Improvements and Generalizations of Two Hardy Type Inequalities and Their Applications to the Rellich Type Inequalities

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Abstract. We give improvements and generalizations of both the classical Hardy inequality and the geometric Hardy inequality based on the divergence theorem. Especially, our improved Hardy type inequality derives both two Hardy type inequalities with best constants. Besides, we improve two Rellich type inequalities by using the improved Hardy type inequality.

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1. Introduction

One dimensional Hardy type inequality

\[ \left( \frac{p-1-a}{p} \right)^p \int_0^\infty \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^p x^a \, dx \leq \int_0^\infty f^p(x) x^a \, dx \]

holds for all measurable nonnegative function \( f \), where \( p > 1 \) and \( a < p - 1 \) (Ref. [20, 21]). Concerning a history of (1), see [30]. On the other hand, there are mainly two inequalities which are higher dimensional cases of (1). One is the classical Hardy inequality with an interior singularity

\[ \left( \frac{N-\alpha}{p} \right)^p \int_{B_R} \frac{|u|^p}{|x|^\alpha} \, dx \leq \int_{B_R} \frac{|
abla u|^p}{|x|^{\alpha-p}} \, dx \]

for \( u \in C^1_c(B_R) \), where \( B_R \subset \mathbb{R}^N, N \geq 2, p > 1 \) and \( \alpha < N \). Especially, in the case where \( \alpha = p \), it is known that Hardy’s best constant \( \left( \frac{N-p}{p} \right)^p \) plays an important role to investigate several properties of solution to elliptic and parabolic partial
differential equations, for example, stability of solution, instantaneous blow-up solution, global-in-time solution, see \[2,9,40\], to name a few. The other is the geometric Hardy type inequality with a boundary singularity

\[
\left(\frac{\beta - 1}{p}\right)^p \int_{B_R} \frac{|u|^p}{\operatorname{dist}(x, \partial B_R)^\beta} dx \leq \int_{B_R} \frac{\nabla u|^p}{\operatorname{dist}(x, \partial B_R)^{\beta - p}} dx. \tag{3}
\]

for \( u \in C^1_c(B_R) \), where \( p, \beta > 1 \), and \( \operatorname{dist}(x, \partial B_R) = R - |x| \). A similar inequality holds for any bounded domain with Lipschitz boundary, see \[5,6,8\].

One of aims in the present paper is to combine two Hardy type inequalities (2), (3), namely, to give a Hardy type inequality which derives both two Hardy type inequalities (2), (3) with best constants. Our first result is as follows.

**Theorem 1.1.** (Improvements of (2) and (3)) Let \( p, \beta > 1, \alpha < N \), and \( \gamma \in (0, \frac{N - \alpha}{\beta - 1}] \). Then the inequality

\[
\left(\frac{\beta - 1}{p}\right)^p \int_{B_R} \frac{|u|^p}{|x|\alpha \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^\beta} dx \leq \int_{B_R} \frac{\nabla u \cdot \frac{x}{|x|}^p}{|x|\alpha - p \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^{\beta - p}} dx \tag{4}
\]

holds for all \( u \in C^1_c(B_R) \). Furthermore, the constant \( \left(\frac{\beta - 1}{p}\right)^p \) in (4) is optimal and is not attained for \( u \neq 0 \) for which the right-hand side is finite.

**Remark 1.2.** \((p = 1)\) In the case where \( p = 1 \), the inequality (4) still holds. However, the result for (non-)existence of extremal function is different from the case where \( p > 1 \). In fact, the best constant \( \beta - 1 \gamma \) of (4) with \( p = 1 \) is attained for any nonnegative functions with \( \nabla u(x) \cdot \frac{x}{|x|} \leq 0 \) for any \( x \in B_R \) when \( \gamma = \frac{N - \alpha}{\beta - 1} \), and is not attained for \( u \neq 0 \) when \( \gamma < \frac{N - \alpha}{\beta - 1} \). See the first part of the proof of Theorem 2.1.

We see that our inequality (4) derives several Hardy type inequalities with their best constants.

**Corollary 1.3.** (I) Let \( \beta = p > 1, \alpha < N \) and \( \gamma = \frac{N - \alpha}{p - 1} \). Then (4) derives (2) as follows.

\[
\left(\frac{N - \alpha}{p}\right)^p \int_{B_R} \frac{|u|^p}{|x|^\alpha dx} \leq \left(\frac{p - 1}{p}\right)^p \int_{B_R} \frac{|u|^p}{|x|\alpha \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^\beta} dx \leq \int_{B_R} \frac{\nabla u \cdot \frac{x}{|x|}^p}{|x|\alpha - p \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^{\beta - p}} dx
\]

(II) Let \( \beta > 1, \alpha = p \) and \( \gamma = 1 \leq \frac{N - p}{\beta - 1} \). Then (4) derives (3) as follows.

\[
\left(\frac{\beta - 1}{p}\right)^p \int_{B_R} \frac{|u|^p}{\operatorname{dist}(x, \partial B_R)^\beta} dx \\
\leq \left(\frac{\beta - 1}{p}\right)^p \int_{B_R} \frac{|u|^p}{|x|^p \left(1 - \left(\frac{|x|}{R}\right)\right)^\beta} dx \\
\leq \int_{B_R} \frac{\nabla u \cdot \frac{x}{|x|}^p}{\operatorname{dist}(x, \partial B_R)^{\beta - p} dx} = \int_{B_R} \frac{\nabla u \cdot \frac{x}{|x|}^p}{\operatorname{dist}(x, \partial B_R)^{\beta - p} dx}
\]
(III) Let $p = \alpha = \beta = \gamma = 2 \leq N - 2$. Then the following holds.

$$
\int_{B_1} \frac{|u|^2}{(1 - |x|^2)^2} \, dx \leq \int_{B_1} \frac{|u|^2}{|x|^2 (1 - |x|^2)^2} \, dx \leq \int_{B_1} \left| \nabla u \cdot \frac{x}{|x|} \right|^2 \, dx
$$

(IV) Letting $\gamma \to 0$, (4) derives the following Hardy type inequalities with logarithmic weights.

$$
\left( \frac{\beta - 1}{p} \right)^p \int_{B_R} \frac{|u|^p}{|x|^\alpha \left( \log \frac{R}{|x|} \right)^\beta} \, dx \leq \int_{B_R} \frac{\left| \nabla u \cdot \frac{x}{|x|} \right|^p}{|x|^\alpha \left( \log \frac{R}{|x|} \right)^\beta} \, dx. \tag{5}
$$

In terms of (III), it is known that the Hardy type inequality

$$
\int_{B_1} \frac{|u|^2}{(1 - |x|^2)^2} \, dx \leq \int_{B_1} |\nabla u|^2 \, dx \tag{6}
$$

is equivalent to the geometric Hardy type inequality with a boundary singularity in the half-space $\mathbb{R}^N_+$

$$
\frac{1}{4} \int_{\mathbb{R}^N_+} \frac{|u|^2}{x_N^2} \, dx \leq \int_{\mathbb{R}^N_+} |\nabla u|^2 \, dx
$$

via Möbius transformation. Also, Hardy-Sobolev-Maz′ya inequality which is the inequality (6) with Sobolev remainder term has been investigated, see e.g. [35] p.139, Corollary 3, [7,51]. In terms of (IV), by taking a limit of (4) as $\gamma \to 0$ and using $1 - r^\varepsilon = \varepsilon \log \frac{1}{r} + o(\varepsilon)$ as $\varepsilon \to 0$, we have

$$
\int_{B_R} \frac{|u|^p}{|x|^\alpha \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)} \, dx = \left( \frac{\beta - 1}{p} \right)^p \int_{B_R} \frac{|u|^p}{|x|^\alpha \left( \log \frac{R}{|x|} \right)^\beta} \, dx + o(\gamma^{p - \beta}),
$$

$$
\int_{B_R} \frac{\left| \nabla u \cdot \frac{x}{|x|} \right|^p}{|x|^\alpha \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)} \, dx = \gamma^{p - \beta} \int_{B_R} \frac{\left| \nabla u \cdot \frac{x}{|x|} \right|^p}{|x|^\alpha \left( \log \frac{R}{|x|} \right)^\beta} \, dx + o(\gamma^{p - \beta})
$$

as $\gamma \to 0$. Therefore, we can obtain (5) as a limiting form of (4) as $\gamma \to 0$. As for the inequalities (5), see e.g. [34,38,42,50].

In Sect. 2.1, we show Theorem 1.1 by using the divergence theorem. Here, we recall the following result. See also [13] Theorem 2.7.

**Theorem A.** ([14] Theorem 3.1) Let $\tilde{a} \in \mathbb{R}$, and let $\rho \in W^{1,p}_{\text{loc}}(\Omega)$ be a nonnegative function satisfying the following properties:

i) $-(p - 1 - \tilde{a}) \Delta_p \rho \geq 0$ on $\Omega$ in weak sense,

ii) $\left| \nabla \rho \right|^p \rho^{p - \tilde{a}}, \rho^{\tilde{a}} \in L^1_{\text{loc}}(\Omega)$. 

Then the following inequality holds
\[
\left(\frac{|p-1-\tilde{a}|}{p}\right)^p \int_{\Omega} \rho^{\tilde{a}} \frac{|
abla u|^p}{\rho^p} |u|^p \, dx \leq \int_{\Omega} \rho^{\tilde{a}} |\nabla u|^p \, dx, \quad u \in C_0^\infty(\Omega). \tag{7}
\]
Although Theorem A is also shown by using the divergence theorem, the inequality (7) does not coincide with the inequality (4). In fact, if we set
\[
\rho(x) = |x|^{-\frac{N-\alpha}{\alpha}} (1 - |x|^{-\gamma})^{-\frac{\beta}{\alpha}} \quad (\gamma > 0),
\]
then we have
\[
\rho^{\tilde{a}} \frac{|
abla u|^p}{\rho^p} = |\tilde{a}|^{-p} \frac{1}{|x|^\alpha (1 - |x|^\gamma)^\beta} |\alpha - p - \{\alpha - p + (\beta - p)\gamma\}| x|^\gamma|^p .
\]
If the left-hand sides of (4) and (7) are the same, it holds that
\[
\alpha - p + (\beta - p)\gamma = 0.
\]
However, for instance, if we set \(\alpha = p\), then \(\beta\) comes to \(p\) due to the above condition. Therefore, we cannot derive (4) with \(\alpha = p, \beta \neq p\) from (7). This difference comes from the difference in terms of how to use the divergence theorem for showing Hardy type inequalities.

In the special cases of the inequality (4), the expression of the best constant and the non-existence of extremals were already known in [14] for \(\alpha = p\) and \(\beta = p\), [25] for \(\beta = p\) and [17] for \(\gamma = \frac{N-\alpha}{\beta - 1}\). In the present paper, we show them for any \(\alpha, \beta\) and \(\gamma\) comprehensively. Furthermore, we study their remainder terms which are discussed in Theorems 2.1 and 2.2 in Sect. 2.1. We emphasize that these are truly new results in the literature. The authors in [17] used the divergence theorem while the author in [25] used the harmonic transplantation from \(B_1\) to \(\mathbb{R}^N\). In Sect. 2.2, we also give another proof of Theorem 1.1 by using the harmonic transplantation.

As we mention a little while ago, we can obtain different kinds of Hardy type inequalities from different ways of using the divergence theorem. In the present paper, we also give generalizations of two Hardy type inequalities (2), (3) by a different way of using the divergence theorem from Theorem 1.1.

**Theorem 1.4.** (Generalizations of (2) and (3)) Let \(p > 1\) and \(\alpha < N\).

(I) If \(\beta \geq 1\) and \(\gamma > 0\), then the inequality
\[
\left(\frac{N-\alpha}{p}\right)^p \int_{B_R} \frac{|u|^p}{|x|^{\alpha \gamma}} \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^{\beta - 1} \, dx \leq \int_{B_R} \frac{|\nabla u \cdot \frac{x}{|x|}|^p}{|x|^{\alpha - p} \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^{\beta - 1}} \, dx \tag{8}
\]
holds for all \(u \in C_c^1(B_R)\). Furthermore, the constant \(\left(\frac{N-\alpha}{p}\right)^p\) in (8) is optimal and is not attained for \(u \neq 0\) for which the right-hand side is finite.

(II) If \(\beta > 1\) and \(\gamma \in (0, \frac{N-\alpha}{p-1}]\), then the inequality
\[
\left(\frac{\beta - 1}{p} \right)^p \int_{B_R} \frac{|u|^p}{|x|^{\alpha - \gamma} \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^{\beta - 1}} \, dx \leq \int_{B_R} \frac{|\nabla u \cdot \frac{x}{|x|}|^p}{|x|^{\alpha - p} \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^{\beta - p}} \, dx \tag{9}
\]
holds for all \(u \in C_c^1(B_R)\). Furthermore, the constant \(\left(\frac{\beta - 1}{p} \right)^p\) in (9) is optimal and is not attained for \(u \neq 0\) for which the right-hand side is finite.
Remark 1.5. The inequality (8) with $\beta = 1$ coincides with the classical Hardy type inequality (2). On the other hand, the inequality (9) with $R = \alpha = \gamma = 1$ coincides with the geometric Hardy type inequality (3).

Remark 1.6. $(\gamma \to 0)$ If we take limits of (8) and (9) as $\gamma \to 0$, then we obtain
\[
\left(\frac{N - \alpha}{p}\right)^p \int_{B_R} \frac{|u|^p}{|x|^{\alpha}} \left(\log \frac{R}{|x|}\right)^\beta dx \leq \int_{B_R} \frac{\nabla u \cdot \frac{x}{|x|}}{|x|^{\alpha - p}} \left(\log \frac{R}{|x|}\right)^\beta dx,
\]
\[
\left(\frac{\beta - 1}{p}\right)^p \int_{B_R} \frac{|u|^p}{|x|^{\alpha}} \left(\log \frac{R}{|x|}\right)^\beta dx \leq \int_{B_R} \frac{\nabla u \cdot \frac{x}{|x|}}{|x|^{\alpha - p}} \left(\log \frac{R}{|x|}\right)^{\beta - p} dx.
\]

Remark 1.7. (Virtual extremals of three Hardy inequalities (4), (8), (9)) It is well-known that each Hardy type inequality has each virtual extremal, which attains the optimal constant formally, however, is not in the suitable functional space since both the integrals in the inequality diverge, see also [12,45] for distance type remainder terms from the virtual extremal. In our inequalities (4), (8), (9), each virtual extremal is as follows.

(4): \( |x|^{-\frac{\gamma - 1}{p}} = |x|^{-\frac{N - \alpha}{p}} (1 - |x|^{\gamma})^{-\frac{1}{p}} \) when $\gamma = \frac{N - \alpha}{\beta - 1}$,

(8): \( |x|^{-\frac{N - \alpha}{p}} \),

(9): \( (1 - |x|^{\gamma})^{-\frac{1}{p}} \).

We see that the virtual extremal of (4) is the multiplication of the virtual extremals of (8), (9). See each proof of Theorems and Remark 2.3.

We also study the improvements of two Rellich type inequalities in Sect. 3.

Theorem 1.1 with $p = \alpha = \beta = 2$ is closely related to the following minimization problem and the eigenvalue problem with a singular potential \( \frac{1}{|x|^2 (1 - |x|^{\gamma})^2} \)
\[
\left(\frac{\gamma}{2}\right)^2 = \inf_{u \in H^1_0(B_1) \setminus \{0\}} \frac{\int_{B_1} |\nabla u|^2 dx}{\int_{B_1} \frac{|u|^2}{|x|^2 (1 - |x|^{\gamma})^2} dx}, \quad \left\{ \begin{array}{ll} -\Delta u = \left(\frac{\gamma}{2}\right)^2 \frac{|u|^2}{|x|^2 (1 - |x|^{\gamma})^2} & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{array} \right.
\]

Theorem 1.1 implies that the above infimum is not attained, and a weak solution of (10) does not exist. Furthermore, the above infimum becomes the threshold of whether a solution of the heat equation with the above singular potential exists, see [10].

This paper is organized as follows: We show Theorem 1.1 in Sect. 2.1 and show Theorem 1.4 in Sect. 2.3. Both Theorems are proved based on the divergence theorem. In Sect. 2.2, we also give another proof of Theorem 1.1 based on transformation approach. We give a generalization of the harmonic transplantation from $B_1$ to $\mathbb{R}^N$ used in [25]. By using this transformation, we also study minimization problems associated with the improved Hardy-Sobolev type inequalities. In Sect. 3, we give an improvement of the Rellich type inequalities as an application of Theorem 1.1.
We consider the case where $p = 2$ for any functions in Sect. 3.1, and the case where $p \neq 2$ for radially symmetric functions in Sect. 3.2. We also give three conjectures for general cases in the end of Sect. 3. These conjectures include the conjecture in [3] p.879. In Sect. 4, we give one dimensional inequalities and calculations to show Theorems.

We fix several notations: $X_{\text{rad}} = \{ u \in X \mid u \text{ is radially symmetric} \}$. $B_R$ denotes a $N$-dimensional ball centered 0 with radius $R$. $\omega_{N-1}$ denotes an area of the unit sphere $S^{N-1}$ in $\mathbb{R}^N$.

2. First Order Inequalities

2.1. The Divergence Theorem and Improvements of Two Hardy Type Inequalities: Proof of Theorem 1.1

We show the following two Theorems. Both Theorems are shown by the divergence theorem and imply Theorem 1.1.

**Theorem 2.1.** Let $p, \beta > 1, \alpha < N$ and $\gamma \in (0, \frac{N-\alpha}{\beta-1}]$. Then the inequality

$$
\left( \frac{\beta - 1}{p} \gamma \right)^p \int_{B_R} \frac{|u|^p}{|x|^\alpha \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^\beta} \, dx + \psi_{N,p,\alpha,\beta}(u) 
$$

$$
\leq \int_{B_R} \frac{|\nabla u \cdot \frac{x}{|x|}|^p}{|x|^\alpha \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^{\beta-\gamma}} \, dx
$$

holds for all $u \in C^1_c(B_R)$, where $C > 0$ depends on $p$ and $N$;

$$
\psi_{N,p,\alpha,\beta}(u) = (N - \alpha - (\beta - 1)\gamma) \left( \frac{\beta - 1}{p} \gamma \right)^{p-1} \int_{B_R} \frac{|u|^p}{|x|^\alpha \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^{\beta-1}} \, dx
$$

for $\gamma < \frac{N-\alpha}{\beta-1}$ and

$$
\psi_{N,p,\alpha,\beta}(u)
$$

$$
= \begin{cases}
C \int_{B_R} |x|^{-N} \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^{p-1} \left| \nabla \left( \frac{u(x)}{\left( \left( \frac{|x|}{R} \right)^\gamma - 1 \right)^{\frac{p-1}{p}}} \right) \right|^p \, dx & \text{if } p \in [2, \infty), \\
C \left( \int_{B_R} |x|^{-N} \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^{p-1} \left| \nabla \left( \frac{u(x)}{\left( \left( \frac{|x|}{R} \right)^\gamma - 1 \right)^{\frac{p-1}{p}}} \right) \right|^p \, dx \right)^{\frac{2}{p-2}} & \text{if } p \in (1, 2)
\end{cases}
$$

for $\gamma = \frac{N-\alpha}{\beta-1}$. 

Theorem 2.2. Let $p, \beta > 1, \alpha < N$ and $\gamma \in \left(0, \frac{N-\alpha}{\beta-1}\right]$. Then the inequality

$$
\left(\frac{\beta - 1}{p}\right)^p \int_{B_R} \frac{|u|^p}{|x|\left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^\beta} dx = \int_{B_R} \frac{|\nabla u \cdot \frac{x}{|x|}|^p}{|x|\left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^{\beta-p}} dx
$$

$$
- p \left(\frac{\beta - 1}{p}\right)^p \int_{B_R} R_p \left[ u, \frac{p}{(\beta-1)\gamma} \left( -\nabla u \cdot \frac{x}{|x|} \right) \left|\nabla u \cdot \frac{x}{|x|}\right|^{-1} \right] dx
$$

$$
\frac{1}{|x|^{\alpha} \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^\beta}
$$

$$
- (N - \alpha - (\beta - 1)\gamma) \left(\frac{\beta - 1}{p}\right)^{p-1} \int_{B_R} \frac{|u|^p}{|x|^{\alpha} \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^{\beta-1}} dx
$$

(12)

holds for all $u \in C^1_c(B_R)$, where

$$
R_p(\xi, \eta) = \frac{1}{p} |\eta|^p + \frac{p-1}{p} |\xi|^p - |\xi|^{p-2} \xi \eta
$$

$$
= (p-1) \int_0^1 |t \xi + (1-t) \eta|^{p-2} dt |\xi - \eta|^2 \geq 0.
$$

Remark 2.3. Note that $R_p(\xi, \eta) = 0$ if and only if $\xi = \eta$. Let $R = 1$ for simplicity. If $R_p = 0$ in (12) for some function $u = u(x) = u(r \omega)$ ($r = |x|, \omega \in S^{N-1}$), then for any $r \in (0, 1), \omega \in S^{N-1}$ we have

$$
- \frac{\partial u}{\partial r}(r \omega) = \frac{(\beta - 1)\gamma}{p} \frac{u(r \omega)}{r (1-r\gamma)}.
$$

Here, for fixed $\omega \in S^{N-1}$, we set $g(r) = u(r \omega)$. Then $g$ satisfies the following ODE:

$$
-g'(r) = \frac{(\beta - 1)\gamma}{p} \frac{g(r)}{r (1-r\gamma)}, \quad g(1) = 0
$$

We can solve it by separation of variables as follows.

$$
g(r) = C_\omega (r^{-\gamma} - 1)^{\frac{\beta-1}{p}} \quad (C_\omega \in \mathbb{R})
$$

This means that $u(x) = f\left(\frac{x}{|x|}\right) \left(|x|^{-\gamma} - 1\right)^{\frac{\beta-1}{p}}$ for some function $f : S^{N-1} \to \mathbb{R}$.

Proof of Theorem 2.1. For the simplicity, we set $R = 1$. Note that

$$
\text{div}\left(\frac{x}{(|x|^{-\gamma} - 1)^{\beta-1}}\right) = \frac{N |x|^{(\beta-1)\gamma}}{(1 - |x|\gamma)^{\beta-1}} + \frac{(\beta - 1)\gamma}{(1 - |x|\gamma)^{\beta}}.
$$

Set $A_\varepsilon = B_1 \setminus B_\varepsilon$ for small $\varepsilon > 0$. Then we have

$$
(\beta - 1)\gamma \int_{A_\varepsilon} \frac{|u|^p}{|x|^{\alpha} (1 - |x|\gamma)^\beta} dx
$$

$$
= \int_{A_\varepsilon} \text{div}\left(\frac{x}{(|x|^{-\gamma} - 1)^{\beta-1}}\right) \frac{|u|^p}{|x|^{\alpha+(\beta-1)\gamma}} - N \frac{|u|^p}{|x|^{\alpha} (1 - |x|\gamma)^{\beta-1}} dx
$$
\begin{align*}
= -p \int_{A_\varepsilon} |u|^{p-2} u (\nabla u \cdot x) \left| \frac{x}{|x|} \right|^{\beta-1} \frac{|x|^\alpha}{|x^\alpha (1 - |x|\gamma)^{\beta-1}} \, dx - (N - \alpha - (\beta - 1)\gamma) \frac{|u|^p}{|x|^{\alpha (1 - |x|\gamma)^{\beta-1}}} \, dx \\
+ \int_{\partial B_\varepsilon} |u|^p \frac{x \cdot (-x)}{|x|^{\alpha-1}(1 - |x|\gamma)^{\beta-1}} \, dS_x \\
\leq p \left( \int_{A_\varepsilon} \frac{|\nabla u \cdot x|}{|x|^{\alpha-p}} \right)^{\frac{1}{p}} \left( \int_{A_\varepsilon} \frac{|u|^p}{|x|^{\alpha (1 - |x|\gamma)^{\beta}}} \, dx \right)^{\frac{1}{p}} \\
- (N - \alpha - (\beta - 1)\gamma) \int_{A_\varepsilon} \frac{|u|^p}{|x|^{\alpha (1 - |x|\gamma)^{\beta-1}}} \, dx
\end{align*}
which implies that for any \( u \neq 0 \)
\begin{align*}
\frac{\beta - 1}{p} \gamma \left( \int_{B_1} \frac{|u|^p}{|x|^{\alpha (1 - |x|\gamma)^{\beta}}} \, dx \right)^{\frac{1}{p}} \\
\leq \left( \int_{B_1} \frac{|\nabla u \cdot \frac{x}{|x|}|^p}{|x|^{\alpha-p (1 - |x|\gamma)^{\beta-p}}} \, dx \right)^{\frac{1}{p}} \\
- \frac{(N - \alpha - (\beta - 1)\gamma)}{p} \int_{B_1} \frac{|u|^p}{|x|^{\alpha (1 - |x|\gamma)^{\beta}}} \, dx \int_{B_1} \frac{|u|^p}{|x|^{\alpha (1 - |x|\gamma)^{\beta}}} \, dx \right)^{\frac{p-1}{p}}.
\end{align*}
by letting \( \varepsilon \to 0 \). Therefore we obtain the inequality (4). Set
\begin{align*}
A &= \left( \int_{B_1} \frac{|\nabla u \cdot \frac{x}{|x|}|^p}{|x|^{\alpha-p (1 - |x|\gamma)^{\beta-p}}} \, dx \right)^{\frac{1}{p}}, \\
B &= \frac{(N - \alpha - (\beta - 1)\gamma)}{p} \int_{B_1} \frac{|u|^p}{|x|^{\alpha (1 - |x|\gamma)^{\beta}}} \, dx \int_{B_1} \frac{|u|^p}{|x|^{\alpha (1 - |x|\gamma)^{\beta}}} \, dx \right)^{\frac{p-1}{p}}.
\end{align*}
By the fundamental inequality \((A - B)^p \leq A^p - p(A - B)^{p-1}B (A \geq B)\) and the inequality \( A - B \geq \frac{\beta-1}{p} \gamma \left( \int_{B_1} \frac{|u|^p}{|x|^{\alpha (1 - |x|\gamma)^{\beta}}} \, dx \right)^{\frac{1}{p}} \) from (13), we have
\begin{align*}
\left( \frac{\beta - 1}{p} \gamma \right)^p \int_{B_1} \frac{|u|^p}{|x|^{\alpha (1 - |x|\gamma)^{\beta}}} \, dx \leq A^p - p(A - B)^{p-1}B \\
\leq \int_{B_1} \frac{|\nabla u \cdot \frac{x}{|x|}|^p}{|x|^{\alpha-p (1 - |x|\gamma)^{\beta-p}}} \, dx - \psi_{N,p,\alpha,\beta}(u)
\end{align*}
which implies (11) for \( \gamma < \frac{N - \alpha}{\beta - 1} \). Next, we assume that \( \gamma = \frac{N - \alpha}{\beta - 1} \). For \( u \in C^1_c(B_1) \), set
\[ v(x) = u(x) \left( |x|^{-\gamma} - 1 \right)^{-\frac{\beta-1}{p}}, \]
\[ J(u) = \int_{B_1} \frac{|\nabla u \cdot \frac{x}{|x|}|^p}{|x|^{\alpha-p} (1 - |x|^{-\gamma})^{\beta-p}} dx - \left( \frac{\beta - 1}{p} \right) \gamma \int_{B_1} \frac{|u|^p}{|x|^{\alpha} (1 - |x|^{-\gamma})^\beta} dx. \]

Then we have

\[ \nabla u \cdot \frac{x}{|x|} = B - A, \text{ where } A = \frac{\beta - 1}{p} \gamma v |x|^{-\gamma-1} (|x|^{-\gamma} - 1)^{\frac{\beta-1}{p}-1}, \]

\[ B = (|x|^{-\gamma} - 1)^{\frac{\beta-1}{p}} \left( \nabla v \cdot \frac{x}{|x|} \right). \]

By the inequality

\[ |a - b|^p - |a|^p + p|a|^{p-2}ab \geq \begin{cases} C_1 |b|^p \quad \text{if } p \in [2, \infty), \\ C_2 \frac{|b|^2}{(|a-b|+|a|)^{2-p}} \quad \text{if } p \in (1, 2) \end{cases} \]

for some \( C_1, C_2 > 0 \) and for any \( a, b \in \mathbb{R} \) (See e.g. [32]), we have

\[ J(u) \geq \int_{B_1} \frac{|A|^p - |A|^{p-2}AB + C_1 |B|^p}{|x|^{\alpha-p} (1 - |x|^{-\gamma})^{\beta-p}} dx - \left( \frac{\beta - 1}{p} \right) \gamma \int_{B_1} \frac{|v|^p}{|x|^{N} (1 - |x|^{-\gamma})} dx \]

\[ = \left( \frac{\beta - 1}{p} \right)^{-1} \int_{B_1} \frac{\nabla (|v|^p) \cdot \frac{x}{|x|}}{|x|^{N-1}} dx + C_1 \int_{B_1} |x|^{N-p} (1 - |x|^{-\gamma})^{p-1} \left| \nabla v \cdot \frac{x}{|x|} \right|^p dx \]

for \( p \in [2, \infty) \). Since

\[ \int_{B_1} \frac{\nabla (|v|^p) \cdot \frac{x}{|x|}}{|x|^{N-1}} dx = \int_{S^{N-1}} \int_0^1 \frac{\partial}{\partial r} \left( |v|^p \right) dr dS_\omega = 0, \]

we have (11) for \( p \in [2, \infty) \) and \( \gamma = \frac{N - \alpha}{\beta - 1} \). On the other hand, we have

\[ J(u) \geq C_2 \int_{B_1} \frac{|B|^2}{|x|^{\alpha-p} (1 - |x|^{-\gamma})^{\beta-p} (|A - B| + |A|)^{2-p}} dx \]

\[ = C_2 \int_{B_1} \left( \left| \nabla u \cdot \frac{x}{|x|} \right| + \frac{\beta - 1}{p} \gamma |v||x|^{-1 - \frac{N - \alpha}{\beta - 1}} (1 - |x|^{-\gamma})^{-\frac{\beta-1}{p}-1} \right)^2 \frac{|\nabla v \cdot \frac{x}{|x|}|^2}{|x|^{\alpha-p} (1 - |x|^{-\gamma})^{\beta-p}} dx \quad (14) \]

for \( p \in (1, 2) \). In the same way as the proof of Theorem 1 in [26], the Hölder inequality and (14) imply

\[ \int_{B_1} |x|^{p-N} (1 - |x|^{-\gamma})^{p-1} \left| \nabla v \cdot \frac{x}{|x|} \right|^p dx \]

\[ \leq \left( \int_{B_1} \left( \left| \nabla u \cdot \frac{x}{|x|} \right| + \frac{\beta - 1}{p} \gamma |v||x|^{-1 - \frac{N - \alpha}{\beta - 1}} (1 - |x|^{-\gamma})^{-\frac{\beta-1}{p}-1} \right)^2 \frac{|\nabla v \cdot \frac{x}{|x|}|^2}{|x|^{\alpha-p} (1 - |x|^{-\gamma})^{\beta-p}} dx \right)^{\frac{p}{2}} \]

\[ \times \left( \int_{B_1} \left| \nabla u \cdot \frac{x}{|x|} \right| + \frac{\beta - 1}{p} \gamma |v||x|^{-1 - \frac{N - \alpha}{\beta - 1}} (1 - |x|^{-\gamma})^{-\frac{\beta-1}{p}-1} \right)^{p} \frac{|x|^{p-N} (1 - |x|^{-\gamma})^{p-1} dx}{|x|^{\alpha-p} (1 - |x|^{-\gamma})^{\beta-p}} \right)^{\frac{2-p}{2}} \]

\[ \leq \left( \frac{J(u)}{C_2} \right)^{\frac{p}{2}} 2^{\frac{(2-p)(2-p)}{2}} \left( \int_{B_1} \left| \nabla u \cdot \frac{x}{|x|} \right|^p \right)^{\frac{2-p}{2}} \frac{\frac{p}{2}}{|x|^{\alpha-p} (1 - |x|^{-\gamma})^{\beta-p}} \right)^{\frac{2-p}{2}}. \]

Therefore we have (11) for \( p \in (1, 2) \) and \( \gamma = \frac{N - \alpha}{\beta - 1} \). \( \square \)
Proof of Theorem 2.2. We refer [28,29]. For the simplicity, we set $R = 1$. Note that
\[
\frac{d}{dr} \left[ (r^{-\gamma} - 1)^{-\beta+1} \right] = (\beta - 1) \gamma r^{-1+(\beta-1)\gamma} (1 - r^\gamma)^{-\beta}
\]
Then we have
\[
\int_{B_1} \frac{|u|^p}{|x|^{\alpha} (1 - |x|^\gamma)^\beta} \, dx \\
= \int_0^1 r^{N-\alpha-1}(1 - r^\gamma)^{-\beta} \int_{S_{N-1}} |u(r\omega)|^p \, drdS_\omega \\
= \frac{1}{(\beta - 1)\gamma} \int_0^1 \int_{S_{N-1}} |u|^{pN-\alpha-(\beta-1)\gamma} \frac{d}{dr} \left[ (r^{-\gamma} - 1)^{-\beta+1} \right] \, drdS_\omega \\
= \frac{p}{(\beta - 1)\gamma} \int |u|^{p-2} u \left( -\frac{\partial u}{\partial r} \right) r^{N-\alpha}(1 - r^\gamma)^{-\beta+1} \, drdS_\omega \\
- \frac{N - \alpha - (\beta - 1)\gamma}{(\beta - 1)\gamma} \int |u|^{pN-\alpha-1}(1 - r^\gamma)^{-\beta+1} \, drdS_\omega \\
= \int \int |\xi|^{p-2} \xi \eta \, drdS_\omega - \frac{N - \alpha - (\beta - 1)\gamma}{(\beta - 1)\gamma} \int_{B_1} \frac{|u|^p}{|x|^{\alpha} (1 - |x|^\gamma)^{\beta-1}} \, dx,
\]
where
\[
\eta = \frac{p}{(\beta - 1)\gamma} \left( -\frac{\partial u}{\partial r} (r\omega) \right) r^{N-\alpha+p-1}(1 - r^\gamma)^{-\beta+p} , \quad \xi = u(r\omega) r^{N-\alpha-1}(1 - r^\gamma)^{-\beta}. 
\]
Since $|\xi|^{p-2} \xi \eta = \frac{1}{p} |\eta|^p + \frac{p-1}{p} |\xi|^p - R_p(\xi, \eta)$, we have
\[
\int_{B_1} \frac{|u|^p}{|x|^{\alpha} (1 - |x|^\gamma)^\beta} \, dx = \frac{1}{p} \left( \frac{p}{(\beta - 1)\gamma} \right)^p \int \left| \frac{\partial u}{\partial r} \right|^p r^{N-\alpha+p-1}(1 - r^\gamma)^{-\beta+p} \, drdS_\omega \\
+ \frac{p-1}{p} \int |u|^{pN-\alpha-1}(1 - r^\gamma)^{-\beta} \, drdS_\omega \\
- \int R_p(\xi, \eta) \, drdS_\omega \\
- \frac{N - \alpha - (\beta - 1)\gamma}{(\beta - 1)\gamma} \int_{B_1} \frac{|u|^p}{|x|^{\alpha} (1 - |x|^\gamma)^{\beta-1}} \, dx
\]
which implies (12).

Here, we show Theorem 1.1 by using Theorem 2.1.

Proof of Theorem 1.1. Let $\gamma \leq \frac{N-\alpha}{\beta-1}$. For the simplicity, we set $R = 1$. We show the optimality of the constant $\left( \frac{\beta-1}{p} \gamma \right)^p$ in (4). For $A > \frac{\beta-1}{p}$ and small $\delta > 0$, set
\[
f_A(x) = \phi_\delta(x) (1 - |x|^\gamma)^A,
\]
where $\phi_\delta$ is a smooth radially symmetric function which satisfies $0 \leq \phi_\delta \leq 1, \phi_\delta \equiv 0$ on $B_{1-2\delta}$ and $\phi_\delta \equiv 1$ on $B_1 \setminus B_{1-\delta}$. Then we have
\[
\left( \frac{\beta-1}{p} \gamma \right)^p \leq \frac{\int_{B_1} \frac{|\nabla f_A|^p}{|x|^{\alpha+p-1} (1 - |x|^\gamma)^{p-p}} \, dx}{\int_{B_1} \frac{|f_A|^p}{|x|^{\alpha} (1 - |x|^\gamma)^{p}} \, dx}
\]
\[ \begin{align*}
&\leq \frac{(A\gamma)^p \int_{1-\delta}^1 (1-r^{\gamma})^{4p-\beta}r^{N-1-\alpha+p}dr + \int_{2-\delta}^1 |(f_A)'|p (1-r^{\gamma})^{4p-\beta}r^{N-1-\alpha+p}dr}{\int_{1-\delta}^1 (1-r^{\gamma})^{4p-\beta}r^{N-1-\alpha}dr} \\
&= \left(\frac{\beta-1}{\gamma}p\right)^p + o(1) \left( A \to \frac{\beta-1}{p} \right).
\end{align*} \]

Therefore the constant \( \left(\frac{\beta-1}{\gamma}p\right)^p \) in (4) is optimal. Since there exists the nonnegative remainder term in Theorem 2.1 when \( \gamma \leq \frac{N-\alpha}{\beta-1} \), we observe that if there exists an extremal function \( U = U(x) \) of the inequality (4), then \( U(x) = c (|x|^{-\gamma} - 1)^{\frac{\beta-1}{p}} = c|x|^{-\frac{\beta-1}{p} \gamma} (1 - |x|^{\gamma})^{\frac{\beta-1}{p}} \) for some \( c \in \mathbb{R} \). However, if \( c \neq 0 \), then the right-hand side of (4) diverges since

\[ \int_{B_1 \setminus B_{1-\varepsilon}} \frac{|\nabla U \cdot \frac{x}{|x|}|^p}{|x|^{\alpha-p} (1 - |x|^{\gamma})^{\beta-p}} dx \geq C(\varepsilon) \int_{B_1 \setminus B_{1-\varepsilon}} \frac{|\nabla (1 - |x|^{\gamma})^{\frac{\beta-1}{p}}|^p}{(1 - |x|^{\gamma})^{\beta-p}} dx + D(\varepsilon)
\]

\[ \geq \tilde{C}(\varepsilon) \int_{B_1 \setminus B_{1-\varepsilon}} (1 - |x|^{\gamma})^{-1} dx + D(\varepsilon) = \infty \]

for any small \( \varepsilon > 0 \), where \( C(\varepsilon) \) and \( \tilde{C}(\varepsilon) \) are positive constants depending on \( \varepsilon \) and \( D(\varepsilon) \) is a constant depending on \( \varepsilon \). The proof of Theorem 1.1 is now complete. \( \square \)

2.2. Transformation Approach and an Improved Hardy–Sobolev Type Inequality

First, we show the inequalities (4), (5) via the following transformation which is a generalization of harmonic transplantation proposed by Hersch [22], see also [1,18]. Concerning a summary of harmonic transplantation, see Sect. 3 in [49]. Consider

\[ u(x) = v(y) = w(z), \quad \text{where} \quad (|x|^{-\gamma} - R^{-\gamma}) \frac{x}{|x|} = |y|^{-\gamma} \frac{y}{|y|} = \left( \log \frac{R}{|z|} \right) \frac{z}{|z|} \] (15)

and set \( \gamma = \frac{N-\alpha}{\beta-1} \). Then we see that

\[ \int_{B_R} \frac{|\nabla u \cdot \frac{x}{|x|}|^p}{|x|^{\alpha-p} \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^{\beta-p}} dx = \int_{\mathbb{R}^N} \frac{|\nabla v \cdot \frac{y}{|y|}|^p}{|y|^{\alpha-p}} dy \]

\[ = \gamma^{-p-1} \int_{B_R} \frac{|\nabla w \cdot \frac{z}{|z|}|^p}{|z|^{N-p} \left( \log \frac{R}{|z|} \right)^{\beta-p}} dz, \]

\[ \int_{B_R} \frac{|u|^p}{|x|^{\alpha} \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^{\beta}} dx = \int_{\mathbb{R}^N} \frac{|v|^p}{|y|^\alpha} dy = \gamma^{-1} \int_{B_R} \frac{|w|^p}{|z|^{N} \left( \log \frac{R}{|z|} \right)^{\gamma}} dz. \]

Therefore the inequality (2) on \( \mathbb{R}^N \) for \( v \) is equivalent to the inequality (4) with \( \gamma = \frac{N-\alpha}{\beta-1} \) for \( u \) and the inequality (5) with \( \alpha = N \) for \( w \). Moreover, since the inequality (2) on \( \mathbb{R}^N \) for \( v \) is invariant under the usual scaling \( v_\lambda(y) = \lambda^{\frac{N-\alpha}{p}} v(\tilde{y}) \), where \( \tilde{y} = \lambda y \) and \( \lambda > 0 \), we obtain scale invariance structures of (4) and (5) thanks to the transformations (15) as follows.
Proposition 2.4. The inequality (4) with \( \gamma = \frac{N-\alpha}{\beta-1} \) for \( u \) is invariant under the scaling \( u_\lambda(x) = \lambda^{\frac{N-\alpha}{\beta-1}} u(\bar{x}), \) where \( \bar{x} = \lambda x \left[ 1 - (1 - \lambda^\gamma) \left( \frac{|x|}{R} \right)^{\gamma} \right]^{-\frac{1}{\gamma}} \) and \( \lambda > 0. \) On the other hand, the inequality (5) with \( \alpha = N \) for \( w \) is invariant under the scaling \( w_\mu(z) = \mu^{-\frac{\beta-1}{\gamma}} w(\bar{z}), \) where \( \bar{z} = \left( \frac{|z|}{R} \right)^{\mu-1} z \) and \( \mu = \lambda^{-\gamma} > 0. \)

Remark 2.5. In the case where \( \gamma < \frac{N-\alpha}{\beta-1}, \) there is no scale invariance structure of (4) with respect to the scaling \( u_\lambda(r\omega) = \lambda^A u(s\omega) \) for \( \lambda > 0, \) so \( s = s(r, \lambda), \) \( |x| = r, |\bar{x}| = s, \frac{r}{|x|} = \frac{s}{|\bar{x}|} = \omega \) and \( \frac{\partial s}{\partial r}(r, \lambda) > 0, s(0, \lambda) = 0, s(R, \lambda) = R. \) In fact, assume that the inequality (4) is invariant, namely, for any \( \lambda > 0 \)

\[
\int_{B_R} \frac{|\nabla u_\lambda \cdot \frac{x}{|x|}|^p}{|x|^{\alpha-p} \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^{\beta-p}} dx = \int_{B_R} \frac{|\nabla u \cdot \frac{\bar{x}}{|\bar{x}|}|^p}{|\bar{x}|^{\alpha-p} \left( 1 - \left( \frac{|\bar{x}|}{R} \right)^\gamma \right)^{\beta-p}} d\bar{x}, \tag{16}
\]

\[
\int_{B_1} \frac{|\nabla u \cdot \frac{x}{|x|}|^p}{|x|^{\alpha-p} \left( 1 - |x|^\gamma \right)^{\beta-p}} dx = \int_{S^{N-1}} \int_0^1 \left| \frac{\partial u}{\partial s} \right|^p s^{N-1-\alpha+p}(1-s^\gamma)^{-\beta+p} ds dS, \tag{17}
\]

For the simplicity, we set \( R = 1. \) Since

\[
\int_{B_1} \frac{|\nabla u \cdot \frac{x}{|x|}|^p}{|x|^{\alpha-p} \left( 1 - |x|^\gamma \right)^{\beta-p}} d\bar{x} = \int_{S^{N-1}} \int_0^1 \left| \frac{\partial u}{\partial s} \right|^p s^{N-1-\alpha+p}(1-s^\gamma)^{-\beta+p} ds dS, \tag{16}
\]

\[
\lambda^{Ap} \left( \frac{\partial s}{\partial r} \right)^{p-1} \left( \frac{\partial s}{\partial r} \right)^{-\beta} = \left( \frac{s(1-s^\gamma)}{r(1-r^\gamma)} \right)^p. \tag{18}
\]

In the same way as above, we have

\[
\frac{\partial s}{\partial r} = \lambda^{Ap} \left( \frac{r^{N-1-\alpha}(1-r^\gamma)^{-\beta}}{s^{N-1-\alpha}(1-s^\gamma)^{-\beta}} \right) \tag{19}
\]

from (17). By (18) and (19), we have

\[
s^{N-\alpha}(1-s^\gamma)^{1-\beta} = \lambda^{Ap} r^{N-\alpha}(1-r^\gamma)^{1-\beta}. \tag{19}
\]

If we differentiate the above equality, then

\[
\frac{\partial s}{\partial r} = \lambda^{Ap} \left[ \frac{N-\alpha-(\beta-1)\gamma}{N-\alpha} \right] \frac{r^{\gamma}}{s^{\gamma}}. \tag{19}
\]

Therefore, we have \( \gamma = \frac{N-\alpha}{\beta-1} \) by comparing it with (19).

In the same way as above, we can also show that the inequality (5) with \( \alpha < N \) is not invariant under the scaling \( u_\lambda(r\omega) = \lambda^A u(s\omega). \)
The following minimization problem associated with the Hardy-Sobolev type inequality is well-known.

**Theorem B.** (Ref. [23] Lemma 3.1 or [24]) Let \( p > 1, N \geq 2, \) and \( W^{1,p}_{A,B,\text{rad}}(\mathbb{R}^N) \) be given by [23, Sect. 2]. Assume that \( p, q, N, A, \) and \( B \) satisfy

\[
(1 - A + B)p < N, 0 < \frac{1}{p} - \frac{1}{q} = \frac{1 - A + B}{N}, -\frac{N}{q} < B. \tag{20}
\]

Under these assumptions we set

\[
S_{\text{rad}} = \inf \left\{ \int_{\mathbb{R}^N} |\nabla v|^p |y|^A \, dx \mid v \in W^{1,p}_{A,B,\text{rad}}(\mathbb{R}^N), \int_{\mathbb{R}^N} |v|^q |y|^B \, dx = 1 \right\}.
\]

Then

\[
S_{\text{rad}} = \pi^{\frac{p(1-A+B)}{N}} N \left( \frac{N - (1 - A + B)p}{p-1} \right)^{p-1} \left( \frac{N - p + pA}{N - (1 - A + B)p} \right)^{p - \frac{p(1-A+B)}{N}} \times \left( \frac{2(p-1)}{(1 - A + B)p} \right)^{\frac{p(1-A+B)}{N}} \left\{ \frac{\Gamma \left( \frac{N}{p(1-A+B)} \right)}{\Gamma \left( \frac{N}{2} \right)} \frac{\Gamma \left( \frac{N(p-1)}{p(1-A+B)} \right)}{\Gamma \left( \frac{N}{1-A+B} \right)} \right\}.
\]

Moreover, \( S_{\text{rad}} \) is attained by functions of the form

\[
V(y) = \left[ a + b |y|^{\frac{pA}{p-1}} \right]^{1 - \frac{N(p-1)}{p(1-A+B)}} (a, b > 0, h = \frac{(1 - A + B)(N - p + pA)}{N - (1 - A + B)p})
\]

Note that the Assumption (20) is equivalent to the following condition.

\[
p < q = q(p) = \frac{Np}{N - (1 - A + B)p} < +\infty, \quad A > \frac{p - N}{p} \tag{21}
\]

By Theorem B and the transformation (15), we can also obtain several results for the minimization problem \( T_{\text{rad}} \) associated with an improved Hardy-Sobolev type inequality for radially symmetric functions. In fact, we set

\[
T_{\text{rad}} := \inf \left\{ \int_{B_R} \frac{|\nabla u|^p}{|x|^{\alpha-p} \left( 1 - \left( \frac{|x|}{R} \right)^{\gamma} \right)^{\beta-p}} \, dx \mid u \in X^{1,p}_{\alpha,\beta,\text{rad}}, \right. \notag\]

\[
\left. \int_{B_R} \frac{|u|^q}{|x|^{N - \frac{2}{p}(N - \alpha)} \left( 1 - \left( \frac{|x|}{R} \right)^{\gamma} \right)^{1 + \frac{(\beta - 1)}{p} \frac{1}{q}}} \, dx = 1 \right\},
\]

where \( q > p, \alpha < N, \beta > 1, \gamma = \frac{N - \alpha}{1 - \beta}, \) and \( X^{1,p}_{\alpha,\beta,\text{rad}} \) be the completion of \( C^\infty_{c,\text{rad}}(B_R) \) with respect to the norm \( \|\nabla(\cdot)\|_{L^p(B_R; |x|^{-(N - \alpha)(1 - (|x|/R)^{\gamma})^{p-\beta}} \, dx). \) By using the transformation (15), we have

\[
\int_{\mathbb{R}^N} |\nabla u|^p |y|^A \, dy = \int_{B_R} \frac{|\nabla u|^p |x|^A}{\left( 1 - \left( \frac{|x|}{R} \right)^{\gamma} \right)^{\beta-p}} \, dx = \int_{B_R} \frac{|\nabla u|^p}{\left( 1 - \left( \frac{|x|}{R} \right)^{\gamma} \right)^{\beta-p}} \, dx,
\]
\[
\int_{\mathbb{R}^N} |u|^q |y|^B q dy = \int_{B_R} \frac{|u|^q}{|x|^{N - \frac{q}{p}(N - \alpha)} (1 - (\frac{|x|}{R})^\gamma)^{1 + \frac{N + Bq}{\gamma}}} dx
\]

where \( \alpha = p - pA, q = \frac{Np}{N - p + Ap - Bp} = \frac{Np}{(\beta - 1)\gamma - Bp}, -Bq = N - \frac{q}{p}(\beta - 1) \gamma = N - \frac{q}{p}(N - \alpha) \) from (21). Therefore, we obtain the following from Theorem B.

**Theorem 2.6.** Let \( q > p > 1, N \geq 2, \alpha < N, \beta > 1, \) and \( \gamma = \frac{N - \alpha}{\beta - 1} \). Then

\[
T_{rad} = \pi^{\frac{n(q-p)}{2q}} N \left( \frac{Np}{q(p-1)} \right)^{p-1} \left( \frac{q(N - \alpha)}{Np} \right)^{p-1+\frac{\beta}{q}} \left( \frac{2q(p-1)}{N(q-p)} \right)^{\frac{q-p}{q}} \cdot
\]

\[
\left\{ \frac{\Gamma \left( \frac{q-p}{q-p} \right)}{\Gamma \left( \frac{q}{q-p} \right)} \right\}^{\frac{2-p}{q}} \cdot
\]

Moreover, \( T_{rad} \) is attained by functions of the form

\[
U(x) = \left[ a + b \left( \left( \frac{|x|}{R} \right)^{-\gamma} - 1 \right) \right]^{-\frac{p}{q-p}} \quad (a, b > 0)
\]

**Remark 2.7.** In the case where \( \beta = p \), Theorem 2.6 coincides with Theorem 1.3 in [25]. In the case where \( \alpha = \beta = p \), a modified minimization problem is also studied without radially symmetry, see [44]. The case where \( q = p \) is studied in Theorem 1.1, see also [27, 48]. In the case where \( \gamma = \frac{N - \alpha}{\beta - 1} = 0 \), minimization problems of Sobolev type inequalities with logarithmic weights are studied by [24, 43, 47].

### 2.3. The Divergence Theorem and Generalizations of Two Hardy Type Inequalities: Proof of Theorem 1.4

We show Theorem 1.4 by using the divergence theorem in a slightly different way from the proof of Theorem 1.1.

**Proof of Theorem 1.4.** For the simplicity, we set \( R = 1 \). Note that

\[
\int_{B_1} \frac{(N - \alpha)|u|^p}{|x|^\alpha (1 - |x|^\gamma)^{\beta - 1}} + \frac{(\beta - 1)\gamma |u|^p}{|x|^{\alpha - \gamma} (1 - |x|^\gamma)^{\beta}} dx = \int_{B_1} \text{div} \left( \frac{x}{|x|^{\alpha} (1 - |x|^\gamma)^{\beta}} \right) |u|^p dx. \quad (22)
\]

(I) If we drop the second term on the left-hand side of (22), then we have

\[
\left( \frac{N - \alpha}{p} \right) \int_{B_1} \frac{|u|^p}{|x|^\alpha (1 - |x|^\gamma)^{\beta - 1}} dx
\]

\[
\leq \int_{B_1} \frac{|u|^p - 1}{\nabla u \cdot \frac{x}{|x|}} \left| \nabla u \cdot \frac{x}{|x|} \right| |x|^\alpha - 1 (1 - |x|^\gamma)^{\beta - 1} dx
\]
which implies the desired inequality (8) for functions $u \in C^1_c(B_1)$. In order to show the optimality of the constant $(\frac{N-\alpha}{p})^p$ in (8), we consider the test function $f_A(x) = \phi_\delta(x) |x|^{A}$ for $A < \frac{N-\alpha}{p}$ and small $\delta > 0$, where $\phi_\delta$ is a smooth radially symmetric function which satisfies $0 \leq \phi_\delta \leq 1$, $\phi_\delta \equiv 0$ on $B_1 \setminus B_{2\delta}$ and $\phi_\delta \equiv 1$ on $B_\delta$. Then we have

$$
\left(\frac{N-\alpha}{p}\right)^p \leq \frac{\int_{B_1} \left|\nabla f_A \cdot \frac{x}{|x|}\right|^p |x|^{\alpha-p(1-|x|\gamma)^{\beta-1}} dx}{\int_{B_1} |x|^\alpha(1-|x|\gamma)^{\beta-1} dx} \\
\leq A^p \int_0^1 (1-r\gamma)^{-\beta+1} r^{A\gamma+N-\alpha-1} dr + \int_0^\beta |(\phi_\delta \cdot r^A')|^p (1-r\gamma)^{-\beta+1} r^{N-1-\alpha+p} dr \\
= \left(\frac{N-\alpha}{p}\right)^p + o(1) \quad \left(A \to \frac{N-\alpha}{p}\right).
$$

Therefore the constant $(\frac{N-\alpha}{p})^p$ in (8) is optimal. If $\beta > 1$, then we can observe that the optimal constant $(\frac{N-\alpha}{p})^p$ is not attained since we drop the second term on the left-hand side of (22). If $\beta = 1$, then the inequality (8) becomes the classical Hardy type inequality in which the non-attainability is well-known. Thus, we omit the proof.

(II) If we drop the first term on the left-hand side of (22), then we have

$$
\left(\frac{\beta-1}{p}\right)^p \int_{B_1} \frac{|u|^p}{|x|^{\alpha-\gamma}(1-|x|\gamma)^{\beta}} dx \leq \int_{B_1} \frac{|u|^{p-1}}{|x|^{\alpha-1}(1-|x|\gamma)^{\beta-1}} dx \\
\leq \left(\int_{B_1} \frac{|u|^p}{|x|^{\alpha-\gamma}(1-|x|\gamma)^{\beta}} dx\right)^{1-\frac{1}{p}} \left(\int_{B_1} \frac{\left|\nabla u \cdot \frac{x}{|x|}\right|^p}{|x|^\alpha(1-|x|\gamma)^{\beta-p}} dx\right)^{\frac{1}{p}}
$$

which implies the desired inequality (9) for functions $u \in C^1_c(B_1 \setminus \{0\})$. We shall show the inequality (9) for functions $u \in C^1_c(B_1)$. For $u \in C^1_c(B_1)$, we consider $u_\varepsilon = u (1 - \varphi_\varepsilon) \in C^1_c(B_1 \setminus \{0\})$, where $\varphi_\varepsilon \in C^\infty_{c,\text{rad}}(B_1)$, $0 \leq \varphi_\varepsilon \leq 1$, $\varphi_\varepsilon \equiv 1$ on $B_{\varepsilon}$, $\varphi_\varepsilon \equiv 0$ on $B_1 \setminus B_{2\varepsilon}$, and $|\nabla \varphi_\varepsilon| \leq C\varepsilon^{-1}$. Letting $\varepsilon \to 0$, we can derive the inequality (9) for $u_\varepsilon \in C^1_c(B_1)$ from the the inequality (9) for $u_\varepsilon \in C^1_c(B_1)$ and the assumption of $\alpha$ since

$$
\int_{B_1} \frac{\left|\nabla u \cdot \frac{x}{|x|}\right|^p}{|x|^\alpha(1-|x|\gamma)^{(\gamma-1)p}} dx \\
\leq \int_{B_{\varepsilon}} \frac{\left|\nabla u \cdot \frac{x}{|x|}\right|^p}{|x|^\alpha(1-|x|\gamma)^{(\gamma-1)p}} dx + \int_{B_1 \setminus B_{2\varepsilon}} \frac{\left|\nabla (u\varphi_\varepsilon) \cdot \frac{x}{|x|}\right|^p}{|x|^\alpha(1-|x|\gamma)^{(\gamma-1)p}} dx
$$
\[
\leq C \| \nabla u \|_\infty^p \int_0^{2\varepsilon} r^{N-1-\alpha+\gamma-(\gamma-1)p} \, dr \\
+C \| u \|_\infty^p \varepsilon^{-p} \int_0^{2\varepsilon} r^{N-1-\alpha+\gamma-(\gamma-1)p} \, dr \to 0 \quad (\varepsilon \to 0), \\
\int_{B_1} \frac{|u - u_\varepsilon|^p}{|x|^\alpha (1-|x|^\gamma)^\beta} \, dx \to 0 \quad (\varepsilon \to 0).
\]

The optimality of the constant \((\frac{\beta-1}{p} \gamma)^p\) in (9) can be shown by the same test function \(f_A\) in the proof of Theorem 1.1. We also see that the optimal constant \((\frac{\beta-1}{p} \gamma)^p\) is not attained since \(\alpha < N\) and we drop the first term on the left-hand side of (22).

\[\square\]

### 3. Higher Order Inequalities

The higher order generalization of Hardy type inequalities (2), (3) are called Rellich type inequalities due to the celebrated work by Rellich [41]. In this section, we consider the higher order generalization of Theorem 1.1.

Let \(k, m \in \mathbb{N}, k \geq 2, p > 1\), and

\[
\nabla^k u = \begin{cases} \\
\Delta^m u & \text{if } k = 2m, \\
\nabla \Delta^m u & \text{if } k = 2m + 1,
\end{cases}
\]

\[
A_{k,p,\alpha} = \begin{cases} \\
\prod_{j=0}^{m-1} \frac{\{N - \alpha + 2jp\} \{N(p-1) + \alpha - 2(j+1)p\}}{N - \alpha + 2mp} \prod_{j=0}^{m-1} \frac{\{N - \alpha + 2jp\} \{N(p-1) + \alpha - 2(j+1)p\}}{p^2} & \text{if } k = 2m, \\
\prod_{j=0}^{m-1} \frac{\{N - \alpha + 2jp\} \{N(p-1) + \alpha - 2(j+1)p\}}{p^2} & \text{if } k = 2m + 1.
\end{cases}
\]

For \(\alpha \in (2 + 2(m-1)p, N)\), the Rellich type inequality

\[
A_{k,p,\alpha}^p \int_{B_R} \frac{|u|^p}{|x|^{\alpha}} \, dx \leq \int_{B_R} \frac{|\nabla^k u|^p}{|x|^{\alpha-kp}} \, dx
\]

holds for all \(u \in C^k_c(B_R)\) (Ref. [15, 19, 33, 36, 41, 46]). If \(p = 2\), it is known that the geometric Rellich inequality

\[
\left( \prod_{j=1}^{k} \frac{jp-1}{p} \right)^p \int_{B_R} \frac{|u|^p}{\text{dist}(x, \partial B_R)^{kp}} \, dx \leq \int_{B_R} |\nabla^k u|^p \, dx
\]

holds for all \(u \in C^k_c(B_R)\) (Ref. [4, 39]). For the case where \(p \neq 2\), see the important remark in [3] p.879 and the end of this section. It is also known that both \(A_{k,p}^p\) and \(\left( \prod_{j=1}^{k} \frac{jp-1}{p} \right)^p\) with \(p = 2\) are the optimal constants.

### 3.1. Improved Rellich Inequalities on \(L^2\)

In this subsection, we treat the case where \(p = 2\).
Theorem 3.1. (I) If $4 - N < \alpha \leq N - \gamma$ and $\gamma > 0$, then the inequality

$$
\left( \frac{N + \alpha - 4}{4} \gamma \right)^2 \int_{B_R} \frac{|u|^2}{|x|^\alpha \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^2} \leq \int_{B_R} \frac{|\Delta u|^2}{|x|^{\alpha-4}} \, dx
$$

(25)

holds for any functions $u \in C^\infty_c(B_R)$. Especially, when $\gamma = N - \alpha$, the constant $(\frac{(N - \alpha)(N + \alpha - 4)}{4})^2$ in (25) is optimal and is not attained for $u \neq 0$ for which the right-hand side is finite.

(II) If $3 \leq \alpha \leq \min\{N - \gamma + 2, N - 3\gamma\}$ and $\gamma > 0$, then the inequality

$$
\left( \frac{3}{4} \gamma^2 \right)^2 \int_{B_R} \frac{|u|^2}{|x|^\alpha \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^4} \leq \int_{B_R} \frac{|\Delta u|^2}{|x|^{\alpha-4}} \, dx
$$

(26)

holds for any functions $u \in C^\infty_c(B_R)$. Furthermore, the constant $(\frac{3}{4} \gamma^2)^2$ in (26) is optimal and is not attained for $u \neq 0$ for which the right-hand side is finite.

Remark 3.2. It seems difficult to show the inequality (26) with the weight $\left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^{-\beta}$ on the right-hand side. This is one of the reasons why we cannot show the higher order case of the inequality (26), see also Remark 4.5.

Our inequalities (25), (26) give improvements of the classical Rellich type inequalities (23), (24) while keeping their best constants as follows.

Corollary 3.3. (I) Let $N \geq 5$. Then the inequalities

$$
\int_{B_R} \frac{|u|^2}{|x|^4} \, dx \leq \int_{B_R} \frac{|u|^2}{|x|^4 \left(1 - \left(\frac{|x|}{R}\right)^{N-4}\right)^2} \, dx \leq \left(\frac{N(N - 4)}{4}\right)^{-2} \int_{B_R} |\Delta u|^2 \, dx
$$

hold for any functions $u \in W^{2,2}_0(B_R)$.

(II) Let $N \geq 7$. Then the inequalities

$$
\int_{B_R} \frac{|u|^2}{\text{dist}(x, \partial B_R)^4} \, dx \leq \int_{B_R} \frac{|u|^2}{|x|^4 \left(1 - \left(\frac{|x|}{R}\right)^{N-4}\right)^4} \, dx \leq \left(\frac{3}{4}\right)^{-2} \int_{B_R} |\Delta u|^2 \, dx
$$

hold for any functions $u \in W^{2,2}_0(B_R)$.

Furthermore, we also obtain two critical Rellich inequalities as limiting forms of our inequalities (25) and (26) as $\gamma \to 0$. For the critical Rellich inequalities (27), (28), see e.g. [11].

Corollary 3.4. (I) If $4 - N < \alpha \leq N$, then the inequality

$$
\left( \frac{N + \alpha - 4}{4} \right)^2 \int_{B_R} \frac{|u|^2}{|x|^\alpha \left(\log \frac{R}{|x|}\right)^2} \leq \int_{B_R} \frac{|\Delta u|^2}{|x|^{\alpha-4}} \, dx
$$

(27)

holds for any functions $u \in C^\infty_c(B_R)$.
(II) If $3 \leq \alpha \leq N$, then the inequality
\[
\left(\frac{3}{4}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^\alpha \left(\log \frac{R}{|x|}\right)^4} \leq \int_{B_R} \frac{|\Delta u|^2}{|x|^\alpha} \ dx
\]
holds for any functions $u \in C^\infty_c(B_R)$.

We shall derive the improved Rellich inequalities (25), (26) simply by integration by parts and the one dimensional inequalities in Sect. 4.

Proof of Theorem 3.1. For the simplicity, we set $R = 1$. We use the polar coordinate $x = r\omega (r = |x|, \omega \in S^{N-1})$ and

\[
\nabla u = \frac{\partial u}{\partial r} \omega + \frac{1}{r} \nabla_{S^{N-1}} u, \quad \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r} \frac{\partial u}{\partial r} + \frac{\Delta_{S^{N-1}} u}{r^2}.
\]

(I) For $u \in C^\infty_c(B_1 \setminus \{0\})$, we have
\[
\int_{B_1} \frac{|\Delta u|^2}{|x|^\alpha} \ dx = \int_0^1 \int_{S^{N-1}} \left( \frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r} \frac{\partial u}{\partial r} + \frac{\Delta_{S^{N-1}} u}{r^2} \right)^2 \ r^{N-\alpha+3} \ dr dS_\omega
\]
\[
= \iint \left( \frac{\partial^2 u}{\partial r^2} \right)^2 \ r^{N-\alpha+3} + (N-1)(\alpha-3) \left| \frac{\partial u}{\partial r} \right|^2 \ r^{N-\alpha+1}
\]
\[
+ |\Delta_{S^{N-1}} u|^2 r^{N-\alpha-1}
\]
\[
- 2(N-1) \nabla_{S^{N-1}} \left( \frac{\partial u}{\partial r} \right) \cdot (\nabla_{S^{N-1}} u) \ r^{N-\alpha}
\]
\[
- 2 \nabla_{S^{N-1}} \left( \frac{\partial^2 u}{\partial r^2} \right) \cdot (\nabla_{S^{N-1}} u) \ r^{N-\alpha+1}.
\]

Since $\frac{\partial}{\partial r}$ and $\nabla_{S^{N-1}}$ are commutative, we have
\[
\int_{B_1} \frac{|\Delta u|^2}{|x|^{\alpha-4}} \ dx = \iint \left( \frac{\partial^2 u}{\partial r^2} \right)^2 \ r^{N-\alpha+3} + (N-1)(\alpha-3) \left| \frac{\partial u}{\partial r} \right|^2 \ r^{N-\alpha+1}
\]
\[
+ |\Delta_{S^{N-1}} u|^2 r^{N-\alpha-1}
\]
\[
- (N-1) \frac{\partial}{\partial r} (|\nabla_{S^{N-1}} u|^2) r^{N-\alpha}
\]
\[
- 2 \frac{\partial}{\partial r} \left( \nabla_{S^{N-1}} \left( \frac{\partial u}{\partial r} \right) \right) \cdot (\nabla_{S^{N-1}} u) r^{N-\alpha+1}
\]
\[
= \iint \left( \frac{\partial^2 u}{\partial r^2} \right)^2 \ r^{N-\alpha+3} + (N-1)(\alpha-3) \left| \frac{\partial u}{\partial r} \right|^2 \ r^{N-\alpha+1}
\]
\[
+ |\Delta_{S^{N-1}} u|^2 r^{N-\alpha-1}
\]
\[
+ (N-\alpha)(\alpha-2) |\nabla_{S^{N-1}} u|^2 r^{N-\alpha-1} + 2 \left| \frac{\partial}{\partial r} (\nabla_{S^{N-1}} u) \right|^2 r^{N-\alpha+1}.
\]

If $4 - N < \alpha$ and $N - \alpha - \gamma \geq 0$, then we have the followings by Proposition 4.1 and Proposition 4.3.
\[
\int_{B_1} \frac{|\Delta u|^2}{|x|^\alpha} \ dx \geq \iint \left( (N-1)(\alpha-3) + \left( \frac{N-\alpha+2}{2} \right)^2 \right) \left| \frac{\partial u}{\partial r} \right|^2 \ r^{N-\alpha+1}
\]
\[
+ |\Delta_{S^{N-1}} u|^2 r^{N-\alpha-1}.
\]
\[
+ \left\{ (N - \alpha)(\alpha - 2) + 2 \left( \frac{N - \alpha}{2} \right)^2 \right\} |\nabla_{S^{N-1}} u|^2 r^{N - \alpha - 1} \geq \int \int \left( \frac{N + \alpha - 4}{2} \right)^2 \left| \frac{\partial u}{\partial r} \right|^2 r^{N - \alpha + 1} + (N - \alpha)(N + \alpha - 4) \left| \nabla_{S^{N-1}} u \right|^2 r^{N - \alpha - 1} \\
\geq \left( \frac{N + \alpha - 4}{2} \right)^2 \left( \frac{\gamma}{2} \right)^2 \int \int |u|^2 r^{N - \alpha - 1} (1 - r^\gamma)^{-2} dr dS_\omega \geq \left( \frac{N + \alpha - 4}{4} \right)^2 \int_{B_1} \frac{|u|^2}{|x|^\alpha (1 - |x|^\gamma)^2} dx
\]

Therefore, we have the inequality (25) for \( u \in C_c^\infty(B_1 \setminus \{0\}) \). By the assumption \( \alpha \leq N - \gamma \), we see that the inequality (25) holds for \( u \in C_c^\infty(B_1) \). In fact, we consider a smooth cut-off function \( \varphi_\varepsilon \in C_c^\infty(B_1 \setminus \{0\}) \) which satisfies \( 0 \leq \varphi_\varepsilon \leq 1 \), \( \varphi_\varepsilon = 1 \) on \( B_1 \setminus B_{2 \varepsilon} \), \( \varphi_\varepsilon = 0 \) on \( B_2 \), \( |\nabla \varphi_\varepsilon| \leq C \varepsilon^{-1} \), and \( |\Delta \varphi_\varepsilon| \leq C \varepsilon^{-2} \). For \( u \in C_c^\infty(B_1) \), set \( u_\varepsilon = u \varphi_\varepsilon \in C_c^\infty(B_1 \setminus \{0\}) \). Thanks to the assumption \( \alpha < N \), we have
\[
\int_{B_1} \frac{|\Delta u_\varepsilon|^2}{|x|^\alpha} dx \to \int_{B_1} \frac{|\Delta u|^2}{|x|^\alpha} dx, \int_{B_1} \frac{|u_\varepsilon|^2}{|x|^\alpha (1 - |x|^\gamma)^2} dx \to \int_{B_1} \frac{|u|^2}{|x|^\alpha (1 - |x|^\gamma)^2} dx,
\]

as \( \varepsilon \to 0 \). Therefore, we obtain the inequality (25) for \( u \in C_c^\infty(B_1) \). Furthermore, the above calculation implies that if \( U \) is an extremal function of the inequality (25), then \( U \) is a radially symmetric function since \( |\nabla_{S^{N-1}} U| = 0 \) a.e. in \( B_1 \). Also, we have \( \gamma = N - \alpha \) and \( U(x) = U(|x|) = C|x|^{-\frac{N-\alpha}{2}} (1 - |x|^{N-\alpha})^\frac{\gamma}{2} \) a.e. in \( B_1 \) for some \( C \in \mathbb{R} \) from the equality condition of Proposition 4.3. However, we have
\[
\int_{B_1} \frac{|\Delta U|^2}{|x|^\alpha} dx = \left( \frac{N + \alpha - 4}{4} \right)^2 \int_{B_1} \frac{|U|^2}{|x|^\alpha (1 - |x|^\gamma)^2} dx = C^2 \left( \frac{N + \alpha - 4}{4} \right)^2 \omega_{N-1} \int_0^1 r^{-1} (1 - r^{N-\alpha})^{-1} dr = \infty
\]

for any \( \alpha \) and any \( C \neq 0 \). The remaining of the proof is to show the optimality of the constant \( \left( \frac{N + \alpha - 4}{4} \right)^2 \) in (25). Since the test function and the calculations are completely the same as them in the proof of Theorem 3.6 (I), we omit here.

(II) For \( u \in C_c^\infty(B_1 \setminus \{0\}) \) and \( \alpha \in [3, \min\{N - \gamma + 2, N - 3\gamma\}] \), we have the followings from the above calculation and Proposition 4.3.
\[
\int_{B_1} \frac{|\Delta u|^2}{|x|^\alpha} dx \geq \int_0^1 \int_{S^{N-1}} \left| \frac{\partial^2 u}{\partial r^2} \right|^2 r^{N - \alpha + 3} + \left( \frac{N - \alpha}{2} \right)^2 \left( \frac{\gamma}{2} \right)^2 \int_{S^{N-1}} \left| \frac{\partial u}{\partial r} \right|^2 r^{N - \alpha + 1} (1 - r^\gamma)^{-2}
\]
Let \( u \) holds for any functions \( u \in C_c^\infty(B_1) \). Furthermore, the above calculation implies that if \( U \) is an extremal function of the inequality (26), then \( \alpha = 3 \) and \( U \) is a radially symmetric function since \( |\nabla_{S^N} U| = 0 \) a.e. in \( B_1 \). Also, we have \( \gamma = \frac{N-3}{2} \) and \( U(x) = U(|x|) = C|x|^{-\frac{N-3}{2}} (1 - |x|^\frac{N-3}{2})^\frac{3}{2} \) a.e. in \( B_1 \) for some \( C \in \mathbb{R} \) from the equality condition of Proposition 4.3. However, we have

\[
\int_{B_1} \frac{|\Delta U|^2}{|x|^{\alpha-4}} \, dx = \left( \frac{3}{4} \gamma^2 \right)^2 \int_{B_1} \frac{|U|^2}{|x|^{\alpha}(1 - |x|^{-\gamma})^4} \, dx
\]

\[
= C^2 \left( \frac{3}{4} \gamma^2 \right)^2 \omega_{N-1} \int_0^1 r^{-1} (1 - r^{\frac{N-3}{2}})^{-1} \, dr = \infty
\]

for any \( C \neq 0 \). The remaining of the proof is to show the optimality of the constant \( \left( \frac{3}{4} \gamma^2 \right)^2 \) in (26). Since the test function and the calculations are completely the same as them in the proof of Theorem 3.6 (II), we omit here.

\[\square\]

From Theorem 3.1 (I) and the classical Rellich type inequality (23), we also obtain higher order case of the inequality (25) as follows.

**Corollary 3.5.** Let \( k \geq 3, k = 2m + 1 \) or \( k = 2m, 1 < p, \beta < \infty, \) and \(-2 + 4m < \alpha \leq N - \gamma\). Then the inequality

\[
\left( \frac{\gamma}{N - \alpha} A_{k,2,\alpha} \right)^2 \int_{B_R} \frac{|u|^2}{|x|^\alpha \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^2} \leq \int_{B_R} \frac{|\nabla^k u|^2}{|x|^\alpha - 2k} \, dx
\]

holds for any functions \( u \in C_c^k(B_R) \). Furthermore, the constant \( \left( \frac{\gamma}{N - \alpha} A_{k,2,\alpha} \right)^2 \) in (29) is optimal and is not attained for \( u \neq 0 \) for which the right-hand side is finite.

As a limiting form of (29) as \( \gamma \to 0 \), we obtain the critical Rellich inequality for \(-2 + 4m < \alpha \leq N\).

\[
\left( \frac{A_{k,2,\alpha}}{N - \alpha} \right)^2 \int_{B_R} \frac{|u|^2}{|x|^\alpha \left( \log \frac{R}{|x|} \right)^2} \leq \int_{B_R} \frac{|\nabla^k u|^2}{|x|^\alpha - 2k} \, dx.
\]

**Proof of Corollary 3.5.** Applying the Rellich type inequality (23) with \( \tilde{\alpha} = \alpha - 4 \) and \( \tilde{k} = k - 2 \), we have

\[
A_{k,2,\tilde{\alpha}}^2 \tilde{\alpha} \int_{B_R} \frac{|\Delta u|^2}{|x|^{\tilde{\alpha}}} \, dx \leq \int_{B_R} \frac{|\nabla^{\tilde{k}}(\Delta u)|^2}{|x|^\tilde{\alpha} - 2\tilde{k}} \, dx = \int_{B_R} \frac{|\nabla^{\tilde{k}} u|^2}{|x|^\tilde{\alpha} - 2\tilde{k}} \, dx.
\]

(30)
On the other hand, from Theorem 3.1 (I), we have
\[
\left( \frac{N + \alpha - 4}{4} \right)^2 \int_{B_R} \frac{|u|^2}{|x|^{\alpha} \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^2} \leq \int_{B_R} \frac{|\Delta u|^2}{|x|^{\alpha - 4}} \, dx. \tag{31}
\]

Note that
\[
A_{k,2,\alpha} \frac{(N - \alpha)(N + \alpha - 4)}{4} = A_{k,2,\alpha}. \tag{32}
\]
From (30), (31), and (32), we obtain the inequality (29). \qed

3.2. Improved Rellich Inequalities for Radially Symmetric Functions on $L^p$

In this subsection, we treat the case where $p \neq 2$ for radially symmetric functions.

**Theorem 3.6.** Let $1 < p, \beta < \infty, \gamma > 0$, and $k \geq 2$.

(I) If $\alpha \leq N - (p - 1)\gamma$, then the inequality
\[
\left( \frac{p - 1}{N - \alpha} \gamma A_{k,p,\alpha} \right)^p \int_{B_R} \frac{|u|^p}{|x|^{\alpha} \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^p} \leq \int_{B_R} \frac{|
abla^k u|^p}{|x|^{\alpha - kp}} \, dx
\]
holds for any radially symmetric functions $u \in C^k_{c,rad}(B_R)$. Especially, when $\gamma = \frac{N - \alpha}{p - 1}$, the constant $A^p_{k,p,\alpha}$ in (33) is optimal and is not attained for $u \not\equiv 0$ for which the right-hand side is finite.

(II) If $\alpha \leq \min \{ N - (\beta - p - 1)\gamma - (N - 2)p, N - (\beta - 1)\gamma \}$, then the inequality
\[
\left( \prod_{j=0}^{k-1} \frac{\beta - jp - 1}{p} \gamma \right)^p \int_{B_R} \frac{|u|^p}{|x|^{\alpha} \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^p} \leq \int_{B_R} \frac{|
abla^k u|^p}{|x|^{\alpha - kp} \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^{\beta - kp}} \, dx
\]
holds for any radially symmetric functions $u \in C^k_{c,rad}(B_R)$. Furthermore, the constant $\left( \prod_{j=0}^{k-1} \frac{\beta - jp - 1}{p} \gamma \right)^p$ in (34) is optimal and is not attained for $u \not\equiv 0$ for which the right-hand side is finite.

**Remark 3.7.** Note that the function $\frac{\gamma}{1 - r^\gamma}$ is monotone-increasing with respect to $\gamma \in (0, \frac{N - \alpha}{p - 1})$ for any $r \in (0, 1)$. Therefore, in the inequality (33), we see that
\[
\left( \frac{p - 1}{N - \alpha} \gamma A_{k,p,\alpha} \right)^p \int_{B_R} \frac{|u|^p}{|x|^{\alpha} \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^p} \leq A^p_{k,p,\alpha} \int_{B_R} \frac{|u|^p}{|x|^{\alpha} \left( 1 - \left( \frac{|x|}{R} \right)^\gamma \right)^{\frac{N - \alpha}{p - 1}}} \, dx
\]
\[
\leq \int_{B_R} \frac{|
abla^k u|^p}{|x|^{\alpha - kp}} \, dx.
\]

**Remark 3.8.** Unfortunately, we cannot derive the geometric Rellich inequality (24) from our inequality (34) due to the assumption $\alpha \leq N - (\beta - p - 1)\gamma - (N - 2)p$. In fact, if we assume $(\alpha, \beta, \gamma) = (kp, kp, 1)$, then we have $p = 1$ from the assumption.
However, the case where $p = 1$ is excluded in Theorem 3.6. See also the end of this section.

We easily see that our inequality (33) with $\gamma = \frac{N - \alpha}{p - 1}$ gives an improvement of the classical Rellich type inequality (23) on $L^p$ for radially symmetric functions. We also obtain the critical Rellich inequality (35) on $L^p$ for radially symmetric functions as a limiting form of our inequality (33) as $\gamma \to 0$. For the critical Rellich inequality (35) on $L^p$, see [16,38,42].

Corollary 3.9. Let $1 < p, \beta < \infty$ and $k \in \mathbb{N}$. If $\alpha \leq N$, then the inequality

$$
\left( \frac{p - 1}{N - \alpha} A_{k,p,\alpha} \right)^p \int_{B_R} \frac{|u|^p}{|x|^\alpha \left( \log \frac{|x|}{R} \right)^\gamma} \leq \int_{B_R} \frac{|
abla^k u|^p}{|x|^{\alpha - kp}} \, dx
$$

(35)

holds for any radially symmetric functions $u \in C^k_{c,\text{rad}}(B_R)$.

To prove Theorem 3.6, we show the improved Rellich type inequality with a remainder term as follows. The remainder term comes from Theorem 2.1.

Theorem 3.10. Let $1 < p, \beta < \infty$, and $k \geq 2$.

$I$ If $\alpha \leq N - (p - 1)\gamma$, then the inequality

$$
\left( \frac{p - 1}{N - \alpha} A_{k,p,\alpha} \right)^p \int_{B_R} \frac{|u|^p}{|x|^\alpha \left( 1 - \left( \frac{|x|}{R} \right) \right)^\gamma} + \left( \frac{p}{N - \alpha} A_{k,p,\alpha} \right)^p \psi_{N,p,\alpha,p}(u)
$$

$$
\leq \int_{B_R} \frac{|
abla^k u|^p}{|x|^{\alpha - kp}} \, dx
$$

(36)

holds for any radially symmetric functions $u \in C^k_{c,\text{rad}}(B_R)$, where $\psi_{N,p,\alpha,\beta}(u)$ is given in Theorem 2.1.

$II$ If $\alpha \leq \min\{N - (\beta - p - 1)\gamma - (N - 2)p, N - (\beta - 1)\gamma\}$, then the inequality

$$
\left( \prod_{j=0}^{k-1} \frac{\beta - jp - 1}{p} \right)^p \int_{B_R} \frac{|u|^p}{|x|^\alpha \left( 1 - \left( \frac{|x|}{R} \right) \right)^\beta} + \left( \prod_{j=1}^{k-1} \frac{\beta - jp - 1}{p} \right)^p \psi_{N,p,\alpha,\beta}(u)
$$

$$
\leq \int_{B_R} \frac{|
abla^k u|^p}{|x|^{\alpha - kp} \left( 1 - \left( \frac{|x|}{R} \right) \right)^{\beta - kp}} \, dx
$$

(37)

holds for any radially symmetric functions $u \in C^k_{c,\text{rad}}(B_R)$, where $\psi_{N,p,\alpha,\beta}(u)$ is given in Theorem 2.1.

Concerning the classical Rellich inequality (23), we need the assumption where $\alpha > 2 + 2(m - 1)p$. In fact, if there is no restriction with respect to $\alpha < N$, it is possible that the best constant of (23) for any functions is less than $A_{k,p,\alpha}^p$. At least when $p = 2$, this is true. Namely, symmetry breaking phenomenon occurs for some $\alpha$. For the details, see [11,52]. However, the best constant of (23) for radially symmetric functions is $A_{k,p,\alpha}^p$ for any $\alpha < N$, see e.g. [37] or Corollary 4.2 in Sect. 4. This is the reason why we do not need any restrictions with respect to $\alpha < N$ in Theorems in this section.
Proof of Theorem 3.10. (I) We obtain the following from Theorem 2.1 with \( \beta = p \) and (39) in Proposition 4.1.

\[
\left( \frac{p - 1}{p} \right)^p \int_{B_R} \frac{|u|^p}{|x|^\alpha \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^p} + \psi_{N, p, \alpha, p}(u)
\leq \int_{B_R} \left| \nabla u \right|^p \frac{dx}{|x|^{\alpha - p}}
\leq \left| \frac{N(p - 1) + \alpha - 2p}{p} \right| \int_{B_R} \left| \Delta u \right|^p \frac{dx}{|x|^{\alpha - 2p}}
\]

This implies the inequality (36) in the case where \( k = 2 \). Combining this with Corollary 4.2, we have

\[
\left( \frac{p - 1}{N - \alpha} \right)^p \frac{A_{2, p, \alpha}}{\alpha} \int_{B_R} \frac{|u|^p}{|x|^\alpha \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^p} + \psi_{N, p, \alpha, p}(u)
\leq \int_{B_R} \frac{|\Delta u|^p}{|x|^{\alpha - 2p}} \frac{dx}{|x|^\alpha} \leq A_{k - 2, p, \alpha - 2p} \int_{B_R} \frac{|\nabla^{k - 2} \Delta u|^p}{|x|^{\alpha - 2p}} \frac{dx}{|x|^\alpha}
\]

Since \( A_{2, p, \alpha} A_{k - 2, p, \alpha - 2p} = A_{k, p, \alpha} \), the inequality (36) holds for any \( k \).

(II) Let \( \alpha \leq \min\{N - (\beta - p - 1)\gamma - (N - 2)p, N - (\beta - 1)\gamma\}. \) Then we obtain the following from Theorem 2.1 and Corollary 4.4.

\[
\left( \frac{\beta - 1}{p} \right)^p \int_{B_R} \frac{|u|^p}{|x|^\alpha \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^p} + \psi_{N, p, \alpha, \beta}(u)
\leq \int_{B_R} \left| \nabla u \right|^p \frac{dx}{|x|^{\alpha - p} \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^{\beta - p}}
\leq \left( \frac{\beta - p - 1}{p} \right)^p \int_{B_R} \frac{|\Delta u|^p}{|x|^{\alpha - 2p}} \frac{dx}{|x|^{\alpha - 2p} \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^{\beta - 2p}}.
\]

This implies the inequality (37) in the case where \( k = 2 \). Note that \( \alpha - jp \leq N - (\beta - jp - 1)\gamma - (N - 1)p \) and \( \alpha - jp \leq N - (\beta - jp - 1)\gamma \) hold for any integer \( j \geq 1 \). Assume that the inequality (37) holds for \( k \geq 2 \). Then, from Corollary 4.4 or Theorem 1.1, we have

\[
\left( \prod_{j=0}^{k} \frac{\beta - jp - 1}{p} \right)^p \int_{B_R} \frac{|u|^p}{|x|^\alpha \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^p} + \left( \prod_{j=1}^{k} \frac{\beta - jp - 1}{p} \right)^p \psi_{N, p, \alpha, \beta}(u)
\leq \left( \frac{\beta - kp - 1}{p} \right)^p \int_{B_R} \frac{|\nabla^k u|^p}{|x|^{\alpha - kp}} \frac{dx}{|x|^{\alpha - kp} \left(1 - \left(\frac{|x|}{R}\right)^\gamma\right)^{\beta - kp}}.
\]
\[ \leq \int_{B_R} \frac{|\nabla^{k+1} u|^p}{|x|^\alpha-(k+1)p} \left( 1 - \left( \frac{|x|}{R} \right) \right)^{\beta-(k+1)p} \, dx. \]

Therefore the inequality (37) holds for \( k + 1 \). Hence, we see that the inequality (37) holds for any \( k \geq 2 \).

Proof of Theorem 3.6. For the simplicity, we set \( R = 1 \).

(I) Let \( \gamma = \frac{N-\alpha}{p-1} \). To show the optimality of the constant \( A_{k,p,\alpha}^p \) in (33), we consider the test function

\[ g_B(x) = \psi_\delta(x) |x|^{-B}, \]

for \( B < \frac{N-\alpha}{p} \) and small \( \delta > 0 \), where \( \psi_\delta \) is a smooth radially symmetric function which satisfies \( 0 \leq \psi_\delta \leq 1 \), \( \psi_\delta \equiv 0 \) on \( B_1 \setminus B_{2\delta} \) and \( \psi_\delta \equiv 1 \) on \( B_\delta \). Since

\[ |\nabla^k |x|^B| = |x|^{A-k} \begin{cases} \prod_{j=0}^{m-1} (A-2j)(N+A-2j-2) & \text{if } k = 2m, \\ (A-2m) \prod_{j=0}^{m-1} (A-2j)(N+A-2j-2) & \text{if } k = 2m+1, \end{cases} \]

we have

\[ A_{k,p,\alpha}^p \leq \frac{\int_{B_1} |\nabla^k g_B|^p |x|^{\alpha-kp} \, dx}{\int_{B_1} |g_B|^p |x|^{\alpha-(1-|x|^\gamma)p} \, dx} \leq A_{k,p,\alpha}^p + o(1) \left( B \to \frac{N-\alpha}{p}, \ \delta \to 0 \right). \]

Therefore the constant \( A_{k,p,\alpha}^p \) in (33) is optimal. Since there exists the non-negative remainder term in Theorem 3.10 (I), we observe that if there exists an extremal function \( U = U(x) \) of the inequality (33), then \( U(x) = c \left( |x|^{-\frac{N-\alpha}{p-1}} - 1 \right)^{\frac{p-1}{p}} = c|x|^{-\frac{N-\alpha}{p}} \left( 1 - |x|^{-\frac{N-\alpha}{p-1}} \right)^{\frac{p-1}{p}} \) for some \( c \in \mathbb{R} \). However, if \( c \neq 0 \), then the right-hand side of (33) diverges since

\[ \int_{B_\varepsilon} |\nabla^k U|^p |x|^{\alpha-kp} \, dx \geq C(\varepsilon) \int_{B_\varepsilon} \frac{|\nabla^k |x|^{-\frac{N-\alpha}{p}}|^p |x|^{\alpha-kp}}{|x|^{\alpha-kp}} \, dx + D(\varepsilon) \geq \tilde{C}(\varepsilon) \int_{B_\varepsilon} |x|^{-N} \, dx + D(\varepsilon) = \infty \]

for any small \( \varepsilon > 0 \), where \( C(\varepsilon), \tilde{C}(\varepsilon) \neq 0, D(\varepsilon) \) are some constants depending on \( \varepsilon \).

(II) Let \( \alpha \leq N - (\beta - p - 1)\gamma - (N - 2)p \). We show the optimality of the constant \( \left( \prod_{j=0}^{k-1} \frac{\beta-jp-1}{p} \right)^{\frac{p}{\gamma}} \) in (34). For \( A > \frac{\beta-1}{p} \) and small \( \delta > 0 \), set

\[ f_A(x) = \phi_\delta(x) \left( 1 - |x|^\gamma \right)^A, \]

where \( \phi_\delta \) is a smooth radially symmetric function which satisfies \( 0 \leq \phi_\delta \leq 1 \), \( \phi_\delta \equiv 0 \) on \( B_1 \setminus 2\delta \) and \( \phi_\delta \equiv 1 \) on \( B_1 \setminus B_{1-\delta} \). Note that

\[ |\nabla^k (1 - |x|^\gamma)^A| = \sum_{j=1}^{k} C_{k,j} |x|^j (1 - |x|^\gamma)^{A-j} \]

\[ = |C_{k,k}| |x|^{k(\gamma-1)} (1 - |x|^\gamma)^{A-k} + o \left( (1 - |x|^\gamma)^{A-k} \right) \quad (|x| \to 1), \]
where $C_{k,k} = (-\gamma)^k \prod_{\ell=1}^{k} (A - \ell + 1)$, see Proposition 4.6 in Sect. 4. Then we have

$$\left(\prod_{j=0}^{k-1} \frac{\beta -jp -1}{p} \gamma\right)^p \leq \frac{\int_{B_1} |\nabla^k u|^p}{\int_{B_1} |u|^p} \leq \frac{\gamma^p \prod_{\ell=1}^{k} (A - \ell + 1)^p \int_{1-\delta}^{1} (1 - r\gamma)^{Ap - \beta} r^{N - 1 - \alpha + k\gamma p} dr + o(1)}{\int_{1-\delta}^{1} (1 - r\gamma)^{Ap - \beta} r^{N - 1 - \alpha} dr} = \left(\prod_{j=0}^{k-1} \frac{\beta -jp -1}{p} \gamma\right)^p + o(1) \left( A \to \frac{\beta -1}{p}, \delta \to 0 \right).$$

Therefore the constant $\left(\prod_{j=0}^{k-1} \frac{\beta -jp -1}{p} \gamma\right)^p$ in (34) is optimal.

The proof of Theorem 3.6 is now complete. □

In the end of this section, we formulate the following three conjectures for general cases (Ref. [3] p.879):

Conjecture 3.11. Let $k \geq 2$, $1 < p < \frac{N}{k}$, and $0 < \gamma \leq \frac{N - kp}{kp - 1}$. Then the inequality

$$\left(\prod_{j=0}^{k-1} \frac{\beta -jp -1}{p} \gamma\right)^p \int_{B_R} |u|^p |x|^{kp} \left(1 - \frac{|x|}{R}\right)^{\gamma kp} \leq \int_{B_R} |\nabla^k u|^p dx$$

holds for any functions $u \in C^k_c(B_R)$. Furthermore, the constant $\left(\prod_{j=0}^{k-1} \frac{\beta -jp -1}{p} \gamma\right)^p$ is optimal and is not attained for $u \neq 0$ for which the right-hand side is finite.

Conjecture 3.12. Let $1 < p < \infty$ and $k \geq 2$. Then the inequality

$$\left(\prod_{j=1}^{k} \frac{jp -1}{p} \gamma\right)^p \int_{B_R} \frac{|u|^p}{\text{dist}(x, \partial B_R)^{kp}} \leq \int_{B_R} |\nabla^k u|^p dx$$

holds for any functions $u \in C^k_c(B_R)$. Furthermore, the constant $\left(\prod_{j=1}^{k} \frac{jp -1}{p} \gamma\right)^p$ is optimal and is not attained for $u \neq 0$ for which the right-hand side is finite.

Conjecture 3.13. Let $N > k \geq 2$. Then the inequality

$$\left(\prod_{j=1}^{k} \frac{jN - k}{N} \frac{N}{N}\right)^\frac{N}{N} \int_{B_R} \frac{|u|^\frac{N}{N}}{|x|^N \left(\log \frac{R}{|x|}\right)^N} \leq \int_{B_R} |\nabla^k u|^\frac{N}{N} dx$$

holds for any functions $u \in C^k_c(B_R)$. Furthermore, the constant $\left(\prod_{j=1}^{k} \frac{jN - k}{N} \frac{N}{N}\right)^\frac{N}{N}$ is optimal and is not attained for $u \neq 0$ for which the right-hand side is finite.
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4. Appendix

First, we list the well-known one dimensional Hardy type inequality (38) and the Hardy-Rellich inequality (39) for radially symmetric functions including their proofs.

Proposition 4.1. The following inequalities hold:

\[
\left| \frac{a+1-p}{p} \right| \int_0^R r^{a-p}|w(r)|^p \, dr \leq \int_0^R r^a|w'(r)|^p \, dr
\]

for any \( w \in C^1(0,R) \) with \( \lim_{r \to 0} r^{a+1-p}|w(r)|^p = 0 = \lim_{r \to R} w(r) \), where \( a \in \mathbb{R}, p \geq 1 \). \hfill (38)

\[
\left| \frac{N(p-1)+\alpha-p}{p} \right| \int_{B_R} |\nabla u|^p \, dx \leq \int_{B_R} |\Delta u|^p \, dx
\]

for any \( u \in C^2_{c,\text{rad}}(B_R) \), where \( p \geq 1, \alpha < N \). \hfill (39)

Proof. If \( a+1-p = 0 \), then the inequality (38) is trivial. Therefore, we assume that \( a+1-p \neq 0 \). Since \( \lim_{r \to 0} r^{a+1-p}|w(r)|^p = 0 = w(R) \), we have

\[
\int_0^R r^{a-p}|w(r)|^p \, dr = \left[ \frac{r^{a+1-p}|w(r)|^p}{a+1-p} \right]_0^R - \frac{p}{a+1-p} \int_0^R r^{a+1-p}|w|^p w' \, dr
\]

\[
\leq \left| \frac{p}{a+1-p} \right| \left( \int_0^R r^{a-p}|w(r)|^p \, dr \right)^{\frac{p-1}{p}} \left( \int_0^R r^a|w'(r)|^p \, dr \right)^{\frac{1}{p}}
\]

which implies (38). Next, we shall show (39) for \( u \in C^2_{c,\text{rad}}(B_R) \). By applying (38) for \( w(r) = r^{N-1}w'(r) \) and \( a = -1+p-N(p-1)-\alpha+p \), we have

\[
\int_{B_R} |\Delta u|^p \, dx = \omega_{N-1} \int_0^R r^a|w'(r)|^p \, dr \geq \omega_{N-1} \left| \frac{a+1-p}{p} \right| \int_0^R r^{a-p}|w(r)|^p \, dr
\]

\[
= \left| \frac{N(p-1)+\alpha-p}{p} \right| \int_{B_R} |\nabla u|^p \, dx.
\]
Here note that \( \lim_{r \to 0} r^{N-\alpha} |w(r)|^p = \lim_{r \to 0} r^{\alpha+1-p} |w(r)|^p = 0 \) since \( \alpha < N \). Therefore we obtain (39).

As a Corollary, we also obtain the following inequality.

**Corollary 4.2.** (Ref. [37]) Let \( \alpha < N \). Then the inequality
\[
A_{\alpha,p}^p \int_{B_R} \frac{|u|^p}{|x|^\alpha} \, dx \leq \int_{B_R} \frac{\nabla u|^{p}}{|x|^\alpha-kp} \, dx
\]
holds for any radially symmetric functions \( u \in C^k_{c,\text{rad}}(B_R) \).

Next, we show one dimensional improved Hardy inequality (40). Although the proof is essentially the same as it of Theorem 2.2, we give the proof here again.

**Proposition 4.3.** Let \( \gamma > 0, b < -1, a + 1 + (b + 1) \gamma \geq 0, \) and \( 1 \leq p < \infty \). The inequality
\[
\left( -\frac{b+1}{p} \gamma \right)^p \int_0^r a \left( 1 - \left( \frac{r}{R} \right)^\gamma \right)^b |w(r)|^p \, dr \leq \int_0^r a^{b+p} \left( 1 - \left( \frac{r}{R} \right)^\gamma \right)^{b+p} |w'(r)|^p \, dr
\]
(40)
holds for any \( w \in C^1(0, R) \) with \( \lim_{r \to R} \frac{a}{R} |w(r)|^p = \lim_{r \to +0} r^{a+1} |w(r)|^p = 0 \). Furthermore, if we assume that the equality of (40) is attained by some non-zero function \( w \), then \( a + 1 + (b + 1) \gamma = 0 \) and \( w(r) = C \left( \left( \frac{r}{R} \right)^{-\gamma} - 1 \right)^{-\frac{b+1}{p}} \) a.e. in \( r \in (0, R) \) for some constant \( C \neq 0 \), whose integrals on both sides in (40) diverge.

**Proof.** For the simplicity, we set \( R = 1 \). Then we have
\[
-(b + 1) \gamma \int_0^1 a \left( 1 - r^\gamma \right)^b |w(r)|^p \, dr = \int_0^1 a^{b+1} \left[ (r^{-\gamma} - 1)^{b+1} \right]' |w(r)|^p \, dr
\]
\[
= -p \int_0^1 a^{b+1} \left( 1 - r^\gamma \right)^{b+1} |w(r)|^{p-2} w(r) w'(r) \, dr
\]
\[
-(a + 1 + (b + 1) \gamma) \int_0^1 a \left( 1 - r^\gamma \right)^{b+1} |w(r)|^p \, dr
\]
On the last equality, we used \( \lim_{r \to 1-0} (1 - r) |w(r)|^p = 0 = \lim_{r \to +0} r^{a+1} |w(r)|^p = 0 \). Since \( a + 1 + (b + 1) \gamma \geq 0 \), we have
\[
\int_0^1 a \left( 1 - r^\gamma \right)^b |w(r)|^p \, dr
\]
\[
\leq \int_0^1 a^{b+1} \left( 1 - r^\gamma \right)^{b+1} |w(r)|^{p-2} w(r) \left( \frac{p}{(b+1) \gamma} w'(r) \right) \, dr
\]
\[
\leq \left( \frac{p}{(b+1) \gamma} \right)^{\frac{1}{p}} \left( \int_0^1 a \left( 1 - r^\gamma \right)^b |w(r)|^p \, dr \right)^{\frac{p}{p-2}} \left( \int_0^1 a^{b+a} \left( 1 - r^\gamma \right)^{b+b} |w'(r)|^p \, dr \right)^{\frac{2}{p}}
\]
(41)
which implies (40). Assume that there exists an extremal function \( w \neq 0 \) which attains the equality of (40). Then we see that \( a + 1 + (b + 1) \gamma = 0 \) from the above calculations. Since \( |w| \) is also an extremal function, we may assume that \( w \) is nonnegative. Besides, if there exists a interval \((A, B) \subset (0, 1)\) such that \( w(r) <
Proposition 4.6. Let

\[ \int_0^1 r^a (1 - r^\gamma)^b |w(r)|^p \, dr < \int_0^1 r^a (1 - r^\gamma)^b |\tilde{w}(r)|^p \, dr, \]

\[ \int_0^1 r^{a+p} (1 - r^\gamma)^{b+p} |w'(r)|^p \, dr > \int_0^1 r^{a+p} (1 - r^\gamma)^{b+p} |\tilde{w}'(r)|^p \, dr \]

which contradicts that \( w \) is an extremal function. Therefore, we may assume that \( w \) is nonnegative and non-increasing. From the equality condition of the Hölder inequality in (41),

\[ w'(r) = \frac{(b+1)\gamma}{p} \frac{w(r)}{r(1-r^\gamma)} \]

which implies that \( w(r) = C (r^{-\gamma} - 1)^{-\frac{b+1}{p}} \) a.e. in \( r \in (0, R) \) and for some constant \( C > 0 \), where \( \gamma = -\frac{b+1}{p} \). Furthermore, we can check that

\[ \int_0^1 r^{a+p} (1 - r^\gamma)^{b+p} |w'(r)|^p \, dr = \left( \frac{(b+1)}{p} \frac{\gamma}{\gamma} \right) \int_0^1 r^a (1 - r^\gamma)^b |w(r)|^p \, dr \]

\[ = \left( \frac{(b+1)}{p} \frac{\gamma}{\gamma} \right) Cp \int_0^1 r^{-1} (1 - r^\gamma)^{-1} \, dr = \infty. \]

As a corollary of Proposition 4.3, we can obtain the following in the same way as the proof of (39).

Corollary 4.4. Let \( p \geq 1, \beta > 1, \gamma > 0, \) and \( \alpha \leq N - (\beta - 1)\gamma - (N - 1)p. \) Then the inequality

\[ \left( \frac{\beta-1}{p} \frac{\gamma}{\gamma} \right)^p \int_{B_R} \frac{|\nabla u|^p}{|x|^\alpha} \left( 1 - \left( \frac{|x|}{p} \right)^\gamma \right)^\beta \, dx \leq \int_{B_R} \frac{|\Delta u|^p}{|x|^\alpha-p} \left( 1 - \left( \frac{|x|}{p} \right)^\gamma \right)^{\beta-p} \, dx \]  

(42)

holds for any \( u \in C^2_{\text{e,rad}}(B_R) \).

Remark 4.5. Due to the assumption \( \alpha \leq N - (\beta - 1)\gamma - (N - 1)p \) in the inequality (42), we cannot show \( L^p(p \neq 2) \) version of the inequality (26) in the same way as the proof in Sect. 3.2.

Finally, we calculate \( \nabla^k \left[ (1 - |x|^\gamma)^A \right] \) to show Theorems in Sect. 3.

Proposition 4.6. Let \( k, m \in \mathbb{N} \).

\[ \nabla^k \left[ (1 - |x|^\gamma)^A \right] = \sum_{j=1}^k C_{k,j} |x|^j \gamma - k (1 - |x|^\gamma)^{A-j} \begin{cases} \frac{1}{|x|} & \text{if } k = 2m, \\ \frac{x}{|x|} & \text{if } k = 2m + 1. \end{cases} \]  

(43)

where \( \{ C_{k,j} \}_{j=1}^k \) is given by

\[ C_{k,k} = (-\gamma)^k \prod_{\ell=1}^{k} (A - \ell + 1) \quad (k \geq 1), \]
$$C_{k,1} = \begin{cases} -A \prod_{\ell=1}^{m} (\gamma - 2\ell + 2)(N + \gamma - 2\ell) & \text{if } k = 2m, \\ -A(\gamma - 2m) \prod_{\ell=1}^{m} (\gamma - 2\ell + 2)(N + \gamma - 2\ell) & \text{if } k = 2m + 1, \end{cases} \quad (k \geq 2)$$

$$C_{k,j} = \begin{cases} C_{k-1,j}(N + j\gamma - k) - C_{k-1,j-1}(A - j + 1) & \text{if } k = 2m, \\ C_{k-1,j}(j\gamma - k + 1) - C_{k-1,j-1}(A - j + 1)\gamma & \text{if } k = 2m + 1, \end{cases} \quad (j = 2, \ldots, k - 1, k \geq 3)$$

**Proof.** Since

$$\nabla \left[(1 - |x|^\gamma)^A\right] = -A\gamma |x|^\gamma - 2x (1 - |x|^\gamma)^{A-1},$$
$$\Delta \left[(1 - |x|^\gamma)^A\right] = -A\gamma(N + \gamma - 2) |x|^\gamma - 2 (1 - |x|^\gamma)^{A-1}$$

$$+ A(A - 1)\gamma^2 |x|^{2\gamma - 2} (1 - |x|^\gamma)^{A-2},$$

we see that (43) holds when $k = 1, 2$. Now we assume that (43) holds for $k = 2m \geq 2$. Then we have

$$\nabla^{k+1} \left[(1 - |x|^\gamma)^A\right] = \nabla \Delta^m \left[(1 - |x|^\gamma)^A\right] = \sum_{j=1}^{k} C_{k,j} \nabla \left[|x|^{j\gamma - k}(1 - |x|^\gamma)^{A-j}\right]$$

$$= \sum_{j=1}^{2m} C_{2m,j} (j\gamma - 2m) |x|^{j\gamma - 2m - 2}x (1 - |x|^\gamma)^{A-j}$$

$$- C_{2m,j} \gamma(A - j) |x|^{j\gamma + \gamma - 2m - 2}x (1 - |x|^\gamma)^{A-j-1}$$

$$= C_{2m,1} (\gamma - 2m) |x|^{\gamma - (k+1)} (1 - |x|^\gamma)^{A-1} \left(\frac{x}{|x|}\right)$$

$$+ \sum_{j=2}^{2m} \{C_{2m,j} (j\gamma - 2m) - C_{2m,j-1} \gamma(A - j + 1)\} |x|^{j\gamma -(k+1)} (1 - |x|^\gamma)^{A-j} \left(\frac{x}{|x|}\right)$$

$$- C_{2m,2m} \gamma(A - 2m) |x|^{(k+1)\gamma -(k+1)} (1 - |x|^\gamma)^{A-(k+1)} \left(\frac{x}{|x|}\right)$$

$$= \sum_{j=1}^{k+1} C_{k+1,j} |x|^{j\gamma -(k+1)} (1 - |x|^\gamma)^{A-j} \left(\frac{x}{|x|}\right)$$

which implies that (43) holds for $k + 1$. In the same way, we can also show (43) in the case where $k = 2m + 1$. Thus (43) holds for any $k \in \mathbb{N}$. \qed

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