On the Stability of Collinear Points of the RTBP with Triaxial and Oblate Primaries and Relativistic Effects

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Author’s contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

Under the influence of some different perturbations, we study the stability of collinear equilibrium points of the Restricted Three Body Problem. More precisely, the perturbations due to the triaxiality of the bigger primary and the oblateness of the smaller primary, in addition to the relativistic effects, are considered. Moreover, the total potential and the mean motion of the problem are obtained. The equations of motion are derived and linearized around the collinear points. For studying the stability of these points, the characteristic equation and its partial derivatives are derived. Two real and two imaginary roots of the characteristic equation are deduced from the plotted figures throughout the manuscript. In addition, the instability of the collinear points is stressed. Finally, we compute some selected roots corresponding to the eigenvalues which are based on some selected values of the perturbing parameters in the Tables 1, 2.

Keywords: Collinear points; triaxiality; oblateness; relativistic RTBP; stability.

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1. INTRODUCTION

The Lagrangian points are very important to the space community as target locations for large space missions, which can be used in many space applications. This needs accurate investigations of the stability of these points. The aim of this paper is to study the linear stability of...
the collinear points under effects due to the triaxiality and oblateness of the massive and less massive primaries, respectively. The concerned Restricted Three Body Problem (in brief RTBP) dynamical system is linearized around the equilibrium points. The RTBP studies the motion of a test particle $m_3$ in the field of two massive bodies $m_1$ and $m_2$, which is assumed move in circular orbits about their center of mass, Szebehely [1]. Euler [2] and Lagrange [3] found interesting solutions to the circular RTBP that describe equilibrium positions of the infinitesimal body when all net forces acting on it are zero. Concerning the relativistic effects, the readers can refer to, Brumberg [4,5]. Miandl and Dovrak [6] calculated the advance of perihelion of Mercury's orbit within the framework of the (RTBP), which is the most relevant relativistic effect in the motion of the planets around the Sun. The following researchers Ragos et al. [7] and Douskos and Perdios [8] showed that all collinear points were unstable in agreement with the non-relativistic collinear points. Many authors, such as Ahmed et al., [9], Ishwar and Kushvah [10], Vishnu Namboori et al. [11], Mittal et al. [12], and Kumar and Ishwar [13], Abd El-Salam and Abd El-Bar [14], Abd El-Bar et al., [15], studied the circular RTBP with(out) the relativistic correction, triaxial and oblateness perturbations, and/or radiating. Elshaboury et al., [16], treated RTBP considering the primaries that are triaxial rigid bodies. They concluded that the three collinear equilibrium points are all unstable. Also, they paid special attention to investigate symmetric periodic orbits. Martínez and Simó [17] obtained the totality of relative equilibria as depending on the parameters $\kappa$ and the mass ratio $\mu$.

The goal of this work is to study the linear stability of collinear equilibrium points with the effects of different combinations of perturbations on stability of collinear points. The rest of this paper is organized as follows: In sec.2, we derived the equations of motion, then we linearized them around the equilibrium points. In sec.3, we discussed the stability of the equilibrium points. While in sec. 4, we outlined the stability of the collinear points. In the subsections 4.1, 4.2, and 4.3 we derived derivatives that are required to study the stability of $L_1$, $L_2$, and $L_3$ respectively. In section 5, we solved the characteristic equation. In sec.6, we gave some stability visualization, and we studied the stability domains in different perturbed cases. Finally, the conclusion was stated in sec.7.

2. RTBP DYNAMICAL EQUATIONS

The motion of an infinitesimal body in the field of our perturbed model of RTBP in dimensionless barycentric-rotating coordinate system are Bhatnagar and Hallan [18].

Where $U$ is the Pseudo-Potential of the problem, $m_1$, $m_2$ ($m_1 > m_2$) and $m$ are the masses of the massive, less massive primaries, and the infinitesimal body, respectively. as shown in the Fig 1.

\[
\ddot{\xi} - 2n\dot{\eta} = \frac{\partial U}{\partial \xi} - \frac{d}{dt} \left( \frac{\partial U}{\partial \xi} \right), \quad \ddot{\eta} + 2n\dot{\xi} = \frac{\partial U}{\partial \eta} - \frac{d}{dt} \left( \frac{\partial U}{\partial \eta} \right)
\]

(1)

Fig. 1. Geometry for the RTBP
Let \( \mu = \frac{m_2}{m_1+m_2} \) is the small parameter of the problem, \( \sigma_i (i = 1, 2) \) and \( A_2 \ll 1 \) be on respective the numerical values of the coefficients of triaxiality of the massive and oblateness of less massive primaries, \( c \) is the speed of light in vacuum.

The Pseudo-Potential function \( U \) of the relativistic RTBP is given by

\[
U = \frac{n^2 r^2}{2} + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{(1-\mu)}{2r_1^3} (2\sigma_1 - \sigma_2) - \frac{3(1-\mu)}{2r_1^3} (\sigma_1 - \sigma_2) \eta^2 - \frac{A_2 \mu}{2r_1^3} \\
+ \frac{1}{c^2} \left( \frac{r_2^2}{2}(\mu(1-\mu)-3) + \frac{1}{8} \left[ (\dot{\xi} + \eta)^2 + (\dot{\eta} - \xi)^2 \right]^2 \right) \\
+ \frac{3}{2} \left( \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} \right) \left[ (\dot{\xi} + \eta)^2 + (\dot{\eta} - \xi)^2 \right] - \frac{1}{2} \left( \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} \right)^2 \\
- \frac{\mu(1-\mu)}{2r_1^2} \left[ \frac{1}{r_1} + \frac{1}{r_2} \right] \left[ (1-3\mu - 7\xi - 8\eta) + \eta^2 \left( \frac{\mu}{r_1^2} + \frac{(1-\mu)}{r_2^2} \right) \right]
\]

(2)

where, \( c \) is the speed of light, \( \xi, \eta \) are the test particle coordinates in synodic frame of reference. The distance of the test particle from the two massive bodies and from the origin respectively are.

\[
r_1 = \sqrt{(\xi+\mu)^2 + \eta^2}, \quad r_2 = \sqrt{(\xi+\mu-1)^2 + \eta^2}, \quad r = \sqrt{\xi^2 + \eta^2}
\]

(3)

The perturbed mean motion \( n \) is given by

\[
n^2 = 1 + \frac{3}{2} (2\sigma_1 - \sigma_2) + \frac{3}{2} A_2 + \frac{1}{2c^2} (\mu(1-\mu) - 3)
\]

(4)

The included dynamical variables are made dimensionless according to the following normalization criteria: the sum of masses in the system is normalized as \( m_1 + m_2 = 1 \), the time is normalized such that unperturbed mean motion \( n_0 = 1 \). The length is normalized according to considering the distance between the two primaries is unity.

### 2.1 The Stability of the Collinear Points \( L_\alpha, \alpha = 1, 2, 3 \)

To study the stability of the orbits near the collinear points, we linearized the equations of motion about the perturbed locations of these points. Let \( (\xi_0, \eta_0) \) be the unperturbed coordinates of \( L_\alpha, \alpha = 1, 2, 3 \). They satisfy the equations

\[
\frac{\partial U}{\partial \xi} \bigg|_{\xi=\xi_0} = 2n\dot{\xi}_0 + \frac{\partial U}{\partial \xi} \bigg|_{\eta=\eta_0}, \quad \frac{\partial U}{\partial \eta} \bigg|_{\xi=\xi_0} = 2n\dot{\eta}_0 + \frac{\partial U}{\partial \eta} \bigg|_{\eta=\eta_0}
\]

(5)

Substituting in equation (1) \( \dot{\xi} = \xi_1 + \xi_0, \dot{\eta} = \eta_1 + \eta_0, \dot{n} = n_1 + n_0 \) yields the linearized version of Eq. (1) as

\[
\frac{\partial U}{\partial \xi} \bigg|_{\xi=\xi_0} = \left( \frac{\partial U}{\partial \xi} \right)_{\xi_0} + \left( \frac{\partial U}{\partial \eta} \right)_{\eta_0} \eta_1 + \ldots
\]

\[
\frac{\partial U}{\partial \eta} \bigg|_{\xi=\xi_0} = \left( \frac{\partial U}{\partial \xi} \right)_{\xi_0} + \left( \frac{\partial U}{\partial \eta} \right)_{\eta_0} \eta_1 + \ldots
\]

(6)
where $U_{\xi\eta} = \frac{\partial^2 U}{\partial \xi \partial \eta} \, U_{\xi\eta} = \frac{\partial^2 U}{\partial \eta \partial \xi}$, $U_{\eta\xi} = \frac{\partial^2 U}{\partial \eta \partial \xi}$, and $U_{\eta\eta} = \frac{\partial^2 U}{\partial \eta^2}$. Retaining 1st order terms in Eq. (6) we obtain linear differential equations with constant. The system (6) has a solution which can be represented as

$$\xi_i = A e^{\lambda t}, \quad \eta_i = B e^{\lambda t}$$

(7)

where A and B are constants, and $\lambda$ are the eigenvalues.

The characteristic equation corresponding to eq. (6) is

$$\lambda^4 - \left( U_{\xi\eta, \xi \eta} + U_{\eta\xi, \xi \eta} - 4 \right) \lambda^2 + \left[ U_{\xi\eta, \xi \eta} - U_{\xi\eta, \xi \eta} \right] = 0, \quad \alpha = 1, 2, 3$$

(8)

where $U_{\xi\eta, \xi \eta}$ and $U_{\eta\xi, \xi \eta}$ are evaluated at the concerned equilibrium point, $\lambda$ is the roots of the eigenvalue equation (8).

In the collinear points $\eta = 0$, hence $U_{\xi\eta, \xi \eta} = 0$, and the characteristic equation of the system is given by

$$\lambda^4 - \left( U_{\xi\xi, \xi \eta} + U_{\eta\eta, \xi \eta} - 4 \right) \lambda^2 + U_{\xi\xi, \xi \eta} - U_{\eta\eta, \xi \eta} = 0, \quad \alpha = 1, 2, 3$$

or

$$\lambda^4 - N_{\xi \eta} \lambda^2 + M_{\xi \eta} = 0, \quad \alpha = 1, 2, 3$$

(9)

where,

$$N_{\xi \eta} = U_{\xi\xi, \xi \eta} + U_{\eta\eta, \xi \eta} - 4, \quad M_{\xi \eta} = U_{\xi\xi, \xi \eta} - U_{\eta\eta, \xi \eta}$$

(10)

So, the roots of equation (9) are

$$\lambda_{1,2} = \pm \sqrt{-N_{\xi \eta} - \frac{\sqrt{(N_{\xi \eta})^2 - 4M_{\xi \eta}}}{2}}, \quad \lambda_{3,4} = \pm \sqrt{-N_{\xi \eta} + \frac{\sqrt{(N_{\xi \eta})^2 - 4M_{\xi \eta}}}{2}}$$

(11)

From equation (11) there are three possible solutions for the $\lambda_{1,2}$, the first one when $\lambda_{1,2}^2$ is real and negative. In this case two purely imaginary roots $\pm \sqrt{\lambda_{1,2}^2}$ exist, which leads to oscillatory stable solutions, so we will only investigate the case when real $\lambda_{1,2}^2 < 0$.

The other two cases when (i) $\lambda_{3,4}^2$ is complex with non-vanishing imaginary part, and (ii) when $\lambda_{1,2}^2$ is real and positive will lead to instability. These roots can be expressed as $\lambda_{1,2} = \pm ib$ and $\lambda_{3,4} = \pm c$ where $b$ and $c$ are real numbers. The product of all root’s equals to the constant term in the characteristic equation (i.e., $M_{\xi \eta}$), this implies that the condition of stability must be

$$M_{\xi \eta} = U_{\xi\xi, \xi \eta} - U_{\eta\eta, \xi \eta} > 0$$

(12)

Substituting $\eta = 0$ into the second order derivatives $U_{\xi\eta, \xi \eta}$ and $U_{\eta\eta, \xi \eta}$ yields,
\[ U_{\xi \xi, \eta_\alpha} = n - \frac{(1 - \mu)}{r_1^3} - \frac{\mu}{r_2^3} + 3 \left[ \frac{(1 - \mu)(\xi + \mu)^2}{r_1^2} + \frac{\mu(\xi + \mu - 1)^2}{r_2^2} \right] - \frac{3A_2 \mu}{2r_1^5} \]
\[ - \frac{3(1 - \mu)(\xi + \mu)(2\sigma_1 - \sigma_2)}{2r_1^7} + \frac{15A_2(\xi + \mu - 1)^2}{2r_1^7} + \frac{15(1 - \mu)(\xi + \mu)^2(2\sigma_1 - \sigma_2)}{2r_1^7} \]
\[ + \frac{1}{c^2} \left[ (\mu - \mu_1^2 - 3) + \frac{3\xi^2}{2} \right] + \frac{(1 - \mu)}{r_1^2}\left[ \frac{(1 - \mu)(\xi + \mu)}{r_1^3} + \frac{\mu(\xi + \mu - 1)}{r_1^3} \right] \]
\[ - 3 \left[ \frac{(1 - \mu)(\xi + \mu)^2}{r_1^5} + \frac{\mu(\xi + \mu - 1)^2}{r_1^2} \right] \]
\[ - 6\xi \left[ \frac{(1 - \mu)(\xi + \mu)}{r_1^3} + \frac{\mu(\xi + \mu - 1)}{r_1^3} \right] - \frac{3\xi^2}{2} \left[ \frac{(1 - \mu)}{r_1^2} + \frac{\mu}{r_1^2} \right] \]
\[ - 3 \left[ \frac{(1 - \mu)(\xi + \mu)^2}{r_1^5} + \frac{\mu(\xi + \mu - 1)^2}{r_1^2} \right] + 3 \left[ \frac{(1 - \mu)}{r_1^2} + \frac{\mu}{r_1^2} \right] \]
\[ = \frac{\mu(1 - \mu)}{2} \left[ \frac{3(\xi + \mu)^2}{r_1^5} + \frac{1}{r_1^3} + 14 \left[ \frac{(\xi + \mu)}{r_1^3} - \frac{(\xi + \mu - 1)}{r_1^3} \right] \right] \]
\[ + (-1 + 3\mu + 7\xi) \left[ \frac{3(\xi + \mu)^2}{r_1^5} - \frac{3(\xi + \mu - 1)^2}{r_1^2} + \frac{1}{r_1^3} + \frac{1}{r_1^3} \right] \] (13)

and

\[ U_{\eta \eta, \mu} = n - \frac{(1 - \mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3A_2 \mu}{2r_1^5} - \frac{3(1 - \mu)(2\sigma_1 - \sigma_2)}{2r_1^7} - \frac{3(1 - \mu)(\sigma_1 - \sigma_2)}{2r_1^7} \]
\[ + \frac{1}{c^2} \left[ \frac{(1 - \mu)}{r_1^2} + \frac{(1 - \mu)}{r_1^3} + \frac{\mu}{r_1^3} \right] \]
\[ - (\mu - \mu_1^2 - 3) + \frac{3\xi^2}{2} \left[ \frac{3(1 - \mu)}{r_1^3} + \frac{\mu}{r_1^3} \right] \]
\[ + 3 \left[ \frac{(1 - \mu)}{r_1^2} + \frac{\mu}{r_1^2} \right] \left[ \frac{1}{r_1^3} - 2 \left[ \frac{(1 - \mu)}{r_2^3} + \frac{\mu}{r_2^3} \right] - \frac{1}{r_1^3} - \frac{1}{r_1^3} \right] \] (14)

2.2 For \( L_1 \)

Since the considered point is a collinear one, then \( \eta = 0. \) The solution of the classical RTBP satisfies

\[ B_1 r_1 + B_2 r_2 = 1, \quad \eta_1 = B_1 (\xi + \mu), \quad r_2 = B_2 (\mu + x - 1) \] (15)

The \( L_1 \) point locates between the two massive primaries and geometry of \( L_1 \) can be visualized as given by Fig. 2.

At the point \( L_1, \) \( B_1 = B_2 = 1 \) equation (15) becomes

\[ r_1 + r_2 = 1, \quad r_1 = \xi + \mu, \quad r_2 = 1 - \xi, \quad \frac{\partial r_1}{\partial \xi} = -\frac{\partial r_2}{\partial \xi} = 1 \] (16)
Fig. 2. The location of $L_i$ and its corresponding parameters

We can assume the position of the $L_i$ is given by

$$r_i = a_i + \delta_L, \quad r_2 = b_i - \delta_L, \quad a_i + b_i = 1$$  \hspace{1cm} (17)

From which we have

$$\eta = (1-b_i) \left( 1 + \frac{\delta_L}{1-h_i} \right), \quad r_2 = b_i \left( 1 - \frac{\delta_L}{b_i} \right),$$  \hspace{1cm} (18)

where $\delta_L$ is very small, $a_i$ and $b_i$ are the classical positions of $r_i$ and $r_2$, respectively, and $h_i$ is given by

$$h_i = \alpha - \frac{1}{3} \alpha^2 - \frac{1}{9} \alpha^3 - \frac{23}{81} \alpha^4 + O(\alpha^5), \quad \alpha = \left( \frac{\mu}{3(1-\mu)} \right)^{1/3}.$$  \hspace{1cm} (19)

After some lengthy algebraic manipulation, the location of $L_i$ is

$$\xi_{0.L_i} = \sum_{k=3}^{9} F^{(1)}_{-k} \left[ -\frac{1}{c^2} \left\{ \frac{1}{3} \mu_i^3 + \frac{5}{9} \mu_i^2 + \frac{4}{3} \mu_i^1 + \frac{245}{486} \left( \frac{\mu_i}{3} \right)^3 - \frac{1729}{486} \left( \frac{\mu_i}{3} \right)^3 + \cdots \right\} ight.$$

$$\left. - \frac{6395}{2187} \left( \frac{\mu_i}{3} \right)^2 + \frac{398435}{36366} \left( \frac{\mu_i}{3} \right)^3 + \frac{422957}{59049} \left( \frac{\mu_i}{3} \right)^4 - \frac{8374501}{354294} \left( \frac{\mu_i}{3} \right)^5 + \cdots \right\}$$  \hspace{1cm} (20)

where non-vanishing coefficients $F^{(1)}_{-k}$ are given Appendix A

Substituting from equations (18) into equations (13) and (14), yields

$$U_{\xi, \xi, L_i} = n + 2 \left[ (1-\mu)(1-3D_i \delta_L) T_i + S_i \mu (1+3E_i \delta_L) \right] + 6 \left[ \mu G_i A_i (1+5E_i \delta_L) + (1-\mu)(2\sigma_1 - \sigma_2)(1-5D_i \delta_L) Q_i \right] + \frac{1}{c^2} \left[ \frac{3(1-h_i - \mu)^2}{2} \right.$$

$$\times \left[ 1 + 2(T_i (1-\mu) + S_i \mu) \right] + (\mu - \mu^2 - 3) + 6(1-h_i - \mu) \left[ F_i \mu - H_i (1-\mu) \right] + 3 \left[ E_i \mu + D_i (1-\mu) \right]$$

$$\left. - J_i \mu^2 - 2\mu H_i + F_i (1-\mu) + W_i (1-\mu)^2 \right] - 2 \left[ S_i \mu - T_i (1-\mu) \right] \left[ E_i \mu + D_i (1-\mu) \right]$$

$$\left. - \frac{\mu(1-\mu)}{2} \left[ 2T_i + 2(-6+4\mu+7h_i)(S_i - T_i) + 14(F_i + H_i) \right] \right\},$$  \hspace{1cm} (21)

and

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\[ U_{\eta \eta, \ell_1} = n \left[ (1 - \mu)(1 - 3D_{\ell_1} \delta_{\ell_1})T_1 + S_1 \mu(1 + 3E_1 \delta_{\ell_1}) \right] \]
\[ + \frac{3}{2} \mu G_1 A_2 (1 + 5E_1 \delta_{\ell_1}) + (1 - \mu)(2\sigma_1 - \sigma_2)(1 - 5D_{\ell_1} \delta_{\ell_1})Q_1 \]
\[ - 3(1 - \mu)(\sigma_1 - \sigma_2)(1 - 5D_2 \delta_{\ell_1})Q_1 \]
\[ + \frac{1}{\varepsilon^2} \left[ (1 - \mu)^2 \left( 1 - 3(T_1(1 - \mu) + S_1 \mu) \right) + (\mu - \mu^2 - 3) \right] \]
\[ + 3 \left[ E_1 \mu - D_1 (1 - \mu) \right] + \left[ S_1 \mu - T_1 (1 - \mu) \right] \left[ E_1 \mu + D_1 (1 - \mu) \right] \]
\[ - \frac{\mu(1 - \mu)}{2} \left[ -T_1 + (-6 + 4\mu + 7h)(S_1 - T_1) + 2(S_1(1 - \mu) + T_1 \mu) \right] \]  

(22)

Where \( D_1, E_1, F_1, G_1, H_1, J_1, Q_1, S_1, T_1 \) and \( W_1 \) are all functions of \( \mu \)

### 2.3 For \( L_2 \)

The \( L_2 \) point lie on x-axis on far side of each primary with respect to the barycenter. The geometry of \( L_2 \) can be visualized as given by Fig. 3.

![Fig. 3. The location of \( L_2 \) and its corresponding parameters](image)

Follow the same procedure as done in \( L_1 \), with the corresponding values of \( B_1 = 1 \), \( B_2 = -1 \) into (15) we get

\[ r_1 - r_2 = 1, \quad r_1 = \xi + \mu, \quad r_2 = \xi + \mu - 1, \quad \frac{\partial r_2}{\partial \xi} = \frac{\partial r_2}{\partial \xi} = 1 \]  

(23)

The perturbed position of \( L_2 \) could be written as a little deviation \( \delta_{\ell_2} \) from the classical position as

\[ r_1 = a_2 + \delta_{\ell_2}, \quad r_2 = b_2 - \delta_{\ell_2}, \quad a_2 + b_2 = 1 \]  

(24)

From which we have

\[ r_1 = (1 + b_2) \left( \frac{1 + \delta_{\ell_2}}{1 + b_2} \right), \quad r_2 = b_2 \left( \frac{1 + \delta_{\ell_2}}{b_2} \right) \]  

(25)

where \( a_2 \) and \( b_2 \) are unperturbed positions of \( r_1 \) and \( r_2 \), respectively, and \( b_2 \) is given by
\[
b_2 = \alpha + \frac{1}{3} \alpha^2 - \frac{1}{9} \alpha^3 - \frac{31}{81} \alpha^4 + O(\alpha^5), \quad \alpha = \left( \frac{\mu}{3(1-\mu)} \right)^{1/3}.
\] (26)

After some lengthy algebraic manipulation, the location of \( L_2 \) is

\[
\xi_{\mu, L_2} = \sum_{k=3}^{9} G^{(2)}_k \frac{1}{c^2} \left[ -\frac{1}{3} \left( \frac{\mu}{3} \right)^3 - \frac{5}{9} \left( \frac{\mu}{3} \right)^2 - \frac{4}{3} \left( \frac{\mu}{3} \right) + \frac{1084}{486} \left( \frac{\mu}{3} \right)^4 + \frac{886}{243} \left( \frac{\mu}{3} \right)^5 \right]
- \frac{8843}{2178} \left( \frac{\mu}{3} \right)^2 - \frac{12796}{19683} \left( \frac{\mu}{3} \right)^3 + \frac{2872}{59049} \left( \frac{\mu}{3} \right)^4 - \frac{1915435}{354294} \left( \frac{\mu}{3} \right)^5 + \ldots \right]
\] (27)

Where non-vanishing coefficients \( G^{(2)}_k \) are given appendix B

Substituting eq. (25) into eq. (13) and (14), yields

\[
U_{\xi, L_2} = n + 2 \left[ (1-\mu)(1-3D_2 \delta_{L_2})T_2 + S_1 \mu(1-3E_2 \delta_{L_2}) \right]
+ 6 \left[ \mu G_2 A_2 (1-5E_2 \delta_{L_2}) + (1-\mu)(2\sigma_1 - \sigma_2)(1-5D_2 \delta_{L_2})Q_2 \right]
+ \frac{1}{c^2} \left[ \frac{3}{2} \left( \frac{b_2 - \mu}{2} \right)^2 \left[ 1 + 2(T_2(1-\mu) + S_2 \mu) \right] + (\mu - \mu^2 - 3) \right]
- 6(1-b_2-\mu)[F_2 \mu - H_2(1-\mu)] + 3[E_2 \mu + D_2 (1-\mu)]
- \left[ J_2 \mu^2 + 2\mu H_2 + F_2 (1-\mu) + W_2 (1-\mu)^2 \right] - 2[S_2 \mu + T_2 (1-\mu) \left[ E_2 \mu + D_2 (1-\mu) \right]
- \frac{\mu(1-\mu)}{2} \left[ 2T_2 + 2(-6 + 4\mu + 7b_2)(S_2 - T_2) + 14(F_2 + H_2) \right]
\] (28)

and

\[
U_{\eta, L_2} = n - \left[ (1-\mu)(1-3D_2 \delta_{L_2})T_2 + S_2 \mu(1-3E_2 \delta_{L_2}) \right]
- \frac{3}{2} \left[ \mu G_2 A_2 (1-5E_2 \delta_{L_2}) - (1-\mu)(2\sigma_1 - \sigma_2)(1-5D_2 \delta_{L_2})Q_2 \right]
+ 3(1-\mu)(\sigma_1 - \sigma_2)(1-5D_2 \delta_{L_2})Q_2
+ \frac{1}{c^2} \left[ \frac{1}{2} \left( \frac{b_2 - \mu}{2} \right)^2 \left[ 1 - 3(T_2(1-\mu) + S_2 \mu) \right] - (\mu - \mu^2 - 3) \right]
+ 3\left[ E_2 \mu + D_2 (1-\mu) \right] + \left[ S_2 \mu + T_2 (1-\mu) \left[ E_2 \mu + D_2 (1-\mu) \right]
+ \frac{\mu(1-\mu)}{2} \left[ T_2 + (6 - 4\mu + 7b_2)(S_2 - T_2) - 2(S_2(1-\mu) + T_2 \mu) \right]
\] (29)

where \( D_2, E_2, F_2, G_2, H_2, J_2, Q_2, S_2, T_2 \) and \( W_2 \) are all functions of \( \mu \).

### 2.4 For \( L_3 \)
The $L_3$ point lie on the negative $x$–axis, the geometry of $L_3$ can be visualized as given by Fig. 4.

Fig. 4. The location of $L_3$ and its corresponding parameters

Follow the same procedure as done in $L_1$, with the corresponding values of $B_1 = -1$, $B_2 = 1$ into (15) we get

$$r_2 - r_1 = 1, \quad r_1 = -\xi - \mu, \quad r_2 = 1 - \mu - \xi, \quad \frac{\partial r_1}{\partial \xi} = \frac{\partial r_2}{\partial \xi} = -1$$

(30)

Proceeding similarly as before following the same steps

$$r_1 = a_1 + \delta_{\xi}, \quad r_2 = b_3 - \delta_{\xi}, \quad a_1 + b_3 = 1$$

(31)

From which we have

$$r_3 = -\left(1-b_3 \right) \left(1 - \frac{\delta_{\xi_3}}{1-b_3} \right), \quad r_2 = b_3 \left(1 + \frac{\delta_{\xi_3}}{b_3} \right)$$

(32)

where $a_3$ and $b_3$ are unperturbed positions of $r_1$ and $r_2$ respectively, and $b_3$ is given by

$$b_3 = 2 - \frac{7}{12} \mu \left(1 + \frac{23}{144} \mu + \frac{25921}{2985984} \mu^4 \right)$$

(33)

After some lengthy algebraic manipulation, the location of $L_3$ is

$$\xi_{0,L_3} = \sum_{k=3}^{9} J^{(3)}_{-k} \frac{1}{c^2} \left( -\frac{3}{4} \mu + \frac{7}{16} \mu^2 + \frac{3227}{41472} \mu^3 + \frac{51037}{497664} \mu^5 + \ldots \right)$$

(34)

Where non-vanishing coefficients $J^{(3)}_{-k}$ are given appendix C

Substituting eq. (32) into eq. (13) and (14), yield
\[ U_{\xi,\xi,\xi} = n + 2 \left( (1 - \mu)(1 - 3D_3\delta_{L_3})T_3 + S_3\mu(1 - 3E_3\delta_{L_3}) \right) + 6\left[ \mu G_3 A_3(1 - 5E_3\delta_{L_3}) + (1 - \mu)(2\sigma_1 - \sigma_2)(1 - 5D_3\delta_{L_3})Q_3 \right] + \frac{1}{c^2} \left[ \frac{3(1 - b_3 - \mu)^2}{2}[1 + 2(T_3(1 - \mu) + S_3\mu)] + (\mu - \mu^2 - 3) \right] + 6(1 - b_3 - \mu)\left[ F_3\mu + H_3(1 - \mu) \right] + 3\left[ E_3\mu + D_3(1 - \mu) \right] - \left[ J_3\mu^2 + 2\mu H_3 + F_3(1 - \mu) + W_3(1 - \mu)^2 \right] - 2\left[ S_3\mu + T_3(1 - \mu) \right] \left[ E_3\mu + D_3(1 - \mu) \right] - \frac{\mu(1 - \mu)}{2} \left[ 2T_3 + 2(-6 + 4\mu + 7b_3)(S_3 - T_3) + 14(F_3 + H_3) \right] \right] \] (35)

where \( D_3, E_3, F_3, G_3, H_3, J_3, Q_3, S_3, T_3 \) and \( W_3 \) are all functions of \( \mu \)

**2.5 Solution of the Characteristic eq. (9)**

Recalling eq. (9)

\[ \lambda^3 - N_{L_\alpha} \lambda^2 + M_{L_\alpha} = 0, \quad \alpha = 1, 2, 3 \]

where \( N_{L_\alpha} \) and \( M_{L_\alpha} \) are given by eq. (14a). Among several methods, we carry out the stability analysis based on the linearized equations by considering roots of the characteristic equation. For \( L_\alpha, \alpha = 1, 2, 3 \), we computed the numerical values of \( M_{L_\alpha} \), in the interval \( \mu \in (0.0.5) \). In all cases we obtained \( M_{L_\alpha} = U_{\xi,\xi,\xi} U_{\eta,\eta,\eta} < 0 \) which leads to two real and two imaginary roots of the characteristic equation.

Therefore, under considered perturbations, the collinear points are unstable as in the classical RTBP. The following tables (1 – 2) show the obtained solutions. Each solution corresponds to one of the collinear points. For several values \( \sigma_1, \sigma_2 \) and \( A_2 \) we also sketched the variations in \( U_{\xi,\xi,\xi} U_{\eta,\eta,\eta} \) versus the mass ratio \( \mu \) in each case.

**2.6 Numerical Representations and Analyses for Stability of \( L_\alpha, \alpha = 1, 2, 3 \)**

A program is constructed using Mathematica 9 software package so as to draw the variations in \( U_{\xi,\xi,\xi} U_{\eta,\eta,\eta} \) of \( L_\alpha, \alpha = 1, 2, 3 \) versus the whole range of the mass ratio \( \mu \) taking into account the oblateness effects \( A_2 \), the triaxial effect \( \sigma_1, \sigma_2 \) and the relativistic corrections.

**Analysis of the Fig. 5 and Fig. 6**

The curve in Fig. 5 shows that the \( U_{\xi,\xi,\xi} U_{\eta,\eta,\eta} < 0 \) for whole domain of the mass ratio, i.e., \( L_\alpha \) is still unstable as is known in RTBP. Its magnitude is
increasing with respect to the increase in the mass ratio. While in Fig. 6, the curve shows that the 
$U_{zz}U_{yy} < 0$ for whole domain of the mass ratio, i.e., $L_3$ is still unstable as is known in RTBP. Its
magnitude is decreasing with respect to the increase in the mass ratio.

**Table 1. The roots of the characteristic equations for points $L_\alpha$, $\alpha = 1, 2, 3$ for different values of $\sigma_1$, $\sigma_2$, $A_2 = 0.001$ and $\mu = 0.35$**

| Triaxiality Coefficients | $\lambda^1$ | Roots $L_1$ | Roots $L_2$ | Roots $L_3$ |
|--------------------------|-------------|-------------|-------------|-------------|
| $\sigma_1 = 0.04$        | $\lambda_i$ | -4.1710     | -1.4950     | -0.9917     |
| $\sigma_2 = 0.04$        | $\lambda_i$ | 4.1710      | 1.4950      | 0.9917      |
| $\sigma_1 = 0.04$        | $\lambda_i$ | -3.0283i    | -1.489i     | -0.9832i    |
| $\sigma_2 = 0.03$        | $\lambda_i$ | 3.0283i     | 1.489i      | 0.9832i     |
| $\sigma_1 = 0.04$        | $\lambda_i$ | -4.2665     | -1.5242     | -1.1761     |
| $\sigma_2 = 0.03$        | $\lambda_i$ | 4.2665      | 1.5242      | 1.1761      |
| $\sigma_1 = 0.04$        | $\lambda_i$ | -3.1153i    | -1.551i     | -1.2237i    |
| $\sigma_2 = 0.02$        | $\lambda_i$ | 3.1153i     | 1.551i      | 1.2237i     |
| $\sigma_1 = 0.03$        | $\lambda_i$ | -4.1252     | -1.4809     | -1.1070     |
| $\sigma_2 = 0.025$       | $\lambda_i$ | 4.1252      | 1.4809      | 1.1070      |
| $\sigma_1 = 0.03$        | $\lambda_i$ | -3.0307i    | -1.4471i    | -1.2156i    |
| $\sigma_2 = 0.015$       | $\lambda_i$ | 3.0307i     | 1.4471i     | 1.2156i     |
| $\sigma_1 = 0.02$        | $\lambda_i$ | -4.0277     | -1.4523     | -1.0697     |
| $\sigma_2 = 0.015$       | $\lambda_i$ | 4.0277      | 1.4523      | 1.0697      |
| $\sigma_1 = 0.02$        | $\lambda_i$ | -2.9898i    | -1.4427i    | -1.2286i    |
| $\sigma_2 = 0.015$       | $\lambda_i$ | 2.9898i     | 1.4427i     | 1.2286i     |
| $\sigma_1 = 0.01$        | $\lambda_i$ | -3.8982     | -1.4167     | -1.0143     |
| $\sigma_2 = 0.0075$      | $\lambda_i$ | 3.8982      | 1.4167      | 1.0143      |
| $\sigma_1 = 0.01$        | $\lambda_i$ | -2.9258i    | -1.4375i    | -1.2332i    |
| $\sigma_2 = 0.0075$      | $\lambda_i$ | 2.9258i     | 1.4375i     | 1.2332i     |

**Table 2. The roots of the characteristic equations for points $L_\alpha$, $\alpha = 1, 2, 3$ for versus $\mu \in (0.04, 0.48)$, $A_2 = 0.001$, $\sigma_1 = 0.03$ and $\sigma_1 = 0.025$**

| $\mu$ | $\lambda^1$ | Roots $L_1$ | Roots $L_2$ | Roots $L_3$ | $\mu$ | Roots $L_1$ | Roots $L_2$ | Roots $L_3$ |
|-------|-------------|-------------|-------------|-------------|-------|-------------|-------------|-------------|
| 0.04  | $\lambda_i$ | -3.5786     | -2.1364     | -0.4477     | 0.24  | -4.0377     | 1.6848      | -0.9094     |
|       | $\lambda_i$ | 3.5786      | 2.1364      | 0.4477      |       | 4.0377      | 1.6848      | 0.9094      |
|       | $\lambda_i$ | -2.6781i    | -1.8009i    | -0.9505i    |       | -2.9809i    | -1.5365i    | -1.1274i    |
|       | $\lambda_i$ | 2.6781i     | 1.8009i     | 0.9505i     |       | 2.9809i     | 1.5365i     | 1.1274i     |
| 0.08  | $\lambda_i$ | -3.7330     | -1.9879     | -0.5743     | 0.32  | -4.1075     | 1.5240      | -1.0487     |
|       | $\lambda_i$ | 3.7330      | 1.9879      | 0.5743      |       | 4.1075      | 1.5240      | 1.0487      |
|       | $\lambda_i$ | -2.7852i    | -1.7218i    | -0.9929i    |       | -3.0214i    | 1.4699i     | -1.1890i    |
|       | $\lambda_i$ | 2.7852i     | 1.7218i     | 0.9929i     |       | 3.0214i     | 1.4699i     | 1.1890i     |
| 0.12  | $\lambda_i$ | -3.8416     | -1.8817     | -0.6742     | 0.36  | -4.1302     | 1.4669      | -1.1290     |
|       | $\lambda_i$ | 3.8416      | 1.8817      | 0.6742      |       | 4.1302      | 1.4669      | 1.1290      |
|       | $\lambda_i$ | -2.8570i    | -1.6638i    | -1.0303i    |       | -3.0331i    | -1.4398i    | -1.2257i    |
|       | $\lambda_i$ | 2.8570i     | 1.6638i     | 1.0303i     |       | 3.0331i     | 1.4398i     | 1.2257i     |
| 0.16  | $\lambda_i$ | -3.9236     | -1.7943     | -0.7601     | 0.4   | -4.1461     | -1.4122     | -1.2476     |
|       | $\lambda_i$ | 3.9236      | 1.7943      | 0.7601      |       | 4.1461      | 1.4122      | 1.2476      |
|       | $\lambda_i$ | -2.9098i    | -1.6159i    | -1.0846i    |       | -3.0401i    | -1.4112i    | -1.2816i    |
|       | $\lambda_i$ | 2.9098i     | 1.6159i     | 1.0846i     |       | 3.0401i     | 1.4112i     | 1.2816i     |
| 0.2   | $\lambda_i$ | -3.9874     | -1.7177     | -0.8375     | 0.48  | -4.1609     | -1.3088     | -2.2503     |
i.e., $L_1$ is still unstable as is known in RTBP. The magnitude of $U_{\xi\xi}U_{\eta\eta}$ is decreasing for a very short interval of mass ratio, then increasing with respect to the increase in the mass ratio. While in Fig. 8, the curve shows that the $U_{\xi\xi}U_{\eta\eta} < 0$ for whole domain of the mass ratio, i.e., $L_2$ is still unstable as is known in RTBP. Its magnitude is decreasing with respect to the increase in the mass ratio.

Analysis of the Fig. 9 and Fig. 10

In Fig. 9, Considering the oblateness effect only and ignoring the triaxiality effects we get the smallest effects in magnitude on $U_{\xi\xi}U_{\eta\eta}$, while ignoring the oblateness and considering only the triaxiality perturbations we get the largest effects in magnitude on $U_{\xi\xi}U_{\eta\eta}$, but it still negatives i.e. $U_{\xi\xi}U_{\eta\eta} < 0$ for whole domain of the mass ratio, i.e., $L_1$ is still unstable as is known in RTBP. The general trend of all curves are as follows: the bigger the gravitational harmonics $\sigma_1, \sigma_2, A_2$ the bigger the perturbations on $U_{\xi\xi}U_{\eta\eta}$. The magnitude of $U_{\xi\xi}U_{\eta\eta}$ is decreasing for a very short interval of mass ratio, then increasing with respect to the increase in the mass ratio. While in Fig. 10, the dynamics of $U_{\xi\xi}U_{\eta\eta}$ seems inverse of Fig. 7, but the dynamical behavior of $L_2$ does not change from.

| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
|------------|------------|------------|
| 3.9874     | 1.7177     | 0.8375     |
| -2.9501i   | -1.5741i   | -1.0967i   |
| 2.9501i    | 1.5741i    | 1.0967i    |

| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
|------------|------------|------------|
| 4.1609     | 1.3088     | 2.2503     |
| -3.0417i   | -1.3580i   | -1.8062i   |
| 3.0417i    | 1.3580i    | 1.8062i    |

Fig. 5. The variations in $U_{\xi\xi}U_{\eta\eta}$ of $L_1$ point versus the mass ratio $\mu$ at $A_2 = 0.0024$, $\sigma_1 = 0.0004$, $\sigma_2 = 0.0003$ and relativistic corrections

Fig. 6. The variations in $U_{\xi\xi}U_{\eta\eta}$ of $L_2$ point versus the mass ratio $\mu$ at $A_2 = 0.0024$, $\sigma_1 = 0.0004$, $\sigma_2 = 0.0003$ and relativistic corrections
Fig. 7. The variations in $U_{\zeta \zeta}^\phi U_{\eta \eta}^\gamma$ of $L_3$ point versus the mass ratio $\mu$, at different values $\sigma_1, \sigma_2, A_2$ and relativistic corrections

Fig. 8. The variations in $U_{\zeta \zeta}^\phi U_{\eta \eta}^\gamma$ of $L_2$ point versus the mass ratio $\mu$, at different values $\sigma_1, \sigma_2, A_2$ and relativistic corrections

Fig. 9. The variations $U_{\zeta \zeta}^\phi U_{\eta \eta}^\gamma$ of $L_1$ point versus the mass ratio $\mu$ at different values $\sigma_1, \sigma_2, A_2$ and relativistic corrections

Fig. 10. The variations in $U_{\zeta \zeta}^\phi U_{\eta \eta}^\gamma$ of $L_2$ point versus the mass ratio $\mu$ at different values $\sigma_1, \sigma_2, A_2$ and relativistic corrections

Fig. 11. The variations in $U_{\zeta \zeta}^\phi U_{\eta \eta}^\gamma$ of $L_3$ point versus the mass ratio $\mu$ at $A_2 = 0.0024$, $\sigma_1 = 0.0004$, $\sigma_2 = 0.0003$ and relativistic corrections

Fig. 12. The variations in $U_{\zeta \zeta}^\phi U_{\eta \eta}^\gamma$ of $L_3$ point versus the mass ratio $\mu$ at different values $\sigma_1, \sigma_2, A_2$ and relativistic corrections
Analysis of the Fig. 11, Fig. 12 and Fig. 13

The curve in Fig. 11 shows that the \( U_{\xi\xi} U_{\eta\eta} < 0 \) for whole domain of the mass ratio, i.e., \( L_3 \) is still unstable as is known in RTBP. Its magnitude is increasing with respect to the increase in the mass ratio. While in Fig. 12, the curve shows that the \( U_{\xi\xi} U_{\eta\eta} < 0 \) for whole domain of the mass ratio, i.e., \( L_3 \) is still unstable as is known in RTBP. Its magnitude is increasing with respect to the increase in the mass ratio. The effect of changing the oblateness and the triaxiality of the primaries are clear from the figures, see Fig. 12. In Fig. 13 Considering the oblateness effect only and ignoring the triaxiality effects we get the smallest effects in magnitude on \( U_{\xi\xi} U_{\eta\eta} \), while ignoring the oblateness and considering only the triaxiality perturbations we get the largest effects in magnitude on \( U_{\xi\xi} U_{\eta\eta} \), but it still negative i.e. \( U_{\xi\xi} U_{\eta\eta} < 0 \) for whole domain of the mass ratio, i.e. \( L_3 \) is still unstable as is known in RTBP.

3. CONCLUSION

In conclusion, firstly, we have treated the problem of the stability of collinear equilibrium points of the RTBP under the influence of triaxiality of the more massive primary, oblateness of the less massive primary and the relativistic corrections. Secondly, we have built up the potential like function of the problem and computed the mean motion of the problem. Moreover, we have constructed the equations of motion of the problem. To study the stability of the current problem, we linearized the equations of motion around the collinear points. In addition, we have derived the characteristic equation of the collinear points. Our study revealed the existence of two real and two imaginary roots of the characteristic equation as deduced from the plotted figures in the manuscript. We have computed some selected roots corresponding to the eigenvalues based on some selected values of the perturbing parameters. These eigenvalues have reflected the instability nature of the collinear points. Finally, as seen from the curves plotted in Figs. 5-13, that the value of \( M_{\xi\xi} = U_{\xi\xi} U_{\eta\eta} \) is negative in the whole domain of the mass ratio \( \mu \in (0,0.5) \) under the considered model of perturbations. i.e., it ensures the negativity of \( U_{\xi\xi} U_{\eta\eta} \) thus the conclusion of instability of the collinear points is true. Also, tables 1 and 2 revealed the existence of two real and two imaginary roots of the characteristic equation that means the collinear points are unstable.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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Appendix A

\[
F_3^{(1)} = \frac{81A_2^2}{4}, \quad F_3^{(2)} = -\frac{9A_1^2}{2} - \frac{2BA_1}{4} + \frac{9n^2A_2}{2} - \frac{567A_1^2}{4} \\
F_1^{(1)} = \frac{93A_2}{2} + 36BA_4 - \frac{327n^2A_2}{8} + \frac{1917A_1^2}{4} \\
F_0^{(1)} = -3B - \frac{9B^2}{4} + \frac{3n^2}{4} + \frac{9Bn^2}{4} + \frac{n^4}{4} - \frac{877A_2}{4} - \frac{87BA_4}{4} + \frac{727n^2A_2}{4} - \frac{5139A_1^2}{4} \\
F_1^{(2)} = \frac{15}{4} + \frac{57B}{4} - \frac{9B^2}{4} + \frac{23n^2}{4} - \frac{27Bn^2}{4} + \frac{11n^4}{4} - \frac{4057A_2}{4} - \frac{715BA_4}{4} - \frac{13345n^2A_2}{24} + 2889A_1^2 \\
F_2^{(1)} = -\frac{67}{2} - \frac{65B}{2} - \frac{3B^2}{4} + \frac{115n^2}{6} + \frac{153Bn^2}{8} + \frac{44n^4}{3} - \frac{59303A_2}{36} - \frac{4399BA_4}{12} + \frac{12059n^2A_2}{9} \\
F_2^{(2)} = \frac{22125A_2^2}{4}
\]
Appendix B

\[ F_1^{(1)} = \frac{802}{9} + \frac{847B}{12} + \frac{53B^2}{4} - \frac{1481n^2}{36} - \frac{327Bn^2}{8} - \frac{1831n^4}{36} + \frac{36262A_A}{108} + \frac{5306BA}{9} + \frac{9042A_A^2}{108} \]

\[ F_2^{(1)} = -200 - 126B + \frac{45B^2}{4} + \frac{413n^2}{6} + \frac{147Bn^2}{2} + \frac{4571n^4}{36} - \frac{925739A_A}{162} - \frac{21113BA}{27} \]

\[ F_3^{(1)} = \frac{1381897n^2A_B}{324} + \frac{154879A_A^2}{12} \]

\[ F_4^{(1)} = \frac{18911}{54} + 167B + 3B^2 - \frac{2426n^2}{27} - \frac{797Bn^2}{8} - \frac{24991n^4}{108} + \frac{1983367A_A}{243} + \frac{312743BA}{324} \]

\[ F_5^{(1)} = \frac{2713313n^2A_A}{486} + \frac{143101A_A^2}{9} \]

\[ F_6^{(1)} = \frac{239344BA}{486} + \frac{4117370n^2A_B}{324} - \frac{1590552A_A^2}{8} \]

\[ F_7^{(1)} = \frac{554072BA}{54} + \frac{6576532n^2A_A}{81} - \frac{12948929A_A^2}{4} \]

\[ F_8^{(1)} = \frac{1750135BA}{729} + \frac{2563681n^2A_B}{2} - \frac{5209148A_A^2}{4} \]

\[ F_9^{(1)} = \frac{28555}{54} + \frac{14774B}{81} + \frac{45B^2}{4} - \frac{27197n^2}{243} - \frac{257Bn^2}{4} - \frac{1273n^4}{12} + \frac{39055031A_A}{4374} \]

\[ G_1^{(2)} = -\frac{9A_A}{2} - \frac{9n^2A_A}{2} - \frac{567A_A^2}{4} \]

\[ G_2^{(2)} = -\frac{93A_A}{2} - 36BA - \frac{345n^2A_B}{8} - \frac{1971A_A^2}{4} \]

\[ G_3^{(2)} = -3B - \frac{9B^2}{4} - \frac{5n^2}{4} - \frac{15Bn^2}{8} - \frac{n^4}{4} - \frac{479A_A}{2} - 87BA - \frac{805n^2A_A}{4} - \frac{3681A_A^2}{4} \]
\[ G^{(2)} = -12 - 21B - \frac{9B^2}{4} - \frac{63n^2}{4} - \frac{45Bn^2}{4} - \frac{11n^4}{4} - \frac{2177A_1}{3} - \frac{283BA_2}{4} - \frac{1359n^2A_2}{24} - \frac{621A^2_2}{9} + \frac{9005n^2A_4}{4} + \frac{5469A^2_4}{9} + 4188A^2_2 \]

\[ G^{(2)} = -71 - 55B - \frac{3B^2}{4} - 86n^2 - \frac{255Bn^2}{8} - \frac{44n^4}{3} - \frac{48017A_1}{36} + \frac{1649BA_2}{12} \]

\[ G^{(2)} = -\frac{4133}{18} - \frac{757B}{12} + \frac{55B^2}{4} - \frac{9931n^2}{36} - \frac{359Bn^2}{8} - \frac{1777n^4}{36} - \frac{65279A_1}{54} + \frac{4090BA_2}{9} - \frac{107995n^2A_4}{108} + \frac{4188A^2_2}{9} \]

\[ G^{(2)} = -\frac{78325n^2A_4}{324} + \frac{55721A^2_4}{12} + \frac{244105n^2A_2}{486} + \frac{6803A^2_2}{9} + \frac{2711}{6} + 22B - \frac{45B^2}{4} - \frac{5156n^2}{9} - \frac{35Bn^2}{2} - \frac{4229n^4}{36} + \frac{166043A_1}{324} + \frac{25139BA_2}{54} \]

\[ G^{(2)} = -\frac{15115}{27} + \frac{1951B}{12} + 3B^2 - \frac{21887n^2}{27} + \frac{905Bn^2}{24} - \frac{23497n^4}{108} + \frac{724733A_1}{243} - \frac{15743BA_2}{324} \]

\[ G^{(2)} = -\frac{177809}{486} + \frac{62437B}{324} - \frac{21B^2}{18} - \frac{429545n^2}{81} + \frac{697Bn^2}{108} - \frac{28960n^4}{2916} + \frac{10534933A_1}{243} + \frac{116210n^2A_4}{729} - \frac{3877025A^2_4}{972} \]

\[ G^{(2)} = -\frac{8141}{162} + \frac{19063B}{324} - \frac{45B^2}{4} - \frac{178025n^2}{162} - \frac{425Bn^2}{12} - \frac{21469n^4}{36} + \frac{3256961A_1}{2187} - \frac{355163BA_2}{729} + \frac{36834215n^2A_2}{17496} + \frac{4233653A^2_2}{972} \]

\[ G^{(2)} = -\frac{100105}{1458} + \frac{16265B}{324} - \frac{3B^2}{2} - \frac{5596727n^2}{2916} - \frac{17113Bn^2}{216} - \frac{27773n^4}{27} + \frac{14087719A_1}{26244} - \frac{127451BA_2}{2187} - \frac{22945408n^2A_2}{6561} + \frac{874352A^2_2}{729} \]

\[ G^{(2)} = \frac{969374}{2187} + \frac{2665B}{2187} - \frac{7145969n^2}{2187} - \frac{26345Bn^2}{486} - \frac{1680113n^4}{972} + \frac{1678094A_1}{19683} - \frac{71BA_2}{13122} + \frac{157236505n^2A_2}{39366} + \frac{863A^2_2}{2916} \]

**Appendix C**

\[ J^{(3)} = -\frac{69}{16} - \frac{651B}{32} - \frac{279B^2}{16} - \frac{79n^2}{16} - \frac{561Bn^2}{32} - \frac{15n^4}{8} + \frac{9A_4}{64} + \frac{81BA_2}{128} + \frac{27n^2A_2}{128} \]
\[ J^{(3)}_{6} = \frac{3661}{128} + \frac{23949B}{256} + \frac{351B^2}{8} + \frac{4291n^2}{128} + \frac{261Bn^2}{4} + \frac{315n^4}{64} - \frac{81A_n^2}{32} - \frac{5535BA_n}{1024} \]
\[ - \frac{261n^2A_n^2}{128} + \frac{81A_n^2}{2048} \]

\[ J^{(3)}_{9} = \frac{13419}{128} + \frac{199339B}{1024} + \frac{39669B^2}{1024} + \frac{172543n^2}{1536} + \frac{256655Bn^2}{2048} + \frac{3087n^4}{512} \]
\[ - \frac{3591A_n^2}{256} - \frac{132489BA_n^2}{8192} - \frac{17325n^2A_n^2}{2048} + \frac{6237A_n^4}{16384} \]

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