Normalization of Twisted Alexander Invariants

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Abstract

Twisted Alexander invariants of knots are well-defined up to multiplication of units. We get rid of this multiplicative ambiguity via a combinatorial method and define normalized twisted Alexander invariants. We can show that the invariants coincide with sign-determined Reidemeister torsion in a normalized setting and refine the duality theorem. As an application, we obtain stronger necessary conditions for a knot to be fibered than those previously known. Finally, we study a behavior of the highest degree of the normalized invariant.

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1 Introduction

Twisted Alexander invariants, which coincide with Reidemeister torsion ([Ki], [KL]), were introduced for knots in the 3-sphere by Lin [L] and generally for finitely presentable groups by Wada [Wad]. They were given a natural topological definition by using twisted homology groups in the notable work of Kirk and Livingston [KL]. Many properties of the classical Alexander polynomial $\Delta_K$ were subsequently extended to the twisted case and it was shown that the invariants have much information on the topological structure of a space. For example, necessary conditions of twisted Alexander invariants for a knot to be fibered were given by Cha [C], Goda-Morifuji [GM], Goda-Kitano-Morifuji [GKM] and Friedl-Kim [FK]. Moreover, even sufficient conditions for a knot with genus 1 to be fibered were obtained by Friedl-Vidussi [FV].

It is well known that $\Delta_K$ can be normalized, for instance, by considering the skein relation. In this paper, we first obtain the corresponding result in twisted settings. The twisted Alexander invariant $\Delta_{K,\rho}$ associated to a linear representation $\rho$ is well-defined up to multiplication of units in a Laurent polynomial ring. We show that the ambiguity can be eliminated via a combinatorial method constructed by Wada and define the normalized twisted Alexander invariant $\tilde{\Delta}_{K,\rho}$ (See Definition 4.4 and Theorem 4.5).

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Turaev [T2] defined sign-determined Reidemeister torsion by refining the sign ambiguity of Reidemeister torsion for an odd-dimensional manifold and showed that the other ambiguity depends on the choice of Euler structures. We also normalize sign-determined Reidemeister torsion $T_{K, \rho}$ for a knot and define $\tilde{T}_{K, \rho}(t)$. Then we prove the equality
\[ \tilde{\Delta}_{K, \rho}(t) = T_{K, \rho}(t) \]
(Theorem 5.7). This shows that $\tilde{\Delta}_{K, \rho}$ is a simple homotopy invariant and give rise to a refined version (Theorem 5.9) of the duality theorem for twisted Alexander invariants.

As an application, we generalize above results for fibered knots. We can define the highest degree and the coefficient of the highest degree term of $\tilde{\Delta}_{K, \rho}$. We show that these values are completely determined for fibered knots (See Theorem 6.3). Finally, we obtain the following inequality which bounds free genus $g_f(K)$ from below by using the highest degree $h$-deg $\tilde{\Delta}_{K, \rho}$:
\[ 2 \text{h-deg} \tilde{\Delta}_{K, \rho} \leq n(2g_f(K) - 1). \] (1.1) (See Theorem 6.6.)

This paper is organized as follows. In the next section, we first review the definition of twisted Alexander invariants for knots. We also describe how to compute them from a presentation of a knot group and the duality theorem for unitary representations. In Section 3, we review Turaev’s sign-determined Reidemeister torsion and the relation with twisted Alexander invariants. In Section 4, we establish normalization of twisted Alexander invariants. In Section 5, we refine the correspondence with sign-determined Reidemeister torsion and the duality theorem for twisted Alexander invariants. Section 6 is devoted to applications. Here we generalize the result of Cha [C], Goda-Kitano-Morifuji [GKM] and Friedl-Kim [FK] for fibered knots and study a behavior of the highest degree and obtain (1.1).

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2 Twisted Alexander invariants

In this section, we review twisted Alexander invariants of $K$ following [C] and [KL]. For a given oriented knot $K$ in $S^3$, let $E_K := S^3 \setminus N(K)$, where $N(K)$ denotes an open tubular neighborhood of $K$ and $G_K := \pi_1 E_K$. We fix an element $\mu \in G_K$ represented by a meridian of $E_K$ and denote by $\alpha: G_K \to \langle t \rangle$ be the abelianization homomorphism which maps $\mu$ to the generator $t$. Let $R$ be a Noetherian unique factorization domain and $Q(R)$ the quotient field of $R$.

We first give a definition of a twisted homology group and a twisted cohomology group. Let $X$ be a connected CW-complex and $\tilde{X}$ the universal covering of $X$. The chain complex $C_*(\tilde{X})$ is
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a left $\mathbb{Z}[\pi_1 X]$-module via the action of $\pi_1 X$ as the deck transformations of $\widetilde{X}$. We regard $C_*(\widetilde{X})$ also as a right $\mathbb{Z}[\pi_1 X]$-module by defining $\sigma \cdot \gamma := \gamma^{-1} \cdot \sigma$, where $\gamma \in \pi_1 X$ and $\sigma \in C_*(\widetilde{X})$. For a linear representation $\rho: \pi_1 X \to GL_n(R)$, $R^{\mathbb{Z}}$ naturally has a left $\mathbb{Z}[\pi_1 X]$-module structure. We define the twisted homology group $H_i(X; R^{\mathbb{Z}}_\rho)$ and the twisted cohomology group $H^i(X; R^{\mathbb{Z}}_\rho)$ of $\rho$ as follows:

$$H_i(X; R^{\mathbb{Z}}_\rho) := H_i(C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} R^\mathbb{Z}),$$
$$H^i(X; R^{\mathbb{Z}}_\rho) := H^i(\text{Hom}_{\mathbb{Z}[\pi_1 X]}(C_*(\widetilde{X}), R^\mathbb{Z})).$$

Definition 2.1. For a representation $\rho: G_K \to GL_n(R)$, we define $\Delta^i_{K, \rho}$ to be the order of the $i$-th twisted homology group $H_i(E_K; R[t, \tau^{-1}]^{\mathbb{Z}\otimes_{\mathbb{Q}}})$, where we consider $R[t, \tau^{-1}]^{\mathbb{Z}} = R[t, \tau^{-1}] \otimes R^\mathbb{Z}$. It is called the $i$-th twisted Alexander polynomial associated to $\rho$, which is well-defined up to multiplication by a unit in $R[t, \tau^{-1}]$. We furthermore define

$$\Delta_{K, \rho} := \Delta^{1}_{K, \rho}/\Delta^{0}_{K, \rho} \in Q(R)(t).$$

It is called the twisted Alexander invariant associated to $\rho$ and well-defined up to a factor $\eta^l$, where $\eta \in R^\times$ and $l \in \mathbb{Z}$.

Remark 2.2. Lin’s twisted Alexander polynomial defined in [L] coincides with $\Delta^1_{K, \rho}$.

The homomorphisms $\alpha$ and $\alpha \otimes \rho$ induce ring homomorphisms $\tilde{\alpha}: \mathbb{Z}[G_K] \to \mathbb{Z}[t, \tau^{-1}]$ and $\Phi: \mathbb{Z}[G_K] \to M_n(R[t, \tau^{-1}])$. For a knot diagram of $K$, we choose and fix a Wirtinger presentation $G_K = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_{m-1} \rangle$. Let us consider the $(m - 1) \times m$ matrix $A_\Phi$ whose component is the $n \times n$ matrix $\Phi(\alpha_{ij}) \in M_n(R[t, \tau^{-1}])$, where $\alpha_{ij}$ denotes Fox’s free derivative with respect to $x_i$. For $1 \leq k \leq m$, let us denote by $A_{\Phi, k}$ the $(m - 1) \times (m - 1)$ matrix obtained from $A_\Phi$ by removing the $k$-th column. We regard $A_{\Phi, k}$ as an $(m - 1)n \times (m - 1)n$ matrix with coefficients in $R[t, \tau^{-1}]$.

The twisted Alexander invariants can be computed from a Wirtinger presentation as follows. This is nothing but Wada’s construction in [Wad].

Theorem 2.3 ([HLN], [KL]). For a representation $\rho: G_K \to GL_n(R)$ and a Wirtinger presentation $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_{m-1} \rangle$ of $G_K$, we have

$$\Delta_{K, \rho} \equiv \frac{\det A_{\Phi, k}}{\det \Phi(x_k - 1)} \mod \langle \eta^l \rangle_{\eta \in R^\times, l \in \mathbb{Z}}$$

for any index $k$.

Remark 2.4. Wada shows in [Wad] that the twisted Alexander invariant is well-defined up to a factor $\eta^{ln}$. He also shows that in case that $\rho$ is a unimodular representation, the twisted Alexander invariant is well-defined up to a factor $\pm t^{ln}$ if $n$ is odd and up to only $t^{ln}$ if $n$ is even.

It is also known that the twisted Alexander invariants have the following duality. We extend complex conjugation to $\mathbb{C}(t)$ by taking $t \mapsto t^{-1}$.

Theorem 2.5 ([Ki], [KL]). Given a representation $\rho: G_K \to U(n)$ (resp. $O(n)$), we have

$$\Delta_{K, \rho}(t) \equiv \overline{\Delta_{K, \rho}(t)} \mod \langle \eta^l \rangle_{\eta \in R^\times, l \in \mathbb{Z}}.$$
3 Sign-determined Reidemeister torsion

In this section, we review the definition of Turaev’s sign-determined Reidemeister torsion. See [T1], [T2] for more details. For two bases $u$ and $v$ of an $n$-dimensional vector space over a field $F$, $[u/v]$ denotes the determinant of the base change matrix from $v$ to $u$.

Let $C_\ast = (0 \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \xrightarrow{\partial_1} C_0 \to 0)$ be a chain complex of finite dimensional vector spaces over $F$. For given bases $b_i$ of $\text{Im} \partial_{i+1}$ and $h_i$ of $H_i(C_\ast)$, we can choose a basis $b_i \cup \bar{h}_i \cup \bar{b}_{i-1}$ of $C_i$ as follows. First, we choose a lift $\bar{h}_i$ of $h_i$ in $C_{i+1}$ and obtain a basis $b_i \cup \bar{h}_i$ of $\text{Ker} \partial_i$. Consider the exact sequence

$$0 \to \text{Im} \partial_{i+1} \to \text{Ker} \partial_i \to H_i(C_\ast) \to 0.$$ 

Then we choose a lift $\bar{b}_{i-1}$ of $b_{i-1}$ in $C_i$ and obtain a basis $(b_i \cup \bar{h}_i) \cup \bar{b}_{i-1}$ of $C_i$. Consider the exact sequence

$$0 \to \text{Ker} \partial_i \to C_i \to \text{Im} \partial_i \to 0.$$

**Definition 3.1.** For given bases $c = (c_i)$ of $C_\ast$ and $h = (h_i)$ of $H_\ast(C_\ast)$, we choose a basis $b = (b_i)$ of $\text{Im} \partial_i$ and define

$$\text{Tor}(C_\ast, c, h) := (-1)^{|C_\ast|} \prod_{i=0}^{n} [b_i \cup \bar{h}_i \cup \bar{b}_{i-1} / c_i]^{(-1)^{j+1}} \in F^\times,$$

where

$$|C_\ast| := \sum_{j=0}^{n} \sum_{i=0}^{j} \dim C_i (\sum_{i=0}^{j} \dim H_i(C_\ast)).$$

**Remark 3.2.** It can be easily checked that $\text{Tor}(C_\ast, c, h)$ does not depend on the choices of $b$, $\bar{b}_i$ and $\bar{h}_i$.

Now let us apply the above algebraic torsion to the geometric situations. Let $X$ be a connected finite CW-complex. By a homology orientation of $X$ we mean an orientation of the homology group $H_\ast(X; \mathbb{R}) = \bigoplus_i H_i(X; \mathbb{R})$ as a real vector space.

**Definition 3.3.** For a representation $\rho: \pi_1 X \to GL_n(F)$ such that the twisted homology group $H_\ast(X; F^\text{\(\otimes\rho\)}}$ vanishes and a homology orientation $\circ$, we define the sign-determined Reidemeister torsion $T_\rho(X, \circ)$ of $\rho$ and $\circ$ as follows. We choose a lift $\bar{e}_i$ of each cell $e_i$ in $\bar{X}$ and bases $h$ of $H_\ast(X; \mathbb{R})$ which is positively oriented with respect to $\circ$ and $\langle f_1, \ldots, f_n \rangle$ of $F^\text{\(\otimes\rho\)}}$. Then,

$$T_\rho(X, \circ) := \tau_0 \text{Tor}(\bar{C}_\ast(X), \bar{C}, F^\text{\(\otimes\rho\)}, \bar{e}) \in F^\times,$$

where

$$\tau_0 := \text{sgn Tor}(C_\ast(X; \mathbb{R}), c, h),$$

$c := \langle e_1, \ldots, e_{\dim C_\ast} \rangle,$

$\bar{e} := \langle \bar{e}_1 \otimes f_1, \ldots, \bar{e}_1 \otimes f_n, \ldots, \bar{e}_{\dim C_\ast} \otimes f_1, \ldots, \bar{e}_{\dim C_\ast} \otimes f_n \rangle.$
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Remark 3.4. It is known that \( T_{\rho}(X, \varnothing) \) does not depend on the choice of \( \tilde{e}, h \) and \( \langle f_1, \ldots, f_n \rangle \) and is well-defined as a simple homotopy invariant up to multiplication of an element in \( \text{Im}(\det \rho) \). See [T1].

Here let us consider the knot exterior \( E_K \). In this case, we can equip \( E_K \) with its canonical homology orientation \( \omega_K \) as follows. We have \( H_*(E_K; \mathbb{R}) = H_0(E_K; \mathbb{R}) \oplus \langle t \rangle \) and define \( \omega_K : [\langle pt \rangle, t] \rightarrow [\mathbb{R}^+, \mathbb{R}] \), where \( \langle pt \rangle \) is the homology class of a point.

Definition 3.5. For a representation \( \rho : G_K \rightarrow GL_n(F) \) such that the twisted homology group \( H_*(X; F(t)^{\text{can}}_{\omega_\rho}) \) vanishes, the sign-determined Reidemeister torsion \( T_{K,\rho}(t) \) of \( \rho \) is defined by \( T_{\alpha \otimes \rho}(E_K, \omega_K) \). Here we consider \( \alpha \otimes \rho : G_K \rightarrow GL_n(F[t, t^{-1}]) \hookrightarrow GL_n(F(t)) \).

In the later section, we generalize the following theorem.

Theorem 3.6 ([Ki], [KL]). For a representation \( \rho : G_K \rightarrow GL_n(F) \) such that the twisted homology group \( H_*(X; F(t)^{\text{can}}_{\omega_\rho}) \) vanishes, we have

\[ \Delta_{K,\rho}(t) \equiv T_{K,\rho}(t) \mod \langle \eta^l \rangle_{\eta \in F^*, l \in \mathbb{Z}}. \]

4 Construction

Now we establish one of our main results. We get rid of the multiplicative ambiguity of twisted Alexander invariants via a combinatorial method. For \( f(t) = p(t)/q(t) \in \mathbb{Q}(R)(t) (p, q \in R[t, t^{-1}]) \), we define

\[ \deg f := \deg p - \deg q, \]
\[ \text{h-deg } f := (\text{the highest degree of } p) - (\text{the highest degree of } q), \]
\[ \text{l-deg } f := (\text{the lowest degree of } p) - (\text{the lowest degree of } q), \]
\[ c(f) := \frac{(\text{the coefficient of the highest degree term of } p)}{(\text{the coefficient of the highest degree term of } q)}. \]

We make use of a combinatorial group theoretical approach constructed by Wada in [Wad].

Definition 4.1. Given a finite presentable group \( G = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle \), the operations of the following types for any word \( w \) in \( x_1, \ldots, x_m \), are called the strong Tietze transformations:

Ia. To replace one of the relators \( r_i \) by its inverse \( r_i^{-1} \).

Ib. To replace one of the relators \( r_i \) by its conjugate \( w r_i w^{-1} \).

Ic. To replace one of the relators \( r_i \) by \( r_i r_j \) for any \( j \neq i \).

II. To add a new generator \( y \) and a new relator \( y w^{-1} \). (Namely, the resulting presentation is \( \langle x_1, \ldots, x_m, y \mid r_1, \ldots, r_n, y w^{-1} \rangle \).)

If a presentation is transformable to another by a finite sequence of operations of above types and their inverse operations, we say that the two presentations are strongly Tietze equivalent.
Remark 4.2. The deficiency of $G$ does not change via the strong Tietze transformations.

Wada shows the following lemma.

Lemma 4.3 (\cite{Wad}). All the Wirtinger presentations of a given link in $S^3$ are strongly Tietze equivalent to each other.

Let $\varphi: \mathbb{Z}[G_K] \rightarrow \mathbb{Z}$ be the augmentation homomorphism. (Namely, $\varphi(\gamma) = 1$ for any element $\gamma$ of $G_K$.) For a fixed presentation $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_{m-1} \rangle$ of $G_K$, we denote $A_{\varphi,k}$ and $A_{\tilde{\varphi},k}$ by $(\varphi\left(\frac{\partial r_i}{\partial x_j}\right))_{j \neq k}$ and $(\tilde{\alpha}\left(\frac{\partial r_i}{\partial x_j}\right))_{j \neq k}$ as in Section 2.

We eliminate the ambiguity of $\eta^l$ in Definition 2.1 as follows.

Definition 4.4. Given a representation $\rho: G_K \rightarrow GL_n(R)$, we choose a presentation $\langle x_1, \ldots, x_m \mid r_1, \ldots, r_{m-1} \rangle$ of $G_K$ which is strongly Tietze equivalent to a Wirtinger presentation and an index $1 \leq k \leq m$ such that $\text{h-deg} \alpha(x_k) \neq 0$. Then we define the normalized twisted Alexander invariant associated to $\rho$ as

$$\bar{\Delta}_{K,\rho} := \frac{\delta^n}{(\epsilon t)^d} \det A_{\varphi,k} \in Q(R)(\epsilon^\frac{1}{2})(t^\frac{1}{2}),$$

where

$$\epsilon := \det \rho(\mu),$$

$$\delta := \text{sgn}(\text{h-deg} \alpha(x_k) \det A_{\varphi,k}),$$

$$d := \frac{1}{2}(\text{h-deg} \det A_{\tilde{\varphi},k} + \text{l-deg} \det A_{\tilde{\varphi},k} - \text{h-deg} \alpha(x_k)).$$

Theorem 4.5. $\bar{\Delta}_{K,\rho}$ is an invariant of a linear representation $\rho$.

Proof. From Lemma 4.3, we have to check (i) the independence of the choice of $k$ and (ii) the invariance for each operation of Definition 4.1.

We assume that we can choose another index $k'$ also satisfying the condition $\text{h-deg} \alpha(x_{k'}) \neq 0$. We set

$$\delta' := \text{sgn}(\text{h-deg} \alpha(x_{k'}) \det A_{\varphi,k'}),$$

$$d' := \frac{1}{2}(\text{h-deg} \det A_{\tilde{\varphi},k'} + \text{l-deg} \det A_{\tilde{\varphi},k'} - \text{h-deg} \alpha(x_{k'})).$$

Since

$$\sum_{j=1}^m \frac{\partial r_i}{\partial x_j}(x_j - 1) = r_i - 1,$$

we have

$$\det A_{\varphi,k'} \det \Phi(x_k - 1) = \det \left(\ldots, \Phi \left(\frac{\partial r_i}{\partial x_k}\right) \Phi(x_k - 1), \ldots\right),$$

$$= \det \left(\ldots, - \sum_{j \neq k} \Phi \left(\frac{\partial r_i}{\partial x_j}\right) \Phi(x_j - 1), \ldots\right),$$

$$= \det \left(\ldots, - \Phi \left(\frac{\partial r_i}{\partial x_{k'}}\right) \Phi(x_{k'} - 1), \ldots\right),$$

$$= (-1)^{n(k-k')} \det A_{\varphi,k} \det \Phi(x_{k'} - 1).$$
Similarly, we obtain
\[ \det A_{\tilde{\alpha},k'} \det \tilde{\alpha}(x_k - 1) = (-1)^{k-k'} \det A_{\tilde{\alpha},k} \det \tilde{\alpha}(x_{k'} - 1). \]

Hence \( d' = d \). Moreover, by dividing this equality by \((t-1)\) and taking \( t \to 1 \), we can see that
\[ \hdeg \alpha(x_k) \det A_{\varphi,k'} = (-1)^{k-k'} \hdeg \alpha(x_{k'}) \det A_{\varphi,k}. \]

Hence \( \delta' = (-1)^{k-k'} \delta \). This concludes the proof of (i).

Next, we consider the strong Tietze transformations. Since
\[ \frac{\partial (r_i^{-1})}{\partial x_j} = -r_i \frac{\partial r_i}{\partial x_j}, \]
\[ \frac{\partial (wr_i^{-1})}{\partial x_j} = w \frac{\partial r_i}{\partial x_j}, \]
\[ \frac{\partial (rr_i)}{\partial x_j} = \frac{\partial r_i}{\partial x_j} + r_i \frac{\partial r_i}{\partial x_j}, \]
the changes of each value by the transformation Ia, Ib and Ic are as follows. By the transformation Ia, \( \det A_{\Phi,k} \mapsto (-1)^n \det A_{\Phi,k}, \delta \mapsto -\delta \) and \( d \) does not change. By the transformation Ib, \( \det A_{\Phi,k} \mapsto (\epsilon t^n)^{\deg \alpha(w)} \det A_{\Phi,k}, \delta \) does not change and \( d \mapsto d + \deg \alpha(w) \). By the transformation Ic and II, it is easy to see that all the values do not change. This concludes the proof of (ii). \( \square \)

From the construction, the following lemma holds.

**Lemma 4.6.** (i) For a representation \( \rho: G_K \to GL_n(R) \),
\[ \Delta_{K,\rho}(t) \equiv \tilde{\Delta}_{K,\rho}(t) \mod \langle \epsilon^\frac{1}{2}, \eta t^\frac{1}{2} \rangle_{\rho \in R', \rho \in \mathbb{Z}}. \]

(ii) If \( \rho \) is trivial (i.e., \( \Phi = \tilde{\alpha} \)),
\[ \nabla_K(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\tilde{\Delta}_{K,\rho}(t), \]
where \( \nabla_K(z) \) is the Conway polynomial of \( K \).

**Proof.** Since (i) is clear from Theorem [2.3] and the definition, we prove (ii). We set
\[ f(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\tilde{\Delta}_{K,\rho}(t). \]

It is easy to see that
\[ f(t) \equiv \Delta_K(t) \mod \langle \pm t \rangle. \]

Moreover, we can check that
\[ f(1) = 1, \]
\[ h-deg f + l-deg f = 0, \]
which establishes the formula. \( \square \)
5 Relation to sign-determined Reidemeister torsion

In this section, we generalize Theorem 2.5 and Theorem 3.6. Here we only consider the case that $R$ is a field $F$.

First, we also normalize sign-determined Reidemeister torsion as twisted Alexander invariants.

**Definition 5.1.** For a representation $\rho : G_K \to GL_n(F)$ such that the twisted homology group $H_*(E_K; F(t)^{\oplus n} \otimes \alpha \otimes \rho)$ vanishes, we define $\tilde{T}_{K, \rho}(t)$ as follows. We choose a lift $\tilde{e}_i$ in $\tilde{E}_K$ of each cell $e_i$, bases $h$ of $H_*(E_K; \mathbb{R})$ which is positively oriented with respect to $\omega_K$ and $\langle f_1, \ldots, f_n \rangle$ of $F(t)^{\oplus n}$. Then

$$\tilde{T}_{K, \rho}(t) := \frac{\tau_0}{(\epsilon t)^{d'}} \text{Tor}(C_*(\tilde{E}_K) \otimes_{\alpha \otimes \rho} F(t)^{\oplus n}, \tilde{c}) \in F(t)^{\times},$$

where

$$\epsilon := \det \rho(\mu),$$
$$\tau_0 := \text{sgn} \text{Tor}(C_*(E_K; \mathbb{R}), c, \hat{h}),$$
$$d' := \frac{1}{2}(h\text{-deg Tor}(C_*(\tilde{E}_K) \otimes_{\alpha} Q(t), \tilde{c}_0) + 1\text{-deg Tor}(C_*(\tilde{E}_K) \otimes_{\alpha} Q(t), \tilde{c}_0)), $$
$$c := \langle e_1, \ldots, e_{\text{dim}C} \rangle,$n
$$\tilde{c}_0 := \langle \tilde{e}_1 \otimes 1, \ldots, \tilde{e}_{\text{dim}C} \otimes 1 \rangle,$n
$$\tilde{c} := \langle \tilde{e}_1 \otimes f_1, \ldots, \tilde{e}_1 \otimes f_n, \ldots, \tilde{e}_{\text{dim}C} \otimes f_1, \ldots, \tilde{e}_{\text{dim}C} \otimes f_n \rangle.$$

**Remark 5.2.** We can also define normalized Reidemeister torsion for a link by a similar method.

One can prove the following lemma by a similar way as in the non-normalized case. As a reference, see [T1].

**Lemma 5.3.** $\tilde{T}_{K, \rho}$ is invariant under homology orientation preserving simple homotopy equivalence.

**Remark 5.4.** From the result of Waldhausen [Wal], the Whitehead group $Wh(G_K)$ is trivial for a knot group $G_K$ in general. Therefore homotopy equivalence between finite CW-complexes whose fundamental groups are knot groups is simple homotopy equivalence.

Let $F$ be a field with (possibly trivial) involution $f \mapsto \bar{f}$. We extend the involution to $F(t)$ by taking $t \mapsto t^{-1}$. We equip $F(t)^{\oplus n}$ with the standard hermitian inner product $(\cdot, \cdot)$ defined by

$$(v, w) := t^*v\bar{w},$$

where $v, w \in F(t)^{\oplus n}$ and $t^*$ is the transpose of $v$. For a representation $\rho : G_K \to GL_n(F)$, we define a representation $\rho^\dagger : G_K \to GL_n(F)$ by

$$\rho^\dagger(\gamma) := \rho(\gamma^{-1})^*,$$

where $\gamma \in G_K$ and $A^* := tA$ for a matrix $A$.

We can also refine the duality theorem for sign-determined Reidemeister torsion as follows.
Theorem 5.5. If the twisted homology group \( H_s(E_K; F(t)_{\alpha \otimes \rho}) \) vanishes for a representation \( \rho: G_K \to GL_n(F) \), then so does \( H_s(E_K; F(t)_{\alpha \otimes \rho}) \) and we have

\[ \widetilde{T}_{K,\rho}(t) = (-1)^n \overline{T}_{K,\rho}(t). \]

The proof is based on the following observation. Let \( (E'_K, \{e'_i\}) \) denote the PL manifold \( E_K \) with the dual cell structure and choose a lift \( \tilde{e}'_i \) which is the dual of \( e'_i \). In the remainder of this section, for abbreviation, we write

\[ C_q := C_q(\tilde{E}_K) \otimes_{\alpha} \mathbb{Q}(t), \quad C'_{q,\rho} := C_q(\tilde{E}_K) \otimes_{\alpha \otimes \rho} F(t)_{\alpha \otimes \rho}^{\otimes n}, \]
\[ C''_{q,\rho} := C_q(\tilde{E}_K) \otimes_{\alpha \otimes \rho} F(t)_{\alpha \otimes \rho}^{\otimes n}, \]
\[ D_q := C_q(\tilde{E}'_K) \otimes_{\alpha} \mathbb{Q}(t), \quad D'_{q,\rho} := C_q(\tilde{E}'_K) \otimes_{\alpha \otimes \rho} F(t)_{\alpha \otimes \rho}^{\otimes n}, \]
\[ B'_q := \text{Im}(\partial: C'_{q+1} \to C'_q), \quad B''_{q,\rho} := \text{Im}(\partial: C''_{q+1} \to C''_{q}). \]

Note that since direct computation gives

\[ H_s(\partial E_K; F(t)_{\alpha \otimes \rho}) = 0 \]

(See, for example, [KL, Subsection 3.3.]), we have

\[ \dim B'_{p,i} = \sum_{j=0}^{i} (-1)^{i-j} \dim C'_{p,j} \]
\[ = \sum_{j=0}^{i} (-1)^{i-j} n \dim C'_j = n \dim B'_i \quad (5.2) \]

Similarly, if \( H_s(E_K; F(t)_{\alpha \otimes \rho}) = 0 \), then from (5.1) and the long exact sequence of the pair \( (E_K, \partial E_K), H_s(E_K, \partial E_K; F(t)_{\alpha \otimes \rho}) = 0 \) and so

\[ \dim B''_{p,i} = n \dim B''_i. \quad (5.3) \]

The well known inner product

\[ [\cdot, \cdot]: C_q(\tilde{E}'_K) \times C_{3-q}(\tilde{E}_K, \partial \tilde{E}_K) \to \mathbb{Z}[G_K] \]

(See, for example, [M, Lemma 2.1.]) defined by

\[ [\tilde{e}'_i, \tilde{e}_j] := \sum_{\gamma \in G_K} (\tilde{e}'_i, \tilde{e}_j \cdot \gamma^{-1}) \gamma, \]

where \( \cdot, \cdot \) denote the intersection number, induces an inner product

\[ \langle \cdot, \cdot \rangle: D_{p,q} \times C''_{p,3-q} \to C(t) \]
defined by
\[ \langle \tilde{e}_i' \otimes v, \tilde{e}_j \otimes w \rangle := \langle v, [\tilde{e}_i', \tilde{e}_j] \cdot w \rangle, \]
where \( v, w \in \mathbb{C}(t)^{\oplus n} \). We see at once that this is well-defined. Thus
\[ D_{\rho, q} \cong (C'_{\rho, 3-q})^*. \quad (5.4) \]
The differential \( \partial_q \) of \( D_{\rho, q} \) corresponds with \((-1)^q \partial_{3-q}' \) of \( (C''_{\rho, 3-q})^* \) under this isomorphism. Similarly, we have
\[ D_q \cong (C''_{3-q})^*. \quad (5.5) \]

**Lemma 5.6.** For any representation \( \rho \colon G_K \to GL_n(F) \),
\[ H_q(E_K; F(t)^{\oplus n}_{\alpha \otimes p}) \cong H_{3-q}(E_K; F(t)^{\oplus n}_{\alpha \otimes p})^*. \]

**Proof.** From (5.4) and the universal coefficient theorem, we can see that
\[ H_q(E_K; F(t)^{\oplus n}_{\alpha \otimes p}) \cong H_{3-q}(E_K, \partial E_K; F(t)^{\oplus n}_{\alpha \otimes p})^*. \]

From (5.1) and the long exact sequence of the pair \( (E_K, \partial E_K) \),
\[ H_* (E_K; F(t)^{\oplus n}_{\alpha \otimes p}) \cong H_*(E_K, \partial E_K; F(t)^{\oplus n}_{\alpha \otimes p}). \]
This completes the proof. \( \square \)

Now we prove the theorem.

**Proof of Theorem [5.5]** Lemma [5.6] gives the first assertion. We use the notation of Definition [5.1]. We choose an orthonormal basis \( \langle f_1, \ldots, f_n \rangle \) of \( F(t)^{\oplus n} \) with respect to the hermitian product \( (\cdot, \cdot) \) defined above. Let \( c', c'', c'_0, c''_0, \tilde{c} \) and \( \tilde{c}'' \) be induced bases of \( C_*(\partial E_K), C_*(E_K, \partial E_K), C'_*, C''_*, C'_{\rho, s} \) and \( C''_{\rho, s} \) by \( c, c_0 \) and \( \tilde{c} \). We set
\[ c^* := \langle e_1', \ldots, e_{\dim C_*} \rangle, \]
\[ c_0^* := \langle e_1' \otimes 1, \ldots, e_{\dim C_*} \otimes 1 \rangle, \]
\[ \tilde{c}^* := \langle \tilde{e}_1' \otimes f_1, \ldots, \tilde{e}_1' \otimes f_n, \ldots, \tilde{e}_{\dim C_*} \otimes f_1, \ldots, \tilde{e}_{\dim C_*} \otimes f_n \rangle. \]

From (5.4) and the duality for algebraic torsion ([12] Theorem 1.9]),
\[ \Tor(D_{\rho, s}, \tilde{c}^*) = (-1)^{\sum \dim B'_{\rho, s-1} \dim B''_{\rho, s}} \Tor(C''_{\rho, s}, \tilde{c}''). \]
On the other hand, from the exact sequence
\[ 0 \to C'_{\rho, s} \to C_{\rho, s} \to C''_{\rho, s} \to 0 \]
and the multiplicativity for algebraic torsion ([12] Theorem 1.5]),
\[ \Tor(C_{\rho, s}, \tilde{c}) = (-1)^{\sum \dim B'_{\rho, s-1} \dim B''_{\rho, s}} \Tor(C'_{\rho, s}, \tilde{c}'). \Tor(C''_{\rho, s}, \tilde{c}''). \]
Therefore, we obtain
\[ \text{Tor}(C_{\rho, \ast}, \tilde{c}) = (-1)^{\sum_i (\dim B^{i' \ast \ast} + \dim B^{i' \ast} + \dim B^{i'})} \text{Tor}(C_{\rho, \ast}, \tilde{c}') \text{Tor}(D_{\rho, \ast}, \tilde{c}). \] (5.6)

Similarly,
\[ \text{Tor}(C_{\ast}, \tilde{c}_0) = (-1)^{\sum_i (\dim B^{i' \ast \ast} + \dim B^{i' \ast} + \dim B^{i'})} \text{Tor}(C_{\ast}, \tilde{c}_0') \text{Tor}(D_{\ast}, \tilde{c}_0). \] (5.7)

We set
\[ d'' := \frac{1}{2} (h\text{-deg Tor}(C_{\ast}, \tilde{c}_0') + 1\text{-deg Tor}(C_{\ast}, \tilde{c}_0)) \]
\[ d^* := \frac{1}{2} (h\text{-deg Tor}(D_{\ast}, \tilde{c}_0') + 1\text{-deg Tor}(D_{\ast}, \tilde{c}_0')). \]

From (5.7), we have
\[ d' = d'' - d^*. \] (5.8)

From Lemma 4.6(ii) and Theorem 5.7,
\[ \lim_{t \to 1} \tau_0(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \text{Tor}(C_{\ast}, \tilde{c}_0) = -\nabla_K(0) \]
\[ = -1. \]

Similarly,
\[ \lim_{t \to 1} \tau_0^*(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \text{Tor}(D_{\ast}, \tilde{c}_0^*) = -1, \]

where
\[ \tau_0^* := \text{sgn Tor}(C_{\ast}(E_{K}; \mathbb{R}), c^*, h). \]

By multiply (5.7) by \((t^{\frac{1}{2}} - t^{-\frac{1}{2}})\) and taking \(t \to 1\), we obtain
\[ \tau_0 = -(-1)^{\sum_i (\dim B^{i' \ast \ast} + \dim B^{i' \ast} + \dim B^{i'})} \tau_0^* \tau_0^*, \] (5.9)

where
\[ \tau_0^* := \lim_{t \to 1} \text{Tor}(C_{\ast}, \tilde{c}_0'). \]

From (5.2), (5.3), (5.6), (5.8) and (5.9),
\[ \widetilde{K}_{\rho}(t) = \frac{\tau_0''}{(et^n)^d} \text{Tor}(C_{\rho, \ast}, \tilde{c}) \]
\[ = (-1)^n (\tau_0')^n \text{Tor}(C_{\rho, \ast}, \tilde{c}') \cdot (\tau_0')^n \text{Tor}(D_{\rho, \ast}, \tilde{c}'). \]

Direct computation gives
\[ \text{Tor}(C_{\ast}, \tilde{c}_0') = \tau_0' d''. \]

(See, for example, [KL, Subsection 3.3.]) Since the normalized invariants do not change via conjugation of representations, we can assume \(\rho(\mu)\) and \(\rho(\lambda)\) are diagonal matrices. This deduces
\[ \text{Tor}(C_{\rho, \ast}, \tilde{c}') = (\tau_0')^n (et^n)^{d''}. \]
Thus
\[ \frac{(\tau'_0)^n}{(\epsilon t_n)^d'} \operatorname{Tor}(C'_{\rho_t}, \check{c}') = 1. \]

It can be easily seen that
\[ \frac{(\tau^*_0)^n}{(\epsilon t_n)^d'} \operatorname{Tor}(D_{\rho_t}, \check{c}^*) = \check{T}_{K,\rho}(t). \]

This proves the theorem. \( \square \)

In the normalized setting, Theorem 3.6 also holds.

**Theorem 5.7.** For a representation \( \rho: G_K \to GL_n(F) \) such that the twisted homology group \( H_*(E_K; F(t)_{\delta \phi^p}) \) vanishes, we have
\[ \check{\Delta}_{K,\rho}(t) = \check{T}_{K,\rho}(t). \]

**Proof.** We choose a Wirtinger presentation \( G_K = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_{m-1} \rangle \) and take the CW-complex \( W \) corresponding with the presentation. Namely, \( W \) has one vertex, \( m \) edges and \( (m - 1) \) 2-cells attached by the relations \( r_1, \ldots, r_{m-1} \). Let the words \( x_1, \ldots, x_m \) and \( r_1, \ldots, r_{m-1} \) also denote the cells. It is easy to see that the exterior \( E_K \) collapses to \( W \). This implies that \( W \) is simple homotopy equivalent to \( E_K \) from Remark 5.4. Thus we can compute the normalized torsion \( \check{T}_{K,\rho} \) as that of \( W \) from Lemma 5.3.

Let \( c_0 = pt, c_1 = \langle x_1, \ldots, x_m \rangle \) and \( c_2 = \langle r_1, \ldots, r_{m-1} \rangle \). We choose \( b_1 = \partial c_2 \) and \( h_0 = [pt], h_1 = [x_k] \) \((1 \leq k \leq m)\). Then
\[ \tau_0 = \operatorname{sgn}(-1)^{C_*(W;\mathbb{R})} \frac{[b_1 \cup \check{h}_1/c_1]}{[h_0/c_0][\check{b}_1/c_2]} \]
\[ = -\operatorname{sgn} \left| \begin{array}{ccc}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\cdots & \cdots & \cdots \\
1 & \cdots & 0 \\
0 & \cdots & 0
\end{array} \right| \]
\[ = (-1)^{k+m+1} \delta. \]
We define an involution \( \tilde{\gamma} : \mathbb{Z}[G_K] \to \mathbb{Z}[G_K] \) by extending the inverse operation \( \gamma \mapsto \gamma^{-1} \) of \( G_K \) linearly. We can choose lifts \( \tilde{pt}, \tilde{x}_i \) and \( \tilde{r}_j \) such that \( C_*(\tilde{W}) \otimes_{\alpha \otimes \rho} F(t)^{\sum} \) is

\[
0 \to \bigoplus_{1 \leq j \leq m-1, 1 \leq i \leq n} F(t)(\tilde{r}_j \otimes f_i) \overset{d_2}{\rightarrow} \bigoplus_{1 \leq j \leq m, 1 \leq i \leq n} F(t)(\tilde{x}_i \otimes f_i) \overset{d_1}{\rightarrow} \bigoplus_{1 \leq i \leq n} F(t)(\tilde{pt} \otimes f_i) \to 0,
\]

where

\[
\tilde{d}_1(\tilde{x}_i \otimes f_i) = \tilde{pt} \otimes \Phi(\tilde{x}_i - 1)f_i,
\]

\[
\tilde{d}_2(\tilde{r}_j \otimes f_i) = \sum_{i=1}^n \tilde{x}_i \otimes \Phi \left( \frac{\partial r_j}{\partial x_i} \right) f_i.
\]

Let \( c'_0 = \langle \tilde{pt} \otimes f_1, \ldots, \tilde{pt} \otimes f_n \rangle \), \( c'_1 = \langle \tilde{x}_1 \otimes f_1, \ldots, \tilde{x}_1 \otimes f_n, \ldots, \tilde{x}_m \otimes f_1, \ldots, \tilde{x}_m \otimes f_n \rangle \) and \( c'_2 = \langle \tilde{r}_1 \otimes f_1, \ldots, \tilde{r}_1 \otimes f_n, \ldots, \tilde{r}_m \otimes f_1, \ldots, \tilde{r}_m \otimes f_n \rangle \). We choose \( b'_0 = \tilde{d}(\tilde{x}_k \otimes f_1, \ldots, \tilde{x}_k \otimes f_n) \) and \( b'_1 = \tilde{d}c'_2 \). Since the twisted homology group \( H_*(W; F(t)^{\sum}) \) vanishes, \( |C_*(\tilde{W}) \otimes_{\alpha \otimes \rho} F(t)^{\sum}| = 0 \) and so

\[
\text{Tor}(C_*(\tilde{W}) \otimes_{\alpha \otimes \rho} F(t)^{\sum}, \langle \tilde{c}_0, \tilde{c}_1, \tilde{c}_2 \rangle) = \frac{[b'_1 \cup \tilde{b}'_0 / c'_1]}{[b'_0 / c'_0][b'_1 / c'_2]}
\]

\[
\det \left( \Phi \left( \frac{\partial r_j}{\partial x_i} \right) \right) I = \frac{\det \Phi(\tilde{x}_k - 1)}{\det \Phi(\tilde{x}_k - 1)}.
\]

Similarly, we have

\[
\text{Tor}(C_*(\tilde{W}) \otimes_{\alpha} \mathbb{Q}(t), \langle \tilde{c}_0, \tilde{c}_1, \tilde{c}_2 \rangle) = (-1)^{(k+m)} \frac{\det \left( \tilde{\alpha} \left( \frac{\partial r_j}{\partial x_i} \right) \right)}{\det \tilde{\alpha}(\tilde{x}_k - 1)}.
\]

Hence \( d' = -d \) and so

\[
\overline{\tilde{T}}_{K,\rho}(t) = (-1)^n \bar{\Delta}_{K,\rho}(t),
\]

where we consider the trivial involution on \( F \). From Theorem 5.5, we obtain the desired formula.

From the above theorems and the following lemma, we have the duality theorem for normalized twisted Alexander invariants.
Lemma 5.8. If $H_*(E_K; F(t)^{a\otimes p})$ does not vanish, then we have
\[
\tilde{\Delta}_{K,\rho}(t) = \tilde{\Delta}_{K,\rho'}(t) = 0.
\]

Proof. If $H_*(E_K; F(t)^{a\otimes p})$ does not vanish, then neither does $H_*(E_K; F(t)^{a\otimes p})$ from Lemma 5.6. Since
\[
\sum_{q=0}^2 \dim H_q(E_K; F(t)^{a\otimes p}) = n\chi(E_K)
\]
from the assumption and (5.1), we have $H_1(E_K; F(t)^{a\otimes p}) \neq 0$ and so $\tilde{\Delta}_{K,\rho}(t) = 0$. Similarly, we obtain $\tilde{\Delta}_{K,\rho'}(t) = 0$, which proves the lemma. □

Theorem 5.9. Given a representation $\rho : G_K \to GL_n(F)$, we have
\[
\tilde{\Delta}_{K,\rho'}(t) = (-1)^q \tilde{\Delta}_{K,\rho}(t).
\]

For a unitary representation $\rho$, the difference between the highest coefficient of $\Delta_{K,\rho}(t)$ and the lowest coefficient of it is not clear from Theorem 2.5 because of the ambiguity. However, this difference is strictly determined from the following corollary.

Corollary 5.10. For a representation $\rho : G_K \to U(n)$ or $O(n)$, we have
\[
\tilde{\Delta}_{K,\rho}(t) = (-1)^q \tilde{\Delta}_{K,\rho}(t).
\]

Example 5.11. Let $K$ be the $(p, q)$ torus knot $(p, q > 1$ and $(p, q) = 1)$. It is well known that the knot group has a presentation
\[G_K = \langle x, y \mid x^p y^{-q}\rangle,\]
where $h$-deg $\alpha(x) = q$ and $h$-deg $\alpha(y) = p$. The 2-dimensional complex $W$ corresponding with this presentation is $K(G_K, 1)$. Therefore we can use this presentation for the computation via Lemma 5.3, Remark 5.4 and Theorem 5.7.

From the result of Klassen [K], all the irreducible $SU(2)$-representations up to conjugation are given as follows:

\[\rho_{a,b,s} : G_K \to SU(2) :\]
\[
x \mapsto \left( \begin{array}{cc}
\cos \frac{aq}{b} + i \sin \frac{aq}{b} & 0 \\
0 & \cos \frac{aq}{p} - i \sin \frac{aq}{p}
\end{array} \right),
\]
\[
y \mapsto \left( \begin{array}{cc}
\cos \frac{bs}{q} + i \sin \frac{bs}{q} \cos \pi s & \sin \frac{bs}{q} \sin \pi s \\
-\sin \frac{bs}{q} \sin \pi s & \cos \frac{bs}{q} - i \sin \frac{bs}{q} \cos \pi s
\end{array} \right),
\]
where $a, b \in \mathbb{N}$, $1 \leq a \leq p - 1$, $1 \leq b \leq q - 1$, $a \equiv b \mod 2$ and $0 < s < 1$. The normalized twisted Alexander invariants of the torus knot for these representations are as follows:
\[
\tilde{\Delta}_{K,\rho_{a,b,s}}(t) = \frac{(t^{\frac{bn}{q}} - (-1)^q t^{-\frac{bn}{q}})^2}{(t^p - 2 \cos \frac{bn}{q} + t^{-p})(t^q - 2 \cos \frac{aq}{p} + t^{-q})}.
\]
6 Applications

Now we consider applications of the normalized invariants. First we generalize the result of Goda-Kitano-Morifuji and Friedl-Kim. We denote by $g(K)$ the genus of $K$.

Their results are as follows.

**Theorem 6.1 ([GKM]).** For a fibered knot $K$ and a unimodular representation $\rho : G_K \to SL_2n(F)$, $c(\Delta_{K,\rho})$ is well-defined and is 1.

**Theorem 6.2 ([C], [FK]).** For a fibered knot $K$ and a representation $\rho : G_K \to GL_n(R)$, $\Delta^1_{K,\rho}$ is a monic polynomial and $\deg \Delta_{K,\rho} = n(2g(K) - 1)$, where “monic” means that the highest and lowest coefficients of a polynomial are units.

In the normalized setting, we have the following theorem.

**Theorem 6.3.** For a fibered knot $K$ and a representation $\rho : G_K \to GL_n(R)$,

$$\deg \tilde{\Delta}_{K,\rho} = 2h \deg \tilde{\Delta}_{K,\rho} = n(2g(K) - 1),$$

$$c(\tilde{\Delta}_{K,\rho}) = c(\nabla)^n \epsilon g(K) - \frac{1}{2}.$$

**Proof.** The equality $\deg \tilde{\Delta}_{K,\rho} = n(2g(K) - 1)$ can be obtained from Theorem 6.2. Since we have $\tilde{\Delta}_{K,\rho} = \Delta_{K,\rho}$, where $i$ is the natural inclusion $GL_n(R) \hookrightarrow GL_n(Q(R))$, we can assume $R$ is a field $F$.

Let $\psi$ denote the automorphism of a surface group induced by the monodromy map. We can take the following presentation of the knot group by using the fibered structure:

$$\langle x_1, \ldots, x_{2g}, h \mid r_i := hx_ih^{-1}\psi(x_i)^{-1}, 1 \leq i \leq 2g(K) \rangle$$

where $\alpha(x_i) = 1$ for all $i$ and $\alpha(h) = t$. It is easy to see that the corresponding CW-complex is homotopy equivalent to the exterior $E_K$. Thus we can compute the invariant by using the presentation as in Example 5.11.

Since

$$\frac{\partial r_i}{\partial x_j} = \begin{cases} h - \frac{\partial \psi_s(x_i)}{\partial x_j} & i = j \\ -\frac{\partial \psi_s(x_i)}{\partial x_j} & i \neq j \end{cases},$$

we have

$$\det A_{h,2g(K)+1} = t^{2g(K)} + \cdots + 1,$$

$$\det A_{\Phi,2g+1} = \epsilon 2^{g(K)}t^{2g(K)} + \cdots + (-1)^n \det(\Phi(\frac{\partial \psi_s(x_i)}{\partial x_j})), $$

$$\det \Phi(h-1) = \epsilon t^n + \cdots + (-1)^n.$$

From the classical theorem of Neuwirth which states that the degree of the Alexander polynomial of a fibered knot equals the twice genus, we can determine that the lowest degree term of
the first equality is 1. Since
\[
\delta = \text{sgn} \ c(\nabla K) \nabla K(t^\frac{1}{2} - t^{-\frac{1}{2}}) \bigg|_{t=1} = c(\nabla K)
\]
\[
d = g(K) - \frac{1}{2},
\]
h-deg $\Delta_{K,\rho} = n(g(K) - \frac{1}{2})$ and $c(\Delta_{K,\rho}) = c(\nabla K)^n e^{2g(K)-1}$. \hfill \Box

Next we study a behavior of the highest degree of a normalized invariant.

**Definition 6.4.** A Seifert surface for a knot $K$ is said to be canonical if it is obtained from a diagram of $K$ by applying the Seifert algorithm. The minimum genus over all canonical Seifert surfaces is called the canonical genus and denoted by $g_c(K)$. A Seifert surface $S$ is said to be free if $\pi_1(S^3 \backslash S)$ is a free group. This condition is equivalent to that $S^3 \backslash N(S)$ is a handlebody, where $N(S)$ is an open regular neighborhood of $S$. The minimum genus over all free Seifert surfaces is called the free genus and denoted by $g_f(K)$.

**Remark 6.5.** Since every canonical Seifert surfaces is free, we have the following fundamental inequality:
\[
g(K) \leq g_f(K) \leq g_c(K).
\]

We obtain an estimate of free genus from below via the highest degree of the invariants.

**Theorem 6.6.** For a representation $\rho: \Gamma_K \to GL_n(\mathbb{R})$, the following inequality holds:
\[
2 \text{h-deg } \Delta_{K,\rho} \leq n(g_f(K) - 1).
\]

**Proof.** We choose a free Seifert surface $S$ with genus $g_f(K)$ and take a bicollar $S \times [-1, 1]$ of $S$ such that $S \times 0 = S$. Let $\iota_{\pm}: S \hookrightarrow S^3 \backslash S$ be the embeddings whose images are $S \times \{\pm 1\}$. Picking generator sets $\{a_1, \ldots, a_{2g_f(K)}\}$ of $\pi_1(S)$ and $\{x_1, \ldots, x_{2g_f(K)}\}$ of $\pi_1(S^3 \backslash S)$ and setting $u_i := (\iota_+)_*(a_i)$ and $v_i := (\iota_-)_*(a_i)$ for all $i$, we have a presentation
\[
\langle x_1, \ldots, x_{2g_f(K)}, h \mid r_i := hu_i h^{-1} v_i^{-1}, 1 \leq i \leq 2g_f(K) \rangle
\]
of $G_K$ where $\alpha(x_i) = 1$ for all $i$ and $\alpha(h) = t$.

Collapsing surfaces $S \times \ast$ and the handlebody $S^3 \backslash (S \times [-1, 1])$ to bouquets, we can realize the 2-dimensional complex corresponding with this presentation as a deformation retract of $E_K$. Therefore we can compute the invariant by using the presentation as in Example 5.11.

Since
\[
\frac{\partial r_i}{\partial x_j} = h \frac{\partial u_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j},
\]
we have
\[
\text{h-deg } \Delta_{K,\rho} = \text{h-deg det } A_{\phi,2g_f(K)+1} - n d - n
\]
\[
\leq 2ng_f(K) - nd - n.
\]
The proof is completed by showing that \( d = g_f(K) - \frac{1}{2} \).

Let \( V \) be the Seifert matrix with respect to the basis \([a_1], \ldots, [a_{2g_f(K)}] \in H_1(S; \mathbb{Z})\) and \([a_1]^*, \ldots, [a_{2g_f(K)}]^* \in H_1(S^3 \setminus S; \mathbb{Z})\) the dual basis, i.e. \( \text{lk}([a_i], [a_j]^*) = \delta_{i,j} \). We denote by \( A_{\pm} \) the matrices representing \((\iota_\pm)^* : H_1(S; \mathbb{Z}) \to H_1(S^3 \setminus S; \mathbb{Z})\) with respect to the bases \([a_1], \ldots, [a_{2g_f(K)}]\) and \([x_1], \ldots, [x_{2g_f(K)}]\) and by \( P \) the base change matrix of \( H_1(S^3 \setminus S; \mathbb{Z}) \) from \([x_1], \ldots, [x_{2g_f(K)}]\) to \([a_1]^*, \ldots, [a_{2g_f(K)}]^*\).

It is well known that the matrices representing \((\iota_+)^*\) and \((\iota_-)^* : H_1(S; \mathbb{Z}) \to H_1(S^3 \setminus S; \mathbb{Z})\) with respect to the bases \([a_1], \ldots, [a_{2g_f(K)}]\) and \([a_1]^*, \ldots, [a_{2g_f(K)}]^*\) are \( V \) and \( 'V \). Hence

\[
\det A_{\pm,2g_f(K)+1} = \det(t'A_+ - 'A_-) \\
= \det(tA_+ - A_-) \\
= \det(tPV - P'V) \\
= \pm \det(tV - 'V),
\]

and \( d = g_f(K) - \frac{1}{2} \) as required.

**Example 6.7.** Let \( K \) be the knot 11_{23} illustrated in Figure 1. The normalized Alexander polynomial of \( K \) is \( t^2 - 2t + 3 - 2t^{-1} + t^{-2} \).

![Figure 1](image-url)
Let $\rho: G_K \rightarrow SL_2(\mathbb{F}_2)$ be a nonabelian representation over $\mathbb{F}_2$ defined as follows:

$$
\rho(x_i) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{if } i = 4, 8 \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \text{if } i = 7, 9 \\
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \text{otherwise}
\end{cases}
$$

From them, we have the following:

$$
\tilde{\Delta}_{K,\rho}(t) = t^5 + t + t^{-1} + t^{-5}.
$$

Since $\deg \tilde{\Delta}_{K,\rho} \neq 2(\deg \Delta_K - 1)$, $K$ is not fibered.

Moreover, from Proposition 6.6, $g_f(K) \geq 3$.

Therefore

$$
g_f(K) \geq 3.
$$

On the other hand, we obtain a canonical Seifert surface with genus 3 by applying the Seifert algorithm to the diagram in Figure 1. Thus

$$
g_f(K) \leq g_c(K) \leq 3.
$$

By these inequalities we conclude

$$
g_f(K) = g_c(K) = 3.
$$

Remark 6.8. From the result of Friedl and Kim [FK],

$$
\deg \Delta_{K,\rho} \leq n(2g(K) - 1).
$$

Therefore $g(K)$ also equals 3 in the above example.

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