SHARP SPECTRAL MULTIPLIERS FOR A NEW CLASS OF GRUSHIN TYPE OPERATORS

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ABSTRACT. We describe weighted Plancherel estimates and sharp Hebisch-Müller-Stein type spectral multiplier result for a new class of Grushin type operators. We also discuss the optimal exponent for Bochner-Riesz summability in this setting.

1. Introduction

On the space $L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ with the standard Lebesgue measure consider a class of Grushin type operators defined by the formula

$$(1) \quad L_\sigma = -\sum_{j=1}^{d_1} \partial_{x_j}^2 - \left( \sum_{j=1}^{d_1} |x_j|^\sigma \right) \sum_{k=1}^{d_2} \partial_{x_{k'}}^2,$$

where exponent $\sigma > 0$. In the case $\sigma = 2$, the spectral properties of these operators were studied by A. Martini and the second author in [13] where sharp spectral multiplier and optimal Bochner-Riesz summability results were obtained. The aim of this paper is to obtain analogous results for the class of Grushin operators corresponding to the exponent $\sigma = 1$. The general strategy of the proof of the sharp spectral multiplier result for $\sigma = 1$ is the same as one described in [13] for $\sigma = 2$. However, the proofs of two most crucial estimates (Proposition 2.2 and Lemma 3.4 below) are new and technically significantly more difficult. The spectral decompositions of the operators $L_2$ and $L_1$ are essentially different. We use results derived in [7] to obtain a description of the spectral decomposition of the operator $L_1$ necessary for the proof of Proposition 2.2 and Lemma 3.4.

The closure of operator $L_\sigma$, $\sigma > 0$ initially defined on $C_0^\infty(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ is a non-negative self-adjoint operator and it admits a spectral resolution $E_{L_\sigma}(\lambda)$ for all $\lambda \geq 0$, see [15]. By spectral theorem for every bounded Borel function $F: \mathbb{R} \to \mathbb{C}$, one can define the operator

$$F(L_\sigma) = \int_\mathbb{R} F(\lambda) dE_{L_\sigma}(\lambda)$$

which is bounded on $L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. This paper is devoted to spectral multipliers that is we investigate sufficient conditions on function $F$ under which the operator $F(L_1)$ extends to bounded operator acting on spaces $L^p(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ for some range of $p$. We also study closely related question of critical exponent $\kappa$ for which the Bochner-Riesz means $(1 - tL_1)^\kappa_+$ are bounded on $L^p(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ uniformly in $t \in [0, \infty)$. In the sequel we shall only discuss the Grushin operator $L_1$ which for simplicity we denote just by $L$.

The motivation and rationale for spectral multiplier results of the type, which we consider here as well as relevant literature and earlier related multiplier results were described in details in the introduction to [13] and we refer readers to this paper for in depth discussion. Here we only want to briefly mention that the theory of spectral multipliers and Bochner-Riesz analysis are central part of harmonic analysis which have attracted a huge amount of
attention, see for example [4, 6, 9, 14, 16] and references within. One especially intriguing and surprising direction in the theory of spectral multipliers is devoted to investigation of sharp results for sub-elliptic or degenerate operators. The main idea in this area is that the sharp results are expected to be determined by the Euclidean dimension of underling ambient space rather than the homogeneous dimension of the space and corresponding heat semigroup. This part of spectral multipliers theory was initiated by results obtained by W. Hebisch [9] and D. Müller and E.M. Stein [14]. Other examples of papers devoted to sharp spectral multipliers for sub-elliptic or degenerate operators include [1, 3, 4, 10, 11].

Our two main results, the sharp spectral multiplier and the corresponding optimal results for convergence of Bochner-Riesz means, are stated in Theorems 1.1 and 1.2 below. Let \( \eta \) be a non-trivial \( C_\infty^\infty \) function with compact support on \( \mathbb{R}^+ \). For function \( F : \mathbb{R} \to \mathbb{C} \) we define \( \delta_t F(x) = F(tx) \) and set \( D = \max\{d_1 + d_2, 3d_2/2\} \). By \( W^s_2 \) we denote \( L^2 \) Sobolev space that is \( \|F\|_{W^s_2} = \|(I - d_2^2)^{s/2}F\|_2 \).

**Theorem 1.1.** Suppose that function \( F : \mathbb{R} \to \mathbb{C} \) satisfies 
\[
\sup_{t > 0} \|\eta \delta_t F\|_{W^s_2} < \infty
\]
for some \( s > D/2 \). Then the operator \( F(L) \) is of weak type \((1, 1)\) and bounded on \( L^p(\mathbb{R}^d_1 \times \mathbb{R}^d_2) \) for all \( p \in (1, \infty) \). In addition
\[
\|F(L)\|_{L^1 -> L^1,w} \leq C \sup_{t > 0} \|\eta \delta_t F\|_{W^s_2} \quad \text{and} \quad \|F(L)\|_{L^p -> L^p} \leq C_p \sup_{t > 0} \|\eta \delta_t F\|_{W^s_2}.
\]

The above result is sharp if \( d_1 \geq d_2/2 \), see discussion in Section 5 below. A version of result essentially equivalent to Theorem 1.1 can be expressed in terms of Bochner-Riesz summability of the operator \( L \). Our approach allows us to obtain the following result which is again optimal if \( d_1 \geq d_2/2 \).

**Theorem 1.2.** Suppose that \( \kappa > (D - 1)/2 \) and \( p \in [1, \infty) \). Then the Bochner-Riesz means \( (1 - tL)^\kappa_+ \) are bounded on \( L^p(\mathbb{R}^d_1 \times \mathbb{R}^d_2) \) uniformly in \( t \in [0, \infty) \).

Proofs of Theorems 1.1 and 1.2 are concluded in Section 4. Similarly as in [13] the key point of proving Theorems 1.1 and 1.2 is to obtain “weighted Plancherel estimate” for spectral multipliers of the considered Grushin type operators. A proof of such estimates is described in Section 3 and constitutes a main original contribution of this paper to the discussed research area. A part of a proof of Theorems 1.1 and 1.2 described in Section 4 below is essentially the same as in [13]. We repeat the short argument here for the sake of completeness. To make it easier to compare the results obtained in [13] and in this paper we try to use the same notation as in [13] whenever it is possible.

2. Notation and preliminaries

A more general class of Grushin type operators which includes operators \( L_\sigma \) for \( \sigma > 0 \) defined above was studied in [15]. In what follows we will need the basic results concerning the Riemannian distance corresponding to Grushin type operators and the standard Gaussian bounds for the corresponding heat kernels which were obtain in [15] and which we recall below.
Proposition 2.1. Let \( \rho \) be Riemannian distance corresponding to the Grushin operator \( L \) and let \( B(x, r) \) be the ball with centre at \( x \) and radius \( r \). Then

\[
\rho(x, y) \sim |x' - y'| + \begin{cases} \\
|\frac{|x'' - y''|}{(|x'| - |y'|)^{3/2}}| & \text{if } |x'' - y''| \leq (|x'| + |y'|)^{3/2}, \\
|\frac{|x'' - y''|^2}{2 |x'|} | & \text{if } |x'' - y''| \geq (|x'| + |y'|)^{3/2}.
\end{cases}
\]

Moreover the volume of \( B(x, r) \) satisfies following estimates

\[
|B(x, r)| \sim r^{d_1+d_2} \max\{r, |x'|\}^{d_2/2},
\]

and in particular, for all \( \lambda \geq 0 \),

\[
|B(x, \lambda r)| \leq C (1 + \lambda)^2 |B(x, r)|
\]

where \( Q = d_1 + \frac{3d_2}{2} \) is a homogenous dimension of the considered metric space. Next, there exist constants \( b, C > 0 \) such that, for all \( t > 0 \), the integral kernel \( p_t \) of the operator \( \exp(-tL) \) satisfies the following Gaussian bounds

\[
|p_t(x, y)| \leq C |B(y, t^{1/2})|^{-1} e^{-b\rho(x,y)^2/t}
\]

for all \( x, y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \).

Proof. For the proof, we refer readers to [15, Proposition 5.1 and Corollary 6.6].

Next, let \( \mathcal{F} : L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \rightarrow L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \) be the partial Fourier transform in variables \( x'' \) defined by

\[
\mathcal{F} \phi(x', \xi) = (2\pi)^{-d_2/2} \int_{\mathbb{R}^{d_2}} \phi(x', x'') e^{-i \xi' \cdot x''} \, dx''.
\]

Then

\[
\mathcal{F} L \phi(x', \xi) = \tilde{L}_\xi \mathcal{F} \phi(x', \xi)
\]

where \( \tilde{L}_\xi \) is Schrödinger type operators defined by

\[
\tilde{L}_\xi = -\Delta_{d_1} + \left( \sum_{j=1}^{d_2} |x'_j| \right) |\xi|^2
\]

acting on \( L^2(\mathbb{R}^{d_1}) \) where \( \xi \in \mathbb{R}^{d_2} \). In what follows we will need the following estimates for the operator \( \tilde{L}_\xi \), compare [9, 41, 3] and [13].

Proposition 2.2. For all \( \gamma \in [0, \infty) \) and \( f \in L^2(\mathbb{R}^{d_1}) \),

\[
\| \left( \sum_{j=1}^{d_1} |x'_j|^\gamma |\xi|^2 \right) f \|_2 \leq C_\gamma \| \tilde{L}_\xi f \|_2.
\]

Proof. Set \( \tilde{L} = -\Delta_{d_1} + \sum_{j=1}^{d_1} |x'_j| \) and next define operator \( L_{x'_i} \) by the following formula

\[
L_{x'_i} = -\partial^2_{x'_i} + |x'_i|.
\]

By Proposition 3.4 of [7]

\[
\| |x'_i|^k f \|_2 \leq C_k \| L_{x'_i} f \|_2
\]

\[
\| |x'_i|^k f \|_2 \leq C_k \| L_{x'_i} f \|_2
\]
for all positive natural numbers $k \in \mathbb{N}$. Hence
\[
\| (\sum_{j=1}^{d_1} |x_j'|)^k f \|_2^2 \leq C \sum_{j=1}^{d_1} \| (x_j')^k f \|_2^2 \leq C_k \sum_{j=1}^{d_1} \| L_{x_j'}^k f \|_2^2.
\]

Note that all $L_{x_j'}$ are non-negative self-adjoint operators and commute strongly, that is, their resolvent commute. Therefore for all $\ell_i \in \mathbb{Z}_+$, operators $\prod_{i=1}^{n} L_{x_j'}^{\ell_i}$ are self-adjoint and non-negative. Hence
\[
\sum_{j=1}^{d_1} L_{x_j'}^{2k} \leq (\sum_{j=1}^{d_1} L_{x_j'}^{2k})^{2k}
\]
for all $k \in \mathbb{N}$ and
\[
\| (\sum_{j=1}^{d_1} |x_j'|)^k f \|_2^2 \leq C_k \sum_{j=1}^{d_1} \| L_{x_j'}^k f \|_2^2 = C_k \langle \sum_{j=1}^{d_1} L_{x_j'}^{2k} f, f \rangle \leq C_k \| L_{x_j'}^{2k} f, f \|_2^2.
\]

Next, for a function $f \in C_c^\infty(\mathbb{R}^{d_1})$ we define function $\delta_t f$ by the formula $\delta_t f(x) = f(tx)$. Note that if $t = |\xi|^{-2/3}$ then
\[
\tilde{L}_k = \left( -\Delta_{d_1} + \left( \sum_{j=1}^{d_1} |x_j'| \right) t^{-3} \right)^k = t^{-2k} \delta_t \tilde{L}_k \delta_t.
\]

Hence
\[
\| \tilde{L}_k f \|_2^2 = \| t^{-2k} \delta_t \tilde{L}_k \delta_t f \|_2^2 = t^{-2k} \delta_t \| \tilde{L}_k f \|_2^2 \geq C_k^\prime \| t^{-2k} \delta_t \|_2^2 \| \sum_{j=1}^{d_1} |x_j'|^k \delta_t f \|_2^2 = C_k^\prime |\xi|^{2k} \| \sum_{j=1}^{d_1} |x_j'|^k f \|_2^2.
\]

This proves Proposition 2.2 for all $\gamma = k \in \mathbb{N}$. Now in virtue of Löwner-Heinz inequality (see, e.g., [2, Section I.5]) we can extend these estimates to all $\gamma \in [0, \infty)$.

3. Crucial estimates

To be able to obtain a required description of spectral decomposition of the operators $\tilde{L}_\xi$ we need the following properties of spectral decomposition of operator $A = -\frac{d^2}{dx^2} + |x|$ acting on $L^2(\mathbb{R})$ and which are essentially based on results from [7].

**Proposition 3.1.** Let $\lambda_n$ and $h_n$ be the $n$-th eigenvalue and normalized eigenfunction of the operator $A = -\frac{d^2}{dx^2} + |x|$. Then its spectral decomposition satisfies following properties:

(i) The operator $A$ has only a pointwise spectrum and its eigenvalues belong to $(1, \infty)$. In particular the first eigenvalue is larger than 1.

(ii) Every eigenvalue of $A$ is simple and the only point of accumulation of the eigenvalue sequence is $\infty$. Thus $\{h_n\}_{n \in \mathbb{N}}$ is a complete orthonormal system of $L^2(\mathbb{R})$. 

(iii) The eigenvalues $\lambda_n$ satisfy the following estimates:

\begin{equation}
C_1 \left( \frac{3\pi}{4} n \right)^{2/3} \leq \lambda_n \leq C_2 \left( \frac{3\pi}{4} n \right)^{2/3},
\end{equation}

\begin{equation}
\frac{\pi}{2} \lambda_{n+1}^{-1/2} \leq \lambda_{n+1} - \lambda_n \leq \frac{\pi}{2} \lambda_n^{-1/2},
\end{equation}

where $C_2 \geq C_1 > 0$ are constants.

(iv) For the eigenfunction $h_n$ corresponding to the eigenvalue $\lambda_n$,

\begin{equation}
\begin{cases} 
C \lambda_n^{-\frac{1}{4}}(1 + |u| - \lambda_n)^{-\frac{1}{4}}, & u \in \mathbb{R}, \\
C \exp(-c|u|^\frac{1}{2}), & u \geq 2\lambda_n.
\end{cases}
\end{equation}

Proof. (i), (ii) and (iii) are just reformulation of Proposition 2.1, Corollary 2.2, Facts 2.3, 2.7 and 2.8 of [7]. (iv) is an easy consequence of Theorem 2.6 of [7] and estimates for Airy function (see for example [8], pp. 213-215). \hfill \Box

Now we are able to describe spectral resolutions of Grushin operator $L = L_1$ and operators $\tilde{L}_\xi$ defined in Section 2. It is interesting to compare it with spectral decomposition of the operator $L_2$ obtained in [13]. Spectral decompositions of $L_1$ and $L_2$ are significantly different even though they share many common features. We also investigate integral kernels of spectral multipliers of $L$ and $\tilde{L}_\xi$. For $T = F(L)$ or $T = F(\tilde{L}_\xi)$, by $K_T$ we denote the integral kernel of the operator $T$, defined by the identity

$$Tf(x) = \int_X K_T(x, y) f(y) dy$$

where $X = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ for $L$ and $X = \mathbb{R}^{d_1}$ for $\tilde{L}_\xi$.

In terms of the eigenvalues and eigenfunctions of the operator $A = -\frac{d^2}{dx^2} + |x|$, one can obtain explicit formula for the integral kernel of the operator $F(L)$, compare also [13, Proposition 5]. Let $\lambda_n$ and $h_n$ be the $n$-th eigenvalue and eigenfunction of the operator $-\frac{d^2}{dx^2} + |x|$ on $L^2(\mathbb{R})$. We know that $\{h_n\}_{n \in \mathbb{N}}$ is a complete orthonormal system of $L^2(\mathbb{R})$. For all positive integers $d_1$, all $n \in \mathbb{N}^{d_1}$ and all $\xi \in \mathbb{R}^{d_2}$, we define function $\tilde{h}_{d_1, n} : \mathbb{R}^{d_1} \rightarrow \mathbb{R}$ by the formula

$$\tilde{h}_{d_1, n}(x', \xi) = |\xi|^{d_1/3} h_{n_1}(|\xi|^{2/3} x'_1) \cdots h_{n_{d_1}}(|\xi|^{2/3} x'_{d_1}).$$

We are now able to describe the kernel $K_{F(L)}$.

**Proposition 3.2.** For all bounded compactly supported Borel functions $F : \mathbb{R} \rightarrow \mathbb{C}$

$$K_{F(L)}(x, y) = (2\pi)^{-d_2} \int_{\mathbb{R}^{d_2}} K_{F(\tilde{L}_\xi)}(x', y') e^{i\xi(x'' - y'')} d\xi$$

$$= (2\pi)^{-d_2} \int_{\mathbb{R}^{d_2}} \sum_{n \in \mathbb{N}^{d_1}} F(\sum_{i=1}^{d_1} |\xi|^{\frac{1}{4}} \lambda_{n_i}) \tilde{h}_{d_1, n}(y', \xi) \tilde{h}_{d_1, n}(x', \xi) e^{i\xi(x'' - y'')} d\xi$$

for almost all $x = (x', x''), y = (y', y'') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. 

Proof. We noticed in Section 2 that $\mathcal{F} L \phi(x', \xi) = \tilde{L}_\xi \mathcal{F} \phi(x', \xi)$ where $\mathcal{F}$ is the partial Fourier transform in variables $x'$. Next note that for all $\xi \neq 0$

$$\tilde{L}_\xi \tilde{h}_{d_1, n}(x', \xi) = \left( \sum_{j=1}^{d_1} |\xi|^j \lambda_{n_j} \right) \tilde{h}_{d_1, n}(x', \xi).$$

Moreover by Proposition 3.1 (ii), the set $\{ \tilde{h}_{d_1, n}(x', \xi) \}_{n \in \mathbb{N}^{d_1}}$ is a complete orthonormal system of $L^2(\mathbb{R}^{d_1})$. Hence if $\mathcal{G} : L^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \to L^2(\mathbb{N}^{d_1} \times \mathbb{R}^{d_2})$ is the isometry defined by

$$\mathcal{G} \psi(n, \xi) = \int_{\mathbb{R}^{d_1}} \psi(x', \xi) \tilde{h}_{d_1, n}(x', \xi) \, dx',$$

then

$$\mathcal{G} \mathcal{F} L \phi(n, \xi) = \sum_{j=1}^{d_1} |\xi|^j \lambda_{n_j} \mathcal{G} \mathcal{F} \phi(n, \xi)$$

and

$$\mathcal{G} \mathcal{F} F(L) \phi(n, \xi) = F(\sum_{j=1}^{d_1} |\xi|^j \lambda_{n_j}) \mathcal{G} \mathcal{F} \phi(n, \xi).$$

However the inverse of $\mathcal{G}$ is given by

$$\mathcal{G}^{-1} \varphi(x', \xi) = \sum_{n \in \mathbb{N}^{d_1}} \varphi(n, \xi) \tilde{h}_{d_1, n}(x', \xi).$$

and inverse of $\mathcal{F}$ can be expressed in terms of partial inverse Fourier transform in $x''$. Applying $\mathcal{G}^{-1}$ and $\mathcal{F}^{-1}$ to both sides of equality (10) shows Proposition 3.2. \qed

Next, for all positive integers $d_1$ and all $n = (n_1, \ldots, n_{d_1}) \in \mathbb{N}^{d_1}$ we define function $H_{d_1, n} : \mathbb{R}^{d_1} \to \mathbb{R}$ by the formula

$$H_{d_1, n}(x') = h_{n_1}^2(x_1') \cdots h_{n_{d_1}}^2(x_{d_1}')$$

As a simple consequence of Proposition 3.2 we obtain following estimates.

Proposition 3.3. For all $\gamma \geq 0$ and for every compactly supported bounded Borel function $F : \mathbb{R} \to \mathbb{C}$,

$$\left\| \left( \sum_{i=1}^{d_1} |x_i'| \right)^\gamma K_{F(L)}(\cdot, y) \right\|_2^2 \leq C_\gamma \int_0^\infty |F(\theta)|^2 \sum_{n \in \mathbb{N}^{d_1}} \theta^{Q/2-\gamma} \frac{\lambda_{n_{Q/2-3\gamma}} h_{d_1, n} \left( \frac{\theta^{1/2} y'}{N_n^{1/2}} \right) \, d\theta}{N_n^{1/2}}$$

for almost all $y = (y', y'') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ where $N_n = \sum_{i=1}^{d_1} \lambda_{n_i}$ and $\lambda_{n_i}$ is the eigenvalue corresponding to eigenfunction $h_{n_i}$.

Proof. By Propositions 2.2 and 3.2

$$\left\| \left( \sum_{i=1}^{d_1} |x_i'| \right)^\gamma K_{F(L)}(\cdot, y) \right\|_2^2 = \int_{\mathbb{R}^{d_2}} \left\| \left( \sum_{i=1}^{d_1} |x_i'| \right)^\gamma K_{F(L)}(x', y') \right\|_2^2 \, d\xi$$

$$\leq \int_{\mathbb{R}^{d_2}} |\xi|^{-4\gamma} \left\| \tilde{L}_\xi K_{F(L)}(x', y') \right\|_{L^2(\mathbb{R}^{d_1})}^2 \, d\xi.$$

Next note that for all $\gamma \geq 0$ and $y' \in \mathbb{R}^{d_1}$

$$\tilde{L}_\xi \left( K_{F(L)}(\cdot, y') \right) = K_{\tilde{L}_\xi F(L)}(\cdot, y')$$
Hence
\[
\|\tilde{L}\xi K_{\tilde{F}(\xi)}(x', y')\|_{L^2(\mathbb{R}^{d_1})}^2 \leq \|K_{\tilde{L}\xi F(\xi)}(x', y')\|_{L^2(\mathbb{R}^{d_1})}^2
\]
\[
\leq \sum_{n \in \mathbb{N}^{d_1}} \left| \left( \sum_{i=1}^{d_1} |\xi|^{\frac{d}{2}} \lambda_n \right)^2 F(\sum_{i=1}^{d_1} |\xi|^{\frac{d}{2}} \lambda_n) \right| |\tilde{h}_{d_1,n}(y', \xi)|^2
\]
\[
\leq C|\xi|^\frac{2d_1 + 8\varepsilon}{3} \sum_{n \in \mathbb{N}^{d_1}} N_n^{2\gamma} |F(|\xi|^{\frac{d}{2}} N_n)|^2 H_{d_1,n}(|\xi|^{\frac{d}{2}} y').
\]
(12)

Now substituting (12) to (11) and simple change of variables proves Proposition 3.3 □

The following lemma is a version of Lemma 9 of [13]. However the proof is more complex and requires a new approach especially when \(d_1 \geq 2\). It is the most essential part of the proof of our main spectral multiplier results.

**Lemma 3.4.** For all \(\varepsilon > 0\) there exists a constant \(C > 0\) which does not depend on \(x' \in \mathbb{R}^{d_1}\) such that

\[
\sum_{n \in \mathbb{N}^{d_1}} \max\{1, |x'|\}^{2\varepsilon} \frac{N_n^{d_1/2 + 3\varepsilon}}{N_n^{1/2}} H_{d_1,n} \left( \frac{x'}{N_n^{1/2}} \right) < C < \infty
\]

where \(N_n = \sum_{i=1}^{d_1} \lambda_{n_i}\) and \(\lambda_{n_i}\) is the eigenvalue corresponding to eigenfunction \(h_{n_i}\).

**Proof.** We split the sum into two parts,

\[
\sum_{n \in \mathbb{N}^{d_1}} \max\{1, |x'|\}^{2\varepsilon} \frac{N_n^{d_1/2 + 3\varepsilon}}{N_n^{1/2}} H_{d_1,n} \left( \frac{x'}{N_n^{1/2}} \right)
\]

\[
\leq \left( \sum_{N_n^{3/2} \leq |x'|/(2d_1)} + \sum_{N_n^{3/2} > |x'|/(2d_1)} \right) \max\{1, |x'|\}^{2\varepsilon} \frac{N_n^{d_1/2 + 3\varepsilon}}{N_n^{1/2}} H_{d_1,n} \left( \frac{x'}{N_n^{1/2}} \right).
\]

**Part 1:** \(N_n^{3/2} \leq |x'|/(2d_1)\). By Proposition 3.1 \(\lambda_{n_i} \geq 1\) so \(N_n > 1\). Hence this part is empty unless \(|x'| > 1\). Note that

\[
\frac{|x'|}{N_n^{1/2}} \geq \frac{|x'|}{d_1 N_n^{1/2}} \geq 2N_n
\]

where \(|x'| = \max\{x'_1, \cdots, x'_{d_1}\}\). By (9) for every natural number \(N \leq |x'|/(2d_1)\)

\[
\sum_{(N-1)^{2/3} < N_n \leq N^{2/3}} H_{d_1,n} \left( \frac{x'}{N_n^{1/2}} \right) \leq C \exp(-c|x'|^{\frac{d}{2}}/N^{\frac{d}{2}}) \leq C \exp(-c|x'|^{\frac{d}{2}}/N^{\frac{d}{2}}).
\]
Thus
\[
\sum_{N_{0,n}^{3/2} \leq |x'|/(2d_1)} \max \{1, |x'|\}^{2\varepsilon} \frac{N_{d_1/2+3\varepsilon}}{N_n^{d_1/2+3\varepsilon}} \frac{H_{d_1,n}}{N_n^{1/2}} \left( \frac{x'}{N_n^{1/2}} \right)
\]
\[
\leq \sum_{N \leq |x'|/(2d_1)} \sum_{(N-1)^{2/3} \leq N \leq N^{2/3}} \max \{1, |x'|\}^{2\varepsilon} \frac{N_{d_1/2+3\varepsilon}}{N_n^{d_1/2+3\varepsilon}} \frac{H_{d_1,n}}{N_n^{1/2}} \left( \frac{x'}{N_n^{1/2}} \right)
\]
\[
\leq C \sum_{N \leq |x'|/2} |x'|^{2\varepsilon} N^{-d_1/3-2\varepsilon} \exp(-c|x'|^3/N^2)
\]
(15)
\[
\leq C \sum_{N \leq t \geq 2N} \sup_{t \geq 2N} t^{4\varepsilon/3} \exp(-ct) \leq C.
\]

Part 2: $N_{0,n}^{3/2} > |x'|/(2d_1)$. Again by (9)
\[
H_{d_1,n} \left( \frac{x'}{N_n^{1/2}} \right) = \prod_{i=1}^{d_1} h_{n_i} \left( \frac{x_i'}{N_n^{1/2}} \right)
\]
\[
\leq C \prod_{i=1}^{d_1} \lambda_{n_i}^{-\frac{1}{2}} \left( 1 + \frac{|x_i'|}{N_n^{1/2} - \lambda_{n_i}} \right)^{-\frac{1}{2}}.
\]
Hence
\[
\sum_{N_{0,n}^{3/2} > |x'|/(2d_1)} \max \{1, |x'|\}^{2\varepsilon} \frac{N_n^{d_1/2+3\varepsilon}}{N_n^{d_1/2+3\varepsilon}} \frac{H_{d_1,n}}{N_n^{1/2}} \left( \frac{x'}{N_n^{1/2}} \right)
\]
\[
\leq C \sum_{N_{0,n}^{3/2} > |x'|/(2d_1)} \max \{1, |x'|\}^{2\varepsilon} \frac{N_n^{d_1/2+3\varepsilon}}{N_n^{d_1/2+3\varepsilon}} \prod_{i=1}^{d_1} \lambda_{n_i}^{-\frac{1}{2}} \left( 1 + \frac{|x_i'|}{N_n^{1/2} - \lambda_{n_i}} \right)^{-\frac{1}{2}}.
\]
(16)

Next, define function $g: [1, \infty)^{d_1} \to \mathbb{R}_+$ by the formula
\[
g(\mu) = g(\mu_1, \ldots, \mu_{d_1}) = \frac{\max \{1, |x'|\}^{2\varepsilon}}{N_{\mu}^{d_1/2+3\varepsilon}} \prod_{i=1}^{d_1} \mu_i^{-\frac{1}{2}} \left( 1 + \frac{|x_i'|}{N_{\mu}^{1/2} - \mu_i^{2/3}} \right)^{-\frac{1}{2}}
\]
where $N_{\mu} = \sum_{i=1}^{d_1} \mu_i^{2/3}$. Note that $g(\mu_1, \ldots, \mu_{d_1}) > 0$ and there exists a constant $C > 0$ such that
\[
|\nabla g(\mu_1, \ldots, \mu_{d_1})| \leq C g(\mu_1, \ldots, \mu_{d_1})
\]
when $\mu = (\mu_1, \ldots, \mu_{d_1}) \in [1, \infty)^{d_1}$ and $N_{\mu} = \sum_{i=1}^{d_1} \mu_i^{2/3} \geq (|x'|/(2d_1))^{2/3}$. By the above estimate for the gradient of $g$
\[
e^{-C|\mu - \bar{\mu}|} \leq \left| \frac{g(\mu)}{g(\bar{\mu})} \right| \leq e^{C|\mu - \bar{\mu}|}
\]
for all $\mu, \bar{\mu}$ in the region described above. Hence
\[
g(\mu_1, \ldots, \mu_{d_1}) \leq C \int_{\prod_{i=1}^{d_1} [\mu_i, \mu_i+1]} g(\xi_1, \ldots, \xi_{d_1}) d\xi_1 \ldots d\xi_{d_1}.
\]
(17)
Set $\mu_n = \lambda_n^{3/2}$. By (16),

$$\sum_{N_n^{3/2} > |x'|/(2d_1)} \max\{1, |x'|\}^{2\varepsilon} \frac{2\varepsilon}{N_n^{d_1/2+3\varepsilon}} H_{d_1,n} \left( \frac{x'}{N_n^{1/2}} \right) \leq \sum_{N_n^{3/2} > |x'|/(2d_1)} g(\mu_{n_1}, \ldots, \mu_{n_{d_1}}).$$

However, by (8) and mean value theorem for each $1 \leq i \leq d_1$,

$$\mu_n - \mu_{n-1} = \lambda_n^{3/2} - \lambda_{n-1}^{3/2} \geq \frac{3\pi}{4} \lambda_n^{-1/2} \lambda_{n-1}^{1/2} \geq \frac{3\pi}{4} \left( \frac{\lambda_n^{3/2}}{\pi^2 + \lambda_n^{3/2}} \right)^{1/2} \geq \frac{3\pi}{8} > 1$$

which means that for all $n \in \mathbb{N}^{d_1}$, cubes $\prod_{i=1}^{d_1} [\mu_n, \mu_n + 1]$ are mutually disjoint. Note again that by Proposition 3.1 $\lambda_n \geq 1$ so $N_n > 1$. Hence by (17), (18) and (19)

$$\sum_{N_n^{3/2} > |x'|/(2d_1)} \max\{1, |x'|\}^{2\varepsilon} \frac{2\varepsilon}{N_n^{d_1/2+3\varepsilon}} H_{d_1,n} \left( \frac{x'}{N_n^{1/2}} \right) \leq C \int_{N_n > \max\{(|x'|/2d_1)^{2/3}, 1\}} g(\mu_1, \ldots, \mu_{d_1}) d\mu_1 \ldots d\mu_{d_1}.$$ 

Using the changes of variables $\mu_i = \xi_i^{3/2}$ we get

$$\sum_{N_n^{3/2} > |x'|/(2d_1)} \max\{1, |x'|\}^{2\varepsilon} \frac{2\varepsilon}{N_n^{d_1/2+3\varepsilon}} H_{d_1,n} \left( \frac{x'}{N_n^{1/2}} \right) \leq C \int_{N_\xi > \max\{(|x'|/2d_1)^{2/3}, 1\}} \left[ \max\{1, |x'|\}^{2\varepsilon} \frac{2\varepsilon}{N_\xi^{d_1/2+3\varepsilon}} \prod_{i=1}^{d_1} \xi_i^{1/2} \left| |x'_i| - \xi_i N_\xi^{1/2} \right|^{-\frac{1}{2}} \right] d\xi_1^{3/4} \ldots d\xi_{d_1}^{3/4}$$

(20) $$\leq C \int_{N_\xi > \max\{(|x'|/2d_1)^{2/3}, 1\}} \left[ \max\{1, |x'|\}^{2\varepsilon} \prod_{i=1}^{d_1} \left| |x'_i| - \xi_i N_\xi^{1/2} \right|^{-\frac{1}{2}} \right] d\xi_1 \ldots d\xi_{d_1} = I$$

where $N_\xi = \sum_{i=1}^{d_1} \xi_i$. To estimate this integral we use the following decomposition

$$\{\xi : N_\xi \geq \max\{(|x'|/2d_1)^{2/3}, 1\}\} = \bigcup_{j=1}^{d_1} E_j = \bigcup_{j=1}^{d_1} \{\xi : N_\xi \geq \max\{(|x'|/2d_1)^{2/3}, 1\}, N_\xi/d_1 \leq \xi_j \leq N_\xi\}.$$ 

Now on each of set $E_j$ we introduce new coordinates

$$\nu_1 = \xi_1, \ldots, \nu_{j-1} = \xi_{j-1}, \nu_j = N_\xi, \nu_{j+1} = \xi_{j+1}, \ldots, \nu_{d_1} = \xi_{d_1}.$$
Then
\[
I \leq C \sum_{j=1}^{d_i} \int_{\max\{|x'|/(2d_i)|^{2/3},1\}}^{\infty} \frac{\max\{1,|x'|\}^{2\varepsilon}}{\nu_j^{d_i/4+3\varepsilon}} \prod_{i \neq j} \left|\left| x_i' \right| - \nu_i \nu_j^{1/2} \right|^{-\frac{1}{2}} \left|\left| x_j' \right| - \nu_j \nu_j^{1/2} \right|^{-\frac{1}{2}} d\nu_1 \ldots d\nu_d
\]
(21)

where \( \nu_j = \nu_j - \sum_{i \neq j} \nu_i \) and \( S_j = \{ \nu : \nu_j/d_i \leq \nu_j \leq \nu_j, 0 \leq \nu_i \leq \nu_j, \forall i \neq j \} \).

Next we split the integral into two parts: \( \nu_j > \max\{(2d_i|x'|)^{2/3},1\} \) and \( (|x'|/(2d_i))^{2/3} \leq \nu_j \leq (2d_i|x'|)^{2/3} \). Note that if \( \nu_j \geq (2d_i|x'|)^{2/3} \) and \( \nu_j/d_i \leq \nu_j \leq \nu_j \) then
\[
\left|\left| x_j' \right| - \nu_j \nu_j^{1/2} \right|^{-\frac{1}{2}} \leq \nu_j^{-3/4}.
\]
Note also that there exists a constant \( C \) such that for all \( A, N > 0 \)
\[
\int_0^N \left| A - x \right|^{-1/2} dx \leq CN^{1/2}.
\]
Hence for \( \nu_j > \max\{(2d_i|x'|)^{2/3},1\} \),
\[
\int_{S_j} \prod_{i \neq j} \left|\left| x_i' \right| - \nu_i \nu_j^{1/2} \right|^{-\frac{1}{2}} \left|\left| x_j' \right| - \nu_j \nu_j^{1/2} \right|^{-\frac{1}{2}} d\nu_1 \ldots d\nu_j \ldots d\nu_d
\]
\[
\leq C \nu_j^{-3/4} \prod_{i \neq j} \left|\left| x_i' \right| - \nu_i \nu_j^{1/2} \right|^{-\frac{1}{2}} d\nu_i
\]
\[
\leq C \nu_j^{-3/4} \nu_j^{-d_i-1} \leq C \nu_j^{d_i/4-1}
\]
and
\[
\int_{\max\{(2d_i|x'|)^{2/3},1\}}^{\infty} \frac{\max\{1,|x'|\}^{2\varepsilon}}{\nu_j^{d_i/4+3\varepsilon}} \prod_{i \neq j} \left|\left| x_i' \right| - \nu_i \nu_j^{1/2} \right|^{-\frac{1}{2}} \left|\left| x_j' \right| - \nu_j \nu_j^{1/2} \right|^{-\frac{1}{2}} d\nu_1 \ldots d\nu_d
\]
\[
\leq C \int_{\max\{(2d_i|x'|)^{2/3},1\}}^{\infty} \frac{\max\{1,|x'|\}^{2\varepsilon}}{\nu_j^{d_i/4-1}} d\nu_j
\]
(22)
\[
\leq C \int_{\max\{(2d_i|x'|)^{2/3},1\}}^{\infty} \frac{\max\{1,|x'|\}^{2\varepsilon}}{\nu_j^{1+3\varepsilon}} d\nu_j \leq C.
\]

If we assume now that \( (|x'|/(2d_i))^{2/3} \leq \nu_j \leq (2d_i|x'|)^{2/3} \) then by the change of variables \( \nu_j^{1/2} = u_i \) one gets
\[
\int_{S_j} \prod_{i \neq j} \left|\left| x_i' \right| - \nu_i \nu_j^{1/2} \right|^{-\frac{1}{2}} \left|\left| x_j' \right| - \nu_j \nu_j^{1/2} \right|^{-\frac{1}{2}} d\nu_1 \ldots d\nu_j \ldots d\nu_d
\]
\[
\leq C \nu_j^{-d_i} \int_{0, \nu_j^{3/2}} \left|\left| x_i' \right| - u_i \right|^{-\frac{1}{2}} \left|\left| x_j' \right| \right| + \sum_{i \neq j} u_i - \nu_j^{1/2} \right|^{-\frac{1}{2}} du
\]
where \( du = du_1 \cdots du_{j-1} du_{j+1} \cdots du_{d_1} \). Hence,
\[
\int \frac{(2d_1 |x'|)^{2/3}}{|x'|/(2d_1)^{2/3}} \max \left\{ 1, \frac{|x'|}{\nu_j^{d_1/4+3c}} \right\} \prod_{i \neq j} |x'_i| - \nu_i \nu_j^{1/2} |x'_j| - \nu_j^{1/2} \, dv_1 \cdots dv_{d_1} \\
\leq C \int \frac{(2d_1 |x'|)^{2/3}}{|x'|/(2d_1)^{2/3}} \nu_j^{2/3} \prod_{i \neq j} |x'_i| - u_i |^{1/2} |x'_j| + \sum_{i \neq j} u_i - \nu_j^{3/2} |^{-1/2} \, du \, dv_j \\
\leq C \int \frac{(2d_1 |x'|)^{2/3}}{|x'|/(2d_1)^{2/3}} \nu_j^{2/3} \prod_{i \neq j} |x'_i| - u_i |^{1/2} \int_{0,2d_1 |x'|} \prod_{i \neq j} |x'_i| - u_i |^{1/2} \, dv_j \, du \\
\leq C \int |x'|^{2-3d_1/6} |x'|^{1/6} \prod_{i \neq j} 0 \int_{0,2d_1 |x'|} |x'_i| - u_i |^{-1/2} \, du_i \\
\leq C \int |x'|^{2-3d_1/6} |x'|^{1/6} |x'|^{(d_1-1)/2} \\
\leq C.
\]

Now (20), (21), (22) and the above estimates yield
\[
\sum_{N_0^{3/2} > |x'|/(2d_1)} \frac{\max \left\{ 1, \frac{|x'|}{\nu_j^{d_1/4+3c}} \right\}^{2\varepsilon}}{N_0^{d_1/2+3c}} H_{d_1,n} \left( \frac{x'}{N_0^{1/2}} \right) \leq C.
\]

Next, for all \( R > 0 \) we define the weight function \( w_R : (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})^2 \to \mathbb{R}_+ \) by the formula
\[
w_R(x, y) = \min \{ R, |y|^{-1} \} |x'|.
\]

The estimates obtained in this section can be summarised in the following proposition.

**Proposition 3.5.** For all \( \gamma \in [0, d_2/4) \) and all bounded compactly supported Borel functions \( F : \mathbb{R} \to \mathbb{C} \),
\[
\left\| \sum_{i=1}^{d_1} |x'_i|^{\gamma} K_{F(L)}(\cdot, y) \right\|_2 \leq C_\gamma \int_0^\infty |F(\lambda)|^2 \lambda^{(d_1+d_2)/2} \min \{ \lambda^{d_2/4 - \gamma}, |y'|^{2\gamma - d_2/2} \} \frac{d\lambda}{\lambda}
\]
for almost all \( y = (y', y'') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \). In particular, for all \( R > 0 \), if \( \text{supp} \, F \subseteq [R^2, 4R^2] \), then
\[
\text{ess sup}_{y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} |B(y, R^{-1})|^{1/2} \| w_R(\cdot, y) \gamma K_{F(L)}(\cdot, y) \|_2 \leq C_\gamma \| \delta_R F \|_{L^2},
\]
where the constant \( C_\gamma \) does not depend on \( R \).

**Proof.** We obtain the first inequality by Proposition 3.3 and Lemma 3.4 with \( \varepsilon = d_2/4 - \gamma \). Next if we assume that \( \text{supp} \, F \subseteq [R^2, 4R^2] \), then in virtue of the definition of the weight \( w_R \) and estimate (3), the first inequality implies the second one. \( \Box \)
4. The multiplier theorems

In the following section we show that Theorems \([1.1]\) and \([1.2]\) are straightforward consequence of Proposition \([3.3]\). The argument is essential the same as in Section 5 of \([13]\) with an obvious adjustment of exponents in some calculations and we quote it here for sake of completeness. An alternative proof based on the wave equation technique can be obtained by a simple modification of the proof of \([4]\) Lemma 3.4.

**Proposition 4.1.** For all \(R > 0\), \(\alpha \geq 0\), \(\beta > \alpha\), and for all functions \(F : \mathbb{R} \to \mathbb{C}\) such that supp\(F \subseteq [-4R^2, 4R^2]\),

\[
\text{ess sup}_{y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} |B(y, R^{-1})|^{1/2} \|(1 + R\beta(\cdot, y))^{\alpha} K_F(\cdot, y)\|_2 \leq C_{\alpha, \beta} \|\delta_R F\|_{W^2_{\beta}},
\]

where the constant \(C_{\alpha, \beta}\) does not depend on \(R\). If in addition \(\beta > \alpha + Q/2\), then

\[
\text{ess sup}_{y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \|(1 + R\beta(\cdot, y))^{\alpha} K_F(\cdot, y)\|_1 \leq C_{\alpha, \beta} \|\delta_R F\|_{W^1_{\beta}},
\]

where again \(C_{\alpha, \beta}\) does not depend on \(R\).

**Proof.** Note that the heat kernel of the operator \(L\) satisfies Gaussian bounds \([5]\) so Proposition \([1.1]\) is a straightforward consequence of \([6]\) Lemmas 4.3 and 4.4. \(\square\)

Recall that the homogeneous dimension of the ambient space is given by \(Q = d_1 + 3d_2/2\).

**Lemma 4.2.** Suppose that \(0 \leq \gamma < \min\{d_1/2, d_2/4\}\) and \(\beta > Q/2 - \gamma\). For all \(y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\) and \(R > 0\),

\[
\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} (1 + w_R(x, y))^{-2\gamma}(1 + R\beta(x, y))^{-2\beta} \, dx \leq C_{\gamma, \beta} |B(y, R^{-1})|.
\]

Moreover, for all \(x, y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\) and \(R > 0\),

\[
w_R(x, y) \leq C(1 + R\beta(x, y)).
\]

**Proof.** By the homogeneity properties of the distance \(\rho\) and the weights \(w_R\), we only prove the case \(R = 1\). For other case, one just dilate them by \(\delta_1(x', x'') = (tx', t^{3/2}x'').\) By \([2]\),

\[
\min\{1, |y'|^{-1}\}|x'| \leq 1 + |x' - y'| \leq C(1 + \rho(x, y)),
\]

which proves \([25]\).

Because of the translation invariance, to prove \([25]\), it is enough to consider the case \(y'' = 0\). By \([3]\) it suffices to show that

\[
\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \left(1 + \frac{|x' - y'|}{1 + |y'|}\right)^{-2\gamma}(1 + \rho(x, y))^{-2\beta} \, dx \leq C_{\gamma, \beta}(1 + |y'|)^{d_2/2}.
\]

Again we split the integral into two parts, according to the asymptotics \([2]\). In the region \(X_1 = \{x \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}: |x''| \geq (|x'| + |y'|)^{3/2}\}\), we choose \(\beta_1\) and \(\beta_2\) in such a way that \(\beta = \beta_1 + \beta_2\), \(\beta_1 > d_1/2 - \gamma\) and \(\beta_2 > 3d_2/4\). Then

\[
\int_{X_1} \left(1 + \frac{|x' - y'|}{1 + |y'|}\right)^{-2\gamma}(1 + \rho(x, y))^{-2\beta} \, dx 
\leq C(1 + |y'|)^{2\gamma} \int_{\mathbb{R}^{d_1}} (1 + |x' - y'|)^{-2(\gamma + \beta_1)} \, dx' \int_{\mathbb{R}^{d_2}} (1 + |x''|^{2/3})^{-2\beta_2} \, dx'' 
\leq C_{\gamma, \beta}(1 + |y'|)^{d_2/2}.
\]
In the region $X_2 = \{ x \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} : |x''| < (|x'| + |y'|)^{3/2} \}$, instead, we choose $\beta_1$ and $\beta_2$ in such a way $\beta = \beta_1 + \beta_2$, $\beta_1 > d_1/2 + d_2/4 - \gamma$ and $\beta_2 > d_2/2$. Then the integral over $X_2$ is estimated by

$$\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \left( 1 + \frac{|x' - y'|}{1 + |y'|} \right)^{-2\gamma} \left( 1 + |x' - y'| \right)^{-2\bar{\beta}_1} \left( 1 + \frac{|x''|}{(|x'| + |y'|)^{1/2}} \right)^{-2\bar{\beta}_2} \, dx$$

$$\leq C_{\gamma, \beta} \int_{\mathbb{R}^{d_1}} \left( 1 + \frac{|u|}{1 + |y'|} \right)^{-2\gamma} \left( 1 + |u| \right)^{-2\bar{\beta}_1} \left( |u + y'| + |y'| \right)^{d_2/2} \, du$$

$$\leq C_{\gamma, \beta} \left( (1 + |y'|)^{2\gamma} \int_{\mathbb{R}^{d_1}} (1 + |u|)^{-2\nu} \, du + |y'|^{d_2/2} \int_{\mathbb{R}^{d_1}} (1 + |u|)^{-2\bar{\beta}_1} \, du \right),$$

where $\nu = \bar{\beta}_1 + \gamma - d_2/4 > d_1/2$. The conclusion follows. \( \square \)

**Proposition 4.3.** For all $R > 0$, $\alpha \geq 0$, $\beta > \alpha$, $\gamma \in [0, d_2/4)$, and for all functions $F : \mathbb{R} \to \mathbb{C}$ such that $\text{supp } F \subseteq [R^2, 4R^2]$, \( \text{ess sup}_{y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} |B(y, R^{-1})|^{1/2} \|(1 + R \rho(\cdot, y))^{\alpha}(1 + w_R(\cdot, y))^{\gamma} K_{F(L)}(\cdot, y)\|_2 \leq C_{\alpha, \beta, \gamma} \delta_{R^2} F \|_{W_2^\beta}, \)  

where the constant $C_{\alpha, \beta, \gamma}$ does not depend on $R$. \( \square \)

**Proof.** The estimate (23), together with (26) and a Sobolev embedding, immediately implies Proposition 4.3 in the case $\beta > \alpha + d_2/2 + 1/2$. On the other hand, in the case $\alpha = 0$, Proposition 4.3 follows from Proposition 3.5 for all $\beta > 0$. We obtain now Proposition 4.3 for the whole range of exponents by interpolation (see [5] and also [6, Lemma 4.3] for similar methods). \( \square \)

For the purpose of the next statement we set $D = Q - \min\{d_1, d_2/2\} = \max\{d_1 + d_2, 3d_2/2\}$. \( \square \)

**Corollary 4.4.** For all $R > 0$, $\alpha \geq 0$, $\beta > \alpha + D/2$, and for all functions $F : \mathbb{R} \to \mathbb{C}$ such that $\text{supp } F \subseteq [R^2, 4R^2]$, \( \text{ess sup}_{y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \|(1 + R \rho(\cdot, y))^{\alpha} K_{F(L)}(\cdot, y)\|_1 \leq C_{\alpha, \beta} \delta_{R^2} F \|_{W_2^\beta}, \)  

where the constant $C_{\alpha, \beta}$ does not depend on $R$. In particular, under the same hypotheses, \( \text{ess sup}_{y \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \setminus B(y, r)} \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} |K_{F(L)}(x, y)| \, dx \leq C_{\alpha, \beta} (1 + rR)^{-\alpha} \delta_{R^2} F \|_{W_2^\beta}. \)  

**Proof.** Corollary 4.4 follows from Proposition 1.3 together with (25) and Hölder's inequality. \( \square \)

We are finally able to prove our main results. \( \square \)

**Proofs of Theorems 1.1 and 1.2.** To prove Theorem 1.1 we can follow the lines of the proof of [6, Theorem 3.1], where the inequality (4.18) there is replaced by our (28). Next we can use that same argument as in [6, Section 6] to conclude the proof of Theorem 1.2 see also [13]. \( \square \)
5. Final remarks

The natural open problem related to the sharp spectral multiplier results which we prove in this paper is to extend them to the class of all operators $L_\sigma$ defined by (5) for $\sigma > 0$. Another interesting problem which arises is to obtain possible precise description of the spectral decompositions of operators $L_\sigma$.

Now we shall show that, if $d_1 \geq d_2 / 2$, then the result in Theorem 1.1 is sharp. More precisely, if $d_1 \geq d_2 / 2$ and $s < D / 2 = (d_1 + d_2) / 2$, then the weak type $(1, 1)$ estimates in Theorem 1.1 cannot hold. Indeed, if we consider the functions $H_t(\lambda) = \lambda^it$, then, for $t > 1$, and any $\eta \in C^\infty_c(\mathbb{R}_+)$

$$\|\eta H_t\|_{W^s_2} \sim t^s.$$  

On the other hand, we make the following observation.

**Proposition 5.1.** Suppose that $L$ is the Grushin operator acting on $X = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then the following lower bounds holds:

$$\|H_t(L)\|_{L^1 \rightarrow L^1, w} = \|L^it\|_{L^1 \rightarrow L^1, w} \geq C(1 + |t|)^{(d_1 + d_2)/2}$$

for all $t > 0$.

**Proof.** Because the Grushin operator is elliptic on $X_0 = \{x \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} : x' \neq 0\}$, one can use the same argument as in [16] to prove that, for all $y \in X_0$,

$$|p_t(x, y) - |y'|^{-d_2}(4\pi t)^{-(d_1 + d_2)/2}e^{-(x, y)^2/4t}| \leq Ct^{1/2}t^{-(d_1 + d_2)/2}$$

for all $x$ in a small neighbourhood of $y$ and all $t \in (0, 1)$. Here $p_t = K_{\exp(-tL)}$ is the heat kernel corresponding to the Grushin operator. The rest of the argument is the same as in [16], so we skip it here. \qed

To show that Theorems 1.1 and 1.2 are sharp one can also use the results described in [12].

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