DERIVED CATEGORIES OF NODAL DEL PEZZO THREEFOLDS

NEBOJSA PAVIC AND EVGENY SHINDER

Dedicated to Yuri Gennadievich Prokhorov

Abstract. We give a complete answer for the existence of Kawamata type semiorthogonal decompositions of derived categories of nodal del Pezzo threefolds. More precisely, we show that nodal del Pezzo threefolds of degree $1 \leq d \leq 4$ have no Kawamata type decomposition and that all nodal del Pezzo threefolds of degree 5 admit a Kawamata decomposition. For the proof we go through the classification of singular del Pezzo threefolds, compute divisor class groups of nodal del Pezzo threefolds of small degree and use projection from a line to construct Kawamata semiorthogonal decompositions for the degree 5 case. An analogous decomposition of the nodal del Pezzo threefold of degree 6 has been recently constructed by Kawamata.

Our construction of the Kawamata decomposition for a singular del Pezzo threefold of degree 5 fits into a family of semiorthogonal decompositions (which we call a relative tilting decomposition) interpolating between a Kawamata decomposition on a singular fiber and a full exceptional collection on the smooth fibers.

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1. Introduction

Derived categories of coherent sheaves, their equivalences and semiorthogonal decompositions provide a homological algebra counterpart of the Minimal Model Program [BO95, BO02]. Namely, flops are expected to give rise to derived equivalences, while flips, divisorial contractions and Mori fiber spaces, as well as Sarkisov links between them often correspond to semiorthogonal decompositions.
Starting from smooth varieties, singularities appear naturally and can not be avoided in the Minimal Model Program. However, derived categories of coherent sheaves are currently mostly understood in the smooth projective case with the singular case providing both technical and conceptual difficulties. For instance, derived categories of smooth Fano threefolds are well-understood, see [Kuz16], but the singular case is much less clear, and constitutes an area of active current research. Derived categories of singular del Pezzo surfaces have recently received a lot of attention [Kuz21, KKS22, Xie21], and for singular Fano threefolds some sporadic nontrivial examples have been constructed by Kawamata [Kaw18, Kaw22b, Kaw22a]. Semiorthogonal decompositions of singular varieties have been related to their smoothings for surfaces [Kaw24] and Fano threefolds [KS25].

One notable conceptual difficulty of constructing semiorthogonal decompositions of singular varieties is appearance of obstructions coming from algebraic K-theory, in the form of the Brauer group [KKS22] or the $K_{−1}$ group [KPS21], which do not appear in the smooth case. In some situations, such as for toric surfaces a semiorthogonal decomposition of a certain type of the derived category can be constructed as soon as the K-theoretic obstruction vanishes [KKS22].

In this paper we study semiorthogonal decompositions of derived categories of nodal projective threefolds in the framework of Kawamata semiorthogonal decompositions [KPS21], which are certain generalizations of full exceptional collections to a singular variety, see Definition 3.3. We concentrate on nodal del Pezzo threefolds, that is projective Fano threefolds $X$ with $-K_X = 2H$ for some $H \in \text{Pic}(X)$, having ordinary double points. Del Pezzo threefolds have been classified by Iskovskikh, as part of the project of classifying smooth Fano threefolds [Isk79] and by Fujita, as part of the project of classifying possibly singular higher-dimensional del Pezzo varieties, see [IP99, Chapter 3] for an overview. The degree $d := H^3$ of del Pezzo threefolds satisfies $1 \leq d \leq 8$, and nodal ones exist in degrees $1 \leq d \leq 6$. Typical examples of del Pezzo threefolds are a cubic hypersurface in $\mathbb{P}^4$ ($d = 3$) and a complete intersection of two quadrics in $\mathbb{P}^5$ ($d = 4$), see Theorem 2.1 for the complete classification. The complexity of the geometry of del Pezzo threefold $X$ depends on its degree which sits in one of three bands, cf. [Pro13]: $1 \leq d \leq 3$ (complicated: smooth $X$ is irrational, but singular degenerations can be rational), $4 \leq d \leq 6$ (interesting and well-behaved: $X$ rational, singularities understood, cf. Corollary 2.12), $d \geq 7$ (trivial: rational, smooth and rigid). These bands roughly match the complexity of the corresponding del Pezzo surface hyperplane section.

We completely characterize nodal del Pezzo threefolds having a Kawamata semiorthogonal decomposition of their derived category. Our main result is the following.

**Theorem 1.1** (see Corollary 3.12 for other equivalent statements). Let $X$ be a nodal (non smooth) del Pezzo threefold. The following condition are equivalent:

1. $\mathbf{D}^b(X)$ admits a Kawamata decomposition
2. $\text{rk}(\text{Cl}(X)) = \text{rk}(\text{Pic}(X)) + |\text{Sing}(X)|$
3. $H^3 \in \{5, 6\}$

Condition (2) says that $X$ has as many Weil divisors as the singularities allow for (see (2.2)).

Theorem 1.1 provides a complete match between obstructions to Kawamata decompositions coming from the $K_{−1}$ group and Weil divisors as developed in [KPS21] and the possibility of constructing such a decomposition when these obstructions are trivial, which is analogous to the case of toric surfaces [KKS22]. Regarding condition (3), we note that a smooth del Pezzo threefold $X$ of degree
$d$ has a full exceptional collection if and only if $d \geq 5$ (equivalently when $h^{1,2}(X) = 0$, cf. Proposition 3.6]), and as we exclude $d \geq 7$ cases which do not have singular degenerations, Kawamata decompositions in (1) can be considered precisely as limits of exceptional collections from the smooth del Pezzo threefolds, see Remark 3.17.

Theorem 1.1 involves a mixture of algebraic K-theory, computing class groups of nodal threefolds and classification of Fano varieties. The proof roughly goes as follows. The implication (1) $\implies$ (2) is proved using vanishing of the negative algebraic K-theory groups \cite{KPS21}, in (2) $\implies$ (3) we need the notion of defect of a linear system for computing class groups \cite{Cyn01, Ram08}, and in (3) $\implies$ (1) a construction of a semiorthogonal decomposition based on the geometry of $X$ is required. Existence of Kawamata decompositions for the nodal del Pezzo threefold of degree 6 goes back to the original work of Kawamata \cite{Kaw22b}. The main new ingredient in this paper is existence of Kawamata decompositions for nodal del Pezzo threefolds of degree 5, which was conjectured in \cite{KPS21, Conjecture 1.3}.

Let us explain our approach to the derived category of nodal del Pezzo threefolds $X$ of degree 5. It is well known that $X$ is a linear section of $\text{Gr}(2, 5) \subset \mathbb{P}^9$ and that there are precisely three such threefolds up to isomorphism, with one, two or three nodes respectively, see e.g. \cite{Pro13}. Following Iskovskikh \cite{Isk79} we project $X$ from a sufficiently general line $L \subset X$. The resulting birational map identifies the blow up of $X$ at $L$ with a blow up of a quadric threefold $Q \subset \mathbb{P}^4$ along a nodal curve $C \subset Q$ of arithmetic genus zero and degree three (see Theorem 2.11 for a more general statement which applies to degenerations of del Pezzo threefolds of degree $4 \leq d \leq 6$). Then the derived category of $X$ will be decomposed using mutations of semiorthogonal decompositions coming from these blow ups. The same approach for degree 4 nodal del Pezzo threefolds leads to a decomposition of their derived category in terms of a copy of a nodal associated curve of genus 2, see Theorem 3.10 for both $d = 4, 5$ cases. Moreover, using this method one can also recover Kawamata’s decomposition of the derived category of the nodal $V_6$ as in \cite{Kaw22b}, see Remark 3.18.

Our construction works in any family of del Pezzo threefolds of degree 5 producing what we call a relative tilting decomposition which restricts to a full exceptional collection of objects on a smooth fibers and to a Kawamata semiorthogonal decomposition on the singular ones. In this sense our Kawamata semiorthogonal decomposition on a singular variety can be understood as a limit of a full exceptional collection from a smoothing, see Remark 3.17.

We remark that in terms of the Minimal Model Program the geometric input for our semiorthogonal decompositions is provided by the so-called Sarkisov links (for us these are simply certain compositions of a blow up and a blow down) with non-smooth lci (nodal) centers, and our point is that from the derived categories perspective, producing nodal threefolds by blowing up and blowing down with centers being nodal curves in the smooth locus is completely analogous to using smooth blow ups. It follows from Theorem 2.11 that all nodal del Pezzo threefolds of degrees 4, 5 and 6 can be produced by this construction. In general, it is not clear which nodal threefolds can be obtained from smooth ones by such nodal blow ups and blow downs, which is a question interesting in its own right, as well as in relation to degenerations of intermediate Jacobians and to rationality problems. Furthermore, it seems to us that existence of Kawamata decompositions has to do with the lci Sarkisov links as explained above, similarly to how full exceptional collections are supposed to correspond to varieties which are rational, or close to rational. We hope to return to these questions in the future.
Relation to other work. The main geometric input in this paper, the idea of blowing up singular curves to produce Gorenstein terminal threefolds, belongs to Yuri Prokhorov [Pro13, Pro17]. The study of semiorthogonal decompositions of singular varieties has been initiated by Kuznetsov and Kawamata, see [Kaw18, Kuz21, KKS22, Xie21, KPS21, Kaw22a, Kaw24] for recent developments. The case of nodal del Pezzo threefolds of degree 5 has been independently done by Fei Xie [Xie23] by analyzing small resolutions rather than performing nodal blow ups. The method of [Xie23] is quite different to ours as we do not resolve singularities but preserve them in birational transformations. A more general notion than Kawamata semiorthogonal decomposition is the so-called categorical absorption [KS23, KS25, KKP24].

Acknowledgements. E.S. was partially supported by the EPSRC grant EP/T019379/1 Derived categories and algebraic K-theory of singularities. We thank Ivan Cheltsov, Martin Kalck, Alexander Kuznetsov, Yuri Prokhorov, Michael Wemyss, Fei Xie for discussions and interest in our work. We thank Alice Rizzardo and Theo Raedschelders for organizing the 2019 Liverpool workshop “The Geometry of Derived Categories” where our previous work [KPS21] has been presented and this work has been conceived.

Conventions and notation. We work over an algebraically closed field $k$ of characteristic zero. Unless specified otherwise, all our varieties are quasiprojective. By $D^b(X)$ we mean the bounded derived category of coherent sheaves on $X$ and $D^{perf}(X)$ is the subcategory of $D^b(X)$ consisting of perfect complexes. More generally, we denote by $D^b(X, R)$ the bounded derived category of coherent sheaves of right $R$-modules, for a locally free sheaf of algebras $R$ on $X$. All functors such as pull-back $\pi^*$, pushforward $\pi_*$ and tensor product $\otimes$ when considered between derived categories are derived functors.

By letters $D, E, W, X, Y$ and similar in the standard font we denote $k$-algebraic varieties. By letters $\mathbf{D}, \mathbf{E}, \mathbf{W}, \mathbf{X}, \mathbf{Y}$ in typed in boldface we denote proper and flat (but not always smooth) families over a smooth base $S$. For $b \in S$, we write $D_b, E_b$ and so on for the corresponding fiber. By calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{T}$ we denote triangulated categories, $\mathcal{F}, \mathcal{G}$ usually denote complexes of coherent sheaves and $\mathcal{R}$ usually stands for a locally free sheaf of algebras on $X$. In the case when $\mathcal{F}$ is a complex of coherent sheaves on an $S$-variety $X$ we write $\mathcal{F}_b$ for the derived restriction of $\mathcal{F}$ to $X_b$.

2. Families of del Pezzo threefolds

2.1. Classification of del Pezzo threefolds. By a del Pezzo threefold we mean a projective threefold $X$ with Gorenstein terminal singularities, such that $-K_X = 2H$, for an ample line bundle $H \in \text{Pic}(X)$. By [Rei83 (1.1)], threefold Gorenstein terminal singularities are precisely the isolated compound du Val singularities. A typical example are the $cA_n$ singularities in which case complete local rings $\hat{O}_{X,x}, x \in \text{Sing}(X)$ are isomorphic to hypersurface singularities

$$k[[x, y, z, w]]/(x^2 + y^2 + g(z, w))$$

with $g(z, w)$ a polynomial having no repeated factors (that is $g(z, w)$ defines a reduced curve singularity). We will pay particular attention to ordinary double points, which is the case $g(z, w) = z^2 + w^2$, and we call projective threefolds with isolated ordinary double points nodal threefolds.
If $X$ is a del Pezzo threefold, the integer $d = H^3$ is called the degree of $X$, and we write $V_d$ for a del Pezzo threefold of degree $d$.

**Theorem 2.1** (Iskovskikh, Fujita). [IP99, Theorem 3.3.1] Del Pezzo threefolds are exactly the following:

- $V_1 \subset \mathbb{P}(1,1,1,2,3)$ a degree 6 hypersurface
- $V_2 \subset \mathbb{P}(1,1,1,1,2)$ a degree 4 hypersurface (equivalently, a double cover of $\mathbb{P}^3$ branched in a quartic)
- $V_3 \subset \mathbb{P}^4$ cubic hypersurface
- $V_4 \subset \mathbb{P}^5$ intersection of two quadrics
- $V_5 \subset \mathbb{P}^6$ codimension three linear section of $\text{Gr}(2,5) \subset \mathbb{P}^9$
- $V_6 \subset \mathbb{P}^7$ hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$
- $V'_6 = (\mathbb{P}^1)^3$
- $V_7 = \text{Bl}_p(\mathbb{P}^3)$, the blow up of a point $p \in \mathbb{P}^3$
- $V_8 = \mathbb{P}^3$

**Remark 2.2.** We note that $V_1$, $V_2$ can not pass through singular points of the ambient weighted projective space; indeed, since our del Pezzo threefolds have only compound du Val singularities hence in particular hypersurface singularities, the tangent space at each point has dimension at most 4, hence the dimension of the tangent space of the ambient space at this point should be at most 5, which is not the case for singular points of $\mathbb{P}(1,1,1,2,3)$ and $\mathbb{P}(1,1,1,1,2)$.

By Bertini’s Theorem, a general element $S \in |H|$ will be a smooth del Pezzo surface of degree $d$ which is convenient e.g. when analyzing the Hilbert scheme of lines on $V_d$. The last three varieties $V'_6, V_7, V_8$ are smooth, and will not be of interest to us. The other types can be nodal. Singularities of del Pezzo threefolds, in particular the maximal number of nodes they can have, have been studied by Prokhorov [Pro13] (see also Corollary 2.12).

**2.2. Class groups of del Pezzo threefolds** $V_1, V_2, V_3$. We recall the relationship between Weil and Cartier divisors on nodal threefolds. For any nodal threefold $X$ we have an exact sequence

$$0 \to \text{Pic}(X) \to \text{Cl}(X) \to \mathbb{Z}^{\text{Sing}(X)}.
$$

(2.2)

Here the last map is given by restricting a Weil divisor to the local class groups of the singular points, see e.g. [KPS21, Corollary 3.8]. In particular the quotient $\text{Cl}(X)/\text{Pic}(X)$ is a free abelian group of finite rank and

$$\delta := \text{rk } (\text{Cl}(X)/\text{Pic}(X))$$

is called defect of $X$. It follows from (2.2) that $\delta \leq |\text{Sing}(X)|$; if $\delta = |\text{Sing}(X)|$ we say that $X$ has maximal defect.

We say that $X$ is maximally nonfactorial if the map $\text{Cl}(X) \to \mathbb{Z}^{\text{Sing}(X)}$ from (2.2) is surjective. It is clear that maximal nonfactoriality of $X$ implies that $X$ has maximal defect, and the converse implication holds for Fano threefolds [KS24, Proposition A.14].

**Remark 2.3.** The importance of the maximal nonfactoriality condition is that it is a simple necessary condition for existence of a Kawamata decomposition for derived categories of nodal threefolds.
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Further, we will show that for nodal del Pezzo threefolds this condition is also sufficient, see Corollary 3.12.

We now use standard computations of defect to rule out del Pezzo threefolds of small degree from being maximally nonfactorial.

Proposition 2.4 ([Cyn01, Ram08]). Let $W$ be a projective toric fourfold with at most isolated quotient singularities, and let $X \subset W$ be a (non smooth) nodal hypersurface which does not intersect the singular locus of $W$. Assume that $O_W(X)$ is ample and that $\omega_W \otimes O_W(2X)$ is basepoint-free in the nonsingular locus of $W$. Then $X$ does not have maximal defect.

Proof. Let $\mu$ be the number of singular points of $X$, and $\delta$ be the defect. Then by [Ram08, (4.2) and Corollary 4.2]

$$\delta = \mu - (h^0(W, L) - h^0(W, L \otimes I_{\text{Sing}(X)})),$$

where $L = \omega_W \otimes O_W(2X)$. Since $L$ is assumed to be basepoint free on $W \setminus \text{Sing}(W)$ and $\text{Sing}(X) \subset W \setminus \text{Sing}(W)$, we have $h^0(W, L \otimes I_{\text{Sing}(X)}) < h^0(W, L)$ so that $\delta < \mu$, and $X$ does not have maximal defect.

Corollary 2.5. Nodal (non smooth) $V_1, V_2, V_3$ do not have maximal defect.

Proof. By Theorem 2.1 in each case $X = V_d$ is a hypersurface in a weighted projective space:

| $d$ | $W$ | $O_W(X)$ | $\omega_W$ | $\omega_W \otimes O_W(2X)$ |
|-----|-----|----------|------------|-----------------|
| 1   | $\mathbb{P}(1,1,1,2,3)$ | $\mathcal{O}(6)$ | $\mathcal{O}(-8)$ | $\mathcal{O}(4)$ |
| 2   | $\mathbb{P}(1,1,1,1,2)$ | $\mathcal{O}(4)$ | $\mathcal{O}(-6)$ | $\mathcal{O}(2)$ |
| 3   | $\mathbb{P}^4$ | $\mathcal{O}(3)$ | $\mathcal{O}(-5)$ | $\mathcal{O}(1)$ |

Here $O_W(X)$ is ample and $\omega_W \otimes O_W(2X)$ is basepoint free on the nonsingular locus of $W$ (the only basepoint is $[0 : 0 : 0 : 0 : 1]$ in $d = 1$ case). By Remark 2.2 $X$ does not pass through the singular points of $W$. All the conditions of Proposition 2.4 are verified and the result follows.

Remark 2.6. It follows from the proof of Proposition 2.4 that $|\text{Sing}(X)| - \delta$ equals the number of linearly independent conditions that the points from $\text{Sing}(X)$ impose on forms of degree $e$ on the corresponding weighted projective space, where $e = 4$ for $V_1$, $e = 2$ for $V_2$ and $e = 1$ for $V_3$.

Note that this approach of computing the defect does not apply to nodal intersections of two singular quadrics $V_4$. We will see in Corollary 3.12 that nodal $V_4$ never have maximal defect, while nodal $V_5$, $V_6$ are always maximally nonfactorial (hence have maximal defect). In the case of $V_4$, its defect can be read off from the associated curve of arithmetic genus two, see Theorem 2.11 and proof of Corollary 3.12.

2.3. Geometry of del Pezzo threefolds $V_4, V_5, V_6$. The key construction in the geometry of del Pezzo threefolds is projection from a line $L \subset X$ which has been used by Iskovskikh [Isk79] in the classification in the smooth case.

Definition 2.7. Let $V$ be a del Pezzo threefold. We call a line $L \subset V$ a standard line if the following conditions hold
(a) $L$ is contained in the smooth locus of $V$
(b) $L$ is not contained in any plane $\Pi \subset V$
(c) $N_{L/V} \cong \mathcal{O}_L^2$

For (b), see [Pro13, 3.7, 3.7.1] for planes on del Pezzo threefolds; for degree $d \geq 3$ a plane is simply a $\mathbb{P}^2$ embedded linearly in the ambient space.

The following example shows that for nodal del Pezzo threefolds the scheme of lines is in general reducible. Reducibility of the Fano variety of lines on singular cubic threefolds in relation to defect have been recently studied [MV23].

**Example 2.8.** Let $V$ be a 3-nodal del Pezzo threefold of degree 5. Then $V$ has a small resolution $\hat{V}$ which is isomorphic to $\text{Bl}_3(\mathbb{P}^3)$, the blow up of $\mathbb{P}^3$ in 3 general points $P_1, P_2, P_3$: the morphism $\hat{V} \to V$ is given by the linear system $|2H - P_1 - P_2 - P_3|$. It contracts three lines passing through pairs of points $P_i, P_j$ [Pro13, 7.1, 7.4]. Lines on $V$ correspond to smooth rational curves $C \subset \hat{V}$ with the property $C \cdot (2H - E_1 - E_2 - E_3) = 1$ where $E_i \subset \hat{V}$ are exceptional divisors of the blow up. These curves can be of the following types:

- proper preimages of lines in $\mathbb{P}^3$ passing through exactly one of the points $P_i$
- proper preimages of conics in $\mathbb{P}^3$ passing through $P_1, P_2, P_3$
- lines on one of the exceptional divisor $\mathbb{P}^2 \cong E_i \subset \hat{V}$

General lines of the first type are standard and the lines of the second and the third type are contained in a plane (see [Pro13, 7.2] for the list of planes on $V$), so they are not standard.

We present projection from a line construction for a degeneration of del Pezzo threefolds:

**Definition 2.9.** A del Pezzo threefold fibration is a flat proper morphism $f : V \to S$ with smooth $V$ and $S$, such that each fiber $V_b$ is a del Pezzo threefold with at most Gorenstein terminal singularities, and such that there exists $H \in \text{Pic}(V)$ which restricts to ample generator of $\text{Pic}(V_b)$ for each $b \in S$.

We call a subscheme $L \subset V$ a standard family of lines if $L \to S$ is a $\mathbb{P}^1$-bundle and for all $b \in S$ $L_b \subset V_b$ is a standard line.

**Lemma 2.10.** (i) Any del Pezzo threefold $V$ of degree $3 \leq d \leq 6$ with terminal Gorenstein singularities contains a standard line.

(ii) A del Pezzo threefold $V$ as in (i) with a standard line $L \subset V$ can be included as a central fiber into a del Pezzo fibration $V \to S$ admitting a standard family of lines $L \to S$ with $L_0 = L$.

**Proof.** (i) Let us show that there exists a line $L \subset V$ in the smooth locus, and not contained in any plane $\Pi \subset V$. Taking a general hyperplane section of $V$ not passing through singular points of $V$ gives rise to a smooth del Pezzo surface $S \subset V$. Let $i : S \to V$ be the embedding morphism; since $S$ lies in the smooth locus of $V$, we have a restriction homomorphism $i^* : \text{Cl}(V) \to \text{Pic}(S)$. If $\Pi \subset V$ is a plane, and $L \subset S \cap \Pi$, then $L = i^*(\Pi)$ by degree reasons. It follows from [Pro13 Corollary 3.9.2] that $i^*$ is not surjective, and since $\text{Pic}(S)$ is generated by lines $L \subset S$, there exists a line not contained in any plane $\Pi \subset V$.

We have a short exact sequence of normal bundles

$$0 \to \mathcal{O}_L(-1) \to N_{L/V} \to \mathcal{O}_L(1) \to 0.$$
Since vector bundles on $L \cong \mathbb{P}^1$ split into a direct sum of line bundles, $\mathcal{N}_{L/V}$ is either $\mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. In both cases $h^0(\mathcal{N}_{L/V}) = 2$, $h^1(\mathcal{N}_{L/V}) = 0$ so that $L$ corresponds to a smooth point of a two-dimensional irreducible component $T \subset F(V)$, where $F(V)$ is the Hilbert scheme of lines on $V$.

We claim that lines parametrized by $t \in T$ cover $V$. Indeed, otherwise they would cover a surface on $V$, and the only integral surface containing a two-parameter family of lines is a plane (because a general point on this surface will be its vertex). This is impossible since by construction $L$ does not lie on a plane $\Pi \subset V$.

We now compute the normal bundle for a general line following a method of Iskovskikh. Let $P$ be the universal line over $T$ so that we have a diagram \[
\begin{array}{ccc}
P & \xrightarrow{\phi} & V \\
\downarrow & & \downarrow \\
T & \xrightarrow{p} & P 
\end{array}
\]
where $\phi$ is generically finite and $p$ is a $\mathbb{P}^1$-bundle. We claim that $\phi$ is etale on $p^{-1}(T \setminus D)$ where $D \subset T$ is a divisor. Let $T^0$ (resp. $P^0$) be the smooth locus of $T$ (resp. $S$). We have $K_{P^0} = -2H + R$, where $R$ is a divisor supported at the ramification locus of $\phi$. On the other hand, by the formula for canonical class of a projective class of a projective bundle $K_{P^0} = p^*(K_{T^0}(\det(\mathcal{E}))) - 2H$, and comparing the two expressions for $K_{P^0}$ we deduce that $R$ is supported over $p^{-1}(D)$, for some divisor $D$. After enlarging $D$ to include $\text{Sing}(T)$, we obtain that $\phi$ is etale away from $p^{-1}(D)$.

Now a general line $L$ parametrized by $t \in T$ does not pass through singular points of $V$, and $\phi$ is etale over $L$. It follows that \[
\mathcal{N}_{L/V} \cong \mathcal{N}_{p^{-1}(t)/S} = \mathcal{O}_L \oplus \mathcal{O}_L,
\]
which finishes the proof.

(ii) Consider a general four-dimensional del Pezzo fourfold $\overline{V}$ of degree $d$ containing $V$ (see Theorem 2.1). A general hyperplane section $V'$ of $\overline{V}$ passing through $L$ will be smooth and not containing any singular points of $V$. Hence blowing up the base locus of a pencil of such hyperplane sections we obtain a flat proper morphism $V := \text{Bl}_{V \cap V'}(\overline{V}) \to \mathbb{P}^1$ with a smooth total space, and removing singular fibers other than $V$ will give the required degeneration. 

\[\square\]

**Theorem 2.11.** Let $V \to S$ be a del Pezzo threefold fibration of degree $4 \leq d \leq 6$. Let $L \subset V$ be a standard family of lines. Let $Y$ be the blow up $Y = \text{Bl}_L(V)$ and $E \subset Y$ be the exceptional divisor.

Then the line bundle $\mathcal{O}(H - E)$ is relatively globally generated. Let $\pi : Y \to W$ be the induced surjective morphism and let $Q = \pi(E)$. Then $\pi|_Q : Q \to S$ is a smooth two-dimensional quadric fibration contained in the smooth locus of $\pi$. Furthermore, there exists a smooth subscheme $C \subset Q$ flat and proper over $S$ such that $\pi : Y \to W$ is the blow up of $W$ along $C$. The possibilities for $W$ and $C$ are given in the table:

| $d$ | $W \to S$ | bidegree of $C_b \subset Q_b$ | description of $C_b$ |
|-----|------------|--------------------------------|---------------------|
| 4   | $\mathbb{P}^3$-fibration | (2,3) | arithmetic genus two curve |
| 5   | $Q^3$-fibration | (1,2) | generalized twisted cubic |
| 6   | $\mathbb{P}^1 \times \mathbb{P}^2$-fibration | (1,1) | conic |
| 6   | $\mathbb{P}^1 \times \mathbb{P}^2$-fibration | (0,2) | two disjoint lines |
The $Q^3$-fibration $W \rightarrow S$ in the case $d = 5$ has smooth or nodal fibers. If $V_b$ is smooth, then the curve $C_b$ and the threefold $W_b$ are also smooth. In the degree $d = 6$ case, in the notation of Theorem 2.7, blowing up the $(1, 1)$ curve case corresponds to $V_b$ and blowing up the $(0, 2)$ curve corresponds to $V'_b$.

The birational maps in Theorem 2.11 are summarized in the following diagram

\[
\begin{array}{ccc}
E & \overset{i}{\rightarrow} & Y \\
p & \sigma & q \\
\downarrow & & \downarrow \\
L & \overset{\pi}{\rightarrow} & V \\
\downarrow & & \downarrow \\
W & \overset{\pi}{\rightarrow} & C \\
\end{array}
\]

\[ (2.3) \]

Proof. By generic smoothness, general fibers of $V \rightarrow S$ are smooth. To simplify the notation let us assume that the only possibly singular fiber is $V_0$ over the point $0 \in S$. Fiberwise the morphism $\pi$ resolves projection from the line $L_0 \subset V_b$ so that $H - E$ is base point free on $Y$. Furthermore, fibers of $\pi$ are residual components obtained by intersecting $V_b$ with planes passing through $L_0$. Since $d \geq 4$, each del Pezzo threefold $V_b$ is an intersection of quadrics [IP99, Theorem 3.2.4(iii)], so that fibers of $\pi$ can be empty, consist of a reduced point or isomorphic to $\mathbb{P}^1$. In other words for each $b \in S$, $\pi$ contracts secant lines to $L_0 \subset V_b$, and induces a birational morphism $Y_b \rightarrow W_b$. We have $h^0(Y_b, H - E_b) = h^0(V_b, I_{L_0}(H)) = d$ and using [IP99] Lemma 2.2.14 for smooth fibers $W_b$

\[ \deg(W_b) = (H - E_b)^3 = H^3 - (3 \cdot H + K_{V_b}) \cdot L_b + 2g - 2 = d - 3 \]

so that $W_b$ is a subvariety of degree $d - 3$ in $\mathbb{P}^{d-1}$; since $(H - E_b)^3$ is independent of $b$, the same is true for the singular fibers as well. As $\deg(W_b) = \text{codim}(W_b) + 1$, it is a so-called variety of minimal degree and there are only the following possibilities for $W_b$ [IP99, Theorem 2.11]:

- $d = 4$: $W_b = \mathbb{P}^3$
- $d = 5$: $W_b \subset \mathbb{P}^4$ is a quadric (possibly singular)
- $d = 6$: $W_b = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ (the Segre embedding), a cone over a cubic scroll or a cone over a rational twisted cubic curve.

The linear system $|H - E|$ restricts to each $E_b \simeq \mathbb{P}^1 \times \mathbb{P}^1$ as $\mathcal{O}(1, 1)$, so that $\pi|_E$ is an isomorphism onto a smooth quadric fibration $Q \subset W$. Thus we have an isomorphism

\[ V \setminus L \simeq Y \setminus E \simeq W \setminus Q \]

and in particular $W \setminus Q$ is smooth, each $W_b \setminus Q_b$ is smooth and $W_Q \setminus Q_0$ has at most terminal Gorenstein singularities. This rules out all types of singular $W_b$ except for a nodal quadric threefolds in the $d = 5$ case. It follows that $Q_b$ is contained in the smooth locus of $W_b$ and since $W \setminus Q$ is also smooth, $W$ is a smooth fourfold.

By construction $\pi$ contracts a divisor onto a relative curve $C \subset Q$. Thus $\pi : Y \rightarrow W$ is a birational projective morphism of smooth varieties, with at most one-dimensional fibers, hence by Danilov’s decomposition theorem [Dan80], $\pi$ is a composition of blow ups with smooth centers. Since all fibers of $\pi$ are irreducible, in fact $\pi$ is a blow up of $C \subset W$ and $C$ is smooth.
It remains to check the type of the curve $C_b \subset Q_b$. Since $C_b$ is a flat family, it suffices to consider $b \neq 0$. The normal bundle of $Q_b$ in $W_b$ is

- $d = 4$: $N = O(2, 2)$
- $d = 5$: $N = O(1, 1)$
- $d = 6$: $N = O(0, 1)$.

Computing the normal bundle of $E_b$ in $Y_b$ in two ways using the blow ups $\sigma$ and $\pi$, we obtain $N(-C_b) \simeq O(0, -1)$. It follows that bidegree and the type of $C_b$ must be as claimed.

In the $d = 6$ case the varieties $V_6$ and $V'_6$ can be distinguished using their Picard rank, which then distinguishes the curve types: the $(0, 2)$ curve consists of two connected components, hence the blow up increases Picard rank by two, while the $(1, 1)$ curve is connected, hence the blow up increases the Picard rank by one. □

Let us consider the restriction of the diagram (2.3) to fibers over $b \in S$. Let us write $V = V_b$, $Y = Y_b$ and so on. Considering the local charts of a blow up $\pi_b$, we see that if the complete local equation for $C$ on a quadric surface is $g(z, w) = 0$, then the blow up $Y$ has an equation $xy + g(z, w) = 0$, and so $Y$, and hence $V$ will have Gorenstein terminal (in fact, $cA_n$ (2.1)) singularities if and only if $C$ is reduced. Furthermore, we see that $V$ is at most nodal if and only if both $W$ and $C$ are at most nodal, in which case

$$|\text{Sing}(V)| = |\text{Sing}(Y)| = |\text{Sing}(W)| + |\text{Sing}(C)|.$$  

(2.4)

We call $C$ an associated curve to $V$ (this curve depends on $V$ and a chosen standard line $L \subset V$).

**Corollary 2.12.** Gorenstein terminal del Pezzo threefolds of degrees listed below can only have the following types of singularities:

- $d = 4$: only $cA_n$ singularities; at most six nodes
- $d = 5$: only nodal; at most three nodes
- $d = 6$: only nodal; at most one node for $V_6$ and $V'_6$ is smooth

**Proof.** Indeed, only these types of singularities can occur when blowing up the curve $C$ of the given type. The number of nodes can be computed using (2.4). □

From diagram (2.3) we obtain the relations between divisor classes in $\text{Pic}(Y)/\text{Pic}(S)$, which we will use later:

- $d = 4$:
  $$h := H - E, \quad E = 2h - D, \quad H = 3h - D$$  
  (2.5)

- $d = 5$:
  $$h := H - E, \quad E = h - D, \quad H = 2h - D$$  
  (2.6)

3. Derived categories of del Pezzo fibrations

3.1. Relative tilting semiorthogonal decompositions. Let $T$ be a $k$-linear triangulated category.

**Definition 3.1** ([BO95], [BKS9]). A collection $A_1, \ldots, A_m$ of full triangulated subcategories of $T$ is called a semiorthogonal decomposition of $T$, if

- for all $1 \leq i < j \leq m$, $\text{Hom}(A_j, A_i) = 0$,
- the smallest triangulated subcategory of $T$ containing $A_1, \ldots, A_m$ coincides with $T$. 


If each $A_i$ is admissible in $T$ that is, the inclusion functor $A_i \subset T$ has both adjoint functors, then we say that the semiorthogonal decomposition is admissible.

We write $T = \langle A_1, \ldots, A_m \rangle$ for a semiorthogonal decomposition; all decompositions we consider will be admissible, and in many cases admissibility is automatic. The following result is a typical starting point for constructing admissible semiorthogonal decompositions.

**Theorem 3.2** (Orlov [Orl92]). Let $Z \subset X$ be a smooth subvariety of pure codimension $c$. Let $\pi: \tilde{X} \to X$ be the blow up of $X$ with center $Z$, and let $i: E \to \tilde{X}$ be the exceptional divisor, with projective bundle structure $p: E \to Z$. Then there is an admissible semiorthogonal decomposition $D^b(\tilde{X}) = \langle D^b(Z)_{-(c-1)}, \ldots, D^b(Z)_{-1}, \pi^*D^b(X) \rangle$, where $D^b(Z)_{-j}$ is the image of the fully faithful functor $(i_*p^*(-)) \otimes \mathcal{O}(jE) : D^b(Z) \to D^b(\tilde{X})$.

We introduce the type of semiorthogonal decompositions of singular varieties that we are interested in.

**Definition 3.3.** [KPS21] Let $X$ be a Gorenstein projective variety. An admissible semiorthogonal decomposition $D^b(X) = \langle B, A_1, \ldots, A_r \rangle$ is called a Kawamata decomposition if $B \subset D^{\text{perf}}(X)$ and each $A_i$ is equivalent to $D^b(R_i)$, with $R_i$ a finite-dimensional $k$-algebra.

For example, a smooth projective variety with a full exceptional collection admits a Kawamata decomposition with each $A_i \cong D^b(k)$ (and $B = 0$), and Kawamata decompositions can be considered as a way of generalizing exceptional collections to singular varieties. For examples of curves admitting Kawamata semiorthogonal decompositions see Remark 3.9 below. For other examples see [KKS22], [KPS21], [Kaw18], [Kaw22a], [Kaw22b], [Kaw24].

We will now introduce a relative version of a Kawamata semiorthogonal decomposition. For that we use semiorthogonal decompositions over a smooth base $S$ defined by Kuznetsov [Kuz11] and reformulated by Perry in a more abstract setting of stable $\infty$-categories [Per19].

An $S$-linear category $\mathcal{T}$ is an appropriately enhanced $k$-linear triangulated category endowed with an action

$$\otimes: D^b(S) \times \mathcal{T} \to \mathcal{T}$$

and it automatically admits relative Hom-objects [Per19, 2.3.1]

$$\mathbf{RHom}_{\mathcal{T}/S}(-, -): \mathcal{T}^{\text{op}} \times \mathcal{T} \to D^b(S)$$

with a functorial isomorphism for all $a \in D^b(S)$, $t, u \in \mathcal{T}$

$$\mathbf{RHom}_S(a, \mathbf{RHom}_{\mathcal{T}/S}(t, u)) = \mathbf{RHom}_\mathcal{T}(a \otimes t, u).$$

For example, if $\mathcal{T} = D^b(X)$ for a scheme $f: X \to S$, $\mathcal{T}$ is $S$-linear with $\otimes$ defined by $f^*(-) \otimes -$ and with $\mathbf{RHom}_{\mathcal{T}/S}(-, -) = f_*\mathbf{RHom}_X(-, -)$.

More generally, if $\mathcal{R}$ is a locally free sheaf of $\mathcal{O}_S$-algebras, we consider the derived category of coherent sheaves of right $\mathcal{R}$-modules $D^b(S, \mathcal{R})$, see [Kuz08, 2.1]. The category $\mathcal{T} = D^b(S, \mathcal{R})$ is...
S-linear via the natural tensor product functor
\[ \otimes : D^b(S) \times D^b(S, \mathcal{R}) \to D^b(S, \mathcal{R}) \]
and \( R\text{Hom}_{\mathcal{T}/S}(-, -) = f_* R\text{Hom}_{\mathcal{R}}(-, -) \). Note that \( D^b(S) = D^b(S, \mathcal{O}) \).

For the rest of this subsection we assume that \( f : X \to S \) is a flat projective morphism between smooth varieties.

**Definition 3.4.** By a relative tilting semiorthogonal decomposition we mean an \( S \)-linear semiorthogonal decomposition
\[ D^b(X) = \langle A_1, \ldots, A_r \rangle \]
such that we have \( S \)-linear equivalences \( A_i \simeq D^b(S, \mathcal{R}_i) \) for some locally free sheaves of algebras \( \mathcal{R}_i \) on \( S \).

The name relative tilting decomposition is due to the fact that the image \( T_i \in D^b(X) \) of each \( \mathcal{R}_i \) satisfies \( f_* R\text{Hom}(T_i, T_i) = \mathcal{R}_i[0] \) so when \( S = \text{Spec}(k) \) we get a so-called pretilting object \([\text{Kaw24}]\) which is tilting (i.e. generating) if \( r = 1 \).

By the standard properties of base change we obtain for every \( b \in S \) an induced Kawamata semiorthogonal decomposition
\[ D^b(X_b) \simeq \langle D^b(\mathcal{R}_1 \otimes k(b)), \ldots, D^b(\mathcal{R}_r \otimes k(b)) \rangle. \]

Before we give examples of relative tilting decompositions we state a simple admissibility property. For an \( S \)-linear category \( \mathcal{T} \) a relative Serre functor \([\text{Per19}, 4.6]\) is an \( S \)-linear equivalence \( SS_{\mathcal{T}/S} : \mathcal{T} \to \mathcal{T} \) with a functorial isomorphism for \( t, u \in \mathcal{T} \)
\[ R\text{Hom}_{\mathcal{T}/S}(t, SS_{\mathcal{T}/S}(u)) \simeq R\text{Hom}_{\mathcal{T}/S}(u, t)^\vee. \]

For example for a morphism \( X \to S \), the \( S \)-linear category \( \mathcal{T} = D^b(X) \) admits a relative Serre functor given by \( - \otimes \omega_{X/S}[\dim(X) - \dim(S)] \). The following result is well-known in the absolute case.

**Lemma 3.5.** Given an \( S \)-linear semiorthogonal decomposition \( \mathcal{T} = \langle A_1, \ldots, A_m \rangle \) if both \( \mathcal{T} \) and all the components \( A_i \) have a relative Serre functor, then the decomposition is admissible.

**Proof.** It is clear that \( A_1 \) is left admissible. Using the standard argument involving the Serre functor for \( \mathcal{T} \) and \( A_i \) we can write
\[ \mathcal{T} = \langle A_2, \ldots, A_m, SS_{\mathcal{T}/S}^{-1} A_1 \rangle \]
which implies that \( A_2 \) is left admissible. By induction we obtain that all \( A_i \) are left admissible. A similar argument shows right admissibility. \( \Box \)

Our first example of a relative tilting decomposition comes from quadric fibrations \([\text{Kuz08}]\).

**Theorem 3.6 (\([\text{Kuz08} \text{ Theorem 4.2}]\)).** Let \( f : X \to S \) be a flat quadric fibration with smooth \( X \) and \( S \). Then there is a relative tilting semiorthogonal decomposition
\[ D^b(X) \simeq \langle D^b(S, \mathcal{B}_0), f^* D^b(S) \otimes \mathcal{O}_X(1), \ldots, f^* D^b(S) \otimes \mathcal{O}_X(n - 2) \rangle, \]
where \( \mathcal{B}_0 \) is the sheaf of even parts of Clifford algebras on \( S \).

Another example of tilting decomposition is given by degenerations of rational curves.
Proposition 3.7. Let $f : X \rightarrow S$ be a flat projective morphism with general members isomorphic to smooth rational curves, and singular fibers having nodal singularities. Then, after passing to an open covering of $S$, we have an $S$-linear semiorthogonal decomposition

$$D^b(X) = \langle D^b(S, R), f^*(D^b(S)) \rangle.$$  (3.1)

Recall that if $X$ is a projective nodal curve of arithmetic genus zero and components $X_1, \ldots, X_m$, we have $\text{Pic}(X) = \mathbb{Z}^m$ generated by isomorphism classes of line bundles $L_i$ such that

$$L_i|_{X_j} \simeq O_{X_j}(-\delta_{ij}).$$  (3.2)

Consider a line bundle on $X$ given by

$$L = \bigoplus_{i=1}^{m} L_i^{\oplus r_i}.$$  (3.3)

with all $r_i > 0$. We will call any such line bundle $L$ a minimally negative bundle. For example if $X = \mathbb{P}^1$, then minimally negative bundles have the form $O(-1)^{\oplus r}$, $r > 0$.

Lemma 3.8. Assume that $f : X \rightarrow S$ is as in Proposition 3.7. Assume that $T$ is a locally free sheaf on $X$ such that for every $b \in B$ the restriction $T|_{X_b}$ is a minimally negative line bundle. Then $\mathcal{R} := f_* R\mathcal{H}om(T, T)$ is a locally free sheaf of algebras on $S$. Define a functor

$$\Phi_T : D^b(S, \mathcal{R}) \rightarrow D^b(X), \quad \mathcal{F} \mapsto f^*(\mathcal{F}) \otimes f^*(\mathcal{R}) T.$$  (3.4)

Then we have an $S$-linear semiorthogonal decomposition

$$D^b(X) = \langle \Phi_T(D^b(S, \mathcal{R})), f^*(D^b(S)) \rangle.$$  (3.5)

Proof. This is a standard argument in the spirit of [Sam07]. The fact that $\mathcal{R}$ is a locally sheaf boils down to a computation on each fiber which uses (3.2). It is then a standard computation that $\Phi_T$ is fully faithful, its image is semiorthogonal to $f^*(D^b(S))$ and that these two subcategories form an $S$-linear semiorthogonal decomposition of $D^b(X)$.

Proof of Proposition 3.7. For every point $b \in S$ there is an open neighborhood $U \subset S$ of $b$ and a relative hyperplane section $D \subset X_U$, which is finite over $U$, and intersects every irreducible component of every fiber. Furthermore we can assume that these intersections are transverse. To simplify the notation let us assume that $S = U$. Let $d$ be the degree of $D$ over $S$.

Then $X' = X \times_S D$ is a degree $d$ cover $p : X' \rightarrow X$ which admits a section $i : D \rightarrow X'$. Let $T := \pi_* (O(-D))$. By construction it restricts to a minimally negative vector bundle on the fibers of $f$. We apply Lemma 3.8 to $T$ to get the result.

Remark 3.9. Restricting the tilting decomposition (3.1) to each genus zero curve $X_b$, for $b \in S$ we obtain a semiorthogonal decomposition

$$D^b(X_b) = \langle D^b(R), O \rangle$$

with a finite-dimensional algebra $R$. This algebra is isomorphic to a matrix algebra when $X_b \simeq \mathbb{P}^1$. On the other hand, if $X_b$ is a nodal curve, then $R$ is the algebra defined by Burban [Bur04]. It can be described explicitly as path algebra of a quiver, see [Bur04, Theorem 2.1] or [KPS21, Remark 4.14] for uniform formulations. For example, if $X_b$ is a chain of two smooth projective lines, that is a nodal
conic, then \( R \) is the path algebra of the quiver
\[
\begin{align*}
1 & \xrightarrow{a} 2 , \quad aa^* = a^*a = 0. \\
1 & \xrightarrow{a^*} 2 \xleftarrow{b} 3 , \quad aa^* = a^*a = bb^* = b^*b = 0.
\end{align*}
\] (3.6)

This algebra is isomorphic to the Clifford algebra of a nodal quadric of odd dimension (cf [Kaw18, Example 5.6]).

Similarly, for a nodal chain of three smooth projective lines, \( R \) is the path algebra of the quiver
\[
\begin{align*}
1 & \xrightarrow{a} 2 \xleftarrow{b} 3 , \quad aa^* = a^*a = bb^* = b^*b = 0.
\end{align*}
\] (3.7)

We call the path algebra for (3.6) the single Burban algebra and the path algebra for (3.7) the double Burban algebra. Both of these algebras will appear in the semiorthogonal decompositions of nodal del Pezzo threefolds.

3.2. Main results. The following theorem generalizes corresponding results in the smooth case to degenerations: semiorthogonal decomposition for smooth \( V_4 \) constructed by Bondal and Orlov [BO02] and an exceptional collection for smooth for \( V_5 \) constructed by Orlov [Orl91; see Kuz16, 2.4] for a uniform treatment of smooth Fano threefolds.

**Theorem 3.10.** Let \( V \to S \) be a nodal del Pezzo fibration of degree \( d \in \{4, 5\} \) with a standard family of lines \( L \subset V \). Let \( C \to S \) be the associated family of curves as in Theorem 2.11, corresponding to \( L \).

(i) If \( d = 4 \), then \( C \to S \) is a family of at most nodal curves of arithmetic genus 2 and we have an admissible \( S \)-linear semiorthogonal decomposition
\[
D^b(V) = \langle D^b(C), D^b(S), D^b(S)(H) \rangle.
\]

(ii) If \( d = 5 \), then \( C \to S \) is a family of generalized twisted cubics and, after passing to an open cover of \( S \), there is an admissible \( S \)-tilting semiorthogonal decomposition
\[
D^b(V) = \langle D^b(S, R_1), D^b(S, R_2), D^b(S), D^b(S)(H) \rangle
\]

where \( R_1 \) is the sheaf of Burban algebras corresponding to \( C \to S \) and \( R_2 \) is the sheaf of Clifford algebras corresponding to the quadric threefold fibration \( W \to S \).

Here the embedding of \( D^b(S) \) and \( D^b(S)(H) \) is obtained by the pullback with respect to \( f : V \to S \) and twisting by \( \mathcal{O}(H) \) in the latter case and the embedding of other categories are specified in the proof of the theorem. As a consequence, we obtain:

**Corollary 3.11.** For a nodal del Pezzo threefold \( V \) of degree \( d \in \{4, 5\} \) we have the following admissible semiorthogonal decompositions.

(i) If \( d = 4 \), then
\[
D^b(V) = \langle D^b(C), \mathcal{O}, \mathcal{O}(H) \rangle
\]

where \( C \) is the associated nodal curve of arithmetic genus two.

(ii) If \( d = 5 \), there is a Kawamata type semiorthogonal decompositions
\[
D^b(V) = \langle D^b(R_1), D^b(R_2), \mathcal{O}, \mathcal{O}(H) \rangle
\]
where $R_1$ and $R_2$ are Burban algebras.

Proof of Corollary 3.11. By Lemma 2.10 (ii) there is a nodal del Pezzo fibration $f: V \to S$ with a standard family of lines $L \subset V$, where $S$ is a smooth affine curve, $0 \in S$ a point and $V_0 = V$. By base change of the decomposition constructed in Theorem 3.10 to $0 \in S$, we get that

$$D^b(V) = \langle A_V, \mathcal{O}, \mathcal{O}(H) \rangle,$$

where $A_V \simeq D^b(C)$ and $A_V \simeq \langle A_C, A_Q \rangle$ for $d = 4$ and $d = 5$ respectively. 

In both cases $d = 4$ and $d = 5$ there is an induced semiorthogonal decomposition of $D^\text{perf}(V)$, see [Orl06, Proposition 1.10 and 1.11] and [KPS21, Theorem 4.4].

Before we prove Theorem 3.10 we use it to deduce a complete structural result about derived categories of nodal del Pezzo threefolds, which implies Theorem 1.1 in the Introduction.

Corollary 3.12. Let $V$ be a nodal (non smooth) del Pezzo threefold of arbitrary degree $d \geq 1$ and $A_V = \langle \mathcal{O}, \mathcal{O}(H) \rangle \perp$ be the main component of the derived category $D^b(V)$. The following conditions are equivalent:

1. $D^b(V)$ admits a Kawamata decomposition
2. $D^b(V)$ has an admissible decomposition with all components equivalent to derived categories of finite-dimensional algebras
3. $A_V$ has an admissible decomposition with all components equivalent to derived categories of finite-dimensional algebras
4. $V$ is maximally nonfactorial
5. $V$ has maximal defect
6. $d \in \{5, 6\}$

Proof. We prove the following chain of implications

$$(3) \implies (2) \implies (1) \implies (4) \implies (5) \implies (6) \implies (3).$$

The first two implications are trivial.

$$(1) \implies (4)$$

is [KPS21] Theorem 1.1.

$$(4) \implies (5)$$

is trivial.

$$(5) \implies (6)$$

We first note that by Corollary 2.5 we have $d \geq 4$. In the $d = 4$ case we can relate the defect to the negative K-group $K_{-1}(V)$ [KPS21] as follows. By Theorem 3.10 (or using the diagram 2.3 directly) we see that $K_{-1}(V) = K_{-1}(C) \neq 0$ (see [KPS21] Corollary 3.3 for $K_{-1}$ of a nodal curve), hence $V$ does not have maximal defect by [KPS21] Proposition 3.5]. Finally if $d \geq 7$, then $V$ can not be singular by Theorem 2.1.

$$(6) \implies (3)$$

follows from Corollary 3.11 for the $d = 5$ case. For the $d = 6$ case the result is [Kaw22b, 7.2] (the variety $X$ in [Kaw22b, 7.2] is the unique nodal $V_6$ by [Pro13, Theorem 7.1]); see also Remark 3.18.

3.3. Proof of Theorem 3.10. The proof relies on projection from a standard family of lines from Theorem 2.11. More concretely, we describe $D^b(Y)$ in terms of the blow up $\sigma: Y \to V$ in Proposition 3.14 and in Proposition 3.15 we describe $D^b(Y)$ in terms of $\pi: Y \to W$ for cases $d = 4, 5$ separately. The proof of Theorem 3.10 follows then readily by comparing these two descriptions of $D^b(Y)$. 

Let $f_E : E \to S$ be the exceptional divisor of $\sigma : Y \to V$, where $i : E \hookrightarrow Y$ denotes the corresponding embedding and let $f_P : D \to S$ be the exceptional divisor of $\pi : Y \to W$ with inclusion morphism $j : D \to W$ as in Theorem \ref{thm:2.11}.

**Lemma 3.13.** (i) We have the following equality of subcategories in $D^b(Y)$:

$$\langle f_Y^* D^b(S)(-E), f_Y^* D^b(S) \rangle = \langle f_Y^* D^b(S), i_* f_E^* D^b(S) \rangle = \langle i_* f_E^* D^b(S), f_Y^* D^b(S)(-E) \rangle$$

(3.8)

and

$$\langle f_Y^* D^b(S)(-h), f_Y^* D^b(S)(D - h) \rangle = \langle j_* f_D^* D^b(S)(D - h), f_Y^* D^b(S)(-h) \rangle.$$  

(3.9)

(ii) We have $\text{Hom}(f_Y^* D^b(S)(H - E), i_* f_E^* D^b(S)(2H)[k]) = 0$ for all $k$.

**Proof.** (i) The pair $i_* f_E^* D^b(S)(E), f_Y^* D^b(S)$ is semiorthogonal (this is part of Theorem \ref{thm:3.2} with $j = 1$) and \ref{eq:3.8} is obtained by standard mutations using the distinguished triangle

$$O_Y(-E) \to O_Y \to O_E.$$

The same argument proves \ref{eq:3.9}.

(ii) By adjunction it suffices to prove vanishing of $\text{Hom}(D^b(S), D^b(S) \otimes f_E^* O_E(H + E)[k])$ and this follows from

$$f_E^* (O_E(H + E)) = f_L^* (p^* (O_E(E))(H)) = 0.$$

\[ \square \]

In what follows we work with the main components of derived categories, in the following cases

- **C** a family of of rational curves: $\mathcal{A}_C = \langle O \rangle^\perp$,
- **Q** a quadric fibration of relative dimension three: $\mathcal{A}_Q = \langle O(-1), O, O(1) \rangle^\perp$,
- **V** a del Pezzo fibration: $\mathcal{A}_V = \langle O, O(H) \rangle^\perp$.

When mutating semiorthogonal decompositions in the proofs below, we use a slight abuse of notation by not specifying the embedding functor of $\mathcal{A}_C$, $\mathcal{A}_Q$ and $\mathcal{A}_V$ in the derived category of $D^b(Y)$ when this embedding is clear from context or not important for us.

**Proposition 3.14.** Let $d \in \{4, 5, 6\}$ and let $f_V : V \to S$ be a degeneration of del Pezzo threefolds containing a standard family of lines $L \subset V$. Let $f_Y : Y \to S$ be the blow up of $V$ along $L$. There is an $S$-linear semiorthogonal decomposition

$$D^b(Y) = \langle \mathcal{A}_V, f_Y^* D^b(S)(E - H), f_Y^* D^b(S)(-E), f_Y^* D^b(S), f_Y^* D^b(S)(H - E) \rangle.$$

**Proof.** We will use the blow up formula (Theorem \ref{thm:3.2}) together with mutations from Lemma \ref{lem:3.13}.

Theorem \ref{thm:3.2} applied to the blow up $\sigma : Y \to V$ along $L$ gives a semiorthogonal decomposition

$$D^b(Y) = \langle D^b(L)_{-1}, \sigma^* D^b(V) \rangle$$

$$= \langle (f_L^* D^b(S) \otimes O_L(-2))_{-1}, (f_L^* D^b(S) \otimes O_L(-1))_{-1}, \sigma^* D^b(V) \rangle$$

$$= \langle i_* f_E^* D^b(S)(E - 2H), i_* f_E^* D^b(S)(E - H), \sigma^* D^b(V) \rangle$$

$$= \langle \sigma^* D^b(V), i_* f_E^* D^b(S)(E - 2H - K_Y), i_* f_E^* D^b(S)(E - H - K_Y) \rangle.$$  

(3.10)

Here we used Orlov’s projective bundle formula [Orl92, Theorem 2.6] in the second equation, that $p^* (O_L(-1)) \simeq O_E(-H)$ (as $H \cdot L_b = 1$ on each fiber of $V_b$, $b \in S$) in the third equality and Serre duality in the fourth equality.
We can write out (3.14) as:

$$K_Y \equiv -2H + F \pmod{\text{Pic}(S)}$$

in (3.10) to obtain

$$D^b(Y) = \langle f_Y^* D^b(S)(-H), \sigma^* A_V, f_Y^* D^b(S), i_* f_E^* D^b(S), i_* f_E^* D^b(S)(H) \rangle. \quad (3.12)$$

Using Serre duality together with (3.8) of Lemma 3.13, we perform the following sequence of mutations:

$$D^b(Y) = \langle f_Y^* D^b(S)(-H), \sigma^* A_V, f_Y^* D^b(S), i_* f_E^* D^b(S), i_* f_E^* D^b(S)(H) \rangle$$

$$= \langle \sigma^* A_V, f_Y^* D^b(S), i_* f_E^* D^b(S), i_* f_E^* D^b(S)(H), f_Y^* D^b(S)(H - E) \rangle$$

$$= \langle \sigma^* A_V, f_Y^* D^b(S)(-E), f_Y^* D^b(S), f_Y^* D^b(S)(H - E), f_Y^* D^b(S)(H) \rangle$$

$$= \langle A_V, f_Y^* D^b(S)(E - H), f_Y^* D^b(S)(-E), f_Y^* D^b(S), f_Y^* D^b(S)(H - E) \rangle. \quad (3.13)$$

In the fourth equality we used Serre duality again and we performed a left mutation of $\sigma^* A_V$ through $f_Y^* D^b(S)(E - H)$ in the fifth equality. This concludes the proof of the proposition.

On the other hand, we can also decompose $D^b(Y)$ semiorthogonally with respect to the blow up $\pi: Y \to W$.

**Proposition 3.15.** Let $d \in \{4, 5\}$ and consider diagram (2.3).

(i) Let $d = 4$. There is an $S$-linear admissible semiorthogonal decomposition

$$D^b(Y) = \langle D^b(C), f_Y^* D^b(S)(-h), f_Y^* D^b(S)(D - 2h), f_Y^* D^b(S)(h) \rangle. \quad (3.14)$$

(ii) Let $d = 5$ so that $W = \mathbb{P}^3$ is a quadric threefold fibration with at most nodal quadrics as fibers. There is an $S$-linear admissible semiorthogonal decomposition

$$D^b(Y) = \langle A_C, A_Q, f_Y^* D^b(S)(-h), f_Y^* D^b(S)(D - h), f_Y^* D^b(S), f_Y^* D^b(S)(h) \rangle. \quad (3.15)$$

**Proof.** Theorem 3.2 applied to the blow up $\pi: Y \to W$ gives a semiorthogonal decomposition

$$D^b(Y) = \langle D^b(C)_{-1}, D^b(W) \rangle. \quad (3.14)$$

We consider the cases $d = 4, 5$ separately.

Case $d = 4$: Here, $W \to S$ is a Zariski locally trivial $\mathbb{P}^3$-bundle with a relative hyperplane class $h$.

We can write out (3.14) as

$$D^b(Y) = \langle D^b(C)_{-1}, f_Y^* D^b(S)(-h), f_Y^* D^b(S), f_Y^* D^b(S)(h), f_Y^* D^b(S)(2h) \rangle$$

$$= \langle f_Y^* D^b(S)(D - 2h), D^b(C)_{-1}, f_Y^* D^b(S)(-h), f_Y^* D^b(S), f_Y^* D^b(S)(h) \rangle$$

$$= \langle D^b(C), f_Y^* D^b(S)(D - 2h), f_Y^* D^b(S)(-h), f_Y^* D^b(S), f_Y^* D^b(S)(h) \rangle, \quad (3.15)$$

where we used the projective bundle formula [Orl92, Theorem 2.6] in the first equation, Serre duality with $K_Y = -4h + D + f_Y^* K_S$ in the second equality and left mutation of $D^b(C)$ through $f_Y^* D^b(S)(D - 2h)$ in the third equality.
Furthermore, using the relations \((2.5)\) we can rewrite the pair \(f_Y^* D^b(S)(D - 2h), f_Y^* D^b(S)(-h)\) as \(f_Y^* D^b(S)(-E), f_Y^* D^b(S)(E - H)\). But from the decomposition of Proposition \(3.14\) we see that this pair is completely orthogonal. Thus we can swap \(f_Y^* D^b(S)(D - 2h)\) and \(f_Y^* D^b(S)(-h)\) in \((3.15)\) and we obtain item \((1)\). Admissibility and \(S\)-linearity of the constructed semiorthogonal decomposition follows as in the proof of Proposition \(3.14\).

Case \(d = 5\): We have \(W = Q^3\), where \(Q^3\) is a quadric fibration in \(\mathbb{P}^3\) and the associated curve \(A \to S\) is a family of curves of arithmetic genus zero. We rewrite \((3.11)\) as
\[
D^b(Y) = \langle A_C, i_* f_D^* D^b(S)(-h + D), A_{Q^3}, f_Y^* D^b(S)(-h), f_Y^* D^b(S), f_Y^* D^b(S)(h) \rangle
\]
\[
= \langle A_C, i_* f_D^* D^b(S)(-h + D), f_Y^* D^b(S)(-h), f_Y^* D^b(S), f_Y^* D^b(S)(h) \rangle
\]
\[
= \langle A_C, i_* f_D^* D^b(S)(-h), f_Y^* D^b(S)(-h + D), f_Y^* D^b(S), f_Y^* D^b(S)(h) \rangle
\]
Here we used \([Kuz08, \text{Theorem 4.2}]\) in the first equation, left mutation of \(A_{Q^3}\) through \(i_* f_D^* D^b(S)(-h + D)\) in the second equation and \((3.9)\) in the third equation. Admissibility of the components is automatic by Lemma \(3.15\).

\[
Proof \text{ of Theorem \(3.14\).} \quad \text{By Proposition \(3.14\) we obtain}
\]
\[
D^b(Y) = \begin{cases} 
\langle A_C, f_Y^* D^b(S)(-h), f_Y^* D^b(S)(D - 2h), f_Y^* D^b(S), f_Y^* D^b(S)(h) \rangle, & \text{if } d = 4 \\
\langle A_V, f_Y^* D^b(S)(-h), f_Y^* D^b(S)(D - h), f_Y^* D^b(S), f_Y^* D^b(S)(h) \rangle, & \text{if } d = 5 
\end{cases}
\]
where we used \((2.5), (2.3)\), to rewrite \(E\) and \(H\) in terms of divisors pulled back from \(W\). Comparing these expressions with Proposition \(3.10\) we obtain that
\[
D^b(C), \quad \text{if } d = 4 \\
\langle A_C, A_{Q^3} \rangle, \quad \text{if } d = 5
\]
The obtained semiorthogonal decompositions are admissible by Lemma \(3.15\) because all the categories involved have relative Serre functors.

\[
\text{Remark 3.16. Both cases } d = 4, 5 \text{ of Corollary \(3.11\) fit into Kuznetsov's framework of homological projective duality, however his interpretation of the category } A_V \text{ is different. Specifically, in } [Kuz06, 6.5], \text{ for } d = 4, A_V \text{ is shown to be equivalent to a derived category of modules over an algebra over a different nodal curve, while in our approach no sheaves of algebras is needed. Using homological projective duality one can decompose } A_V \text{ for } d = 5 \text{ into derived categories of finite dimensional dg-algebras (see } [Kuz06, 6.1] \text{ for the smooth case), which is less restrictive than the Kawamata decomposition we construct for } d = 5.}
\]

\[
\text{Remark 3.17. In the } d = 5 \text{ case the two exceptional objects generating } A_V \text{ in the smooth case degenerate to derived categories of Burban’s algebras } (3.6), (3.7). \text{ Let us explain this in detail. For a 1-nodal degeneration one exceptional object degenerates to a single Burban algebra. For a 2-nodal degeneration one exceptional object degenerates to a double Burban algebra, or both exceptional objects degenerate to single Burban algebras. Finally for a 3-nodal degeneration two exceptional objects degenerate to single and double Burban algebras respectively.}
\]

\[
\text{Remark 3.18. The construction of a Kawamata decomposition for the nodal } V_0 [Kaw22b, 7.2] \text{ can also be done using the projection from a line as for } d = 4, 5 \text{ cases. Indeed, for } d = 6 \text{ we can apply the}
\]
blow up formula (Theorem 3.2) to the blow up \( \pi: Y \to W \), where \( W = \mathbb{P}^2 \times \mathbb{P}^1 \) as in (2.3) and the associated curve \( C \) is of type \((1,1)\). After mutating some line bundles, we can bring \( D^b(Y) \) into the same form as in Proposition 3.14 and we can express \( \mathcal{A}_V \) as
\[
\mathcal{A}_V \cong \langle \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{A}_C \rangle,
\]
where \( \mathcal{E}_i \) are exceptional objects.

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Department of Mathematics and Scientific Computing, University of Graz, Heinrichstrasse 36, 8020 Graz, Austria

Email address: nebojsa.pavic@uni-graz.at

School of Mathematical and Physical Sciences, University of Sheffield, Hounsfield Road, S3 7RH, UK

Email address: eugene.shinder@gmail.com