Bounds on sample size for policy evaluation in Markov environments

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Abstract. Reinforcement learning means finding the optimal course of action in Markovian environments without knowledge of the environment’s dynamics. Stochastic optimization algorithms used in the field rely on estimates of the value of a policy. Typically, the value of a policy is estimated from results of simulating that very policy in the environment. This approach requires a large amount of simulation as different points in the policy space are considered. In this paper, we develop value estimators that utilize data gathered when using one policy to estimate the value of using another policy, resulting in much more data-efficient algorithms. We consider the question of accumulating a sufficient experience and give PAC-style bounds.

1 Introduction

Research in reinforcement learning focuses on designing algorithms for an agent interacting with an environment, to adjust its behavior in such a way as to optimize a long-term return. This means searching for an optimal behavior in a class of behaviors. Success of learning algorithms therefore depends both on the richness of information about various behaviors and how effectively it is used. While the latter aspect has been given a lot of attention, the former aspect has not been addressed scrupulously. This work is the attempt to adapt solutions developed for similar problems in the field of statistical learning theory.

The motivation for this work comes from the fact that, in reality, the process of interaction between the learning agent and the environment is costly in terms of time, money or both. Therefore, it is important to carefully allocate available interactions, to use all available information efficiently and to have an estimate of how informative the experience overall is with respect to the class of possible behaviors. The interaction between agent and environment is modeled by a Markov decision process (MDP) \cite{7, 25}. The learning system does not know the correct behavior, or the true model of the environment it interacts with. Given the sensation of the environment state as an input, the agent chooses the action according to some rule, often called a policy. This action constitutes the output. The effectiveness of the action taken and its effect on the environment is communicated to the agent through a scalar value (reinforcement signal).
The environment undergoes some transformation—changes the current state into a new state. A few important assumptions about the environment are made. In particular, the so-called Markov property is assumed: given the current state and action, the next state is independent of the rest of the history of states and actions. Another assumption is a non-deterministic environment, which means that taking the same action in the same state could lead to a different next state and generate a different payoff signal. It is an objective of the agent to find a behavior which optimizes some long-run measure of payoff, called return.

There are many efficient algorithms for the case when the agent has perfect information about the environment. An optimal policy is described by mapping the last observation into an action and can be computed in polynomial time in the size of the state and action spaces and the effective time horizon [2]. However in many cases the environment state is described by a vector of several variables, which makes the environment state size exponential in the number of variables. Also, under more realistic assumptions, when a model of environment dynamics is unknown and the environment’s state is not observable, many problems arise. The optimal policy could potentially depend on the whole history of interactions and for the undiscounted finite horizon case computing it is PSPACE-complete [18]. In realistic settings, the class of policies is restricted and even among the restricted set of policies, the absolute best policy is not expected to be found due to the difficulty of solving a global multi-variate optimization problem. Rather, the only option is to explore different approaches to finding near-optimal solutions among local optima.

The issue of finding a near-optimal policy from a given class of policies is analogous to a similar issue in supervised learning. There we are looking for a near-optimal hypothesis from a given class of hypotheses [28]. However, there are crucial differences in these two settings. In supervised learning we assume that there is some target function, that labels the examples, and some distribution that generates examples. A crucial property is that the distribution is the same for all the hypotheses. This implies both that the same set of samples can be evaluated on any hypothesis, and that the observed error is a good estimate of the true error.

On the other hand, there is no fixed distribution generating experiences in reinforcement learning. Each policy induces a different distribution over experiences. The choice of a policy defines both a “hypothesis” and a distribution. This raises the question of how one re-uses the experience obtained while following one policy to learn about another. The other policy might generate a very different set of samples (experiences), and in the extreme case the support of the two distributions might be disjoint.

In the pioneering work by Kearns et al. [9], the issue of generating enough information to determine the near-best policy is considered. Using a random policy (selecting actions uniformly at random), they generate a set of history trees. This information is used to define estimates that uniformly converge to the true values. However, this work relies on having a generative model of the environment, which allows simulation of a reset of the environment to any state.
and execute any action to sample an immediate reward. Also, the reuse of information is partial—an estimate of a policy value is built only on a subset of experiences, “consistent” with the estimated policy.

Mansour [11] has addressed the issue of computational complexity in the setting of Kearns et al. [9] by establishing a connection between mistake bounded algorithms (adversarial on-line model [10]) and computing a near-best policy from a given class with respect to a finite-horizon return. Access to an algorithm that learns the policy class with some maximal permissible number of mistakes is assumed. This algorithm is used to generate “informative” histories in the POMDP, following various policies in the class, and determine a near-optimal policy. In this setting a few improvements in bounds are made.

In this work we present a way of reusing all of the accumulated experience without having access to a generative model of the environment. We make use of the technique known as “importance sampling” [26] or “likelihood ratio estimation” [5] to different communities. We discuss properties of different estimators and provide bounds for the uniform convergence of estimates on the policy class. We suggest a way of using these bounds to select among candidate classes of policies with various complexity, similar to structural risk minimization [28].

The rest of this paper is organized as follows. Section 2 discusses reinforcement learning as a stochastic optimization problem. In Section 3 we define our notation. Section 4 presents the necessary background in sampling theory and presents our way of estimating the value of policies. The algorithm and PAC-style bounds are given in Section 5.

## 2 Reinforcement Learning as Stochastic Optimization

There are various approaches to solving RL problems. Value search algorithms find the optimal policy by using dynamic-programming methods to compute the value function—utility of taking a particular action in a particular world state—then deducing the optimal policy from the value function. Policy search algorithms (e.g. REINFORCE [30]) work directly in the policy class, trying to maximize the expected reward without the help of Bellman’s optimality principle.

Policy search methods rely on estimating the value of the policy (or the gradient of the value) at various points in a policy class and attempt to solve the optimization issue. In this paper we ignore the optimization issue and concentrate on the estimation issue—how much and what kind of experience one needs to generate in order to be able to construct uniformly good value estimators over the whole policy class. In particular we would like to know what the relation is between the number of sample experiences and the confidence of value estimates across the policy class.

Two different approaches to optimization can be taken. One involves using an algorithm, driven by newly generated policy value (or gradient thereof) estimates at each iteration to update the hypothesis about the optimal policy after each interaction (or few interactions) with the environment. We will call this on-line optimization. Another is to postpone optimization until all possible interaction
with the environment is exhausted, and combine all information available in order to estimate \textit{(off-line)} the whole “value surface”.

In this paper we are not concerned with the question of optimization. We concentrate on the second case with the goal of building a module that contains a non-parametric model of optimization surface. Given an arbitrary policy such a module outputs an estimate of its value, as if the policy was tried out in the environment. Once such module is built and guarantees on good estimates of policy value are obtained across the policy class, we may use our favorite optimization algorithm. Gradient descent methods, in particular REINFORCE [30, 31] have been used recently in conjunction with policy classes constrained in various ways, e.g., with external memory [21], finite state controllers [14] and in multi-agent settings [20]. Furthermore, the idea of using importance sampling in the reinforcement learning has been explored [13, 24]. However only on-line optimization was considered.

One realistic off-line scenario in reinforcement learning is when the data processing and optimization (learning) module is separated (physically) from the data acquisition module (agent). Say we have an ultra-light micro-sensor connected to a central computer. The agent then has to be instructed initially how to behave when given a chance to interact with the environment for a limited number of times, then bring/transmit the collected data back. Naturally during such limited interaction only a few possible behaviors can be tried out. It is extremely important to be able to generalize from this experience in order to make a judgment about the quality of behaviors which were not tried out. This is possible when some kind of similarity measure in the policy class can be established. If the difference between the values of two policies could be estimated, we could estimate the value of one policy based on experience with the other.

3 Background and Notation

\textbf{MDP} The class of problems described above can be modeled as Markov decision processes (\textsc{mdps}). An \textsc{mdp} is a 4-tuple \( \langle S, A, T, R \rangle \), where: \( S \) is the set of states; \( A \) is the set of actions; \( T : S \times A \rightarrow \mathcal{P}(S) \) is a mapping from states of the environment and actions of the agent to probability distributions\(^1\) over states of the environment; and \( R : S \times A \rightarrow R \) is the payoff function (\textit{reinforcement}), mapping states of the environment and actions of the agent to immediate reward.

\textbf{POMDP} The more complex case is when the agent is no longer able to reliably determine which state of the \textsc{mdp} it is currently in. The process of generating an observation is modeled by an observation function \( B(s(t)) \). The resulting model is a \textit{partially observable Markov decision process} (\textsc{pomdp}). Formally, a \textsc{pomdp} is defined as a tuple \( \langle S, O, A, B, T, R \rangle \) where: \( S \) is the set of states; \( O \) is the set of observations; \( A \) is the set of actions; \( B \) is the observation function \( B : S \rightarrow \mathcal{P}(O) \);

\(^1\) Let \( \mathcal{P}(\Omega) \) denote the set of probability distributions defined on some space \( \Omega \).
\(T : S \times A \rightarrow \mathcal{P}(S)\) is a mapping from states of the environment and actions of the agent to probability distributions over states of the environment; \(R : S \times A \rightarrow \mathcal{R}\) is the payoff function, mapping states of the environment and actions of the agent to immediate reward. In a POMDP, at each time step: an agent observes \(o(t)\) corresponding to \(B(s(t))\) and performs an action \(a(t)\) according to its policy, inducing a state transition of the environment; then receives the reward \(r(t)\). We assume that the rewards \(r(s, a)\) are bounded by \(r_{\text{max}}\) for any \(s\) and \(a\).

**History** We denote by \(H_t\) the set of all possible experience sequences of length \(t\): \(H_t = \{(o(1), a(1), r(1), \ldots, o(t), a(t), r(t), o(t+1))\}\), where \(o(t) \in O\) is the observation of agent at time \(t\); \(a(t) \in A\) is the action the agent has chosen to take at time \(t\); and \(r(t) \in \mathcal{R}\) is the reward received by agent at time \(t\). In order to specify that some element is a part of the history \(h\) at time \(t\), we write, for example, \(r(\tau, h)\) and \(a(\tau, h)\) for the \(\tau^{\text{th}}\) reward and action in the history \(h\).

**Policy** Generally speaking, in a POMDP, a policy \(\pi\) is a rule specifying the action to perform at each time step as a function of the whole previous history: \(\pi : H \rightarrow \mathcal{P}(A)\). Policy class \(\Pi\) is any set of policies. We assume that the probability of the elementary event is bounded away from zero: \(0 \leq \epsilon \leq \Pr(a|h, \pi)\), for any \(a \in A\), \(h \in H\) and \(\pi \in \Pi\).

**Return** A history \(h\) includes several immediate rewards \((r(1) \ldots r(i) \ldots)\), that can be combined to form a return \(R(h)\). In this paper we focus on returns which may be computed (or approximated) using the first \(T\) steps, and are bounded in absolute value by \(R_{\text{max}}\). This includes two well-studied return functions—the undiscounted finite horizon return and the discounted infinite-horizon return. The first is \(R(h) = \sum_{t=0}^{T} r(t, h)\), where \(T\) is the finite-horizon length. In this case \(R_{\text{max}} = Tr_{\text{max}}\). The second is the discounted infinite horizon return \([25]\) \(R(h) = \sum_{t=0}^{\infty} \gamma^t r(t, h)\), with a geometric discounting by the factor \(\gamma \in (0; 1)\). In this case we can approximate \(R\) using the first \(T_\epsilon\) steps we can approximate \(R\) within \(\epsilon\) since \(R_{\text{max}} = \frac{\epsilon}{1-\gamma}\) and \(\sum_{t=0}^{\infty} \gamma^t r(t) = \frac{\sum_{t=0}^{T_\epsilon} \gamma^t r(t)}{1-\gamma} < \epsilon\). It is important to approximate the return in \(T\) steps, since the length of the horizon is a parameter in our bounds.

**Value** Any policy \(\pi \in \Pi\) defines a conditional distribution \(\Pr(h|\pi)\) on the class of all histories \(H\). The value of policy \(\pi\) is the expected return according to the probability induced by this policy on histories space:

\[V(\pi) = E_{\pi} [R(h)] = \sum_{h \in H} [R(h) \Pr(h|\pi)],\]

where for brevity we introduced notation \(E_{\pi}\) for \(E_{\Pr(h|\pi)}\). It is an objective of the agent to find a policy \(\pi^*\) with optimal value: \(\pi^* = \text{argmax}_\pi V(\pi)\). We assume that policy value is bounded by \(V_{\text{max}}\). That means of course that returns are also bounded by \(V_{\text{max}}\) since value is a weighted sum of returns.
4 Sampling

For the sake of clarity we are introducing concepts from sampling theory using functions and notation for relevant reinforcement learning concepts. Rubinstein [26] provides a good overview of this material.

“Crude” sampling If we need to estimate the value $V(\pi)$ of policy $\pi$, from independent, identically distributed (i.i.d.) samples induced by this policy, after taking $N$ samples $h_i, i \in (1..N)$ we have:

$$\hat{V}(\pi) = \frac{1}{N} \sum_i R(h_i).$$

The expected value of this estimator is $V(\pi)$ and it has variance $\text{Var} [\hat{V}(\pi)]$:

$$\frac{1}{N} \sum_{h \in H} R(h)^2 \Pr(h|\pi) - \frac{1}{N} \left[ \sum_{h \in H} R(h) \Pr(h|\pi) \right]^2 = \frac{1}{N} \mathbb{E}_\pi [R(h)^2] - \frac{1}{N} V^2(\pi).$$

Indirect sampling Imagine now that for some reason we are unable to sample from the policy $\pi$ directly, but instead we can sample from another policy $\pi'$. The intuition is that if we knew how “similar” those two policies were to one another, we could use samples drawn according to the distribution $\pi'$ and make an adjustment according to the similarity of the policies. Formally we have:

$$V(\pi) = \sum_{h \in H} R(h_i) \Pr(h_i|\pi) = \sum_{h \in H} R(h_i) \frac{\Pr(h_i|\pi')}{\Pr(h_i|\pi')} \Pr(h_i|\pi') = \mathbb{E}_{\pi'} \left[ R(h_i) \frac{\Pr(h_i|\pi)}{\Pr(h_i|\pi')} \right],$$

where an agent might not be (and most often is not) able to calculate $\Pr(h|\pi)$.

**Lemma 1.** It is possible to calculate $\frac{\Pr(h|\pi)}{\Pr(h|\pi')}$ for any $\pi, \pi' \in \Pi$ and $h \in H$.

**Proof.** The Markov assumption in POMDPs warrants that

$$\Pr(h|\pi) = \Pr(s(0)) \prod_{t=1}^T \Pr(o(t)|s(t)) \Pr(a(t)|o(t), \pi) \Pr(s(t+1)|s(t), a(t))$$

$$= \left[ \Pr(s(0)) \prod_{t=1}^T \Pr(o(t)|s(t)) \Pr(s(t+1)|s(t), a(t)) \right] \prod_{t=1}^T \Pr(a(t)|o(t), \pi))$$

$$= \Pr(h_e) \Pr(h_a|\pi).$$

$\Pr(h_a)$ is the probability of the part of the history, dependent on the environment, that is unknown to the agent and can be only sampled. $\Pr(h_a|\pi)$ is the probability of the part of the history, dependent on the agent, that is known to the agent and can be computed (and differentiated). Therefore we can compute

$$\frac{\Pr(h|\pi)}{\Pr(h|\pi')} = \frac{\Pr(h_e) \Pr(h_a|\pi) \Pr(h_a|\pi)}{\Pr(h_e) \Pr(h_a|\pi')} = \frac{\Pr(h_a|\pi)}{\Pr(h_a|\pi')}.$$

□
We can now construct an indirect estimator $\hat{V}_{\pi'}(\pi)$ from i.i.d. samples $h_i, i \in (1..N)$ according to the distribution $Pr(h|\pi')$

$$\hat{V}_{\pi'}(\pi) = \frac{1}{N} \sum_i R(h_i)w_\pi(h_i, \pi')$$

(1)

where for convenience, we denote the fraction $\frac{Pr(h|\pi)}{Pr(h|\pi')}$ by $w_\pi(h, \pi')$. This is an unbiased estimator of $V(\pi)$ with variance

$$\text{Var}\left[\hat{V}_{\pi'}(\pi)\right] = \frac{1}{N} \left\{ \sum_{h \in H} \left( R(h)w_\pi(h, \pi')^2 Pr(h|\pi') - V(\pi)^2 \right) \right\}$$

$$= \frac{1}{N} \left\{ \sum_{h \in H} \left( \frac{R(h)Pr(h|\pi)}{Pr(h|\pi')} \right)^2 - V(\pi)^2 \right\}$$

$$= \frac{1}{N} E_\pi \left[ R(h)^2 w_\pi(h, \pi') \right] - \frac{1}{N} V(\pi)^2$$

(2)

This estimator $\hat{V}_{\pi'}(\pi)$ is usually called in statistics [26] an importance sampling (is) estimator because the probability $Pr(h|\pi')$ is chosen to emphasize parts of the sampled space that are important in estimating $V$. The technique of is was originally designed to increase the accuracy of Monte Carlo estimates by reducing their variance [26]. Variance reduction is always a result of exploiting some knowledge about the estimated quantity.

**Optimal sampling policy** It can be shown [8], for example by optimizing the expression 2 with Lagrange multipliers, that the optimal sampling distribution is $Pr(h|\pi') = \frac{R(h)Pr(h|\pi)}{V(\pi)}$, which gives an estimator with zero variance. Not surprisingly this distribution can not be used, since it depends on prior knowledge of a model of the environment (transition probabilities, reward function), which contradicts our assumptions, and on the value of the policy which is what we need to calculate. However all is not lost. There are techniques which approximate the optimal distribution, by changing the sampling distribution during the trial, while keeping the resulting estimates unbiased via reweighting of samples, called "adaptive importance sampling" and "effective importance sampling" (see, for example, [15, 32, 17]). In the absence of any information about $R(h)$ or estimated policy, the optimal sampling policy is the one which selects actions uniformly at random: $Pr(a|h) = \frac{1}{2}$. For the horizon $T$, this gives us the upper bound which we denote $\eta$:

$$w_\pi(h, \pi') \leq 2^T (1 - \epsilon) \leq \eta$$

(3)

**Remark.** One interesting observation is that it is possible to get a better estimate of $V(\pi)$ while following another policy $\pi'$. Here is an illustrative example: imagine that reward function $R(h)$ is such that it is zero for all histories in some sub-space $H_0$ of history space $H$. At the same time policy $\pi$, which we are trying to estimate spends almost all the time there, in $H_0$. If we follow $\pi$ in our exploration, we are wasting samples/time! In this case, we can really call what happens "importance sampling", unlike usually when it is just "reweighting", not connected to "importance per se. That is why we advocate using the name "likelihood ratio" rather than "importance sampling". 
Remark. So far, we talked about using a single policy to collect all samples for estimation. We also made an assumption that all considered distributions have equal support. In other words, we assumed that any history has a non-zero probability to be induced by any policy. Obviously it could be beneficial to execute a few different sampling policies, which might have disjoint or overlapping support. There is literature on this so-called stratification sampling technique [26]. Here we just mention that it is possible to extend our analysis by introducing a prior probability on choosing a policy out of a set of sampling policies, then executing this sampling policy. Our sampling probability will become: \( \Pr(h) = \Pr(\pi') \Pr(h|\pi') \).

5 Algorithm and Bounds

Table 1 presents the computational procedure for estimating the value of any policy from the policy class off-line. The sampling stage consists of accumulating histories \( h_i, i \in [1..N] \) induced by a sampling policy \( \pi' \) and calculating returns on these histories \( R(h_i) \). After the first stage is done, the procedure can simulate the interaction with the environment for any policy search algorithm, by returning an estimate for arbitrary policy.

Table 1. Policy evaluation

| Sampling stage: |
|------------------|
| Chose a sampling policy \( \pi' \); |
| Accumulate the set of histories \( h_i, i \in [1..N] \) induced by \( \pi' \); |
| Calculate the set of returns \( R(h_i), i \in [1..N] \); |

| Estimation stage: |
|-------------------|
| Input: policy \( \pi \in \Pi \) |
| Calculate \( w_\pi(h_i, \pi') \) for \( i \in [1..N] \); |
| Output: estimate \( \hat{V}(\pi) \) according to equation 1: \( \frac{1}{N} \sum_i R(h_i)w_\pi(h_i, \pi') \) |

5.1 Sample Complexity

We first compute deviation bounds for the IS estimator for a single policy from its expectation using Bernstein’s inequality:

**Theorem Bernstein [3]** Let \( \xi_1, \xi_2, \ldots \) be independent random variables with identical mean \( \mathbb{E}\xi \), bounded by some constant \( |\xi_i| \leq a \), \( a > 0 \). Also let \( \operatorname{Var}(M_N) = \)
\( E\xi_1^2 + \ldots + E\xi_N^2 \leq L \). Then the partial sums \( M_N = \xi_1 + \ldots + \xi_N \) obey the following inequality for all \( \epsilon > 0 \):

\[
\Pr \left( \left| \frac{1}{N}M_N - E\xi \right| > \epsilon \right) \leq 2 \exp \left( -\frac{\epsilon^2 N}{2L + a\epsilon} \right).
\]

**Lemma 2.** With probability \((1 - \delta)\) the following holds true. The estimated value \( \hat{V}(\pi) \) based on \( N \) samples is close to the true value \( V(\pi) \) for some policy \( \pi \):

\[
\left| V(\pi) - \hat{V}(\pi) \right| \leq \frac{V_{\max}}{N} \left( \log(1/\delta)\eta + \sqrt{2 \log(1/\delta)(\eta - 1) + \log(1/\delta)^2\eta^2} \right).
\]

**Proof.** In our setup, \( \xi_i = R(h_i)w_\pi(h_i, \pi') \), and \( E\xi = E_{\pi'} [R(h_i)w_\pi(h_i, \pi')] = E_\pi [R(h_i)] = V(\pi) \); and \( a = V_{\max} \eta \) by equation 3. According to equation 2 \( L = \text{Var}(M_N) = \text{Var}V_{\pi'}(\pi) \leq \frac{V_{\max}^2}{N}(\eta - 1) \). So we can use Bernstein’s inequality and we get the following deviation bound for a policy \( \pi \):

\[
\Pr \left( \left| V(\pi) - \hat{V}(\pi) \right| > \epsilon \right) \leq 2 \exp \left[ -\frac{\epsilon^2 N}{2 \frac{V_{\max}^2(\eta - 1)}{N} + V_{\max} \eta \epsilon} \right] = \delta,
\]

(4)

After solving for \( \epsilon \), we get the statement of Lemma 2. \( \square \)

Note that this result is for a single policy. We need a convergence result simultaneously for all policies in the class \( \Pi \). We proceed using classical uniform convergence results for covering numbers as a measure of complexity.

**Remark.** We use covering numbers (instead of VC dimension as Kearns et al. [9]) both as a measure of the metric complexity of a policy class in a union bound and as a parameter for bounding the likelihood ratio. Another advantage is that metric entropy is a more refined measure of capacity than VC dimension since the VC dimension is an upper bound on the growth function which is an upper bound on the metric entropy [28].

**Definition 3.** Let \( \Pi \) be class of policies that form a metric space with metric \( D_\infty(\pi, \pi') \) and \( \epsilon > 0 \). The covering number \( \mathcal{N}(\Pi, D, \epsilon) \) is defined as the minimal integer \( \ell \) such that there exist \( \ell \) disks in \( \Pi \) with radius \( \epsilon \) covering \( \Pi \). If no such partition exists for some \( \epsilon > 0 \) then the covering number is infinite. The metric entropy is defined as \( K(\Pi, D, \epsilon) = \log \mathcal{N}(\Pi, D, \epsilon) \).

**Theorem 4.** With probability \( 1 - \delta \) the difference \( |V(\pi) - \hat{V}(\pi)| \) is less than \( \epsilon \) simultaneously for all \( \pi \in \Pi \) for the sample size:

\[
N = O \left( \frac{V_{\max}}{\epsilon} 2^T(1 - \xi)^T \log(1/\delta) + K \right).
\]
Proof. Given a class of policies $\Pi$ with finite covering number $N(\Pi, D, \epsilon)$, the upper bound $\eta = 2T(1 - \epsilon)T$ on the likelihood ratio, and $\epsilon > 0$,

$$\Pr \left( \sup_{\pi \in \Pi} \left| V(\pi) - \hat{V}(\pi) \right| > \epsilon \right) \leq 8N(\Pi, D, \frac{\epsilon}{8}) \exp \left[ -\frac{1}{128} \frac{\epsilon^2 N}{V_{\max}^2(\eta-1)} + \frac{\epsilon V_{\max}}{8} \right].$$

Note the relationship to equation 4. The only essential difference is in the covering number, which takes into account the extension from a single policy $\pi$ to the class $\Pi$. This requires the sample size $N$ to increase accordingly to achieve the given confidence level. The derivation is similar to uniform convergence result of Pollard [22](see pages 24-27), using Bernstein’s inequality instead of Hoeffding’s. Solving for $N$ gives us the statement of the theorem. 

Let us compare our result with a similar result for algorithm by Kearns et al. [9]:

$$N = O \left( \left( \frac{V_{\max}}{\epsilon} \right)^2 2^{2T} VC(\Pi) \log(T) \left( T + \log(V_{\max}/\epsilon) + \log(1/\delta) \right) \right) \quad (5)$$

both dependences are exponential in the horizon, however in our case the dependence on $\left( \frac{V_{\max}}{\epsilon} \right)$ is linear rather than quadratic. The metric entropy $\log(N)$ takes the place of the VC dimension $VC(\Pi)$ in terms of class complexity. This reduction in a sample size could be explained by the fact that the former algorithm uses all trajectories for evaluation of any policy, while the latter uses just a subset of trajectories.

Remark. Let us note that a weaker bound which is remarkably similar to the equation 5 could be obtained [19] using Mc-Diarmid [12] theorem, applicable for a more general case:

$$N = O \left( \left( \frac{V_{\max}}{\epsilon} \right)^2 2^{2T} (1 - \epsilon)^{2T} (K + \log(1/\delta)) \right).$$

The proof is based on the fact that replacing one history $h_i$ in the set of samples $h_i, i \in (1..N)$ for the estimator $\hat{V}_T(\pi)$ of equation 1, can not change the value of the estimator by more than $\frac{V_{\max}}{N}$.

5.2 Bounding the Likelihood Ratio

We would like to find a way to estimate a policy which minimizes sample complexity. Remember that we are free to choose a sampling policy. We have discussed what it means for one sampling policy $\pi$ to be optimal with respect to another. Here we would like to consider what it means for a sampling policy $\pi$ to be optimal with respect to a policy class $\Pi$. Choosing the optimal sampling policy allows us to improve bounds with regard to exponential dependence on the horizon $T$. The idea is that if we are working with a policy class of a finite metric dimension, the likelihood ratio can be upper bounded through the
covering number due to the limit in combinatorial choices. The trick is to consider sample complexity for the case of the sampling policy being optimal in the information-theoretic sense.

This derivation is very similar to the one of an upper bound on the minimax regret for predicting probabilities under logarithmic loss [4, 16]. The upper bounds on logarithmic loss we use were first obtained by Opper and Haussler [16] and then generalized by Cesa-Bianchi and Lugosi [4]. The result of Cesa-Bianchi and Lugosi is more directly related to the reinforcement learning problem since it applies to the case of arbitrary rather than static experts, which corresponds to learning a policy. First, we describe the sequence prediction problem and result of Cesa-Bianchi and Lugosi, then show how to use this result in our setup.

In a sequential prediction game $T$ symbols $h^T_a = \langle a(1), \ldots, a(T) \rangle$ are observed sequentially. After each observation $a(t-1)$, a learner is asked how likely it is for each value $a \in A$ to be the next observation. The learner goal is to assign a probability distribution $\Pr(a(t) | h^{t-1}_a; \pi')$ based on the previous values. When at the next time step $t$, the actual new observation $a(t)$ is revealed, the learner suffers a loss $-\log(\Pr(a(t) | h^{t-1}_a; \pi'))$. At the end of the game, the learner has suffered a total loss $-\sum_{t=1}^{T} \log \Pr(a(t) | h^{t-1}_a; \pi')$. Using the joint distribution $\Pr(h^T_a | \pi') = \prod_{t=1}^{T} \Pr(a(t) | h^{t-1}_a; \pi')$ we are going to write the loss as $-\log \Pr(h^T_a | \pi')$. When it is known that the sequences $h^T_a$ are generated by some probability distribution $\pi$ from the class $\Pi$, we might ask what is the worst regret: the difference in the loss between the learner and the best expert in the target class $\Pi$ on the worst sequence:

$$R_T = \inf_{\pi'} \sup_{h^T_a} \left\{ -\log \Pr(h^T_a | \pi') + \sup_{\pi \in \Pi} \log \Pr(h^T_a | \pi) \right\}.$$ 

Using the explicit solution to the minimax problem due to Shitari[27] Cesa-Bianchi and Lugosi prove the following theorem:

**Theorem Cesa-Bianchi and Lugosi [4] theorem 3** For any policy class $\Pi$:

$$R_T \leq \inf_{\epsilon > 0} \left( \log \mathcal{N}(\Pi, D, \epsilon) + 24 \int_{0}^{\epsilon} \sqrt{\log \mathcal{N}(\Pi, D, \tau)} d\tau \right),$$

where covering number and metric entropy for the class $\Pi$, are defined using the distance measure $D_\infty(\pi, \pi') = \sup_{a \in A} | \log \Pr(a | \pi) - \log \Pr(a | \pi') |$.

It is now easy to relate the problem of bounding the likelihood ratio to the worst case regret. Intuitively, we are asking what is the worst case likelihood ratio if we have the optimal sampling policy. Optimality means that our sampling policy will induce action sequences with probabilities close to the estimated policies. Remember that likelihood ratio depends only on actions sequence $h_a$ in the history $h$ according to the Lemma 1. We need to upper bound the maximum value of the ratio $\frac{\Pr(h_a | \pi)}{\Pr(h_a | \pi')}$, which corresponds to $\inf_{\pi'} \sup_{h_a} \left( \frac{\Pr(h_a | \pi)}{\Pr(h_a | \pi')} \right)$. 

Lemma 5. By the definition of the maximum likelihood policy \( \sup_{\pi \in \Pi} \Pr(h_a|\pi) \) we have:

\[
\inf_{\pi'} \sup_{h_a} \left( \frac{\Pr(h_a|\pi)}{\Pr(h_a|\pi')} \right) \leq \inf_{\pi'} \sup_{h_a} \left\{ \frac{\sup_{\pi \in \Pi} \Pr(h_a|\pi)}{\Pr(h_a|\pi')} \right\}.
\]

Henceforth we can directly apply the results of Cesa-Bianchi and Lugosi and get a bound of \( e^{RT} \). Note the logarithmic dependence of the bound on \( RT \) with respect to the covering number \( N \). Moreover, since actions \( a \) belong to the finite set of actions \( A \), many of the remarks of Cesa-Bianchi and Lugosi regarding finite alphabets apply [4]. In particular, for most “parametric” classes—which can be parametrized by a bounded subset of \( \mathbb{R}^n \) in some “smooth” way [4]—the metric entropy scales as follows: for some positive constants \( k_1 \) and \( k_2 \),

\[
\log \mathcal{N}(\Pi, D, \epsilon) \leq k_1 \log \frac{k_2 \sqrt{T}}{\epsilon}.
\]

For such policies the minimax regret can be bounded by

\[
RT \leq \frac{k_1}{2} \log T + o(\log T),
\]

which makes the likelihood ratio bound of \( \eta = O((T)\frac{k_1}{\epsilon}) \). In this case exponential dependence on the horizon is eliminated and the sample complexity bound becomes

\[
N = O \left( \frac{V_{\max}}{\epsilon} T \frac{k_1}{\epsilon} (\mathcal{K} + \log 1/\delta) \right).
\]

6 Discussion and Future Work

In this paper, we developed value estimators that utilize data gathered when using one policy, to estimate the value of using another policy, resulting in data-efficient algorithms. We considered the question of accumulating a sufficient experience and gave PAC-style bounds. Note that for these bounds to hold the covering number of the class of policies \( \Pi \) should be finite.

Armed with the theorem 4 we are ready to answer a very important question of how to choose among several candidate policy classes. Our reasoning here is similar to that of structural risk minimization principal by Vapnik [28]. The intuition is that given a very limited data, one might prefer to work with a primitive class of hypotheses with good confidence, rather than getting lost in a sophisticated class of hypotheses due to low confidence. Formally, we would have the following method: given a set of policy classes \( \Pi_1, \Pi_2, \ldots \) with corresponding covering numbers \( \mathcal{N}_1, \mathcal{N}_2, \ldots \), a confidence \( \delta \) and a number of available samples \( N \), compare error bounds \( \epsilon_1, \epsilon_2, \ldots \) according to the theorem 4. Another way to utilize the result of theorem 4 is to find what is the minimal experience necessary to be able to provide the estimate for any policy in the class with a given confidence. This work also provides insight for a new optimization technique. Given the value estimate, the number of samples used, and the covering number
of the policy class, one can search for optimal policies in a class using a new cost function $\hat{V}(\pi) + \Phi(N, \delta, N) \leq V(\pi)$. This is similar in spirit to using structural risk minimization instead of empirical risk minimization.

The capacity of the class of policies is measured by bounds on covering numbers in our work or by VC-dimension in the work of Kearns et al. [9]. The worst case assumptions of these bounds often make them far too loose for practical use. An alternative would be to use more empirical or data dependent measures of capacity, e.g. the empirical VC dimension [29] or maximal discrepancy penalties on splits of data [1], which tend to give more accurate results.

We are currently working on extending our results for the \textit{weighted importance sampling} (wis) estimator [23, 26] which is a biased but consistent estimator and has a better variance for the case of small number of samples. This can be done using martingale inequalities by Mc-Diarmid [12] to parallel Bernstein’s result. There is room for employing various alternative sampling techniques, in order to approximate the optimal sampling policy, for example one might want to interrupt uninformative histories, which do not bring any return for a while. Another place for algorithm sophistication is sample pruning for the case when the set of histories gets large. A few most representative samples can reduce the computational cost of estimation.

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