Deficiency indices of Jacobi matrices and Dirac operators with point interactions on a discrete set

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Abstract

The paper concerns with infinite block Jacobi matrices \( J \) with \( p \times p \)-matrix entries. We present new conditions for these block matrices to be selfadjoint and have discrete spectrum. In our previous papers there was established a close relation between a class of such matrices and \( 2p \times 2p \) Dirac operators with point interactions in \( L^2(\mathbb{R}; \mathbb{C}^{2p}) \). For block Jacobi matrices \( J \) of this class we present several conditions ensuring either maximality or intermediatory of their deficiency indices. Applications to Dirac and matrix Schrödinger operators are given. It is worth mentioning that the above mentioned connection is employed here in both directions for the first time. In particular, the property of \( J \) to have maximal deficiency indices was firstly established for Dirac operators.

Contents

1 Introduction 2
2 The selfadjointness conditions for block Jacobi matrices 6
3 Discreteness conditions for Jacobi matrices 11
4 Application to the Schrödinger operator with \( \delta \)-interactions 14
5 Abstract results on deficiency indices of perturbed Jacobi matrices 15
6 Dirac operators on a finite interval with maximal deficiency indices 18
   6.1 Realizations \( D_{X,\alpha} \) with maximal deficiency indices 19
   6.2 Realizations \( D_{X,\beta} \) with maximal deficiency indices 22
7 Jacobi matrices generated by Dirac operators with maximal deficiency indices 23
   7.1 Jacobi matrices \( \tilde{J}_{X,\alpha} \) 23
   7.2 Jacobi matrices \( \tilde{J}_{X,\beta} \) 27
8 Jacobi matrices with intermediate deficiency indices 29
9 Comparison with known results

9.1 Comparison with the results of Kostyuchenko and Mirzoev .......................... 31
9.2 Comparison with Dyukarev’s results ................................................................. 34

A Appendix

1 Introduction

The main object of the paper is the infinite block Jacobi matrix

\[ J = \begin{pmatrix}
A_0 & B_0 & 0 & 0 & \cdots \\
B_0^* & A_1 & B_1 & 0 & \cdots \\
0 & A_2 & B_2 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}, \tag{1.1}
\]

with \( p \times p \)-matrix entries \( A_j, B_j \in \mathbb{C}^{p \times p} \), and \( B_j \in \mathbb{C}^{p \times p} \) is invertible, \( j \in \mathbb{N}_0 \), \( 0 \) (\( \in \mathbb{C}^{p \times p} \)) denotes zero matrix.

Following M.G. Krein (see [39, 40]) matrix \( J \) is also called Jacobi matrix with matrix entries.

Let \( l_0^2(\mathbb{N}; \mathbb{C}^p) \) be a subset of finite sequences in \( l^2(\mathbb{N}; \mathbb{C}^p) \). The mapping \( l_0^2(\mathbb{N}; \mathbb{C}^p) \ni f \rightarrow Jf \) defines a linear symmetric, but not closed operator \( J^0 \). The closure of the operator \( J^0 \) defines a minimal closed symmetric operator \( J_{\text{min}} \) in \( l^2(\mathbb{N}; \mathbb{C}^p) \). In what follows we will identify the minimal operator \( J_{\text{min}} \) with the matrix \( J \) of form (1.1) and write \( J_{\text{min}} = J \). We also put \( J_{\text{max}} = J^* \).

Note that \( J \) is symmetric, \( J \subset J^* \), while it is not necessary self-adjoint, i.e. the deficiency indices \( n_\pm(J) := \dim \mathcal{N}_{\pm i}(J) := \dim \text{ker}(J^* \mp i) \) are non-trivial. The most simple and widely known test of self-adjointness of \( J \) is the matrix version of the Carleman test (see [6, Theorem VII.2.9]), and also [36, 37]), which reads as follows.

**Theorem 1.1** ([6], Carleman Test). If

\[ \sum_{j=0}^{\infty} \|B_j\|^{-1} = +\infty, \tag{1.2} \]

then the (minimal) Jacobi operator \( J \) is self-adjoint, i.e. \( J = J_{\text{min}} = J_{\text{max}} = J^* \).

Condition (1.2) is not necessary for self-adjointness of \( J \) even for scalar matrices (\( p = 1 \)) with real entries, though it is sharp in certain classes of Jacobi matrices. More precisely, it is shown by Berezanskii (see [6, Theorem VII.1.1] and [1]) that (in the case \( p = 1 \)) under additional assumptions on entries \( A_n \) and \( B_n \) the operator \( J \) has non-trivial deficiency indices \( n_\pm(J) = 1 \). A matrix version of his result as well as generalizations can be found in [36, 37].

In general, one has \( 0 \leq n_\pm(J) \leq p \) (see [6, 39, 40]). Besides, there is one more restriction: indices achieve the maximal value only simultaneously, i.e. \( n_+(J) = p \iff n_-(J) = p \) (see [31]). It is shown in [23, 24] that the converse statement is also true: for any pair of numbers \( \{n_-, n_+\} \), satisfying either \( 0 \leq n_-, n_+ < p \), or \( n_\pm = p \), there exists a Jacobi matrix \( J \) such that \( n_\pm(J) = p \).

Following M. Krein [39, 40] one associates to the matrix \( J \) a difference matrix expression

\[ (LU)_n = B_{n-1}^* U_{n-1} + B_n U_{n+1} + A_n U_n, \quad U_0 = I_p, \quad U_{-1} = 0, \quad U_n \in \mathbb{C}^{p \times p}, \quad n \in \mathbb{N}_0. \tag{1.3} \]

It is known (see [39, 41, 5]), that the solution to the Cauchy problem \( (LU)_n = z U_n \) subject to the initial condition (1.3) is a sequence of matrix polynomials \( \{P_n(z)\}_0^\infty \). Subsequently one finds

\[ P_0(z) = I_p, \quad P_1(z) = B_0^{-1}(zI - A_0), \quad P_2(z) = B_1^{-1}((zI - A_1)P_1(z)B_0), \quad \ldots. \tag{1.4} \]
M. Krein [39] (see also [6]) showed, that for all \( z \in \mathbb{C}_\pm \) there is a matrix limit
\[
H(z) = \lim_{k \to \infty} \left( \sum_{n=0}^{k} P_n(z) P_n(z) \right)^{-1} \quad \text{and} \quad \text{rank}(H(z)) = n_\pm(J).
\]  
(1.5)

He also established that certain matrix moment problem is associated to every Jacobi matrix \( J \) and this problem has a unique (normalized in a sense) solution if \( n_-(J) \cdot n_+(J) = 0 \). Moreover, this case is known as a definite case of the matrix moment problem. If \( n_+(J) = p \) (see [39]), then the series
\[
\sum_{n=0}^{\infty} P_n(z) P_n(z) =: H^{-1}(z), \quad z \in \mathbb{C}, \quad (1.6)
\]
converges uniformly on compact subsets of \( \mathbb{C} \). Note, that in this case the defect subspace \( M_z \) is:
\[
M_z := \ker(J^* - zI) = \{ \{P_n(z)h\}_{n=0}^{\infty} : h \in \mathbb{C}^p \}.
\]
In this case, it is said that the matrix \( J \) (and the corresponding matrix moment problem) is in the completely indeterminant case ([39] [40], ch. VII, §2]). In this case for each matrix solution \( \Sigma \) to the respective moment problem the series (1.6) defines a reproducing kernel for the subspace of entire matrix functions generated in \( L^2(\mathbb{R}; \Sigma) \) by the matrix polynomials \( \{P_n(z)\}_{n=0}^{\infty} \).

The problem of computing the deficiency indices of Jacobi matrices is the first main problem naturally arising in the spectral theory of such matrices as well as in the corresponding moment problem. This topic has attracted substantial attention, in particular during the last two decades (see for instance, [9] [10] [12] [13] [15] [17] [23] [24] [25] [32] [33] [34] [35] [36] [37] [38] [39] [40] [46]). We especially mention recent publications [17] (\( p = 1 \)) and [10], [15] (\( p \geq 1 \)) where new different conditions for block Jacobi matrices to be selfadjoint were found. Besides, in [15], [12], [13] several discreteness conditions for these matrices were established. In the scalar case (\( p = 1 \)) one of these conditions (see condition (3.7)) coincides with that discovered by Cojullharn, Janas [19] and Chihara [18].

New applications of Jacobi matrices to Schrödinger operators with \( \delta \)-interactions given by formal differential expression
\[
l_{X, \alpha} := -\frac{d^2}{dx^2} + \sum_{n=1}^{\infty} \alpha_n \delta(x - x_n)
\]  
(1.7)
was recently discovered in [32], [33]. Here \( X = \{ x_n \}_{n=1}^{\infty} \subset \mathcal{I} = (0, b), b \leq \infty, \) is a strictly increasing sequence with \( x_0 := 0, x_{n+1} > x_n, n \in \mathbb{N}, \) and such that \( x_n \to b, \) and \( \{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R} \).

Namely, in [32], [33], there was established a connection between Schrödinger operator \( H_{X, \alpha} \) with \( \delta \)-interactions, associated in \( L^2(\mathcal{I}) \) with expression (1.7) on the one hand and Jacobi matrix
\[
J_{X, \alpha}(H) = \begin{pmatrix}
\frac{1}{r_1} \alpha_1 & -\frac{1}{r_1 r_2 d_2} I_p & \cdots & \cdots \\
-\frac{1}{r_1 r_2 d_2} I_p & \frac{1}{r_2} \alpha_2 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]  
(1.8)
(with \( p = 1 \)) on the other hand. Here \( d_n := x_n - x_{n-1}, \ r_n := \sqrt{d_n + d_{n+1}} \) and
\[
\alpha_n := \alpha_n + \left( \frac{1}{d_n} + \frac{1}{d_{n+1}} \right) I_p, \quad n \in \mathbb{N}.
\]

Let \( \mathcal{J}_{X, \alpha}(H, p) \) denote the class of matrices (1.8).
More precisely, it was shown in [32], [33] that certain spectral properties (deficiency indices, discreteness spectra, semiboundedness, positive definiteness, negative point and singular spectra, etc.) of Hamiltonian $H_{X,\alpha}$ are closely related with the corresponding properties of the (minimal) Jacobi operator $J_{X,\alpha}(H)$ (with $p = 1$). In particular, it was proved in [32], [33], that $n_{\pm}(H_{X,\alpha}) = n_{\pm}(J_{X,\alpha}(H))$, which implies $n_{\pm}(H_{X,\alpha}) \leq 1$. The latter inequality has first been established by different methods in [16], [44]. Another application of the equality $n_{\pm}(H_{X,\alpha}) = n_{\pm}(J_{X,\alpha}(H))$ is immediate when combining it with the Carleman test [12].

**Proposition 1.2.** [32], [33] Schrödinger operator $H_{X,\alpha}$ with $\delta$-interactions is self-adjoint in $L^2(\mathbb{R}_+)$ for any $\alpha = \{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ provided that

$$\sum_{n \in \mathbb{N}} d_n^2 = \infty. \quad (1.9)$$

Note that selfadjointness of $H_{X,\alpha}$ has earlier been established in the case $d_* := \inf_n d_n > 0$ by Gesztesy and Kirsh [27] (see also [2] and review [34]).

The above mentioned connection between $H_{X,\alpha}$ and $J_{X,\alpha}(H)$ was extended in [35] to the case of $p \times p$-matrix differential expressions (1.7) (with $\{\alpha_n\}_1^{\infty} \subset \mathbb{C}^{p \times p}$) and the block Jacobi matrices $J_{X,\alpha}(H)$ with $p > 1$. In particular, it was proved in [35] that

$$n_{\pm}(H_{X,\alpha}) = n_{\pm}(J_{X,\alpha}(H)) \quad \text{for any} \quad p \geq 1. \quad (1.10)$$

It follows that $n_{\pm}(H_{X,\alpha}) \leq p$. Proposition 1.2 has attracted certain attention and has been developed for different classes of (scalar and matrix) Schrödinger operators to ensure their self-adjointness in $L^2(\mathbb{R}_+; \mathbb{C}^p)$ in [8], [11], [45], [46]. In particular, it was shown by different methods in [46] and [35] that condition (1.9) still ensures selfadjointness of $H_{X,\alpha}$ for any $p > 1$. We note only that this result is also immediate by combining equality (1.10) with the matrix Carleman test.

One more application of Jacobi matrices recently occurred in [17] in connection with Dirac operators with $\delta$–interactions given by formal differential expression

$$D_{X,\alpha} := -i c \frac{d}{dx} \otimes \begin{pmatrix} O_p & I_p \\ I_p & O_p \end{pmatrix} + \frac{c^2}{2} \begin{pmatrix} I_p & O_p \\ O_p & -I_p \end{pmatrix} + \sum_{n=1}^{\infty} \alpha_n \delta(x - x_n) \quad (1.11)$$

with $p = 1$ and $\{\alpha_n\}_1^{\infty} \subset \mathbb{R}$, and where $c$ denotes the velocity of light. First rigorous treatment of the operator $D_{X,\alpha}$ associated in $L^2(\mathbb{R}; \mathbb{C}^2)$ with expression (1.11) was done by Gesztesy and Šeba [28] (see formulas (6.2), (6.3) below). Therefore following [17] we call the operator $D_{X,\alpha}$ by Gesztesy–Šeba realization (in short GS-realization) of Dirac differential expression.

Namely, in [17], there was established that like in the Schrödinger case, certain spectral properties of GS-realization $D_{X,\alpha}$ in $L^2(\mathbb{I}; \mathbb{C}^2)$ (deficiency indices, discreteness and other types of spectra, etc.) are closely related to that of Jacobi matrix

$$J'_{X,\alpha} := \begin{pmatrix} O_p & \frac{c}{d_1} I_p & \frac{c}{d_1} I_p & O_p & O_p & \ldots \\ \frac{c}{d_1} I_p & O_p & \frac{c}{d_1^2} I_p & O_p & O_p & \ldots \\ \frac{c}{d_1} I_p & \frac{c}{d_1} I_p & O_p & \frac{c}{d_1} I_p & O_p & \ldots \\ \frac{c}{d_1} I_p & \frac{c}{d_1} I_p & \frac{c}{d_1} I_p & O_p & \frac{c}{d_1^2} I_p & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \quad (1.12)$$

under certain restrictions on $d_n$ (with $p = 1$) and $d_n := x_n - x_{n-1}$. These results have been extended to the matrix case with $\alpha = \{\alpha_n\}_1^{\infty} \subset \mathbb{C}^{p \times p}$, $\alpha_n = \alpha_n^\star$, in [14], [15], [12]. In particular, it was shown in
that formula (1.10) remains valid for the operators $D_{X,a}$ in $L^2(I; \mathbb{C}^{2p})$ and $J'_{X,a}$, i.e.

$$n_\pm (D_{X,a}) = n_\pm (J_{X,a}) = n_\pm (J'_{X,a}) \text{ for any } p \geq 1,$$

(1.13)

where the Jacobi matrix $J_{X,a}$ is defined by formula (7.9).

Denote by $J_{X,a}(p)$ the set of Jacobi matrices (1.12). We also introduce other class $J_{X,\beta}(p)$, consisting of Jacobi matrices $J_{X,\beta}$, given by (7.21). Noting that $\nu(x) < x$ Carleman’s condition (1.2) for $J_{X,a} \in J_{X,\beta}(p)$ becomes $\sum_{n \in \mathbb{N}} d_n = \infty$. Therefore combining (1.13) with the Carleman test yields $n_\pm (D_{X,a}) = n_\pm (J_{X,a}) = 0$, provided that $I = \mathbb{R}_+$.

In the present paper we investigate deficiency indices of Jacobi matrices from the four classes described above assuming that the Carleman condition (1.2) is violated, i.e. $\{d_k\} \in l^1(\mathbb{N})$. We also consider more general matrices $J_{X,a}$ defined by (7.1).

One of our main results reads as follows:

$$n_\pm (J_{X,a}) = n_\pm (J_{X,a}) = p$$

(1.14)

provided that

$$\sum_{n=2}^{\infty} d_n \prod_{k=1}^{n-1} \left( 1 + \frac{1}{c} \|\alpha_k\|_{\mathbb{C}^{2p}} \right)^2 < +\infty$$

(1.15)

(for matrices $J_{X,a}$ some extra conditions are required). In other words, condition (1.15) ensures the complete indeterminacy of the respective moment problems.

Matrices of the classes $J_{X,a}(p)$ have several interesting features.

First, as a byproduct of (1.14) we derive that each selfadjoint extension of $J_{X,a}$ has discrete spectrum whenever (1.15) holds.

Secondary, condition (1.15) is obviously satisfied for $\alpha = 0$, hence Carleman’s condition (1.2) for matrices $J_{X,0}$ becomes also necessary for selfadjointness.

At the same time under other conditions (see (8.2)) depending on $\alpha = \{\alpha_n\}_1^\infty$ matrices $J_{X,a}$ and $J_{X,a}(H)$ may be selfadjoint and even have arbitrary intermediate indices. In particular, matrices $J_{X,a}$ with certain $\alpha$ demonstrate that Carleman’s condition (1.2) is not necessary for selfadjointness.

Moreover, condition (1.15) with $\beta = \{\beta_n\}_1^\infty$ in place of $\alpha = \{\alpha_n\}_1^\infty$ ensures maximal deficiency indices for other Jacobi matrices $J_{X,\beta}$, given by (7.21) (and their perturbations $\hat{J}_{X,\beta}$, defined by (7.20)). Hence, each selfadjoint extension of $J_{X,\beta}$ has discrete spectrum whenever condition (1.15) with $\beta$ in place of $\alpha$ holds.

To the best of our knowledge other conditions for block Jacobi matrices (1.11) to have maximal indices $n_\pm (J) = p$ were obtained by Kostyuchenko and Mirzoev [36–38] (see Theorem 9.1 below) and Dyukarev [24]. Note however that matrices $J_{X,a}$ never meet conditions of Theorem 9.1 (see Lemma 9.2 below). It is worth to note also that one of matrices $J_{X,\beta} \in J_{X,\beta}(p)$ (see (7.21)) with zero diagonal coincides with the block Jacobi matrix constructed by Dyukarev [24] in the framework of another approach. Therefore we obtain his result as a special case of Proposition 7.5 with $\beta_n = -d_n\mathbb{I}_p$ and a special choice of $\{d_n\}_1^\infty$, $d_n \sim \frac{1}{(n+1)^{\nu/2}+1}$, (see Section 7.2 for details).

It is worth to mention that in this paper we employ the close relations between $J_{X,a}$ and $D_{X,a}$ in the opposite direction. Namely, first we find conditions for intensities $\{\alpha_n\}_1^\infty$ that ensure maximal deficiency indices for Dirac operators $D_{X,a}$, and then using formulas (1.13) derive the corresponding statements for matrices $J_{X,a}$, and more general Jacobi matrices $\hat{J}_{X,a}$.

The paper is organized as follows. In the Sections 2–8 conditions for selfadjointness and discreteness properties of block Jacobi matrix (1.11) are obtained.

In Section 6 some previous results of Sections 2–5 are applied to Schrödinger operator $H_{X,a}$. Namely, using equality (1.10) we establish certain selfadjointness and discreteness conditions of $H_{X,a}$.
In Section 5 we prove our main abstract result, Theorem 5.1 on coincidence of the deficiency indices for block Jacobi matrices under certain perturbations.

In Section 4 we prove that condition (1.15) ensures maximal deficiency indices for GS-realization \( D_{X,\alpha} \), i.e. that \( n_\pm(D_{X,\alpha}) = p \).

In Section 7 it is shown that Jacobi matrices \( \tilde{J}_{X,\alpha} \) and \( J_{X,\alpha} \) (of the form (7.1) and (7.9) resp.) have equal deficiency indices under the additional conditions (see (7.4)–(7.7)). In particular, under conditions (7.4)–(7.7) condition (1.15) implies \( n_\pm(\tilde{J}_{X,\alpha}) = p \).

In Section 8 we indicate certain conditions on \( \tilde{J}_{X,\alpha} \) for block Jacobi matrices under certain perturbations.

In Section 9 we compare condition (1.15) and similar condition on \( H \) sets of closed operators and linear relations in \( H \). Finally, in Section 9.2 we show that Dukarev’s result from [24] is a special case of our Proposition 7.5 on matrices \( J_{X,\alpha} \) with zero diagonal.

The main results of the paper were announced without proofs in [12], [13].

Notations. Throughout the paper \( \mathcal{H}, \mathcal{C} \) denote separable Hilbert spaces. \( \mathcal{C}(\mathcal{H}) \) and \( \tilde{\mathcal{C}}(\mathcal{H}) \) are the sets of closed operators and linear relations in \( \mathcal{H} \), respectively; \( \text{dom}(T) \) denotes the domain of \( T \in \mathcal{C}(\mathcal{H}) \). Let \( I_p \) and \( O_p \) be the unit and zero operators in \( \mathcal{C}^p \), respectively. For any \( I \) been a subset of \( Z \) we put \( l^2(I; \mathbb{C}^p) := l^2(I) \otimes \mathbb{C}^p \); \( l^2_0(I; \mathbb{C}^p) \) is a subset of finite sequences in \( l^2(I; \mathbb{C}^p) \). We also put \( L^2([x_{n-1}, x_n]; \mathcal{C}^{2p}) := L^2(x_{n-1}, x_n) \otimes \mathbb{C}^{2p} \) and \( L^2(I; \mathbb{C}^{2p}) := L^2(I) \otimes \mathbb{C}^{2p} \). Let also \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

Part I. Abstract results on the deficiency indices of block Jacobi matrices

2 The selfadjointness conditions for block Jacobi matrices

Consider the block Jacobi matrix

\[
J = \begin{pmatrix}
\mathcal{A}_0 & \mathcal{B}_0 & O_p & O_p & \ldots & O_p & O_p & O_p & \ldots \\
\mathcal{B}_0^* & \mathcal{A}_1 & O_p & O_p & \ldots & O_p & O_p & O_p & \ldots \\
O_p & \mathcal{B}_1^* & \mathcal{A}_2 & O_p & \ldots & O_p & O_p & O_p & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
O_p & O_p & O_p & \ldots & \mathcal{B}_{n-1}^* & \mathcal{A}_n & \mathcal{B}_n & O_p & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}, \quad (2.1)
\]

with \( \mathcal{A}_n = \mathcal{A}_n^* \), \( \mathcal{B}_n \in \mathbb{C}^{p \times p} \) and invertible matrices \( \mathcal{B}_n \), i.e. \( \det \mathcal{B}_n \neq 0 \), \( n \in \mathbb{N}_0 \).

As usual we identify \( J \) with the minimal (closed) symmetric Jacobi operator associated in \( l^2(\mathbb{N}_0; \mathbb{C}^p) \) in a standard way with matrix \( J \) (see [1], [6]).

Definition 2.1 ([30]). Let \( K \) and \( T \) be densely defined linear operators in a Hilbert space \( \mathcal{H} \). Then the operator \( K \) is said to be subordinate to the operator \( T \) if \( \text{dom}T \subset \text{dom}K \) and the following inequality holds

\[
\|K\| \leq a\|Tu\| + b\|u\|, \quad a > 0, \ b \geq 0, \ u \in \text{dom} T. \quad (2.2)
\]
One says that $K$ is strongly subordinate to the operator $T$ if $a < 1$ in (2.2).

By the Kato-Rellich theorem (see [30, Theorem 5.4.3]), if operator $T$ is selfadjoint, and $K$ is symmetric and strongly subordinate to $T$, then the operator $T + K$ is also selfadjoint and $\text{dom}(T + K) = \text{dom} T$. According to Wüst’s theorem (see [50, Theorem X.14]), if $K$ is symmetric and subordinate to $T = T^*$ with $a = 1$ (see (2.2)), then operator $T + K$ is essentially selfadjoint on $\text{dom} T$.

We apply these theorems to the investigation of the selfadjointness of the Jacobi matrix $J$.

**Theorem 2.2.** Let $J$ be the block Jacobi matrix of the form (2.1) and $A := \text{diag}\{A_0, \ldots, A_n, \ldots\}$, $\ker A = \{0\}$. Let for some $N \in \mathbb{N}_0$

$$a_1(N) := \sup_{n \geq N} \left( \|A_n^{-1}B_n\| + \|A_n^{-1}B_n^*\| \right) < \infty,$$

$$a_2(N) := \sup_{n \geq N} \left( \|A_n^{-1}B_n\| + \|A_{n+2}^{-1}B_{n+1}^*\| \right) < \infty.$$  

If

$$\sqrt{a_1(N)a_2(N)} \leq 1,$$

then the operator $J$ is selfadjoint in $l^2(\mathbb{N}_0; \mathbb{C}^p)$. Herewith $\text{dom} J = \text{dom} A$ if the estimate (2.5) is strict, i.e. $\sqrt{a_1(N)a_2(N)} < 1$.

**Proof.** We introduce the block Jacobi submatrix $J_N$ of the matrix $J$, setting

$$J_N := \begin{pmatrix} A_N & B_N & \mathbb{O}_p & \cdots & \mathbb{O}_p & \cdots \\ B_N^* & A_{N+1} & B_{N+1} & \cdots & \mathbb{O}_p & \cdots \\ \mathbb{O}_p & B_{N+1}^* & A_{N+2} & \cdots & \mathbb{O}_p & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}.$$  

(a) First, let $N = 0$ and $K$ be the minimal symmetric operator defined by the Jacobi matrix

$$K = \begin{pmatrix} \mathbb{O}_p & B_0 & \mathbb{O}_p & \mathbb{O}_p & \mathbb{O}_p & \cdots \\ B_0^* & \mathbb{O}_p & B_1 & \mathbb{O}_p & \mathbb{O}_p & \cdots \\ \mathbb{O}_p & B_1^* & \mathbb{O}_p & B_2 & \mathbb{O}_p & \cdots \\ \mathbb{O}_p & \mathbb{O}_p & B_2^* & \mathbb{O}_p & B_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots \end{pmatrix}.$$  

Then on compact vectors the following equality holds

$$J_0 h = J_0 h = A h + K h, \quad h \in l_0^2(\mathbb{N}_0; \mathbb{C}^p).$$  

Moreover, the operator $A^{-1}K$ on the lineal $l_0^2(\mathbb{N}_0; \mathbb{C}^p)$ is given by the matrix

$$A^{-1}K = \begin{pmatrix} \mathbb{O}_p & A_0^{-1}B_0 & \mathbb{O}_p & \mathbb{O}_p & \mathbb{O}_p & \cdots \\ A_0^{-1}B_0^* & \mathbb{O}_p & A_1^{-1}B_1 & \mathbb{O}_p & \mathbb{O}_p & \cdots \\ \mathbb{O}_p & A_1^{-1}B_1^* & \mathbb{O}_p & A_2^{-1}B_2 & \mathbb{O}_p & \cdots \\ \mathbb{O}_p & \mathbb{O}_p & A_2^{-1}B_2^* & \mathbb{O}_p & A_3^{-1}B_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots \end{pmatrix}.$$  

According to the Schur’s test ([7, Theorem 2.10.1]), conditions (2.3) and (2.4) guarantee matrix boundedness (2.9) in $l^2(\mathbb{N}_0; \mathbb{C}^p)$ and estimate $\|A^{-1}K h\| \leq \sqrt{a_1(N)a_2(N)}\|h\|, \quad h \in l_0^2(\mathbb{N}_0; \mathbb{C}^p)$. 


Since $K \subseteq K^*$ and $A = A^*$, then $KA^{-1} \subseteq K^*A^{-1} \subseteq (A^{-1}K)^*$ (see [51, Ch. VIII §2.115]). Therefore, the operator $A^{-1}K$ is bounded in $l^2_0(\mathbb{N}_0; \mathbb{C}^p)$ and its closure $A^{-1}K \in \mathcal{B}(l^2(\mathbb{N}_0; \mathbb{C}^p))$, and $\|KA^{-1}\| = \|(A^{-1}K)^*\| \leq 1$ provided (2.5). Hence $\text{dom} \ A = \text{ran} \ A^{-1} \subseteq \text{dom} \ A$ and

$$\|Kf\| = \|KA^{-1}Af\| \leq \|KA^{-1}\| \cdot \|Af\| \leq \|Af\|, \quad f \in \text{dom} \ A \subset \text{dom} \ K.$$  \hfill (2.10)

Therefore, the operator $K$ is subordinate to the operator $A$ with constants $a = 1$ and $b = 0$ (see (2.2)). By the Wüst’s theorem, operator $A + K (\subseteq J)$ is essentially selfadjoint on $\text{dom} \ A$. Then operator $J = J_{\text{min}} = A + K$ is selfadjoint. For $a < 1$ and $b = 0$, by the Kato-Rellich theorem, operator $A + K (\subseteq J)$ is selfadjoint and $\text{dom} (A + K) = \text{dom} \ A$.

(b) Let $N > 0$. Then operator $J$ admits the following representation

$$J = J'_N \oplus J_N + B'_{N-1},$$  \hfill (2.11)

in which

$$J'_N := \begin{pmatrix} A_0 & B_0 & \mathbb{O}_p & \ldots & \mathbb{O}_p & \mathbb{O}_p & \mathbb{O}_p \\ B_0^* & A_1 & \mathbb{O}_p & \ldots & \mathbb{O}_p & \mathbb{O}_p & \mathbb{O}_p \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ \mathbb{O}_p & \mathbb{O}_p & \mathbb{O}_p & \ldots & B_{N-3}^* & A_{N-2} & B_{N-2} \\ \mathbb{O}_p & \mathbb{O}_p & \mathbb{O}_p & \ldots & \mathbb{O}_p & B_{N-2}^* & A_{N-1} \end{pmatrix}$$  \hfill (2.12)

and $B'_{N-1}$ is a bounded block matrix with unique nontrivial elements $B_{N-1}$ and $B_{N-1}^*$, located in $N$ and $N + 1$ rows. Clearly $J'_N = (J'_N)^*$. Now it follows from (2.11) that

$$n_{\pm} (J) = n_{\pm} (J'_N \oplus J_N) = n_{\pm} (J_N) = 0. \quad \square$$

**Corollary 2.3.** Let that under the conditions of the Theorem 2.2 at least one of the following conditions be satisfied:

(i) $a_1(N) \leq 1$ and $a_2(N) \leq 1$,

(ii) $a_1(N) \leq 1$ and $a_2(N) < 1 \ (a_1(N) < 1 \text{ and } a_2(N) \leq 1)$.

Then the operator $J$ is selfadjoint.

**Corollary 2.4.** Let $J$ be the block Jacobi matrix of the form (2.1) and $\ker A = \{0\}$. Then $J = J^*$, if for some $N \in \mathbb{N}_0$ the following conditions are satisfied

$$\sup_{n \geq N} \|A_n^{-1}B_n\| \leq \frac{1}{2}, \quad \sup_{n \geq N} \|A_n^{-1}B_{n-1}^*\| \leq \frac{1}{2}.$$  \hfill (2.13)

**Proof.** Proof follows from the Theorem 2.2 since in this case $a_1(N) \leq 1$ and $a_2(N) \leq 1. \quad \square$

**Theorem 2.5.** Let $J$ be the block Jacobi matrix of the form (2.1) and $A := \text{diag} \{A_0, \ldots, A_n, \ldots\}$, $\ker A = \{0\}$. Let for some $N \in \mathbb{N}_0$ at least one of the conditions holds:

(i) $$\sup_{n \geq N} \left( \|A_n^{-1}B_n\|^2 + \|A_n^{-1}B_{n-1}^*\|^2 \right) \leq \frac{1}{2},$$  \hfill (2.14)

(ii) $$\sup_{n \geq N} \left( \|A_n^{-1}B_n\|^2 + \|A_{n+2}^{-1}B_{n+1}^*\|^2 \right) \leq \frac{1}{2}. \hfill (2.15)$$

Then the operator $J$ is selfadjoint in $l^2(\mathbb{N}_0; \mathbb{C}^p)$ and $\text{dom} \ A \subseteq \text{dom} \ J$. 

Proof. As in the proof of Theorem 2.2, operator $A^{-1}K$ is defined on finite vectors $h \in l^2_0(N_0; \mathbb{C}^p)$ and is defined by the matrix (2.9).

(i) Therefore for $h = \text{col}(h_0, h_1, \ldots, h_n, \ldots) \in l^2_0(N_0; \mathbb{C}^p)$

$$A^{-1}Kh = \begin{pmatrix} A^{-1}_0B_0h_1 \\ A^{-1}_1B_1h_0 + A^{-1}_1B_1h_2 \\ A^{-1}_2B_2h_1 + A^{-1}_2B_2h_3 \\ A^{-1}_3B_3h_2 + A^{-1}_3B_3h_4 \\ \vdots \end{pmatrix}.$$  

(2.16)

Hence, taking into account the condition (2.14), we obtain

$$\|A^{-1}Kh\|^2 = \sum_{n=1}^{\infty} \|A^{-1}_nB^{-1}_n h_n + A^{-1}_nB_n h_{n+1}\|^2 \leq \sum_{n=1}^{\infty} \left( \|A^{-1}_nB^{-1}_n h_n + A^{-1}_nB_n h_{n+1}\|^2 + \|A^{-1}_nB_n\| \cdot \|h_{n+1}\|^2 \right)$$

$$\leq \sum_{n=1}^{\infty} \left( \|A^{-1}_nB^{-1}_n h_n\|^2 + \|A^{-1}_nB_n h_{n+1}\|^2 \right) \leq \frac{1}{2} \sum_{n=1}^{\infty} \left( \|h_n\|^2 + \|h_{n+1}\|^2 \right) = \frac{1}{2} \cdot 2\|h\|^2 = \|h\|^2.$$  

(2.17)

This inequality is equivalent to the estimate (2.10). Hence, by the W"ust’s theorem, $A + K (\subseteq J)$ is essentially selfadjoint on $\text{dom } A$. Therefore $J = J_{\text{min}} = A + K = J^*.$

(ii) The proof is similar to (i). Consider the operator $KA^{-1}$ instead of the operator $A^{-1}K$. From the condition (2.15), taking into account the equalities $\|B^{-1}_{n-1}A^{-1}_{n-1}\| = \|A^{-1}_{n-1}B^{-1}_{n-1}\|$ and $\|B_nA_{n+1}^{-1}\| = \|A^{-1}_{n+1}B^{-1}_n\|$, similar to (2.17), we obtain

$$\|KA^{-1}h\|^2 = \sum_{n=1}^{\infty} \|B^{-1}_{n-1}A^{-1}_{n-1} h_{n-1} + B_nA_{n+1}^{-1} h_{n+1}\|^2 \leq \sum_{n=1}^{\infty} \left( \|B^{-1}_{n-1}A^{-1}_{n-1} h_{n-1}\|^2 + \|B_nA_{n+1}^{-1} h_{n+1}\|^2 \right)$$

$$\leq \frac{1}{2} \sum_{n=1}^{\infty} \left( \|h_n\|^2 + \|h_{n+1}\|^2 \right) = \frac{1}{2} \cdot 2\|h\|^2 = \|h\|^2.$$  

Therefore, operator $KA^{-1}$ is contracting. Hence, an estimate (2.10) is true, which means the subordination of the operator $K$ to the operator $A$ with constants $a = 1$ and $b = 0$. By the W"ust’s theorem, $A + K (\subseteq J)$ is essentially selfadjoint on $\text{dom } A$ and, therefore, $J = J_{\text{min}} = A + K = J^*.$

\[ \square \]

Remark 2.6. (i) In the case of strict inequality, Theorem 2.2 (ii) was proved differently in [10], the authors of which essentially rely on the theory of matrix orthogonal polynomials of the second kind.

(ii) Condition (2.14) implies condition $a_1(N) \leq 1$, and condition (2.15) implies condition $a_2(N) \leq 1$ (see (2.3) and (2.4)). However, conditions $a_1(N) \leq 1$, $a_2(N) \leq 1$, and even more condition (2.5), weaker set of conditions (2.14) and (2.15).

Recall a version of the KLMN-theorem that is convenient for us (see Theorem X.17)). If $T$ is a positive selfadjoint operator in a Hilbert space $\mathfrak{H}$, and $K$ is symmetric and strongly subordinate to $T$ in the sense of forms, i.e. $\text{dom } K \supset \text{dom } T^{1/2}$ and

$$|(Kf, f)| \leq a(T^{1/2}f, T^{1/2}f) + b(f, f), \quad f \in \text{dom } T^{1/2}, \quad a < 1, \quad b \in \mathbb{R},$$  

(2.18)
Theorem 2.7. Let $\mathbf{J}$ be the block Jacobi matrix of the form (2.1) and $A := \text{diag}(A_0, \ldots, A_n, \ldots)$. Let also operator $A$ be positive definite, i.e. $(Af, f) \geq m\|f\|^2$, $f \in \text{dom}A$, $m > 0$. If for some $N \in \mathbb{N}_0$ at least one of the conditions holds:

\begin{equation}
(i) \quad \sup_{n \geq N} \left(\|A_{n+1/2}B_nA_{n+1}^{-1}\| + \|A_{n+1/2}B_nA_{n+1}^{-1/2}\|\right) < 1, \tag{2.19}
\end{equation}

\begin{equation}
(ii) \quad \sup_{n \geq N} \|A_{n+1/2}B_nA_{n+1}^{-1/2}\| < \frac{1}{2}, \tag{2.20}
\end{equation}

\begin{equation}
(iii) \quad \sup_{n \geq N} \left(\|A_{n+1/2}B_nA_{n+1}^{-1/2}\|^2 + \|A_{n+1/2}B_nA_{n+1}^{-1/2}\|^2\right) < \frac{1}{2}, \tag{2.21}
\end{equation}

then operator $\mathbf{J}$ is selfadjoint in $l_0(N_0; \mathbb{C}^p)$.

Proof. (i) Since the operator $A = A^*$ is positive definite, then the operator $A^{1/2}$ is well defined and selfadjoint. Let $K$ be a matrix of the form (2.7). Then

\[ \mathbf{J}h = \mathbf{A}h + \mathbf{K}h = \mathbf{A}h + A^{1/2}(A^{-1/2}KA^{-1/2})A^{1/2}h, \quad h \in l_0^2(N_0; \mathbb{C}^p). \tag{2.22} \]

Since $\text{dom}K \supset \text{dom}A^{1/2}$, then the operator $A^{-1/2}KA^{-1/2}$ is well defined on finite vectors $h \in l_0^2(N_0; \mathbb{C}^p)$ and is defined by a symmetric matrix

\begin{equation}
A^{-1/2}KA^{-1/2} = \begin{pmatrix}
\mathbb{O}_p & A_0^{-1/2}B_0A_1^{-1/2} & \mathbb{O}_p & \mathbb{O}_p & \cdots \\
A_1^{-1/2}B_0A_0^{-1/2} & \mathbb{O}_p & A_1^{-1/2}B_1A_2^{-1/2} & \mathbb{O}_p & \cdots \\
\mathbb{O}_p & A_2^{-1/2}B_1A_1^{-1/2} & \mathbb{O}_p & A_2^{-1/2}B_2A_3^{-1/2} & \cdots \\
\mathbb{O}_p & \mathbb{O}_p & A_3^{-1/2}B_2A_4^{-1/2} & \mathbb{O}_p & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}. \tag{2.23}
\end{equation}

According to Schur’s test, the condition [2.19] guarantees the boundedness of the minimal (symmetric) operator $A^{-1/2}KA^{-1/2}$, associated with the matrix (2.23), and the estimate $\|A^{-1/2}KA^{-1/2}\| = 1 - \epsilon < 1$.

Therefore, the operator $K$ is strongly subordinate to the operator $A$ in the sense of forms, i.e. $\text{dom}K \supset \text{dom}A^{1/2}$ and

\[ ||(Kf, f)|| = ||(A^{-1/2}(A^{-1/2}KA^{-1/2})A^{1/2}f, f)|| = \left|\left|\left(A^{-1/2}KA^{-1/2}\right)\left(A^{1/2}f, A^{1/2}f\right)\right|\right| \leq \|A^{-1/2}KA^{-1/2}\| \cdot \|A^{1/2}f\|^2 \leq (1 - \epsilon)\|A^{1/2}f\|^2, \quad f \in \text{dom}A^{1/2}. \tag{2.24} \]

According to the KLMN-theorem, the operator $\mathbf{J} = \mathbf{A} + \mathbf{K}$ is selfadjoint.

(ii) Since $\|A_{n+1/2}B_nA_{n+1/2}\| = \|A_{n+1/2}B_nA_{n+1/2}\|$, then (2.20) implies (2.19).

(iii) As in the proof of (i), operator $A^{-1/2}KA^{-1/2}$ is well defined on finite vectors $h \in l_0^2(N_0; \mathbb{C}^p)$ and is defined by matrix (2.23).

Therefore, for $h = \text{col}(h_0, h_1, \ldots, h_n, \ldots) \in l_0^2(N_0; \mathbb{C}^p)$

\begin{equation}
A^{-1/2}KA^{-1/2}h = \begin{pmatrix}
A_0^{-1/2}B_0A_1^{-1/2}h_0 + A_1^{-1/2}B_1A_2^{-1/2}h_2 + A_2^{-1/2}B_2A_3^{-1/2}h_3 \\
A_1^{-1/2}B_0A_1^{-1/2}h_0 + A_1^{-1/2}B_1A_2^{-1/2}h_2 + A_2^{-1/2}B_2A_3^{-1/2}h_3 \\
A_2^{-1/2}B_1A_1^{-1/2}h_0 + A_2^{-1/2}B_2A_3^{-1/2}h_3 \\
\vdots
\end{pmatrix}. \tag{2.25}
\end{equation}
From this, taking into account the condition (2.21), we obtain

\[
\|A^{-1/2}KA^{-1/2}h\|_2^2 = \sum_{n=1}^{\infty} \|A_n^{-1/2}B_n^{-1/2}h_n - A_n^{-1/2}B_n^{-1/2}h_n\|_2^2 \\
\leq \sum_{n=1}^{\infty} \left( \|A_n^{-1/2}B_n^{-1/2}h_n\|_2^2 + \|A_n^{-1/2}B_n^{-1/2}h_n\|_2^2 \right) \\
\leq \left( \frac{1}{2} - \varepsilon \right) \sum_{n=1}^{\infty} \left( \|h_n\|_2^2 + \|h_n+1\|_2^2 \right) = \left( \frac{1}{2} - \varepsilon \right) \|h\|_2^2 = (1 - 2\varepsilon)\|h\|_2^2.
\]
(2.26)

Therefore, operator \(A^{-1/2}KA^{-1/2}\) is contracting. Hence, an estimate (2.24), is true, which means the subordination of the operator \(K\) to the operator \(A\) in the sense of forms. According to the KLMN-theorem, the operator \(J = A + K\) is selfadjoint, and, therefore, \(J = J_{\min} = A + K = J^*\).

3 Discreteness conditions for Jacobi matrices

The following statement complements known results on the spectrum of perturbations (see e.g. [30] Theorems 4.1.16, 4.3.17, 6.3.4).

**Lemma 3.1.** Let \(T = T^* \in C(\mathfrak{h})\) and let \(K(\subset K^*)\) be a symmetric operator in \(\mathfrak{h}\) strongly subordinated to \(T\), i.e. \(\text{dom } K \supset \text{dom } T\) and estimate (2.22) be fulfilled with \(a \in (0, 1)\). If \(T\) has discrete spectrum, then the operator \(S := T + K(= S^*)\) is also discrete.

**Proof.** By the Kato-Rellich theorem \(S = S^*\) and \(\text{dom } S = \text{dom } T\). Clearly, alongside (2.22) similar estimate holds with \(T - \lambda I\) in place of \(T\), the same \(a\), and \(b + |\lambda|\) in place of \(b\).

Besides, it easily follows (and also known) that \(K\) is subordinated to the operator \(S - \lambda I\) with \((S - \lambda I)\)-bound \(a_s - \lambda I(K) \leq a(1 - a)^{-1}\). Hence \(K(S - \lambda)^{-1} \in \mathcal{B}(\mathfrak{h})\) for \(\lambda \in \mathbb{C}_+\) and the resolvent difference admits the factorization

\[
(S - \lambda)^{-1} - (T - \lambda)^{-1} = (T - \lambda)^{-1} (K(S - \lambda)^{-1}), \quad \lambda \in \mathbb{C}_+.
\]
(3.1)

Since \((T - \lambda)^{-1} \in \mathcal{S}_\infty(\mathfrak{h})\) and \(K(S - \lambda)^{-1} \in \mathcal{B}(\mathfrak{h})\), one gets that the operator \((S - \lambda)^{-1}\) is also compact, i.e. \(S\) has discrete spectrum. 

**Lemma 3.2.** Let \(T = T^* \in C(\mathfrak{h})\) be positive definite and let \(K(\subset K^*)\) be a symmetric operator in \(\mathfrak{h}\) and strongly subordinate to \(T\) in the sense of forms, i.e. \(\text{dom } K \supset \text{dom } T^{1/2}\). Let the estimate (2.13) be satisfied with \(a \in (0, 1)\). If \(T\) has a discrete spectrum, then the operator \(S := T + K(= S^*)\) is also discrete.

**Proof.** By the KLMN-theorem \(S = S^*\). Since \(K\) is strongly subordinate to \(T\) in the sense of forms, then operators \(T\) and \(S\) generate the same energy spaces, i.e. \(\mathfrak{h}_T = \mathfrak{h}_S\) algebraically and topologically. If \(T\) has a discrete spectrum, then \(T^{-1} \in \mathcal{S}_\infty(\mathfrak{h})\). Hence, by the Rellich theorem (sufficiency) (see [7] Theorem 10.1.5)) embedding \(\mathfrak{h}_T \subset \mathfrak{h}_S\) is compact.

On the other hand, since the spaces \(\mathfrak{h}_T\) and \(\mathfrak{h}_S\) coincide algebraically and topologically, the embedding \(\mathfrak{h}_S \subset \mathfrak{h}_T\) is also compact. It follows from the Rellich theorem (necessity) that the spectrum of \(S\) is discrete.
Remark 3.3. In fact, the proofs of Lemmas 3.1, 3.1 show that it is proved in passing that the following implication holds
\[(T - \lambda)^{-1} \in S_p(S) \implies (S - \lambda)^{-1} \in S_p(S), \quad p \in (0, \infty).\]

We give conditions for the discreteness of the spectrum of the Jacobi operator.

Theorem 3.4. Let \(J\) be the minimal operator associated in \(l^2(N_0; C^p)\) with a block Jacobi matrix of the form \((2.1)\) and let \(A = \text{diag}\{A_0, A_1, \ldots, A_n, \ldots\}\) has a discrete spectrum. Let at least one of the following conditions also be satisfied:

(i) let \(a_1(N)\) and \(a_2(N)\) be constants of the form \((2.3)\) and \((2.4)\), respectively, and
\[
\sqrt{a_1(N)a_2(N)} < 1; \quad (3.2)
\]

(ii) \(a_1(N) \leq 1\) and \(a_2(N) < 1\) \((a_1(N) < 1\) and \(a_2(N) \leq 1\));

(iii) for some \(N \in \mathbb{N}_0\)
\[
\sup_{n \geq N} \|A_n^{-1}A_{n+1}^{-1}\| < \frac{1}{2}, \quad \sup_{n \geq N} \|A_n^{-1}A_{n+1}^*\| < \frac{1}{2}; \quad (3.3)
\]

(iv) for some \(N \in \mathbb{N}_0\)
\[
\sup_{n \geq N} \left(\|A_n^{-1}A_{n+1}^{-1}\|^2 + \|A_n^{-1}A_{n+1}^*\|^2\right) < \frac{1}{2}; \quad (3.4)
\]

(v) for some \(N \in \mathbb{N}_0\)
\[
\sup_{n \geq N} \left(\|A_n^{-1}A_{n+1}^{-1}\|^2 + \|A_n^{-1}A_{n+1}^*\|^2\right) < \frac{1}{2}; \quad (3.5)
\]

(vi) operator \(A\) be positive definite and for some \(N \in \mathbb{N}_0\)
\[
\sup_{n \geq N} \left(\|A_n^{-1/2}A_{n+1}^{-1/2}\|^2 + \|A_n^{-1/2}A_{n+1}^*\|^2\right) < 1; \quad (3.6)
\]

(vii) operator \(A\) be positive definite and for some \(N \in \mathbb{N}_0\)
\[
\sup_{n \geq N} \|A_n^{-1/2}A_{n+1}^{-1/2}\| < \frac{1}{2}; \quad (3.7)
\]

(viii) operator \(A\) be positive definite and for some \(N \in \mathbb{N}_0\)
\[
\sup_{n \geq N} \left(\|A_n^{-1/2}A_{n+1}^{-1/2}\|^2 + \|A_n^{-1/2}A_{n+1}^*\|^2\right) < \frac{1}{2}. \quad (3.8)
\]

Then the operator \(J\) is self-adjoint and also has a discrete spectrum.

Proof. In accordance with the proofs of Theorems 2.2, 2.3 and Corollaries 2.3, 2.4 operator \(K\) of the form \((2.7)\) is strongly subordinate to \(A = A^*\) under each of the conditions (i) – (v). It remains to apply Lemma 3.1.

In accordance with the proof of Theorem 2.4 operator \(K\) of the form \((2.7)\) is strongly subordinate to \(A = A^*\) in the sense of forms under each of the conditions (vi) – (viii). It remains to apply Lemma 3.2.

We present one more condition for the discreteness of the spectrum of the Jacobi matrix, announced (in a weaker form) in [15].
Theorem 3.5. Let $J$ be the minimal Jacobi operator associated in $l^2(N_0;\mathbb{C}^p)$ with a block Jacobi matrix of the form (2.1), $A := \text{diag}\{A_0, A_1, \ldots, A_n, \ldots\}(= A^*)$, and $|A_n| := \sqrt{A_n^2}$.

(i) Let $0 \in \rho(A)$ and for some $N \in N_0$ condition

$$\sup_{n \geq N} \left\| |A_{n+1}|^{-1/2} \cdot B_n \cdot |A_n|^{-1/2} \right\| < \frac{1}{2}$$

(3.9)

holds. Then the operator $J$ is selfadjoint in $l^2(N_0;\mathbb{C}^p)$ and $0 \in \rho(J)$.

(ii) If in addition $A^{-1} \in S_p(l^2(N_0;\mathbb{C}^p))$, with some $p \in (0;\infty]$, then $J^{-1} \in S_p(l^2(N_0;\mathbb{C}^p))$. In particular, if $A$ has a discrete spectrum, i.e., $A^{-1} \in S_{\infty}(l^2(N_0;\mathbb{C}^p))$, then $J$ also has a discrete spectrum.

Proof. (i) Let for definiteness $N = 0$. The general case ($N > 0$) reduces to the case $N = 0$ in the same way as in Theorem 2.2 (see formulas (2.6), (2.11), (2.12)). We introduce the shift operator in $l^2(N_0)$:

$$U e_n = e_{n+1}, \quad U^* e_n = e_{n-1}, \quad e_{-1} = 0, \quad n \in N_0.$$ (3.10)

Next, we introduce the shift operator in $l^2(N_0;\mathbb{C}^p)$ by setting $U_p := U \otimes I_p$, $U^* = U^* \otimes I_p$, where $I_p$ is the unit operator in $\mathbb{C}^p$.

Consider the polar decomposition of the matrix $A$

$$A = \mathcal{J}|A| = |A|^{1/2}\mathcal{J}|A|^{1/2}, \quad \mathcal{J} = \text{sign}A,$$ (3.11)

and set $B := \text{diag}\{B_0, B_1, \ldots, B_n, \ldots\}$. With this notation, the Jacobi matrix $J$ admits the factorization

$$J = A + U_p B + B U_p^* = \mathcal{J}|A|^{1/2}(I_p + \mathcal{J}|A|^{-1/2}[U_p B + B U_p^*]|A|^{-1/2})|A|^{1/2}.$$ (3.12)

Let further,

$$K_1 := \mathcal{J}|A|^{-1/2}U_p B|A|^{-1/2} \quad \text{and} \quad K_2 := \mathcal{J}|A|^{-1/2}B U_p^*|A|^{-1/2}.$$ 

Hence, $K_1^* := |A|^{-1/2}B U_p^*|A|^{-1/2}\mathcal{J}$ and $K_2 = \mathcal{J}K_1^*\mathcal{J}$. Hence it follows that

$$\|K_1\| = \|K_2\|.$$ (3.13)

It’s obvious that

$$K_1 = \mathcal{J} \begin{pmatrix} 0_p & 0_p & \ldots & 0_p \\ |A_1|^{-1/2}B_0 \cdot |A_0|^{-1/2} & 0_p & \ldots & 0_p \\ 0_p & 0_p & \ldots & 0_p \\ \vdots & \ldots & \ldots & \ldots \\ 0_p & \ldots & |A_{n+1}|^{-1/2}B_n \cdot |A_n|^{-1/2} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$ (3.14)

In the matrix (3.14) only the elements of the first diagonal located below the main diagonal are nonzero. By (3.9) the formulas (3.13) and (3.14) yield the estimates

$$\|K_1\| = \|K_2\| = \sup_{n \geq N} \left\| |A_{n+1}|^{-1/2} \cdot B_n \cdot |A_n|^{-1/2} \right\| < \frac{1}{2}.$$ (3.15)

Assuming $K := K_1 + K_2$, we get $\|K\| < 1$ from (3.15). Hence the operator $I_p + \mathcal{J}K$ is invertible, i.e., $0 \in \rho(I_p + \mathcal{J}K)$. Therefore, it follows from the formulas (3.12) and (3.15) that

$$J^{-1} = |A|^{-1/2}(I_p + \mathcal{J}K)^{-1}|A|^{-1/2}\mathcal{J} \in \mathcal{B}(l^2(N_0;\mathbb{C}^p)).$$
and, hence, \( J = J^* \).

(ii) Let now \( A^{-1} \in S_p(l^2(\mathbb{N}_0; \mathbb{C}^p)) \). Then \( |A|^{-1/2} \in S_{2p}(l^2(\mathbb{N}_0; \mathbb{C}^p)) \) and

\[
J^{-1} = |A|^{-1/2}(I_p + J)K^{-1}|A|^{-1/2}J \in S_p(l^2(\mathbb{N}_0; \mathbb{C}^p)).
\]

In particular, the condition \( A^{-1} \in S_{\infty}(l^2(\mathbb{N}_0; \mathbb{C}^p)) \) implies the inclusion \( J^{-1} \in S_{\infty}(l^2(\mathbb{N}_0; \mathbb{C}^p)) \), which means discreteness of the spectrum of the operator \( J \).

\[
\lim_{n \to \infty} |a_n| = \infty, \quad \lim_{n \to \infty} \frac{b_n^2}{a_n a_{n+1}} < \frac{1}{4}, \quad a_n > 0, \quad b_n \in \mathbb{R}, \quad n \in \mathbb{N}.
\]

The first condition means that the diagonal operator is discrete, while the second condition obviously coincides with condition (3.7) for \( p = 1 \). In other terms, the discreteness condition in the scalar case is contained in (3.7).

4 Application to the Schrödinger operator with \( \delta \)-interactions

We apply Theorem 3.4 to the Schrödinger operator with matrix point interactions associated with the formal differential expression

\[
\ell_{X,\alpha} := -\frac{d^2}{dx^2} + \sum_{x_n \in X} \alpha_n \delta(x - x_n).
\]  

(4.1)

The following differential operator

\[
H_{X,\alpha}^0 := -\frac{d^2}{dx^2} \otimes I_p,
\]

(4.2)

is associated with an expression \( \frac{d^2}{dx^2} \) in \( L^2(\mathbb{R}_+; \mathbb{C}^p) \). Let \( H_{X,\alpha} \) be the closure of \( H_{X,\alpha}^0 \). Here \( X = \{x_n\}_{n=1}^{\infty} \subset \mathcal{I} = (0, b), b \leq \infty \) is a strictly increasing sequence with \( x_0 := 0, x_{n+1} > x_n, x_n \to b \), and \( \{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{C}^{p \times p} \), and \( \delta \) is a Dirac delta-function. In the papers \([33, 32]\) for \( p = 1 \), and in \([35]\) for arbitrary \( p \geq 1 \), it was proved that many spectral properties of the minimal operator \( H_{X,\alpha} \), associated with expression (4.1), are identical to the corresponding spectral properties of the minimal Jacobi operator \( J_{X,\alpha}(H) \), associated in \( l^2(\mathbb{N}_0; \mathbb{C}^p) \) with a block Jacobi matrix of the form

\[
J_{X,\alpha}(H) = \begin{pmatrix}
\frac{1}{r_1} \tilde{\alpha}_1 & -\frac{1}{r_1 r_2 d_2} \mathbb{I}_p & \mathbb{O}_p & \mathbb{O}_p & \mathbb{O}_p & \cdots \\
\frac{1}{r_1 r_2 d_2} \mathbb{I}_p & \frac{1}{r_2} \tilde{\alpha}_2 & -\frac{1}{r_2 r_3 d_2} \mathbb{I}_p & \mathbb{O}_p & \mathbb{O}_p & \cdots \\
\mathbb{O}_p & -\frac{1}{r_2 r_3 d_3} \mathbb{I}_p & \frac{1}{r_3} \tilde{\alpha}_3 & -\frac{1}{r_3 r_4 d_2} \mathbb{I}_p & \mathbb{O}_p & \cdots \\
\mathbb{O}_p & \mathbb{O}_p & -\frac{1}{r_3 r_4 d_3} \mathbb{I}_p & \frac{1}{r_4} \tilde{\alpha}_4 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\
\end{pmatrix}
\]

(4.3)

Here \( d_n := x_n - x_{n-1}, r_n := \sqrt{d_n + d_{n+1}} \) and

\[
\tilde{\alpha}_n := \alpha_n + \left( \frac{1}{d_n} + \frac{1}{d_{n+1}} \right) \mathbb{I}_p, \quad n \in \mathbb{N}.
\]

In particular, it was proved in papers \([33, 32]\) (for \( p = 1 \)) and \([35]\) (for \( p > 1 \)) that: (a) \( n_{\pm}(H_{X,\alpha}) = n_{\pm}(J_{X,\alpha}(H)) \); (b) in the case \( J_{X,\alpha}(H) = J_{X,\alpha}(J) \) for \( \lim_{n \to \infty} d_n = 0 \), operators \( H_{X,\alpha} \) and \( J_{X,\alpha}(H) \) are discrete simultaneously.
Theorem 4.1. Let $\lim_{n \to \infty} d_n = 0$ and let the diagonal part $A := \text{diag} \{ \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots \}$ of the matrix \[4.3\] have a discrete spectrum and matrices $\tilde{\alpha}_n$ are invertible for some $N \in \mathbb{N}$ for $n \geq N$. Let also at least one of the following conditions be satisfied:

(i) $\sup_{n \geq N} \frac{r_n}{k_n} \| \tilde{\alpha}_n^{-1} \| < \frac{1}{2}$, \[4.4\]

(ii) $\sup_{n \geq N} r_n^2 \left( \frac{1}{r_{n-1}^2 d_n^2} + \frac{1}{r_{n+1}^2 d_{n+1}^2} \right) \| \tilde{\alpha}_n^{-1} \|^2 < \frac{1}{2}$; \[4.5\]

(iii) $\sup_{n \geq N} r_n^2 \left( \frac{r_n^2}{d_{n+1}^2} \| \tilde{\alpha}_n^{-1} \|^2 + \frac{r_{n+2}^2}{d_{n+2}^2} \| \tilde{\alpha}_{n+2}^{-1} \|^2 \right) < \frac{1}{2}$; \[4.6\]

(iv) $\sup_{n \geq N} \frac{1}{N d_n} \| \tilde{\alpha}_n^{-1/2} \cdot \tilde{\alpha}_n^{-1/2} \| < \frac{1}{2}$ \[4.7\]

Then the operator $H_{X,\alpha}$ is selfadjoint and has a discrete spectrum.

In addition, equality $H_{X,\alpha} = H_{X,\alpha}^*$ is preserved when replacing in any of relations \[4.4\], \[4.5\], \[4.6\] the inequality sign with the equality sign.

Proof. We only prove item (i). From \[4.4\] we easily obtain

$$\sup_{n \geq N} \| A_n^{-1} \cdot B_n \| = \sup_{n \geq N} r_n^2 \cdot \| \tilde{\alpha}_n^{-1} \| \cdot \frac{1}{r_n r_{n+1} d_{n+1}} = \sup_{n \geq N} r_n \cdot \| \tilde{\alpha}_n^{-1} \| < \frac{1}{2};$$ \[4.8\]

$$\sup_{n \geq N} \| A_n^{-1} \cdot B_{n-1} \| = \sup_{n \geq N} r_n^2 \cdot \| \tilde{\alpha}_n^{-1} \| \cdot \frac{1}{r_{n-1} r_n d_n} = \sup_{n \geq N} r_n \cdot \| \tilde{\alpha}_n^{-1} \| < \frac{1}{2};$$ \[4.9\]

Thus, we have verified the estimate \[4.3\] of Theorem 3.4.

Similarly, conditions \[4.5\], \[4.6\] and \[4.7\] imply the estimates \[3.4\], \[3.5\] and \[3.7\] respectively. \qed

Remark 4.2. (i) The selfadjointness (but not discreteness) of the Hamiltonian $H_{X,\alpha}$ was established in [37] for $\{d_n^2\}_1^\infty \notin l^1$ and in [10] under the condition \[4.6\], where it is removed from Theorem 2.5(ii).

(ii) In the papers [32, 33] for the case $p = 1$, and in [32, 40] for $p \geq 1$ it is shown that the condition

\[4.10\]

guarantees the maximum of the deficiency indices: $n_\pm (H_{X,\alpha}) = p$. Evaluation \[4.10\] is extracted from the results of Kostyuchenko and Mirzoev [27]. The condition \[4.10\] shows the precision of the conditions \[4.4\] - \[4.7\]. Note also that for $p = 1$ the first example of the Hamiltonian $H_{X,\alpha}$ with nontrivial indices $n_\pm (H_{X,\alpha}) = 1$ is given in the paper [53] Shubin Christ and Stolz. Namely, they proved that $n_\pm (H_{X,\alpha}) = 1$ for $d_n = \frac{1}{n}$ and $\alpha_n = -2n - 1$ (in this case $\tilde{\alpha}_n = 0$), $n \in \mathbb{N}$.

5 Abstract results on deficiency indices of perturbed Jacobi matrices

Here, along with the matrix \[2.1\], we consider the Jacobi matrix

$$J = \begin{pmatrix} \hat{A}_0 & \hat{B}_0 & 0_p & 0_p & \ldots & 0_p & 0_p & 0_p & \ldots \\ \hat{B}_0^* & \hat{A}_1 & 0_p & 0_p & \ldots & 0_p & 0_p & 0_p & \ldots \\ 0_p & \hat{B}_1^* & \hat{A}_2 & 0_p & \ldots & 0_p & 0_p & 0_p & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ 0_p & 0_p & 0_p & \ldots & \hat{B}_{n-1}^* & \hat{A}_n & 0_p & 0_p & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$ \[5.1\]
where \( \hat{A}_n = \hat{A}^*_n \), \( \hat{B}_n \in \mathbb{C}^{p \times p} \) and \( \det \hat{B}_n \neq 0 \), \( n \in \mathbb{N}_0 \).

As usual we identify the matrix \( \hat{J} \) with the minimal (closed) symmetric Jacobi operator generated in \( l^2(\mathbb{N}; \mathbb{C}^p) \) by the matrix \( (5.1) \) (see [1, 4]).

**Theorem 5.1.** Let \( J \) and \( \hat{J} \) be the Jacobi matrices given by \( (2.1) \) and \( (5.1) \), respectively. Let for some \( N \in \mathbb{N}_0 \) the following conditions hold:

\[
(i) \quad \sup_{n \geq N} \| \mathbb{I}_p - \hat{B}^*_n (B^*_n)^{-1} \|_{\mathbb{C}^{p \times p}} = a_N < 1;
\]

\[
(ii) \quad \sup_{n \geq N} \| \hat{B}_n - \hat{B}^*_n (B^*_n)^{-1} B_n \|_{\mathbb{C}^{p \times p}} = C_B < \infty;
\]

\[
(iii) \quad \text{Let for each } \varepsilon > 0 \text{ there exist } N_1 = N_1(\varepsilon) \in \mathbb{N} \text{ and } C'_A(\varepsilon) > 0 \text{ such that for each vector } f \in l^2_0(\mathbb{N}_0; \mathbb{C}^p)
\]

\[
\sum_{n \geq N_1} \| (\hat{A}_n - \hat{B}^*_n (B^*_n)^{-1} A_n) f_n \|_{\mathbb{C}^p}^2 \leq \varepsilon \| J f \|_{l^2}^2 + C'_A(\varepsilon) \| f \|_{l^2}^2.
\]

Then:

(a) \( \text{dom} \ J = \text{dom} \hat{J} \) and \( n_\pm(J) = n_\pm(\hat{J}) \).

(b) if moreover \( J = J^* \) the spectrum of \( J \) is discrete, then the spectrum of the perturbed Jacobi operator \( \hat{J} = J^* \) is discrete too.

**Proof.** (a) (1) Let \( \delta_0 := 1 - a_N^2 > 0 \). Choose \( \varepsilon_0 < \frac{\delta_0}{1 - \delta_0} \) and

\[
\varepsilon_1 < \frac{\varepsilon_0}{2} \left( \frac{1}{1 + \varepsilon_0} - a_N^2 \right).
\]

Then we can find \( N_1 = N_1(\varepsilon_1) \) such that condition \( (5.3) \) holds with \( \varepsilon_1 \) instead of \( \varepsilon \) and choose \( M \geq \max\{N, N_1\} \). Finally, we define the block Jacobi submatrix \( J_M \) of the matrix \( J \) of the form \( (2.6) \) with \( M \) instead of \( N \). Similarly, we define the block Jacobi matrix \( \hat{J}_M \) by the same formula \( (2.6) \) with \( \hat{A}_k, \hat{B}_k \) and \( \hat{B}^*_k \) instead of \( A_k, B_k \) and \( B^*_k \), respectively. Summing up we obtain from \( (2.1), (2.6), (5.1) - (5.5) \) that for any finite vector \( f = (f_0 \ f_1 \ \ldots \ f_m) \top \ (\in l^2_0(\mathbb{N}_0; \mathbb{C}^p)) \)

\[
\| (\hat{J}_M - J_M) f \|_{l^2(\mathbb{N}_0; \mathbb{C}^p)}^2
\]

\[
= \sum_{n \geq M} \| (B^*_n - \hat{B}^*_n (B^*_n)^{-1} f_{n-1} + (A_n - \hat{A}_n) f_n + (B_n - \hat{B}_n) f_{n+1}) \|_{\mathbb{C}^p}^2
\]

\[
\leq (1 + \varepsilon_0) \sum_{n \geq M} \| (\mathbb{I}_p - \hat{B}^*_n (B^*_n)^{-1} ) (B^*_n - \hat{B}^*_n (B^*_n)^{-1} f_{n-1} + A_n f_n + B_n f_{n+1}) \|^2_{\mathbb{C}^p}
\]

\[
+ \left( 1 + \frac{1}{\varepsilon_0} \right) \sum_{n \geq M} \| (A_n - \hat{A}_n) f_n - (\mathbb{I}_p - \hat{B}^*_n (B^*_n)^{-1} A_n f_n + (B_n - \hat{B}_n) f_{n+1}
\]

\[
- (\mathbb{I}_p - \hat{B}^*_n (B^*_n)^{-1} B_n f_{n+1}) \|_{\mathbb{C}^p}^2.
\]
Corollary 5.2. Let for some \( N \in \mathbb{N}_0 \) conditions \((5.2)\) and \((5.3)\) of Theorem 5.1 be fulfilled. Assume also the following condition holds

\[
\sup_{n \geq N} \| \hat{A}_n - \hat{B}_{n-1}^*(B_{n-1}^*)^{-1} A_n \|_{C_p \times p} = C_A < \infty. \tag{5.9}
\]
Then $\text{dom } \mathbf{J} = \text{dom } \hat{\mathbf{J}}$ and $n_\pm(\mathbf{J}) = n_\pm(\hat{\mathbf{J}})$.

Proof. Condition (5.9) implies condition (5.4) with $\varepsilon = 0$. \hfill \square

**Corollary 5.3.** Let $\mathbf{J}$ and $\hat{\mathbf{J}}$ be as above and let $\mathcal{A}_n = \hat{\mathcal{A}}_n = \mathbb{O}_p$, $n \in \mathbb{N}$. Then $n_\pm(\mathbf{J}) = n_\pm(\hat{\mathbf{J}})$ provided that conditions (5.2) and (5.3) are satisfied.

**Part II. Dirac operators and associated Jacobi matrices**

### 6 Dirac operators on a finite interval with maximal deficiency indices

Consider one–dimensional Dirac operator associated with the differential expression

$$D := -ic \frac{d}{dx} \otimes \left( \begin{array}{cc} \mathbb{O}_p & \mathbb{I}_p \\ \mathbb{I}_p & \mathbb{O}_p \end{array} \right) + \frac{c^2}{2} \otimes \left( \begin{array}{cc} \mathbb{I}_p & \mathbb{O}_p \\ \mathbb{O}_p & -\mathbb{I}_p \end{array} \right)$$

in $L^2(\mathcal{I}; \mathbb{C}^{2p})$. Here $c > 0$ denotes the velocity of light.

Let $d_n := x_n - x_{n-1} > 0$. We also assume $X = \{x_n\}_{n=0}^\infty (\subset \mathcal{I})$ to be a discrete subset of the interval, $x_{n-1} < x_n$, $n \in \mathbb{N}$. We set $x_0 = 0$, $b := \sup X \equiv \lim_{n \to \infty} x_n < \infty$.

Following [28] (see also [2]) we define two families of symmetric extensions, which turn out to be closely related to their non-relativistic counterparts $\delta$- and $\delta'$–interactions.

To this end we set $f = \{f_1, f_2, \ldots, f_{2p}\}^\top \in L^2(\mathcal{I}; \mathbb{C}^{2p})$ and

$$f = \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right), \quad f_I := \{ f_1, f_2 \ldots f_p \}^\top, \quad f_{II} := \{ f_{p+1}, f_{p+2} \ldots f_{2p} \}^\top.$$  

Here $\top$ means a transpose operation. We define the following Sobolev spaces

$$W^{1,2}(\mathcal{I} \setminus X; \mathbb{C}^{2p}) := \bigoplus_{n=1}^{\infty} W^{1,2}([x_{n-1}, x_n]; \mathbb{C}^{2p}),$$

$$W^{1,2}_{\text{comp}}(\mathcal{I} \setminus X; \mathbb{C}^{2p}) := \{ f \in W^{k,2}(\mathcal{I} \setminus X; \mathbb{C}^{2p}) : \text{supp } f \text{ is compact in } \mathcal{I} \}.$$  

Two families of Gesztesy–Šeb operators (in short, GS-operators or GS-realizations) on the interval $(a, b)$ are defined to be the closures of the operators

$$\text{D}^0_{X,\alpha} = D \upharpoonright \text{dom}(\text{D}^0_{X,\alpha}),$$

$$\text{dom}(\text{D}^0_{X,\alpha}) = \{ f \in W^{1,2}_{\text{comp}}(\mathcal{I} \setminus X; \mathbb{C}^{2p}) : f_I \in AC_{\text{loc}}(\mathcal{I}), \quad f_{II} \in AC_{\text{loc}}(\mathcal{I} \setminus X);$$

$$f_{II}(a+) = 0, \quad f_{II}(x_n^+) - f_{II}(x_n^-) = -\frac{i \alpha}{c} f_I(x_n), \quad n \in \mathbb{N} \}, \quad (6.2)$$

and

$$\text{D}^0_{X,\beta} = D \upharpoonright \text{dom}(\text{D}^0_{X,\beta}),$$

$$\text{dom}(\text{D}^0_{X,\beta}) = \{ f \in W^{1,2}_{\text{comp}}(\mathcal{I} \setminus X; \mathbb{C}^{2p}) : f_I \in AC_{\text{loc}}(\mathcal{I} \setminus X), \quad f_{II} \in AC_{\text{loc}}(\mathcal{I});$$

$$f_{II}(a+) = 0, \quad f_I(x_n^+) - f_I(x_n^-) = i \beta c f_{II}(x_n), \quad n \in \mathbb{N} \}, \quad (6.3)$$

respectively, i.e. \text{D}^0_{X,\alpha} = \overline{\text{D}^0_{X,\alpha}} and \text{D}^0_{X,\beta} = \overline{\text{D}^0_{X,\beta}}. It is easily seen that both operators \text{D}^0_{X,\alpha} and \text{D}^0_{X,\beta} are symmetric, but not necessarily self-adjoint, in general.

In the sequel we need the following proposition established in [17] ($p = 1$) and [14, 15] ($p > 1$).
Proposition 6.1 ([17, 14, 15]). Let $D_{X,\alpha}$ be realization of the Dirac operator in $L^2(\mathcal{I}; \mathbb{C}^{2p})$. Then the operator $D_{X,\alpha}^* := (D_{X,\alpha})^*$ adjoint to the symmetric operator $D_{X,\alpha}$ is given by

$$D_{X,\alpha}^* = D \mid \text{dom}(D_{X,\alpha}^*),$$

$$\text{dom}(D_{X,\alpha}^*) = \left\{ f \in W^{1,2}(\mathcal{I} \setminus X; \mathbb{C}^{2p}) : f_I \in AC_{\text{loc}}(\mathcal{I}), \ f_{II} \in AC_{\text{loc}}(\mathcal{I} \setminus X) ; \right. \left. f_{II}(a+) = 0, \ f_{II}(x_n^+) - f_{II}(x_n^-) = -\frac{i\alpha_n}{c} f_{II}(x_n), \ n \in \mathbb{N} \right\}.$$  \hspace{1cm} (6.4)

Similarly, the operator $D_{X,\beta}^*$ adjoint to $D_{X,\beta}$ is given by the expression [6.3] with $W^{1,2}_{\text{comp}}(\mathcal{I} \setminus X; \mathbb{C}^{2p})$ replaced by $W^{1,2}(\mathcal{I} \setminus X; \mathbb{C}^{2p})$.

6.1 Realizations $D_{X,\alpha}$ with maximal deficiency indices

In this section assuming that $|\mathcal{I}| < \infty$, we construct $2p \times 2p$-matrix GS-realizations $D_{X,\alpha}$ with maximal deficiency indices $n_\pm(D_{X,\alpha}) = p$. Our main result in this direction reads as follows.

Theorem 6.2. Let $\alpha := \{\alpha_n\}_{n=1}^\infty \subset \mathbb{C}^{p \times p}$ be a sequence of selfadjoint $p \times p$-matrices, $\alpha_n = \alpha_n^*, \ n \in \mathbb{N}$. Assume that the following condition is fulfilled:

$$\sum_{n=2}^{\infty} d_n \prod_{k=1}^{n-1} \left( 1 + \frac{1}{c} \|\alpha_k\|_{\mathbb{C}^{p \times p}} \right)^2 < +\infty.$$  \hspace{1cm} (6.5)

Then $n_\pm(D_{X,\alpha}) = p$.

Proof. (i) We examine the operator

$$T_{X,\alpha} := D_{X,\alpha} - \frac{c^2}{2} \otimes \begin{pmatrix} I_p & O_p \\ O_p & -I_p \end{pmatrix} = -i c \frac{d}{dx} \otimes \begin{pmatrix} O_p & I_p \\ I_p & O_p \end{pmatrix}$$  \hspace{1cm} (6.6)

since obviously $n_\pm(D_{X,\alpha}) = n_\pm(T_{X,\alpha})$. It suffices to show that under the assumption [6.5] the equation $(T_{X,\alpha}^* \pm i)F = 0$ has a non-trivial matrix $L^2(\mathcal{I}, \mathbb{C}^{2p \times p})$-solution of full rank, i.e. rank $F(x) = p$.

We restrict ourselves to the case of equation $(T_{X,\alpha}^* + i)F = 0$, which is equivalent to the system

$$\frac{d}{dx} F = \frac{1}{c} A F,$$  \hspace{1cm} (6.7)

Here $F = \begin{pmatrix} F^1 & F^2 & \ldots & F^p \end{pmatrix} \in \mathbb{C}^{2p \times p}$ and

$$F^j = \begin{pmatrix} F^j_I \\ F^j_{II} \end{pmatrix}, \quad F_I^j = \begin{pmatrix} f_1^j \\ \vdots \\ f_p^j \end{pmatrix}, \quad F_{II}^j = \begin{pmatrix} f_{p+1}^j \\ \vdots \\ f_{2p}^j \end{pmatrix}, \quad j \in \{1, \ldots, p\}.$$  \hspace{1cm}

Equation (6.7) has the following piecewise smooth $2p \times p$-matrix solutions

$$F = \bigoplus_{n=1}^{\infty} \begin{pmatrix} F_{I,n} \\ F_{II,n} \end{pmatrix},$$

where

$$F_I = \bigoplus_{n=1}^{\infty} F_{I,n}, \quad F_{I,n}(x) = U_n e^{-(x_n-x)/c} + V_n e^{(x_n-x)/c}, \quad x \in [x_{n-1}, x_n],$$  \hspace{1cm} (6.8)
It follows from (6.15), that for each

\[ F_{II} = \bigoplus_{n=1}^{\infty} F_{II,n}, \quad F_{II,n}(x) = U_n e^{-(x_n-x)/c} - V_n e^{(x_n-x)/c}, \quad x \in [x_{n-1}, x_n], \]  

(6.9)

where \( \{U_n\}_1^\infty \subset \mathbb{C}^{p \times p}, \{V_n\}_1^\infty \subset \mathbb{C}^{p \times p} \).

According to the description of \( \text{dom}(D_{X,\alpha}^*) \) (see Proposition 6.1), each component \( F^j \) of \( F = (F_I, F_{II}) \), \( j \in \{1, \ldots, p\} \), should satisfy boundary conditions (6.4). First we find recursive relations for matrix sequences \( \{U_n\}_1^\infty, \{V_n\}_1^\infty \) that ensure satisfying these conditions.

The condition \( F_{II,n}(x_0^+) = \mathbb{O}_p \) yields \( U_1 e^{-d_1/c} - V_1 e^{d_1/c} = \mathbb{O}_p \). Further, the condition

\[ F_{I,n}(x_n^+) = F_{I,n}(x_n^-), \quad n \in \mathbb{N}, \]

due to the formulas (6.8), (6.9) is equivalent to

\[ U_{n+1} e^{-d_{n+1}/c} + V_{n+1} e^{d_{n+1}/c} = U_n + V_n, \quad n \in \mathbb{N}. \]  

(6.10)

Moreover, the jump condition

\[ F_{II,n}(x_n^+) - F_{II,n}(x_n^-) = -i \frac{\alpha_n}{c} F_{I,n}(x_n), \quad n \in \mathbb{N}, \]

due to the formulas (6.8), (6.9) is transformed into

\[ U_{n+1} e^{-d_{n+1}/c} - V_{n+1} e^{d_{n+1}/c} - (U_n - V_n) = -i \frac{\alpha_n}{c} (U_n + V_n). \]  

(6.11)

Combining (6.10) and (6.11) we arrive to the following recursive equations

\[ U_{n+1} = \left( U_n - i \frac{\alpha_n}{2c} (U_n + V_n) \right) e^{d_{n+1}/c}, \quad n \in \mathbb{N}, \]

(6.12)

\[ V_{n+1} = \left( V_n + i \frac{\alpha_n}{2c} (U_n + V_n) \right) e^{-d_{n+1}/c}, \quad n \in \mathbb{N}, \]

(6.13)

for sequences \( \{U_n\}_1^\infty \) and \( \{V_n\}_1^\infty \) with the following initial data

\[ U_1 = e^{d_1/c} \mathbb{I}_p \quad \text{and} \quad V_1 = e^{-d_1/c} \mathbb{I}_p. \]  

(6.14)

(ii) Let us prove that rank \( F_n(x) = p \) for each \( x \in [x_{n-1}, x_n] \) and \( n \in \mathbb{N} \). It follows from (6.3) and (6.9) that

\[
\begin{pmatrix}
F_{I,n}(x) \\
F_{II,n}(x)
\end{pmatrix} \cdot
\begin{pmatrix}
F_{I,n}(x) \\
F_{II,n}(x)
\end{pmatrix}
= \begin{pmatrix}
U_n e^{-(x_n-x)/c} + V_n e^{(x_n-x)/c} \\
U_n e^{-(x_n-x)/c} - V_n e^{(x_n-x)/c}
\end{pmatrix}
\begin{pmatrix}
U_n e^{-(x_n-x)/c} + V_n e^{(x_n-x)/c} \\
U_n e^{-(x_n-x)/c} - V_n e^{(x_n-x)/c}
\end{pmatrix}
= 2(U_n^* U_n e^{-2(x_n-x)/c} + V_n^* V_n e^{2(x_n-x)/c} - U_n^* V_n - V_n^* U_n) = 2(U_n^* U_n e^{-2(x_n-x)/c} + V_n^* V_n e^{2(x_n-x)/c}).
\]

(6.15)

It follows from (6.15), that for each \( h \in \mathbb{C}^p \) and \( x \in [x_{n-1}, x_n] \)

\[ \left\| \begin{pmatrix}
F_{I,n}(x) \\
F_{II,n}(x)
\end{pmatrix} h \right\|^2_{\mathbb{C}^p} = 2e^{-2(x_n-x)/c} \|U_n h\|^2_{\mathbb{C}^p} + 2e^{2(x_n-x)/c} \|V_n h\|^2_{\mathbb{C}^p}. \]

(6.16)

It follows that for any fixed \( a \in [x_{n-1}, x_n] \) the following important equivalence holds

\[ F_n(a) h = 0 \iff U_n h = V_n h = 0. \]  

(6.17)
Let us prove by induction that rank $F_n(x) = p$ for each $x \in [x_{n-1}, x_n]$. For $n = 1$ it follows from (6.14). Assuming that rank $F_k(x) = p$ for each $x \in [x_{k-1}, x_k]$ let us prove that rank $F_{k+1}(x) = p$ for $x \in [x_k, x_{k+1}]$. Assuming the contrary we find $\hat{x} \in [x_k, x_{k+1}]$ and vector $h \in \mathbb{C}^p$ such that $F_{k+1}(\hat{x})h = 0$. Combining this relation with equivalence (6.17) (with $a = \hat{x}$ and $n = k + 1$) yields $U_{k+1}h = V_{k+1}h = 0$.

In turn, inserting these relations in (6.10) and (6.11) implies

$$U_kh = V_kh = 0.$$ 

Due to equivalence (6.17) these relations are equivalent to $F_k(x)h = 0$ for $x \in [x_{k-1}, x_k]$ which contradicts the induction hypothesis. Thus, rank $F(x) = p$ for each $x \in \mathbb{R}_+$. Note also that in passing we have proved that the matrices $F_1(x)$ and $F_{II}(x)$ are nonsingular for each $x \in \mathbb{R}_+$.

(iii) It remains to check that under condition (6.5) the inclusion $F_1, F_{II} \in L^2(I; \mathbb{C}^{p \times p})$ holds. It follows from (6.8) and (6.9)

$$
\|F_k\|^2 = \sum_{n=1}^{\infty} \|F_{k,n}\|^2 \leq 2 \sum_{n=1}^{\infty} \int_0^{d_n} \left( \|U_n\|^2_{\mathbb{C}^p \times \mathbb{C}^p} e^{-2d_n/x} + \|V_n\|^2_{\mathbb{C}^p \times \mathbb{C}^p} e^{2d_n/x} \right) dx

= c \sum_{n=1}^{\infty} \left( \|U_n\|^2_{\mathbb{C}^p \times \mathbb{C}^p}(1 - e^{-2d_n/x}) + \|V_n\|^2_{\mathbb{C}^p \times \mathbb{C}^p}(e^{2d_n/x} - 1) \right), \quad k = \{I, II\}.
$$

Since $\sum_{n=1}^{\infty} d_n = |I| < +\infty$, $d_n \to 0$ and therefore $(1 - e^{-2d_n/x}) \sim (e^{2d_n/x} - 1) \sim 2d_n/x$ as $n \to \infty$. This implies inequality $\|F_k\|_2 < +\infty$ provided that

$$
\sum_{n=1}^{\infty} \left( \|U_n\|^2_{\mathbb{C}^p \times \mathbb{C}^p} + \|V_n\|^2_{\mathbb{C}^p \times \mathbb{C}^p} \right) d_n < +\infty.
$$

Let us prove by induction the following estimates

$$
\|U_{n+1}\|_{\mathbb{C}^p \times \mathbb{C}^p}, \quad \|V_{n+1}\|_{\mathbb{C}^p \times \mathbb{C}^p} \leq \exp((d_1 + \ldots + d_{n+1})/c) \cdot \prod_{k=1}^{n} \left( 1 + \frac{\|\alpha_k\|_{\mathbb{C}^p \times \mathbb{C}^p}}{c} \right), \quad n \in \mathbb{N}.
$$

For $n = 1$ these estimates are obvious. Assume that (6.19) are proved for $n \leq m - 1$. Then using (6.12) for $n = m$ we obtain

$$
\|U_{m+1}\|_{\mathbb{C}^p \times \mathbb{C}^p} \leq \left( \|U_m\|_{\mathbb{C}^p \times \mathbb{C}^p} \left( 1 + \frac{\|\alpha_m\|_{\mathbb{C}^p \times \mathbb{C}^p}}{2c} \right) + \frac{\|\alpha_m\|_{\mathbb{C}^p \times \mathbb{C}^p}}{2c} \|V_m\|_{\mathbb{C}^p \times \mathbb{C}^p} \right) e^{d_{m+1}/c}

\leq \prod_{k=1}^{m-1} \left( 1 + \frac{\|\alpha_k\|_{\mathbb{C}^p \times \mathbb{C}^p}}{c} \right) \left[ \left( 1 + \frac{\|\alpha_m\|_{\mathbb{C}^p \times \mathbb{C}^p}}{2c} \right) + \frac{\|\alpha_m\|_{\mathbb{C}^p \times \mathbb{C}^p}}{2c} \right] e^{(d_1 + \ldots + d_{m+1})/c}

= \exp((d_1 + \ldots + d_{m+1})/c) \cdot \prod_{k=1}^{m} \left( 1 + \frac{\|\alpha_k\|_{\mathbb{C}^p \times \mathbb{C}^p}}{c} \right).
$$

This inequality proves the inductive hypothesis (6.19) for $U_n$. The estimate for $V_{m+1}$ is proved similarly. Thus, both inequalities (6.19) are established. Combining (6.18) with (6.19) and the assumption (6.5) we conclude that $F_1, F_{II} \in L^2(I; \mathbb{C}^{p \times p})$.

**Corollary 6.3.** If condition (6.5) is fulfilled, then any selfadjoint extension of the operator $D_{X, \alpha}$ has discrete spectrum.
Remark 6.4. Clearly, the condition (6.5) implies the inclusion \( \{d_n\} \in l^1(\mathbb{N}) \), i.e. \( |I| < \infty \). Note in this connection that in the opposite case, \( |I| = \infty \), the realization \( D_{X,\alpha} \) is always selfadjoint (see [17, 14, 12]).

Corollary 6.5. The GS realization \( D_{X,\alpha} \) is symmetric with \( n_\pm(D_{X,\alpha}) = p \) whenever \( \{\alpha_n\}_{1}^{\infty} \in l^1(\mathbb{N}; \mathbb{C}^{p \times p}) \).

Proof. Clearly, for any positive sequence \( \{s_k\}_{1}^{\infty} \)
\[
\prod_{k=1}^{\infty} (1 + s_k) \leq \exp \left( \sum_{k=1}^{\infty} s_k \right).
\]
It follows with account of the inclusion \( \{\alpha_n\}_{1}^{\infty} \in l^1(\mathbb{N}; \mathbb{C}^{p \times p}) \) that
\[
\sum_{n=2}^{\infty} d_n \prod_{k=1}^{n-1} \left( 1 + \frac{1}{c} \|\alpha_k\|_{\mathbb{C}^{p \times p}} \right)^2 \leq \exp \left( \frac{2}{c} \sum_{k=1}^{\infty} \|\alpha_k\|_{\mathbb{C}^{p \times p}} \right) \sum_{n=2}^{\infty} d_n \leq |I| \exp \left( \frac{2}{c} \sum_{k=1}^{\infty} \|\alpha_k\|_{\mathbb{C}^{p \times p}} \right).
\]
It remains to apply Theorem 6.2. \( \square \)

Corollary 6.6. The GS realization \( D_{X,\alpha} \) is symmetric with \( n_\pm(D_{X,\alpha}) = p \) whenever
\[
\limsup_{n \to \infty} \frac{d_{n+1}}{d_n} \left( 1 + \frac{\|\alpha_n\|_{\mathbb{C}^{p \times p}}}{c} \right)^2 < 1.
\]
In particular, \( n_\pm(D_{X,\alpha}) = p \) provided that one of the following conditions is satisfied
(i) \( \limsup_{n \to \infty} (d_{n+1}/d_n) = 0 \) and \( \sup_{n \in \mathbb{N}} \|\alpha_n\|_{\mathbb{C}^{p \times p}} < \infty \);
(ii) \( \limsup_{n \to \infty} (d_{n+1}/d_n) =: (1/d) \) with \( d > 1 \) and \( \sup_{n \in \mathbb{N}} \|\alpha_n\|_{\mathbb{C}^{p \times p}} < c(\sqrt{d} - 1) \).

Proof. By the ratio test condition (6.20) yields the convergence of the series (6.5). It remains to apply Theorem 6.2. \( \square \)

6.2 Realizations \( D_{X,\beta} \) with maximal deficiency indices

Here we present similar results for GS-realizations \( D_{X,\beta} \).

Theorem 6.7. Let \( \beta := \{\beta_n\}_{1}^{\infty} \subset \mathbb{C}^{p \times p} \) be a sequence of selfadjoint \( p \times p \)-matrices, \( \beta_n = \beta_n^* \), \( n \in \mathbb{N} \). Assume also that the following condition is fulfilled:
\[
\sum_{n=2}^{\infty} d_n \prod_{k=1}^{n-1} (1 + c \|\beta_k\|_{\mathbb{C}^{p \times p}})^2 < +\infty.
\]
Then \( n_\pm(D_{X,\beta}) = p \).

Proof. The proof is similar to that of Theorem 6.2 and is omitted. \( \square \)

Corollary 6.8. If condition (6.21) is fulfilled, then any selfadjoint extension of the operator \( D_{X,\beta} \) has discrete spectrum.

Corollary 6.9. The GS realization \( D_{X,\beta} \) is symmetric with \( n_\pm(D_{X,\beta}) = p \) whenever \( \{\beta_n\}_{1}^{\infty} \in l^1(\mathbb{N}; \mathbb{C}^{p \times p}) \).

Remark 6.10. Clearly, the condition (6.21) implies the inclusion \( \{d_n\} \in l^1(\mathbb{N}) \), i.e. \( |I| < \infty \). Note in this connection that in the opposite case, \( |I| = \infty \), the realization \( D_{X,\beta} \) is always selfadjoint (see [17, 14, 12]).
Corollary 6.11. The GS realization $D_{X,\beta}$ is symmetric with $n_\pm(D_{X,\beta}) = p$ whenever

$$\limsup_{n \to \infty} \frac{d_{n+1}}{d_n} (1 + c\|\beta_n\|_{C^p})^2 < 1.$$ (6.22)

In particular, $n_\pm(D_{X,\beta}) = p$ provided that one of the following conditions is satisfied

(i) $\limsup_{n \to \infty}(d_{n+1}/d_n) = 0$ and $\sup_{n \in \mathbb{N}} \|\beta_n\|_{C^p} < \infty$;

(ii) $\limsup_{n \to \infty}(d_{n+1}/d_n) = (1/d)$ with $d > 1$ and $\sup_{n \in \mathbb{N}} \|\beta_n\|_{C^p} < c(\sqrt{d} - 1)$.

7 Jacobi matrices generated by Dirac operators with maximal deficiency indices

7.1 Jacobi matrices $\hat{J}_{X,\alpha}$

Here we apply previous results to block Jacobi matrices

$$\hat{J}_{X,\alpha} = \begin{pmatrix}
A_0 & \tilde{v}_0(1_p + B_0') & \tilde{v}_0(1_p + A_1') & \tilde{v}_0(1_p + B_1') & A_p & A_p & \cdots \\
0_p & \tilde{v}_0(1_p + A_1') & \tilde{v}_0(1_p + B_1') & \tilde{v}_0(1_p + A_2') & \tilde{v}_1(1_p + B_2') & \tilde{v}_1(1_p + A_3') & \cdots \\
0_p & 0_p & \tilde{v}_1(1_p + B_2') & \tilde{v}_1(1_p + A_3') & \tilde{v}_1(1_p + B_3') & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots \\
\end{pmatrix},$$ (7.1)

where $\tilde{v}_n := \frac{\nu(d_n+1)}{d_n+1}$, $\hat{v}_n := \frac{\nu(d_{n+1})}{d_{n+1}}$, $\alpha_n = \alpha_n^*$, $A_\alpha' = (A_n')^*$, $B_\alpha' = (B_n')^* \in \mathbb{C}^{p \times p}$ ($n \in \mathbb{N}_0$) and matrices $1_p + B_n'$ are invertible, i.e. $\det(1_p + B_n') \neq 0$, $n \in \mathbb{N}_0$.

$$\nu(x) := \frac{1}{\sqrt{1 + (c^2x^2)^{-1}}} = \frac{cx}{\sqrt{1 + c^2x^2}}.$$ (7.2)

In what follows we keep also the notation $\hat{J}_{X,\alpha}$ for the minimal Jacobi operator associated in a standard way with the matrix $\hat{J}_{X,\alpha}$ in $l^2(\mathbb{N}; \mathbb{C})$ (see [1] [6], and also [39]). Clearly the operator $\hat{J}_{X,\alpha}$ is symmetric and as it is known $n_\pm(\hat{J}_{X,\alpha}) \leq p$ (see [6] [39] [40]).

Our investigation of the deficiency indices of the Jacobi matrices $\hat{J}_{X,\alpha}$ substantially relies on the following result discovered in [17] for $p = 1$ and extended to the case $p > 1$ in [14] [15]. To state it we introduce the following block Jacobi matrix

$$B_{X,\alpha} = \begin{pmatrix}
0_p & -\frac{\nu(d_1)}{d_1^2}1_p & 0_p & 0_p & 0_p & \cdots \\
-\frac{\nu(d_2)}{d_2^2}1_p & 0_p & \frac{\nu(d_1)}{d_1^2}1_p & \frac{\nu(d_3)}{d_3^2}1_p & 0_p & \cdots \\
0_p & \frac{\alpha_1}{d_1^2}1_p & 0_p & -\frac{\nu(d_2)}{d_2^2}1_p & 0_p & \cdots \\
0_p & 0_p & -\frac{\nu(d_2)}{d_2^2}1_p & 0_p & \frac{\nu(d_4)}{d_4^2}1_p & \cdots \\
0_p & 0_p & 0_p & 0_p & \frac{\nu(d_3)}{d_3^2}1_p & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\end{pmatrix},$$ (7.3)

which non-essentially differs from the matrix $\hat{J}_{X,\alpha}$ with $A_\alpha' = B_{\alpha}' = 0$, $n \in \mathbb{N}_0$.

Proposition 7.1. Let $B_{X,\alpha}$ be the minimal Jacobi operator associated to Jacobi matrix of the form (7.3). Let also a sequence $\alpha := (\alpha_n)_{1}^{\infty}(\subset \mathbb{C}^{p \times p})$ of selfadjoint matrices satisfy condition (6.5). Then $n_\pm(B_{X,\alpha}) = p$. 

Proof. It follows from condition (6.5) that \( \sum_k d_k =: b < \infty \). Consider the minimal Dirac operator \( D_X \) in \( L^2(\mathcal{I}; \mathbb{C}^{2p}) \) with \( \mathcal{I} = [0, b] \) and its GS-realization \( D_{X,\alpha} \). In accordance with Proposition A.4 \( B_{X,\alpha} \) is the boundary operator for \( D_{X,\alpha} \) with respect to the boundary triplet \( \{A.6, A.7\} \). Therefore Proposition A.4 implies \( n_{\pm}(B_{X,\alpha}) = n_{\pm}(D_{X,\alpha}) \). It remains to apply Theorem 6.2 \( \Box \)

Theorem 7.2. Let \( \hat{J}_{X,\alpha} \) be the matrix of the form (7.1) and the sequence \( \alpha := \{\alpha_n\}^\infty_1(\subset \mathbb{C}^{p \times p}) \), \( \alpha_n = \alpha_n^* \), \( n \in \mathbb{N} \), satisfy condition (5.5). Assume also that for some \( N \in \mathbb{N}_0 \) the following conditions hold:

(i) \( \sup_{n \geq N} \|B_n\|_{\mathbb{C}^{p \times p}} = a_N < 1 \). (7.4)

(ii) \( \sup_{j \geq 0, d_{j+1}} \frac{1}{d_{j+1}} \|B'_{2j} - B'_{2j-1}\|_{\mathbb{C}^{p \times p}} = C_B < \infty \). (7.5)

(iii) \( \sup_{j \geq 0, d_{j+1}} \frac{1}{d_{j+1}} \|A'_j - B'_{2j-1}\|_{\mathbb{C}^{p \times p}} = C_A < \infty \). (7.7)

Then the matrix \( \hat{J}_{X,\alpha} \) is in the complete indeterminant case, i.e. \( n_{\pm}(\hat{J}_{X,\alpha}) = p \).

Proof. (1) Note that certain off-diagonal entries of the matrix (7.3) are negative while in the classical theory of Jacobi operators, off-diagonal entries are assumed to be positive definite. However, it is known (see, for instance, [54]) that the (minimal) operator \( B_{X,\alpha} \) is unitarily equivalent to the minimal Jacobi operator associated with the Jacobi matrix

\[
J_{X,\alpha} = \begin{pmatrix}
O_p & \frac{\nu(d_1)}{d_1^2} \bar{p} & O_p & \frac{\nu(d_1)}{d_1^2} \bar{p} & \ldots \\
\frac{\nu(d_1)}{d_1^2} \bar{p} & \frac{\nu(d_1)}{d_1^2} \bar{p} & O_p & \frac{\nu(d_1)}{d_1^2} \bar{p} & \ldots \\
O_p & \frac{\nu(d_1)}{d_1^2} \bar{p} & O_p & \frac{\nu(d_1)}{d_1^2} \bar{p} & \ldots \\
O_p & O_p & \frac{\nu(d_1)}{d_1^2} \bar{p} & O_p & \frac{\nu(d_1)}{d_1^2} \bar{p} & \ldots \\
O_p & O_p & O_p & \frac{\nu(d_1)}{d_1^2} \bar{p} & O_p & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}. \tag{7.9}
\]

Thus,

\[
n_{\pm}(J_{X,\alpha}) = n_{\pm}(B_{X,\alpha}). \tag{7.10}
\]

(2) Let us check conditions of Corollary 5.2 for Jacobi matrices \( \hat{J}_{X,\alpha} \) and \( J_{X,\alpha} \) given by (7.1) and (7.9), respectively. First we check condition (5.2) considering the case \( n = 2j + 1 \) only. Applying condition (7.4) to these matrices we easily derive

\[
\sup_{j \geq N} \|\bar{B}_{2j+1}^{-1}\|_{\mathbb{C}^{p \times p}} = \sup_{j \geq N} \left\| \frac{\nu(d_{j+1})}{d_{j+1}^{3/2} d_{j+2}^{1/2}} \left( \bar{p} + \frac{d_{j+1}^{3/2} d_{j+2}^{1/2}}{\nu(d_{j+1})} \right) \right\|_{\mathbb{C}^{p \times p}} \leq a_N < 1. \tag{7.11}
\]
The case \( n = 2j \) is considered similarly.

Note also that condition (7.14) ensures invertibility of all off-diagonal entries of the Jacobi matrix (7.1).

Next we check condition (5.22). Consider the case \( n = 2j \) only. Applying condition (7.13) to the pair \( \{ J_{X,\alpha}, \tilde{J}_{X,\alpha} \} \) and noting that \( \nu(d_{j+1}) < c d_{j+1} \) and \( \nu(d_{j+1}) \sim c d_{j+1} \) as \( d_{j} \to 0 \), we obtain

\[
\sup_{j \geq 0} \| \tilde{B}_{2j} - \tilde{B}_{2j-1} B_{2j-1} \|_{\mathbb{C}^{p \times p}} =
\]

\[
= \sup_{j \geq 0} \left\| \frac{\nu(d_{j+1})}{d_{j+1}^2} (I_p + B_{2j}^2) - \frac{\nu(d_{j})}{d_{j}^{3/2} d_{j+1}^{1/2}} (I_p + B_{2j-1}^2) \frac{d_{j}^{3/2} d_{j+1}^{1/2} \nu(d_{j+1})}{\nu(d_{j}) d_{j+1}^2} \right\|_{\mathbb{C}^{p \times p}}
\]

\[
= \sup_{j \geq 0} \frac{\nu(d_{j+1})}{d_{j+1}^2} \left\| B_{2j}^2 - B_{2j-1}^2 \right\|_{\mathbb{C}^{p \times p}} \leq \sup_{j \geq 0} \frac{c}{d_{j+1}} \left\| B_{2j}^2 - B_{2j-1}^2 \right\|_{\mathbb{C}^{p \times p}} \leq C_B < \infty.
\]

The case \( n = 2j + 1 \) is treated similarly.

Finally, we check condition (5.22) considering the case \( n = 2j \) only. The case \( n = 2j + 1 \) is considered similarly. Applying condition (7.14) to the pair \( \{ J_{X,\alpha}, \tilde{J}_{X,\alpha} \} \) we derive

\[
\sup_{j \geq 0} \| \tilde{A}_{2j} - \tilde{B}_{2j-1} B_{2j-1} A_{2j} \|_{\mathbb{C}^{p \times p}} = \sup_{j \geq 0} \left\| \frac{\alpha_j}{d_{j+1}} (I_p + A_{2j}) - \frac{\nu(d_{j})}{d_{j}^{3/2} d_{j+1}^{1/2}} (I_p + B_{2j-1}^2) \frac{d_{j}^{3/2} d_{j+1}^{1/2} \nu(d_{j+1})}{\nu(d_{j}) d_{j+1}^2} \right\|_{\mathbb{C}^{p \times p}}
\]

\[
= \sup_{j \geq 0} \frac{1}{d_{j+1}} \| \alpha_j (A_{2j} - B_{2j-1}) \|_{\mathbb{C}^{p \times p}} = C_A < \infty.
\]

Thus, Corollary 5.22 together with equality (7.10) ensures \( n_{\pm}(J_{X,\alpha}) = n_{\pm}(J_{X,\alpha}) = n_{\pm}(B_{X,\alpha}) \). To complete the proof it remains to apply Proposition 7.1.

Alongside with the matrix \( J_{X,\alpha} \) of form (7.9) we consider the following Jacobi matrix

\[
J'_{X,\alpha} = \begin{pmatrix}
O_p & \frac{\nu(d_{j+1})}{d_{j+1}^2} I_p & O_p & O_p & \ldots \\
-\frac{c}{d_{j+1}} I_p & \frac{c}{d_{j+1}^{3/2} d_{j+1}^{1/2}} I_p & O_p & O_p & \ldots \\
O_p & \frac{c}{d_{j+1}^{3/2} d_{j+1}^{1/2}} I_p & \frac{c}{d_{j+1}^{3/2} d_{j+1}^{1/2}} I_p & O_p & \ldots \\
O_p & \frac{c}{d_{j+1}^{3/2} d_{j+1}^{1/2}} I_p & \frac{c}{d_{j+1}^{3/2} d_{j+1}^{1/2}} I_p & \frac{c}{d_{j+1}^{3/2} d_{j+1}^{1/2}} I_p & \ldots \\
O_p & O_p & O_p & \frac{c}{d_{j+1}^{3/2} d_{j+1}^{1/2}} I_p & \frac{c}{d_{j+1}^{3/2} d_{j+1}^{1/2}} I_p & \frac{c}{d_{j+1}^{3/2} d_{j+1}^{1/2}} I_p & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix},
\]

obtained from (7.13) by replacing \( \nu(d_{n}) \) by \( c d_{n} \). Since \( \nu(d_{n}) \sim c d_{n} \) as \( d_{n} \to \infty \) \( d_{n} = 0 \) it is naturally to suppose that \( n_{\pm}(J'_{X,\alpha}) = n_{\pm}(J_{X,\alpha}) \). Below we confirm this hypotheses under an additional assumption on \( d_{n} \) although we don’t know whether it is true in general.

**Proposition 7.3.** Let the Jacobi matrices \( J_{X,\alpha} \) and \( J'_{X,\alpha} \) be of the form (7.9) and (7.14), respectively. Assume that \( \lim_{n \to \infty} \frac{d_{n+1}}{d_{n}} = 0 \). Then \( n_{\pm}(J_{X,\alpha}) = n_{\pm}(J'_{X,\alpha}) \).

In particular, if the sequence \( \alpha := \{ \alpha_n \} \subset \mathbb{C}^{p \times p} \), satisfy condition (6.5), then \( n_{\pm}(J'_{X,\alpha}) = p \).

**Proof.** Let us check the conditions of Theorem 5.1 for the pair \( \{ J_{X,\alpha}, J'_{X,\alpha} \} \).

Now, the entries of the matrices \( J_{X,\alpha} \) and \( J'_{X,\alpha} \) read as follows

\[
\tilde{B}_n = \begin{cases}
\frac{\nu(d_{j+1})}{d_{j+1}^2} I_p, & n = 2j, \\
\frac{\nu(d_{j+1})}{d_{j+1}^{3/2} d_{j+2}^{1/2}} I_p, & n = 2j + 1,
\end{cases}
\]

\[
B_n = \begin{cases}
\frac{c}{d_{j+1}} I_p, & n = 2j, \\
\frac{c}{(d_{j+1} d_{j+2})^{3/2}} I_p, & n = 2j + 1,
\end{cases}
\]
and
\[
\tilde{A}_n = \begin{cases} \frac{\alpha_j}{d_{j+1}} - \frac{c}{d_{j+1}} \mathbb{I}_p, & n = 2j, \\ -\frac{\alpha_j}{d_{j+1}} \mathbb{I}_p, & n = 2j + 1, \end{cases}
\]
\[
A_n = \begin{cases} \frac{\alpha_j}{d_{j+1}} + \frac{c}{d_{j+1}} \mathbb{I}_p, & n = 2j, \\ -\frac{\alpha_j}{d_{j+1}} \mathbb{I}_p, & n = 2j + 1, \end{cases}
\] respectively.

First we check condition (5.2). Consider the case \( n = 2j \). Noting that \( \nu(d_{j+1}) < c d_{j+1} \) and \( \nu(d_{j+1}) \sim c d_{j+1} \) as \( d_j \to 0 \), one easily verifies that \( \lim_{j \to \infty} \|\mathbb{I}_p - \tilde{B}_{2j} \tilde{B}_{2j}^{-1}\|_{\mathbb{C}^p} = 0 \). Hence for sufficiently large \( N \) the condition
\[
\sup_{j \geq N} \|\mathbb{I}_p - \tilde{B}_{2j} \tilde{B}_{2j}^{-1}\|_{\mathbb{C}^p} < 1
\]
holds. The case \( n = 2j + 1 \) is treated similarly.

Next we check condition (5.3). Consider the case \( n = 2j \). Since \( \lim_{j \to \infty} \frac{d_j^2}{d_{j+1}} = 0 \) we get
\[
\|\tilde{B}_{2j} - \tilde{B}_{2j-1} \tilde{B}_{2j-1}^{-1} \tilde{B}_{2j}\|_{\mathbb{C}^p} \leq \frac{c}{d_{j+1}} \left( \frac{1}{\sqrt{1 + c^2 d_{j+1}^2}} - \frac{1}{\sqrt{1 + c^2 d_j^2}} \right)
\leq \frac{c^3 |d_j^2 - d_{j+1}^2|}{d_{j+1}} \leq c^3 \max \left\{ \frac{d_{j+1}}{d_j}, \frac{d_j^2}{d_{j+1}} \right\} \to 0 \text{ as } j \to \infty.
\]
This implies condition (5.3). The case \( n = 2j + 1 \) is considered similarly.

Finally, we check condition (5.4). In the case \( n = 2j + 1 \) this condition is obvious, because \( \tilde{B}_{2j} = -\tilde{A}_{2j+1} \) and \( \tilde{B}_{2j+1}^{-1} \tilde{A}_{2j+1} = -\mathbb{I}_p \).

Consider the case \( n = 2j \). Since \( \lim_{j \to \infty} d_j = 0 \) and \( \lim_{j \to \infty} \frac{d_j^2}{d_{j+1}} = 0 \), for any \( \varepsilon > 0 \) we can find \( N = N(\varepsilon) \), such that \( c^2 d_j^2 < \varepsilon \) and \( \frac{d_j^2}{d_{j+1}} < \varepsilon \) for all \( j \geq N \). Thus, for any \( f \in l_0^p(\mathbb{N}; \mathbb{C}^p) \) with \( f_{2j} = 0 \) for \( j < N \), we obtain
\[
\sum_{j \geq N} \|(\tilde{A}_{2j} - \tilde{B}_{2j-1} \tilde{B}_{2j-1}^{-1} \tilde{A}_{2j}) f_{2j}\|^2_{\mathbb{C}^p} \leq \sum_{j \geq N} \left\| \frac{\alpha_j}{d_{j+1}} \left( 1 - \frac{1}{\sqrt{1 + c^2 d_{j+1}^2}} \right) f_{2j} \right\|^2_{\mathbb{C}^p}
= \sum_{j \geq N} \left( (J'_{X,\alpha} f)_{2j} - c \left( \frac{1}{d_j^{1/2} d_{j+1}^{1/2}} f_{2j-1} + \frac{1}{d_j^{1/2} d_{j+1}^{1/2}} f_{2j+1} \right) \right)^2
\leq \sum_{j \geq N} \|c^2 d_j^2 (J'_{X,\alpha} f)_{2j}\|^2_{\mathbb{C}^p} + c^6 \sum_{j \geq N} \left\| d_j^{1/2} - d_{j+1}^{1/2} \right\|^2_{\mathbb{C}^p} \leq \varepsilon^2 \|J'_{X,\alpha} f\|^2_{l_2} + 4C_1 \|f\|^2_{l_2},
\]
where \( C_1 = \max \{c^5 \varepsilon^{3/2}, c^6 \varepsilon^2 \} \). Thus, Theorem 5.4 ensures that \( n_\pm(J_{X,\alpha}) = n_\pm(J'_{X,\alpha}) \).

Combining this relation with equality (7.10) and Proposition 7.1 yields \( n_\pm(J'_{X,\alpha}) = n_\pm(J_{X,\alpha}) = n_\pm(B_{X,\alpha}) = p \).

\[\square\]

**Remark 7.4.** The analogue of Theorem 7.2 is also valid for the pair \( \{\tilde{J}'_{X,\alpha}, J'_{X,\alpha}\} \), in which the matrix \( \tilde{J}'_{X,\alpha} \) is obtained from \( J'_{X,\alpha} \) by adding bounded perturbations \( A_n' \) and \( B_n' \) similarly to the construction of the matrix \( J_{X,\alpha} \) of the form (7.1).
7.2 Jacobi matrices $\tilde{J}_{X,\beta}$

Here we apply previous results to block Jacobi matrix

$$
\tilde{J}_{X,\beta} = \begin{pmatrix}
A'_0 & \tilde{v}_0(\|p + B'_0) & \mathcal{O}_p & \mathcal{O}_p & \ldots \\
\tilde{v}_0(\|p + B'_1) & -\nu_1(\beta_1 + d_1\|p + A'_1) & \tilde{v}_0(\|p + B'_1) & \mathcal{O}_p & \ldots \\
\mathcal{O}_p & \tilde{v}_0(\|p + B'_2) & A'_2 & \tilde{v}_1(\|p + B'_2) & \ldots \\
\mathcal{O}_p & \mathcal{O}_p & \tilde{v}_1(\|p + B'_3) & -\nu_2(\beta_2 + d_2\|p + A'_3) & \ldots \\
\mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix},
$$

(7.20)

where $\tilde{v}_n := \frac{\nu(d_n+1)}{d_n+2}^2$, $\tilde{v}_n := \frac{\nu(d_n+1)}{d_n+2}^2$, $\tilde{v}_n = \frac{\nu^2(d_n)}{d_n}$, $\alpha_n = \alpha^*$, $A'_n = (A'_n)^*$, $B'_n = (B'_n)^* \in \mathbb{C}^{p \times p}$ and matrices $\|p + B'_n$ are invertible, i.e. $\det(\|p + B'_n) \neq 0$, $n \in \mathbb{N}_0$.

Consider the minimal Jacobi operator associated with the matrix

$$
J_{X,\beta} := \begin{pmatrix}
\mathcal{O}_p & -\frac{\nu(d_1)}{d_1} \|p & \mathcal{O}_p & \mathcal{O}_p & \ldots \\
\frac{\nu(d_2)}{d_2} \|p & -\frac{\nu^2(d_2)(\beta_1 + d_1\|p)}{d_1^{1/2}d_2^{1/2}} \|p & \mathcal{O}_p & \mathcal{O}_p & \ldots \\
\mathcal{O}_p & \mathcal{O}_p & \frac{\nu(d_3)}{d_3} \|p & -\frac{\nu^2(d_3)(\beta_2 + d_2\|p)}{d_2^{1/2}d_3^{1/2}} \|p & \ldots \\
\mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p & \ldots \\
\mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix},
$$

(7.21)

Proposition 7.5. Let $J_{X,\beta}$ be a minimal Jacobi operator associated with $l^2(\mathbb{N}_0; \mathbb{C}^p)$ with the matrix (7.21). If the sequence $\beta := \{\beta_n\}_1^\infty (\subset \mathbb{C}^{p \times p})$ satisfy condition (6.21). Then the deficiency indices of the matrix $J_{X,\beta}$ are maximal, i.e. $n_{\pm}(J_{X,\beta}) = p$.

Proof. Note that the Jacobi operator $J_{X,\beta}$ is unitarily equivalent to the Jacobi operator associated with the matrix

$$
B_{X,\beta} := \begin{pmatrix}
\mathcal{O}_p & -\frac{\nu(d_1)}{d_1} \|p & \mathcal{O}_p & \mathcal{O}_p & \ldots \\
\frac{\nu^2(d_1)(\beta_1 + d_1\|p)}{d_1^{1/2}d_2^{1/2}} \|p & \mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p & \ldots \\
\mathcal{O}_p & \mathcal{O}_p & \frac{\nu(d_2)}{d_2} \|p & -\frac{\nu^2(d_2)(\beta_2 + d_2\|p)}{d_2^{1/2}d_3^{1/2}} \|p & \ldots \\
\mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p & \ldots \\
\mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p & \mathcal{O}_p & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix},
$$

(7.22)

The latter is a boundary operator corresponding to the GS-realization $D_{X,\beta}$ in an appropriate boundary triplet (compare with Proposition (A.4)). It remains to apply Theorem 6.7.

Theorem 7.6. Let $\tilde{J}_{X,\beta}$ be the matrix of the form (7.20) and the sequence $\beta := \{\beta_n\}_1^\infty (\subset \mathbb{C}^{p \times p})$ satisfy condition (6.21). Assume also that for some $N \in \mathbb{N}_0$ the following conditions hold:

(i) $\sup_{n \geq N} \|B'_n\|_{\mathbb{C}^{p \times p}} = a_N < 1$;
Proof. Let us check the conditions of Theorem 5.1 for the pair $(\hat{\mathbf{B}}_{2j}, \hat{\mathbf{B}}_{2j-1})$ obtained from (7.21) by replacing $\nu$ with $\hat{\nu}$.

Proposition 7.7. Let the Jacobi matrices $J_{X,\beta}$ and $J'_{X,\beta}$ be of the form (7.21) and (7.28), respectively. Assume that $\lim_{n \to \infty} \frac{d_j^2}{d_n} = 0$. Then $n_{\pm}(J_{X,\beta}) = n_{\pm}(J'_{X,\beta})$.

In particular, if the sequence $\beta := \left\{ \beta_n \right\}_{n=1}^{\infty} \subset \mathbb{C}^{p \times p}$ satisfy condition (6.21), then $n_{\pm}(J'_{X,\beta}) = p$.

Proof. Let us check the conditions of Theorem 5.1 for the pair $(\mathbf{J}_{X,\beta}, \mathbf{J}'_{X,\beta})$. Now, the entries $\hat{A}_n$ and $\hat{A}_n$ of the matrices $\mathbf{J}_{X,\beta}$ and $\mathbf{J}'_{X,\beta}$, respectively, are given by

$$\hat{A}_n = \begin{cases} \mathbf{0}_p, & n = 2j, \\ \frac{\nu^2(d_{j+1})}{d_{j+1}} (\beta_{j+1} + d_{j+1} \mathbf{I}_p), & n = 2j + 1, \end{cases} \quad \hat{A}_n = \begin{cases} \mathbf{0}_p, & n = 2j, \\ \frac{\nu^2(d_{j+1})}{d_{j+1}} (\beta_{j+1} + d_{j+1} \mathbf{I}_p), & n = 2j + 1, \end{cases}$$

while $\hat{B}_n$ and $\hat{B}_n$ are given by (7.15).

Conditions (5.2) and (5.3) can be checked in just the same way as in Proposition 7.3. Let us check condition (5.4). In the case $n = 2j$ it is obviously satisfied because $\hat{A}_{2j} = A_{2j} = \mathbf{0}_p$. Then the deficiency indices of the minimal Jacobi operator $\hat{J}_{X,\beta}$ are maximal, i.e. $n_{\pm}(\hat{J}_{X,\beta}) = p$. \hfill \square

Next we simplified the matrix $J_{X,\beta}$ of form (7.21) by considering the following Jacobi matrix

$$J'_{X,\beta} = \begin{pmatrix} \mathbf{0}_p & \mathbf{0}_p & \mathbf{0}_p & \mathbf{0}_p & \cdots \\ \frac{c}{d_1} \mathbf{I}_p & \frac{c}{d_1} (\beta_1 + d_1 \mathbf{I}_p) & \frac{c}{(d_1 d_2)^{1/2}} \mathbf{I}_p & \mathbf{0}_p & \mathbf{0}_p & \mathbf{0}_p & \cdots \\ \mathbf{0}_p & \frac{c}{d_1} \mathbf{I}_p & \frac{c}{d_1 d_2} \mathbf{I}_p & \frac{c}{(d_1 d_2)^{1/2}} \mathbf{I}_p & \mathbf{0}_p & \mathbf{0}_p & \cdots \\ \mathbf{0}_p & \mathbf{0}_p & \frac{c}{d_2} \mathbf{I}_p & \frac{c}{d_2 d_3} \mathbf{I}_p & \frac{c}{(d_2 d_3)^{1/2}} \mathbf{I}_p & \mathbf{0}_p & \cdots \\ \mathbf{0}_p & \mathbf{0}_p & \mathbf{0}_p & \frac{c}{d_3} \mathbf{I}_p & \frac{c}{d_3 d_4} \mathbf{I}_p & \frac{c}{(d_3 d_4)^{1/2}} \mathbf{I}_p & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \quad (7.28)$$

It is obtained from (7.21) by replacing $\nu(d_n)$ by $cd_n$. Our next statement is similar to that of Proposition 7.3.

Proposition 7.7. Let the Jacobi matrices $J_{X,\beta}$ and $J'_{X,\beta}$ be of the form (7.21) and (7.28), respectively. Assume that $\lim_{n \to \infty} \frac{d_j^2}{d_n} = 0$. Then $n_{\pm}(J_{X,\beta}) = n_{\pm}(J'_{X,\beta})$.

In particular, if the sequence $\beta := \left\{ \beta_n \right\}_{n=1}^{\infty} \subset \mathbb{C}^{p \times p}$ satisfy condition (6.21), then $n_{\pm}(J'_{X,\beta}) = p$. \hfill \square
Consider the case $n = 2j + 1$. Since $\lim_{j \to \infty} d_{j+1} = 0$ and $\lim_{j \to \infty} d_{j+2}^{3/2} = 0$, for any $\varepsilon > 0$ we can find $N = N(\varepsilon)$, such that $c^2 d_{j+1}^2 < \varepsilon$ and $d_{j+1}^{3/2} < \varepsilon$ for all $j \geq N$. Thus, for any $f \in l^2_0(\mathbb{N}; \mathbb{C}^p)$ with $f_{2j+1} = 0$ for $j < N$, we obtain

\[
\begin{align*}
\sum_{j \geq N} \| (\hat{A}_{2j+1} - \hat{B}_{2j} B_{2j-1} A_{2j+1}) f_{2j+1} \|^2_{\mathbb{C}^p} &\leq \sum_{j \geq N} \left\| \frac{c^2}{d_{j+1}} (\beta_{j+1} + d_{j+1}^2) \cdot \frac{1 - \sqrt{1 + c^2 d_{j+1}^2}}{1 + c^2 d_{j+1}^2} f_{2j+1} \right\|^2_{\mathbb{C}^p} \\
&= \sum_{j \geq N} \left\| (J_{X,\beta}^f)_{2j+1} - \left( \frac{f_{2j}}{d_{j+1}} + \frac{f_{2j+2}}{d_{j+1}^{3/2} d_{j+2}^{1/2}} \right) \right\| \left( 1 + c^2 d_{j+1}^2 \right)^{-1} \left( 1 + \sqrt{1 + c^2 d_{j+1}^2} \right)^{-1} \left\|_{\mathbb{C}^p} \right. \\
&\leq \sum_{j \geq N} \left\| c^2 d_{j+1}^2 (J_{X,\beta}^f)_{2j+1} \right\|^2_{\mathbb{C}^p} + c^6 \sum_{j \geq N} \left\| d_{j+1} f_{2j} + d_{j+1}^{3/2} d_{j+2}^{1/2} f_{2j+2} \right\|^2_{\mathbb{C}^p} \leq \varepsilon^2 \left\| J_{X,\beta}^f \right\|^2_{l^2} + 4C_1 \| f \|^2_{l^2},
\end{align*}
\]

where $C_1 = \max\{c^4 \varepsilon, c^5 \varepsilon^{3/2}\}$. Thus condition (5.4) is verified.

Applying Theorem 5.1 we conclude that $n_\pm (J_{X,\beta}) = n_\pm (J_{X,\beta}^*)$.

The second statement is immediate from Proposition 7.5.

\[\square\]

8 Jacobi matrices with intermediate deficiency indices

First we present results on self-adjointness of Jacobi matrices $J_{X,\alpha}$ of the form (7.9).

**Theorem 8.1.** Let $\alpha = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be diagonal matrices, $n \in \mathbb{N}$, and let $J_{X,\alpha}$ be a minimal Jacobi operator associated with a matrix (7.9) with entries $\alpha$ instead of $\alpha_n$. Let also

\[
|\alpha_{n,1}| \leq |\alpha_{n,2}| \leq \ldots \leq |\alpha_{n,p}|, \quad \text{and} \quad \{d_n\}_{n=1}^{\infty} \in l^1(\mathbb{N}).
\]

(i) If following condition holds

\[
\sum_{n \in \mathbb{N}} \sqrt{d_n d_{n+1}} |\alpha_{n,1}| = +\infty,
\]

then the operator $J_{X,\alpha}$ is selfadjoint, $J_{X,\alpha} = J_{X,\alpha}^*$.

(ii) If an addition following conditions hold

\[
\lim_{n \to \infty} \frac{|\alpha_{n,1}|}{d_{n+1}} = \infty, \quad \lim_{n \to \infty} \frac{c}{|\alpha_{n,1}|} > -\frac{1}{4},
\]

then the spectrum of Jacobi operator $J_{X,\alpha} = J_{X,\alpha}^*$ is discrete.

**Proof.** (i) First we prove selfadjointness of the operator $J_{X,\alpha}$. Since the matrices $\alpha_n = \text{diag}(\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,p})$ are diagonal for each $n \in \mathbb{N}$, the operator $J_{X,\alpha}$ splits into the direct sum of $p$ scalar Jacobi operators $J_{X,\alpha_j}$, where $\alpha_j = \{\alpha_{n,j}\}_{n=1}^{\infty} \subset \mathbb{R}$, $j \in \{1, \ldots, p\}$. By the Dennis-Wall test (see [1], Problem 2, p.25), $J_{X,\alpha_j}$ is self-adjoint whenever

\[
\sum_{n=1}^{\infty} \frac{d_{n+1}^{3/2}}{\nu(d_{n+1})} \left( \frac{d_{n}^{3/2} |\alpha_{n,j}|}{\nu(d_n)} + d_{n+2}^{1/2} \right) = +\infty, \quad j \in \{1, \ldots, p\}.
\]
Since \( \nu(d_n) = \frac{c d_n}{\sqrt{1 + c^2 d_n^2}} \sim c d_n \) as \( n \to \infty \), one gets \( \frac{d_{n/2}}{\nu(d_n)} \sim c^{-1} d_n^{1/2} \) as \( n \to \infty \). Taking these relations into account and noting that

\[
2 \sum_{n \in \mathbb{N}} \sqrt{d_{n+1}} \sqrt{d_{n+2}} \leq \sum_{n \in \mathbb{N}} (d_{n+1} + d_{n+2}) < +\infty,
\]

one concludes that the series (8.2) and (8.3) diverge only simultaneously. Thus series (8.4) diverges and the operator \( J_{X,\tilde{\alpha}} \) is selfadjoint, i.e. \( n_{\pm}(J_{X,\tilde{\alpha}}) = 0 \).

(ii) Since the operator \( J_{X,\tilde{\alpha}} \) is selfadjoint, then according to [15] Theorem 8 (see Remark 3.6) conditions (8.3) guarantee the discreteness of the spectrum of \( J_{X,\tilde{\alpha}} \).

**Corollary 8.2.** Let \( \{d_n\}_{n=1}^{\infty} \in l^1(\mathbb{N}) \) and let \( J_{X,\alpha} \) be the Jacobi matrix of form (7.9). Assume that \( \alpha := \{\alpha_n\}_{1}^{\infty} \subset \mathbb{C}^{p \times p} \), where \( \alpha_n = \alpha_n + \tilde{\alpha}_n = \alpha_n^* \) and matrices \( \tilde{\alpha}_n = \text{diag}(\alpha_{n,1},\alpha_{n,2},\ldots,\alpha_{n,p}) = \tilde{\alpha}_n^* \) are diagonal. If the condition (8.2) holds and

\[
\|\tilde{\alpha}_n\|_{\mathbb{C}^{p \times p}} = O(d_{n+1}) \quad \text{for} \quad n \to \infty,
\]

then the (minimal) Jacobi operator \( J_{X,\alpha} \) is selfadjoint, \( J_{X,\alpha} = J_{X,\alpha}^* \).

Next we construct Jacobi matrices \( J_{X,\alpha} \) with intermediate deficiency indices \( 0 < n_{\pm}(J_{X,\alpha}) = p_1 < p \). Let \( p = p_1 + p_2 \), where \( p_1, p_2 \in \mathbb{N} \). To this end consider block–matrix representation of \( \alpha_n \in \mathbb{C}^{p \times p} \) with respect to the orthogonal decomposition \( \mathbb{C}^p = \mathbb{C}^{p_1} \oplus \mathbb{C}^{p_2} \):

\[
\alpha_n = \left( \begin{array}{cc} \alpha_{11}^{(n)} & \alpha_{12}^{(n)} \\ \alpha_{21}^{(n)} & \alpha_{22}^{(n)} \end{array} \right) = \alpha_n^* \in \mathbb{C}^{p \times p}, \quad \alpha_{ij}^{(n)} \in \mathbb{C}^{p_i \times p_j}, \quad i, j \in \{1, 2\}.
\]

**Corollary 8.3.** Let \( \{d_n\}_{n=1}^{\infty} \in l^1(\mathbb{N}) \) and let \( J_{X,\alpha} \) be the Jacobi matrix of the form (7.9) with \( \alpha = \{\alpha_n\}_{1}^{\infty} \) and \( \alpha_n \) admitting representations (8.6). Assume in addition that:

(i) the matrices \( \alpha_{11}^{n} \), \( n \in \mathbb{N} \) satisfy condition (8.5) with \( p \) replaced by \( p_1 \);

(ii) \( \|\alpha_{12}^{n}\|_{\mathbb{C}^{p_1 \times p_2}} = O(d_{n+1}) \), as \( n \to \infty \);

(iii) the matrices \( \alpha_{22}^{n} = \bar{\alpha}_{22}^{n} + \bar{\alpha}_{22}^{n} \) where \( \bar{\alpha}_{22}^{n} = \text{diag}(\alpha_{n,1},\alpha_{n,2},\ldots,\alpha_{n,p}) \) satisfy condition (8.2) and \( \tilde{\alpha}_{22}^{n} \) satisfy condition (8.5) with \( p \) replaced by \( p_2 \).

Then \( n_{\pm}(J_{X,\alpha}) = p_1 \).

**Proof.** It follows from conditions (ii) and (iii) that \( J_{X,\alpha} \) is a bounded perturbation of the matrix \( J_{X,\tilde{\alpha}} \) with \( \tilde{\alpha}_n = \{\tilde{\alpha}_n\}_{1}^{\infty} \), where \( \tilde{\alpha}_n = \alpha_{11}^{n} \oplus \alpha_{22}^{n} \), and \( \alpha_{11}^{n} = \{\alpha_{11}^{n}\}_{1}^{\infty}, \quad \alpha_{22}^{n} = \{\alpha_{22}^{n}\}_{1}^{\infty} \). Hence \( n_{\pm}(J_{X,\alpha}) = n_{\pm}(J_{X,\tilde{\alpha}}) \).

Note that the decomposition \( \mathbb{C}^p = \mathbb{C}^{p_1} \oplus \mathbb{C}^{p_2} \) yields representation of the operator \( J_{X,\tilde{\alpha}} \) as \( J_{X,\tilde{\alpha}} = J_{X,\alpha_{11}} \oplus J_{X,\alpha_{22}} \). Therefore,

\[
n_{\pm}(J_{X,\alpha}) = n_{\pm}(J_{X,\tilde{\alpha}}) = n_{\pm}(J_{X,\alpha_{11}}) + n_{\pm}(J_{X,\alpha_{22}}) = p_1 + 0 = p_1.
\]

We have used here that in accordance with Corollary 8.2 \( n_{\pm}(J_{X,\alpha_{22}}) = 0 \).

We give a special case of the Corollary 8.3 with diagonal block–matrix sequence \( \alpha_n := \{\alpha_n\}_{1}^{\infty} \)

\[
\alpha_n = \alpha_n^* = \left( \begin{array}{cc} \alpha_{11}^{n} & \alpha_{12}^{n} \\ \alpha_{21}^{n} & \alpha_{22}^{n} \end{array} \right) \in \mathbb{C}^{p \times p}, \quad \alpha_{ij}^{n} \in \mathbb{C}^{p_i \times p_j}, \quad i, j \in \{1, 2\}.
\]

**Corollary 8.4.** Let \( \{d_n\}_{n=1}^{\infty} \in l^1(\mathbb{N}) \) and let \( J_{X,\alpha} \) be the Jacobi matrix of the form (7.9) with \( \alpha = \{\alpha_n\}_{1}^{\infty} \) and \( \alpha_n \) admitting representations (8.7). Assume in addition that:
(i) the matrices $\alpha_{n}^{11}, n \in \mathbb{N}$ satisfy condition \([6.5]\) with \(p\) replaced by \(p_1\);

(ii) the diagonal matrices $\alpha_{n}^{22} = \text{diag}(\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,p_2})$ satisfy condition \([8.2]\) with \(p\) replaced by \(p_2\).

Then $n_{\pm}(J_{X,\alpha}) = p_1$.

Proof. Proof follows from Corollary \([8.3]\).

\[\boxed{\text{Corollary 8.5. Let } \hat{\alpha}_n \text{ be diagonal matrices, } \hat{\alpha}_n = \text{diag}(\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,p}) = \hat{\alpha}_n, \ n \in \mathbb{N}, \text{ and let } \hat{J}_{X,\hat{\alpha}} \text{ be the Jacobi matrix given by } (7.1) \text{ with entries } \hat{\alpha}_n \text{ instead of } \alpha_n. \ Let also } |\alpha_{n,1}| \leq |\alpha_{n,2}| \leq \ldots \leq |\alpha_{n,p}|. \ Assume also that } |\alpha_{n,1}| \leq |\alpha_{n,2}| \leq \ldots \leq |\alpha_{n,p}| \text{ and } \{d_n\}_{n=1}^{\infty} \in l^1(\mathbb{N}).\]

(i) If conditions \((7.4)-(7.7), (8.2)\) are satisfied then the minimal Jacobi operator $\hat{J}_{X,\hat{\alpha}}$ is selfadjoint, $\hat{J}_{X,\hat{\alpha}} = \hat{J}_{X,\hat{\alpha}}^*$.

(ii) If an addition conditions \([8.3]\) hold then the spectrum of Jacobi operator $\hat{J}_{X,\hat{\alpha}}$ is discrete.

Proof. (i) Considering the operator $\hat{J}_{X,\hat{\alpha}}$ as a perturbation of $J_{X,\hat{\alpha}}$ and following reasoning of the proof of Theorem 7.2 we conclude that conditions \((7.4)-(7.7)\) imply conditions \((5.2), (5.3), (5.9)\) of Corollary 5.2. Therefore combining Theorem 8.1 (i) with the conclusion of Corollary 5.2 yields $n_{\pm}(\hat{J}_{X,\hat{\alpha}}) = n_{\pm}(J_{X,\hat{\alpha}}) = 0$, i.e. $\hat{J}_{X,\hat{\alpha}} = \hat{J}_{X,\hat{\alpha}}^*$.

(ii) It follows from Theorems 5.1 (b) and 8.1 (ii), that the Jacobi operator $\hat{J}_{X,\hat{\alpha}}$ has a discrete spectrum. \(\blacklozenge\)

For a Jacobian matrix $J_{X,\beta}$ of the form \((7.2)\) $\beta = \{\beta_n\}_{1}^{\infty}$ similar results on selfadjointness and deficiency indices are valid. We present only the results on selfadjointness.

\[\text{Theorem 8.6. Let } \{d_n\}_{n=1}^{\infty} \in l^1(\mathbb{N}) \text{ and let } J_{X,\beta} \text{ be the Jacobi matrix given by } (7.21) \text{ with entries } \hat{\beta} = \{\hat{\beta}_n\}_{1}^{\infty} \text{ instead of } \beta. \ Here } \hat{\beta}_n = \text{diag}(\beta_{n,1}, \beta_{n,2}, \ldots, \beta_{n,p}) = \hat{\beta}_n^* \text{ are diagonal matrices, and } |\beta_{n,1}| \leq |\beta_{n,2}| \leq \ldots \leq |\beta_{n,p}|, \ n \in \mathbb{N}. \text{ If } \begin{align*}
\sum_{n \in \mathbb{N}} \sqrt{d_n d_{n+1}} |\beta_{n,1}| &= +\infty, \tag{8.8}
\end{align*}\]

then the minimal Jacobi operator $J_{X,\beta}$ is selfadjoint, $J_{X,\beta} = J_{X,\beta}^*$.

\[\text{Corollary 8.7. Let } \{d_n\}_{n=1}^{\infty} \in l^1(\mathbb{N}), \beta_n = \hat{\beta}_n + \beta_n = \beta_n^* \text{ and for } \hat{\beta} = \{\hat{\beta}_n\}_{1}^{\infty} \text{ all conditions of Theorem } 8.6 \text{ are satisfied. If in addition } ||\hat{\beta}_n||_{C_0 \times p} = O(d_n) \text{ for } n \to \infty, \tag{8.9}\]

then the Jacobi operator $J_{X,\beta}$ is selfadjoint, i.e. $J_{X,\beta} = J_{X,\beta}^*$.

9 Comparison with known results

9.1 Comparison with the results of Kostyuchenko and Mirzoev

Consider infinite block Jacobian matrix

$$J = \begin{pmatrix}
A_0 & B_0 & 0 & 0 & 0 & \ldots \\
B_0^* & A_1 & B_1 & 0 & 0 & \ldots \\
0 & B_1^* & A_2 & B_2 & 0 & \ldots \\
0 & 0 & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}, \tag{9.1}$$
where $A_n$, $B_n \in \mathbb{C}^{n \times p}$, $A_n = A_n^*$ and $\det B_n \neq 0$, $n \in \mathbb{N}_0$.

As we have already mentioned in the Introduction other conditions for Jacobi matrices to have maximal deficiency indices were obtained also in [38]. Here we compare Theorems 7.2, 7.6 with results from [38].

Following [38], we introduce the matrices $C_n$ assuming $C_0 := B_1^{-1}$, $C_1 := I_p$, and

$$
C_n = \begin{cases} 
(\alpha)B_{2j-1}^{-1}B_{2j}^* \ldots B_{2}^*B_{1}^{-1}, & \text{if } n = 2j, \\
(\beta)B_{2j}^{-1}B_{2j-1}^* \ldots B_{2}^*B_{1}^{-1}, & \text{if } n = 2j+1, \ j \in \mathbb{N}.
\end{cases}
$$

(9.2)

**Theorem 9.1** ([38]). *Let the sequence $\{C_n\}_{n=0}^{\infty}$ be given by the equalities (9.2). If the conditions

$$
\sum_{n=1}^{\infty} \|C_n\|^2 < +\infty,
$$

(9.3)

$$
\sum_{n=1}^{\infty} \|C_n^* A_n C_n\| < +\infty
$$

(9.4)

hold, then $J$ of the form (9.1) is in completely indeterminate case, i.e. $n_{\pm}(J) = p$.*

This result was applied to the Schrödinger operator with $\delta$–interactions on the semiaxis $(\sum_{n \in \mathbb{N}} d_n = \infty)$ in [33] (scalar case, $p = 1$) and in [35, 46] (matrix case, $p > 1$).

First we show that matrices $J_{X,\alpha}$ never satisfy conditions of Theorem 9.1 for any $\alpha$.

**Proposition 9.2.** *Let $J_{X,\alpha}$ be the Jacobi matrix of form (9.9) and let $C_n$ be the matrices of the form (9.2) constructed from the entries of $J_{X,\alpha}$. Then the series (9.4) diverges, hence $J_{X,\alpha}$ does not satisfy conditions of Theorem 9.1.*

**Proof.** In our case the entries $B_n$ and $A_n$ are

$$
B_n = \begin{cases} 
\frac{\nu(d_{j+1})}{d_{j+1}^2} I_p, & \text{if } n = 2j, \\
\frac{\nu(d_{j+1})}{d_{j+1}^2 d_{j+2}^2} I_p, & \text{if } n = 2j+1,
\end{cases} \quad A_n = \begin{cases} 
\frac{\alpha}{d_{j+1}}, & \text{if } n = 2j, \\
\frac{-\nu(d_{j+1})}{d_{j+1}^2} I_p, & \text{if } n = 2j+1.
\end{cases}
$$

(9.5)

In accordance with (9.2) and (9.5) we derive for $n = 2j+1$

$$
C_{2j+1} = (\alpha)B_{2j}^{-1}B_{2j-1}^* \ldots B_{2}^*B_{1}^{-1}
$$

$$
= (\alpha)^j \frac{d_{j+1}^2}{d_{j}^2} \frac{\nu(d_{j})}{\nu(d_{j+1})} \ldots \frac{d_{j}^2}{d_{j-1}^2} \frac{\nu(d_{j-1})}{\nu(d_{j})} \ldots \frac{d_{2}^2}{d_{1}^2} \frac{\nu(d_{2})}{\nu(d_{1})} \frac{d_{1}^3}{d_{2}^3}
$$

(9.6)

Let us check that (9.4) is violated for the matrix $J_{X,\alpha}$. Combining (9.5) with (9.6) for $n = 2j+1$, $j \in \mathbb{N}$, we obtain

$$
\sum_{j=1}^{\infty} \|C_{2j+1} A_{2j+1} C_{2j+1}\| = \sum_{j=1}^{\infty} \|C_{2j+1}\|^2 \cdot \|A_{2j+1}\| = \sum_{j=1}^{\infty} \frac{\nu^2(d_{j+1}) d_{j+1}^3}{\nu^2(d_{j+1}) d_{j+1}^3} \sum_{j=1}^{\infty} \frac{d_{j+1}^3}{d_{j+1}^3} \sqrt{1 + c^2 d_{j+1}^2}
$$

(9.7)

$$
= \frac{\nu^2(d_{1})}{c d_{1}^3} \sum_{j=1}^{\infty} \sqrt{1 + c^2 d_{j+1}^2} = +\infty.
$$

$$
= \frac{\nu^2(d_{1})}{c d_{1}^3} \sum_{j=1}^{\infty} \sqrt{1 + c^2 d_{j+1}^2} = +\infty.
$$


Thus the series (9.4) diverges.

It follows from (9.2) and (9.5) that for $n = 2j$

$$C_{2j} = (-1)^j B_{2j-1}^{-1} B_{2j-2} \ldots B_4 B_3^{-1} B_2^{-1}$$

$$= (-1)^j \frac{d_{3/2}^j d_{1/2}^j}{\nu(d_j)} \cdot \frac{d_{3/2}^{j-1} d_{1/2}^{j-1}}{\nu(d_{j-1})} \cdot \frac{\nu(d_{j-1})}{d_{j-1}^2} \ldots .$$

Combining this statement with condition (9.7), we conclude that series (9.4) always diverges.

Combining (9.5) with (9.8) for $n = 2j$, $j \in \mathbb{N}$, we obtain

$$\sum_{j=1}^{+\infty} \|C_{2j} A_2 C_{2j}\| = \sum_{j=1}^{+\infty} \frac{d_{j+1}^3}{\nu^2(d_1)} \cdot \|\alpha_j\|_{C^p \times p} = \frac{d_1^3}{\nu^2(d_1)} \sum_{j=1}^{+\infty} \|\alpha_j\|_{C^p \times p}.$$

Thus the series (9.4) for $n = 2j$ converges if and only if $\{\alpha_n\}_1^{+\infty} \in l^1(\mathbb{N}; \mathbb{C}^{p \times p})$.

Combining this statement with condition (9.7), we conclude that series (9.4) always diverges. \qed

Remark 9.3. Clearly Proposition 9.2 is of interest only in the case $\{d_n\}_{n=1}^{+\infty} \subseteq l^1(\mathbb{N})$. Indeed, if $\sum_{n=1}^{+\infty} d_n = +\infty$, then the Carleman test (1.2) implies $n_\pm(J,\alpha) = 0$.

Next we describe the area of applicability of Theorem 9.1 to matrices $J_{X,\beta}$.

Proposition 9.4. Let $\{d_n\}_{n=1}^{+\infty} \subseteq l^1(\mathbb{N})$. Let $J_{X,\beta}$ be the Jacobi matrix of form (7.21) and let $C_n$ be the matrices of the form (9.2) constructed from the entries of $J_{X,\beta}$. Then the series (9.4) converges if and only if $\{\beta_n\}_{n=1}^{+\infty} \subseteq l^1(\mathbb{N}; \mathbb{C}^{p \times p})$.

Proof. In our case the entries $B_n$ and $A_n$ are

$$B_n = \begin{cases} \frac{\nu(d_{j+1})}{d_{j+1}^2} \|p\|_p, & n = 2j, \\ \frac{\nu(d_{j+2})}{d_{j+2}^2 \nu(d_{j+1})} \|p\|_p, & n = 2j+1, \end{cases} \quad A_n = \begin{cases} \frac{\nu^2(d_{j+1})}{d_{j+1}^3} (\beta_{j+1} + d_{j+1}^2), & n = 2j, \\ \frac{\nu^2(d_{j+1})}{d_{j+1}^3} (\beta_{j+2} + d_{j+2}^2), & n = 2j+1. \end{cases} \quad (9.9)$$

For the case $n = 2j$ entries $A_n = \|p\|_p$, hence the convergence of series (9.4) is obvious.

In accordance with (9.2), (9.6) and (9.9) we derive for $n = 2j+1$

$$C_n = (-1)^j \frac{\nu^2(d_{j+1})}{d_{j+1}^3} \|p\|_p. \quad (9.10)$$

Combining (9.9) with (9.10) for $n = 2j+1$, $j \in \mathbb{N}$, we obtain

$$\sum_{j=1}^{+\infty} \|C_{2j+1} A_{2j+1} C_{2j+1}\| = \sum_{j=1}^{+\infty} \|C_{2j+1}\|^2 \cdot \|A_{2j+1}\| = \sum_{j=1}^{+\infty} \frac{\nu^2(d_{j+1})}{d_{j+1}^3} \cdot \frac{\nu^2(d_{j+1})}{d_{j+1}^3} \|\beta_{j+1} + d_{j+1}^2\|_{C^p \times p} \leq$$

$$\leq \frac{\nu^2(d_{j+1})}{d_{j+1}^3} \sum_{j=1}^{+\infty} \|\beta_{j+1} + d_{j+1}^2\|_{C^p \times p}.$$

It suffices to consider the case $\|\beta_j\|_{C^p \times p} > d_j$ for all $j \in \mathbb{N}$. Indeed, for those indices $j_k$, for which $\|\beta_{j_k}\|_{C^p \times p} < d_{j_k}$, he corresponding part of the series (9.11) converges. We have,

$$\sum_{j=1}^{+\infty} \frac{\nu^2(d_{j+1})}{d_{j+1}^3} \sum_{j=1}^{+\infty} (\|\beta_{j+1}\|_{C^p \times p} - d_{j+1}) \leq \sum_{j=1}^{+\infty} \|C_{2j+1} A_{2j+1} C_{2j+1}\| \leq \sum_{j=1}^{+\infty} \frac{\nu^2(d_{j+1})}{d_{j+1}^3} \sum_{j=1}^{+\infty} (\|\beta_{j+1}\|_{C^p \times p} + d_{j+1}).$$

(9.12)
Since \( \{d_n\}_{n=1}^{\infty} \in l^1(\mathbb{N}) \), then due to bilateral assessment \((9.12)\), series \((9.4)\) converges if and only if \( \{\beta_n\}_{n=1}^{\infty} \in l^1(\mathbb{N}; \mathbb{C}^{p \times p}) \).

**Remark 9.5.** Proposition 9.4 shows that conditions of Theorem 9.1 in comparison with the conditions of Proposition 9.4 coincide with that of Corollary 6.9. However, the conditions of Corollary 6.9 (Proposition 9.4) are violated.

Further, suppose that the blocks \( A_n \) and \( B_n \) of the Jacobi matrix \( J_{X,\beta} \) of form (7.28) for a special choice of matrix entries.

**Theorem 9.6 (24, Theorem 2).** Let integers \( p \geq 1 \) and \( p_1 \geq 0 \) satisfy the condition \( 0 \leq p_1 \leq p \) and the diagonal entries of the matrices \( B_n \) and \( R_n \) be defined by the formulae

\[
\tilde{B}_n = \text{diag} \left( \frac{1}{n+1}, \ldots, \frac{1}{n+1}, \frac{1}{m-p} \right), \quad n \geq 0,
\]

\[
R_0 = I_p, \quad R_n = \sqrt{I_p + \tilde{B}^2_{n-1}}, \quad n \geq 1.
\]

Further, suppose that the blocks \( A_n \) and \( B_n \) of the Jacobi matrix \( J_{X,\beta} \) (1.1) have the form

\[
A_n = \mathbb{O}_p, \quad B_0 = \tilde{B}_0, \quad B_n = \tilde{B}_{n-1}^{-1}R_n\tilde{B}_{n-1}^{-1}, \quad n \geq 1.
\]

Then \( n_{\pm}(J_{X,\beta}) = p_1 \).

**Proof.** Below we deduce Theorem 9.6 from our Proposition 7.7.

It is easily seen from (9.13) and (9.15) that entries of \( J_{X,\beta} \) allow representation

\[
A_{n,Dyuk} = \mathbb{O}_p, \quad n \geq 0, \quad B_{0,Dyuk} = I_p,
\]

\[
B_{n,Dyuk} = B'_{n,Dyuk} \oplus B''_{n,Dyuk}, \quad n \geq 1,
\]

where

\[
B'_{n,Dyuk} = \text{diag} \left( (n+1)\sqrt{n^2+1}, \ldots, (n+1)\sqrt{n^2+1} \right) \in \mathbb{C}^{p_1 \times p_1}, \quad n \geq 1,
\]

\[
B''_{n,Dyuk} = \text{diag} \left( \sqrt{2}, \ldots, \sqrt{2} \right) \in \mathbb{C}^{(p-p_1) \times (p-p_1)}, \quad n \geq 1.
\]
Let $J'_{Dyuk}$ be the Jacobi matrix with off-diagonal entries of form (9.17) and let $J''_{Dyuk}$ be the Jacobi matrix with off-diagonal entries of form (9.18). Thus $J_{Dyuk} = J'_{Dyuk} \oplus J''_{Dyuk}$.

Alongside with the Jacobi matrix $J_{Dyuk}$ we consider the Jacobi matrix $J'_{X,\beta} \subset \mathbb{C}^{p_1 \times p_1}$ of form (7.28) with $\beta_n = -d_n \|eta\|_p$ and $d_n = \frac{c}{(n+1)^{\frac{1}{2}n+1}}$, $n \geq 1$. Since $\{\beta_n\}^\infty_1 \in l^1(\mathbb{N}; \mathbb{C}^{p_1 \times p_1})$, Corollary 6.9 ensures that condition (6.21) is satisfied, hence Proposition 7.9 implies maximality indices for the Jacobi matrix $J_{X,\beta}$ given by (7.21), i.e. $n_{\pm}(J_{X,\beta}) = p_1$.

On the other hand, $\lim_{n \to \infty} \frac{d_n}{a_n} = 0$, and the matrix $J_{X,\beta}$ meets conditions of Proposition 7.7. Therefore combining this proposition with the previous equalities yields
\[
n_{\pm}(J'_{X,\beta}) = n_{\pm}(J_{X,\beta}) = p_1.
\]

These equalities allow us to compare Dukarev's matrix $J_{Dyuk}$ with the Jacobi matrix $J'_{X,\beta}$ which is simpler than the original matrix $J_{X,\beta}$. With the above choice of $\beta_n$ and $d_n$ the entries of the matrix $J'_{X,\beta}$ are given by
\[
A_n = \mathbb{O}_{p_1}, \quad n \geq 0,
B_n = \begin{cases}
(j+1)\sqrt{j^2+1}1_{p_1}, & n = 2j, \\
\frac{j+1}{(j+2)\sqrt{j^2+1}}1_{p_1}, & n = 2j+1, \\
\end{cases}, \quad j \geq 0.
\]

To prove the relations $n_{\pm}(J'_{X,\beta}) = n_{\pm}(J'_{Dyuk}) = p$ let us check the conditions of Corollary 5.2 for the pair $J'_{Dyuk}, J'_{X,\beta}$ treating the matrix $J'_{Dyuk}$ of form (9.15) as an unperturbed Jacobi matrix.

For $n = 2j$ condition (5.2) is obvious, because $B_n = B'_{n,Dyuk}$.

For $n = 2j + 1$ direct calculation shows that
\[
\lim_{j \to \infty} \|(B_{2j+1} - B'_{2j+1,Dyuk})(B'_{2j+1,Dyuk})^{-1}\|_{\mathbb{C}^{p_1 \times p_1}} = \frac{3}{4}.
\]

Therefore condition (5.2) of Corollary 5.2 is fulfilled with $N$ big enough, i.e.
\[
\sup_{j \geq N} \|(B_{2j+1} - B'_{2j+1,Dyuk})(B'_{2j+1,Dyuk})^{-1}\|_{\mathbb{C}^{p_1 \times p_1}} < 1.
\]

Further, for even $n = 2j$ direct calculation yields
\[
\lim_{j \to \infty} \|B_{2j} - B_{2j-1}(B'_{2j-1,Dyuk})^{-1}B'_{2j,Dyuk}\|_{\mathbb{C}^{p_1 \times p_1}} = 1,
\]
and the pair $J'_{Dyuk}, J'_{X,\beta}$ meets the condition (5.3) of Corollary 5.2, i.e.
\[
\sup_{j \geq 0} \|B_{2j} - B_{2j-1}(B'_{2j-1,Dyuk})^{-1}B'_{2j,Dyuk}\|_{\mathbb{C}^{p_1 \times p_1}} < \infty.
\]

Similarly, for odd $n = 2j + 1$ we derive
\[
\lim_{j \to \infty} \|B_{2j+1} - B_{2j}(B'_{2j,Dyuk})^{-1}B'_{2j+1,Dyuk}\|_{\mathbb{C}^{p_1 \times p_1}} = \frac{3}{2},
\]
and condition (5.3) for $n = 2j + 1$ is verified too.

Condition (5.9) is obviously satisfied because $A_n = A_{n,Dyuk} = \mathbb{O}_{p_1}$. Thus, the pair $J'_{Dyuk}, J'_{X,\beta}$ meets all conditions of Corollary 5.2 and hence $n_{\pm}(J'_{Dyuk}) = n_{\pm}(J'_{X,\beta})$. Therefore combining the latter with equality (9.19) yields $n_{\pm}(J'_{Dyuk}) = n_{\pm}(J'_{X,\beta}) = n_{\pm}(J_{X,\beta}) = p_1$.

Finally we apply Carleman test (1.2) to conclude that $n_{\pm}(J_{Dyuk}) = 0$ and
\[
n_{\pm}(J_{Dyuk}) = n_{\pm}(J'_{Dyuk} \oplus J''_{Dyuk}) = n_{\pm}(J'_{Dyuk}) + n_{\pm}(J''_{Dyuk}) = p_1 + 0 = p_1.
\]
This proves the result.
A Appendix

1. Boundary triplets. Let $A$ be a densely defined closed symmetric operator in a separable Hilbert space $\mathcal{H}$ with equal deficiency indices $n_{\pm}(A) = \dim \mathcal{N}_{\pm} \leq \infty$, where $\mathcal{N}_z := \ker(A^* - z)$ is the defect subspace.

Definition A.1 ([20, 21, 22, 29]). A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a (ordinary) boundary triplet for the adjoint operator $A^*$ if $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \text{dom}(A^*) \to \mathcal{H}$ are linear mappings such that the second abstract Green identity

$$ (A^*f, g)_{\mathcal{H}} - (f, A^*g)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \text{dom}(A^*), \quad (A.1) $$

holds and the mapping $\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : \text{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

First note that a boundary triplet for $A^*$ exists since the deficiency indices of $A$ are assumed to be equal. Moreover, $n_{\pm}(A) = \dim(\mathcal{H})$ and $A = A^* | (\ker(\Gamma_0) \cap \ker(\Gamma_1))$ hold. Note also that a boundary triplet for $A^*$ is not unique.

A closed extension $\tilde{A}$ of $A$ is called proper if $A \subset \tilde{A} \subset A^*$. The set of all proper extensions of $A$ is denoted by $\text{Ext} A$.

By [21, 22], if $\tilde{A}$ and $A_0 := A^* \upharpoonright \ker \Gamma_0$ are disjoint (i.e. $\ker \tilde{A} \cap \ker A_0 = \text{dom} A$), then the extension $\tilde{A}$ is parameterized in the following way

$$ \tilde{A} = A^* \upharpoonright \text{dom} \tilde{A}, \quad \text{dom} \tilde{A} = \{ f \in \text{dom} A^* : \Gamma_1 f = B \Gamma_0 f \}, \quad B \in C(\mathcal{H}). $$

In this case, $\tilde{A} := A_B$ and the operator $B$ is called the boundary operator of the extension $A_B$.

Proposition A.2 ([21, 22]). Let $A$ be a symmetric operator in $\mathcal{H}$, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet of the operator $A^*$ and $B \in C(\mathcal{H})$. Then:

(i) operator $A_B$ is symmetric (selfadjoint) if and only if so is $B$, and $n_{\pm}(A_B) = n_{\pm}(B)$ holds;

(ii) moreover, if $A_0$ is discrete, then the operator $A_B = A^*_B$ is discrete if and only if $B = B^*$ is discrete.

2. Dirac operator. Let $D_n$ be the minimal operator generated in $L^2([x_{n-1}, x_n]; \mathbb{C}^{2p})$ by the differential expression (6.1)

$$ D_n = D \upharpoonright \text{dom}(D_n), \quad \text{dom}(D_n) = W^{1,2}_0([x_{n-1}, x_n]; \mathbb{C}^{2p}). \quad (A.2) $$

Its adjoint $D_n^*$ is given by

$$ D_n^* = D \upharpoonright \text{dom}(D_n^*), \quad \text{dom}(D_n^*) = W^{1,2}_0([x_{n-1}, x_n]; \mathbb{C}^{2p}). $$

We define the minimal operator $D_X$ on $L^2(I; \mathbb{C}^{2p})$ by

$$ D_X := \bigoplus_{n \in \mathbb{N}} D_n, \quad \text{dom}(D_X) = W^{1,2}_0([x_{n-1}, x_n]; \mathbb{C}^{2p}) = \bigoplus_{n \in \mathbb{N}} W^{1,2}_0([x_{n-1}, x_n]; \mathbb{C}^{2p}) \quad (A.3) $$

where $D_n, n \in \mathbb{N}$, is given by (A.2). Its adjoint $D_X^*$ is given by

$$ D_X^* := \bigoplus_{n \in \mathbb{N}} D_n^*, \quad \text{dom}(D_X^*) = \bigoplus_{n \in \mathbb{N}} W^{1,2}_0([x_{n-1}, x_n]; \mathbb{C}^{2p}). \quad (A.4) $$
The boundary triplet \( \Pi^{(n)} = \{ \mathbb{C}^{2p}, \Gamma_0^{(n)}, \Gamma_1^{(n)} \} \) for the Dirac operator on \([x_{n-1}, x_n]\) is constructed elementarily:

\[
\begin{align*}
\Gamma_0^{(n)} f := \left( \frac{f_I(x_{n-1}^+)}{i c f_{II}(x_{n-1}^-)} \right), \\
\Gamma_1^{(n)} f := \left( \frac{i c f_{II}(x_{n-1}^+)}{f_I(x_{n-1}^-)} \right).
\end{align*}
\] (A.5)

However, the direct sum of boundary triplets, generally, is not a boundary triplet (see examples in [33, 43]).

In papers [17] (scalar case \( p = 1 \)) and [14, 15] (matrix case \( p > 1 \)) the boundary triplets for the operator \( D_X^* \) were constructed using the regularization technique developed in [33, 17] and [43].

**Theorem A.3** ([14] [17] [15]). Let \( X = \{x_n\}_{n=0}^\infty \subset I = (0, b) \) and \( d^*(X) < +\infty \). Define the mappings

\[
\Gamma_j^{(n)} : W^{1,2}([x_{n-1}, x_n]; \mathbb{C}^{2p}) \to \mathbb{C}^{2p}, \quad n \in \mathbb{N}, \quad j \in \{0, 1\},
\]

by setting

\[
\begin{align*}
\Gamma_0^{(n)} f := & \left( \frac{d_n^{1/2} f_I(x_{n-1}^+)}{i c d_n^{1/2} \sqrt{1 + \frac{1}{c^2 d_n^2}} f_{II}(x_{n-1}^-)} \right), \\
\Gamma_1^{(n)} f := & \left( \frac{i c d_n^{-1/2} (f_{II}(x_{n-1}^+) - f_{II}(x_{n-1}^-))}{d_n^{-3/2} \left(1 + \frac{1}{c^2 d_n^2}\right)^{-1/2} (f_I(x_{n-1}^+) - f_I(x_{n-1}^-) - i c d_n f_{II}(x_{n-1}^-))} \right).
\end{align*}
\] (A.6) (A.7)

Then:

(i) for any \( n \in \mathbb{N} \), \( \Pi^{(n)} = \{ \mathbb{C}^{2p}, \Gamma_0^{(n)}, \Gamma_1^{(n)} \} \) is a boundary triplet for \( D_n^* \);

(ii) the direct sum \( \Pi := \bigoplus_{n=1}^\infty \Pi^{(n)} = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) with \( \mathcal{H} = l^2(\mathbb{N}; \mathbb{C}^{2p}) \) and \( \Gamma_j = \bigoplus_{n=1}^\infty \Gamma_j^{(n)}, \ j \in \{0, 1\}, \) is a boundary triplet for the operator \( D_X^* = \bigoplus_{n=1}^\infty D_n^* \).

The next sentence explains the appearance of Jacobi matrices of the form (7.3) and (7.22) in the context of the Dirac operators.

**Proposition A.4** ([14] [17] [15]). Let \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) be the boundary triplet for the operator \( D_X^* \) of the form (A.6), (A.7). Also let \( B_{X, \gamma}, \gamma \in \{\alpha, \beta\} \) be the minimal Jacobi operator associated with a Jacobi matrix of the form (7.3) or (7.22). Then \( B_{X, \gamma} \) is a boundary operator for \( D_{X, \gamma} \), i.e.

\[
D_{X, \gamma} = DB_{X, \gamma} = D_X^* \mid \text{dom}(DB_{X, \gamma}), \quad \gamma \in \{\alpha, \beta\},
\]

\[
\text{dom}(DB_{X, \gamma}) = \{ f \in W^{1,2}(I \setminus X; \mathbb{C}^{2p}) : \Gamma_1 f = B_{X, \gamma} \Gamma_0 f \}.
\]

Moreover \( n_\pm(D_{X, \alpha}) = n_\pm(B_{X, \alpha}) \leq p \) (\( n_\pm(D_{X, \beta}) = n_\pm(B_{X, \beta}) \leq p \)). In particular, \( D_{X, \alpha} = D_{X, \alpha}^* = D_{X, \alpha} \) (\( D_{X, \beta} = D_{X, \beta}^* \) if and only if \( B_{X, \alpha} \) (\( B_{X, \beta} \)) is selfadjoint.

**Proposition A.5** ([14] [17] [15]). Let the operator \( D_{X, \alpha} \) (\( D_{X, \beta} \)) be selfadjoint. Then it is discrete if and only if

(i) \( \lim_{n \to \infty} d_n = 0 \);

(ii) Jacobi matrix \( B_{X, \alpha} = B_{X, \alpha}^* \) (\( B_{X, \beta} = B_{X, \beta}^* \)) has a discrete spectrum.

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