Phase Retrieval by Hyperplanes

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Abstract. We show that a scalable frame does phase retrieval if and only if the hyperplanes of its orthogonal complements do phase retrieval. We then show this result fails in general by giving an example of a frame for \( \mathbb{R}^3 \) which does phase retrieval but its induced hyperplanes fail phase retrieval. Moreover, we show that such frames always exist in \( \mathbb{R}^d \) for any dimension \( d \). We also give an example of a frame in \( \mathbb{R}^3 \) which fails phase retrieval but its perps do phase retrieval. We will also see that a family of hyperplanes doing phase retrieval in \( \mathbb{R}^d \) must contain at least \( 2d - 2 \) hyperplanes. Finally, we provide an example of six hyperplanes in \( \mathbb{R}^4 \) which do phase retrieval.

1. Introduction

In some applications in engineering, the phase of a signal is lost during processing. The problem of retrieving the phase of a signal, given a set of intensity measurements, has been studied by engineers for many years. Signals passing through linear systems often result in lost or distorted phase information. This partial loss of phase information occurs in various applications including speech recognition \([4, 15, 16]\), and optics applications such as X-ray crystallography \([3, 12, 13]\). The concept of phase retrieval for Hilbert space frames was introduced in 2006 by Balan, Casazza, and Edidin \([1]\) and since then it has become an active area of research. Phase retrieval deals with recovering the phase of a signal given intensity measurements from a redundant linear system. In phaseless reconstruction the unknown signal itself is reconstructed from these measurements. In recent literature, the two terms were used interchangeably. However it is not obvious from the definitions that the two are equivalent. Recently, authors in \([6]\) proved that phase retrieval is equivalent to phaseless reconstruction in both the real and complex case.

Phase retrieval has been defined for vectors as well as for projections. Phase retrieval by projections occur in real life problems, such as crystal twinning \([11]\),

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where the signal is projected onto some higher dimensional subspaces and has to be recovered from the norms of the projections of the vectors onto the subspaces. We refer the reader to [8] for a detailed study of phase retrieval by projections. At times these projections are identified with their target spaces. Determining when subspaces \( \{ W_i \}_{i=1}^n \) and \( \{ W_i^\perp \}_{i=1}^n \) both do phase retrieval has given way to the notion of norm retrieval [7], another important area of research.

In this paper we make a detailed study of phase retrieval by hyperplanes. We will see that it takes at least \( 2d - 2 \) hyperplanes to do phase retrieval in \( \mathbb{R}^d \). We will show that scalable frames \( \{ \phi_i \}_{i=1}^n \) do phase retrieval if and only if their induced hyperplanes \( \{ \phi_i^\perp \}_{i=1}^n \) do phase retrieval. We then give examples to show this result fails in general if the frame is not scalable. In particular, we give an example of a frame for \( \mathbb{R}^3 \) which does phase retrieval but its induced hyperplanes fail phase retrieval. Moreover, we show that such frames always exist in \( \mathbb{R}^d \) for any dimension \( d \). We also give an example of a family of hyperplanes in \( \mathbb{R}^3 \) which do phase retrieval but their perp vectors fail phase retrieval. Finally, we give 6 hyperplanes in \( \mathbb{R}^4 \) which do phase retrieval.

2. Preliminaries

In this section we will give the background material needed for the paper. We start with the definition of a frame.

**Definition 2.1.** A family of vectors \( \Phi = \{ \phi_i \}_{i=1}^n \) in \( \mathbb{R}^d \) is a **frame** if there are constants \( 0 < A \leq B < \infty \) so that for all \( x \in \mathbb{R}^d \)

\[
A\|x\|^2 \leq \sum_{i=1}^{n} |\langle x, \phi_i \rangle|^2 \leq B\|x\|^2,
\]

where \( A \) and \( B \) are the **lower and upper frame bounds** of the frame, respectively. The frame is called an **A-tight frame** if \( A = B \) and is a **Parseval frame** if \( A = B = 1 \).

**Definition 2.2.** A frame \( \Phi = \{ \phi_i \}_{i=1}^n \) in \( \mathbb{R}^d \) is called scalable if there exists scalars \( \{ s_i \}_{i=1}^n \) such that \( \{ s_i \phi_i \}_{i=1}^n \) is a tight frame for \( \mathbb{R}^d \).

The main topics here are phase retrieval and norm retrieval in \( \mathbb{R}^d \).

**Definition 2.3.** Let \( \Phi = \{ \phi_i \}_{i=1}^n \subset \mathbb{R}^d \) be such that for \( x, y \in \mathbb{R}^d \)

\[
|\langle x, \phi_i \rangle| = |\langle y, \phi_i \rangle|, \text{ for all } i = 1, 2, \ldots, n.
\]

\( \Phi \) yields

- **(ii) phaseless reconstruction** if \( x = \pm y \).
- **(iii) norm retrieval** if \( \|x\| = \|y\| \).

**Remark 2.4.** It is easy to see that \( \{ \phi_i \}_{i=1}^n \) does phase retrieval (norm retrieval) if and only if \( \{ c_i \phi_i \}_{i=1}^n \) does phase retrieval (norm retrieval), for any non-zero scalars \( \{ c_i \}_{i=1}^n \subset \mathbb{R}^d \).

The paper [1] gives the minimal number of vectors needed in \( \mathbb{R}^d \) to do phase retrieval.

**Theorem 2.5 ([1]).** In order for a frame \( \{ \phi_i \}_{i=1}^n \) in \( \mathbb{R}^d \) to do phase retrieval, it is necessary that \( n \geq 2d - 1 \).
Also [1] presents a fundamental classification of the frames which do phase retrieval in \( \mathbb{R}^d \). For this we need a definition.

**Definition 2.6 ([1])**. A frame \( \Phi = \{ \phi_i \}_{i=1}^n \) in \( \mathbb{R}^d \) satisfies the **complement property** if for all subsets \( I \subset \{1, 2, \ldots, n\} \), either \( \{ \phi_i \}_{i \in I} \) or \( \{ \phi_i \}_{i \in I^c} \) spans \( \mathbb{R}^d \).

A fundamental result from [1] is:

**Theorem 2.7 ([1])**. A frame \( \Phi \) does phaseless reconstruction in \( \mathbb{R}^d \) if and only if it has the complement property.

It follows that if \( \Phi = \{ \phi_i \}_{i=1}^n \) does phase retrieval in \( \mathbb{R}^d \) then \( n \geq 2d - 1 \). Full spark is another important notion of vectors in frame theory. A formal definition is given below:

**Definition 2.8**. Given a family of vectors \( \Phi = \{ \phi_i \}_{i=1}^n \) in \( \mathbb{R}^d \), the **spark** of \( \Phi \) is defined as the cardinality of the smallest linearly dependent subset of \( \Phi \). When \( \text{spark}(\Phi) = d + 1 \), every subset of size \( d \) is linearly independent, and in that case, \( \Phi \) is said to be **full spark**.

We note that from the definitions it follows that full spark frames with \( n \geq 2d - 1 \) vectors have the complement property and hence do phaseless reconstruction. Also, if \( n = 2d - 1 \) then the complement property clearly implies full spark.

We will need a generalization of phase retrieval to phase retrieval by projections.

**Definition 2.9**. A family of subspaces \( \{ W_i \}_{i=1}^n \) (or respectively, their induced projections \( \{ P_i \}_{i=1}^n \)) do

1. **phase retrieval** on \( \mathbb{R}^d \) if whenever \( x, y \in \mathbb{R}^d \) satisfy
   \[
   \| P_i x \| = \| P_i y \|, \text{ for all } i = 1, 2, \ldots, n,
   \]
   then \( x = \pm y \).
2. It does **norm retrieval** if \( \| x \| = \| y \| \).

We will need a result from [8].

**Proposition 2.10**. Let projections \( \{ P_i \}_{i=1}^n \) do phase retrieval on \( \mathbb{R}^d \). Then \( \{ (I - P_i) \}_{i=1}^n \) does phase retrieval if and only if it does norm retrieval.

We note the following result from [8]:

**Theorem 2.11 ([8])**. In \( \mathbb{R}^d \), for any integers \( 1 \leq k_i \leq d - 1 \), there are subspaces \( \{ W_i \}_{i=1}^{2d-1} \) of \( \mathbb{R}^d \) with \( \dim W_i = k_i \) and \( \{ W_i \}_{i=1}^{2d-1} \) does phase retrieval.

The major open problem in the area of real phase retrieval is:

**Problem 2.12**. What is the least number of subspaces needed to do phase retrieval on \( \mathbb{R}^d \)? What are the possible dimensions of these subspaces?

For notation we will use:

**Notation 2.13**. If \( \Phi = \{ \phi_i \}_{i=1}^n \) is a frame in \( \mathbb{R}^d \), we denote the induced hyperplanes as \( \Phi^\perp = \{ \phi_i^\perp \}_{i=1}^n \).
3. Phase Retrieval by Hyperplanes

We will need a result of Edidin [10], which is also generalized in [19].

**Theorem 3.1.** Let \( \{W_i\}_{i=1}^{n} \) be subspaces of \( \mathbb{R}^d \) with respective projections \( \{P_i\}_{i=1}^{n} \). The following are equivalent:

1. \( \{W_i\}_{i=1}^{n} \) does phase retrieval.
2. For every \( 0 \neq x \in \mathbb{R}^d \), \( \text{span}\{P_i x\}_{i=1}^{n} = \mathbb{R}^d \).

We will show that for a scalable frame \( \Phi \), both \( \Phi \) and \( \Phi^\perp \) do norm retrieval. For this we need a proposition.

**Proposition 3.2.** Let \( \{W_i\}_{i=1}^{n} \) be proper subspaces of \( \mathbb{R}^d \) with respective projections \( \{P_i\}_{i=1}^{n} \). Then for any scalars \( \{a_i\}_{i=1}^{n} \subset \mathbb{R} \) and \( 0 < A \in \mathbb{R} \), the following are equivalent.

1. For every orthonormal basis \( \{u_{i,j}\}_{j=1}^{n_i} \) of \( W_i \), the set \( \{a_i u_{i,j}\}_{j=1}^{n_i} \) is a \( A \)-tight frame.
2. For some orthonormal basis \( \{u_{i,j}\}_{j=1}^{n_i} \) of \( W_i \), the set \( \{a_i u_{i,j}\}_{j=1}^{n_i} \) is a \( A \)-tight frame.
3. \( \sum_{i=1}^{n} a_i^2 P_i = A \cdot I \).
4. \( \sum_{i=1}^{n} a_i^2 (I - P_i) = \sum_{i=1}^{n} a_i^2 - A \cdot I \).
5. For every orthonormal basis \( \{v_{i,j}\}_{j=1}^{d-n_i} \) of \( W_i^\perp \), the set \( \{a_i v_{i,j}\}_{j=1}^{d-n_i} \) is a \( (\sum_{i=1}^{n} a_i^2 - A) \)-tight frame.
6. For some orthonormal basis \( \{v_{i,j}\}_{j=1}^{d-n_i} \) of \( W_i^\perp \), the set \( \{a_i v_{i,j}\}_{j=1}^{d-n_i} \) is a \( (\sum_{i=1}^{n} a_i^2 - A) \)-tight frame.

**Proof.** (1) \( \Rightarrow \) (2). Obvious.

(2) \( \Rightarrow \) (3). Let \( \{u_{i,j}\}_{j=1}^{n_i} \) be the orthonormal basis of \( W_i \) in (2). Then for any \( x \in \mathbb{R}^d \),

\[
 a_i^2 P_i x = \sum_{j=1}^{n_i} \langle x, a_i u_{i,j} \rangle a_i u_{i,j}.
\]

Hence

\[
 \sum_{i=1}^{n} a_i^2 P_i x = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \langle x, a_i u_{i,j} \rangle a_i u_{i,j} = Ax.
\]

Therefore, \( \sum_{i=1}^{n} a_i^2 P_i = A \cdot I \).

(3) \( \Rightarrow \) (1) Let \( \{u_{i,j}\}_{j=1}^{n_i} \) be any orthonormal basis of \( W_i \). Then we have

\[
 a_i^2 P_i x = \sum_{j=1}^{n_i} \langle x, a_i u_{i,j} \rangle a_i u_{i,j}.
\]

Hence

\[
 Ax = \sum_{i=1}^{n} a_i^2 P_i x = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \langle x, a_i u_{i,j} \rangle a_i u_{i,j}.
\]

So, \( \{a_i u_{i,j}\}_{i=1}^{n} \) is a \( A \)-tight frame.

(3) \( \Rightarrow \) (4). Obvious.

Similarly, (4), (5), (6) are equivalent, but we need to see that \( \sum_{i=1}^{n} a_i^2 - A > 0 \). This follows immediately from \( \sum_{i=1}^{n} a_i^2 P_i = A \cdot I \) and \( \{W_i\}_{i=1}^{n} \) are proper subspaces. \( \square \)
PROPOSITION 3.3. If \( \{W_i\}_{i=1}^n \) satisfy one of the conditions in Proposition 3.2 then both \( \{W_i\}_{i=1}^n \) and \( \{W_i^\perp\}_{i=1}^n \) do norm retrieval.

PROOF. The results follow from the fact that
\[
\sum_{i=1}^n a_i^2 \|P_i x\|^2 = \sum_{i=1}^n (a_i P_i x, a_i P_i x) = (\sum_{i=1}^n a_i^2 P_i x, x) = A\|x\|^2.
\]
The other case is similar.

COROLLARY 3.4. If \( \{W_i\}_{i=1}^n \) satisfy one of the conditions in Proposition 3.2 then \( \{W_i\}_{i=1}^n \) does phase retrieval if and only if \( \{W_i^\perp\}_{i=1}^n \) does phase retrieval.

PROOF. This follows from Proposition 2.10 and Proposition 3.3.

COROLLARY 3.5. If \( \Phi = \{\phi_i\}_{i=1}^n \) is a scalable frame in \( \mathbb{R}^d \) then \( \Phi \) does phase retrieval if and only if \( \Phi^\perp \) does phase retrieval.

PROOF. If \( P_i \) is the projection onto span\(\{\phi_i\}\) then for any \( x \in \mathbb{R}^d \),
\[
\|\phi_i\|^2 P_i x = \langle x, \phi_i \rangle \phi_i.
\]
Since \( \Phi \) is scalable then there exist scalars \( \{s_i\}_{i=1}^n \) such that \( \{s_i\phi_i\}_{i=1}^n \) is an \( A \)-tight frame.

Therefore, for any \( x \in \mathbb{R}^d \),
\[
A x = \sum_{i=1}^n (x, s_i \phi_i) s_i \phi_i = \sum_{i=1}^n s_i^2 \|\phi_i\|^2 P_i x.
\]
The result follows by Corollary 3.4.

Now we will give examples to show that the Corollary 3.5 does not hold in general without the assumption the frame being scalable. First, let us examine the obvious approach to see why it fails in general. It is known that if \( \{\phi_i\}_{i=1}^n \) does phase retrieval and \( T \) is an invertible operator then \( \{T\phi_i\}_{i=1}^n \) does phase retrieval.

If \( \Phi = \{\phi_i\}_{i=1}^n \) is any frame with frame operator \( S \) which does phase retrieval, \( S^{-1/2} \Phi = \{S^{-1/2}\phi_i\}_{i=1}^n \) is a Parseval frame and so
\[
\left\{ \left( S^{-1/2}\phi_i \right)^\perp \right\}_{i=1}^n = \left\{ S^{1/2}\phi_i^\perp \right\}_{i=1}^n,
\]
does phase retrieval. So we would like to apply the invertible operator \( S^{-1/2} \) to our hyperplanes to conclude that \( \Phi^\perp \) does phase retrieval. The problem is that it is known [5] the invertible operators may not take subspaces doing phase retrieval to subspaces doing phase retrieval.

EXAMPLE 3.6. There is a frame \( \{\phi_i\}_{i=1}^5 \) in \( \mathbb{R}^3 \) which does phase retrieval but the hyperplanes \( \{\phi_i^\perp\}_{i=1}^5 \) fail phase retrieval.

PROOF. Let \( \phi_1 = (0, 0, 1), \phi_2 = (1, 0, 1), \phi_3 = (0, 1, 1), \phi_4 = (1, 1 - \sqrt{2}, 2), \phi_5 = (1, 1, 1) \).

Since \( \{\phi_i\}_{i=1}^5 \) is a full spark frame of 5 vectors in \( \mathbb{R}^3 \) then it does phase retrieval.
We have,

\[ W_1 = \{ \phi_1^\perp \} = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0 \} \]

\[ W_2 = \{ \phi_2^\perp \} = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_3 = 0 \} \]

\[ W_3 = \{ \phi_3^\perp \} = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 + x_3 = 0 \} \]

\[ W_4 = \{ \phi_4^\perp \} = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + (1 - \sqrt{2})x_2 + 2x_3 = 0 \} \]

\[ W_5 = \{ \phi_5^\perp \} = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \} \]

Let \( P_i \) be the orthogonal projection onto \( W_i \), then

\[ P_1(\phi_5) = (1, 1, 0) \]

\[ P_2(\phi_5) = (0, 1, 0) \]

\[ P_3(\phi_5) = (1, 0, 0) \]

\[ P_4(\phi_5) = (1/2, (1 + \sqrt{2})/2, 0) \]

\[ P_5(\phi_5) = (0, 0, 0) \]

Thus, \( \text{span}\{ P_i(\phi_5) \}_{i=1}^5 = W_1 \neq \mathbb{R}^3 \). By Theorem 3.1, \( \{ W_i \}_{i=1}^5 \) cannot do phase retrieval.

**Corollary 3.7.** There exists \( \{ \phi_i \}_{i=1}^5 \) in \( \mathbb{R}^3 \) which does phase retrieval but \( \{ \phi_i^\perp \}_{i=1}^5 \) cannot do norm retrieval.

Now we will generalize this example to all of \( \mathbb{R}^d \). This example looks like it came from nowhere, so we first explain why this is logical by *reverse engineering* the above example in \( \mathbb{R}^d \). We need a full spark set of unit vectors \( \{ \phi_i \}_{i=1}^{2d-1} \) (which therefore do phase retrieval on \( \mathbb{R}^d \)) with projections \( P_i \) onto \( \text{span}\{ \phi_i \} \), and a vector \( x \) so that \( \{(I - P_i)x \}_{i=1}^{2d-1} \) is contained in a hyperplane. So we decide in advance that the vector \( x \) will be \( x = (1, 1, \ldots, 1) \) and the hyperplane will be

\[ H = \{ (c_1, c_2, \ldots, c_{d-1}, 0) : c_i \in \mathbb{R} \} \]

Given a \( \phi = (a_1, a_2, \ldots, a_d) \) of this type, we have:

1. We have:

\[ \| \phi \|^2 = \sum_{i=1}^d a_i^2 = 1. \]

2. We have:

\[ (I - P_i)x = (1, 1, \ldots, 1) - \langle x, \phi \rangle \phi = (1, 1, \ldots, 1) - \left( \sum_{i=1}^d a_i \right) (a_1, a_2, \ldots, a_d). \]

Since this vector is to be in the hyperplane \( H \), we have:

\[ 1 = a_d \sum_{i=1}^d a_i. \]
Combining this with (1) implies:

\[ a_d = \frac{\sum_{i=1}^{d-1} a_i^2}{\sum_{i=1}^{d-1} a_i} \]

Now we can present the example:

**Example 3.8.** There are vectors \( \{ \phi_i \}_{i=1}^{2d-1} \) in \( \mathbb{R}^d \) which do phase retrieval but \( \{ \phi_i \}_{i=1}^{2d-1} \) does not do phase retrieval.

**Proof.** Consider the set

\[ A := \left\{ \left( a_1, a_2, \ldots, a_{d-1}, \frac{\sum_{i=1}^{d-1} a_i^2}{\sum_{i=1}^{d-1} a_i} \right) : a_i \in \mathbb{R}, \sum_{i=1}^{d-1} a_i \neq 0 \right\}. \]

Let \( x = (1, 1, \ldots, 1) \in \mathbb{R}^d \). Let any \( \phi \in A \) and denote \( P_\phi \) the orthogonal projection onto \( \text{span}\{\phi\} \). Then we have

\[ (I - P_\phi)(x) = x - \langle x, \frac{\phi}{\|\phi\|} \|\phi\| \rangle. \]

Denote \( b_d \) the \( d \)-coordinate of \((I - P_\phi)(x)\), then

\[
\begin{align*}
    b_d &= 1 - \frac{1}{\|\phi\|^2} \left( \sum_{i=1}^{d-1} a_i + \frac{\sum_{i=1}^{d-1} a_i^2}{\sum_{i=1}^{d-1} a_i} \sum_{i=1}^{d-1} a_i^2 \sum_{i=1}^{d-1} a_i \right) \\
    &= 1 - \frac{1}{\|\phi\|^2} \left( \sum_{i=1}^{d-1} a_i^2 + \left( \frac{\sum_{i=1}^{d-1} a_i^2}{\sum_{i=1}^{d-1} a_i} \right)^2 \right) = 0.
\end{align*}
\]

Let

\[
\begin{align*}
    \phi_1 &= (1, 0, \ldots, 0, 1) \\
    \phi_2 &= (0, 1, \ldots, 0, 1) \\
    &\vdots \\
    \phi_{d-1} &= (0, 0, \ldots, 1, 1) \\
    \phi_d &= x = (1, 1, \ldots, 1, 1).
\end{align*}
\]

Then \( \{ \phi_i \}_{i=1}^{d} \) is a linearly independent set in \( \mathbb{R}^d \) and \( \{ \phi_i \}_{i=1}^{d} \subset A \).

Now we will show that for any finite hyperplanes \( \{ W_i \}_{i=1}^{k} \) in \( \mathbb{R}^d \), there exists a vector \( \phi \in A \) such that \( \phi \notin \cup_{i=1}^{k} W_i \). Suppose by a contradiction that \( A \subset \cup_{i=1}^{k} W_i \).

Consider the set

\[ B := \left\{ \left( x, x^2, \ldots, x^{d-2}, 1 - \sum_{i=1}^{d-2} x^i, \sum_{i=1}^{d-2} x^{2i} + \left( 1 - \sum_{i=1}^{d-2} x^i \right)^2 \right) : x \in \mathbb{R} \right\}, \]

then \( B \subset A \).

Hence \( B \subset \cup_{i=1}^{k} W_i \). Therefore, there exists \( j \in \{ 1, \ldots, k \} \) such that \( W_j \) contains infinitely many vectors in \( B \).

Let \( u = (u_1, u_2, \ldots, u_d) \in W_j^\perp, u \neq 0 \). Then we have

\[ \langle u, \phi_x \rangle = 0 \]

for infinitely many \( \phi_x \in B \).
Thus,
\[ \sum_{i=1}^{d-2} u_i x^i + u_{d-1} \left( 1 - \sum_{i=1}^{d-2} x^i \right) + u_d \left( \sum_{i=1}^{d-2} x^{2i} + \left( 1 - \sum_{i=1}^{d-2} x^i \right)^2 \right) = 0, \]
for infinitely many \( x \).

This implies \( u_1 = u_2 = \cdots = u_d = 0 \), which is a contradiction.

From above, we can pick \( d - 1 \) vectors \( \{ \phi_i \}_{i=d+1}^{2d-1} \) in \( B \) such that \( \{ \phi_i \}_{i=1}^{2d-1} \) is a full spark of vectors in \( \mathbb{R}^d \). Thus, \( \{ \phi_i \}_{i=1}^{2d-1} \) does phase retrieval in \( \mathbb{R}^d \).

Moreover, since \( \text{span}(\{I - P_i(x)\}_{i=1}^{2d-1}) \neq \mathbb{R}^d \) then \( \{ \phi_i \}_{i=1}^{2d-1} \) cannot do phase retrieval by Theorem 3.1.

In general, if hyperplanes \( \{W_i\}_{i=1}^n \) do phase retrieval in \( \mathbb{R}^d \), it does not ensure that the complement vectors do phase retrieval. The following is an example.

**Example 3.9.** There are 5 vectors \( \{ \phi_i \}_{i=1}^5 \) in \( \mathbb{R}^3 \) which fail phase retrieval but their induced hyperplanes \( \phi_i^\perp \) do phase retrieval.

**Proof.** In \( \mathbb{R}^3 \), let \( W_1 = \text{span}\{e_2, e_3\} \), \( W_2 = \text{span}\{e_1, e_3\} \), \( W_3 = \text{span}\{e_1 + e_2, e_3\} \), \( W_4 = \text{span}\{e_1, e_2 + e_3\} \), \( W_5 = \text{span}\{e_2, e_1 + e_3\} \).

Let \( P_i \) be the projection onto \( W_i \). Then for any \( x = (x_1, x_2, x_3) \), we have
\[
\begin{align*}
P_1 x &= (0, x_2, x_3) \\
P_2 x &= (x_1, 0, x_3) \\
P_3 x &= \left( \frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3 \right) \\
P_4 x &= \left( x_1, \frac{x_2 + x_3}{2}, \frac{x_2 + x_3}{2} \right) \\
P_5 x &= \left( \frac{x_1 + x_3}{2}, x_2, \frac{x_1 + x_3}{2} \right).
\end{align*}
\]

For any \( x \neq 0 \), the rank of the matrix whose the rows are \( P_i x \) equals 3. Therefore, \( \{P_i x\}_{i=1}^5 \) spans \( \mathbb{R}^3 \). By Theorem 3.1, \( \{W_i\}_{i=1}^5 \) does phase retrieval in \( \mathbb{R}^3 \).

We also have \( W_1^\perp = \text{span}\{e_1\} \), \( W_2^\perp = \text{span}\{e_2\} \), \( W_3^\perp = \text{span}\{u_3\} \), \( W_4^\perp = \text{span}\{u_4\} \), \( W_5^\perp = \text{span}\{u_5\} \),

for some \( u_3, u_4, u_5 \in \mathbb{R}^3 \).

Since \( e_1, e_2, u_3 \perp e_3 \) then \( \text{span}\{e_1, e_2, u_3\} \neq \mathbb{R}^3 \). Thus, \( \{e_1, e_2, u_3, u_4, u_5\} \) fails the complement property. Therefore, it cannot do phase retrieval. \( \square \)
4. An Example in $\mathbb{R}^4$

In this section we will give an example of 6 hyperplanes in $\mathbb{R}^4$ which do phase retrieval. First, we will show that this is the minimal number of hyperplanes which can do phase retrieval.

**Theorem 4.1.** If hyperplanes $\{W_i\}_{i=1}^n$ do phase retrieval in $\mathbb{R}^d$ then $n \geq 2d-2$. Moreover, if $n = 2d-2$ then the vectors $\{W_i\}_{i=1}^{2d-2}$ are full spark.

**Proof.** Assume, by way of contradiction, that $n \leq 2d-3$. Choose a vector $0 \neq x \in \cap_{i=1}^{d-1} W_i$.

So $P_i x = x$ for all $i = 1, 2, \ldots, d-1$. It follows that the set $\{P_i x\}_{i=1}^n$ has at most $d-1$ non-zero vectors and hence cannot span $\mathbb{R}^d$, contradicting Theorem 3.1.

For the *moreover* part, we proceed by way of contradiction. Let $W_i = \text{span}\{\phi_i\}$ for $i = 1, 2, \ldots, 2d-2$ and assume there exists $I \subset \{2d-2\}$ with $|I| = d$ and $\{\phi_i\}_{i \in I}$ does not span the whole space. Choose $0 \neq x \perp \phi_i$ for all $i \in I$. It follows that $x \in W_i$ for all $i \in I$ and so $P_i x = x$ for all $i \in I$. But, $|I^c| = d-2$ and so $\{P_i x\}_{i=1}^{2d-2}$ contains at most $d-1$ distinct elements and so cannot span, contradicting Theorem 3.1.

Now we are ready for the main result of this section. In [20] it was shown that there are six 2-dimensional subspaces of $\mathbb{R}^4$ which do phase retrieval. We will now extend this result to hyperplanes in $\mathbb{R}^4$.

**Theorem 4.2.** Suppose that $d = 4$. There exist 6 hyperplanes $W_1, \ldots, W_6 \subset \mathbb{R}^4$ which do phase retrieval on $\mathbb{R}^4$.

**Proof.** Set $W_j := \phi_j^\perp \subset \mathbb{R}^4$, $j = 1, \ldots, 6$, where

$$
\phi_1 = (2, -1,2,2)/\sqrt{13}, \quad \phi_2 = (2, 5,4,1)/\sqrt{46}, \quad \phi_3 = (0, 4, -1, -1)/\sqrt{18},
$$

$$
\phi_4 = (5, 4, -2, -4)/\sqrt{61}, \quad \phi_5 = (4,1,5,3)/\sqrt{51}, \quad \phi_6 = (3, -4, -4, -3)/\sqrt{50}.
$$

Note that $||P_j x||^2 = \text{Tr}(A_j X)$ where $X = xx^T$, $A_j = u_j u_j^T + v_j v_j^T + \omega_j \omega_j^T$ and $u_j, v_j, \omega_j \in \mathbb{R}^3$ is an orthonormal basis of $W_j$. Then $W_1, \ldots, W_6$ do phase retrieval if and only if

$$Z := \{Q \in \mathbb{R}^4 : Q = Q^T, \text{rank}(Q) \leq 2, \text{Tr}(A_j Q) = 0, j = 1, \ldots, 6\}$$

only contains zero matrix. We write $Q$ in the form of

$$Q = \begin{pmatrix}
x_{11} & x_{12} & x_{13} & x_{14} \\
x_{12} & x_{22} & x_{23} & x_{24} \\
x_{13} & x_{23} & x_{33} & x_{34} \\
x_{14} & x_{24} & x_{34} & x_{44}
\end{pmatrix},$$

where $x_{jk}, 1 \leq j \leq k \leq 4$, are 10 variables. The rank($Q$) $\leq 2$ if and only if $m_{j,k} := m_{j,k}(x_{11}, x_{12}, \ldots, x_{44}) = 0$ where $m_{j,k}$ denotes the determinant of the submatrix formed by deleting the $j$th row and $k$th column from the matrix $Q$. Noting that $A_j = I - \phi_j \phi_j^T$, we have

$$\ell_j := \ell_j(x_{11}, \ldots, x_{44}) := \text{Tr}(A_j Q) = \text{Tr}(Q) - \phi_j^T Q \phi_j.$$
The \( Z \) only contain zero matrix if and only if the homogeneous polynomial system
\[(4.1) \quad \ell_1 = \cdots = \ell_6 = m_{1,1} = \cdots = m_{4,4} = 0\]
has no non-trivial (i.e., non-zero) real solutions.

We next verify the polynomial system (4.1) only has real zero solution following the ideas of Vinzant [18, Theorem 1]. Using the computer algebra software Maple, we compute a Gröbner basis of the ideal
\[\langle \ell_1, \ldots, \ell_6, m_{1,1}, \ldots, m_{4,4} \rangle\]
and elimination (see [17]). The result is a polynomial \( f_0 \in \mathbb{Q}[x_{34}, x_{44}] \), which is a homogeneous polynomial of degree 10:

\[
f_0 = 61583681469440125755941750205355957259806055430532979356877900x_{4,4}^{10} - 884972594452387958848562473144241797030697764519228205098183524x_{3,4}^7x_{4,4}^3 \]
\[+ 37549510562762689603032479610577980614684970115180508761212602923x_{4,4}^8 \]
\[- 26178428924525206834251115767386899800307703592293575845456869970x_{4,4}^7x_{4,4}^3 \]
\[+ 131864636101437420380559549371680153746208392291883996543590115151x_{4,4}^6 \]
\[+ 233672503729013471271218615418226060873143131038552226887257194x_{4,4}^5 \]
\[+ 8410996559220253999065870648349938392927420225588274968676284692x_{4,4}^4 \]
\[+ 24531184613874362427247644944973325638226272634695398509857315458204x_{4,4}^3 \]
\[+ 268670263536156020356201268066791183458247644588124478311966009776x_{4,4}^2 \]
\[+ 59872475066978406270800582425071592403273130463063552339405262912x_{4,4}x_{4,4}^3 \]
\[+ 9504840500329006177437929374383632917614227356173754905368787200x_{4,4}^4 \]

We can verify that the univariate polynomial \( f_0(1, x_{4,4}) \) has no real roots using Sturm sequence, and hence
\[
\{ (x_{3,4}, x_{4,4}) \in \mathbb{R}^2 : f_0(x_{3,4}, x_{4,4}) = 0 \} = \{ (0, 0) \},
\]
which implies that if \((x_{1,1}, x_{1,2}, \ldots, x_{3,4}, x_{4,4})\) is a real solution of (4.1) then \(x_{4,4} = 0\). By computing a Gröbner basis of the ideal, we obtain that
\[
1 \in \langle x_{3,4}, x_{4,4}, x_{j,k} - 1, \ell_1, \ldots, \ell_6, m_{1,1}, \ldots, m_{4,4} \rangle, \quad 1 \leq j \leq k \leq 4
\]
which means that (4.1) does not have nonzero real root with \(x_{4,4} = x_{4,4} = 0\). The maple code for these computation is posted at [http://lsec.cc.ac.cn/~xuzq/phase.htm](http://lsec.cc.ac.cn/~xuzq/phase.htm)

Combining results above, we obtain that (4.1) has no non-trivial real solutions.

\[\square\]

**Corollary 4.3.** There are six hyperplanes \( \{ W_i \}_{i=1}^6 \) doing phase retrieval on \( \mathbb{R}^4 \) but \( \{ W_i^\perp \}_{i=1}^6 \) does not do phase retrieval.

**References**

1. R. Balan, P.G. Casazza, and D. Edidin, *On Signal Reconstruction Without Phase*, Appl. Comput. Harmon. Anal., 20 (3) (2006) 345-356.
2. A.S. Bandeira, J. Cahill, D. Mixon and A.A. Nelson, *Saving phase: injectivity and stability for phase retrieval*, Appl. Comput. Harmon. Anal., 37 (1) (2014) 106-125.
3. R. H. Bates and D. Mnyama, *The status of practical Fourier phase retrieval*, Advances in Electronics and Electron Physics, 67 (1986), 1-64.
4. C. Becchetti and L. P. Ricotti, *Speech recognition theory and C++ implementation*, Wiley (1999).
5. B. Bodmann and N. Hammen, *Stable Phase retrieval with low redundancy frames*, Preprint. [arXiv:1302.5487]
6. S. Botelho-Andrade, P. G. Casazza, H. Van Nguyen, J. C. Tremain, *Phase retrieval versus phaseless reconstruction*, J. Math Anal. Appl., 436 (1), (2016) 131-137.
7. J. Cahill, P. G. Casazza, J. Jasper, and L.M. Woodland, *Phase retrieval and norm retrieval*, (2014). arXiv preprint [arXiv:1409.8260]
8. J. Cahill, P. Casazza, K. Peterson, L. Woodland, *Phase retrieval by projections*, Available online: [arXiv:1305.6926]
9. A. Conca, D. Edidin, M. Hering, and C. Vinzant, *An algebraic characterization of injectivity of phase retrieval*, Appl. Comput. Harmon. Anal., 38 (2) (2015) 346-356.
10. D. Edidin, *Projections and phase retrieval*, Appl. Comput. Harmon. Anal., 42 (2)(2017) 350-359.
11. J. Drenth, *Principles of protein x-ray crystallography*, Springer, 2010.
12. J. R. Fienup, *Reconstruction of an object from the modulus of its fourier transform*, Optics Letters, 3 (1978), 27-29.
13. J. R. Fienup, *Phase retrieval algorithms: A comparison*, Applied Optics, 21 (15) (1982), 2758-2768.
14. T. Heinosaaari, L. Maszaarella, and M.M. Wolf, *Quantum tomography under prior information*, Comm. Math. Phys. 318 No. 2 (2013) 355-374.
15. L. Rabiner, and B. H. Juang, *Fundamentals of speech recognition*, Prentice Hall Signal Processing Series (1993).
16. J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, *Symmetric Informationally Complete Quantum Measurements*, J. Math. Phys., 45 (2004), 2171-2180.
17. B. Sturmfels, What is ... a Gröbner basis?, Notices Amer. Math. Soc, 52(10) (2005) 1199-1200.
18. Cynthia Vinzant, A small frame and a certificate of its injectivity, Sampling Theory and Applications (SampTA) Conference Proceedings. (2015)197-200.
19. Yang Wang, Zhiqiang Xu, Generalized phase retrieval : measurement number, matrix recovery and beyond, Available online: [arXiv:1605.08933]
20. Z. Xu, The minimal measurement number for low-rank matrix recovery, Appl. Comput. Harmon. Anal. (2017), [http://dx.doi.org/10.1016/j.acha.2017.01.005].

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