Dimension-free Euler estimates of rough differential equations

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Abstract

We extend the result in [3], [4] and [5], and give a dimension-free Euler estimation of solution of rough differential equations in term of the driving rough path. In the meanwhile, we prove that, the solution of rough differential equation is close to the exponential of a Lie series, with a concrete error bound.

1 Introduction

Suppose that $X$ is a continuous bounded variation path defined on some interval $I$ and taking its values in a Banach space $V$. We view this path as a stream of information and allow that it is highly oscillatory on normal scales. The theory of rough paths considers streams of information, such as $X$, for their effect on other systems and provides quantitative tools to model this interaction. Consider the stream as the input to an automata or controlled differential equation and so impacting on the evolution on the state $Y$ of some controlled system:

$$dY = f(Y) dX, \quad Y_0 = \xi.$$ (1)

A key development of the theory is the development of quantitative tools and estimates that allow one to estimate the response $Y$ from a top down analysis of $X$ and in particular provides a mechanism for directly quantifying the effects of the oscillatory components of $X$ without a detailed analysis of the trajectory of $X$. As a result, the methods apply to equations where $X$ does not have finite length. Differential equations driven by Brownian motion can treated deterministically.

The interest in modelling and understanding such interactions is rather wide. This manuscript is intended to create a useful interface by stating and proving one of the main results in a way that appears to the authors particularly useful for moving out into applications. It deliberately sets out to hide the machinery and implementation of the main proofs in rough path theory and to provide only a useful and rigorous statement of a result that captures the essence of what the machinery delivers and is valid across all Banach spaces (including finite dimensional ones) so that the methods can be used more widely without great initial intellectual investment.

Davie [3] established some high order Euler estimates of solution of rough differential equations, driven by $p$-rough paths, $1 \leq p < 3$. By using geodesic approximations, Friz and Victoir [4], [5] extend Davie’s results to rough differential equations driven by weak geometric $p$-rough paths, $p \geq 3$. The formulation and proof in [3], [4] and [5] are dimension-dependent, and the error bound may explode as the dimension increases.

By modifying the method used in [3], [4] and [5], we give a dimension-free high order Euler estimation of solution of rough differential equations (i.e. both the driving rough path and the solution path live in infinite dimensional spaces). Our estimates are first developed for ordinary differential equations. Then by passing to limit and using universal limit theorem (see [8], [10]), similar estimates holds for rough differential equations.

The main idea of our proof is to compare the solution of (1) (on small interval $[s, t]$) with the solution of another ordinary differential equation (on $[0, 1]$) whose vector field varies with $s, t$. Based on Arous
[1], Hu [7] and Castell [2], the solution of stochastic differential equation can be approximated on fixed small interval by the exponential of a Lie series, and the exponential can equivalently be treated as the solution of an ordinary differential equation. As we demonstrate, their idea is also applicable to rough differential equations in Banach spaces.

2 Background and Notations

We denote $\mathcal{U}$ and $\mathcal{V}$ as two Banach spaces.

2.1 Algebraic Structure

**Definition 1** We select a norm on tensor product of (elements in) $\mathcal{U}$ and $\mathcal{V}$, which satisfies the inequality: (up to an universal constant)

$$\|u \otimes v\|_{\mathcal{U} \otimes \mathcal{V}} \leq \|u\|_{\mathcal{U}} \|v\|_{\mathcal{V}}, \quad \forall u \in \mathcal{U}, \forall v \in \mathcal{V}. \tag{2}$$

Define $\mathcal{U} \otimes \mathcal{V}$ and $[\mathcal{U}, \mathcal{V}]$ as the closure of

$$\left\{ \sum_{k=1}^{m} u_k \otimes v_k, \{u_k\}_{k=1}^{m} \subset \mathcal{U}, \{v_k\}_{k=1}^{m} \subset \mathcal{V}, m \geq 1 \right\},$$

$$\left\{ \sum_{k=1}^{m} (u_k \otimes v_k - v_k \otimes u_k), \{u_k\}_{k=1}^{m} \subset \mathcal{U}, \{v_k\}_{k=1}^{m} \subset \mathcal{V}, m \geq 1 \right\},$$

w.r.t. the norm selected on the tensor product $\otimes$.

As an example, inequality (2) is satisfied by injective and projective tensor norms (Prop 2.1 and Prop 3.1 in [11]).

**Definition 2** For integers $n \geq k \geq 1$, denote $\pi_k$ as the projection of $1 \oplus \mathcal{V} \oplus \cdots \oplus \mathcal{V}^\otimes n$ to $\mathcal{V}^\otimes k$. Define $\exp_n : \mathcal{V} \oplus \cdots \oplus \mathcal{V}^\otimes n \to 1 \oplus \mathcal{V} \oplus \cdots \oplus \mathcal{V}^\otimes n$ as

$$\exp_n(a) := 1 + \sum_{k=1}^{n} \pi_k \left( \sum_{j=1}^{n} \frac{a^{\otimes j}}{j!} \right), \quad \forall a \in \mathcal{V} \oplus \cdots \oplus \mathcal{V}^\otimes n.$$

Define $\log_n : 1 \oplus \mathcal{V} \oplus \cdots \oplus \mathcal{V}^\otimes n \to \mathcal{V} \oplus \cdots \oplus \mathcal{V}^\otimes n$ as

$$\log_n(g) := \sum_{k=1}^{n} \pi_k \left( \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j} (g - 1)^{\otimes j} \right), \quad \forall g \in 1 \oplus \mathcal{V} \oplus \cdots \oplus \mathcal{V}^\otimes n.$$

**Definition 3** ($G^n(\mathcal{V})$) Suppose $\mathcal{V}$ is a Banach space. Then we define recursively

$$[\mathcal{V}]^{k+1} := [\mathcal{V}, [\mathcal{V}]^{k}] \quad \text{with} \quad [\mathcal{V}]^{1} := \mathcal{V}, \tag{3}$$

and define

$$G^n(\mathcal{V}) := \left\{ \exp_n(a) | a \in [\mathcal{V}]^{1} \oplus \cdots \oplus [\mathcal{V}]^{n} \right\}.$$

For $g, h \in G^n(\mathcal{V})$, we define product and inverse as

$$g \otimes h := \sum_{k=0}^{n} \left( \sum_{j=0}^{k} \pi_j (g) \otimes \pi_{k-j} (h) \right) \quad \text{and} \quad g^{-1} := 1 + \sum_{k=1}^{n} \pi_k \left( \sum_{j=1}^{n} (-1)^{j} (g - 1)^{\otimes j} \right).$$

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If we equip $G^n(\mathcal{V})$ with the homogeneous norm
\[
\|g\| := \sum_{k=1}^{n} \|\pi_k(g)\|^\frac{1}{p}, \forall g \in G^n(\mathcal{V}),
\]
then $G^n(\mathcal{V})$ is a topological group, called the step-$n$ nilpotent Lie group over $\mathcal{V}$. 

### 2.2 Rough Path

**Definition 4** ($S_n(x)$) Suppose $x : [0, T] \to \mathcal{V}$ is a continuous bounded variation path. For integer $n \geq 1$, define the step-$n$ signature of $x$, $S_n(x) : [0, T] \to (G^n(\mathcal{V}), \|\cdot\|)$ by assigning that, for $t \in [0, T]$,
\[
S_n(x)_t = \left(1, x_t - x_0, \int_{0 \leq u_1 < u_2 < \cdots < u_n \leq t} dx_{u_1} \otimes dx_{u_2} \cdots \int_{0 \leq u_1 < \cdots < u_n < t} dx_{u_1} \otimes \cdots \otimes dx_{u_n}\right).
\]

**Definition 5** ($d_p$ metric and $p$-variation) For $p \geq 1$, denote $[p]$ as the integer part of $p$. Suppose $X$ and $Y$ are continuous paths defined on $[0, T]$ taking value in $G^{[p]}(\mathcal{V})$. Define
\[
d_p(X, Y) := \max_{1 \leq k \leq [p]} \sup_{0 \leq t \in [0, T]} \left(\sum_{j, t_j \in D} \left\|\pi_k(X_{t_j, t_{j+1}}) - \pi_k(Y_{t_j, t_{j+1}})\right\|^p\right)^{\frac{1}{p}},
\]
where the supremum is taken over all finite partitions $D = \{t_j\}_{j=0}^n$ of $[0, T]$ with $0 = t_0 < t_1 < \cdots < t_n = T, n \geq 1$. With $e$ denotes the identity path (i.e. $e_t = 1 \in G^{[p]}(\mathcal{V}), t \in [0, T]$), we define the $p$-variation of $X$ on $[0, T]$ as
\[
\|X\|_{p-var, [0, T]} := d_p(X, e).
\]

**Definition 6** (geometric $p$-rough path) $X : [0, T] \to (G^{[p]}(\mathcal{V}), \|\cdot\|)$ is called a geometric $p$-rough path, if there exists a sequence of continuous bounded variation paths $x_l : [0, T] \to \mathcal{V}, l \geq 1$, such that
\[
\lim_{l \to \infty} d_p(S_{[p]}(x_l), X) = 0.
\]

**Definition 7** ($C^\gamma(\mathcal{V}, \mathcal{U})$) For $\gamma > 0$, we say $r : \mathcal{V} \to \mathcal{U}$ is Lip($\gamma$) and denote $r \in C^\gamma(\mathcal{V}, \mathcal{U})$, if and only if $r$ is $[\gamma]$-times Fréchet differentiable ($[\gamma]$ denotes the largest integer which is strictly less than $\gamma$), and
\[
|r|_{\text{Lip}(\gamma)} := \max_{0 \leq k \leq [\gamma]} \left\|D^k r\right\|_\infty \vee \left\|D^{[\gamma]} r\right\|_{(\gamma-[\gamma]) - H^0} < \infty,
\]
where $\|\cdot\|_\infty$ denotes the uniform norm and $\|\cdot\|_{(\gamma-[\gamma]) - H^0}$ denotes the $(\gamma-[\gamma])$-Hölder norm. Denote $C^\gamma(\mathcal{V}, \mathcal{U})$ as the space of bounded measurable mappings from $\mathcal{V}$ to $\mathcal{U}$.

**Definition 8** ($L(\mathcal{W}, C^\gamma(\mathcal{V}, \mathcal{U}))$) Suppose $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are Banach spaces. Denote $L(\mathcal{W}, C^\gamma(\mathcal{V}, \mathcal{U}))$ as the space of linear mappings from $\mathcal{W}$ to $C^\gamma(\mathcal{V}, \mathcal{U})$, and denote
\[
|f|_{\text{Lip}(\gamma)} := \sup_{w \in \mathcal{W} \|w\| = 1} \|f(w)\|_{\text{Lip}(\gamma)}, \forall f \in L(\mathcal{W}, C^\gamma(\mathcal{V}, \mathcal{U})).
\]

We define solution of rough differential equation as in Def 5.1 in Lyons [9].

**Definition 9** (solution of RDE) Suppose $\mathcal{U}$ and $\mathcal{V}$ are two Banach spaces, $X : [0, T] \to (G^{[p]}(\mathcal{V}), \|\cdot\|)$ is a geometric $p$-rough path, $f \in L(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U}))$ for some $\gamma > p - 1$ and $\xi \in \mathcal{U}$. Define $h : \mathcal{V} \oplus \mathcal{U} \to L(\mathcal{V} \oplus \mathcal{U}, \mathcal{V} \oplus \mathcal{U})$ as
\[
h(v_1, u_1)(v_2, u_2) = (v_2, f(v_2)(u_1 + \xi)), \forall v_1, v_2 \in \mathcal{V}, \forall u_1, u_2 \in \mathcal{U}.
\]
Then geometric $p$-rough path $Z: [0, T] \to (G[p] (V \oplus U), \|\cdot\|)$ is said to be a solution to the rough differential equation

$$dY = f(Y) dX, \ Y_0 = \xi,$$

if $\pi_{G[p]} (V) (Z) = X$, and $Z$ satisfies the rough integral equation (in sense of Def 4.9 [9]):

$$Z_t = \int_0^t h(Z_u) dZ_u, \ t \in [0, T].$$

For $g \in G^n (V)$ and $\lambda > 0$, we denote $\delta_{\lambda} g := 1 + \sum_{k=1}^{n} \lambda^k \pi_k (g)$.

**Theorem 10 (Lyons)** When $f$ in [5] is in $L (V, C^\gamma (U, U))$ for $\gamma > p$, the solution of [5] exists uniquely. Moreover, there exists constant $C_{p, \gamma}$, such that, for any interval $[s, t] \subseteq [0, T]$ satisfying $|f|_{\text{Lip} (\gamma)} \|X\|_{p\text{-var}, [s, t]} \leq 1$, we have (after rescaling $f$, $X$ and $Y$)

$$\left\| \left( \delta_{|f|_{\text{Lip} (\gamma)}}, X, Y \right) \right\|_{p\text{-var}, [s, t]} \leq C_{p, \gamma} |f|_{\text{Lip} (\gamma)} \left\|X\right\|_{p\text{-var}, [s, t]} .$$

**Remark 11** The unique solution of [5] is recovered by a sequence of rough integrals. Then based on Thm 4.12 and Prop 5.9 in [9] and by using lower semi-continuity of $p$-variation, the constant $C_{p, \gamma}$ in [5] is an absolute constant which only depends on $p$ and $\gamma$, and is finite whenever $\gamma > p - 1$.

When $U$ and $V$ are finite dimensional spaces, for any $f \in L (V, C^\gamma (U, U))$, $\gamma > p - 1$, there exists a solution to [5] which satisfies [5]. Indeed, based on Prop 5.9 [9], when $f$ is $\text{Lip} (\gamma)$ for $\gamma > p - 1$, the sequence of Picard iterations $\{Z^n\}_n : [0, T] \to G[p] (V \oplus U)$, define recursively as rough integrals: (with $h$ defined at [4])

$$Z^0_t = (X_t, 0) , \ t \in [0, T],$$

$$Z^{n+1}_t = \int_0^t h(Z^n_u) dZ^n_u, \ t \in [0, T], \ n \geq 0,$$

are uniformly bounded and equi-continuous. When $V$ and $U$ are finite-dimensional spaces, bounded sets in $G[p] (V \oplus U)$ are relatively compact. Thus, based on Arzelà-Ascoli theorem, there exists a subsequence $\{Z^{n_k}\}_k$ which converge uniformly (denoted the limit as $Z$). Then by spelling out the almost-multiplicative functional (associated with the Picard iteration) and letting $k$ tends to infinity, one can prove that $Z$ is a solution to the rough differential equation [5]. Then based on Thm 4.12 and Prop 5.9 in [9] and by using lower semi-continuity of $p$-variation, the estimate [5] holds for $Z$.

When $U$ is a Banach space and when $f$ in [5] is $\text{Lip} (\gamma)$ for $\gamma \in (p - 1, p)$, there does not always exist a solution to [5]. Godunov [6] proved that, "each Banach space in which Peano’s theorem is true is finite-dimensional". Shkarin [12] (in Cor 1.5) proved that, for any real infinite dimensional Banach space (denoted as $V$), which has a complemented subspace with an unconditional Schauder basis, and for any $\alpha \in (0, 1)$, there exists $\alpha$-Hölder continuous function $f : V \to V$, such that the equation $\dot{z} = f(x)$ has no solution in any interval of the real line. Based on Shkarin [12] (Rrk 1.4), $L_p [0, 1] (1 \leq p < \infty)$ and $C [0, 1]$ are examples of such Banach spaces, and "roughly speaking, all infinite dimensional Banach spaces, which naturally appear in analysis" fall into this category.

### 2.3 Differential Operator

For $\gamma \geq 0$, recall $C^\gamma (U, U)$ in Definition [4] and that $L (V, C^\gamma (U, U))$ denotes the space of linear mappings from $V$ to $C^\gamma (V, U)$. With $f \in L (V, C^\gamma (U, U))$ and integer $k \leq \gamma + 1$, we clarify the meaning of differential operator $f^{\otimes k} (v)$ for $v \in V^{\otimes k}$. For $r \in C^k (U, U)$ and $j = 0, 1, \ldots, k$, $D^j r \in L (U^{\otimes j}, C^{k-j} (U, U))$. 

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Suppose \( f \in \mathcal{U} \), denote \( \mathcal{D}^k(\mathcal{U}) \) as the set of \( k \)th order differential operators (on \( \text{Lip}(k) \) functions from \( \mathcal{U} \) to \( \mathcal{U} \)). More specifically, \( p \in \mathcal{D}^k(\mathcal{U}) \) if and only if \( p : C^k(\mathcal{U}, \mathcal{U}) \to C^0(\mathcal{U}, \mathcal{U}) \) and there exist bounded \( p^j : \mathcal{U} \to \mathcal{U} \), \( j = 0, 1, \ldots, k \), with \( p_k \neq 0 \), such that

\[
p(r)(u) = \sum_{j=0}^{k} (D^j r)(p_j(u))(u), \quad \forall u \in \mathcal{U}, \quad \forall r \in C^k(\mathcal{U}, \mathcal{U}).
\]

We define norm \( |\cdot|_k \) on \( \mathcal{D}^k(\mathcal{U}) \) as

\[
|p|_k := \max_{j=0,1,\ldots,k} \sup_{u \in \mathcal{U}} \|p_j(u)\|, \quad \forall p \in \mathcal{D}^k(\mathcal{U}).
\]

Then \( \mathcal{D}^k(\mathcal{U}) \) can be extended to a Banach space \( (\mathcal{D}^k(\mathcal{U}), |\cdot|_k) \) (with the natural addition and scalar multiplication).

**Definition 13 (composition)** Suppose \( p^1 \in \mathcal{D}^{j_1}(\mathcal{U}) \) and \( p^2 \in \mathcal{D}^{j_2}(\mathcal{U}) \) for integers \( j_1 \geq 0, j_2 \geq 0 \). Define the composition of \( p^1 \circ p^2 \in \mathcal{D}^{j_1+j_2}(\mathcal{U}) \) as

\[
(p^1 \circ p^2)(r) := (p^1 p^2)(r), \quad \forall r \in C^{j_1+j_2}(\mathcal{U}, \mathcal{U}).
\]

When \( p \in \mathcal{D}^j(\mathcal{U}), j \geq 0 \), we define the differential operator \( p^{\circ k} \in \mathcal{D}^{k-j}(\mathcal{U}) \) for integer \( k \geq 1 \) as

\[
p^{\circ 1} := p \quad \text{and} \quad p^{\circ k} := p \circ p^{\circ (k-1)}, \quad k \geq 2.
\]

Composition of differential operators is associative, i.e. \( (p^1 \circ p^2) \circ p^3 = p^1 \circ (p^2 \circ p^3) \).

**Definition 14 (\( f^{\circ k} \))** Suppose \( f \in L(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U})) \) for some \( \gamma \geq 0 \). Then for any \( v \in \mathcal{V} \), we treat \( f(v) \) as a first order differential operator (i.e. in \( \mathcal{D}^1(\mathcal{U}) \)), and define

\[
f(v)(r)(u) := (Dr)(f(v))(u), \quad \forall u \in \mathcal{U}, \quad \forall r \in C^1(\mathcal{U}, \mathcal{U}).
\]

For integer \( k \in 1, 2, \ldots, [\gamma] + 1 \) and \( \{v_j\}_{j=1}^{k} \subset \mathcal{V} \), we define \( f^{\circ k}(v_1 \circ \cdots \circ v_k) \in \mathcal{D}^k(\mathcal{U}) \) as

\[
f^{\circ k}(v_1 \circ \cdots \circ v_k) := f(v_1) \circ f(v_2) \circ \cdots \circ f(v_k).
\]

Then we denote \( f^{\circ k} \in L(\mathcal{V}^{\otimes k}, (\mathcal{D}^k(\mathcal{U}), |\cdot|_k)) \) as the unique continuous linear operator satisfying (8).

## 3 Main Result

In this manuscript, we work with the first level (or "path" level) solution of rough differential equations.

Firstly, we prove a lemma for ordinary differential equations. Then after applying universal limit theorem (Thm 5.3 [9]), this lemma leads to similar estimates of rough differential equations. The proof of this lemma is in the same spirit as Lemma 2.4(a) in [3], Lemma 16 in [4] and Lemma 10.7 in [5], only that we use the ordinary differential equation [10] for the approximation.

We denote \( \mathcal{U} \) and \( \mathcal{V} \) as two Banach spaces. For \( p \geq 1 \), denote \( [p] \) as the integer part of \( p \). For \( \gamma > 0 \), denote \([\gamma]\) as the largest integer which is strictly less than \( \gamma \). Denote \( I_d : \mathcal{U} \to \mathcal{U} \) as the identity function, i.e. \( I_d(u) = u, \forall u \in \mathcal{U} \). For \( f \in L(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U})) \), integer \( k \leq \gamma + 1 \) and \( v \in \mathcal{V}^{\otimes k} \), recall the differential operator \( f^{\circ k}(v) \) defined in Definition [10].

**Lemma 15** Suppose \( \mathcal{U} \) and \( \mathcal{V} \) are two Banach spaces, \( x : [0, T] \to \mathcal{V} \) is a continuous bounded variation path, \( f \in L(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U})) \) for \( \gamma > 1 \), and \( x \in \mathcal{U} \). Denote \( y : [0, T] \to \mathcal{U} \) as the unique solution to the ordinary differential equation

\[
dy = f(y) \, dx, \quad y_0 = x \in \mathcal{U}.
\]
Then for any \( p \in [1, \gamma + 1) \), there exists a constant \( C_{p,\gamma} \), which only depends on \( p \) and \( \gamma \), such that, for any \( 0 \leq s < t \leq T \), if we denote \( y^{s,t} : [0, 1] \to \mathcal{U} \) as the unique solution of the ordinary differential equation: (with \( y_s \) denotes the value of \( y \) in \( \mathcal{U} \) at point \( s \))

\[
dy^{s,t}_u = \left( \sum_{k=1}^{[\gamma]} f^{\circ k} \pi_k \left( \log_{[\gamma]+1} \left( S_{[\gamma]+1} (x)_{s,t} \right) \right) (I_d) (y^{s,t}_u) \right) du, \quad u \in [0, 1],
\]

\[
y^{s,t}_0 = y_s + f^{\circ (\gamma+1)} \pi_{[\gamma]+1} \left( \log_{[\gamma]+1} \left( S_{[\gamma]+1} (x)_{s,t} \right) \right) (I_d) (y_s),
\]

then

\[
(1) \| y_t - y^{s,t}_1 \| \leq C_{p,\gamma} \| f^{[\gamma]+1} \log_{[\gamma]+1} \left( S_{[\gamma]+1} (x) \right) \|_{p\text{-var},[s,t]},
\]

\[
(2) \| y_t - y_s - \sum_{k=1}^{[\gamma]+1} f^{\circ k} \pi_k \left( S_{[\gamma]+1} (x)_{s,t} \right) (I_d) (y_s) \| \leq C_{p,\gamma} \| f^{[\gamma]+1} \log_{[\gamma]+1} \left( S_{[\gamma]+1} (x) \right) \|_{p\text{-var},[s,t]}.
\]

The proof of Lemma 15 starts from page 10. Since \( f \in L (\mathcal{V}, C^\gamma (\mathcal{U}, \mathcal{U})) \), it might be more appropriate to write 10 as \( dy = f (dx) (y) \). We keep it in the current form so that it is consistent with the classical notation of ordinary differential equations.

**Remark 16** We used 10 instead of the ordinary differential equation

\[
dy^{s,t}_u = \left( \sum_{k=1}^{[\gamma]+1} f^{\circ k} \pi_k \left( \log_{[\gamma]+1} \left( S_{[\gamma]+1} (x)_{s,t} \right) \right) (I_d) (y^{s,t}_u) \right) du, \quad u \in [0, 1],
\]

\[
y^{s,t}_0 = y_s,
\]

because \( \mathcal{U} \) is a Banach space, and 12 may not has a solution (Cor 1.5 in Shkarin 12). If 12 has a solution (e.g. when \( \mathcal{U} \) is finite dimensional), then 11 holds with \( y^{s,t}_1 \) replaced by \( y^{s,t}_1 \).

**Remark 17** When \( \mathcal{V} = \mathbb{R}^d \) and \( \mathcal{U} = \mathbb{R}^e \) in Lemma 17, suppose \( f = \left( f^1, \ldots, f^d \right) \in L \left( \mathbb{R}^d, C^\gamma (\mathbb{R}^e, \mathbb{R}^e) \right) \). For \( i = 1, \ldots, d \), we treat \( f^i = \left( f_{i,1}, \ldots, f_{i,d} \right) \) as a first order differential operator: \( \sum_{j=1}^e f_{ij} \frac{\partial}{\partial y_j} \). Then it can be checked that (with \( x = (x^1, \ldots, x^d) : [0, T] \to \mathbb{R}^d \))

\[
f^{\circ k} \pi_k \left( S_{[\gamma]+1} (x)_{s,t} \right) (I_d) = \sum_{i_1, \ldots, i_k = 1, \ldots, d} (f^{i_1} \circ \cdots \circ f^{i_k}) (I_d) \int \cdots \int_{s < u_1 < \cdots < u_k < t} dx_{u_1}^{i_1} \cdots dx_{u_k}^{i_k},
\]

and our formulation coincides with the formulation in 3, 4, and 5.

The theorem below follows from universal limit theorem (Thm 5.3 9) and Lemma 15. Suppose \( X : [0, T] \to G^p (\mathcal{V}) \) is a geometric \( p \)-rough path. For integer \( n \geq [p] \), we denote \( S_n (X) \) as the unique enhancement of \( X \) to a continuous path with finite \( p \)-variation taking value in \( G^n (\mathcal{V}) \) (Thm 3.7 9).

**Theorem 18** Suppose \( f \in L (\mathcal{V}, C^\gamma (\mathcal{U}, \mathcal{U})) \) for \( \gamma > 1 \), \( X : [0, T] \to G^p (\mathcal{V}) \) is a geometric \( p \)-rough path for some \( p \in [1, \gamma) \), and \( \xi \in \mathcal{U} \). Denote \( Z \) as the unique solution (in the sense of Definition 2 of the rough differential equation

\[
dY = f (Y) \, dX, \quad Y_0 = \xi.
\]

Denote \( Y := \pi_{G^p (\mathcal{U})(\mathcal{Z})} (Z) \). Then there exists a constant \( C_{p,\gamma} \), which only depends on \( p \) and \( \gamma \), such that, for any \( 0 \leq s < t \leq T \), if denote \( y^{s,t} : [0, 1] \to \mathcal{U} \) as the unique solution of the ordinary differential equation:

\[
dy^{s,t}_u = \left( \sum_{k=1}^{[\gamma]} f^{\circ k} \pi_k \left( \log_{[\gamma]+1} \left( S_{[\gamma]+1} (X)_{s,t} \right) \right) (I_d) (y^{s,t}_u) \right) du, \quad u \in [0, 1],
\]

\[
y^{s,t}_0 = \pi_1 (y_s) + f^{\circ (\gamma+1)} \pi_{[\gamma]+1} \left( \log_{[\gamma]+1} \left( S_{[\gamma]+1} (X)_{s,t} \right) \right) (I_d) (\pi_1 (y_s)),
\]
then we have
\[
(1), \| \pi_1(Y_t) - y_1^{s,t} \| \leq C_{p,\gamma} |f|_{\text{Lip}(\gamma)} \| X \|_{p\text{-var.}[s,t]},
\]
\[
(2), \left\| \pi_1(Y_{s,t}) - \sum_{k=1}^{|\gamma|+1} f^{\circ k} \pi_k \left( S_{|\gamma|+1} X_{s,t} \right) \left( I_d \right) \left( \pi_1(Y_s) \right) \right\| \leq C_{p,\gamma} |f|_{\text{Lip}(\gamma)} \| X \|_{p\text{-var.}[s,t]}.
\]

The proof of Theorem 18 is on page 19.

**Remark 19** Suppose \( f \in L(V, C^\gamma(U,U)) \) for \( \gamma > 1 \), \( X : [0,T] \to G^{[p]}(V) \) is a weak geometric \( p \)-rough path\(^1\) for some \( p \in [1, \gamma + 1) \), and \( \xi \in U \). Then by following similar reasoning as that of Theorem 18, one can prove that, any solution, in the sense of Def 10.17 in [9], to the rough differential equation
\[
\frac{dy}{dt} = f(y) \, dX, \quad y_0 = \xi,
\]
satisfies the estimates in Theorem 18.

The theorem below follows from Thm 4.12 [9] and Lemma 13.

**Theorem 20** Suppose \( f \in L(V, C^\gamma(U,U)) \) for \( \gamma > 1 \), and \( X \) is a geometric \( p \)-rough path for some \( p \in [1, \gamma + 1) \). Denote \( Y : [0,T] \to G^{[p]}(U) \) as the rough integral (in the sense of Def 4.9 [9])
\[
Y_t = \int_0^t f(X) \, dX, \quad t \in [0,T].
\]
Theorem 20 is almost the same as that of Theorem 18 only that we estimate the rough differential equation
\[
d(Y_t) = (1_V, f(Y)) \, dX, \quad (Y_t)_0 = (Y_0, 0) \in V \oplus U,
\]
and uses Thm 4.12 [9] instead of universal limit theorem.

---

\(^1\)Continuous path \( X : [0,T] \to G^{[p]}(V) \) is called a weak geometric \( p \)-rough path, if \( \| X \|_{p\text{-var.}[0,T]} < \infty \).
4 Proof

We made explicit the dependence of constants (e.g. \( C_{p,\gamma} \)), but the exact value of constants may change from line to line. We denote \( \mathcal{U} \) and \( \mathcal{V} \) as two Banach spaces. Recall \( C^\gamma (\mathcal{U},\mathcal{U}) \) in Definition [7] on page [8].

**Lemma 21** Suppose \( k \geq 1 \) is an integer, and \( f \in L (\mathcal{V}, C^{k-1}(\mathcal{U},\mathcal{U})) \). Recall \( f^{\sigma k} \in L (\mathcal{V}^{\otimes k}, (\mathcal{D}^k (\mathcal{U}), \lVert \cdot \rVert_k)) \) defined in Definition [14] (on page 5). Then for any \( v \in \mathcal{V}^k \) (denoted at [12] on page 2), \( f^{\sigma k} (v) \) is a first order differential operator, which satisfies (with \( I_d : \mathcal{U} \to \mathcal{U} \) denotes the identity function)

\[
f^{\sigma k} (v) (r) = (D_r) \left( f^{\sigma k} (v) (I_d) \right), \quad \forall r \in C^1 (\mathcal{U},\mathcal{U}).
\]

**Proof.** To prove that \( f^{\sigma k} (v) \) is a first order differential operator, we define another first order differential operator, and want to prove that they are the same.

Suppose \( \mathcal{V}^1 \) and \( \mathcal{V}^2 \) are two Banach spaces, and \( f^i \in L (\mathcal{V}^i, (\mathcal{D}^k (\mathcal{U}), \lVert \cdot \rVert_k)) \), \( i = 1, 2 \). Define \( [f^1, f^2] \in L (\mathcal{V}^1 \otimes \mathcal{V}^2, (\mathcal{D}^1 (\mathcal{U}), \lVert \cdot \rVert_1)) \) as the unique continuous linear operator, which satisfies that, for any \( v^1 \in \mathcal{V}^1, \) any \( v^2 \in \mathcal{V}^2, \) and any \( r \in C^1 (\mathcal{U},\mathcal{U}) \),

\[
[f^1, f^2] (v^1 \otimes v^2) (r) = (D_r) \left( f^1 (v^1) \circ f^2 (v^2) - f^2 (v^2) \circ f^1 (v^1) \right) (I_d).
\]

For integer \( k \geq 1 \) and \( f \in L (\mathcal{V}, C^{k-1}(\mathcal{U},\mathcal{U})) \), define \( [f]^{\sigma k} \in L (\mathcal{V}^{\otimes k}, (\mathcal{D}^1 (\mathcal{U}), \lVert \cdot \rVert_1)) \) as (with \( f^{\sigma 1} \) defined in Definition [14])

\[
[f]^{\sigma 1} (v) := f^{\sigma 1} (v), \forall v \in \mathcal{V}, \quad [f]^{\sigma k} := \left[ f^{\sigma 1}, [f]^{\sigma (k-1)} \right] \quad \text{for} \quad k \geq 2.
\]

Then by definition, for any \( k \geq 1 \) and any \( v \in \mathcal{V}^k \), \( [f]^{\sigma k} (v) \) is a first order differential operator.

If we can prove that \( f^{\sigma k} (v^k) \) is a first order differential operator for any \( v^k \) in the form

\[
v^k = \begin{cases} v_1, & \text{if } k = 1; \\
[v_1, \ldots, v_k], & \text{if } k \geq 2,
\end{cases}
\]

then since any \( v \in \mathcal{V}^k \) can be approximated by linear combinations of \( v^k \) in the form of (16), by using that \( f^{\sigma k} : \mathcal{V}^{\otimes k} \to (\mathcal{D}^k (\mathcal{U}), \lVert \cdot \rVert_k) \) is linear and continuous (Definition [14]), and that \( \mathcal{V}^k \) is a closed subspace of \( \mathcal{V}^{\otimes k} \), we can prove that \( f^{\sigma k} (v) \) is a first order differential operator for any \( v \in \mathcal{V}^k \).

We define linear map \( \sigma : [\mathcal{V}]^k \to \mathcal{V}^{\otimes k} \) by assigning

\[
\sigma (v_1) := v_1, \forall v_1 \in \mathcal{V}, \text{if } k = 1,
\]

\[
\sigma ([v_1, \ldots, v_k]) := v_1 \otimes \cdots \otimes v_{k-1} \otimes v_k, \forall \{v_j\}_{j=1}^k \subset \mathcal{V}, \text{if } k \geq 2.
\]

For any \( v^k \) in the form of (16) with \( \sigma \) defined at (14), we want to prove

\[
f^{\sigma k} (v^k) (r) = [f]^{\sigma k} (\sigma (v^k)) (r), \quad \forall r \in C^1 (\mathcal{U},\mathcal{U}).
\]

By definition, \( [f]^{\sigma k} (\sigma (v^k)) \) is a first order differential operator, and

\[
[f]^{\sigma k} (\sigma (v^k)) (r) = (D_r) \left( [f]^{\sigma k} (\sigma (v^k)) (I_d) \right), \quad \forall r \in C^1 (\mathcal{U},\mathcal{U}).
\]

If we can prove (18), then

\[
f^{\sigma k} (v^k) (r) = (D_r) \left( [f]^{\sigma k} (\sigma (v^k)) (I_d) \right) = (D_r) \left( f^{\sigma k} (v^k) (I_d) \right), \quad \forall r \in C^1 (\mathcal{U},\mathcal{U}).
\]

Thus, in the following, we concentrate on proving (18).
It is clear that, \((18)\) is true when \(k = 1\). Indeed, for any \(v^1 \in \mathcal{V}\), since \([f]^{\gamma 1} (v^1) \) := \(f^{\gamma 1} (v^1)\) (see (15)) and \(\sigma (v^1) := v^1\) (see (14)), we have
\[
[f]^{\gamma 1} (v^1) = f^{\gamma 1} (v^1) = f^{\gamma 1} (\sigma (v^1)), \forall v^1 \in \mathcal{V}.
\]
Then we use mathematical induction. Suppose that for integer \(K \geq 1\), \(k = 1, 2, \ldots, K\) and any \(v^k\) in the form of (16), we have
\[
f^{\gamma k} (v^k) = [f]^{\gamma k} (\sigma (v^k)).
\]
We want prove that, for any \(v_0 \in \mathcal{V}\), and any \(v^K\) in the form of (15),
\[
f^{\gamma (K+1)} ([v_0, v^K]) = [f]^{\gamma (K+1)} (v_0 \otimes \sigma (v^K)).
\]
Based on definition (14) and (15), we have, for any \(r \in C^1 (\mathcal{U}, \mathcal{U})\),
\[
[f]^{\gamma (K+1)} (v_0 \otimes \sigma (v^K)) (r) = \left[ f^{\gamma 1} (v_0), \left[ f^{\gamma K} (\sigma (v^K)) \right] (r) \right] (\sigma (v^K)) (\sigma (v^K)) (D^2 r) (f^{\gamma 1} (v_0) (I_d)) (f^{\gamma 1} (v_0) (I_d)),
\]
If we assume in addition that \(r \in C^2 (\mathcal{U}, \mathcal{U})\), then by using that \(f^{\gamma 1} (v_0)\) and \([f]^{\gamma K} (\sigma (v^K))\) are first order differential operators, we have
\[
(Dr) \left( f^{\gamma 1} (v_0) \otimes [f]^{\gamma K} (\sigma (v^K)) \right) (I_d)
\]
\[
= f^{\gamma 1} (v_0) \otimes [f]^{\gamma K} (\sigma (v^K)) (r) - (D^2 r) (f^{\gamma 1} (v_0) (I_d)) (f^{\gamma 1} (v_0) (I_d)),
\]
and
\[
(Dr) \left( [f]^{\gamma K} (\sigma (v^K)) \otimes f^{\gamma 1} (v_0) \right) (I_d)
\]
\[
= \left( [f]^{\gamma K} (\sigma (v^K)) \otimes f^{\gamma 1} (v_0) \right) (r) - (D^2 r) (f^{\gamma 1} (v_0) (I_d)) \left( [f]^{\gamma K} (\sigma (v^K)) \right) (I_d),
\]
Using inductive hypothesis (19) and the definition of \(f^{\gamma (K+1)}\) in Definition (14) we have
\[
\left( f^{\gamma 1} (v_0) \otimes [f]^{\gamma K} (\sigma (v^K)) \right) - \left( [f]^{\gamma K} (\sigma (v^K)) \otimes f^{\gamma 1} (v_0) \right)
\]
\[
= f^{\gamma 1} (v_0) \otimes f^{\gamma K} (v^K) - f^{\gamma K} (v^K) \otimes f^{\gamma 1} (v_0) = f^{\gamma (K+1)} (v_0 \otimes v^K - v^K \otimes v_0) = f^{\gamma (K+1)} ([v_0, v^K]),
\]
Since our differentiability is in Fréchet’s sense and \(r \in C^2 (\mathcal{U}, \mathcal{U})\), we have
\[
(D^2 r) (u_1 \otimes u_2) = (D^2 r) (u_2 \otimes u_1), \forall u_1 \in \mathcal{U}, \forall u_2 \in \mathcal{U}.
\]
Thus, combining (20), (21), (22), (23) and (24), we have,
\[
[f]^{\gamma (K+1)} (v_0 \otimes \sigma (v^K)) (r) = f^{\gamma (K+1)} ([v_0, v^K]) (r), \forall r \in C^2 (\mathcal{U}, \mathcal{U}).
\]
Since \([f]^{\gamma (K+1)} (v_0 \otimes \sigma (v^K)) := [f]^{\gamma 1} (v_0), [f]^{\gamma K} (\sigma (v^K))\) has the explicit form (14), by comparing the ”coefficients” of \(\{D^j r\}_{j=0}^{K+1}\) for any \(r \in C^{K+1} (\mathcal{U}, \mathcal{U})\), we get that (25) holds for any \(r \in C^1 (\mathcal{U}, \mathcal{U})\).

Lemma 22 Suppose \(\mathcal{V}\) and \(\mathcal{U}\) are two Banach spaces, \(f \in L (\mathcal{V}, C^\gamma (\mathcal{U}, \mathcal{U}))\) for some \(\gamma > 1\) and \(\xi \in \mathcal{U}\). Denote \(\lceil \gamma \rceil\) as the largest integer which is strictly less than \(\gamma\). Suppose \(g \in C^{(\gamma)+1} (\mathcal{V})\). Then, there
exists constant $C_{\gamma}$, which only depends on $\gamma$, such that, the unique solution of the ordinary differential equation

$$
\begin{align*}
 dy_u &= \sum_{k=1}^{[\gamma]} f^o \pi_k \left( \log_{[\gamma]+1} g \right) (I_d) (y_u) \, du, \quad u \in [0,1], \\
y_0 &= \xi + f^o([\gamma]+1) \pi_{[\gamma]+1} \left( \log_{[\gamma]+1} (g) \right) (I_d) (\xi),
\end{align*}
$$

(26)

satisfies

$$
\left\| y_1 - \xi - \sum_{k=1}^{[\gamma]+1} f^o \pi_k (g) (I_d) (\xi) \right\| \leq C_{\gamma} \left( |f|_{Lip(\gamma)} \|g\| \right)^{[\gamma]+1}.
$$

Proof. Denote $\{\gamma\} := \gamma - [\gamma]$, and denote $N := [\gamma] + 1$. We assume $|f|_{Lip(\gamma)} = 1$. Otherwise, we replace $f$ by $|f|^{-1}_{Lip(\gamma)} f$ and replace $g$ by $\delta_{|f|_{Lip(\gamma)}} g$ (with $\delta_x g := 1 + \sum_{k=1}^{N} \lambda^k \pi_k (g)$). For integer $k = 1, 2, \ldots, [\gamma] + 1$ and $v \in V^{\otimes k}$, based on Definition of differential operator $f^o (v)$ in [13] on page 5, $f^o (v) (I_d) \in C^{-k+1}(U,U)$ and

$$
|f^o (v) (I_d)|_{Lip(\gamma-k+1)} = \|v\| |f|_{Lip(\gamma)}^o \left( \frac{f}{|f|_{Lip(\gamma)}} \right) (I_d) \leq C_{\gamma} \|v\| |f|_{Lip(\gamma)}^o.
$$

(27)

When $\|g\| > 1$, it can be computed that ($|f|_{Lip(\gamma)} = 1$)

$$
\left\| \sum_{k=1}^{N-1} f^o (I_d) (\xi) \pi_k (g) \right\| \leq C_{\gamma} \|g\|^{N-1} \quad \text{and} \quad \|y_1 - \xi\| \leq \|y_0 - \xi\| + \|y\|_{1-\text{var},[0,1]} \leq C_{\gamma} \|g\|^N.
$$

Thus, ($\|g\| > 1, N \leq \gamma + 1$)

$$
\left\| y_1 - \xi - \sum_{k=1}^{N} f^o (I_d) (\xi) \pi_k (g) \right\| \leq C_{\gamma} \|g\|^N \leq C_{\gamma} \|g\|^{\gamma+1},
$$

and lemma holds. In the following, we assume $|f|_{Lip(\gamma)} = 1$ and $\|g\| \leq 1$.

It is clear that, the solution $y$ of the ordinary differential equation (26) satisfies (since $|f|_{Lip(\gamma)} = 1$ and $\|g\| \leq 1$)

$$
\sup_{u \in [0,1]} \|y_u - \xi\| \leq \|y_0 - \xi\| + \|y\|_{1-\text{var},[0,1]} \leq C_{\gamma} \|g\|.
$$

(28)

For integer $k = 1, \ldots, N$, denote differential operator $F^k$ as

$$
F^k := f^o \pi_k \left( \log_{N} (g) \right).
$$

Based on Lemma [21], $\{F^k\}_{k=1}^{N}$ are first order differential operators, and satisfies

$$
F^k (r) = (Dr) F^k (I_d), \quad \forall r \in C^1 (U,U).
$$

Similar as [27], since we assumed $|f|_{Lip(\gamma)} = 1$, for $1 \leq k \leq N - 1$, we have

$$
\left| D F^k (I_d) \right|_{Lip(\gamma-k)} \leq C_{\gamma} \|g\|^k;
$$

(29)

for $k_i \geq 1$, $\sum_{i=1}^{\gamma} k_i = k \leq N$, we have

$$
\left| (F^{k_1} \circ \ldots \circ F^{k_1}) (I_d) \right|_{Lip(\gamma+1-k)} \leq C_{\gamma} \|g\|^k.
$$

(30)
By using the fact that \( y \) satisfies (26), we have

\[
y_1 - \xi - f^{oN} \pi_N (\log_N (g)) (I_d) (\xi) - \sum_{k=1}^{N-1} F^k (I_d) (\xi) \tag{31}
\]

\[
= \sum_{k=1}^{N-1} \int \int_{0 \leq u_1 \leq u_2 \leq 1} (F^k (I_d) (y_0) - F^k (I_d) (\xi)) \, du_1 \, du_2 \\
+ \sum_{1 \leq k_1 \leq N-1, i=1,2} \int \int_{0 \leq u_1 \leq u_2 \leq 1} DF^{k_1} (I_d) F^{k_2} (I_d) (y_{u_1}) \, du_1 \, du_2.
\]

Since \( y_0 = \xi + f^{oN} \pi_N (\log_N g) (I_d) (\xi) \), by using (30), we have \( \| f \|_{Lip(\gamma)} = 1 \), \( \| g \| \leq 1 \) and \( \gamma \leq N \)

\[
\left\| \sum_{k=1}^{N-1} \int \int_{0 \leq u_1 \leq u_2 \leq 1} F^k (I_d) (y_0) - F^k (I_d) (\xi) \, du_1 \, du_2 \right\| \leq C_{\gamma} \| g \|^{1+N} \leq C_{\gamma} \| g \|^{1+\gamma}.
\]

When \( k_1 \geq 1 \), \( k_2 \geq 1 \), \( k_1 + k_2 \leq N \), using that \( F^{k_2} \) is a first order differential operator, we have \( DF^{k_1} (I_d) F^{k_2} (I_d) = (F^{k_2} \circ F^{k_1}) (I_d) \) . Thus,

\[
\int \int_{0 \leq u_1 \leq u_2 \leq 1} DF^{k_1} (I_d) F^{k_2} (I_d) (y_{u_1}) \, du_1 \, du_2 = \int \int_{0 \leq u_1 \leq u_2 \leq 1} (F^{k_2} \circ F^{k_1}) (I_d) (y_{u_1}) \, du_1 \, du_2. \tag{32}
\]

When \( 1 \leq k_1 \leq N-1 \), \( 1 \leq k_2 \leq N-1 \), \( k_1 + k_2 \geq N+1 \), by combining (29) and (30), we get

\[
\left\| \int \int_{0 \leq u_1 \leq u_2 \leq 1} DF^{k_2} (I_d) F^{k_1} (I_d) (y_{u_1}) \, du_1 \, du_2 \right\| \leq C_{\gamma} \| g \|^{k_1+k_2} \leq C_{\gamma} \| g \|^{N+1} \leq C_{\gamma} \| g \|^{\gamma+1}. \tag{33}
\]

Therefore, combining (30), (29) and (30), we have

\[
\left\| y_1 - \xi - \sum_{k=1}^{N} F^k (I_d) (\xi) - \sum_{1 \leq k_1 \leq N-1, k_1 + k_2 \leq N} \int \int_{0 \leq u_1 \leq u_2 \leq 1} (F^{k_2} \circ F^{k_1}) (I_d) (y_{u_1}) \, du_1 \, du_2 \right\| \leq C_{\gamma} \| g \|^{\gamma+1}.
\]

Then we continue to estimate

\[
\sum_{1 \leq k_1 \leq N-1, k_1 + k_2 \leq N} \int \int_{0 \leq u_1 \leq u_2 \leq 1} (F^{k_2} \circ F^{k_1}) (I_d) (y_{u_1}) \, du_1 \, du_2.
\]

When \( 1 \leq k_1 \leq N-1 \), \( 1 \leq k_2 \leq N-1 \), \( k_1 + k_2 = N \), by using (30) and (29), we have

\[
\left\| \int \int_{0 \leq u_1 \leq u_2 \leq 1} ((F^{k_2} \circ F^{k_1}) (I_d) (y_{u_1}) - (F^{k_2} \circ F^{k_1}) (I_d) (\xi)) \, du_1 \, du_2 \right\| \leq C_{\gamma} \| g \|^{N} \sup_{u \in [0,1]} \| y_u - \xi \|^{\gamma} \leq C_{\gamma} \| g \|^{\gamma+1}.
\]

When \( k_1 \geq 1 \), \( k_2 \geq 1 \) and \( k_1 + k_2 \leq N-1 \), we have

\[
\sum_{k_1 \geq 1, k_1 + k_2 \leq N-1} \int \int_{0 \leq u_1 \leq u_2 \leq 1} ((F^{k_2} \circ F^{k_1}) (I_d) (y_{u_1}) - (F^{k_2} \circ F^{k_1}) (I_d) (\xi)) \, du_1 \, du_2
\]

\[
= \sum_{k_1 \geq 1, k_1 + k_2 \leq N-1} \int \int_{0 \leq u_1 \leq u_2 \leq 1} ((F^{k_2} \circ F^{k_1}) (I_d) (y_0) - (F^{k_2} \circ F^{k_1}) (I_d) (\xi)) \, du_1 \, du_2
\]

\[
+ \sum_{k_1 \geq 1, k_1 + k_2 \leq N-1} \int \int_{0 \leq u_1 \leq u_2 \leq 1} D (F^{k_2} \circ F^{k_1}) (I_d) F^{k_3} (I_d) (y_{u_1}) \, du_1 \, du_2 \, du_3.
\]
Then since \(y_0 = \xi + f^{oN}\pi_N (\log_N g) (I_d) (\xi)\), by using (30) and (28), we have

\[
\left\| \sum_{k_1 \geq 1, k_1 + k_3 \leq N} \left( (F^{k_2} \circ F^{k_1}) (y_0) - (F^{k_2} \circ F^{k_1}) (I_d) (\xi) \right) du_1 du_2 \right\|
\leq C_\gamma \|g\|^2 |y_0 - \xi| \leq C_\gamma \|g\|^{\gamma+1}.
\]

Then similar as in (32) and (33), when \(k_1 + k_2 + k_3 \leq N\), we have

\[
\int \int \int_{0 < u_1 < u_2 < u_3} D (F^{k_2} \circ F^{k_1}) (I_d) F^{k_3} (I_d) (y_{u_1}) \, du_1 du_2 du_3 = \int \int \int_{0 < u_1 < u_2 < u_3} (F^{k_3} \circ F^{k_2} \circ F^{k_1}) (I_d) (y_{u_1}) \, du_1 du_2 du_3;
\]

when \(k_1 + k_2 + k_3 \geq N + 1\), we have

\[
\int \int \int_{0 < u_1 < u_2 < u_3} D (F^{k_2} \circ F^{k_1}) (I_d) F^{k_3} (I_d) (y_{u_1}) \, du_1 du_2 du_3 \leq C_\gamma \|g\|^{\gamma+1}.
\]

Repeating this "subtraction and estimation" process for \(N\) times, we get

\[
\left\| y_1 - \xi - \sum_{j=1}^{N} \frac{1}{j!} \sum_{k_1, k_1 + \cdots + k_j \leq N} (F^{k_j} \circ \cdots \circ F^{k_1}) (I_d) (\xi) \right\| \leq C_\gamma \|g\|^{\gamma+1}.
\]

Since \(f^{o^k}\) is linear in \(V \otimes k\) (Definition 14 on page 5), we have

\[
\sum_{j=1}^{N} \frac{1}{j!} \sum_{k_1, k_1 + \cdots + k_j \leq N} (F^{k_j} \circ \cdots \circ F^{k_1}) (I_d) (\xi)
= \sum_{j=1}^{N} \frac{1}{j!} \sum_{k_1, k_1 + \cdots + k_j \leq N} f^{o(k_1 + \cdots + k_j)} (\pi) \pi_{k_j} (\log_N (g)) \otimes \cdots \otimes \pi_{k_1} (\log_N (g)) (I_d) (\xi)
= \sum_{k=1}^{N} f^{o^k} \pi_k (g) (I_d) (\xi).
\]

Therefore, we have

\[
\left\| y_1 - \xi - \sum_{k=1}^{N} f^{o^k} \pi_k (g) (I_d) (\xi) \right\| \leq C_\gamma \|g\|^{\gamma+1}.
\]

Lemma 23 Suppose \(V\) and \(U\) are two Banach spaces, \(f \in L (V, C^\gamma (U, U))\) for some \(\gamma > 1\) and \(\xi \in U\). Suppose \(g \in C^{[\gamma]+1} (V)\). Then, the unique solution of the ordinary differential equation

\[
\begin{align*}
dy_u &= \sum_{k=1}^{[\gamma]} f^{o^k} \pi_k \left( \log_{[\gamma]+1} (g) \right) (I_d) (y_u) \, du, \ u \in [0, 1], \\
y_0 &= \xi + f^{o([\gamma]+1)} \pi_{[\gamma]+1} \left( \log_{[\gamma]+1} (g) \right) (I_d) (\xi),
\end{align*}
\]

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satisfies, for \( k = 1, 2, \ldots, [\gamma] + 1 \), and any \( v \in V^{\otimes k} \),
\[
\left\| f^{\circ k} (v) \right\|_{L^2(V^*)} \leq C_{\gamma, v} \left\| f \right\|_{L^2(V^*)} \left\| g \right\|^{\gamma + 1 - k}.
\]

\textbf{Proof.} This lemma can be proved similarly as Lemma 22.

\textbf{Lemma 24} Suppose \( V \) and \( U \) are two Banach spaces, \( f \in L^1(V, C^\gamma (U, U)) \) for some \( \gamma > 1 \) and \( \xi \in U \). Denote \( N := [\gamma] + 1 \), and suppose \( g \in G^N (V) \). Denote \( y^\rho \) as the solution of the ordinary differential equation:
\[
dy^\rho_u = \left( \sum_{k=1}^{N-1} f^{\circ k} \pi_k (\log_N (g)) (I_d) (y^\rho_u) \right) du, \quad u \in [0, 1],
\]
\[
y^\rho_0 = \xi + f^{\circ N} \pi_N (\log_N (g)) (I_d) (\xi).
\]

For \( g, h \in G^N (V) \), denote \( y^{g, h} \) as the unique solution to the integral equation:
\[
y^{g, h}_t = \left\{ \begin{array}{c}
\xi + f^{\circ N} \pi_N (\log_N (g)) (I_d) (\xi) + \int_0^t \sum_{k=1}^{N-1} f^{\circ k} \pi_k (\log_N (g)) (I_d) (y^{g, h}_u) du, \quad t \in [0, 1],

y^{g, h}_1 + f^{\circ N} \pi_N (\log_N (h)) (I_d) (y^{g, h}_1) + \int_1^t \sum_{k=1}^{N-1} f^{\circ k} \pi_k (\log_N (h)) (I_d) (y^{g, h}_u) du, \quad t \in (1, 2].
\end{array} \right.
\]

Then we have
\[
\left\| y^{g, h}_1 - y^{g, h}_0 \right\| \leq C_{\gamma, \| g \|, \| h \|} (\| g \|^{\gamma + 1} \| h \|^{\gamma + 1} + \| g \|^{\gamma + 1} + \| h \|^{\gamma + 1}).
\]

\textbf{Proof.} We only prove the Lemma when \( \| f \|_{L^1(V^*)} = 1 \). Otherwise, we replace \( f \) by \( \frac{1}{\| f \|_{L^1(V^*)}} f \), and replace \( g \) and \( h \) by \( \delta f \pi_L (\gamma, \gamma) \) and \( \delta f \pi_L \), respectively.

Since \( \sum_{k=1}^{N-1} f^{\circ k} \pi_k (\log_N (g)) (I_d) \in C^1 (U, U) \), based on the definition of \( y^{g, h} \) and \( y^\rho \), we have \( y^{g, h}_t = y^\rho_t \), \( t \in [0, 1] \). For \( g, h \in G^N (V) \), by using Lemma 22, we get
\[
\| y^{g, h}_1 - y^{g, h}_1 \| = \| y^\rho_1 - \xi + y^{g, h}_1 - y^{g, h}_1 \| = \| y^\rho_1 - \xi \|
\]
\[
\leq \sum_{k=1}^{N} f^{\circ k} \pi_k (g) (I_d) (\xi) + \sum_{k=1}^{N} f^{\circ k} \pi_k (h) (I_d) (y^\rho_1) - \sum_{k=1}^{N} f^{\circ k} \pi_k (g \otimes h) (I_d) (\xi)
\]
\[
+ C_{\gamma, \| g \|, \| h \|, \| g \otimes h \|} \| g \|^{\gamma + 1}.
\]

Based onLemma 24 for \( k = 1, 2, \ldots, N \),
\[
\left\| f^{\circ k} \pi_k (h) (I_d) (y^\rho_1) - \sum_{j=0}^{N-k} f^{\circ j+k} (\pi_j (g) \otimes \pi_k (h)) (I_d) (\xi) \right\| \leq C_{\gamma, \| h \|} \| g \|^{\gamma + 1 - k}.
\]

As a result,
\[
\| y^{g, h}_1 - y^{g, h}_1 \|
\]
\[
\leq \sum_{k=1}^{N} f^{\circ k} \pi_k (g) (I_d) (\xi) + \sum_{k=1}^{N} \sum_{j=0}^{N-k} f^{\circ j+k} (\pi_j (g) \otimes \pi_k (h)) (I_d) (\xi) - \sum_{k=1}^{N} f^{\circ k} \pi_k (g \otimes h) (I_d) (\xi)
\]
\[
+ C_{\gamma, \| g \|, \| h \|, \| g \otimes h \|} \| g \|^{\gamma + 1} + C_{\gamma, \| h \|} \| g \|^{\gamma + 1}.
\]

\textbf{■}
Lemma 25 Suppose $U$ and $V$ are two Banach spaces, $x : [0, T] \rightarrow V$ is a continuous bounded variation path, and $f \in L(V, C^\gamma(U, U))$ for $\gamma \geq 1$. Denote $y : [0, T] \rightarrow U$ as the unique solution of the ordinary differential equation

$$dy = f(y) \, dx, \quad y_0 = \xi \in U.$$  

(35)

Then for any $p \in [1, \gamma + 1]$, there exists constant $C_{p, \gamma}$ (which only depends on $p$ and $\gamma$), such that for any interval $[s, t] \subset [0, T]$ satisfying $\|f|_{Lip(\gamma)} \| S_p(x) \|_{p-var,[s,t]} \leq 1$, we have

$$\|S_p(y)\|_{p-var,[s,t]} \leq C_{p, \gamma} |f|_{Lip(\gamma)} \| S_p(x) \|_{p-var,[s,t]}.$$  

(36)

**Proof.** Define $h : V \oplus U \rightarrow L(V \oplus U, V \oplus U)$ as

$$h(v_1, u_1)(v_2, u_2) = (v_2, f(v_2)(u_1 + \xi)), \quad \forall v_1, v_2 \in V, \forall u_1, u_2 \in U.$$

We define geometric $p$-rough paths $Z(n) : [0, T] \rightarrow G[p](V \oplus U)$, $n \geq 0$, recursively as the rough integral (in the sense of Def 4.9 [9]):

$$Z(0)_t : = (S_p(x)_t, 0) \in G[p](V \oplus U), \quad t \in [0, T],$$  

(37)

$$Z(n+1)_t : = \int_0^t h(Z(n)) \, dZ(n), \quad t \in [0, T], \quad n \geq 0,$$

and define $Y(n) : [0, T] \rightarrow G[p](U)$ as $Y(n) := \pi_{G[p](U)} Z(n)$. Then based on Prop 5.9 [9], there exists constant $C_{p, \gamma}$, which only depends on $p$ and $\gamma$ and is finite whenever $\gamma > p - 1$, such that, for any interval $[s, t]$ satisfying $\|f|_{Lip(\gamma)} \| S_p(x) \|_{p-var,[s,t]} \leq 1$, we have

$$\sup_n \|Y(n)\|_{p-var,[s,t]} \leq C_{p, \gamma} |f|_{Lip(\gamma)} \| S_p(x) \|_{p-var,[s,t]}.$$  

(38)

(Indeed, by properly scaling $f$ and $S_p(x)$, the constant $C_{p, \gamma}$ in (38) can be chosen to be independent of $|f|_{Lip(\gamma)}$ and $\|S_p(x)\|_{p-var,[0,T]}$.) On the other hand, since $x$ is continuous with bounded variation, it can be checked that, if we define continuous bounded variation paths $y(n) : [0, T] \rightarrow U$, $n \geq 1$, recursively as

$$y(0)_t \equiv 0 \in U, \quad t \in [0, T],$$  

(39)

$$y(n+1)_t = \int_0^t f(y(n) + \xi) \, dx, \quad t \in [0, T],$$

then based on the definition of rough integral in Def 4.9 [9], it can be checked that,

$$Y(n) = S_p(y(n)), \quad \forall n \geq 0.$$  

(40)

Combined with (38), for interval $[s, t]$ satisfying $|f|_{Lip(\gamma)} \| S_p(x) \|_{p-var,[s,t]} \leq 1$, we have

$$\sup_n \|S_p(y(n))\|_{p-var,[s,t]} \leq C_{p, \gamma} |f|_{Lip(\gamma)} \| S_p(x) \|_{p-var,[s,t]}.$$  

(41)

On the other hand, since $f$ is $Lip(\gamma)$ for $\gamma \geq 1$, by using (39), we have, for any $[s, t] \subset [0, T]$,

$$\|y(n+2) - y(n+1)\|_{1-var,[s,t]} \leq |f|_{Lip(\gamma)} \|x\|_{1-var,[s,t]} \left(\|y(n+1) - y(n)\|_{1-var,[s,t]} + \|y(n+1)_s - y(n)_s\|\right).$$  

(42)

Then we divide $[0, T] := \cup_{j=0}^{m-1} [t_j, t_{j+1}]$ in such a way that

$$|f|_{Lip(\gamma)} \|x\|_{1-var,[t_j, t_{j+1}]} \leq c < 1, \quad j = 0, 1, \ldots, m - 1.$$
Then, for \([t_j, t_{j+1}], j = 0, 1, \ldots, m - 1\), we let \(n\) tends to infinity (in (42)), and get

\[
\lim_{n \to \infty} \|y(n + 1) - y(n)\|_{1-\text{var},[t_j, t_{j+1}]}
\leq c \lim_{n \to \infty} \|y(n + 1) - y(n)\|_{1-\text{var},[t_{j-1}, t_{j+1}]} + c \lim_{n \to \infty} \|y(n + 1)_{t_j} - y(n)_{t_j}\|.
\]

Since \(y(n)_0 \equiv 0, \forall n \geq 0,\) and \(c \in (0,1),\) we can prove inductively that

\[
\lim_{n \to \infty} \|y(n + 1) - y(n)\|_{1-\text{var},[t_j, t_{j+1}]} = 0, \quad j = 0, 1, \ldots, m - 1.
\]

Thus

\[
\lim_{n \to \infty} \|y(n + 1) - y(n)\|_{1-\text{var},[0,T]} = \sum_{j=0}^{m-1} \lim_{n \to \infty} \|y(n + 1) - y(n)\|_{1-\text{var},[t_j, t_{j+1}]} = 0.
\]

As a result, we have that \(y(n)\) converge in 1-variation as \(n\) tends to infinity (denote the limit as \(\tilde{y}\)), and we have

\[
\lim_{n \to \infty} \max_{1 \leq k \leq |p|} \sup_{0 \leq s \leq t \leq T} \|\pi_k \left( S[p](y(n))_{s,t} \right) - \pi_k \left( S[p](\tilde{y})_{s,t} \right) \| = 0. \tag{43}
\]

Based on (39) and let \(n\) tends to infinity, we have

\[
\tilde{y}_t = \int_0^t f(\bar{y}_u + \xi) \, dx_u.
\]

As a result, if denote \(y\) as the unique solution of the ordinary differential equation (35), then we have

\[
y = \bar{y} + \xi. \tag{44}
\]

Therefore, combine (41), (43), (44) and use lower semi-continuity of \(p\)-variation, we get, for interval \([s, t]\) satisfying \(|f|_{Lip(\gamma)} \|S[p](x)\|_{p-\text{var},[s,t]} \leq 1,

\[
\|S[p](y)\|_{p-\text{var},[s,t]} = \|S[p](\bar{y} + \xi)\|_{p-\text{var},[s,t]} = \|S[p](\tilde{y})\|_{p-\text{var},[s,t]} \\
\leq \lim_{n \to \infty} \|S[p](y(n))\|_{p-\text{var},[s,t]} \leq C_{p,\gamma} |f|_{Lip(\gamma)} \|S[p](x)\|_{p-\text{var},[s,t]}.
\]

Lemma 15 Suppose \(\mathcal{U}\) and \(\mathcal{V}\) are two Banach spaces, \(x : [0, T] \to \mathcal{V}\) is a continuous bounded variation path, \(f \in L(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U}))\) for \(\gamma > 1\), and \(\xi \in \mathcal{U}\). Denote \(y : [0, T] \to \mathcal{U}\) as the unique solution to the ordinary differential equation

\[
dy = f(y) \, dx, \quad y_0 = \xi \in \mathcal{U}. \tag{45}
\]

Then for any \(p \in [1, \gamma + 1]\), there exists a constant \(C_{p,\gamma}\), which only depends on \(p\) and \(\gamma\), such that, for any \(0 \leq s < t \leq T\), if we denote \(y^{s,t} : [0, 1] \to \mathcal{U}\) as the unique solution of the ordinary differential equation: (with \(y_s\) denotes the value of \(y\) in (45) at point \(s\))

\[
dy^{s,t}_{u} = \left( \sum_{k=1}^{[\gamma]} f^{\circ k} \pi_k \left( \log_{[\gamma]+1} \left( S_{[\gamma]+1}(x)_{s,t} \right) \right) (I_d)(y^{s,t}_{u}) \right) \, du, \quad u \in [0, 1], \tag{46}
\]

\[
y^{s,t}_0 = y_s + f^{\circ([\gamma]+1)} \pi_{[\gamma]+1} \left( \log_{[\gamma]+1} \left( S_{[\gamma]+1}(x)_{s,t} \right) \right) (I_d)(y_s),
\]

then

\[
\begin{align*}
(1) & , \|y_t - y^{s,t}_t\| \leq C_{p,\gamma} |f|_{Lip(\gamma)} \|S[p](x)\|_{p-\text{var},[s,t]}^{[\gamma]+1} , \\
(2) & , \|y_t - y_s - \sum_{k=1}^{[\gamma]+1} f^{\circ k} \pi_k \left( S_{[\gamma]+1}(x)_{s,t} \right) (I_d)(y_s)\| \leq C_{p,\gamma} |f|_{Lip(\gamma)} \|S[p](x)\|_{p-\text{var},[s,t]}^{[\gamma]+1} .
\end{align*} \tag{47}
\]

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Proof. We only prove the first estimate in (47); the second follows based on Lemma 22 on page 9. We prove the result when \( |f|_{Lip(\gamma)} = 1 \). The general case can be proved by replacing \( f \) by \( |f|_{Lip(\gamma)} \) \( \hat{f} \) and replacing \( S_{\hat{f}}(x) \) by \( \delta_{Lip(\gamma)}(S_{\hat{f}}(x)) \) (in which case both \( y \) in (45) and \( y^{s,t} \) in (46) would stay unchanged).

Denote \( N := \lfloor \gamma \rfloor + 1 \). Define \( \omega : \{(s,t) | 0 \leq s \leq t \leq T\} \to \mathbb{R}^\gamma \) as

\[
\omega(s,t) := \|S_{\hat{f}}(x)\|^p_{p-var,[s,t]}.
\]

Then it can be checked that, \( \omega \) is continuous and is super-additive, i.e.

\[
\omega(s,u) + \omega(u,t) \leq \omega(s,t), \quad \forall 0 \leq s \leq u \leq t \leq T. \tag{48}
\]

With \( y^{s,t} \) defined at (46), we define \( \Gamma : \{(s,t) | 0 \leq s \leq t \leq T\} \to \mathcal{U} \) as

\[
\Gamma_{s,t} := y_t - y_{s,t} = y_t - y_s - (y_{s,t} - y_s).
\]

For \( 0 \leq s \leq u \leq t \leq T \), with \( y^{s,u} \) defined at (46) and \( x \) in (46), we denote \( \tilde{y}^{s,u} \) as the unique solution of the ordinary differential equation:

\[
d\tilde{y}^{s,u} = \left( \sum_{k=1}^{N-1} \int_{r} f^{\circ k}(p \times (t, \tilde{y}^{s,u})) \pi_k \left( \log_{N} S_N(x)_{\tilde{y}^{s,u}} \right) dr \right), \quad r \in [0,1],
\]

\[
\tilde{y}^{s,u}_0 = y_{s} + f^{\circ N}(p \times (t, y^{s,u})) \pi_N \left( \log_{N} S_N(x)_{\tilde{y}^{s,u}} \right).
\]

For \( 0 \leq s \leq u \leq t \leq T \), we denote piecewise continuous path \( y^{s,u,t} : [0,2] \to \mathcal{U} \) by assigning

\[
y^{s,u,t}_r := y^{s,u} \quad \text{when} \quad r \in [0,1], \quad \text{and} \quad y^{s,u,t}_r := \tilde{y}^{s,u} \quad \text{when} \quad r \in (1,2). \tag{49}
\]

Firstly, suppose \([s,t]\) is an interval satisfying \( \omega(s,t) \leq 1 \) and \( u \in (s,t) \). Then

\[
\|\Gamma_{s,u} + \Gamma_{u,t} - \Gamma_{s,t}\| = \|y^{s,u}_t - y_s + y^{u,t}_1 - y_u - (y^{s,t}_1 - y_s)\| \\
\leq \|y^{s,u}_1 - y^{s,t}_1\| + \|y^{u,t}_1 - y^{u,t}_1\| + \|y^{u,t}_1 - y^{u,t}_1 - (y^{u,t}_1 - y_u)\|.
\]

Then, based on Lemma 24 and Lemma 22 we have (\( |f|_{Lip(\gamma)} = 1 \) and \( \omega(s,t) \leq 1 \), \( \lfloor \gamma \rfloor := \gamma - \lfloor \gamma \rfloor \))

\[
\|\Gamma_{s,u} + \Gamma_{u,t} - \Gamma_{s,t}\| \leq C_{\gamma} \omega(s,t) \tag{50}
\]

Based on the definition of \( y^{s,u} \) (at (46)), when \( \omega(s,t) \leq 1 \), we have

\[
\|y^{s,u}_1 - y_s\| \leq \|y^{s,u}\|_{1-var,[0,1]} + \|y^{s,u}_1 - y_s\| \leq C_{\gamma} \omega(s,t)^\frac{1}{\gamma}.
\]

On the other hand, according to Lemma 23 there exists constant \( C_{p,\gamma} \) (which only depends on \( p \) and \( \gamma \), and is finite whenever \( \gamma > p - 1 \)), such that for any interval \([s,t]\) satisfying \( \omega(s,t) \leq 1 \), we have

\[
\|y_u - y_s\| \leq C_{p,\gamma} \omega(s,t)^\frac{1}{\gamma}.
\]

As a result, combining (51) and (52), we have, when \( \omega(s,t) \leq 1 \),

\[
\|y_u - y_s\| \leq \|y^{s,u}_1 - y_s\|\|\gamma\| \omega(s,t)^\frac{1}{\gamma} \leq C_{p,\gamma} \omega(s,t)^\frac{1}{\gamma}. \tag{53}
\]
Then, continuing with (50), we get, for any interval \([s, t]\) satisfying \(\omega(s, t) \leq 1\) and any \(u \in (s, t)\),
\[
\|\Gamma_{s,t}\| \leq \left(1 + C_{\omega} \omega(s, t) \frac{1}{\delta}\right) \left(\|\Gamma_{s,u}\| + \|\Gamma_{u,t}\|\right) + C_{p,\omega,\gamma} \omega(s, t) \delta \frac{n}{\delta}.
\] (53)

With \(C_{\omega}\) and \(C_{p,\omega,\gamma}\) in (53), suppose \([s, t]\) is an interval satisfying \(\omega(s, t) \leq 1\), denote
\[
\delta := \left(C_{\omega}^{0} \lor C_{p,\omega,\gamma}^{0}\right) \omega(s, t).
\]

Then since \(\omega\) is super-additive (i.e. (45)), by setting \([t^{0}_{0}, t^{0}_{1}] = [s, t]\) and recursively dividing \([t^{n}_{j}, t^{n}_{j+1}] = [t^{n+1}_{2j}, t^{n+1}_{2j+1}] \cup [t^{n+1}_{2j+1}, t^{n+2}_{2j+2}]\) in such a way that
\[
\omega\left(t^{n+1}_{2j}, t^{n+1}_{2j+1}\right) = \omega\left(t^{n+1}_{2j+1}, t^{n+2}_{2j+2}\right) \leq \frac{1}{2} \omega\left(t^{n}_{j}, t^{n}_{j+1}\right), j = 0, 1, \ldots, 2^{n} - 1, n \geq 0,
\]
we have, based on (53),
\[
\|\Gamma_{s,t}\| \leq \lim_{n \to \infty} \left(\sum_{k=0}^{n} \left(\prod_{j=0}^{k} \left(1 + 2^{-\frac{k}{\delta}} \delta \frac{1}{\delta}\right)\right) \left(\frac{1}{2}\right) \left(\frac{n+1}{\delta}\right)^{k}\right) \left(\frac{n}{\delta}\right)^{\frac{2}{\delta}}.
\] (54)

Then we prove that \(\lim_{n \to \infty} \sum_{j=0}^{2^{n}-1} \|\Gamma_{n,t_{j+1}}\| = 0\). Since \(x : [0, T] \to \mathcal{V}\) is a continuous bounded variation path and \(y : [0, T] \to \mathcal{U}\) is the solution of the ordinary differential equation
\[
dy = f(y) \, dx, \quad y_{0} = \xi,
\]
we have that \(|f|_{\text{Lip}^{0}} = 1, \gamma > 1\)
\[
\|y\|_{1-\text{var},[s,t]} \leq \|x\|_{1-\text{var},[s,t]}, \forall 0 \leq s \leq t \leq T.
\]

Thus,
\[
\left\|y_{t} - y_{s} - \sum_{k=1}^{N} f^{0}_{k} \pi_{k} \left(S_{[p]} \left(x\right)_{s, t}\right) \left(I_{d}\right) \left(y_{s}\right)\right\|.
\] (55)

On the other hand, based on Lemma 22 we have
\[
\left\|y_{s}^{1,t} - y_{s} - \sum_{k=1}^{N} f^{0}_{k} \pi_{k} \left(S_{[p]} \left(x\right)_{s,t}\right) \left(I_{d}\right) \left(y_{s}\right)\right\| \leq C_{y} \left\|S_{[p]} \left(x\right)_{s,t}\right\|_{1-\text{var},[s,t]}^{\gamma+1} \leq C_{p,\gamma} \left\|x\right\|_{1-\text{var},[s,t]}^{\gamma+1}.
\] (56)

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Thus, combining (55) and (56), we get
\[ \|\Gamma_{t_j}^{t_{j+1}}\| \leq C_{p,\gamma} \|x\|^{\gamma+1}_{1-\text{var},[t_j^{\cdot},t_{j+1}^{\cdot}]}, \quad j = 0, 1, \ldots, 2^n - 1, \quad n \geq 0, \]
and we have \((\gamma \geq 1)\)
\[ \lim_{n \to \infty} \sum_{j=0}^{2^n-1} \|\Gamma_{t_j}^{t_{j+1}}\| = 0. \]

Thus, continuing with (54), we get that, there exists constant \(C_{p,\gamma}\), which only depends on \(p\) and \(\gamma\), and is finite whenever \(\gamma > p - 1\), such that, for any interval \([s, t]\) satisfying \(\omega(s, t) \leq 1\), we have
\[ \|y_t - y_t^{s,t}\| = \|\Gamma_{s,t}\| \leq C_{p,\gamma}\omega(s, t)^{\frac{n+1}{p}}. \] (57)

For \([s, t]\) satisfying \(\omega(s, t) > 1\), as in Prop 5.10 \([5]\), we decompose \([s, t] = \cup_{j=0}^{n-1}[t_j, t_{j+1}]\) in such a way that \(\omega(t_j, t_{j+1}) = 1, \quad j = 0, 1, \ldots, n - 2, \) and \(\omega(t_{n-1}, t_n) \leq 1\). Then by using super-additivity of \(\omega\), we have \(n - 1 \leq \omega(s, t)\), and
\[ \|y_t - y_s\| \leq \sum_{j=0}^{n-1} \|y_{t_{j+1}} - y_{t_j}\| \leq C_{p,\gamma} \left( n - 1 + \omega(t_{n-1}, t_n)\right) \]
\[ \leq C_{p,\gamma}n \leq C_{p,\gamma} (\omega(s, t) + 1) \leq 2C_{p,\gamma}\omega(s, t). \]

On the other hand, when \(\omega(s, t) \geq 1\),
\[ \|y_1^{s,t} - y_s\| \leq \|y_1^{s,t} - y_0^{s,t}\| + \|y_0^{s,t} - y_s\| \]
\[ \leq C_\gamma \|S_p[x]_{s,t}\|^{N-1} + C_\gamma \|S_p[x]_{s,t}\|^{N} \leq C_\gamma\omega(s, t)^{\frac{N}{p}}. \]

Therefore, when \(\omega(s, t) \geq 1\),
\[ \|y_t - y_1^{s,t}\| = \|y_t - y_s - (y_1^{s,t} - y_s)\| \leq C_{p,\gamma}\omega(s, t) + C_\gamma\omega(s, t)^{\frac{N}{p}} \leq C_{p,\gamma}\omega(s, t)^{\frac{n+1}{p}}. \]

\[ \blacksquare \]

**Lemma 26** Suppose \(f \in \mathcal{L}(\mathcal{V}, C^\gamma(\mathcal{U}, \mathcal{U}))\) for some \(\gamma > 1\). For \(g \in G[\gamma]+1(\mathcal{V})\) and \(\xi \in \mathcal{U}\), define \(y(g, \xi) : [0, 1] \to \mathcal{U}\) as the unique solution of the ordinary differential equation:
\[ dy_u = \sum_{k=1}^{[\gamma]} f^\circ \pi_k \left( \log_{[\gamma]+1}(g) \right) (I_d)(y_u) \, du, \quad u \in [0, 1], \]
\[ y_0 = \xi + f^\circ([\gamma]+1) \pi_{[\gamma]+1} \left( \log_{[\gamma]+1}(g) \right)(I_d)(\xi) \in \mathcal{U}, \]
\[ y_0 = \xi + f^\circ([\gamma]+1) \pi_{[\gamma]+1} \left( \log_{[\gamma]+1}(g) \right)(I_d)(\xi) \in \mathcal{U}, \]
If there exist \(\{g^l\}_{l \geq 1} \subset G[\gamma]+1(\mathcal{V})\) and \(\{\xi^l\}_{l \geq 1} \subset \mathcal{U}\) such that
\[ \lim_{l \to \infty} \max_{1 \leq k \leq [\gamma]+1} \|\pi_k(g^l) - \pi_k(g)\| = 0 \quad \text{and} \quad \lim_{l \to \infty} \|\xi^l - \xi\| = 0, \]
then
\[ \lim_{l \to \infty} \sup_{t \in [0, 1]} \|y(g^l, \xi^l)_t - y(g, \xi)_t\| = 0. \]
Proof. Since \( \sum_{k=1}^{[\gamma]} f^{(k)} (I_d) \in C^1 (\mathcal{U}, \mathcal{U}) \), based on Thm 3.15 \[16\] (their result extends naturally to ordinary differential equations in Banach spaces), we get

\[
\sup_{t \in [0,1]} \left\| y \left( g', \xi \right)_t - y \left( g, \xi \right)_t \right\| \\
\leq C_f \left( \left\| y \left( g', \xi \right)_0 - y \left( g, \xi \right)_0 \right\| + \sum_{k=1}^{[\gamma]} \left\| f^{(k)} \left( \log_{[\gamma]+1} \left( g' \right) \right) - \pi_k \left( \log_{[\gamma]+1} \left( g \right) \right) \right\| \left( I_d \right)_{\infty} \right) \\
\leq C_f \left( \left\| \xi_t - \xi \right\| + \left\| \pi_{[\gamma]+1} \left( \log_{[\gamma]+1} \left( g \right) \right) \right\| \left\| \xi_t - \xi \right\| \left( \gamma \right) + \sum_{k=1}^{[\gamma]+1} \left\| \pi_k \left( \log_{[\gamma]+1} \left( g' \right) \right) - \pi_k \left( \log_{[\gamma]+1} \left( g \right) \right) \right\| \right).
\]

Proof of Theorem \[18\]. Based on the definition of geometric p-rough path (in Definition \[6\] on page \[83\]), there exists a sequence of continuous bounded variation paths \( \{x^l\} : [0,T] \to \mathcal{V} \), such that

\[
\lim_{l \to \infty} d_p \left( S_{[p]} \left( x^l \right), X \right) = 0.
\]

As a result, we have

\[
\lim_{l \to \infty} \left\| S_{[p]} \left( x^l \right) \right\|_{p-var,[s,t]} = \left\| X \right\|_{p-var,[s,t]}, \quad \forall 0 \leq s \leq t \leq T,
\]

and (based on Thm 3.1.3 \[10\])

\[
\lim_{l \to \infty} \max_{n=1,2,\ldots,[\gamma]+1} \left\| \pi_n \left( S_{[\gamma]+1} \left( x^l \right)_s \right) - \pi_n \left( S_{[\gamma]+1} \left( X \right)_s \right) \right\| = 0, \quad \forall 0 \leq s \leq t \leq T.
\]

On the other hand, denote \( y^l : [0,T] \to \mathcal{U} \) as the unique solution of the ordinary differential equation

\[
dy^l = f \left( y' \right) dx^l, \quad y^l_0 = \xi,
\]

and denote \( Y := \pi_{G^{[p]}(\mathcal{U})} (Z) \) with \( Z \) denotes the unique solution (in the sense of Definition \[9\] of the rough differential equation

\[
dY = f \left( Y \right) dX, \quad Y_0 = \xi.
\]

Then based on universal limit theorem (Thm 5.3 \[9\]), we have

\[
\lim_{l \to \infty} \left\| y^l_1 - \pi_1 \left( Y_1 \right) \right\| = 0, \quad \forall t \in [0,T].
\]

For \( 0 \leq s \leq t \leq T \) and \( l \geq 1 \), denote \( y^{s,t,l} : [0,1] \to \mathcal{U} \) as the unique solution of the ordinary differential equation:

\[
dy^{s,t,l}_u = \sum_{k=1}^{[\gamma]} f^{(k)} \pi_k \left( \log_{[\gamma]+1} \left( S_{[\gamma]+1} \left( x^l \right) \right) \right) \left( I_d \right) \left( y^{s,t,l}_u \right) du, \quad u \in [0,1],
\]

\[
y^{s,t,l}_0 = y^l_s + f^{(\left[ \gamma \right]+1)} \pi_{\left[ \gamma \right]+1} \left( \log_{[\gamma]+1} \left( S_{[\gamma]+1} \left( x^l \right) \right) \right) \left( I_d \right) \left( y^l_1 \right) \in \mathcal{U}.
\]

For \( 0 \leq s \leq t \leq T \), denote \( y^{s,t} : [0,1] \to \mathcal{U} \) as the unique solution of the ordinary differential equation:

\[
dy^{s,t}_u = \sum_{k=1}^{[\gamma]} f^{(k)} \pi_k \left( \log_{[\gamma]+1} \left( S_{[\gamma]+1} \left( X \right) \right) \right) \left( I_d \right) \left( y^{s,t}_u \right) du, \quad u \in [0,1],
\]

\[
y^{s,t}_0 = \pi_1 \left( Y_s \right) + f^{(\left[ \gamma \right]+1)} \pi_{\left[ \gamma \right]+1} \left( \log_{[\gamma]+1} \left( S_{[\gamma]+1} \left( X \right) \right) \right) \left( I_d \right) \left( \pi_1 \left( Y_s \right) \right) \in \mathcal{U}.
\]
Then according to Lemma 26, we have
\[ \lim_{l \to \infty} \left\| y_{1,t}^{s,l} - y_{1,t}^{s,l} \right\| = 0. \]  

(61)

Based on Lemma 15 for each \( l \geq 1 \), we have
\[ \left\| y_{l,t}^{s,t} - y_{s,t}^{1} \right\| \leq C_{p,\gamma} \left( \left| f_{\text{Lip}(\gamma)} \right| S_{p} \left( x^{l} \right) \right)^{\gamma+1}, \]  

(62)

Combining (58), (59), (60) and (61), we let \( l \to \infty \) in (62), and get Theorem 18.

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