Quantum Group and $q$-Virasoro Symmetries in Fermion Systems

Haru-Tada Sato$^\dagger$

Department of Physics, Osaka University
Machikaneyama 1-16, Toyonaka 560, Japan

Abstract

We discuss a generalization of the quantum group $\mathcal{U}_q(sl(2))$ to the $q$-Virasoro algebra in two-dimensional electrons system under uniform magnetic field. It is shown that the integral representations of both algebras are reduced to those in a (1+1)-dimensional fermion. As an application of the quantum group symmetry, we discuss a model of quantum group current on the analogy of the Hall current.

$^\dagger$ Fellow of the Japan Society for the Promotion of Science

E-mail address : hsato@phys.wani.osaka-u.ac.jp
1 Introduction

Quantum groups \([1, 2]\) and other deformed algebras have often been studied in some relevance to quantum corrections, anisotropies, discretizations and deformations of original symmetries. Recently in the context of discretizations, in particular, \(q\)-deformations of the Virasoro algebra have been developed intensively \([3]–[6]\). The \(q\)-Virasoro algebras is defined as two-loop Lie algebras which are connected to the lattice deformation of the Liouville theory or to a discrete KdV system \([4]\). Furthermore, one of them is represented in terms of a single Majorana fermion \([3, 5]\) and it might well describe some other fermion physics. We consider the fermion’s phenomenon which possesses at least the quantum subalgebra \(U_q(sl(2))\) accordingly.

There are two other reasons to consider a fermion system. The investigation of \(q\)-deformations of the Virasoro algebra originated in an attempt to deform conformal field theories (CFTs) and string theories \([7]\). In off-critical CFT, the algebra classifying representations is no longer the Virasoro algebra, but some other algebra. One of the typical solvable off-critical CFT models is also a fermion system in an external magnetic field. We are thus interested in whether the \(q\)-deformed Virasoro algebra is related to the fermion system in a constant magnetic field.

Second, in nonrelativistic three dimensional system, we encounter the interesting phenomena, called the quantum Hall effects \([8]\). This is also a phenomenon of electrons in magnetic field and the \(U_q(sl(2))\) symmetry appears. Recently, it has been shown that utilizing the \(U_q(sl(2))\) symmetry, various cases of diagonalization problem of a two-dimensional electron are reduced to the Bethe ansatz equations or solved in terms of the Bethe ansatz equations \([9]\). We then expect some relevance to the \(q\)-Virasoro algebra.

In this paper, we are concerned with the \(q\)-Virasoro symmetry as a generalization of the \(U_q(sl(2))\) in a charged particle system under a constant magnetic field \(B\) perpendicular to the \(x-y\) plane with the Hamiltonian

\[
H = \frac{1}{2m_e}(p - \frac{e}{c}A)^2 .
\]

(1.1)

In abstract stage, the quantum algebra \(U_q(sl(2))\) is defined by four generators \(E^+, E^-, \ldots\)
and $k^{\pm 1}$ which satisfy the following commutation relations
\[
[E^+, E^-] = \frac{k^2 - k^{-2}}{q - q^{-1}}, \quad kE^\pm k^{-1} = q^{\pm 1}E^\pm. \tag{1.2}
\]

The $m$-dimensional matrix representation (spin-$j$ representation) with $m = 2j + 1$ of these generators is
\[
\begin{align*}
\pi(E^+) &= \text{diag}^+[\{2j\}_q, \{2j - 1\}_q, \ldots, \{1\}_q], \\
\pi(E^-) &= \text{diag}^-[\{1\}_q, \{2\}_q, \ldots, \{2j\}_q], \\
\pi(k) &= \text{diag}[q^j, q^{j-1}, \ldots, q^{-j}],
\end{align*} \tag{1.3}
\]

where the notation $\text{diag}^+$ ($\text{diag}^-$) means upper (lower) diagonal matrix and
\[
[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{1.4}
\]

It is known that the generators are realized by the magnetic translations $T_\alpha$
\[
T_{(\alpha_1, \alpha_2)} = \exp\left(\frac{i}{\hbar} \alpha \cdot \beta\right), \tag{1.5}
\]

where
\[
\beta_i = p_i - \frac{e}{c}A_i - \frac{eB}{c}\epsilon_{ij}x^j, \quad A_i = -\frac{1}{2}B\epsilon_{ij}x^j - \partial_i \Lambda, \tag{1.6}
\]

and $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$. The vector $\beta$ is related to the cyclotron center and the scalar function will be fixed for later conveniences.

In sect. 2, we review the $\mathcal{U}_q(sl(2))$ symmetry in the one-body system and survey matrix representations in order to have a prospect how the $q$-Virasoro algebra appears. Sect. 3 is a short note on the case of a many-particle system. In sect. 4, we discuss the integral representations of the $\mathcal{U}_q(sl(2))$ and the $q$-Virasoro generators in the nonrelativistic field theory in three dimensions. We show that these representations correspond to those in two-dimensional chiral fermion theory through a dimensional reduction procedure. In sect. 5, we illustrate a model of quantum group currents on the analogy of the Hall current. In order to complement the argument of sect. 5, we notice the tensor product of two spin-1 representations with $\nu = 1/3$ in sect. 6. In this case, we obtain the 'Hall' current with the filling factor $\nu = 2/3$. 

2
2 Quantum group symmetry

The generators of the quantum group algebra $U_q(sl(2))$ are realized by the following combination of magnetic translations

$$E^+ = \frac{T(\Delta,\bar{\Delta}) - T(-\Delta,\bar{\Delta})}{q - q^{-1}}$$

$$E^- = \frac{T(-\Delta,\bar{\Delta}) - T(\Delta,\bar{\Delta})}{q - q^{-1}}$$

$$k = T(\Delta,0).$$

They satisfy the defining relations of the $U_q(sl(2))$ with the identification

$$q = \exp(i\Delta\bar{\Delta}\lambda^{-2})$$

where $\lambda$ is the magnetic length defined by $\sqrt{hc/eB}$. If we put

$$\Delta = \frac{L_x}{2j+1}, \quad \bar{\Delta} = \frac{L_y}{2j+1}$$

and we assume that the total flux $N_s$ satisfies the condition

$$N_s = \frac{L_x L_y}{2\pi \lambda^2},$$

the Landau states $\psi_l (-j \leq l \leq j$, where $j$ is defined by $N_s = 2j + 1$) behave as a spin-$j$ representation of the $U_q(sl(2))$. Namely, we gain the relations

$$E^\pm \psi_l = \left[\frac{1}{2} \pm l\right]_q \psi_{l \pm 1}, \quad k\psi_l = q^l \psi_l,$$

and

$$q = \exp\left(\frac{2\pi i}{2j+1}\right).$$

When the periodic boundary condition is imposed, $l$ takes an integer value. On the other hand the anti-periodic case, $l$ takes a half-integer value. Accordingly, whether $j$ is an integer or a half-integer is determined from the periodic or anti-periodic boundary condition. The matrix representation $\rho$ of the equations (2.7) becomes

$$\rho(E^+) = \text{diag}^+[-[j - \frac{1}{2}]_q, -[j - \frac{3}{2}]_q, \ldots, [j - \frac{1}{2}]_q],$$

$$\rho(E^-) = \text{diag}^-([[j - \frac{1}{2}]_q, [j - \frac{3}{2}]_q, \ldots, -[j - \frac{1}{2}]_q],$$

$$\rho(k) = \text{diag}[q^j, q^{j-1}, \ldots, q^{-j}].$$
Using the formulae

\[
\begin{align*}
\frac{m - 2l}{2} & = [l]_q, \\
[m - l]_q & = -[l]_q \\
\end{align*}
\]

for \( q^m = 1 \) \( (2.10) \)

we can verify that the representation (2.9) satisfies the commutation relations (1.2) and it turns out that (2.9) coincides with the spin-\( j \) representation (1.3) making use of the relations (2.10). The above representation \( E^\pm \) changes \( l \) into \( l \pm 1 \), however other cases of raising from \( l \) to \( l \pm p \) (\( p \neq 1 \)) are also possible (see appendix A).

When \( j \) is an integer, namely periodic case, the representation (2.9) amounts to

\[
\begin{align*}
\rho(E^+) & = \text{diag}^+[-[1]_q, -[2]_q, \ldots, -[j]_q, [j]_q, [j - 1]_q, \ldots, [1]_q], \\
\rho(E^-) & = \text{diag}^-([[1]_q, [2]_q, \ldots, [j]_q, -[j]_q, -[j - 1]_q, \ldots, -[1]_q]. \quad (2.11) \\
\end{align*}
\]

The other case of \( j \) being a half-integer (anti-periodic case) contrastively includes a zero in the entries of \( \text{diag}^\pm \)

\[
\begin{align*}
\rho(E^+) & = \text{diag}^+[-[1]_q, -[2]_q, \ldots, -[j - \frac{1}{2}]_q, 0, [j - \frac{1}{2}]_q, \ldots, [1]_q], \\
\rho(E^-) & = \text{diag}^-([[1]_q, [2]_q, \ldots, [j - \frac{1}{2}]_q, 0, -[j - \frac{1}{2}]_q, \ldots, -[1]_q]. \quad (2.12) \\
\end{align*}
\]

We note that \( j = 1/2 \) case is excluded in the above representation (2.12) or (2.9) because \( \rho(E^\pm) = 0 \) and \( \rho(k) = \text{diag}^q[q^{1/2}, q^{-1/2}] \). Namely, these neither satisfy the defining relations (1.2) of the quantum algebra \( U_q(sl(2)) \) nor coincide with the representation (1.3).

These matrix representations are related to the cyclic representation of the Curtright-Zachos type of \( q \)-deformed Virasoro algebra[10]. This is a signature of the existence of the \( q \)-Virasoro algebra. Actually, the \( U_q(sl(2)) \) generators (2.1) and (2.2) can be generalized into the following operators

\[
\hat{J}_n^{(k)} = \frac{T_{(k\Delta, n\Delta)} - T_{(-k\Delta, n\Delta)}}{q^k - q^{-k}}. \quad (2.13)
\]

This differential operator satisfies a \( q \)-analogue of the Virasoro algebra (details are in sect.4).
3 Laughlin-Jastrow function

In this section, we comment that the statement of the previous section can be straightforwardly extended to many-particle system. In the following sections, we devote ourselves to the gauge $\Lambda = 0$ and introduce the complex notation $[11]$

$$z = \frac{1}{\lambda \sqrt{2}} (x + iy).$$  \hspace{1cm} (3.1)

In this notation, the one-particle magnetic translation (1.3) becomes

$$T_{(\epsilon, \bar{\epsilon})} = \exp(\epsilon b - \bar{\epsilon} b^\dagger)$$ \hspace{1cm} (3.2)

where

$$b = \frac{1}{2} \bar{z} + \partial, \quad b^\dagger = \frac{1}{2} z - \bar{\partial},$$ \hspace{1cm} (3.3)

and $\epsilon$ and $\bar{\epsilon}$ can be regarded as mutually independent dimensionless constants. The quantum group generators are obtained by the naive replacement of $(\Delta, \bar{\Delta})$ by $(\epsilon, \bar{\epsilon})$ in the eqs.(2.1)-(2.3). The commutation relations (1.2) are satisfied identifying the deformation parameter with

$$q = \exp(-\epsilon \bar{\epsilon}).$$ \hspace{1cm} (3.4)

And the lowest Landau level wavefunction

$$\phi_l(z) = \exp(-\frac{1}{2} z\bar{z}) \exp(ilz)$$ \hspace{1cm} (3.5)

satisfies the relations (2.7) and (2.8) if we choose $\epsilon = L_x L_y / (\lambda N_s)^2$ and $\bar{\epsilon} = -i$.

In a multi-particle system, introducing the total magnetic translation operator defined by

$$T_{(\epsilon, \bar{\epsilon})}^{tot} = \prod_{i=1}^{N_e} T_{(\epsilon, \bar{\epsilon})}^{(i)}(x_i, y_i),$$ \hspace{1cm} (3.6)

the quantum group generators are similarly obtained by replacing $T$ with $T_{(\epsilon, \bar{\epsilon})}^{tot}$ in the above argument. The commutation relations (1.2) are checked using the multiplication formula for $T_{(\epsilon, \bar{\epsilon})}^{tot}$

$$T_{(n_1, n_2)}^{tot} T_{(m_1, m_2)}^{tot} = \exp \left( \frac{i}{2} a \epsilon^{ij} n_i m_j \right) T_{(n_1 + m_1, n_2 + m_2)}^{tot};$$ \hspace{1cm} (3.7)
when identifying
\[ q = \exp(-N_e \epsilon \bar{\epsilon}). \]  (3.8)

In this case also, the deformation parameter \( q \) is given by the filling factor \( \nu = N_e/N_s \) in the lowest Landau level; \( q = \exp(2\pi i \nu) \), which is a natural generalization of (2.8).

Let us consider whether \( N_e \)-body wavefunctions form a representation basis of the \( U_q(sl(2)) \). For that purpose, we deal with the Laughlin-Jastrow wavefunctions [13] multiplied by the center-of-mass factor with filling factor \( \nu = 1/m \) (\( m \) odd)
\[ \psi_l = \exp(il \sum_i N_e z_i) \exp[-\frac{1}{2} \sum_{i=1}^{N_e} z_i \bar{z}_i] \prod_{i<j} (z_i - z_j)^m. \]  (3.9)

Operating the generators of the quantum algebra \( U_q(sl(2)) \) on the above wavefunctions (3.9), we easily recognize that the same relations as (2.7) are satisfied by making use of the relation (3.8). In this consideration, the \( U_q(sl(2)) \) symmetry does not concern the interactive part of the wavefunction \( \prod (z_i - z_j) \). Only the global parts (exponential parts) are relevant [14] and thus we only have to discuss one-particle system in generalization to the \( q \)-Virasoro algebra.

4 \( q \)-Virasoro generators

In this section, generalizing the integral representations of the quantum group \( U_q(sl(2)) \), we derive the \( q \)-Virasoro algebra [3]. The integral representations are obtained with the course of Ref.[11]. After that, we discuss the connection of the representations to those of a relativistic chiral fermion in two dimensions.

In the first quantization of the Hamiltonian
\[ H = 2a^\dagger a + 1, \]  (4.1)
the wavefunctions are given by two commuting harmonic oscillators
\[ \phi_{n,l}(z) = \frac{(b^\dagger)^{n+l} (a^\dagger)^n}{\sqrt{(l+n)!}} \frac{\phi_0(z)}{\sqrt{m^n}}. \]  (4.2)
where $\phi_0(z)$ is the vacuum satisfying $a\phi_0 = b\phi_0 = 0$ and the argument $z$ simply means the two dimensional coordinate $(x, y)$. The quantum numbers $n$ and $l$ correspond to the energy and the angular momentum respectively. In terms of the wavefunctions (4.2) and the fermion oscillators $b_k^{(n)}$ and $b_k^{(n)\dagger}$

$$\{b_k^{(n)}, b_l^{(m)\dagger}\} = \delta_{n,m} \delta_{k,l}, \quad (4.3)$$

the second quantized field operator is defined by

$$\Psi(z, t) = \sum_{n,k=0}^{\infty} b_k^{(n)} \phi_{n,k}(z) e^{-i(2n+1)t} \quad (4.4)$$

and satisfies the canonical anti-commutator

$$\{\Psi(z, t), \Psi(w, t)\} = \delta^{(2)}(z - w). \quad (4.5)$$

When commutation relations of differential operators are given by

$$[\hat{C}_a, \hat{C}_b] = f^{abc} \hat{C}_c, \quad (4.6)$$

it is easy to verify that the following expression satisfies the same commutation relations as (4.6)

$$C_a = \int d^2 w \Psi^{\dagger}(w) \hat{C}_a(w, \bar{w}) \Psi(w), \quad (4.7)$$

and thus (4.7) provides an integral representation of the $\hat{C}$ algebra. If $\hat{C}$ is the Hamiltonian operator (4.1), $C$ becomes the conserved Hamiltonian for the non-relativistic Schrödinger field theory

$$S = \int dt d^2 w \left( i\Psi^{\dagger} \dot{\Psi} - \frac{1}{2} (D\Psi)^{\dagger}(D\Psi) \right) \quad (4.8)$$

where $D_i = \partial_i - iA_i$ (c.e=1). Furthermore, if $\hat{C}_a$ is composed of the harmonic oscillators (3.3), which commute with the covariant derivative $D$ and with the Hamiltonian, $C_a$ is a conserved charge.

Choosing $\hat{C}_a$ to be the $U_q(sl(2))$ operators discussed before, the integral representation of the $U_q(sl(2))$ generators is thus

$$E^+ = \int d^2 w \Psi^{\dagger}(w) T_{(\epsilon, \epsilon)}^{q^{-\epsilon}} \Psi(w)$$

$$E^- = \int d^2 w \Psi^{\dagger}(w) T_{(-\epsilon, -\epsilon)}^{q^{-\epsilon}} \Psi(w)$$

$$k = \int d^2 w \Psi^{\dagger}(w) T_{(\epsilon, 0)} \Psi(w). \quad (4.9)$$
These generators are the nonlocal conserved charges because magnetic translations are given by the harmonic oscillators $b$ and $b^\dagger$ (see (3.3)). We notice that there is an infinite number of nonlocal conserved charges which are the generalization of (4.9)

$$L_n^{(k)} = \int d^2w \Psi^\dagger(w) \frac{T_{(k,\bar{n}\epsilon)} - T_{(-k,\bar{n}\epsilon)}}{q^k - q^{-k}} \Psi(w).$$  (4.10)

These satisfy the $q$-Virasoro algebra (B.7) without a central extension

$$[L^{(i)}_n, L^{(j)}_m] = \sum_{\epsilon = \pm 1} C_{n m}^{i} \epsilon \delta^{(i+\epsilon j)}_n.$$

where the structure constants are

$$C_{n m}^{i} \epsilon = \frac{[(n j - m i)/2]_q [i + j]}{[\epsilon j]_q}.$$  (4.11)

For reference, the two-dimensional integral representation of the $q$-Virasoro algebra is in appendix B. The same algebras as $U_q(sl(2))$ and (4.11) exist in both theories, however eq.(B7) possesses a central extension.

Let us discuss the connection between two- and three-dimensional representations. The generator (4.10) is the form of

$$\int d^2w \delta \mathcal{L}_3 \delta^{(k)}(w, \bar{w}) \Psi(w, t),$$

where $\mathcal{L}_3$ is the Lagrangian density in (4.8). It is therefore convenient to consider the following dimensional reduction procedure

$$\int d^2w \frac{\delta \mathcal{L}_3}{\delta \Psi(w, t)} \rightarrow \frac{1}{2\pi i} \int dw \frac{\delta \mathcal{L}_2}{\delta \psi_H(w)}.$$  (4.13)

where $\mathcal{L}_2$ stands for the two-dimensional Lagrangian density of a dimensionless chiral fermion. This procedure gives the change from a nonrelativistic fermion to relativistic one except the change of dimensions of the fermion field. The dimensions will be counted later. We then consider the following reduced form

$$L_n^{(k)} \rightarrow \frac{1}{2\pi i} \int dw \frac{1}{2} \psi_H(w) \delta^{(k)}_n(w) \psi_H(w),$$  (4.14)

in which $\delta^{(k)}_n(w)$ should be understood as an operator of only holomorphic parts after some projection procedure. In this paper, we follow the method of ref. [15] to extract
holomorphic parts from $\delta_n^{(k)}(w, w)$. After moving all $\bar{w}$ powers to the left of the $w$ powers, we replace $\bar{w} \to 2\partial$ and $\bar{\partial} \to 0$. For example,

$$T_{(k\epsilon,n\epsilon)} = \exp\left(\frac{k\epsilon}{2}\bar{w}\right)\exp(n\epsilon\bar{\partial})\exp\left(-\frac{n\epsilon}{2}w\right)\exp(k\epsilon\bar{\partial}) \to \exp\left(-n\epsilon w/2\right)\exp(2k\epsilon\bar{\partial} - \epsilon\epsilon kn/2).$$

Furthermore, taking account of the coordinate transformation from $w$-cylinder to $z$-plane

$$w = -\frac{2}{\epsilon} \ln z$$

and of the parametrization (3.4), we get the reduced variation operator as

$$\delta_n^{(k)}(z) = zn \left[ k\bar{z}\partial_{\bar{z}} + nk/2\right]_q.$$ (4.16)

Finally substituting $\psi_H \to z^{1/2}\psi(z)$ into (4.14) in order that $\psi$ has the conformal weight $1/2$, we thus obtain the following expression up to the factor $2/\epsilon$

$$L_n^{(k)} \to -\frac{1}{2\pi i} \oint dz \frac{1}{2} \psi(z)z^n \left[ k\bar{z}\partial_{\bar{z}} + \frac{k}{2}(n+1)\right]_q \psi(z).$$ (4.17)

It is easy to show that the RHS of the above equation coincides with the $q$-Virasoro generators of the two-dimensional fermion (B.2). We therefore recognize that the 3-d fermion currents have some share of the two-dimensional fermion’s quantum group symmetries including the $q$-Virasoro algebra.

## 5 Quantum Group current

We have observed the $q$-Virasoro symmetry, however we have not understood the situation on which the $q$-Virasoro symmetry becomes effective. In this paper, we finally discuss a model on the analogy of the simplest model of the Hall current [16, 17].

First we review this model in short. Let us consider the gauge $\Lambda = \frac{1}{2}Bxy + \Phi x/2\pi$ and the cylinder with the radius $r$ defined by $L_x = 2\pi r$. $\Phi$ is now a fixed constant which will be varied adiabatically. The eigenfunctions of the Hamiltonian are degenerated for each Landau level $n$

$$|l,n\rangle = \exp\left\{ i\frac{l}{r}x - \frac{1}{2\lambda^2}(y - y_l)^2\right\} H_n\left(\frac{y - y_l}{\lambda}\right),$$ (5.1)
\[ y_l = \frac{\Phi}{2\pi r B} + \chi^2 \frac{l}{r}, \quad l \in \mathbb{Z}. \] 

(5.2)

All equations in the section 2 are not changed in this case. All the eigenfunctions for fixed value of \( n \), each state is discriminated by the \( l \), i.e., the coordinate \( y_l \) on the cylinder.

When we gradually increase the value of \( \Phi \) up to the flux quantum \( ch/e \) over the time interval \( \Delta T \), the \( y_l \) becomes to \( y_{l+1} \). Accordingly all eigenfunctions \( | l, n \rangle \) are shifted each in their turn (see Fig.1) and finally settle down to the equal set of the eigenfunctions to the original one. If all the states are completely filled with electrons, we observe the Hall current with the filling factor \( \nu = 1 \). Namely,

\[ j_x = \frac{e}{2\pi r \Delta T}, \quad E_y = \frac{1}{2\pi r c} \frac{\Delta \Phi}{\Delta T}, \] 

(5.3)

and we hence see \( \nu = 1 \) in the relation

\[ \sigma_{xy} = \frac{j_x}{E_y} = \nu \frac{e^2}{h}. \] 

(5.4)

Hereafter, we call this mechanism as an ‘entire moving of eigenstates’ (EME) and consider the analogy of the above model. According to the localization theory, there exists a potential which localizes wavefunctions and each Landau level splits into the Landau subband (Fig.2), in which unlocalized states are surrounded by localized states. The Hall current is carried by the electrons not in localized states but in unlocalized states [18] (We can depict the Hall current as in Fig.3 in the words of EME. All the unlocalized states are connected by the arrow lines forming a right moving current but on the other hand the localized states form closed loops).

We furthermore need the following three assumptions [17] to obtain a fractional filling.

(i) The magnetic field should be so strong as to the EME close within the same Landau subband. (ii) The cylinder should be long enough to idealize the states near the both edges of the cylinder, namely, we assume the situation of Fig.1 near the edges. This condition is equivalent to confining the regions of non-vanishing potential into the middle part of the cylinder. (iii) All the states up to the final unlocalized state (point D in Fig.2) within the Fermi surface should be completely filled with electrons and let the Fermi surface be in the upper localized states or in the energy gap (see Fig.2). Because the states between
E and F does not contribute to the Hall current, there can exist unfilled localized states as well as filled localized ones. Hence an unfilled localized state will appear in Fig.3 and this appearance makes the filling factor being a fractional value.

Now we can make use of the above model to construct a EME current associated to the $q$-Virasoro symmetry. For simplicity, we confine ourselves to examine the $U_q(sl(2))$ subalgebra part and consider five states $e_k$ which follows the 5-dimensional representation. According to (1.3), $E^+ e_k = [5 - k]_q e_{k+1}$ and $E^- e_k = [k - 1]_q e_{k-1}$ are satisfied and we should notice that $E^+ e_2 = E^- e_4 = 0$ which forbids the state $e_3$ to be arrived from the states $e_2$ and $e_4$ when $q^3 = 1$. The state $e_3$ is isolated and is therefore regarded as an unfilled ‘localized’ state. In contrast with the $e_1$ moving to the place of $e_2$, the $e_2$ moves into the $e_5$ through other generator of the quantum group $\hat{E}^+$, which will be mentioned in the next section,

$$\hat{E}^\pm = \frac{1}{[3]_q[2]_q[1]_q} (E^\pm)^3 .$$

(5.5)

Then the $e_5$ inevitably goes back to the $e_4$ and the $e_4$ disappears into the vanishing potential region through the action of $\hat{E}^+$. Consequently, the EME is accomplished by the quantum group action.

The occupation ratio of the ‘unlocalized’ states including one state, which is a destination of the outgoing state $e_4$, amounts to

$$\nu = \frac{4}{5+1} = \frac{2}{3} .$$

(5.6)

It should be noticed that this filling ratio is consistent to the relation (3.8). The pattern of the moving of the ‘electrons’ is digested in Fig.4. We should not wonder if the dimension of the representation does not coincide with the number $m$ in $q^m = 1$ contrary to the results in previous sections. The situation like this can be realized from a tensor product decomposition, i.e., $3 \otimes 3 = 5 \oplus 3 \oplus 1$, which will be discussed in the next section.

6 Tensor product representation

The operation rule of the $U_q(sl(2))$ generators on a tensor product representation is
given by the comultiplication mapping $\Delta$ of the Hopf algebra \[}\Delta(E^\pm) = E^\pm \otimes k^{-1} + k \otimes E^\pm, \quad \Delta(k^\pm) = k^\pm \otimes k^\pm.\]\[\text{(6.1)}\]

It follows that the generators on a $N$-fold tensor product representation become

$$E^\pm = \sum_{i=1}^{N} k_1 \ldots k_{i-1} E_i^\pm k_{i+1} \ldots k_{N}^{-1}, \quad k = \prod_{i=1}^{N} k_i$$ \[\text{(6.2)}\]

where $E_i^\pm$ and $k_i$ mean the generators on the $i$-th representation space. And the formula (2.7) is also assumed in each space. Let us consider the tensor product of two spin-1 representations for simplicity

$$|i,j\rangle = |i\rangle \otimes |j\rangle.$$ \[\text{(6.3)}\]

The representation basis is decomposed into three representations, i.e. the singlet $e_9$, the triplet $(e_8, e_7, e_6)$ and the quintet $(e_5, e_4, e_3, e_2, e_1)$;

$$e_5 = |1,1\rangle, \quad e_4 = q^{-1}|0,1\rangle + q|1,0\rangle$$

$$e_3 = q^2|1,-1\rangle + q^{-2}|-1,1\rangle - (q + q^{-1})|0,0\rangle$$

$$e_2 = q|0,-1\rangle + q^{-1}|-1,0\rangle, \quad e_1 = |-1,-1\rangle,$$ \[\text{(6.4)}\]

$$e_8 = q|0,1\rangle - q^{-1}|1,0\rangle, \quad e_6 = q|-1,0\rangle - q^{-1}|0,-1\rangle$$

$$e_7 = |1,-1\rangle - |-1,1\rangle + (q - q^{-1})|0,0\rangle,$$ \[\text{(6.5)}\]

and

$$e_9 = |0,0\rangle + q^{-1}|1,-1\rangle + q|-1,1\rangle.$$ \[\text{(6.6)}\]

The representation matrices are

$$\rho_5(E^+) = \text{diag}[-a,a,-(q^2 + 1 + q^{-1})a,(q^2 + q^{-2})a]$$

$$\rho_5(E^-) = \text{diag}[(q^2 + q^{-2})a,-(q^2 + 1 + q^{-1})a,a,-a]$$

$$\rho_5(k) = \text{diag}[q^2,q,1,q^{-1},q^{-2}]$$ \[\text{(6.7)}\]

and

$$\rho_3(E^+) = \text{diag}[-a,a], \quad \rho_3(E^-) = \text{diag}[a,-a]$$

$$\rho_3(k) = \text{diag}[q^2,1,q^{-2}],$$ \[\text{(6.8)}\]
where \( a = [1/2]_q \). The representations of both (6.7) and (6.8) satisfy the same commutation relations as those of original 9-dimensional representation. When \( q^3 = 1 \), they satisfy the \( U_q(sl(2)) \) commutation relations (1.2) as mentioned in previous sections.

We now notice that some new features happen in the case \( q \) being a root of unity \([19]\). For example when \( q^3 = 1 \), we can easily verify that

\[
(E^\pm)^3 = 0 \tag{6.9}
\]

and \((E^\pm)^3\) commute with \( E^\pm \) and \( k \). As a result, \( e_2 \) and \( e_4 \) can not be arrived from \( e_3 \)

\[
E^\pm e_3 = 0 . \tag{6.10}
\]

In spite of (6.9), however, \( e_2 \) and \( e_4 \) are connected with \( e_5 \) and \( e_1 \) respectively by the operator (5.5)

\[
\hat{E}^\pm = \frac{(E^\pm)^3}{[3]_q} . \tag{6.11}
\]

Namely

\[
\hat{E}^+ e_1 = -ae_4 , \quad \hat{E}^- e_5 = -ae_2 ,
\]

\[
\hat{E}^- e_4 = -ae_1 , \quad \hat{E}^+ e_2 = -ae_5 , \tag{6.12}
\]

in which the value of \( a \) becomes

\[
a = (-1)^k , \quad q = exp(2\pi i \frac{k-1}{3}) , \quad k = 2, 3 . \tag{6.13}
\]

As mentioned in (6.10), \( e_2 \) and \( e_4 \) can not be reached from \( e_3 \) and so the quantum group can not move the state \( e_3 \) whether it is occupied by an localized state which does not contribute to the 'Hall' current and that the pattern of the current is the same as Fig.4 discussed previous section.

7 Conclusion

We have discussed the quantum group \( U_q(sl(2)) \) structure in two-dimensional electron systems and showed that this symmetry can be extended into the \( q \)-deformation of the
Virasoro algebra. These symmetries exist in many-particle systems as the Laughlin system, however they concern only the global translational symmetry and they are essentials in one-body system. Nevertheless, the appearance of the $q$-Virasoro algebra is interesting, because we have not understood the reason why it appears. The dimensionally reduced generators satisfy the centrally extended ($c = 1/2$) $q$-Virasoro algebra. In this relevance, it might be related to the $c = 1/2$ representation.

Owing to the external magnetic field, $su(2)$ subalgebra of the Virasoro algebra is deformed to $U_q(sl(2))$ and then the $q$-Virasoro algebra appears. This feature matches with the picture whether the system is perturbed by some interaction corresponds to whether the conformal symmetry is deformed or not. These quantum group generators are given by the nonlocal charges in a nonrelativistic fermion field theory as well as in solvable off-critical CFT models [20]. These similarities are also interesting features of the appearance of the $q$-Virasoro algebra.

In closing the paper, we would like to expect that the relation of our $q$-deformed algebras or of complete quantum Virasoro algebra to off-critical CFTs would become clear in future. We wish to speculate that our quantum group approach serves us with new aspects of fermion systems, related other topics [21] and the qunatum Virasoro algebra.

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A The case of $p \neq 1$

In this appendix, we note the case of the raising and lowering operators $E^\pm$ changing the quantum number $l$ by the number $p$. When $p = 1$, $(2j + 1)$-fold degeneracy is inevitably built into the $(2j + 1)$-dimensional (spin-$j$) representation and $q$ is determined to (2.8). We however point out that other values of $q$ are possible in the cases of $p \geq 2$.

Let us consider the $(2j + 1)$-fold degeneracy and two integers $k_1$ and $k_2$ defined by

$$2j + 1 = k_1 p + k_2 , \quad p > k_2 . \quad (A.1)$$

Instead of (2.7), we have the relations

$$E^\pm \psi_l = \left[ \frac{1}{2} \pm \frac{l}{p} \right] q^{\psi_l \pm p} , \quad k \psi_l = q^{l/p} \psi_l , \quad (A.2)$$

and thus have $a + 1$ $(k_1 + 1)$-dimensional representations ($0 \leq a < k_2$):

$$\rho(E^+) = diag^+([[-\frac{j+a}{p} + \frac{1}{2}]q, [-\frac{j+a}{p} + \frac{3}{2}]q, \ldots, [-\frac{j+a}{p} + \frac{2k_1-1}{2}]q]) ,$$

$$\rho(E^-) = diag^-([-[-\frac{j+a}{p} + \frac{1}{2}]q, [-[-\frac{j+a}{p} + \frac{3}{2}]q, \ldots, [-[-\frac{j+a}{p} + \frac{2k_1-1}{2}]q]) ,$$

$$\rho(k) = diag(q^{-(j+a)/p+k_1}, q^{-(j+a)/p+k_1-1}, \ldots, q^{-(j+a)/p}) . \quad (A.3)$$

If we put

$$q = exp\left(2\pi i \frac{p}{p + 2j}\right) , \quad (A.6)$$

the representation matrices become simple forms. For example, $E^+$ of $a = 0$ representation reads

$$\rho(E^+) = diag^+([-1]q, [-2]q, \ldots, [-k_1]q) . \quad (A.7)$$

The simplest case is $k_2 = 1$, namely the only possible value of $a$ is zero. Eq.(A.6) becomes

$$q = exp\left(\frac{2\pi i}{k_1 + 1}\right) \quad (A.8)$$

and the representation matrices (A.3) coincides with spin-$(k_1/2)$ representation with the value of (A.8). We then conclude that when the generators label the quantum number $l$ by the $p$ ($\neq 1$), $q$ is related not to the filling factor but to the dimensions of representations only.
B Neveu-Schwarz fermion

We show the realization of the $U_q(sl(2))$ generators by the Neveu-Schwarz fermion in two dimensions\cite{4, 5, 6}:

$$\psi(z) = \sum_r b_r \frac{z^{r+1/2}}{z^r}, \quad r \in \mathbb{Z} + \frac{1}{2}. \tag{B.1}$$

Let us define the following Fourier mode of a nonlocal current

$$L_n^{(k)} = \frac{-1}{2\pi i} \oint dz z^n : \psi(z) q^{k/2} \frac{q^{z\partial} - q^{z\partial}}{q^k - q^{-k}} \psi(z) :. \tag{B.2}$$

The explicit form of $L_n^{(k)}$ in terms of the fermion oscillators is

$$L_n^{(k)} = \frac{1}{2[k]} \sum_r \left[ \frac{n - 2r}{2} \right] : b_r b_{n-r} : \tag{B.3}$$

where $: :$ denotes the normal ordering defined by

$$: b_r b_s := b_r b_s - \theta (r) \{b_r, b_s\}, \quad \{b_r, b_s\} = \delta_{r+s,0}. \tag{B.4}$$

The generators which satisfy the commutation relations (1.2) are

$$E^+ = L_1^{(1)}, \quad E^- = -L_{-1}^{(1)}, \quad k = q^{-L_0}, \tag{B.5}$$

where $L_0$ is the zero mode of the Virasoro operator

$$L_n = \frac{1}{2} \sum_r \left( \frac{n - r}{2} \right) : b_r b_{n-r} :. \tag{B.6}$$

Furthermore, (B.3) satisfies the $q$-Virasoro algebra with a central extension

$$[L_n^{(i)}, L_m^{(j)}] = \sum_{c=\pm 1} C_m^{i, c} L_{n+m}^{(i+cj)} + \frac{1}{2} C_{ij}(n) \delta_{n+m}, \tag{B.7}$$

where $C_m^{i, j}$ is given by (4.12) and

$$C_{ij}(n) = \frac{1}{[i][j]} \sum_{k=1}^n \left[ \frac{(n+1-2k)i}{2} \right] \left[ \frac{(n+1-2k)j}{2} \right]. \tag{B.8}$$
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FIGURE CAPTIONS

**Fig.1:** The Hall current with $\nu = 1$ induced by $\Delta \Phi$.

**Fig.2:** Each Landau level becomes the Landau subband like a mountain. The regions $A$ and $C$ correspond to the localized states and the region $B$ to the unlocalized ones.

The point $F$ means the Fermi energy.

**Fig.3:** In the presence of a localization potential, EME is not like Fig.1 but like Fig.3.

The closed loop corresponds to a localized state.

**Fig.4:** The pattern of the quantum group 'Hall' current with the filling factor $\nu = 2/3$.

$e_3$ is a 'localized' state which does not contribute the current.
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