FINITE DIFFERENCE SCHEME FOR 2D PARABOLIC PROBLEM MODELLING ELECTROSTATIC MICRO-ELECTROMECHANICAL SYSTEMS

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Abstract. This paper is dedicated to study the fully discretized semi implicit and implicit schemes of a 2D parabolic semi linear problem modeling MEMS devices. Starting with the analysis of the semi-implicit scheme, we proved the existence of the discrete solution which converges under certain conditions on the voltage \( \lambda \). On the other hand, we consider a fully implicit scheme, we proved the existence of the discrete solution, which also converges to the stationary solution under certain conditions on the voltage \( \lambda \) and on the time step. Finally, we did some numerical simulations which show the behavior of the solution.

1. Introduction

Micro-Electromechanical Systems (MEMS) combine electronics with micro-size mechanical devices to design various types of microscopic machinery (see [4]). From a very early vision in the early 1950’s, MEMS has gradually made its way out of research laboratories and into everyday products. In the mid-1990’s, MEMS components began appearing in numerous commercial products and applications including accelerometers used to control airbag deployment in vehicles, pressure sensors for medical applications, and inkjet printer heads. Today, MEMS devices are also found in projection displays and for micropositioners in data storage systems. However, the greatest potential for MEMS devices lies in new applications within telecommunications (optical and wireless), biomedical and process control areas. An overview of the developing technology of MEMS devices is given in [14] and the references within.

We consider a simple idealized electrostatic MEMS device (see [4]), which consists of a thin and deformable elastic dielectric membrane with a small finite thickness, coated with a negligibly thin metallic conducting film. Following the electrostatic analysis (see [4] and [14]), we obtain the parabolic problem modeling dynamical
case, given by
\[ \frac{\partial u}{\partial t} - \Delta u = \frac{\lambda f(x)}{(1-u)^2} \text{ in } \Omega, \]
(1.1)
\[ u(t,x) = 0 \quad \text{on } \partial \Omega, \]
\[ u(0,x) = 0 \quad \text{in } \Omega. \]

The steady state of this problem is the solution of the following semi-linear elliptic problem:
\[ -\Delta u = \frac{\lambda f(x)}{(1-u)^2} \text{ in } \Omega, \]
(1.2)
\[ u(x) = 0 \quad \text{on } \partial \Omega, \]
\[ 0 \leq u < 1 \quad \text{in } \Omega. \]

The initial condition in (1.1), demonstrates that initially the elastic membrane was at rest, \( u = 1 - d \) corresponds to the deflection, where \( d \) represents the undeflected gap size, \( \lambda \) is the relative strength of the electrostatic and mechanical forces in the system and is given in terms of the applied voltage \( \lambda = \frac{\epsilon_0 V^2 L^2}{2T L d^3} \), where \( L \) is the length scale of the membrane, \( T \) is the tension of the membrane, \( \epsilon_0 \) is the permittivity of the free space in the gap between the membrane and the bottom plate, and \( f = \frac{\epsilon_0}{\epsilon_2(L,x)} \) represents the varying dielectric permittivity of the elastic membrane(see [4]).

One of the main goals of studying (1.2) is to achieve the maximum possible stable deflection before touchdown occurs, which is referred to as the pull-in distance (see [10], [7]), and one way of achieving larger values of \( \lambda^* \), while simultaneously increasing the pull-in distance, consists of introducing a spatially varying dielectric permittivity \( \epsilon_2(x) \) of the membrane(see [13]), moreover, \( 0 \leq f(x) < 1 \) for all \( x \in \Omega \) (see [4]). Furthermore, a result concerning an estimate on the values of \( \lambda \) for which touchdown occurs was derived in [6], such that there exists some \( \lambda^* \) such that if \( \lambda > \lambda^* \) touchdown occurs.

The pull in voltage is defined as:
\[ \lambda^*(\Omega, f) = \sup\{ \lambda > 0 | (1.2) \text{ possesses at least one solution} \}. \]

A solution \( u(t,x) \) of (1.1) is said to touchdown, i.e. quenching, at finite (infinite) time \( T = T(\lambda, \Omega, f) \), if the maximum value of \( u \) reaches 1 at the time \( T < \infty \) (\( T = \infty \)). More precisely quenching time \( T^* \) is defined by \( \sup\{ t > 0 | \|u(s,\cdot)\|_\infty < 1, \forall s \in [0,t] \} \) (18).

Some proposed modifications on the device, such as tailoring the dielectric properties of device components to achieve a greater stability was established in [13]. Besides, the problems above are studied as well in ([2], [6], [7], [8], [9], [15], [16]) and the references within them.

Indeed, the 1D numerical analysis of problem (1.1) was studied in ([3]), they proved that the solutions for both semi-implicit and implicit semi-discrete schemes are monotonically and pointwise convergent to the minimal solution of the corresponding elliptic partial differential equation under proper assumptions. Besides,
they studied the fully discretized semi-implicit scheme, they proved that the solution of the stationary problem exists and that the discrete solution converges to this steady state. Some numerical simulations were also made, by using the continuation method adopted in [1], they found an upper bound on $\lambda$ namely the pull in voltage $\lambda^*$ and showed graphically the touch down phenomenon for some $\lambda > \lambda^*$.

Consequently, our aim is to extend the result that has been done in [3], we considered a 2D semi-implicit fully discretized scheme. We prove that the steady state of (1.1) exists if $0 \leq \lambda \leq \frac{2\pi^2(1 - \delta)^2}{(b - a)^2}$, for some $\delta \in (0, 1)$, moreover the discrete solution of this problem converges to this steady state if $\lambda < \frac{(1 - \|U^*\|_\infty)^3}{(b - a)^2}$, where $U^*$ denotes the steady state. As well as the study of the semi-implicit scheme, we consider the implicit scheme, we adopted newton method (see [5]), where the jacobian matrix at the steady state is invertible if the time step $\tau < \frac{(b - a)^2}{2\pi^2\delta^2(1 - \delta)}$.

Finally, we give numerical simulations which show the behavior of the solutions, errors, and the touch down phenomenon as well.

2. Semi implicit fully discretized 2D scheme

Let $M > 0$ be an integer such that $h = \frac{b - a}{M + 1}$ is the uniform step on $x$ and $y$ axes ($a = x_0 < \cdots < x_i < \cdots < x_{M+1} = b$) and ($a = y_0 < \cdots < y_i < \cdots < y_{M+1} = b$) where $x_{i+1} = x_i + h$ and $y_{j+1} = y_j + h$. Furthermore, let $\tau$ be the time step such that $t_{n+1} = t_n + \tau$.

For $n \in \mathbb{N} \cup \{0\}$, the fully discretized semi implicit 2D scheme is written as follows:

$$
\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\tau} - \frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{h^2} - \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{h^2} = \frac{\lambda f(x_i, y_j)}{(1 - u_{i,j}^n)^2},
$$

$$
i = 1, \ldots, M \quad \text{and} \quad j = 1, \ldots, M;
$$

$$
u_{i,0}^{n+1} = v_{i,M+1}^{n+1} = v_{0,j}^{n+1} = u_{i+1,0}^{n+1} = 0, \ i, j = 0, \ldots, M + 1,
$$

where $U_{i,j}^n \approx u(t_n, x_i, y_j)$.

This problem can be written in the form:

$$
u_{i,j}^{n+1} - \frac{\tau}{h^2} (u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}) - \frac{\tau}{h^2} (u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}) = \frac{\tau \lambda f(x_i, y_j)}{(1 - u_{i,j}^n)^2}.
$$

Which is equivalent to :

(2.1)

$$
u_{i,j}^{n+1} \left(1 + \frac{4\tau}{h^2}\right) - \frac{\tau}{h^2} u_{i+1,j}^{n+1} - \frac{\tau}{h^2} u_{i-1,j}^{n+1} - \frac{\tau}{h^2} u_{i,j+1}^{n+1} - \frac{\tau}{h^2} u_{i,j-1}^{n+1} = u_{i,j}^n + \frac{\tau \lambda f(x_i, y_j)}{(1 - u_{i,j}^n)^2}.
$$

We can write (2.1) in the vector form as:

$$AU^{n+1} = G(U^n).$$

The initial value of $U$, $U^0$, is null, and

$$U^{n+1} = (u_{1,1}^{n+1}, u_{2,1}^{n+1}, \ldots, u_{M,1}^{n+1}, u_{1,2}^{n+1}, \ldots, u_{M,2}^{n+1}, \ldots, u_{1,M}^{n+1}, \ldots, u_{M,M}^{n+1})^t.$$

\[ A = \begin{pmatrix} B & C & 0 & \cdots & \cdots & 0 \\ C & B & C & 0 & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & 0 & C & B & C & 0 \\ 0 & \cdots & 0 & C & B & C \\ 0 & \cdots & \cdots & 0 & C & B \end{pmatrix}^{M^2 \times M^2} , \]

\[ B \text{ is a tridiagonal matrix of the form:} \]
\[ B = \begin{pmatrix} 1 + \frac{4\tau}{h^2} & -\frac{\tau}{h^2} & 0 & \cdots & \cdots & \cdots \\ -\frac{\tau}{h^2} & 1 + \frac{4\tau}{h^2} & -\frac{\tau}{h^2} & \cdots & \cdots & \cdots \\ 0 & -\frac{\tau}{h^2} & 1 + \frac{4\tau}{h^2} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \\ 0 & \cdots & 0 & -\frac{\tau}{h^2} & 1 + \frac{4\tau}{h^2} & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & -\frac{\tau}{h^2} \end{pmatrix}^{M \times M} , \]
\[ C = -\frac{\tau}{h^2} I^{M \times M} \]

and \[ G(U^n) = \begin{pmatrix} u_{1,1}^n + \frac{\lambda f(x_1, y_1)}{(1 - u_{1,1}^n)^2} \\ \vdots \\ u_{M,1}^n + \frac{\lambda f(x_M, y_1)}{(1 - u_{M,1}^n)^2} \\ \vdots \\ u_{1,M}^n + \frac{\lambda f(x_1, y_M)}{(1 - u_{1,M}^n)^2} \\ \vdots \\ u_{M,M}^n + \frac{\lambda f(x_M, y_M)}{(1 - u_{M,M}^n)^2} \end{pmatrix}^{M^2 \times 1} . \]

**Lemma 2.1.** The matrix \( A \) is positive definite.

**Proof.** Let \( x \in \mathbb{R}^{M^2} \), with \( x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{pmatrix}^{M^2} \), such that \( x^i = \begin{pmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_M^i \end{pmatrix}^M \), \( i = 1, \cdots, M \).

Proceeding with some calculations, it yields
\[
(Ax, x) = \left( 1 + \frac{4\tau}{h^2} \right) \sum_{i=1}^M \sum_{j=1}^M (x_j^i)^2 - \frac{2\tau}{h^2} \sum_{i=1}^M \sum_{j=1}^{M-1} x_j^i x_{j+1}^i - \frac{2\tau}{h^2} \sum_{i=1}^M \sum_{j=1}^M x_j^i x_{j+1}^i
\]
\[ \begin{align*}
&= \sum_{i=1}^{M} \sum_{j=1}^{M} (x_{ij}^j)^2 + \frac{\tau}{h^2} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} (x_{ij}^j)^2 - 2 \sum_{i=1}^{M-1} \sum_{j=1}^{M} x_{ij}^i x_{ij}^j + \sum_{i=1}^{M} \sum_{j=1}^{M} (x_{ij}^j)^2 \right] \\
&\quad + \frac{\tau}{h^2} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} (x_{ij}^j)^2 - 2 \sum_{i=1}^{M-1} \sum_{j=1}^{M} x_{ij}^i x_{ij}^j + \sum_{i=1}^{M} \sum_{j=1}^{M} (x_{ij}^j)^2 \right] \\
&\quad + \frac{\tau}{h^2} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} (x_{ij}^j)^2 - 2 \sum_{i=1}^{M-1} \sum_{j=1}^{M} x_{ij}^i x_{ij}^j + \sum_{i=1}^{M} \sum_{j=1}^{M} (x_{ij}^j)^2 \right] \\
&\quad + \frac{\tau}{h^2} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} (x_{ij}^j)^2 - 2 \sum_{i=1}^{M-1} \sum_{j=1}^{M} x_{ij}^i x_{ij}^j + \sum_{i=1}^{M} \sum_{j=1}^{M} (x_{ij}^j)^2 \right] \\
&\quad + \frac{\tau}{h^2} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} (x_{ij}^j)^2 - 2 \sum_{i=1}^{M-1} \sum_{j=1}^{M} x_{ij}^i x_{ij}^j + \sum_{i=1}^{M} \sum_{j=1}^{M} (x_{ij}^j)^2 \right].
\end{align*} \]

The last equality yields that \( A \) is positive definite. \( \square \)

**Corollary 2.2.** Using the fact that \( A \) is symmetric and positive definite, we deduce that \( A \) is invertible and its inverse is positive definite as well.

### 2.1. Existence of steady state.

Consider the equation \( AU^* = U^* + \lambda \tau F(U^*) \), which is equivalent to

\[(A - I)U^* = \lambda \tau F(U^*),\]

and which in addition could be written as

\[ LU^* = \lambda F(U^*) \quad (2.2) \]
where

\[
L = \frac{1}{h^2} \begin{pmatrix}
4 & -1 & & & & \\
-1 & 4 & -1 & & & \\
& & \ddots & \ddots & \ddots & \\
& & & \ddots & \ddots & \\
& & & & 4 & -1 \\
& & & & -1 & 4 & -1 \\
& & & & & \ddots & \ddots & \\
& & & & & & \ddots & \ddots & \\
\end{pmatrix} - I
\]

and

\[
F(U^*) = \begin{pmatrix}
\frac{f(x_1, y_1)}{(1 - u_{1,1})^2} \\
\vdots \\
\frac{f(x_{M, y_1})}{(1 - u_{M,1})^2} \\
\frac{f(x_1, y_2)}{(1 - u_{1,2})^2} \\
\vdots \\
\frac{f(x_{M, y_2})}{(1 - u_{M,2})^2} \\
\vdots \\
\frac{f(x_1, y_M)}{(1 - u_{1,M})^2} \\
\vdots \\
\frac{f(x_{M, y_M})}{(1 - u_{M,M})^2}
\end{pmatrix}
\]

Theorem 2.3. Assume that \( \lambda \leq \frac{2\pi^2 \delta^2 (1 - \delta)}{(b - a)^2} \), for some \( \delta \in (0, 1) \). Then (2.2) admits a solution \( U^* \) such that \( 0 \leq U^* < E \), where \( E = \{1, 1, \cdots, 1\}_{1 \times M^2} \).

Proof. \( L \) is symmetric and positive definite. Indeed, we have for \( x \in \mathbb{R}^{M^2} \),

\[
(Lx, x) = \frac{1}{h^2} \left( \sum_{j=1}^{M} (x_{1,j})^2 + \sum_{j=1}^{M} (x_{M,j})^2 + \sum_{i=1}^{M} (x_{i,1})^2 + \sum_{i=1}^{M} (x_{i,M})^2 \\
+ \sum_{j=1}^{M-1} \sum_{i=1}^{M} |x_{i,j} - x_{i,j+1}|^2 + \sum_{j=1}^{M} \sum_{i=1}^{M-1} |x_{i,j} - x_{i+1,j}|^2 \right) > 0.
\]
Consequently, \( L \) is invertible and its inverse is positive definite as well.

Noting that \( L \) represents \(-\Delta\) in discrete form, set \( \mu_1 \) as the minimum eigenvalue of \(-\Delta\), then \( \mu_1 = \frac{2\pi^2}{(b-a)^2} \) (see [17]), and using the fact that \( \| L^{-1} \|_2 \) equals to the inverse of the least eigenvalue of the operator \(-\Delta\), we obtain:

\[
\| L^{-1} \|_2 = \frac{1}{\mu_1} = \frac{(b-a)^2}{2\pi^2}.
\]

The problem 2.2 can be written in the form

\[ U^* = \lambda L^{-1} F(U^*) = H(U^*). \]

Consider the sequence

\[ W^{k+1} = H(W^k), \quad W^0 = 0. \]

Suppose that \( W^k \geq 0 \), then \( W^{k+1} = H(W^k) \geq 0 \). So, for every \( k \in \mathbb{N}^* \), we have \( W^k \geq 0 \).

Suppose that \( W^{k+1} \geq W^k \), as \( W^1 \geq 0 = W^0 \), then:

\[
W^{k+2} - W^{k+1} = H(W^{k+1}) - H(W^k) = \lambda h^2 L^{-1} F(W^{k+1} - W^k),
\]

it yields

\[
W^{k+2} \geq W^{k+1}.
\]

Besides, we have that

\[
\| H(\xi) \|_\infty = \| L^{-1} \lambda F(\xi) \|_\infty
\]

\[
\leq \| L^{-1} \|_\infty \lambda \left\| \frac{1}{(1-\xi)^2} \right\|_\infty, \quad 0 \leq f_{i,j} \leq 1, \forall i, j = 1, \cdots, M
\]

\[
\leq \| L^{-1} \|_2 \lambda \left\| \frac{1}{(1-\xi)^2} \right\|_\infty
\]

\[
\leq \frac{(b-a)^2}{2\pi^2} \lambda \max_{i,j} \left| \frac{1}{(1-\xi_{i,j})^2} \right|.
\]

Since, \( 0 \leq \xi_{i,j} < 1 \), let \( \delta \in (0,1) \) such that \( 0 \leq \xi_{i,j} \leq 1 - \delta \), we find

\[
\frac{1}{(1-\xi_{i,j})^2} \leq \frac{1}{\delta^2}.
\]

Thus we obtain

\[
\| H(\xi) \|_\infty \leq \frac{(b-a)^2}{2\pi^2} \lambda \frac{1}{\delta^2}
\]

\[
\leq 1 - \delta
\]

if and only if

\[
\lambda \leq \frac{2\pi^2 \delta^2 (1 - \delta)}{(b-a)^2}.
\]

\( \square \)
2.2. Convergence.

**Theorem 2.4.** Suppose that the assumption (2.3) holds, then the discrete solution \( \{U^n\}_n \) converges to the stationary solution \( U^* \) if \( \lambda < \frac{1 - \|U^*\|_{\infty}}{(b - a)^2} \).

**Proof.** Suppose that the assumption (2.3) holds, so that \( 0 \leq U^1 \leq U^* \). Let \( 0 \leq U^n \leq U^* \), then we have \( 0 \leq U^{n+1} \leq U^* \). In fact, let \( V^{n+1} = U^{n+1} - U^* \), but

\[
AU^{n+1} = U^n + F(U^n)
\]

and

\[
AU^* = U^* + F(U^*).
\]

Hence,

\[
A(U^{n+1} - U^*) = (U^n - U^*) + (F(U^n) - F(U^*)),
\]

which yields

\[
V^{n+1} = A^{-1}(V^n + F(U^n) - F(U^*)).
\]

Since \( F \) is an increasing function, we find that,

\[
F(U^n) \leq F(U^*).
\]

Therefore, \( V^{n+1} \leq 0 \) and \( U^{n+1} \leq U^* \).

In addition, using the fact

\[
AU^{n+1} = U^n + F(U^n),
\]

\( A^{-1} \) is positive definite and \( U^n \geq 0 \), then \( U^{n+1} \geq 0 \). It follows that \( U^n \) is well defined and

\[
0 \leq U^n \leq U^*, \quad n \in \mathbb{N} \cup \{0\}.
\]

We have, for all \( \{V^{n+1}\} \) with \( v_{i,0} = v_{0,j} = 0 \), and \( V^{n+1} = \begin{pmatrix} v_{1,1} \\ \vdots \\ v_{i,j} \\ \vdots \\ v_{M,1} \end{pmatrix} \) where \( v_{i,j} = \)

\[
\begin{pmatrix}
  v_{1,j} \\
  \vdots \\
  v_{M,j}
\end{pmatrix}.
\]

Below, \((.,.)\) denotes the Euclidean scalar product and \(||.||\) its associate norm.

\[
((AV^{n+1}, V^{n+1})) = v_{i,1}^{n+1} B v_{i,1}^{n+1} + v_{i,1}^{n+1} C v_{i,2}^{n+1} + v_{i,2}^{n+1} C v_{i,1}^{n+1} + v_{i,2}^{n+1} B v_{i,2}^{n+1} + \cdots + v_{i,2}^{n+1} C v_{i,3}^{n+1} + v_{i,3}^{n+1} C v_{i,2}^{n+1} + \cdots + v_{i,M}^{n+1} C v_{i,M-1}^{n+1} + v_{i,M}^{n+1} B v_{i,M}^{n+1} + \cdots
\]

\[
= \left(1 + \frac{4\pi}{h^2}\right) \sum_{j=1}^{M} \left( \sum_{i=1}^{M} (v_{i,j}^{n+1})^2 - \frac{2\pi}{h^2} \sum_{j=1}^{M} \sum_{i=1}^{M-1} v_{i,j}^{n+1} v_{i+1,j}^{n+1} \right) - \frac{2\pi}{h^2} \sum_{j=1}^{M-1} \sum_{i=1}^{M} v_{i,j}^{n+1} v_{i,j+1}^{n+1}
\]

\[
= \sum_{j=1}^{M} \sum_{i=1}^{M} (v_{i,j}^{n+1})^2 + \frac{\tau}{h^2} \left[ \sum_{j=1}^{M} \sum_{i=1}^{M} (v_{i,j}^{n+1})^2 - 2 \sum_{j=1}^{M} \sum_{i=1}^{M-1} v_{i,j}^{n+1} v_{i+1,j}^{n+1} \right] + \sum_{j=1}^{M} \sum_{i=1}^{M} (v_{i,j}^{n+1})^2
\]
\[
\begin{align*}
& + \frac{\tau}{h^2} \left[ \sum_{j=1}^{M} \sum_{i=1}^{M} (v_{i,j}^{n+1})^2 - 2 \sum_{j=1}^{M-1} \sum_{i=1}^{M} v_{i,j}^{n+1} v_{i,j+1}^{n+1} + \sum_{j=1}^{M} \sum_{i=1}^{M} (v_{i,j}^{n+1})^2 \right] \\
& = \sum_{j=1}^{M} \sum_{i=1}^{M} (v_{i,j}^{n+1})^2 + \frac{\tau}{h^2} \left[ \sum_{j=1}^{M} \sum_{i=1}^{M} (v_{i,j}^{n+1})^2 - 2 \sum_{j=1}^{M-1} \sum_{i=1}^{M} v_{i,j}^{n+1} v_{i,j+1}^{n+1} + \sum_{j=1}^{M} \sum_{i=1}^{M} (v_{i,j}^{n+1})^2 \right] \\
& + \sum_{j=1}^{M} \sum_{i=1}^{M} (v_{i,j}^{n+1})^2 - \sum_{j=1}^{M} (v_{i,j}^{n+1})^2 \\
\end{align*}
\]

Besides, \(v(x_{i+1}, y_j) \approx v(x_i, y_j) + h \frac{\partial v(x_i, y_j)}{\partial x}\) and \(v(x_i, y_{j+1}) \approx v(x_i, y_j) + h \frac{\partial v(x_i, y_j)}{\partial y}\). Hence,

\[
\frac{\partial v_{i,j}^{n+1}}{\partial x} \approx \frac{v_{i+1,j}^{n+1} - v_{i,j}^{n+1}}{h} \quad \text{and} \quad \frac{\partial v_{i,j}^{n+1}}{\partial y} \approx \frac{v_{i,j+1}^{n+1} - v_{i,j}^{n+1}}{h}.
\]

Therefore,

\[
\left\| \nabla v_{i,j}^{n+1} \right\|^2 = \left( \frac{\partial v_{i,j}^{n+1}}{\partial x} \right)^2 + \left( \frac{\partial v_{i,j}^{n+1}}{\partial y} \right)^2
\]
Suppose that $v_{0,j} = v_{i,0} = 0$, and using discrete Poincare inequality, we have

$$\sum_{j=1}^{M} \sum_{i=0}^{M-1} |v_{i+1,j} - v_{i,j}|^2 \geq \frac{h^2}{(b-a)^2} \sum_{j=1}^{M} \sum_{i=1}^{M} (v_{i,j})^2.$$ 

Furthermore,

$$\sum_{j=0}^{M-1} \sum_{i=1}^{M} |v_{i,j+1} - v_{i,j}|^2 \geq \frac{h^2}{(b-a)^2} \sum_{j=1}^{M} \sum_{i=1}^{M} (v_{i,j})^2.$$ 

Hence,

$$((AV^{n+1}, V^{n+1})) \geq \left( 1 + \frac{2\tau}{(b-a)^2} \right) \|V^{n+1}\|^2.$$ 

Note that, $AV^{n+1} = V^n + F(U^n) - F(U^*)$, and

$$((AV^{n+1}, V^{n+1})) = ((V^n + F(U^n) - F(U^*), V^{n+1}))$$

$$\leq \|V^n + F(U^n) - F(U^*)\| \|V^{n+1}\|.$$ 

And,

$$((AV^{n+1}, V^{n+1})) \geq \left( 1 + \frac{2\tau}{(b-a)^2} \right) \|V^{n+1}\|^2.$$ 

Therefore,

$$\left( 1 + \frac{2\tau}{(b-a)^2} \right) \|V^{n+1}\|^2 \leq \|V^n + F(U^n) - F(U^*)\| \|V^{n+1}\|$$

$$\|V^{n+1}\| \leq \left( 1 + \frac{2\tau}{(b-a)^2} \right)^{-1} \left( \|V^n\| + \|F(U^n) - F(U^*)\| \right)$$

$$\leq \left( 1 + \frac{2\tau}{(b-a)^2} \right)^{-1} \left( \|V^n\| + \lambda \tau \left( \sum_{j=1}^{M} \sum_{i=1}^{M} \left( \frac{v_{i,j}^n(2u_{i,j}^n - u_{i,j}^n)}{(1-u_{i,j}^n)^2(1-u_{i,j}^n)^2} \right)^2 \right)^{1/2} \right)$$

$$\leq \left( 1 + \frac{2\tau}{(b-a)^2} \right)^{-1} \left( \|V^n\| + \lambda \tau \left( \sum_{j=1}^{M} \sum_{i=1}^{M} \left( \frac{v_{i,j}^n((1-u_{i,j}^n) + (1-u_{i,j}^n))}{(1-u_{i,j}^n)^2(1-u_{i,j}^n)^2} \right)^2 \right)^{1/2} \right)$$

$$\leq \left( 1 + \frac{2\tau}{(b-a)^2} \right)^{-1} \left( \|V^n\| + \lambda \tau \left( \sum_{j=1}^{M} \sum_{i=1}^{M} \left( \frac{v_{i,j}^n((1-u_{i,j}^n) + (1-u_{i,j}^n))}{(1-u_{i,j}^n)^2(1-u_{i,j}^n)^2} \right)^2 \right)^{1/2} \right)$$

$$\leq \left( 1 + \frac{2\tau}{(b-a)^2} \right)^{-1} \left( \|V^n\| + \lambda \tau \left( \sum_{j=1}^{M} \sum_{i=1}^{M} \left( \frac{2v_{i,j}^n(1-u_{i,j}^n)}{(1-u_{i,j}^n)^2(1-u_{i,j}^n)^2} \right)^2 \right)^{1/2} \right)$$
We can write the above scheme as:

\[
\begin{aligned}
&\leq \left(1 + \frac{2\tau}{(b-a)^2}\right)^{-1} \left\|V^n\right\| + \lambda\tau \left(\sum_{i=1}^{M} \sum_{j=1}^{M} \left(\frac{2v_{i,j}^n}{(1-u_{i,j}^n)(1-u_{i,j}^n)}\right)^2\right)^{\frac{1}{2}} \\
&\leq \left(1 + \frac{2\tau}{(b-a)^2}\right)^{-1} \left\|V^n\right\| + \lambda\tau \left(\sum_{i=1}^{M} \sum_{j=1}^{M} \left(\frac{2v_{i,j}^n}{(1-u_{i,j}^n)^3}\right)^2\right)^{\frac{1}{2}} \\
&\leq \left(1 + \frac{2\tau}{(b-a)^2}\right)^{-1} \left\|V^n\right\| + \lambda\tau \left(\frac{1}{(1-\|U^*\|_\infty)^3}\left\|V^n\right\|\right) \\
&\leq \left(1 + \frac{2\tau}{(b-a)^2}\right)^{-1} \left(1 + 2\lambda\tau \left(\frac{1}{(1-\|U^*\|_\infty)^3}\left\|V^n\right\|\right)\right) \\
\end{aligned}
\]

Therefore, \(\{V^{n+1}\}\) converges to 0 if

\[
\left(1 + \frac{2\tau}{(b-a)^2}\right)^{-1} \left(1 + 2\lambda\tau \left(\frac{1}{(1-\|U^*\|_\infty)^3}\right)\right) < 1,
\]

in other words, if

\[
\lambda < \frac{(1-\|U^*\|_\infty)^3}{(b-a)^2},
\]

then \(\{U^n\}\) converges to \(U^*\).

\[\square\]

### 3. 2 D Implicit Scheme

System (1.1) can be written as:

\[
u_t - \Delta u - \lambda f (1 - u)^2 = 0,
\]

then its implicit time discretization is given by

\[
\begin{aligned}
\frac{v_{i,j}^{n+1} - v_{i,j}^n}{\tau} &= \frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{h^2} - \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{h^2} - \frac{\lambda f (1 - u_{i,j}^{n+1})^2}{(1 - u_{i,j}^n)^2} = 0 \\
i, j = 1, \ldots, M,
\end{aligned}
\]

and that

\[
u_{i,0}^{n+1} = u_{i,M+1}^{n+1} = u_{0,j}^{n+1} = u_{M+1,j}^{n+1} = 0 \quad \text{where} \quad u_{i,j}^n = u(t^n, x_i, y_j).
\]

The problem (3.1) can be rewritten as, \(\forall i, j = 1, \ldots, M\)

\[
\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\tau} = \frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{h^2} - \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{h^2} - \frac{\tau \lambda f (1 - u_{i,j}^{n+1})^2}{(1 - u_{i,j}^n)^2} = 0.
\]

We can write the above scheme as: \(F(U) = 0\) where

\[
F : [0, 1]^M \longrightarrow \mathbb{R}^M
\]

and

\[
U = (u_{1,1}, \cdots, u_{M,1}, \cdots, u_{1,M}, \cdots, u_{M,M})^t
\]

\[
F(U) = (F_{1,1}(U), \cdots, F_{M,1}(U), \cdots, F_{1,M}(U), \cdots, F_{M,M}(U))^t,
\]
such that
\[ F_{i,j}(U^{n+1}) = \left( 1 + 4 \frac{\tau}{h^2} \right) u_{i,j}^{n+1} - u_{i,j}^{n} - \frac{\tau}{h^2} (u_{i+1,j}^{n+1} + u_{i-1,j}^{n+1} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n+1}) \]
\[ - \frac{\tau \lambda f_i}{(1-u_{i,j}^{n+1})^2}, \quad i,j = 1, \ldots, M, \]
and
\[ 0 = (0, \ldots, 0, \ldots, 0, \ldots, 0)^t_{1 \times M^2}. \]

We know that,
\[ F(U^{n+1}) = F(U^n + \tau) \simeq F(U^n) + \tau DF(U^n) \]
but \( \tau = U^{n+1} - U^n \), so
\[ F(U^{n+1}) \simeq F(U^n) + DF(U^n)(U^{n+1} - U^n), \quad \forall n \in \mathbb{N}, \]
\( U^{n+1} \) is a root of \( F \), then
\[ F(U^{n+1}) = 0 \]
whence \( U^{n+1} \) is an approximate solution of
\[ F(U^n) + DF(U^n)(U^{n+1} - U^n) = 0. \]

Which yields,
\[ U^{n+1} = -DF^{-1}(U^n)F(U^n) + U^n, \]
so, in order to find \( U^{n+1} \) we have to compute the jacobian matrix of \( F \) at each iteration of \( n \).

The jacobian matrix is given by
\[
J_{i,j,r,s} = \frac{\partial F_{i,j}}{\partial u_{r,s}} = \begin{cases} 
- \frac{\tau}{h^2} & \text{if } r = i + 1 \text{ or } r = i - 1 \text{ and } s = j \\
1 + 4 \frac{\tau}{h^2} - \frac{2 \tau f(x_i, y_j)}{(1-u_{i,j}^{n+1})^2} & \text{if } r = i \text{ and } s = j
\end{cases}
\]

3.1. **Existence of root for** \( F \): Let \( 0 \leq U^* < 1 \) be a root of \( F \), then \( F(U^*) = 0 \), thus \( F_i(U^*) = 0, \forall i = 1, \ldots, M \).

Which is equivalent to:
\[ \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \]
\[ - \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} - \frac{\tau \lambda f_{i,j}}{(1-u_{i,j})} = 0, \quad \forall i,j = 1, \ldots, M \]

(3.3) is equivalent to
\[ \frac{4u_{i,j} - u_{i,j+1} + u_{i,j-1}}{h^2} - \frac{\tau \lambda f_{i,j}}{1-u_{i,j}} = 0, \quad \forall i,j = 1, \ldots, M \]

which can be rewritten in vector form as:
\[ LU^* = H(U^*), \]
where \( L \) is equivalent to the matrix in (2.2).

**Theorem 3.1.** Assume that \( \lambda \leq \frac{2\pi^2 \delta^2 (1-\delta)}{(b-a)^2} \), for some \( \delta \in (0,1) \). Then (3.3) admits a solution \( U^* \) such that \( 0 \leq U^* < E \), where \( E = (1,1,\ldots,1)^t_{1 \times M^2} \).
Lemma 3.2. $DF(U^*)$ is invertible if $\tau < \frac{(b-a)^2}{2\pi^2 \delta^2 (1-\delta)}$, for some $\delta \in (0,1)$

Proof. $DF(U^*)$ can be written as: $(1 + 4\frac{\tau}{h^2})I - M$, $\forall i = 1, \cdots, M$,

$$ M = \begin{pmatrix}
H_{i,1} & \frac{\tau}{h^2} I_{M \times M} \\
\frac{\tau}{h^2} I_{M \times M} & H_{i,2} & \frac{\tau}{h^2} I_{M \times M} \\
& \ddots & \ddots & \ddots \\
& & \frac{\tau}{h^2} I_{M \times M} & H_{i,M}
\end{pmatrix}_{M^2 \times M^2} $$

and $\forall j = 1, \cdots, M$

$$ H_{i,j} = \begin{pmatrix}
\frac{2f_{i,j}\lambda \tau}{(1-u_{i,j}^*)^3} & \frac{\tau}{h^2} & 0 & \cdots & \cdots & \cdots \\
\frac{\tau}{h^2} & \frac{2f_{i,j}\lambda \tau}{(1-u_{i,j}^*)^3} & \frac{\tau}{h^2} & 0 & \cdots & \cdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \frac{\tau}{h^2} & \frac{2f_{M-2,j}\lambda \tau}{(1-u_{M-2,j}^*)^3} & \frac{\tau}{h^2} & 0 & \cdots \\
0 & 0 & \frac{\tau}{h^2} & \frac{2f_{M-1,j}\lambda \tau}{(1-u_{M-1,j}^*)^3} & \frac{\tau}{h^2} & \cdots \\
0 & 0 & 0 & \frac{\tau}{h^2} & \frac{2f_{M,j}\lambda \tau}{(1-u_{M,j}^*)^3} & \cdots \\
\end{pmatrix}_{M \times M} $$

Thus,

$$ ||M||_{\infty} = \max \left\{ \frac{2f_{i,j}\lambda \tau}{(1-u_{i,j}^*)^3} + 2\frac{\tau}{h^2} + \frac{2f_{l,k}\lambda \tau}{(1-u_{l,k}^*)^3} + 4\frac{\tau}{h^2} + \frac{2f_{s,t}\lambda \tau}{(1-u_{s,t}^*)^3} + 3\frac{\tau}{h^2} \right\}, $$

$$ i = 1, M \text{ and } j = 1, \cdots, M, s = 1, M, $$

and $t = 2, \cdots, M-1, l = 2, \cdots, M-1, \text{ and } k = 1, \cdots, M$.

$0 \leq U^* < 1$, then $0 \leq U^* \leq 1 - \delta$, for some $\delta \in (0,1)$, $0 \leq u_{i,j}^* \leq 1 - \delta$ $\forall i, j = 1, \cdots, M$. Therefore, $1 \leq \frac{1}{(1-u_{i,j}^*)^3} \leq \frac{1}{\delta^3}$, and in addition to the assumption on the permittivity profile $0 \leq f \leq 1$, we have according to (3.3), $\lambda \leq \frac{2\pi^2 \delta^2 (1-\delta)}{(b-a)^2}$,

so $||M||_{\infty} = 4\frac{\tau}{h^2} + \frac{2\pi^2 \delta^2 (1-\delta)}{(b-a)^2}.$

$DF(U^*)$ is invertible if and only if,

$$ 4\frac{\tau}{h^2} + \frac{2\pi^2 \delta^2 (1-\delta)}{(b-a)^2} < 1 + 4\frac{\tau}{h^2} $$

$$ \tau < \frac{(b-a)^2}{2\pi^2 \delta^2 (1-\delta)}. $$
Moreover,
\[
\|DF(U^*)^{-1}\| = \frac{1}{1 - \left(1 + \frac{4\tau}{h^2}\right)^{-1} \left(\frac{4 \tau}{h^2} + \frac{2\pi^2\delta^2(1 - \delta)\tau}{(b-a)^2}\right)}.
\]

\[\square\]

**Corollary 3.3.** If \(F \in C^2, \forall U \in B(U^*, a)\), then \(DF(U)\) is invertible and
\[\| (DF(U))^{-1} \| \leq 2 \| DF(U^*)^{-1} \| = a_1.\]

**Proof.** Let \(U \in [0, 1)^M\), we have:
\[
DF(U) = DF(U) - DF(U^*) + DF(U^*)
\]
\[
= DF(U^*)(I + [DF(U^*)]^{-1}(DF(U) - DF(U^*)))
\]
\(F \in C^2\); then \(DF\) is \(C^1\), in particular \(DF\) is continuous at \(U^*\), so by the definition of continuity of \(DF\) at \(U^*\), for \(\epsilon = \frac{1}{2\|DF(U^*)^{-1}\|} > 0, \exists a > 0\) such that \(\forall U \in B(U^*, a)\), we have
\[
\|DF(U) - DF(U^*)\| < \epsilon = \frac{1}{2\|DF(U^*)^{-1}\|},
\]
and
\[
\left\| [DF(U^*)]^{-1} (DF(U) - DF(U^*)) \right\| \leq \left\| [DF(U^*)]^{-1}\right\| \|DF(U) - DF(U^*)\| \leq \frac{1}{2} < 1.
\]
By virtue of Von-Neumann lemma; \(I + [DF(U^*)]^{-1}(DF(U) - DF(U^*))\) is invertible, then for all \(U \in B(U^*, a)\), \(DF(U)\) is invertible and
\[
\left\| [DF(U)]^{-1}\right\| = \left\| \left((DF(U^*)(I + [DF(U^*)]^{-1}(DF(U) - DF(U^*)))\right)^{-1}\right\|
\]
\[
= \left\| (I + [DF(U^*)]^{-1}(DF(U) - DF(U^*)))^{-1}[DF(U^*)]^{-1}\right\|
\]
\[
\leq \left\| (I + [DF(U^*)]^{-1}(DF(U) - DF(U^*)))^{-1}\right\| \left\| [DF(U^*)]^{-1}\right\|
\]
\[
\leq \frac{1}{1 - \left\| [DF(U^*)]^{-1}(DF(U) - DF(U^*))\right\|} \left\| [DF(U^*)]^{-1}\right\|
\]
\[
\leq \frac{1}{\frac{1}{2}} \left\| [DF(U^*)]^{-1}\right\|
\]
\[
\leq 2 \left\| [DF(U^*)]^{-1}\right\|
\]
\[\square\]

**Lemma 3.4.** If \(F \in C^2\) and \(U^n \in B(U^*, a)\), then \(\|DF(U^n)(U^{n+1} - U^*)\| \leq a_2 \|U^n - U^*\|^2\), where \(a_2 = \frac{1}{2} \sup_{\xi \in \partial B(U^*, a)} \|D^2F(\xi)\|\)
Proof. Let
\[
\phi(t) = F(U^n + t(U^* - U^n)) - F(U^n) - tDF(U^n)(U^* - U^n)
\]
so that,
\[
\phi'(t) = DF(U^n + t(U^* - U^n))(U^* - U^n) - DF(U^n)(U^* - U^n).
\]
Owing to first fundamental theorem of calculus \(\phi(1) - \phi(0) = \int_0^1 \phi'(t)dt\).
\[
F(U^*) - F(U^n) - DF(U^n)(U^* - U^n)
\]
\[
= \int_0^1 [DF(U^n + t(U^* - U^n))(U^* - U^n) - DF(U^n)(U^* - U^n)] dt.
\]
Because of the fact that, \(F(U^*) = 0\) and applying the norms on the last equality,
we find that
\[
\|F(U^n) - DF(U^n)(U^* - U^n)\|
\]
\[
\leq \int_0^1 \|DF(U^n + t(U^* - U^n))(U^* - U^n) - DF(U^n)(U^* - U^n)\| dt
\]
\[
\leq \int_0^1 \|DF(U^n + t(U^* - U^n)) - DF(U^n)\| \|(U^* - U^n)\| dt.
\]
Noting that \(DF\) is \(C^1\), according to mean value theorem, we obtain
\[
\|DF(U^n + t(U^* - U^n)) - DF(U^n)\| \leq \|t(U^* - U^n)\| \sup_{\xi \in B(U^*,a)} \|D^2F(\xi)\|
\]
and
\[
\|F(U^n) - DF(U^n)(U^* - U^n)\| \leq \|(U^* - U^n)\|^2 \sup_{\xi \in B(U^*,a)} \|D^2F(\xi)\| \int_0^1 t dt
\]
\[
\leq \frac{1}{2} \|(U^* - U^n)\|^2 \sup_{\xi \in B(U^*,a)} \|D^2F(\xi)\|.
\]
Therefore,
\[
\|F(U^n) - DF(U^n)(U^* - U^n)\| \leq a_2 \|(U^* - U^n)\|^2
\]
Thanks to (3.2), we deduce that,
\[
\|DF(U^n)(U^{n+1} - U^n) - DF(U^n)(U^* - U^n)\| \leq a_2 \|(U^* - U^n)\|^2
\]
\[
\|DF(U^n)(U^{n+1} - U^*)\| \leq a_2 \|(U^* - U^n)\|^2
\]
\[\square\]

**Theorem 3.5.** Assume that \(U^0 \in B(U^*, b) \subset B(U^*, a)\), then \(DF(U^{n+1})\) is invertible and \(\{U^n\}\) converges to \(U^*\).

*Proof.* \(U^0 \in B(U^*, b) \subset B(U^*, a)\), then \(DF(U^0)\) is invertible and \(\left\|DF(U^0)^{-1}\right\| \leq a_1\).

Thus \(U^1\) is well defined and we have \(DF(U^0)(U^1 - U^0) = -F(U^0)\) by (3.2). Suppose that this is true up to \(n\), so \(DF(U^n)\) is invertible and \(\left\|DF(U^n)^{-1}\right\| \leq a_1\), \(U^n \in B(U^*, b)\). Then \(U^{n+1}\) is well defined and we have \(DF(U^n)(U^{n+1} - U^n) = -F(U^n)\).
\[
\|U^{n+1} - U^*\| = \left\|DF(U^n)^{-1} DF(U^n)(U^{n+1} - U^*)\right\|
\]
\[
\|DF(U^n) - I\| \leq a_1 a_2 \|U^n - U^*\|^2
\]

but \(U^n \in B(U^*, b)\),
\[
\|U^{n+1} - U^*\| \leq a_1 a_2 b^2
\]

let \(b < \min(a, \frac{1}{a_1 a_2})\), we find,
\[
\|U^{n+1} - U^*\| \leq (a_1 a_2 b) b \leq b,
\]

which yields to, \(U^{n+1} \in B(U^*, b) \subset B(U^*, a)\), \(DF(U^{n+1})\) is invertible and
\[
\left\| \left[DF(U^{n+1})\right]^{-1}\right\| \leq a_1.
\]

Consequently, \(U^n \in B(U^*, b)\) and \(DF(U^n)\) is invertible \(\forall \ n \in \mathbb{N}\).

\[
\|D(U^n)|^{-1}\| \leq a_1.
\]

Therefore, the sequence \(\{U^n\}_n\) is well defined. On the other hand, we have
\[
\|U^{n+1} - U^*\| \leq a_1 a_2 \|U^n - U^*\|^2 = \beta \|U^n - U^*\|^2,
\]

so
\[
\beta \|U^{n+1} - U^*\| \leq \beta^2 \|U^n - U^*\|^2,
\]

then
\[
\|U^{n+1} - U^*\| \leq \frac{1}{\beta} \left(\beta \|U^0 - U^*\|\right)^2 \to 0
\]

since \(U^0 \in B(U^*, b)\), then \(\beta \|U^0 - U^*\| \leq \beta b < 1 \)

4. Numerical Simulations:

In this section we consider the numerical simulations of both semi-implicit and implicit schemes for certain values of the permittivity profile function \(f\), voltage \(\lambda\), time step \(\tau\), number of points \(M\) and with initial value \(u_{\text{initial}} = 0\).

Figure 1 shows for both schemes the behavior of the solution \(u\) with respect to the spatial variables \(x\) and \(y\).

![Figure 1](image1.jpg)

(a) Semi-implicit scheme solution  
(b) Implicit scheme solution

**Figure 1.** \(\lambda = 10, \ f(x, y) = \sqrt{x^2 + y^2}, \ \tau = 0.01, \ M = 29\)

Figure 2 shows the behavior of the error, \(\|u^{n+1} - u^n\|\), which converges to zero with respect to time.

Comparing the rate of convergence of both schemes we confirm the results of convergence stated above, since Newton is quadratic then its rate of convergence is faster.
Furthermore, the touchdown phenomenon for both schemes is represented by the following figures, it is clear that for same value of $\lambda$, $\tau$ and $f(x,y)$ it requires more time to observe touchdown for the implicit scheme Figure 3 illustrates this result.

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