Desingularization of branch points of minimal surfaces in $\mathbb{R}^4$(II)

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Abstract

We desingularize a branch point $p$ of a minimal disk $F_0(\mathbb{D})$ in $\mathbb{R}^4$ through immersions $F_t$’s which have only transverse double points and are branched covers of the plane tangent to $F_0(\mathbb{D})$ at $p$. If $F_0$ is a topological embedding and thus defines a knot in a sphere/cylinder around the branch point, the data of the double points of the $F_t$’s give us a braid representation of this knot as a product of bands.

1 Introduction

1.1 The purpose

Minimal surfaces in $\mathbb{R}^4$ are immersed except at branch points, near which the surface is a $N$-branched covering of the tangent plane at the branch point (for some $N > 1$). In [Vi 2] we looked at a minimal map $F_0 : \mathbb{D} \rightarrow \mathbb{R}^4$ with a branch point at the origin and we described how to desingularize $F_0$ through minimal immersions $F_t$’s with only transverse double points. However, unlike $F_0$ these $F_t$’s were not branched coverings of the disk. We discuss here a desingularization through immersions which are not necessary minimal but which remain $N$-branched coverings of the disk.

If $F_0$ is a topological embedding, we recall that the intersection of $F_0(\mathbb{D})$ with a small sphere (equivalently a small cylinder) centered at the branch point defines a knot which has a representation as a $N$-braid (cf. [S-V] mimicking the construction of [Mi]). In that case, we will use a construction of Rudolph to show how the double points of the immersions $F_t$’s appear in a
band representation of this braid (i.e. an expression of the braid as a product of conjugates of braid generators and of their inverses).

1.2 The setting

We consider a branched immersion

\[ F_0 : \mathbb{D} \rightarrow \mathbb{R}^4 \cong \mathbb{C}^2 \times \mathbb{C}^2 \]

\[ F_0 : z \mapsto (z^N + h_1(z), h_2(z)) \]  \hfill (1)

where, for \( i = 1, 2 \), \( h_i : \mathbb{D} \rightarrow \mathbb{C} \) is a function with \( |h_i(z)| = o(|z|^N) \).

It is standard (cf. for example [G-O-R]) to introduce a function \( w : \mathbb{D} \rightarrow \mathbb{D} \) such that

\[ w(z)^N = z^N + h_1(z) \]  \hfill (2)

and which verifies

\[ w = z + o(|z|) \quad z = w + o(|w|) \]  \hfill (3)

Possibly after restricting ourselves to a smaller disk centered at 0, we reparametrize \( \mathbb{D} \) with \( w \) so we can rewrite \( F \) in terms of \( w \) as

\[ F_0 : w \mapsto (w^N, h(w)) \]  \hfill (4)

where \( h(w) = o(|w|^N) \).

Remark 1. Throughout this paper, we only use the fact that \( F_0 \) is a real analytic branched immersion, not that it is minimal. We could probably also do without the real analytic assumption.

1.3 The construction

For \( \lambda, \mu \) small complex numbers, we will be considering the immersions

\[ F_{\lambda, \mu} : w \mapsto (w^N, h(w) + \lambda w + \mu \bar{w}) \]  \hfill (5)

possibly adding a small correction term if necessary:

\[ F_{\lambda, \mu, \gamma} : w \mapsto (w^N, h(w) + \lambda w + \mu \bar{w} + Re(\gamma w^2)) \]  \hfill (6)

where \( \gamma \) is very small compared to \( \lambda \) and \( \mu \) and is only introduced to give more wiggle room for transversality arguments.
Remark 2. We used immersions similar to (5) in [Vi 1] where we estab-
lished a connection between the algebraic crossing number of the braid and
the normal bundle of the branched disk in an ambient 4-manifold.

The paper is devoted to proving the following:

Theorem 1. For $\lambda, \mu$ generic and small enough, $F_{\lambda, \mu}$ has a finite number of
crossing points $m_1, ..., m_n$, all transverse.
Assume that $F_0$ is a topological embedding and let $K$ be the knot defined by
the branch point. If $\frac{\mu}{\lambda}$ is small enough (resp. large enough), the knot $K$ is
represented by a $N$-braid $\beta$ which is the product of the following pieces:

1. \[
\prod_{2k, 2 \leq 2k \leq N-1} \sigma_{2k}
\] (resp. \[
\prod_{2k, 2 \leq 2k \leq N-1} \sigma_{2k}^{-1}
\])

2. \[
\prod_{2k+1, 1 \leq 2k+1 \leq N-1} \sigma_{2k+1}
\] (resp. \[
\prod_{2k+1, 1 \leq 2k+1 \leq N-1} \sigma_{2k+1}^{-1}
\])

3. for every double point $m_1, ..., m_n$ of $F_{\lambda, \mu}$, one copy of

$$b(m_i)\sigma_{k(m_i)}^{2\epsilon(m_i)} b(m_i)^{-1}$$ (7)

where

• $\epsilon(m_i)$ is the sign of the intersection point $m_i$
• $k(m_i) \in \{1, ..., N - 1\}$
• $b(m_i)$ is some element of the braid group $B_N$.

1.4 Trivial knots

It follows from the expression of the braid that, if $F_{\lambda, \mu}$ is an embedding for
$\frac{\lambda}{\mu}$ large enough or small enough, the knot $K$ is trivial. There exist branched
minimal disks with corresponding knots which are non trivial but have 4-
genus 0, for example $10_{155}$ (cf. [S-V]). For such a knot, the signed number
of double points of the $F_{\lambda, \mu}$ (for $\frac{\lambda}{\mu}$ large enough or small enough) is zero but
$F_{\lambda, \mu}$ has necessarily double points.
1.5 Sketch of the paper

We will first establish some properties of the $F_{\lambda,\mu}$'s, for generic $\lambda$, $\mu$'s; then we will construct a closed loop $\Gamma$ in the plane $\Pi_2$ generated by the first two coordinates. The braid $\beta$ considered in Th. will be defined as

$$\beta = \pi_2^{-1}(\Gamma) \cap F_{\lambda,\mu}(\mathbb{D})$$

for $\lambda, \mu$ small enough and where

$$\pi_2 : \mathbb{R}^4 \rightarrow \Pi_2$$

is the orthogonal projection.

2 The family of immersions

**Lemma 1.** For generic $\lambda$, $\mu$'s, the following is true: if $w_1, w_2 \in \mathbb{D}$ verify $w_1 \neq w_2$ and $F_{\lambda,\mu}(w_1) = F_{\lambda,\mu}(w_2)$, then the two tangent planes $dF_{\lambda,\mu}(w_1)(\mathbb{R}^2)$ and $dF_{\lambda,\mu}(w_2)(\mathbb{R}^2)$ are transverse.

**Remark 3.** Lemma does not exclude the possibility of triple points (i.e. three disks meeting at a point, every two of them transversally): we will see that later.

**Proof.** The proof is based on the Transversality Lemma. We introduce

$$\Phi : \mathbb{C} \times \mathbb{C} \times \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{R}^4 \oplus \mathbb{R}^4$$

$$\Phi(\lambda, \mu, w_1, w_2) = (F_{\lambda,\mu}(w_1), F_{\lambda,\mu}(w_2))$$

and we check that it is transverse to the diagonal $\Delta_8$ of $\mathbb{R}^4 \oplus \mathbb{R}^4$ for $w_1 \neq w_2$. We derive from the basis $(e_1, e_2, e_3, e_4)$ of $\mathbb{R}^4$ (in which (1) is written) a basis

$$(e^{(1)}_1, ..., e^{(1)}_4, e^{(2)}_1, ..., e^{(2)}_4)$$

of $\mathbb{R}^4 \oplus \mathbb{R}^4$; thus the diagonal $\Delta_8$ is generated by $(e^{(1)}_1 + e^{(2)}_1, ..., e^{(1)}_4 + e^{(2)}_4)$. A point in the preimage of $\Delta_8$ via $\Phi$ is of the form $(\lambda, \mu, w_1, w_2)$, where $w_2 = v w_1$ for a complex number $v$ verifying

$$v^N = 1.$$
We introduce real coordinates by setting
\[ \lambda = \lambda_1 + \lambda_2 \quad w_1 = x_1 + iy_1 \quad w_2 = x_2 + iy_2 \quad (11) \]
and we compute the following determinant at points \( w_1, w_2 = \nu w_1 \); the subscripts denote the components in the basis \((e_1, e_2, e_3, e_4)\):

\[
det(\frac{\partial \Phi}{\partial x_1}, \frac{\partial \Phi}{\partial x_2}, \frac{\partial \Phi}{\partial \lambda_1}, \frac{\partial \Phi}{\partial \lambda_2}, e_1^{(1)} + e_1^{(2)}, e_2^{(1)} + e_2^{(2)}, e_3^{(1)} + e_3^{(2)}, e_4^{(1)} + e_4^{(2)}) =
\]

\[
\begin{array}{ccccccc}
(\frac{\partial F}{\partial x_1})_1 & 0 & 0 & 0 & 1 & 0 & 0 \\
(\frac{\partial F}{\partial x_1})_2 & 0 & 0 & 0 & 0 & 1 & 0 \\
(\frac{\partial F}{\partial x_1})_3 & 0 & x_1 & -y_1 & 0 & 0 & 1 \\
(\frac{\partial F}{\partial x_1})_4 & 0 & y_1 & x_1 & 0 & 0 & 1 \\
0 & (\frac{\partial F}{\partial x_2})_1 & 0 & 0 & 1 & 0 & 0 \\
0 & (\frac{\partial F}{\partial x_2})_2 & 0 & 0 & 0 & 1 & 0 \\
0 & (\frac{\partial F}{\partial x_2})_3 & x_2 & -y_2 & 0 & 0 & 1 \\
0 & (\frac{\partial F}{\partial x_2})_4 & y_2 & x_2 & 0 & 0 & 1 \\
\end{array}
\]

\[
= \left[ (x_1 - x_2)^2 + (y_1 - y_2)^2 \right] \left[ -(\frac{\partial F}{\partial x_1})_1 (\frac{\partial F}{\partial x_2})_2 + (\frac{\partial F}{\partial x_2})_1 (\frac{\partial F}{\partial x_1})_2 \right] = |1 - \nu|^{-2N} \Im(\nu). \quad (12)
\]

Similarly we show

\[
det(\frac{\partial \Phi}{\partial x_1}, \frac{\partial \Phi}{\partial y_2}, \frac{\partial \Phi}{\partial \lambda_1}, \frac{\partial \Phi}{\partial \lambda_2}, e_1^{(1)} + e_1^{(2)}, e_2^{(1)} + e_2^{(2)}, e_3^{(1)} + e_3^{(2)}, e_4^{(1)} + e_4^{(2)}) =
\]

\[
= -|1 - \nu|^{-2N} \Re(\nu) \quad (13)
\]

Lemma 1 follows from (12) and (13). \qed
We denote by $\Pi_3$ the 3-plane $\Pi_3$ generated by the first 3 coordinates; we let
\[
\pi_3 : \mathbb{R}^4 \to \Pi_3
\] (14)
be the orthogonal projection and we show a lemma similar to Lemma 1 for $\pi_3 \circ F_{\lambda, \mu}$:

**Lemma 2.** For generic $\lambda, \mu$, the following is true:
if $w_1, w_2 \in D$ verify $w_1 \neq w_2$ and $(\pi_3 \circ F_{\lambda, \mu})(w_1) = (\pi_3 \circ F_{\lambda, \mu})(w_2)$, then the two tangent planes $\pi_3(dF_{\lambda, \mu}(w_1)(\mathbb{R}^2))$ and $\pi_3(dF_{\lambda, \mu}(w_2)(\mathbb{R}^2))$ are transverse.

**Proof.** Similarly to above, we introduce the map
\[
\Psi : C \times C \times D \times D \to \mathbb{R}^3 \oplus \mathbb{R}^3
\]
\[
\Psi : (\lambda, \mu, w_1, w_2) \mapsto \left( (\pi_3 \circ F_{\lambda, \mu})(w_1), (\pi_3 \circ F_{\lambda, \mu})(w_2) \right)
\] (15)
By truncating the determinants appearing in the proof of Lemma 1 we get
\[
det \left( \frac{\partial \Psi}{\partial x_1}, \frac{\partial \Psi}{\partial x_2}, e_1^{(1)} + e_2^{(1)} + e_3^{(1)} + e_3^{(2)} \right) = (x_1 - x_2)N^2|w|^{2N-2}
\]
\[
det \left( \frac{\partial \Psi}{\partial x_1}, \frac{\partial \Psi}{\partial x_2}, e_1^{(1)} + e_2^{(1)} + e_2^{(2)} + e_3^{(1)} + e_3^{(2)} \right) = -(y_1 - y_2)N^2|w|^{2N-2}
\]
Hence $\Psi$ is transverse to the diagonal $\Delta_6$ of $\mathbb{R}^3 \oplus \mathbb{R}^3$ and Lemma 2 follows. \(\square\)

Next we show that $\pi_3 \circ F_{\lambda, \mu}$ has only a finite number of triple points:

**Lemma 3.** Let
\[
\nu = e^{2\pi i}
\] (16)
and let $k, l$ be two different integers in \{1, ..., $N - 1$\}. For generic $\lambda, \mu$, there is a finite number of points $w \in D$ such that
\[
Re(\lambda w + \mu \bar{w} + h(w)) = Re(\nu^k w + \bar{\nu}^k \bar{w} + h(\nu^k w)) = Re(\nu^l w + \bar{\nu}^l \bar{w} + h(\nu^l w))
\] (17)

**Proof.** We let
\[
\psi : C \times D \to \mathbb{R}^2
\]
\[
\psi(\lambda, w) = \left( Re[\lambda(1 - \nu^k)w + \mu(1 - \bar{\nu}^k)\bar{w} + h(w) - h(\nu^k w)], Re[\lambda(1 - \nu^l)w + \mu(1 - \bar{\nu}^l)\bar{w} + h(w) - h(\nu^l w)] \right).
\]
We define
\[(1 - \nu^k)w = w_1^{(k)} + iw_2^{(k)} \quad (1 - \nu^l)w = w_1^{(l)} + iw_2^{(l)}\] (18)
and compute
\[
\det\left(\frac{\partial \psi}{\partial \lambda_1}, \frac{\partial \psi}{\partial \lambda_2}\right) = \begin{vmatrix}
w_1^{(k)} & -w_2^{(k)} \\
w_1^{(l)} & -w_2^{(l)}
\end{vmatrix} = Im[(1 - \nu^k)w(1 - \nu^l)\bar{w}]
\]
which is not zero. We use the Transversality Lemma again and conclude that for a generic \(\lambda\), \(\psi(\lambda, .)\) is transverse to \((0, 0)\), that is, \((0, 0)\) is attained at a finite number of points.

**NOTATIONS.** We remind the reader that \(\pi_2\) is the projection onto the plane \(\Pi_2\) generated by the first two coordinates.

We denote by \(W_{\lambda, \mu}\) the set of \(w\)'s in \(\mathbb{D}\backslash\{0\}\) which verify (17) for some \(k, l\) and we let
\[X_{\lambda, \mu} = (\pi_2 \circ F_{\lambda, \mu})(W_{\lambda, \mu})\] (20)

If \(D_{\lambda, \mu} \in \mathbb{R}^4\) is the set of double points of \(F_{\lambda, \mu}\), we let
\[D_{\lambda, \mu} = \pi_2(D_{\lambda, \mu})\] (21)

In the following lemma, we use \(F_{\lambda, \mu, \gamma}\) defined in (6); nevertheless we keep the notation \(D_{\lambda, \mu}\) and \(X_{\lambda, \mu}\) in order not to burden the notations.

**Lemma 4.** For generic \(\lambda\)'s, \(\mu\)'s, \(\gamma\)
\[D_{\lambda, \mu} \cap X_{\lambda, \mu} = \emptyset\]
In particular \(F_{\lambda, \mu}\) does not have triple points (cf. Remark 3).

**Proof.** We pick a very small positive number \(\eta\) (how small it need to be will be clear from the proof below) and given a 7-uple \(A = (a, b, c, d, e, \alpha, \beta) \in \mathbb{R}^7\), we define
\[H(A, w) = aRe(w) + bIm(w) + \alpha Re(w^2) + \beta Re(e^{i\eta}w^2) + i[dRe(w) + eIm(w)]\] (22)
and we define

\[ F(A, w) = F(w) + (0, H(A, w)) = (w^N, H(A, w) + h(w)) \]  \hspace{1cm} (23)

We let \( j, k, l \) be three different integers in \( \{1, \ldots, N-1\} \) and we define

\[
S(A, w) = \left( \Re(H(A, w) - H(A, e^{2\pi i/j}w)) + \Re(h(w) - h(e^{2\pi i/j}w)), \right.
\]

\[
\Im(H(A, w) - H(A, e^{2\pi i/j}w)) + \Im(h(w) - h(e^{2\pi i/j}w)),
\]

\[
\Re(H(A, e^{2\pi i/k}w) - H(A, e^{2\pi i/l}w)) + \Re(h(e^{2\pi i/k}w) - h(e^{2\pi i/l}w)) \right)
\]

We show

**Sublemma 1.** The map \( S \) is transverse to \((0,0,0)\); thus, for a generic \( A \), \((0,0,0)\) is not attained by \( S(A,.) \).

**Proof.** We set \( w = re^{i\theta} \) and we compute

\[
\det \left( \frac{\partial S}{\partial a}, \frac{\partial S}{\partial b}, \frac{\partial S}{\partial d} \right) = r^3 \left( \cos \theta - \cos \left( \theta + \frac{2j\pi}{N} \right) \right) \Delta = -2 \sin \left( \frac{j\pi}{N} \right) \sin \left( \theta + \frac{j\pi}{N} \right) r^3 \Delta
\]

\[
\det \left( \frac{\partial S}{\partial a}, \frac{\partial S}{\partial b}, \frac{\partial S}{\partial e} \right) = r^3 \left( \sin \theta - \sin \left( \theta + \frac{2j\pi}{N} \right) \right) \Delta = -2 \sin \left( \frac{j\pi}{N} \right) \cos \left( \theta + \frac{j\pi}{N} \right) r^3 \Delta
\]

where

\[
\Delta = \begin{vmatrix}
\cos \theta - \cos \left( \theta + \frac{2j\pi}{N} \right) & \sin \theta - \sin \left( \theta + \frac{2j\pi}{N} \right) \\
\cos \left( \theta + \frac{2k\pi}{N} \right) - \cos \left( \theta + \frac{2l\pi}{N} \right) & \sin \left( \theta + \frac{2k\pi}{N} \right) - \sin \left( \theta + \frac{2l\pi}{N} \right)
\end{vmatrix}
\]

\[
= \begin{vmatrix}
2 \sin \left( \theta + \frac{j\pi}{N} \right) \sin \left( \frac{j\pi}{N} \right) & 2 \cos \left( \theta + \frac{j\pi}{N} \right) \sin \left( \frac{j\pi}{N} \right) \\
-2 \sin \left( \theta + \frac{(k+l)\pi}{N} \right) \sin \left( \frac{(k-l)\pi}{N} \right) & 2 \cos \left( \theta + \frac{(k+l)\pi}{N} \right) \sin \left( \frac{(k-l)\pi}{N} \right)
\end{vmatrix}
\]

\[
= 4 \sin \left( \frac{\pi}{N}j \right) \sin \left( \frac{\pi}{N}(k-l) \right) \sin \left( \frac{\pi}{N}(j-k-l) \right) \quad (24)
\]

If (24) is zero, then

\[ j = k + l \]  \hspace{1cm} (25)
We now assume (25) and we compute

\[ \det \left( \frac{\partial S}{\partial a}, \frac{\partial S}{\partial \alpha}, \frac{\partial S}{\partial d} \right) = r^4 \left( \cos \theta - \cos \left( \theta + \frac{2j\pi}{N} \right) \right) \tilde{\Delta} = -2 \sin \left( \frac{j\pi}{N} \right) \sin \left( \theta + \frac{j\pi}{N} \right) r^4 \tilde{\Delta} \]

\[ \det \left( \frac{\partial S}{\partial a}, \frac{\partial S}{\partial \alpha}, \frac{\partial S}{\partial e} \right) = r^4 \left( \sin \theta - \sin \left( \theta + \frac{2j\pi}{N} \right) \right) \tilde{\Delta} = -2 \sin \left( \frac{j\pi}{N} \right) \cos \left( \theta + \frac{j\pi}{N} \right) r^4 \tilde{\Delta} \]

where

\[ \tilde{\Delta} = \left| \begin{array}{cc} \cos \theta - \cos \left( \theta + \frac{2j\pi}{N} \right) & \cos(2\theta) - \cos(2\theta + \frac{4j\pi}{N}) \\ \cos \left( \theta + \frac{2k\pi}{N} \right) - \cos \left( \theta + \frac{2l\pi}{N} \right) & \cos \left( 2\theta + \frac{4k\pi}{N} \right) - \cos \left( 2\theta + \frac{4l\pi}{N} \right) \end{array} \right| \]

\[ = \left| \begin{array}{cc} 2 \sin \left( \theta + \frac{j\pi}{N} \right) \sin \left( \frac{j\pi}{N} \right) & 2 \sin \left( 2\theta + \frac{2j\pi}{N} \right) \sin \left( \frac{2j\pi}{N} \right) \\ -2 \sin \left( \theta + \frac{j\pi}{N} \right) \sin \left( \frac{(k-l)\pi}{N} \right) & -2 \sin \left( 2\theta + \frac{2j\pi}{N} \right) \sin \left( \frac{2(k-l)\pi}{N} \right) \end{array} \right| \]

\[ = 4 \sin \left( \theta + \frac{j\pi}{N} \right) \sin \left( 2\theta + \frac{2j\pi}{N} \right) \left| \begin{array}{cc} \sin \left( \frac{j\pi}{N} \right) & \sin \left( \frac{2j\pi}{N} \right) \\ -\sin \left( \frac{(k-l)\pi}{N} \right) & -\sin \left( \frac{2(k-l)\pi}{N} \right) \end{array} \right| \]

\[ = -16 \sin \left( \theta + \frac{j\pi}{N} \right) \sin \left( 2\theta + \frac{2j\pi}{N} \right) \sin \left( \frac{j\pi}{N} \right) \sin \left( \frac{(k-l)\pi}{N} \right) \sin \left( \frac{l\pi}{N} \right) \sin \left( \frac{k\pi}{N} \right) \] (26)

The product (26) is not zero unless

\[ \sin \left( \theta + \frac{j\pi}{N} \right) \sin \left( 2\theta + \frac{2j\pi}{N} \right) = 0 \]

in which case we redo the above calculations replacing \( \frac{\partial}{\partial \alpha} \) by \( \frac{\partial}{\partial \beta} \) and get a non-zero determinant. This concludes the proof of Sublemma 1. \( \square \)

Given \( A \), there is a unique \((\lambda, \mu, \gamma)\) such that for every \( w \),

\[ H(A, w) = \lambda w + \mu \bar{w} + \Re \left( \gamma w^2 \right) \] (27)

Moreover the map

\[ A \mapsto (\lambda, \mu, \gamma) \]

defined by (27) is a surjective submersion. Thus Lemma 4 follows from Sublemma 1. \( \square \)

9
3 The 1-complex $A$ in $\Pi_2$

We derive from Lemma 2 that the set

$$A = \{(w_1, w_2) \in \mathbb{D} \times \mathbb{D} / w_1 \neq w_2 \text{ and } \pi_3 \circ F_{\lambda, \mu}(w_1) = \pi_3 \circ F_{\lambda, \mu}(w_2)\}$$  \hspace{1cm} (28)

is a manifold. If $(w_1, w_2) \in A$, then

$$w_1^N = w_2^N$$

and we let $A$ be the subset of $\mathbb{D} \subset \Pi_2$ consisting of the $w_i^N$'s for $(w_1, w_2)$ in $A$. Directly by hand or by standard analytic geometry arguments ($A$ is the projection of an analytic set and is of dimension 1, hence it is semi-analytic and so it is stratified, see [Lo]), we derive

**Lemma 5.** The set $A$ is a 1-submanifold of $\mathbb{D}$ with a finite set of singular points which we denote $\Sigma(A)$.

Moreover we have

**Lemma 6.** The elements of $D_{\lambda, \mu}$ (cf. 21) are regular points of $A$.

**Proof.** If $p \in D_{\lambda, \mu}$, there exists $w \in \mathbb{D}$ and a number $\nu$ with $\nu^N = 1$, $\nu \neq 1$ such that $p = w^N$ and

$$Re(\lambda w + \mu \bar{w} + h(w)) = Re(\lambda \nu w + \mu \bar{\nu} \bar{w} + h(\nu w))$$  \hspace{1cm} (29)

It follows from Lemma 4 that in a neighbourhood of $p$, $A$ identifies with

$$A_\nu = \{w^N / \pi_3(F(w)) = \pi_3(F(\nu w))\}.$$

By the transversality arguments we have been using, we see that, for $\lambda, \mu$ generic, the set of $w$'s which verify (29) is a 1-submanifold, hence $A_\nu$ is one too.

4 The loop $\Gamma$ in $\mathbb{D}$

We now construct a closed loop $\Gamma$ in $\mathbb{D}$. It stays in a small circle around the origin but leaves it to circle around the the points of $D_{\lambda, \mu}$. We require $\Gamma$ to always meet $A$ transversally. The closed loop $\Gamma$ splits $\mathbb{D}$ into two connected components, $U_0$ and $U_1$ and we require
the origin 0 and the elements of $D_{\lambda,\mu}$ are all in $U_0$

- the points of $X_{\lambda,\mu}$ (cf. 20 for the definition of $X_{\lambda,\mu}$) are all in $U_1$: this is possible since $X_{\lambda,\mu}$ and $D_{\lambda,\mu}$ do not intersect.

We can now consider three knots in cylinders, namely

\[ K = F(\mathbb{D}) \cap \pi_2^{-1}(\partial \bar{\mathbb{D}}_2) \quad K_{\lambda,\mu} = F_{\lambda,\mu}(\mathbb{D}) \cap \pi_2^{-1}(\partial \bar{\mathbb{D}}_2) \quad \hat{K}_{\lambda,\mu} = F_{\lambda,\mu}(\bar{\mathbb{D}}) \cap \pi_2^{-1}(\Gamma) \]

(30)

We claim that they are all isotopic. For $K$ and $K_{\lambda,\mu}$ to be isotopic, it is enough to take $\lambda$ and $\mu$ small enough.

Since there are no double points in $\pi_2^{-1}(U_1)$, the set $M_1 = F_{\lambda,\mu}(\mathbb{D}) \cap \pi_2^{-1}(U_1)$ is a submanifold of $\mathbb{R}^4$; moreover, if $m \in M_1$, a vector $T$ in $\Pi_2$ has a unique lift in $T_m M_1$. Thus, if we smoothly deform $\Gamma$ to $\partial \bar{\mathbb{D}}$, we can lift this deformation into an isotopy between $K_{\lambda,\mu}$ and $\hat{K}_{\lambda,\mu}$.

4.1 Construction of $\Gamma$

It is made of three pieces:

4.1.1 The circles $\Gamma_i$’s around the points in $D_{\lambda,\mu}$

We let

\[ D_{\lambda,\mu} = \{p_1, \ldots, p_n\} \]

(31)

The indexing $i$ is chosen so that

\[ \arg(p_1) \geq \arg(p_2) \geq \ldots \geq \arg(p_n) \]

(32)

For every $i = 1, \ldots, n$, the point $p_i$ is a regular point of $A$ (cf. Lemma 6) so we can pick a small circle $\Gamma_i$ in $\mathbb{D} \subset \Pi_2$ centered at $p_i$ and such that

1. the disk bounded by $\Gamma_i$ does not contain any point in $\Sigma(A)$ or a point in $D_{\lambda,\mu}$ different from $p_i$

2. $\Gamma_i$ and $A$ meet transversally at two points:

\[ \Gamma_i \cap A = \{P_i, Q_i\} \]

(33)

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4.1.2 The circle $C_\rho$ around the origin

We pick a small positive number $\rho$; we will indicate below how small we need $\rho$ to be but for the moment we only require

$$\rho < \frac{1}{2} \inf |p_i|$$

and we let $C_\rho$ be the circle in $\mathbb{D}_2$ centered at the origin and of radius $\rho$.

4.1.3 The $T_i$'s between $C_\rho$ and the $\Gamma_i$'s

We pick a point $u_i$ on $\Gamma_i$ different from $P_i, Q_i$. For every $i$, we pick a path $L_i$ between $u_i$ and $C_\rho$ and a small closed tubular neighbourhood $T_i$ of $L_i$.

We pick the $T_i$'s disjoint from one another. Moreover we require for every $i$ that

1. $T_i$ does not contain any point of $\mathcal{D}_{\lambda,\mu}$ or $\Sigma(A)$
2. $T_i \cap C_\rho \cap A = \emptyset$
3. $T_i \cap \Gamma_i$ does not contain $P_i$ and $Q_i$
4. the boundary $\partial T_i$ meets $A$ transversally.

4.1.4 Conclusion: the loop $\Gamma$ and the knot/braid $\hat{K}_\lambda$

To go along the loop $\Gamma$, we start at a point $X_0$ in $C_\rho$ which does not belong to $A$. We follow $C_\rho$ counterclockwise; everytime we meet a $\partial T_i$, we go along it until we meet $\Gamma_i$; then we follow $\Gamma_i$ till we come to the next component of $\partial T_i$ which we follow back to $C_\rho$.

5 The crossing points of $\hat{K}_\lambda$

We now write $\hat{K}_\lambda$ as a braid $\beta$.

We denote by $\Pi_{34}$ the plane in $\mathbb{R}^4$ generated by the last two coordinates. If $\gamma$ is a point of $\Gamma$, there are $N$ points $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_N$ in $\Pi_{34}$ such that for all $i$,

$$(\gamma, \tilde{\gamma}_i) \in \hat{K}_\lambda$$
A crossing point $\gamma^{(0)}$ of $K_\lambda$ is a point where the $\text{Re}(\tilde{\gamma}_i^{(0)})$’s take less than $N$ distinct values, i.e. there are two different points $(\gamma^{(0)}, \tilde{\gamma}_i^{(0)})$ and $(\gamma^{(0)}, \tilde{\gamma}_j^{(0)})$ with the same first three coordinates. In other words, a crossing point occurs when $\Gamma$ meets $A$.

To formalize this, we parametrize $\Gamma$ as

$$\gamma : [0, 2\pi] \rightarrow \mathbb{D}$$

with $\gamma(\theta_0) = \gamma^{(0)}$. We renumber the $i$’s and we reparametrize the $\tilde{\gamma}_i$’s in a neighbourhood of $\theta_0$ so that

$$\text{Re}(\tilde{\gamma}_1^{(\theta_0)}) \geq \text{Re}(\tilde{\gamma}_2^{(\theta_0)}) \geq \ldots \geq \text{Re}(\tilde{\gamma}_k^{(\theta_0)}) = \text{Re}(\tilde{\gamma}_{k+1}^{(\theta_0)}) \geq \ldots \geq \text{Re}(\tilde{\gamma}_N^{(\theta_0)})$$

This gives us the braid generator

$$\sigma_k^{\eta(\gamma_0)}$$

with $\eta(\gamma_0) \in \{ -1, +1 \}$.

Note that if there are two different integers $i, j$, with $1 \leq i, j \leq N - 1$ such that

$$\text{Re}(\tilde{\gamma}_i^{(\theta_0)}) = \text{Re}(\tilde{\gamma}_{i+1}^{(\theta_0)}) \quad \text{and} \quad \text{Re}(\tilde{\gamma}_j^{(\theta_0)}) = \text{Re}(\tilde{\gamma}_{j+1}^{(\theta_0)})$$

then $|i - j| \geq 2$, which implies that the corresponding $\sigma_i^\pm$ and $\sigma_j^\pm$ commute; thus it does not matter in which order we write them in the expression of $\beta$.

The sign $\eta(\gamma_0)$ of the crossing point in (36) is the sign of

$$[\text{Im}(\tilde{\gamma}_{k+1}^{(\theta_0)}) - \text{Im}(\tilde{\gamma}_k^{(\theta_0)})][\text{Re}(\tilde{\gamma}_k^{(\theta_0)}) - \text{Re}(\tilde{\gamma}_{k+1}^{(\theta_0)})]$$

(37)

This sign is well-defined: the first factor in (37) is non zero, otherwise we would have a double point of $F_{\lambda, \mu}$ and we have assumed that none of the double points of $F_{\lambda, \mu}$ project to a point in $\Gamma$.

Let us see why the second factor of (37) is non-zero. The planes $\pi_3(T(\gamma(\theta_0), \tilde{\gamma}_k^{(\theta_0)})F_{\lambda, \mu}(\mathbb{D}))$ and $\pi_3(T(\gamma(\theta_0), \tilde{\gamma}_{k+1}^{(\theta_0)})F_{\lambda, \mu}(\mathbb{D}))$ are transverse (see Lemma 2) so they intersect in a line generated by a vector $X$ which projects to a vector tangent to $A$. The vector $\left( \gamma'(\theta_0), \text{Re}(\tilde{\gamma}_k^{(\theta_0)}) \right)$ - resp. $\left( \gamma'(\theta_0), \text{Re}(\tilde{\gamma}_{k+1}^{(\theta_0)}) \right)$ - completes $X$ in a basis of $\pi_3(T(\gamma(\theta_0), \tilde{\gamma}_k^{(\theta_0)}))$ - resp. $\pi_3(T(\gamma(\theta_0), \tilde{\gamma}_{k+1}^{(\theta_0)}))$. It follows that

$$\text{Re}(\tilde{\gamma}_k^{(\theta_0)}) \neq \text{Re}(\tilde{\gamma}_{k+1}^{(\theta_0)})$$

and the sign (37) is well-defined.

We now examine the three types of crossing points.
5.0.5 On the circle $C_\rho$

We first investigate the crossing points of the braid

$$w \mapsto (w^N, \lambda w) \quad \text{(resp.} \quad w \mapsto (w^N, \mu \bar{w})) \quad (38)$$

Without loss of generality, we assume that $\lambda$ and $\mu$ are real so the crossing points of the braids are given by

$$\cos \frac{2\pi}{N} (\theta + k) = \cos \frac{2\pi}{N} (\theta + l) \quad (39)$$

for $k, l \in \{1, \ldots, N-1\}$ and $\theta \in [\zeta, 1 + \zeta]$, where $\zeta$ is a small positive number which we introduce to avoid crossing points at the endpoints of the interval. We get two values of $\theta$ for (39), namely

$$\theta_1 = \frac{1}{2}, \quad \theta_2 = 1.$$

The integers $k, l$ appearing in (39) verify $k + l = N - 1$ (resp. $k + l = N - 2$) for $\theta_1$ (resp. $\theta_2$). The corresponding values for $\cos \frac{2\pi}{N} (\theta + k)$ are

the $\cos(2s+1)\frac{\pi}{N}$'s with $0 \leq 2s \leq N - 2$

(resp. the $\cos(2s+2)\frac{\pi}{N}$'s with $0 \leq 2s \leq N - 3$)

Thus the $\cos \frac{2\pi}{N} (\theta + k)$'s go through the values of $\cos \frac{\pi}{N} m$ with $1 \leq m \leq N - 1$.

We conclude: the crossing points above $\theta_1$ (resp. $\theta_2$) correspond to the braid generators $\sigma_{\pm 1}^{2k+1}$, $1 \leq 2k + 1 \leq N - 1$ (resp. $\sigma_{\pm 1}^{2k}$, $1 \leq 2k \leq N - 1$).

It follows from (37) that a crossing point $(\theta_1, \theta_2)$ of $w \mapsto (w^N, \lambda w)$ (resp. $w \mapsto (w^N, \mu \bar{w})$) is of the same sign as

$$(\sin \theta_1 - \sin \theta_2)^2 \quad \text{(resp.} \quad - (\sin \theta_1 - \sin \theta_2)^2)$$

hence they are all positive (resp. all negative).

Unlike for the braids (38) the crossing points of $\beta$ on $C_\rho$ will not all occur above the same two points of $C_\rho$; however, if $\rho$ is small enough and $\lambda \mu$ is large enough or small enough, the pieces in $\beta$ corresponding to the crossing points of $C_\rho$ are given by the braids (38): if that is, the crossing points of $\hat{K}$ on $C_\rho$ translate into the two pieces of $\beta$ described in 1. and 2. of Th. 1.
5.0.6 On \( \Gamma_i \)

We recall that \( m_i \) is the double point of \( F_{\lambda,\mu} \) which projects to the center of \( \Gamma_i \). There exist \( w_1, w_2 \in \mathbb{D}, w_1 \neq w_2 \) with

\[
F_{\lambda,\mu}(w_1) = F_{\lambda,\mu}(w_2) = m_i.
\]

We pick a neighbourhood \( V_1 \) of \( w_1 \) (resp. \( V_2 \) of \( w_2 \)) in \( \mathbb{D} \). We know that \( \pi_3(F_{\lambda,\mu}(V_1)) \) and \( \pi_3(F_{\lambda,\mu}(V_2)) \) intersect transversally; the curve \( \pi_3(F_{\lambda,\mu}(V_1)) \cap \pi_3(F_{\lambda,\mu}(V_2)) \) projects to \( A \) on \( \Pi_2 \). We also know that \( \Gamma_i \) meets \( A \) exactly at two points \( P_i, Q_i \) (cf. §4.1.1): the preimages of \( P_i \) and \( Q_i \) on \( \pi_3(F_{\lambda,\mu}(V_1)) \cap \pi_3(F_{\lambda,\mu}(V_2)) \) give us two braid generators \( \sigma^\pm_k \) (with the same \( k \)).

We know from Lemma 4 that these are the only braid generators corresponding to the crossing points \( P_i \) and \( Q_i \); to get a complete picture of the braid above \( \Gamma_i \), we just need to figure out the sign of each of the two \( \sigma^\pm_k \)'s:

**Lemma 7.** Let \( Q \) be one of the crossing points of \( \beta \) on the circle \( \Gamma_i \). Let \( m_i \in \mathbb{R}^4 \) be the double point of \( F_{\lambda,\mu} \) which projects to \( p_i \). The sign of the crossing point \( Q \) is equal to the sign of the double point \( m_i \) as a double point of \( F_{\lambda,\mu} \).

**Proof.** We let \( T_0 \) and \( T_1 \) be the two tangent planes to \( F_{\lambda,\mu}(\mathbb{D}) \) at \( m_i \) and we construct positive bases of \( \mathbb{R}^4, T_0 \) and \( T_1 \).

Since the planes \( T_0 \) and \( T_1 \) intersect transversally, the planes \( \pi_3(T_0) \) and \( \pi_3(T_1) \) also intersect transversally. We let \( U \) be a vector in \( \Pi_3 \) generating \( \pi_3(T_0) \cap \pi_3(T_1) \); since \( \pi_2 \circ F_{\lambda,\mu}(w) = w^N, \pi_2 \circ F_{\lambda,\mu} \) is a local immersion outside of 0 and \( U \) projects to a non-zero vector \( u \) in \( \Pi_2 \) which is tangent to \( A \).

We let \( v \) be a vector tangent to \( \Gamma \) at \( Q \) oriented in the direction of \( \Gamma \); possibly after changing \( u \) in \( -u \), \((u,v)\) is a positive basis of \( \Pi_2 \) (we remind the reader that we have assumed that \( A \) and \( \Gamma \) meet transversally).

Because \( \pi_2 \circ F_{\lambda,\mu} \) is a local immersion outside of 0, there exists a unique \( u_i \in P_i \) and a unique \( v_i \in P_i \) with

\[
\pi_2(u_i) = u \quad \pi_2(v_i) = v
\]

Moreover \( \pi_2 \circ F_{\lambda} \) preserves the orientation, hence the basis \((u_0,v_0)\) (resp. \((u_1,v_1)\)) is a positive basis of \( T_0 \) (resp. \( T_1 \)).
We let \((e_1, e_2, e_3, e_4)\) be an orthonormal positive basis of \(\mathbb{R}^4\) with \(e_1, e_2\) in \(\Pi_2\) and we define another positive basis of \(\mathbb{R}^4\) by

\[
\mathcal{B} = (u, v, e_3, e_4).
\]

We write the coordinates in \(\mathcal{B}\) of the vectors in the bases base vectors of \(T_0\) and \(T_1\), namely

\[
u_0 = (1, 0, \alpha, \gamma) \quad v_0 = (0, 1, \beta, \delta)
\]

\[
u_1 = (1, 0, \alpha', \gamma') \quad v_1 = (0, 1, \beta', \delta')
\]

and we compute the determinant

\[
det(u_0, v_0, u_1, v_1) = \begin{vmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\alpha & \beta & \alpha' & \beta' \\
\gamma & \delta & \gamma' & \delta'
\end{vmatrix} = - (\beta - \beta') (\gamma - \gamma')
\]

We now recover from (41) the sign of the crossing points of the braid given by (37).

For \(i = 0, 1\), we let \(V_i\) be a small disk in \(F_{\lambda,\mu}(\mathbb{D})\) tangent to \(T_i\). We denote again the two strands which meet at the crossing point by the coordinates in \(\mathbb{C} \oplus \mathbb{C}\): \((\gamma(\theta), \tilde{\gamma}_k(\theta))\) and \((\gamma(\theta), \tilde{\gamma}_{k+1}(\theta))\). Locally, one is the lift of \(\Gamma\) to \(V_0\) and the other one is the lift of \(\Gamma\) to \(V_1\).

Since the \(v_i\)'s both project to \(v\), we derive that \(v_0\) (resp. \(v_1\)) is the vector tangent to \(F_{\lambda,\mu} \cap \pi_2^{-1}(\Gamma)\) above \(Q\) on \(V_0\) (resp. \(V_1\)). Hence

\[
Re(\tilde{\gamma}'_k(\theta_0)) - Re(\tilde{\gamma}'_{k+1}(\theta_0))
\]

has the same sign as \(\beta - \beta'\).

We now use the fact that \(Q\) belongs to \(A\). Since \(p_i\) is a regular point of \(A\), \(A\) is parametrized near \(q_i\) by

\[
a : t \mapsto q_i + tu + o(t^2)
\]
Since \((u, v)\) is a positive basis of \(\Pi_2\) and \(v\) is tangent to \(\Gamma\) at \(Q\), the point \(Q\) is on the side of the positive \(t\)'s in \([42]\). The lift of \(A\) to \(V_0\) (resp. \(V_1\)) is parametrized by

\[
\tilde{a}_0(t) = m_i + tu_0 + o(t^2) \quad \tilde{a}_1(t) = m_i + tu_1 + o(t^2)
\]

Thus, if we have taken \(\Gamma_i\) small enough, \(\text{Im}\left(\tilde{\gamma}_k(\theta_0)\right) - \text{Im}\left(\tilde{\gamma}_{k+1}(\theta_0)\right)\) is of the same sign as \((u_0)_4 - (u_1)_4 = \gamma - \gamma'\).

Thus, the circle \(\Gamma_i\) contributes \(\sigma_k^{2\epsilon(Q)}\) to the braid.

5.0.7 On \(\partial T_i\)

We proceed as in [Ru 1].
If \(T_i\) is a small enough neighbourhood, the map \(F_{\lambda,\mu} : F_{\lambda,\mu}^{-1}(T_i) \to T_i\) is a covering, hence \(F_{\lambda,\mu}^{-1}(T_i)\) is a disjoint union of \(N\) copies of \(L_i \times [-\eta, +\eta]\) for a small \(\eta > 0\).
If \(q_0\) is a point in \(L_i \cap A\), there are two points \(q_1\) and \(q_2\) close to \(q_0\) in \(T_i \cap A\), one in each component of \(T_i\). If the \(k\)-th and \((k + 1)\)-th leaf of \(\pi_3 \circ F_{\lambda,\mu}(D)\) coincide above \(q_0\), the same is true for \(q_1\) and \(q_2\). Hence \(q_1\) and \(q_2\) each give us a braid generator \(\sigma_k^\pm\) for \(\beta\).
These two \(\sigma_k^\pm\)'s have opposite signs. Indeed, if we look at formula \([37]\), the factors \(\text{Im}\left(\tilde{\gamma}_{k+1}(\theta_0)\right) - \text{Im}\left(\tilde{\gamma}_k(\theta_0)\right)\) take the same sign for both \(q_1\) and \(q_2\), whereas the factors \(\text{Re}\left(\tilde{\gamma}_k'(\theta_0)\right) - \text{Re}\left(\tilde{\gamma}_{k+1}'(\theta_0)\right)\) take opposite signs.

Putting all the \(\sigma_{k}^{\pm1}\)'s together, we get an element \(b_i \in B_N\) such that the piece of the braid which consists in going along \(T_i\), around \(\Gamma_i\) and back along \(T_i\) can be written as

\[
b_i \sigma_k^{2\epsilon(Q)} b_i^{-1}
\]

where \(k(i)\) is an integer in \(\{1, \ldots, N - 1\}\) and \(\epsilon(Q)\) is the sign of the crossing point.
We get the terms in the braid of Th. 13 and the proof of Th. 1 is completed.

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