Cowen’s class and Thomson’s class

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Abstract: In studying commutants of analytic Toeplitz operators, Thomson [T2] proved a remarkable theorem which states that under a mild condition, the commutant of an analytic Toeplitz operator is equal to that of Toeplitz operator defined by a finite Blaschke product. Cowen [Cow1] gave an significant improvement of Thomson’s result. In this paper, we will present examples in Cowen’s class which does not lie in Thomson’s class.

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1 Introduction

In this paper, $D$ denotes the unit disk in the complex plane $C$, and $H^2(D)$ denotes the Hardy space on $D$, which consists of all holomorphic functions whose Taylor coefficients at 0 are square summable. Let $H^\infty(D)$ denote the set of all bounded holomorphic functions over $D$. For each function $h$ in $H^\infty(D)$, $T_h$ denotes the Toeplitz operator on the Hardy space $H^2(D)$, that is, the multiplication operator defined by the symbol $h$.

In [DW] Deddens and Wong raised several questions about the commutants for analytic Toeplitz operators defined on the Hardy space $H^2(D)$. One of them asks that for a function $\phi \in H^\infty(D)$, whether there is an inner function $\psi$ such that $\{T_\phi\}' = \{T_\psi\}'$ and that $\phi = h \circ \psi$ for some $h \in H^\infty(D)$, where $\{T_\phi\}' = \{A \in B(H^2(D)) : AT_\phi = T_\phi A\}$ is the commutant of $T_\phi$. Baker, Deddens and Ullman [BDU] proved that for an entire function $f$, there is a positive integer $k$ such that $\{T_f\}' = \{T_{z^k}\}'$. Later, by using function-theoretic techniques, Thomson gave the following remarkable result, see [T2].

Theorem 1.1 (Thomson). Suppose that $\phi \in H^\infty(D)$, and there exist uncountably many points $\lambda$ in $D$ such that the inner part of $\phi - \phi(\lambda)$ are finite
Blaschke products. Then there exists a finite Blaschke product \( B \) and an \( H^\infty \)-function \( \psi \) such that \( \phi = \psi(B) \) and \( \{T_\phi\}' = \{T_B\}' \) holds on the Hardy space \( H^2(\mathbb{D}) \).

As a consequence, the following is immediate, see [T1].

**Corollary 1.2.** Let \( \phi \) be a nonconstant function that is holomorphic on the closed unit disk \( \mathbb{D} \). Then there exists a finite Blaschke product \( B \) and a \( \psi \in H^\infty(\mathbb{D}) \) such that \( \phi = \psi(B) \) and \( \{T_\phi\}' = \{T_B\}' \) holds on \( H^2(\mathbb{D}) \). In particular, if \( \phi \) is entire, then \( \psi \) is entire and \( B(z) = z^n \) for some positive integer \( n \).

One related topic is to study \( \{T_h,T_h^*\}' \) for \( h \in H^\infty(\mathbb{D}) \). Now put \( V^*(h) = \{T_h,T_h^*\}' \), which turns out to be a von Neumann algebra. It is interesting because for each \( H^\infty \)-function \( \phi \) satisfying the condition in Theorem 1.1, there is necessarily a finite Blaschke product \( B \) satisfying \( V^*(\phi) = V^*(B) \). It is worthwhile to mention that Theorem 1.1 holds not only on the Hardy space, but also on the Bergman space. On the Hardy space, the structure of \( V^*(B) \) is clear. However, on the Bergman space it is not easy. It is known that \( V^*(B) \) is always nontrivial [HSXY], and very recently, it is shown that \( V^*(B) \) is abelian for each finite Blaschke product [DPW]. Along this line, there are a lot of work, also refer to [Cow1]-[CW],[T1],[T4],[SW],Zhu1,GSZZ, [GH1, GH3, Sim], [SZZ1, SZZ2].

Later, Cowen [Cow1] gave an significant extension of Theorem 1.1, as follows.

**Theorem 1.3 (Cowen).** Suppose that \( \phi \in H^\infty(\mathbb{D}) \), and there exist a point \( \lambda \) in \( \mathbb{D} \) such that the inner part of \( \phi - \phi(\lambda) \) is a finite Blaschke product. Then there exists a finite Blaschke product \( B \) and an \( H^\infty \)-function \( \psi \) such that \( \phi = \psi(B) \) and \( \{T_\phi\}' = \{T_B\}' \) holds on the Hardy space \( H^2(\mathbb{D}) \).

However, as Cowen proved Theorem 1.3 [Cow1], he did not know whether there exists any function \( f \) for which

\[
\{\lambda \in \mathbb{D} | \text{the inner part of } f - f(\lambda) \text{ is a finite Blaschke product}\}
\]

is a nonempty countable set. If there were no functions with this property, then every function which satisfies the hypotheses of Theorem 1.3 would also satisfy the hypotheses of Thomson’s theorem. This paper is to construct a function satisfying the property as mentioned above, and thus explains why Cowen’s generalization is essential.

Before continuing, we present two conditions on functions in \( H^\infty(\mathbb{D}) \). It is convenient to call the assumption in Theorem 1.3 Cowen’s condition. That
is, if \( h \) is a function in \( H^\infty(D) \) such that for some \( \lambda \) in \( D \) the inner part of \( h - h(\lambda) \) is a finite Blaschke product, then \( h \) is said to satisfy Cowen’s condition. Similarly, for a function \( \phi \) in \( H^\infty(D) \), if there are uncountably many \( \lambda \) in \( D \) such that the inner part of \( \phi - \phi(\lambda) \) is a finite Blaschke product, then \( \phi \) is said to satisfy Thomson’s condition. All functions satisfying Thomson’s condition consist of a set, called Thomson’s class, and the set of all functions of satisfying Cowen’s condition, is called Cowen’s class. Clearly, Thomson’s class is contained in Cowen’s class.

One natural question is whether these two classes are the same. Cowen \cite{Cow1} raised it as a question precisely as follows:

Is there a function in Cowen’s class which does not lie in Thomson’s class?

This is to ask whether Thomson’s class is properly contained in Cowen’s class. The following example provides an affirmative answer, and the details will be given in Section 2.

**Example 1.4.** Pick \( a \in D - (-1, 1) \), and denote by \( B \) the thin Blaschke product with only simple zeros: \( a \) and \( 1 - \frac{1}{n^2} \) \((n \geq 2)\). By Riemann mapping theorem, there is a conformal map \( h \) from the unit disk onto \( D - [0, 1) \). Take a such \( h \), define \( \phi = B \circ h \) and put \( b = h^{-1}(a) \). Later one will see that the inner part of \( \phi - \phi(b) \) is a finite Blaschke product; and for any \( \lambda \in D - \{b\} \), the inner part of \( \phi - \phi(\lambda) \) is never a finite Blaschke product. Thus, \( \phi \) is not in Thomson’s class, though it lies in Cowen’s class.

2 The construction of examples

In this section we will provide examples in Cowen’s class, but not lying in Thomson’s class.

Below, \( d \) will denotes the hyperbolic metric on \( D \); that is,

\[
d(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right|, \quad z, w \in D.
\]

First, recall that a Blaschke product with the zero sequence \( \{z_k\} \) is called a thin Blaschke product if

\[
\lim_{k \to \infty} \prod_{j: j \neq k} d(z_k, z_j) = 1.
\]

The following example \cite{GH2} presents the construction for a class of thin Blaschke products.
Example 2.1. This example comes from [GHZ]. For the reader’s convenience, we present its details.

First let us make an observation from [Hof, pp. 203,204]: if \( \{w_n\} \) is a sequence in \( \mathbb{D} \) satisfying
\[
\frac{1 - |w_n|}{1 - |w_{n-1}|} \leq c < 1,
\]
then
\[
\prod_{j:j \neq k} d(w_k, w_j) \geq \left( \prod_{j=1}^{\infty} \frac{1 - c^j}{1 + c^j} \right)^2. \tag{2.2}
\]
Notice that the right hand side tends to 1 as \( c \) tends to 0.

Let \( \{c_n\} \) be a sequence satisfying \( c_n > 0 \) and \( \lim_{n \to \infty} c_n = 0 \). Suppose that \( \{z_n\} \) is a sequence of points in the open unit disk \( \mathbb{D} \) such that
\[
\frac{1 - |z_n|}{1 - |z_{n-1}|} = c_n.
\]
We will show that \( \{z_n\} \) is a thin Blaschke sequence. To see this, given a positive integer \( m \), let \( k > m \) and consider the product
\[
\prod_{j:j \neq k} d(z_k, z_j) \equiv \prod_{1 \leq j \leq m} d(z_k, z_j) \prod_{j > m, j \neq k} d(z_k, z_j). \tag{2.3}
\]
Now write
\[
d_m = \sup\{c_j : j \geq m\}.
\]
Since \( \lim_{n \to \infty} c_n = 0 \), then \( \lim_{m \to \infty} d_m = 0 \). For any \( k > m \),
\[
\frac{1 - |z_k|}{1 - |z_{k-1}|} = c_k \leq d_m,
\]
and thus
\[
\prod_{j > m, j \neq k} d(z_k, z_j) \geq \left( \prod_{j=1}^{\infty} \frac{1 - d_m^j}{1 + d_m^j} \right)^2, \quad k > m,
\]
which implies that
\[
\prod_{j > m, j \neq k} d(z_k, z_j)
\]
tends to 1 as \( m \to \infty \). For any \( \varepsilon > 0 \), there is an \( m_0 \) such that
\[
\prod_{j > m_0, j \neq k} d(z_k, z_j) \geq 1 - \varepsilon.
\]
On the other hand,
\[ \lim_{k \to \infty} \prod_{1 \leq j \leq m_0} d(z_k, z_j) = 1, \]
which, combined with (2.3), implies that
\[ \liminf_{k \to \infty} \prod_{j: j \neq k} d(z_k, z_j) \geq 1 - \varepsilon. \]
Therefore, by the arbitrariness of \( \varepsilon \),
\[ \lim_{k \to \infty} \prod_{j: j \neq k} d(z_k, z_j) = 1, \]
which implies that \( \{z_k\} \) is a thin Blaschke product. For example, \( \{z_n\} \) is a thin Blaschke sequence if we put \( |z_n| = 1 - \frac{1}{n!} \).

The following is the restatement of Example 1.4.

**Theorem 2.2.** There is a holomorphic function \( \phi \) from \( \mathbb{D} \) onto \( \mathbb{D} \), such that the inner part of \( \phi - w \) (\( w \in \mathbb{D} \)) is a finite Blaschke product if and only if \( w = 0 \).

**Corollary 2.3.** There is a holomorphic function \( \phi \) from \( \mathbb{D} \) onto \( \mathbb{D} \), whose zero set \( Z(\phi) \) is a nonempty finite subset of \( \mathbb{D} \), such that the inner part of \( \phi - \phi(\lambda) \) (\( \lambda \in \mathbb{D} \)) is a finite Blaschke product if and only if \( \lambda \) lies in \( Z(\phi) \).

**Remark 2.4.** Corollary 2.3 immediately shows that Thomson’s class is properly contained in Cowen’s class.

This section mainly furnishes the details for Theorem 2.2. In fact, to prove Theorem 2.2, let us verify the details of Example 1.4.

Remind that \( a \) is a point in \( \mathbb{D} - \mathbb{R} \), and \( B \) denotes the Blaschke product with only simple zeros: \( a \) and \( 1 - \frac{1}{n!} (n \geq 2) \). By Example 2.1, this Blaschke product \( B \) is a thin Blaschke product.

We will construct a conformal map \( g \) from \( \mathbb{D} - \{0, 1\} \) onto \( \mathbb{D} \) and set \( h = g^{-1} \). Then define \( \phi = B \circ h \), which turns out to be the function as in Theorem 2.2.

**Lemma 2.5.** With \( \phi \) defined as above, for any \( w \in \mathbb{D} - \{0\} \), the inner part of \( \phi - w \) is never a finite Blaschke product.
Proof. Remind that \( \phi = B \circ h \), where \( h \) is a conformal map from \( \mathbb{D} \) onto \( \mathbb{D} - [0,1) \). To prove Lemma 2.5, it suffices to show that \( B|_{\mathbb{D} - [0,1)} \) attains each nonzero value \( w \) in \( \mathbb{D} \) for infinitely many times. By [GM, Lemma 3.2(3)] each value in \( \mathbb{D} \) can be achieved for infinitely many times by \( B \), and then it is enough to show that \( B|_{[0,1)} \) attains each nonzero value \( w \) in \( B([0,1)) \) for finitely many times.

For this, notice that \( \varphi_a \) maps \([-1,1]\) to a circular arc in \( \overline{\mathbb{D}} \). Then for each fixed \( r \in [-1,1] \), the argument function \( \arg \varphi_a(t)|_{[-1,1]} \) of \( \varphi_a(t) \) attains the value \( \arg \varphi_a(r) \) for at most \( k_0 \) times (say, \( k_0 = 2 \)). Here, the value of \( \arg \) is required to be in \([0,2\pi)\).

Write

\[
B = \varphi_a B_0,
\]

where \( B_0 \) is a Blaschke product, and it is clear that \( B_0(r) \in (-1,1) \). If \( B_0(r) \neq 0 \), then

either \( \arg B(r) = \arg \varphi_a(r) \) or \( \arg B(r) = \arg(\varphi_a(r)) + \pi \mod 2\pi \).

Therefore, for each \( r \in [0,1) - Z(B) \), arg \( B \) attains the value arg \( B(r) \) for at most \( 2k_0 \) times on \([0,1) - Z(B)\). Thus \( B|_{[0,1)} \) attains each nonzero value \( w \) in \( B([0,1)) \) for finitely many times. Since \( h \) is a conformal map from \( \mathbb{D} \) onto \( \mathbb{D} - [0,1) \) and \( \phi = B \circ h \), then for each \( w \in \mathbb{D} - \{0\} \), \( \phi - w \) is never a finite Blaschke product.

\[\square\]

**Remark 2.6.** In the proof of Lemma 2.5, \( \phi = B \circ h \), where \( B = \varphi_a B_0 \). If we replace a finite Blaschke product \( B_1 \) with \( \varphi_a \left( \varphi_a(z) = \frac{a - z}{1 - az} \right) \), such that \( B_1 \) has no zeros on \([0,1)\), and that the restriction of arg \( B_1 \) on \([0,1]\) attains any value in its range for less than \( n \) times for some positive integer \( n \), then Lemma 2.5 also holds. For example, define

\[
B_1 = \prod_{1 \leq j \leq k} \varphi_{a_j},
\]

where all \( a_j \) lie in \( \{ w \in \mathbb{D} : \frac{\pi}{4} < \arg w < \pi \} \). Then \( B_1 \) has the desired property because for each \( a_j \), arg \( \varphi_{a_j} \) can be defined to be a continuous strictly-increasing function on \([0,1]\).

Rewrite \( B = B_0 B_1 \), and put \( \phi = B \circ h \). As one will see later, the inner part of \( \phi - w \) \(( w \in \mathbb{D} )\) is a finite Blaschke product if and only if \( w = 0 \). Thus, the inner part of \( \phi - \phi(\lambda) \) \(( \lambda \in \mathbb{D} )\) is a finite Blaschke product if and only if \( \lambda \) lies in the zero set \( Z(B_1) \) of \( B_1 \), a finite subset of \( \mathbb{D} \).
Proof of Theorem 2.2: The difficulty lies in the remaining part. That is to show the inner part of $\phi$ is a finite Blaschke product. For this, first we will give two computational results.

Put $S_1(z) = \exp(-\frac{1+z}{1-z})$. Clearly, it is continuous on the unit circle except for $z = 1$. We will see that $S_1(z)$ has non-tangential limit 0 at $z = 1$. That is to show, for each $\theta_0$ with $0 < \theta_0 < \frac{\pi}{2}$,

$$\lim_{\varepsilon \to 0^+, |\theta| \leq \theta_0} S_1(1 - \varepsilon e^{i\theta}) = 0. \quad (2.4)$$

For this, write $z = 1 - \varepsilon e^{i\theta}$, where $\varepsilon (\varepsilon > 0)$ is enough small such that $z \in \mathbb{D}$ whenever $|\theta| \leq \theta_0$. By direct computations,

$$\Re\left(-\frac{1+z}{1-z}\right) = -\frac{2\varepsilon \cos \theta - \varepsilon^2}{\varepsilon^2},$$

which shows that

$$|S_1(1 - \varepsilon e^{i\theta})| = \exp\left(-\frac{2\varepsilon \cos \theta - \varepsilon^2}{\varepsilon^2}\right) \leq \exp\left(-\frac{2\cos \theta_0}{\varepsilon} + 1\right) \to 0, \ (\varepsilon \to 0^+).$$

Thus $S_1(z)$ has non-tangential limit 0 at $z = 1$. By the same computations, it follows that for any $t > 0$, $S'_1(z) \triangleq \exp(-t \frac{1+z}{1-z})$ also has the non-tangential limit 0 at $z = 1$.

Next another estimate will be given for $B$. Take a $\theta_1$ with $0 < |\theta_1| < \frac{\pi}{2}$. For example, take $\theta_1 = \pm \frac{\pi}{4}$. Then we have

$$\liminf_{m \to \infty} |B(1 - \frac{1}{m!} e^{i\theta_1})| > 0. \quad (2.5)$$

As before, let $d$ denote the pseudohyperbolic metric defined on $\mathbb{D}$. Since for an enough large integer $m$, $1 - \frac{1}{m!} e^{i\theta_1} \in \mathbb{D}$. Then

$$d\left(1 - \frac{1}{n!}, 1 - \frac{1}{m!} e^{i\theta_1}\right) = \left|\frac{1}{n!} - \frac{1}{m!} e^{i\theta_1}\right| = \left|\frac{1}{n!} - \left(1 - \frac{1}{m!} e^{i\theta_1}\right)\right|$$

$$= \left|\frac{1}{n!} - \left(1 - \frac{1}{n!}\right)(1 - \frac{1}{m!} e^{-i\theta_1})\right|$$

$$= \left|\frac{1}{n!} + \left(\frac{1}{m!} - \frac{1}{n!}\right) e^{-i\theta_1}\right|,$$
and thus,

\[
\prod_{n>m} d(1 - \frac{1}{n!}, 1 - \frac{1}{m!} e^{i\theta_1}) \geq \prod_{n>m} \frac{|\frac{1}{n!} - \frac{1}{m!}|}{\frac{1}{n!} + \frac{1}{m!}}
\]

\[
\geq \frac{1 - \frac{1}{m+1}}{1 + \frac{1}{m+1}} \prod_{n=m+2} \frac{\frac{1}{n!} - \frac{1}{m!}}{\frac{1}{m!} + \frac{1}{n!}}
\]

\[
\geq \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} \prod_{k=2}^{\infty} \frac{1 - \frac{1}{k!}}{1 + \frac{1}{k!}}
\]

\[
> \frac{1}{2} \prod_{k=2}^{\infty} \frac{1 - \frac{1}{k!}}{1 + \frac{1}{k!}} \equiv \frac{1}{2} c > 0. \tag{2.6}
\]

Similarly, we have

\[
\prod_{n<m} d(1 - \frac{1}{n!}, 1 - \frac{1}{m!} e^{i\theta_1}) \geq \prod_{n<m} \frac{|\frac{1}{n!} - \frac{1}{m!}|}{\frac{1}{n!} + \frac{1}{m!}}
\]

\[
\geq \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} \prod_{k=2}^{m-1} \frac{1 - \frac{1}{k!}}{1 + \frac{1}{k!}}
\]

\[
\geq \frac{1}{2} \prod_{k=2}^{\infty} \frac{1 - \frac{1}{k!}}{1 + \frac{1}{k!}}
\]

\[
\geq \frac{1}{2} c > 0. \tag{2.7}
\]

Also, it is easy to see that

\[
\lim_{n \to \infty} d(1 - \frac{1}{n!}, 1 - \frac{1}{n!} e^{i\theta_1}) = \frac{|1 - e^{i\theta_1}|}{|1 + e^{i\theta_1}|} > 0.
\]

Combining the above identity along with (2.6) and (2.7) shows that

\[
\liminf_{m \to \infty} |B(1 - \frac{1}{m!} e^{i\theta_1})| > 0, \tag{2.8}
\]

as desired.

The idea is to compare (2.4) with (2.8) to derive a contradiction. Below we shall give the concrete construction of \( h = g^{-1} \). Precisely, we will construct two conformal maps \( \varphi_2 \) and \( \varphi_1 \), and \( g \triangleq \varphi_2 \circ \varphi_1 \). Now define

\[
\varphi_1(z) = i\sqrt{-z} + 1, z \in \mathbb{D} - [0, 1),
\]
Figure 1: $\varphi_1$

Figure 2: $\varphi_2$
where \( \sqrt{1} = 1 \), see Figure 1. Let us have a look at the geometric property of \( \varphi_1 \). Observe that \( z \mapsto -z \) is a rotation which maps \( \mathbb{D} - [0, 1) \) conformally to \( \mathbb{D} - (-1, 0) \). A map is conformal if it preserves the angle between two differentiable arcs. The map \( z \mapsto \sqrt{-z} \) is conformal on \( \mathbb{C} - [0, +\infty) \), and hence on \( \mathbb{D} - [0, 1) \). One may think that the segment \([0, 1]\) is split into the upper and down parts, which are mapped onto two segments: \( i[-1, 0] \) and \( i[0, 1] \). In particular, the point 1 is mapped to two points: \( -i \) and \( i \). Then with a rotation and a translation, \( \varphi_1 \) maps \( \mathbb{D} - [0, 1) \) conformally to the upper half disk

\[
W_1 \triangleq \{ \text{Re } z > 0; |z - 1| < 1 \}.
\]

Now a biholomorphic map \( \varphi_2 : W_1 \to \mathbb{D} \) will be constructed. First, \( z \mapsto \frac{1}{z} \) maps the upper half-disk \( W_1 \) onto a rectangular domain \( W_2 \) between two half lines: \( \{ \frac{1}{2} + it : t < 0 \} \) and \( \{ \frac{1}{2}, +\infty \} \). Then with a rotation and a translation, \( W_2 \) is mapped onto the first quadrant \( W_3 \). Write \( \varphi_3(z) = z^2 \), which maps \( W_3 \) onto the upper half plane \( \mathbb{D} \). Then one can give a mapping which maps \( \mathbb{D} \) onto the unit disk, say, \( z \mapsto \frac{z^2 - i}{z + 1} \). Define \( \varphi_2 \) to be the composition of the above maps, and we have

\[
\varphi_2(z) = \frac{(\frac{1}{z} - \frac{1}{2})^2 + i}{(\frac{1}{z} - \frac{1}{2})^2 - i}, z \in W_1.
\]

Some words are in order. All the above maps are conformal; and except for the map \( \varphi_3 : z \to z^2 (z \in W_3) \), all maps are conformal at the boundaries of their domains of definition. However, if \( \theta(|\theta| < \frac{\pi}{2}) \) is the angle between two differential arcs beginning at \( z = 0 \), then the angle between their image-arcs under \( \varphi_3 \) equals \( 2\theta \), see Figure 2. Since \( h = g^{-1} \) and \( g = \varphi_2 \circ \varphi_1 \), by some computations

\[
h(z) = \left( \frac{2\lambda(z) - 1}{2\lambda(z) + 1} \right)^2
\]

where

\[
\lambda(z) = \sqrt{-i \frac{z + 1}{z + 1}}.
\]

where \( \sqrt{\cdot} \) denote the branch defined on \( \mathbb{C} - [0, +\infty) \) satisfying \( \sqrt{-1} = -i \).

After some verification, one sees that the map \( h : \mathbb{D} \to \mathbb{D} - [0, 1) \) extends continuously onto \( \partial \mathbb{D} \), which maps exactly two points \( \eta_1 \) and \( \eta_2 \) on \( \mathbb{T} \) to 1; \( \mathbb{T} \) onto \( \partial(\mathbb{D} - [0, 1)) \); one arc \( \eta_1 \eta_2 \) onto \([0, 1]\) for twice. In fact, by some computations we have \( \eta_1 = 1 \) and \( \eta_2 = -1 \). Notice that any non-tangential domain at \( \eta_1 \) or \( \eta_2 \) will be mapped to some non-tangential domain at 1, lying either above or below the real axis, and vice versa. By the latter term
“non-tangential”, we mean the boundary of domain is not tangent to $\mathbb{T}$ nor to the segment $[0, 1]$ at 1.

Now let $S$ denote the inner part of $\phi = B \circ h$. Observe that $h^{-1}(0)$ contains exactly one point on $\mathbb{T}$, say $\eta_0 = h^{-1}(0)$. Since $B$ is holomorphic on $\mathbb{D} - \{1\}$ and $h$ is holomorphic on $\overline{\mathbb{D}}$ except for three possible points:

$$h^{-1}(0, 1) = \{\eta_0, \eta_1, \eta_2\},$$

one sees that $\phi = B \circ h$ is holomorphic at any point $\zeta \in \mathbb{T} - \{\eta_0, \eta_1, \eta_2\}$, and hence so is $S$.

Next one will see that none of $\eta_0, \eta_1$ and $\eta_2$ is a singularity. Let $\phi = SF$ be the inner-outer decomposition of $\phi$, and then $|F| = |\phi|$, a.e. on $\mathbb{T}$. This shows that $F$ is bounded on $\mathbb{D}$. It is easy to check that $\phi$ is continuous at $\eta_0$, and $\phi(\eta_0) = B(0) \neq 0$, by $\phi = SF$ one sees that

$$\lim_{z \to \eta_0} |S(z)| > 0.$$

Then [Gar] p.80, Theorem 6.6], $\eta_0$ is not a singularity of $S$. Therefore, $\eta_1$ and $\eta_2$ are the only possible singularities of $S$, and thus the singular part of $S$ is supported on $\{\eta_1, \eta_2\}$. Remind that $S_1(z) = \exp(-\frac{1 + z}{1 - z})$. Put $b = h^{-1}(a)$, and write

$$S(z) = \varphi_b(z) S_1^{t_1}(\eta_1 z) S_2^{t_2}(\eta_2 z),$$

where $t_1, t_2 \geq 0$. We will show that $t_1 = t_2 = 0$ to finish the proof.

For this, assume conversely that either $t_1 \neq 0$ or $t_2 \neq 0$. Without loss of generality, $t_1 \neq 0$. Then $S$ has non-tangential limit 0 at $\eta_1$, and so does $\phi = SF$, where $F$ is bounded on $\mathbb{D}$. However, with $\theta_1 = \pm \frac{\pi}{4}$, and put

$$\{z_1^j\} = \{h^{-1}(1 - \frac{1}{k!} e^{\frac{\pi}{4}})\} \quad \text{and} \quad \{z_2^j\} = \{h^{-1}(1 - \frac{1}{k!} e^{-\frac{\pi}{4}})\},$$

where $k \geq n_0$ for some enough large integer $n_0$ such that both $\{1 - \frac{1}{k!} e^{\frac{\pi}{4}}\}$ and $\{1 - \frac{1}{k!} e^{-\frac{\pi}{4}}\}$ lie in $\mathbb{D}$. Since $\phi = B \circ h$, by (2.5) we get

$$\lim_{k \to \infty} |\phi(z_k^j)| > 0, \quad j = 1, 2. \quad (2.9)$$

Considering

$$h^{-1}(1) = \{\eta_1, \eta_2\},$$

one notices that $\{z_k^1\}$ and $\{z_k^2\}$ are two non-tangential sequences, one tending to $\eta_1$ and the other to $\eta_2$. By (2.9), this is a contradiction to that $\phi$ has non-tangential limit 0 at $\eta_1$. Therefore, $t_1 = t_2 = 0$, and hence the inner part $S$ of $\phi$ is a M"obius map. The proof of Theorem 2.2 is complete. $\square$
Also, one can present more examples. It is worthwhile to mention that
the sequence \( \{ 1 - \frac{1}{n!} \}_{n \geq 2} \) can be replaced with any thin Blaschke sequence
\( \{ 1 - \varepsilon_n \} \) in \([0, 1]\). The reasoning is as follows. Without loss of generality, let
\( \{ \varepsilon_n \} \) be decreasing to 0. Notice that if \( 0 < |\theta_1| < \frac{\pi}{2} \), then
\[
\begin{align*}
d(1 - \varepsilon_n, 1 - \varepsilon_m e^{i\theta_1}) &= \frac{|1 - \varepsilon_n - (1 - \varepsilon_m e^{i\theta_1})|}{|1 - (1 - \varepsilon_n)(1 - \varepsilon_m e^{-i\theta_1})|} \\
&= \frac{\varepsilon_n - \varepsilon_m e^{i\theta_1}}{\varepsilon_n + (\varepsilon_m - \varepsilon_n \varepsilon_m) e^{i\theta_1}} \\
&\geq \frac{\varepsilon_n - \varepsilon_m}{\varepsilon_n + (\varepsilon_m - \varepsilon_n \varepsilon_m)} \\
&= d(1 - \varepsilon_n, 1 - \varepsilon_m).
\end{align*}
\]
Since \( \{ 1 - \varepsilon_n \} \) is a thin Blaschke sequence; that is,
\[
\lim_{m \to \infty} \prod_{n; n \neq m} d(1 - \varepsilon_n, 1 - \varepsilon_m) = 1,
\]
then it follows that
\[
\lim_{m \to \infty} \prod_{n; n \neq m} d(1 - \varepsilon_n, 1 - \varepsilon_m e^{i\theta_1}) = 1,
\]
forcing
\[
\liminf_{m \to \infty} |B(1 - \varepsilon_m e^{i\theta_1})| > 0.
\]
This is a generalization of (2.5), and the next discussion is just the same.

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