A SHARP $L^{10}$ DECOUPLING FOR THE TWISTED CUBIC

By

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Abstract. We prove a sharp $l^{10}(L^{10})$ decoupling for the moment curve in $\mathbb{R}^3$. The proof involves a two-step decoupling combined with new incidence estimates for planks, tubes and plates.

1 Introduction

Let $\Gamma_d = \{(t, t^2, \ldots, t^d) : t \in [0, 1]\}$ be the moment curve in $\mathbb{R}^d$. For $0 < \delta < 1$, we introduce its isotropic $\delta$-neighborhood (essentially $\Gamma_d + B_d(0, \delta)$)

$$N_{\Gamma_d}(\delta) = \{(t, t^2 + s_2, \ldots, t^d + s_d) : s_2^2 + \cdots + s_d^2 \leq \delta^2\}.$$ 

For $0 < \sigma < 1$, let $I_\sigma$ be the partition of $[0, 1]$ into intervals $I$ of length $\sigma$. For each function $F : \mathbb{R}^d \to \mathbb{C}$ and $I$ lying in the first coordinate, we denote the Fourier projection of $F$ to the infinite strip $I \times \mathbb{R}^{d-1}$ by $P_I F$,

$$P_I F(x) = \int_{I \times \mathbb{R}^{d-1}} \hat{F}(\xi) e(\xi \cdot x) d\xi.$$ 

We will consider functions $F$ with Fourier transform supported in $N_{\Gamma_d}(R^{-1})$, and intervals $I \in I_{R^{-\alpha}}$, with $\frac{1}{d} \leq \alpha \leq 1$. Note that $N_{\Gamma_d}(R^{-1})$ is partitioned by the sets

$$N_I(\delta) = \{(t, t^2 + s_2, \ldots, t^d + s_d) : t \in I\ and\ s_2^2 + \cdots + s_d^2 \leq \delta^2\}.$$ 

So $P_I F$ is in fact the Fourier projection of $F$ onto $N_I(\delta)$. The following decoupling program for curves was initiated in [6].

**Question 1.1** ($l^p(L^p)$ decoupling at frequency scale $R^{-\alpha}$). What is the largest $p_{d, \alpha}$ such that the following $l^p(L^p)$ decoupling holds for all $2 \leq p \leq p_{d, \alpha}$: for each $F : \mathbb{R}^d \to \mathbb{C}$ with Fourier transform supported in $N_{\Gamma_d}(R^{-1})$ we have

$$\|F\|_{L^p(\mathbb{R}^d)} \lesssim_\epsilon R^{\alpha(\frac{1}{2} - \frac{1}{p}) + \epsilon} \left( \sum_{I \in I_{R^{-\alpha}}} \|P_I F\|_{L^p(\mathbb{R}^{d-1})}^p \right)^{\frac{1}{p}}.$$ 

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Until recently, the most relevant case for applications, and the most sought after, was the one corresponding to $\alpha = \frac{1}{2}$. The case $d = 2$ has been solved in [4], showing that $p_{2, \frac{1}{2}} = 6$. After some preliminary progress in [6], the case $d \geq 3$ has also been solved in [7], showing that $p_{d, \frac{1}{2}} = d(d + 1)$. The impact of this result was enormous, as it immediately led to the resolution of the Main Conjecture in Vinogradov’s Mean Value Theorem in number theory.

Perhaps the second most important scale is $\alpha = \frac{1}{2}$, which has played an important role in Bourgain’s progress [3] on the Lindelöf hypothesis regarding the growth of the Riemann zeta on the critical line. The results for this scale are very sparse, and typically not sharp. Since $\alpha = \frac{1}{2}$ will be our main concern here, we will call $p_{d, \frac{1}{2}}$ the critical exponent $p_{d, \frac{1}{2}}$. Using a simple projection argument (the so-called cylindrical decoupling in Exercise 9.22 of [10]), it is easy to see that $p_{d, \frac{1}{2}}$ is nondecreasing in $d$.

In [4] it was proved that $p_{2} = 6$. Testing this inequality with exponential sums shows that $p_{d} \leq 4d - 2$; see for example Section 9 in [13]. In particular, $p_{3} \leq 10$, $p_{4} \leq 14$, $p_{5} \leq 18$.

The following computation for higher values of $d$ is due to Bourgain, [11]. Let $E$ be the extension operator associated with $\Gamma_d$ and let $1$ be a smooth approximation of $1_{[0,1]}$. Then, a result in [9] shows that for each $p \geq 2$,

$$\|E1\|_{L^p(B(0, R))} \gtrsim R^{d - \sigma_{p,d}}$$

with

$$\sigma_{p,d} = \min_{2 \leq k \leq d} \left\{ \frac{1}{k} + \frac{k^2 - k - 2}{2kp} \right\}.$$  

On the other hand, if $J \in \mathbb{I}_{\mathbb{R}^{-1/2}}$,

$$\|E_J1\|_{L^p(B(0, R))} \sim R^{\frac{2d - 1}{2p} - \frac{1}{2}}.$$

So $L^p(L^p)$ decoupling can only have a chance to hold if $\frac{2d - 1}{2p} - \frac{1}{2} + \frac{1}{4} \geq \frac{d}{p} - \sigma_{p,d}$, or $\sigma_{p,d} \geq \frac{1}{4} + \frac{1}{2p}$. This does not give new constraints for $p$ if $d = 3, 4$. But for $d \geq 5$, it implies that

$$\frac{1}{4} + \frac{1}{2p} \leq \min_{2 \leq k \leq d} \left\{ \frac{1}{k} + \frac{k^2 - k - 2}{2kp} \right\},$$

or

$$p \leq \min_{2 \leq k \leq d} \frac{2(k^2 - 2k - 2)}{k - 4}.$$  

Since the smallest value for $\frac{2(k^2 - 2k - 2)}{k - 4}$ when $k \geq 5$ is 22 (for both $k = 6$ and $k = 7$), and since $p_{d}$ is nondecreasing, we conclude that

$$p_{d} \leq 22, \quad \text{for all } d.$$  

This leads to a surprising conclusion. While $p_{d}$ is nondecreasing in $d$, it does not grow to $\infty$. 
There are two types of methods in the literature, let us call them soft and hard, that have been used to produce $L^p$ decouplings for curves for $\alpha > \frac{1}{d}$. Both methods first derive $L^p$ decouplings in the multilinear setting, and then use a standard multilinear-to-linear reduction to get linear decouplings. The soft method proves the multilinear decoupling for $\Gamma_d$ by reducing it to a known decoupling for a higher dimensional manifold in $\mathbb{R}^d$. This approach originates in [6], where a bilinear argument was used for $\Gamma_3$. The sum of two separated pieces of $\Gamma_3$ is a surface in $\mathbb{R}^3$ with nonzero Gaussian curvature. The critical exponent for the decoupling of such a surface (also at scale $\alpha = \frac{1}{2}$) was proved to be 4. This immediately shows that $p_3 \geq 8 = 4 \times 2$ for $\Gamma_3$. Another example is the result in [3]. Using a bilinear approach, the inequality $p_4 \geq 12$ was proved there, as the main step towards improving the state of the art on the Lindelöf hypothesis.

The reproof of $p_4 \geq 12$ in [5] clarifies the nature of the argument, and of the exponent 12. Sums of two separated pieces of $\Gamma_4$ create a surface in $\mathbb{R}^4$, whose critical exponent as far as decoupling at scale $\alpha = \frac{1}{2}$ is concerned, is 6. So again, $12 = 6 \times 2$. The trilinear result at $p = \frac{14}{3}$ in [13] can be used to prove that $p_5 \geq 14$, [12]. Similar applications of the soft method for larger values of $d$ appear in [18].

The soft method is probably never sharp, as there is loss of information and structure in turning curves into higher dimensional manifolds. We illustrate this by proving our main result, the sharp estimate $p_3 = 10$. We find it also likely that $p_4$ is greater than 12, and also that $p_5$ is greater than 14. It is likely that our methods here could be adapted, not without additional significant effort, to prove such estimates.

We now state the main theorem of this paper.

**Theorem 1.2.** Assume that function $F : \mathbb{R}^3 \to \mathbb{C}$ has the Fourier transform supported in $N_{\Gamma_3}(R^{-1})$. Then for $2 \leq p \leq 10$, we have

\[
\|F\|_{L^p(\mathbb{R}^3)} \lesssim \epsilon \left( R^{\frac{d}{2}} + \epsilon \right)^{\frac{1}{p}} \left( \sum_{J \in I_{\mathbb{R}^d}} \|P_J F\|_{L^p(\mathbb{R}^3)}^p \right)^{\frac{1}{p}}.
\]

Combined with the aforementioned inequality $p_3 \leq 10$ proved in [13], this shows the sharp estimate $p_3 = 10$. We get the following essentially sharp exponential sum estimate as a direct consequence.

**Corollary 1.3.** Let $\Lambda \subset [0, 1]$ be a collection of $\sim 1/R^{\frac{1}{2}}$ separated points $\xi$ with $|\Lambda| \sim R^{\frac{1}{2}}$. Let $a_\xi \in \mathbb{C}$, $|a_\xi| \sim 1$ and $Q_R \subset \mathbb{R}^3$ be a cube with side length $R$. Then for $2 \leq p \leq 10$, we have

\[
\left( \frac{1}{R^3} \int_{Q_R} \left| \sum_{\xi \in \Lambda} a_\xi e(\xi x_1 + \xi^2 x_2 + \xi^3 x_3) \right|^p dx \right)^{\frac{1}{p}} \lesssim \epsilon R^{\frac{1}{2} + \epsilon}.
\]
As another measure of the sharpness of our result, we mention that the projection argument mentioned earlier shows that $l^2(L^p)$ decoupling

$$\|F\|_{L^p(\mathbb{R}^d)} \lesssim R^\epsilon \left( \sum_{I \in I_{\epsilon}} \|P_{I}F\|_{L^p(\mathbb{R}^d)}^2 \right)^{1/2}$$

cannot hold if $\alpha > \frac{1}{d}$ and $p > 6$. Indeed, it suffices to test this inequality with functions $F$ Fourier supported in $N_{[0, R^{-1/3}]}(R^{-1})$, and to note that this curved tube is essentially planar.

Our approach is an application of the hard method introduced in [14]. Recall that the general context is about finding $p_{d, \alpha}$, with $\frac{1}{d} \leq \alpha \leq 1$. The index $\alpha = \frac{1}{d}$ is special, since it allows the efficient use of (the canonical version for $\Gamma_d$ of) parabolic rescaling. Because of this, the scale $\alpha = \frac{1}{d}$ is called canonical scale, and decoupling at this scale is sometimes called canonical decoupling. The main result in [7] settles the canonical decoupling for $\Gamma_d$, by induction on scales.

The induction on scales argument makes repeated use of parabolic rescaling for $\Gamma_d$; however, this tool becomes much less efficient in the case $\alpha > \frac{1}{d}$, referred to as small cap decoupling, due to the size $R^{-\alpha} \ll R^{-1/d}$ of the frequency intervals we decouple into.

The approach in [14] circumvents this problem, by introducing a two-step decoupling argument, with a significant emphasis on wave packet decompositions and the incidence geometry of their spatial supports. Parabolic rescaling is only used in the multilinear-to-linear reduction, but not in the main body of the argument. This method has led in [14] to sharp results for special functions $F$ with frequency support near $\Gamma_3$. In particular, inequality (2) was proved there in the case when $F$ is an exponential sum that is periodic in a certain direction.

At large scales, our argument here is structured similarly to the one in [14]. There is a decoupling into larger intervals $I$ of canonical scale $R^{-1/3}$, and then a decoupling of each $I$ into subintervals $J$ of length $R^{-1/2}$. There are however two major new difficulties that are consequences of us considering arbitrary functions $F$ in (2). The first has to do with the complicated nature of each term $P_{I}F$ in our setup. The result in [14] is about the case when each of these functions is a single exponential wave. The (local versions of the) quantities $\|P_{J}F\|_{L^p(\mathbb{R}^3)}$ in this simplified context are easily computable.

The second issue is the complete lack of periodicity in our context. Wave packets in [14] have a lot of structure, due to periodicity in the $x$-direction of the exponential sum.

To address these issues, our argument introduces a few innovations. We need to investigate the complex nature of $P_{J}F$ by introducing a second wave packet
decomposition at scale $R^{-1/2}$, in addition to the one at scale $R^{-1/3}$. There will be two multi-layer pigeonholing arguments in Section 8, one for each decomposition. The lack of periodicity increases the number of rounds of pigeonholing needed in each sequence. The connection between the parameters arising throughout these pigeonholing steps is mainly captured by Proposition 8.4. The function $F$ is split into components that are “uniform” at both scales. Decoupling each $I$ into intervals $J$ will involve $L^2$ orthogonality, the $l^2(L^6)$ canonical decoupling for the parabola, and the $l^4(L^4)$ small cap decoupling from [14].

Most of the work goes into proving a refined decoupling into intervals $I$. The main component of this part is Proposition 8.5. This is an incidence estimate between Vinogradov planks that satisfy a certain spacing assumption. The proof uses new estimates for incidences of tubes and plates, proved in Theorem 4.3 and Theorem 6.3. The proof of the corollary uses the method in the recent papers [16] and [17]. On the other hand, Theorem 6.3 is a new $L^4$ estimate for plates, that we prove using a similar $L^4$ estimate for planks, Theorem 5.2. The proof of this latter result is purely geometric, it exploits the size of the intersections of planks. $L^4$ is optimal in this context.

Similar to [14], we first prove the trilinear version of the desired decoupling; see Theorem 8.2. The only advantage we get from working in the trilinear setup is the access to the $L^6$ trilinear reverse square function estimate for $\Gamma_3$, used in the proof of Theorem 8.6. However, unlike the argument from [14], ours makes no use of either bilinear or trilinear incidence estimates for boxes. All incidence estimates we use for Vinogradov tubes, plates and planks are linear. We caution that while Vinogradov boxes live in $\mathbb{R}^3$, they only have one degree of freedom (essentially their orientation is dictated by the restricted family of tangents/normals to $\Gamma_3$). Because of this, their incidences have planar complexity. Our results are (somewhat complicated) manifestations of the planar Kakeya phenomenon, as opposed to genuinely $\mathbb{R}^3$ results. Nevertheless, we believe that the new incidence estimates are of independent interest, and will serve as tools in future literature.

It seems possible that our inequality (2) will help with progress on various problems concerned with the $L^{10}$ moment of exponential sums. For example, such moments were investigated in [2] using soft decoupling techniques. There the equality $10 = 10/3 \times 3$ is exploited in conjunction with the $L^{10/3}$ decoupling for hypersurfaces in $\mathbb{R}^4$, but the results following this approach are suboptimal.

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2 Notation

We write $X \lesssim Y$ to denote $X \leq CY$ for positive quantities $X, Y$ and some constant $C$. Also $X \sim Y$ denotes $X \lesssim Y$ and $X \gtrsim Y$. We write $\lesssim_{\text{par}}$ if the constant $C$ has dependence on the parameter par. Whenever we encounter quantities $X, Y$ depending on the scale parameter $R$, the notation $X \lesssim Y$ stands for $X \lesssim (\log R)^C Y$ for some $C = O(1)$.

We define $\chi: \mathbb{R}^3 \to \mathbb{R}$ by $\chi(x) = (1 + |x|)^{-100}$. Let $B$ be a box in $\mathbb{R}^3$ and $A_B$ be an affine linear transformation mapping $B$ to $[-1, 1]^3$. We write $\chi_B := \chi \circ A_B$.

We will denote the twisted cubic $\Gamma_3$ simply by $\Gamma$. For each $I = [c, c + \delta] \in \mathbb{I}_\delta$, we denote the unit tangent, binormal and normal vectors at $(c, c^2, c^3) \in \Gamma$ (the Frenet frame) by $t(I), b(I), n(I)$ respectively, and these vectors are parallel to $t(I)\parallel (1, 2c, 3c^2), b(I)\parallel (3c^2, -3c, 1), n(I)\parallel t(I) \times b(I)$.

We will define numerous boxes in $\mathbb{R}^3$ using these directions.

A polytope $B$ (typically the intersection of multiple rectangular boxes) is called an almost rectangular box if $C^{-1}R \subset B \subset CR$ for some genuinely rectangular box $R$ and $C = O(1)$. In our paper all rectangular boxes $R$ will have the three axes oriented along a Frenet frame. The dimensions $(d_1, d_2, d_3)$ of $B$ will be understood to be those of $R$, with respect to the $t, n, b$ axes. Of course, they are well defined only within a multiplicative factor $O(1)$. This slight imprecision is harmless in our argument. We will not distinguish between two boxes $B_1$ and $B_2$ if $C^{-1}B_2 \subset B_1 \subset CB_2$ for $C = O(1)$. The dimensions of various boxes $B$ associated with an interval $I$ will be in such a way that the orientation of $B$ remains essentially unchanged when specified with respect to the Frenet frame at any $\Gamma(t)$ with $t \in I$.

We will focus on four types of boxes: cube, plank, plate and tube. A cube is a box with $d_1 = d_2 = d_3$. A plank has three considerably different dimensions $d_1, d_2, d_3$. A box is called plate if the lengths of two edges are comparable and significantly longer than the remaining edge. On the contrary, a tube has two edges with comparable length while the remaining edge has considerably longer length. We will not distinguish between tubes and cylinders.

We will encounter planks and plates as part of various decompositions both on the spatial and on the frequency side. Typically, cubes and tubes will live on the spatial side. Each of these boxes will be associated with an interval in $[0, 1]$, that determines their orientation via the associated Frenet frame. We will call a collection of such boxes separated if the boxes corresponding to each interval (these boxes are mutually parallel) have bounded overlap.
Let $B_1$ be a $(d_1, d_2, d_3)$-box and $B_2$ be a $(d'_1, d'_2, d'_3)$-box. The boxes $B_1$ and $B_2$ are said to be dual if they have the same orientation and satisfy $d_id'_i = 1$, for each $i = 1, 2, 3$. We sometimes write $B^*$ for a dual box of $B$.

Throughout this paper, we will use the aforementioned parabolic rescaling $T_{\sigma,c}$ which is defined by

$$T_{\sigma,c}(w_1, w_2, w_3) = (w'_1, w'_2, w'_3),$$

$$\begin{cases} 
  w'_1 &= \frac{w_1}{\sigma}, \\
  w'_2 &= \frac{w_2 - 2cw_1}{\sigma^3}, \\
  w'_3 &= \frac{w_3 - 3cw_2 + 3c^2w_1}{\sigma^3}.
\end{cases}$$

We write the family of the inverse transpose linear maps $A_{\sigma,c} = (T_{\sigma,c})^{-1}$ by

$$A_{\sigma,c}(x_1, x_2, x_3) = (x'_1, x'_2, x'_3),$$

$$\begin{cases} 
  x'_1 &= \sigma(x_1 + 2cx_2 + 3c^2x_3), \\
  x'_2 &= \sigma^2(x_2 + 3cx_3), \\
  x'_3 &= \sigma^3x_3.
\end{cases}$$

3 A geometric observation on frequency Vinogradov planks

Recall that $I_{R^{-1/3}}$ is the partition of $[0, 1]$ into intervals $I$ of length $R^{-\frac{1}{3}}$. For each $I$ we define the anisotropic neighborhood of the arc $\Gamma_I$ by

$$\Gamma_I(R^{-1/3}) := \{(\zeta_1, \zeta_2, \zeta_3) : \zeta_1 \in I, |\zeta_2 - \zeta_1^2| \leq R^{-2/3}, |\zeta_3 - 3\zeta_1\zeta_2 + 2\zeta_1^3| \leq R^{-1}\}.$$ 

Note that the sets $\Gamma_I(R^{-1/3})$ with $I \in I_{R^{-1/3}}$ cover $\Gamma(R^{-1/3}) = \Gamma_{[0,1]}(R^{-1/3})$. Each $\Gamma_I(R^{-1/3})$ is an almost rectangular $(R^{-\frac{1}{3}}, R^{-\frac{2}{3}}, R^{-1})$-plank with respect to the $t(I), n(I), b(I)$ axes.

The orientation of the plank is characterized by the direction of the long $R^{-\frac{1}{3}}$-edge and by the normal direction of the wide $(R^{-\frac{2}{3}}, R^{-1})$-face. Using this fact, we define the frequency Vinogradov planks as follows.

**Definition 3.1** (Frequency Vinogradov planks at scale $R$). Let $I = [c, c+R^{-\frac{1}{3}}]$, $c \in [0, 1]$. We call the $(R^{-\frac{1}{3}}, R^{-\frac{2}{3}}, R^{-1})$-plank $\tilde{\theta}_I$

$$\tilde{\theta}_I := \{ w \in \mathbb{R}^3 : |w_1| \leq R^{-\frac{1}{3}}, |w_2 - 2cw_1| \leq R^{-\frac{2}{3}}, |w_3 - 3cw_2 + 3c^2w_1| \leq R^{-1}\}$$

as a frequency Vinogradov plank centered at the origin (at scale $R$) associated with $I$; $\tilde{\theta}_I$ has the long edge pointing in the direction $t(I)$ and the wide $(R^{-\frac{2}{3}}, R^{-1})$-face with normal vector $b(I)$. We call any translation of $\tilde{\theta}_I$ as a frequency Vinogradov plank and denote it by $\theta_I$.  


We replace almost rectangular boxes $\Gamma_I(R^{-1/3})$ by parallelepiped planks $\tilde{\theta}_I := \tilde{\theta}_I + (c, c^2, c^3)$ with dimensions $(R^{-1/2}, R^{-3/2}, R^{-1})$. We can check that the plank $\tilde{\theta}_I$ satisfies $\frac{1}{10}(\tilde{\theta}_I \subset \Gamma_I(R^{-1/4})) \subset 10\tilde{\theta}_I$.

Let $\mathcal{C}_{OL}$ be the truncated light cone in $\mathbb{R}^3$ defined by

$$\mathcal{C}_{OL} := \{ \xi \in \mathbb{R}^3 : \xi_1^2 + \xi_2^2 = \xi_3^2, \ 1/2 \leq \xi_3 \leq 5/2 \}$$

and $\mathcal{C}_{OE}$ be the truncated elliptical cone

$$\mathcal{C}_{OE} := \{ \xi \in \mathbb{R}^3 : 2\xi_1\xi_3 = (\frac{3}{2})\xi_2^2, \ (1/2)R^{-3/2} \leq \xi_1 \leq (5/2)R^{-3/2} \}.$$ 

For $0 < \delta < 1$, we write the $\delta$-neighborhood of $\mathcal{C}_{OL}$ and $\mathcal{C}_{OE}$ by $\mathcal{N}_{\mathcal{C}_{OL}}(\delta)$ and $\mathcal{N}_{\mathcal{C}_{OE}}(\delta)$, respectively.

In the recent paper [17], the authors showed a square function estimate for the function whose Fourier transform is supported in a small neighborhood of the light cone $\mathcal{C}_{OL}$. We will apply that argument to Lemma 3.2 by replacing the light cone with the union of Vinogradov planks. We first show that each Vinogradov plank $\tilde{\theta}$ is associated with the elliptical cone $\mathcal{C}_{OE}$. Then, we prove that a linear transformation maps $\tilde{\theta}$ to a plank associated with the light cone that is centered at the origin.

Let $I = [c, c + R^{-1/3}]$. Then, the plank $\tilde{\theta}_I$ has the long edge pointing in the direction $(1, 2c, 3c^2)$ which coincides the radial direction of elliptical cone $\mathcal{C}_{OE}$. Also, the wide $(R^{-1/2}, R^{-3/2})$-face of $\tilde{\theta}_I$ has the normal vector $(3c^2, -3c, 1)$ which is the same normal vector of $\mathcal{C}_{OE}$ along the radial direction. Moreover, we observe that the intersection of the planks and the infinite strip

$$\tilde{\theta}_I \cap \{ w \in \mathbb{R}^3 : (1/2)R^{-3/2} \leq w_1 \leq R^{-1/2} \}$$

are finitely overlapping and contained in $\mathcal{N}_{\mathcal{C}_{OE}}(R^{-1})$. By symmetry, on the other side of the cone, planks $\tilde{\theta}_I \cap \{ -R^{-3/2} \leq w_1 \leq -(1/2)R^{-3/2} \}$ are finitely overlapping and contained in $\mathcal{N}_{-\mathcal{C}_{OE}}(R^{-1})$ for $-\mathcal{C}_{OE} = \{ \xi \in \mathbb{R}^3 : -\xi \in \mathcal{C}_{OE} \}$. Using convexity, we see that the planks $\tilde{\theta}_I$ are translates of the finite overlapping planks contained in $\mathcal{N}_{\mathcal{C}_{OE}}(R^{-1})$ to the origin.

Let $\mathcal{L}$ be a linear transformation that is a composition of rotation

$$(w_1, w_2, w_3) \mapsto (2^{-1/2}w_1 - w_3, w_2, 2^{-1/2}w_1 + w_3)$$

and scaling $(w_1, w_2, w_3) \mapsto R^{1/3}(w_1, (3/2)^{1/2}w_2, w_3)$. Then we have

$$\mathcal{L}(\mathcal{N}_{\mathcal{C}_{OE}}(R^{-1})) \subset \mathcal{N}_{\mathcal{C}_{OL}}(R^{-3/2}).$$

Similarly, we can check that planks $\mathcal{L}(\tilde{\theta}_I \cap \{ (1/2)R^{-3/2} \leq w_1 \leq R^{-1/2} \})$ are finitely overlapping and contained in $\mathcal{N}_{\mathcal{C}_{OL}}(R^{-3/2})$. On the other side of the cone, the
planks $\mathcal{L}(\tilde{\theta}_l \cap \{-R^{-\frac{2}{3}} \leq w_1 \leq -(1/2)R^{-\frac{4}{3}}\})$ are finitely overlapping and sitting in $\mathcal{N}_{\text{Cone}}(R^{-\frac{3}{2}})$ by symmetry. Again using convexity, we note that $\mathcal{L}(\tilde{\theta})$ are planks associated with the light cone $\mathcal{N}_{\text{Cone}}(R^{-\frac{3}{2}})$ that are translated to the origin.

Therefore, we can apply the square function estimate from Lemma 3.1 in [14] to functions whose Fourier transform is supported in planks $\tilde{\theta}$ by a simple change of variables. In order to state the lemma, we introduce more notations. For $0 \leq s \leq 1$ and dyadic $R^{-\frac{4}{3}} \leq \sigma \leq 1$, we define the small plank $\Theta(\sigma, s)$ at scale $\sigma$ by

$$\Theta(\sigma, s) = \left\{ w \in \mathbb{R}^3 : |w_1| \leq R^{-\frac{4}{3}}\sigma^2, |w_2 - 2sw_1| \leq R^{-\frac{2}{3}}\sigma, |w_3 - 3sw_2 + 3s^2w_1| \leq R^{-1} \right\}.$$ 

Note that two small planks $\Theta(\sigma, s)$ and $\Theta(\sigma, s')$ are distinguishable only if $|s - s'| \geq R^{-\frac{4}{3}}\sigma^{-1}$. For each dyadic $R^{-\frac{4}{3}} \leq \sigma \leq 1$, let $\mathbf{CP}_\sigma$ be the set of $\sim \sigma R^{\frac{1}{3}}$ small planks $\Theta(\sigma, s')$ with $s'$ evenly spaced on the interval $[0, 1]$.

We associate each small plank $\Theta(\sigma, s')$ with $\tilde{\theta}_l$ satisfying $\text{dist}(s', I) \leq R^{-\frac{4}{3}}\sigma^{-1}$. Let $\mathbf{S}_\Theta$ be the set of $\tilde{\theta}$ associated with $\Theta$. Then each $\tilde{\theta}$ is included in $\sim O(1)$ collections $\mathbf{S}_\Theta$. We write the dual box of $\Theta(\sigma, s)$ centered at the origin by

$$U_{\Theta} := \{ x \in \mathbb{R}^3 : |x_1 + 2sx_2 + 3s^2x_3| \leq R^{\frac{1}{3}}\sigma^{-2}, |x_2 + 3sx_3| \leq R^{\frac{1}{3}}\sigma^{-1}, |x_3| \leq R \}.$$ 

Note that $U_{\Theta}$ is a $(R^{\frac{4}{3}}\sigma^{-2}, R^{\frac{1}{3}}\sigma^{-1}, R)$-box. We write $U//U_{\Theta}$ if $U$ is a translation of $U_{\Theta}$.

For each scale $\sigma$, the box $U_{\Theta}$ contains all dual boxes $\tilde{\theta}^*$ centered at the origin for $\tilde{\theta} \in \mathbf{S}_\Theta$. Therefore, any translation of $\tilde{\theta}^*$ is contained in $O(1)$ boxes $U$ such that $U//U_{\Theta}$ for $\tilde{\theta} \in \mathbf{S}_\Theta$. See Lemma 4.4 for the proof.

Lastly, we recall the locally constant property from Lemma 6.1 and Lemma 6.2 in [17]. Suppose $f_{\tilde{\theta}} : \mathbb{R}^3 \to \mathbb{C}$ has a Fourier transform supported in $\tilde{\theta}$. Then, roughly speaking, the magnitude $|f_{\tilde{\theta}}|$ is essentially constant on each translation of dual box $\tilde{\theta}^*$.

Now, we state the main lemma of this section.

**Lemma 3.2** (Lemma 3.1 in [17]). Let each $f_{\tilde{\theta}}$ be a function whose Fourier transform is supported in $\tilde{\theta}$. Then

$$\int_{\mathbb{R}^3} \left( \sum_{\tilde{\theta}} |f_{\tilde{\theta}}|^2 \right)^2 \lesssim \sum_{R^{-1/3} \leq \sigma \leq 1} \sum_{\Theta \in \mathbf{S}_\Theta} \sum_{U//U_{\Theta}} |U|^{-1} \left( \int_U \sum_{\tilde{\theta} \in \mathbf{S}_\Theta} |f_{\tilde{\theta}}|^2 \right)^2. \tag{3}$$

**Proof.** We refer to Section 4 from [17]. The original argument also holds for functions $f_{\mathcal{L}(\tilde{\theta})}$ whose Fourier transform is supported in cone planks centered at the origin $\mathcal{L}(\tilde{\theta})$. Applying the transpose linear map $\mathcal{L}^T$ on both sides of (3), we find the equivalence with Lemma 3.1 in [17].
4 Incidence estimate for well-spaced Vinogradov tubes

In this section, we show an incidence estimate for well-spaced tubes in $\mathbb{R}^3$ associated with $\Gamma$. We first define spatial Vinogradov planks and tubes.

**Definition 4.1** (Spatial Vinogradov planks at scale $R$). Let

$$I = [c, c + R^{-\frac{1}{3}}] \in \mathbb{I}_{R^{-1/3}}.$$ 

Let the $(R^{\frac{1}{3}}, R^{\frac{2}{3}}, R)$-box $P$ have the long edge pointing in the direction $b(I)$ and its wide $(R^{-\frac{1}{3}}, R^{-\frac{2}{3}})$-face have normal direction $t(I)$. We call such a box a spatial Vinogradov plank at scale $R$, associated with the interval $I \in \mathbb{I}_{R^{-1/3}}$. We will denote the plank as $P_I$, or simply as $P$. The collection of all Vinogradov planks in $\mathbb{P}$ associated with $I$ will be denoted by $\mathbb{P}(I)$.

We note that frequency and spatial Vinogradov planks associated with a given $I$ are boxes dual to each other. A general spatial Vinogradov plank $P_I$ is a translate of the following plank at the origin:

$$\{x \in \mathbb{R}^3 : |x_1 + 2cx_2 + 3c^2x_3| \leq R^{\frac{1}{3}}, |x_2 + 3cx_3| \leq R^{\frac{2}{3}}, |x_3| \leq R\}.$$ 

We mention that throughout the rest of the paper we will also encounter rescaled spatial Vinogradov planks. These will be planks of the form $LP$, for some scalar $L > 0$ and $P$ as above.

**Definition 4.2** (Vinogradov tubes at scale $R$). Let $I = [c, c + R^{-\frac{1}{3}}] \in \mathbb{I}_{R^{-1/3}}$. Assume the $(R^{\frac{1}{3}}, R^{\frac{2}{3}}, R)$-tube has the long edge pointing in the direction $b(I)$. We call it a Vinogradov tube associated with $I$, and denote it by $T_I$, or simply as $T$. Families of Vinogradov tubes will typically be denoted by $\mathbb{T}$. The collection of all Vinogradov tubes in $\mathbb{T}$ associated with $I$ will then be denoted by $\mathbb{T}(I)$.

The orientation of a tube is characterized by its long axis. Therefore, we can describe the general Vinogradov tubes $T_I$ as translates of the following tube at the origin:

$$\{x \in \mathbb{R}^3 : |x_1 + 2cx_2 + 3c^2x_3| \leq R^{\frac{1}{3}}, |x_2 + 3cx_3| \leq R^{\frac{2}{3}}, |x_3| \leq R\}.$$ 

Each plank $P_I$ sits naturally inside some tube $T_I$. If we combine $R^{\frac{1}{3}}$ many planks $P_I$, consecutively placed in the $t(I)$ direction, the result is a Vinogradov tube $T_I$.

We use this observation to prove the following incidence result for well-spaced Vinogradov tubes. For $q \subset \mathbb{R}^3$ and a box $B$, we will not distinguish between $q \subset B$ and $q \cap B \neq \emptyset$. 
Theorem 4.3. Let $\mathcal{T}$ be a collection of separated $(R_1^{1/3}, R_2^{1/3}, R)$-Vinogradov tubes $T$ in $[-R, R]^3$, satisfying the following well-spacing condition:

(WS) For each $I \in \mathbb{R}^{1/3}$, we cover $[-R, R]^3$ with $(R_1^{1/3}, R, R)$-plates $S_I$ with normal vector $t(I)$. Note that each $T \in \mathcal{T}$ sits inside one such plate $S_I$.

We assume that each such $S_I$ contains at most $N$ tubes $T \in \mathcal{T}(I)$. (See Figure 1.)

Let $1 \leq r \leq R_1^{1/3}$ and $q$ be the cube with side length $\sim R_1^{1/3}$. We say the cube $q$ is $r$-rich if $q$ intersects at least $r$ tubes $T \in \mathcal{T}$. Let $\Omega_r(\mathcal{T})$ be a collection of pairwise disjoint $r$-rich cubes.

Then for each $r$ we have

$$|\Omega_r| \lesssim \frac{|\mathcal{T}| N R_1^{1/3} + \epsilon}{r^2}.$$  \hspace{1cm} (4)

We see that there is no need for a lower bound on $r$ in contrast to the refined Kakeya estimate in $\mathbb{R}^2$ (Theorem 5.4 in [14]). We mention that this theorem will be used twice in the remaining part of the paper, namely in steps 4 and 10 of the proof of Proposition 8.5.

We compare (4) with the easy estimate

$$|\Omega_r| \lesssim \frac{|\mathcal{T}|^2}{r^2}$$  \hspace{1cm} (5)

that holds in the bilinear setting (when each cube is intersected by at least $r$ tubes from two transverse families with cardinality $|\mathcal{T}|$).
Note that, subject to (WS), the collection $\mathbb{T}$ has size at most $|\mathbb{T}_{\max}| = NR^{\frac{2}{3}}$, when each $S_I$ for each $I$ contains $\sim N$ tubes. For such collections of tubes, the inequality in the corollary above reads

$$|\Omega_r| \lesssim e^{R^2 |\mathbb{T}|^2/W r^2},$$

where $W = R^{1/3}$ is the number of plates $S_I$ inside $[-R, R]^3$, for a fixed $I$. Note the $W$-gain over the estimate in (5).

In order to prove Theorem 4.3, we first observe the simple geometry of the Vinogradov planks centered at the origin. The following lemma is a variation of Lemma 7.3 in [14].

**Lemma 4.4** (Vinogradov planks). Let $R^{-\frac{2}{3}} \leq \rho \leq 1$. Let $H \subset [0, 1]$ be an interval of length $\rho$. Assume that for each $I \in \mathbb{R}^{-1/3}(H)$, the spatial $(R^{\frac{1}{3}}, R^{\frac{2}{3}}, R)$-Vinogradov plank $P_I$ is centered at the origin.

(Intersection) The intersection of all such planks $P_I$ is essentially an $(R^{\frac{1}{3}}, R^{\frac{1}{3}}/\rho, R^{\frac{1}{3}}/\rho^2)$-plank with respect to axes $(t(H), n(H), b(H))$, and centered at the origin.

(Union) The box $B$ centered at the origin with dimensions $(R \rho^2, R \rho, R)$ with respect to axes $(t(H), n(H), b(H))$ contains all such planks $P_I$.

**Proof.** Let $H = [a, a + \rho]$. Using the linear map $A_{1,a}$ introduced in Section 2, both parts can be reduced to the case $H = [0, \rho]$. In this case, the axes $t(H), n(H), b(H)$ are the same as the $x, y, z$ axes.

For the intersection part, we can recall the proof of Lemma 5.2. We can see that the intersection of all planks $P_I$ is an $(R^{\frac{1}{3}}, R^{\frac{1}{3}}/\rho, R^{\frac{1}{3}}/\rho^2)$-box and considering the eccentricity, the box has the orientation associated with $H$.

For the union part, let $I = [c, c + R^{-\frac{2}{3}}]$. Each $P_I$ has the dimensions $(R^{\frac{1}{3}}, R^{\frac{2}{3}}, R)$ with respect to axes $t(c), n(c), b(c)$. Recall that the $(x, y, z)$-coordinates of these axes are given by

$$t(c) = (1, 2c, 3c^2) = (1, O(\rho), O(\rho^2))$$

$$n(c) = (-2c - 9c^3, -c^4 + 1, 3c + 6c^3) = (O(\rho), O(1), O(\rho))$$

and

$$b(c) = (3c^2, -3c, 1) = (O(\rho^2), O(\rho), 1).$$

Therefore, each point inside $P_I$ has $(x, y, z)$-coordinates of the form

$$O(R^{1/3})(1, O(\rho), O(\rho^2)) + O(R^{2/3})(O(\rho), O(1), O(\rho)) + O(R)(O(\rho^2), O(\rho), 1).$$

This is easily seen to be $(O(R \rho^2), O(R \rho), O(R))$, as desired. \qed
We prove Theorem 4.3 by using Lemma 3.2 and the well-spacing condition (WS) of the tubes. The argument is inspired by that from [14]. The notation \( \lesssim \) stands for \( \lesssim (\log R)^C \) with \( C = O(1) \).

**Proof of Theorem 4.3.** Let us substitute each Vinogradov tube \( T \) with \( R^\frac{1}{3} \) Vinogradov planks \( P \) lying in \( T \). We denote by \( \mathbb{P} \) the collection of Vinogradov planks from the substitution. Then we have

\[
|\mathbb{P}| = R^{1/3} |T|.
\]

Let \( \Delta \) be the small cube with side length \( R^{1/3} \). We say the cube \( \Delta \) is \( r \)-rich if \( \Delta \) intersects at least \( r \) planks \( P \in \mathbb{P} \). Let \( \mathcal{Q}_r(\mathbb{P}) \) be the collection of disjoint \( r \)-rich cubes \( \Delta \). Suppose \( q \in \mathcal{Q}_r(\mathbb{T}) \) so that \( R^\frac{2}{3} \)-cube \( q \) intersects at least \( r \) tubes \( T \). Then all \( R^\frac{1}{3} \)-cubes \( \Delta \) lying in \( q \) also intersect at least \( r \) (a slight enlargement of) planks \( P \), consequently \( \Delta \in \mathcal{Q}_r(\mathbb{P}) \). Since the number of \( R^\frac{1}{3} \)-cubes \( \Delta \) contained in \( q \) is \( R \), we have

\[
|\mathcal{Q}_r(\mathbb{T})| \leq |\mathcal{Q}_r(\mathbb{P})|.
\]

Let \( v_p \) be a positive smooth approximation of \( 1_p \) for each \( P \in \mathbb{P} \) so that its Fourier transform \( \hat{v}_p \) is supported in the dual box \( \tilde{\Theta}_I \) centered at the origin. For each \( I \), we define the function \( f_{\tilde{\Theta}_I}(x) = \sum_{P \in \mathbb{P}(I)} v_p(x) \) whose Fourier transform is supported in \( \tilde{\Theta}_I \). Then we have

\[
\sum_{I \in \mathbb{P}} f_{\tilde{\Theta}_I} = \sum_{I \in \mathbb{P}} \sum_{P \in \mathbb{P}(I)} v_p = \sum_{P \in \mathbb{P}} v_p.
\]

Using (7), Chebyshev’s inequality and applying Lemma 3.2 to the functions \( f_{\tilde{\Theta}_I} \), we write

\[
\begin{align*}
\sum_{I \in \mathbb{P}} f_{\tilde{\Theta}_I} &\approx \int_{\mathbb{R}^3} \left( \sum_{I \in \mathbb{P}} \sum_{P \in \mathbb{P}(I)} v_p \right)^2 \\
&\approx \int_{\mathbb{R}^3} \left( \sum_{I \in \mathbb{P}} \sum_{P \in \mathbb{P}(I)} \left| v_p \right|^2 \right)^2 \\
&\lesssim_{\epsilon} R^\epsilon \sum_{R^{-1/3} \leq \sigma \leq 1} \sum_{\Theta \in \mathcal{S}_\Theta} \sum_{U // U_\Theta} |U|^{-1} \left( \int_{\tilde{\Theta}_I} \sum_{P \in \mathbb{P}(I)} |v_p|^2 \right)^2 
\end{align*}
\]

with the notations \( \sigma, \Theta, \mathcal{S}_\Theta, U \) defined in Section 3. Recall that any plank \( P_I \) is contained in \( O(1) \) boxes \( U \) with \( U // U_\Theta \) for \( \tilde{\Theta}_I \in \mathcal{S}_\Theta \). Write for each \( (R^\frac{1}{3} \sigma^{-2}, R^\frac{2}{3} \sigma^{-1}, R) \)-box \( U \),

\[
\int_{U} \sum_{P \in \mathbb{P}(I)} |v_p|^2 \sim \left| \{ P \in \mathbb{P} : P \subset U \} \right| R^2.
\]
For each dyadic $\sigma$, $|U| = \sigma^{-3}R^2$ and define $M_\sigma := \max_U |\{ P \in \mathbb{P} : P \subset U \}|$. We further bound the last inequality by

$$\lesssim R^\epsilon \sum_{R^{-1/3} \leq \sigma \leq 1} \sum_{\Theta \in \mathcal{S}_\Theta} \sum_{U/\Theta_\sigma} (\sigma^{-3}R^2)^{-1}(|\{ P \in \mathbb{P} : P \subset U \}|R^2)^2$$

(8)

$$\lesssim R^\epsilon \sum_{R^{-1/3} \leq \sigma \leq 1} |\mathbb{P}|\sigma^3R^2M_\sigma$$

$$\lesssim R^\epsilon \max_{R^{-1/3} \leq \sigma \leq 1} |\mathbb{P}|\sigma^3R^2M_\sigma.$$

Next, we use Lemma 4.4 with $\rho = R^{-2}R^{-1} \sigma^{-1}$ and well-spacing condition (WS) to prove

$$M_\sigma \lesssim \sigma^{-3}N.\tag{9}$$

Let us pick a $(R^{4/3}R^{-2}, R^2R^{-1}, R)$-box $U$ with the orientation associated with some $H$ of length $R^{-1/3}R^{-1}$. We separate the proof into two cases.

Suppose that $\sigma \geq R^{-4}$. For each $I \subset H$, the box $U$ intersects $\lesssim 1$ plates $S_I$. By the spacing condition (WS), $U$ intersects at most $N$ tubes $T_I \in \mathbb{T}(I)$ from each direction $I \subset H$. Each intersecting tube $T_I$ contains $\lesssim \sigma^{-2}$ planks $P_I$ that are also contained in $U$ by Lemma 4.4. Note from the same lemma that there are $\lesssim \sigma^{-1}$ contributing directions. Therefore, $U$ contains at most $\sigma^{-3}N$ planks $P \in \mathbb{P}$.

Second, we assume that $\sigma < R^{-4}$. For $I \subset H$, the box $U$ intersects $\lesssim \frac{R^{4/3}R^{-2}}{R^2R^{-3}}$ plates $S_I$. The spacing condition (WS) implies $U \cap S_I$ contains at most $N$ tubes $T_I$. Also, there are $\lesssim \sigma^{-1}$ contributing directions by Lemma 4.4. Therefore, $U$ contains at most $\sigma^{-3}R^{-1/3}N$ tubes $T \in \mathbb{T}$. Noting that the tube $T$ contains $R^{4/3}$ planks $P$, we have

$$|\{ P \in \mathbb{P} : P \subset U \}| \lesssim R^{4/3}|\{ T \in \mathbb{T} : T \subset U \}|.$$

Combining these facts, for fixed $\sigma$, a box $U$ contains at most $\sigma^{-3}N$ planks $P \in \mathbb{P}$.

Using (6) and (9), we write

$$\max_{R^{-1/3} \leq \sigma \leq 1} |\mathbb{P}|\sigma^3R^2M_\sigma \lesssim |\mathbb{T}|NR^{2+\frac{1}{3}}.\tag{10}$$

We combine (8) and (10) to finish the proof.

\[ \Box \]

5 An $L^4$ inequality for Vinogradov planks

In this section we introduce another Kakeya type estimate for the Vinogradov planks. It represents one of the main innovations of this paper. We begin with a geometric observation on the intersection of three planks.
Lemma 5.1 (Intersection of three planks). Let $0 < \delta < 1$. Let $\mathbb{I}_\delta$ be the partition of $[0, 1]$ into intervals $I$ of length $\delta$. Let $P_I$ be the Vinogradov plank of dimension $(\delta^{-1}, \delta^{-2}, \delta^{-3})$ associated with $I$. Let us write $I_j = [c_j, c_j + \delta]$ for $j = 1, 2, 3$. Assume that $c_1 \leq c_3 \leq c_2$. Then

$$|P_{I_1} \cap P_{I_2} \cap P_{I_3}| \lesssim \delta^{-1} \times \frac{\delta^{-1}}{D_2 + \delta} \times \frac{\delta^{-1}}{(D_3 + \delta)(D_2 - D_3 + \delta)}.$$ 

Proof. We first suppose that the planks $P_{I_1}, P_{I_2}, P_{I_3}$ are centered at the origin. Recall the linear map $A_{1,c}$ on $\mathbb{R}^3$ introduced in Section 2. If $I = [c, c + \delta]$, then $A_{1,c}(P_I)$ is a Vinogradov plank centered at the origin, associated with the interval $I - c$. Noting that $\det A_{1,c} = 1$ we can assume $I_1 = [0, \delta], I_2 = [D_2, D_2 + \delta]$ and $I_3 = [D_3, D_3 + \delta]$.

In this setting, let us investigate the intersection of two planks $P_{I_1} \cap P_{I_2}$ first. This observation was made in Exercise 5.12 in [10]. From the discussion after Definition 4.1, we see that if $x = (x_1, x_2, x_3) \in P_{I_1} \cap P_{I_2}$, such $x$ satisfies

(11)

$$\begin{cases} |x_1| \leq \delta^{-1}, \\ |x_2| \leq \delta^{-2}, \\ |x_3| \leq \delta^{-3}, \end{cases}$$

and

(12)

$$\begin{cases} |x_1 + 2D_2 x_2 + 3D_2^2 x_3| \leq \delta^{-1}, \\ |x_2 + 3D_2 x_3| \leq \delta^{-2}, \\ |x_3| \leq \delta^{-3}. \end{cases}$$

Combining the second inequalities of (11) and (12), we get $|x_3| \lesssim \frac{\delta^{-2}}{D_2}$. Also, combining the first inequalities we get

$$\left| x_2 + \frac{3}{2}D_2 x_3 \right| \lesssim \frac{\delta^{-1}}{D_2}.$$ 

Therefore, $P_{I_1} \cap P_{I_2}$ is an almost rectangular box of dimensions roughly $(\delta^{-1}, \frac{\delta^{-1}}{D_2}, \frac{\delta^{-1}}{D_2})$ centered at the origin, whose the long side has the direction $t(I_1) \times t(I_2) / / \left(0, -\frac{3D_2}{2}, 1\right)$. In the same way, $P_{I_1} \cap P_{I_3}$ is a box of dimensions $\sim (\delta^{-1}, \frac{\delta^{-1}}{D_3}, \frac{\delta^{-2}}{D_3})$ with the long side pointing in the direction $(0, -\frac{3D_3}{2}, 1)$.

Since these two boxes are subsets of $P_{I_1}$, we see that the intersection $(P_{I_1} \cap P_{I_2}) \cap (P_{I_1} \cap P_{I_3})$ is an almost rectangular box with the length of the edge
parallel to the $x$-axis equal to $\delta^{-1}$. Let us evaluate the length of the other sides. We project the intersection to the $yz$-plane. The image of projection is an intersection of two tubes with dimensions $(\delta^{-1}, \delta^{-2})$, $(\delta^{-1}, \delta^{-2})$ respectively and the angle difference is $\sim D_2 - D_3$. Using planar geometry, the intersection is an almost rectangle with width $\sim \delta^{-1}D_2$ and length $\sim \delta^{-1}D_3(D_2 - D_3)$ with the long edge pointing in the direction $(-\frac{3D_2}{2}, 1)$. Including the simple cases when $D_2 = 0$, $D_3 = 0$ and $D_2 = D_3$, we conclude that the intersection $P_{I_1} \cap P_{I_2} \cap P_{I_3}$ is an almost rectangular box of dimensions

$$
(\delta^{-1}, \delta^{-1}, \delta^{-1}D_2 + \delta, \delta^{-1}(D_3 + \delta)(D_2 - D_3 + \delta)).
$$

If any nonempty intersection $P_{I_1} \cap P_{I_2} \cap P_{I_3}$ is not located at the origin, we can translate the planks $P_{I_1}, P_{I_2}, P_{I_3}$ to the origin and apply the same argument.

We use Lemma 5.1 to prove the following “base case”, when each Vinogradov plank $P_I$ is centered at the origin.

**Theorem 5.2.** Let $0 < \delta < 1$. Let $I_\delta$ be the partition of $[0, 1]$ into intervals $I$ of length $\delta$. Let $P_I$ be the Vinogradov plank of dimension $(\delta^{-1}, \delta^{-2}, \delta^{-3})$ associated with $I$ and centered at the origin. We have

$$
\left\| \sum_{I \in I_\delta} 1_{P_I} \right\|_4^4 \lesssim (\log \delta)^{-2} \left\| \sum_{I \in I_\delta} 1_{P_I} \right\|_1.
$$

**Proof.** Let us write $I_j = [c_j, c_j + \delta] \in I_\delta$ and $D_j = c_j - c_1$ for $j = 1, 2, 3, 4$. A simple observation shows that

$$
\left\| \sum_{I \in I_\delta} 1_{P_I} \right\|_4^4 = \int \sum_{I_1 \in I_\delta} \sum_{I_2 \in I_\delta} \sum_{I_3 \in I_\delta} \sum_{I_4 \in I_\delta} 1_{P_{I_1}} 1_{P_{I_2}} 1_{P_{I_3}} 1_{P_{I_4}}
\lesssim \sum_{I_1 \in I_\delta} \sum_{I_2 : |D_2| < 1} \sum_{I_3 : |D_3| \leq |D_2|} \sum_{I_4 : |D_4| \leq |D_3|} |P_{I_1} \cap P_{I_2} \cap P_{I_3} \cap P_{I_4}|.
$$

It is harmless to assume that $D_j \geq 0$ for $j = 2, 3, 4$. We estimate

$$
|P_{I_1} \cap P_{I_2} \cap P_{I_3} \cap P_{I_4}| \leq |P_{I_1} \cap P_{I_2} \cap P_{I_i}|
$$

so that

$$
|P_{I_1} \cap P_{I_2} \cap P_{I_3} \cap P_{I_4}| \lesssim \delta^{-1} \times \frac{\delta^{-1}}{D_2 + \delta} \times \frac{\delta^{-1}}{(D_3 + \delta)(D_2 - D_3 + \delta)}
$$

by Lemma 5.1.
Using the observation we write

\[ \left\| \sum_{I \in I_\delta} \sum_{I_1, I_2 : D_2 < 1} \sum_{I_3, I_4 : D_3 \leq D_2} |P_{I_1} \cap P_{I_2} \cap P_{I_3} \cap P_{I_4}| \right\|_4 \]

\[ \lesssim \sum_{I_1, I_2 : D_2 < 1} \sum_{I_3, I_4 : D_3 \leq D_2} \sum_{I_5, I_6 : D_4 \leq D_3} \delta^{-1} \times \frac{\delta^{-1}}{D_2 + \delta} \times \frac{\delta^{-1}}{D_3 + \delta(2 - D_3 + \delta)} \]

\[ = \sum_{I_1, I_2 : D_2 < 1} \sum_{I_3, I_4 : D_3 \leq D_2} \sum_{I_5, I_6 : D_4 \leq D_3} \delta^{-5} \frac{\log(1 + D_2)}{D_2 + \delta} \]

\[ \lesssim \sum_{I_1 : D_2 < 1} (\log \delta^{-1})^2 \delta^{-6} \]

\[ \lesssim (\log \delta^{-1})^2 \left\| \sum_{I \in I_\delta} 1_{P_I} \right\|_1. \]

Hölder’s inequality shows that (13) holds true with the exponent 4 replaced with any \( 1 \leq p \leq 4 \). The restriction theorem for \( \Gamma_3 \) from [15] combined with the standard randomization argument (Proposition 5.7 in [10]) proves the case \( p = \frac{7}{2} \).

The example with one plank for each direction centered at the origin shows that \( p \) cannot be taken larger than 4.

There are variants of the previous theorem for planks with different spacing conditions. Let us show one example in the following corollary. We write \( T_I \) for a Vinogradov tube of dimensions \((\delta^{-2} - 2, \delta^{-2}, \delta^{-3})\) with the long edge in the direction \( b(I) \). Also \( P(I) \) and \( T(I) \) will stand for the subsets of \( \mathbb{P} \) and \( \mathbb{T} \) containing all planks \( P_I \in \mathbb{P} \) and all tubes \( T_I \in \mathbb{T} \) associated with \( I \in I_\delta \), respectively. Note that each \( T_I \) can contain at most \( \delta^{-1} \) Vinogradov planks \( P_I \).

**Corollary 5.3.** Let \( \mathbb{P} \) be a collection of separated Vinogradov planks satisfying the following two spacing conditions. For each \( I \in I_\delta \), there are at most \( M \) planks \( P_I \in \mathbb{P}(I) \). Moreover, there are at most \( N \) parallel planks \( P_I \) inside each \( T_I \).

Then we have

\[ \left\| \sum_{P \in \mathbb{P}} 1_{P_I} \right\|_4^4 \lesssim (\log \delta^{-1})^2 MN \left\| \sum_{P \in \mathbb{P}} 1_{P_I} \right\|_1. \]
Proof. A simple observation shows that

$$\left\| \sum_{P_i \in P} 1_{P_i} \right\|_4^4 \lesssim \sum_{P'_1 \in \mathbb{P}} \sum_{I_2 : D_2 < 1} \sum_{I_3 : D_3 \leq D_2} \sum_{I_4 : D_4 \leq D_3} \sum_{P'_2 \in \mathbb{P}(I_2)} \sum_{P'_3 \in \mathbb{P}(I_3)} \sum_{P'_4 \in \mathbb{P}(I_4)} |P'_1 \cap P'_2 \cap P'_3 \cap P'_4|.$$  

The last sum can be estimated by

$$\sum_{P'_4 \in \mathbb{P}(I_4)} |P'_1 \cap P'_2 \cap P'_3 \cap P'_4| = |P'_1 \cap P'_2 \cap P'_3 \cap \bigcup_{P'_4 \in \mathbb{P}(I_4)} P'_4| \leq |P'_1 \cap P'_2 \cap P'_3|.$$  

For each $I_1, I_2, I_3$, we can assume that $P'_1 \cap P'_2 \cap P'_3$ is nonempty, otherwise the summand is 0. In the case of $P'_1 \cap P'_2 \cap P'_3$ nonempty, we can bound the volume $|P'_1 \cap P'_2 \cap P'_3|$ by $\delta^{-1} \times \frac{\delta^{-1}}{D_3(D_2-D_3)}$ as in Lemma 5.1. Using the condition $D_3 \leq D_2$, we see that there exists a unique $T_{I_3}$ such that $(P'_1 \cap P'_2) \subset T_{I_3}$ for fixed $I_3$ and nonempty $P'_1 \cap P'_2$. Combining these facts with the last inequality, the last two sums become

$$\sum_{P'_1 \in \mathbb{P}(I_1)} \sum_{P'_4 \in \mathbb{P}(I_4)} |P'_1 \cap P'_2 | \cap P'_3 \cap P'_4| \leq \sum_{P'_3 \subset T_{I_3}} |P'_1 \cap P'_2 \cap P'_3| \leq N \delta^{-1} \times \frac{\delta^{-1}}{D_2 + \delta} \times \frac{\delta^{-1}}{(D_3 + \delta)(D_2 - D_3 + \delta)}.$$  

Using this observation we find that

$$\left\| \sum_{P_i \in \mathbb{P}} 1_{P_i} \right\|_4^4 \lesssim \sum_{P'_1 \in \mathbb{P}} \sum_{I_2 : D_2 < 1} \sum_{I_3 : D_3 \leq D_2} \sum_{I_4 : D_4 \leq D_3} \sum_{P'_2 \in \mathbb{P}(I_2)} \sum_{P'_3 \in \mathbb{P}(I_3)} \sum_{P'_4 \in \mathbb{P}(I_4)} N \delta^{-1}$$

$$\times \frac{\delta^{-1}}{D_2 + \delta} \times \frac{\delta^{-1}}{(D_3 + \delta)(D_2 - D_3 + \delta)} \leq \sum_{P'_1 \in \mathbb{P}} \sum_{I_2 : D_2 < 1} \sum_{I_3 : D_3 \leq D_2} \sum_{I_4 : D_4 \leq D_3} \sum_{P'_2 \in \mathbb{P}(I_2)} \sum_{P'_3 \in \mathbb{P}(I_3)} \sum_{P'_4 \in \mathbb{P}(I_4)} M \delta^{-1}$$

$$\times \frac{\delta^{-1}}{D_2 + \delta} \times \frac{\delta^{-1}}{(D_3 + \delta)(D_2 - D_3 + \delta)} \leq (\log \delta^{-1})^2 M \| \sum_{P_i \in \mathbb{P}} 1_{P_i} \|_1.$$  

□
6 Incidence estimates for Vinogradov plates

We use the Kakeya estimate for planks from the last section to show incidence estimates for the Vinogradov plates.

**Definition 6.1** (Vinogradov plates). For each $I \in \mathbb{I}_\delta$, the $(\delta^{-1}, \delta^{-2}, \delta^{-2})$-plate $S_I$ is a Vinogradov plate associated with $I$ if it has the normal vector $\mathbf{t}(I)$.

The orientation of each plate is characterized by its normal vector. Therefore, if $I = [c, c + \delta]$, each $S_I$ can be described as a translation of the following Vinogradov plate centered at the origin:

$$S = \{ x \in \mathbb{R}^3 : |x_1 + 2cx_2 + 3c^2x_3| \leq \delta^{-1}, |x_2 + 3cx_3| \leq \delta^{-2}, |x_3| \leq \delta^{-2} \}.$$

We first state the standard $L^2$ estimate for plate incidences, whose proof appears in Proposition 6.4 in [14].

**Theorem 6.2** ($L^2$ Kakeya for plates). Let $\mathcal{S}$ be a collection of separated $(\delta^{-1}, \delta^{-2}, \delta^{-2})$-Vinogradov plates in $[-\delta^{-2}, \delta^{-2}]^3$. Assume that there are at most $N$ plates $S_I$ for each $I$.

Let $\mathcal{Q}_r$ be a collection of pairwise disjoint cubes $q$ with side length $\sim \delta^{-1}$ that intersect at least $r$ plates $S \in \mathcal{S}$. Then for each $r \geq 1$ we have

$$\left\| \sum_{S \in \mathcal{S}} 1_S \right\|_2^2 \lesssim N \left( \sum_{S \in \mathcal{S}} |S| \right)$$

and we get the incidence estimate

$$|\mathcal{Q}_r| \lesssim \frac{|\mathcal{S}|N\delta^{-2}}{r^2}.$$

While we will use this incidence result in step 7 of the proof of Proposition 8.5, we also state it in order to serve as a comparison with the next result. There is a better estimate for $|\mathcal{Q}_r|$ if $r \geq \sqrt{N^2 \delta^{-2}}$ in the same setting. This better estimate will be used twice in the remaining part of the paper, namely in steps 5 and 11 of the proof of Proposition 8.5.

**Theorem 6.3** ($L^4$ Kakeya for plates). Let $\mathcal{S}$ be a collection of separated $(\delta^{-1}, \delta^{-2}, \delta^{-2})$-Vinogradov plates in $[-\delta^{-2}, \delta^{-2}]^3$. Assume that there are at most $N$ plates $S_I$ for each $I$.

Then for each $r \geq 1$ we have

$$|\mathcal{Q}_r| \lesssim \frac{|\mathcal{S}|N^2\delta^{-3}}{r^4}.$$
Proof. We extend each plate $S_I$ in the $b(I)$ direction to get a Vinogradov plank $P_I$ containing $S_I$. Each $P_I$ has dimensions $(\delta^{-1}, \delta^{-2}, \delta^{-3})$ with respect to the axes $(t(I), n(I), b(I))$. Let us denote the collection of Vinogradov planks corresponding to $S$ by $\mathbb{P}$. Each $r$-rich cube with respect to $S$ will also be $r$-rich with respect to $\mathbb{P}$.

Using Corollary 5.3 we write
\[
\sum_{P \in \mathbb{P}} \| \sum_{P \in \mathbb{P}} 1_P \|_4 \lesssim N^2 \| \sum_{P \in \mathbb{P}} 1_P \|_1 \lesssim N^2 |\mathbb{P}| \delta^{-6} = N^2 |S| \delta^{-6}.
\]

7 A few preliminaries

In this section, we record a few simple geometric facts about Vinogradov planks and plates centered at the origin, along the lines of Lemma 7.3 and Lemma 6.3 in [14]. We refer to Lemma 4.4 for similar observations on Vinogradov planks. We also briefly introduce two wave packet decompositions and two decouplings that will be useful in the next section.

7.1 Intersections and unions of tubes and plates. We observe the simple geometry of the Vinogradov tubes (defined in Definition 4.2) centered at the origin. There is a major difference between small angle and large angle.

Lemma 7.1 (Vinogradov tubes). Let $R^{-\frac{1}{3}} \leq \sigma \leq 1$. Let $H \subset [0, 1]$ be an interval of length $\sigma$. For each $I \in \mathbb{P}_{R^{-1/3}}(H)$, let $T_I$ be an $(R^2, R^2, R)$ Vinogradov tube associated with $I$ and centered at the origin. (Intersection) The intersection of all such tubes $T_I$ is essentially an $(R^2, R^2, R^2/\sigma)$-box with respect to axes $(t(H), n(H), b(H))$, and centered at the origin. (Union–small angle) If $\sigma \leq R^{-\frac{1}{6}}$, the box $U$ centered at the origin with dimensions $(R^2, R\sigma, R)$ with respect to the axes $(t(H), n(H), b(H))$ contains all the tubes $T_I$. (Union–large angle) If $\sigma > R^{-\frac{1}{6}}$, the box $U$ centered at the origin with dimensions $(R\sigma^2, R\sigma, R)$ and orientation associated with $H$ contains all the tubes $T_I$.

Proof. Let $H = [a, a+\sigma]$. Noting that the image $A_{1,\sigma}(T_I)$ is also the Vinogradov tube centered at the origin, it is enough to prove the case when $H = [0, \sigma]$. In this case, the axes $(t(H), n(H), b(H))$ of the box $U$ is the same as $x, y, z$ axes.
For the intersection part, note that the intersection of two tubes $T_{[0,R^{-1/3}]}$ and $T_{[\sigma-R^{-1/3},\sigma]}$ has dimensions $(R^2, R^2, R^2/\sigma)$. We observe that this box is contained in all the other $T_I$ for $I \subset H$.

For the union part, let $I = [c, c + R^{-1/3}]$. Each $T_I$ has the dimensions $(R^2, R^2, R)$ with respect to axes $t(c), n(c), b(c)$. Recall that the $(x, y, z)$-coordinates of these axes are given by

$$t(c) = (1, 2c, 3c^2) = (1, O(\sigma), O(\sigma^2)),
$$

$$n(c) = (-2c - 9c^3, -c^4 + 1, 3c + 6c^3) = (O(\sigma), O(1), O(\sigma))$$

and

$$b(c) = (3c^2, -3c, 1) = (O(\sigma^2), O(\sigma), 1).$$

Therefore, each point inside $T_I$ has $(x, y, z)$-coordinates of the form

$$(O(R^2), O(R^2\sigma), O(R^2\sigma^2)) + (O(R^2\sigma), O(R^2), O(R^2\sigma^2)) + (O(R\sigma^2), O(R\sigma), O(R)).$$

For the small angle case $\sigma \leq R^{-\frac{1}{2}}$, the $x$ coordinate is bounded by the first term $O(R^2)$. For the large angle case $\sigma > R^{-\frac{1}{2}}$, the $x$ coordinate is bounded by the last term $O(R\sigma^2)$. This shows the required dimensions of the box $U$. $\square$

**Lemma 7.2** (Vinogradov plates). Let $0 < \delta < 1$.

(Intersection) Let $H \subset [0, 1]$ be an interval of length $\delta^2$. Assume that for each $I \in I_{\delta}(H)$, the $(\delta^{-1}, \delta^{-2}, \delta^{-2})$-Vinogradov plates $S_I$ is centered at the origin. Then the intersection of all the plate $S_I$ is essentially a $(\delta^{-1}, \delta^{-\frac{3}{2}}, \delta^{-2})$-plank centered at the origin, such that the $\delta^{-2}$-edge is pointing in direction $b(H)$ and the $(\delta^{-\frac{3}{2}}, \delta^{-2})$-wide face has normal direction $t(H)$.

(Union) Let $\delta \leq \sigma \leq 1$ and let $H$ be an interval of length $\sigma$. Assume that for each $I \in I_{\delta}(H)$, the $(\delta^{-1}, \delta^{-2}, \delta^{-2})$-Vinogradov plate $S_I$ is centered at the origin. Then the fat plate of dimensions $(\delta^{-2}\sigma, \delta^{-2}, \delta^{-2})$ with the normal vector $t(H)$ contains all the plates $S_I$.

**Proof.** For the intersection part we can find the proof in Lemma 6.3 of [14]. For the union part, we assume that $H = [0, \sigma]$. Let $I = [c, c + \delta]$ and we do the same computation as in the last lemma. The details are left to the reader. $\square$

### 7.2 Wave packet decompositions

For $R > 1$, recall that the isotropic $R^{-1}$-neighborhood of the cubic moment curve $\Gamma$ in $\mathbb{R}^3$ is given by

$$N_\Gamma(R^{-1}) = \{(t, t^2 + s_2, t^3 + s_3) : t \in [0, 1], \ (s_2^2 + s_3^2)^{\frac{1}{2}} \leq R^{-1}\}.$$
For $J \in \mathbb{I}_{R^{1/2}}$, we denote by $\mathcal{N}_J(R^{-1})$ the intersection of $\mathcal{N}_1(R^{-1})$ and the infinite strip $J \times \mathbb{R}^2$. Then, each set $\mathcal{N}_J(R^{-1})$ is an almost rectangular box of dimensions $(R^{-1/2}, R^{-1}, R^{-1})$ with the long side pointing in the direction $t(J)$. Let us consider a partition of $\mathcal{N}_1(R^{-1})$ by the family $\mathcal{N}_J(R^{-1})$. Then we have the following wave packet decomposition at scale $R^{-1/2}$. We mention that this decomposition was not needed in [14], since that paper is only concerned with the case when each $F_J$ is one exponential wave.

Let $\chi : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $\chi(x) = (1 + |x|)^{-100}$. Let $B$ be a box and $A_B$ be an affine linear transformation mapping $B$ to $[-1, 1]^3$. We define $\chi_B := \chi \circ A_B$.

**Theorem 7.3** (Wave packet decomposition at scale $R^{-1/2}$). (Exercise 2.7 in [10], Theorem 7.4 in [14]) Assume that the function $F$ has Fourier transform supported in $\mathcal{N}_1(R^{-1})$. There is a decomposition

$$F = \sum_{J \in \mathbb{I}_{R^{1/2}}} P_J F = \sum_{W \in \mathcal{W}(F)} F_W$$

where $F_W$ is a nonzero function and $\mathcal{W}(F)$ is a collection of separated Vinogradov $(R^{1/2}, R, R)$-plates, such that

1. Each $\widehat{F_W}$ is supported on $2\mathcal{N}_J(R^{-1})$ for some $J \in \mathbb{I}_{R^{1/2}}$, and $W$ is associated with $J$. We denote by $\mathcal{W}_J(F)$ the corresponding plates, so $P_J F = \sum_{W \in \mathcal{W}_J} F_W$.
2. $F_W$ is spatially concentrated near $W$, in the sense that for each $M \geq 1$

$$|F_W(x, y, z)| \lesssim_M \|F_W\|_{L^\infty} \chi^M_P(x, y, z).$$

Moreover, for each $p \geq 1$

$$\|F_W\|_p \sim \|F_W\|_{L^\infty} |W|^{1/p}.$$

3. For each $p \geq 2$ and each $\mathcal{W}_1 \subset \mathcal{W}_2 \subset \mathcal{W}_J(F)$ such that $\|F_W\|_{L^\infty} \sim \text{const}$ for $W \in \mathcal{W}_1$, we have

$$\left\| \sum_{W \in \mathcal{W}_1} F_W \right\|_{L^p(\mathbb{R}^3)} \lesssim \left\| \sum_{W \in \mathcal{W}_2} F_W \right\|_{L^p(\mathbb{R}^3)}.$$

4. For each $p \geq 2$ and each $\mathcal{W}_1 \subset \mathcal{W}_J(F)$ such that $\|F_W\|_{L^\infty} \sim \text{const}$ for $W \in \mathcal{W}_1$, we have

$$\left\| \sum_{W \in \mathcal{W}_1} F_W \right\|_{L^p(\mathbb{R}^3)} \sim \left( \sum_{W \in \mathcal{W}_1} \|F_W\|_{L^p(\mathbb{R}^3)}^p \right)^{1/p}.$$
We note that the magnitude $|F_W|$ decays rapidly outside $W$ from (W2) and is essentially constant on $W$ (Lemma 6.1 and Lemma 6.2 in [17]). Therefore, a somewhat simplified representation of $F_W$ is

$$F_W(x, y, z) \approx A_W W(x, y, z) e((x, y, z) \cdot (a, a^2, a^3))$$

where $a$ is some point in $J$ and $A_W \in \mathbb{C}$ is a constant with $|A_W| = \|F_W\|_\infty$.

Let us consider another decomposition. Recall that we defined the anisotropic neighborhood $\Gamma(R^{-\frac{1}{3}})$ of the cubic moment curve $\Gamma_1$ as follows:

$$\Gamma(R^{-\frac{1}{3}}) = \{ w \in \mathbb{R}^3 : w_1 \in [0, 1], \ |w_2 - w_1| \leq R^{-\frac{2}{3}}, \ |w_3 - 3w_1w_2 + 2w_1^2| \leq R^{-1} \}.$$

For each of $I \in \mathbb{I}_{R^{-1/3}}$, we write the intersection of $\Gamma(R^{-\frac{1}{3}})$ and the infinite strip $I \times \mathbb{R}^2$ as $\Gamma_I(R^{-\frac{1}{3}})$. Then each $\Gamma_I(R^{-\frac{1}{3}})$ is an almost rectangular box with dimensions $(R^{-\frac{1}{3}}, R^{-\frac{2}{3}}, R^{-1})$ with respect to axes $(t(I), n(I), b(I))$. This is essentially a frequency Vinogradov plank that we denote by $\theta_I$. Let us consider a partition of $\Gamma(R^{-1})$ by these $\theta_I$. Then we have the following wave packet decomposition at scale $R^{-\frac{1}{3}}$.

**Theorem 7.4** (Wave packet decomposition at scale $R^{-\frac{1}{3}}$). (Exercise 2.7 in [10], Theorem 7.4 in [14]) Assume that $F$ has Fourier transform supported in the anisotropic neighborhood $\Gamma(R^{-\frac{1}{3}})$. There is a decomposition

$$F = \sum_{\theta_I} \mathcal{P}_{\theta_I} F = \sum_{P \in \mathbb{P}(F)} F_P$$

where $F_P$ is a nonzero function and $\mathbb{P}(F)$ is a collection of spatial Vinogradov $(R^+, R^+, R)$-planks, such that

(P1) Each $\mathcal{F}_P$ is supported on $2\theta_I$ for some $I \in \mathbb{I}_{R^{-1/3}}$, and $P$ is associated with $I$.

We denote by $\mathbb{P}_I(F)$ the corresponding planks, so $\mathcal{P}_I F = \sum_{P \in \mathbb{P}_I(F)} F_P$.

(P2) $F_P$ is spatially concentrated near $P$, in the sense that for each $M \geq 1$

$$|F_P(x, y, z)| \lesssim_M \|F_P\|_\infty \chi_M^P(x, y, z).$$

Moreover, for each $p \geq 1$

$$\|F_P\|_p \sim \|F_P\|_\infty |P|^{1/p}.$$

(P3) For each $p \geq 2$ and each $\mathbb{P}_1 \subset \mathbb{P}_2 \subset \mathbb{P}_I(F)$ such that $\|F_P\|_\infty \sim \text{const}$ for $P \in \mathbb{P}_1$, we have

$$\left\| \sum_{P \in \mathbb{P}_1} F_P \right\|_{L^p(\mathbb{R}^3)} \lesssim \left\| \sum_{P \in \mathbb{P}_2} F_P \right\|_{L^p(\mathbb{R}^3)}.$$

(P4) For each $p \geq 2$ and each $P_1 \subset P_1(F)$ such that $\|F_p\|_\infty \sim \text{const}$ for $P \in P_1$, we have

$$\left\| \sum_{P \in P_1} F_P \right\|_{L^p(R^3)} \sim \left( \sum_{P \in P_1} \|F_P\|_{L^p(R^3)}^p \right)^{1/p}.$$ 

A fair enough representation of $F_P$ is

$$F_P(x, y, z) \approx A_P 1_P(x, y, z) e((x, y, z) \cdot (c, c^2, c^3))$$

where $c$ is some (any!) point in $I$ and $A_P \in \mathbb{C}$ is a constant such that $A_P \sim \|F_P\|_\infty$.

### 7.3 Decouplings

The following refinement of the $l^{12}(L^{12})$-decoupling for the cubic moment curve was proved in Theorem 7.5 of [14].

**Theorem 7.5** (Refined decoupling). Assume the function $F : \mathbb{R}^3 \to \mathbb{C}$ has the Fourier transform supported in $\Gamma(R^{-1/3})$. Let $Q$ be a collection of pairwise disjoint $R^{1/3}$-cubes $q$ in $\mathbb{R}^3$. For some $M \geq 1$, $P \in P(F)$ and $\Delta > 0$, assume that each $q$ intersects at most $M$ fat planks $R^\Delta P$ from different directions. Then for each $2 \leq p \leq 12$ and $\epsilon > 0$ we have

$$\|F\|_{L^p(\sum_{q \in Q} x_q)} \lesssim_{\Delta, \epsilon} R^\epsilon M^{1/2 - 1/p} \left( \sum_{P \in P(F)} \|F_P\|_{L^p(R^3)}^p \right)^{1/p}.$$ 

We will choose $\Delta$ very small depending on $\epsilon$.

Next, we also record the following flat decoupling that will be used to prove the trilinear-linear reduction in the next section. We can refer to Proposition 2.4 of [14].

**Theorem 7.6** (Flat decoupling). Let $B$ be a rectangular box in $\mathbb{R}^n$, and let $B_1, B_2, \ldots, B_L$ be a partition of $B$ into congruent boxes that are translates of each other. Then for each $2 \leq p \leq \infty$ and each $F \in L^p(\mathbb{R}^n)$ we have

$$\|P_B F\|_{L^p(\mathbb{R}^n)} \lesssim L^{1 - \frac{2}{p}} \left( \sum_{i=1}^L \|P_{B_i} F\|_{L^p(\mathbb{R}^n)}^p \right)^{\frac{1}{p}}.$$
8 Proof of the main theorem

Let us recall the main theorem.

**Theorem 8.1.** Assume that function $F : \mathbb{R}^3 \to \mathbb{C}$ has the Fourier transform supported in $N_1(R^{-1})$. Then for $2 \leq p \leq 10$, we have

$$\|F\|_{L^p(\mathbb{R}^3)} \lesssim \epsilon R^{\frac{1}{2} - \frac{1}{p} + \epsilon} \left( \sum_{J \in \mathcal{R}_{R^{-1}/2}} \|P_J F\|_{L^p(\mathbb{R}^3)}^p \right)^{\frac{1}{p}}.$$

The proof will follow in several steps. First, we prove the trilinear-to-linear reduction by the Bourgain–Guth method [8]. We show that the following trilinear estimate implies the main theorem when $p = 10$. Then, applying the decoupling interpolation in Exercise 9.21 of [10], we can prove the main theorem for the full range $2 \leq p \leq 10$.

8.1 Trilinear to linear reduction For each function $F : \mathbb{R}^3 \to \mathbb{C}$ let us write $F_1 = P_{[0,1/6]} F$, $F_2 = P_{[1/3,1/2]} F$, $F_3 = P_{[2/3,1]} F$.

**Theorem 8.2.** Assume that we have the following estimate for all functions $F : \mathbb{R}^3 \to \mathbb{C}$ with $\hat{F}$ supported in $N_{1/R}^\Gamma(R^{-1})$:

$$(14) \quad \|(F_1 F_2 F_3)^\frac{1}{3}\|_{L^{10}(\mathbb{R}^3)} \lesssim \epsilon R^{\frac{1}{2} - \frac{1}{p} + \epsilon} \left( \sum_{J \in \mathcal{R}_{R^{-1}/2}} \|P_J F\|_{L^{10}(\mathbb{R}^3)}^{10} \right)^{\frac{1}{10}}.$$

Then, this estimate implies Theorem 8.1.

**Proof.** Let $K > 100$ be the constant that will be chosen later and $l$ be the number such that $K^l = R^\frac{1}{4}$. Let $I$ be a dyadic interval contained in $[0, 1]$. We write $I_1 \sim I_2 \sim I_3$ if intervals $I_1, I_2, I_3$ are pairwise non-adjacent. We have the following elementary inequality with fixed constant $C = O(1)$:

$$|F(x)| \leq C \max_{|l| = K^{-1}} |P_l F(x)| + K^C \max_{l_1, l_2, l_3 \in \mathcal{R}_K} \|P_{l_1} F P_{l_2} F P_{l_3} F\|^\frac{1}{3}.$$

We iterate the first term $l$ times, raise to power 10 and integrate over $\mathbb{R}^3$

$$\|F\|_{L^{10}(\mathbb{R}^3)}^{10} \lesssim C^l \sum_{l \in \mathcal{R}_{R^{-1}/3}} \int_{\mathbb{R}^3} |P_l F(x)|^{10} dx$$

$$+ C^l K^C \sum_{R^{-1/3} \lesssim |\Delta| \lesssim 1} \sum_{l \in \mathcal{R}_K} \max_{l_1, l_2, l_3 \in \mathcal{R}_K} \int_{\mathbb{R}^3} |P_{l_1} F(x) P_{l_2} F(x) P_{l_3} F(x)|^{10} dx.$$
The first sum with $|I| = R^{-1/3}$ can be estimated by the flat decoupling Theorem 7.6,
\[ \|\mathcal{F}l\|_{L^1_{0}(\mathbb{R}^3)}^{10} \lesssim R^{\left(\frac{1}{2} - \frac{1}{3}\right)(1 - \frac{2}{7})} \sum_{J \in \mathbb{R}_{x=1/2}(I)} \|\mathcal{F}l\|_{L^1_{0}(\mathbb{R}^3)}^{10}. \]
Noting that $l \sim \frac{\log R}{\log K}$, we choose $K$ large enough so that $C^l \lesssim R^\epsilon$. Then the exponent of $R$ satisfies $(\frac{1}{2} - \frac{1}{3})(1 - \frac{2}{7}) < (\frac{1}{2})(\frac{1}{2} - \frac{1}{3})$. Thus the first sum estimate is safe enough.

Next we focus on the second sum. For each $I = [c, c + \Delta]$, we define the affine transformation $\tilde{T}_{\Delta,c}(w_1, w_2, w_3) = (w'_1, w'_2, w'_3)$ by
\[ \begin{cases} w'_1 = \frac{w_1 - c}{\Delta}, \\ w'_2 = \frac{w_2 - 2w_1 + c}{\Delta}, \\ w'_3 = \frac{w_3 - 3w_1 + 3c - c}{\Delta^3}. \end{cases} \]
Let $A_{\Delta,c}(x_1, x_2, x_3) = (x'_1, x'_2, x'_3)$ be the following linear map:
\[ \begin{cases} x'_1 = \Delta(x_1 + 2cx_2 + 3c^2x_3), \\ x'_2 = \Delta^2(x_2 + 3cx_3), \\ x'_3 = \Delta^3x_3. \end{cases} \]
Let $G$ be the function defined by $\tilde{G} = \mathcal{F}l \circ \tilde{T}_{\Delta,c}$ and let $G_i = \mathcal{F}l \circ \tilde{T}_{\Delta,c}$ for $i = 1, 2, 3$. Then $\tilde{G}$ is supported in $N_{\Gamma}(R^{-1})$ and the supports of $G_1, G_2, G_3$ are pairwise non-adjacent neighborhood of $\Gamma$ with length $\frac{1}{\Delta}$. Note that for $G$ and similarly for $G_1, G_2, G_3$, we have
\[ |G(x)| = |\mathcal{F}l(A_{\Delta,c}^{-1}x)|| \det A_{\Delta,c}^{-1}|. \]
Using this fact, we can observe the following:
\[ \int_{\mathbb{R}^3} |\mathcal{F}l \circ \mathcal{F}l \circ \mathcal{F}l \circ \mathcal{F}l|_{\mathbb{R}^3}^{10} dx = \int_{\mathbb{R}^3} |\mathcal{F}l \circ \mathcal{F}l \circ \mathcal{F}l \circ \mathcal{F}l|_{\mathbb{R}^3}^{10} (A_{\Delta,c}^{-1}x)| \det A_{\Delta,c}^{-1}| dx \]
\[ = \int_{\mathbb{R}^3} |G_1G_2G_3|_{\mathbb{R}^3}^{10} (x)| \det A_{\Delta,c}^{-1}| dx \]
\[ \lesssim \epsilon (R\Delta^3)^{\frac{9}{2} - \frac{1}{2}} R^{1/10 + \epsilon} \sum_{J \in \mathbb{R}_{x=1/2}(\tilde{I})} \|\mathcal{F}l \circ \mathcal{F}l \circ \mathcal{F}l \circ \mathcal{F}l\|_{L^1_{0}(\mathbb{R}^3)}^{10} \| \det A_{\Delta,c}^{-1} \|^9 \]
\[ = (R\Delta^3)^{\frac{9}{2} - \frac{1}{2}} R^{1/10 + \epsilon} \sum_{J \in \mathbb{R}_{x=1/2}(\tilde{I})} \|\mathcal{F}l \circ \mathcal{F}l \circ \mathcal{F}l \circ \mathcal{F}l\|_{L^1_{0}(\mathbb{R}^3)}^{10}. \]
For each interval $\tilde{J} \in \mathbb{R}_{x=1/2}(\tilde{J})$, we apply the flat decoupling Theorem 7.6 to decouple further to the intervals $J$ of length $R^{-\frac{1}{2}}$,
\[ \|\mathcal{F}l \circ \mathcal{F}l \circ \mathcal{F}l \circ \mathcal{F}l\|_{L^1_{0}(\mathbb{R}^3)}^{10} \lesssim (\Delta^{-\frac{1}{2}})^{(1 - \frac{2}{7})} R^{10} \sum_{J \in \mathbb{R}_{x=1/2}(\tilde{J})} \|\mathcal{F}l \circ \mathcal{F}l \circ \mathcal{F}l \circ \mathcal{F}l\|_{L^1_{0}(\mathbb{R}^3)}^{10}. \]
We combine the last two inequalities with $C^1 K^C \lesssim_\epsilon R^\epsilon$, and conclude the second sum estimate by

$$
R^{\frac{1}{2} \left( \frac{1}{2} - \frac{1}{10} \right) + \epsilon} \sum_{\Delta \in K^C} \Delta^{\frac{1}{2} + 10} \sum_{J \in \mathbb{I}_{R^{-1/2}}} \| \mathcal{P}_J F \|_{L^{10}(\mathbb{R}^3)}^{10} \lesssim_\epsilon R^{\frac{1}{2} \left( \frac{1}{2} - \frac{1}{10} \right) + \epsilon} \sum_{J \in \mathbb{I}_{R^{-1/2}}} \| \mathcal{P}_J F \|_{L^{10}(\mathbb{R}^3)}^{10}.
$$

This finishes the proof of trilinear to linear reduction. \qed

8.2 Wave packet decomposition. For a given ball $B_R$ of radius $R$ centered at $c$ we define the weight function on $B$ by

$$
w_B(x) = \frac{1}{(1 + \frac{|x-c|}{R})^{300}}.
$$

It will suffice to prove the following local version.

**Theorem 8.3.** Assume that the function $F : \mathbb{R}^3 \to \mathbb{C}$ has the Fourier transform supported in $\mathcal{N}_\Gamma(R^{-1})$. Then we have

$$
\|(F_1 F_2 F_3)^\frac{1}{3}\|_{L^{10}([-R,R]^3)} \lesssim_\epsilon R^{\frac{1}{2} \left( \frac{1}{2} - \frac{1}{10} \right) + \epsilon} \left( \sum_{J \in \mathbb{I}_{R^{-1/2}}} \| \mathcal{P}_J F \|_{L^{10}(\mathcal{W}_R)}^{10} \right)^{\frac{1}{10}}.
$$

We focus on proving the local trilinear version (15). It suffices to assume that $F = F_1 + F_2 + F_3$.

We will perform two rounds of wave packet decomposition at scales $R^{-\frac{1}{2}}$ and $R^{-\frac{1}{3}}$. It is worth recalling that $\mathcal{N}_\Gamma(R^{-1})$ is a subset of anisotropic neighborhood $\Gamma(R^{-1/3})$ introduced in Section 3. Therefore, functions with spectrum in the first set are subject to both decompositions from Section 7.2.

Several steps of partitioning and pigeonholing will follow after the wave packet decompositions.

**Wave packet decomposition at scale $R^{-1/2}$. The first pigeonholing sequence.** We can write

$$
F = \sum_{J \in \mathbb{I}_{R^{-1/2}}} \mathcal{P}_J F.
$$

By splitting $F$ into two parts, we may assume that there are no neighboring intervals $J$ in the sum. Then, we decompose $F$ into wave packets at scale $R^{-\frac{1}{2}}$ as in Theorem 7.3,

$$
F = \sum_{J \in \mathbb{I}_{R^{-1/2}}} \sum_{W \in \mathcal{W}_J(F)} F_W = \sum_{W \in \mathcal{W}(F)} F_W.
$$
The integration domain is $[-R, R]^3$, so we only consider the wave packets associated with plates $W \subset [-R, R]^3$. We can replace the domain $[-R, R]^3$ with $\mathbb{R}^3$ for the rest of the argument, and understand $\mathbb{W}(F)$ as consisting of only the plates $W \subset [-R, R]^3$. Recall that $W \in \mathbb{W}_J$ is a Vinogradov $(R^2, R, R)$-plate with normal vector $\mathbf{t}(J)$.

We partition the set $\mathbb{W}(F)$ by using dyadic parameters $w, n, X, m, l, Y$, into $O((\log R)^C)$ many collections $\mathbb{W}^{(i)}(F)$. We will always use $I$ to denote an element of $\mathbb{I}_{R^{-1/3}}$ and $J$ to denote an element of $\mathbb{I}_{R^{-1/2}}$. Also, we will drop the $F$ dependence and simply write $\mathbb{W}$ for $\mathbb{W}(F)$.

(1) Parameter $w$.

Let $w$ be a dyadic parameter in the range
\[
[R^{-1000} \max_{W \in \mathbb{W}} \|F_W\|_{\infty}, \max_{W \in \mathbb{W}} \|F_W\|_{\infty}].
\]
We partition the set of plates $\mathbb{W}$ into subcollections $\mathbb{W}_w$ so that within each $\mathbb{W}_w$ we have $\|F_W\|_{\infty} \sim w$. There are $O(\log R)$ many such subcollections. We discard all wave packets with weight $w < R^{-1000} \max_{W \in \mathbb{W}} \|F_W\|_{\infty}$, as they contribute negligibly to the $L^p$ norm of $F$.

Therefore, the function $F$ can be written as a sum of wave packets that we keep and a small error term whose contribution is negligible. We fix the parameter $w$ and apply the next pigeonholing step to $\mathbb{W}_w$. We will not change notation when we move to a new step, so we will continue to call $\mathbb{W}_w$ as $\mathbb{W}$.

(2) Parameters $n, X$: definition of $(n, X)$-heavy $J$.

For each $I \in \mathbb{I}_{R^{-1/3}}$, we tile $[-R, R]^3$ with $(R^2, R, R)$-fat plates $\Pi_I$ with normal vector $\mathbf{t}(I)$. Reasoning as in Lemma 7.2, we can assume that each $W \in \mathbb{W}_J$ with $J \subset I$ is uniquely contained in some $\Pi_I$. Each fat plate $\Pi_I$ contains $\lesssim R^2$ parallel plates $W \in \mathbb{W}_J$, for each $J \subset I$. Also, there are $\lesssim R^2$ parallel fat plates $\Pi_I$ inside $[-R, R]^3$.

For fixed dyadic parameters $1 \leq n \leq R^2$ and $1 \leq X \leq R^4$, we call an interval $J$ (subinterval of some $I$) $(n, X)$-heavy, if there are $\sim X$ boxes $\Pi_I$ each containing $\sim n$ plates $W_J$.

Of course, a given $J$ may be heavy with respect to more than one pair $(n, X)$. But at the end of this step, we fix this pair. In the next step, we only consider the heavy $J$ with respect to this pair. Also, $\mathbb{W}_J$ will next refer to the $\sim nX$ plates $W_J$ that contribute to it being $(n, X)$-heavy $J$ with respect to this pair. All other plates from $\mathbb{W}$—both those in $\mathbb{W}_J$ for non-heavy $J$, and those in $\mathbb{W}_J$ for a $(n, X)$-heavy $J$ but not among the special $\sim nX$ ones—will be discarded.

It is worth pointing out that the same $J$ may contribute to more than one collection $\mathbb{W}^{(i)}(F)$, but always with different plates for each collection.
(3) Parameter $m$: definition of heavy $I$.

We partition $\mathbb{I}_{R^{-1/3}}$ into $O(\log R)$ many collections, where each interval $I$ in the collection contains $\sim m (n, X)$-heavy $J \subset I$. We fix the dyadic number $1 \leq m \leq R^{\frac{1}{6}}$ and call $\mathbb{I}_{\text{heavy}}$ the collection of intervals $I$ corresponding to this $m$. All other intervals $I$ will be discarded.

We record the following lower bound for the main theorem

$$\left( \sum_{J \in \mathbb{I}_{k^{-1/2}}} \| P_J F \|_{L^{10}(\mathbb{R}^3)}^{10} \right)^{\frac{1}{10}} = \left( \sum_{I \in \mathbb{I}_{k^{-1/3}}} \sum_{J \in \mathbb{I}_{k^{-1/2}}} \| P_J F \|_{L^{10}(\mathbb{R}^3)}^{10} \right)^{\frac{1}{10}} \geq (|\mathbb{I}_{\text{heavy}}| m w^{10} n X |W|)^{\frac{1}{10}}.$$  \hfill (16)

Also, recall that the volume of the plate $W$ is $|W| \sim R^\frac{5}{3}$.

(4) Parameters $l, Y$: definition of $(n, l)$-heavy $\Pi_I$.

Let $I \in \mathbb{I}_{\text{heavy}}$. We will say that a $(n, X)$-heavy $J \subset I$ contributes to the fat plate $\Pi_I$ if $\Pi_I$ is one of the $\sim X$ boxes that contains $\sim n$ plates $W_J$. Each fat plate $\Pi_I$ can be contributed by at most $m (n, X)$-heavy intervals $J \subset I$. Let

$$1 \leq l \leq m$$

be a dyadic parameter. We split the family of fat plates $\Pi_I$ into $O(\log R)$ collections according to the number $\sim l$ of $(n, X)$-heavy intervals $J$ contributing to $\Pi_I$. We fix $l$, and call all corresponding $\Pi_I$ heavy.

Also, for a fixed dyadic parameter

$$1 \leq Y \leq R^{\frac{1}{6}}.$$

We only retain those $I \in \mathbb{I}_{\text{heavy}}$, for which there are $\sim Y$ heavy $\Pi_I$. We discard all other $I$ and continue to call the smaller collection $\mathbb{I}_{\text{heavy}}$. Also, for each $I$ in this new collection, we only keep those $\sim Y$ heavy boxes $\Pi_I$ that contribute, and call them $(n, l)$-heavy. For each $J$, we only keep those $W_J$ that sit inside one of the selected $\Pi_I$. We note the following simple inequality:

$$l Y \leq m X.$$  \hfill (17)

The pigeonholing is over. We write $(i) := (w, n, X, m, l, Y)$. We denote by $\mathbb{W}^{(i)}$ the collection of these surviving plates.

We write

$$F^{(i)} = \sum_{W \in \mathbb{W}^{(i)}} F_W,$$

so $F$ differs from $\sum_{i \leq 1} F^{(i)}$ by a negligible error term.
Since $F = F_1 + F_2 + F_3$, $\mathcal{W}(F)$ splits as a disjoint union of $\mathcal{W}(F_1)$, $\mathcal{W}(F_2)$, $\mathcal{W}(F_3)$. Thus, we also have the partition for $1 \leq j \leq 3$,

$$\mathcal{W}(F_j) = \bigcup_i (\mathcal{W}^{(i)}(F) \cap \mathcal{W}(F_j)) = \bigcup_i \mathcal{W}^{(i)}(F_j).$$

We can decompose each function $F_1$, $F_2$, $F_3$ as a sum of $O((\log R)^C)$ many restricted functions $F_1^{(i)}$, $F_2^{(i)}$, $F_3^{(i)}$ associated with the families of plates $W \in \mathcal{W}^{(i)}(F_1)$, $\mathcal{W}^{(i)}(F_2)$, $\mathcal{W}^{(i)}(F_3)$, respectively. It follows that

$$\| (F_1 F_2 F_3)^{1/3} \|_{L^p(\mathbb{R}^3)} \lesssim \sup_{i_1, i_2, i_3} \| (F_1^{(i_1)} F_2^{(i_2)} F_3^{(i_3)})^{1/3} \|_{L^p(\mathbb{R}^3)}.$$ 

It remains to estimate each term corresponding to a tuple $(i_1, i_2, i_3)$. To keep the notation simpler, we will consider the case $i_1 = i_2 = i_3 = i$. If the indices $i_1, i_2, i_3$ are different, we replace the estimates depending on index $i$ by three estimates, each depending on $i_1, i_2$ and $i_3$. Then, it suffices to use the geometric mean of these three estimates.

Let us also call $w, n, X, m, l, Y$ the parameters associated with $F^{(i)}$. We call $g_j = F_j^{(i)}$ for $j = 1, 2, 3$. Let us also call $g$ the restricted function $F_1^{(i)} + F_2^{(i)} + F_3^{(i)}$, $\mathcal{W}^{(i)}$ as $\mathcal{W}$ and $\mathcal{W}^{(i)}$ as $\mathcal{W}_j$.

Before we move on to the next wave packet decomposition, let us re-evaluate our goal. Recalling (15), we need to prove that

$$\| (g_1 g_2 g_3)^{1/3} \|_{L^{10}(\mathbb{R}^3)} \lesssim_{\epsilon} R^\frac{1}{3} (\frac{n}{m} - \frac{1}{10} + \epsilon) \left( \sum_{J \in |g|^{-1/2}} \| P_J F \|_{L^{10}(\mathbb{R}^3)}^{10} \right)^{\frac{1}{10}}.$$ 

In light of (16), this will follow if we prove that

$$\| (g_1 g_2 g_3)^{1/3} \|_{L^{10}(\mathbb{R}^3)} \lesssim_{\epsilon} R^\frac{1}{3} (\frac{n}{m} - \frac{1}{10} + \epsilon) (|W_{\text{heavy}}| mu^{10} n X |W|)^{\frac{1}{10}}.$$ 

**Wave packet decomposition at scale $R^{-1/3}$. The second pigeonholing sequence.**

Note that the function $g$ continues to have Fourier transform supported on (a slight enlargement of) $\mathcal{N}(R^{-1})$, due to (W1) in Theorem 7.3.

We can write (recall that the sum is in fact over $I \in \mathbb{N}_{\text{heavy}}$)

$$g = \sum_{l \in |g|^{-1/3}} P_l g.$$ 

We may assume that there are no neighboring intervals $I$ in the sum. We decompose $g$ into wave packets at scale $R^{-\frac{1}{3}}$, as in Theorem 7.4,

$$g = \sum_{P \in \mathcal{P}(g)} g_P.$$
Each $P$ is a $(R^{1/2}, R^{5/2}, R)$-Vinogradov plank in $[-R, R]^3$. We will partition the set of planks $\mathbb{P}(g)$ into $O((\log R)^C)$ many collections $\mathbb{P}_A(g)$, according to the dyadic parameters $A, N, Z_1, Z_2$.

(1) Parameter $A$.

We partition the set of planks $\mathbb{P}(g)$ into $O(\log R)$ many significant collections $\mathbb{P}_A$, with $\|g_P\|_\infty \sim A$ for all $P \in \mathbb{P}_A$. The dyadic parameter $A$ satisfies

$$R^{-1000} \max_{P \in \mathbb{P}(g)} \|g_P\|_\infty \leq A \leq \max_{P \in \mathbb{P}(g)} \|g_P\|_\infty.$$  

We fix such a collection $\mathbb{P}_A$, and move to the next step. Due to Schwartz tail considerations, we may assume that each plank $P_I \in \mathbb{P}_A$ (associated with some $I$) lies inside a $(n, l)$-heavy fat box $\Pi_I$, produced by the previous pigeonholing sequence.

(2) Parameter $N$: definition of $(N)$-heavy $\tau$.

We tile each $(n, l)$-heavy $\Pi_I$ by parallel $(R^{1/2}, R^{5/2}, R)$-boxes $\tau$. Each plank $P_I \in \mathbb{P}_A$ is uniquely contained in one of the boxes $\tau$ and each box $\tau$ can contain at most $R^\frac{1}{4}$ planks $P_I$. Let us partition the family of $\tau$ according to the dyadic parameter $1 \leq N \leq R^{\frac{1}{2}}$, so that $\tau$ in each subfamily contains $\sim N$ parallel planks $P_I \in \mathbb{P}_A$. We fix such $N$ and call the associated boxes $\tau$ $(N)$-heavy. We denote the family of all $(N)$-heavy $\tau$ as $\mathcal{T}$. We write down an explanation of the parameter $N$ for reference:

(19) \hspace{1cm} N := \text{the number of } (R^{1/2}, R^{5/2}, R)-\text{planks } P \text{ inside each } (R^{1/2}, R^{5/2}, R)-\text{box } \tau.

(3) Parameters $Z_1, Z_2$: definition of $(Z_1)$-heavy $\Sigma$, contributing $\Pi_I$ and contributing $I$.

We also tile each $(n, l)$-heavy $\Pi_I$ by parallel $(R^{1/2}, R^{5/2}, R)$-boxes $\Sigma$. Of course, each $(N)$-heavy $\tau$ is uniquely contained in some $\Sigma$. Note that the box $\Sigma$ includes at most $\sim R^{\frac{1}{4}}$ many $\tau$. For a dyadic parameter $1 \leq Z_1 \leq R^{\frac{1}{2}}$, we partition the family of $\Sigma$, so that $\Sigma$ in each subfamily contains $\sim Z_1$ $(N)$-heavy $\tau$. Fixing the parameter $Z_1$, the associated boxes $\Sigma$ will be called $(Z_1)$-heavy. We write

(20) \hspace{1cm} Z_1 := \text{the number of parallel } (R^{1/2}, R^{5/2}, R)-\text{boxes } \tau \text{ inside each } (R^{1/2}, R^{5/2}, R)-\text{box } \Sigma.

Note that each $(n, l)$-heavy $\Pi_I$ can contain at most $\sim R^{1/4}$ many boxes $\Sigma$. For the dyadic parameter $1 \leq Z_2 \leq R^{1/4}$, we partition the family of $(n, l)$-heavy $\Pi_I$ so that $\Pi_I$ in each subfamily contains $\sim Z_2$ $(Z_1)$-heavy $\Sigma$. We call such $\Pi_I$ contributing. Recall that for each $I \in \mathbb{I}_{\text{heavy}}$, there can be $\lesssim Y$ contributing fat plates $\Pi_I$. We call $I$ contributing if there is any contributing fat plate $\Pi_I$. We write the family of contributing intervals $I$ as $\mathbb{I}_{\text{contr}}$. So $|\mathbb{I}_{\text{contr}}| \leq |\mathbb{I}_{\text{heavy}}|$. Lastly, we record the parameter

(21) \hspace{1cm} Z_2 := \text{the number of parallel } (R^{1/2}, R^{5/2}, R)-\text{boxes } \Sigma \text{ inside each } (R^{1/2}, R, R)-\text{plate } \Pi.$
The second pigeonholing sequence is over. We only keep the planks $P \in \mathbb{P}_A$ that are contained in some $(N)$-heavy $\tau$, which itself is contained in a $(Z_1)$-heavy $\Sigma$, contained in some contributing $\Pi_I$. We write $(j) := (A, N, Z_1, Z_2)$ and call $\mathbb{P}(j)$ the collection of these planks. (See Figure 2.)

We write

$$g^{(j)} = \sum_{P \in \mathbb{P}(j)} g_P$$

so that $g$ is $\sum_j g^{(j)}$, apart from a negligible error. We can decompose each function $g_1, g_2, g_3$ as a sum of $O((\log R)^C)$ many restricted functions $g_1^{(j_1)}, g_2^{(j_2)}, g_3^{(j_3)}$ associated with the families of plates $P \in \mathbb{P}(j_1), \mathbb{P}(j_2), \mathbb{P}(j_3)$ respectively. We have as before

$$(22) \quad \|(g_1 g_2 g_3)^{1/3}\|_{L^p(\mathbb{R}^3)} \lesssim \sup_{j_1, j_2, j_3} \|(g_1^{(j_1)} g_2^{(j_2)} g_3^{(j_3)})^{1/3}\|_{L^p(\mathbb{R}^3)}.$$ 

Again we use the same index $j_1 = j_2 = j_3 = j$ and denote $g_1^{(j)} + g_2^{(j)} + g_3^{(j)}$ by $h$, $\mathbb{P}(j)$ by $\mathbb{P}$ and $\mathbb{P}(j)$ by $\mathbb{P}_I$. We also call $A, N, Z_1, Z_2$ the parameters associated with $g$. 

Figure 2.
From the last pigeonholing sequence we have

\[(23) \quad |\mathbb{P}| \lesssim |\mathbb{P}_{\text{contr}}|NZ_1Z_2Y \leq |\mathbb{P}_{\text{heavy}}|NZ_1Z_2Y.\]

This will be used in the proof of Theorem 8.6.

Note that the function \(h\) has Fourier transform supported on the anisotropic neighborhood of the cubic moment curve \(\Gamma(R^{-1/3})\), due (P1) in Theorem 7.4. Let us finish wave packet decomposition by writing (recall that only \(I \in \mathbb{P}_{\text{contr}}\) contribute to the summation)

\[h = \sum_{I \in \mathbb{P}_{\text{contr}}} P_{2I}h = \sum_{I \in \mathbb{P}_{\text{contr}}} \sum_{P \in \mathbb{P}_I} g_P.\]

We have finished introducing and pigeonholing parameters. Combining (18) and (22) our main Theorem 8.1 can be reduced to showing

\[(24) \quad \| (h_1h_2h_3)^{1/3} \|_{L^{10}(R^3)} \lesssim \epsilon R^{\epsilon} (1 - \frac{1}{2})^{1/4}\epsilon (|\mathbb{P}_{\text{heavy}}|mw^{10}nX|W|)^{1/4}.\]

In the next section, we prove this inequality using a two step decoupling approach.

### 8.3 Proof of the main theorem.

The following two propositions make use of the uniformity we obtained from pigeonholing.

The first one decouples intervals \(I\) into smaller intervals \(J\). This will be achieved by using \(L^2\) orthogonality, \(l^k(L^4)\) small cap decoupling and \(l^2(L^6)\) canonical scale decoupling for the parabola. This result offers the main connection between parameters from the two pigeonholing sequences.

**Proposition 8.4.** For the parameters \(w, n, l, A, N, Z_1, Z_2\) from the two pigeonholing sequences, we have the following inequality:

\[A \lesssim \epsilon R^{\epsilon} \min \left( \frac{wl^{1/2}R^{1/4}}{N^{1/2}}, \frac{wl^{2}R^{1/4}}{(NZ_1)^{1/2}}, \frac{wl^{2}n^{1/2}R^{1/4}}{(NZ_1Z_2)^{1/2}} \right).\]

The second proposition is an incidence estimate for Vinogradov planks \(P_I\) under spacing condition from the second pigeonholing sequence. It essentially amounts to decoupling into intervals \(I\) of canonical scale.

**Proposition 8.5.** Let \(\mathbb{P}\) be the collection of planks obtained at the end of the second pigeonholing sequence. Let \(Q_r(\mathbb{P})\) be the collection of \(r\)-rich \(R^{1/3}\)-cubes \(q\) with respect to \(\mathbb{P}\). Then for each \(1 \leq r \leq R^{1/3}\),

\[|Q_r(\mathbb{P})| \lesssim \epsilon \frac{|\mathbb{P}|N^3Z_1R^{2+\epsilon}}{r^3}.\]
Note that parameters $X$ and $Y$ do not appear in either proposition. They will however play a role in the proof of Theorem 8.6 below.

We postpone the proofs of both propositions for later. Let us now prove the reduced version (24) of the main theorem. We use “interpolation” (via Hölder’s inequality) of the trilinear $L^6$ restriction estimate and the refined $l^{12}(L^{12})$ decoupling, as shown in Proposition 8.7 of [14].

**Theorem 8.6.** We have
\[ \| (h_1 h_2 h_3) \|_{L^6(R^3)} \lesssim \epsilon R^{\frac{1}{8} - \frac{1}{4} + \epsilon} (\| \text{heavy} |mu|^{10} nX |W| )^{\frac{1}{8}}. \]

**Proof.** Let us denote $\mathbb{P}(h_1), \mathbb{P}(h_2), \mathbb{P}(h_3)$ by $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$ and $\bigcup_{k=1}^3 \mathbb{P}_k$ by $\mathbb{P}$. Let $1 \leq r_1, r_2, r_3 \leq R^{1/3}$ be dyadic parameters. Let $\mathcal{Q}_{r_1, r_2, r_3}$ be the collection of pairwise disjoint $R^d$-cubes $q$ that intersect $\sim r_1, \sim r_2$ and $\sim r_3$ planks from $\mathbb{P}_1, \mathbb{P}_2$ and $\mathbb{P}_3$ respectively. It suffices to prove that for each $1 \leq r_1, r_2, r_3 \leq R^d$, \[ \| (h_1 h_2 h_3) \|_{L^6(\mathcal{Q}_{r_1, r_2, r_3})} \lesssim \epsilon R^{\frac{1}{8} - \frac{1}{4} + \epsilon} (\| \text{heavy} |mu|^{10} nX |W| )^{\frac{1}{8}}. \]

This is because cubes $q$ intersecting at least one $P$ from each $\mathbb{P}_1, \mathbb{P}_2$ and $\mathbb{P}_3$ will be included in $\mathcal{Q}_{r_1, r_2, r_3}$ for some dyadic parameters $1 \leq r_1, r_2, r_3 \leq R^{1/3}$, while the contribution from those $q$ not intersecting any planks from one of $\mathbb{P}_1, \mathbb{P}_2$ or $\mathbb{P}_3$ can be considered negligible. Note that $\mathcal{Q}_{r_1, r_2, r_3} \subset \mathcal{Q}_{r_k}(\mathbb{P}_k)$ for $k = 1, 2, 3$.

Let us fix $r_1, r_2, r_3 \geq 1$. We exploit the trilinear setup only at this step to apply the $L^6$ restriction theorem for curves from [1]. Note that the functions $h_1, h_2, h_3$ have Fourier transform supported in the anisotropic neighborhood $\Gamma (R^{-1/4})$, which is contained in $\mathcal{N}_1 (R^{-1/4})$. Therefore, we can apply the trilinear $L^6$ restriction theorem for curves on each $q$,
\[ \| (h_1 h_2 h_3) \|_{L^6(q)} \lesssim \left( \left( \sum_{P \in \mathbb{P}_1} |g_P|^2 \right) \left( \sum_{P \in \mathbb{P}_2} |g_P|^2 \right) \left( \sum_{P \in \mathbb{P}_3} |g_P|^2 \right) \right)^{\frac{1}{6}}. \]

Summing this up for all cubes $q \in \mathcal{Q}_{r_1, r_2, r_3}$, we have
\[ \| (h_1 h_2 h_3) \|_{L^6(\mathcal{Q}_{r_1, r_2, r_3})} \lesssim \epsilon A \left( \left( \sum_{P \in \mathbb{P}_1} |\chi_P|^2 \right) \left( \sum_{P \in \mathbb{P}_2} |\chi_P|^2 \right) \left( \sum_{P \in \mathbb{P}_3} |\chi_P|^2 \right) \right)^{\frac{1}{6}} \lesssim \epsilon A (|\mathcal{Q}_{r_1}(\mathbb{P}_1)| |\mathcal{Q}_{r_2}(\mathbb{P}_2)| |\mathcal{Q}_{r_3}(\mathbb{P}_3)|)^{\frac{1}{6}} (R^{1+\epsilon} r_1 r_2 r_3)^{\frac{1}{6}}. \]
The last inequality used the fact that functions $\chi_P$ and $\chi_q$ decay rapidly outside the boxes $P$ and $q$. In order to simplify the notation, let us assume $r_1 = r_2 = r_3 = r$ and replace $Q_{r_1}(P_1)$, $Q_{r_2}(P_2)$, $Q_{r_3}(P_3)$, $Q_{r_1, r_2, r_3}$ with $Q_r(P)$. Then we can rewrite the last inequality as
\[ \lesssim_{\epsilon} A(|Q_r(P)| R^{1+\epsilon} r^3)^{1/5}. \]

Next, we apply the $l^{12}(L^{12})$ refined decoupling Theorem 7.5 to each $h_1, h_2$ and $h_3$.
\[ \| (h_1 h_2 h_3)^{1/2} \|_{L^{12}((\cup_{P_i} Q_i(P)) q)} \leq (\| h_1 \|_{L^{12}((\cup_{P_i} Q_i(P)) q)} \| h_2 \|_{L^{12}((\cup_{P_i} Q_i(P)) q)} \| h_3 \|_{L^{12}((\cup_{P_i} Q_i(P)) q)})^{1/3} \]
\[ \lesssim_{\epsilon} r^{5/2} R^{5/2} \left( \sum_{l \in \mathbb{Z}^{d-1/2}} \| \mathcal{P}_l h \|_{L^{12}(R^3)}^2 \right)^{1/5} \]
\[ \lesssim r^{5/2} R^{5/2} A(|P| R^2)^{1/5}. \]

For the last two sequences of inequalities, we have used that each $q \in Q_r(P)$ intersects at most $r$ planks in each of the families $P_1, P_2, P_3$. This is immediate, since each $P_\xi$ is a subset of $P$.

We combine these two inequalities via Hölder’s inequality,
\[ \| (g_1 g_2 g_3)^{1/2} \|_{L^{10}((\cup_{P_i} Q_i(P)) q)} \lesssim_{\epsilon} A R^{\epsilon} (|Q_r(P)| |R| R^3)^{3/10} (r^{1/2} (|P|^2 R^2)^{1/5}). \]

Therefore, it suffices to show that
\[ A(|Q_r| |R| R^3)^{3/10} (r^{1/2} (|P|^2 R^2)^{1/5}) \lesssim_{\epsilon} R^{\epsilon} r^{5/2} r^{5/2} (|P| R^2)^{1/5} (|P| |P|^2 R^2)^{1/5}. \]

Let us raise both sides to power 30, plug in $|P| \sim R^{3/2}$ and prove
\[ A^{30}(|Q_r| |R| R^3)(|10| |P|^2 R^4) \lesssim_{\epsilon} R^{30} w^{30} (|P| |P|^2 R^2)^{1/5}. \]

We apply the $A$-estimate from Proposition 8.4, the $|Q_r|$-estimate from Proposition 8.5 and the $|P|$-estimate in (23)
\[ A^{30}(|Q_r| |R| R^3)(|10| |P|^2 R^4) \]
\[ \lesssim_{\epsilon} R^{30} \left( \frac{w^{14} R^7 |Z|}{N^8} \right)^{1/8} \left( \frac{w^{14} R^7 |Z|}{N^8} \right)^{1/4} \left( \frac{w^{12} n^{1/2} R^{1/2}}{N^8} \right)^{18} \left( \frac{|P| N^5 R_1^2}{r^{13} |P|^2 R^5} \right) \]
\[ \lesssim_{\epsilon} R^{30} \left( \frac{14 R^7 |Z|}{N^8} \right)^{1/8} \left( \frac{14 R^7 |Z|}{N^8} \right)^{1/4} \left( \frac{n^{1/2} R^{1/2}}{N^8} \right)^{18} \left( \frac{|P| N^5 R_1^2}{r^{13} |P|^2 R^5} \right) \]
\[ = R^{30} w^{30} 14 R^7 |Z|^{1/8} N^{1/4} Y^{3/2} R^7. \]

Finally, we use (17), $l \leq R^{3/2}$, $r \leq R^{3/2}$:
\[ w^{30} 14 R^7 |Z|^{1/8} N^{1/4} Y^{3/2} R^7 \lesssim w^{30} 14 R^7 |Z|^{1/8} N^{1/4} Y^{3/2} R^7 \]
\[ \lesssim w^{30} |P|^{1/2} R^7. \]

This finishes the proof of the main theorem. \qed
8.4 Proof of Proposition 8.4. In this section we prove the inequality

\[ A \lesssim \epsilon R^e \min\left(\frac{wl^{\frac{1}{2}}R^{\frac{1}{2}}}{N^{\frac{1}{2}}}, \frac{wl^{\frac{1}{2}}R^{\frac{1}{2}}}{(NZ_1)^{\frac{1}{2}}}, \frac{wl^{\frac{1}{2}}n^{\frac{1}{2}}R^{\frac{1}{2}}}{(NZ_1Z_2)^{\frac{1}{2}}}\right). \]

**Proof.** We prove the three estimates using different methods. 

**L^2 orthogonality on \( \tau \): the proof of**

\[ A \lesssim \frac{wl^{\frac{1}{2}}R^{\frac{1}{2}}}{N^{\frac{1}{2}}}. \]

We choose one of the \((N)\)-heavy \((R_1^{\frac{1}{2}}, R_2^{\frac{1}{2}}, R)\)-boxes \( \tau \) selected in the second pigeonholing sequence. The box \( \tau \) is associated with some contributing \( I \in \mathbb{I}_{\text{contr}} \) and \( \tau \) is contained in some contributing (in particular also \((n, l)\)-heavy) fat plate \( \Pi_I \). Recall from (19) that there are \( \sim N \) parallel planks \( P_I \) inside the box \( \tau \).

We investigate \( \| P_2Ih \|_{L^2(\tau)} \) where \( P_2Ih = \sum_{P \in P_I} g_P \). Recall from (P2) in Theorem 7.4 that \( g_P \) decays rapidly outside of \( P \). Using the fact that planks \( P \) are parallel, we write

\[ \| P_2Ih \|_{L^2(\tau)}^2 = \int_{\tau} \left| \sum_{P \in P_I} g_P \right|^2 \geq \sum_{P \in P_I, P \subset \tau} \int_{\tau} \left| \sum_{P \in P_I} g_P \right|^2 w_{P}. \]

Invoking the locally constant property (Lemma 6.1 and Lemma 6.2 in [17]) and the geometric observation above, we write

\[ \sum_{P \in P_I, P \subset \tau} \int_{\tau} \left| \sum_{P \in P_I} g_P \right|^2 w_{P} \geq A^2 \sum_{P \in P_I, P \subset \tau} \int_{\tau} 1_P \sim A^2 NR^2. \]

We apply local \( L^2 \) orthogonality (Lemma 4.3 in [17]) on each \( R_1^{\frac{1}{2}} \)-cube \( \Delta \) contained in the box \( \tau \) to show

\[ \| P_2Ih \|_{L^2(\Delta)} \leq \| P_2Ig \|_{L^2(\Delta)} \lesssim \left( \sum_{J \in \mathbb{I}_{l/2}(I)} \| P_2Jg \|_{L^2(w_{J\Delta})}^2 \right)^{\frac{1}{2}}. \]

Sum up these estimates for all \( R_1^{\frac{1}{2}} \)-cubes \( \Delta \) contained in the box \( \tau \) to get

\[ \| P_2Ih \|_{L^2(\tau)} \lesssim \left( \sum_{J \in \mathbb{I}_{l/2}(I)} \| P_2Jg \|_{L^2(w_J)}^2 \right)^{\frac{1}{2}}. \]

Let us recall that the wave packet decomposition of the function \( P_2Ig \) is

\[ P_2Ig = \sum_{J \in \mathbb{I}_{l/2}(I)} P_2Jg = \sum_{J \in \mathbb{I}_{l/2}(I)} \sum_{W \in \mathbb{W}_J} F_W. \]
Since $\Pi_I$ is also $(n,l)$-heavy according to the first pigeonholing sequence, it is contributed by $\sim l$ many intervals $J \subset I$, and contains $\sim n$ many parallel plates $W$ for each of the contributing directions. For each direction $J$, the box $\tau$ intersects at most $O(1)$ plates $W$ and $\|F_W\|_\infty \leq w$. Thus we have

$$\left( \sum_{J \in I_{K^{-1/2}}(I)} \|\mathcal{P}_J g\|^2_{L^2(w_\tau)} \right)^{\frac{1}{2}} \lesssim w l^{\frac{1}{2}} |\tau|^{\frac{1}{2}} \sim w l^{\frac{1}{2}} R^\frac{15}{16}.$$  

Combining (25), (26) and the last two inequalities, we get

$$A(NR^2)^{\frac{1}{2}} \lesssim w l^{\frac{1}{2}} R^\frac{15}{16}.$$  

This finishes the proof of the first estimate.

**Small cap $l^4(L^4)$ decoupling on $\Sigma$: the proof of**

$$A \lesssim \epsilon R^e w l^{\frac{1}{2}} R^\frac{1}{5} (NZ_1)^{\frac{1}{2}}.$$  

Let us choose an arbitrary $(Z_1)$-heavy $(R^2, R^\frac{1}{2}, R)$-box $\Sigma$, selected in (20). Let $I = [c, c + R^{-\frac{1}{2}}]$ be the interval that $\Sigma$ is associated with, and let $\Pi_I$ be the contributing fat plate containing the box $\Sigma$ (so $\Pi_I$ is also $(n,l)$-heavy).

We can translate $\Pi_I$ to the origin and apply the linear transformation $A_{1,c}$. Then $I$ becomes $[0, R^{-\frac{1}{2}}]$ and we can use cylindrical decoupling (Exercise 9.22 of [10]). The Fourier transform of $\mathcal{P}_I g$ is supported in $N_{[0,R^{-1/2}]}(R^{-1})$. This neighborhood is essentially planar, it lies inside $\{(\xi, \xi^2, 0) : \xi \in [0, R^{-\frac{1}{2}}] + B(0, R^{-1})\}$. We apply planar parabolic rescaling $(\xi_1, \xi_2) \mapsto (R^4 \xi_1, R^4 \xi_2)$ to stretch the $I$ to $[0, 1]$, the intervals $J$ to intervals in $\mathbb{I}_{R^{-1/6}}$, and the $(R^{1/2}, R^{5/6})$ horizontal slice of $\Sigma$ into an $R^{1/6}$-square. We use $l^4(L^4)$ small cap decoupling (see Theorem 2.3 in [14]) to get

$$\|\mathcal{P}_I h\|_{L^4(\Sigma)} \lesssim \epsilon R^e \left( \sum_{J \in I_{K^{-1/2}}(I)} \|\mathcal{P}_J g\|^4_{L^4(w_\Sigma)} \right)^{\frac{1}{4}}.$$  

Since $\Pi_I$ is $(n, l)$-heavy, there are $\sim l$ contributing intervals $J \subset I$. Each $\Sigma$ intersects at most $O(1)$ plates $W$ from each contributing direction. Since the wave packet $|F_W|$ is essentially constant on the box $\Sigma$ and since $|\Sigma| \sim R^{7/3}$, we have

$$R^e l^{\frac{1}{2}} (\sum_{J \in I_{K^{-1/2}}(I)} \|\mathcal{P}_J g\|^4_{L^4(w_\Sigma)})^{\frac{1}{2}} \lesssim \epsilon R^e w(I R^\frac{1}{2})^{\frac{1}{2}} = w l^{\frac{1}{2}} R^\frac{1}{5}.$$  

On the other hand, the $(Z_1)$-heavy $\Sigma$ contains $\sim NZ_1$ many parallel planks $P_l$ following (19), (20) and each plank $P_l$ has the volume $\sim R^2$. Using the locally constant property, we have

$$\|\mathcal{P}_I h\|_{L^4(\Sigma)} \gtrsim A(NZ_1 R^2)^{1/4}.$$
We combine the last three inequalities and get
\[ A(NZ_1R^2)^{\frac{1}{2}} \lesssim \epsilon \omega l^{\frac{1}{2}} R^{\frac{5}{2} + \epsilon}. \]
This proves the second estimate.

\[ L^2(L^6) \text{ decoupling: the proof of} \]

\[ A \lesssim \epsilon R^{\frac{1}{2}} n^{\frac{1}{2}} R^{\frac{5}{12}} \frac{1}{(NZ_1 Z_2)^{\frac{1}{6}}}. \]

Let us choose a contributing fat plate \( \Pi_I \) for some \( I = [c, c + R^{-\frac{1}{3}}] \). We translate the fat plate \( \Pi_I \) to the origin and apply the linear transformation \( A_1, c. \) Then the corresponding interval becomes \([0, R^{-\frac{1}{3}}] \). We apply the parabolic rescaling \((\xi_1, \xi_2) \mapsto (R^{\frac{1}{3}} \xi_1, R^{\frac{1}{3}} \xi_2)\) as before, and use \( L^2(L^6) \) decoupling to get
\[
\| P_{2h} \|_{L^6(\Pi_I)} \lesssim \| P_{2g} \|_{L^6(\Pi_I)} \lesssim \epsilon R^{\frac{1}{2}} \left( \sum_{J \in I^{\frac{1}{2}}(I)} \| P_{2jg} \|_{L^6(w\Pi_I)}^{2} \right)^{\frac{1}{2}}.
\]

The fat plate \( \Pi_I \) contains \( \sim n \) many parallel plates \( W \in \mathbb{W}_J \) for each of the contributing \( \sim l \) directions \( J \subset I \). Since each plate \( W \) has a volume \( \sim R^{\frac{5}{2}} \) we have the following:
\[
\left( \sum_{J \in \mathbb{W}_J} \| P_{2jg} \|_{L^6(w\Pi_I)}^{2} \right)^{\frac{1}{2}} \lesssim \epsilon \omega l^{\frac{1}{2}} n^{\frac{1}{2}} R^{\frac{5}{12} + \epsilon}.
\]
On the other hand, since the contributing fat plate \( \Pi_I \) contains \( \sim NZ_1 Z_2 \) many planks \( P_I \) in line with (19), (20) and (21), we have
\[
\| P_{2h} \|_{L^6(\Pi_I)} \gtrsim A(NZ_1 Z_2 R^2)^{\frac{1}{2}}.
\]
Combining the last three inequalities, we have
\[
A(NZ_1 Z_2 R^2)^{\frac{1}{2}} \lesssim \epsilon \omega l^{\frac{1}{2}} n^{\frac{1}{2}} R^{\frac{5}{12} + \epsilon}.
\]
This finishes the proof of Proposition 8.4.

8.5 Proof of Proposition 8.5. We rewrite Proposition 8.5 as follows, spelling out explicitly the spacing conditions satisfied by \( \mathbb{P} \). We recall that for each \( I \in \mathbb{I}_{R^{-1/3}}, \) the \((R^\frac{1}{2}, R^\frac{5}{6}, R)\)-boxes \( \Sigma_I \) tile \([-R, R]^3 \). Also, each \( \Sigma_I \) is tiled with \((R^\frac{1}{2}, R^\frac{5}{6}, R)\)-boxes \( \tau_I \). Finally, each \( \tau_I \) is tiled with \((R^\frac{1}{2}, R^\frac{5}{6}, R)\)-Vinogradov planks \( P_I \).
Proposition 8.7. Let $\mathbb{P}$ be a collection of $(R^{1/3}, R^{2/3}, R)$-Vinogradov planks in $[-R, R]^3$. Assume the following two spacing conditions:

(Sp1) We call $\tau_I (N)$-heavy if it contains $\sim N$ planks $P_I$. We assume that for each $I$, all planks $P_I$ are subsets of $(N)$-heavy boxes $\tau_I$. That is, each $\tau_I$ is either $(N)$-heavy, or otherwise contains no plank $P_I$.

(Sp2) We call $\Sigma_I (Z_1)$-heavy if it contains $\sim Z_1$ heavy boxes $\tau_I$. We assume that for each $I$, all $(N)$-heavy $\tau_I$ are subsets of $(Z_1)$-heavy boxes $\Sigma_I$. That is, each $\Sigma_I$ is either $(Z_1)$-heavy, or otherwise contains no $(N)$-heavy $\tau_I$, so it also contains no planks $P_I \in \mathbb{P}$.

Let $Q_r(\mathbb{P})$ be the collection of $r$-rich $R^{1/3}$-cubes $q$ with respect to $\mathbb{P}$. Then for each $1 \leq r \leq R^{1/3}$ we have

$$|Q_r(\mathbb{P})| \lesssim \frac{|\mathbb{P}|N^5Z_1R^{2+\epsilon}}{r^7}.$$

Proof. In addition to intervals $I \in \mathbb{I}_{R^{-1/6}}$, we will also consider intervals $H \in \mathbb{I}_{R^{-1/6}}$.

We start with several pigeonholing steps that introduce additional structure to the argument. There will be various boxes, dyadic parameters and Kakeya estimates. The main idea is to replace planks $P$ with smaller planks $S$, with dimensions $(R^{1/3}, R^{2/3}, R^2)$, and to consider their incidences inside smaller cubes $Q$ with side length $R^{2/3}$.

We will first need to estimate the number of such planks $S$, in terms of the size $|\mathbb{P}|$ of the collection of large planks. This will be achieved in the first five steps of the argument, by using our earlier results on both tube and plate incidences. These results will be used again inside each $Q$, throughout Steps 6-11 of the argument. There is a finer localization to smaller $R^{1/2}$-cubes $\Delta$. The number of relevant $\Delta$ is counted using incidences for tubes, while the number of $r$-rich $q$ inside each $\Delta$ is counted using incidences for plates. In Step 12, these estimates are summed over all $\Delta$ and $Q$ inside $[-R, R]^3$.

(1) Replacing the planks $P$ with smaller planks $S$.

For each $H \in \mathbb{I}_{R^{-1/6}}$, we tile $[-R, R]^3$ with $(R^{1/3}, R^{2/3}, R)$-boxes $B = B_H$ with corresponding axes $(t(H), n(H), b(H))$. Computations similar to those from Lemma 4.4 and Lemma 7.2 show that for each $I \subset H$, each $\Sigma_I$, $\tau_I$ and $P_I$ are subsets of exactly one of the boxes $B_H$. In fact, the dimensions of $B_H$ are the smallest subject to this property. It is easy to check if dist $(H, H') \gg R^{-1/6}$, then the long side of the almost rectangular box $B_H \cap B_{H'}$ is $\ll R$, so no $P$, $\tau$ or $\Sigma$ would fit inside both boxes, since they all have long side equal to $R$. 


Referring back again to Lemma 4.4 with \( \rho = R^{-1/6} \), we tile each \( B_H \) with smaller planks \( S \) with dimension \( \sim (R^{1/3}, R^{2/3}, R^{2/3}) \) with the same axes as the box \( B_H \). It is immediate that \( R^{-1/6}S \) is a spatial Vinogradov plank associated with \( H \), and that \( R^{1/3}S \) has the same dimensions as \( B_H \). The dimensions and orientation of \( S \) guarantee that each \( P_I \) with \( I \subset H \) can be tiled with translates of \( S \). So we think about the intersection of any number of planks \( P_I \subset B_H \), for various \( I \subset H \), as being the union of pairwise disjoint \( S \). This is similar to understanding the intersection of congruent rectangles in \( \mathbb{R}^2 \) as being essentially a union of squares, whose side length equals the width of the rectangles.

Let us write the set of all small planks \( S \) in \([-R, R]^3 \) as \( \mathcal{S} \), and the set of all \( S \) contained in the box \( B \) as \( \mathcal{S}_B \).

(2) Pigeonholing the parameters \( E_1, E_2 \).

We partition the family of small planks \( \mathcal{S} \) according to the dyadic parameter \( E_2 \); the number of planks \( P \) that each \( S \) belongs to. We only count the planks \( P \) lying inside the same box \( B \) that \( S \) lies inside of. Note that \( E_2 \leq R^{1/3} \). There are \( \lesssim \log R \) such dyadic values of \( E_2 \), and we pick one value for \( E_2 \) in the next paragraph. We will call these small planks \( E_2 \)-planks and from now on \( \mathcal{S}, \mathcal{S}_B \) will refer to only the \( E_2 \)-planks.

Let \( \frac{\rho}{E_2 \log R} \leq E_1 \leq \frac{\rho}{E_2} \) be a dyadic parameter which measures the number of intervals \( H \) that contribute \( R^{1/3} \)-cubes \( q \in \Omega_\rho(\mathbb{P}) \). Note that \( E_1 \leq R^{1/3} \). We can transform the original problem “counting \( r \)-rich \( R^{1/3} \)-cubes \( q \) with respect to \( \mathbb{P} \)” to “counting \( E_1 \)-rich \( R^{1/3} \)-cubes \( q \) with respect to \( \mathcal{S} \)”. We write the family of \( E_1 \)-rich \( R^{1/3} \)-cubes \( q \) as \( \Omega_{E_1}(\mathcal{S}) \) and we choose \( E_2 \) and \( E_1 \) such that

\[
|\Omega_\rho(\mathbb{P})| \lesssim |\Omega_{E_1}(\mathcal{S})|.
\]

(3) Pigeonholing the parameter \( M_2 \) and the boxes \( \Lambda \).

Recall from (Sp1) that each \( P_I \) is a subset of a \((N)\)-heavy \( \tau \in \mathcal{T} \) with the same orientation. Thus the number of all \((N)\)-heavy \( \tau \subset [-R, R]^3 \) is

\[
|\mathcal{T}| \lesssim \frac{|\mathbb{P}|}{N}.
\]

Let us tile each \((R^{1/3}, R^{1/3}, R^{1/3})\)-box \( B_H \) with \((R^{1/3}, R^{1/3}, R^{1/3})\)-boxes \( \Lambda_H \) with the same axes. Note that three boxes \( B_H, R^{1/6} \Lambda_H \) and \( R^{1/3}S \) have the same dimensions. Boxes \( \Lambda_H \) arise as intersections of boxes \( \tau_I \) associated with various intervals \( I \subset H \). We can check this by applying the linear map in step 4 and Lemma 7.1.

Each \( E_2 \)-plank \( S \subset B_H \) can be assumed to be contained inside a unique \( \Lambda_H \). Only those \( \Lambda_H \) that contain at least one such \( S \) will be relevant to us. Note that each such \( \Lambda_H \) must be intersected by at least \( E_2 \) \((N)\)-heavy boxes \( \tau_I \) with \( I \subset H \).
This is because $S$ is intersected by $\sim E_2$ planks $P_I$, and each of these planks lives inside a different $\tau_I$. We classify the relevant boxes $\Lambda_H$ according to the dyadic parameter $M_2 \in [E_2, R^{1/6}]$ that counts the number of $(N)$-heavy $\tau_I$ intersecting $\Lambda_H$.

We fix $M_2$. We write the collection of $M_2$-rich boxes $\Lambda_H$ as $\Omega_{M_2}(\mathcal{J})$ and the collection of such $\Lambda_H$ living inside the box $B$ as $\Omega_{M_2,B}(\mathcal{J})$.

(4) Estimates for $\Omega_{M_2,B}(\mathcal{J})$ using Vinogradov tube incidences.

Let $H = [c, c + R^{-1/6}], I = [a, a + R^{-1/6}]$ and $I \subset H$. We fix a box $B = B_H$, and aim to estimate the number of $M_2$-rich boxes $\Lambda_H$ inside $B$. We translate $B$ to the origin, and apply the linear transformation $A_{R^{-1/6},c}$, where $A_{R^{-1/6},c}(x, y, z) = (x', y', z')$ is as follows:

$$
\begin{aligned}
&x' = (R^{-1/6})(x + 2cy + 3c^2z), \\
y' = (R^{-1/6})^2(y + 3cz), \\
z' = (R^{-1/6})^3z.
\end{aligned}
$$

Let us investigate the images of the boxes $B, \Sigma, \tau, \Lambda$ under this linear map. The image of $B$,

$$\bar{B} := A_{R^{-1/6},c}(B)$$

is an $R^{1/6}$-cube. The image of each $\Sigma_I, \bar{\Sigma}_I := A_{R^{-1/6},c}(\Sigma_I)$ is an $(R^{1/6}, R^{1/6}, R^{1/6})$-plate with normal vector $t(R^{1/6}(a - c))$ contained in the cube $\bar{B}$. The image of $\tau_I, \bar{\tau}_I := A_{R^{-1/6},c}(\tau_I)$ is a $(R^{1/6}, R^{1/6}, R^{1/6})$-Vinogradov tube at scale $R^{1/2}$ (according to Definition 4.2) with long edge in the direction $b(R^{1/6}(a - c))$, contained in one of $\bar{\Sigma}_I$. We note that intervals get stretched by $R^{1/6}$, so that $H$ becomes $[0, 1]$ and each $I$ becomes an interval $\bar{I}$ of length $R^{-1/6}$. So $\bar{\tau}_I$ is a Vinogradov tube associated with the interval $\bar{I}$, according to Definition 4.2. Lastly, the image of $\Lambda, \bar{\Lambda} := A_{R^{-1/6},c}(\Lambda)$ is an $R^{1/6}$-cube. (See Figure 3.)

Using this rescaling, the problem of estimating $|\Omega_{M_2,B}(\mathcal{J})|$ transformed into the problem of counting $R^{1/6}$-cubes which are $M_2$-rich with respect to the Vinogradov tubes $\bar{\tau}_I$ living inside the $R^{1/6}$-cube $\bar{B}$. We note that we can apply (the rescaled $R \mapsto R^{1/2}$ version of the) Theorem 4.3, with $[-R, R]^3$ replaced with $\bar{B}$, $\Sigma_I$ replaced with $\bar{\Sigma}_I$ and tubes replaced with $\bar{\tau}_I$. Also, we replace the parameter $N$ in Theorem 4.3 with $Z_1$, following the spacing condition (Sp2).

Thus we estimate the number of $M_2$-rich boxes $\Lambda$ inside the box $B$ as follows:

$$
|\Omega_{M_2,B}(\mathcal{J})| \lesssim \epsilon \frac{|\{ (N)\text{-heavy} : \tau \subset B \}|Z_1R^{1/6+\epsilon}}{M_2^2}.
$$

(5) Counting small planks $S$ inside each $\Lambda$ by means of Vinogradov plate incidences.
Let $H = [c, c + R^{-\frac{1}{6}}], I = [a, a + R^{-\frac{1}{6}}]$ and $I \subset H$. Recall that each $\Lambda \in Q_{M_2}(I)$ intersects $\sim N$ planks $P_I$ for each of $\sim M_2$ many intervals $I$. This is since $\Lambda$ must (due to dimensional considerations) intersect all $N$ planks $P_I$ contained in each $(N)$-heavy $\tau_I$ that intersects (in fact contains) $\Lambda$. Let us translate each $\Lambda \in Q_{M_2}(I)$ to the origin and apply the same linear transformation $AR^{-1/6,c}$ as in step 4. First, the image of the box $\Lambda$, $\tilde{\Lambda} = AR^{-1/6,c}(\Lambda)$ is an $R^{\frac{1}{6}}$-cube. Next, the image of $P \cap \Lambda$, which we write as $\tilde{P}$, is an $(R^{\frac{1}{6}}, R^{\frac{1}{6}}, R^{\frac{1}{6}})$-Vinogradov plate (see Definition 6.1), with normal vector $t(R^{\frac{1}{6}}(a - c))$, contained inside the $R^{\frac{1}{6}}$ cube $\tilde{\Lambda}$. Lastly, the image of $S$, $\tilde{S} = AR^{-1/6,c}(S)$ is an $R^{\frac{1}{6}}$ cube. (See Figure 4.)

Therefore, the problem of counting $E_2$-planks $S$ inside $\Lambda$ can be transformed into the problem of counting $E_2$-rich cubes $\tilde{S}$ inside the cube $\tilde{\Lambda}$. Note that inside the cube $\tilde{\Lambda}$, there are $\sim N$ many Vinogradov plates $\tilde{P}$ from each of $\sim M_2$ contributing directions. Thus, we can use Theorem 6.3 with $\delta = R^{-\frac{1}{6}}$. We estimate the number of $E_2$-rich planks $S$ inside each $M_2$-rich $\Lambda$ as follows

\begin{equation}
|\{ S \text{ is } E_2 \text{-rich : } S \subset \Lambda \} | \lesssim \frac{N^3 M_2(R^{\frac{1}{6}})^3}{E_2^4}.
\end{equation}

At this point, combining (28), (29), (30) and adding up all the $|S_B|$ estimates, we
get the following $|S|$ estimate in $[-R, R]^3$:

$$
|S| = \sum_{B \subseteq [-R, R]^3} |S_B| \lesssim \sum_{B \subseteq [-R, R]^3} \sum_{\Lambda \in \mathcal{U}_{M_2, \beta}(J)} |\{ S \text{ is } E_2 - \text{rich} : S \subset \Lambda \}| \\
\lesssim \sum_{B \subseteq [-R, R]^3} \sum_{\Lambda \in \mathcal{U}_{M_2, \beta}(J)} \frac{N^3 M_2(R^\frac{1}{2})^3}{E_2^4} \\
\lesssim \epsilon \sum_{B \subseteq [-R, R]^3} \frac{|\{ \tau (N)-\text{heavy} : \tau \subset B \}| Z_1 R^\frac{1}{2} + \epsilon}{M_2^2} \frac{N^3 M_2(R^\frac{1}{2})^3}{E_2^4} \\
\lesssim \frac{|P| Z_1 R^\frac{1}{2} + \epsilon}{M_2^2} \frac{N^3 M_2(R^\frac{1}{2})^3}{E_2^4} = R^\epsilon |P| N^2 Z_1(R^\frac{1}{2})^4 \frac{M_2 E_2^4}{M_2 E_2^4}.
$$

We are half way through the argument. All further boxes will be localized inside smaller cubes $Q$ with side length $R^{2/3}$. Note that each $S$ fits into one of these cubes. The next few steps produce estimates for the number of incidences between small planks $S$ inside such a cube.

(6) Pigeonholing the parameter $U_1$: the tubes $T$.

For each $H \in \mathbb{T}_{R, \epsilon}$, we tile $[-R, R]^3$ with $(R^\frac{1}{2}, R^\frac{1}{2}, R^\frac{1}{2})$-tubes $T$ with direction $b(H)$. Note from Definition 4.2 that $R^{-1/6} T$ is a Vinogradov tube at scale $R^{1/2}$ associated with $H$. We can assume that each small plank $S$ is uniquely contained
in one of the tubes. Let us call the set of $T$ as $\mathbb{T}$ and partition $\mathbb{T}$ according to the dyadic parameter $U_1$, such that each tube in the family contains $\sim U_1$ small $E_2$-planks $S \in \mathbb{S}$. We fix the parameter $U_1 \leq R_1^{\frac{1}{2}}$ and note that $|\mathbb{T}| \lesssim \frac{|\mathbb{S}|}{U_1}$. (See Figure 5.)

(7) Plates $\Phi$ and a bound for $U_2$ via $L^2$ Kakeya for plates.

We tile $[-R, R]^3$ with cubes $Q$ with side length $R_2^{\frac{1}{3}}$. For each $H \in \mathbb{R}^R$, we tile each $Q$ with $(R_1^{\frac{1}{3}}, R_2^{\frac{1}{3}}, R_2^{\frac{1}{3}})$-plates $\Phi_H$ with normal vector $t(H)$. Each tube $T_H \in \mathbb{T}$ associated with $H$ is uniquely contained in one of the plates $\Phi_H$. For each plate $\Phi_H$, we will bound the number $U_2$ of tubes $T_H$ they contain. Recall that the plate $\Phi_H$ containing $T_H$ should intersect $\sim N$ planks $P_I$, for each of $\sim M_2$ contributing $I \subset H$. The plank $P_I$ truncated to $\Phi$ is an $(R_1^{1/3}, R_2^{2/3}, R_2^{2/3})$-Vinogradov plate associated with $I$. We denote such a plate by $\bar{P}_I$. The intersections of such plates $\bar{P}_I$ for $I \subset H$ are unions of small planks $S_H$ (see Lemma 7.2 for the proof).
We combine Chebyshev’s inequality with Theorem 6.2 to get
\[ U_1 U_2 E_2^2 |S_H| \leq \| \sum_{P_{i} \subset \Phi_{H}} 1_{P_{i}} \|_2^2 \lesssim N \left( \sum |P_{i}| \right). \]

The term \( U_1 U_2 \) on the left represents the number of \( E_2 \)-rich small planks \( S_H \) inside \( \Phi_{H} \). We conclude that
\[ (32) \quad U_1 U_2 \lesssim \frac{N^2 M_2 R_{\frac{1}{2}}^6}{E_2^2}. \]

We fix the parameter \( U_2 \), and the corresponding family of plates \( \Phi \).

(8) An upper bound for \( U_1 \) via a double counting argument.

Each plate \( \Phi_{H} \) can intersect \( \sim N \) planks \( P_{i} \) from each of \( M_2 \) contributing directions \( I \subset H \). Each tube \( T_{H} \) inside \( \Phi_{H} \) contains \( U_1 \) many \( E_2 \)-rich planks \( S_{H} \). Therefore, the tube \( T_{H} \) intersects at least \( U_1 E_2 \) many planks \( P_{i} \), since each \( P_{i} \) can intersect at most \( O(1) \) smaller planks \( S_{H} \subset T_{H} \). We conclude that
\[ (33) \quad U_1 E_2 \lesssim N M_2. \]

(9) Pigeonholing the parameter \( M_1 \).

Let us tile each \( Q \) with \( R_{\frac{1}{3}} \)-cubes \( \Delta \). Each small \( R_{\frac{1}{3}} \)-cube \( q \) in \( Q_{E_1}(S) \) lies inside one of \( \Delta \). We will only focus on those \( \Delta \) containing at least one \( q \in Q_{E_1}(S) \).

The cubes \( \Delta \) are the typical intersections between the tubes \( T \). Since each \( q \) is \( E_1 \)-rich with respect to the family of small planks \( S \), and since each \( S \) lies in some tube \( T \), it follows that \( \Delta \) can be assumed to be \( M_1 \)-rich with respect to the family \( T \), for some \( M_1 \geq E_1 \). There are \( \lesssim \log R \) choices of parameter \( M_1 \leq R_{\frac{1}{2}} \).

For fixed \( M_1 \geq E_1 \), we denote the collection of \( M_1 \)-rich cubes \( \Delta \) by \( Q_{M_1}(T) \).

(10) Counting \( M_1 \)-rich \( R_{\frac{1}{2}} \)-cubes \( \Delta \) using Vinogradov tube incidences.

Let us fix an \( R_{\frac{1}{2}} \)-cube \( Q \). For each \( H \), each plate \( \Phi_{H} \) inside \( Q \) contains at most \( U_2 \) many tubes \( T_{H} \). Writing the family of \( M_1 \)-rich \( \Delta \) inside \( Q \) as \( Q_{M_1}(T) \), we get the following estimate by using Theorem 4.3:
\[ (34) \quad |Q_{M_1,Q}(T)| \lesssim \epsilon \frac{|\{ T \in T : T \subset Q \}| U_2 R_{\frac{1}{2}+\epsilon}}{M_1^2}. \]

(11) Counting \( E_1 \)-rich \( R_{\frac{1}{2}} \)-cubes \( q \) inside each \( \Delta \) by Vinogradov plate incidences.

The cube \( \Delta \in Q_{M_1,Q}(T) \) intersects \( \sim U_1 \) many \( E_2 \)-planks \( S \) from each of the \( \sim M_1 \) contributing directions. Let us call \( \bar{S} = S \cap \Delta \). This is a rescaled Vinogradov plate. Therefore, we can use Theorem 6.3 as before to get
\[ (35) \quad |\{ q \in Q_{E_1}(S) : q \subset \Delta \}| \lesssim \frac{U_1^3 M_1(R_{\frac{1}{2}})^3}{E_1^4}. \]
At this point, combining the estimates (34) and (35), we sum up

$$|\{ q \in \Omega_{E_i(S)} : q \subset Q \}|$$

over the cubes $Q \subset [-R, R]^3$ to get the estimate for $|\Omega_{r}(P)|$:

$$|\Omega_{r}(P)| \lesssim |\Omega_{E_i(S)}| = \sum_{Q \subset [-R, R]^3} |\{ q \in \Omega_{E_i(S)} : q \subset Q \}|$$

$$\lesssim \sum_{Q \subset [-R, R]^3} \sum_{\Delta \in \Omega_{M_1}(T)} |\{ q \in \Omega_{E_i(S)} : q \subset \Delta \}|$$

$$\lesssim \sum_{Q \subset [-R, R]^3} \sum_{\Delta \in \Omega_{M_1}(T)} \frac{U_3^3 M_1(\frac{R}{\sqrt{r}})^3}{E_1^4}$$

(36)

$$\lesssim \epsilon \sum_{Q \subset [-R, R]^3} \frac{|T \in \mathbb{T} : T \subset Q| U_2 R^{\frac{1}{2} + \epsilon} U_3 M_1(\frac{R}{\sqrt{r}})^3}{M_1^2}$$

$$\lesssim R^\epsilon \frac{|\mathbb{T}| U_1 N^3 M_2^2(\frac{R}{\sqrt{r}})^5}{M_1 E_1^3 E_2^3}.$$

The last inequality follows from (32) and (33).

(12) Reaching the final estimate.

Recall that $|\mathbb{T}| \lesssim \frac{|S|}{U_1}$. Combining the last inequalities of (31) and (36), we get the following final estimate

$$|\Omega_{r}(P)| \lesssim R^\epsilon \frac{|S| N^3 M_2^2(\frac{R}{\sqrt{r}})^5}{M_1 E_1^3 E_2^3} \lesssim R^\epsilon \frac{|P| N^2 Z_1(\frac{R}{\sqrt{r}})^4 N^3 M_2^2(\frac{R}{\sqrt{r}})^5}{M_1 E_1^4 E_2^4}$$

$$= R^\epsilon \frac{|P| N^2 Z_1 R^2 M_2}{r^4 M_1 E_2^3}$$

$$= R^\epsilon \frac{|P| N^2 Z_1 R^2 M_2 E_1^3}{r^4 M_1 E_2^3 E_1^3}$$

$$\lesssim R^\epsilon \frac{|P| N^5 Z_1 R^{2+\epsilon}}{r^7}.$$

The last inequality follows from $r \approx E_1 E_2, E_1 \leq M_1, E_1 \leq R^\frac{1}{2}, M_2 \leq R^\frac{1}{2}$. This finishes the proof of Proposition 8.5. ∎
SHARP $L^1$ DECOUPLING

REFERENCES

[1] J. Bennett, A. Carbery and T. Tao, On the multilinear restriction and Kakeya conjectures, Acta Math. 196 (2006), 261–302.

[2] J. Bourgain, Decoupling inequalities and some mean-value theorems, J. Anal. Math. 133 (2017), 313–334.

[3] J. Bourgain, Decoupling, exponential sums and the Riemann zeta function J. Amer. Math. Soc. 30 (2017), 205–224.

[4] J. Bourgain and C. Demeter, The proof of the $l^2$ decoupling conjecture, Ann. of Math. (2) 182 (2015), 351–389.

[5] J. Bourgain and C. Demeter, Decouplings for surfaces in $\mathbb{R}^4$, J. Funct. Anal. 270 (2016), 1299–1318.

[6] J. Bourgain and C. Demeter, Decouplings for curves and hypersurfaces with nonzero Gaussian curvature, J. Anal. Math. 133 (2017), 279–311.

[7] J. Bourgain, C. Demeter and L. Guth, Proof of the main conjecture in Vinogradov’s mean value theorem for degrees higher than three, Ann. of Math. (2) 184 (2016), 633–682.

[8] J. Bourgain and L. Guth, Bounds on oscillatory integral operators based on multilinear estimates, Geom. Funct. Anal. 21 (2011), 1239–1295.

[9] L. Brandolini, G. Gigante, A. Greenleaf, A. Iosevich, A. Seeger and G. Travaglini, Average decay estimates for Fourier transforms of measures supported on curves, J. Geom. Anal. 17 (2007), 15–40.

[10] C. Demeter, Fourier Restriction, Decoupling and Applications, Cambridge University Press, Cambridge, 2020.

[11] C. Demeter, private communication.

[12] C. Demeter and S. Guo, unpublished work.

[13] C. Demeter, S. Guo and F. Shi, Sharp decouplings for three dimensional manifolds in $\mathbb{R}^5$, Rev. Mat. Iberoam. 35 (2019), 423–460.

[14] C. Demeter, L. Guth and H. Wang, Small cap decouplings, Geom. Funct. Anal. 30 (2020), 989–1062.

[15] S. W. Drury, Restrictions of Fourier transforms to curves, Ann. Inst. Fourier (Grenoble) 35 (1985), 117–123.

[16] L. Guth, N. Solomon and H. Wang, Incidence estimates for well spaced tubes, Geom. Funct. Anal. 29 (2019), 1844–1863.

[17] L. Guth, H. Wang and R. Zhang, A sharp square function estimate for the cone in $\mathbb{R}^3$, Ann. of Math. (2) 192 (2020), 551–581.

[18] C. Oh, Small cap decoupling inequalities: bilinear methods, Rev. Mat. Iberoam. 38 (2022), 33–52.

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