Algebraic reduced genus one Gromov–Witten invariants for complete intersections in projective spaces, Part 2

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\textbf{Abstract}

In Lee and Oh (Preprint, arXiv:1809.10995, 2018), we provided an algebraic proof of Zinger’s comparison formula (Geom. Topol. 12 (2008), no. 2, 1203–1241; J. Differential Geom. 83 (2009), no. 2, 407–460) between genus one Gromov–Witten invariants and reduced invariants when the target space is a complete intersection of dimension 2 or 3 in a projective space. In this paper, we extend the proof in Lee and Oh (Preprint, arXiv:1809.10995, 2018) to all dimensions and to descendant invariants.

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1  |  INTRODUCTION

Let $Q$ be a smooth projective variety over $\mathbb{C}$. For each $g, k \in \mathbb{Z}_{\geq 0}$ and $d \in H_2(Q; \mathbb{Z})$, the moduli space of stable maps $\overline{M}_{g,k}(Q, d)$ carries the canonical virtual fundamental class $[\overline{M}_{g,k}(Q, d)]^{vir}$ of the virtual dimension $vdim := c_1(T_Q) \cap d + (1 - g) (\dim Q - 3) + k$. In this article, we discuss a
decomposition of \([\overline{M}_{g,k}(Q,d)]^{vir}\) for \(g = 1\), when \(Q\) is embedded in \(\mathbb{P}^n\) as a complete intersection, defined by \(m\) homogeneous equations

\[ Q := \{ f_1 = \cdots = f_m = 0 \} \subset \mathbb{P}^n. \]

For \(d \in H_2(\mathbb{P}^n; \mathbb{Z}) \cong \mathbb{Z}, \overline{M}_{g,k}(Q,d)\) is a disjoint union of all \(\overline{M}_{g,k}(Q,\beta)\), where \(\beta\) pushes forward to \(d\). It leads to an algebraic proof of Zinger’s theorem [22, Theorem 1A] for complete intersections in projective spaces. Here are some notations.

- Let \(\mathcal{M}_{g,k}\) be the moduli space of genus \(g\) prestable curves with \(k\)-marked points.
- Let \(\mathcal{M}_{g,k,d}^w\) be the moduli space of genus \(g\) prestable curves with \(k\)-marked points and non-negative integer weights on each component of curves whose total sum is \(d\). Each object in \(\mathcal{M}_{g,k,d}^w\) is called a weighted curve. We usually abbreviate \(\mathcal{M}_{g,k,d}^w\) by \(\mathcal{M}_{g,k}^w\) when the degree \(d\) is fixed.
- Let \(\overline{M}_{1,k}^{\text{red}}(\mathbb{P}^n, d)\) be the closure of the moduli space of stable maps with smooth domain curves \(M_{1,k}(\mathbb{P}^n, d) \subset \overline{M}_{1,k}^{\text{red}}(\mathbb{P}^n, d)\).
- Let \(\overline{M}_{1,k}^{\text{red}}(Q, d)\) be the closed substack defined by

\[ \overline{M}_{1,k}^{\text{red}}(Q, d) := \overline{M}_{1,k}(Q, d) \cap \overline{M}_{1,k}^{\text{red}}(\mathbb{P}^n, d) \subset \overline{M}_{1,k}^{\text{red}}(Q, d). \]

We introduce other closed substacks in \(\overline{M}_{1,k}(Q, d)\) indexed by numerical information on their rational tails. First, we let \(\mathfrak{S} = \mathfrak{S}_{k,d}\) be the index set consisting of elements

\[ \mu = ( (d_1(\mu), K_1(\mu)), \ldots, (d_{\ell(\mu)}(\mu), K_{\ell(\mu)}(\mu)) ), \]

where \(K_i(\mu)\) are mutually disjoint subsets of \([k] := \{1, \ldots, k\}\), and \(d_i(\mu)\) are positive integers with \(\sum d_i(\mu) = d\). For each \(\mu \in \mathfrak{S}\), let \(K_0(\mu)\) denote \([k] \setminus \bigcup_i K_i(\mu)\). We will abbreviate \(K_0(\mu), K_i(\mu), d_i(\mu), \text{ and } \ell(\mu)\) as \(K_0, K_i, d_i, \text{ and } \ell, \) respectively, when the context is clear. For each \(\mu\) of the form (1.1), we assign the set

\[ \overline{\mu} := \{ (d_1(\mu), K_1(\mu)), \ldots, (d_{\ell(\mu)}(\mu), K_{\ell(\mu)}(\mu)) \}, \]

and we define \(\overline{\mathfrak{S}} := \{ \overline{\mu} | \mu \in \mathfrak{S} \}\).

- Let \(\overline{M}_\mathfrak{P}(\mathbb{P}^n, d)\) be the closed substack of \(\overline{M}_{1,k}(\mathbb{P}^n, d)\) parametrizing \(\overline{\mu}\)-type maps; see [21] for the precise definition.
- Let \(\overline{M}_\mathfrak{P}(Q, d)\) be the closed substack defined by

\[ \overline{M}_\mathfrak{P}(Q, d) := \overline{M}_{1,k}(Q, d) \cap \overline{M}_\mathfrak{P}(\mathbb{P}^n, d). \]

We have a finite, proper node-identifying morphism [21]

\[ t_{\mu_1,Q} : \overline{M}_{1,0,\cup K_1}(Q, d_1) \times_\mathcal{O} \cdots \times_\mathcal{O} \overline{M}_{0,\cup K_{\ell}}(Q, d_{\ell}) \to \overline{M}_\mathfrak{P}(Q, d), \]

where \(\overline{M}_{1,0,\cup K_1}(\cdot)\) is the moduli space of genus one stable curves, and the fiber product is taken by evaluation maps of \(\bullet\). Note that if \(\overline{\mu}_1 = \overline{\mu}_2\), the images of \(t_{\mu_1,Q}\) and \(t_{\mu_2,Q}\) are same in \(\overline{M}_{1,k}(Q, d)\).
Let
\[ M_{0,d}(Q, d) := M_{0, \ast \cup K_1} Q \times \cdots \times Q M_{0, \ast \cup K_\ell} Q, \]

Note that \( M_{0,d}(Q, d) \) has the canonical virtual fundamental class with the virtual dimension \( c_1(T_Q) \cap d - 2\ell + \dim Q + \sum_{i=1}^\ell |K_i| \).

### 1.1 Summary of result

We briefly introduce our main result. In Section 2.2, we construct a decomposition of virtual class

\[ [\overline{M}_{1,k}(Q, d)]_{\text{vir}} = A^\text{red}_{k,d} + \sum_{\mu \in \mathbb{S}} A^\mu_{k,d}, \]

in Chow group \( A_{\text{dim}}(\overline{M}_{1,k}(Q, d)) \). Here and throughout the paper, we consider Chow groups \( A_*(-) \) with \( \mathbb{Q} \)-coefficients. We investigate the cycles, \( A^\text{red}_{k,d} \) and \( A^\mu_{k,d} \) further, and obtain the following, which is our main result.

**Theorem 1.1.** (1) \( A^\text{red}_{k,d} \) is the cycle defined as

\[ e^\text{ref} (\pi_* e^* \bigoplus_{i=1}^n \Theta_{\text{gen}}(\deg f_i)) \cap [\overline{M}^\text{red}_{1,k}(\mathbb{P}^n, d)], \]

where \( \overline{M}^\text{red}_{1,k}(\mathbb{P}^n, d) \) is Vakil–Zinger’s desingularization of the main component \( \overline{M}^\text{red}_{1,k}(\mathbb{P}^n, d) [21] \), \( e^\text{ref} \) denotes the refined Euler class supported on \( \overline{M}^\text{red}_{1,k}(Q, d) \), and \( e^* \) is the universal evaluation map from the universal curve \( \pi : C \to \overline{M}^\text{red}_{1,k}(\mathbb{P}^n, d) \).

(2) \( A^\mu_{k,d} \) is the cycle written by

\[ \frac{1}{\ell !}(\iota_{\mu,Q})_* b_* \left( [\overline{M}_{1,(K_0, l)]}] \times [\overline{M}_{0,\mathcal{P}}(Q, d)]_{\text{vir}} \right) \left( \frac{c(E^* \bigotimes q^* E') \cap C}{c(E^* \bigotimes q^* E')} \right)_{\dim(Q)-1}, \]

which is supported on \( \overline{M}(Q, d) \).

We explain notations above, which will be rigorously defined in Section 3 and 4. Here \( \overline{M}_{1,(K_0, l)]} \) and \( \overline{M}_{0,\mathcal{P}}(Q, d) \) are moduli spaces defined in Section 3 obtained by some modifications from \( \overline{M}_{1,K_0 \cup l] } \) and \( \overline{M}_{0,\mathcal{P}}(Q, d) \). Note that there is an induced node-identifying morphism

\[ \overline{\pi}_{Q} : \overline{M}_{1,(K_0, l)]} \times \overline{M}_{0,\mathcal{P}}(Q, d) \to \overline{M}(Q, d) \times \mathcal{W}_{1,k} \]

where \( \overline{M}_{1,k} \) is defined by a sequence of blow-ups from \( \mathcal{W}_{1,k} \) in [11, p. 931] and [17, Section 2].

Note that there is a blow-down morphism \( \overline{M}_{1,(K_0, l)]} \to \overline{M}_{1,K_0 \cup l] } \). The line bundle \( E^* \) is the pull-back of the Hodge bundle on \( \overline{M}_{1,K_0 \cup l] } \). Instead of defining the line bundle \( q^* E' \) directly, we will
explain its geometric meaning here. As we considered node-identifying morphism of moduli space of maps above, we can consider node-identifying morphism of moduli spaces of base curves:

\[ t_\mu : \mathcal{M}_{1, K_0 | \ell'} \times \mathcal{M}_{0, \mu}^w \to \mathcal{M}_{1, k}^w \]

where \( \mathcal{M}_{0, \mu}^w := \bigsqcup_{\mu' \mid \mu = \mu'} \mathcal{M}_{0, \mu'}^w \) and \( \mathcal{M}_{0, \mu'}^w := \prod_{i=1}^{\ell'} \mathcal{M}_{0, i, K_i, d_i(\mu)} \). Note that \( t_\mu \) is unramified so that the blow-up \( \widetilde{\mathcal{M}}_{1, k}^w \to \mathcal{M}_{1, k}^w \) induces transforms (proper transforms and taking projectivization of a normal bundle) in the left-hand side so that we obtain

\[ \widetilde{t}_\mu : \widetilde{\mathcal{M}}_{1, (K_0, | \ell')} \times \mathbb{P} \mathcal{M}_{0, \mu}^w \to \widetilde{\mathcal{M}}_{1, k}^w \]

the induced node-identifying morphism. This procedure will be explained in Section 3. We will not explain the exact definition of \( \mathbb{P} \mathcal{M}_{0, \mu}^w \) here. Let \( \mathcal{M}_{\mu}^w \) be the image of \( \widetilde{t}_\mu \). Note that \( \{ \mathcal{M}_{\mu}^w \}_{\mu \in \mathcal{S}} \) are normal-crossing divisors of \( \mathcal{M}_{1, k}^w \). When we define \( \mathcal{M}^{rat}_{\mu} := \cup \mathcal{M}_{\mu}^w \), then we have

\[ \mathbb{E}^{\vee} \boxtimes q_\mu^{*} \mathcal{E}' \cong \pi^{*} N_{\mathcal{M}_{\mu}^{rat}/\mathcal{M}_{1, k}^w} \]

where \( \pi \) is the projection

\[ \pi : \mathcal{M}_{1, (K_0, | \ell')} \times \mathcal{M}_{0, \mu}^w(Q, d) \to \mathcal{M}_{1, (K_0, | \ell')} \times \mathbb{P} \mathcal{M}_{0, \mu}^w. \]

The precise construction of the refined Euler class is given in Section 4.3, where we prove condition (1) of Theorem 1.1. We will prove Theorem condition (2) of Theorem 1.1 in Section 4.2. Note that the form in condition (2) of Theorem 1.1 coincides with corresponding term in [22, (3-29)].

In short, we explain the relations of our result and other works. Our definition of the cycle \( A_{k, d}^{red} \) coincides with the reduced virtual cycles defined in [6, 8], by using algebraic geometry. Also the reduced invariants defined by integrations over \( A_{k, d}^{red} \), coincide with the reduced invariants defined in [23] by using symplectic geometry when the target space is a complete intersection in a projective space.

The definition of the cycle \( A_{k, d}^{red} \) in (1.3) says that the reduced invariants satisfy quantum Lefschetz property — the cycle \( A_{k, d}^{red} \) is expressed by Euler class of a vector bundle. It is already studied in the authors’ previous work [17, (4.1)], which follows the idea in [18, 21] (the order follows the timeline on Arxiv). We will review it in Section 4.3. The refined Euler class description of \( A_{k, d}^{red} \) is equivalent to the definition in [8].

The main effort of this article is to prove that the cycle \( A_{k, d}^{red} \) can be expressed as (4.17), which is equivalent to [22, Equation (3-29)], in an algebraic way. On the other hand, the authors already studied invariants defined by integrations on \( A_{k, d}^{rat} \), when \( \dim Q = 2 \) or 3, in our previous work [17]. This case was easier because most integrations became zero by dimension counts and we did not consider descendant invariants but only GW invariants. In contrast, we consider descendant invariants here. We believe that the idea of our algebraic proof has a possibility to be applied for further works related to the quantum Lefschetz property for higher genus case (in a forthcoming paper by Lee, Li and Oh).
1.2 Desingularization and Local equations

In this section, we review the desingularizations of moduli spaces of genus one stable maps studied by Vakil–Zinger [21] and Hu–Li [11] as well as local equations of them studied by Hu–Li [11].

Let $𝔅_{g,k,d}$ be the stack of genus $g$ prestable curves $C$ with $k$-marked points, and line bundles $L$ of degree $d$ on $C.$ Let $𝓜_{g,k,d}^{\text{div}}$ denote the stack of pairs $(C,D),$ where $C$ is a genus $g$ prestable curve with $k$-marked points, and $D$ is a degree $d$ effective divisor on $C$ away from nodal points of the curve $C.$ We abbreviate subscripts on $𝓜_{g,k,d}^w$ (respectively, $𝔅_{g,k,d}$ and $𝓜_{g,k,d}^{\text{div}}$) when the context is clear. Note that $𝓜^w$, $𝔅$, and $𝓜^{\text{div}}$ are smooth Artin stacks.

We now assume that $g = 1$. When $k = 0,$ which is the case without marked points, Hu and Li constructed a (finite) sequence of blow-up $\tilde{𝓜}^w \to 𝓫^w$ to get another smooth Artin stack [11, p. 931]. When $k \neq 0,$ there is also a blow-up $\tilde{𝓜}^w \to 𝓫^w$ defined analogously as the $k = 0$ case [17, Section 2].

In short, we will explain the blow-up construction $\tilde{𝓜}^w \to 𝓫^w$. We blow up subloci $\{𝓟_{\mu}\}_{\mu \in 𝖧}$ following an order on the set $𝖧$. For $\mu, \mu' \in 𝖧$, we define a partial order $≺$ on $𝖧$ by the condition:

$$\mu ≺ \mu' \text{ if and only if } |\mu| < |\mu'| \text{ and } 𝓟_\mu \cap 𝓟_{\mu'} \neq \emptyset.$$  

Then we can choose a total order which is an extension of the partial order. Then we blow up the subloci (and their proper transforms) following the order, so that we obtain $\tilde{𝓜}_{1,k} \to 𝓫_{1,k}$. Note that it does not depend on choices of an order. So it is well defined.

The important property of this successive blow-up is as follows. Consider the morphism $𝓜_{1,k}(ℙ^n, d) \to 𝓫^w$ which assigns a stable map $(C, f)$ to $(C, w_f)$ where the weight $w_f$ on each irreducible component $C' \subset C$ is given by $w_f(C') = \deg(f^*\mathcal{O}(1)|_{C'})$. We define $\tilde{𝓜} := \tilde{𝓜}^w \times_{𝓟^{\text{red}}} 𝓫_{1,k}(ℙ^n, d)$, and $\tilde{𝓜}^{\mu} := \tilde{𝓜}^w \times_{𝓟^{\mu}} 𝓫_{1,k}(ℙ^n, d)$ where $𝓟^{\mu}$ is a closed substack of $𝓟^w$ of $\mu$-type weighted curves, and $\tilde{𝓜}^{\mu}$ is the exceptional divisor of the blow-up along its proper transform in $\tilde{𝓜}^w$. Then we have a decomposition of the space

$$\tilde{𝓟} = \tilde{𝓜}^{\text{red}} \cup \bigcup_{\mu \in 𝖧} \tilde{𝓟}^{\mu}$$  

such that

1. $\tilde{𝓜}^{\text{red}}$ is the proper transform of $𝓜_{1,k}(ℙ^n, d)$ under the blow-up morphism

$$b : \tilde{𝓟} \to 𝓫_{1,k}(ℙ^n, d);$$

2. the image of $\tilde{𝓜}^{\mu}$ under $b$ is supported on $𝓜^{\mu}(ℙ^n, d);$  

3. $\tilde{𝓜}^{\text{red}}$ and $\tilde{𝓜}^{\mu}$ are smooth, and they intersect transversally.

Remark 1.2. $\tilde{𝓜}^{\mu}$ is supported on the image of the gluing morphism

$$𝓜_{1,K_0\cup[\ell]} \times 𝓫_{0,\cup K_1,d_1} \times \cdots \times 𝓫_{0,\cup K_\ell,d_\ell} \to 𝓫^w,$$

which is unramified and has codimension $\ell$ in $𝓟^w$. Note that $𝓟^{\mu}$ is its image.
Furthermore, Hu–Li described $\widetilde{M}$ locally as a zero of a vector-valued function. For each hyperplane $H \subset \mathbb{P}^n$, we can choose an open substack $U_H \subset \overline{M}_{1,k}(\mathbb{P}, d)$ consisting of $(C, f)$ such that $f^*(H) \subset C$ consists of $d$ smooth, distinct points [10, Section 3]. Note that $U_H$ covers $\overline{M}_{1,k}(\mathbb{P}, d)$ as $H$ varies among the hyperplanes in $\mathbb{P}^n$. Let $\overline{U}_H := U_H \times_{\overline{M}_{1,k}(\mathbb{P}, d)} \overline{M}$. Consider the following diagram

$$
\begin{array}{cccccc}
U_H & \rightarrow & M^{\text{div}} & \rightarrow & B & \rightarrow & M^{w} \\
\downarrow & & \downarrow & & \circ & & \downarrow \\
\overline{U}_H & \rightarrow & \overline{M}^{\text{div}} & \rightarrow & \overline{B} & \rightarrow & \overline{M}^{w},
\end{array}
$$

where $\overline{B} := B \times_{\mathfrak{M}^{w}} \overline{M}^{w}$, and $\overline{M}^{\text{div}} := M^{\text{div}} \times_{\mathfrak{M}^{w}} \overline{M}^{w}$ are fiber products; $M^{w} \rightarrow \overline{M}^{w}$ is a smooth affine chart; and $B = \text{Pic}(C_{M^{w}}/M^{w}, d)$ is a relative Picard scheme with the weight $d$. Note that the fiber product of the right-most square of (1.6) is a $C^*$-gerbe over $B$. $M^{\text{div}}$ is a scheme since the morphism $\overline{M}^{\text{div}} \rightarrow \overline{B}$ is representable. The scheme structure on $U_H$ can be described as follows.

We may assume that $M^{w}$ is small enough and it is an affine neighbourhood around a closed point $x \in \overline{M}^{w}$ which projects to a closed point $[C, w] \in \mathfrak{M}^{w}$. Since $C$ is a nodal curve, we can consider node-smoothing variables of the nodes, which are coordinate functions on $M^{w}$.

Next we consider local neighborhood of $\widetilde{M}$ over $M^{\text{div}}$. For a pair $(C, D) \in M^{\text{div}}$, we can express a stable map $(C, f)$ by a choice of $s_1, \ldots, s_n \in H^0(C, \mathcal{O}_C(D))$ so that the morphism $f : C \rightarrow \mathbb{P}^n$ is given by $[1 : s_1 : \cdots : s_n]$ where 1 is a canonical section of $\mathcal{O}(D)$.

Let $\pi : C \rightarrow M^{\text{div}}$ be the universal curve and $D \subset C$ be the universal divisor. By [6, Proposition 3.2], we have

$$\mathbb{R}\pi_* \mathcal{O}_C(D) \cong \left[ \mathcal{O}_{M^{\text{div}}} \xrightarrow{\ast} \mathcal{O}_{M^{\text{div}}} \right] \oplus \left[ \mathcal{O}_{M^{\text{div}}}^{\otimes d} \rightarrow 0 \right],$$

where $t = t_1 t_2 \ldots t_r \in \Gamma(M^{w}, \mathcal{O}_{M^{w}})$ and $t_i$ are node-smoothing variables.

**Remark 1.3.** Each $\overline{M}^{\mu}$, which appears in the local neighborhood $U_H$, is a zero locus of a node-smoothing variable $t_i$. Recall that $\overline{M}^{\mu}$ are proper transforms of exceptional divisors which arise from Vakil–Zinger’s blow-up. So we have $\bigcup_{\mu} \overline{M}^{\mu} = \{ t = 0 \}$.

Since we made a correspondence between stable maps and $n$-sections of $\mathcal{O}_C(D)$ above, the local neighborhood $U_H$ of $\widetilde{M}$ is an open dense subset (which expresses condition to be a stable map) of the kernel space of the following bundle homomorphism [11, Section 5.4] and [17, Section 2.2]

$$\mathcal{O}^{\otimes n}_{M^{\text{div}}} \oplus \mathcal{O}^{\otimes n}_{M^{\text{div}}} \xrightarrow{(t_1, \ldots, t_r) \otimes 0} \mathcal{O}^{\otimes n}_{M^{\text{div}}}.$$

In other words, it is an open dense subset of a zero locus of the following section $F$ of a vector bundle:

$$M^{\text{div}} \times \mathbb{C}^n \times \mathbb{C}^n \xrightarrow{(y_1, \ldots, y_n)} \mathcal{O}^{\otimes n}_{M^{\text{div}} \times \mathbb{C}^n}.$$
where $y_1, \ldots, y_n$ are coordinate functions of $\mathbb{C}^n$ in the left-hand side. We can write $U_H = V_H \cap \{F = 0\}$ for an open dense subset $V_H \subset M^{\text{div}} \times \mathbb{C}^n \times \mathbb{C}^{dn}$.

**Remark 1.4.** By [11, 17], we have $\mathcal{M}^{\text{red}} = \{y_1 = \cdots = y_n = 0\}$ locally.

### 1.3 Relative perfect obstruction theories of the stable map space

In this paper, we will use three kinds of relative perfect obstruction theories, over $\mathcal{B}$ and $\mathcal{M}^w$, and over $\mathcal{M}^{\text{div}}$. We introduce them in this section briefly.

#### 1.3.1 Relative obstruction theory over $\mathcal{M}^w$

Let $\pi : C \to \overline{M}_{1,k}(\mathbb{P}^n, d)$ be the universal curve and $ev : C \to \mathbb{P}^n$ be the evaluation morphism. A natural relative perfect obstruction theory of $\overline{M}_{1,k}(\mathbb{P}^n, d)$ over $\mathcal{M}^w$ is given by

$$E_{\overline{M}_{1,k}(\mathbb{P}^n, d)/\mathcal{M}^w} := (\mathbb{R}\pi_*ev^*\mathcal{T}_{\mathbb{P}^n})^\vee.$$ 

For the blow-up morphism $\widetilde{\mathcal{M}} \to \overline{M}_{1,k}(\mathbb{P}^n)$, the pull-back of the relative perfect obstruction theory gives a relative perfect obstruction theory $E_{\widetilde{\mathcal{M}}/\mathcal{B}}$ of $\widetilde{\mathcal{M}}$ over $\mathcal{M}^w$.

#### 1.3.2 Relative obstruction theory over $\mathcal{B}$

A natural relative perfect obstruction theory of $\overline{M}_{1,k}(\mathbb{P}^n)$ over the stack of line bundles $\mathcal{B}$ is given by

$$E_{\overline{M}_{1,k}(\mathbb{P}^n)/\mathcal{B}} := (\mathbb{R}\pi_*ev^*\mathcal{O}_{\mathbb{P}^n}(1)^{(n+1)})^\vee.$$ 

Again, for the blow-up morphism $\widetilde{\mathcal{M}} \to \overline{M}_{1,k}(\mathbb{P}^n)$, the pull-back of the relative perfect obstruction theory is a relative perfect obstruction theory $E_{\widetilde{\mathcal{M}}/\mathcal{B}}$ of $\widetilde{\mathcal{M}}$ over $\mathcal{B}$.

#### 1.3.3 Relative obstruction theory over $\mathcal{M}^{\text{div}}$

There is no natural global morphisms $\overline{M}_{1,k}(\mathbb{P}^n) \to \mathcal{M}^{\text{div}}$ or $\overline{M}_{1,k}(\mathbb{P}^n) \to \mathcal{M}^{\text{div}}$. But as above, when we fix an hyperplane $H$, there is a morphism in the local charts $\widetilde{U}_H \to \mathcal{M}^{\text{div}}$. In this case, the natural relative perfect obstruction theory of $\widetilde{U}_H^P$ relative to $\mathcal{M}^{\text{div}}$ introduced in [5, Proposition 2.5] is given by

$$E_{\widetilde{U}_H/\mathcal{M}^{\text{div}}}^\vee = \mathbb{R}\pi_*ev^*\mathcal{O}_{\mathbb{P}^n}(H)^{\oplus n}.$$
1.4  Comparison of relative perfect obstruction theories and relative intrinsic normal cones

In Section 1.3, we introduced three relative obstruction theories for the moduli spaces of stable maps and their Vakil–Zinger desingularizations. Here we explain how we can compare these obstruction theories and compare (relative) intrinsic normal cones.

First we compare relative perfect obstruction theories over $\mathcal{M}_{\text{div}}$ and $\mathcal{M}_{\text{w}}$. The above description (1.8) of $U_H$, as a zero locus of a section of a vector bundle, induces relative (dual) perfect obstruction theory:

$$E^\vee_{U_H/M_{\text{div}}} \simeq \left[ T_{U_H/M_{\text{div}}} \xrightarrow{dF} \mathcal{O}^\oplus_{U_H} \right] \simeq \left[ \mathcal{O}^\oplus_{U_H} \oplus \mathcal{O}^\oplus_{U_H} \xrightarrow{(t, \ldots, t) \oplus 0} \mathcal{O}^\oplus_{U_H} \right].$$

It descends to

$$E^\vee_{U_H/\mathcal{M}_{\text{w}}} \simeq \left[ \mathcal{O}^\oplus_{U_H} \oplus \mathcal{O}^\oplus_{U_H} \xrightarrow{(t, \ldots, t) \oplus 0} \mathcal{O}^\oplus_{U_H} \right].$$ (1.9)

Note that that this coincides with the relative perfect obstruction theory of $\tilde{U}_H^p$ relative to $\mathcal{M}_{\text{div}}$ introduced in Section 1.3.3.

Two short exact sequences

$$0 \to \mathcal{O}_{pn} \to \mathcal{O}_{pn}(H)^{\oplus n+1} \to T_{pn} \to 0 \quad \text{(Euler sequence)}$$

$$0 \to \mathcal{O}_{pn} \to \mathcal{O}_{pn}(H) \to \mathcal{O}_H \to 0$$

give rise to an exact triangle

$$\mathbb{R}\pi_* e^* \mathcal{O}_{pn}(H)^{\oplus n} \to \mathbb{R}\pi_* e^* T_{pn} \to \mathbb{R}\pi_* e^*[\mathcal{O}_{pn} \to \mathcal{O}_{pn}(H)] \simeq \mathbb{R}\pi_* e^* \mathcal{O}_H^{+1}. \quad (1.10)$$

Since the effective divisor $e^*H \subset C$ is the sum of distinct, smooth points, $\mathbb{R}\pi_* e^* \mathcal{O}_H \cong \pi_H^* T_{\mathcal{M}_{\text{div}}/\mathcal{M}_{\text{w}}}$, where $\pi_H : \tilde{U}_H \to \mathcal{M}_{\text{div}}$ is the projection morphism. Thus, we have the following diagram of triangles

$$\begin{array}{ccc}
\mathbb{R}\pi_* e^* \mathcal{O}_H[-1] & \rightarrow & E^\vee_{\tilde{U}_H/\mathcal{M}_{\text{div}}} \xrightarrow{\varphi^\vee} E^\vee_{\tilde{U}_H/\mathcal{M}_{\text{w}}} \xrightarrow{+1} \\
\cong & \uparrow & \uparrow \\
\pi_H^* T_{\mathcal{M}_{\text{div}}/\mathcal{M}_{\text{w}}}[{-1}] & \rightarrow & T_{\tilde{U}_H/\mathcal{M}_{\text{div}}} \xrightarrow{\varphi^\vee} T_{\tilde{U}_H/\mathcal{M}_{\text{w}}} \xrightarrow{+1},
\end{array} \quad (1.11)
$$

where the two right-hand vertical morphisms are given by dual of relative perfect obstruction theories. The first row of the diagram (1.11) comes from (1.10). Note that $T_{\mathcal{M}_{\text{div}}/\mathcal{M}_{\text{w}}}$ is a vector bundle, so that it is a complex concentrated on degree 0 since the forgetful morphism $\mathcal{M}_{\text{div}} \to \mathcal{M}_{\text{w}}$ is smooth.
Remark 1.5. By the proof of [15, Proposition 3], we obtain \( \theta^*(\mathcal{C}_{\tilde{U}_i^*/\tilde{M}^w}) = \mathcal{C}_{\tilde{U}_i^*/\tilde{M}_w^w} \). On the other hand, \( \mathcal{C}_{\tilde{U}_i^*/\tilde{M}_w^w} \) is a quotient of \( \mathcal{C}_{\tilde{U}_i^*/\tilde{M}_w^w} \) by the \( \pi^*_i \mathcal{T}_{\tilde{M}_w^w/\tilde{M}_w} \)-action.

Next we compare relative perfect obstruction theories over \( \mathfrak{B} \) and \( \mathfrak{M}_w \). Consider the following commutative diagram in the proof of [5, Lemma 2.8]

\[
\begin{array}{cccccc}
\mathbb{R}\pi_* \mathcal{O}_C & \rightarrow & E^\vee_{\mathcal{M}^p/\mathfrak{B}} & \stackrel{\theta}{\rightarrow} & E^\vee_{\mathcal{M}^p/\mathfrak{M}_w^w} & \rightarrow^+ \\\n\cong & & \uparrow & & \uparrow & \\
\pi^*_{\mathfrak{B}} T_{\mathfrak{B}/\tilde{M}_w^w} [-1] & \rightarrow & T_{\mathfrak{M}^p/\mathfrak{B}} & \stackrel{\theta}{\rightarrow} & T_{\mathfrak{M}^p/\mathfrak{M}_w^w} & \rightarrow^+ ,
\end{array}
\] (1.12)

where \( \pi_{\mathfrak{B}} : \tilde{M}^p \rightarrow \tilde{\mathfrak{B}} \) is the natural projection morphism. Horizontal sequences are distinguished triangles. So we have the induced diagram of bundle stacks:

\[
\begin{array}{cccccc}
h^1/h^0(\mathbb{R}\pi_* \mathcal{O}_C) & \rightarrow & h^1/h^0(E^\vee_{\mathcal{M}^p/\mathfrak{B}}) & \stackrel{\theta}{\rightarrow} & h^1/h^0(E^\vee_{\mathcal{M}^p/\mathfrak{M}_w^w}) & \\
\cong & & \uparrow & & \uparrow & \\
h^1/h^0(\mathbb{R}\pi_* \mathcal{O}_C) & \rightarrow & h^1/h^0(T_{\mathfrak{M}^p/\mathfrak{B}}) & \stackrel{\theta}{\rightarrow} & h^1/h^0(T_{\mathfrak{M}^p/\mathfrak{M}_w^w}).
\end{array}
\] (1.13)

Note that \( \theta \) are smooth morphisms. In the same manner as in the proof of [5, Lemma 2.9], we can check

\[
\theta^*(\mathcal{C}_{\mathfrak{M}_w^w/\mathfrak{M}_w^w}) = \mathcal{C}_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}}. \] (1.14)

In Section 2.2, we will consider restrictions of intrinsic normal cones \( \mathcal{C}_{\mathfrak{M}_w^w/\mathfrak{M}_w^w}|_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}} \), which will turn out to be irreducible component of the intrinsic normal cones. Later, because sometimes it is useful for computation of virtual cycles, we will consider coarse moduli spaces of intrinsic normal cones contained in obstruction sheaves, which are defined in [3, 6]. So let us consider coarse moduli spaces \( C_{\tilde{M}_w^w/\mathfrak{M}_w^w}|_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}} \subset H^1(E^\vee_{\mathfrak{M}_w^w/\mathfrak{M}_w^w}|_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}}) \) and \( C_{\tilde{M}_w^w/\tilde{\mathfrak{B}}}|_{\mathfrak{M}_w^w/\mathfrak{M}_w^w} \subset H^1(E^\vee_{\tilde{M}_w^w/\tilde{\mathfrak{B}}}|_{\tilde{M}_w^w/\mathfrak{M}_w^w}) \).

In Section 2.2, we will check that there is a diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & R^1\pi_* \mathcal{O}_{C_{\tilde{M}_w^w}} & \rightarrow & H^1(E^\vee_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}}|_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}}) & \stackrel{\theta}{\rightarrow} & H^1(E^\vee_{\mathfrak{M}_w^w/\mathfrak{M}_w^w}|_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}}) & \rightarrow 0 \\
0 & \rightarrow & R^1\pi_* \mathcal{O}_{C_{\tilde{M}_w^w}} & \rightarrow & C_{\tilde{M}_w^w/\tilde{\mathfrak{B}}}|_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}} & \rightarrow & C_{\mathfrak{M}_w^w/\mathfrak{M}_w^w}|_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}} & \rightarrow 0,
\end{array}
\]

where \( H^1(E^\vee_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}}|_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}}) \) and \( H^1(E^\vee_{\mathfrak{M}_w^w/\mathfrak{M}_w^w}|_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}}) \) will turn out to be vector bundles. We will also check that

\[
\theta^*(C_{\tilde{M}_w^w/\mathfrak{M}_w^w}|_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}}) = C_{\tilde{M}_w^w/\mathfrak{M}_w^w}|_{\mathfrak{M}_w^w/\tilde{\mathfrak{B}}}. \]
1.5 Moduli space of stable maps to $Q$ versus Moduli space of stable maps to $\mathbb{P}^n$ with fields

Recall that

$$Q = \{f_1 = \cdots = f_m = 0\} \subset \mathbb{P}^n$$

is a complete intersection in $\mathbb{P}^n$ defined by homogeneous polynomials $f_i \in H^0(\mathbb{P}^n, \mathcal{O}(\deg f_i))$. Let $\pi : C \to \overline{M}_{1,k}(Q, d)$ be the universal curve and $ev : C \to Q$ be the evaluation morphism. There is a natural relative perfect obstruction theory of $\overline{M}_{1,k}(Q, d)$ over $\mathcal{M}^w$ given by $(\mathbb{R}_{\pi*} ev^* T_Q)^\vee$. It gives rise to a class

$$[\overline{M}_{1,k}(Q, d)]^{vir} \in A_{\text{udim}}(\overline{M}_{1,k}(Q, d)).$$

By abuse of notation, we denote by $C$, $\pi$ and $ev$ the universal curve, the projection morphism from $C$ and the universal evaluation map, respectively, on any parameter space of maps from curves when the context is clear.

Now we turn our interest to moduli space with fields. Any morphism from a curve $C$ to $\mathbb{P}^n$ can be expressed as a line bundle $L$ on $C$ and a section $u$ in $H^0(C, L^\oplus n+1)$. Hence there is a forgetful morphism $\overline{M}_{1,k}(\mathbb{P}^n, d) \to \mathfrak{B}$. The moduli spaces of stable maps with fields $\overline{M}_{1,k}(\mathbb{P}^n, d)^p$, constructed by Chang–Li [5], is by definition a space parametrizing $(C, L, u) \in \overline{M}_{1,k}(\mathbb{P}^n, d)$ and $p = (p_1, \ldots, p_m) \in H^0(C, (\oplus_i L^\oplus (-\deg f_i)) \otimes \omega_C)$, where $\omega_C$ is a dualizing sheaf of $C$. By using the cosection-localization technique [14] studied by Kiem–Li, one can define a class $[\overline{M}_{1,k}(\mathbb{P}^n, d)^p]^{\text{vir}}_{\text{loc}}$ in $A_{\text{udim}}(\overline{M}_{1,k}(Q, d))$.

The following formula is proved in [5] when $X$ is a quintic 3-fold, and proved in [4, 7, 16, 19] in general,

$$[\overline{M}_{1,k}(\mathbb{P}^n, d)^p]^{\text{vir}}_{\text{loc}} = (-1)^d \sum_i \deg f_i [\overline{M}_{1,k}(Q, d)]^{\text{vir}}.$$

(1.15)

Let $\overline{M}^p := \overline{M}^w \times_{\mathfrak{Y}^w} \overline{M}_{1,k}(\mathbb{P}^n, d)^p$ be the fiber product space. Let $\overline{M}_Q := \overline{M}^w \times_{\mathfrak{Y}^w} \overline{M}_{1,k}(Q, d)$ be a closed substack of $\overline{M}$. Let $b_Q : \overline{M}_Q \to \overline{M}_{1,k}(Q, d)$ be the projection morphism. A pull-back of the relative perfect obstruction theory, defined in [5], of $\overline{M}_{1,k}(\mathbb{P}^n, d)^p$ over $\mathfrak{B}$

$$\left(\mathbb{R}_{\pi*} ev^* \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus (n+1)} \bigoplus \oplus_i \mathbb{R}_{\pi*} (ev^* \mathcal{O}_{\mathbb{P}^n}(-\deg f_i) \otimes \omega_\pi)\right)^\vee$$

is a relative perfect obstruction theory of $\overline{M}^p$ over $\overline{Y}$, denoted by $E_{\overline{M}^p/\overline{Y}}$. The localized virtual class $[\overline{M}^p]^{\text{vir}}_{\text{loc}} \in A_{\text{udim}}(\overline{M}_Q)$ defined by a pull-back of the cosection satisfies the equivalence of classes [17, Lemma 3.2]

$$(b_Q)_* [\overline{M}^p]^{\text{vir}}_{\text{loc}} = [\overline{M}_{1,k}(\mathbb{P}^n, d)^p]^{\text{vir}}_{\text{loc}}.$$

(1.16)

By (1.15) and (1.16), we will use the left-hand side of (1.16) for the proof of Theorem 1.1.

As we explained local description of the moduli space $\overline{M}$ in Section 1.2, we can describe local structure of the moduli space $\overline{M}^p$ with $p$-fields in a similar manner. Recall the smooth affine chart
\( M^{\text{div}} \) of the stack \( \widetilde{\mathfrak{M}}^{\text{div}} \). Over a pair \((C, D) \in M^{\text{div}}\), a stable map with \( p \)-fields is determined by choices of sections \( s_1, \ldots, s_n \in H^0(C, \mathcal{O}_C(D)) \) and \( p_i \in H^0(C, \mathcal{O}_C(\deg f_i \cdot D) \otimes \omega_C) \) for \( 1 \leq i \leq m \).

By [6, Lemma 3.4] and Serre duality, we have the following for \( \ell' > 0 \):

\[
\mathbb{R}\pi_*(\mathcal{O}_C(-\ell' D) \otimes \omega_\pi) \cong (\mathbb{R}\pi_*(\mathcal{O}_C(\ell' \cdot D)))^\vee
\]

\[
\cong [\mathcal{O}_{M^{\text{div}}} \longrightarrow \mathcal{O}_{\tilde{\mathfrak{M}}^{\text{div}}}] \oplus [0 \rightarrow \mathcal{O}^{\text{Bd}d\ell}],
\]

where \( \pi : C \rightarrow M^{\text{div}} \) is the universal curve and \( D \subset C \) is the universal divisor. Therefore, in a same manner as in Section 1.2, we can describe the open chart \( U^p_H := U_H \times_{\tilde{U}^p_H} \tilde{U}^p_H \) as an open dense subset of a zero loci of a section \( F' \) of a vector bundle defined below:

\[
M^{\text{div}} \times (\mathbb{C}^n \times \mathbb{C}^m) \times \mathbb{C}^{dn} \overset{(y_1, \ldots, y_{n+m}) \otimes 0}{\longrightarrow} \mathcal{O}_{M^{\text{div}} \times (\mathbb{C}^n \times \mathbb{C}^m) \times \mathbb{C}^{dn}} \oplus \bigoplus_{i=1}^m \mathcal{O}_{\tilde{\mathcal{M}}^{\text{div}} \times (\mathbb{C}^n \times \mathbb{C}^m) \times \mathbb{C}^{dn}}^{\text{Bd}d\deg f_i}
\]

where \( y_1, \ldots, y_{n+m} \) are coordinate functions of \( \mathbb{C}^n \times \mathbb{C}^m \) in the left-hand side. We can write \( U_H = V^p_H \cap \{F = 0\} \) for an open dense subset \( V^p_H \subset M^{\text{div}} \times (\mathbb{C}^n \times \mathbb{C}^m) \times \mathbb{C}^{dn} \).

We can decompose the virtual cycle \( [\tilde{\mathcal{M}}^p]_{\text{vir}} \) using the above local decomposition of the space \( \tilde{\mathcal{M}}^p \). From the defining equation of \( \tilde{\mathcal{M}}^p \),

\[
U^p_H = \{t y_1 = \ldots t y_{n+m} = 0\},
\]

we have a local decomposition

\[
\tilde{\mathcal{M}}^p_{\text{local}} \cong \{ y_1 = y_2 = \cdots = y_{n+m} = 0\} \cup \bigcup_i \{t_i = 0\}.
\]

It gives rise to a decomposition

\[
\tilde{\mathcal{M}}^p = \tilde{\mathcal{M}}^{p, \text{red}}_\mu \bigcup_{\mu \in \mathfrak{S}} \tilde{\mathcal{M}}^{p, \mu}.
\]

Note that \( \tilde{\mathcal{M}}^{p, \text{red}} \cong \tilde{\mathcal{M}}^{\text{red}} \), which means \( \tilde{\mathcal{M}}^{p, \text{red}} \) is not defined by a fiber product. We will see that the virtual cycle \( [\tilde{\mathcal{M}}^p] \) decompose into cycles supported on components of the decomposition (1.19). The localized virtual cycle \( (b_Q)_* [\tilde{\mathcal{M}}^p]_{\text{vir}} \) decomposes into the cycles supported on \( \tilde{M}^{\text{red}}_{1,k}(Q, d) \) and \( \tilde{M}^{\text{red}}_\mu(Q, d) \) accordingly.

We also want to emphasize the following.

**Remark 1.6.** From (1.18), we have

\[
E^\vee_{U^p_H/\tilde{\mathfrak{M}}^{\text{div}}} \cong \left[ \mathcal{O}^{\otimes n+m}_{U^p_H} \oplus \mathcal{O}^{\otimes d}_{U^p_H} \longrightarrow \mathcal{O}^{\otimes n+m}_{U^p_H} \oplus \bigoplus_{i=1}^m \mathcal{O}_{U^p_H}^{\otimes d \deg f_i} \right].
\]

as we argued in Section 1.4. Also we can compare relative obstruction theories of \( \tilde{\mathcal{M}}^p \) over \( \tilde{\mathfrak{M}}^{\text{div}} \), \( \tilde{\mathfrak{M}}^{\text{red}} \) and \( \mathfrak{B} \) in a parallel way as we argued in Section 1.4.
1.6 Other works on algebraic reduced invariants

In [23], Zinger defined the genus 1 reduced virtual classes and the reduced invariants when target varieties are general projective varieties using symplectic geometry. Zinger’s comparison formula with this definition [22, Theorem 1A] is that for general projective target varieties. As we mentioned above, it is equivalent to our Theorem 1.1 when target varieties are complete intersections. So it is natural to ask whether we can define genus 1 reduced invariants and virtual classes algebraically for any projective varieties. Then we can ask whether Theorem 1.1 holds with this definition.

There are some recent studies on this. In [2], the authors considered Smyth’s 1-stable map spaces and defined reduced invariants by using their virtual cycles. In [20], the authors constructed a moduli space of stable maps in the main component with some additional structures. Also they constructed virtual cycle of the moduli space and reduced invariants.

There are some studies on genus two cases. [1] generalizes [20] in genus two. On the other hand, in [12], the authors generalized Hu–Li’s desingularization [11] so that they obtained desingularization of the main component. This suggests to define reduced invariants as integrations on this desingularized main component.

In a forthcoming paper by Lee, Li and Oh, an ongoing project, we defined reduced quasi-map invariants by using a desingularization of a quasi-map space following [12], which dealt with stable map spaces. With this definition, we suggest and prove a comparison formula between genus two reduced quasi-map invariants and ordinary quasi-map invariants.

1.7 Plan of the paper

In Section 2.2, we discuss decompositions of the relative intrinsic normal cones supported on the decomposition (1.19). Using one of those decompositions, we define the cycles $\mathcal{A}^{red}_{k,d}$ and $\mathcal{A}^{\overline{\mu}}_{k,d}$.

In Section 2.3, we express $\mathcal{A}^{\overline{\mu}}_{k,d}$ by using coarse spaces of the intrinsic normal cones in order to use local descriptions of the cones with coordinates studied in Section 2.1. It leads us to get a description of $\mathcal{A}^{\overline{\mu}}_{k,d}$ by using Chern classes of vector bundles, which will be discussed in Section 4.1. Finally we will prove Theorem 1.1 in Sections 4.2 and 4.3.

The crucial bridge between Sections 2 and 4 is Section 3. Here, we discuss how the normal bundles of node-identifying morphisms (1.5) are modified along the successive blow-up. In Sections 4.1 and 4.2, these normal bundles will be compared to the cones studied in Section 2 in order to obtain a description of $\mathcal{A}^{\overline{\mu}}_{k,d}$.

2 COMPUTATION OF NORMAL CONES

In this section, we study relative intrinsic normal cones of $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}^{P}$, and their coarse moduli spaces; see [3, 6] for the definition of coarse moduli spaces of cone stacks. More precisely, we decompose the relative intrinsic normal cone $\mathcal{C}_{\widetilde{\mathcal{M}}^{P}/\overline{\mathbb{M}}^{w}}$ (respectively, $\mathcal{C}_{\widetilde{\mathcal{M}}^{P}/\overline{\mathbb{M}}^{u}}$) into irreducible components which are supported on irreducible components of $\widetilde{\mathcal{M}}$ (respectively, $\widetilde{\mathcal{M}}^{P}$); see irreducible decompositions (1.4) and (1.19).

Using the decomposition of $\mathcal{C}_{\widetilde{\mathcal{M}}^{P}/\overline{\mathbb{M}}^{w}}$, we will define $\mathcal{A}^{red}_{k,d}$ and $\mathcal{A}^{\overline{\mu}}_{k,d}$. Also we will see that $H^{1}(E^{\nu}_{\widetilde{\mathcal{M}}^{P}/\overline{\mathbb{M}}^{w}}|_{\overline{\mathbb{M}}^{P},\overline{\mathbb{M}}}^{\nu})$ is locally free, and $H^{1}(E^{\nu}_{\widetilde{\mathcal{M}}^{P}/\overline{\mathbb{M}}^{w}}|_{\overline{\mathbb{M}}^{P},\overline{\mathbb{M}}}^{\nu})$ contains the coarse moduli space of an
irreducible component of \( G_{\widetilde{M}^p/\mathbb{D}_w} \) lying on \( \widetilde{M}^{p,\overline{p}} \) so that we can reinterpret \( A_{k,d}^{\overline{p}} \) by using the coarse moduli space of the cone contained in \( H^1(E^\vee_{\widetilde{M}^p/\mathbb{D}_w}|\widetilde{M}^{p,\overline{p}}) \). This interpretation will be helpful for the computation of \( A_{k,d}^{\overline{p}} \) in Section 4.

### 2.1 Decomposition of normal cones: with local coordinates

We compute and decompose normal cones of the moduli spaces \( \widetilde{M} \) and \( \widetilde{M}^p \) locally by using local charts and local equations referred in Sections 1.2 and 1.5.

The following is a general situation. Let \( X \) be an affine scheme \( X = \text{Spec}(R) \) of a commutative \( \mathbb{C} \)-algebra \( R \) and \( t = \prod_i t_i \in R \) be a product element in \( R \). Assume that both \( X \) and each \( \text{Spec}(R/(t_i)) \) are irreducible. Consider the subscheme

\[
Y := \{ty_1 = \cdots = ty_k = 0\} \subset X \times \mathbb{C}^k = \text{Spec}(R[y_1, \ldots, y_k]).
\]

We will apply the computation in this section with this general set-up later

- for \( Y = U_H^{\text{smooth}} \overset{\text{smooth}}{\longrightarrow} \widetilde{M} \), we take \( X = (\widetilde{M}_\text{div} \times \mathbb{C}^d) \cap V_H, \ k = n \), see Section 1.2, and
- for \( Y = U_H^p \overset{\text{smooth}}{\longrightarrow} \widetilde{M}^p \), we take \( X = (\widetilde{M}_\text{div} \times \mathbb{C}^d) \cap V_H^p, \ k = n + m \), see Section 1.5.

From a direct computation, we can check that

\[
C_{Y/X \times \mathbb{C}^k} \cong \text{Spec}\left( \frac{\hat{R}[x_1, \ldots, x_k]}{(y_1 x_j - x_j y_1)_{1 \leq i < j \leq k}} \right), \quad \hat{R} := R[y_1, \ldots, y_k]/(t_1)_{1 \leq i \leq k}.
\]

Note that \( Y = \text{Spec}(\hat{R}) \). Let

\[
Y^\text{red} := \{y_1 = \cdots = y_k = 0\} \text{ and } Y^i := \{t_i = 0\} \text{ in } X \times \mathbb{C}^k.
\]

Then \( X \cong Y^\text{red} \) and \( Y = Y^\text{red} \cup \bigcup_i Y^i \). We have

\[
C_{Y/X \times \mathbb{C}^k} \mid_{Y^\text{red}} \cong \text{Spec}\left( \frac{\hat{R}[x_1, \ldots, x_k]}{(y_1 x_j - x_j y_1)_{1 \leq i < j \leq k}} \otimes_{\hat{R}} \hat{R}/(y_1, \ldots, y_k) \right)
\]

and

\[
C_{Y/X \times \mathbb{C}^k} \mid_i \cong \text{Spec}(R[x_1, \ldots, x_k]), \quad (2.1)
\]

and

\[
C_{Y/X \times \mathbb{C}^k} \mid_{Y^i} \cong \text{Spec}\left( \frac{\hat{R}[x_1, \ldots, x_k]}{(y_i x_j - x_j y_i)_{1 \leq i < j \leq k}} \otimes_{\hat{R}} \hat{R}/(y_1, \ldots, y_k)/(t_i) \right)
\]

and

\[
C_{Y/X \times \mathbb{C}^k} \mid_{Y^i} \cong \text{Spec}\left( \frac{R/(t_i)[y_1, \ldots, y_k][x_1, \ldots, x_k]}{(y_i x_j - x_j y_i)_{1 \leq i < j \leq k}} \right).
\]
Hence $C_{Y/X\times \mathbb{C}^k}|_{Y^{\text{red}}}$ is a rank $k$ vector bundle on $Y^{\text{red}} \cong X = \text{Spec}(R)$, which is isomorphic to the normal bundle $N_{Y^{\text{red}}/X\times \mathbb{C}^k}$, and $C_{Y/X\times \mathbb{C}^k}|_{Y^i}$ is a fiber bundle over $\text{Spec}(R/(t_i))$ whose fibers are isomorphic to the affine cone of $Bt_i \mathbb{C}^k$ in $\mathbb{C}^k \times \mathbb{C}^k$. Since $\text{Spec}(R)$ and $\text{Spec}(R/(t_i))$ are irreducible, so are $C_{Y/X\times \mathbb{C}^k}|_{Y^{\text{red}}}$ and $C_{Y/X\times \mathbb{C}^k}|_{Y^i}$.

**Remark 2.1.** As we mentioned above, $Y$ will be considered as local neighbourhoods for the moduli spaces $\tilde{\mathcal{M}}^\mu$ or $\tilde{\mathcal{M}}$. Since the forgetful morphism $\tilde{\mathcal{M}}^\mu_i \to \tilde{\mathcal{M}}^\mu$ is smooth and also considering Remark 1.5, the above cone computation can be considered as a local computation of relative intrinsic normal cones over the base stack $\tilde{\mathcal{M}}^\mu$.

Now, we consider the local model of $\tilde{\mathcal{M}}$. Let

$$\tilde{U}_H^{\text{red}} := \tilde{U}_H \times_{\tilde{\mathcal{M}}} \tilde{\mathcal{M}}^{\text{red}} \text{ and } \tilde{U}_H^\pi := \tilde{U}_H \times_{\tilde{\mathcal{M}}} \tilde{\mathcal{M}}^{\pi}.$$

Let $\tilde{\mathcal{M}}^{\text{rat}} := \bigcup_{\mu} \tilde{\mathcal{M}}^\mu$. Then $N_{\tilde{\mathcal{M}}^{\text{rat}}/\tilde{U}_H^\pi}$ is locally isomorphic to the conormal bundle $(t)/(t)^2$ on $Y^i$, where $\bar{\mu}$-component is locally defined by $\{t_i = 0\}$. By [6, Proposition 3.2], the obstruction bundle $H^1(E_{\tilde{U}_H/\tilde{\mathcal{M}}^{\text{rat}}}|_{\tilde{U}_H^\pi})$ is locally isomorphic to $Y^i \times \mathbb{C}^n$. Since $Y$ is defined by an ideal $(t y_1, \ldots, t y_n)$, we observe that the natural morphism

$$N_{\tilde{\mathcal{M}}^{\text{rat}}/\tilde{U}_H^\pi} \to H^1(E_{\tilde{U}_H/\tilde{\mathcal{M}}^{\text{rat}}}|_{\tilde{U}_H^\pi})$$

is expressed by

$$Y^i \times \mathbb{C} \to Y^i \times \mathbb{C}^n, \ 1 \mapsto (y_1, \ldots, y_n).$$

One can also see this from the fact that the map (2.2) is given by the differentiation of the section $(t y_1, \ldots, t y_n)$. The local expression above is obtained by observing where the tangent vector $\frac{\partial}{\partial t}$ maps to. From (2.1), we also observe that

$$- \text{ the cone } C_{Y/X\times \mathbb{C}^k}|_{Y^i} \text{ is the closure of the image of the above morphism. -}$$

Note that $N_{\tilde{\mathcal{M}}^{\text{rat}}/\tilde{\mathcal{M}}^{\text{rat}}}$ is locally defined by $(t_i)/(t_i)^2$. Hence, locally we have

$$N_{\tilde{\mathcal{M}}^{\text{rat}}/\tilde{\mathcal{M}}^{\text{rat}}}|_{\tilde{U}_H^\pi} \cong N_{\tilde{\mathcal{M}}^{\text{rat}}/\tilde{U}_H^\pi} \left( \sum_{\mu \neq \bar{\mu}} \tilde{\mathcal{M}}^\mu \cap \tilde{\mathcal{M}}^{\bar{\mu}} \right)|_{\tilde{U}_H^\pi}. $$

By gluing the local isomorphisms, we have a global isomorphism

$$N_{\tilde{\mathcal{M}}^{\text{rat}}/\tilde{\mathcal{M}}^{\text{rat}}}|_{\tilde{U}_H^\pi} \cong N_{\tilde{\mathcal{M}}^{\text{rat}}/\tilde{U}_H^\pi} \left( \sum_{\mu \neq \bar{\mu}} \tilde{\mathcal{M}}^\mu \cap \tilde{\mathcal{M}}^{\bar{\mu}} \right).$$
2.2 Decomposition of intrinsic normal cones of moduli spaces

In Section 2.1, we obtain the irreducible decomposition of the intrinsic normal cone (by taking Remark 2.1 into account)

$$\mathfrak{C} \tilde{U}_H/\tilde{\mathfrak{A}}_w = \mathfrak{C} \tilde{U}_H/\tilde{\mathfrak{A}}_w^{\text{red}} \cup \left( \cup_{\mu} \mathfrak{C} \tilde{U}_H/\tilde{\mathfrak{A}}_w | \tilde{U}_H^{\mu,\tilde{\mathfrak{A}}} \right).$$  \hspace{1cm} (2.5)

Similarly, we obtain the reducible decomposition

$$\mathfrak{C} \tilde{U}_H^p/\tilde{\mathfrak{A}}_w = \mathfrak{C} \tilde{U}_H^p/\tilde{\mathfrak{A}}_w^{\text{red}} \cup \left( \cup_{\mu} \mathfrak{C} \tilde{U}_H^p/\tilde{\mathfrak{A}}_w | \tilde{U}_H^{p,\mu,\tilde{\mathfrak{A}}} \right),$$ \hspace{1cm} (2.6)

where $\tilde{U}_H^{p,\text{red}} := \tilde{U}_H^p \times_{\tilde{\mathfrak{A}}_p} \tilde{\mathcal{M}}_{p,\text{red}}$ and $\tilde{U}_H^{p,\mu,\tilde{\mathfrak{A}}} := \tilde{U}_H^p \times_{\tilde{\mathfrak{A}}_p} \tilde{\mathcal{M}}_{p,\mu,\tilde{\mathfrak{A}}}$. They glue to get the global decompositions of the intrinsic normal cone

$$\mathfrak{C} \tilde{\mathcal{M}}/\tilde{\mathfrak{A}}_w = \mathfrak{C} \tilde{\mathcal{M}}^{\text{red}}/\tilde{\mathfrak{A}}_w \cup \left( \cup_{\mu} \mathfrak{C} \tilde{\mathcal{M}}/\tilde{\mathfrak{A}}_w | \tilde{\mathcal{M}}_{\mu,\tilde{\mathfrak{A}}} \right),$$

$$\mathfrak{C} \tilde{\mathcal{M}}_{p}/\tilde{\mathfrak{A}}_w = \mathfrak{C} \tilde{\mathcal{M}}_{p,\text{red}}/\tilde{\mathfrak{A}}_w \cup \left( \cup_{\mu} \mathfrak{C} \tilde{\mathcal{M}}_{p}/\tilde{\mathfrak{A}}_w | \tilde{\mathcal{M}}_{p,\mu,\tilde{\mathfrak{A}}} \right).$$

In general, there may be more irreducible components on the intersections of $\tilde{\mathcal{M}}_{\mu,\tilde{\mathfrak{A}}}$. But in Section 2.1, by local computations, we showed that there is no such.

Recall that we have $\vartheta^* (\mathfrak{C} \tilde{U}_H/\tilde{\mathfrak{A}}_w) = \mathfrak{C} \tilde{U}_H/\tilde{\mathfrak{B}}$ as we mentioned in (1.14). Therefore the local decompositions (2.5) and (2.6) pull back to the following local decompositions

$$\mathfrak{C} \tilde{U}_H/\tilde{\mathfrak{B}} = \mathfrak{C} \tilde{U}_H^{\text{red}}/\tilde{\mathfrak{B}} \cup \left( \cup_{\mu} \mathfrak{C} \tilde{U}_H/\tilde{\mathfrak{B}} | \tilde{U}_H^{\mu,\tilde{\mathfrak{B}}} \right),$$

$$\mathfrak{C} \tilde{U}_H^p/\tilde{\mathfrak{B}} = \mathfrak{C} \tilde{U}_H^{p,\text{red}}/\tilde{\mathfrak{B}} \cup \left( \cup_{\mu} \mathfrak{C} \tilde{U}_H^p/\tilde{\mathfrak{B}} | \tilde{U}_H^{p,\mu,\tilde{\mathfrak{B}}} \right).$$

They glue to global irreducible decompositions

$$\mathfrak{C} \tilde{\mathcal{M}}/\tilde{\mathfrak{B}} = \mathfrak{C} \tilde{\mathcal{M}}^{\text{red}}/\tilde{\mathfrak{B}} \cup \left( \cup_{\mu} \mathfrak{C} \tilde{\mathcal{M}}/\tilde{\mathfrak{B}} | \tilde{\mathcal{M}}_{\mu,\tilde{\mathfrak{B}}} \right),$$

$$\mathfrak{C} \tilde{\mathcal{M}}_{p}/\tilde{\mathfrak{B}} = \mathfrak{C} \tilde{\mathcal{M}}_{p,\text{red}}/\tilde{\mathfrak{B}} \cup \left( \cup_{\mu} \mathfrak{C} \tilde{\mathcal{M}}_{p}/\tilde{\mathfrak{B}} | \tilde{\mathcal{M}}_{p,\mu,\tilde{\mathfrak{B}}} \right).$$

Now we define the classes $A_{k,d}^{\text{red}}$ and $A_{k,d}^{\tilde{\mathfrak{M}}}$ appeared in Theorem 1.1.

**Definition 2.2.** The classes $A_{k,d}^{\text{red}}$, $A_{k,d}^{\tilde{\mathfrak{M}}}$ $\in A_{\text{vdim}}(\overline{M}_{1,k}(Q, d))$ are defined by

$$A_{k,d}^{\text{red}} := (-1)^d \sum \text{deg} f_i (b_Q) s_0^1 h^1/h^0(F^{\mu,\text{red}}_{\tilde{\mathcal{M}}}/\tilde{\mathfrak{B}}) \text{loc} \left[ \mathfrak{C} \tilde{\mathcal{M}}^{\text{red}}/\tilde{\mathfrak{B}} \right],$$

$$A_{k,d}^{\tilde{\mathfrak{M}}} := (-1)^d \sum \text{deg} f_i (b_Q) s_0^1 h^1/h^0(F^{\mu,\tilde{\mathfrak{M}}}_{\tilde{\mathcal{M}}}/\tilde{\mathfrak{B}}) \text{loc} \left[ \mathfrak{C} \tilde{\mathcal{M}}_{\mu,\tilde{\mathfrak{B}}} \right].$$
They are cycles localized by a cosection; see [14] for the detail of cosection-localized cycles. The cosection will be defined in (4.1).

Remark 2.3. Note that the relative cosection \( h^1(E^g_{\mathcal{X}^p/\mathcal{Y}}) \rightarrow \mathcal{O} \) lifts to the absolute cosection \( h^1(E^g_{\mathcal{X}^{p}/\mathcal{Y}}) \rightarrow \mathcal{O} \) by [5, Proposition 3.5]. Hence the intrinsic normal cone \( C_{\mathcal{X}^{p}/\mathcal{Y}} |_{\mathcal{X}^{p}} \) is contained in the kernel of the cosection by [14, Lemma 4.4]. There is a cosection \( h^1(E^g_{\mathcal{X}^{p}/\mathcal{Y}^{w}}) \rightarrow \mathcal{O} \) which is induced from the absolute cosection. Thus the intrinsic normal cone \( C_{\mathcal{X}^{p}/\mathcal{Y}^{w}} |_{\mathcal{X}^{p}} \) is contained in the kernel of the cosection by the same reason. Therefore, we can apply localized Gysin homomorphisms to the intrinsic normal cones so that we obtain the localized virtual classes.

By (1.15), we have

\[
[M_{1,k}(Q,d)]_{\text{vir}} = (-1)^d \sum_{i} \deg f_i(b_Q) \cdot [\mathcal{M}^p]_{\text{loc}} = A^\text{red}_{k,d} + \sum_{\mu \in \mathcal{S}} A^\mu_{k,d}. \tag{2.7}
\]

This gives the cycle decomposition (1.2) in Theorem 1.1. It remains to show conditions (1) and (2) in Theorem 1.1. We will do this in Section 4.

2.3 Coarse moduli spaces of the cone stacks

In Sections 4.1 and 4.2, we will compute \( A^\mu_{k,d} \) in terms of Chern classes of vector bundles. To do so, we express \( A^\mu_{k,d} \) in terms of the coarse moduli spaces by using [6, Proposition 6.3], namely

\[
0^1_{H^1/h^0(E^g_{\mathcal{X}^{p}/\mathcal{Y}} |_{\mathcal{X}^{p}}),\text{loc}}[\mathcal{C}_{\mathcal{X}^{p}/\mathcal{Y}} |_{\mathcal{X}^{p},\mu}] = 0^1_{H^1(E^g_{\mathcal{X}^{p}/\mathcal{Y}} |_{\mathcal{X}^{p},\mu},\text{loc}}[C_{\mathcal{X}^{p}/\mathcal{Y}} |_{\mathcal{X}^{p},\mu}]. \tag{2.8}
\]

where \( C_{\mathcal{X}^{p}/\mathcal{Y}} |_{\mathcal{X}^{p},\mu} \) is the coarse moduli space of the cone stack \( \mathcal{C}_{\mathcal{X}^{p}/\mathcal{Y}} |_{\mathcal{X}^{p},\mu} \). Note that \( H^1(E^g_{\mathcal{X}^{p}/\mathcal{Y}} |_{\mathcal{X}^{p},\mu}) \), which is the coarse moduli space of the bundle stack \( h^1/h^0(E^g_{\mathcal{X}^{p}/\mathcal{Y}} |_{\mathcal{X}^{p},\mu}) \), is a vector bundle [6, 11]. In Section 4.1, we will study another formula of the right-hand side of (2.8) using

\[
C_{\mathcal{X}^{p}/\mathcal{Y}^{w}} |_{\mathcal{X}^{p},\mu} \subset H^1(E^g_{\mathcal{X}^{p}/\mathcal{Y}^{w}} |_{\mathcal{X}^{p},\mu})
\]

instead of

\[
C_{\mathcal{X}^{p}/\mathcal{Y}^{w}} |_{\mathcal{X}^{p},\mu} \subset H^1(E^g_{\mathcal{X}^{p}/\mathcal{Y}} |_{\mathcal{X}^{p},\mu})
\]

to have an advantage for a Chern class expression of \( A^\mu_{k,d} \). Before doing this, we need to prove \( H^1(E^g_{\mathcal{X}^{p}/\mathcal{Y}^{w}} |_{\mathcal{X}^{p},\mu}) \) is locally free.

Lemma 2.4. The coherent sheaf \( H^1(E^g_{\mathcal{X}^{p}/\mathcal{Y}^{w}} |_{\mathcal{X}^{p},\mu}) \) is locally free of rank \( n + m + \sum_i d \cdot \deg f_i \), and isomorphic to \( H^1(E^g_{\mathcal{X}^{p}/\mathcal{Y}^{w}} |_{\mathcal{X}^{p},\mu}) \).

Proof. For a hyperplane \( H \subset \mathbb{P}^n \), we assign the local chart \( \tilde{U}_H^p \subset \mathcal{M}^p \) as in Section 1.5. It is enough to show that \( H^1(E^\vee_{\tilde{U}_H^p / \mathfrak{m}_w} |_{\tilde{U}_H^p}) \) is locally free of rank \( n + m + \sum_i d \cdot \deg f_i \) where \( \tilde{U}_H^p := \{ t_i = 0 \} \subset \tilde{U}_H^p \).

Recall that the natural relative perfect obstruction theory of \( \tilde{U}_H^p \) relative to \( \mathfrak{m}_{\text{div}} \) introduced in [5, Proposition 2.5] is

\[
E^\vee_{\tilde{U}_H^p / \mathfrak{m}_{\text{div}}} = \left( \mathbb{R} \pi_* e^* \mathcal{O}_{\mathbb{P}^n}(H) \oplus \bigoplus_i \mathbb{R} \pi_* (e^* \mathcal{O}_{\mathbb{P}^n}(-\deg f_i \cdot H) \otimes \omega_{\mathcal{C}}) \right)^\vee.
\]

From Remark 1.6, we have

\[
E^\vee_{\tilde{U}_H^p / \mathfrak{m}_{\text{div}}} \big|_{\tilde{U}_H^p} \cong [\mathcal{O}_{\tilde{U}_H^p} \stackrel{0}{\longrightarrow} \mathcal{O}_{\tilde{U}_H^p}] \oplus \bigoplus_i [\mathcal{O}_{\tilde{U}_H^p} \stackrel{0}{\longrightarrow} \mathcal{O}_{\tilde{U}_H^p}(d \deg f_i + 1)].
\]

Therefore we have \( H^1(E^\vee_{\tilde{U}_H^p / \mathfrak{m}_{\text{div}}} \big|_{\tilde{U}_H^p}) \cong \mathcal{O}(n + m + \sum d \deg f_i) \). On the other hand, from (1.11) and Remark 1.6, we have

\[
\to \pi_* e^* \mathcal{O}_H \to H^1(E^\vee_{\tilde{U}_H^p / \mathfrak{m}_{\text{div}}} \big|_{\tilde{U}_H^p}) \to H^1(E^\vee_{\tilde{U}_H^p / \mathfrak{m}_{\text{w}}} \big|_{\tilde{U}_H^p}) \to 0.
\]

(2.9)

Thus we obtain \( \text{rank}(H^1(E^\vee_{\tilde{U}_H^p / \mathfrak{m}_{\text{w}}} \big|_{\tilde{U}_H^p})) \leq n + m + \sum_i d \cdot \deg f_i \).

From the diagram (1.12), we have the following when we restrict the diagram and take the first cohomology.

\[
\to R^1 \pi_* \mathcal{O}_C \to H^1(E^\vee_{\tilde{U}_H^p / \mathfrak{m}_{\text{w}}} \big|_{\tilde{U}_H^p}) \to H^1(E^\vee_{\tilde{U}_H^p / \mathfrak{m}_{\text{w}}} \big|_{\tilde{U}_H^p}) \to 0.
\]

From [5], we have

\[
E^\vee_{\tilde{U}_H^p / \mathfrak{m}_{\text{w}}} \big|_{\tilde{U}_H^p} \cong \left( \mathbb{R} \pi_* e^* \mathcal{O}_{\mathbb{P}^n}(H) \oplus \bigoplus_i \mathbb{R} \pi_* (e^* \mathcal{O}_{\mathbb{P}^n}(-\deg f_i \cdot H) \otimes \omega_{\mathcal{C}}) \right) \bigg|_{\tilde{U}_H^p}
\]

\[
\cong [\mathcal{O}_{\tilde{U}_H^p} \stackrel{0}{\longrightarrow} \mathcal{O}_{\tilde{U}_H^p}] \oplus \bigoplus_i [\mathcal{O}_{\tilde{U}_H^p} \stackrel{0}{\longrightarrow} \mathcal{O}_{\tilde{U}_H^p}(d \deg f_i + 1)],
\]

so that we conclude that \( \text{rank}(H^1(E^\vee_{\tilde{U}_H^p / \mathfrak{m}_{\text{w}}} \big|_{\tilde{U}_H^p})) \geq n + m + \sum_i d \cdot \deg f_i \). It implies that \( H^1(E^\vee_{\tilde{U}_H^p / \mathfrak{m}_{\text{w}}} \big|_{\tilde{U}_H^p}) \) is a vector bundle of rank \( n + m + \sum_i d \cdot \deg f_i \). The latter part of the lemma comes from (2.2). \( \square \)
Remark 2.5. We can show that the coherent sheaf $H^1(E^\vee_{\tilde{\mathcal{M}}/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{\mathcal{M}}_{\text{rat}}})$ is locally free of rank $n$, and isomorphic to $H^1(E^\vee_{\tilde{U}_H/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{U}_H^{\pi}})$ in a similar manner.

**Corollary 2.6.** Consider the natural morphism
\[
\varphi : N_{\tilde{\mathcal{N}}_{\text{rat}}/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{\mathcal{M}}_{\text{rat}}} \to H^1(E^\vee_{\tilde{\mathcal{M}}/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{\mathcal{M}}_{\text{rat}}}).
\]
Then the zero locus of it is $\tilde{\mathcal{N}}_{\text{rat}} \cap \tilde{\mathcal{M}}_{\text{red}}$.

**Proof.** When we restrict it to the open chart $\tilde{U}_H^{\pi}$ of $\tilde{\mathcal{M}}_{\text{rat}}$, we have
\[
H^1(E^\vee_{\tilde{\mathcal{M}}/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{\mathcal{M}}_{\text{rat}}})|_{\tilde{U}_H^{\pi}} \cong H^1(E^\vee_{\tilde{U}_H/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{U}_H^{\pi}})
\]
by Remark 2.5. Therefore we observe that the restriction of $\varphi$ over $\tilde{U}_H^{\pi}$ is equal to (2.2). Its zero locus is $\{y_1 = \cdots = y_n = 0\}$, which is the local equation of $\tilde{\mathcal{M}}_{\text{rat}} \cap \tilde{\mathcal{M}}_{\text{red}} \subset \tilde{\mathcal{M}}_{\text{rat}}$. □

From (1.12) and the proof of Lemma 2.4, we obtain a diagram of short exact sequences of abelian cones
\[
\begin{array}{cccccc}
0 & \to & R^1\pi_*\mathcal{O}_{C_{\tilde{U}_H^{\pi}}} & \to & H^1(E^\vee_{\tilde{U}_H/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{U}_H^{\pi}}) & \to & H^1(E^\vee_{\tilde{U}_H/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{U}_H^{\pi}}) \to 0 \\
& & \downarrow & & \uparrow & & \square
\end{array}
\]
where $C_{\tilde{U}_H/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{U}_H^{\pi}}$ (respectively, $C_{\tilde{U}_H/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{U}_H^{\pi}}$) is the coarse moduli space of the cone stack $\mathcal{G}_{\tilde{U}_H/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{U}_H^{\pi}}$ (respectively, $\mathcal{G}_{\tilde{U}_H/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{U}_H^{\pi}}$). We glue these local exact sequences to obtain a global diagram
\[
\begin{array}{cccccc}
0 & \to & R^1\pi_*\mathcal{O}_{C_{\tilde{U}_H^{\pi}}} & \to & H^1(E^\vee_{\tilde{\mathcal{M}}_{\text{rat}}/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{\mathcal{M}}_{\text{rat}}^{\pi}}) & \to & H^1(E^\vee_{\tilde{\mathcal{M}}_{\text{rat}}/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{\mathcal{M}}_{\text{rat}}^{\pi}}) \to 0 \\
& & \downarrow & & \uparrow & & \square
\end{array}
\]
where $\theta$ denote the morphism of vector bundles induced by $\theta : h^1/h^0(E_{\tilde{\mathcal{M}}_{\text{rat}}/\tilde{\mathcal{N}}_{\text{rat}}}) \to h^1/h^0(E_{\tilde{\mathcal{M}}_{\text{rat}}/\tilde{\mathcal{N}}_{\text{rat}}})$, by abuse of notation. Note that the right-most square of the above diagram is a fiber diagram. In other words, we have $\theta^*(C_{\tilde{\mathcal{M}}_{\text{rat}}/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{\mathcal{M}}_{\text{rat}}^{\pi}}) = C_{\tilde{\mathcal{M}}_{\text{rat}}/\tilde{\mathcal{N}}_{\text{rat}}}|_{\tilde{\mathcal{M}}_{\text{rat}}^{\pi}}$.

## 3 NORMAL BUNDLES OF NODE-IDENTIFYING MORPHISMS

In this section, we review the results in [21, Section 2.3 and 4] and [22, Section 3.4]. Here is the summary of what we will do in this section. For a fixed $\mu \in \mathfrak{S}$, we consider the following fiber
diagram of the node-identifying morphism $τ_μ$ and the blow-up morphism $\widetilde{M}^w \to M^w$

We will express the space $M_{\text{fib}}$ precisely later. Then:

- the normal bundle $N_{\tau_μ}$ is obtained by some modifications of $N_{i_μ}$.

Note that $N_{\tau_μ} \cong \tau_μ^* N_{\widetilde{M}^w / \widetilde{M}_w}$. It will be combined with (2.3), (2.4), Corollary 2.6, and the explicit description of $N_{\tau_μ}$ (3.2) below through (3.1) for the computation of $A_{k,d}^\tau$.

### 3.1 Successive blow-up and normal bundles

Now, we discuss the detail of (3.1). First, we want to describe a successive blow-up $\widetilde{M}^w \to M^w$, and explain how the relative normal bundle $N_{\tau_μ}$ is modified along the blow-up later. We introduce a partial order on $\overline{𝔖}$ following [21, (2-2)]. For $\overline{μ}_a, \overline{μ}_b \in \overline{𝔖}$,

$$\overline{μ}_a \prec \overline{μ}_b \iff \overline{M}^{\overline{μ}_a} \cap \overline{M}^{\overline{μ}_b} \neq \emptyset, \ l(μ_a) + |K_0(μ_a)| < l(μ_b) + |K_0(μ_b)|.$$  

We then fix any complete order $< \text{on } \overline{𝔖}$ an extension of this partial order. Namely,

$$\overline{𝔖} = \{\overline{μ}_1 < \overline{μ}_2 < \cdots < \overline{μ} = \overline{μ}_{N(\overline{μ})} < \overline{μ}_{N(\overline{μ})+1} < \cdots < \overline{μ}_N\}.$$  

We denote $M^{0} := M^w$, which is an initial space of the sequence of blow-ups. Let $π_1 : M^{1} := Bl_{\overline{M}^{0}} M^{0} \to M^{0}$ be the blow-up morphism, and $\overline{M}^{\overline{μ}}$ be the proper transforms of $\overline{M}^{\overline{μ}}$ via $π_1$. Inductively, we define the blow-up morphism $π_i : M^{i} := Bl_{\overline{M}^{\overline{μ}}} M^{i-1} \to M^{i-1}$ and the proper transforms $\overline{M}^{\overline{μ}}$ of $\overline{M}^{\overline{μ}}$ via $π_i$. In the final step, we obtain $\overline{M}^{w} := M^{N}$ and $\overline{M}^{\overline{μ}} := M^{N | i}$. Let $\overline{π} : \overline{M}^w \to M^w$ be the composition of blow-up morphisms $π_1, \ldots, π_N$. Note that $\overline{π}$ does not depend on choices of complete orders.
At $i$th step of blow-up, we have the node-identifying morphism $t^i_\mu$, obtained by the proper transform of the original node-identifying morphism $t_\mu$. We will get $N_{t^i_\mu} = \overline{t_\mu}$. Now, we want to see how $N_{t^i_\mu}$ is related to the normal bundle of $t^i_\mu$ by an induction on $i$, which finally explains (3.1).

**Initial step of the induction**

The following argument checks the conditions of [21, Lemma 3.5] which is the key idea for the inductive argument. We introduce some notations first. We define an index set $A(\mu_a, \mu_b)$ for any $\mu_a, \mu_b \in \mathfrak{S}$, $A(\mu_a, \mu_b) \colon= \{ \rho : [\ell(\mu_b)] \to [\ell(\mu_a)] | K_i(\rho) \subset K(\mu_a), \text{ for all } i \in [\ell(\mu_b)] \}$

which is a finer index set than the one in [21, Section 4.2]. Let

- $I_j(\rho) \colon= \rho^{-1}(j)$,
- $K_j(\rho) \colon= K_j(\mu_a) \setminus (\cup_{i \in I_j(\rho)} K_i(\mu_b))$, and
- $I_0(\rho) \colon= \rho^{-1}(\{ j \in [\ell(\mu_a)] | \rho^{-1}(j) = [i], K_j(\mu_a) = K_i(\mu_a) \})$.

Note that $|K_j(\rho) \cup I_j(\rho)| \geq 2$ for $j \in [\ell(\mu_a)] \setminus I_0(\rho)$.

Geometrically, each $\rho$ corresponds to an intersection component of $\mathcal{M}_{\mu_a}$ and $\mathcal{M}_{\mu_b}$. Recall that each generic element in $\mathcal{M}_{\mu_a}$ has a genus one component $C_0$ in $\mathcal{M}_{1, \kappa_0(\mu_a)}$ glued to genus zero curves $C_1, \ldots, C_{\ell(\mu_a)}$ with marked points corresponding to elements in $K_j(\mu_a)$ through the nodal points indexed by $j \in [\ell(\mu_a)]$. This element is contained in the intersection component corresponding to $\rho$ when it has the following property. For each $j \in [\ell(\mu_a)]$, $C_j$ has other $|I_j(\rho)|$-nodal points if $|K_j(\rho) \cup I_j(\rho)| \geq 2$, and each attached component has $K_j(\mu_a)$ marked points for $i \in \ell(\mu_a)$. $K_j(\rho)$ denotes the remaining marked points on $C_j$. $I_0(\rho)$ is the index set of attached genus zero curves $C_j$ which are irreducible.

**Remark 3.1.** Figure 1 describes a generic element of the component $\mathcal{M}_{\mu_a}$ satisfying $\ell(\mu_a) = 4$, $K_0(\mu_a) = \{1, 6, 7\}$, $K_1(\mu_a) = \{2, 8, 9\}$, $K_2(\mu_a) = \{3, 5\}$, $K_3(\mu_a) = \{4\}$. Figure 2 describes a generic element of the component $\mathcal{M}_{\mu_b}$ satisfying $\ell(\mu_b) = 6$, $K_0(\mu_b) = \{1, 4, 6, 7, 8\}$, $K_1(\mu_b) = \{2, 3, 5\}$, $K_2(\mu_b) = \{4\}$.
A generic element of the intersection component corresponding to an element $\rho : [6] \to [4] \in A(\mu_a, \mu_b)$

$\{5\}, K_2(\mu_b) = \{2, 9\}, K_4(\mu_b) = \{3\}$. Figure 3 describes the generic element of the intersection component corresponding to $\rho : [6] \to [4]$, such that $\rho^{-1}(1) = \{2, 6\}, \rho^{-1}(2) = \{1, 4\}, \rho^{-1}(3) = \{3\}, \rho^{-1}(4) = \{5\}$. We have $I_0(\rho) = \{4\}$, which corresponds to the non-degenerated tail in Figure 3.

For simplicity, we denote $\mathcal{M}_{0, \mu} := \prod_{\ell \in [\ell]} \mathcal{M}_{0, \ell K_i, \ell}$.

\[
\ell_{\mu}^{-1}(\mathcal{M}_{0, \mu}) = \bigsqcup_{\rho \in A(\mu)} \mathcal{M}_{1, \rho} \times \mathcal{M}_{0, \mu},
\]

where

\[
A(\mu_a, \mu_b) := \bigcup_{\mu' = \mu_a} A(\mu', \mu_b),
\]

and $\overline{\mathcal{M}}_{1, \rho} \subset \overline{\mathcal{M}}_{1, K_0(\mu) \cup [\ell]}(\mu)$ is an image of the node-identifying morphism

\[
\overline{\mathcal{M}}_{1, K_0(\rho) \cup I_0(\rho)} \times \prod_{j \in [\ell(\mu)] \setminus \rho(I_0(\rho))} \overline{\mathcal{M}}_{0, \rho \cup M_j(\rho), j K_j(\rho)} \to \overline{\mathcal{M}}_{1, K_0(\mu) \cup [\ell(\mu)]},
\]
And we have
\[ t^\sigma_{\mu} \left( N_{t^\mu_{(\mu)},(\mathcal{M}_{1,\rho}\times\mathfrak{M}_{0,\mu})/\mathfrak{M}_{1,\rho}} \right) = \bigoplus_{i \in [\mathcal{E}] \setminus I_0(\rho)} L_i \boxtimes L_{i,i} |_{\mathcal{M}_{1,\rho}\times\mathfrak{M}_{0,\mu}}. \] (3.4)

Let
\begin{itemize}
  \item \( L_{i,0} := L_i \)
  \item \( \mathcal{M}^0_{1,(K_0,[\ell])} := \mathcal{M}_{1,K_0,[\ell]} \)
  \item \( \mathcal{M}^1_{1,(K_0,[\ell])} : \) The blow-up of \( \mathcal{M}^0_{1,(K_0,[\ell])} \) along the locus \( \bigsqcup_{\rho \in \mathcal{M}(\mu_1,\mu)} \mathcal{M}^0_{1,\rho} \) where \( \mathcal{M}^1_{1,\rho} := \mathcal{M}^1_{1,\rho} \).
      
Note that we can check \( \mathcal{M}^1_{1,\rho} \) are disjoint from each other by [21, Lemma 2.6].
\end{itemize}

\begin{itemize}
  \item By abuse of notation, we again denote \( \pi_1 \) by the blow-up morphism \( \mathcal{M}^1_{1,(K_0,[\ell])} \rightarrow \mathcal{M}^0_{1,(K_0,[\ell])} \).
  \item \( D_\rho \) : Exceptional divisors of the blow-up \( \pi_1 \) supported on the locus \( \mathcal{M}^1_{1,\rho} \).
\end{itemize}

We note that the collection of unramified morphisms \{\( t^\mu_{\mu_1}, \ldots, t^\mu_{\mu_N} \)\} is properly self-intersecting (see [21, Definition 3.2] for the definition) and the image of \( t^\mu_{\mu_1} \) is smooth by [21, Section 4.3, (I8)]. Hence we can apply [21, Corollary 3.4] to guarantee that the induced node-identifying morphism
\[ t^\mu_{\mu} : \mathcal{M}^1_{1,(K_0,[\ell])} \times \mathfrak{M}_{0,\mu} \rightarrow \mathfrak{M}^{ll} \]
is again an unramified morphism whose image is an open substack of \( \mathfrak{M}^{ll} \). By (3.3) and (3.4), we can apply [21, Lemma 3.5]. So we have
\[ N_{i_1^\mu} = \bigoplus_{i \in [\mathcal{E}]} \pi_1^\sigma L_{i,0} \left( - \sum_{\rho \in \mathcal{M}(\mu_2,\mu)} D_\rho \right) \boxtimes L_{i,i}. \]

\textit{Second step of the induction:} \( j < N(\overline{\mu}) \) case

Inductively, when \( j < N(\overline{\mu}) \), we have the following by applying [21, Corollary 3.4] and [21, Lemma 3.5] repeatedly.

\begin{itemize}
  \item For \( i > j \) the induced node-identifying morphisms
    \[ t^\mu_{\mu_i} : \mathcal{M}^1_{1,(K_0,\mu_i,\mu)} \times \mathfrak{M}_{0,\mu_i} \rightarrow \mathfrak{M}^{lj} \]
    are unramified, and the collection \( \{t^\mu_{\mu_{j+1}}, \ldots, t^\mu_{\mu_N}\} \) are properly self-intersecting. Note that the image of \( t^\mu_{\mu_{j+1}} \) is smooth by [21, Section 4.3, (I8)].
  \item The normal bundle is decomposed into
    \[ N_{i^j_\mu} = \bigoplus_{i \in [\mathcal{E}]} \pi_1^\sigma L_{i,j-1} \left( - \sum_{\rho \in \mathcal{M}(\mu_j,\mu)} D_\rho \right) \boxtimes L_{i,j}. \]
\end{itemize}

Here are the notations above which are defined inductively.
• $\overline{M}_{1,\rho}^j$ : a proper transform of $\overline{M}_{1,\rho}^{j-1}$ via the blow-up

$$\pi_{j-1} : \overline{M}_{1,(K_0,|\ell|)}^{j-1} \to \overline{M}_{1,(K_0,|\ell|)}^{j-2}.$$

• $\overline{M}_{1,(K_0,|\ell|)}^j$ : the blow-up of $\overline{M}_{1,(K_0,|\ell|)}^{j-1}$ along the locus $\bigcup_{\rho \in A(\mu,\mu)} \overline{M}_{j-1,\rho}^{j-1}$.

• $\pi_j : \overline{M}_{1,(K_0,|\ell|)}^j \to \overline{M}_{1,(K_0,|\ell|)}^{j-1}$ : the blow-up morphism.

• $D_\rho$ : Exceptional divisors of the blow-up $\pi_j$ supported on the locus $\overline{M}_{1,\rho}^j$.

We can check the conditions of [21, Lemma 3.5] that we need to apply repeatedly. For $j < N(\mu)$, we have

$$(\iota_{\mu}^j)^{-1}(\mathcal{M}_{i,j}) = \bigcup_{\rho \in A(\mu,\mu)} \overline{M}_{1,\rho}^j \times \mathcal{M}_{0,\mu},$$

where $\overline{M}_{1,\rho}^j$ is the proper transform of $\overline{M}_{1,\rho}^{j-1}$ via the blow-up morphism $\pi_j$. Also, when $j < N(\mu)$ we have the following for each $\rho \in A(\mu,\mu)$

$$(\iota_{\mu}^j)^* \left( N_{\rho}^j(\overline{M}_{1,\rho}^j \times \mathcal{M}_{0,\mu},\mathcal{M}_{j+1,\mu}) \right) = \bigoplus_{i \in [\ell]} L_{i,j} \boxtimes L_{*,i} \mid_{\overline{M}_{1,\rho}^j \times \mathcal{M}_{0,\mu}}$$

where $L_{i,j} := \pi_j^* L_{i,j-1} \left(- \sum_{\rho \in A(\mu,\mu)} D_\rho \right)$.

**Third step of the induction: $j = N(\mu)$ case**

For simplicity, we denote

$$\tilde{M}_{1,(K_0,|\ell|)} := \overline{M}_{1,(K_0,|\ell|)}^{N(\mu)-1}, \quad \tilde{L}_i := L_{i,N(\mu)-1}.$$

In [21], Vakil–Zinger explained that all $\tilde{L}_i$ are isomorphic, so we denote it by $L$. Let $E$ be the Hodge bundle on $\overline{M}_{1,K_0\cup|\ell|}$. Let $E_0 := E$ and we define $E_i$ inductively by $E_i := \pi_i^* E_{i-1} \left( \sum_{\rho \in A(\mu,\mu)} D_\rho \right)$. Let $\tilde{E} := E_{N(\mu)-1}$. Thus we obtain

$$\tilde{E} = \pi^* E \left( \sum_{\rho \in A(\mu,\mu)} \tilde{D}_\rho \right),$$

where $\pi : \tilde{M}_{1,(K_0,|\ell|)} \to \overline{M}_{1,(K_0,|\ell|)}$ is the composition of the blow-ups $\pi_1, \ldots, \pi_{N(\mu)-1}$, $\tilde{D}_\rho$ are pull-backs of $D_\rho$. Note that $L \cong \tilde{E}^\vee$; see [21].

Now we define $\mathbb{P}\mathcal{M}_{0,\mu}^0 := \mathbb{P} \left( \bigoplus_{i \in [\ell]} L_{*,i} \right)$, where $\bigoplus_{i \in [\ell]} L_{*,i}$ is a vector bundle on $\mathcal{M}_{0,\mu}$. Let $\gamma$ be the tautological bundle on $\mathbb{P}\mathcal{M}_{0,\mu}^0$. Then we have the following natural induced
morphism
\[ t_\mu^{N(\bar{\mu})} : \widetilde{\mathcal{M}}_{1,(K_0,|\ell|')} \times \mathbb{P} \mathcal{M}_0^{\mu} \cong \mathbb{P}(N^{N(\bar{\mu})-1}) \]
\[ \cong \mathbb{P}(\langle t_\mu^{N(\bar{\mu})-1} \rangle \times \widetilde{\mathcal{M}}_{1,(K_0,|\ell|')} \times \mathcal{M}_0^{\mu})/\mathcal{M}_1^{N(\bar{\mu})}) \]
\[ \longrightarrow \mathbb{P}(\mathcal{M}_0^{\mu}/\mathcal{M}_0^{\mu}) \cong \mathcal{M}^{N(\bar{\mu})} \].

Since \( \mathcal{M}^{N(\bar{\mu})} \) is the exceptional divisor of \( \text{Bl}_{\mathcal{M}^{N(\bar{\mu})}} \mathcal{M}^{N(\bar{\mu})} \), the normal bundle \( N^{\mathcal{M}^{N(\bar{\mu})}}/\mathcal{M}_0^{\mu} \) is the tautological bundle of the projectivization \( \mathbb{P}(\mathcal{M}_0^{\mu}/\mathcal{M}_0^{\mu}) \). Thus we have
\[ N^{\mathcal{M}^{N(\bar{\mu})}} \cong \mathcal{E} \times \gamma. \]

**Last step of the induction: \( j > N(\bar{\mu}) \) case**

An idea of the last step is again an induction on \( j \). The new inductive argument is pretty much similar to the first and second steps.

First, we consider \( j = N(\bar{\mu}) + 1 \). By [21, Section 4.3 (18)], the image of the unramified morphism \( t_\mu^{N(\bar{\mu})} \), which is equal to \( \mathcal{M}^{N(\bar{\mu})} \subset \mathcal{M}^{N(\bar{\mu})} \), is a smooth divisor and the image of \( t_\mu^{N(\bar{\mu})+1} \) is smooth.

Thus \( \mathcal{M}^{N(\bar{\mu})} \) and \( \mathcal{M}^{N(\bar{\mu})} \) intersect transversally for \( \bar{\mu} \neq \bar{\mu}' \). Hence we observe that the collection \( \{ t_\mu^{N(\bar{\mu})}, t_\mu^{N(\bar{\mu})+1}, \ldots, t_\mu^{N(\bar{\mu})} \} \) of unramified morphisms is properly self-intersecting. Before going further, we introduce some notations.

- For \( \rho \in A(\mu, \mu_j), j > N(\bar{\mu}) \), and for \( i \in [\ell(\mu)] \) let \( \mathcal{M}_{0,\rho_i} \subset \mathcal{M}_0^{w,d_i} \) be the image of the node-identifying morphism
\[ \mathcal{M}_{0,\rho_i} \times \prod_{i' \in I(\rho_i)} \mathcal{M}_{0,\rho_i}^{w,d_i} \rightarrow \mathcal{M}_{0,\rho_i}^{w,d_i}. \]

- Let \( \mathcal{M}_{0,\rho} := \prod_{\rho \in [\ell(\mu)]} \mathcal{M}_{0,\rho_i} \subset \mathcal{M}_0^{w,d_i} \).
- Let \( \mathcal{M}_{0,\rho} := \mathcal{P}(\Theta_{\rho \in [\ell(\mu)]} L_{\rho_i}). \)

Note that \( \mathcal{P}\mathcal{M}_0^{w,d_i} \) are disjoint from each other by [21, Lemma 3.9]. Now we can apply [21, Corollary 3.4] so that we have the following induced unramified node-identifying morphism
\[ t_\mu^{N(\bar{\mu})+1} : \widetilde{\mathcal{M}}_{1,(K_0,|\ell|')} \times \mathcal{P}\mathcal{M}_0^{N(\bar{\mu})+1} \rightarrow \mathcal{M}^{N(\bar{\mu})+1} \]
where \( \mathcal{P}\mathcal{M}_0^{N(\bar{\mu})+1} \) is the blow-up of \( \mathcal{P}\mathcal{M}_0^{N(\bar{\mu})+1} \) along the smooth locus \( \bigcup_{\rho \in A(\mu,\bar{\mu})} \mathcal{P}\mathcal{M}_0^{N(\bar{\mu})+1} \). The domain of \( t_\mu^{N(\bar{\mu})+1} \) has the product form because
\[ (t_{\mu}^{N(\bar{\mu})})^{-1} \mathcal{M}^{N(\bar{\mu})+1} = \bigcup_{\rho \in A(\mu,\bar{\mu})} \mathcal{M}_{1,(K_0,|\ell|')} \times \mathcal{P}\mathcal{M}_0^{N(\bar{\mu})+1}. \]
where

\[ A(\mu_a, \mu_b) := \sqcup_{\mu' | \mu' = \mu_b} A(\mu_a, \mu'); \]

see [21, Section 4.3] for details. Since \( \mathcal{M}^{[N(\mu)]} \) is a smooth divisor in \( \mathcal{M}^{[N]} \), we can apply [21, Lemma 3.5] to obtain

\[ N_{\mu_{N(\mu)+1}} = (id \times \pi'_1)^* N_{\mu_{N(\mu)}}, \]

where \( \pi'_1 : \mathbb{P}\mathcal{M}^{1}_{0, \mu} \to \mathbb{P}\mathcal{M}^{0}_{0, \mu} \) is the blow-up morphism.

Next we consider \( j > N(\mu) + 1 \). We check the following inductively.

- **For** \( j > N(\mu) + 1 \), by [21, Section 4.3 (I5)], we have

  \[ (t^j_{\mu_{j-1}})^{-1}(\mathcal{M}^{[\mu_j]}_{j-1}) = \bigcup_{\rho \in A(\mu_{j}, \mu_{j})} \mathcal{M}_{1, (K_0, [\ell_j])} \times \mathbb{P}\mathcal{M}^{j-N(\mu)-1}_{0, \rho} \]

  where \( \mathbb{P}\mathcal{M}^{j-N(\mu)-1}_{0, \rho} \) are the proper transforms of \( \mathbb{P}\mathcal{M}^{j-N(\mu)-2}_{0, \rho} \) via the blow-up morphism \( \pi'_{j-N(\mu)-1} \).

- **By** [21, Corollary 3.4], the induced node-identifying morphism

  \[ t^j_{\mu_j} : \mathcal{M}_{1, (K_0, [\ell_j])} \times \mathbb{P}\mathcal{M}^{j-N(\mu)}_{0, \mu} \to \mathcal{M}^{[j]}_{j-N(\mu)} \]

  is unramified, where \( \mathbb{P}\mathcal{M}^{j-N(\mu)}_{0, \mu} \) is the blow-up of \( \mathbb{P}\mathcal{M}^{j-N(\mu)-1}_{0, \mu} \) along the smooth locus \( \bigcup_{\rho \in A(\mu_{j}, \mu_{j})} \mathbb{P}\mathcal{M}^{j-N(\mu)-1}_{0, \rho} \). We denote

  \[ \pi'_{j-N(\mu)} : \mathbb{P}\mathcal{M}^{j-N(\mu)}_{0, \mu} \to \mathbb{P}\mathcal{M}^{j-N(\mu)-1}_{0, \mu} \]

  the blow-up morphism.

- **The image of** \( t^j_{\mu_{j+1}} \) is smooth by [21, Section 4.3 (I8)]. Also the image of \( t^j_{\mu} \) is smooth because it is a blow-up of the image of \( t^{j-1}_{\mu} \) by [21, Lemma 3.3]. Hence the collection \( \{t^j_{\mu}, t^j_{\mu_{j+1}}, \ldots, t^j_{\mu_{N}}\} \) is properly self-intersecting.

- **The normal bundle of** \( t^j_{\mu} \) can be obtained by

  \[ N_{t^j_{\mu}} = (id \times \pi'_{j-N(\mu)})^* N_{t^j_{\mu_{j-1}}}. \]

  by [21, Lemma 3.5].

Let us define \( \mathbb{P}\mathcal{M}^{N}_{0, \mu} \) and \( \mathbb{P}\mathcal{M}^{N}_{0, \rho} \). Let \( \overline{\pi}' : \mathbb{P}\mathcal{M}^{0}_{0, \mu} \to \mathbb{P}\mathcal{M}^{0}_{0, \mu} \) be the composition of blow-up morphisms \( \pi'_1, \ldots, \pi'_{N-N(\mu)} \). Also we denote \( t^N_{\mu} \) by \( \overline{t}_{\mu} \). Then we have

\[ N_{t^\mu_{N}} \cong \overline{t}_\mu^* N_{\mathbb{P}\mathcal{M}^{N}_{0, \mu}} \cong E^\vee \boxtimes (\overline{\pi}')^* \gamma. \] (3.6)
By letting
\[ A_1(\mu) := \bigcap_{i=1}^{N(\mu) - 1} A(\mu_i, \mu), \quad A_0(\mu) := \bigcap_{i=N(\mu) + 1}^N A(\mu, \mu_i), \]
we can write
\[
\tilde{\tau}_\mu^* N_{\tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}|\tilde{\mathfrak{h}}} \left( \sum_{\mu' \neq \mu} \overline{\mathcal{M}_{\tilde{L}, \mu}} \cap \overline{\mathcal{M}_{\tilde{L}, \mu}} \right) = \tilde{\tau}_\mu^* N_{\tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}|\tilde{\mathfrak{h}}} \left( \sum_{\rho \in A_1(\mu)} \overline{\mathcal{M}_{\tilde{L}, \rho}} \times \mathbb{P} \overline{\mathcal{M}_{0, \mu}} + \sum_{\rho' \in A_0(\mu)} \overline{\mathcal{M}_{\tilde{L}, (K_0, [\ell])}} \times \mathbb{P} \overline{\mathcal{M}_{0, \rho'}} \right) = N_{\tilde{\mathfrak{h}}} \left( \sum_{\rho \in A_1(\mu)} \overline{\mathcal{M}_{\tilde{L}, \rho}} \times \mathbb{P} \overline{\mathcal{M}_{0, \mu}} + \sum_{\rho' \in A_0(\mu)} \overline{\mathcal{M}_{\tilde{L}, (K_0, [\ell])}} \times \mathbb{P} \overline{\mathcal{M}_{0, \rho'}} \right),
\]
where \( \overline{\mathcal{M}_{\tilde{L}, \rho}} \) is the proper transform of \( D_\rho \) defined in the first and second step of the induction for \( \rho \in A_1(\mu) \). Recall that \( \tilde{D}_\rho \) is the pull-back of \( D_\rho \). We can prove that \( \tilde{D}_\rho = \overline{\mathcal{M}_{\tilde{L}, \rho}} \). For \( \rho' \in A_0(\mu) \), let \( \tilde{D}_{\rho'} := \mathbb{P} \overline{\mathcal{M}_{0, \rho'}} \). Therefore, by using (2.4), (3.7), (3.6), and (3.5) sequentially we obtain
\[
\tilde{\tau}_\mu^* N_{\tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}|\tilde{\mathfrak{h}}} \left( \sum_{\rho \in A_1(\mu)} \overline{\mathcal{M}_{\tilde{L}, \rho}} \times \mathbb{P} \overline{\mathcal{M}_{0, \mu}} + \sum_{\rho' \in A_0(\mu)} \overline{\mathcal{M}_{\tilde{L}, (K_0, [\ell])}} \times \mathbb{P} \overline{\mathcal{M}_{0, \rho'}} \right)
\]
where \( \tilde{\mathcal{E}}' := (\tilde{\pi}')^* \gamma \left( \sum_{\rho' \in A_0(\mu)} \tilde{D}_{\rho'} \right) \).

### 3.2 Connection to perfect obstruction theory

Let \( \overline{\mathcal{M}^r} := \cup_{\mu} \overline{\mathcal{M}^r} \) and
\[
L_{\mu} := N_{\tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}|\tilde{\mathfrak{h}}} \left( \overline{\mathcal{M}^r/\mathfrak{h}} \right), \quad L_{rat} := N_{\tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}|\tilde{\mathfrak{h}}} \left( \overline{\mathcal{M}^r/\mathfrak{h}} \right).
\]
By (2.4), we have
\[
L_{rat} \mid \overline{\mathcal{M}^r} = L_{\mu} \left( \sum_{\mu' \in \mathfrak{h}, \mu' \neq \mu} \overline{\mathcal{M}^r} \cap \overline{\mathcal{M}^r} \right).
\]
In Corollary 2.6, we obtain a morphism
\[
L_{rat} \to H^1 \left( E^\vee_{\overline{\mathcal{M}/\mathfrak{h}}} \mid \overline{\mathcal{M}^r} \right)
\]
whose zero locus is \( \overline{\mathcal{M}^r} \cap \overline{\mathcal{M}^{red}} \). Note that we have
\[
\overline{\tau}_\mu^* H^1 \left( E^\vee_{\overline{\mathcal{M}/\mathfrak{h}}} \mid \overline{\mathcal{M}^r} \right) \cong E^\vee \boxtimes ev^* T_{\mathfrak{g}^r}.
\]
The morphism
\[ \tilde{\iota}_{\mu, \mathbb{P}^n}^*(\mathbb{L}_{\text{rat}} |_{\tilde{\mathcal{M}}}) \cong E^\vee \boxtimes q^* E' \to \tilde{\iota}_{\mu, \mathbb{P}^n}^* H^1(E^\vee_{\tilde{\mathcal{M}} | \tilde{\mathbb{P}}}, \mathcal{M}) \] (3.9)
induced by (3.8) coincides with the morphism of vector bundles \( i_\mathbb{P}^ \oplus 0 \) in [22, p. 1236] after a certain perturbation of the moduli space \( \bar{\mathcal{M}}_{0, (K_0, [\ell])} \times \bar{\mathcal{M}}_{0, \mu}^{p, n}(\mathbb{P}^n, d) \). The perturbed spaces are the virtual fundamental classes in symplectic geometry. Here is the explanation of notations above.

- \( \bar{\mathcal{M}}_{0, \mu}^{p, n}(\mathbb{P}^n, d) := \bar{\mathcal{M}}_{0, \mu}(\mathbb{P}^n, d) \times \mathcal{M}_{0, \mu}^{p, n}(\mathbb{P}^n, d) \) where \( \bar{\mathcal{M}}_{0, \mu}(\mathbb{P}^n, d) \) is defined similar to \( \bar{\mathcal{M}}_{0, \mu}(\mathbb{P}^n, d) \).
- \( \tilde{\iota}_{\mu, \mathbb{P}^n} : \) The node-identifying morphism \( \bar{\mathcal{M}}_{0, (K_0, [\ell])} \times \bar{\mathcal{M}}_{0, \mu}^{p, n}(\mathbb{P}^n, d) \to \bar{\mathcal{M}}_{\mu}^{p, n} \).
- \( q_\mu : \) The forgetful morphism \( \bar{\mathcal{M}}_{0, \mu}^{p, n}(\mathbb{P}^n, d) \to \mathcal{M}_{0, \mu}^{p, n}(\mathbb{P}^n, d) \).

In [22], Zinger proved the morphism \( i_\mathbb{P}^ \oplus 0 \) is injective, whereas the morphism (3.9) has zero locus \((\tilde{\iota}_{\mu, \mathbb{P}^n})^{-1}(\tilde{\mathcal{M}}_{\mu}^{p, n}(\mathbb{P}^n, d) \cap \tilde{\mathcal{M}}_{\mu}^{p, n}(\mathbb{P}^n, d)) \).

4 \  PROOF OF THEOREM 1.1

4.1 \  Chern class expression of \( A_{k,d}^{p, \overline{\mathcal{M}}} \)

We begin this section with introducing some vector bundles on \( \bar{\mathcal{M}}_{\mu}^{p, n} \) (caution: this is not \( \bar{\mathcal{M}}_{p, \overline{\mathcal{M}}}^{\mu} \)). Let
\[ V_1 := \mathbb{R}^1 \pi_* \mathcal{E}_\mathbb{P}(1)^{(1\oplus m+1)}(\mathbb{P}^n), \quad V_2 := \mathbb{R}^1 \pi_* \left( \bigoplus_{i=1}^m \mathcal{E}_\mathbb{P}(\deg f_i) \otimes \omega \right). \]
\[ V := V_1 \oplus V_2, \quad N_{\overline{\mathcal{M}}} := \pi_* \left( \bigoplus_{i=1}^m \mathcal{E}_\mathbb{P}(\deg f_i) \otimes \omega \right). \]
Note that rank \( V_1 = n + 1 \) and rank \( V_2 = d(\sum \deg f_i) + m \). We further note that \( \bar{\mathcal{M}}_{p, \overline{\mathcal{M}}}^{\mu} \) is isomorphic to the total space of \( N_{\overline{\mathcal{M}}} \).

Let \( V_1^p, V_2^p, \) and \( V^p \) be the pull-back vector bundles on \( \bar{\mathcal{M}}_{p, \overline{\mathcal{M}}} \). Let \( \sigma_1 : V_1^p \to \mathcal{O}_{\bar{\mathcal{M}}_{p, \overline{\mathcal{M}}}^{\mu}} \) and \( \sigma_2 : V_2^p \to \mathcal{O}_{\bar{\mathcal{M}}_{p, \overline{\mathcal{M}}}^{\mu}} \) be cosections defined by
\[ \sigma_1 : (u_0', ..., u_n') \mapsto \sum_{i=1}^m p_i \sum_{j=0}^n \partial f_i(u_0, ..., u_n) u_j, \]
\[ \sigma_2 : (p_1', ..., p_m') \mapsto \sum_{i=1}^m p_i f_i(u_0, ..., u_n), \]
and \( \sigma := \sigma_1 \oplus \sigma_2 \). Then the precise statement of Definition 2.2 can be written with the cosection \( \sigma \), for instance,
\[ A_{k,d}^{p, \overline{\mathcal{M}}} = (-1)^d \sum_i \deg f_i(b_Q)_* \partial^i \left( H^1(E^\vee_{\bar{\mathcal{M}}_{p, \overline{\mathcal{M}}}^{\mu}} | \bar{\mathcal{M}}_{p, \overline{\mathcal{M}}}^{\mu}) \sigma \right) \left[ C_{\bar{\mathcal{M}}_{p, \overline{\mathcal{M}}}^{\mu}} \right]. \]

In this section, we will compute \( A_{k,d}^{p, \overline{\mathcal{M}}} \) in terms of Chern classes of vector bundles; see (4.14). To do so, we first introduce some notations.
\[ \begin{align*}
\mathcal{P}_{\mathcal{M}} := \mathbb{P}(N_{\mathcal{M}} \oplus \mathcal{O}_{\mathcal{M}}) \text{ be a completion of } \mathcal{M}_{\mathcal{M}}. \\
\gamma : \mathcal{M}_{\mathcal{M}} = |N_{\mathcal{M}}| \to \mathcal{M}_{\mathcal{M}}, \quad \tilde{\gamma} : \mathcal{P}_{\mathcal{M}} \to \mathcal{M}_{\mathcal{M}} \text{ be the projection morphisms.}
\end{align*} \]

- Let \( D_{\infty} := \mathbb{P}(N_{\mathcal{M}} \oplus 0) \subset \mathcal{P}_{\mathcal{M}} \) be the divisor at infinity.
- \( \overline{V}_1^p := \overline{\gamma}^*V_1(-D_{\infty}), \overline{V}_2^p := \overline{\gamma}^*V_2, \overline{V}_1^p := \overline{V}_1^p \oplus \overline{V}_2^p. \)
- \( \overline{\sigma}_1 \) and \( \overline{\sigma}_2 \) be induced cosections by \( \sigma_1 \) and \( \sigma_2 \), respectively.
- \( \overline{\sigma} := \overline{\sigma}_1 \oplus \overline{\sigma}_2. \)

**Remark 4.1.** Note that we define \( \overline{V}_1^p \) as above by twisting \( \mathcal{O}(-D_{\infty}) \) because it is necessary to induce the cosection. Since we use \( p \)-fields in the definition of the cosection \( \sigma_1 \), and fibers of \( \mathcal{O}_{\mathcal{P}}(-D_{\infty}) \) are \( p \)-fields, there is a naturally induced cosection \( \overline{\gamma}^*V_1 \otimes \mathcal{O}_{\mathcal{P}}(-D_{\infty}) \to \mathcal{O}_{\mathcal{M}} \).

For simplicity, we denote \( C_{\tilde{\mathcal{M}}_{\mathcal{M}}/\overline{\mathcal{M}}_{\mathcal{M}}} \) by \( C_{p,\mathcal{M}} \). Let \( C_{p,\mathcal{M}} := C_{\tilde{\mathcal{M}}_{\mathcal{M}}/\mathcal{M}_{\mathcal{M}}} \cap |0 \oplus V_2^p| \). By [17, Proposition 5.3], we obtain

\[ 0^1_{V_1^p}[C_{p,\mathcal{M}}] = \overline{\gamma}_V^*0^1_{V_2^p}[C_{p,\mathcal{M}}], \tag{4.3} \]

where \( \overline{C}_{p,\mathcal{M}} \) is the closure of \( C_{p,\mathcal{M}} \) in \( V_1^p \) and \( 0^1_{V_1^p} \) is considered as a bivariant class to make sense of a homomorphism \( 0^1_{V_1^p} : A_*(\overline{V}_1^p(\overline{\sigma})) \to A_*(\overline{V}_2^p(\overline{\sigma}_2)) \) here. To make ease of the computation \( 0^1_{V_1^p}[C_{p,\mathcal{M}}] \), it is helpful to approximate \( \overline{C}_{p,\mathcal{M}} \) in the form of a subvector bundle of \( V_1^p \). For this, let us consider \( R_{p,\mathcal{M}} := C_{C_{p,\mathcal{M}}/C_{p,\mathcal{M}}} \), the normal cone to \( C_{p,\mathcal{M}} \) in \( C_{p,\mathcal{M}} \).

Therefore, we have \([C_{p,\mathcal{M}}] = [R_{p,\mathcal{M}}] \) in \( A_*(V_2^p(\sigma)) \). Again by [17, Proposition 5.3], we obtain

\[ 0^1_{V_1^p}[C_{p,\mathcal{M}}] = 0^1_{V_1^p}[R_{p,\mathcal{M}}] = \overline{\gamma}_V^*0^1_{V_2^p}[R_{p,\mathcal{M}}], \]

where \( \overline{R}_{p,\mathcal{M}} \) is the closure of \( R_{p,\mathcal{M}} \) in \( V_1^p \).

Now we investigate the cycle \( R_{p,\mathcal{M}} \) to show that it has a nice property that it can be approximated to a subline bundle of \( V_1^p \). Let \( C_{0,\mathcal{M}} := C_{\mathcal{M}^p/\overline{\mathcal{M}}_{\mathcal{M}}} \) (Caution: \( C_{p,\mathcal{M}} \) was \( C_{\mathcal{M}^p/\overline{\mathcal{M}}_{\mathcal{M}}} \).) Consider the gluing of (2.10)

\[ 0 \to R^1\pi_*\mathcal{O}_{C_{\mathcal{M}^p/\overline{\mathcal{M}}_{\mathcal{M}}}} \xrightarrow{\alpha} H^1(E^\vee_{\mathcal{M}^p/\overline{\mathcal{M}}_{\mathcal{M}}}) \to H^1(E^\vee_{\mathcal{M}^p/\overline{\mathcal{M}}_{\mathcal{M}}}) \to 0. \]

Then the morphism \( \alpha \) factors through

\[ 0 \to V_1^p \xrightarrow{\alpha} H^1(E^\vee_{\mathcal{M}^p/\overline{\mathcal{M}}_{\mathcal{M}}}) \to V_2^p \to 0. \]

The quotient of \( \alpha : R^1\pi_*\mathcal{O}_{C_{\mathcal{M}^p/\overline{\mathcal{M}}_{\mathcal{M}}}} \to V_1^p \) is \( \gamma^*H^1(E^\vee_{\mathcal{M}^p/\overline{\mathcal{M}}_{\mathcal{M}}}). \)
Let $V'_1 := H^1(E(\overline{\mathcal{M}}_{g,n})/\mathfrak{s}_{\mathfrak{g}})$. Then we have
\[ H^1(E(\overline{\mathcal{M}}_{g,n})/\mathfrak{s}_{\mathfrak{g}}) = \gamma^*V'_1 \oplus V^p. \]

Note that $C_{0,0}^{p,\overline{\mathfrak{s}}} \subset \gamma^*V'_1 \oplus V^p$.

Let $(C_{0,0}^{p,\overline{\mathfrak{s}}})_b := C_{0,0}^{p,\overline{\mathfrak{s}}b} \cap |0 \oplus V^p|$ and $\overline{R_0^{p,\overline{\mathfrak{s}}}} := C_{(C_{0,0}^{p,\overline{\mathfrak{s}}})_b}^{p,\overline{\mathfrak{s}}}$. The gluing of (2.10) gives rise to the exact sequence of abelian cones
\[ 0 \to R^1 \pi_* \mathcal{O}_{\mathcal{C}^{p,\overline{\mathfrak{s}}}} \to C^{p,\overline{\mathfrak{s}}}_0 \to 0, \]

and in the middle term, we have $R^1 \pi_* \mathcal{O}_{\mathcal{C}^{p,\overline{\mathfrak{s}}}} \cap C^{p,\overline{\mathfrak{s}}}_b = 0$ since $\alpha$ factors through $V^p_1$. Thus we have the following exact sequence of cones
\[ 0 \to R^1 \pi_* \mathcal{O}_{\mathcal{C}^{p,\overline{\mathfrak{s}}}} \to C^{p,\overline{\mathfrak{s}}}_0 \to 0, \]

where $\overline{R_0^{p,\overline{\mathfrak{s}}}}$ is the closure of $R_0^{p,\overline{\mathfrak{s}}}$ in $\gamma^*V'_1(-D_\infty) \oplus V^p_2$. Thus, we have
\[ B^{\overline{\mathfrak{s}}} := 0^!_{\overline{V}_1} [\overline{R_0^{p,\overline{\mathfrak{s}}}}] = 0^!_{\gamma^*V'_1(-D_\infty)} [\overline{R_0^{p,\overline{\mathfrak{s}}}}] \] 

by the excess intersection formula of the Gysin homomorphism.

To investigate the deformed space $\overline{R_0^{p,\overline{\mathfrak{s}}}}$ and the cycle $B^{\overline{\mathfrak{s}}}$, we investigate its support, $(C_{0,0}^{p,\overline{\mathfrak{s}}})_b$. For this, we recall the notations in Section 2.1. The normal cone $C^{p,\overline{\mathfrak{s}}}_0$ was locally represented by
\[
C^{p,\overline{\mathfrak{s}}}_0 \xrightarrow{loc} C_{U_H^P/M^{div} \times \mathbb{C}^{n+m}} |(M^{div})^k \approx \text{Spec} \left( \frac{R/(t_k)(y_1, \ldots, y_{n+m})[x_1, \ldots, x_{n+m}]}{(y_i x_j - y_j x_i)_{1 \leq i < j \leq n+m}} \right).
\]

Also we recall that the local chart of $\overline{\mathcal{M}}^P$, $U^P_H$ is an open dense subset of the zero locus of the section $F$ defined in (1.18)
\[ M^{div} \times (\mathbb{C}^n \times \mathbb{C}^m) \times \mathbb{C}^d \xrightarrow{(y_1, \ldots, y_{n+m}) \oplus 0 \oplus F} C^{\mathbb{B}^{n+m}}_{M^{div} \times (\mathbb{C}^n \times \mathbb{C}^m) \times \mathbb{C}^d} \oplus \bigoplus_{i=1}^m C^{\mathbb{B}^{d \cdot \text{deg} f_i}}_{M^{div} \times (\mathbb{C}^n \times \mathbb{C}^m) \times \mathbb{C}^d}. \]

Hence the normal cone is contained in the total space
\[ \left| \left( C^{\mathbb{B}^{n+m}}_{M^{div} \times (\mathbb{C}^n \times \mathbb{C}^m) \times \mathbb{C}^d} \right)_{U^P_H} \right| = \left| C^{\mathbb{B}^{n+m}}_{U^P_H} \right| \approx U^P_H \times \mathbb{C}^{n+m}. \]
So the coordinates $x_1, \ldots, x_{n+m}$ are identified with coordinate functions of this $\mathbb{C}^{n+m}$. Then, the local equations of $(C_0^{p, \overline{\mu}})_b \subset C_0^{p, \overline{\mu}}$ are $\{x_1 = \cdots = x_n = 0\}$. Therefore, the (local) coordinate ring of $(C_0^{p, \overline{\mu}})_b$ is given by

$$\bar{R} := R/(t_1)[y_1, \ldots, y_{n+m}]_{x_1, \ldots, x_{n+m}}.$$  

We observe that the defining ideal $(x_1, \ldots, x_n, y_i x_j - y_j x_i)_{1 \leq i < j \leq n+m}$ contains the ideal $(x_1, \ldots, x_n, y_i x_j)_{1 \leq i \leq n, m+1 \leq j \leq n+m}$, whose zero set is decomposed by

$$\{x_1 = \cdots = x_n = 0\} \cup \{x_1 = \cdots = x_n = y_1 = \cdots = y_n = 0\}.$$  

The first component glues to $\hat{\mathcal{M}}^{p, \overline{\mu}}$, which is embedded in $|V|$ by the zero section.

Let us denote the intersection $\hat{\mathcal{M}}^{\mu} \cap \hat{\mathcal{M}}^{\text{red}}$ by $\Delta_{\overline{\mu}}$. Then the loci $\{y_1 = \cdots = y_n = 0\}$ glues to $\hat{\mathcal{M}}^{p, \overline{\mu}}$. By the above local description, we can check that the rank $m$ bundle $F'$ is equal to $\gamma^* F$. Therefore, we have

$$(C_0^{p, \overline{\mu}})_b = C_0^{p, \overline{\mu}} \cap |0 \oplus V^p_2| \subset \hat{\mathcal{M}}^{\mu} \cup \gamma^* F \subset \hat{\mathcal{M}}^{p, \overline{\mu}}. \quad (4.5)$$  

By the definition of $R^{p, \overline{\mu}}$, it is supported in $\hat{\mathcal{M}}^{\mu} \cup |\overline{\gamma}^* F| \subset |\overline{\mathcal{M}}| \subset \mathbb{P}(N^{\overline{\mu}} \oplus \mathcal{O})$. Thus we have the decomposition

$$B^{\overline{\mu}} = B_1^{\overline{\mu}} + B_2^{\overline{\mu}}, \quad B_1^{\overline{\mu}} \in A_*(\hat{\mathcal{M}}^{\mu}) \text{ and } B_2^{\overline{\mu}} \in A_*(|\overline{\gamma}^* F|)$$  

of the cycle $B^{\overline{\mu}} = 0^!_{\overline{\gamma}^* V'_1(\mathcal{D}_\infty) \overline{R}_0^{p, \overline{\mu}}}$.  

We can show that $\overline{\gamma}^* 0^!_{\overline{\gamma}^* V'_2(\mathcal{D}_2)} B^{\overline{\mu}} = 0$ by a dimension reason; see [17, Section 5.0.1]. Combining this with (4.3), we have

$$0^!_{\overline{V}_2(\mathcal{D}_2)} (C^{p, \overline{\mu}}) = \overline{\gamma}^* 0^!_{\overline{\gamma}^* V_2(\mathcal{D}_2)} (B^{\overline{\mu}}) = 0^!_{\overline{V}_2(\mathcal{D}_2)} (B_1^{\overline{\mu}}). \quad (4.6)$$
Now we investigate the cycle $B_{1}^{\mu}$. From the construction of $B_{1}^{\mu}$, we have

$$B_{1}^{\mu} = 0! \left[ R_{0}^{p, \mu} \left| \overline{\Lambda_{\mu}} \right] \right].$$  \hspace{1cm} (4.7)

So we need to investigate the space $R_{0}^{p, \mu} \left| \overline{\Lambda_{\mu}}$. From the above local description of $C_{0}^{p, \mu}$ and $(C_{0}^{p, \mu})_{b}$, the normal cone $R_{0}^{p, \mu} = C_{(C_{0}^{p, \mu})_{b} / C_{0}^{p, \mu}}$ is locally described by

$$C_{(C_{0}^{p, \mu})_{b} / C_{0}^{p, \mu}} \left| \overline{\Lambda_{p, \mu}} \right. \cong \frac{R[z_{1}, \ldots, z_{n}]}{(y_{i}z_{j} - y_{j}z_{i})_{1 \leq i \leq j \leq n}}.$$  \hspace{1cm} (4.8)

Recall that $(C_{0}^{p, \mu})_{b} \subset \overline{\mathcal{M}_{p, \mu}} \cup \gamma^{*}F$. Since the local equation of $\overline{\mathcal{M}_{p, \mu}} \subset |V|$ is $x_{1} = \ldots = x_{n+m} = 0$, we have

$$C_{(C_{0}^{p, \mu})_{b} / C_{0}^{p, \mu}} \left| \overline{\mathcal{M}_{p, \mu}} \right. \cong \frac{R/(t_{1})[y_{1}, \ldots, y_{n+m}]}{(y_{i}z_{j} - y_{j}z_{i})_{1 \leq i \leq j \leq n}}.$$  \hspace{1cm} (4.9)

From the above local computation, we observe that $R_{0}^{p, \mu} \left| \overline{\mathcal{M}_{p, \mu}} = \gamma^{*}C_{0}^{\mu}$ where $C_{0}^{\mu} \subset V_{1}^{\prime}$ is the coarse moduli space of the cone stack $\mathcal{C}_{\mathcal{M}_{/gW}} \mathcal{M}_{\mu} \subset H^{1}/H^{0}(E_{\mathcal{M}_{/gW}}^{\gamma} | \mathcal{M}_{\mu})$ which is irreducible. Thus, we have

$$B_{1}^{\mu} = 0! \left[ R_{0}^{p, \mu} \left| \overline{\mathcal{M}_{\mu}} \right] \right] = 0! \left[ Y_{1}^{\prime} \right].$$  \hspace{1cm} (4.10)

So we need to investigate the cone $C_{0}^{\mu}$. By Corollary 2.6 and notations in Section 3.2, we obtain the morphism of vector bundles

$$i_{0} : \overline{\mathcal{M}_{\mu}} = N_{\mathcal{M}_{/gW}} \mathcal{M}_{\mu} \rightarrow H^{1}(E_{\mathcal{M}_{/gW}}^{\gamma} \mathcal{M}_{\mu}) = V_{1}^{\prime}.$$  \hspace{1cm}

where $\overline{\mathcal{M}_{\mu}}$ denotes $\mathcal{M}_{/gW} \mathcal{M}_{\mu}$. Clearly the morphism $i_{0}$ factors through the cone $C_{0}^{\mu} \subset V^{\prime}$ which is the coarse moduli space of the cone stack $\mathcal{C}_{\mathcal{M}_{/gW}} \mathcal{M}_{\mu}$. When we consider a local chart as in Section 3, the morphism to the cone is induced by the inclusion of ideals

$$(ty_{1}, \ldots, ty_{n}) \subset (t)$$

where $(t)$ is the defining ideal of $\overline{\mathcal{M}_{/gW}}$ and $(ty_{1}, \ldots, ty_{n})$ is the defining ideal of $\overline{\mathcal{M}_{\mu}}$ in the local chart. So the morphism $i_{0}$ is represented by

$$(y_{1}, \ldots, y_{n}) : O \rightarrow O^{g \otimes n}.$$  \hspace{1cm} (4.11)
in the local chart. Combining this observation with the local equation (2.1)(k = n case) of $C_0$, we have $C_0 = \text{Im}(i_0)$ where the closure is taken in $V'$.  

Now we want to modify $C_0$ to a subline bundle. The local expression of the morphism $i_0$, (4.1), says that locally it has a degeneracy locus $\{y_1 = \cdots = y_n = 0\}$. It glues to $\Delta = \tilde{M} \cap \tilde{M}^{red}$. Consider the blow-up morphism

$$q : \tilde{M} := \text{Bl}_{\Delta} \to \tilde{M}$$

be the blow-up morphism. By abuse of notation, we denote the induced morphism $\tilde{M} \times \tilde{M} \to \tilde{M}^{'}$ by $q$ and let $D' \subset \tilde{M} \times \tilde{M}^{'}$ be the exceptional divisor. By Corollary 2.6, we obtain an injective morphism of vector bundles

$$i : q^*L(D') \hookrightarrow q^*H^1(E_{\tilde{M}/\tilde{M}}) = q^*V'_1$$

Moreover, $C_0$ is equal to $q(\text{Im } j)$ by (2.3). Consider the diagram

By abuse of notation, we used notations $q$, $\gamma$, and $\bar{\gamma}$ for pull-backs. Then we observe

$$\gamma^*C_0 = q(\text{Im } j)$$

where $j$ is the injective morphism induced by $i$

$$j : \gamma^*(q^*L(D')) = \bar{\gamma}^*(q^*L(D'))(\sim q^*D_\infty) \hookrightarrow \gamma^*V'_1(\sim q^*d_\infty) = q^*V'_1.$$

Hence we have

$$[\gamma^*C_0] = q_*[\gamma^*(q^*L(D'))(\sim q^*D_\infty)].$$

(4.12)

From (4.7), (4.10) and (4.12), we have

$$B_1^1 = 0^1_{\gamma^*V'_1(\sim D_\infty)}[\gamma^*C_0]$$

$$= 0^1_{\gamma^*V'_1(\sim D_\infty)}q_*[\gamma^*(q^*L(D'))(\sim q^*D_\infty)]$$

$$= q_*c_{top} (\bar{\gamma}^*V'_1(\sim q^*D_\infty))/\bar{\gamma}^*(q^*L(D'))(\sim q^*D_\infty))$$
where $c_{\text{top}}(-)$ stands for the top Chern class. Thus we have

$$
\overline{\nu}_* B_1^{-\nu} = \overline{\nu}_* q_* c_{\text{top}} \left( \overline{\nu}^* q^* V_1'(\omega - q^* D_{\infty}) / \overline{\nu}^* (q^* \overline{\nu}(\overline{D}'))(\omega - q^* D_{\infty}) \right) = q_* \left( c \left( q^* V_1' / \overline{\nu}^* (q^* \overline{\nu}(\overline{D}')) \right) s((q^* N)^\vee) \right)^{\text{rank } V_1' - m - 1}.
$$

(4.13)

by [9, Example 3.2.2]. Here $c(-)$ and $s(-)$ denote the Chern and Segre classes, respectively, and $(-)_i$ indicates the degree $i$ component. For more specific computation, we need the following lemma with a general situation.

**Lemma 4.2.** Let $Y$ be a smooth variety and let $X \subset Y$ be a smooth subvariety of codimension $r$. Let $\pi : \overline{Y} = Bl_X Y \to Y$ be the blow-up morphism and $D$ be the exceptional divisor. Let $E$ and $L$ be a vector bundle and a line bundle on $Y$ respectively. Assume that the line bundle $\pi^* L(D)$ is embedded in $E$ as a subbundle. Then we have

$$
\pi_* c_{r-1} \left( \pi^* E / \pi^* L(D) \right) = \left( \frac{c(E)}{c(L)} \right)^{r-1}.
$$

**Proof.** We have

$$
c_{r-1} \left( \pi^* E / \pi^* L(D) \right) = \left( \frac{c(\pi^* E)}{c(\pi^* L(D))} \right)^{r-1}
= \left( \frac{c(\pi^* E)}{1 + \pi^* c_1(L) + D} \right)^{r-1}
= \left( c(\pi^* E) \left( \sum_{i=0}^{\infty} (-1)^i (\pi^* c_1(L) + D)^i \right) \right)^{r-1}.
$$

Note that $D \cong \mathbb{P}(N_{X/Y})$. For every $1 \leq l \leq k \leq r - 1$, we have

$$
\pi_* \left( c_{r-1-k} (\pi^* E) \cdot D^l \right) = c_{r-1-k} (E) \cdot (\pi_* D^l)
= c_{r-1-k} (E) \cdot \pi_* (c_1(\mathcal{O}_{\mathbb{P}(N_{X/Y})}(-1))^{l-1} \cdot D)
= (-1)^{l-1} \cdot c_{r-1-k} (E) \cdot s_{l-r}(N_{X/Y})[X] = 0.
$$

Hence we have

$$
\pi_* \left( c(\pi^* E) \left( \sum_{i=0}^{\infty} (-1)^i (\pi^* c_1(L) + D)^i \right) \right)^{r-1}
= \pi_* \left( c(\pi^* E) \left( \sum_{i=0}^{\infty} (-1)^i (\pi^* c_1(L))^i \right) \right)^{r-1}
= \left( \frac{c(E)}{c(L)} \right)^{r-1}.
$$

\[\square\]
By (4.6), (4.13) and Lemma 4.2, we have

\[ 0^!_{V_p,\xi_2}[C^p,\mu] = V_2 \cdot 0^!_{V_2,\xi_2}(B_1^\mu) \]

\[ = 0^!_{V_2,\xi_2}(B_1^\mu) \]

\[ = 0^!_{V_2,\xi_2}[\tilde{M}^\mu]\left(\frac{c(V'_1)s((N^\mu)^\nu)}{c(L^\mu)}\right)^{\text{rank } V'_1 - m - 1} \]

where the cosection \( \xi_2 : V_2 \to \mathcal{O}_{\tilde{M}^\mu} \) is defined by

\[(p_1, \ldots, p_m) \mapsto \sum_{i=1}^{m} p_i f_i(u_0, \ldots, u_n), \quad u_0, \ldots, u_n \in \Gamma(C_{\tilde{M}^\mu}, ev^*\mathcal{O}_{\mathbb{P}^n}(1)).\]

By (4.2) and (4.14), we have

\[ A^\mu_{k,d} = (-1)^d \sum_{i=1}^{\deg f_i(b_Q)} 0^!_{V_2,\xi_2}[\tilde{M}^\mu]\left(\frac{c(V'_1)s((N^\mu)^\nu)}{c(L^\mu)}\right)^{\text{rank } V'_1 - m - 1}. \] (4.15)

### 4.2 Condition (2) of Theorem 1.1: computation of \( A^\mu_{k,d} \) in terms of Chern classes of vector bundles and Zinger’s formula

Let

- \( \tilde{M}_{0,\mu}(Q, d) := \tilde{M}_{0,\mu}(Q, d) \times_{\mathbb{P}^n} \mathbb{P} \tilde{M}_{0,\mu} \).
- \( \tilde{M}^\mu(Q) := \tilde{M}^\mu \times_{\tilde{M}_{1,k}(\mathbb{P}^n, d)} \tilde{M}_{1,k}(Q, d) \).

The node-identifying morphism \( \tilde{\tau}_\mu \) lifts to the following morphisms

\[ \tilde{\tau}_{\mu,\mathbb{P}^n} : \tilde{M}_{1,(K_0,\{e\})} \times \tilde{M}_{0,\mu}(\mathbb{P}^n, d) \to \tilde{M}^\mu, \]

\[ \tilde{\tau}_{\mu,Q} : \tilde{M}_{1,(K_0,\{e\})} \times \tilde{M}_{0,\mu}(Q, d) \to \tilde{M}^\mu(Q). \]

We define a subgroup \( S(\mu) \) of the symmetric group \( S_\epsilon \) by

\[ S(\mu) := \{ f \in S_\epsilon \mid d_i = d f(i) \text{ and } |K_i| = |K f(i)| \text{ for all } i \in [\epsilon] \}. \]

Note that \( S(\mu) \) acts on \( \tilde{M}_{1,(K_0,\{e\})} \times \tilde{M}_{0,\mu}(\mathbb{P}^n, d) \) (respectively, \( \tilde{M}_{1,(K_0,\{e\})} \times \tilde{M}_{0,\mu}(Q, d) \)) by permuting marked points on \( \tilde{M}_{1,(K_0,\{e\})} \) and components in \( \tilde{M}_{0,\mu}(\mathbb{P}^n, d) \) (respectively, \( \tilde{M}_{0,\mu}(Q, d) \)). Then \( \tilde{\tau}_{\mu,Q} \) and \( \tilde{\tau}_{\mu,\mathbb{P}^n} \) send an \( S(\mu) \)-orbit to a point. It implies

\[ \deg(\tilde{\tau}_{\mu,\mathbb{P}^n}) = \deg(\tilde{\tau}_{\mu,Q}) = |S(\mu)|. \]
Let us define

\begin{itemize}
  \item $\overline{M}_{0,\mu}(P^n, d) := \bigcup_{\mu' \in \Sigma, \mu' = \mu} \overline{M}_{0,\mu'}(P^n, d)$.
  \item $\overline{M}_{0,\mu}(Q, d) := \bigcup_{\mu' \in \Sigma, \mu' = \mu} \overline{M}_{0,\mu'}(Q, d)$.
  \item $p \overline{M}_{0,\mu} := \bigcup_{\mu' \in \Sigma, \mu' = \mu} p \overline{M}_{0,\mu'}$.
\end{itemize}

Note that $|\{\mu' \in \Sigma \mid \mu' = \mu\}| = \ell'!/|S(\mu)|$. Thus the induced node-identifying morphisms,

\[ \tau_{\mu,p^n} : \overline{M}_{1, (K_0, [\ell])} \times \overline{M}_{0,\mu}(P^n, d) \to \overline{M}_{\mu} \]

\[ \tau_{\mu,Q} : \overline{M}_{1, (K_0, [\ell])} \times \overline{M}_{0,\mu}(Q, d) \to \overline{M}_{\mu}(Q), \]

have degree $\ell'!$. Hence we have

\[ \ell'! \cdot 0'_{V_2, \xi_2} [\overline{M}_{\mu}] = (\tau_{\mu,Q})_* \cdot (\tau_{\mu,p^n})_* \left( \left[ \overline{M}_{1, (K_0, [\ell])} \right] \times [\overline{M}_{0,\mu}(P^n, d)] \right) \quad (4.16) \]

by the bivariant property of localized Gysin map. Therefore, we have

\[ A_{k,d}^\overline{\mu} = (-1)^d \sum_{i} \deg f_i (b_Q)_* \left( 0'_{V_2, \xi_2} [\overline{M}_{\mu}] \frac{c(V'_i) s((N_{\mu'})^\vee)}{c(L_{\mu'})} \right)_{\text{rank } V'_{1-m-1}} \]

\[ = (-1)^d \sum_{i} \deg f_i \frac{1}{\ell'!} (b_Q)_* (\tau_{\mu,Q})_* \left( 0'_{V_2, \xi_2} \left( \left[ \overline{M}_{1, (K_0, [\ell])} \right] \times [\overline{M}_{0,\mu}(Q, d)] \right) \frac{c((\overline{m}, Q)^*)^r V'_i s((\overline{m}, Q)^*)^r (N_{\mu'})^\vee)}{c((\overline{m}, Q)^* L_{\mu'})} \right)_{\text{rank } V'_{1-m-1}} \]

\[ = \frac{1}{\ell'!} (b_Q)_* (\tau_{\mu,Q})_* \left( \left[ \overline{M}_{1, (K_0, [\ell])} \right] \times [\overline{M}_{0,\mu}(Q, d)] \right)_{\text{vir}} \left( \frac{c(V'_i) s(E^\vee \boxtimes e v^* T_{P^n}) (E^\vee \boxtimes e v^* (T_Q))}{c(E^\vee \boxtimes q^*_\mu E')} \right)_{\text{rank } V'_{1-m-1}} \]

\[ = \frac{1}{\ell'!} (b_Q)_* (\tau_{\mu,Q})_* \left( \left[ \overline{M}_{1, (K_0, [\ell])} \right] \times [\overline{M}_{0,\mu}(Q, d)] \right)_{\text{vir}} \left( \frac{c(V'_i) s(E^\vee \boxtimes e v^* T_{Q})}{c(E^\vee \boxtimes q^*_\mu E')} \right)_{\text{rank } V'_{1-m-1}} \quad (4.17) \]

where $q_\mu : \overline{M}_{0,\mu}(Q, d) \to p \overline{M}_{0,\mu}$ is the projection morphism. The first equality comes from (4.15), the second equality comes from (4.16), the third equality comes from [17, Proposition 5.5], (3.9), and the base change theorem, and the fourth equality comes from the exact sequence of tangent bundles

\[ 0 \to T_Q \to T_{P^n} |_Q \to \bigoplus \mathcal{O}_{P^n} (\deg f_i) |_Q \to 0. \]
(4.17) exactly coincides with the formula [22, (3-29)]. Note that an integration over $A_{k,d}^{\mu}$ induces the formula [22, Theorem 1A]; see [22, Section 3.4]. So the second condition (2) of Theorem 1.1 is explained.

### 4.3 | Condition (1) of Theorem 1.1

[11, Theorem 2.11] tells us that

$$\n_{\text{red}} := \pi_* u^*(\mathcal{O}_{P^n}(\deg f_i))$$

is the vector bundle on $\widetilde{\mathcal{M}}_{\text{red}}$. Let $s$ be the section on $\n_{\text{red}}$ induced by $f_i$, which are defining equations of $Q \subset P^n$. By Definition 2.2 and [17, Proposition 4.1], we have

$$A_{k,d}^{\text{red}} = 0^1_{r(\nu^{\nu} \mathcal{M}_{\text{red}} \mathcal{B})} \left[ \mathcal{C} \mathcal{M}_{\text{red}} \mathcal{B} \right] = 0^1_{(\n_{\text{red}})^{\nu,\nu} \left[ \widetilde{\mathcal{M}}_{\text{alg}}^{\text{red}} \right]}.$$

The most right-hand side $0^1_{(\n_{\text{red}})^{\nu,\nu} \left[ \widetilde{\mathcal{M}}_{\text{alg}}^{\text{red}} \right]}$ is a refined Euler class. Thus the first condition (1) of Theorem 1.1 is explained.

### Notations

The below is a table of notations frequently used.

| Symbol | Description |
|---|---|
| $Q$ | a complete intersection in $P^n$ defined by $\{f_1 = \cdots = f_m = 0\}$ |
| $\mathbb{S}_{k,d}$ or $\mathbb{S}$ | the index set of elements $\mu = ((d_1, K_1), \ldots, (d_s, K_s))$ such that $\sum_j d_j = d$, $K_j$ are mutually disjoint subsets of $[k]$ |
| $\mathcal{M}_{g,k}$ or $\mathcal{M}$ | the moduli space of prestable genus $g$ curves with $k$ marked points |
| $\mathcal{M}_{g,k,d}^w$ or $\mathcal{M}^w$ | the moduli space of prestable genus $g$, weight $d$ curves with $k$ marked points |
| $\mathcal{M}^{\nu}$ | the closed substack of $\mathcal{M}^w$ parametrizing $\nu$-type weighted curves |
| $\mathcal{M}_{0,\mu}$ | $\prod_{\ell \in [r]} \mathcal{M}_{0, \nu_\mu, K_{\ell, d_\ell}}$ |
| $\mathcal{M}_{\text{alg}}^w$ | Hu–Li’s desingularization of $\mathcal{M}_{1,1,d}^w$ |
| $\mathcal{M}_{\text{alg}}^{\nu}$ | the exceptional divisor in $\mathcal{M}^w$ lying on a proper transform of $\mathcal{M}^{\nu}$ |
| $\mathcal{M}_{g,k,d}^{\text{div}}$ or $\mathcal{M}^{\text{div}}$ | the moduli space of prestable genus $g$ curves with $k$ marked points and a degree $d$ divisor on $C$ |
| $\mathcal{M}_{\text{alg}}^{\text{div}}$ | $\mathcal{M}_{g,k,d}^w \times_{\mathcal{M}_{1,k,d}} \mathcal{M}_{1,k,d}^{\text{div}}$ |
| $\mathcal{B}_{g,k,d}$ or $\mathcal{B}$ | the moduli space of prestable genus $g$ curves with $k$ marked points and a degree $d$ line bundle on $C$ |
| $\mathcal{B}$ | $\mathcal{M}^w \times_{\mathcal{M}_{1,k,d}} \mathcal{B}_{1,k,d}$ |
\[
\overline{M}_g(P^n, d) = \overline{M}_{1,k}(P^n, d) \times \mathcal{M}
\]

\[
\overline{M}_g(Q, d) = \overline{M}_{1,k}(Q, d) \times \mathcal{M}
\]

\[
\overline{M}_{0,\mu}(P^n, d) = \overline{M}_{0,\mu}(P^n, d) \times \mathcal{M}
\]

\[
\overline{M}_{0,\mu}(Q, d) = \overline{M}_{0,\mu}(Q, d) \times \mathcal{M}
\]

\[
\tilde{\mathcal{M}} \text{ the reduced component of } \overline{\mathcal{M}}
\]

\[
\tilde{\mathcal{M}} \text{ an irreducible component of } \overline{\mathcal{M}} \text{ indexed by } \tilde{\mu}
\]

\[
\tilde{\mathcal{M}}_\mu \times \mathcal{M}_{1,k}(P^n, d)
\]

\[
\tilde{\mathcal{M}}^{\text{red}} \text{ the reduced component of } \tilde{\mathcal{M}}
\]

\[
\tilde{\mathcal{M}}^{\text{red}} \times \mathcal{M}_{1,k}(P^n, d)
\]

\[
\tilde{\mathcal{M}}^{\text{red}} \times \mathcal{M}_{1,k}(P^n, d)
\]

\[
\pi: C \rightarrow X \text{ a universal curve on a stack } X
\]

\[
ev \text{ an evaluation morphism from } C
\]

\[
\mathcal{C}_{A/B} \text{ the relative intrinsic normal cone of } A \text{ relative to } B
\]

\[
C_{A/B} \text{ the coarse moduli space of the intrinsic normal cone } \mathcal{C}_{A/B}
\]

\[
\tilde{h}_\mu^{\text{red}}(E_0 \rightarrow E_1) \text{ a localized stack } [E_1/E_0]
\]

\[
0^\text{red}_{\text{Gysin}} \text{ a localized Gysin map}
\]

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