ON THE DYSON PROCESS OF A MATRIX FRACTIONAL BROWNIAN MOTION

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Abstract. We investigate the process of eigenvalues of a symmetric matrix-valued process which upper diagonal entries are independent one-dimensional Hölder continuous Gaussian processes of order \( \gamma \in (1/2, 1) \). Using the stochastic calculus with respecto to the Young's integral we show that these eigenvalues do not collide at any time with probability one. When the matrix process has entries that are fractional Brownian motions with Hurst parameter \( H \in (1/2, 1) \), we find a stochastic differential equation in a Malliavin calculus sense for the eigenvalues of the corresponding matrix fractional Brownian motion. A new generalized version of the Itô formula for the multidimensional fractional Brownian motion is first established.

Keywords and phrases: eigenvalues process, Young integral, noncolliding process, Hölder continuous Gaussian process.

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1. Introduction

In a pioneering work in 1962, the nuclear physicist Freeman Dyson [12] studied the stochastic process of eigenvalues of an Hermitian matrix Brownian motion. The case of the (real) symmetric matrix Brownian motion was first considered by Mc Kean [23] in 1969. In both cases the corresponding processes of eigenvalues are called Dyson Brownian motion and are governed by a noncolliding system of Itô Stochastic Differential Equations (SDEs) with non-smooth diffusion coefficients; see the modern treatments, for example, in the books by Anderson, Guionnet and Zeitouni [2] and Tao [31].

More specifically, for the symmetric case, let \( \{B(t)\}_{t \geq 0} = \{ (b_{jk}(t)) \}_{t \geq 0} \) be a \( d \times d \) symmetric matrix Brownian motion. That is, \( (b_{jj}(t))_{j=1}^{d}, (b_{jk}(t))_{j<k} \) is a set of \( d(d+1)/2 \) independent one-dimensional Brownian motions with parameter \((1 + \delta_{jk})t\). For each \( t > 0 \), \( B(t) \) is a Gaussian Orthogonal (GO) random matrix with parameter \( t \) ([2], [24]). Let \( \lambda_1(t) \geq \lambda_2(t) \geq \cdots \geq \lambda_d(t), t \geq 0 \), be the \( d \)-dimensional stochastic process of eigenvalues of \( B \). If the eigenvalues start at different positions \( \lambda_1(0) > \lambda_2(0) > \cdots > \lambda_d(0) \), then they never meet at any time \( \lambda_1(t) > \lambda_2(t) > \cdots > \lambda_d(t) \) almost surely \( \forall t > 0 \) and furthermore

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they form a diffusion process satisfying the Itô SDE
\begin{equation}
\lambda_i(t) = \sqrt{2} W^i_t + 2 \sum_{j \neq i} \int_0^t \frac{ds}{\lambda_i(s) - \lambda_j(s)}, \quad 1 \leq i \leq d, \quad t \geq 0,
\end{equation}
where $W^1_t, \ldots, W^d_t$ are independent one-dimensional standard Brownian motions.

Thus, Dyson Brownian motion can be thought as a model for the evolution of $d$ Brownian motions $\lambda_1(t), \ldots, \lambda_d(t)$ that are restricted to never intersect and having a repulsion force which is inversely proportional to the distance between any two eigenvalues. For that reason $\lambda(t) = (\lambda_1(t), \ldots, \lambda_d(t))$ is also called a Dyson non-colliding Brownian motion.

Different aspects of the Dyson Brownian motion have been considered by several authors \cite{3}, \cite{7}, \cite{8}, \cite{13}, \cite{15}, \cite{21}, \cite{19}, \cite{28}, \cite{30}. The corresponding eigenvalues processes of other matrix stochastic processes and their associated Itô SDEs have been studied in \cite{5}, \cite{6}, \cite{10}, \cite{11}, \cite{20}, \cite{22}, \cite{29}, among others.

From now on we call “Dyson process” the $d$-dimensional process of eigenvalues of a $d \times d$ matrix process. This terminology is different from the one used in \cite{32}, where a Dyson process means a $d \times d$ matrix process in which the entries undergo diffusion. The evolution of the singular values of a matrix process is often called Laguerre process, being an important example the Wishart case \cite{5}, \cite{6}, \cite{10}, \cite{22}, \cite{29}.

The purpose of this paper is to study the Dyson process of a symmetric matrix Gaussian process $\{G(t)\}_{t \geq 0}$ which entries in the upper diagonal part are independent one-dimensional zero mean Gaussian processes with Hölder continuous paths of order $\gamma \in (1/2, 1)$. In this case $G(t)$ is still a GO random matrix of parameter $t$, for each $t > 0$, but $\{G(t)\}_{t \geq 0}$ is not a matrix process with independent increments nor a matrix diffusion. We prove that the corresponding eigenvalues do not collide at any time with probability one. The proof of this fact is based on the stochastic calculus with respect to the Young’s integral. Our noncolliding method of proof is different from the case of the Brownian matrix and other matrix diffusions, for which one first has to find a diffusion process governing the Dyson process as done, for example, in \cite{2} for the Hermitian Brownian motion. Our result does not include the latter as a special case.

We study in detail the case when the matrix process has entries that are fractional Brownian motions with Hurst parameter $H \in (1/2, 1)$. For this matrix process we find a stochastic differential equation for the Dyson process, similar to equation (1.1), where instead of $\sqrt{2} W^i_t$ we obtain processes $Y^i_t$ which are expressed as Skorohod indefinite integrals. This equation is derived applying a generalized version of the Itô formula in the Skorohod sense for the multidimensional fractional Brownian motion, which has its own interest. Each process $Y^i_t$ has the same $H$-self-similarity and $1/H$-variation properties as a one-dimensional fractional Brownian motion, although there is no reason for them to be fractional Brownian motions. This phenomenon is related to the representation of fractional Bessel processes, established by Guerra and Nualart in \cite{14}. In the case of the fractional Bessel process, the fact that the indefinite Skorohod integral appearing in the representation is not a one-dimensional standard Brownian motion was proved by Hu and
Nualart in [17]. In our case we conjecture that each \( Y^i \) is not a one-dimensional fractional Brownian motion, but at this moment we are not able to give a proof of this fact.

The paper is organized as follows. Section 2 contains preliminaries on Malliavin calculus for the fractional Brownian motion. Section 3 and establishes a generalized version of the Itô formula for the multidimensional fractional Brownian motion, in the case of functions that are smooth only on a dense subset of the Euclidean space, as needed to describe the evolution of the eigenvalues, a result does not covered in the literature. It also recalls a property on the \( 1/H \)-variation of a divergence integral and establishes a result that allows to compute the \( 1/H \)-variation of a Skorohod integral wrt a multidimensional Brownian motion. Section 4 considers the SDE of the eigenvalues of a matrix fractional Brownian notion similar to (1.1) but in the Malliavin calculus sense. Finally, in Section 5 we prove that if the matrix process \( \{G(t)\}_{t\geq 0} \) has entries that are Hölder continuous Gaussian processes of order \( \gamma \in (1/2, 1) \), then the corresponding eigenvalues processes do not collide at any time with probability one.

2. Malliavin Calculus for the fBm

In this section we present some basic facts on the Malliavin calculus, or stochastic calculus of variations, with respect to the fractional Brownian motion. We refer the reader to [27] for a detailed account of this topic.

Suppose that \( B = \{B_t, t \geq 0\} \) is a fractional Brownian motion with Hurst parameter \( H \in (1/2, 1) \). That is, \( B \) is a zero mean Gaussian process with covariance

\[
R(t, s) = \mathbb{E}(B_tB_s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).
\]

The process \( B \) is \( H \)-self-similar, that is, for any \( a > 0 \), \( \{B_{at}, t \geq 0\} \) and \( \{a^H B_t, t \geq 0\} \) have the same law. On the other hand, it possesses a finite \( 1/H \)-variation on any time interval of length \( t \) (see Definition 3.2) equals to \( t \mathbb{E}(|Z|^{1/H}) \), where \( Z \) is a \( N(0, 1) \) random variable.

Fix a time interval \([0, T]\) and let \( \mathcal{H} \) be the Hilbert space defined as the closure of the set of step functions with respect to the scalar product

\[
\langle 1_{[0,t]} , 1_{[0,s]} \rangle_{\mathcal{H}} = R(s, t).
\]

Given two step functions \( \varphi, \psi \) on \([0, T]\), its inner product can be expressed as

\[
\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T \varphi_t \psi_s |t - s|^{2H-2} \, dt \, ds,
\]

where \( \alpha_H = H(2H - 1) \). We denote by \( |\mathcal{H}| \) the space of measurable functions \( \varphi \) on \([0, T]\) such that

\[
\|\varphi\|^2_{\mathcal{H}} := \alpha_H \int_0^T \int_0^T |\varphi_t| \|\varphi_s\| |t - s|^{2H-2} \, dt \, ds < \infty.
\]

This space is a Banach space, which is isometric to a subspace of \( \mathcal{H} \) and it will be identified with this subspace. Moreover, we have the following continuous embeddings (see [25]).

\[
L^{1/H}([0, T]) \subset |\mathcal{H}| \subset \mathcal{H}.
\]
The mapping \( \mathbf{1}_{[0,t]} \mapsto B_t \) can be extended to a linear isometry between \( \mathcal{H} \) and the Gaussian space generated by \( B \). We denote this isometry by \( h \mapsto B(h) \).

Let \( S \) be the set of smooth and cylindrical random variables of the form
\[
F = f \left( B(h_1), \ldots, B(h_n) \right),
\]
where \( n \geq 1 \), \( f \in C_c^\infty (\mathbb{R}^n) \) ( \( f \) and all its partial derivatives are bounded) and \( h_i \in \mathcal{H} \). The derivative operator is defined in \( S \) as the \( \mathcal{H} \)-valued random variable
\[
DF := \sum_{i=1}^n \frac{\partial f}{\partial x_i} \left( B(h_1), \ldots, B(h_n) \right) h_i.
\]
The derivative operator is a closable operator from \( L^p(\Omega) \) into \( L^p(\Omega; \mathcal{H}) \) for any \( p \geq 1 \).

We denote by \( D_k \) the iteration of \( D \). For any \( p \geq 1 \), we define the Sobolev space \( \mathbb{D}^{k,p}(\mathcal{H}) \) as the closure of \( S \) with respect to the norm
\[
\| F \|_{k,p} := \mathbb{E}[|F|^p] + \mathbb{E} \left[ \sum_{j=1}^k \| D^j F \|_{\mathcal{H} \otimes \mathcal{H}}^p \right].
\]
In a similar way, given a Hilbert space \( V \), we can define the Sobolev space of \( V \)-valued random variables, denoted by \( \mathbb{D}^{k,p}(V) \).

The divergence operator \( \delta \) is the adjoint of the derivative operator, defined by means of the duality relationship
\[
\mathbb{E} [F \delta(u)] = \mathbb{E} [\langle DF, u \rangle_{\mathcal{H}}],
\]
where \( u \) is a random variable in \( L^2(\Omega; \mathcal{H}) \). The domain \( \text{Dom}(\delta) \) is the set of random variables \( u \in L^2(\Omega; \mathcal{H}) \) such that
\[
|\mathbb{E} (\langle DF, u \rangle_{\mathcal{H}})| \leq c \| F \|_2
\]
for all \( F \in \mathbb{D}^{1,2} \). We have the inclusion \( \mathbb{D}^{1,2}(\mathcal{H}) \subset \text{Dom}(\delta) \), and for any \( u \in \mathbb{D}^{1,2}(\mathcal{H}) \), the variance of the divergence of \( u \) can be computed as follows
\[
\mathbb{E} [\delta(u)^2] = \mathbb{E} \left[ \| u \|_{\mathcal{H}}^2 \right] + \mathbb{E} \left[ \langle Du, (Du)^* \rangle_{\mathcal{H} \otimes \mathcal{H}} \right] \leq \| u \|^2_{\mathbb{D}^{1,2}(\mathcal{H})},
\]
where \( (Du)^* \) is the adjoint of \( Du \) in the Hilbert space \( \mathcal{H} \otimes \mathcal{H} \).

By Meyer’s inequalities, for all \( p > 1 \), the divergence operator can be extended to the space \( \mathbb{D}^{1,p}(\mathcal{H}) \) and the mapping
\[
\delta : \mathbb{D}^{1,p}(\mathcal{H}) \to L^p(\Omega)
\]
is a continuous operator, that is,
\[
\| \delta(u) \|_p \leq C_p \| u \|_{\mathbb{D}^{1,p}(\mathcal{H})}.
\]
Denote by \( |\mathcal{H}| \otimes |\mathcal{H}| \) the space of measurable functions \( \varphi \) defined on \([0,T]^2\) such that
\[
\| \varphi \|^2_{|\mathcal{H}| \otimes |\mathcal{H}|} := \alpha_H \int_{[0,T]^4} |\varphi_{r,\theta}| \| \varphi_{u,\eta} \| \| r - u \|^{2H-2} \| \theta - \eta \|^{2H-2} \, dr \, d\theta \, du \, d\eta < \infty.
\]
As in the case of functions of one variable, this space is a Banach space, which is isometric to a subspace of $\mathcal{H} \otimes \mathcal{H}$ and it will be identified with this subspace. For any $p \geq 1$, we denote by $\mathbb{D}^{1,p}(\mathcal{H})$ the subspace of the Sobolev space $\mathbb{D}^{1,p}(\mathcal{H})$ whose elements $u$ are such that $u \in |\mathcal{H}|$ a.s., $Du \in |\mathcal{H}| \otimes |\mathcal{H}|$ a.s. and

$$
\mathbb{E} \left[ \|u\|_{|\mathcal{H}|}^p \right] + \mathbb{E} \left[ \|Du\|_{|\mathcal{H}| \otimes |\mathcal{H}|}^p \right] < \infty.
$$

Then, using the embedding (2.1) one can show that for any $p > 1$, we have the continuous embedding $\mathbb{L}^1_H \subset \mathbb{D}^{1,p}(\mathcal{H})$, where $\mathbb{L}^1_H$ is the set of processes $u \in \mathbb{D}^{1,p}(\mathcal{H})$ such that

$$
\|u\|_{\mathbb{L}^1_H}^p := \mathbb{E} \left[ \|u\|_{L^1/|\mathcal{H}|([0,T])}^p \right] + \mathbb{E} \left[ \|Du\|_{L^1/|\mathcal{H}|([0,T]^2)}^p \right] < \infty.
$$

As a consequence, Meyer inequalities (2.3) imply

$$
\|\delta(u)\|_p \leq C_p^\prime \|u\|_{\mathbb{L}^1_H}^p
$$

if $u \in \mathbb{L}^1_H$ and $p > 1$.

Let $N \in \mathbb{N}$, $d \geq 2$, $H \in (1/2,1)$ and consider the $N$-dimensional fractional Brownian motion

$$
B = \{B_t, t \geq 0\} = \left\{B^{(1)}_t, \ldots, B^{(N)}_t, t \in [0,T]\right\},
$$

with Hurst parameter $H$, defined on the probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is generated by $B$. That is, the components $B^{(i)}$, $i = 1, \ldots, d$, are independent fractional Brownian motions with Hurst parameter $H$. We can define the derivative and divergence operators, $D^{(i)}$ and $\delta^{(i)}$, with respect to each component $B^{(i)}$, as before. Denote by $\mathbb{D}^{1,p}_H(\mathcal{H})$ the associated Sobolev spaces. We assume that these spaces include functionals of all the components of $B$ and not only of component $i$. Similarly, we introduce the spaces $\mathbb{L}^{1,p}_{H,i}$.

That is, $\mathbb{L}^{1,p}_{H,i}$ is the set of processes $u \in \mathbb{D}^{1,p}(\mathcal{H})$ such that

$$
\|u\|_{\mathbb{L}^{1,p}_{H,i}}^p := \mathbb{E} \left[ \|u\|_{L^1/|\mathcal{H}|([0,T])}^p \right] + \mathbb{E} \left[ \|D^{(i)}u\|_{L^1/|\mathcal{H}|([0,T]^2)}^p \right] < \infty.
$$

The processes $u \in \mathbb{L}^{1,p}_{H,i}$ are, in general, functionals of all the components of the process $B$.

If a process $u = \{u_t, t \in [0,T]\}$ belongs to the domain of $\delta^{(i)}$, we call $\delta^{(i)}(u)$ the Skorohod integral of $u$ with respect to the fractional Brownian motion $B^{(i)}$ and we will make use of the notation

$$
\delta^{(i)}(u) = \int_0^T u_s \delta B^{(i)}_s.
$$

3. Stochastic calculus for the fBm

There exist a huge literature on the stochastic calculus for the fBm. There are essentially two types of stochastic integrals: path-wise integrals defined using the Young’s integral in the case $H > 1/2$, and the Skorohod integral which is the adjoint of the divergence operator introduced in the previous section. We refer the reader to the monographs by Biagini, Hu, Øksendal and Zhang [4] and Mishura [26] and the references therein.
In order to describe the evolution of the eigenvalues of a matrix fractional Brownian motion we need a multidimensional version of the Itô formula for the Skorohod integral, in the case of functions that are smooth only on a dense open subset of the Euclidean space and satisfy some growth requirements. This type of formula is not covered by the existing literature on the subject and we provide below a proof based on a duality argument, following the approach developed in the paper [9].

**Theorem 3.1.** Suppose that $B$ is an $N$-dimensional fractional Brownian motion with Hurst parameter $H > 1/2$. Consider a function $F : \mathbb{R}^N \to \mathbb{R}$ such that:

1. There exists an open set $G \subset \mathbb{R}^N$ such that $G^c$ has zero Lebesgue measure and $F$ is twice continuously differentiable in $G$.
2. $|F(x)| + \left| \frac{\partial F}{\partial x_i}(x) \right| \leq C(1 + |x|^M)$, for some constants $C > 0$ and $M > 0$ and for all $x \in G$ and $i = 1, \ldots, N$.
3. For each $i = 1, \ldots, N$ and for each $s > 0$ and $p \geq 1$,
$$
\mathbb{E} \left[ \left| \frac{\partial^2 F}{\partial x_i^2}(B_s) \right|^p \right] \leq Cs^{-pH},
$$
for some constant $C > 0$.

Then, for each $i = 1, \ldots, N$ and $t \in [0, T]$, the process $\{\frac{\partial F}{\partial x_i}(B_s)1_{[0,t]}(s), s \in [0, T]\}$ belongs to the space $L^{1/H}_{H,i}$ and

$$
F(B_t) = F(0) + \sum_{i=1}^{N} \int_{0}^{t} \frac{\partial F}{\partial x_i}(B_s) \delta B_s^i + H \sum_{i=1}^{N} \int_{0}^{t} \frac{\partial^2 F}{\partial x_i^2}(B_s)s^{2H-1} ds.
$$

**Proof.** Notice first that the processes $F(B_t)$, $\frac{\partial F}{\partial x_i}(B_t)$ and $\frac{\partial^2 F}{\partial x_i^2}(B_t)$ are well defined because the probability that $B_t$ belongs to $G^c$ is zero. On the other hand, $F(0)$ is also well defined as the limit in $L^1(\Omega)$ of $F(B_t)$ as $t$ tends to zero. This limit exists because for any $s < t$ we can write

$$
|F(B_t) - F(B_s)| = \left| \int_{0}^{1} \sum_{i=1}^{N} \frac{\partial F}{\partial x_i}(B_s + \sigma(B_t - B_s))(B_t^i - B_s^i) d\sigma \right|
\leq C \left( 1 + \frac{1}{M+1} |B_t - B_s|^M \right).
$$

Conditions (2) and (3) imply that for each $i = 1, \ldots, N$, the process $u_i(s) = \frac{\partial F}{\partial x_i}(B_s)1_{[0,t]}(s)$ belongs to the space $L^{1/H}_{H,i}$. In fact,

$$
\mathbb{E} \left[ \int_{0}^{T} |u_i(s)|^{1/H} ds \right] \leq C^{1/H} \mathbb{E} \left[ \int_{0}^{T} (1 + |B_s|^M)^{1/H} ds \right] < \infty,
$$

and

$$
\mathbb{E} \left[ \int_{0}^{T} \int_{0}^{T} |D^{(j)}_{i;j}(s)\frac{1}{\pi} dr ds \right] = \mathbb{E} \left[ \int_{0}^{T} s \left| \frac{\partial^2 F}{\partial x_i^2}(B_s) \right|^\frac{1}{r} ds \right] \leq CT.
$$
On the other hand, taking \( p = 1 \) in condition (3), we also have for each \( i = 1, \ldots, N \),

\[
\mathbb{E} \left[ \int_0^t \left| \frac{\partial^2 F}{\partial x_i^2}(B_s) \right| s^{2H-1} ds \right] < \infty.
\]

As a consequence, all terms in equation (3.1) are well defined. Then, to prove the equality it suffices to show that for any random variable of the form

\[ G = g(B_{t_1}, \ldots, B_{t_n}), \]

where \( 0 < t_1 < \cdots < t_i < t < t_{i+1} < \cdots t_n \), for some \( i = 0, 1, \ldots, n \), where \( g \) is \( C^\infty \) with compact support, we have

\[
\mathbb{E} [GF(B_t) - GF(0)] = \sum_{i=1}^N \mathbb{E} \left[ G \int_0^t \frac{\partial F}{\partial x_i}(B_s) \delta B_i^s \right] + H \sum_{i=1}^N \mathbb{E} \left[ G \int_0^t \frac{\partial^2 F}{\partial x_i^2}(B_s) s^{2H-1} ds \right].
\]

Denote by \( p_{t,\tau}(y,x) \) the joint density of the vector \( (B_{t_1}, \ldots, B_{t_n}, B_t) \), where \( \tau = (t_1, \ldots, t_n) \). By the duality between the Skorohod integral and the derivative operator (see (2.2)), we can write

\[
\mathbb{E} \left[ G \int_0^t \frac{\partial F}{\partial x_i}(B_s) \delta B_i^s \right] = \mathbb{E} \left[ \left< D^i G, \frac{\partial F}{\partial x_i}(B_s) 1_{[0,t]} \right>_{\mathcal{H}} \right]
\]

\[
= \sum_{j=1}^n \mathbb{E} \left[ \frac{\partial g}{\partial y_{ij}}(B_{t_1}, \ldots, B_{t_n}) \left< 1_{[0,t_j]}, \frac{\partial F}{\partial x_i}(B_s) 1_{[0,t]} \right>_{\mathcal{H}} \right]
\]

\[
= \int_0^t \sum_{j=1}^n \mathbb{E} \left[ \frac{\partial g}{\partial y_{ij}}(B_{t_1}, \ldots, B_{t_n}) \frac{\partial F}{\partial x_i}(B_s) \right] \frac{\partial R}{\partial s}(s,t_j) ds
\]

\[
= \int_0^t \int_{\mathbb{R}^{N(n+1)}} \frac{\partial g}{\partial y_{ij}}(y) \frac{\partial F}{\partial x_i}(x) \frac{\partial R}{\partial s}(s,t_j) p_{s,\tau}(x,y) dy dx ds.
\]

We can integrate by parts in the above expression, for each fixed \( s \not\in \{0, t_1, \ldots, t_n\} \). We know that \( \frac{\partial F}{\partial x_i} \) is only differentiable in \( G \), but using condition (2), and a regularization procedure, we can proof rigorously this integration by parts argument. In that way we obtain

\[
\mathbb{E} \left[ G \int_0^t \frac{\partial F}{\partial x_i}(B_s) \delta B_i^s \right] = - \int_0^t \sum_{j=1}^n \int_{\mathbb{R}^{N(n+1)}} \frac{\partial g}{\partial y_{ij}}(y) F(x) \frac{\partial R}{\partial s}(s,t_j) \frac{\partial p_{s,\tau}}{\partial x_i}(x,y) dy dx ds.
\]

On the other hand,

\[
\mathbb{E}[GF(B_t) - GF(0)] = \int_{\mathbb{R}^{N(n+1)}} F(x)g(y)p_{t,\tau}(x,y) dy dx - \mathbb{E}(G)F(0)
\]

\[
= \int_0^t \int_{\mathbb{R}^{N(n+1)}} F(x)g(y) \frac{\partial p_{s,\tau}}{\partial s}(x,y) dy dx ds.
\]
The second follows by regularizing $F$ with an approximation of the identity. Also, integrating by parts,
\[
\mathbb{E} \left[ G \int_0^t \frac{\partial^2 F}{\partial x_i^2}(B_s)s^{2H-1} \, ds \right] = \int_0^t g(y) \frac{\partial^2 F}{\partial x_i^2}(x)p_{s,\tau}(x,y)s^{2H-1} \, dx \, dy \, ds
\]
\[
= \int_0^t g(y) F(x) \frac{\partial^2 p_{s,\tau}}{\partial x_i^2}(x,y)s^{2H-1} \, dx \, dy \, ds.
\]
Finally, the result follows from
\[
\frac{\partial p_{s,\tau}}{\partial s} = \sum_{i=1}^N \sum_{j=1}^n \frac{\partial R}{\partial s}(s,t_j) \frac{\partial^2 p_{s,\tau}}{\partial y_{ij} \partial x_i} + \sum_{i=1}^N H s^{2H-1} \frac{\partial^2 p_{s,\tau}}{\partial x_i^2}.
\]
\[
\square
\]
In the rest of this section we will recall a property on the $1/H$-variation of the divergence integral established in [14]. Fix $T > 0$ and set $t^n_i := iT/n$, where $n$ is a positive integer and $i = 0, 1, \ldots, n$. Given a stochastic process $X = \{X_t, t \in [0, T]\}$ we define for each $p \geq 1$,
\[
V^n_p(X) := \sum_{i=0}^{n-1} \left| X^n_{t_{i+1}} - X^n_{t_i} \right|^p.
\]

**Definition 3.2.** The $p$-variation, $p \geq 1$, of a stochastic process $X$ is defined as the limit, in $L^1(\Omega)$, if it exists, of $V^n_p(X)$ as $n \to \infty$.

Then, following result (see Theorem 4.8 in [14]), allows one to compute the $1/H$-variation of a Skorohod integral with respect to a multidimensional fractional Brownian motion.

**Theorem 3.3.** Let $1/2 < H < 1$ and $u^i \in L_H^{1/1} \equiv L^{1/1/H}_H$ for each $i = 1, \ldots N$. Set $X_t := \sum_{i=1}^N \int_0^t u^{(i)}_s \, dB^{(i)}_s$, for each $t \in [0, T]$. Then
\[
V^n_{1/H}(X) \xrightarrow{n \to \infty} \int_{\mathbb{R}^N} \left[ \int_0^T |\langle u_s, \xi \rangle|^{1/H} \, ds \right] \nu(d\xi),
\]
where $\nu$ is the standard normal distribution on $\mathbb{R}^N$.

4. **Stochastic differential equation for the eigenvalues**

We first consider several needed facts about eigenvalues as functions of entries of a symmetric matrix. We denote by $\mathcal{H}_d$ the collection of symmetric $d$-dimensional matrices. For a matrix $X = (x_{ij}) \in \mathcal{H}_d$ we use the coordinates $x_{ij}, i \leq j$, and in this way we identify $\mathcal{H}_d$ with $\mathbb{R}^{d(d+1)/2}$. We denote by $\mathcal{H}_d^{vg}$ the set of matrices $X \in \mathcal{H}_d$ such that there is a factorization
\[
X = UDU^*,
\]
where $D$ is a diagonal matrix with entries $\lambda_i = D_{ii}$ such that $\lambda_1 > \lambda_2 > \cdots > \lambda_d$, $U$ is an orthogonal matrix, with $U_{ii} > 0$ for all $i$, $U_{ij} \neq 0$ for all $i, j$ and all minors of $U$ have non zero determinants. The matrices in the set $\mathcal{H}_d^{vg}$ are called *very good* matrices, and we can
As a consequence of these results, $\lambda$ has Lebesgue measure zero.

Denote by $U_d^{\theta}$ the set of all orthogonal matrices $U$, with $U_{ii} > 0$ for all $i$, $U_{ij} \neq 0$ for all $i, j$ and all minors of $U$ have non zero determinants. Let $S_d$ be the set

$$S_d = \{ (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d : \lambda_1 > \lambda_2 > \cdots > \lambda_d \}.$$  

For any $\lambda \in S_d$, let $D_{\lambda}$ the diagonal matrix such that $D_{\lambda i} = \lambda_i$. On the other hand, consider the mapping $T : U_d^{\theta} \to \mathbb{R}^{d(d-1)/2}$ defined by

$$T(U) = \left( \frac{U_{12}}{U_{11}}, \ldots, \frac{U_{1d}}{U_{11}}, \ldots, \frac{U_{d-1,d}}{U_{d-1,d-1}} \right).$$

It is known that $T$ is bijective and smooth. Then, the mapping $\hat{T} : S_d \times T(U_d^{\theta}) \to H_d^{\theta}$ given by $\hat{T}(\lambda, z) = T^{-1}(z)D_{\lambda}T^{-1}(z)^*$ is a smooth bijection. Denote by $\Phi$ the inverse of $\hat{T}$.

Then,

$$\Phi(X) = (\lambda, T(U)).$$

As a consequence of these results, $\lambda(X)$ is a smooth function of $X \in H_d^{\theta}$.

Suppose that $X$ is a smooth function of a parameter $\theta \in \mathbb{R}$. Then, we know that

$$\partial_{\theta} \lambda_i = (U \partial_{\theta} XU^*)_{ii}$$

and

$$\partial_{\theta}^2 \lambda_i = (U \partial_{\theta}^2 XU^*)_{ii} + 2 \sum_{j \neq i} \frac{|(U \partial_{\theta} XU^*)_{ij}|^2}{\lambda_i - \lambda_j}.$$  

In particular if $\theta = x_{kh}$ with $k \leq h$, then

$$\frac{\partial \lambda_i}{\partial x_{kh}} = 2u_{ik}u_{ih}1_{\{k \neq h\}} + u_{ik}^21_{\{k = h\}},$$

and

$$\frac{\partial^2 \lambda_i}{\partial x_{kh}^2} = 2 \sum_{j \neq i} \frac{|u_{ik}u_{jh} + u_{ih}u_{jk}|^2}{\lambda_i - \lambda_j}1_{\{k \neq h\}} + 2 \sum_{j \neq i} \frac{|u_{ik}u_{jk}|^2}{\lambda_i - \lambda_j}1_{\{k = h\}}.$$  

Consider now a family of independent fractional Brownian motions with Hurst parameter $H \in (1/2, 1)$, $b = \{ \{b_{ij}(t), t \geq 0\}, 1 \leq i \leq j \leq d \}$. We define the symmetric matrix fractional Brownian motion (with parameter $H$) $B(t)$ by $B_{i,j}(t) = b_{ij}(t)$ if $i < j$ and $B_{ii}(t) = \sqrt{2}b_{ii}(t)$.

As a consequence of the previous discussion, for any $i = 1, \ldots, d$, there exists a function $\Phi_i : \mathbb{R}^{(d+1)/2} \to \mathbb{R}$, which is $C^\infty$ in an open subset $G \subset \mathbb{R}^{(d+1)/2}$, with $|G^c| = 0$, such that $\lambda_i(t) = \Phi_i(b(t))$.

Moreover, for any $k \leq h$

$$\frac{\partial \Phi_i}{\partial b_{kh}} = 2u_{ik}u_{ih}1_{\{k \neq h\}} + \sqrt{2}u_{ik}^21_{\{k = h\}},$$

and

$$\frac{\partial^2 \Phi_i}{\partial b_{kh}^2} = 2 \sum_{j \neq i} \frac{|u_{ik}u_{jh} + u_{ih}u_{jk}|^2}{\lambda_i - \lambda_j}1_{\{k \neq h\}} + 4 \sum_{j \neq i} \frac{|u_{ik}u_{jk}|^2}{\lambda_i - \lambda_j}1_{\{k = h\}}.$$
On the other hand, the joint density of the eigenvalues $\lambda_1(t) > \cdots > \lambda_d(t)$ of $B(t)$ can be obtained from [1, Th. 13.3.5] as follows

\begin{equation}
(4.2) \quad c_d \prod_{k<h} (\lambda_k - \lambda_h) |t|^{-d(d+1)/2} \exp \left( - \sum_{i=1}^d \frac{\lambda_i^2}{2t^2} \right),
\end{equation}

where $c_d$ is a constant depending only on $d$.

Then we can prove the following analogous of (1.1) for the evolving of the eigenvalues processes of a matrix fractional Brownian motion. It is given in terms of the Skorohod integral of the functions $\partial \Phi_i / \partial b_{kh}, i = 1, \ldots, d$.

**Theorem 4.1.** Let $H \in (1/2, 1)$ and $\{B(t), t \geq 0\}$ be a matrix fractional Brownian motion of parameter $H$ as above. Let $X(0)$ be an arbitrary deterministic symmetric matrix and $B(0) = X(0)$. For each $t \geq 0$, let $\lambda_1(t), \ldots, \lambda_d(t)$ be the eigenvalues of $B(t)$. Then, for any $t > 0$ and $i = 1, \ldots, d$,

\begin{equation}
(4.3) \quad \lambda_i(t) = \lambda_i(0) + Y_i^t + 4H \sum_{j \neq i} \int_0^t \frac{s^{2H-1}}{\lambda_i(s) - \lambda_j(s)} ds,
\end{equation}

where

\begin{equation}
(4.4) \quad Y_i^t = \sum_{k \neq h} \int_0^t \frac{\partial \Phi_i}{\partial b_{kh}} (b(s)) \delta b_{kh}(s).
\end{equation}

**Proof.** Without loss of generality we can consider $X(0) = 0$. We claim that the function $\Phi(t) = \{\lambda_i(t) = \Phi_i(b(t)), i = 1, \ldots, d\}$ satisfies the assumptions of Theorem 1.1. In fact, we know that

\[ \sum_{i=1}^d \Phi_i^2 \leq \frac{1}{d} \sum_{i=1}^d b_{ij}^2 + \frac{1}{2} \sum_{i<j} b_{ij}^2, \]

and

\[ \left| \frac{\partial \Phi_i}{\partial b_{kh}} \right| \leq 2 + \sqrt{2}. \]

On the other hand, using the density of the eigenvalues (4.2), we can write

\[ \mathbb{E} \left[ \left| \frac{\partial^2 \Phi_i}{\partial b_{kh}^2} (b_s) \right|^p \right] \leq C_p \sum_{j \neq i} \mathbb{E} \left[ |\lambda_i(s) - \lambda_j(s)|^{-p} \right] \]

\[ = C_p \sum_{j \neq i} \int S_d \prod_{k<h} (\lambda_k - \lambda_h) |\lambda_i - \lambda_j|^{-p} s^{-H(d+1)/2} \exp \left( - \sum_{i=1}^d \frac{\lambda_i^2}{2s^2} \right) d\lambda \]

\[ \leq C_p s^{-pH}, \]

where we have made the change of variable $\lambda_i = s^H \mu_i$. 
Notice that
\[ \sum_{k \leq h} \frac{\partial^2 \Phi_i}{\partial b_{kh}^2} = 2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}. \]
Therefore, Theorem 1.1 yields
\[ \lambda_i(t) = Y_i^t + 2H \sum_{j \neq i} \int_0^t \frac{s^{2H-1}}{\lambda_i(s) - \lambda_j(s)} ds, \]
where \( Y_i^t \) is given by (4.4).

Notice that equation (4.3) is similar to the Itô SDE (1.1) satisfied for the eigenvalues of the matrix Brownian motion, in the case \( H = 1/2 \). A natural question is to ask whether the processes \( Y_i \) are one-dimensional independent fractional Brownian motions in the general case \( H > 1/2 \). We will show that for each \( i = 1, \ldots, d \), the process \( Y_i \) has the same self-similar and variation properties as the fractional Brownian motion. However, we do not know if these processes are Gaussian and, thus, are fractional Brownian motions. We conjecture that the answer to this question is negative.

**Proposition 4.2.** Assuming \( B(0) = 0 \), the process \( Y = (Y_1, \ldots, Y^d) \) is \( H \)-self-similar.

**Proof.** Let \( a > 0 \). By the self-similarity of the fractional Brownian motion it follows that \( \{\lambda(at), t \geq 0\} \) has the same law as \( \{a^H \lambda(t), t \geq 0\} \). Then, the result follows from the equation
\[ Y_i^t = \lambda_i(t) - 4H \sum_{j \neq i} \int_0^t \frac{s^{2H-1}}{\lambda_i(s) - \lambda_j(s)} ds, \]
for \( i = 1, \ldots, d \). \( \square \)

Finally, as an application of Theorem 3.3, we show that for each \( i = 1, \ldots, d \), the process \( Y_i \) has the same \( 1/H \) variation as a fBm with variance \( 2t^{2H} \).

**Proposition 4.3.** For each \( i = 1, \ldots, d \), the process \( Y^i \) has a \( 1/H \) variation equals to \( \sqrt{2t} \mathbb{E} \left[ |Z|^{1/H} \right] \), where \( Z \) is \( N(0,1) \) random variable.

**Proof.** By Theorem 4.1, we have that for each \( k, h \) the process \( \left\{ \frac{\partial \Phi_k}{\partial b_{kh}} (b(s)), s \in [0,T] \right\} \) belongs to the space \( L_{H,i}^{1,1/H} \) for each \( i = 1, \ldots, N = \frac{d(d+1)}{2} \). Therefore, by Theorem 3.3 the \( 1/H \)-variation of \( Y^i \) in the time interval \([0,t]\) is given by
\[ \mathbb{E} \left[ \int_0^t \left| \sum_{k \leq h} \frac{\partial \Phi_k}{\partial b_{kh}} (b(s)) \Theta_{kh} \right|^{1/H} ds \right], \]
where \( \Theta \) is an \( N \)-dimensional standard normal random variable. Let us denote by \( S^N \) the unitary \( N \)-dimensional sphere and let \( \sigma_N \) be the uniform probability measure defined...
on $S^{N-1}$. Notice that the vector 
\[ \frac{1}{\sqrt{2}} \left( \frac{\partial \Phi}{\partial k_{hh}}(b(s)) \right)_{k \leq h}, \]
denoted $b R_s^i$ takes values in $S^{N-1}$ because from (4.1) we obtain
\[ \sum_{k \leq h} \left| \frac{\partial \Phi}{\partial k_{hh}}(b(s)) \right|^2 = 2. \]
Therefore,
\[ 2\mathbb{E}^{\Theta} \left[ \int_0^T |\langle R_s^i, \Theta \rangle|^{1/H} ds \right] = 2\mathbb{E}[|\Theta|] \int_{S^{N-1}} \left[ \int_0^T |\langle R_s^i, \eta \rangle|^{1/H} ds \right] \sigma_N(d\eta). \]
Moreover, if $e \in S^{N-1}$, the integral \[ \int_{S^{N-1}} |\langle e, \eta \rangle|^{1/H} \sigma_N(d\eta) \] does not depend on the vector $e$. Therefore, choosing $e = (1, 0, \ldots, 0)$, yields
\[ 2\mathbb{E}^{\Theta} \left[ \int_0^T |\langle R_s^i, \Theta \rangle|^{1/H} ds \right] = 2t\mathbb{E}[|\Theta|] \int_{S^{N-1}} |\eta_1|^{1/H} \sigma_N(d\eta) = 2t\mathbb{E} \left[ |Z|^{1/H} \right], \]
where $Z$ is a one-dimensional $N(0, 1)$ random variable. This completes the proof of the proposition.

5. No Colliding of Eigenvalues

In this section we will show that for the random symmetric matrix corresponding to a general Gaussian process with Hölder continuous trajectories of order larger than $1/2$, the eigenvalues do not collide almost surely.

Suppose that $x = \{x(t), t \geq 0\}$ is a zero mean Gaussian process satisfying
\[ \mathbb{E}(|x(t) - x(s)|^2) \leq C_T |t - s|^{2\gamma}, \]
for any $s, t \in [0, T]$, where $\gamma \in (1/2, 1)$. Suppose also that $x(0) = 0$ and the variance of $x(t)$ is positive for any $t > 0$. We know that, by Kolmogorov continuity theorem, the trajectories of $x$ are Hölder continuous of order $\beta$ for any $\beta < \gamma$.

Consider a symmetric random matrix of the form
\[ X(t) = X(0) + \tilde{X}(t), \]
where $X(0)$ is a fixed deterministic symmetric matrix and $\tilde{X}_{i,j}(t) = x_{i,j}(t)$ if $i < j$ and $\tilde{X}_{i,i}(t) = \sqrt{2} x_{i,i}(t)$, where $\{x_{i,j}, i \leq j\}$ are independent copies of the process $x$. For any $t > 0$ the matrix $X(t)$ has full rank a.s. The following is the main result of this section.

**Theorem 5.1.** Denote by $\lambda_i(t)$ the eigenvalues of the random matrix $X(t)$, $i = 1, \ldots, n$. We can assume that $\lambda_1(t) \geq \cdots \geq \lambda_d(t)$. Then,
\[ P(\lambda_1(t) > \cdots > \lambda_d(t), \forall t > 0) = 1. \]

**Proof.** The proof will be done in several steps. Fix $t_0 > 0$. From the well-known results for the Gaussian Orthogonal random matrix we know that $\lambda_1(t_0) > \cdots > \lambda_d(t_0)$ almost surely.
Applying the Hoffman-Weilandt inequality (see [10]), we deduce
\[
\sum_{i=1}^{d} (\lambda_i(t) - \lambda_i(s))^2 \leq \frac{1}{d} \sum_{i,j=1}^{d} (X_{ij}(t) - X_{ij}(s))^2 = \frac{1}{d} \sum_{i,j=1}^{d} (\tilde{X}_{ij}(t) - \tilde{X}_{ij}(s))^2
\]
for any \( s, t \geq 0 \). This implies that for each \( i \) and each real \( p \geq 1 \),
\[
(5.1) \quad \mathbb{E}(|\lambda_i(t) - \lambda_i(s)|^p) \leq C|t - s|^p \gamma,
\]
and choosing \( p \) such that \( p \gamma > 1 \) we deduce that the trajectories of \( \lambda_i(t) \) are H"older continuous of order \( \beta \) for any \( \beta < \gamma \).

Consider the stopping time
\[
\tau = \inf\{t \geq t_0 : \lambda_i(t) = \lambda_j(t) \text{ for some } i \neq j\}.
\]
Notice that \( \tau > t_0 \) almost surely. On the random interval \([t_0, \tau)\) the function \( \log(\lambda_i(t) - \lambda_j(t)) \), where \( i \neq j \), is well defined and we can use the stochastic calculus with respect to the Young’s integral to write for any \( t_0 \leq t < \tau \),
\[
(5.2) \quad \log(\lambda_i(t) - \lambda_j(t)) = \log(\lambda_i(t_0) - \lambda_j(t_0)) + \int_{t_0}^{t} \frac{1}{\lambda_i(s) - \lambda_j(s)} (d\lambda_i(s) - d\lambda_j(s)).
\]

The Riemann-Stieltjes integral
\[
I_{i,j}^t := \int_{t_0}^{t} \frac{1}{\lambda_i(s) - \lambda_j(s)} d\lambda_i(s),
\]
can be expressed in terms of fractional derivative operators, following the approach by Zäble [31]. Choosing \( \alpha \) such that \( 1 - \gamma < \alpha < \frac{1}{2} \), we obtain
\[
I_{i,j}^t = \int_{t_0}^{t} D_{t_0+}^{\alpha}(\lambda_i - \lambda_j)^{-1}(s)D_{t-}^{1-\alpha}\lambda_{i,t-}(s)ds + (\lambda_i(t_0) - \lambda_j(t_0))^{-1} (\lambda_i(t) - \lambda_i(t_0)),
\]
where
\[
I_{i,j}(s) := D_{t_0+}^{\alpha}(\lambda_i - \lambda_j)^{-1}(s) = \left( \frac{1}{\Gamma(1 - \alpha)} \left( s^{-\alpha} \frac{1}{\lambda_i(s) - \lambda_j(s)} - \frac{1}{\lambda_i(t_0) - \lambda_j(t_0)} \right) + \alpha \int_{t_0}^{s} (\lambda_i(s) - \lambda_j(s))^{-1} (\lambda_i(y) - \lambda_j(y))^{-1} dy \right),
\]
and
\[
J_j(s) := D_{t-}^{1-\alpha}\lambda_{i,t-}(s) = \frac{1}{\Gamma(\alpha)} \left( \frac{\lambda_i(s) - \lambda_i(t)}{(t - s)^{1-\alpha}} + (1 - \alpha) \int_{s}^{t} \frac{\lambda_i(s) - \lambda_i(y)}{(y - s)^{2-\alpha}} dy \right).
\]
We claim that
\[
(5.3) \quad P\left( \int_{t_0}^{t} |I_{i,j}(s)||J_j(s)|ds < \infty, \text{ for all } t \geq t_0, i \neq j \right) = 1.
\]
In order to show (5.4), we first apply Hölder’s inequality with exponents $E_i$. Then, if $\lambda$ for all $q > \beta$ such that $1 + \frac{1}{p} = 1$ and we get

$$E\left(\int_{t_0}^t |I_{i,j}(s)||J_j(s)| ds\right) < \infty.$$  

In order to show (5.4), we first apply Hölder’s inequality with exponents $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and we get

$$E(|I_{i,j}(s)||J_j(s)|) \leq \left[ E(|I_{i,j}(s)|^p) \right]^{\frac{1}{p}} \left[ E(|J_j(s)|^q) \right]^{\frac{1}{q}}.$$  

By the estimate (5.1) for any fixed $\beta$ such that $1 - \alpha < \beta < \gamma$ there exists a random variable $G$ with moments of all orders such that for all $i$

$$|\lambda_i(s) - \lambda_i(y)| \leq CG|s - y|^{\beta},$$

for all $s, y \in [t_0, t]$, which leads to the estimate

$$E(|J_j(s)|^q) \leq Cq E(G^q),$$

for all $q > 1$ and for some constant $C > 0$. In order to estimate $E(|I_{i,j}(s)|^p)$ we consider first the second summand in the definition of $I_{i,j}(s)$ denoted by

$$K_{i,j}(s) := \int_{t_0}^t \frac{(\lambda_i(s) - \lambda_j(s))^{-1} - \lambda_i(y) - \lambda_j(y))^{-1}}{(s - y)^{\alpha+1}} dy.$$  

This term can be expressed as

$$K_{i,j}(s) = \int_{t_0}^t \frac{(\lambda_i(y) - \lambda_j(y) - \lambda_i(s) + \lambda_j(s))}{(s - y)^{\alpha+1} \left| (\lambda_i(s) - \lambda_j(s)) (\lambda_i(y) - \lambda_j(y)) \right|} dy.$$  

Then, if $a + b = 1$, using the estimate (5.5), we obtain

$$|K_{i,j}(s)| \leq \int_{t_0}^t \frac{|\lambda_i(y) - \lambda_j(y) - \lambda_i(s) + \lambda_j(s)| a |\lambda_i(y) - \lambda_j(y)|}{(s - y)^{\alpha+1} \left| (\lambda_i(s) - \lambda_j(s)) (\lambda_i(y) - \lambda_j(y)) \right|} dy$$

$$\leq (2G)^a \int_{t_0}^t \frac{|\lambda_i(y) - \lambda_j(y) - \lambda_i(s) + \lambda_j(s)|^b}{|\lambda_i(s) - \lambda_j(s)|} (s - y)^{\alpha-\alpha-1} dy$$

$$\leq (2G)^a \int_{t_0}^t \left(|\lambda_i(y) - \lambda_j(y)|^{b-1} |\lambda_i(s) - \lambda_j(s)|^{-1} + |\lambda_i(y) - \lambda_j(y)|^{-1} |\lambda_i(s) - \lambda_j(s)|^{b-1}\right) (s - y)^{\alpha-\alpha-1} dy.$$  

Therefore

$$\|K_{i,j}(s)\|_p \leq 2^a \|G^a\|_{p_1} \int_{t_0}^t \left(\|\lambda_i(y) - \lambda_j(y)|^{b-1}\|_{p_2} \|\lambda_i(s) - \lambda_j(s)|^{-1}\|_{p_3} + \|\lambda_i(y) - \lambda_j(y)|^{-1}\|_{p_2} \|\lambda_i(s) - \lambda_j(s)|^{b-1}\|_{p_3}\right) (s - y)^{\alpha-\alpha-1} dy,$$
where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$, with $p_i > 1$ for $i = 1, 2, 3$. We choose $a, p_1, p_2$ and $p_3$ such that

$$a > \frac{\alpha}{\beta}, \quad p_3 < 2, \quad p_2 < \frac{2\beta}{\alpha},$$

which is possible by taking $p$ and $p_1$ close to 1 and using the inequality $\alpha < \frac{1}{2} < \beta$. Finally, to complete the proof we need to estimate the expectation

$$\mathbb{E}((|\lambda_i(s) - \lambda_j(s)|^{-q}))$$

when $q < 2$. The joint density of the eigenvalues $\lambda_1(s) > \cdots > \lambda_d(s)$ is now given by

$$c_d \prod_{k<h} (\lambda_k - \lambda_h)|\sigma(s)|^{-d(d+1)/2} \exp \left( - \sum_{i=1}^{d} \frac{\lambda_i^2}{2\sigma^2(s)} \right),$$

where $c_d$ is a constant depending only on $d$ and $\sigma^2(s)$ is the variance of $x(s)$; see [1]. Then, the expectation $\mathbb{E}((|\lambda_i(s) - \lambda_j(s)|^{-q}))$ can be estimated up to a constant by

$$\int_{\mathbb{R}^d} \prod_{k<h} (\lambda_k - \lambda_h)|\lambda_i - \lambda_j|^{-q}\sigma(s)^{-d(d+1)/2} \exp \left( - \sum_{i=1}^{d} \frac{\lambda_i^2}{2\sigma^2(s)} \right) d\lambda.$$ 

Making the change of variable $\lambda_i = \sigma(s)\mu_i$ we get

$$\mathbb{E}((|\lambda_i(s) - \lambda_j(s)|^{-q})) \leq C\sigma(s)^{-q}.$$

Our hypothesis of positivity of the variance, together with its continuity, imply that $\sigma(s)$ is bounded away from zero in the interval $[t_0, t]$. Therefore, $\mathbb{E}((|\lambda_i(s) - \lambda_j(s)|^{-q}))$ is uniformly bounded on $[t_0, t]$. This allows us to complete the proof of (5.4). So, the claim (5.3) holds, and this implies that $P(\tau = \infty) = 1$. Finally, letting $t_0$ tend to zero we obtain the desired result. \qed

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