RELATIVE CARTIER DIVISORS AND K-THEORY

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Abstract. We study the relative Picard group Pic(f) of a map f : X → S of schemes. If f is faithful affine, it is the relative Cartier divisor group I(f). The relative group K₀(f) has a γ-filtration, and Pic(f) is the top quotient for the γ-filtration. When f is induced by a ring homomorphism A → B, we show that the relative “nil” groups NPic(f) and NK₀(f) are continuous W(A)-modules.

Introduction

If f : X → S is a morphism of schemes, the relative Picard group Pic(f) was defined by Bass in [1], and fits into a natural exact sequence

\[ 0 \to \mathcal{O}^×(S) \xrightarrow{f^*} \mathcal{O}^×(X) \xrightarrow{\partial} \text{Pic}(f) \to \text{Pic}(S) \xrightarrow{f^*} \text{Pic}(X). \]

The goal of this paper is to study this group as well as NPic(f), defined to be Pic(f[t])/Pic(f), where f[t] : X × A¹ → S × A¹.

Our first observation is that when f is Spec(B) → Spec(A) for a commutative ring extension A ↪ B, Pic(f) is isomorphic to the relative Cartier divisor group I(f), defined in [13] as the group of invertible A-submodules of B under multiplication and studied in [15, 14, 16]. This definition of I(f) also makes sense (and we still have I(f) ≅ Pic(f)) for scheme maps f : X → S for which \( \mathcal{O}_S^× \to f_*\mathcal{O}_X^× \) is an injection of sheaves. It then follows from [16] that Pic(f) is a contracted functor in the sense of Bass.

We then relate Pic(f) to the relative group K₀(f), which fits into an exact sequence

\[ K_1(S) \xrightarrow{f^*} K_1(X) \xrightarrow{\partial} K_0(f) \to K_0(S) \to K_0(X). \]

For example, if f : A ↪ B is subintegral then K₀(f) ≅ Pic(f) (Proposition 2.5).

Let \( N\mathcal{I} \) denote the Zariski sheaf associated to the presheaf U → NPic(U, f⁻¹U) on S. In Theorem 4.1 and Theorem 4.7, we prove the following:

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Theorem 0.2. Let \( f : X \to S \) be a faithful affine morphism of schemes.

(1) The Zariski sheaf \( \mathcal{N} \mathcal{I} \) is an étale sheaf on \( S \). Moreover,
\[
\text{NPic}(f) \cong H_{\text{et}}^0(S, \mathcal{N} \mathcal{I}) = H_{\text{zar}}^0(S, \mathcal{N} \mathcal{I}).
\]

(2) If \( X \) and \( S \) are schemes then \( H^*_\text{et}(S, \mathcal{N} \mathcal{I}) \cong H^*_\text{zar}(S, \mathcal{N} \mathcal{I}) \).

(3) If \( X \) and \( S \) are both affine schemes then \( H^q_{\text{et}}(S, \mathcal{N} \mathcal{I}) = 0 \) for \( q \neq 0 \).

A secondary goal of this article is to study the relative \( K \)-theory groups \( K_n(f) \) associated to a morphism of schemes \( f : X \to S \). By definition, \( K_n(f) = \pi_n K(S) \). Comparing \( X \to S \) to \( X[t] \to S[t] \) yields groups \( NK_*(f) \).

Theorem 0.3. For each homomorphism \( f : A \to B \):

(1) \( NK_n(f) \) is a continuous \( W(A) \)-module, for all \( n \).
(2) \( \text{NPic}(f) \) is a continuous \( W(A) \)-module.
(3) \( \text{det} : NK_0(f) \to \text{NPic}(f) \) is a \( W(A) \)-module homomorphism.

(See Theorems 3.3 and 5.6, and Proposition 3.2). This implies that if \( \text{char}(A) = p > 0 \) then both \( NK_n(f) \) and \( \text{NPic}(f) \) are \( p \)-groups, while if \( \text{char}(A) = 0 \) the groups have the structure of \( A \)-modules.

We conclude with some remarks about \( K_n(f) \) when \( n \) is negative. If \( X \) and \( S \) have dimension at most \( d \), then \( K_n(S) = K_n(X) = 0 \) for \( n < -d \) in many cases. In such cases, it follows that \( K_n(f) = 0 \) for \( n < -d - 1 \). The cohomological interpretation of the negative \( K \)-theory of a scheme in terms of the cdh-cohomology of the constant sheaf \( \mathbb{Z} \) is given in [4]. In the relative situation, we prove the following (Theorem 6.2 and Theorem 6.3):

Theorem 0.4. Let \( f : X \to S \) be a finite morphism of \( d \)-dimensional noetherian schemes.

(1) If \( X \) and \( S \) are essentially of finite type over a field \( k \) of characteristic 0, \( K_{-d-1}(f) \cong H^d_{\text{cdh}}(S, f_*\mathbb{Z}/\mathbb{Z}). \)
(2) If \( \text{dim } S = 1 \), then \( K_{-2}(f) \cong H^1_{\text{nis}}(S, f_*\mathbb{Z}/\mathbb{Z}) \) and there is an extension
\[
0 \to H^1_{\text{nis}}(S, f_*\mathcal{O}^*_{X}/\mathcal{O}^*_{S}) \to K_{-1}(f) \to H^0_{\text{nis}}(S, f_*\mathbb{Z}/\mathbb{Z}) \to 0.
\]
1. Relative Pic and invertible submodules

In [1], Bass defined $\text{Pic}(f)$ to be the abelian group generated by $[L_1, \alpha, L_2]$, where the $L_i$ are line bundles on $S$ and $\alpha : f^*L_1 \to f^*L_2$ is an isomorphism. The relations are:

1. $[L_1, \alpha, L_2] + [L'_1, \alpha', L'_2] = [L_1 \otimes L'_1, \alpha \otimes \alpha', L_2 \otimes L'_2]$;
2. $[L_1, \alpha, L_2] + [L_2, \beta, L_3] = [L_1, \beta \alpha, L_3]$;
3. $[L_1, \alpha, L_2] = 0$ if $\alpha = f^*(\alpha_0)$ for some $\alpha_0 : L_1 \cong L_2$.

Remark 1.0.1. By (1), every element of $\text{Pic}(f)$ has the form $[L, \alpha, \mathcal{O}_S]$. Writing $[L, \alpha]$ for $[L, \alpha, \mathcal{O}_S]$, an alternative presentation for $\text{Pic}(f)$ is that it is generated by elements $[L, \alpha]$ satisfying: $[L, \alpha] + [L', \alpha'] = [L \otimes L', \alpha \otimes \alpha']$; $[L, \alpha] = 0$ if (and only if) there is an isomorphism $\alpha_0 : L \cong \mathcal{O}_S$ so that $\alpha = f^*(\alpha_0)$. It is easy to see, and observed by Bass, that the map $\text{Pic}(f) \to \text{Pic}(S)$ sending $[L, \alpha]$ to $[L]$ fits into an exact sequence (0.1), where $\partial(b) = [\mathcal{O}_S, b]$.

Proposition 1.1. Bass’ Pic $(f)$ is the hypercohomology group $H^0(S, \mathcal{O}_S^* \to \mathcal{O}_X^*)$.

Proof. Let $C^*$ denote the mapping cone of $\mathcal{O}_S^* \to f_* \mathcal{O}_X^*$. A 0-cocyle of $C^*$ is given by a cover $\{U_i\}$ of $S$, a unit $b_i$ of $f^{-1}(U_i)$ for each $i$, and units $a_{ij}$ of $U_i \cap U_j$ for each $i, j$ satisfying the cocycle condition (so that the $\{a_{ij}\}$ define a line bundle $L$ on $S$) and such that $b_i/b_j = f^\#(a_{ij})$ on each $f^{-1}(U_i \cap U_j)$. Since the $\{b_i\}$ define an isomorphism $f^*L \cong \mathcal{O}_X$, each 0-cocyle defines an element $\lambda = [L, \beta, \mathcal{O}_S]$ of $\text{Pic}(f)$. A 0-coboundary is given by $a_{ij} = a_i/a_j$ and $b_i = f^\#(a_i)$ for units $a_i$ of $U_i$; adding it to a cocycle does not change $\lambda$. Refining the cover does not change $\lambda$ either. The result follows from the 5-lemma applied to the following diagram with exact rows (which is easily checked to be commutative):

\[
\begin{array}{cccccc}
H^0(S, \mathcal{O}^*) & \longrightarrow & H^0(X, \mathcal{O}^*) & \longrightarrow & H^0(S, C^*) & \longrightarrow & H^1(S, \mathcal{O}^*) & \longrightarrow & H^1(X, \mathcal{O}^*) \\
\cong & \downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow & \cong \\
\mathcal{O}^*(S) & \longrightarrow & \mathcal{O}^*(X) & \longrightarrow & \text{Pic}(f) & \longrightarrow & \text{Pic}(S) & \longrightarrow & \text{Pic}(X).
\end{array}
\]

Now suppose that $f$ is faithful and affine. As observed in [16], $\mathcal{I}(f)$ is isomorphic to $H^0(S, f_* \mathcal{O}_X^*/\mathcal{O}_S^*)$. Thus Proposition 1.1 implies that $\mathcal{I}(f) \cong \text{Pic}(f)$. Here is a more elementary proof.

Lemma 1.2. If $f : X \to S$ is a faithful affine map, there is an isomorphism $\rho : \mathcal{I}(f) \cong \text{Pic}(f)$, sending $L$ to $[L, i, \mathcal{O}_S]$, where $i : f^*L \cong \mathcal{O}_X$. 

The isomorphism $f^* L \cong \mathcal{O}_X$ is well defined, because in any affine open $U = \text{Spec}(A)$ of $S$ we have $f^{-1} U = \text{Spec}(B)$ with $A \subset B$; it was proven by Roberts and Singh [13] that $L \subset B$ induces $L \otimes_A B \cong B$.

**Proof.** Since $\rho(LL') = [L \otimes L', i \otimes i', \mathcal{O}_S] = [L, i, \mathcal{O}_S] + [L', i', \mathcal{O}_S]$, $\rho$ is a homomorphism. To define the inverse map, we use the presentation of Pic($f$) and the observation that because $\mathcal{O}_S \to f_* \mathcal{O}_X$ is an injection, so is $L \to L \otimes f_* \mathcal{O}_X$ for every line bundle $L$. Given a triple $[L_1, \alpha, L_2]$, we set $L = L_2^{-1} \otimes L_1$, so that $\alpha$ induces an isomorphism $f^* L \cong f^*(L_2)^{-1} \otimes f^*(L_1) \cong \mathcal{O}_X$, and define $\psi([L_1, \alpha, L_2])$ to be the submodule $L$ of $L \otimes f_* \mathcal{O}_X \cong f_* \mathcal{O}_X$. Since $\psi$ is compatible with the relations of Pic($f$), it descends to a homomorphism $\psi : \text{Pic}(f) \to \mathcal{I}(f)$. Since $[L_1, \alpha, L_2] = [L_2^{-1} \otimes L_1, \alpha, \mathcal{O}_S]$ in Pic($f$) and $f^*(L) = \mathcal{O}_X$ for all $L \in \mathcal{I}(f)$, $\psi$ is an inverse to $\rho$. \qed

2. **Relative $K_0$ and Pic**

Bass gave a presentation of a relative group $K_0(f)$ associated to $f : A \to B$ in [1] and [2, VII.5]; see [29, II.2.10]. It is generated by triples $[P_1, \alpha, P_2]$, where the $P_i$ are finitely generated projective $A$-modules (or vector bundles on $S$) and $\alpha$ is an isomorphism $f^*(P_1) \xrightarrow{\cong} f^*(P_2)$, and agrees with the group $\pi_0 K(f)$ of [29, IV.1.11]. The relations are:

1. $[P_1, \alpha, P_2] + [P'_1, \alpha', P'_2] = [P_1 \oplus P'_1, \alpha \oplus \alpha', P_2 \oplus P'_2]$,
2. $[P_1, \alpha, P_2] + [P_2, \beta, P_3] = [P_1, \beta \alpha, P_3]$,
3. $[P_1, \alpha, P_2] = 0$ if $\alpha = f^*(\alpha_0)$ for some $\alpha_0 : P_1 \cong P_2$.

By (1), every element of $K_0(f)$ has the form $[P, \alpha, A^n]$.

Bass showed [2, VII.5.3] that there is an exact sequence for each $f : A \to B$: \begin{equation} \label{eq:exact-sequence} K_1(A) \xrightarrow{f^*} K_1(B) \xrightarrow{\partial} K_0(f) \to K_0(A) \to K_0(B), \end{equation}

where for $g \in GL_n(B)$ we have $\partial([g]) = [A^n, g, A^n]$. Since we do not know if the corresponding sequence is exact for a quasi-projective map $f : X \to S$, we will restrict to the affine case in this section and the next.

**Lemma 2.2 (Excision).** Let $f : A \to B$ be a ring homomorphism, and let $I$ be an ideal of $A$ mapping isomorphically onto an ideal of $B$; write $\bar{f} : A/I \subset B/I$ for the induced map. Then excision holds for $K_0$ for all $n \leq 0$: $K_n(f) \cong K_n(\bar{f})$.

**Proof.** It suffices to consider the case $n = 0$. Because $K_0(A, I) \cong K_0(B, I)$ [29, Ex. II.2.3] and $K_1(A, I) \to K_1(B, I)$ is onto [29, III.2.2.1], the double-relative group...
vanishes: $K_0(A, B, I) = 0$. Applying contraction, we also have $K_{-1}(A, B, I) = 0$. The result now follows from the exact sequence

$$K_0(A, B, I) \to K_0(\tilde{f}) \to K_0(\tilde{f}) \to K_{-1}(A, B, I).$$

\[ \blacksquare \]

**Remark.** The failure of Lemma 2.2 in the non-affine setting was investigated in [12, A.5–6]. For example, if $X$ is the normalization of $S$ and the support $Y$ of the conductor $c$ is 1-dimensional, the obstruction is $K_0(S, X, Y) \cong H^1(Y, c/c^2 \otimes \Omega_{X/S})$.

As observed by Bass and Murthy long ago [3], the determinant $K_0(S) \to \text{Pic}(S)$ induces a surjective homomorphism

\( (2.3) \quad \det : K_0(f) \to \text{Pic}(f), \quad \det[P_1, \alpha, P_2] = [\det(P_1), \det(\alpha), \det(P_2)]. \)

Since $SK_0(S)$ is the kernel of $\det : K_0(S) \to \text{Pic}(S)$, we write $SK_0(f)$ for the kernel of $\det : K_0(f) \to \text{Pic}(f)$.

Recall [29, II.4.2] that a $\lambda$-ring $K = \mathbb{Z} \oplus \tilde{K}$ has a **positive structure** if it contains a $\lambda$-semiring $P$ (positive elements) including $\mathbb{N}$, such that every element of $\tilde{K}$ can be written as a difference of positive elements, the augmentation $\epsilon : K \to \mathbb{Z}$ sends $P$ to $\mathbb{N}$ and, if $p \in P$ has $\epsilon(p) = n$, then $\lambda^i p = 0$ for $i > n$ and $\lambda^i p$ is a unit. The **line elements** are $\{p \in P : \epsilon(p) = 1\}$; they form a subgroup of the units of $K$.

**Proposition 2.4.** Let $f : A \to B$ be a homomorphism of commutative rings. The operations $\lambda^i[P_1, \alpha, P_2] = [\Lambda^i P_1, \Lambda^i \alpha, \Lambda^i P_2]$ give $\mathbb{Z} \oplus K_0(f)$ the structure of a $\lambda$-ring with a positive structure. The top two ideals in the $\gamma$-filtration are $F^1_\gamma = \tilde{K}_0$ and $F^2_\gamma = SK_0(f)$, and the group of its line elements is $\text{Pic}(f) \cong F^1_\gamma / F^2_\gamma$.

**Proof.** Given $f : A \to B$, choose a surjection $\pi : \mathbb{Z}[X] \to B$ from a polynomial ring $\mathbb{Z}[X]$ in many variables to $B$; let $R$ be the pullback ring $R = \{(p, a) \in \mathbb{Z}[X] \times A : \pi(p) = f(a)\}$, with $\tilde{f} : R \to \mathbb{Z}[X]$ the projection. Since $K_1(\mathbb{Z}[X]) = \pm 1$ and $K_0(\mathbb{Z}[X]) = \mathbb{Z}$, we have $K_0(\tilde{f}) \cong \tilde{K}_0(R)$, and this map is compatible with the operations $\lambda^i$. Similarly, we have $\text{Pic}(\tilde{f}) \cong \text{Pic}(R)$. By Excision 2.2 for $K_0$ and $\text{Pic}$, $K_0(\tilde{f}) \cong K_0(f)$ and $\text{Pic}(\tilde{f}) \cong \text{Pic}(f)$. Hence $\mathbb{Z} \oplus K_0(f) \cong \mathbb{Z} \oplus \tilde{K}_0(R)$ is a $\lambda$-ring. Thus the result follows from the fact that the operations $\lambda^i$ make $K_0(R)$ into a $\lambda$-ring, with $F^2_\gamma = SK_0(R)$, and $\tilde{K}_0(R) / SK_0(R) \cong \text{Pic}(R)$.

Recall (Swan [17]) that an extension $A \subset B$ is said to be **subintegral** if $B$ is integral over $A$, and $\text{Spec}(B) \to \text{Spec}(A)$ is a bijection inducing isomorphisms on all residue fields.
Proposition 2.5. (Ischebeck) If \( f : A \rightarrow B \) is subintegral then \( K_0(f) \cong \text{Pic}(f) \), 
\( K_n(f) = 0 \) for all \( n < 0 \), and there is an exact sequence
\[
1 \rightarrow B^\times/A^\times \rightarrow K_0(f) \rightarrow K_0(A) \rightarrow K_0(B) \rightarrow 0.
\]

Proof. When \( A \subset B \) is subintegral, Ischebeck proved in [9, Prop. 7] that the natural map 
\( K_0(A) \rightarrow K_0(B) \) is surjective and \( SK_1(A) \rightarrow SK_1(B) \) is onto, so the cokernel of 
\( K_1(A) \rightarrow K_1(B) \) is \( B^\times/A^\times \). The exact sequence follows from (2.1). Finally, 
Ischebeck proved in [9, p. 331] that the determinant (2.3) induces an isomorphism 
from the kernel of 
\( K_0(A) \rightarrow K_0(B) \) onto the kernel of \( \text{Pic}(A) \rightarrow \text{Pic}(B) \). The result now follows from (2.1).

Replacing \( A \) and \( B \) by Laurent polynomial extensions, the Fundamental Theorem of 
\( K \)-theory [29, III.4.1] implies that \( LK_n(f) \cong K_{n-1}(f) \) and \( K_1(f) \cong L\text{Pic}(f) \). Since 
\( A[t, 1/t] \subset B[t, 1/t] \) is subintegral, we have \( L\text{Pic}(f) = 0 \) by Proposition 5.6 of [16]. This shows that that \( K_n(f) = 0 \) for all \( n < 0 \). □

Given an extension \( f : A \hookrightarrow B \), let \( i : A \hookrightarrow +A \) be the seminormalization of \( A \) in \( B \) and \( +f : +A \rightarrow B \) the induced map. There is an exact sequence
\[
\cdots \rightarrow K_n(i) \rightarrow K_n(f) \rightarrow K_n(+f) \rightarrow K_{n-1}(i) \rightarrow \cdots.
\]

Corollary 2.6. \( K_n(f) \cong K_n(+f) \) for \( n < 0 \), and the following sequence is exact.
\[
0 \rightarrow K_0(i) \rightarrow K_0(f) \rightarrow K_0(+f) \rightarrow 0.
\]

Proof. By Proposition 2.5 and [16, Lemma 3.3], the map \( K_0(i) \cong \text{Pic}(i) \rightarrow \text{Pic}(f) \) is an injection. Since it factors through \( K_0(i) \rightarrow K_0(f) \), the latter map is an injection. Since \( K_n(i) = 0 \) for \( n < 0 \), again by Proposition 2.5, we are done. □

3. The \( W(A) \)-module structure on \( NK_0(f) \) and \( N\text{Pic}(f) \)

In this section, we fix a ring homomorphism \( f : A \rightarrow B \) and show that \( NK_0(f) \) and \( N\text{Pic}(f) \) are continuous modules over the ring \( W(A) \) of big Witt vectors, so that
\[
NK_1(A) \rightarrow NK_1(B) \xrightarrow{\delta} NK_0(f) \rightarrow NK_0(A) \rightarrow NK_0(B)
\]
is a sequence of \( W(A) \)-modules. Recall that \( (1 + tA[[t]])^\times \) is the underlying abelian group of the ring \( W(A) \); a \( W(A) \)-module is continuous if every element is killed by one of these ideals \( (1 + t^n A[[t]])^\times \).
We first recall the continuous \( W(R) \)-module structure on \( NK_s(A) \) when \( R \) is commutative and \( A \) is an \( R \)-algebra, due to Stienstra [18]. As \( NK_s(A) \) is a continuous module, it suffices to describe multiplication by \( (1 - rt^m) \), \( r \in R \). Setting \( S = R[s]/(s^m - r) \), the inclusion \( i : R \subset S \) induces a base change functor \( i^* : \mathbf{P}(A[t]) \rightarrow \mathbf{P}(A \otimes_R S[t]) \) and a transfer map \( i_* : \mathbf{P}(A \otimes_R S[t]) \rightarrow \mathbf{P}(A[t]) \).

If \( \sigma \) denotes the \( S \)-algebra map \( S[t] \rightarrow S[t] \), \( \sigma(t) = st \), then the composition \( F = i_* \sigma^* i^* \) is an additive self-functor of \( \mathbf{P}(A[t]) \). As noted in [26, 1.5], the composition \( \mathbf{P}(A) \rightarrow \mathbf{P}(A[t]) \xrightarrow{F} \mathbf{P}(A[t]) \rightarrow \mathbf{P}(A) \) is \( \otimes_R S \), so \( F \) induces multiplication by \( m \) on the summand \( K_s(A) \) of \( K_s(A[t]) \); the restriction of \( F \) to \( NK_s(A) \) is multiplication by \( (1 - rt^m)^* \). If \( A \rightarrow B \) is an \( R \)-algebra map, \( NK_s(A) \rightarrow NK_s(B) \) is a homomorphism of continuous \( W(R) \)-modules.

We can adapt these formulas to define a multiplication by \( (1 - at^m)^* \) on \( K_0(f) \) and \( NK_0(f) \) when \( a \in A \): send \([P_1, \alpha, P_2] \) to \([F(P_1), F(\alpha), F(P_2)] \). It is clear from (2.1) that \( (1 - at^m)^* \) is compatible with the exact sequence (3.1). A priori, though, the maps \( (1 - at^m)^* \) do not fit together to make \( NK_0(f) \) into a \( W(A) \)-module.

**Proposition 3.2.** For any homomorphism \( f : A \rightarrow B \), \( NK_0(f) \) is a continuous \( W(A) \)-module, and (3.1) is an exact sequence of continuous \( W(A) \)-modules.

**Proof.** As in the proof of Proposition 2.4, write \( B = Z[X]/I \), where \( Z[X] \) is a polynomial ring. Let \( R \) denote the pullback ring \( A \times_B Z[X] \), and write \( \bar{f} : R \rightarrow Z[X] \) for the quotient map. Since \( NK_s(Z[X]) = 0 \), we have \( NK_n(\bar{f}) \cong NK_n(R) \) for all \( n \). Since \( A = R/I \), Lemma 2.2 and [25] imply that the groups \( NK_0(f) \cong NK_0(\bar{f}) \cong NK_0(R) \) are continuous \( W(R) \)-modules.

Since \( W(A) = W(R)/W(I) \), where \( W(I) = 1 + tI[[t]] \), we are reduced to showing that \( (1 - rt^m) \) acts as zero on \( K_0(f) \) whenever \( r \in I \). When \( r \) is in the kernel \( I \) of \( R \rightarrow A \), the ring \( A \otimes_R S \) is just \( A[s]/(s^m) \), so \( (1 - rt^m) \) and \( (1 - 0t^m) \) act identically on \( K_0(f[t]) \). This shows that \( (1 - rt^m) \) acts as zero on \( K_0(f) \) and proves that the action of \( W(A) \) on \( K_0(f) \) is well defined and continuous. \( \square \)

Applying \( N \) to the determinant described in (2.3), we get an exact sequence

\[
0 \rightarrow NSK_0(f) \rightarrow NK_0(f) \xrightarrow{\det} NPic(f) \rightarrow 0.
\]

If \([P, \alpha, A[t]^n] \) is in \( NK_0(f) \) then \( \det[P, \alpha, A[t]^n] = [\det(P), \det(\alpha), A[t]] \).

**Theorem 3.3.** For any homomorphism \( f : A \rightarrow B \), \( NPic(f) \) is a continuous \( W(A) \)-module, and \( \det : NK_0(f) \rightarrow NPic(f) \) is a \( W(A) \)-module homomorphism.
Proof. Since the group $NK_0(f)$ is a continuous $W(A)$-module by Proposition 3.2, it is enough to show that $NSK_0(f)$ is closed under multiplication by $W(A)$. Since every element of $W(A)$ can be written as $\prod_{m>0}(1-a_m t^m)$, with $a_m \in A$, and for any element $u$ of $NK_0(f)$ there is an $n$ so that $\prod_{m\geq n}(1-a_m t^m) \ast u = 0$, it is enough to show that $NSK_0(f)$ is closed under multiplication by $(1-\alpha t^m)$ for any $\alpha \in A$ and $m \geq 1$.

It is enough to show that $F = \hat{\sigma}^* i^* \sigma^* i^*$ sends $SK_0(f[t])$ to itself. We now modify the argument of [5, 4.1]. Fix $u = [P, \alpha, A[t]^n]$ in $SK_0(f[t])$; By Remark 1.0.1, $\det(u) = 0$ implies that $\det(P) = A[t]$ and $\det(\alpha) \in A$. By naturality of $\det$, $\sigma^* i^*(u) = [P \otimes S, \alpha \otimes S, (S[t]^n)]$, $\det(P \otimes S) = S[t]$, $\det(\alpha \otimes S) \in S$ and $F(u) = [i_*(P \otimes S), i_*(\alpha \otimes S), A[t]^n]$. By Corollary 3.2 of [5] applied to $A[t] \subset S[t]$, $\det(i_*(P \otimes S)) = A[t]$ and $\det(\alpha \otimes S) = \det(\alpha)^m \in A$, so $\det(F(u)) = 0$.

Corollary 3.4. If $\text{char}(A) = p$ then $NPic(f)$ is a $p$-group.
If $\mathbb{Q} \subseteq A$ then $NPic(f)$ is an $A$-module.

Proof. Any continuous $W(A)$-module has these properties; see [25, 3.3].

4. Sheaf properties of $NPic(f)$

When $f : X \to S$ is a faithful affine morphism of schemes, let $\mathcal{I}(f)_{zar}$ denote the Zariski sheaf $f_* \mathcal{O}_X^\times / \mathcal{O}_S^\times$ on the category $Sm/S$ of smooth schemes over $S$; by [16, 4.4], $\mathcal{I}(f)_{zar}$ is also an étale sheaf, and $H^0_{et}(S, \mathcal{I}(f)_{zar}) = H^0_{nis}(S, \mathcal{I}(f)_{zar}) = Pic(f)$.

Our choice of $Sm/S$ is dictated by the need to not only include étale extensions but be closed under product with $A_S^1 \to S$.

Let $\pi^* \mathcal{I}(f)$ denote the restriction of $\mathcal{I}(f)_{zar}$ to $Sm/A_S^1$ along $\pi$. Its direct image $\pi_* (\pi^* \mathcal{I}(f))$ is the Zariski sheaf $\mathcal{I}(f)_{zar} \oplus \mathcal{N} \mathcal{I}(f)$ on $Sm/S$, where $\mathcal{N} \mathcal{I}(f)$ denotes the Zariski sheaf on $Sm/S$ associated to the presheaf $U \mapsto NPic(f \times_S U)$.

Theorem 4.1. Let $f : X \to S$ be a faithful affine morphism of schemes. Then $\mathcal{N} \mathcal{I}(f)$ is an étale sheaf on $S$. Moreover,

$$H^0_{et}(S, \mathcal{N} \mathcal{I}(f)) = H^0_{zar}(S, \mathcal{N} \mathcal{I}(f)) = NPic(f).$$

Proof. Since $\pi^* \mathcal{I}(f)$ is an étale sheaf on $A_S^1$, its direct image $\pi_* (\pi^* \mathcal{I}(f))$ is an étale sheaf on $S$; since $\pi_* (\pi^* \mathcal{I}(f)) \cong \mathcal{I}(f)_{zar} \oplus \mathcal{N} \mathcal{I}(f)$, $\mathcal{N} \mathcal{I}(f)$ is also an étale sheaf. Since

$$H^0_{et}(S, \pi_* (\pi^* \mathcal{I}(f))) = H^0_{et}(A_S^1, \pi^* \mathcal{I}(f)) = Pic(f[t]) = Pic(f) \oplus NPic(f),$$
we see that \(H^0_{\text{et}}(S, \mathcal{NI}(f)) = \text{NPic}(f)\). If \(S_s\) is a Zariski local scheme of \(S\), this shows that the stalk \(\mathcal{NI}(f)_s = H^0_{\text{zar}}(S_s, \mathcal{NI}(f))\) equals \(H^0_{\text{et}}(S_s, \mathcal{NI}(f))\). \(\square\)

**Example 4.2.** If \(f\) is seminormal, the sheaf \(\mathcal{NI}(f)\) vanishes and \(\text{NPic}(f) = 0\). This follows from Theorem 4.1 and [15, 1.5], which states that \(\text{NPic}(A,B) = 0\) when \(A\) is seminormal in \(B\).

We now modify an argument of Vorst [22] and van der Kallen [21]. Suppose that \(\text{Spec}(A) = \bigcup_{i=0}^r U_i\), where \(U_i = \text{Spec}(A_{s_i})\). Given a presheaf \(F\) of abelian groups on \(\text{Spec}(A)\), we write \(C^\bullet(\{U_i\}, F)\) for the augmented Čech complex:

\[
0 \to F(A) \to \prod_{i=0}^r F(A_{s_i}) \to \prod_{0 \leq i < j \leq r} F(A_{s_is_j}) \to \cdots \to F(A_{s_0s_1 \cdots s_r}) \to 0.
\]

Given \(s \in A\), we have an \(A\)-algebra map \(\sigma : A[x] \to A[x]\) determined by \(\sigma(x) = sx\). We write \(NF(A)_[s]\) for the direct limit of \(F(A[x]) \xrightarrow{\sigma} F(A[x]) \xrightarrow{\sigma} \cdots\). Suppose that for all \(0 \leq i_0 < i_1 < \cdots < i_p \leq r\) and \(j \leq p\):

\[
NF(A_{s_{i_0} \cdots s_{i_j} \cdots s_{i_p}}[x]) = NF(A_{s_{i_0} \cdots s_{i_{j-1}} \cdots s_{i_p}}[x])_{[s_{i_j}]}.
\]

In this situation, Vorst proved [22, 1.2] that the sequence \(C^\bullet(\{U_i\}, NF)\) is always exact. He also proved that \(F = NK_n\) satisfied (4.3), so that \(C^\bullet(\{U_i\}, NK_n)\) is exact for all \(n\). (See [22, 1.4] or [29, V.8.5]; the nonzerodivisor hypothesis is unnecessary by [20].)

**Remark 4.4.** It is easy to see (and follows from Vorst’s result [22, 1.2]) that the functor \(NU(A) = (A[t])^\times/A^\times\) satisfies (4.3). From the exact sequence of complexes

\[
0 \to C^\bullet(\{U_i\}, NU) \to C^\bullet(\{U_i\}, NU(- \otimes_A B)) \to C^\bullet(\{U_i\}, NU(- \otimes_A B)/NU) \to 0
\]

we see that \(C^\bullet(\{U_i\}, F)\) is also exact for the functor \(F(A_s) = NU(B_s)/NU(A_s)\).

**Lemma 4.5.** \(C^\bullet(\{U_i\}, \text{NPic})\) is always an exact sequence.

**Proof.** By Theorem 4.2 of [27], given \(s \in A\) we have \(\text{NPic}(A_s) \cong \text{NPic}(A)_[s]\) and hence \(\text{NPic}(A_s[x]) \cong \text{NPic}(A[x])_{[s]}\). This implies that \(\text{NPic}\) satisfies (4.3). Vorst’s result shows that \(C^\bullet(\{U_i\}, \text{NPic})\) is an exact sequence. \(\square\)

We apply these considerations to the presheaf \(\text{NPic}(f) : U \mapsto \text{NPic}(f|_U)\).
Lemma 4.6. Suppose that \( \text{Spec}(A) = \bigcup_{i=0}^{n} U_i \), where \( U_i = \text{Spec}(A_{s_i}) \). If \( f : A \hookrightarrow B \) is a ring extension, the complex \( C^*(\{U_i\}, \text{NPic}(f)) \) is exact.

\[
0 \to \text{NPic}(A, B) \to \prod_{i=0}^{n} \text{NPic}(A_{s_i}, B_{s_i}) \to \prod_{i < i_2} \text{NPic}(A_{s_{i_1}s_{i_2}}, B_{s_{i_1}s_{i_2}}) \to \cdots
\]

Proof. Let \( A^+ \) denote the subintegral closure of \( A \) in \( B \), so \( A^+ \) is seminormal in \( B \), and we have \( A \subset A^+ \subset B \). By [14, Prop. 4.1], we have an exact sequence

\[
1 \to \text{NPic}(A, A^+) \to \text{NPic}(A, B) \to \text{NPic}(A^+, B) \to 1.
\]

By Example 4.2, the third term vanishes and we have \( \text{NPic}(A, A^+) \cong \text{NPic}(A, B) \). Thus we may assume that \( B \) is subintegral over \( A \). In this case, Ischebeck proved [9, Prop. 7] that \( \text{NPic}(A) \to \text{NPic}(B) \) is surjective. Now the result follows from Remark 4.4, Lemma 4.5 and the long exact cohomology sequence associated to

\[
0 \to C^*(\{U_i\}, F) \to C^*(\{U_i\}, \text{NPic}(f)) \to C^*(\{U_i\}, \text{NPic}(f)/F) \to 0,
\]

\[
0 \to C^*(\{U_i\}, \text{NPic}(f)/F) \to C^*(\{U_i\}, \text{NPic}(f)) \to C^*(\{f^{-1}(U_i)\}, \text{NPic}) \to 0.
\]

\[\square\]

Theorem 4.7. Let \( f : A \hookrightarrow B \) be an extension of rings. Then:

\[
H^q_{et}(\text{Spec}(A), \mathcal{N}T) = \begin{cases} 
\text{NPic}(f) & \text{if } q = 0 \\
0 & \text{if } q > 0
\end{cases}
\]

Proof. The case \( q = 0 \) is given by Theorem 4.1. By Lemma 4.6, the Čech cohomology groups \( \check{H}^q(\text{Spec}(A), \mathcal{N}T) \) vanish for \( q > 0 \). Using the Cartan criterion [11, III.2.17], \( H_q^{et}(\text{Spec}(A), \mathcal{N}T) \) equals \( \check{H}^q(\text{Spec}(A), \mathcal{N}T) = 0 \) for \( q > 0 \). \[\square\]

Corollary 4.8. Let \( f : X \to S \) be a faithful affine morphism of schemes. Then

\[
H^*_et(S, \mathcal{N}T) \cong H^*_zar(S, \mathcal{N}T).
\]

Proof. Consider the site change map \( \tau : S_{et} \to S_{zar} \). Then by Theorem 4.7, the higher direct image sheaves \( R^q\tau_* \mathcal{N}T \) vanish for \( q > 0 \). Therefore the Leray spectral sequence degenerates, yielding the result. \[\square\]

Remark. More generally, if \( f : X \to S \) is any morphism of schemes then \( \mathcal{O}_S^{-\infty} \) may not inject into \( f_*\mathcal{O}_X^{-\infty} \). In this case, if we interpret \( f_*\mathcal{O}_X^{-\infty}/\mathcal{O}_S^{-\infty} \) as the mapping cone of \( \mathcal{O}_S^{-\infty} \to f_*\mathcal{O}_X^{-\infty} \) (a complex of Zariski sheaves) and use sheaf hypercohomology, then Theorem 4.1 remains valid. However, Theorem 4.7 may fail in this setting.
5. Module Cartier Divisors on $NK_n(f)$

Given an exact functor $F : \mathcal{P} \to \mathcal{Q}$, the relative $K$-theory groups $K_n(F)$ fit into an exact sequence

$$\cdots \xrightarrow{\mathcal{F}} K_{n+1}\mathcal{Q} \xrightarrow{\partial} K_n(F) \to K_n\mathcal{P} \xrightarrow{\mathcal{F}} K_n\mathcal{Q} \xrightarrow{\partial} \cdots$$

ending in $K_0\mathcal{Q} \xrightarrow{\partial} K_{-1}(F)$. Waldhausen showed that the $K_n(F)$ are the homotopy groups $\pi_{n+2}|wS_*| (F) (n \geq 0)$, where $S_*F$ denotes the category of pairs

$$(P_*, Q_*) = (P_1 \to P_2 \to \cdots \to P_n, Q_0 \to Q_1 \to \cdots \to Q_n)$$

($P_i \in \mathcal{P}$ and $Q_j \in \mathcal{Q}$), together with choices of $Q_i/Q_j$ for $i > j$, such that $F(P_*)$ is $Q_1/Q_0 \to \cdots \to Q_n/Q_0$. (See [23, 1.5.4–7] or [29, IV.8.5.3].)

**Example 5.1.** If $A$ is a ring, we write $\mathbf{P}(A)$ for the category of finitely generated projective $A$-modules. Given a ring homomorphism $f : A \to B$, we have an exact functor $\mathbf{P}(f) : \mathbf{P}(A) \to \mathbf{P}(B)$; by abuse, we write $K_*(f)$ for $K_*\mathbf{P}(f)$. Writing $f[t]$ for $A[t] \to B[t]$, we have $K_*(f[t]) = K_*(f) \oplus NK_*(f)$. The Fundamental Theorem of $K$-theory easily extends to the relative setting, yielding

$$K_*(f[t, 1/t]) \cong K_*(f) \oplus NK_*(f) \oplus NK_*(f) \oplus K_{*-1}(f).$$

Let $A$ be a commutative ring. As in [29], we write $\mathbf{End}(A)$ for the category of pairs $(P, \alpha)$, where $P$ in $\mathbf{P}(A)$ and $P \xrightarrow{\alpha} P$ is an endomorphism, and write $\text{Nil}(A)$ for the full subcategory of $\mathbf{End}(A)$ consisting of all $(P, \alpha)$ with $\alpha$ nilpotent. As pointed out in [29, II.7.4], $K_*\mathbf{End}(A) \cong K_*(A) \oplus \text{End}_*(A)$ and $K_*\text{Nil}(A) \cong K_*(A) \oplus \text{Nil}_*(A)$, where $\text{End}_*(A)$ is a graded-commutative ring and $\text{Nil}_*(A)$ is a graded $\text{End}_*(A)$-module. By naturality, the exact functors $\text{Nil}(f) : \text{Nil}(A) \to \text{Nil}(B)$ yield relative groups $K_*\text{Nil}(f) \cong K_*(f) \oplus \text{Nil}_*(f)$.

The category $\text{Nil}(A)$ is equivalent to the category $H_{1,t}(A[t])$ of $t$-primary torsion $A[t]$-modules $M$ with $pd_{A[t]}M = 1$. Specifically, if $(P, \nu)$ is in $\text{Nil}(A)$, and we write $P_\nu$ for the $A[t]$-module $P$ on which $t$ acts as $\nu$, then $P_\nu$ has projective dimension 1 over $A[t]$. The Fundamental Theorem ([29, V.8.2]) implies that $\text{Nil}_n(A) \cong NK_{n+1}(A)$. We also have $K\mathbf{P}(A[t]) \cong K\mathbf{H}(A[t])$ (see e.g., [29, V.3.2]).

**Proposition 5.2.** There is a natural isomorphism $\text{Nil}_n(f) \cong NK_{n+1}(f)$. 

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Proof. From the diagram of exact categories

\[
\begin{array}{cccc}
\text{Nil}(A) & \xrightarrow{\cong} & \mathbb{H}_{1,t}(A[t]) & \longrightarrow & \mathbb{H}(A[t]) \leftarrow \cong \mathbb{P}(A[t]) & \longrightarrow & \mathbb{P}(A[t, 1/t]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Nil}(B) & \xrightarrow{\cong} & \mathbb{H}_{1,t}(B[t]) & \longrightarrow & \mathbb{H}(B[t]) \leftarrow \cong \mathbb{P}(B[t]) & \longrightarrow & \mathbb{P}(B[t, 1/t])
\end{array}
\]

we get a fibration sequence of $K$-theory spectra

\[
\begin{array}{cccc}
\mathbb{K}\text{Nil}(A) & \longrightarrow & \mathbb{K}(A[t]) & \longrightarrow & \mathbb{K}(A[t, 1/t]) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{K}\text{Nil}(B) & \longrightarrow & \mathbb{K}(B[t]) & \longrightarrow & \mathbb{K}(B[t, 1/t])
\end{array}
\]

Taking vertical fibers, we see that there is a long exact sequence

\[
K_{n+1}(f[t]) \rightarrow K_{n+1}(f[t, 1/t]) \rightarrow K_n\text{Nil}(f) \rightarrow K_n(f[t]) \rightarrow K_n(f[t, 1/t]) \rightarrow
\]

and (using Example 5.1) an isomorphism $\text{Nil}_n(f) \cong N K_{n+1}(f)$. 

Lemma 5.3. For any ring homomorphism $f : A \rightarrow B$, $\text{Nil}_*(f)$ is a graded $\text{End}_*(A)$-module.

Proof. A typical object in the Waldhausen category $S_*\text{Nil}(f)$ is a pair

\[
(\mu_*, \nu_*) = ((M_1, \mu_1) \mapsto \cdots (M_n, \mu_n), (N_0, \nu_0) \mapsto \cdots (N_n, \nu_n)).
\]

There is a pairing $\text{End}(A) \times S_*\text{Nil}(f) \rightarrow S_*\text{Nil}(f)$ of simplicial Waldhausen categories, sending $(P, \alpha) \times (\mu_*, \nu_*)$ to

\[
((P \otimes M_1, \alpha \otimes \mu_1) \mapsto \cdots \mapsto (P \otimes M_n, \alpha \otimes \mu_n), (P \otimes N_0, \alpha \otimes \nu_1) \mapsto \cdots \mapsto (P \otimes N_n, \alpha \otimes \nu_n)).
\]

It induces a pairing $K_*\text{End}(A) \otimes K_*\text{Nil}(f) \rightarrow K_*\text{Nil}(f)$. Since the tensor product $(\alpha \otimes \beta) \otimes \mu \cong \alpha \otimes (\beta \otimes \mu)$ is associative up to natural isomorphism, the two pairings

\[
\text{End}(A) \times \text{End}(A) \times S_*\text{Nil}(f) \rightarrow S_*\text{Nil}(f)
\]

agree up to natural isomorphism, making $K_*\text{Nil}(f)$ a graded $K_*\text{End}(A)$-module. In particular, $\text{Nil}_*(f)$ is a graded module over $\text{End}_*(A)$. 

Recall that the ring $W(A)$ of big Witt vectors has underlying abelian group $(1+tA[[t]])^\times$. Almkvist’s theorem [29, II.7.4.3] states that $[P, \alpha] \mapsto \det(1-t\alpha)$ maps $\text{End}_0(A)$ isomorphically onto the subring of $W(A)$ whose underlying abelian group consists of all quotients $f(t)/g(t)$ of polynomials in $1+tA[t]$. The intersection of the ring $\text{End}_0(A)$ with the ideal $(1+t^nA[[t]])$ of $W(A)$ is the ideal $I_m = \{1+t^m(f/g)\}$
of $\End_0(A)$, and $\End_0(A)/I_m \cong W(A)/(1 + t^m A[[t]])$. In particular, $W(A)$ is the completion of $\End_0(A)$ with respect to the $t$-adic filtration.

We say that an $\End_0(A)$-module $M$ is continuous if for every $x \in M$ there is an $m$ so that $I_m \cdot x = 0$. Thus every continuous $\End_0(A)$-module $M$ is also continuous as a $W(A)$-module: for every $x \in M$ we have $(1 + t^m A[[t]]) \cdot x = 0$ for some $m$.

The exact functors $F_n, V_n : \Nil(A) \to \Nil(A)$, defined by $F_n(P, \nu) = (P, \nu^n)$ and $V_n(Q, \nu) = (Q[t]/(t^n - \nu), t)$, commute with $\Nil(A) \to \Nil(B)$. Hence they induce exact endofunctors $F_n, V_n$ on $\Sing Nil(f)$ by $F_n(\mu_*, \nu_*) = (F_n(\mu_*), F_n(\nu_*))$ and $V_n(\mu_*, \nu_*) = (V_n(\mu_*), V_n(\nu_*))$. For $a \in A$ and $n > 0$, and $\nu$ in $\Nil_*(f)$, Almkvist’s theorem associates $(1 - at^n)$ to $V_n([A, a] - [A, 0])$ and yields the product formula

$$ (1 - at^n) \ast \nu = V_n([A, a] - [A, 0]) \ast \nu. \tag{5.4} $$

Stienstra proved in [18, 19] that the $\Nil_n(A)$ are continuous $\End_0(A)$-modules, and hence $W(A)$-modules. The key step [18, 2.12] was showing that the projection formula holds:

$$(V_n \alpha) \ast \nu = V_n(\alpha \ast F_n(\nu)) \quad \text{for} \quad \alpha \in \End_0(A) \text{ and } \nu \in \Nil_*(A).$$

Here is the corresponding projection formula in the relative setting; we will postpone its proof in order to get to the main result.

**Lemma 5.5.** For all $\alpha \in \End_0(A)$ and $\beta \in \Nil_*(f)$,

$$(V_n \alpha) \ast \beta = V_n(\alpha \ast F_n(\beta)).$$

**Theorem 5.6.** Let $f : A \to B$ be a ring map. Then the product (5.4) makes $\Nil_n(f) \cong NK_{n+1}(f)$ into a continuous $W(A)$-module for every integer $n$.

**Proof.** For each $m > 0$, let $\Nil^m(A)$ denote the exact subcategory of all $(P, \nu)$ in $\Nil(A)$ such that $\nu^m = 0$. Thus we have relative groups $K_*\Nil^m(A)$ associated to

$K_*\Nil^m(A) \to K_*\Nil^m(B)$, and $K_*\Nil(f)$ is the direct limit of the $K_*\Nil^m(f)$.

Suppose that $n \geq m$. Clearly, $F_n$ acts as zero on $\Nil^m(f)$. By the projection formula 5.5, $V_n(\alpha)$ acts as zero on the image $\Nil^m_*(f)$ of $K_*\Nil^m(f) \to K_*\Nil(f) \to \Nil_*(f)$. By (5.4), $(1 - at^n)$ acts as zero on $\Nil^m_*(f)$. Since $\Nil_*(f)$ is the union of the $\Nil^m_*(f)$, for any $\beta \in \Nil_*(f)$ there is an $m$ such that $(1 - at^n) \cdot \beta = 0$ for all $n \geq m$ and $a \in A$. This shows that $\Nil_*(f)$ is a continuous $\End_0(A)$-module, and hence a continuous $W(A)$-module. \qed
Proof of Lemma 5.5. Following Stienstra [18, §6], set \( R = \mathbb{Z}[y_1, y_2] \), and set \( E = \text{End}(R; S_6) \), where \( S_6 \) is the multiplicative subset of \( R[x] \) generated by \( x \) and \( x^n - y_1^ny_2 \). As pointed out in loc. cit., there is a multi-exact pairing

\[
\Theta : E \times \text{End}(A) \times \text{Nil}(B) \rightarrow \text{Nil}(B)
\]

sending \((E, \omega), (P, \alpha)\) and \((N, \nu)\) to \((E \otimes_R (P \otimes_A N), \omega \otimes 1)\), where \( P \otimes_A N \) is regarded as an \( R \)-module by letting \( y_1 \) and \( y_2 \) act as \( \alpha \otimes 1 \) and \( 1 \otimes \nu \). As this pairing is natural in \( B \), we may replace \( \text{Nil}(B) \) by \( S.\text{Nil}(f) \). This yields (among other things) a product

\[
\Theta_\ast : K_0E \otimes \text{End}_0(A) \otimes \text{Nil}_\ast(f) \rightarrow \text{Nil}_\ast(f).
\]

Stienstra proves in loc. cit. that the elements \([R^n, \omega]\) and \([R^n, \omega']\) agree in \( K_0 E \), where

\[
\omega = \begin{pmatrix} 0 & y_1^ny_2 \\ 1 & 0 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \omega' = \begin{pmatrix} 0 & y_1y_2 \\ y_1 & 0 \\ \vdots & \vdots \\ 0 & y_1 \end{pmatrix}.
\]

Therefore the two maps

\[
\Theta_\ast([R^n, \omega], -), \Theta_\ast([R^n, \omega'], -) : \text{End}_0(A) \otimes \text{Nil}_\ast(f) \rightarrow \text{Nil}_\ast(f)
\]

agree. Stienstra also observes that these maps send \([P, \alpha] \otimes \beta\) to \( V_n(\alpha \ast F_n\beta) \) and \((V_n\alpha \ast \beta\), respectively; see also [19, p.14]. The projection formula follows. \( \square \)

6. Negative relative K-theory

Let \( f : X \rightarrow S \) be a morphism of schemes. Then we have a long exact sequence of negative K-groups, part of which is:

\[
(6.1) \quad \cdots \rightarrow K_{-d}(f) \rightarrow K_{-d}(S) \rightarrow K_{-d}(X) \rightarrow K_{-d-1}(f) \rightarrow K_{-d-1}(S) \rightarrow \cdots.
\]

**Theorem 6.2.** Let \( f : X \rightarrow S \) be a morphism of \( d \)-dimensional schemes, essentially of finite type over a field \( k \) of characteristic 0. Then for all \( r > 0 \):

1. \( K_n(f) = K_n(f \times \mathbb{A}^r) = 0 \) for \( n \leq -d-2 \).
2. \( K_{-d-1}(f) \cong K_{-d-1}(f \times \mathbb{A}^r) \) ("\( f \) is \( K_{-d-1} \)-regular.")
3. If \( f \) is a finite map then \( K_{-d-1}(f) \cong H^d_{\text{csh}}(S, f_*\mathcal{O}/\mathcal{O}). \)
Proof. By Corollary 5.9 and Theorem 6.2 of [4], \( K_n(S) \cong K_n(S \times \mathbb{A}^r) \) for all \( n \leq -d \), \( K_n(S) = 0 \) for \( n < -d \) and \( K_{-d}(S) \cong H^d_{\text{cdh}}(S, \mathbb{Z}) \); the analogous assertions hold for \( X \). The exact sequence (6.1) for \( S \) and \( S \times \mathbb{A}^r \) implies the first two assertions. For (3), we have a distinguished triangle cdh sheaves on \( S \),

\[
\mathbb{Z} \to f_*\mathbb{Z} \to f_*\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}[1].
\]

Since the cdh-cohomological dimension of \( S \) is at most \( d \), \( H^{d+1}_{\text{cdh}}(S, \mathbb{Z}) = 0 \). Thus the long exact sequence on cdh-cohomology ends in

\[
\to H^d_{\text{cdh}}(S, \mathbb{Z}) \to H^d_{\text{cdh}}(S, f_*\mathbb{Z}) \to H^d_{\text{cdh}}(S, f_*\mathbb{Z}/\mathbb{Z}) \to 0.
\]

Since \( f \) is finite, we have \( H^\bullet_{\text{cdh}}(S, f_*\mathbb{Z}) \xrightarrow{\sim} H^\bullet_{\text{cdh}}(X, \mathbb{Z}) \); assertion (3) follows. \( \square \)

Remark 6.2.1. Let \( k \) be a perfect field of characteristic \( p \). Kerz and Strunk have shown in [10] that \( K_n(S) \) is a \( p \)-primary torsion group for \( n < -d \). Then Theorem 6.2 holds for \( k \) up to \( p \)-torsion. If in addition \( k \) is a perfect field, over which weak resolution of singularities holds, then Theorem 6.2(1,2) holds for \( k \). This also follows from [10]; if strong resolution of singularities holds, (1) also follows from the Geisser–Hesselholt theorem in [6] that \( K_n(S) = 0 \) for \( n < -d \).

When \( S \) is a curve, not necessarily defined over \( \mathbb{Q} \), we have a similar result.

Theorem 6.3. Let \( f : X \to S \) be a finite map of 1-dimensional noetherian schemes. Then \( K_{-1}(f) \) fits into an exact sequence

\[
0 \to H^1_{\text{nis}}(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times) \to K_{-1}(f) \to H^0_{\text{nis}}(S, f_*\mathbb{Z}/\mathbb{Z}) \to 0.
\]

In addition, \( K_{-2}(f) \cong H^1_{\text{nis}}(S, f_*\mathbb{Z}/\mathbb{Z}) \) and \( K_n(f) = 0 \) for \( n < -2 \).

Proof. By Thomason-Trobaugh [20, 10.8], we have a spectral sequence

\[
E_2^{p,q} = H^p_{\text{nis}}(S, \mathcal{K}_{-q}(f)) \Rightarrow K_{-p-q}(f),
\]

where \( \mathcal{K}_n(f) \) is the Nisnevich sheafification of the presheaf \( U \mapsto K_n(U, f^{-1}U) \). Each stalk \( \mathcal{K}_n(f) \) is \( K_n(A, B) \), where \( A \) is a hensel local ring of dimension \( \leq 1 \). By Lemma 6.4 below, we have

\[
\mathcal{K}_n(f) = \begin{cases} 
0 & \text{if } n \leq -2 \\
f_*\mathbb{Z}/\mathbb{Z} & \text{if } n = -1 \\
f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times & \text{if } n = 0.
\end{cases}
\]
Since $\cd_{\nis}(S) \leq 1$, $E_2^{p,q} \neq 0$ only for $p = 0, 1$ and $q \leq 1$. Thus the spectral sequence degenerates to yield $K_{-2}(f) \cong H_{\nis}^1(S, f_*\mathbb{Z}/\mathbb{Z})$ and $K_n(f) = 0$ for $n < -2$. □

**Lemma 6.4.** Let $A$ be a 1-dimensional hensel local ring and $f : A \hookrightarrow B$ a finite extension. If $B$ has $r$ components, then

$$K_0(f) \cong B^\times/A^\times, \quad K_{-1}(f) \cong \mathbb{Z}^{r-1} \quad \text{and} \quad K_n(f) = 0 \text{ for } n < -1.$$

**Proof.** Since $B$ is a finite $A$-algebra, $B$ is a finite product of $r$ hensel local rings. By [24, 2.8], $K_n(A) = K_n(B) = 0$ for $n < -1$. By a result of Drinfeld [29, III.4.4.3], we have $K_{-1}(A) = K_{-1}(B) = 0$. The result now follows from (6.1). □

**Remark 6.5.** A necessary condition for $K_{-1}(f) = 0$ is that the ring extension $f : A \hookrightarrow B$ is anodal, i.e., if every $b \in B$ such that $(b^2 - b) \in A$ and $(b^3 - b^2) \in A$ belongs to $A$. (See [27, 3.1].) This is because (2.3) induces a surjection $L\det : K_{-1}(f) \to LPic(f)$, and we showed in [16] that $LPic(f) = 0$ implies that $A \subset B$ is anodal. The converse does not hold, even if $f$ is a birational extension of domains, as Example 3.5 in [27] shows.

**Example 6.6.** Here is an example to show why we assume $S$ affine in Proposition 2.5. For each $n$, the scheme $S = \mathbb{P}_k^1$ has a sheaf of algebras $\mathcal{O}_B = \mathcal{O}_S \oplus \mathcal{O}(n)$ with $\mathcal{O}(n)$ a square-zero ideal; fix $n \leq -2$ and set $X = \text{Spec}(\mathcal{O}_B)$. Then $H = H^1(\mathbb{P}_k^1, \mathcal{O}(n))$ is nonzero and $\text{Pic}(X) = \text{Pic}(S) \oplus H$, $K_0(X) \cong K_0(S) \oplus H$. In particular, $K_{-1}(f) = H \neq 0$.

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