Separating points through inclusions of $\mathcal{O}_\infty$

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Abstract. We define a basis property that an inclusion of $C^*$-algebras $\mathcal{O}_\infty \subset A$ may have, and give various conditions for the property to hold. Some applications are considered.

1. Introduction

It occurs frequently in $C^*$-algebraic problems that one has, or wishes to construct, a unitaly embedded copy of $\mathcal{O}_\infty$ in a given $C^*$-algebra. We single out here a rather geometrical aspect of such embeddings, which could be termed the basis problem or the separation of points problem.

Problem 1. Given a unital $C^*$-algebra $A$ and a unitaly embedded copy of $\mathcal{O}_\infty$, is it true for all $x$ in $A$ that $p_ixp_j = 0$ implies $x = 0$?

In the above, $p_i$ denotes the projections of the form $v_iv_i^*$ where the $v_i$ are the generators of the copy of $\mathcal{O}_\infty$. When this problem can be solved affirmatively, we say that the given copy of $\mathcal{O}_\infty$ separates the points of the $C^*$-algebra $A$.

Clearly, the above problem depends on exactly how $\mathcal{O}_\infty$ is embedded in $A$. We point out some useful and tractable special cases. Moreover, we construct interesting examples of embeddings of $\mathcal{O}_\infty$ that separate points.

The paper is organized as follows. In section 2, we prove some lemmas, in section 3 we solve the case of $\mathcal{O}_\infty \subset B(\mathcal{H})$, in section 4 we solve the case of $\mathcal{O}_\infty \subset \mathcal{O}_n$, and in section 5 the case of inclusions of $\mathcal{O}_\infty$ into a corona algebra. Some applications are considered in section 6. Since, for the most part, distinct techniques need to be used for the different cases, each section contains a short introduction of the techniques used in that section.

2. Some useful lemmas

Definition 2. We say a set $S$ of positive elements is strictly positive if for every state $\rho$, there exists some element of $S$ on which $\rho$ is nonzero.

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We recall that for a one-element set \( S = \{ s \} \), the property of being strictly positive can be characterized in either of two ways:

**Lemma 3.** The following are equivalent

1. A one-element set \( S = \{ s \} \) is strictly positive,
2. zero is not an eigenvalue of \( s \) in the double dual \( A^{**} \), and
3. an element \( s \) is strictly positive in a C*-algebra \( A \) if and only if every element \( y \in A \) can be factorized as \( y = f(s)y' \) for some continuous function \( f \in C_0(0, \|s\|) \).

**Proof.** That i and ii are equivalent is [23, pg. 2658]. That i implies iii is [22, Cor. 4.6]. If iii holds, then by considering square roots, it follows that any positive element \( x \) can be written as \( x = f(s)yf(s) \), and Kasparov’s strict positivity criterion [15, par. 7] implies that \( s \) is strictly positive. Thus iii implies i. □

The following elementary lemma connects strict positivity with the property of separating points, showing that if \( \{ p_i \} \) is strictly positive, then the set of projections \( \{ p_i \} \subset B \) separates the points of \( B \).

**Proposition 4.** The basis problem can be solved for \( O_\infty \subset B \) if the subset \( \{ v_i v_i^* \} \) is strictly positive in \( B \), where \( v_i \) denotes the generators of \( O_\infty \).

**Proof.** Denote the projections \( v_i v_i^* \) by \( p_i \). Suppose that the set \( \{ p_i \} \) is strictly positive in \( B \). Let \( m := \sum 2^{-n} p_i \), where the sum converges in norm. If \( \sigma \) is a state, then we note that \( \sigma(m) \geq 2^{-n} \sigma(p_n) \), and strict positivity of the set \( \{ p_i \} \) implies that \( \sigma(p_n) \) is nonzero for at least one value of \( n \), in which case \( \sigma(m) \) is also nonzero. Thus, the element \( m \) is itself strictly positive in \( B \). We are to show that if \( x \in B \) is an element such that \( p_i x p_j = 0 \) then \( x = 0 \). We begin by noting that \( x \in B \) is such that \( p_i x p_j = 0 \), then we have \( m x m = 0 \). Since \( m \) is strictly positive, any element \( y \in B \) can be factored as \( y'' f(m) \), by Lemma[3]part iii. Then for any two elements \( y, y' \in B \) we have the factorization

\[
y x y' = y'' f_1(m) x f_2(m) y''.
\]

Then, approximating the functions \( f_i \) by polynomials with no constant term, we conclude that for \( x \) with \( m x m = 0 \), the above factorization (1) is equal to zero. In other words, we have \( y x y' = 0 \) for any \( y, y' \in B \). But this means that the algebraic ideal generated by the element \( x \) is zero, and this implies that \( x \) is zero. □

The separation of points problem has good behaviour with respect to quotients:
Proposition 5. Suppose that \( O_\infty \subset B \) is such that the set \( \{ v_i v_i^* \} \) is strictly positive in \( B \), where \( v_i \) denotes the generators of \( O_\infty \). Suppose that \( I \) is a two-sided proper ideal of \( B \). If we replace \( B \) by the quotient \( B/I \), then the image of \( O_\infty \) in the quotient will separate the points of \( B/I \).

Proof. Strict positivity in \( B \) implies strict positivity in any nontrivial quotient \( B/I \). This is because a state of \( B/I \) can be composed with the canonical quotient map to give a state of \( B \), and thus is nonzero on the given set. But then by Proposition 4 the image of the set of projections \( \{ v_i v_i^* \} \) separates the points of the quotient \( B/I \).

Semicontinuity. Next we will use some facts about semicontinuity in \( C^* \)-algebras. It is well known that for ordinary real functions, if a function and its negation are both (lower) semicontinuous, then the function is continuous. This fact generalizes nicely to the noncommutative setting as follows. We need only the unital case:

Lemma 6 ([25, th. 3.12.9]). Let \( A \) be a unital \( C^* \)-algebra. Then \( A_{sa} = (U_{sa})^m \cap (U_{sa})_m \) where \( (U_{sa})^m \) (respectively \( (U_{sa})_m \)) denotes the set of limits of increasing (resp. decreasing) nets of self-adjoint elements of \( A \).

The significance of the above lemma is that certain sums that can be defined in the double dual in fact define elements of the \( C^* \)-algebra. In the presence of a solution to the basis problem, we obtain the following theorem:

Theorem 7 (Continuity). Let \( c_i \) denote elements of the unit ball of a unital \( C^* \)-algebra \( Q \), and let \( v_i \) denote the generators of a unitally embedded copy of \( O_\infty \), for which the basis problem holds. Then the sum \( T := \sum_1^\infty v_i v_i^* c_i v_i v_i^* \) defines an element of the \( C^* \)-algebra \( Q \).

Proof. Consider the sum, as above:

\[
T := \sum_1^\infty v_i v_i^* c_i v_i v_i^*
\]

where the \( c_i \) are arbitrary elements of the positive unit ball of a unital \( C^* \)-algebra \( Q \), and the sum converges in the double dual \( Q^{**} \). We note that the sum is monotone increasing, so that \( T \) is a (lower) semi-continuous element of the double dual. Replacing \( c_i \) by \( 1 - c_i \) we have

\[
1 - T = \sum_1^\infty v_i v_i^* (1 - c_i) v_i v_i^*
\]

which shows that \( 1 - T \) is also given by a monotone increasing sum. But then both \( T \) and \(-T\) are lower semicontinuous. Applying Lemma 6 shows
that the element \( T \) defined by the sum is actually in the algebra \( \mathcal{Q} \), as was to be shown. This proof assumed that the \( c_i \) were positive, but we can drop the restriction to positive elements by using the fact that a general element of a \( C^* \)-algebra can be written as a linear combination of four positive elements.

The above theorem provides a conditional expectation from a unital \( C^* \)-algebra \( \mathcal{Q} \) onto the diagonal subalgebra (of diagonal elements) in \( \mathcal{Q} \). These constructions will be used in Section 6.

We remark that a similar proof implies that \( N \)-diagonal infinite matrices also define elements of the \( C^* \)-algebra:

**Corollary 8.** Let \( c_{ij} \) denote elements of the unit ball of a unital \( C^* \)-algebra \( \mathcal{Q} \). Suppose that \( c_{ij} = 0 \) if \( |i - j| > N \), for some fixed integer \( N \), and let \( v_i \) denote the generators of a unitaly embedded copy of \( \mathcal{O}_\infty \) for which the basis problem holds. Then the sum \( T := \sum_{i,j=1}^{\infty} v_i v_i^* c_{ij} v_j v_j^* \), where the \( v_i \) are as in Corollary 1, defines an element of the \( C^* \)-algebra \( \mathcal{Q} \).

### 3. Separating points and the Cuntz-Krieger property for \( \mathcal{O}_\infty \subset B(\mathcal{H}) \)

As we will now show, the problem of separating points reduces to known quantities in the case of \( \mathcal{O}_\infty \subset B(\mathcal{H}) \).

A copy of \( B(\mathcal{H}) \) containing a unital copy of \( \mathcal{O}_\infty \) is equivalent to a representation of the \( C^* \)-algebra \( \mathcal{O}_\infty \) on the given Hilbert space \( \mathcal{H} \). One of the fundamental properties of representations of \( \mathcal{O}_\infty \) is that they may or may not be essential. We define a representation of \( \mathcal{O}_\infty \) to be **essential** if the projections \( v_i v_i^* \) sum strongly to \( 1 \) in the representation. It seems that this property was first described, but not named, by Cuntz and Krieger in [8, Remark 2.15], and the term essential might be due to Arveson. His monograph [3] gives more information on the property of being essential. In the case of \( B(\mathcal{H}) \) we have the following characterization of the separation of points property.

**Theorem 9.** The basis problem for \( \mathcal{O}_\infty \subset B(\mathcal{H}) \) can be solved if and only if, viewing the given inclusion map as a representation, the map is essential.

**Proof.** Let \( p_i \) denote the diagonal projections in \( \mathcal{O}_\infty \). If the representation is not essential, then \( \sum p_i \) does not converge strongly to \( 1_{B(\mathcal{H})} \). But then it converges to some projection \( P \in B(\mathcal{H}) \), other than \( 1 \), and evidently \( P \) acts as \( 1 \) on each of the projections \( p_i \). But then \( p_i (1 - P) p_j = 0 \), while \( 1 - P \neq 0 \), so that the basis problem has a counter-example. On the other hand, if the representation is essential, then it is routine to check that the
set \( \{ p_i \} \) is strictly positive, and we can apply Proposition \([4]\) to solve the basis problem.

Since every \( C^* \)-algebra is a sub-\( C^* \)-algebra of some \( B(\mathcal{H}) \), the above case of \( B(\mathcal{H}) \) is in a sense the most general case of the separation of points problem, and as we have seen, cannot always be solved affirmatively. Given an embedding of \( \mathcal{O}_\infty \) into \( B(\mathcal{H}) \), the above result gives a criterion for being able to solve the separation of points problem. It shows that there do exist embeddings \( \mathcal{O}_\infty \subset B(\mathcal{H}) \) for which this problem can be solved affirmatively.

4. On \( \mathcal{O}_\infty \subset \mathcal{O}_n \)

The case of \( \mathcal{O}_\infty \subset \mathcal{O}_n \) has a more algebraic flavour. In this section, we show the existence of a natural embedding of \( \mathcal{O}_\infty \) into \( \mathcal{O}_n \) such that the diagonal projections of \( \mathcal{O}_\infty \) separate the points of \( \mathcal{O}_n \). Let \( s_1, \ldots, s_n \) and \( t_1, t_2, \ldots \) denote generators of \( \mathcal{O}_n \) and \( \mathcal{O}_\infty \), respectively. Define the embedding \( f_{n,\infty} \) of \( \mathcal{O}_\infty \) into \( \mathcal{O}_n \) by

\[
f_{n,\infty}(t_{(n-1)k+i}) := s_n^k s_i \quad (k \geq 0, \ i = 1, \ldots, n-1)
\]

where we define \( s_n^0 := I \). The existence of this natural embedding \( f_{n,\infty} \) has been observed before, and our description follows \([17]\) Definition 1.4(ii).

Now we are in a position to bring our Theorem \([9]\) to bear. To do this, we shall choose an essential representation of \( \mathcal{O}_\infty \) and show that we can extend it to a representation of \( \mathcal{O}_n \). It will then follow from Theorem \([9]\) that the basis problem can be solved for elements of \( B(\mathcal{H}) \) in \( \mathcal{O}_\infty \subset \mathcal{O}_n \subset B(\mathcal{H}) \), in which case the basis problem is solved for elements of \( \mathcal{O}_n \).

It remains to show that we can in fact extend a representation of \( \mathcal{O}_\infty \) to a representation of \( \mathcal{O}_n \). In the classic theory of \( C^* \)-algebras, there is a standard process \([10]\) prop. 2.10.2] for the extension of representations to a larger algebra, but it requires enlarging the given Hilbert space. In our more algebraic situation given by inclusions of Cuntz algebras, one can use constructions with generators to be able to extend representations of \( \mathcal{O}_\infty \) to representations of \( \mathcal{O}_n \) without enlarging the given Hilbert space. We recall the following known property of representations of Cuntz algebras. Given a representation \( (\mathcal{H}, \pi) \) of \( \mathcal{O}_\infty \), and an embedding \( \mathcal{O}_\infty \subset \mathcal{O}_n \), we can define a representation \( (\mathcal{H}, \pi') \) of \( \mathcal{O}_n \) by

\[
\pi'(s_i) = \begin{cases} 
\pi(t_i) & (i = 1, \ldots, n-1), \\
\pi(I) - \sum_{j=1}^{\infty} \pi(t_j t_j^*) + \sum_{j=1}^{\infty} \pi(t_j+n-1 t_j^*) & (i = n).
\end{cases}
\]

It can be verified that \( \pi'(s_1), \ldots, \pi'(s_n) \) satisfy the Cuntz relations for \( \mathcal{O}_n \).
By definition, if $\pi = 0$, then $\pi' = 0$. We summarize the construction in a lemma, for whose proof see [16, Lemma 3.7].

**Lemma 10.** Let $O_\infty$ be embedded by $f_{n,\infty}$ in $O_n$. For any representation $\pi$ of $O_\infty$ there exists a representation $\pi'$ of $O_n$ that extends the original representation $\pi$ in the sense that $\pi' \circ f_{n,\infty} = \pi$.

This shows that we can indeed extend a representation of the smaller algebra, $O_\infty$, to a representation of the larger algebra, $O_n$, without changing the ambient Hilbert space. Applying this construction to an essential representation of $O_\infty$, the extended representation then separates the points of the larger algebra $O_n$ by Theorem [9]. We thus have the main result of this section:

**Corollary 11.** The basis problem can be solved for $O_\infty \subset f_{n,\infty} O_n$.

5. The corona $Q := \frac{\mathcal{M}(B)}{B}$

Corona algebras provide a huge supply of nontrivial orthogonal elements, but on the other hand, their representation theory is complex, due to the fact that a corona algebra often cannot be represented faithfully on a separable Hilbert space. This makes them an interesting test case for our theory, and moreover there are applications to Kasparov’s $KK$-theory.

We briefly recall [24] the definitions of open, closed, and regular projections. To every hereditary subalgebra $H \subset B$ corresponds a open projection in the double dual, which is a projection $p$ in the double dual such that $pB^{**}p \cap B = H$. Closed projections can be defined to be the complements of open projections. In general, each projection $p$ in $B^{**}$ has a closure $\bar{p}$, which is the smallest closed projection majorizing it. A projection is termed regular if $\|px\| = \|\bar{p}x\|$ for all $x \in B$.

In a von Neumann algebra, every open projection in its double dual is regular [24, pg.28], and although corona algebras are not usually von Neumann algebras, they have the following remarkable property:

**Proposition 12** ([24, Thm 21]). Let $A$ be a $\sigma$-unital $C^*$-algebra with corona algebra $Q$. Then every open $\sigma$-unital projection in $Q^{**}$ is regular.

Now we can improve Proposition 4 to an if and only if characterization.

**Proposition 13.** The separation of points problem can be solved for $O_\infty \subset Q$ if and only if the set $\{v_i v_i^*\}$ is strictly positive in $Q$, where $v_i$ denotes the generators of $O_\infty$.

**Proof.** One direction was already shown in Proposition 4. For the converse, we show that if $\{v_i v_i^*\}$ is not strictly positive then the separation of
points problem cannot be solved. Let $H$ denote the hereditary subalgebra of the corona $Q$ that is generated by the projections $v_i v_i^*$. This hereditary subalgebra is then not all of $Q$. Now let $p$ denote the $\sigma$-unital open projection associated with $H$. Let $\overline{p}$ be the smallest closed projection majorizing $p$. If it were the case that $(1 - \overline{p})x(1 - \overline{p}) = 0$ for all $x$ in $Q$, then $||px|| = ||x||$ for all $x$ in $Q$. But then the previously mentioned regularity of $p$ would imply that $||px|| = ||x||$ for all $x \in Q$, and hence that $H$ is all of $Q$, which is a contradiction. Thus we conclude that $(1 - \overline{p})Q(1 - \overline{p})$ is a nonzero hereditary subalgebra of $Q$. Choosing some nonzero positive element $y$ of $(1 - \overline{p})Q(1 - \overline{p})$, we thus note $y$ is orthogonal to $pQp = H$ and in particular $y$ is orthogonal to the elements $v_i v_i^*$. Hence, $v_i v_i^* yv_j v_j^*$ while $y \neq 0$, providing a counterexample to the separation of points problem.

Actually, in the proof of the above, the product $yv_i v_i^*$ was already zero for all $i$. This implies, for the special case of corona algebras, something slightly stronger than the basis property, so we record this as a corollary.

**Corollary 14.** Given an inclusion $\mathcal{O}_\infty \subset Q$ where $Q$ is the corona algebra of a C*-algebra, the following are equivalent:

1. $\mathcal{O}_\infty$ separates the points of $Q$.
2. For a positive element $x \in Q$, if $xp_i = 0$ for all $i$, then $x = 0$.

Recall that a C*-algebra that is algebraically isomorphic to a von Neumann algebra is called a $W^*$-algebra. As shown by Sakai, $W^*$-algebras stand in the same relationship to von Neumann algebras as abstract C*-algebras do to concrete C*-algebras, and the bridge between the two theories is provided by Sakai’s universal normal representation [26.1.16.7]. The universal normal representation in general is a proper subrepresentation of the C*-algebraic universal GNS representation.

By Sherman’s theorem [10 pg.235] we may regard $U^{**}$ as a C*-algebra in a natural way so that the canonical embedding of $U$ into $U^{**}$ is an algebraic *-isomorphism. $U^{**}$ can be made into a von Neumann algebra, and acts as a universe in which all information about our C*-algebra can be stored. Passing to the abstract (i.e. representation free) setting of $W^*$-algebras, $U^{**}$ is called the enveloping $W^*$-algebra, and has a natural weak topology [26.1.15.2] called the ultraweak topology. This is the topology defined by the predual. The predual is the set of normal states, and normal states can be elegantly defined, in the representation free approach, by a completely additive property on families of orthogonal projections [29 cor. 3.11], i.e.:

$$\omega \left( \sum e_i \right) = \sum \omega(e_i).$$
Normal states are precisely the normal measures introduced by Dixmier when restricted to Stonean abelian subalgebras, as shown in [11] th. 2.

**Lemma 15.** Let $A$ be a unital $C^*$-algebra. Given $O_\infty \subset A$, the basis problem can be solved if the diagonal projections $v_iv_i^*$ sum ultraweakly to $1_A$.

**Proof.** We are given that the sum $\sum v_iv_i^*$, converges ultraweakly to $1_A$. We note that the elements $v_iv_i^*$ belong to $A$, as does the limit $1_A$. To show that we can separate points means that we are to show that given an element $x \in A$ such that $v_iv_i^*xv_jv_j^* = 0$, then $x = 0$. But this is clear, because multiplication is ultraweakly continuous in each variable (separately). Thus, summing over $i$ and taking ultraweak limits, $xv_jv_j^* = 0$ in $A^{**}$. Then, summing over $j$ and taking ultraweak limits, $x = 0$. Thus $x$ is zero, as was to be shown. □

A second lemma will be useful:

**Lemma 16.** Let $B$ denote a $W^*$-algebra that is representable on a separable Hilbert space. Then there exists a countable family of orthogonal projections in $B$ that sum to 1 ultraweakly.

**Proof.** Let $\{p_\lambda\}$ be a maximal family of orthogonal projections. It is not possible to represent uncountably many orthogonal projections on a separable Hilbert space [9] pg.5, so this family is necessarily countable. On the other hand, the increasing sequence of partial sums $\sum p_\lambda$ converges ultraweakly [11] Cor. 2.105, to some projection, $P \in B$. If this projection were not equal to 1, then we could have added $1 - P$ to the family of orthogonal projections, but this would contradict maximality. □

In order to bring these lemmas to bear, let us adapt a standard construction of abelian $W^*$-algebras to our situation.

Let $Q := \mathcal{M}(B \otimes K)/(B \otimes K)$, where $K$ denotes the usual compact operators on a separable Hilbert space, and $B$ is a unital $C^*$-algebra. Evidently, the subalgebra $\frac{1 \otimes B(H)}{1 \otimes K(H)}$ is embedded unitally in $Q$. The Hilbert space $\mathcal{H}$ that the compact operators in the definition of $Q$ are on is separable, and we can therefore suppose that this Hilbert space $\mathcal{H}$ is given by the classic $L^2$ functions $L^2([0, 1], \mu)$ with respect to Lebesque measure $\mu$ on $[0, 1]$. A well-known example of an abelian von Neumann algebra is provided by multiplying $L^2([0, 1], \mu)$ by essentially bounded functions in $L_\infty((0, 1], \mu)$. This construction represents elements of $L_\infty([0, 1], \mu)$ as operators in the above $B(\mathcal{H})$, and as pointed out by Arveson [21] Remark 3.13, the image of the representation does not intersect any nonzero compact operator. In other words, we have an injective $^*$-homomorphism from $L_\infty([0, 1], \mu)$ into the above $B(\mathcal{H})$, see [21] pg. 42, and moreover the canonical quotient map
from $B(H)$ into $B(H)/\mathcal{K}$ is an isomorphism when restricted to this subalgebra of multiplication operators. This is a unital $^*$-isomorphism $\iota$ from $L_\infty([0, 1], \mu)$ into $\mathcal{Q}$, and the image is thus a $W^*$-algebra as defined by Sakai\cite{Sakai}. The bi-transpose $\iota^{**}$ of the map $\iota: L_\infty([0, 1], \mu) \to \mathcal{Q}$ gives a $W^*$-homomorphism from $L_\infty([0, 1], \mu)$ into $\mathcal{Q}^{**}$, and so the map $\iota$ is a (restriction of a) $W^*$-homomorphism.

We have thus constructed a $W^*$-algebra that is a sub-$C^*$-algebra of the corona algebra, and a sub-$W^*$-algebra of the double dual, with the interesting property that this subalgebra is faithfully represented on a separable Hilbert space; without requiring that the corona algebra should be represented on a separable Hilbert space.

The lemmas show:

**Proposition 17.** There is a unitally embedded copy of $\mathcal{O}_\infty$ in the corona $\mathcal{Q}$ of the stabilization of a unital $C^*$-algebra with the property that the projections $v_i v_i^*$ sum ultraweakly to 1 in the double dual.

**Proof.** Recalling that we have a unital $W^*$-homomorphism of $L_\infty([0, 1], \mu)$ into $\mathcal{Q}$, we can use lemma 16 to exhibit a countable family of orthogonal projections, $p_i$, that sums to 1, ultraweakly. Recall that the $p_i$ were constructed as elements of a copy of $B(H)$ that is unitally embedded in the multipliers $\mathcal{M}(B \otimes \mathcal{K})$. But projections which are not compact are all Murray-von Neumann equivalent in $B(H)$, as was shown by Murray and von Neumann themselves. The partial isometries implementing these equivalences generate a copy of $\mathcal{O}_\infty$ in $B(H)$, and applying the natural map into the corona, we obtain a unitally embedded copy of $\mathcal{O}_\infty$ such that the projections on the diagonal coincide with the previously constructed projections $p_i$. But the $p_i$ were constructed to sum to 1, as we wanted. $\square$

Because $\mathcal{O}_\infty$ is semiprojective, a copy of $\mathcal{O}_\infty$ in the corona can be lifted to the multipliers; or we can inspect the proof of the above proposition for an explicit lifting. This comes from the fact that the copy of $B(H)$ used was after all embedded in the multiplier algebra. We notice that the diagonal projections of the lifted copy converge ultraweakly to 1, in the double dual of the multiplier algebra.

**Corollary 18.** There is a unitally embedded copy of $\mathcal{O}_\infty$ in the multiplier algebra of a stable separable $C^*$-algebra that separates the points of the multiplier algebra. Passing to the corona, the image of this copy of $\mathcal{O}_\infty$ separates points in the corona algebra, $\mathcal{Q}$.
6. Applications

6.1. Infinite sums of extensions. Let \( A \) and \( B \) be \( C^* \)-algebras, and let

\[ 0 \to B \to C \to A \to 0 \]

be an extension of \( B \) by \( A \) (i.e. a short exact sequence of \( C^* \)-algebras).

Recall that an extension of \( B \) by \( A \) is determined by its Busby map—the naturally associated map from \( A \) to the quotient multiplier algebra, or corona algebra, of \( B, \frac{\mathcal{M}(B)}{B} \). The \( C^* \)-algebra of the extension is recovered by forming the pullback of the Busby map and the canonical quotient map \( \mathcal{M}(B) \to \frac{\mathcal{M}(B)}{B} \).

Recall (see e.g. [12]) that, if \( B \) is stable, so that the Cuntz algebra \( \mathcal{O}_2 \) may be embedded unitally in \( \mathcal{M}(B) \), then the Brown-Douglas-Fillmore addition of extensions, defined by

\[ \tau_1 \oplus \tau_2 := s_1 \tau_1 s_1^* + s_2 \tau_2 s_2^* , \]

where \( \tau_1 \) and \( \tau_2 \) are (the Busby maps of) two extensions of \( B \) by \( A \), and \( s_1 \) and \( s_2 \) are (the images in \( \frac{\mathcal{M}(B)}{B} \) of) the canonical generators of \( \mathcal{O}_2 \) (which are isometries with range projections summing to 1), is compatible with Brown-Douglas-Fillmore equivalence (defined as unitary equivalence with respect to the unitary group of \( \mathcal{M}(B) \)—or, rather, the image of this group in \( \frac{\mathcal{M}(B)}{B} \), and the resulting binary operation on equivalence classes is independent of the embedding of \( \mathcal{O}_2 \). Indeed, it is sufficient to have a unital copy of \( \mathcal{O}_2 \) in the corona, and this observation is used in Kirchberg’s work [18], see also [12]. With respect to this operation, the equivalence classes of extensions of the \( C^* \)-algebra \( B \) by the \( C^* \)-algebra \( A \) form an abelian semigroup.

Extensions can be regarded as the basic building blocks of Kasparov’s KK-theory. In this theory as originally defined by Kasparov [15], only finite sums of extensions are defined. Actually, Kasparov defined a bivariant group \( KK_1^1(A, B) \) using Fredholm triples, and then pointed out an isomorphism with a group of extensions \( \operatorname{Ext}(A, B) \), that group having the same notation but a coarser equivalence relation than in the related BDF theory [7]. Kasparov remarks that an equivalence relation like that of BDF could be used if the extensions were known to be absorbing. At the time it was not clear which extensions were absorbing. Baaj and Skandalis studied the existence of 6-term exact sequences in KK, and pointed out the importance of a semisplitting map [5]. Skandalis defined a group called \( KK_{nuc}^1 \), which consists of semi-split extensions with a weak nuclearity condition on the semisplitting map [23]. He pointed out that if \( A \) or \( B \) was nuclear, then \( KK_{nuc}^1 \) coincided with \( KK^1 \). Arveson [4] pointed out the good stability properties of the class of semi-split extensions. In [13] it was shown how to
simplify the equivalence relation on most of the above groups, making it more like the relation used in [7], by adding a condition called the purely large property to the extensions considered. The purely large property was in fact a characterization of the previously mentioned property of being absorbing. This gave a picture of $\text{KK}^1_{\text{nuc}}(A, B)$ as purely large semisplit extensions modulo a certain form of unitary equivalence, together with a weak nuclearity condition. Further simplifications are possible with more conditions, thus for example the semisplit property is automatic if either $A$ is separable and has the LLP, and $B$ has the WEP; or if $A$ is exact and $B$ is nuclear.

Both Arveson [4] and Lin [20] have made use of extensions that could be interpreted as infinite sums. Thus, Arveson considered an extension denoted $\pi \otimes \text{Id}$ that was in effect an infinite sum. Lin gave a definition of ‘diagonal extensions’ in [20] para. 1.4], and these were again of the form $m \otimes \text{Id} \in \mathcal{M}(B \otimes K)$. Salinas [27] considered sequences of unital extensions and a weak topology, but he did not consider infinite sums of extensions. Kirchberg used an infinite sum construction in a very specific way in his work on classification [18], and we now use our results to adapt his construction slightly.

**Definition 19.** Let $\tau: A \to \frac{\mathcal{M}(B)}{B}$ be an extension. Define $\tau_\infty(a) := \sum v_i \tau(a) v_i^*$ where the sum is taken in the corona algebra, and the $v_i$ are as in Corollary 18.

The above sum is defined by Theorem 7 applied to the elements $v_i \tau(a) v_i^*$, followed by using the partial isometry equation $v_i v_i^* v_i = v_i$ to simplify. In the above sum all the extensions summed are the same, but it would be possible to allow them to vary. We will mostly be interested in the above sum, but since Corollary 18 also showed that the $v_i$ come from the multiplier algebra and separate points there, we do have an analogous definition at the level of the multiplier algebra:

**Definition 20.** Let $\hat{\tau}: A \to \mathcal{M}(B)$ be the splitting map of a trivial extension. Define $\hat{\tau}_\infty(a) := \sum v_i \tau(a) v_i^*$ where the sum is taken in the multiplier algebra, and the $v_i$ are as in Corollary 18.

To lighten the notation, we use the same symbol for $v_i$ in the multiplier algebra and in the corona algebra.

**Proposition 21.** If $\tau: A \to \frac{\mathcal{M}(B)}{B}$ is a semisplit extension, then so is the corona sum $\tau_\infty(a) = \sum v_i \tau(a) v_i^*$ where the $v_i$ are as in Corollary 18.

**Proof.** First of all, we must show that $\tau_\infty$ takes values in the corona, as implied above. But this follows from Theorem 7 with the elements $c_i$ equal
to $v_i\tau(a)v_i^*$. Since $\tau$ is a *-homomorphism, so is $\tau_\infty$. If $\hat{\tau}: A \rightarrow \mathcal{M}(B)$ is a semi-splitting map for $\tau$, then by a Stinespring theorem this semisplitting map can be taken to have the form

$$\hat{\tau}(a) = w^*\hat{k}(a)w$$

for a suitable homomorphism $\hat{k}: A \rightarrow \mathcal{M}(B \otimes \mathcal{K})$ and a multiplier algebra isometry $w$. The map $\hat{k}$ can be taken to be the splitting map of any trivial absorbing extension. (For these well-known facts, see [4] prop. pg. 353 and [15].) Applying the canonical quotient map $\pi$ into the corona, we have

$$\tau(a) = w^*\pi(\hat{k}(a))w.$$ 

Evidently, by orthogonality of the $v_i$, we have

$$v_i\tau(a)v_i^* = (v_iw^*)\pi(v_i^*\hat{k}_\infty(a)v_i)(wv_i^*) = (v_iw^*v_i^*)\pi(\hat{k}_\infty(a))(v_iwv_i^*).$$

Let us define $\delta_\infty(w)$ to be the infinite sum construction of Definition 19 applied to the image in the corona of the multiplier isometry $w$. Using equation (4) to simplify $\delta_\infty(w^*)\pi(\hat{k}_\infty(a))\delta_\infty(w)$ we conclude that

$$\tau_\infty(a) = \delta_\infty(w^*)\pi(\hat{k}_\infty(a))\delta_\infty(w).$$

Now we lift the corona algebra element $\delta_\infty(w)$ to a contraction $\bar{w}$ in the multiplier algebra, and observe that $\bar{w}^*\hat{k}_\infty(a)\bar{w}$ gives a (completely positive) semisplitting for $\tau_\infty$, as was to be shown. We remark that $\bar{w}$ is approximately diagonal, in the sense that $p_i\bar{w}p_j \in B$ for $i \neq j$. In fact, we could adjust the lifting $\bar{w}$ to be diagonal by making the off-diagonal elements zero (c.f. Theorem 7).

\[\boxeq\]

### 6.2. Remark on weak equivalence

Since, after all, the range of an extension is a subalgebra of the corona, one could wonder if unitary equivalence by unitaries from the multiplier algebra is really the most appropriate notion of equivalence to use. What if, for example, we define absorption with respect to unitary equivalence by unitaries from the corona? Calling this form of equivalence \textit{weak equivalence}, we would then consider what happens if an extension absorbs weakly nuclear trivial extensions with respect to weak equivalence. It is remarkable that this form of absorption is again characterized by the exact same algebraic property as for the more usual sort of absorption (namely, the purely large condition). We now address this point, and at the same time collect some useful facts about extensions.

In the following lemma, the term essential means that the Busby map is injective, the term full means that no nonzero element of the range of
(some) semisplitting map is contained in a proper ideal, and the term absorbing in the nuclear sense means that the extension is unitarily equivalent to its own sum with any weakly nuclear trivial extension.

**Proposition 22.** Let $A$ and $B$ be separable $C^*$-algebras, with $B$ stable. Consider a unital essential extension $\tau$ of $B$ by $A$. The following are equivalent:

(i) the extension is absorbing in the nuclear sense (i.e. absorbs trivial weakly nuclear extensions);

(ii) the extension algebra is purely large;

(iii) the image $\tau(A)$ in the corona $\mathcal{M}(B)/B$ has the property that its positive elements are properly infinite and full; and

(iv) the extension absorbs weakly nuclear trivial extensions with respect to weak equivalence (by corona unitaries).

**Proof.** From [13, Theorem, section 17] we have the equivalence of (i) and (ii). The equivalence of (ii) and (iii) is theorem 1.4 in [19]. It is clear that (i) implies (iv). We now show that (iv) implies (iii). Thus, we are given that $\tau \oplus \sigma \sim_w \tau$ where $\sim_w$ denotes unitary equivalence by a corona unitary, and $\sigma$ is any trivial weakly nuclear extension. The work of Kasparov provides an example of a purely large, weakly nuclear, and trivial extension, see [13, section 12] for a detailed proof. Choosing this extension for $\sigma$, it then follows that the sum $\tau \oplus \sigma$ is also purely large because a sum of extensions is purely large if one of the summands is ([13, lemma, section 13]) and by the already proven direction (ii) implies (iii), we have that the positive elements of the range of $\tau \oplus \sigma$ are purely infinite and full. But unitary equivalence preserves both of these properties, so then it follows that the positive elements of the range of $\tau$ itself are purely infinite and full, as we wanted to show. \qed

As Skandalis already pointed out, in the above weak nuclearity is automatic if $A$ or $B$ is nuclear [28]. The above result shows that the purely large property is insensitive to small changes in the equivalence relation used. Thus $KK(A, B)$ and $KK_w(A, B)$ are, to be sure, distinct groups, but they both have the same basic description as a group given by purely large extensions up to (a form of) unitary equivalence. Similar comments would apply to the related group $KL(A, B)$.

**6.3. Triviality of extensions.** We now turn to the main result of this section, which is the surprising result that even if we sum extensions which are just semisplit, the sum defines a trivial extension. Thus, in this theory, infinite sums are generally equal to zero. The key idea of the proof is that, by a Hilbert hotel argument, we have $\tau_\infty = \tau_\infty + \tau_\infty$, and in an abelian group, an element satisfying such an equation must be the trivial element.
Now we give a generalization of [29] th.1.12.i. We won’t actually use the full strength of the next result, but it seems interesting to provide a criterion for an extension to be absorbing. We start by recalling a technical lemma:

Lemma 23 ([14] Th. 2.1(e)) Let $H$ be a $\sigma$-unital $C^*$-algebra. If there exists a countable family of orthogonal equivalent projections in $\mathcal{M}(H)$ that sums strictly to 1, then $H$ is a stable $C^*$-algebra.

Now we prove the promised criterion:

Lemma 24 (An absorption criterion). Let $B$ a separable simple stable $C^*$-algebra, and let $A$ be a separable $C^*$-algebra. Let $\hat{\tau} : A \to \mathcal{M}(B)$ be the semisplitting map of a not necessarily trivial extension, and suppose that there are countably many orthogonal and equivalent projections $P_i \in \mathcal{M}(B)$ with respect to which $\hat{\tau}(a)$ is diagonal. Then the extension is absorbing in the nuclear sense.

Proof. We verify the purely large condition of Lemma 22: we show that the hereditary subalgebra $\overline{\hat{\tau}(a)B \hat{\tau}(a)}$ contains a nonzero stable hereditary subalgebra that is not contained in any proper ideal of $B$, for positive nonzero elements $a \in A$. The condition of not being contained in a proper ideal is automatic because $B$ is assumed simple. Let $H$ denote the hereditary subalgebra of the multiplier algebra $\mathcal{M}(B)$ that is generated by the projections $\{P_i\}$. Consider the nonempty hereditary subalgebra $H_0 := H \cap (\overline{\hat{\tau}(a)B \hat{\tau}(a)})$. By construction, the projections $P_i$ multiply $H_0$ into $H_0$. Thus, the projections $P_i$ are in $\mathcal{M}(H_0)$ and sum strictly to 1 there. By Lemma 23 this shows that $H_0$ is stable, as we wanted to show. □

Lemma 25. Let $A$ be a separable unital $C^*$-algebra and let $B$ be a separable stable $C^*$-algebra. Let $\tau : A \to \mathcal{M}(B)$ be a full semisplit unital extension. Then the corona sum $\tau_\infty = \sum v_i \tau v_i^*$ is a full semisplit extension with properly infinite range. If we assume that either $A$ or $B$ is a simple $C^*$-algebra then we can drop the assumption that the extension $\tau$ is full.

Proof. The semisplit property was shown in Proposition 21 A positive element $x$ is said to be properly infinite if $x$ is equivalent under generalized Murray von Neumann equivalence to $x \oplus x$. The definition of the sum is basically the same as the BDF sum we mentioned earlier. In the case of a positive element of the form $\tau_\infty(a)$, as in Definition 19 it is thus evident that the properly infinite property holds. It is clear that the sum is full if the summands are. Now let us instead assume that the $C^*$-algebra $B$ is simple. Recall that in the proof of Proposition 21 we found an explicit semisplitting for the given extension $\tau_\infty$, and we remarked that the semisplitting could
be taken to be diagonal. But then since the semisplitting map is diagonal, Lemma 24 applies to show that the given extension $\tau_\infty$ is absorbing, and this implies fullness. Finally, if the $C^*$-algebra $A$ is assumed simple, then the range algebra is a simple unital subalgebra of the corona, and thus every positive element of it is in the same ideal as the unit: in other words, simple range implies fullness if the range is unital. □

Now we obtain our triviality result.

**Theorem 26 (Triviality).** Let $A$ and $B$ be separable $C^*$-algebras, with $B$ nuclear and stable. If $\tau : A \to \frac{M(B)}{B}$ is a semisplit extension, and if either $\tau$ is full or $B$ is simple, then $\tau_\infty$ is a trivial extension.

**Proof.** By Proposition 21 and Lemma 22, the sum $\tau_\infty$ defines a semisplit full extension with properly infinite range. By Lemma 22, the extension $\tau_\infty$ is then absorbing in the nuclear sense. This means that it is an element of $KK(A, B)$, defined as a group of absorbing extensions under unitary equivalence by multiplier unitaries (please see the beginning of this section for a discussion). But from the construction of $\tau_\infty$ as an infinite sum, the BDF sum $\tau_\infty + \tau_\infty$ is equivalent to $\tau_\infty$. Thus, at the level of abelian groups, the element defined by $\tau_\infty$ is trivial. Being trivial in the group means that it is in the same unitary equivalence class as, for example, Kasparov’s trivial extension. But then $\tau_\infty$ is unitarily equivalent—by a multiplier unitary—to a trivial (i.e. split) extension. Thus the extension $\tau_\infty$ is trivial, as was to be shown. □

Because of the fairly simple and general nature of the proof, it seems probable that any reasonable way to define infinite sums of extensions will have the same property as in the above Theorem, and thus will be equivalent in $KK^1_{\text{nuc}}(A, B)$ to the above sum.

6.4. Subalgebras of the corona. We conclude by mentioning that sometimes statements about extensions become simpler if we focus on the range of the extension (in the corona) instead of the Busby map. To further lighten the notation, let us suppose that given a subalgebra $A$ of a corona algebra $\frac{M(B)}{B}$, then $\hat{A}$ denotes the subalgebra of $\frac{M(B \otimes K)}{(B \otimes K)}$ given by the previously defined sum, $\{ \sum v_i a v_i^* : a \in A \}$, in the corona. (As in Definition 19) Then our triviality theorem implies the following:

**Corollary 27.** Let $B$ be a separable, nuclear, and simple $C^*$-algebra. Suppose that $D$ is a separable unital subalgebra, self-adjoint or not, of the corona $\frac{M(B)}{B}$. Then there exists a unitary $U \in \frac{M(B \otimes K)}{(B \otimes K)}$ such that $U^* DU$ is contained in a copy of the Calkin algebra within $\frac{M(B \otimes K)}{(B \otimes K)}$. 
The triviality theorem shows that $U^*DU$ is contained in a copy of the Calkin algebra within $\mathcal{M}(B \otimes K)/(B \otimes K)$.

The above corollary makes precise the statement that corona algebras are locally Calkin algebras.

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