BBW parabolics for simple classical Lie superalgebras

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Lie superalgebras

Definition

A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}/2$-graded vector space with a bilinear operation called the superbracket that satisfies

1. $[a, b] = -(-1)^{|a||b|}[b, a]$
2. $(-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||a|}[b, [c, a]] + (-1)^{|c||b|}[c, [a, b]] = 0$

Example

1. The superalgebra $\mathfrak{gl}(m|n)$ is the set of $(m+n) \times (m+n)$ matrices over $\mathbb{C}$ with $\mathfrak{g}_0 = \{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \}$ and $\mathfrak{g}_1 = \{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \}$. The superbracket is $[E_{ij}, E_{kl}] = E_{ij}E_{ki} - (-1)^{|E_{ij}||E_{kl}|} E_{kl}E_{ij}$.

2. The superalgebra $\mathfrak{q}(n) = \{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in \mathfrak{gl}(n|n) \}$ with

   $\mathfrak{q}(n)_{\bar{0}} = \{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \} \cong \mathfrak{gl}(n)$ and $\mathfrak{q}(n)_{\bar{1}} = \{ \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \}$.
A Lie superalgebra is \textit{classical} if $G_0$ is a connected reductive group and $g_{\bar{1}}$ is completely reducible under the adjoint $G_0$-action. In 1977, Kac classified the finite-dimensional classical “simple” Lie superalgebras:

- $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(n|n)$
- $\mathfrak{osp}(m|n)$
- $D(2, 1, \alpha)$, $G(3)$, $F(4)$
- $\mathfrak{q}(n)$, $\mathfrak{psq}(n)$
- $\mathfrak{p}(n)$, $\tilde{\mathfrak{p}}(n)$
Detecting subalgebras

Theorem (BKN 2008)

Let \( \mathfrak{g} \) be a classical Lie superalgebra with a stable and polar action. Then there exists subalgebras \( \mathfrak{e} \subset \mathfrak{f} \subset \mathfrak{g} \) which induce the following ring isomorphisms via restriction:

\[
\begin{align*}
H^\bullet(\mathfrak{g}, \mathfrak{g}_0, \mathbb{C}) & \longrightarrow H^\bullet(\mathfrak{f}, \mathfrak{f}_0, \mathbb{C})^N \\
& \longrightarrow H^\bullet(\mathfrak{e}, \mathfrak{e}_0, \mathbb{C})^W_e \\
S^\bullet(\mathfrak{g}^*_1)^G_0 & \longrightarrow S^\bullet(\mathfrak{f}^*_1)^N \\
& \longrightarrow S^\bullet(\mathfrak{e}^*_1)^W_e.
\end{align*}
\]

The “detecting subalgebras” \( \mathfrak{f} \) and \( \mathfrak{e} \) are isomorphic to \( \bigoplus \mathfrak{sl}(1|1) \) and \( \bigoplus \mathfrak{q}(1) \), respectively.
Detecting subalgebras

Example

Let $g = gl(2|3)$. Then $\Phi_{\hat{f}_1} = \{\pm(\varepsilon_1 - \delta_1), \pm(\varepsilon_2 - \delta_2)\}$ and

\[
\hat{f}_0 = \begin{pmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{f}_1 = \begin{pmatrix} 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]

The $e$ has the same shape, but the entries are now in $q(1) \oplus 2$ instead of $sl(1|1) \oplus 2$. 
Support Varieties

**Definition**

Let $M$ be a finite-dimensional $g$-module semisimple over $g_0$. Define the *support variety* of $M$ as

$$\mathcal{V}_{(g,g_0)}(M) := \{ P \in \text{Spec}(H^\bullet(g, g_0, C)) : \text{Ext}_{(g,g_0)}^\bullet(M, M)_P \neq 0 \}$$

**Conjecture (BKN 2008)**

Let $M$ be finite-dimensional $g$-module which is semisimple over $g_0$. Then the Chevalley restriction maps induce an isomorphism of support varieties

$$\text{res}^* : \mathcal{V}_{(e,e_0)}(M)/W_e \rightarrow \mathcal{V}_{(f,f_0)}(M)/N \rightarrow \mathcal{V}_{(g,g_0)}(M).$$

When $M = C$, the conjecture reduces to the [BKN 2008] theorem.
BBW Parabolic

Theorem (GGNW 2019)

Let $\mathfrak{g}$ be a classical Lie superalgebra and $\mathfrak{f}$ be a detecting subalgebra. Then there exists a triangular decomposition

$$\mathfrak{g} = \mathfrak{u}^+ \oplus \mathfrak{f} \oplus \mathfrak{u}^-.$$

The parabolic $\mathfrak{b} := \mathfrak{f} \oplus \mathfrak{u}^-$ is called the BBW parabolic.

Remarks:

- The BBW parabolic subalgebras are defined explicitly using roots.
- $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{b}_\overline{1}$ with $\mathfrak{b}_0$ a negative Borel of $\mathfrak{g}_0$.
- $\mathfrak{b}$ is not a Borel (= maximal solvable) and $\mathfrak{f}$ is not a torus (= maximal abelian), but they play analogous roles in the sense of cohomology (Bott-Borel-Weil theorem).
Examples

Let \( g = \mathfrak{gl}(2|3) \). Then

\[
\begin{align*}
\mathfrak{b}_0^- &= \begin{pmatrix}
* & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & * & * & 0 \\
0 & 0 & * & * & * \\
\end{pmatrix}, \\
\mathfrak{b}_1^- &= \begin{pmatrix}
0 & 0 & * & 0 & 0 \\
0 & 0 & * & * & 0 \\
* & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & * & 0 & 0 \\
\end{pmatrix}.
\end{align*}
\]

Let \( g = \mathfrak{gl}(3|3) \) or \( q(3) \). Then

\[
\begin{align*}
\mathfrak{b}_0^- &= \begin{pmatrix}
* & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & 0 & 0 \\
0 & 0 & 0 & * & * & * & 0 \\
\end{pmatrix}, \\
\mathfrak{b}_1^- &= \begin{pmatrix}
0 & 0 & 0 & * & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & 0 & 0 \\
0 & 0 & 0 & * & * & * & 0 \\
* & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\end{align*}
\]
Poincare Polynomials

- Given a supergroup scheme $G$ and parabolic subgroup $B$, we consider the induction functor $\text{ind}_B^G(\cdot)$ on the category of rational $G$ modules. Let $R^n\text{ind}_B^G$ be its higher right derived functors. We can define the Poincare polynomial

$$p_{G,B}(t) := \sum_{n \geq 0} \dim R^i \text{ind}_B^G \mathbb{C} t^i.$$

- Given Lie superalgebra $\mathfrak{a}$, define the Poincare polynomial

$$p_{\mathfrak{a}}(t) := \sum_{n \geq 0} \dim H^i(\mathfrak{a}, \mathfrak{a}_0, \mathbb{C}) t^i.$$

- Given a finite-reflection group $W$, define the Poincare polynomial

$$p_W(t) := \sum_{w \in W} t^{l(w)}$$

where $l(w)$ is the length of $w$. 
Cohomology

Theorem (GGNW 2019)

Let $G$ be algebraic supergroup with $\mathfrak{g} = \text{Lie}(G)$ a classical simple Lie superalgebra not of type $P$. Let $B$ be parabolic subgroup with $\text{Lie}(B) = \mathfrak{b}$ a BBW parabolic. Then there is a finite-reflection group $W_1$ such that as graded vector spaces,

$$H^•(\mathfrak{b}, \mathfrak{b}_0, \mathbb{C}) \cong H^•(\mathfrak{g}, \mathfrak{g}_0, \mathbb{C}) \otimes \mathbb{C}[W_1].$$

Furthermore, there is an equality of Poincare polynomials

$$p_{G,B}(t) = p_{\mathfrak{b}}(t)/p_{\mathfrak{g}}(t) = p_{W_1}(s)$$

with $s = t$ in type $Q$ and $s = t^2$ otherwise.

Example

For $\mathfrak{g} = q(n)$ or $\mathfrak{gl}(n|n)$, then $W_1 = S_n$ and we reach the Pittie-Steinberg theorem:

$$H^•(\mathfrak{b}, \mathfrak{b}_0, \mathbb{C}) = \mathbb{C}[z_1, \ldots, z_n], \quad H^•(\mathfrak{b}, \mathfrak{b}_0, \mathbb{C}) = \mathbb{C}[z_1, \ldots, z_n]^{S_n}$$
Corollaries

- The higher sheaf cohomology groups of the trivial line bundle \( \mathcal{C} \) over \( G/B \) is:
  \[
  \mathcal{H}^j(G/B, \mathcal{L}(0)) := R^j\text{ind}^G_B \mathcal{C} = \mathcal{C}^{\oplus n_j}
  \]
  where \( n_j = \begin{cases} 
  |\{ w \in W_1 : l(w) = j \}| & \text{in type Q} \\
  |\{ w \in W_1 : l(w) = j/2 \}| & \text{in other types}
  \end{cases} \)

- The spectral sequence
  \[
  \text{Ext}^i_{\mathfrak{g}, \mathfrak{g}_0}(M_1, R^j\text{ind}^G_B M_2) \Rightarrow \text{Ext}^{i+j}_{\mathfrak{h}, \mathfrak{h}_0}(M_1, M_2)
  \]
  collapses when \( M_1 = M_2 = \mathcal{C} \).

- This then allows us to verify the support variety conjecture
  \[
  \mathcal{V}_{(e, e_0)}(M)/W_e \cong \mathcal{V}_{(f, f_0)}(M)/N \cong \mathcal{V}_{(\mathfrak{g}, \mathfrak{g}_0)}(M).\]
References I

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