Magnetic Monopole Generated by Spin Damping with Spin-Orbit Coupling

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Abstract. We investigate theoretically the magnetic monopole in magnetic systems. The hedgehog monopole emerges in frustrated magnetic materials. In addition, a novel magnetic monopole is induced by magnetization dynamics in the presence of the spin-orbit interaction. To derive such a magnetic monopole, we calculate an electric current analytically based on the quantum many-body theory. From the result, we define effective electric and magnetic fields which drive the electric current and finally we obtain the Maxwell's equations with the magnetic monopole contribution which these effective fields follow.

1. Introduction
The magnetic monopole was introduced by Dirac [1] as a unique particle arising from singularity. 't Hooft [2] and Polyakov [3] predicted the other magnetic monopole accompanied with no singularity. This magnetic monopole emerges when the U(1) electromagnetic symmetry breaks owing to the unified force having higher asymmetry of SU(5). In particle physics, such a magnetic monopole is created at the high energy more than $10^{17}$ GeV and therefore the evidence of the magnetic monopole has never been found. In condensed matter physics, magnetic monopoles are generated at much less energy (below 1 eV) than one of particle physics and so the experimental observation is possible. In recent years, the magnetic monopole in spin ice has been investigated quite actively in theory [4] and experiment [5]. The magnetic monopole at the surface of topological insulators was also predicted theoretically [6] and today magnetic monopoles in solids become one of the hot topics.

In the present paper, we will theoretically shown two different magnetic monopoles in magnetic materials. One is the hedgehog monopole which arises in topological magnetic textures. The other is the spin damping monopole created by the relaxation of magnetization precession with spin-orbit coupling. Calculating an electric current and its charge density analytically based on the quantum many-body theory, we will define effective electric and magnetic fields inducing the electric current and derive the Maxwell’s equations with magnetic monopoles.

2. Hedgehog monopole
The most well-known magnetic monopole in solids is the hedgehog monopole in frustrated magnetic materials [7], which originates from the SU(2) gauge symmetry breaking. For the hedgehog monopole, the key interaction is the exchange coupling between conduction electron and magnetization. We consider the electron system coupled with the magnetization $M(r,t)$
depending on the space \( r \) and time \( t \), and the total Hamiltonian is given as \( H = H_0 + H_{sd} \),
\[
H_0 = \frac{1}{2m} \int d^3r |p\psi^\dagger(r,t)|^2,
\]
\[
H_{sd} = -J \int d^3r M(r,t) \cdot \psi^\dagger(r,t) \sigma \psi(r,t),
\]
where \( \psi^\dagger (\psi) \) is the annihilation (creation) operator of conduction electrons, \( m \) is the electron mass, \( p \) denotes the electron's momentum, \( J \) is the exchange coupling constant, and \( \sigma \) represents the Pauli matrices. Here we diagonalize the exchange interaction \( H_{sd} \) as the spin quantization direction along the \( z \) axis, \( M \cdot U^\dagger \sigma U = |M|\sigma^z \) when \( M = |M|n \ (n^2 = 1) \), by the unitary transform, \( \psi = U\Psi \). The total Hamiltonian is reduced to
\[
H = \frac{1}{2m} \int d^3r |[p + eA^z(r,t)\sigma^z]\Psi^\dagger(r,t)|^2 - e \int d^3r A^a_i(r,t)\Psi^\dagger(r,t)\sigma^a\Psi(r,t) - J|M| \int d^3r r \cdot \sigma^z \Psi(r,t),
\]
where the SU(2) gauge potential is defined as \( A^a_i = -(i\hbar/e)U^\dagger \partial_i U \) (\( e \) represents the electric charge, \( \hbar \) is the Planck constant divided by \( 2\pi \), and \( \mu = t, x, y, z \) and \( a = x, y, z \) are the indices in the coordinate space and in the spin space, respectively) and is formed in terms of the unit vector of magnetization direction \( \mathbf{n} \),
\[
A^a_i = \frac{\hbar}{2e}(n \times \partial_i n)^a - A^z_i n^a.
\]
The conducting electrons feel the SU(2) fields whose strength is given as \( F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - (2e/\hbar)\epsilon_{abc}A^b_\mu A^c_\nu \), where \( \epsilon_{abc} \) is the antisymmetric tensor and the last term causes the deviation of the U(1) gauge symmetry and induces the hedgehog monopole. In fact, when this filed strength is projected onto the U(1) space in the adiabatic limit (\( J \) is large), we obtain the electromagnetic field tensor as
\[
F^{xz}_{\mu\nu} = \partial_\mu A^z_\nu - \partial_\nu A^z_\mu + \Phi_{\mu\nu},
\]
where \( \Phi_{\mu\nu} = -2e(\epsilon^{xz}A^x_\mu A^y_\nu - A^x_\mu A^y_\nu) \). The hedgehog monopole originates from the anomalous field strength \( \Phi_{\mu\nu} \) and its current \( (j_H) \) and density \( (\rho_H) \) are defined as
\[
j_H = \frac{1}{2} \epsilon_{\mu\nu\sigma} \partial_\mu \Phi_{\nu\sigma} = -\frac{3\hbar}{4e} \epsilon_{ijk} \hat{n} \cdot (\nabla_j \mathbf{n} \times \nabla_k \mathbf{n}),
\]
\[
\rho_H = \frac{1}{2} \epsilon_{\mu\nu\sigma} \partial_\mu \Phi_{\nu\sigma} = \frac{\hbar}{4e} \epsilon_{ijk} \nabla_i \mathbf{n} \cdot (\nabla_j \mathbf{n} \times \nabla_k \mathbf{n}).
\]
respectively. In this case, the Maxwell equations for the magnetic monopole are
\[
\nabla \times \mathbf{E}_H + \dot{\mathbf{B}}_H = -j_H, \\
\nabla \cdot \mathbf{B}_H = \rho_H.
\]
Here the electric field \( \mathbf{E}_H \) and the magnetic field \( \mathbf{B}_H \) are defined as
\[
\mathbf{E}_H = -\nabla i \hat{A}^z_i - \hat{A}^z_i = \frac{\hbar}{2e} \mathbf{n} \cdot (\hat{n} \times \nabla i \mathbf{n}),
\]
\[
\mathbf{B}_H = \epsilon_{ijk} \nabla_j \hat{A}^z_k = \frac{\hbar}{4e} \mathbf{n} \cdot (\nabla_j \mathbf{n} \times \nabla k \mathbf{n}),
\]
respectively. The derivation of this hedgehog monopole based on the SU(2) gauge field was carried out by Volovik [7]. The experimental realization has not been, however, achieved because the hedgehog monopole turns to vanish in usual magnets where the length of magnetization is constant as seen from the forms for \( j_H \) and \( \rho_H \). The hedgehog monopole arises only when the magnetization structure is a hedgehog state (Fig. 1).
Figure 1. Schematic illustration of hedgehog-state magnetization structure. The hedgehog monopole arises only when the magnetization profile is hedgehog.

2.1. Hedgehog monopole in weak exchange coupling regime

The hedgehog monopole emerges even if the exchange coupling \( J \) is small. We here consider the disordered electron system with weak exchange coupling and demonstrate the hedgehog monopole by calculating directly the effective electromagnetic field. The total Hamiltonian is given as

\[
H = H_0 + H_i + H_{sd},
\]

where \( H_i = \int d^3r \psi^\dagger(r, t) v_i(r) \psi(r, t) \) (\( v_i \) being the impurity potential) represents the spin-independent impurity scattering which gives rise to the elastic electron lifetime \( \tau \). In this system, the electric charge density is

\[
\rho(r, t) = -e \text{tr} \left\langle \psi^\dagger(r, t) \psi(r, t) \right\rangle,
\]

where the bracket represents the quantum expectation value and \( \text{tr} \) is the trace over the spin indices, and the electric current density is given as

\[
j_i(r, t) = \frac{eJ^3}{2m} (\nabla_r - \nabla_{r'}) \text{tr} G^<(r, t; r', t') |_{r' = r},
\]

where \( G^<(r, t; r', t') \equiv (i/\hbar) \langle \psi^\dagger_{r'}(r', t') \psi_s(r, t) \rangle \) (\( s \) and \( s' \) are spin indices) is the lesser component of the non-equilibrium Green’s function. Here, we assume slowly varying magnetization profile \( M_{q, \Omega} \) in space and time: the spatially smooth magnetization structure compared to the electron mean free path \( \ell \), \( q\ell \ll 1 \) (\( q \) is a wave number of magnetization texture), and the sufficiently slow dynamics of magnetization, \( \Omega \tau \ll 1 \) (\( \Omega \) is a frequency of magnetization dynamics).

To see the hedgehog monopole, it is enough to discuss the electric current at the third-order exchange coupling, \( J \). This contribution is diagrammatically shown in Fig. 2 and is calculated as

\[
j_i(r, t) = \frac{eJ^3}{\pi m V} \sum_{k, q, q', Q, \omega, \omega'} e^{-iQ \cdot r + iQ \cdot \Omega t} M_{q, \omega} \cdot (M_{q', \omega'} \times M_{Q - q - q', \Omega - \omega - \omega'})
\]

\[
\times \left[ \frac{i \hbar^3}{30m} q_i(Q \cdot q') \text{Im}(g_k^q)^4 - 2\tau^2 \omega q_i' |g_k^q|^2 \right] - D \nabla_i \rho(r, t),
\]

where \( g_k^q = [\varepsilon_F - (\hbar^2 k^2/2m) - (i\hbar/2\tau)]^{-1} \) is the advanced Green’s function (\( \varepsilon_F \) is the Fermi energy), \( \text{Im} \) means taking an imaginary component, \( V \) is the system volume, and \( D \) denotes the diffusion constant. The last term is the diffusive contribution arising from the vertex correction shown in Fig. 2(b) and the electric charge density, \( \rho \), is calculated as

\[
\rho = \frac{4e\nu J^3\tau^4}{\hbar m V} \nabla_i \langle M \cdot (M \times \nabla_i M) \rangle_D,
\]
Figure 2. Diagrammatic representation of the electric current $j$. Solid lines represent the conduction electron’s Green’s function and wavy lines are the interaction with magnetization $\mathbf{M}$. Diagrams (a) describes the contribution of an effective electric and magnetic fields. Diagram (b) contains the diffusion ladder (vertex corrections) denoted by the gray shaded oval and this contribution results in a diffusive current (a gradient of electric charge density).

where $\nu$ represents the density of states and $\langle \cdot \cdot \cdot \rangle_D$ is the average including the electron diffusion, which satisfies $(-D \nabla^2 + \partial_t)\langle F(r, t) \rangle_D = (1/\tau)F(r, t)$ ($F$ is an arbitrary function depending on space $r$ and time $t$). By summing over the wave vectors, we obtain the electric current,

$$
\mathbf{j} = -\frac{e\hbar^2 \nu J^3}{960 m^2 \epsilon_k V} \epsilon_{ijklm} \nabla_j \left[ \mathbf{M} \cdot (\nabla_l \mathbf{M} \times \nabla_m \mathbf{M}) \right] - \frac{4e
u J^3 \tau}{\hbar m V} \mathbf{M} \cdot \left( \dot{\mathbf{M}} \times \nabla_i \mathbf{M} \right) - D \nabla \rho. \tag{12}
$$

This result is rewritten in terms of the effective electric and magnetic fields $(\mathbf{E}_H$ and $\mathbf{B}_H)$ as $\mathbf{j} = (1/\mu)\nabla \times \mathbf{B}_H + \sigma_c \mathbf{E}_H - D \nabla \rho$, where $\mu$ is the magnetic permeability, $\sigma_c$ is the electric conductivity, and electromagnetic fields are defined as

$$
\mathbf{E}_{H,i} = \frac{J^3 \tau}{\epsilon_k} \mathbf{M} \cdot (\nabla_i \mathbf{M}),
$$

$$
\mathbf{B}_{H,i} = -\frac{J^3 \tau}{2e \hbar^2} \epsilon_{ijklm} \mathbf{M} \cdot (\nabla_j \mathbf{M} \times \nabla_k \mathbf{M}). \tag{13}
$$

These effective fields satisfy the Faraday's law and the Gauss's law with the magnetic monopole, $\nabla \times \mathbf{E}_H + \mathbf{B}_H = -j_H$ and $\nabla \cdot \mathbf{B}_H = \rho_H$, and the magnetic monopole contributions are

$$
\dot{j}_{H,i} = \frac{3J^3 \tau}{2e \hbar^2} \epsilon_{ijklm} \mathbf{M} \cdot (\nabla_j \mathbf{M} \times \nabla_k \mathbf{M}),
$$

$$
\rho_H = -\frac{J^3 \tau}{2e \hbar^2} \epsilon_{ijklm} \nabla_i \mathbf{M} \cdot (\nabla_j \mathbf{M} \times \nabla_k \mathbf{M}). \tag{14}
$$

The results [Eqs. (13) and (14)] are consistent with the one based on the gauge theory [Eqs. (8) and (6)], and therefore our derivation formalism is very useful.

3. Spin damping monopole

In this section, we will show the novel magnetic monopole induced by the spin damping with spin-orbit interaction, namely the spin damping monopole [8]. We consider two kinds of spin-orbit interaction. One is the Rashba interaction which appears at the interface and the surface of materials without inversion symmetry. The other is the one caused by random impurity potential. The present system is described as $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_i + \mathbf{H}_{sd} + \mathbf{H}_{so}$ and here the spin-orbit interaction $\mathbf{H}_{so}$ is

$$
\mathbf{H}_{so} = -\frac{1}{\hbar} \int d^3 r \psi^\dagger(r,t) \left[ \lambda_R \mathbf{\alpha}_R - \lambda_i \nabla \psi(r) \right] \cdot (\mathbf{p} \times \mathbf{\sigma}) \psi(r,t), \tag{15}
$$
where the first term is the Rashba effect ($\alpha_R$ being the uniform Rashba field), the last term arises from impurity potential $v_i$, and $\lambda$ represents the coupling constant (the subscript $R$ and $i$ characterize the Rashba and impurity-induced ones, respectively). The electric current in this system is composed of the normal component and the contribution of the anomalous velocity induced by spin-orbit interactions, and is defined as
\[
j(r, t) = e \tau \left( \frac{\hbar^2}{2m} (\nabla_r - \nabla_{r'}) + i [\lambda_R \alpha_R - \lambda_i \nabla v_i(r)] \times \sigma \right) G^< (r, t; r', t) \right|_{r'=r}.
\] (16)

It is well-known that the dynamic magnetization pumps an electric current when the spin-orbit interaction exists [9,10].

By treating spin-orbit couplings perturbatively besides the exchange coupling, we obtain the leading electric current in a similar manner to the previous section,
\[
\hat{j}(r, t) = \frac{1}{\mu} \nabla \times B_s + \sigma_c E_s - D \nabla \rho,
\] (17)
where the effective electric field ($E_s$) and magnetic field ($B_s$) is given as
\[
\begin{align*}
E_s &= -\frac{4e \nu \lambda_R J^2 \tau^2}{\sigma_c \hbar^2 V} \alpha_R \times (M \times \dot{M}), \\
B_s &= -\frac{16e \nu \lambda_i J^2 \varepsilon \tau^2}{3h^2 V} M \times \dot{M},
\end{align*}
\] (18)
respectively, and the electric charge density is
\[
\rho = \frac{4e \nu \lambda_R J^2 \tau^3}{\hbar^2 V} \nabla \cdot \left( \alpha_R \times (M \times \dot{M}) \right) \big|_D.
\] (19)

The effective fields satisfy the Maxwell’s equations with magnetic monopole contribution similar to the hedgehog monopole [Eq. (7)],
\[
\begin{align*}
\nabla \times E_s + \dot{B}_s &= -j_m, \\
\nabla \cdot B_s &= \rho_m,
\end{align*}
\] (20)
where the magnetic monopole current, $j_m$, and density, $\rho_m$, read
\[
\begin{align*}
j_m &= \frac{4e \nu \lambda_R J^2 \tau^2}{\sigma_c \hbar^2 V} \nabla \times \left[ \alpha_R \times (M \times \dot{M}) \right] + \frac{16e \nu \lambda_i J^2 \varepsilon \tau^2}{3h^2 V} \frac{\partial}{\partial t} (M \times \dot{M}), \\
\rho_m &= -\frac{16e \nu \lambda_i J^2 \varepsilon \tau^2}{3h^2 V} \nabla \cdot (M \times \dot{M}).
\end{align*}
\] (21)

This magnetic monopole arises from the spin damping ($M \times \dot{M}$), i.e., the spin damping monopole. The spin damping monopole does not need a specific non-coplanar magnetization structure such as a hedgehog. Therefore, it exists quite generally in magnetic materials.

The derivation of the spin damping monopole can not be carried out based on the gauge theory because the spin-orbit interaction breaks explicitly a SU(2) gauge invariance Our approach is only able to address this problem.
4. Conclusion
We have shown theoretically two different magnetic monopoles in magnetic systems. One is the hedgehog monopole created by the topological magnetic structure. The other is the spin damping monopole induced by magnetization dynamics when coupled to the spin-orbit interaction. We derive these magnetic monopoles by calculating an effective electromagnetic field for the electronic spins on the basis of the electron transport theory. Our formalism is very suitable to describe electromagnetic phenomena in magnetic material and it does not require any knowledge of the gauge theory. This is a natural consequence because a U(1) gauge invariance is equivalent to the electric charge conservation which is imposed in electron transport theory.

Magnetic monopoles are, thus, a common object in magnets. Since a current of magnetic monopoles induces a rotational electric field via the Ampère’s law of magnetic monopoles [shown in the first equation of Eqs. (7) and (20)], the magnetic monopoles act as an anomalous angular momentum source of rotational electron motion. Therefore, control of magnetic monopoles may hold the key to developing the next generation electronics.

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