that this extension is an isomorphism.

Let $D = \Gamma_c(\hat{A}, A)$ and observe that $D$ is dense in $A$. Consider the set of sums of elementary tensors $C_c(H) \otimes D$ and recall [20, Lemma 1.87] that this set is dense in $C_c(H, A)$ with respect to the inductive limit topology. Given $\phi \otimes a \in C_c(H) \otimes D$ define $f(s, \pi) = \phi(s)(a(\pi))$ and observe that $f \in \Gamma_c(G, r^*A)$. Clearly $\Phi(f) = \phi \otimes a$ so that $C_c(H) \otimes D \subset \text{ran } \Phi$. Hence $\text{ran } \Phi$ is dense with respect to the inductive limit topology and therefore with respect to the $I$-norm. In order to show that $\Phi$ is an isomorphism we exhibit the following inverse. First, observe that $\Phi$ is bijective on $\Gamma_c(G, r^*A)$ so that we can define an inverse map $\Psi = \Phi^{-1} : \text{ran } \Phi \to \Gamma_c(G, r^*A)$ by $\Psi(f)(s, \pi) = \Phi(f(s))(\pi)$. It is easy to show that $\Psi$ is a *-homomorphism. Straightforward calculations then show that $\Psi$ is $I$-norm decreasing. Since $\text{ran } \Phi$ is dense in $C_c(H, A)$ with respect to the $I$-norm, we can extend $\Psi$ to $C_c(H, A)$. This extension is still an $I$-norm decreasing *-homomorphism on $C_c(H, A)$ and it follows from [20, Corollary 2.47] that $\Psi$ extends to all of $A \rtimes_\alpha H$. Finally, since $\Psi$ and $\Phi$ are inverses on dense subsets, it follows that they are inverses everywhere and that $A \rtimes H$ is isomorphic to $A \rtimes G$.

The reason we went through the trouble of describing this example in detail is that it is useful to think of groupoid crossed products as generalizing group crossed products in this manner. For instance, in the case where $A$ has Hausdorff spectrum the results of Section 4 can be seen as generalizing corresponding results for group crossed products when the generalization is done via Example 1.4. On the other hand, when we view groups as groupoids with a single unit the results of Section 4 become trivial.

2. Induction

The goal for this section is to show that if we are given a groupoid dynamical system $(A,G,\alpha)$ and a closed subgroupoid $H$ of $G$ with its own Haar system then there is an induction process which takes representations of $A \rtimes_\alpha H$ and creates representations of $A \rtimes_\alpha G$. This induction process has its roots in the Rieffel correspondence and it is assumed that the reader is familiar with representations induced via imprimitivity bimodules as described in [13, Chapters 2,3]. We will apply this theory to groupoid crossed products by constructing a right Hilbert $A \rtimes_\alpha H$-module $Z_H^G$ and proving the following theorem.
Theorem 2.1. Suppose \((A,G,\alpha)\) is a groupoid dynamical system and that \(H\) is a closed subgroupoid of \(G\) with a Haar system. Then given a representation \(R\) of \(A([H(0)] \rtimes_{\alpha|_H} H)\) on \(H\) we may form the induced representation \(\text{Ind}_H^G R\) of \(A \rtimes H\) on \(Z_H^G \otimes_{A \rtimes H} \mathcal{H}\) which is defined for \(f \in \Gamma_c(G, r^* A)\), \(z \in Z_0\) and \(h \in \mathcal{H}\) by
\[
\text{Ind}_H^G R(f)(z \otimes h) = f \cdot z \otimes h
\]
where
\[
f \cdot z(\gamma) = \int_G \alpha_{\gamma}^{-1}(f(\eta)) z(\eta^{-1}\gamma) d\lambda^r(\gamma)(\eta).
\]

Before we prove this theorem we must construct the Hilbert module \(Z_H^G\), which we will do by using Renault’s Equivalence Theorem to build an imprimitivity bimodule. First, however, we need to construct a new action from \(\alpha\) and to do this we must back up even further and discuss groupoid equivalence and imprimitivity groupoids. We assume that the reader is familiar with these notions, although \([10]\) and \([13]\) will serve as references.

Definition 2.2. Suppose \(G\) is a locally compact Hausdorff groupoid and \(H\) is a closed subgroupoid with a Haar system. Let \(X = s^{-1}(H(0))\). We define the imprimitivity groupoid \(G^H\) to be the quotient of \(X \times X\) by \(H\) where \(H\) acts diagonally via right translation. We give \(G^H\) a groupoid structure with the operations
\[
[\gamma, \eta][\eta, \zeta] = [\gamma, \zeta], \quad \text{and} \quad [\gamma, \eta]^{-1} = [\eta, \gamma].
\]

The key facts we need concerning these objects are contained in the following

Proposition 2.3. Suppose \(G\) is a locally compact Hausdorff groupoid and \(H\) is a closed subgroupoid with a Haar system. Then \(G^H\) is a locally compact Hausdorff groupoid with unit space \(X/H\) and Haar system \(\{\mu^H\}\) defined for \(f \in C_c(G^H)\) by
\[
\int_{G^H} f([\xi, \eta]) d\mu^H([\xi, \eta]) = \int_G f([\xi, \eta]) d\lambda s(\xi)(\eta).
\]
Furthermore, \(X\) is a \((G^H, H)\)-equivalence where \(H\) acts on \(X\) by right translation and \(G^H\) acts via the operation \([\gamma, \eta] \cdot \eta = \gamma\).

Proof. This material is all known. The construction of the imprimitivity groupoid and the results concerning the \((G^H, H)\)-equivalence can be found in \([10]\) Section 2. The fact that \(\mu\) is a Haar system for \(G^H\) can be found in \([7]\) Section 2. It is worth observing that in order for \(\mu\) to exist all we really require is that \(H\) have open range and source.

Once we have our imprimitivity groupoid we can couple it with the action of \(G\) on \(A\) to form a new dynamical system.

Proposition 2.4. Let \((A,G,\alpha)\) be a groupoid dynamical system and suppose \(H\) is a closed subgroupoid of \(G\) with a Haar system. Let \(X = s^{-1}(H(0))\), \(G^H\) be the imprimitivity groupoid, define \(\rho : X/H \to G(0)\) by \(\rho([\gamma]) = r(\gamma)\), and let \(\rho^* A\) be the pull back of \(A\). Then the collection \(\sigma = \{\sigma_{[\gamma, \eta]}\}_{[\gamma, \eta] \in G^H}\) where \(\sigma_{[\gamma, \eta]} : A(r(\eta)) \to A(r(\gamma))\) is given for \(a \in A(r(\eta))\) by
\[
\sigma_{[\gamma, \eta]}(a) = \alpha_{\gamma^{-1}}(a)
\]
defines an action of \(G^H\) on \(\rho^* A\).
Proof. First, recall that we identify the unit space of $G^H$ with $X/H$ so that we may view the pull back $\rho^*A$ as a $C_0((G^H)^{(0)}_0)$-algebra. Furthermore, given $[\gamma] \in X/H$ the fibre of $\rho^*A$ over $[\gamma]$ is $\rho^*A([\gamma]) = A(\rho([\gamma])) = A(r(\gamma))$. Thus $\sigma_{[\gamma, \eta]}$ can be viewed as an isomorphism of $\rho^*A([\eta])$ onto $\rho^*A([\gamma])$. It is now straightforward to see that $\sigma$ is well defined, respects the groupoid operations, and is continuous. □

Remark 2.5. Since $(\rho^*A, G^H, \sigma)$ is a groupoid dynamical system, we may form the crossed product $\rho^*A \rtimes_{\sigma} G^H$ as the completion of $\Gamma_c(G^H, r^*\rho^*A)$. Elements of this function algebra can be viewed as continuous, compactly supported maps from $G^H$ into $A$ such that $f([\gamma, \eta]) \in A(r(\gamma))$ for all $[\gamma, \eta] \in G^H$.

We can now use $\sigma$ to build the promised imprimitivity bimodule.

Proposition 2.6. Suppose $(A, G, \alpha)$ is a groupoid dynamical system. Furthermore, suppose $H$ is a closed subgroupoid with Haar system $\lambda_H$. Let $X$, $G^H$, $\rho$ and $\sigma$ be as in Proposition 2.2. Then $Z_0 = \Gamma_c(X, s^*A)$ becomes a $\rho^*A \rtimes_{\alpha|_H} H$-imprimitivity bimodule with respect to the following actions for $f \in \Gamma_c(G^H, r^*\rho^*A)$, $g \in \Gamma_c(H, r^*A)$, and $z, w \in Z_0$:

$$f \cdot z(\gamma) = \int_G a_\gamma^{-1}(f([\gamma, \eta]))z(\eta)d\lambda_{\mathrm{a}}(\eta)$$

$$z \cdot g(\gamma) = \int_H a_\eta(z(\eta\gamma)g(\eta^{-1}))d\lambda_H^\gamma(\eta)$$

$$\langle z, w \rangle_{A \rtimes H}(\gamma) = \int_G z(\eta^{-1}a_\gamma)(w(\eta))d\lambda_{\mathrm{a}}(\eta)$$

The completion of $Z_0$, denoted $Z^G_0$, is a $\rho^*A \rtimes_{\alpha|_H} G^H$-imprimitivity bimodule and $\rho^*A \rtimes_{\alpha} G^H$ is Morita equivalent to $A(H^0) \rtimes H$.

Proof. We will make use of the machinery of equivalence bundles from [11, 15]. In particular we are going to build an equivalence bundle [11, Definition 5.1] and then use Renault’s Equivalence Theorem [11, Theorem 5.5]. First, recall from Proposition 2.2 that $X$ is a $(G^H, H)$-equivalence. The source map for $X$, $s_X : X \to H^0$, is just the restriction of the source map on $G$ to $X$, and the range map, $r_X : X \to X/H$, is the quotient map. Our equivalence bundle is then the pull back $\Upsilon = s_X^*A$. This is an usc-bundle over $X$ and we can identify the fibre over $\gamma \in X$ with $A(s(\gamma))$. This allows us to give $\Upsilon$, an $A(r(\gamma)) - A(s(\gamma))$-imprimitivity bimodule structure via the isomorphism $\alpha_\gamma$ in the usual fashion [13, Example 3.14]. Next, we define actions of $G^H$ and $H$ on $\Upsilon$ for $[\eta, \zeta] \in G^H$, $\xi \in H$ and $(\gamma, a) \in \Upsilon$ by

$$(\gamma, a) \cdot \xi := (\gamma\xi, \alpha_\xi^{-1}(a)), \quad \text{and} \quad [\eta, \zeta] \cdot (\gamma, a) := (\eta\gamma^{-1}\zeta, a).$$

It is then straightforward to show that $\Upsilon$ satisfies the definition of an equivalence bundle and it follows from Renault’s Equivalence Theorem that $Z_0 = \Gamma_c(X, s^*A)$ is a $\rho^*A \rtimes_{\alpha|_H} G^H - A(H^0) \rtimes_H H$-imprimitivity bimodule. It is then a matter of calculating that the operations on $Z_0$ all have the correct form and the rest of the proposition follows. □

Now that we have built the imprimitivity bimodule $Z^G_0$ mentioned in Theorem 2.1 we need to let $A \rtimes H$ act as adjointable operators. However, it follows from the
general theory of imprimitivity bimodules that this is equivalent to letting $A \rtimes_\alpha G$ map into the multipliers on $\rho^*A \rtimes_\sigma G^H$. Theorem 2.1 will then follow immediately from the following

**Proposition 2.7.** Let $(A, G, \alpha)$ be a groupoid dynamical system, $H$ a closed subgroupoid of $G$ with a Haar system and $Z^G_H$ the associated imprimitivity bimodule completed from $Z_0$ as in Proposition 2.6. Then there is a nondegenerate homomorphism $\phi : A \rtimes_\alpha G \to \mathcal{L}(Z^G_H)$ such that for $f \in \Gamma_c(G, r^*A)$ and $z \in Z_0$

\[(4) \quad \phi(f)z(\gamma) = \int_G \alpha_\gamma^{-1}(f(\eta))z(\eta^{-1}\gamma)d\lambda^r(\eta). \]

**Proof.** We start by letting $A \rtimes G$ act as multipliers on $\rho^*A \rtimes_\sigma G^H$. Given $f \in \Gamma_c(G, r^*A)$ and $g \in \Gamma_c(G^H, r^*\rho^*A)$ define

\[(5) \quad M_f(g)([\gamma, \eta]) = \int_G f(\xi)\alpha_\xi(g([\xi^{-1}\gamma, \eta]))d\lambda^r(\xi). \]

The arguments that $M_fg \in \Gamma_c(G^H, r^*\rho^*A)$ and that $(f, g) \mapsto M_fg$ is jointly continuous in the inductive limit topology are similar to the corresponding arguments for convolution and are omitted. In order to show that $M_f$ defines a multiplier we need to show that it extends to an adjointable linear operator when we view $\rho^*A \rtimes_\sigma G^H$ as a right Hilbert module over itself in the usual fashion. It is tedious, yet straightforward, to show that $M$ preserves the following relations for $f, g \in \Gamma_c(G, r^*A)$ and $h, k \in \Gamma_c(G^H, r^*\rho^*A)$

\[M_f(h \ast k) = M_fh \ast M_fk, \quad (M_fh)^* \ast k = h^* \ast (M_f^*k), \quad M_{fg}h = M_fM_gh. \]

This shows that, at least on the appropriate dense function algebras, $M_f$ is an adjointable linear operator and that $M$ is a $*$-homomorphism on $\Gamma_c(G, r^*A)$. Next, we need to show that $M$ satisfies a nondegeneracy condition. Because this portion of the proof is lengthy and somewhat unenlightening we separate it out into the following lemma, which we will prove at the end of the section.

**Lemma 2.8.** Functions of the form $M_fg$ are dense in $\Gamma_c(G^H, r^*\rho^*A)$ with respect to the inductive limit topology.

With this lemma can show that $M_f$ extends to an adjointable operator on $\rho^*A \rtimes_\sigma G^H$ and that $||M_f|| \leq ||f||$ so that as a $*$-homomorphism $M$ extends to $A \rtimes_\alpha G$. Suppose $\tau$ is a state on $\rho^*A \rtimes_\sigma G^H$ and define a pre-inner product on $\rho^*A \rtimes_\sigma G^H$ via $(g, h)_\tau := \tau(g^* \ast h)$. Let $\mathcal{H}_\tau$ denote the resulting Hilbert space and $\mathcal{H}_0$ the dense image of $\Gamma_c(G^H, r^*\rho^*A)$ in $\mathcal{H}_\tau$. Given $f \in \Gamma_c(G, r^*A)$ we define an operator $\pi(f)$ for $g \in \mathcal{H}_0$ by $\pi(f)g = M_fg$. It is a simple matter to show that $\pi(f)$ is well defined and that $\pi$ defines a homomorphism from $\Gamma_c(G, r^*A)$ into the algebra of linear operators on $\mathcal{H}_0$. We will now show that we can apply Renault’s Disintegration Theorem. Since elements of the form $M_fg$ are dense in the inductive limit topology, it is clear that elements of the form $\pi(f)h$ are dense in $\mathcal{H}_\tau$. Furthermore, we have asserted that $M_fg$ is jointly continuous with respect to the inductive limit topology and it follows that $f \mapsto (\pi(f)g, h)_\tau$ is continuous with respect to the inductive limit topology. Finally, it is easy to show, using the fact that $M_f$ is adjointable on $\Gamma_c(G^H, r^*\rho^*A)$ with adjoint $M_{f^*}$ that $(\pi(f)g, h)_\tau = (g, \pi(f^*)h)_\tau$. Thus we may apply the Disintegration Theorem and conclude that $\pi$ extends to a
representation of $A \times G$ on $\mathcal{H}_+$. In particular, this implies that for $f \in \Gamma_c(G, r^*A)$ and $g \in \Gamma_c(G^H, r^*\rho^*A)$ we have
\[
\tau((Mf)g^* \ast (Mfg)) = \|\tau(f)g\|_2^2 \leq \|f\|_2^2\|g\|_2^2 = \|f\|_2^2\tau(g^* \ast g) \leq \|f\|_2^2\|g\|_2^2.
\]
Since $\tau$ is an arbitrary state, we must have $\|Mfg\| \leq \|f\|_2\|g\|_2$. It now follows quickly that $Mf$ extends to a multiplier on $\rho^*A \times_\sigma G^H$ and in turn that $M$ extends to a $*$-homomorphism from $A \times_\sigma G$ into $M(\rho^*A \times_\sigma G^H)$.

At this point we are essentially done. We cite [13 Proposition 3.8] which states that $\rho^*A \times_\sigma G^H \cong \mathcal{K}(Z^H)$ and then extend this isomorphism to $M(\rho^*A \times_\sigma G^H)$.

Furthermore, calculating that $\phi$ is given by (4) on elements of the form $g \cdot z$ where $g \in \Gamma_c(G^H, r^*\rho^*A)$ and $z \in Z^H$ is straightforward. The fact that (4) holds in general now follows quickly using [11 Proposition 6.8] and an inductive limit topology argument. □

Theorem 2.1 is now an immediate consequence of Proposition 2.8 and [13 Proposition 2.66]. We finish the section with the promised nondegeneracy proof.

Proof of Lemma 2.8 Fix $g \in \Gamma_c(G^H, r^*\rho^*A)$ and let $L = supp g$. Let $\{a_i\}_{i \in \Lambda}$ be an approximate identity for $A$ and use [11 Proposition 6.8] to choose, for each 4-tuple $(K, U, l, \epsilon)$ consisting of a compact set $K \subset G^{(0)}$, a conditionally compact neighborhood $U$ of $G^{(0)}$ in $G$, $l \in \Lambda$, and $\epsilon > 0$, a section $e = e(K, U, l, \epsilon) \in \Gamma_c(G, r^*A)$ such that
\begin{enumerate}[(a)]
  \item $\text{supp } e \subset U$,
  \item $\int_G \|e(\gamma)\| d\lambda^u(\gamma) \leq 4$ for all $u \in K$, and
  \item $\|\int_G e(\gamma) d\lambda^u(\gamma) - a_i(u)\| < \epsilon$ for all $u \in K$.
\end{enumerate}
We will show that, when $\kappa = (K, U, l, \epsilon)$ is ordered by increasing $K$ and $l$ and decreasing $U$ and $\epsilon$, we have $M_\kappa g \to g$ with respect to the inductive limit topology.

Fix $\epsilon_1 > 0$ and let $K_1 = \rho(r(L))$. Next, we prove that there exists a conditionally compact neighborhood $U_1$ such that $\xi \in U_1$ implies
\[
\|a_\xi (g(\xi^{-1}\gamma, \eta)) - g(\gamma, \eta)\| < \epsilon_1
\]
for all $[\gamma, \eta] \in G^H$ such that $r(\gamma) = r(\xi)$. Suppose not and fix some conditionally compact neighborhood $W$ of $G^{(0)}$. Then for any conditionally compact neighborhood $G^{(0)} \subset U \subset W$ there exists $\xi_U \in U$ and $[\gamma_U, \eta_U] \in G^H$ such that
\[
\|a_{\xi_U} (g(\xi_U^{-1}\gamma_U, \eta_U)) - g(\gamma_U, \eta_U)\| \geq \epsilon_1.
\]
However, for this to hold one of the terms must be nonzero so that we must have
\[
[\gamma_U, \eta_U] \in \mathcal{L} = \{[\xi, \eta] : \xi \in W, [\gamma, \eta] \in L \text{ and } s(\xi) = r(\gamma)\}.
\]
It is not particularly difficult to show that $\mathcal{L}$ is compact. Thus, ordering $\{[\gamma_U, \eta_U]\}$ by decreasing $U$, we may pass to a subnet, relabel, and find new representations such that there exists $\gamma, \eta \in G$ with $\gamma_U \to \gamma$ and $\eta_U \to \eta$. Next, observe that $r(\xi_U) \in \rho(r(\mathcal{L}))$ for all $U$ so that $\{\xi_U\}$ is contained in the compact set $W \cap r^{-1}(\rho(r(\mathcal{L})))$. Thus we may pass to another subnet, relabel, and find $\xi \in G$ such that $\xi_U \to \xi$. However, by construction, $\xi$ is contained in every conditionally compact neighborhood of $G^{(0)}$. It follows that $\xi \in G^{(0)}$ and therefore $a_{\xi_U} (g(\xi_U^{-1}\gamma_U, \eta_U)) \to g([\gamma, \eta])$. This contradicts (7) which implies that (10) must hold.
In order to complete our 4-tuple we will show there exists \( l_1 \in \Lambda \) such that \( l \geq l_1 \) implies

\[
\|a_l(r(\gamma))g(\gamma, \eta) - g(\gamma, \eta)\| < \epsilon_1.
\]

It clearly suffices to verify (8) on \( L \). Since \( a_l \) factors to an approximate identity on each fibre we have \( a_l(r(\gamma))g(\gamma, \eta) \to g(\gamma, \eta) \) for each \( \gamma, \eta \in L \). Use the fact that the norm is upper-semicontinuous to choose for each \( \gamma, \eta \in L \) some neighborhood \( O_{\gamma, \eta} \) of \( \gamma, \eta \) and some \( b_{\gamma, \eta} \in \{a_l\} \) such that

\[
\|b_{\gamma, \eta}(r(\xi))g(\xi, \zeta) - g(\xi, \zeta)\| < \frac{\epsilon_1}{3}
\]

for all \( [\xi, \zeta] \in O_{\gamma, \eta} \). Since \( L \) is compact, we can find some finite subcover \( \{O_i\} \). Let \( \phi_i \in C_c(G^H) \) be a partition of unity with respect to \( \{O_i\} \) so that \( \text{supp} \phi_i \subset O_i \) and \( \sum \phi_i(\gamma, \eta) = 1 \) if \( \gamma, \eta \in L \). Next define \( h \in \Gamma_c(G^H, \rho^*A) \) to be \( h = \sum_{i=1}^N \phi_i \otimes b_{\gamma, \eta,i} \). Then, by construction, for all \( [\xi, \zeta] \in L \) we have

\[
h([\xi, \zeta])g(\xi, \zeta) - g(\xi, \zeta)\| < \frac{\epsilon_1}{3}.
\]

Moving on, we can find \( l_1 \) such that if \( l \geq l_1 \) then

\[
\|a_l b_{\gamma, \eta,i} \| - b_{\gamma, \eta,i} \| < \frac{\epsilon_1}{3\|g\|_\infty}
\]

for all \( i \). It follows quickly that for all \( [\xi, \zeta] \in L \)

\[
\|a_l(r(\xi))h([\xi, \zeta]) - h(\xi, \zeta)\| < \frac{\epsilon_1}{3\|g\|_\infty}.
\]

Finally, using (10) and (11) and the fact that \( \|a_l\| < 1 \) for all \( l \), we compute for \( l \geq l_1 \) and \( [\xi, \zeta] \in L \)

\[
\|a_l(r(\xi))g(\xi, \zeta) - g(\xi, \zeta)\| \leq \|a_l(r(\xi))(g(\xi, \zeta)) - h(\xi, \zeta))g(\xi, \zeta)\|
\]

\[
+ \|\|a_l(r(\xi))h(\xi, \zeta) - h(\xi, \zeta))g(\xi, \zeta)\|
\]

\[
+ \|h(\xi, \zeta)g(\xi, \zeta) - g(\xi, \zeta)\|
\]

\[
< \|a_l(r(\xi))\| \frac{\epsilon_1}{3} + \|g\|_\infty \frac{\epsilon_1}{3\|g\|_\infty} + \frac{\epsilon_1}{3} \leq \epsilon_1.
\]

Now, suppose we are given \( \epsilon_0 > 0 \) and let \( \epsilon_1 = \epsilon_0/(5 + \|g\|_\infty) \). Choose \( U_1 \) and \( l_1 \) for \( \epsilon_1 \) as above. Then given \( \epsilon = \epsilon_l(K, U_l, l) \) with \( K_1 \subset K, U \subset U_1, l_1 \leq l \), and \( \epsilon < \epsilon_1 \) we compute for \( [\gamma, \eta] \in G^H \)

\[
\|M_\epsilon g(\gamma, \eta) - g(\gamma, \eta)\| \leq \left\| \int_G e(\xi)(a_\xi(g([\xi^{-1}, \gamma, \eta])) - g(\gamma, \eta)))d\lambda^G(\xi) \right\|
\]

\[
+ \left\| \left( \int_G e(\xi)d\lambda^G(\xi) - a_\xi(r(\gamma)) \right) g(\gamma, \eta) \right\|
\]

\[
+ \|a_\xi(r(\gamma))g(\gamma, \eta) - g(\gamma, \eta)\|
\]

\[
< \int_U \|e(\xi)\| \|a_\xi(g([\xi^{-1}, \gamma, \eta])) - g(\gamma, \eta))d\lambda^G(\xi)
\]

\[
+ \epsilon_1 \|g(\gamma, \eta)\| + \epsilon_1
\]

\[
< 4\epsilon_1 + \|g\|_\infty + \epsilon_1 = \epsilon_0.
\]

Thus \( M_\epsilon, g \to g \) uniformly and it is straightforward to show that this convergence takes place with respect to the inductive limit topology. \( \square \)
3. Basic Constructions

3.1. Transitive Groupoid Crossed Products. In this (very short) section we will use an application of Renault’s Equivalence Theorem to identify the Morita equivalence class of transitive groupoid crossed products. This will be used in Section 4 to prove the main result of the paper.

Theorem 3.1. Suppose $(A, G, \alpha)$ is a groupoid dynamical system and that $G$ is transitive. Fix $u \in G^{(0)}$ and let $\beta$ be Haar measure on $S_u$. Then $X_0 = C_c(G_u, A(u))$ becomes a pre-$A \rtimes_\alpha G - A(u) \rtimes_{\alpha|_{S_u}} S_u$-imprimitivity bimodule when equipped with the following operations.

\[
\begin{align*}
    f \cdot z(\gamma) &= \int_G \alpha_\gamma^{-1}(f(\eta))z(\eta^{-1}\gamma)d\lambda(\gamma)(\eta) \\
z \cdot g(\gamma) &= \int_{S_u} \alpha_s(z(\gamma s)g(s^{-1}))d\beta(s) \\
\langle\langle z, w\rangle\rangle_{A(u)\rtimes S_u}(s) &= \int_G z(\eta s^{-1})\alpha_s(w(\eta))d\lambda_u(\eta) \\
A_{A\rtimes G}\langle\langle z, w\rangle\rangle(\gamma) &= \int_{S_u} \alpha_{\gamma^{-1}}(z(\gamma^{-1} s)w(\gamma s))d\beta(s)
\end{align*}
\]

Hence, the completion $X$ of $X_0$ is an imprimitivity bimodule and $A \rtimes_\alpha G$ is Morita equivalent to $A(u) \rtimes_{\alpha|_{S_u}} S_u$.

Proof. Let $(A, G, \alpha)$ be as above and fix $u \in G^{(0)}$. First recall [10, Example 2.2] that because $G$ is a transitive, second countable groupoid the space $X = G_u = s^{-1}(u)$ is a $(G, S_u)$-equivalence with respect to the natural actions of $G$ and $S_u$ on $X$. Consider the trivial bundle $\mathcal{Y} = X \times A(u)$. This is clearly an usc-bundle whose space of compactly supported sections may be identified with $X_0 = C_c(G_u, A(u))$. Given $\gamma \in X$ we equip $\mathcal{Y}_\gamma = (A(u))$ with the $A(r(\gamma)) - A(u)$-imprimitivity bimodule structure arising from the isomorphism $\alpha_\gamma : A(u) \to A(r(\gamma))$. We then define actions of $G$ and $S_u$ on the left and right, respectively, of $\mathcal{Y}$ by

\[
\eta \cdot (\gamma, a) = (\eta\gamma, a), \quad \text{and} \quad (\gamma, a) \cdot s = (\gamma s, \alpha^{-1}_s(a))
\]

for $\eta \in G$, $s \in S_u$ and $(\gamma, a) \in \mathcal{Y}$. It is now straightforward to show that with these operations $\mathcal{Y}$ is an equivalence bundle with respect to the dynamical systems $(A, G, \alpha)$ and $(A(u), \alpha|_{S_u}, S_u)$. The result now follows from Renault’s Equivalence Theorem. 

3.2. Invariant Ideals. In this section we would like to see that crossed products behave well with respect to ideals arising from $G$-invariant sets in $G^{(0)}$. Given a locally compact $G$-invariant subset $Y \subset G^{(0)}$ we define the restriction of $G$ to $Y$ to be $G|_Y = r^{-1}(Y) = s^{-1}(Y)$. It is not difficult to show that $G|_Y$ is a locally compact Hausdorff groupoid and that the restriction of the Haar system to $G|_Y$ is a Haar system. Furthermore, if we have a groupoid dynamical system $(A, G, \alpha)$ then one can easily show that the restriction of the action $\alpha$ to $G|_Y$ defines an action of $G|_Y$ on $A(Y)$. Hence, once can form the restricted crossed product $A(Y) \rtimes_\alpha G|_Y$. The main goal will be to prove the following result.

Definition 3.2. Given a groupoid dynamical system $(A, G, \alpha)$ and an open $G$-invariant subset $U$ of $G^{(0)}$ let $\text{Ex}(U)$ be the closure in $A \rtimes_\alpha G$ of the set

\[
\{f \in \Gamma_c(G, r^*A) : \text{supp } f \subset G|_U\}.
\]
Theorem 3.3. Suppose $(A, G, \alpha)$ be a groupoid dynamical system, $U$ is an open $G$-invariant subset of $G^{(0)}$ and $C$ is the closed $G$-invariant subset $G^{(0)} \setminus U$. Then inclusion and restriction extend to $*$-homomorphisms $\iota : A(U) \times G|_U \to A \rtimes G$ and $\rho : A \rtimes G \to A(C) \rtimes G|_C$ respectively. Furthermore, the following sequence is short exact
\[ 0 \longrightarrow A(U) \rtimes G|_U \stackrel{\iota}{\longrightarrow} A \rtimes G \stackrel{\rho}{\longrightarrow} A(C) \rtimes G|_C \longrightarrow 0 \]
and $\text{ran} \iota = \ker \rho = \text{Ex}(U)$ so that $A(C) \rtimes G|_C$ is isomorphic to the quotient space $A \rtimes G/\text{Ex}(U)$.

Due to the fact that kernels are ill behaved with respect to completions, this theorem is deceptively difficult to prove. We will study the left hand side of the short exact sequence first.

Proposition 3.4. Given a groupoid dynamical system $(A, G, \alpha)$ and an open $G$-invariant set $U \subset G^{(0)}$ then the inclusion map $\iota : \Gamma_c(G|_U, r^*A) \to \Gamma_c(G, r^*A)$ extends to an isomorphism of $A(U) \rtimes \alpha G|_U$ onto $\text{Ex}(U)$. Furthermore, $\text{Ex}(U)$ is an ideal in $A \rtimes G$.

Proof. It is straightforward to show that $\text{Ex}(U)$ is an ideal and that $\iota$ is a homomorphism on $\Gamma_c(G|_U, r^*A)$ which is continuous with respect to the inductive limit topology and which maps onto a dense subset of $\text{Ex}(U)$. It now follows from the Disintegration Theorem that $\iota$ extends to a surjective homomorphism from $A(U) \rtimes \alpha G|_U$ onto $\text{Ex}(U)$. All that is left is to show that $\iota$ is isometric.

Suppose $R$ is a faithful representation of $A(U) \rtimes G|_U$ on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{H}_0 = \text{span}\{\mathcal{R}(f)h : f \in \Gamma_c(G|_U, r^*A), h \in \mathcal{H}\}$ and observe that $\mathcal{H}_0$ is dense in $\mathcal{H}$. If $f \in \text{ran} \iota$ and $g \in \Gamma_c(G, r^*A)$ then it is easy to see that $f * g(\gamma) = 0$ unless $\gamma \in r(\text{supp } f) \subset U$. Thus $r(\text{supp } f * g) \subset U$ so that $\text{supp } f * g \subset G|_U$. In particular, we can view $f * g$ as a function in $\Gamma_c(G|_U, r^*A)$. We now define a representation of $\Gamma_c(G, r^*A)$ on $\mathcal{H}_0$ via

\[ T(f) \sum_{i=1}^{n} R(g_i)h_i = \sum_{i=1}^{n} R(f * g_i)h_i. \]

We need to show $T$ is well defined. Using [11, Proposition 6.8] we can find a net $\{e_\kappa\} \subset \Gamma_c(G|_U, r^*A)$ so that $e_\kappa$ is a left approximate identity with respect to the inductive limit topology. Suppose $\sum_i R(g_i)h_i = 0$ and $f \in \Gamma_c(G, r^*A)$. We then have

\[ \sum_{i=1}^{n} R(f * g_i)h_i = \sum_{i=1}^{n} R(f * \lim_{\kappa} (e_\kappa * g_i)) \]
\[ = \sum_{i=1}^{n} R(\lim_{\kappa} f * e_\kappa * g_i)h_i \]
\[ = \lim_{\kappa} R(f)R(e_\kappa) \sum_{i=1}^{n} R(g_i)h_i = 0. \]

Next, an elementary computation shows that $T$ is a homomorphism into the algebra of linear operators on $\mathcal{H}_0$. Furthermore, observe that if $f \in \Gamma_c(G|_U, r^*A)$ then $T(\iota(f)) = R(f)$. We now show that $T$ satisfies the conditions of the Disintegration Theorem. It follows immediately from the fact that $R$ is nondegenerate and $\{e_\kappa\}$ is a left approximate identity that the set $\{T(f)k : f \in \Gamma_c(G, r^*A), k \in \mathcal{H}_0\}$
has a dense span in \( H \). The fact that \( f \mapsto (Tf)h, k \) is continuous with respect to the inductive limit topology for given \( h, k \in H_0 \) follows immediately from the fact that convolution and \( R \) are both continuous with respect to the inductive limit topology. Finally, a simple computation shows that \( (Tf)h, k = (h, T(f^*)k) \) for all \( f \in \Gamma_c(G, r^*A) \) and \( h, k \in H_0 \). Thus it follows from the Decomposition Theorem that \( T \) is bounded with respect to the universal norm and extends to a representation of \( A \rtimes G \). Furthermore, since \( R = T \circ \iota \) on a dense subset, this identity holds in general. Thus, given \( f \in A(U) \times G|_U \) we have

\[
\|f\| = \|R(f)\| = \|T(\iota(f))\| \leq \|\iota(f)\|.
\]

It follows that \( \iota \) is isometric and we are done. \( \square \)

The complement to Proposition 3.4 is the following

**Proposition 3.5.** Suppose \((A, G, \alpha)\) is a groupoid dynamical system and \( C \) is a closed \( G \)-invariant subset of \( G^{(0)} \). Then the restriction map \( \rho : \Gamma_c(G, r^*A) \to \Gamma_c(G|_C, r^*A) \) extends to a surjective homomorphism from \( A \rtimes G \) onto \( A(C) \rtimes_\alpha G|_C \). Furthermore, \( \rho(\Gamma_c(G, r^*A)) \) is dense in \( \Gamma_c(G|_C, r^*A) \) with respect to the inductive limit topology.

**Proof.** This proposition is almost entirely straightforward. Simple calculations show that \( \rho \) is a homomorphism which is continuous with respect to the inductive limit topology. Hence \( \rho \) extends to a bounded homomorphism on \( A \rtimes G \). Standard arguments using [20] Proposition C.24 then show that \( \rho(\Gamma_c(G, r^*A)) \) is dense with respect to the inductive limit topology and thus \( \rho \) is surjective. \( \square \)

Next we prove the following interesting technical lemma.

**Lemma 3.6.** If \((A, G, \alpha)\) is a groupoid dynamical system then the function

\[
u \mapsto \int_G \|f(\gamma)\| d\lambda^u(\gamma)
\]

is upper-semicontinuous for all \( f \in \Gamma_c(G, r^*A) \).

**Proof.** Given \( f \in \Gamma_c(G, r^*A) \) define \( \lambda(f) : G^{(0)} \to \mathbb{R} \) by

\[
\lambda(f)(u) = \int_G \|f(\gamma)\| d\lambda^u(\gamma).
\]

If \( \phi \otimes a \) is an elementary tensor in \( C_c(G) \otimes A \) then

\[
\lambda(\phi \otimes a)(u) = \int_G |\phi(\gamma)| d\lambda^u(\gamma) \|a(u)\|.
\]

It follows from the continuity of the Haar system and the fact that \( u \mapsto \|a(u)\| \) is upper-semicontinuous that \( \lambda(\phi \otimes a) \) is upper-semicontinuous. Now suppose \( f \in \Gamma_c(G, r^*A) \). Then there exists a set of elementary tensors \( \{\phi_i \otimes a_i\} \) such that \( k_i = \sum_j \phi_j \otimes a_j \) converges to \( f \) with respect to the inductive limit topology and therefore with respect to the \( I \)-norm.

**Remark 3.7.** Recall that for groupoid crossed products the \( I \)-norm is defined for \( f \in \Gamma_c(G, r^*A) \) by

\[
\|f\|_I = \max \left\{ \sup_{u \in G^{(0)}} \int_G \|f(\gamma)\| d\lambda^u(\gamma), \sup_{u \in G^{(0)}} \int_G \|f(\gamma)\| d\lambda_u(\gamma) \right\}.
\]
Now, \( \lambda(k_i) = \sum_j \lambda(a_i' \otimes a_i') \), and it is straightforward to show that sums of upper-semicontinuous functions are upper-semicontinuous. Hence, \( \lambda(k_i) \) is upper-semicontinuous. We then conclude from the computation

\[
\left| \int_G \|f(\gamma)\|d\lambda^u(\gamma) - \int_G \|g(\gamma)\|d\lambda^u(\gamma) \right| \leq \int_G \|f(\gamma)\| - \|g(\gamma)\| \ d\lambda^u(\gamma) \\
\leq \int_G \|f(\gamma) - g(\gamma)\|d\lambda^u(\gamma) \\
\leq \|f - g\|_I
\]

that \( \lambda(k_i) \to \lambda(f) \) uniformly. The result follows from the fact that uniform limits of upper-semicontinuous functions are upper-semicontinuous.

\[\Box\]

We now can finish with a proof of Theorem \[3.3\].

**Proof of Theorem \[3.3\].** All that is left is to show that \( \ker \rho = \text{Ex}(U) = \text{ran} \iota \). It is clear that given \( f \in \Gamma_c(G[U, r^*A]) \) we have \( \rho \circ \iota(f) = 0 \). Hence \( \text{ran} \iota \subseteq \ker \rho \) and we are reduced to proving that \( \ker \rho \subseteq \text{Ex}(U) \). Let \( R \) be a representation of \( A \times G \) such that \( \ker R = \text{Ex}(U) \). Now, suppose we have \( f, g \in \Gamma_c(G, r^*A) \) such that \( \rho(f) = \rho(g) \). Standard approximation arguments show that \( f - g \in \text{Ex}(U) = \ker R \).

Thus the representation \( T \) of \( \Gamma_c(G|C, r^*A) \) given by \( T(\rho(f)) = R(f) \) is well defined. Furthermore, since \( R \) and \( \rho \) are homomorphisms, it follows that \( T \) is as well. We would like to see that \( T \) is \( I \)-norm decreasing.

Suppose \( f \in \Gamma_c(G, r^*A) \) and fix \( \epsilon > 0 \). It follows from Lemma \[3.4\] that we can find for each \( v \in r(\text{supp } f) \cap C \) some relatively compact open neighborhood \( O_v \) of \( v \) such that \( w \in O_v \) implies

\[
\int_G \|f(\gamma)\|d\lambda^w(\gamma) \leq \int_G \|f(\gamma)\|d\lambda^v(\gamma) + \epsilon \leq \|\rho(f)\|_I + \epsilon.
\]

By considering the continuously compactly supported function \( \gamma \mapsto f(\gamma^{-1}) \) we can,

in the same fashion, also find for each \( v \in r(\text{supp } f) \cap C \) some relatively compact open neighborhood \( V_v \) of \( v \) such that \( w \in V_v \) implies

\[
\int_G \|f(\gamma)\|d\lambda_w(\gamma) \leq \int_G \|f(\gamma)\|d\lambda_v(\gamma) + \epsilon \leq \|\rho(f)\|_I + \epsilon.
\]

Since \( \{O_v \cap V_v\} \) is an open cover of the compact set \( r(\text{supp } f) \cap C \) there exists some finite subcover \( \{O_{v_i} \cap V_{v_i}\}_{i=1}^N \). Let \( O = \bigcup_i O_{v_i} \cap V_{v_i} \) and observe that, because the union is finite, \( O \) is relatively compact. Now choose \( \phi \in C_c(G(0)) \) such that \( \phi \) is one on \( r(\text{supp } f) \cap C \), zero off \( O \), and \( 0 \leq \phi \leq 1 \). Define \( g \in \Gamma_c(G, r^*A) \) by \( g(\gamma) = \phi(r(\gamma))f(\gamma) \). It now follows quickly from \[13\] and \[14\] that \( \|g\|_I \leq \|\rho(f)\|_I + \epsilon \).

However, \( g - f \) is zero on \( C \) by construction so that

\[
\|T(\rho(f))\| = \|R(f)\| = \|R(g)\| \leq \|g\| \leq \|\rho(f)\|_I + \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, this shows that \( T \) is \( I \)-norm decreasing, and because \( T \) is an \( I \)-norm decreasing representation it follows from the Disintegration Theorem that \( T \) extends to a representation on \( A(C) \rtimes G|C \). Finally, since \( T \circ \rho = R \) on a dense subset, this identity holds everywhere. Thus \( \ker \rho \subseteq \ker R = \text{Ex}(U) \) and we are done. 

\[\Box\]
4. Stabilizers and Regular Groupoids

The goal for this section, and the main result of this paper, is the following

**Theorem 4.1.** Suppose $(A, G, \alpha)$ is a groupoid dynamical system and that $G$ is a regular groupoid. Then every irreducible representation of $A \rtimes_{\alpha} G$ is equivalent to one of the form $\text{Ind}_{G}^{S_{u}} \phi$ where $u \in G^{(0)}$ and $R$ is an irreducible representation of $A(u) \rtimes_{\alpha} S_{u}$.

**Remark 4.2.** Generalizing this result to non-regular groupoids is difficult. For group crossed products the result is known as the Gootman-Rosenberg-Sauvageot theorem and a proof may be found in [20, Chapter 9]. For groupoid $C^*$-algebras the result is proved in [15], and for general groupoid crossed products the question is still open.

**Remark 4.3.** A concise way of phrasing Theorem 4.1 would be to say that every irreducible representation of $A \rtimes G$ is induced from a stabilizer subgroup. Unfortunately, this would conflict with [20, Definition 8.10]. The problem lies in the meaning of the word stabilizers. In [20] the stabilizers are the stabilizer subgroups with respect to the action of $G$ on $G^{(0)}$ and may be larger. Of course, when $A$ has Hausdorff spectrum these two notions match up, and in this case it is not difficult to show that Theorem 4.1 can be viewed as a generalization of [20, Theorem 8.16] using Example 4.4. Reconciling these differences in the non-Hausdorff case is an open question.

The key to proving Theorem 4.1 will be to reduce to the case where $G^{(0)}/G$ is Hausdorff because in this case we have the following result.

**Proposition 4.4.** Suppose $(A, G, \alpha)$ is a groupoid dynamical system and $G^{(0)}/G$ is Hausdorff. Then $A \rtimes_{\alpha} G$ is a $C_{0}(G^{(0)}/G)$-algebra with the action $\phi \cdot f(\gamma) = \phi(G \cdot r(\gamma)) f(\gamma)$ for $\phi \in C_{0}(G^{(0)}/G)$ and $f \in \Gamma_{c}(G^{(0)}/G)$. Furthermore, restriction factors to an isomorphism of $A \rtimes G(G\cdot u)$ onto $A(G\cdot u) \rtimes G_{G\cdot u}$.

**Proof.** Define the action $\Phi(\phi)f = \phi \cdot f$ as above and extend $\Phi$ to the unitization $C_{0}(G^{(0)}/G)$ of $C_{0}(G^{(0)}/G)$ in the obvious fashion. It is straightforward to show that $\Phi(\xi)$ is an adjointable linear operator on $\Gamma_{c}(G^{(0)}/G)$ for each $\xi \in C_{0}(G^{(0)}/G)^{1}$ and that $\Phi$ is a $*$-homomorphism. We would like to prove that $\|\Phi(\phi)f\| \leq \|\phi\|_{\infty} |f|$ for all $\phi \in C_{0}(G^{(0)}/G)$ and $f \in \Gamma_{c}(G^{(0)}/G)$. It will suffice to show that $\|\phi\|_{\infty}^{2} \langle f, f \rangle - \langle \Phi(\phi)f, \Phi(\phi)f \rangle \geq 0$ as elements of $A \rtimes G$, where $\langle f, g \rangle = f^{*}g$. However, using the fact that $\Phi$ is a homomorphism on $C_{0}(G^{(0)}/G)^{1}$, this amounts to showing

\begin{equation}
\langle \Phi(||\phi||_{\infty}^{2} 1 - \overline{\phi})f, f \rangle \geq 0.
\end{equation}

All elements of the form $||\phi||_{\infty}^{2} 1 - \overline{\phi}$ are positive in $C_{0}(G^{(0)}/G)^{1}$ so there exists $\xi \in C_{0}(G^{(0)}/G)^{1}$ such that $||\phi||_{\infty}^{2} 1 - \overline{\phi} = \xi^{*} \xi$. Therefore, we have

$\langle \Phi(||\phi||_{\infty}^{2} 1 - \overline{\phi})f, f \rangle = \langle \Phi(\xi^{*})f, f \rangle = \langle \Phi(\xi)f, f \rangle \geq 0$.

It follows that $\Phi(\phi)$ is a bounded operator on $\Gamma_{c}(G^{(0)}/G)$ with norm less than $||\phi||_{\infty}$ and that $\Phi(\phi)$ extends to a multiplier on $A \rtimes_{\alpha} G$. It is easy to see that $\Phi$ is a nondegenerate $*$-homomorphism and that it maps into the center of the multiplier algebra. Hence $A \rtimes_{\alpha} G$ is a $C_{0}(G^{(0)}/G)$-algebra.

Next, we identify the fibres of $A \rtimes G$. Fix $u \in G^{(0)}$. Since $G^{(0)}/G$ is Hausdorff, $G\cdot u$ is closed in $G^{(0)}$. Let $O = G^{(0)} \setminus G\cdot u$. It is clear that $G\cdot u$ and $O$ are both
$G$-invariant so that we may apply Theorem 3.3 to conclude that the restriction map $\rho$ factors to an isomorphism of $A \rtimes G/\text{Ex}(O)$ with $A(G \cdot u) \rtimes G|_{G \cdot u}$. Now define

$$I_u = \text{span}\{\phi \cdot f : \phi \in C_0(G^{(0)}/G), f \in \Gamma_c(G, r^*A), \phi(G \cdot u) = 0\}.$$  

An approximation argument then shows that $I_u = \text{Ex}(O)$. The result follows since, by definition, $A \rtimes G(G \cdot u) = A \rtimes G/I_u$.

The reason that this is a useful result is that we know a lot about the fibres when $G^{(0)}/G$ is Hausdorff.

**Corollary 4.5.** Suppose $(A, G, \alpha)$ is a groupoid dynamical system and that the orbit space $G^{(0)}/G$ is Hausdorff. Given $u \in G^{(0)}$ the fibre $A \rtimes G(G \cdot u)$ is Morita equivalent to $A(u) \rtimes S_u$.

**Proof.** The fibre of $A \rtimes G$ over $G \cdot u$ is, by the previous proposition, isomorphic to $A(G \cdot u) \rtimes G|_{G \cdot u}$. However, $G|_{G \cdot u}$ is a transitive groupoid so the result follows from Theorem 3.1.

This allows us to take the first step in proving the main result.

**Proposition 4.6.** Suppose $(A, G, \alpha)$ is a groupoid dynamical system and that the orbit space $G^{(0)}/G$ is Hausdorff. Then every irreducible representation of $A \rtimes G$ is equivalent to one of the form $\text{Ind}_{S_u}^G R$ where $u \in G^{(0)}$ and $R$ is an irreducible representation of $A(u) \rtimes S_u$.

In the case where $G^{(0)}/G$ is Hausdorff, every irreducible representation of $A \rtimes G$ is lifted from a fibre $A \rtimes G(G \cdot u)$, and every irreducible representation of $A \rtimes G(G \cdot u)$ comes from an irreducible representation of $A(u) \rtimes S_u$. It turns out that this two stage description is nothing more than the usual induction process.

**Lemma 4.7.** Suppose $(A, G, \alpha)$ is a groupoid dynamical system and that the orbit space $G^{(0)}/G$ is Hausdorff. Given $u \in G^{(0)}$ let $\rho : A \rtimes G \to A(G \cdot u) \rtimes G|_{G \cdot u}$ be the extension of the restriction map on $\Gamma_c(G, r^*A)$. Furthermore, let $X$ be the $A(G \cdot u) \rtimes G|_{G \cdot u} - A(u) \rtimes S_u$ imprimitivity bimodule from Theorem 3.1. If $R$ is a representation of $A(u) \rtimes S_u$ then $\text{Ind}_{S_u}^G R = X - \text{Ind}(R) \circ \rho$.

**Proof.** Let $Z_{S_u}^G$ be the imprimitivity bimodule coming from Proposition 2.6 and $X$ the imprimitivity bimodule from Theorem 3.1. This lemma is simple enough once we observe that $X$ and $Z$ are equal as right Hilbert modules. Hence if $R$ acts on $\mathcal{H}$ then both $\text{Ind}_{S_u}^G R$ and $X - \text{Ind}(R) \circ \rho$ act on $Z_{S_u}^G \otimes_{A(u) \times S_u} \mathcal{H}$. From here an elementary calculation shows that $\text{Ind}_{S_u}^G R(f) = X - \text{Ind}(R)(\rho(f))$ for $f \in \Gamma_c(G, r^*A)$ and this extends to the entire crossed product by continuity.

With this result at are disposal we are mostly done.

**Proof of Proposition 4.6.** Suppose $(A, G, \alpha)$ is a groupoid dynamical system and that the orbit space $G^{(0)}/G$ is Hausdorff. By Proposition 4.4 $A \rtimes G$ is a $C_0(G^{(0)}/G)$-algebra. It then follows from general $C_0(X)$-algebra theory [29, Proposition C.5] that any irreducible representation $T$ is of the form $T = L \circ \rho$ where $u \in G^{(0)}$, $L$ is an irreducible representation of $A(G \cdot u) \rtimes G|_{G \cdot u}$ and $\rho$ is the canonical extension of the restriction map. However, $A(G \cdot u) \rtimes G|_{G \cdot u}$ is Morita Equivalent to $A(u) \rtimes S_u$ by Corollary 4.5, and therefore there is an irreducible representation $R$ of $A(u) \rtimes S_u$ such that $L$ is equivalent to $X - \text{Ind}(R)$ [13, Section 3.3]. It then follows that $T = L \circ \rho$ and $\text{Ind}_{S_u}^G R = X - \text{Ind}(R) \circ \rho$ are equivalent and we are done.
As an aside, the following corollary is useful and follows quickly from Lemma 4.7, although we will not detail the proof here.

**Corollary 4.8.** Suppose $(A,G,\alpha)$ is a groupoid dynamical system and that the orbit space $G^{(0)}/G$ is Hausdorff. If $R$ is an irreducible representation of $A(u) \rtimes S_u$ then $\text{Ind}^{G}_{S_u} R$ is irreducible. Furthermore, if $L$ and $R$ are irreducible representations of $A(u) \rtimes S_u$ and $\text{Ind}^{G}_{S_u} R$ is equivalent to $\text{Ind}^{G}_{S_u} L$ then $R$ is equivalent to $L$.

We will now prove Theorem 4.1 by extending Proposition 4.6 to groupoids which satisfy the Mackey-Glimm Dichotomy. As mentioned in the introduction, there are a number of conditions which are all equivalent and collectively make up the dichotomy [14]. The most useful condition for our purposes will be the fact that $G$ is regular if and only if $G(0)/G$ is almost Hausdorff.

**Definition 4.9.** A, not necessarily Hausdorff, locally compact space $X$ is said to be almost Hausdorff if each locally compact subspace $V$ contains a relatively open nonempty Hausdorff subset.

The key fact we will use concerning almost Hausdorff spaces is the following proposition, which we cite without proof.

**Proposition 4.10 ([20, Lemma 6.3]).** Suppose $X$ is a, not necessarily Hausdorff, locally compact space. Then $X$ is almost Hausdorff if and only if there is an ordinal $\gamma$ and open sets $\{U_\alpha : \alpha \leq \gamma\}$ such that

(a) $\alpha < \beta < \gamma$ implies that $U_\alpha \subseteq U_\beta$,

(b) $\alpha < \gamma$ implies that $U_\alpha + 1 \setminus U_\alpha$ is a dense Hausdorff subspace of $X \setminus U_\alpha$,

(c) if $\delta \leq \gamma$ is a limit ordinal then

$$U_\delta = \bigcup_{\alpha < \delta} U_\alpha,$$

(d) $U_0 = \emptyset$ and $U_\gamma = X$.

The main reason we care about Proposition 4.10 is that it allows us to build a composition series so that we may make use of the following

**Lemma 4.11 ([20, Lemma 8.13]).** Suppose $\{I_\alpha\}_{\alpha \in \Lambda}$ is a composition series for a $C^*$-algebra $A$. Then every irreducible representation $\pi$ of $A$ lives on a sub-quotient $I_{\alpha+1}/I_\alpha$ for some $\alpha$.

At this point the way forward is clear. If $G^{(0)}/G$ is almost Hausdorff then we will build a composition series of crossed products where the orbit space associated to the sub-quotients is Hausdorff. This will allow us to use Proposition 4.6 to prove Theorem 4.1. The heart of the argument is that the multistage process of taking a representation of $A(u) \rtimes S_u$ to $A \rtimes G$ through the composition series is equivalent to the usual induction process.

**Lemma 4.12.** Suppose $(A,G,\alpha)$ is a groupoid dynamical system and that $U \subset V \subset G^{(0)}$ are open $G$-invariant sets. Then $A(V \setminus U) \rtimes G|_{V \setminus U}$ is naturally isomorphic to the sub-quotient $\text{Ex}(V)/\text{Ex}(U)$. Furthermore, if $u \in V \setminus U$ and $R$ is a representation of $A(u) \rtimes S_u$ then the canonical extension of $\text{Ind}^{G|_{V \setminus U}}_{S_u} R$ to $A \rtimes G$ is equal to $\text{Ind}^{G}_{S_u} R$.

**Proof.** Recall that $A(V) \rtimes G|_V$ is isomorphic to the ideal $\text{Ex}(V)$ in $A \rtimes G$ via the inclusion map $\iota$. Let $\text{Ex}'(U)$ be the isomorphic image of $A(U) \rtimes G|_U$ in $A(V) \rtimes G|_V$
Therefore, there are open sets $\{\text{Proposition 4.10}. \text{Let } G \text{ Since }$

Proof of Theorem 4.1. canonical extension of $\text{Ind}_G \leq \text{for all } 0 \leq \delta \leq \gamma$. Then each $U_\beta$ is an open $G$-invariant subset and we define $I_\beta = \text{Ex}(U_\beta)$. It is clear that $I_0 = \{0\}, I_\gamma = A \times G$, and if $\delta < \beta \leq \gamma$ then $I_\delta \subseteq I_\beta$. Finally, suppose $\delta \leq \gamma$ is a limit ordinal and $f \in \Gamma_c(G[U_\beta], r^*A)$. Then $\{U_\beta\}_{\beta<\delta}$ is an open cover of $r(\text{supp } f)$. Since $r(\text{supp } f)$ is compact there must be a finite subcover, and since the $U_\beta$ are nested, this implies that there exists $\beta' < \delta$ such that $r(\text{supp } f) \subseteq U_{\beta'}$. Hence $f \in \Gamma_c(G[U_{\beta'}], r^*A) \subseteq I_{\beta'}$. It now follows quickly that

$$I_\delta = \bigcup_{\beta<\delta} I_\beta.$$ 

Thus $\{I_\beta\}$ is a composition series for $A \rtimes G$.

Suppose $L$ is an irreducible representation of $A \rtimes G$. Lemma 4.11 implies that there exists $\beta$ such that $L$ lives on $I_{\beta+1}/I_\beta$. In other words, there is an irreducible representation $T$ of $I_{\beta+1}/I_\beta$ such that $L$ is the unique canonical extension of $T$. Next, Lemma 4.12 implies that we can identify $I_{\beta+1}/I_\beta$ with $A(U_{\beta+1} \setminus U_\beta) \rtimes G|_{U_{\beta+1} \setminus U_\beta}$. Furthermore, $q(U_{\beta+1} \setminus U_\beta) = V_{\beta+1} \setminus V_\beta$ is Hausdorff, so that by Proposition 4.9 there exists $u \in U_{\beta+1} \setminus U_\beta$ and an irreducible representation $R$ of $A(u) \rtimes S_u$ such that $T$ is equivalent to $R' = \text{Ind}_{S_u}^{G|_{U_{\beta+1} \setminus U_\beta}} R$. Hence, the extension of $T$ to $A \rtimes G$, which is $L$, is equivalent to the extension of $R'$ to $A \rtimes G$, which is $\text{Ind}_{S_u}^{G} R$ by Lemma 4.12. 

We finish by using Lemma 4.12 to sift out a useful fact.

**Proposition 4.13.** Suppose $(A, G, \alpha)$ is a groupoid dynamical system and $G$ is regular. If $R$ is an irreducible representation of $A(u) \rtimes S_u$ then $\text{Ind}_{S_u}^{G} R$ is irreducible. Furthermore, if $R$ and $L$ are both irreducible representations of $A(u) \rtimes S_u$ and $\text{Ind}_{S_u}^{G} R$ is equivalent to $\text{Ind}_{S_u}^{G} L$ then $R$ is equivalent to $L$.

**Proof.** Suppose $R$ is an irreducible representation of $A(u) \rtimes S_u$. As before, $G^{(0)}/G$ must be locally Hausdorff so that there exists $\{V_\beta\}$ as in Proposition 4.10. Consider
\[ \Gamma = \{ \beta \leq \gamma : G \cdot u \in V_{\gamma} \}. \] If \( \delta = \min \Gamma \) is a limit ordinal then \( G \cdot u \in \bigcup_{\beta < \delta} V_{\beta}. \) However, this is a contradiction since it follows that \( G \cdot u \in V_{\beta} \) for some \( \beta < \delta. \) Therefore, \( \delta \) has an immediate predecessor \( \sigma \) and \( G \cdot u \in V_{\delta} \setminus V_{\sigma}. \) Suppose \( q : G^{(0)} \rightarrow G^{(0)}/G \) is the quotient map, and let \( U_{\delta} = q^{-1}(V_{\delta}) \) and \( U_{\sigma} = q^{-1}(V_{\sigma}). \) Then \( u \in U_{\delta} \setminus U_{\sigma} \) and since \( V_{\delta} \setminus V_{\sigma} \) is Hausdorff it follows from Corollary 4.8 that \( R' = \text{Ind}_{\mathcal{S}_u}^{G} R \) is irreducible. Hence the extension of \( R' \) to \( A \times G \) is irreducible and by Lemma 1.12 this is exactly \( \text{Ind}_{\mathcal{S}_u}^{G} R. \) The rest of the proposition follows quickly via similar concerns. \( \Box \)

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