Limited Information Strategies and Discrete Selectivity

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Pure Math Seminar
Suppose $X$ is a topological space.

- $C_p(X)$ is the space of continuous functions $f : X \rightarrow \mathbb{R}$
- It is endowed with the topology of pointwise convergence.
Suppose $X$ is a topological space.

- $C_p(X)$ is the space of continuous functions $f : X \to \mathbb{R}$
- It is endowed with the topology of pointwise convergence.
- It will be convenient to assume that $X$ is Hausdorff and completely regular (for every closed $C \subseteq X$ and $x \in X \setminus C$, there is a continuous function $f : X \to \mathbb{R}$ so that $f(x) = 0$ and $f[C] = \{1\}$).
Open Neighborhoods of $C_p(X)$

To define the topology on $C_p(X)$ using open sets, we use the following neighborhoods:

**Definition**

If $f : X \to \mathbb{R}$ is continuous, $F = \{x_1, \cdots, x_n\} \subseteq X$, and $\varepsilon > 0$, then

$$[f, F, \varepsilon] = \{ g \in C_p(X) : |g(x_i) - f(x_i)| < \varepsilon \text{ for } 1 \leq i \leq n \}$$

Varying over $F$ and $\varepsilon$ forms a neighborhood basis at $f$. 

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$$[f, F, \varepsilon] = \{g \in C_p(X) : |g(x_i) - f(x_i)| < \varepsilon \text{ for } 1 \leq i \leq n\}$$

Varying over $F$ and $\varepsilon$ forms a neighborhood basis at $f$. 
The concept of a closed discrete selection was isolated by Sanchez and Tkachuk in 2017.

**Closed Discrete Selection**

For every sequence \((U_n : n \in \omega)\) of open subsets of \(X\), there are points \(x_n \in U_n\) so that \(\{x_n : n \in \omega\}\) is closed and discrete.
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Note that if \(X\) is first countable (or even has one point with a countable neighborhood basis), then it cannot have closed discrete selection. The converse is not generally true.
Closed Discrete Selections of Functions

Tkachuk, 2017

Suppose $X$ is Hausdorff and completely regular. Then the following are equivalent:

- $C_p(X)$ fails to satisfy closed discrete selection
- $X$ is countable
- $C_p(X)$ is first-countable
Proof

If $C_p(X)$ fails to satisfy closed discrete selection, then $X$ is countable:

- Suppose $X$ is uncountable.
- Consider a sequence $[f_n, F_n, \varepsilon]$ of basic open sets in $C_p(X)$.
- Since $X$ is uncountable, we can find a point $x^* \in X \setminus F_n$.
- We can find continuous functions $g_n$ so that $g_n|_{F_n} = f_n$ and $g_n(x^*) = n$.
- Then $g_n \in [f_n, F_n, \varepsilon]$ and $\{g_n : n \in \omega\}$ is closed discrete.
Suppose that $\mathcal{A}$ and $\mathcal{B}$ are collections of sets.

**$S_1(\mathcal{A}, \mathcal{B})$**

For every sequence $(A_n : n \in \omega)$ of sets from $\mathcal{A}$, there are $C_n \in A_n$ so that \{ $C_n : n \in \omega$ \} $\in \mathcal{B}$

**$S_{FIN}(\mathcal{A}, \mathcal{B})$**

For every sequence $(A_n : n \in \omega)$ of sets from $\mathcal{A}$, there are finite $F_n \subseteq A_n$ so that $\bigcup_n F_n \in \mathcal{B}$
Common Selection Principles

Let $\mathcal{O}$ be the open covers of $X$. Then $S_1(\mathcal{O}, \mathcal{O})$ and $S_{FIN}(\mathcal{O}, \mathcal{O})$ are both strengthenings of the Lindelof property.

- $S_1(\mathcal{O}, \mathcal{O})$ is called the Rothberger property.
- $S_{FIN}(\mathcal{O}, \mathcal{O})$ is called the Menger property.
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$$S_1(\mathcal{O}, \mathcal{O}) \Rightarrow S_{\text{FIN}}(\mathcal{O}, \mathcal{O}) \Rightarrow \text{Lindelof}$$

and

$$\text{Compact} \Rightarrow \sigma-\text{Compact} \Rightarrow S_{\text{FIN}}(\mathcal{O}, \mathcal{O}) \Rightarrow \text{Lindelof}$$
Selection Games

\(S\Box(\mathcal{A}, \mathcal{B})\) can be turned into a two-player game.

- The game is played over rounds indexed by the naturals.
- At round \(n\), player I plays a set \(A_n\) from \(\mathcal{A}\) and player II responds by playing a selection \(C_n\) from \(A_n\).
- If those selections are singletons, then the game is \(G_1(\mathcal{A}, \mathcal{B})\). If they are finite sets, then the game is \(G_{FIN}(\mathcal{A}, \mathcal{B})\).
- Player II wins a given run of the game \((A_0, C_0, A_1, C_1, \cdots)\) if \(\bigcup_n C_n \in \mathcal{B}\).
- If player II does not win, then player I wins.
Fix a game $G_{\Box}(\mathcal{A}, \mathcal{B})$.

- A perfect information strategy for player I takes in a run of $G_{\Box}(\mathcal{A}, \mathcal{B})$ up to some round $n$ and outputs a set $A_{n+1} \in \mathcal{A}$.
- A perfect information strategy for player II takes in a run of $G_{\Box}(\mathcal{A}, \mathcal{B})$ up to some round $n$ and outputs a selection $C_n$ from I’s most recent move.
Fix a game $G□(\mathcal{A}, \mathcal{B})$.

- A perfect information strategy for player I takes in a run of $G□(\mathcal{A}, \mathcal{B})$ up to some round $n$ and outputs a set $A_{n+1} \in \mathcal{A}$.
- A perfect information strategy for player II takes in a run of $G□(\mathcal{A}, \mathcal{B})$ up to some round $n$ and outputs a selection $C_n$ from I’s most recent move.
- A strategy $\sigma$ for player I is winning if the run $(\sigma(\emptyset), C_0, \sigma(C_0), C_1, \cdots)$ wins for I no matter what selections II makes. If I has a winning strategy we write $I \uparrow G□(\mathcal{A}, \mathcal{B})$.
- A strategy $\tau$ for player II is winning if the run $(A_0, \tau(A_0), A_1, \tau(A_0, A_1), \cdots)$ wins for II no matter what sets player I plays. If II has a winning strategy we write $II \uparrow G□(\mathcal{A}, \mathcal{B})$. 
A Markov tactic for II is a strategy $\tau(A, n)$ which takes in only the round number and the most recent move of I. If II has a winning Markov tactic we write $\text{II} \uparrow_{\text{mark}} G\Box(A, B)$.

A pre-determined strategy for I is a strategy $\sigma(n)$ which takes in only the round number. If I has a winning pre-determined strategy we write $I \uparrow_{\text{pre}} G\Box(A, B)$. 
Easy Implications

\[ I \uparrow_{\text{pre}} G(A, B) \Rightarrow I \uparrow G(A, B) \Rightarrow \neg II \uparrow G(A, B) \]

\[ II \uparrow_{\text{mark}} G(A, B) \Rightarrow II \uparrow G(A, B) \Rightarrow I \not\uparrow G(A, B) \]

\[ II \uparrow G(A, B) \Rightarrow S(A, B) \]

\[ I \not\uparrow_{\text{pre}} G(A, B) \iff S(A, B) \]
Useful Selection Games

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- $G_{FIN}(\mathcal{O}, \mathcal{O})$ is the Menger game
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Useful Selection Games

- $G_1(\mathcal{O}, \mathcal{O})$ is the Rothberger game
- $G_{FIN}(\mathcal{O}, \mathcal{O})$ is the Menger game
- The point-open game is traditionally played with player I choosing points $x_n$ and player II choosing open sets $U_n \ni x_n$. Player I wins if $\{U_n : n \in \omega\}$ is an open cover.
- Cast as a selection game, this looks like $G_1(\mathcal{P}, \neg \mathcal{O})$, where $\mathcal{P}$ is the collection of point bases and $\neg \mathcal{O}$ consists of sequences of opens sets which are not open covers.
- The closed discrete selection game is $G_1(\mathcal{C}, CD)$, where $\mathcal{C}$ is the collection of open sets and $CD$ is the collection of closed discrete subsets of $X$. 
Useful Selection Games

- $I \uparrow G_1(\mathcal{P}, \neg\mathcal{O})$ if and only if $II \uparrow G_1(\mathcal{O}, \mathcal{O})$
- $II \uparrow G_1(\mathcal{P}, \neg\mathcal{O})$ if and only if $I \uparrow G_1(\mathcal{O}, \mathcal{O})$
- $I \uparrow_{pre} G_1(\mathcal{P}, \neg\mathcal{O})$ if and only if $X$ is countable.
Proof

$I \uparrow G_1(\mathcal{P}, \neg \mathcal{O})$ implies $II \uparrow G_1(\mathcal{O}, \mathcal{O})$:

- Let $\sigma$ be a winning strategy for I in $G_1(\mathcal{P}, \neg \mathcal{O})$.
- Say $\sigma(\emptyset) = x_0$.
- We need to define a strategy $\tau$ for II in $G_1(\mathcal{O}, \mathcal{O})$.
- Suppose $\mathcal{U}_0$ has been played by I in $G_1(\mathcal{O}, \mathcal{O})$. Then we can find a $U_0 \in \mathcal{U}$ so that $x_0 \in U_0$. Set $\tau(U_0) = U_0$.
- Pretend II has responded in $G_1(\mathcal{P}, \neg \mathcal{O})$ with $U_0$. Set $x_1 = \sigma(x_0, U_0)$.
- Now suppose I plays $\mathcal{U}_1$ in $G_1(\mathcal{O}, \mathcal{O})$ in response to $\tau(U_0)$. Then there is a $U_1 \in \mathcal{U}_1$ so that $x_1 \in U_1$. Set $\tau(U_1) = U_1$.
- Continue in this way to inductively define $\tau$. Because $\sigma$ is winning for I in $G_1(\mathcal{P}, \neg \mathcal{O})$, $\{\tau(U_n) : n \in \omega\}$ will be an open cover.
Suppose $X$ is Hausdorff and completely regular. Then the following are equivalent:

- $C_p(X)$ fails to satisfy closed discrete selection
- $X$ is countable
- $C_p(X)$ is first-countable

Equivalently, in the language of games:

- $I \uparrow_{pre} G_1(\tau_{C_p(X)}, CD_{C_p(X)})$
- $I \uparrow_{pre} G_1(\mathcal{P}_X, \neg \mathcal{O}_X)$
- $C_p(X)$ is first-countable
Tkachuk was also able to prove a partial result for perfect information strategies:

**Tkachuk, 2017**

Suppose $X$ is Hausdorff and completely regular. Then the following are equivalent:

- $I \uparrow G_1(\mathcal{T}_{C_p}(X), CD_{C_p}(X))$
- $I \uparrow G_1(\mathcal{P}_X, \neg O_X)$

Also, if $II \uparrow G(\mathcal{T}_{C_p}(X), CD_{C_p}(X))$, then $II \uparrow G_1(\mathcal{P}_X, \neg O_X)$. 
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Suppose $X$ is Hausdorff and completely regular. Then the following are equivalent:

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Also, if $II \uparrow G(\mathcal{T}_{Cp}(X), CD_{Cp}(X))$, then $II \uparrow G_1(\mathcal{P}_X, \neg \mathcal{O}_X)$.

Is it true that if $II \uparrow G_1(\mathcal{P}_X, \neg \mathcal{O}_X)$, then $II \uparrow G(\mathcal{T}_{Cp}(X), CD_{Cp}(X))$?
Consider a topological space \((X, \mathcal{T})\).

- \(\mathcal{U}\) is an \(\omega\)-cover of \(X\) if whenever \(F \subseteq X\) is finite, there is a \(U \in \mathcal{U}\) so that \(F \subseteq U\).

- Let \(\Omega_X\) be the collection of \(\omega\)-covers of \(X\).

- For \(F \subseteq X\) finite, let \(\mathcal{N}[F] = \{U \in \mathcal{T} : F \subseteq U\}\). We can then consider \(\mathcal{F}_X = \{\mathcal{N}[F] : F \subseteq X\) is finite\}. 

\(\omega\)-Covers and Games
Consider a topological space \((X, \mathcal{T})\).

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- For \(F \subseteq X\) finite, let \(\mathcal{N}[F] = \{U \in \mathcal{T} : F \subseteq U\}\). We can then consider \(\mathcal{F}_X = \{\mathcal{N}[F] : F \subseteq X\) is finite\}.
- We can now define the finite-open game \(G_1(\mathcal{F}, \neg \mathcal{O})\) and its variant \(G_1(\mathcal{F}, \neg \Omega)\).
- We can also define \(G_1(\Omega, \Omega)\).
The following are equivalent:

- $I \uparrow G_1(\mathcal{P}, \neg \mathcal{O})$
- $I \uparrow G_1(\mathcal{F}, \neg \mathcal{O})$
- $I \uparrow G_1(\mathcal{F}, \neg \Omega)$
- $\Pi \uparrow G_1(\mathcal{O}, \mathcal{O})$
- $\Pi \uparrow G_1(\Omega, \Omega)$
Proposition

The following are equivalent:

- $I \uparrow G_1(\mathcal{P}, \neg O)$
- $I \uparrow G_1(\mathcal{F}, \neg O)$
- $I \uparrow G_1(\mathcal{F}, \neg \Omega)$
- $II \uparrow G_1(O, O)$
- $II \uparrow G_1(\Omega, \Omega)$

What Tkachuk actually showed was that $I \uparrow G_1(\mathcal{T}_{Cp}(X), CD_{Cp}(X))$ if and only if $I \uparrow G_1(\mathcal{F}, \neg \Omega)$ and that if $II \uparrow G_1(\mathcal{T}_{Cp}(X), CD_{Cp}(X))$, then $II \uparrow G_1(\mathcal{F}, \neg \Omega)$. 
Proposition

- If $II \uparrow G_1(\mathcal{F}, \neg \Omega)$, then $II \uparrow G_1(\mathcal{P}, \neg \mathcal{O})$.
- $G_1(\mathcal{F}, \neg \Omega)$ and $G_1(\Omega, \Omega)$ are dual.
- It is consistent with ZFC that there is a space $X$ where $I \uparrow G_1(\mathcal{O}, \mathcal{O})$ but $I \not\uparrow G_1(\Omega, \Omega)$
- Thus it is consistent with ZFC that there is a space $X$ where $II \uparrow G_1(\mathcal{P}, \neg \mathcal{O})$, but $II \not\uparrow G_1(\mathcal{F}, \neg \Omega)$. 
Proposition

- If \( II \uparrow G_1(\mathcal{F}, \neg \Omega) \), then \( II \uparrow G_1(\mathcal{P}, \neg \mathcal{O}) \).
- \( G_1(\mathcal{F}, \neg \Omega) \) and \( G_1(\Omega, \Omega) \) are dual.
- It is consistent with ZFC that there is a space \( X \) where \( I \uparrow G_1(\mathcal{O}, \mathcal{O}) \) but \( I \not\uparrow G_1(\Omega, \Omega) \).
- Thus it is consistent with ZFC that there is a space \( X \) where
  \( II \uparrow G_1(\mathcal{P}, \neg \mathcal{O}) \), but \( II \not\uparrow G_1(\mathcal{F}, \neg \Omega) \).

With this in mind, the proper question is: does \( II \uparrow G_1(\mathcal{F}_X, \neg \Omega_X) \) imply \( II \uparrow G(\mathcal{T}_{Cp}(X), CD_{Cp}(X)) \)?
The following are equivalent.

- $II \uparrow G_1(F_X, \lnot \Omega_X)$
- $I \uparrow G_1(\Omega_X, \Omega_X)$
- $II \uparrow G(T_{C_p(X)}, CD_{C_p(X)})$
- $II \uparrow_{mark} G_1(F_X, \lnot \Omega_X)$
- $I \uparrow_{pre} G_1(\Omega_X, \Omega_X)$, i.e. $X$ is not Rothberger with respect to $\Omega$-covers
- $II \uparrow_{mark} G(T_{C_p(X)}, CD_{C_p(X)})$
Suppose $X$ is a topological space.

- $C_k(X)$ is the space of continuous functions $f : X \to \mathbb{R}$.
- It is endowed with the compact-open topology ("Uniform Convergence on Compact Sets").
- If $f : X \to \mathbb{R}$ is continuous, $K \subseteq X$ is compact, and $\varepsilon > 0$, then

$$[f, K, \varepsilon] = \{ g \in C_p(X) : |g(x) - f(x)| < \varepsilon \text{ for all } x \in K \}$$

These sets form a basis for the topology on $C_k(X)$. 

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$$[f, K, \varepsilon] = \{ g \in C_p(X) : |g(x) - f(x)| < \varepsilon \text{ for all } x \in K \}$$

These sets form a basis for the topology on $C_k(X)$. What happens when we play the closed discrete game on $C_k(X)$?
\section*{\textit{k}-Covers and Games}

Consider a topological space \((X, \mathcal{T})\).

\begin{itemize}
  \item Let \(K(X)\) be the space of compact subsets of \(X\).
  \item \(\mathcal{U}\) is a \(k\)-cover of \(X\) if whenever \(K \subseteq X\) is compact, there is a \(U \in \mathcal{U}\) so that \(K \subseteq U\).
  \item Let \(\mathcal{K}_X\) be the collection of \(k\)-covers of \(X\).
  \item For \(K \subseteq X\) compact, let \(\mathcal{N}[K] = \{U \in \mathcal{T} : K \subseteq U\}\). We can now consider \(\mathcal{N}[K(X)] = \{\mathcal{N}[K] : K \subseteq X\) is compact\}. 
\end{itemize}
$k$-Covers and Games

Consider a topological space $(X, \mathcal{T})$.

- Let $K(X)$ be the space of compact subsets of $X$.
- $\mathcal{U}$ is a $k$-cover of $X$ if whenever $K \subseteq X$ is compact, there is a $U \in \mathcal{U}$ so that $K \subseteq U$.
- Let $\mathcal{K}_X$ be the collection of $k$-covers of $X$.
- For $K \subseteq X$ compact, let $\mathcal{N}[K] = \{U \in \mathcal{T} : K \subseteq U\}$. We can now consider $\mathcal{N}[K(X)] = \{\mathcal{N}[K] : K \subseteq X \text{ is compact}\}$.
- These sets are used to form the compact-open game $G_1(\mathcal{N}[K(X)], \neg \mathcal{O})$ and its variant $G_1(\mathcal{N}[K(X)], \neg \mathcal{K})$.
- We can also play the $k$-Rothberger game $G_1(\mathcal{K}, \mathcal{K})$. 
Hemicompactness

Hemicompact

\( X \) is hemicompact if there are compact sets \( K_n \subseteq X \) for \( n \in \omega \) so that \( X = \bigcup_n K_n \) and whenever \( K \subseteq X \) is compact, there is some \( n \) so that \( K \subseteq K_n \).
Hemicompactness

**Hemicompact**

$X$ is hemicompact if there are compact sets $K_n \subseteq X$ for $n \in \omega$ so that $X = \bigcup_n K_n$ and whenever $K \subseteq X$ is compact, there is some $n$ so that $K \subseteq K_n$.

**Theorem**

The following are equivalent:

- $X$ is hemicompact
- $C_k(X)$ is first countable
- $I \uparrow_{pre} G_1(\mathcal{N}[K(X)], \neg K_X)$
If $\mathcal{R}$ is a reflection of $\mathcal{A}$, then the two games $G_1(\mathcal{A}, \mathcal{B})$ and $G_1(\mathcal{R}, \neg \mathcal{B})$ are dual.
Clontz’ Duality

Clontz 2018

If $\mathcal{R}$ is a reflection of $\mathcal{A}$, then the two games $G_1(\mathcal{A}, \mathcal{B})$ and $G_1(\mathcal{R}, \neg\mathcal{B})$ are dual.

Caruvana/Holshouser 2018

Suppose $\mathcal{A}$ is a collection of subsets of $X$. For $A \in \mathcal{A}$, set $\mathcal{N}[A] = \{U \in \mathcal{T}_X : A \subseteq U\}$. Set

$$\mathcal{N}[A] = \{\mathcal{N}[A] : A \in \mathcal{A}\}.$$ 

and

$${\mathcal{O}}(X, \mathcal{A}) = \{U \in {\mathcal{O}}_X : (\forall A \in \mathcal{A})(\exists U \in U)[A \subseteq U]\}.$$ 

Let $\mathcal{B}$ be a collection of open covers of $X$. Then $\mathcal{N}[A]$ is a reflection of $\mathcal{O}(X, \mathcal{A})$ and therefore $G_1(\mathcal{N}[A], \neg\mathcal{B})$ and $G_1(\mathcal{O}(X, \mathcal{A}), \mathcal{B})$ are dual.
The following are equivalent.

- $I \uparrow G_1(N[K(X)], \neg K_X)$
- $II \uparrow G_1(K_X, K_X)$
- $I \uparrow G(T_{C_k}(X), CD_{C_k}(X))$
The following are equivalent.

\[ I \uparrow G_1(\mathcal{N}[K(X)], \neg K_X) \]
\[ II \uparrow G_1(K_X, K_X) \]
\[ I \uparrow G(\mathcal{T}_{C_k(X)}, CD_{C_k(X)}) \]

The following are equivalent.

\[ I \uparrow_{pre} G_1(\mathcal{N}[K(X)], \neg K_X) \]
\[ II \uparrow_{mark} G_1(K_X, K_X) \]
\[ I \uparrow_{pre} G(\mathcal{T}_{C_k(X)}, CD_{C_k(X)}) \]
Improving Strategies

\[ I \uparrow_{\text{pre}} G □ (O, O) \text{ if and only if } I \uparrow G □ (O, O) \]
Improving Strategies

**Pawlikowski 1994**

\[ I \uparrow^{pre} G\Box(O, O) \text{ if and only if } I \uparrow G\Box(O, O) \]

**Caruvana/Holshouser 2018**

Suppose \( A \subseteq B \). Then \( I \uparrow^{pre} G\Box(O(X, A), O(X, B)) \text{ if and only if } I \uparrow G\Box(O(X, A), O(X, B)) \)
The following are equivalent.

- $\text{II} \uparrow G_1(\mathcal{N}[K(X)], \neg \mathcal{K}X)$
- $\text{I} \uparrow G_1(\mathcal{K}X, \mathcal{K}X)$
- $\text{II} \uparrow G(\mathcal{T}_{C_k}(X), CD_{C_k}(X))$
- $\text{II} \uparrow_{\text{mark}} G_1(\mathcal{N}[K(X)], \neg \mathcal{K}X)$
- $\text{I} \uparrow_{\text{pre}} G_1(\mathcal{K}X, \mathcal{K}X)$, i.e. $X$ is not Rothberger with respect to $k$-covers
- $\text{II} \uparrow_{\text{mark}} G(\mathcal{T}_{C_k}(X), CD_{C_k}(X))$
Questions

- What happens with $C_p(X, [0, 1])$?
- How far can Pawlikowski’s result be generalized?
- How much of this theory can be recovered if the compact sets are replaced with a different ideal?
- How much of this theory can be recovered if instead of choosing subcovers we choose refinements?
Thanks for Listening