Statistical mechanics of nonequilibrium systems of rotators with alternated spins

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Abstract

We consider a finite region of a d-dimensional lattice of nonlinear Hamiltonian rotators, where neighbouring rotators have opposite (alternated) spins and are coupled by a small potential of order $\varepsilon^a$, $a \geq 1/2$. We weakly stochastically perturb the system in such a way that each rotator interacts with its own stochastic thermostat with a force of order $\varepsilon$. Then we introduce action-angle variables for the system of uncoupled rotators ($\varepsilon = 0$) and note that the sum of actions over all nodes is conserved by the purely Hamiltonian dynamics of the system with $\varepsilon > 0$. We investigate the limiting (as $\varepsilon \to 0$) dynamics of actions for solutions of the $\varepsilon$-perturbed system on time intervals of order $\varepsilon^{-1}$. It turns out that the limiting dynamics is governed by a certain stochastic equation for the vector of actions, which we call the transport equation. This equation has a completely non-Hamiltonian nature. This is a consequence of the fact that the system of rotators with alternated spins do not have resonances of the first order.

The $\varepsilon$-perturbed system has a unique stationary measure $\tilde{\mu}^\varepsilon$ and is mixing. Any limiting point of the family $\{\tilde{\mu}^\varepsilon\}$ of stationary measures as $\varepsilon \to 0$ is an invariant measure of the system of uncoupled integrable rotators. There are plenty of such measures. However, it turns out that only one of them describes the limiting dynamics of the $\varepsilon$-perturbed system: we prove that a limiting point of $\{\tilde{\mu}^\varepsilon\}$ is unique, its projection to the space of actions is the unique stationary measure of the transport equation, which turns out to be mixing, and its projection to the space of angles is the normalized Lebesgue measure on the torus $\mathbb{T}^N$.

The results and convergences, which concern the behaviour of actions on long time intervals, are uniform in the number $N$ of rotators. Those, concerning the stationary measures, are uniform in $N$ in some natural case.

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1 Introduction

Investigation of the energy transport in crystals is one of the main problems in the non-equilibrium statistical mechanics (see [BoLeR]). It is closely related to the derivation of autonomous equations which describe a flow of quantities, conserved by the Hamiltonian (for example, the flow of energy and the corresponding heat equation). Following [DL], we call such equations "transport equations". In the classical setting one looks for the energy transport in a Hamiltonian system, coupled with thermal baths which have different temperatures. This coupling is weak in geometrical sense: thermal baths interact with the Hamiltonian system only through its boundary. Unfortunately, for the moment of writing this problem turns out to be too difficult due to the weakness of the coupling. In this case even the existence of a stationary state in the system is not clear (see [EPRB],[RBT], and [Tr],[Dym12] for a similar problem in a deterministic setting). That is why usually one modifies the system in order to get some additional ergodic properties. Two usual ways to achieve that are i) to consider a weak perturbation of the hyperbolic system of independent particles ([DL],[Ru]); ii) to couple each particle of the system with its own thermal bath by the force of order one ([BoLeLu],[BeO],[BaBeO],[BoLeLuO],[LO]).

The problem considered in the present paper is closely related to those studied in [DL] and [LO]. In [DL] the authors consider a finite region of a lattice of weakly interacting geodesic flows on manifolds of negative curvature. In [LO] the authors investigate that of weakly interacting anharmonic oscillators, where each oscillator is coupled with its own energy preserving stochastic thermostat by the force of order one. Then in both papers the authors rescale the time appropriately and, tending the strength of interaction in the
Hamiltonian system to zero, show that the limiting dynamics of energy is governed by a certain transport equation, which turns out to be the same in the both works. Note that the source of the additional ergodic properties (the hyperbolicity of unperturbed system and the coupling between the Hamiltonian system and the thermal baths) stays of order one even when the interaction in the Hamiltonian system tends to zero.

Our goal is to consider a Hamiltonian system, coupled with thermal baths by couplings, which are asymptotically weak in a sense, different from the classical setting, given above. Namely, to consider the situation when, as in ii) above, each particle is coupled with its own thermal bath (so the coupling is not weak geometrically), and to study the energy transport when the strength of the coupling decays to zero. This setting seems to be natural: one can think about a crystal put in some medium and weakly interacting with it. Note that another system, where the coupling with thermal baths is weak in this sense, was considered in [BaOS]. There was studied the FPU-chain with the nonlinearity replaced by an energy preserving stochastic exchange of momentum between neighbouring nodes. The authors investigated the energy transport under the limit when the rate of this exchange tends to zero.

However, as in [DL] and [LO], we have to assume the coupling of particles in the Hamiltonian system also to be sufficiently weak. Namely, we rescale the time and let the strength of interaction in the Hamiltonian system go to zero in the appropriate scaling with the coupling between the Hamiltonian system and the thermal baths. Under this limit we obtain the transport equation and show that the limiting behaviour of steady states of the system is governed by a unique stationary measure of the transport equation, which turns out to be mixing.  

1Compare to [DL] and [LO], we remove, in some sense, the source of the additional ergodic properties. However, our system is different. We consider a d-dimensional lattice of N nonlinear Hamiltonian rotators. The neighbouring rotators have opposite spins and interact weakly via a potential (linear or nonlinear) of order ε^a, a ≥ 1/2. We couple each rotator with its own stochastic thermostat by a coupling of order ε. The thermostats are of rather general form, they do not preserve the energy and have arbitrary nonzero temperatures (similar thermostats was considered, for example, in [BoLeLu], [BoLeLuO]). We introduce action-angle variables for the uncoupled Hamiltonian, corresponding to ε=0, and note that a sum of actions is conserved by the Hamiltonian dynamics with ε > 0. That is why the actions play for us the role of the local energy. In order to feel the interaction between rotators and the influence of thermal baths, we consider time interval of order t ~ ε^{-1}. We let ε go to zero and obtain that the limiting dynamics of actions is given by an equation which describes their autonomous (stochastic) evolution. We call it the transport equation. It has completely non-Hamiltonian nature and describes a non-Hamiltonian flow of actions. However, when a ≥ 1/2, one could expect to obtain a transport equation which has a Hamiltonian nature, since in this case the scalings of interaction in the Hamiltonian system and of time coincide with those in [LO] and [DL], where the transport equations have the Hamiltonian nature.

For readers, interested in the limiting dynamics of energy, we note that it can be easily expressed in terms of the limiting dynamics of actions.

The system in question (i.e. the Hamiltonian system, coupled with thermal baths) is mixing. We show that its stationary measure μ^ε, written in action-angle variables,
converges, as $\varepsilon \to 0$, to the product of the unique stationary measure $\pi$ of the obtained transport equation and the normalized Lebesgue measure on the torus $\mathbb{T}^N$. This result is independent from the scaling of time since a stationary regime does not depend on it. Moreover, the mixing property of the transport equation implies that, when $\varepsilon$ is small and time is large, the behaviour of actions is approximately described by the measure $\pi$.

The convergence as $\varepsilon \to 0$ of the vector of actions to a solution of the transport equation is uniform in the number of rotators $N$ (in the sense of finite-dimensional projections, explained below in Section 2.2). The convergence of the stationary measures is uniform in $N$ in some natural case.

We use technics of averaging and effective equations developed in [KP],[Kuk10],[Kuk13]. For a general Hamiltonian these methods are applied when the interaction potential is of the same order as the coupling with thermal baths, i.e. $a = 1$. However, we find a large natural class of Hamiltonians such that the results stay the same even if $a = 1/2$, i.e. when the interaction potential is stronger. This class consists of Hamiltonians which describe lattices of rotators with alternated spins, when neighbouring rotators rotate in opposite directions. It has to do with the fact that such systems of rotators do not have resonances of the first order. To apply the methods above in the case $1/2 \leq a < 1$ we kill the leading term of the interaction potential by a global canonical transformation which is $\varepsilon^a$-close to the identity. The non-Hamiltonian nature of the transport equation is a consequence of the absence of strong resonances in the system.

### 2 Set up and main results

#### 2.1 Set up

We consider a lattice $\mathcal{C} \subset \mathbb{Z}^d$, $d \in \mathbb{N}$, which consists of $N$ nodes $j \in \mathcal{C}$, $j = (j_1, \ldots, j_d)$. In each node we put an integrable nonlinear Hamiltonian rotator which is coupled through a small potential with rotators in neighbouring positions. The rotators are described by complex variables $u = (u_j)_{j \in \mathcal{C}} \in \mathbb{C}^N$. Introduce the symplectic structure by the 2-form

$$\frac{1}{2} \sum_{j \in \mathcal{C}} du_j \wedge d\bar{u}_j = \sum_{j \in \mathcal{C}} dx_j \wedge dy_j,$$

where $u_j = x_j + iy_j$. Then the system of rotators is given by the Hamiltonian equation

$$\dot{u}_j = i \nabla_j H^\varepsilon(u), \quad j \in \mathcal{C},$$

where the dot means a derivative in time $t$ and $\nabla_j H^\varepsilon = 2\partial_{u_j} H^\varepsilon$ is the gradient of the Hamiltonian $H^\varepsilon$ with respect to the Euclidean scalar product $\cdot$ in $\mathbb{C} \simeq \mathbb{R}^2$:

$$\text{for } z_1, z_2 \in \mathbb{C} \quad z_1 \cdot z_2 := \text{Re} \ z_1 \text{Re} \ z_2 + \text{Im} \ z_1 \text{Im} \ z_2 = \text{Re} \ z_1 \overline{z_2}. \quad (2.2)$$

The Hamiltonian has the form

$$H^\varepsilon = \frac{1}{2} \sum_{j \in \mathcal{C}} F_j(|u_j|^2) + \frac{\varepsilon^a}{4} \sum_{j,k \in \mathcal{C} : |j-k|=1} G(|u_j - u_k|^2), \quad (2.3)$$

where $|j| := |j_1| + \ldots + |j_d|$, $a \geq 1/2$ and $F_j, G : [0, \infty) \to \mathbb{R}$ are sufficiently smooth functions with polynomial bounds on the growth at infinity (precise assumptions are given below). We will usually skip the upper index $\varepsilon$ and write just $H$.

We weakly couple each rotator with its own stochastic thermostat of arbitrary temperature $T_j$, satisfying

$$0 < C^{-1} \leq T_j \leq C < \infty,$$
where the constant $C$ does not depend on $j, N, \varepsilon$. More precisely, we consider the system

$$
\dot{u}_j = i\nabla_j H(u) + \varepsilon g_j(u) + \sqrt{\varepsilon T_j} \beta_j, \quad u_j(0) = u_{0j}, \quad j \in \mathcal{C},
$$

(2.4)

where $\beta = (\beta_j)_{j \in \mathcal{C}} \in \mathbb{C}^N$ are standard complex independent Brownian motions. That is, their real and imaginary parts are standard real independent Wiener processes. Initial conditions $u_0 = (u_{0j})_{j \in \mathcal{C}}$ are random variables, independent from $\beta$. They are the same for all $\varepsilon$. Functions $g_j$, which we call "dissipations", have some dissipative properties, for example, $g_j(u) = -u_j$ (see Remark 2.1 below). They couple only neighbouring rotators, i.e. $g_j(u) = g_j((u_k)_{k \in \mathcal{C} : |k-j| \leq 1})$.

The scaling of the thermostatic term in equation (2.4) is natural since, in view of the dissipative properties of $g_j$, the only possibility for solution of equation $\dot{u}_j = \varepsilon g_j(u) + \varepsilon^b \sqrt{T_j} \beta_j$, $j \in \mathcal{C}$, to stay of the order 1 for all $t \geq 0$ as $\varepsilon \to 0$ is $b = 1/2$.

The case $a = 1/2$ is the most difficult, so further on we consider only it, the other cases are similar. Writing the corresponding equation (2.4) in more details, we obtain

$$
\dot{u}_j = i f_j(|u_j|^2) u_j + i \sqrt{\varepsilon} \sum_{k \in \mathcal{C} : |j-k| = 1} G'(|u_j - u_k|^2)(u_j - u_k) + \varepsilon g_j(u) + \sqrt{\varepsilon T_j} \beta_j,
$$

(2.5)

$$
u_j(0) = u_{0j}, \quad j \in \mathcal{C},
$$

(2.6)

where $f_j(x) := F'_j(x)$ and the prime denotes a derivative in $x$.

**Remark 2.1.** Our principal example is the case of diagonal dissipation, when $g_j(u) = -|u_j|^{p-2} u_j$ for all $j \in \mathcal{C}$ and some $p \in \mathbb{N}, p \geq 2$. In particular, the linear diagonal dissipation when $p = 2$ and $g_j(u) = -u_j$. The diagonal dissipation does not provide any interaction between rotators. In this case each rotator is just coupled with a Langevin-type thermostat. The results become more interesting if we admit functions $g_j$ of a more involved structure which not only introduces dissipation, but also provides some non-Hamiltonian interaction between the rotators. If for the reader the presence of the non-Hamiltonian interaction seems unnatural, he can simply assume that the dissipation is diagonal.

We impose on the system assumptions $HF, HG, Hg$ and $HI$. Their exact statements are given at the end of the section. Now we briefly summarize them. We fix some $p \in \mathbb{N}, p \geq 2$, and assume that $f_j(|u_j|^2) = (-1)^{|j|} f(|u_j|^2)$, where $f(|u_j|^2)$ is separated from zero and has at least a polynomial growth of a power $p$ ($HF$). It means that the leading term of the Hamiltonian $H$ is a nonlinearity which rotates the neighbouring rotators in opposite directions sufficiently fast. We call it the "alternated spins condition“. The function $G'(|u_j|^2)$ is assumed to have at most the polynomial growth of the power $p - 2$, i.e. the interaction term in (2.5) has the growth at most of the power $p - 1$ ($HG$). The functions $g_j(u)$ have some dissipative properties and have the polynomial growth of the power $p - 1$ ($Hg$). The functions $f, G$ and $g_j$ are assumed to be sufficiently smooth. In $HI$ we assume that the initial conditions are "not very bad", this assumption is not restrictive. For an example of functions $f, G$ and $g_j$ satisfying assumptions $HF, HG$ and $Hg$, see Example 2.4. In the case $a \geq 1$ the assumptions get weaker, see Remark 2.5. In particular, the rotators are permitted to rotate in any direction.
2.2 Main results

For a vector \( u = (u_k)_{k \in \mathbb{C}} \in \mathbb{C}^N \) we define the corresponding vectors of actions and angles

\[
I = I(u) = (I_k(u_k))_{k \in \mathbb{C}}, \quad I_k = \frac{1}{2}|u_k|^2 \quad \text{and} \quad \varphi = \varphi(u) = (\varphi_k(u_k))_{k \in \mathbb{C}}, \quad \varphi_k = \text{arg} \ u_k,
\]

where we put \( \varphi_k(0) = 0 \). Thus, \((I, \varphi) \in \mathbb{R}^N_+ \times \mathbb{T}^N \), where \( \mathbb{R}^N_+ = \{I = (I_k)_{k \in \mathbb{C}} \in \mathbb{R}^N : I_k \geq 0 \ \forall k \in \mathbb{C}\}\), and \( u_k = \sqrt{2I_k} e^{i\varphi_k} \). \(^2\) The variables \((I, \varphi)\) form the action-angle coordinates for the uncoupled Hamiltonian \((2.3)\) \(\varepsilon = 0\).

The direct computation shows that the sum of actions \( \sum_{k \in \mathbb{C}} I_k \) is a first integral of the Hamiltonian \( H \) for every \( \varepsilon > 0 \). That is why for our study the actions will play the role of the local energy, and we will examine their limiting behaviour as \( \varepsilon \to 0 \) instead of the limiting behaviour of energy. Moreover, the reader, interested in the limiting dynamics of energy, will easily express it in terms of the limiting dynamics of actions, since in view of \((2.3)\), the energy of a \( j \)-th rotator tends to \( \frac{1}{2}F_j(2I_j) \) as \( \varepsilon \to 0 \).

Let us write a function \( h(u) \) in the action-angle coordinates, \( h(u) = h(I, \varphi) \). Denote its averaging in angles as

\[
\langle h \rangle(I) := \int_{\mathbb{T}^N} h(I, \varphi) \, d\varphi.
\]

Here and further on \( d\varphi \) denotes the normalized Lebesgue measure on the torus \( \mathbb{T}^N \). Let

\[
\mathcal{R}_j(I) := (g_j(u) \cdot u_j), \quad (2.7)
\]

where we recall that the scalar product \( \cdot \) is given by \((2.2)\). It is well known that under our assumptions a solution \( u^\varepsilon(t) \) of system \((2.5)-(2.6)\) exists, is unique and is defined for all \( t \geq 0 \) ([Kha12]). Let \( I^\varepsilon(t) \) and \( \varphi^\varepsilon(t) \) be the corresponding vectors of actions and angles, i.e. \( I^\varepsilon(t) = I(u^\varepsilon(t)) \), \( \varphi^\varepsilon(t) = \varphi(u^\varepsilon(t)) \). We fix arbitrary \( T \geq 0 \) and examine the dynamics of actions \( I^\varepsilon \) on the long-time interval \([0, T/\varepsilon]\) under the limit \( \varepsilon \to 0 \). It is useful to pass to the slow time \( \tau = \varepsilon t \), then the interval \( t \in [0, T/\varepsilon] \) corresponds to \( \tau \in [0, T] \). We prove

**Theorem 2.2.** In the slow time the family of distributions of the actions \( \mathcal{D}(I^\varepsilon(\cdot)) \) with \( \varepsilon \to 0 \) converges weakly on \( C([0, T], \mathbb{R}^N) \) to a distribution \( \mathcal{D}(I^0(\cdot)) \) of a unique weak solution \( I^0(\tau) \) of the system

\[
dI_j = (\mathcal{R}_j(I) + \mathcal{T}_j) \, d\tau + \sqrt{2I_j \mathcal{T}_j} \, d\tilde{\beta}_j, \quad j \in \mathcal{C}, \quad (2.8)
\]

\[
\mathcal{D}(I(0)) = \mathcal{D}(I(u_0)), \quad (2.9)
\]

where \( \tilde{\beta}_j \) are standard real independent Brownian motions. The convergence is uniform in \( N \).

The limiting measure satisfies some estimates, for details see Theorem 4.6. In order to speak about the uniformity in \( N \) of convergence, we assume that \( \mathcal{C} \) depends on the number of rotators \( N \) in such a way that \( \mathcal{C}(N_1) \subset \mathcal{C}(N_2) \) if \( N_1 < N_2 \). The functions \( G, F_j \) and the temperatures \( \mathcal{T}_j \) are assumed to be independent from \( N \), while the functions \( g_j \)

\(^2\)Usually, for a vector from \( \mathbb{C}^N \), denoted by the letter \( u \), we write its actions and angles as above, and for a vector, denoted by \( v \), we write them as \( (J, \psi), J = J(v), \psi = \psi(v) \).
are assumed to be independent from $N$ for $N$ sufficiently large (depending on $j$). \footnote{We can not assume that $g_j$ is independent from $N$ for all $N \in \mathbb{N}$ since for small $N$ the $j$-th rotator may have fewer neighbours then for large $N$.} The initial conditions $u_0$ are assumed to agree in $N$, see assumption $HI(ii)$. The uniformity of convergence of measures through all the text is understood in the sense of finite-dimensional projections. For example, for Theorem 2.2 it means that for any $\Lambda \subset \mathbb{Z}^d$ which does not depend on $N$ and satisfies $\Lambda \subset \mathcal{C}(N)$ for all $N \geq N_\Lambda$, $N_\Lambda \in \mathbb{N}$, we have \footnote{We recall that the weak convergence of measures is metrisable (see [Dud], Theorem 11.3.3), so it makes sense to talk about its uniformity.}

$$\mathcal{D}((I_{j}^\varepsilon(\cdot))_{j \in \Lambda}) \rightarrow \mathcal{D}((I_{j}^0(\cdot))_{j \in \Lambda}) \quad \text{as} \quad \varepsilon \rightarrow 0 \quad \text{uniformly in} \quad N \geq N_\Lambda.$$ 

Note that in the case of diagonal dissipation $g_j(u) = -u_j |u_j|^{p-2}$ equation (2.8) turns out to be diagonal

$$dI_j = \left( -(2I_j)^{p/2} + T_j \right) dt + \sqrt{2I_j T_j} d\beta_j, \quad j \in \mathcal{C}. \quad (2.10)$$

For more examples see Section 4.4.

Relation (2.8) is an autonomous equation on the vector of actions. It describes the transport of actions $I^\varepsilon$ under the limit $\varepsilon \rightarrow 0$ and we call it the transport equation. Note that it does not depend on a precise form of the function $G$. It means that the limiting dynamics does not feel the Hamiltonian interaction between rotators and provides a flow of actions between nodes only if the dissipation is not diagonal.

In Section 4.2 we investigate the limiting behaviour, as $\varepsilon \rightarrow 0$, of averaged in time joint distribution of actions and angles $I^\varepsilon, \varphi^\varepsilon$. See Theorem 4.7.

Recall that a stochastic differential equation is mixing if it has a unique stationary measure and all solutions of this equation (with not "very bad" initial conditions) weakly converge to this stationary measure in distribution. It is well known that equation (2.5) is mixing (see [Kha12],[Ver87],[Ver97]). Denote its stationary measure by $\tilde{\mu}^\varepsilon$. Denote the projections to spaces of actions and angles by $\Pi_{ac} : \mathbb{C}^N \rightarrow \mathbb{R}^N$ and $\Pi_{ang} : \mathbb{C}^N \rightarrow \mathbb{T}^N$ correspondingly. Let

$$\mathcal{C}^\infty := \bigcup_{N \in \mathbb{N}} \mathcal{C}(N).$$

We will call equation (2.8) for the case $N = \infty$, i.e. with $\mathcal{C}$ replaced by $\mathcal{C}^\infty$, the "transport equation for the infinite system of rotators". Let $\mathbb{R}^\infty (\mathcal{C}^\infty)$ be the space of real (complex) sequences provided with the Tikhonov topology.

**Theorem 2.3.** (i) The transport equation (2.8) is mixing.

(ii) For the unique stationary measure $\tilde{\mu}^\varepsilon$ of (2.5), written in the action-angle coordinates, we have

$$(\Pi_{ac} \times \Pi_{ang})_* \tilde{\mu}^\varepsilon \rightarrow \pi \times d\varphi \quad \text{as} \quad \varepsilon \rightarrow 0, \quad (2.11)$$

where $\pi$ is a unique stationary measure of the transport equation (2.8). If the transport equation for the infinite system of rotators has a unique stationary measure on the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R}^\infty)$, then the convergence (2.11) is uniform in $N$.

(iii) The vector of actions $I^\varepsilon(\tau)$, written in the slow time, satisfies

$$\lim_{\tau \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \mathcal{D}(I^\varepsilon(\tau)) = \lim_{\varepsilon \rightarrow 0} \lim_{\tau \rightarrow \infty} \mathcal{D}(I^\varepsilon(\tau)) = \pi. \quad (2.12)$$
We prove this theorem in Section 4.3. Each limiting point (as \( \varepsilon \to 0 \)) of the family of measures \( \{ \tilde{\mu}^\varepsilon, 0 < \varepsilon \leq 1 \} \) is an invariant measure of the system of uncoupled integrable rotators, corresponding to \( (2.1)|_{\varepsilon=0} \). It has plenty of invariant measures. Theorem 2.3 ensures that only one of them is a limiting point, and distinguishes it.

We do not know if a stationary measure of the transport equation for the infinite system of rotators is unique in a general case. However, it is not difficult to show that if this equation is diagonal, then it has a unique stationary measure, and, consequently, the convergence \((2.11)\) holds uniformly in \( N \). In particular, this happens when the dissipation is diagonal. For more examples see Section 4.4.

### 2.3 Strategy

In this section we describe the main steps of proofs of Theorems 2.2 and 2.3.

First we need to obtain uniform in \( \varepsilon, N \) and time \( t \) estimates for solutions of \((2.5)\). For a general system of particles there is no reason why all the energy could not concentrate at a single position, forming a kind of delta-function as \( N \to \infty \). It is remarkable that in our system this does not happen, at least on time intervals of order \( 1/\sqrt{\varepsilon} \), even without alternated spins condition and in absence of dissipation. One can prove it working with the family of norms \( \| \cdot \|_{j,q} \) (see Agreements.6). But for a system with alternated spins the concentration of energy also does not happen as \( t \to \infty \). To see this, we make first one step of the perturbation theory. In Theorem 3.1 we find a global canonical change of variables in \( \mathbb{C}^N \), transforming \( u \to v \), \( (I,\varphi) \to (J,\psi) \), which is \( \sqrt{\varepsilon} \)-close to identity uniformly in \( N \) and kills in the Hamiltonian the term of order \( \sqrt{\varepsilon} \). We rewrite equation \((2.5)\) in the new variables \( v \) and call the result "\( v \)-equation" (see \((3.5)\)). Using the fact that in the new coordinates the interaction potential has the same order as the dissipation and working with the family of norms \( \| \cdot \|_{j,q} \), we obtain desired estimates for solutions of the \( v \)-equation.

Then we pass to the limit \( \varepsilon \to 0 \). In the action-angle coordinates \( (J,\psi) \) the \( v \)-equation takes the form

\[
\begin{align*}
    dJ &= X(J,\psi,\varepsilon)\,d\tau + \sigma(J,\psi,\varepsilon)d\beta + \overline{\sigma}(J,\psi,\varepsilon)d\overline{\beta}, \\
    d\psi &= \varepsilon^{-1}Y(J,\varepsilon)\,d\tau + \ldots,
\end{align*}
\]

where the term \( \ldots \) and \( X,Y,\sigma \) are of order 1. For details see \((4.2)-(4.3)\). So the angles rotate fast, while the actions change slowly. The averaging principle for systems of the type \((2.13)-(2.14)\) was established in [Kha68],[FW98],[FW03] and, more recently, in [KP],[Kuk13]. Our situation is similar to that in [KP],[Kuk13], and we follow the scheme suggested there. Let \( \psi^\varepsilon(\tau) \) be a solution of the \( \psi \)-equation, written in the slow time, and \( J^\varepsilon(\tau) = J(\psi^\varepsilon(\tau)) \) be the corresponding vector of actions. We prove Theorem 4.2, stating that the family of measures \( D(J^\varepsilon(\cdot)) \) converges weakly as \( \varepsilon \to 0 \) to a distribution of a unique weak solution of the averaged in angles equation \((2.13)|_{\varepsilon=0} \), which has the form \((2.8)\). To prove that this convergence is uniform in \( N \), we use the uniformity of estimates obtained above and the fact that the transport equation for the infinite system of rotators has a unique weak solution. Since the change of variables is \( \sqrt{\varepsilon} \)-close to identity, the behaviours of actions \( J^\varepsilon \) and \( I^\varepsilon \) as \( \varepsilon \to 0 \) coincide, and we get Theorem 2.2. The transport equation \((2.8)\) does not feel the Hamiltonian interaction of rotators since the averaging eliminates the Hamiltonian terms.
The transport equation (2.8) is irregular: its dispersion matrix is not Lipschitz continuous. To study it we use the method of effective equation, suggested in [Kuk10],[Kuk13] (in our case its application simplifies). The effective equation (see (4.34)) is defined in the complex coordinates \( v = (v_k)_{k \in \mathbb{C}} \in \mathbb{C}^N \). If \( v(\tau) \) is its solution then the actions \( J(v(\tau)) \) form a weak solution of equation (2.8) and vice versa (see Proposition 4.8). The effective equation is well posed and mixing. This implies item (i) of Theorem 2.3. The proof of item (ii) is based on the averaging technics developed in Theorem 2.2.

Not that the convergence (2.11) is equivalent to
\[
\tilde{\mu}^\varepsilon \rightharpoonup m \quad \text{as} \quad \varepsilon \to 0,
\]
where \( m \) is the unique stationary measure of the effective equation, see Remark 4.12. Item (iii) of Theorem 2.3 follows from the first two items and Theorem 2.2.

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### 2.4 Agreements and assumptions

**Agreements**

1) We refer to item 1 of Theorem 3.1 as Theorem 3.1.1, etc.

2) By \( C, C_1, C_2, \ldots \) we denote various positive constants and by \( C(b), C_1(b), \ldots \) we denote positive constants which depend on the parameter \( b \). We do not indicate their dependence on the dimension \( d \), power \( p \) and time \( T \) which are fixed throughout all the text and always indicate if they depend on the number of rotators \( N \), times \( t, s, \tau, \ldots \), positions \( j, k, l, m, \ldots \in \mathcal{C} \) and small parameter \( \varepsilon \). Constants \( C, C(b), \ldots \) can change from formula to formula.

3) Unless otherwise stated, assertions of the type "\( b \) is sufficiently close to \( c \)" and "\( b \) is sufficiently small/big" always suppose estimates independent from \( N \), positions \( j, k, l, m, \ldots \in \mathcal{C} \) and times \( t, s, \tau, \ldots \).

4) We use notations \( b \wedge c := \min(b, c) \), \( b \vee c = \max(b, c) \).

5) For vectors \( b = (b_k), c = (c_k), b_k, c_k \in \mathbb{C} \), we denote
\[
a \cdot b := \sum a_k \cdot b_k = \sum \operatorname{Re} a_k b_k.
\]

6) For \( 1/2 < \gamma < 1 \), \( j \in \mathcal{C} \) and \( q > 0 \) we introduce a family of scalar products and a family of norms on \( \mathbb{C}^N \) as
\[
(u \cdot u^1)_j := \sum_{k \in \mathcal{C}} \gamma^{j-k}|u_k| u^1_k, \quad \|u\|_{j, q}^q := \sum_{k \in \mathcal{C}} \gamma^{j-k}|u_k|^q, \quad \text{where} \quad u = (u_k)_{k \in \mathcal{C}}, \quad u^1 = (u^1_k)_{k \in \mathcal{C}} \in \mathbb{C}^N.
\]

7) For a metric space \( X \) by \( L_b(X) (L_{loc}(X)) \) we denote the space of bounded Lipschitz continuous (locally Lipschitz continuous) functions from \( X \) to \( \mathbb{R} \).

8) Convergence of measures we always understand in the weak sense.

---

5For details see Section 3.1. We will fix \( \gamma \), so we do not indicate the dependence on it.
9) We suppose \( \varepsilon \) to be sufficiently small, where it is needed.

**Assumptions**

Here we formulate our assumptions. In Example 2.4 we give examples of functions \( F_j, G \) and \( g_j \) satisfying them.

Fix \( p \in \mathbb{N}, p \geq 2 \). Assume that there exists \( \varepsilon > 0 \) such that the following holds.

**HF.** (Alternated spins condition). For every \( j \in \mathbb{C} \) and some function \( f \) we have \( f_j = (-1)^{|j|}f \). Function \( f : (-\varepsilon, \infty) \to \mathbb{R}_+ \) is \( C^3 \)-smooth and its derivative \( f' \) has only isolated zeros. Moreover, assume that for any \( x \geq 0 \)

\[
f(x) \geq C(1 + x^{p/2}) \quad \text{and} \quad |f'(x)|x^{1/2} + |f''(x)|x + |f'''(x)|x^{3/2} \leq C f(x).
\]

**HG.** Function \( G : (-\varepsilon, \infty) \to \mathbb{R} \) is \( C^4 \)-smooth. Moreover, for any \( x \geq 0 \) it satisfies

\[
|G'(x)|x^{1/2} + |G''(x)|x + |G'''(x)|x^{3/2} \leq C(1 + x^{(p-1)/2}).
\]

**Hg.**

(i) Functions \( g_i : \mathbb{C}^N \to \mathbb{R}, l \in \mathbb{C} \) are \( C^2 \)-smooth and depend on \( u = (u_k)_{k \in \mathbb{C}} \) only through \( (u_k)_{k|k-l| \leq 1} \). For any \( u \in \mathbb{C}^N \) and \( l, m \in \mathbb{C} \) they satisfy

\[
|g_i(u)|, |\partial_{um} g_i(u)|, |\partial_{pm} g_i(u)| \leq C \left( 1 + \sum_{k|k-l| \leq 1} |u_k|^{p-1} \right),
\]

while all the second derivatives are assumed to have at most a polynomial growth at infinity, which is uniform in \( l \in \mathbb{C} \).

(ii) (Dissipative condition) There exists a constant \( C_g > 0 \), independent from \( N \), such that for any \( j \in \mathbb{C} \) and \( \gamma < 1 \) sufficiently close to one, for any \( (u_k)_{k \in \mathbb{C}} \in \mathbb{C}^N \)

\[
(g(u) \cdot u)_j \leq -C_g \|u_j\|_{j,p} + C(\gamma), \quad \text{where} \quad g := (g_i)_{i \in \mathbb{C}},
\]

and the scalar product \( \cdot \) and the norm \( \| \cdot \|_{j,p} \) are defined in Agreements.6. Recall that they depend on \( \gamma \).

**HI.**

(i) For some constant \( \alpha_0 > 0 \), independent from \( N \), and every \( j \in \mathbb{C} \) we have

\[
\text{E} e^{\alpha_0 |u_0|} \leq C.
\]

(ii) The initial conditions \( u_0 = u_0^N \) agree in \( N \) in the sense that there exists a \( \mathbb{C}^\infty \)-valued random variable \( u_0^\infty = (u_0^\infty)_{j \in \mathbb{C}} \) satisfying for any \( N \in \mathbb{N} \) the relation

\[
\mathcal{D}((u_0_j)_{j \in \mathbb{C}(N)}) = \mathcal{D}((u_0^\infty)_{j \in \mathbb{C}(N)}).
\]

In what follows, we suppose the assumptions above to be held.

**Example 2.4.** As an example of functions \( f \) and \( G \) satisfying conditions HF and HG, we propose \( f(x) = 1 + x^k \) for any \( \mathbb{N} \ni k \geq p/2 \), and \( G(x) = \tilde{G}(\sqrt{x+C}) \) for any constant \( C > 0 \) and any \( C^4 \)-smooth function \( \tilde{G} : \mathbb{R}_+ \to \mathbb{R} \) satisfying

\[
|\tilde{G}'(x)| + |\tilde{G}''(x)| + |\tilde{G}'''(x)| \leq C(1 + x^{p-1}) \quad \text{for all} \quad x \geq \sqrt{C}.
\]

The simplest example of functions \( g_i \) satisfying assumption Hg is the diagonal dissipation \( g_i(u) = -u_k |u_k|^{p-2} \). As an example of functions \( g_i \) providing non-Hamiltonian interaction between rotators, we propose \( g_i(u) = -u_k |u_k|^{p-2} + \tilde{g}_i(u) \), where \( \tilde{g}_i \) satisfies

\[ Hg(i) \text{ and } |\tilde{g}_i(u)| \leq \tilde{C} \sum_{k|k-l| \leq 1} |u_k|^{p-1} + C, \]

where the constant \( \tilde{C} \) satisfies \( \tilde{C} = \frac{1}{8d(2d+1)^2} \).

---

6This constant is not optimal, one can improve it.
Remark 2.5. In the case \( a \geq 1 \) assumptions \( HF \) and \( HG \) simplify.

\( HF'-HG' \). Functions \( f_j, G : (-\varsigma, \infty) \mapsto \mathbb{R} \) are \( C^1 \)- and \( C^4 \)-smooth correspondingly, \( f'_j \) have only isolated zeros and \( |G'(x)|x^{1/2} \leq C(1 + x^{(p-1)/2}) \) for any \( x \geq 0 \).

3 Preliminaries

3.1 Norms

Since \( \sum_{j \in \mathcal{C}} |u_j|^2 \) conserves by the Hamiltonian flow, it would be natural to work in the \( l_2 \)-norm. However, the \( l_2 \)-norm of solution of (2.5) diverges as \( N \to \infty \). To overcome this difficulty and obtain uniform in \( N \) estimates for the solution, we introduce the family of \( l_q \)-weighted norms with exponential decay: for each \( q \geq 1 \) and every \( j \in \mathcal{C} \), for \( v = (v_k)_{k \in \mathcal{C}} \in \mathbb{C}^N \) we set

\[
\|v\|_{j,q} = \left( \sum_{k \in \mathcal{C}} \gamma^{|k-j|} |v_k|^q \right)^{1/q},
\]

where the constant \( 1/2 < \gamma < 1 \) will be chosen later.

Similar norms was considered, for example, in [DZ], Section 3.12. Define the family of \( l_2 \)-weighted scalar products on \( \mathbb{C}^N \),

\[
(v^1 \cdot v^2) = \sum_{k \in \mathcal{C}} \gamma^{|k-j|} v^1_k \cdot v^2_k,
\]

corresponding to the norms \( \|v\|_{j,2}^2 := \|v\|_{j,2}^2 = (v \cdot v)_j \). It is easy to see that the Hölder inequality holds: for any \( m,n \geq 1 \), satisfying \( m^{-1} + n^{-1} = 1 \), we have

\[
|(v^1 \cdot v^2)_j| \leq \|v^1\|_{j,m} \|v^2\|_{j,n}.
\]

Moreover, since for any \( m \geq n \) we have \( |v_k|^m \leq |v_k|^m + 1 \), then we get

\[
\|v\|^n_{j,n} \leq \|v\|^m_{j,m} + \sum_{k \in \mathcal{C}} \gamma^{|j-k|} \leq \|v\|^m_{j,m} + C(\gamma) \quad \text{for} \quad m \geq n,
\]

where the constant \( C(\gamma) \) does not depend on \( N \) since the geometrical series converges.

3.2 The change of variables

Consider the complex variables \( v = (v_j)_{j \in \mathcal{C}} \in \mathbb{C}^N \) and the corresponding vector of actions and angles \( (I, \varphi) \in \mathbb{R}^{N+0} \times \mathbb{T}^N \). Define a vector \( B := (\beta, \overline{\beta})T \in \mathbb{C}^{2N} \), where \( \beta \) is a complex \( N \)-dimensional Brownian motion as before and \( T \) denotes the transposition. Recall that by \( \langle \cdot \rangle \) we denote the averaging in angles, see Appendix B for its properties. Let \( \nabla := (\nabla_j)_{j \in \mathcal{C}} \) and \( g := (g_j)_{j \in \mathcal{C}} \).

**Theorem 3.1.** There exists a \( C^2 \)-smooth \( \sqrt{\varepsilon} \)-close to identity canonical change of variables of \( \mathbb{C}^N \), transforming \( u \to v, (I, \varphi) \to (J, \psi) \) such that the Hamiltonian \( H^\varepsilon \) in the new coordinates takes the form

\[
H^\varepsilon(J, \psi) = H_0^\varepsilon(J) + \varepsilon H_2(J, \psi) + \varepsilon \sqrt{\varepsilon} H_3^\varepsilon(J, \psi),
\]
where

\[
H_0^\varepsilon(v) = \frac{1}{2} \sum_{j \in \mathcal{C}} F_j(|v_j|^2) + \frac{\sqrt{\varepsilon}}{4} \sum_{|j-k|=1} \langle G(|v_j - v_k|^2) \rangle
\]

(3.4)
is \(C^4\)-smooth and the functions \(H_2(v)\) and \(H_2^\varepsilon(v)\) are \(C^2\)-smooth. System (2.5)-(2.6) written in \(v\)-variables has the form

\[
\dot{v} = i \nabla H_0^\varepsilon(v) + \varepsilon i \nabla H_2(v) + \varepsilon g(v) + \varepsilon \sqrt{\varepsilon} r^\varepsilon(v) + \sqrt{\varepsilon} W^\varepsilon(v) \dot{B},
\]

(3.5)

\[
v(0) = v(u_0) = v_0,
\]

(3.6)
where \(r^\varepsilon = (r_j^\varepsilon)_{j \in \mathcal{C}} : \mathbb{C}^N \rightarrow \mathbb{C}^N\) is a continuous vector-function and \(W^\varepsilon\) is a new dispersion matrix. The latter has the size \(N \times 2N\) and consists of two blocks, \(W^\varepsilon = (W^{\varepsilon 1}, W^{\varepsilon 2})\), so \(W^\varepsilon \dot{B} = W^{\varepsilon 1} \dot{\beta} + W^{\varepsilon 2} \dot{\beta}\). The blocks have the form \(W^{\varepsilon 1,2} = (W^{\varepsilon 1,2})_{k,l \in \mathcal{C}}\), where \(W^{\varepsilon 1}_{kl} = \sqrt{\mathcal{T}_0} \partial_{u_l} v_k, W^{\varepsilon 2}_{kl} = \sqrt{\mathcal{T}_0} \partial_{\gamma} v_k\). Moreover, for any \(j \in \mathcal{C}\) and \(1/2 < \gamma < 1\) we have

1. \(|(i \nabla H_2 \cdot v_j)| \leq (1 - \gamma)C \|v\|_{j,p}^p + C(\gamma).

2. \(\nabla_j H_2\) depends only on \(v_n\) such that \(|n-j| \leq 2\), and \(|\nabla_j H_2| \leq C \sum_{n:|n-j| \leq 2} |v_n|^{p-1} + C\).

b. For any \(q \geq 1\) we have

\[
\|r^\varepsilon\|_{j,q}^q \leq C(\gamma, q) + C(q) \|v\|_{j,q(p-1)}^{q(p-1)}.
\]

3. The functions \(d_{kl}^{1,2}\), defined as in (A.1), satisfy \(|d_{kl}^1 - \delta_{kl} \mathcal{T}_0|, |d_{kl}^2| \leq C \sqrt{\varepsilon}\) for all \(k,l \in \mathcal{C}\).

4. We have \(|u_j - v_j| \leq C \sqrt{\varepsilon}\) and \(|I_j - J_j| \leq C \sqrt{\varepsilon}\).

Further on we will skip the upper index \(\varepsilon\). If \(\gamma = 1\), then the norm \(\|u\|_j\) with \(\gamma\) close to one is an approximate integral of the Hamiltonian flow. Item 1 of Theorem 3.1 means that the change of variables preserves this property in the order \(\varepsilon\), modulo constant \(C(\gamma)\). This is crucial for deriving of uniform in \(N\) estimates for solutions of (3.5).

In equation (2.5) all functions, except the rotating nonlinearity \(i f_j(|u_j|^2) u_j\), have at most a polynomial growth of a power \(p-1\). Item 2 affirms, in particular, that this property is conserved by the transformation.

The proof of the theorem is technically rather complicated and is given in Section 6. Since the potential \(G\) is not a differentiable function of actions, we have to work in the \(v\)-coordinates despite that the transformation is constructed in the action-angle variables. This rises some difficulties since the derivative of \(\psi_j\) with respect to \(v_j\) have a singularity when \(v_j = 0\). Moreover, we have to work in rather inconvenient norms \(\|\cdot\|_{j,q}\) and estimate not only Poisson brackets, but also non-Hamiltonian terms of the \(v\)-equation.

### 3.3 Estimates for solution

System (3.5)-(3.6) has a unique solution since system (2.5)-(2.6) does. Let us denote it by \(v(t) = (v_k(t))_{k \in \mathcal{C}}\).

**Lemma 3.2.** For any \(1/2 < \gamma < 1\) sufficiently close to one there exists \(\alpha = \alpha(\gamma) > 0\) such that for all \(j \in \mathcal{C}, t \geq 0\) and \(\varepsilon\) sufficiently small we have

\[
\mathbb{E} \sup_{s \in [t, t + 1/\varepsilon]} e^{\alpha \|v(s)\|^2_j} < C(\gamma).
\]

(3.7)
Let us emphasize that estimate (3.7) holds uniformly in $N,j,t$ and $\varepsilon$ sufficiently small.

**Corollary 3.3.** There exists $\alpha > 0$ such that for any $m > 0$, $t \geq 0$, $j \in C$ and $\varepsilon$ sufficiently small we have

\[ E \sup_{s \in [t,t+1/\varepsilon]} e^{\alpha |v_j(s)|^2} < C, \quad E \sup_{s \in [t,t+1/\varepsilon]} |r_j(v(s))|^m < C(m), \]

where $r = (r_j)_{j \in C}$ is the reminder in (3.5).

**Proof of Corollary 3.3.** Fix any $\gamma$ and $\alpha$ such that (3.7) holds true. By the definition of $\| \cdot \|^2_j$ we have $|v_j|^2 \leq \|v\|^2$, so Lemma 3.2 implies the first inequality. Let us prove the second one. Without loss of generality we assume that $m \geq 2$. Theorem 3.1.2b implies

\[ |r_j|^m \leq \|r\|_{j,m}^m \leq C(\gamma, m) + C(m)\|v\|_{j,m(p-1)}^{m(p-1)} \leq C(\gamma, m) + C(m, \kappa)e^{\kappa\|v\|^2}_{j,m(p-1)} \]

for any $\kappa > 0$. Using that $2/m(p-1) \leq 1$ and Jensen inequality, we get

\[ e^{\kappa\|v\|^2}_{m(p-1)} \leq e^{\kappa} \sum_{k \in C} \gamma_{j,m(p-1)}^{2(j-k)} |v_k|^2 \leq \sum_{k \in C} \gamma_{j,m(p-1)}^{2(j-k)} (C(\gamma))^{1/\kappa} e^{\kappa\|v\|^2}, \]

where $C(\gamma) = \sum_{k \in C} \gamma_{m(p-1)}^{2(j-k)}$. Choosing $\kappa$ in such a way that $\kappa C(\gamma) \leq \alpha$ and combining (3.8), (3.9) and the first estimate of the corollary, we get the desired inequality. \hfill \Box

**Proof of Lemma 3.2. Step 1.** Take some $1/2 < \gamma < 1$ and $0 < \alpha_1 < 1$. Further on we present only formal computation which could be justified by standard stopping-time arguments (see, e.g., [KaSh]). Applying the Ito formula in complex coordinates (see Appendix A) to $e^{\alpha_1\|v\|^2}_j$ and noting that $i\nabla_j H_0 \cdot v_j = 0$ since $H_0$ depends on $v$ only through $J(v)$, we get

\[
\frac{d}{ds} e^{\alpha_1\|v(s)\|^2}_j = 2\alpha_1 \varepsilon e^{\alpha_1\|v\|^2}_j \left((i\nabla H_2 \cdot v)_j + (g \cdot v)_j + \sqrt{\varepsilon} (r \cdot v)_j + \sum_{k \in C} \gamma_j^{j-k} d_{kl}^{1k} + \alpha_1 \sum_{k,j \in C} \gamma_j^{j-k} d_{kl}^{2k} \left(v_k \overline{v_j} d_{kl}^1 + \Re(\overline{v_k} d_{kl}^2)\right)\right) + 2\alpha_1 \varepsilon M_s, \tag{3.10}
\]

where we recall that $d_{kl}^{1,2}$ are calculated in (A.1), and the martingal

\[ M_s := \int_{s_0}^s e^{\alpha_1\|v\|^2}_j (v \cdot W dB)_j \quad \text{for some} \quad s_0 < s. \tag{3.11} \]

First we estimate $(r \cdot v)_j$. Theorem 3.1.2b implies

\[ \|r\|_{j,p/(p-1)} \leq \left( C\|v\|_{j,p}^p + C(\gamma) \right)^{(p-1)/p} \leq C_1 \|v\|_{j,p}^{p-1} + C_1(\gamma). \]

Then, the Holder inequality (3.1) with $m = p/(p-1)$ and $n = p$, jointly with (3.2) implies

\[ |(r \cdot v)_j| \leq \|r\|_{j,p/(p-1)} \|v\|_{j,p} \leq C_1 \|v\|_{j,p}^{p+p-1} + C_1(\gamma) \|v\|_{j,p} \leq C_2(\gamma)(\|v\|_{j,p}^p + 1). \tag{3.12} \]
Secondly we estimate Ito’s term. By Theorem 3.1.3 we get
\[ \left| \sum_{k \in C} \gamma^{|j-k|} d^1_{kk} \right| \leq C(\gamma). \tag{3.13} \]
Note that
\[ \sum_{k,l \in C} \gamma^{|j-k|+|j-l|} |v_k||v_l| \leq \sum_{k,l \in C} \gamma^{|j-k|+|j-l|}(|v_k|^2 + |v_l|^2) \leq C(\gamma)\|v\|_2^2. \]
Consequently, due to Theorem 3.1.3, we have
\[ \left| \sum_{k,l \in C} \gamma^{|j-k|+|j-l|} (v_k \bar{v}_k d^1_{kl} + \text{Re}(\bar{v}_k v_l d^3_{kl})) \right| \leq \sum_{k \in C} \gamma^{2|j-k|} T_k |v_k|^2 + \sqrt{\varepsilon} C(\gamma)\|v\|_2^2 \leq (C + \sqrt{\varepsilon} C_1(\gamma))\|v\|_p^2 \leq (C + \sqrt{\varepsilon} C_1(\gamma))\|v\|_{j,p}^2 + C_2(\gamma), \tag{3.14} \]
where we have used (3.2). Now Theorem 3.1.1, assumption \( H_g(ii), (3.12), (3.13) \) and (3.14), applied to (3.10), imply that for \( \gamma \) sufficiently close to one we have
\[ \frac{d}{ds} e^{\alpha_1\|v\|_j^2} \leq 2\alpha_1 e^{\alpha_1\|v\|_j^2} \left( -(C_g - (1 - \gamma)C - \alpha_1 C - \sqrt{\varepsilon} C(\gamma))\|v\|_{j,p}^2 + C_1(\gamma) \right) + 2\alpha_1 \sqrt{\varepsilon} \dot{M}_s. \tag{3.15} \]
We take \( 1/2 < \gamma < 1 \) sufficiently close to one, then choose \( \alpha_1(\gamma) > 0 \) and \( \varepsilon_0(\gamma) > 0 \), sufficiently small, in such a way that
\[ \Delta := C_g - (1 - \gamma)C - \alpha_1 C - \sqrt{\varepsilon_0} C(\gamma) > 0. \tag{3.16} \]
For any constant \( C \) there exists a constant \( C_1 \) such that for all \( x \geq 0 \) we have
\[ 2\alpha_1 e^{\alpha_1 x} (-\Delta x + C) \leq -e^{\alpha_1 x} + C_1. \]
Consequently, (3.15) jointly with (3.2) implies that for \( \varepsilon < \varepsilon_0 \) we have
\[ \frac{d}{ds} e^{\alpha_1\|v(s)\|_j^2} \leq -\varepsilon e^{\alpha_1\|v(s)\|_j^2} + \varepsilon C(\gamma) + 2\alpha_1 \sqrt{\varepsilon} \dot{M}_s. \tag{3.17} \]
Fixing \( s_0 = 0 \) (which is defined in (3.11)), taking expectation and applying the Gronwall-Bellman inequality to (3.17), we have
\[ \mathbb{E} e^{\alpha_1\|v(s)\|_j^2} \leq \mathbb{E} e^{\alpha_1\|v_0\|_j^2} e^{-\varepsilon s} + C(\gamma). \]
Due to assumption \( HI(i) \) and Theorem 3.1.4, we have \( \mathbb{E} e^{\alpha_1\|v_0\|_j^2} \leq C \) for all \( j \in C \). Then the Jensen inequality implies that \( \mathbb{E} e^{\alpha_1\|v_0\|_j^2} \leq C(\gamma) \), if \( \alpha_1 \) is sufficiently small. Thus we obtain
\[ \mathbb{E} e^{\alpha_1\|v(s)\|_j^2} \leq C(\gamma) \quad \text{for all } s \geq 0 \text{ and } j \in C. \tag{3.18} \]

**Step 2.** We fix the parameters \( \gamma \) and \( \alpha_1 \) as above. Accordingly, the constants, which depend only on them, will be denoted just \( C, C_1, \ldots \).

Now we will prove (3.7). Take any \( 0 < \alpha < \alpha_1/2 \) and fix \( s_0 = t \). Integrating inequality (3.17) with \( \alpha_1 \) replaced by \( \alpha \) over the interval \( t \leq s \leq t + 1/\varepsilon \) and using (3.18), we have
\[ \mathbb{E} \sup_{s \in [t,t+1/\varepsilon]} e^{\alpha\|v(s)\|_j^2} \leq \mathbb{E} e^{\alpha\|v(t)\|_j^2} + 2\alpha \sqrt{\varepsilon} \mathbb{E} \sup_{s \in [t,t+1/\varepsilon]} M_s \leq C_1 + 2\alpha \sqrt{\varepsilon} \mathbb{E} \sup_{s \in [t,t+1/\varepsilon]} M_s. \tag{3.19} \]
Now we turn to the martingal part. The definition of $M_s$ implies

$$
\sup_{s \in [t,t+1/\varepsilon]} M_s \leq \sum_{k \in \mathcal{C}} \sup_{s \in [t,t+1/\varepsilon]} M_{ks},
$$

where $M_{ks} = \int_t^s e^{\alpha \|v\|^2} \gamma |j^{k-1}| v_k \cdot (WdB)_k$. The Doob-Kolmogorov inequality implies that

$$
\mathbb{E} \sup_{s \in [t,t+1/\varepsilon]} M_s \leq C\mathbb{E} \sqrt{[M_k]_{t+1/\varepsilon}} \leq C\mathbb{E} \sqrt{[M_k]_{t+1/\varepsilon}},
$$

where $[M_k]_s$ denotes the quadratic variation of $M_{ks}$. Similarly to (A.4), we obtain

$$
[M_k]_{t+1/\varepsilon} = \int_t^{t+1/\varepsilon} e^{2\alpha \|v\|^2} 2^{j-1} |S^J_{kk} ds \leq C(\kappa) \gamma^{j-1} \int_t^{t+1/\varepsilon} e^{2(\alpha+\kappa) \|v\|^2} (d^1_{kk} + d^2_{kk}) ds
$$

for any $\kappa > 0$, where $S^J_{kk}$ is defined in (A.3). Take $0 < \kappa < \alpha_1/2 - \alpha$. Then, using Theorem 3.1.3 and (3.18), we get

$$
\mathbb{E} \sup_{s \in [t,t+1/\varepsilon]} M_s \leq C \sum_{k \in \mathcal{C}} \sqrt{\mathbb{E} [M_k]_{t+1/\varepsilon}} \leq C(\kappa) \sum_{k \in \mathcal{C}} \gamma^{j-1} \left( \int_t^{t+1/\varepsilon} \mathbb{E} e^{2(\alpha+\kappa) \|v\|^2} ds \right)^{1/2} \leq \frac{C_1(\kappa)}{\sqrt{\varepsilon}}.
$$

Now (3.7) follows from (3.19). \hfill \Box

## 4 The limiting dynamics

In this section we investigate the limiting (as $\varepsilon \to 0$) behaviour of system (2.5). We prove Theorems 4.6, 4.7 and 2.3 which are our main results.

### 4.1 Transport equation

Here we prove Theorem 4.6, which describes the limiting dynamics of actions on long time intervals of order $\varepsilon^{-1}$. In the slow time $\tau = \varepsilon t$ system (3.5)-(3.6) has the form

$$
dv_j = (\varepsilon^{-1} i \nabla_j H_0 + i \nabla_j H_2 + g_j + \sqrt{\varepsilon} r_j) d\tau + (WdB)_j, \quad v_j(0) = v_{0j}, \quad j \in \mathcal{C}. \quad (4.1)
$$

Let us write equation (4.1) in the action-angle variables $J = J(v), \psi = \psi(v)$. Due to (A.2) and the equalities $i \nabla_j H_0 \cdot v_j$ and $i \nabla_j H_0 \cdot \frac{i v_j}{\|v_j\|^2} = \partial J_j H_0$, we have

$$
dJ_j = A^J_j d\tau + v_j \cdot (WdB)_j, \quad (4.2)
$$

$$
d\psi_j = \left( \varepsilon^{-1} \frac{\partial H_0}{\partial J_j} + \frac{A^\psi_j}{\|v_j\|^2} \right) d\tau + \frac{i v_j}{\|v_j\|^2} \cdot (WdB)_j, \quad j \in \mathcal{C}, \quad (4.3)
$$

where

$$
A^J_j := A_j \cdot v_j + d^1_{jj}, \quad A^\psi_j := A_j \cdot (i v_j) - \text{Im}(\overline{v_j} v_j^{-1} d^2_{jj}), \quad A_j := i \nabla_j H_2 + g_j + \sqrt{\varepsilon} r_j, \quad (4.4)
$$
and \( d_{ij}^{1,2} \) are calculated in (A.1). In view of (3.4), Proposition B.1 implies that

for each \( j \in \mathcal{C} \) the function \( \partial_{I_j} H_0 \) is \( C^1 \)-smooth with respect to \( J = (J_k)_{k \in \mathcal{C}} \). \( (4.5) \)

Theorem 3.1.2a,3 jointly with Corollary 3.3 implies that for all \( j, k, l \in \mathcal{C} \) and every \( m > 0 \) we have

\[
\mathbb{E} \sup_{0 \leq \tau \leq T} \left( |A_j| + |A_j^l| + |A_j^\psi| + |S_{kl}^l| \right)^m \leq C(m),
\]  

(4.6)

where \( S_{kl}^l \) is the element of the diffusion matrix for equation (4.2) with respect to the real Brownian motion; it is calculated in (A.3).

Note that the quadratic vatriations of the martingales from the r.h.s. of (4.2) and (4.3) are calculated in (A.4).

Let \( \psi^\varepsilon(\tau) \) be a solution of (4.1). Then \( J^\varepsilon(\tau) := J(\psi^\varepsilon(\tau)), \psi^\varepsilon(\tau) := \psi(\psi^\varepsilon(\tau)) \) satisfy (4.2)-(4.3). Due to estimate (4.6) and slow equation (4.2), using Arzela-Ascoli theorem, we get

**Proposition 4.1.** The family of measures \( \{D(J^\varepsilon(\cdot)), \ 0 < \varepsilon \leq 1 \} \) is tight on \( C([0, T], \mathbb{R}^N) \).

Let \( Q_0 \) be a weak limiting point of \( D(J^\varepsilon(\cdot)) \):

\[
D(J^{\varepsilon_k}(\cdot)) \rightharpoonup Q_0 \quad \text{as} \quad k \to \infty \quad \text{on} \quad C([0, T], \mathbb{R}^N),
\]  

(4.7)

where \( \varepsilon_k \to 0 \) as \( k \to \infty \) is a suitable sequence. Now we are going to show that the limiting point \( Q_0 \) does not depend on the sequence \( (\varepsilon_k) \) and is governed by the main order in \( \varepsilon \) of the averaging of equation (4.2). Let us begin with writing down this equation. Since by Theorem 3.1.3 we have \( d_{ij}^1 = T_j + O(\sqrt{\varepsilon}) \), the main order of the drift of equation (4.2) is \( i \nabla_j H_2 \cdot v_j + \frac{1}{\varepsilon} g_j(v) \cdot v_j + T_j \). Since for any real-valued \( C^1 \)-smooth function \( \hat{h}(v) \) we have \( i \nabla_j \hat{h} \cdot v_j = -\hat{h}_j \), then \( i \nabla_j H_2 \cdot v_j = 0 \). Thus the main order of the averaged drift takes the form

\[
\langle i \nabla_j H_2 \cdot v_j + g_j(v) \cdot v_j + T_j \rangle = \mathcal{R}_j(J) + T_j,
\]  

(4.8)

where \( \mathcal{R}_j \) is defined in (2.7). Proposition A.2 jointly with Theorem 3.1.3 implies that the main order of the diffusion matrix of (4.2) with respect to the real Brownian motion \( (\text{Re} \beta_k, \text{Im} \beta_k)_k \) is \( \text{diag}(\tau_k, |v_k|^2)_{k \in \mathcal{C}} = \text{diag}(2 \tau_k, J_k)_{k \in \mathcal{C}} \). It does not depend on angles, so the averaging does not change it. Choose its square root as \( \sqrt{\text{diag}(2 \tau_k, J_k)}_{k \in \mathcal{C}} \). Then in the main order the averaging of equation (4.2) takes the form

\[
dJ_j = (\mathcal{R}_j(J) + T_j) d\tau + \sqrt{2 J_j T_j} d\tilde{\beta}_j, \quad j \in \mathcal{C},
\]  

(4.9)

where \( \tilde{\beta}_j \) are independent standard real Brownian motions. We call (4.9) the "transport equation". It has a weak singularity: its dispersion matrix is not Lipschitz continuous. However, its drift is regular: Proposition B.1 implies that

for each \( j \in \mathcal{C} \) the function \( \mathcal{R}_j \) is \( C^1 \)-smooth with respect to \( J = (J_k)_{k \in \mathcal{C}} \). \( (4.10) \)

**Theorem 4.2.** The measure \( Q_0 \) is a law of the process \( J^0(\cdot) \) which is a unique weak solution of the transport equation (4.9) with the initial conditions \( D(J(0)) = D(I(u_0)) \). Moreover,

\[
D(J^\varepsilon(\cdot)) \rightharpoonup D(J^0(\cdot)) \quad \text{as} \quad \varepsilon \to 0 \quad \text{on} \quad C([0, T], \mathbb{R}^N).
\]  

(4.11)
This convergence is uniform in $N$. For all $j \in \mathcal{C}$ we have

$$
\mathbb{E} \sup_{\tau \in [0,T]} e^{2\alpha J_j^0(\tau)} < C \quad \text{and} \quad \int_0^T \mathbb{P} \left( J_j^0(\tau) < \delta \right) d\tau \to 0 \quad \text{as} \quad \delta \to 0,
$$

(4.12)

where the latter convergence is uniform in $N$.

Proof. The proof follows a scheme suggested in [KP], [Kuk13], while the latter works use the averaging method developed in [Kha68], [FW98]. Main difficulties of our situation compared to [Kha68], [FW98] are similar to those in [KP], [Kuk13] and manifest themselves in the proof of Lemma 4.4 below. Equation (4.3) has a singularity when $J_k = 0$, and for $J$, such that the rotating frequencies $\partial_j H_0$ are rationally dependent, system (4.2)-(4.3) enters into resonant regime. To overcome these difficulties we note that singularities and resonances have Lebesgue measure zero and prove the following lemma, which affirms that the probability of the event that actions $J^\varepsilon$ for a long lime belong to a set of small Lebesgue measure is small. A similar idea was used in [FW03], where was established the stochastic averaging principle for a simpler system with weak resonances.

Let $\Lambda \subset \mathbb{Z}^d$ be independent from $N$ and satisfies $\Lambda \subset \mathcal{C}(N)$ for $N \geq N_\Lambda$. Denote by $M$ the number of nodes in $\Lambda$. Further on we assume that $N \geq N_\Lambda$.

(4.13)

**Lemma 4.3.** Let $J^\varepsilon := (J_k^\varepsilon)_{k \in \Lambda}$ and a set $E^\varepsilon \subset \mathbb{R}^M_{+0}$ be such that its Lebesgue measure $|E^\varepsilon| \to 0$ as $\varepsilon \to 0$. Then

$$
\int_0^T \mathbb{P} \left( J^\varepsilon(\tau) \in E^\varepsilon \right) d\tau \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly in} \ N.
$$

(4.14)

The proof of Lemma 4.3 is based on Krylov’s estimates (see [Kry]) and the concept of local time. It follows a scheme suggested in [Shi] (see also [KS], Section 5.2.2).

Another difficulty, which is the principal difference between our case and that in [KP], [Kuk13], is that we need to establish the uniformity in $N$ of the convergence (4.11). For this purpose we use the uniformity of estimates and convergences of Corollary 3.3 and Lemmas 4.3, 4.4, and the fact that the transport equation for the infinite system of rotators has a unique weak solution.

Now let us formulate the following averaging lemma which is the main tools of the proof of the theorem.

**Lemma 4.4.** Take a function $P \in \mathcal{L}_{loc}(\mathbb{C}^N)$ which depends on $v = (v_j)_{j \in \mathcal{C}} \in \mathbb{C}^N$ only through $(v_j)_{j \in \Lambda} \in \mathbb{C}^M$. Let it has at most a polynomial growth at infinity. Then, writing $P(v)$ in the action-angle coordinates $P(v) = P(J, \psi)$, we have

$$
\mathbb{E} \sup_{\tau \in [0,T]} \left| \int_0^\tau P(J^\varepsilon(s), \psi^\varepsilon(s)) \, ds - \langle P \rangle (J^\varepsilon(s)) \right| \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly in} \ N.
$$

Similarly one can prove that

$$
\mathbb{E} \sup_{\tau \in [0,T]} \left| \int_0^\tau P(J^\varepsilon(s), \psi^\varepsilon(s)) \, ds - \langle P \rangle (J^\varepsilon(s)) \right|^2 \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly in} \ N.
$$

(4.15)
We establish Lemmas 4.3 and 4.4 in Section 5.

Now we will prove that \(Q_0\) is a law of a weak solution of (4.9). It suffices to show (see [KaSh], chapter 5.4) that for any \(j, k, l \in \mathcal{C}\) the processes

\[
Z_j(\tau) := J_j(\tau) - \int_0^\tau (\mathcal{R}_j(J(s)) + \mathcal{T}_j) \, ds, \quad Z_kZ_l(\tau) - 2\delta_{kl}T_k \int_0^\tau J_k(s) \, ds
\]

(4.16)

are square-integrable martingales with respect to the measure \(Q_0\) and the natural filtration of \(\sigma\)-algebras in \(C([0, T], \mathbb{R}^N)\). We establish it for the first process, for the second the proof is similar, but one should use (4.15) (for the first one we do not need this). Consider the process

\[
K^{\varepsilon_k}_j(\tau) := J^{\varepsilon_k}_j(\tau) - \int_0^\tau (\mathcal{R}_j(J^{\varepsilon_k}(s)) + \mathcal{T}_j) \, ds.
\]

(4.17)

Then, according to (4.2),

\[
K^{\varepsilon_k}_j(\tau) = M^{\varepsilon_k}_j(\tau) + \Theta^{\varepsilon_k}_j(\tau),
\]

where \(M^{\varepsilon_k}_j\) is a martingal and by (4.8) we have

\[
\Theta^{\varepsilon_k}_j(\tau) = \int_0^\tau ((i\nabla_j H_2 + g_j) \cdot v_j^{\varepsilon_k} - \langle (i\nabla_j H_2 + g_j) \cdot v_j^{\varepsilon_k} \rangle + \sqrt{\varepsilon}r_j \cdot v_j^{\varepsilon_k} + (d^1_{jj} - \mathcal{T}_j)) \, ds.
\]

(4.18)

Due to Corollary 3.3 and Theorem 3.1.3, we have

\[
\mathbb{E} \sup_{0 \leq \tau \leq T} |r_j \cdot v_j^{\varepsilon_k}| \leq C, \quad |d^1_{jj} - \mathcal{T}_j| \leq C\sqrt{\varepsilon}.
\]

(4.19)

Then, applying Lemma 4.4, we get

\[
\mathbb{E} \sup_{0 \leq \tau \leq T} |\Theta^{\varepsilon_k}_j(\tau)| \to 0 \quad \text{as} \quad \varepsilon_k \to 0.
\]

(4.20)

Consequently,

\[
\lim_{\varepsilon_k \to 0} \mathcal{D}(K^{\varepsilon_k}_j(\cdot)) = \lim_{\varepsilon_k \to 0} \mathcal{D}(M^{\varepsilon_k}_j(\cdot))
\]

(4.21)

in the sense that if one limit exists then the another exists as well and the two are equal.

Due to (4.7) and the Skorokhod Theorem, we can find random processes \(L^{\varepsilon_k}(\tau)\) and \(L(\tau), 0 \leq \tau \leq T, \) such that \(\mathcal{D}(L^{\varepsilon_k}(\cdot)) = \mathcal{D}(J^{\varepsilon_k}(\cdot)),\) \(\mathcal{D}(L(\cdot)) = Q_0\) and

\[
L^{\varepsilon_k} \to L \quad \text{in} \quad C([0, T], \mathbb{R}^N) \quad \text{as} \quad \varepsilon_k \to 0 \quad \text{a.s.}
\]

Then by (4.17) the left-hand side limit in (4.21) exists and equals

\[
L_j(\tau) - \int_0^\tau (\mathcal{R}_j(L(s)) + \mathcal{T}_j) \, ds.
\]

(4.22)

Due to (4.6), the family of martingales \(\{M^{\varepsilon_k}_j, k \in \mathbb{N}\}\) is uniformly square integrable. Due to (4.21), they converge in distribution to the process (4.22). Then the latter is a
square integrable martingals as well. Thus, each limiting point $Q_0$ is a weak solution of the transport equation (4.9).

Since the initial conditions $u_0$ are independent from $\varepsilon$, Theorem 3.1.4 implies that $D(J(0)) = D(I(u_0))$. In [YW] Yamada and Watanabe established the uniqueness of a weak solution for an equation with a more general dispersion matrix than that for (4.9), but with a Lipschitz-continuous drift. Their proof can be easily generalized to our case by the stopping time arguments. We will not do this here since in Proposition 4.5 we will consider more difficult infinite-dimensional situation.

The uniqueness of a weak solution of (4.9) implies that all the limiting points (4.7) coincide and we obtain the convergence (4.11). The first estimate in (4.12) follows from Corollary 3.3 and the second one follows from Lemma 4.3.

Now we will prove the uniformity in $N$ of the convergence (4.11). Recall that it is understood in the sense that for any $\Lambda \subset \mathbb{Z}^d$ as in (4.13) we have

$$D((J^\varepsilon_j)_{j \in \Lambda}) \rightarrow D((J^0_j)_{j \in \Lambda}) \quad \text{as} \quad \varepsilon \rightarrow 0 \quad \text{on} \quad C([0, T], \mathbb{R}^M) \quad \text{uniformly in} \quad N. \quad (4.23)$$

It is well known that the weak convergence of probability measures on a separable metric space is equivalent to convergence in the dual-Lipschitz norm, see Theorem 11.3.3 in [Dud]. Analysing the proof of this theorem, we see that in order to establish the uniformity in $N$ of the convergence (4.23) with respect to the dual-Lipschitz norm, it suffices to show that for any bounded continuous functional $h : (J_j(\cdot))_{j \in \Lambda} \in C([0, T], \mathbb{R}^M) \mapsto \mathbb{R}$, we have

$$E \ h(J^\varepsilon) \rightarrow E \ h(J^0) \quad \text{as} \quad \varepsilon \rightarrow 0 \quad \text{uniformly in} \quad N, \quad (4.24)$$

where we have denoted $h(J) := h((J_j(\cdot)))_{j \in \Lambda})$. In order to prove (4.24), first we pass to the limit $N \rightarrow \infty$. Recall that $C^\infty = \cup_{N \in \mathbb{N}} C(N)$. Denote $J^{\varepsilon,N} = (J_0^j,N)_{j \in C^\infty}$, where

$$J_0^j,N := \begin{cases} J_0^j & \text{if} \quad j \in C = C(N), \\
0 & \text{if} \quad j \in C^\infty \setminus C. \quad (4.25) \end{cases}$$

Using the uniformity in $N$ of estimate (4.6), we get that the family of measures

$$\{D(J^{\varepsilon,N}(\cdot)), \quad 0 < \varepsilon \leq 1, \quad N \in \mathbb{N} \}$$

is tight on a space $C([0, T], \mathbb{R}^\infty)$. Take any limiting point $Q_0^\infty$ such that $D(J^{\varepsilon_k,N_k}(\cdot)) \rightarrow Q_0^\infty$ as $\varepsilon_k \rightarrow 0$, $N_k \rightarrow \infty$. Recall that the initial conditions $u_0$ satisfy HI(ii). Denote the vector of actions corresponding to $u_0^\infty$ by $J_0^\infty = I(u_0^\infty) \in \mathbb{R}^{\infty}_{0+}$.

**Proposition 4.5.** The measure $Q_0^\infty$ is a law of the process $J^{0,\infty}(\tau)$ which is a unique weak solution of the transport equation for the infinite system of rotators

$$dJ_j = (R_j(J) + T_j) \, d\tau + \sqrt{2J_0^j T_j} \, d\tilde{\beta}_j, \quad j \in C^\infty, \quad D(J(0)) = D(I^0). \quad (4.26)$$

Moreover, $D(J^{\varepsilon,N}(\cdot)) \rightarrow D(J^{0,\infty}(\cdot))$ as $\varepsilon \rightarrow 0$, $N \rightarrow \infty$ on $C([0, T], \mathbb{R}^\infty)$.

Before proving this proposition we will establish (4.24). Proposition 4.5 implies

$$E \ h(J^\varepsilon) \rightarrow E \ h(J^{0,\infty}) \quad \text{as} \quad \varepsilon \rightarrow 0, \quad N \rightarrow \infty. \quad (4.27)$$

In view of (4.11), the convergence (4.27) implies that $E \ h(J^\varepsilon) \rightarrow E \ h(J^{0,\infty})$ when first $\varepsilon \rightarrow 0$ and then $N \rightarrow \infty$. Consequently, for all $\delta > 0$ there exist $N_1 \in \mathbb{N}$ and $\varepsilon_1 > 0$, such that for every $N \geq N_1$, $0 \leq \varepsilon < \varepsilon_1$, we have

$$|E \ h(J^\varepsilon) - E \ h(J^{0,\infty})| < \delta/2.$$
Then, for $N$ and $\varepsilon$ as above,
\[
|E h(J^\varepsilon) - E h(J^0)| \leq |E h(J^\varepsilon) - E h(J^{0,\infty})| + |E h(J^{0,\infty}) - E h(J^0)| < \delta. \tag{4.28}
\]
Choose $\varepsilon_2 > 0$ such that for every $0 < \varepsilon < \varepsilon_2$ and $N < N_1$ we have
\[
|E h(J^\varepsilon) - E h(J^0)| < \delta. \tag{4.29}
\]
Then, due to (4.28) and (4.29), for all $N$ and $\varepsilon < \varepsilon_1 \wedge \varepsilon_2$ we have
\[
|E h(J^\varepsilon) - E h(J^0)| < \delta.
\]

Thus, we obtain (4.24). The proof of the theorem is completed. \hfill \Box

**Proof of Proposition 4.5.** To prove that $Q_0^\infty$ is a law of a weak solution of (4.26), it suffices to show that the processes (4.16) are square-integrable martingales with respect to the measure $Q_0^\infty$ and the natural filtration of $\sigma$-algebras in $C([0, T], \mathbb{R}^\infty)$ (see [Yor]). The proof of that literally coincides with the corresponding proof for the finite-dimensional case, one should just replace the limit $\varepsilon_k \to 0$ by $\varepsilon_k \to 0, N_k \to \infty$ and the space $C([0, T], \mathbb{R}^N)$ by $C([0, T], \mathbb{R}^\infty)$. To prove that a weak solution of (4.26) is unique, it suffices to show that the pathwise uniqueness of solution holds (see [RSZ]). Let $J(\tau)$ and $\dot{J}(\tau)$ be two solutions of (4.26), defined on the same probability space and corresponding to the same Brownian motions and initial conditions, distributed as $I_0^\infty$. Let $w(\tau) := J(\tau) - \dot{J}(\tau)$.
Following literally the proof of Theorem 1 in [YW], for every $j \in C^\infty$ and any $\tau \geq 0$ we get the estimate
\[
E |w_j(\tau)| \leq E \int_0^\tau |R_j(J(s)) - R_j(\dot{J}(s))| \, ds. \tag{4.30}
\]
Define for $R > 0$ and $q > 0$ a stopping time
\[
\tau_R = \inf \{ \tau \geq 0 : \exists j \in C^\infty \text{ satisfying } J_j(\tau) \vee \dot{J}_j(\tau) \geq R(|j|^q + 1) \}.
\]

Arguing similarly to Lemma 3.2, we get that $J(\tau)$ and $\dot{J}(\tau)$ satisfy the first estimate in (4.12). Then for any $\tau \geq 0$ we have
\[
P(\tau_R \leq \tau) \leq \sum_{j \in C^\infty} P(\sup_{0 \leq s \leq \tau} J_j(s) \geq R(|j|^q + 1)) + \sum_{j \in C^\infty} P(\sup_{0 \leq s \leq \tau} \dot{J}_j(s) \geq R(|j|^q + 1))
\leq C \sum_{j \in C^\infty} e^{-2\alpha R(|j|^q + 1)} \to 0 \text{ as } R \to \infty. \tag{4.31}
\]

For $L \in \mathbb{N}$ denote $|w|_L := \sum_{|j| \leq L} e^{-|j|} |w_j|$. Using the Taylor expansion, it is possible to show that, in view of (4.10) and assumption $Hg(i)$, the derivatives $\partial_{j_k} R_j(J)$ have at most a polynomial growth of some power $m > 0$, which is uniform in $j, k \in C^\infty$. Since for any $\tau < \tau_R$ and $k \in C^\infty$ satisfying $|k| \leq L + 1$ we have $J_k(\tau), \dot{J}_k(\tau) \leq R((L + 1)^q + 1)$, then
estimate (4.30) implies

\[
\mathbb{E}|w(\tau \wedge \tau_R)| \leq C \sum_{|j| \leq L} e^{-|j|} \mathbb{E} \int_0^{\tau \wedge \tau_R} \left(1 + \sum_{k:|k-j| \leq 1} (J_k + \hat{J}_k)^m \right) \sum_{k:|k-j| \leq 1} |w_j| \, ds
\]

\[
\leq C(R)(L+1)^m \mathbb{E} \int_0^{\tau \wedge \tau_R} \left(|w| + e^{-L} \sum_{|k| = L+1} |w_k| \right) \, ds
\]

\[
\leq C_1(R)(L+1)^m \int_0^{\tau} (\mathbb{E} |w(s \wedge \tau_R)| + e^{-L}L^{d-1}) \, ds,
\]

where we used \( \mathbb{E} \sum_{|k| = L+1} |w_k| \leq CL^{d-1} \). Applying the Gronwall-Bellman inequality, we obtain

\[
\mathbb{E}|w(\tau \wedge \tau_R)| \leq L^{d-1} e^{-L+C_1(R)(L+1)^m} \tau.
\]

Choosing \( q < 1/m \), we obtain that \( \mathbb{E}|w(\tau \wedge \tau_R)| \to 0 \) as \( L \to \infty \) and, consequently, \( \mathbb{E}|w_j(\tau \wedge \tau_R)| = 0 \) for all \( j \in \mathcal{C}^\infty \). Sending \( R \to \infty \), in view of (4.31) we get that \( \mathbb{E}|w_j(\tau)| = 0 \) for any \( \tau \geq 0 \) and \( j \in \mathcal{C}^\infty \).

Let us now investigate the dynamics in the original \((I, \varphi)\)-variables. Let \( u^\varepsilon(\tau) \) be a solution of (2.5)-(2.6), written in the slow time and \( I^\varepsilon(\tau) = I(u^\varepsilon(\tau)) \) be the corresponding vector of actions. By Theorems 3.1.4 and 4.2 we have

\[
\lim_{\varepsilon \to 0} \mathcal{D}(I^\varepsilon(\cdot)) = \lim_{\varepsilon \to 0} \mathcal{D}(J^\varepsilon(\cdot)) = Q_0 \quad \text{on} \quad C([0,T], \mathbb{R}^N).
\]

Since the estimate of Theorem 3.1.4 and the convergence (4.11) are uniform in \( N \), then the convergence \( \mathcal{D}(I^\varepsilon(\cdot)) \to Q_0 \) is also uniform in \( N \). Thus, we get

**Theorem 4.6.** The assertion of Theorem 2.2 holds. Moreover, for any \( j \in \mathcal{C} \)

\[
\mathbb{E} \sup_{\tau \in [0,T]} e^{2\alpha t^0_j(\tau)} < C \quad \text{and} \quad \int_0^T P(I^0_j(\tau) < \delta) \, d\tau \to 0 \quad \text{as} \quad \delta \to 0,
\]

where the latter convergence is uniform in \( N \).

### 4.2 Joint distribution of actions and angles

Here we prove Theorem 4.7, which describes the limiting joint dynamics of actions and angles. Let, as usual, \( u^\varepsilon(\tau) \) be a solution of (2.5)-(2.6), written in the slow time, and let \( I^\varepsilon(\tau) = I(u^\varepsilon(\tau)), \varphi^\varepsilon(\tau) = \varphi(u^\varepsilon(\tau)) \). Denote by \( \mu^\varepsilon = \mathcal{D}(I^\varepsilon(\tau), \varphi^\varepsilon(\tau)) \) the law of \( u^\varepsilon(\tau) \) in action-angle coordinates. For any function \( h(\tau) \geq 0 \) satisfying \( \int_0^T h(\tau) \, d\tau = 1 \), set

\[
\mu^\varepsilon(h) := \int_0^T h(\tau) \mu^\varepsilon \, d\tau.
\]

Moreover, denote \( m^0(h) := \int h(\tau) \mathcal{D}(P^0(\tau)) \, d\tau \), where \( P^0(\tau) \) is a weak solution of (2.8)-(2.9).
**Theorem 4.7.** For any continuous function $h$ as above, we have
\[
\mu^\varepsilon(h) \to m^0(h) \times d\varphi \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly in } N.
\]

**Proof.** Let us first consider the case $h = (\tau_2 - \tau_1)^{-1} \mathbb{I}_{[\tau_1, \tau_2]}$, where $\mathbb{I}_{[\tau_1, \tau_2]}$ is an indicator function of the interval $[\tau_1, \tau_2]$. Let $v^\varepsilon(\tau)$ be a solution of (4.1) and $J^\varepsilon(\tau)$, $\psi^\varepsilon(\tau)$ be the corresponding vectors of actions and angles. Take a set $\Lambda$ as in (4.13) and a function $P \in \mathcal{L}_b(\mathbb{C}^N)$ which depends on $u = (u_j)_{j \in \mathcal{C}} \in \mathbb{C}^N$ only through $(u_j)_{j \in \Lambda}$. Due to Theorem 3.1.4, we have
\[
\int_{\tau_1}^{\tau_2} \langle \mu^\varepsilon_\tau, P \rangle \, d\tau = E \int_{\tau_1}^{\tau_2} P(u^\varepsilon(\tau)) \, d\tau \quad \text{is close to} \quad E \int_{\tau_1}^{\tau_2} P(v^\varepsilon(\tau)) \, d\tau \quad \text{uniformly in } N.
\]
Due to Lemma 4.4, the integral $E \int_{\tau_1}^{\tau_2} P(v^\varepsilon(\tau)) \, d\tau$ is close to $E \int_{\tau_1}^{\tau_2} \langle P(J^\varepsilon(\tau)) \rangle \, d\tau$ uniformly in $N$. Due to Theorem 4.2, the last integral is uniformly in $N$ close to
\[
E \int_{\tau_1}^{\tau_2} \langle P(J^0(\tau)) \rangle \, d\tau = E \int_{T^N} \int_{\tau_1}^{\tau_2} P(J^0(\tau), \varphi) \, d\tau \, d\varphi = (\tau_2 - \tau_1) \langle m^0(h) \times d\varphi, P \rangle.
\]
In the case of a continuous function $h$, we approximate it by piecewise constant functions. \hfill \Box

### 4.3 Stationary measures

In this section we prove Theorem 2.3 which describes the limiting behaviour of a stationary regime of (2.5).

**The effective equation and proof of Theorem 2.3.i.** The transport equation (4.9) is irregular: its dispersion matrix is not Lipschitz continuous, so we do not know if (4.9) is mixing or not. We are going to lift it to so-called effective equation which is regular and mixing.

Let us define an operator $\Psi_\theta : v = (v_j)_{j \in \mathcal{C}} \in \mathbb{C}^N \mapsto \mathbb{C}^N$ of rotation by an angle $\theta = (\theta_j)_{j \in \mathcal{C}} \in T^N$, i.e. $(\Psi_\theta v)_j = v_j e^{i\theta_j}$. We rewrite the function $\mathcal{R}_j$ from (2.7) as
\[
\mathcal{R}_j(J) = (g_j(v) \cdot v_j) = \int_{T^N} g_j(\psi_\theta v) \cdot (e^{i\theta_j} v_j) \, d\theta = \mathcal{K}_j(v) \cdot v_j, \quad (4.33)
\]
where $\mathcal{K}_j(v) := \int_{T^N} e^{-i\theta_j} g_j(\psi_\theta v) \, d\theta$ and $d\theta$ is a normalized Lebesgue measure on the torus $T^N$. Consider the effective equation
\[
dv_j = \mathcal{K}_j(v) \, d\tau + \sqrt{T_j} d\beta_j, \quad j \in \mathcal{C}, \quad (4.34)
\]
where $\beta_j$, as usual, are standard complex independent Brownian motions. It is well known that a stochastic equation of the form (4.34) has a unique solution which is defined globally (see [Kha12]), and that it is mixing (see [Kha12],[Ver87],[Ver97]). The following proposition explains the role of the effective equation.
Proposition 4.8. (i) Let \( v(\tau), \tau \geq 0 \) be a weak solution of the effective equation (4.34) and \( J(\tau) = J(v(\tau)) \) be the corresponding vector of actions. Then \( J(\tau), \tau \geq 0 \) is a weak solution of the transport equation (4.9).

(ii) Let \( J^0(\tau), \tau \geq 0 \) be a weak solution of the transport equation (4.9). Then for any vector \( \theta = (\theta_j)_{j \in \mathcal{C}} \in \mathbb{T}^N \) there exists a weak solution \( v(\tau) \) of the effective equation (4.34) such that

\[
\mathcal{D}(J(v(\cdot))) = \mathcal{D}(J^0(\cdot)) \text{ on } C([0, \infty), \mathbb{R}^N) \quad \text{and} \quad v_j(0) = \sqrt{2J^0_j(0)e^{i\theta_j}}, \, j \in \mathcal{C}. \tag{4.35}
\]

Proof. (i) Due to (4.33) and (4.34), the actions \( J(\tau) \) satisfy

\[
dJ_j = (\mathcal{R}_j(J) + \mathcal{T}_j) \, d\tau + \sqrt{T_j} \, v_j \, d\beta_j, \quad j \in \mathcal{C}. \tag{4.36}
\]

The drift and the diffusion matrix of equation (4.36) coincide with those of the transport equation (4.9). Consequently, \( J(\tau) \) is a solution of the (local) martingale problem associated with the transport equation (see [KaSh], Proposition. 5.4.2). So, due to [KaSh], Proposition 5.4.6, we get that \( J(\tau) \) is a weak solution of the transport equation (4.9).

(ii) Let \( v(\tau) \) be a solution of the effective equation with the initial condition as in (4.35). Then, due to (i), the process \( J(\tau) := J(v(\tau)) \) is a weak solution of the transport equation and \( J(0) = J^0(0) \). Since the weak solution of the transport equation is unique, we obtain that \( \mathcal{D}(J(\cdot)) = \mathcal{D}(J^0(\cdot)) \). Consequently, \( v(\tau) \) is the desired process.

Let \( m \) be the unique stationary measure of the effective equation. Denote the projections to the spaces of actions and angles by \( \Pi_{ac} : v \in \mathbb{C}^N \mapsto \mathbb{R}^N_{+0} \ni I \) and \( \Pi_{ang} : v \in \mathbb{C}^N \mapsto \mathbb{T}^N \ni \psi \) correspondingly. Denote

\[
\pi := \Pi_{ac} m. \tag{4.37}
\]

Corollary 4.9. The transport equation (4.9) is mixing, and \( \pi \) is its unique stationary measure. More precisely, for any its solution \( J(\tau) = (J_k(\tau))_{k \in \mathcal{C}} \), satisfying for some \( \beta > 0 \) and all \( k \in \mathcal{C} \) the relation \( \mathbb{E}e^{\beta J_k(0)} < \infty \), we have \( \mathcal{D}(J(\tau)) \rightarrow \pi \) as \( \tau \rightarrow \infty \).

Corollary 4.9 implies Theorem 2.3.i.

Proof. First we claim that \( \pi \) is a stationary measure of the transport equation. Indeed, take a stationary distributed solution \( \tilde{v}(\tau) \) of the effective equation, \( \mathcal{D}(\tilde{v}(\tau)) \equiv m \). By Proposition 4.8.i, the process \( J(\tilde{v}(\tau)) \) is a stationary weak solution of the transport equation. It remains to note that (4.37) implies \( \mathcal{D}(J(\tilde{v}(\tau))) \equiv \pi \).

Now we claim that any solution \( J^0(\tau) \) of the transport equation with the initial conditions as above converges in distribution to \( \pi \) as \( \tau \rightarrow \infty \). For some \( \theta \in \mathbb{T}^N \) take \( v(\tau) \) from Proposition 4.8.ii. Due to the mixing property of the effective equation, \( \mathcal{D}(v(\tau)) \rightarrow m \) as \( \tau \rightarrow \infty \) and, consequently, \( \mathcal{D}(J^0(\tau)) = \mathcal{D}(J(v(\tau))) \rightarrow \Pi_{ac} m = \pi \) as \( \tau \rightarrow \infty \).

Proof of Theorem 2.3.ii. First we will show that

\[
\Pi_{acs} \tilde{\mu}^\varepsilon \rightarrow \pi \quad \text{as} \quad \varepsilon \rightarrow 0. \tag{4.38}
\]

We will work in the \( v \)-variables. Note that equation (4.1) is mixing since it is obtained by a \( C^2 \)-smooth time independent change of variables from equation (2.5), which is mixing. Denote by \( \tilde{\nu}^\varepsilon \) its unique stationary measure. Due to Theorem 3.1.4, to establish (4.38) it suffices to show that

\[
\Pi_{acs} \tilde{\nu}^\varepsilon \rightarrow \pi \quad \text{as} \quad \varepsilon \rightarrow 0. \tag{4.39}
\]
Let $\tilde{v}^\varepsilon(\tau)$ be a stationary solution of equation (4.1), $\mathcal{D}(\tilde{v}^\varepsilon(\tau)) \equiv \tilde{v}^\varepsilon$, and $\tilde{J}^\varepsilon(\tau) = J(\tilde{v}^\varepsilon(\tau))$ be the corresponding vector of actions. Similarly to Proposition 4.1 we get that the set of laws $\{\mathcal{D}(\tilde{J}^\varepsilon(\cdot)), \ 0 < \varepsilon \leq 1\}$ is tight in $C([0, T], \mathbb{R}^N)$. Let $\tilde{Q}_0$ be its limiting point as $\varepsilon_k \rightarrow 0$. Obviously, it is stationary in $\tau$. The same arguments that was used in the proof of Theorem 4.2 imply

**Proposition 4.10.** The measure $\tilde{Q}_0$ is a law of the process $\tilde{J}^0(\tau), \ 0 \leq \tau \leq T$, which is a stationary weak solution of the transport equation (4.9).

Since $\pi$ is the unique stationary measure of the transport equation, we have $\mathcal{D}(\tilde{J}^0(\tau)) \equiv \pi$. Consequently, we get (4.39) which implies (4.38).

Let $\tilde{u}^\varepsilon(\tau)$ be a stationary solution of equation (2.5) and $\tilde{J}^\varepsilon(\tau), \tilde{\varphi}^\varepsilon(\tau)$ be the corresponding vectors of actions and angles. By the same reason as in Theorem 4.7, we have

$$\tilde{\mu}^\varepsilon(h) \rightarrow \tilde{m}^0(h) \times d\varphi \quad \text{as} \quad \varepsilon \rightarrow 0, \quad (4.40)$$

where $\tilde{\mu}^\varepsilon(h)$ and $\tilde{m}^0(h)$ are defined as $\mu^\varepsilon(h)$ and $m^0(h)$, but with the processes $I^\varepsilon(\tau), \varphi^\varepsilon(\tau)$ and $I^0(\tau)$ replaced by the processes $\tilde{I}^\varepsilon(\tau), \tilde{\varphi}^\varepsilon(\tau)$ and $\tilde{J}^0(\tau)$ correspondingly. Since a stationary regime does not depend on time, we get (2.11):

$$\mathcal{D}(\tilde{I}^\varepsilon(\tau), \tilde{\varphi}^\varepsilon(\tau)) \rightarrow \pi \times d\varphi \quad \text{as} \quad \varepsilon \rightarrow 0. \quad (4.41)$$

Assume now that the transport equation for the infinite system of rotators (4.26) has a unique stationary measure $\pi^\infty$. Let us define $\tilde{J}^{\varepsilon,N}$ as in (4.25), but with $J^\varepsilon$ replaced by $\tilde{J}^\varepsilon$. Then the set of laws $\{\mathcal{D}(\tilde{J}^{\varepsilon,N}(\cdot)), \ 0 < \varepsilon \leq 1, N \in \mathbb{N}\}$ is tight in $C([0, T], \mathbb{R}^\infty)$. Let $\tilde{Q}_0^\infty$ be its limiting point as $\varepsilon_k \rightarrow 0, N_k \rightarrow \infty$. Similarly to Proposition 4.1 we get

**Proposition 4.11.** The measure $\tilde{Q}_0^\infty$ is a law of the process $\tilde{J}^{0,\infty}(\tau), \ 0 \leq \tau \leq T$, which is a stationary weak solution of equation (4.26).

Since (4.26) has a unique stationary measure $\pi^\infty$, we get that $\mathcal{D}(\tilde{J}^{\varepsilon,N}(\tau)) \rightarrow \pi^\infty$ as $\varepsilon \rightarrow 0, N \rightarrow \infty$. Then, arguing as in Theorem 4.2, we get that the convergence (4.39) is uniform in $N$. As in the proof of Theorem 4.7, this implies that the convergence (4.40) and, consequently, the convergence (4.41) are also uniform in $N$.

**Proof of Theorem 2.3.iii.** Due to the mixing property of (2.5), we have $\mathcal{D}(I^\varepsilon(\tau)) \rightarrow \prod_{\text{ang}} \tilde{\mu}^\varepsilon$ as $\tau \rightarrow \infty$. Then item (ii) of the theorem implies that $\Pi_{\text{ang}} \tilde{\mu}^\varepsilon \rightarrow \pi$ as $\varepsilon \rightarrow 0$. On the other hand, Theorem 4.6 implies that $\mathcal{D}(I^\varepsilon(\tau)) \rightarrow \mathcal{D}(I^0(\tau))$ as $\varepsilon \rightarrow 0$ for any $\tau \geq 0$, where $I^0(\tau)$ is a weak solution of equation (2.8)-(2.9). Then item (i) of the theorem implies that $\mathcal{D}(I^0(\tau)) \rightarrow \pi$ as $\tau \rightarrow \infty$. The proof of the theorem is completed. \[\square\]

**Remark 4.12.** It is possible to show that the effective equation is rotation invariant: if $v(\tau)$ is its weak solution, then for any $\xi \in \mathbb{T}^N$ we have that $\Psi_\xi v$ is also its weak solution. Since it has the unique stationary measure $m$, we get that $m$ is rotation invariant. Consequently, $\Pi_{\text{ang}} m = d\varphi$. That is why the convergence (2.15) is equivalent to (2.11).

4.4 Examples

1. Consider a system with linear dissipation, i.e. $p = 2$ and $g_j(u) = - u_j + \sum_{k:|k-j|=1} b_{jk} u_k$, where $b_{jk} \in \mathbb{C}$. If $|b_{jk}|$ are sufficiently small uniformly in $j$ and $k$ then assumption $Hg$ is
satisfied (see Example 2.4). Since \( \langle u_k \cdot u_j \rangle = 0 \) for \( k \neq j \), we have \( \mathcal{R}_j(I) = -2I_j \). Then

the transport equation (2.8) turns out to be diagonal and takes the form

\[
dI_j = (-2I_j + T_j)d\tau + \sqrt{2T_j}I_j d\tilde{\beta}_j, \quad j \in \mathcal{C}.
\]

The unique stationary measure of (4.42) is

\[
\pi(dI) = \prod_{j \in \mathcal{C}} \frac{2}{T_j} \mathbb{I}_{\mathbb{R}_+}(I_j)e^{-2I_j/T_j}dI_j.
\]

The transport equation for the infinite system of rotators is diagonal and, consequently, has a unique stationary measure. Thus, the convergence (2.11) holds uniformly in \( N \).

2. Let \( d = 1 \) and \( \mathcal{C} = \{1, 2, \ldots, N\} \). Put for simplicity \( p = 4 \) and choose

\[
g_j(u) = \frac{1}{4}\left(|u_{j+1}|^2u_j - |u_{j-1}|^2u_j - |u_j|^2u_j\right),
\]

where \( 1 \leq j \leq N \), \( u_0 = u_{N+1} = 0 \). By the direct computation one can verify that \( g_j \) satisfies the condition \( Hg \). We have \( R_j(I) = \langle g_j(u) \cdot u_j \rangle = I_{j+1}I_j - I_{j-1}I_j - I_j^2 \), and the transport equation (2.8) takes the form

\[
dI_j = \left(\frac{1}{2}(2I_{j+1}I_j - 2I_{j-1}I_j) - I_j^2 + T_j\right)d\tau + \sqrt{2I_jT_j}d\tilde{\beta}_j.
\]

Its r.h.s. consists of two parts:

\[
dI_j/d\tau = \mathbf{\nabla}\Theta(j) + \text{Ter}(j),
\]

where \( \Theta(j) := 2I_{j+1}I_j \), \( \mathbf{\nabla}\Theta(j) := \frac{1}{2}(\Theta(j) - \Theta(j - 1)) \) is the discrete gradient of \( \Theta \), and \( \text{Ter}(j) := -I_j^2 + T_j + \sqrt{2I_jT_j}d\tilde{\beta}_j/d\tau \). Analogically to the concept of the flow of energy (see [BoLeR], Section 5.2) we call the function \( \Theta(j) \) the \textit{flow of actions}. The term \( \mathbf{\nabla}\Theta(j) \) describes the transport of actions through the \( j \)-th site while the term \( \text{Ter}(j) \) can be considered as an input of a (new) stochastic thermostat interacting with the \( j \)-th node. In the same way one can treat the case \( p = 2q \), where \( q \in \mathbb{N}, q > 2 \).

5 Auxiliary propositions

In this section we prove Lemmas 4.3 and 4.4.

5.1 Proof of Lemma 4.3

For the brevity of notations we skip the index \( \varepsilon \) everywhere, except the set \( E^\varepsilon \). Let us rewrite (4.2) for \( k \in \Lambda \) as an equation with real noise

\[
d\mathcal{J} = A^\mathcal{J}d\tau + \sigma d\tilde{\beta}, \quad \text{where} \quad \mathcal{J} := (J_k)_{k \in \Lambda}, \ A^\mathcal{J} := (A^k_{\mathcal{J}})_{k \in \Lambda}, \ \sigma \in \mathbb{R} \times 2N \text{ matrix with real entries and } \tilde{\beta} = (\text{Re } \beta_k, \text{Im } \beta_k)_{k \in \mathcal{C}}.
\]

Denote by \( a = (a_{kl})_{k,l \in \Lambda} \) the diffusion matrix for (5.1), divided by two, \( a := \frac{1}{2}\sigma^T \sigma \). It is \( M \times M \)-matrix with real
entires $a_{kl} = S_{kl}^j/2$, $k, l \in \Lambda$, where $S_{kl}^j$ is calculated in (A.3). Then Theorem 3.1.3 implies that
\[
|a_{kl} - T_k d_{kl}| |v_k|^2 \leq C \sqrt{\varepsilon} |v_k||v_l|. \tag{5.2}
\]

**Step 1.** For $R > 0$ denote by $\tau_R$ the stopping time
\[
\tau_R = \inf \{ \tau \geq 0 : \|J(\tau)\|_{R^M} \vee \|A^J(\tau)\|_{R^M} \geq R \},
\]
where $\| \cdot \|_{R^M}$ stands for the Euclidean norm in $\mathbb{R}^M$, $J(\tau) = J(v(\tau))$, $A^J(\tau) = A^J(v(\tau))$, and $v(\tau)$ is a solution of (4.1). A particular case of Theorem 2.2.2 in [Kry] provides that
\[
\mathbb{E} \int_0^{\tau_R \wedge T} e^{-\int_0^\tau \|A^J(s)\|_{R^M} ds} \|E^\varepsilon(\Omega_J(\tau))\|_{\pi_{\varepsilon}} (\det a(\tau))^{1/M} d\tau \leq C(R, M)|E^\varepsilon|^{1/M}, \tag{5.3}
\]
where $a(\tau) = a(v(\tau))$. Denote the event $\Omega_{\nu}(\tau) = \{ \det a(\tau) < \nu \}$. We have
\[
\int_0^T \mathbb{P}(J(\tau) \in E^\varepsilon) d\tau = \mathbb{E} \int_0^T \int E^\varepsilon(\Omega_J(\tau)) d\tau \leq \mathbb{E} \int_0^T \int E^\varepsilon(\Omega_J(\tau)) ||E^\varepsilon(\Omega_J(\tau))||_{\pi_{\varepsilon}} (\det a(\tau))^{1/M} d\tau
\]
\[
+ \int_0^T \mathbb{P}(\Omega_{\nu}(\tau)) d\tau + T \mathbb{P}(\tau_R < T) =: \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3. \tag{5.4}
\]
Due to (5.3),
\[
\mathcal{Y}_1 \leq \frac{c^{TR}}{\nu^{1/M}} \mathbb{E} \int_0^{\tau_R \wedge T} e^{-\int_0^\tau \|A^J(s)\|_{R^M} ds} ||E^\varepsilon(\Omega_J(\tau))||_{\pi_{\varepsilon}} (\det a(\tau))^{1/M} d\tau \leq C(R, M)|E^\varepsilon|^{1/M}. \tag{5.5}
\]
Take $\nu = \sqrt{|E^\varepsilon|}$. Choosing $R$ sufficiently large and $\varepsilon$ sufficiently small, we can make the terms $\mathcal{Y}_1$ and $\mathcal{Y}_3$ arbitrary small uniformly in $N$. Indeed, for $\mathcal{Y}_1$ this follows from (5.5), while for $\mathcal{Y}_3$ this follows from Corollary 3.3 and estimate (4.6). So, to finish the proof of the lemma it remains to show that if $\nu(\varepsilon) \to 0$ with $\varepsilon \to 0$ then
\[
\mathcal{Y}_2 = \int_0^T \mathbb{P}(\Omega_{\nu}(\tau)) d\tau \to 0 \text{ when } \varepsilon \to 0 \text{ uniformly in } N. \tag{5.6}
\]

**Step 2.** The rest of the proof is devoted to the last convergence. Note that by (5.2)
\[
\det a = \prod_{k \in \Lambda} (T_k J_k) + \sqrt{\varepsilon} \Delta_1,
\]
where $\mathbb{E} \sup_{0 \leq \tau \leq T} |\Delta_1| \leq C$ by Corollary 3.3. The constant $C$ does not depend on $N$ because the dimension $M$ does not depend on it. Then
\[
\mathbb{P}(\Omega_{\nu}) \leq \mathbb{P} \left( \prod_{k \in \Lambda} (T_k J_k) < \nu + \sqrt{\varepsilon} |\Delta_1| \right) \leq \sum_{k \in \Lambda} \mathbb{P} \left( J_k < T_k^{-1}(\nu + \sqrt{\varepsilon} |\Delta_1|)^{1/M} \right).
\]

Thus, to establish (5.6), it is sufficient to show that
\[
\int_0^T \mathbb{P} \left( \sqrt{J_j(\tau)} < \delta \right) d\tau \to 0 \text{ when } \delta \to 0 \text{ uniformly in } N \text{ and } \varepsilon \text{ sufficiently small.} \tag{5.7}
\]

**Step 3.** To prove the last convergence we use the concept of the local time. Let \( h \in C^2(\mathbb{R}) \) and its second derivative has at most polynomial growth at the infinity. We consider the process \( h_\tau := h(J_j(\tau)) \). Then, by the Ito formula,
\[
dh_\tau = A^h d\tau + \sigma^h d\tilde{\beta},
\]
where
\[
A^h = h'(J_j)A_j^\mathcal{I} + h''(J_j)a_{jj} = h'(J_j)(A_j \cdot v_j + d_{jj}^1) + h''(J_j)a_{jj},
\]
and the \( 1 \times 2N \)-matrix \( \sigma^h(\tau) = (\sigma^h_k(\tau)) \) is out of the interest.

Due to Theorem 3.1.3 and (5.2), for sufficiently small \( \varepsilon \) we have
\[
d_{jj}^1 \geq \frac{7}{8}T_j, \quad |a_{jj}| \leq \frac{3J_j}{2}T_j. \tag{5.8}
\]
Let \( \Theta_{b}(b, \omega) \) be the local time for the process \( h_\tau \). Then for any Borel set \( \mathcal{G} \subset \mathbb{R} \) we have
\[
\int_0^T \mathbb{I}_\mathcal{G}(h_\tau) \sum_k |\sigma^h_k|^2 d\tau = 2 \int_{-\infty}^\infty \mathbb{I}_\mathcal{G}(b) \Theta_T(b, \omega) db.
\]
On the other hand, denoting \( (h_\tau - b)_+ := \max(h_\tau - b, 0) \), we have
\[
(h_T - b)_+ = (h_0 - b)_+ + \int_0^T \mathbb{I}_{(b, \infty)}(h_\tau) \sigma^h d\tilde{\beta} + \int_0^T \mathbb{I}_{(b, \infty)}(h_\tau) A^h d\tau + \Theta_T(b, \omega).
\]
Consequently,
\[
\mathbb{E} \int_0^T \mathbb{I}_\mathcal{G}(h_\tau) \sum_k |\sigma^h_k|^2 d\tau = 2\mathbb{E} \int_{-\infty}^\infty \mathbb{I}_\mathcal{G}(b) \left( (h_T - b)_+ - (h_0 - b)_+ \right) + \int_0^T \mathbb{I}_{(b, \infty)}(h_\tau) A^h d\tau db. \tag{5.9}
\]
The left-hand side is non-negative, so
\[
\mathbb{E} \int_{-\infty}^\infty \mathbb{I}_\mathcal{G}(b) \int_0^T \mathbb{I}_{(b, \infty)}(h_\tau) A^h d\tau db \leq \mathbb{E} \int_{-\infty}^\infty \mathbb{I}_\mathcal{G}(b) \left( (h_T - b)_+ - (h_0 - b)_+ \right) db. \tag{5.9}
\]
Let us apply relation (5.9) with \( \mathcal{G} = (\xi_1, \xi_2) \), \( \xi_2 > \xi_1 > 0 \) and a function \( h(x) \in C^2(\mathbb{R}) \) that coincides with \( \sqrt{x} \) for \( x \geq \xi_1 \) and vanishes for \( x \leq 0 \). Due to Corollary 3.3, the right-hand side of (5.9) is bounded by \( (\xi_2 - \xi_1)C \). Then
\[
\mathbb{E} \int_{\xi_1}^{\xi_2} \int_0^T \mathbb{I}_{(\xi_1, \infty)}(\sqrt{J_j}) \left( \frac{A_j \cdot v_j + d_{jj}^1}{2\sqrt{J_j}} - \frac{a_{jj}}{4J_j^3} \right) d\tau db \leq (\xi_2 - \xi_1)C. \tag{5.10}
\]
In view of estimate (4.6) we have
\[ E \int_{\xi_1}^{\xi_2} \int_0^T \frac{|A_j \cdot v_j|}{2\sqrt{J_j}} d\tau db \leq (\xi_2 - \xi_1)C. \]

Moving this term to the right-hand side of (5.10), applying (5.8) and sending \( \xi_1 \) to \( 0^+ \), we get
\[ E \int_{\xi_1}^{\xi_2} \int_0^T 1_{(\sqrt{J_j})}J_j^{-1/2} d\tau db \leq C\xi_2. \]

Note that
\[ E \int_{\xi_1}^{\xi_2} \int_0^T 1_{(\sqrt{J_j})}J_j^{-1/2} d\tau db \geq \frac{1}{\delta} E \int_{\xi_1}^{\xi_2} \int_0^T 1_{(\sqrt{J_j})} d\tau db \]
\[ = \frac{1}{\delta} \int_{\xi_1}^{\xi_2} \int_0^T P(b < \sqrt{J_j} < \delta) d\tau db. \]

Consequently,
\[ \frac{1}{\xi_2} \int_0^T P(b < \sqrt{J_j} < \delta) d\tau db \leq C\delta. \]

Tending \( \xi_2 \to 0^+ \) we obtain that
\[ \int_0^T P(0 < \sqrt{J_j} < \delta) d\tau \to 0 \text{ when } \delta \to 0 \text{ uniformly in } N \text{ and } \varepsilon \text{ sufficiently small.} \]

**Step 4.** To establish (5.7) it remains to show that
\[ \int_0^T P(|v_j(\tau)| = 0) d\tau = 0 \text{ for all } N, j \in C \text{ and } \varepsilon \text{ sufficiently small.} \quad (5.11) \]

Writing a \( j \)-th component of equation (4.1) in the real coordinates \( v_j^x := \Re v_j \) and \( v_j^y := \Im v_j \), we obtain the following two-dimensional system:
\[ dv_j^x = \Re \tilde{A}_j d\tau + \Re(WdB)_j, \quad dv_j^y = \Im \tilde{A}_j d\tau + \Im(WdB)_j, \quad (5.12) \]
where \( \tilde{A}_j := \varepsilon^{-1}i\nabla_j H_0 + i\nabla_j H_2 + g_j + \sqrt{\varepsilon}v_j \). By the direct computation we get that the diffusion matrix for (5.12) with respect to the real Brownian motion \((\Re \beta_k, \Im \beta_k)_{k \in C}\) is
\[ a^j := \begin{pmatrix} d_{jj}^1 + \Re d_{jj}^2 & \Im d_{jj}^2 \\ \Im d_{jj}^2 & d_{jj}^1 - \Re d_{jj}^2 \end{pmatrix}. \]

Theorem 3.1.3 implies that for \( \varepsilon \) sufficiently small, \( \det a^j(\tau) \) is separated from zero uniformly in \( \tau \). For \( R > 0 \) define a stopping time
\[ \overline{\tau}_R = \inf\{\tau \geq 0 : |v_j(\tau)| \vee |\tilde{A}_j(\tau)| \geq \varepsilon^{-1}R\}. \]
Then, similarly to (5.4) and (5.5), we have
\[ E \int_0^T \mathbb{P}[|v_j(\tau)|] \, d\tau \leq C e^{\varepsilon^{-1}TR} E \int_0^{\bar{\tau}_R \wedge T} e^{-\frac{\tau}{\varepsilon}} \| \hat{A}_j(s) \| \, ds \|\| v_j(\tau) \| (\det a^j(\tau))^{1/2} \, d\tau \]
\[ + TP(\bar{\tau}_R < T) \leq C(R, \varepsilon^{-1}) \sqrt{\delta} + TP(\bar{\tau}_R < T). \] (5.13)
Letting first \( \delta \to 0 \) and then \( R \to \infty \) while \( \varepsilon \) is fixed, we arrive at (5.11).

5.2 Proof of Lemma 4.4

For the purposes of the proof we first introduce some notations. For events \( \Gamma_1, \Gamma_2 \) and a random variable \( \xi \) we denote
\[ E_{\Gamma_1} \xi := E(\xi \mathbb{1}_{\Gamma_1}) \text{ and } P_{\Gamma_1}(\Gamma_2) := P(\Gamma_2 \cap \Gamma_1). \]
Let us emphasize that in these definitions we consider an expectation and a probability on the complement of \( \Gamma_1 \). By \( \kappa(r), \kappa_1(r), \ldots \) we denote various functions of \( r \) such that \( \kappa(r) \to 0 \) as \( r \to \infty \). By \( \kappa_{\infty}(r) \) we denote functions \( \kappa(r) \) such that \( \kappa(r) = o(r^{-m}) \) for each \( m > 0 \). We write \( \kappa(r) = \kappa(r; b) \) to indicate that \( \kappa(r) \) depends on a parameter \( b \). Functions \( \kappa_{\infty}(r), \kappa(r), \kappa(r; b), \ldots \) never depend on \( N \) and may depend on \( \varepsilon \) only through \( r \), and we do not indicate their dependence on the dimension \( d \), power \( p \) and time \( T \). Moreover, they can change from formula to formula.

**Step 1.** For the brevity of notation we skip the index \( \varepsilon \). Denote by \( \tilde{\Lambda} \) the neighbourhood of radius 1 of \( \Lambda \):
\[ \tilde{\Lambda} := \{ n \in \mathbb{C} | \text{there exists } k \in \Lambda \text{ satisfying } |n - k| \leq 1 \}. \] (5.14)
Fix \( R > 0 \). Set
\[ \Omega_R = \{ \max_{k \in \Lambda} \sup_{0 \leq \tau \leq T} |J_k(\tau)| \vee |A^\psi_k(\tau)| \geq R \}. \] (5.15)
Due to Corollary 3.3 and estimate (4.6),
\[ P(\Omega_R) \leq \kappa_{\infty}(R). \] (5.16)
The polynomial growth of the function \( P \) implies
\[ E_{\Omega_R} \sup_{\tau \in [0,T]} \left| \int_0^\tau P(J(s), \psi(s)) \, ds \right| \leq \kappa_{\infty}(R), \]
and the function \( \langle P \rangle(J(s)) \) satisfies a similar relation. Thus it is sufficient to show that for any \( R \geq 0 \)
\[ \mathcal{U} := E_{\Omega_R} \sup_{\tau \in [0,T]} \left| \int_0^\tau P(J(s), \psi(s)) - \langle P \rangle(J(s)) \, ds \right| \to 0 \text{ as } \varepsilon \to 0 \text{ uniformly in } N. \]
For this purpose we consider a partition of the interval \([0,T]\) to subintervals of length \( \nu \) by the points
\[ \tau_l = \tau_0 + l \nu, \quad 0 \leq l \leq L, \quad L = \lfloor T/\nu \rfloor - 1, \]
where the (deterministic) initial point $\tau_0 \in [0, \nu)$ will be chosen later. Choose the diameter of the partition as 
\[ \nu = \varepsilon^{7/8}. \]

Denote
\[ \eta_l = \int_{\tau_l}^{\tau_{l+1}} P(J(s), \psi(s)) - \langle P \rangle(J(s)) \, ds. \]

Then
\[ \mathcal{U} \leq E_{\Omega_R} \sum_{l=0}^{L-1} |\eta_l| + \nu C(R). \]

Denote $Y(J) = (Y_k(J))_{\kappa \in \mathcal{C}} := (\partial \partial_k H_0(J))_{\kappa \in \mathcal{C}} \in \mathbb{R}^N$ and $Y(\tau) := Y(J(\tau))$. (5.17)

We have
\[ |\eta_l| \leq \left| \int_{\tau_l}^{\tau_{l+1}} P(J(s), \psi(s)) - P(J(\tau_l), \psi(\tau_l)) + \varepsilon^{-1} Y(\tau_l)(s - \tau_l) \, ds \right| \]
\[ + \left| \int_{\tau_l}^{\tau_{l+1}} P(J(\tau_l), \psi(\tau_l) + \varepsilon^{-1} Y(\tau_l)(s - \tau_l)) - \langle P \rangle(J(\tau_l)) \, ds \right| \]
\[ + \left| \int_{\tau_l}^{\tau_{l+1}} \langle P \rangle(J(\tau_l)) - \langle P \rangle(J(s)) \, ds \right| =: \mathcal{Y}_l^1 + \mathcal{Y}_l^2 + \mathcal{Y}_l^3. \quad (5.18) \]

**Step 2.** In the next proposition we will introduce "bad" events, outside of which actions are separated from zero, change slowly, and the rotation frequencies $Y(J(\tau_l))$ are not resonant. We will choose the initial point $\tau_0$ in such a way that probabilities of these events will be small, and it will be sufficient to estimate $\mathcal{Y}_l^1, \mathcal{Y}_l^2, \mathcal{Y}_l^3$ only outside these events. Recall that $\Lambda$ is defined in (5.14).

**Proposition 5.1.** There exist events $\mathcal{F}_l, 0 \leq l \leq L - 1$, such that outside $\mathcal{F}_l \cup \Omega_R$

1. \( \forall k \in \Lambda \sup_{\tau_l \leq \tau \leq \tau_{l+1}} J_k(\tau) \geq \frac{1}{2} \varepsilon^{1/24} \),
2. \( \forall k \in \Lambda \sup_{\tau_l \leq \tau \leq \tau_{l+1}} |J_k(\tau) - J_k(\tau_l)| \leq \nu^{1/3} \),
3. \( \frac{1}{\varepsilon^{-1} \nu} \int_0^{\varepsilon^{-1} \nu} P(J(\tau_l), \psi(\tau_l) + Y(\tau_l)s) \, ds - \langle P \rangle(J(\tau_l)) \, ds \leq \kappa(\varepsilon^{-1}; R) \),

where the function $\kappa$ is independent from $0 \leq l \leq L - 1$. There exists $\tau_0$ such that
\[ L^{-1} \sum_{l=0}^{L-1} \mathcal{P}_{\Omega_R}(\mathcal{F}_l) \leq \kappa(\varepsilon^{-1}; R). \quad (5.19) \]

Before proving this proposition we will finish the proof of the lemma. Outside $\Omega_R$ we have $\mathcal{Y}_l^2 \leq \nu C(R) \leq C_1(R)/L$. Fix $\tau_0$ as in Proposition 5.1. Then from (5.19) we obtain
\[ \sum_{l=0}^{L-1} (E_{\Omega_R} - E_{\mathcal{F}_l \cup \Omega_R}) \mathcal{Y}_l^i \leq \frac{C(R)}{L} \sum_{l=0}^{L-1} \mathcal{P}_{\Omega_R}(\mathcal{F}_l) \leq C(R) \kappa(\varepsilon^{-1}; R) = \kappa_1(\varepsilon^{-1}; R), \quad i = 1, 2, 3. \]
Thus, it is sufficient to show that for any $R \geq 0$ we have
\[
\sum_{l=0}^{L-1} \mathbb{E}_{\mathcal{F}_l \cup \Omega_R} (\mathcal{Y}_l^1 + \mathcal{Y}_l^2 + \mathcal{Y}_l^3) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly in} \ N.
\]

**Step 3.** Now we will estimate each term $\mathcal{Y}_l^3$ outside the "bad" event $\mathcal{F}_l \cup \Omega_R$.

**Terms $\mathcal{Y}_l^1$.** We will need the following

**Proposition 5.2.** For every $k \in \Lambda$ and each $0 \leq l \leq L - 1$, we have
\[
P_{\mathcal{F}_l \cup \Omega_R} \left( \sup_{\tau_l \leq \tau \leq \tau_{l+1}} |\psi_k(\tau) - (\psi_k(\tau_l) + \varepsilon^{-1} Y_k(\tau - \tau_l))(\tau - \tau_l)| \geq \varepsilon^{1/24} \right) \leq \kappa_\infty(\varepsilon^{-1}), \quad (5.20)
\]
where the function $\kappa_\infty$ is independent from $k, l$.

**Proof.** Let us denote the event in the left-hand side of (5.20) by $\Gamma$. According to (4.3),
\[
P_{\mathcal{F}_l \cup \Omega_R} (\Gamma) \leq P_{\mathcal{F}_l \cup \Omega_R} \left( \varepsilon^{-1} \sup_{\tau_l \leq \tau \leq \tau_{l+1}} \left| \int_{\tau_l}^{\tau} Y_k(s) - Y_k(\tau_l) \, ds \right| \geq \frac{1}{3} \varepsilon^{1/24} \right) + \mathbb{P}_{\mathcal{F}_l \cup \Omega_R} \left( \sup_{\tau_l \leq \tau \leq \tau_{l+1}} \left| \int_{\tau_l}^{\tau} \frac{A^\psi_k}{|v_k|^2} \, ds \right| \geq \frac{1}{3} \varepsilon^{1/24} \right) + \mathbb{P}_{\mathcal{F}_l \cup \Omega_R} \left( \sup_{\tau_l \leq \tau \leq \tau_{l+1}} \left| \int_{\tau_l}^{\tau} \frac{i v_k}{|v_k|^2} \cdot (W dB)_k \right| \geq \frac{1}{3} \varepsilon^{1/24} \right) = P_{\mathcal{F}_l \cup \Omega_R} (\Gamma_1) + P_{\mathcal{F}_l \cup \Omega_R} (\Gamma_2) + P_{\mathcal{F}_l \cup \Omega_R} (\Gamma_3).
\]

$\Gamma_1$: Due to (4.5), $Y_k(J) \in \mathcal{L}_{loc}(\mathbb{R}^N)$. Since it depends on $J$ only through $J_n$ with $n$ satisfying $|n - k| \leq 1$, we get
\[
P_{\mathcal{F}_l \cup \Omega_R} (\Gamma_1) \leq \mathbb{P}_{\mathcal{F}_l \cup \Omega_R} \left( \max_{n:|n-k| \leq 1} \sup_{\tau_l \leq \tau \leq \tau_{l+1}} |J_n(\tau) - J_n(\tau_l)| \geq C(R) \varepsilon^{1+1/24} \nu^{-1} \right).
\]

If $\varepsilon$ is sufficiently small, we have $C(R) \varepsilon^{1+1/24} \nu^{-1} > \nu^{1/3}$ (recall that $\nu = \varepsilon^{7/8}$). Then, due to Proposition 5.1.ii, we get
\[
P_{\mathcal{F}_l \cup \Omega_R} (\Gamma_1) = 0 \quad \text{for} \quad \varepsilon \ll 1.
\]

$\Gamma_2$: Proposition 5.1.i implies
\[
P_{\mathcal{F}_l \cup \Omega_R} (\Gamma_2) \leq \mathbb{P}_{\mathcal{F}_l \cup \Omega_R} \left( \sup_{\tau_l \leq \tau \leq \tau_{l+1}} |A^\psi_k| \geq \frac{1}{3} \varepsilon^{1/24+1/24 \nu^{-1}} \right) = 0 \quad \text{for} \quad \varepsilon \ll 1,
\]

since outside $\Omega_R$ we have $|A^\psi_k| \leq R$, in view of (5.15).

$\Gamma_3$: In view of (A.4), the Burkholder-Davis-Gandy inequality jointly with Theorem 3.1.3, and Proposition 5.1.i imply that
\[
\mathbb{E}_{\mathcal{F}_l \cup \Omega_R} \sup_{\tau_l \leq \tau \leq \tau_{l+1}} \left| \int_{\tau_l}^{\tau} \frac{i v_k}{|v_k|^2} \cdot (W dB)_k \right|^{2m} \leq C(m) \mathbb{E}_{\mathcal{F}_l \cup \Omega_R} \left( \int_{\tau_l}^{\tau_{l+1}} \frac{1}{|v_k|^2} \, ds \right)^m \leq C(m) \nu^m \varepsilon^{-m/24},
\]

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for any \( m > 0 \). From Chebyshev’s inequality it follows that
\[
P_{\mathcal{F}_t \cup \Omega_R}(\Gamma_3) \leq C(m)\nu^m\varepsilon^{-m(1/24+2/24)} \quad \text{for any } m > 0.
\]
Thus, \( P_{\mathcal{F}_t \cup \Omega_R}(\Gamma_3) = \kappa_{\infty}(\varepsilon^{-1}) \).

Estimates (i) and (ii) of Proposition 5.1 imply that outside \( \mathcal{F}_t \cup \Omega_R \), for any \( k \in \Lambda \)
\[
\sup_{\tau \leq \tau \leq \tau_{n+1}} ||v_k(\tau) - |v_k(\tau)||| \leq \frac{\sqrt{2}|J_k(\tau) - J_k(\tau)|}{\sqrt{J_k(\tau)} + \sqrt{J_k(\tau)}} \leq \nu^{1/3}\varepsilon^{-1/48} = \varepsilon^{13/48}.
\] (5.21)

Since \( P \in \mathcal{L}_{\text{loc}}(\mathbb{C}^N) \), then Proposition 5.2 and (5.21) imply that
\[
P_{\mathcal{F}_t \cup \Omega_R}(\mathcal{Y}_l^{1} \geq \nu C(R)(\varepsilon^{1/24} + \varepsilon^{13/48})) \leq \kappa_{\infty}(\varepsilon^{-1}).
\]

Then we get
\[
\mathbf{E}_{\mathcal{F}_t \cup \Omega_R} \mathcal{Y}_l^{1} \leq \nu C(R)(\varepsilon^{1/24} + \varepsilon^{13/48}) + \nu C(R)\kappa_{\infty}(\varepsilon^{-1}) = \nu \kappa(\varepsilon^{-1}; R).
\]

Terms \( \mathcal{Y}_l^{2} \). Put \( \hat{s} := \varepsilon^{-1}(s - \tau_l) \). Then Proposition 5.1.iii implies that outside \( \mathcal{F}_t \cup \Omega_R \)
\[
\mathcal{Y}_l^{2} = \nu \left| \int_{\varepsilon^{-1}l}^{\varepsilon^{-1}(l+1)} P(J(\tau), \psi(\tau_l) + Y(\tau_l)\hat{s}) d\hat{s} - \langle P \rangle (J(\tau_l)) \right| \leq \nu \kappa(\varepsilon^{-1}; R).
\]

Terms \( \mathcal{Y}_l^{3} \). Proposition B.1.i jointly with (5.21) implies that outside \( \mathcal{F}_t \cup \Omega_R \) we have
\[
\mathcal{Y}_l^{3} \leq \nu C(R)\varepsilon^{13/48}.
\]

Step 4. Summing by \( l \), taking the expectation and noting that \( L \nu \leq T \), we get
\[
\sum_{l=0}^{L-1} \mathbf{E}_{\mathcal{F}_t \cup \Omega_R} (\mathcal{Y}_l^{1} + \mathcal{Y}_l^{2} + \mathcal{Y}_l^{3}) \leq L(\nu \kappa(\varepsilon^{-1}; R) + \nu C(R)\varepsilon^{13/48}) \to 0 \quad \text{as } \varepsilon \to 0,
\]
uniformly in \( N \). The proof of the lemma is complete.

Proof of Proposition 5.1. We will construct the set \( \mathcal{F}_t \) as a union of three parts. The first two are \( \mathcal{E}_t := \bigcup_{k \in \Lambda} \mathcal{E}_l^k \) and \( Q_t := \bigcup_{k \in \Lambda} Q_l^k \), where
\[
\mathcal{E}_l^k := \{ J_k(\tau_l) \leq \varepsilon^{1/24} \}, \quad Q_l^k := \left\{ \sup_{\tau \leq \tau \leq \tau_{n+1}} |J_k(\tau) - J_k(\tau_l)| \geq \nu^{1/3} \right\}.
\] (5.22)

Outside \( Q_t \) we have (ii) and, if \( \varepsilon \) is small, outside \( \mathcal{E}_t \cup Q_t \) we get (i): for every \( k \in \Lambda \)
\[
\sup_{\tau \leq \tau \leq \tau_{n+1}} J_k(\tau) \geq \varepsilon^{1/24} - \nu^{1/3} \geq \frac{1}{2}\varepsilon^{1/24}, \quad \text{if } \varepsilon \ll 1.
\]

Now we will construct the event \( \Omega_{t}^{\varepsilon,R} \), which will form the third part of \( \mathcal{F}_t \). Let us accept the following notation:

for a vector \( Z = (Z_j)_{j \in \Lambda} \in \mathbb{R}^N \) we denote \( Z^{\Lambda} := (Z_j)_{j \in \Lambda} \in \mathbb{R}^M \).

For any fixed \( J \in \mathbb{R}^N_{+} \) the function \( P(J, \psi) \) is Lipschitz-continuous in angles \( \psi \in \mathbb{T}^N \). From [Tm] it follows that the Fourier series of a Lipschitz-continuous function of \( \psi \in \mathbb{T}^N \).
converges uniformly in \( \psi \). Then, using standard method (e.g., see in [MS]), we obtain that for every \( \delta > 0 \) and \( R' > 0 \) there exists a Borel set \( E^{A,R}_R \subset \{ x = (x_k)_{k \in \Lambda} \in \mathbb{R}^M : \|x\|_{\mathbb{R}^M} \leq R \} \) with the Lebesgue measure \( |E^{A,R}_R| \leq \delta \), such that for any \( Z = (Z_k)_{k \in \mathcal{C}} \in \mathbb{R}^N \) satisfying \( Z^A \notin E^{A,R} \) and \( \|Z^A\|_{\mathbb{R}^M} \leq R' \), we have

\[
\left| \frac{1}{t} \int_0^t P(J, \psi + Zs) \, ds - \langle P \rangle(J) \right| \leq \kappa(t; J, \delta, R'), \tag{5.23}
\]

for all \( J \in \mathbb{R}^N_+ \) and \( \psi \in \mathbb{T}^N \). Moreover, since \( P \in \mathcal{L}_{\text{loc}}(\mathbb{C}^N) \), then we can choose the function \( \kappa \) to be independent from \( J \) for \( J \in B^A_R \), where

\[
B^A_R := \{ J = (J_k)_{k \in \mathcal{C}} \in \mathbb{R}^N_{0+} : \max_{k \in \Lambda} J_k \leq R \},
\]

i.e. \( \kappa = \kappa(t; R, \delta, R') \). The rate of convergence in (5.23) depends on \( \delta \). Choose a function \( \delta = \delta(\varepsilon) \), such that \( \delta(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \) so slow that

\[
\left| \frac{1}{\varepsilon R} \int_0^{\varepsilon^{-1} \nu} P(J, \psi + Zs) \, ds - \langle P \rangle(J) \right| \leq \kappa(\varepsilon^{-1}; R, R') \tag{5.24}
\]

for all \( J \in B^A_R, \psi \in \mathbb{T}^N \) and \( Z \) as above.

Let us choose \( R' = R'(R) = \sup_{\Omega_R} \sup_{0 \leq \tau \leq \tau} \|Y^A(\tau)\|_{\mathbb{R}^M} \). Let

\[
\Omega^\varepsilon_{i,R} := \{ Y^A(\tau) \in E^{A(\varepsilon),R'(R)} \}.
\]

Then outside \( \Omega^\varepsilon_{i,R} \cup \Omega_R \) we get \( Y^A(\tau) \notin E^{A(\varepsilon),R'(R)} \) and \( \|Y^A(\tau)\|_{\mathbb{R}^M} \leq R'(R) \). Since outside \( \Omega_R \) we have \( J(\tau) \in B^A_R \), then, due to (5.24), outside \( \Omega^\varepsilon_{i,R} \cup \Omega_R \) we get (iii).

Let \( \mathcal{F}_i := \mathcal{E}_i \cup Q_i \cup \Omega^\varepsilon_{i,R} \). Then outside \( \mathcal{F}_i \cup \Omega_R \) items (i), (ii) and (iii) hold true.

Now we will estimate the probabilities of \( \mathcal{E}_i, Q_i \) and \( \Omega^\varepsilon_{i,R} \).

**Proposition 5.3.** (i) We have \( P(Q_i) \leq \kappa_\infty(\nu^{-1}) \), where \( \kappa_\infty \) is independent from \( l \).

(ii) There exists an initial point \( \tau_0 \in [0, \nu] \) such that

\[
L^{-1} \sum_{i=0}^{L-1} P_{\Omega_R}(\mathcal{E}_i \cup \Omega^\varepsilon_{i,R}) = \kappa(\varepsilon^{-1}; R).
\]

Propositions 5.3 implies (5.19):

\[
L^{-1} \sum_{i=0}^{L-1} P_{\Omega_R}(\mathcal{F}_i) \leq \kappa(\varepsilon^{-1}; R) + \kappa_\infty(\nu^{-1}) = \kappa_1(\varepsilon^{-1}; R).
\]

**Proof of Proposition 5.3.** (i) Let us take \( \rho > \sqrt{\nu} \). Then, due to (4.2), for any \( k \in \tilde{\Lambda} \)

\[
P \left( \sup_{\tau_i \leq \tau \leq \tau_{i+1}} |J_k(\tau) - J_k(\tau_i)| \geq \rho \right) \leq P \left( \sup_{\tau_i \leq \tau \leq \tau_{i+1}} \left| \int_{\tau_i}^{\tau} A^I_k \, ds \right| \geq \rho/2 \right)
\]

\[+ P \left( \sup_{\tau_i \leq \tau \leq \tau_{i+1}} \left| \int_{\tau_i}^{\tau} v_k \cdot (WdB)_k \right| \geq \rho/2 \right) =: P(\Gamma_1) + P(\Gamma_2).
\]

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Due to estimate (4.6), we have

$$\mathbf{P}(\Gamma_1) \leq \mathbf{P}(\nu \sup_{\tau_i \leq \tau \leq \tau_{i+1}} |A_k^j| \geq \rho/2) \leq \kappa(\nu^{-1}).$$

In view of (A.4), the Burkholder-Davis-Gundy inequality jointly with (4.6) implies

$$E \sup_{\tau_i \leq \tau \leq \tau_{i+1}} \left| \int_{\tau_i}^{\tau} v_k \cdot (WdB)_k \right|^{2m} \leq C(m)E \left( \int_{\tau_i}^{\tau_{i+1}} S_{kk}^j d\tau \right)^m \leq C_1(m)\nu^m, \quad (5.25)$$

for every $m > 0$. Consequently, $\mathbf{P}(\Gamma_2) \leq C(m)\nu^m \rho^{-2m}$. Choosing $\rho = \nu^{1/3}$ we get $\mathbf{P}(\Gamma_2) \leq \kappa(\nu^{-1})$. It remains to sum up the probabilities by $k \in \tilde{\Lambda}$.

(ii) Denote $A(\tau) := (\mathcal{E} \cup \Omega^e,R)(\tau)$, where the last set is defined similarly to $\mathcal{E}_l \cup \Omega^e_l$ but at the moment of time $\tau$ instead of $\tau_i$. Recall that $Y^\Lambda(J)$ depends on $J$ only through $J^\Lambda := (J_k)_{k \in \tilde{\Lambda}}$. Denote by $\hat{M}$ the number of nodes in $\tilde{\Lambda}$ and let

$$E^e_j := \{ \exists \tilde{\Lambda} \in \mathbb{R}_{+0}^\hat{M} : Y^\Lambda(J) \in E^\delta(\epsilon,R) \text{ and } J_k \leq R, \forall k \in \tilde{\Lambda} \}.$$

In view of assumption $HF$ which states that the functions $f_j^l$ have only isolated zeros, it is not difficult to show that the convergence $|E^\delta(\epsilon,R)| \to 0$ as $\epsilon \to 0$ implies that $|E^e_j \epsilon \to 0$ as $\epsilon \to 0$. Note that $\Omega_R \cap \Omega^e_j \in \{ J^\Lambda(\tau) \in E^e_j \}$, and Lemma 4.3 implies

$$\int_0^T \mathbf{P}_{\Omega_R}(A(\tau)) \, d\tau \leq \int_0^T \mathbf{P}_{\Omega_R}(\mathcal{E}(\tau)) \, d\tau + \int_0^T \mathbf{P}(J^\Lambda(\tau) \in E^e_j) \, d\tau \to 0 \quad \text{as } \epsilon \to 0,$$

uniformly in $N$. It remains to note that there exists a deterministic point $\tau_0 \in [0, \infty)$ such that

$$\int_0^T \mathbf{P}_{\Omega_R}(A(\tau)) \, d\tau \geq \nu \sum_{l=0}^{L-1} \mathbf{P}_{\Omega_R}(A(l\nu + s)) \, ds$$

$$\geq \nu \sum_{l=0}^{L-1} \mathbf{P}_{\Omega_R}(A(l\nu + \tau_0)) \geq T(L+1)^{-1} \sum_{l=0}^{L-1} \mathbf{P}_{\Omega_R}(A(\tau_l)). \quad \square$$

6 The change of variables: proof of Theorem 3.1

For this section we accept the following notations. Let $a = (a_s) \in \mathbb{C}^M$ be a vector-function $a = a(b)$, where $b = (b_r) \in \mathbb{C}^K$, and $s, r$ are multi-indices. By $\frac{\partial a}{\partial b}$ we denote the $M \times K$-matrix with entires $(\frac{\partial a}{\partial b})_{jn} := \frac{\partial a_j}{\partial b_n}$. By $\frac{\partial a}{\partial b}$ we denote the transposed matrix,

$$\left(\frac{\partial a}{\partial b}\right)^T_{jn} = \frac{\partial a_n}{\partial b_j}. \text{ By } \text{Id}_M \text{ we denote the identity matrix of the size } M \times M.$$
6.1 The Hamiltonian and the equation

Here we find the canonical transformation and calculate the Hamiltonian (2.3) and equation (2.5) in the new variables $v$.

We will find the transformation as the time-1-map $\Gamma$ of the Hamiltonian flow $X_{\sqrt{\varepsilon}\Phi}^s$ given by the Hamiltonian $\sqrt{\varepsilon}\Phi$.

Denoting $\tilde{F}(J) := \frac{1}{2} \sum_{j \in \mathbb{C}} F_j(|v_j|^2)$ and $\tilde{G}(J, \psi) := \frac{1}{4} \sum_{|j-n|=1} G(|v_j - v_n|^2)$ we have

$$\mathcal{H}(J, \psi) = H \circ \Gamma(J, \psi) = \tilde{F}(J) + \sqrt{\varepsilon}\tilde{G}(J, \psi) + \{\tilde{F}, \Phi\}(J, \psi) + \varepsilon\left(\{\tilde{G}, \Phi\}(J, \psi) + \int_0^1 (1-s)\{H, \Phi\}, \Phi\} \circ X_{\sqrt{\varepsilon}\Phi}^s ds\right),$$

where $\{h_1, h_2\} = \sum_{j \in \mathbb{C}} (\partial_{v_j} h_1 \partial_{v_j} h_2 - \partial_{v_j} h_1 \partial_{v_j} h_2) = 2i \sum_{j \in \mathbb{C}} (\partial_{v_j} h_1 \partial_{v_j} h_2 - \partial_{v_j} h_1 \partial_{v_j} h_2)$ denotes the Poisson brackets. We wish to choose $\Phi$ in such a way that the homological equation

$$\tilde{G}(J, \psi) + \{\tilde{F}, \Phi\}(J, \psi) = h(J)$$

holds for some function $h$. Let us denote

$$\theta_{jn} := \psi_j - \psi_n \quad \text{for} \quad j, n \in \mathbb{C}.$$

Note that

$$G(|v_j - v_n|^2) = G(\sqrt{2J}e^{i\theta_{jn}} - \sqrt{2J_n}) =: G(J_j, J_n, \theta_{jn}). \tag{6.3}$$

We seek $\Phi$ in the form $\Phi = \sum_{|j-n|=1} \Phi_{jn}(v_j, v_n) = \sum_{|j-n|=1} \Phi_{jn}(J_j, J_n, \theta_{jn})$. Expanding $G$ and $\Phi$ in the Fourier series, we obtain

$$G(J_j, J_n, \theta_{jn}) = \sum_{k \in \mathbb{Z}} G_k(J_j, J_n)e^{ik\theta_{jn}}, \quad \Phi_{jn}(J_j, J_n, \theta_{jn}) = \sum_{k \in \mathbb{Z}} \Phi_{jnk}(J_j, J_n)e^{ik\theta_{jn}}.$$

Now let us solve (6.2). Calculating the Poisson brackets we have

$$\{\tilde{F}, \Phi\}(J, \psi) = - \sum_{j \in \mathbb{C}} f_j(2J_j) \frac{\partial \Phi}{\partial \psi_j} = - \sum_{|j-n|=1} \partial_{\theta_{jn}} \Phi_{jn}(f_j - f_n).$$

Then the left-hand side of (6.2) is equal to

$$\sum_{k \in \mathbb{Z}} \sum_{|j-n|=1} \left( - ik \Phi_{jnk}(f_j - f_n) + \frac{1}{4} G_k(J_j, J_n) \right) e^{ik\theta_{jn}}. \tag{6.4}$$

We choose $\Phi$ in such a way that each bracket in (6.4) except that with $k = 0$ vanishes. That is

$$\Phi_{jnk} = \frac{1}{4} G_k(J_j, J_n), \quad k \in \mathbb{Z}/\{0\}, \quad \Phi_{jn0} = 0. \tag{6.5}$$
From representation (6.3) it follows that \( G(J_j, J_n, \theta j_n) = G(J_j, J_n, -\theta j_n) \), so \( G_k(J_j, J_n) = G_{-k}(J_j, J_n) \). Then (6.5) implies

\[
\Phi_{jn} = \frac{1}{4} \sum_{k \in \mathbb{Z}/(0)} \frac{G_k(J_j, J_n) e^{ik\theta j_n} }{ik(f_j - f_n)} = 1 \int_0^{\theta j_n} G^0(J_j, J_n, \theta) d\theta, \quad \Phi = \sum_{|j-n|=1} \Phi_{jn}, \tag{6.6}
\]

where \( G^0 := G - \langle G \rangle \). Due to assumption \( HF \), the denominator of (6.6) is separated from zero. Thus, choosing \( \Phi \) as above and putting

\[
h = \langle \tilde{G} \rangle, \tag{6.7}
\]

we get that the homological equation (6.2) holds. Taking the next order term of Taylor’s expansion of Hamiltonian (6.1) and applying (6.2) to the terms of order \( \sqrt{\varepsilon} \) and \( \varepsilon \), we have

\[
\mathcal{H}(J, \psi) = \tilde{F}(J) + \sqrt{\varepsilon}(\tilde{G})(J) + \frac{\varepsilon}{2} \left( \langle \tilde{G} \rangle + G, \Phi \right) + \left( \frac{\varepsilon \sqrt{\varepsilon}}{2} \right) \left( \left\{ \tilde{G}, \Phi \right\} + \mathbf{1} \right) \mathcal{J} \mathbf{2} (1 - s) \mathbf{2} \{H, \Phi\} \mathbf{3} \circ X_{\sqrt{\varepsilon} \Phi} ds \right) =: H_0(J) + \varepsilon H_2(J, \psi) + \varepsilon \sqrt{\varepsilon} H_\varepsilon(J, \psi), \tag{6.8}
\]

where \( \{h, \Phi\}_k := \{\ldots \{h, \Phi\}, \Phi, \ldots, \Phi\} \) denotes the Poisson bracket with \( \Phi \) taken \( k \) times. Thus, we arrive at (3.3).

Since \( G^0(v_j, v_n) \in C(\mathbb{C}^2) \) and \( G^0(0, v_n) = G^0(v_j, 0) = 0 \) for all \( v_j, v_n \in \mathbb{C} \), then \( \Phi(v) \in C(\mathbb{C}^N) \). Moreover,

**Proposition 6.1.** Function \( \Phi(v) \) is \( C^3 \)-smooth. Let \( a, b, c \in \{v, \bar{v}\} \). Then for every \( k, l, m \in \mathbb{C} \), satisfying the relation \( |k - l| \leq 1 \) and \( l = m \), we have

\[
\left| \frac{\partial \Phi}{\partial v_k} \right|, \left| \frac{\partial \Phi}{\partial a_k} \right|, \left| \frac{\partial^2 \Phi}{\partial a_k \partial b_l} \right|, \left| \frac{\partial^3 \Phi}{\partial a_k \partial b_l \partial c_m} \right| \leq C. \tag{6.9}
\]

For other \( k, l, m \in \mathbb{C} \) the second and the third derivatives are equal to zero.

We will prove this proposition after the end of the proof of the theorem. Denote \( v^s := X_{\sqrt{\varepsilon} \Phi}^s (v) \) and \( u^s := X_{-\sqrt{\varepsilon} \Phi}^s (u) \). In particular, \( u = X_{1, \sqrt{\varepsilon} \Phi} (v) = v^1 \), \( v = X_{-1, \sqrt{\varepsilon} \Phi} (u) = u^1 \) and \( v^s = u^{1-s} \). We have

\[
v^s_j = v_j + \sqrt{\varepsilon} \int_0^s i \nabla_j \Phi_{\mid u^s} d\tau, \quad u^s_j = u_j - \sqrt{\varepsilon} \int_0^s i \nabla_j \Phi_{\mid u^s} d\tau. \tag{6.10}
\]

In view of (6.10), Proposition 6.1 implies that the change of variables and functions \( \mathcal{H}(v) \) and \( H_2(v) \) are \( C^2 \)-smooth. Proposition B.1 implies that the function \( H_0(v) \) is \( C^4 \)-smooth. Then, due to (6.8), the function \( H_\varepsilon(v) \) is \( C^2 \)-smooth.

Let us now rewrite equation (2.5) in the \( \nu \)-variables. Applying Ito’s formula to \( v \) we get

\[
\dot{v} = \frac{\partial v}{\partial u} \dot{u} + \frac{\partial v}{\partial \bar{u}} \dot{\bar{u}} + \varepsilon \sum_{k \in \mathbb{C}} T_k \frac{\partial^2 v}{\partial u_k \partial \bar{u}_k}, \tag{6.11}
\]
since, in view of (2.5), the coefficients, corresponding to the second other derivatives vanish. Denote \( B = (\beta, \overline{\beta})^T \), where \( \beta = (\beta_j)_{j \in \mathcal{C}} \) and \( \beta_j \) are, as usual, standard complex independent Brownian motions. Using equation (2.5), we rewrite (6.11) in more details
\[
\dot{v} = i \nabla \mathcal{H}(v) + \varepsilon \frac{\partial v}{\partial u} g(u) + \varepsilon \frac{\partial v}{\partial u} \overline{g}(u) + \varepsilon \sum_{k \in \mathcal{C}} T_k \frac{\partial^2 v}{\partial u_k \partial u_{\bar{k}}} + \sqrt{\varepsilon} W \dot{B},
\]
where the new dispersion matrix \( W \) has the size \( N \times 2N \) and consists of two \( N \times N \) blocks \( W = (W^1, W^2) \), so \( W \dot{B} = W^1 \dot{\beta} + W^2 \dot{\overline{\beta}} \). The blocks have the form \( W^1 := \frac{\partial v}{\partial u} \text{diag}(\sqrt{T}) \) and \( W^2 := \frac{\partial v}{\partial u} \text{diag}(\sqrt{T}) \).

Let us represent the dissipative and Itô’s terms as a leading part and a remainder of higher order in \( \varepsilon \). In view of (6.10), we have \( \frac{\partial v}{\partial u} - 1 \) \( \text{Id} \) \( N \), \( \frac{\partial^2 v}{\partial u_k \partial u_{\bar{k}}} \sim \sqrt{\varepsilon} \). Denote
\[
r^D := \varepsilon^{-1/2} \left( (g(u) - g(v)) + \left( \frac{\partial v}{\partial u} - \text{Id} \right) g(u) + \frac{\partial \overline{v}}{\partial u} \overline{g}(u) \right) \quad \text{and} \quad r^I := \varepsilon^{-1/2} \sum_{k \in \mathcal{C}} T_k \frac{\partial^2 v}{\partial u_k \partial u_{\bar{k}}},
\]
Since the transformation is \( C^2 \)-smooth, \( r^D \) is \( C^1 \)-smooth and \( r^I \) is continuous. Let
\[
r = (r_j)_{j \in \mathcal{C}} := r^D + r^I + i \nabla \mathcal{H} >.
\]
Substituting this relation in (6.12), we arrive at (3.5).

6.2 Some estimates

Here we prove auxiliary propositions essential to establish items 1-4 of the theorem. Recall that the constant \( \gamma \) satisfies the estimate \( 1/2 < \gamma < 1 \). Till the end of Section 6 we always indicate the dependence of constants on \( \gamma \) and do not indicate their dependence on the power \( q \).

First let us establish the following corollary of Proposition 6.1.

**Corollary 6.2.** For any \( q \geq 1 \) there exists a constant \( C(\gamma) \), such that for every \( 0 \leq s \leq 1 \) we have
\[
\|v\|_{j,q} - \sqrt{\varepsilon} C(\gamma) \leq \|v^s\|_{j,q} \leq \|v\|_{j,q} + \sqrt{\varepsilon} C(\gamma).
\]

**Proof.** This follows from Proposition 6.1 by summing in \( k \in \mathcal{C} \) the increments \( |v^s_k - v_k| \), given by (6.10), raised to the power \( q \), and multiplied by coefficients \( \gamma^{j - k} \).

Let us introduce the following notations. By \( \tilde{\mathcal{C}} := \{1, 2\} \times \mathcal{C} \) we denote the set of multi-indices \( \tilde{j} = (j_0, j) = (j_0, \ldots, j_d) \), where \( j_0 \in \{1, 2\} \) and \( j = (j_1, \ldots, j_d) \in \mathcal{C} \). By \( U \) and \( V \) we denote the vectors from \( \mathcal{C}^{2N} \)
\[
U = (U_{\tilde{j}})_{\tilde{j} \in \tilde{\mathcal{C}}} = (u, \overline{u})^T \quad \text{and} \quad V = (V_{\tilde{j}})_{\tilde{j} \in \tilde{\mathcal{C}}} = (v, \overline{v})^T,
\]
where \( U_{(1,j)} = u_j, U_{(2,j)} = \overline{u}_j \) and \( V_{(1,j)} = v_j, V_{(2,j)} = \overline{v}_j \).

Introduce a \( 2N \times 2N \)-matrix \( \frac{\partial U^s}{\partial U} \) with entires \( \left( \frac{\partial U^s}{\partial \tilde{k} \tilde{l}} \right)_j \), where \( \tilde{k}, \tilde{l} \in \mathcal{C} \). It will be convenient to write it in the form
\[
\frac{\partial U^s}{\partial U} = \left( \begin{array}{cc} \frac{\partial u^s}{\partial u} & \frac{\partial v^s}{\partial u} \\ \frac{\partial u^s}{\partial \overline{u}} & \frac{\partial v^s}{\partial \overline{u}} \end{array} \right),
\]
(6.14)
where \( \frac{\partial U^s}{\partial U} \) \((1,k) (1,l)\) = \( \frac{\partial u^s}{\partial u} \) \(k_l\), \( \frac{\partial U^s}{\partial U} \) \((1,k) (2,l)\) = \( \frac{\partial u^s}{\partial u} \) \(k_l\), etc. So the first indices \( k_0, l_0 \) \(\in\) \{1, 2\} numerate the blocks of the matrix (6.14). Using (6.10) it is easy to check that \( \frac{\partial U^s}{\partial U} \) satisfies the equation

\[
\frac{\partial U^s}{\partial U} = -\sqrt{\varepsilon} \int_0^s D^2 \Phi (u^\tau) \frac{\partial U^\tau}{\partial U} d\tau + \text{Id}_{2N}, \quad D^2 \Phi := 2i \left( \frac{\partial^2 \Phi}{\partial u^2} \frac{\partial^2 \Phi}{\partial u^2} \right),
\]

(6.15)

where \( D^2 \Phi = (D^2 \Phi_{kl})_{k,l \in \mathbb{C}} \) and \( (D^2 \Phi)_{(1,k),(1,l)} = \frac{\partial^2 \Phi}{\partial u^k \partial u^l}, \quad (D^2 \Phi)_{(1,k),(2,l)} = \frac{\partial^2 \Phi}{\partial u^k \partial u^l}, \) etc.

Introduce the family of norms on \( V = (V_k)_{k \in \mathbb{C}} \in \mathbb{C}^{2N} \): for \( j \in \mathbb{C} \) and \( q \geq 1 \) we define

\[
| V |_{j,q}^q = \sum_{k \in \mathbb{C}} |\gamma|^{k-j}(|\langle V_{(1,k)} \rangle|^q + |\langle V_{(2,k)} \rangle|^q). \tag{6.16}
\]

For a \( N \times N \)-matrix \( A \) with complex entries by \( |A|_{j,q} \) we will denote its operator norm, corresponding to the norm \( \| \cdot \|_{j,q} \) on \( \mathbb{C}^N \). Likewise for a \( 2N \times 2N \)-matrix \( A \) we define \( |A|_{j,q} \).

**Corollary 6.3.** Let \( a, b \in \{ u, v \} \) or \( a, b \in \{ u, v \} \). Then for any \( q \geq 1 \) there exists a constant \( C \) such that for all \( 0 \leq s \leq 1 \) and \( j \in \mathbb{C} \) we have

\[
| D^2 \Phi |_{j,q} \leq C, \quad \left\| \frac{\partial a^s}{\partial b} - \text{Id}_N \right\|_{j,q}, \quad \left\| \frac{\partial a^s}{\partial b} \right\|_{j,q} \leq C \sqrt{\varepsilon}.
\]

**Proof.** The matrix \( D^2 \Phi \) consists of four blocks of the size \( N \times N \). Consider the upper left-hand block. By Proposition 6.1 the only nonzero entries are \( \frac{\partial^2 \Phi}{\partial u_k \partial u_m} \) with \( |k - m| \leq 1 \) and they are bounded by the same constant. Since the other blocks have a similar structure, we obtain \( | D^2 \Phi |_{j,q} \leq C \). Due to (6.15), we get

\[
\left| \frac{\partial U^s}{\partial U} \right|_{j,q} \leq \sqrt{\varepsilon} C \int_0^s \left| \frac{\partial U^\tau}{\partial U} \right|_{j,q} d\tau + 1 \Rightarrow \left| \frac{\partial U^s}{\partial U} \right|_{j,q} \leq C_1.
\]

Applying (6.15) once more, we obtain

\[
\left| \frac{\partial U^s}{\partial U} - \text{Id}_{2N} \right|_{j,q} \leq \sqrt{\varepsilon} C \int_0^s \left| \frac{\partial U^\tau}{\partial U} \right|_{j,q} d\tau \leq C_1 \sqrt{\varepsilon}. \tag{6.17}
\]

The matrix \( \frac{\partial u^s}{\partial u} - \text{Id}_N \) is the upper left-hand \( N \times N \) block of the matrix \( \frac{\partial U^s}{\partial U} - \text{Id}_{2N} \). Then the wanted estimate for it allows from (6.17). The estimates for the other matrices can be obtained similarly.

We will need the following proposition.

**Proposition 6.4.** Let \( A = (a_{kl})_{k,l \in \mathbb{C}} \) be a matrix with complex entires of the size \( N \times N \). Assume that for some \( q \geq 1 \) we have \( \| A \|_{j,q} \leq C_0 \), where the constant \( C_0 \) is independent from \( j \in \mathbb{C} \). Then \( |a_{kl}| \leq C_0 \) for all \( k, l \in \mathbb{C} \).

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Proof. Let us fix any \( k, l \in \mathcal{C} \) and take the vector \( v = (v_m)_{m \in \mathcal{C}} \) such that \( v_m = \delta_{ml} \). Then we have \( \|v\|_{k,q} = \gamma^{|l-k|} < 1 \). Consequently, \( \|Av\|_{k,q} \leq \sum_{m \in \mathcal{C}} |a_m| \gamma^{|k-m|} \leq C_0 \), and it follows that \( |a_{kl}| \leq C_0 \). \( \square \)

We will need the following reformulation of estimates from assumptions HF and HG.

**Proposition 6.5.** Estimates of assumptions HF and HG imply that for any \( w, z \in \mathcal{C} \) we have

1. \( |\partial_{w_m+w_n}^{m+n} f(|w|^2)| \leq Cf(|w|^2) \), where \( 1 \leq m + n \leq 3 \);
2. \( |\partial_{w_1+w_2+w_3} f(|w-z|^2)| \leq C(1 + |w|^{p-1} + |z|^{p-1}) \), where \( 1 \leq m_1 + l_1 + m_2 + l_2 \leq 3 \);
3. \( |G(x)| \leq C(1 + x^{p/2}) \).

**Proof.** By the direct computation. \( \square \)

### 6.3 Properties of the transformation

Here we prove items 1-4 of the theorem. We give the proofs in the following order: 4, 3, 2a, 1, 2b.

Let us accept the following notations: for functions \( h_1(u), h_2(v) \) we denote \( h_1^* := h_1(u^\star) \), \( h_2^* := h_2(v^\star) \).

**Item 4.** We have

\[
J_j^s = J_j - \sqrt{\varepsilon} \int_0^s \partial_{\psi_j} \Phi|_{J_{\tau},\psi^\tau} d\tau, \quad I_j^s = I_j + \sqrt{\varepsilon} \int_0^s \partial_{\varphi_j} \Phi|_{J_{\tau},\psi^\tau} d\tau. \tag{6.18}
\]

Now the desired estimates follow from (6.10), (6.18) and Proposition 6.1.

**Item 3.** Recall that \( W^1 = \frac{\partial v}{\partial u} \text{diag}(\sqrt{T_k}) \) and \( W^2 = \frac{\partial v}{\partial u} \text{diag}(\sqrt{T_{\xi}}) \). Due to Corollary 6.3, for all \( j \in \mathcal{C} \) and \( q \geq 1 \) we have

\[
W^1 = \text{diag}(\sqrt{T_k}) + \tilde{W}, \text{ where } \|\tilde{W}\|_{j,q} \leq C\sqrt{\varepsilon}, \text{ and } \|W^2\|_{j,q} \leq C\sqrt{\varepsilon}.
\]

Then

\[
\|W^1 W^1^T - \text{diag}(T_k)\|_{j,q}, \|W^2 W^2^T\|_{j,q}, \|W^2 W^1^T\|_{j,q}, \|W^1 W^2^T\|_{j,q} \leq C\sqrt{\varepsilon},
\]

so the desired estimate follows from Proposition 6.4.

**Item 2a.** Recall that, due to (6.8), \( H_2 = \frac{1}{2} \{\tilde{G} + (\tilde{G}, \Phi)\} \). Then, obviously, the first assertion of 2a holds. The second one follows from Propositions 6.5.2 and 6.1: for all \( k \in \mathcal{C} \) we have

\[
\left| \partial_{\kappa} \{\tilde{G}, \Phi\} \right|, \left| \partial_{\kappa} \{(\tilde{G}), \Phi\} \right| \leq C \sum_{n:|n-k| \leq 2} |v_n|^p + C.
\]

**Item 1.** We will need the following

**Proposition 6.6.** Let the function \( h(\psi) = h((\psi_k)_{k \in \mathcal{C}}) \) be \( C^1 \)-smooth and depend on \( \psi \) only through the differences of its neighbouring components, i.e. \( h((\psi_k)_{k \in \mathcal{C}}) = h((\theta_{kn})_{k,n:|k-n|=1}) \), where \( \theta_{kn} = \psi_k - \psi_n \). Then

\[
\sum_{k \in \mathcal{C}} \gamma^{|j-k|} |\partial_{\psi_k} h| \leq 2(1 - \gamma) \sum_{|k-n|=1} \gamma^{|j-k|} |\partial_{\theta_{kn}} h|.
\]

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Proof. Using the equality \( 7 \partial \psi_k = \sum_{n:|k-n|=1} (\partial \theta_{kn} - \partial \theta_{nk}) \), where we formally put \( \partial \theta_{kn} / \partial \theta_{nk} = 0 \), we get
\[
\sum_{k \in \mathcal{C}} \gamma ^{|j-k|} \partial \psi_k h = \sum_{|k-n|=1} \gamma ^{|j-k|} (\partial \theta_{kn} - \partial \theta_{nk}) h = \sum_{|k-n|=1} (\gamma ^{|j-k|} - \gamma ^{|j-n|}) \partial \theta_{kn} h. \tag{6.19}
\]
Since \( 1/2 < \gamma < 1 \), for \( k \) and \( n \) satisfying \( |k-n| = 1 \) we have
\[
|1 - \gamma ^{|j-n|-|j-k|}| = \begin{cases} 
1 - \gamma, & \text{if } |j-n|-|j-k| > 0, \\
\gamma - 1 \leq 2(1 - \gamma), & \text{otherwise.} 
\end{cases} \tag{6.20}
\]
Then (6.19) jointly with (6.20) implies the desired estimate. \( \square \)

Now we will deduce item 1 from item 2a and Proposition 6.6. Passing to the action-angle coordinates and using 2a, we have
\[
\partial \psi_k H_2 = -\{J_k, H_2\} = -(i \nabla_k H_2) \cdot v_k \Rightarrow |\partial \psi_k H_2| \leq C \sum_{n:|n-k| \leq 2} |v_n|^p + C. \tag{6.21}
\]
Denote \( G_{ml} := G(|v_m - v_l|^2) \). Note that
\[
H_2 = \frac{1}{8} \sum_{|m-l|=1} \{ (G_{ml}) + G_{ml} \} + \Phi_{kn} =: \sum_{|m-l|=1} H_{2ml}^m.
\]
Since the functions \( G_{ml} \) and \( \Phi_{kn} \) depend on angles only through the differences of their neighbouring components \( \theta_{kn} \), where \( |k-n| = 1 \), then the function \( H_{2ml}^m \) depends on them in the same way. Thus \( H_{2ml}^m \) satisfies the conditions of Proposition 6.6, and we have
\[
|\langle i \nabla H_{2ml}^m \cdot v \rangle_j| = \left| \sum_{k \in \mathcal{C}} \gamma ^{|j-k|} \partial \psi_k H_{2ml}^m \right| \leq 2(1 - \gamma) \sum_{|k-n|=1} \gamma ^{|j-k|} |\partial \theta_{kn} H_{2ml}^m| \tag{6.22}
\]
Note that \( H_{2ml}^m \) depends on \( v = (v_k)_{k \in \mathcal{C}} \) only through \( v_k \) with \( k \) satisfying
\[
|k - m| \wedge |k - l| \leq 1. \tag{6.23}
\]
Then \( \partial \theta_{kn} H_{2ml}^m, \partial \psi_k H_{2ml}^m \neq 0 \) only if \( k \) and \( n \) satisfy (6.23). Expressing \( \psi_r, r \in \mathcal{C} \) through \( \{ \theta_{kn} : |k-n| = 1 \} \) and \( \psi_s : s \in \partial \mathcal{C} \), we obtain
\[
\gamma ^{|j-k|} |\partial \theta_{kn} H_{2ml}^m| \leq C \gamma ^{|j-k|} \sum_{r:|m-r| \wedge |l-r| \leq 1} |\partial \psi_r H_{2ml}^m| \leq C_1 \sum_{r:|m-r| \wedge |l-r| \leq 1} \gamma ^{|j-r|} |\partial \psi_r H_{2ml}^m|,
\]
since for \( k \) and \( r \) satisfying (6.23) we have \( \gamma ^{|j-k|}/\gamma ^{|j-r|} \leq C \), in view of \( 1/2 < \gamma < 1 \). Then (6.22) implies
\[
|\langle i \nabla H_{2ml}^m \cdot v \rangle_j| \leq (1 - \gamma) C \sum_{k:|m-k| \wedge |l-k| \leq 1} \gamma ^{|j-k|} |\partial \psi_k H_{2ml}^m|. \tag{6.24}
\]
\(^7\text{To make this more clear, let us consider the example } h(\psi) = \psi_k - \psi_n, \text{ where } |k-n| = 1. \text{ In this case we treat the function } h \text{ either as the function } h = \theta_{kn} \text{ which does not depend on } \theta_{nk}, \text{ or as the function } h = -\theta_{nk} \text{ which does not depend on } \theta_{kn}.\)
Clearly, (6.21) holds with $H_2$ replaced by $H_2^{ml}$. Then (6.24) implies
\[
|\langle i\nabla H_2 \cdot v \rangle_j| \leq (1 - \gamma)C \sum_{|m-l|=1} \sum_{k:|m-k|=1, k \leq |l-k| \leq 1} \gamma^{j-k} \left( \sum_{n:|n-k| \leq 2} |v_n|^p + 1 \right) 
\leq (1 - \gamma)C_1 \sum_{|m-l|=1} \sum_{k:|m-k|=1, k \leq |l-k| \leq 3} \gamma^{j-k} |v_k|^p + C(\gamma) 
\leq (1 - \gamma)C_2 \sum_{k \in \mathcal{C}} \gamma^{j-k} |v_k|^p + C(\gamma) = (1 - \gamma)C_2 \|v\|_{j,p}^p + C(\gamma).
\]

Item 2b. Let us remind that $r = i\nabla H_\gamma + r^I + r^D$. Consequently,
\[
\|r\|_{j,q} \leq \|i\nabla H_\gamma\|_{j,q} + \|r^I\|_{j,q} + \|r^D\|_{j,q}.
\]

Step 1. Firstly we will show that for any $j \in \mathcal{C}$ and $q \geq 1$
\[
\|i\nabla H_\gamma\|_{j,q} \leq C\|v\|_{j,q}^{p-1} + C(\gamma). 
\] (6.25)
Due to (6.2) and (6.7), we have $\{\tilde{F}, \Phi\}_3 = \{\tilde{G} - \tilde{G}, \Phi\}_2$. Note that by Taylor’s expansion
\[
\frac{1}{2} \{\tilde{G}, \Phi\}_2 + \frac{1}{2} \int_0^1 (1 - s)^2 \{\tilde{G}, \Phi\}_2 \circ X_s^{\sqrt{\beta}} ds = \int_0^1 (1 - s) \{\tilde{G}, \Phi\}_2 \circ X_s^{\sqrt{\beta}} ds.
\]
Then, due to (6.8), we have
\[
H_\gamma = \frac{1}{2} \int_0^1 (1 - s)^2 \{\tilde{G} - \tilde{G}, \Phi\}_2 \circ X_s^{\sqrt{\beta}} ds + \int_0^1 (1 - s) \{\tilde{G}, \Phi\}_2 \circ X_s^{\sqrt{\beta}} ds = 
\frac{1}{2} \int_0^1 (1 - s^2) \{\tilde{G}, \Phi\}_2 \circ X_s^{\sqrt{\beta}} ds + \frac{1}{2} \int_0^1 (1 - s)^2 \{\tilde{G}, \Phi\}_2 \circ X_s^{\sqrt{\beta}} ds 
=: \int_0^1 Y \circ X_s^{\sqrt{\beta}} ds =: \int_0^1 Y^s ds.
\]
Due to Propositions 6.5.2 and 6.1, for all $k \in \mathcal{C}$ we have
\[
|\partial_{u_k} \{\tilde{G}, \Phi\}_2|, |\partial_{u_k} \{\tilde{G}, \Phi\}_2|, |\partial_{u_k} \{\tilde{G}, \Phi\}_2|, |\partial_{u_k} \{\tilde{G}, \Phi\}_2| \leq C \sum_{l:|k-l| \leq 3} |v_l|^{p-1}.
\]
Thus
\[
\left| \frac{\partial Y}{\partial u_k} \right|, \left| \frac{\partial Y}{\partial u_k} \right| \leq C \sum_{l:|k-l| \leq 3} |v_l|^{p-1} + C \quad \text{for any} \quad k \in \mathcal{C}. 
\] (6.26)
Estimate (6.26) implies
\[
\left\| \frac{\partial Y}{\partial v} \right\|_{j,q}^q, \left\| \frac{\partial Y}{\partial u} \right\|_{j,q}^q \leq C_1 \sum_{k \in \mathcal{C}} \gamma^{j-k} |v_k|^q |v|^q + C(\gamma) = C_1 \|v\|_{j,q}^{q(p-1)} + C(\gamma).
\]
Thus, we get
\[ \left\| \frac{\partial Y}{\partial v} \right\|_{j,q}, \left\| \frac{\partial Y}{\partial \nu} \right\|_{j,q} \leq C \|v\|_{j,q(p-1)}^{p-1} + C(\gamma). \] (6.27)

Exchanging the derivative and the integral we find
\[ \nabla H > = 2 \partial_v \int_0^1 Y^s \, ds = 2 \int_0^1 \left( \frac{\partial Y^s}{\partial v^s} \frac{\partial v^s}{\partial v} + \frac{\partial Y^s}{\partial \nu^s} \frac{\partial \nu^s}{\partial \nu} \right) \, ds. \]

By (6.27), Corollaries 6.3 and 6.2 we obtain (6.25):
\[ \|i\nabla H\|_{j,q} \leq C \int_0^1 \left( \left\| \frac{\partial Y^s}{\partial v^s} \right\|_{j,q} + \left\| \frac{\partial Y^s}{\partial \nu^s} \right\|_{j,q} \right) \, ds \]
\[ \leq C_1 \int_0^1 \|v^s\|_{j,q(p-1)}^{p-1} \, ds + C(\gamma) \leq C_1 \|v\|_{j,q(p-1)}^{p-1} + C_1(\gamma). \]

**Step 2.** Let us show that
\[ \|r^D\|_{j,q} \leq C \|v\|_{j,q(p-1)}^{p-1} + C(\gamma). \] (6.28)

Due to (6.13), we have \( r^D = r^D_1 + r^D_2 \), where
\[ r^D_1 = \varepsilon^{-1/2} \left( g(u) - g(v) \right), \quad r^D_2 = \varepsilon^{-1/2} \left( \frac{\partial v}{\partial u} - \text{Id}_N \right) g(u) + \frac{\partial v}{\partial \nu} g(u) \).

Due to assumption \( Hg(i) \) and item 4 of the theorem, we have
\[ |r^D_{1k}| \leq C \left( 1 + \sum_{l,|k-l| \leq 1} (|u_l|^{p-1} + |v_l|^{p-1}) \right) \text{ for all } k \in \mathcal{C}. \]

Then, Corollary 6.2 implies
\[ \|r^D_{1k}\|_{j,q}^{q} \leq C \sum_{k \in \mathcal{C}} \gamma^{j-k} (|u_k|^{q(p-1)} + |v_k|^{q(p-1)}) + C(\gamma) \leq C_1 \|v\|_{j,q(p-1)}^{q(p-1)} + C_1(\gamma). \] (6.29)

Applying Corollary 6.3 and Corollary 6.2, we get
\[ \|r^D_{2}\|_{j,q}^{q} \leq C \|g(u)\|_{j,q}^{q} \leq C_1 \|v\|_{j,q(p-1)}^{q(p-1)} + C_1(\gamma). \] (6.30)

Combining (6.29) with (6.30), we obtain (6.28).

**Step 3.** Let us show that
\[ \|r^f\|_{j,q} \leq C(\gamma). \] (6.31)
Due to (6.15), we have

$$A^s : = \sum_{k \in \mathcal{C}} T_k \frac{\partial^2 U^s}{\partial u_k \partial u_k} = -\sqrt{\varepsilon} \sum_{k \in \mathcal{C}} T_k \frac{\partial}{\partial u_k} \int_0^s (D^2 \Phi)^\tau \frac{\partial U^\tau}{\partial u_k} \, d\tau$$

$$= -\sqrt{\varepsilon} \sum_{k \in \mathcal{C}} T_k \sum_{\tilde{m} \in \tilde{\mathcal{C}}} \int_0^s \partial_{U_m}^\tau (D^2 \Phi)^\tau \frac{\partial U^\tau}{\partial u_k} \frac{\partial U^\tau}{\partial u_k} \, d\tau - \sqrt{\varepsilon} \sum_{k \in \mathcal{C}} T_k \int_0^s (D^2 \Phi)^\tau \frac{\partial^2 U^\tau}{\partial u_k \partial u_k} \, d\tau$$

$$= \sqrt{\varepsilon} \left( \Gamma - \int_0^s (D^2 \Phi)^\tau A^\tau \, d\tau \right), \quad \text{where } \Gamma = (\Gamma_l)_{l \in \tilde{\mathcal{C}}}.$$  \hspace{1cm} (6.32)

Let us show that

$$\left| \Gamma \right|_{j,q} \leq C(\gamma).$$  \hspace{1cm} (6.33)

Let $\tilde{l} = (l_0, l), \tilde{r} = (r_0, r), \tilde{m} = (m_0, m) \in \tilde{\mathcal{C}}$. Note that $(D^2 \Phi)^\tau_{l \tilde{r}}$ may be nonzero only if $\tilde{l}$ and $\tilde{r}$ satisfy $|l - r| \leq 1$, and $(D^2 \Phi)^\tau_{l \tilde{r}}$ may depend only on components $U_{\tilde{m}}$ with $m$ satisfying $|m - l| \leq 1$. Then for every $\tilde{l} = (l_0, l) \in \tilde{\mathcal{C}}$ we have

$$\Gamma_{\tilde{l}} = - \sum_{\tilde{m}, \tilde{r} : |m-l|, |r-l| \leq 1} \int_0^s \partial_{U_m}^\tau (D^2 \Phi)^\tau_{l \tilde{r}} T_k \frac{\partial U^\tau}{\partial u_k} \frac{\partial U^\tau}{\partial u_k} \, d\tau$$

$$= - \sum_{\tilde{m}, \tilde{r} : |m-l|, |r-l| \leq 1} \int_0^s \partial_{U_m}^\tau (D^2 \Phi)^\tau_{l \tilde{r}} T_k \left[ \frac{\partial U^\tau}{\partial u} \text{diag}(T_k) \left( \frac{\partial U^\tau}{\partial u} \right)^T \right]_{\tilde{m} \tilde{r}} \, d\tau.$$

The $2N \times 2N$ matrix $\frac{\partial U^\tau}{\partial u} \text{diag}(T_k) \left( \frac{\partial U^\tau}{\partial u} \right)^T$ consists of four blocks of the type

$$\frac{\partial a^\tau}{\partial b} \text{diag}(T_k) \left( \frac{\partial c^\tau}{\partial d} \right)^T,$$

where $a, b, c, d \in \{u, \tilde{u}\}$.

By Corollary 6.3, for every $j \in \mathcal{C}$ and $q \geq 1$ we have

$$\left| \left| \frac{\partial a^\tau}{\partial b} \text{diag}(T_k) \left( \frac{\partial c^\tau}{\partial d} \right)^T \right| \right|_{j,q} \leq C.$$

Proposition 6.4 implies that the elements of this matrix are bounded by the same constant. Thus, for every $\tilde{r}, \tilde{m} \in \tilde{\mathcal{C}}$ we have

$$\left| \left| \frac{\partial U^\tau}{\partial u} \text{diag}(T_k) \left( \frac{\partial U^\tau}{\partial u} \right)^T \right| \right|_{\tilde{m} \tilde{r}} \leq C.$$

Proposition 6.1 implies that $|\partial_{U_m}^\tau (D^2 \Phi)^\tau_{l \tilde{r}}| \leq C$. Then for any $\tilde{l} \in \tilde{\mathcal{C}}$ we have $|\Gamma_{\tilde{l}}| \leq C$, and, consequently, we get (6.33)

Due to Corollary 6.3, we have $|D^2 \Phi|_{j,q} \leq C$. Then (6.32) jointly with (6.33) implies

$$\left| A^s \right|_{j,q} \leq \sqrt{\varepsilon} \left| \Gamma \right|_{j,q} + \sqrt{\varepsilon} C \int_0^s \left| A^s \right|_{j,q} \, d\tau \Rightarrow \left| A^s \right|_{j,q} \leq \sqrt{\varepsilon} C(\gamma).$$
Thus, we obtain (6.31). The proof of the theorem is complete. \[\square\]

**Proof of Proposition 6.1.** According to (6.6), we have

\[
\Phi = \frac{1}{4} \sum_{|j-n|=1} \frac{Y_{jn}}{f_j - f_n}, \quad \text{where} \quad Y_{jn} = \int_0^{\theta_{jn}} G^0(j_j, j_n, \theta) \, d\theta, \quad (6.34)
\]

and \(G^0 = G - \langle G \rangle\). Denote

\[
D^q := \partial^q_{v_j^n, v_j^m, v_n^n, \nu_j^n}, \quad \text{where} \quad q := m_j + l_j + m_n + l_n, \quad m_j, l_j, m_n, l_n \in \mathbb{N} \cup \{0\}. \quad (6.35)
\]

Let \(G_{jn}(v_j, v_n) := G(|v_j - v_n|^2)\) and \(G^0_{jn} := G_{jn} - \langle G_{jn} \rangle\). We will need the following proposition.

**Proposition 6.7.** Assume that \(G_{jn} \in C^{q_0+1} \mathbb{C}^2\), where \(q_0 \in \mathbb{N}\). Then

(i) \(Y_{jn} \in C^{q_0} \mathbb{C}^2\).

(ii) Assume that for an operator \(D^q\) as above with \(q \leq q_0\) we have \(|D^q G_{jn}| \leq K(|v_j|, |v_n|)\) for all \(v_j, v_n \in \mathbb{C}\) and some function \(K : \mathbb{R}^2_{+,0} \mapsto \mathbb{R}\). Then \(|D^q Y_{jn}| \leq C K(|v_j|, |v_n|)\) for all \(v_j, v_n \in \mathbb{C}\).

Before proving Proposition 6.7 we will finish the proof of Proposition 6.1. The \(C^3\)-smoothness of \(\Phi\) follows from Proposition 6.7.i and assumptions HF and HG. The assertion stating that the most components of the second and the third derivatives of \(\Phi\) vanish is obvious. By (6.34), assumption HF and Proposition 6.5.3 we have

\[
\left| \partial_{\psi_j} \Phi \right| \leq \frac{1}{2} \sum_{n,|j-n|=1} \left| \frac{G^0(|v_j - v_n|^2)}{f_j - f_n} \right| \leq C,
\]

thus the first estimate of (6.9) is proven. To prove the other, in view of Proposition 6.5.1, it suffices to show that for every \(j, n \in \mathbb{C}\) satisfying \(|j - n| = 1\) and for all \(k, l, m \in \{j, n\}\) we have

\[
\left| \frac{\partial Y_{jn}}{\partial a_k} \right|, \left| \frac{\partial^2 Y_{jn}}{\partial a_k \partial b_l} \right|, \left| \frac{\partial^3 Y_{jn}}{\partial a_k \partial b_l \partial c_m} \right| \leq C (1 + |v_j|^{p-1} + |v_n|^{p-1}), \quad (6.36)
\]

where \(a, b, c \in \{v, \nu\}\). In view of Proposition 6.5.2, estimate (6.36) is an immediate corollary of Proposition 6.7.ii with \(K = C (1 + |v_j|^{p-1} + |v_n|^{p-1})\) and \(q_0 = 3\). \[\square\]

**Proof of Proposition 6.7.** Since we will only use the functions \(G_{jn}, G^0_{jn}, Y_{jn}\) and not \(G, G^0, Y\), then, abusing notations, instead of writing \(G_{jn}, G^0_{jn}, Y_{jn}\), we will write just \(G, G^0, Y\).

(i) The proof is based on the following formulas: if \(v_j \neq 0\) then

\[
\partial_{v_j} = e^{-i \psi_j} \left( \frac{\partial}{2|v_j|} + \frac{1}{2|v_j|^2} \partial_{\psi_j} \right) \quad \text{and} \quad \partial_{\nu_j} = e^{i \psi_j} \left( \frac{\partial}{2|v_j|} - \frac{1}{2|v_j|^2} \partial_{\psi_j} \right). \quad (6.37)
\]

Expanding \(G^0\) in the Fourier series, we get

\[
G^0 = \sum_{k \in \mathbb{Z} \setminus \{0\}} G_k(|v_j|, |v_n|) e^{ik(\psi_j - \psi_n)} \quad \text{and} \quad Y = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{G_k}{ik} e^{ik(\psi_j - \psi_n)},
\]

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where we have used that \( G_k = G_{-k} \), and, consequently, \( \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{G}_k^j \equiv 0 \).

Let us first suppose that \( v_j, v_n \neq 0 \). Take \( D^q \) as in (6.35) with \( 0 < q \leq q_0 \), where by \( D^0 \) we denote the identity operator. It is clear that \( Y \in C^{q_0+1}(\mathbb{C}^2 \setminus \{v_j = 0 \text{ or } v_n = 0\}) \), so \( D^q Y \) is well-defined for the present case. Now our goal is to represent it in an integral form. Using (6.37), we get

\[
\partial_{v_j} (G_k e^{ik(\psi_j - \psi_n)}) = \hat{G}_k^j e^{i[(k-1)\psi_j - k\psi_n]} \quad \text{and} \quad \partial_{\psi_j} (G_k e^{ik(\psi_j - \psi_n)}) = \hat{G}_k^j e^{i[(k+1)\psi_j - k\psi_n]},
\]

for some functions \( \hat{G}_k^j = \hat{G}_k^j([v_j], [v_n]) \) and \( \hat{G}_k^j = \hat{G}_k^j([v_j], [v_n]) \). Similar relations hold for the derivatives with respect to \( v_n, \psi_n \). Arguing by induction, we obtain

\[
D^q(G_k e^{ik(\psi_j - \psi_n)}) = \hat{G}_k^j e^{i(s_j(k)\psi_j + s_n(k)\psi_n)}, \tag{6.38}
\]

where \( s_j(k) = k + l_j - m_j \), \( s_n(k) = l_n - m_n - k \) and \( \hat{G}_k = \hat{G}_k([v_j], [v_k]) \). Since \( G^0 \in C^{q_0+1}(\mathbb{C}^2) \), the Fourier series for \( \partial_{v_j} G^0 \) converges to it. Calculating its coefficients \( A_{k,\nu_n} = (2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} \partial_{v_j} G^0 e^{-i(k\psi_j + \nu_n\psi_n)} d\psi_j d\psi_n \) with help of formula (6.37) and integration by parts, we get that it has the form

\[
\partial_{v_j} G^0 = \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{G}_k^j e^{i[(k-1)\psi_j - k\psi_n]},
\]

i.e. we can differentiate the Fourier series for \( G^0 \) term by term. Calculating similarly the series for \( \partial_{\psi_j} G, \partial_{v_n} G, \partial_{\psi_n} G, \partial_{v_j} Y \ldots \) and arguing by induction, we get

\[
D^q G^0 = \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{G}_k^j e^{i(s_j(k)\psi_j + s_n(k)\psi_n)} \quad \text{and} \quad D^q Y = \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{G}_k^j e^{i(s_j(k)\psi_j + s_n(k)\psi_n)}, \tag{6.39}
\]

if \( v_j, v_n \neq 0 \). By the direct computation, from (6.39) we obtain

\[
D^q Y(v_j, v_n) = \int_0^{\psi_j} \int_0^{\psi_n} D^q G^0([v_j], [v_n], \psi, \psi_n) e^{i(m_j - l_j)(\psi - \psi_j)} d\psi \tag{6.40}
\]

\[
- \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\xi} D^q G^0([v_j], [v_n], \psi, \psi_n) e^{i(m_j - l_j)(\psi - \psi_j)} d\psi d\xi, \quad \text{if} \quad v_j, v_n \neq 0.
\]

and

\[
D^q Y(v_j, v_n) = - \int_0^{\psi_n} \int_0^{\psi_j} D^q G^0([v_j], [v_n], \psi, \psi) e^{i(m_n - l_n)(\psi - \psi_n)} d\psi \tag{6.41}
\]

\[
+ \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\xi} D^q G^0([v_j], [v_n], \psi_j, \psi) e^{i(m_n - l_n)(\psi - \psi_n)} d\psi d\xi, \quad \text{if} \quad v_j, v_n \neq 0.
\]

Denote the r.h.s. of (6.40) and (6.41) by \( A^q_j(v_j, v_n) \) and \( A^q_n(v_j, v_n) \) correspondingly. Since \( D^q G^0 \in C(\mathbb{C}^2) \), then \( A^q_j \in C(\mathbb{C}^2 \setminus \{v_j = 0\}) \) and \( A^q_n \in C(\mathbb{C}^2 \setminus \{v_n = 0\}) \).
Proposition 6.8. For \(0 \leq q \leq q_0\) we have \(A_j^q, A_n^q \in C(\mathbb{C}^2)\).

Before proving Proposition 6.8 we will finish the proof of Proposition 6.7. In view of (6.40) and (6.41) we have

\[
D^qY = A_j^q = A_n^q \quad \text{for} \quad v_j, v_n \neq 0.
\]

Now we will show that \(Y \in C^{q_0}(\mathbb{C}^2)\). We argue by induction. Note that \(D^0Y = Y \in C(\mathbb{C}^2)\) since \(G^0(0, v_n) = G^0(v_j, 0) = 0\). Fix \(0 \leq q < q_0\) and assume that for some \(D^q\) as in (6.35) the function \(D^qY \in C(\mathbb{C}^2)\). We claim that this implies \(D^{q+1}Y \in C(\mathbb{C}^2)\), where \(D^{q+1} := \partial_{v_j} D^q\).

Indeed, in view of (6.42) and Proposition 6.8, the continuity of \(D^qY\) implies that (6.42) holds for all \(v_j, v_n \in \mathbb{C}\). It is clear that \(\partial_{v_j} A_n^q\) is well-defined for all \(v_j, v_n \in \mathbb{C}\) and coincides with \(A_n^{q+1}\). Then we get \(D^{q+1}Y = \partial_{v_j} A_n^q = A_n^{q+1}\) for all \(v_j, v_n \in \mathbb{C}\), and Proposition 6.8 implies that \(D^{q+1}Y \in C(\mathbb{C}^2)\).

The cases \(D^{q+1} = \partial_{v_j} D^q, \partial_{v_n} D^q\) and \(\partial_{\psi_j} D^q\) can be considered similarly. In the last two situations one should differentiate \(A_j^q\) instead of \(A_n^q\). Thus, we obtain that \(D^qY \in C(\mathbb{C}^2)\) for any \(0 \leq q \leq q_0\) and any \(D^q\) as in (6.35). Consequently, \(Y \in C^{q_0}(\mathbb{C}^2)\).

(iii) In the proof of item (i) we have shown that (6.42) holds for all \(v_j, v_n \in \mathbb{C}\) if \(q \leq q_0\). This implies the desired estimate since, obviously, \(|D^qG^0_{j,n}| \leq C K(|v_j|, |v_n|)\).

Proof of Proposition 6.8. We prove the proposition for \(A_j^q, A_n^q\) the proof is similar. We only need to show that \(A_j^q\) is continuous when \(v_j = 0\). Let us take a sequence \((v_j^b, v_n^b) \to (0, v_n)\) as \(b \to \infty\) and let \((\psi_j^b, \psi_n^b)\) be the sequence of the corresponding angles, \(\psi_j^b = \psi_j(v_j^b), \psi_n^b = \psi_n(v_n^b), b \in \mathbb{N}\). It suffices to show that the limit as \(b \to \infty\) of \(A_j^q(|v_j^b|, |v_n^b|, \psi_j^b, \psi_n^b)\) exists and depends only on \(v_n\) and not on the sequence. Since \(D^qG^0 \in C(\mathbb{C}^2)\), we have

\[
\lim_{b \to \infty} A_j^q(|v_j^b|, |v_n^b|, \psi_j^b, \psi_n^b) = \lim_{b \to \infty} A_j^q(0, |v_n^b|, \psi_j^b, \psi_n^b)
\]

in the sense that if one limit exists then the another exists as well and the two are equal. Since \(D^qG^0(0, |v_n^b|, \psi, \psi^b) = D^qG^0(0, v_n^b)\), we obtain

\[
A_j^q(0, |v_n^b|, \psi_j^b, \psi_n^b) = D^qG^0(0, v_n^b)\left(\int_0^{2\pi} e^{i(m_j-l_j)(\psi-\psi^b)} d\psi - \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} e^{i(m_j-l_j)(\psi-\psi^b)} d\psi d\xi \right).
\]

Assume first that \(m_j - l_j \neq 0\). Then, calculating the integrals in the r.h.s. of (6.43), we get \(A_j^q(0, v_n^b) = i(l_j - m_j)^{-1}D^qG^0(0, v_n^b) \to i(l_j - m_j)^{-1}D^qG^0(0, v_n^b)\) as \(b \to \infty\) independently of the sequence \((v_j^b, v_n^b)\).

Let now \(m_j = l_j\). Since \(D^qG^0 \in C^1(\mathbb{C}^2)\), its Fourier series converges to it for all \(v_j, v_n\) and the Fourier coefficients continuously depend on \(|v_j|, |v_n|\). It means that the series for \(D^qG^0\) has the form (6.39) for all \(v_j, v_n \in \mathbb{C}\). Since at the point \((0, v_n^b)\) it does not depend on the angle \(\psi_j\), then \(G_n(0, |v_n^b|) \neq 0\) only if \(k\) satisfies \(s_j(k) = 0\), which is equivalent to \(k = 0\), in view of \(m_j = l_j\). Since in the Fourier series for \(D^qG^0\) there is no term with \(k = 0\), we obtain \(D^qG^0(0, v_n^b) \equiv 0\). Consequently, \(A_j^q(0, |v_n^b|, \psi_j^b, \psi_n^b) \equiv 0\) for all \(b \in \mathbb{N}\). \(\square\)
7 Generalizations

7.1 Non dissipative case

Let us briefly discuss what happens if functions $g_j(u)$ do not have dissipative properties, i.e. if assumption $Hg(ii)$ is not satisfied. In this case for $p = 2$ the estimate, similar to that of Lemma 3.2 holds, but is not uniform in time. Then for $p = 2$ the main results are essentially the same except those concerning stationary measures: Theorems 4.6 and 4.7 hold true but Theorem 2.3 fails. Their proofs do not change.

7.2 Defects

In this section we briefly discuss the situation when some rotators are "defective": there exists a region $C_D \subset \mathbb{Z}^d$ which does not depend on $N$, such that the rotators situated there rotate in arbitrary directions, so their spins are not alternated. In this case the system of rotators has resonances of the first order, and we can not completely remove the leading order of the interaction potential by the canonical change of variables, as we did before. However, we are able to remove its part, responsible to the interaction between non defective rotators.

Let us denote by $\overline{C_D} := \{ k : |k - C_D| \leq 1 \}$ the "closure" of $C_D$, where $|k - C_D| = \inf_{l \in C_D} |k - l|$, and let $M$ be the number of sites in $C \setminus \overline{C_D}$. Denote also

$$C_I := \{ k : |k - \overline{C_D}| = 1 \text{ or } 2 \} \quad \text{and} \quad C_G := \{ k : |k - \overline{C_D}| \geq 3 \},$$

where "I" means "Intermediate" and "G" means "Good". For a vector $a = (a_j)_{j \in C}$ we will write

$$a_{ND} := (a_j)_{j \in C \setminus \overline{C_D}},$$

where "ND" means "Non Defective".

We make a global canonical change of variables, determined by a time-one map of the Hamiltonian flow $X_{\sqrt{\varepsilon} \Phi}^*$ with $\Phi = \sum_{j,k \in C_D, |j-k| = 1} \Phi_{jk}$, where $\Phi_{jk}$ is defined in (6.6). Note that for $j,k \in C \setminus C_D$ the denominator of (6.6) is separated from zero. We obtain a new Hamiltonian

$$\mathcal{H}(J, \psi) = H_0(J) + \sqrt{\varepsilon} H_1(J, \psi) + \varepsilon H_2(J, \psi) + \varepsilon \sqrt{\varepsilon} H_\geq(J, \psi), \quad (7.1)$$

where $H_1(J, \psi) = \frac{1}{2} \sum_{j \text{ or } k \in C_D, |j-k| = 1} G(|v_j - v_k|^2)$.

Let $v^\varepsilon(s)$ be a unique solution of (2.5)-(2.6) with the defective rotators and $I^\varepsilon(s)$, $\varphi^\varepsilon(s)$ be the corresponding vectors of actions and angles. By the change of variables above we obtain the processes $v^\varepsilon(s)$, $J^\varepsilon(s)$ and $\psi^\varepsilon(\tau)$. Arguing similarly to Lemma 3.2 and using that the number of defective rotators does not depend on $N$, we are still able to obtain uniform in $\varepsilon, N, j$ and $s$ estimates on solution $v^\varepsilon(s)$.

In view of (7.1), equation for non defective actions $J_{ND}(\tau)$, written in the slow time, turns out to be slow, and consequently the family of measures $\{ \mathcal{D}(J_{ND}(\cdot)), 0 < \varepsilon \leq 1 \}$ is tight on $C([0, T], \mathbb{R}^M)$. Take a subsequence $\varepsilon_k$ such that $\mathcal{D}(J^{\varepsilon_k}_{ND}(\cdot)) \rightharpoonup Q_{ND}^0$ as $\varepsilon_k \to 0$. Since the transformation is $\sqrt{\varepsilon}$ -close to identity (see Theorem 3.1.4), we get

$$\mathcal{D}(J^{\varepsilon_k}_{ND}(\cdot)) \rightharpoonup Q_{ND}^0 \quad \text{as} \quad \varepsilon_k \to 0. \quad (7.2)$$
We fix this subsequence for the next two theorems.

**Theorem 7.1.** The measure $Q_{ND}^0$ is a distribution $\mathcal{D}(I_{ND}^0(\cdot))$ of a weak solution $I_{ND}^0(\tau) = (I_j^0(\tau))_{j \in \mathcal{C}}$ of the system

$$dI_j = (R_j(I) + T_j) \, d\tau + \sqrt{2I_j T_j} \, d\beta_j, \quad j \in \mathcal{C}_G,$$

(7.3)

with the initial conditions $\mathcal{D}(I_{ND}(0)) = \mathcal{D}(I_{ND}(u_0))$, where $\beta_j$ are standard real independent Brownian motions. Moreover, for any $j \in \mathcal{C} \setminus \overline{\mathcal{C}}_D$, we have

$$E \left( \sup_{\tau \in [0,T]} e^{2\alpha I_j^0(\tau)} \right) < C \quad \text{and} \quad \int_0^T P(I_j^0(s) < \delta) \, ds \to 0 \quad \text{as} \quad \delta \to 0,$$

where the latter convergence is uniform in $N$.

Note that the system of equations (7.3) is not closed: it depends on $I_j$, $j \in \mathcal{C}_I$ for which we cannot obtain a limiting relation. For a general case we can say nothing about the uniqueness of limiting point in (7.2) and about the uniformity in $N$ of convergence in Theorem 7.1. However, if $R_j(I) = R_j(I_j)$ for all $j \in \mathcal{C}_G$, then this holds in some sense, see Example 7.4 below. In the next two theorems we have a similar situation.

Denote $\nu^\varepsilon := \mathcal{D}(I_{ND}^\varepsilon(\tau), \varphi_{ND}^\varepsilon(\tau))$ and for any function $h(\tau) \geq 0$, satisfying

$$\int_0^T h(\tau) \, d\tau = 1,$$

set $\nu^\varepsilon(h) := \int h(\tau) \nu^\varepsilon d\tau$. Moreover, denote $n^0(h) := \int_0^T h(\tau) \mathcal{D}(I_{ND}^0(\tau)) \, d\tau$, where $I_{ND}^0$ is a solution of (7.3), obtained in Theorem 7.1.

**Theorem 7.2.** For any continuous function $h$ as above, we have

$$\nu^{\varepsilon_k}(h) \to n^0(h) \times d\varphi \quad \text{as} \quad \varepsilon_k \to 0.$$

Let $\tilde{\mu}^\varepsilon$ be a unique stationary measure of (2.5) with the defective rotators. Let us denote by $\Pi_{I_{ND}}$ and $\Pi_{\varphi_{ND}}$ the projections to the spaces of actions $I_{ND}$ and angles $\varphi_{ND}$ correspondingly. The family of measures $\{\Pi_{I_{ND} \times \varphi_{ND}} \tilde{\mu}^\varepsilon, 0 < \varepsilon \leq 1\}$ is tight. Take any limiting point $\Pi_{I_{ND} \times \varphi_{ND}} \tilde{\mu}^\varepsilon \to \pi_{ND}$ as $\varepsilon_k \to 0$.

**Theorem 7.3.** The measure $\pi_{ND}$ is a stationary measure of equation (7.3). Moreover,

$$(\Pi_{I_{ND}} \times \Pi_{\varphi_{ND}}) \tilde{\mu}^{\varepsilon_k} \to \pi_{ND} \times d\varphi \quad \text{as} \quad \varepsilon_k \to 0.$$

The proofs of Theorems 7.1, 7.2 and 7.3 repeat the proofs of the corresponding theorems in the non defective case.

**Example 7.4.** Let us consider the situation when $R_j(I)$ depends only on $I_j$ for all $j \in \mathcal{C}_G$. For instance, this happens when the dissipation is diagonal, for more examples see Section 4.4. Then system of equations (7.3) is closed and depends only on $(I_j)_{j \in \mathcal{C}_G}$. It has a unique weak solution and, consequently, the limiting measure $\Pi_{G_\varepsilon} Q_{ND}^0$ is unique. In this case the convergence $\mathcal{D}((I_j(\cdot))_{j \in \mathcal{C}_G}) \to \Pi_{G_\varepsilon} Q_{ND}^0$ as $\varepsilon \to 0$ holds and is uniform in $N$. Similarly, for the projections to $G$ of measures from Theorems 7.2 and 7.3 the corresponding convergences as $\varepsilon \to 0$ hold and are uniform in $N$.  

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8Here $\Pi_G$ denotes the projection on $\mathcal{C}_G$. 

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A Appendix: The Ito formula in complex coordinates

Let \( \{\Omega, \mathcal{F}, P; \mathcal{F}_t\} \) be a filtered probability space and \( v(t) = (v_k(t)) \in \mathbb{C}^N \) be a complex Ito process on this space of the form

\[
dv = b \, dt + W \, dB.
\]

Here \( b(t) = (b_k(t)) \in \mathbb{C}^N \); \( B = (\beta, \overline{\beta})^T \), \( T \) denotes the transposition and \( \beta = (\beta_k) \in \mathbb{C}^N \), \( \beta_k \) are standard independent complex Brownian motions; the \( N \times 2N \)-matrix \( W \) consists of two blocks \((W_1, W_2)\), so \( W \, dB = W_1 \, d\beta + W_2 \, d\overline{\beta} \), and \( W^{1,2}(t) = (W_{kl}^{1,2}(t)) \) are \( N \times N \) matrices with complex entries. The processes \( b_k(t) \), \( W_{kl}^{1,2}(t) \) are \( \mathcal{F}_t \)-adapted and assumed to satisfy usual growth conditions, needed to apply the Ito formula. Let

\[
d_{kl}^1 := (W_1 W_1^T + W_2 W_2^T)_{kl} \quad \text{and} \quad d_{kl}^2 := (W_2 W_1^T + W_1 W_2^T)_{kl}.
\]  

(A.1)

Denote by \((W dB)_k\) the \( k \)-th element of the vector \( W dB \).

**Proposition A.1.** Let \( f : \mathbb{C}^N \to \mathbb{R} \) be a \( C^2 \)-smooth function. Then

\[
df(v(t)) \overline{2} = \sum_k \frac{\partial f}{\partial v_k} \cdot b_k \, dt + \sum_{k,l} \left( \frac{\partial^2 f}{\partial v_k \partial v_l} d_{kl}^1 + \text{Re} \left( \frac{\partial^2 f}{\partial v_k \partial v_l} d_{kl}^2 \right) \right) dt + \sum_k \frac{\partial f}{\partial v_k} \cdot (W dB)_k.
\]

Proof. The result follows from the usual (real) Ito formula. \( \Box \)

Consider the vectors of actions and angles \( J = J(v) \in \mathbb{R}_0^N \) and \( \psi = \psi(v) \in \mathbb{T}^N \). Using formulas \( \partial v_k \psi_k = (2iv_k)^{-1} \) and \( \partial v_k \psi_k = -(2i\overline{v_k})^{-1} \), by Proposition A.1 we get

\[
dJ_k = (b_k \cdot v_k + d_{kk}^1) \, dt + dM_k^J, \quad d\psi_k = \frac{b_k \cdot (iv_k) - \text{Im}(\overline{v_k}v_k^{-1}d_{kk}^2)}{|v_k|^2} \, dt + dM_k^\psi,
\]  

(A.2)

where the martingales \( M_k^J(t) := \int_t^t v_k \cdot (W dB)_k \) and \( M_k^\psi = \int_{t_0}^t \frac{iv_k}{|v_k|^2} \cdot (W dB)_k \) for some \( t_0 < t \).

By the direct computation we obtain

**Proposition A.2.** The diffusion matrices for the \( J \)- and \( \psi \)-equations in (A.2) with respect to the real Brownian motion \((\text{Re} \beta_k, \text{Im} \beta_k)\) have the form \( S^J = (S_{kl}^J) \) and \( S^\psi = (S_{kl}^\psi) \), where

\[
S_{kl}^J = \text{Re}(v_k \overline{v_l} d_{kl}^1 + \overline{v_k} v_l d_{kl}^2) \quad \text{and} \quad S_{kl}^\psi = \text{Re}(v_k \overline{v_l} d_{kl}^1 - \overline{v_k} v_l d_{kl}^2)(|v_k||v_l|)^{-2}.
\]  

(A.3)

The quadratic variations of \( M_k^J \) and \( M_k^\psi \) take the form

\[
[M_k^J]_t = \int_{t_0}^t S_{kk}^J \, ds \quad \text{and} \quad [M_k^\psi]_t = \int_{t_0}^t S_{kk}^\psi \, ds.
\]  

(A.4)

B Appendix: Averaging

Consider a complex coordinates \( v = (v_j) \in \mathbb{C}^N \) and the corresponding vectors of actions \( J = J(v) \) and angles \( \psi = \psi(v) \). Consider a function \( P : \mathbb{C}^N \to \mathbb{R} \) and write it in action-angle coordinates, \( P(v) = P(J, \psi) \). Its averaging

\[
\langle P \rangle := \int_{\mathbb{T}^N} P(J, \psi) \, d\psi
\]

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is independent of angles and can be considered as a function $\langle P \rangle(v)$ of $v$, or as a function $\langle P \rangle(|v_j|)$ of $|v_j|$, or as a function $\langle P \rangle(J)$ of $J$.

**Proposition B.1.** Let $P \in \mathcal{L}_{\text{loc}}(C^N)$. Then

(i) Its averaging $\langle P \rangle \in \mathcal{L}_{\text{loc}}(\mathbb{R}^N_0)$ with respect to $|v_j|$.

(ii) If $P$ is $C^{2s}$-smooth then $\langle P \rangle$ is $C^{2s}$-smooth with respect to $v$ and $C^{s}$-smooth with respect to $J$.

**Proof.** (i) Is obvious.

(ii) The first assertion is obvious. To prove the second consider the function $\hat{P} : x \in \mathbb{R}^N \mapsto \mathbb{R}$, $\hat{P}(x) := \langle P \rangle_{|v=x}$. Then $\hat{P}(x) = \langle P \rangle(J)$, where $J_j = x_j^2/2$. The function $\hat{P}$ is $C^{2s}$-smooth and even in each $x_j$. Any function of finitely many arguments with this property is known to be a $C^{s}$-smooth function of the square arguments $x_j^2$ (see [Whi]).

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