Combinatorial nature of ground state vector of O(1) loop model

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Abstract

Hanging about a hypothetical connections between the ground state vector for some special spin systems and the alternating-sign matrices, we have found a numerical evidence for the fact that the numbers of the states of the fully packed loop model with fixed link-patterns coincide with the components of the ground state vector of the dense O(1) loop model considered by Batchelor, de Gier and Nienhuis. Our conjecture generalizes in a sense the conjecture of Bosley and Fidkowski, refined by Cohn and Propp, and proved by Wieland.

I would say that imagination is a form of memory.
(V. Nabokov)

In paper [1] we made some conjectures related to combinatorial properties of the ground state vector of the XXZ spin chain for the asymmetry parameter $\Delta = -1/2$ and an odd number of cites. In the subsequent paper [2] Batchelor, de Gier and Nienhuis considered two variations of this model along with the corresponding $O(n)$ loop model at $n = 1$ and notably increased the number of models and related combinatorial objects (see also [3]).

During last months we have accumulated a lot of data on relation between the spin systems and alternating sign matrices (ASMs). In this note we limit ourselves to one qualitative conjecture which relates some classes of the states of the fully packed loop (FPL) model with the ground state vector of the dense $O(n)$ loop model (see [4] and references therein). The states the FPL model is in bijective correspondence with the ASMs. Therefore if our conjecture is true then one can investigate the FPL model, and the ASMs, using the methods elaborated in mathematical physics for integrable model (see, for example, [5]).

The background information on the ASMs and their different combinatorial forms can be found in the recent review by Propp [6] and in references therein. For more details on enumerative problems related to the states of the FPL model see the article by Wieland [7].

Following the review paper by Propp [6] we define the “generalized tic-tac-toe” graph as the graph formed by $n$ horizontal lines and $n$ vertical lines meeting $n^2$ intersections of degree 4, with $4n$ vertices of degree 1 at the boundary. Then we number the vertices of degree 1. We start with the left top vertex and number clockwise every other vertex. Now
Consider subgraphs of the underlying tic-tac-toe graph such that each of the $n^2$ internal vertices lies on exactly two of the selected edges and each numbered external vertex lies on a selected edge, while each unnumbered external vertex does not lie on a selected edge (see, for example, figure 1). These subgraphs are the states of the FPL model. The states the FPL model are in bijective correspondence with the states of the square-ice model with the domain-wall boundary conditions, and with the ASM matrices. In particular, the number of such states is equal to the number of ASMs, usually denoted by $A_n$. Each state of the FPL model define a so-called link-pattern describing the pairings of the external vertices. We depict such a pattern as a circle with $2n$ vertices placed on it and connected pairwise inside the circle without intersection (see figure 2). Having in mind the relation to ASMs we denote the number of the states of the FPL model corresponding to the link-pattern $\pi$ by $A_n(\pi)$.

Consider as an example the case of $n = 4$. In this case the FPL model has 42 states and 14 different link-patterns which are given in table 1. The coincidence of the numbers $A_n(\pi)$ for different link-patterns is explained by the conjecture of Bosley and Fidkowski, refined by Cohn and Propp, and proved by Wieland. In accordance with this conjecture, if two link-patterns $\pi$ and $\pi'$ are connected by a rotations or a reflection then $A_n(\pi) = A_n(\pi')$.

Let us now formulate a conjecture on the numbers $A_n(\pi)$. To this end define $2n$ operations $h_i$, $i = 1, \ldots, 2n$, on link-patterns in the following way. For a fixed $i$ consider a

| $\pi$ | $A_n(\pi)$ |
|------|------------|
| ![Link-pattern 1](#) | 7 |
| ![Link-pattern 2](#) | 3 |
| ![Link-pattern 3](#) | 1 |
general link-pattern. Let this link-pattern has the \(i\)-th vertex linked to the \(j\)-th one and the \((i + 1)\)-th vertex linked to the \(k\)-th one. Two different case are possible, the first is when \(j = i + 1\) and \(k = i\). In this case the operation \(h_i\) left the link-pattern unchanged.

In the second case \(j \neq i + 1\) and \(k \neq i\). In this case the operation \(h_i\) replaces the two links under consideration by the link of \(i\)-th vertex and the \((i + 1)\)-th vertex and the link of the \(j\)-th vertex and the \(k\)-th vertex. To be understandable by the widest audience of readers, we decided to formulate our conjecture in three different ways.

The first formulation is in terms of the game theory. Let two players A and B play a game of chance. Both persons randomly choose a state of the FPL model. The player A checks a link-pattern of his state. The player B has to choose randomly one of the operations \(h_i\) and to apply it to the link-pattern of his state. Only the resulting link-pattern is of importance for him. Players aim at obtaining some fixed link-pattern.

**Conjecture 1** Both players have equal chances to win.

Consider, for example, the case of \(n = 4\) with the winning link-pattern be the very first one of table \[I\]. The chances of the player A are \(P_A = \frac{7}{42} = \frac{1}{6}\). Let us evaluate the chances of the player B.

\[
P_B = P \left[ \begin{array}{c} \emptyset \\ \{h_1, h_3, h_5, h_7\} \end{array} \right] P \left[ \begin{array}{c} \{h_5, h_7\} \end{array} \right] + P \left[ \begin{array}{c} \emptyset \\ \{h_1, h_7\} \end{array} \right] P \left[ \begin{array}{c} \{h_1, h_3\} \end{array} \right] + P \left[ \begin{array}{c} \emptyset \\ \{h_3, h_5\} \end{array} \right] P \left[ \begin{array}{c} \{h_3, h_7\} \end{array} \right] + P \left[ \begin{array}{c} \emptyset \\ \{h_1, h_5\} \end{array} \right] P \left[ \begin{array}{c} \{h_1, h_7\} \end{array} \right].
\]

Using the data from table \[I\] one obtains

\[
\frac{7}{42} \times \frac{4}{8} + \frac{3}{42} \times \frac{8}{8} + \frac{3}{42} \times \frac{8}{8} + \frac{3}{42} \times \frac{8}{8} + \frac{1}{42} \times \frac{2}{8} + \frac{1}{42} \times \frac{2}{8} = \frac{1}{6}.
\]

Thus our conjecture in this case is true.

The second formulation is in terms of the combinatorics.

**Conjecture 2** For any \(n = 1, 2, \ldots\), one has

\[
\sum_{i=1}^{2n} \sum_{\pi' \in \Pi_i(\pi)} A_n(\pi') = 2nA_n(\pi),
\]

where \(\Pi_i(\pi)\) is the set of link-patterns \(\pi'\) such that \(h_i(\pi') = \pi\).

The last formulation is in terms of the dense periodic \(O(1)\) model. The state space for this model is the set of all formal linear combinations of the link-patterns. The operations \(h_i\) extended to this space by linearity become linear operators. The Hamiltonian of the model is the sum of the operators \(h_i\):

\[
H = \sum_{i=1}^{2n} h_i,
\]

(1)

Consider the action of this operator on the vector

\[
\Psi = \sum_{\pi} \pi A_n(\pi).
\]

(2)
We have
\[ H \Psi = \sum_{i=1}^{2n} \sum_{\pi} h_i(\pi) A_n(\pi) = \sum_{\pi} \pi \sum_{i=1}^{2n} \sum_{\pi' \in \Pi_i(\pi)} A_n(\pi') = 2n \sum_{\pi} \pi A_n(\pi), \]
where the last equality follows from conjecture \textsuperscript{3}. Hence we can write
\[ H \Psi = 2n \Psi. \]

Thus, the vector \( \Psi \) is an eigenvector of the Hamiltonian of the dense periodic O(1) loop model with an odd number of sites.

Let us consider our usual case of \( n = 4 \). The matrix of the Hamiltonian \( H \) and the component form of the vector \( \Psi \) have in this case the form

\[
H = \begin{pmatrix}
4 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\
0 & 4 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\
1 & 0 & 3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \\
1 & 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 2 & 0 \\
1 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & 2 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 3 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}, \quad \Psi = \begin{pmatrix}
7 \\
7 \\
3 \\
3 \\
3 \\
3 \\
3 \\
3 \\
3 \\
3 \\
1 \\
1 \\
1 \\
1
\end{pmatrix}.
\]

Here we take the basis vectors in the order given in table 1. Note that the sum of the matrix elements of \( H \) belonging to any its fixed column is equal to 8 which coincides with 2\( n \). This is a general property of the Hamiltonian \( H \) valid for any \( n \) which implies that the spectral radius of \( H \) is equal to 2\( n \) (see, for example, [9, section 9.3]). Therefore, the vector \( \Psi \) corresponds to the largest eigenvalue of the Hamiltonian \( H \). Actually this eigenvalue is nondegenerate that can be proved using the positivity of the corresponding transfer matrix. Taking all this into account we give the last formulation of our conjecture.

Conjecture 3 The vector (2) is the unique eigenvector of the Hamiltonian (1) corresponding to its largest eigenvalue 2\( n \).

In paper [2] Batchelor, de Gier and Nienhuis formulated two conjectures on the eigenvector under consideration.\textsuperscript{4} They conjectured that the sum of its components is equal to \( A_n \) and its largest component is equal to \( A_{n-1} \). Our conjecture reveals the combinatorial nature of all components of the eigenvector with the largest eigenvalue.

Note that the Hamiltonian \( H \) has the rotational and reflection invariance. It can be easily shown that the eigenvector with the maximal eigenvalue must have rotational and

\textsuperscript{1}Actually Batchelor, de Gier and Nienhuis take the Hamiltonian with the minus sign. In this case the eigenvector under consideration is the ground state of the model.
reflection invariance. Hence, our conjecture generalizes partially the conjecture proved by Wieland in paper \[7\] and mentioned above.

In conclusion note that we have verified our conjecture up to \( n = 7 \).

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