Delay analysis of a dynamic queue-based random access network

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Abstract

Motivated by design and performance challenges stemming from emerging applications in random-access networks, we focus on the performance of a dynamic two-user slotted-time ALOHA network with a general queue-dependent transmission policy. At the beginning of each slot, each user transmits a packet with a probability that depends on the number of stored packets at both user queues. If both stations transmit at the same slot a collision occurs, and both packets must be retransmitted in a later slot. Each user has external bursty arrivals that are stored in their infinite capacity queues. Arrival processes are independent of each other, but depend also on the state of the network at the beginning of a slot. In such a network of interacting queues, when a user transmits a packet, it causes interference to the nearby user and decreases its successful transmission probability. Each user is aware of the status of the network, and accordingly reconfigures its transmission parameters to improve the network performance. We investigate the ergodicity conditions, and use the generating function approach to investigate the queueing delay. Generating functions for the steady-state distribution are obtained by solving a finite system of linear equations and a functional equation with the aid of the theory of Riemann-Hilbert boundary value problems.

Keywords: ALOHA network, Queue-based transmission, Ergodicity, Delay, Boundary value problem.

1 Introduction

Clearly, due to the need for massive uncoordinated access, next-generation wireless networks are going to have a more decentralized architecture than the current cellular networks. This is why simple random access schemes, such as ALOHA protocol [2] has gained popularity in multiple access communication systems. Random access remains an active research area, and many fundamental questions remain open even for very simple and small-sized networks [10, 33], mainly due to the strong interaction among the wireless nodes.

The vast related literature in random access networks dealt with the investigation of the stability region. However, due to the strong interaction among the queues, the exact characterization of the stability region remains a challenging task. A much more challenging task in such networks is the investigation of the queueing delay, for which the there are very few analytical results even for small random access networks, e.g., [3, 8, 25, 26]. Recently, there is a imperative need on supporting real-time applications, which in turn reveals the need to provide delay-based guarantees [24, 16, 7]. Therefore, the characterization of the delay is of major importance.

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In this work, we consider a two-user ALOHA network with general queue-based transmission policy. Each user receive exogenous arrivals of packets for transmission, which depend on the state of the network. We also seek dynamic distributed random access strategies, based also on the state of the network, with ultimate goal is to keep the network queues stable, and more importantly to derive expressions for the queueing delay. Our main goal is to provide a general framework to analyse network adaptive random access schemes. Our attempt is towards self-aware and computationally intelligent systems, in which each node is capable to obtain knowledge from its operational environment, and use this knowledge to adjust its transmission parameters accordingly. We investigate the stable throughput region, and study the queueing delay by solving a system of linear equations and utilizing the theory of boundary value problems.

1.1 Related work

Aloha-type random-access models have been widely studied in the literature, e.g., \[2, 8, 9, 15, 30, 31, 26, 25, 29, 28\]. The vast related literature is focused on the investigation of the stability, which is a challenging task due to the strong interaction among the queues of the nodes. In such systems, the individual departure rates of the queues cannot be computed separately without knowing the stationary distribution of the joint queue length process \[29\]. This is the reason why the vast majority of previous works has focused on small-sized networks and only bounds or approximations are known for the networks with larger number of sources \[34, 29, 30, 31, 21, 26\]. In \[4\], an approximation of the stability region was obtained based on the mean-field theory for network of nodes having identical arrival rates and transmission probabilities were performed.

In \[19, 32\], dynamic, queue-length based strategies were introduced in order to investigate the stability of a generalized ALOHA-type network. In that work, the actual queue lengths of the flows in each node’s close neighborhood are used to determine the nodes’ channel access probabilities. We also refer to \[12, 18, 5, 23, 35\], in which stability condition for Markov chains both in two and higher dimensions, whose transition structure possess a property of spatial homogeneity was investigated.

Delay analysis of random access networks was studied in \[26, 25, 3, 17\]. More specifically, in \[25\] a two-user network with collision channel was studied and expressions for the average delay were obtained. Quite recently, in \[8, 9\], the authors investigated the stability and the delay of an ALOHA-type network of two users, where the transmission probabilities of each node depend on the state of the neighbour nodes, as well as on the channel state. Stability conditions were investigated based on the stochastic dominance technique introduced in \[29\]. Moreover, based on a relation among the values of the transmission probabilities, the delay analysis was performed with the aid of the theory of Riemann-Hilbert boundary value problems.

1.2 Our contribution

Our contribution is summarized as follows. We consider a two-user wireless network with a common destination and collisions. The nodes/sources access the medium in a random access manner and time is assumed to be slotted. Each user has external arrivals that are based on the state of the network at the beginning of a slot, and they are stored in their infinite capacity queues. The nodes are accessing the wireless channel randomly and they also adapt their transmission probabilities based on the status of the network. To the best of our knowledge this variation of random access has not been reported in the literature. The contribution of this work mainly focused on the detailed analysis of the queueing delay at users nodes.
Note that for such a system we assume that each user node has also cognitive radio capabilities [8, 9, 20, 22], and it is aware of the status of the other nodes. Therefore, each user node adapts its transmission parameters according to its own state as well as the state of the other node. In such a case, we also take into account both the wireless interference [11], and the complex interdependence among users’ nodes due to the shared medium. Clearly, such a protocol leads to substantial performance gains. To the best of our knowledge there is no other work in random access networks that studies the delay of a general random access network where the arrivals and the transmission access channel probabilities of the nodes depend on the queue lengths at each node of the network.

Besides its practical applicability, this work is also theoretically oriented, since it generalizes the model in [13] into the discrete time. More importantly, in [13], the model is described by a random walk in the quarter plane (RWQP) in four directions: west, east, south and north (see Figure 1., p. 296). On the contrary, in this work, the model is described by a RWQP in seven directions, i.e., it further includes transitions to the north-west, south-east and north-east; see Figure 2. Moreover, the model in [13] refers to a two-dimensional birth-death process. This behavior is due to the slotted time setting, which allows the scheduling of multiple events at the same slot. As a result the analysis is complicated considerably. Moreover, our work generalizes the analysis in [13] to non-birth death models; see in Figure 2 where we have transitions to the North-East, which in turn correspond to the simultaneous arrival of two packets.

The paper is organized as follows. In Section 2 we present in detail the mathematical model, while in Section 3 we provide its stability condition. Some preparatory results along with the derivation of the functional equations is presented in Section 4. The solution of the fundamental functional equation in terms of a solution of a Riemann-Hilbert boundary value problem is given in Section 5, while a simple numerical example is given in Section 6.

### 2 Model description

We consider an ALOHA-type wireless network with \( N = 2 \) users that communicate with a common destination node; see Figure 1. Each user is equipped with an infinite capacity buffer for storing arriving and backlogged packets. The packet arrival processes are assumed to be independent from user to user and the channel is slotted in time, with a slot period to be equal the packet length. Denote by \( Q_k(m) \), \( k = 1, 2 \), to be the number of stored packets at the buffer of user \( k \), at the beginning of the \( m \)th slot. Then \( Q(m) = \{(Q_1(m), Q_2(m)) \mid m = 0, 1, \ldots \} \) is a DTMC with state space \( S = \{ \mathbf{n} = (n_1, n_2) \mid n_k \geq 0, k = 1, 2 \} \).

At the beginning of each slot, given that the state of the network is \( \mathbf{n} \), user node \( k, k = 1, 2 \) transmits a packet to the destination node with probability \( a_k(\mathbf{n}) \), i.e., according to a network-aware transmission protocol. If both user nodes transmit at the same slot there is a collision\(^1\) and both packets have to be retransmitted in a later slot. Packet arrivals are assumed i.i.d. random variables from slot to slot, both depended on the state of the network at the beginning of a slot. Let \( A_{k,m}(\mathbf{n}) \) the number of packets that arrive at \( (m, m + 1] \). We assume Bernoulli arrival\(^2\) with the average number of arrivals being \( E(A_{k,m}(\mathbf{n})) = \lambda_k(\mathbf{n}) < \infty \) packets per slot. To keep the mathematical tractability of our model we consider a limited-state dependent queue-based transmission protocol. In particular, we assume that there exist two positive constants, say \( N_1, N_2 \), such that they split the state space \( S \) in

\(^1\)The analysis that follows remains valid even for systems where the nodes have multi-packet reception capabilities, but with (slightly) more complicated expressions.

\(^2\)We can applied a similar methodology for even more general arrival process.
Collision channel model

1. Collision channel model
2. Network state-aware transmission

\[ \lambda_1(n_1, n_2) \rightarrow 1 \quad a_1(n_1, n_2) \]

\[ \lambda_2(n_1, n_2) \rightarrow 2 \quad a_2(n_1, n_2) \]

**Figure 1:** The model.

Four non-intersecting subsets

\[ S_0 = \{(n_1, n_2); n_1 < N_1, n_2 < N_2\}, \quad S_1 = \{(n_1, n_2); n_1 \geq N_1, n_2 < N_2\}, \]
\[ S_2 = \{(n_1, n_2); n_1 < N_1, n_2 \geq N_2\}, \quad S_3 = \{(n_1, n_2); n_1 \geq N_1, n_2 \geq N_2\}, \]

and assume that for \( k = 1, 2, \)

\[ a_k(n) = \begin{cases} 
    a_k(N_1, N_2), & \text{if } n \in S_1, \\
    a_k(n_1, N_2), & \text{if } n \in S_2, \\
    a_k(N_1, n_2), & \text{if } n \in S_3, \\
    a_k(n_1, n_2), & \text{if } n \in S_4, 
\end{cases} \]

\[ \lambda_k(n) = \begin{cases} 
    \lambda_k(N_1, n_2), & \text{if } n \in S_1, \\
    \lambda_k(n_1, N_2), & \text{if } n \in S_2, \\
    \lambda_k(N_1, n_2), & \text{if } n \in S_3, \\
    \lambda_k(n_1, n_2), & \text{if } n \in S_4. 
\end{cases} \]

The one step transition probabilities from \( n = (n_1, n_2) \) to \( (n_1 + i, n_2 + j) \), say \( p_{i,j}(n) \), where, \( n \in S, \)

\[ i, j = -1, 0, 1, \]

are given by:

\[ p_{1,0}(n) = (\bar{a}_1(n)a_2(n) + a_1(n)a_2(n))d_{1,0}(n), \]
\[ p_{0,1}(n) = (\bar{a}_1(n)a_2(n) + a_1(n)a_2(n))d_{0,1}(n), \]
\[ p_{1,1}(n) = (\bar{a}_1(n)a_2(n) + a_1(n)a_2(n))d_{1,1}(n), \]
\[ p_{-1,0}(n) = a_1(n)a_2(n)d_{0,1}(n), \]
\[ p_{1,-1}(n) = a_2(n)ar{a}_1(n)d_{1,0}(n), \]
\[ p_{-1,1}(n) = a_1(n)a_2(n)d_{0,0}(n), \]
\[ p_{0,-1}(n) = a_2(n)a_1(n)d_{0,0}(n), \]
\[ p_{0,0}(n) = (\bar{a}_1(n)a_2(n) + a_1(n)a_2(n))d_{0,0}(n) + \bar{a}_1(n)a_2(n)d_{0,1}(n) + a_1(n)a_2(n)d_{1,0}(n), \]

where

\[ d_{i,j}(n) = \begin{cases} 
    \lambda_1(n)\bar{\lambda}_2(n), & i = 1, j = 0, \\
    \lambda_2(n)\bar{\lambda}_1(n), & i = 0, j = 1, \\
    \lambda_1(n)\bar{\lambda}_2(n), & i = 1, j = 1, \\
    \bar{\lambda}_1(n)\lambda_2(n), & i = 0, j = 0. 
\end{cases} \]

and \( \bar{a}_k(n) = 1 - a_k(n), \quad \bar{\lambda}_k(n) = 1 - \lambda_k(n), \quad k = 1, 2, \quad a_1(0, n_2) = 0 = a_2(n_1, 0). \)

3 **Ergodicity conditions**

Note that our model is described by a two-dimensional Markov with limited state dependency, or equivalently with partial spatial homogeneity. Condition for ergodicity for such random walks in the positive quadrant has been investigated in [14, 35].
For $Q_1(m) > N_1$ (resp. $Q_2(m) > N_2$) the component $Q_2(m)$ (resp. $Q_1(m)$) evolves as a one-dimensional RW. Denote its corresponding stationary distribution by $\psi := (\psi_1, \psi_2, ...)$ (resp. $\phi := (\phi_1, \phi_2, ...)$). Consider now the mean drifts

$$
\gamma_{n_2} := E(Q_1(m + 1) - Q_1(m)|Q(m) = (n_1, n_2)) = \lambda_1(n_1, n_2) - a_1(n_1, n_2)\bar{a}_2(n_1, n_2), \forall n_1 > N_1,
$$

$$
\delta_{n_1} := E(Q_2(m + 1) - Q_2(m)|Q(m) = (n_1, n_2)) = \lambda_2(n_1, N_2) - a_2(n_1, N_2)\bar{a}_1(n_1, N_2), \forall n_2 > N_2.
$$

Since $a_k(n) := a_k$, $\lambda_k(n) = \lambda_k$, $k = 1, 2$, for $n \in S_3 = \{(n_1, n_2) : n_1 \geq N_1, n_2 \geq N_2\}$,

$$
\gamma_{n_2} := \gamma = \lambda_1 - a_1\bar{a}_2, n_2 > N_2,
$$

$$
\delta_{n_1} := \delta = \lambda_2 - a_2\bar{a}_1, n_1 > N_1.
$$

Then, the following theorem provides necessary and sufficient conditions for ergodicity [14]. For a similar approach, see [35].

**Theorem 1.** 1. If $\lambda_1 < a_1\bar{a}_2$, $\lambda_2 < a_2\bar{a}_1$, $Q(m)$ is

(a) ergodic if

$$
\lambda_1(1 - \sum_{k=0}^{N_2-1} \psi_k) < a_1\bar{a}_2(1 - \sum_{k=0}^{N_2-1} \psi_k) - \sum_{k=0}^{N_2-1} \gamma_k \psi_k, \text{ and}
$$

$$
\lambda_2(1 - \sum_{k=0}^{N_1-1} \phi_k) < a_2\bar{a}_1(1 - \sum_{k=0}^{N_1-1} \phi_k) - \sum_{k=0}^{N_1-1} \delta_k \phi_k.
$$

(b) transient if

$$
\lambda_1(1 - \sum_{k=0}^{N_2-1} \psi_k) > a_1\bar{a}_2(1 - \sum_{k=0}^{N_2-1} \psi_k) - \sum_{k=0}^{N_2-1} \gamma_k \psi_k, \text{ or}
$$

$$
\lambda_2(1 - \sum_{k=0}^{N_1-1} \phi_k) > a_2\bar{a}_1(1 - \sum_{k=0}^{N_1-1} \phi_k) - \sum_{k=0}^{N_1-1} \delta_k \phi_k.
$$

\[\text{Note also that for } N_1 = N_2 = 1, \text{ Theorem 1 coincides with the well known ergodicity result presented in Theorem 3.3.1 in [11].}\]
2. If $\lambda_1 \geq a_1 \bar{a}_2$, $\lambda_2 < a_2 \bar{a}_1$, $Q(m)$ is
   (a) ergodic if
   \[ \lambda_1 (1 - \sum_{k=0}^{N_2-1} \psi_k) < a_1 \bar{a}_2 (1 - \sum_{k=0}^{N_2-1} \psi_k) - \sum_{k=0}^{N_2-1} \gamma_k \psi_k. \]
   (b) transient if
   \[ \lambda_1 (1 - \sum_{k=0}^{N_2-1} \psi_k) > a_1 \bar{a}_2 (1 - \sum_{k=0}^{N_2-1} \psi_k) - \sum_{k=0}^{N_2-1} \gamma_k \psi_k, \]
   or when $\lambda_1 > a_1 \bar{a}_2$ and $\lambda_1 (1 - \sum_{k=0}^{N_2-1} \psi_k) = a_1 \bar{a}_2 (1 - \sum_{k=0}^{N_2-1} \psi_k) - \sum_{k=0}^{N_2-1} \gamma_k \psi_k$.

3. If $\lambda_1 < a_1 \bar{a}_2$, $\lambda_2 \geq a_2 \bar{a}_1$, $Q(m)$ is
   (a) ergodic if
   \[ \lambda_2 (1 - \sum_{k=0}^{N_1-1} \phi_k) < a_2 \bar{a}_1 (1 - \sum_{k=0}^{N_1-1} \phi_k) - \sum_{k=0}^{N_1-1} \delta_k \phi_k. \]
   (b) transient if
   \[ \lambda_2 (1 - \sum_{k=0}^{N_1-1} \phi_k) > a_2 \bar{a}_1 (1 - \sum_{k=0}^{N_1-1} \phi_k) - \sum_{k=0}^{N_1-1} \delta_k \phi_k, \]
   or when $\lambda_2 > a_2 \bar{a}_1$ and $\lambda_2 (1 - \sum_{k=0}^{N_1-1} \phi_k) = a_2 \bar{a}_1 (1 - \sum_{k=0}^{N_1-1} \phi_k) - \sum_{k=0}^{N_1-1} \delta_k \phi_k$.

4. If $\lambda_1 \geq a_1 \bar{a}_2$, $\lambda_2 \geq a_2 \bar{a}_1$, $Q(m)$ is transient.

**Proof 1** The proof is based on the construction of quadratic Lyapunov functions following the lines in [??].

### 4 Preparatory analysis and functional equations

Assume hereon that the system is stable, and let the stationary probabilities

\[ \pi_n := \pi_{n_1, n_2} = \lim_{m \to \infty} P(Q_1(m) = n_1, Q_2(m) = n_2). \]

Then, the equilibrium equations are given below

\[ \pi_n = \pi_n \left[ \bar{a}_1(n) a_1(n) + \frac{a_1(n) a_2(n)d_0(n)}{d_0(n) + a_1(n) a_2(n)d_0(n) + a_2(n) a_1(n)d_1(n)} \right] \]
\[ + \sum_{k_1=1}^{n_1+1} \sum_{k_2=n_2+1}^{n_2+1} \pi_{k_1} a_1(k) \bar{a}_2(k) \left[ d_{n_1+1-k_1, n_2+1-k_2}(k) \right], \]
\[ + \pi_0 \left[ d_{0,0}(0) 1_{\{n_1=0, n_2=0\}} + d_{1,0}(0) 1_{\{n_1=1, n_2=0\}} + d_{0,1}(0) 1_{\{n_1=0, n_2=1\}} + d_{1,1}(0) 1_{\{n_1=1, n_2=1\}} \right], \]

such that $\sum_{n_1 \geq 0} \sum_{n_2 \geq 0} \pi_{n_1, n_2} = 1$, $\pi_{n_1, -1} = 0 = \pi_{-1, n_2}$.

Consider first the equations from [??] that corresponds to the region $S_0$. There are $N_1 \times N_2$ equations

\[ (n_1 = 0, 1, ..., N_1-1, n_2 = 0, 1, ..., N_2-1) \]

involving $(N_1 + 1) \times (N_2 + 1) - 1$ unknown probabilities

\[ (\pi_{n_1, n_2}, n_1 = 0, 1, ..., N_1, n_2 = 0, 1, ..., N_2) \]

but not $\pi_{N_1, N_2}$.

In the following we focus on the equations associated with the region $S_1 (n_1 = N_1, N_1 + 1, ..., n_2 = 0, 1, ..., N_2 - 1)$. Let,

\[ g_{n_2}(x) = \sum_{n_1=N_1}^{\infty} \pi_{n_1, n_2} x^{n_1-N_1}, \]

\[ n_2 = 0, 1, ..., \]

\[ n_2 = 0, 1, ..., \]

\[ \text{and} \]

\[ \text{and} \]

\[ \text{and} \]

\[ \text{and} \]
and remember that $a_k(\pi) = a_k(N_1, n_2)$, $\lambda_k(\pi) = \lambda_k(N_1, n_2)$, $n_1 \geq N_1$. Then, we obtain from (3) the following relations,

$$
\begin{cases}
    f_2(N_1, 0, x)g_0(x) - f_3(N_1, 1, x)g_1(x) = b_0(x), \\
    -f_1(N_1, n_2 - 1, x)g_{n_2-1}(x) + f_2(N_1, n_2, x)g_{n_2}(x) - f_3(N_1, n_2 + 1, x)g_{n_2+1}(x) \\
    = b_{n_2}(x), \
\end{cases}
$$

where, for $n_2 = 0, 1, 2, ...$,

$$
\begin{align*}
    f_1(N_1, n_2, x) &= x^2d_{1,1}(N_1, n_2)[\bar{a}_1(N_1, n_2)a_2(N_1, n_2) + a_1(N_1, n_2)a_2(N_1, n_2)] \\
    &+ x^2d_{0,1}(N_1, n_2)(\bar{a}_1(N_1, n_2)a_2(N_1, n_2) + a_1(N_1, n_2)a_2(N_1, n_2)) \\
    &+ a_1(N_1, n_2)a_2(N_1, n_2)d_{1,1}(N_1, n_2) + a_1(N_1, n_2)a_2(N_1, n_2)d_{0,0}(N_1, n_2), \\
    f_2(N_1, n_2, x) &= x[1 - a_1(N_1, n_2)a_2(N_1, n_2)d_{1,0}(N_1, n_2) - \bar{a}_1(N_1, n_2)a_2(N_1, n_2)d_{0,1}(N_1, n_2) \\
    &- \bar{a}_1(N_1, n_2)a_2(N_1, n_2)d_{0,0}(N_1, n_2) + a_1(N_1, n_2)a_2(N_1, n_2)d_{0,0}(N_1, n_2) \\
    &+ x^2[a_2(N_1, n_2)a_1(N_1, n_2)d_{1,1}(N_1, n_2) + \bar{a}_1(N_1, n_2)a_2(N_1, n_2) \\
    &+ a_1(N_1, n_2)a_2(N_1, n_2)] - a_1(N_1, n_2)a_2(N_1, n_2)d_{0,0}(N_1, n_2), \\
    f_3(N_1, n_2, x) &= x\bar{a}_1(N_1, n_2)a_2(N_1, n_2)(d_{0,0}(N_1, n_2) + d_{1,0}(N_1, n_2)x) \\
\end{align*}
$$

$$
b_{n_2}(x) = \frac{\pi_{N_1-1,n_2-1}xd_{1,1}(N_1, n_2)(\bar{a}_1(N_1, n_2)a_2(N_1, n_2) + a_1(N_1, n_2)a_2(N_1, n_2)) + \pi_{N_1-1,n_2+1}xd_{1,0}(N_1, n_2)}{\pi_{N_1,n_2}} \times \bar{a}_1(N_1, n_2 + 1)a_2(N_1, n_2 + 1) - \pi_{N_1,n_2-2}a_1(N_1, n_2 - 1)a_2(N_1, n_2 - 1)d_{0,1}(N_1, n_2) \\
- \pi_{N_1,n_2}a_1(N_1, n_2)a_2(N_1, n_2)d_{0,0}(N_1, n_2) + \pi_{N_1-1,n_2}x(\bar{a}_1(N_1, n_2)a_2(N_1 - 1, n_2) \\
+ a_1(N_1 - 1, n_2)a_2(N_1 - 1, n_2))d_{1,0}(N_1, n_2) + \bar{a}_1(N_1 - 1, n_2)a_2(N_1 - 1, n_2)d_{1,1}(N_1, n_2)].
$$

Relations (4) allow to express $g_{n_2}(x)$, $n_2 = 1, 2, ...$, in terms of $g_0(x)$ and $b_0(x), ..., b_{n_2-1}(x)$. Indeed, starting from the first in (4) and solving recursively, we conclude that\footnote{In [13], similar terms were derived by solving a linear system of equations (similar to the second in [4]).}

$$
g_{n_2}(x) = e_{n_2}(x)g_0(x) + t_{n_2}(x), \quad n_2 = 1, 2, ..., 
$$

where, for $n_2 = 1, 2, ...$,

$$
e_{n_2}(x) = \frac{f_2(N_1, n_2-1, x)e_{n_2-1}(x) - f_1(N_1, n_2-2, x)e_{n_2-2}(x)}{f_3(N_1, n_2, x)}, \\
t_{n_2}(x) = \frac{f_2(N_1, n_2-1, x)t_{n_2-1}(x) - f_1(N_1, n_2-2, x)t_{n_2-2}(x) - b_{n_2-1}(x)}{f_3(N_1, n_2, x)},
$$

where $e_{-1}(x) = 0 = t_{-1}(x) = t_0(x)$ and $e_0(x) = 1$.

Note that up to, and including $n_2 = N_2$, no other new probabilities appear, except those introduced in the equations for $S_0$, i.e., $\pi_{n_1, n_2}$, $n_1 = 0, 1, ..., N_1$, $n_2 = 0, 1, ..., N_2$, but not $\pi_{N_1, N_2}$.

Clearly, region $S_2$ is a mirror image of $S_1$, where index 1 becomes 2 and component $n_1$ becomes $n_2$. Similarly, denote,

$$
h_{n_1}(y) = \sum_{n_2=N_2}^{\infty} \pi_{n_1, n_2}y^{n_2-N_2}, \quad n_1 = 0, 1, ....
$$

By repeating the procedure, we can obtain $h_{n_1}(y)$, as a function of $h_0(y)$. In particular,

$$
h_{n_1}(y) = \tilde{e}_{n_1}(y)h_0(y) + \tilde{t}_{n_1}(y), \quad n_1 = 1, 2, ..., 
$$
where \( \tilde{c}_{n_1}(y) \), are known polynomials and \( \tilde{t}_{n_1}(y) \) contain unknown probabilities, but now new terms except those introduced in the equations for \( S_0 \), i.e., \( \pi_{n_1,n_2}, n_1 = 0,1,\ldots,N_1, n_2 = 0,1,\ldots,N_2 \), but not \( \pi_{N_1,N_2} \).

We now focus on the region \( S_3 \) and denote,

\[
g(x,y) = \sum_{n_1=N_2}^{\infty} \sum_{n_2=N_2}^{n_1} \pi_{n_1,n_2} x^{n_1-N_1} y^{n_2-N_2}
= \sum_{n_2=N_2}^{\infty} g_{n_2}(x) y^{n_2-N_2} = \sum_{n_1=N_1}^{\infty} h_{n_1}(y) x^{n_1-N_1}.
\]

Using (4), noting that \( f_i(N_1,n_2,x) := f_i(N_1,N_2,x) \) for \( n_2 \geq N_2 \), and having in mind (5), (6), we finally obtain after lengthy calculations,

\[
R(x,y)g(x,y) = A(x,y)g_0(x) + B(x,y)h_0(y) + C(x,y),
\]

where, for \( a_k(n_1,n_2) = a_k(N_1,N_2) := a_k, \lambda_k(n_1,n_2) = \lambda_k(N_1,N_2) := \lambda_k, k = 1,2, (n_1,n_2) \in S_3,
\]

\[
R(x,y) = xy - D(x,y)[xy + a_1 \tilde{a}_2 y(1-x) + \tilde{a}_1 a_2 x(1-y)] := xy - \Psi(x,y),
\]

and,

\[
D(x,y) = (\tilde{\lambda}_1 + \lambda_1 x)(\tilde{\lambda}_2 + \lambda_2 y),
A(x,y) = y f_1(N_1,N_2-1,x)e_{N_2-1}(x) - f_3(N_1,N_2,x)e_{N_2}(x),
B(x,y) = x \tilde{f}_1(N_1-1,N_2,y) \tilde{e}_{N_1-1}(y) - \tilde{f}_3(N_1,N_2,y) \tilde{e}_{N_1}(y),
C(x,y) = K(\pi_{N_1-1,N_2-1},\pi_{N_1-1,N_2},\pi_{N_1,N_2-1},x,y) + y f_1(N_1,N_2-1,x) t_{N_2-1}(x)
- f_3(N_1,N_2,x) t_{N_2}(x) + x \tilde{f}_1(N_1-1,N_2,y) \tilde{t}_{N_1-1}(y) - \tilde{f}_3(N_1,N_2,y) \tilde{t}_{N_1}(y),
\]

and,

\[
\tilde{f}_1(n_1,N_2,y) = y^2 d_{1,1}(n_1,N_2)[\tilde{a}_1(n_1,N_2)\tilde{a}_2(n_1,N_2) + a_1(n_1,N_2)a_2(n_1,N_2)]
+ y[d_{1,0}(n_1,N_2)(\tilde{a}_1(n_1,N_2)\tilde{a}_2(n_1,N_2) + a_1(n_1,N_2)a_2(n_1,N_2))
+ \tilde{a}_1(n_1,N_2)a_2(n_1,N_2)d_{1,1}(n_1,N_2)]
+ a_1(n_1,N_2)a_2(n_1,N_2)d_{1,0}(n_1,N_2),
\]

\[
\tilde{f}_3(n_1,N_2,y) = y a_1(n_1,N_2)\tilde{a}_2(n_1,N_2)(d_{0,0}(n_1,N_2) + d_{0,1}(n_1,N_2)y),
\]

\[
K(\pi_{N_1-1,N_2-1},\pi_{N_1-1,N_2},\pi_{N_1,N_2-1},x,y) = \pi_{N_1-1,N_2-1}[\tilde{a}_1(N_1-1,N_2-1)\tilde{a}_2(N_1-1,N_2-1)
+ a_1(N_1-1,N_2-1)a_2(N_1-1,N_2-1)] - y d_{1,0}(N_1,N_2-1)\pi_{N_1-1,N_2-1}a_1(N_1-1,N_2-1)\tilde{a}_2(N_1-1,N_2-1)
- \pi_{N_1-1,N_2}a_1(N_1-1,N_2)a_2(N_1-1,N_2).
\]

Note that for the probabilities of states in region \( S_0 \) we have equations (3), for \( n_1 = 0,1,\ldots,N_1-1, n_2 = 0,1,\ldots,N_2-1 \). For those in region \( S_1 \), equations (5), (6), \( n_1 = 0,1,\ldots,N_2-1 \); for those in \( S_2 \), equations (6), \( n_1 = 0,1,\ldots,N_1-1 \). For region \( S_3 \), all unknown quantities are expressed in terms of

1. \( g_0(x), h_0(y) \),
2. the \( N_1 + N_2 \) probabilities, \( \pi_{N_1,n_2}, n_2 = 0,1,\ldots,N_2-1 \), and \( \pi_{n_1,N_2}, n_1 = 0,1,\ldots,N_1-1 \).

5 Solution of the fundamental functional equation

Our aim in this section is to determine \( g_0(x), h_0(y) \), in terms of the solution of a Riemann-Hilbert boundary value problem. Thus, as a first step, we have to investigate the zeros of the kernel equation \( R(x,y) = 0 \).
5.1 Kernel analysis

Note that the kernel $R(x, y)$ is a quadratic polynomial with respect to $x$, $y$. Indeed,

$$R(x, y) = \hat{a}(x)y^2 + \hat{b}(x)y + \hat{c}(x) = a(y)x^2 + b(y)x + c(y),$$

where,

$$\hat{a}(x) = -\lambda_1[\bar{a}_1a_2 - 2x(\bar{a}_1a_2 - \lambda_1a_1a_2) + \lambda_1x^2(a_1 + a_2)],$$

$$\hat{b}(x) = x[1 - \lambda_1\bar{a}_1a_2 - \lambda_1\bar{a}_1a_2 - \lambda_2\bar{a}_1a_2],$$

$$\hat{c}(x) = -\lambda_2\bar{a}_1a_2(\lambda_1 + \lambda_1x),$$

$$a(y) = -\lambda_1[\bar{a}_2a_2\bar{a}_1 + y(\bar{a}_2a_2 + \bar{a}_1a_2) + \lambda_2\bar{a}_1a_2] + \lambda_2y^2(a_1a_2 + a_1a_2),$$

$$b(y) = y[1 - \lambda_1\bar{a}_2a_2 - \lambda_1\bar{a}_2a_2 - \lambda_2\bar{a}_2a_2],$$

$$c(y) = -\lambda_1\bar{a}_2a_2(\lambda_1 + \lambda_1x).$$

In the following we provide some technical lemmas that are necessary for the formulation of a Riemann-Hilbert boundary value problem, the solution of which provides the unknown partial generating functions $g_0(x)$, $h_0(y)$.

**Lemma 1** For $|y| = 1$, $y \neq 1$, the kernel equation $R(x, y) = 0$ has exactly one root $x = X_0(y)$ such that $|X_0(y)| < 1$. For $\lambda_1 < a_1\bar{a}_2$, $X_0(1) = 1$. Similarly, we can prove that $R(x, y) = 0$ has exactly one root $y = Y_0(x)$, such that $|Y_0(x)| \leq 1$, for $|x| = 1$.

**Proof 2** For $|y| = 1$, $y \neq 1$ and $|x| = 1$ it is clear that $|\Psi(x, y)| < 1 = |xy|$. Thus, from Rouché’s theorem, $xy-\Psi(x, y)$ has exactly one zero inside the unit circle. Therefore, $R(x, y) = 0$ has exactly one root $x = X_0(y)$, such that $|x| < 1$. For $y = 1$, $R(x, 1) = 0$ implies $(x-1)[\lambda_1x + \lambda_1a_1a_2(1-x) - a_1a_2] = 0$. Therefore, for $y = 1$, and since $\lambda_1 < a_1\bar{a}_2$, the only root of $R(x, 1) = 0$ for $|x| \leq 1$, is $x = 1$.

**Lemma 2** The algebraic function $Y(x)$, defined by $R(x, Y(x)) = 0$, has four real branch points $0 < x_1 < x_2 < x_3 < x_4 < \infty$. Moreover, $D_2(x) < 0$, $x \in (x_1, x_2) \cup (x_3, x_4)$. Similarly, $X(y)$, defined by $R(X(y), y) = 0$, has also four real branch points $0 < y_1 < y_2 < 1 < y_3 < y_4 < \infty$, and $D_2(y) < 0$, $y \in (y_1, y_2) \cup (y_3, y_4)$.

**Proof 3** The proof is based on Lemma 2.3.8, pp. 27-28, [L3], and further details are omitted.

To ensure the continuity of the function of the two valued function $Y(x)$ (resp. $X(y)$) we consider the following cut planes: $\tilde{C}_x = C_x - ([x_1, x_2] \cup [x_3, x_4])$, $\tilde{C}_y = C_y - ([y_1, y_2] \cup [y_3, y_4])$, where $C_x$, $C_y$ the complex planes of $x$, $y$, respectively. In $\tilde{C}_x$ (resp. $\tilde{C}_y$), denote by $Y_0(x)$ (resp. $X_0(y)$) the zero of $R(x, Y(x)) = 0$ (resp. $R(Y(x), y) = 0$) with the smallest modulus, and $Y_1(x)$ (resp. $X_1(y)$) the other one. Define also the image contours, $\mathcal{L} = Y_0[\bar{x}_1, \bar{x}_2]$, $\mathcal{M} = X_0[\bar{y}_1, \bar{y}_2]$, where $\bar{u}, \bar{v}$ stands for the contour traversed from $u$ to $v$ along the upper edge of the slit $[u, v]$ and then back to $u$ along the lower edge of the slit. The following lemma shows that the mappings $Y(x)$, $X(y)$, for $x \in [x_1, x_2]$, $y \in [y_1, y_2]$, respectively, give rise to the smooth and closed contours $\mathcal{L}$, $\mathcal{M}$ respectively.

**Lemma 3** 1. For $y \in [y_1, y_2]$, the algebraic function $X(y)$ lies on a closed contour $\mathcal{M}$, which is symmetric with respect to the real line and written as a function of $\text{Re}(x)$, i.e.,

$$|x|^2 = m(\text{Re}(x)), |x|^2 \leq \frac{c(y_2)}{a(y_2)}.$$

Set $\beta_0 := \sqrt{\frac{c(y_2)}{a(y_2)}}$, $\beta_1 = -\sqrt{\frac{c(y_1)}{a(y_1)}}$ the extreme right and left point of $\mathcal{M}$, respectively.
2. For \( x \in [x_1, x_2] \), the algebraic function \( Y(x) \) lies on a closed contour \( \mathcal{L} \), which is symmetric with respect to the real line and written as a function of \( \text{Re}(y) \) as,

\[
|y|^2 = v(\text{Re}(y)), \quad |y|^2 \leq \frac{\hat{C}(x)}{\hat{a}(x)}.
\]

Set \( \eta_0 := \sqrt{\frac{\hat{C}(x)}{\hat{a}(x)}} \), \( \eta_1 = -\sqrt{\frac{\hat{C}(x)}{\hat{a}(x)}} \) the extreme right and left point of \( \mathcal{L} \), respectively.

**Proof 4** We will prove the part related to \( \mathcal{L} \). Similarly, we can also prove part 2. For \( x \in [x_1, x_2] \), \( D_x(x) = b^2(x) - 4\hat{a}(x)\hat{c}(x) \) is negative, so \( X_0(y) \) and \( X_1(y) \) are complex conjugates. Thus, \( |Y(x)|^2 = \frac{\hat{c}(x)}{\hat{a}(x)} = k(x) \). Note that,

\[
\frac{d}{dx}k(x) = \frac{x^2(p_{0,1}p_{1,-1} - p_{1,1}p_{0,-1}) + 2p_{1,-1}p_{1,1}x + p_{1,1}p_{0,-1}}{\hat{a}(x)^2},
\]

where \( p_{0,1}p_{1,-1} - p_{1,1}p_{0,-1} = \lambda_2^2 \lambda_3 \lambda_2 \lambda_3 a_1 a_2 a_2 > 0 \), and thus, \( k(x) \) is a non-negative function for \( x \in (0, \infty) \), which in turn implies that \( k(x) \leq k(x_2) \).

We can further solve \( |y(x)|^2 = \hat{c}(x)/\hat{a}(x) \) as a function of \( x \), and denote the solution that lies within \( [x_1, x_2] \) by \( \hat{x}(y) \), i.e.,

\[
\hat{x}(y) = \frac{p_{0,-1}p_{0,1}|y|^2 - \sqrt{(p_{0,1}|y|^2 - p_{0,-1})^2 - 4p_{1,-1,1}|y|^2[p_{1,1}|y|^2 - p_{1,-1}^2]}}{2[p_{1,1}|y|^2 - p_{1,-1}^2]}.
\]

So \( \hat{x}(y) \) is in fact the one-valued inverse function of \( y(x) \). For each \( y \in \mathcal{L} \) it also follows that

\[
\text{Re}(y(x)) = \frac{-b(\hat{x}(y))}{2\hat{a}(\hat{x}(y))}.
\]

Solving (14) as a function of \( |y(x)|^2 \) then gives an expression for \( |y(x)|^2 \) in terms of \( \text{Re}(y) \).

### 5.2 Formulation and solution of a Riemann-Hilbert boundary value problem

Therefore, for \( y \in \mathcal{D}_y = \{ y \in \mathcal{C} : |y| \leq 1, |X_0(y)| \leq 1 \} \),

\[
A(X_0(y), y)g_0(X_0(y)) + B(X_0(y), y)h_0(y) + C(X_0(y), y) = 0.
\]

(12)

For \( y \in \mathcal{D}_y - [y_1, y_2] \) both \( g(X_0(y)) \), \( h_0(y) \) are analytic and the right-hand side can be analytically continued up to the slit \( [y_1, y_2] \), or equivalently, for \( x \in \mathcal{M} \),

\[
A(x, Y_0(x))g_0(x) + B(x, Y_0(x))h_0(Y_0(x)) + C(x, Y_0(x)) = 0.
\]

(13)

Clearly, \( g_0(x) \) is holomorphic in \( D_x \{ x : |x| < 1 \} \), and continuous in \( D_x \{ x : |x| \leq 1 \} \). However, \( g_0(x) \) might have poles in \( S_x = G_M \cap \bar{D}_x \), where \( \bar{D}_x = \{ x : |x| > 1 \} \), and \( G_U \) denotes the interior domain bounded by the contour \( \mathcal{U} \). These poles (if exist) coincide with the zeros of \( A(x, Y_0(x)) \) in \( S_x \). For \( y \in [y_1, y_2] \), let \( X_0(y) = x \in \mathcal{M} \), and realize that \( Y_0(X_0(y)) = y \). Taking into account the (possible) poles of \( g_0(x) \) (say, \( \xi_1, ..., \xi_k \)), and noticing that \( h_0(Y_0(x)) \) is real for \( x \in \mathcal{M} \) we conclude in,

\[
\text{Re}(i\mathcal{U}f(x)) = w(x), \; x \in \mathcal{M},
\]

(14)

---

5. To improve the readability we set \( p_{i,j} := p_{i,j}(N_1, N_2) = p_{i,j}(n_1, n_2) \) for \( (n_1, n_2) \in S_3 \).

6. Some discussion about \( B(x, Y_0(x)) \neq 0, x \in \mathcal{M} \) is necessary.
Then, we have the following problem: Find a function $\tilde{x}$, satisfying

$$
\text{arg} \{ \tilde{x} \} = \gamma \text{ and } U_i \tilde{x} = \text{const.}
$$

where,

$$
U(x) = \frac{A(x, Y_0(x))}{\prod_{i=1}^k (x - \xi_i)^B(x, Y_0(x))}, \quad f(x) = \prod_{i=1}^k (x - \xi_i)g_0(x), \quad w(x) = \text{Im} \{ C(x, Y_0(x)) \}.
$$

In order to solve (14) we must first transform the problem from $\mathcal{M}$ to the unit circle $\mathcal{C}$, using conformal mappings. Let the mapping, $z = \gamma(x) : G_M \to G_C$, and its inverse $x = \gamma_0(z) : G_C \to G_M$.

Then, we have the following problem: Find a function $\tilde{T}(z) = f(\gamma_0(z))$ regular for $z \in G_C$, and continuous for $z \in \mathcal{C} \cup G_C$ such that,

$$
\text{Re}(iU(\gamma_0(z))\tilde{T}(z)) = w(\gamma_0(z)), \quad z \in \mathcal{C}.
$$

(15)

To obtain the conformal mappings, we need to represent $\mathcal{M}$ in polar coordinates, i.e., $\mathcal{M} = \{ x : x = \rho(\phi) \exp(i\phi), \phi \in [0, 2\pi] \}$. This procedure is described in detail in [3]. We briefly summarized the basic steps: Since $0 \in G_M$, for each $x \in \mathcal{M}$, a relation between its absolute value and its real part is given by $|x|^2 = m(Re(x))$ (see Lemma 3). Given the angle $\phi$ of some point on $\mathcal{M}$, the real part of this point, say $\delta(\phi)$, is the solution of $\delta - \cos(\phi)\sqrt{m(\delta)}, \phi \in [0, 2\pi]$. Since $\mathcal{M}$ is a smooth, egg-shaped contour, the solution is unique. Clearly, $\rho(\phi) = \frac{\delta(\phi)}{\cos(\phi)}$, and the parametrization of $\mathcal{M}$ in polar coordinates is fully specified.

Then, the mapping from $z \in G_C$ to $x \in G_M$, where $z = e^{i\phi}$ and $x = \rho(\psi(\phi))e^{i\psi(\phi)}$, satisfying $\gamma_0(0) = 0$ and $\gamma_0(z) = \gamma_0(\bar{z})$ is uniquely determined by (see [3], Section I.4.4),

$$
\gamma_0(z) = z \exp \left[ \frac{1}{2\pi} \int_0^{2\pi} \log \{ \rho(\psi(\omega)) \} \frac{\omega - z}{\omega - \bar{z}} d\omega \right], \quad |z| < 1,
$$

(16)

$$
\psi(\phi) = \phi - \int_0^{2\pi} \log \{ \rho(\psi(\omega)) \} \cot \left( \frac{\omega - \phi}{2} \right) d\omega, \quad 0 \leq \phi \leq 2\pi,
$$

i.e., $\psi(.)$ is uniquely determined as the solution of a Theodorsen integral equation with $\psi(\phi) = 2\pi - \psi(2\pi - \phi)$. Due to the correspondence-boundaries theorem, $\gamma_0(\bar{z})$ is continuous in $\mathcal{C} \cup G_C$.

The solution of the boundary value problem depends on its index $\chi = \frac{-1}{\pi} \{ \text{arg} \{ U(x) \} \}_{x \in \mathcal{M}}$, where $\{ \text{arg} \{ U(x) \} \}_{x \in \mathcal{M}}$, denotes the variation of the argument of the function $U(x)$ as $x$ moves along $\mathcal{M}$ in the positive direction, provided that $U(x) \neq 0, \, x \in \mathcal{M}$.

The solution of the problem defined in (14) is given by,

$$
g_0(\gamma_0(z)) = \prod_{i=1}^k (\gamma_0(z) - \xi_i)^{-1}e^{i\sigma(z)}z^\chi [iK + \frac{1}{2\pi i} \int_{|t|=1} e^{\omega_1(t)\delta(t)}t^{+\frac{e}{z}} dt], \quad z \in \mathcal{C}^+,
$$

(17)

where $K$ is a constant to be determined, $\sigma(z) = \frac{1}{2\pi i} \int_{|t|=1} (\arctan \frac{b_1(t)}{a_1(t)} - \chi \arg t) \frac{t^{+\frac{e}{z}} dt}{t^\chi}$, $\omega_1(z) = \text{Im}(\sigma(z))$, $\delta(z) = \frac{w(U(\gamma_0(z))}{U(\gamma_0(z))}$, $U(\gamma_0(z)) = b_1(z) + i\alpha_1(z)$. If $\chi \leq 0$ our problem has at most one linearly independent solution. When $\chi > 0$ $K$ can be determined from the solution to $g_0(0)$. If $\chi < 0$, then $K = 0$ and a solution exists if $\frac{1}{2\pi i} \int_{|t|=1} e^{\omega_1(t)}\delta(t)t^{-k-1} dt = 0$ for $k = 0, 1, ..., -\chi - 1$. Note that $g_0(x) = g_0(\gamma(\chi(x)))$. The rest of the procedure is summarized below:

1. Similarly, we obtain $h_0(y)$ by solving another Riemann-Hilbert problem, and substituting back in (7), we obtain $g(x, y)$.

2. Note that $g_0(x), \, h_0(y)$, are expressed in terms of $A(x, y), \, B(x, y)$ and $C(x, y)$. The first two are known, and the third one contains $N_1 + N_2$ unknown probabilities, i.e., $\pi_{N_1, N_2}$, $n_2 = 0, 1, ..., N_2 - 1$ and $\pi_{N_1, N_2}$, $n_1 = 0, 1, ..., N_1 - 1$. Thus, we need some additional equations.

3. Use [5], [6] to express the unknown probabilities in terms of the derivatives of $g_0(x), \, h_0(y)$ at point $0$, i.e.,

$$
n_2!\pi_{N_1, n_2} = \frac{d^{n_2}}{dx^{n_2}} \{ e_{n_2}(x)g_0(x) + t_{n_2}(x) \}, \quad x = 0, \quad n_2 = 1, ..., N_2,
$$

$$
n_1!\pi_{N_1, n_2} = \frac{d^{n_1}}{dy^{n_1}} \{ e_{n_1}(y)h_0(y) + \tilde{t}_{n_1}(y) \}, \quad y = 0, \quad n_1 = 1, ..., N_1.
$$

(18)
Finally, the last unknown term is $\pi_{N_1,N_2}$, which is found by
\[
1 = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \pi_{n_1,n_2} + \sum_{n_1=0}^{N_1-1} h_{n_1}(1) + \sum_{n_2=0}^{N_2-1} g_{n_2}(1) + g(1,1) \quad (19)
\]

6 Numerical example

As we have seen so far, in order to provide the exact information about the stationary joint queue length distribution at users’ queue we have firstly to solve a system of $(N_1 + 1) \times (N_2 + 1)$ linear equations.

1. $N_1 \times N_2$ of them refer to the states in region $S_0$.

2. $N_1 + N_2$ refer to the equations that correspond to the derivatives
\[
\begin{align*}
n_2!\pi_{n_1,n_2} &= \frac{d^{n_2}}{dx^{n_2}} [e_{n_2}(x)g_0(x) + t_{n_2}(x)]|_{x=0}, n_2 = 1, \ldots, N_2, \\
n_1!\pi_{n_1,n_2} &= \frac{d^{n_1}}{dy^{n_1}} [e_{n_1}(y)h_0(y) + \tilde{t}_{n_1}(y)]|_{y=0}, n_1 = 1, \ldots, N_1.
\end{align*}
\]

3. The normalizing equation (19). Moreover, note that each coefficient in the last $N_1 + N_2 + 1$ equations requires the evaluation of complex integrals of type (17). In order to numerically evaluate them, we have firstly to construct the conformal mappings. Note that in most of the cases we are not be able to obtain them explicitly. However, an efficient numerical approach was developed in [6], Sec. IV.1.1. Alternatively, since contours are close to ellipses, we can use the nearly circular approximation, [27]. Function $U(x)$ on which (14) is based, involves determinants of matrices whose elements are polynomials.

4. Solve the functional equation (7).

In the following we provide a simple numerical example to illustrate our theoretical findings. Set $N_1 = N_2 = 2$, and for let $||n|| = n_1 + n_2$, $a_k(n) = a_k ||n||$, $\lambda_k(n) = \lambda_k 2^{-||n||}$, $k = 1, 2$. In Figure 6 we observe the effect of system parameters on the average delay. In particular, in Fig. 6 (left) we can observe the increase on the average delay in queue 2 as a function of $\lambda_2$. As expected, by increasing also $\lambda_1$, the delay in queue 2 will also increase. Similar observations can be deduced by Fig. 6 (right), where the average delay in queue 1 is presented as a function of $\lambda_1$, $a_1$. Definitely, by increasing $a_1$, the delay in queue 1 can be handled as long as $\lambda_1$ remains in small values. However by increasing $\lambda_1$, we observe the increase on the delay, which becomes more apparent when we also increase $\lambda_2$.

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