A study on buckling of piezoelectric circular cylindrical shell

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Abstract. The infinite length piezoelectric circular cylindrical shell is considered in the present study, which is subjected to axisymmetric external pressure and applied electrical voltage in radial direction. The mathematical formulation of the system is adopted simultaneous linear homogeneous ordinary differential equations as governing equations. These governing equations are further solved by using finite difference method. Numerical results of the analysis provide comparative study of conventional method with FDM method and show that piezoelectric effect improves stability of shell.

1. Introduction

Circular cylindrical shell type structures are used extensively in many applications. Some of the important applications of these structures are aero plane fuselage, rocket, missiles, space-vehicles, fuel tank, habitation module of space station, sub-marine, etc. These types of structures are subjected to external pressure while traveling through atmosphere. Under such situations, loss of stability is primary concern for proper functioning of these members. These types of applications require materials possessing certain properties like lightweight, high stiffness, high strength to weight ratio, etc. Due to special characteristics of piezoelectric material, piezoelectric circular cylindrical shell-type structure can function efficiently in these applications as a primary load-carrying member. Owing to these reasons, shell type smart structures attract attention of many researchers.

Kapuria et al [2] have presented three-dimensional axisymmetric solution for a simply supported piezoelectric cylindrical shell. Cheng and Shen [3] have presented a study on stability analysis of piezoelectric circular cylindrical shell subjected to external pressure. Chen et al [3] have presented a study of an orthotropic cylindrical shell containing piezoelectric layers under cylindrical bending. Kapuria et al [5] presented a study for a simply supported finite circular cylindrical hybrid shell with a cross-ply laminate under electromechanical loading. Chen et al [6] have presented an elasticity solution for a simply supported piezoelectric circular cylindrical shell of finite length. Ma et al [7] have studied two-dimensional problem of anisotropic cylindrical piezoelectric tube, bar and shell under pressure loading. Shen [8] has presented a post buckling analysis for a cross-ply laminated cylindrical shell with piezoelectric actuators under combined external pressure and heating. J. Sun, et al [9] established a Hamiltonian system of equations to understand the behaviour of cylindrical shell subjected to external pressure. Cheng and Shen [4] conducted a study on stability analysis of piezoelectric circular cylindrical shell subjected to external pressure as well as uniform electric field by employing Euler 2-point method to solve the governing equations developed. While, in the present work, Finite Difference Method is employed to solve the governing equations for calculating buckling pressure of a piezoelectric circular cylindrical shell subjected to axisymmetric pressure as well as uniform radial electric field. The piezoelectric circular cylindrical shell considered in the present study is of infinite length and is
subjected to axisymmetric external pressure and applied electrical voltage in radial direction. Simultaneous linear homogeneous ordinary differential equations of second order are obtained as governing equations. These governing equations are solved by using Finite Difference Method.

2. Mathematical formulation

The cross-section of piezoelectric circular cylindrical shell as considered in present study is shown in Fig 1. Here, \( P_r \) is axisymmetric radial external pressure. \( E_r \) is uniform radial electrical field. \( R_1 \) and \( R_2 \) are the inner radius and outer radius, respectively.

![Cross-section of piezoelectric circular cylindrical shell](image)

**Figure 1** Cross-section of piezoelectric circular cylindrical shell

By adopting the methodology developed by Cheng and Shen [4] for linear stability analysis of infinitely long orthotropic piezoelectric circular cylindrical shell subjected to axisymmetric external pressure as well as uniform radial electric field while body forces are assumed to be absent, following second order linear homogeneous ordinary differential equations are obtained as governing equations

\[
\begin{align*}
[(\sigma_r^0 + C_{11}) \frac{\partial^2}{\partial r^2} + (r \frac{\partial \sigma_r^0}{\partial r} + C_{11} + \sigma_r^0) \frac{\partial}{\partial r} - (\sigma_0^0 + C_{22}) \frac{1}{r^2} - (\sigma_0^0 + C_{66}) \frac{n^2}{r^2}] F_r \\
+ [n(C_{12} + C_{66}) \frac{\partial}{\partial r} - n(C_{22} + C_{66} + 2\sigma_0^0) \frac{1}{r^2}] F_\theta \\
+ [\varepsilon_{33} - (e_{33} - e_{31}) \frac{\partial}{\partial r} - e_{15} \frac{n^2}{r^2}] F_\theta = 0 \tag{1}
\end{align*}
\]

\[
\begin{align*}
[-n(C_{12} + C_{66}) \frac{\partial}{\partial r} - (C_{22} + C_{66} + 2\sigma_0^0) \frac{n}{r^2}] F_r \\
+ [(\sigma_r^0 + C_{66}) \frac{\partial^2}{\partial r^2} + (r \frac{\partial \sigma_r^0}{\partial r} + C_{66} + \sigma_r^0) \frac{\partial}{\partial r} - (\sigma_0^0 + C_{66}) \frac{1}{r^2} - (\sigma_0^0 + C_{22}) \frac{n^2}{r^2}] F_\theta \\
- n[(e_{31} + e_{15}) \frac{\partial}{\partial r} + e_{15} \frac{1}{r^2}] F_\theta = 0 \tag{2}
\end{align*}
\]
For perturbed state of an infinitely long orthographic piezoelectric circular cylindrical shell subjected to axisymmetric pressure along with electric field, boundary conditions on outer and inner surfaces are obtained as explained by Cheng and Shen [4] and expressed by relations (4) and (5), respectively. Thus, boundary conditions at outer surface are written as

\[
(C_{11} \frac{\partial}{\partial r} + C_{12} \frac{1}{r} - P_r \frac{\partial}{\partial r})F_r + C_{12} \frac{n}{r} F_\theta + e_{33} \frac{\partial}{\partial r} F_\phi = 0
\]

\[
-(C_{66} - P_r) \frac{n}{r} F_r + [C_{66} (\frac{\partial}{\partial r} - \frac{1}{r}) + \frac{P_r}{r} - P_r \frac{\partial}{\partial r}]F_\theta - e_{15} \frac{n}{r} F_\phi = 0
\]

\[
F_\phi = 0
\]  

and boundary conditions at inner surface are written as

\[
(C_{11} \frac{\partial}{\partial r} + C_{12} \frac{1}{r})F_r + C_{12} \frac{n}{r} F_\theta + e_{33} \frac{\partial}{\partial r} F_\phi = 0
\]

\[
-C_{66} \frac{n}{r} F_r + C_{66} (\frac{\partial}{\partial r} - \frac{1}{r})F_\theta - e_{15} \frac{n}{r} F_\phi = 0
\]

\[
F_\phi = 0
\]  

3. Solution Methodology

Present work employed finite difference method (FDM) to solve the governing equations. To apply FDM, the thickness \((R_2 - R_1)\) of cylinder is divided into \((N-1)\) equal intervals. Therefore, total number of nodes is \(N\). The width of interval can be obtained by following relation

\[
h = \frac{R_2 - R_1}{N-1}
\]

Here \(R_2\) and \(R_1\) represent outer and inner radius of shell.

Governing equations after converting into finite difference form for \(i^{th}\) node by using central difference formula, can be expressed in matrix form as

\[
AX = 0
\]

Here, matrices \(A\) and \(X\) of eq. (6) are given by following relations

\[
A = \begin{bmatrix}
A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 \\
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 & B_9 \\
D_1 & D_2 & D_3 & D_4 & D_5 & D_6 & D_7 & D_8 & D_9
\end{bmatrix}
\]

\[
&[X]^T = [F_{r_{i+1}} F_{\theta_{i+1}} F_\phi F_{r_{i+1}} F_{\theta_{i+1}} F_\phi F_{r_{i+1}} F_{\theta_{i+1}} F_\phi]
\]
where \[ A_1 = 2r^2 (C_{11} + \sigma_r^0) - rh(C_{11} + r \frac{\partial \sigma_r^0}{\partial r} + \sigma_r^0) \]
\[ A_2 = -nrh(C_{12} + C_{66}) \]
\[ A_3 = 2r^2 e_{33} - rh(e_{33} - e_{31}) \]
\[ A_4 = -4r^2 (C_{11} + \sigma_r^0) - 2h^2 \{ C_{22} + \sigma_0^0 + n^2 (C_{66} + \sigma_0^0) \} \]
\[ A_5 = -2h^2 n(C_{22} + C_{66} + 2\sigma_0^0) \]
\[ A_6 = -4r^2 e_{33} - 2h^2 n^2 e_{15} \]
\[ A_7 = 2r^2 (C_{11} + \sigma_r^0) + rh(C_{11} + r \frac{\partial \sigma_r^0}{\partial r} + \sigma_r^0) \]
\[ A_8 = nrh(C_{12} + C_{66}) \]
\[ A_9 = 2r^2 e_{33} + rh(e_{33} - e_{31}) \]
\[ B_1 = nrh(C_{12} + C_{66}) \]
\[ B_2 = 2r^2 (C_{66} + \sigma_r^0) - rh(C_{66} + r \frac{\partial \sigma_r^0}{\partial r} + \sigma_r^0) \]
\[ B_3 = nrh(e_{31} + e_{15}) \]
\[ B_4 = -2nh^2 (C_{22} + C_{66} + 2\sigma_0^0) \]
\[ B_5 = -4r^2 (C_{11} + \sigma_r^0) - 2h^2 \{ C_{22} + \sigma_0^0 + n^2 (C_{66} + \sigma_0^0) \} \]
\[ B_6 = -2nh^2 e_{15} \]
\[ B_7 = -nrh(C_{12} + C_{66}) \]
\[ B_8 = 2r^2 (C_{66} + \sigma_r^0) + rh(C_{66} + r \frac{\partial \sigma_r^0}{\partial r} + \sigma_r^0) \]
\[ B_9 = nrh(e_{31} + e_{15}) \]
\[ C_1 = 2r^2 e_{33} - rh(e_{33} + e_{31}) \]
\[ C_2 = -nrh(e_{31} + e_{15}) \]
\[ C_3 = -(2r - h)rk_{33} \]
\[ C_4 = -4r^2 e_{33} - 2n^2 h^2 e_{15} \]
\[ C_5 = -2nh^2 e_{15} \]
\[ C_6 = 4r^2 k_{33} + 2n^2 h^2 k_{14} \]
\[ C_7 = 2r^2 e_{33} + rh(e_{33} + e_{31}) \]
\[ C_8 = nrh(e_{31} + e_{15}) \]
\[ C_9 = -(2r + h)rk_{33} \]

If equation (6) is applied for each node, a set of \((N + 2)\) simultaneous equations is obtained.

On applying boundary conditions, above system of equations reduces to following form

\[ EX = 0 \] \hspace{1cm} (7)

Where \(E\) is the matrix of order \(N\) by \(N\) and \(X\) is the matrix of order of \(N\) by 1. Matrices \(E\) and \(X\) are given in appendix. Equation (7) can also be written as

\[ [Y - P^1Z]X = 0 \] \hspace{1cm} (8)

Here \(Z\) is the matrix of coefficients of \(P^1\) and \(Y\) is the matrix of terms not containing \(P^1\). \(P^1\) is a buckling constant that can be represented by relation given below
\[ p^1 = \frac{P_{cr} R_2^3}{C_{11} (R_2 - R_1)^3} \]

where \( P_{cr} \) represented critical pressure at which buckling takes place. This equation can also be written as

\[ [Z^{-1} Y - P^1]X = 0 \]  \hspace{1cm} (9)

For non-trivial solution of this equation, necessary condition is

\[ |Z^{-1} Y - P^1| = 0 \]  \hspace{1cm} (10)

This is a standard Eigen value problem that is solved by Mat-Lab.

4. Results and discussion

Numerical results are obtained for two cases. In the first case, an infinitely long glass-epoxy fibrous composite circular cylindrical shell subjected to axisymmetric external pressure is studied. In the second case an infinitely long piezoelectric circular cylindrical shell made of piezoelectric ceramics is studied.

4.1 Elastic Composite Circular Cylindrical Shell

For validating the formulation developed, an infinitely long glass-epoxy fiber composite circular cylindrical shell subjected to external pressure is considered here, which was also studied by Kardomateas [1]. The inner radius of cylindrical shell (\( R_1 \)) is taken as 1.0 m and properties of material are given in appendix. To ensure the convergence of results, buckling constant are calculated for increasing number of nodes and figures are plotted. For example, such a graph is shown below in fig.2 for \( R_2 = 1.2 \) and number of buckling half wave, \( n = 2 \).

![Fig 2](image.png)

Variation of buckling pressure with number of nodes for composite cylinder

Similar trends are obtained for other cases and it is observed that results converge at \( N > 60 \). However, numerical results are obtained for \( N = 200 \) to ensure proper convergence.

Now, numerical results are obtained for increasing integer values of number of buckling half waves for \( R_2 = 1.2 \) and same is shown below in graphical form in fig.3.

![Fig.3](image.png)

Variation of buckling pressure with number of buckling half-wave in circumferential direction for composite cylinder
From fig 3, it is clear that minimum Eigen value is obtained when n=2. Similar trends are exhibited by results for other values of outer radius as shown in appendix B. This infers that n=2 give the critical value or buckling value. Thus further calculations are made for n=2. Buckling pressure are calculated for different ratios of outer and inner radius of the composite cylindrical shell and presented in table 1 given below along with the results obtained from different theories for comparison purpose.

| \( \frac{R_2}{R_1} \) | Present Study | Kardomateas (1993) | Chang & Shen (1997) |
|-----------------|---------------|-------------------|---------------------|
| 1.10            | 0.0115        | 0.0117            | 0.0115              |
| 1.15            | 0.0340        | 0.0350            | 0.0341              |
| 1.20            | 0.0702        | 0.0735            | 0.0705              |
| 1.25            | 0.1187        | 0.1268            | 0.1196              |
| 1.30            | 0.1776        | 0.1935            | 0.1794              |
| 1.40            | 0.3158        | 0.3584            | 0.3217              |

These results can be shown in graphical form as

Fig 4 Comparison of buckling pressure obtained by various theories for composite cylinder

It is clear from fig 4 that the simplified elasticity solutions of Kardomateas coincide well with present and Chang & Shen’s results only when shell is not very thick, but results of Kardomateas differs significantly from the results obtained through present study and Chang & Shen study when ratio of outer to inner radius of shell is more than 1.3. The reason may be that an assumption that strain is small compared to the rotation is employed in the simplified elasticity solutions (Kardomateas) while no such assumption is used in both these work. Therefore, the assumption is suitable only for thin to moderately thick shells.

4.2 Piezoelectric circular cylindrical shell

The results are obtained for circular cylindrical shell made from piezoelectric material (Properties are given in appendix) for following two cases in presence of external pressure

4.2.1 Shell Not Subjected To Electric Field

In this case, results are obtained for two conditions; when piezoelectric effect is neglected and when piezoelectric effect considered. Firstly, piezoelectric shell is treated as an elastic orthotropic shell, i.e., piezoelectric effect is neglected during determination of
critical pressure leading to buckling. Critical pressure is calculated for various outer to inner radius ratios and presented in table 2 along with the results obtained by Chang and Shen. It is clear from table 2, that results obtained by two methods are almost identical.

**Table 2** Buckling Pressure when Piezoelectric Effect Neglected

| \( \frac{R_2}{R_1} \) | 1.10 | 1.15 | 1.20 | 1.30 | 1.40 |
|----------------|------|------|------|------|------|
| \( P_{cr} \) in GPa by Present Study | 0.0185 | 0.0562 | 0.1197 | 0.3266 | 0.6276 |
| \( P_{cr} \) in GPa by Chang & Shen | 0.0185 | 0.0562 | 0.1199 | 0.3276 | 0.6311 |

Now, buckling pressure is calculated for various outer radius to inner radius ratio by considering the piezoelectric effect of piezoelectric material. The results are given in table 3 along with the results obtained by Chang and Shen.

**Table 3** Buckling Pressure when Piezoelectric Effect Considered

| \( \frac{R_2}{R_1} \) | 1.10 | 1.15 | 1.20 | 1.30 | 1.40 |
|----------------|------|------|------|------|------|
| \( P_{cr} \) in GPa | 0.0245 | 0.0745 | 0.1590 | 0.4353 | 0.8403 |
| Chang & Shen (1997) | 0.0245 | 0.0746 | 0.1592 | 0.4370 | 0.8465 |

On comparing the results obtained when piezoelectric effect neglected and when piezoelectric effect considered, it can be concluded that piezoelectric effect of material affects significantly critical pressure leading to buckling of shell at higher pressure. This fact is very much clear from the figure given below in which buckling pressure is plotted against ratio of outer to inner radius for both cases simultaneously.

**Fig 5** Effect of piezoelectric effect on buckling pressure

**4.2.2 Shell Subjected to Electric Field** In this section, effect of electricity on critical pressure is studied. Critical pressure for the case of infinitely long circular cylindrical shell made of piezoelectric ceramics is obtained for increasing value of electrical field. The results can be expressed in graphical form as
5. Conclusion
Following conclusions are drawn from the present study:

a) Solutions based on classical elasticity theory are inefficient in predicting stability behaviour of circular cylindrical shell at higher thickness of shell.

b) Piezoelectric effect of material affects significantly critical pressure leading to buckling of shell at higher pressure.

c) Applied electric field does not affect critical pressure value when applied within damage value.

d) Results reveal that the present method gives comparable results with the results obtained by Chang & Shen and from convergence check, it is clear that results converge rapidly. Thus it can be stated that the present method converges faster than the method used by Chang and Shen.

References
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Appendix

Properties of composite cylindrical shell

\[ C_{11}=17.448 \times 10^9 \text{ Pa} \quad C_{12}=6.8938 \times 10^9 \text{ Pa} \]
\[ C_{22}=60.819 \times 10^9 \text{ Pa} \quad C_{66}=5.7 \times 10^9 \text{ Pa} \]

Properties of Piezoelectric Circular Cylindrical Shell

\[ C_{11}=11.5 \times 10^{10} \text{ Pa} \quad C_{12}=7.43 \times 10^{10} \text{ Pa} \]
\[ C_{22}=13.9 \times 10^{10} \text{ Pa} \quad C_{66}=2.56 \times 10^9 \text{ Pa} \]
\[ e_{31}=-5.2 \text{ C-m}^2 \quad e_{33}=15.1 \text{ C-m}^2 \]
\[ e_{15}=12.7 \text{ C-m}^2 \]
\[ k_{31}=6.46 \times 10^{-9} \text{ F-m}^{-1} \]
\[ k_{33}=5.62 \times 10^{-9} \text{ F-m}^{-1} \]

For an orthotropic material, constitutive relation of equation (2) can be written as

\[
\begin{bmatrix}
\sigma_r \\
\sigma_\theta \\
\sigma_z \\
\sigma_{r\theta} \\
\sigma_{r\phi} \\
\end{bmatrix}
= 
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66} \\
\end{bmatrix}
\begin{bmatrix}
\varepsilon_r \\
\varepsilon_\theta \\
\varepsilon_z \\
\gamma_{r\theta} \\
\gamma_{r\phi} \\
0 & e_{24} & e_{15} & 0 \\
\end{bmatrix}
= 
\begin{bmatrix}
E_r \\
E_\theta \\
E_z \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
D_r \\
D_\theta \\
D_z \\
\end{bmatrix}
= 
\begin{bmatrix}
e_{33} & e_{31} & e_{32} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e_{15} \\
0 & 0 & 0 & 0 & 0 & e_{24} \\
\end{bmatrix}
\begin{bmatrix}
\varepsilon_r \\
\varepsilon_\theta \\
\varepsilon_z \\
\gamma_{r\theta} \\
\gamma_{r\phi} \\
0 & k_{33} & 0 & 0 \\
0 & k_{11} & 0 & 0 \\
0 & 0 & k_{22} & 0 \\
\end{bmatrix}
\]

\[ \begin{bmatrix}
E_r \\
E_\theta \\
E_z \\
\end{bmatrix} \]
Matrices $E$ and $X$ of eqs. (7) are given by following relations

$$
E = \begin{bmatrix}
A_1 & A_2 & A_3 & A_4 \\
B_1 & B_2 & B_3 & B_4 \\
C_1 & C_2 & C_3 & C_4 \\
\end{bmatrix}
$$

$$
X = \begin{bmatrix}
A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 \\
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 & B_9 \\
C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 \\
\end{bmatrix}
$$

$$
A^1_4 = A_4 + \frac{2hc_{12}}{R_1 C_{11}} A_1 - \frac{2nh}{R_1} A_2 \\
A^1_5 = A_5 + \frac{2nhc_{12}}{R C_{11}} A_1 - \frac{2h}{R_1} A_2 \\
A^1_6 = A_6 \\
A^1_7 = A_7 + A_1 \\
A^1_8 = A_8 + A_2 \\
A^1_9 = A_9 - A_3 + \frac{2e_{33}}{C_{11}} A_1 \\
B^1_4 = B_4 + \frac{2hc_{12}}{R_1 C_{11}} B_1 - \frac{2nh}{R_1} B_2 \\
B^1_5 = B_5 + \frac{2nhc_{12}}{R C_{11}} B_1 - \frac{2h}{R_1} B_2
$$

$[X]^T = [F_{t_1} F_{0_1} F_{0_2} F_{t_2} F_{0_3} F_{0_4} F_{0_5} F_{0_6} F_{0_7} F_{0_8} F_{0_9}]$

Where-
$B_6^1 = B_6$
$B_7^1 = B_7 + B_1$
$B_8^1 = B_8 + B_2$

$B_9^1 = B_9 - B_3 + \frac{2e_{33}}{C_{11}} B_1$

$C_4^1 = C_4 + \frac{2hC_{12}}{R_1 C_{11}} C_3 - \frac{2nh}{R_1} C_2$

$C_5^1 = C_5 + \frac{2nhC_{12}}{R C_{11}} C_3 - \frac{2h}{R_1} C_2$

$C_6^1 = C_6$
$C_7^1 = C_7 + C_1$
$C_8^1 = C_8 + C_2$

$C_9^1 = C_9 - C_3 + \frac{2e_{33}}{C_{11}} C_1$

$A_1^a = A_1 + A_7$
$A_2^a = A_2 + A_8$

$A_3^a = A_3 - A_9 + \frac{2e_{33}}{C_{11} - P_r} A_7$

$A_4^a = A_4 - \frac{2hC_{12}}{R_2 (C_{11} - P_r)} A_7 + \frac{2nh}{R_2} A_8$

$A_5^a = A_5 - \frac{2nhC_{12}}{R_2 (C_{11} - P_r)} A_3 + \frac{2h}{R_2} A_8$

$A_6^a = A_6$

$B_1^a = B_1 + B_7$
$B_2^a = B_2 + B_8$

$B_3^a = B_3 - B_9 + \frac{2e_{33}}{C_{11} - P_r} B_7$

$B_4^a = B_4 - \frac{2hC_{12}}{R_2 (C_{11} - P_r)} B_7 + \frac{2nh}{R_2} B_8$

$B_5^a = B_5 - \frac{2nhC_{12}}{R_2 (C_{11} - P_r)} B_3 + \frac{2h}{R_2} B_8$

$B_6^a = B_6$
\[ C_1^n = C_1 + C_7 \]
\[ C_2^n = C_2 + C_8 \]
\[ C_3^n = C_3 - C_9 + \frac{2e_{33}}{C_{11} - P_r}C_7 \]
\[ C_4^n = C_4 - \frac{2hC_{12}}{R_2(C_{11} - P_r)}C_7 + \frac{2nh}{R_2}C_8 \]
\[ C_5^n = C_5 - \frac{2nhC_{12}}{R_2(C_{11} - P_r)}C_3 + \frac{2h}{R_2}C_8 \]
\[ C_6^n = C_6 \]