Geometry of Periodic Monopoles

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GandAlF

[based on arXiv: 1212.4481, 1309.7013 (with Richard Ward) and 1311.6354]
Outline

• introduction
  • boundary conditions & topology
  • parameters & moduli
  • Nahm transform & spectral curve

• moduli space
  • identifying the Nahm/Hitchin moduli
  • calculating the metric
  • dynamics on the moduli space
  • the effect of the size parameter

• summary, references
Setup

- Periodic monopoles satisfy Bogomolny equations on $\mathbb{R}^2 \times S^1$ with $\zeta = \rho \, e^{i\theta} = x + iy$ and $z \sim z + 2\pi$

  $$F = \ast D\Phi.$$ 

- $\Phi$ and $A$ are valued in $\mathfrak{su}(2)$, with boundary conditions for $\rho \to \infty$ [Cherkis & Kapustin '01]

  $$\Phi \to i\pi \log(\rho/C)\sigma_3 \quad A_{x,y} \to 0 \quad A_z \to i\chi/\pi\sigma_3.$$ 

- Topological (magnetic) charge $k$ is the number of zeroes of $\Phi$, also

  $$k = \int_{T^2_R} \frac{\text{tr}(F\Phi)}{4\pi|\Phi|}.$$
Parameters and moduli

• There are four parameters we are allowed to vary → moduli:
  • relative $xy$ positions (determined by $K \in \mathbb{C}$),
  • $z$ offset,
  • relative phase.

• Define the $z$-holonomy through

$$\partial_z V(\zeta) = (A_z + i\Phi)V(\zeta) \quad \text{with} \quad V(0) = 1_2.$$ 

This quantity is only sensitive to the $K$ modulus.

• Other parameters have an infinite $L^2$ norm:
  • monopole size $|C|$,  
  • orientation $\text{arg}(C)$.

• In contrast to monopoles in $\mathbb{R}^3$, variations of the centre of mass have infinite $L^2$ norm, so can only consider the relative moduli space, $\mathcal{M}$.  
[Cherkis & Kapustin '02]
Nahm transform

- A generalisation of the ADHM construction of instantons.
- Relates dimensional reductions of the self-dual Yang-Mills equations on reciprocal 4-tori. [Braam & van Baal ’89]
- Here, monopole space $\mathbb{R}^2 \times S^1$ suggests Hitchin equations on $\mathbb{R} \times S^1$ with $r \in \mathbb{R}$, $t \sim t + 1$ and $s = r + it$, 
  
  $$2f_{rt} = i[\phi, \phi^\dagger] \quad D\bar{s}\phi = 0.$$ 

- Spectral data defined from either side of the transform: [Cherkis & Kapustin ’01]
  
  $$\det(e^{\beta s} - V(\zeta)) = 0 \equiv \det(\zeta - \phi(s)) = 0.$$ 

- This fixes gauge invariants of $\phi$,
  
  $$\text{tr}(\phi) = 0 \quad \det(\phi) = -C^2(2 \cosh(2\pi s) - K).$$
Solving the Nahm/Hitchin equations

Charge 2 solutions are gauge-equivalent to [Harland & Ward ’09]

\[
\phi = \begin{pmatrix} 0 & \mu_+ e^{\psi/2} \\ \mu_- e^{-\psi/2} & 0 \end{pmatrix} \quad a_s = a\sigma_3 + \alpha\phi
\]

with \( \mu_+\mu_- = C^2(2\cosh(2\pi s) - K) \), and the Hitchin equations become

\[
\Delta \text{Re}(\psi) = 2 \left( |\mu_+|^2 e^{\text{Re}\psi} - |\mu_-|^2 e^{-\text{Re}\psi} \right) \quad a = -\frac{1}{4} \partial_s \psi
\]

where \( \alpha \) has been set to zero by symmetry (\( \alpha \neq 0 \) encodes \( z \) offset and phase). \( \text{Im}(\psi) \) chosen to ensure \( \phi \) is periodic.

- \( \det(\Phi) \) has two zeroes. Two distinct solutions: we can place both zeroes in \( \mu_+ \) or one in each of \( \mu_\pm \).
- For \( |C| \gg 1 \) and/or \( |K| \gg 2 \), solution is \( \text{Re}(\psi) = \log \left( |\mu_-|/|\mu_+| \right) \).
Lumps on Cylinder

For $K \in \mathbb{R}$, zeroes of $\det(\phi)$ correspond roughly to peaks in $|f_{rt}|$,

- ‘zeroes together’

- ‘zeroes apart’

Here $C = 1$. The size/period ratio now determined by $1/C$. 
For large $|K|$, we have ‘lumps’ at $2\pi s_\pm = \pm \cosh^{-1}(K/2)$.

Treat these as delta-functions.

Approximate fields away from $s_\pm$,
\[
\phi = C \sqrt{2 \cosh(2\pi s) - K}\sigma_3, \quad a_r(r, t) = 0, \quad a_t(0, t) = i\theta\sigma_3.
\]

Peaks have a phase angle in the $\sigma_1/\sigma_2$ plane. Relative phase $\omega$.

The constant $\theta$ gives the $t$-holonomy at $r = 0$,
\[
U = \mathcal{P} \exp \left( \int_0^1 a_t(0, t) dt \right) \quad 2\cos(\theta) = \text{tr}(U).
\]

The moduli $\omega$ and $\theta$ are defined up to a choice of sign.

Take $\text{Re}(K) > 0$ for simplicity.
• The low energy dynamics of monopoles can be understood as geodesic motion on the moduli space, $\mathcal{M}$.

• A tangent vector in $\mathcal{M}$ to the solution $(\phi, a_{\bar{s}})$ is $V_1 = (\delta_1 \phi, \delta_1 a_{\bar{s}})$, which must be orthogonal to the gauge orbits and satisfy Hitchin equations

$$4D_{\bar{s}}(\delta_1 a_s) = [\phi, \delta_1 \phi^\dagger] \quad D_{\bar{s}}(\delta_1 \phi) = [\phi, \delta_1 a_{\bar{s}}].$$

• Three other solutions $V_i = (\delta_i \phi, \delta_i a_{\bar{s}})$,

$$V_2 = iV_1, \quad V_3 = (2\delta_1 a_s, \frac{1}{2}\delta_1 \phi^\dagger), \quad V_4 = iV_3$$

satisfying

$$\langle V_i, V_j \rangle = \frac{1}{2} \text{Re} \int \text{tr} \left( (\delta_i \phi)(\delta_j \phi)^\dagger + 4(\delta_i a_{\bar{s}})(\delta_j a_{\bar{s}})^\dagger \right) \, dr \, dt = p^2 \delta_{ij}$$
A perturbation satisfying all the above is $V_1 = \epsilon (\frac{1}{2} h \sigma_3, 0)$ with

$$h = C(-\text{det}(\phi))^{-1/2} = (2 \cosh(2\pi s) - K)^{-1/2}.$$ 

Change coordinates such that $V_i \equiv (\delta_i \text{Re}(K), \delta_i \text{Im}(K), \delta_i \theta, \delta_i \omega)$, then

$$Q = \frac{1}{p} \begin{pmatrix}
\delta_1 K_r & \delta_2 K_r & \delta_3 K_r & \delta_4 K_r \\
\delta_1 K_i & \delta_2 K_i & \delta_3 K_i & \delta_4 K_i \\
\delta_1 \theta & \delta_2 \theta & \delta_3 \theta & \delta_4 \theta \\
\delta_1 \omega & \delta_2 \omega & \delta_3 \omega & \delta_4 \omega
\end{pmatrix}$$

and

$$g = (QQ^T)^{-1}.$$
A perturbation satisfying all the above is $V_1 = \epsilon \left( \frac{1}{2} h \sigma_3, 0 \right)$ with

$$h = C \left( -\det(\phi) \right)^{-1/2} = \left( 2 \cosh(2\pi s) - K \right)^{-1/2}.$$

Change coordinates such that $V_i \equiv (\delta_i \text{Re}(K), \delta_i \text{Im}(K), \delta_i \theta, \delta_i \omega)$, then

$$Q = \frac{1}{p} \begin{pmatrix} \delta_1 K_r & \delta_2 K_i \\ \delta_2 K_i & \delta_3 \theta & \delta_4 \theta \\ \delta_3 \theta & \delta_3 \omega & \delta_4 \omega \end{pmatrix}$$

and

$$g = (QQ^T)^{-1}.$$
Metric on the Moduli Space

• Recall the solution away from $s_{\pm}$ had $a_t(0, t) = i\theta\sigma_3$

  $\Rightarrow$ determine $\delta\theta$ directly from $\delta a_{\bar{s}}$.

• A change $\delta a_{\bar{s}}$ affects the propagation of $f_-$ to $s_+$ (but leaves $f_\pm$ unchanged), with $\gamma$ a path between $s_{\pm}$ and $\omega = \text{tr}(f_+\hat{f}_-)$.

  $$\partial_\gamma f_- + [a_\gamma + \delta a_\gamma, f_-] = 0 \quad \Rightarrow \quad \delta\omega = 4 \int_\gamma \delta a \cdot d\ell$$

  This quantity is path-dependent due to twisting of $\theta$.

• The constant $p$ is given by

  $$p^2 = \frac{\epsilon^2}{4} \int |h| \, dr \, dt = \frac{\epsilon^2}{4} \int \frac{dr \, dt}{|2 \cosh(2\pi s) - K|} \approx \frac{\epsilon^2}{4\pi} \frac{\log(4|K|)}{|K|}.$$
Performing integrals for $K = |K|e^{2\pi i \eta}$ and $|K| \gg 2$, get metric

$$ds^2 = \frac{\log(4|K|)}{4\pi |K|} \left( C^2 |dK|^2 + 4|K| d\theta^2 \right) + \frac{\pi}{\log(4|K|)} (d\omega - 4\eta d\theta)^2.$$ 

- Agreement with the metric obtained from the monopole side of the Nahm transform using physical arguments. [Cherkis & Kapustin '03]
- Identify
  - $\theta =$ twice the $z$-offset,
  - $\omega =$ twice the relative phase between monopoles.
Surfaces of constant $\theta, \omega$

Recall the two different solutions to the Nahm/Hitchin equations:

- ‘zeroes together’ has $\omega = 0$,
- ‘zeroes apart’ has $\omega = \pi$.

In both cases we can have $\theta = 0$ or $\theta = \pi$. These surfaces are isometric.

- For ‘zeroes together’ simply corresponds to choice of $\text{Im}(\psi) = \pm 2\pi t$. Analogue to Atiyah-Hitchin cone (next slide).

- For ‘zeroes apart’ the surfaces are connected on $K \in [-2, 2]$. “Double trumpet”, 3 dimensional scattering. Double scattering: outgoing chains shifted along $z$ by $\pi$ and rotated.
Surfaces of constant $\theta, \omega$

The metric on surfaces with $\omega = 0$ can be computed numerically for all $K$, for $|C| \to \infty$, the conformal factor is

$$\Omega = \frac{1}{4} \int \frac{dr \, dt}{|2 \cosh(2\pi s) - K|}.$$ 

Agrees with ‘spectral approximation’. Scattering in $xy$ plane.
Surfaces of constant $\theta, \omega$

For $\omega = \pi$, symmetries fix geodesic submanifolds with $K \in \mathbb{R}$ and $K \in i\mathbb{R}$. In terms of the coordinate $K = W + W^{-1}$ there are three distinct geodesics (here $p > 0$),
The trajectory with $W \in \mathbb{R}$ and $W > 0$ describes monopoles incoming along the $x$-axis at $z = 0$ and scattering along the $z$ axis. They then scatter with those in adjacent periods and depart parallel to the $x$-axis and shifted by $\pi$ along $z$.

For $W \in i\mathbb{R}$ monopoles are incoming along $x = y$. They scatter first along $z$, and again along $x = -y$. We thus have a shift and a rotation by $\pi/2$. 

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**Geodesics on the Double Trumpet - radial**
The $|W| = 1$ geodesic describes a chain of equally separated monopoles (at $z = \pm \pi/2$) whose shape oscillates in size. For $C = 1$ and $W = 1$, i: $\text{[Harland & Ward '09]}

\begin{itemize}
  \item $W = \pm 1$ encode a charge one chain taken over two periods.
\end{itemize}
Higher Charge Chains

The above can all be generalised to higher charges, e.g. for charge 3,
Summary & Outlook

- Reproduce the metric describing the dynamics of charge 2 periodic monopoles from Nahm data, identifying asymptotic moduli.

- Investigate novel examples of monopole scattering.

- Small size $C$: describe as small well separated monopoles, large $C$: ‘spectral approximation’. The intermediate region is also interesting.

- Other geodesic submanifolds? e.g. expect a symmetry fixing $\theta = \pi/2$.

- Can we do the same for monopole walls? (work in progress)
• Braam & van Baal, *Nahm’s Transformation for Instantons*, *Commun. Math. Phys.* 122 (1989) 267
• Cherkis & Kapustin, *Nahm transform for periodic monopoles and N=2 super Yang-Mills theory*, hep-th/0006050
• ———, *Hyper-Kähler metrics from periodic monopoles*, hep-th/0109141
• ———, *Periodic monopoles with singularities and N=2 super QCD*, hep-th/0011081
• Harland & Ward, *Dynamics of Periodic Monopoles*, 0901.4428
• Maldonado, *Periodic monopoles from spectral curves*, 1212.4481
• ———, *Higher charge periodic monopoles*, 1311.6354
• Maldonado & Ward, *Geometry of Periodic Monopoles*, 1309.7013
• Ward, *Periodic monopoles*, hep-th/0505254
For larger $|C|$ the picture is more complicated. Here $C = 1, 2$ with $W = i$. 
As $|C|$ is increased the energy density distribution approaches that of the ‘spectral approximation’ (i.e. when approximate monopole fields are read off from the spectral curve). Here, for $W = i (K = 0)$ and $C = 4, 6$. 