Zeno subspaces and interaction-free evolutions with non-Hermitian Hamiltonians

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Effective non-Hermitian Hamiltonians describing decaying systems are considered and analyzed in connection with the occurrence of possible Hilbert space partitioning. This fact can be interpreted as Zeno effect or Zeno dynamics, according to the dimension of the subspace one focuses on. Depending on the complex phases of the diagonal terms of the Hamiltonian, the system reacts in different ways, requiring larger moduli for the Zeno phenomena to occur when the complex phase is close to $\pi/2$. Unperturbed dynamics are also shown to persist in some special conditions where interaction-free subspaces are present.

I. INTRODUCTION

The quantum Zeno effect (QZE), in its original formulation, is the inhibition of the natural time evolution of a physical system due to repeated measurements [1]. In fact, the wave function collapse and the quadratic behavior of the survival probability (the probability to find the system in its initial state) of a quantum system at short time, both avoid any dynamical evolution of a frequently observed system. This paradigmatic effect underlines the active role of measurements in quantum mechanics and has been experimentally demonstrated in connection with Rabi oscillations in trapped ions [2] and tunnel effects in confined atoms [3]. Experimental proofs of the quantum Zeno effect have been provided in the context of Bose-Einstein condensates [4, 5]. The original formulation has been gradually extended including ways to act on a quantum system different from proper measurements. Indeed, for example, a decaying quantum state can be interpreted as a state which is continuously observed by an environment: if a photon is observed that has been emitted in the decay process of a quantum state, then we can say that the system was in the decaying state [6–10]. In fact, a strong decay is proven to play the same role of frequent measurements, hence hindering the time evolution [11]. Of course, in such a situation the inhibition can also be interpreted as a consequence of a dynamical decoupling, which has been predicted in several physical contexts, from the STIRAP manipulation [12–14] to the quantum biological processes [15]. Since a decay is the consequence of an interaction between one level and a continuum of levels, the subsequent most natural extension concerns the case where a coupling induces a Hilbert space partitioning responsible for making ineffective some other interactions [16, 17]. It is interesting to note that the same occurrence can be found in completely classical systems [18]. When the external agents (frequent measurements, strong decays or intense interactions) isolate a degenerate subspace one has the quantum Zeno dynamics. The subspace where the system is repeatedly projected undergoes a dynamics which does not take into account the interactions connecting this subspace to others. The Zeno effect is then a special case of Zeno dynamics with a trivial dynamics.

It is worth mentioning that when the measurements are frequent but not frequent enough an acceleration of the dynamics of the system can occur, instead of an inhibition, leading to the anti-Zeno effect (AZE). The boundary between QZE and AZE has been extensively studied and in the case of a system subjected to an interaction with an environment, the AZE-QZE threshold is traceable back to the spectral properties of the environment [19, 20] and is influenced by the temperature [21] as well as by the bath statistics [22]. Temperature can have an important role in the occurrence of Zeno phenomena. In fact, a certain influence of the detector's temperature on the Zeno effect has been predicted [23], as well as a role of temperature in continuous measurement QZE associated to the system-environment interaction [24, 25]. Moreover, thermodynamic processes can be influenced by quantum Zeno phenomena [26].

There is a link between quantum Zeno effect and non-Hermitian Hamiltonians. Indeed, on the one hand, the QZE induced by strong decays has been studied through non-Hermitian Hamiltonian models, while, on the other hand, the effects of repeated measurements have been proven to be describable via suitable non-Hermitian effective Hamiltonians [27]. In spite of this fact, there is not a systematic study of the quantum Zeno effect in the presence of a non-Hermitian Hamiltonian.

In this paper we analyze a physical scenario where a set of possibly decaying levels are coupled to a set of non-decaying ones. When the gap between the two subspaces (in terms of complex energies or complex diagonal entries) is very large, we get a Zeno dynamics, irrespectively of the phases of the diagonal entries of the Hamiltonian. On the contrary, when the gap is moderately large, the system becomes very sensitive to the phase of the diagonal entries and the occurrence of a proper Zeno dynamics requires higher values of the gap when the phases are close to $\pi/2$. The introduction of proper indicators allows to bring to light Zeno dynamics even in some regimes where is seemingly absent. Insensitivity of some levels to the interaction with other levels is also possible, in connection with special initial conditions, when the requirements for an interaction-free evolution (IFE) are fulfilled [28–30]. In the next section we introduce the Hamiltonian model for a system with a group of levels.
which undergo decays toward levels external to the subspaces we are focusing on. We also apply the perturbation theory to the case where large gaps are present in the complex spectrum of the non-Hermitian Hamiltonian. In sec. [IV] we analyze the Zeno dynamics induced by large gaps between the diagonal entries of the Hamiltonian. We first report on some analytical arguments, then, in section [V] we analyze the Zeno dynamics induced by large gaps between the diagonal entries of the Hamiltonian. In the subsequent section [VI] we analyze the Zeno dynamics induced by large gaps between the diagonal entries of the Hamiltonian. We also apply the perturbation theory to the case where large gaps are present in the complex spectrum of the non-Hermitian Hamiltonian. In sec. [V] we briefly discuss the possibility of having IFE subspaces. Finally, in sec. [VI] we give some conclusive remarks.

II. NON-HERMITIAN HAMILTONIANS

The appearance of non-Hermiticity in Hamiltonian operators is always traceable back to the derivation of an effective Hamiltonian which takes into account the interaction with external degrees of freedom which do not explicitly appear in the description of the reduced system. In the subsequent subsection we consider the effective Hamiltonian description of decay processes, leading to an Hamiltonian with complex diagonal terms, the real parts being the proper energies of the relevant levels, whereas the imaginary parts are the decay rates. In the subsequent subsection we apply the perturbation theory to our non-Hermitian Hamiltonian in a special regime, i.e., when two well separated bands associated to the bare ‘complex energies’, (i.e., the diagonal terms) can be identified.

A. Non-Hermitian Hamiltonian for decaying systems

A system with some states undergoing decay process toward some lower states due to the interaction with a zero-temperature environment, can be described through an effective non-Hermitian Hamiltonian, provided we focus on a subspace not involving the states receiving population from the decaying ones [12] [31] [32]. More precisely, let us consider a physical system governed by the Hamiltonian $\hat{H}^R_S$ and whose Hilbert space can be considered as made of two parts, $R$ and $G$, corresponding to the projectors $\Pi_R$ and $\Pi_G$, and assume that through a zero-temperature Markovian environment such two subspaces are coupled. In other words, we can assume a system-environment interaction term $\hat{H}_{SE} = \lambda \hat{X} \otimes \hat{E}$, with $\hat{X} = \hat{\Pi}_R \hat{X} \hat{\Pi}_G + \hat{\Pi}_G \hat{X} \hat{\Pi}_R$ (i.e., incoherent transitions within $R$ are excluded: $\hat{\Pi}_R \hat{X} \hat{\Pi}_R = 0$). According to the general theory of open quantum systems [33] [34], the relevant Markovian master equation can be written as:

$$\rho = -i[\hat{H}_S, \rho] + \sum_{ij} \gamma_{ij} \left( \hat{X}_{ij} \rho \hat{X}_{ij}^\dagger - \frac{1}{2} \{ \hat{X}_{ij}^\dagger \hat{X}_{ij}, \rho \} \right),$$

where $\hat{X}_{ij} = \hat{\Pi}_G \hat{X}_{ij} \hat{\Pi}_R$ are suitable jump operators connecting states of $R$ with states of $G$, $\gamma_{ij}$ being the relevant decay rates. The Lamb-shifts have been neglected. Terms related to $\hat{X}_{ij} = \hat{\Pi}_R \hat{X}_{ij} \hat{\Pi}_G$ are excluded because of the zero-temperature assumption, also considering that the energies of $G$ are lower than the energies of $R$.

Now, if we assume also that $\hat{H}_S$ does not couple the subspaces $R$ and $G$ (i.e., assume $\hat{H}_S = \hat{\Pi}_R \hat{H}_S \hat{\Pi}_R + \hat{\Pi}_G \hat{H}_S \hat{\Pi}_G$), a closed equation for the density operator restricted to the subspace $R$ can be straightforwardly obtained. Indeed, introducing $\rho^R \equiv \hat{\Pi}_R \rho \hat{\Pi}_R$, one gets:

$$\dot{\rho}^R = -i[\hat{H}^R_S, \rho^R] - \sum_{ij} \gamma_{ij} \frac{1}{2} \{ \hat{X}_{ij}^\dagger \hat{X}_{ij}, \rho^R \},$$

with $\hat{H}^R_S \equiv \hat{\Pi}_R \hat{H}_S \hat{\Pi}_R$ and where we have used both $\hat{\Pi}_R \hat{X}_{ij} \hat{\Pi}_R = 0 \ \forall i, j$ and $\hat{X}_{ij}^\dagger \hat{X}_{ij} = \hat{\Pi}_R \hat{X}_{ij}^\dagger \hat{X}_{ij} \hat{\Pi}_R$.

This equation can be put in the form of a pseudo-Liouville equation, $\dot{\rho}^R = -i[H^R, \rho^R]$, with the non-Hermitian Hamiltonian

$$H = \hat{H}^R_S - i \sum_{ij} \frac{\gamma_{ij}}{2} \hat{X}_{ij}^\dagger \hat{X}_{ij}.$$

It is worth observing that if the system-environment interaction is very strong, then the identification of the jump operators $\hat{X}_{ij}$ should be performed by considering which states of $A$ are connected to which states of $G$, considering the diagonal part of $\hat{H}_S$ with respect to such set of states in order to derive the master equation, and, finally, once the master equation has been obtained, restoring the complete $\hat{H}_S$ for the unitary part of the dynamics. Let us now introduce the eigenstates of the operator $\sum_{ij} \gamma_{ij} \hat{X}_{ij}^\dagger \hat{X}_{ij}$, denote them as $|k\rangle$, and rewrite the Hamiltonian in the following form:

$$H = \sum_k \Delta_k e^{-i\phi_k} |k\rangle \langle k| + \sum_{j \neq k} \hbar \gamma_{jk} |j\rangle \langle k|. $$

Here the diagonal terms $\Delta_k e^{-i\phi_k} = \epsilon_k - i\Gamma_k$ contain both information about the energies of the levels (real parts) and the relevant decay rates (imaginary parts). We can address them as ‘complex energies.’ In Fig. [I] it is shown the paradigmatic case where a three-state system is characterized by two energy levels ($0$ and $\epsilon$) which do not decay and a third level which decays, then having a complex energy $\Delta e^{-i\phi}$. In fact, two non-decaying states and a decaying one are the minimal requirement to have Zeno dynamics (instead of a simple Zeno effect). In sec. [IV] we focus on this specific situation.

B. Perturbation treatment

Let us consider a system whose $R$ subspace consists of $N$ states evolving according to the Hamiltonian in [I]. Moreover, let us assume that a set of $M < N$ diagonal terms are quite close to each other but very different from the remaining $N-M$, which in turn are very close to each
that they correspond to the first $M$ identified and, without loss of generality, we can assume the Hermitian Hamiltonian, some delicate points have to be taken into account (details of this treatment are reported in the Appendix A). The first order-corrected eigenvalues and eigenvectors turn out to be (all the $m$ indexes span the $A$ subspace, then ranging from 1 to $M$, while all the $n$ indexes span the $B$ subspace, then ranging from $M + 1$ to $N$):

$$
\alpha_m = \Delta_m e^{-i\phi_m}, \\
|\alpha_m^R\rangle = |m\rangle + \sum_n \frac{c_{mn}}{\Delta_m e^{-i\phi_m} - E_n} |n\rangle,
$$

$$
\langle \alpha_m| = \langle m| + \sum_n \frac{c_{nm}}{\Delta_m e^{-i\phi_m} - E_n} |n\rangle,
$$

$$
\beta_n = E_n, \\
|\beta_n^R\rangle = |n\rangle + \sum_m \frac{c_{nm}}{E_n - \Delta_m e^{-i\phi_m}} |m\rangle,
$$

$$
\langle \beta_n| = \langle n| + \sum_m \frac{c_{nm}}{E_n - \Delta_m e^{-i\phi_m}} |m\rangle.
$$

It is important to note that $\langle \alpha_m| \neq (|\alpha_m^R\rangle)\dagger$ and $\langle \beta_n| \neq (|\beta_n^R\rangle)\dagger$. Indeed, though $c_{nm} = c_{mn}^*$, the denominators of the first-order correction are the same complex number for $\langle \alpha_m^R| |\alpha_m\rangle$ and $|\alpha_m^R\rangle |\alpha_m\rangle$, i.e., $\Delta_m e^{-i\phi_m} - E_n$ not the complex conjugate to each other.

Concerning the second order correction, we focus on the eigenvalues:

$$
\alpha_m = \Delta_m e^{-i\phi_m} + \sum_n \frac{|c_{nm}|^2}{\Delta_m e^{-i\phi_m} - E_n},
$$

$$
\beta_n = E_n + \sum_m E_n - \Delta_m e^{-i\phi_m}.
$$

It is interesting to note that the corrections to the real energies in the $B$ subspace are complex numbers, meaning that decay processes occur also in the subspace which is subjected to a unitary dynamics in the unperturbed case. In particular,

$$
\text{Im}\beta_n = -\sum_m \frac{|c_{nm}|^2 \Delta_m}{|E_n - \Delta_m e^{-i\phi_m}|^2} \sin \phi_m
$$

is the effective decay rate associated to the state $|\beta_n^R\rangle$ obtained by correcting the state $|n\rangle$ of the subspace $B$. This quantity is expected to be higher when $\phi_m \approx \pi/2$ $\forall m$, and smaller when $\phi_m \approx 0, \pi \forall m$. On this basis, we can expect a role of the phases $\phi_m$ on the appearance of a Zeno dynamics when the moduli $\Delta_m$ are moderately large.

### III. ZENO DYNAMICS

In this section we investigate the occurrence of a Zeno dynamics in a system governed by a non-Hermitian Hamiltonian as in (4) with the assumption of sec. II B that there is a large gap between two subspaces. It is known that a decay can play the role of continuous measurement on a quantum system, and when the relevant
decay rate is large enough (which is the continuous counterpart of getting a larger number of measurements in a given time interval) a partitioning of the Hilbert space can produce either a Zeno effect (freezing the system in its initial condition) or a Zeno dynamics. Therefore, when we have \( M \) levels which have \( \Delta_k e^{-i\phi_k} \) very large with respect to all the other parameters, the states in the subspace \( B \) evolve as if no interaction between the first subspace and the second one were present. Similarly, when there are very large real diagonal elements (a set of \( M \) states with very large \( \Delta_k \)'s and \( \phi_k = 0, \pi \) we have that the dynamics of the relevant states is well separated from the dynamics of the remaining \( N - M \) ones, and again the dynamics of this second subspace is the one obtained in the absence of any interaction with the first subspace.

In the following we investigate the more general situation where a set of \( \Delta_k \)'s \( k = 1, \ldots, M \) are very large while the phases can assume any value. In particular, we want to investigate whether the Zeno dynamics occurs irrespectively of \( \phi_k \)'s.

**Perturbed vs unperturbed dynamics** — Assuming that the system starts in a certain state \( |\psi(0)\rangle \), we evaluate the two evolutions given by the equations \( i\hbar \partial_t |\psi(t)\rangle = H_0 |\psi(t)\rangle \) (unperturbed evolution) and \( i\hbar \partial_t |\psi(t)\rangle = (H_0 + H_I) |\psi(t)\rangle \) (perturbed evolution). The relevant solutions are:

\[
|\psi^0(t)\rangle = \sum_n \langle n|\psi(0)\rangle e^{-iE_nt} |n\rangle ,
\]

\[
|\psi(t)\rangle = \sum_m \langle a_m^R|\psi(0)\rangle e^{-i\omega_m t} |a_m^R\rangle + \sum_n \langle \beta_n^L|\psi(0)\rangle e^{-i\beta_n t} |\beta_n^R\rangle .
\]

When condition in (7) is fulfilled then the perturbation treatment is allowed and, moreover, the smaller \( c/\delta \), the smaller the corrections to the eigenvalues and eigenvectors, the closer the evolutions induced by \( H \) and \( H_0 \) are. When \( c/\delta \) is small though not extremely small, deviations between the two dynamics can be observed. In fact, for extremely small \( c/\delta \) (→ 0) the perturbed and unperturbed eigenvalues and eigenvectors coincide. For small but not extremely small \( c/\delta \) we can consider good the first-order approximation, which leaves unchanged the eigenvalues and slightly changes the eigenvectors. The evolution is then characterized by the same frequencies and phase factors characterizing the unperturbed case, but corresponding to slightly different states. For moderately small values of \( c/\delta \) the second-order correction is more appropriate, leading to corrections of the eigenvalues, which in general become complex also in the \( B \) subspace, as given by (10). In this case, the unitary dynamics in the \( B \) subspace is replaced by a non-unitary one and a general loss of probability in this subspace is predicted. It is worth emphasizing that if the corrected eigenvalues were real numbers, the discrepancy between the unperturbed and the perturbed dynamics would be \( o(c/\delta) \) at any time. The presence of imaginary parts in the eigenvalues makes the gap between the two evolutions increase with time, due to the presence of negative exponentials.

**Indicators for Zeno dynamics** — In order to better analyze the appearance of a Zeno dynamics, we introduce suitable fidelities. In particular we calculate the following quantity:

\[
\mathcal{F}(T) = \min_{t \in [0,T]} \frac{|\langle \psi^0(t) | \Pi_B | \psi(t) \rangle|^2}{\sqrt{\langle \psi^0(0) | \Pi_B | \psi^0(0) \rangle \cdot \langle \psi(0) | \Pi_B | \psi(0) \rangle}},
\]

which gives us the minimum overlap between the unperturbed and the perturbed dynamics in the subspace \( B \) in a time interval \([0,T]\). A Zeno phenomenon (whether freezing or Zeno dynamics) occurs in the time interval when such a quantity approaches unity.

As previously pointed out, since we are in the presence of dissipation, a loss of global probability is generally expected, thus having \( \langle \psi^0(t) | \Pi_B | \psi^0(t) \rangle \leq \langle \psi(0) | \Pi_B | \psi(0) \rangle \) and \( \langle \psi(t) | \Pi_B | \psi(t) \rangle \leq \langle \psi(0) | \Pi_B | \psi(0) \rangle \). It can then happen that the two dynamics are very close but their scalar product is smaller than unity, giving \( \mathcal{F} < 1 \). In order to take into account this fact, we have also considered the functional

\[
\mathcal{F}(T) = \min_{t \in [0,T]} \frac{|\langle \psi^0(t) | \Pi_B | \psi(t) \rangle|^2}{\sqrt{\langle \psi^0(t) | \Pi_B | \psi^0(t) \rangle \cdot \langle \psi(t) | \Pi_B | \psi(t) \rangle}},
\]

which differs from the previous one for the normalization of the two wave functions in the subspace of interest.

**IV. ZENO DYNAMICS IN A THREE-STATE SYSTEM**

We now focus on a three-state system described by the following Hamiltonian written in the basis \( \{|1\rangle, |2\rangle, |3\rangle\} \):

\[
H = \begin{pmatrix} \Delta e^{-i\phi} & g_1 & g_2 \\ g_1 & \epsilon & \Omega \\ g_2 & \Omega & 0 \end{pmatrix}.
\]

In this case the subspace \( A \) consists of \( |1\rangle \) while the subspace \( B \) is generated by \( |2\rangle \) and \( |3\rangle \).

We have evaluated the fidelity \( \mathcal{F}(T) \) and the normalized fidelity \( \mathcal{F}(T) \) in several conditions. In Fig. 2 we show the fidelity \( \mathcal{F}(T) \) in the first line and the corresponding normalized fidelity \( \mathcal{F}(T) \) in the second line. Moving from left to right we consider different initial conditions: \( |2\rangle, (|2\rangle + |3\rangle)/\sqrt{2}, (|2\rangle - |3\rangle)/\sqrt{2} \) and \( |3\rangle \), respectively. In all the plots we have \( g/\epsilon = 0.5, \Omega/\epsilon = 0.25 \) and \( \epsilon T = 2\pi \) (\( T \) being the time window to evaluate the fidelity). As expected, for large values of \( \Delta \) (for example \( \Delta/\epsilon > 10 \)) a Zeno dynamics is predicted, irrespectively of \( \phi \). On the contrary, when \( \Delta \) is moderately larger than \( \epsilon (1 < \Delta/\epsilon < 10) \), a dependence of the fidelity from the phase \( \phi \) is well visible from the figures. In particular,
when \( \phi \) is close to \( \pi/2 \), which means that the diagonal matrix element is essentially a decay rate, higher values of \( \Delta \) are required to have a fidelity \( F \) close to unity. It is anyway interesting to investigate the reason for such a different behavior. To this end, it is useful to consider the normalized fidelity \( \mathcal{F} \) (second line figures) which on the one hand, reaches higher values for lower values of \( \Delta \) and, on the other hand, allows to reveal a good agreement between the complete and the unperturbed dynamics even when \( \phi \approx \pi/2 \). In other words, there is essentially a good agreement between the unperturbed and the perturbed dynamics, the only difference being a general loss of probability due to the presence of dissipation. Therefore, up to a wave function renormalization, the two dynamics essentially coincide. This is in perfect agreement with our theoretical analysis. Indeed, for very large complex energy gaps we can use the first order perturbation treatment, which predicts a dynamics in the subspace \( B \) which is very close to the one obtained in the absence of any interaction with \( A \). When the complex energy gap is only moderately large, it is better to use the second order corrections. Since the eigenvalues associated to the subspace \( B \) acquire imaginary parts, we predict a general decay for the projection of the wave function to the subspace \( B \), which is the reason why the fidelity \( F \) lowers down. Nevertheless, up to such a complessive decay, the dynamics is essentially the one induced by \( B \), leading to higher values for the renormalized fidelity \( \mathcal{F} \).

V. INTERACTION-FREE EVOLUTIONS

It is interesting to observe that in some cases there can be a high fidelity due to the presence of an interaction-free subspace \([28,30]\) instead of the occurrence of a quantum Zeno phenomenon. The presence of IFE subspaces is clearly visible in the range of moderate values of \( \Delta \), where no Zeno effect can be predicted and, moreover, high values of the fidelity are obtained in connection with specific initial conditions. For example, in Fig. 5 we have the fidelity \( F \) as a function of the initial state. In particular, considering \( |\chi(0)\rangle = \cos \theta |2\rangle + e^{i\chi} \sin \theta |3\rangle \), we plotted the fidelity as a function of \( \theta \) and \( \chi \). It is well visible that for moderately large \( \Delta \) and \( \phi = 0 \) (Fig. 5b) we have high fidelity for every initial state, while for the same value of \( \Delta \) but \( \phi = \pi/2 \) (Fig. 5b) one has a lower fidelity almost everywhere (according to the previous analysis, since \( \Delta \) is moderately high, the system is quite sensitive to the phase). For smaller values of \( \Delta \) (Figs. 5a and 5b) we have only a small region where high fidelity is predicted, which is the nearby of a special state. Such a state has the property to be very close to an eigenstate of \( H_0 \) and to belong to kernel of \( H_I \), which is a special case of the general condition to have IFE subspaces \([28,30]\).

When \( \Omega \gg \epsilon \), the eigenstates of \( H_0 \) are \(|1\rangle \) and \(|\pm\rangle \approx (|2\rangle \pm |3\rangle)/\sqrt{2} \). The state \(|-\rangle \) also belongs to the kernel of \( H_I \) when \( g_1 = g_2 \). Therefore, in the limit \( \Omega/\epsilon \to \infty \) the state \(|-\rangle \) is an eigenstate of \( H_0 + H_I \), irrespectively of the values of \( \Delta \), \( \phi \) and \( g_1 \), provided \( g_1 = g_2 \). In the case of Fig. 5 we have \( \Omega/\epsilon = 2 \), which implies

![Fig. 2](http://example.com/fig2.png)
values than in the case where the Hilbert space partitioning is due to very large differences of proper energies. Our theoretical analysis, based on the perturbation treatment for non-Hermitian Hamiltonians, is well supported by numerical calculations. We have also shown that in some special cases, persistence of the unperturbed dynamics (i.e., the dynamics in the absence of interaction between $A$ and $B$) is traceable back to the presence of IFE subspaces, and then is initial state dependent, differently from the occurrence of the Zeno phenomena.

Appendix A: Perturbation Treatment for Non-Hermitian Hamiltonians

In this appendix we consider the perturbation theory for a non-Hermitian Hamiltonian which possesses non-degenerate subspaces. When no degenerations are present, the left and right eigenvector problems can be directly solved in the usual way, with the only peculiarity that the left eigenvectors are not the adjoint of the corresponding right eigenvectors and must be found independently. Let us assume $H = H_0 + \lambda H_1$ (both $H_0$ and $H_1$ can be non-Hermitian) and that we know the left and right eigenvectors of $H_0$:

$$
\left\langle u_k^{(0)} \right| H_0 = \epsilon_k^{(0)} \left| u_k^{(0)} \right\rangle,
$$

(A1a)

$$
H_0 \left| v_k^{(0)} \right\rangle = \epsilon_k^{(0)} \left| v_k^{(0)} \right\rangle,
$$

(A1b)

with the bi-orthogonality condition

$$
\left\langle u_k^{(0)} \right| v_j^{(0)} \rangle = \delta_{kj}.
$$

(A1c)

The eigenvector equations for the total Hamiltonian read,

$$
(H_0 + \lambda H_1) \sum_n \lambda^n \left| v_k^{(n)} \right\rangle = \sum_p \lambda^p \epsilon_k^{(p)} \sum_q \lambda^q \left| v_k^{(q)} \right\rangle,
$$

(A2)

$$
(H_0 + \lambda H_1) \sum_n \lambda^n \langle u_k^{(n)} | = \sum_p \lambda^p \epsilon_k^{(p)} \sum_q \lambda^q \langle u_k^{(q)} |.
$$

(A3)

By projecting the first of such equations through application of $\left\langle u_j^{(m)} \right|$ and the second equation through $\left| v_j^{(m)} \right\rangle$, and equating the terms with the same order in $\lambda$, we straightforwardly obtain the equations for the corrections. In the following we give the correction up to second-order, which resembles the standard one (i.e., the one obtained for Hermitian operators) except for the fact that the ordinary ‘bra’ are replaced by the left eigenvectors ($\left\langle v_k^{(n)} \right| \rightarrow \left\langle u_k^{(n)} \right|$):

$$
\epsilon_k = \epsilon_k^{(0)} + \lambda \left\langle u_j^{(0)} \right| H_1 \left| v_j^{(0)} \right\rangle
$$

$$
+ \lambda^2 \sum_{j \neq k} \frac{\left\langle u_j^{(0)} \right| H_1 \left| v_k^{(0)} \right\rangle \left\langle u_k^{(0)} \right| H_1 \left| v_j^{(0)} \right\rangle}{\epsilon_k^{(0)} - \epsilon_j^{(0)}}
$$

$$
+ o(\lambda^3),
$$

(A4a)
\[ |v_k| = |v_k^{(0)}| + \lambda \sum_{j \neq k} \left\langle u_j^{(0)} | H_I | v_k^{(0)} \right\rangle |v_j^{(0)}| + \lambda^2 \sum_{m,n \neq k} \left\langle u_n^{(0)} | H_I | v_m^{(0)} \right\rangle \left\langle u_m^{(0)} | H_I | v_k^{(0)} \right\rangle |v_m^{(0)}| \right| + a(\lambda^3), \quad (A4b)
\]

\[ |u_k| = \left\langle u_k^{(0)} | \right| + \lambda \sum_{j \neq k} \left\langle u_k^{(0)} | H_I | u_j^{(0)} \right\rangle \left\langle u_j^{(0)} | \right| + \lambda^2 \sum_{m,n \neq k} \left\langle u_n^{(0)} | H_I | u_m^{(0)} \right\rangle \left\langle u_m^{(0)} | H_I | u_k^{(0)} \right\rangle \left\langle u_m^{(0)} | \right\rangle \left\langle u_n^{(0)} | \right\rangle \right| + a(\lambda^3). \quad (A4c)\]

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