A Few Almost Trivial Notes on the Symplectic Radon Transform and the Tomographic Picture of Quantum Mechanics

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Abstract
We emphasize in these pedagogical notes the that the theory of the Radon transform and its applications is best understood using the theory of the metaplectic group and the quadratic Fourier transforms generating metaplectic operator.. Doing this we hope that these notes will be useful to a larger audience, including researchers in time-frequency analysis.

1 Introduction
In many texts studying the tomographic picture of quantum mechanics one finds the following definition of the Radon transform of a quantum state $\hat{\rho}$:

$$R_{\hat{\rho}}(X, \mu, \nu) = \int W(x, p) \delta(X - \mu x - \nu p) dp dx$$

where $\mu$ and $\nu$ are real numbers and $W(x, p)$ is the Wigner distribution of $\hat{\rho}$:

$$W(x, p) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar} p y} (x + \frac{1}{2} y | \hat{\rho} | x - \frac{1}{2} y) dy.$$  

One then also finds the following expression of the “inverse Radon transform”:

$$W\psi(x, p) = \frac{1}{2\pi \hbar} \int R_{\hat{\rho}}(X, \mu, \nu) e^{\frac{i}{\hbar} (X - \mu x - \nu p)} dX d\mu d\nu$$

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1 A short non-exhaustive list is [1, 2, 4, 5, 11, 12]
The expression (1), which should be interpreted as a distributional bracket to have any meaning, is however difficult to justify mathematically. It only makes sense for a restricted class of functions \(W_{x,p}\) (this function has to be at least continuous, but this assumption is not necessarily made in the usual texts). Similarly, the “derivation” of the inversion formula (3) is usually obscure and does not discuss convergence issues. In this short Note we propose a rigorous redefinition of the Radon transform which has the additional advantage of replacing this notion where it belongs, namely rigorous harmonic analysis and the theory of the metaplectic group and its extensions. (We use the definition of the metaplectic group using quadratic Fourier transforms shortly reviewed in APPENDIX A.)

2 The Radon transform

We will only consider the case of a “pure state”, represented by function \(\psi \in L^2(\mathbb{R})\). The general case is then easy to obtain since by the spectral theorem for trace class operators the Wigner distribution of an arbitrary state \(\rho\) is a convex sum of Wigner functions \(W_{\psi,j}\) [9].

Proposition 1 Let \(\hat{\rho}\) be a pure state \(|\psi\rangle\) with \(\psi \in L^2(\mathbb{R})\). We assume that \(\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\) and \(\hat{\psi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\) (\(\hat{\psi}\) the Fourier transform of \(\psi\)).

(i) The (symplectic) Radon transform of the Wigner function \(W_{\psi}(x,p)\) is given by
\[
(4) \quad R_{\hat{\rho}}(X,\mu,\nu) = \lambda^{-1}|\hat{U}_{\mu,\nu}\psi(\lambda^{-1}X)|^2.
\]
where \(\hat{U}_{\mu,\nu} \in Mp(n)\) is anyone of the two metaplectic operators covering the rotation \(U_{\mu,\nu} = \begin{pmatrix} \mu/\lambda & \nu/\lambda \\ -\nu/\lambda & \mu/\lambda \end{pmatrix}\) where \(\lambda = \sqrt{\mu^2 + \nu^2}\).

(ii) The inverse Radon transform is given by the formula:
\[
W_{\psi}(x,p) = \frac{1}{2\pi\hbar} \int R_{\hat{\rho}}(X,\mu,\nu) e^{i(\lambda^{-1}X - \mu x - \nu p)} dXd\mu d\nu
\]
understood as a Fourier transform of the function \(R_{\hat{\rho}}(X,\mu,\nu)\).

Proof. (i) Let us make the change of variables
\[
\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mu/\lambda & \nu/\lambda \\ -\nu/\lambda & \mu/\lambda \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}
\]
in the integral (1). This leads to the expression
\[
R_{\hat{\rho}}(X,\mu,\nu) = \int W_{\psi}(U_{\mu,\nu}^{-1}(u,v)) \delta(X - \lambda u) dudv.
\]
Since $\delta(X - \lambda u) = \lambda^{-1} \delta(\lambda^{-1} X - u)$ this can be rewritten

$$R^\rho(X, \mu, \nu) = \lambda^{-1} \int W\psi(\lambda^{-1}(\mu u - \nu v, \nu u + \mu v))\delta(\lambda^{-1} X - u)dudv \quad (8)$$

In view of the symplectic covariance property [6, 8, 13] of the Wigner transform we have

$$W\psi(U^{-1}_{\mu,\nu}(u, v)) = W(\hat{U}_{\mu,\nu}\psi)(u, v) \quad (9)$$

where $\hat{U}_{\mu,\nu}$ is anyone of the two metaplectic operators (see the APPENDIX A) covering $U$ and hence (8) yields

$$R^\rho(X, \mu, \nu) = \lambda^{-1} \int W(\hat{U}_{\mu,\nu}\psi)(\lambda^{-1} X, v)\delta(\lambda^{-1} X - u)dudv$$

hence (1) using the marginal properties

$$\int W\psi(x, p)dp = |\psi(x)|^2, \quad \int W\psi(x, p)dx = |\hat{\psi}(p)|^2 \quad (10)$$

of the Wigner transform, which are valid [G, S] since $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\hat{\psi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. (ii) Let us denote $A$ the right-hand side of the equality (1). Using the first marginal property (10) we have

$$A = \lambda^{-1} \frac{1}{2\pi\hbar} \int |\hat{U}_{\mu,\nu}\psi(\lambda^{-1} X)|^2 e^{i\frac{\hbar}{\lambda}(X - \mu x - \nu p)}dXd\mu d\nu$$

$$= \lambda^{-1} \frac{1}{2\pi\hbar} \int W(\hat{U}_{\mu,\nu}\psi)(\lambda^{-1} X, P) e^{i\frac{\hbar}{\lambda}(X - \mu x - \nu p)}dXdPd\mu d\nu.$$ 

Replacing $X$ with $\lambda X$ and using the symplectic covariance property (9) we get

$$A = \frac{1}{2\pi\hbar} \int W(\hat{U}_{\mu,\nu}\psi)(X, P) e^{i\frac{\hbar}{\lambda}(\lambda X - \mu x - \nu p)}dXdPd\mu d\nu$$

$$= \frac{1}{2\pi\hbar} \int W\psi(U^{-1}_{\mu,\nu}(X, P)) e^{i\frac{\hbar}{\lambda}(\lambda X - \mu x - \nu p)}dXdPd\mu d\nu$$

$$= \frac{1}{2\pi\hbar} \int W\psi((\mu/\lambda)X - (\nu/\lambda)P, (\nu/\lambda)X + (\mu/\lambda)P) e^{i\frac{\hbar}{\lambda}(\lambda X - \mu x - \nu p)}dXdPd\mu d\nu.$$ 

Setting $Y = (\mu/\lambda)X - (\nu/\lambda)P$ and $Z = (\nu/\lambda)X + (\mu/\lambda)P$ (and hence $\lambda X = \mu Y + \nu Z$) this is

$$A = \frac{1}{2\pi\hbar} \int W\psi(Y, Z) e^{i\frac{\hbar}{\lambda}(\mu(Y - x) + \nu(Z - p))}dXdPd\mu d\nu.$$
In view of the Fourier inversion formula, written formally as
\[ \int e^\frac{i}{\hbar} \mu(Y-x) + \nu(Z-p) \, d\mu d\nu = 2\pi \hbar \delta(Y - x, Z - p) \]
we thus have
\[ A = \int W(x, p) \delta(Y - x, Z - p) \, dx dp = W(x, p) \]
which was to be proven. ■

3 Some explicit calculations

Assume \( \nu \neq 0 \). We have the following explicit form for the metaplectic operator \( \hat{U}_{\mu, \nu} \) [6, 3]:
\[ \hat{U}_{\mu, \nu} \psi(x) = \frac{e^{-i\pi/4}}{\sqrt{2\pi\hbar}} \sqrt{\frac{\lambda}{\nu}} \int_{-\infty}^{\infty} \exp \left[ i \hbar \left( \frac{\mu}{2\nu} x^2 + \frac{\lambda}{\nu} xx' + \frac{\mu}{2\nu} x'^2 \right) \right] \psi(x') dx' \tag{11} \]
where the argument of \( \sqrt{\lambda/\nu} \) can take two possible values. It follows that
\[ |\hat{U}_{\mu, \nu} \psi(x)|^2 = \frac{\lambda}{2\pi\hbar|\nu|} \left| \int_{-\infty}^{\infty} \exp \left[ i \hbar \left( -\frac{\lambda}{\nu} xx' + \frac{\mu}{2\nu} x'^2 \right) \right] \psi(x') dx' \right|^2 \]
so that, taking (4) into account,
\[ R_{\hat{\rho}}(X, \mu, \nu) = \frac{1}{2\pi\hbar|\nu|} \left| \int_{-\infty}^{\infty} \exp \left( -\frac{i}{\hbar \nu} X x' \right) \exp \left( \frac{i\mu}{2\hbar \nu} x'^2 \right) \psi(x') dx' \right|^2. \tag{12} \]
which we can rewrite as a Fourier transform
\[ R_{\hat{\rho}}(X, \mu, \nu) = \frac{1}{|\nu|} \left| \hat{F} \left[ \exp \left( \frac{i\mu}{2\hbar \nu} x'^2 \right) \psi \right] \left( \frac{X}{\nu} \right) \right|^2. \tag{13} \]
where
\[ \hat{F} \phi(x) = \frac{e^{-i\pi/4}}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{i\pi}{4} xx'} \phi(x') dx'. \]

Remark 2 In the language of time-frequency analysis one would say that the Radon transform of \( \psi \) is (up to a rescaling factor) the squared modulus of the Fourier transform of the product of \( \psi \) by a “chirp”.
Denoting by \( \hat{V}_{-\mu/\nu} \) the operator of multiplication by \( \exp(i(\mu/\nu)/2\hbar) \) and by \( \hat{M}_\nu \) the scaling operator

\[
\hat{M}_\nu \phi(x) = \sqrt{\nu} \phi(\nu x)
\]

this formula can be rewritten

\[
\hat{U}_{\mu,\nu} \psi(x) = \hat{V}_{-\mu/\nu} \hat{F}(\hat{M}_{\lambda/\mu} \hat{V}_{-\mu/\nu} \psi)
\]

where \( \hat{F} \) is essentially a Fourier transform: Formula \((13)\) then becomes

\[
R_\rho(X, \mu, \nu) = \left| \hat{M}_\nu \hat{F}(\hat{V}_{-\mu/\nu} \psi)(X) \right|^2.
\]

### 4 The Pauli problem for Gaussians

That the full Radon transform of a state is not necessary to reconstruct it is seen in the elementary example considered in this section, which is a version of the so-called “Pauli problem”. Historically, this problem goes back to the famous question Pauli asked in \[14\], whether the probability densities \( |\psi(x)|^2 \) and \( |\hat{\psi}(p)|^2 \) uniquely determine the wavefunction \( \psi(x) \). The answer is negative, as is seen on the following simple example \[10\]: of Gaussian wavefunction in one spatial dimension

\[
\psi(x) = \left( \frac{1}{2\pi \sigma_{xx}} \right)^{1/4} e^{-\frac{x^2}{4\sigma_{xx}}} e^{i\sigma_{xp}x^2} e^{i\pi \sigma_{xx}}
\]

where \( \sigma_{xx} \) is the variance in the position variable and \( \sigma_{xp} \) the covariance in the position and momentum variables. Using the formula giving the Fourier transform \( \hat{\psi}_{a, b} \) of

\[
\psi_{a, b}(x) = e^{-\frac{1}{\pi}(a+ib)x^2}, \quad a > 0, b \in \mathbb{R}
\]

which is

\[
\hat{\psi}_{a, b}(p) = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} e^{-\frac{p^2}{4\hbar}} \psi_{a, b}(x) dx = \frac{1}{\sqrt{2(a + ib)}} \exp\left(-\frac{1}{4\hbar}(a+ib)^{-1}p^2\right)
\]

we see that the Fourier transform of \( \psi \) is explicitly given by

\[
\hat{\psi}(p) = e^{i\gamma} \left( \frac{1}{2\pi \sigma_{pp}} \right)^{1/4} e^{-\frac{2}{4\sigma_{pp}}} e^{-\frac{i\sigma_{pp}p^2}{2\hbar}}
\]
where $\gamma$ is an unessential constant real phase depending on the covariances. It follows that we have

$$|\psi(x)|^2 = \left(\frac{1}{2\pi \sigma_{xx}}\right)^{1/2} e^{-\frac{x^2}{2\sigma_{xx}}}, \quad |\hat{\psi}(p)|^2 = \left(\frac{1}{2\pi \sigma_{pp}}\right)^{1/2} e^{-\frac{p^2}{2\sigma_{pp}}}$$

and these relations imply the knowledge of $\sigma_{xx}$ and of $\sigma_{pp}$. The covariance $\sigma_{xp}$ is then determined up to a sign because the state $\psi$ saturates the Robertson–Schrödinger inequality: we have

$$\sigma_{xx}\sigma_{pp} - \sigma_{xp}^2 = \frac{1}{4} \hbar^2 \tag{19}$$

and this identity can be solved in $\sigma_{xp}$ yielding $\sigma_{xp} = \pm (\sigma_{xx}\sigma_{pp} - \frac{1}{4} \hbar^2)^{1/2}$.

Let us now apply the Radon transform in its form (12) to $\psi$. For this it suffices to observe that the Fourier formula (17) implies that

$$|\hat{\psi}_{a,b}(p)|^2 = \frac{1}{2\sqrt{a^2 + b^2}} \exp \left[ -\frac{2}{\hbar} \frac{a}{a^2 + b^2} p^2 \right]. \tag{20}$$

Now, formula (12) reads here

$$R_{\hat{\rho}}(X, \mu, \nu) = C \left| \int_{-\infty}^{\infty} \exp \left( -\frac{i}{\hbar \nu} Xx' \right) \exp \left[ i \left( \frac{\mu}{2\hbar \nu} + \frac{i \sigma_{xp}}{2\hbar \sigma_{xx}} \right) x'^2 \right] dx' \right|^2,$$

where

$$C = \frac{1}{2\pi \hbar |\nu|} \left( \frac{1}{2\pi \sigma_{xx}} \right)^{1/2},$$

hence, choosing

$$a = \frac{\hbar}{4\sigma_{xx}}, \quad b = \frac{\mu}{2\nu} + \frac{\sigma_{xp}}{2\sigma_{xx}}.$$  

in (20), we have, using the identity $\sigma_{xx}\sigma_{pp} + \sigma_{xp}^2 = \frac{1}{4} \hbar^2$,

$$a^2 + b^2 = \frac{\sigma_{pp}}{4\sigma_{xx}} + \left( \frac{\mu}{2\nu} \right)^2 + \frac{\mu}{2\nu} \frac{\sigma_{xp}}{\sigma_{xx}}$$

and thus (12) yields

$$R_{\hat{\rho}}(X, \mu, \nu) = \frac{1}{2\pi \hbar |\nu|} \left| \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar \nu} Xx'} \psi_{a,b}(x') dx' \right|^2 = \frac{1}{|\nu|} \left| \hat{\psi}_{a,b} \left( \frac{X}{\nu} \right) \right|^2.$$
that is, by (20),
\[ R_{\hat{\rho}}(X, \mu, \nu) = \frac{1}{2|\nu|} \frac{1}{a^2 + b^2} \exp \left[ -\frac{1}{\hbar} \frac{a}{a^2 + b^2} \left( \frac{X}{\nu} \right)^2 \right] \]  
(21)
where
\[ \frac{a}{a^2 + b^2} = \frac{\hbar}{\sigma_{pp} + 4\varepsilon^2 \sigma_{xx} + 4\varepsilon \sigma_{xp}} , \quad \varepsilon = \frac{\mu}{2\nu}. \]  
(22)
This formula allows the unambiguous determination of the covariance \( \sigma_{xp} \) once the variances \( \sigma_{xx} \) and \( \sigma_{pp} \) are known. Notice that it suffices with one choice of parameters \( \mu, \nu \) to determine the unknown covariance. A geometric explanation will be given below.

5 Geometric interpretation

The Wigner transform of the Gaussian (15) is \[ W\psi(x, p) = \frac{1}{\pi\hbar} e^{-\frac{\hbar}{2} Gx^2} \]
where \( G = \frac{\hbar}{\sigma_{pp} \sigma_{xx} - \sigma_{xp}^2} \). The matrix
\[ \Sigma = \frac{\hbar}{2} G^{-1} = \begin{pmatrix} \sigma_{xx} & \sigma_{xp} \\ \sigma_{xp} & \sigma_{pp} \end{pmatrix} \]
is thus the usual covariance matrix of the state \( \psi \), and to it one associates the covariance ellipse
\[ \Omega : \frac{1}{2}(x, p)\Sigma^{-1}(x) \leq 1 \]
that is, explicitly,
\[ \Omega : \frac{\sigma_{pp}}{2D} x^2 - \frac{\sigma_{xp}}{D} px + \frac{\sigma_{xx}}{2D} p^2 \leq 1 \]
where
\[ D = \sigma_{xx}\sigma_{pp} - \sigma_{xp}^2 = \frac{1}{4}\hbar^2. \]
Consider next the straight lines \( \ell_{\mu, \nu} \) in the phase plane defined by \( \mu x + \nu p = 0 \) and their intersections with \( \Omega \). The set \( \Omega \cap \ell_{0,1} \) is the intersection of \( \Omega \) with the \( x \)-axis and is thus the real interval \([-\hbar/\sqrt{2\sigma_{xx}}, \hbar/\sqrt{2\sigma_{xx}}]\) and, similarly, \( \Omega \cap \ell_{1,0} \) is the intersection of \( \Omega \) with the \( p \)-axis, that is, \([-\hbar/\sqrt{2\sigma_{pp}}, \hbar/\sqrt{2\sigma_{pp}}]\). Thus, the knowledge of these two particular intersections determine the
variances \( \sigma_{xx} \) and \( \sigma_{pp} \). In the general case \( \mu \nu \neq 0 \) the intersection \( \Omega \cap \ell_{\mu,\nu} \) can be parametrized by \( x \) (or \( p \)) and is the interval defined by

\[
\left( \sigma_{pp} + \frac{2\mu}{\nu} \sigma_{xp} + \frac{\mu^2}{\nu^2} \sigma_{xx} \right) x^2 \leq \frac{1}{2} \hbar^2.
\]

We now observe that the coefficient of \( x^2 \) in this formula is exactly the second denominator in formula (22) describing the Radon transform of the Gaussian (15). This is not a mere coincidence. We begin by noting that the marginal conditions (10) can be viewed as line integrals

\[
\int_{\ell_{1,0}} W\psi(x,p)dp = |\psi(x)|^2
\]

\[
\int_{\ell_{0,1}} W\psi(x,p)dx = |\hat{\psi}(p)|^2.
\]

In the general case:

**Proposition 3** Let \( \psi, \hat{\psi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). Let \( \ell^X_{\mu,\nu} \) be the straight line in \( \mathbb{R}^2 \) with equation \( \mu x + \nu p = X \). The Radon transform of \( \psi \) is the line integral

\[
R_{\hat{\rho}}(X,\mu,\nu) = \int_{\ell^X_{\mu,\nu}} W\psi(z_{\mu,\nu})dz_{\mu,\nu} \tag{23}
\]

where \( dz_{\mu,\nu} \) is the push-forward of the Lebesgue measure \( dx \).

**Proof.** Denote by \( W \) the integral on the left-hand side. Parametrizing \( \ell^X_{\mu,\nu} \) by \( x = \nu t + \mu^{-1}X, p = -\mu t \) we have

\[
W = \lambda \int_{-\infty}^{\infty} W\psi(\nu t + \mu^{-1}X, -\mu t)dt \tag{24}
\]

where \( \lambda = \sqrt{\mu^2 + \nu^2} \). Returning to formula (7), we have

\[
R_{\hat{\rho}}(X,\mu,\nu) = \lambda^{-1} \int W\psi(U_{\mu,\nu}^{-1}(u,v))\delta(\lambda^1 X - u)dudv \tag{25}
\]

\[
= \int W\psi(U_{\mu,\nu}^{-1}(\lambda^{-1} X, v))dv \tag{26}
\]

that is, since \( U_{\mu,\nu}^{-1} = \begin{pmatrix} \mu/\lambda & -\nu/\lambda \\ \nu/\lambda & \mu/\lambda \end{pmatrix} \), and replacing \( v \) with \( s \),

\[
R_{\hat{\rho}}(X,\mu,\nu) = \int W\psi(\mu\lambda^{-2}X - v\lambda^{-1}s, v\lambda^{-2}X + \mu\lambda^{-1}s)ds. \tag{27}
\]
6 Extension to higher dimensions: hints

All of the above can be generalized without major difficulties to the case of states in \( L^2(\mathbb{R}^n) \). We only sketch the main modifications here, a detailed account will appear elsewhere.

The definition (1) of the Radon transform should be replaced with

\[
\hat{R}_\rho(X, A, B) = \int_{\mathbb{R}^n \times \mathbb{R}^n} W(x, p) \delta(X - Ax - Bp) dp dx
\]

where \( X \in \mathbb{R}^n \) and \( A, B \) are two real square \( n \times n \) matrices with such that \( A^T B = B A^T \) and \( \text{rank}(A, B) = n \). This ensures us that the subspace \( \ell_{X, A, B} = \{ (x, p) : Ax + Bx = X \} \) of \( \mathbb{R}^{2n} \equiv T^* \mathbb{R}^n \) equipped with its standard symplectic structure is an affine Lagrangian plane [6], which allows us to rewrite the geometric version (23) as a surface integral

\[
\hat{R}_\rho(X, A, B) = \int_{\ell_{X, A, B}} W_\psi(z_{A, B}) dz_{A, B}
\]

(the fact that Lagrangian planes intervene is crucial since the unitary group acts transitively on the Lagrangian Grassmannian). The inversion formula (5) should then be replaced with an expression of the type

\[
W_\psi(x, p) = \left( \frac{1}{2\pi \hbar} \right)^n \int \hat{R}_\rho(X, A, B) e^{ix^T(X - Ax - Bp)} dXdAdB.
\]

APPENDIX A: The metaplectic group \( \text{Mp}(n) \)

For a detailed study of the metaplectic group \( \text{Mp}(n) \) see [6].

Let \( S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) be a real \( 2n \times 2n \) matrix, where the “blocks” \( A, B, C, D \) are \( n \times n \) matrices. Let \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \) the standard symplectic matrix. We have \( S \in \text{Sp}(n) \) (the symplectic group) if and only \( SJS^T = S^T J S = J \). These relations are equivalent to any of the two sets of conditions

\[
\begin{align*}
A^T C, B^T D & \text{ are symmetric, and } A^T D - C^T B = I \quad (30) \\
AB^T, CD^T & \text{ are symmetric, and } AD^T - BC^T = I. \quad (31)
\end{align*}
\]

One says that \( S \) is a free symplectic matrix if \( B \) is invertible, i.e. \( \text{det} B \neq 0 \). To a free symplectic matrix is associated a generating function: it is the
quadratic form

\[ A(x, x') = \frac{1}{2} DB^{-1} x \cdot x - B^{-1} x \cdot x' + \frac{1}{2} B^{-1} A x' \cdot x'. \] (32)

The terminology comes from the fact that the knowledge of \( A(x, x') \) uniquely determines the free symplectic matrix \( S \): we have

\[
\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x' \\ p' \end{pmatrix} \iff \begin{cases} p = \nabla_x A(x, x') \\ p' = -\nabla_{x'} A(x, x') \end{cases}
\]

as can be verified by a direct calculation.

Now, to every free symplectic matrix \( S_A \) we associate two operators \( \hat{S}_{A,m} \) by the formula

\[
\hat{S}_{A,m} \psi(x) = \left( \frac{1}{2\pi \hbar} \right)^{n/2} i^{m-n/2} \sqrt{|\det B^{-1}|} \int e^{i\pi A(x, x')} \psi(x') d^n x'
\] (33)

where \( m \) corresponds to a choice of argument for \( \det B^{-1} \): \( m = 0 \mod 2 \) if \( \det B^{-1} > 0 \) and \( m = 1 \mod 2 \) if \( \det B^{-1} < 0 \). It is not difficult to prove that the generalized Fourier transforms \( \hat{S}_{A,m} \) are unitary operators on \( L^2(\mathbb{R}^n) \).

These operators generate the metaplectic group \( \text{Mp}(n) \). One shows that every \( \hat{S} \in \text{Mp}(n) \) can be written (non uniquely) as a product \( \hat{S}_{A,m} \hat{S}_{A',m'} \).

This group is a double covering of \( \text{Sp}(n) \), the covering projection being simply defined by

\[
\pi_{\text{Mp}} : \text{Mp}(n) \longrightarrow \text{Sp}(n) , \quad \pi_{\text{Mp}}(\hat{S}_{A,m}) = S_A.
\] (34)

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