Computational equivalence of the two inequivalent spinor representations of the braid group in the Ising topological quantum computer

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Abstract. We demonstrate that the two inequivalent spinor representations of the braid group \( \mathbb{B}_{2n+2} \), describing the exchanges of \( 2n + 2 \) non-Abelian Ising anyons in the Pfaffian topological quantum computer, are equivalent from the computational point of view, i.e., the sets of topologically protected quantum gates that could be implemented in both cases by braiding exactly coincide. We give the explicit matrices generating almost all braidings in the spinor representations of the \( 2n + 2 \) Ising anyons, as well as important recurrence relations. Our detailed analysis allows us to understand better the physical difference between the two inequivalent representations and to propose a process that could determine the type of representation for any concrete physical realization of the Pfaffian quantum computer.

Keywords: conformal field theory (theory), fractional QHE (theory)
1. Introduction

Quantum computation [1, 2] has recently become an attractive field of research because of the expected exponential speed-up over the classical computations which could eventually allow us to perform hard computational tasks that are practically impossible for classical computers. One important class of systems in which quantum information is believed to be protected from noise by topology is the class of topological quantum computers [3]–[9]. One of the promising schemes for topological quantum computation (TQC) is based on the non-Abelian statistics of low-energy quasiparticles in low-dimensional strongly correlated systems, such as the p-wave superconductor [10, 11, 9] and the Fractional Quantum Hall (FQH) state at filling factor $\nu = 5/2$ in the second Landau level. There is strong analytical and numerical evidence [12, 13] that the $\nu = 5/2$ FQH state belongs to the universality class of the Moore–Read (MR), or Pfaffian, state [14]–[17], [11]. The main idea of TQC in the Pfaffian state [18, 19, 6, 20], [7]–[9] is to use the non-Abelian statistics of the Ising anyons to execute quantum gates by adiabatically exchanging the quasiparticles comprising the Pfaffian qubits. Because the Berry connection is trivial [21, 22], the entire effect of the adiabatic transport comes from the explicit braiding and monodromy of the Ising conformal blocks. The advantage of using FQH anyons for quantum computation is that the encoded quantum information is naturally protected from noise by topology because of the FQH energy gap which suppresses exponentially all processes leading to noise and decoherence [6].

The elementary exchanges of adjacent quasiparticles in the Pfaffian FQH state with $2n$ Ising quasiholes could be expressed [16] in terms of $\pi/2$-rotations from the orthogonal group $SO(2n)$ and the corresponding braid operators belong to one of the two inequivalent spinor representations $S_{\pm}$ of the covering group [16, 23] $\text{Spin}(2n)$ of $SO(2n)$. The fact that $S_{\pm}$ are inequivalent means that the two sets of matrices generating $S_{\pm}$ differ by more than just a ‘change of basis’, which raises the following reasonable questions: if we assume that an experimental $\nu = 5/2$ FQH sample is indeed in the universality class of the MR
state then in which of the two inequivalent spinor representations is this system and how
do we distinguish between them in a real FQH sample? Next, are these inequivalent
representations equivalent from the computational point of view, i.e., could one construct
the same number and types of quantum gates in the two representations, or, are the sets
of quantum gates significantly different? In this paper we shall emphasize the physical
difference, and shall prove that the two inequivalent representations of the braid group
$B_{2n+2}$ are computationally equivalent.

It is worth stressing that as a mathematical fact the computational equivalence of the
two inequivalent Ising-type representations of the braid group $B_{2n+2}$ is not a new result.
It could be derived from the explicit representation of the monodromy subgroup in [24],
corresponding to the universal $R$-matrix of the Ising model. However, there is no proof
in the literature that the Pfaffian wavefunctions with $2n + 2$ Ising anyons realize precisely
this representation of the braid group $B_{2n+2}$ which is given in [24], though it is intuitively
clear that it should be equivalent to that of the critical Ising model, yet this equivalence
might be nontrivial and is important for the physical implementation of various quantum
gates. Therefore, in this paper we give an independent and self-consistent proof of the
computational equivalence directly in the spinor approach of [16]. A central result in this
paper is the derivation of a set of recurrence relations for the braid generators $R_j^{(n,\pm)}$
of the spinor representations, which are necessary for the consistency of the proof, presented
in [25], of the mathematical equivalence of the braid group representations derived from
the Pfaffian wavefunctions and those obtained in the spinor approach.

The paper is organized as follows: in section 2 we review the Clifford algebra
construction [16] of the two inequivalent spinor representations of the braid group
$B_{2n}$; in section 3 we construct explicitly the two inequivalent spinor representations
of the braid group for 4 Ising anyons, representing 1 qubit, and prove directly that
they are computationally equivalent. In section 4 we derive the recurrence relations
and give explicit formulae for the braid generators in the positive- and negative-
parity representations of the $n$-qubit systems in terms of $2n + 2$ anyons, as well as a
general proof of the computational equivalence of the two inequivalent representations
of the braid group $B_{2n+2}$. Then, in section 5 we describe how to determine the
type of a concrete representation of the Pfaffian system with many non-Abelian
anyons.

2. Braid group representation for Ising anyons in terms of the Clifford algebra

It was suggested in [16] that the Ising-anyon representation of the braid group $B_{2n+2}$ can
be realized by $\pi/2$ rotations from the group $SO(2n + 2)$, however, this statement has not
been completely proven in [16]. From the TQC point of view the most natural approach
to compute the braid generators is to make an analytic continuation of the $(2n + 2)$-
anyon CFT correlation functions which have been used originally to construct the $n$-qubit
Ising system [16, 7, 8]. However, in order to derive consistently all braid matrices for
more Ising anyons one needs to know the Pfaffian wavefunctions in the negative-parity
representation. This difficulty has been overcome in [25], where all braid generators in
both representations with positive and negative parity have been consistently derived by
using the fusion rules in the Neveu–Schwarz and Ramond sectors of the Ising model. In
addition it has been finally proven\(^1\) in [25] that the braid group representation derived by analytic continuation of the multi-anyon Pfaffian wavefunctions is indeed equivalent to one of the spinor representations of \(SO(2n + 2)\) and the equivalence matrices have been constructed explicitly for both representations with positive and negative parity.

The elementary matrices representing the exchanges of 2\(n\) Ising quasiparticles in the Pfaffian FQH state can be expressed [16] in terms of the gamma matrices \(\gamma_i^{(n)}\), \(1 \leq i \leq 2n\), satisfying the anticommutation relations of the Clifford algebra

\[
\{\gamma_i^{(n)}, \gamma_j^{(n)}\} = 2\delta_{ij}, \quad 1 \leq i, j \leq 2n. \tag{1}
\]

In more detail, the elementary operations for the exchange of the \(i\)th and \((i + 1)\)th quasiparticles could be expressed as [16,11]

\[
R_j^{(n)} = e^{i\pi/4} \exp \left( -\frac{\pi}{4} \gamma_j^{(n)} \gamma_{j+1}^{(n)} \right) \equiv e^{i\pi/4} \left( I - \gamma_j^{(n)} \gamma_{j+1}^{(n)} \right), \tag{2}
\]

where the second equality follows from the fact that \((\gamma_j \gamma_{j+1})^2 = -I\) due to the anticommutation relations (1).

The 2\(n\) matrices \(\gamma_i^{(n)}\) have dimension \(2^n \times 2^n\) and can be defined recursively in terms of the Pauli matrices \(\sigma_i\), \(i = 1, 2, 3\) as follows [26]

\[
\begin{align*}
\gamma_j^{(n+1)} &= \gamma_j^{(n)} \otimes \sigma_3, \quad 1 \leq j \leq 2n \\
\gamma_{2n+1}^{(n+1)} &= I_{2^n} \otimes \sigma_1, \\
\gamma_{n+1}^{(n+1)} &= I_{2^n} \otimes \sigma_2.
\end{align*} \tag{3}
\]

Starting with \(n = 0\) as a base, where \(\gamma_1^{(1)} = \sigma_1\) and \(\gamma_2^{(1)} = \sigma_2\), we could write the gamma matrices explicitly as follows [26,16,27]

\[
\begin{align*}
\gamma_1^{(n)} &= \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \\
\vdots \\
\gamma_{2j-1}^{(n)} &= I_{2^{j-1}} \otimes I_{2^{n-j}} \otimes \sigma_1 \otimes I_{2^{j-1}} \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \\
\gamma_{2j}^{(n)} &= I_{2^{j-1}} \otimes I_{2^{n-j}} \otimes \sigma_2 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \\
\vdots \\
\gamma_{2n-1}^{(n)} &= I_{2^{n-1}} \otimes \sigma_1, \\
\gamma_{2n}^{(n)} &= I_{2^{n-1}} \otimes \sigma_2.
\end{align*} \tag{4}
\]

The ‘gamma-five’ matrix \(\gamma_F^{(n)}\), defined by \(\gamma_F^{(n)} = (-i)^n \gamma_1^{(n)} \cdots \gamma_{2n}^{(n)}\), commutes with all matrices (2) and therefore the matrices \(R_j^{(n)}\) cannot change the \(\gamma_F^{(n)}\) eigenvalues \(\pm 1\), which means that the representation (2) is reducible and the two irreducible components, corresponding to eigenvalues \(\pm 1\), can be obtained by projecting with the two projectors

\[
P^\pm_\gamma = \frac{I_{2^n} \pm \gamma_F^{(n)}}{2}, \quad \text{i.e.,} \ (P^\pm_\gamma)^2 = P_\gamma = (P^\pm_\gamma)^\dagger. \tag{5}
\]

\(^1\) Notice that this proof uses relations (17)–(19) given in section 4 below and without them it would be logically incomplete.

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In other words, the generators of the two irreducible spinor representations of the braid group $B_{2n}$ can be obtained by simply projecting\footnote{Note that $P^{(n)}_{\pm}$ are even in the $\gamma$ matrices and therefore commute with $R_j^{(n)}$.} equation (2)

$$
R_j^{(n,\pm)} = e^{i\pi/4} P^{(n)}_{\pm} \exp \left( -\frac{\pi}{4} \gamma_j \gamma_{j+1} \right) P^{(n)}_{\pm} = e^{i\pi/4} \frac{1}{\sqrt{2}} (I - \gamma_j^{(n)} \gamma_{j+1}^{(n)}) P^{(n)}_{\pm}.
$$

The elementary exchange matrices (6) are what we could eventually use to implement topologically protected quantum gates with Ising anyons.

### 3. The single-qubit case: 4 Ising anyons

In this section we shall explicitly construct the two inequivalent spinor representations of the braid group $B_4$ following the general procedure described in section 2. The four-dimensional $\gamma$ matrices in this case are explicitly

$$
\gamma_1^{(2)} = \sigma_1 \otimes \sigma_3, \quad \gamma_2^{(2)} = \sigma_2 \otimes \sigma_3, \quad \gamma_3^{(2)} = I_2 \otimes \sigma_1, \quad \gamma_4^{(2)} = I_2 \otimes \sigma_2
$$

and the diagonal matrix $\gamma_F^{(2)} = -\gamma_1^{(2)} \gamma_2^{(2)} \gamma_3^{(2)} \gamma_4^{(2)} = \text{diag}(1, -1, -1, 1)$ determines the two projectors on the two spinor representations $S_{\pm}$ to be

$$
P^{(2)}_+ = \text{diag}(1, 0, 0, 1) \quad \text{and} \quad P^{(2)}_- = \text{diag}(0, 1, 1, 0).
$$

Applying the two projectors (8), and deleting the (zero) rows and columns with numbers 2, 3 for $P^{(2)}_+$ and 1, 4 for $P^{(2)}_-$, respectively, we obtain the three elementary generators of the two-dimensional spinor representations $S_{\pm}$ of the braid group $B_4$ as follows

$$
R_1^{(2,+)} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad R_2^{(2,+)} = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad R_3^{(2,+)} = \begin{bmatrix} 1 & 0 \end{bmatrix},
$$

and

$$
R_1^{(2,-)} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad R_2^{(2,-)} = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad R_3^{(2,-)} = \begin{bmatrix} 1 & 0 \end{bmatrix}.
$$

**Remark 1.** The positive-parity representation of $B_4$, generated by $R_j^{(2,+)1}$, looks different from that obtained in [25] by analytic continuation of the 4-anyon Pfaffian wavefunctions (with generators $B_j^{(4,+)}$ there), however, as proven in [25], these two positive-parity representations are equivalent and the matrix establishing this equivalence is simply $Z = \text{diag}(1, -1)$.

It is worth stressing that the two inequivalent representations $S_{\pm}$ of the braid group $B_4$, generated from the elementary braid matrices (9) and (10) respectively and their inverses coincide as sets of matrices. This is because as we saw before $R_1^{(2,+)} = R_1^{(2,-)}$, and because

$$
R_2^{(2,+)} R_2^{(2,-)} = R_3^{(2,+)} R_3^{(2,-)} = iI_2.
$$

Note that the matrix $iI_2$ does belong to both representations of the braid group $B_4$, i.e.,

$$
R_1^{(2,\pm)} R_2^{(2,\pm)} (R_3^{(2,\pm)})^2 R_2^{(2,\pm)} R_1^{(2,\pm)} = \pm iI_2.
$$

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In other words all two-dimensional matrices which can be obtained by braiding Ising anyons in the representation $S_+$ can be implemented in the representation $S_-$ as well, so that the two inequivalent representations $S_\pm$ are computationally equivalent.

It is not difficult to see that the diagonal elementary braid matrices in each of the representations $S_\pm$ of $\mathcal{B}_4$ fix the single-qubit computational basis (up to equivalence). Indeed, consider the matrix $R_1^{(2,+)}$: before braiding we can first fuse the quasiholes at positions $\eta_1$ and $\eta_2$ by using the fusion rule $[28,8]$ \[
\psi_{\text{qh}}(\eta_1)\psi_{\text{qh}}(\eta_2) \simeq_{\eta_1,\eta_2} \left( \mathbb{I} + \frac{1}{\sqrt{2}} \sqrt{\eta_{12}} \psi(\eta_2) \right) e^{i(\phi(\eta_2)/\sqrt{2})},
\] where $\mathbb{I}$ corresponds to the fusion channel $\sigma_+ \sigma_+ \simeq |0\rangle$ while the Majorana fermion $\psi$ corresponds to the fusion channel $\sigma_+ \sigma_- \simeq |1\rangle$. Here $\sigma_\pm$ are the chiral spin fields of CFT dimensions $1/16$ of the Ising model $[29,28,30,7,8]$ and the subscript $\pm$ denotes their fermion parity. Executing the braid is now equivalent to the transformation $\eta_{12} \to e^{i\pi\eta_{12}}$ so that the resulting phase is $1$ if the first pair is $\sigma_+(\eta_1)\sigma_+(\eta_2)$ and $i$ if it is $\sigma_+(\eta_1)\sigma_-(\eta_2)$, and this topological phase is independent of how close are the two anyons. Therefore, the braid matrix $R_1^{(2,+)}$ completely determines the type of the $\sigma$ fields with coordinates $\eta_1$ and $\eta_2$. In the same way if we braid $\eta_3$ with $\eta_4$, i.e., $\eta_{34} \to e^{i\pi\eta_{34}}$ the pair $\sigma(\eta_3)\sigma(\eta_4)$ gives a phase $1$ if it is in the channel of the identity or $i$ if it is in the channel of $\psi$. Thus, from the explicit form of the diagonal matrices $R_1^{(2,+)}$ and $R_3^{(2,+)}$, we can unambiguously reconstruct the single-qubit computational basis in terms of the Ising-model correlation functions for the spinor representation $S_+$ as follows \[
|0\rangle_+ \equiv \left( \sigma_+ \sigma_+ \sigma_+ \sigma_+ \prod_{j=1}^{2N} \psi(z_j) \right), \quad |1\rangle_+ \equiv \left( \sigma_+ \sigma_- \sigma_+ \sigma_- \prod_{j=1}^{2N} \psi(z_j) \right). \]
Similarly, from the diagonal matrices $R_1^{(2,-)}$ and $R_3^{(2,-)}$ in equation (10), we can unambiguously reconstruct the single-qubit basis in the spinor representation $S_-$ \[
|0\rangle_- \equiv \left( \sigma_+ \sigma_+ \sigma_+ \sigma_- \prod_{j=1}^{2N+1} \psi(z_j) \right), \quad |1\rangle_- \equiv \left( \sigma_+ \sigma_- \sigma_+ \sigma_+ \prod_{j=1}^{2N+1} \psi(z_j) \right). \]

The above analysis clarifies the physical difference between the two inequivalent spinor representations: the representation $S_+$ is realized (in the large-$N$ limit) with $4$ $\sigma$ fields, with positive total parity and an even number of Majorana fermions, while $S_-$ corresponds to $4$ $\sigma$ fields, with negative total parity and an odd number of Majorana fermions.

4. The $n$-qubit case: projected braid matrices for $2n + 2$ Ising anyons

Using the recursive definition (3) of the gamma matrices one can directly prove that most of the unprojected exchange matrices for $2n + 2$ anyons are simply related to those for $2n$ anyons \[
R_j^{(n+1)} = R_j^{(n)} \otimes \mathbb{I}_2 \quad \text{for } 1 \leq j \leq 2n - 1,
\] where the superscript $(n)$ or $(n + 1)$ now refers to the superscript of the corresponding gamma matrices entering equation (2).
Next, because of the recursive relation $\gamma_{F}^{(n+1)} = \gamma_{F}^{(n)} \oplus (-\gamma_{F}^{(n)})$, where $\oplus$ denotes the direct sum of matrices, it is easy to prove that the projectors (5) are also recursively related by
\[
P_{+}^{(n+1)} = P_{+}^{(n)} \oplus P_{-}^{(n)}, \quad P_{-}^{(n+1)} = P_{-}^{(n)} \oplus P_{+}^{(n)}.
\]
(16)

4.1. Recurrence relations for the projected braid matrices

We can now prove that the projected matrices (6) satisfy the following recurrence relations
\[
R_{j}^{(n+1,+)} = R_{j}^{(n+1,-)} \quad \text{for } 1 \leq j \leq 2n - 1 \quad \text{(17)}
\]
\[
R_{j}^{(n+1,\pm)} = R_{j}^{(n,\pm)} \otimes I_{2} \quad \text{for } 1 \leq j \leq 2n - 3 \quad \text{(18)}
\]
\[
R_{j}^{(n+1,\mp)} = R_{j-2}^{(n,\pm)} \oplus R_{j-2}^{(n,\mp)} \quad \text{for } 3 \leq j \leq 2n + 1, \quad \text{(19)}
\]
which together with the two-qubit case, $n = 2$, as a base case, completely determine all projected matrices (6). To prove equation (18) notice that due to the structure of the projectors (16) we have
\[
P_{\pm}^{(n+1)} = \bigoplus_{i=1}^{2^{n}} P_{\pm \alpha(i)}^{(1)}, \quad \text{where } \alpha(i) = \pm
\]
so that $P_{\pm}^{(n+1)}$ can be written as block-diagonal matrices whose elements on the diagonal are the $2 \times 2$-dimensional matrices
\[
P_{+}^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P_{-}^{(1)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]
(20)

Applying any of these projectors to the unprojected matrices (15) simply removes the matrix $I_{2}$ from the tensor product in equation (15), i.e.,
\[
R_{j}^{(n+1,+,)} = R_{j}^{(n+1,-)} = R_{j}^{(n)} \quad \text{for } 1 \leq j \leq 2n - 1,
\]
which proves equation (17) and expresses the trivial relation between projected matrices for $n$ qubits to the unprojected matrices for $(n - 1)$ qubits for this values of $j$. If there is one more $I_{2}$ in the unprojected matrix (15), as in the case when $1 \leq j \leq 2n - 3$, this relation proves equation (18). On the other hand, when $3 \leq j \leq 2n + 1$ the unprojected matrices are again tensor products in which, however, the unit matrix is to the left
\[
R_{j}^{(n+1)} = I_{2} \otimes R_{j-2}^{(n)} = R_{j-2}^{(n)} \oplus R_{j-2}^{(n)}, \quad 3 \leq j \leq 2n + 1,
\]
so that applying the projectors (16) proves equation (19). Notice the index shift $j \rightarrow j - 2$ which is due to the relabeling $\eta_{j} = \eta_{j-2}$, for $3 \leq j \leq 2n + 1$, after splitting one unit matrix $I_{2}$ to the left corresponding to the first qubit encoded into the pair of anyons with coordinates $\eta_{1}$ and $\eta_{2}$.

Remark 2. The recurrence relations (17)–(19) for $R_{j}^{(n+1,\pm)}$ are identical with relations (28)–(30) in [25], for $B_{j}^{(2n+2,\pm)}$ (see [25] for the notation) despite the fact that $R_{j}^{(n+1,\pm)}$ and $B_{j}^{(2n+2,\pm)}$ generate in principle different representations of the braid group $B_{2n+2}$, and this fact has been used in the proof of the Nayak–Wilczek conjecture there. Without equations (17)–(19) the proof of the Nayak–Wilczek conjecture in [25] would be logically incomplete.

\[^{3}\text{The validity of the recurrence relations (17)–(19) for } n = 2 \text{ could be directly checked from equation (6).}\]
4.2. Explicit formulae for the projected braid matrices

Using equation (4) for the $\gamma$ matrices, and relations (17)–(19), we can obtain most of the projected braid matrices $R_{j}^{(n+1,\pm)}$ explicitly

$$R_{2j-1}^{(n+1,\pm)} = \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2 \otimes \left[ \begin{array}{cc} 1 & 0 \\ 0 & i \end{array} \right] \otimes \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2,$$

for $1 \leq j \leq n$,

$$R_{2j}^{(n+1,\pm)} = \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2 \otimes \frac{e^{i\pi/4}}{\sqrt{2}} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & -i & 1 \end{array} \right] \otimes \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2,$$

for $n \geq 2$ and $1 \leq j \leq n - 1$. These has to be supplemented by the recurrence relations for the last two generators which do not have a tensor product structure, however, still can be constructed recursively from

$$R_{2n}^{(n+1,\pm)} = R_{2n-2}^{(n,\pm)} \oplus R_{2n-2}^{(n,\mp)}, \quad R_{2n+1}^{(n+1,\pm)} = R_{2n-1}^{(n,\pm)} \oplus R_{2n-1}^{(n,\mp)}$$

(23)

using as a base the matrices $R_{2}^{(2,\pm)}$ and $R_{3}^{(2,\pm)}$ defined in equations (9) and (10).

Equations (21)–(23) give the most explicit and practical description of the projected braid generators $R_{j}^{(n+1,\pm)}$.

4.3. Proof of the computational equivalence

It is obvious from equations (21) and (22) that $R_{j}^{(n+1,+) = R_{j}^{(n+1,-)}}$ for $1 \leq j \leq 2n - 1$, as stated in equation (17), so that the two inequivalent spinor representations of $\mathcal{B}_{2n+2}$ differ only in the last two generators, i.e., for $j = 2n, 2n + 1$. It can now be proven that the last two braid generators $R_{2n}^{(n+1,\pm)}$ and $R_{2n+1}^{(n+1,\pm)}$ in the two representations with opposite parity are mutually inverse up to an overall factor of $i$, i.e.,

$$R_{2n}^{(n+1,+) R_{2n}^{(n+1,-)} = i\mathbb{I}_{2^{n}}, \quad and \quad R_{2n+1}^{(n+1,+) R_{2n+1}^{(n+1,-)} = i\mathbb{I}_{2^{n}}.}$$

(24)

Indeed, it follows from equations (31) and (32) in [31] that $(R_{j}^{(n,+)})^{2} = -(R_{j}^{(n,-)})^{2}$ for $j = 2n, 2n + 1$ (while for the other values of $j$ the squares of the braid generators in the two representations coincide). Next, combining equation (35) in [31] with equation (6) in this paper, it is easy to prove that (cf remark 2.5 in [24]),

$$R_{j}^{(n+1,\pm)} = \frac{e^{i\pi/4}}{\sqrt{2}}(\mathbb{I}_{2^{n}} - i(R_{j}^{(n+1,\pm)})^{2}), \quad 1 \leq j \leq 2n + 1.$$
for \( j = 2n, 2n + 1 \). The last equality in the above equation follows from the fact that the unprojected generator could be interpreted as a rotation on \( \pi/2 \) so that \((R_j^{(n+1)})^4 = I_{2n+1}\), hence the fourth power of the projected generators is \((R_j^{(n+1,\pm)})^4 = I_{2n}\) (cf equation (35) in [31]).

The computational equivalence between the two inequivalent spinor representations \( S_\pm \) of \( B_{2n+2} \) formally follows from equations (17) and (24) because the element \( iI_{2n} \) is always an element of the monodromy group \( M_{2n+2} \subset B_{2n+2} \) in the Ising-model representation (see equation (33) in [31]).

Again the explicit construction of the projected braid matrices for \( n \) qubits, from the generators of the spinor representations of \( SO(2n+2) \), assumes a particular basis of computational states. It is not difficult to see that, like in the one-qubit case considered in section 3, the projected diagonal braid matrices (21) and (23) completely fix the \( n \)-qubit’s computational bases in the two inequivalent representations. As obvious from equation (21), the diagonal matrices \( R_{2j-1}^{(n+1,\pm)} \), with \( 1 \leq j \leq n \), represents the phase gate \( S = \text{diag}(1,i) \) on the \( j \)th qubit \((1 \leq j \leq n)\), so that the state of the \( j \)th qubit corresponding to the \( \gamma \)-matrices realization in equation (3), is determined by the pair of Ising anyons with coordinates \( \eta_{2j-1} \) and \( \eta_{2j} \), and the qubits are ordered from left to right as shown in figure 1. The last two anyons, with the coordinates \( \eta_{2n+1} \) and \( \eta_{2n+2} \), form an inert pair which is responsible for restoring the total fermion parity in the two inequivalent representations. In other words, the explicit definition of the \( SO(2n+2) \) \( \gamma \)-matrices, as in equations (3) and (4), already assumes the structure and the ordering of the \( n \)-qubit system as in figure 1.

5. Calibration of the Pfaffian quantum computer

The analysis performed here allows us to unambiguously fix the type of the spinor representation in a real physical sample, calibrating in this way the Pfaffian quantum computer. To this end we propose the following procedure for \( 2n + 2 \) Ising anyons corresponding to \( n \) qubits:

(i) Initialize the \( n \)-qubit system in the state \( |00\cdots 0\rangle \). This could be done by applying the single-qubit initialization scheme of Das Sarma et al [6] for each pair of anyons.

(ii) Measure the total topological charge of the system by Fabry–Perot interferometry [32,33,6]. This charge would be +1 if the system is in the representation \( S_+ \) and -1 if it is in \( S_- \). Because all qubits are in the state \( |0\rangle \) the total topological
charge is equal to the topological charge of the last pair of Ising quasiholes with coordinates $\eta_{2n+1}$ and $\eta_{2n+2}$. We can therefore determine the topological charge of the pair $(\eta_{2n+1}, \eta_{2n+2})$ as shown in figure 2 in the way it was done in the original approach of [6].

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