AN ELEMENTARY CHARACTERISATION OF SIFTED WEIGHTS

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Abstract. Sifted colimits (those that commute with finite products in sets) play a major rôle in categorical universal algebra. For example, varieties of (many-sorted) algebras are precisely the free cocompletions under sifted colimits of (many-sorted) Lawvere theories. Such a characterisation does not depend on the existence of finite products in algebraic theories, but on the above fact that these products commute with sifted colimits and another condition: finite products form a sound class of limits.

In this paper we study the notion of soundness for general classes of weights in enriched category theory. We show that soundness of a given class of weights is equivalent to having a ‘nice’ characterisation of flat weights for that class. As an application, we give an elementary characterisation of sifted weights for the enrichment in categories and in preorders. We also provide a number of examples of sifted weights using our elementary criterion.

1. Introduction

The classical theory of locally presentable [8] and accessible categories [15], [20] (see also the more recent [2]) has been generalised to locally $D$-presentable and $D$-accessible categories for a ‘good’ class $D$ of small categories in [1]. The classical theory hinges a lot upon the interplay of two classes of categories: the class of $\lambda$-small categories for limits and the class of $\lambda$-filtered categories for colimits, where $\lambda$ is a fixed regular cardinal. The precise nature of the interplay is that

$\lambda$-small limits commute with $\lambda$-filtered colimits in the category of all sets and mappings.

The idea of [1] was to develop a more general theory of locally presentable and accessible categories based on the fact that one has a fixed class $D$ of small categories that replaces the class of $\lambda$-small categories. The corresponding class of colimits, called $D$-filtered, is then defined by the requirement that

$D$-limits commute with $D$-filtered colimits in the category of sets and mappings.

It has been shown in [1] that a great deal of the classical theory can be developed for the concept of $D$-filterredness, provided that the class $D$ satisfies a side condition that is called soundness.

Roughly speaking, sound classes $D$ allow for an easier detection of $D$-filteredness: for a sound class $D$ a category $E$ is $D$-filtered if the process of taking colimits over $E$ commutes with taking $D$-limits of representable functors.

For example, the class $D$ consisting of finite discrete categories is sound. The corresponding $D$-filtered colimits turn out to be precisely the sifted colimits of [16]. Free cocompletions of small categories under sifted colimits generalise the notion of a variety, as shown in [3]. In fact, the notion of a sifted colimit turned out to be a cornerstone notion in the categorical treatment of universal algebra, see [4].

One can pass from categories to categories enriched in a suitable monoidal $\mathcal{V}$ and ask whether the above results can be reproduced. Since (co)limits over a class of categories have to be replaced by weighted (co)limits for $\mathcal{V}$-categories, one is naturally forced to define and study soundness of classes of weights. Definitions of soundness of a class of weights have appeared in [14] (Axiom A) and in Section 4 of [7]. The provided definitions of soundness, when specialised to categories enriched in sets, are, however, weaker than the definition in [1].

Results of the paper. For categories enriched in a general $\mathcal{V}$, we give in Section 3 a definition of soundness of a class $\Psi$ of weights that is equivalent to the notion of [1] when $\mathcal{V}$ is the category of sets. We show in Proposition 3.8 that all notions of soundness that have appeared in the literature coincide when the class $\Psi$ satisfies two side conditions that often arise in practice. Namely, the class $\Psi$ has to be locally small in the sense of [12] and saturated in the sense of [5] (there called a closed class).

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Our definition allows us to give, for a sound class \( \Psi \), a characterisation of \( \Psi \)-flat weights (those weights \( \varphi \) such that \( \varphi \)-colimits commute with \( \Psi \)-limits in \( \mathcal{V} \)) by means of a certain coend. We show that this characterisation, when \( \mathcal{V} \) is the category of sets and \( \mathcal{D} \) is sound, boils down to the characterisation of \( \mathcal{D} \)-filteredness of \([1]\) in terms of cocones for certain diagrams.

In Section 4 we turn the coend characterisation to a useful criterion of \( \Psi \)-flat weights when the sound class \( \Psi \) is the class of weights for finite products. Thus \( \Psi \)-flat weights are precisely the (enriched) sifted weights. Specialising to the enrichment in \( \text{Cat} \), we can therefore deal with siftedness for 2-categories. We apply the coend criterion to give elementary proofs of siftedness of various weights used in 2-dimensional universal algebra, see, e.g., \([6]\).

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### 2. Preliminaries on classes of weights

We introduce now the basic notation and results on weighted limits and colimits that we will need later. The material here is standard; for more details we refer to the book \([10]\) and the paper \([12]\).

We fix a complete and cocomplete symmetric monoidal category \( \mathcal{V} = (\mathcal{V}_o, \otimes, I, [-, -]) \). All categories, functors and natural transformations are to be understood as \( \mathcal{V} \)-categories, \( \mathcal{V} \)-functors and \( \mathcal{V} \)-natural transformations.

**Limits and colimits.** A **weight** is a functor \( \varphi : \mathcal{E}^{op} \to \mathcal{V} \), where \( \mathcal{E} \) is small. Given a weight \( \varphi : \mathcal{E}^{op} \to \mathcal{V} \) and a diagram \( D : \mathcal{E} \to \mathcal{K} \), a colimit of \( D \) weighted by \( \varphi \) is an object \( \varphi \cdot D \) together with an isomorphism

\[
\mathcal{K}(\varphi \cdot D, X) \cong [\mathcal{E}^{op}, \mathcal{V}](\varphi, \mathcal{K}(D-, X))
\]

natural in \( X \). Dually, a **limit** of a diagram \( D : \mathcal{E}^{op} \to \mathcal{K} \) weighted by \( \varphi : \mathcal{E}^{op} \to \mathcal{V} \) is an object \( \{\varphi, D\} \) together with an isomorphism

\[
\mathcal{K}(X, \{\varphi, D\}) \cong [\mathcal{E}^{op}, \mathcal{V}](\varphi, \mathcal{K}(X, D-))
\]

natural in \( X \).

Given a weight \( \varphi : \mathcal{E}^{op} \to \mathcal{V} \), a category \( \mathcal{K} \) is called **\( \varphi \)-cocomplete** if it has colimits \( \varphi \cdot D \) of all diagrams \( D : \mathcal{E} \to \mathcal{K} \). Analogously, we define **\( \varphi \)-completeness** of \( \mathcal{K} \).

Suppose \( \mathcal{K} \) and \( \mathcal{L} \) have \( \varphi \)-colimits. A functor \( H : \mathcal{K} \to \mathcal{L} \) between \( \varphi \)-cocomplete categories **preserves** \( \varphi \)-colimits (or, is **\( \varphi \)-continuous**) if the canonical comparison \( \varphi \cdot HD \to H(\varphi \cdot D) \) is an isomorphism, for all \( D : \mathcal{E} \to \mathcal{K} \).

Even more generally, given a class \( \Phi \) of weights, we say that \( \mathcal{K} \) is **\( \Phi \)-cocomplete** (or \( \Phi \)-complete, resp.) if it has \( \varphi \)-colimits (\( \varphi \)-limits, resp.) for all \( \varphi \) in \( \Phi \). Analogously, we define **\( \Phi \)-continuity** (or \( \Phi \)-continuous, resp.) functors.

**Free cocompletions and saturated classes of weights.** Given a class \( \Phi \) of weights, let us write \( U_\Phi : \Phi \text{-}\text{Cocts} \to \text{Cat} \) for the forgetful 2-functor from the 2-category \( \Phi \text{-}\text{Cocts} \) of \( \Phi \)-cocomplete categories, \( \Phi \)-continuous functors and natural transformations. Then \( U_\Phi \) has a left adjoint pseudofunctor, yielding a free \( \Phi \)-cocompletion \( \Phi(\mathcal{K}) \) of any category \( \mathcal{K} \).

For a small category \( \mathcal{E} \), the free \( \Phi \)-cocompletion \( \Phi(\mathcal{E}) \) can be computed via a transfinite process, namely the closure of \( \mathcal{E} \) in \([\mathcal{E}^{op}, \mathcal{V}] \) under \( \Phi \)-colimits.\(^1\) We will need the first step \( \Phi_1(\mathcal{E}) \) of this process: \( \Phi_1(\mathcal{E}) \) is the full subcategory of \([\mathcal{E}^{op}, \mathcal{V}] \) spanned by \( \Phi \)-colimits of representables. Hence a weight \( \alpha : \mathcal{E}^{op} \to \mathcal{V} \) is in \( \Phi_1(\mathcal{E}) \) iff it is of the form \( \text{Lan}_T \varphi \) for some \( T : \mathcal{D} \to \mathcal{E} \) and some weight \( \varphi : \mathcal{D}^{op} \to \mathcal{V} \) in \( \Phi \).

The class \( \Phi \) is **saturated** (the concept introduced in \([5]\), there called **closed**) if, for any small category \( \mathcal{E} \), the free cocompletion \( \Phi(\mathcal{E}) \) consists precisely of all the weights \( \varphi : \mathcal{E}^{op} \to \mathcal{V} \) that belong to \( \Phi \). If we put \( \Phi^* \) to be the largest class such that the 2-categories \( \Phi^* \text{-}\text{Cocts} \) and \( \Phi \text{-}\text{Cocts} \) coincide, then \( \Phi^* \) is saturated and it is the least saturated class containing \( \Phi \).

**Commutation of limits and colimits, flatness.** Let \( \Phi \) and \( \Psi \) be classes of weights. We say that \( \Phi \)-colimits commute with \( \Psi \)-limits in \( \mathcal{V} \), if for any \( \varphi : \mathcal{E}^{op} \to \mathcal{V} \) in \( \Phi \), the functor

\[
\varphi \cdot (-) : \mathcal{E}, \mathcal{V} \to \mathcal{V}
\]

preserves \( \Psi \)-limits. We denote by \( \Psi^+ \) the class of \( \Psi \)-flat weights, i.e., all weights \( \varphi \) such that \( \varphi \)-colimits commute with \( \Psi \)-limits in \( \mathcal{V} \).

\(^1\)In fact, the same transfinite process can be applied to obtain \( \Phi(\mathcal{K}) \) for any category \( \mathcal{K} \). There is, however, a slight technicality concerning size when \( \mathcal{K} \) is not small. Since we will not need \( \Phi(\mathcal{K}) \) for large \( \mathcal{K} \), we refer to \([10]\) for more details.
3. Sound classes of weights

In this section we generalise the definition of a sound class $\mathcal{D}$ of small (ordinary) categories to soundness of a class $\Psi$ of weights for a general $\mathcal{Y}$.

Soundness of a class $\mathcal{D}$ of small categories was defined (in case $\mathcal{Y} = \mathbf{Set}$) in [1] using connectedness of a certain category of cocones. Since cocones are not available for general $\mathcal{Y}$, we use a different phrasing (already implicit in [1]). We prove in Proposition 3.5 below that our definition coincides with that of [1] in case $\mathcal{Y} = \mathbf{Set}$.

For any $\Psi$-flat weight $\varphi$, the functor $\varphi \ast (-)$ is obliged to preserve all $\Psi$-limits by the definition of $\Psi$-flatness (see Section 2). Soundness of $\Psi$ means that we can choose a smaller class of $\Psi$-limits to detect $\Psi$-flatness.

**Definition 3.1.** A class $\Psi$ of weights is called sound if a weight $\varphi : \mathcal{E}^{\text{op}} \to \mathcal{Y}$ is $\Psi$-flat whenever the functor

$$
\varphi \ast (-) : [\mathcal{E}, \mathcal{Y}] \to \mathcal{Y}
$$

preserves $\Psi$-limits of representables.

**Example 3.2.**

1. In case $\mathcal{Y} = \mathbf{Set}$, the list

$$
\mathcal{D} = \text{finite categories, } \mathcal{D} = \text{finite discrete categories, } \mathcal{D} = \text{empty class}
$$

yields a list of sound classes $\Psi_\mathcal{D}$ of weights by Example 2.3 of [1].

By the same example, the one-element class $\mathcal{D}$ consisting of the scheme for pullbacks, or the two-element class $\mathcal{D}$ consisting of the scheme for pullbacks and terminal objects, yield classes $\Psi_\mathcal{D}$ that are not sound.

2. It has been proved in [11] that the class $\mathcal{P}$ of weights for finite products is sound, for every cartesian closed $\mathcal{Y}$.

3. The class $\mathcal{Q}$ of all weights is sound for any $\mathcal{Y}$. Indeed, the class of all $\mathcal{Q}$-flat weights is precisely the class $\mathcal{Q}$ of all small-projective weights by Proposition 6.20 of [12]. By the same proposition, small-projectivity of $\varphi : \mathcal{E}^{\text{op}} \to \mathcal{Y}$ can be detected by the fact that $\varphi \ast (-) : [\mathcal{E}, \mathcal{Y}] \to \mathcal{Y}$ preserves a particular limit of representables, namely the limit $\{ \varphi, Y \}$, where $Y : \mathcal{E}^{\text{op}} \to [\mathcal{E}, \mathcal{Y}]$ is the Yoneda embedding. Thus, $\mathcal{P}$ is sound.

4. The class $\mathcal{Q}$ of all small-projective weights is sound for any $\mathcal{Y}$. By Remark 8.17 of [12], the class of $\mathcal{Q}$-flat weights coincides with the class $\mathcal{P}$ of all weights. Hence the condition on soundness is vacuous.

The following easy result shows that the ‘testing weights’ for $\Psi$-flatness can be taken in a special form:

**Proposition 3.3.** For a class $\Psi$ the following are equivalent:

1. $\Psi$ is sound.
2. The weight $\varphi : \mathcal{E}^{\text{op}} \to \mathcal{Y}$ is $\Psi$-flat, whenever $\varphi \ast (-)$ preserves $\Psi_1(\mathcal{E})$-limits of representables, i.e., whenever the canonical morphism

$$
can : \varphi \ast \{ \psi, Y \} \to \{ \psi, \varphi \}
$$

is an isomorphism, for every $\psi : \mathcal{E}^{\text{op}} \to \mathcal{Y}$ in $\Psi_1(\mathcal{E})$.

**Proof.** Let $Y : \mathcal{E}^{\text{op}} \to [\mathcal{E}, \mathcal{Y}]$ be the Yoneda embedding. Definition 3.1 requires the canonical morphism

$$
\varphi \ast \{ \psi, YT^{\text{op}} \} \to \{ \psi, \varphi \cdot T^{\text{op}} \}
$$

to be an isomorphism, for every $\psi : \mathcal{D}^{\text{op}} \to \mathcal{Y}$ in $\Psi$ and every $T : \mathcal{D} \to \mathcal{E}$. 

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**Example 2.1.** Suppose $\mathcal{Y}$ is cartesian closed. By $\text{const}_1 : \mathcal{D}^{\text{op}} \to \mathcal{Y}$ we denote the weight that is constantly the terminal object 1. Such weights will be called conical. Any class $\mathcal{D}$ of small categories induces a class

$$
\Psi_\mathcal{D}
$$

of conical weights $\text{const}_1 : \mathcal{D}^{\text{op}} \to \mathcal{Y}$ with $\mathcal{D}^{\text{op}}$ in $\mathcal{D}$.

1. Suppose $\mathcal{Y} = \mathbf{Set}$. Then to say that a small category $\mathcal{E}$ is $\mathcal{D}$-filtered in the sense of [1] is to say that the conical weight $\text{const}_1 : \mathcal{E}^{\text{op}} \to \mathbf{Set}$ is $\Psi_\mathcal{D}$-flat. Indeed: (co)limits of diagrams weighted by conical weights yield the usual notions defined by (co)cones.

2. Suppose $\mathcal{Y}$ is arbitrary (but still cartesian closed). The class $\Psi_\mathcal{D}$ for $\mathcal{D}$ consisting of all finite discrete categories will be denoted by $\Pi$. The corresponding class of $\Pi$-flat weights is called the class of sifted weights. We will say more on sifted weights in Section 4.
The weight \( \text{Lan}_{T\to \psi} : \mathcal{E}^{\text{op}} \to \mathcal{V} \) is in \( \Psi_1(\mathcal{E}) \) and every weight in \( \Psi_1(\mathcal{E}) \) has this form, for some \( \psi : \mathcal{D}^{\text{op}} \to \mathcal{V} \) in \( \Psi \) and some \( T : \mathcal{D} \to \mathcal{E} \).

Since there are isomorphisms

\[
\{ \psi, YT^{\text{op}} \} \cong \{ \text{Lan}_{T\to \psi}, Y \}, \quad \{ \psi, \varphi \cdot T^{\text{op}} \} \cong \{ \text{Lan}_{T\to \psi}, \varphi \}
\]

the equivalence of (1) and (2) follows.

The canonical morphism in (3.1) can be rewritten using coends and Yoneda Lemma as the morphism

\[
\text{can} : \int^{e} [\mathcal{E}^{\text{op}, \mathcal{V}}(Ye, e) \otimes [\mathcal{E}^{\text{op}, \mathcal{V}}(\psi, Ye)](\psi, Ye) \to [\mathcal{E}^{\text{op}, \mathcal{V}}(\psi, Ye)) (3.2)
\]

that is given by composition in \( [\mathcal{E}^{\text{op}, \mathcal{V}}] \). We illustrate now on two well-known classes that this coend description yields precisely the ‘classical’ description of flatness by means of the category of cocones.

**Example 3.4 (Sifted weights and flat weights for \( \mathcal{V} = \text{Set} \).)** Suppose \( \mathcal{V} = \text{Set} \). Recall that every \( \varphi : \mathcal{E}^{\text{op}} \to \text{Set} \) has a category of elements \( \text{elts}(\varphi) \): the objects are pairs \( (x, c) \) with \( x \in \varphi e \) and a morphism from \( (x, c) \) to \( (x', c') \) is a morphism \( t : e \to e' \) in \( \mathcal{E} \) such that \( \varphi t(e') = e \) holds.

1. Let \( \Pi \) be the sound class of weights for finite products. The category \( \Pi_1(\mathcal{E}) \) is spanned by finite coproducts of representables in \( [\mathcal{E}^{\text{op}, \text{Set}}] \). Hence a general testing weight \( \psi : \mathcal{E}^{\text{op}} \to \text{Set} \) for \( \Pi \)-flatness by Proposition 3.3 has the form \( \coprod_{e \in I} Ye \) where \( I \) is a finite set.

   We show now that (3.2) yields the well-known characterisation of sifted weights, see [16]. Indeed, given a general weight \( \varphi : \mathcal{E}^{\text{op}} \to \text{Set} \), the mapping \( \text{can} \) has the form

\[
\text{can} : \int^{e} \varphi e \times \prod_{i \in I} \mathcal{E}(e_i, e) \to \prod_{i \in I} \varphi e_i, \quad [(x, (t_i))] \mapsto (\varphi t_i(x))
\]

Hence \( \text{can} \) is a bijection if and only if the following two conditions hold:

(a) The mapping \( \text{can} \) is surjective, i.e., for every element of \( \prod_{i \in I} \varphi e_i \), i.e., for every \( I \)-tuple \( (x_i) \) of elements of \( \varphi \) there is an \( e \), an element \( x \in \varphi e \) and an \( I \)-tuple \( t_i : e_i \to e \) of morphisms in \( \mathcal{E} \) such that \( \varphi t_i(x) = x_i \).

   Briefly: on every \( I \)-tuple of objects of \( \text{elts}(\varphi) \) there is a cocone.

(b) The mapping \( \text{can} \) is injective, i.e., for any pair \( (x, (t_i)), (x', (t'_i)) \) such that \( \varphi t_i(x) = \varphi t'_i(x') \) holds for all \( i \), i.e., for any two cocones of the same \( I \)-tuple of objects of \( \text{elts}(\varphi) \), there is a zig-zag in \( \mathcal{E} \) that connects these cocones in \( \text{elts}(\varphi) \).

   To summarise: a weight \( \varphi : \mathcal{E}^{\text{op}} \to \text{Set} \) is \( \Pi \)-flat iff its category of elements is sifted (every finite family of elements has a cocone and every two cocones for the same finite family are connected by a zig-zag).

2. Let \( \Psi \) be the sound class of finite (conical) limits, i.e., let \( \Psi = \text{Set}_\Delta \) for the class \( \Delta \) of finite categories.

   The category \( \Psi_1(\mathcal{E}) \) is spanned by finite colimits of representable functors in \( [\mathcal{E}^{\text{op}, \text{Set}}] \). Thus, a general testing weight \( \psi : \mathcal{E}^{\text{op}} \to \text{Set} \) for \( \Psi \)-flatness has the form \( \psi = \text{colim} YC \) for a diagram \( C : \mathcal{E} \to \mathcal{E} \) with \( \mathcal{E} \) finite.

   Given a general weight \( \varphi : \mathcal{E}^{\text{op}} \to \text{Set} \), the mapping \( \text{can} \) is a bijection if and only if two conditions hold:

(a) The mapping \( \text{can} \) is surjective, i.e., every finite diagram in \( \text{elts}(\varphi) \) has a cocone.

(b) The mapping \( \text{can} \) is injective, i.e., any two cocones for the same finite diagram in \( \text{elts}(\varphi) \) are connected by a zig-zag in \( \text{elts}(\varphi) \).

   The above two conditions together state that the category of cocones of finite diagrams in \( \text{elts}(\varphi) \) is nonempty and connected. This means that the category \( \text{elts}(\varphi) \) is filtered. As expected, \( \Psi \)-flat weights are precisely the flat ones.

In both cases above, the classes of testing weights can be simplified. For example, for siftedness, one can choose only nullary coproduct of representables and binary coproducts of representables as the testing weights. We will use this fact in Section 4 below.

We prove now that Definition 3.1 coincides with the definition of soundness from [1] (this definition is condition (2) of the proposition).

**Proposition 3.5.** Suppose \( \mathcal{V} = \text{Set} \). For a class \( \Delta \) of small categories, the following conditions are equivalent:

1. The class \( \Psi_\Delta \) of conical weights \( \text{const}_1 : \mathcal{D}^{\text{op}} \to \text{Set} \) with \( \mathcal{D}^{\text{op}} \) in \( \Delta \) is sound.

2. A category \( \mathcal{E} \) is \( \Delta \)-filtered whenever the category of cocones for any functor \( T : \mathcal{D} \to \mathcal{E} \) with \( \mathcal{D}^{\text{op}} \) in \( \Delta \) is nonempty and connected.
Proof. We will use the canonical morphism (3.2). Observe first that \([\mathcal{E}^{\text{op}}, \mathbf{Set}](\psi, \text{const}_1)\) is a one-element set for any small category \(\mathcal{E}\) and any \(\psi: \mathcal{E}^{\text{op}} \to \mathbf{Set}\), since \(\text{const}_1\) is a terminal object in \([\mathcal{E}^{\text{op}}, \mathbf{Set}]\).

By Proposition 3.3 any testing weight \(\psi: \mathcal{E}^{\text{op}} \to \mathbf{Set}\) for \(\Psi_\mathbf{D}\)-flatness of \(\text{const}_1: \mathcal{E}^{\text{op}} \to \mathbf{Set}\) has the form \(\text{Lan}_{T^\text{op}}\text{const}_1\) for some \(T: \mathcal{D} \to \mathcal{E}\), where \(\text{const}_1: \mathcal{D}^{\text{op}} \to \mathbf{Set}\) is in \(\Psi_\mathbf{D}\). The left-hand side of (3.2) therefore has the form

\[
\int^c [\mathcal{E}^{\text{op}}, \mathbf{Set}](\text{Lan}_{T^\text{op}}\text{const}_1, Y) \cong \int^c [\mathcal{D}^{\text{op}}, \mathbf{Set}](\text{const}_1, Y \cdot T^{\text{op}}) \cong \int^c [\mathcal{D}^{\text{op}}, \mathbf{Set}](\text{const}_1, \mathcal{E}(T^-, e))
\]

Observe that the category of elements of \([\mathcal{D}^{\text{op}}, \mathbf{Set}](\text{const}_1, \mathcal{E}(T^-, e))\) is precisely the category of cocones for \(T\) that have \(e\) as a vertex.

Thus (3.2) is a bijection iff

\[
\int^c [\mathcal{D}^{\text{op}}, \mathbf{Set}](\text{const}_1, \mathcal{E}(T^-, e)) \cong 1
\]

holds. From this, the equivalence of (1) and (2) follows immediately.

There is another important issue related to sound classes of weights. Adapting freely the terminology of [4], we may call a small \(\Psi\)-complete category \(\mathcal{F}\) a \(\Psi\)-theory. The category \(\Psi\text{-Alg}(\mathcal{F})\) of \(\Psi\)-algebras for \(\mathcal{F}\) is the full subcategory of \([\mathcal{F}, \mathcal{V}]\) spanned by functors that preserve \(\Psi\)-limits.

By definition, \(\Psi\)-flat colimits commute with \(\Psi\)-limits. Hence the category \(\Psi\text{-Alg}(\mathcal{F})\) is closed in \([\mathcal{F}, \mathcal{V}]\) under \(\Psi\)-flat colimits and it contains the representables, for any \(\Psi\)-theory \(\mathcal{F}\). Therefore, for any class \(\Psi\) and any \(\Psi\)-theory \(\mathcal{F}\), there is an inclusion

\[
\Psi^+(\mathcal{F}^{\text{op}}) \subseteq \Psi\text{-Alg}(\mathcal{F})
\]

since \(\Psi^+(\mathcal{F}^{\text{op}})\) is the closure in \([\mathcal{F}, \mathcal{V}]\) of the representables under \(\Psi\)-flat colimits. We discuss now the case when the above inclusion is an equality.

Lemma 3.6. Suppose the class \(\Psi\) is sound. Then \(\Psi^+(\mathcal{F}^{\text{op}}) = \Psi\text{-Alg}(\mathcal{F})\) holds for any \(\Psi\)-theory \(\mathcal{F}\).

Proof. Suppose \(\varphi: \mathcal{F} \to \mathcal{V}\) preserves \(\Psi\)-limits. Then \(\varphi \ast (-): [\mathcal{F}^{\text{op}}, \mathcal{V}] \to \mathcal{V}\) preserves \(\Psi\)-limits of representables. By soundness of \(\Psi\), this means that \(\varphi\) is \(\Psi\)-flat. Hence \(\varphi\) is in \(\Psi^+(\mathcal{F}^{\text{op}})\).

Remark 3.7. The equality \(\Psi^+(\mathcal{F}^{\text{op}}) = \Psi\text{-Alg}(\mathcal{F})\) for any \(\Psi\)-theory \(\mathcal{F}\) is an important fact that allows for the development of an abstract theory of ‘algebras’ for \(\Psi\).

For example, when \(\mathcal{V} = \mathbf{Set}\) and \(\Psi = \Pi\), much of the theory of varieties of algebras hinges upon the above equality, see [4]. Namely, a \(\Pi\)-theory \(\mathcal{F}\) is then precisely a (many-sorted) Lawvere theory, and a functor \(\varphi: \mathcal{F} \to \mathbf{Set}\) is an algebra for \(\mathcal{F}\) iff \(\varphi\) is a sifted weight.

Another instance of the above is the equality, for any category \(\mathcal{F}\) with finite limits, of the free cocompletion \(\text{Ind}(\mathcal{F}^{\text{op}})\) of \(\mathcal{F}^{\text{op}}\) under filtered colimits and the category \(\text{Lex}(\mathcal{F}, \mathbf{Set})\) of functors preserving finite limits. This coincidence is vital in interpreting locally finitely presentable categories as categories of algebras for essentially algebraic theories, see [2].

In Remark 2.6 of [1], the authors present the class \(\Psi_\mathbf{D}\) for \(\mathbf{D}=(\text{pullbacks}+\text{terminal object})\) as the example of a class that is not sound, yet the equality \(\Psi^+_\mathbf{D}(\mathcal{F}^{\text{op}}) = \Psi_\mathbf{D}\text{-Alg}(\mathcal{F})\) holds. We show now that such a counterexample is essentially due to the fact that \(\Psi_\mathbf{D}\) is not saturated.

In what follows we require a free \(\Psi\)-theory to exist on every small category. More precisely, we require the class \(\Psi\) to be locally small (see [12]): the category \(\Psi(\mathcal{D})\) is small for every small \(\mathcal{D}\).

Proposition 3.8. Suppose the class \(\Psi\) is locally small and saturated. Then the following are equivalent:

1. \(\Psi\) is sound.
2. \(\Psi^+(\mathcal{F}^{\text{op}}) = \Psi\text{-Alg}(\mathcal{F})\) holds for any \(\Psi\)-theory \(\mathcal{F}\).

Proof. It suffices to prove that (2) implies (1). Consider any weight \(\varphi: \mathcal{E}^{\text{op}} \to \mathcal{V}\) such that the functor

\[
F \equiv \varphi \ast (-): [\mathcal{E}, \mathcal{V}] \to \mathcal{V}
\]

preserves \(\Psi\)-limits of representables. We prove that \(F\) preserves all \(\Psi\)-limits.

Denote by

\[
\begin{array}{ccc}
\mathcal{E}^{\text{op}} & \xrightarrow{Y_{\mathcal{E}^{\text{op}}}} & \mathcal{E}^{\text{op}} \\
\mathbf{Z}^{\mathcal{E}^{\text{op}}} & \xrightarrow{W} & [\mathcal{E}, \mathcal{V}] \\
\end{array}
\]

the factorisation of the Yoneda embedding where \(\mathbf{Z}^{\mathcal{E}^{\text{op}}}: \mathcal{E}^{\text{op}} \to \mathcal{E}^{\text{op}}\) is the closure of \(\mathcal{E}^{\text{op}}\) in \([\mathcal{E}, \mathcal{V}]\) under \(\Psi\)-limits. Notice that \(\mathcal{E}^{\text{op}}\) is small since \(\Psi\) is locally small.
By the construction, $\mathcal{C}^{op}$ is a $\Psi$-theory and the functor $W : \mathcal{C}^{op} \rightarrow [\mathcal{E}, \mathcal{Y}]$ preserves $\Psi$-limits. Furthermore, every object of $\mathcal{C}^{op}$ is a $\Psi$-limit of representables, since $\Psi$ is saturated. Hence the composite

$$\mathcal{P} \cong \mathcal{C}^{op} \xrightarrow{W} [\mathcal{E}, \mathcal{Y}] \xrightarrow{\mathcal{P}} \mathcal{Y}$$

preserves $\Psi$-limits. By our assumption (2), the equality $\Psi^+([\mathcal{E}], \mathcal{Y}) = \Psi-Alg(\mathcal{C}^{op})$ holds. Hence the functor $\mathcal{P}(-) : [\mathcal{E}, \mathcal{Y}] \rightarrow \mathcal{Y}$ preserves $\Psi$-limits.

Moreover,

$$\mathcal{P} * (-) \cong (-) * FW$$

holds, since $\mathcal{P} * (-) \cong \text{Lan}_{\mathcal{C}^{op}} \mathcal{P}$, and $\text{Lan}_{\mathcal{C}^{op}} W \cong (-) * W$, and $F$ preserves $\text{Lan}_{\mathcal{C}^{op}} W$ since $F = \mathcal{P} * (-)$ preserves colimits. See the diagram

![Diagram](image)

In the adjunction $(-) * W : [\mathcal{E}, \mathcal{Y}] \rightarrow [\mathcal{E}, \mathcal{Y}]$ the functor $W$ is fully faithful, since $W$ is dense. Thus, the composite

$$[\mathcal{E}, \mathcal{Y}] \xrightarrow{W} [\mathcal{E}, \mathcal{Y}] \xrightarrow{(-) * W} [\mathcal{E}, \mathcal{Y}] \xrightarrow{F} \mathcal{Y}$$

is isomorphic to $F = \mathcal{P} * (-)$ and it preserves all $\Psi$-limits. We proved that $\mathcal{P}$ is $\Psi$-flat; the proof is finished. □

**Example 3.9.** Local smallness of a class $\Psi$ is a nontrivial property.

1. Every small class is locally small. Example 15 of [21] may be modified to show that, when $\mathcal{Y} = \text{Set}$, there exists a locally small class $\Psi$ such that $\Psi$ is not contained in the saturation of any small class of weights.
2. Every subclass of a locally small class is locally small again. The saturation of a locally small class is locally small again.
3. The class $\mathcal{P}$ of all weights is typically not locally small. For example, $\mathcal{P}(\mathcal{D}) = [\mathcal{D}^{op}, \text{Set}]$ in case $\mathcal{Y} = \text{Set}$. In fact, $\mathcal{P}$ is locally small iff $\mathcal{Y}$ is small, i.e., iff $\mathcal{Y}$ is a commutative quantale. If $\mathcal{Y}$ is a commutative quantale, then every class of weights is locally small.
4. Even classes substantially smaller than $\mathcal{P}$ may not be locally small. For example, consider the class $\mathcal{Q}$ of small-projective weights that yields the Cauchy completion $\mathcal{Q}(\mathcal{D})$ of the category $\mathcal{D}$, see [12].
   (a) The class $\mathcal{Q}$ is locally small, whenever $\mathcal{Y}$ is locally presentable as a monoidal category by Theorem 6 of [13].
   (b) However, in case $\mathcal{Y}$ is the monoidal closed category $\text{Sup}$ of complete join-semilattices and join-preserving maps, the class $\mathcal{Q}$ is not locally small by Section 1 of [13]. In this enrichment, the Cauchy completion $\mathcal{Q}(\mathcal{D})$ of a small category $\mathcal{D}$ always contains all small coproducts of representables.

**Remark 3.10.** We do not know whether local smallness of $\Psi$ can be omitted from the assumptions of Proposition 3.8.

We can combine the above with Theorem 8.11 of [12] to obtain further characterisation of soundness of locally small saturated classes.

**Corollary 3.11.** For a locally small saturated class $\Psi$, the following conditions are equivalent:

1. $\Psi$ is sound.
2. $\Psi^+(\mathcal{D}^{op}) = \Psi-Alg(\mathcal{D})$ holds for any $\Psi$-theory $\mathcal{D}$.
3. For any small $\mathcal{D}$, every weight $\varphi : \mathcal{D}^{op} \rightarrow \mathcal{Y}$ is a $\Psi$-flat colimit of a diagram in $\Psi(\mathcal{D})$.
4. For any small $\mathcal{D}$, the closure of $\Psi(\mathcal{D})$ in $[\mathcal{D}^{op}, \mathcal{Y}]$ under $\Psi$-flat colimits is all of $[\mathcal{D}^{op}, \mathcal{Y}]$.

**Remark 3.12.** The class $\Psi_D$ for $\mathcal{D} = (\text{pullbacks+terminal object})$ is locally small and not saturated. It is also not a sound class by Remark 2.6 of [1]. The saturation $\Psi_D$ is the class of finitely presentable weights: this is a locally small, saturated and sound class.
In fact, one can always assume that $\Psi$ is a locally small saturated class, since it is easy to prove the following:

\textit{If $\Psi$ is a locally small and sound class, so is $\Psi^*$.}

Indeed, if $\Psi$ is locally small, then so is $\Psi^*$. Suppose $\Psi$ is sound. For proving soundness of $\Psi^*$, suppose $\mathcal{T}$ is a $\Psi^*$-theory. By Lemma 3.6 we have the equality $\Psi^*(\mathcal{T}^{\text{op}}) = \Psi\text{-Alg}(\mathcal{T})$. Moreover, the equality $\Psi\text{-Alg}(\mathcal{T}) = \Psi^*\text{-Alg}(\mathcal{T})$ holds by the definition of $\Psi^*$. Furthermore, the equality $\Psi^+ = \Psi^{+\circ}$ holds by Proposition 5.4 of [12]. Hence $\Psi^*(\mathcal{T}^{\text{op}}) = \Psi^*\text{-Alg}(\mathcal{T})$ holds for every $\Psi^*$-theory $\mathcal{T}$. The class $\Psi^*$, being locally small and saturated, is sound by Proposition 3.8.

4. An elementary characterisation of sifted weights

In this section we analyse the isomorphism (3.2) in more detail for the enrichment in $\text{Cat}$. We then turn the analysis into a useful elementary criterion of siftedness of weights enriched in $\text{Cat}$. Finally, we comment on similarities and differences in using the criterion for siftedness in another ‘2-dimensional enrichment’, namely that in $\text{Pre}$ (the category of preorders and monotone maps).

An analysis of the coend in (3.2). Suppose that $\mathcal{Y} = \text{Cat}$. Let $\psi, \varphi : \mathcal{E}^{\text{op}} \rightarrow \text{Cat}$ be any weights. Then the coend

\[
\int_e [\mathcal{E}^{\text{op}}, \text{Cat}](Ye, \varphi) \times [\mathcal{E}^{\text{op}}, \text{Cat}](\psi, Ye)
\]

is a category that can be computed as a coequaliser in $\text{Cat}$ of the parallel pair

\[
\prod_{e,e'} [\mathcal{E}^{\text{op}}, \text{Cat}](Ye', \varphi) \times [\mathcal{E}^{\text{op}}, \text{Cat}](\psi, Ye) \overset{L}{\longrightarrow} \prod_{e} [\mathcal{E}^{\text{op}}, \text{Cat}](Ye, \varphi) \times [\mathcal{E}^{\text{op}}, \text{Cat}](\psi, Ye) \overset{R}{\longrightarrow} \prod_{e,e'} [\mathcal{E}^{\text{op}}, \text{Cat}](Ye', \varphi) \times [\mathcal{E}^{\text{op}}, \text{Cat}](\psi, Ye)
\]

of functors

\[
L : (\tilde{x} : Ye' \rightarrow \varphi, f : e \rightarrow e', \tau : \psi \rightarrow Ye) \mapsto (\tilde{x} \cdot Yf : Ye \rightarrow \varphi, \tau : \psi \rightarrow Ye)
\]

\[
R : (\tilde{x} : Ye' \rightarrow \varphi, f : e \rightarrow e', \tau : \psi \rightarrow Ye) \mapsto (\tilde{x} : Ye' \rightarrow \varphi, Yf \cdot \tau : \psi \rightarrow Ye')
\]

Thus the coend (4.1) has the following description (see, e.g., [17]):

1. The objects are equivalence classes

\[
[(\tilde{x}, \tau)]_{\sim}
\]

where $\tilde{x} : Ye \rightarrow \varphi$ and $\tau : \psi \rightarrow Ye$ are natural transformations. The equivalence is generated by

\[
(\tilde{x}, Yf \cdot \tau) \sim (\tilde{x} \cdot Yf, \tau)
\]

for all $\tilde{x} : Ye \rightarrow \varphi$, $\tau : \psi \rightarrow Ye'$, $f : e' \rightarrow e$ in $\mathcal{E}$.

2. The morphisms are equivalence classes

\[
[(u_1, v_1), \ldots, (u_n, v_n)]_{\sim}
\]

of finite sequences $((u_1, v_1), \ldots, (u_n, v_n))$ such that every pair $(u_i, v_i)$ is a morphism in the category $\prod_{e} [\mathcal{E}^{\text{op}}, \text{Cat}](Ye, \varphi) \times [\mathcal{E}^{\text{op}}, \text{Cat}](\psi, Ye)$ and

\[
\text{cod}(u_1, v_1) \sim \text{dom}(u_2, v_2), \quad \text{cod}(u_2, v_2) \sim \text{dom}(u_3, v_3), \ldots, \quad \text{cod}(u_{n-1}, v_{n-1}) \sim \text{dom}(u_n, v_n)
\]

The equivalence relation $\approx$ is generated from the following two conditions

\[
(u \ast Yw, v) \approx (u, Yw \ast v), \quad ((u_1, v_1), (u_2, v_2)) \approx (u_2 \cdot u_1, v_2 \cdot v_1)
\]

by reflexivity, symmetry, transitivity and composition (concatenation). Above, by $\ast$ we denote the horizontal composition of natural transformations.

It will be useful to work with the following graphical representation. The sequence $((u_1, v_1), \ldots, (u_n, v_n))$ as above is going to be depicted as

\[
\begin{align*}
\varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon \\
\tau & \quad \tau & \quad \tau & \quad \tau & \quad \tau & \quad \tau \\
\psi & \quad \psi & \quad \psi & \quad \psi & \quad \psi & \quad \psi \\
\end{align*}
\]

\[
\begin{array}{cccccccc}
\varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon \\
\tau & \quad \tau & \quad \tau & \quad \tau & \quad \tau & \quad \tau \\
\psi & \quad \psi & \quad \psi & \quad \psi & \quad \psi & \quad \psi \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon \\
\tau & \quad \tau & \quad \tau & \quad \tau & \quad \tau & \quad \tau \\
\psi & \quad \psi & \quad \psi & \quad \psi & \quad \psi & \quad \psi \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon & \quad \varepsilon \\
\tau & \quad \tau & \quad \tau & \quad \tau & \quad \tau & \quad \tau \\
\psi & \quad \psi & \quad \psi & \quad \psi & \quad \psi & \quad \psi \\
\end{array}
\]
The above picture is called a hammock from $(\hat{x}, \tau)$ to $(\hat{x}', \tau')$. The wiggly arrow in the above hammock, for example from $(\hat{x}, \tau)$ to $(\hat{x}_1, \tau_1)$, represents a zig-zag connecting $e$ and $e_1$ in $\mathcal{E}$ that witnesses the equivalence $(\hat{x}, \tau) \sim (\hat{x}_1, \tau_1)$.

The whole hammock (4.3) gets evaluated to the composite modification

\[(u_n \ast v_n) \cdot (u_{n-1} \ast v_{n-1}) \cdot \cdots \cdot (u_1 \ast v_1) : \hat{x} : \tau \longrightarrow \hat{x}' : \tau'
\]

in $[\mathcal{E}^{op}, \mathbf{Cat}]$. Up to the equivalence $\approx$, this is how the evaluation functor $\mathrm{can}$ works.

The functor $\mathrm{can}$ is an isomorphism of categories iff it is bijective on objects and fully faithful. Hence, the following two conditions have to hold:

1. The 1-dimensional aspect. To give $\alpha : \psi \longrightarrow \varphi$ is to give a unique $\left[ (\hat{x}, \tau) \right]$ such that $\hat{x} \cdot \tau = \alpha$ holds.
2. The 2-dimensional aspect. To give a modification $\Xi : \alpha \longrightarrow \alpha'$ is to give a unique equivalence class $\left[ ((u_1, v_1), \ldots, (u_n, v_n)) \right]$ such that $\Xi$ is the composite $(u_n \ast v_n) \cdot \cdots \cdot (u_1 \ast v_1)$.

**Siftedness for enrichment in categories.** We are going to fix the class $\Pi$ of (conical) weights for finite products, see Example 4.3. Recall, by Example 2.1 again, that the $\Pi$-flat weights are called sifted.

We are going to fix the class $\Pi$ of (conical) weights for finite products. It follows immediately that the only 2-cells in $\mathbf{Cat}$ can be reduced further to the empty coproduct $\text{const}$ is sifted. Indeed, it suffices to consider the 2-cell $\Delta : \mathcal{E} \longrightarrow \mathcal{E} \times \mathcal{E}$ is sifted iff the following two conditions hold:

1. The unique functor from $\int^\mathcal{E} \varphi \circ \varepsilon$ to the one-morphism category $\mathbb{1}$ is an isomorphism.
2. For any $e_1, e_2$ in $\mathcal{E}$, the canonical morphism

$$\text{can} : \int^\mathcal{E} \varphi \circ \varepsilon \times \mathcal{E}(e_1, e) \times \mathcal{E}(e_2, e) \longrightarrow \varphi e_1 \times \varphi e_2$$

is an isomorphism.

**Remark 4.2.** By analogy to the case $\mathcal{V} = \mathbf{Set}$, we may call the first condition above connectedness of the weight $\varphi : \mathcal{E}^{op} \longrightarrow \mathbf{Cat}$ and the second condition expresses that the diagonal 2-functor $\Delta : \mathcal{E} \longrightarrow \mathcal{E} \times \mathcal{E}$ is cofinal in the sense that the 2-cell

$$\begin{array}{ccc}
\mathcal{E}^{op} & \xrightarrow{\Delta^{op}} & \mathcal{E}^{op} \times \mathcal{E}^{op} \\
\varphi \downarrow & \cong & (e_1, e_2) \mapsto \varphi e_1 \times \varphi e_2 \\
\mathbf{Cat} & & \\
\end{array}$$

where $\delta_e : \varphi e \longrightarrow \varphi e \times \varphi e$ is the diagonal functor, is a left Kan extension. Indeed, it suffices to consider the isomorphism

$$\int^\mathcal{E} \varphi e \times \mathcal{E}(e_1, e) \times \mathcal{E}(e_2, e) \cong \int^\mathcal{E} \varphi e \times (\mathcal{E}^{op} \times \mathcal{E}^{op})(\Delta^{op} e, (e_1, e_2))$$

We apply the criteria of Lemma 4.1, together with the analysis of (3.2) using hammocks, for giving elementary proofs of siftedness of various weights.

**Example 4.3 (A weight that is not sifted).** We start with an example of a weight $\varphi : \mathcal{E}^{op} \longrightarrow \mathbf{Cat}$ that is not sifted, although the ‘underlying’ ordinary functor

$$\begin{array}{ccc}
\mathcal{E}_o^{op} & \xrightarrow{\varphi_o} & \mathbf{Cat}_o \longrightarrow \mathbf{Set} \\
\end{array}$$

is sifted.

Consider the one-morphism category $\mathcal{J}$ with the only object $s$. Denote by $\mathcal{E}_o^{op}$ the free completion of $\mathcal{J}$ under finite products. It follows immediately that the only 2-cells in $\mathcal{E}_o^{op}$ are identities.

Let $\chi : \mathcal{E}_o^{op} \longrightarrow \mathbf{Cat}$ be the product-preserving functor defined by $\chi(s) = 2$, where 2 is the two-element chain, considered as a category. We define $\varphi$ to be the following modification of $\chi$: where $\chi(s^n) = 2^n$, we let $\varphi(s^n) = 2^n$ for every $n > 1$. The structure on $2^n$ is that of an almost discrete preorder with the only nontrivial inequality being $(0, \ldots, 0) \leq (1, \ldots, 1)$. The action of $\varphi$ on morphisms is defined as for $\chi$. Of course, $\varphi$ does not preserve products, but the composite

$$\begin{array}{ccc}
\mathcal{E}_o^{op} & \xrightarrow{\varphi_o} & \mathbf{Cat}_o \longrightarrow \mathbf{Set} \\
\end{array}$$

does; in fact, it is not hard to see that this ordinary functor constitutes an algebra for the ordinary algebraic theory $\mathcal{E}_o^{op}$ and thus it is a sifted weight by [4].
It is enough now to find pairs \((x_1, x_2)\) and \((y_1, y_2)\) from \(\varphi(s) \times \varphi(s)\) such that \((x_1, x_2) \leq (y_1, y_2)\) holds but there is no hammock to witness this inequality. Consider \((x_1, x_2) = (0, 1)\) and \((y_1, y_2) = (1, 1)\). Firstly, we make use of the fact that there are no nontrivial 2-cells in \(\mathcal{E}^{op}\). This implies that the ‘lax’ parts of the hammock consist only of inequalities between the elements of \(\varphi(s^n) = 2^n\) for some \(s^n\). But these are precisely the diagonal inequalities \((0, \ldots, 0) \leq (1, \ldots, 1)\). Together with the fact that the only morphisms of the form \(s^n \to s\) in \(\mathcal{E}^{op}\) are the product projections, it is easy to see that there is no way how any hammock could evaluate its right-hand side to \((1, 1)\) and its left-hand side to \((0, 1)\).

**Remark 4.4.** Siftedness of the composite \(\text{ob} \cdot \varphi : \mathcal{E}^{op} \to \text{Set}\) establishes precisely the 1-dimensional aspect of siftedness: the functor \(\text{can}\) is bijective on objects iff \(\text{ob} \cdot \varphi_o\) is sifted. From this it immediately follows that a weight \(\varphi : \mathcal{E}^{op} \to \text{Cat}\) with \(\mathcal{E}\) locally discrete (i.e., with only the identity 2-cells) and such that every \(\varphi e\) is a discrete category is sifted iff the composite \(\text{ob} \cdot \varphi_o : \mathcal{E}^{op} \to \text{Set}\) is sifted in the ordinary sense.

The 2-dimensional aspect of siftedness of \(\varphi : \mathcal{E}^{op} \to \text{Cat}\) has to be verified in general. Example 4.3 exhibits such a situation when \(\mathcal{E}\) is locally discrete and Example 4.6 shows a conical weight \(\text{const}_1 : \mathcal{E}^{op} \to \text{Cat}\) that is not sifted although the underlying ordinary category \(\mathcal{E}_s\) is sifted in the ordinary sense.

**Example 4.5 (Siftedness for weights based on the simplicial category).** Recall from, e.g., [19], that the simplicial category \(\Delta\) has finite ordinals as objects and monotone maps as morphisms. It can be proved rather easily that the morphisms of \(\Delta\) can be obtained from \(\text{id}_1 : 1 \to 1\), \(\eta : 0 \to 1\) and \(\mu : 2 \to 1\) by ordinal sums subject to monad axioms. Hence we will draw the morphisms of \(\Delta\) as string diagrams that are generated from the following strings

\[
\begin{array}{c}
\bullet & \bullet & \bullet
\end{array}
\]

that represent \(\text{id}_1 : 1 \to 1\), \(\eta : 0 \to 1\) and \(\mu : 2 \to 1\), respectively, by vertical concatenation that is subject to the unit axioms

\[
\begin{array}{c}
\bullet & \bullet & \bullet
\end{array} = \begin{array}{c}
\bullet & \bullet
\end{array} = \begin{array}{c}
\bullet
\end{array}
\]

and the associativity axiom

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet
\end{array} = \begin{array}{c}
\bullet & \bullet
\end{array} \begin{array}{c}
\bullet
\end{array} \begin{array}{c}
\bullet
\end{array}
\]

We show that both the conical weight on \(\Delta\) and the weight given by inclusion of \(\Delta\) into \(\text{Cat}\) are sifted weights. In fact, from our reasoning it will be clear that the same holds of almost any truncation \(\Delta_n\). The truncated category \(\Delta_n\) is just the full subcategory of \(\Delta\) spanned by finite ordinals up to \(n\).

1. It is known that \(\text{const}_1 : \Delta \to \text{Set}\) is an ordinary sifted weight, and therefore even the conical weight \(\text{const}_1 : \Delta \to \text{Cat}\) is sifted due to the fact that there are no non-trivial 2-cells in \(\Delta\), see Remark 4.4. Every truncation \(\Delta_n\) (for \(n \geq 1\)) of the simplicial category \(\Delta\) gives rise to a conical sifted weight as well.
2. Suppose the weight \(\varphi : \Delta \to \text{Cat}\) is given by inclusion. Here

\[
\begin{array}{c}
\Delta_n & \xrightarrow{\varphi_n} & \text{Cat}_n & \xrightarrow{\text{ob}} & \text{Set}
\end{array}
\]

is a representable weight \(\Delta_n(1, -)\). For each object \(n\) of \(\Delta\) the category \(\varphi(n)\) is the free linearly ordered category on an \(n\)-element chain. We will show an elementary proof that \(\varphi\) is a sifted weight. First of all, let us check that the coend \(\int^n \varphi n\) is isomorphic to the one-morphism category \(\mathbb{1}\). Of course, the category \(\int^n \varphi n\) has precisely one object: given any two objects \(x \in \varphi(n)\) and \(y \in \varphi(m)\), they are equivalent by \(\sim\) if there exists a string diagram \(\sigma : \varphi(n) \to \varphi(m)\) such that \(x\) gets mapped to \(y\) by \(\sigma\). A diagram like this always exists; we illustrate this on an example situation with \(n = 4\) and \(m = 3\):

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet
\end{array}
\]

Now given any morphism \(f : x \to x'\) in \(\varphi(n)\), we show that \(f \approx \text{id}_x\), where \(\text{id}_x\) is the identity morphism on the only object \(\ast\) of \(\varphi(1)\). This is again immediate when using the string diagrams:
consider the only string diagram ! : ϕ(n) → ϕ(1). It maps all morphisms in ϕ(n) to the identity morphism, see for example the diagram below.

$$! \times \Delta(n, n_1) \times \Delta(n, n_2) \cong \varphi n_1 \times \varphi n_2$$

So the category $\int^n \varphi n$ indeed has only one morphism. Now we show the isomorphism $\int^n \varphi n \cong \varphi n_1 \times \varphi n_2$ by showing that the canonical morphism is bijective on objects and fully faithful. On objects, the canonical morphism takes an object $x \in \varphi(n)$, two string diagrams $\sigma : \varphi(n) \to \varphi(n_1)$ and $\tau : \varphi(n) \to \varphi(n_2)$, and computes the pair $(\sigma(x), \tau(x))$. It is immediate that for any pair $(y, z)$ in $\varphi n_1 \times \varphi n_2$ there exists a tuple $(x, \sigma, \tau)$ that is mapped to $(y, z)$. More is true: we can always choose $x = * \in \varphi(1)$ and the string diagrams $\sigma, \tau$ are the obvious diagrams choosing $y$ and $z$, respectively.

Thus we have proved fullness and faithfulness of the canonical functor $\text{can}$. The weight $\varphi$ is sifted.

We have actually proved that any truncation $\varphi n \ : \Delta_n \to \text{Cat}$ of the inclusion weight is also sifted for $n \geq 2$.

The 2-dimensional aspect of siftedness is crucial for $\text{Cat}$-enriched weights even in the case of conical weights, as we show in the following easy example.

Example 4.6 (A conical weight that is not sifted). Consider the diagram scheme for reflexive co-equalisers satisfying $\delta_0 \cdot \sigma = \delta_1 \cdot \sigma = \text{id}_1$, and adjoin freely a 2-cell $\alpha$ to it:

The 2-category $E$ is the suspension $\Sigma \Delta$ of the simplicial category $\Delta$. This means that $E$ has a unique object, say $e_0$, and that the hom-category $E(e_0, e_0)$ is the category $\Delta$. Morphisms in $E$ are finite ordinals, and the 2-cells are 'monad-like' string diagrams as described in Example 4.5.

The category $\varphi(e_0)$ is defined as follows: the objects are finite non-zero ordinals, that is, objects of the form $1 + n$ for some $n < \omega$. Every object $1 + n$ is understood as a $(n + 1)$-element chain with a distinguished bottom element. The morphisms in $\varphi(e_0)$ are precisely the monotone maps that preserve the distinguished
bottom element. This definition of $\varphi(e_0)$ again allows a pictorial description in terms of string diagrams. The morphisms in $\varphi(e_0)$ are string diagrams generated by the basic diagrams

subject to monad axioms and the two axioms

that express the fact that the diagram is an algebra for the monad given by the unit and multiplication.

The 2-functor $\varphi : \mathcal{E}^{op} \to \mathbf{Cat}$ is defined on the morphisms and 2-cells of $\mathcal{E}^{op}$ by concatenation: for a given morphism $n : e_0 \to e_0$, the functor $\varphi(n) : \varphi(e_0) \to \varphi(e_0)$ maps an object $1 + m \in \varphi(e_0)$ to the object $1 + m + n$. A string diagram $s$ in $\varphi(e_0)$ is mapped to the diagram $\varphi(n)(s)$, defined as the diagram $s$ concatenated $n$ identity strings. We show an example of this assignment for $n = 1$:

Likewise, given a 2-cell $\theta : m \to n$ in $\mathcal{E}$, the natural transformation $\varphi(\theta)$ is defined componentwise: for an object $1 + m$ in $\varphi(e_0)$, the morphism $\varphi(\theta)_{1+m}$ is the concatenation of the identity diagram on $1 + m$ with the diagram $\theta$. For example, given the diagram as $\theta$ and $m = 2$, the component $\varphi(\theta)_3$ is the following string diagram in $\varphi(e_0)$:

Now to prove that $\varphi$ is a sifted weight, we need to verify that there are canonical isomorphisms

proving that $\varphi_*(-)$ preserves nullary and binary products. We first analyse parts of a general hammock (4.3) for the weight $\varphi$ with the testing weight $\psi = \prod_{i \in I} \mathcal{E}(-, e_i)$. The left-hand side rectangle on the diagram below

represents the information that for each $i \in I$ and the morphisms given in the diagram we have that equalities $f + s_i = t_i$ and $x = y + f$ hold in natural numbers. This situation is depicted on the right-hand side of the above diagram. In general, the tuples $(x, s_i)$ and $(y, t_i)$ are related by the equivalence relation $\sim$ if and only if $x + s_i = y + t_i$ holds for all $i \in I$.

The rectangle of the form

represents the sections $x$ and $e_0$ in $\mathcal{E}$.
is represented by the concatenation of two string diagrams $u$ and $v_i$ for each $i \in I$.

\[
\begin{array}{cc}
\begin{array}{c}
\begin{array}{c}
\text{
\end{array}}
\end{array} & \approx \\
\begin{array}{c}
\begin{array}{c}
\text{
\end{array}}
\end{array}
\end{array}
\]

The above diagram is an example of string diagrams that are equivalent: the ‘sliding’ of the division between the string diagrams generates the equivalence relation $\approx$. Observe moreover that morphisms in the coend are $n$-tuples of composable string diagrams. Any such $n$-tuple is equivalent to a 1-tuple, but the fact that we are allowed to vertically ‘decompose’ any string diagram to $n$ parts is important in the proof of siftedness for $\varphi$. In the following diagram

\[
\begin{array}{cc}
\begin{array}{c}
\begin{array}{c}
\text{
\end{array}}
\end{array} & \approx \\
\begin{array}{c}
\begin{array}{c}
\text{
\end{array}}
\end{array}
\end{array}
\]

we can see such a decomposition of a string diagram into a 2-tuple of shorter string diagrams.

With the complete description of the weight $\varphi$ and of the hammocks, we can conclude that we have the canonical isomorphisms in (4.4):

(1) The weight $\varphi$ satisfies the isomorphism

\[
\int^e \varphi(e) \cong 1.
\]

Indeed, the coend $\int^e \varphi(e)$ has precisely one object: any pair $1+n$ and $1+m$ of objects in $\varphi(e_0)$ is related by a hammock of length 2:

\[
\begin{array}{c}
\begin{array}{c}
\text{
\end{array}}
\end{array}
\]

To show that $\int^e \varphi(e)$ has a unique morphism, we will prove that any string diagram $\sigma : 1+m \to 1+n$ is congruent by the equivalence relation $\approx$ to an identity string diagram $\text{id}_k : k \to k$ for some natural number $k$. We have to distinguish two cases. If the diagram $\sigma$ does not contain as a subdiagram, then it is trivially a concatenation of two string diagrams $\sigma_0 = \text{id}_1 : 1 \to 1$ and $\sigma_1 : m \to n$, and therefore $\sigma \approx \text{id}_1$ holds. If $\sigma$ contains, then it is necessary to factor it into a composition of two diagrams (and denote the red part of the diagram by $\omega$):

\[
\begin{array}{c}
\begin{array}{c}
\text{
\end{array}}
\end{array}
\]

This decomposition is unique. Take the identity morphism $\text{id}_n$ and decompose it in the same way into a concatenation of $\omega$ with $(\tau_1, \text{id}_n)$. By the first case we have that $\tau_1 \approx \sigma_1$, and the equivalence $\text{id}_n \approx \text{id}_n$ is trivial. This decomposition thus witnesses the equivalence $\sigma \approx \text{id}_n$.

(2) The second isomorphism

\[
\int^e \varphi e \times e(0, e) \times e(0, e) \cong \varphi e_0 \times \varphi e_0
\]
is proved similarly. Given two objects \(1 + m\) and \(1 + n\) from \(\varphi(e_0)\), there is a triple \((1, m, n)\) that gets mapped exactly to \((1 + m, 1 + n)\) by the canonical functor. For any other triple \((k, m', n')\) that is mapped to \((1 + m, 1 + n)\) we have the equalities \(k + m' = 1 + m\) and \(k + n' = 1 + n\). Therefore \((1, m, n) \sim (k, m', n')\) holds and the canonical functor is bijective on objects.

To prove that the canonical functor is full, we show that for any two string diagrams \(\sigma : 1 + m \rightarrow 1 + n\) and \(\tau : 1 + p \rightarrow 1 + q\) there is a triple \((\omega, \alpha, \beta)\) getting mapped to \((\sigma, \tau)\). But again, as in the case of the first isomorphism, take \(\omega\) to be the diagram

and factor the diagrams \(\sigma\) and \(\tau\) into pairs \(\alpha = (\sigma_1, \text{id}_n)\) and \(\beta = (\tau_1, \text{id}_q)\) in a way that \(\omega * \alpha = \sigma\) and \(\omega * \beta = \tau\), where \(*\) denotes the horizontal composition. Faithfulness of the canonical functor then comes easily from the fact that the morphisms in the coend have the above mentioned ‘normal form’.

### Siftedness for enrichment in preorders

The enrichment in the category \(\text{Pre}\) of preorders and monotone maps is in many aspects similar to the enrichment in \(\text{Cat}\), but the computations are much simpler. In fact, we will be able to give a full characterisation of sifted conical weights \(\text{const}_1 : \mathcal{E}^{\text{op}} \rightarrow \text{Pre}\), see Example 4.8.

The crucial coend

\[
\int^e \varphi e \times [\mathcal{E}^{\text{op}}, \text{Pre}](\psi, Ye)
\]

is computed as a coequaliser in \(\text{Pre}\) of two monotone maps \(L\) and \(R\) that are defined in the same way as for \(\mathcal{V} = \text{Cat}\), see (4.2). Moreover, the coequaliser of \(L\) and \(R\) can be computed in two steps. First we compute the coequaliser on the level of underlying sets. This yields a set of equivalence classes of the form \([([\widehat{x}, \tau])_\sim])\) w.r.t. the equivalence \(\sim\) generated by \(L\) and \(R\). The set of equivalence classes is then equipped with a least preorder \(\sqsubseteq\) satisfying the following condition:

If \((\widehat{x}, \tau) \leq (\widehat{y}, \sigma)\), then \([([\widehat{x}, \tau])_\sim]) \sqsubseteq ([([\widehat{y}, \sigma])_\sim])\).

where \(\leq\) denotes the preorder of the coproduct \(\coprod_i \varphi e \times [\mathcal{E}^{\text{op}}, \text{Pre}](\psi, Ye)\).

Below, we will also use hammocks for the enrichment in \(\text{Pre}\). These are pictures like (4.3) but the 2-cells \(u_i, v_i\) are replaced by mere inequality signs.

We show now that for conical weights \(\varphi : \mathcal{E}^{\text{op}} \rightarrow \text{Pre}\) the 2-dimensional aspect of siftedness is vacuous. That this is not true for general weights \(\varphi : \mathcal{E}^{\text{op}} \rightarrow \text{Pre}\) is demonstrated by the weight of Example 4.3: all categories there are in fact enriched in \(\text{Pre}\).

**Example 4.8 (Sifted conical weights).** The reasoning is similar to Example 3.4 above. Elements of \(\Psi_1(\mathcal{E})\) are finite coproducts \(\coprod_i Ye_i\) of representables in \([\mathcal{E}^{\text{op}}, \text{Pre}]\). By Yoneda Lemma, every \(\tau : \psi \rightarrow Ye\) can be identified with a cocone \(t_1 : e_i \rightarrow e\). Then the requirement that for any two natural transformations \(\tau : \psi \rightarrow Ye\) and \(\sigma : \psi \rightarrow Ye\) the equivalence \(\tau \sim \sigma\) has to hold, corresponds to the fact that the cocones \(t_1 : e_i \rightarrow e\) and \(s_1 : e_i \rightarrow e\) (corresponding to \(\tau\) and \(\sigma\) respectively) have to be connected by a zig-zag. The 2-dimensional aspect of siftedness is vacuous in this case.

Thus a weight \(\text{const}_4 : \mathcal{E}^{\text{op}} \rightarrow \text{Pre}\) is sifted if and only if the ordinary functor

\[
\mathcal{E}^{\text{op}} \xrightarrow{(\text{const}_4)_e} \text{Pre}_o \xrightarrow{\text{ob}} \text{Set}
\]

is sifted in the ordinary sense.

**Example 4.9 (Sifted weights in general).** Consider a general weight \(\varphi : \mathcal{E}^{\text{op}} \rightarrow \text{Pre}\). To establish the isomorphism

\[
\text{can} : \int^e \varphi e \times \prod_{i \in I} \mathcal{E}(e_i, e) \rightarrow \prod_{i \in I} \varphi e_i,
\]

of preorders we need the monotone map \(\text{can}\) to be bijective and order-reflecting. As we noticed earlier, the coend is computed as a coequaliser in \(\text{Set}\) equipped with a freely generated preorder. More precisely, there are two conditions for a weight to be sifted:

1. To obtain bijectivity of the \(\text{can}\) mapping we demand that

\[
\mathcal{E}^{\text{op}} \xrightarrow{\varphi} \text{Pre}_o \xrightarrow{\text{ob}} \text{Set}
\]

be an ordinary sifted weight.
(2) Order-reflectivity of \( \mathfrak{can} \) means that given any two tuples \((x_i) \leq (x'_i)\) from \( \prod_{i \in I} \varphi_i \) we can form a hammock

\[
\begin{array}{cccc}
\varphi & \varphi & \cdots & \varphi \\
\hat{x} & \hat{x}_1 & \cdots & \hat{x}_n \\
\hat{e} & \cdots & \cdots & \cdots \\
\hat{e}_1 & \cdots & \cdots & \cdots \\
\hat{\tau} & \cdots & \cdots & \cdots \\
(\epsilon_1) & \cdots & \cdots & (\epsilon_n)
\end{array}
\]

such that its left-hand vertical side evaluates to \((x_i)\), and its right-hand vertical side evaluates to \((x'_i)\).

Remark 4.10. Observe that the characterisations of sifted weights for enrichments in \( \text{Cat} \) and \( \text{Pre} \) are strongly related. This is because the computations of coequalisers are essentially the same.

In fact, the requirements for a weight \( \varphi : \mathcal{E}^{\text{op}} \to \text{Pre} \) to be sifted (as enriched in \( \text{Pre} \)) are exactly the requirements of siftedness for the weight \( \varphi' : \mathcal{E}^{\text{op}} \to \text{Cat} \), with \( \varphi' \) being the weight \( \varphi \) considered as enriched in \( \text{Cat} \).

The situation is rather different when considering sifted weights for the enrichment in the category \( \text{Pos} \) of all posets and monotone maps. The computation of a coequaliser in \( \text{Pos} \) runs in two steps: one computes the coequaliser in preorders and then performs the poset-reflection. It is the second step that brings in additional identifications and makes the characterisation of siftedness quite complex.

5. Conclusions and future work

We gave a generalisation of the concept of a sound class of small categories [1] to a sound class \( \Psi \) of weights in the context of enriched category theory. When passing to classes of weights, we showed that an easy analysis of flatness w.r.t. a sound class of weights is possible, providing us, for example, with known characterisations of siftedness and filteredness of ordinary categories. The same result yields elementary proofs of siftedness of various weights for the enrichment in categories or preorders.

A further characterisation of soundness is essentially contained in [7]. Namely, a class \( \Psi \) of weights is sound iff the KZ-monad \( \mathcal{K} \to \mathcal{P}(\mathcal{K}) \) of free cocompletions under all colimits lifts to the 2-category \( \Psi\text{-Cont} \) of all \( \Psi \)-complete categories, all \( \Psi \)-continuous functors and all natural transformations. We thank John Bourke for pointing this out to us.

The theory of lex colimits of [9] works precisely due to the lifting of \( \mathcal{K} \to \mathcal{P}(\mathcal{K}) \) to \( \Psi\text{-Cont} \) for \( \Psi \) being the class of weights for finite limits. Hence a theory of ‘colimits in the \( \Psi \)-world’ can be developed for any sound class \( \Psi \). Since lex colimits are used to understand exactness of enriched categories, one can expect a theory of ‘exactness in the \( \Psi \)-world’ for any sound class \( \Psi \). This is the matter of future research.

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