Abstract

It is well-known that the 5D gauge structure of Stueckelberg-Horwitz-Piron (SHP) electrodynamics permits the exchange of mass between particles and the fields induced by their motion, even at the classical level. This phenomenon presents two closely related problems: (1) What accounts for the stability of the measured masses of the known particles? (2) Under what circumstances can real particles evolve sufficiently off-shell to account for mass changing phenomena such as flavor-changing neutrino interactions and low energy nuclear reactions? To approach these questions, we introduce a constant $c_5$ associated with the invariant time $\tau$, in analogy with the constant $c$ that associates a unit of length with intervals of time $t$ in standard relativity. It follows that electromagnetic mass exchange can be a small effect, in proportion to $c_5/c$. We show that this structure permits a classical self-interaction that tends to restore on-shell propagation. Finally we propose a model in which a particle evolving through a complex charged environment can acquire a significant mass shift for a short time.

1 Introduction

In a formal approach to special relativity that takes Minkowski geometry as its starting point, the constant $c$ is introduced as a means of measuring time in units of spatial distance, and the notion of a speed of light emerges from the role of $c$ in wave equations for U(1) gauge fields. Using natural units ($\hbar = c = 1$) in the development of SHP electrodynamics [1] - [15], no explicit constant was assigned to the invariant time $\tau$ and so the constant $c$ was implicitly assumed to play the same role for $\tau$ that it plays for the coordinate time $t$. In Section 2 we associate a new constant $c_5$ with the invariant time $\tau$, identify the expressions in which it must appear and study its role in the classical electromagnetic theory. Unlike standard 4D special relativity, in which the nonrelativistic limit can be recovered by taking $c \to \infty$, we find that 5D SHP goes over to an equilibrium state of Maxwell theory in the limit $c_5 \to 0$. Thus, the dimensionless ratio $c_5/c$ parameterizes the deviation of SHP from standard electrodynamics, in particular
the coupling of the events that dynamically trace out particle worldlines to the mass changing fields. Put another way, equilibrium Maxwell theory can be understood as the pre-Maxwell fields becoming independent of $c_5 \tau$ as $c_5 \to 0$.

In Section 3 we construct a model for the self-interaction involving an event and the causally retarded field produced by its motion. In numerical solutions [14] it was found that under interactions of this type, the particle mass may asymptotically approach its on-shell value. Here we calculate the classical Lorentz force produced by the self-interaction in the particle rest frame — it is seen to produce a damping force tending to return an off-shell event to on-shell evolution, and vanishing for on-shell evolution. In Section 4 we propose a simple mechanism by which a particle evolving through a complex plasma may acquire a significant mass shift for a short time.

2 Overview of Stueckelberg-Horwitz-Piron (SHP) electrodynamics

2.1 Gauge theory

This section covers familiar territory, reformulated to make each physical constant explicit. The generalized Stueckelberg-Schrodinger equation

$$
\left( i \hbar \partial_\tau + \frac{e_0}{c} \phi \right) \psi(x, \tau) = \frac{1}{2M} \left( p^\mu - \frac{e_0}{c} a^\mu \right) \left( p_\mu + \frac{e_0}{c} a_\mu \right) \psi(x, \tau)
$$

(1)

describes the interaction of an event characterized by the wavefunction $\psi(x, \tau)$ with five gauge fields $a_\mu(x, \tau)$ and $\phi(x, \tau)$. Equation (1) is invariant under the local gauge transformations

$$
\psi(x, \tau) \rightarrow \exp \left[ \frac{ie_0}{\hbar c} \Lambda(x, \tau) \right] \psi(x, \tau)
$$

Vector potential $a_\mu(x, \tau) \rightarrow a_\mu(x, \tau) + \partial_\mu \Lambda(x, \tau)$

Scalar potential $\phi(x, \tau) \rightarrow \phi(x, \tau) + \partial_\tau \Lambda(x, \tau)$

(2)

whose $\tau$-dependence is the essential departure from Stueckelberg’s work, and determines the structure of the resulting theory [5]. The corresponding global gauge invariance leads to the conserved Noether current

$$
\partial_\mu j^\mu + \partial_\tau \rho = 0
$$

(3)
where
\[ j^\mu = -\frac{ih}{2M} \left\{ \psi^* \left( \partial^\mu - \frac{ie_0}{c} a^\mu \right) \psi - \psi \left( \partial^\mu + \frac{ie_0}{c} a^\mu \right) \psi^* \right\} \]
\[ \rho = |\psi(x, \tau)|^2 \] . (4)

In analogy to the notation \( x^0 = ct \) we adopt the formal designations
\[ x^5 = c_5 \tau \quad \partial_5 = \frac{1}{c_5} \partial_\tau \quad \vec{j}^5 = c_5 \rho \quad a_5 = \frac{1}{c_5} \phi \] (5)

and the index convention
\[ \lambda, \mu, \nu = 0, 1, 2, 3 \quad \alpha, \beta, \gamma = 0, 1, 2, 3, 5 \] (6)

so that the gauge and current conditions (2) and (3) can be written
\[ a_\alpha \rightarrow a_\alpha + \partial_\alpha \Lambda \quad \partial_\alpha j^\alpha = 0 \.] (7)

It is convenient to choose the factor \( g_{55} \) and \( g^{55} = 1/g_{55} \) to apply when raising and lowering the 5-index, so that
\[ \partial_\alpha j^\alpha = g^{\alpha\nu} \partial_\mu j_\mu + g^{55} \partial_5 j_5 \.] (8)

### 2.2 Classical event dynamics

The classical mechanics of a relativistic event is found by rewriting the Stueckelberg-Schrodinger equation in the form
\[ i\hbar \partial_\tau \psi(x, \tau) = \left[ \frac{1}{2M} \left( p^\mu - \frac{e_0}{c} a^\mu \right) \left( p_\mu - \frac{e_0}{c} a_\mu \right) - \frac{e_0}{c} \phi \right] \psi(x, \tau) \] (9)

and transforming the classical Hamiltonian to Lagrangian form as
\[ L = \dot{x}^\mu p_\mu - K = \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e_0}{c} \dot{x}^\alpha a_\alpha \] (10)

where
\[ \dot{x}^\mu = \frac{dx^\mu}{d\tau} \quad \dot{x}^5 = \frac{dx^5}{d\tau} = c_5 \] . (11)

The Euler-Lagrange equations
\[ \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}_\mu} - \frac{\partial L}{\partial x_\mu} = 0 \] (12)

are
\[ \frac{d}{d\tau} \left( M \dot{x}^\mu + \frac{e_0}{c} a^\mu \right) - \partial_\mu \left( \frac{e_0}{c} \dot{x}^\alpha a_\alpha \right) = 0 \] (13)
leading to the Lorentz force

\[ M \dddot{x}^\mu = \frac{e_0}{c} [\dot{x}^a \partial^\mu a_a - \dot{x}^a \partial_a \partial^\mu] = \frac{e_0}{c} f_a^\mu (x, \tau) \dot{x}^a \]

\[ = \frac{e_0}{c} f_v^\mu (x, \tau) \dot{x}^v + \frac{e_0}{c} f_{5}^\mu (x, \tau) \dot{x}^5 \]

\[ = \frac{e_0}{c} f_v^\mu (x, \tau) \dot{x}^v - g^{55} \frac{e_0 c_5}{c} f_{5}^\mu (x, \tau) \]

(14)

where

\[ f_a^\mu = \partial^\mu a_a - \partial_a \partial^\mu \].

Because the four components of \( \dot{x}^\mu \) are independent, the event evolution may be off-shell. In this context, on-shell evolution obeys the mass-shell constraint \( \dot{x}^2 = -c^2 \) of standard relativity. In SHP electrodynamics

\[ \dot{x}^2 = \left( \frac{c}{d\tau} \frac{d t}{d \tau}, \frac{d x}{d \tau} \right)^2 = c^2 \dot{t}^2 \left( 1, \frac{1}{c} \left( \frac{d x}{d \tau} \right) \left( \frac{d t}{d \tau} \right)^{-1} \right)^2 = c^2 \dot{t}^2 \left( 1, \frac{1}{c} \frac{d x}{d t} \right)^2 = -c^2 \dot{t}^2 \left( 1 - \frac{v^2}{c^2} \right) \]

(16)

so that an event evolves on-shell when

\[ \frac{|dt|}{d \tau} = |\dot{t}| = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

(17)

and is said to be off-shell when \( |\dot{t}| \) takes any other value. In the SHP formalism, particles may exchange mass with fields through

\[ \frac{d}{d \tau} \left( -\frac{1}{2} M \dot{x}^2 \right) = -M \dddot{x}_\mu \dot{x}_\mu = -\frac{e_0}{c} \dot{x}^\mu (c_5 f_{\mu 5} + f_{\mu \nu} \dddot{x}_\nu) = \frac{e_0 c_5}{c} \dot{x}^\mu f_{\mu 5} = g^{55} \frac{e_0 c_5}{c} f_{5 \mu} \dddot{x}_\mu \]

(18)

and the mass shell is demoted from the status of constraint to that of conservation law for interactions in which \( \dot{x}^\mu f_{5 \mu} = 0 \) (which usually entails \( f_{5 \mu} = 0 \)). However, if the scale of the fields \( f_{5 \mu} \) is small compared to the Maxwell fields \( f_{\mu \nu} \) then the exchange of mass will be correspondingly small. It would be convenient to find that

\[ c_5 \rightarrow 0 \Rightarrow f_{5 \mu} \rightarrow 0 \]

(19)

so that \( c_5 \) can be understood as the scale of dynamic evolution in the microscopic system, approaching an equilibrium equivalent to standard Maxwell theory as \( c_5 \rightarrow 0 \). With this expectation in mind we examine the fields produced by the motions of charged events.
2.3 Electromagnetic action

To write an electromagnetic action requires the choice of a kinetic term for the gauge field; this term must be both gauge and O(3,1) invariant. We write

\[ S_{em} = \int d^4x d\tau \left\{ \frac{e_0}{c} j^a(x, \tau) a_\alpha(x, \tau) - \int ds \frac{\lambda}{4c} \left[ f^{a\beta}(x, \tau) \Phi(\tau - s) f_{a\beta}(x, s) \right] \right\} \]  

(20)

where the five components of the local event current

\[ j^a(x, \tau) = c \dot{X}^a(\tau) \delta^4(x - X(\tau)) \]  

(21)

have support at the spacetime location \( X^\mu(\tau) \) of the event, and we again write \( \dot{X}^5 = c_5 \).

The \( \tau \)-integral of (21) along the worldline concatenates the instantaneous events into the Maxwell particle current in the usual form. The field interaction kernel is defined as

\[ \Phi(\tau) = \delta(\tau) - (a\lambda)^2 \delta''(\tau) = \int \frac{d\kappa}{2\pi} \left[ 1 + (a\lambda\kappa)^2 \right] e^{-i\kappa \tau} \]  

(22)

where

\[ \alpha = \frac{1}{2} \left[ 1 + \left( \frac{c_5}{c} \right)^2 \right] \]  

(23)

is chosen so that the low energy Coulomb force agrees with the standard expression. The inverse function of the interaction kernel

\[ \varphi(\tau) = \Phi^{-1}(\tau) = \int \frac{d\kappa}{2\pi} \frac{e^{-i\kappa \tau}}{1 + (a\lambda\kappa)^2} = \frac{1}{2a\lambda} e^{-|\tau|/a\lambda} \]  

(24)

satisfies

\[ \int d\tau \varphi(\tau) = 1 \]  

(25)

and appears in the field equations as a smoothing of the particle current with respect to the sharp location of each individual event.

Expanding the expression

\[ f^{a\beta} f_{a\beta} = f^{\mu\nu} f_{\mu\nu} + 2 f^{5\mu} f_{5\mu} = f^{\mu\nu} f_{\mu\nu} + 2 g^{55} f_{5\mu} f_{5\mu} \]  

(26)

we may side-step the interpretation of \( g^{55} \) as an element in a 5D metric and rather see its role as equivalent to the choice of sign for the vector contribution \( f_{5\mu} f_{5\mu} \) to the field energy. Using (22) and integrating by parts, the action takes the form

\[ S = \int d\tau \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \int d^4x d\tau \left\{ \frac{e_0}{c} a_\alpha j^\alpha - \frac{\lambda}{4c} f_{a\beta} f^{a\beta} - \frac{a^2\lambda^3}{4c} \left( \partial_\tau f^{a\beta} \right) \left( \partial_\tau f_{a\beta} \right) \right\} \]  

(27)
in which the gauge and O(3,1) invariance are manifest, but the \( \tau \) derivatives in the last term explicitly break any formal 5D symmetry of the terms \( f_{\alpha \beta} f^{\alpha \beta} \).

Varying the action in the form (20) with respect to the fields, and using (24) to remove the kernel \( \Phi \), leads to the field equations

\[
\partial_\beta f^{\alpha \beta} (x, \tau) = \frac{e_0}{\lambda c} \int ds \varphi (\tau - s) j^\alpha (x, s) = \frac{e}{c} \int ds \varphi (x, \tau) \tag{28}
\]

\[
\partial_\alpha f_{\beta \gamma} + \partial_\gamma f_{\alpha \beta} + \partial_\beta f_{\gamma \alpha} = 0 \tag{29}
\]

which are formally similar to 5D Maxwell equations with \( e = e_0 / \lambda \). The source of the field in (28) is

\[
j^\alpha (x, \tau) = \int ds \varphi (\tau - s) j^\alpha (x, s) = c \int ds \varphi (\tau - s) \dot{X}^\alpha (s) \delta^4 (x - X(s)) \tag{30}
\]

formed by smoothing the support of the instantaneous current \( j^\alpha (x, \tau) \) defined in (21) by convolution with the inverse kernel function \( \varphi (\tau) \). For \( \lambda \) very small, \( \varphi \) becomes a delta function which narrows the source to a small neighborhood around the event inducing the current. For \( \lambda \) very large, the convolution becomes a concatenation of the current along the worldline, equivalent to the Maxwell current. The parameter \( \lambda \) thus plays the role of a correlation length, characterizing the range of the electromagnetic interaction.

The field equations (28) and (29) are called pre-Maxwell equations, and together with the Lorentz force (14) describe a microscopic event dynamics for which Maxwell theory can be understood as an equilibrium limit. The connection with Maxwell theory is found by integration over \( \tau \) which concatenates the events along the worldline. With equilibrium boundary conditions

\[
\rho_\varphi (x, \tau) \xrightarrow{\tau \to \pm \infty} 0 \quad f^{5\mu} (x, \tau) \xrightarrow{\tau \to \pm \infty} 0 \tag{31}
\]

we find

\[
\begin{align*}
\partial_\beta f^{\alpha \beta} (x, \tau) &= \frac{e}{c} j^\alpha (x, \tau) \\
\partial_\alpha f_{\beta \gamma} &= 0 \\
\partial_\beta j^\alpha &= 0 \\
\partial_\chi F^\mu \nu (x) &= \frac{e}{c} j^\mu (x) \\
\partial_\mu F^\mu \nu (x) &= 0 \\
\partial_\mu J^\mu (x) &= 0
\end{align*} \tag{32}
\]

where

\[
A^\mu (x) = \int d\tau a^\mu (x, \tau) \quad F^{\mu \nu} (x) = \int d\tau f^{\mu \nu} (x, \tau) \quad J^\mu (x) = \int d\tau j^\mu (x, \tau) \quad \tag{33}
\]
Since $e_0 a^\mu$ must have the dimensions of $e A^\mu$, it follows that $e_0$ and $\lambda$ have the dimension of time and $e = e_0 / \lambda$ is dimensionless.

Rewriting the field equations in vector and scalar components, they take the form

\begin{align*}
\partial_\nu f^{\mu\nu} - \frac{1}{c_5} \partial_\tau f^{5\mu} &= \frac{e}{c} j^\mu \\
\partial_\mu f^{5\nu} &= \frac{c_5}{c} e \rho_\phi \\
\partial_\mu f_{\nu\rho} + \partial_\nu f_{\rho\mu} + \partial_\rho f_{\mu\nu} &= 0
\end{align*}

which may be compared with the 3-vector form of Maxwell equations

\begin{align*}
\nabla \times B - \frac{1}{c} \partial_t E &= \frac{e}{c} J \\
\nabla \cdot E &= \frac{e}{c} j^0 \\
\n\nabla \cdot B &= 0 \\
\nabla \times E + \frac{1}{c} \partial_t B &= 0
\end{align*}

(34)

with $f_{5\mu}$ playing the role of the vector electric field and $f^{\mu\nu}$ playing the role of the magnetic field. We notice that $c_5$ appears three times in the pre-Maxwell equations (34), twice in the form $\frac{1}{c_5} \partial_\tau$ and once multiplying the event density $\rho_\phi$. To make sense of the derivative terms, we first recall that the homogeneous pre-Maxwell equations are automatically satisfied for fields derived from potentials — in this case the fields $f_{5\mu}$ contain terms with $\partial_\tau a_\mu = \frac{1}{c_5} \partial_\tau a_\mu$ that cancel the explicit $\tau$-derivative of $f_{\mu\nu}$. From the second homogeneous equation it follows that

\begin{align*}
0 &= c_5 \left( \partial_\nu f_{5\mu} - \partial_\mu f_{5\nu} \right) + \partial_\tau f_{\mu\nu} \\
&\xrightarrow{c_5 \to 0} \partial_\tau f_{\mu\nu}
\end{align*}

(36)

so that the Maxwell field strength becomes $\tau$-static. The $\tau$-derivative term in the first inhomogeneous pre-Maxwell equation remains finite as long as $f^{5\mu}$ is proportional to $c_5$ and we will see that this is generally the case for fields derived from potentials of the Liénard-Wiechert type. Under the boundary conditions associated with concatenation, the event density $\rho_\phi$ and $f^{5\mu}$ both vanish in equilibrium. Under the slightly weaker assumption that the divergenceless free field $f^{5\mu}$ is $\tau$-independent, it decouples from the Maxwell field, so that $f^{\mu\nu}$ and $j^\mu$ satisfy the standard Maxwell equations.

It is sometimes notationally convenient to further expand the field into 3-vector components as

\begin{align*}
(e)^i &= f^{0i} \\
(h)_{im} &= \epsilon_{ijk} f^{jk} \\
(f^{5})^i &= f^{5i}
\end{align*}

(37)
2.4 Wave equations and induced fields

The pre-Maxwell equations, in Lorenz gauge, lead to the wave equation

$$\partial_\mu \partial^\mu a^a = \left( \partial_\mu \partial^\mu + \partial_\tau \partial^\tau \right) a^a = \frac{1}{c^2} \nabla^2 a^a = -\frac{e}{c} f_\varphi(x, \tau) \tag{38}$$

whose solutions may respect 5D symmetries broken by the O(3,1) symmetry of the event dynamics. A Green’s function solution to

$$\left( \partial_\mu \partial^\mu + \frac{855}{c_5^2} \partial^2_\tau \right) G(x, \tau) = -\delta^4(x) \delta(\tau) \tag{39}$$

can be used to obtain potentials of the form

$$a^a(x, \tau) = -\frac{e}{c} \int d^4 x' d\tau' G(x-x', \tau-\tau') \tilde{f}_\varphi(x', \tau') = -e \int d^4 x' d\tau' ds \left( \phi(\tau'-s) X^a(s) \delta^4(x'-X(s)) \right)$$

$$= -e \int ds \left[ \int d\tau' \phi(\tau'-s) X^a(s) \delta(\tau'-s) \right] \tilde{X}^a(s) = -e \int ds G_\varphi(x-X(s), \tau-\tau) X^a(s). \tag{40}$$

Since $\tilde{X}^5(s) = c_5$, while the 4-vector $\tilde{X}(s) = \tilde{X}^0(s)/(c, v)$ with $|v| < c$, we see that the fifth potential $a^5(x, \tau)$ is in general scaled by $c_5/c$ with respect to $a^\mu(x, \tau)$.

The principal part Green’s function was found \[6\] using Schwinger’s method in the form

$$G_\varphi(x, \tau) = \frac{1}{4\pi} \delta(x^2) \delta(\tau) - \frac{c_5}{2\pi^2} \frac{\partial}{\partial x^2} \theta(-855 g_{a\beta} x^a x^\beta) \frac{1}{\sqrt{-855 g_{a\beta} x^a x^\beta}} \tag{41}$$

$$= G_{\text{Maxwell}} + G_{\text{Correlation}} \tag{42}$$

which recovers the 4D Maxwell Green’s function

$$\int d\tau \ G_{\text{Maxwell}} = D(x) = -\frac{1}{4\pi} \delta(x^2) \quad \int d\tau \ G_{\text{Correlation}} = 0 \tag{43}$$

under concatenation. The support of $G_{\text{Correlation}}$ is

$$-855 g_{a\beta} x^a x^\beta = \begin{cases} -\left( x^2 + c_5^2 \tau^2 \right) = c_5^2 \tau^2 - x^2 - c_5^2 \tau^2 > 0 & , \quad g_{55} = 1 \\ \left( x^2 - c_5^2 \tau^2 \right) = x^2 - c_5^2 \tau^2 - c_5^2 \tau^2 > 0 & , \quad g_{55} = -1 \end{cases} \tag{44}$$

The Green’s function was derived taking $c = c_5 = 1$, so that $x^5 = \tau$. Working through the derivation and replacing $\tau$ with $x^5 = c_5 \tau$ leads to a factor of $c_5$ multiplying the second term, so that both terms have units of distance$^{-2} \times$ time$^{-1}$.\[1\]
leading to causality properties discussed in [6]. In particular, we see that for \( g_{55} = 1 \),
the second term \( G_{\text{Correlation}} \) has timelike support with respect to the event trajectory,
opening the possibility of a self-interaction of a type not present in standard Maxwell theory. In order to exploit this self-interaction, we take \( g_{55} = 1 \) in the remaining sections of this paper.

As required in Schwinger’s method, we take special care in handling the distribution functions and the order of integration. Evaluating the derivative in (41) we find

\[
G_{\text{correlation}}(x, \tau) = -\frac{c_5}{2\pi^2} \frac{\partial}{\partial x^2} \theta(-g_{55} g_{\alpha \beta} x^\alpha x^\beta) \sqrt{-g_{55} g_{\alpha \beta} x^\alpha x^\beta}
\]

\[
= -\frac{c_5}{2\pi^2} \frac{\partial}{\partial x^2} \theta(-x^2 - c_5^2 \tau^2) \left( -x^2 - c_5^2 \tau^2 \right)^{1/2}
\]

\[
G_{\text{correlation}}(x, \tau) = -\frac{c_5}{2\pi^2} \left( \frac{1}{2} \theta(-x^2 - c_5^2 \tau^2) - \delta \left( -x^2 - c_5^2 \tau^2 \right) \right) \left( -x^2 - c_5^2 \tau^2 \right)^{3/2} - \delta \left( -x^2 - c_5^2 \tau^2 \right) \left( -x^2 - c_5^2 \tau^2 \right)^{1/2} \right).
\]

(45)

Although the second term appears highly singular, we will see that when calculating potentials, singularities in the terms of \( G_{\text{correlation}} \) cancel when the subtraction is performed before applying the limits of integration.

The ‘static’ Coulomb potential in this framework is induced by an isolated event moving uniformly along the \( t \) axis. Writing the event as

\[
X(\tau) = (c \tau, 0, 0, 0)
\]

(46)

produces the currents

\[
j^0(x, \tau) = j^5(x, \tau) = c \delta(t - \tau) \delta^3(x) \quad j(x, \tau) = 0
\]

(47)

\[
j^0_\varphi(x, \tau) = j^5_\varphi(x, \tau) = c \varphi(t - \tau) \delta^3(x) \quad j_\varphi(x, \tau) = 0
\]

(48)

and the Maxwell part of the Green’s function induces

\[
a^0(x, \tau) = a^5(x, \tau) = \frac{e}{4\pi |x|} \varphi \left( \tau - \left( t - \frac{|x|}{c} \right) \right) \quad a = 0
\]

(49)

which recovers the standard Coulomb potential

\[
A^0(x) = \int d\tau a^0(x, \tau) = \frac{e}{4\pi |x|} \quad A = 0
\]

under concatenation. In Appendix A we show that the contribution from \( G_{\text{Correlation}} \)
is smaller than the \( G_{\text{Maxwell}} \) contribution by \( c_5/c \) and drops off as \( 1/|x|^2 \). A test event
located at \( x(\tau) = (c\tau, x) \) will see the Yukawa-type potential

\[
\alpha_0^0(x, \tau) = \alpha_5^0(x, \tau) = \frac{e}{4\pi|\mathbf{x}|} \frac{1}{2\alpha\lambda} e^{-|\mathbf{x}|/\alpha\lambda c}
\]

in which \( 1/\lambda \) parameterizes the mass spectrum of the pre-Maxwell field. If \( \lambda \) is small (so that \( \phi \) approaches a delta function and the current narrows to around the event) the mass spectrum becomes wide. If \( \lambda \) is large, the support of the current spreads along the worldline and the potential becomes Coulomb-like. The field strength components are

\[
f_{k0}^k(x, \tau) = f_{50}^k(x, \tau) = \frac{e}{4\pi|\mathbf{x}|} \frac{1}{2\alpha\lambda} e^{-|\mathbf{x}|/\alpha\lambda c} \quad f_{ij}^j(x, \tau) = 0 \quad f_{50}^0 = 0
\]

where we used (24) for \( \phi(\tau) \). The test event will experience the Coulomb force through (14) as

\[
M \ddot{x}^k = \frac{e_0 e}{c} f_{k0}^k \dot{x}^\nu - g_{55} \frac{e_0 c_5}{c} f_{5k}^5 = -\frac{e_0 e}{c} f_{k0}^k \left( \dot{x}^0 - g_{55} c_5 \right)
\]

and since \( \dot{x}(\tau) = (c, 0) \) this becomes

\[
M \ddot{x} = -\frac{e_0 e}{2\alpha\lambda} \left( 1 - g_{55} \frac{c_5}{c} \right) \nabla \left( e^{-|\mathbf{x}|/\alpha\lambda c} \right) = -e^2 \frac{1 - g_{55} \frac{c_5}{c}}{1 + \left( \frac{c_5}{c} \right)^2} \nabla \left( e^{-|\mathbf{x}|/\alpha\lambda c} \right)
\]

where we used (23) for \( \alpha \). This expression for the Coulomb force would vanish for \( c_5 = c \) and \( g_{55} = 1 \), and for this reason it was previously argued \[?\] that \( g_{55} = 1 \) (corresponding to a formal O(4,1) symmetry of the wave equation) is prohibited. However, with \( c_5 < c \) either signature for \( g_{55} \) is permitted. In (53) \( \dot{x}(\tau) < 0 \) for a particle-antiparticle interaction, so that in (54) we will have \(-e^2(1 - g_{55} \frac{c_5}{c}) \rightarrow e^2(1 + g_{55} \frac{c_5}{c})\). This expression therefore leads to an experimental signature for the model, predicting a discrepancy between \( e^-/e^- \) scattering and \( e^+/e^- \) scattering at extremely low energy, and provides an experimental bound on \( c_5/c \).

In order to understand the role of \( c_5 \) in electromagnetic interactions, we study an arbitrary event \( X^\nu(\tau) \), which induces the current

\[
j_\phi^a(x, \tau) = c \int ds \phi(\tau - s) \dot{X}^a(s) \delta^4 [x - X(s)] .
\]

The form of \( G_{\text{Maxwell}} \) allows us apply standard techniques associated with the Liénard-Wiechert potential. Writing

\[
a^a(x, \tau) = -\frac{e}{c} \int d^4x' d\tau' G_{\text{Maxwell}}(x - x', \tau - \tau') j_\phi^a(x', \tau') = \frac{e}{2\pi} \int ds \phi(\tau - s) X^a(s) \delta \left( (x - X(s))^2 \right) \theta^{\text{ret}}
\]

10
and using the identity
\[
\int d\tau f(\tau) \delta[g(\tau)] = \frac{f(\tau_R)}{|g'(\tau_R)|},
\]
where \(\tau_R\) is the retarded time found from
\[
g(\tau) = (x - X(\tau_R))^2 = 0 \quad \theta^{\text{ret}} = \theta \left( x^0 - X^0(\tau_R) \right),
\]
provides
\[
a^\mu(x, \tau) = \frac{e}{4\pi} \phi(\tau - \tau_R) \frac{X^\mu(\tau_R)}{|X^\mu(\tau_R)|},
\]
Using this potential, where we write the event velocity and line of observation as
\[
u^\mu = X^\mu(\tau) \quad z^\mu = x^\mu - X^\mu(\tau)
\]
the potential takes the form
\[
a^\mu(x, \tau) = \frac{e}{4\pi} \phi(\tau - \tau_R) \frac{\nu^\mu}{|u \cdot z|}, \quad a^5(x, \tau) = \frac{e}{4\pi} \phi(\tau - \tau_R) \frac{c_5}{|u \cdot z|}.
\]
By a similar procedure we find the field strengths, separated into the retarded and radiation parts, as
\[
f_{\text{ret}}^{\mu\nu}(x, \tau) = -\frac{e}{4\pi} \phi(\tau - \tau_R) \frac{(z^\mu \nu^\nu - z^\nu \nu^\mu) u^2}{(u \cdot z)^3} \sim \frac{1}{z^2}
\]
\[
f_{\text{ret}}^{5\mu}(x, \tau) = \frac{ec_5}{4\pi} \phi(\tau - \tau_R) \frac{z^\mu u^2 - u^\mu (u \cdot z)}{(u \cdot z)^3} \sim \frac{1}{z^2}
\]
\[
f_{\text{rad}}^{\mu\nu}(x, \tau) = -\frac{e}{4\pi} \phi(\tau - \tau_R) \left[ \frac{(z^\mu \nu^\nu - z^\nu \nu^\mu) (u \cdot z)}{(u \cdot z)^3} \right]
+ \frac{\epsilon(\tau - \tau_R) z^\mu u^\nu - z^\nu u^\mu}{\lambda (u \cdot z)^2} \sim \frac{1}{|z|}
\]
\[
f_{\text{rad}}^{5\mu}(x, \tau) = -\frac{ec_5}{4\pi} \phi(\tau - \tau_R) \left[ \frac{(u \cdot z) z^\mu}{(u \cdot z)^3} - \frac{\epsilon(\tau - \tau_R) z^\mu - u^\mu (u \cdot z)}{\lambda (u \cdot z)^2} \right] \sim \frac{1}{|z|}.
\]
We have used \(|u \cdot z| = -(u \cdot z)\) (easily seen in a co-moving frame) and
\[
\frac{d}{d\tau_R} \phi(\tau - \tau_R) = -\frac{1}{2\alpha \lambda} \frac{d}{d\tau} e^{-|\tau - \tau_R|/\alpha \lambda} = \frac{\epsilon(\tau - \tau_R) z^\mu - u^\mu (u \cdot z)}{\alpha \lambda} \phi(\tau - \tau_R)
\]
where \(\epsilon(\tau) = \text{signum}(\tau)\). Notice that the \(\tau\)-dependence in these expressions is limited to the smoothing function \(\phi(\tau - \tau_R)\) and again \(\lambda\) plays the role of a correlation.
length that localizes the interaction to the neighborhood $\tau_R \pm \lambda$. As expected, concatenation of the potentials and field strengths recovers the expressions found in standard Maxwell theory.

In (62) to (65) we see once again that $c_5$ multiplies $f^{5\mu}$ and so provides a relative scale factor with respect to the components $f^{\mu\nu}$. Using (14), (23) and (24), the Lorentz force on an event moving in the field induced by another event can be written

$$M \ddot{x}^\mu = \frac{e_0}{c} \left[ f^{\mu\nu}(x, \tau) \dot{x}^\nu + f^{5\mu}(x, \tau) \dot{x}^5 \right]$$

$$= \frac{e_0}{c} \frac{e}{4\pi} \varphi (\tau - \tau_R) \left[ F^{\mu\nu}(x, \tau) \dot{x}^\nu + c_5^2 F^{5\mu}(x, \tau) \right]$$

$$= \frac{e_0}{c} \frac{e}{4\pi} \frac{1}{2\alpha \lambda} e^{-|\tau - \tau_R|/\alpha \lambda} \left[ F^{\mu\nu}(x, \tau) \dot{x}^\nu + c_5^2 F^{5\mu}(x, \tau) \right]$$

$$= \frac{e^2}{4\pi c} e^{-|\tau - \tau_R|/\alpha \lambda} \frac{1}{2\alpha} \left[ F^{\mu\nu}(x, \tau) \dot{x}^\nu + c_5^2 F^{5\mu}(x, \tau) \right]$$

$$= \frac{e^2}{4\pi c} e^{-|\tau - \tau_R|/\alpha \lambda} \frac{F^{\mu\nu}(x, \tau) \dot{x}^\nu + c_5^2 F^{5\mu}(x, \tau)}{1 + (c_5/c)^2}$$

(67)

where

$$F^{\mu\nu}(x, \tau) = f^{\mu\nu}(x, \tau) \quad (68)$$

$$F^{5\mu}(x, \tau) = \frac{z^\mu u^2 - u^\mu (u \cdot z)}{(u \cdot z)^3} - \frac{(u \cdot z) z^\mu}{(u \cdot z)^3} + \frac{\epsilon (\tau - \tau_R) z^\mu - u^\mu (u \cdot z)}{\lambda (u \cdot z)^2} \quad (69)$$

are independent of $c_5$. The Lorentz force interaction will be in the range

$$M \ddot{x}^\mu = \frac{e^2}{4\pi c} \times \left\{ \begin{array}{c} e^{-2|\tau - \tau_R|/\lambda} F^{\mu\nu}(x, \tau) \dot{x}^\nu \quad , \quad c_5 \to 0 \\ e^{-|\tau - \tau_R|/\lambda} \frac{1}{2} \left[ F^{\mu\nu}(x, \tau) \dot{x}^\nu + F^{5\mu}(x, \tau) \right] \quad , \quad c_5 \to c \end{array} \right. \quad (70)$$

showing that $c_5/c$ provides a continuous scaling of the Lorentz force. Taking $c_5 \to 0$ reproduces standard Maxwell dynamics in much the way that taking $c \to \infty$ reproduces nonrelativistic mechanics. Unlike the nonrelativistic approximation, in which the speed of light is taken to be infinite and action at a distance instantaneous, in the Maxwell approximation of pre-Maxwell theory the event dynamics evolve so slowly over $\tau$ that the system is essentially in equilibrium, the event density vanishes and in particular, no mass exchange takes place. This equilibrium is the spacetime generalization of a nonrelativistic static system. The contribution associated with $G_{\text{Correlation}}$
is less straightforward, but in Appendix B we gain some insight by studying the $\delta$-function term and assuming that the structure of the $\theta$-function term must be sufficiently similar to permit cancelation of singularities.

## 3 A self-interaction

It was seen in [18] that particles may exchange mass with the fifth electromagnetic field through

$$\frac{d}{d\tau}(-\frac{1}{2}M\dot{x}^2) = \frac{e_0}{c} f^{5\mu} \dot{x}_\mu$$

and despite the scaling of $f^{5\mu}$ by $c_5/c$ this effect cannot be assumed to be insignificant. In order to account for the observed stability of particle masses, we must find some mechanism that tends to enforce on-shell evolution, perhaps by damping off-shell behavior in the manner of air friction producing a terminal velocity. If, for example, some circumstance were to produce a field of the form $f^{5\mu} = \sigma \ddot{x}_\mu$ then

$$\frac{d}{d\tau}(-\frac{1}{2}M\dot{x}^2) = \frac{e_0}{c} \sigma \ddot{x}_\mu \dot{x}_\mu = -\frac{2e_0\sigma}{Mc} \left(-\frac{1}{2}M\dot{x}^2\right)$$

producing mass decay.

In this section, we propose a model for a self-interaction between a moving event and its electromagnetic field that produces a mass decay but vanishes for on-shell propagation. Unlike the self-interaction between a particle and its radiation field, associated with the Abraham-Lorentz-Dirac equation, this model involves the influence of the field induced through $G_{Correlation}$ with retarded timelike support. The event experiences a force along its worldline produced by its earlier motion along that worldline.

### 3.1 Framework

As in Appendix B, we study the motion of an arbitrarily moving event $X^\mu(\tau)$, this time in a co-moving frame, so that

$$X(\tau) = (ct(\tau), 0) \quad \dot{X}(\tau) = (c\dot{t}(\tau), 0).$$

In this frame

$$\dot{X}^2 = -c^2 t^2$$

(73)
and so off-shell propagation is characterized by \( t \neq 1 \) in the rest frame. We are interested in the self-force on the event at time \( \tau^* \) and write the observation point as

\[
X(\tau^*) = (ct(\tau^*), x(\tau^*))
\]  

(74)

so that

\[
X(\tau^*) - X(s) = (ct(\tau^*) - ct(s), x(\tau^*) - x(s)) = c(t(\tau^*) - t(s), 0).
\]  

(75)

Because \( G_{\text{Maxwell}} = 0 \) on this timelike separation, the sole contribution comes from \( G_{\text{Correlation}} \). As in Appendix A, we approximate \( \varphi(\tau' - s) = \delta(\tau' - s) \), introduce the function \( g(s) \) to express terms of the type

\[
c^2 g(s) = -\left((X(\tau) - X(s))^2 + c_5^2 (\tau - s)^2\right) = c^2 \left((t(\tau^*) - t(s))^2 - \frac{c_5^2}{c^2} (\tau^* - s)^2\right),
\]  

(76)

and write

\[
a^\alpha (X(\tau^*), \tau^*) = \frac{ec_5}{2\pi^2 c^3} \int ds \, X^\alpha(s) \left(\frac{1}{2} \frac{\theta(g(s))}{(g(s))^{3/2}} - \frac{\delta(g(s))}{(g(s))^{1/2}}\right) \theta_{\text{ret}}
\]

(77)

for the self-field experienced by the event. We designate the two terms as

\[
a^\alpha (X(\tau^*), \tau^*) = a^\alpha_0 + a^\alpha_5
\]

(78)

### 3.2 Uniform on-shell motion

For an event evolving uniformly on-shell we have

\[
t(\tau^*) = \tau^*, \quad g(s) = \left(1 - \frac{c_5^2}{c^2}\right)(\tau^* - s)^2
\]

(79)

and using identity (57) are led to

\[
a (X(\tau^*), \tau^*) = \frac{ec_5}{2\pi^2 c^3} (c, 0, c_5) \int ds \, \theta(\tau^* - s)
\]

\[
\left(\frac{1}{2} \frac{\theta\left(\left(1 - \frac{c_5^2}{c^2}\right)(\tau^* - s)^2\right)}{\left(\left(1 - \frac{c_5^2}{c^2}\right)(\tau^* - s)^2\right)^{3/2}} - \frac{\delta\left(\left(1 - \frac{c_5^2}{c^2}\right)(\tau^* - s)^2\right)}{\left(\left(1 - \frac{c_5^2}{c^2}\right)(\tau^* - s)^2\right)^{1/2}}\right)
\]

\[
\left. - \frac{ec_5}{2\pi^2 c^3} \left(1 - \frac{c_5^2}{c^2}\right)^{3/2} \int_{-\infty}^{\tau^*} ds \, \left(\frac{1}{2} \frac{1}{(\tau^* - s)^3} - \frac{\delta(\tau^* - s) \theta(\tau^* - s)}{|(\tau^* - s)|^2}\right)\right)
\]

(80)
Since
\[
\int_{-\infty}^{\tau^*} ds \left( \frac{1}{(\tau^* - s)^3} \right) = \left. \frac{1}{2 (\tau^* - s)^2} \right|_{-\infty}^{\tau^*} = \lim_{s \to \tau^*} \frac{1}{2 (\tau^* - s)^2}
\]  
(81)
and
\[
\int_{-\infty}^{\tau^*} ds \, \frac{\delta (\tau^* - s) \theta (\tau^* - s)}{(\tau^* - s)^2} = \lim_{s \to \tau^*} \frac{\theta (\tau^* - s)}{(\tau^* - s)^2} = \lim_{s \to \tau^*} \frac{1}{(\tau^* - s)^2}
\]  
(82)
we find that for uniform on-shell motion
\[
a (X (\tau^*), \tau^*) = \frac{ec_5}{2 \pi^2 c^3} (c, 0, c_5) \lim_{s \to \tau^*} \left( \frac{1}{2 (\tau^* - s)^2} - \frac{1}{(\tau^* - s)^2} \right) = 0.\]  
(83)

### 3.3 Field strengths

From \( \dot{X}' = 0 \) and the form of (77)
\[
a' = 0 \quad \partial_i a^0 = \partial_i a^5 = 0 \quad \Rightarrow \quad f^{\mu \nu} = f^{5i} = 0
\]  
(84)
and so the field reduces to
\[
f^{50} = \delta^5 a^0 - \delta^0 a^5 = g^{55} \frac{1}{c_5} \partial_{\tau^*} a^0 - g^{00} \frac{1}{c} \partial_i a^5 = \frac{1}{c_5} \partial_{\tau^*} a^0 + \frac{1}{c} \partial_i a^5
\]  
(85)
where the partial derivative \( \partial_{\tau^*} \) only acts on the explicit variable (not on \( t (\tau^*) \) or \( \theta^{ret} \)). Similarly, the velocity appears as \( \dot{X}^a (s) \) and is constant with respect to \( \partial_{\tau^*} \).

Working piece-by-piece
\[
\partial_\tau a^0 = \frac{ec_5}{4 \pi^2 c^3 c_5} \partial_{\tau^*} \int ds \, c l (s) \frac{\theta \left( (t (\tau^*) - t (s))^2 - \frac{c^2}{c^2} (\tau^* - s)^2 \right)}{\left[ (t (\tau^*) - t (s))^2 - \frac{c^2}{c^2} (\tau^* - s)^2 \right]^{3/2}} \theta^{ret}
\]  
(86)
contains
\[
\partial_{\tau^*} \theta \left( (t (\tau^*) - t (s))^2 - \frac{c^2}{c^2} (\tau^* - s)^2 \right) = -2 \frac{c^2}{c^2} \delta \left( (t (\tau^*) - t (s))^2 - \frac{c^2}{c^2} (\tau^* - s)^2 \right) (\tau^* - s)
\]  
(87)
and
\[
\partial_{\tau^*} \left[ \frac{1}{\left[ (t (\tau^*) - t (s))^2 - \frac{c^2}{c^2} (\tau^* - s)^2 \right]^{3/2}} \right] = 3 \frac{c^2}{c^2} \left[ (t (\tau^*) - t (s))^2 - \frac{c^2}{c^2} (\tau^* - s)^2 \right]^{5/2}.
\]  
(88)
Similarly,
\[
\frac{1}{c} \partial_t a_0^5 = \frac{ec_5}{4\pi^2 c^3} \partial_{t(t^*)} \int ds \ c_5 \frac{\theta \left( \left( t(t^*) - t(s) \right)^2 - \frac{c_5^2}{c^2}(t^* - s)^2 \right)}{\left( \left( t(t^*) - t(s) \right)^2 - \frac{c_5^2}{c^2}(t^* - s)^2 \right)^{3/2}} \theta^{ret} (\tau, s) \tag{89}
\]
contains
\[
\partial_{t(t^*)} \theta \left( \left( t(t^*) - t(s) \right)^2 - \frac{c_5^2}{c^2}(t^* - s)^2 \right) = 2 \left( t(t^*) - t(s) \right) \times \\
\delta \left( \left( t(t^*) - t(s) \right)^2 - \frac{c_5^2}{c^2}(t^* - s)^2 \right) \tag{90}
\]
\[
\partial_{t(t^*)} \frac{1}{\left[ \left( t(t^*) - t(s) \right)^2 - \frac{c_5^2}{c^2}(t^* - s)^2 \right]^{3/2}} = -3 \frac{t(t^*) - t(s)}{\left[ \left( t(t^*) - t(s) \right)^2 - \frac{c_5^2}{c^2}(t^* - s)^2 \right]^{5/2}} \tag{91}
\]
and
\[
\partial_{t(t^*)} \theta^{ret} = \partial_{t(t^*)} \theta \left( t(t^*) - t(s) \right) = \delta (t(t^*) - t(s)) = 0 \tag{92}
\]
where the last expression vanishes because \( t(t^*) = t(s) \) makes the argument of the \( \theta \)-function negative. Putting the pieces together we find
\[
\partial^5 a_0^5 - \partial^0 a_5 = \frac{3ec_5 c_5}{4\pi^2 c^3} \int ds \ c_5 \frac{\theta \left( \left( t(t^*) - t(s) \right)^2 - \frac{c_5^2}{c^2}(t^* - s)^2 \right)}{\left[ \left( t(t^*) - t(s) \right)^2 - \frac{c_5^2}{c^2}(t^* - s)^2 \right]^{5/2}} \theta^{ret} \Delta (t^*, s) \tag{93}
\]
where
\[
\Delta (t^*, s) = i(s)(t^* - s) - (t(t^*) - t(s)) \tag{94}
\]
Similarly, the derivatives of \( a_\delta \) produce
\[
\partial^5 a_\delta^5 - \partial^0 a_\delta^5 = -\frac{ec_5 c_5}{2\pi^2 c^3} \int ds \ c_5 \frac{\delta \left( \left( t(t^*) - t(s) \right)^2 - \frac{c_5^2}{c^2}(t^* - s)^2 \right)}{\left( \left( t(t^*) - t(s) \right)^2 - \frac{c_5^2}{c^2}(t^* - s)^2 \right)^{3/2}} \theta^{ret} \Delta (t^*, s) \tag{95}
\]
and combining terms we find
\[
f^{50} = f^{50}_\delta + f^{50}_\delta + f^{50}_\delta' \tag{96}
\]
where

\[
\begin{align*}
    f_{\theta}^{50} &= \frac{3e}{4\pi^2 c^4} \int ds \frac{\theta \left( (t (\tau^*) - t (s))^2 - \frac{c^2}{c^4} (\tau^* - s)^2 \right)}{\left[ (t (\tau^*) - t (s))^2 - \frac{c^2}{c^4} (\tau^* - s)^2 \right]^{5/2}} \theta^{\text{ret}} \Delta (\tau^*, s) \\
    f_{\delta}^{50} &= -\frac{e}{\pi^2 c^4} \int ds \frac{\delta \left( (t (\tau^*) - t (s))^2 - \frac{c^2}{c^4} (\tau^* - s)^2 \right)}{\left[ (t (\tau^*) - t (s))^2 - \frac{c^2}{c^4} (\tau^* - s)^2 \right]^{3/2}} \theta^{\text{ret}} \Delta (\tau^*, s) \\
    f_{\varphi}^{50} &= -\frac{e}{\pi^2 c^4} \int ds \frac{\delta' \left( (t (\tau^*) - t (s))^2 - \frac{c^2}{c^4} (\tau^* - s)^2 \right)}{\left[ (t (\tau^*) - t (s))^2 - \frac{c^2}{c^4} (\tau^* - s)^2 \right]^{1/2}} \theta^{\text{ret}} \Delta (\tau^*, s)
\end{align*}
\]

Notice that if the particle remains at constant velocity (in any uniform frame), then

\[
    x^0 (\tau) = u^0 \tau \Rightarrow \Delta (\tau^*, s) = \frac{u^0}{c} (\tau^* - s) - \left( \frac{u^0}{c} \tau^* - \frac{u^0}{c} s \right) = 0
\]

and so \( f^{50} \) vanishes. For any smooth \( t (\tau) \),

\[
    t (\tau^*) - t (s) = t (s) + \dot{t}(s)(\tau^* - s) + \frac{1}{2} \ddot{t}(s)(\tau^* - s)^2 + o \left( (\tau^* - s)^3 \right) - t (s) \\
    = \dot{t}(s)(\tau^* - s) + \frac{1}{2} \ddot{t}(s)(\tau^* - s)^2 + o \left( (\tau^* - s)^3 \right)
\]

so the function

\[
    \Delta (\tau^*, s) = \dot{t}(s)(\tau^* - s) - (t (\tau^*) - t (s)) = -\frac{1}{2} \ddot{t}(s)(\tau^* - s)^2 + o \left( (\tau^* - s)^3 \right)
\]

is nonzero only when the time coordinate accelerates in the rest frame, equivalent to a shift in the particle mass.

### 3.4 Mass jump

As a first order example, we consider a small, sudden jump in mass at \( \tau = 0 \) characterized by

\[
    t (\tau) = \begin{cases} 
    \tau & \tau < 0 \\
    (1 + \beta) \tau & \tau > 0 
    \end{cases} \Rightarrow \dot{t} (\tau) = \begin{cases} 
    1 & \tau < 0 \\
    1 + \beta & \tau > 0 
    \end{cases}
\]

and calculate the self-interaction. Since \( \theta^{\text{ret}} \) enforces \( t(\tau^*) > t(s) \), it follows that

\[
    \tau^* < 0 \Rightarrow s < 0 \Rightarrow \dot{t}(\tau^*) = t(s) = 1 \Rightarrow \Delta (\tau^*, s) = 0.
\]
Similarly,
\[ \tau^* > 0 \text{ and } s > 0 \implies \dot{t}(\tau^*) = t(s) = 1 + \beta \implies \Delta(\tau^*, s) = 0 . \] (105)

But when \( \tau^* > 0 \) and \( s < 0 \),
\[ \Delta(\tau^*, s) = \dot{t}(s)(\tau^* - s) - (t(\tau^*) - t(s)) = (\tau^* - s) - [(1 + \beta)(\tau^*) - s] = -\beta \tau^* \] (106)

and \( f^{50} \) can be found from the contributions (97) – (99). Writing
\[ g(s) = (t(\tau^*) - t(s))^2 - \frac{c_s^2}{c^2}(\tau^* - s)^2 = ((1 + \beta)(\tau^*) - s)^2 - \frac{c_s^2}{c^2}(\tau^* - s)^2 \] (107)

and solving for \( g(s^*) = 0 \), we find
\[ s^* = \left( 1 + \frac{\beta}{1 - \frac{c_s^2}{c^2}} \right) \tau^* > \tau^* \] (108)

so that \( g(s) > 0 \) for \( s < 0 < \tau^* \) and there will be no contribution from (98) or (99). Thus,
\[ f^{50} = f^0 = (-\beta \tau^*) \frac{3e c_s^2}{4\pi^2 c^4} \int_{-\infty}^{0} \frac{1}{(t(\tau^*) - t(s))^2 - \frac{c_s^2}{c^2}(\tau^* - s)^2}^{5/2} ds \]
\[ = (-\beta \tau^*) \frac{3e c_s^2}{4\pi^2 c^4} \int_{-\infty}^{0} \frac{1}{((1 + \beta)(\tau^*) - s)^2 - \frac{c_s^2}{c^2}(\tau^* - s)^2}^{5/2} . \] (109)

Shifting the integration variable as \( x = \tau^* - s \) the integral becomes
\[ \int_{-\infty}^{0} \frac{1}{((1 + \beta)(\tau^*) - s)^2 - \frac{c_s^2}{c^2}(\tau^* - s)^2}^{5/2} ds = -\int_{\tau^*}^{\infty} \frac{dx}{(C x^2 + B x + A)^{5/2}} \] (110)

where
\[ C = 1 - \frac{c_s^2}{c^2} \quad B = 2u \tau^* \quad A = (\beta \tau^*)^2 . \] (111)

This integral can be evaluated using the well-known form [16]
\[ \int \frac{dx}{(C x^2 + B x + A)^{5/2}} = \frac{2(2C x + B)}{3q\sqrt{C x^2 + B x + A}} \left( \frac{1}{C x^2 + B x + A} + \frac{8C}{q} \right) \] (112)

where
\[ q = 4AC - B^2 \] (113)
leading to

\[- \int_{\tau^*}^{\infty} dx \frac{dx}{(Cx^2 + Bx + A)^{5/2}} = - \frac{1}{3 (\beta \tau^*)^4} \times \]

\[
\left[ 2 \frac{c^4}{c_5^2} \left( 1 - \frac{c_5^2}{c^2} \right)^{3/2} \left( \frac{1 - \frac{c_5^2}{c^2}}{1 - \frac{1}{c_5^2}} \right)^{1/2} \left( \frac{1 + \frac{\beta}{\left( 1 - \frac{c_5^2}{c^2} \right)^{1/2}}}{1 + \frac{2\beta}{1 - \frac{c_5^2}{c^2}} + \frac{\beta^2}{1 - \frac{c_5^2}{c^2}}} \right)^{3/2} \right] \]

(114)

and the field strength in the form

\[ f^{50} = \frac{e}{4\pi^2} \frac{1}{c_5^2 (\beta \tau^*)^3} Q \left( \beta, \frac{c_5^2}{c^2} \right) \]

(115)

where \( Q \left( \beta, \frac{c_5^2}{c^2} \right) \) is the positive, dimensionless factor.
\[ Q \left( \beta, \frac{c_5}{c^2} \right) = 2 \left( 1 - \frac{c_5}{c^2} \right)^{3/2} \left( 1 - \frac{1}{2} \frac{\beta}{1 - \frac{c_5}{c^2}} \right)^{1/2} \left( 1 + \frac{1}{1 + \frac{2\beta c_5}{c^2} + \frac{\beta^2}{1 - \frac{c_5}{c^2}}} \right)^{1/2} + \beta^2 \frac{c_5^2}{c^2} \left( \frac{1 - \frac{c_5}{c^2}}{1 - \frac{c_5}{c^2}} \right)^{1/2} \left( \frac{1 + \frac{2\beta c_5}{c^2} + \frac{\beta^2}{1 - \frac{c_5}{c^2}}}{1 + \frac{2\beta c_5}{c^2} + \frac{\beta^2}{1 - \frac{c_5}{c^2}}} \right)^{3/2} \]

(116)

which is seen to be finite for \( c_5 < c \)

\[ Q \left( \beta, \frac{c_5}{c^2} \right) \rightharpoonup_{c_5 \to 0} 2 \left( 1 - \frac{1 + \beta}{1 + 2\beta + \beta^2} \right)^{1/2} = 0 . \]  

(117)

Since \( f^{\mu\nu} = 0 \), the Lorentz force induced by this field strength is then

\[ M \ddot{x}^\mu = e_0 f^{\mu\alpha} \dot{x}_\alpha = e_0 f^{5\mu} \dot{x}_5 = -e_0 f^{5\mu} \dot{x}_5 = -g_{55} e_0 f^{5\mu} \dot{x}_5 = -e_0 f^{5\mu} c_5 \]  

(118)

and since \( f^{5i} = 0 \)

\[ M \ddot{x}^i = 0 \]  

(119)

\[ M \ddot{x}^0 = -c_5 e_0 f^{50} = \begin{cases} 0 & , \tau^* < 0 \\ -\frac{\lambda e^2}{4\pi^2} \frac{1}{c_5 (\beta \tau^*)^3} Q \left( \beta, \frac{c_5^2}{c^2} \right) & , \tau^* > 0 \end{cases} \]  

(120)

in which the factor \( \lambda \) is an artifact of the approximation \( \varphi(\tau' - s) = \delta(\tau' - s) \). Under the influence of a negative Lorentz force, the 0-coordinate will decelerate until the event returns to on-shell propagation, so that the function \( \Delta(\tau^*, s) \) and field strength \( f^{50} \) again vanish. Similarly,

\[ \frac{d}{d\tau} \left( -\frac{1}{2} M \dot{x}^2 \right) = e_0 f^{5\mu} \dot{x}_\mu = e_0 f^{50} \dot{x}_0 = -e_0 c f^{50} i = -\frac{\lambda e^2}{4\pi^2} \frac{c}{c_5^2 (\beta \tau^*)^3} Q \left( \beta, \frac{c_5^2}{c^2} \right) i \]  

(121)
so that the mass will damp back to the on-shell value. Notice also that if $\beta < 0$ then $f^{50}$ changes sign so that the self-interaction results in damping or anti-damping to restore on-shell behavior.

We see that the Lorentz force is singular at $\tau^* = 0$, the moment at which the velocity and mass jump discontinuously. We expect that the force will be smooth for a smooth mass increase. Although this model is approximate, it seems to indicate that the self-interaction of the event with the field generated by its mass shift will restore the event to on-shell propagation. Additional work is needed to provide a more complete solution.

4 A simple model for mass shift

We consider an event propagating uniformly on-shell as

$$x(\tau) = u\tau = (u^0, u) \quad u^2 = -c^2$$

(122)

until it passes through a dense region of charged particles that induce a small stochastic perturbation $X(\tau)$ such that

$$x(\tau) = u\tau + X(\tau) .$$

(123)

If the typical distance scale between force centers is $d$ then the perturbation will be roughly periodic with characteristic period

$$d \left| \frac{u}{u} \right| = \text{very short distance} \quad \text{moderate velocity} = \text{very short time},$$

(124)

fundamental frequency

$$\omega_0 = 2\pi \frac{|u|}{d} = \text{very high frequency},$$

(125)

and amplitude on the order of

$$|X''(\tau)| \sim \alpha d$$

(126)

for some macroscopic factor $\alpha < 1$. We may expand the perturbation in a Fourier series

$$X(\tau) = \Re \sum_n a_n e^{i\omega_0 \tau}$$

(127)
and write the four-vector coefficients as
\[ a_n = \alpha d s_n = \alpha d \left( s_n^0, s_n \right) = \alpha d \left( c s_n^t, s_n \right) \]  \hfill (128)
where the \( s_n \) represent a normalized Fourier series \( s_n^H \sim 1 \). The perturbed motion
\[ X (\tau) = \alpha d \Re \sum_n s_n^H e^{i n \omega_0 \tau} \]  \hfill (129)
is seen to be of scale \( d \), but the perturbed velocity
\[
\dot{x}^H (\tau) = u^H + \dot{X}^H (\tau)
\]
\[ = u^H + \alpha d \Re \sum_n n \omega_0 s_n^H i e^{i n \omega_0 \tau} \]
\[ = u^H + \alpha d \Re \sum_n \left( 2 \pi \frac{|u|}{d} \right) s_n^H i e^{i n \omega_0 \tau} \]
\[ = u^H + \alpha |u| \Re \sum_n 2 \pi n s_n^H i e^{i n \omega_0 \tau} \]  \hfill (130)
is of macroscopic scale. The unperturbed, on-shell mass is
\[ m = - \frac{M \dot{x}^2 (\tau)}{c^2} = M \]  \hfill (131)
and the perturbed mass is
\[ m = - \frac{M \dot{x}^2 (\tau)}{c^2} = - \frac{M}{c^2} \left( u + \alpha |u| \Re \sum_n 2 \pi n s_n^H i e^{i n \omega_0 \tau} \right)^2 \]
\[ = - \frac{M}{c^2} \left( u^2 + \left( \alpha |u| \Re \sum_n 2 \pi n s_n^H i e^{i n \omega_0 \tau} \right)^2 + 2 \alpha |u| \Re \sum_n 2 \pi n (u \cdot s_n) i e^{i n \omega_0 \tau} \right) \]
\[ = - \frac{M}{c^2} \left( -c^2 + 2 \alpha |u| \Re \sum_n 2 \pi n (u \cdot s_n) i e^{i n \omega_0 \tau} \right) \]
\[ m = M \left( 1 - \frac{2 \alpha |u|}{c^2} \Re \sum_n 2 \pi n (u \cdot s_n) i e^{i n \omega_0 \tau} \right) \]
\[ + \frac{\alpha^2 u^2}{c^2} \Re \sum_{n,m} (2 \pi)^2 n m s_n \cdot s_m e^{i (n+m) \omega_0 \tau} \]  \hfill (132)
Evaluating the typical coefficients in the rest frame of the unperturbed motion
\[ \frac{2 \alpha |u|}{c^2} 2 \pi n (u \cdot s_n) = \frac{4 \pi \alpha}{c^2} \frac{|u|}{n} (c, 0) \cdot (c s_n^t, s_n) = -4 \pi \alpha |u| n s_n^t \]  \hfill (133)
\[ \frac{\alpha^2 u^2}{c^2} (2 \pi)^2 n m s_n \cdot s_m = (2 \pi)^2 \alpha^2 u^2 n m \left( s_n^t s_m^t - \frac{s_n \cdot s_m}{c^2} \right) \]  \hfill (134)
and neglecting the $\alpha^2$ term, we find

$$m \simeq M \left(1 + 4\pi\alpha |u| \text{Re} \sum_n n s_n^t e^{in\omega_0\tau}\right)$$  \hspace{1cm} (135)$$

which expresses a mass shift as

$$m \rightarrow m \left(1 + \frac{\Delta m}{m}\right) \quad \frac{\Delta m}{m} = 4\pi\alpha |u| \text{Re} \sum_n n s_n^t e^{i n \omega_0 \tau}. \hspace{1cm} (136)$$

Larger mass shifts can be observed if $\alpha > 1$ and the second order term in $\alpha^2$ becomes significant.

5 Summary

In Section 2, we obtained the SHP electromagnetic theory in a form that explicitly includes the constants $c$ and $c_5$ associated with the Einstein time $t$ and the invariant $\tau$, and considers phenomenology in the case of $c_5 < c$. We see that the field $f_5^{\mu}$ that permits exchange of mass between particles and fields generally appears in proportion to $c_5$, effectively scaling this effect. We also that at very low-energy the scale of particle-particle scattering is proportional to $1 - g_{5\tau}(c_5/c)$ while particle-antiparticle scattering scales as $1 + g_{5\tau}(c_5/c)$, providing an experimental limit on $c_5$. The implicit assumption that $c_5 = c$ would have precluded $g_{5\tau} = +1$ because it would prohibit the static Coulomb force. The possibility that $c_5 < c$ thus permits either signature $g_{5\tau} = \pm 1$.

In Section 3, we consider a classical self-interaction in which an propagating event experiences a force as it passes through the electromagnetic field induced by its earlier motion along its worldline. Such an interaction is prohibited by the lightlike support of $G_{\text{Maxwell}}$, the Maxwell part of the SHP Green’s function, and by the spacelike support of $G_{\text{Correlation}}$ for $g_{5\tau} = -1$. However, for $g_{5\tau} = +1$ the support of $G_{\text{Correlation}}$ is timelike and includes the particle’s own future worldline. We found that for uniform motion this self-interaction vanishes — it depends on the time acceleration $\ddot{x}^0$ in the rest frame of the particle, associated with shifting mass. The Lorentz force acting on a particle that undergoes a discrete jump in $x^0$ in the rest frame was found to be negative, and so acts on the particle motion to oppose the time acceleration and restore the particle to on-shell propagation (for which the interaction again vanishes). This self-interaction would appear to provide an underlying mechanism for the asymptotic
on-shell behavior found by Aharonovich and Horwitz in numerical solutions \[14\], and perhaps an explanation of the observed mass stability of the known particles.

In Section 4, we discuss a simple model in which a uniformly moving on-shell particle enters a region of densely packed charges and experiences a mass shift induced by a very small perturbation, which nevertheless contributes a very high frequency stochastic velocity.

Considerable work is still required to work out the details of these simple models.

**Appendix A — Coulomb potential from $G_{\text{Correlation}}$**

We are interested in an event moving as

$$X = (ct, 0) \quad u^2 = -c^2$$  \hspace{1cm} (137)

where we approximate

$$\varphi(\tau' - s) = \delta(\tau' - s)$$  \hspace{1cm} (138)

so that

$$a^a(x, \tau) = -e \int ds \, G(x - X(s), \tau - s) \, X^a(s) = e \frac{e}{2\pi^2} \, \varphi^a(s) \int ds \, G(x - X(s), \tau - s) \cdot$$  \hspace{1cm} (139)

We introduce the function $g(s)$ to express terms of the type

$$- \left( (x - X(s))^2 + c_s^2(\tau - s)^2 \right) = - \left( (ct, x) - (cs, 0))^2 + c_s^2(\tau - s)^2 \right) = c^2 g(s)$$  \hspace{1cm} (140)

where

$$g(s) = (t - s)^2 - \frac{R^2}{c^2} - c_s^2(\tau - s)^2 = C s^2 + B s + A$$  \hspace{1cm} (141)

and

$$\zeta^2 = \frac{c_s^2}{c^2} \quad C = \left( 1-\zeta^2 \right) \quad B = -2 \left( t - \zeta^2 \tau \right) \quad A = t^2 - \frac{R^2}{c^2} - \zeta^2 \tau^2$$  \hspace{1cm} (142)

so that the potential can be written as

$$\varphi(x, \tau) = \frac{ec_s}{2\pi^2 c^3} (c, 0, c_s) \int ds \, \left[ \frac{1}{2} \left( \frac{\varphi(g(s))}{g^{3/2}(s)} - \frac{\delta(g(s))}{g^{1/2}(s)} \right) \right] \theta(t - s) \cdot$$  \hspace{1cm} (143)

The zeros of $g(s)$ are found to be

$$s_{\pm} = -B \pm \frac{\sqrt{B^2 - 4AC}}{2C} = \frac{t - \zeta^2 \tau}{(1-\zeta^2)^2} \pm \sqrt{\frac{R^2}{c^4} \left( 1 - \zeta^2 \right) + \zeta^2 (t - \tau)^2}$$  \hspace{1cm} (144)
and since we assume $\zeta^2 < 1$ there will be roots for any values of $t$ and $R$. In addition, the condition $\theta^{ret} = \theta(t-s)$ requires $t > s$.

If $t < s_-$ then

$$t < \frac{(t - \zeta^2 \tau) - \sqrt{\frac{R^2}{c^2}} (1 - \zeta^2) + \zeta^2 (t - \tau)^2}{1 - \zeta^2} \Rightarrow -\zeta^2 (t - \tau)^2 > \frac{R^2}{c^2} \quad (145)$$

and so $t \geq s_-$ becomes a condition of integration for the $\theta$ term. Similarly, if $t > s_+$ then

$$t > \frac{(t - \zeta^2 \tau) + \sqrt{\frac{R^2}{c^2}} (1 - \zeta^2) + \zeta^2 (t - \tau)^2}{1 - \zeta^2} \Rightarrow -\zeta^2 (t - \tau) > \frac{R^2}{c^2} \quad (146)$$

leading to the condition

$$s_- \leq t \leq s_+ \quad (147)$$

from which

$$a(x, \tau) = \frac{ec_5}{2\pi^2 c^5} \left( 1, 0, \frac{c_5}{c} \right) \left( \frac{1}{2} \int_{-\infty}^{s_-} ds \frac{1}{g^{3/2}(s)} + \int_{-\infty}^{\infty} ds \frac{\delta(g(s))}{g^{1/2}(s)} \theta(t-s) \right) \quad (148)$$

Using the well-known form [16]

$$\int \frac{dx}{(Cx^2 + Bx + A)^{3/2}} = \frac{2(2Cs + B)}{q(Cx^2 + Bx + A)^{1/2}} \quad (149)$$

where

$$q = 4AC - B^2 \quad (150)$$

we notice from (144) that

$$s_- = \frac{-B - \sqrt{B^2 - 4AC}}{2C} = \frac{-B - \sqrt{-q}}{2C} \Rightarrow -\sqrt{-q} = 2Cs_- + B \quad (151)$$

and so

$$\frac{1}{2} \int_{-\infty}^{s_-} ds \frac{1}{g^{3/2}(s)} = \frac{2C_s_- + B}{qg^{1/2}(s_-)} - \frac{2C_s + B}{qg^{1/2}(s)} \bigg|_{-\infty} \quad (152)$$
The second term is
\[ \int_{-\infty}^{\infty} ds \frac{\delta(g(s))}{g^{1/2}(s)} \theta(t-s) \] (153)
and using the identity
\[ \int ds \, f(s) \, \delta(g(s)) = f(s^-) \left| \frac{g'(s^-)}{|g'(s^-)|} \right|_{s^-=g^{-1}(0)} \] (154)
we can evaluate
\[ \int_{-\infty}^{\infty} ds \frac{\delta(g(s))}{g^{1/2}(s)} \theta(t-s) = \frac{\theta(t-s)}{|g'(s_-)| g^{1/2}(s_-)} = \frac{1}{|g'(s_-)| g^{1/2}(s_-)} \] (155)
Since
\[ g'(s_-) = \left( Cs_-^2 + Bs_- + A \right)' = 2Cs_- + B = -\sqrt{-q} \] (156)
we see that this term cancels the singularity in the first term, leaving
\[ \frac{1}{2} \int_{-\infty}^{\infty} ds \frac{1}{g^{3/2}(s)} - \frac{1}{2} \int_{-\infty}^{\infty} ds \frac{\delta(g(s))}{g^{1/2}(s)} \theta(t-s) = \frac{1}{2} \frac{\sqrt{1-\zeta^2}}{R^2 \left( 1 - c_5 \right) + \frac{c_5}{c} c^2 (t-\tau)^2} \] (157)
and
\[ a(x, \tau) = \frac{e}{4\pi^2} \left( c, 0, c_5 \right) \frac{c_5}{c} \frac{\sqrt{1 - \frac{c_5}{c}}}{R^2 \left( 1 - c_5 \right) + \frac{c_5}{c} c^2 (t-\tau)^2} \] (158)
We notice that the potential has units of $c$/distance$^2 = 1$/time $\times$ distance, as does the potential associated with $G_{\text{Maxwell}}$. On concatenation — integration over $\tau$ — we recover the 1/distance units of the Maxwell potential. This contribution to the potential is smaller by a factor of $c_5/c$ than the Yukawa potential found in (51), and drops off faster with distance.

**Appendix B — $c_5$-dependence of general potential from $G_{\text{Correlation}}$**

We are interested in an arbitrary event moving as
\[ X(\tau) = (ct(\tau), x(\tau)) \quad X^5 = c_5 \tau \] (159)
and the induced field
\[ a^\alpha(x, \tau) = -e \int ds \, G_\phi(x-X(s), \tau-s) \, \dot{X}^\alpha(s) \] (160)
Making the approximation
\[ \varphi(\tau' - s) = \delta(\tau' - s) \] (161)
leads to
\[ a^\alpha (x, \tau) = \frac{ec_5}{2\pi^2} \int ds \, X^\alpha (s) \left( \frac{1}{2} \theta \left( - (x - X(s))^2 - c_5^2(\tau - s)^2 \right) \right. \]
\[ \left. \frac{3}{2} \left[ - (x - X(s))^2 - c_5^2(\tau - s)^2 \right]^{3/2} \delta \left( - (x - X(s))^2 - c_5^2(\tau - s)^2 \right) \right) \theta^{ret} \] (162)

We designate
\[ g(s) = - (x - X(s))^2 - c_5^2(\tau - s)^2 \quad s^\pm = g^{-1}(0) \] (163)
and assume that the observation point \( x \) is in the timelike future of \( X (-\infty) \) so that
\[ a^\alpha (x, \tau) = \frac{e}{2\pi^2} \int^{-\infty}_{s^=} ds \, \frac{1}{2} \hat{X}^\alpha (s) \theta \left( - (x - X(s))^2 - c_5^2(\tau - s)^2 \right) \]
\[ \left[ - (x - X(s))^2 - c_5^2(\tau - s)^2 \right]^{3/2} \delta \left( - (x - X(s))^2 - c_5^2(\tau - s)^2 \right) \theta \left( ct - X^0(s) \right) \] (164)

Using the identity
\[ \int ds \, f(s) \delta[g(s)] = \sum_{s^\pm = g^{-1}(0)} \frac{f(s)}{|g'(s)|} \] (165)
the second term in the integral becomes
\[ \int_{-\infty}^{\infty} ds \, X(s) \frac{\delta(g(s))}{(g(s))^{1/2}} \theta \left( ct - X^0(s) \right) \frac{\hat{X}(s^\pm) \theta \left( ct - X^0(s^\pm) \right)}{(g(s^\pm))^{1/2} |g'(s^\pm)|} . \] (166)

At the observation point \( (x, x^5) = (x, c_5\tau) \) we define a 5D line of observation as
\[ Z = (z, z^5) = (x, x^5) - (X(s), X^5) = (x - X(s), c_5\tau - c_5s) \] (167)
and a 5-velocity
\[ U = (u, u^5) \quad u^\mu = \dot{X}^\mu (s) \quad u^5 = \dot{X}^5 \] (168)
leading to a generalization of the denominator of (61) in the form
\[ g'(s) = 2U \cdot Z \] (169)
so that (166) becomes

\[
\frac{X(s)\theta(ct - X^0(s))}{(g(s))^{1/2} |2U\cdot Z|} \bigg|_{s \rightarrow s^{\pm}}
\]

which is singular as \( s \rightarrow s^{\pm} \). As seen in Appendix A, we expect that this singularity is canceled by a corresponding singularity in the \( \theta \)-term of (164). Nevertheless, for \( s \neq s^{\pm} \) this expression remains finite if we take \( c_5 \rightarrow 0 \). Since we expect the \( \delta \)-term to have a similar structure to the \( \theta \)-term, it seems that the contribution of \( G_{\text{correlation}} \) to the field induced by a general event will split as

\[
\frac{\mathcal{F}_\mu^\nu(x, \tau) \dot{x}^\nu + c_5^2 \mathcal{F}_5^\mu(x, \tau)}{1 + (c_5/c)^2}
\]

where \( \mathcal{F}_\mu^\nu(x, \tau) \) and \( \mathcal{F}_5^\mu(x, \tau) \) remain finite as \( c_5 \rightarrow 0 \).

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