NON-NOETHER SYMMETRIES IN SINGULAR DYNAMICAL SYSTEMS

George Chavchanidze
Department of Theoretical Physics
A. Razmadze Institute Mathematics
1 Aleksidze Street, Ge 380093
Tbilisi, Georgia
e-mail:gch@rmi.acnet.ge

Abstract

In the present paper geometric aspects of relationship between non-Noether symmetries and conservation laws in Hamiltonian systems is discussed. Case of irregular/constrained dynamical systems on presymplectic and Poisson manifolds is considered.

2000 Mathematical Subject Classification: 70H33, 70H06, 53Z05

1 Introduction

Noether’s theorem associates conservation laws with particular continuous symmetries of the Lagrangian. According to the Hojman’s theorem [1-3] there exists the definite correspondence between non-Noether symmetries and conserved quantities. In 1998 M. Lutzky showed that several integrals of motion might correspond to a single one-parameter group of non-Noether transformations [4]. In the present paper, the extension of Hojman-Lutzky theorem to singular dynamical systems is considered.

First of all let us recall some basic knowledge of description of the regular dynamical systems (see, e.g.,[5]). In this case time evolution is governed by Hamilton’s equation

\[ i_{X_h} \omega + dh = 0, \]

(1)

where \( \omega \) is the closed (\( d\omega = 0 \)) and non-degenerate (\( i_{X} \omega = 0 \rightarrow X = 0 \)) 2-form, \( h \) is the Hamiltonian and \( i_{X} \omega \) denotes contraction of \( X \) with \( \omega \). Since \( \omega \) is non-degenerate, this gives rise to an isomorphism between the vector fields and 1-forms given by \( i_{X} \omega + \alpha = 0 \). The vector field is said to be Hamiltonian if it corresponds to exact form

\[ i_{X} \omega + df = 0. \]

(2)
The Poisson bracket is defined as follows:

\[
\{f, g\} = X_f g = -X_g f = i_{X_f} i_{X_g} \omega.
\] (3)

By introducing a bivector field \( W \) satisfying

\[
\omega^{ij} = \omega^{ij} + i_{X_f} i_{X_g} \omega,
\]

Poisson bracket can be rewritten as

\[
\{f, g\} = i_W df \land dg.
\] (5)

It’s easy to show that

\[
i_{X_f} i_{X_g} L_Z \omega = i_{[X_f, W]} i_{X_f} \omega \land i_{X_g} \omega,
\]

where the bracket \([,]\) is actually a supercommutator (for an arbitrary bivector field \( W = \Sigma_i V^i \land U^i \) we have \([X, W] = \Sigma_i [X, V^i] \land U^i + \Sigma_i V^i \land [X, U^i]\)).

Equation (6) is based on the following useful property of the Lie derivative

\[
L_X i_W \omega = i_{[X, W]} i_X \omega + i_W L_X \omega,
\] (7)

Indeed, for an arbitrary bivector field \( W = \Sigma_i V^i \land U^i \) we have

\[
L_X i_W \omega = L_X i_{\Sigma_i V^i \land U^i} \omega = L_X \Sigma_i i_{V^i} i_U \omega = \\
= \Sigma_i i_{[X, U^i]} i_{V^i} \omega + \Sigma_i i_{U^i} i_{[X, V^i]} \omega + \Sigma_i i_{U^i} i_{V^i} L_X \omega = i_{[X, W]} \omega + i_W L_X \omega
\]

where \( L_Z \) denotes the Lie derivative along the vector field \( Z \). According to Liouville’s theorem Hamiltonian vector field preserves \( \omega \)

\[
L_X \omega = 0;
\] (9)

therefore it commutes with \( W \):

\[
[X_f, W] = 0.
\] (10)

In the local coordinates \( z_i \) where \( \omega = \omega^{ij} dz_i \land z_j \) bivector field \( W \) has the following form \( W = W^{ij} \partial_{z_i} \land \partial_{z_j} \) where \( W^{ij} \) is matrix inverted to \( \omega^{ij} \).
2 Case of regular Lagrangian systems

We can say that a group of transformations $g(a) = e^{aL_E}$ generated by the vector field $E$ maps the space of solutions of equation onto itself if

$$i_{X_h}g_*(\omega) + g_*(dh) = 0$$

(11)

For $X_h$ satisfying

$$i_{X_h}\omega + dh = 0$$

(12)

Hamilton’s equation. It’s easy to show that the vector field $E$ should satisfy

$$[E, X_h] = 0$$

Indeed, $i_{X_h}L_E\omega + dL_Eh = L_E(i_{X_h}\omega + dh) = 0$ since $[E, X_h] = 0$.

When $E$ is not Hamiltonian, the group of transformations $g(a) = e^{aL_E}$ is non-Noether symmetry (in a sense that it maps solutions onto solutions but does not preserve action).

**Theorem:** (Lutzky, 1998) If the vector field $E$ generates non-Noether symmetry, then the following functions are constant along solutions:

$$I^{(k)} = i_{W^k}\omega_E^k \quad k = 1...n,$$

(13)

where $W^k$ and $\omega_E^k$ are outer powers of $W$ and $L_E\omega$.

**Proof:** We have to prove that $I^{(k)}$ is constant along the flow generated by the Hamiltonian. In other words, we should find that $L_{X_h}I^{(k)} = 0$ is fulfilled. Let us consider $L_{X_h}I^{(1)}$

$$L_{X_h}I^{(1)} = L_{X_h}(i_{W}\omega_E) = i_{[X_h,W]}\omega_E + i_{W}L_{X_h}\omega_E,$$

(14)

where according to Liouville’s theorem both terms ($[[X_h, W] = 0$ and $i_{W}L_{X_h}L_E\omega = i_{W}L_EL_{X_h}\omega = 0$ since $[E, X_h] = 0$ and $L_{X_h}\omega = 0$) vanish. In the same manner one can verify that $L_{X_h}I^{(k)} = 0$

**Note 1:** Theorem is valid for a larger class of generators $E$. Namely, if $[E, X_h] = X_f$ where $X_f$ is an arbitrary Hamiltonian vector field, then $I^{(k)}$ is still conserved. Such a symmetries map the solutions of the equation $i_{X_h}\omega + dh = 0$ on solutions of $i_{X_h}g_*(\omega) + d(g_*(h + f) = 0$.

**Note 2:** Discrete non-Noether symmetries give rise to the conservation of

$I^{(k)} = i_{W^k}g_*(\omega)^k$ where $g_*(\omega)$ is transformed $\omega$.

**Note 3:** If $I^{(k)}$ is a set of conserved quantities associated with $E$ and $f$ is any conserved quantity, then the set of functions $\{I^{(k)}, f\}$ (which due to the
Poisson theorem are integrals of motion) is associated with \([X_h, E]\). Namely it is easy to show by taking the Lie derivative of (13) along vector field \(E\) that \(\{I^{(k)}, f\} = i_{W^k} \omega^k_{[X_f, E]}\) is fulfilled. As a result conserved quantities associated with Non-Noether symmetries form Lie algebra under the Poisson bracket.

**Note 4:** If generator of symmetry satisfies Yang-Baxter equation \([[E[E, W]]W] = 0\) Lutzky’s conservation laws are in involution [7] \(\{Y^{(l)}, Y^{(k)}\} = 0\)

### 3 Case of irregular Lagrangian systems

The singular Lagrangian (Lagrangian with vanishing Hessian) leads to degenerate 2-form \(\omega\) and we no longer have isomorphism between vector fields and 1-forms. Since there exists a set of ”null vectors” \(u^k\) such that \(i_{u^k} \omega = 0\) \(k = 1, 2...n – \text{rank}(\omega)\), every Hamiltonian vector field is defined up to linear combination of vectors \(u^k\). By identifying \(X_f\) with \(X_f + C_k u^k\), we can introduce equivalence class \(X_f^\bullet\) (then all \(u^k\) belong to \(0^\bullet\)). The bivector field \(W\) is also far from being unique, but if \(W_1\) and \(W_2\) both satisfy

\[
i_{X} i_{Y} \omega = i_{W_{1,2}} i_{X} \omega \wedge i_{Y} \omega, \tag{15}\]

then

\[
i_{(W_1-W_2)} i_{X} \omega \wedge i_{Y} \omega = 0 \quad \forall X, Y \tag{16}\]

is fulfilled. It is possible only when

\[W_1 - W_2 = v_k \wedge u^k \tag{17}\]

where \(v_k\) are some vector fields and \(i_{u^k} \omega = 0\) (in other words when \(W_1 - W_2\) belongs to the class \(0^\bullet\))

**Theorem:** If the non-Hamiltonian vector field \(E\) satisfies \([E, X_f^\bullet = 0^\bullet\) commutation relation (generates non-Noether symmetry), then the functions

\[I^{(k)} = i_{W^k} \omega^k_{E} \quad k = 1...\text{rank}(\omega) \tag{18}\]

(where \(\omega_E = L_E \omega\)) are constant along trajectories.

**Proof:** Let’s consider \(I^{(1)}\)

\[L_{X_h^\bullet} I^{(1)} = L_{X_h^\bullet}(i_{W} \omega_E) = i_{[X_h^\bullet, W]} \omega_E + i_{W} L_{X_h^\bullet} \omega_E = 0 \tag{19}\]
The second term vanishes since \([E, X^\bullet] = 0^\bullet\) and \(L_{X^\bullet}\omega = 0\). The first one is zero as far as \([X^\bullet, W^\bullet] = 0^\bullet\) and \([E, 0^\bullet] = 0^\bullet\) are satisfied. So \(I^{(1)}\) is conserved. Similarly one can show that \(L_{X^\bullet}I^{(k)} = 0\) is fulfilled.

**Note 1:** \(W\) is not unique, but \(I^{(k)}\) doesn’t depend on choosing representative from the class \(W^\bullet\).

**Note 2:** Theorem is also valid for generators \(E\) satisfying \([E, X^\bullet] = X^\bullet\).

**Example:** Hamiltonian description of the relativistic particle leads to the following action

\[
A = \int (p^2 + m^2)^{\frac{1}{2}} dx_0 + p_k dx_k
\]  

(20)

with vanishing canonical Hamiltonian and degenerate 2-form

\[
\omega = (p^2 + m^2)^{-\frac{1}{2}} (p_k dp_k \wedge dx_0 + (p^2 + m^2)^{\frac{1}{2}} dp_k \wedge dx_k).
\]  

(21)

\(\omega\) possesses the "null vector field" \(i_u \omega = 0\)

\[
u = (p^2 + m^2)^{\frac{1}{2}} \partial_{x_0} + p_k \partial_{x_k}.
\]

(22)

One can check that the following non-Hamiltonian vector field

\[
E = (p^2 + m^2)^{\frac{1}{2}} x_0 \partial_{x_0} + p_1 x_1 \partial_{x_1} + ... + p_k x_n \partial_{x_n}
\]  

(23)

generates non-Noether symmetry. Indeed, \(E\) satisfies \([E, X^\bullet] = 0^\bullet\) because of \(X^\bullet = 0^\bullet\) and \([E, u] = u\). Corresponding integrals of motion are combinations of momenta:

\[
I^{(1)} = (p^2 + m^2)^{\frac{1}{2}} + p_1 + ... + p_k = \sum_a p_a;
\]

(24)

\[
I^{(2)} = \sum_{a \neq b} p_a p_b;
\]

(25)

... 

(26)

\[
I^{(n)} = \prod_a p_a
\]

(27)

This example shows that the set of conserved quantities can be obtained from a single one-parameter group of non-Noether transformations.
4 Case of dynamical systems on Poisson Manifold

The previous two sections dealt with dynamical systems on symplectic and presymplectic manifolds. Now let us consider the case of dynamical systems on the Poisson manifold. In general, the Poisson manifold is an even dimensional manifold equipped with the Poisson bracket which can be defined by means of the bivector field $W$ satisfying $[W, W] = 0$ as follows:

$$\{f, g\} = i_W df \wedge dg$$  \hfill (28)

Due to skewsymmetry of the $W$ Poisson bracket is also skewsymmetric and, in general, it is degenerate. The commutation relation $[W, W] = 0$ (where $[\ , \ ]$ denotes the supercommutator of vector fields) ensures that the Poisson bracket satisfies the Jacobi identity. As well as in case of symplectic (presymplectic) manifold, we have correspondence between vector fields and 1-forms governed by equation:

$$\beta(X) + \alpha \wedge \beta(W) = 0 \quad \forall \beta$$  \hfill (29)

The classification of vector fields is based on this correspondence. The vector field is called the (locally) Hamiltonian if it corresponds to the (closed) exact 1-form

$$\beta(X_h) + dh \wedge \beta(W) = 0 \quad \forall \beta$$  \hfill (30)

and the non-Hamiltonian if the corresponding 1-form is not closed (Note that, when $W$ is degenerate there are vector fields that are not associated with 1-forms and lay beyond our classification). Note also that, according to the Liouville’s theorem, Hamiltonian vector field preserves the bivector field $W$, i. e. , $L_{X_h} W = [X_h, W] = 0$.

Now let’s consider one-parameter group of transformations $g(a) = e^{aL_E}$ generated by the non-Hamiltonian vector field $E$. Like in the regular case, $E$ generates symmetry of Hamilton’s equation (maps space of solutions onto itself) if $[E, X_h] = 0$ and the correspondence between non-Noether symmetries and conservation laws is governed by the following theorem:

**Theorem:** If the non-Hamiltonian vector field $E$ generates the symmetry of Hamilton’s equation, then the set of functions

$$I^{(k)} = \frac{[E, W]^{r-k} \wedge W^k}{W^r}$$  \hfill (31)
(where \( r \) is the rank of the bivector field \( W \) and \( k = 1, 2, \ldots r \)) is conserved.

**Proof:** It is clear that \( I^k \) are conserved, since \( W \) and \([E, W]\) are invariant bivector fields \( (L_{X_h} = [X_h, W] = 0) \), according to the Liouville’s theorem and

\[
L_{X_h}[E, W] = [X_h[E, W]] = [[X_h, E]W] = [E[X_h, W]] = 0, \tag{32}
\]

since \([E, X_h] = 0\). Now we have to show that the ratio of \( r \)-vector fields

\[
[E, W]^{r-k} \wedge W^k = 0, 1, \ldots r, \tag{33}
\]

is defined correctly. Explicitly this fact can be demonstrated in (local) canonical coordinates where \( W = \sum_{i=1}^r \partial_{p_i} \wedge \partial_{q_i} \) and every non-Hamiltonian vector field can be represented as \( E = \sum_{i=1}^r E^{(p_i)} \partial_{p_i} + E^{(q_i)} \partial_{q_i} \) and as a result every \( r \)-vector field of the form (33) is proportional to \( w = \partial_{p_1} \wedge \partial_{q_1} \wedge \partial_{p_2} \wedge \partial_{q_2} \wedge \ldots \partial_{p_r} \wedge \partial_{q_r} \). q. e. d.

5 Acknowledgements

Author is grateful to Z. Giunashvili and M. Maziashvili for constructive discussions and particularly grateful to George Jorjadze for invaluable help. This work was supported by INTAS (00-00561) and Scholarship from World Federation of Scientists.

References

[1] S. Hojman, A new conservation law constructed without using either Lagrangians or Hamiltonians, 1992 J. Phys. A: Math. Gen. 25 L291-295

[2] F. González-Gascón, Geometric foundations of a new conservation law discovered by Hojman, 1994 J. Phys. A: Math. Gen. 27 L59-60

[3] M. Lutzky, Remarks on a recent theorem about conserved quantities, 1995 J. Phys. A: Math. Gen. 28 L637-638

[4] M. Lutzky, New derivation of a conserved quantity for Lagrangian systems, 1998 J. Phys. A: Math. Gen. 15 L721-722
[5] N.M.J. Woodhouse, Geometric Quantization, Claredon, Oxford, 1992.

[6] G. Chavchanidze, Bi-Hamiltonian structure as a shadow of non-Noether symmetry 2001, math-ph/0106018

[7] G. Chavchanidze, Remark on conservation laws associated with of non-Noether symmetries 2002, math-ph/0207021