Covering random graphs by monochromatic trees and Helly-type results in hypergraphs

Matija Bucić
joint work with Daniel Korándi and Benny Sudakov

ETH Zürich
Definition

$\text{tc}_r(G)$ is the smallest $m$ s.t. in any $r$-edge colouring of a graph $G$ we can find $m$ monochromatic trees which cover all vertices of $G$.  

Conjecture (Lovasz '75, Ryser '70)

$\text{tc}_r(K_n) \leq r - 1$

For specific trees: Gyárfás; Gerencsér and Gyárfás; Pokrovskiy; Gyárfás, Ruszinkó, Sárközy and Szemerédi; Erdős, Gyárfás and Pyber.
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Covering by trees

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Claim

There are graphs missing only a few edges with arbitrarily large $t_{c_r}$.
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There are graphs missing only a few edges with arbitrarily large $tc_r$.

An independent set of size $t$\hspace{1cm} $K_{n-t}$

$\forall v \in B$ sends exactly $r-1$ edges to $A$.

Colour all edges in $B$ in colour $r$.

$\forall v \in B$ colour its $r-1$ edges to $A$ rainbowly using first $r-1$ colours.

Any monochromatic tree in first $r-1$ colours is a star with a centre in $A$. 

Proposition

If $G$ contains an independent set of size $t$ with no $r$ vertices having a common neighbour then $tc_r(G) \geq t$. 

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An independent set of size $t$

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**Theorem (Bal and DeBiasio ’15)**

a) If $p \ll \left(\frac{\log n}{n}\right)^{1/r}$ then w.h.p. $t_{c_r}(G(n,p)) \to \infty$

$b)$ If $p \gg \left(\frac{\log n}{n}\right)^{1/(r+1)}$ then w.h.p. $t_{c_r}(G(n,p)) \leq r^2$. 

If $\alpha(G) \geq r$ then $t_{c_r}(G) \geq r$. 

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For \( r \geq 3 \) if \( p \ll \left( \frac{\log n}{n} \right)^{1/(r+1)} \) then w.h.p. \( \text{tc}_r(G(n, p)) \geq r + 1 \).
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Theorem (B., Korándi, Sudakov)

a) If \( p \ll \left( \frac{\log n}{n} \right)^{\sqrt{r}/2^{r-2}} \) then w.h.p. \( tc_r(G(n, p)) > r \).
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Theorem (B., Korándi, Sudakov)

Let $d > 1$, $\left(\frac{\log n}{n}\right)^{\frac{1}{r}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{d(r+1)}}$ then w.h.p. $tc_r(G(n, p)) = \Theta(r^2)$. 

Established in weaker form by Korándi, Mousset, Nenadov, Škorić and Sudakov. 

Answers a question of Lang and Lo.
Theorem (Bal and DeBiasio ’15)

a) If \( p \ll \left( \frac{\log n}{n} \right)^{1/r} \) then w.h.p. \( \text{tc}_r(G(n, p)) \rightarrow \infty \) and

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Theorem (B., Korándi, Sudakov)

If \( \left( \frac{\log n}{n} \right)^{1/k} \ll p \ll \left( \frac{\log n}{n} \right)^{1/(k+1)} \) then w.h.p. \( \frac{r^2}{20 \log k} \leq \text{tc}_r(G(n, p)) \leq \frac{16r^2 \log r}{\log k} \).
Question (Erdős, Hajnal and Tuza ’90)

Given an $r$-uniform hypergraph $H$ in which any $k$ edges have a cover of size at most $\ell$, how big can a cover of $H$ be?
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- A transversal cover in an \( r \)-partite \( r \)-uniform hypergraph \( H \) is a cover of \( H \) which has at most one vertex in each part of the \( r \)-partition.
Covering in hypergraphs

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- A *transversal cover* in an $r$-partite $r$-uniform hypergraph $H$ is a cover of $H$ which has at most one vertex in each part of the $r$-partition.

**Definition**

Let $h_{pr}(k)$ be the maximum possible size of a cover of an $r$-partite, $r$-uniform $H$ in which any $k$ edges have a transversal cover.
The connection

Theorem (B., Korándi, Sudakov)

a) Let $k > r \geq 2$, $np^k \gg \log n$ then w.h.p. $\text{tc}_r(G(n, p)) \leq hp_r(k)$.

b) Let $k > r \geq 2$, $np^{k+1} \ll \log n$ then w.h.p. $\text{tc}_{r+1}(G(n, p)) > hp_r(k)$. 
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- If $\left(\frac{\log n}{n}\right)^{\frac{1}{k}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{k+1}}$ then w.h.p. $\text{tc}_r(G(n, p)) \approx hp_r(k)$. 

Best possible in terms of $\delta(G)$. Proved for $r \leq 3$ by Girão, Letzter and Sahasrabudhe.
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Conjecture (Bal and DeBiasio)

Let $G$ be an $r$-coloured graph on $n$ vertices with $\delta(G) \geq (1 - \frac{1}{2r})n$. Then vertices of $G$ can be covered by monochromatic trees of distinct colours.
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If any $k$ vertices in $G$ have a common neighbour then $tc_r(G) \leq hp_r(k)$. 
Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $\text{tc}_r(G) \leq h_p r(k)$.

Proof.

Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:

Vertices of $H$ are monochromatic components of $G$.

Let $e_v$ be the set of monochromatic components of $G$ containing $v$.

$E(H) := \{e_v \mid v \in V(G)\}$.

Parts correspond to monochromatic components of the same colour.

Given $k$ edges $e_{v_1}, \ldots, e_{v_k}$, let $w$ be a common neighbour of $v_1, \ldots, v_k$.

$e_w \cap e_{v_i} \neq \emptyset$ so $e_w$ is a transversal cover of $e_{v_1}, \ldots, e_{v_k}$ so $\tau(H) \leq h_p r(k)$.

If monochromatic components $C_1, \ldots, C_t$ cover $H$ then $C_1 \cup \cdots \cup C_t = V(G)$.

For any $v \in V(G)$ there exists $i$ such that $C_i \in e_v = \Rightarrow v \in C_i$. 
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Proof.

Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
Theorem (B., Korándi, Sudakov)

If any \( k \) vertices in \( G \) have a common neighbour then \( tc_r(G) \leq hp_r(k) \).

Proof.

- Given \( r \)-colouring of \( G \) we build \( r \)-partite \( r \)-uniform hypergraph \( H \):
  - Vertices of \( H \) are monochromatic components of \( G \).
Theorem (B., Korándi, Sudakov)

If any \( k \) vertices in \( G \) have a common neighbour then \( t_{cr}(G) \leq h_{pr}(k) \).

Proof.

- Given \( r \)-colouring of \( G \) we build \( r \)-partite \( r \)-uniform hypergraph \( H \):
  - Vertices of \( H \) are monochromatic components of \( G \).
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Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $\text{tc}_r(G) \leq \text{hp}_r(k)$.

Proof.

- Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
  - Vertices of $H$ are monochromatic components of $G$.
  - Let $e_v$ be the set of monochromatic components of $G$ containing $v$.
  - $E(H) := \{e_v \mid v \in V(G)\}$.

$$G$$

![Graph Diagram]

For any $v \in V(G)$, $\exists i : C_i \in e_v \Rightarrow v \in C_i$. 
Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $tc_r(G) \leq hp_r(k)$.

Proof.

Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:

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Given $k$ edges $e_{v_1}, \ldots, e_{v_k}$, let $w$ be a common neighbour of $v_1, \ldots, v_k$. $e_w \cap e_{v_i} \neq \emptyset$ so $e_w$ is a transversal cover of $e_{v_1}, \ldots, e_{v_k}$ so $\tau(H) \leq hp_r(k)$.

If monochromatic components $C_1, \ldots, C_t$ cover $H$ then $C_1 \cup \cdots \cup C_t = V(G)$.

For any $v \in V(G)$ there exists $i$ such that $C_i \in e_v$ so $v \in C_i$. 
Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $tcr(G) \leq hp_r(k)$.

Proof.

Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
- Vertices of $H$ are monochromatic components of $G$.
- Let $e_v$ be the set of monochromatic components of $G$ containing $v$.
- $E(H) := \{e_v \mid v \in V(G)\}$.
Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $\text{tc}_r(G) \leq h_{\text{p}_r}(k)$.

Proof.

- Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
  - Vertices of $H$ are monochromatic components of $G$.
  - Let $e_v$ be the set of monochromatic components of $G$ containing $v$.
  - $E(H) := \{e_v \mid v \in V(G)\}$.

![Diagram of hypergraph $H$ with $r$-partite structure and monochromatic components associated with each part.](image)
Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $\tau_{c_r}(G) \leq h_{p_r}(k)$.

Proof.

Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
- Vertices of $H$ are monochromatic components of $G$.
- Let $e_v$ be the set of monochromatic components of $G$ containing $v$.
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For any $v \in V(G)$ exists $i$ such that $C_i \in e_v \Rightarrow v \in C_i$. 

![Diagram](image.png)
Theorem (B., Korándi, Sudakov)

*If any* $k$ *vertices in* $G$ *have a common neighbour then* $\text{tc}_r(G) \leq h_p r(k)$.

**Proof.**

- Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
  - Vertices of $H$ are monochromatic components of $G$.
  - Let $e_v$ be the set of monochromatic components of $G$ containing $v$.
  - $E(H) := \{e_v \mid v \in V(G)\}$.

For any $v \in V(G)$ there exists $i$ such that $C_i \in e_v \Rightarrow v \in C_i$.
Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $t_{c_r}(G) \leq h_{p_r}(k)$. 

**Proof.**

- Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
  - Vertices of $H$ are monochromatic components of $G$.
  - Let $e_v$ be the set of monochromatic components of $G$ containing $v$.
  - $E(H) := \{e_v \mid v \in V(G)\}$.

![Diagram of the hypergraph $H$ with vertices and edges labeled appropriately.]
Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $t_{cr}(G) \leq h_{p_r}(k)$.

Proof.

Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
- Vertices of $H$ are monochromatic components of $G$.
- Let $e_v$ be the set of monochromatic components of $G$ containing $v$.
- $E(H) := \{e_v \mid v \in V(G)\}$.

$G$ and $H$ with monochromatic components and parts correspond to monochromatic components of the same colour.
Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $tc_r(G) \leq hp_r(k)$.

Proof.

Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:

- Vertices of $H$ are monochromatic components of $G$.
- Let $e_v$ be the set of monochromatic components of $G$ containing $v$.
- $E(H) := \{e_v \mid v \in V(G)\}$.

For any $v \in V(G)$ there exists $i$ such that $C_i \in e_v \Rightarrow v \in C_i$. 

```
G   R1,B3   R1,B1   R3,B1
    R1,B2   R1,B1   R2,B1
H   B1      B2      B3
    R1      R2      R3
```
Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $tc_r(G) \leq hp_r(k)$.

Proof.

- Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
  - Vertices of $H$ are monochromatic components of $G$.
  - Let $e_v$ be the set of monochromatic components of $G$ containing $v$.
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- Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
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Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $\text{tc}_r(G) \leq h p_r(k)$.

Proof.

- Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
  - Vertices of $H$ are monochromatic components of $G$.
  - Let $e_v$ be the set of monochromatic components of $G$ containing $v$.
  - $E(H) := \{ e_v \mid v \in V(G) \}$.

![Diagram of G and H](image)

For any $v \in V(G)$ there exists $i$ such that $C_i \in e_v$ implies $v \in C_i$. 

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Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $t_{cr}(G) \leq h_{pr}(k)$.

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Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
- Vertices of $H$ are monochromatic components of $G$.
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- $E(H) := \{e_v \mid v \in V(G)\}$.

For any $v \in V(G)$, $\exists i$: $C_i \in e_v \Rightarrow v \in C_i$. 

![Diagram](image-url)
Theorem (B., Korándi, Sudakov)

If any \( k \) vertices in \( G \) have a common neighbour then \( \text{tc}_r(G) \leq \text{hp}_r(k) \).

Proof.

- Given \( r \)-colouring of \( G \) we build \( r \)-partite \( r \)-uniform hypergraph \( H \):
  - Vertices of \( H \) are monochromatic components of \( G \).
  - Let \( e_v \) be the set of monochromatic components of \( G \) containing \( v \).
  - \( E(H) := \{ e_v \mid v \in V(G) \} \).

\[
\begin{align*}
\text{G} & \quad R_1, B_3 & R_1, B_1 & R_3, B_1 \\
& \quad R_1, B_2 & R_1, B_1 & R_2, B_1 \\
\text{H} & \quad B_1 & B_2 & B_3 \\
& \quad R_1 & R_2 & R_3
\end{align*}
\]
Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $\text{tc}_r(G) \leq h_p r(k)$.

Proof.

- Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
  - Vertices of $H$ are monochromatic components of $G$.
  - Let $e_v$ be the set of monochromatic components of $G$ containing $v$.
  - $E(H) := \{e_v \mid v \in V(G)\}$.

- Parts correspond to monochromatic components of the same colour.
Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $\text{tc}_r(G) \leq h_p r(k)$.

Proof.

- Given $r$-colouring of $G$ we build $r$-partite $r$-uniform hypergraph $H$:
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  - $E(H) := \{e_v \mid v \in V(G)\}$.

| $G$ | $R_1, B_3$ | $R_1, B_1$ | $R_3, B_1$ |
|-----|-------------|-------------|-------------|
| $R_1, B_2$ | $R_1, B_1$ | $R_2, B_1$ |

| $H$ |
|-----|
| $B_1$ |
| $B_2$ |
| $B_3$ |

- Parts correspond to monochromatic components of the same colour.
- Given $k$ edges $e_{v_1}, \ldots, e_{v_k}$, let $w$ be a common neighbour of $v_1, \ldots, v_k$. 
Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $tc_r(G) \leq hp_r(k)$.

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  - Vertices of $H$ are monochromatic components of $G$.
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  - $e_w \cap e_{v_i} \neq \emptyset$ so $e_w$ is a transversal cover of $e_{v_1}, \ldots, e_{v_k}$ so $\tau(H) \leq hp_r(k)$. 
Theorem (B., Korándi, Sudakov)

If any \( k \) vertices in \( G \) have a common neighbour then \( \text{tc}_r(G) \leq h\pi_r(k) \).

Proof.

- Given \( r \)-colouring of \( G \) we build \( r \)-partite \( r \)-uniform hypergraph \( H \):
  - Vertices of \( H \) are monochromatic components of \( G \).
  - Let \( e_v \) be the set of monochromatic components of \( G \) containing \( v \).
  - \( E(H) := \{ e_v \mid v \in V(G) \} \).

\[
\begin{align*}
G & \quad R_1, B_3 & \quad R_1, B_1 & \quad R_3, B_1 \\
R_1, B_2 & \quad R_1, B_1 & \quad R_2, B_1 \\
\end{align*}
\]

Parts correspond to monochromatic components of the same colour.

- Given \( k \) edges \( e_{v_1}, \ldots, e_{v_k} \), let \( w \) be a common neighbour of \( v_1, \ldots, v_k \).
- \( e_w \cap e_{v_i} \neq \emptyset \) so \( e_w \) is a transversal cover of \( e_{v_1}, \ldots, e_{v_k} \) so \( \tau(H) \leq h\pi_r(k) \).
- If monochromatic components \( C_1, \ldots, C_t \) cover \( H \) then \( C_1 \cup \cdots \cup C_t = V(G) \).
Theorem (B., Korándi, Sudakov)

If any \( k \) vertices in \( G \) have a common neighbour then \( \text{tc}_r(G) \leq \text{hp}_r(k) \).

Proof.

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- Parts correspond to monochromatic components of the same colour.

- Given \( k \) edges \( e_{v_1}, \ldots, e_{v_k} \), let \( w \) be a common neighbour of \( v_1, \ldots, v_k \).
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- If monochromatic components \( C_1, \ldots, C_t \) cover \( H \) then \( C_1 \cup \cdots \cup C_t = V(G) \)

- For any \( v \in V(G) \) \( \exists i : C_i \in e_v \)
Theorem (B., Korándi, Sudakov)

If any $k$ vertices in $G$ have a common neighbour then $\text{tc}_r(G) \leq \text{hp}_r(k)$.

Proof.

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  - Vertices of $H$ are monochromatic components of $G$.
  - Let $e_v$ be the set of monochromatic components of $G$ containing $v$.
  - $E(H) := \{e_v | v \in V(G)\}$.

- Parts correspond to monochromatic components of the same colour.
- Given $k$ edges $e_{v_1}, \ldots, e_{v_k}$, let $w$ be a common neighbour of $v_1, \ldots, v_k$.
  - $e_w \cap e_{v_i} \neq \emptyset$ so $e_w$ is a transversal cover of $e_{v_1}, \ldots, e_{v_k}$ so $\tau(H) \leq \text{hp}_r(k)$.
- If monochromatic components $C_1, \ldots, C_t$ cover $H$ then $C_1 \cup \cdots \cup C_t = V(G)$.
- For any $v \in V(G)$ $\exists i : C_i \in e_v \implies v \in C_i$. 

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Theorem (B., Korándi, Sudakov)

\[ \text{hp}_r(2^r) = r. \]
Theorem (B., Korándi, Sudakov)

\[ h_p(r^{2r}) = r. \]

Proof.

AFSOC there is an \( r \)-uniform, \( r \)-partite hypergraph without a transversal cover but with any \( 2r \) of its edges having a transversal cover. Let \( H \) be minimal satisfying above conditions. Let \( V_1, \ldots, V_r \) be parts and \( A_1, \ldots, A_k \) edges of \( H \). Let \( B_i \) be a transversal cover of \( H - A_i \). Then:

\[ |A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j. \]

\[ A_i \cap B_j \neq \emptyset, \forall i \neq j \text{ since } B_j \text{ is a cover of } H - A_j. \]

\[ A_i \cap B_i = \emptyset, \forall i \text{ since otherwise } B_i \text{ is a transversal cover of } H, \text{ a contradiction.} \]

This implies \( |E(H)| = k \leq \prod_{i=1}^{r} (1 + \frac{1}{1}) = 2r \), So \( H \) has a transversal cover, a contradiction.
Theorem (B., Korándi, Sudakov)

\[ \text{hp}_r(2^r) = r. \]

Proof.

- AFSOC there is an \( r \)-uniform, \( r \)-partite hypergraph without a transversal cover but with any \( 2^r \) of its edges having a transversal cover.
Theorem (B., Korándi, Sudakov)

\[ h^r_p(2^r) = r. \]

Proof.

- AFSOC there is an \( r \)-uniform, \( r \)-partite hypergraph without a transversal cover but with any \( 2^r \) of its edges having a transversal cover.
- Let \( H \) be minimal satisfying above conditions.
Theorem (B., Korándi, Sudakov)

\[ hp_r(2^r) = r. \]

Proof.

- AFSOC there is an \( r \)-uniform, \( r \)-partite hypergraph without a transversal cover but with any \( 2^r \) of its edges having a transversal cover.
- Let \( H \) be minimal satisfying above conditions.
- Let \( V_1, \ldots, V_r \) be parts and \( A_1, \ldots, A_k \) edges of \( H \).
Theorem (B., Korándi, Sudakov)

$$hp_r(2^r) = r.$$  

Proof.

- AFSOC there is an $$r$$-uniform, $$r$$-partite hypergraph without a transversal cover but with any $$2^r$$ of its edges having a transversal cover.
- Let $$H$$ be minimal satisfying above conditions.
- Let $$V_1, \ldots, V_r$$ be parts and $$A_1, \ldots, A_k$$ edges of $$H$$.
- Let $$B_i$$ be a transversal cover of $$H - A_i$$. Then:
Theorem (B., Korándi, Sudakov)
\[ h_p(2^r) = r. \]

Proof.
- AFSOC there is an \( r \)-uniform, \( r \)-partite hypergraph without a transversal cover but with any \( 2^r \) of its edges having a transversal cover.
- Let \( H \) be minimal satisfying above conditions.
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  - \( |A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i,j. \)
Theorem (B., Korándi, Sudakov)

\[ h_{r}(2^r) = r. \]

Proof.

- AFSOC there is an \( r \)-uniform, \( r \)-partite hypergraph without a transversal cover but with any \( 2^r \) of its edges having a transversal cover.

- Let \( H \) be minimal satisfying above conditions.

- Let \( V_1, \ldots, V_r \) be parts and \( A_1, \ldots, A_k \) edges of \( H \).

- Let \( B_i \) be a transversal cover of \( H - A_i \). Then:
  - \( |A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j. \)
  - \( A_i \cap B_j \neq \emptyset, \forall i \neq j \)

So \( H \) has a transversal cover, a contradiction.
Theorem (B., Korándi, Sudakov)

\[ h_{p_r}(2^r) = r. \]

Proof.
- AFSOC there is an \( r \)-uniform, \( r \)-partite hypergraph without a transversal cover but with any \( 2^r \) of its edges having a transversal cover.
- Let \( H \) be minimal satisfying above conditions.
- Let \( V_1, \ldots, V_r \) be parts and \( A_1, \ldots, A_k \) edges of \( H \).
- Let \( B_i \) be a transversal cover of \( H - A_i \). Then:
  - \( |A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j. \)
  - \( A_i \cap B_j \neq \emptyset, \forall i \neq j \) since \( B_j \) is a cover of \( H - A_j \).
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Proof.
- AFSOC there is an \( r \)-uniform, \( r \)-partite hypergraph without a transversal cover but with any \( 2^r \) of its edges having a transversal cover.
- Let \( H \) be minimal satisfying above conditions.
- Let \( V_1, \ldots, V_r \) be parts and \( A_1, \ldots, A_k \) edges of \( H \).
- Let \( B_i \) be a transversal cover of \( H - A_i \). Then:
  - \( |A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j. \)
  - \( A_i \cap B_j \neq \emptyset, \forall i \neq j \) since \( B_j \) is a cover of \( H - A_j \).
  - \( A_i \cap B_i = \emptyset, \forall i \)
Theorem (B., Korándi, Sudakov)

$$h_{p_r}(2^r) = r.$$  

Proof.

- AFSOC there is an $r$-uniform, $r$-partite hypergraph without a transversal cover but with any $2^r$ of its edges having a transversal cover.
- Let $H$ be minimal satisfying above conditions.
- Let $V_1, \ldots, V_r$ be parts and $A_1, \ldots, A_k$ edges of $H$.
- Let $B_i$ be a transversal cover of $H - A_i$. Then:
  - $|A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j.$
  - $A_i \cap B_j \neq \emptyset, \forall i \neq j$ since $B_j$ is a cover of $H - A_j$.
  - $A_i \cap B_i = \emptyset, \forall i$ since otherwise $B_i$ is a transversal cover of $H$, a contradiction.
Theorem (B., Korándi, Sudakov)
\[ h_p(r)(2^r) = r. \]

Proof.
- AFSOC there is an \( r \)-uniform, \( r \)-partite hypergraph without a transversal cover but with any \( 2^r \) of its edges having a transversal cover.
- Let \( H \) be minimal satisfying above conditions.
- Let \( V_1, \ldots, V_r \) be parts and \( A_1, \ldots, A_k \) edges of \( H \).
- Let \( B_i \) be a transversal cover of \( H - A_i \). Then:
  - \( |A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j. \)
  - \( A_i \cap B_j \neq \emptyset, \forall i \neq j \) since \( B_j \) is a cover of \( H - A_j \).
  - \( A_i \cap B_i = \emptyset, \forall i \) since otherwise \( B_i \) is a transversal cover of \( H \), a contradiction.
- This implies \( |E(H)| = k \leq \prod_{i=1}^{r} \left( \frac{1+1}{1} \right) = 2^r, \)
Theorem (B., Korándi, Sudakov)

$$hp_r(2^r) = r.$$ 

Proof.

- AFSOC there is an $r$-uniform, $r$-partite hypergraph without a transversal cover but with any $2^r$ of its edges having a transversal cover.
- Let $H$ be minimal satisfying above conditions.
- Let $V_1, \ldots, V_r$ be parts and $A_1, \ldots, A_k$ edges of $H$.
- Let $B_i$ be a transversal cover of $H - A_i$. Then:
  - $|A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j.
  - $A_i \cap B_j \neq \emptyset, \forall i \neq j$ since $B_j$ is a cover of $H - A_j$.
  - $A_i \cap B_i = \emptyset, \forall i$ since otherwise $B_i$ is a transversal cover of $H$, a contradiction.
- This implies $|E(H)| = k \leq \prod_{i=1}^{r} \left( \frac{1+1}{1} \right) = 2^r$.
- So $H$ has a transversal cover, a contradiction.$\blacksquare$
Concluding remarks and open problems

Theorem (B., Korándi, Sudakov)

\[ h_p(r(2^r)) = r \text{ and } h_p(r(2^r/\sqrt{r})) > r. \]
Theorem (B., Korándi, Sudakov)

\[ h_{p_r}(2^r) = r \text{ and } h_{p_r}(2^r / \sqrt{r}) > r. \]

Question

Is the \( \sqrt{r} \) necessary?
Concluding remarks and open problems

Theorem (B., Korándi, Sudakov)

\[ h_p(r, 2^r) = r \text{ and } h_p(r, 2^r / \sqrt{r}) > r. \]

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Is the \( \sqrt{r} \) necessary?

Theorem (B., Korándi, Sudakov)

\[ \frac{r^2}{12 \log k} \leq h_p(k) \leq \frac{16 r^2 \log r}{\log k}. \]
Concluding remarks and open problems

**Theorem (B., Korándi, Sudakov)**

\[ h_p(r) = r \text{ and } h_p(r) (2r/\sqrt{r}) > r. \]

**Question**

*Is the \( \sqrt{r} \) necessary?*

**Theorem (B., Korándi, Sudakov)**

\[ \frac{r^2}{12 \log k} \leq h_p(k) \leq \frac{16r^2 \log r}{\log k}. \]

**Question**

*Is the \( \log r \) necessary?*
Concluding remarks and open problems

**Theorem (B., Korándi, Sudakov)**

\[ h_{p_r}(2^r) = r \text{ and } h_{p_r}(2^r / \sqrt{r}) > r. \]

**Question**

*Is the \( \sqrt{r} \) necessary?*

**Theorem (B., Korándi, Sudakov)**

\[ \frac{r^2}{12 \log k} \leq h_{p_r}(k) \leq \frac{16r^2 \log r}{\log k}. \]

**Question**

*Is the \( \log r \) necessary? Same question without \( r \)-partiteness condition.*
