On Gauge couplings, “Large” extra-dimensions and the limit $\alpha' \to 0$ of the String.

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Abstract

Using an effective field theory (EFT) approach for a generic model with two additional dimensions compactified on a two-torus, we compute the total one-loop radiative corrections to the gauge couplings due to associated Kaluza-Klein massive and massless states. A consistent treatment of both infrared (IR) and ultraviolet (UV) divergences shows a connection via infrared regulator effects between the massless and massive sectors of a compactified theory. A new correction to the gauge couplings is found such that their UV behaviour is sensitive to IR regulator dependent effects in the sector of (infinitely many) massive modes. This correction is a one-loop UV-IR mixing effect due to infinitely many Kaluza-Klein modes and exists for two compact dimensions. The link with string theory is addressed to show that this correction, logarithmic in the UV scale, cannot be recovered from known (infrared regularised) string calculations in the field theory “limit” $\alpha' \to 0$. We explain the origin of this discrepancy and address some of its implications.

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Introduction.

String theory is usually thought of as a more fundamental theory of particle physics than current effective field theory descriptions are. The latter are very successful in describing accurately many aspects of particle physics, but it is thought that ultimately, at some (very) high energy scale, a string description is supposed to provide a fully consistent theoretical framework of high energy physics. In the low energy limit, one would hope to recover the effective field theory description. However, the precise link between these two theories is not always clear. To clarify this relationship, physicists try to reproduce previous string-derived results using effective field theory approaches.

It is the purpose of this work to address such a link for the problem of radiative corrections to the gauge couplings in the context of an effective field theory model compactified on a two-torus. Although we seek to clarify the link with (heterotic) string theory, the calculation is also relevant in the context of models with “large” extra-dimensions, without reference to string theory. The model is assumed to be a 4D $\mathcal{N}=1$ supersymmetric orbifold with an $\mathcal{N}=2$ sub-sector (e.g. $Z_4$ orbifold) which “survives” after compactification. The two extra dimensions compactified on the two-torus have associated Kaluza-Klein states – and at string level winding states as well – which fall into $\mathcal{N}=2$ multiplets, charged under the gauge group of the model. These states bring significant one-loop corrections to the gauge couplings. At (heterotic) string level, such corrections are well-known \cite{1} (see also \cite{2}, \cite{3}, \cite{4}), and will also be the subject of the discussion below.

An effective field theory (EFT) analysis of such one-loop corrections can only account for the effects of momentum states. Such analysis would correspond at string level to the limit of an infinite string scale or $\alpha' \to 0$ (with $M_s^2 \propto 1/\alpha'$ \cite{5}), when winding modes’ effects are suppressed. It was shown in \cite{6} that such an EFT approach can recover the (gauge dependent part of the) string result for the radiative corrections to the gauge couplings due to massive momentum and winding modes only, in the limit $\alpha' \to 0$ (and up to worldsheet instantons contribution, vanishing in this limit). This is important for the EFT approach does not respect the full symmetries and spectrum of the string. For further details on this EFT approach and model description see \cite{6}. Here we continue this EFT analysis, to reveal an additional, interesting effect.

The one-loop corrections to the gauge coupling $\alpha_i$ in a $\mathcal{N}=1$ orbifold are usually separated as

$$\alpha_i^{-1}(Q) = \alpha_o^{-1}(M_s) + \frac{b_i}{4\pi} \ln \frac{M_s^2}{Q^2} + \Omega_i,$$

$i$: gauge group index (1)

$Q$ is the low energy scale, $M_s$ is the string/UV scale, $\alpha_o$ is the “bare” coupling, the “log” term is due to $\mathcal{N}=1$ massless modes correction, usually identified with MSSM-like spectrum, with $b_i$ one-loop beta function. $\Omega_i$ is the (gauge dependent part of the) one-loop correction due to massive $\mathcal{N}=2$ multiplets/states (momentum and in string case winding states as well).
A (heterotic) string calculation of $\Omega_i$ gives

$$
\Omega_i^H = -\frac{b_i}{4\pi} \ln \left[ C_{\text{reg.}} U_2 |\eta(U)|^4 T_2 |\eta(T)|^4 \right], \quad \eta(T) \equiv e^{\frac{2\pi i T}{\Lambda}} \prod_{k=1}^{\infty} \left( 1 - e^{2\pi i kT} \right). \quad (2)
$$

$b_i$ is the beta function coefficient in the $N=2$ sub-sector, $C_{\text{reg.}}$ is an infrared regularisation constant. $T = T_1 + iT_2$, $U = U_1 + iU_2$ are moduli expressed in function of the metric $G_{ij}$ (of determinant $G$) and anti-symmetric tensor background $B_{ij} = e^{\epsilon_{ij}}$ as $T = 2[B + i\sqrt{G}] = 2[B + iR_1 R_2 \sin \theta/(2\alpha')]$, and $U = 1/G_{11}[G_{12} + i\sqrt{G}] = R_2/R_1 \exp(i\theta)$. $T_2$ is expressed in $\alpha'$ units and describes the area of compactification while $U$ describes the shape of the torus ($\alpha'$ independent).

The EFT result for $\Omega_i$, of summing one-loop effects of massive Kaluza-Klein modes only is

$$
\Omega_i^{\text{EFT}} = -\frac{b_i}{4\pi} \ln \left[ C_{\text{reg.}} U_2 |\eta(U)|^4 e^{-T_2^*} T_2^* \right], \quad T_2^* \equiv \Lambda^2 R_1 R_2 \sin \theta; \quad \Lambda: \text{UV scale} \quad (3)
$$

Up to a UV regulator re-scaling/re-definition\(^2\), eq.\(^3\) has identical (UV divergent) behaviour to the limit $\alpha' \to 0$ of the string result\(^2\) which includes the effects of the winding modes as well. If one uses the same regularisation in both cases, the regularisation constants $C_{\text{reg.}}$ in (2) and (3) are equal, but unlike the string case, $C_{\text{reg.}}$ of (3) is due to an UV regularisation (not IR as in eq.\(^2\)).

In the analysis below we address further subtleties of the gauge couplings corrections in this two-torus compactification. So far we mentioned the one-loop effects of the massive (momentum, and in string case also winding) modes on the gauge couplings. Orbifold models also have massless modes, for example (but not only) the twisted modes “living” at the fixed points of the orbifold\(^3\). Considered separately, the massless states bring a logarithmic correction as in eq.\(^1\), in addition to the massive modes’ correction. In this work we analyse more closely effects related to these states.

To compute the massless modes corrections to $\alpha_i$ one needs an IR regularisation. An UV regularisation is also required for their logarithmic correction. Surprisingly, the IR regularisation may have implications for the radiative effects of the massive Kaluza-Klein modes as well, in the EFT approach that we adopt in this work. Apparently it would seem that these two mass sectors are not connected. However, consistency of the regularisation requires us keep an IR regulator $\chi$ in the massive sector as well (only UV divergent in the EFT case), and only in the end remove this regulator, $\chi \to 0$. The result is the presence of an EFT correction to the gauge couplings which is a “mixing” term $\chi \ln \xi$ with UV regulator $\xi \to 0$ as well. This brings a “non-decoupling” of the IR\(^2\)

\(^2\)This re-scaling by a constant $(3/\pi)$ is ultimately an effect of winding modes (modular invariance) and has no physical meaning at EFT level; it only relates the EFT regulator to modular invariance “regularisation”.

\(^3\)Other massless states exist, for example if Wilson lines are present. The effect discussed is not (necessarily) due to (massless) twisted states, but more generally to the existence of massless states in a infinite Kaluza-Klein tower.
effects from a UV logarithmically divergent correction. This correction must be added in the r.h.s. of eq. (1). This is a combined effect of the massless and massive sectors (via infrared effects).

This correction is not recovered from string calculations. In the one-loop string calculation of $\Omega^H_i$ (only) an IR regularisation is needed [1]. The aforementioned EFT correction can have a string correspondent of type $\epsilon \ln \alpha'$, with $\epsilon \to 0$ acting now as IR regulator. As we will discuss, such terms do appear in string calculations. This term vanishes in the limit $\epsilon \to 0$ because $\alpha'$ is non-zero and the limit of removing the IR string regulator in such terms is then possible/allowed. If we then take the field theory “limit” $\alpha' \to 0$ on the string, terms like $\epsilon \ln \alpha'$ are not encountered/kept. Thus the limit $\alpha' \to 0$ on the (regularised) string result will not recover the aforementioned EFT correction. This seems generic for the limit $\alpha' \to 0$ of string calculations with infrared regularisation.

String symmetries (e.g. modular invariance) ensure that calculations done in this framework show a better (finite) UV behaviour than the EFT does, though the string does require IR regularisation. When $\alpha' \to 0$ such symmetries do not necessarily survive, UV divergences (in $\alpha'$) appear, may “interfere” with the IR regularisation and string symmetries (now broken) cannot teach us anymore in which order to remove the IR/UV regulators. If correct, the presence of the new EFT correction also tells us that one cannot address the UV behaviour (the $\alpha' \to 0$ limit) of a string model without simultaneously addressing its IR behaviour and the limit $\alpha' \to 0$ of the (regularised) string may not always recover the UV behaviour as found in the corresponding EFT theory!

In the next section one-loop corrections are computed in the EFT approach with particular attention to the effect of massless states. The link with string theory and its $\alpha' \to 0$ limit is then discussed, followed by conclusions. Some mathematical details are presented in the Appendix.

**One-loop corrections to the gauge couplings.**

To give quantitative support to our statements, we use the Coleman-Weinberg formula for one-loop radiative effects to the gauge couplings, see for example [5]. The one-loop corrected coupling is

$$\alpha_i^{-1}\big|_{\text{one-loop}} = \alpha_i^{-1}\big|_{\text{tree-level}} + \Omega_i^T, \quad i: \text{gauge group index}$$

(4)

In the context of two extra dimensions compactified on a two-torus the Coleman-Weinberg expression for $\Omega_i^T$ due to summing the effects of all states in theory is

$$\Omega_i^T = \frac{1}{4\pi} \sum_R T_i(R) \sum_{m_1,2 \in \mathbb{Z}} \int_0^\infty dt e^{-\pi tM_{m_1,m_2}^2/\mu^2} \bigg|_{\text{reg.}}$$

(5)

$\mu$ is a finite, non-zero mass parameter introduced to set this equation dimensionless. $T_i(R)$ is the beta function contribution of a state - charged under the symmetry group of the model - and which
has an associated Kaluza-Klein tower. The sum over the integers \( m_{1,2} \) runs over all Kaluza-Klein states of mass \( M_{m_{1,2}} \) and the subscript “reg” emphasizes that the above expression only makes sense in the presence of a regularisation. Indeed, integral (5) is divergent in the UV \((t \to 0)\) so a regulator \( \xi \) is introduced \((\xi \to 0)\). For the massless states/modes the above integral is also IR divergent \((t \to \infty)\), so an IR regulator \( \chi \) \((\chi \to 0)\) is also required. A well defined formula is then\(^4\)

\[
\Omega^T_i = \frac{1}{4\pi} \sum_{R} T_i(R) \sum_{m_{1,2} \in \mathbb{Z}} \int_{\xi}^{\infty} \frac{dt}{t} e^{-\pi t M_{m_{1,2}}^2 / \mu^2} e^{-\pi \chi t}\]

To evaluate the above integral we must specify the compactification manifold/spectrum. To keep the analysis model independent, we note that for a two-torus compactification (radii \( R_{1,2} \), cycles’ angle \( \theta \), \( \theta = \pi/2 \) for orthogonal torus) the masses of Kaluza-Klein states have the generic structure (see details in e.g. [7])

\[
M_{m_{1,2}}^2 = \frac{\sin^2 \theta}{R_1^2 R_2^2} \left[ \frac{m_1^2}{R_1^2} + \frac{m_2^2}{R_2^2} - \frac{2m_1 m_2 \cos \theta}{R_1 R_2} \right] = \frac{|m_2 - U m_1|^2}{(\mu T_2^2) U_2}.
\]

In the EFT approach we use the following definitions for the (dimensionless) quantities \( T \) and \( U \)

\[
T(\mu) \equiv i T_2(\mu) = i \mu^2 R_1 R_2 \sin \theta, \quad U = U_1 + i U_2 = \frac{R_2}{R_1} e^{i \theta}
\]

In eq.(6) one can isolate the contribution of the massless mode (in our case \((0,0)\)) from that of massive modes, in our case \((m_1, m_2) \neq (0,0)\) as

\[
\Omega^T_i = \Omega^s_i + \Omega^0_i
\]

where\(^5\)

\[
\Omega^0_i = \frac{b_i}{4\pi} \int_{\xi}^{\infty} \frac{dt}{t} e^{-\pi \chi t} = \frac{b_i}{4\pi} \Gamma(0, \pi \chi \xi) = \frac{b_i}{4\pi} \ln \left( \frac{1}{\xi (\pi e^\gamma)} \right) \chi
\]

\[
\Omega^s_i = \frac{\overline{b}_i}{4\pi} \sum_{m_{1,2} \in \mathbb{Z}} \int_{\xi}^{\infty} \frac{dt}{t} e^{-\pi t M_{m_{1,2}}^2 / \mu^2} e^{-\pi \chi t}
\]

A “prime” on the double sum stands for the absence of \((0,0)\) mode. \( \Omega^0_i \) is due to massless modes, and \( \Omega^s_i \) is due to massive modes only. In general \( \Omega^0_i \) and \( \Omega^s_i \) have in front different beta functions coefficients \((b_i, \overline{b}_i)\). The exact value of the latter, a sum (of \( T_i(R) \)) over a model dependent spectrum, is not relevant for our purposes. In general \( b_i \neq \overline{b}_i \) since in the class of models we address the massless modes include “twisted” states (MSSM-like matter states) living at the fixed

\(^4\)Other regularisations are possible. For a discussion on this see [6].

\(^5\)We denote by \( \gamma \) the Euler constant, \( \gamma = 0.57721 \)
points of the orbifold considered, where the spectrum is different from that of the "bulk" modes \((m_1, m_2)\neq (0, 0)\).

\(\Omega_i^0\) is both IR and UV (logarithmically) divergent as the regulators dependence shows. Ultimately the logarithm in \((10)\) is just the log term of \((11)\), as we will see upon relating the regulators \((\chi, \xi)\) to some IR and UV physical scales \((Q, \Lambda)\) respectively. We now address the massive/"bulk" modes' correction \(\Omega_i^*\). Except its \(\chi\) additional dependence, \(\Omega_i^*\) is identical to that in \(\text{(3)}\) and in \(\text{(8)}\), where the effect of massive modes alone was analysed. Since \(\Omega_i^*\) is IR finite\(^6\), it only needs UV regularisation. However, we cannot remove its \(\chi\) dependence i.e. set \(\chi = 0\) in \(\Omega_i^*\) of \((11)\) before performing the integral. This is because of the massless sector (\(\chi\) dependent) which prevents us from doing so at this stage, and because the limits \(\xi \to 0\) and \(\chi \to 0\) of \(\Omega_i^*\) may not commute.

The \(\chi\) independent part of the integrand of eq. \((11)\) can be written, using eq. \((A.5)\):

\[
\mathcal{I} \equiv \sum_{m_1, m_2 \in \mathbb{Z}} e^{-\frac{\pi}{T_2^{m_2}} |U_{m_1-m_2}|^2} = \sum_{m_2 \in \mathbb{Z}} e^{-\frac{\pi}{T_2^{m_2}} m_2^2} + \sum_{m_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} e^{-\frac{\pi}{T_2^{m_2}} |U_{m_1-m_2}|^2}
\]

\[
= \sum_{m_2 \in \mathbb{Z}} e^{-\frac{\pi}{T_2^{m_2}} m_2^2} + \left[\frac{T_2 U_2}{t}\right] \sum_{m_1 \in \mathbb{Z}} e^{-\frac{\pi}{T_2} m_1^2} + \left[\frac{T_2 U_2}{t}\right] \sum_{m_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} e^{-\frac{\pi}{T_2} m_1^2 - \frac{\pi}{T_2} T_2 U_2 \tilde{m}_2^2 - 2i\pi m_1 \tilde{m}_2 U_1} \tag{12}
\]

A "prime" on a single sum stands for the absence of the zero-mode. \(m_2\) was replaced in \((12)\) by the Poisson re-summed index \(\tilde{m}_2\). Further, each of the sums above (multiplied by \(\exp(-\pi \chi t)\)) can be integrated over \(t \in [\xi, \infty)\) separately. From \((11)\), \((12)\) we have

\[
\Omega_i^* = \frac{b_i}{4\pi} (J_1 + J_2 + J_3), \tag{13}
\]

where

\[
J_1 \equiv \int_{\xi}^{\infty} \frac{dt}{t} \sum_{m_2 \in \mathbb{Z}} e^{-\frac{\pi}{T_2^{m_2}} m_2^2} e^{-\pi \chi t} = -\ln \left[4\pi e^{-\gamma} U_2 \frac{T_2}{\xi}\right] + 2 \left[\frac{T_2 U_2}{\xi}\right]^{\frac{1}{2}} \tag{14}
\]

\[
J_2 \equiv \int_{\xi}^{\infty} \frac{dt}{t} \left[\frac{T_2 U_2}{t}\right] \sum_{m_1 \in \mathbb{Z}} e^{-\frac{\pi}{T_2} m_1^2} e^{-\pi \chi t} = \frac{\pi}{3} U_2 + \frac{T_2}{\xi} + \pi T_2 \chi \ln \left[4\pi e^{-\gamma} U_2 \frac{T_2}{\xi}\right] - 2 \left[\frac{T_2 U_2}{\xi}\right]^{\frac{1}{2}} \tag{15}
\]

\[
J_3 \equiv \int_{\xi}^{\infty} \frac{dt}{t} \left[\frac{T_2 U_2}{t}\right] \sum_{m_1 \in \mathbb{Z}} \sum_{\tilde{m}_2 \in \mathbb{Z}} e^{-\frac{\pi}{T_2} m_1^2 - \frac{\pi}{T_2} T_2 U_2 \tilde{m}_2^2 - 2i\pi m_1 \tilde{m}_2 U_1} e^{-\pi \chi t}
\]

\[
= -\ln \prod_{m_1 \geq 1} \left[1 - e^{2i\pi m_1 U_1}\right]^4. \tag{16}
\]

\(J_1, J_2\) and \(J_3\) are evaluated in detail in the Appendix, see eqs. \((A.1)\), \((B.1)\) and \((C.3)\). The above results are only valid in the limit of removing the UV and IR regulators, \(\xi \to 0\) and \(\chi \to 0\). Particular

\(^6\)If Wilson lines exist, "bulk" modes may be massless and \(\Omega_i^*\) is not IR finite, requiring itself an IR regularisation.
care must be taken when the two limits do not “commute”, in which case such terms are kept in
the final result. The term \( \chi \ln \xi \) in (15) is such an example. This is the only correction in the
massive modes contributions \( J_1, J_2, J_3 \) compared to their value when \( \chi = 0 \) of [6], where only the
radiative effects of the massive sector alone were addressed.

The final result for the total one-loop correction to the tree level coupling of eq.(4) is then
(using eqs.(9) to (17))

\[
\Omega_i^T = \frac{b_i}{4\pi} \ln \frac{\Lambda^2}{Q^2} - \frac{b_i}{4\pi} \ln \left[ 4\pi e^{-\gamma} e^{-T_2} T_2 U_2 |\eta(U)|^4 \right] + \frac{b_i}{4\pi} Q^2 R_1 R_2 \sin \theta e^{-\gamma} \ln \left[ 4\pi e^{-\gamma} \frac{U_2}{T_2} \right]
\]  

(18)

We denoted

\[
T_2^* \equiv \frac{T_2}{\xi} \bigg|_{\xi \to 0} = \Lambda^2 R_1 R_2 \sin \theta, \quad \text{and} \quad \Lambda^2 \equiv \frac{\mu^2}{\xi} \bigg|_{\xi \to 0}, \quad Q^2 \equiv \pi e^\gamma \mu^2 \chi \bigg|_{\chi \to 0}
\]  

(19)

Q is associated with a low-energy scale and \( \Lambda \) is the UV scale. Eq.(19) also clarifies the link between
the UV/IR regulators and their associated mass scales.

The main result (18) is valid in the limit of removing the regulators when higher order corrections
in these are vanishing (see Appendix D eqs.(D-8), (D-9), (D-10)). For the mass scales \( Q \) and \( \Lambda \) of
the model, this means that

\[
Q \ll \frac{1}{R_2 \sin \theta} \leq \frac{1}{R_1} \ll \Lambda, \quad (U_2 \geq 1) \quad \text{or} \quad Q \ll \frac{1}{R_1} \leq \frac{1}{R_2 \sin \theta} \ll \left[ \frac{\Lambda}{R_1} \right]^{1/2} \ll \Lambda, \quad (U_2 < 1)
\]  

(20)

from which one can also find \( T_2^* \equiv \Lambda^2 R_1 R_2 \sin \theta \gg 1 \). \( Q \) is the lowest mass scale of the theory.

Eq.(18) shows an interesting result. The first term is just the massless modes logarithmic
contribution from the UV cut-off (\( \Lambda \)) to the low energy scale (\( Q \)) where such states decouple. It
corresponds to the second term in the r.h.s. of (1). The second term in (18) is the contribution (IR
finite, UV regularised) of massive Kaluza-Klein modes alone mentioned in the introduction, eq.(2)
in the string case and eq.(3) in the EFT approach (with \( C_{\text{reg.}} = 4\pi \exp(-\gamma) \)). Its relationship to
the result of the heterotic string case [1] was extensively discussed in [6].

The last term in (18) is a new correction logarithmic in the UV scale \( \Lambda \) and is not present in the
limit \( \alpha' \to 0 \) of the string calculation eq.(2) or previous effective field theory calculations [6], [8]. It
is due to the term \( \pi T_2 \chi \ln \xi \) of eq.(15) which is relating/mixing the IR and UV scales through their
associated regulators \( \chi \) and \( \xi \) respectively. This term arises from the sector of massive Kaluza-Klein
modes in the presence of the IR regulator \( \chi \). The infrared dependent part of the massive (“bulk”)
modes is controlling (part of) the UV behaviour of the gauge couplings. The effect is related to the
infinite number of momentum states that we summed over, see also its origin in the first integral in
(18), and in (B-3), (B-8) of Appendix B. These equations also show the effect is present in models
with *two* additional compact dimensions (more generally, for an even number of these). The effect arises from Poisson re-summed \((0,0)\) levels\(^7\) with respect to both dimensions compactified on the two-torus. These are the levels which also bring the leading power-like correction to the couplings.

The last term in (18) also shows that even though the massive Kaluza-Klein modes may have a large mass, of the order the compactification scale, they can still bring a contribution proportional to the (much) lower scale \(Q\) (where they are actually decoupled!). The logarithm can be large since \(U_2 \ll T_2^\ast\) (equivalently \(1/R_1 \ll \Lambda\)). However the coefficient in front is small \(Q^2 R_1 R_2 \sin \theta \ll 1\) for our result to hold, see (20), and vanishes when the two cycles of the torus collapse onto each other \(\theta \ll 1\), (one extra dimension case). The new correction implies the existence in some compactified effective field theories of a connection between the IR and UV sectors. This effect is due to the presence of the IR regulator (required by the massless sector) combined with the UV divergence due to infinitely many massive states. It seems that the overall effect of the *infinite* Kaluza-Klein tower on the gauge couplings cannot be split into massive and massless contributions only, as done in eq. (11), and an additional term is present, the last term in eq. (18). This term is ultimately a combined effect of these two mass sectors, through infrared effects.

There remains the difficult question of regularisation independence. The UV regularisation employed here is supported by previous works [6] which match the string result [11] for the corrections due to massive modes alone (no massless state present). The IR regularisation using \(\exp(-\chi t)\), \(\chi \rightarrow 0\) is similar to the IR regularisation used in some string theory approaches [11]. The discussion in the next section also gives support to the regularisation choice independence of the result obtained.

The new correction found may have interesting implications. At the phenomenological level an example would be its effects on the exact value of the unification scale in the class of models addressed here, and the matching of the MSSM unification scale vs. the heterotic string scale.

**The limit \(\alpha' \rightarrow 0\) of the string.**

The presence of the new correction found at the EFT level may have additional implications, in particular on the link between this approach and string theory. At the (heterotic) string level the role of the UV scale \(\Lambda\) (or equivalently \(\xi \rightarrow 0\)) is replaced by the string scale \(M_s\) or equivalently \(\alpha' \propto 1/M_s^2\). In the limit \(\alpha' \rightarrow 0\) one would expect that the string result and the effective field theory result (18) would match. This does actually happen for the effects of the *massive* modes alone, \(\Omega^H_i\) and \(\Omega^{EFT}_i\) while ignoring the massless modes role in the theory, as discussed in the introduction, eqs. (11), (2), (3) and in reference [6].

\(^7\)Note that these levels are different from the “original” \((0,0)\) Kaluza-Klein modes.
The difference between the (limit $\alpha' \to 0$ of the) string results and those of the EFT approach appears when considering the role of the massless states. At the EFT level they introduced the IR regulator whose presence in the massive sector brought in the last term in (18). This term is not recovered from available string results in the limit $\alpha' \to 0$, (see eq.(1) and (2)). To understand why this is so, it is important to mention that the string evaluation of the corrections to the gauge couplings requires an infrared regularisation [1]. This is because the contribution of the degenerate orbits of the modular group $SL(2,Z)$ contributing to the gauge couplings (see Appendix of [1]) is IR divergent. In the dimensional regularisation version [9] of the (IR divergent) string calculation, the total string correction to the “bare” gauge couplings is evaluated to (the beta functions are not shown explicitly)

$$\Omega_T^{\text{string}} \propto \lim_{\epsilon \to 0} \int_{\Gamma} \frac{d^2 \tau}{T_2 + \epsilon} Z \propto \left[ \frac{1}{\epsilon} + \Omega^H + \epsilon \ln(T_2/\alpha') + O(\epsilon) \right]_{\epsilon \to 0}$$  \hspace{1cm} (21)

$\epsilon$ is the infrared regulator in string case ($\epsilon \to 0$), $Z$ the string partition function including the massless (“zero”) modes contribution, and with the $\alpha'$ dependence of $T_2$ as defined in string case, shown explicitly. $1/\epsilon$ accounts for the massless modes infrared divergence, which is logarithmic in scale, and included in eq.(11). Finally $\Omega^H$ is just $\Omega_i^H/\bar{b}_i$ with $\Omega_i^H$ given in eq.(2), and stands for the massive modes contribution. Additional terms like $\epsilon \ln(T_2/\alpha')$ with $\epsilon \to 0$ do appear in (21) during the IR regularisation of the string (see details in Appendix A of [9]) and are similar in structure to $\chi \ln \xi$ of eq.(15) leading to the last term in (18). Other terms relevant for the field theory limit $\alpha' \to 0$, may be present in eq.(21) e.g. $\epsilon(T_2/\alpha')$ or may have a more complicated structure like $\epsilon \times g(T)$ with $g(T)$ being some $SL(2,Z)_T$ invariant function and $\epsilon \to 0$.

In the limit of removing the IR string regulator $\epsilon \to 0$, the term $\epsilon \ln(T_2/\alpha')$ of eq.(21) vanishes and is not present in the final string correction to the gauge couplings given by eqs.(11), (2). This is true only if $\alpha'$ is non-zero as assumed by string calculations. However, in the limit $\alpha' \to 0$ supposed to give the field theory regime, such term can bring corrections to the gauge couplings as it was the case in the EFT analysis, and should be added to the final result. Therefore the limits $\alpha' \to 0$ and $\epsilon \to 0$ do not commute (similar to the EFT case where $\xi \to 0$ and $\chi \to 0$ limits on $\chi \ln \xi$ did not

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8In this work we always refer to the gauge dependent part of the string correction to the gauge couplings. The universal part (gauge independent) is “absorbed” into the definition of the “bare” or string coupling $\alpha_o$ of eq.(11) to ensure this is invariant (as it should) under the symmetries of the string, $SL(2,Z)_T \times SL(2,Z)_U \times Z_2^{T+U}$ [10].

9There are other IR string regularisations, considered to be modular invariant [11] which introduce a mass gap parameter $\mu^2/M_2^2 \ll 1$ as IR regulator. The correspondent of this regulator in our EFT approach is $\chi$. In this IR string regularisation it would be useful to know if a calculation of the massive modes effects gives [in addition to $\Omega_i^H$ of (2)], correction terms $O(\mu^2/M_2^2)$ of structure $\mu^2/M_2^2(T_2/\alpha') \ln(T_2/\alpha') \propto \mu^2 R_1 R_2 \sin \theta \ln(T_2/\alpha')$. Such $O(\mu^2/M_2^2)$ terms were not computed in this regularisation scheme.
commute). This also requires an analysis at the string level, to explain if any string symmetry can still prescribe (upon compactification) the order to take such limits in eq. (21). In a sense this tells us that to understand the UV behaviour of an effective field theory model from the limit $\alpha' \to 0$ of its string embedding, one may have to address the IR behaviour of the string!

We think that one can recover in the string case the additional correction found in the EFT approach, the last term in eq. (18), using the regularisation approach of [11]. At string level this correction may have a more general form, invariant under string symmetries left after compactification. The limit $\alpha' \to 0$ on such additional string correction should recover the last term in eq. (18) computed in the EFT approach.

The above discussion explains why there is no string counterpart available for the last term in (18), even though terms of structure similar to it can arise in the string case. This supports our expectation that the last term in (18) is not a result of the regularisation choice, but a generic presence in the EFT approach. This gives an example where one cannot recover from the $\alpha' \to 0$ limit of available string results the UV behaviour as obtained on pure EFT grounds. The effect may be associated with the non-renormalisability of the EFT approach and the infinite number of momentum states. The extra term found in the EFT case raises questions on the exact matching procedure of the $\alpha' \to 0$ limit of some (infrared regularised) string calculations with effective field theory results.

Conclusions

The effects of two extra dimensions compactified on a two-torus were studied in an effective field theory approach by summing one-loop radiative corrections to the gauge couplings, due to massless and massive Kaluza-Klein modes. First, if one considered the one-loop correction to the gauge couplings of the massive modes alone (i.e. if no massless states existed in the theory) one would only need an UV regularisation. The result obtained would be equal to that of the heterotic string in the limit $\alpha' \to 0$, as previous works showed. However when including in the field theory approach the effects of the massless states of the theory an IR regularisation is required in addition to the UV regularisation. A consistent treatment of these regularisations brings an additional correction to the gauge couplings, which is logarithmic in the UV scale. This (one-loop) correction originates from a UV-IR mixing term $\chi \ln \xi$ (with $\chi \to 0$, $\xi \to 0$) between the UV and IR sectors through their associated regulators $\xi$ and $\chi$ respectively. Consequently, one obtains the surprising result that these sectors do not decouple from each other and this is due to the presence of infinitely many Kaluza-Klein states. This also shows that the UV behaviour of the couplings is sensitive to IR
regulator effects associated with massive ("bulk") modes. The effect is present for two compact dimensions.

Regarding the link with string theory, we discussed why the new field theory correction is not recovered from known heterotic string results in the limit $\alpha' \to 0$. While evaluating this limit, one should note that string calculations for radiative corrections (to gauge couplings) usually require an IR regularisation. In the limit of removing the IR string regulator $\epsilon \to 0$, one discards those $\alpha'$ dependent terms which are multiplied by the regulator, for example $\epsilon \ln \alpha'$, ($\alpha'$ non-zero). Such terms are relevant for the gauge couplings, at least in the limit $\alpha' \to 0$ of the string. This is because such $\alpha'$ dependent terms considered on their own, without the $\epsilon$ dependence are divergent in the limit $\alpha' \to 0$ supposed to give the field theory regime. The discrepancy between the EFT and the limit $\alpha' \to 0$ of the string is ultimately due to the fact that the limits $\alpha' \to 0$ and $\epsilon \to 0$ of the string do not commute. This issue and its implications deserve further investigation.

The effective field theory result applies to models with "large" extra dimensions and is valid without reference to string theory. The additional radiative correction to the gauge couplings that was found shows a connection via infrared regulator effects between the massless and massive sectors (of infinitely many states) of a compactified theory. The example discussed tells us that the limit $\alpha' \to 0$ on the results of the (IR regularised) string as well as its matching with corresponding effective field theory results is more subtle than initially thought, if massless states are present in the theory.

Acknowledgements:
It is a pleasure to thank H.P. Nilles and F. Quevedo for helpful discussions and observations on this work. The author thanks S. Stieberger for his many helpful suggestions and comments on the manuscript. He also thanks S. Groot-Nibbelink for discussions on related topics. The author acknowledges the financial support from PPARC (UK).
Appendix

A. Kaluza-Klein Integrals: $J_1$

General integrals present in Kaluza-Klein models are evaluated below (see also Appendix C in [12]).

• We evaluate the integral: ($\beta > 0$, $y > 0$)

$$
J_1^* (\beta, y) \equiv \int_{\xi}^{\infty} \frac{dt}{t} \sum_{\tilde{m}_2} e^{-\pi \beta t (\tilde{m}_2^2 + y)}
$$

(A-1)

in the limit $\xi \to 0$ and $y \to 0$.

For $\beta = 1/(T_2 U_2)$, $y = \chi/\beta$ one recovers the integral $J_1$ of eq.(14) in the text. $J_1^*$ does not have divergences in $y \to 0$, (see eq.(A-1) in [6] where $y = 0$ case is evaluated) thus terms $\xi/y$ or $\xi \ln y$ and alike are not present or “missed” when in the second integral in (A-3) we set $\xi = 0$. For the error introduced by doing so, see eq.(D-1). We have ($\xi \to 0$)

$$
J_1^* = \int_{\xi}^{\infty} \frac{dt}{t} \left[ -1 + \frac{1}{\sqrt{\beta t}} + \frac{1}{\sqrt{\beta t}} \sum_{\tilde{m}_2} e^{-\pi \tilde{m}_2^2} e^{-\pi \beta y t} \right]
$$

(A-2)

$$
= \int_{\xi}^{\infty} \frac{dt}{t} \left[ -1 + \frac{1}{\sqrt{\beta t}} \right] e^{-\pi \beta y t} + \frac{1}{\sqrt{\beta t}} \sum_{\tilde{m}_2} \int_{0}^{\infty} \frac{dt}{t^{3/2}} e^{-\pi \tilde{m}_2^2 - \pi \beta t y}
$$

(A-3)

$$
= -\Gamma[0, \pi \beta \xi y] + \frac{2}{\sqrt{\beta \xi}} e^{-\pi \beta \xi y} - 2\pi \sqrt{y} \left[ 1 - \text{Erf} \sqrt{\pi \beta \xi y} \right] - 2 \ln \left[ 1 - e^{-2\pi \sqrt{y}} \right]
$$

(A-4)

In (A-2) we used

$$
\sum_{n \in \mathbb{Z}} e^{-\pi A (n+\sigma)^2} = \frac{1}{\sqrt{A}} \sum_{n \in \mathbb{Z}} e^{-\pi (n^2 + 2i\pi \sigma n)}
$$

(A-5)

We also used

$$
\text{Erf}[x] \equiv \frac{2}{\sqrt{\pi}} \int_{0}^{x} dt e^{-t^2} ; \quad \text{Erf}[x] = \frac{2x}{\sqrt{\pi}} - \frac{2x^3}{3 \sqrt{\pi}} + O(x^5), \quad \text{if } (x \ll 1)
$$

(A-6)

and [13]

$$
\int_{0}^{\infty} dx x^{\nu-1} e^{-bx^p - ax^{-p}} = \frac{2}{p} \frac{a}{b} \frac{\Gamma \left( \frac{\nu}{p} \right)}{K_{-\frac{\nu}{p}} (2\sqrt{ab})}, \quad \text{Re}(a), (b) > 0, \quad K_{-\frac{\nu}{p}} (z) = \sqrt{\frac{\pi}{2z}} e^{-z}
$$

(A-7)

From (A-4) we find the leading behaviour

$$
J_1^* (\beta, y) = -\ln \left[ 4\pi e^{-\gamma} \frac{1}{\beta \xi} \right] + \frac{2}{\sqrt{\beta \xi}}, \quad \xi \to 0, \ y \to 0.
$$

(A-8)

(A-8) is a good approximation to (A-4) if $\xi$ and $y$ respect additional conditions, converted into constraints on the mass scales, see Appendix D eq.(D-9). The two limits $\xi \to 0$ and $y \to 0$ on (A-1) commute (see also eq.(A-1) of the Appendix of [6]). Eqs.(A-1), (A-8) give $J_1$ used in the text.
B. Kaluza-Klein Integrals: \( J_2 \)

- We evaluate the integral \((\beta > 0, y > 0)\)

\[
J_2^*(\beta, y) \equiv \int_\xi^\infty \frac{dt}{t^{3/2}} \sum_{m_1} e^{-\pi \beta t (m_1^2 + y)}
\]

in the limits \( \xi \to 0, y \to 0 \).

For \( \beta = U_2/T_2 \) and \( y = \chi/\beta \) one recovers \( J_2 \) of eq.\( (15) \) up to an overall factor, \( J_2 = \sqrt{T_2 U_2} J_2^* \).

\( J_2^* \) does not have divergences in \( y \to 0 \), (see eq.\( (A-12) \) of \cite{[6]} where \( y = 0 \) case is evaluated); thus terms divergent in \( y \) for example \( \xi/y \) or \( \xi \ln y \) are not present or “missed” when in the second integral in \( (B-3) \) we set \( \xi = 0 \). For the error introduced by doing so see eq.\( (B-2) \). We have \( (\xi \to 0) \):

\[
J_2^* = \int_\xi^1 \frac{dt}{t^{3/2}} \left[ -1 + \frac{1}{\sqrt{\beta t}} \sum_{m_1} e^{-\pi \beta t/m_1^2} \right] e^{-\pi \beta y} + \int_1^\infty \frac{dt}{t^{3/2}} \sum_{m_1} e^{-\pi \beta t (m_1^2 + y)}
\]

(\text{B-2})

\[
= \int_\xi^1 \frac{dt}{t^{3/2}} \left[ \frac{1}{\sqrt{\beta t}} - 1 \right] e^{-\pi \beta y} t + \int_1^\xi \frac{dt}{\beta t^{3/2}} \sum_{m_1} e^{-\pi \beta t/m_1^2} + \int_1^\infty \frac{dt}{t^{3/2}} \sum_{m_1} e^{-\pi \beta t (m_1^2 + y)}
\]

(\text{B-3})

\[
= \int_\xi^1 \frac{dt}{t^{3/2}} \left[ \frac{1}{\sqrt{\beta t}} - 1 \right] e^{-\pi \beta y} t + \int_1^\infty \frac{dt}{\beta t^{3/2}} \sum_{m_1} e^{-\pi \beta y/m_1^2} + \int_1^\infty \frac{dt}{t^{3/2}} \sum_{m_1} e^{-\pi \beta t (m_1^2 + y)}
\]

(\text{B-4})

We denote the first integral in \( (B-4) \) by \( J_{2,\xi}^* \) while the second integral (finite) can be written as a well defined limit \( \lim_{\epsilon \to 0} J_{2,\epsilon} \) with \( J_{2,\epsilon} \) defined/computed in \( (B-9) \). For \( J_{2,\xi}^* \) we have

\[
J_{2,\xi}^* = \int_\xi^1 \frac{dt}{t^{3/2}} \left[ \frac{1}{\sqrt{\beta t}} - 1 \right] e^{-\pi \beta y} t
\]

(\text{B-5})

\[
= (\pi \beta y)^{\frac{3}{2}} \left[ \Gamma[-1/2, \pi \beta y] - \Gamma[-1/2, \pi \beta y \xi] \right] - \frac{e^{-\pi \beta y \xi}}{\beta^{1/2}} - \frac{e^{-\pi \beta y}}{\xi^{1/2}}
\]

(\text{B-6})

Using that \( \text{Ei}[z] = \gamma + \ln z + \sum_{k \geq 1} \frac{(-z)^k}{k! k} \), for \( z > 0 \)

\[
\Gamma[\alpha] - \Gamma[\alpha, x] = x^\alpha \sum_{n \geq 0} \frac{(-1)^n x^n}{n! (\alpha + n)}, \quad \alpha \neq 0, -1, -2, \cdots
\]

(\text{B-7})

one can expand \( \text{Ei} \) for \( \xi \) and \( y \) small enough, to find the leading behaviour

\[
J_{2,\xi}^* = \beta^2 \left[ \frac{1}{\xi \beta} - \frac{1}{\sqrt{\xi \beta}} + \pi y \ln \xi - \frac{1}{\beta} + \frac{2}{\sqrt{\beta}} \right], \quad \xi \to 0, y \to 0.
\]

(\text{B-8})
For the term $y \ln \xi$ the limits $y \to 0$ and $\xi \to 0$ do not commute and it is for this reason that this term is kept in (B-8). This is the only such case. This term is a mixing of IR/UV contributions, with implications discussed in the text. Its origin is traced to $\int_{\xi}^{1} dt/t^2 \exp(-\pi \beta t y)$ of eq.(B-4) arising from “Poisson re-summed (0,0)” Kaluza-Klein modes with respect to both dimensions.

In the step from (B-4) to (B-8) additional corrections (convergent series) are discarded. These can be neglected if $\xi$ and $y$ respect additional conditions converted into bounds on the mass scales of the theory in Appendix D eq.(D-10).

We now compute the last integral in eq.(B-4) which is equal to the finite limit $\lim_{\epsilon \to 0} J_{2, \epsilon}$ with

$$J_{2, \epsilon} = \int_{1}^{\infty} dt \left[ \frac{t}{\sqrt{\beta}} \sum_{m_{1}} e^{-\pi \frac{m_{1}^2}{t^2}} e^{-\pi \beta y t} \right] + \frac{1}{t^{3/2+\epsilon}} \sum_{m_{1}} e^{-\pi \beta (m_{1}^2+y)}$$

$$= \frac{1}{\sqrt{\beta}} \int_{0}^{1} \frac{dt}{t^{2+\epsilon}} \sum_{m_{1}} e^{-\pi \frac{m_{1}^2}{t^2}} e^{-\pi \beta y t} + \int_{1}^{\infty} \frac{dt}{t^{3/2+\epsilon}} \sum_{m_{1}} e^{-\pi \beta (m_{1}^2+y)}$$

$$= \frac{1}{\sqrt{\beta}} \int_{0}^{1} \frac{dt}{t^{2+\epsilon}} \left[ -1 + (\beta t)^{\frac{1}{2}} \right] e^{-\pi \beta y t} + \int_{0}^{\infty} \frac{dt}{t^{3/2+\epsilon}} \sum_{m_{1}} e^{-\pi \beta (m_{1}^2+y)}$$

$$= M_{1} + M_{2} \quad \text{(B-9)}$$

where $M_{1,2}$ denote the first/second integral respectively. The second integral is similar to a DR regularised version of the initial integral (B-1), for details see also Appendix B, C of [6].

To evaluate the first integral ($M_{1}$) of (B-9) we have

$$M_{1} \equiv \frac{1}{\sqrt{\beta}} \int_{0}^{1} \frac{dt}{t^{2+\epsilon}} \left[ -1 + (\beta t)^{\frac{1}{2}} \right] e^{-\pi \beta y t}$$

$$= (\pi \beta y)^{\frac{1}{2}+\frac{\epsilon}{2}} \left\{ (\pi y)^{\frac{1}{2}} \left[ \Gamma[-1, -\epsilon, \pi \beta y] - \Gamma[-1] \right] + \Gamma[-1/2, -\epsilon, \pi \beta y] - \Gamma[-1/2 - \epsilon, \pi \beta y] \right\}$$

$$= \beta^{\frac{1}{2}} \frac{1}{\beta} - \frac{2}{\sqrt{\beta}} - \frac{\pi y}{\epsilon} - \frac{1}{\sqrt{\beta}} \sum_{n=2}^{\infty} \frac{(-\pi \beta y)^n}{n! (n-1)} + \sum_{n=1}^{\infty} \frac{(-\pi \beta y)^n}{n! (n-1/2)} + O(\epsilon) \quad \text{(B-10)}$$

We used eq.(B-7) to isolate the only “mixing” term $\pi y/\epsilon$ from the remaining (convergent) series.

To evaluate the second integral ($M_{2}$) of (B-9) we have:

$$M_{2} \equiv \int_{0}^{\infty} \frac{dt}{t^{3/2+\epsilon}} \sum_{m_{1}} e^{-\pi \beta (m_{1}^2+y)} = 2 \Gamma[-1/2 - \epsilon, (\pi \beta)^{\frac{1}{2}+\epsilon}] \sum_{m_{1} > 0}^{\infty} \frac{1}{(m_{1}^2 + y)^{1/2-\epsilon}}$$

$$= 2 (\pi \beta)^{\frac{1}{2}+\epsilon} \sum_{m=0}^{\infty} \frac{(-y)^m}{m!} \Gamma[m - 1/2 - \epsilon] \zeta(2m - 1 - 2\epsilon) \quad \text{(B-11)}$$
In the last step we used the convergent expansion \[14\] \((0 < y < 1)\)
\[
\sum_{n>0} \frac{1}{(n^2 + y)^s} = \sum_{m=0}^{\infty} \frac{\Gamma[m + s]}{\Gamma[s] m!} (-y)^m \zeta(2s + 2m)
\]
\[(B-12)\]

In \((B-11)\) the terms with \(m \neq 1\) give finite contributions for \(\epsilon \to 0\) and \(0 < y < 1\) (in these we can set \(\epsilon = 0\)). However, the term with \(m = 1\) brings \(\zeta(1 - 2\epsilon)\), singular in \(\epsilon \to 0\). Expanding in \(\epsilon\) the factors of this term leads to
\[
\zeta(1 - 2\epsilon) = -\frac{1}{2\epsilon} + \gamma + O(\epsilon)
\]
\[
\Gamma[1/2 - \epsilon] = \pi^{1/2} (1 - \epsilon \psi(1/2) + O(\epsilon^2)), \quad \psi(x) \equiv d\ln\Gamma[x]/dx
\]
\[
(\pi\beta)^\epsilon = 1 + \epsilon \ln(\pi\beta) + O(\epsilon^2)
\]
\[(B-13)\]

Altogether we find for \(\mathcal{M}_2\)
\[
\mathcal{M}_2 = \beta^{\frac{1}{2}} \left[ \frac{\pi}{3} + \frac{\pi y}{\epsilon} \right] + \pi\beta^{\frac{1}{2}} y \ln (4\pi e^{-\gamma}\beta)
\]
\[
+ 2(\pi\beta)^{\frac{1}{2}} \sum_{m=2}^{\infty} \frac{(-y)^m}{m!} \Gamma[m - 1/2] \zeta(2m - 1) + O(\epsilon)
\]
\[(B-14)\]

We find again a term \(\pi y/\epsilon\), cancelled in \(\mathcal{M}_1 + \mathcal{M}_2\). This is a DR “equivalent” of \(y \ln \xi\) of \((B-8)\).

Adding \(\mathcal{M}_1\) of \((B-10)\) and \(\mathcal{M}_2\) of \((B-14)\) we find the finite value of \(\lim_{\epsilon \to 0} J_{2,\epsilon}^\ast\) of eq.\((B-9)\)
\[
\lim_{\epsilon \to 0} J_{2,\epsilon}^\ast = \mathcal{M}_1 + \mathcal{M}_2
\]
\[
= \beta^{\frac{1}{2}} \left[ \frac{\pi}{3} + \frac{\pi y}{\epsilon} \right] - \frac{1}{\sqrt{\beta}} \sum_{n=2}^{\infty} \frac{(-\pi\beta y)^n}{n!(n-1)} + \sum_{n=1}^{\infty} \frac{(-\pi\beta y)^n}{n!(n-1/2)}
\]
\[
+ \pi\beta^{\frac{1}{2}} y \ln (4\pi e^{-\gamma}\beta) + 2(\pi\beta)^{\frac{1}{2}} \sum_{m=2}^{\infty} \frac{(-y)^m}{m!} \Gamma[m - 1/2] \zeta(2m - 1), \ 0 < y < 1
\]
\[(B-15)\]

Eqs.\((B-5)\) and \((B-15)\) give the result for \(J_2^\ast\) when \(0 < y < 1\).

The leading term of eq.\((B-15)\) is
\[
\lim_{\epsilon \to 0} J_{2,\epsilon}^\ast = \beta^{\frac{1}{2}} \left[ \frac{1}{\beta} - \frac{2}{\sqrt{\beta}} + \frac{\pi}{3} + \pi y \ln (4\pi e^{-\gamma}\beta) \right], \quad y \to 0
\]
\[(B-16)\]

The leading term in \(J_2^\ast\) is (using \((B-8), \ (B-16)\))
\[
J_2^\ast \equiv J_{2,\epsilon}^\ast + \lim_{\epsilon \to 0} J_{2,\epsilon}^\ast = \beta^{\frac{1}{2}} \left[ \frac{1}{\xi\beta} - \frac{2}{\sqrt{\xi\beta}} + \pi y \ln (4\pi e^{-\gamma}\beta\xi) + \frac{\pi}{3} \right], \quad y \to 0, \ \xi \to 0
\]
\[(B-17)\]

This result holds if \(y\) and \(\xi\) respect additional conditions, converted into constraints on the mass scales in Appendix D. Eqs.\((B-1), \ (B-17)\) give \(J_2\) used in the text eq.\((15)\). For comparison see eqs.\((A-12)\) to \((A-19)\) in ref.\([6]\).
C. Kaluza-Klein Integrals: \( J_3 \)

- We introduce \( J_3^* (y > 0) \)

\[
J_3^*(\beta,y) = \sum_{m_1} \sum_{\tilde{m}_2} \frac{1}{\sqrt{\beta}} \int_0^\infty \frac{dt}{t^{3/2}} e^{-\frac{\pi \tilde{m}_2^2}{\beta t}} e^{- \pi \beta t (U_2^2 m_1^2 + y)} e^{-2\pi i \tilde{m}_2 U_1 m_1} \tag{C-1}
\]

and evaluate it in the limit \( \xi \to 0, y \to 0 \).

For \( \beta = 1/(T_2 U_2) \) and \( y = \chi/\beta \) one recovers the integral of \( J_3 \) of eq.(16) in the text.

\( J_3^* \) is well defined and finite in the UV limit, \( t \to 0 \) due to the non-vanishing exponentially suppressed term \( \exp(-\tilde{m}_2^2/t) \). Thus no UV divergence (in \( \xi \)) can appear and one can then set \( \xi = 0 \). To evaluate the error introduced while doing so, see eq.(D-3). Note that \( J_3^* \) does not have divergences in \( y \to 0 \) (we assume \( U_2 \neq 0 \)), see eq.(14) of [6] where \( y = 0 \) case is evaluated; thus terms divergent in \( y \) e.g. \( \xi/y \) or \( \xi \ln y \) are not present or “missed” when we set \( \xi = 0 \).

Using the integral representation of Bessel functions, eq.(A-7) we find

\[
J_3^* = \sum_{m_1=1}^\infty \sum_{\tilde{m}_2=1}^\infty \frac{2}{\tilde{m}_2} e^{-2\pi \tilde{m}_2 (y+m_1^2 U_2^2)^{1/2}} e^{2\pi i U_1 \tilde{m}_2 m_1} + c.c. \\
= -2 \ln \prod_{m_1 \geq 1} \left| 1 - e^{-2\pi (y+m_1^2 U_2^2)^{1/2}} e^{2\pi i U_1 m_1} \right|^2 \tag{C-2}
\]

The product above may further be written as \( \prod (1 + a_{m_1}) \) and converges uniformly since the associated series \( \sum |a_{m_1}| \) converges uniformly. One can therefore take the limit \( y \to 0 \) on \( J_3^* \) to find

\[
J_3^* = -2 \ln \prod_{m_1 \geq 1} \left| 1 - e^{2\pi i m_1 U} \right|^2 \tag{C-3}
\]

Eq.(C-1) and (C-3) give \( J_3 \) in the text eq.(16) for \( \chi \ll \beta \).

D. Vanishing errors in the regularised field theory result.

While computing \( J_1 \) of eq.(14), \( J_2 \) of eq.(15) and \( J_3 \) of eq.(16) we introduced additional errors for each of these. We ensure these errors are negligible for “small enough” \( \xi \) and \( \chi \equiv \beta y \) and would like to convert this condition into bounds on the mass scales of the theory.
For $J_1$ an error $\delta_1$ arises in the step from $({A-2})$ to $({A-3})$; for $J_2$ an error $\delta_2$ arises in the step from $({B-3})$ to $({B-4})$; for $J_3$ an error $\delta_3$ arises from $({C-1})$ (by setting $\xi = 0$). Their expressions are

$$\delta_1 \equiv \frac{1}{\sqrt{\beta}} \int_0^\xi \frac{dt}{t^{\frac{3}{2}}} \sum_{\tilde{m}_2} e^{-\pi \tilde{m}_2^2/(\beta t) - \pi \beta y}, \quad \beta = \frac{1}{T_2U_2}, \quad y = \chi(T_2U_2) \tag{D-1}$$

$$\delta_2 \equiv \sqrt{\frac{T_2U_2}{\beta}} \int_{1/\xi}^\infty \frac{dt}{t} \sum_{m_1} e^{-\pi t \tilde{m}_1^2/\beta - \pi \beta y/t}, \quad \beta = \frac{U_2}{T_2}, \quad y = \frac{T_2}{U_2} \tag{D-2}$$

$$\delta_3 \equiv \frac{1}{\sqrt{\beta}} \int_0^\xi \frac{dt}{t^{\frac{3}{2}}} \sum_{m_1} \sum_{\tilde{m}_2} e^{-\pi \tilde{m}_2^2/(\beta t) - \pi \beta t(m_1^2U_2^2 + y)} - 2\pi it \tilde{m}_2 m_1, \quad \beta = \frac{1}{T_2U_2}, \quad y = \chi(T_2U_2) \tag{D-3}$$

This gives, for any positive $y$

$$|\delta_1| \leq \int_{T_2U_2/\xi}^\infty \frac{dt}{t^{\frac{1}{2}}} \sum_{\tilde{m}_2} e^{-\pi \tilde{m}_2^2/t} \tag{D-4}$$

$$|\delta_2| \leq U_2 \int_0^{\xi U_2/T_2} \frac{dt}{t^2} \sum_{m_1} e^{-\pi \tilde{m}_1^2/t} \tag{D-5}$$

$$|\delta_3| \leq \sqrt{\frac{T_2U_2}{\beta}} \int_0^\xi \frac{dt}{t^{\frac{3}{2}}} \sum_{m_1} \sum_{\tilde{m}_2} e^{-\pi \tilde{m}_2^2 T_2 U_2/t - \pi \beta t m_2 m_1} \tag{D-6}$$

A sufficient condition for the $\delta_i$ to vanish is (using eqs. $(A-20)$ to $(A-34)$ of Appendix A.1 of ref. [6])

$$\frac{T_2U_2}{\xi} \gg \max \left\{ \frac{U_2^2}{U_2^2}, \frac{1}{U_2^2} \right\} \tag{D-7}$$

With the notation $\Lambda^2 \equiv \mu^2/\xi$ this means

$$\Lambda R_2 \sin \theta > \Lambda R_1 \gg 1, \quad (U_2 > 1),$$

$$\Lambda R_1 > \Lambda R_2 \sin \theta \gg (\Lambda R_1)^\frac{3}{2}, \quad (U_2 < 1) \tag{D-8}$$

used in the text, eq. (19). These also lead to a large compactification area $T_2^\star \equiv \Lambda^2 R_1 R_2 \sin \theta \gg 1$.

- The limit of “removing” the infrared regulator $\chi \equiv y\beta$, provides further bounds on the mass scales of the theory:
  
  (a). While computing $J_1$ in the text using $J_1^\star$ of eq. $(A-1)$ in the step from eq. $(A-4)$ to $(A-8)$ a (vanishing) correction (in $\xi$ and $\chi \equiv \beta y$) is discarded when the regulators are removed. This is converted into the following bounds on the mass scale $Q^2 \equiv \pi e^\gamma \chi \mu^2$ associated with the infrared regulator $\chi$

$$Q^2 \ll e^\gamma \Lambda^2, \quad Q^2 \ll \frac{e^\gamma}{4 (R_2 \sin \theta)^2} \tag{D-9}$$
In computing $\mathcal{J}_2$ in the text eq. (15) using $\mathcal{J}_2^*$ of eq. (B-1) in the steps from eq. (B-5) to eq. (B-8) and from eq. (B-15) to eq. (B-16) respectively, we discarded additional corrections (in $\xi$ and $y \equiv \chi/\beta$) while removing the regulators. Ensuring that these corrections are negligible provides the following (sufficient) conditions for the mass scales of the theory

$$Q^2 \ll e^\gamma \Lambda^2, \quad Q^2 \ll e^\gamma \frac{1}{R_1 R_2 \sin \theta},$$

$$Q^2 \ll e^\gamma \mu^2, \quad Q^2 \ll \pi e^\gamma \frac{1}{R_1^2}$$

(Eqns. (D-8), (D-9), (D-10) combined together lead to eq. (19) up to (regularisation dependent) coefficients of order unity, not shown there explicitly.

References

[1] L. J. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. B 355 (1991) 649.

[2] P. Mayr and S. Stieberger, Nucl. Phys. B 407 (1993) 725 [arXiv:hep-th/9303017].

D. Bailin, A. Love, W. A. Sabra, S. Thomas, Mod. Phys. Lett. A 10, 337 (1995).

D. Bailin, A. Love, W. A. Sabra, S. Thomas, Mod. Phys. Lett. A 9 (1994) 2543.

C. Kokorelis, Nucl. Phys. B 579 (2000) 267 [arXiv:hep-th/0001217].

[3] V. Kaplunovsky and J. Louis, Nucl. Phys. B 444 (1995) 191 [arXiv:hep-th/9502077].

[4] P. Mayr and S. Stieberger, Phys. Lett. B 355 (1995) 107 [arXiv:hep-th/9504129].

H. P. Nilles and S. Stieberger, Phys. Lett. B 367 (1996) 126 [arXiv:hep-th/9510009].

P. Mayr, H. P. Nilles, S. Stieberger, Phys.Lett. B 317 (1993) 53 [arXiv:hep-th/9307171].

H. P. Nilles and S. Stieberger, Nucl. Phys. B 499 (1997) 3 [arXiv:hep-th/9702110].

J. A. Harvey and G. W. Moore, Nucl. Phys. B 463 (1996) 315 [arXiv:hep-th/9510182].

S. Stieberger, Ph.D. thesis., TUM-HEP-220-95.

[5] V. S. Kaplunovsky, Nucl. Phys. B 307 (1988) 145; [Erratum-ibid. B 382 (1988) 436] arXiv:hep-th/9205068; (For a completely revised version see hep-th/9205070).

[6] D. Ghilencea, S. Groot-Nibbelink, Nucl. Phys. B 641 (2002) 35 [arXiv:hep-th/0204094].

[7] K. R. Dienes, Phys. Rev. Lett. 88 (2002) 011601 [arXiv:hep-ph/0108115].
[8] T. R. Taylor and G. Veneziano, Phys. Lett. B 212 (1988) 147.
    M. Lanzagorta and G. G. Ross, Phys. Lett. B 349 (1995) 319 [arXiv:hep-ph/9501394].
    K. R. Dienes, E. Dudas, T. Gherghetta, Nucl. Phys. B 537 (1999) 47 [arXiv:hep-ph/9806292].

[9] K. Foerger and S. Stieberger, Nucl. Phys. B 559 (1999) 277 [arXiv:hep-th/9901020].

[10] H. P. Nilles and S. Stieberger, Nucl. Phys. B 499 (1997) 3 [arXiv:hep-th/9702110].
    B. de Wit, V. Kaplunovsky, J. Louis, D. Lust, Nucl. Phys. B 451 (1995) 53
    E. Kiritsis, C. Kounnas, P. M. Petropoulos, J. Rizos, Nucl. Phys. B 483 (1997) 141
    S. Stieberger, Nucl. Phys. B 541, 109 (1999) [arXiv:hep-th/9807124].

[11] E. Kiritsis and C. Kounnas, Nucl. Phys. Proc. Suppl. 41 (1995) 331 [arXiv:hep-th/9410212].
    E. Kiritsis and C. Kounnas, Nucl. Phys. B 442 (1995) 472 [arXiv:hep-th/9501020].
    E. Kiritsis and C. Kounnas, [arXiv:hep-th/9507051]
    E. Kiritsis and C. Kounnas, Nucl. Phys. Proc. Suppl. 45BC (1996) 207.
    P. M. Petropoulos and J. Rizos, Phys. Lett. B 374 (1996) 49 [arXiv:hep-th/9601037].

[12] D. M. Ghilencea, H. P. Nilles, S. Stieberger, New J. Phys. 4 (2002) 15 [arXiv:hep-th/0108183].

[13] I.S. Gradshteyn, I.M.Ryzhik, “Table of Integrals, Series and Products”, Academic Press Inc.,
    New York/London, 1965.

[14] E. Elizalde, “Ten Physical Applications of Spectral Zeta functions”, Springer, Berlin, 1995.