Null cosmological singularities
and free strings: II

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Abstract

In arXiv:0909.4731 [hep-th], we argued that the free string lightcone Schrodinger wave
functional in the vicinity of null Kasner-like cosmological singularities has nonsingular
time-dependence if the Kasner exponents satisfy certain relations. These backgrounds
are anisotropic plane waves with singularities. We first show here that only certain sin-
gularities admit a Rosen-Kasner frame with exponents satisfying relations leading to a
wavefunctional with nonsingular time-dependence. Then we build on the (Rosen) de-
scription further and study various physical observables for a time-dependent harmonic
oscillator toy model and then the free string, reconciling this with the corresponding de-
scription in the conventional plane wave variables. We find that observables containing
no time derivatives are identical in these variables while those with time derivatives are
different. Various free string observables are still divergent, perhaps consistent with string
oscillator states becoming light in the vicinity of the singularity.
1 Introduction

In this paper, we continue exploring free string propagation in the background of null cosmological singularities, following [1, 2], and motivated by [3, 4, 5, 6], and earlier related investigations, e.g. [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

Time-dependent generalizations of AdS/CFT were studied in [3, 4, 5], the bulk containing null or spacelike cosmological singularities (curvatures as $R_{MN} \sim \partial_M \Phi \partial_N \Phi$, with $e^\Phi$ a nontrivial dilaton vanishing at the singularity), the gauge theory duals being $\mathcal{N}=4$ Super Yang-Mills theories with a time-dependent gauge coupling $g_{YM}^2 = e^\Phi$. In the vicinity of the bulk cosmological singularity, supergravity breaks down with possible resolutions arising from stringy effects (extrapolating from the usual AdS/CFT dictionary corroborates this, with $\alpha' \sim \frac{1}{g_{YM} \sqrt{\mathcal{N}}} \sim \frac{1}{e^{\Phi/2} \sqrt{\mathcal{N}}}$, suggesting vanishing effective tension for stringy excitations as $e^\Phi \to 0$ near the singularity).

In particular, for the case of null singularities, the gauge theory duals were argued to be weakly coupled [3], with a well-defined lightcone “near-singularity” Schrodinger wavefunctional (barring possible renormalization effects, as discussed in [5]). With a view to understanding these stringy effects but without the complications of holography and RR-backgrounds, we studied string propagation in purely gravitational spacetimes containing null Kasner-like singularities in [1, 2]. These are essentially anisotropic plane-waves in Rosen coordinates. Building on intuition from the Schrodinger wavefunctional as a response of the gauge theory to a time-dependent coupling source [5], we studied the response of the free string lightcone Schrodinger wavefunctional to the external time-dependence induced by these cosmological singularities. The wavefunctional was found to have nonsingular time-dependence for certain classes of singularities satisfying certain relations among the Kasner exponents. Furthermore, a detailed free string quantization can be performed using the exact mode functions that can be solved for...
in these backgrounds: this shows various string oscillator states becoming light in the vicinity of the singularity (e.g. on a cutoff constant-(null)time surface), relative to the local curvature scale there.

The presence of multiple Rosen patches (i.e. \( \{ A_I \} \)) for a given singularity specified by \( \{ \chi_I \} \) means that the conditions \( |A_I| \leq 1 \) for each \( I \) are nontrivial and not automatically satisfied for a generic singularity. Indeed, using the \( \chi_I, A_I \)-relations, we first analyse which singularities admit such a Rosen frame with a nonsingular wavefunctional and find that only certain anisotropic plane wave singularities with \( \{ \chi_I \} \) lying in specific windows qualify. Since the equation of motion gives one condition between the various \( \chi_I \), the space of allowed \( \chi_I \)-windows increases as the number of \( \chi_I \) increases, i.e. as the anisotropy increases.

The nonsingular time-dependence of the free string Schrödinger wavefunctional for some singularities in the Rosen variables suggests that string propagation might be well-defined across these singularities in the Rosen variables. However it is essential to understand physical observables, especially with a view to reconciling the description in the Rosen and Brinkman frames. This is our primary objective in this work.

We first explore a 1-dimensional time-dependent harmonic oscillator propagating in these cosmological backgrounds in Rosen and Brinkman variables, thinking of this as a (1-dim) single momentum mode of a string. As for the string discussed in [1], we see that the harmonic oscillator wavefunction for a generic state acquires a wildly oscillating phase in Brinkman variables, suggesting that the Rosen coordinates with a nonsingular wavefunctional above provide a better-defined set of variables. Being a quantum mechanical system, this can be analyzed in great detail and we study the wavefunctions, observables and asymptotic behaviour. While the wavefunction itself is ill-defined in the Brinkman variables but well-defined in Rosen variables for certain values \( (A \leq 1) \) of the single Kasner exponent in this model, the probability density is quite similar in both frames, consistent with reconciling probability conservation in both frames. While expectation values of observables not containing time derivatives are identical in both frames as expected, those of observables involving time derivatives are quite different: for instance the natural momentum-squared expectation value in Rosen variables is quite different from the corresponding Brinkman ones, although still divergent.

This quantum mechanical analysis then paves the way for an investigation of the free string lightcone quantization, in part reviewing [1,2]. The lightcone string Schrödinger wavefunctional has regular evolution near the singularity if the Kasner exponents satisfy \( |A_I| \leq 1, \) for each dimension \( I \). The main new feature here is the presence of various string oscillator states that are light near the singularity, as was argued already in [1]. This apart, various features of the observables are similar to the harmonic oscillator case above and the expectation values of observables containing time derivatives are different in both frames. In a sense, this is analogous to the differences in the variables \( A_\mu \) and \( \tilde{A}_\mu \) and various observables made from them in the
analysis of null cosmological generalizations of AdS/CFT studied in [3]: we recall that the $A_\mu$-variables are dual to local bulk supergravity fields which are singular, while the $\tilde{A}_\mu$-variables are likely to not have local bulk duals, but stringy ones.

Our analysis thus shows that although the wavefunctional has nonsingular time-dependence in the Rosen variables, physical observables in fact are still divergent for the free string, strictly speaking, although the degree of divergence is milder than for the Brinkman ones. It is not clear to us at this point what the significance is, if any, of the detailed differences in the degree of divergence of various observables in the two frames. However the presence of light string oscillator states suggests that the free string description is breaking down in the vicinity of the singularity, perhaps consistent with the divergence of the free string observables.

In sec. 2, we discuss some key features of the spacetime backgrounds in question and the nonsingular time-dependence of the free string lightcone Schrödinger wavefunctional, in part reviewing aspects of [1, 2]. In sec. 2.1, we identify conditions for the null Rosen-Kasner exponents to admit a well-defined spacetime and wavefunctional with nonsingular time-dependence. In sec. 3, we discuss various aspects of the time-dependent harmonic oscillator, with a description of the string in sec. 4. Finally sec. 5 summarises some conclusions, with a brief discussion.

2 Plane waves, null cosmological singularities and strings

Consider singular plane-wave spacetimes with metric in the usual Brinkman coordinates

$$ds^2 = -2dy^+dy^- - \sum I \chi_I (y^I)^2 \left(\frac{dy^+}{y^+}\right)^2 + (dy^I)^2 ,$$

where $\chi_I$ are real numbers characterizing the plane wave. As is well known, the coordinate transformation $y^I = (y^+)^{A_I/2} x^I$, $y^- = x^- + (\sum I A_I (y^I)^2)$, recasts these spacetimes in Rosen coordinate form, or manifest null cosmology form,

$$ds^2 = -2dy^+dx^- + (y^+)^{A_I} (dx^I)^2 , \quad A_I = 1 \pm \sqrt{1 - 4\chi_I} .$$

These are thus null Kasner cosmologies, the $A_I$ being Kasner exponents.

For this spacetime to be a solution to supergravity with no other background fields turned on (these in fact preserve half lightcone supersymmetry [2]), we require Ricci-flatness, *i.e.*

$$R_{++} = \frac{1}{(y^+)^2} \sum I \chi_I = \frac{1}{(y^+)^2} \sum I A_I (2 - A_I) = 0 .$$

Thus, completely homogenous singular plane-waves, *i.e.* all $\chi_I$ equal, are solutions only in the presence of additional matter fields sourcing the system, for instance a scalar field (dilaton). For purely gravitational systems with no additional matter, interesting but still sufficiently simple
spacetimes arise for just two distinct $\chi_I$, or $A_I$. These are then solutions if $2\chi_1 + (D - 4)\chi_2 = 2A_1(2 - A_1) + (D - 4)A_2(2 - A_2) = 0$. Null geodesic congruences stretching solely along $y^+$ (with cross-section along any of the other directions) have an affine parameter $y^+$ (which can be seen from the geodesic equation noting that all $\Gamma^+_ij$ vanish). As we approach the singularity $y^+ \to 0$, such congruences exhibit diverging geodesic deviation, stemming from $e.g.$ $R_{+IJ+} \sim \frac{y^+}{(y^+)^2}$, so that these spacetimes exhibit diverging tidal forces as $y^+ \to 0$, although all curvature invariants are finite.

Alternative convenient forms\footnote{Indeed, we had originally found these in \cite{2, 1} as purely gravitational spacetimes with null scale factors $e^{f_I(x^+)}$ developing null Kasner-like Big-Crunch singularities $e^{f_I(x^+)} \to (x^+)^{a_I}$ at some location $x^+=0$. For the case of two scale factors, one scale factor essentially simulates the dilaton in corresponding AdS/CFT cosmology models \cite{3, 4, 5}, driving the crunch of the 4-dim part of the spacetime.} of these spacetimes arise by using the coordinate $x^+$, with the Rosen coordinate metric

$$ds^2 = -2(x^+)^a dx^+ dx^- + (x^+)^{a_I} dx^I, \quad a > 0,$$

where $I = 1, 2, \ldots, D - 2$. Solutions with $a < 0$ can be transformed to ones with $a > 0$. The variable $y^+ = \frac{(x^+)^{a+1}}{a+1}$ is the affine parameter for null geodesics stretched solely along $x^+$. The corresponding plane wave Brinkman form of the metric is

$$ds^2 = -2(x^+)^a dx^+ dy^- + \sum_I \left( \frac{a_i^2}{4} - \frac{a_I(a + 1)}{2} \right) (y^I)^2) \left( \frac{dx^I}{(x^+)^2} + (dy^I)^2 \right),$$

with $a_I = a, b, \text{ distinct, and } A_I = \frac{a_i}{a+1}$. The form of the metric \footnote{Indeed, we had originally found these in \cite{2, 1} as purely gravitational spacetimes with null scale factors $e^{f_I(x^+)}$ developing null Kasner-like Big-Crunch singularities $e^{f_I(x^+)} \to (x^+)^{a_I}$ at some location $x^+=0$. For the case of two scale factors, one scale factor essentially simulates the dilaton in corresponding AdS/CFT cosmology models \cite{3, 4, 5}, driving the crunch of the 4-dim part of the spacetime.} has the noteworthy feature that $e.g.$ for two Kasner exponents $a, b$, integer-values of $a, b$ exist, allowing for manifest analytic continuation of the metric across the singularity at $x^+ = 0$. In what follows, we will be analysing string propagation and we will not focus much on this feature: we will primarily use the metric variables in \footnote{Indeed, we had originally found these in \cite{2, 1} as purely gravitational spacetimes with null scale factors $e^{f_I(x^+)}$ developing null Kasner-like Big-Crunch singularities $e^{f_I(x^+)} \to (x^+)^{a_I}$ at some location $x^+=0$. For the case of two scale factors, one scale factor essentially simulates the dilaton in corresponding AdS/CFT cosmology models \cite{3, 4, 5}, driving the crunch of the 4-dim part of the spacetime.}.

Free strings can be quantized in these backgrounds in considerable detail since the mode functions can be exactly solved: here we review the analysis of \cite{1, 2} (which we refer to for details) but adapt that to the variables in \footnote{Indeed, we had originally found these in \cite{2, 1} as purely gravitational spacetimes with null scale factors $e^{f_I(x^+)}$ developing null Kasner-like Big-Crunch singularities $e^{f_I(x^+)} \to (x^+)^{a_I}$ at some location $x^+=0$. For the case of two scale factors, one scale factor essentially simulates the dilaton in corresponding AdS/CFT cosmology models \cite{3, 4, 5}, driving the crunch of the 4-dim part of the spacetime.} \cite{1, 2}. Starting with the closed string worldsheet action $S = -\int \frac{d^2x}{2\pi\alpha'} \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X)$, we use lightcone gauge $y^+ = \tau$ to reduce the system to the physical transverse degrees of freedom. We will review some details in sec. 4.

The lightcone string worldsheet Hamiltonian in Brinkman variables is

$$H_B = \frac{1}{4\pi\alpha'} \int d\sigma \left( (2\pi\alpha')^2 (\Pi_y')^2 + (\partial_\sigma y^I)^2 + \sum_I \frac{\chi_I}{\tau^2} (y^I)^2 \right),$$

where the conjugate momentum is $\Pi_y' = \partial_\sigma y^I$, which we elevate to the operator $\Pi_y'[\sigma] = -i\frac{\delta}{\delta y'[\sigma]}$. We see that the Hamiltonian in these variables $y^I$ contains a mass-term which diverges as $\tau \to 0$. \footnote{Indeed, we had originally found these in \cite{2, 1} as purely gravitational spacetimes with null scale factors $e^{f_I(x^+)}$ developing null Kasner-like Big-Crunch singularities $e^{f_I(x^+)} \to (x^+)^{a_I}$ at some location $x^+=0$. For the case of two scale factors, one scale factor essentially simulates the dilaton in corresponding AdS/CFT cosmology models \cite{3, 4, 5}, driving the crunch of the 4-dim part of the spacetime.}
The wavefunctional $\Psi[y^I(\sigma), \tau]$ for string fields $y^I(\sigma)$ then acquires a “wildly” oscillating phase on approaching the singularity as $\tau \to 0^-$,
\[ \Psi[y^I, \tau] \sim e^{-i \tau \sum I \chi_I(y^I)^2} \Psi[y^I]. \quad (7) \]

Thus the wavefunction does not have a well-defined limit there. This renders a well-defined Schrodinger wavefunctional interpretation near the singularity difficult in Brinkman coordinates.

By comparison, the Rosen coordinates (2) above naturally give well-defined variables for string propagation. The lightcone string worldsheet Hamiltonian becomes
\[ H_R = \frac{1}{4 \pi \alpha'} \int d\sigma \left( \frac{(2\pi \alpha')^2 (\Pi^I)^2}{g_{II}} + g_{II}(\partial_\sigma x^I)^2 \right) = \frac{1}{4 \pi \alpha'} \int d\sigma \left( \frac{(2\pi \alpha')^2 (\Pi^I)^2}{\tau A_I} + \tau A_I (\partial_\sigma x^I)^2 \right), \quad (8) \]

with the conjugate momentum $\Pi^I = \frac{\tau A_I}{2 \pi \alpha'} \partial_\sigma x^I$, which we elevate to the operator $\Pi^I[\sigma] = -i \frac{\delta}{\delta x^I[\sigma]}$. Now consider $A_I > 0$. Then as $\tau \to 0$, the kinetic terms dominate and the Schrodinger equation for the wavefunctional $\Psi[x^I(\sigma), \tau]$ becomes
\[ i \partial_\tau \Psi[x^I, \tau] = -\pi \alpha' \tau^{-A_I} \int d\sigma \frac{\delta^2}{\delta x^I} \Psi[x^I, \tau] \quad (9) \]
giving for the time-dependence
\[ \Psi[x^I, \tau] \sim e^{i \tau \frac{1}{2} \tau^{-A_I} \int d\sigma \frac{\delta^2}{\delta x^I} \Psi[x^I]}, \quad (10) \]

The phase in the functional operator is well-defined if $A_I \leq 1$. Alternatively, we can recast (9) as a free Schrodinger equation in terms of the time parameter $\tau^{1-A_I}$ (with the dominant exponent $A_I$) which for $A_I \leq 1$ is well-defined (vanishing as $\tau \to 0$; by comparison, for $A_I > 1$, this time parameter $\tau^{1-A_I} \to \infty$ as $\tau \to 0$). This free string Hamiltonian and associated Schrodinger equation essentially has a product structure with the wavefunctional itself factorizing as $\Psi = \prod_I \Psi_I[x^I]$ over the various coordinate dimensions.

For spacetimes with $A_I < 0$, the potential terms dominate and we have
\[ i \partial_\tau \Psi[x^I, \tau] = \frac{\tau^{-|A_I|}}{4 \pi \alpha'} \int d\sigma (\partial_\sigma x^I)^2 \Psi[x^I, \tau], \quad (11) \]
giving for the time-dependence
\[ \Psi[x^I, \tau] \sim e^{i \frac{1}{4 \pi \alpha'(1-|A_I|)} \int d\sigma (\partial_\sigma x^I)^2 \Psi[x^I]}, \quad (12) \]

which is well-defined for $|A_I| \leq 1$.

We see that the lightcone string Schrodinger wavefunctional for the generic state is well-defined near null cosmological singularities with Kasner exponents satisfying $|A_I| \leq 1$. A more
“nuts-and-bolts” traditional quantization can be performed using the explicitly solvable mode functions and corroborates this, as argued in [1,2], and discussed at length in sec. 4.

From [2], we see that for each \( \chi_I \), there exist two Kasner exponents \( A_I^\pm \), i.e. two Rosen “patches” each. Thus we need to be careful in identifying the precise regime where the string wavefunctional is apparently well-defined, and in particular whether such Rosen patches exist for a given singularity \( \{ \chi_I \} \) at all: we will now discuss and elaborate on this. Then in sec. 3, we will first discuss the time-dependent harmonic oscillator in detail, studying the analogs of Rosen and Brinkman frames and wavefunctions/observables therein, after which we discuss string quantization (sec. 4), wavefunctionals and observables.

2.1 Plane wave singularities, Rosen patches and strings

We have seen that the Rosen frames where \( |A_I| \leq 1 \) encode a wavefunctional that has non-singular time-dependence in the vicinity of the singularity. However since there are multiple such Rosen-Kasner exponents, it is not obvious if there exists a Rosen frame where \( |A_I| \leq 1 \) for each of the coordinates \( x^I \). With a view to understanding this, note that interesting but still sufficiently simple plane wave spacetimes arise in the absence of matter fields (e.g. a dilaton scalar) for just two distinct \( \chi_I \) (and the corresponding \( A_I \)), as we have seen. The metric is

\[
ds^2 = -2dy^+dx^- + \tau A_1(dx_1^2 + dx_3^2) + \tau A_2(dx_4^2 + \ldots + dx_{D-2}^2) .
\]

From sec. 2, these are then solutions if \( 2\chi_1 + (D-4)\chi_2 = 2A_1(2-A_1) + (D-4)A_2(2-A_2) = 0 \). From (2), we see that one of the \( A_I \)s is positive, while the other is negative. Indeed, given \( \chi_I = \frac{A_I}{4}(2-A_I) \), the two Rosen-Kasner exponents are \( A_I = 1 \pm \sqrt{1-4\chi_I} \), for each \( \chi_I \), so that \( A_I^- < 1 \) while \( A_I^+ > 1 \) for each \( I \) (note that \( \chi_I \) is invariant under the exchange \( A_I \leftrightarrow 2-A_I \), i.e. \( 1-A_I \rightarrow A_I-1 \)). This in all gives four Rosen frames or patches, \( (A_1^-, A_2^-), (A_1^+, A_2^-), (A_1^+, A_2^+), (A_1^-, A_2^+) \).\footnote{Note that for flat space, we have \( \chi_I = 0 \), giving \( A_I = 0 \) or \( A_I = 2 \), for each \( I \). Thus the condition \( |A_I| \leq 1 \) singles out the patch \( (A_1^-, A_2^-) \) which in this case is trivially flat space again. The apparent singularities in the other patches would seem to be coordinate artifacts: this is also possibly the case for the general plane wave.}

We would like a crunch happening in the “noncompact” directions \( x_{2,3} \), i.e. we impose \( A_1 > 0 \). Then the choice (with \( \chi_1 > 0, \chi_2 < 0 \))

\[
0 < \chi_1 \leq \frac{1}{4} , \quad 0 < A_1 = 1 - \sqrt{1-4\chi_1} \leq 1 , \quad -\frac{3}{4} \leq -\frac{1}{2(D-4)} \leq \chi_2 < 0 , \quad -1 \leq A_2 = 1 - \sqrt{1-4\chi_2} < 0 , \tag{13}
\]
gives a Rosen patch \( (A_1^-, A_2^-) \) which satisfies \( 0 < |A_1|, |A_2| \leq 1 \), and consistent with the equation of motion \( 2\chi_1 + (D-4)\chi_2 = 0 \). To elaborate, the restriction \( \chi_1 \leq \frac{1}{4} \) for reality of \( A_1 \) translates using the equation of motion to \( \chi_2 \geq \frac{1}{2(D-4)} \), while \( A_2 \equiv A_2^- \geq -1 \) (from the regularity of the Rosen wavefunctional) gives \( \chi_2 \geq -\frac{3}{4} \): for \( D > 4 \), these are compatible and satisfied if...
\( \chi_2 \geq \frac{1}{2(D-4)} \). We see thus that the parameters \( \chi_1, \chi_2 \), must satisfy nontrivial conditions, lying in a particular window of parameter space: in other words, only certain singular plane wave spacetimes admit a Rosen patch of this sort. For the critical dimension \( D = 10 \), this gives \( 0 < \chi_1 \leq \frac{1}{4} \), \( -\frac{1}{12} \leq \chi_2 < 0 \).

Correspondingly, we see that \( \chi_1 < 0 \) with \( A_1 > 0 \) requires \( A_1 > 2 \), which violates our requirement of regularity of the Rosen wavefunctional. Thus \( \chi_1 < 0, \chi_2 > 0 \), does not give any further solutions.

If we have more than 2 Kasner exponents, \( i.e. \) more than two \( \chi_I \), the space of possible \( \chi_I \) increases, as expected. For instance, with say three exponents \( A_1, A_2, A_3 \), or \( \chi_1, \chi_2, \chi_3 \), arising from a metric

\[
ds^2 = -2dy^+dx^- + \tau^{A_1}(dx_2^2 + dx_3^2) + \tau^{A_2}(dx_4^2 + dx_5^2) + \tau^{A_3}(dx_6^2 + \ldots + dx_{D-2}^2),
\]

we have the equation of motion \( 2\chi_1 + 2\chi_2 + (D-6)\chi_3 = 0 \), with \( \chi_I = \frac{A_I(2-A_I)}{4} \). Requiring \( A_1 > 0 \) as before gives \( A_1 = 1 - \sqrt{1-4\chi_1} \leq 1 \): we give the various distinct allowed \( \{ \chi_I \} \) windows below —

- \( \chi_2 > 0, \chi_3 < 0 \): we have \( 0 < A_2 = 1 - \sqrt{1-4\chi_2} \leq 1 \) and \( A_3 = 1 - \sqrt{1+4|\chi_3|} < 0 \) (note that \( e.g. \) \( A_2 < 0 \) requires \( A_2 > 2 \) for \( \chi_2 > 0 \), as before, which is disallowed from the regularity of the Rosen wavefunctional). This is well-defined if \( \chi_1, \chi_2 \leq \frac{1}{4} \) which translates, using the equation of motion, to \( -\frac{1}{D-6} \leq \chi_3 < 0 \). The regularity of the Rosen wavefunctional in addition requires that \( |\chi_3| \leq 1, i.e. \chi_3 \geq -\frac{3}{4} \) as before, which is automatically satisfied if \( D > 7 \).

- \( \chi_2 < 0, \chi_3 > 0 \): now we have \( A_2 = 1 - \sqrt{1+4|\chi_2|} < 0 \) and \( 0 < A_3 = 1 - \sqrt{1-4\chi_3} \leq 1 \). Requiring \( \chi_1, \chi_3 \leq \frac{1}{4} \) translates using the equation of motion, to \( -\frac{D-4}{8} \leq \chi_2 < 0 \). Also, \( A_2 \geq -1 \) gives as before \( \chi_2 \geq -\frac{3}{4} \); these are identical for \( D = 10 \), while \( -\frac{3}{4} \leq \chi_2 < 0 \) is the stronger inequality if \( D > 10 \) and vice versa for \( D < 10 \).

- \( \chi_2, \chi_3 < 0 \): here we have \( A_2 = 1 - \sqrt{1+4|\chi_2|} < 0 \) and \( A_3 = 1 - \sqrt{1+4|\chi_3|} < 0 \). Then \( A_2, A_3 \geq -1 \) gives \( \chi_2, \chi_3 \geq -\frac{3}{4} \), while \( 0 < \chi_1 \leq \frac{1}{4} \) translates to \( 2|\chi_2| + (D-6)|\chi_3| \leq \frac{1}{4} \).

We thus see that three Kasner exponents allows several possibilities. Clearly the story becomes richer as the spacetime becomes more anisotropic with multiple Kasner exponents.

Likewise, with a dilaton driving the system, there are various possibilities: the equation of motion in this case is \( R_{++} = \frac{1}{2}(\partial_+ \Phi)^2 \) which simplifies to \( \sum_I \chi_I = \frac{\alpha^2}{2} \), where the dilaton time-dependence is \( e^\Phi = t^\alpha \). In this case, we have the possibility of an isotropic crunch with all \( A_I \) (or \( \chi_I \)) equal to say \( A_I = A \): this gives \( \chi = \frac{\alpha^2}{2(D-2)} > 0 \), giving \( 0 < A = 1 - \sqrt{1-\frac{2\alpha^2}{D-2}} \leq 1 \). This is well-defined if the dilaton exponent satisfies \( \alpha \leq \sqrt{\frac{D-2}{2}} \).
We now discuss the nature of the singularity locus in these various patches, recalling that $y^I = (y^+)^{A_I/2} x^I$. Thus as $\tau \equiv y^+ \to 0$, in the Rosen null cosmology description, the spacetime is clearly seen to either crunch or expand depending on whether the corresponding Kasner exponent is $A_I > 0$ or $A_I < 0$. The singularity locus is thus the entire $\{x_I\}$-plane. In the Brinkman description, we see that $y^I \to 0$ for fixed $x^I$ in the Rosen patch with $A_I > 0$, while $y^I \to \infty$ for $A_I < 0$. Thus e.g. we have:

- Two exponents $\{\chi_1, \chi_2\}$: here, the singularity locus in Brinkman frame is $y_{2,3} \to 0$, $y_{4,\ldots} \to \infty$, for the Rosen patch $A_1 > 0, A_2 < 0$ ($\chi_1 > 0, \chi_2 < 0$), which has the potentially well-defined wavefunctional. This corresponds to the $x_{2,3}$-space crunching and the $x_{4,\ldots}$-space growing as $y^+ \to 0$. The other patch has $y_{2,3}, y_{4,\ldots} \to 0$ ($A_1, A_2 > 0$).

- Three exponents $\{\chi_1, \chi_2, \chi_3\}$: the singularity loci in Brinkman frame likewise are (i) $y_{2,3}, y_{4,5} \to 0$, $y_{6,\ldots} \to \infty$ ($\chi_1, \chi_2 > 0, \chi_3 < 0$ or $A_1, A_2 > 0, A_3 < 0$), (ii) $y_{2,3}, y_{6,\ldots} \to 0$, $y_{4,5} \to \infty$ ($\chi_1, \chi_2 > 0, \chi_3 < 0$ or $A_1, A_2 > 0, A_3 < 0$), (iii) $y_{2,3} \to 0$, $y_{4,5}, y_{6,\ldots} \to \infty$ ($\chi_1 > 0, \chi_2, \chi_3 < 0$ or $A_1 > 0, A_2, A_3 < 0$).

### 3 Toy model: a time-dependent harmonic oscillator

#### 3.1 Classical and preliminary quantum analysis

We consider a 1-dim harmonic oscillator subjected to external time-dependence: this can be regarded as an oscillator propagating in a time-dependent background cosmological spacetime of the sort we have discussed earlier, and is in fact a 1-dim, single momentum mode of a string propagating in the null cosmological spacetime in Rosen coordinates and lightcone gauge $\tau = y^+$. We will first study the classical mechanics of this system: the (Rosen) action is

$$S_R = \frac{m}{2} \int d\tau \left[ g_{xx} \dot{x}^2 - n^2 g_{xx} x^2 \right] = \frac{m}{2} \int d\tau \left[ -2 \dot{x}^- + \tau^A (\dot{x}^2 - n^2 x^2) \right]. \quad (15)$$

The conjugate momenta are $p_x = m \tau^A \dot{x}$, $p_- = \frac{\partial L_R}{\partial (\partial_x \dot{x}^-)} = -m$. The lightcone momentum $p_-$ is conserved. The Hamiltonian then is (using $p_- = -m$)

$$H_R = p_\dot{x}^- + p_x \dot{x} - L_R = -\frac{p_x^2}{2p_- \tau^A} - \frac{n^2 p_-}{2} \tau^A x^2. \quad (16)$$

The equations of motion then become

$$\partial_\tau (\tau^A \partial_\tau x) + n^2 \tau^A x = 0, \quad \dot{x}^- = \frac{\partial H_R}{\partial p_-} = \frac{1}{2} \tau^A (\dot{x}^2 - n^2 x^2), \quad (17)$$

re-expressing in terms of $x, \dot{x}$. 

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Let us now consider a redefinition to Brinkman variables
\[ y = \tau^{A/2} x, \quad \dot{x} = \tau^{-A/2} \left( \dot{y} - \frac{A}{2\tau} y \right), \]  
which have canonical kinetic terms. This, after defining
\[ y^- = x^- + \frac{A y^2}{4\tau}, \quad \chi = \frac{A(2 - A)}{4}, \quad A = 1 \pm \sqrt{1 - 4\chi}, \]
then transforms the action to
\[ S_B = \frac{m}{2} \int d\tau \left[ -2\partial_\tau x^- - \frac{d}{d\tau} \left( \frac{A y^2}{2\tau} \right) + y^2 - \left( n^2 + \frac{A(2 - A)}{4\tau^2} \right) y^2 \right] \]
\[ = \frac{m}{2} \int d\tau \left[ -2\partial_\tau y^- + y^2 - \left( n^2 + \frac{\chi}{\tau^2} \right) y^2 \right]. \]
This is in fact precisely the action for a 1-dim oscillator obtained from a single momentum mode of the string propagating in the plane wave background in Brinkman coordinates. Using this Brinkman action, we note the conjugate momenta
\[ p_y = m \dot{y}, \quad p_\tau = \partial L_B / \partial \dot{p}_\tau = -m = p_-: \]
then the (conserved) lightcone momentum \( p_- \) is the same in both frames. The Hamiltonian here is (using \( p_- = -m \))
\[ H_B = p_\tau \dot{y}^- + p_y \dot{y}^- - L_B = -\frac{p_y^2}{2p_-} - \frac{p_-}{2} \left( n^2 + \frac{\chi}{\tau^2} \right) y^2. \]
The equations of motion are
\[ \ddot{y} + (n^2 + \frac{\chi}{\tau^2}) y = 0, \quad \dot{y}^- = \frac{\partial H_B}{\partial p_-} = \frac{1}{2} y^2 - \frac{1}{2} \left( n^2 + \frac{\chi}{\tau^2} \right) y^2, \]
re-expressing in terms of \( y, \dot{y} \).

It can be seen using (17), (22), that the expressions for \( \dot{x}^- \) and \( \dot{y}^- \) are consistent with (19). Starting with a Rosen action for just the variables \( x, \dot{x} \) (i.e. without \( x^- \)), the redefinition to \( y, \dot{y} \) gives the Brinkman action (without \( y^- \)) up to a total derivative term: the presence of the variable \( x^- \) in the action allows us to absorb the total derivative by redefining a new variable \( y^- \). This procedure is thus equivalent to the coordinate transformation between the Rosen and Brinkman spacetime variables: the field redefinition automatically implements the coordinate transformation as it should.

We will now analyze the quantum theory of these oscillators. To begin, we describe some basic observations, elaborating on them later. First we consider Brinkman variables. The Hamiltonian is dominated by the potential term which diverges as \( \tau \to 0 \), for fixed momenta \( p_y \) and oscillator number \( n \). The corresponding Schrodinger equation in Brinkman variables thus becomes
\[ i\partial_\tau \psi_y = H_B \psi_y \sim -\frac{p_-\chi}{2\tau^2} y^2 \psi_y \quad \Rightarrow \quad i\partial_{1/\tau} \psi \sim \chi y^2 \psi, \]
which is a potential-dominated time-independent harmonic oscillator equation in the time variable $T = \frac{1}{\tau}$. However this time variable with $T \to \infty$ as $\tau \to 0$ is ill-defined: for instance, it takes an infinite amount of time to reach $\tau = 0$ in the $\{T, y\}$-variables so that continuing past $\tau = 0$ is not possible. Alternatively we can solve for the time-dependence of the wavefunction $\psi_y \sim e^{i\frac{\tau}{2} p y^2}$, which has a wildly oscillating phase as $\tau \to 0$, rendering it ill-defined there. This leads us to expect that the quantum system in these variables is ill-defined near $\tau \sim 0$.

Now we consider Rosen variables: then for $A > 0$, we see that the Hamiltonian $H_R$ is dominated by the kinetic term as $\tau \to 0$. The Schrödinger equation becomes

$$i\partial_\tau \psi_x(\tau, x) = H_R \psi(\tau, x) \sim -\frac{p_x^2}{2p_\tau^A} \psi(\tau, x) = \frac{1}{2p_\tau^A} \partial^2_x \psi(\tau, x).$$ (24)

The time-dependence of the wavefunction near $\tau \to 0$ can be solved for to give

$$\psi_x(\tau, x) \sim e^{-i\frac{1-A}{4} \partial^2_x \psi_x(0, x)},$$

which has a well-defined phase if $A < 1$. Alternatively, we can recast this as a free Schrödinger equation in terms of the variable $\tau^{1-A} \sim \lambda$. More precisely, consider recasting this system by redefining in (15) a new time variable given by $d\lambda = \tau^A d\tau$, i.e. $\tau = ((1-A)\lambda)^{\frac{1}{1-A}}$. This recasts the system as a harmonic oscillator subjected solely to an external time-dependent frequency: the associated Schrödinger equation is

$$i\partial_\lambda \psi(\lambda, x) = H_\lambda \psi(\lambda, x) = \frac{1}{2} \left( \frac{p_\lambda^2}{2p_-} + \frac{n^2p_-}{2} \lambda^{-\frac{2A}{1-A}} x^2 \right) \psi(\lambda, x).$$ (25)

As the time $\lambda$ approaches $\lambda \to 0$, for $A < 1$ the potential terms are subdominant and this system approaches

$$i\partial_\lambda \psi(\lambda, x) \sim \frac{p_\lambda^2}{2p_-} \psi(\lambda, x),$$ (26)

which is a free Schrödinger equation, completely regular near $\lambda = 0$ if $A < 1$. We thus choose $A = 1 - \sqrt{1-4\chi}$, since the other value is necessarily $A^+ > 1$. We are also assuming that $0 < \chi \leq \frac{1}{4}$ for $A$ to be well-defined.

In the $\tau$-variables, although the wavefunction is regular near $\tau = 0$ for $A < 1$, the Hamiltonian $H_R$ appears to be ill-defined, with a diverging expectation value $\langle H_\tau \rangle \sim \frac{1}{\tau^A} \int dx \psi^*(\tau) \partial^2_x \psi(\tau)$.

This is the 1-dim harmonic oscillator analog of the string wavefunctional described previously, with two different sets of variables describing the physical system. While the Rosen formulation with $H_\lambda, H_R$, appears well-defined at the level of the wavefunctional, the Brinkman one in terms of $H_y$ appears to not be: we will study this in greater detail in what follows.

### 3.2 A more detailed analysis: wavefunctions, observables

We will now describe in greater detail the quantization of the harmonic oscillator in the two observer frames. There are some parallels between this analysis and that in [5] for a toy oscillator adapted from gauge theories dual to AdS cosmologies with spacelike singularities.
In “Brinkman” coordinates, the equation of motion and its solution, \( i.e. \) the mode function, are
\[
\ddot{y} + \left(n^2 + \frac{1}{r^2}\right)y = 0 \quad \Rightarrow \quad f_y = \sqrt{n\tau}(c_1 J_{\frac{\lambda}{2\sqrt{\tau}}} (n\tau) + c_2 Y_{\frac{\lambda}{2\sqrt{\tau}}} (n\tau)) .
\]  
Choosing Hankel-like functions, we have early time asymptotics \( f \sim e^{-im\tau} \). Quantization proceeds by taking
\[
y = k(a_y f_y + a_y^* f_y^*) , \quad p_y = -kp_y = -p_y (a_y f_y + a_y^* f_y^*) \equiv -i\partial_y .
\]  
The constant \( k \) is fixed by the commutation relations. This gives
\[
a_y = \frac{\dot{f}_y y - f_y p_y}{k (f_y \dot{f}_y - f_y^* \dot{f}_y^*)} , \quad a_y^* = -\frac{\dot{f}_y y - f_y p_y}{k (f_y \dot{f}_y - f_y^* \dot{f}_y^*)} ,
\]  
so that the ground state wavefunction is given by \( a_y \psi_y = 0, \ i.e. \)
\[
p_y \psi_y (y, t) = -i\partial_y \psi_y (y, t) = \frac{\dot{f}_y}{f_y} y \psi_y (y, t) .
\]  
The near-singularity behaviour of this wavefunction is dictated by the asymptotics of the mode function \( f_y \sim \lambda_1 \tau^{1-\sqrt{1-4\chi}}/2 + \lambda_2 \tau^{1-\sqrt{1-4\chi}}/2 \), where \( \lambda_{1,2} \) are linear combinations\(^3\) of \( c_{1,2} \). Using \( A = 1 - \sqrt{1-4\chi} \), this is recast as \( f_y \sim \lambda_1 \tau^{1-A/2} + \lambda_2 \tau^{A/2} \). Since \( A < 1 \), we have \( f_y \rightarrow c_2 \tau^{A/2} \) as \( \tau \rightarrow 0 \), so that \( \psi_y (y, \tau) \sim e^{i\frac{\chi}{2\sqrt{\tau}}} \psi(y) \), with an ill-defined wildly oscillating phase. In more detail, we have \( \psi_y (\tau, y) = C_y (\tau) e^{i\dot{f}_y y^2 / f_y} / \tau \), where the coefficient \( C_y (\tau) \) is fixed by demanding that \( \psi_y \) solve the time-dependent Schrodinger equation. Noting that \( f_y \) solves the equation of motion, we find \( \frac{c_y}{C_y} = \frac{1}{2} \frac{\dot{f}_y}{f_y} \), giving the Brinkman wavefunction
\[
\psi_y (\tau, y) = \frac{c}{\sqrt{f_y}} e^{i\dot{f}_y y^2 / f_y} .
\]  
Excited states constructed by acting on \( \psi_y \) with \( a_y^* \) can be seen to have similar singular time-dependence in the wavefunction phase.

Now let us analyze this system in the Rosen variables: the equation of motion and the mode function solution are
\[
\partial_\tau (\tau^A \partial_\tau x) + n^2 \tau^A x = 0 \quad \Rightarrow \quad f_x = \sqrt{n\tau^{\frac{1-A}{2}}} (c_1 J_{\frac{1-A}{2\sqrt{\tau}}} (n\tau) + c_2 Y_{\frac{1-A}{2\sqrt{\tau}}} (n\tau)) .
\]  
These mode functions are in fact related as \( f_y = \tau^{A/2} f_x \), given the \( A, \chi \)-relation. The Rosen mode function has asymptotics \( f_x \sim \lambda_2 + \lambda_1 \tau^{1-A} \). These asymptotics as \( \tau \rightarrow 0 \) are of course
\[^3\text{We have } \lambda_1 = -c_2 \frac{n^{1-A/2}}{2\sqrt{\tau}} csc(\pi^{1-A/2} / 2) \), \( \lambda_2 = \frac{n^{A/2}}{2\sqrt{\tau}} (c_1 + c_2 cot(\pi^{1-A/2} / 2)) \), from the Bessel series expansions.\]
consistent with \( f_y = \tau^{A/2} f_x \): this is a crucial difference between the mode function asymptotics in these two coordinate systems and translates to the striking difference between the corresponding wavefunctions. For what follows, we note the near singularity behaviour,

\[
f_x \sim \lambda_2 + \lambda_1 \tau^{-A}, \quad \dot{f}_x \sim \lambda_1 (1-A) \tau^{-A}, \quad f_y \sim \lambda_1 \tau^{-A/2} + \lambda_2 \tau^{-A/2}, \quad \dot{f}_y \sim \lambda_1 \tau^{-A/2} + \lambda_2 A \tau^{-A/2-1},
\]

so that the Wronskian is \( f_x \dot{f}_x^* - f_x^* \dot{f}_x \sim (1-A) (c_2 c_1^* - c_1 c_2^*) \tau^{-A} \equiv \tilde{c}_0 \tau^{-A} \). With \( A < 1 \), the asymptotics above simplifies to \( f_x \sim \lambda_2, \quad \dot{f}_x \sim \lambda_1 (1-A) \tau^{-A}, \quad f_y \sim \lambda_2 \tau^{-A/2}, \quad \dot{f}_y \sim \lambda_2 A \tau^{-A/2-1} \).

Quantization in the Rosen variables proceeds as

\[
x = k (a_x f_x + a_x^\dagger \dot{f}_x^*), \quad p_x = -k p_- \tau^A \dot{x} = -k p_- \tau^A (a_x \dot{f}_x + a_x^\dagger \dot{f}_x^*).
\]

The constant \( k \) is fixed by the canonical commutation relations as

\[
[x, p_x] = i = -p_- k^2 \tau^A (f_x \dot{f}_x^* - f_x^* \dot{f}_x) [a_x, a_x^\dagger] = -p_- k^2 c_0 \quad \Rightarrow \quad k = \frac{1}{\sqrt{|c_0 p_-|}},
\]

where we have used the \( f_x \)-Wronskian \((= \tilde{c}_0 \tau^{-A})\). We thus have

\[
a_x = \frac{\dot{f}_x^* x - f_x^* \tau^A p_x}{f_x \dot{f}_x^* - f_x^* \dot{f}_x}, \quad a_x^\dagger = -\frac{\dot{f}_x^* x - f_x^* \tau^A p_x}{f_x \dot{f}_x^* - f_x^* \dot{f}_x}.
\]

The ground state wavefunction then, from \( a_x \psi_x = 0 \), is

\[
p_x \psi_x (x, \tau) = -i \partial_x \psi_x (x, \tau) = \frac{\dot{f}_x^*}{f_x^*} \tau^A x \psi_x (x, \tau) \quad \Rightarrow \quad \psi_x (x, \tau) \sim e^{i \tau^A \frac{f_x^* x^2}{2 \tau^2}} \rightarrow^{\tau \rightarrow 0} e^{i \tau^A \frac{x^2}{2 \tau^2}}.
\]

which has a well-defined phase, as \( \tau \rightarrow 0 \). In more detail, the Rosen wavefunction is

\[
\psi_x (\tau, x) = \frac{c}{\sqrt{f_x^*}} e^{i \tau^A \frac{f_x^* x^2}{2 \tau^2}},
\]

with the coefficient fixed, as for \( \psi_y \), by demanding that \( \psi_x \) solve the time-dependent Schrödinger equation. For the case \( A = 0 \), i.e. no time-dependence, this wavefunction is the usual harmonic oscillator gaussian ground state wavefunction (with \( f \sim e^{-i \tau^A} \)). Excited states can be constructed by acting with \( a_x^\dagger \): e.g. the lowest excited state can be easily seen to have the form

\[
a_x^\dagger \psi_x = \frac{x}{k f_x^*} \psi_x, \quad \text{using (36)},
\]

with the same nonsingular time-dependence in the phase of the wavefunction. This is also clearly true for generic excited states.

The Hilbert spaces would appear to be the same in both observer frames, since we have, using (29), (36),

\[
a_x = \frac{1}{k (f_x \dot{f}_x^* - f_x^* \dot{f}_x)} \left( \dot{f}_x^* \tau^{-A/2} \dot{y} - f_x^* \tau^{-A/2} (p_y - A \tau y) \right) = \frac{\tau^A}{k (f_y \dot{f}_y^* - f_y^* \dot{f}_y)} \left( \dot{f}_y^* \dot{y} - f_y^* p_y \right) = a_y,
\]

\( (39) \)
using
\[ \dot{x} = \tau^{-A/2}(y - \frac{A}{2\tau}y) \quad \Rightarrow \quad p_x = \tau^{A/2}(p_y - \frac{A}{2\tau}y) , \]
and \( f_y = \tau^{A/2}f_x \), giving \( f_x\dot{f}_x - f_x\dot{f}_x = \tau^{-A}(f_y\dot{f}_y - f_y\dot{f}_y) \).

**Observables:** We will now calculate some observables in these two frames: first, we note using \( f_y = \tau^{A/2}f_x \), \( \dot{f}_y = \tau^{A/2}(\dot{f}_x + \frac{A}{2\tau}f_x) \), that the Brinkman and Rosen wavefunctions are related as
\[ \psi_y(\tau, y) = \frac{c}{\tau^{A/4}\sqrt{f_x}} e^{i(f_x\dot{f}_x f_y\dot{f}_y)\frac{A^2}{2}} = \frac{1}{\tau^{A/4}} e^{i\tau^{-A/2}A^2} \psi_x(\tau, x) , \]
the extra phase perhaps interpretable as the reflection of the canonical transformation \( (y = \tau^{A/2}x) \) between the classical Brinkman and Rosen variables. Note that the phase prefactor above diverging as \( \tau \to 0 \) so that the Rosen wavefunction in Brinkman variables is apparently ill-defined.

Now we calculate the probability density: in Brinkman variables, we have
\[ |\psi_y(\tau, y)|^2 = \frac{|c|^2}{|f_y|} e^{i(f_x\dot{f}_x f_y\dot{f}_y)\frac{A^2}{2|f_y|^2}} = \frac{|c|^2}{|f_y|^2} e^{i(f_x\dot{f}_x - f_x\dot{f}_x)\frac{A^2}{2|f_y|^2}} = \frac{|c|^2}{|f_y|^2} e^{-i|\psi_y|^2} , \]
while in Rosen variables, we have similarly
\[ |\psi_x(\tau, x)|^2 = \frac{|c|^2}{|f_x|^2} e^{i\tau^{-A}(f_x\dot{f}_x - f_x\dot{f}_x)\frac{A^2}{2|f_x|^2}} = \frac{|c|^2}{|f_x|^2} e^{-i|\psi_x|^2} , \]
so that
\[ |\psi_y(\tau, y)|^2 = \frac{1}{\tau^{A/2}} |\psi_x(\tau, x)|^2 . \]

Using this relation (41) and \( y = \tau^{A/2}x \), we find on a constant-time surface
\[ 1 = \int dy \ |\psi_y(\tau, y)|^2 = \int \tau^{A/2}dx \ \frac{|\psi_x(\tau, x)|^2}{\tau^{A/2}} = \int dx \ |\psi_x(\tau, x)|^2 , \]
which is an important consistency check with probability conservation of our calculations in both observer frames.

Let us now calculate the position-squared expectation value. In Rosen variables, the invariant position-squared operator expectation value in the ground state, using (38), is
\[ g_{xx}\langle x^2 \rangle_x = \int dx \ \tau^A x^2 \ |\psi_x|^2 = \int dx \ \tau^A x^2 \ e^{-c_0\tau^{-A}\frac{A^2}{2|f_x|^2}} \sim \tau^A |f_x|^2 , \]

using \( \int dx x^2 e^{-\alpha x^2} = \frac{\sqrt{\pi}}{2\alpha^{3/2}} \), while that in Brinkman variables, using (31), is
\[ \langle y^2 \rangle_y = \int dy \ \frac{y^2}{|f_y|^2} e^{-c_0\frac{y^2}{2|f_y|^2}} \equiv \int dx \ \tau^{A/2} \ \frac{\tau^A x^2}{\tau^{A/2}|f_x|^2} \ e^{-c_0\frac{\tau^A x^2}{2|f_x|^2}} , \]
which is identical to the Rosen one. Correspondingly, we have
\[ g_{xx}(x^2)_x = \tau^A |f_x|^2 \langle a_x a_x^\dagger \rangle_0 = |f_y|^2 \langle a_y a_y^\dagger \rangle_0 = \langle y^2 \rangle_y . \tag{48} \]

We see that \( g_{xx}(x^2) \to 0 \) as \( \tau \to 0 \).

The fact that the position vevs are identical should not be surprising given the transformation between the variables: we expect observables containing derivatives might differ significantly in the two frames. Indeed observables such as momentum-squared and correspondingly energy are different: e.g. the natural Rosen frame momentum-squared expectation value is
\[ g^{xx}(p_x^2)_x = \tau^{-A} \tau^{2A} |\dot{f}_x|^2 \langle a_x a_x^\dagger \rangle_0 = \tau^A |\dot{f}_x|^2 \langle a_x a_x^\dagger \rangle_0 , \tag{49} \]
while the natural Brinkman frame momentum-squared expectation value is
\[ \langle p_y^2 \rangle_y = \langle |\dot{f}_y|^2 a_y a_y^\dagger \rangle_0 = \tau^A |\dot{f}_x|^2 \langle a_x a_x^\dagger \rangle_0 \tag{50} \]
which are quite different. In particular in the vicinity of the singularity \( \tau \to 0 \), the asymptotics are
\[ \text{Rosen}: \ \sim \ \tau^{-A} \langle a_x a_x^\dagger \rangle_0 , \quad \text{Brinkman}: \ \sim \ \tau^{A-2} \langle a_x a_x^\dagger \rangle_0 . \tag{51} \]

For \( A < 1 \), as \( \tau \to 0 \), we see that the Brinkman momentum-squared is more divergent. Note however that there exist a set of Brinkman observables that simulates the natural Rosen observables.

The expressions (16), (21), for the Hamiltonian in the Rosen and Brinkman frames can be simplified to obtain
\[ H_R = -\frac{k^2 p_y - \tau^A}{2} \left[ a_x^2 (f_x^2 + n^2 f_x^2) + (a_x^\dagger)^2 (f_x^2 + n^2 f_x^2) + (a_x a_x^\dagger + a_x^\dagger a_x)(|\dot{f}_x|^2 + n^2 |f_x|^2) \right] , \tag{52} \]
and
\[ H_B = -\frac{k^2 p_y - \tau^A}{2} \left[ (a_y a_y^\dagger + a_y^\dagger a_y)(|\dot{f}_y|^2 + (n^2 + \frac{X}{\tau^2}) |f_y|^2) + a_y^2 (f_y^2 + (n^2 + \frac{X}{\tau^2}) f_y^2) + (a_y^\dagger)^2 (f_y^2 + (n^2 + \frac{X}{\tau^2}) f_y^2) \right] . \tag{53} \]

Note that the Rosen and Brinkman Hamiltonians do not transform into each other through the field redefinitions: e.g. the coefficients of \{a, a^\dagger\} in the two expressions differ as \( \tau^A (|\dot{f}_x|^2 + n^2 |f_x|^2) = |\dot{f}_y|^2 + (n^2 + \frac{X}{\tau^2}) |f_y|^2 - \frac{d}{d\tau} (\frac{A|f_y|^2}{2\tau}) \), consistent with the fact that the Lagrangians differ by the total derivative term. The ground state energy expectation values are
\[ \langle H_R \rangle \sim \tau^A \tau^{-2A} k^2 |p_-| , \quad \langle H_B \rangle \sim \tau^{A-2} k^2 |p_-| . \tag{54} \]

To see this in dimensionful detail, we note that \( \text{dim}[x^+] = \text{dim}[\tau] = \text{dim}[x^-] = \text{length}(L) \), \( \text{dim}[(x^+)A^2] = \text{dim}[y] = L \), \( \text{dim}[m] = L^{-1} \), \( \text{dim}[p_-] = L^{-1} \). Then the potential term in
(15) (and correspondingly in (20)) is really \( \frac{n^2}{l^2} x^2 \), where \( l \) is the string coordinate length with \( \text{dim}[l] = L \). This gives e.g. the dimensionless modes \( f_x = \sqrt{n} \left( \frac{1}{2} - \frac{A}{2} \right) \left( c_1 J_{1-A} \left( \frac{\sqrt{2}}{l} x \right) + c_2 Y_{2-A} \left( \frac{\sqrt{2}}{l} x \right) \right) \), and \([x,p_x] = i = \frac{p_x \cosh^2 \left( \frac{A}{2} \right) [a_x, a_x^+], \) so that \( k = \sqrt{\frac{1-A}{\text{dim}[\lambda]}} \), which has \( \text{dim}[k] = \text{dim}[x] \). Finally this gives \( H_R \sim |p_-| \left[ \frac{2}{l^2} \text{dim}^{-A} \text{dim}^{-A} \right] (a_x^+ a_x + \ldots) \), so that \( \text{dim}[H_R] = \text{dim}[\frac{1}{l}] \).

### 3.3 The free particle

This is essentially the \( n = 0 \) limit of the harmonic oscillator we have already discussed: however it is instructive. So consider the free particle with (Rosen) action, momenta and Hamiltonian

\[
S_R = \frac{m}{2} \int \! dt \left[ -2 \partial_\tau x^+ + \tau^A \dot{x}^2 \right], \quad H_R = -\frac{p_x^2}{2p_- \tau^A}, \quad p_- = \frac{\partial L_R}{\partial (\partial_\tau x^-)} = -m, \quad p_x = m \tau^A \dot{x}.
\]

(55)

Then the equation of motion is \( \partial_\tau (\tau^A \partial_\tau x) = 0 \), giving \( x = c_1 \tau^{1-A} + c_2 \) and \( \dot{x}^- = \frac{\partial H_R}{\partial p_-} = \frac{1}{2} \tau^{A-2} \sim c_1 \tau^{1-A} \), \( i.e. \) \( x^- \sim \tau^1 \). If \( A < 1 \), these are well-defined particle trajectories, with finite \( x \to c_2 = \text{const}, \) \( i.e. \) non-diverging as \( \tau \to 0 \). The Schrodinger time evolution is well-defined for fixed \( x, p_x \).

Now consider the redefinition to the Brinkman variable \( y = \tau^{A/2} x \). We have \( \dot{x} = \tau^{-A/2} \left( \dot{y} - \frac{A}{2\tau} y \right) \), giving

\[
S_B = \frac{m}{2} \int \! dt \left[ -2 \partial_\tau y^- + \dot{y}^2 - \frac{\chi}{2} \tau y^2 \right], \quad H_B = -\frac{p_y^2}{2p_-} - \frac{p_- \chi}{2\tau^2} y^2.
\]

(56)

where we have used the conjugate momenta \( p_y = m \dot{y}, \) \( p_- = \frac{\partial L_B}{\partial \dot{y}^-} = -m = p_R \). For this Brinkman \( S_B \) system, we have the equation of motion \( \partial_\tau^2 y + \frac{\chi}{2\tau} y = 0 \), giving the solutions \( y = c_1 \tau^{1-A/2} + c_2 \tau^{A/2} = \tau^{A/2} x \), where \( A = 1 - \sqrt{1 - 4\chi} \). These trajectories all have \( y \to 0 \) as \( \tau \to 0 \). Also we have \( \dot{y}^- = \frac{1}{2} \dot{y}^2 - \frac{1}{2} \frac{\chi}{\tau^2} y^2 \sim \tau^{A-2} \), giving \( y^- \sim \tau^{A-1} \), which diverges.

While the Brinkman system looks like a time-dependent harmonic oscillator with a divergent frequency \( \frac{\chi}{\tau} \), the classical solutions are not oscillatory. The Hamiltonians here are in fact those obtained from free particle propagation seen by Rosen and Brinkman observers respectively. The Schrodinger time evolution is ill-defined for fixed \( y, p_y \).

### 4 String quantization: Rosen and Brinkman variables

We now continue our discussion of string propagation following from sec. 2 earlier, where we have described the Hamiltonians and the lightcone string Schrodinger wavefunctionals in the two frames. We saw there that the lightcone string Schrodinger wavefunctional for the generic state has well-defined evolution in the vicinity of the singularity in Rosen variables if the Kasner exponents satisfy \( |A_j| \leq 1 \). We will now discuss string quantization in greater detail in Rosen.
and Brinkman variables: in part this will review the analysis of [1, 2]. There are close parallels with our earlier discussion on the time-dependent harmonic oscillator which is a 1-dim single momentum mode of the string.

The closed string worldsheet action \( S = -\int \frac{d\tau d\sigma}{4\pi\alpha'} \sqrt{-g} h^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) \) takes the form \( S_R = -\frac{1}{4\pi\alpha'} \int d^2\sigma (-2E g_+ \partial_\tau X^- - E g_{11}(\partial_\tau x^I)^2 + \frac{1}{2} g_{11}(\partial_\sigma x^I)^2) \), in Rosen variables, upon using lightcone gauge \( y^+ = \tau \) and setting \( h_{\tau\sigma} = 0 \), \( E(\tau, \sigma) = \sqrt{\frac{\nu_{aa}}{h_{\tau\tau}}} \), as in [26] (see also [27]). The momentum conjugate to \( x^- \) can be fixed to a \( \tau \)-independent constant \( p_- = \frac{E g_+}{2\pi\alpha' l} = -\frac{1}{2\pi\alpha' l} \) by a \( \tau \)-independent \( \sigma \)-reparametrization invariance (not fixed by the gauge fixing above), giving \( E = -\frac{1}{g_{++}} = 1 \). The quantity \( l \) here is the string coordinate length, which from above becomes \( l = -2\pi p_+ \alpha' = 2\pi |p_-| \alpha' \) (note that \( p_- \leq 0 \) always, and we will sometimes use \( p_- \) to denote \( |p_-| \)). [Our conventions here agree with [28] for flat space.] Thus we see that for the spacetime backgrounds we consider (with \( g_{++} = -1 \)), lightcone gauge \( y^+ = \tau \) is compatible with conformal gauge \( h_{ab} = \eta_{ab} \). The string Hamiltonian, \( H = -p_+ \), in Rosen variables [5], after re-expressing the momenta in terms of derivatives, becomes

\[
H_R = \frac{1}{4\pi\alpha'} \int_0^l d\sigma \, \tau A^I \left( (\partial_\tau x^I)^2 + (\partial_\sigma x^I)^2 \right), \tag{57}
\]

containing only the physical transverse modes \( x^I(\tau, \sigma) \). The range of \( \int d\sigma \) is \( \int_0^{2\pi p_- |\alpha'|} d\sigma \), involving the lightcone momentum \( p_- \). This Hamiltonian is the physical Hamiltonian \( H = -p_+ \) satisfying the physical state condition \( m^2 = -2g^+ p_+ p_- - g_{11}(p_{10})^2 \). In lightcone gauge \( y^+ = \tau \), the worldsheet Schrödinger equation also essentially governs the evolution in spacetime \( i\frac{\partial}{\partial\tau}\Psi \) of the string wavefunctional. From the Hamiltonian \( H[x^-, p^-, x^I, \Pi^I] \), we can solve for \( x^- \) using \( \partial_\tau x^- = \frac{\partial H}{\partial p_-} \).

Similarly, the string action in Brinkman variables is \( S_B = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left( 2\partial_\tau y^- - (\partial_\tau y^I)^2 + (\partial_\sigma y^I)^2 + \frac{\chi_I}{\tau^2}(y^I)^2 \right) \), with the Brinkman Hamiltonian [6] becoming

\[
H_B = \frac{1}{4\pi\alpha'} \int_0^l d\sigma \left( (\partial_\tau y^I)^2 + (\partial_\sigma y^I)^2 + \frac{\chi_I}{\tau^2}(y^I)^2 \right). \tag{58}
\]

As should be clear, single oscillator/momentum modes of the string are essentially identical in description to the time-dependent harmonic oscillator discussed earlier. In what follows, we will discuss the explicit quantization of the string.

The mode expansion for the spacetime coordinate fields of the string is

\[
X^I(\tau, \sigma) = X^I_0(\tau) + \sum_{n=-\infty}^{\infty} \left( k_n^I f_n^I(\tau)(a_n^I e^{in\sigma/l} + \bar{a}_n^I e^{-in\sigma/l}) + k_n^I f_n^I(\tau)(a_n^I e^{-in\sigma/l} + \bar{a}_n^I e^{in\sigma/l}) \right), \tag{59}
\]

where \( f_n^I(\tau) \) are the mode functions in either Rosen or Brinkman coordinate frames, following from the worldsheet equations of motion. The conjugate momentum is \( \Pi^I = \frac{\tau A^I}{2\pi\alpha'} \partial_\tau x^I \) and
\( \Pi^I_y = \frac{\partial_x y^I}{2\pi \alpha'} \) in Rosen and Brinkman variables respectively. The constants \( k^I_n \) are fixed as \( k^I_{Bn} = \frac{i}{\hbar} \sqrt{\frac{2\alpha' \lambda}{2\pi \alpha'}} \) (Brinkman variables) and \( k^I_{Rn} = \frac{i}{\hbar} \sqrt{\frac{2\alpha' \lambda}{2\pi \alpha'}} \) (Rosen variables) by demanding that the canonical commutation relations for the fields be consistent with the creation-annihilation operator algebra \([a^I_n, a^I_{-m}] = [\tilde{a}^I_n, \tilde{a}^I_{-m}] = n\delta^{I,I'}\delta_{nm}\) (or equivalently, the canonical commutation relations for the \( X^I, \Pi^I \)). This effectively makes the mode functions \( f^I_n \) dimensionless. The no-scale property of these spacetimes, i.e. requiring no explicit length scale, is manifest in the Brinkman form, so that the dimensions of the Rosen variables \( x^I \) are nontrivial for consistency. Nontrivial factors of \( l \) enter in the string quantization in accord with this.

To calculate the string Hamiltonian, let us first use the mode expansion (59) to evaluate

\[
\frac{1}{l} \int_0^l d\sigma (\partial_\sigma X^I)^2 = (\dot{X}_0^I)^2 + \sum_n |k^I_n|^2 \left( |\tilde{f}^I_n|^2 (\{a^I_n, a^I_{-n}\} + \{\tilde{a}^I_n, \tilde{a}^I_{-n}\}) - (\dot{f}^I_n)^2 \{a^I_n, \tilde{a}^I_n\} \right),
\]

\[
\frac{1}{l} \int_0^l d\sigma (\partial_\sigma X^I)^2 = \sum_n n^2 |k^I_n|^2 \left( |f^I_n|^2 (\{a^I_n, a^I_{-n}\} + \{\tilde{a}^I_n, \tilde{a}^I_{-n}\}) - (\dot{f}^I_n)^2 \{a^I_n, \tilde{a}^I_n\} \right),
\]

\[
\frac{1}{l} \int_0^l d\sigma X^I)^2 = (X_0^I)^2 + \sum_n |k^I_n|^2 \left( |f^I_n|^2 (\{a^I_n, a^I_{-n}\} + \{\tilde{a}^I_n, \tilde{a}^I_{-n}\}) - (\dot{f}^I_n)^2 \{a^I_n, \tilde{a}^I_n\} \right).
\]

The Rosen Hamiltonian then simplifies to

\[
H_R = \frac{l}{2\alpha'} \langle \dot{X}_0^I \rangle^2 \left( \frac{\dot{\chi}_0}{2} + \frac{\chi_0^2}{2} \right) + \frac{l}{2\alpha'} \sum_n |k^I_{Rn}|^2 \left( \left( \{a^I_n, a^I_{-n}\} + \{\tilde{a}^I_n, \tilde{a}^I_{-n}\} \right) \left( |\tilde{f}^I_{Rn}|^2 + \frac{n^2}{l^2} |f^I_{Rn}|^2 \right) - \{a^I_n, \tilde{a}^I_n\} \left( (\dot{f}^I_{Rn})^2 + \frac{n^2}{l^2} (f^I_{Rn})^2 \right) \right),
\]

while the Brinkman Hamiltonian becomes

\[
H_B = \frac{l}{2\alpha'} \left( \frac{\langle \dot{y}_0^I \rangle^2}{2} + \frac{\chi_0^2}{2} \right) + \frac{l}{2\alpha'} \sum_n |k^I_{Bn}|^2 \left[ \left( \{a^I_n, a^I_{-n}\} + \{\tilde{a}^I_n, \tilde{a}^I_{-n}\} \right) \left( |\tilde{f}^I_{Bn}|^2 + \frac{n^2}{l^2} \sum_i \frac{\chi_i}{\tau^2} |f^I_{Bn}|^2 \right) - \{a^I_n, \tilde{a}^I_n\} \left( (\dot{f}^I_{Bn})^2 + \frac{n^2}{l^2} \sum_i \frac{\chi_i}{\tau^2} (f^I_{Bn})^2 \right) \right].
\]
The mode functions in the mode expansion above are

\[
R : \quad f^I_{Rn}(\tau) = \frac{\sqrt{\lambda_{\nu}}}{l^{\frac{3}{2}}} \left( c_{n1} J_{\frac{1}{2} - \Delta I} \left( \frac{n\tau}{l} \right) + c_{n2} Y_{\frac{1}{2} - \Delta I} \left( \frac{n\tau}{l} \right) \right),
\]

\[
B : \quad f^I_{Bn}(\tau) = \frac{\sqrt{\lambda_{\nu}}}{l^{\frac{3}{2}}} \left( c_{n1} J_{\frac{1}{2} + \Delta I} \left( \frac{n\tau}{l} \right) + c_{n2} Y_{\frac{1}{2} + \Delta I} \left( \frac{n\tau}{l} \right) \right),
\]

which, using the relation (2) between \( A_I, \chi_I, \) are related as \( f^I_{Bn} = \left( \frac{\tau}{l} \right)^{\Delta I/2} f^I_{Rn}, \) and correspondingly for the asymptotic forms too. Using this relation between the modes, we see that the Rosen and Brinkman Hamiltonians do not transform into each other through the field redefinitions: e.g. the coefficient of \( \{a^I_n, a^I_{-n}\} \) in the Rosen Hamiltonian is

\[
\frac{|k^I_{Rn}|^2}{|k^I_{Bn}|^2} T^A_I \left( \left| f^I_{Rn} \right|^2 + \frac{n^2}{l^2} \left| f^I_{Rn} \right|^2 \right) = \left| f^I_{Bn} \right|^2 + \left( \frac{n^2}{l^2} - \frac{\chi_I}{r^2} \right) \left| f^I_{Bn} \right|^2 - \frac{d}{d\tau} \left( \frac{A_I |f^I_{Bn}|^2}{2\tau} \right).
\]

This is consistent with the fact that the field redefinition \( y^I = \tau^{\Delta I/2} x^I \) transforms \( x^-, y^- \) in the corresponding Rosen and Brinkman Lagrangians by the total time derivative term.

The string oscillator masses are then given by \( m^2 = -2g^+ - H^- p_- - g^I (p^0)^2, \) which simplifies to the terms containing the oscillator operators in the Hamiltonians. Thus we see that the detailed string spectra in the Rosen and Brinkman frames are different.

To find the mode asymptotics in the vicinity of the singularity, define a cutoff surface at constant \( y^+ \equiv \tau = \tau_c \) a little away from the singularity at \( y^+ = 0. \) Then the low lying states \((n \ll \frac{1}{\tau_c})\) have mode asymptotics essentially identical to the harmonic oscillator described earlier, with power law behaviour \( f^I_{Rn} \sim \lambda_{2n} + \lambda_{1n} (\frac{\tau}{\tau_c})^{1-A_I}, \) with corresponding asymptotics for the Brinkman modes (the \( \lambda^I_n \)'s arise from the Bessel series expansions; see footnote 3 in the harmonic oscillator case). A single low lying oscillator mode \(-\{I, n\} (n \ll \frac{1}{\tau_c})\) has mass \( m^2 \sim g^+ - H^- p_- \sim \frac{\lambda^I_{2n}}{(\frac{\tau}{\tau_c})^{\Delta I}} \) in Rosen variables, which is light relative to the local curvature scale \( \frac{1}{\tau_c}, \) if \( \frac{\lambda^I_{2n}}{(\frac{\tau}{\tau_c})^{\Delta I}} \ll \frac{1}{\tau_i}. \) Likewise in Brinkman variables, a single low lying oscillator mode has mass \( m^2 \sim g^+ - H^- p_- \sim \frac{\lambda^I_{2n}}{(\frac{\tau}{\tau_c})^{\Delta I}}, \) different in form from the Rosen one: this is light relative to the local curvature scale if \( \frac{\tau^A_I p}_{l^2} \ll 1. \) These two conditions look a priori different: they are consistent if \( \frac{\tau^A_I p}{l^2} \ll \frac{(\tau_c)^{\Delta I}}{l^2} \ll \frac{\tau_i}{l}, \) which using \( l \sim p_- \alpha', \) simplifies to \( p_- \ll \frac{\tau_i}{\tau_c} \equiv \frac{1}{\beta^2} \). This condition also arises from consistency for other modes being light as we will see later.

We now see that for any cutoff \( \tau_c \) no matter how small, there exist highly stringy modes defined by the limit of small \( \tau_c, \) large \( n, \) with \( n\tau_c \gg 1. \) These are modes that effectively do not see the singularity, thus behaving like flat space modes. In the Rosen frame, these modes have \( f^I_{Rn} \sim \frac{e^{-in\tau_c/\tau_c}}{\tau_c l^3/2}, \) for \( c^I_{n1} = 1, c^I_{n2} = -i, \) and the Hamiltonian simplifies to

\[
H_R \sim \frac{1}{l} \sum_{n \gg 1/\tau_c} (a_{-n} a_{n}^I + a_{-n} a_{n}^I + n).
\]
The corresponding oscillator mass for a single highly stringy mode is (using \( l = 2\pi |p_-|\alpha' \)) \( m^2 \sim -g^{+\mbox{-}} H_{BP_-} \sim \frac{n}{\alpha'} \). \( (66) \)

Similarly in the Brinkman frame, we have \( f_B^I n \sim e^{-inr_c/l} \) for \( c^I_{n1} = 1, c^I_{n2} = -i \), with the Hamiltonian simplifying to \( H_B \sim \frac{1}{l} \sum_{n \gg 1/r_c} (1 - \frac{\chi_l}{2n^2r_c^2})(a^I_{-n}a^I_n + a^I_{-n}\tilde{a}^I_n + n) - \mathcal{O}(\frac{1}{(n\tau_c)^2}) \), \( (67) \)

the same as \( (65) \) to leading order, and the mass of a single highly stringy mode becoming \( m^2 \sim -g^{+\mbox{-}} H_{BP_-} \sim \frac{n}{\alpha'} \). Note that the norms of the mode amplitudes are \( g_{II}|f|^2 \), which is the same in both observer frames, \( i.e. \tau^A(\frac{1}{\tau_c^2})^2 \sim \text{const} \) (Rosen) and likewise for Brinkman. We see that the highly stringy modes have similar structure in either frame, perhaps not surprisingly since these are essentially modes that are sufficiently high frequency that they effectively do not see the approaching singularity at the hypersurface \( \tau = \tau_c \). Now note that in the vicinity of the cutoff surface \( y^+ = y_c^+ \), the local curvature scale is \( \frac{1}{\alpha'(y_c^+)^2} \). Thus various highly stringy single oscillator states are light on a near-singularity cutoff surface if \( m^2 \ll \frac{1}{(y_c^+)^2} \). This gives \( \frac{p_-\alpha'}{(y_c^+)^2} \ll n \ll \frac{\alpha'}{(y_c^+)^2} \), \( (68) \)

the first inequality arising from our definition of highly stringy modes. This implicitly implies \( p_- \ll \frac{1}{y_c^+} \). Note that the coordinate length \( l = 2\pi |p_-|\alpha' \) of the string increases as the lightcone momentum \( p_- \) increases. Thus we have \( l \ll l_s(\frac{1}{y_c^+}) \). For a Planck scale cutoff \( y_c^+ \sim l_P \), we thus have \( l \ll \frac{l}{g_s^2(\frac{1}{y_c^+})} \), using the naive relation for the Newton constant \( G_D = l_P^{D-2} = g_s^2 l_s^{D-2} \). Thus in the weakly coupled (or free) string limit, the string coordinate length can be large for these states in the vicinity of the singularity, \( i.e. \) the string can effectively be a large floppy object. The number of such oscillator levels excited is \( \frac{\alpha'}{(y_c^+)^2}(1-p_-y_c^+) \). In the singular limit \( y_c^+ \rightarrow 0 \), all oscillator states are light and the number of excited oscillator states diverges. Conversely in the sector \( p_- \sim \frac{1}{y_c^+} \), the window of light highly stringy states pinches off. On a string scale cutoff surface \( y_c^+ \sim l_s \), we see that no string oscillators are turned on, \( i.e. n \sim 1 \) is already not a light state from \( (68) \). On a Planck scale cutoff surface, the highest oscillator level turned on is of order \( n \sim (\frac{l}{l_P})^2 \sim \frac{1}{g_s^2(\frac{1}{y_c^+})} \), \( i.e. \) in the free string limit \( g_s \rightarrow 0 \), we have a large number \( n \gg 1 \) of highly stringy oscillator states \( [1] \).

Thus various (highly stringy) light oscillator states arise near the singularity, in all observer frames, suggesting string interactions are non-negligible near the singularity.

**Observables:** Let us now calculate the expectation value of the position-coordinate-squared for the string. Considering the string to be a discretized set of oscillators, this would be
\[ \langle \sum_k g_{II}(X^I_{\sigma_k})^2 \rangle, \text{ which in the continuum limit becomes } \langle \frac{1}{l} \int d\sigma \ g_{II}(X^I)^2 \rangle. \] Since different coordinate directions behave differently due to the anisotropy of the spacetime, we calculate this expectation value for each direction \( I \) separately, i.e. the index \( I \) is not summed over. We then have, using (60),

\[ \langle \frac{1}{l} \int d\sigma \ g_{II}(X^I)^2 \rangle = (X^I_0)^2 + \sum_n |k_n|^2 g_{II}|f_n^I|^2(\{a_n^I, a_{-n}^I\} + \{\tilde{a}_n^I, \tilde{a}_{-n}^I\}), \tag{69} \]

where we are considering states with single excitations, e.g. \( a_{-m}^I \tilde{a}_{-m}^I \ket{} \). It is clear that this observable is identical in both Rosen and Brinkman frames, as for the harmonic oscillator discussed previously, noting that the mode functions are related as \( f^I_{Br} = (\frac{2}{\tau})^{A_I/2} f^I_{Rn} \). In the Rosen variables, since \( f^I \to \text{constant} \) for the low lying oscillator modes, we see that the spatial directions with \( A_I > 0 \) have the string shrinking, while those with \( A_I < 0 \) have the string elongating.

For the highly stringy oscillator modes, we have \( f^I \to \frac{\epsilon^{-m_I}}{\tau A_I} \), so that \( \langle \frac{1}{l} \int d\sigma \ g_{II}(X^I)^2 \rangle \to \text{const.} \)

It is natural to define the momentum-squared expectation value likewise as the continuum version of the discretized observable \( \langle \sum_k g_{II}(\Pi^I_{\sigma_k})^2 \rangle = \langle \sum_k g_{II}(\partial_\tau X^I_{\sigma_k})^2 \rangle \), which becomes \( \langle \frac{1}{l} \int d\sigma \ g^{II}(\Pi^I)^2 \rangle = \langle \frac{1}{l} \int d\sigma \ g_{II}(\partial_\tau X^I)^2 \rangle \). Then, using (60), we have

\[ \langle \frac{1}{l} \int d\sigma \ g^{II}(\Pi^I)^2 \rangle = g_{II}(X^I_0)^2 + \sum_n |k_n|^2 g_{II}|f_n^I|^2(\{a_n^I, a_{-n}^I\} + \{\tilde{a}_n^I, \tilde{a}_{-n}^I\}), \tag{70} \]

again considering states with single excitations, e.g. \( a_{-m}^I \tilde{a}_{-m}^I \ket{} \). We see now that the natural momentum observable in the Rosen frame is \( \langle \frac{1}{l} \int d\sigma \ \tau^{A_I}(\partial_\tau x^I)^2 \rangle \), whereas it is \( \langle \frac{1}{l} \int d\sigma \ (\partial_\tau y^I)^2 \rangle \) in the Brinkman frame. These however are not the same clearly: we have

\[ \frac{(\Pi^I_y)^2}{\tau A_I} = \left( \Pi^I_y - \frac{A_I}{2\tau} y^I \right)^2. \tag{71} \]

Part of the kinetic energy of the string in the Rosen frame looks like a potential energy in the Brinkman frame with a time-dependent prefactor that diverges near the singularity. In detail, we see from the asymptotics of the mode functions \( f^I_{Br}, f^I_{Rn} \), (similar to (33) for the time-dependent harmonic oscillator) that the Rosen momentum-squared expectation value is less divergent than the Brinkman one (for \( |A_I| \leq 1 \))

\[
\begin{align*}
\text{Rosen} : & \quad \langle \frac{1}{l} \int d\sigma \ \tau^{-A_I}(\Pi^I_y)^2 \rangle \sim \tau^{A_I}|f^I_{Br}|^2 \sim \tau^{-A_I}, \\
\text{Brinkman} : & \quad \langle \frac{1}{l} \int d\sigma \ (\Pi^I_y)^2 \rangle \sim |f^I_{Br}|^2 \sim \tau^{A_I-2},
\end{align*}
\tag{72}
\]

very similar to (51) for the harmonic oscillator.

The expectation value of the energy can be evaluated in similar fashion. Focussing on the singularities with \( |A_I| \leq 1 \) for each \( I \) (in the next section, we show that such singularities
exist but only in certain windows for the \( \{ \chi_I \} \), the mode function asymptotics as \( \tau \to 0 \) are 
\[ f_I^{R_n} \sim \lambda_{I2n}^{2}, \quad \dot{f}_I^{R_n} \sim \frac{\lambda_I}{\tau} (\dot{\tau})^{-A_I}. \]
Since the time-dependent factor (in the Rosen variables) in the Hamiltonian appears as \( \tau^{\pm A_I} \) in the kinetic and potential energy terms respectively, for both \( A_I > 0 \) and \( A_I < 0 \) one factor will dominate as \( \tau \to 0 \) and diverge as \( \tau^{-|A_I|} \). Thus the Hamiltonian expectation value in a state with some oscillators in Rosen variables, using (61) and (65), on a cutoff surface \( \tau = \tau_c \), is
\[
\langle H_R \rangle \sim \frac{l}{2\alpha'} \tau_c^{A_I} \left( \frac{\dot{x}_I^0}{2} \right)^2 + \frac{l}{2\alpha'} \sum_{n \ll \frac{1}{\tau_c}} \left| k_{Rn}^I \right|^2 (N_n^I + \tilde{N}_n^I + n) C_n^I + \frac{1}{l} \sum_{n \gg \frac{1}{\tau_c}} (N_n^I + \tilde{N}_n^I + n),
\]
where the constants \( C_n^I \) are either \( \frac{\alpha^2}{\tau_c} \lambda_{2n} |^2 \) or \( \frac{|A_I|^2}{\tau_c} \) depending on whether \( A_I < 0 \) or \( A_I > 0 \) respectively, the \( N_n^I, \tilde{N}_n^I \) being the number of oscillators turned on. We have approximated the energy contributions from the low lying and highly stringy states on the cutoff surface. Since the number of such oscillator levels (both low lying and highly stringy) excited increases as \( y^+ \to 0 \), we expect that the total energy imparted to the string also increases, and the expression above indicates a divergence as we approach the singularity.

The Brinkman Hamiltonian expectation value \( \langle H_B \rangle \) has similar structure, except with the zero mode contribution being \( \frac{l}{2\alpha'} \left( \frac{\dot{x}_0^0}{2} \right)^2 + \frac{\chi}{\tau_c} \left( y_0^0 \right)^2 \), and the low lying oscillator contribution having a time-dependent factor \( \sim \tau_c^{-A_I} \).

5 Discussion

We have studied spacetimes with null cosmological singularities and string propagation in their background: these are essentially anisotropic plane waves with singularities. As we have discussed (argued in [1]), the free string lightcone Schrodinger wavefunctional is regular in the vicinity of the singularity in the Rosen frame where the null Kasner exponents satisfy \( |A_I| \leq 1 \), while ill-defined in the Brinkman and other Rosen frames. Only certain singularities admit a Rosen frame of this sort with a well-defined wavefunctional, as we have shown by analysing the \( A_I, \chi_I \)-relations between the Kasner exponents and the plane wave parameters. Although the wavefunctional is well-defined in these Rosen frames suggesting well-defined time evolution across the singularity, various physical observables for the free string, in particular the energy, are still divergent in this Rosen frame as we have seen. This makes the significance of the apparent regularity of the Schrodinger wavefunctional in the “nice” Rosen frames less clear and suggests that in fact the free string limit is singular in any frame. Finally we note that various single string oscillator states become light in the vicinity of the singularity, perhaps consistent with the fact that the free string limit is breaking down, suggesting that string interactions become important in the vicinity of the singularity. Perhaps a second quantized description, say
string field theory, is the appropriate framework for investigating if string evolution is smooth across these singularities.

Perhaps it is worth noting that the wavefunctions discussed here are not Hamiltonian eigenstates. What we have done is to construct a cutoff constant-(null)time surface in the vicinity of the singularity (akin to a stretched horizon outside a black hole), identify mode asymptotics and thereby construct “near-singularity” stringy states living on this cutoff surface. This is somewhat different in spirit from following the time evolution towards the singularity of a given eigenstate. Likewise the oscillator masses are the “instantaneous” masses, perhaps best interpreted as arising from the spectrum of string fluctuations in the vicinity of the singularity. It would be interesting to develop a deeper understanding of these near-singularity string states/wavefunctionals. Part of our motivation here stems from intuition arising from the investigations of cosmological singularities in AdS/CFT [3,4,5]. The dual gauge theory effective action is subject to renormalization effects: defining this precisely in a Wilsonian fashion by say constructing the effective potential on a cutoff constant-time surface in the vicinity of the singularity might be dual to understanding stringy effects on a corresponding cutoff surface in the bulk. The present case of purely gravitational plane wave spacetimes is entirely closed string, but presumably the bulk intuition should hold nevertheless.

The spacetimes considered here can be thought of near singularity limits of spacetimes of the general form $ds^2 = -2dx^+dx^- + e^{f_I(x^+)}(dx^I)^2$, where the $e^{f_I}$ are null scale factors which crunch at some location say $x^+ = 0$, possibly as $e^{f_I} \rightarrow (x^+)^{A_I}$. These are in Rosen form: the redefinition $y^I = e^{f_I/2}x^I$, $y^- = x^- + f_I(y^I)^2$, transforms this to a Brinkman-like form $ds^2 = -2dx^+dy^- + (dy^I)^2 + \frac{1}{4}((f'_I)^2 + 2f''_I)(y^I)^2(dx^+)^2$ (with $f'_I \equiv \partial_+ f_I$). The Rosen form spacetimes are solutions if $R_{++} = 0$, giving $\sum_I (\frac{1}{2}(f'_I)^2 + f''_I) = 0$, which is one equation for $D-2$ scale factors $f_I$, for a $D$-dim spacetime. Thus the space of such cosmologies is large and we can choose the scale factors in various ways, to suit the physical question we are interested in. In particular, choosing the $e^{f_I} \rightarrow 1$ at early times $x^+ \rightarrow -\infty$ renders the spacetimes asymptotically flat at early times, in both Rosen and Brinkman frames. Note also that spacetimes with $e^{f_I} \rightarrow (x^+)^{A_I}$ near $x^+ = 0$ acquire a Rosen-like null Kasner cosmology form we have elaborated on earlier, with the corresponding Brinkman form discussed earlier.

Looking more closely at the null Kasner solutions, we note that the equation of motion can be written as $2\sum_I A_I = \sum_I A^2_I$, very similar to the condition $\sum_i p_i = \sum_i p_i^2$ for the usual Kasner cosmologies with exponents $p_i$. However while the $p_i$ there also were required to satisfy $\sum_i p_i = 1$ (from the two equations for $R_{tt}, R_{ii}$), the single equation of motion here stemming from $R_{++}$ gives just one condition on the $A_I$, and more freedom in the space of such cosmologies. It might be interesting to understand more general null cosmologies where the spatial slices are curved: these would be null analogs of the well-known BKL cosmologies (discussed in the AdS/CFT context in [5]), and it would be interesting to explore the role of spatial curvatures,
and the approach to the singularity.

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