Some $q$-supercongruences modulo the fifth and sixth powers of a cyclotomic polynomial

Chuanan Wei

School of Biomedical Information and Engineering, Hainan Medical University, Haikou 571199, China
Email address: weichuanan78@163.com

Abstract. In this paper, we establish some $q$-supercongruences modulo the fifth and sixth powers of a cyclotomic polynomial in terms of several summation and transformation formulas for basic hypergeometric series, the creative microscoping method recently introduced by Guo and Zudilin, and the Chinese remainder theorem for coprime polynomials. More concretely, we give a $q$-analogue of Swisher’s result, a $q$-analogue of Liu’s conjecture, a $q$-analogue of a nice formula due to Long and Ramakrishna [Adv. Math. 290 (2016), 773–808], and some $q$-supercongruences involving double series.

Keywords: $q$-supercongruence; creative microscoping method; Chinese remainder theorem for coprime polynomials; basic hypergeometric series; a $q$-analogue of Whipple’s $3F_2$ summation formula; Jackson’s $8\phi_7$ summation formula; Watson’s $8\phi_7$ transformation formula

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1 Introduction

For any nonnegative integer $n$ and complex number $x$, define the shifted-factorial to be

$$(x)_n = \frac{\Gamma(x + n)}{\Gamma(x)},$$

where $\Gamma(x)$ is the famous Gamma function. In his second letter to Hardy on February 27, 1913, Ramanujan mentioned the identity

$$\sum_{k=0}^{\infty} (-1)^k (4k + 1) \left(\frac{1}{2}\right)_k^5 \frac{1}{k!^5} = \frac{2}{\Gamma(3/4)^4}. \quad (1.1)$$

Let $p$ be an odd prime throughout the paper and $\mathbb{Z}_p$ denote the ring of all $p$-adic integers. Define Morita’s $p$-adic Gamma function (cf. [19, Chapter 7]) by

$$\Gamma_p(0) = 1 \quad \text{and} \quad \Gamma_p(n) = (-1)^n \prod_{\substack{1 \leq k < n \\text{p|k}}} k, \quad \text{when} \quad n \in \mathbb{Z}^+.$$
Noting that $N$ is a dense subset of $\mathbb{Z}_p$ associated with the $p$-adic norm $| \cdot |_p$, for each $x \in \mathbb{Z}_p$, the definition of $p$-adic Gamma function can be extended as

$$\Gamma_p(x) = \lim_{n \in N \atop |x-n|_p \to 0} \Gamma_p(n).$$

Two important properties of the function may be expressed as follows:

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } p \nmid x, \\ -1, & \text{if } p | x, \end{cases} \quad \Gamma_p(x)\Gamma_p(1-x) = (-1)^{\langle -x \rangle} p^{-1},$$

where $\langle x \rangle_p$ indicates the least nonnegative residue of $x$ modulo $p$, i.e., $\langle x \rangle_p \equiv x \pmod{p}$ and $\langle x \rangle_p \in \{0, 1, \ldots, p-1\}$.

In 1997, Van Hamme [22, (A.2) and (D. 2)] conjectured a nice $p$-adic analogue of (1.1):

$$\sum_{k=0}^{(p-1)/2} (-1)^k (4k+1) \frac{(1/2)_k^5}{k!^5} \equiv \begin{cases} -\frac{p}{\Gamma_p(3/4)^4} \pmod{p^3}, & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^3}, & \text{if } p \equiv 3 \pmod{4}, \end{cases} \quad (1.2)$$

and the following supercongruence: for $p \equiv 1 \pmod{6}$,

$$\sum_{k=0}^{(p-1)/3} (6k+1) \frac{(1/3)_k^6}{k!^6} \equiv -p\Gamma_p(1/3)^9 \pmod{p^4}. \quad (1.3)$$

In 2015, Swisher [20] showed that (1.2) also holds modulo $p^5$ for $p > 5$ and $p \equiv 1 \pmod{4}$. Recently, Liu [12] proved that, for $p > 5$ and $p \equiv 3 \pmod{4}$,

$$\sum_{k=0}^{(p-1)/2} (-1)^k (4k+1) \frac{(1/2)_k^5}{k!^5} \equiv -\frac{p^3}{16} \Gamma_p(1/4)^4 \pmod{p^4}. \quad (1.4)$$

He further conjectured that (1.4) is true modulo $p^5$. In 2016, Long and Ramakrishna [14, Theorem 2] obtained the generalization of (1.3):

$$\sum_{k=0}^{p-1} (6k+1) \frac{(1/3)_k^6}{k!^6} \equiv \begin{cases} -p\Gamma_p(1/3)^9 \pmod{p^6}, & \text{if } p \equiv 1 \pmod{6}, \\ -\frac{10}{27} p^4 \Gamma_p(1/3)^9 \pmod{p^6}, & \text{if } p \equiv 5 \pmod{6}. \end{cases} \quad (1.5)$$

Some results and conjectures related to (1.5) can be seen in Guo, Liu, and Schlosser [6].

For any complex numbers $x$ and $q$, define the $q$-shifted factorial to be

$$(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = (1-x)(1-xq) \cdots (1-xq^{n-1}) \quad \text{when } n \in \mathbb{Z}^+. $$

For simplicity, we usually adopt the compact notation
Guo’s conjecture (cf. [5]): for any positive integer \( n \) and the Chinese remainder theorem for polynomials. Similarly, the author [26] proved fourth power of a cyclotomic polynomial by using the method of creative microscoping where

\[
\Phi_n(q) = \prod_{1 \leq k \leq n, \gcd(k,n)=1} (q - \zeta^k),
\]

where \( \zeta \) is an \( n \)-th primitive root of unity. Through the creative microscoping method recently introduced by Guo and Zudilin [9], Guo [3] and Wang and Yue [24] gave a \( q \)-analogue of (1.2); for any positive integer \( n \),

\[
\sum_{k=0}^{M} (-1)^k[4k+1] \frac{(q;q^2)_k}{(q^2;q^2)_k} \frac{(q^2;q^4)_k}{(q^4;q^4)_k} q^k \equiv \begin{cases} \left[ n \right] (q^2;q^4)^2_{(n-1)/4} (q^4;q^4)^2_{(n-1)/4} & \text{if } n \equiv 1 \pmod{4}, \\ 0 & \text{if } n \equiv 3 \pmod{4}, \end{cases}
\]

where \( M = (n-1)/2 \) or \( n-1 \). Guo [4] proved three \( q \)-supercongruences modulo the fourth power of a cyclotomic polynomial by using the method of creative microscoping and the Chinese remainder theorem for polynomials. Similarly, the author [26] proved Guo’s conjecture (cf. [5]): for any positive integer \( n \equiv 3 \pmod{4} \) and \( M = (n-1)/2 \) or \( n-1 \),

\[
\sum_{k=0}^{M} (-1)^k[4k+1] \frac{(q;q^2)_k}{(q^2;q^2)_k} \frac{(q^2;q^4)_k}{(q^4;q^4)_k} q^k \equiv \left[ n \right]^2 q^{(1+n)/2} \frac{(q^3;q^4)^{(n-1)/2}}{(q^5;q^4)^{(n-1)/2}} (q;q^4)^{(n-1)/2} \quad \text{(mod } [n]\Phi_n(q)^3)\]

and further conjectured that, for a positive integer \( n \) with \( n \equiv 3 \pmod{4} \) and \( M = (n-1)/2 \) or \( n-1 \),

\[
\sum_{k=0}^{M} (-1)^k[4k+1] \frac{(q;q^2)_k}{(q^2;q^2)_k} \frac{(q^2;q^4)_k}{(q^4;q^4)_k} q^k \equiv \left[ n \right]^2 q^{(1-n)/2} \frac{(q^3;q^4)^{(n-1)/2}}{(q^5;q^4)^{(n-1)/2}} \quad \text{(mod } [n]\Phi_n(q)^4)\]

In this paper, we shall establish the \( q \)-supercongruence including the above conjecture.

**Theorem 1.1.** Let \( n \) be a positive integer. Then, modulo \( [n]\Phi_n(q)^4 \),

\[
\sum_{k=0}^{M} (-1)^k[4k+1] \frac{(q;q^2)_k}{(q^2;q^2)_k} \frac{(q^2;q^4)_k}{(q^4;q^4)_k} q^k \equiv \begin{cases} \left[ n \right] (q^2;q^4)^2_{(n-1)/4} (q^4;q^4)^2_{(n-1)/4} \frac{1}{1 + [n]^2 \sum_{j=1}^{(n-1)/2} \frac{(-1)^{j+1}q^{2j-n}}{[2j]^2}}, & \text{if } n \equiv 1 \pmod{4}, \\ \left[ n \right]^2 q^{(1-n)/2} \frac{(q^3;q^4)^{(n-1)/2}}{(q^5;q^4)^{(n-1)/2}}, & \text{if } n \equiv 3 \pmod{4}, \end{cases}
\]

where \( M = (n-1)/2 \) or \( n-1 \).
Guo and Schlosser [3, Theorem 2.3] discovered a partial \( q \)-analogue of (1.5): for any positive integer \( n \),

\[
\sum_{k=0}^{n-1} [6k + 1] \frac{(q; q^3)_k^6}{(q^3; q^3)_k^6} q^{3k} \equiv \begin{cases} 
0 \mod [n], & \text{if } n \equiv 1 \mod 3, \\
\Phi_n(q), & \text{if } n \equiv 2 \mod 3.
\end{cases} \tag{1.6}
\]

They also proposed the following two conjectures (cf. [8, Conjectures 4.2 and 5.11]):

\[
\sum_{k=0}^{n-1} [2dk + 1] \frac{(aq, q/a, bq, q/b; q^d)_k(q; q^d)_k^2}{(aq^d, q^d/a, bq^d, q^d/b; q^d)_k(q; q^d)_k^2} q^{(2d-3)k} \\
\equiv \begin{cases} 
0 \mod [n]\Phi_n(q), & \text{if } n \equiv -1 \mod d, \\
0 \mod [n], & \text{otherwise},
\end{cases}
\]

where \( n > 0, d \geq 3 \) are integers with \( \gcd(n, d) = 1 \), and

\[
\sum_{k=0}^{n-1} [2dk - 1] \frac{(aq^{-1}, q^{-1}/a, bq^{-1}, q^{-1}/b; q^d)_k(q^{-1}; q^d)_k^2}{(aq^d, q^d/a, bq^d, q^d/b; q^d)_k(q; q^d)_k^2} q^{(2d+3)k} \\
\equiv \begin{cases} 
0 \mod [n]\Phi_n(q), & \text{if } n \equiv 1 \mod d, \\
0 \mod [n], & \text{otherwise},
\end{cases}
\]

where \( n > 1, d \geq 3 \) are integers satisfying \( \gcd(n, d) = 1 \). It should be pointed out that the two conjectures have been proved by Ni and Wang [15]. There are more \( q \)-analogues of supercongruences in the literature, we refer the reader to [2, 7, 10, 11, 13, 21, 25, 27].

In this paper, we shall establish the following two theorems, which extend (1.6) and can be regarded as the further partial \( q \)-analogue of (1.5).

**Theorem 1.2.** Let \( n \) be a positive integer such that \( n \equiv 1 \mod 3 \). Then, modulo \( [n]\Phi_n(q)^4 \),

\[
\sum_{k=0}^{M} [6k + 1] \frac{(q; q^3)_k^6}{(q^3; q^3)_k^6} q^{3k} \equiv [n] \frac{(q^2; q^3)_k^{(n-1)/3}}{(q^3; q^3)_k^{(n-1)/3}} \\
\times \left\{ 1 + [n]^2(2 - q^n) \sum_{j=1}^{(n-1)/3} \frac{q^{3j-1}}{[3j - 1]^2} - \frac{q^{3j}}{[3j]^2} \right\},
\]

where \( M = (n - 1)/3 \) or \( n - 1 \).

**Theorem 1.3.** Let \( n \) be a positive integer such that \( n \equiv 2 \mod 3 \). Then, modulo \( [n]\Phi_n(q)^5 \),

\[
\sum_{k=0}^{M} [6k + 1] \frac{(q; q^3)_k^6}{(q^3; q^3)_k^6} q^{3k} \equiv 5[2n] \frac{(q^2; q^3)_{(2n-1)/3}}{(q^3; q^3)_{(2n-1)/3}}
\]

where \( M = (2n - 1)/3 \) or \( n - 1 \).
Choosing \( n = p^s \) and taking \( q \to 1 \) in Theorem 1.1, we get the supercongruence.

**Corollary 1.4.** Let \( p \) be an odd prime and \( s \) a positive integer. Then, modulo \( p^{s+4} \),

\[
\sum_{k=0}^{m} (-1)^k (4k + 1) \frac{(1/2)^5}{k!^5} = \begin{cases} \\
\frac{(1/2)^2}{(1)^{(p^s-1)/4}} \left\{ p^s + \frac{p^{3s}}{4} H(2)_{(p^s-1)/2} - \frac{p^{3s}}{8} H(2)_{(p^s-1)/4} \right\}, & \text{if } p^s \equiv 1 \pmod{4}, \\
\frac{p^{2s} (3/4)(p^s-1)/2}{(5/4)(p^s-1)/2}, & \text{if } p^s \equiv 3 \pmod{4}, \\
\end{cases}
\]

where \( m = (p^s - 1)/2 \) or \( p^s - 1 \) and the harmonic numbers of \( \ell \)-order are defined by

\[
H(\ell)_m = \sum_{k=1}^{m} \frac{1}{k^\ell} \quad \text{with} \quad \ell, m \in \mathbb{Z}^+.
\]

Fixing \( n = p^s \) and taking \( q \to 1 \) in Theorem 1.2, we arrive at the conclusion.

**Corollary 1.5.** Let \( p \) be an odd prime and \( s \) a positive integer such that \( p^s \equiv 1 \pmod{3} \). Then, modulo \( p^{s+4} \),

\[
\sum_{k=0}^{m} (6k + 1) \frac{(1/3)^6}{k!^6} \equiv \frac{(2/3)^3}{(1)^3(p^s-1)/3} \left\{ p^s + p^{3s} \sum_{j=1}^{(p^s-1)/3} \left( \frac{1}{(3j-1)^2} - \frac{1}{(3j)^2} \right) \right\},
\]

where \( m = (p^s - 1)/3 \) or \( p^s - 1 \).

Setting \( n = p^s \) and taking \( q \to 1 \) in Theorem 1.3, we are led to the formula.

**Corollary 1.6.** Let \( p \) be an odd prime and \( s \) a positive integer such that \( p^s \equiv 2 \pmod{3} \). Then, modulo \( p^{s+5} \),

\[
\sum_{k=0}^{m} (6k + 1) \frac{(1/3)^6}{k!^6} \equiv 10p^s \frac{(2/3)^3}{(1)^3(2p^s-1)/3},
\]

where \( m = (2p^s - 1)/3 \) or \( p^s - 1 \).

In order to explain the equivalence of the \( s = 1 \) case of Corollary 1.4 and Swisher’s result and Liu’s conjecture associated with 1.2, we need to verify the following proposition.

**Proposition 1.7.** Let \( p > 5 \) be an odd prime. If \( p \equiv 1 \pmod{4} \), then

\[
\frac{(1/2)^2}{((p^s-1)/4)^2} \left\{ 1 + \frac{p^2}{4} H(2)_{(p^s-1)/2} - \frac{p^2}{8} H(2)_{(p^s-1)/4} \right\} \equiv -\Gamma_p(1/4)^4 \pmod{p^4}.
\]

If \( p \equiv 3 \pmod{4} \), then

\[
\frac{(3/4)(p^s-1)/2}{(5/4)(p^s-1)/2} \equiv -\frac{p}{16} \Gamma_p(1/4)^4 \pmod{p^3}.
\]
For illustrating the relation of the $s = 1$ case of Corollaries 1.5 and 1.6 and (1.5), we demand to certify the following supercongruences.

**Proposition 1.8.** Let $p$ be an odd prime. If $p \equiv 1 \pmod{6}$, then
\[
\frac{(2/3)^3}{(1)^3} \left\{ 1 + p^2 \sum_{j=1}^{(p-1)/3} \left( \frac{1}{(3j-1)^2} - \frac{1}{(3j)^2} \right) \right\} \equiv -\Gamma_p(1/3)^9 \pmod{p^4}.
\]

If $p \equiv 5 \pmod{6}$, then
\[
\frac{(2/3)^3}{(1)^3} \equiv -\frac{p^3}{27} \Gamma_p(1/3)^9 \pmod{p^5}.
\]

The rest of the paper is arranged as follows. Via the creative microscoping method and the Chinese remainder theorem for coprime polynomials, we shall derive a parametric generalization of Theorem 1.1 and then prove this theorem in Section 2. Similarly, we shall prove Theorems 1.2 and 1.3 in Section 3. The proof of Propositions 1.7 and 1.8 will be provided in Section 4 and two $q$-supercongruences involving double series will be established in Sections 5 and 6.

## 2 Proof of Theorem 1.1

For the sake of proving Theorem 1.1, we require the following proposition.

**Proposition 2.1.** Let $n$ be a positive integer. Then, modulo $(1 - aq^n)(a - q^n)(1 - bq^n)(b - q^n)$,
\[
\sum_{k=0}^{M} (-1)^k [4k + 1] \frac{(aq, q/a, bq, q/b; q^2)_k(q^2; q^4)_k q^k}{(q^2/a, aq^2; q^2/b, bq^2; q^2)_k(q^4; q^4)_k q^k} - \Omega(a, b, n) \frac{(aq^2, bq^2; q^4)(n-1)/4}{(q^4/b; bq^4; q^4)(n-1)/4} + \Omega(b, a, n) \frac{(aq^2, bq^2; q^4)(n-1)/4}{(q^4/a; aq^4; q^4)(n-1)/4} \equiv 0,
\]
\[
\begin{cases}
\Omega(a, b, n) \frac{(aq^2, bq^2; q^4)(n-1)/4}{(q^4/b; bq^4; q^4)(n-1)/4} \\
+ \Omega(b, a, n) \frac{(aq^2, bq^2; q^4)(n-1)/4}{(q^4/a; aq^4; q^4)(n-1)/4} \\
\Omega(a, b, n)(-q) \frac{(b, 1/b; q^4)(n+1)/4}{(q^2/b; bq^2; q^4)(n+1)/4} \\
+ \Omega(b, a, n)(-q) \frac{(a, 1/a; q^4)(n+1)/4}{(q^2/a; aq^2; q^4)(n+1)/4},
\end{cases}
\]

if $n \equiv 1 \pmod{4}$,

if $n \equiv 3 \pmod{4}$,

where $M = (n - 1)/2$ or $n - 1$ and
\[
\Omega(a, b, n) = \left\lfloor \frac{n}{4} \right\rfloor \frac{(aq^{-n})(1 - bq^n)(b - q^n)}{(a - b)(1 - ab)}.
\]
Proof. Firstly, we shall prove the result: modulo \((1 - aq^n)(a - q^n)\),

\[
\sum_{k=0}^{M} (-1)^k [4k + 1] \frac{(aq, q/a, bq, q/b; q^2)_{k}(q^4; q^4)_{k}}{(q^2/a, aq^2, q^2/b, bq^2; q^2)_{k}(q^4; q^4)_{k}} q^k
\]

\[
\equiv \begin{cases} 
[n](bq^2, q^2/b; q^4)_{(n-1)/4}, & \text{if } n \equiv 1 \pmod{4}, \\
[n](-q)(b, 1/b; q^4)_{(n+1)/4}, & \text{if } n \equiv 3 \pmod{4}.
\end{cases} \quad (2.1)
\]

When \(a = q^{-n}\) or \(a = q^n\), the left-hand side of \((2.1)\) is equal to

\[
\sum_{k=0}^{M} (-1)^k [4k + 1] \frac{(q^{1-n}, q^{1+n}, bq, q/b; q^2)_{k}(q^4; q^4)_{k}}{(q^{2-n}, q^{2+n}, q^2/b, bq^2; q^2)_{k}(q^4; q^4)_{k}} q^k
\]

\[
= s\phi_7 \left[ \begin{array}{c} q, \; q^{1/2}, \; q^{-1/2}, \; q^1, \; a, \; q/a, \; c, \; -d, \; -q/d \\
\end{array} \right],
\]

where the basic hypergeometric series (cf. [1]) has been defined by

\[
s_{r+1}\phi_r \left[ \begin{array}{c} a_0, a_1, \ldots, a_r \\
\end{array} \right] = \sum_{k=0}^{\infty} \frac{(a_0, a_1, \ldots, a_r; q)_k}{(q, b_1, \ldots, b_r; q)_k} z^k.
\]

By means of a \(q\)-analogue of Whipple’s \(3F_2\) summation formula (cf. [1] Appendix (II.18)):

\[
s\phi_7 \left[ \begin{array}{c} -c, \; q(-c)^{1/2}, \; -q(-c)^{1/2}, \; a, \; q/a, \; c, \; -d, \; -q/d \\
\end{array} \right] = \frac{(cq/a, cq/d, cdq/a, cq^2/ad; q^2)_{\infty}}{(cd, cq/d, -ac, -cq/a; q)_{\infty}},
\]

the right-hand side of \((2.2)\) can be written as

\[
\begin{cases} 
[n](bq^2, q^2/b; q^4)_{(n-1)/4}, & \text{if } n \equiv 1 \pmod{4}, \\
[n](-q)(b, 1/b; q^4)_{(n+1)/4}, & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

This shows that \((2.1)\) holds modulo \((1 - aq^n)\) or \((a - q^n)\). Since \((1 - aq^n)\) and \((a - q^n)\) are relatively prime polynomials, we obtain \((2.1)\).

Secondly, interchanging the parameters \(a\) and \(b\) in \((2.1)\), we have the formula: modulo
where relatively prime. Noting the relations: 

\[ n \left( \frac{aq^2, q^2/a; q^4}{(q^2/a; aq^2; q^2/bq^2; q^2; q^4)} \right)_k, \quad \text{if } n \equiv 1 \pmod{4}, \]

\[ n \left( -q \frac{(a, 1/a; q^4)_{(n+1)/4}}{(q^2/a; aq^2; q^4)} \right), \quad \text{if } n \equiv 3 \pmod{4}. \]

Finally, it is not difficult to understand that, modulo \((1 - aq^n)(a - q^n)(1 - bq^n)(b - q^n)\),

\[
\sum_{k=0}^{M} (-1)^k [4k + 1] \frac{(aq, q/a, bq, q/b; q^2)_{k}(q^2; q^4)_{k}q^k}{(q^2/a, aq^2, q^2/bq^2; q^2; q^4)_{k}} \equiv \begin{cases} 
\Theta(a, b, n) \left( \frac{bq^2, q^2/b; q^4}{(q^2/b; bq^2; q^4)} \right)_{(n-1)/4}, & \text{if } n \equiv 1 \pmod{4}, \\
\Theta(a, b, n) \left( \frac{aq^2, q^2/a; q^4}{(q^2/a; aq^2; q^4)} \right)_{(n-1)/4}, & \text{if } n \equiv 3 \pmod{4}, 
\end{cases}
\]

where

\[ \Theta(a, b, n) = [n] \frac{(1 - bq^n)(b - q^n)(1 - a^2 + aq^n)}{(a - b)(1 - ab)}. \]

Thirdly, it is clear that the polynomials \((1 - aq^n)(a - q^n)\) and \((1 - bq^n)(b - q^n)\) are relatively prime. Noting the relations:

\[
\frac{(1 - bq^n)(b - q^n)(-1 - a^2 + aq^n)}{(a - b)(1 - ab)} \equiv 1 \pmod{(1 - aq^n)(a - q^n),} \]

\[
\frac{(1 - aq^n)(a - q^n)(-1 - b^2 + bq^n)}{(b - a)(1 - ba)} \equiv 1 \pmod{(1 - bq^n)(b - q^n)}. \]

and employing the Chinese remainder theorem for coprime polynomials, we can deduce, from (2.1) and (2.3), the following \(q\)-supercongruence: modulo \((1 - aq^n)(a - q^n)(1 - bq^n)(b - q^n)\),

\[
\sum_{k=0}^{M} (-1)^k [4k + 1] \frac{(aq, q/a, bq, q/b; q^2)_{k}(q^2; q^4)_{k}q^k}{(q^2/a, aq^2, q^2/bq^2; q^2; q^4)_{k}} \equiv \begin{cases} 
\Theta(a, b, n) \left( \frac{bq^2, q^2/b; q^4}{(q^2/b; bq^2; q^4)} \right)_{(n-1)/4} + \Theta(b, a, n) \left( \frac{aq^2, q^2/a; q^4}{(q^4/a; aq^4; q^4)} \right)_{(n-1)/4}, & \text{if } n \equiv 1 \pmod{4}, \\
\Theta(a, b, n) \left( \frac{-q}{(q^2/b; bq^2; q^4)} \right)_{(n+1)/4} + \Theta(b, a, n) \left( \frac{-q}{(q^2/a; aq^2; q^4)} \right)_{(n+1)/4}, & \text{if } n \equiv 3 \pmod{4}, 
\end{cases}
\]

where

\[ \Theta(a, b, n) = [n] \frac{(1 - bq^n)(b - q^n)(-1 - a^2 + aq^n)}{(a - b)(1 - ab)}. \]
with \( n \equiv 1 \pmod{4} \),

\[
\Theta(a, b, n)(-q)\frac{(b, 1/b; q^4)_{(n+1)/4}}{(q^2/b; bq^2; q^4)_{(n+1)/4}} \equiv \Theta(b, a, n)(-q)\frac{(a, 1/a; q^4)_{(n+1)/4}}{(q^2/a; aq^2; q^4)_{(n+1)/4}}
\]

\[
\equiv \Omega(a, b, n)(-q)\frac{(b, 1/b; q^4)_{(n+1)/4}}{(q^2/b; bq^2; q^4)_{(n+1)/4}} + \Omega(b, a, n)(-q)\frac{(a, 1/a; q^4)_{(n+1)/4}}{(q^2/a; aq^2; q^4)_{(n+1)/4}} \tag{2.8}
\]

with \( n \equiv 3 \pmod{4} \). Utilizing (2.6)–(2.8), we get Proposition 2.1.

Now we display a parametric generalization of Theorem 1.1.

**Theorem 2.2.** Let \( n \) be a positive integer. Then, modulo \([n](1-qa^n)(a-q^n)(1-bq^n)(b-q^n)\),

\[
\sum_{k=0}^{M} (-1)^k[4k + 1] \frac{(aq, q/a, bq, q/b; q^2)_k(q^4)_k}{(q^2/a, aq^2, q^2/b, bq^2; q^4)_k} q^k
\]

\[
= \begin{cases} 
\Omega(a, b, n)\frac{(aq^2, q^2/a; q^4)_{(n-1)/4}}{(q^4/a; aq^2; q^4)_{(n-1)/4}} & \text{if } n \equiv 1 \pmod{4}, \\
+ \Omega(b, a, n)(-q)\frac{(a, 1/a; q^4)_{(n+1)/4}}{(q^2/a; aq^2; q^4)_{(n+1)/4}} & \text{if } n \equiv 3 \pmod{4}, 
\end{cases}
\]

where \( M \) and \( \Omega(a, b, n) \) have appeared in Proposition 2.1.

**Proof.** Ni and Wang [15, Lemma 2.2]) provides

\[
\sum_{k=0}^{n} [2dk + r] \frac{(aq^r, q^r/a, bq^r, q^r/b, q^r/c, q^r; q^d)_k(q^2d-3r)_k}{(q^d/a, aq^d, q^d/b, bq^d, cq^d, q^d)_k} (cq^{2d-3r}_k)^k \equiv 0 \pmod{[n]}, \tag{2.9}
\]

\[
\sum_{k=0}^{n-1} [2dk + r] \frac{(aq^r, q^r/a, bq^r, q^r/b, q^r/c, q^r; q^d)_k(q^2d-3r)_k}{(q^d/a, aq^d, q^d/b, bq^d, cq^d, q^d)_k} (cq^{2d-3r}_k)^k \equiv 0 \pmod{[n]}, \tag{2.10}
\]

where \( n, d \) are positive integers and \( r \) is an integer such that \( 0 \leq \mu \leq n-1 \), \( \gcd(n, d) = 1 \), and \( d \mu = -r \pmod{n} \).

Selecting \( c = -1, d = 2, \mu = (n-1)/2, r = 1 \) in (2.9) and (2.10), we arrive at the conclusion:

\[
\sum_{k=0}^{M} (-1)^k[4k + 1] \frac{(aq, q/a, bq, q/b; q^2)_k(q^4)_k}{(q^2/a, aq^2, q^2/b, bq^2; q^4)_k} q^k \equiv 0 \pmod{[n]} \tag{2.11}
\]

Considering that \((1-qa^n)(1-q^n)(1-bq^n)(b-q^n)\) and \([n]\) are relatively prime polynomials, we derive Theorem 2.2 from Proposition 2.1 and (2.11).
Subsequently, we begin to prove Theorem 1.1.

Proof of Theorem 1.1. When \( n \equiv 1 \pmod{4} \), the \( b \to 1 \) case of Theorem 2.2 reads

\[
\sum_{k=0}^{M} (-1)^k [4k + 1] \left( \frac{(aq, q/a)_k (q; q^2)^2 (q^4)_k}{(q^2/a, aq^2; q^2)_k (q^4; q^4)_k} \right) q^k \\
\equiv \beta(a, n) \pmod{\lfloor n \rfloor \Phi_n(q^2)(1 - aq^n)(a - q^n)), \tag{2.12}
\]

where

\[
\beta(a, n) = [n] \frac{aq^{-n}(1 - q^n)^2 (q^2; q^4)^2_{(n-1)/4}}{(1 - a)^2 (q^4; q^4)^2_{(n-1)/4}} \\
- [n] \frac{q^{-n}(1 - aq^n)(a - q^n) (aq^2, q^2/a, q^4)(n-1)/4}{(1 - a)^2 (q^4/a, aq^4; q^4)(n-1)/4}.
\]

By the L'Hôpital rule, we have

\[
\lim_{a \to 1} \beta(a, n) = [n] \frac{(q^2; q^4)^2_{(n-1)/4}}{(q^4; q^4)^2_{(n-1)/4}} \left\{ 1 + [n] \sum_{i=1}^{(n-1)/2} \left( \frac{-1}{q^2} \right)^{2i-n} \right\} \tag{2.13}
\]

For two nonnegative integer \( s, t \) with \( s \leq t \), it is well known that the \( q \)-binomial coefficient

\[
\left[ \begin{array}{c} t \\ s \end{array} \right] = \frac{(q; q)_t}{(q; q)_s (q; q)_{t-s}}
\]

is a polynomial in \( q \) and

\[
\frac{(q; q^2)_t}{(q^2; q^2)_t} = \frac{1}{(-1; q^2)^2} \left[ \begin{array}{c} 2t \\ t \end{array} \right]. \tag{2.14}
\]

Letting \( a \to 1 \) in (2.12) and using (2.13) and (2.14), we prove that Theorem 1.1 is correct for \( n \equiv 1 \pmod{4} \).

When \( n \equiv 3 \pmod{4} \), the \( b \to 1 \) case of Theorem 2.2 reads

\[
\sum_{k=0}^{M} (-1)^k [4k + 1] \left( \frac{(aq, q/a)_k (q; q^2)^2 (q^4)_k}{(q^2/a, aq^2; q^2)_k (q^4; q^4)_k} \right) q^k \\
\equiv \lambda(a, n) \pmod{\lfloor n \rfloor \Phi_n(q^2)(1 - aq^n)(a - q^n)), \tag{2.15}
\]

where

\[
\lambda(a, n) = [n] \frac{q^{1-n}(1 - aq^n)(a - q^n)(a, 1/a, q^4)(n+1)/4}{(1 - a)^2 (q^2/a, aq^2; q^4)(n+1)/4}.
\]
With the help of the \(q\)-Chu–Vandermonde summation formula (cf. \cite{[1]} Appendix (II.7)):
\[
2\phi_1 \left[ \frac{q^{-n}}{c}; q; \frac{cq^n}{b} \right] = \frac{(c/b; q)_n}{(c; q)_n},
\]
we have
\[
[n]^3 \frac{(q^4; q^4)^2_{(n-3)/4}}{(q^6; q^4)^2_{(n-3)/4}} = [n] \left\{ \sum_{j=0}^{(n-3)/4} (-1)^j q^{2j+2j} \left[ \begin{array}{c} (n - 3)/4 \\ j \end{array} \right] q^j + q^{1+2j} [1 + 2j] \right\}^2,
\]
where
\[
\left[ (n - 3)/4 \right]_q^j = \frac{(q^4; q^4)_{(n-3)/4}}{(q^4; q^4)^j q^4_{(n-3)/4-j}}.
\]
It leads to the relation:
\[
-[n]^3 q^{1-n} \frac{(q^4; q^4)^2_{(n-3)/4}}{(1 + q)^2 (q^6; q^4)^2_{(n-3)/4}} \equiv 0 \pmod {[n]}.
\]
(2.16)
Letting \(a \to 1\) in (2.15) and employing (2.16), we obtain
\[
\sum_{k=0}^{M} (-1)^k [4k + 1] \frac{(q; q^2)^k (q^4; q^4)_k}{(q^2; q^2)^k (q^4; q^4)_k} q^{2k} \equiv -[n]^3 \frac{q^{1-n} (q^4; q^4)^2_{(n-3)/4}}{(1 + q)^2 (q^6; q^4)^2_{(n-3)/4}} \pmod {[n] \Phi_n(q^4)}.
\]
(2.17)
For a positive odd integer \(n\), a known relation due to Wei \cite{[26]} Lemma 2.1] reads
\[
[n]^2 \frac{(q^3; q^4)_{(n-1)/2}}{(q^5; q^4)_{(n-1)/2}} \equiv 0 \pmod {[n]}.
\]
(2.18)
Through (2.16), (2.18), and the \(q\)-supercongruence
\[
\frac{(q^3; q^4)_{(n-1)/2}}{(q^5; q^4)_{(n-1)/2}} = [n] \frac{(1 - q)^2}{(1 - q^{n+2})(1 - q^{n-2})} \frac{(q^3; q^{n+4}; q^4)_{(n-3)/4}}{(q; q^{n+6}; q^4)_{(n-3)/4}}
\]
\[
\equiv [n] \frac{(1 - q)^2}{(1 - q^2)(1 - q^{-2})} \frac{(q^3-n, q^4; q^4)_{(n-3)/4}}{(q^{1-n}, q^6; q^4)_{(n-3)/4}}
\]
\[
= -[n]^3 \frac{q^{(1+n)/2} (q^4; q^4)^2_{(n-3)/4}}{(1 + q)^2 (q^6; q^4)^2_{(n-3)/4}} \pmod {\Phi_n(q^2)},
\]
it is routine to confirm the formula
\[
-[n]^3 q^{1-n} \frac{(q^4; q^4)^2_{(n-3)/4}}{(1 + q)^2 (q^6; q^4)^2_{(n-3)/4}} \equiv [n]^2 q^{(1-n)/2} (q^3; q^4)_{(n-1)/2} \frac{(q^3; q^4)_{(n-1)/2}}{(q^5; q^4)_{(n-1)/2}} \pmod {[n] \Phi_n(q^4)}.
\]
(2.19)
Combining (2.17) and (2.19), we prove that Theorem 1.1 is true for \(n \equiv 3 \pmod{4}\). \(\square\)
3 Proof of Theorems 1.2 and 1.3

In order to prove Theorems 1.2 and 1.3, we need the following proposition.

Proposition 3.1. Let \( n \) be a positive integer such that \( n \equiv t \pmod{3} \). Then, modulo \((1 - aq^n)(a - q^n)(1 - bq^n)(b - q^n)\),

\[
\sum_{k=0}^{T} [6k + 1] \frac{(aq, q/a, bq, q/b; q^3)_k(q; q^3)^2_k}{(q^3/a, aq^3, q^3/b, bq^3; q^3)_k(q; q^3)^2_k} q^{3k} \\
\equiv [tn] \frac{(1 - bq^n)(b - q^n)(-1 - a^2 + aq^n) (bq^2, q^2/b, q^2; q^3)^{(tn-1)/3}}{(a - b)(1 - ab)} \\
+ [tn] \frac{(1 - aq^n)(a - q^n)(-1 - b^2 + bq^n) (aq^2, q^2/a, q^2; q^3)^{(tn-1)/3}}{(b - a)(1 - ba)} \quad (3.1)
\]

where \( T = (tn - 1)/3 \) or \( n - 1 \) and \( t \in \{1, 2\} \).

Proof. When \( a = q^{-tn} \) or \( a = q^n \), the left-hand side of (3.1) is equal to

\[
\sum_{k=0}^{T} [6k + 1] \frac{(q^{1-tn}, q^{1+tn}, bq, q/b; q^3)_k(q; q^3)^2_k}{(q^3+tn, q^3-tn, q^3/b, bq^3; q^3)_k(q; q^3)^2_k} q^{3k} \\
= s\phi_7 \left[ q, q^2, -q^2, q, -q^2, bq, q/b; q^{1+tn}, q^{1-tn} \right] \\
q^2, -q^2, q^3, q^3/b, bq^3, q^{3-tn}, q^{3+tn}; q^3, q^3 \right]. \quad (3.2)
\]

Via Jackson’s \( s\phi_7 \) summation formula (cf. [1] Appendix (II.22)):

\[
s\phi_7 \left[ a, qa^{1/2}, -q^{-1/2}a, b, c, d, e, q^{-n}, \frac{1}{a}, \frac{1}{-a}; \frac{aq/b, aq/c, aq/d, aq/e, aq^{n+1}}{aq/b, aq/c, aq/d, aq/bcd; q_n} \right] \\
= \frac{(aq, aq/bc, aq/bd, aq/cd; q_n)}{(aq/b, aq/c, aq/d, aq/bcd; q_n)},
\]

where \( a^2q = bcdeq^{-n} \), the right-hand side of (3.2) can be stated as

\[
[tn] \frac{(bq^2, q^2/b, q^2; q^3)^{(tn-1)/3}}{(q^3/b, bq^3, q^3; q^3)^{(tn-1)/3}}.
\]

Because \((1 - aq^n)\) and \((a - q^n)\) are relatively prime polynomials, we get the following result: modulo \((1 - aq^n)(a - q^n)\),

\[
\sum_{k=0}^{T} [6k + 1] \frac{(aq, q/a, bq, q/b; q^3)_k(q; q^3)^2_k}{(q^3/a, aq^3, q^3/b, bq^3; q^3)_k(q; q^3)^2_k} q^{3k} \\
\equiv [tn] \frac{(bq^2, q^2/b, q^2; q^3)^{(tn-1)/3}}{(q^3/b, bq^3, q^3; q^3)^{(tn-1)/3}}. \quad (3.3)
\]
Proof. Letting \( n \) be a positive integer such that \( n \equiv 1 \pmod{3} \). Then, modulo \( [n](1 - aq^n)(a - q^n)(1 - bq^n)(b - q^n) \),

\[
\sum_{k=0}^{M}[6k + 1]\frac{(aq, q/a, bq, q/b; q^3)_{k}(q; q^3)^2_k}{(q^3/a, aq^3, q^3/b, bq^3; q^3)_{k}(q^3; q^3)^2_k}q^{3k} \equiv [t_n]\frac{(aq^2, q^2/a, q^2; q^3)_{(tn-1)/3}}{(q^3/a, aq^3, q^3; q^3)_{(tn-1)/3}}. \tag{3.4}
\]

It is clear that the polynomials \((1 - aq^n)(a - q^n)\) and \((1 - bq^n)(b - q^n)\) are relatively prime. Utilizing (2.4) and (2.5) with \( q \rightarrow q^t \) and the Chinese remainder theorem for coprime polynomials, from (3.3) and (3.4) we can deduce the \( q \)-supercongruence (3.1). \( \square \)

Now we are ready to give the parametric generalizations of Theorems 1.2 and 1.3.

**Theorem 3.2.** Let \( n \) be a positive integer such that \( n \equiv 1 \pmod{3} \). Then, modulo \( [n](1 - aq^n)(a - q^n)(1 - bq^n)(b - q^n) \),

\[
\sum_{k=0}^{M}[6k + 1]\frac{(aq, q/a, bq, q/b; q^3)_{k}(q; q^3)^2_k}{(q^3/a, aq^3, q^3/b, bq^3; q^3)_{k}(q^3; q^3)^2_k}q^{3k} \equiv [n]\frac{(1 - bq^n)(b - q^n)(1 - a^2 + aq^n)(bq^2, q^2/b, q^2; q^3)_{(n-1)/3}}{(a - b)(1 - ab)} + [n]\frac{(1 - ak^n)(a - q^n)(1 - b^2 + bq^n)(aq^2, q^2/a, q^2; q^3)_{(n-1)/3}}{(b - a)(1 - ba)}, \tag{3.5}
\]

where \( M = (n - 1)/3 \) or \( n - 1 \).

**Proof.** Letting \( c = 1, d = 3, \mu = (n - 1)/3, r = 1 \) in (2.9) and (2.10), we have

\[
\sum_{k=0}^{M}[6k + 1]\frac{(aq, q/a, bq, q/b; q^3)_{k}(q; q^3)^2_k}{(q^3/a, aq^3, q^3/b, bq^3; q^3)_{k}(q^3; q^3)^2_k}q^{3k} \equiv 0 \pmod{[n]}, \tag{3.5}
\]

where \( n \) is a positive integer satisfying \( n \equiv 1 \pmod{3} \).

Since \((1 - aq^n)(a - q^n)(1 - bq^n)(b - q^n)\) and \([n]\) are relatively prime polynomials, we can find Theorem 3.2 by the \( t = 1 \) case of Proposition 3.1 and (3.5). \( \square \)

**Theorem 3.3.** Let \( n \) be a positive integer such that \( n \equiv 2 \pmod{3} \). Then, modulo \( [n]\Phi_n(q)(1 - aq^{2n})(a - q^{2n})(1 - bq^{2n})(b - q^{2n}) \),

\[
\sum_{k=0}^{M}[6k + 1]\frac{(aq, q/a, bq, q/b; q^3)_{k}(q; q^3)^2_k}{(q^3/a, aq^3, q^3/b, bq^3; q^3)_{k}(q^3; q^3)^2_k}q^{3k} \equiv [2n]\frac{(1 - bq^{2n})(b - q^{2n})(1 - a^2 + aq^{2n})(bq^2, q^2/b, q^2; q^3)_{(2n-1)/3}}{(a - b)(1 - ab)} + [2n]\frac{(1 - ak^{2n})(a - q^{2n})(1 - b^2 + bq^{2n})(aq^2, q^2/a, q^2; q^3)_{(2n-1)/3}}{(b - a)(1 - ba)}, \tag{3.6}
\]

where \( M = (2n - 1)/3 \) or \( n - 1 \).
Proof. A known result due to Ni and Wang [15, Theorem 2.3]) is
\[
\sum_{k=0}^{\nu} [2dk + r] (aq^r, q^r/a, bq^r, q^r/b; q^d)_k (q^d/a, q^d/q, q^d/b, bq^d; q^d)_k (2d-3r)q^3k \equiv 0 \pmod{[n] \Phi_n(q)},
\]
(3.7)
where \( n > 1, d \geq 3 \) are integers, \( r = \pm 1 \), and \( \nu = (dn - n - r)/d \) or \( n - 1 \) such that \( n \geq d - r \), \( \gcd(n, d) = 1 \), and \( n \equiv -r \pmod{d} \). Letting \( d = 3, r = 1 \) in (3.7), we arrive at
\[
\sum_{k=0}^{M} [6k + 1] (aq, q/a, bq, q/b; q^3)_k (q^3/q, q^3/q, q^3/q)_k q^3k \equiv 0 \pmod{[n] \Phi_n(q)},
\]
(3.8)
where \( n \) is a positive integer with \( n \equiv 2 \pmod{3} \). According to the method, which is used to prove Guo [2, Lemma 1], and noting that the factor \((1 - q^n)\) appears in \((q^2; q^3)^{(2n-1)/3}\), it is routine to see that
\[
[2n] (q^2; q^3)^{(2n-1)/3} (q^3; q^3)^{(2n-1)/3} \equiv 0 \pmod{[n] \Phi_n(q)}.
\]
So we prove that (3.6) is correct modulo \([n] \Phi_n(q)\). Some similar discuss will be omitted elsewhere in the paper.

Because \((1 - aq^{2n})(a - q^{2n})(1 - bq^{2n})(b - q^{2n})\) and \([n] \Phi_n(q)\) are relatively prime polynomials, we can establish (3.6) by the \( t = 2 \) case of Proposition 3.1 and the upward conclusion. \( \square \)

Subsequently, we shall display the proof of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. The \( b \to 1 \) case of Theorem 3.2 yields the formula: modulo \([n] \Phi_n(q)^2 (1 - aq^n)/(a - q^n)\),
\[
\sum_{k=0}^{M} [6k + 1] (aq, q/a; q^3)_k (q^3/q)_k q^3k \equiv [n] (1 - q^n)^2 (1 + a^2 - aq^n) (q^2; q^3)^{(n-1)/3} (q^3; q^3)^{(n-1)/3} \]
\[
- [n] (1 - aq^n)(a - q^n) (2 - q^n) (aq^2, q^2/a, q^2; q^3)^{(n-1)/3} \]
\[
- [n] (1 - q^n)^2 (q^2; q^3)^{(n-1)/3} (q^3; q^3)^{(n-1)/3} \]
\[
+ [n] a(1 - q^n)^2 (2 - q^n) (q^2; q^3)^{(n-1)/3} \]
\[
- [n] (1 - aq^n)(a - q^n)(2 - q^n) (aq^2, q^2/a, q^2; q^3)^{(n-1)/3}. \]
(3.9)
By the L'Hôpital rule, we have
\[
\lim_{a \to 1} \left\{ \frac{a(1 - q^n)^2 (q^2; q^3)^{(n-1)/3}}{(1 - a)^2 (q^3; q^3)^{(n-1)/3}} - \frac{(1 - a q^n)(a - q^n)(aq^2, q^2/a; q^3)^{(n-1)/3}}{(1 - a)^2 (q^3/a, aq^3; q^3)^{(n-1)/3}} \right\}
= \left\{ \frac{(q^2; q^3)^2 (n-1)/3}{(q^3; q^3)^{(n-1)/3}} \right\} q^n + [n]^2 \sum_{j=1}^{(n-1)/3} \left( \frac{q^{3j-1}}{[3j - 1]^2} - \frac{q^{3j}}{[3j]^2} \right).
\]

Letting \( a \to 1 \) in (3.9) and employing the above limit, we obtain Theorem 1.2.

**Proof of Theorem 1.3.** The \( b \to 1 \) case of Theorem 3.3 results in the conclusion: modulo \([n]\Phi_n(q)^3(1 - a q^{2n})(a - q^n),\)
\[
\sum_{k=0}^{M} [6k + 1] \frac{(aq, q/a; q^3)_k (q; q^3)_k}{(q^3/a, aq^3; q^3)_k} q^{3k} = [2n](1 - q^{2n})^2 \left( \frac{q^2; q^3}{(2n - 1)/3} \right)^{3/2} \left( \frac{q^3}{(2n - 1)/3} \right)^{3/2} \left( \frac{q^3}{(2n - 1)/3} \right)^{3/2} + [2n] \frac{a(1 - q^{2n})^2 (2 - q^{2n})(q^2, q^3)^{(2n-1)/3}}{(1 - a)^2 (q^3/a, aq^3, q^3)^{(2n-1)/3}}.
\]

By the L'Hôpital rule, we have
\[
\lim_{a \to 1} \left\{ \frac{a(1 - q^{2n})^2 (q^2; q^3)^{(2n-1)/3}}{(1 - a)^2 (q^3; q^3)^{(2n-1)/3}} - \frac{(1 - a q^{2n})(a - q^{2n})(aq^2, q^2/a; q^3)^{(2n-1)/3}}{(1 - a)^2 (q^3/a, aq^3; q^3)^{(2n-1)/3}} \right\}
= \left\{ \frac{(q^2; q^3)^2 (2n-1)/3}{(q^3; q^3)^{(2n-1)/3}} \right\} q^{2n} + [n]^2 \sum_{j=1}^{(2n-1)/3} \left( \frac{q^{3j-1}}{[3j - 1]^2} - \frac{q^{3j}}{[3j]^2} \right).
\]

Letting \( a \to 1 \) in (3.10) and utilizing the upper limit, we get the \( q \)-supercongruence: modulo \([n]\Phi_n(q)^5),\)
\[
\sum_{k=0}^{M} [6k + 1] \frac{(q; q^3)_k^6}{(q^3; q^3)_k^6} q^{3k} \equiv [2n] \frac{(q^2; q^3)^{(2n-1)/3}}{(q^3; q^3)^{(2n-1)/3}} \left( \frac{q^{3j-1}}{[3j - 1]^2} - \frac{q^{3j}}{[3j]^2} \right). \quad (3.11)
\]
It is ordinary to certify the relation:

\[
1 + [2n]^2(2 - q^{2n}) \sum_{j=1}^{(2n-1)/3} \left( \frac{q^{3j-1}}{[3j-1]^2} - \frac{q^{3j}}{[3j]^2} \right)
\]

\[
\equiv 1 + [2n]^2(2 - q^{2n}) q^n [n]^2
\]

\[
\equiv 5 \pmod{\Phi_n(q)^2}.
\]

(3.12)

Considering that the factor \((1 - q^n)\) appears in \((q^2; q^3)_{(2n-1)/3}\), the combination of (3.11) and (3.12) produces Theorem 1.3.

\[\Box\]

4 Proof of Propositions 1.7 and 1.8

Let \(\Gamma_p'(x), \Gamma_p''(x), \) and \(\Gamma_p'''(x)\) respectively be the first derivative, second derivative, and third derivative of \(\Gamma_p(x)\). Now we are going to prove Propositions 1.7 and 1.8.

**Proof of Proposition 1.7** In terms of the properties of the \(p\)-adic Gamma function, we arrive at

\[
\frac{(1/2)^2}{((p-1)/4)!^2} = \left\{ \frac{\Gamma_p((1+p)/4)\Gamma_p(1)}{\Gamma_p((1/2)/4)\Gamma_p(3+p)/4) } \right\}^2
\]

\[
\equiv -\left\{ \Gamma_p(1/4) + \Gamma_p(1/4)\frac{p}{4} + \Gamma_p''(1/4)\frac{p^2}{32} + \Gamma_p'''(1/4)\frac{p^3}{384} \right\}^2
\]

\[
\times \left\{ \Gamma_p(1/4) - \Gamma_p(1/4)\frac{p}{4} + \Gamma_p''(1/4)\frac{p^2}{32} - \Gamma_p'''(1/4)\frac{p^3}{384} \right\}^2
\]

\[
\equiv -\Gamma_p(1/4)\left\{ 1 - \frac{p^2}{8}G_1(1/4)^2 + \frac{p^2}{8}G_2(1/4) \right\} \pmod{p}, \quad (4.1)
\]

where \(G_1(x) = \Gamma_p'(x)/\Gamma_p(x)\) and \(G_2(x) = \Gamma_p''(x)/\Gamma_p(x)\).

Set

\[
S_n^{(0)}(p) = 1, \quad S_n^{(\ell)}(p) = \sum_{1 \leq k_1 < k_2 < \cdots < k_\ell \leq n} \frac{1}{k_1k_2 \cdots k_\ell}, \quad H_n^{(\ell)}(p) = \sum_{1 \leq k \leq n} \frac{1}{k^\ell},
\]

where \(\ell \in \mathbb{Z}^+\). By means of the two relations from H. Pan, Tauraso and Wang [16, Theorem 4.1]:

\[
G_1(1/4) \equiv G_1(0) + S_n^{(1)}(3p^2-3)/4(p) \pmod{p^2},
\]

\[
G_2(1/4) \equiv G_2(0) + 2G_1(0)S_n^{(1)}(3p^2-3)/4(p) + 2S_n^{(2)}(3p^3-3)/4(p) \pmod{p^3}
\]

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and the equation (cf. [23, Lemma 4.3]):
\[ G_2(0) = G_1(0)^2, \]
we are led to
\[ G_2(1/4) - G_1(1/4)^2 \equiv 2\delta_{(3p^2-3)/4}(p) - \delta_{(3p^2-3)/4}(p)^2 \]
\[ \equiv 2\delta_{(3p^2-3)/4}(p) - \delta_{(3p^2-3)/4}(p)^2 \]
\[ \equiv -\mathcal{H}_{(3p^2-3)/4}(p) \]
\[ = - \left( \frac{(3p-7)/4}{p-1} \right) \sum_{j=1}^{p-1} \frac{1}{(pk+j)^2} - \sum_{j=(3p^2-3p)/4+1}^{(3p^2-3p)/4} \frac{1}{j^2} \]
\[ \equiv \frac{3}{4} \mathcal{H}_{p-1} - \mathcal{H}_{(3p^2-3)/4} - \frac{3p}{2} \mathcal{H}_{(3p^2-3)/4} \]
\[ \equiv -\frac{1}{4} \mathcal{H}_{p-1} + \mathcal{H}_{(p-1)/4} + \frac{p}{2} \mathcal{H}_{(p-1)/4} \pmod{p^2}. \quad (4.2) \]

On the basis of (4.1) and (4.2), we can proceed as follows:
\[ \left( \frac{1/2}{(p-1)/4} \right)^2 \left\{ 1 + \frac{p^2}{4} \mathcal{H}_{(p-1)/2} - \frac{p^2}{8} \mathcal{H}_{(p-1)/4} \right\} \]
\[ \equiv -\Gamma_{p}(1/4)^4 \left\{ 1 - \frac{p^2}{32} \mathcal{H}_{p-1} + \frac{p^2}{8} \mathcal{H}_{(p-1)/4} + \frac{p^3}{16} \mathcal{H}_{(p-1)/4} \right\} \]
\[ \times \left\{ 1 + \frac{p^2}{4} \mathcal{H}_{(p-1)/2} - \frac{p^2}{8} \mathcal{H}_{(p-1)/4} \right\} \]
\[ \equiv -\Gamma_{p}(1/4)^4 \left\{ 1 - \frac{p^2}{32} \mathcal{H}_{p-1} + \frac{p^2}{4} \mathcal{H}_{(p-1)/2} + \frac{p^3}{16} \mathcal{H}_{(p-1)/4} \right\} \pmod{p^4}. \quad (4.3) \]

Combing (4.3) and three known supercongruences (cf. [17, Corollaries 5.1 and 5.2] and [18, Corollary 3.4]):
\[ \mathcal{H}_{p-1}^{(2)} \equiv \frac{2p}{3} B_{p-3} \pmod{p^2}, \]
\[ \mathcal{H}_{(p-1)/2}^{(2)} \equiv \frac{7p}{3} B_{p-3} \pmod{p^2}, \]
\[ \mathcal{H}_{(p-1)/4}^{(3)} \equiv -9 B_{p-3} \pmod{p} \text{ for } p > 5, \]
where the Bernoulli numbers \( B_0, B_1, B_2, \ldots \) are integers defined by
\[ B_0 = 1 \text{ and } \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \text{ when } n > 1, \]
we can discover
\[
\frac{(1/2)^{(p-1)/4}}{(p-1)/4)!^2 \left\{ 1 + \frac{p^2}{4} H_{(p-1)/2}^{(2)} - \frac{p^2}{8} H_{(p-1)/4}^{(2)} \right\}
\]
\[\equiv -\Gamma_p(1/4)^4 \left\{ 1 - \frac{p^3}{48} B_{p-3} + \frac{7p^3}{12} B_{p-3} - \frac{9p^3}{16} B_{p-3} \right\}
\]
\[= -\Gamma_p(1/4)^4 \pmod{p^4}.
\]
This completes the proof of (1.7).

With the help of the properties of the \(p\)-adic Gamma function, we have
\[
\frac{(3/4)^{(p-1)/2}}{(5/4)^{(p-1)/2}} = \frac{p}{4} \frac{\Gamma_p((1+2p)/4)\Gamma_p(5/4)}{\Gamma_p((3+2p)/4)\Gamma_p((1-2p)/4)}
\]
\[\equiv -\frac{p}{16} \Gamma_p(1/4)^2 \Gamma_p((1+2p)/4)\Gamma_p((1-2p)/4)
\]
\[\equiv -\frac{p}{16} \Gamma_p(1/4)^2 \left\{ \Gamma_p(1/4) + \Gamma_p'(1/4) \frac{p}{2} \right\} \left\{ \Gamma_p(1/4) - \Gamma_p'(1/4) \frac{p}{2} \right\}
\]
\[\equiv -\frac{p}{16} \Gamma_p(1/4)^4 \pmod{p^3}.
\]
This finishes the proof of (1.8). \(\square\)

The proof of Proposition 1.8 is similar to that of Proposition 1.7. The corresponding details have been omitted here.

5 Generalizations of Theorems 1.2 and 1.3

The main results of this section are the following two theorems.

**Theorem 5.1.** Let \(n, d\) be positive integers and \(r\) an integer such that \(d + n - d \leq r \leq n, \ \gcd(n, d) = 1, \) and \(n \equiv r \pmod{d}.\) Then, modulo \([n]\Phi_n(q)^4,
\[
\sum_{k=0}^{M} [2dk + r] \frac{(q^r; q^d)_k^5 (cq^r; q^d)_k^5}{(q^d; q^d)_k^5 (q^d/c; q^d)_k^5} \left( \frac{q^{2d-3r}}{c} \right)^k
\]
\[\equiv [n](cq^r)^{(r-n)/d} (cq^r)^{(n-r)/d} (q^{d}/c; q^d)^{(n-r)/d}
\]
\[\times \sum_{k=0}^{(n-r)/d} \frac{(q^r; q^d)_k (q^{d-r}; q^d)_k}{(q^d/c; q^d)^{(n-r)/d}} q^{dk}
\]
\[\times \left\{ 1 - [n]^2(2 - q^n) \sum_{j=1}^{k} \left( \frac{q^dj}{[dj]^2} + \frac{q^{dj-d+r}}{[dj-d+r]^2} \right) \right\},
\]
where $M = (n - r)/d$ or $n - 1$.

**Theorem 5.2.** Let $n, d$ be integers such that $n + r \geq d \geq 3$, $\gcd(n, d) = 1$, and $n \equiv -r \pmod{d}$. Then, modulo $[n] \Phi_n(q)^5$,

$$\sum_{k=0}^{M} [2dk + r] \left( \frac{q^r; q^d} {q^d; q^d} \right)_k^n q^{(2d-3r)k}$$

$$\equiv [dn - n]q^{r(n+1-dn)/d} \left( \frac{q^{2r}; q^d} {q^d; q^d} \right)_{dn-1-r}/d$$

$$\times \sum_{k=0}^{(dn-n-r)/d} \left( \frac{q^r; q^d} {q^d; q^d} \right)_k^n q^{dk}$$

$$\times \left\{ 1 - [dn - n] q^{d(r-n-dn)/d} \sum_{j=1}^{k} \left( \frac{q^{dj}} {dj^2} + \frac{q^{dj-d+r}} {dj-d+r} \right) \right\},$$

where $r = \pm 1$ and $M = (n - r)/d$ or $n - 1$.

Theorems 5.1 and 5.2 are respectively the generalizations of Theorems 1.2 and 1.3. Some special cases from them may be displayed as follows.

Choosing $n = p^s$ and taking $c \to 1, q \to 1$ in Theorem 5.1, we obtain the conclusion.

**Corollary 5.3.** Let $d, s$ be positive integers, $p$ an odd prime, and $r$ an integer such that $d + p^s - dp^s \leq r \leq p^s$, $\gcd(p, d) = 1$, and $p^s \equiv r \pmod{d}$. Then, modulo $p^{s+4}$,

$$\sum_{k=0}^{m} (2dk + r) \left( \frac{r/d} {k!^6} \right) \equiv \left( \frac{2r/d} {(p^s - r)/d} \right) (1)_{(p^s - r)/d}$$

$$\times \sum_{k=0}^{(p^s-r)/d} \left( \frac{r/d} {k!^3(2r/d)_k} \right) \left\{ p^s - p^{3s} \sum_{j=1}^{k} \left( \frac{1} {dj^2} + \frac{1} {(dj-d+r)^2} \right) \right\},$$

where $m = (p^s - r)/d$ or $p^s - 1$.

Fixing $n = p^s$ and taking $c \to -1, q \to 1$ in Theorem 5.1 we get the supercongruence.

**Corollary 5.4.** Let $d, s$ be positive integers, $p$ an odd prime, and $r$ an integer such that $d + p^s - dp^s \leq r \leq p^s$, $\gcd(p, d) = 1$, and $p^s \equiv r \pmod{d}$. Then, modulo $p^{s+4}$,

$$\sum_{k=0}^{m} (-1)^k (2dk + r) \left( \frac{r/d} {k!^5} \right) \equiv (-1)^{(r-p^s)/d}$$

$$\times \sum_{k=0}^{(p^s-r)/d} \left( \frac{r/d} {k!^3(1-r/d)_k} \right) \left\{ p^s - p^{3s} \sum_{j=1}^{k} \left( \frac{1} {dj^2} + \frac{1} {(dj-d+r)^2} \right) \right\},$$

where $m = (p^s - r)/d$ or $p^s - 1$.
Setting \( n = p^s \) and taking \( q \to 1 \) in Theorem 5.2, we arrive at the formula.

**Corollary 5.5.** Let \( d, s \) be positive integers and \( p \) an odd prime such that \( p^s + r \geq d \geq 3 \), \( \gcd(p, d) = 1 \), and \( p^s \equiv -r \pmod{d} \). Then, modulo \( p^{s+5} \),

\[
\sum_{k=0}^{m} \frac{(2dk + r)(r/d)^k}{k!^6} \equiv \frac{(2r/d)(dp^r-p^r-1/d)}{(1)(dp^r-p^r-1/d)} \sum_{k=0}^{(dp^r-p^r-1/d)} \frac{(r/d)^3(1-1/d)^k}{k!^3(2r/d)^k}
\]

\[
\times \left\{ (d-1)p^s - (d-1)^3p^{3s} \sum_{j=1}^{k} \left( \frac{1}{(dj)^2} + \frac{1}{(dj-d+r)^2} \right) \right\},
\]

where \( r = \pm 1 \) and \( m = (dp^s - p^s - r)/d \) or \( p^s - 1 \).

To achieve the goal of proving Theorems 5.1 and 5.2 we shall establish the following proposition above all.

**Proposition 5.6.** Let \( n, d \) be positive integers and \( r \) an integer such that \( d + tn - dn \leq r \leq tn \), \( \gcd(n, d) = 1 \), and \( tn \equiv r \pmod{d} \). Then, modulo \( (1-aqtn)(a-qtn)(1-bqtn)(b-qtn) \),

\[
\sum_{k=0}^{T} \frac{(2dk + r)(aq^r/a, bq^r/b, cq^r/c, dq^r/d)}{(q^r/c, q^r/b, dq^r/d)} \left( \frac{q^2d-3r}{c} \right)^k
\]

\[
\equiv [tn](cq^r)^{(r-tn)/d} \frac{(cq^2r; q^d)^{(tn-r)/d}}{(q^r/c; q^r/b)^{(tn-1)/d}}
\]

\[
\times \left\{ \sum_{k=0}^{(tn-r)/d} \frac{(aq^r/a, cq^r/c, dq^r/d)^k}{(bq^r/b, cq^r/c, dq^r/d)^k} \right\},
\]

where \( T = (tn - r)/d \) or \( n - 1 \) and \( t \in \{1, d-1\} \).

**Proof.** When \( a = q^{-tn} \) or \( a = q^{tn} \), the left-hand side of (5.1) is equal to

\[
\sum_{k=0}^{T} \frac{(2dk + r)(aq^r/a, bq^r/b, cq^r/c, dq^r/d)^k}{(q^r/c, q^r/b, dq^r/d)^k} \left( \frac{q^2d-3r}{c} \right)^k
\]

\[
= [r]s \phi_7 \left[ q^r, \ q^{d+r}, \ -q^r, \ aq^r/b, \ cq^r/c, \ dq^r/d \right],
\]

\[
= (aq, aq/de, q)^n \phi_3 \left[ \right].
\]

According to Watson’s \( s \phi_7 \) transformation (cf. [1] Appendix (III.18)):

\[
\phi_7 \left[ a, \ aq^r/b, -aq^r/c, \ b, \ c, \ d, \ e, \ q^{-n}, \ aq^r/b, \ aq^r/c, \ aq^r/d, \ aq^r/e, \ aq^{n-1} ; q, \ a^2q^{n+2} \right]
\]

\[
= (aq/d, aq/e ; q)_n \frac{(aq/b, c, d, e, q^{-n})}{(aq/c, aq/d, aq/e, aq^{n+1} ; q, a^2q^{n+2}; bcde)}.
\]
the right-hand side of (5.2) can be expressed as

\[ [tn](cq^r)^{(r-tn)/d}(cq^{2r}; q^d)_{(tn-r)/d} \sum_{k=0}^{(tn-r)/d} (q^{d-r}, cq^r, q^{r+tn}; q^d)_k q^{dk}. \]

Since \((1 - aq^{tn})\) and \((a - q^{tn})\) are relatively prime polynomials, we find the result: modulo \((1 - aq^{tn})(a - q^{tn})\),

\[
\sum_{k=0}^{T}[2dk + r]\frac{(aq^r, q^r/a, bq^r, q^r/b, cq^r, q^r; q^d)_k}{(q^d/a, aq^d, q^d/b, bq^d, q^d/c, c; q^d)_k} \left( \frac{q^{2d-3r}}{c} \right)^k \equiv [tn](cq^r)^{(r-tn)/d}(cq^{2r}; q^d)_{(tn-r)/d} \sum_{k=0}^{(tn-r)/d} (aq^r, q^r/a, cq^r, q^{d-r}; q^d)_k q^{dk}. \quad (5.3)
\]

Interchanging the parameters \(a\) and \(b\) in (5.3), we are led to the relation: modulo \((1 - bq^{tn})(b - q^{tn})\),

\[
\sum_{k=0}^{T}[2dk + r]\frac{(aq^r, q^r/a, bq^r, q^r/b, cq^r, q^r; q^d)_k}{(q^d/a, aq^d, q^d/b, bq^d, q^d/c, c; q^d)_k} \left( \frac{q^{2d-3r}}{c} \right)^k \equiv [tn](cq^r)^{(r-tn)/d}(cq^{2r}; q^d)_{(tn-r)/d} \sum_{k=0}^{(tn-r)/d} (bq^r, q^r/b, cq^r, q^{d-r}; q^d)_k q^{dk}. \quad (5.4)
\]

Employing (2.4) and (2.5) with \(q \mapsto q^t\) and the Chinese remainder theorem for coprime polynomials, we can derive, from (5.3) and (5.4), the \(q\)-supercongruence (5.1).

Whereafter, we shall give the following parametric generalizations of Theorems 5.1 and 5.2.

**Theorem 5.7.** Let \(n, d\) be positive integers and \(r\) an integer such that \(d + n - dn \leq r \leq tn\), \(\gcd(n, d) = 1\), and \(n \equiv r \pmod{d}\). Then, modulo \([n](1 - aq^n)(a - q^n)(1 - bq^n)(b - q^n)\),

\[
\sum_{k=0}^{M}[2dk + r]\frac{(aq^r, q^r/a, bq^r, q^r/b, cq^r, q^r; q^d)_k}{(q^d/a, aq^d, q^d/b, bq^d, q^d/c, c; q^d)_k} \left( \frac{q^{2d-3r}}{c} \right)^k \equiv [n](cq^r)^{(r-n)/d}(cq^{2r}; q^d)_{(n-r)/d} \sum_{k=0}^{(n-r)/d} (aq^r, q^r/a, cq^r, q^{d-r}; q^d)_k q^{dk}
\]

\[
\times \left\{ \frac{(1 - bq^n)(b - q^n)(-1 - a^2 + aq^n)}{(a - b)(1 - ab)} \right\}^{(n-r)/d} \sum_{k=0}^{(n-r)/d} (bq^r, q^r/b, cq^r, q^{d-r}; q^d)_k q^{dk}
\]

\[
+ \frac{(1 - aq^n)(a - q^n)(-1 - b^2 + bq^n)}{(b - a)(1 - ba)} \sum_{k=0}^{(n-r)/d} (aq^r, q^r/a, cq^r, q^d; q^d)_k q^{dk}
\]

where \(M = (n - r)/d\) or \(n - 1\).
Proof. Because \((1 - ag^n)(a - q^n)(1 - bq^n)(b - q^n)\) and \([n]\) are relatively prime polynomials, we can prove Theorem 5.7 through (2.9), (2.10), and the \(t = 1\) case of Proposition 5.6. □

**Theorem 5.8.** Let \(n, d\) be integers such that \(n + r \geq d \geq 3\), \(\gcd(n, d) = 1\), and \(n \equiv -r \pmod{d}\). Then, modulo \([n]\Phi_n(q)(1 - aq^{dn-n})(a - q^{dn-n})(1 - bq^{dn-n})(b - q^{dn-n})\),

\[
\sum_{k=0}^{M}[2dk + r] \frac{(aq^r, q^r/a, bq^r, q^r/b; q^d)_k(q^r; q^d)_k^2}{(q^d/a, aq^d, q^d/b, bq^d; q^d)_k(q^d; q^d)_k^2} \left(\frac{q^{2d-3r}}{c}\right)^k
\]

\[
\equiv [dn - n]q^{r(r + n - dn)/d} \left(\frac{q^{2r}; q^d}{q^d/c; q^d(dn - 1)/d}\right)
\]

\[
\times \left\{ (1 - bq^{dn-n})(b - q^{dn-n})(-1 - a^2 + aq^{dn-n}) \frac{(dn-n-r)/d}{(a-b)(1-ab)} \sum_{k=0}^{(dn-n-r)/d} \frac{(aq^r, q^r/a, q^r, q^{d-r}; q^d)_k}{(bq^d, q^d/b, q^{2r}, q^d; q^d)_k} q^{dk}
\right. 
\]

\[
+ \frac{(1 - aq^{dn-n})(a - q^{dn-n})(-1 - b^2 + bq^{dn-n})}{(b-a)(1-ba)} \sum_{k=0}^{(dn-n-r)/d} \frac{(aq^r, q^r/b, q^r, q^{d-r}; q^d)_k}{(aq^d, q^d/a, q^{2r}, q^d; q^d)_k} q^{dk}
\right\}.
\]

where \(r = \pm 1\) and \(M = (dn - n - r)/d\) or \(n - 1\).

**Proof.** Since \((1 - aq^{dn-n})(a - q^{dn-n})(1 - bq^{dn-n})(b - q^{dn-n})\) and \([n]\Phi_n(q)\) are relatively prime polynomials, we can establish Theorem 5.8 via (3.7) and the \(c = 1, r = \pm 1, t = d - 1\) case of Proposition 5.6. □

Now we prepare to prove Theorems 5.1 and 5.2.

**Proof of Theorem 5.7.** Letting \(b \to 1\) in Theorem 5.7 we obtain the conclusion: modulo \([n]\Phi_n(q)^2(1 - aq^n)(a - q^n)\),

\[
\sum_{k=0}^{M}[2dk + r] \frac{(aq^r, q^r/a, cq^r, q^d)_k(q^r; q^d)_k^3}{(q^d/a, aq^d, q^d/c; q^d)_k(q^d; q^d)_k^2} \left(\frac{q^{2d-3r}}{c}\right)^k
\]

\[
\equiv [n](cq^r)^{(r-n)/d} \frac{(cq^{2r}; q^d)^{(n-r)/d}}{(q^d/c; q^d)^{(n-r)/d}}
\]

\[
\times \left\{ (1 - q^n)^{(n-r)/d} \sum_{k=0}^{(n-r)/d} \frac{(aq^r, q^r/a, cq^r, q^{d-r}; q^d)_k}{(q^d; q^d)_k^3(cq^{2r}; q^d)_k} q^{dk}
\right. 
\]

\[
+ \frac{a(1 - q^n)^2(2 - q^n)}{(1 - a)^2} \sum_{k=0}^{(n-r)/d} \frac{(aq^r, q^r/a, cq^r, q^{d-r}; q^d)_k}{(q^d; q^d)_k^3(cq^{2r}; q^d)_k} q^{dk}
\]

\[
- \frac{(1 - aq^n)(a - q^n)(2 - q^n)}{(1 - a)^2} \sum_{k=0}^{(n-r)/d} \frac{(q^r; q^d)_k^2(cq^r, q^{d-r}; q^d)_k}{(aq^d, q^d/a, cq^{2r}, q^d; q^d)_k} q^{dk}
\right\}.
\]  \hspace{1cm} (5.5)
By the L'Hôpital rule, we have

\[
\lim_{a \to 1} \left\{ \frac{a(1 - q^n)^2}{(1 - a)^2} \sum_{k=0}^{(n-r)/d} \frac{(aq^r, q^r/a, cq^r, q^{d-r}; q^d)_k}{(q^d; q^d)_k^3(cq^{2r}; q^d)_k} q^{dk} \right\} - \frac{(1 - aq^n)(a - q^n)}{(1 - a)^2} \sum_{k=0}^{(n-r)/d} \frac{(q^r; q^d)_k^2(q^{d-r}; q^d)_k}{(aq^d, q^d/a, cq^{2r}; q^d)_k} q^{dk} \right\} \\
= \sum_{k=0}^{(n-r)/d} \frac{(q^r; q^d)_k^2(q^{d-r}; q^d)_k}{(q^d; q^d)_k^3(cq^{2r}; q^d)_k} q^{dk} \\
\times \left\{ q^n - [n]^2 \sum_{j=1}^{k} \left( \frac{q^{dj}}{[dj]^2} + \frac{q^{dj-d+r}}{[dj-d+r]^2} \right) \right\}.
\]

Letting \(a \to 1\) in (5.5) and using the above limit, we get Theorem 5.1 \(\square\)

**Proof of Theorem 5.2.** Letting \(b \to 1\) in Theorem 5.8, we discover the formula: modulo \([n]\Phi_n(q)^3(1 - aq^{dn-n})(a - q^{dn-n}),\]

\[
\sum_{k=0}^{M} [2dk + r](aq^r, q^r/a; q^d)_k(q^r; q^d)_k^4(q^{2d-3r})_k(k(q^2; q^d)_{(dn-n-r)/d})^2 \\
\equiv [dn - n]q^{r(r+n+dn-n)/d}(q^{2r}; q^d)_{(dn-n-r)/d}(q^d; q^d)_{(dn-n-r)/d} \\
\times \left\{ (1 - q^{dn-n})^2 \sum_{k=0}^{(dn-n-r)/d} \frac{(aq^r, q^r/a, cq^r, q^{d-r}; q^d)_k}{(q^d; q^d)_k^3(cq^{2r}; q^d)_k} q^{dk} \right\} \\
+ \frac{a(1 - q^{dn-n})^2(2 - q^{dn-n})}{(1 - a)^2} \sum_{k=0}^{(dn-n-r)/d} \frac{(aq^r, q^r/a, cq^r, q^{d-r}; q^d)_k}{(q^d; q^d)_k^3(cq^{2r}; q^d)_k} q^{dk} \\
- \frac{(1 - aq^{dn-n})(a - q^{dn-n})(2 - q^{dn-n})}{(1 - a)^2} \sum_{k=0}^{(dn-n-r)/d} \frac{(q^r; q^d)_k^3(q^{d-r}; q^d)_k}{(aq^d, q^d/a, q^{2r}; q^d; q^d)_k} q^{dk} \right\}. \quad (5.6)
\]

By the L'Hôpital rule, we arrive at

\[
\lim_{a \to 1} \left\{ \frac{a(1 - q^{dn-n})^2}{(1 - a)^2} \sum_{k=0}^{(dn-n-r)/d} \frac{(aq^r, q^r/a, cq^r, q^{d-r}; q^d)_k}{(q^d; q^d)_k^3(cq^{2r}; q^d)_k} q^{dk} \\
- \frac{(1 - aq^{dn-n})(a - q^{dn-n})}{(1 - a)^2} \sum_{k=0}^{(dn-n-r)/d} \frac{(q^r; q^d)_k^3(q^{d-r}; q^d)_k}{(aq^d, q^d/a, q^{2r}; q^d; q^d)_k} q^{dk} \right\} \\
\]

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\[(dn-n-r)/d \sum_{k=0} (q^n; q^d)_{2k} (q^{d-r}; q^d)_k q^{dk} \]
\[\times \left\{ q^{dn-n} - [dn-n]^2 \sum_{j=1}^k \left( \frac{q^{dj}}{|dj|^2} + \frac{q^{dj-d+r}}{|dj-d+r|^2} \right) \right\} \]  

Letting \(a \rightarrow 1\) in (5.6) and utilizing the upper limit, we are led to Theorem 5.2.  

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