Loop quantum gravity and quanta of space: a primer

Carlo Rovelli, Peush Upadhya
Physics Department, University of Pittsburgh, Pittsburgh PA 15260, USA
rovelli@pitt.edu
(March 24, 2022)

We present a straightforward and self-contained introduction to the basics of the loop approach to quantum gravity, and a derivation of what is arguably its key result, namely the spectral analysis of the area operator. We also discuss the arguments supporting the physical prediction following this result: that physical geometrical quantities are quantized in a non-trivial, computable, fashion. These results are not new; we present them here in a simple form that avoids the many non-essential complications of the first derivations.

I. INTRODUCTION

Combining quantum mechanics and general relativity, and understanding the quantum properties of the spacetime geometry, is a central open problem in fundamental physics. Among the various approaches presently pursued to address this problem (see [1] for an overview), is loop quantum gravity (see [2] for a recent overview).

The theoretical framework of loop quantum gravity has evolved through numerous twists and re-foundations [3–9], and it is easy for the non-experts to lose track of the basics of the theory. Furthermore, the basic structures have been found via very roundabout paths, involving mathematical tools such as Penrose’s spinor and binor calculus, Temperley-Lieb algebras, the Kauffman-axioms, C*-algebraic techniques, Gelfand representation theory, infinite dimensional measures, generalized connections, projective limits and so on. Each of these tools adds a valuable new perspective and possible new handles to the overall picture, but it is now clear that none of these tools is strictly necessary for the definition of the basics theory and the derivation of the main results. Therefore, it is now a good time for a simple presentation of the basics of the loop representation and a simple derivation of its main results. We provide here such a derivation, confining ourselves to the definition of the (unconstrained) state space of the theory, and the quantization of the area. The results we present were obtained in References [3–8]. The derivation here is slightly original, but our aim is mainly to provide a simple introduction to the subject; we refer to the original works for some proofs and details.

We organize the presentation in two parts. The first part (Section II) is purely mathematical: we construct an Hilbert space $H$ and certain operators on $H$, and we solve the spectral problem for these operators. In the second part (Section III), we present the motivations for a physical interpretation of the structures defined in the first part. We add two appendices. In the first, we recall a few relevant facts from $SU(2)$ representation theory, which clarify and make more explicit the construction presented in the main text; in the second, we discuss a special case disregarded in the main text.

II. MATHEMATICS

In this section, we define the basic tools of the kinematics of the loop representation. These will get a physical interpretation in Section II.

A. The Hilbert space

Let $M$ be a fixed three-dimensional compact smooth manifold. For concreteness, we choose $M$ to be (topologically) a three-sphere. Let $A$ be an $SU(2)$ connection on $M$; that is, $A$ is a smooth 1-form with values in $su(2)$, the Lie algebra of $SU(2)$. We denote the space of the smooth $su(2)$ valued 1-forms $A$ on $M$ as $A$. The space $A$, equipped with the sup norm is a topological space. We denote the space of the continuous functions $\Psi(A)$ on $A$ as $L$. Equipped with the pointwise topology, $L$ is a topological vector space.

An important class of functions in $L$ is given by the cylindrical functions defined as follows. Let $\Gamma$ be a graph embedded in $M$. That is, $\Gamma$ is a collection of a finite number $n$ of (oriented) piecewise analytic curves embedded in $M$, which we call links and denote as $\gamma_1, \ldots, \gamma_n$, which may overlap only at their end-points, which we call nodes, and denote as $p_1, \ldots, p_m$. The number of lines meeting in a node $p$ is called the valence of the node. A node may
have any valence equal or larger than 1. Given a curve $\gamma$ and a connection $A$ defined on a manifold, the holonomy (or parallel transport matrix) of $A$ along $\gamma$ is defined (See Appendix A); it is an element of $SU(2)$, and we denote it as $U(\gamma, A)$. Given a graph $\Gamma$ with links $\gamma_1, \ldots, \gamma_n$, and a (Haar-integrable) complex function $f$ on $(SU(2))^n$, we define the function

$$\Psi_{\Gamma,f}(A) \equiv f(U(\gamma_1, A), \ldots, U(\gamma_n, A)).$$

We call functions of the form (1) cylindrical. The cylindrical functions are dense in $L$.

Any cylindrical function $\Psi_{\Gamma,f}(A)$, based on a graph $\Gamma'$, can be rewritten as as a cylindrical function $\Psi_{\Gamma,f}(A)$ based on a larger graph $\Gamma$ containing $\Gamma'$, by simply choosing $f$ to be independent from the links in $\Gamma$ but not in $\Gamma'$. Furthermore, any link of a graph can be broken in two links, separated by a (bivalent) node. Therefore it is clear that any two given cylindrical functions $\Psi_{\Gamma',f}$ and $\Psi_{\Gamma'',g'}$ can always be rewritten as based on the same graph $\Gamma$ (choosing $\Gamma$ as the smallest graph having both $\Gamma'$ and $\Gamma''$ as subgraphs). Using this, we define the following quadratic form, defined for any two cylindrical functions.

$$(\Psi_{\Gamma,f}, \Psi_{\Gamma,g}) \equiv \int_{(SU(2))^n} dU_1 \ldots dU_n \overline{f(U_1, \ldots, U_n)} g(U_1, \ldots, U_n).$$

We write this in the suggestive form

$$(\Psi_{\Gamma,f}, \Psi_{\Gamma,g}) = \int_{A} DA \overline{\Psi_{\Gamma,f}(A)} \Psi_{\Gamma,g}(A),$$

where the meaning of the infinite dimensional integration is given only by equation (2). The quadratic form (2) defines the norm $||\Psi_{\Gamma,f}||^2 \equiv (\Psi_{\Gamma,f}, \Psi_{\Gamma,f})$ and can be extended to $L$ by continuity. By factoring away the zero-norm subspace and closing $L$ in norm, we obtain a (non-separable) Hilbert space $\mathcal{H}$. This Hilbert space plays a key role in the following.

### B. Gauge transformations and Diffeomorphisms

The connection $A$ transforms under local $SU(2)$ gauge transformations $A \rightarrow AV = V^{-1}AV + V^{-1}dV$, where $V$ is a smooth map from $M$ to $SU(2)$. The Hilbert space $\mathcal{H}$ carries a natural representation of the group of these gauge transformations $\Psi(A) \rightarrow \Psi(AV)$. This representation is unitary, because the scalar product (2) is invariant. In fact, the parallel transport matrices transform as $U(\gamma, A) \rightarrow U(\gamma, AV) = V(x_i)U(\gamma, A)V(x_f)$, where $x_i$ and $x_f$ are the initial and final points of $\gamma$, and the Haar integral in (2) is $SU(2)$ invariant.

The gauge invariant functions in $\mathcal{H}$ (which satisfy $\Psi(A) = \Psi(AV)$) form a proper subspace of $\mathcal{H}$, which we denote $\mathcal{H}_0$.

The connection $A$ transforms under a diffeomorphism $\phi : M \rightarrow M$ as a one form $A \rightarrow \phi^*A$. The Hilbert space $\mathcal{H}$ carries a natural representation of the diffeomorphism group $\Psi(A) \rightarrow \Psi(\phi^*A)$. This representation is unitary, because the scalar product (3) is invariant. In fact, the parallel transport matrices transform as $U(\gamma, A) \rightarrow U(\gamma, \phi^*A) = U(\phi^{-1}\cdot \gamma, A)$, where $[\phi^{-1}\cdot \gamma](x) = \gamma(\phi(x))$, and this transformation has no effect on the right hand side of (3). Thus, the Hilbert space $\mathcal{H}$ carries a natural unitary representation of the group of the diffeomorphisms of $M$.

### C. An orthonormal basis

Consider a graph $\Gamma$ in $M$. Associate an irreducible representation $j_i$ of $SU(2)$ to each link $\gamma_i$ of the graph. This is called coloring of the links. We denote the Hilbert space on which the representation $j_i$ is defined as $H_i$.

Let $p$ be a $n$-valent node of $\Gamma$ and let $\gamma_1 \ldots \gamma_n$ be the $n$ links that meet in $p$, and $j_1 \ldots j_n$ their associated representations. Consider the tensor product of the Hilbert spaces of these representations $H_{(j_1)} \otimes \ldots \otimes H_{(j_n)}$. Let $H_p$ be the invariant subspace (the spin-zero component) of this tensor product. Assume that $H_p$ has dimension equal or larger

---

1 There are no finite norm diffeomorphism invariant functions in $\mathcal{H}$ (satisfying $\Psi(A) = \Psi(\phi^*A)$), but a space of diffeomorphism invariant (infinite norm) “generalized functions” can be naturally defined, and can be equipped by a natural Hilbert structure, using general techniques.
can be obtained, for instance, by taking the first variation of the differential equation (38) defining the holonomy (see "punctures". To begin with, let us consider the simplest case in which the surface Σ and the spin network is geometrically well defined, that is, it is independent of the coordinates. It can be viewed as an operator from the Σ, associated to its initial point to the Hₗ, associated to its final point. These operators can be contracted with the invariant tensors vₗ at the nodes (see Appendix A), obtaining a number Ψₗ(A) that depends only on the spin network and the connection.

We now have the remarkable result that the spin network states Ψₗ (for a fixed choice of a basis in Hₗ at each node) form an orthogonal basis in H₀. This can be proven by direct computation, using standard SU(2) representation theory machinery. The normalization of the spin network states, to obtain an orthonormal basis, is simple; it is given explicitly in [8].

D. The operator E(Σ)

We now introduce some operators on H. It is easier to work with indices. Let xₐ be local coordinates on M, a, b, . . . = 1, 2, 3 be tangent indices and i, j, . . . = 1, 2, 3 be indices on the Lie algebra of SU(2). Thus the components of the connection are given by A(x) = Aₐ(x)dxₐ = Aₐᵢ(τ)τᵢdxₐ, where the τᵢ are the three generators of SU(2). The first operator we define is a functional derivative " operator. A functional derivative is a distribution and needs to be suitably smeared. Because of the peculiar structure of the functions in H, which are (limits of sequences of) functions with support on one dimension, it is sufficient to smear the functional derivative in two dimensions only, in order to have a well defined operator. Let Σ : σ → xₐ(σ) be an oriented (2d) surface in M, and let σ = (σ¹, σ²) be coordinates on Σ. We define the operator

\[
E(Σ) = -i \int dσ₁dσ₂εᵦᵦ \frac{∂xᵦ(σ)}{∂σ₁} \frac{∂xᵦ(σ)}{∂σ₂} \frac{δ}{δAᵢ₂(σ)},
\]

where εᵦᵦ is the completely antisymmetric tensor-density with ε₁₂₃ = 1. It is easy to verify that the operator E(Σ) is geometrically well defined, that is, it is independent of the coordinates σ and xₐ chosen.

The operator E(Σ) is well defined on the cylindrical functions. To see how this may happen, consider its action on the spin network state Ψₗ(A). We denote the intersection points between the spin network s and the surface Σ as "punctures". To begin with, let us consider the simplest case in which the surface Σ and the spin network s intersect on single puncture p, where p lies on (the interior of) the link γ. Let j be the color of γ. A standard result, which can be obtained, for instance, by taking the first variation of the differential equation (38) defining the holonomy (see for instance [10]), is

\[
\frac{δ}{δAᵢ₂(σ)} U(γ, A) = \int_γ ds \frac{dxᵦ(σ)}{ds} δᵦγ(σ, s) U(γ(0, s), A) \ τᵢ U(γ(s, 1), A).
\]

Here, s ∈ [0, 1] is a coordinate along the curve γ : s → xₐ(s) and the curves γ(0, s) and γ(s, 1) are the two segments in which the point with coordinate s cuts γ. It follows that the derivative of a matrix in the representation j is

\[
\frac{δ}{δAᵢ₂(σ)} j[U(γ, A)] = \int_γ ds \frac{dxᵦ(σ)}{ds} δᵦγ(σ, s) \ j[U(γ(0, s), A)] \ τᵢ(j) \ j[U(γ(s, 1), A)],
\]

where τᵢ(j) are the generators of the spin j representation of SU(2). Let us isolate j[U(γ, A)] (γ being the link that crosses Σ) in the state and write the state Ψₗ(A) as

\[
Ψₗ(A) = Ψₗₘ(lₗ₋γ) A) j[U(γ, A)]ₘₗ,
\]

where l and m are indices in the Hilbert space of the representation associated to γ. From [4], the action of the operator E(Σ) on Ψₗ is

\[
E(Σ) Ψₗ(A) = -i \int_Σ dσ₁dσ₂εᵦᵦ \frac{∂xᵦ}{∂σ₁} \frac{∂xᵦ}{∂σ₂} Ψₗₘ(lₗ₋γ) A) \frac{δ}{δAᵢ₂(σ)} j[U(γ, A)]ₘₗ.
\]
Using (9), we obtain
\[ E^i(\Sigma)\Psi_s(A) = -i \int_{\Sigma} d\sigma^1 d\sigma^2 \int_{\gamma} ds \ \epsilon_{abc} \frac{\partial x^a(\bar{\sigma})}{\partial \sigma^1} \frac{\partial x^b(\bar{\sigma})}{\partial \sigma^2} \frac{dx^c(s)}{ds} \delta^3(\gamma(s), x(\bar{\sigma}))) \times \Psi_{(s-\gamma)}^{lm}(A) \left( j[U(\gamma(0, s), A)]_{(j)} \tau^i_{(j)} j[U(\gamma(s, 1), A)] \right)_{lm}. \]

Remarkably, the three partial derivatives combine to produce the Jacobian for the change of integration coordinates from $(\sigma^1, \sigma^2, s)$ to $x^1, x^2, x^3$. If this Jacobian is non vanishing, we perform the change of integration coordinates and then we can integrate away the delta function, obtaining
\[ E^i(\Sigma) \Psi_s(A) = -i \Psi_{(s-\gamma)}^{lm}(A) \left( j[U(\gamma(0, s), A)]_{(j)} \tau^i_{(j)} j[U(\gamma(s, 1), A)] \right)_{lm}. \]

Thus, the effect of the operator $E^2(\Sigma)$ on the state $\Psi_s(A)$ is simply the insertion of the matrix $-i\tau^i_{(j)}$ in the point corresponding to the puncture. If, on the other hand, the Jacobian vanishes, then the entire integral vanishes. This happens if the tangent to the loop $\frac{dx^a(s)}{ds}$ is tangent to the surface. In particular, for instance, this happens if the loop $\gamma$ lies entirely on the surface, in which case the puncture is not an isolated point. Therefore only isolated punctures contribute to $E^2(\Sigma)\Psi_s(A)$.

The key of the above computation is the analytical expression for the (integer) intersection number $I[\Sigma, \gamma]$ between a surface $\Sigma$ and a loop $\gamma$
\[ I[\Sigma, \gamma] = \int_{\Sigma} d\sigma^1 d\sigma^2 \int_{\gamma} ds \ \epsilon_{abc} \frac{\partial x^a(\bar{\sigma})}{\partial \sigma^1} \frac{\partial x^b(\bar{\sigma})}{\partial \sigma^2} \frac{dx^c(s)}{ds} \delta^3(\gamma(s), x(\bar{\sigma}))). \]

This integral is independent of the coordinates $\bar{\sigma}, s$ and $x^a$, and yields an integer: the (oriented) number of punctures. (The sign is determined by the relative orientation of surface and loop.) Finally, it is easy to see that what happens if the surface $\Sigma$ and the spin network $s$ intersect in more than one puncture (along the same or different links). In this case $E^2(\Sigma)\Psi_s(A)$ is a sum of one term per puncture – each term being given by the insertion of a $\tau$ matrix. Finally, a bit more care is required for the computation of $E^i(\Sigma)\Psi_s(A)$ when the punctures are also nodes of $s$. This case is discussed in Appendix B.

**E. The operator $A(\Sigma)$**

Clearly $E^i(\Sigma)$ is not an $SU(2)$ gauge invariant operator. In fact, we have seen in the previous section that acting on spin network states, which are in $\mathcal{H}_0$, it gives states which, in general, are not in $\mathcal{H}_0$. Consider the operator
\[ E(\Sigma) = \sqrt{E^i(\Sigma)E^i(\Sigma)}. \]

Acting on a state $\Psi_s$ that intersects $\Sigma$ only once, it gives
\[ E^2(\Sigma)\Psi_s(A) = -\Psi_{(s-\gamma)}^{lm}(A) \left( j[U(\gamma(0, s), A)]_{(j)} \tau^i_{(j)} \right)_{lm} = \Psi_{(s-\gamma)}^{lm}(A) \left( j[U(\gamma(0, s), A)]_{(j)} \right)_{lm} = j(j+1) \Psi_s(A). \]

(The relative orientation of the surface and the link becomes irrelevant because of the square.) Thus
\[ E(\Sigma) \Psi_s(A) = \sqrt{j(j+1)} \Psi_s(A). \]

This is a nice result: not only $E(\Sigma)\Psi_s$ is in $\mathcal{H}_0$, the space of the $SU(2)$ gauge invariant states, but we even have that the spin network state $\Psi_s$ is an eigenstate. Both properties, however, hold only because there is a single puncture. It is easy to see that if there are two (or more) punctures, the cross terms spoil the gauge invariance of $E^2(\Sigma)\Psi_s$. We are now going to define a $SU(2)$ invariant operator by eliminating the cross terms. To this purpose, let $\Sigma^{(n)} = (\Sigma_k^{(n)}, k = 1 \ldots n)$ be a sequence of increasingly fine partitions of $\Sigma$ in $n$ small surfaces. We define
\[ A(\Sigma) = \lim_{n \to \infty} \sum_k E(\Sigma_k^{(n)}). \]
For every given spin network state \( \Psi_s \), there is a partition \((n)\) sufficiently fine for which each puncture \( i \) falls on a different small surface \( \Sigma_k \). For these \( k \), equation (14) holds, while the other \( E^i(\Sigma_k) \) vanish. From that \((n)\) on, the limit is trivial. And therefore

\[
A(\Sigma) \, \Psi_s = \sum_{i \in \{n(\Sigma)\}} \sqrt{j_i(j_i + 1)} \, \Psi_s,
\]

where the index \( i \) labels the punctures. This formula holds for all cases except for the case in which a node of \( p \) is on \( \Sigma \), for which see Appendix B.

The operator \( A(\Sigma) \) is well defined on \( \mathcal{H}_0 \). It is diagonalized by spin network states. It is self-adjoint. Its spectrum includes the main sequence given by (14), namely

\[
A = \sum_i \sqrt{j_i(j_i + 1)}.
\]

The full spectrum is computed taking also the cases with nodes into account (see Appendix B). It is given by

\[
A = \sum_{j_i, j_i', j_i''} \sqrt{j_i''(j_i'' + 1) + j_i'(j_i' + 1) - 1/2j_i(j_i + 1)}.
\]

In conclusion, for each surface \( \Sigma \) in \( M \) there is a self-adjoint operator \( A(\Sigma) \), with spectrum (18), which is defined by equations (1), (12) and (15), and which is diagonalized by a basis of spin network states.

III. PHYSICS

We now provide a physical interpretation to the construction of the previous section. General relativity can be seen as a diffeomorphism invariant theory for a \( SU(2) \) connection \( A^i_a(x) \) \( \mathbb{I} \). In this formalism, the conjugate momentum observables \( E^i_a(x) \) must be interpreted as the inverse densitized triad. That is, the three-dimensional metric \( g_{ab}(x) \) is defined by

\[
(\det g)g^{ab}(x) = E^i_a(x)E^i_b(x).
\]

The space \( \mathcal{A} \) defined in Section I has thus a natural interpretation as the configuration space of general relativity in this formalism. The two (kinematical) gauge groups of the theory, namely local \( SU(2) \) and the group of the diffeomorphisms act naturally in \( \mathcal{A} \).

To quantize the theory canonically, we need a Hilbert space of states on which an algebra of observables is represented. The simplest possibility is to consider a space of functions over configuration space. The Hilbert space \( \mathcal{H} \) is an example of such a space. It represents an interesting candidate particularly because it carries natural unitary representations of the two invariance groups of the theory.

The analogous choice would not be viable in a conventional Yang-Mills theory, because \( \mathcal{H} \) is a “far too big” state space (in fact, non-separable), obtained by assigning finite norm to far “too singular” functions, having domain of dependence on one-dimensional curves. But the presence of the diffeomorphism group changes the game drastically, because the physical states invariant under (a suitable quantum version of) the diffeomorphisms are de facto “smeared all over the manifold”, and form a separable Hilbert space. On the other hand, the state spaces on which conventional quantum field theory is built (say, Fock space) do not carry a sensible unitary representation of the diffeomorphism group. Their structure relies heavily on the existence of a background metric, and the difficulty of making sense of background independent notions in field theory (or string theory) is well known (see for instance the discussion in \( \mathbb{I} \)). The physical hypothesis that loop quantum gravity explores is that the correct mathematics for describing in a non-perturbative and background independent fashion a diffeomorphism invariant theory –a theory of spacetime, and not a theory in spacetime– must be searched in rather different realms than the usual splendor of conventional quantum field theory.

Thus we interpret the states \( \Psi(A) \) in \( \mathcal{H} \) as quantum states of the gravitational field (prior to imposing the quantum constraints). The conjugate momentum operator \( \frac{\delta}{\delta A^i_a(x)} \) is then to be interpreted as the quantum operator corresponding to the inverse densitized triad \( E^i_a(x) \) (the two have correctly, of course, the same geometrical transformation properties). More precisely, the canonical commutation relations between \( A \) and \( E \) are

\[
\{A^i_a(x), E^b_j(y)\} = G \, \delta^b_a \, \delta^i_j \, \delta^3(x, y).
\]
where we have used (19). Using the formula for the inverse of a matrix, we have
\[ \delta_{ij} = \frac{\delta a_i}{\delta a_j}. \]

For a sufficiently fine triangulation \((n)\), \(\Sigma^{(n)}_k\) is arbitrarily small. We can thus replace the integration in (22) with the value of the integrand in a point \(x_k\) in \(\Sigma^{(n)}_k\) times the coordinate area \(a(\Sigma^{(n)}_k)\) of the small surface
\[ E^i(\Sigma^{(n)}_k) = a(\Sigma^{(n)}_k) \epsilon_{abc} \frac{\partial x^a}{\partial \sigma^i} \frac{\partial x^b}{\partial \sigma^i} E^{ci}(x_k). \]

Inserting this in (21), we obtain precisely the definition of Riemann integral, yielding
\[ A(\Sigma) = \int \frac{da^3}{d\sigma^2} \sqrt{E^{a_3}(x)} E^{a_3}(x) = \int \frac{da^3}{d\sigma^2} \sqrt{g} \det g_{a_3 a_3}. \]

where we have used (14). Using the formula for the inverse of a matrix, we have
\[ A(\Sigma) = \int \frac{da^3}{d\sigma^2} \sqrt{\frac{g_{i_1 j_1}}{g_{i_2 j_2}}} = \int \frac{da^3}{d\sigma^2} \sqrt{\det \frac{g}{2}}, \]

where \(\frac{g}{2}\) is the 2d metric induced by \(g_{a b}\) on \(\Sigma\). The last equation shows that the result is covariant and is immediately recognized as the physical area of \(\Sigma\). In conclusion, and restoring physical units, we have
\[ hG A(\Sigma) = \text{Area of } \Sigma. \]

The area depends on the metric and the metric is the gravitational field. In quantum gravity, the gravitational field is given by a physical operator, and therefore the area is represented by an operator as well. According to standard quantum mechanics rules, the spectrum of the operator corresponds to the possible outcomes of individual measurements of the corresponding physical quantity. Therefore the hypothesis that the quantum theory of gravity can be based on the Hilbert space \(\mathcal{H}\) –which is the basic hypothesis of loop quantum gravity– yields the physical result that Planck scale measurements of the area can yield only quantized outcomes. These are labeled (see (17)) by \(n\)-tuplets of triplets of half integers \((j_i^a, j_i^d, j_i^f), i = 1, \ldots, n\) and given by
\[ A = \frac{1}{2} hG \sum_i \sqrt{2j_i^a(j_i^a + 1) + 2j_i^d(j_i^d + 1) - j_i^f(j_i^f + 1)}. \]

For \(j_i^f = 0\) and \(j_i^a = j_i^d = j_i\), this reduces to the spectrum
\[ A = hG \sum_i \sqrt{j_i(j_i + 1)}. \]

A few comments are needed, in order to qualify this result.
The overall multiplicative factor in front of the right hand side of (28) is uncertain. In fact, it has been noted that the loop quantization of a SU(2) theory contains an arbitrary parameter, the Immirzi parameter, which rescales the spectrum. See [13] for details.

The operator \( A(\Sigma) \) is invariant under SU(2) gauge transformations, but not under three or four dimensional diffeomorphisms. Therefore, strictly speaking it is not an observable of the theory, and we cannot directly give its spectrum physical meaning. The failure of \( A(\Sigma) \) to be diff-invariant is a consequence of the fact that the area of an abstract surface defined in terms of coordinates is not a diff invariant concept. In fact, physical measurable areas in general relativity correspond to surfaces defined by physical degrees of freedom, for instance matter (the area of the surface this table) or the gravitational field itself (the area of an event horizon). However, it is reasonable to expect that the fully gauge invariant operator corresponding to a physically defined area (say defined by matter) has precisely the same mathematical form as the non gauge invariant operator studied here. The reason is that one can use matter degrees of freedom to gauge-fix the diffeomorphisms – so that a non-diff-invariant quantity in pure gravity corresponds to a diff-invariant quantity in a gravity+matter theory. This expectation has been confirmed explicitly in a number of cases [4,14].

Objections have been raised about the last point. See in particular [15]. Some objections are based on the intuition that the position of the matter defining the surface could be subjected to quantum fluctuations, preventing the possibility of defining a sharp surface. This objection is incorrect. Neither the position of matter, nor the area of a surface, have physical independent reality. It is only the gravitational field in the location determined by the matter, or, the other way around, the location of the matter in the gravitational field, that have physical reality. The two do not form independent sets of degrees of freedom subjected to independent quantum fluctuations. See [16] for a detailed discussion of this point.

A result analogous to the one for the area holds for the volume.

There are many spin network bases, because a basis in each \( H_p \) must be selected in order to have a fully determined choice. For fixed \( \Sigma \), we can choose a basis (in fact, many bases) that diagonalizes \( A(\Sigma) \). However, we cannot choose a basis that diagonalizes \( A(\Sigma) \) for every \( \Sigma \). To see how this may happen, consider a the 4-valent node described in Appendix A. The area of a surface \( \Sigma_1 \) that separates \( \gamma_1 \) and \( \gamma_2 \) from \( \gamma_3 \) and \( \gamma_4 \) is diagonalized by the basis (36), while the area of a surface \( \Sigma_2 \) that separates \( \gamma_1 \) and \( \gamma_3 \) from \( \gamma_2 \) and \( \gamma_4 \) is diagonalized by the basis (57). Thus, \( A(\Sigma_1) \) and \( A(\Sigma_2) \) do not commute, and there is no basis that diagonalizes all area operators.

The spectrum of the area operator is composed by two sequences. A main sequence (29), and a secondary sequence, given by (28) with \( j^{(t)} \neq 0 \), generated by the states with nodes on the surface. Based on the intuition that such states are –in a sense– degenerate, doubts have been raised on whether the secondary sequence must be considered a prediction of the theory on the same ground as the main sequence (29).

Obviously, direct experimental verification of the predicted spectrum (28) is far outside the present possibilities. Possibilities of empirical corroboration, if any, are likely to be indirect. For instance, future observations of the cosmic background gravitational radiation might perhaps reveal an imprint of the spectrum. This possibility has not yet been investigated. On the other hand, the result (29) is the basis of all applications of loop quantum gravity to black hole thermodynamics [8,17].

In summary, we have defined the unconstrained state space of quantum gravity in the loop representation, and the basic operators of the theory. We have then defined an operator \( A(\Sigma) \), associated to each surface \( \Sigma \), and computed its spectrum. We have finally shown that the classical limit of \( A(\Sigma) \) is the geometrical area of the surface \( \Sigma \). This leads to the physical hypothesis that the spectrum of \( A(\Sigma) \) describes the quantizations of the physically measurable area. This constructions puts the old suggestion that at the Planck scale geometry is discrete (in the quantum sense) in a precise framework and gives it a precise and non trivial quantitative form.

For a discussion of the rest of the loop quantum gravity and the problems still open in the theory, and for detailed references, see [4].

ACKNOWLEDGMENTS

We thank Don Marolf for a useful exchange. This work has been supported by NSF Grant PHY-95-15506.
APPENDIX A: DETAILS ON SPIN NETWORK STATES

We add a few details in order to clarify the construction of the previous subsection.

A. SU(2)

We begin by recalling some basic facts about SU(2) representation theory, in order to fix our notation. Let \( U_{AB} \), where \( A, B = 0, 1 \) be a SU(2) matrix. These matrices act on \( C^2 \) vectors \( v^A \), and leave the unit antisymmetric objects \( \epsilon^{AB} \) and \( \epsilon_{AB} \) (where \( \epsilon^{01} = \epsilon_{01} = 1 \)) invariant. Using \( \epsilon_{AB} \) (from, say, the left), we can lower the first index of the matrix \( U \), defining \( U_{AB} = \epsilon_{AC} U^C_B \). The irreducible representations of SU(2) are labeled by their spin \( j \), which is a nonnegative half-integer. The spin-\( j \) representation has dimension \( 2j + 1 \) and can be obtained as the symmetrized tensor product of \( 2j \) copies of the fundamental (2-dimensional) representation. Thus the Hilbert space \( H_j \) of the \( j \) representation is the symmetrized tensor product of \( 2j \) copies of \( C^2 \). We write vectors in \( H_{ij} \) (in “spinor” notation) as completely symmetric tensors with \( 2j \) indices \( u^{A_1 \ldots A_{2j}} \) (there are \( 2j + 1 \) unordered combinations of \( 2j \) zero’s and one’s). For every \( U \in SU(2) \), \( j[U] \) is then obtained by symmetrizing the indices of the (tensor) product of \( 2j \) times \( U_{AB} \).

\[
 j[U]^{A_1 \ldots A_{2j} B_1 \ldots B_{2j}} = U^{(A_1 (B_1 \ldots U^{A_{2j}) B_{2j}).}\]

Again, we can lower all upper indices of \( j[U] \) with \( \epsilon_{AB} \)’s.

B. The invariant tensors \( v \)

The invariant tensors \( v \) defined in Section \( \text{II} \) are SU(2) invariant vectors in the tensor product \( H_{j_1} \otimes \ldots \otimes H_{j_n} \) of the representations \( j_1, \ldots, j_n \) associated to the links \( \gamma_1, \ldots, \gamma_n \) that meet at a \( n \)-valent node \( p \). Thus they have the form \( v^{(A_1 \ldots A_{2j_1}) \ldots (B_1 \ldots B_{2j_n})} \). Since the only invariant object is \( \epsilon^{AB} \), they are given by suitably symmetrized sequences of \( \epsilon^{AB} \)’s. Recall we denote the space of such vectors (the spin zero subspace of the tensor product) as \( H_p \).

\( H_p \) has dimension zero if the valence of \( p \) is one. If \( p \) is bi-valent, \( H_p \) has dimension zero unless the two links are associated to the same representation. In this case, the only invariant tensor is

\[
 v^{A_1 \ldots A_{2j_1} B_1 \ldots B_{2j_2}} = \epsilon^{(A_1 (B_1 \ldots \epsilon^{A_{2j_1}) B_{2j_2})}.\]

The most interesting case is if \( p \) is trivalent, then \( H_p \) has dimension zero unless the three representations satisfy the Clebsh-Gordon condition \( |j_2 - j_3| \leq j_1 \leq j_2 + j_3 \) (the angular momentum addition condition: each one of three representations is in the tensor product of the other two). If they do satisfy these conditions, then \( v \) is the Wigner 3-j symbol, namely the symmetric form of the Clebsh-Gordon coefficients, which can be written as

\[
 v^{A_1 \ldots A_{2j_1} B_1 \ldots B_{2j_2} C_1 \ldots C_{2j_3}} = \epsilon^{A_1 B_1 \ldots \epsilon^{A_{2j_1}) B_{2j_2}) \epsilon^{B_{2j_2} C_1 \ldots \epsilon^{B_{2j_2}) C_{2j_3}) \epsilon^{C_{2j_3}) A_{2j_3}) \epsilon^{A_{2j_3}) C_{2j_3}) \epsilon^{A_{2j_3}) B_{2j_3})}.\]

(32)

where complete symmetrization in the \( A \) indices, in the \( B \) indices and in the \( C \) indices is understood, and

\[
 2j_1 = a + c, \quad 2j_2 = a + b, \quad 2j_3 = b + c.\]

(33)

Notice that the existence of three integers \( a, b \) and \( c \) satisfying (33) is equivalent to the Clebsh-Gordon condition on the three spins \( j_1, j_2, j_3 \). Indeed, the Clebsh-Gordon condition is equivalent to the possibility of rooting lines across a fork: if we have a triple node with \( 2j_1, 2j_2, 2j_3 \) lines coming along each of the three links \( \gamma_1, \gamma_2, \gamma_3 \), then \( j_1, j_2, j_3 \) satisfy the Clebsh-Gordon condition if and only if the lines can be consistently rooted. Then \( a \) lines connect \( \gamma_1 \) and \( \gamma_2 \), \( b \) lines connect \( \gamma_2 \) and \( \gamma_3 \) and \( c \) lines connect \( \gamma_3 \) and \( \gamma_1 \).

For higher valence \( n \), the \( n \)-indices tensor \( v \) can be constructed by contracting 3-j symbols. It is easier at this point to shift to vector notation. That is, to write \( u^{A_1 \ldots A_{2j_1}} \) as \( u^{im}_{(ij)} \) where \( m = 1 \ldots (2j + 1) \). Then we write the representation matrices as \( j[U]^{ab} \) (and \( j[U]_{ab} \)) and we write (31) and (32) as \( u^{im}_{(ij)(j_1)(j_2)(j_3)} \) and \( u^{lm}_{(ij)(j_1)(j_2)(j_3)} \). The 4-index \( v \) have the form \( \epsilon^{lmnp}_{(ij)(j_1)(j_2)(j_3)(j_4)} \). A basis in the space of these objects is given as

\[
 v^{lmnp}_{(ij)(j_1)(j_2)(j_3)(j_4)} = c^j \epsilon^{lmnp}_{(ij)(j_1)(j_2)(j_3)(j_4)}\]

(34)

where the basis elements are
Here \( j \) is any representation such that the Clebsh-Gordon condition are satisfied (between \( j_1, j_2, j \) as well as between \( j, j_3, j_4 \)). Since there is a finite number of representations \( j \) satisfying these Clebsh Gordon conditions, the \( v \) span a finite dimensional space, which is \( H_p \). A direct calculation shows that the basis vectors (with different \( j \)'s) (35) are orthogonal to each other. We can view equation (35) as representing the expansion of a 4-valent node into two 3-valent nodes, joined by a “virtual” link with color \( j \). Notice that in (35) we could have grouped the 4 representations in couples differently. Namely, there exists another basis

\[
e_j^{lmnp}(j_1)(j_2)(j_3)(j_4) = \delta_{lj}^{mr} \delta_{pj}^{sn} v^{(j_1)(j_2)(j_3)(j_4)}.
\]

According to the recoupling theorem, the matrix of the change of basis is given by the Wigner 6-\( j \) symbols

\[
a_{j}^{abcd}(j_1)(j_3)(j_2)(j_4) = \sum_{k} \left( \begin{array}{cccc} j_1 & j_2 & j & j_3 \end{array} \right) e_{k}^{abcd}(j_1)(j_2)(j_3)(j_4).
\]

The technique of expanding a node in 3-valent nodes joined by virtual links can be used for constructing the basis of \( H_p \) in general. For every \( n \)-valent node, we write a “virtual” tree-like (no closed loops) graph with \( n \) open ends. The dimension of \( H_p \) is given by counting the number of colorings of the virtual links which are (Clebsh-Gordon-) compatible at all virtual 3-valent nodes. Each topologically distinct virtual decomposition defines a basis in \( H_p \).

C. Holonomies

We recall that given a loop \( \gamma : s \in [0,1] \rightarrow \gamma(s) \in M \) and a connection \( A \) on \( M \), the parallel propagator \( U(\gamma, A) \) is the value for \( s = 1 \) of the \( SU(2) \) matrix \( U(s) \) that solves the differential equation

\[
dU(s) = A_\alpha(\gamma(s))U(s) = 0
\]

with the initial condition \( U(0) = 1 \), the identity element of \( SU(2) \). The solution of this form is formally written as

\[
U(\gamma, A) = \mathcal{P} e^{\int_{0}^s A_\alpha(\gamma(s)) \frac{ds}{ds}}.
\]

and can be given explicitly as a power series in \( A \) by expanding the exponential in powers and path-ordering the powers of \( A \) along \( \gamma \). (That is \( \mathcal{P} A(\gamma(s'))A(\gamma(s)) \equiv A(\gamma(s))A(\gamma(s')) \) if \( s < s' \).)

D. Spin network states

Given a spin network \( s = (\Gamma, j_i, v_r) \), the (representative in the representation \( j_i \) of the) parallel propagator of the connection \( A \) along each link \( \gamma_i \) defines the matrix \( j[U(\gamma_i, A)]_{lm} \). By contracting these matrices with the invariant tensors \( v_r \) at each node \( p_r \), we obtain the spin network state \( \Psi_s(A) \).

As a simple example, consider a graph \( \Gamma \) formed by two nodes \( p \) and \( q \) in \( M \) joined by three links \( \gamma_1, \gamma_2, \gamma_3 \). Then

\[
\Psi(\Gamma, j_1j_2j_3) = v_{lj_1j_2j_3}^{mr} J[U(\gamma_1, A)]_{lp} J[U(\gamma_2, A)]_{mq} J[U(\gamma_3, A)]_{nr} v_{lj_1j_2j_3}^{mr} v_{lj_1j_2j_3}^{mr}.
\]

(In this example, there is no need of indicating explicitly the coloring of the nodes, because they are both trivalent and therefore admit a unique coloring.) Notice that in (38), the pattern of the indices reflects the topology of the graph \( \Gamma \). In fact, spin network originated from Penrose’s graphical notation for tensors. For more details, see [6].

APPENDIX B: NODES ON \( \Sigma \)

In this Appendix we discuss the action of \( E^s(\Sigma) \) and \( A(\Sigma) \) on \( \Psi_s \) in the case that we have neglected in the main text, namely the case in which \( s \) includes nodes which lies on \( \Sigma \). Let us assume that there is a single node \( p \) of \( s \) on \( \Sigma \); the case in which there is more than one is then trivial. Assume that \( p \) is an \( n \)-valent node. We must distinguish the
links that meet in \( p \) in three classes, which we denote “up”, “down” and “tangential”. The tangential links are the ones that overlap with \( \Sigma \) for a finite interval. The others links are naturally separated into two classes according to the side of \( \Sigma \) they lie (which we arbitrarily label “up” and “down”). We chose a basis in \( H_p \) as follows. We decompose the \( n \) valent node \( p \) into a tree-like virtual graph (see Appendix A) by combining all “up” links into a virtual link “up”, all “down” links into a virtual link “down”, and all “tangential” links into a virtual link “tangential”. We call \( j^u, j^d \) and \( j^t \), respectively, the colors of these three virtual links. These three virtual links then meet together, at the node \( \psi_{(j^u)(j^d)(j^t)} \).

By inspecting equation \( (3) \), we see that the operator \( E_i(\Sigma) \) fails to “see” the tangential links, because for these links the three derivatives in

\[
\epsilon_{abc} \frac{\partial x^a(\vec{\sigma})}{\partial \sigma^1} \frac{\partial x^b(\vec{\sigma})}{\partial \sigma^2} \frac{\partial x^c(s)}{\partial s}
\]

are coplanar, and the above expression vanishes. Therefore the tangential links do not contribute. On the other hand, the other two set of links contribute (by symmetry) in equal fashion. Thus \( E^i(\Sigma) \) inserts a \( \tau_{(j^*)} \) matrix in the “up” link plus a \( \tau_{(j^t)} \) matrix in the “down” virtual link. In other words, if we write \( \Psi_s = \psi_{slmn} \psi_{(j^u)(j^d)(j^t)} \), we have

\[
E^i(\Sigma) \Psi_s = \psi_{slmn} \frac{1}{2} \left( \tau_{(j^u)}^i \delta_{pq}^m - \delta_{pq}^i \tau_{(j^d)}^m \right) \psi_{(j^u)(j^d)(j^t)}.
\]

Notice the minus sign coming from the inverse orientation of the jacobian \( (3) \) in the two cases. For a more complete derivation of \( (3) \), see \[8\]. Since \( \psi \) is an invariant tensor,

\[
\left( \tau_{(j^u)}^i + \tau_{(j^d)}^i + \tau_{(j^t)}^i \right) \psi_{(j^u)(j^d)(j^t)} = 0,
\]

from which

\[
(\tau_{(j^u)}^i + \tau_{(j^d)}^i)^2 = (\tau_{(j^t)}^i)^2
\]

when acting on \( \psi_{(j^u)(j^d)(j^t)} \). Using this, simple algebra yields

\[
(\tau_{(j^u)}^i - \tau_{(j^d)}^i)^2 = 2(\tau_{(j^u)}^i)^2 + 2(\tau_{(j^d)}^i)^2 - (\tau_{(j^t)}^i)^2 = 2j^u(j^u + 1) + 2j^d(j^d + 1) - j^t(j^t + 1).
\]

In conclusion

\[
A(\Sigma) \Psi_s = \frac{1}{2} \sqrt{2j^u(j^u + 1) + 2j^d(j^d + 1) - j^t(j^t + 1)} \Psi_s.
\]

And in general

\[
A(\Sigma) \Psi_s = \frac{1}{2} \sum_{\tau \in \{0,1,2\}} \sqrt{2j^u(j^u + 1) + 2j^d(j^d + 1) - j^t(j^t + 1)} \Psi_s.
\]

which is valid for all spin network states. It includes also the case of a link \( \gamma \) crossing \( \Sigma \), because we can always rewrite \( \gamma \) as two links joined by a 2-valent node on \( \Sigma \), with \( j^t = 0 \) and \( j^u = j^d \). In this case, \( (47) \) reduces to \( (7) \).

---

[1] C Rovelli, “Strings, Loops and Others: a critical survey of the present approaches to Quantum Gravity”, to appear in the Proceedings of the GR15, Pune, India, 1997; [gr-qc/9803024](http://arxiv.org/abs/gr-qc/9803024).
[2] C Rovelli, “Loop Quantum Gravity”, Living Reviews in Relativity, [http://www.livingreviews.org/Articles/Volume1/1998-1rovelli](http://www.livingreviews.org/Articles/Volume1/1998-1rovelli) and [gr-qc/9709008](http://arxiv.org/abs/gr-qc/9709008).
[3] T Jacobson, L Smolin: “Nonperturbative quantum geometries”, Nucl Phys B299 (1988) 295.
C Rovelli, L Smolin: “Knot theory and quantum gravity”, Phys Rev Lett 61 (1988) 1155.
C Rovelli, L Smolin: “Loop space representation of quantum general relativity”, Nucl Phys B 331 (1989) 80.
A Ashtekar, C Rovelli, L Smolin: “Weaving a classical geometry with quantum threads”, Phys Rev Lett 69 (1992) 237–240.
A Ashtekar, CJ Isham: “Representations of the holonomy algebras of gravity and non-abelian gauge theories”, Class Quant
C Rovelli: “Basis of the Ponzano-Regge-Turaev-Viro-Ooguri quantum-gravity model is the loop representation basis”, Phys Rev D48 (1993) 2702–1707.
A Ashtekar, J Lewandowski: “Representation theory of analytic holonomy $C^\ast$ algebras”, in Knots and quantum gravity J Baez ed (Oxford University Press, Oxford, 1994).
J Baez: “Generalized measures in gauge theory”, Lett Math Phys 31 (1994) 213.
C Rovelli, L Smolin: “Spin networks and quantum gravity”, Phys Rev D52 (1995) 5743.
C Rovelli, L Smolin: “Discreteness of area and volume in quantum gravity”, Nucl Phys B442 (1995) 593; Erratum Nucl Phys B456 (1995) 753.
JC Baez: “Spin networks in nonperturbative quantum gravity”, in The interface of knots and physics L Kauffman ed (Providence, Rhode Island: American mathematical society 1996).
A Ashtekar, J Lewandowski, D Marolf, J Mourao, T Thiemann: “Quantization of gauge invariant theories of connections with local degrees of freedom”, J Math Phys 36 (1995) 6456.
R Loll: “Spectrum of the volume operator in quantum gravity”, Nucl Phys B 460 (1996) 143.
J Lewandowski: “Volume and quantizations”, Class Quantum Grav 14 (1997) 71.
R DePietri: “On the relation between the connection and the loop representation of quantum gravity”, Class Quant Grav 14 (1997) 53–69.
G Immirzi: “Quantum gravity and Regge calculus”, Nucl Phys B (Proc Suppl) 1997) 57 65.
[4] C Rovelli: “A generally covariant quantum field theory and a prediction on quantum measurements of geometry”, Nucl Phys B405 (1993) 797–815.
[5] JF Barbero: “Real Ashtekar variables for Lorentzian signature Space–times”, Phys Rev D51 (1995) 5507.
[6] R De Pietri, C Rovelli: “Geometry eigenvalues and the scalar product from recoupling theory in loop quantum gravity”, Phys Rev D54 (1996) 2664.
[7] S Frittelli, L Lehner, C Rovelli: “The complete spectrum of the area from recoupling theory in loop quantum gravity”, Class Quant Grav 13 (1996) 2921.
A Ashtekar, J Lewandowski: “Quantum theory of geometry I: Area operators”, Class Quantum Grav 14 (1997) A55.
A Ashtekar, J Lewandowski: “Quantum theory of geometry II: Volume operators”, gr-qc/9711031.
[8] C Rovelli: “Loop quantum gravity and black hole physics”, Helv Phys Acta 69 (1997) 582.
[9] C Rovelli: “Ashtekar’s formulation of general relativity and loop-space non-perturbative quantum gravity: a report”, Class Quant Grav 8 (1991) 1613–1675.
A Ashtekar, C Rovelli, L Smolin: “Gravitons and loops”, Phys Rev D44 (1991) 1740–55.
A Ashtekar, C Rovelli: “A loop representation for the quantum Maxwell field”, Class Quant Grav 9 (1992) 1121.
L Smolin: “Recent developments in nonperturbative quantum gravity”, in Proceedings of the XXII Gift International Seminar on Theoretical Physics, QuantumGravity and Cosmology, June 1991, Catalonia, Spain J Perez-Mercader, J Sola, E Verdaguer ed (World Scientific, Singapore, 1992).
C Di Bartolo, R Gambini, J Griego, J Pullin “Extended loops: A new arena for nonperturbative quantum gravity”, Phys Rev Lett 72 (1994) 3638–3641.
R Gambini, J Pullin: “The Gauss linking number in quantum gravity”, in Knots and quantum gravity J Baez ed (Oxford University Press, Oxford, 1994).
C Rovelli, L Smolin: “The physical Hamiltonian in nonperturbative quantum gravity”, Phys Rev Lett 72 (1994) 446–49.
C Rovelli: “Outline of a generally covariant quantum field theory and a quantum theory of geometry”, J Math Phys 36 (1995) 6529–6547.
JC Baez: “Spin networks in gauge theory”, Adv Math 117 (1995) 253.
A Ashtekar: “Mathematical problems of non-perturbative quantum general relativity”, in Gravitation and Quantization, Les Houches, Session LVIII, 1992 B Julia ed (Elsevier, Amsterdam 1995).
N Grot, C Rovelli: “Moduli-space of knots with intersections”, J Math Phys 37 (1996) 3014–3021.
T Thiemann: “Anomaly–free formulation of non–perturbative, four–dimensional Lorentzian quantum gravity”, Phys Lett B 380 (1996) 257.
R Gambini, J Pullin Loops, Knots, Gauge Theory and Quantum Gravity (Cambridge University Press, Cambridge, 1996).
J Lewandowski, D Marolf: “Loop constraints: A habitat and their algebra”, gr-qc/9710016.
R Gambini, J Lewandowski, D Marolf, J Pullin: “On the consistency os the constraint algebra in spin network quantum gravity”, gr-qc/9710015.
M Reisenberger, C Rovelli “Sum over surfaces form of loop quantum gravity”, Phys Rev D 56 (1997) 3490.
C Rovelli: “Quantum gravity as a ‘sum over surfaces’ ”, Nucl Phys B (Proc Suppl ) 57 (1997) 28.
JC Baez: “Spin foam models”, gr-qc/9709052 M Reisenberger: “A lattice worldsheet sum for 4–d Euclidean general relativity”, gr-qc/9711053.
F Markopoulou, L Smolin: “Quantum geometry with intrinsic local causality”, gr-qc/9712067.
F Markopoulou, L Smolin: “Nonperturbative dynamics for abstract $(p,q)$ string networks”, hep-th/9712148.
JA Zapata: “Combinatorial space from loop quantum gravity”, gr-qc/9703038.
T Thiemann: “Quantum spin dynamics (QSD)”, Class Quantum Grav 15 (1998) 839.
T Thiemann: “Quantum spin dynamics (QSD): II The kernel of the Wheeler–DeWitt constraint operator”, Class Quantum Grav 15 (1998) 875.

R Gambini, J Griego, J Pullin: “Vassiliev invariants: a new framework for quantum gravity”, gr-qc/9803018.

M Reisenberger: “Classical Euclidean general relativity from “left–handed area = right–handed area”, gr-qc/9804061.

[10] J Lewandowski, ET Newman, C Rovelli: “Variation of the parallel propagator and holonomy operator and the Gauss law constraint”, J Math Phys 34 (1993) 4646.

[11] A Sen: “Gravity as a spin system”, Phys Lett B119 (1982) 89–91.

A Ashtekar: “New variables for classical and quantum gravity”, Phys Rev Lett 57 (1986) 2244–2247.

A Ashtekar: “New Hamiltonian formulation of general relativity”, Phys Rev D36 (1987) 1587–1602.

A Ashtekar: Lecture notes on non-perturbative canonical gravity. Advanced Series in Astrophysics and Cosmology, Volume 6. (World Scientific, Singapore, 1991).

[12] E Witten: “Quantum background independence in string theory”, hep-th/9306122.

[13] C Rovelli, T Thiemann: “The Immirzi parameter in quantum general relativity”, Phys Rev D 57 (1998) 1009.

KV Krasnov: “On the constant that fixes the area spectrum in canonical quantum gravity”, Class Quantum Grav 15 (1998) L1.

[14] L Smolin: “Finite, diffeomorphism invariant observables in quantum gravity”, Phys Rev D49 (1994) 4028-4040.

[15] G Amelino-Camelia: “On the area operator of the Husain-Kuchar-Rovelli model and Canonical/Loop Quantum Gravity”, gr-qc/9804063.

[16] C Rovelli: “What is observable in classical and quantum gravity?”, Class Quant Grav 8, 297 (1991).

C Rovelli: “Quantum reference systems”, Class Quant Grav 8, w 317 (1991).

[17] C Rovelli: “Loop quantum gravity and black hole physics”, Helv Phys Acta 69 (1996) 582.

KV Krasnov: “Geometrical entropy from loop quantum gravity”, Phys Rev D55 (1997) 3505.

KV Krasnov: “On quantum statistical mechanics of a Schwarzschild black hole”, Gen Rel Grav 30 (1998) 53.

A Ashtekar, J Baez, A Corichi, KV Krasnov: “Quantum geometry and black hole entropy”, Phys Rev Lett 80 (1998) 904.