Noncommutative Moduli for Multi-Instantons

Tatiana A. Ivanova†, Olaf Lechtenfeld* and Helge Müller-Ebhardt*

†Bogoliubov Laboratory of Theoretical Physics, JINR
141980 Dubna, Moscow Region, Russia
Email: ita@thsun1.jinr.ru

*Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
Email: lechtenf, ebhardt@itp.uni-hannover.de

Abstract

There exists a recursive algorithm for constructing BPST-type multi-instantons on commutative \( \mathbb{R}^4 \). When deformed noncommutatively, however, it becomes difficult to write down non-singular instanton configurations with topological charge greater than one in explicit form. We circumvent this difficulty by allowing for the translational instanton moduli to become noncommutative as well. Such a scenario is natural in the self-dual Yang-Mills hierarchy of integrable equations where the moduli of solutions are seen as extended space-time coordinates associated with higher flows. By judicious adjustment of the moduli-noncommutativity we achieve the ADHM construction of generalized 't Hooft multi-instanton solutions with everywhere self-dual field strengths on noncommutative \( \mathbb{R}^4 \).
1 Introduction

In recent years, many important nonperturbative field configurations, like solitons, vortices, monopoles and instantons, have been generalized in various dimensions to Moyal-type noncommutative spaces (see e.g. [1]–[13] and reviews [14] for further references). Specializing to instantons on $\mathbb{R}^4$, the self-dual Yang-Mills equations [15] are solved systematically through the ADHM method [16]. Its noncommutative extension to $\mathbb{R}^4_\theta$, as developed in [1]–[3] and [17]–[20], is straightforward for self-dual $\theta$ but needs modification in case of anti-self-dual $\theta$.

A singular subclass of solutions are the ’t Hooft multi-instantons, which have also been deformed noncommutatively by way of the splitting and ADHM approaches [19, 20]. With the help of Murray-von-Neumann transformations – the noncommutative analog of singular gauge transformations – one can remove the singularities and arrive at a non-singular gauge [19]. A direct path to non-singular ’t Hooft multi-instantons is again offered by the ADHM construction. Its noncommutative extension (for self-dual $\theta$) yields a recursive algorithm for generating $n$-instanton configurations on $\mathbb{R}^4_\theta$; yet, the explicit realization is rather technical beyond $n=1$.

In this letter we propose a generalization of noncommutative multi-instantons by rendering part of the instanton moduli noncommutative. This is a logical step in the self-dual Yang-Mills hierarchy where moduli are naturally regarded as additional space-time coordinates. Surprisingly, with a noncommutative (translational) moduli space it is possible to write down explicit ’t Hooft $n$-instanton configurations with field strengths being self-dual everywhere, and we will do this here.

2 Commutative non-singular multi-instantons

Notation. Instantons are localized finite-action solutions to the classical equations of motion for a Euclidean field theory [15, 21]. In this paper we specialize to four-dimensional Euclidean Yang-Mills theory with the gauge group $\text{U}(2)$. Hence, we have four $\text{u}(2)$-valued gauge potentials $A_\mu$ and the field strengths $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Possible solutions to the field equations $D_\mu F_{\mu\nu} = 0$ are obtained by demanding the field strength to be self-dual,

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} F_{\lambda\rho} = F_{\mu\nu} \quad \text{for} \quad \mu, \nu, \ldots = 1, 2, 3, 4,$$

which enjoy the properties

$$e_{\mu} e_{\nu} = \delta_{\mu\nu} 1_2 + \eta^a_{\mu\nu} i \sigma_a =: \delta_{\mu\nu} 1_2 + \eta_{\mu\nu}, \quad e^\dagger_{\mu} e_{\nu} = \delta_{\mu\nu} 1_2 + \bar{\eta}^a_{\mu\nu} i \sigma_a =: \delta_{\mu\nu} 1_2 + \bar{\eta}_{\mu\nu},$$

where $\sigma_a, a = 1, 2, 3$, are the Pauli matrices while $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$ denote the self-dual and anti-self-dual ’t Hooft tensors [22], respectively, which satisfy the identities

$$\eta^a_{\mu\nu} \bar{\eta}^b_{\mu\nu} = 0 \quad \text{and} \quad \eta^a_{\mu\nu} \eta^b_{\mu\nu} = 4 \delta^{ab}.$$
With the help of the matrices (2.2), one forms the quaternions
\[ x := x^\mu e^\dagger_\mu \quad \text{and} \quad x^\dagger = x^\mu e_\mu \quad \text{with} \quad \{x^\mu\} \in \mathbb{R}^4. \]  
(2.5)

We will frequently have to shift \( x^\mu \) by some real constants \( a^\mu_i, i = 1, \ldots, n \), and so define
\[ x^\mu_i := x^\mu - a^\mu_i \quad \implies \quad x_i = x^\mu_i e^\dagger_\mu \quad \text{and} \quad a_i = a^\mu_\mu e^\dagger_\mu. \]  
(2.6)

Likewise, in addition to the radius-squared \( r^2 := x^\mu x^\mu \) we introduce the distance-squared to the point \( a_i \) as
\[ r^2_i := x^\mu_i x^\mu_i = (x^\mu_i - a^\mu_i)(x^\mu_i - a^\mu_i) \quad \text{(no sum over } i) . \]  
(2.7)

**ADHM construction.** The most systematic way to generate instanton solutions is via the ADHM approach. The construction (see [16, 22]) of an \( n \)-instanton solution is based on a
\[ (2n+2) \times 2 \text{ matrix } \Psi \quad \text{and a} \quad (2n+2) \times 2n \text{ matrix } \Delta = a + b(x \otimes 1_n), \]  
(2.8)

where \( a \) and \( b \) are constant \( (2n+2) \times 2n \) matrices. These matrices must satisfy the following conditions:
\[ \Delta^\dagger \Delta \quad \text{is invertible}, \]  
(2.9)
\[ [\Delta^\dagger \Delta, e_\mu \otimes 1_n] = 0 \quad \forall x, \]  
(2.10)
\[ \Delta^\dagger \Psi = 0, \]  
(2.11)
\[ \Psi^\dagger \Psi = 1_2. \]  
(2.12)

It is not difficult to see that conditions (2.9) and (2.10) are met if
\[ \Delta^\dagger \Delta = 1_2 \otimes h^{-1}_{n \times n} . \]  
(2.13)

For \((\Delta, \Psi)\) satisfying (2.9)–(2.12) the gauge potential is chosen in the form
\[ A = \Psi^\dagger d\Psi. \]  
(2.14)

The resulting gauge field \( F \) will be self-dual if \( \Delta \) and \( \Psi \) obey the completeness relation
\[ \Psi \Psi^\dagger + \Delta (\Delta^\dagger \Delta)^{-1} \Delta^\dagger = 1_{2n+2}. \]  
(2.15)

Namely, using (2.11), (2.12) and (2.15), we find
\[ F_{\mu\nu} = \partial_\mu(\Psi^\dagger \partial_\nu \Psi) - \partial_\nu(\Psi^\dagger \partial_\mu \Psi) + [\Psi^\dagger \partial_\mu \Psi, \Psi^\dagger \partial_\nu \Psi] = 2 \Psi^\dagger b (\Delta^\dagger \Delta)^{-1} \eta_{\mu\nu} b^\dagger \Psi, \]  
(2.16)
i.e. the anti-self-dual part of \( F_{\mu\nu} \) is zero.

To become more concrete, let us take the following ansatz (see e.g. [23, 19]):
\[ \Psi = \begin{pmatrix} \Psi_0 \\ \Psi_1 \\ \vdots \\ \Psi_n \end{pmatrix}, \quad a = \begin{pmatrix} A_1 1_2 & \cdots & A_n 1_2 \\ -a_1 & \ddots & 0_2 \\ \vdots & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0_2 & \cdots & 0_2 \\ 1_2 & \cdots & 0_2 \\ \vdots & \ddots & \ddots \end{pmatrix} \]  
(2.17)
where \( a_i = a_i^\mu e_\mu^i \) and the \( \Lambda_i \) are positive constants (scale parameters). From (2.17) we get

\[
\Delta = \begin{pmatrix}
\Lambda_11_2 & \cdots & \Lambda_n1_2 \\
x_1 & & 0_2 \\
\vdots & & \ddots \\
0_2 & & & & x_n
\end{pmatrix}
\quad \text{and} \quad
\Delta^\dagger = \begin{pmatrix}
\Lambda_11_2 & x_1^\dagger & 0_2 \\
\vdots & \ddots & \vdots \\
0_2 & & x_n^\dagger
\end{pmatrix},
\tag{2.18}
\]

and arrive at

\[
\Delta^\dagger\Delta = 1_2 \otimes (\delta_{ij}r_i^2 + \Lambda_i\Lambda_j) =: 1_2 \otimes (R + \Lambda\Lambda^T),
\tag{2.19}
\]

where

\[
R = \begin{pmatrix}
r_1^2 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & r_n^2
\end{pmatrix}
\quad \text{and} \quad
\Lambda = \begin{pmatrix}
\Lambda_1 \\
\vdots \\
\Lambda_n
\end{pmatrix}.
\tag{2.20}
\]

From (2.19) we see that the condition (2.13) (and thus also (2.9) and (2.10)) is satisfied. Indeed, by direct calculation one finds

\[
1_2 \otimes h_{n \times n} = (\Delta^\dagger\Delta)^{-1} = 1_2 \otimes (R^{-1} - R^{-1}\Lambda\phi_n^{-1}\Lambda^T R^{-1})
\tag{2.21}
\]

with

\[
\phi_n = 1 + \sum_{i=1}^n \frac{\Lambda_i^2}{r_i^2}.
\tag{2.22}
\]

For the given form (2.18) of \( \Delta \), the remaining conditions (2.11) and (2.12) become

\[
\Lambda_i\Psi_0 + x_i^\dagger\Psi_i = 0_2 \quad \text{for} \quad i = 1, \ldots, n,
\tag{2.23}
\]

\[
\Psi_0^\dagger \Psi_0 + \Psi_1^\dagger \Psi_1 + \cdots + \Psi_n^\dagger \Psi_n = 1_2.
\tag{2.24}
\]

The task is to solve these two equations. If successful one can evaluate the gauge potential (2.14) and its field strength (2.16). The latter is guaranteed to be self-dual since the completeness relation (2.15) is automatically satisfied in the commutative case.

**One instanton.** One starts with the ansatz

\[
\Psi_0 = x_1^\dagger f_1 \quad \text{and} \quad \Psi_1 = -\Lambda_1 f_1
\tag{2.25}
\]

which solves (2.23) for an arbitrary matrix-valued function \( f_1 \). With \( x_1 x_1^\dagger = r_1^2 1_2 \), the normalization (2.24) then determines this function (up to a constant unitary matrix) as

\[
f_1 = \frac{1}{\sqrt{r_1^2 + \Lambda_1^2}} 1_2.
\tag{2.26}
\]

This solution is obviously non-singular at finite values of \( r_1^2 \). Using (2.14) one arrives at

\[
A_\mu = \Psi_0^\dagger \partial_\mu \Psi_0 + \Psi_1^\dagger \partial_\mu \Psi_1 = -\eta_\mu^\nu \frac{x_1^\nu}{r_1^2 + \Lambda_1^2},
\tag{2.27}
\]

which is known as the BPST solution [15].
Two instantons. In this case one takes the ansatz
\[
\Psi_0 = x_1^\dagger (a_2-a_1) x_2^\dagger f_2, \quad \Psi_1 = -\Lambda_1(a_2-a_1) x_2^\dagger f_2 \quad \text{and} \quad \Psi_2 = -\Lambda_2(a_2-a_1) x_1^\dagger f_2,
\] (2.28)
which again solves (2.23) for any matrix-valued function \(f_2\). With this, (2.24) yields the function
\[
f_2 = \frac{1}{\sqrt{(a_2-a_1)^2 (r_1^2 r_2^2 + \Lambda_1^2 r_2^2 + \Lambda_2^2 r_1^2)}} \mathbf{1}_2.
\] (2.29)
This configuration too is non-singular since \(r_1^2\) and \(r_2^2\) cannot vanish simultaneously.

Recursion for \(n\) instantons. It is possible to systematize the above sequence of ansätze and write down a recursive formula for the \(n\)-instanton solution. Because its explicit form is somewhat intricate and we will not make use of it later on, there is no point displaying it here. It may be remarked, however, that the proof of regularity (at finite points) is rather non-trivial. Nevertheless, it is possible in this way to generate explicit non-singular multi-instanton configurations for any instanton number. To be sure, the singular ’t Hooft configurations are also easily obtained within the ADHM approach, with appropriate ansätze for \(\Psi_i\) (see, e.g. [23]).

3 Noncommutative non-singular multi-instantons

In this section we shall give a short account of the status of explicit noncommutative multi-instanton configurations for the (simpler) case of self-dual noncommutativity.

A Moyal deformation of Euclidean \(\mathbb{R}^4\) is achieved by replacing the ordinary pointwise product of functions on it by the nonlocal but associative Moyal star product. The latter is characterized by a constant antisymmetric matrix \((\theta^{\mu\nu})\) which prominently appears in the star commutation relation between the coordinates,
\[
[x^\mu, x^\nu] = i \theta^{\mu\nu}.
\] (3.1)
A different realization of this algebraic structure keeps the standard product but promotes the coordinates (and thus all their functions) to noncommuting operators acting in an auxiliary Fock space \(\mathcal{H}\). The two formulations are tightly connected through the Moyal-Weyl map. When dealing with noncommutative U(2) Yang-Mills theory from now on, we shall not denote the noncommutativity by either inserting stars in all products or by putting hats on all operator-valued objects, but simply by agreeing that our coordinates are subject to (3.1). The existence of \((\theta^{\mu\nu})\) breaks the Euclidean SO(4) symmetry to SO(4) \(\cap\) Sp(4,\(\mathbb{R}\)) = U(2), but we may employ SO(4) rotations to pick a basis in which \((\theta^{\mu\nu})\) takes Darboux form, i.e. the only nonzero entries are \(\theta^{12} = -\theta^{21}\) and \(\theta^{34} = -\theta^{43}\). Such a matrix is a linear combination of the self-dual \((\eta^{3\mu\nu})\) and the anti-self-dual \((\bar{\eta}^{3\mu\nu})\). In this work, we restrict ourselves to the special case of a purely self-dual noncommutativity tensor given by
\[
\theta^{\mu\nu} = \theta \eta^{3\mu\nu}.
\] (3.2)

Let us try to generalize the ADHM construction of the previous section to the noncommutative case. We take the same multi-instanton ansätze as in the commutative case but must take care of ordering now. It is quickly verified that (for the the above choice of \((\theta^{\mu\nu})\)) one actually ends up with the same equations (2.8)–(2.16) and (2.23)–(2.24), of course now holding for noncommutative coordinates. In contrast to the previous section, the completeness relation (2.15) is no longer automatic, and so one needs to show that it holds as well.
**One instanton.** This was already calculated by Furuuchi [3]. Irrespective of the noncommutativity, the ansatz (2.25) solves (2.23) for any matrix $f_1$. The determination of $f_1$ again proceeds via (2.24) but one must take into account a modified relation,

$$x_i^\dagger x_i = r_i^2 1_2 \quad \text{but} \quad x_i x_i^\dagger = r_i^2 1_2 - 2\theta \sigma_3 \quad \text{(no sum over $i$)}, \quad (3.3)$$

since coordinate products now feature antisymmetric parts. The result is

$$f_1 = \left( \begin{array}{cc} \frac{1}{\sqrt{r_1^2 - 2\theta + \Lambda_1^2}} & 0 \\ 0 & \frac{1}{\sqrt{r_1^2 + 2\theta + \Lambda_1^2}} \end{array} \right). \quad (3.4)$$

This is non-singular because the spectrum of the operator $r_1^2$ is bounded by $2\theta$. Furthermore,

$$\Psi \Psi^\dagger = \left( \begin{array}{cc} x_1^\dagger f_1^2 x_1 & -\Lambda_1 x_1^\dagger f_1^2 \\ -\Lambda_1 f_1^2 x_1 & \Lambda_2 f_1^2 \end{array} \right) \quad \text{and} \quad \Delta (\Delta^\dagger \Delta)^{-1} \Delta^\dagger = \left( \begin{array}{cc} \frac{\Lambda_2^2}{r_1^2 + \Lambda_1^2} & \frac{\Lambda_1}{r_1^2 + \Lambda_1^2} x_1^\dagger \\ x_1 \frac{\Lambda_1}{r_1^2 + \Lambda_1^2} & x_1 \frac{1}{r_1^2 + \Lambda_1^2} x_1^\dagger \end{array} \right) \quad (3.5)$$

so that, due to $f_2^2 x_1 = x_1 \frac{1}{r_1^2 + \Lambda_1^2}$, the completeness relation (2.15) is indeed satisfied. Finally one can calculate from (2.14) the gauge potential, which is entirely regular and merges with the BPST solution for $\theta \to 0$. The explicit expression coincides with the one obtained in the dressing approach by Horváth et al in [9].

**Two instantons.** As in commutative case one takes the ansatz (2.28) which fulfils (2.23) also when read noncommutatively. In order to find $f_2$ one then computes $\Psi^\dagger_0 \Psi_0 + \Psi^\dagger_1 \Psi_1 + \Psi^\dagger_2 \Psi_2$ which yields a non-diagonal $2 \times 2$ matrix with noncommuting matrix elements. Equating it to unity and solving for $f_2$ turns out to be technically difficult, and we will not try to do this here. Nevertheless, we expect this solution to be non-singular and the completeness relation (2.15) to hold for it.

**Multi-instantons.** The recursive construction mentioned in the previous section can be carried over to the noncommutative domain. Yet, the determination of the matrices $f_n$ and the verification of the completeness relation gets increasingly complicated due to the noncommutativity. To summarize this section, for the gauge group U(2) BPST-type instanton solutions on noncommutative $\mathbb{R}^4$ are known only for charge one. It remains a computational challenge to present an explicit noncommutative U(2) two-instanton configuration.

### 4 Multi-instantons with noncommutative translational moduli

**Noncommutative moduli.** In this section we propose an unorthodox alternative which avoids all previously mentioned difficulties. It employs the noncommutative ’t Hooft multi-instantons (via ADHM) but allows their translational moduli to become noncommutative as well! The result is, of course, a rather non-standard generalization of multi-instantons.

To be specific, we now modify the commutation relations to

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \, , \quad [a_i^\mu, a_j^\nu] = -i\theta^{\mu\nu} \delta_{ij} \quad \text{and} \quad [x^\mu, a_i^\nu] = 0 \, , \quad (4.1)$$

where $\theta^{\mu\nu}$ is given in (3.2). As a consequence,

$$[x_i^\mu, x_j^\nu] = i\theta^{\mu\nu} (1 - \delta_{ij}) \quad \text{for} \quad i, j = 1, \ldots, n \, . \quad (4.2)$$
Using (2.3), (2.4) and (4.2) we obtain – in distinction to (3.3) –
\[ x_i^\dagger x_i = x_i^\dagger x_i = x_i^\mu x_i^\mu \mathbf{1}_2 = r_i^2 \mathbf{1}_2 \quad \text{(no sum over } i) \quad . \tag{4.3} \]

**Invertibility of** \( r_i^2 \). The commutation relations (4.1) can be realized in terms of annihilation and creation operators,
\[ \{x^\mu\} \rightarrow \{\alpha_0, \beta_0, \alpha_0^\dagger, \beta_0^\dagger\} \quad \text{and} \quad \{a_i^\mu\} \rightarrow \{\alpha_i, \beta_i, \alpha_i^\dagger, \beta_i^\dagger\} , \tag{4.4} \]
acting in the tensor product \( \mathcal{H} := \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n \) of \( n+1 \) copies of the two-oscillator Fock space. In this formulation one finds that
\[ r_i^2 = 2\theta [(\alpha_i^\dagger - \alpha_0)(\alpha_i - \alpha_0^\dagger) + (\beta_i^\dagger - \beta_0)(\beta_i - \beta_0^\dagger)] = 2\theta (\tilde{\alpha}_i^\dagger \tilde{\alpha}_i + \tilde{\beta}_i^\dagger \tilde{\beta}_i) \tag{4.5} \]
with new annihilation operators
\[ \tilde{\alpha}_i := \alpha_i - \alpha_0^\dagger \quad \text{and} \quad \tilde{\beta}_i := \beta_i - \beta_0^\dagger \tag{4.6} \]
created by a Bogoliubov transformation. The general form of such a transformation is (see e.g. [24])
\[
\begin{pmatrix} \tilde{\alpha}_0 \\ \tilde{\alpha}_i \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_i \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} \alpha_0^\dagger \\ \alpha_i^\dagger \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{\beta}_0 \\ \tilde{\beta}_i \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_i \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} \beta_0^\dagger \\ \beta_i^\dagger \end{pmatrix} \tag{4.7} \]
where \((b_{ij}) =: B\) is a symmetric \(2 \times 2\) matrix with complex entries. Note that the new annihilation operators \(\tilde{\alpha}_0\) and \(\tilde{\alpha}_i\) have normalizable vectors in their kernel only if the hermitian matrix \(1_2 - BB^\dagger\) is positive definite. The case under consideration, however, is degenerate because
\[
B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \Rightarrow \quad 1_2 - BB^\dagger = 0_2 . \tag{4.8} \]

Hence, the operators \(\alpha_i - \alpha_0^\dagger\) and \(\beta_i - \beta_0^\dagger\) as well as their hermitian conjugates have no zero modes either on \(\mathcal{H}\) or on its dual \(\mathcal{H}^*\). We conclude that, for any \(i = 1, \ldots, n\), the operator \(r_i^2\) is invertible on all finite-norm states, i.e. on \(\mathcal{H}\) as well as on \(\mathcal{H}^*\).

**Ansatz.** Since for self-dual \((\theta_{\mu \nu})\) the ADHM scheme described in the previous section is unaltered, we take over the ansatz (2.17) unchanged, in order to construct noncommutative ’t Hooft \(n\)-instantons with noncommutative moduli parameters (4.1). The commutative computation can be literally copied until arriving at (2.21) with (2.22), after having assured that \(r_i^2\) is indeed invertible in the noncommutative case.

The task to solve the equations (2.23) and (2.24) is accomplished with
\[
\Psi_0 = \phi_n^{-\frac{1}{2}} \mathbf{1}_2 \quad \text{and} \quad \Psi_i = -x_i \frac{\Lambda_i}{r_i^2} \phi_n^{-\frac{1}{2}} , \tag{4.9} \]
where the factor \(\phi_n^{-\frac{1}{2}}\) was introduced to achieve the normalization
\[
\Psi^\dagger \Psi = \phi_n^{-\frac{1}{2}} \left( 1 + \sum_{i=1}^n \frac{\Lambda_i^2}{r_i^4} \right) 1_2 \phi_n^{-\frac{1}{2}} = 1_2 . \tag{4.10} \]
Hence, our \((\Delta, \Psi)\) satisfies all conditions (2.9)–(2.13), and we can define the gauge potential via (2.14). Note that this configuration with noncommutative translational moduli can be interpreted as a four-dimensional “slice” of a solution to the generalized self-dual Yang-Mills equations on \(\mathbb{R}^{4+4n}\) [25] in which both \(x^\mu\) and \(a^\mu_i\) play the role of (noncommutative) coordinates.

**Completeness relation.** To be sure that we have indeed constructed self-dual field configurations, we must still check the completeness relation (2.15). Actually, it is known that for commuting moduli the latter is easily violated [18, 19] unless an additional effort [20] is made. In fact, the main point for adjusting the noncommutativity of the translational moduli as in (4.1) is to make the completeness relation work out. To our satisfaction, after (lengthy) computations we can indeed confirm the validity of (2.15). In more detail, the matrix \(\Delta (\Delta^\dagger \Delta)^{-1} \Delta^\dagger\) takes the form

\[
\begin{pmatrix}
1_2 - \phi^{-1}_n 1_2 & \phi^{-1}_n \frac{A}{r_1} x^\dagger_1 & \phi^{-1}_n \frac{A}{r_2} x^\dagger_2 & \ldots & \phi^{-1}_n \frac{A}{r_n} x^\dagger_n \\
x_1 \frac{A}{r_1} \phi^{-1}_n & 1_2 - x_1 \frac{A}{r_1 r_2} x^\dagger_1 & -x_1 \frac{A}{r_1} \phi^{-1}_n \frac{A}{r_2} x^\dagger_2 & \ldots & -x_1 \frac{A}{r_1} \phi^{-1}_n \frac{A}{r_n} x^\dagger_2 \\
x_2 \frac{A}{r_2} \phi^{-1}_n & -x_2 \frac{A}{r_2} \phi^{-1}_n \frac{A}{r_1} x^\dagger_2 & 1_2 - x_2 \frac{A}{r_2 r_3} x^\dagger_2 & \ldots & -x_2 \frac{A}{r_2} \phi^{-1}_n \frac{A}{r_n} x^\dagger_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_n \frac{A}{r_n} \phi^{-1}_n & -x_n \frac{A}{r_n} \phi^{-1}_n \frac{A}{r_1} x^\dagger_1 & \ldots & -x_n \frac{A}{r_n} \phi^{-1}_n \frac{A}{r_n-1} x^\dagger_{n-1} & 1_2 - x_n \frac{A}{r_n r_{n+1}} x^\dagger_n
\end{pmatrix}
\]

In this computation the relation (4.2) was important. Likewise, the calculation of \(\Psi \Psi^\dagger\) yields

\[
\begin{pmatrix}
\phi^{-1}_n 1_2 & -\phi^{-1}_n \frac{A}{r_1} x^\dagger_1 & -\phi^{-1}_n \frac{A}{r_2} x^\dagger_2 & \ldots & -\phi^{-1}_n \frac{A}{r_n} x^\dagger_n \\
x_1 \frac{A}{r_1} \phi^{-1}_n & 1_2 - x_1 \frac{A}{r_1 r_2} x^\dagger_1 & -x_1 \frac{A}{r_1} \phi^{-1}_n \frac{A}{r_2} x^\dagger_2 & \ldots & -x_1 \frac{A}{r_1} \phi^{-1}_n \frac{A}{r_n} x^\dagger_2 \\
x_2 \frac{A}{r_2} \phi^{-1}_n & -x_2 \frac{A}{r_2} \phi^{-1}_n \frac{A}{r_1} x^\dagger_2 & 1_2 - x_2 \frac{A}{r_2 r_3} x^\dagger_2 & \ldots & -x_2 \frac{A}{r_2} \phi^{-1}_n \frac{A}{r_n} x^\dagger_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_n \frac{A}{r_n} \phi^{-1}_n & -x_n \frac{A}{r_n} \phi^{-1}_n \frac{A}{r_1} x^\dagger_1 & \ldots & -x_n \frac{A}{r_n} \phi^{-1}_n \frac{A}{r_n-1} x^\dagger_{n-1} & 1_2 - x_n \frac{A}{r_n r_{n+1}} x^\dagger_n
\end{pmatrix}
\]

It is not hard to see that the latter two matrices add up to \(1_{2n+2}\), as the completeness relation (2.15) demands. So, for our choice of \((\Delta, \Psi)\) and commutation relations (4.1) the tensor \(F_{\mu \nu}\) is self-dual.

**Topological charge.** In a number of papers (see e.g. [18, 26]) it was argued that in \(\mathbb{R}^4\) the topological charge of noncommutative ADHM \(n\)-instantons for the gauge group \(U(N)\) is an integer since this is encoded in the dimensions of the ADHM matrices even in the noncommutative case. Noncommutativity of translational moduli does not alter this conclusion. However, due to the noncommutativity of not only \(x^\mu\) but also \(a^\mu_i\), our operators are defined on the larger Fock space \(\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n\). That is why the topological charge of our solution will be \(n\) times the identity operator in \(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n\). In the commutative limit this identity operator becomes unity, and we recover the standard ’t Hooft multi-instanton with commutative moduli featuring a topological charge equal to \(n\).

**Acknowledgements.** The authors are grateful to A.D. Popov for fruitful discussions and for reading the manuscript. T.A.I. acknowledges the Heisenberg-Landau Program for partial support and the Institut für Theoretische Physik der Universität Hannover for its hospitality. This work was partially supported by the Deutsche Forschungsgemeinschaft (DFG).
References

[1] N. Nekrasov and A. Schwarz, Commun. Math. Phys. 198 (1998) 689 [hep-th/9802068].

[2] K. Furuuchi, Prog. Theor. Phys. 103 (2000) 1043 [hep-th/9912047];
    Commun. Math. Phys. 217 (2001) 579 [hep-th/0005199].

[3] K. Furuuchi, JHEP 0103 (2001) 033 [hep-th/0010119].

[4] D.J. Gross and N.A. Nekrasov, JHEP 0007 (2000) 034 [hep-th/0005204];
    JHEP 0103 (2001) 044 [hep-th/0010090];
    A.P. Polychronakos, Phys. Lett. B 495 (2000) 407 [hep-th/0007043];
    D. Bak, Phys Lett. B 495 (2000) 251 [hep-th/0008204].

[5] M. Hamanaka and S. Terashima, JHEP 0103 (2001) 034 [hep-th/0010221];
    K. Hashimoto, JHEP 0012 (2000) 023 [hep-th/0010251];
    O. Lechtenfeld, A.D. Popov and B. Spendig, JHEP 0106 (2001) 011 [hep-th/0103196];
    R. Gopakumar, M. Headrick and M. Spradlin,
    Commun. Math. Phys. 233 (2003) 355 [hep-th/0103256].

[6] D.H. Correa, G.S. Lozano, E.F. Moreno and F.A. Schaposnik,
    Phys. Lett. B 515 (2001) 206 [hep-th/0105085];
    D.H. Correa, E.F. Moreno and F.A. Schaposnik,
    Phys. Lett. B 543 (2002) 235 [hep-th/0207180];
    F.A. Schaposnik, “Noncommutative solitons and instantons,” hep-th/0310202;
    D.H. Correa, P.Forgacs, E.F. Moreno, F.A. Schaposnik and G.A. Silva,
    “Noncommutative 3 dimensional soliton from multi-instantons,” hep-th/0404015.

[7] A. Dimakis and F. Müller-Hoissen,
    “A noncommutative version of the nonlinear Schrödinger equation,” hep-th/0007015;
    J. Phys. A 34 (2001) 9163 [nlin.si/0104071]; J. Phys. A 37 (2004) 4069 [hep-th/0401142].

[8] L.D. Paniak, “Exact Noncommutative KP and KdV Multi-solitons,” hep-th/0105185;
    O. Lechtenfeld and A.D. Popov, JHEP 0111 (2001) 040 [hep-th/0106213];
    Phys. Lett. B 523 (2001) 178 [hep-th/0108118]; JHEP 0401 (2004) 069 [hep-th/0306263];
    K. Furuta, T. Inami, H. Nakajima and M. Yamamoto,
    Phys. Lett. B 537 (2002) 165 [hep-th/0203125].

[9] M. Wolf, JHEP 0206 (2002) 055 [hep-th/0204185];
    F. Franco-Sollova and T.A. Ivanova, J. Phys. A 36 (2003) 4207 [hep-th/0209153];
    Z. Horváth, O. Lechtenfeld and M. Wolf, JHEP 0212 (2002) 060 [hep-th/0211041];
    M. Ihl and S. Uhlmann, Int. J. Mod. Phys. A 18 (2003) 4889 [hep-th/0211263];
    O. Lechtenfeld, “Noncommutative instantons and solitons,” hep-th/0401158.

[10] J. Murugan and R. Adams, JHEP 0212 (2002) 073 [hep-th/0211171];
    I. Cabrera-Carnero and M. Moriconi, Nucl. Phys. B 673 (2003) 437 [hep-th/0211193];
    M.T. Grisaru, L. Mazzanti, S. Penati and L. Tamassia,
    “Some properties of the integrable noncommutative sine-Gordon system,” hep-th/0310214;
    J. Murugan and A. Millner, “Transmogrifying fuzzy vortices,” hep-th/0403105.
[11] M. Hamanaka and K. Toda, J. Phys. A 36 (2003) 11981 [hep-th/0301213];
“Towards noncommutative integrable equations,” hep-th/0309265;
M. Hamanaka, “Commuting flows and conservation laws for noncommutative Lax hierarchies,”
hep-th/0311206.

[12] N. Wang and M. Wadati, J. Phys. Soc. Japan 72 (2003) 1366; 1881; 3055.

[13] A.D. Popov, A.G. Sergeev and M. Wolf, J. Math. Phys. 44 (2003) 4527 [hep-th/0304263];
T.A. Ivanova and O. Lechtenfeld, Phys. Lett. B 567 (2003) 107 [hep-th/0305195];
O. Lechtenfeld, A.D. Popov and R.J. Szabo, JHEP 0312 (2003) 022 [hep-th/0310267].

[14] M.R. Douglas and N.A. Nekrasov, Rev.Mod.Phys. 73 (2002) 977 [hep-th/0106048];
A. Konechny and A. Schwarz, Phys. Rept. 360 (2002) 353 [hep-th/0107251];
R.J. Szabo, Phys. Rept. 378 (2003) 207 [hep-th/0109162];
M. Hamanaka, “Noncommutative solitons and D-branes,” hep-th/0303256.

[15] A.A. Belavin, A.M. Polyakov, A.S. Schwarz and Y.S. Tyupkin, Phys. Lett. B 59 (1975) 85.

[16] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, Phys. Lett. A 65 (1978) 185.

[17] N.A. Nekrasov, “Trieste lectures on solitons in noncommutative gauge theories,”
hep-th/0011095; Commun. Math. Phys. 241 (2003) 143 [hep-th/0010017].

[18] C.S. Chu, V.V. Khoze and G. Travaglini, Nucl. Phys. B 621 (2002) 101 [hep-th/0108007].

[19] O. Lechtenfeld and A. D. Popov, JHEP 0203 (2002) 040 [hep-th/0109209].

[20] Y. Tian and C.J. Zhu, Phys. Rev. D 67 (2003) 045016 [hep-th/0210163].

[21] R. Rajaraman, “Solitons and instantons. An introduction to solitons and instantons in quantum field theory,” Amsterdam, North-Holland (1982).

[22] M.K. Prasad, Physica D 1 (1980) 167.

[23] E. Corrigan, D.B. Fairlie, S. Templeton and P. Goddard, Nucl. Phys. B 140 (1978) 31;
H. Osborn, Nucl. Phys. B 140 (1978) 45.

[24] A.M. Perelomov, “Generalized coherent states and their applications,” Berlin, Springer (1986).

[25] R.S. Ward, Nucl. Phys. B 236 (1984) 381;
E. Corrigan, P. Goddard and P. Goddard, Nucl. Phys. B 140 (1978) 31;
T.A. Ivanova and A.D. Popov,
Theor. Math. Phys. 94 (1993) 225 [Teor. Mat. Fiz. 94 (1993) 316].

[26] A. Sako, JHEP 0304 (2003) 023 [hep-th/0209139].