AN ALTERNATE APPROACH TO THE LIE BRACKET ON HOCHSCHILD COHOMOLOGY

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ABSTRACT. We define Gerstenhaber’s graded Lie bracket directly on complexes other than the bar complex, under some conditions. The Koszul complex of a Koszul algebra in particular satisfies our conditions. As examples we recover the Schouten-Nijenhuis bracket for a polynomial ring and the Gerstenhaber bracket for a group algebra of a cyclic group of prime order.

1. INTRODUCTION

Hochschild cohomology incorporates useful information about an algebra. In low degrees one finds the center of the algebra, derivations, infinitesimal deformations, and obstructions in algebraic deformation theory. Vanishing in high degrees is equivalent to smoothness in a commutative setting, and analogous notions have been explored in noncommutative settings. Hochschild cohomology is used in support variety theory, a tool for studying representations of some types of finite dimensional algebras.

In spite of its many uses, some of the structure of Hochschild cohomology remains elusive. It is a Gerstenhaber algebra, that is, it has both a cup product and a graded Lie bracket, and the bracket induces graded derivations with respect to the product. Products are defined on arbitrary resolutions in any number of equivalent ways, making them quite tractable. Brackets have not been so amenable to study on resolutions other than the bar resolution where they were historically defined, and thus they are more difficult to compute and to use. One wishes to rectify this unfortunate circumstance. In this paper we do so for a large class of algebras that includes Koszul algebras.

We begin with the observation that there is more than one way to define the graded Lie bracket on the bar resolution $B$ of an algebra: We show in Section 2 that a particular class of chain maps, of graded degree 1, from the tensor product of three copies of $B$ to $B$, gives rise to many brackets at the chain level. These all induce the Gerstenhaber bracket on cohomology. We mimic this construction in Section 3 for an arbitrary resolution and define brackets there. Under some hypotheses we prove that these brackets also induce Gerstenhaber brackets on cohomology. One useful condition in particular is when the resolution embeds into

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the bar resolution in such a way that the diagonal maps commute with the embedding, and the strongest results follow from this condition (Subsection 3.2). Koszul resolutions of Koszul algebras in particular satisfy this hypothesis. We illustrate by recovering the Schouten-Nijenhuis bracket on polynomial rings in Section 4. We also give some results under weaker conditions (Subsection 3.4) that still may be useful but have the disadvantage of requiring a more detailed comparison with the bar resolution. In Section 5 we show that these techniques may be used to recover Gerstenhaber brackets for a group algebra of a cyclic group of prime order $p$ over a field of characteristic $p$.

Many questions remain. Does this technique provide a more efficient, or simply a different, way to compute brackets for Koszul algebras? Can the theory be improved for algebras that are not Koszul, that is, can we eliminate altogether the use of maps comparing to the bar resolution, as we effectively have for Koszul algebras? Even if the answer is no, does the apparent parallel theory (not involving the bar complex) produce a well-defined bracket with useful properties? For example, does such a bracket vanish whenever the Gerstenhaber bracket vanishes, at least in degree 2 where vanishing is the condition for a noncommutative Poisson structure?

2. Alternate brackets on the Hochschild complex

Let $k$ be a field of arbitrary characteristic and let $A$ be a $k$-algebra. Let us recall the definitions of the bar resolution $B$ of $A$ and the Hochschild cochain complex. We write $\otimes$ to mean $\otimes_k$.

Let $TA = T(A)$ denote the graded tensor coalgebra, that is, $TA = \bigoplus_{r \geq 0} (TA)_r$ and the coproduct $\Delta : TA \to TA \otimes TA$ is the $k$-linear map defined by

$$\Delta(a_1 \otimes \cdots \otimes a_r) = \sum_{i=0}^{r} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_r)$$

for $a_1, \ldots, a_r \in A$. As a graded $A$-bimodule, we have $B = A \otimes TA \otimes A$, with $B_r = A^{\otimes (r+2)}$ for each $r \geq 0$. We may use the notation $a \otimes x \otimes a'$ to denote monomials in $B$, where $a, a' \in A$, and $x \in TA$. The differential on $B$ is

$$a_0 \otimes \cdots \otimes a_{r+1} \mapsto \sum_{0 \leq i \leq r} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{r+1}. \quad (2.0.1)$$

Note that the comultiplication on $TA$ induces a quasi-isomorphism

$$\Delta : B \to B \otimes_A B$$

$$a_0 \otimes \cdots \otimes a_{r+1} \mapsto \sum_{i} (a_0 \otimes \cdots \otimes a_i \otimes 1) \otimes (1 \otimes a_{i+1} \otimes \cdots \otimes a_{r+1}). \quad (2.0.2)$$

The map $\Delta$ is coassociative by construction, that is, $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ as chain maps from $B$ to $B \otimes_A B \otimes_A B$. On monomials, we can write this map symbolically as

$$a \otimes x \otimes a' \mapsto \sum (a \otimes x_1 \otimes 1) \otimes (1 \otimes x_2 \otimes a')$$
where the sum runs over all possible ways to factor the monomial $x$. Let

$$C(A) := \text{Hom}_{A^c}(B, A),$$

where $A^c = A \otimes A^{op}$. The cup product may be defined at the cochain level via the diagonal map $\Delta$: If $f \in \text{Hom}_{A^c}(B_r, A)$, $g \in \text{Hom}_{A^c}(B_s, A)$, then

$$(f \smile g)(a_0 \otimes \ldots \otimes a_{r+s+1}) = f(a_0 \otimes \ldots \otimes a_r \otimes 1)g(1 \otimes a_{r+1} \otimes \ldots \otimes a_{r+s+1}).$$

We use the notation $|f| = r$ for the degree of $f$ in this case. The Gerstenhaber bracket is defined as follows, where we replace tensor products by commas in function notation for convenience.

**Definition 2.0.3** (Standard Gerstenhaber Bracket [5]). Let $\circ$ denote the operation $C(A) \otimes C(A) \to C(A)$ given on homogeneous elements $f$ and $g$ by

$$f \circ g(a_1 \otimes \ldots \otimes a_n) = \sum_{j=1}^{|f|} (-1)^{|f|(|g|-1)(j-1)} f(a_1, \ldots, a_{j-1}, g(a_j, \ldots, a_{j+|g|-1}), \ldots, a_n),$$

and define the bracket $[\cdot, \cdot]$ by

$$[f, g] = f \circ g - (-1)^{|f|(|g|-1)} g \circ f.$$

The cup product and bracket induce operations on Hochschild cohomology that enjoy many useful properties, for example,

$$[\bar{f} \smile \bar{g}, \bar{h}] = [\bar{f}, \bar{h}] \smile \bar{g} + (-1)^{|\bar{f}|(|\bar{h}|-1)} \bar{f} \smile [\bar{g}, \bar{h}],$$

(2.0.4)

where $f, g, h$ are homogeneous cocycles and $\bar{f}, \bar{g}, \bar{h}$ are their images in Hochschild cohomology. See [5] for this and other properties.

**Lemma 2.0.5.** As a graded $A$-bimodule, $B \otimes_A B \cong A \otimes TA \otimes A \otimes TA \otimes A$. Under this identification, the differential is given by

$$(a_0 \otimes \ldots \otimes a_{j-1}) \otimes (a_j) \otimes (a_{j+1} \otimes \ldots \otimes a_{n+1})$$

$$\sum_{i<j-1} (-1)^i (a_0 \otimes \ldots \otimes a_{i+1} \ldots) \otimes (a_j) \otimes (\ldots \otimes a_{n+1})$$

$$+ (-1)^{j-1} (a_0 \otimes \ldots) \otimes (a_{j-1}a_j) \otimes (\ldots \otimes a_{n+1})$$

$$+ (-1)^{j+1} (a_0 \otimes \ldots) \otimes (a_j a_{j+1}) \otimes (\ldots \otimes a_{n+1})$$

$$+ \sum_{k>j} (-1)^{k-1} (a_0 \otimes \ldots) \otimes (a_j) \otimes (\ldots \otimes a_ka_{k+1} \ldots \otimes a_{n+1}).$$

**Proof.** The first portion of the statement is clear. The second is an easy check from the fact that the differential on the tensor complex $X \otimes_A Y$, of any two $A$-bimodule complexes $X$ and $Y$, is given by $d(x \otimes y) = d(x) \otimes y + (-1)^{|x|} x \otimes d(y)$.

We will deconstruct the bracket operation, realize it as a composition of several maps, and make some changes in the apparent choices involved. We will observe that these choices do not matter at the level of cohomology, giving us some freedom in the definition. It is this freedom that will allow us, in the next section, to define the bracket independently on other cochain complexes satisfying certain conditions.
We first define a chain map $F_B : B \otimes_A B \to B$. By the isomorphism of Lemma 2.0.5, elements in the tensor product $B \otimes_A B$ may be identified with sums of elements of the form

$$a \otimes x \otimes a' \otimes y \otimes a'' ,$$

with $x \in A^0$, $y \in A^0$ and $a,a',a'' \in A$. We define $F_B : B \otimes_A B \to B$ on such monomials as follows: If $i > 0$ and $j > 0$, then

$$F_B(a \otimes x \otimes a' \otimes y \otimes a'') = 0,$$

$$F_B(a \otimes a' \otimes y \otimes a'') = aa' \otimes y \otimes a'',$$

$$F_B(a \otimes x \otimes a' \otimes a'') = -a \otimes x \otimes a'a'' .$$

In degree 0,

$$F_B(a \otimes a' \otimes a'') = aa' \otimes a'' - a \otimes a'a'' .$$

As one can see from the definition, $F_B$ is 0 on most of the tensor complex $B \otimes_A B$, and is simply given by the actions of $A$ on $B$ for the extremal terms $B^0 \otimes_A B^0$ and $B^i \otimes_A B^0$. One may check directly that $F_B$ is a chain map. Alternatively, this follows from Proposition 2.0.7 or the general Lemma 3.2.2 below.

In the remainder of this article, we use the isomorphism of Lemma 2.0.5, without comment, to identify $B \otimes_A B$ with $A \otimes TA \otimes A \otimes TA \otimes A$.

**Notation 2.0.6.**

1. Let $\Delta^{(2)}(\otimes id)\Delta = (id \otimes \Delta)\Delta : B \to B \otimes_A B \otimes_A B$.

2. Let $G : B \otimes_A B \to B$ denote the map given on monomials by

$$G((a_0 \otimes \ldots \otimes a_{j-1}) \otimes (a_j) \otimes (a_{j+1} \otimes \ldots \otimes a_{n+1})) = (-1)^{j-1}a_0 \otimes \ldots \otimes a_{j-1}a_ja_{j+1} \otimes \ldots \otimes a_{n+1} .$$

Notice that the circle operation $f \circ g$ of Definition 2.0.3, on Hochschild cochains $f$ and $g$ is precisely the composition

$$B \xrightarrow{\Delta^{(2)}} B \otimes_A B \otimes_A B \xrightarrow{id_B \otimes_A g \otimes_A id_B} B \otimes_A B \xrightarrow{G} B \xrightarrow{f} A .$$

To be clear, in the definition of the map $id_B \otimes_A g \otimes_A id_B$ one includes “Koszul signs” so that on elements the map is given by

$$(a \otimes x \otimes a' \otimes y \otimes a'' \otimes z \otimes a''' ) \mapsto (-1)^{[x][y]}(a \otimes x \otimes y \otimes a'') \otimes (z \otimes a''').$$

This observation inspires our alternate definition of brackets below (Definition 2.1.1). First we record a crucial property of the map $G$.

**Proposition 2.0.7.** Let $d$ be the differential on the complex $\text{Hom}_A(B \otimes_A B, B)$. The map $G \in \text{Hom}_A(B \otimes_A B, B)$ is a contracting homotopy for $F_B$, that is, $d(G) := d_BG + Gd_B \otimes_A B = F_B$.

**Proof.** Take a monomial

$$(a_0 \otimes \ldots \otimes a_{j-1}) \otimes (a_j) \otimes (a_{j+1} \otimes \ldots \otimes a_{n+1}) \in B_{j-1} \otimes_A B_{n-j}$$
with $j - 1, n - j > 0$. Applying the formulas given in Lemma 2.0.5 and Notation 2.0.6(2), the function $Gd_B \otimes_A B$ sends $(a_0 \otimes \ldots \otimes a_{j-1}) \otimes (a_j) \otimes (a_{j+1} \otimes \ldots \otimes a_{n+1})$ to the element
\[
\sum_{i<j-1} (-1)^{(i+j-2)}(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \\
+ (-1)^{(j-1+j-2)}(a_0 \otimes \ldots \otimes a_{j-1} a_j \otimes \ldots \otimes a_{n+1}) \\
+ (-1)^{(j-1+j-1)}(a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_{n+1}) \\
+ \sum_{j<k} (-1)^{(k-1+j-1)}(a_0 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_{n+1})
\] (2.0.8)
in $B$. Now $d_B G$ will send that same element in $B \otimes_A B$ to
\[
\sum_{i<j-1} (-1)^{(j-1+i)}(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \\
+ (-1)^{(j-1+j-1)}(a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_{n+1}) \\
+ (-1)^{(j-1+j)}(a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_{n+1}) \\
+ \sum_{j<k} (-1)^{(j-1+k)}(a_0 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_{n+1})
\] (2.0.9)
Comparing the exponents of $-1$, we see that $Gd_B \otimes_A B = -d_B G$ so that $d(G) = 0$ on $B_{>0} \otimes_A B_{>0}$.

Now consider an element $(a_0) \otimes (a_1) \otimes (a_2 \otimes \ldots \otimes a_{n+1}) \in B_0 \otimes_A B_{n-1}$, with $n-1 > 0$. Applying $Gd_B \otimes_A B$ to this element yields (2.0.8) where $j = 1$, minus the first two summands, and applying $d_B G$ yields (2.0.9) where $j = 1$, minus the first summand. So applying $d(G)$ to this element yields
\[
a_0 a_1 \otimes \ldots \otimes a_{n+1} \in B.
\]
Similarly, for the elements $(a_0 \otimes \ldots \otimes a_{n-1} \otimes (a_n) \otimes (a_{n+1})$, applying $d(G)$ yields
\[
(-1)^{n-1+n} a_0 \otimes \ldots \otimes a_n a_{n+1} = -a_0 \otimes \ldots \otimes a_n a_{n+1}.
\]
In degree 0 we have
\[
d(G)((a_0) \otimes (a_1) \otimes (a_2)) = a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2.
\]
Comparing these values in the different cases to our definition of $F_B$ above, we see that $d(G) = F_B$. \hfill $\square$

2.1. Alternate definition of bracket on the Hochschild complex. We call a map $\phi : B \otimes_A B \to B$ for which $d(\phi) := d_B \phi + \phi d_B \otimes_A B = F_B$ a contracting homotopy for $F_B$.

Definition 2.1.1 ($\phi$-circle operation, $\phi$-bracket). Let $\phi \in \text{Hom}_{A^e}(B \otimes_A B, B)$ be any contracting homotopy for $F_B$. The $\phi$-circle operation $f \circ \phi g$ on Hochschild cochains is defined as the composite
\[
f \circ \phi g := f(\phi(id_B \otimes_A g \otimes_A id_B)) \Delta^{(2)}.
\]
The $\phi$-bracket is then defined as the graded commutator
\[
[f, g]_\phi := f \circ \phi g - (-1)^{(|f|-1)(|g|-1)} g \circ \phi f.
\]
Note that the $G$-circle operation $\circ_G$ is the standard circle operation and the $G$-bracket $[,]_G$ is the standard Gerstenhaber bracket.

Lemma 2.1.2. Let $\phi$ be any contracting homotopy for $F_B$. Then the difference $(\phi - G) : B \otimes_A B \to B$ is a boundary in the Hom complex.
The difference is a cycle, since \( d(\phi) = d(G) = F_B \). Recall that the following map is a quasi-isomorphism, where \( \text{proj}_* \) is induced by the canonical projection of \( B \) onto \( A \) (considered as a complex in degree 0, with 0 in all other degrees):

\[
\text{Hom}_{A^e}(B \otimes_A B, B) \xrightarrow{\text{proj}_*} \text{Hom}_{A^e}(B \otimes_A B, A).
\]

Note that \( B \otimes_A B \) is also a bimodule resolution of \( A \), by the Künneth formula, and so the homology of the right hand side is \( \text{Ext}_{A^e}(A, A) \). Since \( \text{Ext}_{A^e}(A, A) = 0 \) in negative degrees, we have \( H_{-1}(\text{Hom}_{A^e}(B \otimes_A B, B)) = 0 \). So any cycle in degree \(-1\) is a boundary. Consequently, the difference \( \phi - G \) is a boundary. □

**Proposition 2.1.3.** Let \( f \) and \( g \) be cocycles in the Hochschild cochain complex \( C(A) = \text{Hom}_{A^e}(B, A) \). Let \( \phi \) be a contracting homotopy for \( F_B \). Then the difference

\[
f \circ \phi \ g - f \circ g
\]

is a boundary, as is the difference

\[
[f, g]_\phi - [f, g].
\]

**Proof.** Take \( \tilde{g} \) to be the function \( \text{id}_B \otimes_A g \otimes_A \text{id}_B \) in \( \text{Hom}_{A^e}(B \otimes_A B \otimes_A B, B \otimes_A B) \).

By this notation \( \tilde{g} \), in degree \( n \), we mean the sum of all maps on \( (B \otimes_A B \otimes_A B)_n \) of the form \( \text{id}_i \otimes_A g \otimes_A \text{id}_j \), where \( \text{id}_i \) is the identity map on \( B_i \) and \( i + j = n - \vert g \vert \).

Note that the map \( \tilde{g} \) is still a cocycle since \( g \) is a cocycle.

Then

\[
f \circ \phi \ g = f \phi \tilde{g} \Delta(2), \quad \text{while} \quad f \circ g = fG \tilde{g} \Delta(2).
\]

The difference is given by

\[
f \circ \phi \ g - f \circ g = f \phi \tilde{g} \Delta(2) - fG \tilde{g} \Delta(2) = f(\phi - G) \tilde{g} \Delta(2).
\]

By Lemma 2.1.2, there exists some map \( \psi \) with \( d(\psi) = \phi - G \). Then, since \( f \) and \( \tilde{g} \) are cocycles,

\[
(-1)^{\vert f \vert} d(f \psi \tilde{g} \Delta(2)) = f d(\psi) \tilde{g} \Delta(2) = f(\phi - G) \tilde{g} \Delta(2) = f \circ \phi \ g - f \circ g,
\]

whence \( f \circ \phi \ g - f \circ g \) is seen to be a boundary, as claimed. The second statement follows from the first. □

**Corollary 2.1.4.**

(1) For any two cocycles \( f \) and \( g \), the \( \phi \)-bracket \( [f, g]_\phi \) is yet another cocycle.

(2) If \( f \) or \( g \) is a boundary, then so is \( [f, g]_\phi \).

(3) On cocycles, the \( \phi \)-bracket is graded anti-commutative up to a boundary and also satisfies the Jacobi identity up to a boundary.

**Proof.** All of these statements follow from the previous proposition and the fact that these conditions are satisfied by the Gerstenhaber bracket. □

**Corollary 2.1.5.** For any contracting homotopy \( \phi \) for \( F_B \), the \( \phi \)-bracket \( [\ , \ ]_\phi \) induces a graded Lie bracket on the shifted cohomology

\[
[\ , \ ]_\phi : \text{HH}(A)[1] \otimes \text{HH}(A)[1] \to \text{HH}(A)[1].
\]
This bracket agrees with the standard Gerstenhaber bracket on cohomology.

3. BRACKETS ON OTHER COCHAIN COMPLEXES

In this section we define brackets at the cochain level on complexes other than the Hochschild complex. We show that under some conditions, these brackets induce precisely the Gerstenhaber bracket on cohomology. Koszul algebras over \( k \) will satisfy these conditions.

Let \( K \to A \) be a projective \( A \)-bimodule resolution of \( A \). For most of this section we will want \( K \) to satisfy some hypotheses which we outline next.

3.1. Hypotheses on the bimodule resolution \( K \to A \). We assume that the \( A \)-bimodule resolution \( K \to A \) satisfies the following conditions:

(a) \( K_0 = A \otimes A \).

(b) \( K \) admits an embedding \( \iota : K \to B \) of complexes of \( A \)-bimodules that restricts to the identity map on \( A \otimes A \) in degree 0.

(c) The embedding \( \iota \) admits a section \( \pi : B \to K \), i.e. an \( A^e \)-chain map \( \pi \) with \( \pi \iota = \text{id}_K \).

(d) The diagonal map \( \Delta_B : B \to B \otimes_A B \) preserves \( K \), and hence induces a diagonal quasi-isomorphism \( \Delta_K : K \to K \otimes_A K \). Equivalently, \( K \) comes equipped with a diagonal quasi-isomorphism \( \Delta_K : K \to K \otimes_A K \) satisfying \( \Delta_B = (\iota \otimes_A \iota) \Delta_K \).

Note that condition (b) implies the quasi-isomorphism \( K \to A \) is given by the multiplication map \( K_0 = A \otimes A \to A \). Practically speaking, the easiest way for condition (c) to be satisfied is for \( K \) to be free on some graded base space \( W \subset K \) with \( \iota(W) \subset TA \subset B \) under \( \iota \). Indeed, one can verify that condition (c) holds if and only if the cokernel of each map \( \iota_l : K_l \to B_l \) is projective over \( A^e \). So we could, alternately, require that \( K \) satisfy the slightly stronger condition

\[(c') \quad K \text{ is free on a graded base } W \subset K \text{ with } \iota(W) \subset TA \text{ in } B.\]

Conditions (a)–(c) can be seen as relatively mild restrictions. In contrast, condition (d) holds a great deal of significance. Indeed, it can be shown that if the minimal free bimodule resolution of a connected graded algebra can be made to satisfy (d), then the algebra is Koszul. This does not mean, however, that non-Koszul algebras have no resolutions satisfying the above conditions, or that the minimal resolution can not be used in some way to compute the Lie bracket. We will see in Section 5 that we can still use the minimal resolution for (the group algebras of) the cyclic \( p \)-group in characteristic \( p \) to compute the Lie bracket on Hochschild cohomology.

As the above discussion suggests, the Koszul complex of a Koszul algebra does satisfy our conditions (a)–(d). See, e.g., [1] or [8] for a discussion of diagonal maps in the case of a Koszul algebra. Verification of the other conditions is more straightforward. We will not need the definition of a Koszul algebra however, as we work in the general setting of a complex satisfying conditions (a)–(d). In the next section we give explicitly the example of a polynomial ring, which is a Koszul algebra. One can also show that the Koszul resolution of a PBW deformation of
a Koszul algebra fits into our framework ([4, Lemma 4.1], [8, Lemma 6.2], [9]). In this case, the diagonal map on $K$ will be induced by the natural comultiplication on the base $W \subset K$ (denoted $\Lambda^*$ in [8]). Localizations of such algebras will also fit into our scheme.

**Remark 3.1.1.** One actually has to replace $B$ with the reduced bar resolution to get (a)–(d) to hold in the case of a non-augmented PBW deformation of a Koszul algebra. This is, however, a straightforward process.

### 3.2. $\phi$-brackets on $\text{Hom}_A(K, A)$. For this subsection, let us fix a resolution $K \to A$ satisfying the hypotheses 3.1(a)–(d). In the case that $K$ is free, we may write $K = A \otimes W \otimes A$ and denote elements in this space as sums of monomials $a \otimes x \otimes a'$, with $x \in W$, $a, a' \in A$. In degree 0 we simply have monomials $a \otimes a'$. Consequently, elements in the tensor product $K \otimes_A K$ may be identified with sums of elements of the form

$$a \otimes x \otimes a' \otimes y \otimes a''$$

with $x, y \in W$ and $a, a', a'' \in A$, via an isomorphism $K \otimes_A K \cong A \otimes W \otimes A \otimes W \otimes A$, analogous to that of Lemma 2.0.5. We define the map $F_K : K \otimes_A K \to K$ just as in the special case $K = B$: If $|x|, |y| > 0$, then

$$F_K(a \otimes x \otimes a' \otimes y \otimes a'') = 0,$$

$$F_K(a \otimes a' \otimes y \otimes a'') = aa' \otimes y \otimes a'',$$

$$F_K(a \otimes x \otimes a' \otimes a'') = -a \otimes x \otimes a''.$$

In degree 0,

$$F_K(a \otimes a' \otimes a'') = aa' \otimes a'' - a \otimes a'a''.$$  

(3.2.1)

In general, for a non-free resolution, we can define $F_K$ as the action map on $K_0 \otimes_A K_{\geq 0} \cong A \otimes K_{\geq 0}$ and the negated action map on $K_{> 0} \otimes_A K_0 \cong K_{> 0} \otimes A$. On $K_0 \otimes_A K_0 \cong A \otimes A \otimes A$ we still use the formula (3.2.1), and take $F_K$ to be 0 on all the intermediate terms $K_{> 0} \otimes K_{> 0}$.

**Lemma 3.2.2.** The map $F_K : K \otimes_A K \to K$ is a morphism of chain complexes.

**Proof.** Suppose first that $K$ is free. We start by checking that in degree 1, $F_K$ commutes with the differentials: Let $a \otimes a' \otimes y \otimes a'' \in K_0 \otimes_A K_1 \cong A \otimes A \otimes W_1 \otimes A$. Then

$$d_1F_K(a \otimes a' \otimes y \otimes a'') = d_1(aa' \otimes y \otimes a'') = aa'd_1(1 \otimes y \otimes 1)a'',$$

$$F_Kd_1(a \otimes a' \otimes y \otimes a'') = F_K((a \otimes a') \otimes_A d_1(1 \otimes y \otimes 1)a'') = F_K(a \otimes a'd_1(1 \otimes y \otimes 1)a'').$$

Letting $m : A \otimes A \to A$ denote the multiplication map, since $m \circ d_1 = 0$, we have $F_K(a \otimes a'd_1(1 \otimes y \otimes 1)a'') = aa'd_1(1 \otimes y \otimes 1)a''$, as desired. A similar calculation applies to elements of $K_1 \otimes_K K_0$. So $d_1F_K = F_Kd_1$.  

Now let $n > 1$. We will show that $d_n F_K = F_K d_n$. First let $a \otimes a' \otimes y \otimes a'' \in K_0 \otimes_A K_n \cong A \otimes A \otimes W_n \otimes A$. Then
\begin{align*}
d_n F_K (a \otimes a' \otimes y \otimes a'') &= d_n (aa' \otimes y \otimes a''), \\
F_K d_n (a \otimes a' \otimes y \otimes a'') &= F_K ((a \otimes a') \otimes_A d_n (1 \otimes y \otimes a'')) \\
&= aa' d_n (1 \otimes y \otimes a'') \\
&= d_n (aa' \otimes y \otimes a''),
\end{align*}
since in degree 0 it is given by $F$. The map $d$ restricted to the kernel of $F$ is the composition

\begin{align*}
K_0 &= A \otimes A, \\
d_0 : A \otimes A &\rightarrow A \otimes A \\
\phi \otimes \phi &\mapsto \phi \phi + \phi d_{K \otimes A} K = F_K.
\end{align*}

so $F_K$ and $F_K d_n$ restrict to the same map on $K_0 \otimes_A K_n$. A similar calculation applies to $K_n \otimes_A K_0$. Now let $i, j > 0$, $i + j = n$, $x \in W_i$, $y \in W_j$. Then
\begin{align*}
d_n F_K (a \otimes x \otimes a' \otimes y \otimes a'') &= 0, \\
F_K d_n (a \otimes x \otimes a' \otimes y \otimes a'') &= F_K (d_i (a \otimes x \otimes a') \otimes_A (1 \otimes y \otimes a'')) + \phi (a \otimes x \otimes a') \otimes_A d_j (1 \otimes y \otimes a'') \\
&= 0.
\end{align*}

(The last equality is immediate if $i > 1$ and $j > 1$, by the definition of $F_K$. If $i = 1$ or $j = 1$, we also obtain 0 since for example $d_1 (a \otimes x \otimes a') = ad_1 (1 \otimes x \otimes 1) a'$, which is in the kernel of the multiplication map $m$. This results in an element in the kernel of $F_K$, and similarly for $d_1 (1 \otimes y \otimes a'')$.)

In the case that $K$ is not free, we can embed it in a free resolution $K'$ with $K_0' = A \otimes A = K_0$. Since we have $F_K = F_{K'}|K$, and $F_{K'}$ is a chain map (by the above calculation), we conclude that $F_K$ is a chain map as well. \hfill \Box

**Lemma 3.2.3.** The map $F_K : K \otimes_A K \rightarrow K$ is a boundary in $\text{Hom}_{A^e} (K \otimes_A K, K)$.

**Proof.** One can easily check that $F_K$ maps to 0 under the quasi-isomorphism
\[ \text{Hom}_{A^e} (K \otimes_A K, K) \xrightarrow{\text{proj}} \text{Hom}_{A^e} (K \otimes_A K, A), \]
since in degree 0 it is given by
\[ K_0 \otimes_A K_0 = A \otimes A \otimes A \rightarrow A \otimes A = K_0 \]
\[ a \otimes a' \otimes a'' \mapsto aa' \otimes a'' - a \otimes a' a''. \]

When composed with the multiplication map, this yields 0. So $F_K$ must be a boundary in the original complex. \hfill \Box

Recall that a contracting homotopy for $F_K$ is a map $\phi : K \otimes_A K \rightarrow K$ with $d(\phi) := d_K \phi + \phi d_{K \otimes_A K} = F_K$. The lemma allows us to make the following definition.

**Definition 3.2.4** (General $\phi$-circle operation, $\phi$-bracket). Let $\phi$ be a contracting homotopy for $F_K$, and let $\Delta_K^{(2)}$ be a chain map from $K$ to $K \otimes_A K \otimes_A K$. (Under hypothesis 3.1(d), we take $\Delta_K^{(2)} := (\Delta_K \otimes id_K) \Delta_K = (id_K \otimes \Delta_K) \Delta_K$.) The $\phi$-circle product $f \circ \phi g$ is the composition
\[ f \circ \phi g := f \phi (id_K \otimes_A g \otimes_A id_K) \Delta_K^{(2)}. \]
The $\phi$-bracket is the graded commutator
\[ [f, g]_\phi := f \circ_\phi g - (-1)^{(|f|-1)(|g|-1)} g \circ_\phi f. \]

Suppose that our resolution $K \to A$ satisfies the freeness property $(c')$. For example, we could take $K$ to be the Koszul resolution of a PBW deformation of a Koszul algebra. (See, e.g., [4, 9].) We can then express the $\phi$-circle operation and bracket on elements in the generating set $x \in W \subset K$ as
\[ (f \circ_\phi g)(x) = \sum (-1)^{|g||x_1|} f(\phi(x_1 \otimes g(x_2) \otimes x_3)) \]
and
\[ [f, g]_\phi(x) = \sum (-1)^{|g||x_1|} f\left(\phi(x_1 \otimes g(x_2) \otimes x_3)\right) \]
\[ -(-1)^{|f|-1} \sum (-1)^{|f||x_1|} g\left(\phi(x_1 \otimes f(x_2) \otimes x_3)\right). \]

Here the sum $\sum x_1 \otimes x_2 \otimes x_3$ denotes the element $\Delta(2)(x)$, which lies in $W \otimes W \otimes W \subset K \otimes A$ by hypothesis. In the case that $K = B$ and $\phi = G$, the map $\phi$ simply inserts the apparent missing factor $(-1)^{|x_1|}$ in the above expressions.

We will see in Theorem 3.2.7 that the $\phi$-bracket operation preserves cocycles and coboundaries, and that the induced operation on cohomology is precisely the Gerstenhaber bracket. The following lemma will be of significance in a moment.

**Lemma 3.2.5.** Let us take $G_K := \pi G(\iota \otimes_A \iota) : K \otimes_A K \to K$, where $G$ is the standard contracting homotopy for $F_B$ given in Notation 2.0.6(2). Then

1. $F_K = \pi F_B(\iota \otimes_A \iota)$.
2. $d(G_K) = F_K$.

**Proof.** Statement (1) follows directly from the definitions of $F_K$ and $F_B$ given above, and the fact that $\pi \iota = \text{id}_K$. Statement (2) follows from (1) since we have
\[ d(G_K) = d(\pi G(\iota \otimes_A \iota)) = \pi d(G)(\iota \otimes_A \iota) = \pi F_B(\iota \otimes_A \iota) = F_K. \]

Note that, since $\pi : B \to K$ and $\iota : K \to B$ are quasi-isomorphisms, they induce quasi-isomorphisms on the Hom complexes
\[ \pi^* : \text{Hom}_{A^e}(K, A) \to \text{Hom}_{A^e}(B, A) \quad \text{and} \quad \iota^* : \text{Hom}_{A^e}(B, A) \to \text{Hom}_{A^e}(K, A). \]

The latter map is simply restriction to $K$.

**Proposition 3.2.6.** Assume hypotheses 3.1(a)–(d). Given $f$ and $g$ in $\text{Hom}_{A^e}(K, A)$ we have an equality of functions
\[ f \circ_{G_K} g = \iota^*(\pi^* f \circ \pi^* g) \]
and subsequent equality
\[ [f, g]_{G_K} = \iota^*[\pi^* f, \pi^* g]. \]
Proof. Let us simply expand the functions.

\[
(f \pi \circ g \pi) \iota = f \pi G(id_B \otimes_A g \pi \otimes_A id_B) \Delta^{(2)}_B \\
= f \pi G(id_B \otimes_A g \pi \otimes_A id_B) \iota \otimes (2) \Delta^{(2)}_K \\
= f(\pi G(\iota \otimes_A \iota))(id_K \otimes_A g \otimes_A id_K) \Delta^{(2)}_K \\
= f G(id_K \otimes_A g \otimes_A id_K) \Delta^{(2)}_K \\
= f \circ G_K g.
\]

The equality of brackets follows from the fact that the bracket is defined as the graded o-commutator. \square

Let \( \phi \) be any contracting homotopy for \( F_K \). By the same proof as the one given for Proposition 2.1.3, the differences \( f \circ \phi g - f \circ G_K g \) and \([f, g]_\phi - [f, g]_{G_K}\) will be boundaries whenever \( f \) and \( g \) are cocycles in \( \text{Hom}_{\mathcal{A}^e}(K, A) \).

**Theorem 3.2.7.** Suppose \( K \) is a bimodule resolution of \( A \) satisfying hypotheses 3.1(a)–(d), and let \( \phi \) be any contracting homotopy for \( F_K \). Let \( f \) and \( g \) be cocycles in \( \text{Hom}_{\mathcal{A}^e}(K, A) \).

1. The bracket \([f, g]_\phi\) is a cocycle.
2. If \( f \) or \( g \) is a boundary, then \([f, g]_\phi\) is a boundary.
3. The induced bracket \([\cdot, \cdot]_\phi : \text{HH}(A)[1] \otimes \text{HH}(A)[1] \rightarrow \text{HH}(A)[1]\) on cohomology agrees with the Gerstenhaber bracket.

**Proof.** By the discussion preceding the statement of the theorem, we may assume without loss of generality that \( \phi = G_K = \pi \pi(i \circ A i) \). Now (1) and (2) follow directly from Proposition 3.2.6 and the fact that \( \pi^* \) and \( \iota^* \) are quasi-isomorphisms. Since \( id_{\text{HH}(\text{Hom}(K, A))} = (\pi \iota)^* = \iota^* \pi^* \), we see that the induced isomorphisms on homology are in fact mutually inverse. So we have

\[
(\iota^*)^{-1} = \pi^* : \text{H}(\text{Hom}_{\mathcal{A}^e}(K, A)) \rightarrow \text{H}(\text{Hom}_{\mathcal{A}^e}(B, A)).
\]

This isomorphism is one of graded Lie algebras since, according to Proposition 3.2.6, we will have an equality

\[
\pi^*(\pi^* g) = \pi^*(\pi^* f, \pi^* g) = (\pi^* \iota^* \pi^* f, \pi^* g) = [\pi^* f, \pi^* g]
\]
on cohomology. Finally, the homologies \( \text{H}(\text{Hom}_{\mathcal{A}^e}(K, A)) \) and \( \text{H}(\text{Hom}_{\mathcal{A}^e}(B, A)) \) are precisely the Hochschild cohomology \( \text{HH}(A) \).

3.3 Formula for \( \phi \). In general, it may be difficult to find a map \( \phi \) satisfying \( d(\phi) := d_K \phi + \phi d_{K \otimes_A K} = F_K \). Let us give one method for constructing such a homotopy that is related to constructions of chain maps via contracting homotopies (for example, as described in Mac Lane [7]). Consider the extended complex \( K \rightarrow A \rightarrow 0 \), by which we mean the complex \( \cdots \rightarrow K_1 \rightarrow K_0 \rightarrow A \rightarrow 0 \). This complex is acyclic and can, alternatively, be described as the mapping cone of the quasi-isomorphism \( K \rightarrow A \).
The following lemma is general, that is, it does not require hypotheses 3.1, only that $K$ be a free $A$-bimodule resolution of $A$, as well as the further hypotheses stated in the lemma.

**Lemma 3.3.1.** Suppose $K$ is free on a graded subspace $W \subset K$. Let $h$ be any $k$-linear contracting homotopy for the identity map on the extended complex $K \to A \to 0$. Take $\phi_{-1} = 0$. Define $\phi$, in each degree $i \geq 0$, as the $A^i$-linear map $\phi_i : (K \otimes_A K)_i \to K_{i+1}$ given inductively by the formula

$$\phi_i|_{W \otimes_A W} := h_i((F_K)_i - \phi_{i-1}(d_{K \otimes_A K}))|_{W \otimes_A W}.$$  

Then $d(\phi) = F_K$.

**Proof.** To simplify notation, take $F = F_K$ and $d = d_K$, or $d_{K \otimes_A K}$ when appropriate. Let us consider $F$ and $\phi$ as maps to the extended complex. Note that, since $F_0$ has image in the space of degree 0 cycles $Z_0(K \to A \to 0)$, this new version of $F$ will still be a chain map. Take $\phi_j = 0$ for all negative $j$. Then for all negative $j$ the equality $d_{j+1} \phi_j + \phi_{j-1}d_j = F_j$ holds, since both sides are just zero. Now suppose, for a given $i$, that for all $j < i$ the formula $d_{j+1} \phi_j + \phi_{j-1}d_j = F_j$ holds. Then after restricting to the generating subspace $W \otimes A \otimes W \subset K \otimes_A K$,

$$(d_{i+1} \phi_i + \phi_{i-1}d_i) = d_{i+1} h_i(F_i - \phi_{i-1}d_i) + \phi_{i-1}d_i$$

$$= (F_i - \phi_{i-1}d_i) - h_{i-1}d_i(F_i - \phi_{i-1}d_i) + \phi_{i-1}d_i$$

$$= F_i - h_{i-1}d_iF_i + h_{i-1}d_i \phi_{i-1}d_i$$

$$= F_i - h_{i-1}d_iF_i + h_{i-1}d_i \phi_{i-1}d_i$$

whence $d(\phi) = F_K$. \hfill $\square$

We will use the formula of Lemma 3.3.1 in Sections 4 and 5.

### 3.4. $\phi$-brackets under weaker conditions.

In the remainder of this section, we describe some weaker conditions under which the conclusion of Theorem 3.2.7 still holds. We will need this more general statement in Section 5 below. For the following lemma, we assume only hypotheses 3.1(a)–(c), and we let $\Delta^{(2)}_K$ be a chain map from $K$ to $K \otimes_A K \otimes_A K$. (We do not assume that there is a coassociative chain map $\Delta : K \to K \otimes_A K$ from which $\Delta^{(2)}_K$ is defined.)

Recall that we have the canonical contracting homotopy $G_K := \pi G(\iota \otimes_A \iota) : K \otimes_A K \to K$ for $F_K$ (as Lemma 3.2.5 does not require hypothesis 3.1(d)).

**Lemma 3.4.1.** Assume hypotheses 3.1(a)–(c). For cocycles $f, g \in \text{Hom}_{A^*}(K, A)$, the difference

$$\iota^*(\pi^* f \circ \pi^* g) - f G_K(id_K \otimes_A g \otimes_A id_K)(\pi \otimes_A \pi \otimes_A \pi) \Delta^{(2)}_K \iota$$

is a boundary. If $\Delta^{(2)}_K = (\pi \otimes_A \pi \otimes_A \pi) \Delta^{(2)}_B \iota$, then

$$\iota^*(\pi^* f \circ \pi^* g) - f \circ \phi g$$

and

$$\iota^*[\pi^* f, \pi^* g] - [f, g]_\phi$$

are boundaries.
Proof. We have
\[ \iota^*(\pi^* f \circ \pi^* g) = f \pi G(id \otimes g \pi \otimes id) \Delta_B^{(2)} \iota \]
and
\[ f G_K(id_K \otimes g \otimes id_K)(\pi \otimes \pi \otimes \pi) \Delta_B^{(2)} \iota = f \pi G(\iota \otimes \iota)(\pi \otimes g \pi \otimes \pi) \Delta_B^{(2)} \iota \]
\[ = f \pi G(\iota \pi \otimes \iota \pi)(id \otimes g \pi \otimes id) \Delta_B^{(2)} \iota. \]
Now one can check that \( \pi F_B(\iota \pi \otimes \iota \pi) = \pi F_B \) (since \( \pi \iota = id_K \) and by the definition of \( F_B \)). So
\[ d(\pi G(\iota \pi \otimes \iota \pi)) = \pi d(G)(\iota \pi \otimes \iota \pi) = \pi F_B(\iota \pi \otimes \iota \pi) = \pi F_B = d(\pi G), \]
and, since cohomology vanishes in negative degrees, \( \pi G(\iota \pi \otimes \iota \pi) - \pi G \) is a boundary. It follows that this difference
\[ \iota^*(\pi^* f \circ \pi^* g) - f G_K(id_K \otimes A \otimes A id_K)(\pi \otimes A \pi \otimes A \pi) \Delta_B^{(2)} \iota \]
\[ = f \pi G(id \otimes g \pi \otimes id) \Delta_B^{(2)} \iota - f \pi G(\iota \pi \otimes \iota \pi)(id \otimes g \pi \otimes id) \Delta_B^{(2)} \iota \]
is a boundary as well, since all of \( f, \pi, \iota, g, \Delta_B^{(2)} \) are cycles. □

It follows that under hypotheses 3.1(a)–(c), taking \( \Delta_K^{(2)} := (\pi \otimes A \pi \otimes A \pi) \Delta_B^{(2)} \iota \) in Definition 3.2.4, the conclusion of Theorem 3.2.7 holds. Thus \( \phi \)-brackets may be defined in a fairly general setting, at the expense of dealing more directly with maps \( \pi, \iota \) comparing to the bar resolution. Note that Definition 3.2.4 can be used to define other versions of \( \phi \)-bracket, given other choices of chain map \( \Delta_K^{(2)} \). We do not know if these other \( \phi \)-brackets are well-defined on cohomology, nor whether they have useful properties.

4. Recovering Schouten-Nijenhuis brackets for polynomial rings

4.1. Review of the Koszul resolution. Let \( A = k[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables. We take \( V \) to be the \( k \)-vector space with basis \( \{x_1, \ldots, x_n\} \). As a formality, let \( x_0 = 1 \).

Definition 4.1.1. Let \( S_i \) denote the symmetric group on \( i \) symbols. For any \( v_1, \ldots, v_i \in V \), let \( o(v_1, \ldots, v_i) \) denote the \( S_i \)-orbit sum
\[ o(v_1, \ldots, v_i) = \sum_{\sigma \in S_i} sgn(\sigma)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)} \]
in \( V^\otimes n \subset A^\otimes n \). We take \( o(\varnothing) := 1 \).

Let \( W \) denote the graded subspace \( \oplus_{i \geq 0} o(V, \ldots, V) \) in \( TA \). One can check that \( W \) is a subcoalgebra of \( TA \) and that \( K = K(A) := A \otimes W \otimes A \) is a subcomplex of the bar resolution \( B = (A \otimes TA \otimes A, d) \). (See also [1], [8], and (4.2.2) below.) It is well known that the embedding \( K \to B \) is a quasi-isomorphism, i.e. that \( A \) is Koszul. In the following lemma, the notation \( \hat{v}_i \) indicates that \( v_i \) has been removed.
Lemma 4.1.2. The differential on $K$ is given on monomials by

$$a \otimes o(v_1, \ldots, v_i) \otimes a' \mapsto \sum_l(-1)^{l+1}av_l \otimes o(v_1, \ldots, \hat{v}_l, \ldots, v_i) \otimes a' - (-1)^{l+1}a \otimes o(v_1, \ldots, \hat{v}_l, \ldots, v_i) \otimes v_la'.$$

Proof. This follows by direct computation and the fact that $v_lv_m - v_mv_l = 0$ in $A$ for each $v_l, v_m \in V$. $\square$

We choose the ordering $x_1 < x_2 < \cdots < x_n$ on the generators of $A$ and call an element

$$1 \otimes o(x_{i_1}, \ldots, x_{i_{u}}) \otimes x_{j_1} \ldots x_{j_{t}} \otimes o(x_{k_1}, \ldots, x_{k_u}) \otimes 1$$

in $k \otimes W \otimes A \otimes W \otimes k$ an ordered “monomial” if $x_{i_1} < x_{i_{i+1}}$, $x_{j_1} \leq x_{j_{t+1}}$, and $x_{k_{l}} < x_{k_{l+1}}$, for all $l$. We define ordered monomials in $A \otimes W \otimes k$ and in $k \otimes W \otimes A$ similarly. The $A^e$ generating subspaces $k \otimes W \otimes k$ and $k \otimes W \otimes A \otimes W \otimes k$, of $K$ and $K \otimes_A K$ respectively, are spanned over $k$ by the respective sets of ordered monomials.

We employ a slight variation of a left $k$-linear contracting homotopy for the identity on the extended complex $K \rightarrow A \rightarrow 0$ given in [12]. In homological degrees $-1$ and $0$, which are $A$ and $A \otimes A$ respectively, $h$ is given by the formula

$$h: a \mapsto 1 \otimes a \quad x_{j_1} \ldots x_{j_{t}} \otimes a \mapsto \sum_{1 \leq \nu \leq t} x_{j_1} \ldots x_{j_{\nu-1}} \otimes o(x_{j_{\nu}}) \otimes x_{j_{\nu+1}} \ldots x_{j_{t}}a,$$

for any ordered monomial $x_{j_1} \ldots x_{j_{t}} \otimes 1$ in $A \otimes A = K_0$ and $a$ in $A$. In higher degrees we define $h$ to be the right $A$-linear map specified on ordered monomials by the formulae

$$h: x_{j_1} \ldots x_{j_{t}} \otimes o(x_{k_1}, \ldots, x_{k_u}) \otimes 1 \mapsto (-1)^{u} \sum_{x_{j_{\nu}} > x_{k_u}} x_{j_1} \ldots, x_{j_{\nu-1}} \otimes o(x_k, \ldots, x_{k_u}, x_{j_{\nu}}) \otimes x_{j_{\nu+1}} \ldots x_{j_{t}},$$

When the indexing set $\{x_{j_{\nu}} : x_{j_{\nu}} > x_{k_u}\}$ is empty, the sum is indeed taken to be $0$.

Using Lemma 3.3.1 and the contracting homotopy $h$ given above, one can easily construct a contracting homotopy $\phi: K \otimes_A K \rightarrow K$ for $F_K$ in low degrees. One then deduces from this information the following general formula.

Definition 4.1.3. We define the $A^e$-linear map $\phi: K \otimes_A K \rightarrow K$ on ordered monomials by the formulas

$$\phi: 1 \otimes x_{j_1} \ldots x_{j_{t}} \otimes o(x_{k_1}, \ldots, x_{k_u}) \otimes 1 \mapsto (-1)^{u} \sum_{x_{k_u} < x_{j_{\nu}}} x_{j_1} \ldots x_{j_{\nu-1}} \otimes o(x_{k_1}, \ldots, x_{k_u}, x_{j_{\nu}}) \otimes x_{j_{\nu+1}} \ldots x_{j_{t}},$$

$$\phi: 1 \otimes o(x_{i_1}, \ldots, x_{i_{u}}) \otimes x_{j_1} \ldots x_{j_{t}} \otimes 1 \mapsto \sum_{x_{j_{\nu}} < x_{i_1}} x_{j_1} \ldots x_{j_{\nu-1}} \otimes o(x_{j_{\nu}}, x_{i_1}, \ldots, x_{i_{u}}) \otimes x_{j_{\nu+1}} \ldots x_{j_{t}}.$$
on $K_0 \otimes_A K_u$ and on $K_s \otimes_A K_0$ ($u, s \geq 0$), and

$$\phi : 1 \otimes o(x_{i_1}, \ldots, x_{i_s}) \otimes x_{j_1} \ldots x_{j_t} \otimes o(x_{k_1}, \ldots, x_{k_u}) \otimes 1 \mapsto \sum_{x_{k_u} < x_{j_1} < x_{i_1}} (-1)^{su + u} x_{j_1} \ldots x_{j_{t-1}} \otimes o(x_{k_1}, \ldots, x_{k_u}, x_{j_1}, \ldots, x_{i_s}) \otimes x_{j_{t+1}} \ldots x_{j_t}.$$  

(4.1.4)

Proposition 4.1.5. The map $\phi : K \otimes_A K \to K$ satisfies $d(\phi) := d_K \phi + \phi d_{K \otimes_A K} = F_K$.

We omit the proof of this proposition, which is a delicate, but straightforward calculation.

4.2. Computing the bracket directly from $\phi$ and Theorem 3.2.7. Before we begin let us make a remark. In general, one wants to be strategic in computing the Lie bracket. One should probably use some additional structures on Hochschild cohomology, such as the cup product in combination with (2.0.4), additional gradings, etc. (One can see the computation of the Gerstenhaber bracket on the Hochschild cohomology of cyclic $p$-groups given in the following section for an example of such a strategic approach.) In what follows, we give a bare bones direct computation of the bracket, on cocycles of arbitrary degree, via Theorem 3.2.7.

We will employ the standard isomorphism

$$A[\partial_1, \ldots, \partial_n] \to \text{Hom}_{A^e}(K, A) = \text{HH}(A)$$

$$\partial_i \mapsto (a \otimes o(x_j) \otimes a' \mapsto \delta_{ij} aa').$$

Here the generators $\partial_i$ are given degree 1, and $A[\partial_1, \ldots, \partial_n]$ denotes the free graded commutative $A$-algebra on these generators. We identify the monomial $\partial_{i_1} \ldots \partial_{i_s}$ with the function dual to the orbit sum $(-1)^{\sum_{r=1}^s \ell o(x_{i_1}, \ldots, x_{i_s})}$ in $\text{Hom}_{A^e}(K, A) = \text{Hom}_k(W, A)$.

It will be convenient to have a bit more notation for the statement of the next proposition.

Notation 4.2.1. For any ordered set $I = \{i_1, \ldots, i_s\}$ of integers satisfying $1 \leq i_k \leq n$ for all $k$, we take

$$\partial_I := \partial_{i_1} \ldots \partial_{i_s} \in \text{Hom}_{A^e}(K, A)$$

and

$$o(x_I) = o(x_{i_1}, \ldots, x_{i_s}) \in W.$$  

For $i_k \in I$, we take

$$I(k) := \{i_1, \ldots, i_{k-1}\} \text{ and } I'(k) := \{i_{k+1}, \ldots, i_l\}.$$  

For ordered sets $I$ and $J$ we give $I \sqcup J$ the natural ordering with $i < j$ for each $i \in I, j \in J$.

In these notations we do not require that the ordering on $I$ is such that $i_k < i_{k+1}$ as integers. We will always let $i_k$ denote the $k$th element of $I$, as determined by $I$'s given order. If we take $\underline{n} = \{1, \ldots, n\}$, then the standard $A$-basis for $A[\partial_1, \ldots, \partial_n] = \text{Hom}_{A^e}(K, A)$ can now be written as the set $\{\partial_I : I \text{ an ordered subset of } \underline{n}\}$.  

Via indexing by ordered sets, we can give a clear expression of the comultiplication on $W$, and the corresponding map $\Delta : K \to K \otimes_A K$. We have

$$\Delta(1 \otimes o(x_I) \otimes 1) = \sum_{I_1, I_2} \pm(1 \otimes o(x_{I_1}) \otimes 1) \otimes (1 \otimes o(x_{I_2}) \otimes 1), \quad (4.2.2)$$

where the sum is indexed by all ordered disjoint subsets $I_1, I_2 \subset I$ with $I_1 \cup I_2 = I$, and $\pm$ is the sign of $\sigma$, where $\sigma$ is the unique permutation with $\{i_{\sigma(1)}, \ldots, i_{\sigma(|I|)}\} = I_1 \sqcup I_2$ as an ordered set.

**Proposition 4.2.3.** The $\circ_{\phi}$ operation is given by

$$(a \partial_I) \circ_{\phi} (b \partial_J) = \sum_{1 \leq k \leq |I|} (-1)^{|I|-k} a \frac{\partial}{\partial x_{ik}} (b) \partial (I_k) \cup (J_{I,k} I') (k)$$

and the bracket $[ , ]_{\phi}$ is given by

$$[a \partial_I, b \partial_J]_{\phi} = \sum_{1 \leq k \leq |I|} (-1)^{|I|-k} a \frac{\partial}{\partial x_{ik}} (b) \partial (I_k) \cup (J_{I,k} I') (k) - \sum_{1 \leq I \leq |J|} (-1)^{|I|-1} b \frac{\partial}{\partial x_{ik}} (a) \partial (J_I) \cup (I_{J,I} I') (I).$$

Note that if $I$ and $J$ share some indices, many of the terms $a \frac{\partial}{\partial x_{ik}} (b) \partial (I_k) \cup (J_{I,k} I') (k)$ may be 0.

**Proof.** Take $f = (a \partial_I)$ and $g = (b \partial_J)$ with $I$ and $J$ ordered subsets of $\mathbb{N}$. We may assume $b$ is an ordered monomial $b = x_{i_1} \ldots x_{i_{|J|}}$. We first provide a computation with symbols $(-1)^{\ell_i}$ in place of significant signs. We will then go back and provide the appropriate signs.

Suppose we have a nonzero monomial $1 \otimes o(x_{I(k)} \cup J_{I,k} I') (k) \otimes 1$. This implies, in particular, that $J$ and $I(k) \cup J'(k)$ share no indices. Then

$$\Delta^{(2)}(1 \otimes o(x_{I(k)} \cup J_{I,k} I') (k) \otimes 1) =$$

$$= (1 \otimes o(x_{I(k)} \otimes 1) \otimes (1 \otimes o(x_J) \otimes 1) \otimes (1 \otimes o(x_{I(k)} \otimes 1))$$

$$+ \sum_{J'} \pm(1 \otimes o(x_{I_2}) \otimes 1) \otimes (1 \otimes o(x_J) \otimes 1) \otimes (1 \otimes o(x_{I_1}) \otimes 1)$$

$$+ \sum_{J' \neq J} \pm(1 \otimes o(x_{I_2'}) \otimes 1) \otimes (1 \otimes o(x_{J'}) \otimes 1) \otimes (1 \otimes o(x_I') \otimes 1),$$

where $J'$ is the set of all pairs of subsets $I_1, I_2 \subset I(k) \cup J'(k) = I \setminus \{i_k\}$ with $I_1 \cup I_2 = I \setminus \{i_k\}$, minus the pair $\{I(k), I'(k)\}$. We do not specify the indexing set of the final sum, except to say that $J' \neq J$. Since $b \partial_J (1 \otimes o(x_J) \otimes 1) = 0$ whenever $J' \neq J$, the above expression gives

$$(-1)^{\ell_1} 1 \otimes o(x_{I(k)} \otimes 1) \otimes b \otimes o(x_{I(k)} \otimes 1) + \sum_{J'} \pm(1 \otimes o(x_{I_2}) \otimes 1) \otimes b \otimes (1 \otimes o(x_I') \otimes 1).$$

Now, one can conclude from the description of $J'$ that the maximal element of each $I_1$ is greater than the minimal element of $I_2$. So

$$\phi(1 \otimes o(x_{I_1}) \otimes 1) \otimes b \otimes (1 \otimes o(x_{I_2}) \otimes 1) = 0$$
and
\[
\Phi(1 \otimes (b \partial J) \otimes 1) \Delta^{(2)}(1 \otimes o(x_i') \otimes 1)
= (-1)^{i} \Phi(1 \otimes o(x_i') \otimes b \otimes o(x_i) \otimes 1)
= (-1)^{i} \Phi(1 \otimes o(x_i') \otimes x_j \ldots x_j \otimes o(x_i) \otimes 1)
= (-1)^{i} \sum_{j < k, j \neq k} (x_j \otimes x_j \otimes o(x_i) \otimes 1) \otimes x_{j+1} \ldots x_{j'}.
\]

Finally, since \( \partial I(1 \otimes o(x_i') \otimes 1) = 0 \) whenever \( I(k) \not\subset I' \), i.e. whenever \( j \neq i_k \), we have
\[
((a \partial I) \circ \phi (b \partial J))(1 \otimes o(x_i') \otimes 1) = (a \partial I)(1 \otimes (b \partial J) \otimes 1) \Delta^{(2)}(1 \otimes o(x_i') \otimes 1) = (-1)^{i} a(\{ \nu : x_j = x_i \}) \frac{1}{x_i}
= (-1)^{i} a \frac{\partial}{\partial x_i}(b).
\]

One can check that \( (a \partial I) \circ \phi (b \partial J) \) vanishes on all monomials \( 1 \otimes o(x_i) \otimes 1 \) not of the form \( I' \otimes J \not\subset I \), after some permutation. So we have
\[
(a \partial I) \circ \phi (b \partial J) = \sum_k (-1)^{i} a \frac{\partial}{\partial x_i}(b) \partial o(x_i) \otimes 1,
\]
and, after switching the position of \( I(k) \) and \( I'(k) \),
\[
(a \partial I) \circ \phi (b \partial J) = \sum_k (-1)^{i} a \frac{\partial}{\partial x_i}(b) \partial o(x_i) \otimes 1.
\]

As for the signs, we have
\[
\begin{align*}
\epsilon_1 &= (|I| - k) |J| + \sum_{\ell=1}^{\ell} \ell \\
\epsilon_2 &= \epsilon_1 + (|I| - k)(k - 1) + (k - 1) \\
\epsilon_3 &= \epsilon_2 + \sum_{\ell=1}^{\ell} \ell \\
\epsilon_4 &= \epsilon_3 - \sum_{\ell=1}^{\ell} \ell \\
\epsilon_5 &= \epsilon_4 + (|I| - k)(|J| - 1)
\end{align*}
\]

The formula for the bracket now follows by the definition of \([f, g]_\phi\). \hfill \Box

For elements \( a \partial I, b \partial J \) the above formula gives
\[
[a \partial I, b \partial J]_\phi = a \frac{\partial}{\partial x_i}(b) \partial_j - b \frac{\partial}{\partial x_j}(a) \partial_i.
\]

One can also verify the identity
\[
[a \partial I, b_1 \partial J_1, b_2 \partial J_2]_\phi = [a \partial I, b_1 \partial J_1]_\phi b_2 \partial J_2 + (-1)^{(|I| - 1)|J_1|} b_1 \partial J_1 [a \partial I, b_2 \partial J_2]_\phi.
\]

So the bracket given in Proposition 4.2.3 is seen to recover the Schouten-Nijenhuis bracket, which is generally expressed using the formula (4.2.4) for the bracket on degree 1 cocycles and extended to all of \( A[\partial_1, \ldots, \partial_k] = HH(A) \) using the graded derivation identity (4.2.5).
5. Recovering Gerstenhaber Brackets for Groups of Prime Order

Assume in this section that the characteristic of the field \( k \) is \( p > 0 \). Let \( G \) be a cyclic group of order \( p \), with generator \( g \), and \( A = kG \), the group algebra. Let \( x := g - 1 \) in \( A \), so that \( A \cong k[x]/(x^p) \). The Hochschild cohomology of \( A \) is well-known. See [3] for the algebra structure of \( \text{HH}(kG) \) in the more general case that \( G \) is abelian. In particular, in that case, \( \text{HH}(kG) \cong \text{H}(G,k) \otimes kG \) as algebras, where \( \text{H}(G,k) \) denotes group cohomology. See Sanchez-Flores [10] for the Gerstenhaber brackets when \( G \) is cyclic; using our new techniques, we will recover her results in our case (i.e. \( G \) has order \( p \)). While the minimal resolution that we use here does not satisfy all the hypotheses 3.1(a)-(d) assumed in Theorem 3.2.7, it does satisfy 3.1(a)–(c) (the weaker conditions assumed in Subsection 3.4). We will show that our alternative approach yields the Gerstenhaber bracket for these examples.

We will use the following \( A^e \)-module resolution of \( A \) (see, e.g., [11, Exer. 9.1.4]):

\[
K : \quad \cdots \rightarrow A^e \xrightarrow{u} A^e \xrightarrow{v} A^e \xrightarrow{u} A^e \xrightarrow{m} A \rightarrow 0,
\]

where \( u = x \otimes 1 - 1 \otimes x \), \( v = x^{p-1} \otimes 1 + x^{p-2} \otimes x + \cdots + 1 \otimes x^{p-1} \), and \( m \) denotes multiplication. For each \( i \), let \( \xi_i \) denote the element \( 1 \otimes 1 \) of \( A^e \) in degree \( i \).

The following maps \( h_n : K_n \rightarrow K_{n+1} \) constitute a contracting homotopy for \( K \), as may be verified by direct calculation.

\[
\begin{align*}
    h_{-1}(x^i) & = \xi_0 x^i, \\
h_0(x^i \xi_0 x^j) & = \sum_{l=0}^{i-1} x^l \xi_1 x^{i-j-1-l}, \\
h_1(x^i \xi_1 x^j) & = \delta_{i,p-1} x^2 \xi_2, \\
h_{2n}(x^i \xi_2 x^j) & = - \sum_{l=0}^{j-1} x^{i+j-1-l} \xi_{2n+1} x^l \quad (n \geq 1), \\
h_{2n+1}(x^i \xi_{2n+1} x^j) & = \delta_{j,p-1} x^i \xi_{2n+1} \quad (n \geq 1).
\end{align*}
\]

Applying Lemma 3.3.1, we may obtain maps \( \phi_n : (K \otimes_A K)_n \rightarrow K_{n+1} \), from the maps \( h_n \), for which \( d(\phi) = F_K \). We only need these maps in degrees 0 and 1:

\[
\begin{align*}
    \phi_0(\xi_0 \otimes x^i \xi_0) & = \sum_{l=0}^{i-1} x^l \xi_1 x^{i-1-l}, \\
    \phi_1(\xi_1 \otimes x^i \xi_0) & = -\delta_{i,p-1} \xi_2, \\
    \phi_1(\xi_0 \otimes x^i \xi_1) & = \delta_{i,p-1} \xi_2.
\end{align*}
\]
Next we record a diagonal map $\Delta : K \to K \otimes_A K$. It may be checked directly that the following map is a chain map:

\[
\begin{align*}
\Delta_0(\xi_0) &= \xi_0 \otimes \xi_0, \\
\Delta_1(\xi_1) &= \xi_1 \otimes \xi_0 + \xi_0 \otimes \xi_1, \\
\Delta_2(\xi_2) &= \xi_2 \otimes \xi_0 + \xi_0 \otimes \xi_2 + \sum_{a+b+c=p-2} x^a \xi_1 \otimes x^b \xi_1 x^c, \\
\Delta_3(\xi_3) &= \xi_3 \otimes \xi_0 + \xi_2 \otimes \xi_1 + \xi_1 \otimes \xi_2 + \xi_0 \otimes \xi_3,
\end{align*}
\]

and generally for $n \geq 1$,

\[
\Delta_{2n}(\xi_{2n}) = \sum_{i=0}^{n} \xi_{2i} \otimes \xi_{2n-2i} + \sum_{i=0}^{n-1} \sum_{a+b+c=p-2} x^a \xi_{2i+1} \otimes x^b \xi_{2n-2i-1} x^c,
\]

\[
\Delta_{2n+1}(\xi_{2n+1}) = \sum_{i=0}^{2n+1} \xi_i \otimes \xi_{2n+1-i}.
\]

Let

\[
\Delta^{(2)}_K := (id \otimes \Delta)\Delta
\]

for the purpose of computing $\phi$-brackets under Definition 3.2.4. (Note that $\Delta$ is coassociative if and only if $p = 2$.)

We will compute $\phi$-brackets on cohomology in low degrees. Applying $\text{Hom}_{A^e}(-, A)$ to $K$, the differentials all are 0. In each degree, the cohomology is the free $A$-module $A$. Let $x^j \xi^*_i \in \text{Hom}_{A^e}(A^e, A)$ denote the function that takes $\xi_i$ to $x^j$. Cup products are known: If $p = 2$, then $\xi_i^*$ generates the Hochschild cohomology as an $A$-algebra (recall $\text{HH}(A) \cong \text{H}(G, k) \otimes A$), while if $p > 2$, it is generated by $\xi_1^*$ and $\xi_2^*$. By applying the identities (2.0.4), we need only compute brackets of pairs of elements of degrees 1 and 2. The $\phi$-circle product of $x^j \xi^*_1$ and $x^j \xi^*_1$ in degree 1 is given by

\[
(x^j \xi^*_1 \circ_\phi x^j \xi^*_1)(\xi_1) = x^j \xi^*_1(\phi_0(x^j \xi_0 \otimes \xi_0 + \xi_0 \otimes x^j \xi_0 + \xi_0 \otimes \xi_0 x^j))
\]

\[
= x^j \xi^*_1(\xi_1 x^{j-1} + x \xi_1 x^{j-2} + \ldots + x^{j-1} \xi_1)
\]

\[
= j x^{i+j-1}.
\]

Therefore, by symmetry, we obtain

\[
[x^j \xi^*_1, x^j \xi^*_1]_\phi = (j - i)x^{i+j-1} \xi^*_1.
\]

The $\phi$-circle product of elements in degrees 1 and 2 is given similarly by

\[
(x^j \xi^*_1 \circ_\phi x^j \xi^*_2)(\xi_2) = j x^{i+j-1},
\]

while in the reverse order we have

\[
(x^j \xi^*_2 \circ_\phi x^j \xi^*_1)(\xi_2) = x^j \xi^*_2(\sum_{a+b+c=p-2} \phi_1(-x^a \xi_1 x^{b+i} \otimes \xi_0 x^c) + \phi_1(\xi_0 \otimes x^{a+b+i} \otimes \xi_1 x^c))
\]

\[
= \sum_{a+c=i-1} x^{i+j-1} - \sum_{a+b=p-1-i} x^{i+j-1}
\]

\[
= (i + p - i)x^{i+j-1} = 0.
\]
3.4.1 to 6

(π must be at least 1.) The chain map

(In the above expressions, any term involving a negative exp onent of
π interpreted as 0.) Using these maps, we may check directly that the map ∆

(Note that in the above sum, α

So [x

(1)

So [x

These computations agree with the Gerstenhaber bracket as computed by Sanchez-Flores [10], as well as with direct computations of the Gerstenhaber bracket using standard chain maps ι, π. Next we will verify the conditions of Lemma 3.4.1 to explain why these φ-brackets agree with Gerstenhaber brackets.

Let υ : K → B and ω : B → K be defined as follows. (See [6] for a more general setting and [2, Section 3] for the maps as below in this specific case.) The chain map υ is given by

where α

(Note that in the above sum, i

l can take on the value 0 while each of i

l must be at least 1.) The chain map π is given by

(In the above expressions, any term involving a negative exponent of x should be interpreted as 0.) Using these maps, we may check directly that the map ∆(2)

defined by (5.0.1) satisfies ∆(2)

K = (π ⊗ π)∆(2)

K. Consequently, Lemma 3.4.1 implies that φ-brackets, defined as above, coincide with Gerstenhaber brackets, as we have observed.

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