Index Function and Minimal Cycles

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Abstract

Let $P$ be a closed triangulated manifold, $\dim P = n$. We consider the group of simplicial 1-chains $C_1(P) = C_1(P, \mathbb{Z}_2)$ and the homology group $H_1(P) = H_1(P, \mathbb{Z}_2)$. We also use some nonnegative weighting function $L : C_1(P) \to \mathbb{R}$. For any homological class $[x] \in H_1(P)$ method proposed in article builds a cycle $z \in [x]$ with minimal weight $L(z)$. The main idea is in using a simplicial scheme of space of the regular covering $\hat{P} \to P$ with automorphism group $G \cong H_1(P)$. We construct this covering applying index function $J : C_1(P) \to \mathbb{Z}_2^r$ relative to any basis of group $H_{n-1}(P)$, $r = \text{rank} H_{n-1}(P)$.

Keywords. Triangulated manifold, homology group, minimal cycle, intersection index, regular covering.

1 Index Function

Consider a triangulated closed manifold $P$, $\dim P = n$, and a basis $[z_1^{n-1}], \ldots, [z_r^{n-1}]$ of homology group $H_{n-1}(P) = H_{n-1}(P, \mathbb{Z}_2)$. Let $\text{Ind} : H_1(P) \times H_{n-1}(P) \to \mathbb{Z}_2$ be intersection index.

Definition 1. Define the homomorphism $J_0 : Z_1(P) \to \mathbb{Z}_2^r$ by the formulas $J_k^0(y) = \text{Ind}([y], [z_k^{n-1}])$, $k = 1, \ldots, r$, $J_0 = (J_1^0, \ldots, J_r^0)$. We call its arbitrary extension $J : C_1(P) \to \mathbb{Z}_2^r$ index function. For any chain $x \in C_1(P)$ the value $J(x)$ is called its index relative to the basis $[z_1^{n-1}], \ldots, [z_r^{n-1}]$.

Remark 1. Index function $J : C_1(P) \to \mathbb{Z}_2^r$ is not uniquely defined, however we can use this function for solve our problems.

Proposition 1. If $J : C_1(P) \to \mathbb{Z}_2^r$ is index function relative to the basis $\{[z_1^{n-1}], \ldots, [z_r^{n-1}]\}$ of group $H_{n-1}(P)$, $x, y \in C_1(P)$ and $\partial x = \partial y$, then $J(x) = J(y)$ if and only if $x \sim y$. 
Proof. Let \([ [z_1], \ldots, [z_r] ] \) be a basis of group \( H_1(P) = H_1(P, \mathbb{Z}_2) \), that is dual to the given basis \( \{ [z^n_1], \ldots, [z^n_r] \} \). Assume now that \( z = x + y \). Then \( z \in Z_1(P) \) and \( [z] = \sum_{i=1}^{r} l_i [z_i] \), where \( l_i \in \mathbb{Z}_2 \). This implies that \( J^k(z) = \text{Ind}([z], [z^n_k]) = l_k \) for all \( k = 1, \ldots, r \). So \( J(x) = J(y) \) if and only if \( l^1 = \cdots = l^r = 0 \). And this latter expression is equivalent to equality \([z] = 0\).  

ALGORITHM 1. Construction of index function relative to the basis of group \( H_{n-1}(P) \).

Input:
1) simple basis cycles \( z_1^{n-1}, z_2^{n-1}, \ldots, z_r^{n-1} \) which are lists of \( (n-1) \)-dimensional simplices;
2) list \( K^1(P) \) of edges for polyhedron \( P \);
3) lists \( K^1_1(P, z_1^{n-1}), \ldots, K^1_r(P, z_r^{n-1}) \) consisting of \( n \)-dimensional simplices from neighbourhoods of cycles \( z_1^{n-1}, \ldots, z_r^{n-1} \) respectively;

Output:
1) vectors \( J(a) = (J^1(a), \ldots, J^r(a)) \in \mathbb{Z}_2^r \) for all edges \( a \in K^1(P) \);
2) chains \( M_1, \ldots, M_r \) of edges indexed relative to cycles \( z_1^{n-1}, \ldots, z_r^{n-1} \) respectively;
3) lists \( M_k(u), k = 1, \ldots, r \) of edges, that we add to \( M_k \) when considering vertex \( u \) of cycle \( z_k^{n-1} \);
4) sets \( \Sigma_k(u), k = 1, \ldots, r \) of \( n \)-simplices incident to edges from \( M_k(u) \).

Algorithm Description.

Step 0. For all \( k = 1, \ldots, r \) execute steps 1 – 3.

Step 1. Start operations. Assume \( M_k = \emptyset \), \( J^k(a) := 0 \) for all \( a \in K^1(P) \). We denote \( z_k^{n-1} \) by \( X \) and \( K^n_k(P, z_k^{n-1}) \) by \( K^n(X) \). We create then lists of vertices and edges for all simplices of cycle \( X \), \( K^n(X) \) and \( K^1(X) \) respectively.

Step 2. Indexing edges that do not belong to the cycle. For each vertex \( u \in K^n(X) \) execute steps 2.1 – 2.4.

Step 2.1. Initializing vertex neighbourhood. Create a list \( K^n(P, u) \subset K^n(P, X) \) of \( n \)-dimensional simplices of the polyhedron \( P \), that contain \( u \), and a list \( K^{n-1}(P, u) \) of all \( (n-1) \)-dimensional faces of simplices from \( K^n(P, u) \). At the same time, for each simplex \( \sigma^{n-1} \in K^{n-1}(P, u) \) we get a list \( \partial^{n-1}(\sigma^{n-1}, u) \) of \( n \)-dimensional simplices from \( K^n(P, u) \) those are incident to \( \sigma^{n-1} \), and assume \( \mu(\sigma^{n-1}) := 0 \). Then we create empty lists \( M_k(u) := \emptyset \) and \( \Sigma_k(u) := \emptyset \).
Step 2.2. Creating the queue to keep \( n \)-simplices. We chose a simplex \( \sigma^n_0 \in K^n(P, u) \), create a queue \( R := \{ \sigma^n_0 \} \) and remove \( \sigma^n_0 \) from \( K^n(P, u) \).

Step 2.3. Main procedure of the Algorithm. While the queue \( R \) is not empty we will do the following actions. Take the first simplex \( \sigma^n \in R \) and remove it from the queue \( R \). For each \((n-1)\)-dimensional face \( \sigma^{n-1} \) of the simplex \( \sigma^n \) we check the following: whether it belongs to the cycle \( X \), whether \( \mu(\sigma^{n-1}) \) is equal to zero, whether the list \( \partial^{n-1}(\sigma^{n-1}, u) \) contains any simplices different from \( \sigma^n \). If all above conditions are satisfied we will execute steps 2.3.1 - 2.3.2.

**Step 2.3.1.** Take the simplex \( \sigma^n \in \partial^{n-1}(\sigma^{n-1}, u) \setminus \{ \sigma^n \} \), remove it from \( K^n(P, u) \) and enqueue to \( R \); set \( \mu(\sigma^{n-1}) := 1 \) and \( \Sigma_k(u) = \Sigma_k(u) \cup \{ \sigma^n \} \).

**Step 2.3.2.** For all vertices \( w \neq u \) of the simplex \( \sigma^{n-1} \) we check whether the edge \( a = [uv] \) is in the list \( K^1(X) \); having \( a \notin K^1(X) \) set \( J^k(a) := J^k(a) + 1 \mod 2 \), \( M_k(u) = M_k(u) \cup \{ a \} \), \( M_k := M_k + a \mod 2 \).

**Step 2.4. Main procedure repeated.** If the list \( M_k(u) \) is empty then go back to step 2.2.

**Step 3. Indexing the edges of cycle.** For each edge \( a = [uv] \in K^1(X) \) we search any edges \( b \in M_k(u) \) and \( c \in M_k(v) \) such that \( b \cap c \neq \emptyset \) and that \( a, b \) and \( c \) are sides of some triangle of polyhedron \( P \). If the edges \( b \) and \( c \) do not exist then we set \( J^k(a) := 1 \) and \( M_k = M_k + a \mod 2 \).

**End of algorithm.**

**Theorem 1.** If \( P \) is a closed \( n \)-dimensional manifold, \( z^{n-1}_1, \ldots, z^{n-1}_r \) are simple cycles, \( x = a_1 + \cdots + a_i \in C_1(P) \) and \( J(x) = \sum_{i=1}^r J(a_i) \), then the vector \( J(x) = (J^1(x), \ldots, J^r(x)) \in \mathbb{Z}^r_2 \) is index of the chain \( x \in C_1(P) \) relative to the basis \([z^{n-1}_1], \ldots, [z^{n-1}_r]\) of group \( H_{n-1}(P) \).

**Proof.** Let \( x \in Z_1(P) \). We will prove that \( J^k(x) = \text{Ind}([x], [z^{n-1}_k]) \) for all \( k = 1, \ldots, r \).

Set \( z^*_0 = z^{n-1}_k \). For all \( p = 1, \ldots, N \) we will make the following constructions; here \( N \) is power of the set \( K^0(z^{n-1}_k) \).

Consider vertex \( u_p \in K^0(z^{n-1}_k) \) and its barycentric star \( \text{bst}(u_p, P) \).

Let \( \Sigma^*_k(u_p) \) be the set of all \( n \)-simplices from the barycentric subdivision of \( \Sigma_k(u_p) \). Construct the chain \( c(u_p) \) of simplices \( \sigma_1 \in \text{bst}(u_p, P) \cap \Sigma^*_k(u_p) \).

Then we write the chain boundary \( \partial c(u_p) \) as a sum \( Y_1 + Y_2 \), where \( Y_1 \) is the sum of all its \((n-1)\)-dimensional simplices, that belong to cycle \( z^{n-1}_k \) and \( Y_2 \) is the sum of all remaining simplices from the chain \( \partial c(u_p) \). Set \( z^*_p = z^*_p - 1 + Y_1 + Y_2 \mod 2 \).
By construction $z^*_p \sim z^*_{p-1}$ for all $p = 1, \ldots, N$. Hence, the cycle $z^* = z^*_N$ is homologous to the cycle $z^*_{n-1} = z^*_0$.

Let now prove that for any edge $a = [uv] \in K^1(P)$ and $\sigma_b \in \text{bst}(a)$ the simplex $\sigma_b$ belong to $z^*$ if and only if $a \in M_k$.

Let view all possible positions of the edge $a$. At the same time we also agree to think that $M_k(u) = \emptyset$ and that $\Sigma_k(u) = \emptyset$ for all $u \notin K^0(z^*_{n-1})$.

0. If $a \notin M_k(u) \cup M_k(v)$ and $a \notin K^1(z^*_{n-1})$, then according to the algorithm $a \notin M_k$. On the other hand, the edge $a$ can not be incident to simplices from the lists $\Sigma_k(u)$ and $\Sigma_k(v)$ and hence $\sigma_b \notin z^*$.

1. Let $u \in K^0(z^*_{n-1})$, $a \in M_k(u)$ and $v \notin K^0(z^*_{n-1})$. Then the edge $a$ will be still in the chain $M_k$ when algorithm \[ is completed. At the same time the barycentric star $\text{bst}(a)$ belongs to the boundary of the chain $c(u)$ and does not belong to the cycle $z^*_{n-1}$. Thus in this case $a \in M_k$ and the chain $\text{bst}(a)$ belongs to the cycle $z^*$.

2. Further, assume that $u, v \in K^0(z^*_{n-1})$ and $a \in M_k(u)$. At that, $a \notin K^1(z^*_{n-1})$.

2.1. If $a \in M_k(v)$, then $a \notin M_k$, and simplices of its barycentric star will be added twice to the initial cycle $z^*_{n-1}$ and will not be in the resulting cycle $z^*$.

2.2. If $a \notin M_k(v)$, then $a \in M_k$ and any simplex $\sigma_b \in \text{bst}(a)$ is added to the cycle $z^*$ exactly once. So $\sigma_b \in z^*$.

3. Finally, let $a \in K^1(z^*_{n-1})$.

3.1. Let assume that the condition from step 3 of algorithm \[ is satisfied, i.e.:

(*) there exist edges $b \in M_k(u)$ and $c \in M_k(v)$ such that $b \cap c \neq \emptyset$ and that $a$, $b$ and $c$ are sides of some triangle $\sigma' \in K^2(P)$.

In this case, according to the algorithm $a \notin M_k$.

Let view all triangles $\sigma'$ from (*), and all $n$-dimensional simplices incident to them. The such $n$-simplices belong both to $\Sigma_k(u)$ and $\Sigma_k(v)$. Consider $n$-dimensional simplex $\sigma$, $\sigma_b \in \sigma$. If $\sigma_b \in \text{bst}(a)$, then $\sigma$ either belong to the both sets $\Sigma_k(u)$ and $\Sigma_k(v)$ or does not belong to them. Hence, the simplex $\sigma_b$ either is not added to the cycle $z^*$ or is added twice. Therefore $\sigma_b \notin z^*$.

3.2. Assume now that condition (*) is not satisfied. Then according to step 3 of algorithm \[ $a \in M_k$.

Barycentric star $\text{bst}(a)$ of the edge $a = [uv]$ belongs to the union $D(a)$ of all $n$-simplices that contain the edge $a$. We will prove that the sub-polyhedron $D(a)$ belongs to the union of simplices from the sets $\Sigma_k(u)$ and $\Sigma_k(v)$. 


Cycle \( z_k^{n-1} \) divides \( D(a) \) into two components of strong connectivity \( D^+(a) \) and \( D^-(a) \).

By construction the set \( \Sigma_k(u) \) can not be empty. Moreover, if the simplex \( \sigma \in z_k^{n-1} \) is incident to the vertex \( u \), then \( \sigma \) is a face of some \( n \)-simplex from \( \Sigma_k(u) \). So there exists a simplex \( \sigma^n \in \Sigma_k(u) \) that contains the edge \( a \).

Let the simplex \( \sigma^n \) belongs to \( D^+(a) \). Then under the strong connectivity \( D^+(a) \) and according to algorithm \( \square \) all \( n \)-simplices from \( D^+(a) \) also belong to \( \Sigma_k(u) \).

This implies, in accordance with our assumption, that no \( n \)-simplex from \( D^+(a) \) can belong to the set \( \Sigma_k(v) \).

The set \( \Sigma_k(v) \) can not be empty also. Since each simplex of \( z_k^{n-1} \) incident to the vertex \( v \) is a face of some \( n \)-simplex from \( \Sigma_k(v) \), it follows that there exists a simplex \( \sigma^*_n \in \Sigma_k(v) \) that contains the edge \( a \). By the above proof \( \sigma^*_n \) belongs to \( D^-(a) \). Then all \( n \)-simplices from \( D^-(a) \) belong to the set \( \Sigma_k(v) \) too. Consequently all \( n \)-simplices of the polyhedron \( D(a) = D^+(a) \cup D^-(a) \) belong either to the set \( \Sigma_k(u) \) or to \( \Sigma_k(v) \).

Consider \( \sigma_b \in \text{bst}(a) \). If there exists a simplex \( \sigma \in \Sigma_k(u) \) containing \( \sigma_b \), then \( \sigma \notin \Sigma_k(v) \). Otherwise, in accordance to the above proof, there is a simplex \( \tilde{\sigma} \in \Sigma_k(v) \) such that \( \sigma_b \subset \tilde{\sigma} \). It follows that \( \sigma_b \) is involved in the cycle \( z^* \) exactly once, so \( \sigma_b \in z^* \).

Thus we have proved that the cycle \( z^* \sim z_k^{n-1} \) consists of barycentric stars of the edges from chain \( M_k \). That means that this cycle intersects transversally only the edges of the cycle \( x \), that are in the list \( M_k \). According to algorithm \( \square \), \( J^k(a) = 1 \) for all \( a \in M_k \) and \( J^k(b) = 0 \) for all edges \( b \notin M_k \). So

\[
\text{Ind}([x], [z_k^{n-1}]) = \text{Ind}([x], [z^*]) = \sum_{a \in x} J^k(a) \mod 2 = J^k(x).
\]

\( \square \)

Remark 2. The fact that \([z_1^{n-1}], \ldots, [z_n^{n-1}]\) is a basis of group \( H_{n-1}(P) \) has no impact on the behaviour of algorithm \( \square \). So we can apply this algorithm to an arbitrary set of simple \((n-1)\)-dimensional cycles of the manifold \( P \). In particular this set may consist of only one cycle \( z^{n-1} \). Then we will get a function \( J : C_1(P) \to \mathbb{Z}_2 \) such that \( \sum_{i=1}^{l} J(a_i) = \text{Ind}([x], [z^{n-1}]) \) for \( x = a_1 + \cdots + a_l \in Z_1(P) \). So we can use algorithm \( \square \) to compute the intersection index of a given \((n-1)\)-cycle \( z^{n-1} \in Z_{n-1}(P) \) with any one-dimensional cycle of the manifold \( P \).
Remark 3. We can find any basis \([z_1^{n-1}], \ldots, [z_r^{n-1}]\) of group \(H_{n-1}(P)\) using standard matrix algorithm (see, for example, [3]). If \(n = 2\), we also can apply algorithms that don’t use incidence matrices (see [4, 5]).

2 Regular Covering with the Automorphism Group \(H_1(P)\)

Let \(P\) be a \(n\)-dimensional triangulated closed manifold and \(S = (V, K)\) be its simplicial scheme. We will construct an abstract simplicial scheme \(\hat{S} = (\hat{V}, \hat{K})\) as follows.

Set \(\hat{V} = V \times G\), where \(G = \mathbb{Z}_2^r\). Let \(\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_m \in \hat{V}\), where \(\hat{v}_i = (v_i, b_i)\) for all \(i = 0, 1, \ldots, m\). We will think that \(\{\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_m\} \in \hat{K}\) if the below conditions are satisfied:

(U1) \(\{v_0, v_1, \ldots, v_m\} \in K;\)

(U2) \(g_0 + g_i = J([v_0v_i])\) for any \(i = 1, \ldots, m\); here \(J([v_0v_i])\) is the index of the edge \([v_0v_i]\).

Remark 4. When the conditions (U1) and (U2) are satisfied the equalities \(g_i + g_j = J([v_iv_j])\) are also true for all \(i, j = 1, \ldots, m\). In fact, according to (U1), the cycle \(z = [v_jv_i] + [v_iv_0] + [v_0v_j]\) is homologous to zero. So \(J([v_iv_j]) = J([v_0v_i]) + J([v_0v_j])\). By invoking (U2) we can have these equalities \(J([v_iv_j]) = g_i + g_0 + g_0 + g_j = g_i + g_j\).

Let define now a mapping \(p^0 : \hat{V} \to V\) and a left action \(\lambda^0 : G \times \hat{V} \to \hat{V}\) of group \(G\) on \(\hat{V}\), assuming

\[p^0((v, g)) = v \quad \text{and} \quad \lambda^0(g', (v, g)) = g' \cdot (v, g) = (v, g' + g)\]  (1)

for all \((v, g) \in \hat{V}\) and \(g' \in G\).

Let \(\hat{P}\) define some realization of the scheme \(\hat{S} = (\hat{V}, \hat{K})\). At that we identify the set of vertices of the polyhedron \(\hat{P}\) with \(\hat{V}\).

Proposition 2. For the mapping \(p^0 : \hat{V} \to V\) there exists the unique continuation \(p : \hat{P} \to P\) that is simplicial regular covering with a group of covering transformations \(G \cong H_1(P)\).
Proof. Simplicial and surjective properties of the mapping $p^0$ follow directly from its definition and from the construction of the complex $\hat{K}$. If $\hat{s} = \{(v_0, g_0), (v_1, g_1), \ldots, (v_m, g_m)\} \in \hat{K}$, then $\{v_0, v_1, \ldots, v_m\} \in K$ and $g_0 + g_i = J([v_0 v_1])$ for all $i = 1, \ldots, m$. On the other hand, $g \cdot \hat{s} = \{(v_0, g + g_0), (v_1, g + g_1), \ldots, (v_m, g + g_m)\}$ for an arbitrary $g \in G$. Since $g + g_0 + g + g_i = g_0 + g_i = J([v_0 v_1])$, then $g \cdot \hat{s} \in \hat{K}$. So the action $\lambda^0$ is also simplicial.

Let $s = \{v_0, v_1, \ldots, v_m\} \in K$ and $\hat{v}_0 \in (p^0)^{-1}(v_0)$. Then $\hat{v}_0 = (v_0, g_0)$, where $g_0 \in G$. Set $g_i = g_0 + J([v_0 v_1])$ and $\hat{v}_i = (v_i, g_i)$ for all $i = 1, \ldots, m$. At that $\hat{s} = \{\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_m\} \in \hat{K}$, $\hat{v}_0 \in \hat{s}$ and $p^0(\hat{s}) = s$. Hence, the mapping $p^0$ has the following property:

(C1) for each abstract simplex $s \in K$ and for any vertex $\hat{v} \in (p^0)^{-1}(s)$ there is the unique abstract simplex $\hat{s} \in \hat{K}$ containing the vertex $\hat{v}$ and satisfying the equality $p^0(\hat{s}) = s$.

Let choose an abstract simplex $\hat{s} = \{\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_m\} \in \hat{K}$, and an element $g$ of group $G$ and assume that $g \cdot \hat{s} = \hat{s}$. Then $\hat{v}_i = (v_i, g_i)$ and $g \cdot \hat{v}_i = (v_i, g + g_i)$ for all $i = 1, \ldots, m$. At the same time it follows from the equality $g \cdot \hat{s} = \hat{s}$ that $(v_0, g + g_0) = (v_k, g_k)$ for some $k \in \{0, 1, \ldots, m\}$. The latter is possible only if $k = 0$ and $g = 0$. Thus the action $\lambda^0$ has the following property:

(C2) if $g \cdot \hat{s} = \hat{s}$ for at least one non-empty simplex $\hat{s} \in \hat{K}$, then $g$ is the neutral element of the group $G$.

Let now consider the simplices $\hat{s} = \{(v_0, g_0), (v_1, g_1), \ldots, (v_m, g_m)\}$ and $\hat{s}'$ of the complex $\hat{K}$.

First, if $g \in G$ and $\hat{s}' = g \cdot \hat{s}$, then $\hat{s}' = \{(v_0, g + g_0), (v_1, g + g_1), \ldots, (v_m, g + g_m)\}$. At that $p^0(\hat{s}') = \{v_0, v_1, \ldots, v_m\} = p^0(\hat{s})$.

Further, assume that $p^0(\hat{s}') = p^0(\hat{s}) = \{v_0, v_1, \ldots, v_m\}$. Then according to (1), $\hat{s}' = \{(v_0, g'_0), (v_1, g'_1), \ldots, (v_m, g'_m)\}$, where $g'_0, g'_1, \ldots, g'_m$ are some elements of group $G$, and $g_0 + g_i = J([v_0 v_1]) = g'_0 + g'_i$ for $i = 1, \ldots, m$.

Set $g = g'_0 + g_0$. Then according to the above equalities $g'_i = g + g_i$ for all $i = 0, 1, \ldots, m$ and hence $\hat{s}' = g \cdot \hat{s}$.

This proves that $p^0$ and $\lambda^0$ have the following property:

(C3) for arbitrary abstract simplices $\hat{s}, \hat{s}' \in \hat{K}$ the equality $p^0(\hat{s}) = p^0(\hat{s}')$ is equivalent to the existence of an element $g \in G$ such that $g \cdot \hat{s} = \hat{s}'$.

It is known that $p^0$ and $\lambda^0$ may have the unique continuation to the simplicial mapping $p : \hat{P} \rightarrow P$ and the simplicial action $\lambda : G \times \hat{P} \rightarrow \hat{P}$ of
group $G$ on $\hat{P}$. It also follows from (C1) – (C3) that $p$ is a regular covering, and $G$ is a corresponding group of covering transformations (see, for example, [4]).

**Proposition 3.** Let $x = [v_0v_1] + [v_1v_2] + \cdots + [v_{s-1}v_s]$ and $y = [u_0u_1] + [u_1u_2] + \cdots + [u_{t-1}u_t]$ be edge paths of the polyhedron $P$, that run from the vertex $v_0 = u_0$ to the vertex $v_s = u_t$, $\tilde{x} = [\hat{v}_0\hat{v}_1] + [\hat{v}_1\hat{v}_2] + \cdots + [\hat{v}_{s-1}\hat{v}_s]$ and $\hat{y} = [\hat{u}_0\hat{u}_1] + [\hat{u}_1\hat{u}_2] + \cdots + [\hat{u}_{t-1}\hat{u}_t]$ paths of $\hat{P}$, that cover the paths $x$ and $y$ respectively and have the same beginning $\hat{v}_0 = \hat{u}_0$. Then $\hat{v}_s = \hat{u}_t$ if and only if $x \sim y$.

**Proof.** Let $z = [w_0w_1] + [w_1w_2] + \cdots + [w_{s-1}w_s]$ be a path in the polyhedron $P$ and $g_0 \in G = \mathbb{Z}_2$. Then the unique path $\hat{z}$ of the polyhedron $\hat{P}$, starting in the vertex $\hat{w}_0 = (w_0, g_0)$ and covering the path $z$, is defined by the formulas

\[
\hat{w}_i = (w_i, g_0 + J(z_i)), \quad i = 1, \ldots, s,
\]

where $z_i = [w_0w_1] + [w_1w_2] + \cdots + [w_{i-1}w_i]$, and

\[
\hat{z} = [\hat{w}_0\hat{w}_1] + [\hat{w}_1\hat{w}_2] + \cdots + [\hat{w}_{s-1}\hat{w}_s].
\]

Set $g_i = g_0 + J(z_i)$ for $i = 1, \ldots, s$ and $z_0 = 0$. Then $J(z_i) = J(z_{i-1}) + J([w_{i-1}w_i])$ for all $i = 1, \ldots, s$. At the same time $g_i = g_{i-1} + J([w_{i-1}w_i])$ and the vertices $\hat{w}_{i-1}$ and $\hat{w}_i$ from $\hat{V}$, defined by the formula (2), are connected by the edge $[\hat{w}_{i-1}\hat{w}_i] \in \hat{K}$. Then in the polyhedron $\hat{P}$ there is defined a path $\hat{z}$ starting at the vertex $\hat{w}_0 = (w_0, b_0)$. As $p(\hat{w}_i) = p((w_i, g_0 + J(z_i))) = w_i$ for all $i = 0, 1, \ldots, s$, then $\hat{z}$ covers the path $z$. Since $p$ is a covering then the path $\hat{z}$ is unique.

Assume now that $\hat{v}_0 = (v_0, g_0)$, where $g_0 \in G$. By the above proof, the equalities $p(\hat{x}) = x$, $p(\hat{y}) = y$ and $\hat{v}_0 = \hat{u}_0$ imply that $\hat{v}_s = (v_s, g_0 + J(x))$ and $\hat{u}_t = (u_t, g_0 + J(y))$. So $\hat{v}_s = \hat{u}_t$ if and only if $J(x) = J(y)$. According to proposition [4], the last equality is equivalent to the homology of the chains $x$ and $y$.

\[\square\]

### 3 Minimal Cycles Searching

Let $E(P) = K^1(P)$ be the set of edges of the polyhedron $P$, and $L : E(P) \to \mathbb{R}$ be a non-negative function. Using the formulas

\[L(0) = 0 \text{ and } L(\{a_1, \ldots, a_s\}) = \sum_{i=1}^{s} L(a_i).
\]

(4)
we can extend $L$ to the function $L : C_1(R) \rightarrow (R)$. This function is often called weight function. And for an arbitrary chain $x \in C_1(P)$ the value $L(x)$ is called its weight (see, for example, [2]).

Let define a weight function $\hat{L} : C_1(\hat{P}) \rightarrow \mathbb{R}$ assuming that

$$\hat{L}(\hat{x}) = L(p(\hat{x}))$$

for an arbitrary chain $\hat{x} \in C_1(\hat{P})$.

**ALGORITHM 2.** Searching for the minimal cycle with fixed vertex and index.

**Input:**
1) list $V(P)$ of vertices for polyhedron $P$;
2) lists $U(v, P)$ of vertices incident to $v$ for all vertices $v \in V(P)$;
3) index function $J : C_1(P) \rightarrow \mathbb{Z}_2$ relative to some basis of group $H_{n-1}(P)$;
4) weight function $L : C_1(P) \rightarrow \mathbb{R}$;
5) vector $i \in G = \mathbb{Z}_2$;
6) vertex $u \in V(P)$.

**Output:**
1-chain $z \in C_1(P)$.

**Algorithm Description.**

**Step 1. Initializing cycle $z$.** Set $z := \emptyset$.

**Step 2. Initializing sets $\hat{T} \subset V(P) \times G$, $\hat{P}^* \subset V(P) \times G$ and a mapping $\tilde{D} : V(P) \times G \rightarrow \mathbb{R}$**. Let $\hat{T} := \{(u, 0)\}$, where $0$ – null vector of space $G = \mathbb{Z}_2$, $\hat{P}^* := \emptyset$ and $\tilde{D}(u, 0) := 0$.

**Step 3. First extension of $\hat{P}^*$ and $\tilde{D}$.** For each vertex $v \in U(u, P)$ set $j := J([uv])$ and add the pair $(v, j)$ into the list $\hat{P}^*$. At the same time set $\tilde{D}(v, j) := L([uv])$, $F(v, j) := (u, 0)$.

**Step 4. Choosing a next element to add to $\hat{T}$.** Find the pair $(w, k) \in (\hat{P}^* \setminus \hat{T})$ such that $\tilde{D}(w, k) = \min_{(v, j) \in (\hat{P}^* \setminus \hat{T})} \tilde{D}(v, j)$.

**Step 5. Stop criterion of $\hat{T}$, $\hat{P}^*$, $\tilde{D}$ construction.** If $w = u \ k = i$, then go to step 9.

**Step 6. Extension of the set $\hat{T}$.** Add the pair $(w, k)$ into the list $\hat{T}$.

**Step 7. Next extension of $\hat{P}^*$ and $\tilde{D}$.** For each vertex $v \in U(w, P)$ set $j := k+J([uv])$. If the pair $(v, j) \notin \hat{P}^*$, then set $\tilde{D}(v, j) := \tilde{D}(w, k)+L([uv])$, $F(v, j) := (w, k)$ and add the pair $(v, j)$ into $\hat{P}^*$. If $(v, j) \in (\hat{P}^* \setminus \hat{T})$ and
\[ \hat{D}(w, k) + L([wv]) < \hat{D}(v, j), \text{ then set } \hat{D}(v, j) = \hat{D}(w, k) + L([wv]) \text{ and } F(v, j) := (w, k). \]

**Step 8. Continuation of \( T, \hat{P}^*, \hat{D} \) construction.** Go back to step 4.

**Step 9. Construction of chain \( z \).**

**Step 9.1.** Take a pair \((v, j) = F(w, k)\) and set \( z := z + [vw] \).

**Step 9.2.** If \((v, j) \neq (u, i)\), then set \((w, k)\) equal to \((v, j)\) and go back to step 9.1.

**End of algorithm.**

**Theorem 2.** The chain \( z \in C_1(P) \) computed by the algorithm \( \square \) has the following properties:

- \( z \in Z_1(P) \);
- \( J(z) = i \);
- \( u \in V(z) \), where \( V(z) \) is the vertex set of the chain \( z \);
- \( L(z) \leq L(x) \) for all cycles \( x \in Z_1(P) \) that satisfy conditions \( J(x) = i \) and \( u \in V(x) \).

**Proof.** Let \( T^* \) be the result set of Dijkstra’s algorithm for a one-dimensional skeleton \( \hat{P}^1 \) of the polyhedron \( \hat{P} \) if we choose the pair \( \hat{u} = (u, 0) \) as the start point, and the pair \((u, i)\) as the end point (see, for example, [1]).

According to the definition of the complex \( \hat{K} \), the pairs \( \hat{v} = (v, j) \) and \( \hat{u} = (u, 0) \) in step 3, as well as the pairs \( \hat{v} = (v, j) \) and \( \hat{w} = (w, k) \) in step 7 are connected by the edges \([\hat{v}\hat{u}] \in E(\hat{P})\) and \([\hat{v}\hat{w}] \in E(\hat{P})\) respectively.

Also, according to (3), we have the equalities \( \hat{L}([\hat{v}\hat{u}]) = L([vu]) \) in step 3 and \( \hat{L}([\hat{v}\hat{w}]) = L([vw]) \) in step 7. This implies that the set \( \hat{T} \) constructed by step 9 is the same that \( T^* \).

Let note that step 9 is not limited to compute \( z = [v_0 v_1] + \cdots + [v_{q-1} v_q] \) starting at \( v_0 = u \) and ending at \( v_q = u \), but it also gives us the possibility to construct the vector sequence \( j_0, j_1, \ldots, j_q \in \mathbb{Z}_2 \), that will satisfy the equalities \( j_0 = 0, j_q = i \) and \( j_s = j_{s-1} + J([v_{s-1} v_s]) \).

Set \( \hat{v}_s = (v_s, j_s) \) for all \( s = 0, 1, \ldots, q \). Then \([\hat{v}_{s-1} \hat{v}_s] \in E(\hat{P})\) for the same \( s \) and \( z = [\hat{v}_0 \hat{v}_1] + [\hat{v}_1 \hat{v}_2] + \cdots + [\hat{v}_{q-1} \hat{v}_q] \) is a path in the skeleton \( \hat{P}^1 \), starting at \( \hat{u} = (u, 0) \) and ending at \( (u, i) \). Since it can be computed by Dijkstra’s algorithm, \( \hat{L}(\hat{z}) \) is not over than weight of any other path in \( \hat{P}^1 \), running from \( \hat{u} = (u, 0) \) to \( (u, i) \).
By the construction of the path \( \hat{z} \) and according to (5), \( L(z) = \hat{L}(\hat{z}) \). Now, in the polyhedron \( P \), let consider another cycle \( z' \) containing the vertex \( u \) and having the index \( J(z') = i \). According to proposition 1, \( \left[ z' \right] = [x] \) in \( H_1(P) \). Since \( p: \hat{P} \to P \) is a covering, there exists the unique path \( \hat{z}' \) in \( \hat{P} \), that covers \( z' \) and starts at the vertex \( \hat{u} = (u,0) \). At the same time, by statements 1 and 3 the end points of these paths \( \hat{z} \) and \( \hat{z}' \) coincide. But then according to the above proof, \( L(z) = \hat{L}(\hat{z}) \leq \hat{L}(\hat{z}') = L(z') \).

ALGORITHM 3. Searching for the minimal cycle from fixed homology class.

**Input:**
1) list \( V(P) \) of vertices for polyhedron \( P \);
2) lists \( U(v, P) \) of vertices incident to \( v \) for all vertices \( v \in V(P) \);
3) simple basis cycles \( z_{n-1}^1, z_{n-1}^2, \ldots, z_{n-1}^r \) of homology group \( H_{n-1}(P) \);
4) lists \( V(z_{n-1}^1), \ldots, V(z_{n-1}^r) \) of vertices from cycles \( z_{n-1}^1, \ldots, z_{n-1}^r \) respectively;
5) index function \( J: C_1(P) \to \mathbb{Z}_2 \) relative to basis \( [z_{n-1}^1], \ldots, [z_{n-1}^r] \) of group \( H_{n-1}(P) \);
6) weight function \( L: C_1(P) \to \mathbb{R} \);
7) cycle \( x \in Z_1(P) \).

**Output:**
1-cycle \( z \in Z_1(P) \).

**Algorithm Description.**

**Step 1.** Set \( Z := \emptyset \).

**Step 2.** Determine the vector \( i = J(x) \).

**Step 3.** If \( i = 0 \), then set \( z = 0 \) and go to step 7.

**Step 4.** Find a number \( k \in \{1, \ldots, r\} \) such that coordinate \( i^k \) of the vector \( i \) is equal 1.

**Step 5.** For each vertex \( v \in V(z_{n-1}^k) \) execute steps 5.1 – 5.3.

**Step 5.1.** Using algorithm 2 we find containing \( v \) cycle \( z_v \in Z_1(P) \) with index \( J(z_v) = i \) having minimal weight \( L(z_v) \) in set of all cycles with the same properties.

**Step 5.2.** Add the cycle \( z_v \) into the list \( Z \).

**Step 5.3.** Take the next vertex \( v \in V(z_{n-1}^k) \).

**Step 6.** Choose the cycle \( z \in Z \) such that \( \bar{L}(z) = \min_{z' \in Z} L(z') \).

**Step 7.** Quit.

End of algorithm.
Theorem 3. Let \( z \) be the cycle found by the algorithm. Then

- \( z \sim x \);
- \( L(z) = \min_{y \in [x]} L(y) \).

Proof. First, if \( i = 0 \), then according to proposition, cycle \( x \) is homologous to zero. At the same time we assume in step 3 that \( z = 0 \). According to \( L(0) = 0 \). Thus, in this case \( z \sim x \) and \( L(z) = \min_{y \in [x]} L(y) \).

Further, let \( i \neq 0 \). Then according to step 4 \( i^k = 1 \) for \( k \in \{1, \ldots, r\} \).

Let now consider an arbitrary element \( z_v \) in the list \( Z \). It is chosen in step 5, and according to this step \( J(z_v) = i = J(x) \). According to proposition, it follows that \( z_v \sim x \). Since \( z = z_v \) for some \( v \in V(z_n) \) then \( z \sim x \) too.

Let assume that some one-dimensional cycle \( y \) of the polyhedron \( P \) belongs to the class \( [x] \). Then \( J(y) = J(x) = i \). Hence, \( \text{Ind}([y],[z_n^{-1}]) = J^k(y) = 1 \), and therefore the cycles \( y \) and \( z_n^{-1} \) have at least one common vertex \( u \in V(z_n^{-1}) \). In this case, according to the selection of cycle \( z_u \) in step 5.1 of algorithm, \( L(z_u) \leq L(y) \). This implies according to step 6, that \( L(z) \leq L(z_u) \leq L(y) \).

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