Quantum Field Theory without divergences: Quantum Spacetime

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Abstract: A fundamental length is introduced into physics in a way which respects the principles of relativity and quantum field theory. This improves the properties of quantum field theory: divergences are removed. How to quantize gravity is also indicated. When the fundamental length tends to zero the present version of quantum field theory is recovered.

PACS 11. General Theory of fields and particles.
In this paper the problem of divergences in quantum field theory is solved. This is done by a simple modification of the concept of spacetime. Mathematically what is being done is called a deformation [1]. The idea is to introduce a fundamental length into physics. There are compelling reasons to do this. Masters like Heisenberg, Dirac, Landau and Schwinger [2] have on various occasions talked about the logical inconsistencies of quantum field theory. Further the problem of quantizing gravity remains [3]. Wheeler [4] talks of the possibility of fluctuations at the Planck length.

The origin of this work is as follows. Weinberg [5] has shown that the equivalence principle and the General Theory of Relativity can be derived from special relativity and quantum field theory. However the problem of renormalization remains. Thus the author was tempted to play with the axioms underlying the concepts of spacetime and pursue the consequences. The article contains no exhaustive references to the literature. This is because the results were arrived at without reference to the literature and attempts to give references would be misleading.

Mathematicians call the process of introducing a fundamental constant into a structure as deformation. Thus relativity which introduces the velocity of light $c$ and quantum mechanics which introduces the Planck constant $\hbar$ into the older structures, namely Galilean relativity and classical mechanics, are called deformations. In each case the two structures will be formally and mathematically similar to the older structures though conceptually severe
changes are needed.

Let us attempt to deform quantum field theory. It is immediate that the Planck length $L$ is the only parameter available. To introduce the parameter let us introduce a pair of operators $\hat{x}_\mu$ and $\hat{x}_\mu^#$ satisfying the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu^#] = i\eta_{\mu\nu}L^2.$$ 

Further the interval is now defined as

$$\hat{s}^2 = \frac{1}{2} \left( \hat{x}_\mu \eta^\mu_{\nu} \hat{x}_\nu + \hat{x}_\mu^# \eta^\mu_{\nu} \hat{x}_\nu^# \right).$$

From the traditional algebra of operators it follows that the interval is quantized in units of $L^2$. Further the commutation relations are compatible with the other relation namely,

$$[\hat{x}_\mu, \hat{p}_\nu] = i\hbar \eta_{\mu\nu}.$$ 

But there is one problem: we have doubled the number of spacetime coordinates and further we would like to have classical spacetime in the limit of $L \to 0$. We therefore construct coherent states of the annihilation operator constructed from $x^\mu$ and $x^{\mu#}$. To halve the number of variables we take those coherent states $|x> = |x^\mu + ix^{\mu#}>$ for which $x^\mu = x^{\#\mu}$. It can now be checked that $<x|y> = e^{-\frac{(x-y)^2}{2L^2}}$ for these states. This corresponds to taking a subset of the overcomplete set.

To fix ideas take only one coordinate $x$. Then

$$[\hat{x}, \hat{x}^#] = iL^2.$$
and

\[ \hat{s}^2 = \frac{1}{2} (\hat{x}^2 + \hat{x}^\#^2). \]

Set

\[ d = \frac{\hat{x} + i\hat{x}^\#}{\sqrt{2L}}, \]
\[ d^\dagger = \frac{\hat{x} - i\hat{x}^\#}{\sqrt{2L}}, \]
\[ [d, d^\dagger] = 1, \]
\[ \hat{s}^2 = \frac{L^2}{2} (dd^\dagger + d^\dagger d). \]

Now comes the halving of coordinates. We cannot let \( \hat{x} \) equal \( \hat{x}^\# \) as operators. So we set \( x = x^\# \) on the coherent states which we assume as the state of spacetime. Then

\[ |x > = e^{-\frac{x^2}{2L^2}} \sum_{0}^{\infty} \frac{[(1+i)^{\frac{x}{\sqrt{2L}}}]^n}{\sqrt{n!}} |n > \]
\[ < x|x' > = e^{-\frac{(x-x')^2}{2L^2}}. \]

We normalize differently to

\[ < x|x' > = \frac{1}{\sqrt{2\pi L}} e^{-\frac{(x-x')^2}{2L^2}}. \]

This goes over to \( \delta(x-x') \) as \( L \to 0 \). The reason for doing this will become clear shortly.

We now go over to quantum field theory. We have so far introduced two ideas. We have doubled the number of coordinates to introduce a fundamental length and changed the definition of interval. This leads to quantization
of the interval. We have next introduced the idea that spacetime is in a coherent state and halved the number of coordinates which finally appear. These two ideas leave us with a fine balance between discreteness and continuity as the following analysis will show.

Thus we see that functions of $x$ will now be functions of annihilation operators acting on states $|x>$. This is just like in optics. To make contact with quantum field theory is our next task.

Since there are spacetime fluctuations, a little thought shows that we should take the action and average over the spacetime fluctuations. Thus we are led to replace the action for the scalar field $\int \partial^{\mu} \phi \partial_{\mu} \phi \, d^4x$ by

$$\int d^4x d^4y \partial_{\mu} \phi(x) K(x - y) \partial^{\mu} \phi(y)$$

where

$$K(x - y) = \left( \frac{1}{\sqrt{2\pi L}} \right)^4 e^{-\frac{(x - y)^2}{2L^2}}.$$  

This is got by sandwiching of the operator corresponding to $\partial_{\mu} \phi \partial^{\mu} \phi$ between the states $<x|$ and $|y>$ and averaging over the fluctuations. It will turn out that the factor $K(x - y)$ will appear in several places.

To carry out the path integral will involve replacing

$$Z = \int \mathcal{D}\phi \, e^{-\frac{1}{2} \partial \phi \partial \phi + J\phi}$$

where

$$\partial \phi \partial \phi = \int d^4x \partial_{\mu} \phi(x) \partial^{\mu} \phi(x)$$

and

$$J\phi = \int d^4x \, J(x) \phi(x)$$
by

$$Z = \int D\phi \ e^{-\frac{1}{2}\partial\phi K \partial\phi + JK\phi}$$

where

$$\partial\phi K \partial\phi = \int d^4x d^4y \partial_\mu \phi(x) K(x-y) \partial^\mu \phi(y)$$

and

$$JK\phi = \int d^4x d^4y J(x) K(x-y) \phi(y).$$

Thus the propagators in momentum space $\frac{1}{p^2}$ are replaced by $\frac{e^{-p^2L^2}}{p^2}$. This corresponds to a regularization of the theory. It is remarkable that this means something very simple in the parametric representation [6] of the propagator.

$$\frac{e^{-\frac{p^2L^2}{2}}}{p^2} = e^{-\frac{p^2L^2}{2}} \int_0^\infty e^{-\alpha p^2} d\alpha$$

$$= \int_0^\infty e^{-(\alpha + \frac{L^2}{2})p^2} d\alpha$$

$$= \int_{\frac{L^2}{2}}^\infty e^{-\alpha p^2} d\alpha$$

by replacing $\alpha + \frac{L^2}{2}$ by $\alpha$ and changing limits of integration. Thus the theory is automatically regularized. To do quantum electrodynamics is now simple. It turns out that the propagator and vertices will be modified. However the gauge principle survives. Nonlocal field theory thus arises naturally from our assumptions. The parametric representation of Feynman amplitudes can be utilized with a natural cutoff. Thus no new techniques need be invented for calculations. Further the results of QED will be reproduced but now with a natural cutoff.
We recapitulate what has happened. We assumed that spacetime is in a coherent state and averaged over the fluctuations and this caused the propagators to be regulated. Thus a natural regularization comes about. Further the parametric representation makes the calculations very simple. This is because we have deliberately deformed the original theory and such results are bound to appear.

To quantize gravity is now fairly simple. The work of Weinberg [5] shows the way. However with the structure of spacetime modified the infinities which plague the older theory will go away. The spin-two graviton will couple to a modified energy momentum tensor which will have nonlocal structure. Physically one can say that point particles are replaced by objects which are roughly of the order of $L$ in size due to the new structure of spacetime.

Notice that modifying the action leads to modified classical equations of motion as well. Thus the hope that Dirac expressed of modifying the classical and quantum theory together is fulfilled in an unexpected way.

There are several conceptual issues involved. The reason is as follows. Whenever one deforms an older theory the newer theory will have mathematically similar structure. Witness the fact that the Lie algebra structure survives in both classical and quantum mechanics. Thus the language of the original theory and its deformation will be similar but conceptual issues will be thorny. The author refrain from any discussion of conceptual issues as this is a luxury he cannot afford at present.
Acknowledgements: Thanks to Prof. E.C.G. Sudarshan for frequently emphasizing the structural similarity between classical and quantum mechanics and the beauty of coherent states. Thanks to Dr. H.S. Sharatchandra for short, illuminating discussions in the initial stages of the work. Thanks to L. Kannan of PPST Foundation and Shambu Prasad of Dastkar for encouraging new modes of thinking.

The author wishes to dedicate this work to Paramahamsa Yogananda whose centenary is being celebrated.
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