AN EXPOSITION ON FINITE DIMENSIONALITY OF CHOW GROUPS

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ABSTRACT. In this exposition we understand when the natural map from the Chow variety parametrizing codimension \( p \) cycles on a smooth projective variety \( X \) to the Chow group \( \text{CH}^p(X) \) is surjective. We derive some consequences when the map is surjective.

1. INTRODUCTION

The representability question in the theory of Chow groups is an important question. Precisely it means the following: let \( X \) be a smooth projective variety and let \( \text{CH}^p(X) \) denote the Chow group of codimension \( p \) algebraic cycles on \( X \) modulo rational equivalence. Let \( \text{CH}^p(X)_{\text{hom}} \) denote the subgroup of \( \text{CH}^p(X) \) consisting of homologically trivial cycles. Suppose that there exists a smooth projective curve \( C \) and a correspondence \( \Gamma \) on \( C \times X \) such that \( \Gamma_* \) from \( J(C) \cong \text{CH}^1(C)_{\text{hom}} \) to \( \text{CH}^p(X)_{\text{hom}} \) is surjective. Then we say that the group \( \text{CH}^p(X)_{\text{hom}} \) is representable. The most interesting and intriguing is the case of highest codimensional cycle on a smooth projective variety \( X \), i.e. the zero cycles on \( X \). The first breakthrough result in this direction is the result by Mumford: the group \( \text{CH}^2(S)_{\text{hom}} \) of a smooth projective complex algebraic surface with geometric genus greater than zero is not representable. It was further generalized by Roitman [R1] for higher dimensional varieties proving that: if the variety \( X \) has a globally holomorphic \( i \)-form on it, then the group \( \text{CH}^n(X)_{\text{hom}} \) is not representable. Here \( n \) is the dimension of \( X \) and we have \( i \leq n \). Then there is the famous converse question due to Spencer Bloch saying that: for a smooth projective complex algebraic surface \( S \) with geometric genus equal to zero, the group \( \text{CH}^2(S)_{\text{hom}} \) is representable. This question has been answered for the surfaces not of general type with geometric genus equal to zero by Bloch-Kas-Lieberman, in [BKL]. The conjecture is still open in general for surfaces of general type but it has been solved in many examples in [B], [IM], [V], [VC].
In the case of highest codimensional cycles on a smooth projective variety $X$, the notion of representability can also be defined in another way. That is we consider the natural map from the symmetric power $\text{Sym}^d X$ to $\text{CH}^n(X)_{\text{hom}}$, for $d$ positive and $n$ being the dimension of $X$. Suppose that this map is surjective for some $d$ then we say that the group $\text{CH}^p(X)_{\text{hom}}$ is representable. It can be proved as in [Vo], that this notion of representability implies the first notion of representability and vice-versa.

So following the approach of Voisin as in [Vo], it is natural to ask is there a second notion of representability for lower co-dimensions. Precisely it means the following: Let us consider the two-fold product of the Chow variety $C^p_d (X) \times C^p_d (X)$. Consider the natural map from this product two the group $\text{CH}^p(X)_{\text{hom}}$ and we ask the question: does the surjectivity of this map implies the representability in the first sense of $\text{CH}^p(X)_{\text{hom}}$.

First we prove the following in this direction:

**The Chow group of codimension $p$-cycles are generated by linear subspaces.** That is there exists a surjective map from $\text{CH}_0(F(X))$ to $\text{CH}^p(X)$. Here $F(X)$ is the Fano variety of linear subspaces of codimension $p$. Suppose that $\text{CH}^p(X)_{\text{hom}}$ is representable in the sense that the map from the two-fold product of the Chow variety to $\text{CH}^p(X)_{\text{hom}}$ is surjective. Then there exists a smooth projective curve $C$ and a correspondence $\Gamma$ on $C \times X$, such that $\Gamma_*: \text{CH}^1(C)_{\text{hom}} \to \text{CH}^p(X)_{\text{hom}}$ is surjective.

As an application we show that the natural map from $C^3_d (X) \times C^3_d (X)$ to $\text{CH}^3(X)_{\text{hom}}$ is not surjective for any $d$, where $X$ is a cubic fourfold embedded in $\mathbb{P}^5$.

Our argument in this direction is minor modification of the argument present in the approach of Voisin in [Vo], where she deals with the case of zero cycles. First we recall various notions representability in the second sense, denoted as "finite dimensionality" of Chow groups of codimension $p$-cycles and show their equivalence. The key point is to use the Roitman’s result on the map from the two-fold product of the Chow variety to $\text{CH}^p(X)_{\text{hom}}$ saying that the fibers of this map is a countable union od Zariski closed subsets in the product of Chow varieties. Then after having this equivalent notions of "finite dimensionality" in hand we proceed to the main theorem.

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2. FINITE DIMENSIONALITY OF THE CHOW GROUP OF CODIMENSION \(i\) CYCLES

Let \(X\) be a smooth projective variety defined over the ground field \(k\). Let \(C^p_d(X)\) denote the Chow variety of \(X\) parametrizing all codimension \(p\) cycles on \(X\) of a certain degree \(d\). To consider the degree we fix an embedding of \(X\) into some projective space \(\mathbb{P}^N\). Consider the \(k\)-points of the variety \(C^p_d(X)\). Then consider the map

\[
\theta^d_p : C^p_d(X) \times C^p_d(X) \to \text{CH}^p(X)
\]

given by

\[
(Z_1, Z_2) \mapsto [Z_1 - Z_2]
\]

where \([Z_1 - Z_2]\) is the class of the cycle \(Z_1 - Z_2\) in the Chow group. By abusing notation we will denote the class \([Z_1 - Z_2]\) as \(Z_1 - Z_2\).

**Definition 2.1.** We say that the group \(\text{CH}^p(X)\) is representable if there exists \(d\) such that \(\theta^d_p\) is surjective.

Now the natural question is that, what are the fibers of \(\theta^d_p\), for a fixed \(p, d\). Here is a theorem about that:

**Theorem 2.2.** The fibers of the map \(\theta^d_p\) are countable union of Zariski closed subsets of \(C^p_d(X) \times C^p_d(X)\).

**Proof.** To prove this we consider the following reformulation of the definition of rational equivalence. Let \(Z_1, Z_2\) be two codimension \(p\)-cycles. They are rationally equivalent if there exists a positive cycle \(Z'\), such that \(Z_1 + Z', Z_2 + Z'\) belong to \(C^p_d(X)\) for some fixed \(d\), and there exists a regular morphism \(f\) from \(\mathbb{P}^1\) to \(C^p_d(X)\), such that

\[
f(0) = Z_1 + Z', \quad f(\infty) = Z_2 + Z'.
\]

Let us consider two cycles \(Z_1, Z_2\) belonging to \(\theta^{p-1}_d(z)\) for some rational equivalence class \(z\). Then \(Z_1, Z_2\) belong to \(\theta^{p-1}_d(z)\) means that \(Z_1, Z_2\) are rationally equivalent. So there exists \(Z', f\) as above such that

\[
f(0) = Z_1 + Z', \quad f(\infty) = Z_2 + Z'.
\]

So it is natural to consider the following subvarieties of \(C^p_d(X) \times C^p_d(X)\) denoted by \(W^u_d\), given as the collection of all \((Z_1, Z_2)\) so that there exist \(Z' \in C^p_d(X)\) \(f\) in \(\text{Hom}^u(\mathbb{P}^1, C^p_d(X))\), for some positive integer \(u\) satisfying

\[
f(0) = Z_1 + Z', \quad f(\infty) = Z_2 + Z'.
\]
Here $\text{Hom}^v(\mathbb{P}^1, C^p_{d+u}(X))$ is the Hom scheme of degree $v$ morphisms from $\mathbb{P}^1$ to $C^p_{d+u}(X)$. For working purpose denote $\prod_{i=1}^n C^p_{d_i}(X)$ as $C^p_{d_1,\ldots,d_n}(X)$.

Let $e : \text{Hom}^v(\mathbb{P}^1, C^p_{d+u,d+u}(X)) \to C^p_{d+u,d+u}(X)$ be the evaluation morphism sending $f : \mathbb{P}^1 \to C^p_{d+u,d+u}(X)$ to the ordered pair $(f(0), f(\infty))$, and let us consider the diagonal in $C^p_{d,u}(X)$ and multiply with $C^p_{d,d}(X)$, call it $F$ and consider:

$$s : F \to C^p_{d+u,d+u}(X)$$

be the regular morphism sending $(Z_1, Z_2)$ to $(Z_1 + Z', Z_2 + Z')$. The two morphisms $e$ and $s$ allow to consider the fibred product

$$V = \text{Hom}^v(\mathbb{P}^1, C^p_{d+u,d+u}(X)) \times C^p_{d+u,u,d+u}(X) F .$$

This $V$ is a closed subvariety in the product

$$\text{Hom}^v(\mathbb{P}^1, C^p_{d+u,d+u}(X)) \times C^p_{d+u,u,d+u}(X)$$

over $\text{Spec}(k)$ consisting of quintuples $(f, Z_1, Z_2, Z')$ such that

$$e(f) = s(Z, Z_2, Z') ,$$

i.e.

$$(f(0), f(\infty)) = (Z_1 + Z', Z_2 + Z') .$$

The latter equality gives

$$V = W^u_{d,v} .$$

Vice versa, if $(Z_1, Z_2)$ is a closed point of $W^u_{d,v}$, there exists a regular morphism

$$f \in \text{Hom}^v(\mathbb{P}^1, C^p_{d+u,d+u}(X))$$

with $f(0) = Z_1 + Z'$ and $f(\infty) = Z_2 + Z'$. Then $(f, Z_1, Z_2, Z')$ belongs to $V$.

So the set $W^u_{d,v}$ is itself a quasi-projective variety.

Suppose that $(Z_1, Z_2)$ is in $W^u_{d,v}$. Then there exists $f$ in $\text{Hom}^v(\mathbb{P}^1, C^p_{d+u}(X))$, $Z'$ in $C^p_{d,u}(X)$ such that

$$f(0) = Z_1 + Z', \quad f(\infty) = Z_2 + Z' .$$

Then this immediately imply that $(Z_1 + Z', Z_2 + Z')$ is in $W^{0,v}_{d+u}$. On the other consider the map

$$\bar{s} : C^p_{d,d}(X) \times \Delta_{C^p_{d,u}(X)} \to C^p_{d+u,d+u}(X) .$$
given by

\[(Z_1, Z_2, Z') \mapsto (Z_1 + Z', Z_2 + Z').\]

By the above we have that

\[W^u,v_d \subset \text{pr}_{1,2}(\tilde{s}^{-1}(W^0,v_{d+u})).\]

Conversely suppose that \((Z'_1, Z'_2)\) belongs to \(\text{pr}_{1,2}(\tilde{s}^{-1}(W^0,v_{d+u}))\). Then \((Z'_1, Z'_2)\) is of the form \((Z_1 + Z', Z_2 + Z')\), such that there exists \(f \in \text{Hom}^v(\mathbb{P}^1, C_{d+u}^{0}(X))\) satisfying

\[f(0) = Z_1 + Z', \quad f(\infty) = Z_2 + Z'.\]

This tell us that \((Z_1, Z_2)\) belongs to \(W^u,v_d\). Hence we have that

\[W^u,v_d = \text{pr}_{1,2}(\tilde{s}^{-1}(W^0,v_{d+u})).\]

Since \(\tilde{s}\) is continuous and \(\text{pr}_{1,2}\) is proper,

\[\bar{W}^u,v_d = \text{pr}_{1,2}(\tilde{s}^{-1}(\bar{W}^0,v_{d+u})).\]

This gives that to prove the second assertion of the proposition it is enough to show that \(\bar{W}^0,v_d\) is contained in \(W_d\).

Let \((Z_1, Z_2)\) be a closed point of \(\bar{W}^0,v_d\). If \((Z_1, Z_2)\) is in \(W^0,v_d\), then it is also in \(W_d\). Suppose \((Z_1, Z_2) \in \bar{W}^0,v_d \setminus W^0,v_d\).

Let \(W\) be an irreducible component of the quasi-projective variety \(W^0,v_d\) whose Zariski closure \(\bar{W}\) contains the point \((Z_1, Z_2)\). Let \(U\) be an affine neighbourhood of \((Z_1, Z_2)\) in \(\bar{W}\). Since \((Z_1, Z_2)\) is in the closure of \(W\), the set \(U \cap W\) is non-empty.

Let us show that we can always take an irreducible curve \(C\) passing through \((Z_1, Z_2)\) in \(U\). Indeed, write \(U\) as \(\text{Spec}(A)\). It is enough to show that there exists a prime ideal in \(\text{Spec}(A)\) of height \(n - 1\), where \(n\) is the dimension of \(\text{Spec}(A)\), where \(A\) is Noetherian. Since \(A\) is of dimension \(n\) there exists a chain of prime ideals

\[p_0 \subset p_1 \subset \cdots \subset p_n = p\]

such that this chain can not be extended further. Now consider the sub-chain

\[p_0 \subset p_1 \subset \cdots \subset p_{n-1}.\]

This is a chain of prime ideals and \(p_{n-1}\) is a prime ideal of height \(n - 1\), so we get an irreducible curve.
Let \( \hat{C} \) be the Zariski closure of \( C \) in \( \hat{W} \). Two evaluation regular morphisms \( e_0 \) and \( e_\infty \) from \( \text{Hom}^\nu(\mathbb{P}^1, C_d^p(X)) \) to \( C_d^p(X) \) give the regular morphism
\[
e_0, \infty : \text{Hom}^\nu(\mathbb{P}^1, C_d^p(X)) \to C_d^p(X). \]
Then \( W_{d, d}^\nu \) is exactly the image of the regular morphism \( e_{0, \infty} \), and we can choose a quasi-projective curve \( T \) in \( \text{Hom}^\nu(\mathbb{P}^1, C_d^p(X)) \), such that the closure of the image \( e_{0, \infty}(T) \) is \( \hat{C} \).

For that consider the curve \( C \) in \( W \) so it is contained in \( W_{0, d}^\nu \). We know that the image of \( e_{0, \infty} \) is \( W_{0, d}^\nu \). Consider the inverse image of \( \hat{C} \) under the morphism \( e_{0, \infty} \). Since \( \hat{C} \) is a curve, the dimension of \( e_{0, \infty}^{-1}(C) \) is greater than or equal than 1. So it contains a curve. Consider two points on \( \hat{C} \), consider their inverse images under \( e_{0, \infty} \). Since \( \text{Hom}^\nu(\mathbb{P}^1, C_d^p(X)) \) is a quasi-projective variety, \( e_{0, \infty}^{-1}(\hat{C}) \) is also projective, we can embed it into some \( \mathbb{P}^m \) and consider a smooth hyperplane section through the two points fixed above. Continuing this procedure we get a curve containing these two points and contained in \( e_{0, \infty}^{-1}(C) \). Therefore we get a curve \( T \) mapping onto \( \hat{C} \). So the closure of the image of \( T \) is \( \hat{C} \).

Now, as we have mentioned above, \( \text{Hom}^\nu(\mathbb{P}^1, C_d^p(X)) \) is a quasi-projective variety. This is why we can embed it into some projective space \( \mathbb{P}^m \). Let \( \tilde{T} \) be the closure of \( T \) in \( \mathbb{P}^m \), let \( \tilde{T} \) be the normalization of \( \tilde{T} \) and let \( \tilde{T}_0 \) be the pre-image of \( T \) in \( \tilde{T} \). Consider the composition
\[
f_0 : \tilde{T}_0 \times \mathbb{P}^1 \to T \times \mathbb{P}^1 \subset \text{Hom}^\nu(\mathbb{P}^1, C_d^p(X)) \times \mathbb{P}^1 \xrightarrow{e} C_d^p(X),
\]
where \( e \) is the evaluation morphism \( e_{\mathbb{P}^1, C_d^p(X)} \). The regular morphism \( f_0 \) defines a rational map
\[f : \tilde{T} \times \mathbb{P}^1 \dashrightarrow C_d^p(X)\]
Then by resolution of singularities we get that \( f \) could be extended to a regular map from \( (\tilde{T} \times \mathbb{P}^1)' \) to \( C_d^p(X) \), where \( (\tilde{T} \times \mathbb{P}^1)' \) denote the blow up of \( \tilde{T} \times \mathbb{P}^1 \) along the indeterminacy locus which is a finite set of points. Continue to call the strict transform of \( \tilde{T} \) in the blow up as \( \tilde{T} \), and the pre-image of \( T \) as \( \tilde{T}_0 \).

The regular morphism \( \tilde{T}_0 \to T \to \hat{C} \) extends to the regular morphism \( \tilde{T} \to \hat{C} \). Let \( P \) be a point in the fibre of this morphism at \((Z_1, Z_2)\). For any closed point \( Q \) on \( \mathbb{P}^1 \) the restriction \( f|_{\tilde{T} \times \{Q\}} \) of the rational map \( f \) onto \( \tilde{T} \times \{Q\} \) is regular on the whole curve \( \tilde{T} \), because \( \tilde{T} \) is non-singular.
Then
\[(f|_{T \times \{0\}})(P) = Z_1 \quad \text{and} \quad (f|_{T \times \{\infty\}})(P) = Z_2 .\]

has the property that
\[f(0) = Z_1, \quad f(\infty) = Z_2 .\]

Hence we have that \(W_{0,v}^{u,v}\) is Zariski closed. So \(W_{u,v}^{u,v}\) is Zariski closed, and hence we have that \(\theta^{-1}_d(Z)\) is a countable union of Zariski closed subsets in the product \(C^p_d(X)\). □

By the previous Theorem 2.2 we can define the dimension of the fiber of \(\theta^p_d\) to be the maximum of the dimensions of the Zariski closed subsets occurring in \(\theta^{-1}_d(Z)\), for \(Z\) in \(\text{CH}^p(X)\). Now consider the subset of \(C^p_{d,d}(X)\) consisting of points such that the dimension of \(\theta^{-1}_d(\theta^p_d(Z))\) is not constant as \(Z\) varies. By the existence of Hilbert schemes this is a countable union of Zariski closed subsets of \(C^p_{d,d}(X)\). Call this subset \(B\). Then for \(Z\) in the complement of \(B\), we have that the dimension of \(\theta^{-1}_d(\theta^p_d(Z))\) is constant and say \(r\).

**Definition 2.3.** The dimension of the image of \(\theta^p_d\) is defined to be equal to \(2\dim(C^p_d(X)) - r\).

Suppose that there exists a codimension \(p\) prime cycle on \(X\) of degree \(e\). Then this prime cycle gives rise to an embedding of \(C^p_d(X)\) into \(C^p_{d+e}(X)\). Hence we have
\[\dim(C^p_d(X)) \leq \dim(C^p_{d+e}(X)) .\]

Hence we can define the limit superior of the
\[\dim(\text{im}(\theta^p_d)) .\]

We say that \(\text{CH}^p(X)\) is infinite dimensional if
\[\limsup_d \dim(\text{im}(\theta^p_d)) = \infty\]

and finite dimensional otherwise.

**Theorem 2.4.** The group \(\text{CH}^p(X)\) is representable if and only if it is finite dimensional.
Proof. Suppose that \( \text{CH}^p(D) \) is representable. Then there exists \( d \) such that \( \theta^p_d \) is surjective. For every integer \( n \) consider the subset

\[
R \subset C^p_n(X) \times C^p_n(X) \times C^p_d(X) \times C^p_d(X)
\]

consisting of quadruples

\[
(Z_1, Z_2, Z'_1, Z'_2)
\]

such that

\[
\theta^p_n(Z_1, Z_2) = \theta^p_d(Z'_1, Z'_2).
\]

As we have that \( \theta^p_d \) is surjective, it follows that the projection

\[
\text{pr}_1 : R \to C^p_n(X) \times C^p_n(X)
\]

is surjective. Now by Theorem 2.2, \( R \) is a countable union of Zariski closed subsets in the ambient variety \( C^p_n(X) \times C^p_n(X) \times C^p_d(X) \times C^p_d(X) \). We prove it as a separate lemma:

**Lemma 2.5.** The set \( R \) is a countable union of Zariski closed subsets in \( C^p_n(X) \times C^p_n(X) \times C^p_d(X) \times C^p_d(X) \).

Proof. To prove this we consider the following reformulation of the definition of rational equivalence. Let \( Z_1, Z_2 \) be two codimension \( p \)-cycles. They are rationally equivalent if there exists a positive cycle \( Z' \), such that \( Z_1 + Z', Z_2 + Z' \) belong to \( C^p_d(X) \) for some fixed \( d \), and there exists a regular morphism \( f \) from \( \mathbb{P}^1 \) to \( C^p_d(X) \), such that

\[
f(0) = Z_1 + Z', \quad f(\infty) = Z_2 + Z'.
\]

Let us consider two cycles \( (Z_1, Z_2, Z'_1, Z'_2) \) belonging to \( R \). That would mean that the cycle class of \( Z_1 - Z_2 \) is rationally equivalent to that of \( Z'_1 - Z'_2 \). So there exists a positive cycle \( Z \) and a regular map \( f \) from \( \mathbb{P}^1 \) to \( C^p_d(X) \), such that

\[
f(0) = Z_1 + Z_2 + Z', \quad f(\infty) = Z'_1 + Z_2 + Z'.
\]

So it is natural to consider the following subvarieties of \( C^p_{d+n+u}(X) \times C^p_{d+n+u}(X) \) denoted by \( W^{u,v}_{d+n} \). It is given by the collection of all elements in the image of \( C^p_d(X) \times C^p_d(X) \times C^p_n(X) \times C^p_n(X) \) in \( C^p_{d+n+u}(X) \times C^p_{d+n+u}(X) \), under the natural map, given by

\[
(Z_1, Z_2, Z'_1, Z'_2) \mapsto (Z_1 + Z_2, Z'_1 + Z'_2)
\]

Call this image as \( F \).
so that there exist \( Z \in C^p_{\nu u}(X) \) and \( f \) in \( \text{Hom}^\nu(\mathbb{P}^1, C^p_{\nu u+d+n} X) \), for some positive integer \( u \) satisfying

\[
f(0) = Z_1 + Z'_2 + Z, \quad f(\infty) = Z_2 + Z'_1 + Z'.
\]

Here \( \text{Hom}^\nu(\mathbb{P}^1, C^p_{\nu u+d+n} X) \) is the Hom scheme of degree \( \nu \) morphisms from \( \mathbb{P}^1 \) to \( C^p_{\nu u+d+n} X \). For working purpose denote \( \prod_{i=1}^n C^p_{d_i} X \) as \( C^p_{d_1, \ldots, d_n} X \).

Let

\[
e: \text{Hom}^\nu(\mathbb{P}^1, C^p_{\nu u+d+n} X) \to C^p_{\nu u+d+n} X
\]

be the evaluation morphism sending \( f: \mathbb{P}^1 \to C^p_{\nu u+d+n} X \) to the ordered pair \( (f(0), f(\infty)) \), and let us consider the diagonal in \( C^p_{\nu u+d+n} X \) and multiply with \( F \) and consider:

\[
s: F \times \Delta_{C^p_{\nu u+d+n} X} \to C^p_{\nu u+d+n} X
\]

be the regular morphism sending \((Z_1 + Z'_2, Z_2 + Z'_1, Z)\) to \((Z_1 + Z'_2 + Z, Z_2 + Z'_1 + Z')\). The two morphisms \( e \) and \( s \) allow to consider the fibred product

\[
V = \text{Hom}^\nu(\mathbb{P}^1, C^p_{\nu u+d+n} X) \times_{C^p_{\nu u+d+n} X} (F \times \Delta_{C^p_{\nu u+d+n} X} X).
\]

This \( V \) is a closed subvariety in the product

\[
\text{Hom}^\nu(\mathbb{P}^1, C^p_{\nu u+d+n} X) \times F \times \Delta_{C^p_{\nu u+d+n} X}
\]

over \( \text{Spec}(k) \) consisting of tuples \((f, Z_1 + Z'_2, Z_2 + Z'_1, Z)\) such that

\[
e(f) = s(Z_1 + Z'_2, Z_2 + Z'_1, Z),
\]

i.e.

\[
(f(0), f(\infty)) = (Z_1 + Z'_2 + Z, Z_2 + Z'_1 + Z).
\]

The latter equality gives

\[
V = W_{d+n}^{\nu u}.
\]

Vice versa, if \((Z_1 + Z'_2, Z_2 + Z'_1)\) is a closed point of \( W_{d+n}^{\nu u} \), there exists a regular morphism

\[
f \in \text{Hom}^\nu(\mathbb{P}^1, C^p_{\nu u+d+n} X)
\]

with \( f(0) = Z_1 + Z'_2 + Z \) and \( f(\infty) = Z_2 + Z'_1 + Z \). Then \((f(0), f(Z_1 + Z'_2, Z_2 + Z'_1, Z))\) belongs to \( V \).

So the set \( W_{d+n}^{\nu u} \) is itself a quasi-projective variety.

Suppose that \((Z_1 + Z'_2, Z_2 + Z'_1)\) is in \( W_{d+n}^{\nu u} \). Then there exists \( f \) in \( \text{Hom}^\nu(\mathbb{P}^1, C^p_{\nu u+d+n} X) \), \( Z \) in \( C^p_{\nu u} X \) such that

\[
f(0) = Z_1 + Z'_2 + Z, \quad f(\infty) = Z_2 + Z'_1 + Z.
\]
Then this immediately imply that \((Z_1 + Z_2' + Z, Z_2 + Z_1') = W_{d+n}^{0,\nu}\).

On the other consider the map

\[ \tilde{s} : F \times \Delta_{c_{1,1}}(X) \to C^p_{d+n+u,d+n+u}(X) \]

given by

\[ (Z_1 + Z_2', Z_2 + Z_1', Z) \mapsto (Z_1 + Z_2' + Z, Z_2 + Z_1' + Z). \]

By the above we have that

\[ W_{d+n}^{u,\nu} \subset \text{pr}_{1,2}(\tilde{s}^{-1}(W_{d+n+u}^{0,\nu})). \]

Conversely suppose that \((Z_1 + Z_2', Z_2 + Z_1') = \text{pr}_{1,2}(\tilde{s}^{-1}(W_{d+n+u}^{0,\nu})).\)

Then there exists \(f \in \text{Hom}^{\nu}(\mathbb{P}^1, C^p_{d+n+u}(X))\) satisfying

\[ f(0) = Z_1 + Z_2' + Z', \quad f(\infty) = Z_2 + Z_1' + Z'. \]

This tells us that \((Z_1 + Z_2', Z_2 + Z_1') \in W_{d+n}^{u,\nu}.\)

Hence we have that

\[ W_{d+n}^{u,\nu} = \text{pr}_{1,2}(\tilde{s}^{-1}(W_{d+n+u}^{0,\nu})). \]

Since \(\tilde{s}\) is continuous and \(\text{pr}_{1,2}\) is proper,

\[ \tilde{W}_{d+n}^{u,\nu} = \text{pr}_{1,2}(\tilde{s}^{-1}(\tilde{W}_{d+n+u}^{0,\nu})). \]

This gives that to prove the second assertion of the proposition it is enough to show that \(\tilde{W}_{d+n}^{0,\nu}\) is contained in \(W_{d+n}^{0,\nu}\).

Let \((Z_1 + Z_2', Z_1' + Z_2)\) be a closed point of \(\tilde{W}_{d+n}^{0,\nu}\) (here the closure is taken with respect to \(F\)). Suppose

\[ (Z_1 + Z_2', Z_1' + Z_2) \in \tilde{W}_{d+n}^{0,\nu} \setminus W_{d+n}^{0,\nu}. \]

Let \(W\) be an irreducible component of the quasi-projective variety \(W_{d+n}^{0,\nu}\) whose Zariski closure \(\tilde{W}\) contains the point \((Z_1 + Z_2', Z_1' + Z_2)\). Let \(U\) be an affine neighbourhood of \((Z_1 + Z_2', Z_2 + Z_1')\) in \(\tilde{W}\). Since \((Z_1 + Z_2', Z_1' + Z_2)\) is in the closure of \(W\), the set \(U \cap W\) is non-empty.

Let us show that we can always take an irreducible curve \(C\) passing through \((Z_1 + Z_2', Z_1' + Z_2)\) in \(U\). Indeed, write \(U\) as \(\text{Spec}(A)\). It is enough to show that there exists a prime ideal in \(\text{Spec}(A)\) of height \(n - 1\), where \(n\) is the dimension of \(\text{Spec}(A)\), where \(A\) is Noetherian. Since \(A\) is of dimension \(n\) there exists a chain of prime ideals

\[ p_0 \subset p_1 \subset \cdots \subset p_n = \mathfrak{p} \]
such that this chain can not be extended further. Now consider the sub-chain
\[ \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{n-1}. \]
This is a chain of prime ideals and \( \mathfrak{p}_{n-1} \) is a prime ideal of height \( n-1 \), so we get an irreducible curve.

Let \( \tilde{C} \) be the Zariski closure of \( C \) in \( \tilde{W} \). Two evaluation regular morphisms \( e_0 \) and \( e_\infty \) from \( \text{Hom}^v(\mathbb{P}^1, C_{d+n}^p(X)) \) to \( C_{d+n}^p(X) \) give the regular morphism
\[ e_{0,\infty} : \text{Hom}^v(\mathbb{P}^1, C_{d+n}^p(X)) \to C_{d+n,d+n}^p(X). \]
Then \( W_{d+n}^{0,v} \) is exactly the image of the regular morphism \( e_{0,\infty} \), and we can choose a quasi-projective curve \( T \) in \( \text{Hom}^v(\mathbb{P}^1, C_{d+n}^p(X)) \), such that the closure of the image \( e_{0,\infty}(T) \) is \( \tilde{C} \).

For that consider the curve \( C \) in \( W \) so it is contained in \( W_{d+n}^{0,v} \). We know that the image of \( e_{0,\infty} \) is \( W_{d+n}^{0,v} \). Consider the inverse image of \( \tilde{C} \) under the morphism \( e_{0,\infty} \). Since \( \tilde{C} \) is a curve, the dimension of \( e_{0,\infty}^{-1}(C) \) is greater than or equal than 1. So it contains a curve. Consider two points on \( \tilde{C} \), consider their inverse images under \( e_{0,\infty} \). Since \( \text{Hom}^v(\mathbb{P}^1, C_{d+n}^p(X)) \) is a quasi-projective variety, \( e_{0,\infty}^{-1}(\tilde{C}) \) is also quasi-projective, we can embed it into some \( \mathbb{P}^m \) and consider a smooth hyperplane section through the two points fixed above. Continuing this procedure we get a curve containing these two points and contained in \( e_{0,\infty}^{-1}(C) \). Therefore we get a curve \( T \) mapping onto \( \tilde{C} \). So the closure of the image of \( T \) is \( \tilde{C} \).

Now, as we have mentioned above, \( \text{Hom}^v(\mathbb{P}^1, C_{d+n}^p(X)) \) is a quasi-projective variety. This is why we can embed it into some projective space \( \mathbb{P}^m \). Let \( \tilde{T} \) be the closure of \( T \) in \( \mathbb{P}^m \), let \( \bar{T} \) be the normalization of \( \tilde{T} \) and let \( \tilde{T}_0 \) be the pre-image of \( T \) in \( \tilde{T} \). Consider the composition
\[ f_0 : \tilde{T}_0 \times \mathbb{P}^1 \to T \times \mathbb{P}^1 \subset \text{Hom}^v(\mathbb{P}^1, C_{d+n}^p(X)) \times \mathbb{P}^1 \xleftarrow{e} C_{d+n}^p(X), \]
where \( e \) is the evaluation morphism \( e_{p^1,C_{d+n}^p(X)} \). The regular morphism \( f_0 \) defines a rational map
\[ f : \tilde{T} \times \mathbb{P}^1 \dashrightarrow C_{d+n}^p(X) \]
Then by resolution of singularities we get that \( f \) could be extended to a regular map from \( (\tilde{T} \times \mathbb{P}^1)' \) to \( C_{d+n}^p(X) \), where \( (\tilde{T} \times \mathbb{P}^1)' \) denote the blow up of \( \tilde{T} \times \mathbb{P}^1 \) along the indeterminacy locus which is a finite set of points. Continue to call the strict transform of \( \tilde{T} \) in the blow up as \( \tilde{T} \), and the pre-image of \( T \) as \( \tilde{T}_0 \).
The regular morphism \( \tilde{T}_0 \to T \to \tilde{C} \) extends to the regular morphism \( \tilde{T} \to \tilde{C} \). Let \( P \) be a point in the fibre of this morphism at \((Z_1 + Z'_1, Z_2 + Z'_2)\). For any closed point \( Q \) on \( \mathbb{P}^1 \) the restriction \( f|_{\tilde{T} \times \{ Q \}} \) of the rational map \( f \) onto \( \tilde{T} \times \{ Q \} \equiv \tilde{T} \) is regular on the whole curve \( \tilde{T} \), because \( \tilde{T} \) is non-singular. Then

\[
(f|_{\tilde{T} \times \{ Q \}})(P) = Z_1 + Z'_2 \quad \text{and} \quad (f|_{\tilde{T} \times \{ \infty \}})(P) = Z_2 + Z'_1.
\]

\[
f : \{ P \} \times \mathbb{P}^1 \to C^p_{d+n}(X)
\]

has the property that

\[
f(0) = Z_1 + Z'_2, \quad f(\infty) = Z_2 + Z'_1.
\]

Hence we have that \( W^0_{d+n} \) is Zariski closed. So \( W^i_{d+n} \) is Zariski closed. Therefore we have a countable union of Zariski closed subsets in \( F \), whose pull-back to \( C^p_d(X) \times C^p_d(X) \times C^p_n(X) \times C^p_n(X) \) is exactly \( R \). Therefore \( R \) itself is a countable union of Zariski closed subsets in \( C^p_d(X) \times C^p_d(X) \times C^p_n(X) \times C^p_n(X) \).

\( \Box \)

Write \( R = \bigcup_i R_i \), where each \( R_i \) is a Zariski closed subset in \( C^p_d(X) \times C^p_n(X) \times C^p_d(X) \times C^p_n(X) \). Considering the projection from \( \text{pr}_1 \) from \( R \) to \( C^p_d(X) \times C^p_n(X) \), we have \( \bigcup_i \text{pr}_1(R_i) = C^p_n(X) \times C^p_n(X) \). But \( C^p_d(X) \times C^p_n(X) \) can be uniquely decomposed into finitely many Zariski closed irreducible subsets of maximal dimension. Using the fact that the ground field \( k \) is uncountable, it will follow that there exists finitely many components \( R_1, \ldots, R_m \) of \( R \), such that \( \bigcup_i R_i = R' \) surjects onto \( C^p_n(X) \times C^p_n(X) \). So we have that

\[
\dim R' \geq 2 \dim C^p_d(X).
\]

Now consider \((Z_1, Z_2, Z'_1, Z'_2)\) in \( \bigcup R_i \), then we have

\[
\dim_{(Z_1, Z_2)} R' \cap C^p_d(X) \times C^p_n(X) \times (Z'_1, Z'_2) \geq 2 \dim C^p_n(X) - 2 \dim C^p_d(X),
\]

this number on the right hand side is bigger than zero if we take sufficiently large \( n \) such that \( C^p_n(X) \) contains \( C^p_d(X) \). Then the above is an algebraic set contained in

\[
\theta_n^{p-1}(\theta_d(Z'_1, Z'_2)).
\]

As \((Z_1, Z_2)\) is arbitrary and the projection

\[
R' \to C^p_n(X) \times C^p_n(X)
\]

at 12
is surjective, we have that dimension of $\theta_{n-1}^p(Z_1, Z_2)$ is at least $2(\dim(C^p_d(X)) - \dim(C^p_d(X)))$. Hence the image of $\theta^p$ is bounded by $2 \dim C^p_d(X)$. Hence $C^p_d(X)$ is finite dimensional. Now suppose that $C^p_d(X)$ is finite dimensional. We have to prove that there exists $d$ such that $\theta^p_d$ is surjective. Let $d$ be such that

$$\dim(\text{im}(\theta^p_d)) = \dim(\text{im}(\theta^p_{d+e}))$$

for some positive integer $e$. Let $V$ be a subvariety of degree $e$ and codimension $p$ giving an embedding of $C^p_d(X) \times C^p_d(X)$ into $C^p_{d+e}(X) \times C^p_{d+e}(X)$. Call the embedding as $i_V$, then we have

$$\theta^p_{d+e} \circ i_V = \theta^p_d.$$

Let $F$ be the fiber of $\theta^p_d$ passing through a general point $(Z_1, Z_2)$ of $C^p_d(X) \times C^p_d(X)$, let $F'$ be the fiber of $\theta^p_{d+e}$ through a general point in $C^p_{d+e}(X) \times C^p_{d+e}(X)$. By assumption we have that

$$2 \dim(C^p_d(X)) - r_1 = 2 \dim(C^p_{d+e}(X)) - r_2$$

where $r_1, r_2$ are dimensions of $F, F'$. So we have

$$r_2 - r_1 = 2n$$

where $n = \dim(C^p_{d+e}(X)) - \dim(C^p_d(X))$. Let $F''$ be the fiber of a $\theta^p_{d+e}$ such that it passes through $(Z_1 + V, Z_2 + V) = i_V(Z_1, Z_2)$. Then by the definition of dimension of the fiber of $\theta^p_{d+e}$ we have that

$$\dim(F'') \geq \dim(F').$$

Now consider the subset

$$R = \{(Z_1, Z_2, Z'_1, Z'_2) : \theta^p_{d+e}(Z_1, Z_2) = \theta^p_d(Z'_1, Z'_2)\}.$$

The projection from $R$ to $C^p_d(X) \times C^p_d(X)$ is surjective, so there exists finitely many irreducible subsets containing $(Z_1 + V, Z_2 + V, Z_1, Z_2)$ such that there union $R'$ dominates $C^p_d(X) \times C^p_d(X)$. Fiber of this projection from $R'$ is of dimension greater or equal than that of $F''$. So it is greater than or equal to $\dim(F) + 2 \dim C^p_d(X)$. The fibers of the first projection

$$p : R' \rightarrow C^p_{d+e}(X) \times C^p_{d+e}(X)$$

are of dimension atmost $\dim(F)$. So we have that

$$\dim p(R') \geq \dim R' - \dim F \geq \dim F'' + 2 \dim(C^p_d(X)) - \dim(F)$$

$$\geq \dim F' - \dim F + 2 \dim(C^p_d(X) = \dim(C^p_{d+e}(X)) \times C^p_{d+e}(X))$$

\qed
Now our aim is to detect the kernel of the Abel-Jacobi map for higher dimensional cycles. Let’s recall that the Abel-Jacobi map has domain $\text{CH}_p(X)_{\text{hom}}$ and target the Intermediate Jacobian $IJ^p(X)$ given by

$$H^{2p-1}(X, \mathbb{C})/F^p H^{2p-1}(X, \mathbb{C}) \oplus H^{2p-1}(X, \mathbb{Z}) .$$

The first theorem in this direction is to relate the representability of the Chow group of codimension $p$ cycles with zero cycles on Jacobian.

**Theorem 2.6.** Suppose that there exists a smooth projective curve $C$, and a correspondence $\Gamma$ on $C \times X$ such that

$$\Gamma_* : \text{CH}^1(C)_{\text{hom}} \to \text{CH}^p(X)_{\text{hom}}$$

is surjective. Then $\text{CH}^p(X)_{\text{hom}}$ is representable.

**Proof.** Note that the natural map from $\text{Sym}^g C \times \text{Sym}^g C$ to $\text{CH}^1(C)_{\text{hom}}$ is surjective, where $g$ is the genus of the curve $C$. Therefore $\text{CH}^1(C)_{\text{hom}}$ is finite dimensional. Therefore the image of $\Gamma_*$ is finite dimensional. But $\Gamma_*$ is surjective. So $\text{CH}^p(X)_{\text{hom}}$ is finite dimensional hence representable. $\square$

It is difficult to prove the converse, that is: suppose $\text{CH}^p(X)_{\text{hom}}$ is representable then does there exist a curve $C$ and a correspondence $\Gamma$ on $C \times X$, such that $\Gamma_*$ is onto. Let us consider the following situation when:

1) The Chow group of codimension $p$-cycles are generated by linear subspaces. That is there exists a surjective map from $\text{CH}_0(F(X))$ to $\text{CH}^p(X)_{\text{hom}}$.

Theorem 2.7. Let $X$ be as above. Suppose that $\text{CH}^p(X)_{\text{hom}}$ is representable. Then there exists a smooth projective curve $C$ and a correspondence $\Gamma$ on $C \times X$, such that

$$\Gamma_* : \text{CH}^1(C)_{\text{hom}} \to \text{CH}^p(X)_{\text{hom}}$$

is surjective.

**Proof.** Consider the map $\theta^p_d$ from $C^p_d(X)$ to $\text{CH}^p(X)_{\text{hom}}$ given by

$$Z \mapsto Z - dL_0$$

where $L_0$ is a fixed linear subspace of $X$. Since $\text{CH}^p(X)_{\text{hom}}$ is actually generated by linear subspaces. The above map restricted to $\text{Sym}^d F(X)$ is surjective, continue to call it $\theta^p_d$. Let us consider large $d$ such that
\[ \dim(\text{im}(\theta^p_d)) \] is constant and equal to \( K \). Then the dimension of a general fiber is equal to
\[ \dim(\text{Sym}^d F(X)) - K \]

Now we prove that an irreducible component \( Z \) of maximal dimension of a general fiber of \( \theta^p_d \) cannot be contained in a set of the form
\[ \text{Sym}^{d-i} F(X) + W \]
where \( W \) is in \( \text{Sym}^i F(X) \), \( \dim W < i \) and the above + means the image of the natural map from
\[ \text{Sym}^{d-i} F(X) \times W \to \text{Sym}^d F(X). \]

If possible assume that \( Z \) is contained in such a set. Note that the dimension of \( Z \) is \( \dim(\text{Sym}^d F(X)) - K \). So we have
\[ \dim(\text{Sym}^{d-i} F(X)) \geq \dim(\text{Sym}^d F(X)) - K - i + 1. \]

Let the dimension of \( F(X) \) be \( n \). Then the above says
\[ n(d - i) \geq nd - K - i + 1 \]
which implies
\[ i < K/n - 1. \]

Consider the subset
\[ Z' = \{(z, w) \mid z + w \in Z\} \subset \text{Sym}^{d-i} F(X) \times W. \]
By definition this set dominates \( Z \), hence is of dimension greater or equal than \( nd - K \). So the general fibers of the second projection \( \text{pr}_2 : Z' \to W \)
are of dimension at least
\[ nd - K - i + 1. \]

Also note that
\[ \theta^p_d(z + w) = \theta^p_{d-i}(z) + \theta^p_i(w). \]
Since \( \theta^p_d \) is constant along \( Z \) we have \( \theta^p_{d-i} \) is constant along \( Z'_w \). So if \( Z \) passes through a very general point of \( \text{Sym}^d F(X) \), then \( Z'_w \) passes through a very general point of \( \text{Sym}^{d-i} F(X) \). So we have \( \dim(Z'_w) \) is less than the dimension of a general fiber of \( \theta^p_{d-i} \) for generic \( w \). Now the dimension of \( Z'_w \) is greater than or equal to
\[ nd - K - i + 1 \]
but the dimension of the fiber of \( \theta_{d-i}^p \) is equal to \((d - i)n - K\) because \(d - i > d - K/n - 1\) can be chosen to be arbitrarily large.

Let us assume that \(d \geq 2\), and we have \(nd - K \geq d\). Consider the following lemma.

**Lemma 2.8.** Let \(Y\) be an ample hypersurface of \(F(X)\) and let \(Z\) be an irreducible subset of \(\text{Sym}^d F(X)\) not contained in any subset of the form

\[ \text{Sym}^{d-i} F(X) + W \]

with \(W \subset \text{Sym}^i F(X)\) and dimension of \(W\) is less than \(i\). Then \(Z\) intersects \(\text{Sym}^d Y\), provided that \(\dim(Z) \geq m\).

Therefore by applying the lemma we see that a general fiber of \(\theta_d^p\) intersects \(\text{Sym}^d Y\), for sufficiently large \(d\) and provided that \(n \geq 2\). Therefore \(\theta_d^p\) and \(\theta_d^p|_{\text{Sym}^d Y}\) has same image and the later has image of bounded dimension. So we can apply the lemma again and finally get that \(\theta_d^p\) and \(\theta_d^p|_{\text{Sym}^d C}\) has same image, where \(C\) is a smooth projective curve obtained by intersecting \(n - 1\) many ample hypersurfaces. This proves the theorem.

**Proof of Lemma 2.8.** Consider the quotient map \(r : F(X)^d \to \text{Sym}^d F(X)\). Let \(r^{-1}(Z) = \widetilde{Z}\), let \(\widetilde{Z}_0\) be a component of \(\widetilde{Z}\) dominating \(Z\). By the hypothesis we have the following:

for every \(i \geq 1\) and every subset \(I\) of cardinality \(i\), we have \(\dim p_I(\widetilde{Z}_0) \geq i\), where \(p_I\) is the projection from \(F(X)^d\) to \(F(X)^I\) corresponding to the set of indices. Since \(\widetilde{Z}_0\) dominates \(Z\), it is sufficient to prove that \(\widetilde{Z}_0\) intersect \(Y^d\), for an ample hypersurface \(Y\) in \(F(X)\). Consider a complete intersection \(V\) in \(\widetilde{Z}_0\), which is obtained by intersecting \(\widetilde{Z}_0\) with finitely many ample hypersurfaces, so that dimension of \(V\) is \(d\). Then the hypotheses on \(\widetilde{Z}_0\) implies that same would be true for \(V\). So without loss of generality we can assume that dimension of \(\widetilde{Z}_0 = d\). Consider a desingularization \(Z'\) of \(\widetilde{Z}_0\). Consider the divisors

\[ D_i := (pr_i \circ \tau)^{-1}(Y) \]

where \(\tau\) is the natural map from \(Z'\) to \(\widetilde{Z}_0\). Now \(Y\) is ample, so we have

\[ (pr_i \circ \tau)_* ((pr_i \circ \tau)^*(Y).C) = Y.(pr_i \tau)_* C \geq 0 \]

which means that \(D_i\)'s are numerically effective. Our claim will follow from the fact that \(D_1 \cap \cdots \cap D_d\) is non-empty. So we prove that \(D_1 \cap \cdots \cap \)
$D_d$ is non-empty. First suppose that $d = 2$. We have $D_1, D_2$ two divisors numerically effective. Hence we have

$$D_1^2 \geq 0; \quad D_2^2 \geq 0$$

Suppose that $D_1.D_2 = 0$. Then the intersection matrix of $(D_1, D_2)$ is semi-positive. So by the Hodge index theorem we have $D_1 = rD_2$ for some integer $r$. Hence $(D_1 + D_2)^2 = 0$. But $D_1 + D_2$ is the pull-back of an ample divisor on $F(X) \times F(X)$, under a generically finite map. So it is ample. Therefore $(D_1 + D_2)^2 > 0$, which is a contradiction. The general case follows from this.

So when, $\text{CH}^p(X)$ is representable and $X$ satisfies the assumption of Theorem 2.7, then the above Theorem 2.7 and arguments present in [Vo] give that there exists an abelian variety $A$ and a correspondence $\Gamma$ supported on $A \times F(X)$, such that $L_* \Gamma_* : A \to \text{CH}_0(F(X))_{\text{hom}} \to \text{CH}^p(X)_{\text{hom}}$ is surjective, where $L_*$ is the universal incidence correspondence given by

$$\{(x, L) : x \in L\}.$$ 

This leads us to the following result.

**Theorem 2.9.** Let $X$ be smooth projective and it satisfies the hypotheses of Theorem 2.7. Suppose that $\text{CH}^2(X)_{\text{hom}}$ is representable. Then the kernel of the Abel-Jacobi map is torsion.

**Proof.** By Theorem 2.7 there exists a smooth projective curve $C$ in $F(X)$ and a correspondence $\Gamma$ on $C \times F(X)$ such that $L_* \Gamma_* : J(C) \to \text{CH}^2(X)_{\text{hom}}$ is surjective. This yields further, a correspondence on $J(C) \times F(X)$, such that

$$L_* \Gamma_* : J(C) \to \text{CH}^2(X)_{\text{hom}}$$

is surjective. By Theorem 2.2 we have that kernel of $L_* \Gamma_*$ is a countable union of Zariski closed subsets of $J(C)$. Since kernel of $L_* \Gamma_*$ is a subgroup of $J(C)$ and we work over an uncountable ground field, the kernel is a countable union of translates of an abelian variety $A$ sitting in $\ker(L_* \Gamma_*)$. Now consider the supplementary abelian variety $B$, such that $A \times B \to J(C)$ is an isogeny. Now replacing $J(C)$ by $B$, and $L \circ \Gamma$ by $(L \circ \Gamma)_{B \times X}$, we get that the kernel of $L_* \Gamma_*$ is countable.

Fix $x_0$ in $F(X)$, consider the subset $R$ of $F(X) \times B$ given by

$$\{(x, a) : L_* \Gamma_*(a) = L_*(x) - L_*(x_0)\}.$$
By Theorem 2.2, we have \( R \) is a countable union of Zariski closed subsets in \( F(X) \times B \). Since \( \Gamma_* L_* \) is surjective, the projection from \( R \) onto \( F(X) \) is onto. Hence there exists a component \( R_0 \) of \( R \) surjecting onto \( F(X) \). Since \( \ker(L_* \circ \Gamma_*) \) is countable, the map is actually finite of say degree \( r \). Thus \( R_0 \) gives rise to a correspondence of dimension equal to \( \dim(F(X)) \) between \( F(X) \) and \( B \), this provides a morphism

\[
\alpha : F(X) \to B
\]
given by

\[
\alpha(x) = \text{alb}_B(R_0^*(x - x_0)).
\]

By definition of \( R_0 \) we have that

\[
L_* \Gamma_* \alpha(x) = r(L_* (x) - L_* (x_0)).
\]

Now this \( \alpha \) gives rise to a regular homomorphism from \( \text{CH}^2(X)_{\text{hom}} \) to \( B \). Hence by the universality the intermediate Jacobian \( I^2(J(X)) \), there exists a unique regular map \( \beta : I^2(J(X)) \to B \), such that

\[
\alpha = \beta \circ \Phi_2
\]
where \( \Phi_2 \) is the Abel-Jacobi map. So we have

\[
L_* \Gamma_* \beta \Phi_2(z) = rz.
\]

This proves that kernel of \( \Phi_2 \) is torsion.

\[\square\]

2.10. **Application of the above result.** Consider \( X \) to be a smooth cubic fourfold embedded in \( \mathbb{P}^5 \). Then we know that the group of homologically trivial one cycles \( \text{CH}^3(X)_{\text{hom}} \) is generated by lines on \( X \). So the criterion for Theorem 2.7 is satisfied. We know, by [Sc], that there does not exists a smooth projective curve \( C \) and a correspondence \( \Gamma \) on \( C \times X \), such that \( \Gamma_* \) is surjective. Hence it follows that the natural map from the Chow varieties parametrising one cycles on \( X \) does not surject onto \( \text{CH}^3(X)_{\text{hom}} \).

**Theorem 2.11.** The group \( \text{CH}^3(X)_{\text{hom}} \) is not representable. That is, the natural map from the product \( C^3_d(X) \times C^3_d(X) \) to \( \text{CH}^3(X)_{\text{hom}} \) is not surjective for any \( d \).

The above Theorem 2.9 gives us a criterion by which we can detect the representability of \( \text{CH}^2(X)_{\text{hom}} \). Precisely when, the Abel-Jacobi kernel is non-trivial, \( \text{CH}^2(X)_{\text{hom}} \) is not representable, in the sense that the natural map from the Chow varieties (infact from the symmetric powers of \( F(X) \))
to $\text{CH}^2(X)_{\text{hom}}$ are not surjective. There are some examples of higher dimensional varieties with non-trivial Abel-Jacobi mappings given in [GG]. So the Theorem 2.9 forces that such varieties cannot have:

$\text{Sym}^d F(X)$ surjecting onto $\text{CH}^2(X)_{\text{hom}}$. So for a threefold $X$, the non-triviality of the Abel-Jacobi kernel implies that the $\text{CH}^2(X)_{\text{hom}}$ cannot be generated by integral linear combination of lines on the threefold of any fixed degree.

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