Moderate deviations for the range of planar random walks

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Abstract

Given a symmetric random walk in $\mathbb{Z}^2$ with finite second moments, let $R_n$ be the range of the random walk up to time $n$. We study moderate deviations for $R_n - \mathbb{E}R_n$ and $\mathbb{E}R_n - R_n$. We also derive the corresponding laws of the iterated logarithm.

1 Introduction

Let $X_i$ be symmetric i.i.d. random vectors taking values in $\mathbb{Z}^2$ with mean 0 and finite covariance matrix $\Gamma$, set $S_n = \sum_{i=1}^n X_i$, and suppose that no proper subgroup of $\mathbb{Z}^2$ supports the random walk $S_n$. For any random variable $Y$ we will use the notation

$$\overline{Y} = Y - \mathbb{E}Y.$$ 

Let

$$(1.1) \quad R_n = \# \{ S_1, \ldots, S_n \}$$

be the range of the random walk up to time $n$. The purpose of this paper is to obtain moderate deviation results for $\overline{R_n}$ and $-\overline{R_n}$.

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For moderate deviations of $\bar{R}_n$ we have the following. Let

(1.2) \[ \mathcal{H}(n) = \sum_{k=0}^{n} \mathbb{P}^0(S_k = 0). \]

Since the $X_i$ have two moments, then by \[23\], Section 2,

\[ \mathcal{H}(n) = \sum_{k=0}^{n} \mathbb{P}^0(S_k = 0) \sim \frac{\log n}{2\pi \sqrt{\det \Gamma}}, \]

and

\[ \mathcal{H}(n) - \mathcal{H}(\lfloor n/b_n \rfloor) = \sum_{k=\lfloor n/b_n \rfloor+1}^{n} \mathbb{P}^0(S_k = 0) \sim \frac{\log b_n}{2\pi \sqrt{\det \Gamma}}. \]

**Theorem 1.1** Let $\{b_n\}$ be a positive sequence satisfying $b_n \to \infty$ and $\log b_n = o((\log n)^{1/2})$ as $n \to \infty$. There are two constants $C_1, C_2 > 0$ independent of the choice of the sequence $\{b_n\}$ such that

\[ -C_1 \leq \liminf_{n \to \infty} b_n^{-1} \log \mathbb{P} \left\{ \bar{R}_n \geq \frac{n}{\mathcal{H}(n)^2} (\mathcal{H}(n) - \mathcal{H}(\lfloor n/b_n \rfloor)) \right\} \]

(1.3)

\[ \leq \limsup_{n \to \infty} b_n^{-1} \log \mathbb{P} \left\{ \bar{R}_n \geq \frac{n}{\mathcal{H}(n)^2} (\mathcal{H}(n) - \mathcal{H}(\lfloor n/b_n \rfloor)) \right\} \leq -C_2. \]

**Remark 1.2** The proof will show that $C_2$ in the statement of Theorem \[1.1\] is equal to the constant $L$ given in Theorem 1.3 in \[2\]. We believe that $C_1$ is also equal to $L$, but we do not have a proof of this fact.

A more precise statement than Theorem \[1.1\] is possible when the $X_i$ have slightly more than two moments.

**Corollary 1.3** Suppose $\mathbb{E}[|X_i|^2(\log^+ (|X_i|))^{\frac{1}{2}+\delta}] < \infty$ for some $\delta > 0$. Let $\{b_n\}$ be a positive sequence satisfying $b_n \to \infty$ and $\log b_n = o((\log n)^{1/2})$ as $n \to \infty$. There are two constants $C_1, C_2 > 0$ independent of the choice of the sequence $\{b_n\}$ such that

\[ -C_1 \leq \liminf_{n \to \infty} b_n^{-\theta} \log \mathbb{P} \left\{ \bar{R}_n \geq 2\theta \pi \sqrt{\det \Gamma} \frac{n}{(\log n)^2} \log b_n \right\} \]

(1.4)

\[ \leq \limsup_{n \to \infty} b_n^{-\theta} \log \mathbb{P} \left\{ \bar{R}_n \geq 2\theta \pi \sqrt{\det \Gamma} \frac{n}{(\log n)^2} \log b_n \right\} \leq -C_2 \]

for any $\theta > 0$.  

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Remark 1.4 The constants $C_1$, $C_2$ are the same as in the statement of Theorem 1.1. See Remark 1.2

For $b_n$ tending to infinity faster than the rate given in Theorem 1.1, e.g., \( \log b_n = (\log n)^2 \), then we are in the realm of large deviations. For results on large deviations of the range, see [14], [18], [19].

For the moderate deviations of \( -\overline{R}_n = \mathbb{E}R_n - R_n \) we have the following. Let $\kappa(2, 2)$ be the smallest $A$ such that
\[
\|f\|_4 \leq A \|\nabla f\|_2^{1/2} \|f\|_2^{1/2}
\]
for all $f \in C^1$ with compact support. (This constant appeared in [2].)

**Theorem 1.5** Suppose $b_n \to \infty$ and $b_n = o((\log n)^{1/5})$ as $n \to \infty$. For $\lambda > 0$
\[
\lim_{n \to \infty} \frac{1}{b_n} \log \mathbb{P}\left(-\overline{R}_n > \lambda \frac{nb_n}{\log^2 n}\right) = -(2\pi)^{-2} (\det \Gamma)^{-1/2} \kappa(2, 2)^4 \lambda.
\]

Comparing Theorems 1.1 and 1.5, we see that the upper and lower tails of $\overline{R}_n$ are quite different. This is similar to the behavior of the distribution of the self-intersection local time of planar Brownian motion. This is not surprising, since LeGall, [21, Theorem 6.1], shows that $\overline{R}_n$, properly normalized, converges in distribution to the self-intersection local time.

The moderate deviations of $\overline{R}_n$ are quite similar in nature to those of $-\overline{L}_n$, where $L_n$ is the number of self-intersections of the random walk $S_n$; see [4]. Again, [21, Theorem 6.1] gives a partial explanation of this. However the case of the range is much more difficult than the corresponding results for intersection local times. The latter case can be represented as a quadratic functional of the path, which is amenable to the techniques of large deviation theory, while the range cannot be so represented. This has necessitated the development of several new tools, see in particular Sections 5 and 6, which we expect will have further applications in the study of the range of random walks.

Theorem 1.1 gives rise to the following LIL for $\overline{R}_n$. 

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Theorem 1.6

\[(1.5) \limsup_{n \to \infty} \frac{R_n}{n \log \log \log n/ \log^2 n} = 2\pi \sqrt{|\det \Gamma|}, \quad \text{a.s.}\]

This result is an improvement of that in [6]; there it was required that the \(X_i\) be bounded random variables and the constant was not identified. Theorem 1.1 is a more precise estimate than is needed for Theorem 1.6; this is why Theorem 1.1 needs to be stated in terms of \(\mathcal{H}(n)\) while Theorem 1.6 does not.

For an LIL for \(-R_n\) we have a different rate.

Theorem 1.7 We have

\[\limsup_{n \to \infty} \frac{-R_n}{n \log \log n/ \log^2 n} = (2\pi)^{-2} \sqrt{|\det \Gamma|} \kappa(2, 2)^4, \quad \text{a.s.}\]

The study of the range of a lattice-valued (or \(\mathbb{Z}^d\)-valued) random walk has a long history in probability and the results show a strong dependence on the dimension \(d\). See [15], [20], [21], [23], [18], [13], [14], and [6] and the references in these papers, to cite only a few. The two dimensional case seems to be the most difficult; in one dimension no renormalization is needed (see [9]), while for \(d \geq 3\) the tails are sub-Gaussian and have asymptotically symmetric behavior. In two dimensions, renormalization is needed and the tails have non-symmetric behavior. In this case, the central limit theorem was proved in 1986 in [21], while the first law of the iterated logarithm was not proved until a few years ago in [6].

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2 Moments of the range

In this section we first give an estimate for the expectation of the range.

By [23], Theorem 6.9, we have

\[\mathbb{E}R_n = \frac{n}{\mathcal{H}(n)} + \frac{1}{2\pi \sqrt{|\det \Gamma|} \mathcal{H}(n)^2} \frac{n}{(1 + o(1))},\]
where $\mathcal{H}$ is defined in (1.2). By [23], Section 2,

$$
\mathcal{H}(n) \sim \frac{\log n}{2\pi \sqrt{\det \Gamma}}
$$

and

$$
\mathcal{H}(n) - \mathcal{H}(m) \sim \frac{\log(n/m)}{2\pi \sqrt{\det \Gamma}}
$$
as $n$ and $m$ tend to infinity.

Throughout this paper we will mostly be concerned with random walks that have only second moments. The exception is the following proposition, which supposes slightly more than two moments, and Corollary 1.3.

**Proposition 2.1** Suppose $\{X_i\}$ is a sequence of i.i.d. mean zero random vectors taking values in $\mathbb{Z}^2$ with

$$
\mathbb{E}\left(|X|^2(\log^+|X|)^{\frac{3}{2}+\delta}\right) < \infty
$$

for some $\delta > 0$ and nondegenerate covariance matrix $\Gamma$. Let $S_n = \sum_{i=1}^n X_i$ and suppose $S_n$ is strongly aperiodic. Then

$$
\mathbb{P}(S_n = 0) = \frac{1}{2\pi n\sqrt{\det \Gamma}} + O\left(\frac{1}{n(\log n)^{(1+\delta)/2}}\right).
$$

**Proof.** Let $\varphi$ be the characteristic function of $X_i$, let $x \cdot y$ denote the inner product in $\mathbb{R}^2$, let $Q(u) = u \cdot \Gamma u$, and let $C = [-\pi, \pi]^2$. We observe that

$$
|1 - \varphi(u) - Q(u)| = |\mathbb{E}\left(1 - e^{iu \cdot X} + iu \cdot X + (1/2)(iu \cdot X)^2\right)|
\leq c_1 |u|^3 \mathbb{E}\left(1_{\{|X| \leq 1/|u|\}}|X|^3\right) + c_1 |u|^2 \mathbb{E}\left(1_{\{|X| > 1/|u|\}}|X|^2\right)
$$
and consequently for any fixed $M > 0$

(2.7)\[
|1 - \varphi(u/\sqrt{n}) - Q(u/\sqrt{n})| \\
\leq c_2 \left( \frac{1}{n^{3/2}} \right) \mathbb{E} \left( 1_{\{|u||X| \leq \sqrt{n}\}} (|u||X|)^3 \right) + c_2 \left( \frac{1}{n} \right) \mathbb{E} \left( 1_{\{|u||X| > \sqrt{n}\}} (|u||X|)^2 \right) \\
\leq c_3 \frac{1}{n^{3/2}} + c_3 \left( \frac{1}{n^{3/2}} \right) \mathbb{E} \left( 1_{\{|M| < |u||X| \leq \sqrt{n}\}} (|u||X|)^3 \right) \\
+ c_3 \left( \frac{1}{n} \right) \mathbb{E} \left( 1_{\{|u||X| > \sqrt{n}\}} (|u||X|)^2 \right).
\]

Choose $M$ so that $x/\log^{1/2+\delta}(x)$ is monotone increasing on $x \geq M$, and therefore

(2.8) \[
\mathbb{E} \left( 1_{\{|M| < |u||X| \leq \sqrt{n}\}} (|u||X|)^3 \right) \\
\leq \mathbb{E} \left( 1_{\{|M| < |u||X| \leq \sqrt{n}\}} (|u||X|)^2 \log^{1/2+\delta}(|u||X|) \cdot \frac{|u||X|}{\log^{1/2+\delta}(|u||X|)} \right) \\
\leq \left( \frac{\sqrt{n}}{\log^{1/2+\delta}(\sqrt{n})} \right) \mathbb{E} \left( 1_{\{|M| < |u||X| \leq \sqrt{n}\}} (|u||X|)^2 \log^{1/2+\delta}(|u||X|) \right).
\]

Also

(2.9) \[
\mathbb{E} \left( 1_{\{|u||X| > \sqrt{n}\}} (|u||X|)^2 \right) \\
\leq \left( \frac{1}{\log^{1/2+\delta}(\sqrt{n})} \right) \mathbb{E} \left( 1_{\{|u||X| > \sqrt{n}\}} (|u||X|)^2 \log^{1/2+\delta}(|u||X|) \right).
\]

Then implies that

(2.10) \[
|1 - \varphi(u/\sqrt{n}) - Q(u/\sqrt{n})| \leq c \frac{|u|^2 \log^{1/2+\delta}(|u|)}{n \log^{1/2+\delta}(n)}.
\]

Following the proof in Spitzer [34], pp. 76–77,

\[
2\pi n \mathbb{P}(S_n = 0) = (2\pi)^{-1} \int_{\sqrt{nC}} \varphi(u/\sqrt{n})^n du \\
= I_0 + I_1(n, A_n) + I_2(n, A_n) + I_3(n, A_n, r) + I_4(n, r),
\]
where
\[
I_0 = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-Q(u)/2} du = (\det Q)^{-1/2},
\]
\[
I_1(n, A_n) = (2\pi)^{-1} \int_{|u| \leq A_n} [\varphi(u/\sqrt{n})^n - e^{-Q(u)/2}] du,
\]
\[
I_2(n, A_n) = -(2\pi)^{-1} \int_{|u| > A_n} e^{-Q(u)/2} du,
\]
\[
I_3(n, A_n, r) = (2\pi)^{-1} \int_{A_n < |u| < r \sqrt{n}} \varphi(u/\sqrt{n})^n du,
\]
\[
I_4(n, r) = (2\pi)^{-1} \int_{|u| \geq r \sqrt{n}, u \in \sqrt{n} C} \varphi(u/\sqrt{n})^n du.
\]

Still following \[34\], we can choose \(r\) such that \(|\varphi(u/\sqrt{n})^n| \leq e^{-Q(u)/4}\) if \(|u| \leq r \sqrt{n}\) and by the strong aperiodicity there exists \(\gamma > 0\) such that \(|\varphi(u/\sqrt{n})| \leq 1 - \gamma\) if \(|u| > r \sqrt{n}\) and \(u \in \sqrt{n} C\). Set \(A_n = c_4 \sqrt{\log \log n}\). We have
\[
|I_4(n, r)| \leq (2\pi)^{-1} \int_{u \in \sqrt{n} C} (1 - \gamma)^n du = O(n^{-p})
\]
for every positive integer \(p\). Next
\[
|I_3(n, A_n, r)| \leq \int_{|u| > c_4 \sqrt{\log \log n}} e^{-Q(u)/4} du = O((\log n)^{-2})
\]
for \(c_4\) large and similarly we have the same bound for \(|I_2(n, A_n)|\). To estimate \(I_1(n, A_n)\) we use the inequality \(|a^n - b^n| \leq n|a - b|\) if \(|a|, |b| \leq 1\) with \(a = \varphi(u/\sqrt{n})\) and \(b = e^{-Q(u)/2n}\). Using \((2.10)\) and the analogous expansion for \(e^{-Q(u)/2n}\) we have
\[
|\varphi(u/\sqrt{n})^n - e^{-Q(u)/2^n}| \leq n|\varphi(u/\sqrt{n}) - e^{-Q(u)/2n}|
\]
\[
\leq c_5 n \frac{|u|^2 \log^{1/2+\delta}(|u|)}{n \log^{1/2+\delta} (n)} = c_5 \frac{|u|^2 \log^{1/2+\delta}(|u|)}{\log^{1/2+\delta} (n)}.
\]

Integrating this over the set \(|u| \leq A_n\), we see
\[
|I_1(n, A_n)| = O((\log \log n)^{2+\delta/2}/(\log n)^{1/2+\delta}) = O(1/(\log n)^{(1+\delta)/2}).
\]

Summing \(I_0\) through \(I_4\), we obtain
\[
2\pi n \mathbb{P}(S_n = 0) = (\det \Gamma)^{-1/2} + O(1/(\log n)^{(1+\delta)/2}).
\]
Next we establish some sharp exponential estimates for the range and intersection of ranges. Aside from their intrinsic interest, they will be used to estimate the tail probabilities in our first main theorem.

We write $S(I)$ for $\{S_k : k \in I\}$. Let $S^{(i)}$, $i = 1, \ldots, p$ be $p$ independent copies of $S$. First, by Corollary 1 of [8], for any integers $a \geq 1$, $n_1, \ldots, n_a \geq 1$,

\begin{equation}
(\mathbb{E} J_{n_1+\cdots+n_a}^m)^{1/p} \leq \sum_{k_1+\cdots+k_a=m \atop k_1,\ldots,k_a \geq 0} \frac{m!}{k_1! \cdots k_a!} \left(\frac{\mathbb{E} J_{n_1}^{k_1}}{p} \cdots \frac{\mathbb{E} J_{n_a}^{k_a}}{p}\right)^{1/p},
\end{equation}

where

$$J_n = \#\left\{S^{(1)}[1,n] \cap \cdots \cap S^{(p)}[1,n]\right\} \quad n = 1, 2, \ldots.$$ 

In the next Theorem we deduce from this the exponential integrability of $J_n$, which was established in [6] in the special case $p = 2$ and under the condition that $S$ had bounded increments.

**Theorem 2.2** Assume that the planar random walk $S$ has finite second moments and zero mean. There exists $\theta > 0$ such that

\begin{equation}
\sup_n \sup_{y_1,\ldots,y_p} \mathbb{E}(y_1,\ldots,y_p) \exp \left\{ \theta \left(\frac{(\log n)^p}{n}\right)^{1/(p-1)} J_n^{1/(p-1)} \right\} < \infty.
\end{equation}

**Proof.** We recall the fact (see Remarks, p. 664, in [23]) that

\begin{equation}
\mathbb{E} J_n^k \leq (k!)^p (\mathbb{E} J_n)^k, \quad k = 0, 1, \ldots,
\end{equation}

and for some $C < \infty$

\begin{equation}
\mathbb{E} J_n \leq \frac{C n}{(\log n)^p}, \quad n = 1, \ldots.
\end{equation}

The proof of (2.12) is a modification of the approach used in Lemma 1 of [8]. We begin by showing that there is a constant $C > 0$ such that

\begin{equation}
\sup_n \mathbb{E} J_n^m \leq C^m (m!)^{p-1} \left(\frac{n}{(\log n)^p}\right)^m, \quad m, n = 1, 2, \ldots.
\end{equation}
We first consider the case $m \leq (\log n)^{(p-1)/p}$. Write $l(n, m) = \lfloor n/m \rfloor + 1$. Then by (2.11) and (2.14),

\[
(\mathbb{E}J_n^m)^{1/p} \leq \sum_{k_1 + \cdots + k_m = m, k_1, \ldots, k_m \geq 0} \frac{m!}{k_1! \cdots k_m!} \left( \mathbb{E}J_{l(n,m)}^{k_1} \cdots \mathbb{E}J_{l(n,m)}^{k_m} \right)^{1/p} \\
\leq \sum_{k_1 + \cdots + k_m = m, k_1, \ldots, k_m \geq 0} \frac{m!}{k_1! \cdots k_m!} k_1! \cdots k_m! \left( \mathbb{E}J_{l(n,m)}^{k_1} \cdots \mathbb{E}J_{l(n,m)}^{k_m} \right)^{k_1/p} \cdots \left( \mathbb{E}J_{l(n,m)}^{k_1} \cdots \mathbb{E}J_{l(n,m)}^{k_m} \right)^{k_m/p} \\
= \binom{2m-1}{m} m! \left( \mathbb{E}J_{l(n,m)} \right)^{m/p} \leq \left( \binom{2m-1}{m} m! \left( \frac{n}{(\log n)^p} \right)^{m/p} \\
\leq \left( \binom{2m}{m} \right)^{\frac{p-1}{p}} C^m \left( \frac{n}{(\log n)^p} \right)^{m/p},
\]

where the second inequality follows from (2.13) and the third from (2.14) using the fact that $m = O(\log n)$ so that $\log n = O(\log(n/m))$. Hence, taking $p$-th powers we obtain

\[
\mathbb{E}J_n^m \leq \left( \binom{2m}{m} \right)^{p} C^{pm} (m!)^{p-1} \left( \frac{n}{(\log n)^p} \right)^m,
\]

and (2.15) for the case of $m \leq (\log n)^{(p-1)/p}$ follows from the fact

\[
\binom{2m}{m} \leq 4^m.
\]

For the case $m > (\log n)^{(p-1)/p}$, notice from the definition of $J_n$ that $J_n \leq n$. So we have

\[
\mathbb{E}J_n^m \leq n^m = (\log n)^{pm} \left( \frac{n}{(\log n)^p} \right)^m \leq m^{(p-1)m} \left( \frac{n}{(\log n)^p} \right)^m \\
\leq (m!)^{p-1} C^m \left( \frac{n}{(\log n)^p} \right)^m,
\]

where the last step follows from Stirling’s formula. This completes the proof of (2.15).
By Hölder’s inequality this shows that

\[
(\log n)^{p \frac{m}{(p-1)}} \sup_{y_1, \ldots, y_p} \mathbb{E}^{(y_1, \ldots, y_p)}(J_{n}^{m/(p-1)}) \leq \left(\frac{\log n}{n}\right)^{p \frac{m}{(p-1)}} \sup_{y_1, \ldots, y_p} \left\{ \mathbb{E}^{(y_1, \ldots, y_p)}(J_{n}^{m}) \right\}^{1/(p-1)} \leq C^m m!
\]

where the second inequality used [8], p.1053. Our theorem then follows from a Taylor expansion. \(\square\)

**Remark.** Theorem 2.2 is sharp in the sense that (2.12) does not hold if \(\theta\) is too large. Indeed, by [21], for any \(m = 1, 2, \ldots,\)

\[
\left(\frac{\log n}{n}\right)^{p m} \mathbb{E} J_{n}^{m} \rightarrow (2\pi)^{p m} \det(\Gamma)^{m/2} \mathbb{E} \alpha([0, 1]^p)^m
\]
as \(n \rightarrow \infty\), where \(\alpha([0, 1]^p)\) is the Brownian intersection local time formally defined by

\[
\alpha([0, 1]^p) = \int_{\mathbb{R}^d} \left[ \prod_{j=1}^{p} \int_{0}^{1} \delta_x(W_j(s)) ds \right] dx,
\]

and by Theorem 2.1 in [7]

\[
\mathbb{E} \exp \left\{ \theta \alpha([0, 1]^p)^{(p-1)^{-1}} \right\} = \infty
\]

for large \(\theta\). The following theorem is sharp in the same sense.

**Theorem 2.3** Assume that the planar random walk \(S\) has finite second moments and zero mean. Then there exists \(\theta > 0\) such that

\[
\sup_n \mathbb{E} \exp \left\{ \theta \frac{(\log n)^2}{n} |R_n| \right\} < \infty.
\]

**Proof.** We first consider the case where \(n\) is replaced by \(2^n\). Let

\[
N = [2(\log 2)^{-1} \log n]
\]
so that $2^N \sim n^2$ and note that

$$\sum_{k=1}^{2^N} \# \{ S((k-1)2^{n-N}, k2^{n-N}) \} - \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \# \{ S((2k-2)2^{n-j}, (2k-1)2^{n-j}] \cap S((2k-1)2^{n-j}, (2k)2^{n-j}) \}.$$

Setting

$$\beta_k = \# \{ S((k-1)2^{n-N}, k2^{n-N}) \}$$

and

$$\alpha_{j,k} = \# \{ S((2k-2)2^{n-j}, (2k-1)2^{n-j}] \cap S((2k-1)2^{n-j}, (2k)2^{n-j}) \}$$

leads to the decomposition

$$R_{2^n} \approx \sum_{k=1}^{2^N} \beta_k - \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \alpha_{j,k}.$$

Recall that (Lemma 3 in [8]),

$$\sup_n \mathbb{E} \exp \left\{ \lambda \frac{\log n}{n} \# \{ S[1, n] \} \right\} < \infty$$

for all $\lambda > 0$. In particular,

$$\sup_n \mathbb{E} \exp \left\{ \lambda \frac{\log 2^{n-N}}{2^{n-N}} |\beta_1| \right\} < \infty.$$

Notice that $\beta_1, \ldots, \beta_{2^N}$ is an i.i.d. sequence with $\mathbb{E} \beta_1 = 0$. By Lemma 1 in [3], there is a $\theta > 0$ such that

$$\sup_n \mathbb{E} \exp \left\{ \theta 2^{-N/2} \frac{\log 2^{n-N}}{2^{n-N}} \left| \sum_{k=1}^{2^N} \beta_k \right| \right\} < \infty.$$

By the choice of $N$ one can see that there is a $c > 0$ independent of $n$ such that

$$2^{-N/2} \frac{\log 2^{n-N}}{2^{n-N}} \geq c \frac{(\log 2^n)^2}{2^n}.$$
So there is some $\theta > 0$ such that

$$\sup_n \mathbb{E} \exp \left\{ \frac{\theta (\log 2^n)^2}{2^n} \left| \sum_{k=1}^{2N} \beta_k \right| \right\} < \infty.$$ 

We need to show that for some $\theta > 0$,

(2.20) $$\sup_n \mathbb{E} \exp \left\{ \frac{\theta (\log 2^n)^2}{2^n} \left| \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \alpha_{j,k} \right| \right\} < \infty.$$ 

Set

(2.21) $$\tilde{J}_n = \# \{ S[1,n] \cap S'[1,n] \} \quad n = 1, 2, \cdots,$$

where $S'$ is an independent copy of the random walk $S$. In our notation, for each $1 \leq j \leq N$, $\{\overline{\pi}_{j,1}, \cdots, \overline{\pi}_{j,2^{j-1}}\}$ is an i.i.d. sequence with the same distribution as $\tilde{J}_{2^n-j}$. By Theorem 2.2 (with $p = 2$), there is a $\delta > 0$ such that

$$\sup_n \sup_{j \leq N} \mathbb{E} \exp \left\{ \delta \frac{(\log 2^{n-j})^2}{2^{n-j}} \left| \overline{\alpha}_{j,1} \right| \right\} < \infty.$$ 

By Lemma 1 in [3] again, there is a $\overline{\theta} > 0$ such that

$$\sup_n \sup_{j \leq N} \mathbb{E} \exp \left\{ \overline{\theta} 2^{-j/2} \frac{(\log 2^n)^2}{2^n} \left| \sum_{k=1}^{2^{j-1}} \overline{\alpha}_{j,k} \right| \right\} < \infty.$$ 

Hence for some $\theta > 0$

$$\lambda_N = \prod_{j=1}^{N} \left( 1 - 2^{-j/2} \right) \quad \text{and} \quad \lambda_\infty = \prod_{j=1}^{\infty} \left( 1 - 2^{-j/2} \right).$$
Using Hölder’s inequality with $1/p = 1 - 2^{-N/2}$, $1/q = 2^{-N/2}$ we have

$$
\mathbb{E} \exp \left\{ \lambda_N \theta \frac{(\log 2^n)^2}{2^n} \left| \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \alpha_{j,k} \right| \right\}
\leq \left( \mathbb{E} \exp \left\{ \lambda_{N-1} \theta \frac{(\log 2^n)^2}{2^n} \left| \sum_{j=1}^{N-1} \sum_{k=1}^{2^{j-1}} \alpha_{j,k} \right| \right\} \right)^{1 - 2^{-N/2}}
\times \left( \mathbb{E} \exp \left\{ \lambda_N \theta 2^{N/2} \frac{(\log 2^n)^2}{2^n} \left| \sum_{k=1}^{2^{N-1}} \alpha_{N,k} \right| \right\} \right)^{2^{-N/2}}
\leq \mathbb{E} \exp \left\{ \lambda_{N-1} \theta \frac{(\log 2^n)^2}{2^n} \left| \sum_{j=1}^{N-1} \sum_{k=1}^{2^{j-1}} \alpha_{j,k} \right| \right\} \cdot C(\theta)^{2^{-N/2}}
esince \lambda_N < 1. Repeating this procedure,

$$
\mathbb{E} \exp \left\{ \lambda_N \theta \frac{(\log 2^n)^2}{2^n} \left| \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \alpha_{j,k} \right| \right\}
\leq C(\theta) 2^{-1/2 + \cdots + 2^{-N/2}} \leq C(\theta) 2^{-1/2(1-2^{-1/2})^{-1}}.
$$

So we have

$$
\sup_n \mathbb{E} \exp \left\{ \lambda_{\infty} \theta \frac{(\log 2^n)^2}{2^n} \left| \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \alpha_{j,k} \right| \right\} \leq C(\theta) 2^{-1/2(1-2^{-1/2})^{-1}}.
$$

We have proved (2.20) and therefore (2.17) when $n$ is the power of 2. We now prove Theorem 2.3 for general $n$. Given an integer $n \geq 2$, we have the following unique representation:

$$
n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_l}
$$

where $m_1 > m_2 > \cdots m_l \geq 0$ are integers. Write

$$
n_0 = 0 \text{ and } n_i = 2^{m_1} + \cdots + 2^{m_i} \quad i = 1, \cdots, l.
$$

Then

$$
\# \{ S[1, n] \} = \sum_{i=1}^{l} \# \{ S(n_{i-1}, n_i) \} - \sum_{i=1}^{l-1} \# \{ S(n_{i-1}, n_i) \cap S(n_i, n) \}
= \sum_{i=1}^{l} B_i - \sum_{i=1}^{l-1} A_i.
$$

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Write
\[
\sum_{i=1}^{l} B_i = \sum' B_i + \sum'' B_i
\]
where \(\sum'\) is the summation over \(i\) with \(2^{m_i} \geq \sqrt{n}\) and \(\sum''\) is the summation over \(i\) with \(2^{m_i} < \sqrt{n}\). We also define the products \(\prod'\) and \(\prod''\) in a similar manner. Then

\[
\mathbb{E} \exp \left\{ \theta \frac{(\log n)^2}{n} \left| \sum' (B_i - \mathbb{E} B_i) \right| \right\}
\]

\[
\leq \prod' \left( \mathbb{E} \exp \left\{ \theta \frac{(\log n)^2}{n} 2^{-m_i} \left( \sum' 2^{m_j} \right) |R_{2^{m_i}}| \right\} \right)^{2^{m_i} \left( \sum' 2^{m_j} \right)^{-1}}
\]

\[
\leq \prod' \left( \mathbb{E} \exp \left\{ 4 \theta \frac{(\log 2^{m_i})^2}{2^{m_i}} |R_{2^{m_i}}| \right\} \right)^{2^{m_i} \left( \sum' 2^{m_j} \right)^{-1}}
\]

\[
\leq \sup_m \mathbb{E} \exp \left\{ 4 \theta \frac{(\log 2^m)^2}{2^m} |R_{2^m}| \right\}.
\]

Assume that the set \(\{1 \leq i \leq l; 2^{m_i} < \sqrt{n}\}\) is non-empty. We have

\[
\sum'' 2^{m_i} \leq 2 \sqrt{n}.
\]

So we have

\[
\frac{(\log n)^2}{n} \leq \frac{1}{\sqrt{n}} \leq 2 \left( \sum'' 2^{m_i} \right)^{-1}.
\]
Hence

\[ \mathbb{E} \exp \left\{ \frac{\theta (\log n)^2}{n} \left| \sum_i'' (B_i - \mathbb{E}B_i) \right| \right\} \]

\[ \leq \prod_i'' \left( \mathbb{E} \exp \left\{ \frac{\theta (\log n)^2}{n} 2^{-m_i} \left( \sum_j'' 2^{m_j} |R_{2^m_i}| \right) \right\} 2^{m_i} \left( \sum_j'' 2^{m_j} \right)^{-1} \]

\[ \leq \prod_i'' \left( \mathbb{E} \exp \left\{ 2\theta \frac{1}{2^{m_i}} |R_{2^m_i}| \right\} \right) \left( \sum_j'' 2^{m_j} \right)^{-1} \]

\[ \leq \sup_m \mathbb{E} \exp \left\{ 2\theta \frac{1}{2^{m_i}} |R_{2^m_i}| \right\}. \]

By the Cauchy-Schwarz inequality and what we have proved in the previous step, there exists \( \theta > 0 \) such that

\[ \mathbb{E} \exp \left\{ \frac{\theta (\log n)^2}{n} \left| \sum_{i=1}^l (B_i - \mathbb{E}B_i) \right| \right\} \]

is bounded uniformly in \( n \). By the fact that

(2.22) \[ n - n_i = 2^{m_{i+1}} + \cdots + 2^{m_i} \leq 2^{m_i} \]

we have

(2.23) \[ A_i \overset{d}{=} \# \{ S[1, 2^{m_i}] \cap S'[1, n - n_i] \} \leq J_{2^{m_i}}. \]

By (2.14) there is a constant \( C > 0 \) independent of \( n \) such that

(2.24) \[ \sum_{i=1}^{l-1} \mathbb{E}A_i \leq \sum_{i=1}^{l-1} \mathbb{E}J_{2^m_i} \leq C \sum_{i=1}^l \frac{2^{m_i}}{m_i^2} \]

\[ \leq C \sum_{m_i < l/2} \frac{2^{m_i}}{m_i^2} + C \sum_{m_i \geq l/2} \frac{2^{m_i}}{m_i^2} \]

\[ \leq C 2^{l/2} + C \frac{n}{(\log n)^2} \]

\[ \leq C \frac{n}{(\log n)^2}. \]
It remains to show that

\begin{equation}
(2.25) \quad \sup_n \mathbb{E} \exp \left\{ \theta \frac{(\log n)^2}{n} \sum_{i=1}^{l-1} A_i \right\} < \infty.
\end{equation}

Using (2.24), this follows from (2.12), (with \( p = 2 \)), and the same argument used for \( B_1 - \mathbb{E} B_1, \ldots, B_l - \mathbb{E} B_l \).

In view of the remark prior to Theorem 2.3, the next result shows that \( \overline{R}_n \) has a non-symmetric tail behavior.

**Theorem 2.4** Under the assumptions of Theorem 2.3

\begin{equation}
(2.26) \quad \sup_n \mathbb{E} \exp \left\{ \theta \frac{(\log n)^2}{n} \overline{R}_n \right\} < \infty
\end{equation}

for all \( \theta > 0 \).

**Proof.** By Theorem 2.3, (2.26) holds for some \( \theta_0 > 0 \). For \( \theta > \theta_0 \), take an integer \( m \geq 1 \) such that \( m^{-1} \theta < \theta_0 \). It is easy to see that it suffices to prove

\begin{equation}
(2.27) \quad \sup_n \mathbb{E} \exp \left\{ \theta \frac{(\log n)^2}{mn} \overline{R}_{nm} \right\} < \infty.
\end{equation}

Set \( \zeta_{jn} = \# \{ S((j-1)n, jn] \} \). By the facts that

\[ \overline{R}_{nm} \leq \sum_{j=1}^{m} \zeta_{jn} + \left( \sum_{j=1}^{m} \mathbb{E} \zeta_{jn} \right) - \mathbb{E} \overline{R}_{nm} \]

and that by (2.2) and (2.3),

\[ \left( \sum_{j=1}^{m} \mathbb{E} \zeta_{jn} \right) - \mathbb{E} \overline{R}_{nm} = m \mathbb{E} R_n - \mathbb{E} R_{nm} \]

\[ = \frac{mn}{\mathcal{H}(n)} + O\left( \frac{mn}{\mathcal{H}(n)^2} - \frac{mn}{\mathcal{H}(mn)} + O\left( \frac{mn}{\mathcal{H}(mn)^2} \right) \right) \]

\[ = \frac{mn}{\mathcal{H}(n)\mathcal{H}(mn)}((\mathcal{H}(mn) - \mathcal{H}(n)) + O\left( \frac{n}{\log^2 n} \right) \]

\[ = O\left( \frac{n}{(\log n)^2} \right) \]
as \( n \to \infty \) (note \( m \) is fixed), there is a constant \( C_{m, \theta} > 0 \) depending only on \( m \) and \( \theta \) such that

\[
\mathbb{E} \exp \left\{ \theta \frac{(\log n)^2}{mn} \overline{R}_{nm} \right\} \leq C_m \left( \mathbb{E} \exp \left\{ \theta \frac{(\log n)^2}{mn} \overline{R}_n \right\} \right)^m.
\]

So we have \( (2.27) \). \( \square \)

### 3 Moderate deviations for \( R_n - \mathbb{E} R_n \)

We can now prove Theorem 1.1.

**Proof.** We first prove the upper bound. Let \( t > 0 \) and write \( K = \lceil t^{-1} b_n \rceil \).

Divide \([1, n]\) into \( K \) disjoint subintervals, each of length \([n/K]\) or \([n/K]+1\). Call the \( i^{th} \) subinterval \( I_i \). Let \( E_i = \#\{ S(I_i) \} \). Then

\[
\overline{R}_n \leq \sum_{j=1}^K E_j + \left( \sum_{j=1}^K \mathbb{E}E_j \right) - \mathbb{E}R_n
\]

From \( (2.1) \) we have

\[
\sum_{j=1}^K \mathbb{E}E_j - \mathbb{E}R_n
\]

\[
= K \frac{n/K}{\mathcal{H}([n/K])} - \frac{n}{\mathcal{H}(n)} + \frac{1}{2\pi \sqrt{\det \Gamma}} \left\{ K \frac{n/K}{\mathcal{H}^2([n/K])} - \frac{n}{\mathcal{H}^2(n)} \right\} + o\left( \frac{n}{\mathcal{H}^2(n)} \right)
\]

\[
= \frac{n(\mathcal{H}(n) - \mathcal{H}([n/K]))}{\mathcal{H}^2(n)} \left\{ 1 + \frac{\mathcal{H}(n) - \mathcal{H}([n/K])}{\mathcal{H}([n/K])} \right\} + o\left( \frac{n}{\mathcal{H}^2(n)} \right),
\]

where the error term can be taken to be independent of \( \{b_n\} \). (This is where the hypothesis \( \log b_n = o((\log n)^{1/2}) \) is used.) Since

\[
\mathcal{H}(n) - \mathcal{H}([n/K]) = \sum_{k=[n/K]+1}^n \mathbb{P}\{ S_k = 0 \} \sim \frac{\log K}{2\pi \sqrt{\det \Gamma}},
\]
we have

\[ (3.2) \quad \sum_{j=1}^{K} \mathbb{E}E_j - \mathbb{E}R_n = \frac{n(\mathcal{H}(n) - \mathcal{H}([n/K]))}{\mathcal{H}(n)} + o\left(\frac{n}{\mathcal{H}(n)}\right). \]

Hence for any \( \lambda > 0 \),

\[ \mathbb{P}\left\{ \mathcal{R}_n \geq \frac{n}{\mathcal{H}(n)}(\mathcal{H}(n) - \mathcal{H}([n/b_n])) \right\} \]

\[ \leq \exp \left\{ -\lambda b_n (\mathcal{H}(n) - \mathcal{H}([n/b_n])) \right\} \mathbb{E} \exp \left\{ \lambda \frac{\mathcal{H}(n)/b_n R_n}{n} \right\} \]

\[ \leq \exp \left\{ -\lambda b_n (\mathcal{H}([n/K]) - \mathcal{H}([n/b_n])) + o(b_n) \right\} \left( \mathbb{E} \exp \left\{ \lambda \frac{\mathcal{H}(n)/b_n E_1}{n} \right\} \right)^K. \]

Notice that

\[ \lim_{n \to \infty} \left( \mathcal{H}([n/K]) - \mathcal{H}([n/b_n]) \right) = \log \frac{2}{\sqrt{\det \Gamma}} \]

and that by \([21, \text{Theorem 6.1}]\),

\[ \frac{\mathcal{H}(n)/b_n E_1}{n} \xrightarrow{d} - \frac{2\pi t}{2\pi \sqrt{\det \Gamma}} \gamma_1, \]

where \( \gamma_t \) is the renormalized self-intersection local time of a planar Brownian motion. By Theorem \([24]\) and the dominated convergence theorem,

\[ \mathbb{E} \exp \left\{ \lambda \frac{\mathcal{H}(n)/b_n E_1}{n} \right\} \longrightarrow \mathbb{E} \exp \left\{ -\lambda \frac{2\pi t}{2\pi \sqrt{\det \Gamma}} \gamma_1 \right\} \]

Consequently,

\[ (3.3) \quad \lim_{n \to \infty} \sup b_n^{-1} \log \mathbb{P}\left\{ \mathcal{R}_n \geq \frac{n}{\mathcal{H}(n)}(\mathcal{H}(n) - \mathcal{H}([n/b_n])) \right\} \]

\[ \leq -\lambda \frac{\log t}{2\pi \sqrt{\det \Gamma}} + \frac{1}{t} \frac{\log \mathbb{E} \exp \left\{ -\lambda \frac{2\pi t}{2\pi \sqrt{\det \Gamma}} \gamma_1 \right\}}{\log \frac{2\pi \sqrt{\det \Gamma}}{\lambda}} \]

\[ = \frac{\lambda}{2\pi \sqrt{\det \Gamma}} \log \frac{\lambda t}{\left(2\pi \sqrt{\det \Gamma}\right)^{1/2}} \log \frac{\lambda t}{2\pi \sqrt{\det \Gamma}} - \lambda \frac{2\pi t}{2\pi \sqrt{\det \Gamma}} \gamma_1. \]
By [2], see the proof of Theorem 3.2, the limit
\[
C \equiv \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \exp \left\{ -t \log t - 2\pi t \gamma_1 \right\}
\]
exists. Set
\[L = \exp(-1 - C).\]
Letting \(t \to \infty\) in (3.3) gives
\[
\limsup_{n \to \infty} b^{-1}_n \log \mathbb{P} \left\{ \mathcal{R}_n \geq \frac{n}{H^2(n)} \left( \mathcal{H}(n) - \mathcal{H}([n/b_n]) \right) \right\} 
\leq \frac{\lambda}{2\pi \sqrt{\det \Gamma}} \log \frac{\lambda}{2\pi \sqrt{\det \Gamma}} + C \frac{\lambda}{2\pi \sqrt{\det \Gamma}}.
\]
Taking
\[\frac{\lambda}{2\pi \sqrt{\det \Gamma}} = \exp \left\{ -1 - C \right\}
\]
than yields
\[
\limsup_{n \to \infty} b^{-1}_n \log \mathbb{P} \left\{ \mathcal{R}_n \geq \frac{n}{H^2(n)} \left( \mathcal{H}(n) - \mathcal{H}([n/b_n]) \right) \right\} 
\leq - \exp \left\{ -1 - C \right\} = -L.
\]

We now prove the lower bound. The proof is similar to that of Proposition 4.4 of [3]. Fix \(n\) and let \(K = [b_n]\). Let \(M = [n/b_n]\). Let \(I_j\) be the interval \((m_j, m_{j+1}]\), where the \(m_j\) are integers such that \(m_0 = 0, m_K = n,\) and \(m_{j+1} - m_j\) is equal to either \(M\) or \(M + 1\).

Let \(e\) be a vector of length \(\sqrt{M}\) and let \(B(x, r)\) be the ball of radius \(r\) about \(x\). Set
\[E_j = \#\{S(I_j)\}, \quad H_j = \#\{S(I_j) \cap S(I_{j-1})\}.
\]
Let
\[
A_j = \{S_{m_{j+1}} \in B((j+1)e, \frac{1}{8}\sqrt{M}) \cap \{S(I_j) \subset B((j+\frac{1}{2})e, \sqrt{M})\}
\]
and
\[
B_j = \{E_j \log^2 M/M \geq -c_1\}
\]
where we will select \( c_1 \) in a moment. By the central limit theorem, we know \( \mathbb{P}^{S_{m-1}}(A_j) \geq c_2 \) on the event \( A_{j-1} \) if \( n \) is large. By [21, Theorem 6.1], \( \mathbb{P}^{S_{m-1}}(A_j \cap B_j) > c_2/2 \) on the event \( A_{j-1} \) if we take \( c_1 \) sufficiently large. If we let
\[
F = \bigcap_{j=0}^{K-1} (A_j \cap B_j),
\]
then by the Markov property applied \( K - 1 \) times we have
\[
\mathbb{P}(F) \geq (c_2/2)^{K-1}. \tag{3.7}
\]

On the set \( F \) we have that \( S(I_j) \) is disjoint from \( S(I_i) \) if \( |i - j| > 1 \), and so on \( F \)
\[
R_n = \sum_{j=1}^{K} E_j + \left( \sum_{j=1}^{K} EE_j - \mathbb{E}R_n \right) - \sum_{j=1}^{K} H_j. \tag{3.8}
\]

On the set \( F \) the event \( B_j \) holds for each \( j \), and so
\[
\sum_{j=1}^{K} E_j \geq -\frac{c_1 KM}{\log^2 M} \geq -\frac{c_3 n}{\log^2 n}. \tag{3.9}
\]

As in (3.2),
\[
\left( \sum_{j=1}^{K} EE_j \right) - \mathbb{E}R_n = \frac{n(\mathcal{H}(n) - \mathcal{H}([n/K]))}{\mathcal{H}(n)^2} + o\left( \frac{n}{\mathcal{H}(n)^2} \right) \tag{3.10}
\]
if \( n \) is large.

Let \( \Lambda > 0 \) be chosen in a moment. Let
\[
C_1 = \left\{ \sum_{\{j \text{ odd}\}} H_j \geq \frac{n\Lambda}{\log^2 n} \right\}, \quad C_2 = \left\{ \sum_{\{j \text{ even}\}} H_j \geq \frac{n\Lambda}{\log^2 n} \right\}.
\]
Set \( G = F \cap C_1 \cap C_2^c \). For \( j \) odd the \( H_j \) are independent, and by Lemma 4.6 of [6],
\[
\mathbb{P}(C_1) = \mathbb{P}\left( \sum_{\{j \text{ odd}\}} \frac{H_j}{M/\log^2 M} \geq c_4 K \Lambda \right)
\leq e^{-c_4 c_3 K \Lambda} \mathbb{E}e^{c_3 H_j \log^2 M/M}
\leq e^{-c_4 c_5 K \Lambda c_6 K},
\]
where \(c_4, c_5, c_6\) do not depend on \(\Lambda\) and without loss of generality we may assume \(c_6 > 1\). Choose \(\Lambda\) large so that \(e^{-c_4c_5\Lambda} \leq c_6^{-2}\). When \(n\) is large, \(K\) will be large, and then \(P(C_1) \leq P(F)/3\). We have a similar estimate for \(P(C_2)\), so
\[
P(G) \geq (c_2/2)^{K-1}/3.
\]
Set \(v_n = \mathcal{H}(n) - \mathcal{H}([n/b_n])\). On the event \(G\)
\[
(3.11) \sum_{j=1}^{K} H_j \leq 2 \frac{n\Lambda}{\log^2 n},
\]
and so combining (3.9), (3.10), and (3.11), on the event \(G\)
\[
(3.12) \mathcal{R}_n \geq \left(1 - \frac{c_7}{v_n}\right)nv_n/\mathcal{H}(n)^2.
\]
Therefore
\[
(3.13) \quad P\left(\mathcal{R}_n \geq \left(1 - \frac{c_7}{v_n}\right)nv_n/\mathcal{H}(n)^2\right) \geq c_8c_9^{b_n}.
\]
Define \(b_n'\) by \(v_n' = \mathcal{H}(n) - \mathcal{H}([n/b_n']) = v_n + c_7\). If we apply (3.13) with \(b_n\) replaced by \(b_n'\), we have
\[
P(\mathcal{R}_n \geq nv_n/\mathcal{H}(n)^2) \\
= P\left(\mathcal{R}_n \geq \left(1 - \frac{c_7}{v_n'}\right)nv_n'/\mathcal{H}(n)^2\right) \\
\geq c_8c_9^{b_n'}.
\]
We now take the logarithms of both sides, divide by \(b_n\), and use the fact that the ratio \(b_n/b_n'\) is bounded above and below by positive constants to obtain the lower bound. \(\square\)

**Proof of Corollary 1.3** Assume first that \(S_n\) is strongly aperiodic. We have by Proposition 2.1 that
\[
P(S_n = 0) = \frac{1}{2\pi n \sqrt{\det \Gamma}} + O\left(\frac{1}{n(\log n)^{1/2}}\right).
\]
Then, if \(\gamma\) denotes Euler’s constant
\[
(3.15) \sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + O\left(\frac{1}{n}\right)
\]
and

\[(3.16)\quad \sum_{k=3}^{n} \frac{1}{k^{(\log k)^{1/2}}} \leq \int_{2}^{n} \frac{dx}{x^{(\log x)^{1/2}}} \leq c_1 (\log n)^{1/2}\]

so that

\[(3.17)\quad H(n) = \sum_{k=0}^{n} p^0(S_k = 0) = 1 + \frac{1}{2\pi \sqrt{\det \mathbf{\Gamma}}} \sum_{k=1}^{n} \left( \frac{1}{k} + O\left( \frac{1}{k^{(\log k)^{1/2}}} \right) \right)
\]

\[= \frac{1}{2\pi \sqrt{\det \mathbf{\Gamma}}} \left( \log n + \gamma + O\left( (\log n)^{1/2} \right) \right)\]

\[= \frac{\log n}{2\pi \sqrt{\det \mathbf{\Gamma}}} \left( 1 + O\left( \frac{1}{(\log n)^{1/2}} \right) \right) .\]

Similarly we

\[(3.18)\quad \sum_{k=[n/b_n]}^{n} \frac{1}{k^{(\log k)^{1/2}}} \leq c_2 \left( (\log n)^{1/2} - (\log (n/b_n))^{1/2} \right) .\]

To evaluate this note that

\[(3.19)\quad (\log (n/b_n))^{1/2} = (\log n - \log b_n)^{1/2}
\]

\[= (\log n)^{1/2} (1 - \log b_n / \log n)^{1/2}
\]

\[= (\log n)^{1/2} (1 + O(\log b_n / \log n))
\]

\[= (\log n)^{1/2} + O(\log b_n / (\log n)^{1/2})\]

by our assumption that \(\log b_n = o((\log n)^{1/2})\). It follows that

\[(3.20)\quad H(n) - H([n/b_n]) = \frac{1}{2\pi \sqrt{\det \mathbf{\Gamma}}} \sum_{k=[n/b_n]+1}^{n} \left( \frac{1}{k} + O\left( \frac{1}{k^{(\log k)^{1/2}}} \right) \right)
\]

\[= \frac{1}{2\pi \sqrt{\det \mathbf{\Gamma}}} \left( \log b_n + O\left( \frac{\log b_n}{(\log n)^{1/2}} \right) \right)
\]

\[= \frac{\log b_n}{2\pi \sqrt{\det \mathbf{\Gamma}}} \left( 1 + O\left( \frac{1}{(\log n)^{1/2}} \right) \right) .\]
We then have that

\[(3.21) \quad \frac{n}{\mathcal{H}(n)^2}(\mathcal{H}(n) - \mathcal{H}([n/b_n])) = 2\pi \sqrt{\det \Gamma} \frac{n \log b_n}{(\log n)^2} \left(1 + O\left(\frac{1}{(\log n)^{1/2}}\right)\right) = 2\pi \sqrt{\det \Gamma} \frac{n \log b_n}{(\log n)^2} (1 + a_n),\]

where we use the last equality to define \(a_n\). Let

\[(3.22) \quad 1 + \hat{a}_n = (1 + a_n)^{-1} = 1 + O\left(\frac{1}{(\log n)^{1/2}}\right).\]

Then if we set

\[(3.23) \quad \hat{b}_n := b_n^{1+\hat{a}_n} = b_n^{(1+a_n)^{-1}}\]

we see from (3.21) that

\[(3.24) \quad \frac{n}{\mathcal{H}(n)^2}(\mathcal{H}(n) - \mathcal{H}([n/\hat{b}_n])) = 2\pi \sqrt{\det \Gamma} \frac{n \log b_n}{(\log n)^2}.\]

Also, \(\log \hat{b}_n = (1 + \hat{a}_n) \log b_n = o((\log n)^{1/2})\), so that Theorem 1.1 applies to \(\hat{b}_n\), and indeed to \(\hat{b}_n^\theta\) for any \(\theta > 0\).

Note that

\[(3.25) \quad \hat{b}_n^\theta = b_n^{\theta \left(1+O\left(\frac{1}{(\log n)^{1/2}}\right)\right)} = b_n^{\theta \exp \left(O\left(\frac{\log b_n}{(\log n)^{1/2}}\right)\right)} = b_n^{\theta (1 + o(1))}\]

by our assumption that \(\log b_n = o((\log n)^{1/2})\). Hence by (3.24) and (3.25)

\[(3.26) \quad \hat{b}_n^{-\theta} \log \mathbb{P}\left\{ \mathcal{R}_n \geq \frac{\theta n}{\mathcal{H}(n)^2}(\mathcal{H}(n) - \mathcal{H}([n/\hat{b}_n])) \right\} = (1 + o(1))b_n^{-\theta} \log \mathbb{P}\left\{ \mathcal{R}_n \geq 2\pi \theta \sqrt{\det \Gamma} \frac{n \log b_n}{(\log n)^2} \right\}\]
Together with Proposition 2.1, Theorem 1.1 applied to \( \hat{\theta}_n \) proves the corollary in the strongly aperiodic case. The modifications to handle the case where \( S_n \) is not strongly aperiodic are very similar to those in Section 2 of [23]. \( \square \)

4 Moderate deviations for \( \mathbb{E} R_n - R_n \)

To avoid difficulties connected with subdividing time intervals, it is more convenient to look at the continuous time analogue of \( S_n \). We let \( T_1, T_2, \ldots \) be i.i.d. exponential random variables with parameter 1 that are independent of the sequence \( S_n \). Define \( Z_t = S_n \) if \( \sum_{i=1}^{n} T_i \leq t < \sum_{i=1}^{n+1} T_i \). \( Z_t \) is a Lévy process that waits an exponential length of time, then jumps according to \( X_1 \), and then repeats the procedure. Define \( N_t = n \) if \( \sum_{i=1}^{n} T_i \leq t < \sum_{i=1}^{n+1} T_i \). Note that \( N_t \) is a Poisson process with \( \mathbb{E} N_t = t \) and that \( Z_t = S_{N_t} \). We write \(|Z[a,b]|\) for the cardinality of \( \{Z_s : s \in [a,b]\} \).

Theorems 2.2 and 2.3 have the following analogues for continuous time processes. We omit the proofs, which are almost identical to the proofs given for the discrete time random walks.

**Lemma 4.1** Let \( Z_1(t), \ldots, Z_p(t) \) be independent copies of \( Z(t) \). There is \( C > 0 \) such that

\[
\sup_{y_1, \ldots, y_p} \mathbb{E} \left| Z_1[0,t] \cap \cdots \cap Z_p[0,t] \right|^m \leq C^m (m!)^{p-1} \left( \frac{t}{(\log t)^p} \right)^m.
\]

Consequently, there is \( \theta > 0 \) such that

\[
\sup_t \sup_{y_1, \ldots, y_p} \mathbb{E} \left( \left( \frac{t}{(\log t)^{p-1}} \right)^{p-1} \right) < \infty.
\]

**Lemma 4.2** There is \( \theta > 0 \) such that

\[
\sup_t \exp \left\{ \theta \left( \frac{\log t}{t} \right)^2 \left| \mathbb{E}[Z[0,t]] - |Z[0,t]| \right| \right\} < \infty.
\]

Consequently

\[
\limsup_{t \to \infty} \frac{1}{bt} \log \mathbb{P} \left\{ \left| \mathbb{E}[Z[0,t]] - |Z[0,t]| \right| \geq \lambda \frac{tb}{(\log t)} \right\} \leq -\theta \lambda.
\]
We will prove Theorem 1.5 by first proving the following analogue for $Z_t$.

**Theorem 4.3** For any $\lambda > 0$ and for any $b_t$ satisfying $b_t \to \infty$ and $b_t = o \left( (\log t)^{1/5} \right)$ as $t \to \infty$, we have

\[
\lim_{t \to \infty} \frac{1}{b_t} \log \mathbb{P} \left\{ \left| \mathbb{E}|Z[0, t]| - |Z[0, t]| \right| \geq \lambda \frac{tb_t}{(\log t)^2} \right\} = -(2\pi)^{-2} \det(\Gamma)^{-1/2} \kappa(2, 2)^{-4}\lambda.
\]

The next proposition shows that Theorem 1.5 follows from Theorem 4.3 and Theorem 1.1.

**Proposition 4.4** For any $\varepsilon > 0$,

\[
\lim_{n \to \infty} \frac{1}{b_n} \log \mathbb{P} \left\{ \left| \mathbb{E}|S[0, n]| - |S[0, n]| \right| \geq \varepsilon \frac{nb_n}{(\log n)^2} \right\} = -\infty.
\]

**Remark 4.5** Our proof actually gives a stronger result, but this is all we need.

**Proof.** Observe that if $n > m$, then

\[
\left| \mathbb{E}|S[0, n]| - \mathbb{E}|S[0, m]| \right| \leq \mathbb{E}|S[m, n]| = \mathbb{E}|S[0, n - m]| \leq n - m.
\]

Consequently,

\[
\left| \mathbb{E}|Z[0, n]| - \mathbb{E}|Z[0, n]| \right| = \left| \mathbb{E}|S[0, N_n]| - \mathbb{E}|S[0, n]| \right| \\
\leq \mathbb{E}|N_n - n| \leq C\sqrt{n}.
\]

Hence, it suffices to show that for any $\varepsilon > 0$

\[
\lim_{n \to \infty} \frac{1}{b_n} \log \mathbb{P} \left\{ \left| |Z[0, n]| - |S[0, n]| \right| \geq \varepsilon \frac{\sqrt{nb_n}3/2}{\log n} \right\} = -\infty.
\]
Let $M > 0$ be fixed. On the event $\{|N_n - n| \leq M\sqrt{n}b_n\}$

\begin{equation}
|Z[0, n] - |S[0, n]|| \leq |S[N_n \land n, N_n \lor n]| \\
\quad \overset{d}{=} |S[0, N_n \lor n - N_n \land n]| \leq |S[0, 2M\sqrt{n}b_n]|.
\end{equation}

So we have

\[
P\left\{ |Z[0, n] - |S[0, n]|| \geq \varepsilon \frac{\sqrt{n}b_n^{3/2}}{\log n} \right\}
\leq P\left\{ |S[0, 2M\sqrt{n}b_n]| \geq \varepsilon \frac{\sqrt{n}b_n^{3/2}}{\log n} \right\} + P\{|N_n - n| \geq M\sqrt{n}b_n\}.
\]

By Lemma 3 in [8],

\[
\sup_n \mathbb{E} \exp \left\{ \theta \frac{\log n}{\sqrt{n}b_n} S[0, 2M\sqrt{n}b_n] \right\} < \infty, \quad \theta > 0.
\]

By the Chebyshev inequality one can see that

\[
\lim_{n \to \infty} \frac{1}{b_n} \log P\left\{ |S[0, 2M\sqrt{n}b_n]| \geq \varepsilon \frac{\sqrt{n}b_n^{3/2}}{\log n} \right\} = -\infty.
\]

By the classical moderate deviation principle ([13, Theorem 3.7.1]),

\[
\lim_{n \to \infty} \frac{1}{b_n} \log P\{|N_n - n| \geq M\sqrt{n}b_n\} = -\frac{M^2}{2}.
\]

Thus,

\[
\limsup_{n \to \infty} \frac{1}{b_n} \log P\left\{ |Z[0, n]| - |S[0, n]| \geq \varepsilon \frac{\sqrt{n}b_n^{3/2}}{\log n} \right\} \leq -\frac{M^2}{2}.
\]

Letting $M \to \infty$ proves the proposition.

Thus we need to prove Theorem 4.3. By the Gärtner-Ellis theorem ([13, Theorem 2.3.6]), to prove Theorem 4.3 it suffices to prove

\[
\lim_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} (\log t) \mathbb{E} |Z[0, t]| - |Z[0, t]|^{1/2} \right\} \\
= (\theta \pi)^2 \sqrt{\det(\Gamma)\kappa(2, 2)^4}.
\]
Let \( h(x) \) be a smooth symmetric probability density on \( \mathbb{R}^2 \) with compact support and write \( h_{\varepsilon}(x) = \varepsilon^{-2} h(\varepsilon^{-1} x) \). We have

\[
(4.17) \quad \Lambda_{\varepsilon}(t) \equiv \sum_{x \in \mathbb{Z}^2} h_{\varepsilon}(x) \approx t, \quad t \to \infty.
\]

The following lemma describing exponential asymptotics for the smoothed range will be proved in Section 5.

**Lemma 4.6** Let

\[
(4.18) \quad A_t(\varepsilon) \equiv \Lambda_{\varepsilon}(b_t t)^{-2} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}[0,t]} h_{\varepsilon}(\sqrt{b_t t}(x-y)) \right]^2.
\]

For any \( \theta > 0 \),

\[
(4.19) \quad \lim_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{b_t (\log t)} \left| A_t(\varepsilon) \right|^{1/2} \right\} = \sup_{g \in \mathcal{F}} \left\{ 2\pi \theta \sqrt{\det(\Gamma)} \left( \int_{\mathbb{R}^2} \| (g^2 * h_{\varepsilon})(x) \|^2 dx \right)^{1/2} \right.
\]

\[
- \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \Gamma \nabla g(x) \rangle^2 dx \}
\]

where

\[
\mathcal{F} = \{ g \in W^{1,2}(\mathbb{R}^2); \ |g|_2 = 1 \}. \]

Furthermore, for any \( N = 0, 1, \ldots \) and any \( \varepsilon > 0 \),

\[
(4.20) \quad \lim_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{b_t (\log t)} \right. \times \left( \Lambda_{\varepsilon}(b_t t)^{-2} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}[0,2^{-N} t]} h_{\varepsilon}(\sqrt{b_t t}(x-y)) \right]^2 \right)^{1/2} \}
\]

\[
\leq 2^{-N+2} \pi^2 \theta^2 \sqrt{\det(\Gamma)} \kappa(2,2)^4.
\]

The following lemma on exponential approximation will be proved in Section 6. In this lemma \( Z' \) denotes an independent copy of \( Z \).
Lemma 4.7  Let
\[ B_t^{(j)}(\varepsilon) \equiv \Lambda_{\varepsilon}\left(\frac{t}{b_t}\right)^{-2} \]
\[ \times \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}[0,2^{-j}t]} h_{\varepsilon}\left(\sqrt{\frac{b_t}{t}}(x-y)\right) \right] \left[ \sum_{y' \in \mathbb{Z}'[0,2^{-j}t]} h_{\varepsilon}\left(\sqrt{\frac{b_t}{t}}(x-y')\right) \right]. \]  
(4.21)

Then for any \( \theta > 0 \) and any \( j = 0, 1, \ldots, \)
\[ \limsup_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} (\log t) |Z[0,2^{-j}t] \cap Z'[0,2^{-j}t]| - B_t^{(j)}(\varepsilon) \right\}^{1/2} = 0. \]  
(4.22)

These lemmas will be the key to proving Theorem 4.3. Before proving this theorem, we present a simple lemma which will be used several times in the proof of Theorem 4.3.

Lemma 4.8  Let \( l \geq 2 \) be a fixed integer and let \( \{\xi_1(\rho); \rho > 0\}, \ldots, \{\xi_l(\rho); \rho > 0\} \) be \( l \) independent non-negative stochastic processes.

(a) If there is a constant \( C_1 > 0 \) such that for any \( 1 \leq j \leq l, \)
\[ \limsup_{\rho \to 0^+} \rho \log \mathbb{P}\{\xi_j(\rho) \geq \lambda\} \leq -C_1 \lambda, \quad \lambda > 0, \]  
(4.23)

then
\[ \limsup_{\rho \to 0^+} \rho \log \mathbb{P}\{\xi_1(\rho) + \cdots + \xi_l(\rho) \geq \lambda\} \leq -C_1 \lambda, \quad \lambda > 0. \]  
(4.24)

(b) If there is a constant \( C_2 > 0 \) such that for any \( 1 \leq j \leq l, \)
\[ \limsup_{\rho \to 0^+} \rho \log \mathbb{E}\exp\left\{\rho^{-1} \theta \sqrt{\xi_j(\rho)}\right\} \leq C_2 \theta^2, \quad \theta > 0, \]  
(4.25)

then
\[ \limsup_{\rho \to 0^+} \rho \log \mathbb{E}\exp\left\{\rho^{-1} \theta \sqrt{\xi_1(\rho) + \cdots + \xi_l(\rho)}\right\} \leq C_2 \theta^2, \quad \theta > 0. \]  
(4.26)
Proof. Clearly, part (a) needs only to be proved in the case \( l = 2 \). Given \( 0 < \delta < \lambda \), let \( 0 = a_0 < a_1 < \cdots < a_N = \lambda \) be a partition of \([0, \lambda]\) such that \( a_k - a_{k-1} < \delta \). Then

\[
\mathbb{P}\{\xi_1(\rho) + \xi_2(\rho) \geq \lambda\} \leq \sum_{k=1}^{N} \mathbb{P}\{\xi_1(\rho) \in [a_{k-1}, a_k]\}\mathbb{P}\{\xi_2(\rho) \geq \lambda - a_k\}
\]

(4.27)

\leq \sum_{k=1}^{N} \mathbb{P}\{\xi_1(\rho) \geq a_{k-1}\}\mathbb{P}\{\xi_2(\rho) \geq \lambda - a_k\}.

Hence

(4.28)

\[
\limsup_{\rho \to 0^+} \rho \log \mathbb{P}\{\xi_1(\rho) + \xi_2(\rho) \geq \lambda\} \\
\leq \max_{1 \leq k \leq N} \left\{- C_1 a_{k-1} - C_1 (\lambda - a_k)\right\} = -C_1 (\lambda - \delta).
\]

Letting \( \delta \to 0^+ \) proves part (a).

We now prove part (b). By Chebyshev’s inequality, for any \( \lambda > 0 \)

(4.29)

\[
\limsup_{\rho \to 0^+} \rho \log \mathbb{P}\{\xi_j(\rho) \geq \lambda\} \leq - \sup_{\theta > 0} \{\theta \sqrt{\lambda} - C_2 \theta^2\} = -\frac{\lambda}{4C_2}.
\]

By part (a)

(4.30)

\[
\limsup_{\rho \to 0^+} \rho \log \mathbb{P}\{\xi_1(\rho) + \cdots + \xi_l(\rho) \geq \lambda\} \leq -\frac{\lambda}{4C_2}, \quad \lambda > 0.
\]

In addition, by the triangle inequality and by independence,

(4.31)

\[
\mathbb{E}\exp\left\{\rho^{-1} \theta \sqrt{\xi_1(\rho)} + \cdots + \xi_l(\rho)\right\} \leq \prod_{j=1}^{l} \mathbb{E}\exp\left\{\rho^{-1} \theta \sqrt{\xi_j(\rho)}\right\}.
\]

So by assumption, for any \( \theta > 0 \),

(4.32)

\[
\limsup_{\rho \to 0^+} \rho \log \mathbb{E}\exp\left\{\rho^{-1} \theta \sqrt{\xi_1(\rho)} + \cdots + \xi_l(\rho)\right\} < \infty.
\]

By [13, Lemma 4.3.6],

(4.33)

\[
\limsup_{\rho \to 0^+} \rho \log \mathbb{E}\exp\left\{\rho^{-1} \theta \sqrt{\xi_1(\rho)} + \cdots + \xi_l(\rho)\right\} \\
\leq \sup_{\lambda > 0} \left\{\theta \sqrt{\lambda} - \frac{\lambda}{4C_2}\right\} = C_2 \theta^2.
\]

29
**Proof of Theorem 4.3** We begin with the decomposition

\[
|Z[0, t]| = \sum_{k=1}^{2^N} \left| Z\left[ \frac{k-1}{2^N} t, \frac{k}{2^N} t \right] \right| - \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \left| Z\left[ \frac{2k - 2}{2^j} t, \frac{2k - 1}{2^j} t \right] \cap Z\left[ \frac{2k - 1}{2^j} t, \frac{2k}{2^j} t \right] \right| \]

(4.34)

\[=: I_t - J_t.\]

We first establish the upper bound. Let \(\epsilon > 0\) be fixed. Since

\[E|Z[0, t]| - |Z[0, t]| = (E I_t - I_t) + J_t - E J_t \leq (E I_t - I_t) + J_t,\]

it follows that

\[
P\left\{ |E I_t - I_t| \geq \epsilon b_t \left( \log t \right)^2 \right\} + P\left\{ J_t \geq (\lambda - \epsilon) b_t \left( \log t \right)^2 \right\}\]

(4.36)

Notice that

\[
|E I_t - I_t| \leq \sum_{k=1}^{2^N} \left| E \left| Z\left[ \frac{k-1}{2^N} t, \frac{k}{2^N} t \right] \right| - \left| Z\left[ \frac{k-1}{2^N} t, \frac{k}{2^N} t \right] \right| \right|.

(4.37)

Replacing \(t\) by \(2^{-N} t\), \(\lambda\) by \(2^N \lambda\) and \(b_t\) by \(\tilde{b}_t =: b_{2^N t}\) in (4.4) we obtain

\[
\limsup_{t \to \infty} \frac{1}{b_t} \log P\left\{ \left| E Z[0, 2^{-N} t] \right| - \left| Z[0, 2^{-N} t] \right| \geq \frac{\lambda t b_t \left( \log t \right)^2}{(\log t)^2} \right\} \leq -2^N C\lambda.
\]

Hence by Lemma 4.8

\[
\limsup_{t \to \infty} \frac{1}{b_t} \log P\left\{ |E I_t - I_t| \geq \frac{\epsilon b_t \left( \log t \right)^2}{(\log t)^2} \right\} \leq -\epsilon C 2^N.
\]

(4.39)

By the triangle inequality,

\[
P\left\{ J_t \geq \frac{(\lambda - \epsilon) b_t \left( \log t \right)^2}{(\log t)^2} \right\} \leq \sum_{j=1}^{N} P\left\{ \sum_{k=1}^{2^{j-1}} \xi_{j,k} \geq 2^{-j} \frac{(\lambda - \epsilon) b_t \left( \log t \right)^2}{(\log t)^2} \right\}
\]

(4.40)
where for each $1 \leq j \leq N$,
\begin{equation}
\xi_{j,k}(t) = \left| Z\left[\frac{2k-2}{2j}t, \frac{2k-1}{2j}t\right] \cap Z\left[\frac{2k-1}{2j}t, \frac{2k}{2j}t\right]\right|, \quad k = 1, \ldots, 2^{j-1},
\end{equation}
forms an i.i.d. sequence with the same distribution as
\begin{equation}
|Z[0, 2^{-j}t] \cap Z'[0, 2^{-j}t]|.
\end{equation}
By Theorem 1 in [8] (with $2^{-j}t$ instead of $t$), for any $\lambda > 0$,
\begin{equation}
\lim_{t \to \infty} \frac{1}{b_t} \log \mathbb{P}\left\{ |Z[0, 2^{-j}t] \cap Z'[0, 2^{-j}t]| \geq \frac{\lambda t b_t}{(\log t)^2} \right\} = -2^j (2\pi)^{-2} \det(\Gamma)^{-1/2} \kappa(2, 2)^{-4} \lambda.
\end{equation}
Therefore, by Lemma 4.8,
\begin{equation}
\lim_{t \to \infty} \frac{1}{b_t} \log \mathbb{P}\left\{ \sum_{k=1}^{2^{j-1}} \xi_{j,k} \geq \frac{\lambda t b_t}{(\log t)^2} \right\} = -2^j (2\pi)^{-2} \det(\Gamma)^{-1/2} \kappa(2, 2)^{-4} \lambda.
\end{equation}
In particular,
\begin{equation}
\lim_{t \to \infty} \frac{1}{b_t} \log \mathbb{P}\left\{ \sum_{k=1}^{2^{j-1}} \xi_{j,k} \geq 2^{-j} \frac{(\lambda - \varepsilon) t b_t}{(\log t)^2} \right\} = -(2\pi)^{-2} \det(\Gamma)^{-1/2} \kappa(2, 2)^{-4} (\lambda - \varepsilon)
\end{equation}
and therefore by (4.40)
\begin{equation}
\lim_{t \to \infty} \frac{1}{b_t} \log \mathbb{P}\left\{ J_t \geq \frac{(\lambda - \varepsilon) t b_t}{(\log t)^2} \right\} = -(2\pi)^{-2} \det(\Gamma)^{-1/2} \kappa(2, 2)^{-4} (\lambda - \varepsilon).
\end{equation}
Combining (4.36), (4.39) and (4.46) and letting $\varepsilon \to 0$ we obtain
\begin{equation}
\limsup_{t \to \infty} \frac{1}{b_t} \log \mathbb{P}\left\{ \left| \mathbb{E}[Z[0, t]] - |Z[0, t]| \right| \geq \frac{\lambda t b_t}{(\log t)^2} \right\} \leq -(2\pi)^{-2} \det(\Gamma)^{-1/2} \kappa(2, 2)^{-4} \lambda.
\end{equation}
By Varadhan’s integral lemma [13, Section 4.3]

\[
(4.48) \quad \limsup_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{\frac{t}{b_t}} (\log t) \left| \mathbb{E} |Z[0, t]| - |Z[0, t]| \right|^{1/2} \right\}
\]

\[
\leq \sup_{\lambda > 0} \left\{ \theta \lambda^{1/2} - (2\pi)^{-2} \det(\Gamma)^{-1/2} \kappa(2, 2)^{-4} \lambda \right\}
\]

\[
= (\theta \pi)^2 \sqrt{\det(\Gamma)} \kappa(2, 2)^4.
\]

(The uniform exponential integrability is provided by Lemma 4.2.)

We now prove the lower bound. Using induction on \(N\), one can see that

\[
A_t(\varepsilon) =: \Lambda_{\varepsilon} \left( \frac{t}{b_t} \right)^{-2} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in Z[0, t]} h_{\varepsilon} \left( \sqrt{\frac{b_t}{t}} (x - y) \right) \right]^2
\]

\[
\leq \Lambda_{\varepsilon} \left( \frac{t}{b_t} \right)^{-2} \sum_{k=1}^{2N} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in Z \left[ \frac{k-1}{2N} t, \frac{k}{2N} t \right]} h_{\varepsilon} \left( \sqrt{\frac{b_t}{t}} (x - y) \right) \right]^2
\]

\[
+ 2\Lambda_{\varepsilon} \left( \frac{t}{b_t} \right)^{-2} \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in Z \left[ \frac{2k-1}{2^j} t, \frac{2k}{2^j} t \right]} h_{\varepsilon} \left( \sqrt{\frac{b_t}{t}} (x - y) \right) \right]
\]

\[
\times \left[ \sum_{y' \in Z \left[ \frac{2k-1}{2^j} t, \frac{2k}{2^j} t \right]} h_{\varepsilon} \left( \sqrt{\frac{b_t}{t}} (x - y') \right) \right]
\]

\[
=: I_t(\varepsilon) + 2J_t(\varepsilon).
\]

(4.49)

Therefore, with \(I_t, J_t\) given by (4.50)

\[
(4.50) \quad \mathbb{E} |Z[0, t]| - |Z[0, t]| = (\mathbb{E} I_t - I_t) + (\mathbb{E} J_t - J_t - J_t(\varepsilon)) - \mathbb{E} J_t
\]

\[
\geq (\mathbb{E} I_t - I_t) + J_t(\varepsilon) - |J_t - J_t(\varepsilon)| - \mathbb{E} J_t
\]

\[
\geq (\mathbb{E} I_t - I_t) - \frac{1}{2} I_t(\varepsilon) - |J_t - J_t(\varepsilon)| - \mathbb{E} J_t + \frac{1}{2} A_t(\varepsilon).
\]

We will see that the dominant contribution to the lower bound comes from \(A_t(\varepsilon)\). By the last display we see that

\[
(4.51) \quad \frac{1}{2} A_t(\varepsilon) \leq |\mathbb{E} |Z[0, t]| - |Z[0, t]|| + |\mathbb{E} I_t - I_t| + \frac{1}{2} I_t(\varepsilon) + |J_t - J_t(\varepsilon)| + \mathbb{E} J_t.
\]
and consequently
\[
\left| \frac{1}{2} A_t(\varepsilon) \right|^{1/2} \leq \left| \mathbb{E} [Z[0, t]] - |Z[0, t]| \right|^{1/2} + |\mathbb{E} I_t - I_t|^{1/2} \\
+ \left| \frac{1}{2} I_t(\varepsilon) \right|^{1/2} + |J_t - J_t(\varepsilon)|^{1/2} + |\mathbb{E} J_t|^{1/2}.
\]
(4.52)

Notice that it follows from (4.1) that
\[
\mathbb{E} J_t \leq C_N \frac{t}{(\log t)^2}.
\]
(4.53)

If \( \bar{p} \) is such that \( p^{-1} + \bar{p}^{-1} = 1 \), then by the generalized Hölder inequality with \( f = \theta \sqrt{\frac{b_t}{t}} \log t \) we have
\[
\left\| \exp \frac{f}{p} \left| \frac{1}{2} A_t(\varepsilon) \right|^{1/2} \right\|_1 \\
\leq e^{C_N \sqrt{\bar{p}}} \left\| \exp \frac{f}{p} \left| \mathbb{E} [Z[0, t]] - |Z[0, t]| \right|^{1/2} \right\|_p \cdot \left\| \exp \frac{f}{p} |\mathbb{E} I_t - I_t|^{1/2} \right\|_{3\bar{p}} \\
\cdot \left\| \exp \frac{f}{p} \left| \frac{1}{2} I_t(\varepsilon) \right|^{1/2} \right\|_{3\bar{p}} \cdot \left\| \exp \frac{f}{p} |J_t - J_t(\varepsilon)|^{1/2} \right\|_{3\bar{p}}.
\]
(4.54)

Taking the \( p \)-th power and noting that \( \bar{p}/p = 1/(p - 1) \), this can be rewritten as
\[
\mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} (\log t) \left| \mathbb{E} [Z[0, t]] - |Z[0, t]| \right|^{1/2} \right\} \\
\geq e^{-C_N \sqrt{\bar{p}}} \left[ \mathbb{E} \exp \left\{ \frac{3\theta}{p - 1} \sqrt{\frac{b_t}{t}} (\log t) |\mathbb{E} I_t - I_t|^{1/2} \right\} \right]^{-\frac{p-1}{p}} \\
\times \left[ \mathbb{E} \exp \left\{ \frac{3\theta}{p - 1} \sqrt{\frac{b_t}{t}} (\log t) I_t(\varepsilon)^{1/2} \right\} \right]^{-\frac{p-1}{p}} \\
\times \left[ \mathbb{E} \exp \left\{ \frac{3\theta}{p - 1} \sqrt{\frac{b_t}{t}} (\log t) |J_t - J_t(\varepsilon)|^{1/2} \right\} \right]^{-\frac{p-1}{p}} \\
\times \left[ \mathbb{E} \exp \left\{ \frac{\theta}{2p} \sqrt{\frac{b_t}{t}} (\log t) |A_t(\varepsilon)|^{1/2} \right\} \right]^p.
\]
(4.55)
By Lemma 4.6

\[
\lim_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \frac{\theta}{2p} \sqrt{\frac{b_t}{t}(\log t)|A_t(\varepsilon)|} \right\}
\]

\[
= \sup_{g \in \mathcal{F}} \left\{ \frac{\pi \theta}{p} \sqrt{\det(\Gamma)} \left( \int_{\mathbb{R}^2} |(g^2 * h_\varepsilon)(x)|^2 dx \right)^{1/2}
\]

\[
- \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \Gamma \nabla g(x) \rangle dx \right\}.
\]

This will give the main contribution to (4.55). We now bound the other factors in (4.55).

Using Lemma 4.8 together with (4.48) (with \(t\) replaced by \(2^{-N} t\), \(\theta\) by \(2^{-N/2} \theta\), and \(b_t\) by \(\tilde{b}_t = b_{2Nt}\)) we can prove that for any \(\theta > 0\),

\[
\limsup_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}(\log t)|\mathbb{E}I_t - I_t|^2} \right\} \leq 2^{-N} C \theta^2.
\]

Using (4.20) and Lemma 4.8 we see that

\[
\limsup_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}(\log t)|I_t(\varepsilon)|^2} \right\} \leq 2^{-N} C \theta^2,
\]

where \(C > 0\) does not depend on \(\varepsilon\). Notice that

\[
|J_t - J_t(\varepsilon)| \leq \sum_{j=1}^{2N} \sum_{k=1}^{2^{j-1}} |K_{j,k}(\varepsilon)|,
\]

where

\[
K_{j,k}(\varepsilon) = \left| Z \left[ 2k - 2, \frac{2k - 1}{2} t \right] \right| \cup \left| Z \left[ 2k - 1, \frac{2k - 1}{2} t \right] \right|
\]

\[
- \Lambda_\varepsilon \left( \frac{t}{b_t} \right)^{-2} \sum_{x \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z} \left[ \frac{2k-2}{2} t, \frac{2k-1}{2} t \right]} h_\varepsilon \left( \sqrt{\frac{b_t}{t}} (x - y) \right)
\]

\[
	imes \left| \sum_{y' \in \mathbb{Z} \left[ \frac{2k-1}{2} t, \frac{2k}{2} t \right]} h_\varepsilon \left( \sqrt{\frac{b_t}{t}} (x - y') \right) \right|.
\]
For each $1 \leq j \leq N$, $K_{j,1}(\varepsilon), \ldots, K_{j,2^{N-1}}(\varepsilon)$ forms an i.i.d sequence with the same distribution as $B_t^{(j)}(\varepsilon)$. It then follows from Lemma 4.7 and Hölder’s inequality that

$$
\limsup_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} (\log t) |J_t - J_t(\varepsilon)|^{1/2} \right\} = 0.
$$

Hence

$$
\liminf_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} (\log t) \mathbb{E} \left| Z[0, t] - |Z[0, t]| \right|^{1/2} \right\}
\geq -2^{-N+1} C' \frac{p - 1}{3} \left( \frac{3\theta}{p - 1} \right)^2
- \frac{p - 1}{3} \limsup_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \frac{3\theta}{p - 1} \sqrt{\frac{b_t}{t}} \log t |J_t - J_t(\varepsilon)|^{1/2} \right\}
+ p \sup_{g \in \mathcal{F}} \left\{ \frac{\pi \theta}{p} \sqrt{\text{det}(\Gamma)} \left( \int_{\mathbb{R}^2} |(g^2 \ast h_{\varepsilon})(x)|^2 dx \right)^{1/2} \right. \\
\left. - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \Gamma \nabla g(x) \rangle dx \right\}.
$$

Take limits on the right hand side in the following order: let $\varepsilon \to 0^+$, (using (4.61)), $N \to \infty$, and then $p \to 1^+$. We obtain

$$
\liminf_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} (\log t) \mathbb{E} \left| Z[0, t] - |Z[0, t]| \right|^{1/2} \right\}
\geq \sup_{g \in \mathcal{F}} \left\{ \pi \theta \sqrt{\text{det}(\Gamma)} \left( \int_{\mathbb{R}^2} |g(x)|^4 dx \right)^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \Gamma \nabla g(x) \rangle dx \right\}
= (\pi \theta)^2 \sqrt{\text{det}(\Gamma)} \sup_{f \in \mathcal{F}} \left\{ \left( \int_{\mathbb{R}^2} |f(x)|^4 dx \right)^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla f(x)|^2 dx \right\}
= (\pi \theta)^2 \sqrt{\text{det}(\Gamma)} \kappa(2, 2)^4,
$$

where the second step follows from the substitution $g(x) = \sqrt{\text{det}(A)} f(Ax)$ with the $2 \times 2$ matrix $A$ satisfying

$$
A^T \Gamma A = (\pi \theta)^2 \sqrt{\text{det}(\Gamma)} I_{2 \times 2}
$$
\((I_{2 \times 2} \text{ is the } 2 \times 2 \text{ identity matrix})\), and where the last step follows from Lemma A.2 in [7]. \(\Box\)

5 Exponential asymptotics for the smoothed range

In order to prove Lemma 4.6 we first obtain a weak convergence result.

Let \(\beta > 0\) and write

\[
A_{t,\beta}(\varepsilon) = \Lambda_{\varepsilon}(t)^{-2} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}[0,\beta t]} h_{\varepsilon}\left(\frac{x-y}{\sqrt{t}}\right) \right]^2
\]

and

\[
B_{t,\beta}(\varepsilon) = \Lambda_{\varepsilon}(t)^{-2} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}[0,\beta t]} h_{\varepsilon}\left(\frac{x-y}{\sqrt{t}}\right) \right] \left[ \sum_{y' \in \mathbb{Z}'[0,\beta t]} h_{\varepsilon}\left(\frac{x-y'}{\sqrt{t}}\right) \right].
\]

Let \(W(t), \tilde{W}(t)\) be independent planar Brownian motions, each with covariance matrix \(\Gamma\) and write

\[
\alpha_{\varepsilon}([0,\beta]^2) = \int_0^t \int_0^t (h_{\varepsilon} \ast h_{\varepsilon})(W(s) - W'(r)) dr ds
\]

and

\[
\alpha([0,\beta]^2) = \lim_{\varepsilon \to 0} \alpha_{\varepsilon}([0,\beta]^2).
\]

Lemma 5.1

\[
\frac{(\log t)^2}{t} \left[ |Z[0,\beta t] \cap \mathbb{Z}'[0,\beta t]| - B_{t,\beta}(\varepsilon) \right] \xrightarrow{\text{d}} (2\pi)^2 \det(\Gamma) \left[ \alpha([0,\beta]^2) - \alpha_{\varepsilon}([0,\beta]^2) \right]
\]

and

\[
\frac{(\log t)^2}{t} A_{t,\beta}(\varepsilon) \xrightarrow{\text{d}} (2\pi)^2 \det(\Gamma) \int_{\mathbb{R}^2} \left( \int_0^\beta h_{\varepsilon}(W(s) - x) ds \right)^2 dx.
\]

as \(t \to \infty\).
Proof. To prove (5.5), we consider the following result given on p.697 of [23]: if \( Z(t) =: \frac{Z(ts)}{\sqrt{t}} \) then

\[
\left( Z(t) \cdot (Z'(t) \cdot \frac{(\log t)^2}{t} |Z[0, \beta t] \cap Z'[0, \beta t]| \right)
\]

in the Skorohod topology as \( t \to \infty \). Actually, the proof in [23] is for the discrete time random walk, but a similar proof works for \( Z \).

Let \( M > 0 \) be fixed for a moment. Notice that

\[
\psi_t(x) \equiv \Lambda_\varepsilon(t) \cdot \frac{1}{h_\varepsilon(x)} \exp \left\{ i \lambda \cdot \frac{x}{\sqrt{t}} \right\} \to \tilde{h}_\varepsilon(\lambda)
\]

uniformly on \([-M, M]^2\) as \( t \to \infty \). Consequently the family

\[
\psi_t(x, y) = \int_{[-M,M]^2} \left| \frac{\lambda}{\sqrt{t}} \right|^2 \left[ \int_0^\beta e^{i \lambda \cdot x(s)} ds \right] \left[ \int_0^\beta e^{-i \lambda \cdot y(s')} ds' \right] d\lambda
\]

are convergent continuous functionals on \( D([0, \beta], \mathbb{R}^2) \otimes D([0, \beta], \mathbb{R}^2) \). There-
fore
\[ \left( \frac{1}{t^2} \int_{[-M,M]^2} |\hat{p}_{t,\epsilon}(\frac{\lambda}{\sqrt{t}})|^2 \right)^2 \left[ \int_0^{t^\beta} \exp \left\{ i\lambda \cdot \frac{Z(s)}{\sqrt{t}} \right\} ds \right] \left[ \int_0^{t^\beta} \exp \left\{ -i\lambda \cdot \frac{Z'(s')}{\sqrt{t}} \right\} ds' \right] d\lambda, \]
\[ \frac{(\log t)^2}{t} \left| Z[0, \beta t] \cap Z'[0, \beta t] \right| \]
\[ = \left( \int_{[-M,M]^2} |\hat{p}_{t,\epsilon}(\frac{\lambda}{\sqrt{t}})|^2 \right)^2 \left[ \int_0^{t^\beta} \exp \left\{ i\lambda \cdot Z^{(t)}(s) \right\} ds \right] \left[ \int_0^{t^\beta} \exp \left\{ -i\lambda \cdot (Z')^{(t)}(s') \right\} ds' \right] d\lambda, \]
\[ \frac{(\log t)^2}{t} \left| Z[0, \beta t] \cap Z'[0, \beta t] \right| \]
\[ \rightarrow \left( \int_{[-M,M]^2} \left| \hat{h}_{\epsilon}(\lambda) \right|^2 \left[ \int_0^{t^\beta} e^{i\lambda \cdot W(s)} ds \right] \left[ \int_0^{t^\beta} e^{-i\lambda \cdot W'(s')} ds' \right] d\lambda, \]
\[ (2\pi)^2 \det(\Gamma) \alpha([0, \beta]^2) \).

(5.11)

Recall that by Lemma 3 in [8],
\[ \sup_t \mathbb{E} \exp \left\{ \theta \frac{\log t}{t} |Z[0, t]| \right\} < \infty \]
for all \( \theta > 0 \). We will show that uniformly in \( \lambda \in [-M, M]^2 \)
\[ \lim_{t \to \infty} \frac{1}{t^2} \mathbb{E} \left| \int_0^{t^\beta} \exp \left\{ i\lambda \cdot \frac{Z(s)}{\sqrt{t}} \right\} ds \right| \]
\[ - \frac{\log t}{2\pi \sqrt{\det(\Gamma)}} \sum_{x \in Z[0, \beta t]} \exp \left\{ i\lambda \cdot \frac{x}{\sqrt{t}} \right\} = 0. \]
(5.13)

Using the inequality
\[ |AA' - BB'| \leq |A(B - B')| + |(A - B)B'|, \]
the Cauchy-Schwarz inequality and (5.12), we see from (5.13) that uniformly
in \( \lambda \in [-M, M]^2 \)

\[
\lim_{t \to \infty} \frac{1}{t^2} E \left[ \left[ \int_0^{\beta t} \exp \left\{ i \lambda \cdot \frac{Z(s)}{\sqrt{t}} \right\} ds \right] \left[ \int_0^{\beta t} \exp \left\{ -i \lambda \cdot \frac{Z'(s')}{\sqrt{t}} \right\} ds' \right] \right]
\]

\( (5.14) \)

\[= 0. \]

Together with (5.11) this shows that

\[
\left( \frac{\log t}{2 \pi \sqrt{\det(\Gamma)}} \right)^2 \left[ \sum_{x \in Z[0, \beta t]} \exp \left\{ i \lambda \cdot \frac{x}{\sqrt{t}} \right\} \right] \left[ \sum_{x' \in Z'[0, \beta t]} \exp \left\{ -i \lambda \cdot \frac{x'}{\sqrt{t}} \right\} \right] d\lambda,
\]

\[
\left( \frac{\log t}{2 \pi \sqrt{\det(\Gamma)}} \right)^2 \left[ \sum_{x \in Z[0, \beta t]} \exp \left\{ i \lambda \cdot \frac{x}{\sqrt{t}} \right\} \right] \left[ \sum_{x' \in Z'[0, \beta t]} \exp \left\{ -i \lambda \cdot \frac{x'}{\sqrt{t}} \right\} \right] d\lambda,
\]

\( (5.15) \)

Notice by (5.9) that for any \( \delta > 0 \), one can take \( M > 0 \) sufficiently large so that

\[
|\tilde{p}_{t, \varepsilon} \left( \frac{\lambda}{\sqrt{t}} \right)| < \delta, \quad \lambda \in [-\sqrt{t} \pi, \sqrt{t} \pi]^2 \setminus [M, M]^2,
\]

if \( t \) is sufficiently large. Consequently

\[
H_t =: \left[ \int_{\sqrt{t} \pi, \sqrt{t} \pi}^{2 \pi} \left| \tilde{p}_{t, \varepsilon} \left( \frac{\lambda}{\sqrt{t}} \right) \right|^2 \left[ \sum_{x \in Z[0, \beta t]} \exp \left\{ i \lambda \cdot \frac{x}{\sqrt{t}} \right\} \right] \right] \left[ \sum_{x' \in Z'[0, \beta t]} \exp \left\{ -i \lambda \cdot \frac{x'}{\sqrt{t}} \right\} \right] d\lambda
\]

\( (5.17) \)

\[\leq (2\pi)^2 \delta t |Z[0, \beta t] \cap Z'[0, \beta t]|.\]
It follows from (4.1) that \( \left( \frac{\log t}{2\pi t} \right)^2 H_t \to 0 \) in \( L^1 \) uniformly in large \( t \) as \( M \to \infty \). Therefore, using (5.15) and the fact that \( \hat{h} \in L^2 \), we obtain

\[
\left( \frac{\log t}{2\pi t} \right)^2 \int_{[\sqrt{\pi\pi}, \sqrt{\pi\pi}]} \left| \hat{p}_{t,\varepsilon} \left( \frac{\lambda}{\sqrt{t}} \right) \right|^2 \left[ \sum_{x \in Z[0,\beta t]} \exp \left\{ i\lambda \cdot \frac{x}{\sqrt{t}} \right\} \right] \times \left[ \sum_{x' \in Z'[0,\beta t]} \exp \left\{ -i\lambda \cdot \frac{x'}{\sqrt{t}} \right\} \right] d\lambda, \quad \frac{(\log t)^2}{t} \left| \mathbb{Z}[0,\beta t] \cap \mathbb{Z'}[0,\beta t] \right|
\]

\[
\xrightarrow{d} \left( \det(\Gamma) \int_{\mathbb{R}^2} |\hat{h}_\varepsilon(\lambda)|^2 \left[ \int_0^\beta e^{i\lambda W(s)} ds \right] \left[ \int_0^\beta e^{-i\lambda W'(s')} ds' \right] d\lambda, \right)
\]

(5.18) \( (2\pi)^2 \det(\Gamma) \alpha([0, \beta]^2) \).

Note that

\[
B_{t,\beta}(\varepsilon) = \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in Z[0,\beta t]} \Lambda_\varepsilon(t)^{-1} h_\varepsilon \left( \frac{x - y}{\sqrt{t}} \right) \right] \left[ \sum_{y' \in Z'[0,\beta t]} \Lambda_\varepsilon(t)^{-1} h_\varepsilon \left( \frac{x - y'}{\sqrt{t}} \right) \right]
\]

(5.19)

It then follows from Parseval’s identity that

(5.20) \( (2\pi)^2 t B_{t,\beta}(\varepsilon) \).

\[
= t \int_{[-\pi,\pi]^2} \left| \hat{p}_{t,\varepsilon}(\lambda) \right|^2 \left[ \sum_{y \in Z[0,\beta t]} e^{i\lambda y} \right] \left[ \sum_{y' \in Z'[0,\beta t]} e^{-i\lambda y'} \right] d\lambda
\]

\[
= \int_{[-\sqrt{\pi\pi}, \sqrt{\pi\pi}]^2} \left| \hat{p}_{t,\varepsilon} \left( \frac{\lambda}{\sqrt{t}} \right) \right|^2 \left[ \sum_{y \in Z[0,\beta t]} \exp \left\{ i\lambda \cdot \frac{y}{\sqrt{t}} \right\} \right] \left[ \sum_{y' \in Z'[0,\beta t]} \exp \left\{ -i\lambda \cdot \frac{y'}{\sqrt{t}} \right\} \right] d\lambda.
\]

Similarly, using the fact that \( h_\varepsilon \) is symmetric so that \( \hat{h}_\varepsilon(\lambda) \) is real

(5.21) \( \int_{\mathbb{R}^2} |\hat{h}_\varepsilon(\lambda)|^2 \left[ \int_0^\beta e^{i\lambda W(s)} ds \right] \left[ \int_0^\beta e^{-i\lambda W'(s')} ds' \right] d\lambda = \alpha_\varepsilon([0, \beta]^2) \).
Thus, we have proved
\[
\left( \frac{\log t}{t} \right)^2 B_{t, \beta}(\varepsilon), \quad \left( \frac{\log t}{t} \right)^2 \left| Z[0, \beta t] \cap Z'[0, \beta t] \right| \xrightarrow{d} \left( 2\pi \right)^2 \det(\Gamma) \alpha_\varepsilon(0, \beta \varepsilon), \quad \left( 2\pi \right)^2 \det(\Gamma) \alpha((0, \beta \varepsilon)^2). \tag{5.22}
\]

(5.23) follows from this.

Thus to complete the proof of (5.5) it only remains to show (5.13) uniformly in \( \lambda \in [-M, M]^2 \). We will show that for any \( \delta > 0 \) we can find \( \delta' > 0 \) and \( t_0 < \infty \) such that
\[
\left| \frac{1}{t^2} \mathbb{E} \int_0^{\beta t} \exp \left\{ i \lambda \cdot \frac{Z(s)}{\sqrt{t}} \right\} ds - \int_0^{\beta t} \exp \left\{ i \gamma \cdot \frac{Z(s)}{\sqrt{t}} \right\} ds \right|^2 < \delta \tag{5.23}
\]
and
\[
\left| \frac{1}{t^2} \mathbb{E} \sum_{x \in Z[0, \beta t]} \exp \left\{ i \lambda \cdot \frac{x}{\sqrt{t}} \right\} - \sum_{x \in Z[0, \beta t]} \exp \left\{ i \gamma \cdot \frac{x}{\sqrt{t}} \right\} \right|^2 < \delta \tag{5.24}
\]
for all \( t \geq t_0 \) and \( |\lambda - \gamma| \leq \delta' \). We then cover \([-M, M]^2\) by a finite number of discs \( B(\lambda_k, \delta') \) of radius \( \delta' \) centered at \( \lambda_k, k = 1, \ldots, N \). Define \( \tau(\lambda) = \lambda_k \) where \( k \) is the smallest integer with \( \lambda \in B(\lambda_k, \delta') \). By \([3] (4.11)\), we can choose \( t_1 < \infty \) such that for all \( t \geq t_1 \) and \( k = 1, \ldots, N \),
\[
\left| \frac{1}{t^2} \mathbb{E} \int_0^{\beta t} \exp \left\{ i \lambda_k \cdot \frac{Z(s)}{\sqrt{t}} \right\} ds - \frac{\log t}{2\pi \sqrt{\det(\Gamma)}} \sum_{x \in Z[0, \beta t]} \exp \left\{ i \lambda_k \cdot \frac{x}{\sqrt{t}} \right\} \right|^2 \leq \delta. \tag{5.25}
\]
Hence, uniformly in \( \lambda \in [-M, M]^2 \) we have that for all \( t \geq t_0 \lor t_1 \)
\[
\left| \frac{1}{t^2} \mathbb{E} \int_0^{\beta t} \exp \left\{ i \lambda \cdot \frac{Z(s)}{\sqrt{t}} \right\} ds - \frac{\log t}{2\pi \sqrt{\det(\Gamma)}} \sum_{x \in Z[0, \beta t]} \exp \left\{ i \lambda \cdot \frac{x}{\sqrt{t}} \right\} \right|^2 \leq 3\delta \tag{5.26}
\]
proving that (5.13) holds uniformly in \( \lambda \in [-M, M]^2 \).
(5.27) \[
\frac{1}{t^2} \mathbb{E} \left| \int_0^{\beta t} \exp \left\{ i\lambda \cdot \frac{Z(s)}{\sqrt{t}} \right\} ds - \int_0^{\beta t} \exp \left\{ i\gamma \cdot \frac{Z(s)}{\sqrt{t}} \right\} ds \right|^2 \\
\leq \frac{1}{t^2} \mathbb{E} \left| \int_0^{\beta t} |\lambda - \gamma| \frac{|Z(s)|}{\sqrt{t}} ds \right|^2 \\
= \frac{|\lambda - \gamma|^2}{t^3} \mathbb{E} \int_0^{\beta t} \int_0^{\beta t} |Z(s)||Z(r)| ds dr \\
\leq C\frac{|\lambda - \gamma|^2}{t^3} \int_0^{\beta t} \int_0^{\beta t} s^{1/2} r^{1/2} ds dr \leq C'|\lambda - \gamma|^2.
\]

As for (5.24),
\[
(5.28) \mathbb{E} \left| \sum_{x \in Z[0, \beta t]} \exp \left\{ i\lambda \cdot \frac{x}{\sqrt{t}} \right\} - \sum_{x \in Z[0, \beta t]} \exp \left\{ i\gamma \cdot \frac{x}{\sqrt{t}} \right\} \right|^2 \\
\leq 4\mathbb{E} \left\{ |Z[0, \beta t]|^2 \mathbb{1}_{\{\sup_{s \leq \beta t} |Z(s)| \geq C\sqrt{t}\}} \right\} \\
+ |\lambda - \gamma|^2 \mathbb{E} \left\{ \sum_{x \in Z[0, \beta t]} \frac{|x|^2}{\sqrt{t}} \mathbb{1}_{\{\sup_{s \leq \beta t} |Z(s)| \leq C\sqrt{t}\}} \right\} \\
\leq 4\mathbb{E} \left\{ |Z[0, \beta t]|^2 \mathbb{1}_{\{\sup_{s \leq \beta t} |Z(s)| \geq C\sqrt{t}\}} \right\} + C^2|\lambda - \gamma|^2 \mathbb{E}|Z[0, \beta t]|^2
\]
and by (5.12)
\[
(5.29) 4\mathbb{E} \left\{ |Z[0, \beta t]|^2 \mathbb{1}_{\{\sup_{s \leq \beta t} |Z(s)| \geq C\sqrt{t}\}} \right\} + C^2|\lambda - \gamma|^2 \mathbb{E}|Z[0, \beta t]|^2 \\
\leq 4 \left\{ \mathbb{E}(|Z[0, \beta t]|^4) P(\sup_{s \leq \beta t} |Z(s)| \geq C\sqrt{t}) \right\}^{1/2} + C^2|\lambda - \gamma|^2 \mathbb{E}|Z[0, \beta t]|^2 \\
\leq \left( \frac{c\lambda}{\log t} \right)^2 \left( 4 \left\{ P(\sup_{s \leq \beta t} |Z(s)| \geq C\sqrt{t}) \right\}^{1/2} + C^2|\lambda - \gamma|^2 \right).
\]

Taking \( C \) large and then choosing \( \delta' > 0 \) sufficiently small completes the proof of (5.24) and hence of (5.5).
We now prove (5.6). Using the facts that $\Lambda_\varepsilon(t) \sim t$, that
\[
\frac{1}{t} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in [0, \beta t]} h_\varepsilon \left( x - \frac{y}{\sqrt{t}} \right) \right]^2 - \int_{\mathbb{R}^2} \left[ \sum_{y \in [0, \beta t]} h_\varepsilon \left( x - \frac{y}{\sqrt{t}} \right) \right]^2 \, dx
= o(1) \lvert [0, \beta t] \rvert^2,
\]
(where the boundedness and continuity of $h_\varepsilon$ is used), and (5.12) we need only show that
\[
(\log t)^2 \int_{\mathbb{R}^2} \left[ \sum_{y \in [0, \beta t]} h_\varepsilon \left( x - \frac{y}{\sqrt{t}} \right) \right]^2 \, dx
\overset{d}{\longrightarrow} (2\pi)^2 \det(\Gamma) \int_{\mathbb{R}^2} \left( \int_{0}^{\beta} h_\varepsilon(W(s) - x) \, ds \right)^2 \, dx.
\]
By the Parseval identity,
\[
\int_{\mathbb{R}^2} \left[ \sum_{y \in [0, \beta t]} \exp \{ i \lambda \cdot y \} \right]^2 \, dx
= (2\pi)^{-2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} e^{i \lambda x} \sum_{y \in [0, \beta t]} h_\varepsilon \left( x - \frac{y}{\sqrt{t}} \right) \, dx \right)^2 \, d\lambda
= (2\pi)^{-2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} h_\varepsilon(x) e^{i \lambda x} \, dx \right)^2 \left( \sum_{y \in [0, \beta t]} \exp \left\{ i \lambda \cdot \frac{y}{\sqrt{t}} \right\} \right)^2 \, d\lambda
= \int_{\mathbb{R}^2} |\tilde{h}_\varepsilon(\lambda)|^2 \sum_{y \in [0, \beta t]} \exp \left\{ i \lambda \cdot \frac{y}{\sqrt{t}} \right\}^2 \, d\lambda.
\]
Let $M > 0$ be fixed and $\lambda_1, \ldots, \lambda_N$ and $\tau$ be defined as above. By [S] Theorem 7, 
\[
\frac{\log t}{t} \left( \sum_{y \in [0, \beta t]} \exp \left\{ i \lambda_1 \cdot \frac{y}{\sqrt{t}} \right\}, \ldots, \sum_{y \in [0, \beta t]} \exp \left\{ i \lambda_N \cdot \frac{y}{\sqrt{t}} \right\} \right)
\overset{d}{\longrightarrow} (2\pi)^\sqrt{\det(\Gamma)} \left( \int_{0}^{\beta} e^{i \lambda_1 \cdot W(s)} \, ds, \ldots, \int_{0}^{\beta} e^{i \lambda_N \cdot W(s)} \, ds \right).
\]
In particular,

\[
(5.34) \quad \left( \frac{\log t}{t} \right)^2 \int_{[-M,M]^2} |\hat{h}_\varepsilon(\lambda)|^2 \left| \sum_{y \in \mathbb{Z}[0,\beta t]} \exp \left\{ i\tau(\lambda) : \frac{y}{\sqrt{t}} \right\} \right|^2 d\lambda
\]

\[
= \sum_{k=1}^N \int_{B_k} |\hat{h}_\varepsilon(\lambda)|^2 \left| \frac{\log t}{t} \sum_{y \in \mathbb{Z}[0,\beta t]} \exp \left\{ i\lambda_k : \frac{y}{\sqrt{t}} \right\} \right|^2 d\lambda
\]

\[
\overset{\rightarrow}{\rightarrow} (2\pi)^2 \det(\Gamma) \sum_{k=1}^N \int_{B_k} |\hat{h}_\varepsilon(\lambda)|^2 \left| \int_0^\beta e^{i\lambda_k W(s)} ds \right|^2 d\lambda
\]

\[
= (2\pi)^2 \det(\Gamma) \int_{[-M,M]^2} |\hat{h}_\varepsilon(\lambda)|^2 \left| \int_0^\beta e^{i\tau(\lambda) W(s)} ds \right|^2 d\lambda.
\]

Notice that the right hand side of (5.34) converges to

\[
(5.35) \quad (2\pi)^2 \det(\Gamma) \int_{[-M,M]^2} |\hat{h}_\varepsilon(\lambda)|^2 \left| \int_0^\beta e^{i\lambda W(s)} ds \right|^2 d\lambda
\]

as \(N \to \infty\). Applying (5.24) to the left hand side of (5.34) gives

\[
(5.36) \quad \left( \frac{\log t}{t} \right)^2 \int_{[-M,M]^2} |\hat{h}_\varepsilon(\lambda)|^2 \left| \sum_{y \in \mathbb{Z}[0,\beta t]} \exp \left\{ i\lambda : \frac{y}{\sqrt{t}} \right\} \right|^2 d\lambda
\]

\[
\overset{\rightarrow}{\rightarrow} (2\pi)^2 \det(\Gamma) \int_{[-M,M]^2} |\hat{h}_\varepsilon(\lambda)|^2 \left| \int_0^\beta e^{i\lambda W(s)} ds \right|^2 d\lambda.
\]

As \(M \to \infty\), the right hand side of (5.36) converges to

\[
(5.37) \quad (2\pi)^2 \det(\Gamma) \int_{\mathbb{R}^2} |\hat{h}_\varepsilon(\lambda)|^2 \left| \int_0^\beta e^{i\lambda W(s)} ds \right|^2 d\lambda
\]

\[
= \det(\Gamma) \int_{\mathbb{R}^2} \left( \int_0^\beta h_\varepsilon(W(s) - x) ds \right)^2 dx
\]

by Parseval’s identity. Note

\[
(5.38) \quad H'_{t,M} =: \int_{\mathbb{R}^2 \setminus [-M,M]^2} |\hat{h}_\varepsilon(\lambda)|^2 \left| \sum_{y \in \mathbb{Z}[0,\beta t]} \exp \left\{ i\lambda : \frac{y}{\sqrt{t}} \right\} \right|^2 d\lambda
\]

\[
\leq |Z[0,\beta t]|^2 \int_{\mathbb{R}^2 \setminus [-M,M]^2} |\hat{h}_\varepsilon(\lambda)|^2 d\lambda.
\]
It follows from (5.12) and the fact that \( \hat{h}_\varepsilon \in L^2 \) that 
\[
\left( \log \frac{t}{2\pi} \right)^2 H'_{t,M} \to 0 \text{ in } L^1 \text{ as } M \to \infty \text{ uniformly in } t. 
\]
Therefore, using the last three displays, we obtain
\[
(5.39) \quad \left( \log \frac{t}{2\pi} \right)^2 \int_{\mathbb{R}^2} |\hat{h}_\varepsilon(\lambda)|^2 \left| \sum_{y \in \mathbb{Z}[0,\beta]} \exp \left\{ i\lambda \cdot \frac{y}{t} \right\} \right|^2 d\lambda 
\overset{d}{\to} \det(\Gamma) \int_{\mathbb{R}^2} \left( \int_0^\beta h_\varepsilon(W(s) - x) ds \right)^2 dx.
\]

**Proof of Lemma 4.6** Let \( T > 0 \) be fixed for the moment. Write \( \gamma_t = t/[T^{-1}b_t] \). We have
\[
\mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} (\log t) \left( \Lambda_\varepsilon \left( \frac{t}{b_t} \right)^{-2} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}[0,\gamma_t]} h_\varepsilon \left( \sqrt{\frac{b_t}{t}}(x - y) \right) \right]^2 \right) \right\}^{1/2} 
\leq \left[ \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} (\log t) \right. \times \left( \Lambda_\varepsilon \left( \frac{t}{b_t} \right)^{-2} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}[0,\gamma_t]} h_\varepsilon \left( \sqrt{\frac{b_t}{t}}(x - y) \right) \right]^2 \right) \right]^{1/2} \right]^{[T^{-1}b_t]}.
\]

We obtain from Lemma 5.1 (with \( t \) being replaced by \( t/b_t \) and \( \beta = T \))
\[
\frac{b_t}{t} (\log t)^2 \Lambda_\varepsilon \left( \frac{t}{b_t} \right)^{-2} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}[0,\gamma_t]} h_\varepsilon \left( \sqrt{\frac{b_t}{t}}(x - y) \right) \right]^2
\overset{d}{\to} (2\pi)^2 \det(\Gamma) \int_{\mathbb{R}^2} \left( \int_0^T h_\varepsilon(W(s) - x) ds \right)^2 dx, \quad t \to \infty.
\]

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In addition,

\[
\frac{b_t}{t} (\log t)^2 \Lambda_\varepsilon \left( \frac{t}{b_t} \right)^{-2} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}[0, \gamma t]} h_\varepsilon \left( \sqrt{\frac{b_t}{t}} (x - y) \right) \right]^2 \\
\leq \frac{b_t}{t} (\log t)^2 \Lambda_\varepsilon \left( \frac{t}{b_t} \right)^{-2} \|h\|_{\infty} |\mathbb{Z}[0, \gamma t]| \sum_{x \in \mathbb{Z}^2} h_\varepsilon \left( \sqrt{\frac{b_t}{t}} x \right) \\
= \frac{b_t}{t} (\log t)^2 \Lambda_\varepsilon \left( \frac{t}{b_t} \right)^{-2} \|h\|_{\infty} |\mathbb{Z}[0, \gamma t]|^2 \sum_{x \in \mathbb{Z}^2} h_\varepsilon \left( \sqrt{\frac{b_t}{t}} x \right)
\]

(5.42)

\[
\leq C \left( \frac{b_t}{t} \right)^2 (\log t)^2 |\mathbb{Z}[0, \gamma t]|^2,
\]

where in the last step we used (4.17). (5.12) together with (5.41) then implies that

\[
\mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} \Lambda_\varepsilon \left( \frac{t}{b_t} \right)^{-1} (\log t) \left( \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}[0, \gamma t]} h_\varepsilon \left( \sqrt{\frac{b_t}{t}} (x - y) \right) \right]^2 \right)^{1/2} \right\}
\]

(5.43)

\[
\rightarrow \mathbb{E} \exp \left\{ 2 \pi \theta \sqrt{\det(\Gamma)} \left( \int_{\mathbb{R}^2} \left( \int_0^T h_\varepsilon (W(s) - x) ds \right)^2 dx \right)^{1/2} \right\}.
\]

Combining (5.40) and (5.43) we see that

\[
\limsup_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} (\log t)|A_t(\varepsilon)|^{1/2} \right\}
\leq \frac{1}{T} \log \mathbb{E} \exp \left\{ 2 \pi \theta \sqrt{\det(\Gamma)} \left( \int_{\mathbb{R}^2} \left( \int_0^T h_\varepsilon (W(s) - x) ds \right)^2 dx \right)^{1/2} \right\}.
\]

(5.44)

Then the upper bound for (4.19) follows from the fact that

\[
\lim_{T \to \infty} \frac{1}{T} \log \mathbb{E} \exp \left\{ 2 \pi \theta \sqrt{\det(\Gamma)} \left( \int_{\mathbb{R}^2} \left( \int_0^T h_\varepsilon (W(s) - x) ds \right)^2 dx \right)^{1/2} \right\}
\]

\[
= \sup_{g \in \mathcal{F}} \left\{ 2 \pi \theta \sqrt{\det(\Gamma)} \left( \int_{\mathbb{R}^2} \left( \int_0^T \left( |(g^2 * h_\varepsilon)(x)|^2 dx \right)^{1/2} \right) \right) \right\}.
\]

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This is \[12\] Theorem 7]. (Or see the earlier \[7\] Theorem 3.1], which uses a slightly different smoothing).

We now prove the lower bound for (4.19). Let \( f \) be a smooth function on \( \mathbb{R}^2 \) with compact support and

\[
||f||_2 = \left( \int_{\mathbb{R}^2} |f(x)|^2 dx \right)^{1/2} = 1.
\]

We can write

\[
\sqrt{\frac{b_t}{t}} \left( \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}[0,t]} h_\varepsilon \left( \sqrt{\frac{b_t}{t}} (x - y) \right) \right]^2 \right)^{1/2}
\]

\[
= \sqrt{\frac{b_t}{t}} \left( \int_{\mathbb{R}^2} \left[ \sum_{y \in \mathbb{Z}[0,t]} h_\varepsilon \left( \sqrt{\frac{b_t}{t}} (|x| - y) \right) \right]^2 dx \right)^{1/2}
\]

\[
= \left( \int_{\mathbb{R}^2} \left[ \sum_{y \in \mathbb{Z}[0,t]} h_\varepsilon \left( \sqrt{\frac{b_t}{t}} (|x| - y) \right) \right]^2 dx \right)^{1/2}.
\]

Hence by the Cauchy-Schwarz inequality,

\[
\sqrt{\frac{b_t}{t}} \left( \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}[0,t]} h_\varepsilon \left( \sqrt{\frac{b_t}{t}} (x - y) \right) \right]^2 \right)^{1/2}
\]

\[
= \left( \int_{\mathbb{R}^2} h_\varepsilon \left( \sqrt{\frac{b_t}{t}} (|x| - y) \right) dx \right)^{1/2}
\]

\[
\geq \int_{\mathbb{R}^2} f(x) \sum_{y \in \mathbb{Z}[0,t]} h_\varepsilon \left( x - \sqrt{\frac{b_t}{t}} y \right) dx + O(1) |\mathbb{Z}[0,t]|, \quad t \to \infty,
\]

where \( O(1) \) is bounded by a constant. In view of (4.12), recalling that

\[
\sqrt{\frac{b_t}{t}} |A_t(\varepsilon)|^{1/2} \sim \frac{b_t}{t} \sqrt{\frac{b_t}{t}} \left( \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}[0,t]} h_\varepsilon \left( \sqrt{\frac{b_t}{t}} (x - y) \right) \right]^2 \right)^{1/2},
\]

and using Hölder’s inequality one can see that the term \( O(1) |\mathbb{Z}[0,t]| \) does not contribute anything to (4.19).

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By [3, Theorem 8],

\[
\liminf_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \frac{b_t \log t}{t} \sum_{y \in \mathbb{Z}[0,t]} (f \ast h_{\varepsilon}) \left( \sqrt{\frac{b_t}{t} y} \right) \right\}
\]

\[
\geq \sup_{g \in \mathcal{F}} \left\{ 2\pi \theta \sqrt{\det(\Gamma)} \int_{\mathbb{R}^2} (f \ast h_{\varepsilon})(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \Gamma \nabla g(x) \rangle dx \right\}
\]

\[
= \sup_{g \in \mathcal{F}} \left\{ 2\pi \theta \sqrt{\det(\Gamma)} \int_{\mathbb{R}^2} f(x) (g^2 \ast h_{\varepsilon})(x) dx - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \Gamma \nabla g(x) \rangle dx \right\}.
\]

We see from (5.47) and (5.48) that

\[
\liminf_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \sqrt{\frac{b_t}{t}} (\log t) |A_t(\varepsilon)|^{1/2} \right\}
\]

\[
\geq \sup_{g \in \mathcal{F}} \left\{ 2\pi \theta \sqrt{\det(\Gamma)} \int_{\mathbb{R}^2} f(x) (g^2 \ast h_{\varepsilon})(x) dx - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \Gamma \nabla g(x) \rangle dx \right\}.
\]

Taking the supremum over \( f \) on the right gives

\[
\liminf_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \sqrt{\frac{b_t}{t}} (\log t) |A_t(\varepsilon)|^{1/2} \right\}
\]

\[
\geq \sup_{g \in \mathcal{F}} \left\{ 2\pi \theta \sqrt{\det(\Gamma)} \left( \int_{\mathbb{R}^2} |(g^2 \ast h_{\varepsilon})(x)|^2 dx \right)^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \Gamma \nabla g(x) \rangle dx \right\}.
\]

This completes the proof of (4.19).

To prove (4.20), in (4.19), we replace \( t \) by \( 2^{-N} t \), \( \theta \) by \( 2^{-N/2} \theta \), \( b_t \) by \( \tilde{b}_t =: b_{2^N t} \).
and \( \varepsilon \) by \( 2^{N/2}\varepsilon \) to find that

\[
(5.51)
\]

\[
\lim_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} (\log t) \right\} 
\times \left( \Delta_2 \left( \frac{t}{b_t} \right)^{-2} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}^2} h_2 \left( \sqrt{\frac{b_t}{t}} (x - y) \right)^2 \right] \right)^{1/2}
\]

\[
= \lim_{t \to \infty} \frac{1}{b_{2^{-N}t}} \log \mathbb{E} \exp \left\{ 2^{-N/2} \theta \sqrt{\frac{b_{2^{-N}t}}{2^{-N}t}} (\log t) \right\} 
\times \left( \Delta_{2/2} \left( \frac{2^{-N}t}{b_{2^{-N}t}} \right)^{-2} \sum_{x \in \mathbb{Z}^2} \left[ \sum_{y \in \mathbb{Z}^2} h_{2/2} \left( \sqrt{\frac{b_{2^{-N}t}}{2^{-N}t}} (x - y) \right)^2 \right] \right)^{1/2}
\]

\[
= \sup_{g \in \mathcal{F}} \left\{ 2\pi 2^{-N/2} \theta \sqrt{\text{det} (\Gamma)} \left( \int_{\mathbb{R}^2} |(g^2 \ast h_{2/2}) (x)|^2 \, dx \right)^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \Gamma \nabla g(x) \rangle^2 \, dx \right\}
\]

\[
\leq \sup_{g \in \mathcal{F}} \left\{ 2\pi 2^{-N/2} \theta \sqrt{\text{det} (\Gamma)} \left( \int_{\mathbb{R}^2} |g(x)|^4 \, dx \right)^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \Gamma \nabla g(x) \rangle^2 \, dx \right\}.
\]

\[
= (2\pi 2^{-N/2} \theta)^2 \sqrt{\text{det} (\Gamma)} \sup_{f \in \mathcal{F}} \left\{ \left( \int_{\mathbb{R}^2} |f(x)|^4 \, dx \right)^{1/2} - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla f(x)|^2 \, dx \right\}.
\]

\[
= 2^{-N + 2} \pi^2 \theta^2 \sqrt{\text{det} (\Gamma)} \kappa(2, 2)^4,
\]

where the third step follows from Jensen’s inequality, the fourth step follows from the substitution \( g(x) = \sqrt{\text{det} (A)} \) \( f(Ax) \) with the \( 2 \times 2 \) matrix \( A \) satisfying

\[
A^T \Gamma A = (2\pi 2^{-N/2} \theta)^2 \sqrt{\text{det} (\Gamma)} I_{2 \times 2},
\]

and the last step follows from Lemma 7.2 in [7].
6 Exponential approximation

Let \( t_1, \ldots, t_a \geq 0 \) and write

\[
\Delta_1 = [0, t_1], \quad \Delta_k = \left[ \sum_{j=1}^{k-1} t_j, \sum_{j=1}^{k} t_j \right] \quad k = 2, \ldots, a.
\]

Let \( p(x) \) be a positive symmetric function on \( \mathbb{Z}^2 \) with \( \sum_{x \in \mathbb{Z}^2} p(x) = 1 \) and define

\[
L = \sum_{j,k=1}^{a} \left[ |Z(\Delta_j) \cap Z'(\Delta_k)| \right. \\
\left. - \sum_{x \in \mathbb{Z}^2} p(x) |Z(\Delta_j) \cap (Z'(\Delta_k) + x)| \right],
\]

and

\[
L_j = \left[ |Z[0, t_j] \cap Z'[0, t_j]| \right. \\
\left. - \sum_{x \in \mathbb{Z}^2} p(x) |Z[0, t_j] \cap (Z'[0, t_j] + x)|, \quad j = 1, \ldots, a. \right)
\]

**Lemma 6.1** For any \( m \geq 1 \),

\[
\mathbb{E} L^m \geq 0
\]

and

\[
\left\{ \mathbb{E} L^m \right\}^{1/2} \leq \sum_{k_1 + \cdots + k_a = m \atop k_1, \ldots, k_a \geq 0} \frac{m!}{k_1! \cdots k_a!} \left\{ \mathbb{E} |L_1|^{k_1} \right\}^{1/2} \cdots \left\{ \mathbb{E} |L_a|^{k_a} \right\}^{1/2}.
\]

Consequently, for any \( \theta > 0 \)

\[
\sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left\{ \mathbb{E} L^m \right\}^{1/2} \leq \prod_{j=1}^{a} \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left\{ \mathbb{E} |L_j|^m \right\}^{1/2}.
\]

**Proof.** Write

\[
\hat{p}(\lambda) = \sum_{x \in \mathbb{Z}^2} p(x) e^{i\lambda \cdot x}.
\]
We note that
\[ |\hat{p}(\lambda)| \leq \hat{p}(0) = 1. \] (6.8)

Notice also that
\[ L = \frac{1}{(2\pi)^2} \int_{[-\pi, \pi]^2} \left[ 1 - \hat{p}(\lambda) \right] \left[ \sum_{j=1}^{a} \sum_{x \in Z(\Delta_j)} e^{i\lambda \cdot x} \right] \left[ \sum_{j' = 1}^{a} \sum_{x' \in Z'(\Delta_{j'})} e^{-i\lambda \cdot x'} \right] d\lambda. \] (6.9)

We therefore have
\[
\mathbb{E} L^m = \frac{1}{(2\pi)^{2m}} \int_{([-\pi, \pi]^2)^m} \left| \mathbb{E} \prod_{k=1}^{m} \sum_{j=1}^{a} \sum_{x_k \in Z(\Delta_j)} e^{i\lambda_k \cdot x_k} \right|^2 \left( \prod_{k=1}^{m} [1 - \hat{p}(\lambda_k)] d\lambda_k \right) \\
= \frac{1}{(2\pi)^{2m}} \int_{([-\pi, \pi]^2)^m} \left| \sum_{l_1, \ldots, l_m = 1}^{a} \mathbb{E} \left( H_{l_1}(\lambda_1) \cdots H_{l_m}(\lambda_m) \right) \right|^2 \\
\left( \prod_{k=1}^{m} [1 - \hat{p}(\lambda_k)] d\lambda_k \right),
\] (6.10)

where
\[ H_j(\lambda) = \sum_{x \in Z(\Delta_j)} e^{i\lambda \cdot x}. \] (6.11)

This proves (6.4) and implies that
\[
\left\{ \mathbb{E} L^m \right\}^{1/2} \leq \frac{1}{(2\pi)^{m}} \sum_{l_1, \ldots, l_m = 1}^{a} \left\{ \int_{([-\pi, \pi]^2)^m} \left| \mathbb{E} \left( H_{l_1}(\lambda_1) \cdots H_{l_m}(\lambda_m) \right) \right|^2 \left( \prod_{k=1}^{m} [1 - \hat{p}(\lambda_k)] d\lambda_k \right) \right\}^{1/2}. 
\] (6.12)

Note that for any \( k > j \) we can write
\[ H_k(\lambda) = \sum_{x \in Z(\Delta_k)} e^{i\lambda \cdot x} = e^{i\lambda Z(t_j)} H_k^{(j)}(\lambda), \] (6.13)
where
\begin{equation}
H_k^{(j)}(\lambda) = \sum_{x \in Z(\Delta_k) - Z(t_j)} e^{i\lambda x}
\end{equation}

is independent of \( \mathcal{F}_{t_j} \).

Let \( l_1, \ldots, l_m \) be fixed and let \( k_j = \sum_{i=1}^m \delta(l_i, j) \) be the number of \( l \)’s which are equal to \( j \), for each \( 1 \leq j \leq a \). Then using independence
\begin{equation}
\int_{([-\pi, \pi]^2)^m} \left| \mathbb{E} \left( H_{l_1}(\lambda_1) \cdots H_{l_m}(\lambda_m) \right) \right|^2 \left( \prod_{k=1}^m [1 - \tilde{p}(\lambda_k)] d\lambda_k \right)
= \int_{([-\pi, \pi]^2)^m} \left| \mathbb{E} \left( \prod_{j=1}^a (H_j(\lambda_{j,1}) \cdots H_j(\lambda_{j,k_j})) \right) \right|^2 \left( \prod_{j=1}^a \prod_{l=1}^{k_j} [1 - \tilde{p}(\lambda_{j,l})] d\lambda_{j,l} \right)
= \int_{([-\pi, \pi]^2)^{m-k_1}} \left| \mathbb{E} \left( \prod_{j=2}^a (H_j^{(1)}(\lambda_{j,1}) \cdots H_j^{(1)}(\lambda_{j,k_j})) \right) \right|^2 \left( \prod_{j=2}^a \prod_{l=1}^{k_j} [1 - \tilde{p}(\lambda_{j,l})] d\lambda_{j,l} \right)
\end{equation}

\[ F(\lambda_{2,1}, \ldots, \lambda_{2,k_2}; \ldots; \lambda_{a,1}, \ldots, \lambda_{a,k_a}) \left( \prod_{j=2}^a \prod_{l=1}^{k_j} [1 - \tilde{p}(\lambda_{j,l})] d\lambda_{j,l} \right), \]

where
\begin{equation}
F(\lambda_{2,1}, \ldots, \lambda_{2,k_2}; \ldots; \lambda_{a,1}, \ldots, \lambda_{a,k_a})
= \int_{([-\pi, \pi]^2)^{k_1}} \left| \mathbb{E} \left\{ \exp \left\{ i \left( \sum_{j=2}^a \sum_{l=1}^{k_j} \lambda_{j,l} \right) \right\} \right| \left( H_1(\lambda_{1,1}) \cdots H_1(\lambda_{1,k_1}) \right) \right|^2 \left( \prod_{l=1}^{k_1} [1 - \tilde{p}(\lambda_{1,l})] d\lambda_{1,l} \right).
\end{equation}
Notice that by symmetry

\[(6.17) \quad \mathbb{E} \left[ \exp \left\{ i \left( \sum_{j=2}^{a} \sum_{l=1}^{k_j} \lambda_{j,l} \right) \cdot Z(t_1) \right\} \left( H_1(\lambda_{1,1}) \cdots H_1(\lambda_{1,k_1}) \right) \right] \]

is real valued. Hence if \( Z' \) denotes an independent copy of \( Z \), and \( H'_1 \) is obtained from \( H_1 \) by replacing \( Z \) by \( Z' \),

\[(6.18) \quad F(\lambda_{2,1}, \ldots, \lambda_{2,k_2}; \ldots; \lambda_{a,1}, \ldots, \lambda_{a,k_a}) \]

\[= \int_{([-\pi,\pi]^2)^{k_1}} \mathbb{E} \left[ \exp \left\{ i \left( \sum_{j=2}^{a} \sum_{l=1}^{k_j} \lambda_{j,l} \right) \cdot (Z(t_1) + Z'(t_1)) \right\} \times \prod_{l=1}^{k_1} \left( H_1(\lambda_{1,l}) H'_1(\lambda_{1,l}) \right) \left( \prod_{l=1}^{k_1} [1 - \hat{p}(\lambda_{1,l})] d\lambda_{1,l} \right) \]

\[= \mathbb{E} \left[ \exp \left\{ i \left( \sum_{j=2}^{a} \sum_{l=1}^{k_j} \lambda_{j,l} \right) \cdot (Z(t_1) + Z'(t_1)) \right\} \times \int_{([-\pi,\pi]^2)^{k_1}} \prod_{l=1}^{k_1} \left( H_1(\lambda_{1,l}) H'_1(\lambda_{1,l}) \right) \left( \prod_{l=1}^{k_1} [1 - \hat{p}(\lambda_{1,l})] d\lambda_{1,l} \right) \right]. \]

By the fact that

\[(6.19) \quad \int_{([-\pi,\pi]^2)^{k_1}} \prod_{l=1}^{k_1} \left( H_1(\lambda_{1,l}) H'_1(\lambda_{1,l}) \right) \left( \prod_{l=1}^{k_1} [1 - \hat{p}(\lambda_{1,l})] d\lambda_{1,l} \right) \]

\[= \left[ \int_{[-\pi,\pi]^2} [1 - \hat{p}(\lambda)] H_1(\lambda) H'_1(\lambda) d\lambda \right]^{k_1} = (2\pi)^{2k_1} L_1^{k_1}, \]

we have proved that

\[
\int_{([-\pi,\pi]^2)^{m}} \left| \mathbb{E}\left( H_{l_1}(\lambda_1) \cdots H_{l_m}(\lambda_m) \right) \right|^2 \left( \prod_{k=1}^{m} [1 - \hat{p}(\lambda_k)] d\lambda_k \right) \\
\leq (2\pi)^{2k_1} \mathbb{E}|L_1|^{k_1} \int_{([-\pi,\pi]^2)^{m-k_1}} \left| \mathbb{E}\left( \prod_{j=2}^{a} \left( H_j^{(1)}(\lambda_{j,1}) \cdots H_j^{(1)}(\lambda_{j,k_j}) \right) \right) \right|^2 \\
(6.20) \quad \left( \prod_{j=2}^{a} \prod_{l=1}^{k_j} [1 - \hat{p}(\lambda_{j,l})] d\lambda_{j,l} \right). 
\]
Repeating the above procedure,
\[
\int_{[-\pi, \pi]^2} \left| \mathbb{E}\left( H_{t_1}(\lambda_1) \cdots H_{t_m}(\lambda_m) \right) \right|^2 \left( \prod_{k=1}^{m} [1 - \tilde{p}(\lambda_k)] d\lambda_k \right) \leq \prod_{j=1}^{a} \left\{ (2\pi)^{2k} \mathbb{E}|L_j|^{k_j} \right\} = (2\pi)^{2m} \prod_{j=1}^{a} \mathbb{E}|L_j|^{k_j}.
\]
(6.21)

Our Lemma now follows from (6.12).

**Proof of Lemma 4.7** Define
\[
q_{t, \varepsilon}(x) = \mathbb{E}_\varepsilon \left( \left( \frac{t}{b_t} \right)^{-2} \sum_{x \in \mathbb{Z}^2} h_\varepsilon \left( \frac{b_t}{t} (x - z) \right) \right) \left( \frac{b_t}{t} z \right), \quad x \in \mathbb{Z}^2.
\]
(6.22)

Then \(q_{t, \varepsilon}(x)\) is a probability density on \(\mathbb{Z}^2\). We claim that
\[
B_t^{(0)}(\varepsilon) = \sum_{x \in \mathbb{Z}^2} q_{t, \varepsilon}(x) |Z[0, t] \cap (x + Z'[0, t])|.
\]
(6.23)

This follows from the fact that
\[
\sum_{x \in \mathbb{Z}^2} \sum_{y \in Z[0, t]} h_\varepsilon \left( \frac{b_t}{t} (x - y) \right) \sum_{y' \in Z'[0, t]} h_\varepsilon \left( \frac{b_t}{t} (x - y') \right) = \sum_{x \in \mathbb{Z}^2} \sum_{y' \in Z'[0, t]} h_\varepsilon \left( \frac{b_t}{t} x \right) \sum_{y \in \mathbb{Z}^2} h_\varepsilon \left( \frac{b_t}{t} (x + y' - y) \right) \mathbb{1}_{\{y \in Z[0, t]\}}
\]
\[
= \sum_{x \in \mathbb{Z}^2} \sum_{y' \in Z'[0, t]} h_\varepsilon \left( \frac{b_t}{t} x \right) \sum_{y \in \mathbb{Z}^2} h_\varepsilon \left( \frac{b_t}{t} (x - y) \right) \mathbb{1}_{\{y + y' \notin Z[0, t]\}}
\]
\[
= \sum_{y' \in Z'[0, t]} h_\varepsilon \left( \frac{b_t}{t} x \right) |Z'[0, t] \cap (Z[0, t] - y')| Z[0, t] \cap (Z[0, t] - y)
\]
and
\[
|Z'[0, t] \cap (Z[0, t] - y)| = |Z[0, t] \cap (Z'[0, t] + y)|.
\]
(6.24)

Write \(\gamma_t = t/[b_t]\) and \(\Delta_j = [(j - 1)\gamma_t, j\gamma_t], j = 1, \ldots, [b_t]\). Note that
\[
\sum_{j=1}^{[b_t]} |Z(\Delta_j) \cap Z[0, t]| - \sum_{1 \leq j < k \leq [b_t]} |Z(\Delta_j) \cap Z(\Delta_k) \cap Z'[0, t]| \leq |Z[0, t] \cap Z'[0, t]| \leq \sum_{j=1}^{[b_t]} |Z(\Delta_j) \cap Z'[0, t]|.
\]
(6.25)
and similarly

\[ (6.27) \quad \sum_{j=1}^{[b_t]} \sum_{x \in \mathbb{Z}^2} q_{t, \varepsilon}(x) |Z(\Delta_j) \cap (x + Z'[0, t])| \]

\[- \sum_{1 \leq j < k \leq [b_t]} \sum_{x \in \mathbb{Z}^2} q_{t, \varepsilon}(x) |Z(\Delta_j) \cap Z(\Delta_k) \cap (x + Z'[0, t])| \]

\[ \leq \sum_{x \in \mathbb{Z}^2} q_{t, \varepsilon}(x) |Z[0, t] \cap (x + Z'[0, t])| \]

\[ \leq \sum_{j=1}^{[b_t]} \sum_{x \in \mathbb{Z}^2} q_{t, \varepsilon}(x) |Z(\Delta_j) \cap (x + Z'[0, t])|. \]

Hence,

\[ \left| |Z[0, t] \cap Z'[0, t]| - \sum_{x \in \mathbb{Z}^2} q_{t, \varepsilon}(x) |Z[0, t] \cap (x + Z'[0, t])| \right| \]

\[ \leq \left| \sum_{j=1}^{[b_t]} \left[ |Z(\Delta_j) \cap Z'[0, t]| - \sum_{x \in \mathbb{Z}^2} q_{t, \varepsilon}(x) |Z(\Delta_j) \cap (x + Z'[0, t])| \right] \right| \]

\[ + \sum_{1 \leq j < k \leq [b_t]} |Z(\Delta_j) \cap Z(\Delta_k) \cap Z'[0, t]| \]

\[ (6.28) \quad + \sum_{1 \leq j < k \leq [b_t]} \sum_{x \in \mathbb{Z}^2} q_{t, \varepsilon}(x) |Z(\Delta_j) \cap Z(\Delta_k) \cap (x + Z'[0, t])|. \]

We first take care of the last two terms. This is the easy step. Write

\[ (6.29) \quad \eta(t, \varepsilon) = \sum_{1 \leq j < k \leq [b_t]} |Z(\Delta_j) \cap Z(\Delta_k) \cap Z'[0, t]| \]

\[ + \sum_{1 \leq j < k \leq [b_t]} \sum_{x \in \mathbb{Z}^2} q_{t, \varepsilon}(x) |Z(\Delta_j) \cap Z(\Delta_k) \cap (x + Z'[0, t])|. \]

It follows from (4.2) that

\[ (6.30) \quad \sup_{t, j, k, x} \mathbb{E} \exp \left\{ c \frac{(\log t)^{3/2}}{\sqrt{t}} |Z(\Delta_j) \cap Z(\Delta_k) \cap (x + Z'[0, t])|^{1/2} \right\} < \infty. \]
for some $c > 0$. Hence, if $b_t = o((\log t)^{1/5})$, then for any $\theta > 0$ we can find $t_0 < \infty$ such that

\begin{equation}
\sup_{t \geq t_0} \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} (\log t) \eta(t, \varepsilon)^{1/2} \right\}
\leq \sup_{t \geq t_0} \sup_{j,k,x} \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} b^2_t |Z(\Delta_j) \cap Z(\Delta_k) \cap (x + Z'[0, t])|^{1/2} \right\}
< \infty.
\end{equation}

Hence

\begin{equation}
\limsup_{t \to \infty} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \theta \sqrt{\frac{b_t}{t}} (\log t) \eta(t, \varepsilon)^{1/2} \right\} = 0.
\end{equation}

To handle the first term on the right hand side of (6.28) set

\begin{equation}
\xi(t, \varepsilon) = \sum_{j=1}^{[b_t]} \left[ |Z(\Delta_j) \cap Z'[0, t]| - \sum_{x \in \mathbb{Z}^2} q_{t, \varepsilon}(x) |Z(\Delta_j) \cap (x + Z'[0, t])| \right].
\end{equation}

Using Fubini, independence and then the Cauchy-Schwarz inequality we have

\begin{equation}
\mathbb{E} \xi^m(t, \varepsilon) = (2\pi)^{-2m} \mathbb{E} \int_{([-\pi, \pi]^2)^m} \left( \prod_{k=1}^{m} \left[ 1 - \hat{q}_{t, \varepsilon}(\lambda_k) \right] \right) d\lambda_1 \cdots d\lambda_m \times \left[ \prod_{k=1}^{[b_t]} \sum_{x_k' \in \mathbb{Z}[0, t]} e^{i\lambda_k \cdot x_k'} \right] \left[ \prod_{k=1}^{m} \sum_{x_k \in Z(\Delta_j)} e^{-i\lambda_k \cdot x_k} \right] d\lambda_1 \cdots d\lambda_m
\leq (2\pi)^{-2m} \left\{ \int_{([-\pi, \pi]^2)^m} \left( \prod_{k=1}^{m} \left[ 1 - \hat{q}_{t, \varepsilon}(\lambda_k) \right] \right) \mathbb{E} \prod_{k=1}^{m} \sum_{x_k \in Z[0, t]} e^{i\lambda_k \cdot x_k} 1/2 \right\}^{1/2}
\times \left\{ \int_{([-\pi, \pi]^2)^m} \left( \prod_{k=1}^{m} \left[ 1 - \hat{q}_{t, \varepsilon}(\lambda_k) \right] \right) \mathbb{E} \prod_{k=1}^{[b_t]} \sum_{x_k \in Z(\Delta_j)} e^{i\lambda_k \cdot x_k} 1/2 \right\}^{1/2}
\leq \mathbb{E} |Z[0, t] \cap Z'[0, t]|^{m} \left\{ \mathbb{E} \xi^m(t, \varepsilon) \right\}^{1/2}.
\end{equation}
where

\begin{equation}
(6.35) \quad \zeta(t, \epsilon) = \sum_{j,k=1}^{[b_t]} \left[ |Z(\Delta_j) \cap Z'(\Delta_k)| - \sum_{x \in \mathbb{Z}^2} q_{t,\epsilon}(x) |Z(\Delta_j) \cap (x + Z'(\Delta_k))| \right]
\end{equation}

and we have used the fact that \(1 - \tilde{q}_{t,\epsilon}(\lambda) \leq 1\) in the last step. Note that in the notation of (6.2), \(\zeta(t, \epsilon) = L\) with \(p(x) = q_{t,\epsilon}(x)\), so that by (6.4), for all \(m \geq 1\)

\begin{equation}
(6.36) \quad \mathbb{E}\zeta^m(t, \epsilon) \geq 0.
\end{equation}

Let \(\delta > 0\) be fixed for a while. By Cauchy-Schwarz and then (6.34)

\begin{equation}
(6.37) \quad \mathbb{E} \cosh \left\{ \theta \sqrt{\frac{b_t}{t}} (\log t) |\xi(t, \epsilon)|^{1/2} \right\}
= \sum_{m=0}^{\infty} \frac{\theta^{2m}}{(2m)!} \left( \sqrt{\frac{b_t}{t}} (\log t) \right)^{2m} \mathbb{E} |\xi^m(t, \epsilon)|
\leq \sum_{m=0}^{\infty} \frac{\theta^{2m}}{(2m)!} \left( \sqrt{\frac{b_t}{t}} (\log t) \right)^{2m} \left\{ \mathbb{E} \xi^2m(t, \epsilon) \right\}^{1/2}
\leq \left\{ \sum_{m=0}^{\infty} \frac{(\delta \theta)^{2m}}{(2m)!} \left( \sqrt{\frac{b_t}{t}} (\log t) \right)^{2m} \left\{ \mathbb{E} |Z[0,t] \cap Z'[0,t]|^{2m} \right\}^{1/2} \right\}^{1/2}
\times \left\{ \sum_{m=0}^{\infty} \frac{(\delta^{-1}\theta)^{2m}}{(2m)!} \left( \sqrt{\frac{b_t}{t}} (\log t) \right)^{2m} \left\{ \mathbb{E} \xi^2m(t, \epsilon) \right\}^{1/2} \right\}^{1/2}
\leq \left\{ \sum_{m=0}^{\infty} \frac{(\delta \theta)^{m}}{m!} \left( \sqrt{\frac{b_t}{t}} (\log t) \right)^{m} \left\{ \mathbb{E} |Z[0,t] \cap Z'[0,t]|^m \right\}^{1/2} \right\}
\times \left\{ \sum_{m=0}^{\infty} \frac{(\delta^{-1}\theta)^{m}}{m!} \left( \sqrt{\frac{b_t}{t}} (\log t) \right)^{m} \left\{ \mathbb{E} \xi^m(t, \epsilon) \right\}^{1/2} \right\},
\end{equation}

where in the last step we used (6.36) and the fact that \(|Z[0,t] \cap Z'[0,t]| \geq 0\).

By (2.11), there is a \(C > 0\) independent of \(\delta\) and \(\theta\) such that

\begin{equation}
(6.38) \quad \lim_{t \to \infty} \frac{1}{b_t} \log \sum_{m=0}^{\infty} \frac{(\delta \theta)^{m}}{m!} \left( \sqrt{\frac{b_t}{t}} (\log t) \right)^{m} \left\{ \mathbb{E} |Z[0,t] \cap Z'[0,t]|^m \right\}^{1/2} = C(\delta \theta)^2.
\end{equation}
In addition, by Lemma 6.1

\[
\sum_{m=0}^{\infty} \frac{(\delta - 1)^m}{m!} \left( \sqrt{\frac{b_t}{T} \log t} \right)^m \left\{ \mathbb{E} \zeta^m(t, \varepsilon) \right\}^{1/2} \leq \left\{ \sum_{m=0}^{\infty} \frac{(\delta - 1)^m}{m!} \left( \sqrt{\frac{b_t}{T} \log t} \right)^m \left\{ \mathbb{E} |\beta(t, \varepsilon)|^m \right\}^{1/2} \right\}^{1/2},
\]

where

\[
\beta(t, \varepsilon) = |Z[0, \gamma_t] \cap Z'[0, \gamma_t]| - \sum_{x \in \mathbb{Z}^2} q_{t, \varepsilon}(x)|Z[0, \gamma_t] \cap (x + Z'[0, \gamma_t])|. 
\]

Recall that \(q_{t, \varepsilon}(x)\) is defined by (6.22) and \(\gamma_t = t/[b_t]\). As in the proof of (6.23) we can check that

\[
\sum_{x \in \mathbb{Z}^2} q_{t, \varepsilon}(x)|Z[0, \gamma_t] \cap (x + Z'[0, \gamma_t])| = B_{\gamma_t, 1},
\]

see (5.2). By Lemma 5.1 (with \(t\) replaced by \(\gamma_t\)),

\[
\frac{b_t}{t} \log \sum_{m=0}^{\infty} \frac{(\delta - 1)^m}{m!} \left( \sqrt{\frac{b_t}{T} \log t} \right)^m \left\{ \mathbb{E} \zeta^m(t, \varepsilon) \right\}^{1/2} \leq \log \sum_{m=0}^{\infty} \frac{(\delta - 1)^m}{m!} \left( (2\pi) \sqrt{\det(\Gamma)} \right)^m \left\{ \mathbb{E} |\alpha([0, 1]^2) - \alpha_x([0, 1]^2)| \right\}^{1/2}.
\]

Hence,

\[
\lim_{t \to \infty} \sum_{m=0}^{\infty} \frac{(\delta - 1)^m}{m!} \left( \sqrt{\frac{b_t}{T} \log t} \right)^m \left\{ \mathbb{E} |\beta(t, \varepsilon)|^m \right\}^{1/2} = \sum_{m=0}^{\infty} \frac{(\delta - 1)^m}{m!} \left( (2\pi) \sqrt{\det(\Gamma)} \right)^m \left\{ \mathbb{E} |\alpha([0, 1]^2) - \alpha_x([0, 1]^2)| \right\}^{1/2}.
\]

So by (6.39) we have

\[
\limsup_{t \to \infty} \frac{1}{b_t} \log \sum_{m=0}^{\infty} \frac{(\delta - 1)^m}{m!} \left( \sqrt{\frac{b_t}{T} \log t} \right)^m \left\{ \mathbb{E} \zeta^m(t, \varepsilon) \right\}^{1/2} \leq \log \sum_{m=0}^{\infty} \frac{(\delta - 1)^m}{m!} \left( (2\pi) \sqrt{\det(\Gamma)} \right)^m \left\{ \mathbb{E} |\alpha([0, 1]^2) - \alpha_x([0, 1]^2)| \right\}^{1/2}.
\]
By [24, Theorem 1, p.183],

\[(6.45) \quad \mathbb{E}\left| \alpha([0, 1]^2) - \alpha_{\varepsilon}([0, 1]^2) \right|^m \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+ ,\]

for all \(m \geq 1\). In addition, by [7, (1.12)], there is a constant \(C > 0\) such that

\[(6.46) \quad \mathbb{E}\left| \alpha([0, 1]^2) - \alpha_{\varepsilon}([0, 1]^2) \right|^m \leq \mathbb{E}\alpha^m([0, 1]^2) \leq m!C^m\]

for all \(m \geq 1\). By dominated convergence, therefore,

\[(6.47) \quad \sum_{m=0}^{\infty} \frac{(\delta^{-1}\theta)^m}{m!} \left( \frac{2\pi}{\sqrt{\det(\Gamma)}} \right)^m \left\{ \mathbb{E}\left| \alpha([0, 1]^2) - \alpha_{\varepsilon}([0, 1]^2) \right|^m \right\}^{1/2} \rightarrow 1\]

as \(\varepsilon \rightarrow 0^+\). (Alternatively, this follows immediately from [12, (6.29)]). Thus

\[(6.48) \quad \lim_{\varepsilon \rightarrow 0^+} \limsup_{t \rightarrow \infty} \frac{1}{b_t} \log \sum_{m=0}^{\infty} \frac{(\delta^{-1}\theta)^m}{m!} \left( \sqrt{\frac{b_t}{t}} \log t \right)^m \left\{ \mathbb{E}\xi^m(t, \varepsilon) \right\}^{1/2} = 0.\]

Summarizing what we have,

\[(6.49) \quad \limsup \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{b_t} \log \mathbb{E} \left\{ \sqrt{\frac{b_t}{t}} (\log t) \left| \xi(t, \varepsilon) \right|^{1/2} \right\} \leq C(\delta \theta)^2.\]

Letting \(\delta \rightarrow 0^+\) gives

\[(6.50) \quad \limsup \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{b_t} \log \mathbb{E} \left\{ \sqrt{\frac{b_t}{t}} (\log t) \left| \xi(t, \varepsilon) \right|^{1/2} \right\} = 0.\]

Since \(\exp(x) \leq 2 \cosh(x)\) we see from (6.50) that

\[(6.51) \quad \limsup \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{b_t} \log \mathbb{E} \exp \left\{ \sqrt{\frac{b_t}{t}} (\log t) \left| \xi(t, \varepsilon) \right|^{1/2} \right\} = 0.\]

By (6.28) and (6.23) we have thus completed the proof of Lemma 4.7 when \(j = 0\). If in (4.22) with \(j = 0\) we replace \(t\) by \(2^{-j}t\), \(\theta\) by \(2^{-j/2}\theta\), \(b_t\) by \(b_{2^j t}\) and \(\varepsilon\) by \(2^{j/2}\varepsilon\), we obtain (4.22) for any \(j\) (compare the proof of (4.20)).
7  Laws of the iterated logarithm

We first prove some lemmas in preparation for the proof of Theorem 1.6. Define
\[ \tilde{\phi}_j = \frac{j}{\mathcal{H}(j)}, \quad \tilde{G}_j = (R_j - \tilde{\phi}_j) \frac{\log^2 n}{n}, \]
and \( K = [\log \log n] + 1. \)

Lemma 7.1 There exists a constant \( c_1 \) such that if \( A \) and \( B \) are positive integers and \( C = A + B \), then
\[ |\tilde{\phi}_C - \tilde{\phi}_A - \tilde{\phi}_B| \leq c_1 \frac{A \land B}{C^{1/2} \log^2 C}. \]

Proof. The cases when \( A \) or \( B \) equal 1 are easy, so we suppose \( A, B > 1 \). Write
\[ \tilde{\phi}_C - \tilde{\phi}_A - \tilde{\phi}_B = \frac{C}{\mathcal{H}(C)} \left[ - \frac{A \mathcal{H}(C) - \mathcal{H}(A)}{C} - \frac{B \mathcal{H}(C) - \mathcal{H}(B)}{C} \right]. \]
By (2.2) and (2.3), the right hand side is bounded in absolute value by
\[ c_2 \frac{C}{\log C} \left[ - \frac{A \log C - \log A}{C} - \frac{B \log C - \log B}{C} \right] = c_2 |\tilde{\phi}_C - \varphi - \tilde{\phi}_B|, \]
where \( \varphi_j = j/\log j \). Our result now follows by Lemma 4.2 of [6]. \( \square \)

Lemma 7.2 There exists \( \lambda_0 \) such that if \( \lambda \geq \lambda_0 \), then
\[ \mathbb{P}(\max_{m \leq n} \tilde{R}_m > \lambda \log \log \log n/\log^2 n) \leq (\log n)^{-2}. \]

Proof. Using Lemma 7.1 in place of Lemma 4.2 of [6] and with \( \tilde{\phi}_j, \tilde{G}_j \) replacing \( \varphi_j, G_j \) resp., we have by [6], Lemma 4.3 and the proof of Proposition 4.1 (up through the display in the middle of p. 1390), that
\[ \mathbb{P}(\max_{m \leq n} \tilde{G}_m > A \log \log \log n) \leq (\log n)^{-2}. \]
If $A$ is large enough. By (2.1) and (2.2), we see that
\[
\max_{m \leq n}|R_m - (R_m - \bar{\sigma}_m)| \leq c_1 \frac{n}{\log^2 n} = o(n \log \log n/ \log^2 n),
\]
and our result now follows immediately.

Proof of Theorem 1.6: Let $\xi = 2\pi \sqrt{\det \Gamma}$. We begin with the upper bound. Let $\eta, \varepsilon > 0$ be small and let $q > 1$ be very close to 1. Let $t = \lfloor q^i \rfloor$.

If
\[
A_i = \{ R_{t_i} \geq (1 + \eta) \xi t_i \log \log t_i/ \log^2 t_i \},
\]
then it follows from Theorem 1.1 that $\sum_i \mathbb{P}(A_i) < \infty$, and so by Borel-Cantelli, $\mathbb{P}(A_i \text{ i.o.}) = 0$.

Next, if $\lambda$ is sufficiently large,
\[
\mathbb{P}(\max_{m \leq n} R_m > \lambda n \log \log n/ \log^2 n) \leq (\log n)^{-2};
\]
by Lemma 7.2, let
\[
B_i = \left\{ \max_{t_i \leq k \leq t_i + 1} [R_k - R_{t_i}] > \varepsilon t_i \log \log t_i/ \log^2 t_i \right\}.
\]
By subadditivity $R_k - R_{t_i} \leq R_{k-t_i} \circ \theta_{t_i}$, where $\theta_{t_i}$ is the usual shift operator of Markov theory. By Lemma 7.1,
\[
\mathbb{E}R_k - \mathbb{E}R_{t_i} \geq \mathbb{E}R_{k-t_i} - c \frac{t_i}{\log^2 t_i}.
\]
So by the Markov property, and using the fact that the $\mathbb{P}^x$ law of $R_{k-t_i}$ does not depend on $x$, for $i$ large
\[
\mathbb{P}(B_i)
\]
\[
= \mathbb{P}( \max_{t_i \leq k \leq t_i + 1} [R_k - R_{t_i} - (\mathbb{E}R_k - \mathbb{E}R_{t_i})] > \varepsilon t_i \log \log t_i/ \log^2 t_i )
\]
\[
\leq \mathbb{P}( \max_{t_i \leq k \leq t_i + 1} [R_k - R_{t_i} - \mathbb{E}R_{k-t_i}] + c \frac{t_i}{\log^2 t_i} > \varepsilon t_i \log \log t_i/ \log^2 t_i )
\]
\[
\leq \mathbb{P}(\max_{t_i \leq k \leq t_i + 1} [R_k - t_i] > \varepsilon t_i \log \log t_i/ \log^2 t_i - c \frac{t_i}{\log^2 t_i})
\]
\[
\leq \mathbb{P}(\max_{k \leq t_i + 1 - t_i} R_k \geq \frac{\varepsilon}{2} t_i \log \log t_i/ \log^2 t_i )
\]
If $q$ is sufficiently small, then $\sum_i \mathbb{P}(B_i)$ will be summable by (7.1). So with probability one, for $i$ large enough

$$\max_{k \leq t_{i+1}} \overline{R}_k \leq ((1 + \eta) \xi + \varepsilon) qt_i \log \log t_i / \log^2 t_i.$$  

Since $\eta$ and $\varepsilon$ are arbitrary, and we can take $q$ as close to 1 as we like, this implies the upper bound.

Let $\eta > 0$, $t_i = \exp(i \exp(\eta))$, $V_i = \#S((t_i, t_{i+1}])$, and set

$$C_i = \{ V_i > (1 - \eta) \xi (t_{i+1} - t_i) \log \log (t_{i+1} - t_i) / \log^2 (t_{i+1} - t_i) \}.$$  

Note that the events $C_i$ are independent. By Theorem 1.1 and Borel-Cantelli, $\mathbb{P}(C_i \text{i.o.}) = 1$. Note

$$\frac{(t_{i+1} - t_i) \log \log (t_{i+1} - t_i)}{\log^2 (t_{i+1} - t_i)} = \frac{t_{i+1} \log \log t_{i+1}}{\log^2 t_{i+1}} (1 + o(1)).$$

Also

$$|V_i - R_{t_{i+1}}| + |E V_i - ER_{t_{i+1}}| \leq 2t_i = o \left( \frac{t_{i+1} \log \log t_{i+1}}{\log^2 t_{i+1}} \right).$$

Therefore with probability one, infinitely often

$$\overline{R}_{t_{i+1}} > (1 - \eta / 2) \xi t_{i+1} \log \log t_{i+1} / \log^2 t_{i+1}.$$  

This proves the lower bound.

We now turn to the LIL for $-\overline{R}_n$. First we prove

**Lemma 7.3** Let $\varepsilon > 0$. There exists $q_0(\varepsilon)$ such that if $1 < q < q_0(\varepsilon)$, then

$$\mathbb{P}(\max_{[q^{-1} n] \leq k \leq n} (\overline{R}_n - \overline{R}_k) > \varepsilon n \log \log n / \log^2 n) \leq \frac{1}{\log^2 n}$$

for $n$ large.

**Proof.** Let

$$G_k = (R_n - R_k) \frac{\log^2 n}{n}.$$  

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Let
\[ A_i = \left\{ [q^{-1}n] + \left\lfloor \frac{n\ell}{2^i} \right\rfloor : \ell \in \mathbb{Z}_+ \right\} \cap [0, n], \quad i \leq \log_2 n + 1. \]

Given \( k \), let \( k_i = \max\{ j \in A_i : j \leq k \} \). We write
\[ G_k = G_{k_1} + (G_{k_2} - G_{k_1}) + (G_{k_3} - G_{k_2}) + \cdots, \]
where the sum is actually a finite one. If \( G_k > \varepsilon \log \log n \) for some \([q^{-1}n] \leq k \leq n\), then either
\[ (7.4) \quad G_{[q^{-1}n]} > \frac{\varepsilon}{2} \log \log n \]
or for some \( i \) there exist consecutive elements \( \ell, m \) of \( A_i \) such that
\[ (7.5) \quad G_m - G_\ell > \frac{\varepsilon}{10i^2} \log \log n. \]

By subadditivity \( R_n - R_k \leq R_{n-k} \circ \theta_k \) for \( k \leq n \), while by Lemma 7.1
\[ \mathbb{E}R_n - \mathbb{E}R_k \geq \mathbb{E}R_{n-k} - c_1 (1 - q^{-1})^{1/2} \frac{n}{\log^2 n}. \]

Then setting \( k = [q^{-1}n] \),
\[ \mathbb{P}(G_{[q^{-1}n]} > \frac{\varepsilon}{2} \log \log n) = \mathbb{P}\left( (R_n - R_k) \frac{\log^2 n}{n} - (\mathbb{E}R_n - \mathbb{E}R_k) \frac{\log^2 n}{n} > \frac{\varepsilon}{2} \log \log n \right) \leq \mathbb{P}^S_k \left( R_{n-k} \frac{\log^2 n}{n} - \mathbb{E}R_{n-k} \frac{\log^2 n}{n} + c_1 (1 - q^{-1})^{1/2} > \frac{\varepsilon}{2} \log \log n \right). \]

Using the fact that the \( \mathbb{P}^x \) law of \( R_{n-k} \) does not depend on \( x \), this is the same as
\[ \mathbb{P} \left( \frac{\overline{R}_{n-k}}{(n-k)/\log^2(n-k)} > \frac{n}{n-k} \frac{\log^2(n-k)}{\log^2 n} \left( \frac{\varepsilon}{2} \log \log n - c_1 (1 - q^{-1})^{1/2} \right) \right). \]

If \( q > 1 \) is close enough to 1 and \( n \) is large enough, by Theorem 1.5 this is bounded by
\[ (7.6) \quad c_2 \exp \left( -c_3 \frac{\varepsilon}{2} \frac{1}{1 - q^{-1}} \log \log n \right) \leq \frac{1}{2 \log^2 n}. \]
This bounds the probability of the event described in (7.4).

Similarly, \( R_m - R_\ell \leq R_{m-\ell} \circ \theta_\ell \) and by Lemma 7.1

\[
\mathbb{E}R_m - \mathbb{E}R_\ell \geq \mathbb{E}R_{m-\ell} - c_1 \left( \frac{m-\ell}{n} \right)^{1/2} \frac{n}{\log^2 n}.
\]

So if \( \ell \) and \( m \) are consecutive elements of \( \mathcal{A}_i \), similarly to (7.6) we obtain (7.7)

\[
P(\overline{G}_m - \overline{G}_\ell \geq \frac{\varepsilon}{10i^2 \log \log n}) \leq \mathbb{P}\left( \frac{\overline{R}_{m-\ell}}{n/\log^2 n} \geq \frac{\varepsilon}{10i^2 \log \log n - c_1 2^{-i/2}} \right).
\]

For \( n \) large, \( c_1 2^{-i/2} \leq \frac{c_2 \varepsilon}{20i^2} \log \log n \) for all \( i \) and \( n/(m-\ell) = 2^i \), so by Theorem 1.5 the left hand side of (7.7) is less than

\[
P\left( \frac{\overline{R}_{m-\ell}}{(m-\ell)/\log^2 (m-\ell)} \geq \frac{\varepsilon}{40i^2 \log \log n} \right) \leq c_2 \exp \left( -c_3 \frac{\log \log n}{40i^2 2^i} \right).
\]

There are at most \( 2^{i+1} \) such pairs \( \ell, m \), so

\[
w_i := P(\text{for some consecutive elements } \ell, m \in \mathcal{A}_i : \overline{G}_m - \overline{G}_\ell > \frac{\varepsilon}{10i^2 \log \log n})
\]

\[
\leq c_2 2^{i+1} \exp \left( -c_3 \frac{\log \log n}{40i^2 2^i} \right).
\]

Since \( c_3 2^i/40i^2 > 2(\log 2 + i) \log 2 / i \) for \( i \) large, then for \( n \) large enough

\[
w_i \leq c_2 \exp \left( -c_3 \frac{2^i \log \log n}{40i^2} \right).
\]

So then

\[
\sum_{i=1}^{\infty} w_i \leq \frac{1}{2 \log^2 n}
\]

for large \( n \), and this bounds the event that for some \( i \) there exist consecutive elements \( \ell, m \) of \( \mathcal{A}_i \) such that (7.5) holds. Combining with the bound for (7.4), the result follows.

Proof of Theorem 1.7: Let

\[
\Theta = (2\pi)^2 \det(\Gamma)^{-1/2} \kappa(2,2)^{-4}.
\]
Upper bound. Let $\eta, \varepsilon > 0$ and choose $q \in (1, q_0(\varepsilon))$ where $q_0(\varepsilon)$ is as in Lemma 7.3. Let $t_i = [q^i]$. If

$$A_i = \left\{ -\overline{R}_{t_i} > (1 + \eta)\Theta^{-1} t_i \log \log t_i \right\},$$

then by Theorem 1.5 $\sum_i \mathbb{P}(A_i) < \infty$, and hence by Borel-Cantelli, $\mathbb{P}(A_i \text{ i.o.}) = 0$. Let

$$B_i = \left\{ \max_{t_i \leq k \leq t_{i+1}} (\overline{R}_{k+1} - \overline{R}_k) > \varepsilon \frac{t_{i+1} \log \log t_{i+1}}{\log^2 t_{i+1}} \right\}.$$

By Lemma 7.3 $\sum_i \mathbb{P}(B_i) < \infty$, and again $\mathbb{P}(B_i \text{ i.o.}) = 0$. So with probability one, for $k$ large we have $t_i \leq k \leq t_{i+1}$ for some $i$ large, and then

$$-\overline{R}_k = -\overline{R}_{t_{i+1}} + (\overline{R}_{t_{i+1}} - \overline{R}_k) \leq \Theta^{-1}(1 + \eta) \frac{t_{i+1} \log \log t_{i+1}}{\log^2 t_{i+1}} + \varepsilon \frac{t_{i+1} \log \log t_{i+1}}{\log^2 t_{i+1}} \leq q(\Theta^{-1}(1 + 2\eta) + 2\varepsilon) \frac{k \log \log k}{\log^2 k}.$$

Since $\varepsilon, \eta$ can be made as small as we like and we can take $q$ as close to 1 as we like, this gives the upper bound.

Lower bound. Let $\eta > 0$, $t_i = \lfloor \exp(i^{1+\eta}) \rfloor$, $V_i = \#S((t_i, t_{i+1}))$. Let

$$C_i = \left\{ -\overline{V}_i \geq \Theta^{-1}(1 - \eta) \frac{(t_{i+1} - t_i) \log \log(t_{i+1} - t_i)}{\log^2(t_{i+1} - t_i)} \right\}.$$

By Theorem 1.5 $\sum_i \mathbb{P}(C_i) = \infty$. The $C_i$ are independent, and so by Borel-Cantelli, $\mathbb{P}(C_i \text{ i.o.}) = 1$.

Since $R_{t_{i+1}} \leq V_i + R_{t_i}$ and $\mathbb{E}R_{t_{i+1}} \geq \mathbb{E}V_i$, then

$$-\overline{R}_{t_{i+1}} \geq -\overline{V}_i - R_{t_i}.$$

Now

$$R_{t_i} \leq t_i = o\left( \frac{t_{i+1} \log \log t_{i+1}}{\log^2 t_{i+1}} \right)$$

and

$$\frac{(t_{i+1} - t_i) \log \log(t_{i+1} - t_i)}{\log^2(t_{i+1} - t_i)} \sim \frac{t_{i+1} \log \log t_{i+1}}{\log^2 t_{i+1}},$$
so 

\[ -\overline{R_{t+1}} \geq \Theta^{-1}(1 - 2\eta) \frac{t_{i+1} \log \log t_{i+1}}{\log^2 t_{i+1}}, \quad i.o. \]

This implies the lower bound. \( \square \)

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