A brief review of the pion-nucleon sigma-term is given. Aspects of both chiral perturbation theory and phenomenology are discussed.

1 Introduction

The pion-nucleon sigma-term is defined as

$$\sigma = \frac{\hat{m}}{2m_p} \langle p|\bar{u}u + \bar{d}d|p\rangle, \quad \hat{m} = \frac{1}{2}(m_u + m_d),$$

i.e. as the proton matrix element of the $u$- and $d$- quark mass term of the QCD Hamiltonian ($m_p$ is the mass of the proton). More generally, sigma-terms are proportional to the scalar quark currents

$$\langle A|m_q\bar{q}q|A\rangle ; q = u, d, s ; A = \pi, K, N.$$ These are of interest, because they are related to the hadron mass spectrum, to the scattering amplitudes through Ward identities, to the strangeness content of $A$, to the quark mass ratios and to the question of dark matter. For an early review of the topic, see ref. [1].

The pion-nucleon sigma-term is the $t = 0$ value of the scalar form factor

$$\bar{u}'\sigma(t)u = \hat{m} \langle p'|\bar{u}u + \bar{d}d|p\rangle, \quad t = (p' - p)^2,$$

i.e. $\sigma = \sigma(t = 0)$. The strangeness content of the proton can then be defined as

$$y = \frac{2 \langle p|\bar{s}s|p\rangle}{\langle p|\bar{u}u + \bar{d}d|p\rangle}$$

(the OZI rule would imply $y = 0$).

Algebraically the $\sigma$ can be written in the form

$$\sigma = \frac{\hat{m}}{2m_p} \frac{\langle p|\bar{u}u + \bar{d}d - 2\bar{s}s|p\rangle}{1 - y},$$

where the numerator is proportional to the octet breaking piece in the Hamiltonian. To first order in SU(3) breaking we have now

$$\sigma \simeq \frac{\hat{m}}{m_s - \hat{m}} \frac{m_\Xi + m_\Sigma - 2m_N}{1 - y} \simeq \frac{26 \text{ MeV}}{1 - y},$$

where the quark mass ratio

$$\frac{m_s}{\hat{m}} = 2 \frac{M_K^2}{M_\pi^2} - 1 \simeq 25$$

has been used.
Chiral perturbation theory (ChPT) allows for the determination of the combination
\[ \hat{\sigma} = \sigma(1 - y) \]
from the baryon spectrum. Therefore, if the sigma-term can be determined from data, the strangeness content \( y \) can be estimated.

Section 2 will be dealing with the ChPT aspects of the sigma. The phenomenological discussion will follow in section 3. A brief summary is given in section 4 together with reference to recent developments in the lattice frontier.

2 The \( \sigma \)-term

ChPT gives in leading order
\[ \hat{\sigma} \simeq 26 \text{ MeV} \]
as indicated above. The \( O(m_q^{3/2}) \) calculation of Gasser and Leutwyler \[2, 3\] yields
\[ \hat{\sigma} = 35 \pm 5 \text{ MeV}. \]
Borasoy and Meißner \[4\] have made the calculation in the heavy baryon framework of ChPT to order \( O(m_q^2) \) with the result
\[ \hat{\sigma} = 36 \pm 7 \text{ MeV}. \]

2.1 Scalar form factor

Contact to pion-nucleon scattering can be made at the unphysical Cheng-Dashen point \((s = u = \sqrt{m_N^2}, t = 2M_{\pi}^2)\) and, therefore, it is of interest to determine the difference of the scalar form factor
\[ \Delta_{\sigma} \equiv \sigma(2M_{\pi}^2) - \sigma(0). \]

In leading order we have \[3, 5\]
\[ \Delta_{\sigma} = \frac{3g^2A_{\pi}^3}{64\pi F_{\pi}^2} + O(M_{\pi}^4 \ln M_{\pi}^2), \]
which is numerically about 7 MeV. ChPT to one loop yields \( \Delta_{\sigma} \simeq 5 \text{ MeV} \[3\]. In heavy baryon ChPT (HBChPT) including the \( O(p^4) \) pieces due to the low-lying spin-3/2 baryons the result \( \Delta_{\sigma} \simeq 15 \) MeV is obtained \[7\]. A dispersion analysis where particular emphasis was on the treatment of the \( \pi\pi \) interaction dominating the curvature of the \( \sigma(t) \) yields \[8\]
\[ \Delta_{\sigma} = 15.2 \pm 0.4 \text{ MeV}. \]

More recently, Becher and Leutwyler have calculated \[9\] the scalar form factor to order \( p^4 \) in a formulation of the baryon ChPT which keeps the Lorentz and chiral invariance explicit at all stages. The result for \( \Delta_{\sigma} \) is
\[ \Delta_{\sigma} = 14.0 \text{ MeV} + 2M_{\pi}^4\tilde{e}_2, \]
where \( M \) is the leading order result for \( M_{\pi} (M^2 = 2mB) \) and \( \tilde{e}_2 \) is a renormalized coupling constant due to the \( \mathcal{L}_N^{(4)} \) lagrangian. Comparison with the result of the dispersive calculation shows that the piece proportional to \( \tilde{e}_2 \) is small as it should be.
The value of the form factor at \( t = 0 \), i.e. the \( \sigma \), can be calculated from the quark mass expansion of the nucleon mass by making use of the Feynman-Hellmann theorem

\[
\sigma = \hat{m} \frac{\partial m_N}{\partial \hat{m}}
\]

or equivalently

\[
\sigma = M^2 \frac{\partial m_N}{\partial M^2}.
\]

The physical mass of the nucleon to order \( p^4 \) is \([8, 10]\)

\[
m_N = m_0 + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln \frac{M^2}{m_0^2} + k_4 M^4 + O(M^5),
\]

where \( m_0 \) is the nucleon mass in the chiral limit and the factors \( k_i \) contain the low-energy constants. This yields for \( \sigma \)

\[
\sigma = k_1 M^2 + 3 k_2 M^3 + k_3 M^4 \{2 \ln \frac{M^2}{m_0^2} + 1\} + 2 k_4 M^4 + O(M^5)
\]

and numerically

\[
\sigma = (75 - 23 - 7 + 0) \text{ MeV} = 45 \text{ MeV},
\]

where the leading term, 75 MeV, is fixed by requiring the result \( \sigma = 45 \) MeV for the sigma-term \([13]\). Also, the term \( k_4 M^4 \) is put equal to 0, because it is expected to be very small.

A \( O(p^3) \) calculation in HBChPT, where the low-energy constants are fixed inside the Mandelstam triangle, gives \([14]\) the value \( \sigma = 40 \) MeV, if Karlsruhe phase shifts (KA84) are used as input. The VPI input (SP99) would yield \( \sigma \simeq 200 \) MeV.

### 2.2 Cheng-Dashen point

As mentioned earlier, contact to the pion-nucleon interaction can be made at the Cheng-Dashen point: A low-energy theorem of chiral symmetry states

\[
\Sigma \equiv F_2^2 \bar{D}^+(\nu = 0, t = 2M^2) = \sigma(2M^2) + \Delta_R NOC,
\]

where \( \nu = (s - u)/4m_N \), \( \bar{D}^+ \) is the isoscalar \( D \)-amplitude with the pseudovector Born term subtracted and \( \Delta_R \) is the remainder. The quantity \( \Delta_R \) is formally of the order \( M^4 \), and the one-loop result [a \( O(p^3) \) value] is \( \Delta_R = 0.35 \) MeV \([9]\). In HBChPT it has been shown that no logarithmic contribution to order \( M^4 \) appears \([11]\). This result is verified in the \( O(p^4) \) calculation of the pion-nucleon amplitude \([12]\). Numerically, with the low-energy constants estimated with the resonance exchange saturation, the result is \( \Delta_R \simeq 2 \) MeV \([11]\) which is considered to be the upper limit for \( \Delta_R \). Therefore, it can well be approximated as

\[
\Sigma \simeq \sigma(2M^2).
\]

One may, of course, ask how this result would change, if the fact \( m_u - m_d \neq 0 \) would be taken into account.
3 \( \Sigma \) phenomenology

The standard expression for the \( \pi N \) amplitude is

\[
T_{\pi N} = \bar{u}[A(\nu, t) + \frac{1}{2} \gamma^\mu (q + q')_\mu B(\nu, t)]u,
\]

where \( q \) and \( q' \) are the initial and final pion momentum respectively. The \( D \)-amplitude is

\[
D(\nu, t) = A(\nu, t) + \nu B(\nu, t)
\]

and, through the optical theorem,

\[
\text{Im} D(\omega, t = 0) = k_{\text{lab}} \sigma.
\]

Its imaginary part in the forward direction is directly fixed by the cross section data (\( \omega \) is the initial pion laboratory energy). The isospin components are simply related to the amplitudes in the particle basis

\[
D^\pm = \frac{1}{2}(D_{\pi-p} \pm D_{\pi+p}).
\]

The relevant combination for the \( \Sigma \) -term discussion is the isoscalar piece, \( D^+ \), at the Cheng-Dashen point.

The standard value with the Karlsruhe input has been the result [15]

\[
\Sigma = 64 \pm 8 \text{ MeV}
\]

based on hyperbolic dispersion relations. The error reflects the internal consistency of the method. An attempt to include an estimate of the error in \( \Sigma \) generated by the errors of the low-energy data was published in ref. [13]. The numerical result there was \( \Sigma \approx 60 \text{ MeV} \) with the Karlsruhe input.

The \( \Sigma \) can also be related to the threshold parameters [14]

\[
\Sigma = F_\pi^2 [L(a_{1+}^+, \tau) + (1 + \frac{M_\pi}{m_N})\tau J^+] + \delta_{\text{ChPT}},
\]

where \( L \) is a linear combination of the threshold parameters and \( \tau \) is a free parameter, \( J^+ \) is the integral over the total cross section

\[
J^+ = \frac{2M_\pi^2}{\pi} \int_0^\infty \frac{\sigma^+(k')}{\omega(k')^2} dk'
\]

and \( \delta_{\text{ChPT}} \) is the remainder from ChPT, see also ref. [17], where references to earlier work in a similar spirit can be found. In such a formulation the contribution from \( a_{1+}^+ \) to \( \Sigma \) may vary from -150 MeV to 250 MeV for \( \tau \in [-1, 1] \). Olsson has recently [18] written a sum rule for \( \Sigma \) which includes an expansion in terms of threshold parameters. With the Karlsruhe input the consistent result, \( \Sigma = 55 \pm 6 \text{ MeV} \), follows. With input from ref. [19] the value \( \Sigma = 71 \pm 9 \text{ MeV} \) is obtained.

3.1 Low-energy analysis

At low energy the pion-nucleon interaction is dominated by six partial waves, 2 \( s \)-waves and 4 \( p \)-waves. Therefore, six relations are needed to pin down the six partial waves. Such relations can be obtained by writing six dispersion relations for the \( D^\pm \), \( B^\pm \) and \( E^\pm \) where

\[
E^\pm = \frac{\partial}{\partial t}(A^\pm + \omega B^\pm)|_{t=0}.
\]
There are two subtraction constants in the 6 dispersion relations, one for $D^+$ and $E^+$ ($x = M_\pi/m_N$):

$$
\bar{D}^+(\mu) = 4\pi(1 + x)a_{0+}^+ + \frac{g^2x^3}{M_\pi(4 - x^2)}
$$

$$
\bar{E}^+(\mu) = 6\pi(1 + x)a_{1+}^+ - \frac{g^2x^2}{M_\pi^2(2 - x^2)}.
$$

As described in ref. [13] the six dispersion relations can be solved iteratively with input for the invariant amplitudes from high energy (here $k_{lab} \geq 185$ MeV/c) and for the high partial waves at low energy. The method allows for fixing two of the constants in the subthreshold expansion for $\bar{D}^+$ in powers of $\nu^2$ and $t$

$$
\bar{D}^+ = d_{00}^+ + d_{10}^+\nu^2 + d_{01}^+ t + d_{20}^+\nu^4 + d_{11}^+\nu^2 t + ..., 
$$

where

$$
d_{00}^+ = \bar{D}^+(0), \quad d_{01}^+ = \bar{E}^+(0).
$$

The curvature term $\Delta_D$ is defined by

$$
\Sigma = F_\pi^2(d_{00}^+ + 2M_\pi^2d_{01}^+) + \Delta_D \equiv \Sigma_d + \Delta_D,
$$

where $\Delta_D$ is dominated by the $\pi\pi$ cut giving [8]

$$
\Delta_D = 11.9 \pm 0.6 \text{ MeV}.
$$

$\Sigma_d$ is a sensitive quantity as is demonstrated by the numerical values for the two solutions, A and B, of ref. [13], where $\Sigma_d = 48 - 50$ MeV with an error of about 10 MeV. Now the question is how the value for $\Sigma$ would change, if amplitudes based on modern meson factory data would be used as input instead of the Karlsruhe amplitudes where input mostly consisted of data before the meson factory era. With the VPI/GWU input (SM99 and SM01) [20] results typically in the range

$$
\Sigma_d = ([-80. \text{ to } -77.]) + [146. \text{ to } 157.]) \text{ MeV} = 65. \text{ to } 80. \text{ MeV}
$$

follow, i.e. a considerably larger value than the Karlsruhe input would give. The corresponding $\pi^-p$ scattering length

$$
a_{\pi^-p} = 0.0857 - 0.0899 M_{\pi}^{-1}
$$

is to be compared with the experimental value $0.0883 \pm 0.0008 M_{\pi}^{-1}$ [21].

4 Summary

In a lattice calculation the value for $\sigma$ can be obtained from the quark mass expansion of the nucleon mass by making use of the Feynman-Hellmann theorem as given above. A new development has recently been the inclusion of dynamical quarks [22] giving $\sigma = 18 \pm 5$ MeV. In general, there is, however, the problem that the value for $m_q$ is still quite large and the extrapolation to small quark mass values is uncertain.

The nucleon mass in full (two-flavour) QCD as a function of the pion mass has been calculated by UKQCD [23] and CP-PACS [24] collaborations. The values so found have been fitted with a ChPT-inspired expression [25]

$$
m_N = \alpha + \beta M_{\pi}^2 + \sigma_{NN}(M_\pi, \Lambda) + \sigma_{N\Delta}(M_\pi, \Lambda)
$$
for the quark mass dependence of the nucleon mass leading to the result $\sigma = 45 - 55$ MeV. The functions $\sigma_{NN}$ and $\sigma_{\Delta N}$ are due to the nucleon self-energy diagrams with an intermediate nucleon and delta respectively.

Promising steps have been made in the lattice frontier, but still more work is needed. For the phenomenological part questions remain. It turns out that $\Sigma$ is a quite sensitive quantity and, therefore, requirements of consistency of the low-energy data and analysis are of particular importance. E.g., $\Sigma_d$ is sensitive to the high partial waves at low energy. An additional problem is the question of electromagnetic corrections close to the physical threshold, see [26, 27].

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