Generating the superpotential on a D-brane category:

C. I. Lazaroiu

Trinity College Dublin
Dublin 2
Ireland
calin@maths.tcd.ie

ABSTRACT: I study $A_\infty$ enhancements of algebraic Calabi-Yau triangulated categories admitting a (triangle) generator, showing that the Serre pairing on such categories determines and is determined by a cyclic pairing on an enhancement of the generator. Using this result, I construct a formal topological string field action inducing an extended D-brane superpotential for such categories. I also give a procedure for lifting certain 2d boundary topological field theories to open topological string theories generated by a single D-brane.
Contents

1. Shift-equivariant open topological field theories in two dimensions 10
   1.1 The mathematical description of open topological field theories in two dimensions 10
   1.2 Two-dimensional open topological field theories with shifts 11
   1.3 Basic extension operations 13
   1.4 The triangulated case 14

2. Background on $A_\infty$ categories 15
   2.1 Basics 16
   2.2 Sector decomposition 20
   2.3 The $A_\infty$ categories $\mathbb{Z}A$, $\Sigma A$ and $\text{tw}(A)$ 23
   2.4 The triangulated categories $D(A)$, $\text{tria}(A)$ and $\text{per}(A)$ 25

3. Cyclic $A_\infty$ categories 27
   3.1 Basics 27
   3.2 Extension of cyclic pairings 31
   3.3 Minimal models induced by a cohomological splitting 33
   3.4 Interpretation through formal open string field theory 41
   3.5 The case of $A_\infty$ algebras 44

4. Cyclic differential graded algebras and their minimal models 45
   4.1 dG modules and bimodules over a dGA 45
   4.2 Cyclic structures on a dGA 46
   4.3 Homological algebra over a dGA 48
   4.4 Serre duality on $\text{tria}(A)$ and $\text{per}(A)$ 49
   4.5 Reconstruction of Serre pairings 53

5. Generating the superpotential 55
   5.1 $A_\infty$ generators of a triangulated category 55
   5.2 The open string field action determined by a cyclic minimal $A_\infty$ generator 56
   5.3 The induced prolongation and superpotential 57
A. Categories with shifts, duality structures and cyclic structures 58
   A.1 Associative and graded associative categories with shifts 58
   A.2 Duality structures and cyclic structures 63
   A.3 Graded duality structures 66
   A.4 Shift-equivariant cyclic structures 67
   A.5 Equivalence of cyclic structures. 69
   A.6 Transport of cyclic structures 70
   A.7 The triangulated case 72

B. Symmetric $\infty$-inner products on an $A_\infty$ algebra 74
   B.1 $A_\infty$ algebras as formal noncommutative $Q$-manifolds 74
   B.2 Symmetric $\infty$ -inner products 76
   B.3 Pull-back of symmetric $\infty$ -inner products 79
Introduction

A basic problem in open topological string theory is the construction of D-brane superpotentials on topological D-brane categories. At first sight, this might seem hopeless since current techniques approach the question one D-brane at a time — while any interesting example involves a D-brane category with an uncountable set of objects.

It is apparent that one needs a method for ‘generating’ this quantity starting from a finite collection of D-branes — in the sense that the superpotential of any finite D-brane system in the category should be determined by the generating branes.

To formulate this clearly, we must distinguish from the outset between (oriented) open-closed topological field theory in two dimensions and (oriented) open-closed topological conformal field theory, also known as topological string theory. The boundary sector of such theories can be described as follows.

For a 2d topological field theory, the boundary sector is given[1, 2] by a graded associative category $G$ (enriched over vector spaces) endowed with nondegenerate bilinear pairings $\langle \ , \rangle_{ab} : \text{Hom}_G(a, b) \times \text{Hom}_G(b, a) \to \mathbb{C}$ which are homogeneous of some common degree $-D$ and satisfy certain compatibility conditions with respect to the composition of morphisms. The physical interpretation of such data is as follows. The objects of $\mathcal{T}$ define the boundary sectors of the theory in the sense of [1], while the morphism spaces $\text{Hom}_G(a, b)$ identify with the spaces of boundary and boundary condition changing topological observables. The morphism compositions describe the associative product of boundary observables, while the bilinear pairings correspond to the boundary topological metrics and thus come from the two-point functions on the disk. When the graded category $\mathcal{G}$ admits a shift functor which preserves the pairings up to sign, then one can also describe this data through the associative category $\mathcal{T}$ obtained from $\mathcal{G}$ by keeping only degree zero morphisms. In this equivalent language, the bilinear pairings give nondegenerate maps $\langle \ , \rangle_{ab} : \text{Hom}_\mathcal{T}(a, b) \times \text{Hom}_\mathcal{T}(b, a[D]) \to \mathbb{C}$ compatible with morphism compositions, which define a nondegenerate ‘invariant’ pairing on $\mathcal{T}$. The original graded category can be recovered from $\mathcal{T}$ as the category $\mathcal{G} = \mathcal{T}^\bullet$ which has the same objects as $\mathcal{T}$, morphism spaces $\text{Hom}_{\mathcal{T}^\bullet}(a, b) = \oplus_{n \in \mathbb{Z}} \text{Hom}_\mathcal{T}(a, b[n])$ and morphism compositions given by:

$$g * f = g[m] \circ f \quad \forall f \in \text{Hom}_\mathcal{T}(a, b[m]), \quad \forall g \in \text{Hom}_\mathcal{T}(b, c[n]).$$

For a topological string theory, the boundary sector is described [3, 4] by a minimal, cyclic and strictly unital $A_\infty$ category $(\mathcal{B}, \langle \ , \rangle)$, whose bilinear pairings $\langle \ , \rangle_{ab} : \text{Hom}_\mathcal{B}(a, b) \times \text{Hom}_\mathcal{B}(b, a) \to \mathbb{C}$ are nondegenerate and homogeneous of common degree $-D$. The ‘suspended forward $A_\infty$ compositions’ give degree one linear maps $\rho_{a_0 \ldots a_n} : \text{Hom}_\mathcal{B}(a_0, a_1)[1] \otimes \ldots \otimes \text{Hom}_\mathcal{B}(a_{n-1}, a_n)[1] \to \text{Hom}_\mathcal{B}(a_0, a_n)[1]$ such that the
quantities \( \langle u_0, \rho_{a_0 \ldots a_n} (u_1 \otimes \ldots \otimes u_n) \rangle_{a_na_0} \) (where \( u_j \in \text{Hom}_B(a_{j-1}, a_j) \) with \( a_{-1} := a_n \)) are graded cyclically symmetric and coincide up to sign with the integrated boundary \( n \)-point functions on the disk. The \( A_\infty \), cyclicity and unitality constraints for \( \rho_{a_0 \ldots a_n} \) were derived in [3, 4] from the axioms of open-closed topological conformal field theory. In such models, the theory possesses a (worldsheet) boundary BRST charge such that the morphism spaces \( \text{Hom}_B(a, b) \) arise as the BRST cohomology on the space of boundary operators for strings stretching from \( a \) to \( b \). The units of the \( A_\infty \) category are the unit boundary observables (=BRST cohomology classes of boundary operators) in the various boundary sectors. The bilinear pairings are induced from the BPZ form by passage to BRST cohomology. One has\(^1\):

\[
\langle u_0, \rho_{a_0 \ldots a_n} (u_1 \otimes \ldots \otimes u_n) \rangle_{a_na_0} = \mathcal{F}_{u_0 \ldots u_n} \; ,
\]

with

\[
\mathcal{F}_{u_0 \ldots u_n} = (-1)^{\tilde{u}_1 + \ldots + \tilde{u}_{n-1}} \left\langle \mathcal{O}_{u_0} \mathcal{O}_{u_1} \mathcal{P} \int \mathcal{O}_{u_2}^{(1)} \ldots \int \mathcal{O}_{u_{n-1}}^{(1)} \mathcal{O}_{u_n} \right\rangle \; ,
\]

(2)

where the big brackets in the right hand side denote the expectation value in the worldsheet theory on the disk. Here \( \mathcal{O}_{u_j} \) are the boundary (condition changing) observables associated with \( u_j \in \text{Hom}_B(a_{j-1}, a_j) \), and \( \mathcal{O}_{u_j}^{(1)} \) are their boundary topological descendants, which are inserted in the clockwise order on the boundary of the disk. The portion of the disk’s boundary lying between the insertion of \( \mathcal{O}_{u_j} \) and \( \mathcal{O}_{u_{j+1}} \) carries the boundary label \( a_j \) and corresponds to the boundary condition associated with that D-brane. The integrals stand for path ordered integration (hence the symbol \( \mathcal{P} \)) over the positions of insertions of boundary descendants. The naive amplitude (2) receives divergent contributions when two or more boundary insertions approach each other. These can be regularized either as in [3] (a version of cutoff regularization) or geometrically by considering the moduli space of stable punctured disks. The second regularization corresponds to the modular functor approach of [4]. Both methods lead to the same constraints on amplitudes, which are encoded by the nondegenerate cyclic, minimal and strictly unital \( A_\infty \) category \( \mathcal{B} \).

Given an open topological string theory, one defines a tree-level extended potential \( W_e \) as follows. Consider the (typically infinite-dimensional) graded vector space \( \mathcal{H}_B := \bigoplus_{a,b \in \text{Ob} \mathcal{A}} \text{Hom}_\mathcal{A}(a, b) \). This carries a cyclic minimal \( A_\infty \) structure with products \( \rho_n \) and pairing \( \langle \rangle \) obtained from \( \rho_{a_0 \ldots a_n} \) and \( \langle \rangle_{ab} \) by ‘summing over sectors’[5]. Picking a Grassmann algebra \( G \), one considers the right \( G \)-module \( (\mathcal{H}_B)_e := \mathcal{H}_B \otimes G \) and the natural extensions \( \langle \rangle_e \) and \( \rho_n^e \) of the pairing and \( A_\infty \) products of \( \mathcal{H}_B \) to \( (\mathcal{H}_B)_e \). The tree-level extended potential is the \( G \)-valued function \( W_e : (\mathcal{H}_B)_e^{\text{odd}} \rightarrow G \) defined through

\(^1\)Notice that the boundary topological metric \( (\; , \; ) \) is denoted by \( \omega \) in reference [3].
the formal expression:

\[ W_e(\psi) = \sum_{n \geq 2} \frac{1}{n+1} \langle \psi, \rho_n^e(\psi^{\otimes n}) \rangle_e \tag{3} \]

where \( \psi \in (\mathcal{H}_B)_{\text{odd}} \). When \( D = 3 \), then one can interpret the restriction of \( W_e \) to the subspace \( \mathcal{H}_B^1 \otimes (\text{Cid}_G) \approx \mathcal{H}_B^1 \) as the superpotential of an \( N = 1 \) supersymmetric field theory in four dimensions obtained from an abstract ‘compactification’ of the string theory associated with the untwisted conformal field theory on which we base our topological model.

Given an open topological string theory, one recovers a 2d topological field theory by keeping only the binary \( A_\infty \) products \( r_{abc} \) and forgetting all higher \( A_\infty \) compositions. Indeed, the \( A_\infty \) constraints imply that the ‘desuspensions’ \( m_{abc} : \text{Hom}_B(a,b) \otimes \text{Hom}_B(b,c) \rightarrow \text{Hom}_B(a,c) \) of \( r_{abc} \) give (degree zero) associative compositions \( * \) on \( B \) via the formula \( m_{abc}(f,g) = (-1)^{\deg f \deg g} g * f \). When endowed only with these compositions, \( B \) becomes a graded associative category with ‘invariant’ nondegenerate bilinear pairings, which can be identified with the category \( G \) of a 2d topological field theory. We say that \( (\mathcal{B}, \langle \, , \, \rangle) \) prolongs \( (\mathcal{G}, \langle \, , \, \rangle) \). A shift functor for the \( A_\infty \) category \( B \) induces a shift functor of the graded category \( G \), and we require that the two shift functors agree. Since \( W_e \) is determined by such a prolongation, the problem of ‘describing the superpotential on \( T \)’ can be strengthened as follows:

**Problem**  
Given a 2d open topological field theory whose boundary sector is described by \( (\mathcal{T}, \langle \, , \, \rangle) \), find an open topological string theory whose boundary sector \( (\mathcal{B}, \langle \, , \, \rangle) \) prolongs \( (\mathcal{T}^\bullet, \langle \, , \, \rangle) \).

In the present paper, I give one solution of this problem under the assumption that \( \mathcal{T} \) is an algebraic triangulated category which is triangle generated by one object (in which case the pairings on \( \mathcal{T} \) are Serre pairings), and show that in this simple situation a superpotential is determined by a single D-brane. The construction I discuss is as follows. Assume that \( \mathcal{T} \) is triangulated and algebraic (i.e. equivalent with the stable category of a Frobenius category, see [16, 21]). Also assume given \( g \in \text{Ob}\mathcal{T} \) such that the smallest triangulated subcategory of \( \mathcal{T} \) containing \( g \) (and its shifts) coincides with \( \mathcal{T} \). Then I construct an ‘off shell model’ (cyclic \( A_\infty \) enhancement) of \( (\mathcal{T}^\bullet, \langle \, , \, \rangle) \), i.e. a non-minimal but strictly unital nondegenerate cyclic \( A_\infty \) category with shifts \( (\mathcal{A}, \langle \, , \, \rangle, \mathcal{A}) \) such that \( H(\mathcal{A}) \approx \mathcal{T}^\bullet \) and such that the nondegenerate pairings of \( \mathcal{A} \) induce the pairings of \( \mathcal{T}^\bullet \) (up to an uninteresting equivalence) when passing to cohomology. Via the results of [4], the data \( (\mathcal{A}, \langle \, , \, \rangle, \mathcal{A}) \) defines a topological D-brane system, which gives a formal extended string field action \( S_e \). The extended tree-level potential of
this system gives an extended ‘superpotential’ $W_e$ on $T^\bullet$, which also carries a cyclic $A_\infty$ prolongation. Both of these are obtained by constructing a strictly unital and shift-invariant cyclic minimal model of $(\mathcal{A}, \langle , \rangle_{\mathcal{A}})$ associated with an appropriate choice of gauge for $S_e$.

In detail, the argument proceeds as follows. Since $T$ is algebraic and triangle generated by $g$, the results of [16, 9] imply that one can find a minimal and strictly unital $A_\infty$ algebra $A_{\text{min}}$ such that $H(A_{\text{min}}) = \text{Hom}_{T^\bullet}(g, g)$ and $T^\bullet = H(\mathcal{A})$, where $\mathcal{A} := \text{tw}(A_{\text{min}})$ is the $A_\infty$ category of twisted complexes over $A_{\text{min}}$ [8, 10, 9]. Any strictly unital and shift-invariant minimal model of $\mathcal{A}$ gives a candidate prolongation of $T^\bullet$, but such a minimal model need not be cyclic. To insure cyclicity, we proceed in two steps:

1. We show that existence of a nondegenerate pairing on $T$ implies that one can choose $A_{\text{min}}$ such that it carries a nondegenerate cyclic pairing. To avoid computational morass, we do this by using a quasi-isomorphic dG model, showing that it carries an invariant and homologically nondegenerate bilinear pairing, then transport this to a nondegenerate cyclic pairing on a minimal model via an $A_\infty$ quasi-isomorphism.

2. We show that any nondegenerate cyclic pairing on $A_{\text{min}}$ induces a nondegenerate cyclic pairing on $\text{tw}(A_{\text{min}})$ via a natural extension process. In turn, the latter descends to a Serre pairing on $H(\text{tw}(A_{\text{min}})) \approx T^\bullet$, which induces the original Serre pairing on $T$ up to an uninteresting transformation.

When endowed with the induced pairing, the cyclic $A_\infty$ category $\mathcal{A} = \text{tw}(A_{\text{min}})$ provides the cyclic off-shell model of $T$ promised above. The space $H_{\mathcal{A}} := \bigoplus_{a, b \in T} \text{Hom}_{\mathcal{A}}(a, b)$ carries the structure of a unital and cyclic $A_\infty$ algebra induced from $\mathcal{A}$, whose bilinear pairing is nondegenerate. Using this data, we construct a formal extended string field action $S_e$ describing a topological D-brane system whose D-branes are the objects of $\mathcal{A}$. Since $(\mathcal{A}, \langle , \rangle_{\mathcal{A}})$ is completely determined by $A_{\text{min}}$ and its pairing, the physics of this topological D-brane system is determined by the latter data. Studying the extremum conditions for $S_e$, one finds that any twisted complex in $\mathcal{A} = \text{tw}(A_{\text{min}})$ can be obtained as the result of ‘topological tachyon condensation’ in a system of open strings stretching between a finite number of shifted copies of a single D-brane $a \in \text{Ob}\mathcal{A}$ such that $A_{\text{min}} = \text{Hom}_{\mathcal{A}}(a, a)$. In this sense, the single D-brane $a$ and its shifts generate the entire $A_\infty$ D-brane category $\mathcal{A}$, as well as its pairings. It therefore also generates all string field amplitudes in this D-brane category.

An extended D-brane potential on $T^\bullet = H(\mathcal{A})$ can now be obtained as the effective tree-level potential of this formal string field action. The superpotential has the form (3), where the minimal products $\rho$ correspond to a cyclic, strictly unital and shift-invariant minimal model of the cyclic, strictly unital $A_\infty$ category with shifts $(\mathcal{A}, \langle , \rangle_{\mathcal{A}})$. For example, one can pick a ‘standard’ gauge fixing condition, leading to a minimal
Since $S_e$ is determined by $A_{\text{min}}$ and its pairing in the sense explained above, the extended potential is entirely determined by this data. In this sense, such a superpotential is generated by the single D-brane $a$.

The paper is organized as follows. Section 1 recalls the mathematical description of oriented open 2d topological field theories, paying special attention to the shift-equivariant and triangulated cases. In Section 2, we recall some background on $A_{\infty}$ categories and their homological algebra, fixing the notation and conventions used throughout the paper. The important point of this section is the choice of signs in the construction of the $A_{\infty}$ category of twisted complexes. In Section 3, we discuss cyclic $A_{\infty}$ categories and give an extension procedure which starts with a cyclic pairing on an $A_{\infty}$ category $\mathcal{A}$ and induces a cyclic pairing on its category of twisted complexes $\text{tw}(\mathcal{A})$ (this generalizes a construction originally discussed in [13, 14, 6] for the dG case with $D = 3$). We also give an explicit construction of cyclic minimal models, which enriches a result of [15], addressing the issues of unitality and shift-equivariance for such models. Finally, we discuss the formal string field theory interpretation of cyclic $A_{\infty}$ categories and give the construction of extended D-brane ‘superpotentials’ starting from a formal string field action. In Section 4, we consider the case of cyclic differential graded algebras $A$. After recalling an equivalent construction of homological algebra over $A$, we show that the category $H^0(\text{tw}(A))$ is Calabi-Yau iff $A$ admits a homologically nondegenerate cyclic pairing (which in this case amounts to the more familiar notion of a graded-symmetric invariant pairing). We also show that a similar statement holds when $A$ is replaced by a minimal $A_{\infty}$ algebra $A_{\text{min}}$. Section 5 gives the construction of our formal string field action. After recalling a generation result due to [16] and [9], we use the results of Section 4 to show that any generator of an algebraic Calabi-Yau triangulated category $\mathcal{T}$ can be enhanced to a cyclic and unital minimal $A_{\infty}$ generator. Using this fact and the extension procedure of Section 3, we build the desired string field action and show that it defines a topological D-brane system enriching the 2d topological field theory described by $\mathcal{T}$. We also show that all D-branes in this system can be obtained from a single D-brane and its shifts through the process of topological tachyon condensation. Finally, we use the construction of Section 3 to induce a superpotential on $\mathcal{T}$ as well as a prolongation of $\mathcal{T}^\bullet$.

Certain technical results used in the paper can be found in appendices. In Appendix A, we discuss categories with shifts, duality structures and cyclic structures. Appendix B provides a different perspective on cyclic minimal models, which is afforded by the geometric formalism of cyclic $A_{\infty}$ algebras [17, 18, 19, 5].

**Conventions, notation and terminology.** Throughout this paper, we work over
the field $\mathbb{C}$ of complex numbers\(^2\) and consider the following tensor (=symmetric monoidal) categories:

- The category vect of vector spaces over $\mathbb{C}$, whose morphism spaces we denote by $\text{Hom}_\mathbb{C}(V, W) := \text{Hom}_{\text{vect}}(V, W)$.

- The category gr of $\mathbb{Z}$-graded vector spaces over $\mathbb{C}$. The morphism spaces are
  \[
  \text{Hom}_{\text{gr}}(V, W) := \{ f \in \text{Hom}_\mathbb{C}(V, W) \mid f(V^n) \subset W^n \quad \forall n \in \mathbb{Z} \},
  \]
  where $V = \bigoplus_{n \in \mathbb{Z}} V^n$ and $W = \bigoplus_{n \in \mathbb{Z}} W^n$.

- The category dif of (possibly unbounded) cochain complexes of vector spaces over $\mathbb{C}$. Viewing complexes as pairs $(V, d_V)$ with $V \in \text{Ob}[\text{gr}]$ and $d_V$ a differential of degree +1 on $V$, the morphism spaces are:
  \[
  \text{Hom}_{\text{dif}}(V, W) := \{ f \in \text{Hom}_{\text{gr}}(V, W) \mid d_W \circ f_k = f \circ d_V \} .
  \]

In the present paper, an **associative category** means a small associative category enriched over vect. A **graded associative category** means a small associative category enriched over gr. A **differential graded (dG) category** means a small associative category enriched over dif. An $\mathcal{A}_\infty$ category means an $\mathcal{A}_\infty$ category enriched over gr. Similar enrichment conventions apply to functors between such categories. We make systematic use of the Koszul sign rule for graded quantities, unless explicitly mentioned otherwise. We also use the convention of writing various equations only for homogeneous elements (in order to indicate the signs). We will make use of the following enriched categories:

- Vect is the $\mathbb{C}$-category of vector spaces over $\mathbb{C}$. This is the same as vect except that $\text{Hom}_\mathbb{C}(V, W)$ are viewed as vector spaces rather than as sets.

- Gr is the graded associative category of $\mathbb{Z}$-graded vector spaces over $\mathbb{C}$. This has morphisms spaces $\text{Hom}_{\text{Gr}}(V, W) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}^n_{\text{Gr}}(V, W)$, where:
  \[
  \text{Hom}^n_{\text{Gr}}(V, W) := \{ f \in \text{Hom}_\mathbb{C}(V, W) \mid f(V^k) \subset W^{k+n} \quad \forall k \in \mathbb{Z} \} .
  \]
  A graded vector space $V$ is called **degreewise finite** if $\dim_{\mathbb{C}} V^n < \infty$ for all $n \in \mathbb{Z}$. Degreewise finite graded vector spaces form a full graded subcategory $\text{Gr}_{\text{df}}$ of Gr.

\(^2\)All results extend trivially to base fields of characteristic zero.
• Dif is the dG category of (possibly unbounded) cochain complexes of vector spaces over \( \mathbb{C} \). This has morphism spaces:

\[
\text{Hom}_{\text{Dif}}(V, W) := \text{Hom}_{\text{Gr}}(V, W)
\]

with differentials \( d_{V,W} \in \text{Hom}^1_{\text{Gr}}(\text{Hom}_{\text{Dif}}(V, W)) \) given by:

\[
d_{V,W}(f) = d_W \circ f - (-1)^{\deg f} f \circ d_V .
\]  

Notice that \( Z^0(\text{Dif}) = \text{dif} \) and the forgetful functor gives an embedding \( \text{Dif} \subset \text{Gr} \). Here and below, we let \( Z(\ldots), B(\ldots), H(\ldots) \) denote passage to cocycles, coboundaries and cohomology.

The graded category \( \text{Gr} \) is endowed with a shift functor [1], which is defined through:

\[
V[1]^n := V^{n+1} ,
\]

with the obvious action on morphisms. This gives an automorphism of \( \text{Gr} \) as a graded category. Setting \([n] := [1]^n\), we have \( \text{Hom}^n_{\text{Gr}}(V, W) = \text{Hom}_{\text{Gr}}(V, W[n]) \). Let \( \text{id}_{\text{Gr}} \) be the identity endofunctor. The suspension of \( V \in \text{ObGr} \) is the map \( s_V : V \to V[1] \) (the identity map of \( V \) viewed as a map of degree \(-1\) from \( V \) to \( V[1] \)). The signed suspension is the map \( \sigma_V : V \to V[1] \) of degree \(-1\) given by \( \sigma_V(x) = (-1)^{\deg x} x \). The signed suspensions give a natural transformation \( \sigma : \text{id}_{\text{Gr}} \to [1] \) of degree \(-1\) since one has \( \sigma_W f = (-1)^{\deg f} f \sigma_V \) for homogeneous \( f \in \text{Hom}_{\text{Gr}}(V, W) \) (notice the sign factor which is required by the Koszul rule).

Similarly, the dG category \( \text{Dif} \) has a shift functor which acts on objects \((V, d_V)\) as \((V, d_V)[1] := (V[1], d_V)\) and on morphisms in the same way as in \( \text{Gr} \). The signed suspensions give maps of complexes \( \sigma_V : (V, d_V) \to (V, d_V)[1] \) of degree \(-1\). Together, they define a natural transformation \( \sigma : \text{id}_{\text{Dif}} \to [1] \) of degree \(-1\).

The dualization functor is the contravariant endofunctor \( \text{v} \) of \( \text{Gr} \) defined as follows. For any \( V \in \text{ObGr} \), set \( V^\text{v} := \text{Hom}_{\text{Gr}}(V, \mathbb{C}) \), where \( \mathbb{C} \) is viewed as a graded vector space concentrated in degree zero. We have \((V^\text{v})^n = \text{Hom}_{\text{Gr}}(V, \mathbb{C}[n]) = \{ \eta \in \text{Hom}_\mathbb{C}(V, \mathbb{C}) | \eta(x) = 0 \text{ unless } \deg x = -n \} \). This gives isomorphisms \((V^\text{v})^n \approx \text{Hom}_\mathbb{C}(V^{-n}, \mathbb{C}) \). The functor \( \text{v} \) acts on homogeneous morphisms \( f \in \text{Hom}_{\text{Gr}}(V, W) \) by:

\[
(f^\text{v})(\eta) := (-1)^{\deg f \deg \eta} \eta \circ f ,
\]  

which implies the graded contravariance condition \((f \circ g)^\text{v} = (-1)^{\deg f \deg g} g^\text{v} \circ f^\text{v} \) for composable \( f, g \). The dualization functor preserves \( \text{Gr}_{\text{df}} \) and squares to the identity on this subcategory.
The dualization functor induces a contravariant dG functor $v : \text{Dif} \to \text{Dif}$ as follows. For any complex $(V, d_V)$, endow $V^v$ with the differential $d_{V^v} = -d_V$, i.e.:

$$d_{V^v} \eta := (1 + \text{deg} \eta) \circ d_V$$

and let $v$ act on morphisms as in (5). Notice the $d$-compatibility relations $d_{W^v, V^v}(f^v) = d_{V, W}(f)^v$. The natural isomorphism

$$H(V^v) \approx H(V)^v,$$

implies that dualization preserves the full subcategory of acyclic complexes.

An associative category $\mathcal{A}$ will be called Hom-finite if $\text{Hom}_\mathcal{A}(a, b)$ is finite-dimensional for all objects $a, b$. A graded associative category $\mathcal{G}$ is called Hom-finite if its underlying associative category is Hom-finite. It is called degreewise Hom-finite if all $\text{Hom}_\mathcal{G}(a, b)$ are degree-wise finite.

For any unital associative ring $R$, we let $R\text{Mod}$, $\text{Mod}_R$ and $R\text{GrMod}$ denote the categories of (unital) left, right and bi-modules over $R$ and $R\text{GrMod}$, $\text{GrMod}_R$, $R\text{GrMod}_R$ the categories of (unital) graded left, right and bi-modules over $R$.

For a unital dG algebra $A$, we let $A\text{dGMod}$, $\text{dGMod}_A$ and $A\text{dGMod}_A$ denote the dG categories of (strictly) unital dG left, right and bi-modules over $A$.

For an $A_{\infty}$ algebra $A$, we let $A\text{Mod}$, $\text{Mod}_A$ and $A\text{Mod}_A$ denote the dG categories of strictly unital $A_{\infty}$ left, right and bi-modules over $A$. Similar notation applies when $A$ is replaced with an $A_{\infty}$ category $\mathcal{A}$.

1. Shift-equivariant open topological field theories in two dimensions

In this Section, we discuss the mathematical description of open topological field theories in two dimensions, paying special attention to the case when the category of boundary sectors has a shift functor. A more detailed account of certain aspects can be found in Appendix A, which also discusses the relation with the usual theory of Serre functors.

1.1 The mathematical description of open topological field theories in two dimensions

Given an integer $D$, a $D$-cyclic structure on a unital graded associative category $\mathcal{G}$ is a family of degree zero linear maps $\text{tr}_a : \text{Hom}_\mathcal{G}(a, a) \to \mathbb{C}[-D]$ indexed by the objects

\[^3\]This follows from $B(V^v) = \text{im}(d_{V^v}) = (\ker d_{V^v})^o$, $Z(V^v) = \ker(d_{V^v}) = (\text{im } d_V)^o$ and $(\text{im } d_V)^o/(\ker d_V)^o \approx (\ker d_V/\text{im } d_V)^v$, where $S^o := \oplus_{n \in \mathbb{Z}} \{ \eta \in (V^v)^o | \eta|_{s-n} = 0 \} \subset V^v$ is the degree-wise polar of a homogeneous linear subspace $S = \oplus_{n \in \mathbb{Z}} S^n \subset V^v$. These relations do not require finite-dimensionality.
A graded category \(G\), which satisfy the relations:

\[
\text{tr}_a(uv) = (-1)^\deg_u \deg_v \text{tr}_b(vu), \quad \text{for } v \in \text{Hom}_G(a,b), \forall u \in \text{Hom}_G(b,a).
\]

(1.1)

Defining degree zero bilinear pairings \(\langle \rangle_{ab} : \text{Hom}_G(a,b) \times \text{Hom}_G(b,a) \to \mathbb{C}[-D]\) via 

\[
\langle u, v \rangle_{ab} := \text{tr}_b(uv),
\]

this corresponds to the conditions:

\[
\langle uf, v \rangle_{a'b} = \langle u, fv \rangle_{a,b}, \forall f \in \text{Hom}_G(a',a), \ u \in \text{Hom}_G(a,b), \ v \in \text{Hom}_G(b,a').
\]

(1.2)

\[
\langle u, v \rangle_{a,b} = (-1)^{\deg_u \deg_v} \langle v, u \rangle_{b,a}, \forall u \in \text{Hom}_G(a,b), \forall v \in \text{Hom}_G(b,a).
\]

(1.3)

A graded category \(G\) endowed with a \(D\)-cyclic structure will be called a \(D\)-cyclic graded category. The cyclic structure is called non-degenerate if \(G\) is degreewise \(\text{Hom}\)-finite and all pairings \(\langle u, v \rangle_{ab}\) are nondegenerate as bilinear forms. The following is a trivial extension of a result proved in [1]:

An oriented open topological field theory in two-dimensions (=the boundary sector of an oriented open-closed 2d topological field theory) is described by a unital graded associative category \(G\) endowed with a nondegenerate cyclic structure.

As in [1], this follows from the modular functor approach, except that we allow for a \(\mathbb{Z}\)-grading on the space of worldsheet fields. This is possible provided that the worldsheet model admits an unbroken \(U(1)\) symmetry. The objects of \(G\) are interpreted as boundary sector labels, while the composition of morphisms in \(G\) gives the boundary products.

1.2 Two-dimensional open topological field theories with shifts

In this paper, we are interested in the case when \(G\) admits a shift functor, which we denote by \([1]\). By definition, this is an automorphism of \(G\) together with isomorphisms of graded vector spaces \(\text{Hom}_G(a,b[1]) \approx \text{Hom}_G(a,b)[1]\), which are natural in \(a\) and \(b\). A \(D\)-cyclic structure on \((G,[1])\) is called shift-equivariant if the following conditions are satisfied:

\[
\text{tr}_{a[1]}(u[1]) = (-1)^{D+1} \text{tr}_a(u) \Leftrightarrow \langle u[1], v[1] \rangle_{a[1][b][1]} = (-1)^{D+1} \langle u, v \rangle_{ab}.
\]

(1.4)

In this case, we also say that the associated open 2d topological field theory is shift-equivariant.

Let \(\mathcal{T} = G^0\) be the null restriction of \(G\), i.e. the subcategory of \(G\) obtained by keeping only morphisms of degree zero. This admits an automorphism (again denoted by \([1]\)) obtained by restricting the shift functor of \(G\). Then \(G\) can be reconstructed as
the graded completion $\mathcal{G} = \mathcal{T}^\bullet$ of the unital associative category $\mathcal{T}$. This is the category having the same objects as $\mathcal{T}$ and morphism spaces $\text{Hom}_{\mathcal{T}^\bullet}(a, b) = \oplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(a, b[n])$, with compositions defined as in (1). In fact, graded completion and null restriction give inverse equivalences between the categories of small associative categories with shifts and small graded associative categories with shifts (see Appendix A).

Let us define a $D$-cyclic structure on a unital (not graded) associative category with shifts $(\mathcal{T}, [1])$ to be a family of linear maps $tr_a : \text{Hom}_\mathcal{T}(a, a[D]) \to \mathbb{C}$ indexed by the objects of $\mathcal{T}$, which satisfy the relations:

$$tr_a(u \circ v) = tr_b(v[D] \circ u) , \text{ for } v \in \text{Hom}_\mathcal{T}(a, b) , u \in \text{Hom}_\mathcal{T}(b, a[D]) .$$

(1.5)

Defining pairings $(, )_{ab} : \text{Hom}_\mathcal{T}(a, b) \times \text{Hom}_\mathcal{T}(b, a[D]) \to \mathbb{C}$ via $(u,v)_{ab} := tr_b(u[D] \circ v)$, this corresponds to the conditions:

$$(u \circ f, v)_{a'b} = (u, f[D] \circ v)_{a,b} \forall f \in \text{Hom}_\mathcal{T}(a', a) , u \in \text{Hom}_\mathcal{T}(a, b) , v \in \text{Hom}_\mathcal{T}(b, a'[D])$$

(1.6)

$$(u, v)_{a,b} = (v, u[D])_{b,a[D]} \forall u \in \text{Hom}_\mathcal{T}(a, b) \forall v \in \text{Hom}_\mathcal{T}(b, a[D])$$

(1.7)

We say that the $D$-cyclic structure is shift-equivariant if the following relations are satisfied:

$$tr_{a[1]}(u[1]) = (-1)^{D+1} tr_a(u) \Leftrightarrow \langle u[1], v[1] \rangle_{a[1][b[1]]} = (-1)^{D+1} \langle u, v \rangle_{ab} .$$

(1.8)

We say that it is nondegenerate if $\mathcal{T}$ is Hom finite and all pairings $(, )_{ab}$ are nondegenerate as bilinear forms. One has the following correspondence (see Appendix A):

Let $(\mathcal{G}, [1])$ be a unital graded category with shifts and $(\mathcal{T}, [1])$ the corresponding unital associative category with shifts (thus $\mathcal{T} = \mathcal{G}^0$ and $\mathcal{G} = \mathcal{T}^\bullet$). The following data are equivalent:

(a) A shift-equivariant $D$-cyclic structure on $(\mathcal{G}, [1])$.

(b) A shift-equivariant $D$-cyclic structure on $(\mathcal{T}, [1])$.

Moreover, one is nondegenerate iff the other is. In this case, a shift-equivariant topological field theory in two dimensions can be described by either datum.

It is often convenient to work with the twisted shift functor $[[1]]$ of $\mathcal{G}$, an automorphisms of $\mathcal{G}$ which acts on objects through $a[[1]] := a[1]$ and on homogeneous morphisms $f$ through $f[[1]] = (-1)^{\text{deg} f} f[1]$. In terms of this, the shift-equivariance conditions (1.4) take the form:

$$\text{tr}_{a[[1]]}(u[[1]]) = -\text{tr}_a(u) \Leftrightarrow \langle u[1], v[1] \rangle_{a[[1]][b[1]]} = -\langle u, v \rangle_{ab} .$$

(1.9)
The isomorphisms $\text{Hom}_G(a, b[1]) \approx \text{Hom}_G(a, b)[1]$ become isomorphisms $\text{Hom}_G(a, b[[1]]) \approx \text{Hom}_G(a, b)[1]$ which are natural up to missing Koszul signs (see Appendix A).

Since their restrictions to $\mathcal{T} = \mathcal{G}^0$ coincide, $[1]$ and $[[1]]$ can be viewed as different extensions of the shift functor of $\mathcal{T}$ to $\mathcal{G}$. It is clear that the shift functor of $\mathcal{G}$ can be recovered from $[[1]]$ by a further twist, i.e. we have $[[[[1]]]] = [1]$.

Two shift-equivariant $D$-cyclic structures $\text{tr}$ and $\text{tr}'$ on $\mathcal{G}$ are called equivalent if there exists an automorphism $f$ of the identity functor $\text{id}_\mathcal{G}$ of $\mathcal{G}$ such that $f_{a[1]} = f_a[1]$ and $\text{tr}'_a(u) = \text{tr}_a(uf_a)$ for all $a \in \text{Ob}\mathcal{G}$ and all $u \in \text{Hom}_\mathcal{G}(a, a)$. Equivalently, $\text{tr}'_a(u) = \text{tr}_a(u \circ f_a)$ for all $a$ and all $u \in \text{Hom}_\mathcal{T}(a, a[D])$ (see Appendix A). The Yoneda lemma implies that any two nondegenerate $D$-cyclic structures on $\mathcal{T}$ are equivalent in this sense. It follows that a shift-equivariant 2d topological field theory whose boundary sectors and products are specified by $\mathcal{T}$ is determined up to such a transformation.

### 1.3 Basic extension operations

Define a shift-equivariant $D$-cyclic category to be a triplet $(\mathcal{T}, [1], \text{tr})$ where $\mathcal{T}$ is a unital associative category, $[1]$ is a shift functor on $\mathcal{T}$ and $\text{tr}$ is a shift-equivariant $D$-cyclic structure on $(\mathcal{T}, [1])$. One has two basic unary operations on such objects, namely additive completion and idempotent completion. Both of them preserve the nondegeneracy condition on cyclic structures and thus induce operations on shift-equivariant two-dimensional open topological field theories.

Recall that the additive completion of $\mathcal{T}$ is the smallest additive category $\mathcal{T}^{\text{add}}$ containing $\mathcal{T}$ as a full subcategory. Its objects are finite direct sums $A = \bigoplus_{i=1}^n a_i$ of objects $a_i$ of $\mathcal{T}$, while its morphism spaces are given by $\text{Hom}_{\mathcal{T}^{\text{add}}}(A, A') = \bigoplus_{i,j} \text{Hom}_{\mathcal{T}}(a_i, a'_j)$, where $A' = \bigoplus_{j=1}^n a'_j \in \text{Ob}\mathcal{T}^{\text{add}}$. When $\mathcal{T}$ admits a shift functor $[1]$, then $\mathcal{T}^{\text{add}}$ admits the shift functor $[1]^{\text{add}}$ given by $A = \bigoplus_{i=1}^n a_i \rightarrow A[1]^{\text{add}} := \bigoplus_{i=1}^n a_i[1]$ on objects and by $u = \bigoplus_{i,j} u_{ij} \rightarrow u[1]^{\text{add}} := \bigoplus_{i,j} u_{ij}[1]$ on morphisms $u \in \text{Hom}_{\mathcal{T}^{\text{add}}}(A, A')$. $\mathcal{T}$ embeds in the obvious manner as a full subcategory of $\mathcal{T}^{\text{add}}$. It is clear that $\mathcal{T}^{\text{add}}$ is Hom finite iff $\mathcal{T}$ is. When $(\mathcal{T}, [1], \text{tr})$ is a shift-equivariant $D$-cyclic category, then $(\mathcal{T}^{\text{add}}, [1]^{\text{add}})$ admits a shift-equivariant $D$-cyclic structure $\text{tr}^{\text{add}}$ given by:

$$\text{tr}^{\text{add}}_A(u) = \sum_i \text{tr}_a(u_{ii}) \quad \forall u = \bigoplus_{i,j} u_{ij} \in \text{Hom}_{\mathcal{T}^{\text{add}}}(A, A[D]) \quad , \quad A = \bigoplus_{i=1}^n a_i .$$

We say that $(\mathcal{T}^{\text{add}}, [1]^{\text{add}}, \text{tr}^{\text{add}})$ is the additive completion of $(\mathcal{T}, [1], \text{tr})$. It is easy to see that the former is nondegenerate iff the later is.

Recall that the idempotent completion of the category $\mathcal{T}$ is the category $\mathcal{T}^\pi$ defined as follows. Its objects are pairs $(a, e)$ with $a \in \text{Ob}\mathcal{A}$ and $e \in \text{Hom}_{\mathcal{T}}(a, a)$ such that $e^2 = e$. Its morphism spaces are $\text{Hom}_{\mathcal{T}^\pi}((a, e), (b, e')) := e' \circ \text{Hom}_{\mathcal{T}}(a, b) \circ e$. When $\mathcal{T}$ has a shift functor $[1]$, then $\mathcal{T}^\pi$ has the shift functor $[1]^\pi$ which acts on objects by
\((a,e)[1]^\pi := (a[1], e[1])\) and on morphisms \(u \in \text{Hom}_T((a,e), (b,e'))\) by \(a[1]^\pi := u[1]\). Notice that \(\text{Hom}_T((a,e), (b,e'))\) is a subspace of \(\text{Hom}_T(a,b)\). It is clear that \(\mathcal{T}^\pi\) is Hom finite iff \(\mathcal{T}\) is. \(\mathcal{T}\) embeds in the obvious manner as a full subcategory of \(\mathcal{T}^\pi\). When \((\mathcal{T}, [1], tr)\) is a shift-equivariant \(D\)-cyclic category, then \((\mathcal{T}^\pi, [1]^\pi)\) admits a shift-equivariant \(D\)-cyclic structure \(tr^\pi\) defined by restricting \(tr\):

\[
tr^\pi_{(a,e)}(u) := tr_a(u) \quad \forall u \in \text{Hom}_{\mathcal{T}^\pi}((a,e), (a,e)[D]) \subset \text{Hom}_T(a,a[D]).
\]

We say that \((\mathcal{T}^\pi, [1]^\pi, tr^\pi)\) is the idempotent completion of \((\mathcal{T}, [1], tr)\). It is easy to check that the former is nondegenerate iff the later is. More details about idempotent completion can be found in Appendix A.

1.4 The triangulated case

In this paper, we are interested in the case when the category \(\mathcal{T}\) is triangulated. As argued in [13, 31], this condition must be imposed if our 2d topological field theory is to admit a lift to a ‘dynamically closed’ topological string theory, a.k.a a 2d topological conformal field theory. In this case, we let \([1]\) be the shift functor of \(\mathcal{T}\) as a triangulated category. The functor \([D]\) becomes exact when endowed with the isomorphism of functors \([D] \circ [1] \xrightarrow{\sim} [1] \circ [D]\) which acts trivially on objects but acts on morphisms through multiplication by \((-1)^D\).

Assuming \(\mathcal{T}\) to be triangulated, a shift-equivariant and nondegenerate \(D\)-cyclic structure on \((\mathcal{T}, [1])\) corresponds to a Serre duality structure whose Serre functor equals \([D]\); the pairings of the \(D\)-cyclic structure are the usual Serre pairings of \(\mathcal{T}\). We say that \(\mathcal{T}\) is a Calabi-Yau category of dimension \(D\) (or \(D\)-Calabi-Yau category) if it admits a non-degenerate shift-equivariant \(D\)-cyclic structure; in this case, all such \(D\)-cyclic structures are equivalent.

The triangulated structure of \(\mathcal{T}\) allows one to introduce various generation properties, which — when present — allow for an explicit characterization of \(\mathcal{T}\). We recall these below for later use.

Generators of triangulated categories Let \(\mathcal{T}\) be a triangulated category and \(\mathcal{U}\) a set of objects of \(\mathcal{T}\). We let \(\text{add}(\mathcal{U})\) be the full subcategory of \(\mathcal{T}\) whose objects are finite direct sums of shifts of objects lying in \(\mathcal{U}\). The smallest strictly full triangulated subcategory\(^4\) of \(\mathcal{T}\) containing \(\mathcal{U}\) will be denoted \(\text{tria}_\mathcal{T}(\mathcal{U})\). It consists of successive extensions of objects of \(\text{add}(\mathcal{U})\). Explicitly, the objects of \(\text{tria}_\mathcal{T}(\mathcal{U})\) are those objects of \(\mathcal{T}\) which admit a finite filtration whose associated graded belongs to \(\text{add}(\mathcal{U})\) (the graded is defined by taking triangles on each morphism of the filtration, and is unique up to non-canonical isomorphism). When \(\text{tria}_\mathcal{T}(\mathcal{U}) = \mathcal{T}\), we say that \(\mathcal{U}\) triangle generates \(\mathcal{T}\).

\(^4\)In particular, this is assumed closed under shifts and thus contains \(a[n]\) for all \(a \in \mathcal{U}\) and \(n \in \mathbb{Z}\).
The smallest thick\(^5\) and strictly full triangulated category of \( \mathcal{T} \) containing \( \mathcal{U} \) will be denoted \( \text{tria}_T(\mathcal{U}) \); it consists of direct summands of objects of \( \text{tria}_T(\mathcal{U}) \). When \( \mathcal{T} \) is idempotent complete, we have a natural isomorphism \( \text{tria}_T(\mathcal{U}) \cong \text{tria}_T(\mathcal{U})^\pi \). When \( \text{tria}_T(\mathcal{U}) = \mathcal{T} \), we say that \( \mathcal{U} \) is a Karoubian generating set for \( \mathcal{T} \).

If \( \mathcal{T} \) has arbitrary coproducts, we let \( \text{add}(\mathcal{U}) \) be the full subcategory of \( \mathcal{T} \) whose objects are arbitrary direct sums of shifts of objects lying in \( \mathcal{U} \). In this case, we define \( \text{Tria}_T(\mathcal{U}) \) to be the smallest strictly full triangulated subcategory of \( \mathcal{T} \) containing \( \mathcal{U} \) and closed under arbitrary coproducts. It consists of successive extensions of objects of \( \text{add}(\mathcal{U}) \), i.e. of those objects of \( \mathcal{T} \) admitting a finite filtration whose graded belongs to \( \text{add}(\mathcal{U}) \). We say that \( \mathcal{U} \) compactly generates \( \mathcal{T} \) if \( \text{Tria}_T(\mathcal{U}) = \mathcal{T} \) and moreover each object \( a \) of \( \mathcal{U} \) is compact (a.k.a small) in \( \mathcal{T} \), i.e. \( \text{Hom}_\mathcal{T}(a, \cdot) \) commutes with all coproducts on \( \mathcal{T} \). When \( \mathcal{T} \) is clear from the context, we write \( \text{Tria}(\mathcal{U}) \) instead of \( \text{Tria}_T(\mathcal{U}) \) etc. When \( \mathcal{U} \) consists of a single object \( a \), we write \( \text{Tria}(a) \) instead of \( \text{Tria}(\{a\}) \) etc.

Relation with cyclic structures Let \( (\mathcal{T}, [1], tr) \) be a shift-equivariant \( D \)-cyclic triangulated category, and \( \mathcal{U} \subset \text{Ob}\mathcal{T} \) a non-void set of objects as above. Setting \( \mathbb{Z}\mathcal{U} := \{a[n] | a \in \mathcal{U}, n \in \mathbb{Z}\} \), we let \( \mathcal{A} \) be the full subcategory of \( \mathcal{T} \) on the set of objects \( \mathbb{Z}\mathcal{U} \). When endowed with the induced shift functor and traces, this becomes a \( D \)-cyclic category \( (\mathcal{A}, [1]^\mathcal{A}, \text{tr}^\mathcal{A}) \). Let us assume that \( \mathcal{U} \) triangle generates \( \mathcal{T} \) or that \( \mathcal{T} \) is idempotent complete and \( \mathcal{U} \) Karoubi generates \( \mathcal{T} \). Then one has the following non-degeneracy criterion, which shows that nondegeneracy of \( \text{tr} \) is equivalent with nondegeneracy of \( \text{tr}^\mathcal{A} \).

Proposition (Appendix A) Assume that \( \mathcal{U} \) triangle generates \( \mathcal{T} \) or that \( \mathcal{T} \) is idempotent complete and \( \mathcal{U} \) Karoubi generates \( \mathcal{T} \). Then a shift-equivariant \( D \)-cyclic structure \( tr \) on \( (\mathcal{T}, [1]) \) is non-degenerate iff the bilinear forms \( (\ , \ )_{ab} : \text{Hom}_\mathcal{T}(a,b) \times \text{Hom}_\mathcal{T}(b,a[D]) \to \mathbb{C} \) are non-degenerate for all \( a,b \in \mathbb{Z}\mathcal{U} \).

2. Background on \( A_\infty \) categories

This section recalls the basics of \( A_\infty \) categories, fixing notations and sign conventions. The reader can consult [20, 21, 22] for reviews and [9, 10, 23, 24] for in-depth discussion. We use ‘forward suspended compositions’ in order to simplify sign factors. The sign conventions in the definition of shift functors and of the shift completion are somewhat non-standard, being motivated by the application to enhanced triangulated categories.

\(^5\)i.e. closed under taking direct summands (epaisse).
These are chosen consistently with those of Appendix A, which the reader might wish to consult while reading this section.

2.1 Basics

Recall that a small $A_\infty$ category $\mathcal{A}$ is specified by a set of objects $\text{Ob}\mathcal{A}$ and by graded vector spaces $\text{Hom}_\mathcal{A}(a,b)$ for any $a, b \in \text{Ob}\mathcal{A}$, together with linear maps $\mu_{a_0...a_n} : \text{Hom}_\mathcal{A}(a_{n-1},a_n) \otimes \ldots \otimes \text{Hom}_\mathcal{A}(a_1,a_2) \otimes \text{Hom}_\mathcal{A}(a_0,a_1) \to \text{Hom}_\mathcal{A}(a_0,a_n)$ of degree $2-n$ subject to $A_\infty$ constraints (see eqs. (2.1) below). Denoting the degree of homogeneous elements $x \in \text{Hom}_\mathcal{A}(a,b)$ by $|x|$, the homogeneity constraints on the $A_\infty$ products take the form:

$$|\mu_{a_0...a_n}(x_n \otimes \ldots \otimes x_1)| = |x_1| + \ldots + |x_n| + 2 - n.$$ 

In particular, $\mu_{ab} : \text{Hom}_\mathcal{A}(a,b) \to \text{Hom}_\mathcal{A}(a,b)[1]$ have degree one, while $\mu_{cba} : \text{Hom}_\mathcal{A}(b,c) \otimes \text{Hom}_\mathcal{A}(a,b) \to \text{Hom}_\mathcal{A}(a,c)$ have degree zero.

For any objects $a, b$ of $\mathcal{A}$, let $s_{ab} : \text{Hom}_\mathcal{A}(a,b) \to \text{Hom}_\mathcal{A}(a,b)[1]$ be the suspension operator of the graded vector space $\text{Hom}_\mathcal{A}(a,b)$. Denoting the degree of elements $x \in \text{Hom}_\mathcal{A}(a,b)[1]$ by $\tilde{x} = |x| - 1$, we have $s(x) = x$ and $s$ is a map of degree $-1$. Notice that $\text{Hom}_\mathcal{A}(a,b)[1]$ is the same vector space as $\text{Hom}_\mathcal{A}(a,b)$, except that we use the ‘tilde grading’ instead of the grading given by $|\ |$. To simplify notation, we will often use $s_{ab}$ instead of $s_{ab}[n]$ to denote the induced map $s_{ab}[n] : \text{Hom}_\mathcal{A}(a,b)[n] \to \text{Hom}_\mathcal{A}(a,b)[n + 1]$. Accordingly, we write $s^0_{ab} : \text{Hom}_\mathcal{A}(a,b) \to \text{Hom}_\mathcal{A}(a,b)[n]$ for the iteration $s_{ab}[n - 1] \circ \ldots \circ s_{ab}[1] \circ s_{ab}$ with $n$ a positive integer and set $s^0_{ab} := \text{id}_{\text{Hom}_\mathcal{A}(a,b)}$ and $s^0_{ab} := s^{-1}_{ab}[n + 1] \circ \ldots \circ s^{-1}_{ab}[1] \circ s^{-1}_{ab}$ for $n$ a negative integer.

The $A_\infty$ constraints can be written in a few equivalent forms. In order to obtain the maximum simplification of sign factors, it is convenient to work not with the traditional compositions $\mu$, but rather with equivalent maps defined as follows. First, introduce ‘forward compositions’ $m_{a_0...a_n} : \text{Hom}_\mathcal{A}(a_0,a_1) \otimes \text{Hom}_\mathcal{A}(a_1,a_2) \ldots \otimes \text{Hom}_\mathcal{A}(a_{n-1},a_n) \to \text{Hom}_\mathcal{A}(a_0,a_n)$ via \(^6\):

$$m_{a_0...a_n}(x_1 \otimes \ldots \otimes x_n) := (-1)^{\sum_{1 \leq i < j \leq n} |x_i||x_j|} \mu_{a_0...a_0}(x_n \otimes \ldots \otimes x_1) .$$

Next, introduce ‘suspended forward compositions’ $r_{a_0...a_n} := s_{a_0a_n} \circ m_{a_0...a_n} \circ (s^{-1}_{a_n1} \otimes \ldots \otimes s^{-1}_{a_{n-1}a_n}) : \text{Hom}_\mathcal{A}(a_0,a_1)[1] \otimes \text{Hom}_\mathcal{A}(a_1,a_2)[1] \otimes \ldots \otimes \text{Hom}_\mathcal{A}(a_{n-1},a_n)[1] \to \text{Hom}_\mathcal{A}(a_0,a_n)[1]$, which have degree $+1$. Of course, we can also view these as maps $r_n : \text{Hom}_\mathcal{A}(a_0,a_1) \otimes \text{Hom}_\mathcal{A}(a_1,a_2) \otimes \ldots \otimes \text{Hom}_\mathcal{A}(a_{n-1},a_n) \to \text{Hom}_\mathcal{A}(a_0,a_n)$, of degree $+1$ with respect to the ‘tilde grading’. To keep notation manageable, we will often tacitly change between

\(^6\)The relation between $\mu$ and $m$ is similar to but not quite the same as passing to the opposite $A_\infty$ category, since we do not reverse the sense of arrows in $\mathcal{A}$. 

16
these points of view. For the first few compositions, we find:

\[
\begin{align*}
    m_{ab}(x) &= r_{ab}(x), \quad m_{abc}(x_1 \otimes x_2) = (-1)^{\bar{x}_1} r_{abc}(x_1 \otimes x_2) \\
    m_{abcd}(x_1 \otimes x_2 \otimes x_3) &= (-1)^{\bar{x}_2} r_{abcd}(x_1 \otimes x_2 \otimes x_3)
\end{align*}
\]

and

\[
\begin{align*}
    \mu_{ab}(x) &= r_{ba}(x), \quad \mu_{abc}(x_1 \otimes x_2) = (-1)^{|x_1||x_2|} m_{cba}(x_2 \otimes x_1) = (-1)^{\bar{x}_1 \bar{x}_2 + \bar{x}_1 + 1} r_{cba}(x_2 \otimes x_1) \\
    \mu_{abcd}(x_1 \otimes x_2 \otimes x_3) &= (-1)^{\bar{x}_2 + \sum_{i<j}(\bar{x}_i + 1)(\bar{x}_j + 1)} r_{dcba}(x_3 \otimes x_2 \otimes x_1)
\end{align*}
\]

In terms of the suspended forward compositions, the \(A_\infty\) constraints take the relatively simple form:

\[
\sum_{i \geq 0, j \geq 1 \atop 1 \leq i+j \leq n} (-1)^{\bar{x}_1 + \ldots + \bar{x}_i} r_{a_0 \ldots a_i a_{i+1} \ldots a_n} (x_1 \otimes \ldots \otimes x_i \otimes r_{a_{i+1} \ldots a_{i+j}} (x_{i+1} \otimes \ldots \otimes x_{i+j}) \otimes x_{i+j+1} \otimes \ldots \otimes x_n) = 0 \quad \forall n \geq 1 \tag{2.1}
\]

where \(x_j \in \text{Hom}_\mathcal{A}(a_{j-1}, a_j)[1]\) is any sequence of ‘forward-composable’ morphisms. Using the Koszul rule, these can also be written as\(^7\):

\[
\sum_{i \geq 0, j \geq 1 \atop 1 \leq i+j \leq n} r_{a_0 \ldots a_i a_{i+1} \ldots a_n} \circ (\text{id}_{a_0} a_1 \otimes \ldots \otimes \text{id}_{a_{i-1}} a_i \otimes r_{a_{i+1} \ldots a_{i+j}} \otimes \text{id}_{a_{i+j+1}} a_{i+j+1} \otimes \ldots \otimes \text{id}_{a_{n-1}} a_n) = 0 \quad \forall n \geq 1 \tag{2.2}
\]

where \(\text{id}_{ab} : \text{Hom}_\mathcal{A}(a, b)[1] \rightarrow \text{Hom}_\mathcal{A}(a, b)[1]\) is the identity endomorphism of the vector space \(\text{Hom}_\mathcal{A}(a, b)[1]\) (which of course can be identified with the identity endomorphism of \(\text{Hom}_\mathcal{A}(a, b)\) by applying the shift functor \([1]\) of the category of graded vector spaces to the latter).

The first three constraints imply that \(m_{ab}\) square to zero and act as derivations of \(m_{abc}\), which in turn are associative up to homotopy. This also amounts to the conditions that \(\mu_{ab}\) square to zero and act as derivations of \(\mu_{abc}\), which in turn are associative up to homotopy. In this paper, we will make systematic use of the compositions \(r_{a_0 \ldots a_n}\). However, we stress that the \(A_\infty\) structure is defined by the backward compositions \(\mu_{a_n \ldots a_0}\); in particular an \(A_\infty\) module over \(\mathcal{A}\) is understood with respect to the structure given by \(\mu\); this is important when distinguishing between left and right \(A_\infty\) modules — a right \(A_\infty\) module in this paper is the same as a right \(A_\infty\) module in the sense of \([9]\) (even though it looks like a ‘left’ module when written with respect to the forward compositions \(r\)).

\(^7\)Such a simple formula does not seem to exist for the traditional ‘backward’ compositions.
**Observation** The suspended forward compositions considered in this paper are related to the suspended backward compositions $r^S_{a_n \ldots a_0}$ of [23] (denoted by $\mu_A$ in loc. cit.) via:

$$r_{a_0 \ldots a_n}(x_1 \otimes \cdots \otimes x_n) = r^S_{a_n \ldots a_0}(x_n \otimes \cdots \otimes x_0),$$

without any sign prefactors. However the $A_\infty$ constraints for $r^S$ are not as nice as (2.2).

**Observation** It might seem more natural to define the suspended $A_\infty$ products by using the signed suspensions $\sigma_{ab}$ of the spaces $\text{Hom}_A(a,b)$ (see the introduction). However, this introduces unwanted sign factors in other formulas. This is why we define $r$ as above.

**The cohomology category.** The cohomology category $H(\mathcal{A})$ is the (possibly non-unital) graded associative category having the same objects as $\mathcal{A}$, morphism spaces given by $\text{Hom}_{H(\mathcal{A})}(a,b) := H_{\mu_{ab}}(\text{Hom}_A(a,b)) := \ker(\mu_{ab})/\im(\mu_{ab})$ and morphism compositions $\text{Hom}_{H(\mathcal{A})}(b,c) \otimes \text{Hom}_{H(\mathcal{A})}(a,b) \to \text{Hom}_{H(\mathcal{A})}(a,c)$ induced by $\mu_{cba}$:

$$[x]*[y] := [\mu_{cba}(x \otimes y)] = (-1)^{|x||y|}[m_{abc}(y \otimes x)] \quad \forall x \in Z_{\mu_{bc}}(\text{Hom}_A(b,c)), \quad \forall y \in Z_{\mu_{ab}}(\text{Hom}_A(a,b)).$$

We let $H^0(\mathcal{A})$ be the associative subcategory obtained from $H(\mathcal{A})$ by considering only morphisms of degree zero.

**Unitality and finiteness conditions.** The $A_\infty$ category $\mathcal{A}$ is called strictly unital if every object $a$ admits a degree zero endomorphism $u_a \in \text{Hom}_{\mathcal{A}}^0(a,a)$ such that the following relations are satisfied:

$$r_{a_0 \ldots a_{j-2},a_j,a_{j+1} \ldots a_n}(x_1 \otimes \cdots \otimes x_{j-1} \otimes u_{a_j} \otimes x_{j+1} \otimes \cdots \otimes x_n) = 0 \quad \text{for all } n \neq 2 \quad \text{and all } j$$

$$r_{a,a,b}(u_a \otimes x) = -x \quad , \quad r_{a,b,b}(x \otimes u_b) = (-1)^{\tilde{x}} x \quad , \quad (2.3)$$

where $x_j \in \text{Hom}_A(a_{j-1},a_j)$ etc. It is called homologically unital if every object $a$ admits a degree zero $r_{aa}$-closed endomorphism $u_a$ which induces an identity morphism in the graded associative category $H(\mathcal{A})$. It is easy to check that the units $u_a$ of a strictly unital $A_\infty$ category are uniquely determined, as are the cohomology classes $[u_a]$ in the homologically unital case.

An $A_\infty$ category $\mathcal{A}$ is called degreewise Hom-finite if $\dim_{\mathbb{C}} \text{Hom}_\mathcal{A}^n(a,b) < \infty$ for all $a,b \in \text{Ob}\mathcal{A}$ and all $n \in \mathbb{Z}$. It is compact if $H(\mathcal{A})$ is degreewise Hom-finite, i.e. $\dim_{\mathbb{C}} H^n(\text{Hom}_\mathcal{A}(a,b)) < \infty$ for all $a,b \in \text{Ob}\mathcal{A}$ and all $n \in \mathbb{Z}$.

**$A_\infty$ functors.** Given two $A_\infty$ categories $\mathcal{A}$, $\mathcal{B}$, an $A_\infty$ functor $F : \mathcal{A} \to \mathcal{B}$ is given by a map $F : \text{Ob}\mathcal{A} \to \text{Ob}\mathcal{B}$ together with linear maps $F_{a_0 \ldots a_n} : \text{Hom}_\mathcal{A}(a_0,a_1) \otimes \cdots \otimes \text{Hom}_\mathcal{A}(a_{n-1},a_n) \to \text{Hom}_\mathcal{B}(F(a_0),F(a_n))$ homogeneous of degree $1-n$ (here $n \geq 1$) such
that the suspended maps \( F_{a_0 \ldots a_n}^s := s^B_{F(a_0)F(a_n)} \circ F_{a_0 \ldots a_n} \circ ((s^A_{a_0 a_1})^{-1} \otimes \ldots \otimes (s^A_{a_{n-1} a_n})^{-1}) : \text{Hom}_A(a_0, a_1)[1] \otimes \ldots \otimes \text{Hom}_A(a_{n-1}, a_n)[1] \to \text{Hom}_B(F(a_0), F(a_n))[1] \) — which are homogeneous of degree 0 — satisfy the conditions:

\[
\sum_{p=1}^{n} \sum_{0 < i_1 < i_2 < \ldots < i_{p-1} < n} r^B_{F(a_0)F(a_{i_1})F(a_{i_2}) \ldots F(a_{i_{p-1}})F(a_n)} \circ (F_{a_0 \ldots a_{i_1}, a_{i_1} \ldots a_{i_2}, a_{i_2} \ldots a_{i_{p-1}}, a_{i_{p-1}} \ldots a_n}) = \sum_{0 \leq i < j \leq n} F_{a_0 \ldots a_i, a_{i+1} \ldots a_j, a_{i+1} \ldots a_j, a_{i+1} \ldots a_j} \circ (\text{id}_A^{a_0 a_1} \otimes \ldots \otimes \text{id}_A^{a_{n-1} a_n} \otimes r^A_{a_i a_j} \otimes \text{id}_A^{a_{j+1} a_{j+1}} \otimes \ldots \otimes \text{id}_A^{a_{n-1} a_n}) \quad \forall n \geq 1 \quad (2.4)
\]

Together with \( F_{ab} \), the map on objects induces a (possibly non-unital) functor \( H(F) : H(\mathcal{A}) \to H(\mathcal{B}) \) of graded associative categories. \( F \) is called a quasi-isomorphism if \( H(F) \) is an isomorphism. It is called strict if \( F_{a_0 \ldots a_n} = 0 \) unless \( n = 1 \). In this case, equations (2.4) reduce to:

\[
r^B_{F(a_0)F(a_1), F(a_n)} \circ (F_{a_0 a_1} \otimes F_{a_1 a_2} \otimes \ldots \otimes F_{a_{n-1} a_n}) = F_{a_0 a_n} \circ r^A_{a_0 \ldots a_n} \quad \forall n \geq 1
\]

We will often not indicate the object subscripts on the maps \( F_{ab} \).

An \( A_\infty \) endomorphism of \( \mathcal{A} \) is an \( A_\infty \) functor \( F : \mathcal{A} \to \mathcal{A} \). An \( A_\infty \) endomorphism is an automorphism if the map on objects is bijective and \( F_{ab} \) are bijective for all \( a, b \). As in the case of \( A_\infty \) algebras, one has a notion of strictly unital \( A_\infty \) functor between strictly unital \( A_\infty \) categories, as well as a notion of \( A_\infty \) equivalence of such categories, which amounts to an \( A_\infty \) functor for which \( H(F) \) is an equivalence between the graded cohomology categories. Finally, one has a notion of \( A_\infty \) natural transformations etc. Instead of reviewing these here, we refer the reader to [9, 10, 23] for details.

**Twisted shift functors.** A twisted shift functor on \( \mathcal{A} \) is a strict automorphism \([1]\) of \( \mathcal{A} \) together with isomorphisms of complexes \( \text{Hom}_A(a, b[[1]]) \xrightarrow{\rho_{ab}} \text{Hom}_A(a, b)[1] \) for all \( a, b \in \text{Ob}\mathcal{A} \) which are natural up to signs in \( a \) and \( b \). The last condition means the following. Endowing \( \text{Hom}_A(a, b) \) with the differential \( \mu_{ba} \), we can view \( \text{Hom}_A(\cdot, \cdot) \) as an \( A_\infty \) bifunctor \( \text{Hom}_A : \mathcal{A}^{\text{opp}} \times \mathcal{A} \to \text{Dif} \). Then we require that the maps \( \gamma_{ab} := s^{-1}_{ab} \circ \rho_{ab} : \text{Hom}_A(a, b[[1]]) \to \text{Hom}_A(a, b) \) give a morphism \( \gamma : \text{Hom}_A \circ (\text{id}_A \times [1]) \to \text{Hom}_A \) of degree +1 in the associative category of \( A_\infty \) bifunctors \( \mathcal{A}^{\text{opp}} \times \mathcal{A} \to \text{Dif} \) (whose morphisms are the strict natural transformations of degree zero, see [23, paragraph (1d)]). We have \( \text{Hom}_A(a[[m]], b[[n]]) \approx \text{Hom}_A(a, b)[n - m] \) for all \( a, b \in \text{Ob}\mathcal{A} \) and \( m, n \in \mathbb{Z} \). When \( \mathcal{A} \) is strictly unital, the twisted shift functor automatically preserves all units, i.e. \( u_a[[1]] = u_a[[1]] \) for all \( a \in \text{Ob}\mathcal{A} \); in the homologically unital case, only the cohomology classes must agree, i.e. \( [u_a[[1]]] = [u_a[[1]]] \).

The strict automorphism conditions take the form:

\[
r_{a_0[[1]]a_1[[1]] \ldots a_n[[1]]} \circ ([1]^s \otimes \ldots \otimes [1]^s) = [1]^s \circ r_{a_0 \ldots a_n} \quad \forall n \geq 1
\]
where \([1])^s = s_{a[1][b[1]]} \circ ([1]) \circ s_{ab}^{-1} : \text{Hom}_A(a, b)[1] \to \text{Hom}_A(a[[1]], b[[1]])[1]\] are the suspended maps on morphisms as in the previous paragraph. Notice that these conditions are equivalent with:

\[
m_{a_0[[1]]a_1[[1]]...a_n[[1]]} \circ ([[1]] \otimes ... \otimes [[1]]) = [[1]] \circ m_{a_0...a_n}, \quad \forall n \geq 1.
\]

Passing to cohomology, we find that \(([1])\) induces a twisted shift functor (see Appendix A) \([1])^H\) of the graded associative category \(H(\mathcal{A})\); this acts on objects in the same way as \([1])\). \([1])^H\) is an automorphism of \(H(C)\) endowed with isomorphisms of graded vector spaces \(\text{Hom}_{H(C)}(a, b[[1]])^H \approx \text{Hom}_{H(C)}(a, b)[1]\) which are natural up to missing Koszul signs. These isomorphisms of graded vector spaces are induced by the isomorphisms of complexes \(\text{Hom}_A(a, b[[1]])\approx \text{Hom}_A(a, b)[1]\).

The functor \([1])^H\) restricts to a shift functor for the ungraded subcategory \(H^0(C) \subset H(C)\). Following the notations of Appendix A, we denote this restriction by \([1]\) (of course, this again acts on objects in the same way as \([1])\). The relation \(\text{Hom}_{H^0(C)}(a, b[[n]]) \approx H^0(\text{Hom}_A(a, b))\) implies that the graded completion \(H^0(C)^\bullet\) of the associative category with shifts \((H^0(C), [1])\) is isomorphic with \(H(C)\), and that \([1])^H\) is the twisted shift functor of \(H^0(C)^\bullet\) in the sense of Appendix A. To simplify notation, we will write \([1])\) instead of \([1])^H\) — which of them is meant should be clear from the context.

Minimal \(A_{\infty}\) categories and minimal models. An \(A_{\infty}\) category is called minimal if all unary compositions \(r_{ab}\) vanish. Given an \(A_{\infty}\) category \(\mathcal{A}\), a minimal model of \(\mathcal{A}\) is a minimal \(A_{\infty}\) category \(\mathcal{B}\) which is quasi-isomorphic with \(\mathcal{A}\). Any \(A_{\infty}\) category admits a minimal model [9, 10].

2.2 Sector decomposition. The total Hom space. When working with \(A_{\infty}\) categories, many formulas can be simplified by using the following trick [5]. Consider the commutative associative algebra \(R := R_\mathcal{A}\) on generators \((\epsilon_a)_{a \in \text{Ob}A}\) and with relations \(\epsilon_a \epsilon_b = \delta_{ab} \epsilon_a\) where \(\delta_{ab}\) is the Kronecker symbol. Notice that \(R\) is unital iff \(\mathcal{A}\) has a finite number of objects (in which case \(\sum_{a \in \text{Ob}A} \epsilon_a\) is the unit). Since \(\epsilon_a\) are commuting idempotents, we have a decomposition \(R \approx \oplus_{a \in \text{Ob}A} \mathbb{C}\) as an associative algebra. Consider the vector space:

\[
\mathcal{H} := \mathcal{H}_\mathcal{A} := \oplus_{a,b \in \text{Ob}A} \text{Hom}_\mathcal{A}(a, b),
\]

with the grading:

\[
\mathcal{H}^n := \oplus_{a,b \in \text{Ob}A} \text{Hom}_\mathcal{A}^n(a, b).
\]

We let \(P_{ab} : \mathcal{H} \to \text{Hom}_\mathcal{A}(a, b)\) be the projectors of \(\mathcal{H}\) on the subspaces \(\text{Hom}_\mathcal{A}(a, b)\) defined by this decomposition. As in [5], the ‘binary decomposition’ (2.5) gives a graded
$R$-bimodule structure on $\mathcal{H}$, obtained by defining the outer left and right multiplications with $\epsilon_a$ respectively $\epsilon_b$ to be given by the projectors $aP, P_b$ of $\mathcal{H}$ on the subspaces $a\mathcal{H} := \oplus_{b \in \text{Ob}_\mathcal{A}} \text{Hom}_\mathcal{A}(a,b)$ respectively $\mathcal{H}_b := \oplus_{a \in \text{Ob}_\mathcal{A}} \text{Hom}_\mathcal{A}(a,b)$:

$$
\epsilon_a x := aP(x) \quad , \quad x \epsilon_b := P_b(x) \quad \forall x \in \mathcal{H} .
$$

**Total $A_\infty$ products.** Let us define total products $r_n : \mathcal{H}[1]^{\otimes n} \to \mathcal{H}[1]$ via:

$$
r_n(x^{(1)} \otimes \ldots \otimes x^{(n)}) := \oplus_{a_0 \ldots a_n} \sum_{a_1 \ldots a_{n-1}} r_{a_0 \ldots a_n}(x^{(1)}_{a_0 a_1} \otimes \ldots \otimes x^{(n)}_{a_{n-1} a_n})
$$

(2.6)

where $x^{(j)} = \oplus_{a,b \in \text{Ob}_\mathcal{A}} x^{(j)}_{ab} \in \mathcal{H}[1]$ with $x^{(j)}_{ab} \in \text{Hom}_\mathcal{A}(a,b)[1]$. The sum in the right hand side has a finite number of nonzero terms since $u^{(j)}$ have finite support in $\text{Ob}_\mathcal{A} \times \text{Ob}_\mathcal{A}$. It is clear from (2.6) that we can view $r_n$ as elements of $\text{Hom}^1_{R\text{Mod}_R}(\mathcal{H}[1]^{\otimes R^{n}}, \mathcal{H}[1])$. Moreover, they obey the $A_\infty$ relations:

$$
\sum_{i \geq 0, j \geq 1 \atop 1 \leq i + j \leq n} (-1)^{\bar{x}_i + \ldots + \bar{x}_j} r_{n-j+1}(x_1 \otimes \ldots \otimes x_i \otimes r_j(x_{i+1} \otimes \ldots \otimes x_{i+j}) \otimes x_{i+j+1} \otimes \ldots \otimes x_n) = 0 .
$$

(2.7)

The categorical $A_\infty$ compositions $r_{a_0 \ldots a_n}$ can be recovered from $r_n$, since $R$-multilinearity implies the decomposition (2.6), while (2.7) imply (2.1).

**Description of $A_\infty$ functors.** Similarly, an $A_\infty$ functor $F : \mathcal{A} \to \mathcal{B}$ induces maps $F^s_n \in \text{Hom}_{A_\infty \text{GrMod}_R}(\mathcal{H}_\mathcal{A}[1]^{\otimes R^{n}}, \mathcal{H}_{\mathcal{B}}[1])$ defined through:

$$
F^s_n(x^{(1)} \otimes \ldots \otimes x^{(n)}) := \oplus_{a_0 \ldots a_n} \sum_{a_1 \ldots a_{n-1}} F^s_{a_0 \ldots a_n}(x^{(1)}_{a_0 a_1} \otimes \ldots \otimes x^{(n)}_{a_{n-1} a_n}) ,
$$

such that conditions (2.4) amount to the constraints:

$$
\sum_{p=1}^n \sum_{0 \leq i_1 < i_2 < \ldots < i_{p-1} < n} r^R_p \circ (F^s_{i_1} \otimes F^s_{i_2-i_1} \otimes \ldots \otimes F^s_{n-i_{p-1}})
$$

$$
= \sum_{0 \leq i < j \leq n} F^s_{n+i-j+1} \circ (\text{id}_{\mathcal{H}_\mathcal{A}} \otimes r^A_j \otimes \text{id}_{\mathcal{H}_\mathcal{A}}^{\otimes n-j}) , \quad \forall n \geq 1 .
$$

(2.8)

Once again, the $A_\infty$ functor $F$ can be recovered form the maps $F^s_n$.

---

$^8$Thus $(\mathcal{H}, (r_n)_{n \geq 1})$ is an $A_\infty$ algebra over $\mathbb{C}$. However it is not quite an $A_\infty$ algebra over the commutative ring $R$ since the left and right module structures on $\mathcal{H}$ need not agree.
Description of twisted shift functors  Let us assume that $\mathcal{A}$ has twisted shifts. Then the bijection on objects $[[1]] : \text{Ob}\mathcal{A} \to \text{Ob}\mathcal{A}$ induces an algebra automorphism $[[1]]^R : R \to R$ given by:

$$
\lambda = \sum_{a \in \text{Ob}\mathcal{A}} \lambda_a \epsilon_a \to \lambda[[1]]^R := \sum_{a \in \text{Ob}\mathcal{A}} \lambda_a \epsilon_{a[[1]]},
$$
i.e. $(\lambda[[1]]^R)_a = \lambda_{a[-1]}$. Here the coefficients $\lambda_a \in \mathbb{C}$ vanish except for a finite number of objects of $\mathcal{A}$. The automorphism $[[1]]^R$ is unital when $\mathcal{A}$ has a finite number of objects.

The maps on morphisms $[[1]] : \text{Hom}_\mathcal{A}(a,b) \to \text{Hom}_\mathcal{A}(a[[1]], b[[1]])$ induce a graded vector space automorphism $[[1]]^H$ of $\mathcal{H}$ defined through:

$$
x = \oplus_{a,b} x_{ab} \to x[[1]]^H := \oplus_{a,b} x_{ab}[1],
$$
i.e. $(x[[1]]^H)_{ab} = x_{a[-1][b[-1]]}[1]$. This is compatible with $[[1]]^R$ in the following sense:

$$(\lambda x \mu)[[1]]^H = \lambda[[1]]^R x[[1]]^H \mu[[1]]^R \quad \forall \lambda, \mu \in R \quad \forall x \in \mathcal{H}.$$  

The suspended twisted shift functor $[[1]]^s$ corresponds to the suspended graded vector space automorphism:

$$([[1]]^H)^s := s \circ [[1]] \circ s^{-1},$$
where $s : \mathcal{H} \to \mathcal{H}[1]$ is the suspension map of $\mathcal{H}$. For simplicity, we will denote $[[1]]^R$ and $[[1]]^H$ simply by $[[1]]$, and the suspended map $(([[1]]^H)^s$ simply by $[[1]]^s$.

It is not hard to check that the strict $A_\infty$ automorphism conditions for the twisted shift functor are equivalent with the relations:

$$r_n \circ ([1]^s \otimes \ldots \otimes [1]^s) = [1]^s \circ r_n \quad \forall n \geq 1. \quad (2.9)$$

Observation  When $\mathcal{A}$ has a finite number of objects, strict unitality amounts to the existence of a central degree zero element $u$ of the $R$-bimodule $\mathcal{H}$ such that the following relations are satisfied:

$$r_n(x_1 \otimes \ldots \otimes x_{j-1} \otimes u \otimes x_{j+1} \otimes \ldots \otimes x_n) = 0 \quad \text{for all } n \neq 2 \text{ and all } j$$
$$r_2(u \otimes x) = -x, \quad r_2(x \otimes u) = (-1)^{\hat{x}} x, \quad (2.10)$$

In this case, the categorical units are recovered as $u_a := P_{aa}(u)$, thus $u = \oplus_{a \in \text{Ob}\mathcal{A}} u_a$.  

22
2.3 The $A_\infty$ categories $\mathcal{ZA}$, $\Sigma \mathcal{A}$ and $\text{tw}(\mathcal{A})$

Twisted shift completion. The twisted shift completion $\mathcal{ZA}$ of $\mathcal{A}$ is an $A_\infty$ category whose objects are pairs $(a, n)$ where $a \in \text{Ob}\mathcal{A}$ and $n \in \mathbb{Z}$, with $\text{Hom}_{\mathcal{ZA}}((a, m), (b, n)) := \text{Hom}_{\mathcal{A}}(a, b)[n-m]$. Define a bijective map $[[1]]$ on $\text{Ob}\mathcal{A}$ via $(a, n)[[1]] := (a, n+1)$ and denote its $n$-th iteration by $[[n]]$, where $[[0]] = \text{id}_{\text{Ob}\mathcal{ZA}}$ (for $n < 0$, set $[[n]] := [[-n]]^{-1}$).

Identifying $a$ with $(a, 0)$ allows us to write $(a, n) \equiv a[[n]]$. For $A, B \in \text{Ob}\mathcal{ZA}$, define a bijection $[[1]]_{AB} : \text{Hom}_{\mathcal{ZA}}(A, B) \to \text{Hom}_{\mathcal{ZA}}(A[[1]], B[[1]]) = \text{Hom}_{\mathcal{ZA}}(A, B)$ via $f[[1]]_{AB} = (-1)^{|f|}f$ for homogeneous $f$. Let $[[n]]_{AB}$ be the $n$-th iteration for all $n \in \mathbb{Z}$ (with the obvious notational conventions). Thus $[[n]]_{AB} : \text{Hom}_{\mathcal{ZA}}(A, B) \to \text{Hom}_{\mathcal{ZA}}(A[[n]], B[[n]]) = \text{Hom}_{\mathcal{ZA}}(A, B)$ acts as $f[[n]]_{AB} = (-1)^{|n|f}f$ for homogeneous $f$.

We will usually not write the subscripts $A$ and $B$. Similarly, we define $[[n]]_{AB}^* := s_A[[n]]B[[n]] \circ [[n]]_{AB} \circ s_{AB}^{-1} : \text{Hom}_{\mathcal{ZA}}(A, B)[1] \to \text{Hom}_{\mathcal{ZA}}(A[[n]], B[[n]])[1] = \text{Hom}_{\mathcal{ZA}}(A, B)[1]$, which acts as $f[[n]]_{AB} = (-1)^{|n|f}f$ for homogeneous $f$. We will often not write the object subscripts of $[[n]]_{AB}$ and $[[n]]_{AB}^*$. With this convention, we have $[[m]] \circ [[n]] = [[m+n]]$ and $[[m]]^* \circ [[n]]^* = [[m+n]]^*$ on both objects and morphisms.

For all $a, b \in \text{Ob}\mathcal{A}$, we have the bijection $[[[-m]]^* : \text{Hom}_{\mathcal{ZA}}(a[[m]], b[[n]])[1] \to \text{Hom}_{\mathcal{ZA}}(a[[n]], b[[m]])[1] = \text{Hom}_{\mathcal{A}}(a, b)[n-m] = \text{Hom}_{\mathcal{A}}(a, b)[n-m+1]$.

Hence $s_{ab}^{-m-n}[1] \circ [[-m]]^*$ (which we write simply as $s_{ab}^{-m-n} \circ [[-m]]^*$) gives a bijection from $\text{Hom}_{\mathcal{ZA}}(a[[m]], b[[n]])[1]$ to $\text{Hom}_{\mathcal{A}}(a[[n]], b[[m]])[1]$, whose inverse is $[[m]]^* \circ s_{ab}^{n-m}[1]$ (written simply as $[[m]]^* \circ s_{ab}^{n-m}$). Using these maps, we define suspended forward compositions

$$r_{a_0[[k_0]], \ldots, a_n[[k_n]]}^{ZA} : \text{Hom}_{\mathcal{ZA}}(a_0[[k_0]], a_1[[k_1]])[1] \otimes \cdots \otimes \text{Hom}_{\mathcal{ZA}}(a_{n-1}[[k_{n-1}]], a_n[[k_n]])[1] \to \text{Hom}_{\mathcal{ZA}}(a_0[[k_0]], a_n[[k_n]])[1]$$

via the expressions:

$$r_{a_0[[k_0]], \ldots, a_n[[k_n]]}^{ZA} := (-1)^{k_n-k_0} \circ s_{a_0, \ldots, a_n}^{k_n-k_0} \circ r_{a_0, \ldots, a_n} \circ (s_{a_0a_1}^{k_0-k_1} \otimes \cdots \otimes s_{a_{n-1}a_n}^{k_{n-1}-k_n}) \circ ([[-k_0]]^* \otimes \cdots \otimes [[-k_{n-1}]]^*)$$

These are easily seen to satisfy the $A_\infty$ constraints, thus making $\mathcal{ZA}$ into an $A_\infty$ category. Moreover, $[[1]]$ becomes a twisted shift functor for $\mathcal{ZA}$, so $(\mathcal{ZA}, [[1]])$ is an $A_\infty$ category with (twisted) shifts. If $\mathcal{A}$ is strongly unital with identity morphisms $u_a$, then $\mathcal{ZA}$ is strongly unital with identity morphisms $u_a[[n]] := u_{a[[n]]}$ (this follows from an easy computation). $\mathcal{ZA}$ contains $\mathcal{A}$ as the full $A_\infty$ subcategory on the objects $a[[0]]$ $(a \in \text{Ob}\mathcal{A})$. When $\mathcal{A}$ is degreewise Hom-finite respectively compact, then $\mathcal{ZA}$ has the same property.

Additive completion of the shift completion. The additive completion $\Sigma \mathcal{A}$ of $\mathcal{ZA}$ is the smallest additive $A_\infty$ category containing $\mathcal{ZA}$. Its objects are finite direct
sums of objects of $\mathbb{Z}\mathcal{A}$, with morphisms defined accordingly. Thus any object $A$ of $\Sigma\mathcal{A}$ decomposes as $A = \oplus_{i=1}^n A_i[[n_i]]$ for some $A_i \in \text{Ob}\mathcal{A}$ and some $n_i$ in $\mathbb{Z}$. Given $B = \oplus_{j=1}^m b_j[[m_j]] \in \text{Ob}\Sigma\mathcal{A}$, we have $\text{Hom}_{\Sigma\mathcal{A}}(A,B) = \oplus_{i,j} \text{Hom}_{\mathcal{A}}(a_i[[n_i]], b_j[[m_j]]) = \oplus_{i,j} \text{Hom}_{\mathcal{A}}(a_i, b_j)[m_j - n_i]$. The compositions $r^{\Sigma\mathcal{A}}$ extend to $A_\infty$ compositions on $\Sigma\mathcal{A}$ in the obvious manner, making $\Sigma\mathcal{A}$ into an $A_\infty$ category. Explicitly, we have:

$$r^{\Sigma\mathcal{A}}_{A^0 \ldots A^n}(x^{(1)} \otimes \ldots \otimes x^{(n)}) = \oplus_{i_0 \ldots i_n} \sum_{i_1 \ldots i_{n-1}} r^{\Sigma\mathcal{A}}_{a_i^0 \ldots a_i^n}[[n_0 \ldots n_n]](x^{(1)}_{i_0i_1} \otimes x^{(2)}_{i_1i_2} \otimes \ldots \otimes x^{(n)}_{i_{n-1}i_n})$$

where $x^{(k)} = \oplus_{i,j} x^{(k)}_{ij} \in \text{Hom}_{\Sigma\mathcal{A}}(A^{(k-1)}, A^{(k)})$ with $A^{(k)} = \oplus_{i} a_i^{(k)}[[m_i^{(k)}]]$ and $x^{(k)}_{ij} \in \text{Hom}_{\mathcal{A}}(a_i^{(k-1)}[[m_i^{(k-1)}]], a_j^{(k)}[[m_j^{(k)}]])$. The twisted shift functor of $\mathbb{Z}\mathcal{A}$ extends to a strict automorphism of $\Sigma\mathcal{A}$ given by:

$$A = \oplus_{i=1}^n A_i \rightarrow A[[1]] := \oplus_{i=1}^n A_i[[1]]$$

where $A_i \in \text{Ob}\mathbb{Z}\mathcal{A}$; one takes the obvious action on morphisms. $[[1]]$ is a twisted shift functor, so $(\Sigma\mathcal{A}, [[1]])$ is an $A_\infty$ category with (twisted) shifts, which is strictly unital when $\mathcal{A}$ is. The units are given by $u_{\oplus a_i[[n_i]]} = \oplus_i u_{a_i}[[n_i]]$, where $u_a$ are the units of $\mathcal{A}$. $\Sigma\mathcal{A}$ contains $\mathbb{Z}\mathcal{A}$ as a full $A_\infty$ subcategory in the obvious manner. When $\mathcal{A}$ is degreewise Hom-finite respectively compact, then $\Sigma\mathcal{A}$ has the same property.

**Bounded twisted complexes.** A (strictly one-sided) bounded twisted complex $q$ over $\mathcal{A}$ is a finite collection of morphisms $q_{ij} \in \text{Hom}_{\mathbb{Z}\mathcal{A}}^1(A_i, A_j) = \text{Hom}_{\mathcal{A}}(a_i[[n_i]], a_j[[n_j]])$ of the shift-completed category $\mathbb{Z}\mathcal{A}$ with $1 \leq i, j \leq l_q$ and $q_{ij} = 0$ unless $i < j$, which are required to obey the generalized Maurer-Cartan equations:

$$\sum_{n=1}^{j-i} \sum_{i_1 \ldots i_{n-1}} r^{\Sigma\mathcal{A}}_{A_{i_1} \ldots A_{i_{n-1}} A_j}(q_{i i_1} \otimes q_{i_1 i_2} \otimes \ldots \otimes q_{i_{n-2} i_{n-1}} \otimes q_{i_{n-1} j}) = 0 \quad \forall 1 \leq i < j \leq n$$

(2.11)

where the term for $n = 1$ is defined to be $r^{\Sigma\mathcal{A}}_{A_i A_j}(q_{ij})$. Notice that we denote the twisted complex by $q$, with the understanding that the objects $A_1 \ldots A_l_q$ of $\Sigma\mathcal{A}$ are implicitly given as the domains/codomains of the morphisms $q_{ij}$. This is done to simplify notation. The positive integer $l_q$ is called the length of $q$. For later convenience, we set $A_q := \oplus_{i=1}^{l_q} A_i \in \text{Ob}\Sigma\mathcal{A}$. The morphisms $q_{ij}$ can be combined into a single (endo)morphism $\hat{q} := \oplus_{i,j} q_{ij} \in \text{Hom}_{\Sigma\mathcal{A}}(A_q, A_q)$ of $\Sigma\mathcal{A}$. Recall that $A_i := a_i[[n_i]]$ for some $a_i \in \text{Ob}\mathcal{A}$ and some $n_i \in \mathbb{Z}$.

Bounded twisted complexes form an $A_\infty$ category $\text{tw}(\mathcal{A})$ if one sets $\text{Hom}_{\text{tw}(\mathcal{A})}(q, q') := \text{Hom}_{\Sigma\mathcal{A}}(A_q, A_{q'}) = \oplus_{i,j} \text{Hom}_{\mathcal{A}}(a_i[[n_i]], a_j'[[n_j']])$ and defines $A_\infty$ products as follows:

$$r^{\text{tw}(\mathcal{A})}_{q_0 \ldots q_n}(x_1 \otimes \ldots \otimes x_n) := \sum_{t_0 \ldots t_n \geq 0} r^{\Sigma\mathcal{A}}_{q_0 \otimes t_1 \ldots \otimes q_n}(\hat{q}_0^\otimes t_0 \otimes x_1 \otimes \hat{q}_1^\otimes t_1 \otimes x_2 \otimes \ldots \otimes x_n \otimes \hat{q}_n^\otimes t_n)$$

(2.12)
In the expression above, we take \(x_i \in \text{Hom}_{\text{tw}(\mathcal{A})}(q_{i-1}, q_i) = \text{Hom}_{\Sigma \mathcal{A}}(A_{q_{i-1}}, A_{q_i})\), with \(\hat{q}_i \in \text{Hom}_{\Sigma \mathcal{A}}(A_{q_i}, A_{q_i})\) defined as above. The notation \((\mathcal{A})_k\) stands for the sequence \(A, A, \ldots A\) consisting of \(k\) copies of \(A\). The \(A_\infty\) relations for (2.12) follow from the generalized Maurer-Cartan equations (2.11).

The twisted shift functor \([1]\) of \(\Sigma \mathcal{A}\) extends to \(\text{tw}(\mathcal{A})\) as follows. Given a twisted complex \(q\) with \(q_{ij} : A_i \to A_j\), we let \(q[[1]]\) be the twisted complex \(q'\) given by \(A'_i := A_i[[1]]\) and \(q'_{ij} = q_{ij}[[1]]\). Shift-invariance of \(r^{\Sigma \mathcal{A}}\) implies that \(q[[1]]\) satisfies the generalized Maurer-Cartan equations. We let \([1]\) act on morphisms in \(\text{tw}(\mathcal{A})\) in the same way as in \(\Sigma \mathcal{A}\). Using definition (2.12) and shift-invariance of \(\Sigma \mathcal{A}\), we find that \((\text{tw}(\mathcal{A}), [[1]])\) is an \(A_\infty\) category with (twisted) shifts, which is strictly unital if \(\mathcal{A}\) is. The units are \(u_q = u_{A_q}\) where \(u_A\) are the units of \(\Sigma \mathcal{A}\) (this again follows by an easy computation). Notice that \(\text{tw}(\mathcal{A})\) contains \(\Sigma \mathcal{A}\) as the full subcategory on those twisted complexes \(q\) for which all \(q_{ij}\) vanish (such a twisted complex is called degenerate and identifies with the object \(A_q\) of \(\Sigma \mathcal{A}\)). When \(\mathcal{A}\) is degreewise Hom-finite then \(\text{tw}(\mathcal{A})\) has the same property. When \(\mathcal{A}\) is compact, the same is true\(^9\) of \(\text{tw}(\mathcal{A})\).

### 2.4 The triangulated categories \(D(\mathcal{A}), \text{tria}(\mathcal{A})\) and \(\text{per}(\mathcal{A})\)

**The derived category** \(D(\mathcal{A})\). Let \(\mathcal{A}\) be a strictly unital \(A_\infty\) category and \(\text{Mod}_\mathcal{A}\) be the dG category\(^10\) of strictly unital right \(A_\infty\) modules over \(\mathcal{A}\) [9], i.e. contravariant unital \(A_\infty\) functors from \(\mathcal{A}\) to \(\text{Dif}\). We can define the derived category of \(\mathcal{A}\) via \(D(\mathcal{A}) := H^0(\text{Mod}_\mathcal{A})\) (this is one possible description of \(D(\mathcal{A})\), see Remark 5.2.0.2 ref. [9], which however uses different notations). For any object \(a \in \text{Ob}\mathcal{A}\), let \(\hat{a} \in \text{Mod}_\mathcal{A}\) denote its image through the (first component of the) Yoneda \(A_\infty\) functor \(y : \mathcal{A} \to \text{Mod}_\mathcal{A}\) constructed in [9, 10] (\(\hat{a}\) is the \(A_\infty\) functor \(\text{Hom}_\mathcal{A}(\cdot, a)\)). We let \(\mathcal{A}\) denote the full dG subcategory of \(\text{Mod}_\mathcal{A}\) determined by the set of objects \(\mathcal{U} := \{\hat{a} | a \in \text{Ob}\mathcal{A}\}\) and \(\mathcal{A} = H^0(\mathcal{A})\) the full associative subcategory of \(D(\mathcal{A})\) determined by \(\mathcal{U}\). The Yoneda functor induces isomorphisms \(H(\text{Hom}_{\text{Mod}_\mathcal{A}}(\hat{a}, \hat{b})) \approx H(\text{Hom}_\mathcal{A}(a, b))\) for all \(a, b \in \text{Ob}\mathcal{A}\), which imply \(\mathcal{A} = H^0(\mathcal{A}) \approx H^0(\mathcal{A})\). It is shown in [9] that \(D(\mathcal{A})\) is a triangulated category with infinite coproducts, compactly generated by \(\mathcal{U}\) — in particular we have \(D(\mathcal{A}) = \text{Tria}_{D(\mathcal{A})}(\mathcal{U})\) in the notation of Appendix 1.4.

**The categories** \(\text{tria}(\mathcal{A})\) and \(\text{per}(\mathcal{A})\). One defines \(\text{tria}(\mathcal{A}) = \text{tria}_{D(\mathcal{A})}(\mathcal{U})\) to be the smallest strictly full triangulated subcategory containing \(\mathcal{U}\), and \(\text{per}(\mathcal{A}) = k\text{tria}_{D(\mathcal{A})}(\mathcal{U})\) to be the smallest strictly full triangulated and idempotent-complete subcategory con-

---

9 This follows from the spectral sequence of [23, Section 3, paragraph (31)], which computes \(H(\text{tw}(\mathcal{A}))\) starting from \(H(\Sigma \mathcal{A})\) by using the obvious finite filtration possessed by each twisted complex.

10 The morphisms in \(Z^0(\text{Mod}_\mathcal{A})\) are strictly unital \(A_\infty\) morphisms of \(A_\infty\) right modules over \(\mathcal{A}\).
taining the same set of objects (see Appendix 1.4 for notation). The category $\text{per}(\mathcal{A})$ is called the perfect derived category of $\mathcal{A}$. It follows from [9] that $\text{per}(\mathcal{A})$ coincides with the full category of all compact objects of $D(\mathcal{A})$. We have obvious inclusions $\text{tria}(\mathcal{A}) \subset \text{per}(\mathcal{A}) \subset D(\mathcal{A})$.

As explained in [9, 10], the Yoneda $A_\infty$ functor factorizes as $y = y'' \circ y'$, where $y' : \mathcal{A} \to \text{tw}(\mathcal{A})$ is the obvious embedding and $y'' : \text{tw}(\mathcal{A}) \to \text{Mod}_{\mathcal{A}}$ induces an equivalence $H^0(\text{tw}(\mathcal{A})) \approx \text{tria}(\mathcal{A})$. The latter gives an explicit description of $\text{tria}(\mathcal{A})$ through twisted complexes, presenting it as an $A_\infty$-enhanced triangulated category\(^\text{11}\). Since $\text{tw}(\mathcal{A})$ is an $A_\infty$ category with shifts, we also have $H(\text{tw}(\mathcal{A})) = H^0(\text{tw}(\mathcal{A}))^\bullet = \text{tria}(\mathcal{A})^\bullet$.

The case of $A_\infty$ algebras. When $\mathcal{A}$ has a single object $a$, then $\mathcal{A}$ can be identified with the $A_\infty$ algebra $A := \text{Hom}_{\mathcal{A}}(a,a)$. We have suspended products $r_n : A[1]^{\otimes n} \to A[1]$ $(n \geq 1)$ subject to conditions (2.7), as well as desuspended products $m_n : A^{\otimes n} \to A$ given by $r_n = s \circ m_n \circ (s^{-1})^{\otimes n}$, where $s : A \to A[1]$ is the suspension operator. As per our conventions, the classical $A_\infty$ structure of $A$ is defined by the products $\mu_n$, so $m_n$ define a classical $A_\infty$ algebra structure on the opposite $A_\infty$ algebra $A^{\text{op}}$. Strict unitality amounts to existence of an element $u \in A^0$ satisfying (2.10). An $A_\infty$ functor between two $A_\infty$ categories with one object corresponds to an $A_\infty$ morphism of $A_\infty$ algebras, given by maps $F_n : A^{\otimes n} \to B$ $(n \geq 1)$ satisfying (2.8). The cohomology category $H(\mathcal{A})$ reduces to the $\mu_1$-cohomology $H(\mathcal{A})$, which is a graded associative algebra. In this case, $\mathcal{A}$ is degreewise Hom-finite iff $A$ is degreewise finite as a graded vector space, i.e. iff $A^n$ is finite-dimensional for all $n$; we then say that $A$ is degreewise finite. $\mathcal{A}$ is compact iff the graded vector space $H(\mathcal{A})$ is degreewise finite i.e. iff $H^n(\mathcal{A})$ is finite-dimensional for all $n$. In this case, we say that $A$ is compact.

We use the notation $D(\mathcal{A}) := D(\mathcal{A})$, $\text{tria}(\mathcal{A}) := \text{tria}(\mathcal{A})$, $\text{tw}(\mathcal{A}) := \text{tw}(\mathcal{A})$ etc. The category $D(\mathcal{A})$ is compactly generated by the single object $\hat{a}$, which in this case we denote by $A$; this object of $\text{Mod}_{A_\infty} = \text{Mod}_{(A,\mu)} = (A^{\text{op}},m)$ Mod can be identified with $(A,\mu)$ viewed as a right $A_\infty$ module over itself. Then $\text{tria}(\mathcal{A})$, $\text{per}(\mathcal{A})$ are the strictly full triangulated subcategories of $D(\mathcal{A})$ generated by $\hat{A}$ (the second in the Karoubi sense).

A basic result [9] states that any strictly unital $A_\infty$ morphism $\varphi : A \to B$ between $A_\infty$ algebras $A, B$ induces a ‘restriction’ exact functor $D(B) \to D(A)$ mapping $\hat{B}$ into $\hat{A}$ and thus $\text{tria}(B)$ into $\text{tria}(A)$ and $\text{per}(B)$ into $\text{per}(A)$. When $\phi$ is an $A_\infty$ quasi-isomorphism, this functor is an exact equivalence $D(B) \xrightarrow{\approx} D(A)$, whose restrictions give equivalences $\text{tria}(B) \xrightarrow{\approx} \text{tria}(A)$ and $\text{per}(B) \xrightarrow{\approx} \text{per}(A)$. Hence the triangulated categories $\text{tria}(A)$, $\text{per}(A)$ and $D(A)$ are determined up to exact equivalence by the

\(^{11}\)This notion is essentially equivalent with that considered in [11].
A∞ quasi-isomorphism class of A. This allows one to replace A, for example, by a minimal or antiminimal (dG) model [9].

3. Cyclic A∞ categories

In this section we discuss cyclic A∞ categories, then explain how a cyclic pairing extends from an A∞ category to its category of twisted complexes. We also discuss a class of cyclic minimal models of A∞ categories obtained by adapting a procedure due to [15] (see also [26]). After addressing the issues of unitality and existence of shifts for such minimal models, we give a string field theory interpretation of this construction.

3.1 Basics

Let D be an integer. A D-cyclic structure on an A∞ category A consists of morphisms of graded vector spaces ⟨ , ⟩ab : HomA(a, b) × HomA(b, a) → C[−D] for all a, b ∈ ObA which satisfy the graded symmetry condition ⟨u ⊗ v⟩ab = (−1)|u||v|⟨v ⊗ u⟩ba and the graded cyclicity relations:

\[ \langle x_0 \otimes r_{a_0} \ldots a_n (x_1 \otimes \ldots \otimes x_n) \rangle_{a_n, a_0} = (−1)^{\bar{x}_0 + \ldots + \bar{x}_n} \langle x_1 \otimes r_{a_1} \ldots a_n a_0 (x_2 \otimes \ldots \otimes x_n \otimes x_0) \rangle_{a_0, a_1} . \]  

(3.1)

The doublet (A, ⟨ , ⟩) is called a D-cyclic A∞ category. We will often view ⟨ , ⟩ as bilinear forms ⟨ , ⟩ : HomA(a, b) × HomA(b, a) → C. Homogeneity of the pairings amounts to the ‘selection rule’:

\[ \langle x, y \rangle = 0 \text{ unless } |x| + |y| = D . \]

For reader’s convenience, we list the first two cyclicity relations:

\[ \langle x_0 \otimes r_{a_0 a_2} (x_1) \rangle_{a_1 a_0} = (−1)^{\bar{x}_0} \langle r_{a_1 a_0} (x_0) \otimes x_1 \rangle_{a_1 a_0} , \quad \langle x_0 \otimes r_{a_0 a_2} (x_1 \otimes x_2) \rangle_{a_2 a_0} = (−1)^{\bar{x}_0 + \bar{x}_1} \langle r_{a_2 a_0 a_1} (x_0 \otimes x_1) \otimes x_2 \rangle_{a_2 a_1} \]

and their translation in terms of the products m1, m2:

\[ \langle m_{a_1 a_0} (x_0) \otimes x_1 \rangle_{a_1 a_0} = 0 , \quad \langle m_{a_2 a_0 a_1} (x_0 \otimes x_1) \otimes x_2 \rangle_{a_2 a_0} = \langle x_0 \otimes m_{a_0 a_1 a_2} (x_1 \otimes x_2) \rangle_{a_0 a_2} , \]

and in terms of µ1, µ2:

\[ \langle µ_{a_0 a_1} (x_0) \otimes x_1 \rangle_{a_1 a_0} = 0 , \quad \langle µ_{a_2 a_0 a_1} (x_2 \otimes x_1) \otimes x_0 \rangle_{a_0 a_2} = \langle x_2 \otimes µ_{a_1 a_0 a_2} (x_1 \otimes x_0) \rangle_{a_1 a_2} . \]  

(3.2)

Here \( x_i ∈ \text{Hom}_A(a_{i−1 \mod 2}, a_i) \) for the first equation of each pair and \( x_i ∈ \text{Hom}_A(a_{i−1 \mod 3}, a_i) \) for the second equation of each pair.
Cyclic structure induced on cohomology. The first equations in (3.2) imply that \( \langle \rangle_{ab} \) descend to well-defined pairings on the cohomology category \( H(\mathcal{A}) \), which we denoted by \( \langle \rangle_{ab}^H \). These are given by

\[
\langle [x] \otimes [y] \rangle_{ab}^H = \langle x \otimes y \rangle_{ab} \quad \forall x \in Z(\text{Hom}_A(a, b)) \; \forall y \in Z(\text{Hom}_A(b, a))
\]

and define a \( D \)-cyclic structure on the graded category \( H(\mathcal{A}) \). Indeed, the associated bilinear forms \( \langle \; , \; \rangle_{ab}^H \) are obviously graded-symmetric (with respect to the grading \( | | \)) and satisfy \( \langle u \ast v, w \rangle_{ac}^H = \langle u, v \ast w \rangle_{bc}^H \) for all \( u \in \text{Hom}_{H(\mathcal{A})}(b, c) \), \( v \in \text{Hom}_{H(\mathcal{A})}(a, b) \) and \( w \in \text{Hom}_{H(\mathcal{A})}(c, a) \) (as implied by the second equation in (3.2)).

Nondegeneracy conditions. A \( D \)-cyclic structure on \( \mathcal{A} \) will be called strictly nondegenerate if \( \mathcal{A} \) is degreewise Hom-finite and the bilinear forms \( \langle \; , \; \rangle_{ab} \) are non-degenerate. It is called homologically non-degenerate if \( \mathcal{A} \) is compact and the bilinear pairings induced on cohomology are nondegenerate; equivalently, the \( D \)-cyclic structure induced on the graded associative category \( H(\mathcal{A}) \) is nondegenerate. We say that \( \mathcal{A} \) is \( D \)-Calabi-Yau if it admits at least one homologically nondegenerate \( D \)-cyclic structure.

A cyclic structure on \( \mathcal{A} \) is nondegenerate iff the pairings \( \langle \; , \; \rangle_{ab} \) induce isomorphisms of graded vector spaces \( \text{Hom}_A(a, b)[D] \to \text{Hom}_A(b, a) \). Notice that the latter condition implies\footnote{Indeed, an ungraded vector space \( V \) can be isomorphic with its linear dual \( V' = \text{Hom}_\mathbb{C}(V, \mathbb{C}) \) iff it is finite-dimensional. This follows from the inequality \( \dim_\mathbb{C}V \leq \dim_\mathbb{C}V' \) for the transfinite dimension (cardinal of any basis), which is strict unless \( \dim_\mathbb{C}V \) is finite.} degreewise Hom-finiteness of \( \mathcal{A} \).

The cyclic structure is homologically nondegenerate iff the bilinear forms induce isomorphisms of graded vector spaces \( \text{Hom}_{H(\mathcal{A})}(a, b)[D] \to \text{Hom}_{H(\mathcal{A})}(b, a) \). Notice that the latter condition implies compactness of \( \mathcal{A} \).

Compatibility with shifts. Let us assume that \( \mathcal{A} \) has a twisted shift functor \([ [1] ]\). A \( D \)-cyclic structure on \( \mathcal{A} \) is called shift-equivariant if the following condition holds (cf. relation (1.9)):

\[
\langle [x[[1]]] \otimes [y[[1]]] \rangle_{a[[1]], b[[1]]} = -\langle x \otimes y \rangle_{ab}, \quad \forall a, b \in \text{Ob}\mathcal{A}, \; \forall x \in \text{Hom}_A(a, b), \; \forall y \in \text{Hom}_A(b, a),
\]

i.e. \( \langle a[[1]], b[[1]] \rangle \circ ([1] \otimes [1]) = \langle \rangle_{ab} \). In this case, we also say that \( \langle \mathcal{A}, [[1]], \langle \; , \; \rangle \rangle \) is a cyclic \( A_\infty \) category with shifts. Given such a pairing on \( \mathcal{A} \), the cyclic pairing induced on \( H(\mathcal{A}) \) is shift-equivariant.

The strictly unital case. Let us assume that \( \mathcal{A} \) is strictly unital. Then all information carried by the cyclic pairings is encoded by the linear maps \( \text{tr}_a : \text{Hom}_A(a, a) \to \)
$\mathbb{C}[-D]$ of degree zero defined through:

$$\text{tr}_a(x) := \langle u_a \otimes x \rangle_{aa} \quad \forall x \in \text{Hom}_A(a, a) .$$

Indeed, the second cyclicity condition in (3.1) gives

$$\langle x \otimes y \rangle_{ab} = \text{tr}_a(m_{aba}(x \otimes y)) \quad \forall x \in \text{Hom}_A(a, b) \quad \forall y \in \text{Hom}_A(b, a) ,$$

(3.4)

while graded symmetry of the pairings reduces to:

$$\text{tr}_a(m_{aba}(x \otimes y)) = (-1)^{|x||y|} \text{tr}_b(m_{bab}(y \otimes x)) \quad \forall x \in \text{Hom}_A(a, b) \quad \forall y \in \text{Hom}_A(b, a) .$$

(3.5)

The remaining cyclicity conditions (3.1) become:

$$\text{tr}_a \circ m_{aa...a_{n-1}a} = 0 \quad \forall n \neq 2 .$$

(3.6)

Conversely, equations (3.4), (3.5) and (3.6) imply (3.1) upon using the $A_\infty$ constraints. When $\mathcal{A}$ admits a twisted shift functor $[[1]]$, the shift-equivariance condition (3.3) becomes:

$$\text{tr}_a[[1]]x[[1]] = -\text{tr}_a(x) \quad \forall x \in \text{Hom}_A(a, a) \quad \forall a \in \text{Ob}A .$$

In terms of the original $A_\infty$ compositions $\mu$, equations (3.4) take the form:

$$\langle x \otimes y \rangle_{ab} = (-1)^{|x||y|} \text{tr}_a(\mu_{aba}(y \otimes x)) \quad \text{for } x \in \text{Hom}_A(a, b) \quad \text{and } y \in \text{Hom}_A(b, a) ,$$

(3.7)

while relations (3.5) and (3.6) read:

$$\text{tr}_a(\mu_{aba}(y \otimes x)) = (-1)^{|x||y|} \text{tr}_b(\mu_{bab}(x \otimes y)) \quad , \quad \text{tr}_a \circ \mu_{aa...a_{n-1}a} = 0 \quad \forall n \neq 2 .$$

(3.8)

The last relation in (3.8) implies $\text{tr}_a \circ \mu_{aa} = 0$, which shows that $\text{tr}_a$ descend to well-defined functionals on $H(\mathcal{A})$, which we denote by $\text{tr}_a^H$:

$$\text{tr}_a^H([x]) := \text{tr}_a(x) \quad \forall x \in Z(\text{Hom}_A(a, a)) .$$

These satisfy:

$$\text{tr}_a^H(x \ast y) = (-1)^{|x||y|} \text{tr}_b^H(y \ast x) \quad \forall x \in \text{Hom}_{H(\mathcal{A})}(b, a) \quad \forall y \in \text{Hom}_{H(\mathcal{A})}(a, b)$$

and:

$$\langle x, y \rangle_{ab}^H = (-1)^{|x||y|} \text{tr}_a^H(y \ast x) = \text{tr}_b^H(x \ast y) \quad \forall x \in \text{Hom}_A(a, b) \quad \forall y \in \text{Hom}_A(b, a) .$$

Therefore, they correspond to the traces defined by $\langle \rangle^H$. 

29
Suspended pairings  It is sometimes convenient to use suspended pairings $\omega_{ab}$:

$$
\omega_{ab} := \langle \rangle_{ab} \circ (s_{ab}^{-1} \otimes s_{ba}^{-1}) \ ,
$$

(3.9)
i.e. $\omega_{ab}(x \otimes y) = (-1)^{\bar{x}} \langle x \otimes y \rangle_{ab}$. These are graded antisymmetric:

$$
\omega_{ab}(x \otimes y) = (-1)^{\bar{x} + 1} \omega_{ba}(y \otimes x)
$$

and satisfy the modified cyclicity conditions:

$$
\omega_{aa_0a_0}(x_0 \otimes r_{a_0} \cdots a_0(x_1 \otimes \cdots \otimes x_n)) = (-1)^{\bar{x}_0 + \bar{x}_1 + \cdots + \bar{x}_n} \omega_{a_0a_1}(x_1 \otimes r_{a_1} \cdots a_0(x_2 \otimes \cdots \otimes x_n \otimes x_0)) \ .
$$

(3.10)

When $\mathcal{A}$ is strictly unital, $\omega_{ab}$ correspond to the suspended traces:

$$
\text{Tr}_a := \text{tr}_a \circ s_{aa}^{-1} : \text{Hom}_\mathcal{A}(a,a)[1] \to \mathbb{C}[1 - D]
$$
to which they are related through the equations:

$$
\omega_{ab} = \text{Tr}_a \circ r_{aba} \ .
$$

We also have $\text{Tr}_a(x) = -\omega_{ab}(u_a \otimes x)$. The modified cyclicity conditions (3.10) amount to:

$$
\text{Tr}_a \circ r_{aa_1 \cdots a_{n-1}a} = 0 \ \forall n \neq 2
$$
together with:

$$
\text{Tr}_a (r_{aba}(x \otimes y)) = (-1)^{\bar{x} + 1} \text{Tr}_b(r_{bab}(y \otimes x)) \ .
$$

When $\mathcal{A}$ admits a twisted shift functor, the shift-equivariance condition (3.3) becomes $\omega_{a[[1]]b[[1]]} \circ ([[1]]^* \otimes [[1]]^*) = -\omega_{ab}$, i.e.:

$$
\omega_{a[[1]]b[[1]]}(x[[1]]^* \otimes y[[1]]^*) = -\omega_{ab}(x \otimes y) \ ,
$$

(3.11)

which in the strictly unital case amounts to:

$$
\text{Tr}_a [[1]]^* = -\text{Tr}_a \ .
$$

Sector decomposition.  In terms of the sector decomposition considered in Section 2.2, a $D$-cyclic pairing on $\mathcal{A}$ amounts to a morphism of graded $R$-bimodules $\langle \rangle \in \text{Hom}_{R\text{GrMod}_R}(\mathcal{H} \otimes_R \mathcal{H}, R[-D])$ where $R$ is endowed with the obvious bimodule structure over itself. This map takes the form $\langle x \otimes_R y \rangle = \oplus_{a,b \in \text{Ob}} \langle x_{ab} \otimes y_{ba} \rangle_{ab}^{e_a}$ for all $x, y \in \mathcal{H}$. The cyclicity conditions (3.1) reduce to:

$$
\langle x_0 \otimes_R r_n(x_1 \otimes_R \cdots \otimes_R x_n) \rangle = (-1)^{\bar{x}_0 + \bar{x}_1 + \cdots + \bar{x}_n} \langle x_1 \otimes_R r_n(x_2 \otimes_R \cdots \otimes_R x_n \otimes_R x_0) \rangle \ .
$$

(3.12)
These can also be written as follows. Since $R$ is commutative, one has natural inclusions\footnote{Given $f \in \text{Hom}_{\text{GrMod}_R}(H^{\otimes Rn}, R[-D])$ we have $f(x) = \sum_{a,b} f(\epsilon_a x \epsilon_b) = \sum_{a,b} \epsilon_a f(x) \epsilon_b = \sum_a \epsilon_a f(x) \epsilon_a = f(\sum_a \epsilon_a x \epsilon_a)$, where we used commutativity of $R$ and the identities $\epsilon_a \epsilon_b = \delta_{ab} \epsilon_a$. Thus $f(x)$ is determined by its restriction to $[H^{\otimes Rn}]^R$. This means that $f$ can be viewed as a degree zero $\mathbb{C}$-linear map from $[H^{\otimes Rn}]^R$ to $R[-D]$.} $\text{Hom}_{\text{GrMod}_R}(H^{\otimes Rn}, R[-D]) \subset \text{Hom}_R([H^{\otimes Rn}]^R, R[-D])$, where $[H^{\otimes Rn}]^R$ is the center of the $R$-bimodule $H^{\otimes Rn}$. In particular, $\langle \rangle$ can be viewed as a morphism of graded vector spaces $\langle \rangle \in \text{Hom}_R([H \otimes_R H]^R, R[-D])$ which commutes with the action of $\epsilon_a$. Consider the maps $P_{n+1} \in \text{Hom}_R(H^{\otimes (n+1)}, R[-D])$ defined through:

\[ P_{n+1} := \langle \rangle \circ (\text{id}_H \otimes_R r_n) \]

which we view as elements of $\text{Hom}_R([H^{\otimes (n+1)}]^R, R[-D])$ as explained above. Let $\Pi_{n+1} : [H^{\otimes (n+1)}]^R \to [H^{\otimes (n+1)}]^R$ be the $\mathbb{C}$-linear automorphism given by\footnote{This is easily seen to be well-defined upon considering the sector decomposition of $x_0 \ldots x_n$.}:

\[ \Pi_{n+1}(x_0 \otimes_R x_1 \otimes_R \ldots \otimes_R x_n) = (-1)^{\bar{x}_0(\bar{x}_1 + \ldots + \bar{x}_n)} x_1 \otimes_R \ldots \otimes_R x_n \otimes_R x_0 \]

Then equations (3.12) amount to:

\[ P_{n+1} \circ \Pi_{n+1} = P_{n+1} \quad \forall n \geq 1 \]

When $A$ admits a twisted shift functor, the shift-equivariance condition (3.3) for the bilinear pairings reduces to:

\[ \langle \rangle \circ ([1] \otimes [1]) = -\langle \rangle \]

where $[[1]] : H \to H$ is the total shift operator defined in the previous section. When $A$ is also unital, we have total traces $\text{tr} : H^R \to R[-D]$ given by $\text{tr}(x) = \sum_a x_{aa}$, and the shift-equivariance condition takes the form:

\[ \text{tr} \circ [[1]] = -\text{tr} \]

### 3.2 Extension of cyclic pairings

Let $A$ be an $A_\infty$ category endowed with a $D$-cyclic structure $\langle \rangle$. In this subsection, we show that $\langle \rangle$ extends naturally to a shift-equivariant pairing on $\text{tw}(A)$, thereby inducing a shift-equivariant pairing on the triangulated category $\text{tria}(A)$. When the pairing of $A$ is nondegenerate or homologically nondegenerate, the same is true of the pairing induced on $\text{tw}(A)$. In each of these cases, the triangulated category $\text{tria}(A) = H^0(\text{tw}(A))$ is $D$-Calabi-Yau.
**Extension to $\Sigma A$.** Consider the linear maps $\omega_{a[[m]],b[[n]]} : \text{Hom}_{\Sigma A}(a[[m]], b[[n]])[1] \otimes \text{Hom}_{\Sigma A}(b[[n]], a[[m]])[1] \to \mathbb{C}[2 - D]$ given by:

$$\omega_{a[[m]],b[[n]]} := (-1)^m \omega_{ab} \circ (s^{-n}_{ab} \otimes s^{n-m}_{ba}) \circ ([[-m]]^s \otimes [[-n]]^s) \quad (3.16)$$

and the desuspended maps $\langle \rangle_{a[[m]],b[[n]]} : \text{Hom}_{\Sigma A}(a[[m]], b[[n]]) \otimes \text{Hom}_{\Sigma A}(b[[n]], a[[m]]) \to \mathbb{C}[-D]$ defined through $\omega_{a[[m]],b[[n]]} = \langle \rangle_{a[[m]],b[[n]]} = (s^{-1}_{a[[m]]} b[[n]] \otimes s^{-1}_{b[[n]] a[[m]]})$, i.e.:

$$\langle \rangle_{a[[m]],b[[n]]} = (-1)^m \langle \rangle_{ab} \circ (s^{-n}_{ab} \otimes s^{n-m}_{ba}) \circ ([[-m]] \otimes [[-n]]) \quad (3.17)$$

An easy computation shows that $\langle \rangle_{\Sigma A}$ is a cyclic pairing on the $A_\infty$ category $\Sigma A$, of the same degree as the original pairing on $A$. It is also clear from (3.17) that $\langle \rangle_{\Sigma A}$ is shift-equivariant. Hence $(\Sigma A, [[1]], \langle \rangle_{\Sigma A})$ is a cyclic $A_\infty$ category with twisted shifts.

When $A$ (and thus $\Sigma A$) is strictly unital, we define traces $\text{tr}_{\Sigma A}$ and $\text{tr}$ associated with $\langle \rangle_{\Sigma A}$ and $\langle \rangle$ and suspended traces $\text{Tr}_{\Sigma A}$ and $\text{Tr}$ associated with $\omega_{\Sigma A}$ and $\omega$ as in the previous subsection:

$$\omega_{ab} = \text{Tr} \circ r_{ab} \quad , \quad \omega_{a[[m]],b[[n]]} = \text{Tr}_{\Sigma A} \circ r_{a[[m]],b[[n]],a[[m]]}$$

$$\langle \rangle_{ab} = \text{tr} \circ m_{ab} \quad , \quad \langle \rangle_{a[[m]],b[[n]]} = \text{Tr}_{\Sigma A} \circ m_{a[[m]],b[[n]],a[[m]]}$$

and we have $\text{Tr} \circ s^{-1}_{a[[m]]}$ and $\text{Tr}_{\Sigma A} \circ s^{-1}_{a[[m]]}$. Equations (3.16) and (3.17) amount to:

$$\text{Tr}_{a[[m]]} = (-1)^m \text{Tr} \circ [[-m]]^s \iff \text{tr}_{a[[m]]} = (-1)^m \text{tr} \circ [[-m]] \quad (3.18)$$

i.e.

$$\text{Tr}_{a[[m]]}(x) = (-1)^m \text{Tr}(x) \quad \text{and} \quad \text{tr}_{a[[m]]}(x) = (-1)^m (x) \quad (3.19)$$

**Extension to $\Sigma A$.** The pairing $\langle \rangle_{\Sigma A}$ extends by additivity to a pairing $\langle \rangle_{\Sigma A}$ on $\Sigma A$:

$$\langle f, g \rangle_{\Sigma A} = \sum_{i,j} (f_{ij} \otimes g_{ji})_{A_i,B_j}$$

where $f = \oplus_{i,j} f_{ij} \in \text{Hom}_{\Sigma A}(A, B), g = \oplus_{i,j} g_{ji} \in \text{Hom}_{\Sigma A}(B, A)$ with $A = \oplus_i A_i, B = \oplus_j B_j \in \text{Ob} \Sigma A$ and $f_{ij} \in \text{Hom}_{\Sigma A}(A_i, B_j), g_{ji} \in \text{Hom}_{\Sigma A}(B_j, A_i)$ (here $A_i, B_j \in \text{Ob} \Sigma A$). This pairing on $\Sigma A$ obeys the cyclicity conditions with respect to $r_{\Sigma A}$ and is obviously shift-equivariant. Thus $(\Sigma A, [[1]], \langle \rangle_{\Sigma A})$ is a cyclic $A_\infty$ category with twisted shifts.
**Extension to** $\text{tw}(\mathcal{A})$. Recalling that the morphism spaces of $\text{tw}(\mathcal{A})$ coincide with those of $\Sigma \mathcal{A}$, one checks by direct computation that the products $r^{\text{tw}(\mathcal{A})}$ are cyclic with respect to $\langle \rangle^{\Sigma \mathcal{A}}$. Defining $\langle \rangle^{\text{tw}(\mathcal{A})} := \langle \rangle^{\Sigma \mathcal{A}}$, we conclude that $(\text{tw}(\mathcal{A}), [1], \langle \rangle^{\text{tw}(\mathcal{A})})$ is a cyclic $A_{\infty}$ category with twisted shifts. Recall that $\text{tw}(\mathcal{A})$ is degreewise Hom-finite, respectively compact, iff $\mathcal{A}$ has the same property. Tracing through the steps above, it is clear $\langle \rangle^{\text{tw}(\mathcal{A})}$ is strictly nondegenerate iff the original pairing on $\mathcal{A}$ is strictly nondegenerate. As we will see below, a similar statement holds for homological nondegeneracy.

**The cyclic structure induced on** $\text{tria}(\mathcal{A})$. Passing to the cohomology category, the pairing on $\text{tw}(\mathcal{A})$ induces a shift-equivariant $D$-cyclic structure $\langle \rangle^{H}$ on $H(\text{tw}(\mathcal{A}))$. Since $H(\text{tw}(\mathcal{A})) = H^0(\text{tw}(\mathcal{A}))^*$, this corresponds to a shift-equivariant $D$-cyclic structure on the triangulated category $H^0(\text{tw}(\mathcal{A})) = \text{tria}(\mathcal{A})$. The results of Appendix A.7 show that the latter is non-degenerate iff $\langle \rangle^{H}_{ab} : H(H_{\mathcal{A}}(a,b)) \otimes H(H_{\mathcal{A}}(b,a)) \rightarrow \mathbb{C}$ are nondegenerate for all $a, b \in \text{Ob}\mathcal{A}$, which amounts to homological nondegeneracy of the cyclic pairing of $\mathcal{A}$. Thus:

**Proposition** A $D$-cyclic structure on $\mathcal{A}$ induces shift-equivariant $D$-cyclic structures on $\text{tw}(\mathcal{A})$ and $\text{tria}(\mathcal{A}) = H^0(\text{tw}(\mathcal{A}))$. Moreover:

1. $\text{tw}(\mathcal{A})$ is degreewise Hom-finite iff $\mathcal{A}$ is and the cyclic structure induced on $\text{tw}(\mathcal{A})$ is strictly nondegenerate iff the original cyclic structure on $\mathcal{A}$ is strictly nondegenerate
2. $\text{tw}(\mathcal{A})$ is compact iff $\mathcal{A}$ is and the cyclic structure induced on $\text{tw}(\mathcal{A})$ is homologically nondegenerate (in particular, $\text{tria}(\mathcal{A})$ is $D$-Calabi-Yau) iff the original cyclic structure on $\mathcal{A}$ is homologically nondegenerate

### 3.3 Minimal models induced by a cohomological splitting

Fixing an $A_{\infty}$ category $\mathcal{A}$, let $R, \mathcal{H} := \mathcal{H}_{\mathcal{A}}$ and $r_1$ be defined as in Section 2.2. Recall that a minimal model of $\mathcal{A}$ is a minimal $A_{\infty}$ category $\mathcal{B}$ which is quasi-isomorphic with $\mathcal{A}$. The work of [26, 15, 10] provides an explicit construction of a particular class of minimal models, which we recall below. Adapting this will allow us to build a special class of cyclic minimal models for a cyclic $A_{\infty}$ category. In this section, we view $r$ as defined on the space $\mathcal{H}$ endowed with the tilde grading.

**Retracts.** Define a *strict homotopy retraction* of $\mathcal{A}$ to be a homotopy retraction of the $R$-complex $(\mathcal{H}, r_1)$ (notice that $r_1 = m_1$), i.e. a pair $(P, G)$ with $P \in \text{Hom}_{\text{GrMod}_R}(\mathcal{H}, \mathcal{H})$ and $G \in \text{Hom}_{\text{GrMod}_R}(\mathcal{H}, \mathcal{H}[-1])$ such that:

1. $P^2 = P$
2. $\text{id}_{\mathcal{H}} - P = r_1 \circ G + G \circ r_1$. 33
In category-theoretic language, this corresponds to morphisms of graded vector spaces $P_{ab} : \text{Hom}_A(a,b) \to \text{Hom}_A(a,b)$ and $G_{ab} : \text{Hom}_A(a,b) \to \text{Hom}_A(a,b)[-1]$ such that $(P_{ab})^2 = P_{ab}$ and $\text{id}_{ab} - P_{ab} = r_{ab} \circ G_{ab} + G_{ab} \circ r_{ab}$ (recall that $\text{id}_{ab} := \text{id}_{\text{Hom}_A(a,b)}$). The submodule $B := \text{im} P \subset \mathcal{H}$ corresponds to the subspaces $B_{ab} = \text{im} P_{ab} \subset \text{Hom}_A(a,b)$ (we have $B = \bigoplus_{a,b \in \text{Ob} A} B_{ab}$).

We let $i : B \to \mathcal{H}$ be the inclusion and $p : \mathcal{H} \to B$ be the corestriction of $P$ to $B$ (thus $P = i \circ p$). These correspond to the inclusions $i_{ab} : B_{ab} \to \text{Hom}_A(a,b)$ and surjections $p_{ab} : \text{Hom}_A(a,b) \to B_{ab}$. Using the identity $r_1 \circ r_1 = 0$, condition (2) above implies $r_1 \circ P = P \circ r_1$, which in turn shows that $r_1(B) \subset B$.

For every $n \geq 2$, consider the set $\mathcal{T}_n$ of all oriented and connected planar trees $T$ such that:

(I) $T$ has exactly $n+1$ vertices of valency one (called external vertices), all other vertices having valency at least 3 (these are called internal vertices). The edges meeting an external vertex are called external edges, the other being called internal edges. An external edge is called incoming if it leaves the corresponding external vertex, and outgoing if it enters the corresponding external vertex.

(II) Exactly one external edge is outgoing (being called the root of $T$); the other $n$ external edges are incoming (being called the leaves of $T$).

(III) For each internal vertex of $T$, exactly one of the edges it meets leaves that vertex; the others enter it.

We let $E(T)$ respectively $E_i(T)$, $E_e(T)$ be the sets of all edges, respectively all internal and external edges of $T$. We also let $e_i(T) := \text{Card}E_i(T)$ be the number of internal edges. For each $T \in \mathcal{T}_n$ we define a morphism of graded $R$-bimodules $\rho_T \in \text{Hom}_{R\text{-Mod}_R}(B^\otimes_R n, B)$ as follows:

(a) Associate the inclusion $i$ with every leaf of $T$

(b) Associate the surjection $p$ with the root of $T$

(c) Associate the product $r_k$ with each internal vertex of $T$ of valency $k+1$

(d) Associate $G$ with each internal edge of $T$.

Following the tree from its root toward its leaves, consider the composition of the operators associated to each edge and vertex, using tensor products and insertions.

---

15Orienting the plane clockwise, this means that all edges meeting any given vertex of $T$ are cyclically ordered in the clockwise direction on the plane. Such an ordering is called a ribbon structure in [10], where a planar graph is called a ribbon graph.
of the identity map $\text{id}_B$ wherever needed (always arranged in clockwise order in the plane). Finally, multiply the result by the sign factor $(-1)^{e_i(T)}$. For the example shown in figure 1, this gives the product:

$$\rho_T = +p \circ r_4 \circ (i \otimes G \otimes G \otimes i) \circ (\text{id}_B \otimes r_3 \otimes r_2 \otimes \text{id}_B) \circ (\text{id}_B \otimes i \otimes i \otimes i \otimes i \otimes \text{id}_B) : B^\otimes T \to B.$$ 

We now define $\rho_1 := r_1|_B = p \circ r_1 \circ i$ (recall that $r_1(B) \subset B$) and $\rho_n := \sum_{T \in \mathcal{T}_n} \rho_T \in \text{Hom}_{B\text{-Mod}}(B^\otimes R^n, B)$ for all $n \geq 2$. Notice that $\rho_2 = p \circ r_2 \circ (i \otimes 2) = p \circ r_2|_{B \otimes B}$. Expanding into sectors, one can write the compositions $\rho_{a_0...a_n}$ as sums over decorated trees, i.e., trees $T \in \mathcal{T}_n$ together with labels chosen coherently for the two sides of each edge. One can visualize this by considering the ribbon associated with $T$, and placing labels in the obvious manner. This is entirely trivial and we leave it as an exercise for the reader. One has the following result:

**Theorem[15, 10]** The products $(\rho_n)_{n \geq 1}$ satisfy the (forward) $A_\infty$ relations. Hence they define an $A_\infty$ category $\mathcal{B}$ having the same objects as $\mathcal{A}$ and morphism spaces $\text{Hom}_{\mathcal{B}}(a, b) := B_{ab}$.

The $A_\infty$ category $\mathcal{B}$ will be called the retract of $\mathcal{A}$ along $(P, G)$. Its $A_\infty$ compositions $\rho_{a_0...a_n} : \text{Hom}_{\mathcal{B}}(a_0, a_1) \otimes \ldots \otimes \text{Hom}_{\mathcal{B}}(a_{n-1}, a_n) \to \text{Hom}_{\mathcal{B}}(a_0, a_n)$ are obtained by decomposing:

$$\rho_n = \bigoplus_{a_0, a_n} \sum_{a_1, \ldots, a_{n-1}} \rho_{a_0...a_n},$$

which is possible by $R$-multilinearity. In the notation of Section 2.2, we have $B = \mathcal{H}_B = \bigoplus_{a, b \in \text{Obj} \mathcal{B}} \text{Hom}_{\mathcal{B}}(a, b)$. 

---

**Figure 1:** An oriented tree $T \in \mathcal{T}_7$ with $e_i(T) = 2$. 

---

35
Observation  Notice the relations \( \rho_T = p \circ \lambda_T \) and \( \rho_n = p \circ \lambda_n \) where \( \lambda_n = \sum_{T \in \mathcal{T}_n} \lambda_T \), with \( \lambda_T \in \text{Hom}_{\text{Mod}_{R}}(B^{\otimes R_n}, \mathcal{H}) \) \( (n \geq 2) \) defined exactly as \( \rho_T \) except that we insert \( \text{id}_\mathcal{H} \) instead of the map \( p \) along the root of the tree \( T \in \mathcal{T}_n \). Consider the maps \( \iota_n \in \text{Hom}_{\text{Mod}_{R}}(B^{\otimes R_n}, \mathcal{H}) \) given by \( \iota_1 := i \) and \( \iota_n = G \circ \lambda_n \) for \( n \geq 2 \). It was shown in [10] that \( i = (i_n)_{n \geq 1} \) gives an \( A_\infty \) quasi-isomorphism between \( (B, \rho) \) and \( (\mathcal{H}, r) \). Of course, this corresponds to an \( A_\infty \) quasi-isomorphism between the \( A_\infty \) categories \( \mathcal{B} \) and \( \mathcal{A} \). In particular, \( i \) induces an isomorphism of graded \( R \)-bimodules \( i_* : H(\mathcal{H}) \to H(B) \), i.e. an isomorphism between the graded associative categories \( H(B) \) and \( H(\mathcal{A}) \).

A strict homotopy retraction \((P, G)\) of \( \mathcal{A} \) is called a cohomological splitting if \( r_1|_B = 0 \), i.e. \( p_1 = 0 \). In this case, \( i_* \) and \( p_* \) give inverse isomorphisms between \( B \) and \( H_{r_1}(\mathcal{H}) \). Moreover, \((B, \rho)\) is a minimal model of \((\mathcal{H}, r)\) and the category \( \mathcal{B} \) is a minimal model of \( \mathcal{A} \).

Proof. We give the proof of the proposition for completeness\(^{16}\). Let \((r)^n_i := r_1 \circ r_n + \sum_{i=0}^{n-1} r_n \circ (\text{id}_\mathcal{H}^{\otimes i} \otimes r_1 \otimes \text{id}_\mathcal{H}^{\otimes (n-i-1)})\), with a similar notation \((\rho)^n_i\) for the products \( \rho \). The \( A_\infty \) relations (2.7) are equivalent with \( r_1^2 = 0 \) together with the equations:

\[
(r)^n_i = -\sum_{i \geq 0} \sum_{j = 2}^{n-i} r_{n-j+1} \circ (\text{id}_\mathcal{H}^{\otimes i} \otimes r_j \otimes \text{id}_\mathcal{H}^{\otimes (n-j-i)}) \quad \forall n \geq 2 .
\]

(3.20)

Since \( r_1 \) preserves the subspace \( B \), we have \( \rho_1^2 = r_1^2|_B = 0 \), so it suffices to prove that \( \rho_n \) satisfy:

\[
(\rho)^n_i = -\sum_{i \geq 0} \sum_{j = 2}^{n-i} \rho_{n-j+1} \circ (\text{id}_B^{\otimes i} \otimes \rho_j \otimes \text{id}_B^{\otimes (n-j-i)}) \quad \forall n \geq 2 .
\]

(3.21)

Given \( f \in \text{End}_{\text{Mod}_R}(\mathcal{H}) \), a tree \( T \in \mathcal{T}_n \) having at least one internal edge and \( e \in E_i(T) \), we let \( \rho^f_{T,e} \) be the product obtained from \( \rho_T \) upon replacing the insertion of \( G \) along the internal edge \( e \) with the operator \( f \). We define \( \rho^f_n = \sum_{T \in \mathcal{T}_n, e_i(T) \geq 1} \sum_{e \in E_i(T)} \rho^f_{T,e} \in \text{Hom}_{\text{Mod}_R}(B^{\otimes R_n}, B) \). We also let \( \hat{\rho}_{T,e} \) be the maps obtained from \( \rho_T \) upon inserting the operator \( r_1 \) before, respectively after the insertion of \( G \) along that edge and define \( \hat{\rho}_{T,e} := \hat{\rho}_{T,e+} + \hat{\rho}_{T,e-} \). Given an arbitrary tree \( T \in \mathcal{T}_n \) and \( e \in E_i(T) \), we let \( \hat{\rho}_{T,e} \) be the map obtained from \( \rho_T \) upon inserting the operator \( r_1 = r_1|_B \) near the external vertex lying on the edge \( e \). Since \( r_1 \circ P = P \circ r_1 \) and \( r_1(B) \subset B \), we have \( \rho_1 \circ p = p \circ r_1 \) and \( i \circ \rho_1 = r_1 \circ i \), so \( \hat{\rho}_{T,e} \) coincides with the map obtained by inserting \( r_1 \) next to the

\(^{16}\)This is essentially the proof given in [15], except that we give a clear accounting of the signs.
internal vertex meeting \( e \). Finally, define \( \hat{\rho}_n := \sum_{T \in \mathcal{T}_n} \sum_{e \in E(T)} \hat{\rho}_{T,e} \). Using equation \( r_1 \circ G + G \circ r_1 = \text{id}_\mathcal{H} - P \), we find:

\[
\hat{\rho}_n = (\rho)^n_1 + \rho^{\text{id}_\mathcal{H}}_n - \rho^P_n ,
\]

(3.22)

where the first term comes from the \( \rho_1 \)-insertions along external edges.

The map \( \hat{\rho}_n \) can also be computed by using equations (3.20). Indeed, consider the sum of those contributions to \( \hat{\rho}_T \) coming from insertions of \( r_1 \) immediately next to the output or immediately next to one of the inputs of the product \( r_k \) associated with a fixed internal vertex \( v \) of \( T \) of valency \( k + 1 \). Equation (3.20) (with \( n \) replaced by \( k \)) allows us to replace the sum of such contributions with the sum of those contributions to the product \( \rho_{\text{id}_\mathcal{H}} \) which arise from trees \( T' \) obtained from \( T \) upon replacing the vertex \( v \) with two vertices of valency \( k - j + 2 \) and \( j + 1 \) connected by an internal edge (here \( j \) runs from 2 to \( k \)). This edge can be chosen in \( k - j + 1 \) distinct ways which correspond to the sum over \( i \) in (3.20). This edge of \( T' \) carries the insertion of the identity operator \( \text{id}_\mathcal{H} \) required by the definition of \( \rho_{\text{id}_\mathcal{H}} \). Since \( e_i(T') = e_i(T) + 1 \), the minus sign from (3.20) produces the extra minus sign required by the sign prefactor in the definition of \( \rho_{\text{id}_\mathcal{H}} \). Applying this procedure to all internal vertices, it is clear that \( \hat{\rho}_n \) can be expressed as:

\[
\hat{\rho}_n = \rho_{\text{id}_\mathcal{H}}^n .
\]

(3.23)

Combining this with equation (3.22) gives \( (\rho)^n_1 = \rho^P_n \). On the other hand, a moment’s thought shows that \( \rho^P_n = -\sum_{i \geq 0} \sum_{j=2}^{n-i} \rho_{n-j+1} \circ (\text{id}_{\mathcal{B}}^i \otimes \rho_j \otimes \text{id}_{\mathcal{B}}^{(n-j-i)}) \), where we used the decomposition \( P = i \circ p \) and the minus sign is again due to the prefactor used in the definition of \( \rho_T \). This shows that (3.21) are satisfied.

The strictly unital case. Now assume that \( \mathcal{A} \) is strictly unital. A strict homotopy retraction \( (P,G) \) of \( \mathcal{A} \) is called strictly unital if it satisfies the supplementary conditions:

\[
(3) \quad P_{ab} \circ G_{ab} = 0 \quad \text{for all } a,b \in \text{Ob}\mathcal{A} \quad \text{and} \quad G_{aa}(u_a) = 0 \quad \forall a \in \text{Ob}\mathcal{A} .
\]

Since \( r_{aa}(u_a) = 0 \), conditions (3) imply \( P_{aa}(u_a) = u_a \) i.e. \( u_a \in B_{aa} \). They also imply \( p_{ab} \circ G_{ab} = 0 \). We have the following:

**Proposition** Assume that \( \mathcal{A} \) is strictly unital and let \( (P,G) \) be a strict homotopy retraction of \( \mathcal{A} \). Then the \( A_\infty \) category \( \mathcal{B} \) is strictly unital with the same units as \( \mathcal{A} \).

**Proof.** To avoid notational morass, let us first assume that \( \mathcal{A} \) has a finite number of objects. With this assumption, set \( u := \bigoplus_{a \in \text{Ob}\mathcal{A}} u_a \in \mathcal{H}^\mathbb{R} \) and notice that conditions (3) amounts to \( G(u) = 0 \) and \( p \circ G = 0 \) and that we have \( u \in B \). It suffices to show
that the products $\rho_n$ satisfy (2.10). For this, let $T \in \mathcal{T}_n$. Condition (3) and unitality of $r_n$ imply that $\rho_T(x_1 \otimes \ldots \otimes u \otimes \ldots \otimes x_n)$ vanishes unless the internal vertex of $T$ which meets the root of $T$ has valency 3 and $u$ is inserted along an incoming edge flowing directly into this vertex (otherwise $u$ is killed either by a product $r_k$ with $k \neq 2$ or by an insertion of $G$). If $T$ satisfies these conditions, then the unitality condition for $r_2$ implies that the insertion of $r_2$ at this vertex of $T$ reproduces whatever flows into it from the other branch up to a sign. Since $p \circ G = 0$, the result is killed by the final insertion of $p$ at the root unless this other branch is again reduced to an external edge. Thus $\rho_T(x_1 \otimes \ldots \otimes u \otimes \ldots \otimes x_n)$ vanishes unless $T$ has a single internal vertex, which is of valency 2 (i.e. unless $T$ is the unique tree in $\mathcal{T}_2$ having exactly one internal vertex). In particular, this requires $n = 2$, which gives the unitality property $\rho_n(x_1 \otimes \ldots \otimes u \otimes \ldots \otimes x_n) = 0$ for $n \neq 2$. The unitality constraint for $\rho_2$ follows from $\rho_2 = p \circ r_2|_{B^\otimes 2}$ by using the unitality property of $r_2$ and the fact that $u \in B$.

If $\mathcal{A}$ has an infinity of objects, a trivial adaptation of proof above goes through if one adds object labels to all trees and maps involved, as alluded to above. We leave this as an exercise for the reader.

The cyclic case. Now suppose that $\mathcal{A}$ is endowed with a cyclic pairing $\langle \rangle$. A strict homotopy retraction $(P,G)$ of $\mathcal{A}$ is called cyclic if the following condition is satisfied:

\begin{equation}
\langle G(x) \otimes y \rangle = (-1)^{|x|} \langle x \otimes G(y) \rangle \quad \forall x,y \in \mathcal{H}.
\end{equation}

Combining (4) and (2) above and the cyclicity condition $\langle r_1(x) \otimes y \rangle = (-1)^2 \langle x \otimes r_1(y) \rangle$ (see Section 3.1) gives:

\begin{equation}
\langle P(x) \otimes y \rangle = \langle x \otimes P(y) \rangle.
\end{equation}

The following proposition generalizes a result of [25] and [6].

**Proposition** Let $(P,G)$ be a cyclic strict homotopy retraction of a $D$-cyclic $A_\infty$ category $(\mathcal{A}, \langle \rangle)$. Then the $A_\infty$ products $\rho_n$ on $B$ defined above are cyclic with respect to the restriction of $\langle \rangle$ to $B$. Together with this restricted pairing, they define a $D$-cyclic $A_\infty$ structure on the category $B$.

The restricted pairing $\langle \rangle^B = \langle \rangle \circ (i \otimes i)$ corresponds to pairings $\langle \rangle^B_{ab}: \text{Hom}_B(a,b) \otimes \text{Hom}_B(b,a) \to \mathbb{C}$ as explained in Section 3. The cyclic $A_\infty$ category $(B, \langle \rangle^B)$ will be called the retract of $(\mathcal{A}, \langle \rangle)$ along $(P,G)$. A cohomological splitting $(P,G)$ is called cyclic if it is cyclic as a strict homotopy retraction. In this case, the cyclic $A_\infty$ category $(B, \langle \rangle^B)$ is minimal.
Proof. Writing \((3.24)\) as \(\langle \rangle \circ (P \otimes \text{id}_H) = \langle \rangle \circ (\text{id}_H \otimes P)\) and using \(P \circ i = i\) and the decomposition \(P = i \circ p\), we find:

\[
\langle \rangle^B \circ (\text{id}_B \otimes p) = \langle \rangle \circ (i \otimes \text{id}_H),
\]

i.e. \((x \otimes p(y))^B = (x \otimes y)\) for all \(x \in B\) and \(y \in H\). To prove cyclicity of \(\langle \rangle^B\), we must show that the maps \(P_{n+1} := \langle \rangle^B \circ (\text{id}_B \otimes \rho_n) \in \text{Hom}_C([B \otimes_R (n+1)]^R, R)\) satisfy \(P_{n+1} \circ \Pi_{n+1} = P_{n+1}\) (see equations \((3.14)\)). This follows from the following argument.

\textbf{Figure 2:} A graph \(\theta \in \Theta_8\) (left) and one of the contributions it brings to \(\mathcal{P}_8\). The pairing \(\langle \rangle\) is associated with the empty circle, which also indicates the choice of external vertex \(e\) which determines the presentation \(\theta = T'_e\) and thus the contribution shown to the right.

Using the definition of \(\rho_n\), we write \(\rho_n = p \circ \lambda_n\). Equation \((3.25)\) shows that \(\mathcal{P}_{n+1} = \langle \rangle \circ (i \otimes \lambda_n)\). Thus \(\mathcal{P}_{n+1} = \sum_{T \in \mathcal{T}_n} \mathcal{P}_T\), where \(\mathcal{P}_T = \langle \rangle \circ (i \otimes \lambda_T)\) have the following graphical description.

Let \(\Theta_{n+1}\) be the set of all unoriented simply connected \textit{planar} graphs having \(n+1\) vertices of valency one (=external vertices) and such that all other vertices have valency at least 3 (=internal vertices). Given a tree \(T \in \mathcal{T}_n\), we let \(T' \in \Theta_{n+1}\) be the unoriented graph obtained from \(T\) by forgetting the orientation of all edges. This gives a surjection \(\mathcal{T}_n \xrightarrow{\pi} \Theta_{n+1}, \pi(T) = T'\), which is an \(n+1\)-fold cover. Let us fix \(\tau \in \Theta_{n+1}\). Picking any external edge \(e\) of \(\tau\) gives a presentation \(\tau = T'_e\) where \(T_e \in \mathcal{T}_n\) is obtained by orienting \(e\) outwards (and viewing it as the root edge) and orienting all other edges toward the root edge (in fact all trees in the preimage \(\pi^{-1}(\tau)\) are obtained in this way). Now split the edge \(e\) in the middle by inserting a new vertex (depicted as an empty circle), thus
creating two edges $e', e''$, where $e''$ is the new external edge. We give $e'$ the orientation originally carried by $e$ and $e''$ the opposite orientation (see figure 2). Forgetting $e''$ for a moment, we obtain a tree in $T_n$ which is isotopic with $T_e$. To its incoming external edges and internal vertices we associate the maps $i$ and $r_k$ as before. To its root edge $e'$ we associate the map $\text{id}_H$. Finally, to the edge $e''$ of the new tree we associate the map $i$. Reading the diagram in the obvious way gives the contribution $P_{\tau,e} := P_{T_e}$ to $P_{n+1}$. These observations allow us to write $P_{n+1} = \sum_{\tau \in \Theta_n+1} \sum_{e \in E_{\tau}} P_{\tau,e}$. Equations (3.14) now follow essentially from the fact that the last expression is invariant under cyclic permutations of the $n + 1$ external edges (we leave the details of this last step to the reader).

**Shift equivariance.** Let us assume that $A$ has a twisted shift functor $[[1]]$ and let $[[1]] : \mathcal{H} \to \mathcal{H}$ be the total shift operator defined in the previous section. A strict homotopy retraction $(P,G)$ is called *shift-invariant* if the following condition is satisfied:

$$ (5) \ G \circ [[1]] = [[1]] \circ G $$

In this case, equations (1) and shift-invariance of $m_1 = r_1$ imply $P \circ [[1]] = [[1]] \circ P$.

**Proposition** Let $A$ be a cyclic $A_{\infty}$ category with shifts whose pairing $\langle \cdot \rangle$ is shift-equivariant, and let $(P,G)$ be a shift-invariant and cyclic strict homotopy retraction. Then the retract category $B$ constructed as above has a twisted shift functor and its pairing is shift-equivariant.

**Proof.** Equations $P \circ [[1]] = [[1]] \circ P$ show that $[[1]]$ preserves the subspace $B = \text{im} P$, on which it restricts to a total shift functor. Thus $[[1]]_{ab} : \text{Hom}_A(a,b) \to \text{Hom}_A(a[[1]], b[[1]])$ map the subspace $\text{Hom}_B(a,b)$ of $\text{Hom}_A(a,b)$ into the subspace $\text{Hom}_B(a[[1]], b[[1]])$ of $\text{Hom}_A(a[[1]], b[[1]])$. Since the total pairing $\langle \cdot \rangle$ of $A$ satisfies $\langle \cdot \rangle \circ (((1] \otimes [1])) = -\langle \cdot \rangle$, it is clear that the restricted bilinear pairing $\langle \cdot \rangle^B$ satisfies $\langle \cdot \rangle^B \circ (((1] \otimes [1])) = -\langle \cdot \rangle^B$. Since both $P$ and $G$ commute with $[[1]]$, and since $r_n \circ (((1] \otimes \cdots \otimes [1])^*) = [[1]]^* \otimes r_n$, it is clear from the definition of $\rho_n$ that $\rho_n \circ (((1] \otimes \cdots \otimes [1])^*) = [[1]]^* \otimes \rho_n$.

**The cyclic and strictly unital case.** Combining everything, we find:

**Corollary** Any strictly unital and cyclic strict homotopy retraction $(P,G)$ of a cyclic and strictly unital $A_{\infty}$ category $(A, \langle \cdot \rangle)$ determines a strictly unital and cyclic $A_{\infty}$ category $(B, \langle \cdot \rangle^B)$. Moreover:
Assume that $\mathcal{A}$ has a twisted shift functor and its cyclic structure is shift-equivariant, and that $(P,G)$ is shift-invariant. Then $\mathcal{B}$ has a twisted shift functor induced from $\mathcal{A}$ and its cyclic structure is shift-equivariant.

When $(P,G)$ is a cohomological splitting, then $\mathcal{B}$ is a minimal model of $\mathcal{A}$.

**Observation** Assume that the cyclic structure on $\mathcal{A}$ is homologically nondegenerate and let $(P,G)$ be a cyclic cohomological splitting of $\mathcal{A}$. Then the cyclic structure induced on $\mathcal{B}$ is strictly nondegenerate. Indeed, we have $\langle \rangle = \langle \rangle \circ (i \otimes i)$, which gives $\langle \rangle = \langle \rangle \circ (i_* \otimes i_*)$, where $i_* : B \to H_{r_1}(\mathcal{H})$ is the map induced by $i$ on cohomology and $\langle \rangle$ is the pairing induced by $\langle \rangle$ on $r_1$-cohomology. Since $i_*$ is bijective and $\langle \rangle$ is nondegenerate, we have the desired conclusion.

### 3.4 Interpretation through formal open string field theory

**The formal extended action.** In the nondegenerate cyclic case, the construction given above has a string field theory interpretation, which generalizes a result of [6]. Fixing a strictly unital, nondegenerate $D$-cyclic $A_\infty$ category $(\mathcal{A}, \langle \rangle)$, let $R = \oplus_{a \in \text{Ob} \mathcal{A}} \mathbb{C} \epsilon_a$ and consider the graded $R$-bimodule $\mathcal{H} := \mathcal{H}_A := \oplus_{a,b \in \text{Ob} \mathcal{A}} \text{Hom}_A(a,b)$ of Section 2.2, together with its $A_\infty$ products $r_n$.

Fixing a unital Grassmann $\mathbb{C}$-algebra $G$, consider the $G$-supermodule $\mathcal{H}_e := \mathcal{H} \otimes G$, and notice that it carries a graded $R$-bimodule structure induced in the obvious manner from that of $\mathcal{H}$. Consider the natural extensions of the pairing and $A_\infty$ products of $\mathcal{H}$ to maps $\langle \rangle : \mathcal{H}_e \otimes \mathcal{H}_e \to G$ and $r_n^e : \mathcal{H}_e^{\otimes n} \to G$:

$$\langle x \otimes \alpha, y \otimes \beta \rangle_e = (-1)^{\deg \alpha |y|} \langle x, y \rangle_{\alpha \beta}$$

and:

$$r_n^e((x_1 \otimes \alpha_1) \ldots (x_n \otimes \alpha)n) = (-1)^{\sum_{i<j} \deg \alpha_i \otimes \alpha_j} r_n(x_1 \ldots x_n)_{\alpha_1 \ldots \alpha_n} .$$

Following [7, 3, 5], we define a (Grassmann-valued) formal action $S_e : \mathcal{H}_e^{\text{odd}} = (\Pi \mathcal{H}_e)^{\text{even}} \to G$ by the formal sum:

$$S_e(\varphi) := \sum_{n \geq 1} \frac{1}{n + 1} \langle \varphi \otimes r_n^e(\varphi^{\otimes n}) \rangle_e , \quad (3.26)$$

where $\varphi \in \mathcal{H}_e^{\text{odd}}$ is the dynamical variable. The term $\frac{1}{2} \langle \varphi \otimes r_n^e(\varphi) \rangle_e$ in (3.26) plays the role of kinetic term. To make sense of the sum in (3.26), one can introduce a topology on $\mathcal{H}_e$ or simply restrict to the subspace $\mathcal{H}_e,\text{tors} := \{ \varphi \in \mathcal{H}_e^{\text{odd}} | \exists N_\varphi \in \mathbb{Z}_+ : \varphi^{\otimes n} = 0 \ \forall n \geq N_\varphi \}$. The extremum conditions of (3.26) amount to the equations:

$$\sum_{n \geq 1} r_n^e(\varphi^{\otimes n}) = 0 \quad (3.27)$$
which also read:

$$\sum_{n \geq 1} r^{e}_{\alpha_1 \ldots \alpha_{n-1} \beta}(\varphi_{\alpha_1} \otimes \ldots \otimes \varphi_{\alpha_{n-1}}) = 0 \quad \forall a, b \in \text{Ob}A ,$$  \hspace{1cm} (3.28)$$

with implicit summation over $a_1 \ldots a_{n-1}$. The formal sums above make sense at least for $\varphi \in \mathcal{H}_{\text{odd}}^{e, \text{tors}}$. In particular, taking $\varphi$ in (3.28) to have the form $\varphi = \phi \otimes 1_G$ with $\phi \in \mathcal{H}^1$ and $1_G$ the unit of $G$ gives the equations:

$$\sum_{n \geq 1} r^{e}_{\alpha_1 \ldots \alpha_{n-1} \beta}(\phi_{\alpha_1} \otimes \ldots \otimes \phi_{\alpha_{n-1}}) = 0 \quad \forall a, b \in \text{Ob}A ,$$  \hspace{1cm} (3.29)$$

which make sense at least when $\phi$ belongs to the subspace $\mathcal{H}_{\text{odd}}^{1, \text{tors}} := \{ \phi \in \mathcal{H}^1 \mid \exists N_{\phi} \in \mathbb{Z}_+ : \phi \otimes n = 0 \forall n \geq N_{\phi} \}$. It follows that any solution of (3.29) induces a solution of (3.28).

**The tree-level potential induced by a cohomological splitting.** When $r_1 \neq 0$, the action (3.26) has to be gauge-fixed. Any consistent gauge-fixing procedure determines a low energy, $G$-valued, formal potential via the semiclassical (WKB) approximation. A particular class of gauges is provided by strictly unital cyclic cohomological splittings $(P, G)$ of $(A, \langle \rangle)$. Defining $P_e := P \otimes \text{id}_G$ and $G_e := G \otimes \text{id}_G$, we can consider the gauge condition:

$$(\text{id}_{\mathcal{H}_e} - P_e)(\varphi) = 0 \iff \varphi \in B_{e}^{\text{odd}} ,$$  \hspace{1cm} (3.30)$$

where $B_e := \text{im} P_e$. Working out the Feynman rules as in [6, 27], one finds that $G$ plays the role of propagator in the gauge (3.30). The tree-level Feynman diagrams are given by graphs $\theta \in \sqcup_{n \geq 2} \Theta_{n+1}$ (in the notation of Section 3.3); see the first diagram in figure 2 for an example. The higher terms in (3.26) give vertices in the perturbative expansion. One finds that the tree-level potential defined in the gauge (3.30) takes the following form up to an uninteresting prefactor (this generalizes a result of [6]):

$$W_e(\varphi) := \sum_{n \geq 2} \frac{1}{n + 1} \langle \varphi \otimes \rho_n(\varphi^{\otimes n}) \rangle^B_e \quad \text{for} \quad \varphi \in B_{e}^{\text{odd}} .$$  \hspace{1cm} (3.31)$$

Here $\rho_n$ are the unital minimal $A_\infty$ products of Section 3.3 (and $\rho_n^e$ their extensions to $B_e$), while $\langle \rangle^B_e$ is the restriction of the pairing $\langle \rangle_e$ to the subspace $B$. The former pairing can be viewed as the extension of $\langle \rangle^B$ to $B_e$. By the results of the previous subsection, the products $\rho_n$ are cyclic with respect to $\langle \rangle^B_e$, which in turn is nondegenerate. In categorical language, $(B, \rho, \langle \rangle^B_e)$ corresponds to a cyclic, minimal, strictly

\footnote{One can in fact study the gauge-fixing procedure in the BV formalism, as done for the dG case in [28, 29].}
unital $A_\infty$ category $(\mathcal{B}, \langle \rangle^\mathcal{B})$ having the same objects as $\mathcal{A}$ and whose cyclic structure is nondegenerate; as explained in the previous subsection, $\mathcal{B}$ is a minimal model of $\mathcal{A}$. Since $\mathcal{B}$ is isomorphic with $H_{r_1}(\mathcal{H})$, we can identify $\mathcal{B}$ with $H(\mathcal{A})$ and view the minimal $A_\infty$ structure determined by $(P,G)$ as a cyclic $A_\infty$ prolongation of the cyclic graded associative category $(H(\mathcal{A}), \langle \rangle^\mathcal{H})$. By the results of the previous subsection, $\mathcal{B}$ has a twisted shift functor and its pairings are shift-equivariant provided that $\mathcal{A}$ has the same properties.

**Topological string field theory interpretation.** Using the modular functor approach initiated in [2, 1], it was proved in [4] that any strictly unital, minimal, non-degenerate cyclic $A_\infty$ category defines an oriented open string theory (=oriented open topological conformal field theory); this provides a converse to the work of [3]. Applying this to our case, we find that $(\mathcal{B}, \langle \rangle^\mathcal{B})$ describes an open topological string theory, allowing us to view (3.26) as a formal string field theory description of the latter.

With this interpretation, the objects of $\mathcal{A}$ (which are the same as the objects of $\mathcal{B}$) are topological D-branes. The space $H_{r_{ab}}(\text{Hom}_\mathcal{A}(a,b)) \approx \text{Hom}_\mathcal{B}(a,b)$ becomes the spaces of topological boundary observables for the open string stretching from $a$ to $b$. The minimal $A_\infty$ compositions $\rho_{a_0...a_n}$ are the string products (associated with the integrated $n+1$-point functions on the disk), while the non-minimal $A_\infty$ compositions of $\mathcal{A}$ are string field products. The latter correspond to geometric string field vertices constructed as in [30].

Solutions of (3.28) describe classical vacua of the formal action (3.26). As in [13, 14], a Grassmann-even solution $\varphi$ can be viewed as the result of ‘condensing target space fields’ in the finite D-brane system described by the elements of the set $\{a \in \text{Ob}\mathcal{A} | \exists b \in \text{Ob}\mathcal{A} : \varphi_{ab} \neq 0 \text{ or } \exists b \in \text{Ob}\mathcal{A} : \varphi_{ba} \neq 0\}$, where the nonzero components $\varphi_{ab}$ are associated with those ‘target space fields’ which acquire a VEV. Notice that in topological string theory there is no way to determine which elements $\varphi_{ab}$ correspond to massless, massive or tachyonic ‘target space’ excitations. To do this one needs a stability condition, which is not visible at the level of topological string theory.

**The generation property.** Considering an $A_\infty$ subcategory $\mathcal{A}_0 \subset \mathcal{A}$ endowed with the restricted D-cyclic structure, let us assume that $\mathcal{A} = \text{tw}(\mathcal{A}_0)$. In this case, it turns out that any object of $\mathcal{A}$ (=twisted complex over $\mathcal{A}_0$) can be viewed as the result of a condensation process taking place between appropriately shifted copies of D-branes belonging to $\mathcal{A}_0$.

Indeed, let $q \in \text{Ob}[\text{tw}(\mathcal{A}_0)]$ be given by morphisms $q_{ij} \in \text{Hom}_{\mathcal{Z}\mathcal{A}_0}(a_i[[n_i]], a_j[[n_j]])$, where $i,j \in I := \{1...l_q\}$, $a_i \in \text{Ob}\mathcal{A}_0$ and $q_{ij} = 0$ unless $i < j$. Consider the sets $S := \{a_i[[n_i]] | 1 \leq i \leq l_q\} \subset \text{Ob}\mathcal{Z}\mathcal{A}_0$ and $I_\alpha := \{i \in I | a_i[[n_i]] = \alpha\}$ for each $\alpha \in S$. 43
For every $\alpha \in S$, we set $s_\alpha := \text{Card}(I_\alpha)$ and define $A_\alpha := \alpha \oplus s_\alpha \in \text{Ob}\Sigma A_0$. For every $\alpha, \beta \in S$, let $\phi_{A_\alpha, A_\beta} := \oplus_{i \in I_\alpha, j \in I_\beta} q_{ij} \in \text{Hom}^1_{\Sigma A_0}(A_\alpha, A_\beta)$, where we view $A_\alpha$ as degenerate twisted complexes via the canonical embedding of $\Sigma A_0$ into $\text{tw}(A_0)$ (i.e. $A_\alpha$ are viewed as twisted complexes with zero maps). Then $\phi := \oplus_{\alpha, \beta \in S} \phi_{A_\alpha, A_\beta}$ is an element of $H^1_{\text{tors}}$ and it is easy to check that equations (3.29) for $\phi$ amount to the generalized Maurer-Cartan equations (2.11) for $q$. Thus $q$ can be viewed as the result of condensing $\phi_{A_\alpha, A_\beta} \otimes 1_G$, which arise from strings stretching between $A_\alpha$ and $A_\beta$. Since $\text{Hom}^1_{\Sigma A_0}(A_\alpha, A_\beta) = \text{Hom}^1_{\text{tw}(A_0)}(A_\alpha, A_\beta)$, this can also be viewed as a condensation process taking place between the D-branes of $S \subset \mathbb{Z}A_0$. Summarizing this discussion, we obtain:

Let $A_0$ be a strictly unital $A_\infty$ category endowed with a nondegenerate $D$-cyclic structure and assume that $A = \text{tw}(A_0)$, endowed with the nondegenerate $D$-cyclic structure induced from $A_0$. Then every twisted complex $q \in A$ is the result of a condensation process involving open strings stretching between a finite number of D-branes belonging to $\mathbb{Z}A_0$. Hence the category $\mathbb{Z}A_0$ generates $A$ via condensation processes.

**Observation** For the case of 3-cyclic dG categories, the arguments of this section are due to [13, 14] (see also [31]). Since any $A_\infty$ category admits a quasi-equivalent dG model[9, 10], the treatment given in [13, 14] is essentially equivalent with the more general discussion above.

### 3.5 The case of $A_\infty$ algebras

Recall that a strictly unital $A_\infty$ category $A$ with a single object $a$ identifies with a strictly unital $A_\infty$ algebra $A$. A $D$-cyclic pairing on $A$ amounts to a bilinear and graded-symmetric form $\langle \cdot, \cdot \rangle : A \times A \to \mathbb{C}$ of degree $-D$ satisfying relations (3.12). A cyclic structure $\langle \cdot \rangle$ on $A$ induces a cyclic pairing on $H(A)$, as well as a cyclic pairing on $\text{tw}(A)$ and therefore on $\text{tria}(A)$. The latter is nondegenerate iff the pairing on $A$ is homologically nondegenerate. All constructions described above apply with trivial simplifications.

When $(A, \langle \cdot, \cdot \rangle)$ is homologically nondegenerate (and thus compact), one can show by direct computation that the cyclic minimal categories induced by cohomological splittings of $A$ give explicit representatives of the isomorphism class of cyclic minimal models considered from a different perspective in Appendix B.

---

18This amounts to checking that the restricted pairing $\langle \cdot \rangle^B$ can be obtained from $\langle \cdot \rangle$ by pullback through the $A_\infty$ quasi-isomorphism induced by $i$. We omit the proof since we have no need for it in the present paper.
4. Cyclic differential graded algebras and their minimal models

A differential graded algebra $A$ corresponds to an $A_\infty$ algebra having $\mu_n = 0$ for all $n \geq 3$. Setting $d := \mu_1$ and $xy := \mu_2(x, y)$ for all $x, y \in A$, the $A_\infty$ constraints show that $\mu_2$ is an associative composition and $d$ a degree one derivation which squares to zero. We will assume that $A$ is strictly unital, which amounts to the existence of a unit 1 for the associative multiplication such that $d(1) = 0$. In this section, we give an equivalent description of cyclic pairings on $A$ and briefly recall the basics of homological algebra over $A$, following [16, 21]. We then show that $A$ admits a nondegenerate $D$-cyclic structure iff $\text{tria}(A)$ is $D$-Calabi-Yau, and prove a similar result for a minimal model of $A$.

4.1 dG modules and bimodules over a dGA

Recall that a unital right dG module over $A$ is a unital $\mathbb{Z}$-graded right module $M$ over the unital graded associative algebra underlying $A$, together with a differential $d_M : M \to M$ of degree +1 which satisfies the compatibility conditions:

$$d_M(mx) = (d_Mm)x + (-1)^{|m|} m \ dx$$

for homogeneous elements $x \in A$ and $m \in M$. A unital left dG module is a (unital) left $\mathbb{Z}$-graded right module $M$ over the unital graded associative algebra $A$, together with a differential $d_M : M \to M$ of degree +1 which satisfies:

$$d_M(xm) = (dx)m + (-1)^{|x|} x \ d_Mm .$$

Of course, a unital right dG module over $A$ is the same as a unital left dG module over the opposite dG algebra $A^{\op}$, which is defined on the underlying set of $A$ by the differential and multiplication:

$$d^{\op}(x) := d(x) , \quad x \cdot^{\op} y := (-1)^{|x||y|}yx .$$

A unital dG bimodule is a unital $\mathbb{Z}$-graded bimodule $M$ over the unital graded associative algebra underlying $A$, together with a differential $d_M : M \to M$ of degree +1 which satisfies the compatibility conditions:

$$d_M(xm) = (dx)m + (-1)^{|x|} x \ d_Mm \quad , \quad d_M(mx) = (d_Mm)x + (-1)^{|m|} m \ dx .$$

This is the same as a right dG module over the unital dG algebra $A^{\op} \otimes A$, whose differential and multiplication are defined through:

$$d^{A^{\op} \otimes A}(x \otimes y) := (d^{\op}x) \otimes y + (-1)^{|x|} x \otimes dy \quad , \quad (x_1 \otimes y_1) \cdot^{A^{\op} \otimes A}(x_2 \otimes y_2) := (-1)^{|y_1|} x_1x_2 \otimes y_1y_2 ;$$
the outer multiplications of $M$ are recovered as $xmy = (-1)^{|x||m|}m(x \otimes y)$.

We let $\text{dGMod}_A, \text{dGMod}$ and $\text{dGMod}_A$ denote the $\text{dG}$ categories of unital right, left and bi-$\text{dG}$ modules over $A$; their morphisms are those morphisms in $\text{Gr}$ which are compatible with the module structures (though not necessarily with the differentials); the differentials on morphisms are defined as in (4).

**Dualization.** One has dualization functors $\text{dGMod}_A \leftrightarrow_A \text{dGMod}$ and $\text{dGMod}_A \rightarrow_A \text{dGMod}_A$ defined as follows. Given a unital right $\text{dG}$ module $M$ over $A$, consider the dual complex $(M^\vee, d_M^\vee)$, endowed with the outer left multiplication $(x\eta)(m) := (-1)^{|x||\eta|+|m|}\eta(mx)$. This makes $M^\vee$ into a unital left $\text{dG}$-module over $A$. For a unital left $\text{dG}$ module $M$, endow $M^\vee$ with the outer right multiplication $(\eta x)(m) := \eta(mx)$ and with the same differential as above; this makes it into a unital right $\text{dG}$-module over $A$. Given an $A$-bimodule $M$, endow $M^\vee$ with the outer multiplications $(x\eta y)(m) := (-1)^{|x||\eta|+|y||m|}\eta(ymx)$ for all $x, y \in A$ and $m \in M$ and with the differential $d_{M^\vee}$; this make it into a unital $\text{dG}$-bimodule over $A$. The functors $^\vee$ act on morphisms as in equation (5) of the introduction. They square to the identity on the corresponding subcategories of finite-dimensional $\text{dG}$ modules.

**Tensor product.** Given a unital right $\text{dG}$ module $M$ and a unital left $\text{dG}$ module $N$, the usual tensor product as modules $M \otimes_A N$ becomes a complex when endowed with the differential $d_{M \otimes_A N}(m \otimes_A n) = (d_M m) \otimes_A n + (-1)^{|m|}m \otimes_A (d_N n)$. When $M$ (respectively $N$) is a unital $\text{dG}$ $A$-bimodule, this complex is a unital left (resp. right) $A$-module when endowed with the outer multiplication induced from the left outer multiplication of $M$ (resp. the right outer multiplication of $N$). It is a unital $\text{dG}$ $A$-bimodule when both $M$ and $N$ are $\text{dG}$ bimodules over $A$.

**Center of a $\text{dG}$ bimodule.** For any $\text{dG}$ bimodule $M$ over $A$, we let $M^A$ denote its center as a graded $A$-bimodule, i.e. the linear subspace of all elements of $M$ which graded-commute with all elements of $A$. Notice that $M^A$ is a subcomplex of $M$, as well as a graded central bimodule over the graded associative algebra underlying $A$ (together with the induced differential, $M^A$ is a ‘central $\text{dG}$ bimodule’ over $A$).

### 4.2 Cyclic structures on a $\text{dGA}$

Given a pairing $\langle \cdot, \cdot \rangle : A \otimes A \rightarrow \mathbb{C}[-D]$, the cyclicity conditions of Section 3 reduce to:

$$\langle x, y \rangle = (-1)^{|x||y|}\langle y, x \rangle, \quad \langle dx, y \rangle + (-1)^{|x|}\langle x, dy \rangle = 0, \quad \langle x, yz \rangle = \langle x, yz \rangle \ \forall x, y, z \in A, \quad \text{(4.1)}$$

where we identified the pairing with the corresponding bilinear form. Thus a cyclic structure on $A$ is the same as a homogeneous ‘invariant bilinear form’, where invariance
is understood as compatibility with both the differential and multiplication. Since $A$ is unital, we can also describe this through the linear map $\text{tr} : A \to \mathbb{C}$ defined through $\text{tr}(x) := \langle 1, x \rangle = \langle x, 1 \rangle$. The last condition in (4.1) reduces to $\langle x, y \rangle = \text{tr}(xy)$, while the remaining constraints state that $\text{tr}$ is an invariant trace:

$$\text{tr}(xy) = (-1)^{|x||y|}\text{tr}(yx) \quad \text{and} \quad \text{tr}(dx) = 0 . \quad (4.2)$$

The cohomology $H(A) := H_d(A)$ is a unital graded associative algebra with respect to the multiplication induced from $A$. If $\text{tr}^H : H(A) \to \mathbb{C}$ and $\langle \cdot, \cdot \rangle^H : H(A) \times H(A) \to \mathbb{C}$ denote the maps induced on cohomology, then $\text{tr}^H$ is a (possibly degenerate) invariant trace on the graded associative algebra $H(A)$, and we have $\langle u, v \rangle^H = \text{tr}^H(uv)$.

Viewing $A$ as a dG-bimodule over itself, consider its dual dG bimodule $A^\vee$. Then equations (4.2) state that $\text{tr}$ is a $d_A^\vee$-closed central element of this bimodule. Generalizing this, we define:

**Definition** Let $M$ be a dG bimodule over $A$. A $D$-trace on $M$ is an element $\text{tr} \in Z^{-D}((M^\vee)^A)$. Explicitly, this is a degree zero linear map $\text{tr} : A \to \mathbb{C}[-D]$ which obeys:

1. $\text{tr}(dm) = 0 \quad \forall m \in M$
2. $\text{tr}(xm) = (-1)^{|x||m|}\text{tr}(mx) \quad \forall$ homogeneous $x \in A$ and $m \in M$.

With this definition, we can describe cyclic structures on $A$ as follows.

**Proposition** Giving a $D$-cyclic structure on $A$ amounts to giving a trace $\text{tr} \in Z^{-D}((A^\vee)^A)$.

The pairing $\langle \cdot, \cdot \rangle$ induces a morphism of graded vector spaces $\Phi : A[D] \to A^\vee$ via the relation:

$$\Phi(x)(y) := \langle x, y \rangle = \text{tr}(xy) . \quad (4.3)$$

Notice that the trace can be recovered as $\text{tr} := \Phi(1)$, where $1$ is the unit of $A$. Equations (4.1) amount to the condition that $\Phi$ is a morphism of dG $A$-bimodules from $A[D]$ to $A^\vee$. We have $\text{tr}^H = \Phi_*([1])$, where $\Phi_* : H(A) \to H(A^\vee)$ is the map induced by $\Phi$ on cohomology. It is clear that $\langle \cdot, \cdot \rangle$ is nondegenerate in the sense of Section 3 iff $A$ is degreewise finite and $\Phi$ is bijective. It is homologically nondegenerate iff $A$ is compact and $\Phi$ is a quasi-isomorphism. Thus:

**Proposition** Giving a $D$-cyclic structure on a differential graded algebra $A$ amounts to giving a morphism of dG bimodules $\Phi : A[D] \to A^\vee$. Moreover, the cyclic structure is nondegenerate iff $\Phi$ is an isomorphism, and homologically nondegenerate iff $\Phi$ is a quasi-isomorphism.
Observation  The dG bimodule $A^v$ carries a canonical degree zero trace $\theta_A : A^v \to \mathbb{C}$

given by evaluation at the unit of $A$, i.e. $\theta_A(\eta) = \eta(1)$ for all $\eta \in A^v$. Given a dG-
bimodule morphism $\Phi : A[D] \to A^v$, the trace $\text{tr}_\Phi = \Phi(1)$ it induces on $A$ can be
expressed as $\text{tr}_\Phi = \theta \circ \Phi$.

4.3 Homological algebra over a dGA

In this subsection, we recall a few basic results about the derived category of a dGA, which
will be used later to give a homological characterization of cyclic differential
graded algebras. Let $A$ be a unital dGA. Since $A$ is a particular type of $A_\infty$ algebra,
its homological algebra can be treated as in the $A_\infty$ case; this amounts to considering
$A_\infty$ modules over $A$, $A_\infty$ morphisms of $A_\infty$ modules etc. Because this is rather
complicated, it is advantageous to follow the direct approach of [16], which works in-
stead with dG modules over $A$. The price one pays in the direct approach is that a
quasi-isomorphism of dG modules need not be a homotopy equivalence of dG modules,
hence the dG derived category of $A$ does not coincide with the homotopy category of
dG modules over $A$.

The dG derived category.  Recall that $\text{dGMod}_A$ denotes the dG category of unital
right dG modules over $A$. We let $\mathcal{C}_{dG}(A) := Z^0(\text{dGMod}_A)$ be the Abelian category of
right dG modules, and $\mathcal{H}_{dG}(A) = H^0(\text{dGMod}_A)$ be the homotopy category taken in
the dG sense; the latter is a triangulated category. The dG derived category $D_{dG}(A)$
of $A$ is the triangulated category obtained by localizing $\mathcal{H}_{dG}(A)$ with respect to quasi-

isomorphisms of dG-modules [16]. Since unital dG modules over $A$ are particular
instances of strictly unital $A_\infty$ modules, one has a faithful non-full functor $\text{dGMod}_A \to
\text{Mod}_A$. It was shown in [9] that this functor induces an equivalence between $D_{dG}(A)$ and
the $A_\infty$ derived category $D(A)$ of $A$. Hence we can view $D_{dG}(A)$ as an equivalent model
of $D(A)$, and throughout this paper we shall identify it with $D(A)$ via the equivalence
above. Due to this identification, we denote $D_{dG}(A)$ simply by $D(A)$.

(P)-resolutions and (I)-resolutions.  The category $\mathcal{C}_{dG}(A)$ admits two Quillen
model structures whose weak equivalences are the quasi-isomorphisms. In the ‘pro-
jective’ model structure, a dG module $M$ is cofibrant iff it has property (P) [16], which
amounts to $\mathcal{H}_{dG}(M, G) = 0$ for all acyclic dG modules $G$. In the ‘injective’ model
structure, $M$ is cofibrant iff it has property (I) [16], which amounts to $\mathcal{H}_{dG}(G, M) = 0$
for all acyclic $G$. An explicit description of the cofibrant objects of these two model
structures can be found in [16]. The full subcategories $\mathcal{H}_p(A)$ (resp. $\mathcal{H}_i(A)$) of $\mathcal{H}_{dG}(A)$

\footnote{This should not be confused with the homotopy category taken in the $A_\infty$ sense. The latter is
obtained by working with $A_\infty$ homotopy classes of $A_\infty$ morphisms of dG modules.}
formed by all dG modules having property (P) (resp. (I)) are triangulated subcategories of $\mathcal{H}_{dG}$. For each right dG-module $M$, there exist triangles in $\mathcal{H}_{dG}(A)$ (unique up to isomorphism):

$$pM \to M \to aM \to (pM)[1], \quad a'M \to M \to iM \to (a'M)[1]$$

where $aM, a'M$ are acyclic, $pM$ has property (P) and $iM$ has property (I). Then $pM \to M$ is called a (P)-resolution of $M$, while $M \to iM$ is called an (I)-resolution. The exact functors $p : \mathcal{H}_{dG}(A) \to \mathcal{H}_p(A)$ and $i : \mathcal{H}_{dG}(A) \to \mathcal{H}_i(A)$ commute with arbitrary coproducts and are right respectively left adjoint to the inclusion $\mathcal{H}_p(A) \subset \mathcal{H}_{dG}(A)$. The localization functor $\mathcal{H}_{dG}(A) \to D(A)$ restricts to exact equivalences $\mathcal{H}_p(A) \xrightarrow{\sim} D(A)$ and $\mathcal{H}_i(A) \xrightarrow{\sim} D(A)$, with inverses given by the functors induced by $p$ and $i$, which we denote by the same letters. Thus $\text{Hom}_{D(A)}(M,N) = \text{Hom}_{\mathcal{H}_{dG}(A)}(pM,N) = \text{Hom}_{\mathcal{H}_{dG}(A)}(M,iN)$ for all right dG modules $M,N$. This allows one to construct a homological calculus much as one does for modules over a unital associative algebra (see [16] for details). Each of the categories $C_{dG}(A), \mathcal{H}_{dG}(A), D(A), \mathcal{H}_p(A)$ and $\mathcal{H}_i(A)$ has infinite coproducts.

**Description of tria($A$) and per($A$) through dG modules.** Let $\hat{A}$ be $A$ viewed as a (unital) right dG module over itself. As in the $A_\infty$ case define $\text{tria}(A) := \text{tria}_{D(A)}(\hat{A})$ and $\text{per}(A) := k\text{tria}_{D(A)}(\hat{A})$ (see Appendix 1.4) to be the smallest triangulated (resp triangulated and idempotent complete) strictly full subcategories of $D(A)$ containing $\hat{A}$; these can be identified with the categories denoted by the same symbols but defined in the $A_\infty$ sense. Thus $\hat{A}$ is a compact generator of $D(A)$, in particular $D(A) = \text{Tria}_{D(A)}(\hat{A})$ etc. When $A$ is concentrated in degree zero, $\text{per}(A)$ is the category of perfect complexes (= complexes quasi-isomorphic with bounded complexes of finitely generated projective modules), while $\text{tria}(A)$ consists of complexes quasi-isomorphic with bounded complexes of finitely-generated free modules.

### 4.4 Serre duality on tria($A$) and per($A$)

Fixing a unital differential graded algebra $A$, we let $\nu := (,) \otimes_A^L A^\vee : D(A) \to D(A)$ be the left derived functor [16] of tensorization from the right with the $A$-bimodule $A^\vee$. This is defined by $\nu(M) = M \otimes_A^L A^\vee := \pi[(pM) \otimes_A A^\vee]$, where $\pi$ is the canonical surjection $\mathcal{H}_{dG}(A) \to D(A)$. Notice that $\nu$ commutes with arbitrary coproducts.

In this section, we often write $\hat{A}$ simply as $A$ in order to simplify notation. Consider the linear maps $\beta_n := \nu_{A,\hat{A}[n]} : \text{Hom}_{D(A)}(A,A[n]) \to \text{Hom}_{D(A)}(\nu(A),\nu(A[n]))$ defined by the functor $\nu$. Recall that $A$ is called compact if $H^n(A)$ is finite-dimensional for all $n \in \mathbb{Z}$. 

49
Proposition A is compact iff each of the maps $\beta_n$ is bijective.

Proof. For any $M$ in dGMod$_A$, we have a natural isomorphism of complexes $\text{Hom}_{\text{dGMod}_A}(A, M) \cong M$ (given by evaluation at 1) and a natural isomorphism of complexes $\text{Hom}_{\text{dGMod}_A}(A, A^v) \cong M^v$ given by $\phi \to \eta_\phi$, where $\eta_\phi$ is the functional on $M$ given by $\eta_\phi(m) = \phi(m)(1)$. For any acyclic right dG module $G$ over $A$, these isomorphisms show that $\text{Hom}_{\text{dGMod}_A}(A, G)$ and $\text{Hom}_{\text{dGMod}_A}(G, A^v)$ are acyclic. Thus $\text{Hom}_{H_{\text{dg}}(A)}(A, G) = \text{Hom}_{H_{\text{dg}}(A)}(G, A^v) = 0$, which show that $A$ has property (P) and $A^v$ has property (I). Of course, the same is true for $A[n]$ and $A^v[n]$ given any integer $n$.

Since $A$ has property (P), we have $pA = A$ and $\nu(A[n]) = A[n] \otimes_A A^v \cong A^v[n]$ for all $n \in \mathbb{Z}$. Hence $\beta_n$ can be viewed as linear maps $\beta_n : \text{Hom}_{D(A)}(A, A[n]) \to \text{Hom}_{D(A)}(A^v, A^v[n])$. Moreover, we have:

$$\text{Hom}_{D(A)}(A, A[n]) = \text{Hom}_{H_{\text{dg}}(A)}(A, A[n]) = H^0(\text{Hom}_{\text{dGMod}_A}(A, A[n])) \cong H^0(A[n]) = H^n(A)$$

and:

$$\text{Hom}_{D(A)}(A^v, A^v[n]) = \text{Hom}_{H_{\text{dg}}(A)}(A^v, A^v[n]) = H^0(\text{Hom}_{\text{dGMod}_A}(A^v, A^v[n])) = H^0(\text{Hom}_{\text{dGMod}_A}(A^v[-n], A^v)) \cong H^0(A[n]^v) \cong H^n(A)^{v^v},$$

where the first equalities in each chain follow from $pA = A$ and $i(A^v[n]) = A^v[n]$.

Combining everything, we see that $\beta_n$ identify with the linear maps $\gamma_n^H : H^n(A) \to H^n(A)^{v^v} \cong [H^n(A)]^{v^v}$ induced by the dG bimodule morphism $\gamma : A \to A^v$, $\gamma(a)(\eta) = (-1)^{|a||\eta|}\eta(a)$. Since $\gamma_n^H$ coincides up to sign with the bidualization morphism of the vector space $H^n(A)$, we know that it is bijective iff $H^n(A)$ is finite-dimensional for all $n \in \mathbb{Z}$. It follows that all $\beta_n$ are bijective iff $A$ is compact.

Observation The following are equivalent:

(a) $A$ is compact
(b) $\text{tria}(A)$ is $\text{Hom}$-finite
(c) $\text{per}(A)$ is $\text{Hom}$-finite

Proof. The equivalence $(b) \Leftrightarrow (c)$ follows trivially from $\text{per}(A) = \text{tria}(A)^\pi$. The equivalence $(a) \Leftrightarrow (b)$ follows from $\text{tria}(A) = H^0(\text{tw}(A))$ upon using the equivalence $A = \text{compact} \Leftrightarrow \text{tw}(A) = \text{compact}$. One can also prove the equivalence $(a) \Leftrightarrow (b)$ directly by using $\text{Hom}_{D(A)} = H^{n-m}(A)$ and the fact that $\text{Hom}(...)$ is a cohomological functor in the first variable and a homological functor in the second.
Lemma The following conditions are equivalent:
(a) $\nu : D(A) \to D(A)$ is fully faithful
(b) $A$ is compact and $A^v$ belongs to $\text{per}(A)$ (in particular, $\nu$ preserves $\text{per}(A)$).
In this case, $\nu$ restricts to an autoequivalence of $\text{per}(A)$ iff $A^v$ is a Karoubian generator of $\text{per}(A)$.

Proof. Recall from [16] that an object of $D(A)$ is compact iff it belongs to $\text{per}(A)$. Using this fact, a result of [16, paragraph 4.2] states that an exact functor $F : D(A) \to D(A)$ which commutes with arbitrary coproducts is fully faithful iff:
(α) $F(\hat{A})$ belongs to $\text{per}(A)$
(β) the maps $F_{A,A[n]} : \text{Hom}_{D(A)}(\hat{A}, \hat{A}[n]) \to \text{Hom}_{D(A)}(F(\hat{A}), F(\hat{A}[n]))$ are bijective for all $n \in \mathbb{Z}$.
Applying this to $F = \nu$, the previous Proposition shows that condition (β) is equivalent with compactness of $A$, while condition (α) amounts to the requirement that $A^v = \nu(\hat{A})$ belongs to $\text{per}(A)$. In this case, $\nu$ preserves $\text{per}(A)$ since $\hat{A}$ is a Karoubian generator of the latter. The last statement of the lemma is obvious.

Recall that a Serre functor on a Hom-finite triangulated category $\mathcal{T}$ is an exact autoequivalence $S$ of $\mathcal{T}$ together with isomorphisms $\text{Hom}_\mathcal{T}(a,b) \approx \text{Hom}_\mathcal{T}(b,S(a))$. In this case, $S$ is unique up to isomorphism of functors [33].

Proposition The category $\text{per}(A)$ is Hom-finite and has a Serre functor $S$ iff the following conditions are satisfied:
(1) $A$ is compact
(2) $A^v$ belongs to $\text{per}(A)$ and is a Karoubian generator of the latter.
In this case, we have $S \approx \nu|_{\text{per}(A)} = (.) \otimes_A A^v$.

Proof. ($\Leftarrow$) Assume that (1) and (2) hold. Since a Serre functor is unique up to isomorphism, it suffices to show that $\nu|_{\text{per}(A)}$ is a Serre functor on $\text{per}(A)$. By the Lemma, assumptions (1) and (2) imply that $\nu$ restricts to an autoequivalence of $\text{per}(A)$. Given $P$ in $\text{per}(A)$ and $M$ in $D(A)$, we have natural isomorphisms:
$$R\text{Hom}_A(P,M)^v \approx R\text{Hom}_A(M,R\text{Hom}_A(P,A)^v) , \quad R\text{Hom}_A(P,A)^v \approx P \otimes_A^L A^v ,$$
which it suffices to check for $P = A[n]$, when they hold trivially. Combining these gives:
$$R\text{Hom}_A(P,M)^v \approx R\text{Hom}_A(M,P \otimes_A^L A^v) \quad \forall P \in \text{Ob}[\text{per}(A)] , \quad \forall M \in \text{Ob}D(A) .$$
Taking $P = P_1 \in \text{Ob}[\text{per}(A)]$ and $M = P_2 \in \text{Ob}[\text{per}(A)]$ and applying $H^0$ gives natural isomorphisms:
$$\text{Hom}_{D(A)}(P_1, P_2)^v \approx \text{Hom}_{D(A)}(P_2, \nu(P_1)) .$$
Since $A$ is compact, the category $\text{per}(A)$ is Hom-finite. Hence dualizing the last equation gives:

$$\text{Hom}_{D(A)}(P_1, P_2) \approx \text{Hom}_{D(A)}(P_2, \nu(P_1))^\vee,$$

which shows that $\nu|_{\text{per}(A)}$ is a Serre functor.

$(\Rightarrow)$ Assume that $\text{per}(A)$ is Hom-finite with Serre functor $S$. Hom-finiteness of $\text{per}(A)$ implies (1) by the observation above. To prove (2), start by combining (4.4) with the Serre isomorphism:

$$\forall P_1, P_2 \in \text{Ob}[\text{per}(A)]$$

which gives natural isomorphisms:

$$\text{Hom}_{D(A)}(P_1, P_2) \approx \text{Hom}_{D(A)}(P_2, S(P_1))^\vee \quad \forall P_1, P_2 \in \text{Ob}[\text{per}(A)]$$

Applying this for $P_2 := P$ and $P_1 = A$ gives isomorphisms:

$$\text{Hom}_{D(A)}(P, S(A)) \xrightarrow{\phi_P} \text{Hom}_{D(A)}(P, A^\vee) \quad \forall P \in \text{Ob}[\text{per}(A)]$$

which are natural in $P$. Setting $P = S(A)$ in (4.7) gives a morphism $\theta = \phi_{S(A)}(\text{id}_{S(A)}) \in \text{Hom}_{D(A)}(S(A), A^\vee)$. Using this, we define maps

$$\psi_M : \text{Hom}_{D(A)}(M, S(A)) \to \text{Hom}_{D(A)}(M, A^\vee) \quad \forall M \in \text{Ob}D(A)$$

by setting $\psi_M(u) := \theta \circ u$ for all $u \in \text{Hom}_{D(A)}(M, S(A))$; these are clearly natural in $M$. Given $u \in \text{Hom}_{D(A)}(P, S(A))$, we have $\phi_P(u) = \phi_P(\text{id}_{S(A)} \circ u) = \phi_P(\text{id}_{S(A)}) \circ u = \theta \circ u = \psi_P(u)$ by naturality of $\phi_P$. Thus $\psi_P = \phi_P$ for all $P \in \text{Ob}[\text{per}(A)]$.

Consider the full subcategory $\mathcal{T}$ of $D(A)$ whose objects are those $M \in \text{Ob}D(A)$ for which $\psi_M$ is an isomorphism. This is a triangulated subcategory by the 5-lemma. Moreover, it is closed under taking arbitrary coproducts. Indeed, given $M_\alpha \in \text{Ob}\mathcal{T}$, we have

$$\text{Hom}_{D(A)}(\sqcup_\alpha M_\alpha, S(A)) \approx \prod_\alpha \text{Hom}_{D(A)}(M_\alpha, S(A)) \xrightarrow{\prod_\alpha \psi_{M_\alpha}} \prod_\alpha \text{Hom}_{D(A)}(M_\alpha, A^\vee) \approx \text{Hom}_{D(A)}(\sqcup_\alpha M_\alpha, A^\vee)$$

where the first and last isomorphisms follow from the definition of the categorical coproduct and the map in the middle is bijective because $\psi_{M_\alpha}$ are.

Since $D(A) = \text{Tri}A$ and $\mathcal{T}$ contains $A[n]$, it follows that $\mathcal{T} = D(A)$, so $\psi_M$ are bijective for all objects $M$ of $D(A)$. The Yoneda lemma now shows that $\theta$ is an isomorphism, so $A^\vee$ is isomorphic with $S(A)$ in $D(A)$. This implies that $A^\vee = \nu(A)$ belongs to $\text{per}(A)$ because the latter is a strictly full subcategory of $D(A)$. Since $\nu$ is exact and $A^\vee$ a Karoubi generator of $\text{per}(A)$, we find that $\nu$ preserves $\text{per}(A)$. We can now apply the Yoneda lemma to (4.6). This gives an isomorphism of functors $\nu|_{\text{per}(A)} \approx S$, showing that $\nu|_{\text{per}(A)}$ is an autoequivalence of $\text{per}(A)$. Thus $A^\vee$ is a Karoubi generator of $\text{per}(A)$, which completes the proof of (1).
Lemma The following conditions are equivalent:
(a) $\nu$ preserves $\text{tria}(A)$ and restricts to an autoequivalence of $\text{tria}(A)$
(b) $A$ is compact and $A^\nu$ belongs to $\text{tria}(A)$ and is a triangle generator of the latter.

Proof. A result of [16, paragraph 4.2] shows that $\nu|_{\text{tria}(A)} : \text{tria}(A) \to D(A)$ is fully faithful iff $\beta_n$ are bijective for all $n \in \mathbb{Z}$, which amounts to compactness of $A$. On the other hand, $\nu$ preserves $\text{tria}(A)$ iff $\nu(A) = A^\nu$ belongs to $\text{tria}(A)$. The conclusion now follows.

Proposition The category $\text{tria}(A)$ is compact and has a Serre functor $S$ iff the following conditions are satisfied:
(1) $A$ is compact
(2) $A^\nu$ belongs to $\text{tria}(A)$ and is a triangle generator of the latter.
In this case, we have $S \approx (.) \otimes_{\mathbb{A}} A^\nu$.

Proof. Virtually identical to that of the previous proposition.

Corollary The following are equivalent:
(a) The category $\text{per}(A)$ is $D$-Calabi-Yau
(b) The category $\text{tria}(A)$ is $D$-Calabi-Yau
(c) $A$ is compact and admits a homologically nondegenerate $D$-cyclic pairing.

Proof. $(a) \Rightarrow (b)$ Obvious.

$(b) \Rightarrow (c)$ If $\text{tria}(A)$ is $D$-Calabi-Yau, then the previous proposition implies that $A$ is compact and $\nu|_{\text{tria}(A)} \approx [D]$. Applying this to the generator $A$, we find $\nu(A) = A^\nu \approx A[D]$ in $D(A)$. Since $\text{Hom}_{D(A)}(A[D], A^\nu) = H^0(\text{Hom}_{A_{dGMod}}(A[D], A^\nu))$ (because $A[D]$ has property (P)), this means that there exists a quasi-isomorphism from $A[D]$ to $A^\nu$. The results of Section 4.2 imply that $A$ carries a homologically nondegenerate $D$-cyclic structure (which induces a Serre pairing on $\text{tria}(A)$).

$(c) \Rightarrow (a)$ If $A$ is homologically nondegenerate $D$-cyclic and compact, then $A[D]$ is quasi-isomorphic with $A^\nu$ as shown in Section 4.2. Thus $A^\nu \approx A[D]$ in $D(A)$, which implies that $A^\nu$ belongs to $\text{per}(A)$ and is a Karoubian generator of the latter. By a previous proposition, we find that $\text{per}(A)$ has Serre duality with Serre functor $\nu|_{\text{per}(A)} = (. \otimes_{A} A^\nu \approx (.) \otimes_{A} A[D] \approx [D]$, where we used the fact that perfect dG modules have property (P). Thus $\text{per}(A)$ is $D$-Calabi-Yau.

4.5 Reconstruction of Serre pairings

53
The results of Sections 3.2 and 4.4 give:

**Proposition** Let $A$ be a unital differential graded algebra. Then the following statements are equivalent:

(a) $A$ is compact and admits a homologically nondegenerate $D$-cyclic structure
(b) tria$(A)$ is Hom-finite and $D$-Calabi-Yau.

In this case, any homologically nondegenerate $D$-cyclic structure on $A$ induces a Serre duality structure on tria$(A)$ via the construction of Section 3.2.

**Proof.** The implication $(a) \Rightarrow (b)$ follows from the results of Section 3.2, while the inverse implication follows from the previous subsection. The rest follows from Section 3.2.

On the other hand, Section 3.3 (see Appendix B for a different point of view) gives the following:

**Proposition** Let $(A, \langle \rangle)$ be a compact $D$-cyclic and strictly unital dGA whose cyclic pairing is homologically nondegenerate. Then there exists a finite-dimensional unital $A_\infty$ minimal model $A_{\text{min}}$ of $A$ which admits a nondegenerate $D$-cyclic pairing. For example, one can pick the cyclic minimal model induced by a strictly unital and cyclic cohomological splitting of $A$.

Combining with the previous result, we obtain:

**Proposition** Let $A_{\text{min}}$ be a unital and minimal $A_\infty$ algebra such that tria$(A_{\text{min}})$ is Hom-finite and $D$-Calabi-Yau. Then $A_{\text{min}}$ is finite-dimensional and there exists a unital and minimal $A_\infty$ algebra $A'_{\text{min}}$ which is $A_\infty$ isomorphic with $A_{\text{min}}$ and admits a nondegenerate $D$-cyclic structure which induces a Serre duality structure on tria$(A'_{\text{min}}) \approx$ tria$(A_{\text{min}})$ via the construction of Section 3.2.

**Proof.** It follows from [9, Proposition 7.5.0.2] that there exists a strictly unital differential graded algebra $A$ such that $A_{\text{min}}$ is a minimal model of $A$. By the first proposition above, $A$ is compact and admits a homologically nondegenerate $D$-cyclic structure. The second proposition gives a finite-dimensional, unital minimal model $A'_{\text{min}}$ of $A$ which admits a nondegenerate $D$-cyclic structure. This must be $A_\infty$ isomorphic with $A_{\text{min}}$ since the minimal model of $A$ is determined up to $A_\infty$ isomorphism. The extension procedure of Section 3.2 shows that the cyclic structure of $A'_{\text{min}}$ induces a shift-invariant and nondegenerate $D$-cyclic structure on tw$(A_{\text{min}})$ and thus a Serre pairing on tria$(A'_{\text{min}}) \approx$ tria$(A_{\text{min}})$. 

54
Observation As explained in Appendix B, the $D$-cyclic structure of the dG algebra $A$ considered in the proof pulls back to a nondegenerate ‘symmetric $\infty$-inner product’ on $A_{\text{min}}$ via any quasi-isomorphism $A_{\text{min}} \to A$; this is a generally infinite collection of multilinear forms on $A_{\text{min}}$, the first of which is a nondegenerate bilinear pairing. The ‘noncommutative Darboux theorem’ discussed in Appendix (B) implies that one can find a $D$-cyclic minimal model $A'_{\text{min}}$ such that the pull-back of the $\infty$-inner product through an $A_{\infty}$ isomorphism $A'_{\text{min}} \to A_{\text{min}}$ reduces to a $D$-cyclic structure on $A'_{\text{min}}$, i.e. all of its higher multilinear forms vanish. This gives a different proof of the theorem above (see Appendix B).

Recall that any two Serre pairings on $\mathcal{T} := \text{tria}(A_{\text{min}})$ are equivalent in the sense that they are related by a shift-invariant automorphism of the identity functor of $\mathcal{T}$ (see Appendix A). It follows that any Serre pairing on $\text{tria}(A_{\text{min}})$ is induced from a nondegenerate cyclic structure on $A_{\text{min}}$ up to such an equivalence. The equivalence $\text{tria}(A) \approx \text{tria}(A'_{\text{min}})$ in the proposition above also induces an equivalence of cyclic structures. We refer the reader to Appendix A for details about the transport of cyclic structures through functors.

5. Generating the superpotential

In this section we give the construction of the $A_{\infty}$ prolongation promised in the introduction. We start by discussing a result of [16] and its generalization due to [9].

5.1 $A_{\infty}$ generators of a triangulated category

Following [21], we say that a triangulated category $\mathcal{T}$ is algebraic if it is triangle equivalent with the stable category of a Frobenius category. Given $g \in \text{Ob}\mathcal{T}$, we set $H_g := \text{Hom}_{\mathcal{T}}(g, g) = \oplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(g, g[n])$, viewed as a unital graded associative algebra with the composition induced from $\mathcal{T}^\bullet$. Let $\text{GrMod}_{H_g}$ be the category of graded right modules over $H_g$. Consider the functor $\bar{F}_g : \mathcal{T} \to \text{GrMod}_{H_g}$ which sends $a \in \text{Ob}\mathcal{T}$ to the right graded $H_g$-module $\bar{F}_g(a) = \text{Hom}_{\mathcal{T}}(g, a)$. One has the following result:

Proposition [16, 21] Assume that $\mathcal{T}$ is algebraic and let $g \in \text{Ob}\mathcal{T}$. Then there exists a unital dG algebra $\hat{A}$ such that $H(A)$ is isomorphic to $H_g$ and an exact functor $F : \mathcal{T} \to D(A)$ mapping $g$ into $\hat{A}$ such that $H \circ F$ is isomorphic with $\bar{F}$ (here $H : D(A) \to \text{GrMod}_{H(A)}$ is the functor obtained by taking total cohomology). Moreover, $F$ induces an equivalence from $\mathcal{T}$ to $\text{tria}(A)$ iff $g$ triangle generates $\mathcal{T}$, i.e. tria$_{\mathcal{T}}(g) = \mathcal{T}$.
Recall from [9, Proposition 3.2.4.1] that a strictly unital $A_\infty$ algebra admits a strictly unital minimal model related to the original algebra by a strictly unital quasi-isomorphism. Since pullback through quasi-isomorphism induces an equivalence of derived categories [9, Theorem 4.1.2.4], one finds that a strictly unital minimal model $A_{\min}$ of $A$ gives triangle equivalences $D(A) \approx D(A_{\min})$, $\text{per}(A) \approx \text{per}(A_{\min})$ and $\text{tria}(A) \approx \text{tria}(A_{\min})$. This gives the following version of the result above:

**Proposition [9]** Assume that $T$ is algebraic and let $g \in \text{Ob} \ T$ such that $\text{tria}_T(g) = T$. Then there exists a unital minimal $A_\infty$ algebra $(A_{\min}, (\mu_n)_{n \geq 2})$ such that the associative algebra $(A_{\min}, \mu_2)$ is isomorphic with $H_g$ and an exact functor $F : T \to D(A_{\min})$ mapping $g$ into $\hat{A}_{\min}$, which corestricts to an equivalence from $T$ to $\text{tria}(A_{\min})$.

A unital minimal $A_\infty$ algebra $A_{\min}$ as in the proposition will be called a *minimal* $A_\infty$ *generator* of $T$. Given such a generator, any minimal and unital $A_\infty$ algebra $A'_{\min}$ isomorphic with $A_{\min}$ is again an $A_\infty$ generator. Using the results of Section 4.5, we obtain the following ‘cyclic’ variant of the previous proposition.

**Proposition** Assume that $T$ is algebraic and Calabi-Yau of dimension $D$ and let $g \in \text{Ob} \ T$. If $\text{tria}_T(g) = T$, then there exists a unital minimal $D$-cyclic $A_\infty$ algebra $(A_{\min}, \langle \rangle)$ whose pairing is nondegenerate such that $\text{tria}(A_{\min}) \approx T$ via a triangle equivalence mapping $\hat{A}_{\min}$ to $g$. Moreover, any Serre pairing on $T$ is equivalent with the Serre pairing on $\text{tria}(A_{\min})$ induced by $\langle \rangle$ via the procedure of Section 3.2.

A unital, minimal and nondegenerate cyclic $A_\infty$ algebra $A_{\min}$ as in the proposition will be called a *cyclic minimal* $A_\infty$ *generator* for $T$.

### 5.2 The open string field action determined by a cyclic minimal $A_\infty$ generator

Let $T$ be a triangulated algebraic category which is Calabi-Yau of dimension $D$, and assume given $g \in \text{Ob} \ T$ such that $\text{tria}_T(g) = T$. Fix a cyclic and unital minimal $A_\infty$ generator $(A_{\min}, \langle \rangle)$ associated with $g$, and view $A_{\min}$ as the morphism space $\text{Hom}_{A_0}(a,a)$ for an $A_\infty$ category $A_0$ having a single object $a$. Then $\hat{a}$ is identified with $\hat{A}_{\min}$ etc. Because the pairing on $A_{\min}$ is (strictly) nondegenerate, the same is true of the shift-equivariant cyclic structure induced on $\text{tw}(A_{\min}) = \text{tw}(A_0)$ via the extension procedure of Section 3.2. Thus $A := \text{tw}(A_{\min}) = \text{tw}(A_0)$ is a unital and strictly nondegenerate $D$-cyclic $A_\infty$ category with a twisted shift functor (which of course need not be minimal). This allows us to define a formal topological string field action as in section 3.4. Consider the total boundary state space $\mathcal{H}_A := \bigoplus_{q,q' \in \text{Ob}[\text{tw}(A_{\min})]} \text{Hom}_{\text{tw}(A_{\min})}(q,q')$. 

56
endowed with the total bilinear pairing \( \langle \rangle \) and total \( A_\infty \) products \( r_n \). As in Section 3.4, pick a Grassmann algebra \( G \) and define a formal extended open string field action \( S_e : (\mathcal{H}_A)_e^{\text{odd}} \to G \) by the formal sum (3.26).

**The generation property.** From Section 3.4, we know that \( \text{Hom}_{\text{tw}(A_{\text{min}})}(q, q') \) provides an ‘off-shell model’ for the space of boundary observables \( H(\text{Hom}_{\text{tw}(A_{\text{min}})}(q, q')) \approx \text{Hom}_{\mathcal{T}}(q, q') \) of the open string stretching from \( q \) to \( q' \), while the \( A_\infty \) products \( r^{\text{tw}(A_{\text{min}})}_{q_0, \ldots, q_n} \) are string field products. The discussion of Section 3.4 implies:

\[
\text{Every twisted complex } q \in \mathcal{A} = \text{tw}(A_0) = \text{tw}(A_{\text{min}}) \text{ is the result of a condensation process involving open strings stretching between a finite number of shifted copies } a[[n_k]] \text{ of the D-brane described by } a. \text{ In this sense, condensation processes among the D-branes } a[[n]] \text{ generate the entire } A_\infty \text{ category } \mathcal{A}.
\]

Notice that the string field action (3.26) is entirely determined by the minimal cyclic \( A_\infty \) generator \( (A_{\text{min}}, \langle \rangle) \) of \( \mathcal{T} \). Indeed, we just showed that \( a \) generates our D-brane category \( \mathcal{A} \). On the other hand, the cyclic pairing on \( \mathcal{A} \) is induced by the pairing on \( A_{\text{min}} = \text{Hom}_A(a, a) \) through the extension procedure of Section 3.2. Hence the entire open string field theory is determined by \( (A_{\text{min}}, \langle \rangle) \). Thus:

\[
\text{Every minimal cyclic and unital } A_\infty \text{ generator } A_{\text{min}} \text{ of an algebraic Calabi-Yau triangulated category defines a topological open string field theory governing the dynamics of a topological D-brane system whose zeroth cohomology as an } A_\infty \text{ category recovers the triangulated category } \mathcal{T}. \text{ This topological D-brane system consists of topological D-brane composites which can be obtained as condensates between a finite number of shifted copies of a single topological D-brane.}
\]

### 5.3 The induced prolongation and superpotential

The formal string field action introduced above determines a cyclic minimal model of \( (\text{tw}(A_{\text{min}}), \langle \rangle) \) and thus an extended ‘superpotential’ for \( H(\text{tw}(A_{\text{min}})) = \text{tria}(A_{\text{min}})^* \approx \mathcal{T}^* \) via the construction of Sections 3.3 and 3.4. As explained in Section 3.4, the shift-

equivariant pairing \( \langle \rangle^{\text{tria}(A_{\text{min}})} \) determined by the pairing induced by \( \langle \rangle \) on \( \text{tw}(A_{\text{min}}) \) is equivalent with the shift-equivariant pairing \( \langle \rangle \) induced on \( \mathcal{T}^* \) by the original Serre pairing of \( \mathcal{T} \). Since everything is determined by the cyclic \( A_\infty \) generator \( (A_{\text{min}}, \langle \rangle) \), we conclude:

\[
\text{Every minimal cyclic } A_\infty \text{ generator } A_{\text{min}} \text{ of an algebraic Calabi-Yau triangulated category } \mathcal{T} \text{ defines a Serre pairing on } \mathcal{T} \text{ together with a cyclic } A_\infty \text{ prolongation of}
\]
the resulting cyclic graded associative category $\mathcal{T}^•$ (and thus determines an extended ‘superpotential’ for $\mathcal{T}$). These can be constructed explicitly via the extension procedure of Section 3.2 and the procedure of Sections 3.3.

This result allows one to lift the open 2d topological field theory described by $\mathcal{T}$ to a topological open string theory.

A. Categories with shifts, duality structures and cyclic structures

In this Appendix, we discuss duality structures and cyclic structures on associative categories with shifts. This is a slight generalization of the usual theory of Serre functors [33], obtained by relaxing the nondegeneracy condition. We are interested in the description through pairings and traces, which affords a direct link with physics. The treatment of signs is inspired by [32]. In order to keep the discussion reasonably short, we leave the proof of most statements to the reader – they are straightforward diagram arguments, though a few are somewhat lengthy.

A.1 Associative and graded associative categories with shifts

**Associative categories with shifts.** An associative category with shifts is a pair $(\mathcal{A}, [1])$ where $\mathcal{A}$ is a (possibly non-unital) associative category and $[1] : \mathcal{A} \to \mathcal{A}$ is a fixed automorphism of $\mathcal{A}$, called the shift functor. Given a category with shifts, we identify $\text{Hom}_\mathcal{A}(a,b)$ and $\text{Hom}_\mathcal{A}(a[1], b[1])$ through the linear isomorphism induced by $[1]$. We let $[n] := [1]^n$ for all $n \in \mathbb{Z}$, where $[1]^0 = [0] = \text{id}_\mathcal{A}$ is the identity automorphism of $\mathcal{A}$ and $[1]^{-1} = [-1]$ is the inverse of $[1]$. When $\mathcal{A}$ is unital, the shift functor satisfies $\text{id}_a[n] = \text{id}_{a[n]}$ for all objects $a$ and all $n \in \mathbb{Z}$. Small associative categories with shifts form an associative category $\text{SCat}$ whose morphisms $(\mathcal{A}, [1]_\mathcal{A}) \to (\mathcal{B}, [1]_\mathcal{B})$ are the shift-invariant functors, i.e. functors $F : \mathcal{A} \to \mathcal{B}$ obeying $F \circ [1]_\mathcal{A} = [1]_\mathcal{B} \circ F$.

**Graded associative categories with shifts.** Let $\mathcal{G}$ be a graded associative category. A shift functor (in the sense of graded categories) on $\mathcal{G}$ is an automorphism $[1]$ of $\mathcal{G}$ together with isomorphisms $\text{Hom}_\mathcal{G}(a, b[1]) \xrightarrow{\rho} \text{Hom}_\mathcal{G}(a, b)[1]$ for all $a, b \in \text{Ob}\mathcal{G}$, which are natural in $a$ and $b$. Equivalently, $\rho : \text{Hom}_\mathcal{G}(\text{id}_\mathcal{A} \times [1]) \to [1]_{\text{Gr}} \circ \text{Hom}_\mathcal{G}$ is an isomorphism of functors, where $[1]_{\text{Gr}}$ is the shift functor of the category of graded vector spaces $\text{Gr}$. In this case, the pair $(\mathcal{G}, [1])$ is called a graded category with shifts. We let $[n] := [1]^n$ as before. Small graded associative categories with shifts form an associative category $\text{SGrCat}$ whose morphisms are the shift-invariant functors of graded categories, i.e. functors of graded categories which strictly commute with shifts.
**Equivalence of SCat and SGrCat.** It is easy to see that the categories SCat and SGrCat are equivalent. A pair of quasi-inverse equivalences is given by the functors $0 : \text{SCat} \rightarrow \text{SGrCat}$ and $\bullet : \text{SGrCat} \rightarrow \text{SCat}$ constructed as follows. Given $\langle \mathcal{G}, [1] \rangle \in \text{SGrCat}$, we let $\mathcal{G}^0$ be the null restriction of $\mathcal{G}$, i.e. the category obtained from $\mathcal{G}$ by keeping only morphisms of degree zero. Thus $\text{Ob}\mathcal{G}^0 = \text{Ob}\mathcal{G}$ and $\text{Hom}_{\mathcal{G}^0}(a, b) = \text{Hom}_0\mathcal{G}(a, b)$ for all objects $a, b$. Then $\mathcal{G}^0$ is an associative category with shifts, with shift functor given by restricting the action of $[1]$ on morphisms. We let $0$ act in the obvious manner on morphisms of SGrCat. Conversely, given $\langle \mathcal{A}, [1] \rangle \in \text{SCat}$, we define its graded completion $\mathcal{A}^\bullet \in \text{SGrCat}$ as follows. We take $\text{Ob}\mathcal{A}^\bullet = \text{Ob}\mathcal{A}$ and $\text{Hom}_{\mathcal{A}^\bullet}(a, b) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}_\mathcal{A}(a, b[k])$ for all objects $a, b$, with the composition of morphisms:

$$g \ast f = \bigoplus_{n \in \mathbb{Z}} \sum_{k+l=n} g_l[k] \circ f_k \quad \forall f = \bigoplus_{k+l \in \mathbb{Z}} f_k \in \text{Hom}_{\mathcal{A}^\bullet}(a, b) \quad \forall g = \bigoplus_{l \in \mathbb{Z}} g_l \in \text{Hom}_{\mathcal{A}^\bullet}(b, c),$$

(A.1)

where $f_k \in \text{Hom}_{\mathcal{A}}(a, b[k])$ and $g_l \in \text{Hom}_{\mathcal{A}}(b, c[l])$. The morphism spaces of $\mathcal{A}^\bullet$ are graded with homogeneous components $\text{Hom}^k_\mathcal{A^\bullet}(a, b) := \text{Hom}_{\mathcal{A}}(a, b[k])$. The compositions (A.1) have degree zero, so $\mathcal{A}^\bullet$ is a graded associative category. The relations $\text{Hom}_{\mathcal{A}^\bullet}(a[1], b[1]) \approx \text{Hom}_{\mathcal{A}}(a, b)$ imply that $\mathcal{A}^\bullet$ is a graded category with shifts. We let $\bullet$ act in the obvious manner on morphisms of SGrCat. The category $\mathcal{A}^\bullet$ is sometimes denoted by $\mathcal{A}/[1]$ (the ‘quotient’ of $\mathcal{A}$ by the group of automorphisms generated by $[1]$). It is clear that $0$ and $\bullet$ interchange unital associative categories with unital graded associative categories.

**Twisting the shift functor of a graded associative category with shifts.** Given a graded associative category with shifts $\langle \mathcal{G}, [1] \rangle$, we can define a new endofunctor $[[1]]$ of $\mathcal{G}$ as follows. We let $[[1]]$ act on objects via $a[[1]] := a[1]$ and on morphisms $f \in \text{Hom}_\mathcal{G}(a, b)$ through:

$$f[[1]] := (-1)^{\text{deg} f} f[1].$$

It is clear that $[[1]]$ is an automorphism of $\mathcal{G}$, which we call the twist of $[1]$. When $\mathcal{G}$ is unital, we have $\text{id}_a[[1]] = \text{id}_a[1]$ because $\text{deg}(\text{id}_a) = 0$.

Notice that the restrictions of $[1]$ and $[[1]]$ to the subcategory $\mathcal{G}^0$ coincide. When $\mathcal{G} = \mathcal{A}^\bullet$ for some associative category with shifts, this remark allows us to view $[1]$ and $[[1]]$ as different extensions of the shift functor of $\mathcal{A}$. Also notice that the twist of $[[1]]$ recovers $[1]$, i.e. $[[[[[1]]]]] = [1]$.

**Observation** Let $\rho : \text{Hom}\mathcal{G} \circ (\text{id}\mathcal{G} \times [1]) \rightarrow [1]_{\mathcal{G}^0} \circ \text{Hom}\mathcal{G}$ be the isomorphism of functors defined by $[1]$ and let $\beta = \sigma^{-1} \circ \rho : \text{Hom}\mathcal{G} \circ (\text{id}\mathcal{A} \times [1]) \rightarrow \text{Hom}\mathcal{G}$ be the morphism of
functors of degree +1 obtained by composing with the inverse of the signed suspension \( \sigma : \text{id}_A \to [1]_{\mathcal{G}} \). Functoriality of \( \rho \) amounts to the conditions:

\[
\beta(uv) = \beta(u)v \ , \ \beta(u[1]v) = (-1)^{\text{deg}u}u\beta(v)
\]

for all composable morphisms \( u, v \) of \( \mathcal{G} \). These show that \([1]\) satisfies:

\[
\beta(uv) = \beta(u)v \ , \ \beta(u[1]|v) = u\beta(v) \ ,
\]

which means that \( \rho : \text{Hom}_G \circ (\text{id}_G \times [[1]]) \to [1]_{\mathcal{G}} \circ \text{Hom}_G \) is a `twisted natural transformation`, i.e. it satisfies the naturality conditions without Koszul signs. Equivalently, the maps \( \gamma_{ab} := s^{-1}_{ab} \circ \rho_{ab} \) give a morphism of functors \( \gamma : \text{Hom}_G \circ (\text{id}_A \times [[1]]) \to \text{Hom}_G \) of degree +1. An automorphism \([1]\) of \( \mathcal{G} \) endowed with isomorphisms of graded vector spaces \( \text{Hom}_G(a,b[[1]]) \overset{\rho_{ab}}{\to} \text{Hom}_G(a,b)[1] \) which are natural up to missing Koszul signs will be called a twisted shift functor of \( \mathcal{G} \).

**Graded functors between unital associative categories with shifts.** Let \((\mathcal{A}, [1])\) and \((\mathcal{B}, [1])\) be unital categories with shifts. A covariant graded functor from \((\mathcal{A}, [1])\) to \((\mathcal{B}, [1])\) is a pair \((F, \eta)\) where \( F : \mathcal{A} \to \mathcal{B} \) is a covariant functor and \( \eta : F \circ [1] \to [1] \circ F \) is an isomorphism of functors. Notice that \( \eta \) induces an isomorphism of functors \( F \circ [k] \overset{\eta_k}{\to} [k] \circ F \), given by the compositions \( \eta^k_a := \eta_a[k - 1] \circ \eta_{a[1]}[k - 2] \circ \ldots \circ \eta_{a[k-1]} : F(a[k]) \to F(a)[k] \) for all \( a \in \text{Ob}\mathcal{A} \) (we have \( \eta^1 = \eta \)). Also notice that the morphisms of \( \mathbf{SCat} \) are those graded endofunctors for which \( \eta \) is the identity. Given two graded functors \( F : (\mathcal{A}, [1],_A) \to (\mathcal{B}, [1],_B) \) and \( G : (\mathcal{B}, [1],_B) \to (\mathcal{C}, [1],_C) \), their composition \( G \circ F \) is a graded functor from \((\mathcal{A}, [1],_A)\) to \((\mathcal{C}, [1],_C)\) whose grading is given by \( \eta^{G \circ F} = \eta^G \circ G(\eta^F) \), where \( \eta^{F,G} \) are the gradings of \( F \) and \( G \). Using naturality of \( \eta^F \) and \( \eta^G \), this implies \( (\eta^{G \circ F})^k = (\eta^G)^k \circ G(\eta^F)^k \) for all \( k \in \mathbb{Z} \).

**Examples** A simple example is given by the shift functor \([1]\) of \( \mathcal{A} \), graded through the trivial isomorphism \( \text{id} : [1] \circ [1] \to [1] \circ [1] \). The resulting graded functor \( \sigma := ([1], \text{id}) \) will be called the unsigned graded shift functor of \( \mathcal{A} \). Another useful example is the signed graded shift functor \( s = ([1], \eta^{(s)}) \), where \( \eta^{(s)} : [1] \circ [1] \to [1] \circ [1] \) is the isomorphism of functors given by \( \eta^{(s)}_a := -a[2] : a[1][1] = a[2] \to a[1][1] = a[2] \). In this case, we have \( (\eta^{(s)}_a)^k = (-1)^{k}a[k+1] \) for all \( a \in \text{Ob}\mathcal{A} \), which gives an isomorphism of functors \( (\eta^{(s)}_a)^k : [1] \circ [k] \to [k] \circ [1] \).

**Observation** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two triangulated categories. Then an exact functor \( F : \mathcal{A} \to \mathcal{B} \) is in particular a graded functor from \((\mathcal{A}, [1])\) to \((\mathcal{B}, [1])\). Given a triangulated category \( \mathcal{A} \), the signed graded shift functor \( s \) is exact, while the unsigned graded shift functor \( \sigma \) fails to be exact when endowed with the trivial grading [32]. This application
motivates our interest in twisted shift functors, as well as the choice of sign in equation (A.9) below.

**Graded functors between unital graded associative categories with shifts.** Given unital graded categories with shifts \((\mathcal{F}, [1])\) and \((\mathcal{G}, [1])\), a graded functor from \((\mathcal{F}, [1])\) to \((\mathcal{G}, [1])\) is a pair \((F, \eta)\) where \(F : \mathcal{A} \to \mathcal{B}\) is a covariant functor of graded categories and \(\eta : F \circ [1] \to [1] \circ F\) is an isomorphism of functors.

**Observation** A grading \([1] \circ [1] \to [1] \circ [1]\) of the shift functor \([1]\) of \(\mathcal{G}\) obviously corresponds to the same grading \([(1)] \circ [1] \to [1] \circ [1]\) of the twisted shift functor \([1][1]\).

As for ungraded categories, we let \(\eta(s)\) be the grading given by \(\eta(s) := -id_{a[2]} : a[1][1] = a[2] \to a[1][1] = a[2]\).

**Graded completion of graded functors.** Let \((\mathcal{A}, [1]_{\mathcal{A}})\) and \((\mathcal{B}, [1]_{\mathcal{B}})\) be unital categories with shifts. Given a covariant graded functor \((F, \eta) : (\mathcal{A}, [1]_{\mathcal{A}}) \to (\mathcal{B}, [1]_{\mathcal{B}})\), its *graded completion* is the covariant graded functor of graded categories \((F^{\bullet}, \eta^{\bullet}) : (\mathcal{A}^{\bullet}, [1]_{\mathcal{A}^{\bullet}}) \to (\mathcal{B}^{\bullet}, [1]_{\mathcal{B}^{\bullet}})\) defined as follows. We set \(F^{\bullet}(a) = F(a)\) for all objects \(a\) and:

\[
F^{\bullet}(u) := \eta_{a}^{k} \circ F(u) \in Hom_{B}(F(a), F(b)[k]) = Hom_{B}^{k}(F^{\bullet}(a), F^{\bullet}(b))
\]

for all \(u \in Hom_{A}^{k}(a, b) = Hom_{A}(a, b[k]).\) Finally, we set \(\eta^{\bullet} = \eta\). A somewhat lengthy diagrammatic argument using the definition of \(\eta_{a}^{k}\) shows that \(\eta^{\bullet} : F^{\bullet} \circ [1]_{\mathcal{A}^{\bullet}} \to [1]_{\mathcal{B}^{\bullet}} \circ F^{\bullet}\) is an isomorphism of functors. Here \([1]_{\mathcal{A}^{\bullet}}\) and \([1]_{\mathcal{B}^{\bullet}}\) are the shift functors of \(\mathcal{A}^{\bullet}\) and \(\mathcal{B}^{\bullet}\) induced by the shift functors of \(\mathcal{A}\) and \(\mathcal{B}\).

**Examples** The graded completion of the unsigned graded shift functor \(\sigma = ([1]_{\mathcal{A}}, \text{id})\) of \(\mathcal{A}\) is the unsigned graded shift functor \(\sigma^{\bullet} = ([1]_{\mathcal{A}^{\bullet}}, \text{id})\) of \(\mathcal{A}^{\bullet}\). The graded completion of the signed graded shift functor \(s = ([1]_{\mathcal{A}}, \eta^{(s)})\) is the signed graded twisted shift functor \(s^{\bullet} = ([1][1], \eta^{(s)})\) of \(\mathcal{A}^{\bullet}\).

**Observation** Recall that \([1]_{\mathcal{A}^{\bullet}}\) is defined through the identifications \(\text{Hom}_{A}^{k}(a, b) \overset{\text{def}}{=} \text{Hom}_{A}(a, b[k][1]) \overset{[1]}{=} \text{Hom}_{A}(a[1], b[1][k]) \overset{\text{def}}{=} \text{Hom}_{A}^{k}(a[1], b[1]),\) where the isomorphism in the middle is trivial since \([k] \circ [1] = [1] \circ [k]\). On the other hand, \([1][1]_{\mathcal{A}^{\bullet}}\) results by performing the identification in the middle through the nontrivial isomorphism \((\eta^{(s)})^{-1} : [k] \circ [1] \approx [1] \circ [k]\).

**Idempotent completion.** Given a unital associative category \(\mathcal{A}\) and an object \(a \in \text{Ob}\mathcal{A}\), an idempotent endomorphism of \(a\) is an element \(e \in \text{Hom}_{\mathcal{A}}(a, a)\) satisfying \(e^{2} = e\). We say that \(e\) is *split* if there exists an object \(b \in \text{Ob}\mathcal{A}\) and morphisms \(s \in \text{Hom}_{\mathcal{A}}(b, a), r \in \text{Hom}_{\mathcal{A}}(a, b)\) such that \(r \circ s = \text{id}_{b}\) and \(e = s \circ r\). In this case, \(b\) is called a retraction
of $a$; two retractions of $a$ are easily seen to be isomorphic. When $\mathcal{A}$ is additive, this condition amounts to the existence of a direct sum decomposition $a = \ker e \oplus \text{im} e$ for any idempotent endomorphism $e$.

We say that $\mathcal{A}$ is **idempotent complete**$^{20}$ if any idempotent of $\mathcal{A}$ is split. Given an associative category $\mathcal{A}$, its **idempotent completion**$^{21}$ is the smallest idempotent complete category $\mathcal{A}^\pi$ which contains $\mathcal{A}$ as a full subcategory; it has the property that any object of $\mathcal{A}^\pi$ is a retract of an object of $\mathcal{A}$. Any associative category $\mathcal{A}$ admits an idempotent completion, determined up to an equivalence which restricts to the identity on $\mathcal{A}$. In particular, $\mathcal{A}$ is idempotent complete iff $\mathcal{A}^\pi \approx \mathcal{A}$. A canonical representative can be constructed by taking the objects of $\mathcal{A}^\pi$ to be the pairs $(a,e)$ where $a$ is an object of $\mathcal{A}$ and $e$ an idempotent endomorphism of $a$, and setting $\text{Hom}_{\mathcal{A}^\pi}((a,e),(a',e')) := e' \circ \text{Hom}_A(a,a') \circ e \subset \text{Hom}_A(a,a')$, with the composition of morphisms induced from $\mathcal{A}$. We will always understand $\mathcal{A}^\pi$ to be this canonical representative. It is clear that $\mathcal{A}^\pi$ is Hom-finite iff $\mathcal{A}$ is.

Any functor $F : \mathcal{A} \to \mathcal{B}$ extends to a functor $F^\pi : \mathcal{A}^\pi \to \mathcal{B}^\pi$ defined through $F^\pi(a,e) = (F(a), F_{aa}(e))$ and $F^\pi_f = F_{a_1a_2} f)$ for all $f \in \text{Hom}_{\mathcal{A}^\pi}((a_1,e_1),(a_2,e_2))$. Given a functor $G : \mathcal{B} \to \mathcal{C}$, we have $(G \circ F)^\pi = G^\pi \circ F^\pi$. Given $F,G : \mathcal{A} \to \mathcal{B}$ and a natural transformation $\phi : F \to G$, one has a natural transformation $\phi^\pi : F^\pi \to G^\pi$ given by $\phi^\pi_{(a,e)} = G_{aa}(e) \circ \phi_a \circ F_{aa}(e) = G_{aa}(e) \circ \phi_a = \phi_a \circ F_{aa}(e)$, where the last two equalities follow from naturality of $\phi$ and $e^2 = e$.

The idempotent completion of an additive category is additive. Less obviously [35], the idempotent completion of a triangulated category is canonically triangulated. It is also known [34] that a triangulated category with countable coproducts is idempotent complete.

An almost identical discussion holds for graded categories. In this case, an idempotent $e$ is required by definition to be a homogeneous morphism of degree zero, and the same condition is imposed on the maps $r,s$ for a split idempotent. The idempotent completion is now a graded category, which is constructed the same way as above.

**Idempotent completion of (graded) categories with shifts.** Given an associative category with shifts $(\mathcal{A},[1])$, the idempotent completion of the shift functor $[1]^\pi$ is a shift functor for $\mathcal{A}^\pi$; thus $(\mathcal{A}^\pi,[1]^\pi)$ is a category with shifts. Given a graded functor $(F,\eta) : (\mathcal{A},[1]_A) \to (\mathcal{B},[1]_B)$, its idempotent completion $F^\pi$ is a graded functor from $(\mathcal{A},[1]_A)$ to $(\mathcal{B},[1]_B)$ when endowed with the grading $\eta^\pi$. Similar statements holds for graded categories with shifts.

---

$^{20}$One also says that $\mathcal{A}$ is Karoubi closed, Karoubian or split-closed and also that $\mathcal{A}$ has split idempotents.

$^{21}$Also called the Karoubi closure, or split closure of $\mathcal{A}$. 
A.2 Duality structures and cyclic structures

**Duality structures.** Let $\mathcal{A}$ be an associative category. A *duality structure* on $\mathcal{A}$ is a pair $(S, \phi)$ where $S$ is an automorphism of $\mathcal{A}$ and $\phi$ is a family of linear maps $\phi_{ab} : \text{Hom}_\mathcal{A}(a, b) \to \text{Hom}_\mathcal{A}(b, S(a))^\vee$ which are natural in $a$ and $b$. We say that $(S, \phi)$ is *nondegenerate* if $\mathcal{A}$ is Hom-finite and all $\phi_{ab}$ are bijective.

**Observation** One can generalize the notion of duality structure by allowing $S$ to be an autoequivalence of $\mathcal{A}$. This doesn’t give anything essentially new since such $S$ correspond to automorphisms of a skeleton of $\mathcal{A}$. In this paper, we are interested mostly in the case $S = [D] = [1]^D$ for some shift functor $[1]$; then $S$ is an automorphism.

Fixing a duality structure $(S, \phi)$ on $\mathcal{A}$, consider the pairings $(\ )_{ab} : \text{Hom}_\mathcal{A}(a, b) \otimes \text{Hom}_\mathcal{A}(b, S(a)) \to \mathbb{C}$ defined through $(u \otimes v)_{ab} := \phi_{ab}(u)(v)$ (we will tacitly identify $(\ )_{ab}$ with the associated bilinear form). Then naturality of $\phi$ amounts to the conditions:

$$(u \circ f, v)_{a', b} = (u, S(f) \circ v)_{a, b} \quad \forall f \in \text{Hom}_\mathcal{A}(a', a), \ u \in \text{Hom}_\mathcal{A}(a, b), \ v \in \text{Hom}_\mathcal{A}(b, S(a'))$$

and

$$(g \circ u, v)_{a, b'} = (u, v \circ g)_{a, b'} \quad \forall g \in \text{Hom}_\mathcal{A}(b, b'), \ u \in \text{Hom}_\mathcal{A}(a, b), \ v \in \text{Hom}_\mathcal{A}(b', S(a)).$$

Hence a duality structure on $\mathcal{A}$ amounts to an automorphism $S$ together with pairings $(\ )_{ab}$ obeying the conditions above. Nondegeneracy of $(S, \phi)$ amounts to Hom-finiteness of $\mathcal{A}$ plus nondegeneracy of these pairings as bilinear forms.

When $\mathcal{A}$ is unital, a duality structure can also be described as follows. Defining linear maps $tr_a : \text{Hom}_\mathcal{A}(a, S(a)) \to \mathbb{C}$ via $tr_a(u) := (\text{id}_{a}, u)_{a, a}$, the first condition above is equivalent with:

$$(u, v)_{a, b} = tr_b(S(u) \circ v) \quad ,$$

while the second becomes:

$$tr_a(u \circ v) = tr_b(S(v) \circ u) \quad \forall v \in \text{Hom}_\mathcal{A}(a, b), \ u \in \text{Hom}_\mathcal{A}(b, S(a)) \quad .$$

Hence a duality structure on $\mathcal{A}$ amounts to an autoequivalence $S$ together with linear maps $tr_a$ obeying (A.3). The information carried by the traces is equivalent with that carried by $\phi_{ab}$, which can be recovered as $\phi_{ab}(u)(v) = tr_b(S(u) \circ v)$.

The notion of duality structure has an obvious graded analogue, which we spell out in detail for later reference. Thus a duality structure on a graded associative category $\mathcal{G}$ is a pair $(S, \phi)$ where $S$ is an automorphism of $\mathcal{G}$ as a graded associative category and $\phi_{ab} : \text{Hom}_\mathcal{G}(a, b) \to \text{Hom}_\mathcal{G}(b, S(a))^\vee$ are morphisms of graded vector spaces.
which are natural in $a$ and $b$. Equivalently, this is specified by degree zero pairings 
$$(u, v)_{a,b} : \text{Hom}_{\mathcal{G}}(a, b) \otimes \text{Hom}_{\mathcal{G}}(b, S(a)) \to \mathbb{C}$$ 
satisfying the graded analogues of the conditions above:
$$(uf, v)_{a',b} = (u, S(f)v)_{a,b} \quad \text{for} \quad f \in \text{Hom}_{\mathcal{G}}(a', a), \ u \in \text{Hom}_{\mathcal{G}}(a, b), \ v \in \text{Hom}_{\mathcal{G}}(b, S(a'))$$
and
$$(gu, v)_{a',b} = (-1)^{\deg(u) + \deg(v)}(u, vg)_{a,b} \quad \text{for} \quad g \in \text{Hom}_{\mathcal{G}}(b, b'), \ u \in \text{Hom}_{\mathcal{G}}(a, b), \ v \in \text{Hom}_{\mathcal{G}}(b', S(a)).$$
When $\mathcal{G}$ is unital, this data is encoded by degree zero linear maps $tr_a : \text{Hom}_{\mathcal{G}}(a, S(a)) \to \mathbb{C}$ subject to the graded analogue of conditions (A.3):
$$tr_a(uv) = (-1)^{\deg(u) + \deg(v)}tr_b(S(v)u) \quad \text{for} \quad v \in \text{Hom}_{\mathcal{G}}(a, b), \ u \in \text{Hom}_{\mathcal{G}}(b, S(a)). \quad \text{(A.4)}$$
Once again, the traces and bilinear pairings are related through (A.2). We also have $tr_a(u) := (\text{id}_a, u)_{a,a}$. We say that $(S, \phi)$ is nondegenerate if $\mathcal{G}$ is degreewise Hom-finite and all $\phi_{ab}$ are bijective.

**Idempotent completion of duality structures.** Consider a unital associative category with shifts $(\mathcal{A}, [1])$. A duality structure $(S, \phi)$ on $\mathcal{A}$ extends to a duality structure $(S^{\pi}, \phi^{\pi})$ on the shift completion $\mathcal{A}^{\pi}$, where $\phi^{\pi}_{(a_1,e_1),(a_2,e_2)} : \text{Hom}_{\mathcal{A}^{\pi}}((a_1, e_1), (a_2, e_2)) = e_2 \circ \text{Hom}_{\mathcal{A}}(a_1, a_2) \circ e_1 \to \text{Hom}_{\mathcal{A}^{\pi}}((a_2, e_2), S^{\pi}(a_1, e_1))^{\pi} = [S(e_1) \circ \text{Hom}_{\mathcal{A}}(a_2, S(a_1)) \circ e_2]^{\pi}$ is defined by the restriction:
$$\phi^{\pi}_{(a_1,e_1),(a_2,e_2)}(x) = \phi_{a_1,a_2}(x)|_{S(e_1)\circ\text{Hom}_{\mathcal{A}}(a_2,S(a_1))\circ e_2} \quad \forall x \in \text{Hom}_{\mathcal{A}^{\pi}}((a_1, e_1), (a_2, e_2))$$
This amounts to defining pairings and traces $tr^{\pi}$ on $\mathcal{A}^{\pi}$ by restricting the pairings of $\mathcal{A}$:
$$(u, v)^{\pi}_{(a_1,e_1),(a_2,e_2)} := (u, v)_{a,b}$$
for all $u \in \text{Hom}_{\mathcal{A}^{\pi}}((a_1, e_1), (a_2, e_2)) \subset \text{Hom}_{\mathcal{A}}(a_1, a_2)$ and $v \in \text{Hom}_{\mathcal{A}^{\pi}}((a_2, e_2), S^{\pi}(a_1, e_1)) \subset \text{Hom}_{\mathcal{A}}(a_2, S(a_1))$, i.e.
$$tr^{\pi}_{a,e}(u) = tr_a(u) \quad \forall u \in \text{Hom}_{\mathcal{A}^{\pi}}((a, e), (S(a), S(e))) = S(e)\text{Hom}_{\mathcal{A}}(a, S(a))e \subset \text{Hom}_{\mathcal{A}}(a, S(a)).$$

Recall that $\mathcal{A}^{\pi}$ is Hom-finite iff $\mathcal{A}$ is. In this case, it is easy to see that $(S^{\pi}, \phi^{\pi})$ is nondegenerate iff $(S, \phi)$ is. Indeed, the direct implication is obvious while the inverse implication follow from the relations $tr^{\pi}_{a,e}(uexe) = tr_a(uexe) = tr_a(S(e)uex) = tr_a(ux)$, which hold for all $u \in S(e)\text{Hom}_{\mathcal{A}}(a, S(a))e$ and all $x \in \text{Hom}_{\mathcal{A}}(a, a)$. These show that $tr^{\pi}_{a}(uv)$ vanishes for all $v \in e\text{Hom}_{\mathcal{A}}(a, a)e$ iff $tr_a(ux)$ vanishes for all $x \in \text{Hom}_{\mathcal{A}}(a, a)$, which requires $x = 0$ by non-degeneracy of $tr_a$.

An almost identical discussion holds for duality structures on graded associative categories.
Cyclic structures on graded associative categories. Given an integer $D$, a $D$-cyclic structure on a graded associative category $G$ is a collection of morphisms of graded vector spaces $\psi_{ab} : \text{Hom}_G(a, b) \to \text{Hom}_G(b, a)[D]^\vee$ which are natural in $a$ and $b$. Equivalently, it is specified by a collection of degree zero pairings $\langle \cdot, \cdot \rangle_{ab} : \text{Hom}_G(a, b) \otimes \text{Hom}_G(b, a) \to \mathbb{C}[-D]$ (viewed as linear maps of degree $-D$ from $\text{Hom}_G(a, b) \otimes \text{Hom}_G(b, a)$ to $\mathbb{C}$) which satisfy:

$$\langle uf, v \rangle_{ab} = \langle u, fv \rangle_{a,b} \quad \forall f \in \text{Hom}_G(a', a), u \in \text{Hom}_G(a, b), v \in \text{Hom}_G(b, d') \quad (A.5)$$

as well as

$$\langle f, g \rangle_{a,b} = (-1)^{\text{deg} f \text{deg} g} \langle g, f \rangle_{b,a} \quad \text{for} \quad f \in \text{Hom}_G(a, b), g \in \text{Hom}_G(b, a) \quad (A.6)$$

The homogeneity condition on the pairings amounts to the selection rule:

$$\langle f, g \rangle = 0 \quad \text{unless} \quad \text{deg} f + \text{deg} g = D \quad (A.7)$$

Let $s^D_{ab} : \text{Hom}_G(a, b) \to \text{Hom}_G(a, b)[D]$ be the map of degree $-D$ induced by the suspension operator; we denote its inverse by $s^{-D}_{ab}$ through a slight abuse of notation. Then the relation with the maps $\psi_{ab}$ is given by $\langle u, v \rangle_{ab} = \psi_{ab}(u)(s^{-D}_{ba}(v))$. When $G$ is unital, we can also describe this in terms of traces $\text{tr}_a : \text{Hom}_G(a, a) \to \mathbb{C}[-D]$ (viewed as homogeneous linear maps of degree $-D$ from $\text{Hom}_G(a, a)$ to $\mathbb{C}$) defined through $\text{tr}_a(u) = \langle \text{id}_a, u \rangle_{aa}$. These satisfy:

$$\text{tr}_a(uv) = (-1)^{\text{deg} u \text{deg} v} \text{tr}_b(vu) \quad \text{for} \quad v \in \text{Hom}_G(a, b), u \in \text{Hom}_G(b, a) \quad .$$

The bilinear pairings can be recovered as $\langle u, v \rangle_{ab} = \text{tr}_b(uv) = (-1)^{\text{deg} u \text{deg} v} \text{tr}_a(vu)$.

The cyclic structure is called non-degenerate when $G$ is degreewise Hom-finite and $\psi_{ab}$ are bijective for all $a, b \in \text{Ob}A$; the latter condition amounts to nondegeneracy of the bilinear pairings $\langle \cdot, \cdot \rangle_{ab}$.

Idempotent completion of cyclic structures on graded categories. Given a $D$-cyclic structure $\text{tr}$ on a unital graded category $G$, we define traces $\text{tr}^\pi$ on $G^\pi$ via:

$$\text{tr}^\pi_{(a,e)}(u) := \text{tr}_a(u) \quad \forall u \in \text{Hom}_{G^\pi}((a, e), (b, e')) = e' \text{Hom}_G(a, b)e \subset \text{Hom}_G(a, b) \quad .$$

These define a $D$-cyclic structure on $G^\pi$, called the idempotent completion of $\text{tr}$.

Recall that $G^\pi$ is degreewise Hom-finite iff $G$ is. In this case, it is easy to see that $\text{tr}^\pi$ is nondegenerate iff $\text{tr}$ is.

\footnote{Of course, one can also work with the morphisms $\psi_{ab}[D] : \text{Hom}_G(a, b)[D] \to \text{Hom}_G(b, a)^\vee$.}
Cyclic structures on categories with shifts. Consider a graded associative category with shifts \((\mathcal{G}, [1])\), and let \(\rho_{ab} : \text{Hom}_\mathcal{G}(a, b[1]) \xrightarrow{\cong} \text{Hom}_\mathcal{G}(a, b)[1]\) be the isomorphism determined by the shift functor of \(\mathcal{G}\). For any integer \(n\), let \(\rho_{ab}^n\) denote the induced isomorphism \(\text{Hom}_\mathcal{G}(a, b[n]) \xrightarrow{\cong} \text{Hom}_\mathcal{G}(a, b)[n]\) (when \(n > 0\), we have \(\rho_{ab}^n := \rho_{ab}[n-1] \circ \ldots \circ \rho_{ab}[n-2][1] \circ \rho_{ab}[n-1]\) etc).

Using these isomorphisms, a \(D\)-cyclic structure \(\psi\) on \(\mathcal{G}\) can be identified with a duality structure \((S, \phi)\) on \(\mathcal{G}\) having \(S = [[D]]\) and \(\phi_{ab} = (\rho_{ba}^D)\circ \psi_{ab} : \text{Hom}_\mathcal{G}(a, b) \to \text{Hom}_\mathcal{G}(b, a[D])\)\(^\vee\). This corresponds to setting \((u, v)_{ab} = \langle u, (s_{ba}^D \circ \rho_{ba}^D)(v)\rangle_{ab}\) for the bilinear pairings \((u, v)_{ab} : \text{Hom}_\mathcal{G}(a, b) \times \text{Hom}_\mathcal{G}(b, a[D]) \to \mathbb{C}\) of \((S, \phi)\). When \(\mathcal{G}\) is unital, the traces \(tr_a\) of \((S, \phi)\) are related to those of \(\psi\) via \(tr_a = tr_a \circ s_{aa}^{-D} \circ \rho_{aa}^D\). Thus:

\[
\text{Given a graded associative category with shifts } (\mathcal{G}, [1]), \text{ a } D\text{-cyclic structure on } \mathcal{G} \text{ amounts to a duality structure } (S, \phi) \text{ on } \mathcal{G} \text{ having } S = [[D]].
\]

Restricting \(\phi\) to morphisms of degree zero gives a duality structure \((S_0, \phi_0)\) on the associative category \(\mathcal{G}^0\) having \(S_0 = [D]\). This justifies the following:

**Definition** A duality structure \((S, \phi)\) on an associative category with shifts \((\mathcal{A}, [1])\) is called a \(D\)-cyclic structure if \(S = [D]\).

**Observation** Since \([1]\)^{\pi} is the shift functor of \(\mathcal{A}^{\pi}\), it is clear that the idempotent completion of a \(D\)-cyclic structure on \(\mathcal{A}\) (viewed as a duality structure) is again a \(D\)-cyclic structure.

### A.3 Graded duality structures

**Graded duality structures on unital associative categories with shifts.** Let \((\mathcal{A}, [1])\) be a unital associative category with shifts. A graded duality structure on \((\mathcal{A}, [1])\) is a triple \((S, \phi, \eta)\) where \((S, \phi)\) is a duality structure on \(\mathcal{A}\) and \((S, \eta)\) is a graded functor, subject to the compatibility conditions:

\[
(u[1], v[1])_{a[1][1]} = -(u, \eta_a[-1] \circ v)_{ab} \text{ for } u \in \text{Hom}_\mathcal{A}(a, b), v \in \text{Hom}_\mathcal{A}(b, S(a[1])[-1])
\]

\[(A.8)\]

for all \(a, b \in \text{Ob}\mathcal{A}\). Naturality of \(\eta\) implies that these conditions are equivalent with:

\[
tr_{a[1]}(u[1]) = -tr_a(\eta_a[-1] \circ u) \quad \forall a \in \text{Ob}\mathcal{A}, \forall u \in \text{Hom}_\mathcal{A}(a, S(a[1])[-1])
\]

\[(A.9)\]

**Graded duality structures on unital graded associative categories with shifts.** A graded duality structure on a unital graded associative category with shifts \((\mathcal{G}, [1])\) is a triple \((S, \phi, \eta)\) where \((S, \phi)\) is a duality structure on \(\mathcal{A}\) and \((S, \eta)\) is a graded functor, subject to the same compatibility conditions as above. In this case, the maps \(\phi_{ab}\), bilinear pairings \((u, v)_{ab}\) and traces \(tr_a\) are homogeneous of degree zero.
Idempotent completion of graded duality structures. Let \((\mathcal{A}, [1])\) be a unital associative category with shifts. The idempotent completion of a duality structure \((S, \phi, \eta)\) on \((\mathcal{A}, [1])\) is the triple \((S^\pi, \phi^\pi, \eta^\pi)\), which is easily seen to be a duality structure on \((\mathcal{A}^\pi, [1]^\pi)\). A similar definition works for graded associative categories.

Graded completion of graded duality structures. Consider a unital associative category with shifts \((\mathcal{A}, [1])\) endowed with a graded duality structure \((S, \phi, \eta)\). The graded completion of \((S, \phi, \eta)\) is the graded duality structure \((S^{\bullet}, \phi^{\bullet}, \eta^{\bullet})\), where \((S^{\bullet}, \eta^{\bullet})\) is the graded completion of the graded functor \((S, \eta)\), while \(\phi^{\bullet}_{ab} : \text{Hom}_{\mathcal{A}^{\bullet}}(a, b) \to \text{Hom}_{\mathcal{A}^{\bullet}}(b, S^{\bullet}(a))^v\) are defined through the compositions:

\[
\text{Hom}_{\mathcal{A}^{\bullet}}^k(a, b) = \text{Hom}_{\mathcal{A}}(a, b[k]) \xrightarrow{\phi^{\bullet}_{ab}} \text{Hom}_{\mathcal{A}}(b[k], S(a))^v \xrightarrow{|k|v} \text{Hom}_{\mathcal{A}}(b, S(a)[-k])^v = ([\text{Hom}_{\mathcal{A}^{\bullet}}(b, S^{\bullet}(a))]^v)^k ,
\]

i.e.

\[
\phi^{\bullet}_{ab}(u, v) = \phi_{a,b[k]}(u)(v[k]) \quad \text{for} \quad u \in \text{Hom}_{\mathcal{A}}(a, b[k]) \quad \text{and} \quad v \in \text{Hom}_{\mathcal{A}}(b, S(a)[-k]) .
\]

Hence the pairings of \((S^{\bullet}, \phi^{\bullet}, \eta^{\bullet})\) are the homogeneous bilinear maps of degree zero \((\cdot, \cdot)^{\bullet}_{ab} : \text{Hom}_{\mathcal{A}^{\bullet}}(a, b) \times \text{Hom}_{\mathcal{A}^{\bullet}}(b, S^{\bullet}(a)) \to \mathbb{C}\) given by:

\[
(u, v)^{\bullet}_{ab} = (u, v[k])_{\text{ab}[k]} \quad \text{for} \quad u \in \text{Hom}_{\mathcal{A}^{\bullet}}^k(a, b) \quad \text{and} \quad v \in \text{Hom}_{\mathcal{A}^{\bullet}}^k(b, S(a)) ,
\]

while the traces are the homogeneous degree zero linear maps \(\text{tr}^{\bullet}_a : \text{Hom}_{\mathcal{A}^{\bullet}}(a, S^{\bullet}(a)) \to \mathbb{C}\) obtained from \(\text{tr}_a\) through extension by zero. It is clear that \((S^{\bullet}, \phi^{\bullet}, \eta^{\bullet})\) is nondegenerate iff \((S, \phi, \eta)\) is.

Observation To check condition (A.4), it suffices to notice that (A.9) implies the relation:

\[
\text{tr}_a(g \circ f) = (-1)^k \text{tr}_b(S^{\bullet}(f) \circ g) \quad \forall f \in \text{Hom}_{\mathcal{A}}(a, b[k]) , \quad \forall g \in \text{Hom}_{\mathcal{A}}(b, S(a)[-k]) ,
\]

where \(\circ\) is the composition of morphisms in \(\mathcal{A}^{\bullet}\).

A.4 Shift-equivariant cyclic structures

When \((\mathcal{G}, [1])\) is a graded category with shifts, a \(D\)-cyclic structure \(\psi\) on \(\mathcal{G}\) is called shift-equivariant if its pairings satisfy:

\[
\langle f[1], g[1]\rangle_{a[1], b[1]} = (-1)^{D+1}\langle f, g\rangle_{ab} \quad \forall f \in \text{Hom}_{\mathcal{G}}(a, b) , \quad \forall g \in \text{Hom}_{\mathcal{G}}(b, a) .
\]

Using the selection rule (A.7), this becomes:

\[
\langle f[[1]], g[[1]]\rangle_{a[[1]], b[[1]]} = -\langle f, g\rangle_{ab} \quad \forall f \in \text{Hom}_{\mathcal{G}}(a, b) , \quad \forall g \in \text{Hom}_{\mathcal{G}}(b, a) .
\]
When \( \mathcal{G} \) is unital, the shift equivariance condition takes the following form in terms of traces:

\[
\text{tr}_{a[1]}(f[1]) = (-1)^{D+1}\text{tr}_a(f) \Leftrightarrow \text{tr}_{a[1]}(f[1]) = -\text{tr}_a(f)
\]

Recall that \( \psi \) can be identified with a duality structure \( ([D], \phi) \) on \( \mathcal{G} \), whose pairings and traces we denote by \( (\ )_{ab} \) and \( \text{tr}_a \) (the traces are defined only in the unital case). Naturality and shift-equivariance amount to the conditions:

\[
(u \circ f, v)_{a'b} = (u, f[[D]] \circ v)_{a,b} \quad \forall f \in \text{Hom}_\mathcal{G}(a', a), \ u \in \text{Hom}_\mathcal{G}(a, b), \ v \in \text{Hom}_\mathcal{G}(b, a'[D]) \tag{A.13}
\]

\[
(u, v)_{a,b} = (v, u[[D]]b,a[D]) , \quad \forall u \in \text{Hom}_\mathcal{G}(a, b) \quad \forall v \in \text{Hom}_\mathcal{G}(b, a[D]) \tag{A.14}
\]

\[
(u[1], v[1])_{a[1][1]} = (-1)^{D+1}(u, v)_{ab} \quad \forall u \in \text{Hom}_\mathcal{G}(a, b) \quad \forall v \in \text{Hom}_\mathcal{G}(b, a[D]) \ , \tag{A.15}
\]

or, in terms of traces:

\[
\text{tr}_a(u \circ v) = (-1)^{deg_u \deg_v}\text{tr}_b(v[[D]] \circ u) , \quad \forall v \in \text{Hom}_\mathcal{G}(a, b) , \quad \forall u \in \text{Hom}_\mathcal{G}(b, a[D]) \tag{A.16}
\]

\[
\text{tr}_{a[1]}(u[1]) = (-1)^{D+1}\text{tr}_a(u) , \quad \forall u \in \text{Hom}_\mathcal{G}(a, a[D]) \ . \tag{A.17}
\]

When \( \mathcal{G} \) is unital, the shift-equivariance condition \( (A.17) \) means that \( ([D], \phi, (\eta^{(a)})^D) \) is a graded duality structure on \( (\mathcal{G}, [1]) \). Thus:

A shift-equivariant \( D \)-cyclic structure on a unital graded category with shifts \( (\mathcal{G}, [1]) \) amounts to a graded duality structure of the form \( ([D], \phi, (\eta^{(a)})^D) \) on \( (\mathcal{G}, [1]) \).

Restricting to degree zero morphisms gives a \( D \)-cyclic structure on \( \mathcal{G}^0 \) which satisfies the shift-equivariance condition:

\[
(u[1], v[1])_{a[1][1]} = (-1)^{D+1}(u, v)_{ab} \Leftrightarrow \text{tr}_{a[1]}(u[1]) = (-1)^{D+1}\text{tr}_a(u) .
\]

This justifies the following:

**Definition** A shift-equivariant \( D \)-cyclic structure on a unital associative category with shifts is a graded duality structure \( (S, \phi, \eta) \) such that \( (S, \eta) = s^D = ([D], (\eta^{(a)})^D) \) as graded functors. In this case, \( (S, \phi) \) is a \( D \)-cyclic structure on \( (\mathcal{A}, [1]) \).

Giving a shift-equivariant \( D \)-cyclic structure on \( (\mathcal{A}, [1]) \) amounts to giving bilinear forms \( (\ )_{ab} : \text{Hom}_\mathcal{A}(a, b) \otimes \text{Hom}_\mathcal{A}(b, a[D]) \rightarrow \mathbb{C} \) which obey the conditions:

\[
(u \circ f, v)_{a'b} = (u, f[D] \circ v)_{a,b} \quad \forall f \in \text{Hom}_\mathcal{G}(a', a), \ u \in \text{Hom}_\mathcal{G}(a, b), \ v \in \text{Hom}_\mathcal{G}(b, a'[D]) \tag{A.18}
\]

\[
(u, v)_{a,b} = (v, u[D]b,a[D]) , \quad \forall u \in \text{Hom}_\mathcal{G}(a, b) \quad \forall v \in \text{Hom}_\mathcal{G}(b, a[D]) \tag{A.19}
\]

\[
(u[1], v[1])_{a[1][1]} = (-1)^{D+1}(u, v)_{ab} \quad \forall u \in \text{Hom}_\mathcal{G}(a, b) \quad \forall v \in \text{Hom}_\mathcal{G}(b, a[D]) \ , \tag{A.20}
\]
or, equivalently, to giving traces \( \text{tr}_a : \text{Hom}_A(a, b[D]) \to \mathbb{C} \) which satisfy:

\[
\text{tr}_a(u \circ v) = (-1)^{\text{deg}_u \text{deg}_v} \text{tr}_b(v[D] \circ u), \quad \forall v \in \text{Hom}_A(a, b), \quad \forall u \in \text{Hom}_A(b, a[D])
\]

(A.21)

\[
\text{tr}_a([1]) = (-1)^{D+1} \text{tr}_a(u), \quad \forall u \in \text{Hom}_A(a, a[D])
\]

(A.22)

Given a shift-equivariant \( D \)-cyclic structure \( ([D], \phi, (\eta^{(s)})^D) \) on \((A, [1])\), its graded completion gives a shift-equivariant \( D \)-cyclic structure on \((A^\bullet, [1])\). Indeed, we have \( s^\bullet = ([D], (\eta^{(s)})^D)^\bullet = ([D], (\eta^{(s)})^D) \). Thus:

**Observation** Given a shift-equivariant \( D \)-cyclic structure on an associative category with shifts \((A, [1])\) amounts to giving a shift-equivariant \( D \)-cyclic structure on the graded category with shifts \((A^\bullet, [1])\). Moreover, the cyclic structure on \((A, [1])\) is non-degenerate iff the cyclic structure on \((A^\bullet, [1])\) is nondegenerate.

Hence the notion of shift-equivariant cyclic structure is well-behaved under the inverse equivalences \( ^\bullet \) and \(^0\) of Subsection A.3.

### A.5 Equivalence of cyclic structures.

Given a unital category \( A \), we let \( Z(A) \) denote its center, defined as the unital associative \( \mathbb{C} \)-algebra of endomorphisms of the identity functor \( \text{id}_A \). Its elements are given by collections \( f = (f_a)_{a \in \text{Ob}_A} \) with \( f_a \in \text{Hom}_A(a, a) \) such that \( f_b \circ u = u \circ f_a \) for all \( u \in \text{Hom}_A(a, b) \) and any objects \( a, b \) of \( A \). The invertible elements under multiplication form the group of automorphisms \( \text{Aut}(\text{id}_A) \). They are given by collections \( f \) as above with the supplementary condition that \( f_a \in \text{Aut}_A(a) \) for all \( a \). When \( A \) has shifts, an element \( f \in \text{Aut}(\text{id}_A) \) is called shift-invariant if \( f_{a[1]} = f_a \) for all \( a \); such elements form a subgroup \( \text{Aut}_{\text{id}}(\text{id}_A) \).

Let \( T^D(A, [1], A) \) be the set of all \( D \)-cyclic structures on \((A, [1], A)\) and \( T^D_{\text{se}}(A, [1], A) \)
\( T^D_{\text{id}}(A, [1], A) \) be the subsets of those \( D \)-cyclic structures which are shift-equivariant respectively nondegenerate (the latter is defined when \( A \) is Hom-finite).

We say that two \( D \)-cyclic structures \( tr, tr' \) on \((A, [1])\) are equivalent if there exists \( f \in \text{Aut}(\text{id}_A) \) such that \( tr'_a(u) := tr_a(u \circ f_a) = tr_a(f_{a[D]} \circ u) \) for all \( u \in \text{Hom}_A(a, a[D]) \).

In this case, we write \( tr' \approx tr \), which defines an equivalence relation on \( T(A, [1]) \).
When \( tr \) and \( tr' \) are shift-equivariant, we say that they are \textit{graded equivalent} if there exists \( f \in \text{Aut}_{si}(\text{id}_A) \) with the property above. In this case, we write \( tr' \approx_{gr} tr \), giving an equivalence relation on \( T_{se}(A, [1]) \). It is clear that \( \approx \) and \( \approx_{gr} \) are compatible with nondegeneracy and that \( tr \approx_{gr} tr' \Rightarrow tr \approx tr' \). We set \( T^D(A, [1]) := T^D(A, [1])/\approx \) as well as \( T^D_{se}(A, [1]) := T^D_{se}(A, [1])/\approx_{gr} \) and \( T^D_{nd}(A, [1]) := T^D_{nd}(A, [1])/\approx \).

The Yoneda lemma implies that any two non-degenerate \( D \)-cyclic structures \( tr, tr' \) on \( (A, [1]) \) are equivalent through a uniquely-determined \( f \in \text{Aut}(\text{id}_A) \). When the cyclic structures are also shift-invariant, we must have \( f \in \text{Aut}_{si}(\text{id}_A) \), so in this case equivalence implies graded equivalence. Thus \( T^D_{nd}(A, [1]) \) and \( [T^D_{nd}(A, [1]) \cap T^D_{se}(A, [1])]/\approx_{gr} \) have a single element.

\[ A.6 \text{ Transport of cyclic structures} \]

\textbf{Morphisms of graded functors.} Let \( (A, [1]_A) \) and \( (B, [1]_B) \) be two unital associative categories with shifts. Given two graded functors \( F, G : (A, [1]_A) \to (B, [1]_B) \), a \textit{morphism of graded functors} from \( F \) to \( G \) is a natural transformation \( \phi : F \to G \) such that the following diagram commutes for all \( a \in \text{Ob}A \):

\[
\begin{array}{ccc}
F(a[1]) & \xrightarrow{\phi_a} & F(a)[1] \\
\phi_{a[1]} \downarrow & & \downarrow \phi_a[1] \\
G(a[1]) & \xrightarrow{\phi^G_a} & G(a)[1]
\end{array}
\]

An isomorphism of graded functors is a morphism of graded functors such that all \( \phi_a \) are isomorphisms. We say that two graded functors \( F, G \) are \textit{graded isomorphic} if there exists an isomorphism of graded functors from \( F \) to \( G \); in this case, we write \( F \approx_{gr} G \). This defines an equivalence relation on the class of all graded functors from \( (A, [1]_A) \) to \( (B, [1]_B) \). We say that \( F \) and \( G \) are \textit{ungraded isomorphic} if they are isomorphic as usual functors; we write this weaker equivalence relation as \( F \approx G \). Of course, we have \( F \approx_{gr} G \Rightarrow F \approx G \).

\textbf{Graded equivalence and weak graded equivalence.} A graded functor \( F : (A, [1]_A) \to (B, [1]_B) \) is called an \textit{equivalence of graded categories} (or graded equivalence) if there exists a graded functor \( H : (B, [1]_B) \to (A, [1]_A) \) such that \( F \circ H \approx_{gr} \text{id}_B \) and \( H \circ F \approx_{gr} \text{id}_A \). If there exists a graded equivalence from \( (A, [1]_A) \) to \( (B, [1]_B) \), then we write \( (A, [1]_A) \approx_{gr} (B, [1]_B) \) and say that \( (A, [1]_A) \) and \( (B, [1]_B) \) are \textit{graded-equivalent}.

A \textit{weak graded equivalence} is a graded functor \( F : (A, [1]) \to (B, [1]) \) with the property that that there exist a graded functor \( H : (B, [1]) \to (A, [1]) \) such that \( F \circ H \approx \text{id}_B \) and \( H \circ F \approx \text{id}_A \) (notice that these relations involve usual isomorphism of functors rather than isomorphism of graded functors). In particular, \( F \) is an equivalence of
unital associative categories. Given two graded functors \( F, G \) from \( A \) to \( B \), we have the obvious implication \( F \approx_{gr} G \Rightarrow F \approx G \).

**Pull-back of cyclic structures**  Let \((A, [1]_A)\) and \((B, [1]_B)\) be two unital associative categories with shifts and \( F : (A, [1]) \rightarrow (B, [1]) \) a graded functor, with grading isomorphism \( \eta_F : F \circ [1]_A \rightarrow [1]_B \circ F \). Given a \( D \)-cyclic structure \( tr \) on \( B \), define maps \( tr^D_a : \text{Hom}_A(a,a[D]) \rightarrow C \) through:

\[
tr_a^F(u) := tr_{F(a)}((\eta_a^F)^D \circ F(u))
\]

Using naturality of \( \eta^F \) and the cyclicity property of \( tr \), it is not hard to check that \( tr^F_a \) define a \( D \)-cyclic structure on \((A, [1]_A)\). We call this the pull-back of \( tr \) through \( F \). When \( tr \) is shift-equivariant, then it is not hard to check that \( tr^F \) is again shift-equivariant. When \( F \) is fully faithful and \( tr \) is nondegenerate, then \( tr^F \) is nondegenerate. Define a map \( F^* : T^D(B, [1]_B) \rightarrow T^D(A, [1]_A) \) through \( F^*(tr) := tr^F \). The remarks above show that \( F^*([T^D_{se}(B, [1]_B)]) \subset T^D_{se}(A, [1]_A) \); when \( F \) is fully faithful, we also have \( F^*([T^D_{nd}(B, [1]_B)]) \subset T^D_{nd}(A, [1]_A) \). The pull-back of cyclic structures has the following properties, whose detailed proofs we leave to the reader:

I. Given graded functors \( F : (A, [1]_A) \rightarrow (B, [1]_B) \) and \( G : (B, [1]_B) \rightarrow (C, [1]_C) \) and a \( D \)-cyclic structure \( tr \) on \((C, [1]_C)\), we have:

\[
tr^{G \circ F} = (tr^G)^F.
\]

Thus \((G \circ F)^* = F^* \circ G^*\).

II. Given two graded functors \( F, G : (A, [1]_A) \rightarrow (B, [1]_B) \) and a \( D \)-cyclic structure on \((B, [1]_B)\), we have the implication:

\[
F \approx_{gr} G \Rightarrow tr^F = tr^G.
\]

In particular, a graded equivalence \((A, [1]_A) \approx_{gr} (B, [1]_B)\) induces a bijection\(^{23}\) \( T^D(A, [1]_A) \approx T^D(B, [1]_B) \). This bijection is compatible with restriction to the subsets of shift-equivariant respectively nondegenerate cyclic structures, giving bijections \( T^D_{se}(A, [1]_A) \approx T^D_{se}(B, [1]_B) \) and \( T^D_{nd}(A, [1]_A) \approx T^D_{nd}(B, [1]_B) \).

III. Let \( F : (A, [1]) \rightarrow (B, [1]) \) be a faithful graded functor and \( tr, tr' \) be two \( D \)-cyclic structures on \((B, [1])\). Then:

\[
tr \approx tr' \Rightarrow tr^F \approx tr'^F.
\]

\(^{23}\)Namely \( F^* \circ H^* = \text{id}_{T^D(B, [1]_B)} \) and \( H^* \circ F^* = \text{id}_{T^D(A, [1]_A)} \) for a graded quasi-inverse pair of graded functors \( A \xrightarrow{F} B \rightarrow C \).
It follows that $F$ and $G$ determine the same map $\hat{F} = \hat{G}$ from $T^D(B, [1]_B)$ to $T^D(A, [1]_A)$.

**Sketch of proof.** Indeed, if $g \in \text{Aut}(\text{id}_A)$ satisfies $tr'_A(U) = tr_A(U \circ g_A)$ for all $U \in \text{Hom}_B(A, A[D])$, then the element $f$ of $\text{Aut}(\text{id}_A)$ defined through:

$$F(f_a) = g_{F(a)} \\forall a \in \text{Ob} A$$

satisfies $tr'_a(U) = tr_F(U \circ f_a)$ for all $u \in \text{Hom}_A(a, a[D])$. It follows that $F^*$ descends to a well-defined map $\hat{F} : T^D(B, [1]_B) \to T^D(A, [1]_A)$.

IV. Let $F, G : (A, [1]) \to (B, [1])$ be two weak graded equivalences and $tr$ a $D$-cyclic structure on $(B, [1]_B)$. Then:

$$F \approx G \Rightarrow tr^F \approx tr^G .$$

**Sketch of proof.** Fixing a usual isomorphism of functors $\phi : F \to G$ (which need not be an isomorphism of graded functors), we have $tr^F_a(u) = tr^G_a(u \circ f_a) = tr^G_a(f_{a[D]} \circ u)$, where $f \in \text{Aut}(\text{id}_A)$ is determined by the formula:

$$G(f_{a[D]}) = \phi_{a[D]} \circ (\eta^{F}_a)^D - 1 \circ (\phi_a[D]) - 1 \circ (\eta^{G}_a)^D .$$

V. A weak graded equivalence $F : (A, [1]) \to (B, [1])$ induces a bijection $\hat{F} : T^D(B, [1]_B) \xrightarrow{\approx} T^D(A, [1]_A)$. Indeed, we have $F \circ H = \text{id}_B$ and $H \circ F = \text{id}_A$ for some graded functor $H : (B, [1]) \to (A, [1])$, which gives $\hat{F} \circ \hat{G} = \text{id}_{T^D(B, [1])}$ and $\hat{G} \circ \hat{F} = \text{id}_{T^D(A, [1])}$.

**A.7 The triangulated case**

Let $\mathcal{T}$ be a triangulated category, endowed with its shift functor $[1]$. A pre-Serre duality structure on $\mathcal{T}$ is a graded duality structure $(S, \phi, \eta)$ such that $S$ is exact (in particular, it preserves finite direct sums). When $\mathcal{T}$ is Hom-finite, a Serre duality structure on $\mathcal{T}$ is a pre-Serre structure which is nondegenerate as a duality structure. In this case, $S$ is called a Serre functor [33]. Given an integer $D$, the category $\mathcal{T}$ is called $D$-Calabi-Yau if it admits a Serre structure $(S, \phi, \eta)$ such that $(S, \eta) \approx (s^D, (\eta^s)^D)$ as graded functors. In this case, $(S, \phi, \eta)$ is a nondegenerate and shift-equivariant $D$-cyclic structure on $\mathcal{T}$. In this subsection, a (co)homological functor on $\mathcal{T}$ means a linear Vect-valued (co)homological functor which preserves direct sums.

Let $\mathcal{T}$ be a triangulated category and $\mathcal{U} \subset \text{Ob} \mathcal{T}$, and set $\mathbb{Z} \mathcal{U} := \{ a[n] | a \in \mathcal{U}, n \in \mathbb{Z} \}$. We recall the following:
Lemma  Assume that $\mathcal{U}$ triangle generates $\mathcal{T}$ or that $\mathcal{T}$ is idempotent complete and $\mathcal{U}$ Karoubi generates $\mathcal{T}$. Let $F, G : \mathcal{T} \to \text{Vect}$ be two homological or two cohomological functors on $\mathcal{T}$ and $\phi : F \to G$ a natural transformation. Then $\phi$ is an isomorphism of functors if the linear map $\phi_a \in \text{Hom}_C(F(a), G(a))$ is bijective for all $a \in \mathbb{Z}_U$.

Proof. The direct implication is obvious, while the inverse implication is a trivial application of the five-lemma. In the Karoubi case we also make use of the additivity of $F$ and $G$.

This implies the following criterion:

Proposition  Assume that $\mathcal{U}$ triangle generates $\mathcal{T}$ or that $\mathcal{T}$ is idempotent complete and $\mathcal{U}$ Karoubi generates $\mathcal{T}$. Then a graded duality structure $(S, \phi)$ on $\mathcal{T}$ is strictly non-degenerate iff the linear map $\phi_{ab} : \text{Hom}_\mathcal{T}(a, b) \to \text{Hom}_\mathcal{T}(b, S(a))^\vee$ is bijective for all $a, b \in \mathbb{Z}_U$.

Proof. The direct implication is obvious. For the inverse implication, let $F, G : \mathcal{T} \to \text{Vect}$ be the linear bifunctors defined through $F(a, b) = \text{Hom}_\mathcal{T}(a, b)$ and $G(a, b) = \text{Hom}_\mathcal{T}(b, S(a))^\vee$ (with the obvious actions on morphisms). Fixing $a \in \mathbb{Z}_U$ gives homological functors $F_a := F(a, \cdot)$ and $G_a := \text{Hom}_\mathcal{T}(\cdot, S(a))^\vee$ on $\mathcal{T}$, related by the natural transformation $\phi_a := \phi_a$. Applying the lemma, we find that $\phi_a$ is an isomorphism of functors for all $a \in \mathbb{Z}_U$ so $\phi_{ab}$ is bijective for all $a \in \mathbb{Z}_U$ and $b \in \mathcal{T}$. Now fix $b \in \mathcal{T}$ and consider the cohomological functors $F^b := F(\cdot, b)$ and $G^b := \text{Hom}_\mathcal{T}(b, S(\cdot))^\vee$ on $\mathcal{T}$. These are related by the natural transformation $\phi_{\cdot, b}$, which gives gives bijections $\phi_{ab}$ when $a \in \mathbb{Z}_U$. A second application of the lemma shows that $\phi_{ab}$ is a bijection for all $a, b \in \text{Ob}\mathcal{T}$.

Taking $S = s^D$ (with $s$ the twisted shift functor of $\mathcal{T}$), the proposition translates as follows into the language of cyclic structures.

Corollary  Assume that $\mathcal{U}$ triangle generates $\mathcal{T}$ or that $\mathcal{T}$ is idempotent complete and $\mathcal{U}$ Karoubi generates $\mathcal{T}$. Then a shift-equivariant $D$-cyclic structure on $(\mathcal{T}, [1])$ specified by the bilinear forms $(\cdot, \cdot)_{ab}$ is non-degenerate iff $(\cdot, \cdot)_{ab} : \text{Hom}_\mathcal{T}(a, b) \times \text{Hom}_\mathcal{T}(b, a[D]) \to \mathbb{C}$ are non-degenerate for all $a, b \in \mathbb{Z}_U$.

Notice that the condition in the last proposition is equivalent to nondegeneracy of the pairing $(\cdot, \cdot)_{ab} : \text{Hom}_\mathcal{T^*}(a, b) \times \text{Hom}_\mathcal{T^*}(b, a) \to \mathbb{C}$ for all $a, b \in \mathcal{U}$.
B. Symmetric $\infty$-inner products on an $A_\infty$ algebra

In this appendix, we consider the notion of symmetric $\infty$-inner product on an $A_\infty$ algebra $A$. After reviewing the geometric description of $A_\infty$ algebras in terms of formal noncommutative Q-manifolds (FNQ-manifolds) [17, 19, 5], we define a symmetric $\infty$-inner product on $A$ as a noncommutative pre-symplectic form on the associated FNQ-manifold. A symmetric $\infty$-inner product can also be viewed as a countable sequence of multilinear maps on $A$ which are compatible with the $A_\infty$ products and obey certain graded symmetry conditions, showing that this notion is a particular case of that considered in [36]. The $\infty$-inner product is called flat when all higher multilinear maps vanish, so one is left with a bilinear pairing on $A$, which is graded symmetric and compatible with the $A_\infty$ products; this corresponds to a cyclic pairing as considered in the body of the paper as well as in [3, 4, 5]. Hence $\infty$-inner products generalize cyclic pairings. Their main advantage over the latter is that a morphism of $A_\infty$ algebras pulls-back $\infty$-inner products to $\infty$-inner products, a statement which fails in general for cyclic pairings. After discussing pull-backs of symmetric $\infty$-inner products through $A_\infty$ morphisms, we recall the noncommutative Darboux theorem of [17] and introduce nondegeneracy and homological nondegeneracy conditions, showing that homological nondegeneracy is preserved when pulling back through $A_\infty$ quasi-isomorphisms. We also show that a dGA endowed with a cyclic pairing admits a cyclic minimal model, i.e a minimal model on which the pairing transports to a flat $\infty$-product. This result generalizes the ‘gauge-fixing’ construction of a cyclic minimal model (Section 3.3) and is used in Section 4.

B.1 $A_\infty$ algebras as formal noncommutative Q-manifolds

The bar construction. Consider a $\mathbb{Z}$-graded $A_\infty$ algebra $A$ with (degree one) suspended infinity products $r_n : A[1]^{\otimes n} \to A[1]$ ($n \geq 1$). As usual, we let $|x|$ denote the degree of homogeneous elements $x \in A$, and $\tilde{x} := |x| - 1$ denote the suspended degree (the degree of $x$ as an element of $A[1]$). We also let $m_n$ be defined through $r_n := s \circ m_n \circ (s^{-1})^{\otimes n}$, where $s : A \to A[1]$ is the suspension operator.

It is well-known that the $A_\infty$ products are equivalently encoded by a degree +1 codifferential $\delta$ on the graded coassociative coalgebra $\bar{T}(A[1]) := \oplus_{n \geq 1} A[1]^{\otimes n}$, known as the reduced tensor coalgebra of the vector space $A[1]$. The suspended products $r_n$ can be recovered through:

$$r_n := p_1 \circ \delta|_{A[1]^{\otimes n}},$$

where $p_1 : T(A[1]) \to A[1]$ is the projection on the first component of $TA[1]$. The differential graded coalgebra $BA := (\bar{T}(A[1]), \delta)$ is known as the bar dual of $A$. As explained in [9], the bar construction provides a functor $B : \text{Alg}_{\infty} \to \text{Cgc}$ from the
category $\text{Alg}_{\infty}$ of $A_{\infty}$ algebras to the category $\text{Cogc}$ of cocomplete\textsuperscript{24} differential graded coalgebras. This functor induces an equivalence between $\text{Alg}_{\infty}$ and the full subcategory $\text{Cogtr}$ of $\text{Cogc}$ consisting of those objects whose underlying graded coalgebra is isomorphic with the reduced tensor coalgebra of a graded vector space. The category $\text{Cogc}$ admits a Quillen model structure with respect to which all objects are cofibrant. With respect to this structure, an object of $\text{Cogc}$ is fibrant iff it lies in $\text{Cogtr}$. Hence the category $\text{Alg}_{\infty}$ of $A_{\infty}$ algebras identifies via the bar construction with the full subcategory $\text{Cogtr}$ of fibrant objects in $\text{Cogc}$. Similarly, the category $\text{Alg}_{\infty}$ admits a ‘model structure without limits’\textsuperscript{25} whose weak equivalences are the $A_{\infty}$ quasi-isomorphisms, and whose cofibrations/fibrations are the $A_{\infty}$ morphisms $\varphi : A_1 \to A_2$ such that $\varphi_1$ is a monomorphism/epimorphism. The bar construction maps these into weak equivalences, respectively into cofibrations/fibrations of $\text{Cogc}$ [9].

Two morphisms $F, G : C_1 \to C_2$ of differential graded coalgebras are called \textit{(classically) homotopy equivalent} if there exists a degree -1 linear map $H : C_1 \to C_2$ such that:

$$\Delta_2 \circ H = (F \otimes H + H \otimes G) \circ \Delta_1 \quad \text{and} \quad F - G = \delta_2 \circ H + H \circ \delta_1,$$

where $\Delta_i$ are the comultiplications on $C_i$. When $C_1$ and $C_2$ belong to $\text{Cogtr}$, it is shown in [9] that $F$ and $G$ are homotopy equivalent iff they are left homotopy equivalent in the sense of model categories. Moreover, the bar functor $B$ interchanges homotopy equivalent morphisms of $A_{\infty}$ algebras with homotopy equivalent morphisms of coalgebras. These results imply that homotopy equivalence of $A_{\infty}$ algebras is an equivalence relation, and that a morphism of $A_{\infty}$ algebras is a quasi-isomorphism iff it is a homotopy equivalence. In particular, any quasi-isomorphism of $A_{\infty}$ algebras admits an inverse up to homotopy. Moreover, the bar dual of the minimal model of an $A_{\infty}$ algebra is a minimal model in $\text{Cogc}$ (in the sense of model categories) of the bar dual of the original $A_{\infty}$ algebra. These results give a complete description of $A_{\infty}$ algebras in the language of cocomplete differential graded coalgebras.

\textbf{Description through formal noncommutative $Q$-manifolds.} The geometric interpretation of $A_{\infty}$ algebras arises by further dualizing this picture [17, 19]. We say that a graded associative $\mathbb{C}$-algebra is \textit{formal} if it is the inverse limit of an inverse system of nilpotent and finite-dimensional associative graded $\mathbb{C}$-algebras. Such algebras are topological algebras with respect to the inverse limit topology. They form a category whose

\begin{itemize}
  \item \textsuperscript{24}A coassociative coalgebra $(C, \delta)$ is called cocomplete if $\cup_n C_n = C$, where $C_n = \ker \Delta^{(n+1)}$ are the components of the so-called primitive filtration $C_1 \subset C_2 \subset \ldots \subset C$. Here $\Delta^{(2)} = \Delta$ is the comultiplication of $C$ and $\Delta^{(n+1)} := (\mathbb{id} \otimes^{n-1} \otimes \Delta) \circ \Delta^{(n)} : C \to C \otimes^{(n+1)}$ for all $n \geq 2$.
  \item \textsuperscript{25}I.e. all axioms of a model category are satisfied except for the existence of finite limits and colimits, which is replaced by a weaker axiom (see [9]).
\end{itemize}
morphisms are continuous morphisms of graded associative algebras. Given such an algebra $B$ we let $\text{Der}_l(B)$ denote the space of its \textit{continuous} left derivations. As explained in the appendix of [19], the category of cocomplete graded coalgebras is antiequivalent with the category of formal graded algebras, an antiequivalence being given by taking the vector space dual $C \to C'' := \text{Hom}_\mathbb{C}(C, \mathbb{C})$, and inverse antiequivalence given by taking the dual as a topological vector space, an operation which we denote by $\ast$. A \textit{formal noncommutative $Q$-manifold} (FNQ-manifold) is a pair $(B, Q)$ where $B$ is a formal graded $\mathbb{C}$-algebra whose dual coalgebra $B^\ast$ belongs to $\text{Cogtr}$ (i.e. is a reduced tensor coalgebra) and $Q \in \text{Der}_l(B)$ is a \textit{homological derivation}, i.e. a continuous left derivation of degree $-1$ which squares to zero. A morphism of FNQ-manifolds is a morphism of formal graded algebras which commutes with the homological derivations. Applying the dualization functor, we find that the category $\text{Cogtr}$ of fibrant cocomplete differential graded coalgebras is antiequivalent with the category of FNQ-manifolds, with inverse anti-equivalence given by taking the topological dual. The results of [9] recalled above now translate trivially into the dual language of formal NQ-manifolds. In particular, we find the following dual description of quasi-isomorphisms of $A_\infty$ algebras. Given two formal noncommutative $Q$-manifolds $B_2$ and $B_1$, two morphisms $f, g : B_2 \to B_1$ of noncommutative $Q$-manifolds are called \textit{homotopy equivalent} if there exists a degree $+1$ linear map $h : B_2 \to B_1$ such that:

$$h(xy) = (-1)^{\tilde{x}} f(x) h(y) + h(x) g(y) \quad \text{and} \quad f - g = Q_1 \circ h + h \circ Q_2.$$ 

The morphism $f$ is called a homotopy equivalence if it becomes invertible in the category obtained by taking homotopy equivalence classes of morphisms. The discussion above shows that quasi-isomorphisms (a.k.a homotopy equivalences) of $A_\infty$ algebras from $A_1$ to $A_2$ correspond to homotopy equivalences of formal noncommutative $Q$-manifolds from $(BA_2[1])'$ to $(BA_1[1])'$.

**B.2 Symmetric $\infty$ -inner products**

\textbf{Noncommutative Cartan calculus.} Given an $A_\infty$ algebra $A$, let $B := B'A[1] := (BA[1])' = \prod_{n \geq 1} \text{Hom}_\mathbb{C}(A[1] \otimes^n, \mathbb{C})$ denote the corresponding formal NQ-manifold, endowed with the homological vector field $Q := \delta^v$. Notice that $B$ is bigraded\(^{26}\); we denote the grading induced from $A[1]$ by a tilde and place it in first position; the ‘tensor grading’ is placed in the second position. Let $\Omega B$ be the formal dGA of noncommutative forms over $B$, whose differential we denote by $D$. Consider the Karoubi complex $\mathcal{C}(B) = \Omega B/\left[\Omega B, \Omega B\right]$ of $B$, whose differential (induced from $B$) we again denote by $d$. Notice that $\Omega B$ is trigraded, with the rank grading placed in first position,

\(^{26}\)All gradings are in the sense of direct \textit{product} decompositions.
grading induced from $A[1]$ (tilde grading) in second position and grading induced from the tensor grading of $B$ in third position. We let $C^n(B)$ be the homogenous components of $C(B)$ with respect to the form rank grading. Notice that each $C^n(B)$ carries the bigrading induced from $B$. We let $\pi : \Omega B \to C(B)$ be the canonical surjection, and use the notation $(\omega)_c := \pi(\omega)$ for any $\omega \in \Omega B$. We also let $\text{Der}_h^g(B)$ (resp. $\text{Der}_i^{(h,g)}(\Omega B)$) be the spaces of continuous left derivations of $B$ (resp $\Omega B$) which are homogeneous of tilde degree $g$ respectively of bidegree $(h, g)$ with respect to the (rank, tilde) bigrading. Given a homogeneous derivation $\theta \in \text{Der}^g(B)$, we have well-defined derivations $i_\theta \in \text{Der}_i^{(1, \tilde{g})}(\Omega B)$ and $L_\theta \in \text{Der}_i^{(0, \tilde{g})}(\Omega B)$ which play the role of contraction and Lie derivative. These are uniquely determined by the conditions (in the conventions of [5]):

$$i_\theta(x) = 0 \ , \ i_\theta(dx) = \theta(x) \ \forall x \in B$$

and:

$$L_\theta(x) = \theta(x) \ , \ L_\theta(dx) = d\theta(x) \ \forall x \in B \ .$$

They descend to well-defined operators on the Karoubi complex, which we denote by the same letters. The operators $d, L_\theta$ and $i_\theta$ satisfy the classical identities:

$$L_\theta = [i_\theta, d] \ , \ [L_\theta, i_\gamma] = i_{[\theta, \gamma]} \ , \ [L_\theta, L_\gamma] = L_{[\theta, \gamma]} \ , \ [i_\theta, i_\gamma] = [L_\theta, d] = 0 \ .$$

In particular, one finds $[L_Q, d] = 0$ and $L_Q^2 = \frac{1}{2}[L_Q, L_Q] = \frac{1}{2}L_{[Q, Q]} = 0$, where $[,]$ denotes the graded commutator of continuous left derivations (which is again a continuous left derivation). Since $L_Q$ preserves form rank, this allows us to consider the homology of $L_Q$ on $C^n(B)$ and on the subspaces $C^n(B)_{cl}$ of closed $n$-forms; we will denote these by $H_{L_Q}(\ldots)$, where $\ldots$ stands for the corresponding complex. We let $Z_{L_Q}(C^n(B))$ etc. denote the corresponding spaces of $L_Q$-cycles. A morphism $\phi : B_1 \to B_2$ of formal non-commutative Q-manifolds (i.e. a continuous morphism of the underlying topological algebras) induces a morphism of differential graded algebras $\phi_* : \Omega B_1 \to \Omega B_2$. It is easy to check that this obeys all expected properties, in particular $d_2 \circ \phi_* = \phi_* \circ d_1$ and $L_{Q_2} \circ \phi_* = \phi_* \circ L_{Q_1}$. The last identities imply that $\phi_*$ descends to a well-defined linear map from $H_{L_{Q_1}}(C(B_1))$ to $H_{L_{Q_2}}(C(B_2))$, denoted by $\hat{\phi}$. Moreover, $\hat{\phi}$ maps $H_{L_{Q_1}}(C(B_1)_{cl})$ into $H_{L_{Q_2}}(C(B_2)_{cl})$. We also have:

$$L_{\hat{\phi} \circ \theta \circ \phi_*^{-1}} = \phi_* \circ L_\theta \circ \phi_*^{-1}$$
$$i_{\hat{\phi} \circ \theta \circ \phi_*^{-1}} = \phi_* \circ i_\theta \circ \phi_*^{-1} \ ,$$

for any associative algebra automorphism $\phi$ of $B$. 

77
Geometric approach to symmetric $\infty$-inner products. A noncommutative presymplectic form on $B$ is a closed two-form $\omega \in C^2(B)_{cl}$ which is homogeneous (of degree $-\tilde{\omega}$) with respect to the grading induced from $A[1]$ (tilde grading). Given such a form, the pair $(B, \omega)$ is called a presymplectic $FNQ$-manifold. A symplectomorphism $\phi : (B_1, \omega^{(1)}) \to (B_1, \omega^{(2)})$ between presymplectic $FNQ$-manifolds is a morphism of $NQ$-manifolds such that $\phi^*(\omega^{(1)}) = \omega^{(2)}$.

Given a pre-symplectic form $\omega$, we can expand it as $\omega = \oplus_{n \geq 0} \omega_n$, where $\omega_n$ are its homogeneous components with respect to the tensor product grading. Explicitly, one has:

$$\omega_{n-2} = \frac{1}{n} \omega_{a_1 \ldots a_i-1 a_i+1 \ldots a_{n-1} a_n} (s^{a_1} \ldots s^{a_i-1} ds^{a_i} s^{a_{i+1}} \ldots s^{a_{n-1}} ds^{a_n})$$

where $(s^a)$ is a topological basis of $A[1]'$, suspended dual to a basis $(e_a)$ of $A$ (we use implicit summation over repeated indices). Here $\omega_{a_1 \ldots a_i-1 a_i+1 \ldots a_{n-1} a_n}$ are graded-cyclic complex coefficients, and the underline on the indices $a_i$ and $a_n$ indicates the position of $ds^{a_i}$ and $ds^{a_n}$ in the noncommutative differential monomial appearing in (B.1). These coefficients can be used to define homogeneous linear maps $\omega_{i,n} : A[1]^n \to \mathbb{C}[\tilde{\omega}]$ via the relations:

$$\omega_{i,n}(e_{a_1} \otimes \ldots \otimes e_{a_n}) := -\omega_{a_1 \ldots a_i-1 a_i+1 \ldots a_{n-1} a_n}$$

where the sign is chosen for agreement with [5]. Notice that $\omega_{i,n}$ satisfy the graded antisymmetry properties:

$$\omega_{i,n}(x_1 \otimes \ldots \otimes x_n) = (-1)^{1+(\tilde{x}_1+\ldots+\tilde{x}_i)(\tilde{x}_{i+1}+\ldots+\tilde{x}_n)} \omega_{n-i,n}(x_{i+1} \otimes \ldots \otimes x_n \otimes x_1 \otimes \ldots \otimes x_n)$$

Defining $\langle \langle \rangle_{i,n} : A^n \to \mathbb{C}[\tilde{\omega}+n]$ via $\omega_{i,n} = \langle \langle \rangle_{i,n} (s^{-1})^n$ for all $n \geq 2$ and $1 \leq i \leq n-1$ gives a countable sequence of homogeneous multilinear maps on $A$ which satisfy a graded symmetry property derived from (B.2). The integer $D := \tilde{\omega} + 2$ will be called the dimension of this collection of multilinear maps.

For $n = 2$, this gives a single pairing $\langle \langle \rangle_{1,2}$ which is a graded-symmetric bilinear form of degree $-D$ on $A$ and can be described invariantly as follows [5]. Expand $\omega_0 = -\frac{1}{2} \sum_i (df_i dg_i)$, with $f_i, g_i \in A[1]'$. Then $\langle \langle \rangle_{1,2}$ is given by:

$$\langle x, y \rangle_{1,2} = (-1)^{\tilde{x}} \sum_i f_i (x g_i(y)) \quad \forall x, y \in A$$

A pre-symplectic form is called $Q$-compatible if $L_Q \omega = 0$. In this case, the collection of maps $(\langle \langle \rangle_{i,n}$ satisfies a complicated series of compatibility conditions with the products $r_n$, which can be obtained by computing $L_Q \omega$. The collection $(\langle \langle \rangle_{i,n}$ associated to a $Q$-compatible presymplectic form will be called a symmetric $\infty$-inner product on $A$. The pair $(A, \langle \langle \rangle_{:.}) will be called a symmetric $A_{\infty}$ algebra.
A lengthy direct computation shows that a symmetric $\infty$-inner product is an $\infty$-inner product in the sense of [36], subject to the graded symmetry conditions derived from (B.2) (which are not required in loc. cit.) As explained there, giving such data amounts to giving a morphism of $A_\infty$ bimodules from $A$ to $A^\vee$, where $A$ and $A^\vee$ are viewed as $A$-bimodules in the obvious manner.

**Cyclic structures as flat symmetric $\infty$-inner products.** We say that $\omega$ and its $\infty$-inner product are flat if $\omega = \omega_0$, i.e. $\omega_n = 0$ for all $n \geq 1$. In this case, all multilinear forms $\langle \rangle_{i,n}$ vanish except for the pairing $\langle \rangle_{1,2}$, which for simplicity we shall denote by $\langle \rangle$. Moreover, equation $L_Q \omega = 0$ reduces [5] to the cyclicity conditions (3.12). Hence:

Giving a $D$-cyclic pairing on $A$ amounts to giving the following equivalent data:

(a) a flat symmetric $\infty$-inner product on $A$ of dimension $D$.

(b) a flat pre-symplectic form $\omega \in C^2(A)_d$, homogeneous of degree $-\tilde{\omega} = 2 - D$, which satisfies $L_Q \omega = 0$.

**Nondegeneracy conditions.** A symmetric $\infty$-inner product and its associated presymplectic form $\omega$ are called strictly nondegenerate if $A$ is finite-dimensional and the map $\theta \in \text{Der}_l(B) \rightarrow i_\theta \omega \in C^1(B)$ is a bijection. One can check by direct computation that the second condition amounts to the requirement that $\langle \rangle_{1,2}$ is nondegenerate. In this case, we say that the $\infty$-inner product is strictly nondegenerate and that the triplet $(B' A[1], Q, \omega)$ is a formal noncommutative symplectic manifold. A basic result for this case (originally due to [17]) is the noncommutative Darboux theorem, which states that there exists a (continuous) algebra automorphism $\phi : B' A[1] \rightarrow B' A[1]$ such that $\phi_* (\omega)$ is a flat symplectic form (notice that $\phi$ need not commute with $Q$). Defining $Q_1 := \phi \circ Q \circ \phi^{-1}$, this shows that $(B' A[1], Q, \omega)$ is isomorphic with $(B' A[1], Q_1, \phi_* (\omega))$ as noncommutative formal symplectic manifolds, i.e. as symmetric $A_\infty$ algebras. Because of this result, the theory of strictly nondegenerate symmetric $A_\infty$ algebras reduces to the theory of strictly nondegenerate cyclic $A_\infty$ algebras.

In terms of the products $m_n$ (defined through $r_n = s \circ m_n \circ (s^{-1})^\otimes n$), the first cyclicity condition in (B.2) takes the form $\langle m_1(x), y \rangle_{1,2} + (-1)^{|x|} \langle x, m_1(y) \rangle_{1,2} = 0$, which implies that $\langle \ , \ \rangle_{1,2}$ descends to a graded symmetric pairing on $H(A)$. We say that $\omega$ and the corresponding symmetric $\infty$-inner product are homologically nondegenerate if $A$ is compact and this pairing induced on $H(A)$ is nondegenerate. In the flat case, this notion reduces to homological nondegeneracy of cyclic pairings.

**B.3 Pull-back of symmetric $\infty$-inner products**

Any morphism of $A_\infty$ algebras $\varphi : A_2 \rightarrow A_1$ allows one to pull back a symmetric $\infty$-inner product $\langle \ , \ \rangle_\cdot$ from $A_1$ to $A_2$. Indeed, $\varphi$ corresponds to a morphism $\phi$ of formal
noncommutative $Q$-manifolds from $B'A_1[1]$ to $B'A_2[1]$, and $\phi_*$ takes closed forms to closed forms and $L_Q$-closed forms to $L_Q$-closed forms. We define the pullback of $\langle , \rangle_\omega$ through $\varphi$ to be the symmetric $\infty$-inner product $\langle , \rangle_{\omega}^{(\varphi)}$ associated with $\phi_*(\omega)$, where $\omega$ is the presymplectic form on $B'A_1[1]$ associated with $\langle , \rangle_\omega$. A symmetric $A_\infty$ morphism (or morphism of symmetric $A_\infty$ algebras) $\varphi : (A_2, \langle , \rangle_{\omega}^{(2)}) \to (A_1, \langle , \rangle_{\omega}^{(1)})$ is an $A_\infty$ morphism such that $\langle , \rangle_{\omega}^{(2)} = [\langle , \rangle_{\omega}^{(1)}]_{(\varphi)}$. This corresponds to a symplectomorphism (in the opposite direction) between the associated presymplectic $\text{FNQ}$-manifolds. A symmetric $A_\infty$ quasi-isomorphism is a morphism of symmetric $A_\infty$ algebras which is an $A_\infty$ quasi-isomorphism. The proposition below shows that pull-back through an $A_\infty$ quasi-isomorphism preserves homological nondegeneracy.

**Proposition** Let $A_1$ and $A_2$ be two $A_\infty$ algebras and $\varphi : A_2 \to A_1$ an $A_\infty$ quasi-isomorphism. If $\langle , \rangle_{\omega}^{(1)}$ is a homologically nondegenerate symmetric $\infty$-inner product on $A_1$, then its pullback through $\varphi$ is homologically nondegenerate.

*Proof.* Let $\omega^{(1)} \in Z_{LQ_1}(C^2(B'A_1[1])_d)$ and $\omega^{(2)} := \phi_*(\omega^{(1)}) \in Z_{LQ_2}(C^2(B'A_2[1])_d)$ be the $Q_j$-compatible presymplectic forms associated with $\langle , \rangle_{\omega}^{(1)}$ and its pull-back. We have to show that $\omega^{(2)}$ is homologically nondegenerate if $\omega^{(1)}$ is. Consider the expansion in components homogeneous with respect to the tensor grading $\omega^{(1)} = \sum_{n \geq 0} \omega^{(1)}_n$, where:

$$
\omega^{(2)}_n = \omega^{(2)}_{a_1 \ldots a_n} (s^{a_1} \ldots s^{a_n})
$$

is further expanded in a topological basis $(s^a)$ of $A_1[1]'$. Then $\omega^{(2)} = \sum_{n \geq 0} \omega^{(2)}_n$ with:

$$
\omega^{(2)}_n = \omega^{(2)}_{a_1 \ldots a_n} (\phi(s^{a_1}) \ldots \phi(s^{a_n}))
$$

Expanding: $\phi(s^a) = \sum_{m \geq 0} \phi_{a_1 \ldots a_m} \sigma^{a_1} \ldots \sigma^{a_m}$ with respect to a topological basis $(\sigma^a)$ of $A_2[1]'$, we find $\omega^{(1)} = \frac{1}{2} \omega^{(2)}_{ab} (d\sigma^a) (d\sigma^b)$, where $\omega^{(2)}_{ab} = \omega_{ab}^{(1)} \phi_{a}^{b}$. Hence the bilinear form on $A_2$ induced by the ‘constant term’ of $\omega^{(2)}$ takes the form:

$$
\langle x, y \rangle^{(2)}_0 = \langle \varphi_1(x), \varphi_1(y) \rangle^{(1)}_0 \quad \forall x, y \in A_2
$$

(B.3)

where $\langle x, y \rangle^{(1)}_0$ is the corresponding form on $A[1]$ induced by $\omega^{(1)}$ and $\varphi_1 : A_2 \to A_1$ is the first component of the $A_\infty$ morphism $\varphi$. The latter has the expansion $\varphi_1(e_a) = \phi_a^a$ in the bases $(e_a), (\epsilon_a)$ of $A_1, A_2$ suspended dual to $(s^a), (\sigma^a)$. Since $\varphi$ is an $A_\infty$ quasi-isomorphism, we know that $\varphi_1$ is a quasi-isomorphism of cochain complexes from $(A_2, m_1^{(2)})$ to $(A_1, m_1^{(1)})$. Moreover, the pairing $\langle , \rangle^{(1)}$ descends to a non-degenerate pairing on $H_{m_1^{(1)}}(A_1)$ since $\omega^{(1)}$ is homologically non-degenerate. Together with relation (B.3), these observations imply that $\langle , \rangle^{(2)}$ descends to a non-degenerate pairing on $H_{m_1^{(2)}}(A_2)$. Thus $\omega^{(2)}$ is homologically non-degenerate.
Cyclic minimal models as flat symmetric minimal models. A symmetric minimal model of the symmetric $A_{\infty}$ algebra $(A, \langle , \rangle, \cdot)$ is a minimal flat symmetric $A_{\infty}$ algebra $(A_{\text{min}}, \langle , \rangle, \cdot)$, together with a symmetric $A_{\infty}$ quasi-isomorphism $\varphi : (A_{\text{min}}, \langle , \rangle, \cdot) \to (A, \langle , \rangle, \cdot)$. The minimal model theorem for $A_{\infty}$ algebras implies that any symmetric $A_{\infty}$ algebra has a symmetric minimal model. The Darboux theorem implies that any flat compact and homologically nondegenerate symmetric $A_{\infty}$ algebra admits a flat symmetric minimal model:

**Proposition** Let $A$ be a compact $A_{\infty}$ algebra endowed with a homologically nondegenerate cyclic pairing $\langle \rangle$. Then there exists a finite-dimensional minimal model $A_{\text{min}}$ of $A$ and an $A_{\infty}$ quasi-isomorphism $\varphi : A_{\text{min}} \to A$ such that the pulled back symmetric $\infty$-inner product is a nondegenerate cyclic pairing on $A_{\text{min}}$.

**Proof.** Let $\omega$ be the presymplectic form on $B'A[1]$ determined by $\langle \rangle$. Pick any minimal model $A_m$ of $A$ and any quasi-isomorphism $\varphi_m : A_m \to A$ (for example, take any inverse up to homotopy of a quasi-isomorphism from $A$ to $A_m$). By the previous proposition, pulling back through $\varphi_m$ gives a $Q_m$-compatible presymplectic form $\omega_m := \phi_m^*(\omega)$ which is homologically nondegenerate. Since $A_m$ is minimal, $\omega_m$ is in fact symplectic. Applying the noncommutative Darboux theorem, we pick any $\varphi_0 \in \text{Aut}(A_m)$ and define $\varphi := \varphi_0 \circ \varphi_m$, $Q_{\text{min}} := \phi_0 \circ Q_m \circ \phi_0^{-1}$, $\omega_{\text{min}} := \phi_0^*(\omega_m) = \phi^*(\omega)$. Then the $A_{\infty}$ algebra $A_{\text{min}}$ defined by the formal noncommutative $Q$-manifold $(B'A_{\text{min}}[1], Q_{\text{min}})$ is symplectic with flat symplectic form $\omega_{\text{min}}$.

A minimal model endowed with a constant pre-symplectic form as in the proposition will be called a cyclic minimal model of $(A, \langle , \rangle)$. Since minimal models are unique up to $A_{\infty}$ isomorphism, it is clear that two cyclic minimal models are related by an isomorphism of cyclic $A_{\infty}$ algebras, i.e. an $A_{\infty}$ isomorphism which interchanges cyclic structures under pull-back.

**Observation** It follows from the results of [24] that a cyclic minimal model is determined by the $L_Q$-homology class of $\omega$ up to isomorphism of cyclic $A_{\infty}$ algebras.

**References**

[1] C. I. Lazaroiu, “On the structure of open-closed topological field theory in two dimensions,” Nucl. Phys. B 603 (2001) 497 [arXiv:hep-th/0010269].

\[27\text{Minimal as an } A_{\infty} \text{ algebra.}\]
[2] G. W. Moore, “Some comments on branes, G-flux, and K-theory,” Int. J. Mod. Phys. A 16 (2001) 936 [arXiv:hep-th/0012007].

[3] M. Herbst, C. I. Lazaroiu and W. Lerche, “Superpotentials, A(infinity) relations and WDVV equations for open topological strings,” JHEP 0502 (2005) 071 [arXiv:hep-th/0402110].

[4] K. Costello, “Topological conformal field theories and Calabi-Yau categories”, math.QA/0412149.

[5] C. I. Lazaroiu, “On the non-commutative geometry of topological D-branes”, JHEP 0511 (2005) 032 [hep-th/0507222]

[6] C. I. Lazaroiu, “String field theory and brane superpotentials,” JHEP 0110 (2001) 018 [arXiv:hep-th/0107162].

[7] C. I. Lazaroiu, “D-brane categories,” Int. J. Mod. Phys. A 18 (2003) 5299 [arXiv:hep-th/0305095].

[8] M. Kontsevich, “Homological algebra of mirror symmetry”, Proc. Internat. Congr. Math., Zürich, Switzerland 1994 (Basel), vol. 1, Birkhäuser Verlag, 1995, 120–139.

[9] K. Lefevre-Hasegawa, “Sur les A-infini catégories”, Ph.D. Thesis, Universite Paris 7 (2002) [math.CT/0310337]

[10] K. Fukaya, “Floer homology and mirror symmetry. II”, in Minimal surfaces, geometric analysis and symplectic geometry (Baltimore, MD, 1999), Adv. Stud. Pure Math. 34, Math. Soc. Japan, Tokyo, 2002, pp. 31–127.

[11] A. I. Bondal, M. M. Kapranov, “Enhanced triangulated categories”, Math USSR Sbornik, 1991, 70 (1), 93-107.

[12] M. R. Gaberdiel, B. Zwiebach, “Tensor constructions of open string theories I: Foundations,” Nucl. Phys. B 505 (1997) 569 [arXiv:hep-th/9705038].

[13] C. I. Lazaroiu, “Generalized complexes and string field theory,” JHEP 0106 (2001) 052 [arXiv:hep-th/0102122].

[14] C. I. Lazaroiu, “Unitarity, D-brane dynamics and D-brane categories,” JHEP 0112 (2001) 031 [arXiv:hep-th/0102183].

[15] M. Kontsevich, Y. Soibelman, “Homological mirror symmetry and torus fibrations”, math.SG/0011041.

[16] B. Keller, “Deriving dG categories”, Ann. Scient. Ec. Norm Sup, 4e série, 27 (1994) 63-102
[17] M. Kontsevich, “Formal (non)commutative symplectic geometry,”, The Gelfand Mathematical Seminars, 1990–1992, 173–187, Birkhäuser Boston, Boston, MA, 1993.

[18] V. Ginzburg, “Noncommutative symplectic geometry, quiver varieties and operads”, Math. Res. Lett 8 (2001) 3, pp 377-400.

[19] A. Hamilton, A. Lazarev, “Homotopy algebras and noncommutative geometry”, math.QA/0410621.

[20] B. Keller, “Introduction to A-infinity algebras and modules”, Homology, Homotopy and Applications 3 (2001)1, pp 1–35 [math.RA/9910179]

[21] B. Keller, “On differential graded categories “, [math.KT/0601185]

[22] B. Keller, “A-infinity algebras, modules and functor categories”, [math.RT/0510508]

[23] P. Seidel, “ Fukaya categories and Picard-Lefschetz theory”, preprint

[24] M. Kontsevich, Y. Soibelman, “Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. I”, [math.RA/0606241]

[25] A. Polishchuk, “Homological mirror symmetry with higher products”, [math.AG/9901025].

[26] S. A. Merkulov, “Strongly homotopy algebras of a Kähler manifold”, Internat. Math. Res. Notices 3 (1999) 153-164 [math.AG/9809172].

[27] C. I. Lazaroiu, R. Roiban, “Holomorphic potentials for graded D-branes,” JHEP 0202 (2002) 038 [arXiv:hep-th/0110288].

[28] C. I. Lazaroiu, R. Roiban and D. Vaman, “Graded Chern-Simons field theory and graded topological D-branes,” JHEP 0204 (2002)023 [arXiv:hep-th/0107063].

[29] C. I. Lazaroiu, R. Roiban, “Gauge-fixing, semiclassical approximation and potentials for graded Chern-Simons theories,” JHEP 0203 (2002) 022 [arXiv:hep-th/0112029].

[30] B. Zwiebach, “Oriented Open-Closed String Theory Revisited”, Annals Phys. 267 (1998) 193-248 [hep-th/9705241].

[31] D.-E. Diaconescu, “Enhanced D-Brane Categories from String Field Theory”, JHEP 0106 (2001) 016 [hep-th/0104200]

[32] R. Bocklandt, “Graded Calabi Yau Algebras of dimension 3”, [math.RA/0603558]

[33] A. I. Bondal, M. M. Kapranov, “Representable functors, Serre functors and mutations”, Math. USSR Izv. 1990, 35 (3), 519-541.

83
[34] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies **148**, Princeton Univ. Press, 2001.

[35] P. Balmer, M. Schlichting, *Idempotent completion of triangulated categories*, J. Algebra 236(2001) 819-834.

[36] T. Tradler, “Infinity-Inner-Products on A-Infinity-Algebras”, [math.AT/0108027]