Bandits with Delayed Anonymous Feedback

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Abstract

We study the bandits with delayed anonymous feedback problem, a variant of the stochastic $K$-armed bandit problem, in which the reward from each play of an arm is no longer obtained instantaneously but received after some stochastic delay. Furthermore, the learner is not told which arm an observation corresponds to, nor do they observe the delay associated with a play. Instead, at each time step, the learner selects an arm to play and receives a reward which could be from any combination of past plays. This is a very natural problem; however, due to the delay and anonymity of the observations, it is considerably harder than the standard bandit problem. Despite this, we demonstrate it is still possible to achieve logarithmic regret, but with additional lower order terms. In particular, we provide an algorithm with regret $O(\log(T) + \sqrt{g(\tau)\log(T)} + g(\tau))$ where $g(\tau)$ is some function of the delay distribution. This is of the same order as that achieved in \cite{9} for the simpler problem where the observations are not anonymous. We support our theoretical observation equating the two orders of regret with experiments.

1 Introduction

Online advertising is one of the main application areas of multi-armed bandit algorithms. A typical interaction between the system and the user consists of the user being regularly shown adverts on social media or other websites by a bandit algorithm. The algorithm is responsible for selecting adverts that resonate with the user to make them buy a particular product or visit a retailer’s website. Once in a while, the user visits the website or buys the product and the bandit algorithm is informed of this event. It is not normally possible to distinguish which advert caused this event so the algorithm does not have any information on which advert was actually responsible for attracting the user’s interest. Furthermore, the feedback is not instantaneous but might be received some time after the algorithm presented the advert. Even though this setting arises naturally in many applications of bandit algorithms, it has not yet been well-studied and understood. This paper is about filling this gap in our understanding and about providing efficient algorithms that perform - surprisingly - nearly as well as standard multi-armed bandit algorithms under instantaneous feedback.

In the standard multi-armed bandit problem \cite{14,10,3,2}, at each time step, a player selects one of $K$ actions and then receives some reward drawn from the associated unknown probability distribution. This sampled value, along with information on other previous arm-reward pairs, is used to inform future decisions about which action to play. The player’s goal is to maximize their cumulative reward from selecting actions over $T$ plays of the bandit, or equivalently minimize their regret. It is known that the regret must be at least of order $\log T$. We are interested in a related, but arguably more practical formulation. In this problem, which we will refer to as bandits with delayed anonymous feedback, we no longer assume that the player receives instantaneous feedback on their actions, nor do they observe the outcome of a specific action. Instead, at each time step, $t$, a player selects an action $j_t$ and then receives reward $X_t$, which could be the cumulative reward from any of the $t$ past plays of the bandit. Thus, compared to the standard multi-armed bandit
Multi-Armed Bandits

\[ O(\log T) \]

Delayed Feedback Bandits

\[ O(\log T + E[\tau] + \sqrt{E[\tau] \log T}) \]

Bandits with Delayed Anonymous Feedback

\[ O(\log T + E[\tau] + \sqrt{E[\tau] \log T}) \]

### Difficulty

Figure 1: The relative difficulties of the different problems. All the regret bounds are given in terms of the horizon, \( T \), and mean delay, \( E[\tau] \), and are for the case of unbounded delays.

problem, this is a more difficult problem since at each time step the user has to deal with both the potentially missing rewards and additionally the anonymous nature of the received observations.

A significantly simpler problem, where the payoff is delayed but not anonymous, has been studied in \([9, 11]\). There, the player receives delayed feedback in the shape of arm-reward pairs, in which the player is informed of both the individual payoff and which arm generated it. To avoid confusion, we will refer to that problem as the delayed feedback bandit problem, and our problem as the bandits with delayed anonymous feedback problem. In the anonymous feedback problem the algorithm has to deal with assigning the feedback to arms on top of learning from the delayed feedback, so at best one can hope to achieve regret comparable to that of \([9, 11]\). At first glance, one would expect algorithms for bandits with delayed anonymous feedback to perform significantly worse than those for delayed feedback bandits. Remarkably, it turns out that in the delayed anonymous feedback setting one can achieve regret of similar order to the simpler delayed feedback setting, despite the anonymity of the observations. In particular, we provide an algorithm for bandits with delayed anonymous feedback that matches the regret of \([9]\) up to constant factors when the delay distribution is unbounded, and performs comparably in experiments (see Figure 2 on pg.9). The relationship between these settings, and the leading terms in the regret are shown in Figure 1. Note that in all three cases, the dominant term is \( \log T \) and asymptotically, the delayed anonymous feedback problem is no more difficult than the standard multi-armed bandit problem.

The key observation that allows us to derive an efficient algorithm for the bandits with delayed anonymous feedback problem is that playing an arm consecutively for long stretches allows us to obtain an accurate estimate of the mean reward of that arm. This is because the observations received during that period will predominately be from the arm of interest. Hence, we are interested in strategies that do not switch arms often. Note that by \([8]\), an algorithm that switches arms rarely cannot achieve the optimal constants of regret indicated in \([10]\) for multi-armed bandits. In our setting, if we have minimal knowledge of the random delay, \( \tau \), in the form of an upper bound on its mean, then we are able to quantify the distortion of the observations by other arms. This enables us to construct confidence bounds that take into account this distortion. We leverage this insight to develop an adaptation of Improved UCB from \([2]\) that is tailored to the bandits with delayed anonymous feedback problem, and in this setting achieves regret of \( O(\log T + E[\tau] + \sqrt{E[\tau] \log T}) \) in terms of \( T \) and \( E[\tau] \), matching the regret of \([9]\) for the simpler delayed feedback bandit problem.

### 1.1 Related work

One of the earliest investigations of delayed feedback appears in \([1]\) where sequential testing with constant delays was studied empirically. This was then extended in \([13]\) to cover the Bayes risk. More recently, the problem of online learning under delayed feedback where the learner receives the action-reward pairs after some delay, has attracted considerable attention (see \([9]\) and references therein). We are interested in the case of learning with bandit feedback in the presence of delays. In \([12]\) it was shown that for a fixed constant delay and adversarial bandit feedback, the regret is penalized multiplicatively. This is in line with the results summarized in \([9]\) where delay causes an additive penalty in the stochastic case and a multiplicative one in the adversarial case. For stochastic contextual bandits and a fixed delay \( D \), \([7]\) show that the regret increases by an additive factor of \( O(\sqrt{K \log(T)D}) \). For Gaussian Process bandits, \([4]\) show that if the stochastic delay is bounded by \( d \), with correct initialization, the regret is increased by an additive term of \( O(d \log(d)) \). \([3]\) considered the empirical performance of some standard multi-armed bandit algorithms in the presence of...
delay.

The most relevant work to ours is [9] where stochastic bandits with unbounded random delays are studied. This is done in the delayed feedback setting, so the learner receives action-reward pairs after some delay and knows which pull of an arm generated a specific reward. In this setting, they show regret bounds of \( O(\log(T) + E[G_T^2]) \) where \( G_T^2 \) is the maximal number of missing observations at any time \( t \leq T \). For bounded delays, \( E[G_T^2] \leq d \), where \( d \) is the bound on the delay, whereas if the delay has infinite support and \( \tau \) is the delay of all arms, \( E[G_T^2] \) is of order \( O(E[\tau] + \sqrt{E[\tau] \log T + \log T}) \). [11] consider only bounded delays and provide a Bayesian algorithm for this problem, proving the same regret bound as [9] for this case. They also show good empirical performance of their approach. Both the QPM-D algorithm of [9] and SBD algorithm of [11] are queue based and as such rely heavily upon the assumption that after the delay we receive arm-reward pairs and so know exactly which rewards correspond to which arms. This allows for queues of the rewards for each arm to be formed. Furthermore, this assumption is necessary for this sort of ‘black box’ approach where action-reward pairs are fed into an algorithm for standard multi-armed bandits.

2 Problem definition

There are \( K \) arms in the set \( A \). For each arm \( j \), let \( \nu_j \) be its reward distribution with expected value \( \mu_j \), and denote by \( \mu^* = \max_{1 \leq j \leq K} \mu_j \) the maximal expected reward of the \( K \) arms. For any suboptimal arm \( j \), we denote by \( \Delta_j = \mu^* - \mu_j \). At each time \( t \) when we play an arm, we wait some random amount of time, \( \tau_l \), before receiving the reward generated from that play. For each arm \( j \), assume there is a random variable \( \tau_j \) governing its delay. Define the reward generated from playing arm \( j \) at time \( l \) as \( R_{l,j} \) and the corresponding delay as \( \tau_{l,j} \). Consequently, we receive payoff \( R_{l,j} \) at time \( l + \tau_{l,j} \). We assume that \( R_{l,j} \in [0,1] \) and \( \tau_{l,j} \in \{0,1,2,\ldots\} \) are independent for all times \( l \) and arms \( j \).

Define our observations \( X_t \) to be the reward received at time \( t \). \( X_t \) could be the sum of the reward generated by several plays at or before time \( t \), possibly of different arms. Define \( j_l \) to be the arm played at time \( l \), then,

\[
X_t = \sum_{l=1}^{t} \sum_{j=1}^{K} R_{l,j} \chi\{\tau_{l,j} = t - l, j_l = j\},
\]

where \( \chi\{\cdot\} \) is the indicator function. This formulation can be easily extended to continuous delay distributions by redefining the event \( \{\tau_{l,j} = t - l\} \) as \( \{t - l - 1 < \tau_{l,j} \leq t - l\} \). We will generally consider discrete delay distributions, however all the results carry across to continuous delays.

In the bandits with delayed anonymous feedback problem, there are multiple ways to define the regret. In line with [9], we define the regret up to time \( T \) as the sum of the difference in expected reward between the arm \( j_l \) played by the algorithm at time \( t \) and the optimal,

\[
\text{Reg}_T = T \mu^* - \sum_{l=1}^{T} \mu_{j_l}.
\]

This includes the rewards received after the horizon \( T \). In many applications such as web advertising, this is the most natural definition of regret as we still receive the reward from a user clicking on an advert, even if this happens after we have finished our experiment. A good algorithm will learn to play the optimal arm(s) and so we would expect \( \text{Reg}_T \rightarrow 0 \) as \( T \rightarrow \infty \). In [10] it was shown that the regret of any algorithm for the stochastic \( K \) armed bandit problem must satisfy

\[
\lim_{T \to \infty} \inf \frac{E[\text{Reg}_T]}{\log(T)} \geq \sum_{j: \Delta_j > 0} \frac{\Delta_j}{KL(\nu_j, \nu^*)},
\]

(1)

where \( KL(\nu_j, \nu^*) \) is the KL-divergence between the reward distribution of arm \( j \) and that of the optimal arm. Consequently, the gold standard for any algorithm for the multi-armed bandit problem is regret that is logarithmic in \( T \). Since the bandits with delayed anonymous feedback problem generalizes the standard multi-armed bandit problem, the lower bound in (1) holds in our setting.
3 Improved UCB for bandits with delayed anonymous feedback

The Improved UCB algorithm from [2] is an algorithm that lends itself naturally to the bandits with delayed anonymous feedback setting since it achieves logarithmic regret and does not switch arms often. Here, we develop an adaptation of Improved UCB [2] tailored to this problem. We call this adaptation Improved UCB for Delayed Anonymous Feedback (IUCB-DAF).

For a known horizon $T$, the IUCB-DAF algorithm is as given in Algorithm 1. Like Improved UCB, the algorithm runs in rounds. We divide each round into phases, where in each phase only one arm is played. The algorithm maintains a set, $L$, of active arms which is updated at the end of every round. In round $m$, each active arm $j$ is sampled $n_{m,j} - n_{m-1,j}$ times consecutively to ensure that it has been played $n_{m,j}$ times by the end of round $m$. The $X_t$ received during these $n_{m,j}$ plays are then used to construct confidence bounds on the expected reward of arm $j$. These incorporate both the uncertainty of our estimates and the additional uncertainty due to the anonymity of the observations. At the end of each round, an arm is removed from the active set if its upper confidence bound is lower than the largest lower confidence bound of the other arms. The confidence bounds use an approximation $\hat{\Delta}_m$ of the unknown $\Delta_j$'s. At the end of each round, we set $\hat{\Delta}_m = \hat{\Delta}_{m-1} + \hat{\Delta}_0 + \hat{\Delta}_m$. If all confidence bounds hold, arm $j$ will be eliminated in the first round where $\Delta_j < \hat{\Delta}_m/2$. Intuitively, if arm $j$ is still active in a later round, $\Delta_j$ should be small. If at any point, the algorithm has eliminated $K - 1$ arms from the active set, we play the remaining arm until the horizon $T$.

The IUCB-DAF algorithm relies on parameters $n_{m,j}$ and $d_{m,j}$ which are chosen in relation to the assumption on the delay. Generally, $d_{m,j}$ is chosen to ensure that with high probability the difference between the reward generated by playing arm $j$ in $m$ phases and the reward obtained in these $m$ phases does not exceed $md_{j,m}$. Then, $n_{m,j}$ is then chosen to guarantee that the width of the confidence intervals after $m$ rounds of playing arm $j$ is less than $\hat{\Delta}_m/2$. Finally, $\lambda^*$ is the minimum value $\hat{\Delta}_m$ can reach in the final round $m^*$ and is chosen such that $n_{m^*,j} \leq T$ for any arm $j$.

4 Bounded delay

We first consider the case where the delay is bounded by some constant $d > 0$. That is, $\tau_j \leq d$ for all arms $1 \leq j \leq K$. In this setting, we show IUCB-DAF achieves logarithmic regret.
4.1 High confidence bounds

For an arm \(j \in \{1, \ldots, K\}\), we begin by looking at just the \(i\)th phase in which it is played. Let \(X_t\) be the reward obtained at time \(t\), \(R_{j,t}\) be the reward generated by playing arm \(j\) at time \(t\) (which we do not observe directly) and denote by \(s_{i,j}\) the first time point in the \(i\)th phase of playing arm \(j\), and \(t_{i,j}\) the last. During this phase, only arm \(j\) will be played, however, we are interested in the sum of \(X_t\) over this whole phase, which may include observations which have seeped in from the plays in the previous phase. Observe that the sum of \(X_t\) over the phase will differ from the sum of \(R_{j,t}\) by at most \(d\), since in the worst case, we either have an extra \(d\) observations that have come in from the previous arm, or the last \(d\) observations from this phase are lost into another, and all rewards are bounded in \([0, 1]\). More concretely,

\[
\left| \sum_{t=s_{i,j}}^{t_{i,j}} R_{j,t} - \sum_{t=s_{i,j}}^{t_{i,j}} X_t \right| \leq d,
\]

and so,

\[
\frac{1}{n_{m,j}} \left| \sum_{t \in T_{j}(m)} R_{j,t} - \sum_{t \in T_{j}(m)} X_t \right| \leq \frac{dm}{n_{m,j}},
\]

where \(T_{j}(m)\) is the set of time steps where arm \(j\) was played in the first \(m\) phases and \(n_{m,j} = |T_{j}(m)|\) is the number of times arm \(j\) has been played by the end of round \(m\). Combining this with Hoeffding’s inequality (see Appendix B.1 for details) gives that with probability \(1 - \frac{1}{(T \Delta_m^2)}\),

\[
|\bar{X}_{m,j} - \mu_j| < \sqrt{\frac{\log(T \Delta_m^2)}{2 n_{m,j}}} + \frac{md}{n_{m,j}}.
\]

Repeating this for all arms \(j\), we get the confidence bounds in Algorithm 1. The first term in these confidence bounds is due to the uncertainty in \(\bar{X}_{m,j}\) as in the standard bandit case, whereas the second term captures the additional uncertainty due to the anonymity of observations.

4.2 Parameter choices for bounded delay

In this setting, we know that the delay is bounded by \(d > 0\) for all arms \((\tau_{j,t} \leq d\) for all \(t\)), so we set \(d_{m,j} = d\) for all arms \(j\) and rounds \(m\). We then choose

\[
n_{m,j} = \left\lfloor \frac{1}{2 \Delta_m^2} \left( \sqrt{\log(T \Delta_m^2)} + \sqrt{\log(T \Delta_m^2) + 4 \Delta_m md} \right)^2 \right\rfloor, \quad \text{and} \quad \lambda^* = \sqrt{\frac{4de}{T}}.
\]

Here \(n_{m,j}\) is chosen such that the width of the confidence intervals is less than \(\frac{\Delta_m}{2}\), and \(\lambda^*\) such that in the final round \(m^*\), \(n_{m^*,j} \leq T\). A full discussion of these choices is given in Appendix B.2.

4.3 Regret bounds for bounded delay

The regret from \(T\) plays of the IUCB-DAF algorithm with bounded delays and parameters \(d_{m,j} = d\) and \(n_{m,j}\) as in 2 is logarithmic in \(T\) and, in terms of \(T\) and \(d\), is \(O(\log T + \sqrt{d \log T + d})\).

**Theorem 1** Let \(d\) be a constant such that \(\tau_{j,t} \leq d\) for all arms \(1 \leq j \leq K\). Then, for all \(\lambda \geq \lambda^*\), the total expected regret of the IUCB-DAF algorithm up to time \(T\) can be upper bounded by

\[
\sum_{j \in A: \Delta_j > \lambda} \left( \Delta_j + \frac{32 \log(T \Delta_j^2)}{\Delta_j} + \frac{16 \sqrt{2 \log(T \Delta_j^2)} d}{\Delta_j} + \frac{16d + 96}{\Delta_j} \right) + \sum_{0 < \Delta_j \leq \lambda} \sum_{j \in A} \frac{64}{\Delta_j} + \frac{\max \Delta_j T}{\Delta_j}.
\]

**Proof:** See Section B.3 for a complete proof. The general idea is the same as 2. □

In particular, taking \(\lambda = \lambda^*\) in Theorem 1, we can replace the last two terms of the bound in Theorem 1 by \(\sum_{j \in A: 0 < \Delta_j \leq \lambda} \frac{64}{\Delta_j} + \max \sum_{j \in A: \Delta_j \leq \lambda} \frac{4de}{\Delta_j}\), which gives the result that the regret is logarithmic in \(T\). In the
delayed feedback bandit setting with bounded delays, both QPM-D \cite{9} and SBD \cite{11} achieve regret of order $O(\log(T) + E[\tau])$. This comes from noticing that if you play each arm $n + E[\tau]$ times, in expectation we will have the same information as if we had played an arm in the standard bandit problem $n$ times. Due to the anonymity of rewards in the bandits with delayed anonymous feedback problem, this is not the case. Particularly, no matter how many times an arm is played consecutively, in the first $d$ plays there will still be identifiability issues.

5 Unbounded delay

We now consider the more general case where the delay has unbounded support. This is a more natural setting since most commonly occurring delay distributions will be unbounded. Let $T_j(m)$ be the set of all time points where arm $j$ was played in the first $m$ rounds of the algorithm. Assume we are given $C_{1,j}(m)$ and $C_{2,j}(m)$ for each arm $j$ and round $m$, which are upper bounds on the expected number of rewards from other arms received in $T_j(m)$, and the expected number of rewards from arm $j$ received outside of $T_j(m)$. Note that $C_{1,j}(m)$ and $C_{2,j}(m)$ appear naturally in Lemma 2 and may depend on the start and end times of each phase. We show how to compute $C_{1,j}(m)$ and $C_{2,j}(m)$ in Section 5.4. Throughout, we assume that the mean delay of each arm is finite.

5.1 General framework

For every arm $j$ and round $m$, define $C_{1,j}(m)$ and $C_{2,j}(m)$ as bounds such that,

$$\sum_{t \in T_j(m) \cap \tau_{t,j}} P(\tau_{t,j} + t \in T_j(m)) \leq C_{1,j}(m), \quad \sum_{t \not\in T_j(m)} P(\tau_{t,j} + t \not\in T_j(m)) \leq C_{2,j}(m).$$

In the following lemma these are used to bound with high probability the difference between the reward from any other arm received in $T_j(m)$ and the reward generated in $T_j(m)$.

Lemma 2 For a given arm $j$, define by $T_j(m)$ the set of all time points in which arm $j$ was played in $m$ phases. Then for any $d_{m,j} > 0$,

$$P\left(\left| \sum_{t \in T_j(m)} X_t - \sum_{t \not\in T_j(m)} R_{t,j} \right| > md_{m,j} \right) \leq \exp\left\{-\frac{1}{2}(md_{m,j} - 3C_{1,j}(m) - C_{2,j}(m))\right\} + \exp\left\{-\frac{1}{2}(md_{m,j} - C_{1,j}(m) - 3C_{2,j}(m))\right\}.$$  

Proof: Full proof in Appendix C.1 The idea is to bound $|\sum_{t \in T_j(m)} X_t - \sum_{t \not\in T_j(m)} R_{t,j}|$ by the absolute difference in the rewards of observations received in $T_j(m)$ from other arms (which is related to $C_{1,j}(m)$) and the rewards generated by arm $j$ and received outside $T_j(m)$ (which is related to $C_{2,j}(m)$). Bernstein bounds are then used to control the deviations from the mean.

5.2 Parameter choices for unbounded delay

IUCB-DAF relies on parameters $d_{m,j}$ (used in the confidence bounds to control $|\sum_{t \in T_j(m)} X_t - \sum_{t \in T_j(m)} R_{t,j}|$) and $n_{m,j}$ (the number of times arm $j$ is played by the end of round $m$). These are chosen to ensure that the confidence intervals hold with high enough probability and are tight enough for a sub-optimal arm to be eliminated in the correct round. For the analysis, it is also necessary to define a bound, $\lambda^*$, on the smallest $\Delta_j$ the algorithm can be expected to eliminate. Let $c$ be a constant independent of $m$ and $j$, such that for all arms $j$ and phases $m$, $C_{1,j}(m), C_{2,j}(m) \leq mc$ and $c$ is an upper bound on the expected number of rewards entering or leaving any phase. Then, we set,

$$n_{m,j} = \left[ \frac{1}{2\Delta^2_m} \left( \sqrt{\log(T\Delta^2_m)} + \sqrt{\log(T\Delta^2_m)} + 4md_{m,j}\Delta_m \right) \right]^2, \quad \lambda^* = \frac{(1 + \sqrt{T + 16c})^2}{2\sqrt{T}}.$$  

6
and, 
\[ d_{m,j} = 4 \log(T \Delta_m^2) + C_1(j(m) + C_2(j(m) + 2 \max\{C_1(j(m), C_2(j(m))} \). \]

A full discussion of these parameter choices is given in Appendix C.2.

5.3 Regret bounds for unbounded delay

We bound the regret of IUCB-DAF for unbounded delays and parameters \( d_{m,j} \) and \( n_{m,j} \) defined in (3) in terms of the horizon \( T \), the gaps \( \Delta_j \), and \( c \), the bound on \( C_1(j(m)) \) and \( C_2(j(m)) \).

**Theorem 3** Assume that there is an upper bound \( c \) on the expected number of rewards entering or leaving any phase. Then, for all \( \lambda \geq \lambda^* \), the total expected regret of the IUCB-DAF algorithm up to time \( T \) in the case of unbounded delays can be upper bounded by

\[
\sum_{j \in A: \Delta_j > \lambda} \left( \frac{8(2 + 8\Delta_j + 2\sqrt{1 + 8\Delta_j}) \log(T \Delta_j^2)}{\Delta_j} + \frac{64 \sqrt{c \log(T \Delta_j^2)}}{\Delta_j} + \frac{128c}{\Delta_j} + \Delta_j \right) + \sum_{j \in A: \Delta_j > \lambda} \frac{288}{\Delta_j} + \sum_{j \in A: 0 < \Delta_j \leq \lambda} \frac{192}{\lambda} + \max_{j \in A: \Delta_j \leq \lambda} \Delta_j T.
\]

**Proof:** Full proof in Appendix C.1. The proof is similar to that of [2], except that we must now also consider events \( F_j(m) = \{\sum_{t \in T_j(m)} X_t - \sum_{t \in T_j(m)} R_{t,j} > md_j, m\} \) to deal with the possibility that, due to distortion, our estimated sample means are far from the true sample means.

In particular, setting \( \lambda = \lambda^* \) here will mean that the last two terms of the regret bound can be bounded by \( \sum_{j \in A:0 < \Delta_j \leq \lambda} \frac{192}{\lambda} \max_{j \in A: \Delta_j \leq \lambda} \frac{1}{2}(1 + \sqrt{T^2 + 16c})^2 \) which do not depend on \( T \). Thus, for large horizons, we can conclude that the regret is logarithmic in \( T \) and \( O(\log T + \sqrt{c \log T} + c) \).

5.4 Selecting C

It is important to choose \( C \) (and consequently \( c \)) carefully to ensure our confidence bounds are tight. If the delays are all iid with \( \tau_j = \tau \) for all arms \( 1 \leq j \leq K \) and \( E[\tau] < \infty \), we can use the fact that \( \sum_{t=0}^{\infty} P(\tau > t) = E[\tau] \) to define \( C = (C_1(j(m), C_2(j(m))) \) in terms of \( E[\tau] \). More concretely,

\[
\sum_{t \in T_j(m)} P(\tau_{t,j} > t) \leq \sum_{i=1}^m \sum_{t=s_{i,j}^{-1}}^{s_{i,j}} P(\tau_{t,j} > s_{i,j} - t) = \sum_{i=1}^m \sum_{t=s_{i,j}}^{s_{i,j}-1} P(\tau > l) \leq \sum_{i=1}^m E[\tau],
\]

and, \( \sum_{t \in T_j(m)} P(\tau_{t,j} + t \notin T_j(m)) \leq \sum_{i=1}^m \sum_{t=s_{i,j}}^{t_{i,j}} P(\tau_{t,j} > t_{i,j} - t) \leq \sum_{i=1}^m E[\tau] \),

We then define \( C \) and \( c \) (chosen such that \( C_1(j), C_2(j) \leq mc \) for all \( 1 \leq j \leq K \)) as

\[
C_1(j) = C_2(j) = mc \quad \text{and} \quad c = E[\tau].
\]

Hence, from Theorem 3, it can be seen that for unbounded delays, the regret of IUCB-DAF in terms of \( E[\tau] \) and \( T \) is \( O(\log T + \sqrt{E[\tau]} \log T + E[\tau]) \), matching that of [2]. Note that if we do not know \( E[\tau] \) but have an upper bound \( B \) such that \( E[\tau] \leq B \), we can use \( B \) instead of \( E[\tau] \) in \( C \) and \( c \).
5.4.1 Arm dependent delays

If each arm \( j \) has a different delay distribution, we use a different bound for \( C_{1,j}(m) \) to take into account the possibility that different distributions influence the probability of a reward from an arm \( a \neq j \) being received in \( T_j(m) \). Note that \( C_{2,j}(m) \) is not affected by this. We set,

\[
C'_{1,j}(m) = \sum_{a=1; a \neq j}^K mE[\tau_a], \quad \text{then}, \quad c' = \max\left\{ \sum_{a=1; a \neq j}^K E[\tau_a], E[\tau_j] \right\}.
\]

Again, if \( E[\tau] \) is unknown, but upper bounded by \( B < \infty \), we can use \( B \) instead of \( E[\tau] \) in \( c' \) and \( c' \). These bounds may depend on the number of arms, \( K \). However, if we know the \( \{\tau_j\}_{j=1}^K \) are sub-Gaussian or have finite variance, we can exploit their similarity to remove this dependence on \( K \). We keep \( C_{2,j}(m) \) the same, then, for constants \( 0 \leq \zeta, B < \infty \) such that \( \text{var}(\tau_j) \leq \zeta \) and \( E[\tau_j] \leq B \) for all arms \( 1 \leq j \leq K \), Chebychev’s inequality (Lemma 8, Appendix D.2) gives

\[
C'_{1,j}(m) = \sum_{i=2}^m \chi\{s_{i,j} > 1\} \left( \zeta \left( 1 + \frac{1}{\zeta + 1} - \frac{1}{s_{i,j} - 1 - \lfloor B \rfloor} \right) + \lfloor B \rfloor \right) \leq mc',
\]

where \( c' = m(\zeta + 1 - \frac{1}{\zeta + 1} + \lfloor B \rfloor) \). Similarly, if \( \tau_j \) is \( \rho \)-sub-Gaussian for all arms \( j \) with \( \rho_j \leq \rho < \infty \), define

\[
c' = ((1 - \exp(-\frac{1}{2\rho})))^{-1} + \lfloor B \rfloor - 1),
\]

then Lemma 7 (Appendix D.1) gives,

\[
C_{1,j}(m) = \sum_{i=1}^m \chi\{s_{i,j} > 1\} \left( 1 - \exp\left( -\frac{s_{i,j} - \lfloor B \rfloor}{2\rho} \right) \right) \left( 1 - \exp\left( -\frac{\lfloor B \rfloor}{2\rho} \right) \right) \leq mc'.
\]

6 Experimental results

We investigated the performance of IUCB-DAF and QPM-D [9] in various experimental settings with unbounded delays. Throughout, we used two arms with Bernoulli reward distributions and success probabilities \( (0.5, 0.6) \). This simple reward setup allowed us to focus on the effects of the delay. In every experiment, we ran each algorithm to horizon \( T = 250,000 \). In the first two cases, in IUCB-DAF we used \( C_{1,j}(m) = m(K-1)B \) and \( C_{2,j}(m) = mB \) for a bound \( B \) on the mean delay. In the third case, we were interested in the impact of the mean delay so we used \( C_{1,j}(m) = C_{2,j}(m) = mE[\tau] \). For QPM-D, UCB1 [3] was used as the base algorithm. The results of these experiments are shown in Figure 2, where Normal_+ is the Normal distribution truncated at 0.

From Figure 2a we can see that in all cases the ratio of the regret of IUCB-DAF to the regret of QPM-D converges to a constant. This shows that the regret of IUCB-DAF is of the same order as that of QPM-D. IUCB-DAF predetermines the number of times to play each arm per round based on the bound on the mean delay, so the jumps in the figure correspond to the algorithm changing arm which happens at the same points in all 200 replications. However, the location of the jumps and the constants that the ratios converge to vary with the distribution and \( B \).

From Figure 2b it is unclear whether it is the delay distribution or value of \( B \) that affects the regret. In Figure 2c we investigated the effect of the distribution by changing the delay distributions but keeping the mean delay and \( B \) the same. The plot again shows the ratio of the regret of IUCB-DAF to the regret of QPM-D. It can be seen here that changing the distribution has very little effect on the performance of both algorithms. In Figures 2a and 2b the regret was averaged over 200 replications.

In the first two experiments, we have also investigated using an alternative definition of \( C \),

\[
C_{1,j}(m) = (K-1)(\lfloor \frac{1}{2}(m(m+1)) \rfloor + B) \quad \text{and} \quad C_{2,j}(m) = \lfloor \frac{1}{2}(m(m+1)) \rfloor + B.
\]

This separates the effects of \( \tau \) and \( m \), and as such is more appropriate for large delays (\( E[\tau] > \log T \)). The derivation is in Appendix D.3. As can be seen in both Figure 2a and Figure 2b where the dotted lines
The ratio of \( \frac{\text{Reg}(\text{IUCB-DAF})}{\text{Reg}(\text{QPM-D})} \) under various delay distributions with different values of \( E[\tau] \) and \( B \).

The dotted lines are the results with the alternative \( C \) from (4).

The increase in regret at time \( T = 250000 \) with increasing mean delay when the delay is Normal with variance \( \sigma^2 = 100 \). The increase is calculated as \( \frac{\text{Reg}(\text{IUCB-DAF})}{\text{Reg}(\text{QPM-D})} \) for increasing values of \( E[\tau] \) where \( \tilde{\mu}_0 \) is the mean of the \( \text{N}(0, 100) \) distribution truncated at 0 and \( \text{Reg}(\text{IUCB-DAF}) \) represents the regret when the mean delay is \( E[\tau] \). The regret was averaged over 1000 replications for IUCB-DAF and 5000 for QPM-D (the variance in the regret of QPM-D was significant). From this, it can be seen that increasing the mean delay causes the regret of both QPM-D and IUCB-DAF to increase linearly. This is in accordance with the regret bounds for QPM-D and IUCB-DAF which both include a linear factor of \( E[\tau] \).

7 Conclusion

In this paper we have studied an extension of the multi-armed bandit problem to bandits with delayed anonymous feedback, where observations are received after some stochastic delay and we do not learn which plays correspond to which observations. In this more difficult setting, we have proven that an adaptation of Improved UCB from [2] can still achieve logarithmic regret, matching the rate of QPM-D [9] for the easier problem when we know the assignment of rewards to plays. We have supported this in several experiments. The constants in the regret of our algorithm do not match those of QPM-D, so it is an open problem to find an algorithm for bandits with delayed anonymous feedback that improves these constants. Furthermore, to the best of our knowledge, it remains to prove lower bounds for delayed feedback bandits and bandits with delayed anonymous feedback.
References

[1] T. Anderson. Sequential analysis with delayed observations. *Journal of the American Statistical Association*, 59(308):1006–1015, 1964.

[2] P. Auer and R. Ortner. Ucb revisited: Improved regret bounds for the stochastic multi-armed bandit problem. *Periodica Mathematica Hungarica*, 61(1-2):55–65, 2010.

[3] P. Auer, N. Cesa-Bianchi, and P. Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine Learning*, 47(2-3):235–256, 2002.

[4] S. Boucheron, G. Lugosi, and P. Massart. *Concentration inequalities: A nonasymptotic theory of independence*. Oxford University Press, 2013.

[5] O. Chapelle and L. Li. An empirical evaluation of thompson sampling. In *Advances in Neural Information Processing Systems*, pages 2249–2257, 2011.

[6] T. Desautels, A. Krause, and J. W. Burdick. Parallelizing exploration-exploitation tradeoffs in gaussian process bandit optimization. *Journal of Machine Learning Research*, 15(1):3873–3923, 2014.

[7] M. Dudík, D. Hsu, S. Kale, N. Karampatziakis, J. Langford, L. Reyzin, and T. Zhang. Efficient optimal learning for contextual bandits. In *Proceedings of the Twenty-Seventh Conference on Uncertainty in Artificial Intelligence*, pages 169–178, 2011.

[8] A. Garivier, T. Lattimore, and E. Kaufmann. On explore-then-commit strategies. In *Advances in Neural Information Processing Systems*, pages 784–792, 2016.

[9] P. Joulani, A. György, and C. Szepesvári. Online learning under delayed feedback. In *International Conference on Machine Learning*, pages 1453–1461, 2013.

[10] T. L. Lai and H. Robbins. Asymptotically efficient adaptive allocation rules. *Advances in Applied Mathematics*, 6(1):4–22, 1985.

[11] T. Mandel, Y.-E. Liu, E. Brunskill, and Z. Popovic. The queue method: Handling delay, heuristics, prior data, and evaluation in bandits. In *AAAI*, pages 2849–2856, 2015.

[12] G. Neu, A. Antos, A. György, and C. Szepesvári. Online markov decision processes under bandit feedback. In *Advances in Neural Information Processing Systems*, pages 1804–1812, 2010.

[13] Y. Suzuki. On sequential decision problems with delayed observations. *Annals of the Institute of Statistical Mathematics*, 18(1):229, 1966.

[14] W. R. Thompson. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. *Biometrika*, 25(3–4):285–294, 1933.
Appendix

A Table of Notation

For ease of reading, we define here key notation that will be used in this Appendix.

- \( T \): The horizon.
- \( \Delta_j \): The gap between the mean of the optimal arm and the mean of arm \( j \),
  \( \Delta_j = \mu^* - \mu_j \).
- \( \tilde{\Delta}_m \): The approximation to \( \Delta_j \) at round \( m \) of the IUCB-DAF algorithm,
  \( \tilde{\Delta}_m = 1/2^m \).
- \( n_{m,j} \): The number of samples of arm \( j \) IUCB-DAF needs by the end of round \( m \).
- \( d \): The bound on the delay in the case of bounded delay.
- \( d_{m,j} \): A parameter of the IUCB-DAF algorithm used to control the width of the bounds in the case of unbounded delay.
- \( m_j \): The first round of the IUCB-DAF algorithm where \( \tilde{\Delta}_m < \Delta_j / 2 \).
- \( T_j(m) \): The set of all time point where arm \( j \) is played up to (and including) round \( m \).
- \( X_t \): The reward received at time \( t \) (from any possible past plays of the algorithm).
- \( R_{t,j} \): The reward generated by playing arm \( j \) at time \( t \).
- \( s_{m,j} \): The start point of phase \( m \) of playing arm \( j \). \( s_j \) is the vector \((s_{1,j}, \ldots, s_{m,j})\).
- \( t_{m,j} \): The end point of phase \( m \) of playing arm \( j \). \( t_j \) is the vector \((t_{1,j}, \ldots, t_{m,j})\).
- \( C_{1,j}(m) \): The upper bound on the expected number of rewards from other arms received in \( T_j(m) \).
- \( C_{2,j}(m) \): The upper bound on the expected number of rewards generated by arm \( j \) received outside of \( T_j(m) \).
- \( c \): A constant not depending on \( s_j \) and \( t_j \) such that \( C_{1,j}(m) \leq c \) and \( C_{2,j}(m) \leq c \).
- \( B \): An upper bound on the mean delay of any arm, \( E[\tau_j] \leq B \forall 1 \leq j \leq K \).
- \( \rho \): In the case of sub-Gaussian delay, a bound on the maximal variance parameter of the delay of any arm.
- \( \zeta \): In the case of delay with finite second moment, an upper bound on the variance of the delay of any arm.

B Bounded Delay

B.1 Confidence Bounds

For the construction of the confidence bounds in the bounded delay setting, first note that since the delay is bounded by \( d \), at most \( d \) rewards from other arms can seep into phase \( i \) of playing arm \( j \) and at most \( d \) rewards from arm \( j \) can be lost. Hence,

\[
\left| \sum_{t=s_{i,j}}^{t_{i,j}} R_{j,t} - \sum_{t=s_{i,j}}^{t_{i,j}} X_t \right| \leq d,
\]

where \( s_{i,j} \) and \( t_{i,j} \) are the start and end points of phase \( i \) of playing arm \( j \). We now consider all \( m \) phases during which arm \( j \) has been played a total of \( n_{m,j} \) times. Let \( T_j(m) \) denote the set of time points arm \( j \) was
played in these \( m \) phases and note that \( |T_j(m)| = n_{m,j} \). Then,

\[
\left| \sum_{t \in T_j(m)} R_{j,t} - \sum_{t \in T_j(m)} X_t \right| \leq dm \quad \Rightarrow \quad \frac{1}{n_{m,j}} \left| \sum_{t \in T_j(m)} R_{j,t} - \sum_{t \in T_j(m)} X_t \right| \leq \frac{dm}{n_{m,j}}.
\]

Define \( \bar{R}_{m,j} = \frac{1}{n_{m,j}} \sum_{t \in T_j(m)} R_{j,t} \) and \( \bar{X}_{m,j} = \frac{1}{n_{m,j}} \sum_{t \in T_j(m)} X_t \), then \( \bar{X}_j \) is a biased but consistent estimator of \( \mu_j \), since it may contain observations from different distributions but in the limit (assuming \( d \) is fixed), the proportion of these observations from another distribution will tend to 0. We are interested in bounding the difference between \( \bar{X}_{m,j} \) and \( \mu_j \). For any \( a > \frac{dm}{n_{m,j}} \),

\[
P(|\bar{X}_{m,j} - \mu_j| > a) \leq P(|\bar{X}_{m,j} - \bar{R}_{m,j}| + |\bar{R}_{m,j} - \mu_j| > a) \leq P\left(|\bar{R}_{m,j} - \mu_j| > a - \frac{dm}{n_{m,j}}\right)
\]

\[
\leq \exp\left\{-2n_{m,j} \left(a - \frac{dm}{n_{m,j}}\right)^2\right\},
\]

where the first inequality is from the triangle inequality and the last from Hoeffding’s inequality since \( R_{j,t} \in [0,1] \) are independent samples from \( \nu_j \), the reward distribution of arm \( j \). In particular, taking \( a = \sqrt{\frac{\log(T\Delta_j^2)}{2n_{m,j}}} + \frac{md}{n_{m,j}} \) ensures that \( P(|\bar{X}_j - \mu_j| > a) \leq \frac{1}{\Delta_m^2} \) where \( \Delta_m \) is the approximation of \( \Delta_j \) which is used by the algorithm. This is then repeated for every arm \( j \) to get the confidence bounds used in the IUCB-DAF algorithm.

### B.2 Parameter Choices for Bounded Delay

We first discuss the choice of \( n_{m,j} \). For the analysis of the algorithm, if \( a \) is the width of the confidence interval, we require that \( 4a < \Delta_i \) and show that as long as the confidence intervals hold, a suboptimal arm is eliminated as soon as \( \Delta_m < \frac{\Delta_i}{4} \). Hence, we must pick \( n_{m,j} \), the number of plays of an active arm, \( j \), required by the end of round \( m \), such that the width of the confidence interval, \( a \), satisfies \( a \leq \frac{\Delta_m}{2} \).

\[
\sqrt{\frac{\log(T\Delta_m^2)}{n_{m,j}}} + \frac{md}{n_{m,j}} \leq \frac{\Delta_m}{2}
\]

\[
\Rightarrow \frac{\Delta_m}{2}n_{m,j} - \sqrt{n_{m,j}} \sqrt{\frac{\log(T\Delta_m^2)}{2}} - dm \geq 0
\]

\[
\Rightarrow \sqrt{n_{m,j}} \geq \sqrt{\frac{\log(T\Delta_m^2)}{2}} + \frac{\sqrt{\log(T\Delta_m^2) + 4\Delta_m dm}}{\sqrt{2\Delta_m}}
\]

\[
\Rightarrow n_{m,j} \geq \frac{\left(\sqrt{\log(T\Delta_m^2)} + \sqrt{\log(T\Delta_m^2) + 4\Delta_m md}\right)^2}{2\Delta_m^2}.
\]

Hence, we take

\[
n_{m,j} = \left[\frac{\left(\sqrt{\log(T\Delta_m^2)} + \sqrt{\log(T\Delta_m^2) + 4\Delta_m md}\right)^2}{2\Delta_m^2}\right],
\]

for all arms \( 1 \leq j \leq K \). This number of samples increases at each round and so we want to make sure that the maximum number of rounds is such that the total number of samples required after the last round, \( m^* \),
is not more than the horizon $T$. If we set $\tilde{\Delta}_m^* = \sqrt{\frac{4d}{T}}$ then,

$$n_m^* = \frac{1}{2} \left\{ \sqrt{\log(4de)} \sqrt{\frac{T}{4de}} + \sqrt{T \log(4de) + 4 \sqrt{4de/T} \log_2(\sqrt{T/4de})} \right\}^2$$

$$\leq \frac{1}{2} \left( \sqrt{\frac{T}{e}} + \sqrt{T \log(4d^2/3)} \log(4de/T) \right)^2$$

$$= \frac{T}{2e} \left( 1 + \sqrt{1 + \frac{2}{3}} \right)^2 \leq T$$

where the first inequality follows since $\log_2(x) \leq x/1.5$ and $\log(x) \leq x$ for all $x > 0$, and the last since $\frac{1}{2} \left( 1 + \sqrt{1 + \frac{2}{3}} \right)^2 \approx 0.96$. Thus we set $\lambda^* = \sqrt{\frac{4de}{T}}$.

### B.3 Regret Bounds for Bounded Delay

**Theorem 4** (Theorem 1 in main text) *For all $\lambda \geq \lambda^*$, the total expected regret of the IDF-UCB algorithm up to time $T$ can be upper bounded by*

$$\sum_{j \in A; \Delta_j > \lambda} \left( \Delta_j + \frac{32 \log(T \Delta_j^2)}{\Delta_j} + \frac{16 \sqrt{2 \log(T \Delta_j^2)} d}{\Delta_j} + \frac{16d + 96}{\Delta_j} \right) + \sum_{0 < \Delta_j < \lambda} \frac{64}{\lambda} + \max_{\Delta_j \leq \lambda} \Delta_j T.$$

**Proof:** We use the results of Theorem 3 from [2] for the cases (a) and (b2). We can use the same notation of $A' = \{ j \in A | \Delta_j > \lambda \}$ for some fixed $\lambda$, and $m_j = \min\{m|\tilde{\Delta}_m < \frac{\Delta_j}{2}\}$. Note that

$$2^{m_j} = \frac{1}{\tilde{\Delta}_m} \leq \frac{4}{\Delta_j} < \frac{1}{\tilde{\Delta}_m + 1} \implies \frac{\Delta_j}{4} \leq \tilde{\Delta}_m \leq \frac{\Delta_j}{2}. \quad (5)$$

For case (b1), we consider the case where the optimal arm $\ast \in L_m$, for all $j \in A'$ so arm $j \in A'$ is eliminated in or before round $m_j$. By the relationships in (5),

$$n_{m_j,j} \leq 1 + \frac{8 \left( \sqrt{\log(T \Delta_j^2)} + \sqrt{\log(T \Delta_j^2)} + d \Delta_j m_j \right)^2}{\Delta_j^2}.$$  

We then use the same reasoning as [2] along with the fact that $m_j \leq \log_2(2/\Delta_j) \leq \frac{2}{\Delta_j}$ by (5) to get that the regret contribution over arms in $A'$ in this case can be bounded by,

$$\sum_{j \in A'} \left( \Delta_j + \frac{32 \log(T \Delta_j^2)}{\Delta_j} + \frac{16 \sqrt{2 \log(T \Delta_j^2)} d}{\Delta_j} + \frac{16d}{\Delta_j} \right).$$

Combining this with the results of cases (a) and (b2), and using the trivial regret bound of $\max_{j \in A; \Delta_j \leq \lambda} \Delta_j T$ for suboptimal arms not in $A'$ as in [2] gives the result. \qed
C Unbounded Delay

C.1 General Results for Unbounded Delay

Lemma 5 (Lemma 2 in main text) For a given arm \( j \), define by \( T_j(m) \) the set of all time points in which arm \( j \) was played in \( m \) phases, \( |T_j(m)| = n_{m,j} \), and \( s_j = (s_{1,j}, \ldots, s_{m,j}) \) and \( t_j = (t_{1,j}, \ldots, t_{m,j}) \) to be the start and end points respectively of each of the \( m \) phases in which arm \( j \) was played. Then for any \( d_{m,j} > 0 \),

\[
P \left( \left| \sum_{t \in T_j(m)} X_t - \sum_{t \in T_j(m)} R_{t,j} \right| > md_{m,j} \right) \leq \exp \{-1/4(md - 3C_{1,j}(m) - C_{2,j}(m))\} + \exp \{-1/4(md - C_{1,j}(m) - 3C_{2,j}(m))\}.
\]

Proof: The proof of this result begins by noting that \( |\sum_{t \in T_j(m)} X_t - \sum_{t \in T_j(m)} R_{t,j}| \) can be bounded by the difference between the reward of samples from previous phases that are obtained in \( T_j(m) \) and the rewards generated in each phase of \( T_j(m) \) that are obtained after the phase has finished. Formally, define

\[
Q = \sum_{t \notin T_j(m)} R_{t,j} \chi\{t_{t,j} + t \in T_j(m)\} \quad \text{and} \quad Q' = \sum_{t \in T_j(m)} R_{t,j} \chi\{t_{t,j} + t \notin T_j(m)\}
\]

then,

\[
P \left( \left| \sum_{t \in T_j(m)} X_t - \sum_{t \in T_j(m)} R_{t,j} \right| > md \right) \leq P \left( \left| \sum_{t \notin T_j(m)} R_{t,j} \chi\{t_{t,j} + t \in T_j(m)\} - \sum_{t \in T_j(m)} R_{t,j} \chi\{t_{t,j} + t \notin T_j(m)\} \right| > md \right)
\]

\[
= P \left( |(Q - E[Q]) + E[Q']| > md \right)
\]

\[
\leq P \left( |(Q - E[Q])| + |(Q' - E[Q'])| \geq md - |E[Q] - E[Q']| \right)
\]

We now consider the right hand side of (7), and note that,

\[
|E[Q] - E[Q']| = \sum_{t \notin T_j(m)} E[R_{t,j} \chi\{t_{t,j} + t \in T_j(m)\}] - \sum_{t \in T_j(m)} E[R_{t,j} \chi\{t_{t,j} \geq t_{i,j} - t + 1\}]
\]

\[
\leq \sum_{t \notin T_j(m)} E[R_{t,j} \chi\{t_{t,j} + t \in T_j(m)\}] + \sum_{t \in T_j(m)} E[R_{t,j} \chi\{t_{t,j} + t \notin T_j(m)\}]
\]

\[
\leq \sum_{t \in T_j(m)} E[\chi\{t_{t,j} + t \in T_j(m)\}] + \sum_{t \in T_j(m)} E[\chi\{t_{t,j} + t \notin T_j(m)\}]
\]

\[
\leq C_{1,j}(m) + C_{2,j}(m),
\]

where we have use the fact that \( R_{t,j} \in [0, 1] \) and the last line follows from the definition of \( C_{1,j}(m) \) and \( C_{2,j}(m) \). Substituting this into (7) and using the shorthand \( C_1 = C_{1,j}(m) \) and \( C_2 = C_{2,j}(m) \), we can use
Bernstein’s Inequality (see for example [4]) to get,

\[
P \left( \left| \sum_{t \in T_j} X_t - \sum_{t \in T_j} R_{t,j} \right| > md \right) \\
\leq P \left( \left| \sum_{t \notin T_j} \left( R_{t,j} \chi \{ \tau_{t,j} + t \in T_j(\bar{m}) \} - E[R_{t,j} \chi \{ \tau_{t,j} + t \in T_j(\bar{m}) \}] \right) \right| \geq \frac{md - C_1 - C_2}{2} \right) \\
+ P \left( \left| \sum_{t \notin T_j} \left( R_{t,j} \chi \{ \tau_{t,j} + t \notin T_j(\bar{m}) \} - E[R_{t,j} \chi \{ \tau_{t,j} + t \notin T_j(\bar{m}) \}] \right) \right| \geq \frac{md - C_1 - C_2}{2} \right) \\
\leq \exp \left\{ -\frac{(md - C_1 - C_2)^2}{4} \frac{1}{2} \sum_{t \notin T_j} \var(R_{t,j} \chi \{ \tau_{t,j} + t \in T_j(\bar{m}) \}) \right\} \\
+ \exp \left\{ -\frac{(md - C_1 - C_2)^2}{4} \frac{1}{2} \sum_{t \notin T_j} \var(R_{t,j} \chi \{ \tau_{t,j} + t \notin T_j(\bar{m}) \}) \right\}. \quad (8)
\]

However, using the fact that \( R_{t,j} \in [0, 1] \),

\[
\sum_{t \notin T_j} \var(R_{t,j} \chi \{ \tau_{t,j} + t \in T_j(\bar{m}) \}) \leq \sum_{t \notin T_j} E[R_{t,j}^2 \chi \{ \tau_{t,j} + t \in T_j(\bar{m}) \}]
\leq \sum_{t \notin T_j} E[\chi \{ \tau_{t,j} + t \in T_j(\bar{m}) \}] = \sum_{t \notin T_j} P(\tau_{t,j} + t \in T_j(\bar{m})) \leq C_1,
\]

and likewise,

\[
\sum_{t \notin T_j} \var(R_{t,j} \chi \{ \tau_{t,j} + t \notin T_j(\bar{m}) \}) \leq C_2.
\]

Substituting this into (8) gives,

\[
P \left( \left| \sum_{t \in T_j} X_t - \sum_{t \in T_j} R_{t,j} \right| > md \right) \\
\leq \exp \left\{ -\frac{(md - C_1 - C_2)^2}{4(2C_1 + md - C_1 - C_2)} \right\} + \exp \left\{ -\frac{(md - C_1 - C_2)^2}{4(2C_2 + md - C_1 - C_2)} \right\}
\leq \exp \left\{ -\frac{(md - C_1 - C_2)^2}{4(2C_1 + md - C_1 - C_2)} \right\} + \exp \left\{ -\frac{(md - C_1 - C_2)^2}{4(2C_2 + md - C_1 - C_2)} \right\}
\]

15
\[
\leq \exp \left\{ -\frac{(md + C_1 - C_2)^2 - 4C_1(md + C_1 - C_2) + 4C_2^2}{4(md + C_1 - C_2)} \right\} + \exp \left\{ -\frac{(md + C_2 - C_1)^2 - 4C_2(md + C_2 - C_1) + 4C_1^2}{4(md - C_1 + C_2)} \right\}
\]

\[
= \exp\{-\frac{1}{4}(md - 3C_1 - C_2)\} + \exp\{-\frac{1}{4}(md - C_1 - 3C_2)\}
\]

\[\square\]

**Theorem 6** (Theorem 3 in main text) For all \(\lambda \geq \lambda^*\), the total expected regret of the IUCB-DAF algorithm (Algorithm 4) up to time \(T\) in the case of unbounded delays can be upper bounded by

\[
\sum_{j \in A : \Delta_j > \lambda} \left( \frac{8(2 + 8\Delta_j) + 2\sqrt{1 + 8\Delta_j} \log(T\Delta_j^2)}{\Delta_j} + \frac{64e\log(T\Delta_j^2)}{\Delta_j} + 128c + \Delta_j \right)
\]

\[
+ \sum_{j \in A : \Delta_j > \lambda} \frac{288}{\Delta_j} + \sum_{j \in A : 0 < \Delta_j \leq \lambda} \frac{192}{\lambda} + \max_{j \in A : \Delta_j \leq \lambda} \Delta_j T.
\]

**Proof:** As in [2], for each suboptimal arm \(j\), define \(m_j = \min\{m|\tilde{\Delta}_m < \frac{\Delta_j}{T}\}\). We also define events

\[
E_j(m) = \left\{ \hat{r}_j - \mu_j > \sqrt{\frac{\log(T\Delta_m^2)}{2m} + \frac{md_{m_j}}{n_{m,j}}} \right\}
\]

and

\[
F_j(m) = \left\{ \left| \sum_{t \in T_{j}(m)} X_t - \sum_{t \in T_{j}(m)} R_{t,j} \right| > md_{m,j} \right\}
\]

respectively to be the events that the reward confidence bounds and the delay bounds fail for each arm \(j\) in round \(m\). Then, for all \(m\) and \(j\), it holds that

\[
P(E_j(m)) \leq \frac{1}{T\Delta_m} \quad \text{and} \quad P(F_j(m)) \leq \frac{2}{T\Delta_m}.
\]

Then, as in [2], we consider suboptimal arms in \(A' = \{j \in A|\Delta_j > \lambda\}\) and analyze the regret in three cases. Remember that \(L_m\) denotes the set of active arms at round \(m\) of IUCB-DAF.

**Case (a):** Some suboptimal arm \(j\) is not eliminated in round \(m_j\) or before, with an optimal arm \(\ast \in L_{m_j}\). This case can occur for one of three reasons; either the reward confidence bounds fail for arm \(i\) or \(\ast\) (or both) in round \(m_j\), or the delay bounds fail for arm \(j\) or \(\ast\) (or both) in round \(m_j\), or both the reward and delay confidence bounds fail for arm \(j\) or \(\ast\) (or both) in round \(m_j\). Hence, the probability that this occurs is,

\[
P(E_j(m_j) \cup E_{\ast}(m_j) \cup F_j(m_j) \cup F_{\ast}(m_j))
\]

\[
\leq P(E_j(m_j)) + P(E_{\ast}(m_j)) + P(F_j(m_j)) + P(F_{\ast}(m_j)) \leq \frac{6}{T\Delta_m^2}.
\]

We then take the trivial regret bound of \(T\Delta_j\) for each arm \(j \in A'\) and sum over all arms in this set to get a regret bound for this case of \(\sum_{j \in A'} \frac{6T\Delta_j}{T\Delta_m^2} = \sum_{j \in A'} \frac{96}{\Delta_m^2}\).

**Case (b):** For every suboptimal arm \(j \in A'\), either \(j\) is eliminated in round \(m_j\) or before, or the optimal arm \(\ast \not\in L_{m_j}\).

**Case (b1):** Every suboptimal arm \(j \in A'\) is eliminated in or before round \(m_j\) and \(\ast \in L_{m_j}\).

For a fixed suboptimal arm \(j\), this will occur when \(E_j(m_j)^C\) and \(E_{\ast}(m_j)\) occur, i.e. when the confidence bounds on the reward hold. We would expect that for the confidence bounds to hold, \(F_j(m_j)^C\) would occur, however, even if \(F_j(m_j)\) occurs, it is still possible that the confidence bounds on the reward will hold.
Therefore, we do not need to consider \( F_j(m_j) \) in order to calculate the regret in this case. To get the regret, using the bound on \( n_{m,j} \) in terms of \( c \), we can bound the contribution of arm \( j \) to the regret by,

\[
\frac{1}{2\Delta_j^2} \left( \sqrt{\log(T\Delta_j^2)} + \sqrt{(1+16\Delta_j) \log(T\Delta_j^2) + 16\Delta_j mc} \right)^2 \Delta_j
\]

\[
\leq \left( \frac{8}{\Delta_j} \left( \sqrt{\log(T\Delta_j^2/4)} + \sqrt{(1+8\Delta_j) \log(T\Delta_j^2/4) + 8\Delta_j log_2(2/\Delta_j) c} \right)^2 + 1 \right) \Delta_j
\]

\[
\leq \frac{8}{\Delta_j} \left( \sqrt{\log(T\Delta_j^2)} + \sqrt{(1+8\Delta_j) \log(T\Delta_j^2) + 16c} \right)^2 + \Delta_j
\]

\[
\leq \frac{8(2+8\Delta_j + 2\sqrt{1+8\Delta_j}) \log(T\Delta_j^2)}{\Delta_j} + \frac{64clog(T\Delta_j^2)}{\Delta_j} + \frac{128c}{\Delta_j} + \Delta_j,
\]

where we have used \( m \leq \log_2(2/\Delta_i) \) from [5]. Hence, summing over all suboptimal arms \( j \) gives regret for this case of

\[
\sum_{j \in A'} \left( \frac{8(2+8\Delta_j + 2\sqrt{1+8\Delta_j}) \log(T\Delta_j^2)}{\Delta_j} + \frac{64clog(T\Delta_j^2)}{\Delta_j} + \frac{128c}{\Delta_j} + \Delta_j \right).
\]

**Case (b2):** The optimal arm * is eliminated by some suboptimal arm \( j \in A'' = \{ j \in A | \Delta_j > 0 \} \) in some round \( m^* \).

Similar to Case (a), for the optimal arm * to be eliminated by a suboptimal arm \( i \), one of the events \( E_j(m^*), E_i(m^*), F_j(m^*), F_i(m^*) \) must occur, and \( P(E_j(m^* \cup E_i(m^* \cup F_j(m^* \cup F_i(m^*)) \leq \frac{6}{T\Delta_{m^*}} \). Using the same argument as [2] for Case (b2) gives a bound on the regret for this case of

\[
\max_{m^* = 0} \sum_{j \in A''; m^* \geq m^*} 6T\Delta_{m^*} \max_{i \in A''; m^* \geq m^*} \Delta_i \leq \sum_{j \in A'} \frac{192}{\Delta_j} + \sum_{j \in A'' \setminus A'} \frac{192}{\lambda}.
\]

Summing up the regret over these three cases and using the trivial regret bound of \( \max_{j \in A, 0 < \Delta_j \leq \lambda \Delta_j T} \) for \( j \notin A' \) gives the total regret bound of

\[
\sum_{j \in A; \Delta_j > \lambda} \left( \frac{8(2+8\Delta_j + 2\sqrt{1+8\Delta_j}) \log(T\Delta_j^2)}{\Delta_j} + \frac{64clog(T\Delta_j^2)}{\Delta_j} + \frac{128c}{\Delta_j} + \Delta_j \right)
\]

\[
+ \sum_{j \in A; \Delta_j > \lambda} 288 \Delta_j + \sum_{j \in A; 0 < \Delta_j \leq \lambda} \frac{192}{\lambda} + \max_{j \in A; \Delta_j \leq \lambda} \Delta_j T.
\]

\[\square\]

### C.2 Parameter Choices for General Unbounded Delay

#### C.2.1 Choice of \( d_{m,j} \)

In Algorithm 1, we need to select \( d_{m,j} \) to ensure that after each phase \( m \), the probability of \( \{ | \sum_{t \in T_j(m)} X_t - \sum_{t \in T_j(m)} R_{t,j} | > md_{m,j} \} \) is less than \( \frac{1}{T\Delta_m^2} \). This we select \( d = d_{m,j} \) such that,

\[
\exp\left\{ -1/4(md - 3C_1 - C_2) \right\} \leq \frac{1}{T\Delta_m^2}
\]

\[\implies d \geq \frac{4\log(T\Delta_m^2) + 3C_{1,m} + C_{2,m}}{m},\]

17
and,
\[ \exp \left\{ -\frac{1}{4} (md - C_1 - 2C_2) \right\} \leq \frac{1}{T\Delta_m^2} \]
\[ \Rightarrow d \geq \frac{4 \log(T\hat{\Delta}_m^2) + C_{1,j}(m) + 3C_{2,j}(m)}{m} \]

Thus,
\[ d_{m,j} = \max \left\{ \frac{4 \log(T\hat{\Delta}_m^2) + C_{1,j}(m) + C_{2,j}(m)}{m}, \right. \]
\[ \left. \quad \frac{4 \log(T\hat{\Delta}_m^2) + C_{1,j}(m) + 3C_{2,j}(m)}{m} \right\} = \frac{4 \log(T\hat{\Delta}_m^2) + C_{1,j}(m) + C_{2,j}(m) + 2 \max\{C_{1,j}(m), C_{2,j}(m)\}}{m} \]

C.2.2 Choice of \( n_{m,j} \)

In IUCB-DAF, \( n_{m,j} \) is the total number of samples required by the algorithm of arm \( j \) after it has been played in \( m \) rounds. In the regret analysis, we require that \( n_{m,j} \) is chosen such that the width of the confidence interval after \( m \) rounds is less than \( \tilde{\Delta}_m \). This ensures that, as long as the confidence intervals hold, any arm \( j \) is eliminated in the round \( m \) where \( \tilde{\Delta}_m < \frac{\Delta_j}{2} \). To calculate this value of \( n_{m,j} \), first note that,
\[ md_{m,j} = 4 \log(T\hat{\Delta}_m^2) + C_{1,j}(m) + C_{2,j}(m) + 2 \max\{C_{1,j}(m), C_{2,j}(m)\} \]

(9)

Hence, since the width of the confidence interval is given by \( \sqrt{\frac{\log(T\hat{\Delta}_m^2)}{2n_{m,j}} + \frac{md_{m,j}}{n_{m,j}}} \), we find \( n_{m,j} \) by solving the inequality
\[ \sqrt{\frac{\log(T\hat{\Delta}_m^2)}{2n_{m,j}}} + \frac{md_{m,j}}{n_{m,j}} \leq \frac{\tilde{\Delta}_m}{2} \]
\[ \Rightarrow \frac{\tilde{\Delta}_m}{2} n_{m,j} - \sqrt{n_{m,j} \frac{\log(T\hat{\Delta}_m^2)}{2}} - md_{m,j} \geq 0 \]
\[ \Rightarrow \sqrt{n_{m,j}} \geq \frac{1}{\Delta_m} \left( \sqrt{\frac{\log(T\hat{\Delta}_m^2)}{2}} + \sqrt{\frac{\log(T\hat{\Delta}_m^2)}{2}} + 2md_{m,j}\Delta_m \right) \]
\[ \Rightarrow n_{m,j} \geq \frac{1}{2\Delta_m^2} \left( \sqrt{\log(T\hat{\Delta}_m^2)} + \sqrt{\frac{\log(T\hat{\Delta}_m^2)}{2}} + 4md_{m,j}\Delta_m \right)^2 . \]

Substituting in (9) gives
\[ n_{m,j} \geq \frac{1}{2\Delta_m^2} \left( \sqrt{\log(T\hat{\Delta}_m^2)} + \sqrt{(1 + 16\tilde{\Delta}_m) \log(T\hat{\Delta}_m^2) + 4\Delta_m(C_{1,j}(m) + C_{2,j}(m)} \right) \]
\[ + 2 \max\{C_{1,j}(m), C_{2,j}(m)\} \right)^2 . \]

Hence, we define
\[ n_{m,j} = \left[ \frac{1}{2\Delta_m^2} \left( \sqrt{\log(T\hat{\Delta}_m^2)} + \sqrt{(1 + 16\tilde{\Delta}_m) \log(T\hat{\Delta}_m^2) + 4\Delta_m(C_{1,j}(m) + C_{2,j}(m)} \right) \right)^2 . \]
If there exists a constant $c$ such that,

$$C_1(s_j, t_j, m) \leq mc \quad \text{and} \quad C_2(s_j, t_j, m) \leq mc,$$

we can define

$$n_{m,j} = \left\lfloor \frac{1}{2\Delta_m^2} \left( \sqrt{\log(T\Delta_m^2)} + \sqrt{(1 + 16\Delta_m) \log(T\Delta_m^2) + 16\Delta_m mc} \right)^2 \right\rfloor.$$

**C.2.3 Choice of $\lambda^*$**

In the regret bound of IUCB-DAF, $\lambda^*$ represents the minimum value $\Delta_m$ can take. This ensures that the total number of samples in any one round is less than the horizon, $T$. We use the bounds $C_{1,j}(m), C_{2,j}(m) \leq mc$ and select $m$ (and $\Delta_m = \frac{1}{2^m}$) such that,

$$\frac{1}{2\Delta_m^2} \left( \sqrt{\log(T\Delta_m^2)} + \sqrt{(1 + 16\Delta_m) \log(T\Delta_m^2) + 16\Delta_m mc} \right)^2 \leq T.$$

Substituting in $\Delta_m = \sqrt{\frac{a}{T}}$ (and consequently $m = \log_2 \sqrt{T/a}$) for some constant $a > 1$ gives,

$$\frac{2T}{\sqrt{a}} \left( \sqrt{\log(a)} + \sqrt{\left( 1 + 16\sqrt{\frac{a}{T}} \right) \log(a) + 16\sqrt{\frac{a}{T}} \log_2 \sqrt{\frac{T}{a}}} \right)^2 \leq T.$$

$$\leq \frac{2T}{\sqrt{a}} \left( a^{1/4} + \sqrt{\left( 1 + 16\sqrt{\frac{a}{T}} \right) \sqrt{a} + 16\sqrt{\frac{a}{T}} \sqrt{\frac{T}{a}}} \right)^2 \leq \frac{2T}{\sqrt{a}} \left( 1 + \sqrt{\left( 1 + 16\sqrt{\frac{a}{T}} \right) + 16c} \right)^2 \leq \frac{2T}{\sqrt{a}} (1 + \sqrt{17 + 16c})^2 \leq T$$

$$\implies \sqrt{a} \geq \frac{(1 + \sqrt{17 + 16c})^2}{2}.$$

Where we have used the facts that $\log x \leq \sqrt{x}, \log_2 x \leq x, a > 1$ and $\sqrt{a/T} = \Delta_m < 1$ by definition of $\Delta_m$. Hence, we set,

$$\lambda^* = \frac{(1 + \sqrt{17 + 16c})^2}{2\sqrt{T}}.$$
D Alternative choices of $C$

D.1 Calculation of $C$ in the Sub-Gaussian Setting

Lemma 7 Assume that $\tau_j$ is $\rho_j$-sub-Gaussian for some $\rho_j \leq \rho$ and $E[\tau_j] \leq B$ for all $1 \leq j \leq K$ and let $s_j = (s_{1,j}, \ldots, s_{m,j})$ be the time points when we started playing arm $j$ in each phase $1 \leq i \leq m$. Then, for any arm $j$,

$$\sum_{\substack{t \in T_j(m) \ni \tau_j \in t \in T_j(m) \ni \sum_{i=2}^{m} \left(1 - \exp\left\{-\frac{s_{i,j} - |B|}{2 \rho_j}\right\} \right) + [B] - 1$$

$$+ \chi\{s_{1,j} > 1\} \left(1 - \exp\left\{-\frac{s_{1,j} - |B|}{2 \rho_j}\right\} \right) + [B] - 1).$$

Proof: First note that the expected number of rewards from other arms obtained in $T_j(m)$ can be decomposed by phase,

$$\sum_{\substack{t \in T_j(m) \ni \tau_j \in t \in T_j(m) \ni \sum_{i=1}^{m} \sum_{t=1}^{s_{i,j}-1} P(\tau_{t,j} > s_{i,j} - t).$$

Then, for all $1 < i \leq m$,

$$\sum_{t=1}^{s_{i,j}-1} P(\tau_{t,j} > s_{i,j} - t) = \sum_{t=1}^{s_{i,j}-1} P(\tau_{t,j} - E[\tau_j] \geq s_{i,j} - t - E[\tau_j])$$

$$\leq \sum_{t=1}^{s_{i,j}-1} P(\tau_{t,j} - E[\tau_j] \geq s_{i,j} - t - B)$$

$$= \sum_{t=1}^{s_{i,j}-1} P(\tau_{t,j} - E[\tau_j] \geq s_{i,j} - t - B)$$

$$+ \sum_{t=s_{i,j}-1}^{s_{i,j}-1} P(\tau_{t,j} - E[\tau_j] \geq s_{i,j} - t - B)$$

$$\leq \sum_{t=1}^{s_{i,j}-1} \exp\left\{-\frac{(s_{i,j} - t - B)^2}{2 \rho_j}\right\} + [B]$$

$$\leq \sum_{t=1}^{s_{i,j}-1} \exp\left\{-\frac{t^2}{2 \rho}\right\} + [B]$$

$$\leq \sum_{t=0}^{s_{i,j}-1} \left(\exp\left\{-\frac{1}{2 \rho}\right\}\right) + [B]$$

$$= \frac{1 - \exp\left\{-\frac{s_{i,j} - |B|}{2 \rho_j}\right\} \right) + [B] - 1.$$
D.2 Calculation of $C$ in the Setting with Bounded Second Moment

**Lemma 8** Assume that $\tau_j$ is $\rho_j$-sub-Gaussian for some $\rho_j \leq \rho$ and $E[\tau_j] \leq B$ for all $1 \leq j \leq K$ and let $s_j = (s_{1,j}, \ldots, s_{m,j})$ be the time points when we started playing arm $j$ in each phase $1 \leq i \leq m$. Then, for any arm $j$,

$$
\sum_{t \in T_j(m)} P(\tau_{t,j} + t \in T_j(m)) \leq \sum_{i=2}^{m} \left( \zeta \left(1 + \frac{1}{\zeta + 1} - \frac{1}{s_{i,j} - 1 - [B]} \right) + [B] \right) + \chi\{s_{1,j} > 1\} \zeta \left(1 + \frac{1}{\zeta + 1} - \frac{1}{s_{1,j} - 1 - [B]} \right).
$$

**Proof:** First note that the expected number of rewards from other arms obtained in $T_j(m)$ can be decomposed by phase,

$$
\sum_{t \in T_j(m)} P(\tau_{t,j} + t \in T_j(m)) \leq \sum_{i=1}^{m} \sum_{l=1}^{s_{i,j} - 1} P(\tau_{t,j} > s_{i,j} - t).
$$

Then, for all $1 < i \leq m$,

$$
\sum_{t=1}^{s_{i,j} - 1} P(\tau_{t,j} \geq s_{i,j} - t) = \sum_{t=1}^{s_{i,j} - 1} P(\tau_{t,j} - E[\tau_j] \geq s_{i,j} - t - E[\tau_j])
\leq \sum_{t=1}^{s_{i,j} - 1} P(\tau_{t,j} - E[\tau_j] \geq s_{i,j} - t - B)
\leq \sum_{t=1}^{s_{i,j} - [B] - 1} P(\tau_{t,j} - E[\tau_j] \geq s_{i,j} - t - B) + \sum_{t=s_{i,j} - [B]}^{s_{i,j} - 1} \frac{\var(\tau_{t,j})}{\zeta} + \var(\tau_{t,j}) + (s_{i,j} - t - B)^2 + [B]
\leq \sum_{t=1}^{s_{i,j} - [B] - 1} \frac{\var(\tau_{t,j})}{\zeta + (s_{i,j} - t - B)^2} + [B]
\leq \sum_{t=1}^{s_{i,j} - [B] - 1} \frac{\zeta}{\zeta + t^2} + [B]
\leq \frac{\zeta}{\zeta + 1} + \sum_{l=2}^{s_{i,j} - [B] - 1} \frac{1}{\zeta + l^2} + [B]
\leq \frac{\zeta}{\zeta + 1} + \zeta \sum_{l=2}^{s_{i,j} - [B] - 1} \frac{1}{l(l - 1)} + [B]
\leq \frac{\zeta}{\zeta + 1} + \zeta \left(1 - \frac{1}{s_{i,j} - [B] - 1} \right) + [B]
\leq \frac{\zeta}{\zeta + 1} + \zeta \left(1 - \frac{1}{s_{i,j} - [B] - 1} \right) + [B].
$$

21
Then, if

\[ \text{Proof: } \]

For the first inequality,

In the case where the delays are large, it is practically beneficial to separate the effects of the round number bound \( s \) and \( k \). \( E[k] \)

\[ \sum_{k=2}^{a} \left( \frac{1}{k-1} - \frac{1}{k} \right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{a-1} - \frac{1}{a} = 1 - \frac{1}{a}. \]

For the first phase where arm \( j \) is played \( (i = 1) \) if \( s_{1,j} > 1 \), arm \( j \) is not the first arm played, and we can bound \( \sum_{i=1}^{s_{1,j}^{-1}} P(\tau_{i,j} > s_{i,j} - t) \) as above. However, if arm \( j \) is the first arm played and \( s_{1,j} = 1 \), there will be no chance of rewards of other arms being received in the first phase of playing arm \( j \). This gives the result.

\[ \square \]

### D.3 Alternative choice of \( C \) for large delays

In the case where the delays are large, it is practically beneficial to separate the effects of the round number \( m \) and \( E[\tau] \) in the definition of \( C \). This can be done by means of the following lemma.

**Lemma 9** Assume there is some random variable \( \tau \) governing the delay of all arms and that \( E[\tau] < \infty \). Then, if \( s_{i,j} \) and \( t_{i,j} \) are the start and end points of phase \( i \) of playing arm \( j \) for \( 1 \leq j \leq K, 1 \leq i \leq m \),

\[ \sum_{t \notin T_j(m)} P(\tau_{i,j} + t \in T_j(m)) \leq \lfloor \nicefrac{1}{2}m(m + 1) \rfloor + E[\tau] \]

and

\[ \sum_{t \in T_j(m)} P(\tau_{i,j} + t \notin T_j(m)) \leq \lfloor \nicefrac{1}{2}m(m + 1) \rfloor + E[\tau] \]

**Proof:** For the first inequality,

\[
\sum_{t \notin T_j(m)} P(\tau_{i,j} + t \in T_j(m)) \leq \sum_{i=1}^{m} \sum_{t=1}^{s_{i,j}^{-1}} P(\tau_{i,j} > s_{i,j} - t) \\
= \sum_{i=1}^{m} \sum_{l=1}^{s_{i,j}^{-1}} P(\tau > l) \\
= m \sum_{l=1}^{s_{i,j}^{-1}} P(\tau > l) + (m - 1) \sum_{l=s_{1,j}}^{s_{2,j}^{-1}} P(\tau > l) + \cdots + \sum_{l=s_{m,j}^{-1}}^{s_{m,j}^{-1}} P(\tau > l) \\
= \sum_{i=1}^{m} \left( \sum_{l=s_{i,j}^{-1}} \left( \sum_{q=1}^{m} (m - q + 1) \right) P(\tau = l) \right) \\
= \sum_{i=1}^{m} \left( \sum_{l=s_{i,j}^{-1}} \left( \sum_{q=m-i+1}^{m} q \right) P(\tau = l) \right) \\
\leq \sum_{i=1}^{m} \left( \sum_{l=s_{i,j}^{-1}} \left( \lfloor \nicefrac{1}{2}m(m + 1) \rfloor P(\tau = l) \right) \\
= \sum_{l=1}^{1/2m(m + 1)} P(\tau = l)
\]

22
\[
\begin{align*}
&= \lfloor \frac{1}{2m}(m+1) \rfloor P(\tau \leq \lfloor \frac{1}{2m}(m+1) \rfloor) \\
&\quad + \sum_{l=\lfloor \frac{1}{2m}(m+1) \rfloor}^{s_{m,j} \cdot -1} \frac{1}{2m}(m+1)P(\tau = l) \\
&\leq \lfloor \frac{1}{2m}(m+1) \rfloor + \sum_{l=\lfloor \frac{1}{2m}(m+1) \rfloor}^{s_{m,j} \cdot -1} lP(\tau = l) \\
&\leq \lfloor \frac{1}{2m}(m+1) \rfloor + E[\tau].
\end{align*}
\]

Similarly, for the second inequality,

\[
\sum_{t \in T_j(m)} P(\tau_{t,j} + t \notin T_j(m)) \leq \sum_{i=1}^{m} \sum_{l=\lfloor s_{i,j} \rfloor}^{t_{i,j}} P(\tau_{t,j} > t_{i,j} - t) \\
= \sum_{i=1}^{m} \sum_{l=0}^{s_{i,j} - t_{i,j}} P(\tau > l) \\
\leq \sum_{i=1}^{m} \sum_{l=0}^{n_{i,j}} P(\tau > l) \\
= m \sum_{l=1}^{n_{1,j}} P(\tau > l) + (m - 1) \sum_{l=n_{1,j}+1}^{n_{2,j}} P(\tau > l) + \ldots \\
\ldots + \sum_{l=n_{m-1,j}+1}^{n_{m,j}} P(\tau > l) \\
= \sum_{i=1}^{m} \left( \sum_{l=n_{i-1,j}+1}^{n_{i,j}} \left( \sum_{q=1}^{i} (m - q + 1) P(\tau = l) \right) \right) \\
= \lfloor \frac{1}{2m}(m+1) \rfloor P(\tau \leq \lfloor \frac{1}{2m}(m+1) \rfloor) \\
\quad + \sum_{l=\lfloor \frac{1}{2m}(m+1) \rfloor}^{s_{m,j} \cdot -1} lP(\tau = l) \\
\leq \lfloor \frac{1}{2m}(m+1) \rfloor + E[\tau].
\]

We can then define \( C_{1,j}(m) = C_{2,j}(m) = \lfloor \frac{1}{2m}(m+1) \rfloor + E[\tau] \) to be used in the IUCB-DAF algorithm. Like with the standard definition of \( C_{1,j}(m) \) and \( C_{2,j}(m) \), we can amend this slightly to use a bound \( B \) on \( E[\tau] \) or to include different delay distributions for each arm. As demonstrated in Section 8, this leads to improved regret in many experimental settings. However, since there is no constant \( c \) such that \( C_{1,j}(m), C_{2,j}(m) \leq mc \), we cannot directly substitute \( c \) into Theorem 3 to get the theoretical regret bounds for IUCB-DAF with this choice of \( C_{1,j}(m) \) and \( C_{2,j}(m) \). Instead, we amend Theorem 3 slightly to get slightly improved dependence on \( E[\tau] \) (in terms of constants) but worse dependence on \( \sqrt{\log(T \Delta^2_j)} \) and constants as demonstrated in Theorem 10.
Theorem 10  For all $\lambda \geq \lambda^*$, the total expected regret of the IUCB-DAF algorithm up to time $T$ in the case of unbounded delays with this alternative definition of $C$ can be upper bounded by

$$\sum_{j \in A; \Delta_j > \lambda} \left( \frac{8(2 + 8\Delta_j + 2\sqrt{1 + 8\Delta_j}) \log(T\Delta_j^2)}{\Delta_j} + \frac{32\sqrt{2E[\tau] \log(T\Delta_j^2)}}{\sqrt{\Delta_j}} + 64E[\tau] + \Delta_j + \frac{128}{\Delta_j^2} \right)$$

$$+ \frac{32\sqrt{\frac{2}{\Delta_j}} + 1\sqrt{2\log(T\Delta_j^2)}}{\Delta_j} + \frac{64}{\Delta_j} \right) + \sum_{j \in A; \Delta_j > \lambda} \frac{288}{\Delta_j} + \sum_{j \in A; 0 < \Delta_j \leq \lambda} \frac{192}{\lambda} + \max_{j \in A; \Delta_i \leq \lambda} \Delta_j T.$$ 

Proof: The proof follows exactly as the proof of Theorem 3, the only difference comes from the fact that in Case (b1) we need to use the alternative definition of $C_{1,j}, C_{2,j}$ to bound $n_{m,j}$ as follows.

$$\left\lceil \frac{1}{2\Delta_m^2} \left( \sqrt{\log(T\Delta_m^2)} + \sqrt{(1 + 16\Delta_m) \log(T\Delta_m^2) + 16\Delta_m C_{1,j}} \right)^2 \right\rceil$$

$$= \left\lceil \frac{1}{2\Delta_m^2} \left( \sqrt{\log(T\Delta_m^2)} + \sqrt{(1 + 16\Delta_m) \log(T\Delta_m^2) + 16\Delta_m ([1/2m(m + 1)] + E[\tau])} \right)^2 \right\rceil$$

$$\leq \left\lceil \frac{8}{\Delta_j^2} \left( \sqrt{\log(T\Delta_j^2)} + \sqrt{(1 + 8\Delta_j) \log(T\Delta_j^2) + 8\Delta_j (1/2 \log_2(2/\Delta_j) + 1/2 \log_2(2/\Delta_j) + E[\tau])} \right)^2 \right\rceil$$

$$\leq \frac{8}{\Delta_j^2} \left( \sqrt{\log(T\Delta_j^2)} + \sqrt{(1 + 8\Delta_j) \log(T\Delta_j^2) + 8\Delta_j \left( \frac{1}{\Delta_j^2} + 1 + E[\tau] \right)} \right)^2 + 1$$

$$\leq \frac{8(2 + 8\Delta_j + 2\sqrt{1 + 8\Delta_j}) \log(T\Delta_j^2)}{\Delta_j^2} + \frac{64}{\Delta_j^2} \left( \frac{2}{\Delta_j^2} + 1 + \Delta_j E[\tau] \right)$$

$$+ \frac{32\sqrt{2\log(T\Delta_j^2)}}{\sqrt{\Delta_j}} + \frac{1 + \Delta_j E[\tau]}{\Delta_j} + 1.$$ 

Thus the regret bound in the case with this alternative definition of $C$ becomes

$$\sum_{j \in A; \Delta_j > \lambda} \left( \frac{8(2 + 8\Delta_j + 2\sqrt{1 + 8\Delta_j}) \log(T\Delta_j^2)}{\Delta_j} + \frac{32\sqrt{2E[\tau] \log(T\Delta_j^2)}}{\sqrt{\Delta_j}} + 64E[\tau] + \Delta_j + \frac{128}{\Delta_j^2} \right)$$

$$+ \frac{32\sqrt{\frac{2}{\Delta_j}} + 1\sqrt{2\log(T\Delta_j^2)}}{\Delta_j} + \frac{64}{\Delta_j} \right) + \sum_{j \in A; \Delta_j > \lambda} \frac{288}{\Delta_j} + \sum_{j \in A; 0 < \Delta_j \leq \lambda} \frac{192}{\lambda} + \max_{j \in A; \Delta_i \leq \lambda} \Delta_j T.$$ 

□