Low-momentum dynamic structure factor of a strongly interacting Fermi gas at finite temperature: The Goldstone phonon and its Landau damping

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We develop a microscopic theory of dynamic structure factor to describe the Bogoliubov-Anderson-Goldstone phonon mode and its damping rate in a strongly interacting Fermi gas at finite temperature. It is based on a density functional approach - the so-called superfluid local density approximation. The accuracy of the theory is quantitatively examined by comparing the theoretical predictions with the recent experimental measurements for the local dynamic structure factor of a nearly homogeneous unitary Fermi gas at low transferred momentum [S. Hoinka et al., Nat. Phys. 13, 943 (2017)], without any free parameters. We calculate the dynamic structure factor as functions of temperature and transferred momentum, and determine the temperature evolution of the phonon damping rate, by considering the dominant decay process of the phonon mode via scatterings off fermionic quasiparticles. These predictions can be confronted with future Bragg scattering experiments on a unitary Fermi gas near the superfluid transition.

**Introduction.** — The understanding of the density fluctuation spectrum of superfluid $^4$He plays a central role in the early development of quantum many-body physics [1,2]. The modern concept of quasiparticles began with Landau’s original theory of superfluid $^4$He [3]. The extensive studies of either Brillouin and Raman light scattering [4] or inelastic neutron scattering [2] in such systems lead to the discovery of phonons and rotons. In particular, the measurements of the sound attenuation reveal the underlying decay mechanism of quasiparticles. At low temperature (i.e., $T \lesssim 0.6\text{ K} \sim 0.37\text{He}$) in the collisionless regime, the decay rate of phonons $\Gamma$ is due to the three-phonon Landau-Beliaev process and exhibits a characteristic $\omega T^4$ dependence on the phonon frequency $\omega$ [1]. At higher temperature ($T \gtrsim 1.0\text{ K} \sim 0.57\text{He}$), superfluid $^4$He crosses over to the hydrodynamic regime and the damping rate of phonons instead shows a quadratic $\omega^2$ dependence [3].

After nearly fifty years, the community of quantum physics welcomes the arrival of another strongly interacting many-body system [5], a unitary Fermi gas at the cusp of the crossover from Bose-Einstein condensates (BEC) to Bardeen-Cooper-Schrieffer (BCS) superfluids [6]. This novel fermionic superfluid is unique, owing to the unprecedented accuracy in tuning almost all the controlling parameters of the system [7]. To date, there are already a number of milestone observations of a unitary Fermi gas, confirming its high-temperature superfluidity [8], measuring the zero-temperature equation of state [8], revealing the universal thermodynamics [10–12] and probing the second sound [14]. The density fluctuation spectrum is also measured, however, restricted to collective oscillations with discrete frequencies [15–18], due to the very existence of a harmonic trapping potential that is necessary to hold atoms from escaping. Only most recently, the density excitation spectrum of a nearly homogeneous unitary Fermi gas has been obtained at Swinburne University of Technology, by applying the low-momentum two-photon Bragg spectroscopy to determine the local dynamic structure factor near the trap center [19]. The purpose of this work is to present a microscopic theory that quantitatively explains the observed Bogoliubov-Anderson-Goldstone phonon mode and to provide reliable theoretical predictions on the phonon damping for future experimental confirmation.

The development of a quantitative description of the density response of a strongly interacting Fermi superfluid is by no means an easy task [20–29]. There is no small parameter to control the precision of the theory due to the divergent scattering length $a_s$ in the unitary limit [6]. For the experimental work at Swinburne, the data have been qualitatively understood using a standard random-phase-approximation (RPA) theory, with a modified chemical potential as a fitting parameter [19]. The calculated spectrum overestimates the phonon peak (i.e., more than twice in height) and accounts for only two-thirds the measured width. This good but somewhat unsatisfactory agreement is partly because of the violation in the $f$-sum rule, as a result of the large modification to the mean-field chemical potential [19]. The quantitative agreement between our microscopic theory and experiment without any adjustable parameters, as found in this work, is therefore highly nontrivial.

The establishment of an accurate density response theory also allows us to identify the main decay mechanism for Goldstone phonons. In contrast to the three-phonon Landau-Beliaev processes as previously suggested [30], we clarify that the phonon damping is dominated by the inelastic process of absorption or emission by fermionic quasiparticles [31,32]. We determine the inverse quality factor $\Gamma/\omega$ of a unitary Fermi gas as functions of temperature and transferred momentum. These predictions could be readily examined in state-of-art experiments in cold-atom laboratories.
Density response theory within SLDA. — We start by briefly reviewing the superfluid local density approximation (SLDA) of a unitary Fermi gas \[ \frac{34}{34}, \frac{35}{35} \] and the resulting improved SLDA-RPA theory for density response functions \[ \frac{28}{28} \]. As the s-wave scattering length \( a_s \) diverges in the unitary limit, the low-energy physics of the system can be well-governed by a regularized energy density functional \( \mathcal{E}[\tau_c(r,t), n(r,t), \nu_c(r,t)] \),

\[
\mathcal{E}[\tau_c, n, \nu_c] = \frac{\tau_c}{2m} + \frac{3}{10m} \int \frac{d^3 p}{(2\pi)^3} \frac{\beta}{n^{5/3}} + g_{\text{eff}} |\nu_c|^2, \tag{1}
\]

where \( \tau_c = 2 \sum_{|k|<\Lambda} \gamma_{nk}^2 \) is the kinetic density, \( n = \sum_{|k|<\Lambda} \gamma_{nk}^2 \) the number density, \( \nu_c = \sum_{|k|<\Lambda} u_k v_k^* \) the anomalous Cooper-pair density, \( u_k(r,t) \) and \( v_k(r,t) \) are the Bogoliubov quasiparticle wavefunctions, to be determined by solving a generalized Bogoliubov-de Gennes equation for momentum \( k \) below the cut-off momentum \( \Lambda \), and \( g_{\text{eff}}^{-1} = m n^{5/3}/\gamma - \sum_{|k|<\Lambda} m/k^2 \) is the inverse effective coupling constant. The form of the above energy density functional is motivated by the scale invariance that is satisfied at unitarity and the two parameters \( \beta \) and \( \gamma \) can be uniquely fixed by requiring that the resulting chemical potential \( \mu \) and pairing gap \( \Delta \) agree with those calculated by microscopic theories or measured experimentally \[ \frac{37}{37} \].

As the pairing gap is related to the anomalous density \( \nu_c \) by \( \Delta(r,t) = -g_{\text{eff}} \nu_c(r,t) \), we may rewrite the interaction part of the density functional as,

\[
\mathcal{E}_{\text{int}} = \beta \frac{3}{10m} \left[ n(r,t) \right]^{5/3} + \frac{|\Delta(r,t)|^2}{g_{\text{eff}}}. \tag{2}
\]

Let us consider the fluctuations in the number densities \( n_\uparrow(r,t), n_\downarrow(r,t) \), and Cooper-pair density \( \nu_c(r,t) \) and its complex conjugate \( \nu_c^*(r,t) \), to be collectively denoted as \( \delta n_i \) or \( \delta n_{ij} \) \( (i,j = \uparrow, \downarrow, c, c^*) \). Here, we have split the total density \( n(r,t) \) into the spin-up and spin-down components to allow the calculation of spin-density dynamic structure factor. These local fluctuations induce a self-generated mean-field potential \( \sum_i E_{ij} \delta n_j \), where \( E_{ij} = (\delta^2 \mathcal{E}_{\text{int}}/\delta n_i \delta n_j) \). As a result, the dynamic response function takes the standard RPA form,

\[
\chi = \frac{\chi^{(0)}(\mu, \nu_c)}{1 - \chi^{(0)}E^{\dagger}} \chi^{(0)} = \chi^{(0)}(1 - \chi^{(0)}E^{\dagger})^{-1},
\]

where \( \chi^{(0)} \) is the bare response function without the inclusion of the induced potential \[ \frac{20}{20} \]. The density response function is a summation of \( \chi_{ij} \) in the density channel, i.e., \( \chi^{(0)}(\mu, \nu_c + i0^+) = \chi_{11} + \chi_{12} + \chi_{21} + \chi_{22} = 2(\chi_{11} + \chi_{12}) \).

In our previous work \[ \frac{28}{28} \], we have derived the expression for the matrix \( E^{\dagger} \) and calculated the dynamic and static structure factor at zero temperature. We have found that the SLDA-RPA dynamic structure factor satisfies the important f-sum rule and compressibility sum rule and the static structure factor \( S(k) \) agrees very well with the latest quantum Monte Carlo result for \( k < k_F \).

\[
\text{Im} \chi^{(0)}_{nn}(k,\omega) = \int \frac{d\omega'}{\pi \sigma_B} \text{Im} \chi^{(0)}_{nn}(k,\omega') \sin^2 \left( \frac{\omega - \omega'}{\sigma_B} \right), \tag{3}
\]

where \( k_F \) is the Fermi wavevector. The excellent agreement strongly indicates that our SLDA-RPA theory could be quantitatively reliable near the unitary limit at low temperature. Here, we confirm this anticipation by the more stringent comparison with the recent experimental measurements at finite but low temperature \[ \frac{19}{19} \], without any free parameters.

Quantitative comparison. — To foster the comparison, it should be noted that the experimentally measured density fluctuation spectrum includes instrumental broadening due to the finite duration of the Bragg pulse. It can be viewed as the intrinsic response function convoluted with a sinc line shape \[ \frac{39}{39}, \frac{40}{40} \]:

\[
\text{Im} \chi^{(0)}_{nn}(k,\omega) = \int \frac{d\omega'}{\pi \sigma_B} \text{Im} \chi^{(0)}_{nn}(k,\omega') \sin^2 \left( \frac{\omega - \omega'}{\sigma_B} \right),
\]

FIG. 1. (color online). The comparison between the SLDA-RPA theoretical predictions and the experimental data for \( -\omega \text{Im} \chi^{(0)}_{nn}(k,\omega)/\pi \equiv \omega(1 - e^{-\omega/k_BT})^{38} \) at three sets of experimental conditions \[ \frac{19}{19} \]: (a) the unitary limit with \( 1/(k_F a_s) = 0, T = 0.097T_F \) and \( k_F = 0.55k_F \), (b) \( 1/(k_F a_s) = -0.11, T = 0.082T_F \) and \( k_F = 0.60k_F \), and (c) \( 1/(k_F a_s) = -0.21, T = 0.078T_F \) and \( k_F = 0.59k_F \).
with a Lorentzian line shape, induced broadening by further convoluting the spectrum the Fermi energy mogeneity near the trap center ([19]). In addition, the experimental spectrum also includes broadening arising from the slight density inhomogeneity ([19]).

\[ \delta n/n \sim 0.08 \]  

As the Fermi energy \( \varepsilon_F \propto n^{2/3} \) at unitarity, we estimate the variation \( \delta \varepsilon_F \approx 0.06 \varepsilon_F \). We account for this trap-induced broadening by further convoluting the spectrum with a Lorentzian line shape,

\[ \text{Im} \chi^{(\text{exp})}_{nn}(k, \omega) = \int_{-\infty}^{\infty} d\omega' \frac{\sigma_T}{\pi} \frac{\text{Im} \chi^{(\text{BR})}_{nn}(k, \omega')}{(\omega - \omega')^2 + \sigma_T^2} \]  

where \( 2\sigma_T = \delta \varepsilon_F \) is the full width at half maximum.

Fig. 1 presents the comparison between the experimental spectra (symbols) and theoretical predictions obtained after performing the two convolutions (solid lines) ([41]), for three sets of parameters. We have used the zero-temperature equations of state given by a Gaussian pair fluctuation theory ([12]) as the inputs to determine the parameters \( \beta \) and \( \gamma \) at different interaction strengths ([28]). The results for density response function before the convolutions are also shown by dashed lines with shadow. There is an excellent agreement for the entire spectrum, both at the unitary limit (a) and near unitarity (b, c).

Our theory greatly improves the previous RPA explanation ([19]), in the sense that (i) it removes the necessity of introducing a fitting parameter, to fit the measured peak position \( \omega_0(k) \) of the Goldstone phonon mode; (ii) it fully accounts for the observed width and height of the peak; and (iii) it does not need to scale, in order to match the amplitude of the measured pair-breaking excitations at \( \omega \sim 2\Delta \). The last point is particularly clear in Fig. 1 as the broad single-particle excitations are essentially unaffected by the instrumental broadening and trap inhomogeneity. The agreement is therefore highly non-trivial and actually it emphasizes the importance of the significant renormalization of single-particle behavior, due to the strong pairing effect, which is indeed taken into account in our theory via a density functional approach.

\[ T \text{-dependence of the Goldstone mode.} \]  

By establishing the reliability of our SLDA-RPA theory, we turn to consider the temperature evolution of the dynamic structure factor in the unitary limit, as shown in Fig. 2(a). The experimentally measurable density response (after convolutions) is reported in Fig. 2(b). Here and in the following, we have taken the experimentally determined chemical potential \( \mu = 0.376 \varepsilon_F \) ([13]) and pairing gap \( \Delta = 0.47 \varepsilon_F \) ([19]) to fix the parameters \( \beta \) and \( \gamma \). As temperature increases, it is apparent that the phonon peak in the spectrum becomes wider and lower, suggesting that the intrinsic width of the phonon mode becomes significant. Moreover, the single-particle excitations at around \( \omega = 2\Delta \) are enhanced (see the inset in Fig. 2(b)).

Focusing on the Goldstone mode, we present the temperature dependence of its width and peak height in Fig. 3. It is interesting that, by subtracting a background width \( \Gamma_0 \sim 0.1 \varepsilon_F \) due to the instrumental and inhomogeneity
broadenings, the width to be experimentally measured (symbols with dot-dashed line) is roughly equal to the intrinsic width of the phonon peak in the dynamic structure factor (solid line). This simply indicates that the intrinsic width of the Goldstone mode could be directly read from the measured width, with reasonable accuracy.

**Landau damping.** — We now turn to discuss the intrinsic width or damping rate of the Goldstone phonon mode in greater detail. What is the main mechanism responsible for damping? Physically, there are three possible sources that we may consider: (1) the three-phonon Landau-Beliaev process \( \phi \leftrightarrow \phi \phi \), where \( \phi \) is the annihilation field operator of phonons; (2) the four-phonon Landau-Khalatnikov process \( \phi \phi \leftrightarrow \phi \phi \); and (3) the inelastic process of absorption or emission by the single-particle excitations. All these processes are responsible in the case of superfluid \(^4\)He. For example, at low temperature the three-phonon process is kinematically allowed by the anomalous dispersion of the phonon mode (i.e., \( \omega_0(k) = c_\phi k (1 + \zeta k^2) \) with a positive \( \zeta > 0 \)) at \( k < k_c \sim 0.55 \text{A}^{-1} \). The much weaker four-phonon process is possible at \( k > k_c \). For temperatures above 1 K, the last inelastic process becomes favorable by scattering from thermally excited rotons.

For a unitary Fermi gas, Kurkjian and co-workers suggested that the three-phonon process is the dominant decay mechanism, since the standard RPA theory predicts a positive \( \zeta \) [43], and derived an elegant expression for the inverse quality factor at low temperature [30]:

\[
\frac{\Gamma}{\omega_0} \approx \frac{16 \pi^5 \sqrt{3}}{405} \xi^{3/2} \left( \frac{k_B T}{mc^2} \right)^4 \approx 1.2 \times 10^4 \left( \frac{T}{T_F} \right)^4,
\]

where we have used the Bertsch parameter \( \xi = 0.376 \) [13]. This observation, however, is not conclusive, since our more accurate SLDA-RPA theory gives a negative \( \zeta \) at both zero and finite temperatures, as shown in Fig. 4. Another qualitative \( \epsilon \)-expansion theory provides a similar negative \( \zeta \) at unitarity [44]. On the other hand, it is known that the three-phonon process in \(^4\)He is only relevant at low temperature (i.e., \( T < 1.0 \text{ K} \sim 0.5T_{\text{He}} \)) [24]. It is thus unlikely to be the main damping source in a unitary Fermi gas for \( T > 0.5T_c \sim 0.09T_F \). From the above considerations, we would like to argue that the inelastic scatterings of phonons from fermionic quasiparticles causes their damping at the temperature region \( T \gtrsim 0.1T_F \), which is of great experimental interest.

If this is true, we expect that the damping rate \( \Gamma \) will be approximately proportional to the number of fermionic quasiparticles present, i.e., \( \Gamma \propto e^{-E_{\text{max}}/k_B T} \),
where $E_{\text{min}} \equiv \min\{E(k)\}$ is the minimum energy of single particles that satisfies some momentum and energy conservation requirements to allow inelastic scatterings. More quantitatively, an analytic expression for $\Gamma$ was first derived by Zhang and Liu by considering the phase fluctuations at small momentum $k \to 0$ within RPA [31]. Most recently, it was improved by Kurkjian et al. by taking into account the modified single-particle dispersion [32] and/or the amplitude fluctuations [33]. Our SDLA-RPA theory may give a better prediction, starting from a more fundamental microscopic approach, although we cannot obtain an analytic expression for $\Gamma/\omega_0$. Our results at $k \to 0$ and $k = 0.5k_F$ are reported in Fig. 5(a) using solid and dashed lines, respectively. The result at small momentum qualitatively agrees with the earlier prediction by Zhang and Liu (dot-dashed line), as one may anticipate. The damping rate decreases steadily with increasing transferred momentum $k$ as predicted earlier [33], as can be seen in Fig. 5(b).

It is interesting to note that the damping rate of phonons in superfluid $^4$He at $k = 0.4 \, \text{Å}^{-1}$ (circles in Fig. 5(a) [46, 47]) closely follows our prediction at small momentum. This similarity between superfluid $^4$He and a unitary Fermi gas suggests that any strongly interacting quantum fluids may share a universal damping rate for phonons, independent of their entirely different internal structures and quantum statistics.

Conclusions. — In summary, we have developed a finite-temperature microscopic theory of the density response of a unitary Fermi gas at small transferred momentum, and have quantitatively examined its reliability by comparing our theoretical results with the latest Bragg scattering measurements [19]. We have clarified that the damping rate of the Goldstone phonon mode is largely due to the inelastic scatterings from fermionic quasiparticles and have predicted its universal temperature dependence, which is to be confronted with future experimental confirmation.

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so we set \( m^* = m \) for simplicity. This simple choice also ensures that the \( f \)-sum rule of the dynamic structure factor is strictly satisfied.

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   where the wavevector \( p \) satisfies \( \hbar p \xi_p = m c_s E_p \) with \( \xi_p \equiv \hbar^2 p^2/(2m) - \mu \) and \( E_p = \sqrt{\xi_p^2 + \Delta^2} \). By setting \( \mu = 0.376\varepsilon_F \) and \( \Delta = 0.47\varepsilon_F \), we find that \( p \approx 0.784k_F, E(p) \approx 0.527\varepsilon_F, \) and \( \Gamma/\omega_0 \approx 5.03e^{-0.527T_F/T} \). The last expression gives the blue dotted-dashed line in Fig. 5(a).

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