EULER SYSTEMS OF $K_2$ OF CM ELLIPTIC CURVES

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Abstract. We construct certain systems of elements in $K_2$ of some CM elliptic curves. When the classnumber of the field of CM is 1, The image of this system under the regulator map forms an euler system in the sense of Rubin [Ru].

§1. Systems of $K_2$ of CM elliptic curves.

Let $K$ be an imaginary quadratic field. Take a Hecke character $\phi$ of $K$ of type $(1,0)$ and let $f_\phi$ be its conductor.
Let $H = K(f_\phi)$ be the ray class field of $K$ modulo $f_\phi$. Then by ([dS], lemma 1.4, Ch. 2) there is an elliptic curve $E$ defined over $H$ with complex multiplication by the ring of integers $O_K$ of $K$ such that the Hecke character $\psi$ associated to $E$ is of the form
$$\psi = \phi \circ N_{H/K}.$$ We assume that $E(\mathbb{C})$ is isomorphic to $\mathbb{C}/O_K$.
Let $f$ be the conductor of $E$ and write $f = N_{H/K}(f)$. For any ideal $n$ of $O_K$ we write $H(n)$ for $H(E[n]).$
Let $p > 3$ be a rational prime which splits in $K/\mathbb{Q}$, say $p = \overline{p} \overline{p}$. We assume that $p$ is relatively prime to $f$. Let $a$ be an integer relatively prime to $p$. Take a function $g_a \in H(E)$ such that $\text{div} g_a = a^2(0) - E[a]$.
Let $L$ be the set of principal prime ideals of $O_K$ relatively prime to $f\overline{p}a$ and which split completely in $H$.
Let $R$ be the set of ideals of $O_K$ which are divisible only by primes in $L$.
For each $m \in R$ we fix a generator $x_m$ of $E[m]$ as an $O_K$ module so that they satisfy the relation $[\phi(n)]x_{mn} = x_m$ for any $n \in R$. Here $[\phi(n)]$ is $\phi(n)$ regarded as an element of $\text{End}(E)$.
Fix a generator $x_f$ of $E[f]$ as an $O_K$ module. Since any ideal $m \in R$ is prime to $f$, $[\phi(m)]$ is an automorphism of $E[f]$.
For each $m \in R$ let $y_m := x_m + [\phi(m)]^{-1}x_f \in E(\overline{f})$ and take a function $s_m \in H(mf)(E)$ (the function field of $E \otimes_H H(mf)$) such that
\[
\text{div } s_m = N(\mathfrak{mf})(y_m) - N(\mathfrak{mf})(0).
\]
For each \( \gamma \in E[a] - \{0\} \) take a function \( t_\gamma \in H(a)(E) \) such that \( \text{div } t_\gamma = a(\gamma) - a(0) \).

For any ideal \( \mathfrak{m} \in R \) consider the element
\[
\alpha'_m := N(\mathfrak{mf})^{-1} \left( a\{g_a(y_m)^{-1}g_a, s_m\} - \sum_{\gamma \in E[a]-\{0\}} \{s_m(\gamma), t_\gamma\} \right)
\in \Gamma(E_{H(\mathfrak{mf})\text{zar}}, \mathcal{K}_2) \otimes \mathbb{Q}
\]
where \( \mathcal{K}_2 \) is the Zariski sheaf associated to presheaf
\[
E \supset_{\text{open}} U \mapsto K_2(U)
\]
and let
\[
\alpha_m := \begin{cases} 
[\phi(\mathfrak{m})]_*N_{H(\mathfrak{mf})/H(\mathfrak{m})}\alpha'_m \quad \mathfrak{p} | \mathfrak{m} \\
[\phi(\mathfrak{mp})]_*N_{H(\mathfrak{mpmf})/H(\mathfrak{m})}\alpha'_{\mathfrak{mp}} \quad \mathfrak{p} \nmid \mathfrak{m}.
\end{cases}
\]
Noting that \( K_2 \) of number fields is torsion, it can be checked that the definition of \( \alpha_m \) is independent of choice of \( g_a, s_m \) and \( t_\gamma \). This definition is similar to the one in Chap. 7 of [Bl-Ka].

We will state the main result.

**Theorem.** Let \( \mathfrak{m} \in R \) and let \( \mathfrak{l} \in \mathcal{L} \). Then the element \( \alpha_{\mathfrak{ml}} \) satisfies the following equality:

\[
(E1) \quad N_{H(\mathfrak{ml})/H(\mathfrak{m})}\alpha_{\mathfrak{ml}} = \alpha_m \quad \text{if} \quad \mathfrak{l} | \mathfrak{m} \text{ or } \mathfrak{l} = \mathfrak{p}
\]
\[
(E2) \quad N_{H(\mathfrak{ml})/H(\mathfrak{m})}\alpha_{\mathfrak{ml}} = (1 - [\phi(\mathfrak{l})]_*Fr_{\mathfrak{l}}^{-1})\alpha_m \quad \text{if} \quad \mathfrak{l} \nmid \mathfrak{mp}.
\]

Here \( Fr_{\mathfrak{l}} \) is the Frobenius element of any prime of \( H \) over \( \mathfrak{l} \).

**proof.**

First we prove (E1). When \( \mathfrak{l} = \mathfrak{p} \) and \( \mathfrak{p} \nmid \mathfrak{m} \) this holds by definition. So we assume that \( \mathfrak{l} | \mathfrak{m} \). We will show the equality
\[
[\phi(\mathfrak{l})]_*(N_{H(\mathfrak{mlmf})/H(\mathfrak{mf})}\alpha'_{\mathfrak{ml}}) = \alpha'_m.
\]
Take functions \( s_{\mathfrak{ml}} \in H(\mathfrak{mf})(E)^* \otimes \mathbb{Q} \) and \( s_m \in H(\mathfrak{mf})(E)^* \otimes \mathbb{Q} \) such that \( \text{div } s_{\mathfrak{ml}} = (y_{\mathfrak{ml}}) - (0) \), \( \text{div } s_m = (y_m) - (0) \) respectively and such that \( [\phi(\mathfrak{l})]_*s_{\mathfrak{ml}} = s_m \).

Since \( \text{Gal}(H(\mathfrak{mlmf})/H(\mathfrak{mf})) \simeq \mathcal{O}_K/\mathfrak{l} \) we see that
\[
[\phi(\mathfrak{l})]^{-1}(y_m) = \bigcup_{\tau \in \text{Gal}(H(\mathfrak{mlmf})/H(\mathfrak{mf}))} y_{\mathfrak{ml}}^\tau.
\]
Take a function \( g_l \in H(\mathfrak{m}f)(E)^* \otimes \mathbb{Q} \) such that \( N_{H(\mathfrak{mf})/H(\mathfrak{m})} s_{\mathfrak{mt}} = [\phi(l)]^*(s_m)g_l \).

Note that \( \text{div} \ g_l = \sum_{c \in E[1]} (c) - N(l)(0) \). Then we have the equality

\[
N_{H(\mathfrak{mf})/H(\mathfrak{mf})} a_{\mathfrak{mf}} \\
= a\{g_a, N_{H(\mathfrak{mf})/H(\mathfrak{mf})} s_{\mathfrak{mt}}\} - \sum_{\gamma \in E[a]-\{0\}} \{N_{H(\mathfrak{mf})/H(\mathfrak{mf})} s_{\mathfrak{mt}}(\gamma), t_\gamma\} \\
+ \sum_{\tau \in Gal(H(\mathfrak{mf})/H(\mathfrak{mf}))} a\{g_a^{-1}(y_{\mathfrak{mt}}^\tau), s_{\mathfrak{mt}}^\tau\} \\
= a\{g_a, [\phi(l)]^*(s_m)g_l\} - \sum_{\gamma \in E[a]-\{0\}} \{([\phi(l)]^*(s_m)g_l)(\gamma), t_\gamma\} \\
+ \sum_{\tau \in Gal(H(\mathfrak{mf})/H(\mathfrak{mf}))} a\{g_a^{-1}(y_{\mathfrak{mt}}^\tau), s_{\mathfrak{mt}}^\tau\}.
\]

Since \([\phi(l)]_s s_{\mathfrak{mt}}^\tau = s_{\mathfrak{mt}} = s_m \) for \( \tau \in Gal(H(\mathfrak{mf})/H(\mathfrak{mf})) \), the equality

\[
[\phi(l)]_s \sum_{\tau \in Gal(H(\mathfrak{mf})/H(\mathfrak{mf}))} \{g_a(y_{\mathfrak{mt}}^\tau)^{-1}, s_{\mathfrak{mt}}^\tau\} \\
= \{ \prod_{\tau \in Gal(H(\mathfrak{mf})/H(\mathfrak{mf}))} g_a(y_{\mathfrak{mt}}^\tau)^{-1}, s_m\} \\
= \{([\phi(l)]_s g_a(y_m)^{-1}, s_m\}
\]

holds. Here the first equality holds by the projection formula.

Let \( c \in E[1] \) be a nonzero point and take a function \( u \in H(l)(E)^* \otimes \mathbb{Q} \) which has the divisor \((c) - (0)\). Let

\[
A := a\{g_a, g_l\} - \sum_{\gamma \in E[a]-\{0\}} \{g_l(\gamma), t_\gamma\} \quad \text{and} \quad B := \sum_{\xi \in Gal(H(l)/H)} a\{g_a(c^\xi), u^\xi\}
\]

be the elements of \( K_2(H(\mathfrak{mf})(E)) \otimes \mathbb{Q} \). It can be seen that \( A - B \in \Gamma(E_{H(\mathfrak{mf})}, \mathcal{K}_2) \otimes \mathbb{Q} \). We know that \([-1]\) acts on this group by \(-1\). However, since \([-1]t_\gamma/t_{-\gamma} \in H(\mathfrak{a}), [-1]g_a = \pm g_a \) and \([-1]g_l = \pm g_l, A - B \) is invariant under \([-1]\). From this we see that \( A = B \). Since \( B \in Ker([\phi(l)]_s) \), it follows that \( A \in Ker([\phi(l)]_s) \).

Using this fact and the projection formula we get the equality \((E1)\).

Next we prove \((E2)\). We will show that

\[
[\phi(l)]_s (N_{H(\mathfrak{mf})/H(\mathfrak{mf})} a_{\mathfrak{mf}} + Fr_{\mathfrak{m}}^{-1}(\alpha_{\mathfrak{m}}^{'}) = \alpha_{\mathfrak{m}}' \).
\]

Since \( (\mathfrak{m}, 1) = 1 \), \([\phi(l)]_s \) is an automorphism of \( E[\mathfrak{mf}] \). Let \([\phi(l)]^{-1}y_{\mathfrak{m}} =: n \in E[\mathfrak{m}] \) and take \( s_n \in H(\mathfrak{mf})(E)^* \otimes \mathbb{Q} \) such that \( \text{div} s_n = (n) - (0) \) and \([\phi(l)]_s s_n = s_m \).

Then the equality
\[ Fr_1^{-1} \alpha_m = a\{g_a(n)^{-1}g_a, s_n\} - \sum_{\gamma \in E[a] \setminus \{0\}} \{s_n(\gamma), t_\gamma\} \]
holds.

Since \( \text{Gal}(H(mmf)/H(maf)) \simeq (\mathcal{O}_K/l)^* \), we see that
\[
[\phi(l)]^{-1}(y_m) = \bigcup_{\tau \in \text{Gal}(H(mmf)/H(maf))} y_m^\tau \cup \{n\}.
\]

Similarly as in the case (E1) we take \( s_{mf} \) such that \([\phi(l)]*s_{mf} = s_m\) and let \( g_l \in H(mf)(E)^* \otimes \mathbb{Q} \) be the function satisfying the relation
\[
N_{H(mmf)/H(mf)}(s_{mf})s_n = [\phi(l)]*(s_m)g_l.
\]

The rest of the proof is similar to that of (E1). \(\Box\)

§2. Images under the regulator map.

Let \( T_p(E) \) be the Tate module of \( E \). There is a decomposition
\[
T_p(E) = T_p(E) \oplus T_{\bar{p}}(E)
\]
where
\[
T_p(E) = \lim_{n} E[p^n]
\]
and
\[
T_{\bar{p}}(E) = \lim_{n} E[\bar{p}^n].
\]

We will define a map
\[
r_G : \Gamma(E_{H(m)}, K_2) \rightarrow H^1(H(m), T_p(E)(1))
\]
for each \( m \in R \).

Soulé defined the Chern class map
\[
K_2(E_{H(m)}) \rightarrow H^2(E_{H(m)}, \mathbb{Z}/p^n(2))
\]
in [So].

Since \( p \) is relatively prime to \( f \), the prime \( \bar{p} \) of \( K \) is unramified in \( H(m) \) so that the field \( H(m) \) and the cyclotomic field \( K(\mu_{p^n}) \) is linearly disjoint over \( K \).

Hence the group \( H^0(H(m), H^2(E_{\bar{p}}, \mathbb{Z}_p/p^n(2))) = 0 \).

By Hochschild-Serre spectral sequence there is a map
\[
Ch_n : K_2(E_{H(m)}) \rightarrow H^1(H(m), H^1(E_{\bar{p}}, \mathbb{Z}_p/p^n(2)))
\]
and taking projective limit we get the map

\[ Ch : K_2(E_{H(m)}) \to \lim_{\leftarrow n} H^1(H(m), H^1(E_{HH}, \mathbb{Z}_p/p^n(2))) = H^1(H(m), H^1(E_{HH}, \mathbb{Z}_p(2))). \]

Here the last equality follows from ([Ta], Proposition 2.2).

Take an element \( \alpha \in \Gamma(E_{H(m)}, K_2) \). Since \( K_2(H(m)) = \lim_{\leftarrow E \supset \text{open } U} K_2(U) \)

there is an open set \( U \) of \( E \) and an element \( \alpha_U \in K_2(U) \) such that the image of \( \alpha_U \) in \( \Gamma(E_{H(m)}, K_2) \) is \( \alpha \).

Since \( K_2 \) of number fields is torsion, \( \alpha_U \) is well defined modulo torsion elements.

Noting that there is an exact sequence

\[ \bigoplus_{t \in E-U} K_2(\kappa(t)) \to K_2(E) \to K_2(U) \to \bigoplus_{t \in E-U} K_1(\kappa(t)) \]

and that \( \alpha \) is in the kernel of Tame symbol, we see that there is an element \( \tilde{\alpha} \in K_2(E_{H(m)}) \) such that \( \tilde{\alpha}|_U = \alpha_U \).

Since the group \( H^1(H(m), T_p(E)(1)) \) has no torsion we can define \( r_G(\alpha) \) to be the projection of \( Ch(\tilde{\alpha}) \) to \( H^1(H(m), T_p(E)(1)) \).

Since \( \phi(m)/N(m) \in (\mathcal{O}_K)^*_p \) for each \( m \in R \), \( \alpha_m \) defines a class \( r_G(\alpha_m) \in H^1(H(m), T_p(E)(1)) \).

When the class number of \( K = 1 \), it can be checked that this family of cohomology classes forms an “euler system” in the sense of ([Ru], definition 2.1.1).

We let the field \( K \) resp. \( K_\infty \) in loc.cit be the union of fields \( K(E[m]) \) for \( m \in R \) resp. the maximal \( \mathbb{Z}_p \) extension of \( K \) in \( K(E[p^\infty]) \).

The condition (i) in loc.cit can be checked using Proposition 1.6 in Chap. II of [dS] and the fact that the action of inertia group of a prime of \( K \) on \( E[q] \) for any prime \( q \nmid f \) factors through the roots of unity in \( K \).

The second condition of (ii) in loc.cit follows from Proposition 1.9 in Chap. II of [dS].

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