Leveraging vague prior information in general models via iteratively constructed Gamma-minimax estimators

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Abstract

Gamma-minimax estimation is an approach to incorporate prior information into an estimation procedure when it is implausible to specify one particular prior distribution. In this approach, we aim for an estimator that minimizes the worst-case Bayes risk over a set $\Gamma$ of prior distributions. Traditionally, Gamma-minimax estimation is defined for parametric models. In this paper, we define Gamma-minimaxity for general models and propose iterative algorithms with convergence guarantees to compute Gamma-minimax estimators for a general model space and a set of prior distributions constrained by generalized moments. We also propose encoding the space of candidate estimators by neural networks to enable flexible estimation. We illustrate our method in two settings, namely entropy estimation and a problem that arises in biodiversity studies.

1 Introduction

It is often of scientific interest to estimate an aspect of the data-generating mechanism underlying the data at hand. To obtain a sensible estimator, we often use certain principles to guide the search for a good estimator. Asymptotic efficiency [Pfanzagl 1990], minimaxity and Bayes optimality [Berger 1985] are popular examples of such principles. Defining the performance criteria underlying these principles requires specifying a model space, that is, a collection of possible data-generating mechanisms that contains the true, underlying distribution.
It is often desirable to incorporate prior information about the true data-generating mechanism into a statistical procedure. This might be done differently in different statistical paradigms. For example, prior information is often incorporated by specifying the model space. For frequentist methods such as those based on the asymptotic efficiency or minimax principle, this is the primary way to incorporate prior information. However, in some applications, there is more vague prior information that cannot be accurately represented in this manner. In the Bayesian paradigm, such information may be represented by further specifying a prior distribution (or prior for short) over the model space and aiming for an estimator that minimizes the induced Bayes risk. However, in many cases, there may be several priors that are compatible with the available information; this is especially likely to occur if the model space is rich. The Gamma-minimax paradigm provides a principled means to overcome this challenge. Under this paradigm, the statistician first specifies the set \( \Gamma \) of all priors that are all consistent with available prior information and subsequently aims for an estimator that minimizes the worst-case Bayes risk over the set of priors. The Gamma-minimax paradigm may be viewed as a robust version of the Bayesian paradigm [Vidakovic 2000]. Moreover, the Gamma-minimax paradigm is closely related to Bayes and minimax paradigms: when the set of priors consists of one prior, a Gamma-minimax estimator is Bayes with respect to that prior; when the set \( \Gamma \) of priors is the entire set of possible prior distributions, under mild conditions, a Gamma-minimax estimator is minimax [Wald 1945]. In this paper, we focus on Gamma-minimax estimation.

Gamma-minimax estimators have been studied in a variety of problems. Some explicit forms of Gamma-minimax estimators have been obtained. For example, [Olman and Shmundak 1985] studied Gamma-minimax estimation of the mean of a normal distribution for the set of symmetric and unimodal priors on an interval and obtained an explicit form when this interval is sufficiently small. [Eichenauer-Herrmann 1990] generalized this result to more general parametric models and [Eichenauer-Herrmann et al. 1994] obtained a further generalization with the requirement of symmetry on the priors dropped. [Chen et al. 1988] studied Gamma-minimax estimation for multinomial distributions and the set of priors with bounded mean. [Chen et al. 1991] studied Gamma-minimax estimation for one-parameter exponential families and the set of priors that place certain bounds on the first two moments. These results do not deal with general model spaces, such as semiparametric or nonparametric models, and general forms of the set of priors that may not directly impose bounds on prior moments of the parameters of interest. One reason for this lack of generality might be that, in the existing literature, Gamma-
minimaxity is usually defined only for parametric models. Another possible explanation is that it is typically intractable to analytically derive Gamma-minimax estimators.

To overcome this lack of analytical tractability, algorithms to compute a minimax or Gamma-minimax estimator have been proposed. Still, most of these works focus on parametric models. For example, Nelson (1966) and Kempthorne (1987) each proposed an algorithm to compute a minimax estimator. Bryan et al. (2007) and Schafer and Stark (2009) proposed an algorithm to compute an approximate confidence region of optimal expected size in the minimax sense. Noubiap and Seidel (2001) proposed an iterative algorithm to compute a Gamma-minimax decision for the set of priors constrained by generalized moment conditions. More recent works explored computing estimators under more general models. For example, Luedtke et al. (2020a) introduced an approach, termed Adversarial Monte Carlo meta-learning (AMC), for constructing minimax estimators. In the special case of prediction problems with mean-squared error, Luedtke et al. (2020b) studied the invariance properties of the decision problem and their implications for AMC.

In this paper, we make the following contributions.

1. We define Gamma-minimaxity in general models. Our general definition suggests an approach for leveraging potentially-vague prior information even when the statistical model is infinite dimensional.

2. We propose iterative algorithms to compute Gamma-minimax estimators for a general model space and a set of priors constrained by generalized moments. Such constraints provide a natural means to represent prior information (Berger, 1990). To the best of our knowledge, this is the first algorithm to compute Gamma-minimax estimators under general models, including infinite-dimensional models. We also show that, for certain problems, there is a unique Gamma-minimax estimator and, moreover, our computed estimator converges to this estimator as the number of iteration increases to infinity.

3. Similarly to the approach proposed in Noubiap and Seidel (2001), our proposed iterative algorithm involves solving a minimax optimization problem in each intermediate step. However, we explicitly describe algorithms to solve these minimax problems. Moreover, these algorithms are not nested optimizations and therefore may take less time to converge. When the space of estimators can be parameterized by a Euclidean space and gradients are available, we propose to use a gradient-based
algorithm or a stochastic variant thereof. When gradients are unavailable, we propose to instead use fictitious play \cite{Brown1951, Robinson1951} and provide a convergence result that, unlike the results in \cite{Robinson1951}, is applicable even when the space of estimators is an infinite set.

4. Like the approach proposed in \cite{NoubiapSeidel2001}, our proposed iterative algorithm relies on increasingly fine finite grids over the model space. However, since we allow the model space to be high or even infinite dimensional, randomly adding grid points to the grid may lead to unacceptably slow convergence. To overcome this challenge, we propose an algorithm that is similar to a Markov chain Monte Carlo (MCMC) method to efficiently construct such grids.

5. We utilize recent advances in neural networks, especially adversarial learning \cite[e.g.,][]{Goodfellow2014, Luedtke2020a,b}, when specifying the space of estimators. We also discuss an alternative parameterization using extreme learning machines \cite{Huang2006a} and show that, if the Gamma-minimax estimator is unique, our computed estimator converges to the Gamma-minimax estimator under this parameterization. Thanks to the universal approximation properties of neural networks \cite[e.g.,][]{Hornik1991, Csaji2001} and extreme learning machines \cite{Huang2006a}, we also show that both of these parameterizations can achieve good performance for sufficiently large networks. Furthermore, inspired by pre-training \cite[e.g.,][]{Erhan2010} and transfer learning \cite[e.g.,][]{Torrey2009}, we recommend leveraging knowledge of existing estimators as inputs to the network in settings where this is possible. Under such choices of the space of estimators, we can expect to obtain a reasonably good estimator even if the associated nonconvex-concave minimax problems prove to be difficult.

This paper is organized as follows. In Section \ref{sec:framework} we introduce the framework of Gamma-minimax estimation and regularity conditions that we assume throughout the paper. In Section \ref{sec:algorithm} we describe our proposed algorithms. Our proposal involves two layers of iterations, and we describe algorithms for the first and second layer in Sections \ref{sec:inner_algorithm} and \ref{sec:outer_algorithm} respectively. In Section \ref{sec:hyperparameters} we discuss considerations when choosing hyperparameters in the algorithms. In Section \ref{sec:simulation} we demonstrate our method in three simulation studies. We conclude with a discussion in Section \ref{sec:discussion}. Proof sketches of key results are provided in the main text, and complete proofs can be found in the appendix. All simulation codes are available at \url{https://github.com/QIU-Hongxiang-David/Gamma-minimax-learning}. 
2 Problem setup

Let $\mathcal{M}$ be a space of data-generating mechanisms $P$ that contains the truth, $P_0$, and let $\rho$ be a metric on $\mathcal{M}$. Under a data-generating mechanism $P \in \mathcal{M}$, let $X^* \in X^*$ denote the random data being generated. Let $C$ denote a known coarsening mechanism such that the observed data $X = C(X^*)$ belongs to $X$. In some cases, the coarsening mechanism will be the identity map, whereas in other settings, such as those in which there is censored or missing data, the coarsening mechanism will be more involved (e.g., Birmingham et al., 2003; Gill et al., 1997; Heitjan and Rubin, 1991; Heitjan, 1993, 1994). Let $D$ denote the space of estimators (or decision functions) equipped with a metric $\varrho$. Let $R : D \times \mathcal{M} \to \mathbb{R}$ denote a risk function that measures the performance of an estimator under a data-generating mechanism such that smaller risks are preferable. We suppose throughout that $\mathcal{M}$ and $D$ are equipped with the topologies induced by $\rho$ and $\varrho$, respectively.

We now present two examples in which we formulate statistical decision problems in the above form.

Example 1. (Point estimation) Suppose that $\mathcal{M}$ statistical model, which may be parametric, semi-parametric, or locally nonparametric (Bickel et al., 1993). The data $X^*$ consists of independently and identically distributed random variables following the true distribution $P_0 \in \mathcal{M}$. We set $C$ to be the identity function so that $X = X^*$. We wish to estimate an aspect $\Psi(P_0) \in \mathbb{R}$ of $P_0$. Then, we can consider $D$ being a set of functions $X \to \mathbb{R}$ and set the risk to be induced by the quadratic loss, that is,

$$R(d, P) = \mathbb{E}_{P_0}[d(X) - \Psi(P_0)]^2. $$

Example 2. (Predicting the expected number of new categories) Suppose that $\mathcal{M}$ consists of multinomial distributions with an unknown number of categories. Let an independent and identically distributed (iid) random sample of size $n$ be generated from the true multinomial distribution, so that $X^*$ is a multiset containing the number of observations $X_k$ in each category $k$. Let the observed data be the multiset containing only the nonzero entries of $X^*$, so that $X = C(X^*) = \{X_k : X_k \neq 0\}$. Hence, only categories with nonzero occurrence are observed. Then, we may wish to predict the number of new categories that would be observed if a new sample of size $m$ were collected. This problem has been extensively studied in the literature with applications in microbiome data, species taxonomic surveys, assessment of vocabulary size, etc. (e.g., Shen et al., 2003; Bunge et al., 2014; Orlitsky et al., 2016). We now show how to formulate this prediction problem in our framework. For each $P \in \mathcal{M}$, let $p_k (k = 1, \ldots, K)$ be the probability of category $k$, and let $\Psi(P) : X^* \mapsto \sum_{k=1}^{K} I(X_k = 0)(1 - (1 - p_k)^m)$ be the expected
number of new observed categories given the current full data $X^*$. We consider $D$ to be a set of $X \rightarrow \mathbb{R}$ functions and set the risk to be the mean-squared error, that is, $R(d, P) = \mathbb{E}_P[\{d(X) - \Psi(P)(X^*)\}^2]$.

We now define Gamma-minimaxity within our decision theoretic framework. We assume that $\mathcal{M}$ is equipped with the Borel $\sigma$-field and let $\Pi$ denote the set of all probability distributions on the measurable space $(\mathcal{M}, \mathcal{B})$. We also assume that, for any $d \in D$ and any $\pi \in \Pi$, $P \mapsto R(d, P)$ is $\pi$-integrable. The Bayes risk corresponding to an estimator $d$ and a prior $\pi$ is defined as $r : (d, \pi) \mapsto \int R(d, P) \pi(dP)$. Let $\Gamma \subseteq \Pi$ be the set of priors such that all $\pi \in \Gamma$ are consistent with prior information. An estimator is called a $\Gamma$-minimax estimator if it is in the set $\operatorname{argmin}_{d \in D} \sup_{\pi \in \Gamma} r(d, \pi)$.

In this paper, we consider the case in which $\Gamma$ is characterized by finitely many generalized moment conditions, that is, $\Gamma = \{\pi \in \Pi : \Phi_k \in L^1(\pi), \int \Phi_k(P) \pi(dP) \leq c_k, k = 1, \ldots, K\}$ where each $\Phi_k : \mathcal{M} \rightarrow \mathbb{R}$ is a prespecified function that extracts an aspect of a data-generating mechanism and $c_k \in \mathbb{R}$ is a prespecified constant. Such constraints can represent a variety of forms of prior information. For example, with $\Phi_k = \pm \Psi^\kappa$ for some $\kappa \geq 1$, $\Gamma$ imposes bounds on prior moments of $\Psi(P)$; with $\Phi_k(P) = \pm 1(\Psi(P) \in I)$ for some known interval $I$, $\Gamma$ imposes bounds on the prior probability of $\Psi(P)$ lying in $A$. Similar prior information on aspects of $P_0$ other than $\Psi(P_0)$ can also be represented. In addition, note that an equality can be equivalently expressed by two inequalities, $\Gamma$ may also impose equality constraints on prior generalized moments.

We assume that the following conditions hold throughout the rest of the paper.

**Condition 1.** $\mathcal{M}$ is separable.

**Condition 2.** $D$ is compact.

**Condition 3.** (i) $R : D \times \mathcal{M} \rightarrow \mathbb{R}$ is a bounded function and (ii) $d \mapsto R(d, P)$ is Lipschitz continuous with a universal Lipschitz constant $L \in (0, \infty)$ independent of $P \in \mathcal{M}$, that is, there exists an $L$ so that $|R(d_1, P) - R(d_2, P)| \leq L\rho(d_1, d_2)$ for any $d_1, d_2 \in D$ and any $P \in \mathcal{M}$.

Condition 2 is satisfied by many interesting classes of estimators. For example, we may choose $D$ to be a space of neural networks whose indexing parameters fall in some specified compact set. We now illustrate the plausibility of the other two conditions in Example 1. For Condition 3, if the metric $\rho$ on $\mathcal{M}$ is chosen as the supremum norm of the difference in cumulative distribution functions, then a countable dense subset of $\mathcal{M}$ can be the set of all empirical distributions with support contained in a countable dense subset of $\mathcal{X}$. If we instead assume that $\mathcal{X}$ is contained in a Euclidean space and
all distributions in $M$ have a differentiable Lebesgue density, then we may choose the metric to be the supremum norm of the difference of density functions. A countable dense subset of $M$ is then the set of all kernel densities with locations being rational points in $X$ and scales being positive rational numbers. For Condition 3, suppose that all distributions in $M$ are dominated by a measure $\mu$ and their density functions are uniformly bounded. If $\int d(X)^2 \mu(dX)$ is uniformly bounded and $\Psi$ is bounded, then $E_P[|d_1(X) - d_2(X)|^2] \lesssim \|d_1 - d_2\|_{L^2(\mu)}$, where $\lesssim$ stands for less than or equal to up to a multiplicative constant and $\|\cdot\|_{L^2(\mu)}$ and $\|\cdot\|_{L^2(P)}$ denote the $L^2(\mu)$- and $L^2(P)$-distance, respectively. Therefore, Condition 3 holds for $\varrho$ being the $L^2(\mu)$-distance.

Example 2 is similar.

3 Proposed algorithm to compute a $\Gamma$-minimax estimator

Our proposed iterative algorithm consists of two layers of iterations: the outer layer described in Section 3.1 is used to approximate the minimax problem with its discretized version on an increasingly fine grid; the inner layer described in Section 3.2 is used to solve the discretized minimax problem constructed in the outer layer. We now describe these two layers separately.

3.1 Grid-based approximation of $\Gamma$-minimax estimators

In this section, we present an algorithm that is similar to that in Noubiap and Seidel (2001) but can be applied to richer model spaces, including to nonparametric models. Let $\{M_\ell\}_{\ell=1}^\infty$ be an increasing sequence of finite subsets of $M$ such that $\bigcup_{\ell=1}^\infty M_\ell$ is dense in $M$. That is, $\{M_\ell\}_{\ell=1}^\infty$ is an increasingly fine grid over $M$. By Condition 1 such an $\{M_\ell\}_{\ell=1}^\infty$ necessarily exists. Define $\Gamma_\ell := \{\pi \in \Gamma : \pi \text{ has support in } M_\ell\}$.

We also define $r_{\sup}(d, \Gamma') := \sup_{\pi \in \Gamma'} r(d, \pi)$ for any $d \in D$ and $\Gamma' \subseteq \Pi$.

In this section, we propose an algorithm (Algorithm 1) that approximates a $\Gamma$-minimax estimator with a $\Gamma_\ell$-minimax estimator. We will show that the approximation error decays to zero as $\ell$ grows to infinity. We note that, under Condition 3 $d \mapsto r_{\sup}(d, \Gamma_\ell)$ is continuous for all $\ell$ by Lemma 2, and hence $d_\ell^*$ exists. Here and in the rest of the algorithms in the paper, for any real-valued function $f$, when we assign $\arg\min_x f(x)$ or $\arg\max_x f(x)$ to a variable, we arbitrarily pick a minimizer or maximizer if there are multiple optimizers. We note that the minimax problem in Line 3 of Algorithm 1 is nontrivial to
solve, and therefore we propose two algorithms to solve it in Section 3.2.

**Algorithm 1** Iteratively approximate a $\Gamma$-minimax estimator over an increasingly fine grid.

1: for $\ell = 1, 2, \ldots$ do
2: Construct a grid $\mathcal{M}_\ell \subseteq \mathcal{M}$ such that $\mathcal{M}_{\ell-1} \subseteq \mathcal{M}_\ell$
3: $d^*_\ell \leftarrow \arg\min_{d \in \mathcal{D}} \sup_{\pi \in \mathcal{\Gamma}_\ell} r(d, \pi)$

We will present algorithms to find $\Gamma_\ell$-minimax estimators $d^*_\ell$ in Section 3.2.

Let $d^* \in \mathcal{D}$ be an accumulation point of the sequence $\{d^*_\ell\}_{\ell=1}^\infty$, which is guaranteed to exist by Condition 2. We next present a sufficient condition to ensure that $d^*$ is $\Gamma$-minimax, so that $d^*_\ell$ is approximately $\Gamma$-minimax for sufficiently large $\ell$.

**Condition 4.** We assume that there exists an increasing sequence $\{\Omega_\ell\}_{\ell=1}^\infty$ of subsets of $\mathcal{M}$ such that

1. $\bigcup_{\ell=1}^\infty \Omega_\ell = \mathcal{M}$;
2. for all $\ell = 1, 2, \ldots$ and all $d \in \mathcal{D}$, it holds that
   \[
   \lim_{i \to \infty} r_{\sup}(d, \Gamma_{i|\ell}) = r_{\sup}(d, \tilde{\Gamma}_\ell),
   \]

   where $\tilde{\Gamma}_\ell := \{\pi \in \Gamma : \pi \text{ has support in } \Omega_\ell\}$ and $\Gamma_{i|\ell} := \{\pi \in \Gamma : \pi \text{ has support in } \mathcal{M}_i \cap \Omega_\ell\}$.

We note that, in contrast to $\mathcal{M}_\ell$, $\Omega_\ell$ may be an infinite set. We may expect Condition 4 to hold in many cases. Exceptions may be caused by $P \mapsto R(d, P)$ being discontinuous. Another cause could be that $\Gamma$ imposes a constraint on $\mathcal{M}$ such that no prior in $\Gamma$ has support contained in $\mathcal{M}_i$ and hence $\Gamma_{i|\ell} = \emptyset$, but this can be resolved by rewriting the problem such that the constraint is incorporated into the specification of $\mathcal{M}$. We now present the theorem on $\Gamma$-minimaxity of $d^*$.

**Theorem 1.** Under Conditions 1–4, $d^*$ is $\Gamma$-minimax and

\[
r_{\sup}(d^*_\ell, \Gamma_\ell) \nearrow \min_{d \in \mathcal{D}} r_{\sup}(d, \Gamma)
\]
as $\ell \to \infty$.

To prove Theorem 1, we utilize a result in Pinelis (2016) to establish that $r_{\sup}(d, \Gamma)$ can be approximated arbitrarily well by a discrete prior in $\Gamma$ for any $d \in \mathcal{D}$. This is a key ingredient in the proof of Lemma 1 which states that, for any $d \in \mathcal{D}$, $r_{\sup}(d, \tilde{\Gamma}_\ell)$ converges to $r_{\sup}(d, \Gamma)$. Then, we show that the
sequence \( \{r_{\sup}(d^*_\ell, \Gamma_\ell)\}_{\ell=1}^\infty \) is nondecreasing and upper bounded by \( \inf_{d \in \mathcal{D}} r_{\sup}(d, \Gamma) \), which is less than or equal to the \( \Gamma \)-maximal Bayes risk \( r_{\sup}(d^*, \Gamma) \) of the earlier-defined accumulation point \( d^* \) of \( \{d^*_\ell\}_{\ell=1}^\infty \).

Therefore, \( r_{\sup}(d^*_\ell, \Gamma_\ell) \) converges to a limit. We finally use a contradiction argument to prove that this limit is greater than or equal to \( r_{\sup}(d^*, \Gamma) \), which implies Theorem 1.

We have the following corollary on the uniqueness of the \( \Gamma \)-minimax estimator and the convergence of \( \{d^*_\ell\}_{\ell=1}^\infty \) for certain problems.

**Corollary 1.** Suppose that \( \mathcal{D} \) is a convex subset of a vector space, \( d \mapsto R(d, P) \) is strictly convex for each \( P \in \mathcal{M} \), and \( r_{\sup}(d, \Gamma) \) is attainable for each \( d \in \mathcal{D} \) in the sense that, for all \( d \in \mathcal{D} \), there exists a \( \pi \in \Gamma \) such that \( r(d, \pi) = r_{\sup}(d, \Gamma) \). Under Conditions 1–4, \( d^* \) is the unique \( \Gamma \)-minimax estimator and \( d^*_\ell \rightarrow d^* \) as \( \ell \rightarrow \infty \).

We prove Corollary 1 by establishing that \( d \mapsto r_{\sup}(d, \Gamma) \) is strictly convex.

In practice, the user also needs to specify a stopping criterion for Algorithm 1. In Noubiap and Seidel (2001), the authors proposed to compute or approximate \( r_{\sup}(d^*_\ell, \Gamma_\ell) \) and stop if \( r_{\sup}(d^*_\ell, \Gamma_\ell) \) is sufficiently close to \( r_{\sup}(d^*, \Gamma_\ell) \). However, the procedure to approximate \( r_{\sup}(d^*_\ell, \Gamma_\ell) \) in that work relies on the compactness of \( \mathcal{M} \), but we do not want to assume this condition because it may restrict the applicability of the method. Therefore, we propose to use the following alternative criterion: stop if \( r_{\sup}(d^*_\ell, \Gamma_{\ell+1}) - r_{\sup}(d^*_\ell, \Gamma_\ell) \leq \epsilon \) for a prespecified tolerance level \( \epsilon > 0 \). Note that this criterion does not guarantee that \( r_{\sup}(d^*_\ell, \Gamma_\ell) \) is close to \( r_{\sup}(d^*, \Gamma) \). For example, if \( \mathcal{M}_{\ell+1} \setminus \mathcal{M}_\ell \) is small, it is even possible that \( r_{\sup}(d^*_\ell, \Gamma_{\ell+1}) - r_{\sup}(d^*_\ell, \Gamma_\ell) = 0 \), but \( d^*_\ell \) is far from being \( \Gamma \)-minimax. We discuss this issue in more detail in Section 4.1.

We finally remark that \( r_{\sup}(d, \Gamma_\ell) \) may be difficult to evaluate exactly. Since the risk is often an expectation, we recommend approximating \( r_{\sup}(d, \Gamma_\ell) \) for any given \( d \) via Monte Carlo as follows: first, estimate risks \( R(d, P) \) for all \( P \in \mathcal{M}_\ell \) with a large number of Monte Carlo runs; second, estimate the corresponding least favorable prior \( \pi_{d, \ell} \in \arg\max_{\pi \in \Gamma_\ell} r(d, \pi) \) using the estimated risks; third, estimate the risks \( R(d, P) \) (\( P \in \mathcal{M}_\ell \)) again with independent Monte Carlo runs, and, finally, calculate \( r(d, \pi_{d, \ell}) \) with the estimated risks and the estimated least favorable prior. Using two independent estimates of the risk can remove the positive bias that would otherwise arise due to using the same data to estimate the risks and the least favorable prior.
3.2 Computation of a $\Gamma_r$-minimax estimator

In this section, we present two candidate algorithms to compute a $\Gamma_r$-minimax estimator, which corresponds to Line 3 in Algorithm 1. One is based on gradient and generally more computationally feasible, but requires differentiability of $R$ in the parameters indexing the estimators; the other is more computationally intensive by requiring computing a sequence of Bayes estimators for a sequence of priors, but is also more general in that it does not rely on any differentiability conditions.

3.2.1 (Stochastic) gradient descent with max-oracle

Gradient descent with max-oracle (GDmax) and its stochastic variant (SGDmax), which were presented in Lin et al. (2019), can be used to solve general minimax problems in Euclidean spaces. To apply these algorithms to find a $\Gamma_r$-minimax estimator, we need to assume that $D$ can be parameterized by a subset of a Euclidean space, that is, that for any $d \in D$, there exists a real vector-valued coefficient $\beta$ in a compact set $H \subseteq \mathbb{R}^D$ such that $d$ may be written as $d(\beta)$. For example, $D$ may be a neural network class. More discussions on the parameterization of $D$ are in Section 4.2. In this section, in a slight abuse of notation, we define $R(\beta, P) := R(d(\beta), P)$, $r(\beta, \pi) := r(d(\beta), \pi)$ and $r_{\sup}(\beta, \Gamma_r) := r_{\sup}(d(\beta), \Gamma_r)$ for a coefficient $\beta \in \mathbb{R}^D$, data-generating mechanism $P \in \mathcal{M}$ and prior $\pi \in \Gamma$. We assume that $\beta \mapsto R(\beta, P)$ is differentiable for all $P \in \mathcal{M}$, and hence so is $\beta \mapsto r(\beta, \pi)$ for all $\pi \in \Gamma$. We also assume that a coefficient $\beta^*_r \in \arg\min_{\beta \in H} r_{\sup}(\beta, \Gamma_r)$ also minimizes the same function over $\mathbb{R}^D$, so that we may solve the minimax problem over the unbounded space $\mathbb{R}^D$ ignoring the specification of $H$.

We now present GDmax and SGDmax in our context of finding a $\Gamma_r$-minimax estimator. If we can evaluate $R(\beta, P)$ exactly for all $\beta \in H$ and $P \in \mathcal{M}_r$, then the GDmax algorithm (Algorithm 2) may be used. Note that the Line 3 can be formulated into a linear program, which can always be solved in polynomial time with an interior point method (e.g., Jiang et al., 2020) and often be solved in polynomial time with a simplex method (Spielman and Teng, 2004).

**Algorithm 2** Gradient descent with max-oracle (GDmax) to compute a $\Gamma_r$-minimax estimator

1: Initialize $\beta_{(0)} \in \mathbb{R}^D$. Set learning rate $\eta > 0$ and max-oracle accuracy $\zeta > 0$.
2: for $t = 1, 2, \ldots$ do
3:     Maximization: find $\pi_{(t)} \in \Gamma_r$ such that $r(\beta_{(t-1)}, \pi_{(t)}) \geq \max_{\pi \in \Gamma_r} r(\beta_{(t-1)}, \pi) - \zeta$
4:     Gradient descent: $\beta_{(t)} \leftarrow \beta_{(t-1)} - \eta \nabla_r R(\beta_{(t-1)}, \pi_{(t)})|_{\beta=\beta_{(t-1)}}$

In many cases, it is difficult to evaluate $R(\beta, P)$ exactly. When $R(\beta, P)$ is expressed as an expect-
tion, \( R(\beta, P) \) may instead be approximated using Monte Carlo techniques. With \( \xi \) being an exogenous source of randomness according to law \( \Xi \), let \( \hat{R}(\beta, P, \xi) \) be an unbiased approximation of \( R(\beta, P) \) with 
\[
\mathbb{E}[\|\nabla_\beta \hat{R}(\beta, P, \xi) - R(\beta, P)\|^2] \leq \sigma^2 < \infty,
\]
where \( \| \cdot \| \) denotes the \( \ell_2 \)-norm in Euclidean spaces. Let 
\[
\hat{r}(\beta, \pi, \xi) := \int \hat{R}(\beta, P, \xi) \pi(dP)
\]
for \( \pi \in \Gamma_\ell \). In this case, SGDmax (Algorithm 3) may be used to find a (locally) \( \Gamma_\ell \)-minimax estimator. Note that Algorithm 3 represents a generalization of the nested minimax AMC strategy in Luedtke et al. (2020a) to \( \Gamma_\ell \)-minimax problems.

**Algorithm 3**

Stochastic gradient descent with max-oracle (SGDmax) to compute a \( \Gamma_\ell \)-minimax estimator

1. Initialize \( \beta(0) \in \mathbb{R}^D \). Set learning rate \( \eta > 0 \), max-oracle accuracy \( \zeta > 0 \) and batch size \( J \).
2. for \( t = 1, 2, \ldots \) do
3. Stochastic maximization: use a stochastic procedure to find \( \pi(t) \in \Gamma_\ell \) such that
\[
\mathbb{E}[\hat{r}(\beta(t-1), \pi(t))] \geq \max_{\pi \in \Gamma_\ell} r(\beta(t-1), \pi) - \zeta,
\]
where the expectation is over the randomness in stochastic maximization (e.g., variants of stochastic gradient ascent).
4. Generate iid copies \( \xi_1, \ldots, \xi_J \) of \( \xi \).
5. Stochastic gradient descent: \( \beta(t) \leftarrow \beta(t-1) - \eta \sum_{j=1}^J \nabla_\beta \hat{r}(\beta(t-1), \xi_j) \). 

We now present further conditions needed for the convergence result for Algorithms 2 and 3.

**Condition 5.** For each \( \ell = 1, 2, \ldots \), \( \beta \mapsto R(\beta, P) \) is Lipschitz continuous with a universal Lipschitz constant \( L_1 \) independent of \( P \in M_\ell \).

Note that Condition 5 differs from Condition 3 in that the former relies on the parameterization of \( \mathcal{D} \) in a Euclidean space equipped with the Euclidean norm, while the latter may rely on a different metric on \( \mathcal{D} \) such as an \( L^2 \)-distance. In addition, the Lipschitz constant in Condition 5 may depend on \( \ell \), while that in Condition 3 must not.

**Condition 6.** For each \( \ell = 1, 2, \ldots \), \( \nabla_\beta R(\beta, P) \) is bounded; \( \beta \mapsto \nabla_\beta R(\beta, P) \) is Lipschitz continuous with a universal Lipschitz constant \( L_2 \) independent of \( P \in M_\ell \).

Under these conditions, using the results in Lin et al. (2019), we can show that, in general, GDmax and SGDmax can yield a local minimum of \( \beta \mapsto r_{\sup}(\beta, \Gamma_\ell) \) when the algorithms’ hyperparameters are suitably chosen. Before we formally present the theorem, we introduce some definitions related to locally optimality of a potentially nondifferentiable and nonconvex function. A real-valued function \( f \) is called \( q \)-weakly convex if 
\[
x \mapsto f(x) + (q/2)\|x\|^2
\]
is convex \( (q > 0) \). The Moreau envelope of a real-valued function \( f \) with parameter \( q > 0 \) is 
\[
f_q : x \mapsto \min_{x'} f(x') + \|x' - x\|^2/(2q).
\]
A point \( x \) is an \( \epsilon \)-stationary point \( (\epsilon \geq 0) \) of a \( q \)-weakly convex function \( f \) if \( \|\nabla f_1/(2q)(x)\| \leq \epsilon \). Similarly, a random point \( x \) is an
\(\epsilon\)-stationary point \((\epsilon \geq 0)\) of a \(q\)-weakly convex function \(f\) in expectation if \(E[\|\nabla f_1/(2q)(x)\|] \leq \epsilon\). If \(x\) is an \(\epsilon\)-stationary point in expectation, we may conclude that it is an \(\epsilon\)-stationary point with high probability by Markov’s inequality. Lemma 3.8 in [Lin et al. 2019] shows that an \(\epsilon\)-stationary point of \(f\) is close to a point \(x'\) at which \(f\) has at least one small subgradient for small \(\epsilon\).

We next present the convergence result for Algorithms 2 and 3.

**Theorem 2.** Suppose that Conditions 1–3, 5–6 hold. Let \(\epsilon > 0\) be fixed and define \(\Delta := (r_{\sup})_{1/(2L_1)}(\beta(0)) - \min_{\beta \in \mathbb{R}} (r_{\sup})_{1/(2L_1)}(\beta)\), where we recall that \((r_{\sup})_{1/(2L_1)}\) is the Moreau envelope of \(r_{\sup}\) with parameter \(1/(2L_1)\).

\[\square\] In Algorithm 3 with \(\eta = \epsilon^2/(L_1L_2^2)\) and \(\zeta = \epsilon^2/(24L_1)\), \(\beta(t)\) is an \(\epsilon\)-stationary point of \(\beta \mapsto r_{\sup}(\beta, \Gamma_t)\) for \(t = O(L_1L_2\Delta/\epsilon^4)\).

\[\square\] In Algorithm 3 with \(\eta = \epsilon^2/[L_1(L_2^2 + \sigma^2)]\), \(\zeta = \epsilon^2/(24L_1)\) and \(J = 1\), \(\beta(t)\) is an \(\epsilon\)-stationary point of \(\beta \mapsto r_{\sup}(\beta, \Gamma_t)\) in expectation for \(t = O(L_1(L_2^2 + \sigma^2)\Delta/\epsilon^4)\).

It may be inconvenient to implement Line 3 in Algorithm 3 because linear program solvers often do not use stochastic optimization. Therefore, we propose a variant (Algorithm 4) by replacing this line with Lines 3–4 so that ordinary linear program solvers can be directly applied. The following theorem justifies this variant.

**Algorithm 4** Convenient variant of SGDmax (Algorithm 3) to compute a \(\Gamma_\ell\)-minimax estimator

1. Initialize \(\beta(0) \in \mathbb{R}^D\). Set learning rate \(\eta > 0\) and batch sizes \(J, J'\).
2. for \(t = 1, 2, \ldots\) do
3. Generate iid copies \(\xi_1, \ldots, \xi_{J'}\) of \(\xi\).
4. Stochastic maximization: \(\pi(t) \leftarrow \text{argmax}_{\pi \in \Gamma_t} \frac{1}{J} \sum_{j=1}^{J'} \hat{r}(\beta(t-1), \pi, \xi_j)\).
5. Generate iid copies of \(\xi_{J'+1}, \ldots, \xi_{J'+J}\) of \(\xi\).
6. Stochastic gradient descent: \(\beta(t) \leftarrow \beta(t-1) - \frac{\eta}{J} \sum_{j=J'+1}^{J'+J} \nabla_{\beta} \hat{r}(\beta, \pi(t), \xi_j)\big|_{\beta=\beta(t-1)}\).

**Theorem 3.** Suppose that \(\{\xi \mapsto \hat{r}(\beta, \pi, \xi) : \beta \in \mathbb{R}^D, \pi \in \Gamma_t\}\) is a \(\Xi\)-Glivenko-Centelli class. For any \(\zeta > 0\), there exists a sufficiently large \(J'\) such that \(E[r(\beta(t-1), \pi(t))] \geq \max_{\pi \in \Gamma_t} r(\beta(t-1), \pi) - \zeta\) for all \(t\), where the expectation is taken over \(\pi(t)\) and \(\beta(t-1)\) is fixed. Therefore, with the chosen parameters in Theorem 2, we may choose a sufficiently large \(J'\) so that \(\beta(t)\) is an \(\epsilon\)-stationary point of \(\beta \mapsto r_{\sup}(\beta, \Gamma_t)\) in expectation for \(t = O(L_1(L_2^2 + \sigma^2)\Delta/\epsilon^4)\).
We prove Theorem 3 by showing that \( \max_{\pi \in \Gamma} r(\beta_{(t-1)}, \pi) - E[r(\beta_{(t-1)}, \pi_{(t)})] \) converges to 0 as \( J' \to \infty \). The proof is essentially an application of empirical process theory to the study of an M-estimator.

### 3.2.2 Fictitious play

Brown (1951) introduced fictitious play as a means to find the value of a zero-sum game. Robinson (1951) then proved that fictitious play can be used to iteratively solve a two-player zero-sum game for a saddle point that is a pair of mixed strategies where both players have finitely many pure strategies. Our problem of finding a \( \Gamma \)-minimax estimator may also be viewed as a two-player zero-sum game where one player chooses a prior from \( \Gamma \) and the other player chooses an estimator from \( D \). If we assume that, for the \( \Gamma \)-minimax problem at hand, the pair of both players’ optimal strategies is a saddle point, which holds in many minimax problems (e.g., v. Neumann 1928, Fan 1953, Sion 1958), then fictitious play may also be used to find a \( \Gamma \)-minimax estimator. Since \( \Gamma \) may be too rich to allow for feasible implementation of fictitious play, we propose to use this algorithm to find a \( \Gamma \)-minimax estimator. In this section, we present how to use this algorithm to find a \( \Gamma \)\( ^{\ell} \)-minimax estimator and its convergence results.

In the fictitious play algorithm in Robinson (1951), the two players take turns to play the best pure strategy against the mixture of the opponent’s historic pure strategies, and the final output is a pair of mixtures of the two players’ historic pure strategies. Since this algorithm aims to find minimax mixed strategies, we consider stochastic estimators. That is, consider the Borel \( \sigma \)-field \( F \) over \( D \) and let \( \Pi \) denote the set of all probability distributions on the measurable space \( (D, F) \). We define \( \mathcal{D} \) to be the space of stochastic estimators with each element taking the following form: first draw an estimator from \( D \) according to a distribution \( \varpi \in \Pi \) with an exogenous random mechanism and then use the estimator to obtain an estimate based on the data. Note that we may write any \( \vec{d} \in \mathcal{D} \) as \( \vec{d}(\varpi) \) for some \( \varpi \in \Pi \).

We consider estimators in \( \mathcal{D} \) throughout this section, with the definition of \( \Gamma \)\( ^{\ell} \)-minimaxity extended in the natural way, so that \( \vec{d}' = \vec{d}(\varpi^*) \in \mathcal{D} \) is \( \Gamma \)\( ^{\ell} \)-minimax if \( r_{\sup}(\vec{d}', \Gamma) = \min_{\vec{d} \in \mathcal{D}} r_{\sup}(\vec{d}, \Gamma) \); we similarly extend all other definitions from Section 2. We assume that there exists \( \pi_{t}^* \in \Gamma_{\ell} \) (\( \ell = 1, 2, \ldots \)) such that

\[
 r(\vec{d}', \pi_{t}^*) = \sup_{\pi \in \Gamma_{t}} \inf_{\vec{d} \in \mathcal{D}} r(\vec{d}, \pi) = \inf_{\vec{d} \in \mathcal{D}} \sup_{\pi \in \Gamma_{t}} r(\vec{d}, \pi).
\]  

(1)

In other words, \( (\vec{d}', \pi_{t}^*) \) is a saddle point of \( r \) in \( \mathcal{D} \times \Gamma_{\ell} \). Under this condition and the further conditions
that $\mathcal{D}$ is convex and $d \mapsto R(d, P)$ is convex for all $P \in \mathcal{M}$, it is possible to use a $\Gamma$-minimax estimator over the richer class $\overline{\mathcal{D}}$ of stochastic estimators to derive a $\Gamma$-minimax estimator over the original class $\mathcal{D}$. Indeed, for any $\overline{d}(\varpi) \in \overline{\mathcal{D}}$ and $P \in \mathcal{M}$, by Jensen’s inequality, $R(\overline{d}(\varpi), P) = \int R(d, P) \varpi(dd) \geq R(\overline{d}(\varpi), P)$ where $\overline{d}(\varpi) := \int d \varpi(dd) \in \mathcal{D}$ is the average of the stochastic estimator $\overline{d}(\varpi)$; that is, the risk of $\overline{d}(\varpi)$ is never greater than that of $\overline{d}(\varpi)$. Therefore, we may use the fictitious play algorithm to compute $\overline{d}(\varpi^*_\ell)$ for each $\ell$ and further apply Algorithm 1 to compute $\overline{d}(\varpi^*)$. After that, we may take $\overline{d}(\varpi^*)$ as the final output deterministic estimator.

Algorithm 5) presents the fictitious play algorithm for finding a $\Gamma_\ell$-minimax estimator in $\overline{\mathcal{D}}$. Note that $\Gamma_\ell$ is convex, and hence $\pi$ always lies in $\Gamma_\ell$ throughout the iterations. In practice, we may initialize $\varpi$ as a point mass at an initial estimator in $\mathcal{D}$. In addition, similarly to Robinson (1951), we may replace Line 5 with $d^\dagger_{(t)} \leftarrow \arg\min_{d \in \mathcal{D}} r(d, \pi_{(t-1)})$, that is, minimizing the Bayes risk with the most recently updated prior rather than with the previous prior.

Algorithm 5 Fictitious play to compute a $\Gamma_\ell$-minimax stochastic estimator

1: Initialize $\varpi(0) \in \Pi$ and $\pi(0) \in \Gamma_\ell$.
2: for $t=1,2, \ldots$ do
3: $\pi^*_{(t)} \leftarrow \arg\max_{\pi \in \Gamma_\ell} r(\overline{d}(\varpi_{(t-1)}), \pi)$
4: $\pi_{(t)} \leftarrow \frac{t-1}{t} \pi_{(t-1)} + \frac{1}{t} \pi^*_{(t)}$
5: $d^\dagger_{(t)} \leftarrow \arg\min_{d \in \mathcal{D}} r(d, \pi_{(t-1)})$
6: $\varpi_{(t)} \leftarrow \frac{t-1}{t} \varpi_{(t-1)} + \frac{1}{t} \delta(d^\dagger_{(t)})$, where $\delta(d)$ denotes a point mass at $d \in \mathcal{D}$.

We next present a convergence result for this algorithm.

Theorem 4. Using Algorithm 5 under Conditions 1–3 it holds that

$$r(d^\dagger_{(t)}, \pi_{(t-1)}) \leq r(\overline{d}(\varpi^*_{(t)}), \pi^*_{(t)}) \leq r(\overline{d}(\varpi_{(t-1)}), \pi^*_{(t)})$$

for all $t$ and

$$\lim_{t \to \infty} \left[ r(\overline{d}(\varpi_{(t-1)}), \pi^*_{(t)}) - r(d^\dagger_{(t)}, \pi_{(t-1)}) \right] = 0.$$ 

Consequently, the $\Gamma_\ell$-maximal risk of $\overline{d}(\varpi_{(t)})$ converges to the $\Gamma_\ell$-minimax risk, that is, $r_{\sup}(\overline{d}(\varpi_{(t-1)}), \Gamma_\ell) \rightarrow r_{\sup}(\overline{d}(\varpi^*_{(t)}), \Gamma_\ell)$ as $t \rightarrow \infty$.

Robinson (1951) proved a similar case for two-player zero-sum games where each player has finitely many pure strategies. In contrast, in our problem, each player may have infinitely many pure strategies.
A natural approach to use to attempt to prove Theorem 4 would be to consider finite covers of $D$ and $\Gamma_\ell$, i.e., $D = \bigcup_{i=1}^I D_i$ and $\Gamma_\ell = \bigcup_{j=1}^J \Pi_j$, such that the range of $r(d, \pi)$ in each $D_i$ and $\Pi_j$ is small (say less than $\epsilon$), bin pure strategies into these subsets, and then apply the argument in Robinson (1951) to these bins. The collection of $D_i$ and $\Pi_j$ may be viewed as finitely many approximated pure strategies to $\Gamma_\ell$ and $D$ up to accuracy $\epsilon$, respectively. Unfortunately, we found that this approach fails. The problem arises because Robinson (1951) inducted on $I$ and $J$, and, after each induction step, the corresponding upper bound becomes twice as large. Unlike the case with finitely many pure strategies that was considered in Brown (1951) and Robinson (1951), as the desired approximation accuracy $\epsilon$ approaches zero, the numbers of approximated pure strategies, $I$ and $J$, may diverge to infinity, and so does the number of induction steps. Therefore, the resulting final upper bound is of order $2^{I+J}\epsilon$ and generally does not converge to zero as $\epsilon$ tends to zero. To overcome this challenge, we instead control the increase in the relevant upper bound after each induction step more carefully so that the final upper bound converges to zero as $\epsilon$ decreases to zero, despite the fact that $I$ and $J$ may diverge to infinity.

We remark that, because Line 5 of Algorithm 5 typically involves another layer of iteration in addition to that over $t$, this algorithm will often be more computationally intensive than are Algorithms 2–4. Nevertheless, Algorithm 5 provides an approach to construct $\Gamma_\ell$-minimax estimators in cases where these other algorithms cannot be applied, for example, in settings where the risk is not differentiable in the parameters indexing the estimator.

4 Considerations in implementation

4.1 Considerations when constructing the grid over the model space

By Theorem 1, $r_{\sup}(d_\ell^*, \Gamma_\ell) \nearrow \min_{d \in D} r_{\sup}(d, \Gamma)$ whenever Conditions 1–4 hold and the increasing sequence $\{M_\ell\}_{\ell=1}^\infty$ is such that $\bigcup_{\ell=1}^\infty M_\ell$ is dense in $M$. Though this guarantee holds for all such sequences $\{M_\ell\}_{\ell=1}^\infty$, in practice, judiciously choosing this sequence of grids of distributions can lead to faster convergence. In particular, it is desirable that the least favorable prior $\Gamma_\ell$ puts mass on some of the distributions in $M_\ell \setminus M_{\ell-1}$ since, if this is not the case, then $d_\ell^*$ will be the same as $d_{\ell-1}^*$. While we may try to arrange for this to occur by adding many new points when enlarging $M_{\ell-1}$ to $M_\ell$, it may not be likely that any of these points will actually modify the least favorable prior unless they are carefully chosen.
To better address this issue, we propose to add grid points using a method that is similar to Markov chain Monte Carlo (MCMC). Our intuition is that, given an estimator $d$, the maximal Bayes risk is likely to significantly increase if we add distributions that (i) have high risk for $d$, and (ii) are consistent with prior information so that there exists some prior such that these distributions lie in a high-probability region. We propose to use an MCMC-like algorithm to bias the selection of distributions in favor of those with the above characteristics. Let $\tau : M \rightarrow [0, \infty)$ denote a function such that $\tau(P) > \tau(P')$ if $P$ is more consistent with prior information than $P'$. For example, given a prior mean $\mu$ of some real-valued summary $\Psi(P)$ of $P$ and an interval $I$ that contains $\Psi(P)$ with prior probability at least 95%, we may choose $\tau : P \mapsto \phi(\Psi(P))$, where $\phi$ is the density of a normal distribution that has mean $\mu$ and places 95% of its probability mass in $I$. We call $\tau$ a pseudo-prior. Then, with the current estimator being $d$, we wish to select distributions $P$ for which $R(d, P) \tau(P)$ is large. We may use the Metropolis-Hastings-Green algorithm (Metropolis et al., 1953; Hastings, 1970; Green, 1995) to draw samples from a density proportional to $P \mapsto R(d, P) \tau(P)$. We then let $\mathcal{M}_\ell$ be equal to the union of $\mathcal{M}_{\ell-1}$ and the set containing all unique distributions in this sample.

Details of the proposed scheme are provided in Algorithm 6. To use this proposed algorithm, we rely on it being possible to define a sequence of parametric models $\{\Omega_\ell\}_{\ell=1}^{\infty}$ such that $\mathcal{M} := \cup_{\ell=1}^{\infty}\Omega_\ell$ is dense in $\mathcal{M}_\ell$ — this is possible in many interesting examples (see, e.g., Chen, 2007). When combined with Condition 1 this condition enables the definition of an increasing sequence of grids of distributions $\{\mathcal{M}_\ell\}_{\ell=1}^{\infty}$ such that, for each $\ell$, $\mathcal{M}_\ell \subseteq \mathcal{M}$.

**Algorithm 6** MCMC-like algorithm to construct $\mathcal{M}_\ell$

**Require:** Previous grid $\mathcal{M}_{\ell-1}$, current estimator $d_{\ell-1}^*$ and number $T$ of iterations. We define $\mathcal{M}_{-1} := \emptyset$. An initial estimator $d_0^*$ must be available if $\ell = 1$.

1: Initialize $P(0) \in \mathcal{M}$.
2: for $t = 1, 2, \ldots, T$ do
3: Propose a distribution $P' \in \mathcal{M}$ from $P(t-1)$
4: Calculate the MCMC acceptance probability $p_{\text{accept}}$ of $P'$ for target density $P \mapsto R(d_{\ell-1}^*, P) \tau(P)$
5: With probability $p_{\text{accept}}$, accept $P'$ and $P(t) \leftarrow P'$
6: if $P'$ is not accepted then
7: $P(t) \leftarrow P(t-1)$
8: $\mathcal{M}_\ell \leftarrow$ unique elements of $\mathcal{M}_{\ell-1} \cup\{P(1), P(2), \ldots, P(T)\}$

The following theorem on distributional convergence follows from that for Metropolis-Hastings-Green algorithm (see Section 3.2 and 3.3 of Green 1995).
Theorem 5. Suppose that \( P \mapsto R(d^*_\ell, P) \tau(P) \) is bounded and integrable with respect to some measure \( \mu \) on \( \tilde{\mathcal{M}} \) and let \( \mathcal{L} \) denote the probability law on \( \tilde{\mathcal{M}} \) whose density function with respect to \( \mu \) is proportional to this function. Then, in Algorithm \( \tilde{\mathcal{A}} \), \( P_{(t)} \) converges weakly to \( \mathcal{L} \) as \( t \to \infty \).

Implementing Algorithm \( \tilde{\mathcal{A}} \) relies on the user making several decisions. These decisions include the choice of the pseudo-prior \( \tau \) and the technique used to approximate the risk \( R(d, P) \) to a reasonable accuracy. Fortunately, regardless of the decisions made, Theorem 1 suggests that \( r_{\sup}(d^*_\ell, \Gamma) \to \min_{d \in \mathcal{D}} r_{\sup}(d, \Gamma) \) for a wide range of sequences \( \{\mathcal{M}_\ell\}_{\ell=1}^\infty \). Indeed, all that theorem requires on this sequence is that the grid \( \mathcal{M}_\ell \) become arbitrarily fine as \( \ell \) increases. Though the final decisions made are not important when \( \ell \) is large, we still comment briefly on the decisions that we have made in our experiments. First, we have found it effective to approximate \( R(d, P) \) via a large number of Monte Carlo draws. Second, in a variety of settings, we have also identified, via numerical experiments, candidate pseudo-priors that balance high risk and consistency with prior information (see Sections 5.2 and 5.3 for details).

4.2 Considerations when choosing the space of estimators

It is desirable to consider a rich space \( \tilde{\mathcal{D}} \) of estimators to obtain an estimator with low maximal Bayes risk, and thus good general performance. However, to make numerically constructing these estimators computationally feasible, we usually have to consider a restricted space \( \mathcal{D} \) of estimators. In the upcoming theorem, we provide an upper bound on the increment of the maximal Bayes risk induced by making this restriction. This result shows that, if estimators in \( \mathcal{D} \) can approximate estimators in \( \tilde{\mathcal{D}} \) well, then the resulting excess maximal Bayes risk is small. This result relies on what we call Condition 3', which is the same as Condition 3 except that each instance of \( \mathcal{D} \) in that condition is replaced by \( \tilde{\mathcal{D}} \).

Theorem 6. Fix \( \mathcal{D} \subseteq \tilde{\mathcal{D}} \). Let \( d^* \) be a \( \Gamma \)-minimax estimator in \( \mathcal{D} \) and \( \tilde{d}^* \) be a \( \Gamma \)-minimax estimator in \( \tilde{\mathcal{D}} \), so that \( r_{\sup}(d^*, \Gamma) = \min_{d \in \mathcal{D}} r_{\sup}(d, \Gamma) \) and \( r_{\sup}(\tilde{d}^*, \Gamma) = \min_{d \in \tilde{\mathcal{D}}} r_{\sup}(d, \Gamma) \). Under Condition 3', \( r_{\sup}(d^*, \Gamma) - r_{\sup}(\tilde{d}^*, \Gamma) \leq L \inf_{d' \in \mathcal{D}} \vartheta(d', \tilde{d}^*) \).

Proof of Theorem 6. By the definition of \( d^* \), for any \( d' \in \mathcal{D} \), \( r_{\sup}(d^*, \Gamma) - r_{\sup}(\tilde{d}^*, \Gamma) \leq r_{\sup}(d', \Gamma) - r_{\sup}(\tilde{d}^*, \Gamma) \), and so \( r_{\sup}(d^*, \Gamma) - r_{\sup}(\tilde{d}^*, \Gamma) \leq \inf_{d' \in \mathcal{D}} [r_{\sup}(d', \Gamma) - r_{\sup}(\tilde{d}^*, \Gamma)] \). By Lemma 2 in Appendix A.1, \( d' \mapsto r_{\sup}(d', \Gamma) \) is Lipschitz continuous with Lipschitz constant \( L \). Therefore, the right hand side is upper bounded by \( L \inf_{d' \in \mathcal{D}} \vartheta(d', \tilde{d}^*) \). \( \square \)
Feedforward neural networks (or neural networks for short) are natural options for the space of estimators because of their universal approximation property (e.g., Hornik, 1991; Csáji, 2001; Hanin and Sellke, 2017; Kidger and Lyons, 2020). However, training commonly used neural networks can be computationally intensive. Moreover, a space of neural networks is typically nonconvex, and hence it may be difficult to find a global minimizer of the maximal Bayes risk even if the risk is convex in the estimator. Therefore, the learned estimator might not perform well.

To help overcome this challenge, we advocate for utilizing available statistical knowledge when designing the space of estimators. We call estimators that take this form statistical knowledge networks. In particular, if a sensible simple estimator is already available, we propose to use neural networks with such an estimator as a node connected to the output node. An example of such an architecture is presented in Fig. 1. In this sample architecture, each node is an activation function such as the sigmoid or the rectified linear unit (ReLU) function applied to an affine transformation of the vector containing the ancestors of the node. The only exception is the output node, which is again an affine transformation of its ancestors, but uses the identity activation function. When training the neural network, we may initialize the affine transformation in the output layer to only give weight to the simple estimator. Under this approach, the space of estimators is a set of perturbations of a sensible simple estimator. Although we may still face the challenge of nonconvexity and local optimality, we can at least expect to improve the initial simple estimator.

We note that we might overcome the challenge of nonconvexity and local optimality by using an
extreme learning machine (ELM) (Huang et al., 2006b) to parameterize the estimator. ELMs are neural networks for which the weights in hidden nodes are randomly generated and are held fixed, and only the weights in the output layer are trained. Thus, the space of ELMs with a fixed architecture and fixed hidden layer weights is convex. Like traditional neural networks, ELMs have the universal approximation property (Huang et al., 2006a). In addition, Corollary 1 may be applied to an ELM so that the $\Gamma$-minimax estimator may converge to the $\Gamma$-minimax estimator. As for traditional neural networks, we may incorporate knowledge of existing statistical estimators into an ELM.

Next, we present a corollary of Theorem 6 for some special cases of neural networks and ELMs based on their universal approximation results. We expect similar results to hold for more general architectures of neural networks and ELMs, for example, with other activation functions, more hidden layers or more complicated architectures. Indeed, whenever universal approximation results are available over the space $\bar{\mathcal{D}}$, Theorem 6 can be immediately applied to obtain an upper bound for the excess maximal Bayes risk $r_{\sup}(d^*, \Gamma) - r_{\sup}(\tilde{d}^*, \Gamma)$ due to restriction of the space of estimators.

**Corollary 2.** Suppose that $\mathcal{X}$ is a compact subset of a Euclidean space $\mathbb{R}^\alpha$. Let $\bar{\mathcal{D}}$ be the collection of all continuous functions defined on $\mathcal{X}$ that are square-integrable with respect to Lebesgue measure. Let the metric $\varrho$ on $\bar{\mathcal{D}}$ be the $L^2$ distance with respect to Lebesgue measure. Suppose that Condition 3 holds.

1. Suppose that $\mathcal{D}$ is a space of estimators parameterized as neural networks with identity activation for the output layer and ReLU activation for all hidden layers. Then, for any $\epsilon > 0$, it holds that $\inf_{d \in \mathcal{D}} r_{\sup}(d, \Gamma) - \inf_{\tilde{d} \in \bar{\mathcal{D}}} r_{\sup}(\tilde{d}, \Gamma) \leq \epsilon$ provided that networks in $\mathcal{D}$ have a sufficiently large number of hidden layers and a sufficiently large number of hidden nodes in each hidden layer.

2. Suppose that $\mathcal{D}$ is a space of estimators parameterized as ELMs with one hidden layer, identity activation for the output layer and a bounded nonconstant piecewise continuous $\mathbb{R} \rightarrow \mathbb{R}$ activation function for the hidden layer. Suppose that the values of the hidden weights and hidden biases in the ELM are independently drawn from a continuous distribution with support $\mathbb{R}^{\alpha+1}$. Then, for any $\epsilon > 0$, $P\{\inf_{d \in \mathcal{D}} r_{\sup}(d, \Gamma) - \inf_{\tilde{d} \in \bar{\mathcal{D}}} r_{\sup}(\tilde{d}, \Gamma) \leq \epsilon\} \rightarrow 1$ as the number of hidden nodes tends to infinity.

**Proof.** The result follows from the universal approximation results (Theorem 4.16 in Kidger and Lyons (2020) and Theorem II.1 in Huang et al. (2006a), respectively) and Theorem 6.
Under Condition 3 the above result can be generalized to a variety of collections of estimators \( D_1 \) that are richer than the space \( \tilde{D} \) of continuous functions considered in the above lemma. Indeed, if \( D_1 \) is such that \( \tilde{D} \) is dense in \( D_1 \), then Lemma 2 in Appendix A.1 shows that the same conclusion will hold. This shows that the same conclusions of the above theorem hold when the collection of estimators \( \tilde{D} \) is enriched to contain all \( \mathcal{X} \to \mathbb{R} \) functions that are square integrable with respect to Lebesgue measure (e.g., Theorem 1.15 in [Evans and Gariepy, 2015]).

We finally remark that, besides computational intensity when constructing (i.e., learning) a \( \Gamma \)-minimax estimator, another important factor to be considered when choosing \( D \) is the computational intensity to evaluate the learned estimator at the observed data set. This is another reason for our choosing neural networks or ELMs as the space of estimators. Indeed, existing software packages (e.g., [Paszke et al., 2019]) make it easy to leverage graphics processing units to efficiently evaluate the output of neural networks for any given input. Therefore, if the existing estimator being used is not too difficult to compute, then estimators parameterized using similar architectures to that displayed in Figure 1 will be able to be computed efficiently in practice. This efficiency may be especially important in settings where the estimator will be applied to many datasets, so that the cost of learning the estimator is amortized and the main computational expense is evaluating the learned estimator.

5 Simulation

5.1 Estimation of the mean

We start by illustrating our proposed method in a special case of Example 1, namely for estimating the mean of a distribution. We assume that \( \mathcal{M} \) consists of all probability distributions defined on the Borel \( \sigma \)-algebra on \([0, 1]\) and we observe \( \mathbf{X} = (X_1, X_2, \ldots, X_n) \), where \( X_1, \ldots, X_n \overset{iid}{\sim} P_0 \in \mathcal{M} \). Here we take \( n = 10 \). The estimand is \( \Psi(P_0) = \int x P_0(dx) \). We use the mean squared error risk introduced in Example 1. Suppose that we represent the prior information by \( \Gamma = \{ \pi \in \Pi : \int \Psi(P) \pi(dP) = 0.3 \} \), which corresponds to the set of prior distributions in \( \Pi \) that satisfy an equality constraint on the prior mean of \( \Psi(P) \).

We apply our method to three spaces of estimators separately. The first space, \( D_{\text{linear}} \), is the set of affine transformations of the sample mean, that is, \( D_{\text{linear}} = \{ d : d(\mathbf{X}) = \beta_0 + \beta_1 \sum_{i=1}^{n} X_i/n, \beta_0, \beta_1 \in \mathbb{R} \} \).
As shown in Proposition \[1\] in Appendix \[A.5\] there is an estimator \(d^\ast\) in \(D_{\text{linear}}\) that is \(\Gamma\)-minimax in the space of all estimators that are square-integrable with respect to all \(P \in \mathcal{M}\), so we consider this simple space to better compare our computed estimator with that theoretical \(\Gamma\)-minimax estimator. When computing a \(\Gamma\)-minimax estimator in \(D_{\text{linear}}\), we initialize the estimator to be the sample mean, that is, we let \(\beta_0 = 0\) and \(\beta_1 = 1\).

The second space, \(D_{\text{skn}}\) (statistical knowledge network), is a set of neural networks designed based on statistical knowledge that includes the sample mean as an input. We consider this space to illustrate our proposal in Section \[4.2\]. More precisely, we use an architecture in Fig.\[2\] that is similar to the deep set architecture \[\text{Zaheer et al., 2017} - \text{Maron et al., 2019}\], which is a permutation invariant neural network.

We use such an architecture to account for the fact that the sample is iid. In this architecture, the sample mean node is used as an augmenting node to an ordinary deep set network and is combined with the output of that ordinary network in the fourth hidden layer to obtain the final output. Note that \(D_{\text{skn}} \supset D_{\text{linear}}\). When computing a \(\Gamma\)-minimax estimator for this class, we also initialize the network to be exactly the sample mean, which is a reasonable choice given that the sample mean is known to be a sensible estimator. In this simulation experiment, we choose the dimensionality of nodes in each hidden layer in Fig.\[2\] as follows: each node in the first, second, third and fourth hidden layer represents a vector in \(\mathbb{R}^{10}\), \(\mathbb{R}^5\), \(\mathbb{R}^{10}\) and \(\mathbb{R}\), respectively. We do not use larger architectures because usually the sample mean is already a good estimator, and we expect to obtain a useful estimator as a small perturbation of this estimator. We also use the ReLU as the activation function. We did not use ELMs in this and the following simulations because we found that neural networks perform well.

The third space, \(D_{\text{nn}}\), is a set of neural networks that does not utilize knowledge of the sample mean. We consider this space to illustrate our method without utilizing existing sensible estimators. These estimators are also deep set networks with a similar architecture as \(D_{\text{skn}}\) in Fig.\[2\]. The main difference is that the explicit sample mean node and the fourth hidden layer are removed. When computing a \(\Gamma\)-minimax estimator in \(D_{\text{nn}}\), we also randomly initialize the network, unlike \(D_{\text{linear}}\) and \(D_{\text{skn}}\), in order not to input statistical knowledge. Because the ReLU activation function is used, \(D_{\text{nn}} \supset D_{\text{linear}}\), and we do not expect that optimizing over \(D_{\text{nn}}\) should not lead to a \(\Gamma\)-minimax estimator with worse performance than those in \(D_{\text{linear}}\) and \(D_{\text{skn}}\).

To construct the grid \(\mathcal{M}_T\) for this problem, we use a simpler method than Algorithm \[6\]. As indicated by Lemma \[5\] in Appendix \[A.5\] for estimators in \(D_{\text{linear}}\), Bernoulli distributions tend to have high risks.
Figure 2: Architecture of the permutation invariant neural network estimator of the mean in $D_{\text{skn}}$. $X_i$: observation $i$ in the sample; $\sum$: the node that sums up all ancestor nodes. In the first two hidden layers, all inputs nodes are transformed by the same function. The arrows from the input nodes to the sample mean estimator are omitted from this graph. Each node in the hidden layers represent a vector.

since all probability weights lie on the boundary of $[0, 1]$; in addition, a prior $\pi^*$ for which $d^*$ is Bayes is a Beta prior over Bernoulli distributions. Therefore, we randomly generate 2,000 Bernoulli distributions as grid points in $M_1$. We also include two degenerate distributions in this grid, namely the distribution that places all of its mass at 0 and that which places all of its mass at 1. When constructing $M_\ell$ from $M_{\ell-1}$, we still add in more complicated distributions to make the grid dense in the limit: we first randomly generate 500 discrete distributions with support being those in $M_{\ell-1}$; then we randomly generate 10 new support points in $[0, 1]$ and 1,000 distributions with support points being the union of the new support points and the existing support points in $M_{\ell-1}$.

When computing the $\Gamma$-minimax estimator, for each grid $M_\ell$, we compute the $\Gamma_\ell$-minimax estimator for all three estimator spaces with Algorithm $\Gamma$. We set the learning rate $\eta = 0.005$, the batch size $J = 50$ and the number of iterations to be 200 for $\Gamma_\ell$ ($\ell > 1$). The number of iterations for $\Gamma_1$ is larger because, in our experiments, we saw that a $\Gamma_1$-minimax estimator is already close to a $\Gamma$-minimax estimator, and using a large number of iterations in this step can improve the initial estimator substantially. For $D_{\text{linear}}$ and $D_{\text{skn}}$, the number of iterations for $\Gamma_1$ is 2,000; the corresponding number for $D_{\text{nn}}$ is 6,000 to account for the lack of human knowledge input. We also use Algorithm $\Gamma$ with 10,000 iterations to compute a $\Gamma_\ell$-minimax estimator for $D_{\text{linear}}$ for illustration. In this setup, as described in Section 3.2.2, we take the average of the computed $\Gamma$-minimax stochastic estimator as the final output estimator in $D_{\text{linear}}$. We do
Table 1: Coefficients and Bayes risks of estimators of the mean. Unrestricted space: the space
of all estimators that are square-integrable with respect to all $P \in \mathcal{M}$.

| Estimator space | Method to obtain $d^*$ | $\beta_0$ | $\beta_1$ | $r(d, \pi^*)$ |
|-----------------|------------------------|-----------|-----------|----------------|
| Unrestricted space | Theoretical derivation | 0.072 | 0.760 | 0.012 |
| $\mathcal{D}_{\text{linear}}$ | Algorithms 1 & 4 | 0.072 | 0.763 | 0.012 |
| $\mathcal{D}_{\text{skn}}$ | Algorithms 1 & 4 | 0.071 | 0.767 | 0.012 |
| $\mathcal{D}_{\text{nn}}$ | Algorithms 1 & 4 | — | — | 0.012 |
| $\mathcal{D}_{\text{linear}}$ | Algorithms 1 & 5 | 0.072 | 0.760 | 0.012 |

not apply Algorithm 5 to $\mathcal{D}_{\text{skn}}$ or $\mathcal{D}_{\text{nn}}$ because it is computationally intractable.

We set the stopping criterion in Algorithm 1 as follows. When Algorithm 4 is used to compute$
\Gamma_\ell$-minimax estimators, we estimate $r_{\text{sup}}(d_\ell^*, \Gamma_\ell)$ and $r_{\text{sup}}(d_{\ell-1}^*, \Gamma_{\ell-1})$ with 2,000 Monte Carlo runs as described in Section 3.1. When Algorithm 6 is used, $r_{\text{sup}}(d_{\ell-1}^*, \Gamma_\ell)$ and $r_{\text{sup}}(d_{\ell-1}^*, \Gamma_{\ell-1})$ are computed exactly because $R(d, \Gamma)$ has a closed-form expression for all $d \in \mathcal{D}_{\text{linear}}$ and $P \in \mathcal{M}_\ell$. We set the tolerance $\epsilon$ to be equal to 0.0001 so that we stop Algorithm 1 if $r_{\text{sup}}(d_\ell^*, \Gamma_\ell) - r_{\text{sup}}(d_{\ell-1}^*, \Gamma_{\ell-1}) \leq \epsilon$.

After computation, we report the Bayes risk of the computed and theoretical $\Gamma$-minimax estimators under $\pi^*$, the prior such that $r(d^*, \pi^*) = \inf_{d \in \mathcal{D}} r_{\text{sup}}(d, \Gamma)$. For the estimators in $\mathcal{D}_{\text{linear}}$, we further report their coefficients. We also report two coefficients of the computed estimator in $\mathcal{D}_{\text{skn}}$ as follows. Since $\mathcal{D}_{\text{linear}} \subseteq \mathcal{D}_{\text{skn}}$ and we initialize the estimator to be the sample mean for $\mathcal{D}_{\text{skn}}$, we would expect that the bias $\beta_0$ and the weight of the sample mean $\beta_1$ in the output layer for the computed $\Gamma$-minimax estimator in $\mathcal{D}_{\text{skn}}$ may correspond to those in $\mathcal{D}_{\text{linear}}$. Therefore, we also report these two coefficients $\beta_0$ and $\beta_1$ for $\mathcal{D}_{\text{skn}}$. This may not be the case for $\mathcal{D}_{\text{nn}}$ because sample mean is not explicit in its parameterization and all coefficients are randomly initialized, so we do not report any coefficients for $\mathcal{D}_{\text{nn}}$.

Table 1 presents the computation results. By Theorem 7 in Appendix A.5, these computed estimators are all approximately $\Gamma$-minimax since their Bayes risks for $\pi^*$ are all close to that of a theoretical $\Gamma$-minimax estimator. The coefficients $\beta_0$ and $\beta_1$ of the computed estimators in $\mathcal{D}_{\text{linear}}$ and $\mathcal{D}_{\text{skn}}$ are also close to a theoretically derived estimator. For the computed estimator in $\mathcal{D}_{\text{skn}}$, the weight of the other ancestor node in the output layer (i.e., the node in the 4th hidden layer in Fig 2) is 0.999. Therefore, our computed $\Gamma$-minimax estimator in $\mathcal{D}_{\text{skn}}$ is also close to a theoretically derived $\Gamma$-minimax estimator.

In our experiments, Algorithm 1 converged after computing a $\Gamma_1$-minimax estimator except when using Algorithm 5 for $\mathcal{D}_{\text{linear}}$. Even in this exceptional case, the computed $\Gamma_1$-minimax estimator is still approximately $\Gamma$-minimax. We think the algorithm does not stop then in these cases because of Monte
Figure 3: Estimated Bayes risks of the estimator over iterations when computing a $\Gamma_1$-minimax estimator. The lines are the current Bayes risks (y-axis) over iterations (x-axis) (unbiased estimates with 50 Monte Carlo runs for Algorithm 4; exact values for Algorithm 5). The solid lines are the Bayes risks after an update in the estimator to decrease the Bayes risk. The dashed lines are the Bayes risks after an update in the prior to increase the Bayes risk. The two horizontal lines are the Bayes risk of the sample mean (dashed) and $d^*$ (solid), respectively, for $\pi^*$. For ease of visualization, in subfigures (a) and (b), the Bayes risks are plotted every 50 iterations; in subfigures (c) and (d), the Bayes risks are plotted every 200 iterations; subfigure (d) contains the part in subfigure (c) after 500 iterations.

Carlo errors when computing $r_{\sup}(d^*_{\ell-1}, \Gamma_\ell)$ and $r_{\sup}(d^*_{\ell-1}, \Gamma_{\ell-1})$.

Fig 3 presents the Bayes risks (or its unbiased estimates) over iterations when computing a $\Gamma_1$-minimax estimator. In all cases using Algorithm 4, the Bayes risks appear to decrease and converge. When using Algorithm 5, the upper and lower bounds both converge to the same limit. The limiting values of the Bayes risks in all cases are close to $r(d^*, \pi^*)$ because $\Gamma_1$ can approximate $\pi^*$ well.

5.2 Prediction of the expected number of new categories

We apply our proposed method to Example 2. We set the true population to be an infinite population with the same categories and same proportions as the sample studied in Miller and Wiegert (1989), which consists of 1088 observations in 188 categories. This is the same as the simulation setting in Shen et al. (2003). We set the sample size to be $n = 100$ and the size of the new sample to be $m = 200$. In this setting, the expected number of new categories in the new sample unconditionally on the observed sample, namely $\Phi(P_0) := \mathbb{E}_{P_0}[\Psi(P_0)(X^*)]$, can be analytically computed and equals 48.02. We note that this quantity can also be computed via simulation: (i) sample $n$ and $m$ individuals with replacement from the data set in Miller and Wiegert (1989), (ii) count the number of new categories in the second
sample, and (iii) repeat steps (i) and (ii) many times and compute the average.

We consider three sets of prior information:

1. strongly informative: prior mean of $\Phi(P)$ in $[45, 50]$, $\geq 95\%$ prior probability that $\Phi(P)$ lies in $[40, 55]$;

2. weakly informative: prior mean of $\Phi(P)$ in $[40, 55]$, $\geq 95\%$ prior probability that $\Phi(P)$ lies in $[30, 65]$; and

3. almost noninformative: prior mean of $\Phi(P)$ in $[35, 60]$, $\geq 95\%$ prior probability that $\Phi(P)$ lies in $[20, 75]$.

We note that a traditional Bayesian approach would require specifying a prior on $\mathcal{M}$, including the total number of categories and the proportion of each category, which may be difficult in practice.

We design the architecture of the neural network estimator as in Fig 4. We choose two existing estimators (referred to as OSW and SCL estimators, respectively) proposed by Orlitsky et al. (2016) and Shen et al. (2003) as human knowledge inputs to the architecture. As in Section 5.1, we use the ReLU activation function. There are 50 hidden nodes in the first hidden layer. We initialize the neural network that we train to output the average of these two existing estimators.

We use Algorithm 6 to construct $\mathcal{M}_\ell$. There are 2,000 grid points in $\mathcal{M}_1$, and we add 1,000 grid points
each time we enlarge the grid. When generating $\mathcal{M}_1$, we chose the starting point to be a distribution $P_{(0)}$ with 146 categories and $\Phi(P_{(0)}) = 49.9$. We selected the log pseudo-prior as a weighted sum of two log density functions: (i) a normal distribution with mean being the midpoint of the interval constraint on prior mean of $\Phi(P)$ and central 95% probability interval being the interval with at least 95% prior probability, (ii) a negative-binomial distribution of the total number of categories with success probability 0.995 and 2 failures until the Bernoulli trial is stopped so that the mode and the variance are approximately 200 and $8 \times 10^4$, respectively. These log-densities are provided weight 30 and 10, respectively. We selected the weights based on the empirical observation that distributions with only a few categories tend to have high risks, but these distributions are relatively inconsistent with prior information and may well be given almost negligible probability weight in a computed least favorable prior, thus contributing little to computing a $\Gamma$-minimax estimator. We chose the aforementioned weights so that Algorithm 6 can explore a fairly large range of distributions and does not generate too many distributions with too few categories.

We use Algorithm 4 with learning rate $\eta = 0.005$ and batch size $J = 30$ to compute $\Gamma_\ell$-minimax estimators. The number of iterations is 4,000 for $\Gamma_1$ and 200 for $\Gamma_\ell$ ($\ell > 1$). The stopping criterion in Algorithm 1 is that the estimated maximal Bayes risk with 2,000 Monte Carlo runs does not relatively increase by more than 2% or absolutely increase by more than 0.0001.

We finally examine the performance of OSW estimator, SCL estimator and our trained $\Gamma$-minimax estimator by comparing their risks under our set data-generating mechanism computed with 20,000 Monte Carlo runs. We also compare their Bayes risks under the computed prior from Algorithm 4 using the last and finest grid in the computation with 20,000 Monte Carlo runs. We present the results in Table 2. In this simulation experiment, our $\Gamma$-minimax estimator significantly reduces the risk compared to two existing estimators. The $\Gamma$-minimax estimator also has the lowest Bayes risk in all cases. Therefore, incorporating the fairly informative prior knowledge into the estimator may lead to a significant improvement in performance.

Fig 5 presents the unbiased estimated of Bayes risks over iterations when computing a $\Gamma_1$-minimax estimator. The Bayes risks appear to have a decreasing trend and to approach a limiting value. Over iterations, the Bayes risks decrease by a considerable amount. The limiting value of the Bayes risks appears to be slightly higher than the risk of the computed $\Gamma$-minimax estimator under $P_0$. This might indicate that $P_0$ is not an extreme distribution that yields a high risk.
Table 2: Risks and Bayes risks of estimators. $R(d, P_0)$: risk of the estimator under the true data-generating mechanism $P_0$. $r(d, \hat{\pi}^*)$: Bayes risk under prior $\hat{\pi}^*$, the computed prior from Algorithm 4 in the last and finest grid in the computation.

| Strength of prior |Estimator | $R(d, P_0)$ | $r(d, \hat{\pi}^*)$ |
|-------------------|----------|-------------|---------------------|
| strong            | OSW      | 265         | 300                 |
|                   | SCL      | 146         | 179                 |
|                   | $\Gamma$-minimax | 22     | 36                  |
| weak              | OSW      | 265         | 252                 |
|                   | SCL      | 146         | 142                 |
|                   | $\Gamma$-minimax | 56     | 85                  |
| almost none       | OSW      | 265         | 220                 |
|                   | SCL      | 146         | 119                 |
|                   | $\Gamma$-minimax | 76     | 108                 |

Figure 5: Estimated Bayes risks of the estimator over iterations when computing a $\Gamma_1$-minimax estimator. The lines are unbiased estimates of the current Bayes risks (y-axis) with 30 Monte Carlo runs over iterations (x-axis). The two dashed horizontal lines are the risks of OSW (upper) and SCL (lower) estimators, respectively, under $P_0$ in the simulation. The solid horizontal line is the risk of the computed $\Gamma$-minimax estimator under $P_0$. For clearness of visualization, the estimated Bayes risks are plotted every 50 iterations.
5.3 Estimation of the entropy

We also apply our method to estimate the entropy of a multinomial distribution. The setup of data-generating mechanism is the same as in Example 2, and the estimand of interest is the entropy, that is, \( \Psi(P_0) = \sum_{k=1}^{K} -p_k \log p_k \). We choose the same true population and the same sample size \( n = 100 \) as in Section 5.2. We take the same risk function as in Example 1. The true entropy \( \Psi(P_0) \) is 4.57. As a reference, the entropy of the uniform distribution with the same number of categories — which corresponds to the maximum entropy of multinomial distributions with the same total number of categories — is 5.24.

As in Section 5.2 we consider three sets of prior information:

1. Strongly informative: Prior mean of \( \Psi(P) \) in \([4.3, 4.7]\), \( \geq 95\% \) probability that \( \Psi(P) \) lies in \([4, 5]\);
2. Weakly informative: Prior mean of \( \Psi(P) \) in \([4, 5]\), \( \geq 95\% \) probability that \( \Psi(P) \) lies in \([3.5, 5.5]\);
3. Almost noninformative: Prior mean of \( \Psi(P) \) in \([3.7, 5.3]\), \( \geq 95\% \) probability that \( \Psi(P) \) lies in \([3, 6]\).

The architecture of our neural network estimator is almost identical to that in Section 5.2 except that the existing estimator being used is the one proposed in Jiao et al. (2015) (referred to as JVHW estimator), and we initialize the network to return the JVHW estimator. We use Algorithm 6 to construct \( \mathcal{M}_\ell \) and Algorithm 4 to compute a \( \Gamma \)-minimax estimator. The tuning parameters in the algorithms are identical to those used in Section 5.2 except that, in Algorithm 4 (i) the learning rate is \( \eta = 0.001 \), and (ii) the number of iterations is 6,000 for \( \Gamma_1 \). We change these tuning parameters because JVHW estimator is already minimax in terms of its convergence rate (Jiao et al., 2015), and we think we need to update the estimator more carefully in Algorithm 4 to obtain any possible improvement.

We finally compare the risk of JVHW and our trained \( \Gamma \)-minimax estimator estimator under our set data-generating mechanism computed with 20,000 Monte Carlo runs. We also compare their Bayes risk under the computed prior from Algorithm 4 using the last and finest grid in the computation with 20,000 Monte Carlo runs. The results are summarized in Table 3. In this simulation experiment, our \( \Gamma \)-minimax estimator reduces the risk by a fair percentage compared with JVHW estimator with somewhat informative prior knowledge. With almost noninformative prior knowledge, the risk of our \( \Gamma \)-minimax under \( P_0 \) is slightly higher than JVHW estimator, but the Bayes risk is still lower. The elevated risk under \( P_0 \) in this case is not surprising given that \( \Gamma \)-minimax estimators generally do not achieve optimal
Table 3: Risks and Bayes risks of estimators. $R(d, P_0)$: risk of the estimator under the true data-generating mechanism $P_0$. $r(d, \hat{\pi}^*)$: Bayes risk under prior $\hat{\pi}^*$, the computed prior from Algorithm 4 in the last and finest grid in the computation.

| Strength of prior | Estimator | $R(d, P_0)$ | $r(d, \hat{\pi}^*)$ |
|-------------------|-----------|-------------|---------------------|
| strong            | JVHW      | 0.041       | 0.045               |
|                   | $\Gamma$-minimax | 0.033 | 0.033              |
| weak              | JVHW      | 0.041       | 0.056               |
|                   | $\Gamma$-minimax | 0.040 | 0.048              |
| almost none       | JVHW      | 0.041       | 0.063               |
|                   | $\Gamma$-minimax | 0.046 | 0.055              |

Performance under every data-generating mechanism, but rather achieve optimal performance under the least favorable prior that is consistent with available knowledge. According to these simulation results, we conclude that incorporating weakly or strongly informative prior knowledge into the estimator may result in some improvement.

Fig 6 presents the unbiased estimated of Bayes risks over iterations when computing a $\Gamma_1$-minimax estimator. With somewhat informative prior information present, the Bayes risks appear to fluctuate without an increasing or decreasing trend at the beginning and decrease after several thousand iterations. With almost no prior information, the Bayes risks appear to fluctuate with no trend. A reason may be that JVHW estimator is already minimax rate optimal ([Jiao et al., 2015]). The computed $\Gamma$-minimax estimators also appear to be somewhat similar to JVHW estimator: in the output layer of the three settings with different prior information, the coefficients for JVHW estimator are 0.96, 0.95 and 0.95, respectively; the coefficients for the previous hidden layer are 0.17, 0.09 and 0.02, respectively; the intercepts are 0.09, 0.13 and 0.16, respectively.

6 Discussion

We mainly focus on estimation. Nevertheless, our framework can be immediately applied to prediction. In this setup, an estimator may take in the observed data and output a function (e.g., coefficients of a feedforward neural network), whose prediction performance may be evaluated by an appropriately chosen risk function. Studying the performance of our algorithms in this setting is an interesting area for future work.
We propose algorithms to compute a Gamma-minimax estimator with theoretical guarantees under fairly general settings. These algorithms still leave room for improvement. As we discussed in Section 3.1, the stopping criterion we employ does not necessarily indicate that the maximal Bayes risk is close to the true minimax Bayes risk. In future work, it would be interesting to derive a better criterion that necessarily does indicate this near optimality. Our algorithms also require the user to choose increasingly fine approximating grids to the model space. Although we propose a heuristic algorithm for this procedure that performed well in our experiments, at this point, we have not provided optimality guarantees for this scheme. It may also possible to improve our proposed algorithms to solve intermediate minimax problems in Section 3.2 by utilizing recent and ongoing advances from the machine learning literature that can be used to improve the training of generative adversarial networks.

We do not explicitly consider uncertainty quantification such as confidence intervals or credible intervals under a Gamma-minimax framework. Uncertainty quantification is important in practice since it provides more information than a point estimator and can be used for decision making. In theory, our method may be directly applied if such a problem can be formulated into a Gamma-minimax problem. However, such a formulation remains unclear. The most challenging part is to identify a suitable risk function that correctly balances the level of uncertainty and the size of the output interval/region. Though the risk function used in Schafer and Stark (2009) appears to provide one possible starting point, it is not clear how to extend this approach to nonparametric settings.

In conclusion, we propose algorithms to compute a Gamma-minimax estimator under general models...
that can incorporate prior information in the form of generalized moment conditions. They can be useful when a parametric model is undesirable, semi-parametric efficiency theory does not apply, or we wish to utilize prior information to improve estimation.

Appendix A  Proofs

A.1  Proof of Theorem 1 and Corollary 1

Lemma 1. If \( \{\Omega_\ell\}_{\ell=1}^\infty \) is an increasing sequence of subsets of \( \mathcal{M} \) such that \( \bigcup_{\ell=1}^\infty \Omega_\ell = \mathcal{M} \), then, for any \( d \in \mathcal{D} \), \( r_{\text{sup}}(d, \hat{\Gamma}_\ell) \to r_{\text{sup}}(d, \Gamma) \ (\ell \to \infty) \).

Proof of Lemma 1. Since \( \hat{\Gamma}_\ell \subseteq \hat{\Gamma}_{\ell+1} \subseteq \Gamma \), it holds that \( r_{\text{sup}}(d, \hat{\Gamma}_\ell) \leq r_{\text{sup}}(d, \hat{\Gamma}_{\ell+1}) \leq r_{\text{sup}}(d, \Gamma) \), and so we only need to lower bound \( r_{\text{sup}}(d, \hat{\Gamma}_\ell) \). Fix \( \epsilon > 0 \). By Corollary 5 of Pinelis (2016), \( r_{\text{sup}}(d, \Gamma) \) can be approximated by \( r(d, \nu) \) arbitrarily well for priors \( \nu \in \Gamma \) with a finite support; that is, there exists \( \nu \in \Gamma \) with finite support such that \( r(d, \nu) \geq r_{\text{sup}}(d, \Gamma) - \epsilon \). For sufficiently large \( \ell \), \( \Omega_\ell \) contains all support points of \( \nu \) and hence \( r_{\text{sup}}(d, \hat{\Gamma}_\ell) \geq r(d, \nu) \geq r_{\text{sup}}(d, \Gamma) - \epsilon \). The desired result follows. \( \square \)

Lemma 2. Under Condition 3, \( d \mapsto r(d, \pi) \) is Lipschitz continuous with Lipschitz constant \( L \); moreover, \( d \mapsto r_{\text{sup}}(d, \Gamma') \) is Lipschitz continuous with Lipschitz constant \( L \) for any \( \Gamma' \subseteq \Gamma \).

Proof of Lemma 2. By Condition 3, \( |R(d_1, P) - R(d_2, P)| \leq L\varrho(d_1, d_2) \) for any \( d_1, d_2 \in \mathcal{D} \) and any \( P \in \mathcal{M} \). Then, for any \( \pi \in \Gamma \) and any \( d_1, d_2 \in \mathcal{D} \),

\[
|r(d_1, \pi) - r(d_2, \pi)| = \left| \int [R(d_1, P) - R(d_2, P)] \pi(dP) \right| \\
\leq \int |R(d_1, P) - R(d_2, P)| \pi(dP) \\
\leq L\varrho(d_1, d_2).
\]

This proves that \( d \mapsto r(d, \pi) \) is Lipschitz continuous with a universal Lipschitz constant \( L \). We now prove that \( d \mapsto r_{\text{sup}}(d, \Gamma) \) is Lipschitz continuous with Lipschitz constant \( L \). Let \( \epsilon > 0 \). For any \( d_1 \in \mathcal{D} \), there exists \( \pi_1 \in \Gamma' \) such that \( r_{\text{sup}}(d_1, \Gamma') \leq r(d_1, \pi_1) + \epsilon \). Then, for any \( d_2 \in \mathcal{D} \),

\[
r_{\text{sup}}(d_1, \Gamma') - r_{\text{sup}}(d_2, \Gamma') \leq r(d_1, \pi_1) + \epsilon - r(d_2, \pi_1) \leq L\varrho(d_1, d_2) + \epsilon.
\]
Since $\epsilon$ is arbitrary, we have that $r_{\sup}(d_1, \Gamma') - r_{\sup}(d_2, \Gamma') \leq L_\varrho(d_1, d_2)$. Reversing the role of $d_1$ and $d_2$, we derive that $r_{\sup}(d_2, \Gamma') - r_{\sup}(d_1, \Gamma') \leq L_\varrho(d_1, d_2)$. Therefore, $|r_{\sup}(d_1, \Gamma') - r_{\sup}(d_2, \Gamma')| \leq L_\varrho(d_1, d_2)$ for any $d_1, d_2 \in D$.

**Proof of Theorem 4** Let $\epsilon > 0$. There exists $d' \in D$ such that

$$r_{\sup}(d', \Gamma) \leq \inf_{d \in D} r_{\sup}(d, \Gamma) + \epsilon.$$ 

Moreover, there exists $\pi_{\ell} \in \Gamma_{\ell}$ such that

$$r_{\sup}(d', \Gamma_{\ell}) \leq r(d', \pi_{\ell}) + \epsilon.$$ 

Using the fact that $d'_\ell$ is $\Gamma_{\ell}$-minimax and the definition of $r_{\sup}$, it holds that

$$r_{\sup}(d'_\ell, \Gamma_{\ell}) \leq r_{\sup}(d', \Gamma_{\ell}) \leq r(d', \pi_{\ell}) + \epsilon$$

$$\leq r_{\sup}(d', \Gamma) + \epsilon \leq \inf_{d \in D} r_{\sup}(d, \Gamma) + 2\epsilon.$$ 

Since this inequality holds for any $\epsilon > 0$, we have that $r_{\sup}(d'_\ell, \Gamma_{\ell}) \leq \inf_{d \in D} r_{\sup}(d, \Gamma)$. An almost identical argument shows that the sequence $\{r_{\sup}(d'_\ell, \Gamma_{\ell})\}_{\ell=1}^{\infty}$ is nondecreasing. Therefore, this sequence converges to some limit $\mathcal{R} \leq \inf_{d \in D} r_{\sup}(d, \Gamma) \leq r_{\sup}(d^*, \Gamma)$.

We next prove that $r_{\sup}(d^*, \Gamma) \leq \mathcal{R}$. Let $\epsilon > 0$. Without loss of generality, we may assume that $\mathcal{M}_\ell \subseteq \Omega_\ell$ for all $\ell = 1, 2, \ldots$ in Condition 4 (Otherwise, we may instead consider the sequence $\{\Omega_\ell\}_{\ell=1}^{\infty}$). Note that Condition 4 also holds for $\{\Omega_{\ell'}\}_{\ell'=1}^{\infty}$.) By Lemma 1 there exists $\ell_0$ such that $r_{\sup}(d^*, \bar{\Gamma}_{i_0}) \geq r_{\sup}(d^*, \Gamma) - \epsilon/3$. By Condition 4 there exists $i_1$ such that $r_{\sup}(d^*, \Gamma_{i_1|\ell_0}) \geq r_{\sup}(d^*, \bar{\Gamma}_{i_0}) - \epsilon/3$. Without loss of generality, suppose that $d^*_i \to d^*$ (otherwise, take a convergent subsequence to this accumulation point). This then implies that there exists $i_2 > i_1$ such that $\varrho(d^*_i, d^*) \leq \epsilon/(3L)$. By Lemma 2 $r_{\sup}(d_{i_2|\ell_0}, \Gamma_{i_1|\ell_0}) \geq r_{\sup}(d^*, \Gamma_{i_1|\ell_0}) - \epsilon/3$. Moreover, since $\Gamma_{i_1|\ell_0} \subseteq \Gamma_{i_1} \subseteq \Gamma_{i_2}$, it holds that $r_{\sup}(d_{i_2}, \bar{\Gamma}_{i_2}) \geq r_{\sup}(d^*, \Gamma_{i_1|\ell_0})$. Therefore, $r_{\sup}(d_{i_2}, \Gamma_{i_2}) \geq r_{\sup}(d^*, \Gamma) - \epsilon$. Since the sequence $\{r_{\sup}(d^*_i, \Gamma_{\ell})\}_{\ell=1}^{\infty}$ is nondecreasing, it holds that $r_{\sup}(d^*_i, \Gamma_{\ell}) \geq r_{\sup}(d^*, \Gamma) - \epsilon$ for all $\ell \geq i_2$. Therefore, $\liminf_{\ell \to \infty} r_{\sup}(d^*_i, \Gamma_{\ell}) \geq r_{\sup}(d^*, \Gamma)$, and hence $\mathcal{R} \geq r_{\sup}(d^*, \Gamma)$.

Combining the results from the preceding two paragraphs, $\mathcal{R} = \inf_{d \in D} r_{\sup}(d, \Gamma) = r_{\sup}(d^*, \Gamma)$. 

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Consequently, $d^*$ is $\Gamma$-minimax. Moreover, as $\{r_{\sup}(d^*_\ell, \Gamma)\}_{\ell=1}^{\infty}$ increases to $R$, this sequence also increases to $r_{\sup}(d^*, \Gamma)$. This concludes the proof.

\[ \square \]

**Proof of Corollary 1.** We first establish the strict convexity of $d \mapsto r(d, \pi)$ for any $\pi \in \Gamma$. We then establish the strict convexity of $d \mapsto r_{\sup}(d, \Gamma)$. We then establish that there is a unique minimizer of $d \mapsto r_{\sup}(d, \Gamma)$ and show that the desired result follows from Theorem 1.

Let $d_1, d_2 \in \mathcal{D}$ and $c \in (0,1)$ be arbitrary, then by the convexity of $\mathcal{D}$ and the strict convexity of $d \mapsto R(d, P)$ for each $P \in \mathcal{M}$,

$$ r(cd_1 + (1-c)d_2, \pi) = \int R(cd_1 + (1-c)d_2, P) \pi(dP) < \int \{cR(d_1, P) + (1-c)R(d_2, P)\} \pi(dP) = cr(d_1, \pi) + (1-c)r(d_2, \pi). $$

Therefore, $d \mapsto r(d, \pi)$ is strictly convex for any $\pi \in \Gamma$.

Let $d_1, d_2 \in \mathcal{D}$ and $c \in (0,1)$ be arbitrary. Since $r_{\sup}(d, \Gamma)$ is attainable for any $d \in \mathcal{D}$, there exists $\tilde{\pi} \in \Gamma$ such that

$$ r_{\sup}(cd_1 + (1-c)d_2, \Gamma) = r_{\sup}(cd_1 + (1-c)d_2, \tilde{\pi}) < cr(d_1, \tilde{\pi}) + (1-c)r(d_2, \tilde{\pi}) \leq cr_{\sup}(d_1, \Gamma) + (1-c)r_{\sup}(d_2, \Gamma). $$

Thus, $d \mapsto r_{\sup}(d, \Gamma)$ is strictly convex.

As $d \mapsto r_{\sup}(d, \Gamma)$ is continuous by Condition 3 and $\mathcal{D}$ is compact by Condition 2, $d \mapsto r_{\sup}(d, \Gamma)$ achieves at least one minimum on $\mathcal{D}$. As $d \mapsto r_{\sup}(d, \Gamma)$ is strictly convex and $\mathcal{D}$ is convex, this function achieves exactly one minimum on $\mathcal{D}$. By Theorem 1 any accumulation point $d^*$ of $\{d^*_\ell\}_{\ell=1}^{\infty}$ is a minimizer of $d \mapsto r_{\sup}(d, \Gamma)$, and so the sequence has a limit point, which is also the unique $\Gamma$-minimax estimator.

\[ \square \]

**A.2 Proof of Theorem 2**

We prove Theorem 2 by checking that Assumptions 3.1 and 3.6 in Lin et al. (2019) are satisfied and using Theorem E.3 and E.4 in Lin et al. (2019), respectively. Since Assumption 3.1 is satisfied by our
construction of \( \hat{R} \), we focus on Assumption 3.6 for the rest of this section.

Let \( \mathcal{M}_\ell = \{P_1, P_2, \ldots, P_\Lambda\} \subseteq \mathcal{M} \). For any \( \pi \in \Gamma_\ell \), let \( \pi_\lambda \) denote the probability weight of \( \pi \) on \( P_\lambda \) (\( \lambda = 1, \ldots, \Lambda \)). For the rest of this section, we also use \( \pi \) to denote the vector \((\pi_1, \ldots, \pi_\Lambda)\). We also use \( \lesssim \) to denote less than equal to up to a universal positive constant that may depend on \( \beta \). Therefore, straightforward calculations imply that \( \nabla_\beta r(\beta, \pi) = \sum_{\lambda=1}^\Lambda \pi_\lambda \nabla_\beta R(\beta, P_\lambda) \) and \( \nabla_\pi r(\beta, \pi) = (R(\beta, P_1), \ldots, R(\beta, P_\Lambda))^\top \).

For each \( \ell = 1, 2, \ldots \), for any \( \beta^1, \beta^2 \in \mathcal{H} \) and \( \pi^1, \pi^2 \in \Gamma_\ell \), by Conditions 5 and 6,

\[
\left\| \nabla_\beta r(\beta, \pi)|_{\beta=\beta^1, \pi=\pi^1} - \nabla_\beta r(\beta, \pi)|_{\beta=\beta^2, \pi=\pi^2} \right\| \\
= \left\| \sum_{\lambda=1}^\Lambda \left\{ \pi_\lambda \nabla_\beta R(\beta, P_\lambda)|_{\beta=\beta^1} - \pi_\lambda \nabla_\beta R(\beta, P_\lambda)|_{\beta=\beta^2} \right\} \right\| \\
\leq \sum_{\lambda=1}^\Lambda \pi_\lambda \left\| \nabla_\beta R(\beta, P_\lambda)|_{\beta=\beta^1} - \nabla_\beta R(\beta, P_\lambda)|_{\beta=\beta^2} \right\| + \left\| \sum_{\lambda=1}^\Lambda (\pi_\lambda - \pi_\lambda^2) \nabla_\beta R(\beta, P_\lambda)|_{\beta=\beta^2} \right\| \\
\lesssim \|\beta^1 - \beta^2\| + \|\pi^1 - \pi^2\| \\
\lesssim \|(\beta^1, \pi^1) - (\beta^2, \pi^2)\|,
\]

and similarly for \( \nabla_\pi r(\beta, \pi) \),

\[
\left\| \nabla_\pi r(\beta, \pi)|_{\beta=\beta^1, \pi=\pi^1} - \nabla_\pi r(\beta, \pi)|_{\beta=\beta^2, \pi=\pi^2} \right\| \\
= \left\| (R(\beta^1, P_1) - R(\beta^2, P_1), R(\beta^1, P_2) - R(\beta^2, P_2), \ldots, R(\beta^1, P_\Lambda) - R(\beta^2, P_\Lambda))^\top \right\| \\
\lesssim \|\beta^1 - \beta^2\| \leq \| (\beta^1, \pi^1) - (\beta^2, \pi^2) \|.
\]

This implies that for each \( \ell \), the gradient of \( r(\beta, \pi) \) (\( \beta \in \mathcal{H}, \pi \in \Gamma_\ell \)) is Lipschitz continuous.

For each \( \ell = 1, 2, \ldots \), for any \( \beta^1, \beta^2 \in \mathcal{H} \) and \( \pi \in \Gamma_\ell \), Condition 6 implies that

\[
|r(\beta^1, \pi) - r(\beta^2, \pi)| = \left| \sum_{\lambda=1}^\Lambda \pi_\lambda \left[ R(\beta^1, P_\lambda) - R(\beta^2, P_\lambda) \right] \right| \\
\leq \sum_{\lambda=1}^\Lambda \pi_\lambda \left| R(\beta^1, P_\lambda) - R(\beta^2, P_\lambda) \right| \lesssim \|\beta^1 - \beta^2\|.
\]

Therefore, \( \beta \mapsto r(\beta, \pi) \) is Lipschitz continuous with a universal Lipschitz constant independent of \( \pi \in \Gamma_\ell \).

Finally, it is straightforward to check that (i) \( \pi \mapsto r(\beta, \pi) \) is concave for any \( \beta \in \mathcal{H} \), and (ii) \( \Gamma_\ell \) is
parameterized by a convex subset of a simplex in a Euclidean space, which is a convex and bounded set. These results show that Assumption 3.6 in Lin et al. (2019) is satisfied for Algorithm 2 and 3.

A.3 Proof of Theorem 3

Proof of Theorem 3 Let \( \pi(t), 0 \) denote a maximizer of \( \pi \mapsto r(\beta_{(t-1)}, \pi) \). It holds that

\[
0 \leq r(\beta_{(t-1)}, \pi(t), 0) - r(\beta_{(t-1)}, \pi(t)) \\
\leq \frac{1}{J'} \sum_{j=1}^{J'} \hat{r}(\beta_{(t-1)}, \pi(t), \xi_j) - \frac{1}{J'} \sum_{j=1}^{J'} \hat{r}(\beta_{(t-1)}, \pi(t), 0, \xi_j) \\
+ r(\beta_{(t-1)}, \pi(t), 0) - r(\beta_{(t-1)}, \pi(t)) \\
= \frac{1}{J'} \sum_{j=1}^{J'} \left\{ \hat{r}(\beta_{(t-1)}, \pi(t), \xi_j) - \hat{r}(\beta_{(t-1)}, \pi(t), 0, \xi_j) \right\} \\
- \mathbb{E} \left[ \hat{r}(\beta_{(t-1)}, \pi(t), \xi) - \hat{r}(\beta_{(t-1)}, \pi(t), 0, \xi) \right] \\
\leq \sup_{\beta \in \mathbb{R}^D, \pi_1, \pi_2 \in \Gamma_{\ell}} \left\{ \frac{1}{J'} \sum_{j=1}^{J'} \left[ \hat{r}(\beta, \pi_1, \xi_j) - \hat{r}(\beta, \pi_2, \xi_j) \right] \\
- \mathbb{E} \left[ \hat{r}(\beta, \pi_1, \xi) - \hat{r}(\beta, \pi_2, \xi) \right] \right\}.
\]

Note that the right hand side does not depend on \( t \). Therefore,

\[
0 \leq \sup_t \left\{ r(\beta_{(t-1)}, \pi(t), 0) - \mathbb{E}[r(\beta_{(t-1)}, \pi(t))] \right\}
\leq \mathbb{E}^* \sup_{\beta \in \mathbb{R}^D, \pi_1, \pi_2 \in \Gamma_{\ell}} \left\{ \frac{1}{J'} \sum_{j=1}^{J'} \left[ \hat{r}(\beta, \pi_1, \xi_j) - \hat{r}(\beta, \pi_2, \xi_j) \right] \\
- \mathbb{E} \left[ \hat{r}(\beta, \pi_1, \xi) - \hat{r}(\beta, \pi_2, \xi) \right] \right\}.
\]

where \( \mathbb{E}^* \) stands for outer expectation. We may apply Corollary 9.27 in Kosorok (2008) to \( \mathcal{F} := \{ \xi \mapsto \hat{r}(\beta, \pi, \xi) : \beta \in \mathbb{R}^D, \pi \in \Gamma_{\ell} \} \) and show that \( \mathcal{F} - \mathcal{F} := \{ f_1 - f_2 : f_1, f_2 \in \mathcal{F} \} \supset \{ \xi \mapsto \hat{r}(\beta, \pi_1, \xi) - \hat{r}(\beta, \pi_2, \xi) :
\[ \beta \in \mathbb{R}^{\mathcal{D}}, \pi_1, \pi_2 \in \Gamma \} \] is a \( \Xi \)-Glivenko-Cantelli class. Therefore,

\[
\left\{ \sup_{\beta \in \mathbb{R}^{\mathcal{D}}, \pi_1, \pi_2 \in \Gamma} \left| \frac{1}{J'} \sum_{j=1}^{J'} \left[ \hat{r}(\beta, \pi_1, \xi_j) - \hat{r}(\beta, \pi_2, \xi_j) \right] \right| \right\}^* \\
- \mathbb{E} \left[ \hat{r}(\beta, \pi_1, \xi_j) - \hat{r}(\beta, \pi_2, \xi_j) \right] \right\}^* \\
\leq \left\{ \sup_{f \in \mathcal{F} - \mathcal{F}} \left| \frac{1}{J'} \sum_{j=1}^{J'} \left( f(\xi_j) - \mathbb{E}[f(\xi_j)] \right) \right| \right\}^* \overset{a.s.}{\rightarrow} 0,
\]

as \( J' \to \infty \). Here, \( X^* \) stands for the minimal measurable majorant with respect to \( \Xi \) of a (possibly non-measurable) mapping \( X \) (van der Vaart and Wellner, 2000).

By Problem 1 of Section 2.4 in [van der Vaart and Wellner (2000)], there exists a random variable \( F \) such that \( F \geq \sup_{f \in \mathcal{F} - \mathcal{F}} |f(\xi) - \mathbb{E}[f(\xi)]| \) \( \Xi \)-almost surely and \( \mathbb{E}[F] < \infty \). Then,

\[
\sup_{f \in \mathcal{F} - \mathcal{F}} \left| \frac{1}{J'} \sum_{j=1}^{J'} \left( f(\xi_j) - \mathbb{E}[f(\xi_j)] \right) \right| \leq F
\]

\( \Xi \)-almost surely. By dominated convergence theorem,

\[
\mathbb{E}^* \sup_{\beta \in \mathbb{R}^{\mathcal{D}}, \pi_1, \pi_2 \in \Gamma} \left| \frac{1}{J'} \sum_{j=1}^{J'} \left[ \hat{r}(\beta, \pi_1, \xi_j) - \hat{r}(\beta, \pi_2, \xi_j) \right] \right| \\
- \mathbb{E} \left[ \hat{r}(\beta, \pi_1, \xi_j) - \hat{r}(\beta, \pi_2, \xi_j) \right] \right\} \rightarrow 0
\]

as \( J' \to \infty \), and so does \( \sup \{ r(\beta(t-1), \pi(t), 0) - \mathbb{E}[r(\beta(t-1), \pi(t))] \} \). Thus, for any \( \zeta > 0 \), there exists a sufficiently large \( J' \) such that \( \mathbb{E}[r(\beta(t-1), \pi(t))] - \zeta \) for all \( t \).

\[ \mathbb{A.4 \ Proof \ of \ Theorem} \]

Our proof of Theorem \[ \] builds on that of Robinson (1951). Major modifications are needed to allow for more general definitions that can accommodate for potentially infinite spaces of pure strategies and a more careful control on a bound on \( r(\tilde{d}(\pi(t)), \pi^*_t) - r(d^*_t, \pi(t-1)) \) towards the end of the proof.

We first introduce the notion of cumulative Bayes risk functions. Under Algorithm \[ \] we let \( U_0 : \]
\( \mathcal{D} \to \mathbb{R} \) and \( V_0 : \Gamma_\ell \to \mathbb{R} \) be any two continuous functions such that

\[
\min_{d \in \mathcal{D}} U_0(d) = \max_{\pi \in \Gamma_\ell} V_0(\pi)
\]

and recursively define

\[
U_{t+1}(d) := U_t(d) + r(d, \pi^{\dagger}_t), \quad V_{t+1}(\pi) := V_t(\pi) + r(d_t^{\dagger}, \pi)
\]

for \( d \in \mathcal{D} \) and \( \pi \in \Gamma_\ell \). Here, we let \( \pi^{\dagger}_t \in \operatorname{argmax}_{\pi \in \Gamma_\ell} V_{t-1}(\pi) \) and \( d_t^{\dagger} \in \operatorname{argmin}_{d \in \mathcal{D}} U_{t-1}(d) \). Note that the choices of \( \pi^{\dagger}_t \) and \( d_t^{\dagger} \) in Algorithm 5 corresponds to setting \( U_0 \equiv 0 \) and \( V_0 \equiv 0 \), in which case

\[
U_t(d) = t \cdot r(d, \pi_t), \quad V_t(\pi) = t \cdot r(\mathcal{D}(\pi_t), \pi).
\]

for some \( \pi_t \in \Gamma \) and \( \mathcal{D}(\pi_t) \in \mathcal{D} \). Later in this section, we will also make use of \( U_t \) and \( V_t \) with other initializations \( U_0 \) and \( V_0 \).

To make notations concise, we define \( \min_{d \in \mathcal{D}'} U_t := \min_{d \in \mathcal{D}'} U_t(d) \) for any \( \mathcal{D}' \subseteq \mathcal{D} \), and define \( \max_{\mathcal{D}'} U_t, \min_{\Pi'} V_t \) and \( \max_{\Pi'} V_t \) (\( \Pi' \subseteq \Gamma_\ell \)) similarly. We also drop the subscript denoting the set when the set is the whole space we consider, i.e., \( \mathcal{D} \) or \( \Gamma_\ell \). Note that for any \( t_1, t_2 = 1, 2, \ldots \), under the setting of Algorithm 5, it holds that

\[
\min U_{t_1} / t_1 = \min_{\eta \in \mathcal{D}} r(\mathcal{D}, \pi_{t_1})
\]

\[
\leq \max_{\pi \in \Gamma_\ell} \min_{\eta \in \mathcal{D}} r(\mathcal{D}, \pi) = r(\mathcal{D}(\pi^{\ast}_t), \pi^{\ast}_t) = \min_{\eta \in \mathcal{D}} r(\mathcal{D}, \pi)
\]

\[
\leq \max_{\pi \in \Gamma_\ell} r(\mathcal{D}(\pi_{t_2}), \pi) = V_{t_2} / t_2
\]

Therefore, to prove the first result in Theorem 4, it suffices to show that \( \limsup_{t \to \infty} (\max V_t - \min U_t) / t \leq 0 \).

We next introduce additional definitions related to iterations. We say that \( \pi \in \Gamma_\ell \) is eligible in the interval \( [t_1, t_2] \) if there exists \( t \in [t_1, t_2] \) such that \( V_t(\pi) = \max V_t \); we say that \( d \in \mathcal{D} \) is eligible in the interval \( [t_1, t_2] \) if there exists \( t \in [t_1, t_2] \) such that \( U_t(d) = \min U_t \). We also define eligibility for sets.
say that $\Pi' \subseteq \Gamma$ is eligible in the interval $[t_1, t_2]$ if there exists $\pi \in \Pi'$ that is eligible in that interval; we say that $\mathcal{D}' \subseteq \mathcal{D}$ is eligible in the interval $[t_1, t_2]$ if there exists $d \in \mathcal{D}'$ that is eligible in the interval $[t_1, t_2]$. In addition, for any $\mathcal{D}' \subseteq \mathcal{D}$, we define maximum variation $\text{MV}_t(\mathcal{D}') := \sup_{d \in \mathcal{D}'} U_t(d) - \inf_{d \in \mathcal{D}'} U_t(d)$ and $\text{MV}_t(\Pi')$ similarly for any $\Pi' \subseteq \Gamma$. By Condition [3], there exists $B \in (0, \infty)$ such that $R \in [-B, B]$. Note that by Condition [2] and Lemma [2] given an arbitrary desired approximation accuracy $\epsilon > 0$, $\mathcal{D}$ can be covered by finitely many compact subsets with the maximum variation of each subset bounded by $\epsilon t$ for all $t$; by Condition [3], since $\Gamma$ is parameterized by a compact subset of a simplex in a Euclidean space, $\Gamma$ can also be covered by finitely many compact subsets with the maximum variation of each subset bounded by $\epsilon t$ for all $t$. These covers can be viewed as discrete finite approximations to $\mathcal{D}$ and $\Gamma$, respectively.

All of the above definitions are associated with the space of estimators $\mathcal{D}$ and the set of priors $\Gamma$. We call $\{(U_t, V_t)\}_t$ a pair of cumulative Bayes risk functions constructed from the pair $(\mathcal{D}, \Gamma)$ of the space of estimators and the set of priors, and will consider pairs of cumulative Bayes risk functions constructed from other pairs $(\mathcal{D}', \Pi')$ of the space of estimators and the set of priors in the subsequent proof. We can define the above quantities similarly for such cases.

The following lemma gives an upper bound on the maximum variation of $U_{s+t}$ and $V_{s+t}$ over the corresponding entire space from which they are constructed after $t$ iterations from $s$ when essentially all parts of these spaces are eligible in $[s, s + t]$.

**Lemma 3.** Suppose that $\{(U_t, V_t)\}_t$ is a pair of cumulative Bayes risk functions constructed from $(\mathcal{D}', \Pi')$. Suppose that $\mathcal{D}' = \bigcup_{i=1}^{j} \mathcal{D}_i$ and $\Pi' = \bigcup_{i=1}^{j} \Pi_j$ where

$$\sup_{i,t} \text{MV}_t(\mathcal{D}_i)/t \leq A, \quad \sup_{j,t} \text{MV}_t(\Pi_j)/t \leq A$$

for $A < \infty$. If all $\mathcal{D}_i$ and $\Pi_j$ are eligible in $[s, s + t]$, then $\max_{\mathcal{D}'} U_{s+t} - \min_{\mathcal{D}'} U_{s+t} \leq (2B + A)t$ and $\max_{\Pi'} V_{s+t} - \min_{\Pi'} V_{s+t} \leq (2B + A)t$.

**Proof of Lemma 3.** Without loss of generality, assume that $\tilde{d} \in (\arg\max_{d \in \mathcal{D}'} U_{s+t}) \cap \mathcal{D}_1$. Since $\mathcal{D}_1$ is eligible in $[s, t]$, there exists $\tilde{t} \in [s, s + t]$ such that $(\arg\min_{d \in \mathcal{D}'} U_{\tilde{t}}) \cap \mathcal{D}_1 \neq \emptyset$. By the recursive definition of the sequence $\{U_t\}_t$ in [3], the bound on the risk, and the assumption that $\sup_{i,t} \text{MV}_t(\mathcal{D}_i)/t \leq A$, we have that $\max_{\mathcal{D}'} U_{s+t} = U_{s+t}(\tilde{d}) \leq U_{\tilde{t}}(\tilde{d}) + B(s + t - \tilde{t}) \leq \min_{\mathcal{D}'} U_{\tilde{t}} + At + B(s + t - \tilde{t}) \leq \min_{\mathcal{D}'} U_{\tilde{t}} + (A + B)t$. Letting $\tilde{d}' \in \arg\min_{d \in \mathcal{D}'} U_{s+t}$, by the bound on the risk, we can derive that
\[
\min_{D'} U_{s+t} = U_{s+t}(\tilde{d}') \geq U_f(\tilde{d}') - B(s + t - \tilde{t}) \geq \min_{D'} U_f - Bt. \]
Combine these two inequalities and we have that \(\max_{D'} U_{s+t} - \min_{D'} U_{s+t} \leq (2B + A)t\). An identical argument applied to the sequence \(\{V_t\}_t\) shows that \(\max_{\Pi'} V_{s+t} - \min_{\Pi'} V_{s+t} \leq (2B + A)t\).

The next lemma builds on the previous lemma and provides an upper bound on \(\max V_{s+t} - \min U_{s+t}\) under the same conditions.

**Lemma 4.** Under the same setup and conditions as in Lemma 3, \(\max_{\Pi'} V_{s+t} - \min_{\Pi'} U_{s+t} \leq (4B+2A)t\).

**Proof of Lemma 4.** Summing the two inequalities in Lemma 3 and rearranging the terms, we have that \(\max_{\Pi'} V_{s+t} - \min_{\Pi'} U_{s+t} \leq (4B + 2A)t + \max_{\Pi'} V_{s+t} - \max_{D'} U_{s+t}\). It therefore suffices to show that \(\min_{\Pi'} V_{s+t} \leq \max_{D'} U_{s+t}\).

Let \(\tau := s + t\). There exists \(\pi' \in \Pi'\) and a stochastic strategy \(\tilde{d}' \in D'\) such that \(U_f(d) = U_0(d) + \tau \cdot r(d, \pi')\) and \(V_f(\pi) = V_0(\pi) + \tau \cdot r(\tilde{d}', \pi)\) for all \(d \in D'\) and all \(\pi \in \Pi'\). Therefore, for this choice of \(\pi'\) and \(\tilde{d}'\), using (2) \(\min_{\Pi'} V_\tau(\pi') = V_0(\pi') + \tau \cdot r(\tilde{d}', \pi') \leq \max_{\Pi'} V_0 + \tau \cdot r(\tilde{d}', \pi') \leq U_0(\tilde{d}') + \tau \cdot r(\tilde{d}', \pi') = V_\tau(\tilde{d}') \leq \max_{D'} U_\tau\).

**Proof of Theorem 4.** It suffices to show that \(\limsup_{t \to \infty} (\max V_t - \min U_t)/t \leq 0\) by letting \(U_0 \equiv 0\) and \(V_0 \equiv 0\), which corresponds to Algorithm 5. Let \(\epsilon > 0\). Note that \(r\) is Lipschitz continuous by Lemma 2 and the fact that \(r(d, \pi)\) is an average of bounded risks with weights \(\pi\). Furthermore, \(D\) and \(\Gamma_t\) are both compact. In addition, \(U_0\) and \(V_0\) are both continuous. Therefore, there exist covers \(D = \bigcup_{i=1}^{I} D_i\) and \(\Gamma_t = \bigcup_{j=1}^{J} \Pi_j\) such that (i) \(D_i\) and \(\Pi_j\) are all compact, and (ii) \(\sup_{t \in \mathbb{I}} MV_t(D_i)/t \leq \epsilon\), \(\sup_{t \in \mathbb{J}} MV_t(\Pi_j)/t \leq \epsilon\). (Note that \(I\) and \(J\) may depend on \(\epsilon\).) For index sets \(\mathbb{I} \subseteq \{1, 2, \ldots, I\}\) and \(\mathbb{J} \subseteq \{1, 2, \ldots, J\}\), define \(\mathbb{D}_I := \bigcup_{i \in \mathbb{I}} D_i\) and \(\Pi_{\mathbb{I}_J} := \bigcup_{j \in \mathbb{J}_J} \Pi_j\). We show that \(\max V_t - \min U_t \leq Ct\) for an absolute constant \(C\) and all sufficiently large \(t\) via induction on the sizes of \(\mathbb{I}\) and \(\mathbb{J}\).

Let \(\{(U_t, V_t)\}_t\) be a pair of cumulative Bayes risk functions constructed from \((\mathbb{D}_I, \Pi_{\mathbb{I}_J})\) where \(\mathbb{I}_J = \ldots = \mathbb{I}_J = \mathbb{J}_J = \ldots = \mathbb{J}_J = \ldots\).
$|\mathcal{J}| = 1$. By (4) and the fact that $MV_t(D_I) \leq \epsilon t$ and $MV_t(\Pi_{\mathcal{J}}) \leq \epsilon t$, we have that

$$\min_{D_I} U_t = \min_{d \in D_I} [U_0(d) + t \cdot r(d, \pi(d))] \geq E_{d \sim \pi_d} [U_0(d)] + t \cdot r(\bar{d}(\pi_0), \pi_0) - \epsilon t$$

$$\geq \min_{d \in D_I} U_0(d) + t \cdot r(\bar{d}(\pi_0), \pi_0) - \epsilon t$$

$$= \max_{\pi \in \Pi_{\mathcal{J}}} V_0(\pi) + t \cdot r(\bar{d}(\pi_0), \pi_0) - \epsilon t$$

$$\geq \max_{\pi \in \Pi_{\mathcal{J}}} [V_0(\pi) + t \cdot r(\bar{d}(\pi_0), \pi)] - 2\epsilon t = \max_{\Pi_{\mathcal{J}}} V_t - 2\epsilon t.$$

Therefore, $\max_{\Pi_{\mathcal{J}}} V_t - \min_{D_I} U_t \leq 2\epsilon t$.

Let $\epsilon' > 0$ be arbitrary. Suppose that there exists $t_0$ such that, for any $\mathcal{I}' \subseteq \mathcal{I}$ and $\mathcal{J}' \subseteq \mathcal{J}$ such that $\mathcal{I}' \neq \mathcal{I}$ or $\mathcal{J}' \neq \mathcal{J}$, for any pair of cumulative Bayes risk functions $\{(U_t, V_t)\}_t$ constructed from $(D_{\mathcal{I}'}, \Pi_{\mathcal{J}'})$, it holds that $\max_{\Pi_{\mathcal{J}'}} V_t - \min_{D_{\mathcal{I}'}} U_t \leq \epsilon'$ for all $t \geq t_0$. We next obtain a slightly greater bound on $\max_{\Pi_{\mathcal{J}'}} V_t - \min_{D_{\mathcal{I}'}} U_t$ for all sufficiently large $t$.

We first prove that if, for a given pair of cumulative Bayes risk functions $\{(U_t, V_t)\}_t$ constructed from $(D_{\mathcal{I}}, \Pi_{\mathcal{J}})$, there exists $i' \in \mathcal{I}$ or $j' \in \mathcal{J}$ such that $D_{i'}$ or $\Pi_{j'}$ is not eligible in an interval $[s, s + t_0]$, then

$$\max_{\Pi_{\mathcal{J}}} V_{s+t_0} - \min_{D_{\mathcal{I}}} U_{s+t_0} \leq \max_{\Pi_{\mathcal{J}}} V_s - \min_{D_{\mathcal{I}}} U_s + \epsilon't_0. \tag{5}$$

Suppose that $D_{i'}$ is not eligible in $[s, s + t_0]$, then define $U_t' := U_{s+t}$ and $V_t' := V_{s+t} - \max_{\Pi_{\mathcal{J}}} V_s + \min_{D_{\mathcal{I}}} U_s$ for all $t \geq 0$. It is straightforward to check that $\{(U_t', V_t')\}_{t=0}^{t_0}$ satisfies the recursive definition of a pair of cumulative Bayes risk functions constructed from $(D_{\mathcal{I}\setminus\{i'\}}, \Pi_{\mathcal{J}})$. By the induction hypothesis, $\max_{\Pi_{\mathcal{J}}} V_{t_0}' - \min_{D_{\mathcal{I}\setminus\{i'\}}} U_{t_0}' \leq \epsilon't_0$. Therefore, $\max_{\Pi_{\mathcal{J}}} V_{s+t_0} - \min_{D_{\mathcal{I}}} U_{s+t_0} = \max_{\Pi_{\mathcal{J}}} V_{t_0}' - \min_{D_{\mathcal{I}\setminus\{i'\}}} U_{t_0}' + \max_{\Pi_{\mathcal{J}}} V_s - \min_{D_{\mathcal{I}}} U_s + \epsilon't_0$. Similar argument can be applied if $\Pi_{j'}$ is not eligible in $[s, s + t_0]$.

Now we obtain a bound on $\max_{\Pi_{\mathcal{J}}'} V_t - \min_{D_{\mathcal{I}}} U_t$. Let $t > t_0$, $\Omega := \lfloor t/t_0 \rfloor \geq 1$ and $\mathcal{R} := t/t_0 - \Omega \in [0, 1)$. There are two cases.

**Case 1:** There exists $s_0 \leq \Omega$ such that $D_i$ and $\Pi_j$ are eligible in $[(s_0 - 1 + \mathcal{R})t_0, (s_0 + \mathcal{R})t_0]$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Take $s_0$ to be the largest such integer. Then, repeatedly apply (5) to intervals $[(s_0 + \mathcal{R})t_0, (s_0 + 1 + \mathcal{R})t_0], [(s_0 + 1 + \mathcal{R})t_0, (s_0 + 2 + \mathcal{R})t_0], \ldots, [(\Omega - 1 + \mathcal{R})t_0, (\Omega + \mathcal{R})t_0] = [t - t_0, t]$ and
we derive that

\[
\max_{\pi_{i,j}} V_t - \min_{\pi_{i,j}} U_t \leq \max_{\pi_{i,j}} V_{(s_0 + \mathbb{R})t_0} - \min_{\pi_{i,j}} U_{(s_0 + \mathbb{R})t_0} + \epsilon'(Q - s_0)t_0.
\]

By Lemma 4, \( \max_{\pi_{i,j}} V_{(s_0 + \mathbb{R})t_0} - \min_{\pi_{i,j}} U_{(s_0 + \mathbb{R})t_0} \leq (4B + \epsilon)t_0 \). Therefore,

\[
\max_{\pi_{i,j}} V_t - \min_{\pi_{i,j}} U_t \leq (4B + \epsilon)t_0 + \epsilon'(Q - s_0)t_0 \leq (4B + \epsilon)t_0 + \epsilon't.
\]

**Case 2:** There is no integer \( s_0 \) satisfying the condition in Case 1. Then, repeatedly apply (5) to intervals \([\mathbb{R}t_0, (1 + \mathbb{R})t_0], [(1 + \mathbb{R})t_0, (2 + \mathbb{R})t_0], \ldots, [(Q - 1 + \mathbb{R})t_0, (Q + \mathbb{R})t_0] = [t - t_0, t] \), we derive that

\[
\max_{\pi_{i,j}} V_t - \min_{\pi_{i,j}} U_t \leq \max_{\pi_{i,j}} V_{\mathbb{R}t_0} - \min_{\pi_{i,j}} U_{\mathbb{R}t_0} + \epsilon'\Omega t_0.
\]

By the bound on the risk, \( \max_{\pi_{i,j}} V_{\mathbb{R}t_0} \leq B\mathbb{R}t_0 \) and \( \min_{\pi_{i,j}} U_{\mathbb{R}t_0} \geq -B\mathbb{R}t_0 \). Hence,

\[
\max_{\pi_{i,j}} V_t - \min_{\pi_{i,j}} U_t \leq 2B\mathbb{R}t_0 + \epsilon'\Omega t_0 \leq (4B + \epsilon)t_0 + \epsilon't.
\]

Thus, in both cases, it holds that \( \max_{\pi_{i,j}} V_t - \min_{\pi_{i,j}} U_t \leq (4B + \epsilon)t_0 + \epsilon't \) for \( t > t_0 \). Let \( C > 0 \) be any constant (which may depend on \( \epsilon \), the approximation error of the covers, that is, the bound on \( \text{MV}_t/t \)). The following holds for any sufficiently large \( t \),

\[
\max_{\pi_{i,j}} V_t - \min_{\pi_{i,j}} U_t \leq (4B + \epsilon)t_0 + \epsilon't \leq (1 + C)\epsilon't.
\]

In other words, we show that after increasing the size of either index set by 1, for all sufficiently large \( t \), we obtain a bound on \( \max_{\pi_{i,j}} V_t - \min_{\pi_{i,j}} U_t \) that grows by a multiplicative factor of \( (1 + C) \) relative to the original bound.

It takes finitely many, say \( N \), steps to induct from the initial case where the sizes of both index sets are one to the case of interest with index sets \( \{1, \ldots, I\} \) and \( \{1, \ldots, J\} \). (Note that \( N \) may also depend on \( \epsilon \) through its dependence on \( I \) and \( J \).) Take \( C = 1/N \) in (6) and we derive that, for all sufficiently large \( t \),

\[
\max_{\pi_{i,j}} V_t - \min_{\pi_{i,j}} U_t = \max_{\pi_{(1, \ldots, j)}} V_t - \min_{\pi_{(1, \ldots, i)}} U_t \leq (1 + 1/N)^N \cdot 2\epsilon t \leq 2\epsilon t
\]

where \( e \) is the base of natural logarithm. Since \( \epsilon \) is arbitrary, we show that \( \limsup_{t \to \infty} (\max_{\pi_{i,j}} V_t - \min_{\pi_{i,j}} U_t) \leq 2\epsilon t \) for all sufficiently large \( t \).
\[ \min U_i / t \leq 0. \]

A.5 Derivation of \( \Gamma \)-minimax estimator of the mean in Section 5.1

In this section, we show that, for the problem of estimating the mean in Section 5.1, one \( \Gamma \)-minimax estimator lies in \( D_{\text{linear}} \). This is formally presented below.

**Proposition 1.** Let \( \mathcal{M} \) consist of all probability distributions defined on the Borel \( \sigma \)-algebra on \([0,1]\). Let \( X_1, \ldots, X_n \overset{iid}{\sim} P_0 \in \mathcal{M} \) and \( X = (X_1, X_2, \ldots, X_n) \) be the observed data. Let \( \Psi : P \mapsto \int xP(dx) \) denote the mean parameter and \( \Gamma = \{ \pi \in \Pi : \int \Psi(P)\pi(dP) = \mu \} \) be the set of priors that represent prior information. Let \( \mathcal{D} \) denote the space of estimators that are square-integrable with respect to all \( P \in \mathcal{M} \). Consider the risk in Example 2: \( R : (d,P) \mapsto \mathbb{E}_P[(d(X) - \Psi(P))^2] \). Define \( \bar{X} = \sum_{i=1}^n X_i / n \) and \( d_0 : X \mapsto (\mu + \sqrt{n}\bar{X})/(1 + \sqrt{n}) \). Then \( d_0 \in D_{\text{linear}} \) is \( \Gamma \)-minimax over \( \mathcal{D} \).

We first present a theorem on a criterion of \( \Gamma \)-minimality.

**Theorem 7.** Suppose that \( d_0 \in \mathcal{D} \) is a Bayes estimator for \( \pi_0 \in \Gamma \) and \( r(d_0, \pi_0) = r_{\sup}(d_0, \Gamma) \). Then \( d_0 \) is a \( \Gamma \)-minimax estimator in \( \mathcal{D} \).

**Proof of Theorem 7.** Clearly \( r_{\sup}(d_0, \Gamma) \geq \inf_{d' \in \mathcal{D}} r_{\sup}(d', \Gamma) \). Fix \( d' \in \mathcal{D} \). Then, \( r_{\sup}(d', \Gamma) \geq r(d', \pi_0) \geq r(d_0, \pi_0) = r_{\sup}(d_0, \Gamma) \). Since \( d' \) is arbitrary, this shows that \( \inf_{d' \in \mathcal{D}} r_{\sup}(d', \Gamma) \geq r_{\sup}(d_0, \Gamma) \). Thus, \( r_{\sup}(d_0, \Gamma) = \inf_{d' \in \mathcal{D}} r_{\sup}(d', \Gamma) \) and \( d_0 \) is \( \Gamma \)-minimax.

We now present a lemma that is used to prove Proposition 1.

**Lemma 5.** Let \( a < b \) and suppose that \( \mathcal{M} \) denotes the model space that consists of all probability distributions defined on the Borel \( \sigma \)-algebra on \([a,b] \subseteq \mathbb{R} \) with mean \( \mu \in [a,b] \). Let \( X \) denote a generic random variable generated from some \( P \in \mathcal{M} \). Then \( \max_{P \in \mathcal{M}} \text{Var}_P(X) = \text{Var}_{P^*}(X) = (b - \mu)(\mu - a) \), where \( P^* \) is defined by \( P^*(X = a) = (b - \mu)/(b - a) \) and \( P^*(X = b) = (\mu - a)/(b - a) \).

**Proof of Lemma 5.** Without loss of generality, we may assume that \( a = -1 \) and \( b = 1 \). Note that for any \( P \in \mathcal{M} \), it holds that \( \text{Var}_P(X) = \mathbb{E}_P[X^2] - \mathbb{E}_P[X]^2 = \mathbb{E}_P[X^2] - \mu^2 \leq 1 - \mu^2 \), where the equality is attained if \( P(X \in \{-1,1\}) = 1 \). Therefore, the maximum variance is achieved at the distribution with the specified mean \( \mu \) and support being \( \{a,b\} \), that is, at the distribution \( P^* \) defined in the lemma statement. Straightforward calculations show that \( \text{Var}_{P^*}(X) = (b - \mu)(\mu - a) \).
Proof of Proposition 1. Let $\mathcal{M}' := \{\text{Bernoulli}(\theta) : \theta \in (0,1)\} \subseteq \mathcal{M}$ and let $\pi_0$ be a prior distribution over $\mathcal{M}'$ such that the prior distribution on the success probability $\theta$ is Beta$(\mu\sqrt{n}, (1 - \mu)\sqrt{n})$. By Theorem 1.1 in Chapter 4 of Lehmann and Casella (1998), a Bayes estimator for $\pi_0$ minimizes the risk under the posterior distribution, whose minimizer over $\mathcal{D}$ is the posterior mean $d_0$ for our choice of risk. That is, $d_0$ is a Bayes estimator in $\mathcal{D}$ for $\pi_0$.

We next show that $r(d_0, \pi_0) = \sup_{\pi \in \Gamma} r(d_0, \pi)$. Let $\pi \in \Gamma$ be arbitrary. Since $\mathbb{E}_P[\bar{X}] = \Psi(P)$ and $\text{Var}_P(\bar{X}) = \text{Var}_P(X_1)/n$, we can derive that

$$r(d_0, \pi) = \int \mathbb{E}_P \left[ \{\mu + \sqrt{n}\bar{X} - \Psi(P)\}^2 \right] \pi(dP)$$

$$= \int \mathbb{E}_P \left[ \{\frac{\sqrt{n}}{1 + \sqrt{n}} (\bar{X} - \Psi(P)) + \frac{\mu - \Psi(P)}{1 + \sqrt{n}}\}^2 \right] \pi(dP)$$

$$= \int \left\{ \frac{1}{(1 + \sqrt{n})^2} \text{Var}_P(X_1) + \frac{(\mu - \Psi(P))^2}{(1 + \sqrt{n})^2} \right\} \pi(dP)$$

Apply Lemma 5 to $\text{Var}_P(X_1)$ and the display continues as

$$\leq \int \left\{ \frac{1}{(1 + \sqrt{n})^2} \Psi(P)(1 - \Psi(P)) + \frac{(\mu - \Psi(P))^2}{(1 + \sqrt{n})^2} \right\} \pi(dP)$$

$$= \int \frac{1}{(1 + \sqrt{n})^2} \left\{ \mu^2 + (1 - 2\mu)\Psi(P) \right\} \pi(dP) = \frac{\mu(1 - \mu)}{(1 + \sqrt{n})^2}.$$  

This upper bound can be attained by any $\pi$ with support contained in $\mathcal{M}'$, for example, $\pi_0$. Therefore, $r_{\text{sup}}(d_0, \Gamma) = r(d_0, \pi_0)$. By Theorem 7, $d_0$ is $\Gamma$-minimax over $\mathcal{D}$. 

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