Online Stochastic Matching with Edge Arrivals

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Abstract

Online bipartite matching with edge arrivals is an important extension of the classic vertex-arrival model of Karp, Vazirani and Vazirani. Finding an algorithm better than the straightforward greedy strategy remained a major open question for a long time until a recent negative result by Gamlath et al. (forthcoming FOCS 2019), who showed that no online algorithm has a competitive ratio better than 0.5 in the worst case.

In this work, we consider the bipartite matching problem with edge arrivals in the stochastic framework. We find an online algorithm that on average is 0.506-competitive. We give a supplementary upper bound of $\frac{2}{3}$ and also obtain along the way an interesting auxiliary result that if the initial instance has a 2-regular stochastic subgraph, then a natural prune & greedy algorithm matches at least 0.532-fraction of all vertices.

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1 Introduction

Matching theory is a central area in combinatorial optimization with a big range of applications [26]. Many market models for jobs, commercial products, dating, healthcare, etc., rely on matching as a fundamental mathematical primitive. These examples often aim to describe environments that evolve in real time and thus are relevant to the area of online bipartite matching initiated by a seminal paper of Karp, Vazirani and Vazirani [24]. In this work Karp et al. consider the one-sided vertex-arrival model within the competitive analysis framework, i.e., vertices only on one side of a bipartite graph appear online and each new vertex reveals all its incident edges. The algorithm immediately and irrevocably decides to which vertex (if any) the new arrival is matched. They studied the worst-case performance of online algorithms and solved the problem optimally with an elegant \( \frac{1}{e} \)-competitive algorithm, named **Ranking**. Later, the proof of the result has been simplified by a series of papers [6, 17, 12].

The interest in matching models and online bipartite matching problems in particular has been on the rise since a decade ago due to emergence of the internet advertisement industry and online market platforms [30]. Justified by large amount of data available to the online platforms from day-to-day user activities and prevalence of Bayesian approach in theoretical economics, in more recent works on online bipartite matching, there has been a significant shift towards stochastic models. In particular, Feldman et al. [14] proposed the stochastic i.i.d. model in which online vertices are drawn i.i.d. from a known distribution and improved the competitive ratio of the classic result by Karp et al. to 0.67. The competitive ratio has been further improved by a series of papers [3, 28, 22] to 0.706. Another line of work [23, 27] studied the model in which online vertices arrive in a random order and showed that the **Ranking** algorithm is 0.696-competitive.

The aforementioned results and other works, e.g., [31, 7, 11, 33, 16, 20, 21, 2], have made remarkable progress on different online matching settings with vertex arrivals, i.e., models where all incident edges of a new vertex are reported to the algorithm. However, more general arrival models are much less understood. E.g., one of the most natural and nonrestrictive extensions of online bipartite matching to the model where edges appear online and must be immediately matched or discarded was not known to have a competitive ratio better than the greedy algorithm for a long time. Only the very recent negative result by Gamlath et al. [16] closed this tantalizing question showing that no online algorithm can be better than 0.5-competitive in the worst case. Algorithms with better performance are only known for quite special family of graphs, e.g., bounded-degree graphs [8] and forests [30, 8], or under strong assumptions on the edge arrival order, e.g., random arrival order [19].

It might seem that the edge-arrival model is too weak to allow non-trivial theoretical results without strong assumptions on the instance. On the other hand, besides pure theoretical interest and clean mathematical formulation, the edge-arrival model also has relevant practical significance not unlike the examples we discuss below.

**Practical Motivation: Edge Arrivals.** Imagine any online matching platform for job search, property market, or even online dating. All these instances can be viewed as online matching processes in bipartite graphs. They also share a common trait that the realization of any particular edge is not instantaneous, often consumes significant effort and time from one or both sides of the potential match, and may exhibit complex concurrent behavior across different parties of the market. The platform can be thought of as an online matching algorithm, if it has any degree of

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\(^1\)It is also sometimes called Bayesian.

\(^2\)Their result holds under the assumption that the expected number of vertices for each type is an integer.
control to intervene in the process of edge formation at any point. However, the platform does not have enough power to control the order in which edges are realized. Hence, using arbitrary edge arrival order seems to be an appropriate modeling choice in these situations.

Another notable feature of these instances is the vast amount of historical data accumulated over time. The data enables the platform to estimate the probability of a potential match between any pair of given agents. Thus the Bayesian (stochastic) approach widely adapted in economics seems to be another reasonable modeling choice. This raises the following natural question that to the best of our knowledge has not been considered before:

Is there an online matching algorithm for stochastic bipartite graphs with edge arrivals that is better than greedy?

This question is the main focus of our work. Let us first specify the model in more details.

Our Model: Edge Arrivals in Stochastic Graphs. We call our model online stochastic matching with edge arrivals. It is a relaxation of the standard edge-arrival model that performs on a random bipartite graph. In particular, we assume the input graph $G$ is stochastic. That is, each edge $e$ exists (is realized) in $G$ independently with probability $p_e$ and the probabilities $(p_e)_{e \in E(G)}$ are known to the online algorithm. The algorithm observes a sequence of edges arriving online in a certain (unknown) order. Upon the arrival of an edge $e$, we observe the realization of $e$ and if $e$ exists, then the algorithm immediately and irrevocably decides whether to add $e$ to the matching.

We assume that the arrival order of the edges is chosen by an oblivious adversary, i.e., an adversary who does not observe the realization of the edges and algorithm’s decisions, which is a standard assumption in the literature on online algorithms in stochastic settings (see, e.g., [25]). We compare the expected performance of our algorithm with the maximum matching in hindsight, i.e., the expected size of a maximum matching over the randomness of all edges.

1.1 Comparison with Other Stochastic Models

Our model is closely connected to two existing theoretical lines of works on stochastic bipartite matching and prophet inequality in algorithmic game theory. Below we compare our model with the most relevant results in each of these lines of works.

Stochastic Probing Model. It has the same ingredient as our model: the underlying stochastic graph. That is, the input is also a bipartite graph with the stochastic information on existence probability of every edge $e$. On the other hand, it is an offline model under the query-commit framework, i.e., the algorithm can check the existence of the edges in any order. However, if an edge exists, it has to be included into the solution. For this model, an adaptation of the Ranking algorithm by Karp et al. is $(1 - 1/e)$-competitive. Costello et al. [10] provided a 0.573-approximation algorithm on general (non-bipartite) graphs and showed that no algorithm can have an approximation ratio larger than 0.898. Recently, Gamlath et al. [15] designed a $(1 - 1/e)$-approximation algorithm for the weighted version of this problem.

Prophet Inequality for Bipartite Matching. Consider a bipartite graph, where all edges have random values independently sampled from given probability distributions. Upon the arrival of an

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3Even if the platform cannot directly prohibit an edge formation or disallow certain matches, it usually can affect outcome indirectly by restricting access/information exchange between certain pairs of agents, so that they never consider each other as potential matches.
edge, we see the realization of its value and decide immediately whether to include this edge if possible in the matching. This model was originally proposed by Kleinberg and Weinberg [25] for a more general setting of intersection of $k$ matroids. Gravin and Wang [18] studied explicitly the setting of bipartite matching and provided $\frac{1}{2}$-approximation. Our model can be viewed as an unweighted version of this prophet setting. Indeed, we assume that each edge has value either 0 or 1 and, hence, the probability distribution is a product of Bernoulli random variables summarized by existence probabilities $(p_e)_{e \in E(G)}$. Note that the weighted case is strictly harder than the unweighted one. Gravin and Wang [18] provided a 1/2.25 hardness result for the weighted setting while our goal is to design an online algorithm with a competitive ratio strictly better than 1/2. After all, the simple greedy algorithm achieves a competitive ratio of 1/2 for unweighted graphs.

1.2 Our Results and Techniques

We find an online 0.506-competitive matching algorithm for any bipartite stochastic graph. This result confirms that the edge-arrival model is theoretically interesting in the stochastic framework. A complementary hardness result shows that no online algorithm can be better than 2/3-competitive.

To obtain our main result and build useful intuition, we consider an interesting special case of regular stochastic graphs. In the beginning of this project, we were looking for a class of instances where the greedy algorithm may be more than $\frac{1}{2}$-competitive. We chose to study the case of $c$-regular stochastic graphs motivated by the remarkable guarantees on maximum matching for the regular deterministic bipartite graphs. In this important case, it is natural to use a stronger benchmark equal to the number of vertices (on one side) instead of the expected maximum matching. Although, positive results are harder to get for a stronger benchmark, that allowed us to focus on the analysis of online algorithm by removing the non-trivial comparison with stochastic maximum matching.

We derived a guarantee of $f(c)$ on the fraction of vertices matched by the greedy algorithm for any $c$-regular stochastic graph, where the function $f(c)$ has a single peak around $c = 2$ with $f(2) \approx 0.532$, and developed analytically tractable relaxation on the performance of greedy that we later generalized to non-regular case. Interestingly, the greedy algorithm may perform worse on $c$-regular graphs and is not better than 0.5-competitive as $c \to \infty$. This prompted us to the following two-stage approach: (i) prune initial $G$ to a regular instance with smaller probabilities $\hat{p}_e \leq p_e$; (ii) run greedy on the pruned stochastic graph. In particular, if we could find a 2-regular stochastic subgraph, then the prune & greedy algorithm is 0.53-competitive (against a stronger benchmark).

We note that the family of prune & greedy algorithms may be of independent interest due to their practical relevance. Indeed, in those market applications we discussed above, the online platform cannot always prevent the matching between two parties (pair of vertices) once they realized their compatibility. But the platform usually possesses all information about the graph and thus is fully capable of implementing pruning step by restricting information to its users. After that participants naturally implement greedy matching by exploring compatibilities with the other side of the graph exposed to them by the platform in an arbitrary order.

On the technical side, the analysis of our algorithms uses the relaxation in which each vertex on the left side makes only one proposal to a vertex on the right side, i.e., one attempt to match, and if this attempt fails the vertex does not try any other edge. This relaxation allows us to break the analysis into independent optimization problems per each vertex on the right side of the graph. However, this approach alone is not sufficient for non-regular graphs, as we do not make any use of the expected optimal matching. To this end, we consider an LP relaxation (an upper bound) on the expected optimal matching in stochastic graphs recently proposed in [15]. This LP gives a set of values $(\bar{x}_e)_{e \in E(G)}$ with the objective $\sum_{e \in E(G)} \bar{x}_e$ which satisfy a set of constraints that could be conveniently added to our optimization problem for each right side vertex of $G$. We modified
the nonadaptive prune & greedy algorithm to adaptively make use of the existing information on the realized edges, so that the proposal probability for edge \(e\) gets closer to the corresponding LP variable \(x_e\). Interestingly, our analysis did not require to actually solve this LP, i.e., the same analysis works as is for \(x_e\) being equal to the probability of \(e\) appearing in the optimal matching.

### 1.3 Other Related Works

The edge-arrival setting is also studied under the free-disposal assumption, i.e., the algorithm is able to dispose of previously accepted edges. McGregor \[29\] gave a deterministic \(\frac{1}{3+2\sqrt{2}} \approx 0.171\)-competitive algorithm for weighted graphs. Varadaraja \[32\] proved the optimality of this result among deterministic algorithms. Later, Epstein et al. \[13\] gave a \(\frac{1}{3.356} \approx 0.186\)-competitive randomized algorithm and proved a hardness result of \(\frac{1}{1+\ln 2} \approx 0.591\) for unweighted graphs. Recently, the bound is improved to \(2 - \sqrt{2} \approx 0.585\) by Huang et al. \[21\]. We remark that the question of designing an algorithm that beats 0.5-competitive remains open.

One of the earlier work on stochastic matching is due to Chen et al. \[9\]. They proposed stochastic model with edge probing motivated by real life matching applications such as kidney exchange. This model is more complex than the stochastic probing model we discussed before, since it has an additional constraint per each vertex \(v\) on how many times edges incident to \(v\) can be queried. Another difference is that a weaker benchmark than the optimal offline matching has to be used in this setting. Chen et al. developed a \(\frac{1}{4}\)-approximation algorithm. Bansal et al. \[4\] considered the weighted version and provided a \(\frac{1}{3}\)-approximation and a \(\frac{1}{4}\)-approximation for bipartite graphs and general graphs respectively. The ratio for general graphs was further improved to \(\frac{1}{3.709}\) by Adamczyk et al. \[1\], and then to \(\frac{1}{3.224}\) by Baveja et al. \[5\].

### 2 Preliminaries

The bipartite graph \(G = (L, R, E)\) consists of left and right sides denoted respectively \(L\) and \(R\). The graph \(G\) is a multigraph, i.e., \(E\) is a multiset that may have multiple edges between the same pair of vertices. We use \(E_v\) to denote the multiset of edges incident to the vertex \(v\) and \(E_{uv}\) to denote the multiset of edges connecting \(u\) and \(v\). We consider the Bayesian model, where each edge \(e \in E\) is realized with probability \(p_e \in [0,1]\), which is known in advance. The realizations of different edges are independent. We are interested in online matching algorithms with the objective of maximizing the expected size of the matching. We assume that all edges in \(E\) arrive one by one according to some fixed unknown order (i.e., oblivious adversarial order). Upon arrival of the edge \(e\), the algorithm observes whether or not \(e\) is realized. If the edge exists, the algorithm immediately and irrevocably decides whether to include \(e\) into the matching; the algorithm does nothing, if the edge is not realized. We compare the performance of the algorithm with the performance of the optimal offline algorithm, also known as the prophet, who knows the realization of the whole graph in hindsight, i.e., \(\text{OPT} = E[\text{size of maximum matching}]\).

A natural online matching strategy is the greedy algorithm: Take every available edge \(e = (u,v)\) whenever both vertices \(u\) and \(v\) have not yet been matched. Obviously, the greedy algorithm is a 0.5-approximation, since it selects a maximal matching in all possible realizations of the graph, which is always a 0.5-approximation to the maximum matching.

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\[4\] We only need the constraints from the LP which are satisfied by the expected optimal matching. Such a result would be for the actual expected size of the optimal matching instead of a slightly stronger LP \[15\] benchmark.
3 GREEDY Algorithm on Regular Graphs

A regular graph is a graph whose vertices have the same degree. But how do we define vertex degrees in a stochastic graph? One standard way is to use the expected vertex degree, i.e., $\sum_{e \in E_u} p_e$ for the degree of a vertex $u \in L$. However, the expectation alone does not contain all the important information about a degree distribution. Consider for example a vertex $a$ having only one incident edge $(a, b)$ with $p_{(a, b)} = 1$ and a vertex $u$ having 2 incident parallel edges $e = (u, v)$ with probability $p_e = 0.5$ for each $e \in E_u$. Both vertices $a$ and $u$ have the same expected degree, but while $a$ always has exactly one incident edge, $u$ gets no incident edges with 0.25 probability. On the other hand, $u$ may have 2 incident edges in some realizations, which is almost the same for our purposes as having only a single incident edge.

A good way to reconcile this difference is to substitute each edge $(u, v)$ by multiple parallel edges $e_i = (u, v)$ with small probabilities such that $p_{(u, v)}$ matches the probability that at least one of $e_i$ edges exists. Alternatively, we can define a log-normalized weight for each edge $e$ as $w_e \overset{\text{def}}{=} -\ln(1 - p_e)$, i.e., given an input instance $G = (L, R, E)$ we construct a one-to-one correspondence between vectors of probabilities $p = (p_e)_{e \in E}$ and vectors of log-normalized weights $w = (w_e)_{e \in E}$. In particular, if we split an edge with log-normalized weight $w_e$ into two edges $e_1 = e_2 = (u, v)$ with $w_{e_1} + w_{e_2} = w_e$, then the new instance gets only harder, i.e., any online algorithm for the new instance can be easily adapted to the original instance with the same or better performance. Indeed, notice that the probability that at least one of the edges $e_1, e_2$ exists equals $1 - e^{-w_{e_1}} \cdot e^{-w_{e_2}} = 1 - e^{-w_e}$, the probability that $e$ exists, i.e., there is a probability coupling between the event that $e$ exists with the event that at least one of $e_1, e_2$ exists. Then, we can substitute $e$ in any arrival order with a pair of consecutive edges $e_1$ and $e_2$ and match $e$ whenever the online algorithm matches $e_1$ or $e_2$ in the modified instance. Thus the log-normalized weight is the correct notion for us to do additive operations over the existence probabilities and leads to the following definition of the regular stochastic graph.

**Definition 1.** A graph $G$ is a log-normalized $c$-regular graph if for every $v \in L \cup R$, $\sum_{e \in E_v} w_e = c$.

We restrict our attention to log-normalized regular graphs in the remainder of this section. Our next goal is to analyze the performance of the GREEDY algorithm on log-normalized $c$-regular graphs for a small constant $c$. Remarkably, it is not easy to give a precise answer and produce a tight worst-case estimate even for a specific value $c = 1$. We instead will look at a surrogate version of GREEDY with an equal or worse performance than GREEDY, but which admits tight analysis. Specifically, we will treat vertices on one side (the left part) of the graph differently from the vertices on the other side. We allow each vertex $u \in L$ only to be matched with the vertex $v$ in the first realized edge incident to $u$, and even if $v$ was previously matched to another vertex in $L$, we would simply reject all further edges from $E_u$ and treat $u$ as a matched vertex. The following lemma shows why the surrogate algorithm is no better than GREEDY.

**Definition 2.** Let $u \in L, v \in R$. For each edge $e \in E_{uv}$, define $Q_e$ to be the event that $e$ exists and is the first realized edge of $E_u$ in the given arrival order. Let $q_e \overset{\text{def}}{=} \Pr[Q_e]$.

**Lemma 1.** For all $v \in R$, $\Pr[v \text{ is matched}] \geq \Pr[\bigcup_{e \in E_v} Q_e] \geq 1 - \exp \left(-\sum_{e \in E_v} q_e\right)$.

**Proof.** For each edge $e \in E_v$, consider the case when $e$ arrives and $Q_e$ happens. At this moment, either $v$ is already matched, or $e$ will be included in the matching by GREEDY. Therefore, whenever $Q_e$ is true for an $e \in E_v$, $v$ is covered by GREEDY.
Next, we observe that the events \( \{ \cup_{e \in E_{uv}} Q_e \}_{u \in L} \) are mutually independent, because (i) the event \( \bigcup_{e \in E_{uv}} Q_e \) solely depends on the random realization of the edges in \( E_u \) and (ii) \( E_u \cap E_{u'} = \emptyset \) when \( u \neq u' \).

Furthermore, for any \( e_1, e_2 \in E_{uv} \), we have \( Q_{e_1} \cap Q_{e_2} = \emptyset \). Hence, \( \Pr[\bigcup_{e \in E_{uv}} Q_e] = \sum_{e \in E_{uv}} q_e \).

Putting the above observations together, we have

\[
\Pr[\text{v is matched}] \geq \Pr[\bigcup_{e \in E_v} Q_e] = 1 - \prod_{u \in E_v} \left( 1 - \sum_{e \in E_{uv}} q_e \right) \geq 1 - \prod_{u \in E_v} \exp \left( - \sum_{e \in E_{uv}} q_e \right) = 1 - \exp \left( - \sum_{e \in E_v} q_e \right),
\]

where the last inequality follows from the fact that \( e^{-\sum_{i=1}^n x_i} \leq 1 - \sum_{i=1}^n x_i \).

\[ \square \]

**Theorem 1.** The competitive ratio of Greedy on log-normalized \( c \)-regular graphs is at least

\[
f(c) \overset{\text{def}}{=} \int_0^1 1 - e^{-ce^{-cx}} \, dx.
\]

\[ \text{Proof.} \] We write the lower bound on the performance of Greedy using Lemma 1.

\[
\text{ALG} = \sum_{v \in R} \Pr[\text{v is matched}] \geq \sum_{v \in R} 1 - \exp \left( - \sum_{e \in E_v} q_e \right) \quad \text{(by Lemma 1)}
\]

\[
\geq \sum_{v \in R} \sum_{e \in E_v} \frac{w_e}{c} \cdot \left( 1 - \exp \left( - \frac{cq_e}{w_e} \right) \right) \quad \text{(by Jensen’s inequality)}
\]

\[
= \sum_{e \in E} \frac{w_e}{c} \cdot \left( 1 - \exp \left( - \frac{cq_e}{w_e} \right) \right) = \sum_{u \in E_u} \sum_{e \in E_{uv}} \frac{w_e}{c} \cdot \left( 1 - \exp \left( - \frac{cq_e}{w_e} \right) \right),
\]

where we use the concavity of \( 1 - \exp(-x) \) in the second inequality. Let \( u \) be any fixed vertex in \( L \) and \( e_1, e_2, \ldots, e_k \) be the edges of \( E_u \) enumerated according to their arrival order. Then we have

\[
Q_{e_i} = \{ \exists e_i \text{ and } \forall j < i \text{ (not } \exists e_j \})\;
\]

\[
q_{e_i} = \Pr[\exists e_i] \cdot \prod_{j < i} \Pr[\text{not } \exists e_j] = (1 - \exp(-w_{e_i})) \cdot \exp \left( - \sum_{j < i} w_{e_j} \right).
\]

Without loss of generality, we can assume that all \( w_e \)’s are small, as per the definition of log-normalized weight each edge can be split into smaller edges with the same log-normalized total weight. And specifically, Greedy has the same performance for the split instance. Then

\[
\sum_{e \in E_u} \frac{w_e}{c} \cdot \left( 1 - \exp \left( - \frac{cq_e}{w_e} \right) \right) = \sum_{i=1}^k \frac{w_{e_i}}{c} \cdot \left( 1 - \exp \left( - \frac{cq_{e_i}}{w_{e_i}} \right) \right)
\]

\[
= \sum_{i=1}^k \frac{w_{e_i}}{c} \cdot \left( 1 - \exp \left( - c \exp \left( - \sum_{j < i} w_{e_j} \frac{1 - \exp(-w_{e_j})}{w_{e_j}} \right) \right) \right)
\]

\[
\approx \sum_{i=1}^k \frac{w_{e_i}}{c} \cdot \left( 1 - \exp \left( - c \exp \left( - \sum_{j < i} w_{e_j} \right) \right) \right) \geq \int_0^1 1 - e^{-ce^{-cx}} \, dx.
\]

We conclude our proof by noticing that \( |L| \) is an upper bound of OPT. \[ \square \]

\[ ^5 \text{This is the only place where we use the fact that } G \text{ is a bipartite graph. Indeed, the result in this section can be generalized to triangle-free graphs.} \]
Remark. When \( c = 2 \), the competitive ratio is at least \( f(c) \geq 0.532 \). Notice that \( f \) has a peak at \( c \approx 2.1 \), but it gets smaller again for \( c > 3 \) and our analysis gives relatively weak results for large \( c \). One reason is because of the relaxation from Lemma 1. On the other hand, GREEDY indeed does not perform well on \( c \)-regular graphs when \( c \) is large. In particular, GREEDY is no better than \( 0.5 \)-competitive on \( c \)-regular graphs when \( c \) goes to infinity.

**Theorem 2.** GREEDY is not better than \( 0.5 \)-competitive on log-normalized \( c \)-regular graphs when \( c \to \infty \).

**Proof.** Consider the graph shown in Figure 1. We use \( L_1 = \{u_i\}_{i=1}^{n+1}, R_1 = \{v_j\}_{j=1}^{n+1}, L_2 = \{u'_i\}_{i=1}^{n} \) and \( R_2 = \{v'_j\}_{j=1}^{n} \) to denote the vertices in the graph. The edges are defined as the following:

1. For each \( i \in [n+1] \), there is a (red solid) edge \( (u_i, v_i) \) with existence probability \( 1 - \varepsilon \).

2. For each pair of \( (u, v) \in (L_2 \times R_1) \cup (L_1 \times R_2) \), there is a (green/blue dashed) edge \( (u, v) \) with existence probability \( 1 - \varepsilon \).

It is easy to verify the graph is log-normalized regular. When \( \varepsilon \to 0 \), with high probability, the graph admits a perfect matching with size \( 2n + 1 \). On the other hand, consider when the red edges arrive first. With high probability, all these edges exist and GREEDY matches \( n + 1 \) edges. This finishes the proof since \( \frac{n+1}{2n+1} \to \frac{1}{2} \) when \( n \to \infty \). \( \square \)

Note that we only count the probability of \( \bigcup_{e \in E_u} Q_e \) in Lemma 1 to lower bound the probability that \( v \in R \) is matched. If we take the perspective of a vertex \( u \in L \), then we effectively only try to match the first realized edge of \( u \). Consider an alternative interpretation of our analysis. Initially, all vertices in \( L \) are available. When an edge \( (u, v) \) with \( u \in L, v \in R \) comes,

- \( u \) proposes to \( v \) if the edge \( (u, v) \) exists and \( u \) is available. Once \( u \) made a proposal, \( u \) becomes committed, so all future edges in \( E_u \) are ignored.
- \( v \) accepts the first proposal.

We refer to this algorithm as the COMMITMENT algorithm. Noticeably, the ratio \( f(c) \) is tight for this algorithm on log-normalized \( c \)-regular graphs.

**Theorem 3.** COMMITMENT matches in expectation \( f(c) \) fraction of all vertices on log-normalized \( c \)-regular graphs and the ratio is tight.
Proof. The first part of the theorem follows immediately from the proof of Theorem [1]. To see the tightness of the ratio, we consider a complete bipartite graph with \( n \) vertices on each side and every edge has log-normalized weight \( \frac{c}{n} \). Let \( L = \{u_1, \ldots, u_n\} \) and \( R = \{v_1, \ldots, v_n\} \). The edges \((u_i, v_j)\)'s arrive in the alphabetical order of \((i, j)\). We now consider the performance of COMMITMENT for this instance. Note that

\[
\mathbb{P}(v_j \text{ is matched}) = \mathbb{P}\left[ \bigcup_i (u_i, v_j) \text{ is proposed} \right]
\]

\[
= \mathbb{P}\left[ \bigcup_i (u_i, v_j) \text{ is the first realized edge of } u_i \right]
\]

\[
= 1 - \prod_i \mathbb{P}\left[ (u_i, v_j) \text{ is not the first edge of } u_i \right]
\]

\[
= 1 - \prod_i \left( 1 - \exp\left( -\frac{(j - 1)c}{n} \right) \cdot \left( 1 - \exp\left( -\frac{c}{n} \right) \right) \right)
\]

\[
\approx 1 - \left( 1 - \frac{c}{n} \cdot \exp\left( -\frac{(j - 1)c}{n} \right) \right)^n \approx 1 - \exp\left( -c \exp\left( -\frac{(j - 1)c}{n} \right) \right),
\]

where the last two steps we assume \( n \to \infty \). In other words, the above analysis shows that the Jensen’s inequality step in the proof of Theorem [1] is tight for this instance. To sum up, we have

\[
\text{ALG} = \sum_j \mathbb{P}(v_j \text{ is matched}) = \sum_j \left( 1 - \exp\left( -c \exp\left( -\frac{(i - 1)c}{n} \right) \right) \right) \approx n \cdot \int_0^1 1 - e^{-c_\epsilon - \epsilon x} \, dx. \quad \square
\]

4 General Input Graphs

The fact that Greedy beats half on log-normalized \( c = 2 \) regular graphs lends itself to the following natural two step adaptation for general graphs: (i) prune (remove or decrease probabilities of certain edges in \( G \)) such that the log-normalized degree of each vertex in the remaining graph is 2; (ii) greedily take every edge in the pruned instance \( G_c \). Specifically, upon the arrival of an edge \( e \), we first adjust its probability by dropping \( e \) so that its realization probability is consistent with \( e \)'s log-normalized weight in \( G_c \), then we match the realized edge if none of \( e \)'s endpoints are currently matched. This approach would already yield the desired result for the dense graphs that can be pruned to the log-normalized 2-regular graph \( G_c \). However, such a direct strategy fails for the graphs that have a few small degree vertices.

Before we proceed with the fix for the general graphs, let us take a closer look at the proof of Theorem [1]. Note that in the theorem we actually compare our algorithm with a stronger benchmark, half the total number of vertices in \( G \). The problem with such a benchmark, is that it may be too strong for any algorithm to approximate. To address this issue, we have to adjust our algorithm and analysis to handle low degree vertices. To this end, we can calculate \( x_e \), the probability that \( e \) appears in the maximum matching of the random graph for every \( e \), as the first step of our algorithm. By definition, \( \text{OPT} = \sum_{e \in E} x_e \) is the right benchmark to compare with. Alternatively, we can solve the following LP introduced by Gamlath et al. [15] \footnote{The LP is polynomial-time solvable. See [15] for the details.}

\[
\begin{align*}
\text{maximize} & \quad \sum_{(x_e \geq 0) \in E} x_e \\
\text{subject to} & \quad \sum_{e \in F} x_e \leq 1 - \prod_{e \in F} (1 - p_e), \quad \forall v \in L \cup R, \quad \forall F \subseteq E_v.
\end{align*}
\]

(2)
The constraints of the LP simply state that for each \( v \) and \( F \subseteq E_v \), the probability that an edge of \( F \) appears in the maximum matching is at most the probability that at least one edge of \( F \) is realized. Note that the value of each variable \( x_e \) in the LP (2) does not necessarily match the exact probability of \( e \) to appear in the maximum matching. However, \( \sum_{e \in E} x_e \) still serves as a valid upper bound on \( \text{OPT} \). As a matter of fact, our analysis works for either benchmark: the solution to LP (2), or for each \( x_e \) being the probability of \( e \) to appear in the optimal matching. To obtain the desired competitive ratio we will only need LP (2) constraints on \( x = (x_e)_{e \in E} \), which hold for the former and the latter benchmark. We choose the LP (2) formulation in the description of the algorithm and the following analysis, since the LP optimal solution is a stronger benchmark and important constraints are explicitly stated in the LP.

Let us consider the following natural algorithm tailored to the LP solution \((x_e)_{e \in E}\):

1. prune the graph by decreasing the probabilities of each edge from \( p_e \) to \( x_e \),
2. run \textsc{Greedy} on the pruned instance.

Unfortunately, this attempt also fails to beat 0.5. Consider a special case when the graph always admits a perfect matching in any realization. Then the pruned graph after the first phase will have the maximum expected degree 1. In the extreme case when all \( x_e \)'s are small, log-normalized weights are equal to the real weights, i.e., \(-\ln(1-x) \approx x \) when \( x \) is small. Furthermore, since we are comparing to an upper bound on the maximum matching size \( \sum_{e \in E} x_e \), the benchmark is equal to half of the total number of vertices in the graph. The problem boils down to analyzing performance of \textsc{Greedy} on log-normalized 1-regular graphs as in the previous section. However, not only our analysis from Theorem 1 gives \( f(1) \approx 0.459 < 0.5 \), our experiment suggests that \textsc{Greedy} matches no more than 0.5 fraction of all vertices for log-normalized 1-regular graphs.

Consider the following log-normalized 1-regular graph \( G \) with \(|L| = |R| = n\): There is one edge between each \((u \in L, v \in R)\) pair with log-normalized weight of \( \frac{1}{n} \). The edges arrive in random order. We did computer-assisted experiment on the performance of \textsc{Greedy} on \( G \), whose result is summarized in Table 1. For each \( n \), we generate \( T(n) \) new instances and run \textsc{Greedy} on them, and then calculate the average \( \text{ALG}/n \) value in those \( T(n) \) runs. We observe in our experiment that our \( T(n) \)'s are large enough that the average \( \text{ALG}/n \) almost converges, and that as \( n \) increases, \( \text{ALG}/n \) becomes closer to 0.5.

| \( n \) | \( T(n) \) | \( \text{ALG}/n \) | \( n \) | \( T(n) \) | \( \text{ALG}/n \) |
|---|---|---|---|---|---|
| 3 | \( 10^{11} \) | 0.53132 | 300 | \( 10^7 \) | 0.50029 |
| 10 | \( 10^{10} \) | 0.50862 | 1000 | \( 10^6 \) | 0.50009 |
| 30 | \( 10^9 \) | 0.50281 | 3000 | \( 10^5 \) | 0.50002 |
| 100 | \( 10^8 \) | 0.50084 | 10000 | \( 10^4 \) | 0.49997 |

Table 1: Empirical performance of \textsc{Greedy} on the 1-regular graph \( G \)

We potentially can do a more conservative pruning step to fix the algorithm: instead of decreasing all \( p_e \)'s to \( x_e \)'s, we can decrease the probabilities less so that the graph has larger average degree. On the other hand, for some of the edges, \( p_e \) can be as small as \( x_e \), which prevents us using a simple rule in the pruning step and makes analysis challenging. We leave it as an interesting open question to find a “greedy algorithm with offline pruning” that is better than 0.5-competitive.

\textbf{Our Algorithm.} Our final solution is inspired by \textsc{Commitment} in the previous section, which allows us to significantly simplify the analysis. We treat the left-hand side of the graph \( L \) as the \textit{proposing} side and the right-hand side \( R \) as the \textit{receiving} side. Any given arrival order of edges in \textsc{Commitment} creates a probability distribution for each vertex \( u \in L \) over disjoint events in
which \( u \) proposes to the other end of an \( e \in E_u \) edge. On the other hand, the value of the variables \( x_e \) for \( e \in E_u \) are ideal targets for these probabilities. So it stands to a reason to adjust proposal probability of each edge \( e \in E_u \) to match as closely as possible \( x_e \) when running COMMITMENT. Specifically, we keep track of the probability that \( u \) is not yet committed during the execution and make the commitment decision adaptively. We give below a formal pseudo-code description of our Adaptive Commitment algorithm.

**Algorithm 1: Adaptive Commitment**

```plaintext
Input: \( G = (L, R, E) \), where every \( e \in E \) is associated with an existence probability \( p_e \)
1 \( (x_e)_{e \in E} \leftarrow \text{solve Linear Program (2)} \)
2 forall \( u \in L \) do
3 \( r_u \leftarrow 0 \) // \( r_u \) def current commitment probability (given edge arrival order)
4 \( \text{commit}[u] \leftarrow \text{false} \) // \( \text{commit}[u] \) def if \( u \) is committed (given realized edges)
5 while edge \( e = (u, v) \) arrives do
6 if \( e \) is realized and \! \text{commit}[u] then
7 With probability \( \min \left( \frac{x_e}{(1-r_u)p_e}, 1 \right) \)
8 Try to match \( e \) // \( u \) proposes edge \( e \)
9 \( \text{commit}[u] \leftarrow \text{true} \)
10 \( r_u \leftarrow r_u + \min(x_e, (1-r_u)p_e) \) // Update \( r_u \)
```

4.1 Analysis

In this section, we analyze the Adaptive Commitment algorithm and prove the main result of this paper. Our analysis has a similar structure to Theorem [1]. However, the derived optimization problem even after appropriate relaxations turns out to be not as simple as for log-normalized regular graphs. One reason for this is that we have to carefully compare the algorithm’s performance and the benchmark value and use LP (2) constraints.

**Theorem 4.** Adaptive Commitment is \( 0.5067 \)-competitive.

**Proof.** We begin by recalling Definition [2] in terms of the (Adaptive) Commitment algorithm.

**Definition 3.** For \( e \in E_{uv} \), where \( u \in L \), \( Q_e \) def \( \{ u \) proposes \( e \) to \( v \} \). Let \( q_e \) def \( \text{Pr}[Q_e] \).

As in Theorem [1] we get a similar to (1) lower bound on the performance \( \text{ALG} \) of Adaptive Commitment, where \( \frac{w_e}{e} \) is simply substituted by \( x_e \).

**Lemma 2.** The expected size of matching is at least \( \sum_{u \in L} \sum_{e \in E_u} x_e \cdot \left( 1 - e^{-\frac{q_e}{x_e}} \right) \).

**Proof.** Lemma [1] holds for Adaptive Commitment exactly as is. We obtain the desired estimate by applying Jensen inequality for \( 1 - e^{-x} \), as in derivation (1). The only difference being that \( \sum_{e \in E_u} x_e \leq 1 \) instead \( \sum_{e \in E_u} \frac{w_e}{e} = 1 \), which does not affect our derivations, since \( 1 - e^{-0} = 0 \).

Then the competitive ratio of Algorithm [3] is at least

\[
\frac{\text{ALG}}{\text{OPT}} \geq \frac{\sum_{u \in L} \sum_{e \in E_u} x_e \cdot \left( 1 - \exp\left(-\frac{q_e}{x_e}\right)\right)}{\sum_{u \in L} \sum_{e \in E_u} x_e} \geq \min_{u \in L} \frac{\sum_{e \in E_u} x_e \cdot \left( 1 - \exp\left(-\frac{q_e}{x_e}\right)\right)}{\sum_{e \in E_u} x_e}.
\]
It is sufficient to prove the desired lower bound on the RHS of the above equation for any given \( u \in L \). In the following we fix \( u \in L \). Let \( \{e_1, e_2, \ldots, e_k\} = E_u \) be all edges incident to \( u \) and enumerated in their arrival order. To simplify notations, let \( p_i, x_i, q_i \) denote respectively \( p_{e_i}, x_{e_i}, \) and \( q_{e_i} \). The optimization problem (3) below gives a lower bound on \( \text{ALG}_{OPT} \).

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{k} x_i \cdot \left(1 - e^{-\frac{q_i}{x_i}}\right) / \sum_{i=1}^{k} x_i \\
\text{s.t.} & \quad q_i = \min \left[x_i, \ p_i \cdot \left(1 - \sum_{j<i} q_j\right)\right], \quad \forall i \in [k] \\
& \quad \sum_{i \in S} x_i \leq 1 - \prod_{i \in S} (1 - p_i), \quad \forall S \subseteq [k]
\end{align*}
\]

where we optimize over \( x \) and \( p \); the first family of constraints in (3) captures that \( q_i \) (the probability of \( e_i \) being proposed) only depends on the arrival order of the edges in \( E_u \), and the equality is due to Line 7 and 10 of Algorithm 1 the second family of constraints in (3) follows from LP (2). We only use the LP (2) constraints for \( u \in L \) and drop those for \( v \in R \).

We further decrease the value of optimization problem (3) by increasing the size \( k \) of the instance and getting a simpler optimization problem (4). The fact that we can construct more regular instance with all \( x_i = x \) and smaller or equal objective value follows from the “subdivision” Lemma 3 below.

**Lemma 3.** Let \( (x_j, p_j, q_j)_{j \in [k]} \) be any feasible solution to (3). Let \( e_i \) be any edge \( i \in [k] \) which we subdivide into two consecutive edges \( e' \) and \( e'' \). Then there is a feasible solution to the new instance of (3) with the same \( (x_j, p_j, q_j)_{j \neq i} \) and \( x_{e'} + x_{e''} = x_i \), proportional \( q_{e'} = \frac{x_{e'}}{x_i} q_i \), \( q_{e''} = \frac{x_{e''}}{x_i} q_i \), where \( x_{e'} \geq 0 \) and \( x_{e''} \geq 0 \) can be set to have any values subject to \( x_{e'} + x_{e''} = x_i \).

**Proof.** Consider the optimal solution \( (x_j, p_j, q_j)_{j \in [k]} \) to the optimization problem (3) and a particular edge \( e_i \). We construct another instance (3) (we call it “subdivision”) and a feasible solution to it by replacing \( e_i \) with two consecutive edges \( e' \) and \( e'' \) with \( x_{e'} + x_{e''} = x_i \) and \( q_{e'} = \frac{x_{e'}}{x_i} q_i \), \( q_{e''} = \frac{x_{e''}}{x_i} q_i \). Let \( \beta = \frac{x_{e'}}{x_i} \).

To prove the lemma we define \( p_{e'}, p_{e''} \). Consider two cases depending on the value of \( q_i \):

- **When** \( q_i = x_i \), let \( p_{e'} = p_{e''} = 1 \). We verify that

\[
q_{e'} = \min \left(x_{e'}, \left(1 - \sum_{j<i} q_j\right) \cdot p_{e'}\right) = \min \left(\beta x_i, 1 - \sum_{j<i} q_j\right) = \beta x_i = \beta q_i;
\]

\[
q'_{j+1} = \min \left(x_{e''}, \left(1 - \sum_{j<i} q_j - q_{e'}\right) \cdot p_{e''}\right) = \min \left((1 - \beta) x_i, 1 - \sum_{j<i} q_j - \beta q_i\right)
\]

\[
= (1 - \beta) x_i = (1 - \beta) q_i,
\]

where the second to last equality follows from the fact that \( 1 - \sum_{j<i} q_j - \beta q_i \geq q_i - \beta q_i = (1 - \beta) x_i \). It is easy to see that the inequality constraints of the program (3) are satisfied in this case, since whenever an edge \( e' \) or \( e'' \) is involved, the right-hand side of the constraint becomes 1.

\[\text{Since } k \text{ can be arbitrary number in (3), we use the same } k \text{ in (4) as well.}\]
• When $q_i = (1 - \sum_{j<i} q_j) \cdot p_i$, let $p_{e'} = \beta p_i$. Then

$$q_{e'} = \min \left( x_{e'}, \left( 1 - \sum_{j<i} q_j \right) \cdot p_{e'} \right) = \min \left( \beta x_i, \left( 1 - \sum_{j<i} q_j \right) \cdot \beta p_i \right) = \beta q_i.$$  

Define $p_{e''}$ so that $(1 - p_{e'})(1 - p_{e''}) = 1 - p_i$, implying $1 \geq p_{e''} = \frac{(1 - \beta) p_i}{1 - \beta p_i} \geq (1 - \beta)p_i$. We have

$$q_{e''} = \min \left( x_{e''}, \left( 1 - \sum_{j<i} q_j - q_{e'} \right) \cdot p_{e''} \right) = \min \left( (1 - \beta)x_i, \left( 1 - \sum_{j<i} q_j \right) (1 - \beta p_i) p_{e''} \right) = \min \left( (1 - \beta)x_i, \left( 1 - \sum_{j<i} q_j \right) (1 - \beta) p_i \right) = (1 - \beta)q_i.$$

Next, we prove that the inequality constraints in (3) are still satisfied.

- When neither of $e'$, $e''$ is involved, the constraint trivially holds.
- When both $e'$, $e''$ are involved, the inequality holds since $x_{e'} + x_{e''} = x_i$ and $(1 - p_{e'})(1 - p_{e''}) = 1 - p_i$.
- When only $e'$ is involved. Let $S \subseteq [k] - \{i\}$ be any set of indexes.

$$1 - (1 - p_{e'}) \prod_{j \in S} (1 - p_j) = 1 - (1 - \beta p_i) \prod_{j \in S} (1 - p_j) \quad \text{(since } p_{e'} = \beta p_i)$$

$$= \beta \left( 1 - (1 - p_i) \prod_{j \in S} (1 - p_j) \right) + (1 - \beta) \left( 1 - \prod_{j \in S} (1 - p_j) \right)$$

$$\geq \beta \left( x_i + \sum_{j \in S} x_j \right) + (1 - \beta) \sum_{j \in S} x_j = \beta x_i + \sum_{j \in S} x_j = x_{e'} + \sum_{j \in S} x_j.$$

- When only $e''$ is involved, we have similar to the previous case for any $S \subseteq [k] - \{i\}$

$$1 - (1 - p_{e''}) \prod_{j \in S} (1 - p_j) \geq 1 - (1 - (1 - \beta)p_i) \prod_{j \in S} (1 - p_j) \quad \text{(since } p_{e''} \geq (1 - \beta)p_i)$$

$$= (1 - \beta) \left( 1 - (1 - p_i) \prod_{j \in S} (1 - p_j) \right) + \beta \left( 1 - \prod_{j \in S} (1 - p_j) \right)$$

$$\geq (1 - \beta) \left( x_i + \sum_{j \in S} x_j \right) + \beta \sum_{j \in S} x_j = (1 - \beta)x_i + \sum_{j \in S} x_j = x_{e''} + \sum_{j \in S} x_j.$$  

Therefore, the lemma holds in both cases we discussed above. This concludes our proof.  

Indeed, we can start with the optimal solution to (3) for any given $k$, then apply multiple times Lemma 3 to every edge $e_i$, $i \in [k]$ getting an instance with $k' \gg k$ edges and a feasible solution with the same value, where almost all $x_j = x$ and at most $k$ edges have $x_e < x$ ($x$ may depend on $k'$). Finally, we can remove all edges with $x_e < x$, keep the rest $x_j$ and $p_j$ untouched and redefine ($q_j$) according to the recurrent formula. The impact of the change to $q_j$’s before the removal of edges.
Proof. For each given $k$ we consider the optimal solution to Problem 4. We prove (ii) first, then (i), and finally (iii). Suppose there exists $i \in [k]$ such that $q_i = (1 - \sum_{j<i} q_j) \cdot p_i < x$ and $q_{i+1} = x$. Consider substituting $p_i, p_{i+1}$ with $p'_i = 1$ and $p'_{i+1} = \frac{q_i}{1 - x - \sum_{j<i} q_j}$ (since $\sum_{j<i} q_j \leq 1$). The other $(p_j)_{j \neq i, i+1}$, $x$ remain the same while $(q'_j)_{j \in [k]}$ are adjusted accordingly. Consequently, we have

$$q'_i = \min \left( x, 1 - \sum_{j<i} q_j \right) = x \quad \text{and} \quad q'_{i+1} = \min \left( x, \left( 1 - \sum_{j<i} q_j - x \right) \cdot p'_{i+1} \right) = q_i.$$  

Notice that $q'_i + q'_{i+1} = q_i + q_{i+1}$, this substitution shall not change $q_j$’s for $j > i + 1$. Furthermore, $p'_i \geq p_{i+1}$ and $p'_{i+1} = \frac{(1 - \sum_{j<i} q_j) \cdot p_i}{1 - x - \sum_{j<i} q_j}$, hence, the constraints of the program are preserved, which finishes the proof of statement (ii).

Let $\ell$ be the cut-off point, i.e., $q_i$ equals $x$ when $i \leq \ell$ and equals $(1 - \sum_{j<i} q_j) \cdot p_i$ when $i > \ell$. Then we can get explicit expression for $q_i$,

$$q_i = p_i (1 - \ell x) \cdot \prod_{\ell < j < i} (1 - p_j), \quad \forall i \geq \ell \quad (5)$$

by solving a simple recursion $q_{i+1} = (1 - \sum_{j<i+1} q_j) \cdot p_{i+1} = \left( \frac{q_i}{p_i} - q_i \right) p_{i+1}$. Without loss of generality, we assume that $p_i = 1$ for all $i \leq \ell$, which does not change the objective. Next, if $p_i < p_{i+1}$ for an $i > \ell$, we can decrease the objective by swapping $p_i$ and $p_{i+1}$ in the instance (the rest of $p_j$’s and $x$ remain the same). We note that the new instance $x, p'$ satisfies all inequality constraints in Problem 4, because the set of constraints is invariant under any permutation of $p_j$’s. Let $t \overset{\text{def}}{=} \left( 1 - \ell x \right) \cdot \prod_{\ell < j < i} (1 - p_j) \in [0, 1]$. In the original instance, $q_i = p_i \cdot t$ and $q_{i+1} = (1 - p_i) p_{i+1} \cdot t$ by (5). After the swap, we have $q'_i = p_{i+1} \cdot t$ and $q'_{i+1} = (1 - p_{i+1}) p_i \cdot t$. Notice that $q'_i + q'_{i+1} = q_i + q_{i+1}$, and hence, the above inequality is true due to the convexity of the function $\exp(-y)$. A contradiction that concludes the proof of (i), the monotonicity of $p_i$’s.

We are left to prove (iii). First, given the monotonicity of $p_i$’s, we note that “critical” inequality constraints $|S| \cdot x \leq 1 - \prod_{i \in S} (1 - p_i)$ are those where $S = \{j, j + 1, \cdots, k\}$, i.e., the remaining (non-critical) inequality constraints for other sets $S$ are automatically satisfied, if the constraints for $S = \{j, j + 1, \cdots, k\}$ hold. Indeed, when restricting to $S$ with a fixed cardinality $s$, the left-hand side of each constraint is the same $|S| \cdot x$, while the right-hand side is minimized when $S$ consists of
the $s$ smallest $p_i$; i.e., $\{p_{k-s+1}, \ldots, p_k\}$. Note that any critical constraint trivially holds for $j \leq \ell$, since in this case the right-hand side of the constraint equals 1. We are going to prove (iii), that all other critical constraints for $j > \ell$ are tight. First we write a few useful identities for $(q_i)_{i \in [k]}$.

$$q_i = p_i \cdot \left(1 - \sum_{\ell < i} q_\ell\right) \quad \text{for } i \in (\ell..k],$$

(6)

$$q_i = p_i (1-\ell x) \prod_{\ell < j < i} (1-p_j) \quad \text{for } i \in (\ell..k],$$

(7)

$$\sum_{i \in [\ell..k]} q_i = \frac{q_i}{p_i} \left(1 - \prod_{i \in [\ell..k]}(1-p_i)\right) + q_{i+1} = \frac{q_i}{p_i} \left(1 - \prod_{i \in [\ell..k]}(1-p_i)\right) + \frac{q_{i+1}}{p_{i+1} \cdot \prod_{i \in [\ell..k]}(1-p_i)} = \frac{q_i}{p_i} \left(1 - \prod_{i \in [\ell..k]}(1-p_i) \cdot (1-p_{i+1})\right).$$

Equation (6) holds since $i > \ell$ and definition of $\ell$; we already have (7) as (5); to derive (8) we use induction on $i$.

Now, we suppose to the contrary that a critical constraint is not tight for an $S = \{i, i+1, \ldots, k\}$ for $\ell > 1$, while all critical constraints for each $S = \{i, i+1, \ldots, k\}$ where $i > \ell$ are tight (if $i = k$, we don’t require any constraints to be tight). We first consider a non-degenerate case when $1 > p_{\ell-1}$, which also means that $\ell - 1 > \ell$ (otherwise $q_{\ell-1} = x$ and we would set $p_{\ell-1} = 1$).

**Case 1 $(1 > p_{\ell-1})$.** We will provide the instance with a strictly smaller objective’s value. Before that we prove the following fact.

**Claim 1.** If $1 > p_i = p_{i+1}$ for $i \in (\ell..k)$, then inequality $|S| \cdot x < 1 - \prod_{j \in S}(1-p_j)$ for $S = \{i+1, \ldots, k\}$ is strict.

**Proof.** Suppose to the contrary that the inequality is an equality, that is

$$(1 - p_{i+1}) = \frac{\prod_{j > i}(1-p_j)}{\prod_{j > i+1}(1-p_j)} = \frac{1 - (k-i)x}{\prod_{j > i+1}(1-p_j)}.$$

The inequality constraint for $S = \{i, \ldots, k\}$ in (4) gives:

$$(1 - p_i)(1 - p_{i+1}) = \frac{\prod_{j > i}(1-p_j)}{\prod_{j > i+1}(1-p_j)} \leq \frac{1 - (k-i+1)x}{\prod_{j > i+1}(1-p_j)}.$$

Putting the two equations together and by the assumption that $p_i = p_{i+1}$, we have

$$\left(\frac{1 - (k-i)x}{\prod_{j > i+1}(1-p_j)}\right)^2 \leq \frac{1 - (k-i+1)x}{\prod_{j > i+1}(1-p_j)} \Rightarrow \frac{(1 - (k-i)x)^2}{(1 - (k-i+1)x)^2} \leq \prod_{j > i+1}(1-p_j) \leq 1 - (k-i-1)x,$$
where the last inequality follows from the constraint for \( S = \{i + 2, \ldots, k\} \). Thus
\[
(1 - (k - i + 1)x)(1 - (k - i - 1)x) = (1 - (k - i)x)^2 - x^2 \geq (1 - (k - i)x)^2,
\]
a contradiction. \( \square \)

Let \( i \) be the smallest index so that \( p_1 = p_{i+1} = \cdots = p_{i-1} \). So \( p_{i-1} > p_i \) if \( i > 1 \). Recall that we consider the case \( 1 > p_{i-1} \). Thus \( \ell \leq i \) (otherwise \( q_i = x \) and we should have set \( p_i = 1 \)). By Claim 1 each inequality in (4) for \( S = \{i, i + 1, \ldots, k\} \) with \( i + 1 \leq i \leq \bar{i} - 1 \) must be strict. On the other hand, a contra-positive statement to Claim 1 gives us that \( p_i \) cannot be equal to \( p_{i+1} \) (if \( \bar{i} = k \), this also is true). Thus \( p_i > p_{i+1} \) (if \( i > k \)).

We consider the following modification \((p', x)\) of (4)'s feasible solution: slightly increase \( p_{i} \) and decrease \( p_i \) so that \((1 - p_i)(1 - p_i)\) remains the same; all other \( p_i \) for \( i \neq i, \bar{i} \) and \( x \) are the same in \((p', x)\) and original optimum \((p, x)\); \( q' \) are redefined according to the recurrent formula in (4). Note that we can always do such modification when \( p_i > 0 \) (we can assume that \( p_i > 0 \), as otherwise \( \bar{i} = k \) due to the monotonicity of \( p \) and we should have considered (4) for a smaller \( k \)).

For any sufficiently small such perturbation of \( p_i \) and \( p_{i+1} \), \((p', i)_{i \in [k]}\) remain monotone and all constraints in (4) are satisfied. Indeed, we only need to check the critical constraints in (4) for monotone \( p' \): \( p' \) and \( p \) are the same for \( S = \{i, i + 1, \ldots, x\} \) for \( i \in (\ell, \bar{i}] \); all inequalities for \( S = \{i, i + 1, \ldots, k\} \) where \( i \in (\ell + 1, \bar{i}] \) are strict and, therefore, for sufficiently small perturbation of \( p_i \) and \( p_{i+1} \) they still hold; for \( S = \{i, i + 1, \ldots, x\} \) where \( i \in (\ell, \bar{i}] \), the right-hand side of each critical constraint does not change, because \((1 - p_i)(1 - p_i) = (1 - p_{i+1})(1 - p_{i+1})\).

Now we examine the changes to \( q_i \). Observe that each \( q'_i = q_i \) for any \( i < \bar{i} \) as \((p', i)_{i < \bar{i}}\) and \((p, i)_{i < \bar{i}}\) are the same. Since \( i \geq \bar{i}, q_i \) is given by (6) for each \( i \in [\bar{i}, \bar{i}] \). By (8) we notice that
\[
\sum_{i \in [\bar{i}, \bar{i}]} q_i = \frac{q_{\bar{i}}}{p_{\bar{i}}} \left(1 - \prod_{i \in [\bar{i}, \bar{i}]} (1 - p_i)\right) = \sum_{i \in [\bar{i}, \bar{i}]} q'_i.
\]
This means that \( q'_i = q_i \) for \( i > \bar{i} \) due to the formula (6) for \( q_i \). In the interval \( i \in [\bar{i}, \bar{i}] \), we notice that by increasing \( p_{\bar{i}} \) and decreasing \( p_i \) we increase \( q_\bar{i} \) and decrease each \( q_i \) for \( i \in (\bar{i}, \bar{i}] \) by (7). Moreover, by (7) we have
\[
q_\bar{i}' = \left[(1 - \ell x) \prod_{j \in (\bar{i}, \bar{i}]} (1 - p_j) \prod_{j \in (\bar{i}, \bar{i}]} (1 - p_j)\right] \cdot (1 - p_{\bar{i}}) \cdot p_{\bar{i}}'
\]
\[
= \left[(1 - \ell x) \prod_{j \in (\bar{i}, \bar{i}]} (1 - p_j) \prod_{j \in (\bar{i}, \bar{i}]} (1 - p_j)\right] \cdot (1 - p_{\bar{i}} - (1 - p_{\bar{i}})(1 - p_{\bar{i}})) < q_{\bar{i}},
\]
since \( p_{\bar{i}}' > p_\bar{i} \) while \((1 - p_{\bar{i}}')(1 - p_{\bar{i}}') = (1 - p_\bar{i})(1 - p_\bar{i}) \). Hence, due to convexity of the function \( \exp(-y) \), we conclude that the objective \( \sum_i (1 - \exp(-\frac{y_i}{x_i})) \) decreases when we substitute \( q \) with \( q' \). Indeed, \( q_{\bar{i}} \), the largest number among \( \{q_i\}_{i = \bar{i}} \), increases, while all other affected \( q_i \) in \([\bar{i}, \bar{i}] \) decrease.

**Case 2** \((p_{i-1} = 1 \text{ and } 1 > p_i)\). Now we consider a degenerate case when \( p_{i-1} = 1 \). Recall that constraint of (4) for \( S = \{\bar{i}, \bar{i} + 1, \ldots, k\} \) is not tight, while the constraints for \( S = \{j, j + 1, \ldots, k\} \) with \( j > \bar{i} \) are tight. We will construct another instance \((p', x)\) with a strictly smaller objective's value. Now, we only modify \( p_i \) in the optimal instance \((p, i)_{i \in [k]} \) by slightly increasing \( p_i \). To be able to do so we also assume that \( 1 = p_{i-1} > p_i \) (we consider the case \( p_i = 1 \) later).
We ensure that such small modification of \( p_\ell \) preserves monotonicity of \( p \). Then we can easily satisfy all critical constraints in \( [4] \) (the only non-trivial constraint is for \( S = \{7, 7 + 1, \ldots, k\} \), which is strict inequality by definition of \( i \)). Thus all inequality constraints in \( [4] \) are satisfied. We observe that \( q' \) is the same as \( q \) for \( i < i \), \( q' > q_\ell \) (by \( [5] \)), \( q' < q_i \) for \( i \in [i, k] \) (by \( [7] \)), and that \( \sum_{i \in [i, k]} q'_i > \sum_{i \in [i, k]} q_i \) (by \( [8] \)). Similar to the previous case, due to concavity and monotonicity of the function \( 1 - \exp(-y) \), we conclude that the objective \( \sum_i (1 - \exp(-y_i)) \) decreases when we substitute \( q \) with \( q' \). Indeed, \( q'_\ell \), the largest number among \( \{q_i\}_{i \in [i, k]} \), increases, while all other affected \( q_i \)'s in \([i, 7]\) decrease. This contradicts the optimality of \((p, x)\) and concludes the proof of this case.

**Case 3** \((1 = p_{\ell - 1} = p_\ell)\). In this case, by definition of \( \ell \) and \( i \ell > \ell \) we should have \( x > q_\ell = p_\ell \cdot (1 - \sum_{i < \ell} q_i) = 1 - \sum_{i < \ell} q_i \). Moreover, \( x \cdot \ell \leq x \cdot k \leq 1 \) according to the constraint of \( [4] \) for \( S = \{1, 2, \ldots, k\} \). Thus \( x \leq 1 - (\ell - 1)x \leq 1 - \sum_{i < \ell} q_i < x \) which is a contradiction.

Given Lemma 3, we can explicitly calculate all \( \{q_i\}_{i \in [k]} \). Specifically, \( q_i = x \) for \( i \leq \ell \); for all \( i > \ell \), \( \prod_{j=i+1}^k (1 - p_j) = 1 - (k - i)x \) and \( \prod_{j=i}^k (1 - p_j) = 1 - (k - i + 1)x \) by (iii) of Lemma 3. Thus, \( p_i = \frac{1}{1 - (k - i)x} \).

Hence, \[
q_i = (1 - \ell x) \cdot \prod_{j=\ell+1}^{i-1} (1 - p_j) \cdot p_i = (1 - \ell x) \cdot \frac{\prod_{j=\ell+1}^k (1 - p_j)}{\prod_{j=i}^k (1 - p_j)} \cdot p_i = (1 - \ell x) \cdot \frac{1 - (k - \ell)x}{1 - (k - i + 1)x} \cdot \frac{x}{1 - (k - i)x},
\]
for all \( i > \ell \). To simplify notations let \( s \overset{\text{def}}{=} kx \) and \( \alpha \overset{\text{def}}{=} \frac{\ell}{k} \).

Then \[
\frac{q_i}{x} = \frac{(1 - \alpha s) \cdot (1 - (1 - \alpha)s)}{(1 - s (1 - \frac{1}{k})) \cdot (1 - s (1 - \frac{1}{k}))},
\]
for all \( i > \ell \). Consequently, we have that \[
\frac{\text{ALG}}{\text{OPT}} \geq \frac{\sum_{j=1}^k x \cdot (1 - e^{-\frac{q_j}{x}})}{k x} = \frac{\ell}{k} \cdot \left( 1 - \frac{1}{e} \right) + \sum_{i=\ell+1}^k \frac{1}{k} \cdot \left( 1 - e^{-\frac{(1-\alpha s)(1-(1-\alpha)s)}{(1-s)(1-s)}} \right) \geq \alpha \cdot \left( 1 - \frac{1}{e} \right) + \int_{\alpha}^1 \left( 1 - e^{-\frac{(1-\alpha s)(1-(1-\alpha)s)}{(1-s)(1-s)}}(1-s)^2 \right) \, dt.
\]
Furthermore, since \( \frac{q_{i+1}}{x} = \frac{1 - \alpha s}{1 - s + \alpha s + x} \leq 1 \), we have \( \alpha \geq \frac{k-1}{2k} \). Recall that we can apply the "subdivision" Lemma 3 to have an arbitrarily large \( k \). This gives us the constraint that
\[
\alpha \geq \frac{1}{2}.
\]

Finally, we calculate the minimum of \( g(s, \alpha) \overset{\text{def}}{=} \alpha \cdot \left( 1 - \frac{1}{e} \right) + \int_{\alpha}^1 \left( 1 - e^{-\frac{(1-\alpha s)(1-(1-\alpha)s)}{(1-s)(1-s)}}(1-s)^2 \right) \, dt \).

**Lemma 5.** We have \( g(s, \alpha) \geq 0.5067 \) for all \( s \in [0, 1] \) and \( \alpha \in \left[\frac{1}{2}, 1\right] \).
Proof. We start by proving that function

$$g(s, \alpha) = \alpha \cdot \left(1 - \frac{1}{e}\right) + \int_{\alpha}^{1} \left(1 - e^{-\frac{(1-\alpha)s (1-(1-\alpha)s)}{(1-s+t)^2}}\right) dt$$

is monotonically decreasing in $s$. It suffices to show $\frac{(1-\alpha)s (1-(1-\alpha)s)}{(1-s+t)^2}$ is a decreasing function in $s$ for all $t \in [\alpha, 1]$ and $\alpha \in [0.5, 1]$.

Observe that $\alpha \geq 1 - t$ and $1 - \alpha \geq 1 - t$. This implies that $\frac{1-\alpha s}{1-(1-\alpha)s}$ and $\frac{1-(1-\alpha)s}{1-(1-\alpha)s}$ are both decreasing in $s$. Therefore $g(s, \alpha) \geq g(1, \alpha)$.

Next, we prove the monotonicity of $g(1, \alpha)$ by taking the derivative of $g(1, \alpha)$ with respect to $\alpha$.

$$\frac{\partial g(1, \alpha)}{\partial \alpha} = (1 - e^{-1}) - \left(1 - e^{-\frac{\alpha(1-\alpha)}{\alpha^2}}\right) - \int_{\alpha}^{1} \left(\frac{2\alpha - 1}{t^2} \cdot e^{-\frac{\alpha(1-\alpha)}{\alpha}}\right) dt$$

$$\geq (1 - e^{-1}) - \left(1 - e^{-\frac{(1-\alpha)}{\alpha^2}}\right) - \int_{\alpha}^{1} \left(\frac{2\alpha - 1}{\alpha^2} \cdot e^{-\frac{\alpha(1-\alpha)}{\alpha^2}}\right) dt$$

where the inequality comes from the fact that the function $\frac{1}{t^2} \cdot e^{-\frac{\alpha(1-\alpha)}{\alpha}}$ is monotonically decreasing in $t$ for $t \geq \alpha \geq 0.5$. Further, we have

$$\frac{\partial g(1, \alpha)}{\partial \alpha} \geq \left(e^{-\frac{1-\alpha}{\alpha}} - e^{-1}\right) - (1 - \alpha) \cdot \left(\frac{2\alpha - 1}{\alpha^2} \cdot e^{-\frac{1-\alpha}{\alpha^2}}\right)$$

$$= -e^{-1} + \frac{3\alpha^2 - 3\alpha + 1}{\alpha^2} \cdot e^{-\frac{1-\alpha}{\alpha}} = e^{-1} \cdot \left(\frac{3\alpha^2 - 3\alpha + 1}{\alpha^2} \cdot e^{\frac{2\alpha - 1}{\alpha}} - 1\right)$$

$$\geq e^{-1} \cdot \left(\frac{3\alpha^2 - 3\alpha + 1}{\alpha^2} \cdot \left(\frac{2\alpha - 1}{\alpha^2} + 1\right) - 1\right) = \frac{(2\alpha - 1)^3}{e\alpha^3} \geq 0.$$ 

Finally, $g(s, \alpha) \geq g(1, 0.5) \approx 0.5067$.

This concludes the proof of Theorem 4 as $g(s, \alpha) \geq 0.5067$ in the entire relevant $(s, \alpha)$ range.

5 Problem Hardness

In this section, we present an upper bound of $\frac{2}{3} \approx 0.667$ for all online algorithms. Consider the graph shown in Figure 2. We use $L_1 = \{u_i\}_{i=1}^{n}$, $R_1 = \{v_j\}_{j=1}^{n}$, $L_2 = \{u'_i\}_{i=1}^{n}$ and $R_2 = \{v'_j\}_{j=1}^{n}$ to denote the vertices in the graph. The edges are defined as the following:

1. For each pair of $(u, v) \in L_1 \times R_1$, let there be an edge $(u, v)$ with existence probability 1. We call them type-1 edges (red solid edges).
2. For each $i \in [n]$, let there be an edge $(u_i, v_j)$ with existence probability $\frac{1}{2}$. We call them type-2 edges (blue dashed edges).
3. For each $i \in [n]$, let there be an edge $(u'_i, v_i)$ with existence probability $\frac{1}{2}$. We call them type-3 edges (green dashed edges).

Let the type-1 edges arrive first and then type-2 and type-3 edges.

**Theorem 5.** No algorithm is better than $\frac{2}{3}$-competitive.

**Proof.** Note that there is no randomness for type-1 edges. If an algorithm matches $k$ of them, there will be $\frac{n-k}{2}$ possible type-2 edges and $\frac{n-k}{2}$ type-3 edges in expectation. Thus any online algorithm matches no more than $k + \frac{n-k}{2} + \frac{n-k}{2} = n$ in expectation.
On the other hand, with high probability, there are at least \( (0.5 - o(1)) \cdot n \) realized type-2 edges and at least \( (0.5 - o(1)) \cdot n \) realized type-3 edges. In this case, the prophet can match \((0.5 - o(1)) \cdot n\) type-2 and type-3 edges respectively and then \(0.5 \cdot n\) type-1 edges. In total, the prophet matches \((1.5 - o(1)) \cdot n\) edges with high probability. That is, \( \text{OPT} \geq (1.5 - o(1)) \cdot n \) when \( n \to \infty \).

6 Open Problems

Our results indicate that edge-arrival model is theoretically interesting in the stochastic framework. They can be considered as just a first step, but one that suggests a promising research direction. Below are a few immediate open questions left from our work.

1. Our analysis for general bipartite graphs does not seem to be tight. It would be interesting to either improve the analysis of our algorithm, or find a new online algorithm with a better competitive ratio.
2. Is it possible to do better than \textsc{Greedy} for non-bipartite graphs in the stochastic setting?
3. Find a better and cleaner analysis of \textsc{Greedy} on bipartite regular graphs.
4. Explore prune & greedy online algorithms and more generally the family of non-adaptive online algorithms. In particular, find a prune & greedy algorithm which is better than \(0.5\)-competitive.

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