LEVEL SETS AND THE UNIQUENESS OF MEASURES

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A result of Nymann is extended to show that a positive $\sigma$-finite measure with range an interval is determined by its level sets. An example is given of two finite positive measures with range the same finite union of intervals but with the property that one is determined by its level sets and the other is not.

0. Introduction

The purpose of this note is to extend some results of Leth [L], Malitz [M], and Nymann [N] and answer some questions raised in these papers. Those papers deal with a uniqueness property exhibited by positive measures whose range is an interval and used techniques from real analysis. The main idea in this paper to change from a purely real analysis approach to the problems to a functional analytic one. This approach reveals and clarifies the issues involved.

The basic problem is the following. Suppose that $\mu$ is a $\sigma$-finite signed measure on some measurable space $(\Omega, \mathcal{B})$. We will say that a signed measure $\nu$ on $(\Omega, \mathcal{B})$ satisfies $(\mathcal{L})$ if for every $A$ and $B \in \mathcal{B}$ such that $\mu(A) = \mu(B) \neq \pm \infty$, $\nu(A) = \nu(B) \neq \pm \infty$.

Question 1. Under what assumptions on $\mu$ is it true that if $\nu$ satisfies $(\mathcal{L})$, $\nu = \alpha \mu$ for some $\alpha \in \mathbb{R}$?

We will also consider the stronger condition $(\mathcal{O})$ which requires that condition $(\mathcal{L})$ hold and for every $A$ and $B \in \mathcal{B}$ such that $\mu(A) \leq \mu(B)$, $\nu(A) \leq \nu(B)$.

Question 2. Under what assumptions on $\mu$ is it true that if $\nu$ satisfies $(\mathcal{O})$, $\nu = \alpha \mu$ for some $\alpha \in \mathbb{R}$?

As a convenience in stating results we will say that $\mu$ is uniquely determined by $(\mathcal{L})$, respectively, $(\mathcal{O})$, if $\mu$ satisfies some property $P$ and for any $\nu$ satisfying $(\mathcal{L})$, respectively, $(\mathcal{O})$, $\nu$ is a constant times $\mu$.

The condition $(\mathcal{L})$ was used in [N] where it was shown that for finite positive measures with range equal to an interval that $\mu$ was uniquely determined by $(\mathcal{L})$. The symmetric form of $(\mathcal{O})$ was considered by Leth for the case of positive purely atomic measures such that for each $\varepsilon > 0$, there are only finitely many atoms of mass greater than $\varepsilon$. He called measures satisfying the symmetric form of $(\mathcal{O})$ sympathetic. This condition arose from some problems in qualitative measure theory. (See the references in [L] for more on this.) Malitz [M] showed that for this same...
class of atomic measures the symmetric form of \((L)\) implies the symmetric form of \((O)\).

It is easy to give examples where even under the symmetric form of \((O)\) \(\mu\) is not uniquely determined. As pointed about by Leth \([L]\), if \(0 < r < \frac{1}{2}\) and \(\mu\) is purely atomic with atoms \(\{A_n : n = 0, 1, ...\}\) such that \(\mu(A_n) = r^n\), there are no sets \(A \neq B\) such that \(\mu(A) = \mu(B) < \infty\). In order to generalize this observation, Leth makes the following definition. An atom \(A\) is a bully if

\[
\sup \left\{ \sum \mu(B_j) : (B_j) \subset \mathcal{B}, B_j \cap B_k = \emptyset, j \neq k \text{ and } \mu(B_j) < \mu(A) \text{ for all } j \right\} < \mu(A).
\]

In the example just given every atom is a bully. Clearly if \(\mu\) is purely atomic and every atom is a bully, the condition \((L)\) in vacuous and any purely atomic measure \(\nu\) for which every atom is a bully will satisfy \((O)\). On the other hand, Leth gives examples to show that bullies can be present but under \((L)\) \(\mu\) is uniquely determined. Leth (as improved by Malitz) also showed that if the space consists of a sequence of atoms with measures converging to 0 and there are no bullies then \(\mu\) is uniquely determined by the symmetric form of \((L)\). It is not hard to see that the no bullies condition is equivalent for positive measures to the condition that the range of the measure is an interval and therefore Nymann’s result can be viewed as an extension of Leth’s in the case of finite measures. Thus the question remains as to what conditions on \(\mu\) imply that \((L)\) or \((O)\) uniquely determines \(\mu\).

In this paper we will complete the work of \([N]\) by showing that a positive \(\sigma\)-finite measure with range an interval is uniquely determined by \((L)\). We will also prove some results on signed measures. Leth asked if it is possible to construct a measure so that if the atoms are listed in decreasing order every second atom is a bully, but the measure is uniquely determined. We will show that this is impossible. We also will give several examples which show that the range of measure is not a good indicator of whether the measure is uniquely determined. In particular we present an example of two positive measures \(\mu\) and \(\nu\) such that the range of \(\mu\) is a finite union of intervals, the range of \(\mu\) equals the range of \(\nu\), \(\mu\) is uniquely determined by \((L)\) but \(\nu\) is not uniquely determined by \((O)\).

We will use standard notation and terminology from real analysis and elementary functional analysis as may be found in \([F]\) or \([R]\). By a \(\sigma\)-finite signed measure \(\mu\) we mean an extended real valued measure on a measurable space \((\Omega, \mathcal{B})\) taking on only one of the two infinite values \(\pm \infty\) such that \(|\mu|\) is \(\sigma\)-finite. If \(\mu\) is a \(\sigma\)-finite signed measure we will denote the atoms of \(\mu\) by \(\{A_i\}\) and the complement of the union of the atoms by \(C\). If for every \(\varepsilon > 0\) there only finitely many atoms \(A_i\) with \(|\mu|(A_i) > \varepsilon\), we will assume that the atoms are ordered so that \(|\mu|(A_i) \geq |\mu|(A_{i+1})\) for all \(i\). \(L_p\) will always refer to \(L_p(\Omega, \mathcal{B}, |\mu|), 1 \leq p \leq \infty\). If \(R\) is a subset of a vector space \(X\) then \(\text{sp } R\) will denote the span of \(R\).
1. A Functional Analytic Approach.

Before we reformulate the problem in functional analytic terms, we will present two simple lemmas which will allow us to assume that the measure \( \mu \) is positive in most of our considerations. We would like to thank the referee who suggested these to us and made several other suggestions to improve the exposition.

**Lemma 1.1.** Suppose that \( \mu \) is a \( \sigma \)-finite signed measure on \((\Omega, \mathcal{B})\) and \( \Omega^+ \) and \( \Omega^- \) are the positive and negative sets of the Hahn decomposition of \( \mu \), i.e., \( \mu|_{\Omega^+} \geq 0 \), \( \mu|_{\Omega^-} \leq 0 \) and \( \Omega^+ \cup \Omega^- = \Omega \). Suppose that \( \nu \) is another \( \sigma \)-finite signed measure on \((\Omega, \mathcal{B})\) and let

\[
\nu'(A) = \nu(A \cap \Omega^+) - \nu(A \cap \Omega^-)
\]

for all \( A \in \mathcal{B} \). Then \( \nu' \) is a \( \sigma \)-finite signed measure on \((\Omega, \mathcal{B})\) such that \( \nu \) satisfies \((\mathcal{L})\), respectively, \((\mathcal{O})\), with respect to \( \mu \) if and only if \( \nu' \) satisfies \((\mathcal{L})\), respectively, \((\mathcal{O})\), with respect to \(|\mu|\).

**Proof.** Suppose that \((\mathcal{O})\) is satisfied for \( \mu \) and \( \nu \) and that \( A, B \in \mathcal{B} \) such that

\[
|\mu|(A) = \mu(A \cap \Omega^+) - \mu(A \cap \Omega^-) \leq \mu(B \cap \Omega^+) - \mu(B \cap \Omega^-) = |\mu|(B).
\]

Then

\[
\mu(A \cap \Omega^+) + \mu(B \cap \Omega^-) \leq \mu(B \cap \Omega^+) + \mu(A \cap \Omega^-).
\]

By \((\mathcal{O})\),

\[
\nu(A \cap \Omega^+) + \nu(B \cap \Omega^-) \leq \nu(B \cap \Omega^+) + \nu(A \cap \Omega^-)
\]

and hence

\[
\nu'(A) = \nu(A \cap \Omega^+) - \nu(A \cap \Omega^-) \leq \nu(B \cap \Omega^+) - \nu(B \cap \Omega^-) = \nu'(B).
\]

The proofs of the converse and the case of \((\mathcal{L})\) are similar. \( \square \)

Thus \( \mu \) is uniquely determined by \((\mathcal{L})\), respectively, \((\mathcal{O})\), if and only if \(|\mu|\) is uniquely determined by \((\mathcal{L})\), respectively, \((\mathcal{O})\).

Most of the conditions we will impose on \( \mu \) are on the range of \( \mu \) and the next lemma shows that the conditions are easily transferred to the range of \(|\mu|\).

**Lemma 1.2.** If \( \mu \) is a \( \sigma \)-finite signed measure on \((\Omega, \mathcal{B})\) and \( \Omega^+ \) and \( \Omega^- \) are the positive and negative sets of the Hahn decomposition of \( \mu \), then

\[
\text{range } |\mu| = \text{range } \mu - \mu(\Omega^-), \quad \text{if } \mu(\Omega^-) > -\infty,
\]

and

\[
\text{range } |\mu| = -\text{range } \mu + \mu(\Omega^+), \quad \text{if } \mu(\Omega^+) < +\infty.
\]

**Proof.** Assume that \( \mu(\Omega^-) > -\infty \). If \( A \in \mathcal{B} \),

\[
\mu(A) = \mu(A \cap \Omega^+) + \mu(A \cap \Omega^-) = \mu(A \cap \Omega^+) + \mu(\Omega^-) - \mu(\Omega^- \setminus A)
\]

\[
= |\mu|((A \cap \Omega^+) \cup (\Omega^- \setminus A)) + \mu(\Omega^-).
\]
and

\[ |\mu|(A) = \mu(A \cap \Omega^+) - \mu(A \cap \Omega^-) = \mu(A \cap \Omega^+) - \mu(\Omega^-) + \mu(\Omega^- \setminus A) = \mu((A \cap \Omega^+) \cup (\Omega^- \setminus A)) - \mu(\Omega^-). \]

Thus range $|\mu| = \text{range } \mu - \mu(\Omega^-)$. The other case is similar. \qed

We will now formulate the problem in functional analytic terms. Let $\mu$ be a positive $\sigma$-finite measure and consider the following linear subspace $X$ of $L_0$, the space of equivalence classes (a.e. $\mu$) of measurable functions on $(\Omega, \mathcal{B})$.

\[ X = \text{sp}\{1_A - 1_B : \mu(A) = \mu(B) \neq +\infty, A, B \in \mathcal{B}\}. \]

From a functional analytic viewpoint condition (\mathcal{L}) describes a subset of the kernel of $\mu$ as a linear functional on some subspace of $L_0$. Two possibilities for the subspace come immediately to mind, $L_1$ and $L_\infty$. If $\nu$ is a $\sigma$-finite signed measure satisfying (\mathcal{L}), then taking $A$ to be the empty set we see that $\nu \ll \mu$ and that if $g = dv/d\mu$ then $\int f g d\mu = 0$ for all $f \in X$. If $\nu$ has finite variation then $g \in L_1$ and the uniqueness question for condition (\mathcal{L}) is equivalent to determining if $X^{\sigma^*} = L_0^\infty$, the set of functions $h$ in $L_\infty$ with $\int h d\mu = 0$. If $dv/d\mu \in L_\infty$ then uniqueness under (\mathcal{L}) is equivalent to $X = L_1^0$. (Here the overbar denotes norm closure.)

If $\mu$ is $\sigma$-finite but not finite then we need to consider the space of functions with finite $L_1$ norm on each set of finite measure with the (locally convex) topology of $L_1$ convergence on sets of finite measure which we denote by $FL_1$. The dual of $FL_1$ is $FL_\infty = \{f \in L_\infty : \mu(\{f \neq 0\}) < \infty\}$. Note that $X \subset FL_\infty$ and (\mathcal{L}) determines $\mu$ uniquely if and only if the $\sigma(FL_\infty, FL_1)$ closure of $X$ in $FL_\infty$ is $FL_1^0$. Let $X_1$ denote the norm closure of $X$ in $L_1$ and let $X_\infty$ denote the $\sigma(FL_\infty, FL_1)$ closure of $X$ in $FL_\infty$. If $\mu$ is finite $X_\infty = X^{\sigma^*}$, i.e., the $\sigma(L_\infty, L_1)$ closure.

To see the power of this viewpoint consider the following question posed in [L]. Is it possible to construct a purely atomic measure with atoms $(A_i)$ such that $\mu(A_i) \downarrow 0$ and every second atom is a bully but (\mathcal{O}) does uniquely determine $\mu$? Actually we can show that neither $A_1$ and $A_2$ can be a bully and have $\mu$ uniquely determined. Indeed, $A_1$ cannot be a bully since we need only choose $\nu(A_1) > \mu(A_1)$ and $\nu(A_i) = \mu(A_i)$ for $i > 1$. If $A_1$ is not a bully but $A_2$ is, define a map from $L_\infty^0$ into $\mathbb{R}^2$ by $T f = (\int_{A_1} f d\mu, \int_{A_2} f d\mu)$. Then the range of $T$ is $\mathbb{R}^2$ but if $f \in X$ then $f = c(1_{A_1} - 1_{A_2}) + g$ where $\text{supp } g \subset \bigcup_{i \geq 2} A_i$. Thus the range of $T|_{X_\infty}$ is a subspace properly contained in $\mathbb{R}^2$. This immediately implies that (\mathcal{L}) does not uniquely determine $\mu$. It will follow from Corollary 1.7 (c) below that (\mathcal{O}) cannot determine $\mu$ uniquely, but we can see this directly as follows. Fix $b$, $0 < b < 1$ and define $\nu(A_i) = b\mu(A_i)$ for $i \geq 2$, $\nu(A_1) = \mu(A_1)$, and $\nu(A_2) = (1 - b)\mu(A_1) + b\mu(A_2)$. A simple computation shows that $\nu$ satisfies (\mathcal{O}). The point of the using functional analysis is that it often reduces questions about (\mathcal{L}) to counting dimensions.

Next we will explore the role of the non-atomic part of $\mu$.

**Lemma 1.3.** If $\mu$ is a finite purely non-atomic positive measure, then $X_1 = L_1^0$ and $X_\infty = L_0^\infty$.

**Proof.** First note that because $\mu$ is finite $L_\infty \subset L_1$ and thus by the Hahn-Banach theorem if $X_\infty = L_1^0$, $X_\infty = L_0^\infty$. Thus we need only consider the case of $X_1$.

Let $A \subset \Omega$ be such that $\mu(A) > 0$. From Corollary 1.7 (c) we see that $X_\infty = L_0^\infty$.

If $\mathcal{O}$ is finitely additive then it is enough to consider $A = \bigcup_{i=1}^n A_i$ where $\mu(A_i) > 0$ if $i > 1$ and $\mu(A_1) > (1 - b)\mu(A_1) + b\mu(A_2)$. A simple computation shows that $\nu$ satisfies (\mathcal{O}). The point of the using functional analysis is that it often reduces questions about (\mathcal{L}) to counting dimensions.
Observe that $L^0_\infty$ is a $w^*$ closed subspace of $L_\infty$ and thus the unit ball of $L^0_\infty$ is $w^*$ compact. It is easy to see that an extreme point of the unit ball of $L^0_\infty$ is of the form $1_A - 1_B$ where $\mu(A) = \mu(B)$, $A \cup B = \Omega$, and $A \cap B = \emptyset$. Thus all of these extreme points are in $X_\infty$ and it follows from the Krein-Milman theorem that $X_\infty$ is $w^*$ dense in $L^0_\infty$. □

The argument above used the extreme points which happened to be in $X$. If the measure space contains atoms then the extreme points of the ball of $L^0_\infty$ are of the form $1_D - 1_E + \lambda 1_A$ where $D$, $E$ and $A$ are disjoint, $D \cup E \cup A = \Omega$, $A$ is an atom, $|\lambda| \leq 1$, and $\lambda \mu(A) + \mu(D) = \mu(E)$. Whether or not such things are in $X_\infty$ is a matter of the combinatorics of the measures of the atoms, however, we can still say something about the non-atomic part.

**Proposition 1.4.** If $\mu$ is a $\sigma$-finite positive measure on $(\Omega, \mathcal{B})$, $\mu = \mu_c + \mu_a$, where $\mu_c$ has no atoms and $\mu_a$ is purely atomic, and $\nu$ is a $\sigma$-finite signed measure which satisfies (L) then $\frac{d\nu}{d\mu_c}$ is constant.

**Proof.** The hypothesis implies that $\nu \ll \mu$ and thus if $A \in \mathcal{B}$ such that $\mu(A)$ and $\nu(A)$ are finite, $g1_A \in L_1$, where $g = \frac{d\nu}{d\mu}$. Because both $\mu$ and $\nu$ are $\sigma$-finite we can find a sequence of sets $D_n \in \mathcal{B}$ such that $\mu(D_n)$ is finite and non-zero, $\cup_n D_n$ contains the complement of the union of the atoms, and $g|1_{D_n} \in L_\infty(\mu(D_n))$ for all $n$. Now by Lemma 1.3, $X \cap L^0_1(\mu_c|1_{D_n} \cup D_j)$ is dense in $L^0_1(\mu_c|1_{D_n} \cup D_j)$ for each $j$. Thus $g$ is constant on $D_1 \cup D_j$ for all $j$ and therefore $g$ is constant on $\cup D_n$, establishing the result. □

Proposition 1.4 shows that the difficulty really is in the atomic part of $\mu$. Next we show that uniqueness can be established provided it holds for a rich enough family of restrictions of $\mu$.

**Proposition 1.5.** Suppose that $\mu$ is a $\sigma$-finite measure and for every pair of disjoint sets $D$ and $E$ of finite measure there is a measurable set $F$ such that $X_\infty \cap FL_\infty(\mu|F) = FL^0_\infty(\mu|F)$, respectively, $X_1 \cap FL_1(\mu|F) = FL^0_1(\mu|F)$, and there exist $D'$ and $E'$ contained in $F$ such that $\mu(D) = \mu(D')$ and $\mu(E) = \mu(E')$. Then if $g \in FL_1$, respectively, $g \in L_\infty(\mu)$, and $g|X = 0$, then $g$ is constant.

**Proof.** Because $sp\{\mu(D)1_E - \mu(E)1_D : D, E$ disjoint and finite measure\} is dense in $FL^0_\infty$ and in $FL^0_1$, in either case it is sufficient to show that $\mu(E) \int_D gd\mu = \mu(D) \int_E gd\mu$ for all disjoint sets and $D$ and $E$ of finite measure. So given $D$ and $E$ let $\tilde{F}$, $\tilde{D}'$, and $\tilde{E}'$ satisfy the hypothesis of the proposition. Because $\mu(D')1_{E'} - \mu(E')1_{D'}$, $1_{D'} - 1_D$, and $1_{E'} - 1_E$ are in $X$, $\mu(E') \int_{\tilde{D}'} gd\mu = \mu(D') \int_{\tilde{E}'} gd\mu$, $\int_D gd\mu = \int_D gd\mu$, and $\int_{\tilde{E}'} gd\mu = \int_{\tilde{D}'} gd\mu$. Hence

$$\mu(E) \int_D gd\mu = \mu(E') \int_{\tilde{D}'} gd\mu = \mu(D') \int_{\tilde{E}'} gd\mu = \mu(D) \int_{\tilde{E}'} gd\mu.$$ □

In the papers [L] and [N] the following auxiliary function plays an important role. Let $W = \text{range of } \mu$ and suppose that $\nu$ satisfies (L). For each $w \in W$ define $f(w) = \nu(A)$ if $\mu(A) = w$. The absolute continuity of $\nu$ with respect to $\mu$ implies that $f(0) = 0$. The lemma below summarizes the important properties of $f$. Versions of a)–c) are in [L] and d) and e) are in [N]. Because the arguments are short we include them for the sake of completeness.
Lemma 1.6. Suppose that \( \mu \) is a \( \sigma \)-finite positive measure and \( \nu \) is a \( \sigma \)-finite signed measure on \((\Omega, \mathcal{B})\). If \( \nu \) satisfies (L), then with \( f \) as above

a) \( f \) is well defined and finite for all \( w \) finite in \( W \).

b) (O) is equivalent to the assertion that \( f \) is non-decreasing.

c) If there exists sets \( A \in \mathcal{B} \) of arbitrarily small but positive measure, and \( f'(w) \) exists at some \( w \in W \), then \( \lim_{\mu(A) \to 0} \frac{\nu(A)}{\mu(A)} \) exists and equals \( f'(w) \). (\( f' \) is defined only at limit points of \( W \).) In particular, if (O) holds and \( W \) has positive measure then \( \lim_{\mu(A) \to 0} \frac{\nu(A)}{\mu(A)} \) exists.

If range \( \mu \) is a finite interval, we have in addition

d) \( f \) is continuous.

e) If \( \nu \geq 0 \), then \( f \) is nondecreasing. Consequently, (O) is satisfied and \( f'(w) \) exists and is constant a.e. \( \mu \).

Proof. a) and b) are obvious. Let \( w \) be a point in the range of \( \mu \) where the derivative exists and suppose that \( \mu(B) = w \). If \( A_n \subset B \), \( \mu(A_n) \neq 0 \), and \( \mu(A_n) \to 0 \) then

\[
f'(w) = \lim_{\mu(A_n) \to 0} \frac{\nu(B \setminus A_n) - \nu(B)}{\mu(B \setminus A_n) - \mu(B)} = -\frac{\nu(A_n)}{\mu(A_n)}.
\]

If \( A_n \cap B = \emptyset \), \( \mu(A_n) \neq 0 \) and \( \mu(A_n) \to 0 \), then

\[
f'(w) = \lim_{\mu(A_n) \to 0} \frac{\nu(B \cup A_n) - \nu(B)}{\mu(B \cup A_n) - \mu(B)} = \frac{\nu(A_n)}{\mu(A_n)}.
\]

Finally if \( (A_n) \) is any sequence of sets with \( \mu(A_n) \to 0 \),

\[
\nu(A_n) - f'(w)\mu(A_n) = \nu(A_n \cap B) - f'(w)\mu(A_n \cap B) + \nu(A_n \cap B^c) - f'(w)\mu(A_n B^c)
\]

\[
= o(\mu(A_n \cap B)) + o(\mu(A_n \cap B^c)) = o(\mu(A_n)).
\]

If \( W \) has positive measure there must be a sequence of sets \( (A_n) \) with \( \mu(A_n) \to 0 \). Indeed, if not, \( \mu \) would be purely atomic with the measures of the atoms bounded away from 0 and thus the range of \( \mu \) would be a countable set. If (O) is satisfied then \( f' \) exists at almost every point of \( W \).

For d) we will show that continuity at \( \mu(B) = x \) follows from the absolute continuity of \( \nu \) with respect to \( \mu \). First if \( \mu_c \neq 0 \), by Proposition 1.4 there is a constant \( \kappa > 0 \) such that \( \nu = \kappa \mu_c \) or \( -\kappa \mu_c \) on \( \Omega \setminus \cup A_i \), where \( (A_i) \) is the sequence of atoms of \( \mu \). Given \( \varepsilon > 0 \) find \( \delta \), \( \varepsilon/(3\kappa) > \delta > 0 \) such that \( |\nu|(A) < \varepsilon/6 \) if \( \mu(A) < \delta \). Choose \( n \) such that \( \sum_{i \geq n} \mu(A_i) < \delta/2 \). Let \( \{r_j\} \) be the range of \( \mu|_{\cup_{i < n} A_i} \) and let \( K \) be the range of \( \mu|_{\cup_{i \geq n} A_i} \). Because \( K \) is compact there is a number \( \delta' > 0 \) such that if \( x \notin r_i + K \), \( (x - \delta', x + \delta') \cap r_i + K = \emptyset \). Let \( \rho = \min\{\delta, \varepsilon/(3\kappa), \delta'\} \).

If \( y \in W \) and \( |x - y| < \rho \), then for some \( i \), \( x \) and \( y \) are in \( r_i + K \). Therefore

\[
|f(x) - f(y)| = |f(r_i) + \nu(\cup_{i \in I} A_i) + \nu(D) - (f(r_i) + \nu(\cup_{i \in J} A_j) + \nu(E))|
\]

where \( I \cup J \subset \{i \geq n\}, D \cup E \subset C \), \( x = r_i + \mu(\cup_{i \in I} A_i \cup D) \), and \( y = r_i + \mu(\cup_{i \in J} A_j \cup E) \). Because \( \mu(\cup_{i \in I \cup J} A_i) < \delta/2 \) and \( |\nu(D - \nu(E)| = \kappa|\mu(D) - \mu(E)|\),

\[
|f(x) - f(y)| < \varepsilon/3 + \nu(D) + \nu(E)| < \varepsilon/3 + \nu[r_i - \rho] + \delta \leq \varepsilon/3 + \varepsilon/3 < \varepsilon.
\]
To see that $f$ is nondecreasing when $\mu, \nu \geq 0$, suppose that this is false. Then $f$ must have a local minimum at some point $x$, $0 < x < \mu(\Omega)$, and we may assume that $f(y) > f(x)$ for all $y$ in some interval $(x - \delta, x)$, $\delta > 0$. Because the range of a finite measure is compact, either $x = \mu(\bigcup_{i \in I} A_i) + \mu(D)$ with $D \subset C$ and $\mu(D) > 0$, or $x = \sup\{\sum_{i \in I} \mu(A_i) : \sum_{i \in I} \mu(A_i) < x\}$ for some infinite set $I \subset \mathbb{N}$. In either case $f(x) = f(y_n) + \nu(B_n)$ for some sequence $(y_n)$ increasing to $x$. Since $\nu(B_n) \geq 0$, $f(x)$ cannot be a strict local (left) minimum.

The results in [L] and [N] were obtained by showing that in the case in which $W$ is an interval $f'$ is constant and there is no singular part to $\nu$ and thus $f(x) = x$. The function $f$ is unsatisfactory for generalizing the results. We will usually use the Radon-Nikodym derivative of $\nu$ with respect to $\mu$ in our arguments. The Radon-Nikodym derivative is more indicative of the situation because it retains the information about the underlying sets whereas $f$ mostly reflects properties of the range.

**Corollary 1.7.**

a) If $\mu$ is a $\sigma$-finite measure with range of positive Lebesgue measure such that for every $\varepsilon > 0$, $\mu(\bigcup\{A : A \text{ is an atom and } \mu(A) > \varepsilon\}) < \infty$ and $\nu$ is a $\sigma$-finite signed measure satisfying (O), then $\frac{d\nu}{d\mu} \in L_\infty(\mu)$.

b) If $\mu$ is a positive measure with range a finite interval and $\nu$ is a positive measure satisfying (L), then $\frac{d\nu}{d\mu} \in L_\infty(\mu)$.

c) If $\mu$ is a positive $\sigma$-finite measure, $\nu$ satisfies (L) and $\frac{d\nu}{d\mu} \in L_\infty(\mu)$ and is not constant then there is a $\sigma$-finite positive measure $\nu'$ satisfying (L) and $\frac{d\nu}{d\mu}$ is not constant. If, in addition, the range of $\mu$ is a finite interval, then $\nu'$ satisfies (O).

**Proof.** By Proposition 1.4, $\frac{d\nu}{d\mu}$ is constant on $C$, the complement of the atoms, and by c) of Lemma 1.6, $\lim_{\mu(A) \to 0} \frac{\nu(A)}{\mu(A)}$ exists. Thus $\frac{\nu(A)}{\mu(A)}$ is bounded for all atoms with $\mu(A)$ sufficiently small. The hypothesis implies that there are only finitely many other atoms, and thus $\frac{d\nu}{d\mu}$ is bounded. Part b) follows from e) of Lemma 1.6 and a). If $\mu \geq 0$ and $\frac{d\nu}{d\mu} \in L_\infty(\mu)$ is not constant and (L) holds then there is a nonzero constant $\gamma$ such that $\nu' = \mu + \gamma \nu \geq 0$. Clearly $\nu'$ also satisfies (L). If the range of $\mu$ is an interval, by Lemma 1.6 e) $\nu'$ satisfies (O). □

Note that Corollary 1.7 c) says that if the range of $\mu$ is a finite interval, then the uniqueness question for signed measures with bounded Radon-Nikodym derivative is the same for (L) and (O).

2. Finite Signed Measures

Next we will prove some technical results which are useful for establishing uniqueness for the case of signed measures.
Lemma 2.1. Suppose that \( \mu \) is a finite positive measure with atoms \( \{ A_n \} \) arranged so that \( \mu(A_n) \geq \mu(A_{n+1}) \) and let \( C \) be the complement of the union of the atoms. If \( Z \) is a subspace of \( L_\infty \) which contains \( \{ 1_{A_n} - \mu(A_n)/(\mu(C \cup \cup_{j>n} A_j)1_{C \cup \cup_{j>n} A_j} : n = 1, 2, ... \} \)
and \( L_\infty^0(C, \mu) \), then \( \overline{Z}^{w^*} \supset L_\infty^0 \).

Proof. It is sufficient to show that

\[
\mathcal{D} = \{ \mu(A_1)1_{A_k} - \mu(A_k)1_{A_1} : k = 1, 2, ... \} \cup \{ \mu(C)1_{A_1} - \mu(A_1)1_C \} \subset \overline{Z}^{w^*}.
\]

Indeed if \( g \in L_1 \) and \( z(g) = 0 \) for all \( z \in \mathcal{D} \), then \( \mu(A_k)g(A_1) = \mu(A_1)g(A_k) \), for all \( k \) and \( g(A_1)\mu(C) = \int_C gd\mu \). Because \( L_\infty^0(C, \mu) \subset Z \), \( g \) is a constant \( \gamma \) on \( C \). It follows that \( g = \gamma \) on \( \Omega \).

For each \( n \) let \( B_n = C \cup \cup_{k>n} A_n \). In this notation we have from the hypothesis that \( 1_{A_n} - \mu(A_n)/(\mu(B_n)1_{B_n} \in Z \) for all \( n \). Now observe that since \( B_1 = A_2 \cup B_2 \),

\[
1_{A_1} - \mu(A_1)/\mu(B_1)1_{B_1} + \mu(A_1)/\mu(B_1)(1_{A_2} - \mu(A_2)/\mu(B_2)1_{B_2})
= 1_{A_1} - \mu(A_1)/\mu(B_2)1_{B_2} \in Z.
\]

An easy induction argument shows that \( 1_{A_1} - \mu(A_1)/\mu(B_k)1_{B_k} \in Z \) for all \( k \).

Therefore

\[
1_{A_1} - \mu(A_1)/\mu(A_n)1_{A_n} = 1_{A_1} - \mu(A_1)/\mu(B_{n-1}1_{B_{n-1}}
+ (\mu(A_1)/\mu(B_{n-1}) - \mu(A_1)/\mu(A_n))(1_{A_n} - \mu(A_n)/\mu(B_n)1_{B_n}) \in Z
\]

for all \( n \). If \( \mu(C) = 0 \) or there are only finitely many atoms, then our initial observation shows that \( \overline{Z}^{w^*} \supset L_\infty^0 \). If \( \mu(C) \neq 0 \) and there are infinitely many atoms, then note that \( \|1_{A_1} - \mu(A_1)/\mu(B_n)1_{B_n}\|_\infty \leq \max\{1, \mu(A_1)/\mu(C)\} \) for all \( n \) and that \( 1_{A_1} - \mu(A_1)/\mu(B_n)1_{B_n} \to 1_{A_1} - \mu(A_1)/\mu(C)1_C \) in the \( w^* \) topology. Again the conclusion follows from our initial observation.

\[ \square \]

Theorem 2.2. Suppose that \( \mu \) is a finite positive measure with atoms \( \{ A_i \}, \mu(A_i) \geq \mu(A_{i+1}) \), and there is a constant \( K \) such that for each \( i, m \in \mathbb{N} \) there is an \( h \in X_1 \) with supp \( h - 1_A \subset \Omega \cup \cup_{j \leq m} A_j \) and \( \|h\|_1 \leq K\mu(A_i) \). If \( F \in X_1^+ \cap L_\infty \) and \( \lim_{n \to \infty} F(A_n) \) exists, then \( F \) is a constant.

Proof. We will show the hypothesis of Lemma 2.1 is satisfied where \( Z = \{ z \in L_\infty : \int zF d\mu = 0 \} \).

Let \( C \) be the complement of the union of the atoms. Fix \( n \in \mathbb{N} \). Let \( g = 1_{A_n} - \mu(A_n)/\mu(C \cup \cup_{j > n} A_j)1_{C \cup \cup_{j > n} A_j} \). For each integer \( m > n \) let

\[
g_m = h_{mn} - \mu(A_n)/\mu(C \cup \cup_{j > n} A_j) \sum_{s = n+1}^{m} h_{ms}
\]

where \( h_{ms} \in X_\infty \) satisfies \( \|h_{ms}\|_1 \leq K\mu(A_s) \), \( h_{ms}|\cup_{s \leq n} A_j = 0 \), \( h_{ms}|A_s = 1_{A_s} \), and \( h_{ms}|\cup_{s \leq m} A_j = 0 \) for \( n \leq s \leq m \). Therefore

\[
\|g_m\|_1 \leq K\left[ \mu(A_n) + \mu(A_n)/\mu(C \cup \cup_{j > n} A_j) \sum_{s = n+1}^{m} \mu(A_s) \right] \leq 2K\mu(A_n).
\]
Suppose that $F \in L_\infty \cap X_1^\perp$ and $\lim F(A_n) = \rho$. If $\mu(C) > 0$, by Proposition 1.4, $L_1^0(C, \mu)$ is contained in $X_1$ and thus $F$ is constant on $C$. Also note that $F(c) = \rho$ for all $c \in C$, because $F$ is constant on $C$ and if $0 < \mu(A_i) \leq \mu(C)$, $1_{C'} - 1_{A_i} \in X_1$ where $C' \subset C$ such that $\mu(C') = \mu(A_i)$. Hence

$$\left| \int (g - g_m)Fd\mu \right| = \left| \int_{\cup j > m A_j \cup C} (g - g_m)Fd\mu \right| \leq \| (F - \rho) (\cup j > m A_j \cup C) \| \infty \| g - g_m \| 1 + \left| \int (g - g_m) \rho d\mu \right|.$$

The second term is zero because $g$ and $g_m \in L_\infty^0$ and the first term clearly converges to zero. Since $g_m \in X_1$, $\int g_m F d\mu = 0$, for each $m$. Thus $\int g F d\mu = 0$ and therefore $\{F\}^\perp \supset L_\infty^0$ and $F$ is constant as claimed. □

As an application of Theorem 2.2 we will prove Nymann’s Theorem. We will need the following lemma whose proof we leave to the reader.

**Lemma 2.3.** If $f$ and $g \in L_1(\mu)$ for some measure $\mu$ and there is a set $A$ such that $f|_A = \gamma g|_A$ for some $\gamma \in [-1, 0]$ and $\|f|_A\|_1 \geq \|f - f|_A\|_1$, then $\|f + g\|_1 \leq \|g\|_1$.

**Theorem 2.4.** If $\nu$ is a positive measure with range equal to a finite interval, $\nu$ is a measure satisfying (L), and $\nu$ is positive or $\nu$ is a signed measure with $d\nu/d\mu \in L_\infty(\mu)$, then $\nu$ is a constant times $\mu$.

**Proof.** Assume first that $\nu$ is positive. We will verify the hypotheses of Theorem 2.2 for $F = d\nu/d\mu$. By Lemma 1.6, $\lim_{\mu(A) \to 0} \nu(A)/\mu(A) = c$ so $\lim_{n \to \infty} F(t_n) = c$, if $t_n \in A_n$ and $\mu(A_n) \to 0$. Also $F$ is a constant on $C$ by Proposition 1.4 and therefore $F \in L_\infty(\mu)$. Thus we need only produce the functions $h$ in $X_1$.

Because the range of $\mu$ is an interval for each $n$ there exists a subset $M_n$ of $\{n + 1, n + 2, \ldots\}$ and $C_n \subset C$ such that $\mu(A_n) = \mu(C_n \cup \bigcup j \in M_n A_j)$, that is, $h_n = 1_{A_n} - 1_{C_n \cup \bigcup j \in M_n A_j} \in X_1$ and $\|h_n|_{A_n}\| = \|h_n - h_n\|_{A_n}$. (See [L, Proposition 1].)

**CLAIM.** If $k$ and $m$ are integers, $k < m$, and $g \in X_1$, then there is an $h \in X_1$ such that $h|_{\bigcup \cup k A_n} = g|_{\bigcup \cup k A_n}$, $h|_{\bigcup k < n \leq m A_n} = 0$, and $\|g\|_{L_1(\mu)} \geq \|h\|_{L_1(\mu)}$.

Indeed, suppose that $g|_{A_{k+1}} = a1_{A_{k+1}}$. Let $f = -ah_n$ in Lemma 2.3. Then if $g_1 = g + f, g_1|_{\bigcup \cup k A_n} = g|_{\bigcup \cup k A_n}, g|_{A_{k+1}} = 0$, and $\|g_1\| \leq \|g\|$. Induction finishes the proof of the claim. Applying the claim to $g = h_n$ shows that the hypothesis of Theorem 2.2 is satisfied and thus $d\nu/d\mu$ is constant.

For $\nu$ signed, we apply the previous argument to $\nu'$ from Corollary 1.7 c). It follows that $\nu'$ and hence $\nu$ is a multiple of $\mu$. □

**Remark 2.5.** This theorem includes Nymann’s result. The proof given by Nymann uses the result of Malitz [M] which in turn uses [L]. Thus one virtue of the proof we have given is that it is direct. More important our argument shows that there are two main ingredients to the proof. The ability to solve equations and the “continuity at $\emptyset$” of the Radon-Nikodym derivative.

This second ingredient is also the main obstacle to be overcome if the requirements on signed measures are to be relaxed. It would be nice to find a replacement for the argument used in the proof of Corollary 1.7 so that signed measures without
bounded Radon-Nikodym derivative could also be treated. In particular we do not know whether a finite signed measure with range an interval is determined uniquely by \( (L) \).

### 3. \( \sigma \)-Finite Measures

In the paper \([N]\) only finite positive measures were considered. Here we are allowing \( \sigma \)-finite measures. At first glance it may appear that the \( \sigma \)-finite case should reduce to the finite case. However we wish to observe that given \( t \in \text{range } \mu \) there may be no set \( C \) of finite measure such that range \( \mu|_C \) is an interval containing \( t \). Consider the following example.

**Example.** Let \( \Omega \) be \([0,1]\) union a sequence of disjoint sets \( \{A_n\} \) (also disjoint from \([0,1]\)) and let \( \mu \) restricted to \([0,1]\) be Lebesque measure and let the \( A_n \)'s be atoms of measure \( 1 + 2^{-n} \). Note that the range of \( \mu \) is \( \mathbb{R}^+ \) but for every \( C \) of finite measure greater than one the range of \( \mu|_C \) misses some interval \((1, 1 + 2^{-n})\), with \( n \in \mathbb{N} \).

Thus it is not possible to reduce the problem to the finite case directly. The next few lemmas will show that the example given above is the prototype for this difficulty.

**Lemma 3.1.** Let \( \mu \) be a purely atomic measure with range equal to an interval such that for every \( \varepsilon > 0, \mu(\{A_n : A_n \text{ is an atom and } \mu(A_n) > \varepsilon\}) < \infty \). Then for every \( t < \mu(\Omega) \) there is a subset \( C \) of \( \Omega \) such that \( \mu(C) \geq t \) and the range of \( \mu|_C \) is a finite interval.

**Proof.** There is nothing to prove if \( \mu(\Omega) < \infty \), so assume that \( \mu(\Omega) = \infty \). The requirement that the range is equal to a finite interval is equivalent to the assumption that there are no bullies. Let \( (a_n) \) be the masses at the atoms arranged in decreasing order. Assume that \( a_1 = 1 \). We will inductively construct a subsequence of the \( a_n \)'s without bullies so that the sum is finite.

As a convenience we will say that if \( E \) and \( F \) are subsets of \( \mathbb{N} \) and every element of \( E \) is less than every element of \( F \) then \( F > E \). Let \( E_0 = \{1, 2, \ldots, N\} \) where \( N \) satisfies \( \sum_{i \leq N} a_i > t \). Choose \( E_1 > E_0 \) such that \( \frac{1}{3} \leq \sum_{n \in E_1} a_n \leq \frac{2}{3} \) and \( a_n \leq \frac{1}{3} \) for all \( n \in E_1 \). This is possible because \( \sum_{n \geq k} a_n = \infty \) for all \( k \) and \( (a_k) \) decreases to 0. Next choose \( E_2 > E_1 \) such that \( \frac{1}{3} \geq \sum_{n \in E_2} a_n \geq \frac{1}{4} \) and \( a_n \leq \frac{1}{4} \) for all \( n \in E_2 \). Thus we can continue in this way to construct a sequence of finite subsets of \( \mathbb{N} \), \((E_j)\), such that for all \( j \geq 1 \)

a) if \( n \in E_j, a_n \leq 2^{-j} \)

b) \( 2^{-j+2}/3 \geq \sum_{n \in E_j} a_n \geq 2^{-j} \)

c) \( E_{j+1} > E_j \)

It is easy to see that \( C = \cup \{A_n : n \in E_j, \text{ some } j\} \) is the required set. If \( n \in E_j \), then \( a_n = \mu(A_n) \leq 2^{-j} \) and \( \sum_{k \geq j} \sum_{n \in E_k} a_n \geq \sum_{k \geq j} 2^{-k} = 2^{-j} \). Thus \( A_n \) is not a bully. \( \square \)

**Lemma 3.2.** If \( \mu \) is a positive \( \sigma \)-finite measure with range equal to an interval, then there is a possibly infinite number \( \beta \) and a subset \( B \) of \( \Omega \) such that

a) The range of \( \mu|_B \) is the interval \([0, \beta]\) and for every \( t < \beta \) there is a set \( B' \subset B \) with the range of \( \mu|_{B'} \) an interval of finite length at least \( t \).

b) either
i) $B = \Omega$

or

ii) $\mu_{|\Omega \setminus B}$ is purely atomic, every atom in $\Omega \setminus B$ has measure strictly greater than $\beta$ and $\beta$ is a limit point of the range of $\mu_{|\Omega \setminus B}$.

Proof. The result is trivial if $\mu(\Omega) < \infty$. If $\mu(\Omega) = \infty$ there are two cases to consider.

1) there is a sequence of atoms $(A_i)$ such that $\lim \mu(A_i) = 0$ and $\sum \mu(A_i) = \infty$

2) there is no such sequence.

If Case 1 occurs, let $A = \bigcup A_i$, then range of $\mu_{|A} = \mathbb{R}^+$ and by Lemma 3.1 we can take $B = \Omega$ and $\beta = \infty$.

Now we will treat Case 2. As usual let $C$ be the complement of the atoms and let $\kappa = \mu(C)$, which we assume is finite. (If $\kappa = \infty$, we can again let $B = \Omega$.) Let $B = \{G : G \subset \Omega \setminus C, \mu(G) < \infty$ and range $\mu_{|G}$ omits no intervals of length greater than $\kappa\}$.

Note that if $B = \bigcup_{i \in M} G_i, G_i \in B$, for each $i \in M$, and $\mu(B) < \infty$, then range $\mu_{|B}$ omits no intervals of length greater than $\kappa$. Indeed, if $H \subset B$ and $s = \sup \{\mu(D) : D \subset B, \mu(D) < \mu(H)\} < \mu(H) - \kappa$, then every non-null $E \subset H$ has measure greater than $\kappa$. Let $E$ be an atom of minimal measure contained in $H$, then there exists $G_i \supset E$ with range $\mu_{|G_i}$, missing no interval of length greater than $\kappa$. But for some $t \in [\mu(E) - \kappa, \mu(E))$ there is a subset $F$ of $G_i$ so that $\mu(F) = t$. Clearly $F \cap H = \emptyset$ so $\{\mu((H \setminus E) \cup F) : F \subset G_i\}$ contains a point in $(s, \mu(H))$. Hence $\mu_{|B}$ omits no intervals of length greater than $\kappa$.

It follows that if $B_1 = B \cup C$, then if $t < \mu(B) + \kappa$ there is a subset $B'$ of $B \cup C$ such that range $\mu_{|B' \cup C}$ is an interval containing $[0, t]$ and $\mu(B' \cup C) < \infty$. If $\mu(B_1) = \infty$, then let $B = \Omega$. Otherwise let $B = B_1$ and $\beta = \mu(B) + \kappa$. Also note that $\beta > 0$. Indeed if $\beta = 0, 1)$ Case 1 does not occur and there is some $\epsilon > 0$ so that $\mu(F) < \infty$ where $F = \bigcup \{A : A$ is an atom and $\mu(A) < \epsilon\}$. Because the range of $\mu$ is an interval, $F \neq \emptyset$. The range of $\mu_{|F}$ cannot be an interval because $\beta = 0$, so there must be a bully in $F$. However this implies that the range of $\mu$ is not an interval.

Clearly the range of $\mu_{|B} = [0, \beta]$. Observe that if $A$ is an atom with $\mu(A) \leq \beta < \infty$ then $A \cup B \in B$ and hence $A \subset B$. Therefore if $\beta < \infty$ and the range of $\mu$ is $[0, \infty)$, every atom $A \subset \Omega \setminus B$ has measure greater than $\beta$ and there must be an infinite sequence of atoms $(A_i)$ such that $\mu(A_i)$ decreases to $\beta$. $\Box$

We are now in a position to extend the result in $[N]$ to all positive $\sigma$-finite measures with range an interval.

**Proposition 3.3.** Suppose that $\mu$ is a positive $\sigma$-finite measure with range an interval and that $\nu$ is a $\sigma$-finite measure satisfying (L). If $\nu$ is positive or if there exist $\epsilon > 0$ and $M < \infty$ such that $\frac{\nu(A)}{\mu(A)} < M$ if $\mu(A) < \epsilon$, then $\nu$ is a multiple of $\mu$.

**Proof.** By Theorem 2.4, either condition implies that if $B \in \mathcal{B}$, $\mu(B) < \infty$ and range $\mu_{|B}$ is an interval, then $\frac{d\nu}{d\mu}$ is constant on $B$. Hence we assume that $\mu(\Omega) = \infty$.

From Lemma 3.2 it follows that if $\beta = \infty$ then for any pair of disjoint sets of finite measure there is a subset $B' \subset B$ such $\mu_{|B'}$ has range a finite interval containing $\mu(D)$ and $\mu(E)$. By Proposition 1.5 and the finite case this implies that $d\nu/d\mu$ is constant.
Now assume that $\beta < \infty$. In view of the Proposition 1.5 it is sufficient to show that for $r > s > 0$ there is a set $F$ such that $X \cap L_\infty(\mu_1 F)^{w^*} = L_\infty^0(\mu_1 F)$, the range of $\mu_1 F$ contains $\{s, r\}$ and $\mu(F) < \infty$. If $\beta \geq r$ we can take $F = B$ and proceed as above. If $r > \beta$, we will add some atoms to $B$ to get the required set $F$. From Lemma 3.2 we know that there is an infinite sequence of atoms $(A_i)$ such that $\mu(A_i)$ decreases to $\beta$. It follows that there is a finite set of the $A_i$’s, $B_1$, $B_2$, ..., $B_n$, such that $\mu(B_i) > \mu(B_{i+1})$ for all $i$, $\mu(B_1) + \beta > \mu(B_2) + \mu(B_3) > \mu(B_1)$, and both $r$ and $s$ are in the range of $\mu_1 F$, where $F = B \cup \cup_{i \leq n} B_i$. We will show that $X_\infty$ is $w^*$ dense in $L_\infty^0(\mu_1 F)$. Because the range of $\mu_1 B$ is a finite interval, $X_\infty \cap L_\infty^0(\mu_1 B)$ is $w^*$ dense in $L_\infty^0(\mu_1 B)$. Also $\dim L_\infty^0(\mu_1 F)/L_\infty^0(\mu_1 B) = n$, so it is sufficient to find $n$ linearly independent elements of this quotient which arise from $X$. For each $i < n$ there is a subset $C_i$ of $B$ with positive measure such that $1_{B_i} - 1_{B_n} - 1_{C_i} \in X$, and there is a set $C_n \subset B$ such that $1_{B_1} + 1_{C_n} - 1_{B_2} - 1_{B_3} \in X$. It is easy to see that these are in $n$ linearly independent cosets of $X \cap L_\infty(\mu_1 F)^{w^*}/L_\infty^0(\mu_1 B)$. \(\Box\)

Remark 3.4. Recently Khamsi and Nymann [KN] have shown that if $\mu$ is a finite measure with range equal to an interval then any $\sigma$-finite measure $\nu$ such that if $\mu(A) = \mu(B)$ then $\nu(A) = \nu(B)$, i.e., (L) without the finiteness requirement is satisfied, then (L) holds. It follows that Proposition 3.3 holds with the weakened version of (L) as well.

Theorem 3.5. Suppose that $\mu$ is a $\sigma$-finite signed measure with range an interval and that $\nu$ is a $\sigma$-finite measure satisfying (L). If there exist $\epsilon > 0$ and $M < \infty$ such that $|\frac{\nu(A)}{\mu(A)}| < M$ if $\mu(A) < \epsilon$ and $A$ is an atom, then $\nu$ is a multiple of $\mu$.

Proof. By Lemmas 1.1 and 1.2, $|\mu|$ and $\nu'$ also satisfy the hypothesis. By Proposition 3.3, $\nu'$ is a multiple of $|\mu|$ and hence $\nu$ is a multiple of $\mu$. \(\Box\)

Corollary 3.6. Suppose that $\mu$ and $\nu$ are $\sigma$-finite signed measures satisfying (O). If the range of $\mu$ is an interval, then $\nu$ is a multiple of $\mu$.

Proof. Proceeding as in the previous proof, $|\mu|$ and $\nu'$ satisfy (O) and by Lemma 1.6 b), $\nu'$ is positive. Therefore by Proposition 3.3, $\nu'$ is a multiple of $|\mu|$ and hence $\nu$ is a multiple of $\mu$. \(\Box\)

4. Examples.

In order to construct examples we will use the finite dimensional case to control the combinatorics in the same spirit as the examples constructed by Leth.

Let us now consider the finite dimensional case, i.e., suppose that there are finitely many atoms $(A_i)_{i=1}^n$ with $\mu(A_i) = a_i$ and assume that $\Omega = \cup A_i$. The statement of the problem can be rephrased in terms of dimension and thus the uniqueness question for (L) reduces to linear algebra. To make the statement more succinct let us introduce the following notation. Let $R = \{\nu \in \{-1, 0, 1\}^n : \langle \nu, (a_i) \rangle = 0\}$.

Proposition 4.1. Condition (L) uniquely determines $\mu$ if and only if $\dim sp R = n - 1$.

It would be nice if there were some easy characterization of the finite sequences $(a_i)$ so that $\dim sp R = n - 1$, but as we will see below there is no obvious one.
description. One easy observation is that the \( a_i \)'s must all belong to the same rational equivalence class. (This same observation also applies in the case where \( \mu \) is purely atomic \( \sigma \)-finite but the measures of the atoms are bounded away from 0.) Note also that uniqueness for \((O)\) implies uniqueness for \((L)\) in the case of finitely many atoms because the Radon-Nikodym derivative is automatically bounded.

For our first example we will produce a finite dimensional example with range not an arithmetic progression but such that \((L)\) uniquely determines \( \mu \). (In the closing remarks in [N] it was claimed that no such example exists.) For convenience we will make our measure integer valued.

Example 1. Let \( \mu \) have nine atoms with measures 1, 2, 5, 6, 7, 8, 9, 10, 11. With the notation above and \( a_1 = 1, a_2 = 2, \ldots, a_9 = 11 \), the set \( R \) contains the following elements.

\[
\begin{align*}
(1, 0, 1, -1, 0, 0, 0, 0, 0) \\
(1, 0, 0, 1, -1, 0, 0, 0, 0) \\
(1, 0, 0, 0, 1, -1, 0, 0, 0) \\
(1, 0, 0, 0, 0, 1, -1, 0, 0) \\
(1, 0, 0, 0, 0, 0, 1, -1, 0) \\
(1, 0, 0, 0, 0, 0, 0, 1, -1) \\
(0, 1, 1, 0, -1, 0, 0, 0, 0) \\
(0, 0, 1, 1, 0, 0, 0, 0, -1)
\end{align*}
\]

It is not hard to see that these are linearly independent and therefore by Proposition 4.1 \( \mu \) is uniquely determined by \((L)\). Clearly 4 is omitted from the range of \( \mu \) but 1, 2 and 3 are included, so the range is not an arithmetic progression.

Our next two examples show that properties of the range are not good indicators of whether uniqueness holds.

Example 2. We begin with two finite dimensional examples. Let the five atoms of \( \mu \) be of size 1, 2, 2, 2, 5 and let the four atoms of \( \mu' \) be of size 1, 2, 4, 5. The range of \( \mu \) is the same as the range \( \mu' \), namely \( \{1, 2, \ldots, 12\} \). For \( \mu \) note that the set \( R \) contains the four linearly independent elements \( (1, 1, 1, 0, -1), (0, 1, -1, 0, 0), (0, 0, 1, -1, 0), \) and \( (1, -1, -1, 1) \). Thus \( \mu \) is uniquely determined by \((L)\). On the other hand for \( \mu' \) we have \( R = \{\pm(1, 0, 1, -1), \pm(1, -1, -1, 1)\} \) and thus \( \dim \text{sp} R = 2 < 3 \). Thus \( \mu' \) is not uniquely determined by \((L)\). (Take \( \nu \) to have atoms of size 1, 2, 6, and 7, for example.)

Example 3. We will modify the measures in Example 2 to get two measures with range the same finite union of intervals, but so that uniqueness holds for the first but not the second. To do this observe that if we omit the first atom from each of the measures in Example 2, the range of the restricted measure in each case is \( \{2, 4, 5, 6, 7, 9, 11\} \). Let \( \gamma \) be any probability measure on a measurable space \((\Omega_1, C)\) with range \([0,1]\). Replace the atom \( A_1 \) by \( \Omega_1 \) and enlarge the \( \sigma \)-algebra to contain \( C \) for each of \( \mu \) and \( \mu' \) and define \( \mu(A) = \mu'(A) = \gamma(A) \) for \( A \in C \). (Extend \( \mu \) and \( \mu' \) to the \( \sigma \)-algebra in the obvious way.) Then the range of \( \mu \) and \( \mu' \) is now \([0, 12] \setminus \{(1,2) \cup (3,4) \cup (8,9) \cup (10,11)\} \). Proposition 1.2 implies that if \( \nu \) is another finite measure on \( \Omega_1 \cup \cup_{i \geq 1} A_i \) satisfying \((L)\) for either \( \mu \) or \( \mu' \) then \( \nu \) is a multiple of \( \gamma \) on \( \Omega_1 \). Thus the uniqueness problem is the same here as in Example 2, i.e., for \( \mu \) condition \((L)\) implies uniqueness but for \( \mu' \) it does not.

These examples indicate that a characterization of the measures for which \((L)\) is sufficient for uniqueness would require a similar result for the finite dimensional
case and thus seems to be difficult. Finally we present an example of a finite signed measure with range the interval $[-1,1]$ which is uniquely determined by $(\mathcal{L})$ but for which neither the positive nor negative parts are uniquely determined by $(\mathcal{O})$.

**Example 4.** For each natural number $n$ let $A_{2n-1}$ and $A_{2n}$ be atoms of a signed measure $\mu$ with $\mu(A_{2n-1}) = \frac{2}{3^n} = -\mu(A_{2n})$. It is not hard to see that the ranges of $\mu^+$ and $\mu^-$ are both the usual Cantor set $C$ in $[0,1]$. Moreover it is known and elementary to see that $C - C = [-1,1]$. By Lemma 2.1, $|\mu|$ is uniquely determined by $(\mathcal{L})$. Indeed, for each $n$, $1_{A_{2n-1}} - 1_{\cup_{j>2n} A_j} = 1_{A_{2n}} - 1_{\cup_{j>2n} A_j} \in X$. Also $1_{A_{2n-1}} - \frac{1}{2} 1_{\cup_{j>2n} A_j} = 1_{A_{2n-1}} - 1_{\cup_{j>2n} A_j} - \frac{1}{2}(1_{A_{2n}} - 1_{\cup_{j>2n} A_j}) \in X$. Thus $X_\infty \supset L_0^0(|\mu|)$. Lemma 1.1 implies that $\mu$ is also uniquely determined by $(\mathcal{L})$. On the other hand $\frac{2}{3^n} > \frac{1}{3^n} = \sum_{i>n} \frac{2}{3^i}$ and thus for $\mu^+$ and $\mu^-$ each atom is a bully.

Notice that there is no restriction imposed on $\frac{d\nu}{d\mu}$ in this example and thus Theorem 3.5 does not apply. It seems likely that the restriction is not necessary in general.

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