POINTWISE LONG TIME BEHAVIOR FOR THE MIXED DAMPED NONLINEAR WAVE EQUATION IN $\mathbb{R}^n_+$

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(Communicated by Seok-Bae Yun)

Abstract. In this paper, we investigate the long time behavior of the solution for the nonlinear wave equation with frictional and visco-elastic damping terms in $\mathbb{R}^n_+$. It is shown that for the long time, the frictional damped effect is dominated. The Green’s functions for the linear initial boundary value problem can be described in terms of the fundamental solutions for the full space problem and reflected fundamental solutions coupled with the boundary operator. Using the Duhamel’s principle, we get the pointwise long time behavior of the solution $\partial_\alpha^\gamma x u$ for $|\alpha| \leq 1$.

1. Introduction. In this paper, we study the pointwise long time behavior of the solution for the nonlinear wave equation with frictional and visco-elastic damping terms

\[
\left\{ \begin{array}{l}
\partial_t^2 u - c^2 \Delta u + \nu_1 \partial_t u - \nu_2 \partial_t \Delta u = f(u), \\
u|_{t=0} = u_0(x), \\
u_t|_{t=0} = u_1(x),
\end{array} \right.
\]

(1)

in multi-dimensional half space $\mathbb{R}^n_+ := \mathbb{R}_+ \times \mathbb{R}^{n-1}$, with absorbing and radiative boundary condition

\[
(a_1 \partial_{x_1} u + a_2 u) (x_1 = 0, x', t) = 0.
\]

(2)

$x = (x_1, x')$ is the space variable with $x_1 \in \mathbb{R}_+ := (0, \infty)$, $x' = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$, $t > 0$ is the time variable. $\nu_1$ and $\nu_2$ are positive constant viscosities, $a_1$ and $a_2$ are constants. The Laplacian $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$, $f(u)$ is the smooth nonlinear term and $f(u) = O(|u|^k)$ when $k > 0$.

2020 Mathematics Subject Classification. Primary: 35B40; Secondary: 35A08.

Key words and phrases. Initial boundary value problem, mixed damped, nonlinear wave equation, Green’s function method, pointwise estimate.

The first author is supported by the Fundamental Research Funds for the Central Universities, National Natural Science Foundation of China (No. 11671075, No.12001097) and Natural Science Foundation of Shanghai (No. 18ZR1401300).

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Over the past few decades, many mathematicians have concentrated on solving different kinds of damped nonlinear wave equations. The first kind is called the frictional damped wave equation, which is given as follows

\[
\begin{aligned}
\partial_t^2 u - c^2 \Delta u + \nu \partial_t u &= f(u), \\
\frac{\partial}{\partial t} u |_{t=0} &= u_0(x), \\
\frac{\partial}{\partial t} u |_{t=0} &= u_1(x),
\end{aligned}
\]  

(3)

which is given by the following

\[
\begin{aligned}
\partial_t^2 u - c^2 \Delta u + \nu \partial_t u &= f(u), \\
\frac{\partial}{\partial t} u |_{t=0} &= u_0(x), \\
\frac{\partial}{\partial t} u |_{t=0} &= u_1(x).
\end{aligned}
\]

(4)

The first kind is called the frictional damped wave, which is given by the following

\[
\begin{aligned}
\partial_t^2 u - c^2 \Delta u + \nu \partial_t u &= f(u), \\
\frac{\partial}{\partial t} u |_{t=0} &= u_0(x), \\
\frac{\partial}{\partial t} u |_{t=0} &= u_1(x),
\end{aligned}
\]  

see [9, 19, 20, 23] for the references. It is showed that for the long time, the fundamental solution for the linear system of (3) behaves like the Gauss kernel \( \frac{e^{- |c(t-t_0)|^2}}{(t+1)^2} \). The second kind is called the visco-elastic damped wave, which is given by the following

\[
\begin{aligned}
\partial_t^2 u - c^2 \Delta u - \nu \partial_t \Delta u &= f(u), \\
\frac{\partial}{\partial t} u |_{t=0} &= u_0(x), \\
\frac{\partial}{\partial t} u |_{t=0} &= u_1(x).
\end{aligned}
\]

(4)

One can refer to [22] for the decaying rate of the linear solution, [11, 12] for the asymptotic profiles of the linear problem, [4, 21] for the nonlinear equation, etc. In [9], the authors studied the fundamental solution for the linear system of (4). The results show that the hyperbolic wave transport mechanism and the visco-elastic damped mechanism interact with each other so that the solution behaves like the convected heat kernel, i.e., \( \frac{e^{- |c(t-t_0)|^2}}{(t+1)^2} \) for the odd dimensional case and \( \frac{e^{- |c(t-t_0)|^2}}{(t+1)^{n+1}} + \frac{H(ct-|x|)}{(1+1)^{n+1}} \) for the even dimensional case. The solution exhibits the generalized Huygens principle. For other damped wave equations, one can refer to [2, 27] for the damped abstract wave equation, and [14, 15, 16] for the existence and large time behavior of the solutions for the Cauchy problem of mixed damping (both frictional and visco-elastic damping terms are involved) wave equation.

For the initial-boundary value problem of the different damped wave equations, many authors studied the global well-posedness existence, long time behaviors, global attractors and decaying rate estimates of some elementary wave by using delicate energy estimate method, for example [1, 13, 25, 26, 28, 29]. In this paper, we will use the pointwise estimate technique to give the long time behavior of the solution for system (1) with boundary condition (2). The main part of this technique is the construction and estimation of the Green’s functions for the following linear systems:

\[
\begin{aligned}
\partial_t^2 G_1 - c^2 \Delta G_1 + \nu_1 \partial_t G_1 - \nu_2 \partial_t \Delta G_1 &= 0, x_1, y_1 > 0, x' \in \mathbb{R}^{n-1}, \quad t > 0, \\
G_1(x_1, x', 0; y_1) &= \delta(x_1 - y_1)\delta(x'), G_{1t}(x_1, x', 0; y_1) = 0, \\
a_1 \partial_{x_1} G_1(0, x', t; y_1) + a_2 G_1(0, x', t; y_1) &= 0;
\end{aligned}
\]  

(5)

\[
\begin{aligned}
\partial_t^2 G_2 - c^2 \Delta G_2 + \nu_2 \partial_t G_2 - \nu_2 \partial_t \Delta G_2 &= 0, x_1, y_1 > 0, x' \in \mathbb{R}^{n-1}, \quad t > 0, \\
G_2(x_1, x', 0; y_1) &= 0, G_{2t}(x_1, x', 0; y_1) = \delta(x_1 - y_1)\delta(x'), \\
a_1 \partial_{x_1} G_2(0, x', t; y_1) + a_2 G_2(0, x', t; y_1) &= 0.
\end{aligned}
\]  

(6)

The way of estimating the Green’s functions \( G_i \) was initiated by [17] and developed by [3, 5, 6, 8, 10, 18, 24] and the reference therein. Following the scheme
in [10], we will find the relations between the fundamental solutions for the linear Cauchy problem and Green’s functions for the linear half space problem, by comparing their symbols in the transformed tangential-spatial and time variables. Then the Green’s functions can be described in terms of the fundamental solutions and the boundary surface operator.

With the help of the accurate expression of Green’s functions for the linear half space problem and the Duhamel’s principle, we get the pointwise long time behavior for the nonlinear solution $\partial^\alpha u$, $|\alpha| \leq 1$. We only treat the case $a_1a_2 < 0$. The boundary condition of Dirichlet type ($a_1 = 0$) and Neumann type ($a_2 = 0$) are much simpler. For the case of $a_1a_2 > 0$, the linear problem is unstable.

The main results of our paper are given as follows:

**Theorem 1.1.** Let $n = 2, 3$ be the spatial dimension, $k > 1 + \frac{2}{n}$. Assume the initial data $(u_0(x), u_1(x)) \in (H^{l+1} \cap W^{1,1}) \times (H^l \cap W^{l,1})$, $l \geq \lceil \frac{n}{2} \rceil + 1$, and satisfy

$$\|\partial_x^\alpha u_0, \partial_t^\alpha u_1\| \leq O(1)\varepsilon(1 + |x|^2)^{-r}, \quad r > \frac{n}{2}, \quad |\alpha| \leq 1,$$

$\varepsilon$ sufficiently small, then there exists a unique global classical solution to the problem (1) with the mixed boundary condition (2) while $a_1a_2 < 0$. The solution has the following pointwise estimate:

$$|\partial_x^\alpha u(x, t)| \leq O(1)\varepsilon(1 + t)^{-|\alpha|/2}(1 + t + |x|^2)^{-\frac{n}{2}}.$$

Moreover, we get the following optimal $L^p(\mathbb{R}^n_+)$ estimate of the solution

$$\|\partial_x^\alpha u(\cdot, t)\|_{L^p(\mathbb{R}^n_+)} \leq O(1)\varepsilon(1 + t)^{-\frac{n}{2}(1-l)}(1 + t + |x|^2)^{-\frac{n}{2}}, \quad p \in (1, \infty].$$

**Remark 1.** We can develop a similar theorem for the case of higher space dimension with a suitable choice of $k$ which guarantees the existence of the solution. In Section 2.2, the approximation used in the calculation of the singular part depends on the space dimension. One could modify the short wave part expression of Green’s functions for the linear half space problem to prove the results for the general case.

**Notations.** Let $C$ and $O(1)$ be denoted as generic positive constants. For multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $|\alpha| = \sum_{i=1}^n \alpha_i$. Let $L^p$ denotes the usual $L^p$ space on $x \in \mathbb{R}^n_x$. For nonnegative integer $l$, we denote by $W^{l,p}(1 \leq p < \infty)$ the usual $L^p$-Sobolev space of order $l$: $W^{l,p} = \{u \in L^p(\mathbb{R}^n_x) : \partial_x^\alpha u \in L^p(|\alpha| \leq l)\}(l \geq 1), W^{0,p} = L^p$. The norm is denoted by $\|\cdot\|_{W^{l,p}} = \|u\|_{W^{l,p}} = \sum_{|\alpha| \leq l} \|\partial_x^\alpha u\|_{L^p}$. When $p = 2$, we define $W^{l,2} = H^l$ for all $l \geq 0$. We denote $D_\delta := \{\xi \in \mathbb{C}^n | |Im(\xi_i)| \leq \delta, i = 1, 2, \cdots, n\}$. Introduce the Fourier transform and Laplace transform of $f(x, t)$ as follows:

$$f(\xi, t) := \mathcal{F}[f](\xi, t) = \int_{\mathbb{R}^n_x} e^{-i\xi \cdot x} f(x, t) dx,$$

$$f(x, s) := \mathcal{L}[f](x, s) = \int_0^\infty e^{-st} f(x, t) dt.$$

The rest of paper is arranged as follows: in Section 2, we study the fundamental solutions for the linear Cauchy problem and give a pointwise description of the fundamental solutions in $(x, t)$ variables. We also describe the fundamental solutions in other transformed variables. In Section 3, the Green’s functions for the half space problem are constructed in the transformed tangential-spatial and time domain. By comparing the symbols in the transformed space, we get the relationship between the fundamental solutions and the Green’s functions. Finally in Section 4, we give
the long time behavior of the solution for the nonlinear problem. Some useful lemmas are given in Appendix.

2. Fundamental solutions for the linear Cauchy problem.

2.1. Fundamental solutions in \((\xi, t)\) variables. The fundamental solutions for the linear damped wave equations are defined by

\[
\begin{align*}
\partial_t^2 G_1 - c^2 \Delta G_1 + \nu_1 \partial_t G_1 - \nu_2 \partial_t \Delta G_1 &= 0 \\
G_1 (x, 0) &= \delta (x), G_{1t} (x, 0) = 0,
\end{align*}
\]

(7)

\[
\begin{align*}
\partial_t^2 G_2 - c^2 \Delta G_2 + \nu_1 \partial_t G_2 - \nu_2 \partial_t \Delta G_2 &= 0 \\
G_2 (x, 0) &= 0, G_{2t} (x, 0) = \delta (x).
\end{align*}
\]

(8)

Applying the Fourier transform to (7) and (8) in the space variable \(x\), one can compute the fundamental solutions \(G_i(\xi, t) (i = 1, 2)\) in the Fourier space,

\[
G_1(\xi, t) = \frac{\sigma_+ e^{\sigma_+ t} - \sigma_- e^{\sigma_- t}}{\sigma_+ - \sigma_-}, \quad G_2(\xi, t) = \frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-},
\]

\[
\sigma_\pm = -\frac{\nu_1 + \nu_2 |\xi|^2}{2} \pm \frac{1}{2} \sqrt{\nu_1^2 + (2\nu_1 \nu_2 - 4c^2)|\xi|^2 + \nu_2^2|\xi|^4}.
\]

2.2. Fundamental solutions in \((x, t)\) variables. In [16], authors have studied the pointwise estimates of the fundamental solutions by long wave-short wave decomposition in the Fourier space. Here we will use the local analysis and inverse Fourier transform to get the pointwise structures of the fundamental solutions in the physical variables \((x, t)\). Outside the finite Mach number region \(|x| \geq 3(t + 1)\), one can use the weighted energy estimates to get the exponentially decaying estimates of solution in time and space. Inside the finite Mach number region \(|x| \leq 4(t + 1)\), we will use the long wave short wave decomposition to get the long wave regular parts and short wave singular parts. Here the long wave and short wave are defined as follows:

\[
f(x, t) = f^L(x, t) + f^S(x, t),
\]

\[
\mathcal{F}[f^L] = H \left( 1 - \frac{|\xi|}{\varepsilon_0} \right) \mathcal{F}[f](\xi, t),
\]

\[
\mathcal{F}[f^S] = \left( 1 - H \left( 1 - \frac{|\xi|}{\varepsilon_0} \right) \right) \mathcal{F}[f](\xi, t),
\]

with the parameter \(\varepsilon_0 \ll 1\), the Heaviside function \(H(x)\) is defined by

\[
H(x) = \begin{cases} 
1, & x > 0, \\
0, & x < 0.
\end{cases}
\]

**Long wave component.** When \(|\xi| \leq \varepsilon_0 \ll 1\), we have the following Taylor expansion for \(\sigma_\pm \) and \(\sigma_+ - \sigma_-\):

\[
\begin{align*}
\sigma_+ &= -\frac{c^2 |\xi|^2}{\nu_1} + O(|\xi|^2), \\
\sigma_- &= -\nu_1 + (\nu_2 + \frac{c^2}{\nu_1})|\xi|^2 + O(|\xi|^2),
\end{align*}
\]

\[
\sigma_+ - \sigma_- = \nu_1 + \frac{(\nu_1 \nu_2 - 2c^2)}{\nu_1} |\xi|^2 + O(|\xi|^2).
\]
This non-decaying property results in the singularities of the fundamental solution. So we can approximate the fundamental solutions as follows.

\[
\sigma_+ e^{-\tau} = \left(-\frac{c^2\tau^2}{\nu_1} + o(\tau^2)\right) e^{t(-\nu_1 + (-\nu_2 + \frac{c^2}{\nu_1})\tau^2 + o(\tau^2))} = -\frac{c^2}{\nu_1} \tau^2 e^{-\nu_1 t} + o(\tau^2) e^{-C\tau^2},
\]

\[
\sigma_- e^{-\tau} = -\nu_1 + (-\nu_2 + \frac{c^2}{\nu_1}) \tau^2 + o(\tau^2) e^{t\left(-\frac{c^2\tau^2}{\nu_1} + o(\tau^2)\right)} = -\nu_1 e^{-\frac{c^2}{\nu_1} \tau^2} + O(\tau^2) e^{-C\tau^2},
\]

\[
\frac{1}{\sigma_+ - \sigma_-} = \frac{1}{\nu_1} + O(\tau^2).
\]

So we can approximate the fundamental solutions as follows.

\[
\frac{\sigma_+ e^{\sigma_+ t} - \sigma_- e^{\sigma_- t}}{\sigma_+ - \sigma_-} = -\frac{c^2\tau^2}{\nu_1} e^{-\nu_1 t} + \frac{c^2}{\nu_1} \tau^2 e^{-\nu_1 t} + o(\tau^2) e^{-C\tau^2} + O(\tau^2) e^{-C\tau^2},
\]

\[
\frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-} = \frac{1}{\nu_1} e^{-\frac{c^2}{\nu_1} \tau^2} e^{-\nu_1 t} - \frac{1}{\nu_1} e^{-\nu_1 t} + O(\tau^2) e^{-C\tau^2} + o(\tau^2) e^{-C\tau^2}.
\]

Using Lemma 5.1 in Appendix, for \(|\alpha| \geq 0\) we have

\[
|D_\alpha \tilde{G}_1^L(x, t)| \leq O(1) \left(\frac{e^{-\frac{|\alpha|^2}{1+(1+t)^{\alpha/2}}}}{(1+t)^{\alpha/2} + e^{-\frac{|\alpha|^2}{1+t}}}ight),
\]

\[
|D_\alpha \tilde{G}_2^L(x, t)| \leq O(1) \left(\frac{e^{-\frac{|\alpha|^2}{1+(1+t)^{\alpha/2}}}}{(1+t)^{\alpha/2} + e^{-\frac{|\alpha|^2}{1+t}}}ight).
\]

**Short wave component.** We adopt the local analysis method to give a description about all types of singular functions for the short wave component of the fundamental solutions. When \(|\tau| \geq N\) for \(N\) sufficiently large, we have the following Taylor expansion for \(\sigma_\pm\):

\[
\begin{aligned}
\sigma_+ &= -\frac{c^2}{\nu_2} + \frac{c^2(-\nu_2 - 1)}{\nu_2} \frac{1}{|\tau|^4} + O(|\tau|^{-4}), \\
\sigma_- &= -\sigma_+ - (\nu_1 + \nu_2) |\tau|^2.
\end{aligned}
\]

This non-decaying property results in the singularities of the fundamental solution \(G_i\) in spatial variable. To investigate the singularities, we approximate the spectra \(\sigma_\pm\) by \(\sigma^\pm_\pm\):

\[
\begin{aligned}
\sigma_+^\pm &= \frac{c^2}{\nu_2} + \frac{c^2(-\nu_2 - 1)}{\nu_2} \left(1 + \frac{1}{(1+|\tau|^2)^2}\right) + \frac{c^2(-\nu_2 - 1)}{\nu_2} O((1 + |\tau|^4)^{-1}), \\
\sigma_-^\pm &= -\sigma_+^\pm - (\nu_1 + \nu_2) |\tau|^2,
\end{aligned}
\]

\[
\inf_{\xi \in D_{r_0}} |\sigma_\pm(\xi) - \sigma_+^\pm(\xi)| > 0, \quad \sup_{\xi \in D_{r_0}} Re(\sigma_\pm(\xi)) \leq -J_0, \\
\sup_{\xi \in D_{r_0}} |\tau|^3 |\sigma_\pm(\xi) - \sigma_+^\pm(\xi)| < \infty \text{ as } |\tau| \to \infty.
\]

Therefore, the approximated analytic spectra \(\sigma^\pm_\pm\) given above satisfy

\[
\frac{\sigma_+ e^{\sigma_+ t} - \sigma_- e^{\sigma_- t}}{\sigma_+ - \sigma_-} - \frac{\sigma_+^\pm e^{\sigma_+^\pm t} - \sigma_-^\pm e^{\sigma_-^\pm t}}{\sigma_+^\pm - \sigma_-^\pm} = \frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-} - \frac{e^{\sigma_+^\pm t} - e^{\sigma_-^\pm t}}{\sigma_+^\pm - \sigma_-^\pm} \leq O(1) \left(1 + |\tau|^2\right)^4.
\]
By Lemma 5.4 in the Appendix, we have
\[
\left\| \mathcal{F}^{-1} \left[ \frac{\sigma_+ e^{\sigma_+ t} - \sigma_- e^{\sigma_- t}}{\sigma_+ - \sigma_-} - 2 \frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-} \right] \right\|_{L^\infty(\mathbb{R}^n)} = O(1),
\]
\[
\left\| \mathcal{F}^{-1} \left[ \frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-} - 2 \frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-} \right] \right\|_{L^\infty(\mathbb{R}^n)} = O(1),
\]
which asserts that all singularities are contained in \( \frac{\sigma_+ e^{\sigma_+ t} - \sigma_- e^{\sigma_- t}}{\sigma_+ - \sigma_-}, \ \frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-} \). Moreover, one can also prove that the errors of this approximation decay exponentially fast in the space-time domain, just like the proof in [7].

Now we seek out all the singularities. For the short wave part of \( G_1(\xi, t) \), one breaks
\[
\frac{\sigma_+^* e^{\sigma_+^* t} - \sigma_-^* e^{\sigma_-^* t}}{\sigma_+^* - \sigma_-^*} = e^{\sigma_+^* t} - \frac{\sigma_+^* e^{\sigma_+^* t} + \sigma_-^* e^{\sigma_-^* t}}{\sigma_+^* - \sigma_-^*}.
\]
The first term is
\[
e^{\sigma_+^* t} = e^{-\frac{\nu_1^2}{\nu^2}} e^{-\frac{c^2(\nu_1 \nu_2 - c^2) t}{\nu_2^2} \frac{1}{1 + |\xi|^2}} \frac{c^2(\nu_1 \nu_2 - c^2) t}{\nu_2^2} \frac{1}{(1 + |\xi|^2)^2} O \left( \frac{1}{(1 + |\xi|^2)^3} \right)
\]
\[
+ e^{-\frac{\nu_1^2}{\nu^2}} c^2(\nu_1 \nu_2 - c^2) t \frac{1}{1 + |\xi|^2} \frac{c^2(\nu_1 \nu_2 - c^2) t}{\nu_2^2} \frac{1}{(1 + |\xi|^2)^2} O \left( \frac{1}{(1 + |\xi|^2)^3} \right)
\]
\[
+ te^{-\frac{\nu_1^2}{\nu^2}} c^2(\nu_1 \nu_2 - c^2) t \frac{1}{1 + |\xi|^2} \frac{c^2(\nu_1 \nu_2 - c^2) t}{\nu_2^2} \frac{1}{(1 + |\xi|^2)^2} O \left( \frac{1}{(1 + |\xi|^2)^3} \right).
\]
It can be estimated as follows
\[
\left| \mathcal{F}^{-1} \left[ e^{\sigma_+^* t} \right] - e^{-\frac{\nu_1^2}{\nu^2} \delta(\xi)} - tc^2(\nu_1 \nu_2 - c^2) \nu_2^{-3} e^{-c^2 t/\nu_2} Y_n(x) \right| \leq C e^{-\frac{|\xi|^2}{\nu^2}}.
\]
The second term contains no singularities and we have
\[
\frac{\sigma_+^* e^{\sigma_+^* t}}{\sigma_+^* - \sigma_-^*} = -c^2 \nu_2^{-2} e^{-c^2 t/\nu_2} \frac{1}{1 + |\xi|^2} + e^{-\frac{\nu_1^2}{\nu^2}} O \left( \frac{1}{(1 + |\xi|^2)^2} \right),
\]
so
\[
\left| \mathcal{F}^{-1} \left[ \frac{\sigma_+^* e^{\sigma_+^* t}}{\sigma_+^* - \sigma_-^*} \right] \right| + c^2 \nu_2^{-2} e^{-c^2 t/\nu_2} Y_n(x) \leq C e^{-\frac{|\xi|^2}{\nu^2}}.
\]
For the third term, the function \( \mathcal{F}^{-1} \left[ \frac{\sigma_+^* e^{\sigma_+^* t}}{\sigma_+^* - \sigma_-^*} \right] \) does not contain singularities in \( x \) variable due to its asymptotic when \( |\xi| \to \infty \) for \( t > 0 \):
\[
\left| \frac{\sigma_+^* e^{\sigma_+^* t}}{\sigma_+^* - \sigma_-^*} \right| \leq K_0 e^{-|\xi|^2 / C_1 t} \frac{1}{1 + |\xi|^2},
\]
and we have the following estimate for $G_0^*$, then there exists a constant $C > 0$ such that for $\delta = (\varepsilon_0, \varepsilon_0)$,

$$\int_{l_m(\xi^*) = \delta} \left| \frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-} \right|^2 d\xi \leq C \int_{\mathbb{R}^n} \frac{e^{-|\xi|^2 C - J_0^* t}}{(1 + |\xi|)^2} d\xi$$

$$= C \Gamma(n) \int_0^\infty e^{-r^2/C - J_0^* t} t^{-n-1} dr \leq Ce^{-t/C} L_n(t),$$

where

$$L_n(t) \equiv \begin{cases} 1, & n = 1, \\ \log(t), & n = 2, \\ t^{-\frac{n}{n-2}}, & n \geq 3. \end{cases}$$

We denote

$$j_1(x, t) := \mathcal{F}^{-1} \left[ \frac{\sigma_+ e^{\sigma_+ t} - \sigma_- e^{\sigma_- t}}{\sigma_+ - \sigma_-} \right],$$

following the way of proof for Lemma 5.4, we get

$$|j_1(x, t)| \leq Ce^{-|x|+t/C} L_n(t)$$

from (9). So the following estimate for $G^*_1(x, t)$ hold,

$$|G^*_1(x, t) - j_1(x, t) - e^{-c^2 t/v_2} \delta_n(x) - (t(\nu_2^2/3(\nu_1^2 v_2 - c^2) + c^2 v_2^2) e^{-c^2 t/v_2} Y_n(x)|$$

$$\leq e^{-|x|+t}. $$

For the short wave part of $G_2(\xi, t)$, one breaks

$$\frac{e^{\sigma_+ t} - e^{\sigma_- t}}{\sigma_+ - \sigma_-} = \frac{e^{\sigma_+ t}}{\sigma_+ - \sigma_-} - \frac{e^{\sigma_- t}}{\sigma_+ - \sigma_-}.$$ 

The first term is

$$\frac{e^{\sigma_+ t}}{\sigma_+ - \sigma_-} = \nu_2^{-1} e^{-c^2 t/v_2} + e^{\frac{c^2 t}{v_2}} O \left( \frac{1}{(1 + |\xi|^2)^2} \right),$$

and we have

$$|\mathcal{F}^{-1} \left[ \frac{e^{\sigma_+ t}}{\sigma_+ - \sigma_-} \right] - \nu_2^{-1} e^{-c^2 t/v_2} Y_n(x)| \leq Ce^{-|x|+t}. $$

The second term contains no singularities. If denoting

$$j_2(x, t) \equiv -\mathcal{F}^{-1} \left( \frac{e^{\sigma_- t}}{\sigma_+ - \sigma_-} \right),$$

then there exists $C > 0$ such that

$$|j_2(x, t)| \leq Ce^{-|x|+t/C} L_n(t),$$

and we have the following estimate for $G^*_2(x, t)$,

$$|G^*_2(x, t) - j_2(x, t) - \nu_1^{-1} e^{-c^2 t/v_2} Y_n(x)| \leq Ce^{-|x|+t}. $$
Hence the short wave components have the following estimates in the finite Mach number region $|x| \leq 4(t+1)$:

$$
\begin{align*}
G_1^1(x,t) - j_1(x,t) - e^{-\frac{c^2_1}{\nu^2_1} \delta_n(x)} &= \left( \frac{tc^2_2(v_1^2 - \nu^2_1)}{\nu^2_2} + c^2_1 e^{-\frac{c^2_1}{\nu^2_1} \delta_n(x)} \right) \leq Ce^{-\frac{|x|^{t+1}}{4(t+1)}}.
G_2^2(x,t) - j_2(x,t) - \nu_2^{-1} e^{-\frac{c^2_1}{\nu^2_2} \delta_n(x)} &= \left( \frac{tc^2_1(v_1^2 - \nu^2_2)}{\nu^2_1} + c^2_2 e^{-\frac{c^2_1}{\nu^2_2} \delta_n(x)} \right) \leq Ce^{-\frac{|x|^{t+1}}{4(t+1)}}.
\end{align*}
$$

**Outside the finite Mach number region** $|x| \geq 3(t+1)$.

We choose the weighted function $w$ to be $w = e^{|x|-a(t)/M}$, $M$ and $a$ will be determined later. It satisfies

$$
\begin{align*}
w_t &= -\frac{a}{M}w, \quad \nabla w = \frac{x}{M|x|}w, \quad \Delta w = \frac{w}{M^2}.
\end{align*}
$$

Consider the linear damped wave equation outside the finite Mach number region:

$$
\begin{align*}
\begin{cases}
\partial^2_t u_i - c^2 \Delta u_i + \nu_1 \partial_t u_i - \nu_2 \partial_i \Delta u_i = 0, |x| \geq 3(t+1),
\end{cases}
\end{align*}
$$

Denote the outside finite Mach number region $\{x \in \mathbb{R}^n, |x| \geq 3(t+1)\}$ by $D_t$ and its boundary by $\partial D_t$. Multiplying each side of the equation in (10) by $w w_t$ and integrating with respect to $x$ on $D_t$, choosing $2 < a < 3$, $M$ sufficiently large such that $\nu_1 > \frac{c^2}{2M}$ and $\frac{a}{2} + \frac{a}{2M} - \frac{\nu^2}{2M^2} > 0$, we have

$$
\begin{align*}
&c^2 \int_{\partial D_t} w \partial_t u_i \nabla u_i \cdot d\mathbf{S}_x + \nu_2 \int_{\partial D_t} w \partial_t u_i \partial_i \nabla u_i \cdot d\mathbf{S}_x
\end{align*}
$$

$$
\begin{align*}
&= \frac{1}{2} \frac{d}{dt} \int_{D_t} w((\partial_t u_i)^2 + c^2 |\nabla u_i|^2)dx + \int_{D_t} (\nu_1 w - \frac{1}{2} w_t - \frac{1}{2} \nu_2 \Delta w)(\partial_t u_i)^2 dx
\end{align*}
$$

$$
\begin{align*}
&\leq \frac{1}{2} \frac{d}{dt} \int_{D_t} w((\partial_t u_i)^2 + c^2 |\nabla u_i|^2)dx + \int_{D_t} \left( \nu_1 + \frac{a}{2M} - \frac{\nu_2}{2M^2} \right) (\partial_t u_i)^2 dx

&\geq \frac{1}{2} \frac{d}{dt} \int_{D_t} w((\partial_t u_i)^2 + c^2 |\nabla u_i|^2)dx
\end{align*}
$$

On the boundary $\partial D_t$, by the structures of the fundamental solutions in the finite Mach number region $|x| \leq 4(t+1)$, we have

$$
|\partial_t u_i|, |\nabla u_i|, |\partial_i \nabla u_i| \leq Ce^{-c_1 t}, \; x \in \partial D_t,
$$

So

$$
\begin{align*}
\frac{d}{dt} \int_{D_t} w((\partial_t u_i)^2 + c^2 |\nabla u_i|^2)dx + 2\delta_0 \int_{D_t} w((\partial_t u_i)^2 + c^2 |\nabla u_i|^2)dx \leq Ce^{-c_1 t},
\end{align*}
$$

$$
\delta_0 = \min\{\frac{a}{2M}, \frac{\nu_1}{2M} - \frac{\nu_2}{2M^2}\}.
$$
One can also get similar estimates for any higher order derivatives $l$:

$$
\sum_{|\alpha|=1}^{l} \left( \frac{d}{dt} \int_{\mathbb{R}^n} w((\partial_t \partial_{\xi}^\alpha u_i)^2 + c^2 |\nabla \partial_{\xi}^\alpha u_i|^2) dx \right)
+ \delta_{|\alpha|} \int_{\mathbb{R}^n} w((\partial_t \partial_{\xi}^\alpha u_i)^2 + c^2 |\nabla \partial_{\xi}^\alpha u_i|^2) dx
\leq C e^{-Ct}.
$$

(12)

Integrating (11) and (12) over $t$, using Sobolev’s inequality, we have

$$
\sup_{(x,t) \in D_t} ((\partial_t \partial_{\xi}^\alpha u_i)^2 + c^2 |\nabla \partial_{\xi}^\alpha u_i|^2) \leq Ce^{-|x|-at}/C \leq Ce^{-|x|+t}/C, \text{ for } |\alpha| < l - \frac{n}{2},
$$

since $|x| \geq 3(t + 1)$. This means that the fundamental solutions $G_i(i = 1, 2)$ satisfy the following estimate outside the finite Mach number region $D_t$:

$$
|D_x^\alpha G_i(x, t)| \leq Ce^{-|x|+t}/C, \text{ for } |\alpha| < l - \frac{n}{2}.
$$

To summarize, we have the following pointwise estimates for the fundamental solutions:

**Lemma 2.1.** The fundamental solutions have the following estimates for all $x \in \mathbb{R}^n$, $|\alpha| \geq 0$:

$$
\left| D_x^\alpha \left( G_1(x, t) - j_1(x, t) - e^{-ct/\nu_2} \delta_n(x) - ((t \nu_2^{-1} c \nu_2 - c^2) + c^2 \nu_2^{-2}) e^{-ct/\nu_2} Y_n(x) \right) \right|
\leq O(1) \left( \frac{e^{-|x|^2/(t + 1)^2}}{(t + 1)^{\frac{n-2}{2}}} + e^{-(|x|+t)/C} \right),
$$

$$
\left| D_x^\alpha \left( G_2(x, t) - j_2(x, t) - \nu_2^{-1} e^{-ct/\nu_2} Y_n(x) \right) \right|
\leq O(1) \left( \frac{e^{-|x|^2/(t + 1)^2}}{(t + 1)^{\frac{n-2}{2}}} + e^{-(|x|+t)/C} \right).
$$

Here

$$
|j_1(x, t), j_2(x, t)| \leq O(1) L_n(t) e^{-(|x|+t)/C},
$$

$$
L_2(t) = \log(t), \quad L_n(t) = t^{\frac{n-2}{2}} \text{ for } n \geq 3,
$$

$$
Y_2(x) = O(1) \frac{1}{2\pi} BesselK_0(|x|), \quad Y_n(x) = O(1) \frac{e^{-|x|}}{|x|^{n-2}} \text{ for } n \geq 3.
$$

*BesselK_0(|x|)* is the modified Bessel function of the second kind with degree 0.

### 2.3. Fundamental solutions in $(x_1, \xi', s)$ variables

Applying Laplace transform in $t$ and Fourier transform in $x$ to the equations in (7) and (8), denoting the transformed variables by $s$ and $\xi$ respectively, we get the transformed fundamental solutions in $(\xi, s)$ variables:

$$
G_1(\xi, s) = \frac{s + \nu_1 \nu_2 |\xi|^2}{s^2 + \nu_1 s + (c^2 + \nu_2 s) |\xi|^2}, \quad G_2(\xi, s) = \frac{1}{s^2 + \nu_1 s + (c^2 + \nu_2 s) |\xi|^2}.
$$

Now we give a lemma:

**Lemma 2.2.**

$$
\frac{1}{2\pi} \int \frac{e^{i\xi_1 x_1}}{s^2 + \nu_1 s + \nu_2 s |\xi|^2 + c^2 |\xi|^2} d\xi_1 = \frac{1}{\nu_2 s + c^2} \frac{e^{-\lambda |x_1|}}{2\lambda},
$$

where $\lambda = \lambda (\xi', s) = \sqrt{\nu_2 s + c^2} |\xi|^2 + \nu_1 s$. 


Proof. We prove it by using the contour integral and the residue theorem. Note that
\[
\frac{1}{2\pi} \int_{\Gamma} e^{i\xi_1 x_1} \left( \frac{1}{\nu_2 s + c^2} \int_{\mathbb{R}} e^{i\xi_1 z} d\xi_1 \right) d\xi_1 = \frac{1}{2\pi} \int_{\Gamma} e^{i\xi_1 x_1} \left( \frac{1}{\nu_2 s + c^2} \int_{\mathbb{R}} e^{i\xi_1 z} d\xi_1 \right) d\xi_1.
\]
Define a closed path \( \gamma \) containing \( \Gamma' := [-R, R] \) while \( R \) is a positive constant, \( \Omega = \gamma - \Gamma' = \{ z | z = Re^{i\theta} \} \).

If \( x_1 > 0 \), set \( 0 \leq \theta \leq \pi \), \( R \) is chosen to be sufficiently large such that \( \lambda \) is contained in the domain surrounded by \( \gamma \). Consider the contour integral over path \( \gamma \). The contribution of the integration over \( \Omega \) approaches to 0 when \( R \to \infty \), therefore by the residue theorem, we have for \( x_1 > 0 \),
\[
\frac{1}{2\pi} \int_{\Gamma} e^{i\xi_1 x_1} \left( \frac{1}{\nu_2 s + c^2} \int_{\mathbb{R}} e^{i\xi_1 z} d\xi_1 \right) d\xi_1 = \frac{1}{2\pi} \int_{\Gamma} e^{i\xi_1 x_1} \left( \frac{1}{\nu_2 s + c^2} \int_{\mathbb{R}} e^{i\xi_1 z} d\xi_1 \right) d\xi_1.
\]
The computation for the case \( x_1 < 0 \) is similar. Set \( \pi \leq \theta \leq 2\pi \),
\[
\frac{1}{2\pi} \int_{\Gamma} e^{i\xi_1 x_1} \left( \frac{1}{\nu_2 s + c^2} \int_{\mathbb{R}} e^{i\xi_1 z} d\xi_1 \right) d\xi_1 = -\frac{1}{2\pi} \int_{\Gamma} e^{i\xi_1 x_1} \left( \frac{1}{\nu_2 s + c^2} \int_{\mathbb{R}} e^{i\xi_1 z} d\xi_1 \right) d\xi_1.
\]
Hence we prove this lemma.

With the help of Lemma 2.2, we get the expression of fundamental solutions \( G_1 \) and \( G_2 \) in \( (x_1, \xi', s) \) variables:
\[
G_1 \left( x_1, \xi', s \right) = \frac{1}{\nu_2 s + c^2} \left( \nu_2 \delta \left( x_1 \right) + \frac{c^2(s + \nu_1)}{\nu_2 s + c^2} \frac{e^{-\lambda |x_1|}}{2\lambda} \right),
\]
\[
G_2 \left( x_1, \xi', s \right) = \frac{e^{-\lambda |x_1|}}{2\lambda(\nu_2 s + c^2)}.
\]
In particular, when \( \bar{x}_1 > 0 \), we have
\[
G_1 \left( \bar{x}_1, \xi', s \right) = \frac{c^2(s + \nu_1)}{(\nu_2 s + c^2)^2} e^{-\lambda\bar{x}_1}, \quad G_2 \left( \bar{x}_1, \xi', s \right) = \frac{e^{-\lambda\bar{x}_1}}{2\lambda(\nu_2 s + c^2)}.
\]

3. The Green’s functions for the initial boundary value problem. In this section, we will give the pointwise estimates of the Green’s functions for the initial boundary value problem. Firstly, we compute the transformed Green’s functions in the partial-Fourier and Laplace transformed space. Then by comparing the symbols of the fundamental solutions and the Green’s functions in this transformed space, we get the simplified expressions of Green’s functions for the initial-boundary value problem. With the help of the pointwise estimates of the fundamental solutions and boundary operator, we finally get the sharp estimates of Green functions for the half space linear problem.
Before computing, we make the initial value zero by considering the error function $R_i(x_1, x', t; y_1) = G_i(x_1, x', t; y_1) - G_i(x_1 - y_1, x', t)$, which satisfies the following system:

$$
\begin{align*}
\begin{cases}
\partial_t^2 R_i - c^2 \Delta R_i + \nu_1 \partial_t R_i - \nu_2 \partial_t \Delta R_i = 0, \quad \text{x} \in \mathbb{R}^n_+, \quad t > 0, \\
R_i|_{t=0} = 0, \quad R_i|_{t=0} = 0, \\
(a_1 \partial_{x_1} + a_2) R_i(0, x', t; y_1) = -(a_1 \partial_{x_1} + a_2) G_i(x_1 - y_1, x', t)|_{x_1=0}.
\end{cases}
\end{align*}
$$

Taking Fourier transform only with respect to the tangential spatial variable $x'$, Laplace transform with respect to time variable $t$, the following ODE system can be obtained:

$$
\begin{align*}
\begin{cases}
(s^2 + \nu_1 s) R_i - (c^2 + \nu_2 s) R_i x_1 x_1 + (c^2 + \nu_2 s)|\xi|^2 R_i = 0, \\
(a_1 \partial_{x_1} + a_2) R_i(0, \xi', s; y_1) = (a_1 \partial_{y_1} - a_2) G_i(-y_1, \xi', s) = -(a_1 \lambda + a_2) G_i(-y_1, \xi', s).
\end{cases}
\end{align*}
$$

Solving it and dropping out the divergent mode as $x_1 \to +\infty$, using the boundary relationship, we have

$$
R_i(x_1, \xi', s; y_1) = -\frac{a_1 \lambda + a_2}{a_2 - a_1 \lambda} G_i(-y_1, \xi', s) = -\frac{a_1 \lambda + a_2}{a_2 - a_1 \lambda} G_i(x_1 + y_1, \xi', s),
$$

where $\lambda$ is defined in Lemma 2.2.

Therefore the transformed Green’s functions $G_i\left(x_1, \xi', s; y_1\right) \left(i = 1, 2\right)$ are

$$
G_i\left(x_1, \xi', s; y_1\right) = G_i\left(x_1 - y_1, \xi', s\right) - \frac{a_1 \lambda + a_2}{a_2 - a_1 \lambda} G_i\left(x_1 + y_1, \xi', s\right)
$$

$$
= G_i\left(x_1 - y_1, \xi', s\right) + G_i\left(x_1 + y_1, \xi', s\right) - \frac{2a_2}{a_2 - a_1 \lambda} G_i\left(x_1 + y_1, \xi', s\right),
$$

which reveal the connection between fundamental solutions and the Green’s functions.

Hence,

$$
G_i\left(x_1, x', t; y_1\right) = G_i\left(x_1 - y_1, x', t\right) + G_i\left(x_1 + y_1, x', t\right)
$$

$$
= \mathcal{F}_{x' \to x, \xi' \to \xi}^{-1} \mathcal{L}_{s \to t}^{-1} \left[ \frac{2a_2}{a_2 - a_1 \lambda} \right] * G_i\left(x_1 + y_1, x', t\right).
$$

Now we estimate the boundary operator $\mathcal{F}_{x' \to x, \xi' \to \xi}^{-1} \mathcal{L}_{s \to t}^{-1} \left[ \frac{2a_2}{a_2 - a_1 \lambda} \right]$. The function $\frac{1}{a_2 - a_1 \lambda}$ has the poles in the right half time space if $a_1 a_2 > 0$, which suggests that the boundary term will grow exponentially in time. In the following we only consider the case $a_1 a_2 < 0$.

Instead of inverting the boundary symbol, we follow the differential equation method. Notice that

$$
\mathcal{F}_{x' \to x, \xi' \to \xi}^{-1} \mathcal{L}_{s \to t}^{-1} \left[ \frac{2a_2}{a_2 - a_1 \lambda} \right] G_i\left(x_1 + y_1, \xi', s\right)
$$

$$
= 2 \frac{a_2}{a_1 \partial_{x_1} + a_2} G_i\left(x_1 + y_1, x', t\right),
$$

setting

$$
g\left(x_1, x', t\right) \equiv 2 \frac{a_2}{a_1 \partial_{x_1} + a_2} G_i\left(x_1, x', t\right),
$$

then the function $g\left(x_1, x', t\right)$ satisfies

$$
(a_2 + a_1 \partial_{x_1}) g = 2a_2 G_i\left(x_1, x', t\right).$$
Solving this ODE gives
\[ g(x_1, x', t) = 2\gamma \int_{x_1}^{\infty} e^{-\gamma(z-x_1)} G_i(z, x', t) \, dz = 2\gamma \int_{0}^{\infty} e^{-\gamma z} G_i(x_1 + z, x', t) \, dz. \]  

(13)

Summarizing previous results we obtain

**Lemma 3.1.** The Green’s functions \( G_i(x_1, x', t; y_1) \) (\( i = 1, 2 \)) of the linear initial-boundary value problem (5) and (6) can be represented as follows

\[ G_i(x_1, x', t; y_1) = G^L_i(x_1, x', t; y_1) + G^S_i(x_1, x', t; y_1). \]

Meanwhile, the following estimates hold:

\[ |D^\alpha_x G^L_i(x_1, x', t; y_1)| \leq O(1) \left( e^{-\frac{(x_1-y_1)^2+(x'-y')^2}{c(t+1)}} + e^{-\frac{(x_1+y_1)^2+(x'-y')^2}{c(t+1)}} \right), |\alpha| \geq 0; \]

\[ |G^S_i(x_1, x', t; y_1)| \leq O(1) \left( j_1(x_1-y_1, x', t) + j_2(x_1+y_1, x', t) + e^{-\frac{x_1^2}{2}} (\delta_n(x_1-y_1, x') + \delta_n(x_1+y_1, x')) + e^{-\frac{x_2^2}{2}} (\nu_1 \nu_2 - c^2 + c^2 \nu_2^{-2}) (Y_n(x_1-y_1, x') + Y_n(x_1+y_1, x')) \right) \]

and

\[ |G^S_i(x_1, x', t; y_1)| \leq O(1) \left( j_1(x_1-y_1, x', t) + j_2(x_1+y_1, x', t) + \nu_1^{-1} e^{-\frac{x_1^2}{2}} (Y_n(x_1-y_1, x') + Y_n(x_1+y_1, x')) \right). \]

**Proof.** Note that

\[ G_i(x_1, x', t; y_1) = G_i(x_1 - y_1, x', t) + G_i(x_1 + y_1, x', t) - g(x_1 + y_1, x', t), \]

based on the long-wave short-wave decomposition of the fundamental solutions

\[ G_i(x, t) = G^L_i(x, t) + G^S_i(x, t), \]

we can write

\[ G^L_i(x_1, x', t; y_1) = O(1) \left( G^L_i(x_1 - y_1, x', t) + G^L_i(x_1 + y_1, x', t) \right), \]

\[ G^S_i(x_1, x', t; y_1) = O(1) \left( G^S_i(x_1 - y_1, x', t) + G^S_i(x_1 + y_1, x', t) \right), \]

and get the estimates directly from Lemma 2.1 and (13).

\[ \square \]

4. **Long time behavior of solution for the initial-boundary value problem (Proof of Theorem 1.1).** The study of boundary operator in the last section suggests that we can only consider the case \( a_1 a_2 < 0 \) for the nonlinear stability. In [15, 16], they proved a threshold \( k = 1 + \frac{a_2}{a_1} \) between global and non-global existence of small data solutions. Here under the assumption of \( k > 1 + \frac{a_2}{a_1} \), the global in time existence of solution for the initial-boundary value problem can be proved using the fixed point theorem of Banach, which is similar to the proof given by [16], we omit the details.

Now we give the pointwise long time behavior of the solution for the nonlinear problem and prove the Theorem 1.1. The Green’s functions \( G_i(x_1, x', t; y_1)(i = 1, 2) \)
Hence we combine (15) and (16) to get the estimate of the first part in (14) when give the representation of the solution $u(x, t)$:

$$
\partial^0_x u(x, t) = \partial^0_x \int_{\mathbb{R}^n_+} (G_1(x, x' - y', t; y_1) u_0(y) + G_2(x, x' - y', t; y_1) u_1(y)) dy
$$

$$
+ \partial^0_x \int_0^t \int_{\mathbb{R}^n_+} G_2(x, x' - y', t - \tau; y_1) f(u(y, \tau)) dy d\tau
$$

$$
= \partial^0_x \mathcal{I}(x, t) + \partial^0_x N(x, t).
$$

The initial part $\partial^0_x \mathcal{I}(x, t)$ contains two parts:

$$
\partial^0_x \mathcal{I}(x, t) = \partial^0_x \mathcal{I}^L(x, t) + \partial^0_x \mathcal{I}^S(x, t),
$$

where

$$
\partial^0_x \mathcal{I}^L(x, t) = \partial^0_x \int_{\mathbb{R}^n_+} (G_1^L(x, x' - y', t; y_1) u_0(y) + G_2^L(x, x' - y', t; y_1) u_1(y)) dy
$$

$$
\partial^0_x \mathcal{I}^S(x, t) = \partial^0_x \int_{\mathbb{R}^n_+} (G_1^S(x, x' - y', t; y_1) u_0(y) + G_2^S(x, x' - y', t; y_1) u_1(y)) dy.
$$

By lemma 5.2, we have the following estimates in the finite Mach number region $|x| \leq 4(t + 1)$,

$$
|\mathcal{I}^L(x, t)| \leq O(1) \varepsilon \int_{\mathbb{R}^n_+} e^{-\frac{|x-y|^2}{(t+1)^2}} (1 + |y|^2)^{-\tau} dy
$$

$$
\leq O(1) \varepsilon \left( \frac{e^{-\frac{x^2}{(t+1)^2}}}{(t+1)^2} + \left( 1 + t + |x|^2 \right)^{-\frac{\nu_2}{2}} \right),
$$

(15)

$$
|\mathcal{I}^S(x, t)|
$$

$$
\leq O(1) \varepsilon e^{-\frac{(|x|+t)^2}{c^2}} \int_{\mathbb{R}^n} \left( L_n(t) + \delta_n(x - y) + \left[ \frac{tc^2}{\nu_1^2} (\nu_1 c^2 - c^2) + \frac{c^2}{\nu_2^2} \right] Y_n(x - y) \right) (1 + |y|^2)^{-\tau} dy
$$

$$
+ O(1) \varepsilon e^{-\frac{(|x|+t)^2}{c^2}} \int_{\mathbb{R}^n} \left( L_n(t) + \nu_2^{-1} Y_n(x - y) \right) (1 + |y|^2)^{-\tau} dy
$$

$$
\leq O(1) \varepsilon \left( \frac{e^{-\frac{|x|^2}{(t+1)^2}}}{(t+1)^2} + \left( 1 + t + |x|^2 \right)^{-\frac{\nu_2}{2}} \right).
$$

(16)

Hence we combine (15) and (16) to get the estimate of the first part in (14) when $|\alpha| = 0$

$$
|\mathcal{I}(x, t)| \leq O(1) \varepsilon \left( \frac{e^{-\frac{|x|^2}{(t+1)^2}}}{(t+1)^2} + \left( 1 + t + |x|^2 \right)^{-\frac{\nu_2}{2}} \right),
$$

(17)
Similarly, when $|\alpha| = 1$, we have
\[
|\partial_{\xi}^{\alpha}I(x, t)| = |\partial_{\xi}^{\alpha}I_{L}(x, t) + \partial_{\xi}^{\alpha}I_{S}(x, t)|
\leq O(1)\varepsilon \int_{\mathbb{R}^n} \left( e^{-(\frac{(x_1-y_1)^2}{(t+1)^{\frac{\sigma}{2}}} + \frac{(x_1+y_1)^2}{(t+1)^{\frac{\sigma}{2}}})} (1 + |y|)^{-r} dy \right.
\]
\[
+ \left. 1_{\{\partial_{\xi}^{\alpha}=\partial_{\varepsilon_1}\}} O(1)\varepsilon e^{-\frac{|x|+1}{C}} \left| \int_{\mathbb{R}^n} L_n(t) + \delta_n(x_1-y_1, \mathbf{x}'-\mathbf{y}', t) + \delta_n(x_1+y_1, \mathbf{x}'-\mathbf{y}', t) + (t\varepsilon^2\nu_2^3(\nu_1\nu_2 - c^2) + c^2\nu_2^2) \right. \right.
\]
\[
\left. \left( Y_n(x_1-y_1, \mathbf{x}'-\mathbf{y}') + Y_n(x_1+y_1, \mathbf{x}'-\mathbf{y}') \right) (1 + |y|)^{-r} dy \right|_{y_1=0} \right.
\]
\[
+ \left. O(1)\varepsilon e^{-\frac{|x|+1}{C}} \left| \int_{\mathbb{R}^n} L_n(t) + \delta_n(x_1-y_1, \mathbf{x}'-\mathbf{y}', t) + \delta_n(x_1+y_1, \mathbf{x}'-\mathbf{y}', t) + (t\varepsilon^2\nu_2^3(\nu_1\nu_2 - c^2) + c^2\nu_2^2) \right. \right.
\]
\[
\left. \left( Y_n(x_1-y_1, \mathbf{x}'-\mathbf{y}') + Y_n(x_1+y_1, \mathbf{x}'-\mathbf{y}') \right) (1 + |y|)^{-r} dy \right|_{y_1=0} \right.
\]
\[
+ \left. 1_{\{\partial_{\xi}^{\alpha}=\partial_{\varepsilon_1}\}} O(1)\varepsilon e^{-\frac{|x|+1}{C}} \left| \int_{\mathbb{R}^n} L_n(t) + \nu_1^{-1}Y_n(x_1-y_1, \mathbf{x}'-\mathbf{y}') + \nu_1^{-1}Y_n(x_1+y_1, \mathbf{x}'-\mathbf{y}') \right) (1 + |y|)^{-r} dy \right|_{y_1=0} \right.
\]
\[
+ \left. O(1)\varepsilon e^{-\frac{|x|+1}{C}} \left| \int_{\mathbb{R}^n} L_n(t) + \nu_1^{-1}Y_n(x_1-y_1, \mathbf{x}'-\mathbf{y}') + \nu_1^{-1}Y_n(x_1+y_1, \mathbf{x}'-\mathbf{y}') \right) (1 + |y|)^{-r} dy \right|_{y_1=0} \right.
\]
\[
\leq O(1)\varepsilon (1 + t)^{-\frac{|\alpha|}{2}} \left( e^{-(\frac{2|\mathbf{x}|^2}{(t+1)^{\frac{\sigma}{2}}} + \frac{2|\mathbf{x}|^2}{(t+1)^{\frac{\sigma}{2}}})} (1 + t + |\mathbf{x}|^2)^{-r} \right) + O(1)\varepsilon e^{-\frac{|x|+1}{C}}.
\]

where
\[
1_{\{\partial_{\xi}^{\alpha}=\partial_{\varepsilon_1}\}} = \begin{cases} 1, & \text{if } \partial_{\xi}^{\alpha} = \partial_{\varepsilon_1}, \\ 0, & \text{otherwise.} \end{cases}
\]

Here we use the integration by parts to estimate the short wave component part. Outside the finite Mach number region, we have
\[
|\partial_{\xi}^{\alpha}I(x, t)| \leq O(1)\varepsilon e^{-\nu_1 t} \int_{\mathbb{R}^n} e^{-|x-y|} (1 + |y|)^{-r} dy
\leq O(1)\varepsilon e^{-\nu_1 t} (1 + |x|^2)^{-r}, \quad |\alpha| \leq 1.
\]

Based on the estimates of (17)-(18), the ansatz is posed for the solution as follows:
\[
|\partial_{\xi}^{\alpha}u(x, t)| \leq O(1)\varepsilon (1 + t)^{-\frac{|\alpha|}{2}} (1 + t + |x|^2)^{-\frac{\sigma}{2}}, \quad |\alpha| \leq 1.
\]
Straightforward computations show that
\[ |f(u)(x,t)| \leq O(1)\varepsilon^k (1 + t + |x|^2)^{-\frac{\alpha}{2}}. \]

Now we justify the ansatz for the nonlinear term. For \( N(x,t) \), we have
\[
|N(x,t)| = \left| \int_0^t \int_{\mathbb{R}^n_+} G_2(x_1, x' - y', t - \tau; y_1) f(u(y, \tau)) \, dy \, d\tau \right|
\leq \left| \int_0^t \int_{\mathbb{R}^n_+} G_2^I(x_1, x' - y', t - \tau; y_1) f(u(y, \tau)) \, dy \, d\tau \right|
+ \left| \int_0^t \int_{\mathbb{R}^n_+} G_2^S(x_1, x' - y', t - \tau; y_1) f(u(y, \tau)) \, dy \, d\tau \right|
= N_1 + N_2.
\]

Using Lemma 5.3, one gets
\[
N_1 \leq O(1)\varepsilon^k \left| \int_0^t \int_0^{\infty} \int_{\mathbb{R}^{n-1}} \left( e^{-\frac{(x_1 - y_1)^2 + |x' - y'|^2}{C_1(t + \tau + 1)^2}} + e^{-\frac{(x_1 + y_1)^2 + |x' - y'|^2}{C_2(t + \tau + 1)^2}} \right) \right|
(1 + \tau + |y|^2)^{\frac{-\alpha}{2}} \, dy \, dy \, d\tau
\leq O(1)\varepsilon^k \left| \int_0^t \int_{\mathbb{R}^n_+} e^{-\frac{1}{\nu_2^2} \frac{\epsilon^2}{(t-\tau)^2}} \left( L_n(t - \tau) + \nu_2^{-1} Y_n(x_1, x' - y'; y_1) \right) \right|
(1 + \tau + |y|^2)^{\frac{-\alpha}{2}} \, dy \, d\tau
\leq O(1)\varepsilon^k (1 + t + |x|^2)^{-\frac{\alpha}{2}}.
\]

Now we compute the estimate of \( \partial_x^\alpha N \) when \( |\alpha| = 1 \):
\[
|\partial_x^\alpha N(x,t)| = \left| \partial_x^\alpha \int_0^t \int_{\mathbb{R}^n_+} G_2(x_1, x' - y', t - \tau; y_1) f(u(y, \tau)) \, dy \, d\tau \right|
\leq \left| \int_0^t \int_{\mathbb{R}^n_+} \partial_x^\alpha G_2^I(x_1, x' - y', t - \tau; y_1) f(u(y, \tau)) \, dy \, d\tau \right|
+ \left| \int_0^t \int_{\mathbb{R}^n_+} \partial_x^\alpha G_2^S(x_1, x' - y', t - \tau; y_1) f(u(y, \tau)) \, dy \, d\tau \right|
= \partial_x^\alpha N_1 + \partial_x^\alpha N_2.
\]
Similarly we have
\[
\partial_N^\alpha N_1 = O(1)\epsilon^k \left| \int_0^t \int_{R^{n-1}} \partial_N^\alpha \mathcal{G}^S_2(x_1, x'-y', t-\tau; y_1) f(u(y, \tau)) dy \, d\tau \right|
\]
\[
\leq O(1)\epsilon^k \left| \int_0^t \int_{R^{n-1}} e^{-\frac{(x_1-y_1)^2+(x'-y')^2}{(t-\tau+1)^2 + |y|^2}} \alpha \right|
\]
\[
\leq |\alpha| \left| \int_0^t \int_{R^{n-1}} e^{-\frac{(x_1-y_1)^2+(x'-y')^2}{(t-\tau+1)^2 + |y|^2}} \alpha \right|
\]
\[
\leq |\alpha| \left| \int_0^t \int_{R^{n-1}} e^{-\frac{(x_1-y_1)^2+(x'-y')^2}{(t-\tau+1)^2 + |y|^2}} \alpha \right|
\]
\[
\leq O(1)\epsilon^k (1 + t)^{-|\alpha|} (1 + t + |x|^2)^{-\frac{n}{2}}.
\]

The second term in (19) satisfies
\[
\left| \int_0^t \int_{R^{n-1}} \mathcal{G}^S_2(x_1, x'-y', t-\tau; y_1) \partial_N^\alpha f(u(y, \tau)) dy \, d\tau \right|
\]
\[
\leq O(1)\epsilon^k \left| \int_0^t \int_{R^{n-1}} e^{-\frac{(x_1-y_1)^2+(x'-y')^2}{(t-\tau+1)^2 + |y|^2}} \alpha \right|
\]
\[
\leq O(1)\epsilon^k \left| \int_0^t \int_{R^{n-1}} e^{-\frac{(x_1-y_1)^2+(x'-y')^2}{(t-\tau+1)^2 + |y|^2}} \alpha \right|
\]
\[
\leq O(1)\epsilon^k (1 + t)^{-|\alpha|} (1 + t + |x|^2)^{-\frac{n}{2}}.
\]

The boundary term in (19) has the following estimates:
\[
\int_0^t \int_{R^{n-1}} \mathcal{G}^S_2(x_1, x'-y', t-\tau; y_1) f(u(y, \tau)) dy' \big|_{y_1=0} d\tau
\]
\[
\leq \left( \int_0^{t/2} + \int_{t/2}^t \right) \int_{R^{n-1}} \mathcal{G}^S_2(x_1, x'-y', t-\tau; y_1) f(u(y, \tau)) dy' \big|_{y_1=0} d\tau
\]
\[
\leq O(1)\epsilon^k (1 + t)^{-|\alpha|} (1 + t + |x|^2)^{-\frac{n}{2}}.
\]

Therefore one has the following estimate for the nonlinear term
\[
|\partial_N^\alpha N| \leq O(1)\epsilon^k (1 + t)^{-|\alpha|} (1 + t + |x|^2)^{-\frac{n}{2}}, \quad |\alpha| \leq 1.
\]

Outside the finite Mach number region,
\[
|\partial_N^\alpha N| \leq O(1)\epsilon^k \left| \int_0^t \int_{R^{n-1}} e^{-\frac{(x_1-y_1)^2+(x'-y')^2}{(t-\tau+1)^2 + |y|^2}} \alpha \right|
\]
\[
\leq O(1)\epsilon^k (1 + t + |x|^2)^{-\frac{n}{2}}, \quad |\alpha| \leq 1.
\]
Thus, we verify the ansatz and finish the proof of pointwise estimates of the solution. The $L^p$ ($p > 1$) estimate can be easily proved using the following equalities:

\[
\left( \int_{\mathbb{R}^n} \left(1 + t + |x|^2\right)^{-\frac{n}{2}} d\mathbf{x} \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^n} \left(1 + t + \frac{|x|^2}{1 + t}\right)^{-\frac{n}{2p}} d\mathbf{x} \right)^{\frac{1}{p}}
\]

\[
= (1 + t)^{-\frac{n}{2}} (1 + t)^{\frac{n}{2p}} = (1 + t)^{-\frac{n}{2}(1 - \frac{1}{p})}.
\]

Hence we finish the proof of Theorem 1.1.

5. Appendix.

Lemma 5.1. [10] In the finite Mach number region $|x| \leq 4(t + 1)$, we have the following estimate for the inverse Fourier transform:

\[
\left| \frac{1}{(2\pi)^n} \int_{|\xi| \leq 1} (i\xi)^\alpha e^{i\xi \cdot x} e^{-\frac{|\xi|^2}{2t}} d\xi \right| \leq O(1) \frac{e^{-\frac{|x|^2}{2(t + 1)^{n-|\alpha|}}}}{(1 + t)^{\frac{n-|\alpha|}{2}}} + O(1) e^{-\frac{|x|^2}{2(t + 1)^{n-|\alpha|}}} , |\alpha| \geq 0.
\]

Lemma 5.2. [9] We have the follow estimate for $|\alpha| \leq 1$ and $r > \frac{n}{2}$,

\[
\int_{\mathbb{R}^n} \frac{e^{-\frac{(x-y)^2}{2(t+\tau)^{n-|\alpha|}}}}{(1 + |y|^2)^{-r}} d\mathbf{y} \leq O(1)(1 + t)^{-\frac{|\alpha|}{2}} \left( \frac{e^{-\frac{|x|^2}{2(t+\tau)^{n-|\alpha|}}}}{(1 + t)^{\frac{n-|\alpha|}{2}}} + (1 + t + |x|^2)^{-r} \right).
\]

Lemma 5.3. [9] For $x \in \mathbb{R}^n$, $|\alpha| \leq 1$, we have

\[
\int_0^t \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{2(t+\tau)^{n-|\alpha|}}} \chi_n(x - y)(1 + \tau)^{-|\alpha|} (1 + \tau + |y|^2)^{-nk} d\mathbf{y} d\tau \leq O(1)(1 + t)^{-\frac{|\alpha|}{2}} (1 + t + |x|)^{-nk/2},
\]

\[
\int_0^t \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{2(t+\tau)^{n-|\alpha|}}} (1 + \tau + |y|^2)^{-nk} d\mathbf{y} d\tau \leq O(1)(1 + t)^{-\frac{|\alpha|}{2}} (1 + t + |x|^2)^{-\frac{nk}{2}}.
\]

Lemma 5.4. [7] Suppose a function $f \in L^1(\mathbb{R}^n)$ and its Fourier transform $\mathcal{F}[f](\xi)$ is analytic in $D_\delta$ and satisfies

\[
|\mathcal{F}[f](\xi)| \leq \frac{E}{(1 + |\xi|)^{n+1}}, \text{ for } |\text{Im}(\xi_i)| \leq \delta, \text{ and } i = 1, 2, \ldots, n.
\]

Then, the function $f(x)$ satisfies

\[
|f(x)| \leq E e^{-\delta|x|/C},
\]

for any positive constant $C > 1$. 

Acknowledgments. The authors would like to thank the referees very much for their valuable comments and suggestions which improve the presentation of paper-significantly.
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Received May 2020; revised August 2020.

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