ALMOST-EXPONENTIAL MOMENTS OF THE INELASTIC BOLTZMANN EQUATION FOR HARD POTENTIALS WITHOUT ANGULAR CUTOFF

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Abstract. This paper is concerned with the inelastic Boltzmann equation without angular cutoff. We work in the spatially homogeneous case. We establish the global-in-time existence of measure-valued solutions under the generic hard potential long-range interaction on the collision kernel. In addition, we provide a rigorous proof for the creation of almost exponential moments of the measure-valued solutions, which is a special property that can only be expected from hard potential collisional cross-sections. The proofs rely crucially on the establishment of a sharper Povzner-type inequality. The class of initial data that we require is general in the sense that we only require the boundedness of $(2 + \kappa)$-moment for $\kappa > 0$ and do not assume any entropy bound.

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1. Introduction

1.1. Problem statement and motivations. This paper is concerned with the inelastic Boltzmann equation which describes the dynamics of particles that undergo the inelastic collision process such as granular media. Though studies on the Boltzmann equation have a long history since its first appearance, the global well-posedness of the equation for inelastic collisional processes for general initial-boundary values has only been answered partially and is still highly open. In this paper, we are interested in working on the inelastic regime without any angular cutoff assumption. Especially, we are interested in the hard potential case whose range can cover the standard hard-sphere case (with an angular singularity though the angular kernel for the actual hard-sphere model has a cutoff). The motivation behind the collision cross-section with an angular singularity goes back to the very early prototype of Maxwell even before Boltzmann’s formulation of the equation. Regarding the hard-potential equation without angular cutoff, we would first like to establish the global existence of measure-valued solutions in the spatially homogeneous case under a general class of initial conditions. The class of initial conditions requires a slightly higher moment bound than the energy bound, and no entropy bound is needed. In addition, we would also like to prove the moment-creation property under the same class of initial conditions. We note that this moment-creation property is a special property that only the hard potential regime can retain. These problems have been in general considered to be very difficult, since we have been lacking a sufficiently sharp \textit{a priori} Povzner-type estimate. Hence, we put the establishment of a \textit{sharper} Povzner-type estimate as the main goal of this paper. Then after we obtain the \textit{almost sharp} Povzner inequality, we will provide a proof for the global well-posedness and the moment-creation property for the inelastic Boltzmann equation without angular cutoff.

The establishment of the existence theory of the measure-valued solution in the hard-potential case completes our journey for the global existence of measure-valued solutions complementing our preceding work \cite{27, 28}, in which the inelastic Boltzmann equation associated with Maxwellian molecule and soft-potential collision kernels have been studied. Furthermore, another significant novelty of this paper is on the proof of the creation of almost exponential moments of the solution obtained for the inelastic Boltzmann equation under a certain class of initial conditions. In fact, as a distinctive property of the solution for the hard-potential collision kernel (for instance, hard sphere), the moment-creation property of the elastic Boltzmann equation has been figured out for a long period of time. We would like to introduce that Desvillettes in \cite{13} showed that if some moments of the initial data of more than two order is bounded, then all moments of the solution are bounded for any positive time, which essentially extended the results of Elmroth in \cite{15} that all moments remain bounded uniformly in time if they are initially bounded. Moreover, the result of Desvillettes has further been proved to be true by Wennberg in \cite{29, 30} only if the energy of the initial data is bounded. Later, Morimoto-Wang-Yang showed that the same result also holds for the measure-valued solution in \cite{25} where they use a sharpen form of the so-called Povzner inequality originated by Mischler-Wennberg in \cite{23}. Meanwhile, thanks to the moment-estimates mentioned above, more intricate studies about of tail behavior of the solution have been derived, for which we refer to the seminal work \cite{5, 20} by Bobylev and Gamba-Panferov-Villani as well.
as recent progresses \cite{2,17} by Alonso-Caño-Gamba-Mouhot and Fournier for an exponential moment-estimate.

On the other hand, compared to the elastic case, such a moment-estimate for the inelastic Boltzmann equation is still very rare, especially under the non-cutoff assumption. In \cite{9}, Bobylev-Gamba-Panferov first established the moment-inequalities of the inelastic Boltzmann equation with cutoff hard-potential collision kernel, where a sharper version of the Povzner inequality for the inelastic collision was proved generalizing the previous elastic counterpart \cite{5} by Bobylev. Besides, the pseudo-Maxwellian model, regarded as an approximated equation via the replacement of the collision kernel by a certain mean-value independent on the relative velocity, was studied in \cite{6} by Bobylev-Carrillo-Gamba; in addition, for the self-similar scaling problem, solution with “flat” tail, decaying like an inverse power law, was first conjectured by Ernst-Brito in \cite{16} and was rigorously justified by Bobylev-Cercignani-Toscani in \cite{8}. We also mention the moment-equation for the inelastic model with a heat bath by Carrillo-Cercignani-Gamba in \cite{11}, Bobylev-Cercignani in \cite{7} and Gamba-Panferov-Villani in \cite{19} for references. Recently, the rigorous derivation of the system of hydrodynamic equations from the inelastic Boltzmann equation has been introduced in \cite{3} and \cite{4}.

1.2. The inelastic Boltzmann equation. Unlike the classical elastic collision between particles, the loss of energy occurs in the impact direction during the inelastic collision process. If we let \( e \) be the restitution coefficient, the post-collisional velocities \( v', v_\#' \) can be represented as

\[
\begin{align*}
  v' &= \frac{v + v_\#}{2} + \frac{1 - e}{4}(v - v_\#) + \frac{1 + e}{4}|v - v_\#|\sigma, \\
  v_\#' &= \frac{v + v_\#}{2} - \frac{1 - e}{4}(v - v_\#) - \frac{1 + e}{4}|v - v_\#|\sigma,
\end{align*}
\]

where \( v, v_\# \) are the pre-collisional velocities and \( \sigma \) is a vector on the unit sphere \( S^2 \). Then, without considering the dependence on a spatial variable, we will study the following spatially homogeneous Boltzmann equation in the inelastic case

\[
\partial_t f(t, v) = Q_e(f, f)(t, v),
\]

associated with the non-negative initial condition

\[
f(0, v) = F_0(v).
\]

Here \( f = f(t, v) \) is an unknown density function of a probability density function, and \( Q_e(f, f) \) is called inelastic Boltzmann collision operator.

The weak formulation of the equation can then be defined as

\[
\int_{\mathbb{R}^3} f(t, v)\psi(v) \, dv - \int_{\mathbb{R}^3} f(0, v)\psi(v) \, dv = \int_0^t \int_{\mathbb{R}^3} Q_e(g, f)(s, v)\psi(v) \, dv \, ds
\]

\[
= \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B(|v - v_\#|, \sigma)g(s, v_\#)f(s, v)
\]

\[
\times \left[ \psi(v') + \psi(v_\#') - \psi(v) - \psi(v_\#) \right] \, d\sigma \, dv_\# \, ds \, dv,
\]

for any test function \( \psi(v) \in C^0_c(\mathbb{R}^3) \). The reason why the weak form (1.4) is usually preferred in the study of the inelastic equation is that the post-collisional mechanism of the inelastic collision has been merely manifested in the test function \( \psi(v') \) and \( \psi(v_\#') \). Although the restitution coefficient \( e \) might depend on the relative velocity of the colliding particles \cite{12}, we will consider coefficient \( e \) as a constant in this paper.
Lastly, we also introduce the conservation of momentum and the dissipation of energy during each inelastic collision, which are the direct consequences of (1.1):

\[ v' + v'_* = v + v_* \text{ and } |v'|^2 + |v'_*|^2 \leq |v|^2 + |v_*|^2. \]

This further implies that the inelastic collision operator \( Q_e \) satisfies

\[
\int_{\mathbb{R}^3} \frac{1}{v}|Q_e(f, f)(v)|dv = 0, \quad \text{but} \quad \int_{\mathbb{R}^3} |v|^2 Q_e(f, f)(v)dv \leq 0. \tag{1.5}
\]

\[
\int_{\mathbb{R}^3} |v|^2 Q_e(f, f)(v)dv \leq 0. \tag{1.6}
\]

### 1.3. Hard potential collision kernel without angular cutoff.

The Boltzmann collision operator (1.4) contains a collision kernel \( B(|v - v_*|, \sigma) \) which further describes the types of the interactions of colliding particles. It is defined for measuring the intensity of the collision between interacting particles, which is not only relevant with the deviation angle \( \theta \), but also depends on the relative velocity \( |v - v_*| \). From the physical perspective, \( B(|v - v_*|, \sigma) \) is taken in the product form; i.e.,

\[
B(|v - v_*|, \sigma) = b(\cos \theta) \Phi(|v - v_*|) \text{ with } \cos \theta = \hat{q} \cdot \sigma \text{ and } \hat{q} = \frac{v - v_*}{|v - v_*|}. \tag{1.7}
\]

Thanks to the standard symmetrization, we can restrict the range of \( \theta \) into \([0, \pi/2]\) without loss of generality in the sense that

\[
b(\cos \theta) = [b(\cos \theta) + b(\cos (\pi - \theta))] 1_{0 \leq \theta \leq \pi/2}.
\]

In general, the kernel that is derived from the inverse-power-law interaction potential involves an asymptotic singularity in \( b(\cos \theta) \) as the deviation angle \( \theta \) approaches to zero:

\[
\sin \theta b(\cos \theta)|_{\theta \to 0^+} \sim K \theta^{-1-2s}, \quad \text{for } 0 < s < 1 \text{ and } K > 0. \tag{1.8}
\]

In order to overcome the difficulties from the singular behavior, the so-called Grad’s cutoff assumption has been introduced cutting off a small angle nearby the singularity such that

\[
\int_{S^2} b_n(\hat{q} \cdot \sigma) d\sigma < \infty. \tag{1.9}
\]

Here, the angular kernel \( b_n \) can be general enough as long as it is integrable in \( \sigma \) and \( b_n \to b \) as \( n \to \infty \). One of the possible examples is the one with the standard mollification that we provide in (3.2) for the existence theory.

For the hard potential collisions without cutoff, we consider \( B(|v - v_*|, \sigma) \) in the form of (1.7) with the following assumptions throughout the rest of the paper:

- The angular kernel \( b(\cos \theta) \) is not locally integrable but satisfies

\[
\exists \text{ some } \alpha_0 \in (0, 2] \text{ such that } \int_0^{\pi/2} \sin^{\alpha_0} \left( \frac{\theta}{2} \right) b(\cos \theta) \sin \theta d\theta < \infty. \tag{1.10}
\]

- The kinetic kernel \( \Phi(|v - v_*|) \) is in a power form and satisfies the hard potential assumption that

\[
\Phi(|v - v_*|) = |v - v_*|^\gamma \quad \text{with} \quad 0 < \gamma \leq 2. \tag{1.11}
\]

**Remark 1.1.** Several previous results in the case without angular cutoff such as [1,21,24] refer the hard potential model to the case with an assumption that \( \gamma + 2s \geq 0 \). In this paper, we only consider the case when \( 0 < \gamma \leq 2 \) in which one can expect the moment-creation property as well. For the remaining case of \(-2s \leq \gamma \leq 0\) we refer to [27,28].
1.4. Statement of our main results. Throughout this paper, we shall use the
Japanese bracket notation $\langle \cdot \rangle = \sqrt{1 + \cdot^2}$, and $A \lesssim B$ (or $B \lesssim A$) means that
there is a generic constant $C > 0$ such that $A \leq CB$ (or $B \leq CA$, resp.).

We would first like to introduce the global existence of a measure-valued solution
to the inelastic Boltzmann equation without angular cutoff. We define the space
$P_\alpha(\mathbb{R}^3)$ as the set of probability measures on $\mathbb{R}^3$ with finite moments up to the
order $\alpha$; i.e.,

$$
P_\alpha(\mathbb{R}^3) \overset{\text{def}}{=} \left\{ F(v) \in P_0(\mathbb{R}^3) \middle| \int_{\mathbb{R}^3} dF(v) = 1, \int_{\mathbb{R}^3} |v|^\alpha dF(v) < \infty, \right. \quad (1.12)
$$

and if $\alpha > 1$, $F$ further satisfies $\int_{\mathbb{R}^3} v_j dF(v) = 0, j = 1, 2, 3 \}$.

Then we establish the following existence theorem under a general class of initial
conditions without the smallness requirement nor an entropy bound.

**Theorem 1.2** (Global existence). Assume that $\varepsilon \in (0, 1]$ and the collision kernel
$B(|v - v_\ast|, \sigma)$ satisfies the non-cutoff hard potential assumptions (1.10) and (1.11)
with $0 < \gamma \leq 2$. For any initial datum $F_0(v) \in P_{2+\kappa}(\mathbb{R}^3)$ with $\kappa > 0$, then
there exists a measure-valued solution $F_t(v) \in C \left( [0, \infty), P_{2+\kappa}(\mathbb{R}^3) \right)$ to the Cauchy
problem (1.2)-(1.3).

**Remark 1.3.** The uniqueness for a measure-valued solution for the inelastic case
without angular cutoff has not been known so far in general to the best of the authors’
knowledge. In the elastic case for non-cutoff hard potentials, one can obtain the
uniqueness results by assuming the initial boundedness of either the exponential
moment

$$
\int_{\mathbb{R}^d} e^{\varepsilon|v|^\gamma} f_0(du) < +\infty,
$$
as in [18, Corollary 2.3] or a more regular norm

$$
\|f_0\|_{W_q^{1,1}} < +\infty, \text{ for } q \geq 2,
$$
as in [14, Theorem 2].

In addition, we also present the moment-creation property for any measure-valued
solution $F_t$ with a suitable dissipation condition.

**Theorem 1.4** (Creation of almost-exponential moments). Assume that $\varepsilon \in (0, 1]$, $\kappa > 0$, and the collision kernel $B(|v - v_\ast|, \sigma)$ satisfies the non-cutoff hard potential
assumptions (1.10) and (1.11) with $0 < \gamma \leq 2$. For any measure-valued solution
$F_t(v) \in C \left( [0, \infty), P_{2+\kappa}(\mathbb{R}^3) \right)$ to problem (1.2)-(1.3) with the initial datum $F_0(v) \in P_{2+\kappa}(\mathbb{R}^3)$, the following moment propagation property holds; for any $l \in \mathbb{R}$ and $t_0 \in (0, \infty)$, we have

$$
\sup_{t > t_0} \int_{\mathbb{R}^3} \langle v \rangle^l dF_t(v) < \infty. \quad (1.13)
$$

**Remark 1.5.** Here we would like to emphasize that the proofs of the global well-
posedness and the moment-creation property heavily depend on the establishment
of a sharper version of the Povzner inequality for inelastic collision under both
cutoff and non-cutoff assumption. For instance, the establishment of an improved
Povzner inequality relieves the moment-conditions to guarantee the moment-gain
property in Theorem 1.4 above. One of the key ideas for the establishment of a
sharper inequality is to work on the center-of-momentum coordinate system \((2.4)\) instead of working with the standard inelastic post-collisional representations \((1.1)\). Our work under the presence of an angular singularity is motivated by the work in the elastic case \([23, 25]\). Since the inelastic case involves additional parameters that directly affect the decomposition of collisional scattering angle, this requires more technical (but careful) analysis on the computations of the Jacobian determinants for the varied changes of variables during the journey of the proof of the sharper Povzner inequality \((Proposition \ 2.1)\).

1.5. Outline of the paper. The rest of the paper is organized as follows. In Section 2, we will first establish a sharper Povzner-type estimate for the inelastic Boltzmann equation under both cutoff and non-cutoff assumptions, which works as a powerful tool in the following sections. Using the Povzner estimate, we establish the existence theory of the measure-valued solution to the inelastic Boltzmann equation with hard potential in Section 3. Then the moment-creation property of the measure-valued solution will further be provided in Section 4.

2. New non-cutoff Povzner estimates for the inelastic hard-potentials

One of the main properties that arise in the hard potential case is the moment-creation property. For the proofs of global wellposedness and the moment-creation property in the noncutoff situation, the establishment of an improved Povzner-type estimate is crucial. Unfortunately, all the existing Povzner-type estimates are not sharp enough to prove the well-posedness in the non-cutoff case and the moment-creation property to the best of the authors’ knowledge. Therefore, we establish in this section an almost sharp version of the Povzner inequality for the proof of the moment-creation property.

**Proposition 2.1.** Assume that \(e \in (0, 1]\) and the angular cross-section \(b\) satisfies \((1.8)\) and \((1.10)\). Let \(\psi\) denote a convex function
\[
\psi_1(x) = \psi_{1, \kappa}(x) \overset{\text{def}}{=} x^{1 + \frac{\kappa}{2}} \quad \text{or} \quad \psi_2(x) = \psi_{2, \kappa}(x) \overset{\text{def}}{=} (1 + x)^{1 + \frac{\kappa}{2}} - 1,
\]
with \(\kappa > 0\). Let \(K^\kappa(v, v_\ast)\) be defined for every \(\kappa > 0\) as follows:
\[
K^\kappa(v, v_\ast) = \int_{S^2} b(\hat{q} \cdot \sigma) \left[ \psi(|v'|^2) + \psi(|v_\ast|^2) - \psi(|v|^2) - \psi(|v_\ast|^2) \right] d\sigma. \tag{2.2}
\]
Then, we can represent \(K^\kappa\) as \(K^\kappa(v, v_\ast) = -H(v, v_\ast) + G(v, v_\ast)\) where \(H\) and \(G\) further satisfy the following bounds:
\[
-H(v, v_\ast) \leq -C_1 \left( \langle v \rangle^{2 + \kappa} 1_{\langle v \rangle \geq 2\langle v_\ast \rangle} + \langle v_\ast \rangle^{2 + \kappa} 1_{\langle v_\ast \rangle \geq 2\langle v \rangle} \right),
\]
and
\[
G(v, v_\ast) \left\{ \begin{array}{l}
\leq C_2 |v| |v_\ast|^2, \quad \text{if } \kappa < 2, \\
\leq C_3 \left( |v| |v_\ast|^\kappa + |v|^2 |v_\ast|^\kappa \right), \quad \text{if } \kappa \geq 2,
\end{array} \right.
\]
where \(C_1, C_2, C_3\) are constants that depend on \(e\) and \(\kappa\).

Before we move onto the proof, we first introduce several different coordinates that we use. Our key steps are on the several series of changes of angular variables. The first one is that we decompose the angular variable into the standard polar coordinates \(\sigma \mapsto (\theta, \phi)\) with a specific choice of the \(z\)-axis motivated by \([23, 25]\) where the authors improve the Povzner estimates sharper in the elastic case. Then
the key step is that we further proceed defining our own special angles $\chi, \mu$ and take $(\theta, \phi) \longrightarrow (\chi, \mu)$ to further improve the estimates even in the non-cutoff inelastic situation.

To this end, we first define and write the cutoff version $K_n^e(v, v_*)$ of the operator $K^e$ as

$$K_n^e(v, v_*) = \int_{S^2} b_n(\hat{\sigma} \cdot \sigma) \left[ \psi(||v'||^2) + \psi(||v'_*||^2) - \psi(||v||^2) - \psi(||v_*||^2) \right] \, d\sigma, \quad (2.3)$$

where $b_n$ is a mollified angular kernel with Grad’s cutoff that satisfies (1.9). This includes the specific one (3.2) that we use. In terms of distinguishing inelastic collisions, it would be more advantageous to parametrize the post-collisional velocities $v'$ and $v'_*$ in the center-of-momentum coordinate system as illustrated in Figure 1, where the similar coordinate system used to be applied in [9, 19] as well.

We begin with setting

$$v' = \frac{v_+ + \lambda v_- |\omega|}{2} \quad \text{and} \quad v'_* = \frac{v_+ - \lambda v_- |\omega|}{2}, \quad (2.4)$$

where $v_+ = v + v_*$, $v_- = v - v_*$, and $\omega \in S^2$ is a unit vector. Then we observe

$$\lambda \omega = a_+ \sigma + a_- \hat{v}_-, \quad (2.5)$$
with \( a_+ = \frac{1 + e}{2}, \quad a_- = \frac{1 - e}{2} \) and \( \dot{v}_- = \frac{v_0}{|v_-|} \). Therefore, one can see that

\[
\lambda = \lambda(\cos \chi) = a_- \cos \chi + \sqrt{a_-^2 (\cos^2 \chi - 1) + a_+^2},
\]

where \( \chi \) is the angle between \( v_- \) and \( \omega \). Then we note that

\[
0 < e \leq \lambda(\cos \chi) \leq 1,
\]

for all \( \chi \). With this parametrization, we can represent the size of the post-collisional velocities as

\[
|v'|^2 = \frac{|v_+|^2 + \lambda^2 |v_-|^2 + 2 \lambda |v_+||v_-| \cos \mu}{4} = Y(\chi) + Z(\chi) \cos \mu, \quad \text{and}
\]

\[
|v'_\perp|^2 = \frac{|v_+|^2 + \lambda^2 |v_-|^2 - 2 \lambda |v_+||v_-| \cos \mu}{4} = Y(\chi) - Z(\chi) \cos \mu,
\]

where

\[
Y(\chi) = \left( |v_+|^2 + \lambda^2 |v_-|^2 \right)/4, \quad Z(\chi) = (\lambda |v_+||v_-|)/2. \tag{2.6}
\]

Here \( \mu \) is the angle between the vector \( v_+ \) and \( \omega \). Moreover, by taking an inner-product with \( \dot{v}_- \) on both sides of relation (2.5), we can also derive that

\[
\lambda \cos \chi = a_+ \cos \theta + a_-.
\]

Then this provides

\[
\cos \theta = \frac{\lambda \cos \chi - a_-}{a_+} = \left[ a_- \cos \chi + \sqrt{a_-^2 (\cos^2 \chi - 1) + a_+^2} \right] \cos \chi - a_-, \tag{2.7}
\]

where \( \theta \) is the deviation angle between \( \sigma \) and \( \frac{v_0 - v_\perp}{|v_0 - v_\perp|} \), manifested in the standard decomposition of \( \sigma \in S^2 \).

We would like to mention that the estimate with respect to \( \sigma \) is now transferred to the estimates of \( \theta \) and \( \phi \), and the singularity of \( \theta \) can now be canceled by the “good” terms provided from the integration by part in \( \phi \). Consequently, by noticing that \( |v'|^2 \) and \( |v'_\perp|^2 \) have been represented in the angle \( \chi \) and \( \mu \) instead of \( \theta \) and \( \phi \), we can have a sharper version of the inelastic Povzner inequality which follows from the Jacobian determinant of the transformation \( \sigma \rightarrow (\theta,\phi) \rightarrow (\chi,\mu) \). This Jacobian determinant will be computed in the following proof. The proof also requires the use of the convexity of specifically chosen functions \( \psi \), as well as the use of the monotonicity of \( \psi \) for \( x > 0 \) and \( \psi(0) \leq 0 \). These conditions will be used to provide an additional room to cancel the angular singularity.

**Proof of Proposition 2.1.** As the first step of the standard polar coordinate change \( \sigma \rightarrow (\theta,\phi) \) is now clear, our focus will now be on the second step \( (\theta,\phi) \rightarrow (\chi,\mu) \). We first attempt to find the relation between \( \cos \theta = A \) and \( \cos \chi = B \) such that,
by (2.7),
\[
\frac{dA}{dB} = \frac{1}{a_+} \left[ \frac{a_+^2 B^2}{\sqrt{a_+^2(B^2 - 1) + a_+^2}} + 2a_- B + \sqrt{a_+^2(B^2 - 1) + a_+^2} \right]
\]
\[
= \left[ a_- B + \sqrt{a_+^2(B^2 - 1) + a_+^2} \right]^2 a_+ \sqrt{a_+^2(B^2 - 1) + a_+^2}.
\]

In the original representation in the spherical coordinates
\[
\sigma = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi),
\]
the singularity of the collision kernel \( b(\cos \theta) \) appears when \( \theta \rightarrow 0 \); i.e., \( A = \cos \theta \rightarrow 1 \). By (2.7), this corresponds to \( B = \cos \chi \rightarrow 1 \) as well. Furthermore, when \( A = \cos \theta \rightarrow 0 \), it again follows from the direct calculation via (2.7) that,
\[
0 = A = \frac{a_- B + \sqrt{a_+^2(B^2 - 1) + a_+^2}}{a_+} B - a_-,\]
\[
\Rightarrow a_- B + \sqrt{a_+^2(B^2 - 1) + a_+^2} B = a_-,\]
\[
\Rightarrow \sqrt{a_+^2(B^2 - 1) + a_+^2} = \frac{a_-}{B} - a_- B,\]
\[
\Rightarrow a_+^2(B^2 - 1) + a_+^2 = a_-^2 \left( \frac{1}{B^2} - 2 + B^2 \right),\]
\[
\Rightarrow a_+^2 + a_-^2 = \frac{a_-^2}{B^2},\]
\[
\Rightarrow B = \frac{a_-}{\sqrt{a_+^2 + a_-^2}}.
\]

Thus the limit \( A \rightarrow 0 \) corresponds to the limit \( B = \cos \chi \rightarrow \frac{a_-}{\sqrt{a_+^2 + a_-^2}} \). Hence, the transformation \((\theta, \phi) \rightarrow (\chi, \mu)\) can be re-written as \((A, \phi) \rightarrow (B, \mu)\), and we further note that the Jacobian determinant \[\frac{\partial (A, \phi)}{\partial (B, \mu)}\] is
\[
\left| \frac{\partial (A, \phi)}{\partial (B, \mu)} \right| = \frac{\partial A}{\partial B} \frac{\partial \phi}{\partial B} - \frac{\partial A}{\partial \phi} \frac{\partial \phi}{\partial B} = 0.
\]

Thus in order to compute the Jacobian determinant \[\frac{\partial (A, \phi)}{\partial (B, \mu)}\], we need to calculate \[\frac{\partial \phi}{\partial B}\].

In order to express the relationship between \( \mu \) and \( \phi \), we devise a different coordinate system as follows. In the coordinate system \((\hat{v}_-, \hat{j}, \hat{h})\), considering the fact that \( \hat{v}_+ \) lies on the plane spanned by \((\hat{v}_-, \hat{j}, \hat{h})\), we have
\[
\hat{v}_+ = (\hat{v}_- \cdot \hat{v}_-) \hat{v}_- + (\hat{v}_+ \cdot \hat{h}) \hat{h}
\]
\[
\cos \mu = \omega \cdot \hat{v}_+ = \frac{a_+ \sigma + a_- \hat{v}_-}{\lambda} \implies \cos \mu = \cos \beta \frac{a_+ \cos \theta + a_-}{\lambda} + \sin \beta \frac{a_+ \sigma \cdot \hat{h}}{\lambda},
\]
where \( \beta \) is the angle between \( \hat{v}_+ \) and \( \hat{v}_- \). By further noticing (2.7), we obtain the expression of \( \cos \mu \) that

\[
\cos \mu = \cos \beta \cos \chi + \sin \beta \frac{a_+ \sin \theta \sin \phi}{\lambda}
\]

\[
= \cos \chi \left( \cos \beta + \frac{a_+ \sin \beta \sin \theta}{a_+ \cos \theta + a_-} \sin \phi \right),
\]

(2.10)
to which we take the derivative with respect to \( \phi \) such that

\[
- \sin \mu \frac{\partial \mu}{\partial \phi} = \frac{a_+ \sin \beta \sin \theta \cos \chi}{a_+ \cos \theta + a_-} \cos \phi.
\]

Therefore, we have

\[
\frac{\partial \phi}{\partial \mu} = - \frac{(a_+ \cos \theta + a_-) \sin \mu}{a_+ \sin \beta \sin \theta \cos \chi \cos \phi}
\]

(2.11)

\[
= - \frac{\lambda \sin \mu}{a_+ \sin \beta \sin \theta \cos \phi}
\]

\[
= - \frac{\sin \mu \tan \phi}{\cos \mu - \cos \beta \cos \chi},
\]

by (2.7) and (2.10). To obtain \( \tan \phi \), we recall (2.10) and find that

\[
\sin \phi = \frac{\lambda (\cos \mu - \cos \beta \cos \chi)}{a_+ \sin \beta \sin \theta}
\]

\[
= \frac{\lambda (\cos \mu - \cos \beta \cos \chi)}{\sin \beta \sqrt{a_+^2 - (\lambda \cos \chi - a_-)^2}}
\]

(2.12)

from which, we derive that

\[
\tan \phi = \frac{\sin \phi}{\sqrt{1 - \sin^2 \phi}} = \frac{\lambda (\cos \mu - \cos \beta \cos \chi)}{\sqrt{\sin^2 \beta [a_+^2 - (\lambda \cos \chi - a_-)^2] - \lambda^2 (\cos \mu - \cos \beta \cos \chi)^2}}
\]

Combing (2.11) and (2.12), we finally obtain

\[
\frac{\partial \phi}{\partial \mu} = - \frac{\lambda \sin \mu}{\sqrt{\sin^2 \beta [a_+^2 - (\lambda \cos \chi - a_-)^2] - \lambda^2 (\cos \mu - \cos \beta \cos \chi)^2}}
\]

(2.13)

Then, in order to have a better reduced representation, we apply another change of variable

\[
\eta = \cos \mu - \cos \beta \cos \chi,
\]

(2.14)
such that \( d\eta = - \sin \mu d\mu \). Then from (2.10), we have

\[
\sin \phi = \frac{\lambda}{\sin \beta \sqrt{a_+^2 - (\lambda \cos \chi - a_-)^2}} \eta
\]

which implies that

\[
\begin{align*}
\phi = \frac{\pi}{2} & \iff \mu \iff \eta = \eta_0, \\
\phi = \pi & \iff \mu \iff \eta = 0, \\
\phi = \frac{3\pi}{2} & \iff \mu \iff \eta = -\eta_0,
\end{align*}
\]
where
\[ \eta_0 = \frac{\sin \beta \sqrt{a_+^2 - (\lambda \cos \chi - a_-)^2}}{\lambda}. \] (2.15)

Then (2.13) is finally reduced into a much simpler form of
\[ \frac{\partial \phi}{\partial \mu} = -\frac{\sin \mu}{\sqrt{\eta_0^2 - \eta^2}}. \] (2.16)

Finally, we have
\[ \int_{S^2} b_n(\hat{\nu} \cdot \sigma) \, d\sigma = 2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{0}^{1} b_n(A) \, dA \, d\phi \]
\[ = 2 \int_{0}^{1} b_n(B) \left\{ \frac{dA}{dB} \right\}_{\eta_0}^{\eta} \left[ \psi(|v'|^2) + \psi(|v|^2) - \psi(|v_+|^2) \right] \left\{ \frac{d\phi}{d\mu} \right\}_{\eta_0}^{\eta} \left\{ \frac{d\mu}{d\eta} \right\}_{\eta_0}^{\eta} \, d\eta \, dB. \]

We put this into the estimate for \( K_n(v, v_+) \) and obtain
\[ K_n(v, v_+) = \int_{S^2} b_n(\hat{\nu} \cdot \sigma) \left[ \psi(|v'|^2) + \psi(|v|^2) - \psi(|v_+|^2) \right] \, d\sigma \]
\[ = 2 \int_{0}^{1} b_n(A) \left\{ \frac{dA}{dB} \right\}_{\eta_0}^{\eta} \left[ \psi(|v'|^2) + \psi(|v|^2) - \psi(|v_+|^2) \right] \left\{ \frac{d\phi}{d\mu} \right\}_{\eta_0}^{\eta} \left\{ \frac{d\mu}{d\eta} \right\}_{\eta_0}^{\eta} \, d\eta \, dB. \]

Now our job is to compute each integral \( \int_{-\eta_0}^{\eta_0} \psi(|u|^2) \left\{ \frac{d\phi}{d\mu} \right\}_{\eta_0}^{\eta} \left\{ \frac{d\mu}{d\eta} \right\}_{\eta_0}^{\eta} \, d\eta \) for \( u = v' \) and \( u = v_+ \).

Noting (2.14) and (2.16), we first observe that
\[ \int_{-\eta_0}^{\eta_0} \psi(|u|^2) \left\{ \frac{d\phi}{d\mu} \right\}_{\eta_0}^{\eta} \left\{ \frac{d\mu}{d\eta} \right\}_{\eta_0}^{\eta} \, d\eta \]
\[ = \left( \int_{\eta_0}^{\eta_0} + \int_{-\eta_0}^{0} \right) \psi(Y(\chi) + Z(\chi)(\cos \beta B + \eta)) \frac{1}{\sqrt{\eta_0^2 - \eta^2}} \, d\eta \]
\[ = \int_{0}^{\eta_0} \left[ \psi(Y(\chi) + Z(\chi)(\cos \beta B + \eta)) + \psi(Y(\chi) + Z(\chi)(\cos \beta B - \eta)) \right] \frac{1}{\sqrt{\eta_0^2 - \eta^2}} \, d\eta \]
\[ = \int_{0}^{\eta_0} \left[ \psi(Y(\chi) + Z(\chi)(\cos \beta B + \eta)) + \psi(Y(\chi) + Z(\chi)(\cos \beta B) + \eta) \right] \frac{1}{\sqrt{\eta_0^2 - \eta^2}} \, d\eta + \pi \psi(Y(\chi) + Z(\chi) \cos \beta B), \]
where $Y$ and $Z$ are defined as in (2.6). Taking the integration by parts twice and denoting $Y = Y(\chi)$, $Z = Z(\chi)$ for the sake of simplicity, we have

$$\int_{-\eta_0}^{\eta_0} \psi(|v'|^2) \frac{d\phi}{d\mu} \left| \frac{d\mu}{d\eta} \right| d\eta$$

$$= - \int_{0}^{\eta_0} \left[ \psi(Y + Z(\cos \beta B + \eta)) + \psi(Y + Z(\cos \beta B - \eta)) - \psi(Y + Z \cos \beta B) \right]$$

$$\times d \left[ \cos^{-1} \left( \frac{\eta}{\eta_0} \right) \right] + \pi \psi(Y + Z \cos \beta B)$$

$$= \left\{ - \cos^{-1} \left( \frac{\eta}{\eta_0} \right) \left[ \psi(Y + Z(\cos \beta B + \eta)) + \psi(Y + Z(\cos \beta B - \eta)) \right] \right\}^{\eta_0}_{0}$$

$$+ Z \int_{0}^{\eta_0} \cos^{-1} \left( \frac{\eta}{\eta_0} \right) \left[ \psi'(Y + Z(\cos \beta B + \eta)) - \psi'(Y + Z(\cos \beta B - \eta)) \right] d\eta$$

$$+ \pi \psi(Y + Z \cos \beta B)$$

$$= Z \int_{0}^{\eta_0} \left[ \psi'(Y + Z(\cos \beta B + \eta)) - \psi'(Y + Z(\cos \beta B - \eta)) \right]$$

$$\times d \left[ \eta \cos^{-1} \left( \frac{\eta}{\eta_0} \right) - \sqrt{\eta_0^2 - \eta^2} \right] + \pi \psi(Y + Z \cos \beta B)$$

$$= \left\{ Z \left[ \eta \cos^{-1} \left( \frac{\eta}{\eta_0} \right) - \sqrt{\eta_0^2 - \eta^2} \right] \right\}^{\eta_0}_{0}$$

$$- Z^2 \int_{0}^{\eta_0} \left[ \eta \cos^{-1} \left( \frac{\eta}{\eta_0} \right) - \sqrt{\eta_0^2 - \eta^2} \right]$$

$$\times \left[ \psi''(Y + Z(\cos \beta B + \eta)) + \psi''(Y + Z(\cos \beta B - \eta)) \right] d\eta$$

$$+ \pi \psi(Y + Z \cos \beta B)$$

$$= Z^2 \int_{0}^{\eta_0} \left[ \sqrt{\eta_0^2 - \eta^2 - \eta \cos^{-1} \left( \frac{\eta}{\eta_0} \right)} \right]$$

$$\times \left[ \psi''(Y + Z(\cos \beta B + \eta)) + \psi''(Y + Z(\cos \beta B - \eta)) \right] d\eta$$

$$+ \pi \psi(Y + Z \cos \beta B).$$

Similarly, for $\psi(|v'_w|^2)$, we have

$$\int_{-\eta_0}^{\eta_0} \psi(|v'_w|^2) \frac{d\phi}{d\mu} \left| \frac{d\mu}{d\eta} \right| d\eta$$

$$= \pi \psi(Y - Z \cos \beta B) + Z^2 \int_{0}^{\eta_0} \left[ \sqrt{\eta_0^2 - \eta^2 - \eta \cos^{-1} \left( \frac{\eta}{\eta_0} \right)} \right]$$

$$\times \left[ \psi''(Y - Z(\cos \beta B + \eta)) + \psi''(Y - Z(\cos \beta B - \eta)) \right] d\eta.$$

With the help of the formula above, $K_n(v, v_w)$ can now be divided into two parts

$$K_n(v, v_w) = -H_n(v, v_w) + G_n(v, v_w),$$
where
\[-H_n(v, v_*) = 2\pi \int_{\sqrt{\nu_*^2 + \rho^2}}^{1} b_n(B) \left[ \psi(Y + Z \cos \beta B) + \psi(Y - Z \cos \beta B) \right. \]
\[-\psi(|v|^2) - \psi(|v_*|^2)] \left. \right| \frac{dA}{dB} \right| dB \]
\[(2.17)\]
with
\[-H_n^1 \overset{\text{def}}{=} 2\pi \int_{\sqrt{\nu_*^2 + \rho^2}}^{1} b_n(B) \left[ \psi(Y + Z \cos \beta B) + \psi(Y - Z \cos \beta B) \right. \]
\[-\psi\left( |v|^2 \cos^2 \frac{\theta}{2} + |v_*|^2 \sin^2 \frac{\theta}{2} \right) - \psi\left( |v_*|^2 \cos^2 \frac{\theta}{2} + |v|^2 \sin^2 \frac{\theta}{2} \right) \left. \right| \frac{dA}{dB} \right| dB, \]
and
\[-H_n^2 \overset{\text{def}}{=} 2\pi \int_{\sqrt{\nu_*^2 + \rho^2}}^{1} b_n(B) \]
\[\times \left\{ \psi\left( |v|^2 \cos^2 \frac{\theta}{2} + |v_*|^2 \sin^2 \frac{\theta}{2} \right) - \cos^2 \frac{\theta}{2} \psi(|v|^2) - \sin^2 \frac{\theta}{2} \psi(|v_*|^2) \right\} \left. \right| \frac{dA}{dB} \right| dB. \]

Regarding the estimate for $H_n^1$, we consider the fact that $\psi(x)$ is convex and $\psi(0) \leq 0$ and obtain that $\psi(x)$ has the super-additive property such that
\[\psi(Y + Z \cos \beta B) + \psi(Y - Z \cos \beta B) \leq \psi(2Y). \]

Furthermore, noting that
\[\frac{|v + v_*|^2}{2} \leq 2Y = \frac{|v_+|^2 + \lambda_2 |v_-|^2}{2} \leq \frac{|v_+|^2 + |v_-|^2}{2} = |v|^2 + |v_*|^2, \]
and the monotonicity of $\psi(x)$ for $x > 0$, we obtain that
\[\psi(Y + Z \cos \beta B) + \psi(Y - Z \cos \beta B) \leq \psi(2Y) \leq \psi(|v|^2 + |v_*|^2). \]
(2.18)

Then by the convexity of $\psi$, on the other hand, we obtain that
\[\psi\left( |v|^2 \cos^2 \frac{\theta}{2} + |v_*|^2 \sin^2 \frac{\theta}{2} \right) \leq \psi\left( |v|^2 \cos^2 \frac{\theta}{2} + |v|^2 \sin^2 \frac{\theta}{2} \right) \]
\[\geq 2\psi\left( \frac{|v|^2 + |v_*|^2}{2} \right) \geq \psi(|v|^2 + |v_*|^2), \]
(2.19)
\[\implies - \left[ \psi\left( |v|^2 \cos^2 \frac{\theta}{2} + |v_*|^2 \sin^2 \frac{\theta}{2} \right) + \psi\left( |v_*|^2 \cos^2 \frac{\theta}{2} + |v|^2 \sin^2 \frac{\theta}{2} \right) \right] \leq -\psi(|v|^2 + |v_*|^2). \]

Therefore, combining (2.18) and (2.19), we have $-H_n^1 \leq 0$.

Regarding $H_n^2$, we follow [25, Proposition 4.1] that there exists a constant $C_1 > 0$ independent of $n$ such that
\[-H_n^2 \leq -C_1 \left( \langle v \rangle^{2+\kappa} 1_{\langle v \rangle \geq 2\langle v_* \rangle} + \langle v_* \rangle^{2+\kappa} 1_{\langle v_* \rangle \geq 2\langle v \rangle} \right). \]
Consequently, by summing up $H_1^2$ and $H_2^2$, we obtain that there exists a constant $C_1 > 0$ independent of $n$ such that
\[
-H_n(v, v_*) \leq -C_1 \left( \langle v \rangle^{2+\kappa} 1_{\langle v \rangle \geq 2\langle v_* \rangle} + \langle v_* \rangle^{2+\kappa} 1_{\langle v_* \rangle \geq 2\langle v \rangle} \right).
\]

On the other hand, for $\psi(x) = \psi_1(x)$ or $\psi_2(x)$ as in (2.1), by noting that $Z \leq Y$, we further have
\[
Z^2 \int_0^{\eta_0} \left[ \sqrt{\eta_0^2 - \eta^2 - \eta \cos^{-1} \left( \frac{\eta}{\eta_0} \right)} \right] \times \left[ \psi''(Y + Z (\cos \beta B + \eta)) + \psi''(Y + Z (\cos \beta B - \eta)) \right] \, d\eta
\]
\[
\leq Z^2 \eta_0^2 (1 + 2Y)^{\frac{\kappa}{2} - 1} \int_0^{\eta_0} \left[ \left( \frac{Y + Z (\cos \beta B)}{1 + 2Y} + \frac{Z}{1 + 2Y^2} \right)^{\frac{\kappa}{2} - 1} + \left( \frac{Y + Z (\cos \beta B)}{1 + 2Y} - \frac{Z}{1 + 2Y^2} \right)^{\frac{\kappa}{2} - 1} \right] \, d\eta
\]
\[
\leq Z^2 \eta_0^2 (1 + 2Y)^{\frac{\kappa}{2} - 1} \int_1^1 \left[ \sqrt{1 - \eta^2 - \eta \cos^{-1} \left( \frac{\eta}{\eta_0} \right)} \right] \left[ \left( 1 + \eta_0 \eta \right)^{\frac{\kappa}{2} - 1} + \left( 1 - \eta_0 \eta \right)^{\frac{\kappa}{2} - 1} \right] \, d\eta
\]
\[
\leq \begin{cases} 
Z^2 \eta_0^2, & \text{if } \kappa < 2; \\
Z^2 \eta_0^2 (1 + 2Y)^{\frac{\kappa}{2} - 1}, & \text{if } \kappa \geq 2.
\end{cases}
\]

By noting that $Z = (\lambda|v_+||v_-|)/2$ and that $\eta_0$ is defined as in (2.15), we observe that
\[
Z^2 \eta_0^2 = \left( \frac{|v_+||v_-| \sin \beta}{2} \right)^2 \left[ a_+^2 - (\lambda \cos \chi - a_-)^2 \right]
\]
\[
= \left( \frac{|v_+ \times v_-|}{2} \right)^2 \left[ a_+^2 - a_-^2 \cos^2 \theta \right]
\]
\[
= a_+^2 |v_* \times v|^2 \sin^2 \theta
\]
\[
\leq a_+^2 |v|^2 |v_*|^2 \sin^2 \theta,
\]
where in the second equality, we notice that $\beta$ is the angle between $v_+$ and $v_-$ and also we used the relationship (2.7). Consequently, we have the following estimate for $G_n(v, v_*)$ that,
\[
G_n(v, v_*) \begin{cases} 
\leq C_{0.2} a_+^2 |v|^2 |v_*|^2 \int_0^{\frac{\pi}{2}} b_n(\cos \theta) \sin^3 \theta d\theta \leq C_2 |v|^2 |v_*|^2, & \text{if } \kappa < 2; \\
\leq C_{0.3} a_+^2 |v|^2 |v_*|^2 \left( 1 + |v|^2 + |v_*|^2 \right)^{\frac{\kappa}{2} - 1} \int_0^{\frac{\pi}{2}} b_n(\cos \theta) \sin^3 \theta d\theta
\end{cases}
\]
\[
\leq C_3 \left( |v_*|^2 \langle v \rangle^\kappa + |v|^2 \langle v_* \rangle^\kappa \right), & \text{if } \kappa \geq 2;
\]
where $C_{0.2}$, $C_2$, $C_{0.3}$, $C_3$ are constants that are independent of $n$. This completes the proof of the sharper Povzner inequality.

We are now equipped with an improved version of the Povzner-type inequality, which can cover the non-cutoff regime with inelastic hard potential interaction up to $\kappa > 0$. Using this, we will provide the well-posedness theory and the moment-creation property in the next sections.
3. Existence of the measure-valued solution

In this section, we establish the existence theory of the measure-valued solution to the Cauchy problem (1.2)-(1.3) using the Fourier transform. The main idea is to first prove the well-posedness for the “cutoff” model by a fixed point theorem, with the help of which, we can further construct a sequence of the approximated solution to the “non-cutoff” equation such that the existence of the “non-cutoff” solution can be guaranteed by a compactness argument.

3.1. Well-posedness theory for the cutoff model. The cutoff model reads

\[ \partial_t F_t(v) = Q^n(F_t, F_t)(v), \]  

(3.1)

with the initial condition defined as a non-negative probability measure \( F_0(v) \), where \( Q^n \) is obtained by replacing the collision kernel \( B(|v - v_*|, \sigma) = b(\hat{\sigma} \cdot \sigma)\Phi(|v - v_*|) \) of (1.7) by its “cutoff” counterpart. More precisely, the angular part \( b(\hat{\sigma} \cdot \sigma) \) is being replaced by \( b_n(\hat{\sigma} \cdot \sigma) \) as follows:

\[ b_n(\hat{\sigma} \cdot \sigma) \overset{\text{def}}{=} \min \{ b(\hat{\sigma} \cdot \sigma) , n \} \leq b(\hat{\sigma} \cdot \sigma) , \ n \in \mathbb{N}, \]  

(3.2)

where \( \Phi_n(|v - v_*|) \) is defined as

\[ \Phi_n(|v - v_*|) = \Phi(|v - v_*|) \phi_n(|v - v_*|) \overset{\text{def}}{=} |v - v_*| \gamma \phi_c \left( \frac{|v - v_*|}{n} \right), \]  

(3.3)

Such that both \( b_n \) and \( \Phi_n \) will approach to the non-cutoff kernel \( b \) and \( \Phi \) as \( n \to \infty \).

Throughout the paper, we denote the Fourier(-in-v) transform of \( \Phi_n \) as \( \hat{\Phi}_n \), which is defined as

\[ \hat{\Phi}_n(\zeta) = \int_{\mathbb{R}^3} \Phi_n(|v - v_*|) e^{-iv \cdot \zeta} dv. \]

Then regarding the Fourier transform, we have the following preliminary lemma:

**Lemma 3.1** (Lemma 2.5 of [25]). Let \( \Phi_n \) be in (3.3). For the hard potential case \( \gamma > 0 \), we have

\[ \forall k \geq 0, \ k \in \mathbb{N}, \ \left| \hat{\partial}_r^k \hat{\Phi}_n(\zeta) \right| \lesssim_n \frac{1}{(\zeta)^{3+\gamma+k}}. \]

On the other hand, by applying the Fourier(-in-v) transform to (3.1), the Fourier transform \( \varphi = \varphi(t, \xi) \overset{\text{def}}{=} \mathcal{F}(F_t)(\xi) \) of \( F_t(v) \) satisfies

\[ \partial_t \varphi(t, \xi) = \int_{\mathbb{S}^2} b_n \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \int_{\mathbb{R}^3} \hat{\Phi}_n(\zeta) \left[ \varphi(t, \xi^+ - \zeta) \varphi(t, \xi^- + \zeta) - \varphi(t, \zeta) \varphi(t, \xi - \zeta) \right] d\zeta d\sigma, \]  

(3.4)

where

\[ \xi^+ = \left( \frac{1}{2} + \frac{a_*}{2} \right) \xi + \frac{a_*}{2} |\xi| \sigma, \]  

(3.5)

\[ \xi^- = \left( \frac{1}{2} - \frac{a_*}{2} \right) \xi - \frac{a_*}{2} |\xi| \sigma. \]  

(3.6)
By reformulating (3.4), we have
\[ \partial_t \varphi(t, \xi) + A \varphi(t, \xi) = G_1^\varphi(t, \xi) + G_2^\varphi(t, \xi), \]
where \( A = \sup_{q \in \mathbb{R}^3} |\Phi_n(|q|)| \) and
\[ G_1^\varphi(t, \xi) = \int_{\mathbb{R}^3} b_n \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \int_{\mathbb{R}^3} \hat{\Phi}_n(\zeta) \left[ \varphi(t, \xi_e^- - \zeta) \varphi(t, \xi_e^- + \zeta) \right] d\zeta d\sigma, \]
\[ G_2^\varphi(t, \xi) = A \varphi(t, \xi) - \int_{\mathbb{R}^2} b_n \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \int_{\mathbb{R}^3} \hat{\Phi}_n(\zeta) \left[ \varphi(t, \xi) \varphi(t, \xi - \zeta) \right] d\zeta d\sigma. \]
Then we can further obtain the following integral form of the solution \( \varphi(t, \xi) \),
\[ \varphi(t, \xi) = e^{-A t} \varphi(t, \xi) + \int_0^t e^{-A (t - \tau)} \left( G_1^\varphi(\tau, \xi) + G_2^\varphi(\tau, \xi) \right) d\tau. \]
where \( \varphi(0) \) is the Fourier transform of the initial condition \( F_0 \) in (1.3).

The proof of the well-posedness of the cutoff equation with hard potential above can directly follow the counterpart of the soft-potential case [27, Theorem 3.1], as long as one can prove the following compactness lemma (Lemma 3.2) of \( G_1^\varphi \) and \( G_2^\varphi \) in the case of hard potential. Before we introduce the lemma, we define the following space \( K \) and \( K^\alpha \) of characteristic functions:
\[ K \overset{\text{def}}{=} F(P_0(\mathbb{R}^3)) = \left\{ \int_{\mathbb{R}^3} e^{-iv \cdot \xi} dF(v) \mid F \in P_0(\mathbb{R}^3) \right\}, \]
and
\[ K^\alpha = \left\{ \varphi \in K \mid \| \varphi - 1 \|_\alpha = \sup_{\xi \neq 0} \frac{|\varphi(\xi) - 1|}{|\xi|^{\alpha}} < \infty \right\}. \]
Note that \( K^\alpha \) is a subspace of the characteristic function space \( K \) and is a complete metric space endowed with the \( C^{0,\alpha} \)-norm
\[ \| \varphi - \tilde{\varphi} \|_{\alpha} \overset{\text{def}}{=} \sup_{\xi \neq 0} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^{\alpha}}. \]
It follows from [10, Lemma 3.12] that \( \{1\} \subset K^\alpha \subset K^0 = K \) for \( 2 \geq \alpha \geq 0 \), and \( K^\alpha = \{1\} \) for all \( \alpha > 2 \). Note that a characteristic function \( \varphi \) satisfies \( \varphi(0) = 1 \) and \( |\varphi| \leq 1 \). See [10, 25, 27] for more details about the space of characteristic function.

**Lemma 3.2.** For any restitution coefficient \( e \in (0, 1) \) and characteristic function \( \varphi \in K \), both of \( G_1^\varphi \) and \( G_2^\varphi \), defined by (3.8) and (3.9) respectively, are continuous and positive-definite. Furthermore, if \( 0 < \gamma \leq 2 \) and \( 0 < \alpha \leq \min(\gamma, 1) \), then for any characteristic functions \( \varphi, \tilde{\varphi} \in K^\alpha \), there exists a constant \( C_{e,\alpha} > 0 \) such that
\[ \left| G_1^\varphi + G_2^\varphi - G_1^\tilde{\varphi} - G_2^\tilde{\varphi} \right| \leq (A + C_{e,\alpha}) \| \varphi - \tilde{\varphi} \|_\alpha |\xi|^\alpha, \]
for all \( \xi \in \mathbb{R}^3 \), where we define \( C^{0,\alpha} \)-Hölder norm \( \| \cdot \|_\alpha \) as
\[ \| \varphi - \tilde{\varphi} \|_{\alpha} \overset{\text{def}}{=} \sup_{\xi \neq 0} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^{\alpha}}. \]

**Proof.** (Continuity and positive-definiteness) For the proof of the continuity and positive-definite property of \( G_1^\varphi \) and \( G_2^\varphi \), it suffices to show that \( G_1^\varphi \) and \( G_2^\varphi \) are characteristic functions. For \( G_1^\varphi \), we note that \( \hat{\Phi}_n \) is integrable with respect to \( \zeta \) in (3.8), and hence the rest of the proof follows from the construction of
a mollified characteristic function $G^\in_{\in,m}[\varphi]$ and the use of the Lebesgue dominated convergence theorem as in [27, Lemma 3.4], [25, Lemma 2.6], and [26, Lemma 2.1]. On the other hand, $G^\in_{\in,m}[\varphi]$ in (3.9) is defined the same as its elastic counterpart in [25, Lemma 2.6], and the same proof is applied for $0 < \gamma \leq 2$ and $0 < \alpha \leq \min\{\gamma, 1\}$.

**Proof of the estimates (3.11)**] To prove the estimate (3.11) in hard potential case ($0 < \gamma \leq 2$), we begin with substituting $\varphi$ and $\hat{\varphi}$ into (3.8)-(3.9) and taking the subtraction and obtain

$$
\left|G^\in_{\in,m}[\varphi] + G^\in_{\in,m}[\hat{\varphi}] - G^\in_{\in,m}[\varphi] - G^\in_{\in,m}[\hat{\varphi}]\right|
$$

$$
\leq \int_{\mathbb{R}^d} b_n \left(\frac{\xi - \sigma}{|\xi|}\right) \int_{\mathbb{R}^d} \left|\Phi_n(\xi - \xi_c^+) - \Phi_n(\xi_c^-)\right| |\varphi(\xi)\varphi(\xi - \zeta) - \hat{\varphi}(\xi)\hat{\varphi}(\xi - \zeta)| d\zeta d\sigma
$$

$$
+ A \|\varphi - \hat{\varphi}\|_\alpha |\xi|^\alpha,
$$

where we utilize the change of variable $\xi_c^+ + \xi_c^- = \xi$.

Then we observe that

$$
|\varphi(\xi)\varphi(\xi - \zeta) - \hat{\varphi}(\xi)\hat{\varphi}(\xi - \zeta)|
$$

$$
\leq |\varphi(\xi)\varphi(\xi - \zeta) - \varphi(\xi)\hat{\varphi}(\xi - \zeta) + \varphi(\xi)\hat{\varphi}(\xi - \zeta) - \hat{\varphi}(\xi)\hat{\varphi}(\xi - \zeta)|
$$

$$
\leq |\varphi(\xi)| \|\varphi - \hat{\varphi}\|_\alpha |\xi - \zeta|^\alpha + |\varphi - \hat{\varphi}\|_\alpha |\zeta - \xi |^\alpha |\hat{\varphi}(\xi - \zeta)|
$$

$$
\leq \frac{|\xi - \zeta|^\alpha + |\zeta - \xi|^\alpha}{|\xi|^\alpha} \|\varphi - \hat{\varphi}\|_\alpha |\xi|^\alpha,
$$

where we used the property of the characteristic functions that $|\varphi(\xi)| < 1$ and $|\hat{\varphi}(\xi - \zeta)| < 1$ for the last inequality. Hence, combining the analysis above, we realize that it suffices to prove the following estimate:

$$
\int_{\mathbb{R}^d} b_n \left(\frac{\xi - \sigma}{|\xi|}\right) \int_{\mathbb{R}^d} \left|\Phi_n(\xi - \xi_c^+) - \Phi_n(\xi_c^-)\right| \frac{|\xi - \zeta|^\alpha + |\zeta - \xi|^\alpha}{|\xi|^\alpha} d\zeta d\sigma \leq C_{\in,n}.
$$

(3.12)

In fact, for $|\xi| \leq 1$, by considering Lemma 3.1 and the following estimates

$$
\frac{|\xi_c^-|}{|\xi|^\alpha} = \left(\frac{a_+}{2}\right)^\frac{1}{2} \sin \frac{\theta}{2} |\xi| \leq \left(\frac{a_+}{2}\right)^\frac{1}{2} \sin \frac{\theta}{2},
$$

(3.13)

and

$$
|\xi - \zeta|^\alpha + |\zeta - \xi|\alpha = |\xi - \zeta|^\alpha + |\zeta - \tau \xi_e^- + \tau \xi_e^-|^\alpha
$$

$$
\leq |\zeta - \tau \xi_e^-|^\alpha + |\zeta - \tau \xi_e^-|^\alpha + |\tau \xi_e^-|^\alpha
$$

$$
\leq 2|\zeta - \tau \xi_e^-|^\alpha + 1, \text{ for any } \tau \in [0, 1],
$$

(3.14)
we obtain that
\[
\int_{\mathbb{R}^3} \left| \hat{\Phi}_n(\zeta - \xi^-) - \hat{\Phi}_n(\zeta) \right| \frac{|\xi - \zeta|^\alpha + |\zeta|^\alpha}{|\xi|^\alpha} \, d\zeta \\
\leq \int_{\mathbb{R}^3} \left| \frac{\partial \hat{\Phi}_n}{\partial \zeta}(\zeta - \tau \xi^-) \right| \, d\tau \frac{|\xi - \zeta|^\alpha + |\zeta|^\alpha}{|\xi|^\alpha} \, d\zeta \\
\lesssim_n \left( \frac{a_n}{2} \right)^{\frac{7}{2}} \sin \frac{\theta}{2} \int_0^1 \int_{\mathbb{R}^3} \frac{|\xi - \zeta|^\alpha + |\zeta|^\alpha}{\langle \zeta - \tau \xi^- \rangle^{3+\gamma+1}} \, d\zeta \, d\tau \\
\lesssim_n \left( \frac{a_n}{2} \right)^{\frac{7}{2}} \sin \frac{\theta}{2} \int_0^1 \int_{\mathbb{R}^3} \frac{2|\xi - \tau \xi^-|^\alpha + 1}{\langle \zeta - \tau \xi^- \rangle^{3+\gamma+1}} \, d\zeta \, d\tau 
\lesssim_n \left( \frac{a_n}{2} \right)^{\frac{7}{2}} \sin \frac{\theta}{2}.
\]

Here we utilize Lemma 3.1 and (3.13) in the second inequality and (3.14) in the third inequality. On the other hand, for $|\xi| > 1$, we use Lemma 3.1 and obtain
\[
\int_{\mathbb{R}^3} \left| \hat{\Phi}_n(\zeta - \xi^-) - \hat{\Phi}_n(\zeta) \right| \frac{|\xi - \zeta|^\alpha + |\zeta|^\alpha}{|\xi|^\alpha} \, d\zeta \\
\leq \int_{\mathbb{R}^3} \hat{\Phi}_n(\zeta - \xi^-) \left| \frac{|\xi - \zeta|^\alpha + |\zeta|^\alpha}{|\xi|^\alpha} \right| \, d\zeta + \int_{\mathbb{R}^3} \hat{\Phi}_n(\zeta) \left| \frac{|\xi - \zeta|^\alpha + |\zeta|^\alpha}{|\xi|^\alpha} \right| \, d\zeta \\
\lesssim_n \int_{\mathbb{R}^3} \frac{2|\xi - \xi^-|^\alpha + 2}{\langle \zeta - \xi^- \rangle^{3+\gamma+1}} \, d\zeta + \int_{\mathbb{R}^3} \frac{2|\xi|^\alpha + 1}{\langle \zeta \rangle^{3+\gamma+1}} \, d\zeta \lesssim_n 1.
\]

This completes the proof of the desired estimate (3.12) and the proof of the lemma.

Consequently, we use Lemma 3.2 and apply the standard Banach fixed-point theorem and continuation argument towards (3.10) in the space $K^\alpha$. After applying the inverse Fourier transform $F^\alpha_t = \mathcal{F}^{-1}(\hat{\varphi}^\alpha)$, we obtain the following well-posedness result for the cutoff model:

**Proposition 3.3.** Let $e \in (0, 1]$ and $\alpha_0 \in (0, 2]$ and let the collision kernel $B$ be a cutoff kernel $B = b_n \Phi_n$. Then, for any initial datum $F_0(v) \in P_{\alpha_0}(\mathbb{R}^3)$, there exists a unique solution $F^\alpha_t(v) \in C([0, \infty), P_n)$ with $\alpha \in (0, \alpha_0)$ to the cutoff model (3.1) in the measure-valued sense.

**Remark 3.4.** Note that the energy boundedness $\int_{\mathbb{R}^3} |v|^2 \, dF(v) < \infty$ and the energy dissipation property (3.21) exclude the possible non-uniqueness scenario that is caused by the instantaneous energy jump [31] in the elastic hard-sphere model with an angular cutoff.

In addition, we can also obtain the following moment-estimate of the sequence of measure-valued solution $F^\alpha_t(v)$ as a direct application of the Povzner estimate that we proved in Proposition 2.1, if initial datum $F_0(v)$ has finite $(2 + \kappa)$-moments for any given $\kappa > 0$.

**Corollary 3.5.** Assume that $F_0$ satisfies, for a given $\kappa > 0$,
\[
\int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa} \, dF_0(v) < \infty.
\]
Then, for any fixed $T > 0$, there exists a constant $C_{\kappa, e, T} > 0$ independent of $n$ such that,

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa} \, dF_t^n(v) + \int_0^T \int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa} \min\{\langle v \rangle^\gamma, n^\gamma\} \, dF^n_{\tau}(v) \, d\tau \leq C_{\kappa, e, T}. \tag{3.15}$$

**Proof.** By (1.6), we first have

$$\int_{\mathbb{R}^3} |v|^2 \, dF^n_t(v) \leq \int_{\mathbb{R}^3} |v|^2 \, dF_0(v), \tag{3.16}$$

for any $t > 0$. By considering the cutoff equation (3.1) and integrating over $v \in \mathbb{R}^3$ with the test function $\psi(x) = \psi_\kappa(x) = (1 + x)^{1+\kappa/2} - 1$ as well as the conservation of mass, we observe that

$$\int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa} \, dF^n_t(v) - \int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa} \, dF^n_0(v)$$

$$= \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi_n(|v - v_\kappa|) K_n^\kappa(v, v_\kappa) \, dF^n_{\tau}(v) \, dF^n_{\tau}(v_\kappa) \, d\tau,$$

where $K_n^\kappa(v, v_\kappa)$ is defined as in (2.3). Now, we use the Povzner inequality (Proposition 2.1) and obtain

$$\int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa} \, dF^n_t(v) + C_1 \int_0^t \int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa} \min\{\langle v \rangle^\gamma, n^\gamma\} \, dF^n_{\tau}(v) \, d\tau$$

$$\leq \int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa} \, dF_0(v)$$

$$+ C_{2,3} \left( \int_{\mathbb{R}^3} \langle v \rangle^2 \, dF_0(v) \right) \int_0^t \int_{\mathbb{R}^3} \langle v \rangle^{\max\{2, \kappa\}} \min\{\langle v \rangle^\gamma, n^\gamma\} \, dF^n_{\tau}(v) \, d\tau. \tag{3.17}$$

Here the constant $C_1$ is the same constant of Proposition 2.1 and $C_{2,3}$ depends only on the constants $C_2$ and $C_3$ of Proposition 2.1. Now, we split the domain $v \in \mathbb{R}^3$ into $|v| \leq R_0$ and $|v| \geq R_0$ for a sufficiently large $R_0 > 0$. By further selecting $R_0 > 0$ large such that

$$C_1 (1 + R_0^{\frac{1}{2} \min\{2, \kappa\}}) \geq 2C_{2,3} \int_{\mathbb{R}^3} \langle v \rangle^2 \, dF_0(v),$$

we obtain

$$C_1 \int_{|v| \geq R_0} \langle v \rangle^{2+\kappa} \min\{\langle v \rangle^\gamma, n^\gamma\} \, dF^n_{\tau}(v)$$

$$\geq 2C_{2,3} \left( \int_{\mathbb{R}^3} \langle v \rangle^2 \, dF_0(v) \right) \int_{|v| \geq R_0} \langle v \rangle^{\max\{2, \kappa\}} \min\{\langle v \rangle^\gamma, n^\gamma\} \, dF^n_{\tau}(v),$$

and

$$C_{2,3} \left( \int_{\mathbb{R}^3} \langle v \rangle^2 \, dF_0(v) \right) \int_{|v| \leq R_0} \langle v \rangle^{\max\{2, \kappa\}} \min\{\langle v \rangle^\gamma, n^\gamma\} \, dF^n_{\tau}(v) \lesssim R_0^{2+\kappa+\gamma} \lesssim 1.$$ 

Thus, they yield (3.15) together with (3.17), and this completes the proof. □

This closes the discussions on the existence and the moment-estimates under an angular cutoff. In the next subsection, we introduce the rather standard approximation of a non-cutoff model by cutoff models.
3.2. Non-cutoff model as an approximation of cutoff model. To obtain the existence of the solution to non-cutoff equation, in this section, we will regard the non-cutoff model as an approximation of cutoff models, of which the well-posedness has already been studied as in the last subsection. In fact, by directly applying Proposition 3.3, we can construct a sequence of approximated solutions \( \{ F^n_t(v) \} = \{ \mathcal{F}^{-1}[\varphi^n(t, \zeta)] \} \) where \( \{ \varphi^n(t, \zeta) \} \) is the sequence of solutions to the cutoff equation (3.1) as before.

In order to pass to the limit for approximated solutions by the compactness argument, we need to have the uniform boundedness and equicontinuity of the sequence of solutions \( \{ \varphi^n \} \). In fact, we note that the uniform-boundedness property \( |\varphi^n| \leq 1 \) and the equicontinuity in the Fourier variable \( \xi \) of the sequence of solution \( \{ \varphi^n \} \) are directly inherited from the fact that all of them are found in the space of characteristic function. Hence, we only need to prove the equicontinuity in the temporal variable \( t \).

To this end, we observe that for any \( 0 \leq s < t \leq T \), there exist constants \( C_{\epsilon, \psi}, C'_{\epsilon, \psi} > 0 \) (independent of \( n \)) such that, for any \( \psi \in C^2_c(\mathbb{R}^3) \),

\[
|\varphi^n_t(\xi) - \varphi^n_s(\xi)| \\
\leq \frac{1}{2} \int_s^t \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b_n(\sigma \cdot \hat{q}) \left( \psi'_v + \psi'_u - \psi \right) d\sigma \times |v - u|^\gamma \phi_n(|v - u|) dF^n_\tau(v) dF^n_\tau(v_u) d\tau \\
\leq \frac{C_{\epsilon, \psi}}{2} \int_s^t \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - u|^{\gamma+2} \phi_n(|v - u|) dF^n_\tau(v) dF^n_\tau(v_u) d\tau \\
\leq C'_{\epsilon, \psi} \left[ |t - s| \sup_{s \leq \tau \leq t} \left( \int_{\mathbb{R}^3} \langle v \rangle^2 dF^n_\tau(v) \right) \left( \int_{\mathbb{R}^3} \langle v_u \rangle^2 dF^n_\tau(v_u) \right) \\
+ \int_s^t \int_{\mathbb{R}^3} \chi_{\{|v| \geq 2|v_u|\}} \cup \{|v_u| \geq 2|v|\} |v - u|^2 \right] \\
\times \min\{|v - u|^\gamma, \tau \} \quad dF^n_\tau(v) dF^n_\tau(v_u) d\tau \right], \\
(3.18)
\]

where \( \chi_U(v) \in C^\infty_c(\mathbb{R}^3) \) is the indicator function

\[
\chi_U(v) \overset{\text{def}}{=} \begin{cases} 
1, & \text{if } v \in U, \\
0, & \text{if } v \notin U,
\end{cases}
(3.19)
\]

\( \phi_n \) is defined as in (3.3), and we utilized the estimate [27, Eq. (4.15)]

\[
\left| \int_{\mathbb{S}^2} b_n(\sigma \cdot \hat{q}) \left( \psi'_v + \psi'_u - \psi \right) d\sigma \right| \leq C_{\epsilon, \psi} |v - u|^2,
\]

in the second inequality and the condition \( 0 < \gamma \leq 2 \) in the first in the third inequality. For the second term in (3.18), we take advantage of (3.15) in Corollary 3.5 and obtain that for any \( T > 0 \), there exists a \( C_{\kappa, \epsilon, T} > 0 \) independent of \( n \) for any
given $\kappa > 0$ such that
\[
\int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \chi(|v| > R) \langle v \rangle^2 \min\{\langle v \rangle^\gamma, n^\gamma\} \, dF^n(v) \right)^2 \, d\tau \lesssim \frac{C_{\kappa,e,T}}{R^8},
\]
for a sufficiently large $R \gg 1$, since we observe that
\[
R^\kappa \int_{S} \left( \int_{\mathbb{R}^3} \chi(|v| > R) \langle v \rangle^2 \min\{\langle v \rangle^\gamma, n^\gamma\} \, dF^n(v) \right)^2 \, d\tau
\]
\[
\leq \int_{S} \left( \int_{\mathbb{R}^3} \chi(|v| > R) \langle v \rangle^{2+\kappa} \min\{\langle v \rangle^\gamma, n^\gamma\} \, dF^n(v) \right)^2 \, d\tau \lesssim C_{\kappa,e,T},
\]
for $|v| \geq R$. On the other hand, since $0 < \gamma \leq 2$, we also have
\[
\int_{S} \left( \int_{\mathbb{R}^3} \chi(|v| < R) \langle v \rangle^2 \min\{\langle v \rangle^\gamma, n^\gamma\} \, dF^n(v) \right)^2 \, d\tau \lesssim \int_{S} \left( \int_{\mathbb{R}^3} \chi(|v| < R) \langle v \rangle^2 \, dF^n(v) \right)^2 \, d\tau \lesssim C_{\kappa,e,T} R^8 |t - s|.
\]
Then this leads to the equicontinuity in the time variable $t$.

Thus, thanks to the Arzelà-Ascoli Theorem, we are able to take the limit of $\{\varphi^n\}$ (up to a subsequence) in the sense that, on every compact subset of $[0, \infty) \times \mathbb{R}^3$,
\[
\varphi(t, \xi) = \lim_{n \to \infty} \varphi^n(t, \xi),
\]
such that the limit function $\varphi(t, \xi)$ is a characteristic function as well.

To obtain the moment-estimate of the measure-valued solution
\[
F_t(v) = F^{-1}[\varphi(t, \xi)]
\]
in the limit $n \to \infty$, we also introduce the following Lemma 3.6 to illustrate that some moments properties will be maintained in the limiting process. The proof of the following lemma below in the inelastic case is the same as the counterpart [25, Lemma 3.2] of the elastic case, since the estimates are only associated with the variables $v$ and $v_\ast$ in the limiting process, which have no distinction between the elastic and inelastic cases. We omit the proof.

**Lemma 3.6.** [25, Lemma 3.2] Assume that $F^n_t(v) = F^{-1}[\varphi^n(t, \xi)]$ is a sequence of measure-valued solution to cutoff equation and $F_t(v) = F^{-1}[\varphi(t, \xi)]$ is the limit of sequence with $\varphi(t, \xi)$ obtained in (3.20). Then,

(i) For every $t > 0$, we have,
\[
\int_{\mathbb{R}^3} |v|^2 \, dF_t(v) \leq \int_{\mathbb{R}^3} |v|^2 \, dF_0(v).
\]

(ii) For any $T > 0$ and $\psi(v) \in C(\mathbb{R}^3)$ with $|\psi(v)| \lesssim \langle v \rangle^l$ for some $0 < l < 2$, we have,
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} \psi(v) \, dF^n_t(v) = \int_{\mathbb{R}^3} \psi(v) \, dF_t(v),
\]
uniformly for $t \in [0, T]$.

(iii) For any $T > 0$ and $\psi(v, v_\ast) \in C(\mathbb{R}^3 \times \mathbb{R}^3)$ with $|\psi(v, v_\ast)| \lesssim \langle v \rangle^l \langle v_\ast \rangle^l$ for some $0 < l < 2$, we have,
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(v, v_\ast) \, dF^n_t(v) \, dF^n_t(v_\ast) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(v, v_\ast) \, dF_t(v) \, dF_t(v_\ast),
\]
uniformly for \( t \in [0, T] \).

Based on the Lemma 3.6, we finally obtain the moment estimate for the limit \( F_t(v) \) if initial datum \( F_0(v) \) has finite \((2 + \kappa)\)-moments.

**Corollary 3.7.** Assume that \( F_0 \) satisfies, for a given \( \kappa > 0 \),

\[
\int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa} \, dF_0(v) < \infty.
\]

Let \( F_t(v) \) satisfy Lemma 3.6. Then, for any fixed \( T > 0 \), there exists a constant \( C_{\kappa,e,T} > 0 \) independent of \( n \) such that

\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa} \, dF_t(v) + \int_0^T \int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa+\gamma} \, dF_\tau(v) \, d\tau \leq C_{\kappa,e,T}. \tag{3.24}
\]

**Proof.** We obtain (3.24) by taking the limit of (3.15) as \( n \to \infty \) with the help of Lemma 3.6. This completes the proof. \( \square \)

Now we are ready to pass to the limit \( n \to \infty \) and discuss the non-cutoff case. In the next subsection, prove the main theorem (Theorem 1.2) in the non-cutoff situation.

### 3.3. Existence theory for the non-cutoff model.

In this subsection, we will complete the main existence theorem by showing that the limit function \( F_t(v) \) is indeed the measure-valued solution to the original non-cutoff equation (3.4).

**Proof of Theorem 1.2.** Let \( \Psi^e_n(v, v_\ast) \) and \( \Psi^e(v, v_\ast) \) denote

\[
\Psi^e_n(v, v_\ast) = |v - v_\ast|^\gamma \phi_n(|v - v_\ast|) \int_{S^2} b_n(\sigma \cdot \hat{q}) \left( \psi'_\ast + \psi' - \psi_\ast - \psi \right) \, d\sigma,
\]

and

\[
\Psi^e(v, v_\ast) = |v - v_\ast|^\gamma \int_{S^2} b(\sigma \cdot \hat{q}) \left( \psi'_\ast + \psi' - \psi_\ast - \psi \right) \, d\sigma.
\]

Then, to establish the limit \( n \to \infty \) in the measure-valued sense, it suffices to calculate the following difference:

\[
\int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} b_n(\sigma \cdot \hat{q}) |v - v_\ast|^\gamma \phi_n(|v - v_\ast|) \times \left( \psi'_\ast + \psi' - \psi_\ast - \psi \right) \, d\sigma \, dF_n^e(v) \, dF_n^e(v_\ast) \, d\tau
\]

\[
- \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} b(\sigma \cdot \hat{q}) |v - v_\ast|^\gamma \left( \psi'_\ast + \psi' - \psi_\ast - \psi \right) \, d\sigma \, dF_e(v) \, dF_e(v_\ast) \, d\tau
\]

\[
= \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Psi^e_n(v, v_\ast) \, dF_n^e(v) \, dF_n^e(v_\ast) \, d\tau - \int_0^t \int_{\mathbb{R}^3} \Psi^e(v, v_\ast) \, dF_e(v) \, dF_e(v_\ast) \, d\tau,
\]

\[
\overset{\text{def}}{=} I_1 - I_2 + I_3 + I_4. \tag{3.25}
\]
where the integrals \( I_j \) with \( j = 1, 2, 3, 4 \) are defined as follows. For some \( R > 0 \), we define

\[
I_1 \overset{\text{def}}{=} \int_0^t \int \int \chi_{\{|v| > R\} \cup \{|v_*| > R\}} \Psi_n^c(v, v_*) \, dF_{\tau}^{n}(v) \, dF_{\tau}^{n}(v_*) \, d\tau,
\]

\[
I_2 \overset{\text{def}}{=} \int_0^t \int \int \chi_{\{|v| > R\} \cup \{|v_*| > R\}} \Psi^c(v, v_*) \, dF_{\tau}^c(v) \, dF_{\tau}^c(v_*) \, d\tau,
\]

\[
I_3 \overset{\text{def}}{=} \int_0^t \int \int \chi_{\{|v| \leq R\} \cap \{|v_*| \leq R\}} \left[ \Psi_n^c(v, v_*) - \Psi^c(v, v_*) \right] \, dF_{\tau}^{n}(v) \, dF_{\tau}^{n}(v_*) \, d\tau,
\]

\[
I_4 \overset{\text{def}}{=} \int_0^t \int \int \chi_{\{|v| \leq R\} \cap \{|v_*| \leq R\}} \Psi^c(v, v_*) \, dF_{\tau}^c(v) \, dF_{\tau}^c(v_*) \, d\tau
\]

\[- \int d \int \chi_{\{|v| \leq R\} \cap \{|v_*| \leq R\}} \Psi^c(v, v_*) \, dF_{\tau}^c(v) \, dF_{\tau}^c(v_*) \, d\tau.\]

Note that

\[
|\Psi_n^c(v, v_*)| \lesssim \langle v \rangle^2 \langle v_* \rangle^2 \chi_{\{|v| < 2|v_*| < 4|v|\}} + \langle v \rangle^2 \min\{\langle v \rangle^\gamma, n \gamma\} \chi_{\{|v| > 2|v_*|\}} + \langle v_* \rangle^2 \min\{\langle v \rangle^\gamma, n \gamma\} \chi_{\{|v_*| > 2|v|\}},
\]

and

\[
|\Psi^c(v, v_*)| \lesssim |v - v_*|^{\gamma+2} \lesssim \langle v \rangle^2 \langle v_* \rangle^2 \chi_{\{|v| < 2|v_*| < 4|v|\}} + \langle v \rangle^{\gamma+2} \chi_{\{|v| > 2|v_*|\}} + \langle v_* \rangle^{\gamma+2} \chi_{\{|v_*| > 2|v|\}}.
\]

By applying the moment-estimates (3.15) and (3.24), we first obtain that

\[
I_1 \lesssim TR^{-\kappa} \left( \int \langle v \rangle^{2+\kappa} \, dF_0 \right)^2 + R^{-\kappa} \int_0^T \left( \int \langle v \rangle^{2+\kappa} \min\{\langle v \rangle^\gamma, n \gamma\} \, dF_{\tau}^{n}(v) \right) \, d\tau \lesssim R^{-\kappa}.
\]

By the similar estimate, we also have

\[
I_2 \lesssim \left| \int_0^t \int \int \chi_{\{|v| > R\} \cup \{|v_*| > R\}} \Psi^c(v, v_*) \, dF_{\tau}^c(v) \, dF_{\tau}^c(v_*) \, d\tau \right| \lesssim R^{-\kappa}.
\]

On the other hand, the third term \( I_3 \) of (3.25) converges to zero uniformly for \( t \to 0 \), as \( n \to \infty \), since the function \( \Psi_n^c(v, v_*) \) converges to \( \Psi^c(v, v_*) \) uniformly on a compact set of \( (v, v_*) \in \mathbb{R}_v^3 \times \mathbb{R}_v^3 \). Moreover, by Lemma 3.6 (iii), the fourth term \( I_4 \) of (3.25) also converges to zero uniformly on a compact set of \( t \in [0, \infty) \), as \( n \to \infty \). Then by taking \( R \to \infty \), we have \( I_1 - I_2 + I_3 + I_4 \to 0 \) and we obtain that \( F_t \) is indeed a measure-valued solution. This completes the proof of Theorem 1.2.

\[\square\]

**Remark 3.8.** As we observed above, the proof of our main theorem (Theorem 1.2) relies on the fact that \( \kappa > 0 \) and this cannot be relaxed to \( \kappa = 0 \). Note that since we assume the initial datum \( F_0 \) has finite \((2 + \kappa)\)-moment, the case \( \kappa = 0 \) corresponds to the energy bound \( F_0 \in P_2(\mathbb{R}^3) \). In order to relax our assumption to the sharp energy bound, we believe that one must establish a more improved Povzner estimate than our almost-sharp Povzner estimate (Proposition 2.1).
This completes the proof of the global existence of a measure-valued solution in the case without angular cutoff. In the next section, we prove that the solution produces additional moments, which are also bounded.

4. Moment-creation property

In this section, we will present the proof of our main theorem (Theorem 1.4) on the moment production of the corresponding measure-valued solution in Theorem 1.2.

Proof of Theorem 1.4. We first introduce the outline of the proof. We will basically apply the inductive strategy to prove the moment-creation property of the measure-valued solution. During the induction process, we will prove that for each iteration, the solution gains an additional $\gamma$-order moments in an arbitrarily small time interval such that any higher order moments will instantly become finite after the time evolution.

Motivated by [23, pp. 479], to achieve the moment-estimate in each time interval, we will first try to construct the convex approximation $\psi_{\kappa,m}^{\kappa}(x)$ of the weight function $\psi_\kappa(x) = (1 + x^{1+\kappa/2}) - 1$ with $\kappa > 0$. However, in our case, we remark that the collision process is inelastic and it conserves the total mass and momentum but not the energy as in (1.5) and (1.6). Therefore, we have to design a different type of approximating functions $p_{\kappa,m}$ defined as follows:

$$\psi_{\kappa,m}(x) \equiv \begin{cases} \psi_\kappa(x), & \text{if } x \leq m, \\ p_{\kappa,m}(x), & \text{if } x > m, \end{cases} \quad (4.1)$$

where

$$p_{\kappa,m}(x) = C_\kappa(m)x^{1+\kappa/2} + \psi_\kappa(m) - C_\kappa(m)m^{1+\kappa/2} \quad (4.2)$$

for $\kappa > 0$ and all $m \in \mathbb{N}$. Then, if we further define

$$\psi_{\kappa,m}^\kappa(|v|^2) \equiv \psi_{\kappa,m}(|v|^2) - p_{\kappa,m}(|v|^2)$$

$$= \begin{cases} \psi_\kappa(|v|^2) - \psi_\kappa(m) - C_\kappa(m)\left[(|v|^2)^{1+\kappa/2} - m^{1+\kappa/2}\right], & \text{if } |v|^2 \leq m, \\ 0, & \text{if } |v|^2 > m, \end{cases} \quad (4.3)$$

where $C_\kappa(m) = (1 + \frac{1}{m})^{\kappa/2}$ can be determined by the continuity requirement of $\partial \psi_\kappa^\kappa$ at $|v|^2 = m$ such that $\psi_{\kappa,m}^\kappa$ is a bounded and continuously differentiable approximation of $\psi_\kappa$ with compact support, where $\psi_{\kappa,m}^\kappa(v) \in C_0^1(\mathbb{R}^3)$, and $\partial \psi_{\kappa,m}^\kappa(v) \in C_0(\mathbb{R}^3)$ is Lipschitz continuous with a uniform Lipschitz constant.

Then by the conservation of mass (1.5), we first observe that,

$$\int_{\mathbb{R}^3} \psi_{\kappa,m}(|v|^2) \, dF_t(v) - \int_{\mathbb{R}^3} \psi_{\kappa,m}(|v|^2) \, dF_0(v)$$

$$= \int_{\mathbb{R}^3} \psi_{\kappa,m}^\kappa(|v|^2) \, dF_t(v) - \int_{\mathbb{R}^3} \psi_{\kappa,m}^\kappa(|v|^2) \, dF_0(v)$$

$$+ \int_{\mathbb{R}^3} p_{\kappa,m}(|v|^2) \, dF_t(v) - \int_{\mathbb{R}^3} p_{\kappa,m}(|v|^2) \, dF_0(v), \quad (4.4)$$
which leads that

\[ \int_{\mathbb{R}^3} \psi_m^\kappa(|v|^2) \, dF_t(v) - \int_{\mathbb{R}^3} \psi_m^\kappa(|v|^2) \, dF_0(v) = \int_{\mathbb{R}^3} \psi_m^\kappa(|v|^2) \, dF_t(v) - \int_{\mathbb{R}^3} \psi_m^\kappa(|v|^2) \, dF_0(v) + C_k(m) \int_{\mathbb{R}^3} |v|^{2+\kappa} \, dF_t(v). \]

After substituting \( \psi = \psi_m^\kappa \) by approximating \( C_\kappa \) functions into the weak formulation (1.4), we find, for \( t \geq 0 \),

\[
\int_{\mathbb{R}^3} \psi_m^\kappa(|v|^2) \, dF_t(v) - \int_{\mathbb{R}^3} \psi_m^\kappa(|v|^2) \, dF_0(v) = \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} dF_{\tau}(v_*) \int_{\mathbb{R}^3} dF_{\tau}(v) \, |v - v_*|^\gamma \\
\times \int_{\mathbb{R}^2} b(\hat{q} \cdot \sigma) \left[ \psi_m^\kappa(|v'|^2) + \psi_m^\kappa(|v'_*|^2) - \psi_m^\kappa(|v|^2) - \psi_m^\kappa(|v_*|^2) \right] \, d\sigma \\
+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} dF_{\tau}(v_*) \int_{\mathbb{R}^3} dF_{\tau}(v) \, |v - v_*|^\gamma \\
\times \int_{\mathbb{R}^2} b(\hat{q} \cdot \sigma) \left[ p_m^\kappa(|v'|^2) + p_m^\kappa(|v'_*|^2) - p_m^\kappa(|v|^2) - p_m^\kappa(|v_*|^2) \right] \, d\sigma \tag{4.5}
\]

where, for the last term in (4.5) above, we have, by noting (4.2) and the collision invariance of 1,

\[
\frac{1}{2} \int_0^t \int_{\mathbb{R}^3} dF_{\tau}(v_*) \int_{\mathbb{R}^3} dF_{\tau}(v) \, |v - v_*|^\gamma \\
\times \int_{\mathbb{R}^2} b(\hat{q} \cdot \sigma) \left[ p_m^\kappa(|v'|^2) + p_m^\kappa(|v'_*|^2) - p_m^\kappa(|v|^2) - p_m^\kappa(|v_*|^2) \right] \, d\sigma \\
= C_\kappa(m) \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} dF_{\tau}(v_*) \int_{\mathbb{R}^3} dF_{\tau}(v) \, |v - v_*|^\gamma \\
\times \int_{\mathbb{R}^2} b(\hat{q} \cdot \sigma) \left[ \psi_1^\kappa(|v'|^2) + \psi_1^\kappa(|v'_*|^2) - \psi_1^\kappa(|v|^2) - \psi_1^\kappa(|v_*|^2) \right] \, d\sigma, \tag{4.6}
\]

where the right-hand side of which can also be well-estimated by the Povzner inequality in Proposition 2.1.
Then, combing (4.4)-(4.6) and applying the Povzner inequality to the right-hand side of (4.5), see [25, Proposition 1.4] for a similar operation, we obtain,

\[
\int_{\mathbb{R}^3} \psi_{\kappa,m}(|v|^2) \, dF_t(v) + C_k(m) \int_{\mathbb{R}^3} |v|^{2+\kappa} \, dF_0(v)
\]

\[
\leq \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^\gamma \left[ -H_m^\kappa(v, v_\ast) + G_m^\kappa(v, v_\ast) \right] dF_t(v) dF_t(v_\ast) \, d\tau
\]

\[
+ \frac{C_k(m)}{2} \int_0^t \int_{\mathbb{R}^3} dF_t(v_\ast) \int_{\mathbb{R}^3} dF_t(v) |v - v_*|^\gamma \times \int_{\mathbb{R}^3} b(\hat{q} \cdot \sigma) \left[ \psi_{1,\kappa}(|v|^2) + \psi_{1,\kappa}(|v_*|^2) - \psi_{1,\kappa}(|v|^2) - \psi_{1,\kappa}(|v_*|^2) \right] \, d\sigma
\]

\[
+ C_k(m) \int_{\mathbb{R}^3} |v|^{2+\kappa} \, dF_t(v),
\]

(4.7)

where \(H_m^\kappa(v, v_\ast), G_m^\kappa(v, v_\ast)\) are defined in the same way as in Proposition 2.1 except for now replacing \(\psi_{\kappa} by \psi_{\kappa,m}\) and changing \(b_n\) into \(b\). By further letting \(m \to \infty\) and noticing \(C_k(m)\) will approach to 1, we find that, for suitable \(C_1, C_2 > 0\) and arbitrarily small \(t_0 > 0\),

\[
\int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa} \, dF_{t_0}(v) + C_1 \int_0^{t_0} \int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa+\gamma} \, dF_{t_\tau}(v) \, d\tau
\]

\[
\leq \int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa} \, dF_0(v) + \int_{\mathbb{R}^3} |v|^{2+\kappa} \, dF_{t_0}(v) + C_2 \int_0^{t_0} \int_{\mathbb{R}^3} \langle v \rangle^{2+\gamma} \, dF_{t_\tau}(v) \, d\tau,
\]

(4.8)

where the last two terms on the right-hand side are finite by Corollary 3.7. As the second term of the left-hand side (4.8) is finite, it then implies that, there exists \(t_1 \in (0, t_0)\) such that,

\[
\int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa_1} \, dF_{t_1}(v) < \infty.
\]

(4.9)

where we denote \(\kappa_1 = \kappa + \gamma\). By repeating the same procedures as above and replacing \(\kappa\) and \(t = 0\) by \(\kappa_1\) and \(t_1\) respectively, we have, for suitable \(C_1' > 0\) and \(t \in (t_1, t_0)\),

\[
\int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa_1} \, dF_t(v) + C_1' \int_{t_1}^t \int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa_1+\gamma} \, dF_{t_\tau}(v) \, d\tau
\]

\[
\leq \int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa_1} \, dF_{t_1}(v) + \int_{\mathbb{R}^3} |v|^{2+\kappa} \, dF_t(v) + C_2' \int_{t_1}^t \int_{\mathbb{R}^3} \langle v \rangle^{2+\gamma} \, dF_{t_\tau}(v) \, d\tau.
\]

Here the right-hand side is also finite by noticing (4.9). Applying the induction by setting \(\kappa_{j+1} = \kappa_j + \gamma = \gamma(j + 1) + \kappa\), we obtain that, for any \(T > t_0\) and \(l > 0\), there exists a constant \(C_{T,l}\) such that,

\[
\sup_{t_0 \leq t \leq T} \int_{\mathbb{R}^3} \langle v \rangle^l \, dF_t(v) \leq C_{T,l}.
\]

Thus, we conclude

\[
\sup_{t \geq t_0} M_l(t) < \infty,
\]

by applying the similar argument as in [25, Proposition 1.4] and [22, Lemma 3.8]. This completes the proof of Theorem 1.4. \(\square\)
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References

[1] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang. Regularizing effect and local existence for the non-cutoff Boltzmann equation. *Arch. Ration. Mech. Anal.*, 198(1):39–123, 2010.
[2] R. J. Alonso, J. Cañizo, I. M. Gamba, and C. Mouhot. A new approach to the creation and propagation of exponential moments in the Boltzmann equation. *Comm. Partial Differential Equations*, 38(1):155–169, 2013.
[3] R. J. Alonso, B. Lods, and I. Tristani. Fluid dynamic limit of boltzmann equation for granular hard-spheres in a nearly elastic regime, 2020.
[4] R. J. Alonso, B. Lods, and I. Tristani. From Boltzmann Equation for Granular Gases to a Modified Navier-Stokes-Fourier System. *J. Stat. Phys.*, 106(3-4):547–567, 2002.
[5] A. V. Bobylev. Moment inequalities for the Boltzmann equation and applications to spatially homogeneous problems. *J. Stat. Phys.*, 88(5-6):1183–1214, 1997.
[6] A. V. Bobylev, J. A. Carrillo, and I. M. Gamba. On some properties of kinetic and hydrodynamic equations for inelastic interactions. *J. Stat. Phys.*, 98(3-4):743–773, 2000.
[7] A. V. Bobylev and C. Cercignani. Moment equations for a granular material in a thermal bath. *J. Stat. Phys.*, 106(3-4):547–567, 2002.
[8] A. V. Bobylev, C. Cercignani, and G. Toscani. Proof of an asymptotic property of self-similar solutions of the Boltzmann equation for granular materials. *J. Stat. Phys.*, 111(1-2):403–417, 2003.
[9] A. V. Bobylev, I. M. Gamba, and V. A. Panferov. Moment inequalities and high-energy tails for Boltzmann equations with inelastic interactions. *J. Stat. Phys.*, 116(5-6):1651–1682, 2004.
[10] M. Cannone and G. Karch. Infinite energy solutions to the homogeneous Boltzmann equation. *Comm. Pure Appl. Math.*, 63(6):747–778, 2010.
[11] J. A. Carrillo, C. Cercignani, and I. M. Gamba. Steady states of a Boltzmann equation for driven granular media. *Phys. Rev. E (3)*, 62(6, part A):7700–7707, 2000.
[12] J. A. Carrillo, J. Hu, Z. Ma, and T. Rey. Recent development in kinetic theory of granular materials: analysis and numerical methods. In *Trails in kinetic theory—foundational aspects and numerical methods*, volume 25 of *SEMA SIMAI Springer Ser.*, pages 1–36. Springer, Cham, 2021.
[13] L. Desvillettes. Some applications of the method of moments for the homogeneous Boltzmann and Kac equations. *Arch. Ration. Mech. Anal.*, 123(4):387–404, 1993.
[14] L. Desvillettes and C. Mouhot. Stability and uniqueness for the spatially homogeneous Boltzmann equation with long-range interactions. *Arch. Ration. Mech. Anal.*, 193(2):227–253, 2009.
[15] T. Elmoço. Global boundedness of moments of solutions of the Boltzmann equation for forces of infinite range. *Arch. Ration. Mech. Anal.*, 82(1):1–12, 1983.
[16] M. H. Ernst and R. Brito. Scaling solutions of inelastic Boltzmann equations with overpopulated high energy tails. *J. Stat. Phys.*, 109(3-4):407–432, 2002.
[17] N. Fournier. On exponential moments of the homogeneous Boltzmann equation for hard potentials without cutoff. *Comm. Math. Phys.*, 387(2):973–994, 2021.
[18] Nicolas Fournier and Clément Mouhot. On the well-posedness of the spatially homogeneous Boltzmann equation with a moderate angular singularity. *Comm. Math. Phys.*, 289(3):803–824, 2009.
[19] I. M. Gamba, V. Panferov, and C. Villani. On the Boltzmann equation for diffusively excited granular media. *Comm. Math. Phys.*, 246(3):503–541, 2004.
[20] I. M. Gamba, V. Panferov, and C. Villani. Upper Maxwellian bounds for the spatially homogeneous Boltzmann equation. *Arch. Ration. Mech. Anal.*, 194(1):253–282, 2009.
[21] Philip T. Gressman and Robert M. Strain. Global classical solutions of the Boltzmann equation without angular cut-off. *J. Amer. Math. Soc.*, 24(3):771–847, 2011.

[22] X. Lu and C. Mouhot. On measure solutions of the Boltzmann equation, part I: moment production and stability estimates. *J. Differential Equations*, 252(4):3305–3363, 2012.

[23] S. Mischler and B. Wennberg. On the spatially homogeneous Boltzmann equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 16(4):467–501, 1999.

[24] Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang. Regularity of solutions to the spatially homogeneous Boltzmann equation without angular cutoff. *Discrete Contin. Dyn. Syst.*, 24(1):187–212, 2009.

[25] Y. Morimoto, S. Wang, and T. Yang. Measure valued solutions to the spatially homogeneous Boltzmann equation without angular cutoff. *J. Stat. Phys.*, 165(5):866–906, 2016.

[26] A. Pulvirenti and G. Toscani. The theory of the nonlinear Boltzmann equation for Maxwell molecules in Fourier representation. *Ann. Mat. Pura Appl. (4)*, 171:181–204, 1996.

[27] K. Qi. On the measure valued solution to the inelastic Boltzmann equation with soft potentials. *J. Stat. Phys.*, 183(27), 2021.

[28] K. Qi. Measure valued solution to the spatially homogeneous Boltzmann equation with inelastic long-range interactions. *J. Math. Phys.*, 63(2):021503, 22, 2022.

[29] B. Wennberg. On moments and uniqueness for solutions to the space homogeneous Boltzmann equation. *Transport Theory Statist. Phys.*, 23(4):533–539, 1994.

[30] B. Wennberg. Entropy dissipation and moment production for the Boltzmann equation. *J. Stat. Phys.*, 86(5-6):1053–1066, 1997.

[31] B. Wennberg. An example of nonuniqueness for solutions to the homogeneous Boltzmann equation. *J. Stat. Phys.*, 95(1-2):469–477, 1999.