Topological quantum optical states in quasiperiodic cold atomic chains

B. X. Wang and C. Y. Zhao

Institute of Engineering Thermophysics, School of Mechanical Engineering, Shanghai Jiao Tong University, Shanghai, 200240, China

(Dated: January 16, 2020)

Topological quantum optical states in one-dimensional (1D) quasiperiodic cold atomic chains are studied in this work. We propose that by introducing incommensurate modulations on the interatomic distances of 1D periodic atomic chains, the off-diagonal Aubry-André-Harper (AAH) model can be mimicked, although the crucial difference is the existence of long-range dipole-dipole interactions. The discrete band structures with respect to the modulation phase, playing the role of a dimension extension parameter, are calculated for finite chains beyond the nearest-neighbor approximation. It is found that the present system indeed supports nontrivial topological states localized over the boundaries. Despite the long-range dipole-dipole interactions that lead to an asymmetric band structure, it is demonstrated that the present system inherits the topological properties of two-dimensional integer quantum Hall systems. The spectral position, for both real and imaginary frequencies, and number of these topologically protected edge states are still governed by the gap labeling theorem and characterized by the topological invariant, namely, the (first) Chern number, indicating the validity of bulk-boundary correspondence. Due to the fractal spectrum arising from quasiperiodicity, the present system provides a large number of topological gaps and quantum optical states readily for practical use. It is also revealed that a substantial proportion of the topological edge states are highly subradiant with extremely low decay rates, which therefore provide an appealing route to control single atom emission and achieve high-fidelity quantum state storage.

I. INTRODUCTION

Topological phases of matter have received a great deal of attention in recent years, since they can support unidirectional edge states that are immune against backscattering from disorder and imperfections. They have been demonstrated for electronic [1], optical [2], acoustic [3], cold atomic [4] and mechanical [5] systems. Among them, topological photonics is one of the most fast-growing subfields in achieving the analog of topological phases of matter [6–8]. Specially designed topological photonic systems are able create topologically protected optical states [7, 8], which show promising applications in novel photonic devices, such as unidirectional waveguides [9], optical isolators [10, 11], and topological lasers [12], etc.

Conventional topological optical states are usually created by using dielectric and metallic materials with artificially carved micro/nanostructures [2]. On the other hand, topological optical states can also be engineered in ultracold atom settings by utilizing the state-of-art versatile control of light-atom interactions [13], especially using the cooperative quantum optical states in cold atom arrays [14–17]. Current laser cooling and trapping technologies allow the creation of almost arbitrary geometries of atom arrays in a relatively large scale [18–20]. On this basis, topological quantum optical states in cold atomic arrays loaded in optical lattices have been theoretically proposed recently [21–24]. The advantage is that the quantum nature of these topological optical states is promising for high-fidelity quantum state transfer and quantum information storage [25, 26]. In addition, the intrinsic optical nonlinearity of ultracold atoms can be exploited to induce strong photon-photon interactions and thus provides a route to achieve many-body topological photonic states such as the fractional quantum Hall effect for photons [27–30].

By now topological quantum optical states are mainly studied for one- (1D) and two-dimensional (2D) cold atomic arrays by mimicking the 1D Su-Schrieffer-Heeger (SSH) model [24] and 2D quantum Hall system [22, 23], which are all periodic structures. On the other hand, it is known that quasicrystals, an intermediate phase with long-range order between periodic and fully disordered lattices, can also exhibit nontrivial topological properties [31–39]. A paradigmatic example of quasiperiodic lattices is the Aubry-André-Harper (AAH) model [36–40], which is a 1D tight-binding lattice model with on-site (namely, diagonal) or/and hopping terms (off-diagonal) being cosine modulated. When the cosine modulation is incommensurate (commensurate) with the lattice, this system becomes quasiperiodic (periodic). Due to this modulation, the AAH model interestingly possesses nontrivial topological properties that can be mapped to the 2D quantum Hall system (namely, the Harper-Hofstadter model in square lattice), without the need to apply a magnetic field [2]. In particular, although the system is 1D, the modulation phase shift \( \phi \) plays the role of momentum in a perpendicular synthetic dimension, leading to a dimensional extension [36, 37]. A notable example is the realization of the Hofstadter butterfly in the 1D AAH lattice by varying the modulation periodicity [41–43]. Therefore, this model provides a playground to study profound quantum topological phase transitions and topological states.

In this work, we show that topological quantum optical states can be realized in 1D quasiperiodic cold atomic chains by introducing incommensurate modulations on the interatomic distances, as an extension of the off-diagonal AAH model. We calculate the discrete band structures with respect to the modulation phase \( \phi \) beyond the nearest-neighbor approximation, since the Hamiltonian of the present sys-

* changying.zhao@sjtu.edu.cn
tem demonstrates substantial long-range dipole-dipole interactions, vastly from the conventional AAH model. In spite of this significant difference, we find the present system still inherits the topological properties of 2D integer quantum Hall systems, and the spectral position (for both real and imaginary frequencies) and number of these topologically protected edge states are actually governed by the gap labeling theorem and characterized by the topological invariant, namely, the Chern number, indicating the validity of bulk-boundary correspondence. Due to the fractal nature of the spectrum, the present system provides a large number of topological gaps and quantum optical states for practical use. Moreover, by investigating the imaginary parts (decay rates of eigenstates) of the band structure, we reveal a substantial proportion of the topological edge states are highly subradiant, which therefore provide an appealing route to control single atom emission and achieve high-fidelity quantum information storage. We expect the present proposal will offer possibilities for engineering quantum states of light and matter.

II. MODEL

Consider a one-dimensional (1D) quasiperiodic chain composed of two-level ultracold atoms aligned along the $x$-axis. The cold atoms are assumed to be well-trapped in their positions upon interaction with photons, and the tunneling of atoms between sites is negligible [13, 21, 44]. The quasiperiodicity is introduced via incommensurate modulations of the spacings between cold atoms with the distance between adjacent atoms given by

$$x_{n+1} - x_n = d[1 + \eta \cos(2\pi \beta n + \phi)],$$

where $x_n$ denotes the position of the $n$-th atom, $d$ introduces the on-average interatomic distance (or the lattice constant before modulation), $\eta$ determines the amplitude of modulation, $\beta$ is an irrational number that controls the quasiperiodicity of the modulation and $\phi$ stands for the modulation phase that corresponds to the momentum in a synthetic orthogonal dimension that will be discussed below.

The two-level atom, for simplicity, is assumed to have three degenerate excited states denoted by $|e_\alpha\rangle$ polarized along different directions, where $\alpha = x, y, z$ stands for the Cartesian coordinates, with a ground state denoted by $|g\rangle$. By applying the single excitation approximation (which is valid for sufficiently weakly excited system) [45–47], we can work in the subspace spanned by the ground states $|G\rangle \equiv |g...g\rangle$ and the single excited states $|i\rangle \equiv |g...e_\alpha...g\rangle$ of the atoms [45–47]. Moreover, by adiabatically eliminating the photonic degrees of freedom in the reservoir (i.e., the quantized electromagnetic field), we obtain the effective Hamiltonian describing light-atom interaction in the absence of any external driving field as [16, 17, 21, 23, 45–49]

$$H = \frac{\hbar}{2N} \sum_{i=1}^{N} \sum_{\alpha=x,y,z} (\omega_0 - i\gamma/2)|e_{i,\alpha}\rangle\langle e_{i,\alpha}| + \frac{3\pi\hbar c}{\omega_0} \sum_{i=1,i\neq j}^{N} \sum_{\alpha,\beta=x,y,z} G_{\alpha\beta}(r_{ij})|e_{i,\alpha}\rangle\langle e_{j,\beta}|,$$

(2)

which acts on the single excited states of the atoms. Here $\hbar$ is the Planck’s constant, $\omega_0$ is angular frequency of the dipole transition from $|g\rangle$ to $|e\rangle$ in a single atom in free space with a radiative linewidth of $\gamma$, and $c$ is the speed of light in vacuum. $G_{\alpha\beta}(r_{ij}, r_{ij})$ is the free-space dyadic Green’s function describing the propagation of photons emitting from the $i$-th atom to $j$-th atom, where $r_{ij}$ and $r_{ij}$ indicate their positions [47, 50]:

$$G_{\alpha\beta}(r_{ij}, r_{ij}) = -\frac{\exp(ikr)}{4\pi r} \left[ (1 + \frac{i}{kr} - \frac{1}{(kr)^2}) \delta_{\alpha\beta} + \frac{3i}{kr} + \frac{3}{(kr)^2} \right] r_{ij}$$

(3)

where $k = \omega_0/c$ is the wavenumber in vacuum, $r = |r|$, $r = r_{ij} - r_{ij}$, and $r_{ij} = r_{ij}/r$.

In this low-excitation picture of light-atom interactions, since the quasiperiodic modulation of interatomic distances in Eq.(1) leads to quasiperiodic “hopping” amplitudes of excited states along the atom chain, the present Hamiltonian is quite similar to the conventional AAH model with off-diagonal modulations (i.e., modulations over the inter-site hopping amplitudes for electrons). However, due to the long-range photon-mediated dipole-dipole interactions between the excited states which can induce long-range hoppings (see the algebraic decayings interactions reflected in the Green’s function), as well as the retardation effect of electromagnetic fields (see the $\exp(ikr)$ factor in the Green’s function), the present system exhibits more complexities than the conventional off-diagonal AAH model with only nearest-neighbor hoppings, as will be seen below.

The single excitation eigenstates of the 1D atomic chains can be classified into two categories according to the polarization directions of the excited states of atoms [51]: transverse eigenstates if atoms are excited to the $|e_y\rangle$ or $|e_z\rangle$ states, and longitudinal ones if atoms are excited to $|e_x\rangle$ states. For the present system without periodicity, Bloch theorem is not applicable and the photonic band structure can only be calculated directly for a finite-sized chain. This can be done by calculating the eigenstates of the Hamiltonian in Eq.(2) with respect to the wave function constructed as a linear combination of single-atom excited states, here taking longitudinal eigenstates as an example,

$$|\psi\rangle = \sum_{j=1}^{N} p_j |e_{j,x}\rangle,$$

(4)

where $p_j$ is the expansion coefficient denoting the probability amplitude of each excited state $|e_{j,x}\rangle$ [45–47, 51], whose physical significance in classical electrodynamics is the (normalized) dipole moment of the $j$-th atom. This equation specifies a set of solutions in the form $E = \omega - i\Gamma/2$ ($\Gamma > 0$)
in the lower complex plane denoting to the eigenstates of the dimerized chain (Here we use the $e^{-iωt}$ convention for harmonic oscillations). ω amounts to the angular frequency of an eigenstate while Γ refers to its radiative linewidth (decay rate), where the corresponding right eigenvector $|p_j⟩ = [p_1, p_2, ..., p_N]$ then indicates the dipole moment distribution of an eigenstate in a classical interpretation. Moreover, to facilitate the analysis, we adopt the inverse participation ratio (IPR) of an eigenstate from its eigenvector as [52]

$$\text{IPR} = \frac{\sum_{j=1}^{N} |p_j|^4}{\sum_{j=1}^{N} |p_j|^2}^2. \quad (5)$$

The IPR can be used to indicate the spatial confinement (localization degree) of an eigenstate [52, 53]. For instance, for an IPR approaches $1/M$, where $M$ is an integer, the corresponding eigenstate involves the excitation of $M$ atoms [52, 53].

### III. BAND STRUCTURES AND TOPOLOGICAL EDGE STATES

In this paper, we choose a quasiperiodicity $\beta = (\sqrt{5} - 1)/2$ that is most commonly investigated in the conventional AAH model. Similar to the conventional AAH model, the modulation phase $\phi$ here can be regarded as playing the role of momentum in a perpendicular synthetic dimension, and it is therefore straightforward to demonstrate the calculated eigenstate spectra (band structures) for the present system as a function of the modulation phase $\phi$ varying in the range from 0 to $2\pi$.

![FIG. 1. Longitudinal band structures of quasiperiodic lattices with $\beta = (\sqrt{5} - 1)/2$ and $d = 0.1λ_0$. (a) $\eta = 0.1$ with 1000 atoms. (b) $\eta = 0.3$ with 1000 atoms. (c) $\eta = 0.5$ with 100 atoms. (d) $\eta = 0.3$ with 1001 atoms.](image)

In Figs. 1a to 1c, the longitudinal band structures of a lattice with $N = 1000$ atoms and $d = 0.1λ_0$ under different modulation amplitudes of $\eta = 0.1, 0.3, 0.5$ are presented, respectively, where the color of the eigenstates indicates the value of eigenstate IPR. Owing to the irrational nature of the interatomic distance modulation, the band structures thus break into a set of fractal bands and gaps (more precisely, for an infinitely long chain, the spectrum constitutes a Cantor set [54]), with several main gaps clearly visible, while many minigaps in the spectra can only be seen in an enlarged figure. With the increase of the modulation amplitude $\eta$, the main gaps become wider because the near-field dipole-dipole interactions between atoms can give rise to very strong frequency shift thus open larger band gaps [17].

It is clearly observed that for all modulation amplitudes, there are midgap states residing in the band gaps. The midgap states in the two main gaps (e.g., the band gaps covering $2.8 \lesssim \Delta/\gamma \lesssim 13$ and $-12 \lesssim \Delta/\gamma \lesssim 0.8$ in Fig.1b) are highly localized as indicated by their large IPR values. In fact, these states are topologically protected edge states described by a nonzero Chern number, similar to the behavior of the conventional AAH model, as will be explained in Section IV. It should be noted that, although the IPRs of the midgap states in the minigaps (e.g., the band gaps covering $-17 \lesssim \Delta/\gamma \lesssim -13$) are not significantly large, these midgap states are still highly localized over the boundaries and topologically protected, as will also be discussed below. Moreover, it is found that with the increase of modulation amplitude, the IPRs of the eigenstates within the bulk bands are also increased which can approach 0.5 or higher (especially for the case of $\eta = 0.5$ where most of bulk eigenstates are localized), resulting in highly localized bulk states. This phenomenon is a consequence of localization transition, a similar behavior to the conventional off-diagonal AAH model at large modulations [38, 55]. In addition, another feature to note is the even-odd effect presented in Fig.1 for a lattice containing 1001 atoms, which shows a distinct distribution of midgap edge states as a result of the sublattice symmetry of the off-diagonal AAH model [56, 57].

In Fig.2, more details on the midgap states are given. The eigenstate spectrum at $\phi = 1.1\pi$ that contains a pair of midgap states in both main gaps is shown in Fig.2a, where the two main band gaps along with several minigaps are more clearly observed. The state number is assigned according to the detuning of an eigenstate. In each of the two main gaps, there are two highly localized midgap states, and the state numbers are denoted by 382, 393, 618 and 619 respectively. The excitation probability amplitude distributions of these midgap states $p_j$ (or classically, dipole moment distributions) are presented in Fig.2b, which show that states No. 382 and No. 619 are highly localized over the left edge while states No. 383 and No. 618 are localized over the right edge. As a comparison, the dipole moment distributions of states No. 794 and 900 located in upper bulk bands are also given, implying that these eigenstates are localized in the bulk as a result of localization transition. Furthermore, the eigenstate spectra at the $\phi = 0.4\pi$ and $\phi = 1.6\pi$ are presented in Figs.2c and 2e, which both consist of only one midgap state in the main gaps. For the $\phi = 0.4\pi$ case, this midgap state localizes over the left edge while for the $\phi = 1.6\pi$ case, the midgap state is bound to the right edge. In fact, by further investigating the eigenstate spectra at different modulation phases, it can be found that in each of the two main gaps, by varying the modulation phase $\phi$, the midgap edge states keep localized over the same edge as
long as they remain in the band gap. The midgap states localized over the same edge in the same band gap therefore belong to the same mode when considering φ plays the role of an additional momentum. Therefore, for the two main gaps, there are two edge modes traversing the spectral gap, one localized over the left edge and the other localized over the right edge. This property is a manifestation of the topological nature of the band gaps [37], as will be discussed below. In addition, it is noted that despite the even-odd effect for finite-size lattices, there is always one edge mode in the band gap that does not vary with the number of atoms [38].

In a similar manner, we further calculate the band structures of transverse eigenstates shown in Fig. 3 with the same parameters with the longitudinal case. The band gaps in the transverse band structures are substantially narrower as a consequence of weaker dipole-dipole interactions, which involve a long-range interaction term which slowly decays with the distance r as 1/r. Such long-range interactions can result in long-range hoppings of excited states and thus reduce near-field interactions and then the band gap width. In spite of these differences, the qualitative behavior is still quite similar to the conventional AAH model, including the fractal bands and gaps, midgap states and localization transition at large modulations, as in longitudinal band structures.

In Fig. 4, transverse eigenstate spectra for several typical modulation phases are also presented. Firstly, Fig. 4a shows the eigenstate spectrum at φ = 1.1π, along with the dipole moment distributions of midgap edge states with state numbers 382, 618 and 619 plotted in Fig. 4b, in which states No. 104 and 172 are localized bulk states. The eigenstate spectra at φ = 0.6π and φ = 1.4π are presented in Figs. 4c and 4d, both of which contain one midgap state in each of the two main gaps. The two midgap gap states in the two main gaps are localized over the left and right edges, respectively, as indicated by Figs. 4d and 4f. Note the state numbers of the two midgap states under the two different modulation phases are the same, i.e., 382 and 618. This is indeed a signal of topological protection, as will be discussed below.

IV. TOPOLOGICAL INVARIANTS

Above results indicate that in spite of the long-range nature of dipole-dipole interactions beyond the nearest-neighbor approximation, the features of band structures of the present quasiperiodic lattice are largely similar to the conventional AAH model. And we have noticed several phenomena that hint the topological nature of the lattice, including the highly localized midgap edge states, quantized number of edge modes in the band gaps, fixed state number of midgap states. In this section, we continue to a theoretical description of the band topology and demonstrate the validity of the bulk-boundary correspondence, in order to verify these midgap edge states are indeed topologically protected.

As mentioned before, in the conventional AAH model, the modulation phase φ plays the role of momentum in a perpendicular synthetic dimension. On this basis, it is shown that this model can be mapped to the 2D Harper-Hofstadter model in the presence of a perpendicular magnetic field (the discrete lattice version of the Landau level problem) [2, 36, 37]. To be more precisely, in such mapping, the modulation (quasi)periodicity β actually corresponds to the magnetic flux quanta per unit cell and an irrational β stands...
for an incommensurate magnetic field. The AAH Hamiltonian \( H(\phi) \) at a specific \( \phi \) can be regarded as the \( k \)-th Fourier component of the 2D Harper-Hofstadter Hamiltonian, where \( k = \phi/\alpha \) and \( \alpha \) is the lattice constant in the synthetic dimension \([37, 59]\). Therefore the conventional AAH model inherits nontrivial topological properties from a 2D quantum Hall system, and the midgap edges states are thus of topological origin from the robust chiral states in 2D integer quantum Hall effect (IQHE), without the need of a real magnetic flux and the breaking of the time-reversal symmetry.

As a result, the band topology of the conventional AAH model can be well characterized by an integer known as the Chern number like that in 2D IQHE systems even when the modulation is incommensurate with the lattice, which satisfies the following Diophantine equation \([41, 60–62]\):

\[
\mathcal{N} = \mu + \nu \beta, 
\]

in which \( \mu \) is an integer, \( \nu \) is the Chern number of the band gap and \( \mathcal{N} \) is the normalized integrated density of states (IDOS) in a band gap. It is a general result following from magnetic translational symmetry \([62]\). It is shown that for an irrational \( \beta \), this equation has only one solution \((\mu, \nu)\), which is therefore universal \([62, 63]\). In this sense, the bands gaps with the same \( \mathcal{N} \) and \( \beta \) can be labeled by the same set of integers \((\mu, \nu)\), independent of system details \([36, 63, 64]\), and this equation is also called the gap-labeling theorem \([31, 32, 34, 35, 52, 54, 65–71]\).

Here we attempt to examine the applicability of the gap-labeling theorem in describing the topology of band gaps and verify the bulk-boundary correspondence in the present system which differently exhibits long-range dipole-dipole interactions and non-Hermitian nature (due to the coupling with the free-space modes). The normalized IDOS \( \mathcal{N} \) of a band gap is simply given by the number of eigenstates below the gap divided by the total number of eigenstates. It is noted that the bands are invariant as a function of \( \phi \) (namely, flat band) and therefore by only considering the eigenstate spectrum at a fixed \( \phi \), one can readily obtain the normalized IDOS in a band gap of the whole band structure \([37]\). Taking the longitudinal band structures as an example, we can find the lower main gap (around \(-12 < \Delta/\gamma < 0.8\)) has a normalized IDOS \( \mathcal{N} \approx 382/1000 = 0.382 \). Therefore, we have \( \mu = 1, \nu = -1 \) for this main gap. Similarly, the second main gap spanning \(2.8 < \Delta/\gamma < 13\) has a normalized IDOS of \( \mathcal{N} \approx 618/1000 = 0.618 \), which leads to \( \mu = 0, \nu = 1 \). Therefore, in the bulk side, we have the topological invariant, i.e., the Chern number of the lower (upper) main gap as \( \nu = -1 \) (\( \nu = +1 \)). According to the bulk-boundary correspondence in 2D IQHE, for a gap Chern number of \( \nu \), there should be exactly \(|\nu|\) edge mode(s) on each edge, whose energy (frequency) traverses the gap when the modulation phase \( \phi \) varies from 0 to \( 2\pi \) \([36, 37, 40]\). And the sign of gap Chern number determines the chirality (group velocity) of the edge modes \([40]\). Considering the observed edge modes in the main gaps in Figs.1 and 2, we can conclude the bulk-boundary correspondence is valid for these two main band gaps.

![FIG. 4. Transverse eigenstate spectra for different modulation phases. There are 1000 atoms in the chain with \( d = 0.1\lambda_0 \) and \( \eta = 0.3 \). (a) \( \phi = 1.1\pi \). (b) Dipole moment distributions of three midgap states, along with two arbitrarily selected trivially localized states. (c) \( \phi = 0.6\pi \). (d) Dipole moment distributions of the midgap states in (c), (e) \( \phi = 1.4\pi \). (f) Dipole moment distributions of two midgap states in (e).](image)

![FIG. 5. Dipole moment distributions of topological edge states in two minibands in the longitudinal band structure. (a-b) The \( \mathcal{N} = 0.236 \) gap. (a) Left edge state under the modulation of \( \phi = 0.1\pi \) and right edge state under \( \phi = 1.1\pi \). (b) Left edge state under the modulation of \( \phi = 0.8\pi \) and right edge state under \( \phi = 1.8\pi \). (c-e) The \( \mathcal{N} = 0.146 \) gap. (c) Right edge state under the modulation of \( \phi = 0.2\pi \) and left edge state under \( 0.4\pi \). (d) Right edge state under the modulation of \( \phi = 0.8\pi \) and left edge state under \( \phi = 1.2\pi \) (e) Right edge state under the modulation of \( \phi = 1.6\pi \) and left edge state under \( \phi = 1.8\pi \). There are 1000 atoms in the chain with \( d = 0.1\lambda_0 \) and \( \eta = 0.3 \).](image)
are given. It is noted that these transverse edge states have the dipole moment distributions of representative midgap traversing the gap. In Fig.6, similar to the longitudinal case, the gap-labeling theorem can also describe the band topology of the figure legends. Similarly, the dipole moment distributions of representative midgap edge states belonging to the gap are presented, selected from the longitudinal eigenstate distributions of representative midgap edge states belonging to the gap. (c-e) The right edge state under the modulation of \( \phi = 0.01\pi \) and left edge state under \( \phi = 0.15\pi \). (b) Left edge state under the modulation of \( \phi = 0.75\pi \) and right edge state under \( \phi = 1.95\pi \). (c) Right edge state under the modulation of \( \phi = 0.02\pi \) and left edge state under \( \phi = 0.38\pi \). (d) Right edge state under the modulation of \( \phi = 0.78\pi \) and left edge state under \( \phi = 1.14\pi \). (e) Right edge state under the modulation of \( \phi = 1.54\pi \) and left edge state under \( \phi = 1.8\pi \). There are 1000 atoms in the chain with \( d = 0.1\lambda_0 \) and \( \eta = 0.3 \).

boundary correspondence, the topological edge states in the minigaps with larger topological numbers are studied. A larger topological number indicates a narrower band gap. For doing this, the band structures for the 1000-atom lattice with \( d = 0.1\lambda_0 \) and \( \eta = 0.3 \) under the nearest-neighbor (NN) approximation are first calculated, as presented in Figs.8a and 8b for longitudinal and transverse eigenstates, respectively. After a comparison with the band structures considering full dipole-dipole interactions (Figs.1b and 3b), it is found that at small atomic distances, long-range interactions are not prominent because strong nearest-neighbor coupling dominates. However, it is revealed that even for such small lattice constants the long-range dipole-dipole interactions lead to an asymmetric band structure. More precisely, the band structure under NN is exactly symmetric with respect to \( \Delta = 0 \) and the IPR distribution of eigenstates is also symmetric [24]. Moreover, this asymmetry is most prominent at large atom distances for transverse eigenstates, as presented in Figs.8c and 8d, which compare the band structures considering full dipole-dipole interactions and only NN interactions for a lattice with \( d = 0.5\lambda_0 \). This is because for transverse eigenstates, there is a long-range component in the dipole-dipole interactions.

The present system is in essence non-Hermitian [21–23]. Recent theoretical progresses in non-Hermitian topological physics, including the topological classification of non-Hermitian systems [74–76], theoretical investigation [77, 78] and experimental observation [79, 80] of non-Hermitian bulk-boundary correspondence including non-Hermitian skin effect (NHSE), etc., has received a lot of attention. Therefore, it would be instructive to further investigate the imaginary parts of the band structures, which are shown in Figs.7a and 7b under a specific modulation phase \( \phi = 1.1\pi \) for longitudinal and transverse eigenstates, respectively. It is observed that the imaginary spectra also exhibit clear band gaps, which, as we have verified, can also be characterized by the gap-labeling theorem. The normalized IDOS and topological numbers \((\mu, \nu)\) of typical band gaps are also marked in the figure. An impressive feature of the longitudinal spectra is that due to the logarithmic scale minigaps with large gap Chern numbers can be distinguished, e.g., a gap with a Chern number of \( \nu = -6 \) is identified in Fig.7a. Another interesting feature is the in the longitudinal band structure, the topological edge states at high IDOS gaps are highly subradiant (e.g., the topological edge state at the \( N \approx 0.854 \) gap can reach \( \Gamma/\gamma \sim 0.016 \)), and in the transverse band structure, the topological edge states at low IDOS gaps are also highly subradiant (e.g., the topological edge state at the \( N \approx 0.146 \) gap can reach \( \Gamma/\gamma \sim 0.005 \)). Therefore these topological edge states provide an appealing route to achieve highly protected, long-lived quantum optical states, which are promising for controlling the emission of individual atoms [23] and high-efficiency robust quantum storage [26, 81].

V. EFFECTS OF LONG-RANGE DIPOLE-DIPOLE INTERACTIONS

For completeness, the effects of long-range dipole-dipole interactions are further discussed in this section. Recently it was found that for certain lattices, long-range electromagnetic interactions can induce new class of topological corner states [82]. For doing this, the band structures for the 1000-atom lattice with \( d = 0.1\lambda_0 \) and \( \eta = 0.3 \) under the nearest-neighbor (NN) approximation are first calculated, as presented in Figs.8a and 8b for longitudinal and transverse eigenstates, respectively. After a comparison with the band structures considering full dipole-dipole interactions (Figs.1b and 3b), it is found that at small atomic distances, long-range interactions are not prominent because strong nearest-neighbor coupling dominates. However, it is revealed that even for such small lattice constants the long-range dipole-dipole interactions lead to an asymmetric band structure. More precisely, the band structure under NN is exactly symmetric with respect to \( \Delta = 0 \) and the IPR distribution of eigenstates is also symmetric [24]. Moreover, this asymmetry is most prominent at large atom distances for transverse eigenstates, as presented in Figs.8c and 8d, which compare the band structures considering full dipole-dipole interactions and only NN interactions for a lattice with \( d = 0.5\lambda_0 \). This is because for transverse eigenstates, there is a long-range component in the dipole-dipole interactions.
interactions which decays with the distance $r$ as $1/r$.

Nevertheless, we note that the strong long-range dipole-dipole interactions do not alter the topological properties of the system qualitatively. To demonstrate this, in Fig. 8e, the upper-in-frequency band structure of Fig. 8e ($\Delta > -0.3$) is presented in an enlarged fashion, from which midgap edge modes are clearly observed. Furthermore, the number of edge modes in a gap is consistent with the gap Chern number obtained from the gap-labeling theorem. In particular, the eigenstate distribution at $\phi = 0.4\pi$ is given in Fig. 8f, where the normalized IDOS and topological integers are labeled for several typical gaps. Therefore, it can be confirmed that the gap-labeling theorem can give a suitable topological characterization of the present system and the bulk-boundary correspondence is valid, even when large long-range interactions dominate and NN approximation breaks down [73].

VI. CONCLUSION

In conclusion, topological quantum optical states in 1D quasiperiodic cold atomic chains are studied, which can be regarded as an extension of the off-diagonal AAH model, despite the existence of long-range dipole-dipole interactions in the Hamiltonian. The discrete band structures are investigated for finite chains beyond the NN approximation. It is found that the present system indeed supports nontrivial topological states localized over the boundaries. Despite the long-range dipole-dipole interactions, it is demonstrated that, for both longitudinal and transverse eigenstates, the present system inherits the topological properties of two-dimensional integer quantum Hall systems, and the spectral position (for both real and imaginary frequencies) and number of these edge states are governed by the gap labeling theorem for quasicrystals and protected by the nonzero Chern number. These results indicate the validity of bulk-boundary correspondence in spite of long-range dipole-dipole interactions that can lead to asymmetric band structures. Due to the fractal nature of the spectrum, the present system readily provides a large number of topological gaps and quantum optical states. Moreover, it is noted that a substantial proportion of these topologically nontrivial states are highly subradiant and thus are promising for controlling the emission of individual atoms and robust quantum state storage. This work thus provides useful implications.
for the design of efficient interfaces between quantum states of light and matter.

The proposed quasiperiodic cold atom chain is within reach of current quantum simulation techniques, e.g., by superimposing two optical lattices with incommensurate wavelengths [18, 19, 42] or applying the cut-and-project procedure to a 2D optical lattice [83]. Moreover, cutting-edge developments in the one-by-one assembling of atoms based on optical tweezer arrays make the fabrication of such aperiodic atomic chains feasible [84–86]. Nanophotonic atom lattices using dielectric photonic crystals [87] or plasmonic nanoparticle arrays [88] also provide possible routes.

ACKNOWLEDGMENTS

We thank the financial support from the National Natural Science Foundation of China (No. 51636004 and No. 51906144), Shanghai Key Fundamental Research Grant (No. 18JC1413300), China Postdoctoral Science Foundation (No. BX20180187 and No. 2019M651493) and the Foundation for Innovative Research Groups of the National Natural Science Foundation of China (No. 51521004).

[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
[2] T. Ozawa, H. M. Price, A. Amo, N. Goldman, M. Hafezi, L. Lu, M. C. Rechtsman, D. Schuster, J. Simon, O. Zilberberg, and I. Carusotto, Rev. Mod. Phys. 91, 015006 (2019).
[3] C. He, X. Ni, H. Ge, X.-C. Sun, Y.-B. Chen, H.-M. Lu, X.-P. Liu, and Y.-F. Chen, Nature Physics 12, 1124 EP (2016).
[4] M. Atala, M. Aидelsburger, J. T. Barreiro, D. Abanin, T. Kita-gawa, E. Demler, and I. Bloch, Nature Physics 9, 795 (2013).
[5] R. Susstrunk and S. D. Huber, Science 349, 47 (2015), https://science.sciencemag.org/content/349/6243/47.full.pdf.
[6] A. B. Khanikaev, S. H. Mousavi, W.-K. Tse, M. Kargarian, A. H. MacDonald, and G. Shvets, Nature materials 12, 233 (2013).
[7] L. Lu, J. D. Joannopoulos, and M. Soljačić, Nature Photonics 8, 821 (2014).
[8] A. B. Khanikaev and G. Shvets, Nature Photonics 11, 763 (2017).
[9] C. Poli, M. Bellec, U. Kuhl, F. Mortessagne, and H. Schomerus, Nature communications 6, 6710 (2015).
[10] R. El-Ganainy and M. Levy, Opt. Lett. 40, 5275 (2015).
[11] D. Karaki, R. El-Ganainy, and M. Levy, Phys. Rev. Applied 11, 034045 (2019).
[12] M. Parto, S. Wittek, H. Hodaei, G. Harari, M. A. Bandres, J. Ren, M. C. Rechtsman, M. Segev, D. N. Christodoulides, and M. Khajavikhan, Phys. Rev. Lett. 120, 033601 (2018).
[13] C. Cohen-Tannoudji and D. Gury-Odelin, Advances in Atomic Physics (WORLD SCIENTIFIC, 2011) https://www.worldscientific.com/doi/pdf/10.1142/6631.
[14] J. Rui, D. Wei, A. Rubio-Abadal, S. Hollerith, J. Zeiher, D. M. Stamper-Kurn, C. Gross, and I. Bloch, arXiv e-prints, arXiv:2001.00795 (2020), arXiv:2001.00795 [quant-ph].
[15] R. J. Bettes, S. A. Gardiner, and C. S. Adams, Phys. Rev. Lett. 116, 103602 (2016).
[16] E. Shahmoon, D. S. Wild, M. D. Lukin, and S. F. Yelin, Phys. Rev. Lett. 118, 113601 (2017).
[17] B. X. Wang, C. Y. Zhao, Y. H. Kan, and T. C. Huang, Opt. Express 25, 18760 (2017).
[18] G. Roati, C. D’Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Nature 453, 895 (2008).
[19] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Science, aaa7432 (2015).
[20] P. Bordia, H. Lüschen, U. Schneider, M. Knap, and I. Bloch, Nature Physics 13, 460 (2017).
[21] J. Perczel, J. Borregaard, D. E. Chang, H. Pichler, S. F. Yelin, P. Zoller, and M. D. Lukin, Phys. Rev. A 96, 063801 (2017).
[22] R. J. Bettes, J. c. v. Minář, C. S. Adams, I. Lesanovsky, and B. Olmos, Phys. Rev. A 96, 041603 (2017).
[23] J. Perczel, J. Borregaard, D. E. Chang, H. Pichler, S. F. Yelin, P. Zoller, and M. D. Lukin, Phys. Rev. Lett. 119, 023603 (2017).
[24] B. X. Wang and C. Y. Zhao, Phys. Rev. A 98, 023808 (2018).
[25] Y. Wang, X.-L. Pang, Y.-H. Lu, J. Gao, Y.-J. Chang, L.-F. Qiao, Z.-Q. Jiao, H. Tang, and X.-M. Jin, Optica 6, 955 (2019).
[26] F.-O. Guimond, A. Grundk, D. V. Vasilyev, B. Vermersch, and P. Zoller, Phys. Rev. Lett. 122, 093601 (2019).
[27] R. O. Umeulilar and I. Carusotto, Phys. Rev. Lett. 108, 206809 (2012).
[28] I. Carusotto and C. Ciuti, Rev. Mod. Phys. 85, 299 (2013).
[29] M. F. Maghrehbi, N. Y. Yao, M. Hafezi, T. Pohl, O. Firstenberg, and A. V. Gorshkov, Phys. Rev. A 91, 033838 (2015).
[30] P. Roushan, C. Neill, A. MeGrant, Y. Chen, R. Babbush, R. Barends, B. Campbell, Z. Chen, B. Chiaro, A. Dunsworth, A. Fowler, E. Jeffrey, J. Kelly, E. Lucero, J. Mutus, P. J. J. O’Malley, M. Neeley, C. Quintana, D. Sank, A. Vainsencher, J. Wenner, T. White, E. Kapit, H. Neven, and J. Martinis, Nature Physics 13, 146 (2017).
[31] D. Tanese, E. Gurevich, F. Baboux, T. Jacqmün, A. Lemaître, E. Galopin, I. Sagnes, A. Amo, I. Bloch, and E. Akkermans, Phys. Rev. Lett. 112, 146404 (2014).
[32] E. Levy, A. Barak, A. Fisher, and E. Akkermans, arXiv e-prints, arXiv:1509.04028 (2015), arXiv:1509.04028 [physics.optics].
[33] M. A. Bandres, M. C. Rechtsman, and M. Segev, Phys. Rev. X 6, 011016 (2016).
[34] A. Dareau, E. Levy, M. B. Aguilera, R. Bouganne, E. Akkermans, F. Gerbier, and J. Beugnon, Phys. Rev. Lett. 119, 215304 (2017).
[35] F. Baboux, E. Levy, A. Lemaître, C. Gómez, E. Galopin, L. Gréatiet, I. Sagnes, A. Amo, I. Bloch, and E. Akkermans, Phys. Rev. B 95, 161114 (2017).
[36] Y. E. Kraus and O. Zilberberg, Phys. Rev. Lett. 109, 116404 (2012).
[37] Y. E. Kraus, Y. Lahini, Z. Ringel, M. Verbin, and O. Zilberberg, Phys. Rev. Lett. 109, 106402 (2012).
[38] S. Ganeshan, K. Sun, and S. Das Sarma, Phys. Rev. Lett. 110, 180403 (2013).
[39] M. Verbin, O. Zilberberg, Y. E. Kraus, Y. Lahini, and Y. Zilberberg, Phys. Rev. Lett. 110, 076403 (2013).
[40] A. V. Poshakinskiy, A. N. Poddubny, L. Pilozzi, and E. L. Ivchenko, Phys. Rev. Lett. 112, 107403 (2014).
[41] G. Amit and I. Dana, Phys. Rev. B 97, 075137 (2018).
[42] S. V. Rajagopal, T. Shimakawa, P. Dotti, M. Račiunas,
[43] X. Ni, K. Chen, M. Werner, D. J. Apigo, C. Prodan, A. Alù, E. Prodan, and A. B. Khanikaev, Communications Physics 2, 55 (2019).

[44] I. Bloch, Nature Physics 1, 23 (2005).

[45] T. Bienaim, M. Petruzzo, D. Bigerni, N. Piovella, and R. Kaiser, Journal of Modern Optics 58, 1942 (2011).

[46] W. Guerin, M. O. Araújo, and R. Kaiser, Phys. Rev. Lett. 116, 083601 (2016).

[47] M. Antezza and Y. Castin, Phys. Rev. A 80, 013816 (2009).

[48] A. A. Svidzinsky, J.-T. Chang, and M. O. Scully, Phys. Rev. A 59, 013816 (2009).

[49] S. E. Skipetrov and I. M. Sokolov, Phys. Rev. Lett. 112, 023905 (2014).

[50] J. M. Luck, Phys. Rev. B 49, 5834 (1989).

[51] J. Cao, Y. Xing, L. Qi, D.-Y. Wang, C.-H. Bai, A.-D. Zhu, Q.-B. Zeng, Y.-B. Yang, and Y. Xu, arXiv preprint 1901.08060v1 (2019).

[52] J. Cao, Y. Xing, L. Qi, D.-Y. Wang, C.-H. Bai, A.-D. Zhu, S. Zhang, and H.-F. Wang, Laser Physics Letters 15, 015211 (2017).

[53] Z. Guo, H. Jiang, Y. Sun, Y. Li, and H. Chen, Opt. Lett. 43, 5142 (2018).

[54] Q.-B. Zeng, Y.-B. Yang, and Y. Xu, arXiv preprint arXiv:1901.08060v2 (2019).

[55] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. 49, 405 (1982).

[56] A. H. MacDonald, Phys. Rev. B 29, 3057 (1984).

[57] I. Dana, Y. Avron, and J. Zak, Journal of Physics C: Solid State Physics 18, L679 (1985).

[58] I. Dana, Phys. Rev. B 89, 205111 (2014).

[59] I. Dana and J. Zak, Phys. Rev. B 32, 3612 (1985).

[60] J. Bellissard, B. Iochum, E. Scoppola, and D. Testard, Communications in Mathematical Physics 125, 527 (1989).

[61] Y. Liu, X. Fu, W. Deng, and S. Wang, Phys. Rev. B 46, 9216 (1992).

[62] X. Fu, Y. Liu, P. Zhou, and W. Sritrakool, Phys. Rev. B 55, 2882 (1997).

[63] N. X. A. Rivolta, H. Benisty, and B. Maes, Phys. Rev. A 96, 023864 (2017).

[64] D. J. Apigo, K. Qian, C. Prodan, and E. Prodan, Phys. Rev. Materials 2, 1223201 (2019).

[65] G. Rai, S. Haas, and A. Jagannathan, Phys. Rev. B 100, 165121 (2019).

[66] D. J. Apigo, W. Cheng, K. F. Dobiszewski, E. Prodan, and C. Prodan, Phys. Rev. Lett. 122, 095501 (2019).

[67] F. Piéchon, M. Benakli, and A. Jagannathan, Phys. Rev. Lett. 74, 5248 (1995).

[68] B. X. Wang and C. Y. Zhao, Phys. Rev. B 98, 165435 (2018).

[69] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X 8, 031079 (2018).

[70] C.-H. Liu, H. Jiang, and S. Chen, Phys. Rev. B 99, 125103 (2019).

[71] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Phys. Rev. X 9, 041015 (2019).

[72] D. S. Borgia, A. J. Kruchkov, and R.-J. Slager, arXiv e-prints , arXiv:1902.07217 (2019), arXiv:1902.07217 [cond-mat.mes-hall].

[73] F. Song, S. Yao, and Z. Wang, Phys. Rev. Lett. 123, 170401 (2019).

[74] L. Xiao, T. Deng, K. Wang, G. Zhu, Z. Wang, W. Yi, and P. Xue, arXiv e-prints , arXiv:1907.12566 (2019), arXiv:1907.12566 [cond-mat.mes-hall].

[75] A. Ghiatak, M. Brandenbourger, J. van Wezel, and C. Coulaire, arXiv e-prints , arXiv:1907.11619 (2019), arXiv:1907.11619 [cond-mat.mes-hall].

[76] A. Zhang, L. Wang, X. Chen, V. V. Yakovlev, and L. Yuan, Communications Physics 2, 157 (2019).

[77] M. Li, D. Zhirihin, M. Gorlach, X. Ni, D. Filonov, A. Slobozhanyuk, A. Alù, and A. B. Khanikaev, Nature Photonics 10.1038/s41566-019-0561-9 (2019).

[78] K. Singh, K. Saha, S. A. Parameswaran, and D. M. Weld, Phys. Rev. A 92, 063426 (2015).

[79] M. Endres, H. Bernien, A. Keesling, H. Levine, E. R. Anschuetz, A. Krajenbrink, C. Senko, V. Vuletic, M. Greiner, and M. D. Lukin, Science , aah3752 (2016).

[80] D. Barredo, S. De Léséleuc, V. Lienhard, T. Lahaye, and A. Browaeys, Science 354, 1021 (2016).

[81] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner, V. Vuletic, and M. D. Lukin, Nature 551, 579 (2017).

[82] A. González-Tudela, C.-L. Hung, D. E. Chang, J. I. Cirac, and H. Kimble, Nature Photonics 9, 320 (2015).

[83] M. Gullans, T. G. Tiecke, D. E. Chang, J. Feist, J. D. Thompson, J. I. Cirac, P. Zoller, and M. D. Lukin, Phys. Rev. Lett. 109, 235309 (2012).