Nonrenormalizability of (Last Hope) D=11 Supergravity, with a Terse Survey of Divergences in Quantum Gravities

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Abstract: Before turning to the new result that D=11 supergravity is 2-loop nonrenormalizable, we give a very brief history of the ultraviolet problems of ordinary quantum gravity and of supergravities in general D.

1. Introduction

The organizers have asked me to preface my presentation of the 2-loop nonrenormalizability of D=11 supergravity (SUGRA) with a historical survey of the ultraviolet problems in Einstein theory and in lower dimensional SUGRAs. I am happy to comply, as this will help in understanding why I called D=11 SUGRA the last hope: it was the only local Quantum Field Theory that contains general relativity whose non-renormalizability properties had not yet been established. Then, as an introduction to our results, I will motivate the analysis and its implications. Naturally, I will be brief both in my survey and references, given the space and time constraints of this conference; by a happy coincidence, the new work with D. Seminara appears in the synchronous issue of Phys. Rev. Lett. [2].

2. The Ultraviolet Problems of General Relativity

This is a subject with little prehistory, aside from an old remark of Heisenberg that theories with (positive) dimensional coupling constants would be ill-behaved at high energies, one that will be amply borne out by the following considerations.

We will be working throughout in the standard perturbative formulation of GR about a flat space vacuum, expanding the metric in powers of $\kappa h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$. This displays the D-dimensional Einstein action,

$$I_E = \kappa^{-2} \int d^D x \sqrt{-g} R$$

as an infinitely self-coupled QFT,

$$I_E = \int d^D x (\partial h)^2 \sum_{n=0}^{\infty} (\kappa h)^n ,$$

homogeneous in second derivatives. The quadratic ($n=0$) terms describe the usual $g^{-2}$ propagator, the cubic ($n=1$) the 3-point vertex describing the lowest self-interaction of the $h$-field with its own
stress-(pseudo) tensor $T_{\mu\nu}(h) \sim (\partial h)^2$, and so on. The dimensions of $\kappa^2$ are, by (2.1a), $\sim L^{D-2}$, which will be important in what follows.

Let us dispose of some special cases. The D=3 theory is well-known to have no propagating modes \[5\] and correspondingly it is finite \[4, 5\] despite having a dimensional $\kappa$. For completeness, we mention that at D=3 there is a third derivative order but unitary model with local excitations, topologically massive gravity \[6\], for which the verdict is still not known \[7\]. For D=2, the Lagrangian is the Euler density and no purely gravitational Einstein theory exists. We will also not deal with higher derivative order theories, with terms quadratic in curvatures; they are renormalizable but at the price of ghost modes, since they typically have propagator denominators $\sim p^{-4}$ or $(m^2 p^2 + p^4)^{-1}$. Viewed as fundamental actions, they are thus really regulator terms; if, instead, the $R^2$ parts are themselves viewed as effective perturbative additions to the Einstein action, they cannot be used to “improve” the Einstein propagator.

Consider the simplest one-loop self-energy diagrams using dimensional regularization – which organizes divergences most transparently. Since the vertex and propagator are of reciprocal powers $(p^2, p^{-2})$ in momentum, and since each external line $\sim (\kappa h)$ must acquire two $p$’s to maintain gauge invariance by becoming a curvature, we see that each loop order acquires potential divergent contributions proportional to

$$\Delta I_{\ell} = \int d^D x \, R^{D/2} (\kappa^2 R^{D/2-1})^{\ell-1}$$

where $\ell$ represents the loop order and $R$ stands for generic curvatures or two covariant derivatives. [In odd dimensions there can only be even loop divergences.] Thus, in principle, unless there is a reason for the vanishing of an infinite number of coefficients, the theory loses predictability at all orders – it is non-renormalizable. The above result is for pure gravity. When the latter is coupled to normal matter, there will be further counterterms: graviton loops will generate matter-dependent ones, while matter loops also contribute curvature-dependent divergences. [Recall that (boson, fermion) propagators go as $(q^{-2}, q^{-1})$ and their minimal coupling to gravity through their stress tensors go as $(q^{+2}, q^{+1})$, expressing the universality of gravitational couplings.] This will lead to an expansion analogous to (2.2), involving powers both of $R$ and of $\kappa^2 T_{\mu\nu}$–like terms.

What are the concrete outcomes of this general framework? In the early seventies, a transparent algorithm to calculate generic one-loop graphs was presented \[8\] and exploited to make several important conclusions for D=4: In pure gravity, while the quadratic curvature’s coefficients in (2.2) do not vanish, this is actually easily remedied by a (divergent) field redefinition because the Gauss–Bonnet identity, $\int d^4 x [R^2_{\mu\nu\alpha\beta} - 4 R^2_{\mu\nu} + R^2] = 0$, made these terms proportional to the field equation, $\delta I_E / \delta g_{\mu\nu} \sim \int d^4 x G_{\mu\nu} X^{\mu\nu}$. The gravity-scalar field system does contain infinite and non-removable terms, i.e., they are not proportional to the field equations, $\Delta I \neq \int d^4 x (G_{\mu\nu} - \kappa^2 T_{\mu\nu}) X^{\mu\nu}$. In the wake of these results, other relevant matter couplings, including fermions, Yang–Mills and QED were systematically explored \[9\]. In all cases, their gauge or fermionic character was of no help; they all failed the one-loop test. Beyond one loop, pure GR can exhibit invariant counterterms proportional to cubic and higher powers of the Weyl tensor that do not vanish on-shell. The much harder job of explicitly calculating that pure GR failed at 2-loop order was successfully undertaken in \[10, 11\]; the coefficients of the $R^3_{\mu\nu\alpha\beta}$ counterterms were indeed non-zero.

Because the above failures of GR were at the perturbative level, I should note for balance that
already in the late '50s it was suggested that GR might be a universal regulator for all QFT in some nonperturbative way, a development that was to materialize much later and in an unexpected way through strings. Indeed, that closed strings are both finite and contain \( s=2, m=0 \) excitations, providing an acceptable quantum version of GR, is an essential part of their (and their successors') central role.

3. D<11 Supergravities

One of the hopes following the discovery of D=4 supergravity in 1976 was that, despite also describing gravity plus “matter”, it would be more convergent than the systems described earlier. The reason was of course the additional supersymmetry (SUSY), which combined spin 3/2 “matter” and gravitons into a single (super) multiplet. Indeed, that closed strings are both finite and contain \( \kappa^2 \), \( m=0 \) excitations, providing an acceptable quantum version of GR, is an essential part of their (and their successors') central role.

\[
I \sim \int d^4x [T_{\mu\nu}^2 - i\bar{J}_\mu \beta J^\mu + 3C_\mu \Box C^\mu]
\]

(3.1a)
is constructible. Now as is well-understood, there is no invariant stress-tensor for the gravitational field itself, and the supercurrent is of higher derivative (and 3-index) order, \( J \sim (Rf) \) where \( f_{\mu\nu} = D_\mu \psi_\nu - D_\nu \psi_\mu \) is the field strength of the potential \( \psi_\mu \). It is thus natural to seek a higher derivative analog of \( T_{\mu\nu} \), and there is one, namely the well-known Bel–Robinson tensor \( B_{\mu\nu\alpha\beta} \) defined uniquely in D=4 according to

\[
B_{\mu\nu\alpha\beta} = R^\rho_\mu \sigma_\alpha R_{\rho\sigma\beta\epsilon} + \frac{1}{2}g_{\mu\nu}R_\alpha^\rho \sigma R_{\beta\rho\sigma\epsilon}.
\]

(3.2)

On-shell \( (R_{\mu\nu} = 0) \) \( B \) is totally symmetric, (covariantly) conserved and traceless and there is again an invariant like (3.1a) namely

\[
\Delta I_3 = \kappa^4 \int d^4x [B^2 - i\bar{J}_\mu \beta J^\mu + 3C_\mu \Box C^\mu]
\]

(3.1b)

where I have omitted all indices (\( C \) is a chiral current bilinear in \( f_{\mu\nu} \)) and added the \( \kappa^4 \) to show eligibility of \( \Delta I_3 \) as a 3-loop counterterm. The existence of similar invariants for N>1 was soon confirmed as well \[8\] and their full nonlinear completions were found by superspace methods that are, unfortunately, not available in our D=11 world. Surprisingly, it was even possible to learn something about the actual coefficients of such counterterms: For the non-maximal 1≤N<8 models, where covariant superspace quantization is possible, it was concluded \[14\] that they are uniformly non-vanishing. For the special maximal N=8 case, very beautiful recent work \[15\] (to which we shall return for its impact on D=11) suggests that, while the three-loop coefficient may vanish, its 5-loop(!) coefficient probably does not.

To summarize, then, the D=4 SUGRA situation is essentially that the one-loop counterterms are accidentally protected by the supersymmetric extension of the Gauss–Bonnet identity, \( i.e. \), they
vanish on shell and hence are safe, whatever their coefficients. At two loops, whose bosonic part
must start as \( \Delta f_2 \sim \kappa^2 \int d^4 x [R_{\mu \nu \alpha \beta}^4] \), there exists no SUSY completion; this is the improvement
over pure gravity, where this \( R^4 \) term is both allowed and has nonzero coefficient. On the other
hand, at 3 loops there is indeed the allowed N=1 term we have displayed in (3.1b) and no miracles
protect its coefficient or that of its N<8 counterparts; even for the maximal N=8, five loops prove fatal.

I will not repeat this general SUGRA argument for rising dimensions, 4<\( D <11 \), (again even
and odd ones differ a bit) except to say that it is possible to construct SUSY counterterms explicitly
where a suitable superspace formulation exists and that the construction and negative conclusions
of \[13\] seem to apply. Instead, I now move to my main topic, \( D=11 \) SUGRA.

4. \( D=11 \) SUGRA

This section represents the work in \[3\] to which I refer for details. In the period of rapid con-
struction of \( D>4 \) SUGRAs following the initial \( D=4 \) theory, it became clear from the mathematics
of graded algebras \[16\] that if one wanted these models to obey the following four criteria: inclu-
sion of \( s=2 \), \( m=0 \) excitations, \( \text{i.e.} \), the graviton (of which Einstein theory is the essentially unique
local model); only one such graviton; no massless excitations with spin exceeding 2, and only one
timelike dimension, then there is a highest dimension, \( D=11 \), for SUGRA. This theorem can be
understood in various ways, in particular from the requirements that the numbers of boson/fermion
excitations must match. The \( s < 5/2 \) requirement is connected with the fact that massless fields
with \( s \geq 2 \) do not consistently interact with gravity itself \[17\]. The \( D=11 \) theory \[1\] is indeed a very
special one, even in the universe of SUGRAs. Some examples: it allows only \( N=1 \), so “maximal =
minimal”; there is no SUSY “matter” in \( D=10 \) to provide a source; while (negative) cosmological
constants can be included in lower-dimensional SUGRA, (unbroken) \( D=11 \) SUGRA is also unique
in forbidding a cosmological term \[18\].

Before going into the description of our construction and its implications for the theory’s
nonrenormalizability, let me turn to our motivations. They are actually twofold. The immediate
aim was to construct on-shell non-vanishing local invariants, as potential counterterms. Now this
was already a very nontrivial task, because in this theory there is no technology to test the SUSY
of, let alone construct, candidate invariants. Even assuming these would be likely to exist, this was
the last remaining SUGRA model and (with its mysterious other uniqueness properties) the only
one with a chance at staving off infinities. Besides, having been reenthroned (from its previous
anomalous position in the old \( D=10 \) string world) as a basic cornerstone/QFT limit of M-theory, it
is not only “there”, but of prime interest! Secondly, quite apart from renormalizability, any higher
invariant that could be constructed would automatically be a candidate (finite) correction term
in some M-theory expansion and as such be a window on that mysterious region, precisely in the
same way that \( D=10 \) string models yielded (also finite) corrections to the \( D=10 \) supergravities. So
much for why – now we need the how.

To know where to focus, let us first re-count dimensions, which is best done by looking at
the SUGRA action. I will only write the purely bosonic part here, because that is all we will
(fortunately!) need:

\[
I^B_{11} = \int d^{11}x \left[ -\frac{\sqrt{g}}{4\kappa^2} R(g) - \frac{\sqrt{g}}{48} F^2 + \frac{2\kappa}{144^2} \epsilon^{1 \ldots 11} F_1 \ldots F_5 \ldots A_{11} \right],
\]  

(4.1)
This is not the place for a review; let me just recall that in addition to the graviton, there is a 3-form potential $A_{\mu \nu \alpha}$ with associated field strength $F_{\mu \nu \alpha \beta} \equiv \partial_{[\beta} A_{\mu \nu \alpha]}$ and a cubic Chern–Simons (CS) term in which the 11 indices of the epsilon symbol are saturated by two $F$'s and one $A$. The dimensions of $\kappa^2$ are here $L^{+9}$; note the explicit $\kappa$ also in the CS part since $A \sim L^{-9/2}$. Among the fermionic terms, there are non-minimal ones $\sim \bar{\psi} R \psi$, $\bar{\psi} F \psi$ but we can avoid this whole sector here. It is clear, because dimension is odd, that no 1-loop $\sim \kappa^0 \int d^{11}x$ candidate $\Delta I_1$ exist – one cannot make gravitational scalars with odd numbers of derivatives, except irrelevant (because parity-violating) $\epsilon R^n \partial R^m$ terms. At 2-loop order, $$\Delta I_2 \sim \kappa^2 \int d^{11}x \Delta L_{20} \quad (4.2)$$ where $\Delta L_{20}$ has dimension 20.

To construct the leading purely gravitational term here requires, at first sight, $R^{+10}$. While such terms are undoubtedly possible and present, they are impracticable to obtain. Related to this is the question of regularization scheme. If we use some dimensional (energy) cutoff, it alters the allowed dimension of candidate $\Delta L$'s. Instead, the dimensional regularization we use here uniformly has logarithmic cutoff so all $\Delta L$ are $\Delta L_{20}$'s. But all we have to do is to present one that exists and show that its coefficient fails to vanish; if we can accomplish that as we (essentially, see below) will, other possible regularization schemes are irrelevant. Coming back to finding a more tractable $\Delta L_{20}$, we recall that a curvature is dimensionally equivalent to two covariant derivatives $D_\mu$, which means that candidate terms are schematically of the form $\Delta L_{20} \sim \sum_{n=4}^{10} R^n D^{2(10-n)}$. We start at $n=4$ because clearly the lower $n$'s are either (like $R^2$) not parts of super-invariants or are leading order trivial (like $R^2$ which obeys Gauss–Bonnet to quadratic order in $h_{\mu \nu}$). Thus, our lowest possible choice is $\Delta I_2 \sim \kappa^2 \int d^{11}x [R^4 D^{12} + \ldots]$ where the ellipsis represents the SUSY completion, if any. How to find a suitable candidate in absence of any guiding super-calculus? Our procedure was the following. As was also recognized in [15], there is certainly one on-shell nonvanishing lowest order SUSY invariant that starts out quartic in $h_{\mu \nu}$, and that is the tree-level 4-point scattering amplitude generated by the $D=11$ action (4.1) itself. It has the enormous advantage that, since there are no loops, and SUSY transformations are linear at our level, the purely bosonic terms are guaranteed to be part of the overall SUSY invariant that is the total 4-point amplitude. However, it presents two a priori obstacles: First, we want a local invariant, whereas the amplitude has a denominator, from virtual particle exchanges; can we extract a local but still SUSY residue? We can, simply because each term in the amplitude is in fact proportional to the product $(1/stu)$ of the Mandelstam variables. Second, we want to have 12 explicit derivatives in the $R^4$ and other terms; can those be inserted without losing SUSY or having everything vanish on-shell? The answer is again yes, e.g., by further multiplication with $(stu)^2$ or $(s^6 + t^6 + u^6)$, after the initial $stu$ one.

Now we have cleared the decks for the actual computation. It consists of applying the Feynman rules in terms of the propagators and vertices needed up to 4-point level. Let’s review the ingredients. Expanding the action (4.1) about flat space gives us first the free particle propagator terms, symbolically, $$I^{(2)}_{11} \sim \int d^{11}x (h D_{g}^{-1} h + A D_{F}^{-1} A) \quad , \quad (4.3)$$ where $(D_F, D_g)$ are the respective free ($\sim \Box^{-1}$) propagators. The cubic terms are of three types. First, the purely topological (metric-free) CS term furnishes the 3-form self-coupling vertex $\kappa \epsilon F \bar{F} A \equiv C_F \cdot A$ where the three-index current $C_F \sim \kappa (\epsilon F F)$ is both gauge invariant and
identically conserved. The other relevant vertex involving $F$ is its gravitational coupling $\kappa h_{\mu\nu} T^\mu_\nu$ where $T^\mu_\nu \sim F F$ is the form’s stress-tensor. Finally the purely gravitational contribution is of the same $\kappa h_{\mu\nu} T^\mu_\nu (h)$ form, where $T^\mu_\nu (h) \sim (\partial h \partial h)^{\mu\nu}$ is the (gauge-variant) stress pseudo-tensor of the gravitational field itself. This term, upon contracting one of its graviton lines with that of another, yields the graviton-graviton exchange $\sim \kappa^2 T_{\mu\nu} (h) D^\mu_{\rho\sigma\theta} T^\rho_{\sigma\theta} (h)$ just as self-contracting the $AC_F$ term contributes $C_F D^F C_F \sim \kappa^2 F^4$ to form-form scattering, as does the $h T F - h T F$ graviton contraction. However, the four-graviton term is not even abelian gauge invariant by itself but requires inclusion of the local quartic 4-point vertex $\kappa^2 h h \partial h \partial h$ to restore it. Besides the above pure $h^4$ and $F^4$ amplitudes, there are mixed terms: If we contract the form field in $h T F - h T F$, we get $\sim F^2 R^2$, form-graviton Compton scattering. Finally we can get a form-graviton bremsstrahlung term $\sim R F^3$ when contracting the $AC_F$ and $h T F$ vertices (only form exchange contributes here). Single-form creation $\sim R^3 F$, is of course forbidden.

It is a straightforward, if index-intensive, procedure to perform these calculations. The worst, graviton-graviton scattering has fortunately been done earlier in arbitrary dimension [19] which is a very useful check; the answer is as in D=10, namely it is proportional to the famous lowest string correction to the Einstein action in D=10:

$$L_g = stu M_g \sim t_8 t_8 R R R R$$

(4.4)

where $t_8$ is a constant 8-index tensor; it has a piece proportional to the D=8 Levi-Civita symbol $\epsilon_8$ and hence the $\epsilon_8 \epsilon_8$ part of $L_g$, proportional to the D=8 Gauss–Bonnet invariant, cannot be seen at our lowest, $O(h^4)$, order (at its lowest order, the G–B term of any dimension is a total divergence in all D). Remember that we are always on (linearized) shell, so that the letter $R$ really means the Weyl tensor; also, we have emphasized that multiplication of $M$ by $stu$ yields the local scalar $L$.

Let me now discuss more explicitly our bosonic component of the full SUSY invariant; as mentioned earlier, it is interesting in its own right as the correction to the D=11 SUGRA action from M-theory, of which we mostly know that it contains SUGRA as the local limit. Below, I will summarize the appearance of the various “localized” on-shell 4-point amplitudes. They are to be multiplied by the required twelve derivatives, say with $(stu)^2$ to make dimension 20. Schematically (see [2] for details), with $B$ a Bel–Robinson-like curvature quadratic and $F$ always appearing with a gradient,

$$\Delta I^B_4 (g, F) = \kappa^2 \int d^{11} x [B^2 + (\partial F)^4 + B (\partial F)^2 + R (\partial F)^3] .$$

(4.5)

Although everything looks very coordinate invariant, these terms are only accurate to lowest, $4^{th}$, order in the combined $R’s$ and $F’s$, and of course are to be supplemented by fermion-dependent terms that we do not write down. Nevertheless linearized SUSY is guaranteed by our construction.

What about the coefficient, is it non-zero? As I mentioned, a powerful tool for answering this question was provided by the amazing correspondence between SUGRA and Super-Yang–Mills models established and exploited in [13] to obtain otherwise “impossible” results. The only catch is that SYM is only defined for $D \leq 10$; however one can argue, quite convincingly, that the results as provided really do not depend directly on D, and extend analytically also to D=11. At any rate, once their D=11 extension has been made, it is of course independent of its origin and it may not be all that difficult to verify intrinsically; thus, the strong odds are against even this maximal theory, already at leading possible level.
5. Summary

We have tried to outline in a very short space, the history of the ultraviolet problems of any QFT that contains general relativity, but (to preserve unitarity) no higher derivative kinetic terms. These models include pure gravity in any $D>3$, where the problems arise uniformly at 2-loop level because the 1-loop terms, while still infinite are accidentally harmless [8]. For $D=4$ in particular the coefficient of the 2-loop counterterm has been explicitly and reliably shown not to vanish [10, 11]. For gravity plus lower spin ($0 \leq s \leq 1$) matter the one-loop terms are already known to be non-zero, also explicitly [9]. All these problems are traceable to the positive dimensionality of the gravitational coupling constant, $\kappa^2 \sim L^{D-2}$.

The addition “gravity-matter” symmetry that is the hallmark of supergravity was shown early on, for the $D=4$, $N=1$ model, to improve things but only marginally: there the danger comes from 3 loops onward [12]; things are not improved by higher $N$ [14] although it may be [15] that the maximal, $N=8$, model is only destroyed at 5-loop order. Increasing dimension is no help either, and indeed our main new result is that the most unique and extreme, $D=11$, SUGRA is also sick already at minimal 2-loop order, despite its relation to the M-theory unification of string theories. More specifically, a local SUSY invariant counterterm was constructed from the 4-point tree-level scattering amplitude generated by the $D=11$ action. Together with the compelling argument of [15] that its coefficient is non-null, this conclusion is all but inevitable. Hence Heisenberg’s curse on every QFT that includes gravity holds universally and forces us beyond locality, to strings and their generalizations.

Acknowledgements: This work was supported by the National Science Foundation under grant PHY93-15811. I am grateful to my $D=11$ coauthor D. Seminara, as well as to Z. Bern and L. Dixon, for useful comments.
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