Natural vibration of stepped nanoplate with crack on an elastic foundation

M Hossain and J Lellep
Institute of Mathematics and Statistics, University of Tartu, Tartu, Estonia
E-mail: mainul.hossain@ut.ee

Abstract. The small scale effect on the vibrational characteristic of isotropic, rectangular nanoplate embedded in an elastic medium is investigated. The formulation is based on the plate theory on aggregate with the nonlocal elasticity theory. The solution procedure is derived using the governing differential equations of physical phase that are converted into set of linear algebraic equations. Latter these are solved by computer code. The effects of aspect ratio, step, crack and rotatory inertia on the different modal vibrations of nanoplate are explored. The results show the significant effect of different physical and geometrical parameters on the vibration of nanoplate.

1. Introduction
Nanoplate is a two dimensional structural form of nanomaterial which is commonly used as a component in nano-electromechanical systems [1]. At nanoscale, the laws of physics operate in an unfamiliar way because of two important reasons: high surface-to-volume ratio and atomic attraction. When the size of building blocks gets smaller, the surface area of the material increases by six orders of magnitude [2]. To encounter these effects non local theory of elasticity [3] is very popular among the researchers.

It is obvious, cracks or cracks like defects form on the plate during the processes of manufacturing. The cracks are modelled by a massless rotational springs that connect two neighbouring segments together. On the other hand, steps are the sudden changes in geometry or height of the nanoplate. Sometime designer fabricates stepped plate for using material effectively. In addition, rotatory inertia is the ability to resist the change of rotational speed on a specific axis. The effect of rotatory inertia is small on natural frequency. When the natural frequency is very large, the effect of rotatory inertia is significant. It is reasonable to assume that the elastic foundation is formed by means of identical, independent, closely spaced, discrete and linearly elastic springs. Elastic foundation model is developed by Winkler. According to his assumption, the vertical displacement of a point of the elastic foundation is proportional to the pressure at that point and does not depend on the pressure at the adjacent points.

There are only a few of studies on the vibration of nanoplate on an elastic foundation considering the effect of steps, cracks and rotatory inertia. Whereas, some researchers showed their interest on nanoplate on elastic foundation. Daneshmehr et al.[4], Ansari et al.[5], Sobhy[6] investigated the free vibrations of the functionally graded nanoplate on an elastic foundation. They showed the frequency ratio decreases with the increase in size scale parameter value. They also analyzed elastic foundation stiffness, plate aspect ratio and side-to-thickness ratio on the behavior of nanoplates. In addition, Poure saga eli et al. [7], Murmu and Adhikari [8], T Natsuki and J Natsuki [9] analyzed the vibration of double nanoplates on an elastic foundation. They presented analytical scale-based nonlocal approach which is useful for n-nanoplates system of graphene based nanocomposites. The influences of small scale coefficient, stiffness of the external and internal mediums and aspect ratio on the frequencies of the double
nanoplates were also elucidated. Furthermore, Ruocco and Mallardo [10] investigated the buckling and vibration behaviour of imperfect nanoplates via nonlocal Mindlin plate theory. Their numerical results showed that the influence of the imperfection on the buckling load depends on the boundary conditions. Similarly, Ebrahimi et al. [11] presented magneto-electro-elastic heterogeneous porous material plates resting on elastic foundations. They discussed the effect of material graduation exponent, porosity volume fraction, magnetic potential, electric voltage, various boundary conditions, elastic foundation parameters and plate side-to-thickness ratio on natural frequencies.

In this paper, a modified Euler-Bernoulli plate theory and Eringen nonlocal theory of elasticity are used. The main purpose of this paper is to discuss the effect of aspect ratio, steps, cracks and rotatory inertia on the vibration of nanoplate which is placed on an elastic foundation with different support systems. Elastic foundation is considered as Winkler one parameter model. It is also assumed that the nanoplate has stepped cross section and that the plate is weakened with crack located along the line of step. The influence of the crack is modelled as simple elastic spring that connects two adjacent segments of the nanoplate. Most of the authors negated the rotatory inertia of the nanoplate for simplifying their calculation. However, our concern is to measure the effect of rotatory inertia on the vibration of nanoplate. The simply supported and clamped supported nanoplates are considered and analysed in greater detail.

2. Method and Analysis
Consider a linearly elastic, isotropic, stepped plate resting on elastic foundation, undergoing linear vibrations. Euler-Bernoulli modified plate theory and Eringen nonlocal theory are used to derive the governing differential equations of motion. Exact solution techniques are used to solve these governing equations. The aim of the study is to determine the eigenfrequencies of natural vibrations of the nanoplate and to clarify the sensitivity of eigenfrequencies on the geometrical parameters of the plate and the physical parameters of the material.

![Figure 1. Stepped nanoplate with crack on an elastic foundation.](image)

Let us consider a nanoplate of length $A$ along $x$ axis and width $B$ along $y$ axis where the edges of the plate are placed at $x = 0$, $x = A$, $y = 0$ and $y = B$ respectively. Simply supported and fully clamped nanoplates are studied separately in different cases. The coordinate axis $0x$ and $0y$ coincide with the axis of corresponding plate and the origin of coordinates is located at the left-hand end of the plate. It is assumed that the nanoplate has rectangular cross sections with the height

$$h = \begin{cases} h_0, & x \in (0,a) \\ h_1, & x \in (a,A) \end{cases}$$

(1)

In Eq. (1) the quantities $h_0$, $h_1$ and $a$ are considered as given numbers. The nanoplate is weakened with a crack of length $c$ at $x = a$. The crack length is expected to be constant. Thus the crack area is
\[ S_x = cB \]  

According to Eringen’s nonlocal theory, the stresses in the nanoplate can be written as

\[ \left(1-(e_0a)^2\right)^2\sigma_{ij} = \sigma^*_j \]  

where \( \sigma_{ij}, \sigma^*_j \) and \( v^2 \) are the stress tensor of the nonlocal elasticity, the classical local stress tensor and the Laplace operator respectively. The quantity \( e_0a \) is a nano length scale where \( a \) is an internal characteristic length (e.g. C-C bond length, granular distance, lattice parameter) and \( e_0 \) is a calibration constant appropriate to each material. The magnitude of \( e_0 \) is determined experimentally using the atomic dispersion curves of the plane. A conservative estimate of the scale coefficient \( e_0a < 2.0 \text{ nm} \) for a single-walled carbon nanoelement has been supported by many researchers. The nonlocal constitutive relations (Eq. (3)) for two-dimensional case can be written as:

\[ \sigma(X, Y) - (e_0a)^2 v^2 \sigma(X, Y) = D\delta(X, Y) \]  

where \( D = \frac{EI}{1 - v^2} \), \( E \) is the Young’s modulus and \( \delta(X, Y) \) is the local strain. Eq. (4) can be expressed in terms of moment and deflection as below

\[ M\left(1-(e_0a)^2 v^2\right) = -\left[D \frac{\partial^4 W}{\partial X^4} + 2D \frac{\partial^4 W}{\partial X^2 \partial Y^2} + D \frac{\partial^4 W}{\partial Y^4}\right] \]  

In the absence of any applied transverse mechanical and thermal loads, dynamic equation of plate can be written as

\[ D \frac{\partial^4 W}{\partial X^4} + 2D \frac{\partial^4 W}{\partial X^2 \partial Y^2} + D \frac{\partial^4 W}{\partial Y^4} = -I_0 \frac{\partial^2 W}{\partial t^2} + I_2 \left( \frac{\partial^4 W}{\partial X^2 \partial t^2} + \frac{\partial^4 W}{\partial Y^2 \partial t^2} \right) - KW \]  

where \( K \) stands for the elasticity modulus of the foundation. Assume that \( D \) is constant. Substituting Eq. (5) into Eq. (6), we obtain the following equations:

\[ \left[D \frac{\partial^4 W}{\partial X^4} + 2D \frac{\partial^4 W}{\partial X^2 \partial Y^2} + D \frac{\partial^4 W}{\partial Y^4}\right] + \left[1-(e_0a)^2 v^2\right] \left[I_0 \frac{\partial^2 W}{\partial t^2} - I_2 \left( \frac{\partial^4 W}{\partial X^2 \partial t^2} + \frac{\partial^4 W}{\partial Y^2 \partial t^2} \right) + KW \right] = 0 \]  

For free vibration, the transverse displacement is assumed to be of the form \( W(X, Y, t) = W(X) \sin\left(\frac{Y}{B}\right)e^{j\omega t} \), where \( \omega \) is the angular frequency. We can write (7) as below

\[ D \left( \frac{\partial^4 W}{\partial X^4} - 2\frac{\partial^4 W}{\partial X^2 \partial Y^2} + \frac{\partial^4 W}{\partial Y^4}\right) - \omega^2 I_2 W + KW + I_2 \left( \frac{\partial^4 W}{\partial X^2 \partial t^2} - \frac{\partial^4 W}{\partial Y^2 \partial t^2} \right) \left( \omega^2 I_0 - \frac{\partial^2 W}{\partial X^2} - \frac{\partial^2 W}{\partial Y^2} \right) = 0 \]  

The dimensionless variables are defined as follows:

\[ x = \frac{X}{A}; \zeta = \frac{X}{A}; \omega = \frac{\omega a}{A}; \beta = \frac{W}{A}; \phi = \frac{\phi}{A}; \theta = \frac{\theta}{A}; I_0 = \frac{\rho h_0^3}{12}; I_2 = \frac{h_0}{12}; \rho_0 = 1; \beta = \frac{\beta}{A}; c = \frac{c A^4}{D}; \gamma = \frac{\gamma h_0}{h_0}; \epsilon_0 = \frac{\epsilon_0}{a}; \]  

Using the dimensionless variable, Eq. (8) can be written as below

\[ \frac{\partial^4 W}{\partial \zeta^4} - 2\zeta^2 + 2\zeta^4 - 2\zeta^2 - \zeta^4 \]  

\[ = 0 \]  

The solution of Eq. (10) has the exponential form:

\[ w(x) = A e^{j\lambda x} \]  

Inserting Eq. (11) into Eq. (10) and dividing through by \( A e^{j\lambda x} \), we obtain the characteristic equation:
$P = 1 - \varepsilon^2 n_{0} \omega^2$

$Q = 2 \zeta^2 - n_{0} \omega^2 - \varepsilon^2 \omega^2 - 2 \zeta^2 \varepsilon^2 n_{0} \omega^2 + \varepsilon^2 k$

$R = \omega^2 - \zeta^2 - k + \varepsilon^2 \omega^2 \xi^2 - n_{0} \omega^2 \xi^2 + \varepsilon^2 n_{0} \omega^2 \xi^2 - \varepsilon^2 k \xi^2$

The equation (12) has the solutions:

$$\lambda_{1,2} = \pm i \sqrt{\frac{Q^2 + 4PR - Q}{2P}} = \pm i \alpha$$

$$\lambda_{3,4} = \pm 1 \sqrt{\frac{Q^2 + 4PR - Q}{2P}} = \pm i \beta$$

Thus, the solution can be presented for the two different parts of the plate separately

For $0 \leq x \leq a$

$$w(x) = C_{1} \sinh(\alpha_{0} x) + C_{2} \cosh(\alpha_{0} x) + C_{3} \sin(\beta_{0} x) + C_{4} \cos(\beta_{0} x)$$

(13)

For $a \leq x \leq A$

$$w(x) = C_{5} \sin(\alpha_{0} x) + C_{6} \cos(\alpha_{0} x) + C_{7} \sin(\beta_{0} x) + C_{8} \cos(\beta_{0} x)$$

(14)

Boundary conditions for simply supported edges

$$w_{1}(0) = 0, w_{1}''(0) = 0, w_{1}(A) = 0, w_{1}''(A) = 0$$

(15)

Boundary conditions for clamped supported edges

$$w_{1}(0) = 0, w_{1}'(0) = 0, w_{1}(A) = 0, w_{1}'(A) = 0$$

(16)

Intermediate conditions for steps are

$$w_{i}(x_{j}) = w_{i+1}(x_{j})$$

$$w_{i}'(x_{j}) = w_{i+1}'(x_{j})$$

(17)

$$w_{i}'(x_{j}) - w_{i}'(x_{j}) = K_{w} w_{i}'(x_{j})$$

$$w_{i}'(x_{j}) + \psi_{i} w_{i}(x_{j}) = w_{i}'(x_{j}) + \psi_{i+1} w_{i+1}(x_{j})$$

Where

$$\psi_{i} = \varepsilon^2 \omega^2$$

For crack, the following relations are used

$$K = \frac{72 \pi}{E b h^6} f(s)$$

(18)

where

$$f(s) = \int \frac{F(z)}{2} dz$$

$$F(s) = 1.93 - 3.07(s) + 14.53(s)^2 - 25.11(s)^3 + 25.80(s)^4$$

A linear system of algebraic equations is formed considering the solutions of differential equations (13), (14) and the total set of boundary conditions with respect to unknown constants $C_{1} - C_{8}$. Since it is a linear homogeneous system, the non-trivial solution exists under the condition where the determinant of the system vanishes. This essential condition leads to the system for determination of natural frequencies. This set of linear homogeneous equations is solved by determining the eigenvalues using computer code developed in Maple, various classical boundary conditions are also applied within the Maple code. Thus, the natural frequencies for different physical and geometrical parameters of the plate are evaluated.

3. Results and Discussion

In this section, the practical applicability of the presented method are demonstrated by various tabular and graphical examples. Our results show decent correlation with multiple cracks in nanobeam [12],
stepped nanobeam [13] and other available results in the open literature. However, the literature on the investigation of the effects of rotary inertia and crack on the vibration of nanoplate on an elastic foundation is very limited. The key objective is to study the influences of nonlocal parameter, crack and step location, step height and rotary inertia on the natural frequency of the nanoplate on an elastic foundation.

Table 1. Natural frequency versus aspect ratio for different nonlocal parameter and frequency mode.

| Mode | $\varepsilon = 0$ | $\varepsilon = 1$ | $\varepsilon = 0$ | $\varepsilon = 1$ |
|------|------------------|------------------|------------------|------------------|
|      | S-S              | C-C              | S-S              | C-C              |
| 0    | 0.6              | 1                | 1                | 1                |
| 0.6  | 0.1              | 0.0              | 0.0              | 0.0              |
| 1    | 0.2              | 0.1              | 0.1              | 0.1              |
| 2    | 0.3              | 0.2              | 0.2              | 0.2              |
| 3    | 0.4              | 0.3              | 0.3              | 0.3              |

Table 2. Natural frequency versus aspect ratio and height length ratio for different nonlocal parameter and frequency mode.

| Mode | $\varepsilon = 0.5$ | $\varepsilon = 1$ | $\varepsilon = 0.5$ | $\varepsilon = 1$ |
|------|------------------|------------------|------------------|------------------|
|      | S-S              | C-C              | S-S              | C-C              |
| 0.1  | 0                | 0                | 0                | 0                |
| 0.2  | 0.0              | 0.0              | 0.0              | 0.0              |
| 0.3  | 0.0              | 0.0              | 0.0              | 0.0              |
| 0.4  | 0.0              | 0.0              | 0.0              | 0.0              |

The natural frequencies of stepped nanoplate with a crack on an elastic foundation are presented in Table 1-2 and Figure 2-9, where $A = 1nm, E = 180GPa, \nu = 0.3$ are considered. Table 1 describes the natural frequency in different modes versus aspect ratio, nonlocal parameter and support system. Table 1 shows natural frequency increases with the increase of aspect ratio and decreases with the increase of nonlocal parameter. Table 2 presents the natural frequency versus aspect ratio and height length ratio for different values of nonlocal parameter, frequency mode and support system. Where natural frequency decreases with the increase of height length ratio. The height length ratio or rotary inertia is more effective in the higher mode of frequency. At the same time, it is less effective in the higher aspect ratio.

In Figure 2, the relationship between the natural frequency and the elastic foundation coefficient is shown for the nanoplate without step and crack in simply supported, clamped supported plate...
respectively. The natural frequency increases with the increase of elastic foundation coefficient. In Figure 4, 5 illustrate the relationship between natural frequency and step height ratio in different support systems. It is very obvious that natural frequency increases with the increase of step height ratio. The effect of aspect ratio is more effective in simply supported plate than claimed supported plate.

Figure 2. Natural frequency versus elastic foundation coefficient for different aspect ratio (S-S).

Figure 3. Natural frequency versus elastic foundation coefficient for different aspect ratio (C-C).

Figure 4. Natural frequency versus step heights ratio for different aspect ratio (S-S).

Figure 5. Natural frequency versus step heights ratio for different aspect ratio (C-C).

Figure 6, 7 represent the natural frequency versus crack depth ratio for different aspect ratios and support systems where natural frequency decreases with the increase of crack depth ratio. Finally, in Figure 8, 9 reveal the relationship between natural frequency and step location for different aspect ratio and different support systems. When step location is \( a = 0 \), it represents thinner plate and its natural
frequency is lower. On the other hand, when step location is \( a = 1 \), it represents thicker plate and its natural frequency is higher.

**Figure 6.** Natural frequency versus crack depth ratio for different aspect ratio (S-S).

**Figure 7.** Natural frequency versus crack depth ratio for different aspect ratio (C-C).

**Figure 8.** Natural frequency versus step location for different aspect ratio (S-S).

**Figure 9.** Natural frequency versus step location for different aspect ratio (C-C).

It can be summarized that the effects of different geometrical parameters such as aspect ratio, step, crack and rotatory inertia are noticeable in dynamic nanoplate. Similarly, the different physical parameters such as nonlocal coefficient and spring constant have significant effect on the vibration of nanoplate.
4. Conclusion
The nonlocal effect on the vibration of isotropic, rectangular, single layer nanoplate embedded on an elastic foundation has been illustrated. The solution technique is based on the classical plate theory combined with nonlocal elasticity theory for encountering the small scale effect. The elastic foundation is considered according to Winkler one parameter model. The governing equations are solved using exact solution technique where matrix of coefficients is formed and resolved with the help of computer code. The effects of physical parameter such as nonlocal parameter, elastic medium parameter and geometrical parameter such as step, crack, rotatory inertia on the vibration of nanoplate have been investigated. It is shown that increasing the aspect ratio, coefficient of elastic foundation and step height ratio, the natural frequency of nanoplate increases. On the other hand, increasing the rotatory inertia and crack depth ratio, the natural frequency of nanoplate decreases. The clamped supported plate is less influenced than simply supported plate due to the effect of different parameters. This solution can be used for designing nanoelectromechanical system effectively.

Acknowledgements
The partial support from the Institutional Research Funding IUT2057 of Estonian Ministry of Education and Research is gratefully acknowledged. Also, the support from Estonian Doctoral School in Mathematics is acknowledged.

References
[1] Tantra R 2016 Nanomaterial Characterization, An Introduction John Wiley & Sons, Inc.
[2] Kumar N and Kumbhat S 2016 Essentials in nanoscience and nanotechnology John Wiley & Sons, Inc.
[3] Chakraverty S and Behera 2017 Static and Dynamic Problems of Nanobeams and Nanoplates World Scientific Publishing Co. Pte. Ltd.
[4] Daneshmehr A, Rajabpoor A and Hadi A 2015 Size dependent free vibration analysis of nanoplates made of functionally graded materials based on nonlocal elasticity theory with high order theories International Journal of Engineering Science 95 23–35.
[5] Ansari R N, Shahabodini A and Shojaei M F 2016 Nonlocal three-dimensional theory of elasticity with application to free vibration of functionally graded nanoplates on elastic foundations Physica E 76 70–81.
[6] Sobhy M 2015 A comprehensive study on FGM nanoplates embedded in an elastic medium Composite Structures 134 966–980.
[7] Natsuki T and Natsuki J 2018 Transverse impact analysis of double-layered graphene sheets on an elastic foundation International Journal of Engineering Science 124 41–48.
[8] Murmu T and Adhikari S 2011 Nonlocal vibration of bonded double-nanoplate-systems Composites: Part B 42 1901–1911.
[9] Pouresmaeeli S, Fazelzadeh S A and Ghavanloo E 2012 Exact solution for nonlocal vibration of double-orthotropic nanoplates embedded in elastic medium Composites: Part B 43 3384–3390.
[10] Ruocco E and Mallardo V 2019 Buckling and vibration analysis nanoplates with imperfections Applied Mathematics and Computation 357 282–296.
[11] Ebrahimi F, Jafari A and Barati M R 2017 Vibration analysis of magneto-electro-elastic heterogeneous porous material plates resting on elastic foundations Thin-Walled Structures 119 33–46.
[12] Roostai H and Haghpanahi M 2014 Vibration of nanobeams of different boundary conditions with multiple cracks based on nonlocal elasticity theory Appl. Math. Modell. 38 1159-1169.
[13] Lellep J and Lenbaum A 2018 Free vibrations of stepped nanobeams Int. J. Comp. Meth. Exp. Meas. 6 (4) 716-715.