Tau-functions as highest weight vectors for $W_{1+\infty}$ algebra

B. Bakalov * E. Horozov † M. Yakimov ‡

Department of Mathematics and Informatics,
Sofia University, 5 J. Bourchier Blvd., Sofia 1126, Bulgaria

Abstract

For each $\mathbf{r} = (r_1, r_2, \ldots, r_N) \in \mathbb{C}^N$ we construct a highest weight module $M_\mathbf{r}$ of the Lie algebra $W_{1+\infty}$. The highest weight vectors are specific tau-functions of the $N$-th Gelfand–Dickey hierarchy. We show that these modules are quasifinite and we give a complete description of the reducible ones together with a formula for the singular vectors.

1 Introduction

The remarkable connection between the infinite-dimensional Lie algebras and the soliton equations was noticed by M. Sato [23] and further developed by Date et al. in [6]. In particular it was found that the important for the conformal field theory Kac–Moody algebras and the Virasoro algebra play a substantial role in soliton theory (see [2, 25, 7], etc. for more details). We would only mention the work of several authors (see [20, 21, 28] and references therein) where it was discovered that the partition function of 2D-quantum gravity is a tau-function for the KdV-hierarchy and also satisfies the so called Virasoro constraints. This result can also be interpreted as a construction of a certain highest weight representation of the Virasoro algebra. Later a whole class of representations of the Virasoro algebra in terms

*E-mail: bbakalov@fmi.uni-sofia.bg
†E-mail: horozov@fmi.uni-sofia.bg
‡E-mail: myakimov@fmi.uni-sofia.bg
of tau-functions was built in [13, 14]. Certain special functions like Airy or Bessel functions and Hermite or Laguerre polynomials play an important role in all above mentioned results.

The present paper deals with similar questions but for the Lie algebra $W_{1+\infty}$. This algebra is the unique central extension of the Lie algebra of regular differential operators on the circle [16]. In recent works (see e.g. references in [10, 3]) this algebra and its reductions $W_N$ were found to play an important role in quantum field theory. $W_{1+\infty}$ is also the algebra of the additional symmetries of the KP tau-functions [27]. The representation theory of $W_{1+\infty}$ was recently initiated in [18, 10, 3], etc. In particular V. Kac and A. Radul isolated a class of of $W_{1+\infty}$-modules and classified them. These are graded modules with finite-dimensional level spaces, called in [18] quasifinite.

In contrast to the general theory we are interested in concrete representations connected to classical special functions – this time – Meijer’s $G$-functions (see [5, 22]). Our construction uses a simple but beautiful idea of Kac and Schwarz [20]. We recall it briefly. Each tau-function corresponds to a plane $W$ from the Sato Grassmannian $Gr$ which can be considered as an infinite wedge product $|W\rangle$. Assume that an operator $A$ leaves the plane $W$ invariant. Then under the boson-fermion correspondence $\sigma$ the image $\tau_W(t) = \sigma(|W\rangle)$ is an eigenvector of $A$ in the (completed) bosonic Fock space. We take $A$ to be $\zeta \partial_\zeta$ (recall that $Gr$ is built from the space of formal Laurent series in $\zeta$) and impose on $W$ to be invariant under the multiplication by $\zeta^N$ (hence $\tau_W$ is a solution of the $N$-th reduction of KP hierarchy). These restrictions yield a compatibility condition, satisfied by $W$. Thus we come to the other classical object – Meijer’s differential equation $[3]$ (see eq. (8) below) which is connected to the above modules. In the last section of the paper we give explicit formulas for the singular vectors in these modules and point out the embeddings among reducible ones.

Although we consider that these representations have their own value we have to point out that our first motivation in their construction was the solution of the so called bispectral problem (see [8, 27, 12] and references therein). Starting with the highest weight vectors of these modules we build broad classes of solutions of any rank to this problem (see [4] and references therein). But what we find more important is that these modules provide a natural representation-theoretic setting for many results in the bispectral problem including those of [3]. In this way we exhibit a completely new area of applications of the crucial idea of Sato: the interplay between representation theory of infinite-dimensional Lie algebras and soliton equations.
2 Preliminaries on $W_{1+\infty}$ and Sato’s Grassmannian

An adequate representation-theoretic model for the Dirac sea is the infinite wedge space $F = \oplus_{m \in \mathbb{Z}} F^{(m)}$, defined as follows [10, 17, 19]. Let $V = \oplus_{j \in \mathbb{Z}} \mathbb{C}v_j$ be infinite-dimensional vector space with basis $v_j$. Then $F^{(m)}$ for $m \in \mathbb{Z}$ is the linear span of all semi-infinite monomials $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots$ such that $i_0 > i_1 > \ldots$ and $i_k = m - k$ for $k \gg 0$. The free fermions can be realized as wedging and contracting operators:

$$
\psi_{-j+\frac{1}{2}}(v_{i_0} \wedge v_{i_1} \wedge \ldots) = v_j \wedge v_{i_0} \wedge v_{i_1} \wedge \ldots
$$

$$
\psi^*_{j-\frac{1}{2}}(v_j \wedge v_{i_0} \wedge v_{i_1} \wedge \ldots) = v_{i_0} \wedge v_{i_1} \wedge \ldots
$$

The Lie algebra $gl_\infty$ of all $\mathbb{Z} \times \mathbb{Z}$ matrices, having only a finite number of non-zero entries, can be represented in $F$ via

$$
r(E_{ij}) = \psi_{-i+\frac{1}{2}} \psi^*_{j-\frac{1}{2}}
$$

(where $E_{ij}v_k = \delta_{jk}v_i$). We shall need, however, a larger Lie algebra $\widetilde{gl}_\infty$ of $\mathbb{Z} \times \mathbb{Z}$ matrices with finite number of non-zero diagonals. The above representation does not make sense for $\widetilde{gl}_\infty$. It must be regularized and leads to a representation

$$
\hat{r}(E_{ij}) = :\psi_{-i+\frac{1}{2}} \psi^*_{j-\frac{1}{2}}:
$$

of a central extension $\hat{gl}_\infty = \widetilde{gl}_\infty \oplus \mathbb{C}c$ with central charge $c = 1$ (as usual $:\psi_\mu \psi^*_\nu : = \psi_\mu \psi^*_\nu$ for $\nu > 0$ and $= -\psi^*_\nu \psi_\mu$ for $\nu < 0$ ). Introduce free fermionic fields

$$
\psi(z) = \sum_{j \in \mathbb{Z}} \psi_{j-\frac{1}{2}} z^{-j}, \quad \psi^*(z) = \sum_{j \in \mathbb{Z}} \psi^*_{j-\frac{1}{2}} z^{-j}
$$

and $U(1)$ current

$$
J(z) = :\psi^*(z)\psi(z) : = \sum_{n \in \mathbb{Z}} J_n z^{-n-1}.
$$

The modes $J_n$ generate the Heisenberg algebra and each $F^{(m)}$ is its irreducible representation with charge $m$ and central charge $c = 1$. This
gives an isomorphism, known as the boson-fermion correspondence (see e.g. [17, 14, 15])

$$\sigma: F \rightarrow B = \mathbb{C}[t_1, t_2, t_3, \ldots; Q, Q^{-1}],$$

$$J_n = \frac{\partial}{\partial t_n}, J_{-n} = nt_n \text{ for } n > 0 \text{ and } J_0 = Q \frac{\partial}{\partial Q}.$$ (3)

Introducing the states $$|m\rangle = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots$$ and the operator

$$H(t) = -\sum_{k=1}^{\infty} t_k J_k,$$

we have for $$|\varphi\rangle \in F$$

$$\sigma(|\varphi\rangle) = \sum_{m \in \mathbb{Z}} \langle m | e^{H(t)} | \varphi \rangle Q^m.$$ (4)

Let

$$\phi(z) = \hat{q} + J_0 \log z + \sum_{n \neq 0} J_n z^{-n}$$

be a scalar bosonic field with OPE $$\phi(z_1)\phi(z_2) \sim \log(z_1 - z_2),$$ such that

$$J(z) = \partial \phi(z), Q^m = \lim_{z \to 0} :e^{m\phi(z)}:|0\rangle.$$ (5)

Then the fermionic fields $$\psi(z), \psi^*(z)$$ act on $$B$$ via

$$\psi^*(z) = :e^{\phi(z)}:, \psi(z) = :e^{-\phi(z)}:$$

(as usual $$:J_n J_m: = J_n J_m$$ if $$m > n$$ and $$= J_m J_n$$ if $$m < n$$, $$:\hat{q} J_0 = :J_0 \hat{q}: = \hat{q} J_0$$).

The Lie algebra $$W_{1+\infty} = \mathcal{D}$$ is the unique central extension of the Lie algebra $$\mathcal{D}$$ of complex regular differential operators on the circle [10, 18]. Denoting by $$c$$ the central element of $$W_{1+\infty}$$ we introduce a basis

$$c, J_k^l = W(-z^{k+l} D_z^l), k \in \mathbb{Z}, l \geq 0$$

and another basis

$$c, L_k^l = W(-z^k D_z^l), k \in \mathbb{Z}, l \geq 0$$

where $$D_z \equiv D = z \partial_z$$. Here and further for $$A \in \mathcal{D}$$ we denote by $$W(A)$$ the corresponding element from $$W_{1+\infty}$$. The commutator in $$W_{1+\infty}$$ can be written most conveniently by the generating series [18]

$$\left[W(z^k e^{xD}), W(z^m e^{yD})\right] =$$

$$\left(e^{xm} - e^{yk}\right)W(z^{k+m} e^{(x+y)D}) + \delta_{k,-m} \frac{e^{xm} - e^{yk}}{1 - e^{x+y}} c.$$ (6)
If we fix the basis \( v_j = z^{-j} \) of \( V \) we can consider any operator \( A \in D \) as an element of \( \tilde{gl}_\infty \). This gives an embedding \( \phi_0: D \hookrightarrow \tilde{gl}_\infty \) which can be extended to an embedding \( \hat{\phi}_0: W_{1+\infty} \hookrightarrow \hat{gl}_\infty \) with \( \hat{\phi}_0(c) = c \). Using \( (1) \) we obtain for \( c = 1 \) a free-field realization

\[
W(A) = \text{Res}_{z=0} \psi(z) A \psi^*(z);
\]

for \( A \in D \). Introducing the fields

\[
J^l(z) = \sum_{k \in \mathbb{Z}} J^l_k z^{-k-l-1}
\]

we get

\[
J^l(z) = :\left( \partial^l \psi^* \right)(z) \psi(z):
\]

and we have a bosonic realization

\[
J^l(z) = :e^{-\phi(z)} \frac{\partial^{l+1}}{l+1} e^{\phi(z)}:.
\] (5)

Later we shall need a slight modification of this realization. Let \( u(z) = u_0 \log z + \sum_{n \neq 0} u_n \frac{z}{n} \) be a constant series and replace in \( (1) \) \( \phi(z) \) by \( \phi(z) + u(z) \). Considering \( A \in D \) in the basis \( v_j = e^{u(z)} z^{-j} \) gives an embedding \( \phi_u: D \hookrightarrow \tilde{gl}_\infty \) (i.e. \( \phi_u(A) = \phi_0(e^{-u} A e^u) \)) and its extension \( \hat{\phi}_u: W_{1+\infty} \hookrightarrow \hat{gl}_\infty \) is

\[
(\hat{r} \circ \hat{\phi}_u) J^l(z) = :e^{-\phi(z)-u(z)} \frac{\partial^{l+1}}{l+1} e^{\phi(z)+u(z)}:.
\]

For example when \( u(z) = s \log z \) we obtain the embedding \( \hat{\phi}_s \) from \( (18) \):

\[
\hat{\phi}_{s \log z} \left( W \left( z^k e^{xD} \right) \right) = \sum_{j \in \mathbb{Z}} e^{x(s-j)} E_{j-k,j} - \delta_{k,0} \frac{e^{sx} - 1}{e^x - 1} c.
\] (6)

Now we shall briefly recall the Plücker embedding of the Sato Grassmannian in the projectivization of the infinite wedge space \([17, 24]\). Let

\[
\tilde{V} = \left\{ \sum_{k \in \mathbb{Z}} a_k v_k \middle| a_k = 0 \text{ for } k \ll 0 \right\}
\]

be the space of formal series. The Sato Grassmannian \( Gr \) consists of all subspaces \( W \subset \tilde{V} \) which have an admissible basis

\[
w_k = v_k + \sum_{i > k} w_{ik} v_k, \quad k = 0, -1, -2, \ldots
\]
We consider only transversal subspaces.

Denote by \( \tilde{F} \) and \( \tilde{B} \) the formal completions of \( F \) and \( B \). Then to the plane \( W \in Gr \) we associate a state \( |W\rangle \in \tilde{F}(0) \) as follows

\[
|W\rangle = w_0 \wedge w_{-1} \wedge w_{-2} \wedge \ldots
\]

A change of admissible basis results in multiplication of \( |W\rangle \) by a nonzero constant. The tau-function of \( W \) is the image of \( |W\rangle \) under the boson-fermion correspondence:

\[
\tau_W(t) = \sigma(|W\rangle) = \langle 0| e^{H(t)} |W\rangle.
\]

It is a formal power series in \( t = (t_1, t_2, t_3, \ldots) \).

Finally, recall that the Baker (or wave) function of \( W \in Gr \) is a formal series of the form

\[
\psi_W(x, z) = e^{xz} \left( 1 + \sum_{i=1}^{\infty} a_i(x) z^{-i} \right)
\]

such that

\[
w_{-k} = \partial_x^k \psi_W(x, z) \big|_{x=0}, \quad k = 0, 1, 2, \ldots
\]

is an admissible basis of \( W \) when \( v_j = z^{-j} \). It is expressed by the tau-function via

\[
\psi_W(x, z) = e^{xz} \frac{\tau_W(x - [z^{-1}])}{\tau_W(x)}
\]

where \( \tau_W(x) = \tau_W(x, 0, 0, \ldots), \ [z^{-1}] = (z^{-1}, z^{-2}/2, z^{-3}/3, \ldots) \).

### 3 Generalized Bessel functions

Fix \( r \in \mathbb{C}^N \) and let

\[
P_t(D) = (D - r_1)(D - r_2) \ldots (D - r_N).
\]

Consider the differential equation

\[
P_t(D_z) \Phi(z) = z \Phi(z).
\]

After the substitution \( z = \zeta^N \), it becomes an equation in \( \zeta \) with two singular points: regular at \( \zeta = 0 \) and irregular of rank 1 at \( \zeta = \infty \) (see e.g. [23]).
every sector $S$ with a center at $\zeta = \infty$ and an angle less than $2\pi$, it has a solution with asymptotics
\[
\Phi(z) \sim \zeta^s e^{N\zeta} \left(1 + \sum_{i=1}^{\infty} a_i(r) \zeta^{-i}\right)
\]  
for $|\zeta| \to \infty$, $\zeta \in S$, $\zeta = z^{1/N}$.

The formal (divergent) series $\Psi_r(\zeta)$ is uniquely determined by eq. (8) and does not depend on the sector $S$. The other solutions of (8) are obtained by replacing $\zeta$ by $e^{2\pi ik/N} \zeta$ ($0 \leq k \leq N - 1$). We have
\[
s(r) = \sum_{i=1}^{N} r_i - \frac{N-1}{2}, \quad a_0(r) = 1,
\]
\[
a_1(r) = \sum_{i<j} r_ir_j - \frac{N-1}{2N} \left(\sum_{i} r_i\right)^2 + \frac{N^2-1}{24N}
\]
and all $a_i(r)$ are symmetric polynomials in $r_1, \ldots, r_N$.

**Example 1** (i) When $N = 2$, $r = (\alpha/2, -\alpha/2)$, $x = 2z^{1/2}$, eq. (8) becomes the classical Bessel’s equation
\[
x^2 \partial^2_\Phi + x \partial_x \Phi - (x^2 + \alpha^2) \Phi = 0
\]
and the solution with asymptotics (3) can be taken the Bessel’s function of the third kind $K_\alpha$ (see e.g. [5]).

(ii) For all $N$ we can take $\Phi(z) = G_0^N((-1)^N z|r)$ be the Meijer’s G-function (see e.g. [3, 22]). When $r_i - r_j \not\in \mathbb{Z}$ for all $i \neq j$ it can be expressed in terms of generalized hypergeometric functions.

We shall need some elementary properties of $\Psi_r(\zeta)$, which follow directly from eq. (8). They correspond to classical properties of Meijer’s G-function (see [3], §5.3). Here and further we denote by $e_i$ the vector from $\mathbb{C}^N$ with 1 on the $i$-th place and 0’s elsewhere, and $e = \frac{1}{N} \sum_{i=1}^{N} e_i$. Then we have
\[
\zeta^s \Psi_r(\zeta) = \Psi_{r+se}(\zeta), \quad (D_z - r_i) \Psi_r(\zeta) = \Psi_{r+e_i}(\zeta).
\]

4 Construction of highest weight modules

From now on we fix $s \in \mathbb{C}$, $N \in \mathbb{N}$ and consider $r \in \mathbb{C}^N$ such that $s(r) = s$. Recall that $z = \zeta^N$. We choose a basis in $V$
\[
v_k = e^{N\zeta} \zeta^{s-k}, \quad k \in \mathbb{Z}.
\]
Then $\Psi_r(\zeta)$ corresponds to the following element of $\tilde{V}$

$$w_0 = \Psi_r(\zeta) = v_0 + \sum_{i=1}^{\infty} a_i(r)v_i. \quad (13)$$

We define Bessel’s plane $W_r \in Gr$ to be the unique plane, containing $\Psi_r(\zeta)$ and invariant under $D_z$, and denote by $\tau_r$ its tau-function. By (11) we can construct an admissible basis $\{w^{-k}\}_{k \geq 0}$ of $W_r$ as follows. Choose arbitrary vectors $e^{(k)} = \sum_{i=1}^{N} k_i e_i$ with $k_i \in \mathbb{Z}_{\geq 0}$, $\sum_{i=1}^{N} k_i = k$ and set

$$w_{-k} = \Psi_{r+e^{(k)}}(\zeta) = v_{-k} + \sum_{i>0} a_i(r + e^{(k)})v_{i-k}. \quad (14)$$

The Baker function of the plane $W_r$ is

$$\psi_r(x, z) = e^{-N\zeta} \left( \zeta \left(1 + \frac{x}{N} \right) \right)^{-s} \Psi_r \left( \zeta \left(1 + \frac{x}{N} \right) \right). \quad (15)$$

By eq. (8) it follows that $W_r$ is invariant under the operators $z$ and $D_z$, i.e.

$$zW_r \subset W_r, \quad D_z W_r \subset W_r. \quad (16)$$

The algebra $W_{1+\infty}$ is isomorphic to its subalgebra consisting of elements of degrees divisible by $N$ [11, 25]. The isomorphism is given explicitly by

$$\pi_N: W(z^k e^{xD_z}) \mapsto W(\zeta^N e^{\frac{x}{N}D_\zeta}) + \delta_{k,0} \left( \frac{1}{1 - e^{x/N}} - \frac{N}{\frac{N}{1 - e^x}} \right) c, \quad (17)$$

$$\pi_N: c \mapsto Nc.$$

Combining it with (5) we obtain a bosonic representation $\hat{r}_{s,N} = \hat{r} \circ \hat{\phi}_{N\zeta + s \log \zeta} \circ \pi_N$ of $W_{1+\infty}$ with central charge $c = N$. It is easy to see that

$$\hat{r}_{s,N}(W(z^k e^{xD_z})) = \hat{r}(z^k e^{xD_z}) + c_k(x) \quad (18)$$

where the action of $z^k e^{xD_z}$ is taken in the basis (12), $c_k(x) = 0$ for $k > 0$ and

$$c_0(x) = \frac{1}{1 - e^{x/N}} - \frac{N}{\frac{N}{1 - e^x}} + \frac{e^{sx} - 1}{1 - e^x}. \quad (19)$$

Note that when $k \neq 0$ in (18) $\hat{r}$ can be replaced by $r$. From now on we shall consider only this representation $(s, N$ being fixed) and skip the symbol $\hat{r}_{s,N}$. Denote by $\mathcal{M}_r$ be the module generated by $\tau_r$. 
Theorem 2  (i) $\tau_r$ is a highest weight vector with highest weight $\lambda_r$, i.e.

$$L_k^r \tau_r = 0 \quad \text{for} \ k > 0, \quad L_0^r \tau_r = \lambda_r(L_0^r) \tau_r. \tag{20}$$

Here $\lambda_r(L_0^r)$ are certain symmetric polynomials in $r_1, \ldots, r_N$, for example $\lambda_r(L_0^0) = \frac{1}{2} \sum_{i=1}^N r_i^2 - \frac{1}{2} \sum_{i=1}^N r_i$.

(ii) The module $\mathcal{M}_r$ is quasifinite with characteristic polynomials

$$P_{r,k}(D) = P_r(D)P_r(D-1) \cdots P_r(D-k+1)$$

($k \in \mathbb{N}$), i.e. we have

$$W(z^{-k}P_{r,k}(D))\tau_r = 0$$

and $P_{r,k}$ are polynomials of minimal degree with this property.

Proof. By (8) and (16) $W_r$ is invariant under the operators $z^k D_l^r$ for $k, l \geq 0$ and $z^{-k}P_{r,k}(D) = (z^{-1}P_r(D))^k$ for $k \geq 1$. We use the result of [20] that for $A \in \hat{gl}_{\infty}$, $W \in Gr$

$$AW \subset W \text{ iff } \hat{r}(A)\tau_W = \text{const.}\tau_W$$

and the fact that

$$W(z^k f(D)) = \frac{1}{k} W(D), W(z^k f(D))$$

for $k \neq 0$. The polynomials $\lambda_r(L_0^r)$ can be computed comparing the coefficient of $|0\rangle$ in both sides of (20) and using (14) and (19).

Example 3 For $N = 2$ and $r = (\alpha/2, -\alpha/2)$ the Virasoro modules $M_{\infty}^r$ introduced in [14] are reductions of the modules $\mathcal{M}_r$ (obtained by putting $t_2 = t_4 = t_6 = \ldots = 0$).

To describe $\lambda_r$ introduce, following Kac and Radul [18], the generating series

$$\Delta_{\lambda_r}(x) = \lambda_r(W(-e^{xD}))$$

It is proved in [18] that for a quasifinite representation with first characteristic polynomial $P_r(D)$ the function

$$F(x) = (e^x - 1)\Delta_{\lambda_r}(x) + c$$

satisfies the equation

$$P_r(\partial_x)F(x) = 0.$$

When all $r_i$ are distinct, it follows that $F(x) = \sum_{i=1}^N a_i e^{r_i x}$. Because of the symmetry $a_1 = \ldots = a_N$. Using the value of $\lambda_r(L_0^0)$ given by Theorem 2 we get $a_1 = \ldots = a_N = 1$. But $\lambda_r(L_0^r)$ are polynomials in $r$, thus we have proved
Theorem 4 The generating series $\Delta_{\lambda_\tau}(x)$ is given by

$$\Delta_{\lambda_\tau}(x) = \sum_{i=1}^{N} \frac{e^{r_i x} - 1}{e^{x} - 1}.$$ 

The irreducible module with such generating series is called in [10] a primitive module. However, our modules are irreducible only when $r_i - r_j \notin \mathbb{Z}$ for $i \neq j$ [13].

We shall show that $\tau_\tau$ is characterized among all formal power series by the constraints (20).

Proposition 5 There exists only one (up to a constant) formal power series $\tau$ in $t_1, t_2, \ldots$ satisfying

$$J^l_k \tau = 0, \quad J^0_0 \tau = c_l \tau$$

for $0 < k$, $0 \leq l \leq N - 1$ and some constants $c_l$.

Proof. (cf. [1, 14]) After taking derivatives of (21) and letting $t_1 = t_2 = \ldots = 0$ one sees inductively that all derivatives of $\tau$ at $t = 0$ vanish, hence it is determined by $\tau(0)$.

5 Embeddings of the modules $\mathcal{M}_r$

Theorem 6 Let $r \in \mathbb{C}^N$, $r_1 - r_2 = \alpha \in \mathbb{Z}_{\geq 0}$. Then $\tau_{r+e_1-e_2}$ is a singular vector in the module $\mathcal{M}_r$ and is given by the formula

$$\tau_{r+e_1-e_2} = W(A) \tau_r + \text{const} \cdot \tau_r$$

where

$$A = -(\alpha + 1) \frac{z^{-1}P_r(D_z)}{D_z - r_2} \left( z^{-1}P_r(D_z) \right)^{\alpha}$$

Therefore $\mathcal{M}_{r+e_1-e_2} \hookrightarrow \mathcal{M}_r$.

Proof. Because $L_0^{l_1}_{t_1} = t_2 = t_3 = \ldots = 0 = \frac{N}{2N} \partial_x + \frac{s(s-1)}{2N} + \frac{1-N^2}{2N^2} \cdot (20)$ implies

$$\tau_r(x) = \left(1 + \frac{x}{N} \right)^{N \sum_i r_i^2 + \text{const}}$$

(the inessential constant depends on $s$ and $N$ but not on $r$). For $\lambda \in \mathbb{C}$ we consider the sum $\tau_{\lambda}(t) = \tau_r(t) + \lambda \tau_{r+e_1-e_2}(t)$ (cf. [3]). Then (14) implies that

$$\tau_{\lambda}(t) = \sigma((\Psi_r + \lambda \Psi_{r+e_1-e_2}) \wedge \Psi_{r+e_1} \wedge \Psi_{r+e_1+e_1}) \wedge \ldots)$$
is a tau-function. By (7) its Baker function $\psi_\lambda(x, \zeta)$ is equal to

$$
\frac{\tau_r(x)\psi_r(x, \zeta) + \lambda \tau_{r+e_1-e_2}(x)\psi_{r+e_1-e_2}(x, \zeta)}{\tau_r(x) + \lambda \tau_{r+e_1-e_2}(x)} = \psi_r(x, \zeta) + \lambda \frac{(1 + \frac{x}{N})^{N(\alpha+1)}}{1 + \lambda (1 + \frac{x}{N})^{N(\alpha+1)}} \psi_{r+e_1-e_2}(x, \zeta).
$$

Using (10) and the eq. (8) it is easy to see that

$$
(1 + \frac{x}{N})^{N(\alpha+1)} \psi_{r+e_1-e_2}(x, \zeta) = e^{-N\zeta} \zeta^{s-A(z, D_z)z} e^{N\zeta} \psi_r(x, \zeta).
$$

Therefore in the basis (12)

$$
\tau_\lambda(t) = \sigma((1 + \lambda A)\Psi_r \wedge (1 + \lambda A)\Psi_{r+e_1} \wedge (1 + \lambda A)\Psi_{r+e_2} \wedge \ldots)
$$

and comparing the coefficient of $\lambda$ gives

$$
\tau_{r+e_1-e_2}(t) = r(A)\tau_r(t)
$$

The formula (22) now follows from (18).

Theorem 3 gives in practice all possible embeddings among the modules $\mathcal{M}_r$.

**Acknowledgements**

We thank Prof. I. T. Todorov for useful discussions and for his interest in this work. This paper is partially supported by Grant MM–402/94 of Bulgarian Ministry of Education, Science and Technologies.

**References**

[1] M. Adler, P. van Moerbeke, Commun. Math. Phys. 147 (1992) 25.

[2] M. Adler, P. van Moerbeke, Comm. Pure Appl. Math. 47 (1994) 5.

[3] H. Awata, M. Fukuma, Y. Matsuo, S. Odake, Representation theory of the $W_{1+\infty}$ algebra, Kyoto preprint (1994), hep-th/9408158.
[4] B. Bakalov, E. Horozov, M. Yakimov, *Highest weight modules of \( W_{1+\infty} \), Darboux transformations and the bispectral problem*, to appear in Proc. Conf. Geom. and Math. Phys., Zlatograd 95, Bulgaria, q-alg/9601017.

[5] H. Bateman, A. Erdélyi, *Higher transcendental functions* (McGraw-Hill, New York, 1953).

[6] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, *Transformation groups for soliton equations*, in: Proc. RIMS Symp. Nonlinear integrable systems (Kyoto 1981), (World Scientific, 1983).

[7] L. A. Dickey, *Soliton equations and hamiltonian systems*, Adv. Ser. Math. Phys. 12 (World Scientific, 1991).

[8] J. J. Duistermaat, F. A. Grünbaum, Commun. Math. Phys. 103 (1986) 177.

[9] J. Fastré, doctoral dissertation, Univ. of Louvain, 1993.

[10] E. Frenkel, V. Kac, A. Radul, W. Wang, Commun. Math. Phys. 170 (1995) 337, hep-th/9405121.

[11] M. Fukuma, H. Kawai, R. Nakayama, Commun. Math. Phys. 143 (1992) 371.

[12] F. A. Grünbaum, Comm. Pure Appl. Math. 47 (1994) 307.

[13] L. Haine, E. Horozov, Bull. Sci. Math. (2) 177 (1993) 485.

[14] L. Haine, E. Horozov, in: *Abelian varieties*, eds. Barth et al. (Walter de Gruyter, 1995).

[15] V. G. Kac, *Infinite-dimensional Lie algebras*, 3-d edition, (Cambridge Univ. Press, 1990).

[16] V. G. Kac, D. H. Peterson, Proc. Natl. Acad. Sci. USA 78 (1981) 3308.

[17] V. G. Kac, D. H. Peterson, *Lectures on the infinite wedge representation and the MKP hierarchy*, Sem. Math. Sup. 102 (Presses Univ. de Montréal, 1986).

[18] V. G. Kac, A. Radul, Commun. Math. Phys. 157 (1993) 429, hep-th/9308153.

[19] V. G. Kac, A. Raina, *Bombay lectures on highest weight representations of infinite dimensional Lie algebras*, Adv. Ser. Math. Phys. 2 (World Scientific, 1987).

[20] V. G. Kac, A. Schwarz, Phys. Lett. B257 (1991) 329.

[21] M. Kontsevich, Commun. Math. Phys. 147 (1992) 1.

[22] Y. Luke, *Mathematical functions and their approximations*, (Academic Press, 1975).

[23] M. Sato, RIMS Kokyuroku 439 (1981) 30.

[24] G. Segal, G. Wilson, Publ. Math. IHES 61 (1985) 5.
[25] P. van Moerbeke, *Integrable foundations of string theory*, CIMPA–Summer school at Sophia–Antipolis (1991), (World Scientific, 1994).

[26] W. Wasow, *Asymptotic expansions for ordinary differential equations* (Interscience, 1965).

[27] G. Wilson, J. reine angew. Math. 442 (1993) 177.

[28] E. Witten, Surveys in Diff. Geom. 1 (1991) 243.