Order separability.

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Abstract

This paper is devoted to the investigation of the property of order separability for free products of groups.

Key words: free products, residual properties.

MSC: 20E26, 20E06.

1 Introduction.

Definition. A group \( G \) is called order separable if for each elements \( u \) and \( v \) of \( G \) such that \( u \) is conjugate to neither \( v \) nor \( v^{-1} \) there exists a homomorphism \( \varphi \) of \( G \) onto a finite group such that the orders of \( \varphi(u) \) and \( \varphi(v) \) are different.

In [1] it was proved that free groups are order separable. In this work we prove that this property is inherited by free products:

**Theorem 1.** The group \( G = A \ast B \) is order separable if and only if \( A \) and \( B \) are order separable.

Note that the property of order separability for free groups was generalized in [2] where it was proved that free groups are actually omnipotent.

2 Notations and definitions.

Investigate the graph \( \Gamma \) satisfying the following properties:

1) \( \Gamma \) is an oriented graph whose positively oriented edges are labelled by elements of groups \( A \) and \( B \) so that for each vertex \( p \) of \( \Gamma \) and for each \( a \in A \) and \( b \in B \) there exist exactly one edge with label \( a \) and exactly one edge with label \( b \) ending at \( p \) and there exist exactly one edge with label \( a \) and exactly one edge with label \( b \) starting at \( p \);

2) for each vertex \( p \) of \( \Gamma \) we define the subgraph \( A(p) \) of the graph \( \Gamma \) as the maximal connected graph which contains \( p \) and whose positively oriented edges are labelled by the elements of \( A \); it is required that \( A(p) \) is the Cayley graph of \( A \) with the set of generators \( \{A\} \). The graph \( B(p) \) is defined analogically.

We shall use the following notations. The symbols \( \text{Lab}(e), \alpha(f), \omega(f), \alpha(S), \omega(S) \) will denote correspondingly the label of the positively oriented edge \( e \), the beginning and the end of the edge \( f \) and the beginning and the end of the path \( S \). Having a path \( S = e_1...e_k \) we define its label \( \text{Lab}(S) = \text{Lab}(e_1)... \text{Lab}(e_n) \).

**Definition 1.** Consider the graph \( \Gamma \) satisfying the properties 1), 2) and the cyclically reduced element \( u \in A \ast B \). The closed path \( S = e_1...e_n \) is called \( u \)-cycle if \( \text{Lab}(e_{i+l+1}...e_{i+l+k}) = u \) where \( l \) is the length of the element \( u \), \( k \) is an arbitrary natural number and subscripts are modulo \( n \).

If a label of the \( u \)-cycle \( S \) is \( u^k \) then we shall say that the length of the \( u \)-cycle \( S \) equals \( k \).
The group \( A \ast B \) acts on the right on the set of vertices of the graph \( \Gamma \) by the following way. Consider the vertex \( p \) of \( \Gamma \) and the elements \( c \in (A \cup B) \setminus \{1\} \). Then according to the property 1) there exist the edge \( u \) with label \( c \) starting at \( p \) and the edge \( v \) ending at \( p \) and the labels of \( u \) and \( v \) coincide with \( c \). Then we put \( p \circ c = \omega(u), p \circ c^{-1} = \alpha(v) \).

**Definition 2.** We say that the cycle \( S = e_1...e_n \) of a graph \( \Gamma \) with properties 1), 2) does not have near edges if there are no distinct edges of \( S \) belonging to one subgraph \( A(p) \) or \( B(p) \) for some \( p \).

**Definition 3.** A group \( G \) is called subgroup separable if each finitely generated subgroup of \( G \) coincides with the intersection of finite index subgroups of \( G \).

In [3] the following theorem was proved.

**Theorem 2.** The class of subgroup separable groups is closed with respect to the operation of the free product of groups.

**Corollary.** The free product of finite groups is cyclic subgroup separable.

### 3 Proof of theorem 1.

If \( A \ast B \) is order separable then it is obvious that \( A \) and \( B \) are order separable. Consider order separable groups \( A \) and \( B \) and prove that \( A \ast B \) is order separable. Put \( G = A \ast B \). Consider cyclically reduced elements \( u \) and \( v \) of \( G \) such that \( u \) is not conjugate to \( v^{\pm 1} \). If \( u \) and \( v \) belong to free factors then we use the natural homomorphism of \( G \) onto \( A \) or \( B \) and use the order separability of free factors.

Suppose that \( u \notin A \cup B \). Consider the case when \( u \) and \( v \) belong to the Cartesian subgroup \( C = \{[a,b]|a \in A, b \in B\} \) and do not equal to unit. Consider that for each homomorphism of \( G \) onto a finite group the images of \( u \) and \( v \) have equal orders. It is possible to consider that the normal forms for \( u \) and \( v \) have the following presentations: \( u = a_1b_1...a_nb_n, v = a'_1b'_1...a'_mb'_m \). Since order separability involves residual finiteness we may deduce that there exists a homomorphism of \( G \) onto a group \( A_1 \ast B_1 \) such that \( A_1 \) is the image of \( A \) and \( B_1 \) is the image of \( B \) besides \( a_i, b_i, a'_j, b'_j \) have nonunit images and each element presented as \( a_1a_1b_1b_1, a'_1a'_1b'_1b'_1 \) which differs from unit has a nonunit image too, \( i,k = 1,...,n, j,l = 1,...,m \). Thereby we may consider that the groups \( A \) and \( B \) are finite. For each number \( n = 0, 1, 2,... \) construct the graph \( \Gamma_n \) with properties 1), 2) which satisfies also the following properties:

1) the length of each \( u \)-cycle divides the length of a maximal \( u \)-cycle; the same is true for \( v \)-cycles;

4) in \( \Gamma_n \) (when \( n > 0 \)) there exists the path \( R_n \) of length \( n \) which is contained in a maximal \( u \)-cycle and in all maximal \( v \)-cycles;

5) all \( u \)- and \( v \)-cycle of \( \Gamma_n \) have no near edges;

6) the length of a maximal \( u \)-cycle coincides with the length of a maximal \( v \)-cycle.

The construction of \( \Gamma_0 \). Due to the corollary there exists the homomorphism \( \varphi \) of \( G \) onto a finite group such that the elements \( ax \) and \( bx \) which are not conjugate to elements from \( \langle u \rangle \) and \( \langle v \rangle \) have nonunit images where \( x \) and \( z \) are
the subwords of words $u^k, v^k$, $k = 0, 1, 2, \ldots, a \in A, b \in B$. We may also consider that $u$ and $v$ do not belong to the kernel of $\varphi$. Then we may take the Cayley graph of $\varphi(G)$ with the generating set $\varphi(A \cup B)$ in the capacity of $\Gamma_0$ (labels $\varphi(a), \varphi(b)$ are identified with $a$ and $b$ correspondingly). Conditions 1), 2) and 3) are held because of the definition of the Cayley graph; conditions 4), 5) are held due to the properties of the homomorphism; the property 6) is true by the supposition about the orders of images of $u$ and $v$.

The construction of $\Gamma_{n+1}$ from $\Gamma_n$. Let $t$ be the length of the maximal $u$-cycle in $\Gamma_n$. Consider $t$ copies of $\Gamma_n : \Delta_1, \ldots, \Delta_t$. Put $q_k = \omega(R_{n,k})$ where $R_{n,k}$ is the path in $\Delta_k$ corresponding to the path $R_n$ of $\Gamma_n$. $p_k$ is the vertex following after $q_k$ on the maximal $u$-cycle passing through $R_{n,k}$ (it is supposed that $p_k$ does not belong to $R_{n,k}$ and vertices $p_k$ in graphs $\Delta_k$ and chosen maximal $u$-cycles passing through $R_{n,k}$ correspond to each other). If $n = 0$ then $q_k$ is an arbitrary vertex and the edge $(q_k, p_k)$ belongs to a $u$-cycle. Otherwise $(q_k, p_k)$ is the edge connecting $p_k$ and $q_k$ and belonging to the chosen maximal $u$-cycle. Consider that $\text{Lab}(q_k, p_k) \in A$. In order to construct the graph $K_{n,1}$ from $\Delta_1, \ldots, \Delta_t$ we delete all edges from $A(q_k)$ which are incident to $q_k$. Let $s$ be an arbitrary vertex of the subgraph $A(q)$ of the graph $\Gamma_n$ which differs from $q$ and $s$ is connected with $q$ be the edge $e \in A(q)$. The vertex $s_k \in \Delta_k$ corresponds to the vertex $s$. Connect the vertex $q_k$ by the edge with the vertex $s_{k+1}$ (if $k = t$ we consider that $k + 1 = 1$). The label of this new edge $f_k$ equals $\text{Lab}(e)$. Also if $\alpha(e) = q$ in $\Gamma_n$ then $\alpha(f_k) = q_k$; if $\omega(e) = q$ in $\Gamma_n$ then $\omega(f_k) = q_k$. Put $S_n = R_{n,1} \cup d_1$ where $d_1$ is the edge which is appended instead of the edge $(q_1, p_1)$. The graph $K_{n,1}$ satisfies properties 1), 2). The property 3) is fulfilled since the lengths of each $u$- or $v$-cycle either does not change or becomes $t$ times greater than it was. The condition 5) is true because otherwise it is not held for $\Gamma_n$. If in the graph $K_{n,1} \Delta_n$ all maximal $v$-cycles pass through $S_n$ then we put $\Gamma_{n+1} = K_{n,1}, R_{n+1} = S_n$. Otherwise consider $t^2$ copies of the graph $K_{n,1} : \Omega_1, \ldots, \Omega_{t^2}$. Let $r_k$ be the vertex next to $\omega(S_{n,k})$ on the maximal $u$-cycle passing through $S_{n,k}$ where $S_{n,k}$ is a path of $\Omega_k$ corresponding to $S_n$ in the graph $K_{n,1}, r_k$ does not belong to $S_{n,k})$. Vertices $r_k$ and maximal $u$-cycles passing through $S_{n,k}$ correspond to each other in $\Omega_k$. Construct the graph $K_{n,2}$ from $\Omega_1, \ldots, \Omega_{t^2}$ the same way as the graph $K_{n,1}$ is constructed from $\Delta_1, \ldots, \Delta_t$ but we consider $\omega(S_{n,k}), r_k, B(\omega(S_{n,k}))$ instead of vertices $q_k, p_k$ and the subgraph $A(q_k)$ correspondingly. The graph $K_{n,2}$ satisfies the properties 1), 2), 3), 5). This is established in similar way as for the graph $K_{n,1}$. The path $S_{n,1}$ is contained in a maximal $u$-cycle and in all maximal $v$-cycles of the graph $K_{n,2}$ because of the property 5) and 6) for $K_{n,2}$. Thus $\Gamma_{n+1} = K_{n,2}, R_{n+1} = S_{n,1}$.

Since $n$ is an arbitrary natural number then conjugating $u$ and $v$ we may consider that $u = w^k, v = w^l$. It is possible to consider that $(k, l) = 1$, that is $k$ and $l$ are coprime. Hence $w \in C$. Suppose that $|k| > 1$. Then there exists a prime number $p$ such that $p \mid k, p \nmid l$. There exists a homomorphism $\psi$ of $C$ onto a finite $p$-group $P$ such that $w$ has a nonunit image $[4]$. Put $N = \ker \psi$. Then the group $N' = \cap_{g \in G} g^{-1}Ng$ is a finite index normal divisor of $G$. Besides in the quotient-group $G/N'$ the image of $w$ has the order which equals the nonzero power of $p$. Denote by $\psi_1$ the natural homomorphism of $G$ onto a finite group
Because of the conditions on the order of $\psi_1(w)$ we conclude that the order of elements $\psi_1(u), \psi_1(v)$ are different. So $|k| = |l| = 1$ and this involves the violation.

Consider now the case when $u$ and $v$ belong to $A * B \setminus (C \cup A \cup B)$. It was shown in [5] that this condition involves that $u$ and $v$ have infinite orders. Since the groups $A$ and $B$ are finite there exists the natural number $q$ such that $u^q$ and $v^q$ belong to $C$. Besides since $u$ and $v$ are cyclically reduced and $u$ and $v$ do not belong to $A * V \setminus (C \cup A \cup B)$ and due to the conjugacy theorem for free products [5] we deduce that $u^n$ is not conjugate to $v^{kn}$. If the orders of images of $u^q$ and $v^q$ are different after some homomorphism then the orders of images of $u$ and $v$ are also different after the same homomorphism. The case when $u \notin A * B \setminus \{ \cup g \in G \{ g^{-1}(A \cup B)g \}$ and $v \in h^{-1}(A \cup B)h, h \in A * B$ can be solved with the usage of the residual finiteness of $G$.

Theorem 1 is proved.

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References.

1. Klyachko A. A. Equations over groups, quasivarieties, and a residual property of a free group // J. Group Theory. 1999. 2. 319–327.

2. Wise, Daniel T. Subgroup separability of graphs of free groups with cyclic edge groups. Q. J. Math. 51, No.1, 107-129 (2000). [ISSN 0033-5606; ISSN 1464-3847]

3. Romanovskii N. S. On the residual finiteness of free products with respect to membership. // Izv. AN SSSR. Ser. matem., 1969, 33, 1324-1329.

4. Kargapolov, M. I., Merzlyakov, Yu. I. (1977). Foundations of group theory. Nauka.

5. Lyndon, R. C., Schupp, P. E. (1977). Combinatorial group theory. Springer-Verlag.