Exact Rosenthal-type inequalities for $p = 3$, and related results

Iosif Pinelis

Department of Mathematical Sciences
Michigan Technological University
Houghton, Michigan 49931, USA
E-mail: ipinelis@mtu.edu

Abstract: An exact Rosenthal-type inequality for the third absolute moments is given, as well as a number of related results. Such results are useful in applications to Berry–Esseen bounds.

AMS 2010 subject classifications: 60E15.
Keywords and phrases: Rosenthal inequality, bounds on moments, sums of independent random variables, probability inequalities.

1. Introduction, summary, and discussion

Let $X_1, \ldots, X_n$ be independent random variables (r.v.'s), with the sum $S := X_1 + \cdots + X_n$, such that for some real positive constant $\beta$ and all $i$ one has

$$\mathbb{E} X_i \leq 0, \quad \sum \mathbb{E} X_i^2 \leq 1, \quad \text{and} \quad \sum \mathbb{E} (X_i)^3 \leq \beta; \quad (1)$$

as usual, we let $x_+ := 0 \vee x$ and $x_+^p := (x_+)^p$ for all real $x$ and all real $p > 0$.

Consider the following class of functions:

$$\mathcal{F}^3 := \{ f \in C^2 : f \text{ and } f'' \text{ are nondecreasing and convex}\}$$
$$= \{ f \in C^2 : f, f', f'', f''' \text{ are nondecreasing}\}, \quad (2)$$

where $C^2$ denotes the class of all twice continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $f'''$ denotes the right derivative of the convex function $f''$. For example, functions $x \mapsto a + b x + c (x - t)^\alpha$ and $x \mapsto a + b x + c e^{\lambda x}$ belong to $\mathcal{F}^3$ for all $a \in \mathbb{R}$, $b \geq 0$, $c \geq 0$, $t \in \mathbb{R}$, $\alpha \geq 3$, and $\lambda \geq 0$.

Remark. If a r.v. $X$ has a finite expectation and a function $f : \mathbb{R} \to \mathbb{R}$ is in $\mathcal{F}^3$ or, more generally, is any convex function, then, by Jensen’s inequality, $\mathbb{E} f(X)$ always exists in $(-\infty, \infty]$.

The main result of this note is

Theorem 1. For any function $f \in \mathcal{F}^3$

$$\mathbb{E} f(S) \leq \mathbb{E} f(Z) + \frac{f'''(\infty^-)}{3!} \beta, \quad (3)$$

*Supported by NSA grant H98230-12-1-0237
where $Z$ is a standard normal r.v. Moreover, for each function $f \in \mathcal{F}^3$ the upper bound in (3) is exact, in the sense that it is equal to the supremum of $E f(S)$ over all independent $X_i$'s satisfying conditions (1).

Of course, in the case when $f'''(\infty-) = \infty$, the inequality (3) is trivial. Theorem 1 is based on the main result of [10].

It follows immediately from Theorem 1 that for all real $x$

$$E(S - x)^3_+ \leq E(Z - x)^3_+ + \beta.$$  (4)

If it is additionally assumed that $E X_i = 0$ for all $i$, then (4) in turn yields

$$E|S - x|^3 \leq E|Z - x|^3 + \sum E|X_i|^3;$$  (5)

moreover, one can similarly show that the upper bound in (5) is exact, for each real $x$; the special case $x = 0$ of (5) is also a special case of Rosenthal’s inequality [16]:

$$E|S|^p \leq c_p (1 + \sum E|X_i|^p),$$  (6)

for all $p \geq 2$, where $c_p$ is a positive constant depending only on $p$ (inequality (6) too needs the assumption that the $X_i$’s be zero-mean). In the case when $x = 0$ and the $X_i$’s are symmetric, inequality (5) was obtained by Ibragimov and Sharakhmetov [4], who at that considered arbitrary real $p > 2$. Besides, inequality (5) follows from Tyurin’s result [17, Theorem 2], which also implies (4) but with $\sum E|X_i|^3$ in place of $\beta$. More on Rosenthal-type inequalities and related results can be found, among other papers, in [1–3, 5–8, 11, 15, 18].

Theorem 1 admits

**Corollary 2.** For any $p \in (0, 3)$ and any real $a > 0$

$$E S^p_+ \leq \frac{p^p(3 - p)^3 - p}{3^3} E(Z + a)^3_+ + \frac{1}{a^{3-p}};$$

in particular, taking here $(p, a) = (1, \frac{1746}{1000})$ and $(p, a) = (2, \frac{639}{1000})$, one obtains, respectively, the inequalities

$$E S_+ \leq 0.514 + 0.0486 \beta \quad \text{and} \quad E S^2_+ \leq 0.555 + 0.232 \beta.$$  (7)

One may compare the latter two bounds with the “naive” ones, obtained using the inequalities $(E S_+)^2 \leq E S^2_+ \leq E S^2 \leq 1$; here one may note that $\beta$ will rather typically be small. One can similarly bound $E(S - x)^p_+$ for any real $x$ and any $p \in (0, 3)$. The first inequality in (7) can in fact be improved:

$$E S_+ \leq \frac{1}{2},$$  (8)

which follows because $4u^+_+ \leq u^2 + 2u + 1$ for all real $u$; the bound $\frac{1}{2}$ on $E S_+$ in (8) is obviously attained when $P(S = \pm1) = \frac{1}{2}$.

The case $p = 3$ of Rosenthal-type inequalities, including the results stated above, is especially important in applications to Berry–Esseen bounds; see e.g. [14], Remark 3.4 in [12], and the “quick proofs” of Nagaev’s nonuniform Berry–Esseen bound in [12, 13].
2. Proofs

Proof of Theorem 1. Take indeed any \( f \in F^3 \). Next, take any real \( y > \beta \) and introduce the r.v.'s

\[
X_{i,y} := X_i \wedge y \quad \text{and} \quad S_y := \sum_i X_{i,y}.
\]

Then the conditions (1) hold for the \( X_{i,y} \)'s in place of \( X_i \). Also, \( X_{i,y} \leq y \) for all \( i \). So, by the main result of [10],

\[
E f(S_y) \leq E f(\sqrt{1-\beta/y} Z + y\tilde{\Pi}_{\beta/y}^\theta) \leq \sum_{j=0}^{\infty} E f(\sqrt{1-\beta/y} Z + yj - \beta/y^2) \frac{(\beta/y^2)^j}{j!} e^{-\beta/y^3}, \tag{10}
\]

where \( \tilde{\Pi}_{\theta} := \Pi_{\theta} - E \Pi_{\theta} = \Pi_{\theta} - \theta \) and \( \Pi_{\theta} \) is any r.v. which is independent of \( Z \) and has the Poisson distribution with parameter \( \theta \), for any real \( \theta > 0 \). Moreover, by [10, Proposition 2.3], for any given triple \( (f, \beta, y) \in F^3 \times (0, \infty) \times (0, \infty) \) with \( y > \beta \) the bound in (9) is exact, in the sense that it is equal to the supremum of \( E f(S_y) \) over all independent \( X_i \)'s satisfying conditions (1).

Now let

\[
y \to \infty.
\]

Then, by the monotone convergence theorem,

\[
E f(S_y) \to E f(S). \tag{11}
\]

As was mentioned earlier, in the case when \( f'''(\infty-) = \infty \) the inequality (3) is trivial. Consider now the case when \( f'''(\infty-) < \infty \). Then, by a l'Hospital-type rule, \( f(x)/x^3 \to f'''(\infty-)/3! \) as \( x \to \infty \), which also leads to \( |f(x)| = O(1 + |x|^3) \) over all real \( x \) (for negative real \( x \), one even has \( |f(x)| = O(1 + |x|) \), since \( f \) is nondecreasing and convex; cf. e.g. [9, Lemma 7]). Therefore, by the dominated convergence theorem,

\[
E f(\sqrt{1-\beta/y} Z + yj - \beta/y^2) \to f'''(\infty-)/3! j^3 \quad \text{if} \ j > 0,
\]

and so, again by the dominated convergence theorem (say), the sum in (10) converges to \( E f(Z) + f'''(\infty-) \beta/3! \). In view of (9)–(11), this proves the inequality (3); the exactness of the bound in (3) follows from that of the bound in (9)–(10).

Proof of Corollary 2. This follows from (4), since \( \sup_{u \geq 0} \frac{u^p}{(u+a)^{3-p}} = \frac{p^p(3-p)^{3-p}}{3^a a^{3-p}} \) for any \( p \in (0, 3) \) and any real \( a > 0 \).
References

[1] S. Boucheron, O. Bousquet, G. Lugosi, and P. Massart. Moment inequalities for functions of independent random variables. *Ann. Probab.*, 33(2):514–560, 2005.

[2] D. L. Burkholder. Distribution function inequalities for martingales. *Ann. Probability*, 1:19–42, 1973.

[3] E. Giné, R. Latala, and J. Zinn. Exponential and moment inequalities for U-statistics. In *High dimensional probability, II (Seattle, WA, 1999)*, volume 47 of *Progr. Probab.*, pages 13–38. Birkhäuser Boston, Boston, MA, 2000.

[4] R. Ibragimov and S. Sharakhmetov. On an exact constant for the Rosenthal inequality. *Teor. Veroyatnost. i Primenen.*, 42(2):341–350, 1997.

[5] R. Ibragimov and S. Sharakhmetov. On extremal problems and best constants in moment inequalities. *Sankhyā Ser. A*, 64(1):42–56, 2002.

[6] R. Latala. Estimation of moments of sums of independent real random variables. *Ann. Probab.*, 25(3):1502–1513, 1997.

[7] S. V. Nagaev and I. F. Pinelis. Some inequalities for the distributions of sums of independent random variables. *Teor. Veroyatnost. i Primenen.*, 22(2):254–263, 1977. MR0443034.

[8] I. Pinelis. Optimum bounds on moments of sums of independent random vectors. *Siberian Adv. Math.*, 5(3):141–150, 1995.

[9] I. Pinelis. Exact inequalities for sums of asymmetric random variables, with applications. *Probab. Theory Related Fields*, 139(3-4):605–635, 2007.

[10] I. Pinelis. On the Bennett-Hoeffding inequality, a shorter version to appear in *Annales de l’Institut Henri Poincaré*. [http://arxiv.org/abs/0902.4058](http://arxiv.org/abs/0902.4058), 2012.

[11] I. Pinelis. Rosenthal-type inequalities for martingales in 2-smooth Banach spaces. [http://arxiv.org/abs/1212.1912](http://arxiv.org/abs/1212.1912), 2012.

[12] I. Pinelis. More on the nonuniform Berry–Esseen bound. [http://arxiv.org/abs/1302.0516](http://arxiv.org/abs/1302.0516), 2013.

[13] I. Pinelis. On the nonuniform Berry–Esseen bound. [http://arxiv.org/abs/1301.2828](http://arxiv.org/abs/1301.2828), 2013.

[14] I. Pinelis and R. Molzon. Berry-Esséen bounds for general nonlinear statistics, with applications to Pearson’s and non-central Student’s and Hotelling’s (preprint), arXiv:0906.0177 [math.ST].

[15] I. F. Pinelis and S. A. Utev. Estimates of moments of sums of independent random variables. *Theory Probab. Appl.*, 29(3):574–577, 1984.

[16] H. P. Rosenthal. On the subspaces of $L^p$ ($p > 2$) spanned by sequences of independent random variables. *Israel J. Math.*, 8:273–303, 1970.

[17] I. S. Tyurin. Some optimal bounds in the central limit theorem using zero biasing. *Statist. Probab. Lett.*, 82(3):514–518, 2012.

[18] S. A. Utev. Extremal problems in moment inequalities. In *Limit theorems of probability theory*, volume 5 of *Trudy Inst. Mat.*, pages 56–75, 175. “Nauka” Sibirsk. Otdel., Novosibirsk, 1985.