Scaling of Hadronic Form Factors in Point Form Kinematics

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The general features of baryon form factors calculated with point form kinematics are derived. With point form kinematics and spectator currents hadronic form factors are functions of \( \eta := \frac{1}{4}(v_{\text{out}} - v_{\text{in}})^2 \) and, over a range of \( \eta \) values, are insensitive to unitary scale transformations of the model wave functions when the extent of the wave function is small compared to the scale defined by the constituent mass, \( \langle r^2 \rangle \ll 1/m^2 \). The form factors are sensitive to the shape of such compact wave functions. Simple 3-quark proton wave functions are employed to illustrate these features. Rational and algebraic model wave functions lead to a reasonable representation of the empirical form factors, while Gaussian wave functions fail. For large values of \( \eta \) point form kinematics with spectator currents leads to power law behavior of the form factors.

1. Introduction

The calculation of hadronic form factors within the framework of Poincaré covariant quantum mechanics involves the following two separate ingredients: (1) Bound-state wave functions, which represent vectors in the representation space of the little group of the Poincaré group, and (2) current operators that are covariant under a kinematic subgroup. Covariant conserved currents are generated from these ingredients by the dynamics, while the choice of the kinematic subgroup specifies the form of kinematics. Spectator currents, by definition, commute with quark spectator momenta, the definition of which depends on the choice of a form of kinematics.

With instant and front form kinematics spectator currents provide an impulse approximation, in which the spatial structure of the bound-state wave function determines the quantitative features of the form factors. When the spatial extent of the bound-state wave function is scaled unitarily to zero both instant and front form kinematics yield form factors, which are independent of momentum transfer.

Point-form kinematics employs the full Lorentz group as the kinematic subgroup. In this case the Lorentz covariant spectator currents probe the velocity structure specified by the bound-state wave function. When the extent of the wave function is scaled unitarily to zero, point-form kinematics yields a nontrivial scaling limit for the form factors, which depends on the shape of the wave function. At high values of momentum transfer the
scaled form factors decrease with an inverse power of the momentum transfer. The power is determined by the current operator and is independent of the shape of the wave function. A derivation of these features is presented in sections 2 and 3 below. The purpose here is to illuminate the unfamiliar qualitative features of relativistic quantum mechanics with point form kinematics. Relations to quantum field theory are beyond the scope of this article.

Recently point form kinematics [1,2] was applied to a constituent quark model calculation of the form factors of the nucleon, which achieved remarkable agreement with experimental data [3,4,5]. The calculation employed a fairly compact 3 quark wave function \((\langle r^2 \rangle \sim 0.1 \text{ fm}^2)\) and point-like quark currents. A comparably compact wave function employed with instant form kinematics requires either that the constituent quarks are spatially extended or implementation of vector meson dominance for agreement with the empirical form factors [6]. The qualitative difference between form factors calculated with point form kinematics and those calculated with instant and front form kinematics has been recently emphasized in ref.[7].

The general features of the point form kinematics of the confined quark description of hadron states and current matrix elements are described in section 2 below. Unitary scale transformations of the bound state wave functions are described in section 3. Section 4 contains the illustration of the shape dependence and the insensitivity of the proton form factors to scale changes with simple 3-quark wave functions for the proton. Section 5 contains a summarizing discussion. Explicit expressions for the Wigner rotation operators, the boost relations of the spectator momenta to the constituent momenta, and Dirac current kernels are listed in 3 appendices.

2. Constituent quark representations of single-hadron states and current density operators

2.1. Point form kinematics of confined quark.

Single hadron eigenstates with \(n\) constituents with four-momentum \(P = Mv\), may be represented by functions of the form

\[
\Psi_{M,j,v,\sigma}(\vec{v}; \vec{k}_1, \ldots, \vec{k}_n; \sigma_1, \ldots, \sigma_n) = \phi_\sigma(\vec{k}_1, \ldots, \vec{k}_n; \sigma_1, \ldots, \sigma_n) \delta^{(3)}(\vec{v} - \vec{v}_a),
\]

where \(\vec{k}_i\) and \(\sigma_i\) are constituent momenta and spin variables. Flavor and color variables are implied. The norm of the wave function \(\phi_\sigma\) is specified by

\[
\|\phi_\sigma\|^2 = \sum_{\sigma_1, \ldots, \sigma_n} \int d^3k_1 \ldots \int d^3k_n \delta \left( \sum_i \vec{k}_i \right) |\phi_\sigma(\vec{k}_1, \ldots, \vec{k}_n; \sigma_1, \ldots, \sigma_n)|^2,
\]

which implies that

\[
(\Psi_{M,j,v,\sigma}, \Psi_{M,j',v,\sigma'}) = \delta^{(3)}(\vec{v}_f - \vec{v}_a) \delta_{\sigma', \sigma}.
\]

Under Poincaré transformations the velocity \(v\) transforms as a four-vector, while the total spin operator \(\vec{j}\) undergoes Wigner rotations, \(R_W(\Lambda, v) := B^{-1}(\Lambda v) \Lambda B(v)\) as:

\[
U^\dagger(\Lambda, d) v U(\Lambda, d) = \Lambda v, \quad v^2 = -1, \quad U^\dagger(\Lambda, d) \vec{j}_i U(\Lambda, d) = R_W(\Lambda, v) \vec{j}_i .
\]
Here the boost $B(v)$ is the rotationless Lorentz transformation, which satisfies the defining relation $B(v)\{1, 0, 0, 0\} = v$.

The constituent momenta and spins undergo the same Wigner rotations:

$$U^\dagger(\Lambda) \vec{k}_i U(\Lambda) = R_W(\Lambda, v) \vec{k}_i, \quad U^\dagger(\Lambda) \vec{j}_i U(\Lambda) = R_W(\Lambda, v) \vec{j}_i.$$  \hfill (5)

The Poincaré covariance of the bound-state wave function $\phi_\sigma$, is realized by its covariance under rotations, invariance under translations and independence of the velocity $v$. By assumption $\phi$ is an eigenfunction of an invariant mass operator. No constituent quark masses are required in this representation, but they provide an essential scale in the definition of the current operators.

Some features of the unobservable wave functions are indirectly observable through the matrix elements of the covariant current operators $I^\mu(x; v_f, v_a)$, which satisfy the relation

$$I^\mu(x; v_f, v_a) = e^{i\mathcal{M}v_f \cdot x} I^\mu(0; v_f, v_a) e^{-i\mathcal{M}v_a \cdot x},$$  \hfill (6)

where $\mathcal{M}$ is the mass operator. The mass operator of confined quark may be defined by the eigenvalues $M_n$ and assumed wave functions, $\phi_n$,

$$\mathcal{M} := \sum_n \phi_n M_n \phi_n^\dagger,$$  \hfill (7)

or by the conventional assumption that either $\mathcal{M}$ or $\mathcal{M}^2$ may be expressed as the sum of a kinetic term, which is a function of the internal momenta and a confining term, which is function of the operators conjugate to the internal momenta. The basic mass operator of confined quark need not involve constituent quark masses and the formal structure of the dynamics is simpler if it does not \[8,9\]. This implies that the gross features of the mass spectrum and the spatial extent of the wave function are related. Since the kinetic part of the mass operator is repulsive it follows with this convention that the use of $\mathcal{M}$ leads to more compact wave functions than the use of $\mathcal{M}^2$ \[9\].

The current operators are represented by the kernels

$$\langle \sigma_1', \ldots, \sigma_n', \vec{k}_2, \ldots, \vec{k}_n | I^\mu(0; v_f, v_a) | \vec{k}_n, \ldots, \vec{k}_2, \sigma_n, \ldots, \sigma_1 \rangle,$$

from which the dependent momentum $\vec{k}_1 := -(\vec{k}_2 + \ldots + \vec{k}_n)$ has been omitted. The electric and the magnetic currents, $I^\mu_e(v_f, v_a)$ and $I^\mu_m(v_f, v_a)$, are defined respectively by the projection into the plane defined by $v_f$ and $v_a$ and the projection perpendicular to that plane. The magnetic current is then conserved by definition:

$$\mathcal{M} v_f \cdot I_m(v_f, v_a) - v_a \cdot I_m(v_f, v_a) \mathcal{M} = 0,$$  \hfill (8)

since $v_f \cdot I_m(v_f, v_a) = v_a \cdot I_m(v_f, v_a) = 0$.

For the electric current the conservation requirement can be satisfied by the expression

$$I^\mu_e(v_f, v_a) = \frac{1}{2} \left( \mathcal{M}^\dagger \mathcal{I}_e(\eta) \mathcal{M}^{-\frac{1}{2}} + \mathcal{M}^{-\frac{1}{2}} \mathcal{I}_e(\eta) \mathcal{M}^\dagger \right) \frac{v^\mu_f + v^\mu_a}{2\sqrt{1 + \eta}}$$

$$+ \frac{1}{2} \left( \mathcal{M}^\dagger \mathcal{I}_e(\eta) \mathcal{M}^{-\frac{1}{2}} - \mathcal{M}^{-\frac{1}{2}} \mathcal{I}_e(\eta) \mathcal{M}^\dagger \right) \frac{v^\mu_f - v^\mu_a}{2\eta} \sqrt{1 + \eta},$$  \hfill (9)

where $\eta$ is defined as

$$\eta := \frac{1}{4} (v_f - v_a)^2, \quad -\frac{1}{4} (v_f + v_a)^2 = 1 + \eta.$$  \hfill (10)
The Lorentz invariant operator \( \mathcal{I}_e(\eta) \) is a functional of the current:

\[
\mathcal{I}_e(\eta) = \frac{1}{2\sqrt{1+\eta}} \left\{ \mathcal{M}^{-\frac{1}{2}} I \cdot v_f \mathcal{M}^{\frac{1}{2}} + \mathcal{M}^{\frac{1}{2}} I \cdot v_a \mathcal{M}^{-\frac{1}{2}} \right\}.
\] (11)

The expression (9) may be viewed as a quantum mechanical analog of the Ward identity. For convenience, without loss of generality, we may assume

\[
v_a = \{ \sqrt{1+\eta}, 0, 0, -\sqrt{\eta} \}, \quad v_f = \{ \sqrt{1+\eta}, 0, 0, \sqrt{\eta} \}.
\] (12)

The magnetic current then has the components \( \{0, I_m x(\eta), I_m y(\eta), 0\} \) and magnetic form factors are proportional to the invariant reduced matrix element \( \vec{I}_m(\eta) \). Electric form factors are proportional to the invariant reduced matrix element \( \vec{I}_e(\eta) \). These operator relations simplify significantly when projections onto eigenstates of \( \mathcal{M} \) with eigenvalues \( M_f \) and \( M_a \) are considered. The following treatment is restricted to elastic transitions, \( M_f = M_a \). Electric and magnetic form factors are invariant reduced matrix elements of the operators \( \mathcal{I}_e(\eta) \) and \( \mathcal{I}_m(\eta) \).

### 2.2. Spectator currents

Changes in the representation of initial and final states are convenient to accommodate the construction of simple current operators. Individual four-momenta \( p_i \) for the spectator constituents may be defined as functions of the \( n-1 \) constituent momenta \( \vec{k}_2, \ldots, \vec{k}_n \), a constituent quark mass \( m \) and the velocity \( v \) as:

\[
p_i := B(v)k_i, \quad k_i := \{ \omega_i, \vec{k}_i \}, \quad \omega_i := \sqrt{m^2 + |\vec{k}_i|^2}.
\] (13)

It follows from this definition and eq.(5) that the momenta \( p_i \) transform as four-vectors,

\[
U^\dagger(\Lambda) p_i U(\Lambda) = \Lambda p_i.
\] (14)

The parameters, which specify the boost are different variables in different forms of kinematics. With Lorentz kinematics, as used here, the components of the velocity \( \vec{v} \) are the independent kinematic variables. With other forms of kinematics the velocity, which specifies the boost, is a function of the kinematic components of the total momentum, the internal momenta \( \vec{k}_i \) and the constituent quark masses.

Free-particle spin operators \( \vec{s}_i \) which transform according to

\[
U^\dagger(\Lambda) \vec{s}_i U(\Lambda) = \mathcal{R}_W(\Lambda, p_i) \vec{s}_i,
\] (15)

are related to the constituent spins \( \vec{j}_i \) by

\[
\vec{s}_i = \mathcal{R}_W[B(v), \vec{k}_i] \vec{j}_i.
\] (16)

Since the relations (13) and (16) are invertible one may choose the spectator momenta \( \vec{p}_i \) and spin components \( \lambda_i \) as the independent variables. The transformation involves multiplication of the wave function by the square root of the Jacobian:

\[
J(v, \vec{p}_2, \ldots, \vec{p}_n) = \frac{\partial(k_2, \ldots, k_n)}{\partial(p_2, \ldots, \vec{p}_n)} = \frac{\omega_2 \cdots \omega_n}{E_2 \cdots E_n}, \quad E_i := \sqrt{m^2 + |\vec{p}_i|^2},
\] (17)
and products of Wigner rotations $\mathcal{R}_W[B(v), k_i]$. In terms of these variables the wave function takes the form

$$
\psi(\vec{v}; \sigma_1, \vec{p}_2, \lambda_2, \ldots, \vec{p}_n, \lambda_n) := 
\sum_{\sigma_2 \ldots \sigma_n} \prod_{i=2}^n D_{\lambda_i, \sigma_i}^{1/2} (\mathcal{R}_W[B(v), k_i]) \sqrt{\mathcal{J}(v, \vec{p}_2, \ldots, \vec{p}_n)} \phi(\sigma_1, \vec{k}_2[\vec{v}, \vec{p}_2], \sigma_2, \ldots, \vec{k}_n[\vec{v}, \vec{p}_n], \sigma_n),
$$

(18)

with the norm

$$
\|\psi\|^2 = \sum_{\sigma_1} \sum_{\lambda_2 \ldots \lambda_n} \int d^3p_2 \ldots \int d^3p_n |\psi(\vec{v}; \vec{p}_2, \ldots, \vec{p}_n)|^2 = \|\phi\|^2 = 1.
$$

(19)

In this representation spectator currents have the general form

$$(\sigma_1, \vec{p}_2', \lambda_2', \ldots, \vec{p}_n', \lambda_n', |I^\mu(0; v', v)|\vec{p}_n, \lambda_n, \ldots, \vec{p}_2, \lambda_2, \sigma_1) :=$$

$$(\sigma_1' | I^\mu_1(v', v; \vec{p}_2, \ldots, \vec{p}_n) | \sigma_1) \delta(\vec{p}_2' - \vec{p}_2) \ldots \delta(\vec{p}_n' - \vec{p}_n) \delta_{\lambda_2', \lambda_2} \ldots \delta_{\lambda_n', \lambda_n}. \quad (20)$$

The dependence of the current $I^\mu_1(\vec{p}_2, \ldots, \vec{p}_n) = I^\mu_{1s}(\vec{p}_2, \ldots, \vec{p}_n) + I^\mu_{1m}(\vec{p}_2, \ldots, \vec{p}_n)$ on the spectator momenta is subject to model assumptions. The simplest form is independent of the spectator momenta, which implies structureless fermionic constituents:

$$(v_f, \sigma_1' | I^\mu_{1m}(\vec{p}_2, \ldots, \vec{p}_n) | \sigma_1, v_a) := \tilde{u}_{\sigma_1'}(v_f) \left( \gamma^\mu - \frac{(v_f + v_a)\mu}{2(1 + \eta)} \right) u_{\sigma_1}(v_a),$$

$$(v_f, \sigma_1' | I^\mu_{1s}(\vec{p}_2, \ldots, \vec{p}_n) | \sigma_1, v_a) := \tilde{u}_{\sigma_1'}(v_f) \frac{(v_f + v_a)\mu}{2(1 + \eta)} u_{\sigma_1}(v_a) = \frac{(v_f + v_a)\mu}{2\sqrt{1 + \eta}} \delta_{\sigma_1', \sigma_1}. \quad (21)$$

In that case the charge form factor is the overlap integral of the initial and final state wave functions. The model independent $\eta$ dependence of the square of the product of the Jacobians, which appears in the overlap integral, is given by

$$
\mathcal{J}_{fa} := \mathcal{J}(v_f, \vec{p}_2, \ldots, \vec{p}_n) \mathcal{J}(v_a, \vec{p}_2, \ldots, \vec{p}_n) = \prod_{i=2}^n \frac{(v_f \cdot p_i)(v_a \cdot p_i)}{E_i^2} = \prod_{i=2}^n \left( 1 + \frac{\eta (m^2 + p_{i1}^2)}{m^2 + |\vec{p}_i|^2} \right). \quad (22)
$$

The Jacobian factor has an obvious zero-mass limit

$$
(\mathcal{J}_{fa})_{m=0} = \prod_{i=2}^n \left( 1 + \eta (1 - z_i^2) \right), \quad z_i := p_{iz}/|\vec{p}_i|, \quad (23)
$$

which is independent of the magnitudes of the momenta. For large values of $\eta$ the Jacobian factor $\sqrt{\mathcal{J}_{fa}}$ is proportional to $\eta^{(n-1)/2}$.

Since the spectator constraints (20) imply the relations

$$
k_i' = B^{-1}(v_f)B(v_a)k_i = B(v_a)^2 k_i, \quad i = 2, \ldots, n, \quad (24)
$$

between the initial and final constituent momenta, the spectator Wigner rotations are $\mathcal{R}_W[B(v_a)^2, k_i]$. Explicit expressions for these operators are given in Appendix A.
For small values of $\eta$ the spectator Wigner rotations reduce to the identity.

Single-quark current kernels, $\langle \tilde{p}_1', \lambda_1'|I^\mu(0)|\lambda_1, p_1 \rangle$ and the associated Wigner rotations introduce an additional $\eta$ dependence in the spectator current:

$$(v_f; \sigma_1'|I^\mu_1(\vec{p}_2, \cdots, \vec{p}_n)|\sigma_1, v_a) := \sum_{\lambda_1', \lambda_1} \frac{1}{\lambda_1', \lambda_1} D^{\lambda_1', \lambda_1}_{\lambda_1, \sigma_1}(\mathcal{R}_{Wf}) \langle p_1', \lambda_1'|I^\mu(0)|\lambda_1, p_1 \rangle D^{\lambda_1, \sigma_1}_{\lambda_1', \sigma_1}(\mathcal{R}_{Wa}),$$

with $\mathcal{R}_{Wa} := B^{-1}(p_1)B(v_a)B(k_1)$ and $\mathcal{R}_{Wf} := B^{-1}(p_1')B(v_f)B(k_1')$ and

$$\langle p_1'I^\mu_1(0)p_1 \rangle = \bar{u}(p_1')\gamma^\mu u(p_1).$$

The boost relations (2.13) relate the quark momenta $p_1$ and $p_1'$ to internal momenta $\vec{k}_1$ and $\vec{k}_1'$, which are functions of the spectator momenta $\vec{k}$ and $\vec{k}'$. Thus the initial and final momenta $p_1$ and $p_1'$ are boost dependent functions of $\eta$ and the spectator momenta $p_2, \ldots, p_n$. Explicit expressions for these relations are given in Appendix B. The details of the spinor currents are given in Appendix C. The explicit representations of the Wigner rotations $\mathcal{R}_{Wa}$ and $\mathcal{R}_{Wf}$ are in Appendix A.

3. **Unitary scale transformations**

The rms radius $r_0$ of the matter distribution represented by the wave function $\phi$ is defined by

$$r_0^2 := -6 \left( \frac{dF_0(Q^2)}{dQ^2} \right)_{Q^2=0},$$

where $F_0$ is the overlap integral

$$F_0(Q^2) := \int d^3k_2 \cdots \int d^3k_n \phi(\vec{k}_2 - \vec{Q}/2n, \ldots, \vec{k}_n - \vec{Q}/2n)^* \phi(\vec{k}_2 + \vec{Q}/2n, \ldots, \vec{k}_n + \vec{Q}/2n).$$

For any given wave function the radius $r_0$ can be varied to any positive value by the unitary transformation

$$U_\beta \phi(\vec{k}_2, \ldots, \vec{k}_n) = \beta^{-3(n-1)/2} \phi(\vec{k}_2/\beta, \ldots, \vec{k}_n/\beta).$$

The radius $r_0$ is a measure of the extent of the Fourier transform of the wave function. In the limit $\beta \to \infty$ the radius $r_0$ is reduced to zero. Quark masses appear as scale parameters in the currents. With zero mass constituents the current operators commute with the unitary scale transformations $U_\beta$. With point-form kinematics the relevant structure is the distribution of internal velocities $\vec{k}_i/m$. The dimensionless form factors are functions of $\eta$ and $m r_0$, and in the zero mass limit are invariant under unitary scale transformations. The point limit $r_0 \to 0$ and the zero-mass limit are identical. However, when the mass operator $\mathcal{M}$ (or $\mathcal{M}^2$) of confined quark is the sum of a kinetic term plus a confining potential the scale $r_0$ is related to the mass spectrum independently of a constituent mass.

With “Galilean relativity” $r_0$ is equal to the observable charge radius. With instant and front-form kinematics, and $m r_0 > 1$, the relation between the radius $r_0$ and the charge operator involves “relativistic corrections”, which arise from the boosts and the
spinor structure. At this point the qualitative difference between point and instant-form
kinematics is readily apparent. In the latter form the kinematic quantity that specifies
the total momentum is \( \vec{P} \) instead of \( \vec{v} \). Constituent momenta \( p_i \) are specified as functions
of \( \vec{P} \) and \( \vec{k}_2, \ldots, \vec{k}_n \) by

\[
p_i := - B(-\vec{Q}/2M_0)\{\omega_i, \vec{k}_i\}, \quad M_0 := \sum_{i=1}^n \omega_i,
\]
\[
p_i' := B(\vec{Q}/2M_0')\{\omega'_i, \vec{k}'_i\} \quad M'_0 := \sum_{i=1}^n \omega'_i.
\]

(30)

It follows from this definition that

\[
\sum_i \vec{p}_i = -\frac{1}{2}\vec{Q}, \quad \text{and} \quad \sum_i \vec{p}_i' = \frac{1}{2}\vec{Q}
\]

(31)

The spectator constraints are \( \vec{p}_i' = \vec{p}_i \) for \( i = 2, \ldots, n \). Since

\[
\lim_{\beta \to \infty} U^{-1}(\beta) \frac{\vec{Q}}{M_0} U(\beta) = 0,
\]

(32)

the spectator constraints reduce to \( \vec{k}_i' - \vec{k}_i \) and the form factors are independent of \( \vec{Q} \).

With Lorentz kinematics the charge form factors decrease with a power of \( \eta \) in the
scaling limit, \( \beta \to \infty \), when \( \eta \gg 1 \). The exponent is independent of the shape of the
wave function. This becomes evident with the change of variables of integration variable
\( \vec{p}_i \to \sqrt{\eta} \vec{p}_i \), which brings a factor of \( \eta^{-3(n-1)/2} \) in front of the integral. Combined with
the asymptotic \( \eta \) dependence of the Jacobian factor \( [22] \) the overlap integral is proportional to \( \eta^{-(n-1)} \) at large values of \( \eta \). Dirac spinor currents \( [25] \) introduce an additional
asymptotic \( \eta \) dependence of the form factor, proportional to \( 1/\eta^{3/2} \). Consequently point
form kinematics with the single particle current kernel \( [25] \) implies that the electric form
factors behave as \( \eta^{-(n+1/2)} \) at large values of \( \eta \).

4. Proton form factor illustrations

4.1. Expressions for \( G_E \) and \( G_M \)

In order to illustrate the behavior of the form factors we employ a conventional com-
pletely symmetric spin-isospin amplitude \( \chi_{\sigma,\tau} \) represented by a function of Clebsch-Gordan
coefficients:

\[
\chi_{\sigma,\tau}(\sigma_1, \tau_1, \sigma_2, \tau_2, \sigma_3, \tau_3) := \frac{1}{\sqrt{2}} \left\{ \delta_{\sigma,\sigma_1}(\frac{1}{2}, \frac{1}{2}, \sigma_2, \sigma_3|0, 0)(\frac{1}{2}, \frac{1}{2}, \tau_2, \tau_3|0, 0)
\right.
\]
\[
+ \left( \frac{1}{2}, \frac{1}{2}, \sigma_2, \sigma_3|1, \sigma_2 + \sigma_3)(1, \frac{1}{2}, \sigma_2 + \sigma_3, \sigma_1|\frac{1}{2}, \sigma \right)
\]
\[
\times \left( \frac{1}{2}, \frac{1}{2}, \tau_2, \tau_3|1, \tau_2 + \tau_3)(1, \frac{1}{2}, \tau_2 + \tau_3, \tau_1|\frac{1}{2}, \tau \right) \right\},
\]

(33)

and two different permutation symmetric radial wave function models. These have the
Gaussian and rational shapes:

\[
\phi_G \left( \frac{\kappa^2 + q^2}{2b^2} \right) := \frac{1}{(b\sqrt{\pi})^3} \exp \left( -\frac{\kappa^2 + q^2}{2b^2} \right),
\]

(34)
\[ \phi_R \left( \frac{\kappa^2 + q^2}{2b^2} \right) := b^{-3} \sqrt{\frac{3}{4\pi^3}} \left( 1 + \frac{\kappa^2 + q^2}{2b^2} \right)^{-2}, \tag{35} \]

where
\[
\tilde{\kappa} := \sqrt{\frac{3}{2}} \kappa_1 \equiv -\sqrt{\frac{3}{2}} (\bar{k}_2 + \bar{k}_3), \quad \tilde{q} := \sqrt{\frac{1}{2}} (\bar{k}_2 - \bar{k}_3), \quad \frac{\partial(\tilde{\kappa}, \tilde{q})}{\partial(\bar{k}_2, \bar{k}_3)} = \sqrt{27}, \tag{36} \]

and
\[
\frac{1}{2}(\tilde{\kappa}^2 + q^2) = \bar{k}_2^2 + \bar{k}_3^2 + \bar{k}_2 \cdot \bar{k}_3 \equiv \frac{1}{4}(\bar{k}_1^2 + \bar{k}_2^2 + \bar{k}_3^2). \tag{37} \]

Changes of the scale parameter \( b \) represent unitary transformations. The overlap integral \( F_0(Q^2) \) takes the convenient form
\[
F_0(Q^2) = \int d^3 \kappa' \int d^3 \kappa \int d^3 q \phi \left( \frac{\kappa'^2 + q^2}{2b^2} \right) \phi \left( \frac{\kappa^2 + q^2}{2b^2} \right) \delta^{(3)} \left( \tilde{\kappa}' - \tilde{\kappa} - \sqrt{\frac{2}{3}} \tilde{q} \right). \tag{38} \]

By definition the proton form factors \( G_E(\eta) \) and \( G_M(\eta) \) are related to the electric and magnetic current matrices by
\[
G_E(\eta) = \frac{1}{2} \text{Tr}(\psi_f \mathcal{I}_e(\eta) \psi_a),
\]
\[
G_M(\eta) = \frac{1}{2} \text{Tr}[(\sigma_x - i\sigma_y)(\psi_f \mathcal{I}_m + \psi_a)]. \tag{39} \]

After summation over the spin-isospin indices the expressions for the factors of the proton reduce to the integrals
\[
G_E(\eta) = \int d^3 p_2 d^3 p_3 \phi \left( \frac{\kappa'^2 + q^2}{2b^2} \right) \phi \left( \frac{\kappa^2 + q^2}{2b^2} \right) \sqrt{27} J_{fa}(p_2, p_3) C_{23}(\eta, p_2, p_3) S_e(\eta, p_2, p_3),
\]
\[
G_M(\eta) = \int d^3 p_2 d^3 p_3 \phi \left( \frac{\kappa'^2 + q^2}{2b^2} \right) \phi \left( \frac{\kappa^2 + q^2}{2b^2} \right) \sqrt{27} J_{fa}(p_2, p_3) C_{23}(\eta, p_2, p_3) S_m(\eta, p_2, p_3). \tag{40} \]

For zero-mass constituents and large \( \eta \) the Jacobian factor (23) is proportional to \( \eta^2 \):
\[
J_{fa}(\eta, p_2, p_3) \approx \eta^2 (1 - z_2^2)(1 - z_3^2). \tag{41} \]

The coefficient \( C_{23}(\eta, p_2, p_3) \) is determined by the spectator Wigner rotations,
\[
C_{23}(\eta, p_2, p_3) = \frac{1}{2} \sum_{\sigma',\sigma} \frac{1}{2} D_{\sigma', \sigma}^2 \left( \mathcal{R}_W[B(v_a)^2, k_2] \right) D_{-\sigma', -\sigma}^2 \left( \mathcal{R}_W[B(v_a)^2, k_3] \right) \]
\[
= \cos \frac{\theta_2}{2} \cos \frac{\theta_3}{2} + \frac{\bar{p}_{2\perp} \cdot \bar{p}_{3\perp}}{|\bar{p}_{2\perp}| |\bar{p}_{3\perp}|} \sin \frac{\theta_2}{2} \sin \frac{\theta_3}{2}. \tag{42} \]
For zero-mass constituents and large $\eta$ this coefficient is independent of $\eta$ (33):

$$C_{23}(\eta, \vec{p}_2, \vec{p}_3) \approx \frac{\vec{p}_{2\perp} \cdot \vec{p}_{3\perp}}{|\vec{p}_{2\perp}||\vec{p}_{3\perp}|}. \quad (43)$$

For zero-mass constituents and $\eta \gg 1$ the arguments of the wave function have the approximate forms:

$$|\vec{k}_i|^2 \approx \eta |\vec{p}_i|^2 (1 + z_i)^2, \quad |\vec{k}_i'|^2 \approx \eta |\vec{p}_i'|^2 (1 - z_i)^2 \quad i = 2, 3, \quad (44)$$

and

$$\vec{k}_2 \cdot \vec{k}_3 \approx \eta |\vec{p}_2| |\vec{p}_3| \sqrt{(1 + z_2)(1 + z_3)}, \quad \vec{k}_2' \cdot \vec{k}_3' \approx \eta |\vec{p}_2'| |\vec{p}_3'| \sqrt{(1 - z_2)(1 - z_3)}. \quad (45)$$

When the current is specified by (65) and (66) the current factors $S_e$ and $S_m$ are

$$S_e = \sqrt{\frac{(E'_1 + m)(E_1 + m)(1 + \eta)}{4E_1'E_1}} \left\{ (1 + \frac{\vec{p}_{1\perp}' \cdot \vec{p}_1}{(E'_1 + m)(E_1 + m)}) \cos \left( \frac{\theta_1 - \theta'_1}{2} \right) 
+ \frac{|p_{1\perp}|(|p_{1z}' - p_{1z}|)}{(E'_1 + m)(E_1 + m)} \sin \left( \frac{\theta_1 - \theta'_1}{2} \right) \right\}, \quad (46)$$

and

$$S_m = \sqrt{\frac{1 + \eta}{4\eta E_1'E_1(E_1 + m)(E_1 + m)}} \left\{ [p_{1z}'(E_1 + m) - p_{1z}(E'_1 + m)] \cos \frac{\theta_1'}{2} \cos \frac{\theta_1}{2} 
+ |p_{1\perp}|(E'_1 + E_1 + 2m) \sin \frac{\theta_1' - \theta_1}{2} + |p_{1\perp}|(E_1 - E'_1) \sin \frac{\theta_1' + \theta_1}{2} \right\}. \quad (47)$$

For zero-mass constituents these expressions reduce to

$$S_e(m = 0) = \sqrt{\frac{1 + \eta}{4}} \left\{ (1 + \hat{\vec{p}}_1' \cdot \hat{\vec{p}}_1) \cos \left( \frac{\theta_1 - \theta'_1}{2} \right) 
+ \left( \sqrt{1 - z^2_1} z'_1 - \sqrt{1 - z^2_1} z_1 \right) \sin \left( \frac{\theta_1 - \theta'_1}{2} \right) \right\},$$

and

$$S_m(m = 0) = \sqrt{\frac{1 + \eta}{4\eta}} \left\{ (z'_1 - z_1) \cos \frac{\theta_1'}{2} \cos \frac{\theta_1}{2} 
+ \left( \sqrt{1 - z^2_1} + \sqrt{1 - z^2_1} \right) \sin \frac{\theta_1' - \theta_1}{2} + \left( \sqrt{1 - z^2_1} - \sqrt{1 - z^2_1} \right) \sin \frac{\theta_1' + \theta_1}{2} \right\}. \quad (48)$$

For $\eta \gg 1$ the factor $S_e$ is proportional to $\eta^{-3/2}$ and the factor $S_m$ is independent of $\eta$. It then follows from eq. (40) that electric and the magnetic form factor behave as $\eta^{-7/2}$ and $\eta^{-2}$ respectively for large values of $\eta$. With the current (21) one has $S_e = S_m = 1$. In this case both form factors behave as $\eta^{-2}$. From the Rosenbluth formula for the elastic cross section,

$$\frac{1}{\sigma_{Mott}} \frac{d\sigma}{d\Omega} = \frac{1}{1 + \eta} G_{ep}^2 + \left\{ \frac{1}{1 + \eta} + 2 \tan^2 \theta_e/2 \right\} \eta G_{mp}^2 \quad (49)$$

it follows that the magnetic form factor dominates for large $\eta$. 

4.2. Numerical results

For the two shapes of the wave function considered here the rms radius \( r_0 \) is \( r_0 = 1/b \) for the Gaussian model \( \text{Eq. (31)} \), and \( r_0 = \sqrt{2/5}/b \) for the rational wave function \( \text{Eq. (35)} \). With \( b = 650 \text{ MeV} \) the Gaussian model \( \text{Eq. (31)} \) gives same mean square radius \( \sim 0.1 \text{ fm}^2 \) as the wave function derived in ref. \[11\] by diagonalization of a 3 quark mass operator, which gives a satisfactory description of the empirical nucleon spectrum. This wave function was used in refs. \[3,4,5\] to calculate the nucleon form factors with point form kinematics. In the case of the rational wave function \( \text{Eq. (35)} \) the same value obtains with \( b = 410 \text{ MeV} \). With the quark mass \( m = 340 \text{ MeV} \) used in refs. \[3,4,5\] the relevant dimensionless parameter is \( mr_0 = .52 \).

To illustrate the dependence of the form factor on unitary scale transformations of the wave function we show numerical results for \( mr_0 = .52, .33, 0 \) and both wave function shapes in Fig. 1. The results reveal the relative insensitivity of the form factors to unitary scale transformations of the wave function when \( (mr_0)^2 \ll 1 \). A recent parameterization of form factor data ref. \[12\] provides a bench mark for comparison.

With the spinor currents the magnetic form factors, Fig. 2, show a similar more compact pattern. In that case the Gaussian wave function in the point limit and the rational wave function with \( mr_0 = .52 \) are in rough agreement with each other and the data parameterization. The rational wave function with \( mr_0 = .52 \) gives the magnetic moment as \( 2.86 \text{ nm} \), which is close to the empirical value (2.79 nm).

The rational wave function with \( mr_0 = .52 \) provides a reasonable representation of the data, which is comparable to the results obtained with the wave function employed in refs. \[3,4\]. Even in the point limit the Gaussian shape does not yield both form factors close to the data.

With \( S_e = 1 \) a regular scaling pattern obtains as shown in Figs. 3 and 4, which depends only on the shape of the wave function and the point-form spectator constraint. For \( mr_0 < .5 \) the two shapes provide form factors that range over limited non-overlapping regions of size. These figures illustrate the effects of the Lorentz-kinematic spectator constraints without the single-quark spinor currents. With the spinor current the electric form factors converge more rapidly to the point limit and show a more drastic dependence on the shape of the wave function.

5. Discussion

The main results of the present investigation can be summarized as follows: With zero-mass constituent quarks relativistic quantum mechanics with point-form kinematics yields hadron form factors, which are invariant under unitary scale transformations, but which depend strongly on the shape of the wave function. With non-vanishing constituent quark masses the form factors depend but weakly on the scale when \( \langle r^2 \rangle m^2 \) is about one fourth or less. Within that range it is possible to achieve realistic features of the proton form factors with simple rational wave function shapes. It has also been shown is that a realistic mass spectrum is compatible with these features \[3,4\]. With the simple dynamical structure discussed in ref. \[9\] adjustment of the confinement shape and the quark masses may provide a realistic description of both the spectrum and the form factors.

The quark momenta or velocities used in the definition of currents are related to the
internal relative momenta by boost relations, which depend the form of kinematics. The special features of Lorentz kinematics are numerically significant, and lead to the non-trivial point limits of the form factors. The numerical significance of the Wigner rotations of the spins is small compared to that of the boost dependence of the momenta. When \( \eta \gg 1 \) the form factors in the point limit asymptotically attain power law behavior. The exponent of the leading power depends on the current model, but not on the wave function.

The results shown in Fig. 1 suggest that it should be possible to approximate electric form factor data with a simple wave function in the point limit. Indeed we find that with the spinor current considered above the wave function

\[
\phi_I(\kappa^2 + q^2) = C_a \left( 1 + \frac{\kappa^2 + q^2}{2b^2} \right)^{-a},
\]

where \( C_a \) is a normalization constant, and \( a = 11/4 \) yields the electric form factor shown in Fig. 5., which is quite close to the parameterization of ref. [12]. The magnetic form calculated with this wave function in the point limit is however not as close to the corresponding parameterization given in ref. [12]. The results shown in Fig 1 suggest that better representations of the empirical magnetic form factor call for moderate finite values of \( m/b \). As examples the results for both \( G_E \) and \( G_M \) as obtained with \( m/b = 0.52 \) with the wave function (50) with \( a = 9/4 \) are also shown in Fig. 5. In this case the calculated electric form factor is effectively indistinguishable from the corresponding parameterization of the empirical values, and the calculated magnetic moment 2.80 nm coincides with the empirical value. The calculated magnetic form factor falls slightly faster than the parameterization of the empirical form factor.

The main qualitative features of relativistic quantum mechanics with Lorentz kinematics have been illustrated above. These features appear to be appropriate for a phenomenology of confined quark. It appears that the point limit, which is characteristic of Lorentz kinematics may provide a useful zero-order description to be refined by finite quark masses, which enter as scale parameters.

6. Acknowledgment

We are grateful for instructive discussions and correspondence with L.Ya. Glozman, W. H. Klink and W. Plessas on the subject matter. D. O. R. is indebted to R. D. McKeown for hospitality at the W. K. Kellogg Radiation Laboratory. Research supported in part by the Academy of Finland through grant 54038 and by the U.S. Department of Energy, Nuclear Physics Division, contract W-31-109-ENG-38.

Appendices

A. Wigner rotations

The explicit representations of the spectator Wigner rotations \( \mathcal{R}_W[B(v_a)^2, k_i] \), are

\[
D_{i-1}^I \left( \mathcal{R}_W[B(v_a)^2, k_i] \right) = \cos \frac{\theta_i}{2} - i \sin \frac{\theta_i}{2} \frac{(\vec{p}_i \times \vec{\sigma}_i)_z}{|p_i \perp|}, \quad i = 2, \ldots, n.
\]
where the angles $\theta_i$ are defined as
\[
\sin \frac{\theta_i}{2} := \frac{-\sqrt{\eta}|p_{i\perp}|}{\sqrt{2m(m + \sqrt{1 + \eta E_i}) + |p_{i\perp}|^2 + \eta(m^2 + p_{i\perp}^2)}}. \tag{52}
\]
For zero-mass constituents ($m=0$) the Wigner rotations are independent of the magnitudes of the momenta,
\[
\left(\sin^2 \frac{\theta_i}{2}\right)_{m=0} = \frac{\eta(1 - z_i^2)}{1 + \eta(1 - z_i^2)} = 1 - \frac{1}{1 + \eta(1 - z_i^2)}, \tag{53}
\]
with $z_i := p_{iz}/|\vec{p}_i|$. The explicit representations of the Wigner rotations $\mathcal{R}_{W_a}$ and $\mathcal{R}_{W_f}$ associated with the single-quark spinor current are
\[
D^\frac{1}{2}(\mathcal{R}_{W}[B(v_a), \vec{k}_1]) = \cos \frac{\theta_1}{2} - i \sin \frac{\theta_1}{2} \frac{(p_1 \times \vec{\sigma})_z}{|p_{1\perp}|}, \quad D^\frac{1}{2}(\mathcal{R}_{W}[B(v_f), \vec{k}_1']) = \cos \frac{\theta_1'}{2} - i \sin \frac{\theta_1'}{2} \frac{(p_1' \times \vec{\sigma})_z}{|p_{1\perp}|}. \tag{54}
\]
Here the angles $\theta_1, \theta_1'$ are defined as
\[
\sin \frac{\theta_1}{2} = \frac{\sqrt{\eta}|p_{1\perp}|}{\sqrt{2(1 + \sqrt{1 + \eta})(m + E_1)(m + \omega_1)}}, \\
\sin \frac{\theta_1'}{2} = -\frac{\sqrt{\eta}|p_{1\perp}|}{\sqrt{2(1 + \sqrt{1 + \eta})(m + E_1')(m + \omega_1')}}. \tag{55}
\]
For $m = 0$ and $\eta \gg 1$ these expressions reduce to
\[
\left(\sin \frac{\theta_1}{2}\right)^2_{m=0} \approx 1 + \frac{k_{1z}}{|k_1|}, \quad \left(\sin \frac{\theta_1'}{2}\right)^2_{m=0} \approx 1 - \frac{k_{1z}'}{|k_1'|}. \tag{56}
\]

**B. Boost relations**

According to the definition (13) the Breit-frame (12) components of the spectator momenta $p_i$ are related to the initial and final constituent momenta $\vec{k}_i$ and $\vec{k}_i'$ by
\[
\{E_i, \vec{p}_i\} = \{\sqrt{1 + \eta \omega_i} - \sqrt{\eta} k_{iz}, k_{i\perp}, \sqrt{1 + \eta} k_{iz} - \sqrt{\eta} \omega_i\} = \{\sqrt{1 + \eta} \omega_i' + \sqrt{\eta} k_{iz}', k_{i\perp}', \sqrt{1 + \eta} k_{iz}' + \sqrt{\eta} \omega_i'\}. \tag{57}
\]
These relations can be inverted to give
\[
k_{i\perp} = p_{i\perp}, \quad k_{iz} = \sqrt{1 + \eta} p_{iz} + \sqrt{\eta} E_i, \quad \omega_i = \sqrt{\eta} p_{iz} + \sqrt{1 + \eta} E_i, \\
k_{i\perp}' = p_{i\perp}', \quad k_{iz}' = \sqrt{1 + \eta} p_{iz}' - \sqrt{\eta} E_i', \quad \omega_i' = -\sqrt{\eta} p_{iz}' + \sqrt{1 + \eta} E_i'. \tag{58}
\]
The momenta $p_1$ and $p_1'$ of the active quark are functions of $\eta$ and the spectator momenta. The formal relation to the internal momenta $\vec{k}_1$ and $\vec{k}_1'$ are the same as those given in eq. (57),
\[
p_1 := B(v_a)\{\omega_1, \vec{k}_1\} = \left\{\sqrt{1 + \eta} \omega_1 - \sqrt{\eta} k_{1z}, \ k_{1\perp}, \sqrt{1 + \eta} k_{1z} - \sqrt{\eta} \omega_1\right\}. \tag{59}
\]
\[ p'_1 := B(v_f)\{\omega'_1, \vec{k}_1'\} = \left\{ \sqrt{1 + \eta \omega'_1 + \sqrt{\eta} k'_{1z}}, \ k_{1\perp}, \sqrt{1 + \eta k'_{1z} + \sqrt{\eta} \omega'_1} \right\}, \quad (59) \]

where

\[ \vec{k}_1 := -\sum_{i=2}^{n} \vec{k}_i, \quad \text{and} \quad \vec{k}_1' := -\sum_{i=2}^{n} \vec{k}_i'. \quad (60) \]

It follows that

\[
\frac{p_{1z}}{E_1} = \frac{\sqrt{1 + \eta k_{1z} - \sqrt{\eta} \omega_1}}{\sqrt{1 + \eta \omega_1 - \sqrt{\eta} k_{1z}}} \approx -1 + \frac{1}{2\eta} \frac{\omega_1 + k_{1z}}{\omega_1 - k_{1z}}, \quad \text{for} \ \eta \gg 1, \quad (61)
\]

and

\[
\frac{p_{1z}'}{E_1'} = \frac{\sqrt{1 + \eta k'_{1z} + \sqrt{\eta} \omega'_1}}{\sqrt{1 + \eta \omega'_1 + \sqrt{\eta} k'_{1z}}} \approx 1 - \frac{1}{2\eta} \frac{\omega'_1 - k'_{1z}}{\omega'_1 + k'_{1z}}, \quad \text{for} \ \eta \gg 1. \quad (62)
\]

The momenta \(\vec{k}_1\) and \(\vec{k}_1'\) are related to the spectator momentum

\[ p_{sp} := \{E_{sp}, \vec{p}_{sp}\} = \sum_{i=2}^{n} \{E_i, \vec{p}_i\}, \quad (63) \]

by the boost relations

\[ -k_{1z} = \sqrt{1 + \eta p_{sz} + \sqrt{\eta} E_{sp}}, \quad -k_{1z}' = \sqrt{1 + \eta p_{sz}' - \sqrt{\eta} E_{sp}}. \quad (64) \]

C. Spinor currents

For the single-quark Dirac current we have

\[
\frac{\langle \vec{p}_1 | \mathcal{I}_1 | \vec{p}_1 \rangle}{\sqrt{1 + \eta}} = \frac{1 + \beta_1}{2} \frac{(\vec{\alpha}_1 \cdot \vec{p}_1' + E_1' + m)(\vec{\alpha}_1 \cdot \vec{p}_1 + E_1 + m)}{\sqrt{4E_1'(E_1' + m)E_1(E_1 + m)}} 2 \quad (59)
\]

\[
\frac{\langle \vec{p}_i | \mathcal{I}_{1m} | \vec{p}_1 \rangle}{\sqrt{1 + \eta}} = \frac{1 + \beta_1}{2} \frac{(\vec{\alpha}_1 \cdot \vec{p}_1' + E_1' + m)(\vec{\alpha}_1 \cdot \vec{p}_1 + E_1 + m)}{\sqrt{4E_1'(E_1' + m)E_1(E_1 + m)}} 2 \quad (60)
\]

and

\[
\frac{\langle \vec{p}_1 | \mathcal{I}_{1m+} | \vec{p}_1 \rangle}{\sqrt{1 + \eta}} = \frac{\sigma_{1+}}{2} \frac{[p_{1z}(E_1 + m) - p_{1z}(E_1' + m)]}{\sqrt{E_1(E_1' + m)E_1(E_1 + m)}} - \frac{\sigma_{1z}p_{1z}(E_1 - E_1')}{2\sqrt{E_1(E_1' + m)E_1(E_1 + m)}} + \frac{p_{1+}(E_1 + E_1' + 2m)}{2\sqrt{E_1(E_1' + m)E_1(E_1 + m)}}. \quad (66)
\]
In the zero-mass limit the current kernels reduce to

\[
\left( \frac{\langle \vec{p}_1' | I_{el} | \vec{p}_1 \rangle}{\sqrt{1 + \eta}} \right)_{m=0} = \frac{1}{2} \left\{ (1 + \hat{p}_1 \cdot \hat{p}_1) + i (\hat{p}_1' \times \hat{p}_1) \cdot \vec{\sigma}_1 \right\}, \quad \hat{p}_1 := \vec{p}_1 / |\vec{p}_1|,
\]

\[
\left( \frac{\langle \vec{p}_1' | I_{m} | \vec{p}_1 \rangle}{\sqrt{1 + \eta}} \right)_{m=0} = -\frac{i}{2} \left\{ \sigma_{1+}(\hat{p}'_{1z} - \hat{p}_{1z}) + p_{1z} + \sigma_{1z} \frac{1}{|\vec{p}_1'| |\vec{p}_1|} \left| \vec{p}_1' \right| - \left| \vec{p}_1 \right| \right\}.
\]

(67)

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Figure Captions

Fig.1 The proton electric form factor calculated with point form kinematics with the Gaussian (“G”) \textbf{(34)} and rational (“R”) \textbf{(35)} wave function models for different values of $m r_0$. The value $m r_0 = 0$ is the point limit. The parameterization (“LOMON”) of the data is taken from ref.\textbf{[12]}.

Fig.2 Illustration of the dependence of the proton magnetic form factors, with the spinor spectator current, on the shape and scale of the wave function. The notation is the same as in Fig.1. The parameterization (“LOMON”) of the data is taken from ref.\textbf{[12]}.

Fig.3 Comparison of form electric proton form factors as calculated with the rational wave function \textbf{(35)} with and without (“$S_e = 1$”) the spinor factor in the current operator. The parameterization of the data “LOMON” is that given in ref.\textbf{[12]}.

Fig.4 Comparison of electric proton form factors as calculated with the Gaussian wave function \textbf{(34)} with and without (“$S_e = 1$”) the spinor factor in the current operator. The parameterization “LOMON” of the data is that given in ref.\textbf{[12]}.

Fig.5 The electric and magnetic form factors given by the wave function model \textbf{(30)} with $a = 11/4$ in the point limit and with $a = 9/4$ with $m/b = 0.52$. The parameterization (“LOMON”) of the data is from ref.\textbf{[12]}.
