AUTOMORPHISMS OF A SYMMETRIC PRODUCT OF A CURVE

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Abstract. We show that all the automorphisms of the symmetric product $\text{Sym}^d(X)$, $d > 2g - 2$, of a smooth projective curve $X$ of genus $g > 2$ are induced by automorphisms of $X$.

1. Introduction

Let $X$ be a smooth projective curve of genus $g$, with $g > 2$, over an algebraically closed field. Take any integer $d > 2g - 2$. Let $\text{Sym}^d(X)$ be the $d$-fold symmetric product of $X$. Our aim here is to study the group $\text{Aut}(\text{Sym}^d(X))$ of automorphisms of the algebraic variety $\text{Sym}^d(X)$. An automorphism $f$ of the algebraic curve $X$ produces an algebraic automorphism $\rho(f)$ of $\text{Sym}^d(X)$ that sends any $\{x_1, \ldots, x_d\} \in \text{Sym}^d(X)$ to $\{f(x_1), \ldots, f(x_d)\}$. This map 
$$\rho : \text{Aut}(X) \longrightarrow \text{Aut}(\text{Sym}^d(X)), \ f \longmapsto \rho(f)$$

is a homomorphism of groups.

Theorem 1.1. The natural homomorphism
$$\rho : \text{Aut}(X) \longrightarrow \text{Aut}(\text{Sym}^d(X))$$
is an isomorphism.

The idea of the proof is as follows. The homomorphism $\rho$ is clearly injective, so we have to show that it is also surjective. The Albanese variety of $\text{Sym}^d(X)$ is the Jacobian $J(X)$ of $X$. So an automorphism of $\text{Sym}^d(X)$ induces an automorphism of $J(X)$. Using results of Fakhruddin and Collino–Ran, we show that the induced automorphisms of $J(X)$ respects the theta divisor up to translation. Invoking the strong form of the Torelli theorem for the Jacobian, it follows that such automorphisms are generated by automorphisms of the curve $X$, translations of $J(X)$, and the inversion of $J(X)$ that sends each line bundle to its dual. Using a result of Kempf we show that if an automorphism $\alpha$ of $J(X)$ lifts to $\text{Sym}^d(X)$, then $\alpha$ is induced by an automorphism of $X$, and this finishes the proof.

It should be clarified that we need a slight generalisation of the result of Kempf [Ke]; this is proved in Section 2. The proof of Theorem 1.1 is in Section 3.

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2. Some properties of the Picard bundle

A branding of a Picard variety $P^d = \text{Pic}^d(X)$ is a Poincaré line bundle $Q$ on $X \times P^d$ [Ke, p. 245]. Two brandings differ by the pullback of a line bundle on $P^d$.

The degree of a line bundle $\xi$ over a smooth projective variety $Z$ is the class of the first Chern class $c_1(\xi)$ in the Néron-Severi group $\text{NS}(Z)$, so the line bundles of degree zero on $Z$ are classified by the Jacobian $J(Z)$. A normalised branding is a branding such that $Q_{|\{x\} \times P^d}$ has degree zero for one point $x \in X$ (equivalently, for all points of $X$). Two normalised brandings differ by the pullback of a degree zero line bundle on $P^d$.

The natural projection of $X \times P^d$ to $P^d$ will be denoted by $\pi_{P^d}$. A normalised branding $Q$ induces an embedding

$$I_Q : X \rightarrow J(P^d) =: J$$

that sends any $x \in X$ to the point of $J$ corresponding to the line bundle $Q_{|\{x\} \times P^d}$. If $Q' = Q \otimes \pi_{P^d}^* L_j$, where $L_j$ is the line bundle corresponding to a point $j \in J(P^d) = J$, then we have $I_{Q'} = I_Q + j$.

Assume that $d > 2g - 2$. A Picard bundle $W(Q)$ on $P^d$ is the vector bundle $\pi_{P^d*} Q$, where $Q$ is a normalised branding. From the projection formula it follows that two Picard bundles differ by tensoring with a degree zero line bundle on $P^d$.

There is a version of the following proposition for $d < 0$ in [Ke, Corollary 4.4] (for negative degree, the Picard bundle is defined using the first direct image).

Proposition 2.1. Let $d > 2g - 2$.

1. $H^1(P^d, W(Q))$ is non-zero (in fact, it is one-dimensional if it is non-zero) if and only if $0 \in I_Q(X)$.

2. Let $L_j$ be the line bundle on $P^d$ corresponding to a point $j \in J$. Then $H^1(P^d, L_j \otimes W(Q))$ is non-zero (in fact, it is one-dimensional if it is non-zero) if and only if $-j \in I_Q(X)$.

Proof. Part (1). If $0 \notin I_Q(X)$, then $H^1(P^d, W(Q)) = 0$ by [Ke, p. 252, Theorem 4.3(c)]. Fix a line bundle $M$ on $X$ of degree one, and consider the associated Abel-Jacobi map

$$X \rightarrow J(X), \quad x \mapsto M \otimes \mathcal{O}_X(-x).$$

Let $N_{X/J(X)}$ be the normal bundle of the image of $X$ under this Abel-Jacobi map. If $0 \in I_Q(X)$, then using [Ke, p. 252, Theorem 4.3(d)] it follows that $H^1(P^d, W(Q))$ is isomorphic to the space of sections of the skyscraper sheaf on $X$

$$K_X^{-1} \otimes \wedge^0 N_{X/J(X)} \otimes M^d|_{I_Q^{-1}(0)},$$

where $I_Q$ is constructed in (2.1). But the space of sections of this skyscraper sheaf is clearly one-dimensional, because $I_Q^{-1}(0)$ consists of one point of $X$.

Part (2) follows from part (1) because $L_j \otimes W(Q) = W(Q \otimes \pi_{P^d}^* L_j)$, and $I_{W(Q \otimes \pi_{P^d}^* L_j)} = I_{W(Q)} + j$. \qed
Proposition 2.2. Assume \( g(X) > 1 \) and \( d > 2g - 2 \). Let \( j \) be a point of \( \text{Pic}^0(X) \), and let \( T_j : P^d \rightarrow P^d \) be the translation by \( j \). Let \( M \) be a degree zero line bundle on \( P^d \). If
\[
T_j^*(M \otimes W(Q)) \cong W(Q),
\]
then \( j = 0 \) and \( M = \mathcal{O}_{P^d} \).

Let \( i : P^d \rightarrow P^d \) be the inversion given by \( z \mapsto -z + 2z_0 \), where \( z_0 \) is a fixed point in \( P^d \). If
\[
i^*T_j^*(M \otimes W(Q)) \cong W(Q),
\]
then \( X \) is a hyperelliptic curve.

Proof. The first part is [Ke] Proposition 9.1] except that there it is assumed that \( d < 0 \); the proof of Proposition 9.1 uses [Ke] Corollary 4.4] which requires this hypothesis. However, the case \( d > 2g - 2 \) can be proved similarly; for the convenience of the reader we give the details.

Let \( y \in J \) be the point corresponding to the line bundle \( M \). The line bundle on \( P^d \) corresponding to any \( t \in J \) will be denoted by \( L_t \). In particular, \( M = L_y \). For every \( t \in J \), using the hypothesis, we have
\[
T_j^*(L_{t+y} \otimes W(Q)) = T_j^*L_t \otimes T_j^*(M \otimes W(Q)) = L_t \otimes W(Q); \tag{2.2}
\]
note that the fact that a degree zero line bundle on an Abelian variety is translation invariant is used above. Combining (2.2) and the fact that \( T_j \) is an isomorphism, we have
\[
H^1(L_t \otimes W(Q)) \cong H^1(T^*(L_{t+y} \otimes W(Q))) \cong H^1(L_{t+y} \otimes W(Q)).
\]
Using [Ke] Corollary 4.4] it follows that \( t \in -I_Q(X) \) if and only if \( t + y \in -I_Q(X) \). Hence \( I_Q(X) = y + I_Q(X) \). If \( g(X) > 1 \), this implies that \( y = 0 \). Therefore, we have \( W(Q) = T_j^*(W(Q)) \). Using the fact that \( c_1(W(Q)) = \theta \), a theta divisor, it follows that \( \theta \) is rationally equivalent to the translate \( \theta - j \), hence \( j = 0 \).

The proof of the second part is similar. We have
\[
i^*T_j^*(L_{y-t} \otimes W(Q)) = i^*T_j^*L_{-t} \otimes i^*T_j^*(M \otimes W(Q)) = i^*L_{-t} \otimes W(Q) = L_t \otimes W(Q); \tag{2.3}
\]
the fact that \( i^*L_{-t} = L_t \) is used above. Consequently,
\[
H^1(L_t \otimes W(Q)) \cong H^1(i^*T^*(L_{y-t} \otimes W(Q))) \cong H^1(L_{y-t} \otimes W(Q)),
\]
and using [Ke] Corollary 4.4] it follows that \( t \in -I_Q(X) \) if and only if \( y - t \in -I_Q(X) \). Hence \( I_Q(X) = -I_Q(X) - y \). Let
\[
f : X \rightarrow X
\]
be the morphism uniquely determined by the condition
\[
I_Q(x) = -I_Q(f(x)) - y.
\]
We note that \( f \) is well defined because \( -I_Q \) and \( y + I_Q \) are two embeddings of \( X \) in \( J \) with the same image, so they differ by an automorphism of \( X \) which is \( f \). In other words, if we identify \( X \) with its image under \( I_Q \), then \( f \) is induced from the automorphism \( T_{-y} \circ i \) of \( J \). This automorphism \( T_{-y} \circ i \) is clearly an involution. Let \( \omega \in H^0(C, \Omega_C) \) be a holomorphic 1-form on \( X \). Then \( f^*\omega = -\omega \), because of the isomorphism \( H^0(X, \Omega_X) = H^0(J, \Omega_J) \) induced by \( I_Q \), and the fact that \( i^* \) acts as multiplication by \( -1 \) on the 1-forms on \( J \). It now follows by Lemma 2.3 that \( f \) is a hyperelliptic involution. \( \Box \)
Lemma 2.3. Let $g > 1$. Let $f : X \to X$ be an involution satisfying the condition that $f^\ast \omega = -\omega$ for every 1-form $\omega$. Then $X$ is hyperelliptic with $f$ being the hyperelliptic involution.

Proof. Consider the canonical morphism

$$F : X \to \mathbb{P}(H^0(X, K_X))$$

that sends any $x \in X$ to the hyperplane $H^0(K_X(-x))$ in $H^0(K_X)$. By definition,

$$H^0(\Omega_X(-x)) = \{ \omega \in H^0(\Omega_X) \mid \omega(x) = 0 \},$$

but the hypothesis implies that $\omega(x) = 0$ if and only if $\omega(f(x)) = f^\ast(\omega)(x) = 0$. Therefore, we have

$$H^0(\Omega_X(-x)) = H^0(\Omega_X(-f(x))),$$

and it follows that $F(x) = F(f(x))$, implying that the canonical morphism is not an embedding; note that $f$ is not the identity because there are nonzero holomorphic 1-forms. Therefore, $X$ is hyperelliptic, and $f$ is the hyperelliptic involution. □

3. Proof of Theorem

Using the morphism $X \to \text{Sym}^d(X)$, $y \mapsto dy$, it follows that the homomorphism $\rho$ in Theorem is injective.

Fix a point $x \in X$. Let $L$ be the normalised Poincaré line bundle on $X \times J(X)$, i.e., it is trivial when restricted to the slice $\{x\} \times J(X)$. Let

$$E := q_\ast(L \otimes p^\ast \mathcal{O}_X(dx))$$

be the Picard bundle, where $p$ and $q$ are the projections from $X \times J(X)$ to $X$ and $J(X)$ respectively. Since $d > 2g - 2$, it follows that $E$ is a vector bundle of rank $d - g + 1$.

We will identify $\text{Sym}^d(X)$ with the projective bundle $P(E) = \mathbb{P}(E^\vee)$.

Let $\theta$ be the theta divisor of $J(X)$; in particular, we have $\theta^g = g!$. The Chern classes of $E$ are given by $c_i(E) = \theta^i$. [ACGH].

The Albanese variety of $P(E)$ is the Jacobian $J(X)$, and the Albanese map sends an effective divisor $\sum_{i=1}^d P_i$ of degree $d$ to the degree zero line bundle $\mathcal{O}_X((\sum_{i=1}^d P_i) - dx)$. Given an automorphism

$$\varphi : P(E) \to P(E),$$

the universal property of the Albanese variety yields a commutative diagram

$$\begin{array}{ccc}
P(E) & \xrightarrow{\varphi} & P(E) \\
\downarrow & & \downarrow \\
J(X) & \xrightarrow{\alpha} & J(X)
\end{array}$$

and this produces an automorphism of projective bundles

$$\begin{array}{ccc}
P(E) & \xrightarrow{\psi} & P(\alpha^\ast E) \\
\downarrow & & \downarrow \\
J(X) & & 
\end{array}$$
Therefore, there is a line bundle $L$ on $J(X)$ such that there is an isomorphism
\[ \alpha^* E \cong E \otimes L. \] (3.2)

There is a commutative diagram of groups
\[ \begin{array}{ccc}
\text{Aut}(P(E)) & \xrightarrow{\lambda} & \text{Aut}(J(X)) \\
\rho & & \mu \\
\text{Aut}(X) & &
\end{array} \] (3.3)

where $\lambda$ is constructed as above using the universal property of the Albanese variety given in (3.1), and $\rho$ is the homomorphism in Theorem 1.1. To construct $\mu$, note that the commutativity of the diagram (3.1) implies that $\mu(f)$, $f \in \text{Aut}(X)$, has to send
\[ O_X((\sum_{i=1}^{d} P_i) - dx) \]

\[ \rightarrow \]
\[ O_X((\sum_{i=1}^{d} f(P_i)) - dx). \]

A short calculation yields
\[ \mu(f) = (f^{-1})^* \circ T_{dx-\theta^{-1}(x)}, \] (3.4)

where $T_a, a \in J(X)$, is translation on $J(X)$ by $a$.

Let $\theta' = c_1(\alpha^* E) = \theta + L$. Then
\[ c_i(\alpha^* E) = \alpha^* c_i(E) = \frac{\alpha^* \theta^i}{i!} = \frac{\theta^i}{i!}, \]

and $\theta^g = \alpha^* \theta^g = g!$. Now we apply [Fa, Lemma 1]; here the condition $g > 2$ is used. We obtain $\theta^i = \theta^{g!}$ for all $i > 1$.

We identify $X$ with the image in $J(X)$ of the Abel-Jacobi map. In particular $X$ is numerically equivalent to $\theta^g - 1/(g - 1)!$. We calculate the intersection (note that the condition $g > 2$ is again used, because we need $g - 1 > 1$)
\[ \theta' X = \theta' \frac{\theta^g - 1}{(g - 1)!} = \theta' \frac{\theta^{g!}}{(g - 1)!} = g. \]

Invoking a characterisation of a Jacobian variety due to Collino and Ran, [Co], [Ra], it follows that $(J(X), \theta', X)$ is a Jacobian triple, i.e., $\theta'$ is a theta divisor of the Jacobian variety $J(X)$ up to translation. This means that $\theta$ and $\theta'$ differ by translation, in other words, the class of $c_1(L)$ in the Néron-Severi group $\text{NS}(J(X))$ is zero. Consequently, $\alpha$ is an isomorphism of polarised Abelian varieties, i.e., it sends $\theta'$ to a translate of it.

The strong form of the classical Torelli theorem ([LS, Théorème 1 and 2 of Appendix]) tells us that such an automorphism $\alpha$ is of the form
\[ \alpha = F \circ \sigma \circ T_a, \quad \sigma \in \{1, \iota\}, \]

where $F = (f^{-1})^*$ for an automorphism $f$ of $X$, while $T_a$ is translation by an element $a \in J(X)$ and $\iota$ sends each element of $J(X)$ to its inverse. If $X$ is hyperelliptic, then $\iota$ is induced by the hyperelliptic involution, so we may assume that $\sigma$ is the identity map of $X$ when $X$ is hyperelliptic.

Let $f$ be an automorphism of $X$ with $F = (f^{-1})^*$ being the induced isomorphism on $J(X)$. Using the definition of $E$, it is easy to check that
\[ F^* E \cong T_{dx'} - dx E, \]

where $x' = f^{-1}(x)$.

We claim that $\alpha = F \circ T_a$. 

To prove this, assume that $\alpha \neq F \circ T_a$. Then $X$ is not hyperelliptic, and $\alpha = F \circ \iota \circ T_a$. Hence

$$\alpha^* E = T^*_{a} \circ T^*_{dx'-dx} E = T^*_{a} \circ T^*_{dx'-dx} E,$$

and using (3.2),

$$E \cong T^*_{dx'-dx-a}(E \otimes L).$$

Now from Proposition 2.2 it follows that $X$ is hyperelliptic, and we arrive at a contradiction. This proves the claim.

Summing up, we can assume that $\alpha = F \circ T_a$. Using (3.2),

$$E \cong T^*_{dx-dx'}(E \otimes L).$$

From Proposition 2.2 it follows that $L$ is the trivial line bundle, and $a = dx - dx'$. Therefore,

$$\alpha = (f^{-1})^* \circ T^*_{dx-dx'}(x)$$

for some automorphism $f$ of $X$, and hence, by (3.4),

$$\text{Image}(\lambda) \subset \text{Image}(\mu). \quad (3.5)$$

We will now show that the morphism $\lambda$ is injective.

Suppose $\alpha = \lambda(\varphi) = \text{Id}_{J(X)}$. Using (3.2), the morphism $\varphi$ is induced by an isomorphism between $E$ and $E \otimes L$. We have just seen that $L$ is trivial, the morphism $\varphi$ is induced by an automorphism of $E$, and this automorphism has to be multiplication by a nonzero scalar, because $E$ is stable with respect to the polarisation given by the theta divisor (cf. [EL]). Therefore, the morphism $\varphi$ is the identity. This proves that the morphism $\lambda$ is injective.

The homomorphism $\mu$ is also injective, since it is a composition of a translation and the pullback induced by an automorphism of $X$.

Combining these it follows that the morphism $\rho$ is also injective (this can also be checked directly), and hence all the homomorphisms in the diagram (3.3) are injective. This, combined with (3.5), shows that $\rho$ is an isomorphism.

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