EULER CHARACTERISTIC OF THE TRUNCATED ORDER COMPLEX OF GENERALIZED NONCROSSING PARTITIONS

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Abstract. The purpose of this note is to complete the study, begun in the first author’s PhD thesis, of the topology of the poset of generalized noncrossing partitions associated to real reflection groups. In particular, we calculate the Euler characteristic of this poset with the maximal and minimal elements deleted. As we show, the result on the Euler characteristic extends to generalized noncrossing partitions associated to well-generated complex reflection groups.

1. Introduction

We say that a partition of the set \([n] := \{1, 2, \ldots, n\}\) is noncrossing if, whenever we have \(\{a, c\}\) in block \(A\) and \(\{b, d\}\) in block \(B\) of the partition with \(a < b < c < d\), it follows that \(A = B\). For an introduction to the rich history of this subject, see [1, Chapter 4.1]. We say that a noncrossing partition of \([mn]\) is \(m\)-divisible if each of its blocks has cardinality divisible by \(m\). The collection of \(k\)-divisible noncrossing partitions of \([kn]\) — which we will denote by \(NC^{(m)}(n)\) — forms a join-semilattice under the refinement partial order. This structure was first studied by Edelman in his PhD thesis; see [7].

Twenty-six year later, in his own PhD thesis [1], the first author defined a generalization of Edelman’s poset to all finite real reflection groups. (We refer the reader to [8] for all terminology related to real reflection groups.) Let \(W\) be a finite group generated by reflections in Euclidean space, and let \(T \subseteq W\) denote the set of all reflections in the group. Let \(\ell_T : W \to \mathbb{Z}\) denote the word length in terms of the generators \(T\). Now fix a Coxeter element \(c \in W\) and a positive integer \(m\). We define the set of \(m\)-divisible noncrossing partitions as follows:

\[
NC^{(m)}(W) = \left\{ (w_0; w_1, \ldots, w_m) \in W^{m+1} : w_0 w_1 \cdots w_m = c \quad \text{and} \quad \sum_{i=0}^{m} \ell_T(w_i) = \ell_T(c) \right\}.
\]

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That is, $NC^{(m)}(W)$ consists of the minimal factorizations of $c$ into $m+1$ group elements. We define a partial order on $NC^{(m)}(W)$ by setting

$$(w_0; w_1, \ldots, w_m) \leq (u_0; u_1, \ldots, u_m)$$

if and only if $\ell_T(u_i) + \ell_T(u_i^{-1} w_i) = \ell_T(w_i)$ for $1 \leq i \leq m$. (1.2)

In other words, we set $(w_0; w_1, \ldots, w_m) \leq (u_0; u_1, \ldots, u_m)$ if for each $1 \leq i \leq m$ the element $u_i$ lies on a geodesic from the identity to $w_i$ in the Cayley graph $(W, T)$. We place no a priori restriction on the elements $w_0$, $u_0$, however it follows from the other conditions that $\ell_T(w_0) + \ell_T(w_0^{-1} u_0) = \ell_T(u_0)$. We note that the poset is graded with rank function

$$\operatorname{rk}(w_0; w_1, \ldots, w_m) = \ell_T(w_0),$$

hence the element $(c; \varepsilon, \ldots, \varepsilon) \in W^{m+1}$ — where $\varepsilon \in W$ is the identity — is the unique maximum element. There is no unique minimum. It turns out that the isomorphism class of the poset $NC^{(m)}(W)$ is independent of the choice of Coxeter element $c$. Furthermore, when $W$ is the symmetric group $\mathfrak{S}_n$ we recover Edelman’s poset $NC^{(m)}(n)$.

In this note we are concerned with the order complex $\Delta (NC^{(m)}(W))$ — that is, with the abstract simplicial complex whose $d$-dimensional faces are the chains $\pi_0 < \pi_1 < \cdots < \pi_d$ in the poset $NC^{(m)}(W)$. In particular, we wish to compute the Euler characteristic of this complex when the maximal and minimal elements of the poset have been deleted. The answer will involve the following quantity, called the positive Fuß–Catalan number:

$$\operatorname{Cat}^{(m)}_+(W) := \prod_{i=1}^{n} \frac{mh + d_i - 2}{d_i}. \quad (1.4)$$

Here, $n$ is the rank of the group $W$ (the dimension of the Euclidean space on which it acts), $h$ is the Coxeter number (the order of a Coxeter element), and the integers $d_1, d_2, \ldots, d_n$ are the degrees of $W$ (the degrees of the fundamental $W$-invariant polynomials). Our main theorem is the following, which settles Conjecture 3.7.9 from [1].

**Theorem 1.** Let $W$ be a finite real reflection group of rank $n$ and let $m$ be a positive integer. The order complex of the poset $NC^{(m)}(W)$ with maximal and minimal elements deleted has reduced Euler characteristic

$$(-1)^n \left( \operatorname{Cat}^{(m)}_+(W) - \operatorname{Cat}^{(m-1)}_+(W) \right), \quad (1.5)$$

and it is homotopy equivalent to a wedge of this many $(n-2)$-dimensional spheres.

In Section 2 we will collect some auxiliary results and in Section 3 we will prove the main theorem.

In [3, 4], Bessis and Corran have shown that the notion of noncrossing partitions extends rather straightforwardly to well-generated complex reflection groups. It is not done explicitly in [1], but from [3, 4] it is obvious that the definition of generalized noncrossing partitions in [1] can be extended without any effort to well-generated complex reflection groups, the same being true for many (most?) of the results from [1] (cf. [1, Disclaimer 1.3.1]). In Section 4 we show that the assertion in Theorem 1 on the Euler characteristic of the truncated order complex of generalized noncrossing partitions continues to hold for well-generated complex reflection groups. We suspect that this is also true for the topology part of Theorem 1 but what is missing here is the extension
to well-generated complex reflection groups of the result of Hugh Thomas and the first
author [1, Cor. 3.7.3] that the poset of generalized noncrossing partitions associated to
real reflection groups is shellable. This extension has so far not even been done for [2],
the special case of the poset of noncrossing partitions.

2. Auxiliary results

In this section we record some results that are needed in the proof of the main
theorem. The first result is Theorem 3.5.3 from [1].

Theorem 2. The cardinality of \( NC^{(m)}(W) \) is given by the Fuß–Catalan number for
reflection groups

\[
\text{Cat}^{(m)}(W) := \prod_{i=1}^{n} \frac{mh + d_i}{d_i},
\]

where, as before, \( n \) is the rank, \( h \) is the Coxeter number and the \( d_i \) are the degrees of
\( W \). Equivalently, given a Coxeter element \( c \), the number of minimal decompositions
\( w_0w_1 \cdots w_m = c \) with \( \ell_T(w_0) + \ell_T(w_1) + \cdots + \ell_T(w_m) = \ell_T(c) \)
is given by \( \text{Cat}^{(m)}(W) \).

Since the numbers \( h - d_i + 2 \) are a permutation of the degrees [8, Lemma 3.16], we
have an alternate formula for the positive Fuß–Catalan number:

\[
\text{Cat}^{(m)}_+(W) = \prod_{i=1}^{n} \frac{mh + d_i - 2}{d_i} = (-1)^n \text{Cat}^{(-m-1)}(W).
\]

Our next result is Theorem 3.6.9(1) from [1].

Theorem 3. The total number of (multi-)chains

\[
\pi_1 \leq \pi_2 \leq \cdots \leq \pi_l
\]
in \( NC^{(m)}(W) \) is equal to \( \text{Cat}^{(ml)}(W) \).

And, moreover, we have the following.

Lemma 4. The number of (multi-)chains \( \pi_1 \leq \pi_2 \leq \cdots \leq \pi_l \) in \( NC^{(m)}(W) \) with
\( \text{rk}(\pi_1) = 0 \) is equal to \( \text{Cat}^{(ml-1)}(W) \).

Proof. If \( \pi_1 = (w_0^{(1)}; w_1^{(1)}, \ldots, w_m^{(1)}) \) then the condition \( \text{rk}(\pi_1) = 0 \) is equivalent to
\( w_0^{(1)} = \varepsilon \). We note that Theorem 3.6.7 of [1], together with the fundamental map
between multichains and minimal factorizations [1, Definition 3.2.3], establishes a bi-
jection between multichains \( \pi_1 \leq \cdots \leq \pi_l \) in \( NC^{(m)}(W) \) and elements \( (u_0; u_1, \ldots, u_{ml}) \)
of \( NC^{(ml)}(W) \) for which \( u_0 = w_0^{(1)} \). Since \( \ell_T(\varepsilon) = 0 \), we wish to count factorizations
\( u_1u_2\cdots u_{ml} = c \) in which \( \ell_T(u_1) + \cdots + \ell_T(u_{ml}) = \ell_T(c) \). By the second part of Theo-
rem 2, this number is equal to \( \text{Cat}^{(ml-1)}(W) \), as desired. \( \square \)

A stronger version of rank-selected chain enumeration will be important in the proof
of our main theorem in Section 3. Given a finite reflection group \( W \) of rank \( n \), let
\( R_W(s_1, s_2, \ldots, s_l) \) denote the number of (multi-)chains

\[
\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{l-1}
\]
in $NC^{(m)}(W)$, such that $\text{rk}(\pi_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \ldots, l-1$, and $s_1 + s_2 + \cdots + s_l = n$. The following lemma says that zeroes in the argument of $R_W(.)$ can be suppressed except for a zero in the first argument.

**Lemma 5.** Let $W$ be a finite real reflection group of rank $n$ and let $s_1, s_2, \ldots, s_l$ be non-negative integers with $s_1 + s_2 + \cdots + s_l = n$. Then

$$R_W(s_1, \ldots, s_i, 0, s_{i+1}, \ldots, s_l) = R_W(s_1, \ldots, s_i, s_{i+1}, \ldots, s_l)$$

(2.2)

for $i = 1, 2, \ldots, l$. If $i = l$, equation (2.2) must be interpreted as

$$R_W(s_1, \ldots, s_i, 0) = R_W(s_1, \ldots, s_l).$$

**Proof.** This is obvious as long as $i < l$. If $i = l$, then, by definition, $R_W(s_1, \ldots, s_l, 0)$ counts all multi-chains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_l$ with $\text{rk}(\pi_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \ldots, l$. In particular, $\text{rk}(\pi_i) = s_1 + s_2 + \cdots + s_l = n$, so that $\pi_i$ must be the unique maximal element $(c; \varepsilon, \ldots, \varepsilon)$ of $NC^{(m)}(W)$. Thus we are counting multi-chains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{l-1}$ with $\text{rk}(\pi_i) = s_1 + s_2 + \cdots + s_i$, $i = 1, 2, \ldots, l-1$, and, again by definition, this number is given by $R_W(s_1, \ldots, s_l)$. \qed

Finally we quote the version of inclusion-exclusion given in [2, Sec. 2.1, Eq. (4)] that will be relevant to us.

**Proposition 6.** Let $A$ be a finite set and $w : A \rightarrow \mathbb{C}$ a weight function on $A$. Furthermore, let $S$ be a set of properties an element of $A$ may or may not have. Given a subset $Y$ of $S$, we define the functions $f_\leq(Y)$ and $f_\geq(Y)$ by

$$f_\leq(Y) := \sum'_a w(a),$$

where $\sum'$ is taken over all $a \in A$ which have exactly the properties $Y$, and by

$$f_\geq(Y) = \sum_{X \supseteq Y} F_\leq(X).$$

Then

$$f_\leq(\emptyset) = \sum_{Y \subseteq S} (-1)^{|Y|} f_\geq(Y).$$

(2.3)

3. Proof of Main Theorem

Let $c$ denote the unique maximum element $(c; \varepsilon, \ldots, \varepsilon)$ of $NC^{(m)}(W)$ and let $\text{mins}$ denote its set of minimal elements, the cardinality of which is $\text{Cat}^{(m-1)}(W)$. The truncated poset

$$NC^{(m)}(W) \setminus \{(c) \cup \text{mins}\}$$

is a rank-selected subposet of $NC^{(m)}(W)$, the latter being shellable due to [1, Cor. 3.7.3]. If we combine this observation with the fact (see [3, Theorem 4.1]) that rank-selected subposets of shellable posets are also shellable, we conclude that $NC^{(m)}(W) \setminus \{(c) \cup \text{mins}\}$ is shellable. Since it is known that a pure $d$-dimensional shellable simplicial complex $\Delta$ is homotopy equivalent to a wedge of $\chi(\Delta)$ $d$-dimensional spheres (this follows from Fact 9.19 in [4] and the fact that shellability implies the property of being...
homotopy-Cohen-Macaulay [6, Sections 11.2, 11.5]), it remains only to compute the reduced Euler characteristic \( \chi \) of \( NC^{(m)}(W) \backslash \{ c \} \cup \text{mins} \).

For a finite real reflection group \( W \) of rank \( n \), let us again write \( R_W(s_1, s_2, \ldots, s_l) \) for the number of \((\text{multi-})\)chains

\[
\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{l-1}
\]
in \( NC^{(m)}(W) \) with \( \text{rk}(\pi_i) = s_1 + s_2 + \cdots + s_i, i = 1, 2, \ldots, l-1 \), and \( s_1 + s_2 + \cdots + s_l = n \). By definition, the reduced Euler characteristic is

\[
-1 + \sum_{l=2}^{n} (-1)^l \sum_{s_1 + \cdots + s_l = n \atop s_1, \ldots, s_l \geq 0} R_W(s_1, s_2, \ldots, s_l).
\] (3.1)

The sum over \( s_1, s_2, \ldots, s_l \) in (3.1) could be easily calculated from Theorem 3 if there were not the restriction \( s_1, s_2, \ldots, s_l > 0 \). In order to overcome this difficulty, we appeal to the principle of inclusion-exclusion. More precisely, for a fixed \( l \), in Proposition 6 choose \( A = \{(s_1, s_2, \ldots, s_l) : s_1 + s_2 + \cdots + s_l = n\} \), \( w((s_1, s_2, \ldots, s_l)) = R_W(s_1, s_2, \ldots, s_l) \), and \( S = \{S_i : i = 1, 2, \ldots, l\} \), where \( S_i \) is the property of an element \((s_1, s_2, \ldots, s_l) \in A \) to satisfy \( s_i = 0 \). Then (2,3) becomes

\[
\sum_{s_1 + \cdots + s_l = n \atop s_1, \ldots, s_l \geq 0} R_W(s_1, s_2, \ldots, s_l) = \sum_{I \subseteq \{1, \ldots, l\}} (-1)^{|I|} \sum_{s_1 + \cdots + s_l = n \atop s_1, \ldots, s_l \geq 0 \atop s_i = 0 \text{ for } i \in I} R_W(s_1, s_2, \ldots, s_l).
\]

In view of Lemma 5 the right-hand side may be simplified, so that we obtain the equation

\[
\sum_{s_1 + \cdots + s_l = n \atop s_1, \ldots, s_l \geq 0} R_W(s_1, s_2, \ldots, s_l) = \sum_{I \subseteq \{1, \ldots, l\}} (-1)^{|I|} \sum_{s_1 + \cdots + s_{l-|I|} = n \atop s_2, \ldots, s_{l-|I|+1} \geq 0} R_W(0, s_2, \ldots, s_{l-|I|+1})
\]

\[
+ \sum_{I \subseteq \{1, \ldots, l\}} (-1)^{|I|} \sum_{s_1 + \cdots + s_{l-|I|} = n \atop s_1, \ldots, s_{l-|I|} \geq 0} R_W(s_1, s_2, \ldots, s_{l-|I|})
\]

\[
= \sum_{j=1}^{l} (-1)^j \binom{l-1}{j-1} \sum_{s_2 + \cdots + s_{l-j+1} = n \atop s_2, \ldots, s_{l-j+1} \geq 0} R_W(0, s_2, \ldots, s_{l-j+1})
\]

\[
+ \sum_{j=0}^{l} (-1)^j \binom{l-1}{j} \sum_{s_1 + \cdots + s_{l-j} = n \atop s_1, \ldots, s_{l-j} \geq 0} R_W(s_1, s_2, \ldots, s_{l-j}).
\]

By Lemma 4 the sum over \( s_2, \ldots, s_{l-j+1} \) on the right-hand side is equal to \( \text{Cat}^{((l-j)m-1)}(W) \), while by Theorem 3 the sum over \( s_1, \ldots, s_{l-j} \) is equal to
Cat\((l-j-1)m\)(W). If we substitute all this in (3.1), we arrive at the expression

\[-1 + \sum_{l=2}^{n} (-1)^l \left( \sum_{j=1}^{l} (-1)^j \binom{l-1}{j-1} \right) \text{Cat}^{(l-j)m-1}(W) + \sum_{l=0}^{l-1} (-1)^j \binom{l-1}{j} \text{Cat}^{(l-j-1)m}(W) \right) \tag{3.2}

for the reduced Euler characteristics that we want to compute. We now perform the replacement \(l = j + k\) in both sums. Thereby we obtain the expression

\[-1 - \text{Cat}^{(-1)}(W) - \text{Cat}^{(-m)}(W) + \text{Cat}^{(0)}(W) + \sum_{k=0}^{n} (-1)^k \left( \sum_{j=1}^{n-k} \binom{j+k-1}{j-1} \right) \text{Cat}^{(km-1)}(W) + \sum_{j=0}^{n-k} \binom{j+k-1}{j} \text{Cat}^{((k-1)m)}(W) \right), \tag{3.3}

the various terms in the first line being correction terms that cancel terms in the sums in the second line violating the condition \(l = j + k \geq 2\), which is present in (3.2). Since we shall make use of it below, the reader should observe that, by the definition (2.1) of Fuß–Catalan numbers, both \(\text{Cat}^{(km-1)}(W)\) and \(\text{Cat}^{((k-1)m)}(W)\) are polynomials in \(k\) of degree \(n\) with leading coefficient \((mh)^n\).

Again by (2.1), we have \(\text{Cat}^{(-1)}(W) = 0\) and \(\text{Cat}^{(0)}(W) = 1\). Therefore, if we evaluate the sums over \(j\) in (3.3) (this is a special instance of the Chu–Vandermonde summation), then we obtain the expression

\[-\text{Cat}^{(-m)}(W) + \sum_{k=0}^{n} (-1)^k \left( \binom{n}{k+1} \text{Cat}^{(km-1)}(W) + \binom{n}{k} \text{Cat}^{((k-1)m)}(W) \right) = -\text{Cat}^{(-m)}(W) + \text{Cat}^{(-m-1)}(W) - \sum_{k=0}^{n} (-1)^k \binom{n}{k} \text{Cat}^{((k-1)m)}(W) \right.

\[= -\text{Cat}^{(-m)}(W) + \text{Cat}^{(-m-1)}(W) - (-1)^n n!(mh)^n + (-1)^n n!(mh)^n \]

\[= -\text{Cat}^{(-m)}(W) + \text{Cat}^{(-m-1)}(W) - (-1)^n \text{Cat}^{(m-1)}+ (W) + (-1)^n \text{Cat}^{(m)}(W), \]

where, to go from the second to the third line, we used the well-known fact from finite difference calculus (cf. [11, Sec. 1.4, Eq. (26) and Prop. 1.4.2]), that, for any polynomial \(p(k)\) in \(k\) of degree \(n\) and leading coefficient \(p_n\), we have

\[\sum_{k=0}^{n} (-1)^k \binom{n}{k} p(k) = (-1)^n n! p_n.\]
4. The case of well-generated complex reflection groups

We conclude the paper by pointing out that our result in Theorem 1 on the Euler characteristic of the truncated poset of generalized noncrossing partitions extends naturally to well-generated complex reflection groups. We refer the reader to [9, 10] for all terminology related to complex reflection groups.

Let $W$ be a finite group generated by (complex) reflections in $\mathbb{C}^n$, and let $T \subseteq W$ denote the set of all reflections in the group. (Here, a reflection is a non-trivial element of $GL(\mathbb{C}^n)$ which fixes a hyperplane pointwise and which has finite order.) As in Section 1, let $\ell_T : W \to \mathbb{Z}$ denote the word length in terms of the generators $T$. Now fix a regular element $c \in W$ in the sense of Springer [10] and a positive integer $m$. (If $W$ is a real reflection group, that is, if all generators in $T$ have order 2, then the notion of “regular element” reduces to that of a “Coxeter element.”) As in the case of Coxeter elements, it can be shown that any two regular elements are conjugate to each other. A further assumption that we need is that $W$ is well-generated, that is, that it is generated by $n$ reflections given that $n$ is minimal such that $W$ can be realized as reflection group on $\mathbb{C}^n$.

Given these extended definitions of $\ell_T$ and $c$, we define the set of $m$-divisible noncrossing partitions by (1.1), and its partial order by (1.2), as before. In the extension of Theorem 1 to well-generated complex reflection groups, we need the Fuß-Catalan number for $W$, which is again defined by (2.1), where the $d_i$’s are the degrees of (homogeneous polynomial generators of the invariants of) $W$, and where $h$ is the largest of the degrees.

**Theorem 7.** Let $W$ be a finite well-generated (complex) reflection group of rank $n$ and let $m$ be a positive integer. The order complex of the poset $NC^{(m)}(W)$ with maximal and minimal elements deleted has reduced Euler characteristic

$$\text{Cat}^{(-m-1)}(W) - \text{Cat}^{(-m)}(W).$$

(4.1)

In order to prove this theorem, we may use the proof of Theorem 1 given in Sections 2 and 3 essentially verbatim. The only difference is that all notions (such as the reflections $T$ or the order $\ell_T$, for example), have to be interpreted in the extended sense explained above, and that “Coxeter element” has to be replaced by “regular element” everywhere. In particular, the extension of Theorem 2 to well-generated complex reflection groups is Proposition 13.1 in [3], and the proofs of Theorems 3.6.7 and Theorems 3.6.9(1) in [1] (which we used in order to establish Lemma 4 respectively Theorem 3) carry over essentially verbatim to the case of well-generated complex reflection groups.

**References**

[1] D. Armstrong, *Generalized noncrossing partitions and combinatorics of Coxeter groups*, Ph.D. thesis, Cornell University, 2006; to appear in Mem. Amer. Math. Soc.; arXiv:math/0611106.

[2] C. A. Athanasiadis, T. Brady and C. Watt, *Shellability of noncrossing partition lattices*, Proc. Amer. Math. Soc. 135 (2007), 939–949.

[3] D. Bessis, *Finite complex reflection groups are $K(\pi,1)$*, preprint, arXiv:math/0610777.

[4] D. Bessis and R. Corran, *Non-crossing partitions of type $(e,e,r)$*, Adv. Math. 202 (2006), 1–49.

[5] A. Björner, *Shellable and Cohen–Macaulay partially ordered sets*, Trans. Amer. Math. Soc. 260 (1980), 159–193.

[6] A. Björner, *Topological methods*, Ch. 34 in Handbook of Combinatorics, Vol. 2, Elsevier, Amsterdam, 1995, pp. 1819–1872.
[7] P. H. Edelman, Chain enumeration and non-crossing partitions, Discrete Math. 31 (1980), 171–180.
[8] J. E. Humphreys, Reflection groups and Coxeter groups, Cambridge University Press, Cambridge, 1990.
[9] G. C. Shephard and J. A. Todd, Finite unitary reflection groups, Canad. J. Math. 6 (1954), 274–304.
[10] T. A. Springer, Regular elements of finite reflection groups, Invent. Math. 25 (1974), 159–198.
[11] R. P. Stanley, Enumerative Combinatorics, Vol. 1, Wadsworth & Brooks/Cole, Pacific Grove, California, 1986; reprinted by Cambridge University Press, Cambridge, 1998.

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