Higher Dimensional Homology Algebra IV: Projective Resolutions and Derived 2-Functors in (\(\mathcal{R}\)-2-Mod)

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Abstract: In this paper, we will construct the projective resolution of any \(\mathcal{R}\)-2-module, define the derived 2-functor and give some related properties of the derived 2-functor.

Keywords: \(\mathcal{R}\)-2-Module; Projective Resolution; Derived 2-Functor

1 Introduction

A 2-ring \(\mathcal{R}\) is a category with categorical ringed structure (see [8]). As 1-dimensional algebra, we defined \(\mathcal{R}\)-2-modules [5] in a different way with M. Dupont’s 2-modules in his PhD. thesis [2]. An \(\mathcal{R}\)-2-module we mentioned in this paper is \((\mathcal{A}, I, \cdot, a, b, i, z)\), where \(\mathcal{A}\) is a symmetric 2-group with \(\mathcal{R}\)-2-module structure \(\cdot\), \(I\) is the unit object under \(\cdot\), \(a, b, i, z\) are natural isomorphisms satisfying canonical properties [5].

Based on the works of A. del Río, J. Martínez-Moreno and E. M. Vitale [3], we defined the left derived 2-functor in the 2-category (2-SGp) and gave a fundamental property of the derived 2-functor in our third paper [7] of the series of higher dimensional homology algebra. In [2, 5], the authors showed that

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the 2-category \((\mathcal{R}\text{-}2\text{-Mod})\) is an abelian 2-category which has enough projective(injective) objects\([6, 10]\). Naturally, we will consider the higher dimensional homological theory in \((\mathcal{R}\text{-}2\text{-Mod})\).

The aim of this paper is to develop a homological theory in the 2-category \((\mathcal{R}\text{-}2\text{-Mod})\) just like the 1-dimensional case. We will construct the projective resolution of any \(\mathcal{R}\text{-}2\text{-module},\) which is unique up to 2-chain homotopy(Definition 3) and give the definition of the left derived 2-functor in \((\mathcal{R}\text{-}2\text{-Mod})\). Moreover, we shall give a fundamental property of the derived 2-functor. In our paper, most results are similar to \([7]\), just replacing the morphisms of symmetric 2-groups by morphisms of \(\mathcal{R}\text{-}2\text{-modules}.\) The most different and difficult are to give the \(\mathcal{R}\text{-}2\text{-module}\) structures of relative kernel and cokernel.

The present paper is organized as follows. In section 2, we give some basic facts on \(\mathcal{R}\text{-}2\text{-modules}\) such as the relative kernel and cokernel which are appeared in \([2, 3, ?]\) for symmetric 2-group case. The homology \(\mathcal{R}\text{-}2\text{-modules}\) of a complex of \(\mathcal{R}\text{-}2\text{-modules}\) appear in this section, too. In the last section, we mainly give the definition of projective resolution of an \(\mathcal{R}\text{-}2\text{-module}\) and give its construction(Proposition 2). After the basic definition of derived 2-functors from abelian 2-category \((\mathcal{R}\text{-}2\text{-Mod})\) to \((\mathcal{S}\text{-}2\text{-Mod})\)(\([2, 5]\)), we obtain our main result(Theorem 2).

This is the fourth paper of the series works on higher dimensional homological algebra.

2 Preliminary

In this section, we give the definitions and constructions of the relative (co)kernel in \((\mathcal{R}\text{-}2\text{-Mod})\) from the definitions of them given in \([2, 3]\), and then give the homology \(\mathcal{R}\text{-}2\text{-modules}\) of a complex of \(\mathcal{R}\text{-}2\text{-modules}\) which is similar to the homology symmetric 2-groups given in \([7]\), where \(\mathcal{R}\) is a 2-ring. In this paper, we will omit the composition symbol \(\circ\) in our diagrams.

Definition 1. The relative kernel of the sequence \((F, \varphi, G) : \mathcal{A} \to \mathcal{B} \to \mathcal{C}\) in \((\mathcal{R}\text{-}2\text{-Mod})\) is the triple \((\text{Ker}(F, \varphi), e_{(F, \varphi)}, \varepsilon_{(F, \varphi)})\) in \((\mathcal{R}\text{-}2\text{-Mod})\) as in the following diagram
with $\varepsilon_{(F,\varphi)}$ compatible with $\varphi$, i.e. the following diagram commutes

\[
\begin{array}{c}
\text{Ker}(F,\varphi) \\ \downarrow \varepsilon_{(F,\varphi)} \\
A \\
\downarrow F \\
B \\
\downarrow G \\
C
\end{array}
\]

and satisfies the following universal property:

Given a diagram in $(\mathcal{R} \text{-2-Mod})$

\[
\begin{array}{c}
\mathcal{K} \\ \downarrow \psi \\
A \\
\downarrow F \\
B \\
\downarrow G \\
C
\end{array}
\]

with $\psi$ compatible with $\varphi$, there is a factorization

\[(E' : \mathcal{K} \to \text{Ker}(F,\varphi), \psi' : e_{(F,\varphi)} \circ E' \Rightarrow E)\]

in $(\mathcal{R} \text{-2-Mod})$ through $(e_{(F,\varphi)}, \varepsilon_{(F,\varphi)})$, that is the following diagram commutes

\[
\begin{array}{c}
FE_{(F,\varphi)}E \\ \downarrow F_{\psi} \\
FE \\
\downarrow \text{can} \\
0
\end{array}
\]

and if $(E'', \psi'')$ is another factorization of $(E, \psi)$ through $(e_{(F,\varphi)}, \varepsilon_{(F,\varphi)})$, then there is a unique 2-morphism $e : E' \Rightarrow E''$, such that
The existence of relative kernel is given similarly to the general kernel\([5]\).

First, \(\text{Ker}(F, \varphi)\) is a symmetric 2-group (see \([3, 7]\)) consisting of:

\cdot An object is a pair \((A \in \text{obj}(\mathcal{A}), a : F(A) \to 0)\) such that the following diagram commutes

\[
\begin{array}{ccc}
G(F(A)) & \xrightarrow{G(a)} & G(0) \\
\downarrow{\varphi_A} & & \downarrow{=} \\
0 & \overset{=}\longleftarrow & 
\end{array}
\]

\cdot A morphism \(f : (A, a) \to (A', a')\) is a morphism \(f : A \to A'\) of \(\mathcal{A}\) such that the following diagram commutes

\[
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(A') \\
\downarrow{a} & & \downarrow{a} \\
0 & \overset{=}\longleftarrow & 
\end{array}
\]

Second, \(\text{Ker}(F, \varphi)\) is an \(\mathcal{R}\)-2-module:

There is a bifunctor

\[
\cdot : \mathcal{R} \times \text{Ker}(F, \varphi) \to \text{Ker}(F, \varphi)
\]

\((r, (A, a)) \mapsto r \cdot (A, a) \defeq (r \cdot A, r \cdot a), \quad (x, f) \mapsto x \cdot f,
\]

where \(r \cdot A\) and \(x \cdot f\) are the object and morphism in \(\mathcal{A}\) under its \(\mathcal{R}\)-2-module structure, respectively, \(r \cdot a\) is the composition morphism \(F(r \cdot A) \cong r \cdot F(A) \xrightarrow{\tau_a} r \cdot 0 \cong 0\). The above bifunctor is well-defined. In fact, for \((A, a) \in \text{obj}(\text{Ker}(F, \varphi))\) with \(G(a) = \varphi_A\), there is \(G(r \cdot a) = r \cdot G(a) = r \cdot \varphi_A = \varphi_{r \cdot A}\) from the basic properties.
of \( \mathcal{R}\)-2-modules. Moreover, the natural isomorphisms in the definition of \( \mathcal{R}\)-2-modules and the universal property are given as general kernels (more details see \[5\]).

**Definition 2.** The relative cokernel of the sequence \((F, \varphi, G): \mathcal{A} \to \mathcal{B} \to \mathcal{C}\) in \((\mathcal{R}\)-2-Mod\) is the triple \((\text{Coker}(\varphi, G), p(\varphi, G), \pi(\varphi, G))\) in \((\mathcal{R}\)-2-Mod\) as in the following diagram

\[
\begin{array}{cccccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} & \xrightarrow{p(\varphi, G)} \text{Coker}(\varphi, G) \\
\downarrow \varphi & & \downarrow \pi(\varphi, G) & & \downarrow 0 \\
\mathcal{A} & \xrightarrow{\text{Coker}(\varphi, G)} & \text{Coker}(\varphi, G) & & \end{array}
\]

with \(\pi(\varphi, G)\) compatible with \(\varphi\), i.e. the following diagram commutes

\[
\begin{array}{cccccc}
p(\varphi, G)F & \xrightarrow{\pi(\varphi, G)^0} & p(\varphi, G)0 \\
\downarrow \pi(\varphi, G)F & & \downarrow \text{can} \\
0F & \xrightarrow{\text{can}} & 0
\end{array}
\]

and satisfies the following universal property:

Given a diagram in \((\mathcal{R}\)-2-Mod\)

\[
\begin{array}{cccccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} & \xrightarrow{E} \mathcal{K} \\
\downarrow \psi^0 & & \downarrow \psi & & \downarrow 0 \\
\mathcal{A} & \xrightarrow{\psi^0} & \mathcal{K} & & \end{array}
\]

with \(\psi\) compatible with \(\varphi\), there is a factorization

\((E': \text{Coker}(\varphi, G) \to \mathcal{K}, \psi^0 : E' \circ p(\varphi, G) \Rightarrow E)\)

in \((\mathcal{R}\)-2-Mod\) through \((p(\varphi, G), \pi(\varphi, G))\), that is the following diagram commutes
and if \((E'', \psi'')\) is another factorization of \((E, \psi)\) through \((p_{(\varphi, G)}, \pi_{(\varphi, G)})\), then there is a unique 2-morphism \(e : E' \Rightarrow E''\), such that

\[
\begin{array}{c}
E' \xrightarrow{p_{(\varphi, G)}} E  \\
\downarrow \psi  \\
E  \\
\end{array}
\xRightarrow{e} 
\begin{array}{c}
E''  \\
\downarrow \psi  \\
\end{array}
\]

commutes.

The existence of relative cokernel is also similar to the general cokernel\(^5\).

First, \(\text{Coker}(\varphi, G)\) is a symmetric 2-group (see \([3, 7]\)) consisting of:

\[
\begin{array}{c}
- \text{Objects are those of } \mathcal{C}. \\
- \text{A morphism from } X \text{ to } Y \text{ is an equivalent class of a pair } (B, f) : X \to Y \text{ with } B \in \text{obj}(\mathcal{B}) \text{ and } f : X \to G(B) + Y. \text{ For two morphisms } (B, f), (B', f') : X \to Y \text{ are equivalent if there is } A \in \text{obj}(\mathcal{A}) \text{ and } a : B \to F(A) + B' \text{ such that the following diagram commutes}
\end{array}
\]

Second, \(\text{Coker}(\varphi, G)\) is an \(\mathcal{R}\)-2-module.
There is a bifunctor

$$
\cdot : \mathcal{R} \times \text{Coker}(\varphi, G) \to \text{Coker}(\varphi, G)
$$

$$(r, X) \mapsto r \cdot X,$$

$$(r_1 \xrightarrow{\varphi} r_2, X \xrightarrow{[B, f]} Y) \mapsto r_1 \cdot X \xrightarrow{[r_1, B, \mathcal{F}]} r_2 \cdot Y,$$

where $r \cdot X$ and $r_1 \cdot B$ are the objects in $\mathcal{C}$ and $\mathcal{B}$ under the $\mathcal{R}$-2-module structures of them, respectively, $\mathcal{F}$ is the composition morphism $r_1 \cdot X \xrightarrow{\tau_1} r_1 \cdot (G(B) + Y) \simeq r_1 \cdot G(B) + r_1 \cdot Y \xrightarrow{1 + x \cdot Y} G(r_1 \cdot B) + r_2 \cdot Y$. This bifunctor is well-defined. In fact, if $[B, f] = [B', f'] : X \to Y$, i.e. there exist $A \in \text{obj}(A)$ and $a : B \to F(A) + B'$ such that the following diagram commutes

$$
\begin{array}{ccc}
X & \xrightarrow{f} & G(B) + Y \\
\downarrow{\gamma} & & \downarrow{G(\alpha) + 1} \\
G(B) + Y & \xrightarrow{G(F(A) + B') + Y} & G(F(A) + B') + Y \\
\downarrow{=} & & \downarrow{=} \\
0 + G(B) + Y & \xleftarrow{p_x + 1 + 1} & GF(A) + G(B') + Y
\end{array}
$$

Hence, there exist $r_1 \cdot A \in \text{obj}(A)$ and $r_1 \cdot a : r_1 \cdot B \to F(r_1 \cdot A) + r_1 \cdot B'$ such that the following diagram commutes

Then $x \cdot [B, f] = [r_1 \cdot B, \mathcal{F}] = [r_1 \cdot B', \mathcal{F}] = r_1 \cdot [B', f'] : r_1 \cdot X \to r_2 \cdot Y.$
Moreover, the natural isomorphisms in the definitions of $\mathcal{R}$-$2$-module and the universal property are given as general cokernel (more details see [5]).

**Remark 1.** Just like symmetric 2-group discussed in [1, 2, 3, 7], we can give the definitions of (relative-)2-exact, cohomology $\mathcal{R}$-2-modules in $(\mathcal{R}$-$2$-$\text{Mod})$.

A complex of $\mathcal{R}$-2-modules in $(\mathcal{R}$-$2$-$\text{Mod})$ is a sequence

$$\mathcal{A} = \cdots \xrightarrow{L_{n+1}} \mathcal{A}_n \xrightarrow{L_n} \mathcal{A}_{n-1} \xrightarrow{L_{n-1}} \mathcal{A}_{n-2} \xrightarrow{L_{n-2}} \cdots \xrightarrow{L_2} \mathcal{A}_1 \xrightarrow{L_1} \mathcal{A}_0$$

together with a family of 2-morphisms $\{\alpha_n : L_{n-1} \circ L_n \Rightarrow 0\}_{n \geq 2}$ in $(\mathcal{R}$-$2$-$\text{Mod})$ such that, for all $n$, the following diagram commutes

$$
\begin{array}{ccc}
L_{n+1}L_nL_{n+1} & \xrightarrow{a_nL_{n+1}} & 0L_{n+1} \\
\downarrow & & \downarrow \\
L_{n+1} & \xrightarrow{\text{cov}} & 0
\end{array}
$$

Consider part of the complex

$$
\begin{array}{ccc}
\mathcal{A}_{n+2} & \xrightarrow{L_{n+2}} & \mathcal{A}_{n+1} \\
\uparrow & \uparrow & \uparrow \\
\mathcal{A}_n & \xrightarrow{L_n} & \mathcal{A}_{n-1} \\
\downarrow & \downarrow & \downarrow \\
0 & \xrightarrow{a_{n+1}} & 0
\end{array}
$$

Based on the universal properties of relative kernel $\text{Ker}(L_n, \alpha_n)$, we have the following diagram

$$
\begin{array}{ccc}
\mathcal{A}_{n+2} & \xrightarrow{L_{n+2}} & \mathcal{A}_{n+1} \\
\uparrow & \uparrow & \uparrow \\
\mathcal{A}_n & \xrightarrow{L_n} & \mathcal{A}_{n-1} \\
\downarrow & \downarrow & \downarrow \\
0 & \xrightarrow{a_{n+1}} & 0
\end{array}
\quad
\begin{array}{ccc}
\mathcal{A}_{n+1} & \xrightarrow{L_n} & \mathcal{A}_{n-1} \\
\uparrow & \uparrow & \uparrow \\
\mathcal{A}_{n-2} & \xrightarrow{L_{n-1}} & \mathcal{A}_{n-3} \\
\downarrow & \downarrow & \downarrow \\
\text{Ker}(L_n, \alpha_n) & \xrightarrow{\text{cov}} & \text{Ker}(L_{n-1}, \alpha_{n-1})
\end{array}
$$

Similarly as the definition of (co)homology 2-group in [3, 7], the $n$th homology $\mathcal{R}$-2-module $\mathcal{H}_n(\mathcal{A})$ of the complex $\mathcal{A}$ defined as the relative cokernel $\text{Coker}(\tau_{n+2}, L'_{n+1})$. 
Note that, to get $\mathcal{H}_0(\mathcal{A})$ and $\mathcal{H}_1(\mathcal{A})$, we have to complete the complex $\mathcal{A}$ on the right with the two zero morphisms and two canonical 2-morphisms

\[ \cdots \to A_i \xrightarrow{L_i} A_{i-1} \to 0 \to 0, \text{ can: } 0 \circ L_i \Rightarrow 0, \text{ can: } 0 \circ 0 \Rightarrow 0. \]

The explicit description of $\mathcal{H}_n(\mathcal{A})$ can also be given from the existence of relative kernel and relative cokernel in $(R\text{-}2\text{-}\text{Mod})$ like the symmetric 2-group case in $[3,7]$.

A morphism $(F, \lambda) : \mathcal{A} \to \mathcal{B}$ of complexes in $(R\text{-}2\text{-}\text{Mod})$ is a picture in the following diagram such that the following diagram commutes

![Diagram]

where $F_n : A_n \to B_n$ is 1-morphism in $(R\text{-}2\text{-}\text{Mod})$, $\lambda_n : F_{n-1} \circ L_n \Rightarrow M_n \circ F_n$ is 2-morphism in $(R\text{-}2\text{-}\text{Mod})$, for each $n$, making the following diagram commutative

\[
\begin{array}{c}
F_{n-1} L_n L_{n+1} \xrightarrow{\lambda_n L_{n+1}} M_n F_n L_{n+1} \xrightarrow{M_n \lambda_{n+1}} M_n M_{n+1} F_{n+1} \\
F_{n-1} 0 \xrightarrow{\text{can}} 0 \xleftarrow{\text{can}} 0 F_{n+1}
\end{array}
\]

Such a morphism induces, for each $n$, a morphism of homology $R\text{-}2\text{-}\text{modules}$ $\mathcal{H}_n(F) : \mathcal{H}_n(\mathcal{A}) \to \mathcal{H}_n(\mathcal{B})$ from the universal properties of relative kernels and cokernels. It can be described as follows:(more details see $[3,7]$).

Given an object $(A_n \in \text{obj}(\mathcal{A}_n), a_n : L_n(A_n) \to 0)$ of $\mathcal{H}_n(\mathcal{A})$, we have $\mathcal{H}_n(F)(A_n, a_n) = (F_n(A_n) \in \text{obj}(B_n), b_n : M_n(F_n(A_n)) \to 0)$, where $b_n$ is the composition $M_n(F_n(A_n)) \xrightarrow{(\lambda_n)_{A_n}} F_{n-1} L_n(A_n) \xrightarrow{F_{n-1}(a_n)} F_{n-1}(0) \simeq 0$.

Given a morphism $[X_{n+1} \in \text{obj}(\mathcal{A}_{n+1}), x_{n+1} : A_n \to L_{n-1}(X_{n+1}) + A'_n]$ : $(A_n, a_n) \to (A'_n, a'_n)$ in $\mathcal{H}_n(\mathcal{A})$, we have $\mathcal{H}_n(F)[X_{n+1}, x_{n+1}] = [F_{n+1}(X_{n+1}) \in \cdots]$
Let \((\lambda, \mu) : (\mathcal{A}, L, \alpha) \to (\mathcal{B}, M, \beta)\) be two morphisms of complexes of \(\mathcal{R}\)-2-modules. If there is a family of 1-morphisms \(\{H_n : \mathcal{A}_n \to \mathcal{B}_{n+1}\}_{n \in \mathbb{Z}}\) and a family of 2-morphisms \(\{\tau_n : F_n \Rightarrow M_{n+1} \circ H_n + H_{n-1} \circ L_n + G_n : \mathcal{A}_n \to \mathcal{B}_n\}_{n \in \mathbb{Z}}\) satisfying the obvious compatible conditions, i.e. the following diagram commutes

\[
\begin{array}{ccc}
F_{n+1}L_n & \xrightarrow{\lambda_n} & M_nF_n \\
\downarrow & & \downarrow \mu_n \\
(M_nH_{n+1} + H_{n-2}L_{n+1} + G_{n+1})L_n & \xrightarrow{\text{can}} & M_n(M_{n+1}H_n + H_{n-1}L_n + G_n) \\
\downarrow & & \downarrow \text{can} \\
M_nH_{n+1}L_n + H_{n-2}L_{n+1}L_n + G_{n+1}L_n & \xrightarrow{1 + H_{n-2}L_n + 1} & M_nM_{n+1}H_n + M_nH_{n-1}L_n + M_nG_n \\
\downarrow & & \downarrow \text{can} \\
M_nH_{n+1}L_n + H_{n-2}(0) + G_{n+1}L_n & \xrightarrow{1 + H_{n-2}L_n + 1} & 0 + M_nH_{n-1}L_n + M_nG_n \\
\downarrow & & \downarrow \text{can} \\
M_nH_{n+1}L_n + 0 + G_{n+1}L_n & \xrightarrow{\text{can}} & M_nH_{n-1}L_n + G_{n+1}L_n \\
\end{array}
\]

We call the above morphisms \((F, \lambda), (G, \mu)\) are 2-chain homotopy in \((\mathcal{R}-2\text{-Mod})\).

Like the symmetric 2-group case, we have

**Proposition 1.** Let \((F, \lambda), (G, \mu) : (\mathcal{A}, L, \alpha) \to (\mathcal{B}, M, \beta)\) be two morphisms of complexes of \(\mathcal{R}\)-2-modules. If they are 2-chain homotopy, there is an equivalence \(\mathcal{H}_n(F) \simeq \mathcal{H}_n(G)\) between induced morphisms.
Lemma 1. Let $\mathcal{R}, \mathcal{S}$ be 2-rings, $(F, \lambda), (G, \mu) : (\mathcal{A}, L, \alpha) \to (\mathcal{B}, M, \beta)$ be two 2-chain homotopy morphisms of complexes of $\mathcal{R}$-2-modules and $T : (\mathcal{R}$-$2$-Mod)$\to (\mathcal{S}$-$2$-Mod)$ be a 2-functor. Then $T(F, \lambda)$ is 2-chain homotopic to $T(G, \mu)$ in $(\mathcal{S}$-$2$-Mod).

3 Projective Resolution and Derived 2-Functor in $(\mathcal{R}$-$2$-Mod)

In this section we will construct a projective resolution of any $\mathcal{R}$-2-module, define the left derived 2-functor and then give the basic property of this derived 2-functor.

Definition 4. Let $\mathcal{M}$ be an $\mathcal{R}$-2-module. A projective resolution of $\mathcal{M}$ in $(\mathcal{R}$-$2$-Mod) is a complex of $\mathcal{R}$-2-modules which is relative 2-exact in each point as in the following diagram

with $\mathcal{P}_n$ ($n \geq 0$) projective objects in $(\mathcal{R}$-$2$-Mod). i.e. the above complex is relative 2-exact in each $\mathcal{P}_i$ and $\mathcal{M}$.

Proposition 2. Every $\mathcal{R}$-2-module $\mathcal{M}$ has a projective resolution in $(\mathcal{R}$-$2$-Mod).

Sketch of proof. The construction of projective resolution of $\mathcal{M}$ is similar to symmetric 2-group case.

For $\mathcal{M}$, there is an essentially surjective morphism $F_0 : \mathcal{P}_0 \to \mathcal{M}$, with $\mathcal{P}_0$ projective object in $(\mathcal{R}$-$2$-Mod)([10]). Then we get a sequence as follows

\[
\begin{array}{c}
\mathcal{P}_0 \xrightarrow{F_0} \mathcal{M} \xrightarrow{0} 0 \\
\uparrow \text{can} \\
\mathcal{P}_0 \end{array}
\]

S.1.
where \( 0 : \mathcal{M} \to 0 \) is the zero morphism in \((\mathcal{R}\text{-}2\text{-Mod})\), 0 is the \( \mathcal{R}\text{-}2\text{-module} \) with only one object and one morphism, \( \text{can} \) is the canonical 2-morphism in \((\mathcal{R}\text{-}2\text{-Mod})\), which is given by the identity morphism of only one object of 0.

From the existence of the relative kernel in \((\mathcal{R}\text{-}2\text{-Mod})\), we have the relative kernel \((\text{Ker}(F_0, \text{can}), e_{(F_0, \text{can})}, \varepsilon_{(F_0, \text{can})})\) of the sequence S.1, which is in fact the general kernel \((\text{Ker}F_0, e_{F_0}, \varepsilon_{F_0})\) [5]. For \( \mathcal{R}\text{-}2\text{-module} \text{Ker}F_0 \), there exists an essentially surjective morphism \( G_1 : \mathcal{P}_1 \to \text{Ker}F_0 \), with \( \mathcal{P}_1 \) projective object in \((\mathcal{R}\text{-}2\text{-Mod})[6, 10])\). Let \( F_1 = e_{F_0} \circ G_1 : \mathcal{P}_1 \to \mathcal{P}_0 \). Then we get the following sequence

\[
\begin{array}{cccccc}
\mathcal{P} & \xrightarrow{\alpha} & \mathcal{P}_0 & \xrightarrow{\varepsilon} & \mathcal{M} & \xrightarrow{\circ} & 0 \\
\downarrow{\alpha_1} & & \downarrow{\varepsilon_1} & & \downarrow{\circ_0} & & \\
\text{Ker}F_0 & & & & \\
\end{array}
\]

where \( \alpha_1 \) is the composition \( F_0 \circ F_1 = F_0 \circ e_{F_0} \circ G_1 \Rightarrow 0 \circ G_1 \Rightarrow 0 \) and compatible with \( \text{can} \).

Consider the above sequence, there exists the relative kernel \((\text{Ker}(F_1, \alpha_1), e_{(F_1, \alpha_1)}, \varepsilon_{(F_1, \alpha_1)})\) in \((2\text{-SGp})\). For the \( \mathcal{R}\text{-}2\text{-module} \text{Ker}(F_1, \alpha_1) \), there is an essentially surjective morphism \( G_2 : \mathcal{P}_2 \to \text{Ker}(F_1, \alpha_1) \), with \( \mathcal{P}_2 \) projective object in \((\mathcal{R}\text{-}2\text{-Mod})[6, 10])\). Let \( F_2 = e_{(F_1, \alpha_1)} \circ G_2 : \mathcal{P}_2 \to \mathcal{P}_1 \). Then we get a sequence

\[
\begin{array}{cccccc}
\mathcal{Ker}(F_1, \alpha_1) & \xrightarrow{\alpha_2} & \mathcal{P}_2 & \xrightarrow{\varepsilon_{(F_2, \alpha_2)}} & \mathcal{P}_1 & \xrightarrow{\varepsilon_1} & \mathcal{M} & \xrightarrow{\circ} & 0 \\
\downarrow{\alpha_3} & \downarrow{\varepsilon_{(F_2, \alpha_2)}} & \downarrow{\varepsilon_1} & & \downarrow{\circ_0} & & \\
\mathcal{P}_2 & \xrightarrow{\alpha} & \mathcal{P}_1 & & & & \\
\end{array}
\]

where \( \alpha_2 \) is the composition \( F_1 \circ F_2 = F_1 \circ e_{(F_1, \alpha_1)} \circ G_2 \Rightarrow 0 \circ G_2 \Rightarrow 0 \) and compatible with \( \alpha_1 \).

Using the same method, we get a complex of \( \mathcal{R}\text{-}2\text{-modules} \)
The proof of relative 2-exactness of the sequence is the same as symmetric 2-group case.

**Theorem 1.** Let \((F : \mathcal{P} \to \mathcal{M}, \alpha)\) be a projective resolution of \(\mathcal{R}\)-2-module \(\mathcal{M}\), and \(H : \mathcal{M} \to \mathcal{N}\) a morphism in \((\mathcal{R}\text{-2-Mod})\). Then for any projective resolution \((G : \mathcal{Q} \to \mathcal{N}, \beta)\), there is a morphism \(H : \mathcal{P} \to \mathcal{Q}\) of complexes in \((\mathcal{R}\text{-2-Mod})\) together with the family of 2-morphisms \(\{\varepsilon_n : G_n \circ H_n \Rightarrow H_{n-1} \circ F_n\}_{n \geq 0}\) as in the following diagram:

![Diagram](image)

If there is another morphism between projective resolutions, they are 2-chain homotopy.

The proof of this Theorem is also similar to symmetric 2-group case in [7]. The difference is that the existence of 1-morphisms and 2-morphisms is from the properties of projective \(\mathcal{R}\)-2-modules in \((\mathcal{R}\text{-2-Mod})\).

**Definition 5.** Let \(\mathcal{R}, \mathcal{S}\) be 2-rings. An additive 2-functor \([2] T : (\mathcal{R}\text{-2-Mod}) \to (\mathcal{S}\text{-2-Mod})\) is called right relative 2-exact if the relative 2-exactness of

![Diagram](image)
in \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{C} \) implies relative 2-exactness of

\[
\begin{array}{c c c}
T(\mathcal{A}) & T(\mathcal{B}) & T(\mathcal{C}) \\
\downarrow T(P) & 0 \quad \text{and} \quad T(Q) & 0 \\
& \downarrow T(Q) & 0
\end{array}
\]

in \( T(\mathcal{B}) \) and \( T(\mathcal{C}) \).

The left relative 2-exact 2-functor can be defined dually.

By Remark 2 and Proposition 1, Theorem 1, there is

\textbf{Corollary 1.} Let \( \mathcal{R}, \mathcal{S} \) be 2-rings, and \( T: (\mathcal{R}-2\text{-Mod}) \to (\mathcal{S}-2\text{-Mod}) \) be an additive 2-functor, and \( \mathcal{A} \) be any object of \( (\mathcal{R}-2\text{-Mod}) \). For two projective resolutions \( P, \ Q \) of \( \mathcal{A} \), there is an equivalence between homology 2-modules \( H_i(T(P)) \) and \( H_i(T(Q)) \).

Let \( T: (\mathcal{R}-2\text{-Mod}) \to (\mathcal{S}-2\text{-Mod}) \) be an additive 2-functor. There is a 2-functor

\[
\begin{align*}
\mathcal{L}_iT : (\mathcal{R}-2\text{-Mod}) & \to (\mathcal{S}-2\text{-Mod}) \\
\mathcal{A} & \mapsto \mathcal{L}_iT(\mathcal{A}), \\
\mathcal{A} \xrightarrow{F} \mathcal{B} & \mapsto \mathcal{L}_iT(\mathcal{A}) \xrightarrow{\mathcal{L}_iT(F)} \mathcal{L}_iT(\mathcal{B}),
\end{align*}
\]

where \( \mathcal{L}_iT(\mathcal{A}) \) is defined by \( H_i(T(P)) \), and \( P \) is the projective resolution of \( \mathcal{A} \). \( \mathcal{L}_iT \) is a well-defined 2-functor from the properties of additive 2-functor and Corollary 1.

\textbf{Corollary 2.} Let \( T: (\mathcal{R}-2\text{-Mod}) \to (\mathcal{S}-2\text{-Mod}) \) be a right relative 2-exact 2-functor, and \( \mathcal{A} \) be a projective object in \( (\mathcal{R}-2\text{-Mod}) \). Then \( \mathcal{L}_iT(\mathcal{A}) = 0 \) for \( i \neq 0 \).

The following is a basic property of derived functors.

\textbf{Theorem 2.} Let \( T: (\mathcal{R}-2\text{-Mod}) \to (\mathcal{S}-2\text{-Mod}) \) be a right relative 2-exact 2-functor.

Then the left derived 2-functor \( \mathcal{L}_sT \) takes the sequence of \( \mathcal{R}-2\text{-modules} \)

\[
\begin{array}{c c c}
\mathcal{A} & \xrightarrow{0} & \mathcal{C} \\
\downarrow F & \uparrow 0 & \downarrow G \\
\mathcal{B} & \xrightarrow{0} & \mathcal{C}
\end{array}
\]

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which is relative 2-exact in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ to a long sequence 2-exact(similar [1, ?]) in each point

\[ \cdots \rightarrow \mathcal{L}_n^{T_i}(\mathcal{A}) \xrightarrow{\mathcal{L}_n^{T_i}(\varphi)} \mathcal{L}_n^{T_i}(\mathcal{B}) \rightarrow \mathcal{L}_n^{T_i}(\mathcal{C}) \rightarrow \mathcal{L}_{n+1}^{T_i}(\mathcal{A}) \xrightarrow{\mathcal{L}_{n+1}^{T_i}(\varphi)} \mathcal{L}_{n+1}^{T_i}(\mathcal{B}) \rightarrow \cdots \]

In order to prove this theorem, we need the following Lemmas.

Similar to the proofs of symmetric 2-group case. We have

**Lemma 2.** Let $\mathcal{P}$ and $\mathcal{Q}$ be projective objects in $(\mathcal{R}^{-2}\text{-Mod})$. Then the product category $\mathcal{P} \times \mathcal{Q}$ is a projective object in $(\mathcal{R}^{-2}\text{-Mod})$.

**Lemma 3.** Let $(F, \varphi, G) : \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ be an extension of $\mathcal{R}^{-2}$-modules in $(\mathcal{R}^{-2}\text{-Mod})$(similar to [1, 2]), $(\mathcal{P}, L, \alpha)$ $(\mathcal{Q}, N, \beta)$ be projective resolutions of $\mathcal{A}$ and $\mathcal{C}$, respectively. Then there is a projective resolution $(\mathcal{K}, M, \varphi)$ of $\mathcal{B}$, such that $\mathcal{P}. \rightarrow \mathcal{K}. \rightarrow \mathcal{Q}.$ forms an extension of complexes in $(\mathcal{R}^{-2}\text{-Mod})$.

By the universal property of (bi)product of $\mathcal{R}^{-2}$-modules and the property of additive 2-functor([2, 3]). We get

**Lemma 4.** Let $T$: $(\mathcal{R}^{-2}\text{-Mod}) \rightarrow (\mathcal{S}^{-2}\text{-Mod})$ be an additive 2-functor, and $\mathcal{A}$, $\mathcal{B}$ be objects in $(\mathcal{R}^{-2}\text{-Mod})$. Then there is an equivalence between $T(\mathcal{A} \times \mathcal{B})$ and $T(\mathcal{A}) \times T(\mathcal{B})$ in $(\mathcal{S}^{-2}\text{-Mod})$.

Proof of Theorem 2. For $\mathcal{R}^{-2}$-modules $\mathcal{A}$ and $\mathcal{C}$, choose projective resolutions $\mathcal{P}. \rightarrow \mathcal{A}$ and $\mathcal{Q}. \rightarrow \mathcal{C}$. By Lemma 2 and Lemma 3, there is a projective resolution $\mathcal{P}. \times \mathcal{Q}. \rightarrow \mathcal{B}$ fitting into an extension $\mathcal{P}. \rightarrow \mathcal{P}. \times \mathcal{Q}. \xrightarrow{p} \mathcal{Q}$. of projective complexes in $(2\text{-SGp})([1])$. By Lemma 4, we obtain a complexes of extension

\[ T(\mathcal{P}.) \xrightarrow{T(i)} T(\mathcal{P}. \times \mathcal{Q}.) \xrightarrow{T(p)} T(\mathcal{Q}.). \]

Similar to Theorem 4.2 in [3], the long sequence
is 2-exact in each point.

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**References**

[1] D. Bourn, E.M. Vitale, Extensions of symmetric c at-groups, Homol. Homotopy Appl. 4 (2002) 103C162

[2] M. Dupont. Abelian categories in dimension 2, PhD.Thesis. arxiv:hep-th/0809.1760v1.

[3] A.del Río, J. Martínez-Moreno, and E. M. Vitale, Chain complexes of symmetric categorical groups, J. Pure Appl. Algebra, 196 (2005).

[4] I.P.Freyd. Abelian categories[M], New York: Harper&Row, 1964.

[5] F.Huang, S.H,Chen, W.Chen, Z.J.Zheng, 2-Modules and the Representation of 2-Rings. arxiv:hep-th/1005.2831 18 May 2010

[6] F.Huang, S.H,Chen, W.Chen, Z.J.Zheng, Higher Dimensional Homology Algebra II:Projectivity. arXiv:1006.4677

[7] F.Huang, S.H,Chen, W.Chen, Z.J.Zheng. Higher Dimensional Homology Algebra III:Projective Resolutions and Derived 2-Functors in (2-S Gp) arXiv:1006.4677
[8] M. Jibladze and T. Pirashvili, Third Mac Lane cohomology via categorical rings [J]. J. Homotopy Relat. Struct., 2 (2007), pp.187–216.

[9] H.-J. Baues and M. Jibladze. Secondary derived functors and the Admas spectral sequence. Topology. 45(2006) 295-324.

[10] T. Pirashvili. Projective and injective objects in symmetric categorical groups. arxiv:hep-th/1007.0121v1.

[11] T. Pirashvili. On abelian 2-categories and derived 2-functors. arxiv:hep-th/1007.0121v1.

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