Research Article

On 4-colorable robust critical graphs

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Abstract

Given a proper k-coloring of a graph G, a vertex v is locally recolorable if there is a proper k-coloring of the graph that changes the color of v and limits any other color changes to the neighbors of v. The coloring is robust if every vertex is locally recolorable. The robust chromatic number of G, \( \chi_R(G) \), is the smallest number k for which G has a robust k-coloring. If \( \chi_R(G) = \chi(G) \), the graph is \( \chi \)-robust and if deleting any vertex of a \( \chi \)-robust graph decreases \( \chi_R(G) \), the graph is \( \chi \)-robust-critical. We conjecture that only complete graphs are \( \chi \)-robust-critical. This paper investigates this conjecture for \( \chi = 4 \) and supports the conjecture for a large class of such graphs. Furthermore, conditions that must be satisfied for such graphs are determined.

Keywords: robust coloring; chromatic number; \( \chi \)-robust-critical.

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1. Introduction

As introduced by Anderson, Brigham, Dutton and Vitray in 2014 [1], if c is a proper k-coloring of a graph G, a vertex v is locally recolorable with respect to c if there is a proper k-coloring \( c^v \) of G such that \( c^v(v) \neq c(v) \) and \( c^v(x) = c(x) \) for all \( x \in V(G) - N[v] \) (\( N(v) \) and \( N[v] \) are the open and closed neighborhoods, respectively, of a vertex v). The coloring \( c^v \) is called a local recoloring of v with respect to c. A proper coloring c is robust if every vertex of G is locally recolorable with respect to c, and G is k-robust if it has a robust k-coloring. The smallest k such that G has a robust k-coloring is the robust chromatic number of G, denoted \( \chi_R(G) \).

Any proper k-coloring of a graph is also a robust \((k + 1)\)-coloring of that graph, since the extra color can be used to locally recolor any vertex. Therefore,

\[
\chi(G) \leq \chi_R(G) \leq \chi(G) + 1
\]

where \( \chi(G) \) is the chromatic number of G. There are a number of results throughout graph theory where one parameter is known to be one of two consecutive numbers. Because of these inequalities, all graphs G fall into one of two classes. A graph G is \( \chi \)-robust if \( \chi_R(G) = \chi(G) \), and G is \( \chi \)-robust-critical if G is both \( \chi \)-robust and \( \chi_R(G - v) < \chi_R(G) \) for all \( v \in V(G) \). Note that, if G is \( \chi \)-robust-critical, then it is vertex \( \chi \)-critical, that is, \( \chi(G - v) < \chi(G) \) for all \( v \in V(G) \).

The complete graph \( K_n \) is \( \chi \)-robust-critical for \( n \geq 3 \), since \( \chi(K_n) = \chi_R(K_n) = n \) and \( \chi_R(K_n - v) = n - 1 \) for any vertex v. A natural and useful relation between G and its induced subgraphs is expressed in the following proposition.

**Proposition 1.1.** If H is an induced subgraph of G, then \( \chi_R(H) \leq \chi_R(G) \).

**Proof.** If H is an induced subgraph of G and c is a proper k-coloring of G, then the restriction \( c|_H \) of c to vertices in H is a proper coloring of H. Furthermore, for any vertex v of H, if \( c^v \) is a local recoloring of v with respect to c, then \( c^v|_H \) is a local recoloring of v with respect to \( c|_H \). \( \square \)

The previous proposition is not true without the word “induced.” For the graph \( K_2 \times K_3 \) in Figure 1, removing an edge can increase the robust chromatic number from three to four. This graph is vertex transitive, and, up to symmetry, has only one proper 3-coloring, illustrated as shown in Figure 1. Figure 1 also shows a local recoloring of the white vertex. So \( \chi_R(K_2 \times K_3) = 3 \). The non-induced subgraph shown in Figure 2 has two proper 3-colorings, up to symmetry, both shown. Neither is robust; the white vertex in each graph is not locally recolorable.

*This paper is dedicated to the memory of Frank Harary.
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Figure 1: Every proper 3-coloring of $K_2 \times K_3$ is robust.

Figure 2: The two proper 3-colorings of a subgraph of $K_2 \times K_3$. In each, the white vertex is not locally recolorable.

We use $K_n^-$ to denote the graph $K_n$ minus an edge. The robust chromatic number of $K_n^-$ is $n$, implying the following result.

**Proposition 1.2.** If a graph $G$ is $(n-1)$-robust, then $G$ does not contain the graph $K_n^-$ as a subgraph.

**Proof.** If $G$ contains $K_n^-$ as a subgraph, then either $K_n^-$ or $K_n$ is an induced subgraph of $G$ and so, by Proposition 1.1, $\chi_R(G) \geq n$. This leads to a result about $\chi$-robust-critical graphs.

**Corollary 1.1.** If a $\chi$-robust-critical graph $G$ with chromatic number $n$ contains $K_n^-$ as a subgraph, then $G = K_n$.

If a bipartite graph $G$ has a vertex $v$ of degree two or larger, then for any 2-coloring of $G$, no neighbor of $v$ is locally-recolorable. Therefore, the only connected graphs which have robust chromatic number 2 are $K_1$ and $K_2$. This implies there are no $\chi$-robust-critical graphs with chromatic number 2. It also implies that deleting a vertex from a $\chi$-robust-critical graph with chromatic number 3 will result in a graph with maximum degree less than or equal to 1. Since $K_3$ is the only connected, 3-chromatic graph with this property, it is the only $\chi$-robust-critical graph with chromatic number 3. Failure to find any $\chi$-robust-critical graphs other than complete graphs leads us to propose the following conjecture.

**Conjecture 1.1.** A graph is $\chi$-robust-critical if and only if it is a complete graph on 3 or more vertices.

This paper supports Conjecture 1.1 for a large collection of graphs having chromatic number 4. If $\chi(G) = 4$, $G$ is a $\chi$-robust-critical graph, and $v \in V(G)$, then $G - v$ is 3-robust. Accordingly, the next section establishes results for 3-robust graphs.

2. Graphs with robust chromatic number 3

Throughout this section, we assume that $H$ is a graph with $\chi_R(H) = 3$.

**Definition 2.1.** Let $v$ be a vertex in a triangle $T$ and $z$ an adjacent vertex not in $T$. Vertex $z$ is a sidekick of $T$ at $v$ if $v$ is the only vertex in $T$ adjacent to $z$. A set of sidekicks of $T$ is free if no two are adjacent to the same vertex in $T$.

In the graph in Figure 3, vertices $t$, $x$, $y$, and $z$ are all sidekicks of triangle $T$ at $u$, $u$, $w$, and $v$, respectively. The sets $\{x, y, z\}$, and $\{t, y, z\}$ are free, whereas $\{x, t\}$ is not. More informally, we say $y$ and $z$ are free sidekicks of $T$, whereas $x$ and $t$ are not.

**Proposition 2.1.** If $c$ is a robust 3-coloring of a triangle with two free sidekicks, then the two sidekicks must have different colors.

**Proof.** Let $c$ be a robust 3-coloring of a triangle $T$ with two free sidekicks, $x$ and $y$, as shown in Figure 4. Assume, by way of contradiction, that $c(x) = c(y)$. Since $c(v) \neq c(u)$ and $c(v) \neq c(w)$, $c(v) = c(x) = c(y)$. It follows that, $v$ is not locally recolorable, contradicting the robustness of $c$. 

\[\square\]
Proposition 2.2. If $c$ is a robust 3-coloring of $H$ and $\delta(H) \geq 3$, then
(i) any three free sidekicks of a triangle are assigned three different colors,
(ii) any set of sidekicks of a triangle at the same vertex have the same color,
(iii) no two triangles share a vertex, and
(iv) the set of sidekicks of a triangle at a single vertex forms an independent set.

Proof. (i) This is an immediate consequence of Proposition 2.1.
(ii) Let $T$ be a triangle with vertices $u, v,$ and $w$, and suppose $t$ and $x$ are sidekicks of $T$ at vertex $u$, as in Figure 3. Since $\delta(H) \geq 3$, there is at least one sidekick $y$ of $T$ at $w$ and one sidekick $z$ at $v$, and by Proposition 1.2, $y \neq z$. By (i), $t, y$ and $z$ have different colors and $x, y$ and $z$ have different colors. Therefore, since $c$ is proper 3-coloring, $c(t) = c(x)$.
(iii) Suppose $T_1$ and $T_2$ are two different triangles containing $u$. By Proposition 1.2, no subgraph of $H$ is isomorphic to $K_4^-$, and therefore, $u$ is the only vertex common to both triangles. However, the two vertices in $V(T_2) - \{u\}$ are both sidekicks of $T_1$ at $u$ and therefore, by (ii), have the same color. This contradicts that $c$ is a proper coloring.
(iv) This follows from (iii).

Due to Proposition 2.2, triangles in $H$ come in two varieties.

Definition 2.2. Suppose $c$ is a robust 3-coloring of $H$ with colors 0, 1, 2. Let $T$ be a triangle in $H$. If for every $u \in V(T)$, and every sidekick $x$ of $T$ at $u$, $c(x) = c(u) + 1 \mod 3$, we say $T$ is a + triangle. Similarly, $T$ is a − triangle when $c(x) = c(u) - 1 \mod 3$.

Proposition 2.3. If $c$ is a robust 3-coloring of $H$ and $\delta(H) \geq 3$, then
(i) every triangle in $H$ is either + or − and
(ii) any two triangles in $H$ containing adjacent vertices must be of opposite signs.

Proof. (i) Let $T$ be a triangle in $H$ with vertices $u_0, u_1, u_2$ where $c(u_i) = i$ for $i \in \{0, 1, 2\}$. Since $\delta(H) \geq 3$, all three vertices in $T$ are adjacent to some vertex not in $T$. Since $H$ does not contain $K_4^-$ as a subgraph, these vertices must be sidekicks. Suppose that $x_0$ is a sidekick of $T$ at $u_0$ with $c(x_0) = 1$. By Proposition 2.1, a sidekick of $T$ at $u_2$ cannot be colored 1, hence, it must be colored 0. This implies, by Proposition 2.2(i), that a sidekick of $T$ at $u_1$ is colored 2. By Proposition 2.2(ii), $T$ is +. Similarly, if $c(x_0) = 2$, then $T$ is −.

(ii) Let $u$ and $v$ be adjacent vertices in distinct triangles $T_u$ and $T_v$, respectively. If $T_u$ is +, then $c(v) = c(u) + 1 \mod 3$, which implies $c(u) = c(v) - 1 \mod 3$. Therefore, by (i), $T_v$ is −. Similarly, if $T_u$ is −, then $T_v$ is +.

3. The main result

We will assume throughout this section that $\chi(G) = 4$, $G$ is a $\chi$-robust-critical graph not equal to $K_4$, and all colorings are proper.
The following lemma establishes a lower bound for the minimum degree of $G$. Since $G$ is vertex \( \chi \)-critical, \( \delta(G) \geq \chi(G) - 1 = \chi(G) - 1 \). Furthermore, $G \neq K_4$ implies $G$ does not contain a subgraph isomorphic to $K_4^\ast$, by Corollary 1.1.

**Lemma 3.1.** If $G$ is a $\chi$-robust-critical graph not equal to $K_4$ and $\chi(G) = 4$, then $\delta(G) \geq 4$.

**Proof.** By the comment preceding the lemma, $\delta(G) \geq 3$. Suppose $v \in V(G)$ has degree 3. By definition of $\chi$-robust-critical, there is a robust 3-coloring of $G - v$. Since $N[v]$ cannot be $K_4^\ast$, some vertex of $N(v)$ is not adjacent to either of the others. By locally recoloring that vertex, if necessary, we can obtain a 3-coloring of $G - v$ which uses no more than 2 colors for the vertices in $N(v)$. By assigning the third color to $v$ we obtain a 3-coloring of $G$ which contradicts $\chi(G) = 4$.

**Lemma 3.2.** If $G$ is a $\chi$-robust-critical graph not equal to $K_4$ and $\chi(G) = 4$, then $|V(G)| \geq 10$.

**Proof.** The Grötzsch graph $[2]$, which has 11 vertices, is the smallest triangle-free graph with chromatic number 4; so, we may assume $G$ contains a triangle $T$ with vertices $u_0$, $u_1$, and $u_2$. By Lemma 3.1, there are two vertices $x_0$ and $y_0$ in $N(u_0) - V(T)$. Since $K_4^\ast$ is not a subgraph of $G$, $x_0$ and $y_0$ are both sidekicks of $T$ at $u_0$. Similarly, there exist sidekicks $x_1$ and $y_1$ of $T$ at $u_1$, and sidekicks $x_2$ and $y_2$ at $u_2$. Since $G - x_1$ has a robust 3-coloring, $x_0$ and $y_0$ are not adjacent, by Lemma 2.2(iv). Likewise, $x_1$ and $y_1$ are not adjacent, and $x_2$ and $y_2$ are not adjacent. There may, however, be edges between pairs of free sidekicks of $T$ (between $x_1$ and $x_2$, for example). In any event, $c(u_0) = i$ and $c(x_i) = c(y_i) = i + 1 \mod 3$ for $i \in \{0, 1, 2\}$ and $z$ is a sidekick of the subgraph $H$ induced by $N(u_0) \cup N(u_1) \cup N(u_2)$. However, if $|V(G)| \leq 9$ then $G = H$, which contradicts $\chi(G) = 4$.

We will use the results of Section 2 by deleting a vertex $z$ of $G$ to produce a subgraph with a robust 3-coloring.

**Proposition 3.1.** If $G$ is a $\chi$-robust-critical graph not equal to $K_4$ and $\chi(G) = 4$, then

(i) every vertex of $G$ is in at most one triangle, and

(ii) the set of sidekicks of a triangle at a single vertex is independent.

**Proof.** (i) Suppose vertex $u$ is on two triangles. By Lemma 3.2, there is a vertex $z$ that is not on either triangle. Since $G$ is $\chi$-robust-critical, $G - z$ has a robust 3-coloring, and by Lemma 3.1, $\delta(G - z) \geq 3$. However, $u$ is on two triangles in $G - z$, contradicting Proposition 2.2(iii).

(ii) This follows immediately from (i).

Note that if $x$ is a sidekick of a triangle, then a local recoloring of $x$ cannot alter the colors assigned to the vertices of that triangle.

**Definition 3.1.** A vertex is mono-triangular if it is in exactly one triangle and each of its neighbors is also in exactly one triangle.

When a vertex $v$ is in exactly one triangle, we use $T_v$ to designate that triangle. If $G$ has a mono-triangular vertex $v$, then by Lemma 3.1, the triangle $T_v$ has at least two sidekicks $x$ and $y$ at $v$. The next lemma shows that there is always a vertex $z$, not in any of the triangles $T_v$, $T_x$, and $T_y$, whose deletion maintains the mono-triangularity of $v$.

**Lemma 3.3.** Suppose $G$ is a $\chi$-robust-critical graph and $\chi(G) = 4$. If $v$ is a mono-triangular vertex of $G$ and $\{x, y\} \subseteq N(v) - V(T_v)$ then there exists a vertex $z \in V(G) - (V(T_v) \cup V(T_x) \cup V(T_y))$ such that $z$ is a mono-triangular vertex of $G - z$.

**Proof.** By Lemma 3.2, $|V(G)| \geq 10$, which implies $V(G) - (V(T_v) \cup V(T_x) \cup V(T_y)) \neq \emptyset$. Suppose $N[v] \subseteq V(T_v) \cup V(T_x) \cup V(T_y)$ and let $z \in V(G) - (V(T_v) \cup V(T_x) \cup V(T_y))$. Since $T_v$, $T_x$, and $T_y$ are the only triangles containing a vertex in $N[v]$, $v$ is a mono-triangular vertex of $G - z$. On the other hand, if $N[v] \nsubseteq V(T_v) \cup V(T_x) \cup V(T_y)$, there exists $z \in N(v) - (V(T_v) \cup V(T_x) \cup V(T_y))$. Since $z$ is a sidekick of $T_v$ at $v$ and, by Proposition 3.1(iii), the set of all sidekicks of $T_v$ at $v$ is independent, $z \notin V(T_v)$, for any $u \in N[v] - \{z\}$ and so $v$ is mono-triangular in $G - z$.

Using Proposition 2.3, we impose a structure on the set of triangles which contain vertices in the neighborhood of a mono-triangular vertex.

**Corollary 3.1.** Suppose $G$ is a $\chi$-robust-critical graph with $\chi(G) = 4$ and $v$ is a mono-triangular vertex of $G$. If $x$ and $y$ are distinct sidekicks of $T_v$ at $v$, then $V(T_x) \cap V(T_y) = \emptyset$ and no vertex in $T_x$ is adjacent to a vertex in $T_y$.

**Proof.** By Lemma 3.3, there exists $z \in V(G) - (V(T_v) \cup V(T_x) \cup V(T_y))$ such that $v$ is a mono-triangular vertex of $G - z$. Let $c$ be a robust 3-coloring of $G - z$. Proposition 2.2(iii), with $H = G - z$, implies $V(T_x) \cap V(T_y) = \emptyset$. Now, suppose a vertex in $T_x$ is adjacent to a vertex in $T_y$. By Proposition 2.3(ii), one of $T_x$ and $T_y$ must be $+$ and the other must be $-$. Since $x$ and $y$ are in $N(v)$, Proposition 2.3(ii) also implies $T_v$ is both $-$ and $+$, an impossibility.
If \( T_v \) is \(+\) for some 3-coloring of \( G \), then we can obtain a local recoloring of \( v \) without changing the colors of any vertices not in \( V(T_v) \) (see Figure 5) by subtracting 1 from each of the colors on the triangle. Similarly, if \( T_v \) is \(-\), we can add 1 to each color on the triangle. As indicated in the figure, these recolorings change a triangle from \(+\) to \(-\) and vice versa.

![Figure 5: Recoloring signed triangles.](image)

The colorings in Figure 5 play a key role in the proof of the next theorem.

**Theorem 3.1.** If \( G \) contains a mono-triangular vertex \( v \), then \( G \) is not \( \chi \)-robust-critical with chromatic number 4.

**Proof.** Suppose \( G \) has a mono-triangular vertex \( v \) and is \( \chi \)-robust-critical with chromatic number 4. Let \( c \) be a robust 3-coloring of \( G - v \). Let \( u \) and \( w \) be the other two vertices of \( T_v \). Without loss of generality, assume \( c(u) = 0 \) and \( c(w) = 1 \). Let \( P = \{ r \in N(v) - \{ u, w \} : c(r) = 2 \) and \( T_r \) is \(+\) \} and \( M = \{ r \in N(v) - \{ u, w \} : c(r) = 2 \) and \( T_r \) is \(-\) \}. See Figure 6(a).

We define a coloring \( c' \) of \( G \) by,

\[
c'(z) = \begin{cases} 
2 & \text{if } z = v \\
c(z) - 1 \pmod{3} & \text{if } z \in T_r, \text{ where } r \in P \\
c(z) + 1 \pmod{3} & \text{if } z \in T_r, \text{ where } r \in M \\
c(z) & \text{otherwise.}
\end{cases}
\]

See Figure 6(b). Notice \( c' \) assigns a color to \( v \) and changes the colors only of vertices in the triangles containing a vertex of \( N(v) \) colored 2 by \( c \).

![Figure 6: Constructing a 3-coloring of \( G \).](image)

We now show that \( c' \) is a proper 3-coloring of \( G \), contradicting the hypothesis that \( \chi(G) = 4 \). Suppose \( x \) and \( y \) are adjacent vertices in \( G \). All the arithmetic in the cases below is modulo 3.

Case 1. One of the vertices, say \( y \), is equal to \( v \). This implies \( x \in N(v) \). If \( c(x) = 2 \), then \( x \in P \cup M \) and \( c'(x) \neq 2 = c'(v) \). On the other hand, if \( c(x) \neq 2 \), then \( x \notin P \cup M \) and \( c'(x) = c(x) \neq 2 = c'(v) \).

Case 2. Neither \( x \) nor \( y \) is \( v \).

Subcase 2a. Neither \( x \) nor \( y \) is in \( V(T_r) \) for any \( r \in P \cup M \). By definition of \( c' \), \( c'(x) = c(x) \neq c(y) = c'(y) \).
Subcase 2b. Both \(x\) and \(y\) are in \(V(T_r)\) for some \(r \in P \cup M\). If \(r \in P\), then \(c'(x) = c(x) - 1\) and \(c'(y) = c(y) - 1\). If \(r \in M\), \(c'(x) = c(x) + 1\) and \(c'(y) = c(y) + 1\). In either case, we have \(c'(x) \neq c'(y)\), since \(c(x) \neq c(y)\).

Subcase 2c. For some \(r \in P \cup M\), \(x \in V(T_r)\), but \(y \notin V(T_r)\) (i.e., \(y\) is a sidekick of \(T_r\) at \(x\)). By Corollary 3.1, \(y \notin V(T_{r'})\) for any \(r' \in P \cup M\), and hence, \(c'(y) = c(y)\). If \(r \in P\), then \(c(y) = c(x) + 1\) and \(c'(x) = c(x) - 1\). Thus, \(c'(x) \neq c'(y)\). Similarly, \(c'(x) \neq c'(y)\) if \(r \in M\).

Since \(c'(x) \neq c'(y)\) in all cases, \(c'\) is a proper 3-coloring of \(G\), which contradicts \(G\) having chromatic number 4.

**Definition 3.2.** A graph \(G\) is called an MT-graph if every vertex in \(G\) is mono-triangular.

Theorem 3.1 shows that no MT-graph of chromatic number 4 is \(\chi\)-robust-critical. The next theorem shows that all \(\chi\)-robust-critical graphs with small maximum degree are MT-graphs.

**Theorem 3.2.** If \(G \neq K_4\) is a \(\chi\)-robust-critical graph with \(\chi(G) = 4\) and \(\Delta(G) \leq 5\), then \(G\) is an MT-graph.

**Proof.** Let \(v\) be a vertex in \(G\). By Proposition 3.1(i), \(v\) is in at most one triangle. Thus, it suffices to show that \(v\) is in at least one triangle, that is, that \(N(v)\) is not an independent set. For any proper 3-coloring of \(G - v\), every color in \(\{0, 1, 2\}\) is assigned to at least one vertex in \(N(v)\), or else the coloring could be extended to a proper 3-coloring of \(G\). Let \(c\) be a robust 3-coloring of \(G - v\). Since \(\deg(v) < 6\), there is some color \(i\) with \(|c_i \cap N(v)| = 1\). Let \(u\) be the vertex in \(N(v)\) with \(c(u) = i\). There is a local recoloring \(c^u\) of \(u\) with respect to \(c\) and some vertex \(w\) in \(N(v)\) has \(c^u(w) = i\), otherwise \(c^u\) could be extended to a 3-coloring of \(G\). Therefore, \(u\) and \(w\) are adjacent and \(N(v)\) is not independent.

Theorems 3.1 and 3.2 imply the following.

**Theorem 3.3.** If \(G\) is not \(K_4\) and \(G\) is \(\chi\)-robust-critical with chromatic number 4, then \(\Delta(G) \geq 6\).

4. Open problems

1. Prove or disprove: there are no triangle-free \(\chi\)-robust-critical graphs with chromatic number 4.
2. Prove or disprove: there are no \(\chi\)-robust-critical graphs with \(\Delta(G) \geq 6\) and chromatic number 4.
3. Prove or disprove that do or do not contain \(\chi\)-robust-critical graphs.
4. Prove or disprove: there are no \(\chi\)-robust-critical graphs with chromatic number 5 other than \(K_5\).
5. Prove or disprove Conjecture 1.1.

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