RECENT DEVELOPMENTS IN TORIC GEOMETRY

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ABSTRACT. This paper will survey some recent developments in the theory of toric varieties, including new constructions of toric varieties and relations to symplectic geometry, combinatorics and mirror symmetry.

INTRODUCTION

A toric variety over $\mathbb{C}$ is a $n$-dimensional normal variety $X$ containing $(\mathbb{C}^*)^n$ as a Zariski open set in such a way that the natural action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on $X$. This seemingly simple definition leads to a fascinating combinatorial structure and some surprisingly rich mathematics. In this article, we will discuss some recent developments in toric varieties, including novel applications and new foundations for the entire theory.

Toric varieties were discovered in the early 1970's independently by several groups of people. From the beginning, the theory of toric varieties has led to some notable applications, including the following:

- The characterization of algebraic subgroups of maximal rank of Cremona groups, in Demazure's 1970 paper [Dem].
- The stable reduction theorem, in the 1973 book [KKMS] by Knudsen, Kempf, Mumford and Saint-Donat.
- The construction of nice (meaning “toroidal”) compactifications of discrete quotients of bounded symmetric domains, begun in the 1973 paper [Sat] by Satake and the 1975 book [AMRT] by Ash, Mumford, Rapaport and Tai.
- The rich connections between Newton polytopes, toric varieties and singularities, first explored by Kushnirenko [Kus] in 1976 and Khovanskii [Kho] in 1977.
- The use of Hard Lefschetz for simplicial toric varieties to prove McMullen’s conjectures for the number of vertices, edges, faces, etc. of convex simplicial polytopes, in Stanley’s 1980 paper [Sta].

This brief list, of course, does not do justice to the work of many other people who have written about toric varieties.

Besides these applications of toric varieties, many people have come to the realization, as noted by Fulton in his 1993 book [Ful], that “toric varieties have provided a remarkably fertile testing ground for general theories.” An example of this philosophy can be found in Reid’s 1983 paper [Rei2] which studies Mori theory in the context of toric varieties.

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The theory of toric varieties has also had the benefit of some superb exposition. We’ve already mentioned Fulton’s book [Ful], and two other classic references in the field are Danilov’s 1978 survey [Dan2] and Oda’s 1988 book [Oda2]. We warmly recommend these to anyone who wants to learn more about this fascinating part of algebraic geometry.

Although the classical theory seems fairly complete, the last few years have seen an explosion of new ideas and tools for studying toric varieties as well as significant new applications and relations to other areas of mathematics (and physics!). This paper will survey some of these new developments. We begin in §1 with a review of the notation and terminology we will use. Then §2 introduces a new construction of toric varieties, similar to way that $\mathbb{P}^n$ is realized as a quotient of $\mathbb{C}^{n+1} - \{0\}$. It follows that simplicial toric varieties have “homogeneous coordinates” which enable us to define subvarieties by global equations. In §3, we briefly discuss the Kähler cone of a toric variety. Then §4 describes a related construction of smooth simplicial toric varieties coming from symplectic geometry.

In §5, we discuss a yet another method for constructing toric varieties which has gained prominence recently. Here, a toric variety is defined to be the closure of an equivariant map from a torus. Such toric varieties need not be normal, which gives the theory a slightly different flavor. Then §6 shows how classical results of Griffiths on the cohomology of projective hypersurfaces can be generalized to the toric case. The secondary fan is the subject of §7. This fan, which has some remarkable applications, begins with a collection of rays and asks how many fans have these rays as their 1-dimensional cones. In §8 we will discuss reflexive polytopes and Calabi-Yau hypersurfaces, and then §9 will touch briefly on the work of Gelfand, Kapranov and Zelevinsky on resultants, discriminants and hypergeometric equations.

Mirror symmetry is the topic of §10. This field represents a fascinating interaction between mathematics and physics. Toric geometry plays a prominent role in many mirror symmetry constructions, and there are even physical theories specially designed for toric varieties. In particular, we will see that mirror symmetry makes use of virtually everything in §§2–9. Finally, §11 will discuss some other research being done on toric varieties.

In this survey, we will assume that the reader is familiar with basic theory of toric varieties as presented in [Dan2], [Ful] or [Oda2]. For simplicity, we will usually work over the complex numbers $\mathbb{C}$.

To keep the bibliography from getting too large, references to some topics are not complete—it was often more convenient to refer to later papers rather than the original ones. A fuller picture of recent work on toric varieties can be obtained by checking the references in the papers mentioned in this survey. The reader may also want to consult the 1989 toric survey of Oda [Oda3], which has an extensive bibliography.

§1. Notation and terminology for toric varieties

The basic combinatorial object associated to a toric variety is a fan. One starts with an integer lattice $M \cong \mathbb{Z}^n$ and its dual lattice $N$. Then a fan $\Sigma$ in $N_\mathbb{R} = N \otimes \mathbb{R}$ consists of a finite collection of strongly convex rational polyhedral cones $\sigma \subset N_\mathbb{R}$ which is closed under intersection and taking faces. The 1-dimensional cones $\rho$ play a prominent role in the theory, and it is customary to denote the unique generator of $\rho \cap N$ by the same letter $\rho$. Finally, we say that $\Sigma$ is simplicial if the minimal
The classical construction. Given a fan $\Sigma$, each cone $\sigma \in \Sigma$ has a dual cone $\sigma^\vee = \{ v \in M_\mathbb{R} : \langle v, \sigma \rangle \geq 0 \}$, which determines the semigroup algebra $\mathbb{C}[\sigma^\vee \cap M]$. In the classical formulation of the theory, the toric variety $X = X_\Sigma$ is obtained from $\Sigma$ by gluing together the affine toric varieties $X_\sigma = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ for $\sigma \in \Sigma$. We will see in §2 that there are now other ways to construct $X$ from $\Sigma$.

The torus $T = N \otimes \mathbb{C}^*$ sits inside $X$ as described in the introduction. Hence $N$ gives the 1-parameter subgroups of $T$ and $M$ is the character group. So each $m \in M$ gives $\chi^m : T \rightarrow \mathbb{C}^*$, which can be regarded as a rational function on $X$.

Although a toric variety $X$ can be singular, it is always Cohen-Macaulay (this is proved in [Hoc] and [Dan2]), so that duality theory works nicely. In particular, the dualizing sheaf $\omega_X$ coincides with the sheaf $\Omega_X^n$ of Zariski $n$-forms on $X$. An especially nice case is when $X$ is simplicial (meaning that $\Sigma$ is simplicial). In this case, $X$ is a $V$-manifold and for many purposes (including cohomology over $\mathbb{Q}$ and Hodge theory) behaves like a manifold.

The role of polyhedra. One way to see the connection with polyhedra is via divisors on $X$. Each 1-dimensional cone $\rho \in \Sigma$ corresponds to a Weil divisor $D_\rho \subset X$. A divisor $D = \sum_\rho a_\rho D_\rho$ gives a (possibly unbounded) convex polyhedron

$$\Delta_D = \{ m \in M_\mathbb{R} : \langle m, \rho \rangle \geq -a_\rho \} \subset M_\mathbb{R}. \tag{1.1}$$

To see how $\Delta_D$ relates to $D$, consider the reflexive sheaf $\mathcal{O}_X(D)$ whose sections over $U \subset X$ are those rational functions $f$ such that $\text{div}(f) + D \geq 0$ on $U$. Then

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{m \in \Delta_D \cap M} \mathbb{C} \cdot \chi^m. \tag{1.2}$$

When $D = \sum_\rho a_\rho D_\rho$ is Cartier, there is a support function $\psi_D$ with the property that for each $\sigma \in \Sigma$, there is $m_\sigma \in M$ such that $\psi_D(\rho) = \langle m_\sigma, \rho \rangle = -a_\rho$ for all $\rho \subset \sigma$. In particular, $\psi_D$ is linear and integral on each cone in $\Sigma$.

When $X$ is complete, a classic fact is that $D$ is ample if and only if $\psi_D$ is strictly convex. In this case, $\Delta_D$ is a $n$-dimensional integral convex polytope (= bounded polyhedron) which is combinatorially dual to $\Sigma$, i.e., facets of $\Delta_D$ (faces of dimension $n-1$) correspond to 1-dimensional cones $\rho \in \Sigma$ and, more generally, $i$-dimensional faces of $\Delta_D$ correspond to $(n-i)$-dimensional cones of $\Sigma$.

Conversely, given a $n$-dimensional integral convex polytope $\Delta \subset M_\mathbb{R}$, there is a unique fan (sometimes called the normal fan of $\Delta$) such that corresponding toric variety $X_\Delta$ has a Cartier divisor which gives $\Delta$ exactly (see [Oda1, Sect. 2.4]). In §5 we will give a method (due to Batyrev) for obtaining $X_\Delta$ directly from $\Delta$.

Other remarks. A minor omission in the classic references for toric varieties is that line bundles are discussed in detail, but not the reflexive sheaves coming from Weil divisors $\sum_\rho a_\rho D_\rho$. Basic facts about reflexive sheaves on normal varieties can be found in [Rei1]. We should also mention that there is a standard conflict of notation: some authors use $\Delta$ for the fan (see [Ful], [Oda2], [Stu2]), while others use $\Sigma$ for the fan and $\Delta$ for a polytope (see [Dan2], [Baty1]).

It is also possible to consider infinite fans. Gluing together affine toric varieties for cones in such a fan would lead to a scheme which is only locally of finite type.
However, given an appropriate discrete group action, one can take a quotient to get a variety. This is the approach used in [KMMS] and [AMRT]. A nice example is the resolution of a 2-dimensional Hilbert cusp singularity, which is discussed in [Oda1, Sect. 4.1].

§2. Global coordinates for toric varieties

Projective space is one of the simplest examples of a toric variety. The way \( \mathbb{P}^n \) is obtained by gluing together affine spaces is a special case of the classic construction of a toric variety. But \( \mathbb{P}^n \) can also be constructed as a quotient

\[ \mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*, \]

which is where we get the homogeneous coordinates on projective space. Recently, this construction has been generalized to most toric varieties. We will describe this construction and some of its consequences.

The construction. We begin by fixing a fan \( \Sigma \) in \( \mathbb{N} \mathbb{R} \cong \mathbb{R}^n \) and letting \( \Sigma(1) \) be the set of 1-dimensional cones in \( \Sigma \). We will assume that

\[(2.1) \quad \text{the 1-dimensional cones } \rho \in \Sigma(1) \text{ span } \mathbb{N} \mathbb{R} \]

(where as usual we regard \( \rho \) as the integral generator of its cone). Any complete fan satisfies this condition. Then consider the affine space \( \mathbb{C}^{\Sigma(1)} \) with variables \( x_\rho \) for \( \rho \in \Sigma(1) \). We need to remove a certain exceptional subset from \( \mathbb{C}^{\Sigma(1)} \).

These generate the monomial ideal \( B(\Sigma) = \langle \hat{x}_\sigma : \sigma \in \Sigma \rangle \), and the exceptional set \( Z(\Sigma) \subset \mathbb{C}^{\Sigma(1)} \) is the subvariety defined by \( B(\Sigma) \).

The toric variety \( X = X_\Sigma \) will be a quotient of \( \mathbb{C}^{\Sigma(1)} - Z(\Sigma) \) by the group

\[ G = \text{Hom}_\mathbb{Z}(A_{n-1}(X), \mathbb{C}^*), \]

where \( A_{n-1}(X) \) is the Chow group of Weil divisors modulo rational equivalence. To see how this group acts on \( \mathbb{C}^{\Sigma(1)} - Z(\Sigma) \), recall the exact sequence

\[(2.2) \quad 0 \longrightarrow M \overset{\alpha}{\longrightarrow} \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_\rho \overset{\beta}{\longrightarrow} A_{n-1}(X) \longrightarrow 0 \]

where \( \alpha \) sends \( m \in M \) to \( \text{div}(\chi^m) = \sum_\rho \langle m, \rho \rangle D_\rho \) and \( \beta \) is the obvious map from Weil divisors to the Chow group. The injectivity of \( \alpha \) is equivalent to (2.1).

Applying \( \text{Hom}_\mathbb{Z}(-, \mathbb{C}^*) \) yields the exact sequence

\[(2.3) \quad 1 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow N \otimes \mathbb{C}^* \longrightarrow 1. \]

This gives a natural action of \( G \) on \( \mathbb{C}^{\Sigma(1)} \), and since \( Z(\Sigma) \) is a union of coordinate subspaces, \( G \) preserves \( \mathbb{C}^{\Sigma(1)} - Z(\Sigma) \).

A nice example of what this looks like is given by \( \mathbb{P}^n \). The fan \( \Sigma \) of \( \mathbb{P}^n \) is well-known, and we leave it to the reader to check that in this case, the monomial ideal \( B(\Sigma) \) is the “irrelevant” ideal \( \langle x_0, \ldots, x_n \rangle \), so that the exceptional set \( Z(\Sigma) \) consists of the origin. Furthermore, the exact sequence (2.2) becomes

\[ 0 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n+1} \overset{\beta}{\longrightarrow} \mathbb{Z} \longrightarrow 0, \]
where $\beta(a_0, \ldots, a_n) = \sum_{i=0}^{n} a_i$. By (2.3), the action of $G$ on $C^\Sigma(1) - Z(\Sigma)$ is the usual action of $C^\ast$ on $C^{n+1} - \{0\}$. The quotient, of course, is $\mathbb{P}^n$.

Returning to the general case, assume for the moment that $X$ is given by the quotient $(C^\Sigma(1) - Z(\Sigma))/G$. Then the natural inclusion $(C^\ast)^{\Sigma(1)} \subset C^\Sigma(1) - Z(\Sigma)$, combined with (2.3), gives an inclusion $T = N \otimes C^\ast \subset X_{\Sigma}$ as a dense open set. The action of $T$ on $X$ is also easy to see in this picture: it is inherited from the action of the “big torus” $(C^\ast)^{\Sigma(1)}$ on $C^\Sigma(1) - Z(\Sigma)$.

Taking quotients in algebraic geometry can be a bit subtle, and the “quotient” $(C^\Sigma(1) - Z(\Sigma))/G$ is no exception. The precise relation between this quotient and the toric variety is as follows.

**Theorem 2.1.** Let $X$ be a toric variety whose fan $\Sigma$ satisfies (2.1). Then:

1. $X$ is the universal categorical quotient $(C^\Sigma(1) - Z(\Sigma))/G$.
2. $X$ is a geometric quotient $(C^\Sigma(1) - Z(\Sigma))/G$ if and only if $\Sigma$ is simplicial.

**Remarks on Theorem 2.1.** A universal categorical quotient is a $G$-equivariant map $\pi : C^\Sigma(1) - Z(\Sigma) \to X$ (where the action on $X$ is trivial) which is universal in the obvious sense. It is a geometric quotient precisely when the fibers of $\pi$ coincide with the $G$-orbits. Hence the simplicial case is the one closest to the way we think about projective space.

To get a better idea of what the quotient is like in the general case, observe that $C^\Sigma(1) - Z(\Sigma)$ is the union of affine open sets $U_\sigma = \{ \hat{x}_\sigma \neq 0 \}$ for $\sigma \in \Sigma$. Then the categorical quotient $U_\sigma/G$ is determined by the ring of invariants of $G$ acting on the coordinate ring of $U_\sigma$. In [Cox2], this ring of invariants is identified with the semigroup algebra $\mathbb{C}[\sigma^\vee \cap M]$, which explains why the quotients $U_\sigma/G$ patch together to give $X$.

An interesting aspect of this construction is that it was found by several people independently at approximately the same time. Audin [Aud] (following ideas of Delzant [Del] and Kirwan [Kir]) described the construction in the context of symplectic geometry, while Musson [Mus] was studying differential operators on toric varieties, and Cox [Cox2] was more interested in the algebraic aspects of the situation. This result was also discovered by Batyrev [Bat3] and Fine [Fin].

It is possible to develop the entire theory of toric varieties using Theorem 2.1 as the definition of toric variety.

**The exceptional set.** In $X = (C^\Sigma(1) - Z(\Sigma))/G$, note that the group $G$ depends only on the 1-dimensional cones of $\Sigma$, while the exceptional set $Z(\Sigma)$ depends on the full fan. The combinatorics of $Z(\Sigma)$ are quite interesting. For projective space, this set is very small, but it is usually bigger as the following result of [BC, Sect. 2] shows.

**Proposition 2.2.** Let $X$ be a $n$-dimensional complete simplicial toric variety with fan $\Sigma$. Then either

1. $2 \leq \text{codim } Z(\Sigma) \leq \lfloor n/2 \rfloor + 1$, or
2. $Z(\Sigma) = \{0\}$ and $X$ is a finite quotient of a weighted projective space.

When $X$ is simplicial, the coordinate subspaces making up $Z(\Sigma)$ can be described in terms of Batyrev’s notion of a primitive collection, which is a subset $P \subset \Sigma(1)$ with the property that $P$ is not the set of generators of a cone in $\Sigma$ while every
proper subset of \( \mathcal{P} \) is. Then the decomposition of \( Z(\Sigma) \) into irreducible components is given by

\[
Z(\Sigma) = \bigcup_{\mathcal{P}} A(\mathcal{P}),
\]

where \( A(\mathcal{P}) \) is the coordinate subspace determined by \( x_\rho \) for \( \rho \in \mathcal{P} \) and the union is over all primitive collections \( \mathcal{P} \). When \( X \) is smooth and complete, Batyrev has conjectured \([\text{Bat}2]\) that the number of irreducible components of \( Z(\Sigma) \) (= the number of primitive collections) is bounded by a constant depending only on the Picard number of \( X \).

**The homogeneous coordinate ring.** We next explore the algebraic consequences of \( X = (\mathbb{C}^{\Sigma(1)} - Z(\Sigma))/G \). The basic idea is that like projective space, homogeneous coordinates allow us to define subvarieties using global equations. When \( X \) is a simplicial toric variety, a point of \( X \) has “homogeneous coordinates” \((t_\rho) \in \mathbb{C}^{\Sigma(1)} - Z(\Sigma)\), which are well-defined up to the action of \( G \). The corresponding polynomial ring is

\[
S = \mathbb{C}[x_\rho : \rho \in \Sigma(1)]
\]

with a grading induced by the action of \( G \) on \( \mathbb{C}^{\Sigma(1)} \). The character group of \( G \) is \( \mathbb{A}_{n-1}(X) \) and the grading on \( S \) can be viewed as defining the “degree” of a monomial \( x^a = \Pi_\rho x_\rho^{a_\rho} \) to be \( \text{deg}(x^a) = [\sum_\rho a_\rho D_\rho] \in \mathbb{A}_{n-1}(X) \). With this grading, \( S = \mathbb{C}[x_\rho] \) is called the homogeneous coordinate ring of \( X \).

**Examples.** 1. In the case of \( \mathbb{P}^n \) or \( \mathbb{P}^n \times \mathbb{P}^m \), the coordinate rings are the classical rings of homogeneous or bihomogeneous polynomials.

2. Another good example is the blow-up of \( \mathbb{C}^n \) at the origin. We leave it as an exercise for the reader to show that the coordinate ring is \( \mathbb{C}[t, x_1, \ldots, x_n] \), where \( \text{deg}(t) = -1 \), \( \text{deg}(x_i) = 1 \). Furthermore, the exceptional set in \( \mathbb{C}^{n+1} \) is \( Z = \mathbb{C} \times \{(0, \ldots, 0)\} \) and \( G = \mathbb{C}^* \) acts on \( \mathbb{C}^{n+1} = \mathbb{C} \times \mathbb{C}^n \) by \( \mu \cdot (t, x) = (\mu^{-1} t, \mu x) \). Then, given a point \((t, x) \in \mathbb{C}^{n+1} - Z\), we have

\[
(t, x) \sim_G (1, tx) \quad \text{if} \ t \neq 0
\]

\[
(0, x) \sim_G (0, \mu x) \quad \text{if} \ \mu \neq 0.
\]

From this, it should be clear that \( X = (\mathbb{C}^{n+1} - Z)/G \) is the blow-up of \( \mathbb{C}^n \) at the origin. Notice also that the blow-up map \( X \to \mathbb{C}^n \) is given by \( (t, x) \mapsto tx \).

The coordinate ring \( S = \mathbb{C}[x_\rho] \) allows us to define subvarieties of \( X \) using homogeneous ideals of \( S \). The relation between ideals and varieties is similar to what happens in \( \mathbb{P}^n \), with the ideal \( B(\Sigma) = (x_\rho : \rho \in \Sigma(1)) \) playing the role of the irrelevant ideal (see \([\text{Cox2}]\)). One can also define sheaves on \( X \) using graded \( S \)-modules. Here is an example of how this works.

**Example.** Consider the graded \( S \)-module \( \Omega_S^p \) defined as the kernel of the map

\[
\gamma : S \otimes \Lambda^p M \longrightarrow \bigoplus_{\rho} S/(x_\rho) \otimes \Lambda^{p-1} M,
\]

where the \( p \)th component of \( \gamma \) is \( \gamma_p(f \otimes \omega) = f \mod x_\rho \otimes i_\rho(\omega) \) and \( i_\rho(\omega) \) is interior product. Then \([\text{BC}, \text{Sect. 8}]\) shows that the sheaf corresponding to \( \Omega_S^p \) is the sheaf
of Zariski \( p \)-forms \( \hat{\Omega}^p_X \) on \( X \). Furthermore, given \( \alpha \in A_{n-1}(X) \), we can define a shifted module \( \hat{\Omega}^p_S(\alpha) \) in the usual way. This gives a sheaf \( \hat{\Omega}^p_S(\alpha) \) with global sections
\[
H^0(X, \hat{\Omega}^p_S(\alpha)) \simeq (\hat{\Omega}^p_S)_\alpha
\]
(where the subscript refers to the graded piece in degree \( \alpha \)).

If \( L \) is a line bundle (or, more generally, a rank one torsion-free reflexive sheaf) on \( X \), then we get \( \alpha = [L] \in A_{n-1}(X) \), and one can prove that
\[
(2.4) \quad H^0(X, L) \simeq S^\alpha
\]
(see [Cox2]). When \( L = O_X(D) \) and \( X \) is complete, (1.2) then shows that \( \dim S^\alpha \) is the number \( l(\Delta) \) of integer points in the polytope \( \Delta_D \). From (2.4) we also obtain a ring isomorphism
\[
(2.5) \quad \bigoplus_{k=0}^\infty H^0(X, L^\otimes k) \simeq \bigoplus_{k=0}^\infty S^{k\alpha} \subset S.
\]

In particular, if \( X \) is projective, the “coordinate ring” \( S \) contains the coordinate rings (in the usual sense) of all possible projective embeddings of \( X \).

**Applications.** There have been several recent applications of global coordinates for toric varieties. We’ve already mentioned [Mus], which studies differential operators on toric varieties. In [Per1] and [Per2], homogeneous coordinates on a smooth toric variety \( X \) and the Euler sequence
\[
0 \rightarrow \Omega_X^1 \rightarrow \bigoplus_{\rho} O_X(-D_\rho) \rightarrow A_{n-1}(X) \otimes O_X \rightarrow 0
\]
(see [BC] and [Jac]) are used to compute the principal parts of line bundles and to study highly inflected toric varieties in dimensions \( \leq 3 \).

Using homogeneous coordinates, maps to a toric variety can be studied in much the same way one describes maps to projective space—see, for example, [Cox1], [Gue] and [Jac]. Homogeneous coordinates were also used in [Cox2] to show that Demazure’s results on the automorphism group of a smooth complete toric variety (see [Oda1, Sect. 3.4] for a description) remain valid in the simplicial case.

Further applications of homogeneous coordinates will be given in §§4 and 6, and in §10, we will also see how homogeneous coordinates are used in mirror symmetry.

### §3. The Kähler cone

The Kähler classes of a smooth projective variety \( X = X_\Sigma \) form an open cone in \( H^{1,1}(X, \mathbb{R}) \) called the Kähler cone. This cone can be complicated in general, but it is pleasantly simple when \( X \) is a smooth projective toric variety.

In this situation, the cohomology classes \([D_\rho]\) span \( H^2(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R}) = A_{n-1}(X) \otimes \mathbb{R} \). Since \( X \) is smooth, a class \( \mathbf{a} = [\sum_{\rho} a_\rho D_\rho] \) (with \( a_\rho \in \mathbb{R} \)) has a support function \( \psi : N_\mathbb{R} \rightarrow \mathbb{R} \) with the property that for each \( \sigma \in \Sigma \), there is \( m_\sigma \in M_\mathbb{R} \) with the property that \( \psi(\rho) = (m_\sigma, \rho) = -a_\rho \). This is similar to the support functions considered in §2 except that \( \psi \) is only well defined up to a linear function on \( N_\mathbb{R} \) (this follows from (2.2)) and the \( m_\sigma \) need not be integral. Then we say that \( \mathbf{a} \) is convex if \( \psi \) is a convex function on \( N_\mathbb{R} \). The convex classes form a cone \( \text{cpl}(\Sigma) \subset A_{n-1}(X) \otimes \mathbb{R} \) which has the following nice structure.
Proposition 3.1. If $X$ is a simplicial projective toric variety, then $\text{cpl}(\Sigma) \subset A_{n-1}(X) \otimes \mathbb{R} = H^{1,1}(X, \mathbb{R})$ is a strongly convex polyhedral cone with nonempty interior in $H^{1,1}(X, \mathbb{R})$. Furthermore, the interior of this cone is precisely the Kähler cone of $X$.

As observed in [Bat3], the first part of the proposition follows from [OP] (see also [Rei2]), and the second part follows easily in the smooth case. When $X$ is simplicial, [AGM2] gives a careful definition the Kähler cone and shows that Proposition 3.1 continues to hold in this case. We will see later that this proposition has implications for both symplectic geometry and mirror symmetry.

Support functions $\psi$ corresponding to Kähler classes are strictly convex and are described in [Bat3] using the primitive collections from §2 as follows.

Proposition 3.2. If $X$ is a simplicial projective toric variety, then a support function $\psi$ coming from $[\sum_{\rho} a_{\rho} D_{\rho}] \in A_{n-1}(X) \otimes \mathbb{R}$ is strictly convex if and only if for every primitive collection $\mathcal{P} = \{\rho_1, \ldots, \rho_k\}$, we have

$$
\psi(\rho_1 + \cdots + \rho_k) > \psi(\rho_1) + \cdots + \psi(\rho_k).
$$

The dual of the Kähler cone is the Mori cone of effective 1-cycles modulo numerical equivalence. Then Proposition 3.2 can be interpreted as describing generators for the Mori cone (see [Bat2], [OP] and [Rei2] for more on the Mori cone).

Example. Consider fans $\Sigma$ in $\mathbb{R}^3$ whose 1-dimensional cone generators are

$$e_0 = (0, 0, -2), \ e_1 = (1, 1, 1), \ e_2 = (1, -1, 1), \ e_3 = (-1, -1, 1), \ e_4 = (-1, 1, 1).$$

Think of $e_1, e_2, e_3, e_4$ as the upper vertices of a cube and $e_0$ as lying on the negative $z$-axis. We will use the integer lattice generated by $e_1, e_2, e_3$. Note that $e_0 = -e_1 - e_3$ and $e_4 = e_1 - e_2 + e_3$.

There are several ways to get a complete fan from these generators. For example, the cones $\sigma_{1234}, \sigma_{012}, \sigma_{023}, \sigma_{034}$ and $\sigma_{041}$ (where $\sigma_{1234}$ is the cone with generators $e_1, e_2, e_3, e_4$, etc.) and their faces determine a singular fan $\Sigma$. But if we subdivide $\sigma_{1234}$ into $\sigma_{123}$ and $\sigma_{341}$, then we get a smooth fan $\Sigma_1$. Similarly, we can subdivide $\sigma_{1234}$ into $\sigma_{124}$ and $\sigma_{234}$ to get another smooth fan $\Sigma_2$. The toric varieties corresponding to $\Sigma, \Sigma_1$ and $\Sigma_2$ will be denoted $X, X_1$ and $X_2$ respectively.

The primitive collections for $\Sigma_1$ are $\{e_2, e_4\}$ and $\{e_0, e_1, e_3\}$. Hence, using Proposition 3.2, we see that a support function $\psi$ is strictly convex if and only if

$$
\psi(e_2 + e_4) > \psi(e_2) + \psi(e_4)
$$

$$
\psi(e_0 + e_1 + e_3) > \psi(e_0) + \psi(e_1) + \psi(e_3).
$$

If we let $\psi(e_i) = -a_i$ and use the relations $e_2 + e_4 = e_1 + e_3$ and $e_0 + e_1 + e_3 = 0$, these inequalities are equivalent to

$$
a_2 + a_4 > a_1 + a_3
$$

$$
a_0 + a_1 + a_3 > 0.
$$

(3.1)

To determine the Kähler cone of $X_1$, we have to interpret (3.1) in terms of the Chow group $A_2(X_1) \otimes \mathbb{R}$. However, for $X_1$, the exact sequence (2.2) can be written

$$
0 \to \mathbb{Z}^3 \overset{\alpha}{\to} \mathbb{Z}^5 \overset{\beta}{\to} \mathbb{Z}^2 \to 0,
$$

(3.2)
where $\beta$ maps $(a_0, a_1, a_2, a_3, a_4)$ to $(s, t) = (a_0 + a_1 + a_3, a_0 + a_2 + a_4)$. Using $s, t$ as coordinates on $A_2(X_1) \otimes \mathbb{R} \simeq \mathbb{R}^2$, the inequalities (3.1) can be written $t > s > 0$. By Proposition 3.1, this is the Kähler cone of $X_1$. Thus $cpl(\Sigma_1)$ is $t \geq s \geq 0$.

We now turn our attention to $X_2$. The primitive collections for $\Sigma_2$ are \{e_1, e_3\} and \{e_0, e_2, e_4\}, which gives inequalities similar to (3.1) (just interchange $a_1, a_3$ with $a_2, a_4$). Since (3.2) depends only on the 1-dimensional cones of a fan, we see that $A_2(X_1) \otimes \mathbb{R}$ is the same $\mathbb{R}^2$ with the same coordinates $s, t$. The only difference is that the Kähler cone of $X_2$ is given by $s > t > 0$ and $cpl(\Sigma_2)$ is $s \geq t \geq 0$.

Hence the first quadrant in $\mathbb{R}^2$ is divided into cones $t \geq s \geq 0$ and $s \geq t \geq 0$ whose interiors are the Kähler cones of the smooth toric varieties $X_1$ and $X_2$. Also, the ray $s = t > 0$ corresponds to ample divisors on the singular toric variety $X$.

As we will see in §7, this is an example of the secondary fan or GKZ decomposition. We should also mention that $X_1$ and $X_2$ are related by a flop (see [Rei2]).

§4. SYMPLECTIC GEOMETRY

Besides the quotient $\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$ considered in §2, there is the related quotient

$$\mathbb{P}^n = S^{2n+1}/\mathbb{S}^1$$

where $\mathbb{S}^1 \subset \mathbb{C}^*$ acts on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ in the usual way. In this section, we will use symplectic reduction to generalize this construction to simplicial projective toric varieties.

The construction. Let $X = X_\Sigma$ be the toric variety determined by a fan $\Sigma$ in $N_\mathbb{R} \simeq \mathbb{R}^n$. We will assume that $X$ is simplicial and projective. To simplify notation, let $r = |\Sigma(1)|$. As in §2, we have the group $G = \text{Hom}_\mathbb{Z}(A_{n-1}(X), \mathbb{C}^*)$. The maximal compact subgroup of $G$ is

$$(4.1) \quad G_\mathbb{R} = \text{Hom}_\mathbb{Z}(A_{n-1}(X), \mathbb{S}^1).$$

The inclusion $G \subset (\mathbb{C}^*)^r$ gives an action of $G_\mathbb{R}$ on $\mathbb{C}^r$.

Now consider the map (called the moment map)

$$(4.2) \quad \mu_\Sigma : \mathbb{C}^r \xrightarrow{\mu} \mathbb{R}^r \xrightarrow{\beta_\mathbb{R}} A_{n-1}(X) \otimes \mathbb{R},$$

where $\mu$ defined by $\mu(z_1, \ldots, z_r) = \frac{1}{2}(|z_1|^2, \ldots, |z_r|^2)$ and $\beta_\mathbb{R}$ comes from the exact sequence

$$0 \rightarrow M_\mathbb{R} \rightarrow \mathbb{R}^r \xrightarrow{\beta_\mathbb{R}} A_{n-1}(X) \otimes \mathbb{R} \rightarrow 0$$

obtained by tensoring (2.2) with $\mathbb{R}$. Note that $\mu_\Sigma$ is constant on $G_\mathbb{R}$-orbits.

Since $X$ is projective and simplicial, $A_{n-1}(X) \otimes \mathbb{R} \simeq H^2(X, \mathbb{R}) \simeq H^{1,1}(X, \mathbb{R})$. Recall from §3 that the Kähler cone in $H^{1,1}(X, \mathbb{R})$ consists of all possible Kähler classes on $X$. Under these isomorphisms, we get a cone in $A_{n-1}(X) \otimes \mathbb{R}$, also called the Kähler cone. Then we modify the construction $X = (\mathbb{C}^r - Z(\Sigma))/G$ from §2 as follows.

**Theorem 4.1.** Let $X = X_\Sigma$ be a projective simplicial toric variety, and assume that $a \in A_{n-1}(X) \otimes \mathbb{R}$ is in the Kähler cone. Then $\mu_\Sigma^{-1}(a) \subset \mathbb{C}^r - Z(\Sigma)$, and the natural map

$$\mu_\Sigma^{-1}(a)/G_\mathbb{R} \rightarrow (\mathbb{C}^r - Z(\Sigma))/G = X$$
is a diffeomorphism which preserves the class of the symplectic form (to be explained below).

Proof. When \( X \) is smooth, a proof that we have a diffeomorphism can be found in Guillemin’s recent book [Gui, Appendix 1], and the statement about the class of the symplectic form follows from equation (1.6) of [Gui, Appendix 2] (the \( \lambda_i \) in [Gui] are \(-a_i\) in our notation). This proof can be modified to work in the simplicial case. One can also use results in [Kir] to prove the theorem. The version in [Aud] is somewhat incomplete since the Kähler cone is not mentioned. \( \square \)

Examples. 1. When \( X = \mathbb{P}^n \), the map \( \beta_\mathbb{R} \) in (4.2) is the map \( \mathbb{R}^{n+1} \to \mathbb{R} \) defined by \( (a_0, \ldots, a_n) \mapsto \sum_{i=0}^n a_i \). Thus \( \mu_\Sigma(z_0, \ldots, z_n) = \frac{1}{2} \sum_{i=0}^n |z_i|^2 \). Since the Kähler cone is \( \mathbb{R}^+ \), we see that \( \mu_\Sigma^{-1}(a) \) is a sphere for any \( a > 0 \). Hence we recover the usual description of \( \mathbb{P}^n \) as a quotient of the \((2n + 1)\)-sphere.

2. Consider the toric varieties \( X_1 \) and \( X_2 \) from the example in §3. These have the same group \( G_\mathbb{R} \) acting on \( \mathbb{C}^5 \) and the same moment map \( \mu_{\Sigma_1} = \mu_{\Sigma_2} \), which we write as \( \mu : \mathbb{C}^5 \to \mathbb{R}^2 \). Then \( \mu \) is given by

\[
\mu(z_0, z_1, z_2, z_3, z_4) = \frac{1}{2}(|z_0|^2 + |z_1|^2 + |z_3|^2, |z_0|^2 + |z_2|^2 + |z_4|^2)
\]

(this follows from the description of \( \beta \) in (3.2)). Since we determined the Kähler cones of \( X_1 \) and \( X_2 \) in §3, it follows that

\[
\mu^{-1}(s, t)/G_\mathbb{R} \simeq \begin{cases} X_1, & \text{if } t > s > 0 \\ X_2, & \text{if } s > t > 0 \end{cases}
\]

Symplectic manifolds and Hamiltonian actions. To understand the construction given in Theorem 4.1, one needs to discuss symplectic geometry. A symplectic structure on a real manifold \( M \) is a closed, nondegenerate 2-form \( \omega \). The symplectic form \( \omega \) converts functions into vector fields as follows: if \( f \) is a \( C^\infty \) function on \( M \), then there is a unique vector field \( X_f \) on \( M \) with the property that \( \omega(X, X_f) = X(f) \) for any vector field \( X \). We call \( X_f \) the Hamiltonian of \( f \). Basic references on symplectic geometry are [Aud] and [Kir], and the reader might also want to consult [Ati1], [Ati2] and [GS].

Now suppose that a compact connected Lie group \( G_\mathbb{R} \) acts on \( M \). This action induces an infinitesimal action of the Lie algebra \( g_\mathbb{R} \) where every \( \lambda \in g_\mathbb{R} \) gives a vector field \( X_\lambda \) on \( M \). Then the action is Hamiltonian if:

1. The symplectic form \( \omega \) is invariant under the group action.
2. For each \( \lambda \in g_\mathbb{R} \), the vector field \( X_\lambda \) is Hamiltonian (i.e., is the Hamiltonian vector field of some \( C^\infty \) function on \( M \)).

A basic property of a Hamiltonian action is that it has a moment map

\[
\mu : M \to g_\mathbb{R}^* 
\]

which has the property that for every \( \lambda \in g_\mathbb{R} \), the vector field \( X_\lambda \) is the Hamiltonian of the function \( \lambda \circ \mu : M \to \mathbb{R} \).

Examples. 1. The most basic example is \( \mathbb{C}^r \) endowed with the symplectic form

\[
\omega = \sum_{j=1}^r dx_j \wedge dy_j,
\]

where \( z_j = x_j + iy_j \). It is easy to check that the natural action of \( (S^1)^r \) on \( \mathbb{C}^r \) is Hamiltonian and the moment map

\[
\mu : \mathbb{C}^r \to (\mathbb{R}^r)^*
\]
is defined by \( \mu(z_1, \ldots, z_r) = \frac{1}{2}(|z_1|^2, \ldots, |z_r|^2) \) (this uses the basis of \((\mathbb{R}^r)^*\) dual to the standard basis of the Lie algebra \(\mathbb{R}^r\) of \((S^1)^r\)).

2. When \(G_\mathbb{R}\) is the group coming from a toric variety \(X\) as described in (4.1), then the action of \(G_\mathbb{R}\) on \(\mathbb{C}^r\) is Hamiltonian and the moment map is exactly as described in (4.2) provided we identify \(\mathbb{R}^r\) with its dual \((\mathbb{R}^r)^*\). Thus Theorem 4.1 tells us how to construct a toric variety using the moment map of a Hamiltonian action.

**Symplectic reduction.** Given a symplectic manifold \(M\) with a Hamiltonian action of \(G_\mathbb{R}\) and moment map \(\mu: M \to \mathfrak{g}_\mathbb{R}_0\), we can use this data to construct other symplectic manifolds by the process of symplectic reduction. If \(a \in \mathfrak{g}_\mathbb{R}\) is a regular value of \(\mu\), then \(\mu^{-1}(a)\) is a manifold, but the restriction of the symplectic form \(\omega\) to \(\mu^{-1}(a)\) will fail to be symplectic (it won’t be nondegenerate). However, if \(G_\mathbb{R}\) acts freely on \(\mu^{-1}(a)\), then the restriction of \(\omega\) descends to the quotient \(\mu^{-1}(a)/G_\mathbb{R}\) as a symplectic form. This is what we mean by symplectic reduction.

If we look back at Theorem 4.1, we see that the basic assertion of the theorem is that we can construct smooth projective toric varieties by symplectic reduction. Furthermore, we can now explain what it means for the diffeomorphism

\[
\mu_{\Sigma}^{-1}(a)/G_\mathbb{R} \longrightarrow (\mathbb{C}^r - Z(\Sigma))/G = X
\]

to preserve the class of the symplectic form: the symplectic reduction \(\mu_{\Sigma}^{-1}(a)/G_\mathbb{R}\) has a natural symplectic structure coming from \(\omega\) on \(\mathbb{C}^r\), and \(X\) has a symplectic structure coming from any Kähler form whose cohomology class is \(a \in A_{n-1}(X) \otimes \mathbb{R}\). The above diffeomorphism need not map one symplectic form to the other, but it does preserve their cohomology classes.

**Delzant polytopes.** In addition to the Hamiltonian action used to construct a smooth projective toric variety \(X\), we also have an action of the real torus \(T_\mathbb{R} = (S^1)^n\) on \(X\). It is well-known that if we give \(X\) the symplectic structure coming from an ample divisor \(D = \sum a_\rho D_\rho\) (so \(a_\rho \in \mathbb{Z}\)), then the action of \(T_\mathbb{R}\) is Hamiltonian, and the moment map

\[
\mu_X: X \to M_\mathbb{R}
\]

(note that \(M_\mathbb{R} = N_\mathbb{R}^*\) and \(N_\mathbb{R}\) is the Lie algebra of \(T_\mathbb{R}\)) can be described explicitly (see [Ful, Sect. 4.2] or [Oda1, Sect. 2.4]). Furthermore, the image of the moment map is, up to translation, precisely the polytope \(\Delta_D\) defined in (1.1), and the induced map

\[
X/T_\mathbb{R} \to \Delta_D
\]

is a homeomorphism.

With the publication of Delzant’s thesis [Del], it became possible to view the above results in a broader context. Namely, given a smooth projective toric \(X\) and a class \([D] = [\sum a_\rho D_\rho]\) in its Kähler cone (so \(a_\rho \in \mathbb{R}\)), we get the polytope

\[
\Delta_D = \{m \in M_\mathbb{R} : \langle m, \rho \rangle \geq -a_\rho \} \subset M_\mathbb{R}.
\]

This polytope is no longer integral, but the normals to its facets are integral, and since \(X\) is smooth, the normals to the facets meeting at each vertex of \(\Delta_D\) form a basis of the lattice \(N\) (this is just another way of saying that the normal fan of \(\Delta_D\) is the fan of \(X\)). Such a polytope is called a Delzant polytope.
Example. Consider the toric varieties $X_1$ and $X_2$ from the example in §3. We leave it for the reader to show that, up to translation, the Delzant polytope in $\mathbb{R}^3$ corresponding to $D = a_0D_0 + \cdots + a_4D_4$ is defined by the inequalities

$$x, y, z \geq 0, \quad x + z \leq s, \quad x - y + z \geq s - t,$$

where as usual $s = a_0 + a_1 + a_3$ and $t = a_0 + a_2 + a_4$. For $t > s > 0$, this gives the Delzant polytope for $X_1$, and for $s > t > 0$ we get the polytope for $X_2$. It is a good exercise to draw these polytopes so you can see what happens when $s = t$. (For the best picture, have the first octant face away from you and let the $xz$-plane be horizontal.)

If we give the toric variety $X$ a symplectic structure whose class lies in $[D]$, then the action of $T_{\mathbb{R}}$ is still Hamiltonian and the Delzant polytope $\Delta_D$ is again the image of the moment map. Conversely, given any Delzant polytope $\Delta$, one can construct a smooth projective toric variety $X_\Delta$ with $\Delta$ as the image of the moment map (see [Gui]).

What is more remarkable is Delzant’s purely symplectic characterization of smooth projective toric varieties (see [Del]).

**Theorem 4.2.** Let $X$ be a real $2n$-dimensional compact connected symplectic manifold with an effective Hamiltonian action of $(S^1)^n$. Then the image of the moment map is a Delzant polytope $\Delta$ and $X$ is diffeomorphic (as a Hamiltonian $(S^1)^n$-space) to the smooth projective toric variety $X_\Delta$ determined by $\Delta$.

One way to understand this theorem is to observe that for an effective Hamiltonian action of $(S^1)^m$ on a connected symplectic manifold $M$, we always have $2m \leq \dim M$. Thus Theorem 4.2 characterizes what happens when $M$ is compact and the dimension of the torus $(S^1)^m$ is as large as possible.

§5. Torus coordinates and toric ideals

Besides the homogeneous coordinates of §2, toric varieties have intrinsic coordinates living on the torus. This is because a basis $e_1, \ldots, e_n$ of $M$ induces an isomorphism $T = N \otimes \mathbb{Z} \simeq (\mathbb{C}^*)^n$, giving coordinates $t_1, \ldots, t_n$ on $T$. Then, for $m = \sum_{i=1}^n a_i e_i \in M$, the character $\chi^m$ from §1 is the Laurent monomial $t^m = \Pi_{i=1}^n t_i^{a_i}$, and the coordinate ring of $T$ is $\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. These coordinates don’t extend to the whole toric variety, but they are still very useful.

The toric variety of a polytope. Given an $n$-dimensional integral convex polytope $\Delta \subset M_\mathbb{R}$, we get the toric variety $X_\Delta$ and an ample divisor $D$ as mentioned in §1. Then (1.2) can be written as

$$H^0(X_\Delta, \mathcal{O}_X(D)) = \bigoplus_{m \in \Delta \cap M} \mathbb{C} \cdot t^m = L(\Delta),$$

so that we can think of the global sections of $D$ in terms of Laurent polynomials.

The most concrete way to get $D$ from $\Delta$ is to represent $\Delta \subset M_\mathbb{R}$ by inequalities $\langle m, \rho \rangle \geq -a_\rho$ (so the $\rho$’s are normals to the facets of $\Delta$), and then $D = \sum_{\rho} a_\rho D_\rho$. From a more sophisticated point of view, $D$ is the divisor associated to the support function $\psi : N_\mathbb{R} \to \mathbb{R}$ defined by

$$\psi(u) = \min\{\langle m, u \rangle : m \in \Delta\}.$$
Since $D$ is ample, the global sections of some multiple $kD$ give a projective embedding of $X_\Delta$. We can use this to construct $X_\Delta$ as follows. Let $l(k\Delta) = \dim L(k\Delta)$ be the number of integer points of $k\Delta$, and consider the map
\[ \Psi : (\mathbb{C}^*)^n \longrightarrow \mathbb{P}^{l(k\Delta)-1} \]
defined by
\[ \Psi(t_1, \ldots, t_n) = (t^{m_1}, \ldots, t^{m_{l(k\Delta)}}) \]
where $k\Delta \cap M = \{m_1, \ldots, m_{l(k\Delta)}\}$. Then, since $\Psi$ extends to an embedding of $X_\Delta$, it is clear that $X_\Delta$ is the closure of $\Psi((\mathbb{C}^*)^n)$ in $\mathbb{P}^{l(k\Delta)-1}$. Later in this section we will use this approach to define non-normal toric varieties.

A more algebraic method of constructing $X_\Delta$ is due to Batyrev [Bat4]. Given $\Delta$, consider the cone over $\Delta \times \{1\} \subseteq M_\mathbb{R} \oplus \mathbb{R}$ (in the terminology of [BB1], this is a Gorenstein cone). The integer points of the cone give a semigroup algebra $S_\Delta$. Since $(m, k) \in M \oplus \mathbb{Z}$ is in the cone if and only if $m \in k\Delta$, $S_\Delta$ is the subring of $\mathbb{C}[t_0, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ spanned by Laurent monomials $t_0^k t^m$ with $k \geq 0$ and $m \in k\Delta$. This ring can be graded by setting $\deg(t_0^k t^m) = k$, and one can show that
\[ X_\Delta = \text{Proj}(S_\Delta). \]

Since $S_\Delta$ is the coordinate ring of an affine toric variety, it is Cohen-Macaulay and hence $X_\Delta$ is arithmetically Cohen-Macaulay. This ring will play an important role in the next section.

We can also relate $S_\Delta$ to the coordinate ring $S$ of §2. If $\alpha = [D] \in A_{n-1}(X_\Delta)$ is the class of $D = \sum \rho \alpha D_\rho$, then by [BC], we can define a ring isomorphism
\[ S_\Delta \simeq \bigoplus_{k=0}^\infty S_{k\alpha} \subset S \]
by sending the Laurent monomial $t_0^k t^m$ to the monomial $\Pi_{\rho \alpha} t_0^{k_\rho m_{\rho} + \langle m, \rho \rangle}$. This is a special case of the isomorphism (2.5).

**Newton polytopes.** There are many situations where instead of a polytope, the initial data is a Laurent polynomial corresponding to a finite set $\mathcal{A} \subset M = \mathbb{Z}^n$ of exponents, which we write as
\[ f = \sum_{m \in \mathcal{A}} c_m t^m, \quad c_m \neq 0. \]
The convex hull $\Delta = \text{Conv}(\mathcal{A})$ is called the Newton polytope of $f$ and is used in many contexts (see, for example, [AVG], [Dan3], [DL], [GKZ1], [Kho], [Kus], [McD], [Var]). This polytope might not be $n$-dimensional, but if we use the lattice $\mathbb{Z} \mathcal{A}$ generated by $\mathcal{A}$, then we get a toric variety $X_\Delta$ of dimension $\text{rank}(\mathbb{Z} \mathcal{A})$ and, as in (5.1), an ample line bundle on $X_\Delta$ whose global sections are $L(\Delta) = \oplus_{m \in \Delta \cap \mathbb{Z} \mathcal{A}} \mathbb{C} \cdot t^m$. In particular, our Laurent polynomial $f$ is a global section.

A variant of this is that sometimes one is given Laurent polynomials $f_1, \ldots, f_s$ corresponding to possibly different sets of exponents $\mathcal{A}_i, \ldots, \mathcal{A}_s \subset \mathbb{Z}^n$. In this situation, we get polytopes $\Delta_i = \text{Conv}(\mathcal{A}_i)$, and we let
\[ \Delta = \Delta_1 + \cdots + \Delta_s \]
be their Minkowski sum. Also let $L(\Delta) = \oplus_{m \in \Delta \cap \mathbb{Z} \mathcal{A}} \mathbb{C} \cdot t^m$, where $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_s$. From Proposition 2.4 of [BB3], we get the following result which is useful when studying complete intersections in toric varieties.
**Proposition 5.1.** Given $f_1, \ldots, f_s$ and $\Delta$ as above, the toric variety $X_\Delta$ has divisors $D_i$ whose global sections are $L(\Delta_i)$, and $\mathcal{O}_{X_\Delta}(D_i)$ is generated by these sections. In particular, each $f_i$ is a global section of $\mathcal{O}_{X_\Delta}(D_i)$.

**Non-normal toric varieties.** We can generalize the construction (5.2) as follows. Given a finite set of exponents $A = \{m_1, \ldots, m_\ell\} \subset \mathbb{Z}^n$, we get a map

$$\Psi : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}^\ell$$

defined by

$$(5.5) \quad \Psi(t_1, \ldots, t_n) = (t^{m_1}, \ldots, t^{m_\ell}).$$

The Zariski closure of $\Psi((\mathbb{C}^*)^n) \subset \mathbb{C}^\ell$ is an affine variety denoted $^aX_A$ (the superscript $^a$ refers to affine).

If $d = \text{rank}(\mathbb{Z}A)$, then one can show that $T = ^aX_A \cap (\mathbb{C}^*)^\ell$ is isomorphic to $(\mathbb{C}^*)^d$ and is Zariski open in $^aX_A$. Furthermore, the natural action of $T$ on itself extends to an action on $^aX_A$ (see [Stu2, Lemma 13.4]). This sounds like the definition of toric variety, but normality is missing. Nevertheless, we will refer to $^aX_A$ as a toric variety, and we will see in this section and in §9 that these toric varieties are very useful. Basic references for non-normal toric varieties are [GKZ1] and [Stu2].

**Example.** When $A = \{2, 3\} \subset \mathbb{Z}$, we get the map $\Psi(t) = (t^2, t^3)$, which leads to the cuspidal cubic $y^2 = x^3$ in $\mathbb{C}$. This is clearly a non-normal toric variety.

The following proposition explains how $^aX_A$ relates to the usual kind of affine toric variety. See [Stu2] for the proof.

**Proposition 5.2.** The normalization of $^aX_A$ is the affine toric variety $X_\sigma$, where $M = \mathbb{Z}A$ and $\sigma \subset N_\mathbb{R}$ is dual to the convex polyhedral cone $\text{Cone}(A)$ generated by $A$. Hence $^aX_A$ is normal if and only if $\mathbb{N}A = \mathbb{Z}A \cap \text{Cone}(A)$, where $\mathbb{N}A$ is the set of all non-negative integer linear combinations of $A$.

We next consider projective toric varieties defined using $A \subset \mathbb{Z}^n$. In practice, there are two ways of doing this. The first method assumes that $A$ lies in an affine hyperplane in $\mathbb{Z}^n$ not passing through the origin. In this case, it is easy to see that $^aX_A \subset \mathbb{C}^\ell$ is defined by homogeneous polynomials. Then $^aX_A$ is the affine cone of a projective variety in $\mathbb{P}^{\ell-1}$ denoted $X_A$. One can prove that $X_A$ is a toric variety (possibly non-normal) of dimension equal to the dimension of the affine span of $A$ (= the dimension of the convex hull $\text{Conv}(A)$).

A second method for creating projective toric varieties starts with an arbitrary subset $A = \{m_1, \ldots, m_\ell\} \subset \mathbb{Z}^n$ and considers the map

$$(5.6) \quad \Psi(t_1, \ldots, t_n) = (t^{m_1}, \ldots, t^{m_\ell}) \in \mathbb{P}^{\ell-1}.$$

Then, as in [GKZ1], we define $X_A$ to be the closure in $\mathbb{P}^{\ell-1}$ of the image of $\Psi$. The dimension of $X_A$ again equals the dimension of $\text{Conv}(A)$.

These two approaches are related as follows. Given $A$ as in the second method, $A \times \{1\} \subset \mathbb{Z}^{n+1}$ lies in an affine hyperplane not passing through the origin. Then the affine cone of $X_A$ is easily seen to be $^aX_A \times \{1\}$. Thus $X_A$, as defined by the second method, equals $X_A \times \{1\}$, as defined by the first.

The normalization of the projective toric variety $X_A$ can be computed using Proposition 5.2. This is a bit delicate because of the distinction between normality and projective normality—it is possible for $X_A$ to be normal without $^aX_A$ being so (see [Har, Ex. 3.18 on p. 23] for an example).
Proposition 5.3. The normalization of $X_A$ is $X_\Delta$, where $\Delta = \text{Conv}(A)$.

Proof. We regard $X_A$ as arising from (5.6). As noted above, its affine cone is $aX_{A \times \{1\}}$, so by Proposition 5.2, the normalization of $aX_{A \times \{1\}}$ comes from the cone over $A \times \{1\}$. This equals the cone over $\Delta \times \{1\}$, which is exactly the cone used in Batyrev’s construction of $X_\Delta$ earlier in this section. Thus, the normalization of $aX_A$ is the affine cone of $X_\Delta$, which implies that the map $X_\Delta \to X_A$ is finite and birational. The proposition now follows from Zariski’s Main Theorem. □

Example. To see how non-normal projective toric varieties can occur, consider a complete toric variety $X$ (in the usual sense) with an ample divisor $D$. If we let $A = \Delta_D \cap M$ and use the basis of $H^0(X, \mathcal{O}_X(D))$ given by Laurent monomials, we get a map $\Psi : X \to \mathbb{P}^{d-1}$ as above. Since $D$ need not be very ample, $\Psi$ need not be an embedding. But (5.6) shows that the image $\Psi(X)$ is precisely the toric variety $X_A$. It follows from Proposition 5.3 that $X$ is the normalization of the image of the map to projective space given by an ample line bundle on $X$.

There is a nice criterion for normality which involves the Hilbert polynomial of $X_A$ and the Ehrhart polynomial of the polytope $\Delta = \text{Conv}(A)$. By [Stu2, Ch. 13], the Hilbert polynomial of $X_A$ is given by

$$H_A(k) = |\{m_{i_1} + \cdots + m_{i_k} : m_{i_1}, \ldots, m_{i_k} \in A\}|, \quad k \geq 0,$$

and, as usual, the Ehrhart polynomial of $\Delta$ is

$$E_\Delta(k) = |Z_A \cap k\Delta|, \quad k \geq 0$$

These polynomials have the same leading term, which implies that the degree of $X_A \subset \mathbb{P}^{d-1}$ is the normalized volume of $\Delta$ (see [Stu2, Thm. 4.16]). Then we have the following result of Sturmfels [Stu2, Thm. 13.11].

Theorem 5.4. The toric variety $X_A \subset \mathbb{P}^{d-1}$ is normal if and only if the Hilbert polynomial $H_A$ equals the Ehrhart polynomial $E_A$.

Toric ideals. A familiar example from algebraic geometry is the twisted cubic $(x, y, z) = (t, t^2, t^3)$ in $\mathbb{C}^3$. This is a special case of the construction (5.5). The ideal of the twisted cubic is $(y-x^2, z-x^3) \subset \mathbb{C}[x, y, z]$ and is our first example of a toric ideal. As we will soon see, the simple form of its generators is no accident.

Given $A = \{m_1, \ldots, m_\ell\} \subset \mathbb{Z}^d$, the affine toric variety $aX_A$ is defined by an ideal $I_A \subset \mathbb{C}[x_1, \ldots, x_\ell]$, which we call a toric ideal. In terms of elimination theory, the ideal $I_A$ arises from the equations $x_i - t^{m_i} = 0$, $1 - yt_1 \cdots t_n = 0$ by eliminating $y, t_1, \ldots, t_n$ (the equation $1 - yt_1 \cdots t_n = 0$ guarantees that $(t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$).

A toric ideal $I_A$ is homogeneous when $A$ lies in an affine hyperplane missing the origin, in which case $I_A$ defines the projective toric variety $X_A$. Also, if $X_A$ is projectively normal, then $\mathbb{C}[x_1, \ldots, x_\ell]/I_A$ is isomorphic to the ring $S_\Delta$ of (5.3). So facts about toric ideals give useful information about the coordinate rings of projective toric varieties.

As with the twisted cubic, toric ideals are generated by binomials, which are differences of monomials. To state the result, note that a vector $a \in \mathbb{Z}^\ell$ can be written $a^+ - a^-$, where $a^+$ and $a^-$ have non-negative entries and disjoint support.
Lemma 5.5. If $\mathcal{A} = \{m_1, \ldots, m_\ell\} \subset \mathbb{Z}^n$, then the toric ideal $I_{\mathcal{A}}$ can be written

$$I_{\mathcal{A}} = \langle x^{a^+} - x^{a^-} : a = (a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell, \sum_{i=1}^\ell a_i m_i = 0 \rangle.$$ 

In [Stu1] and [Stu2], one can find a wealth of results about toric ideals, including facts about Gröbner bases and relations to secondary fans. We will discuss these ideas briefly in §7. Applications to enumeration, sampling, integer programming, and primitive partition identities are given in [Stu2, Ch. 5 and 6].

We close this section with an unsolved problem about toric ideals.

Conjecture 5.6. If $X_{\mathcal{A}}$ is a smooth projectively normal toric variety, then the toric ideal $I_{\mathcal{A}}$ is generated by quadratic binomials.

§6. Cohomology of toric hypersurfaces

A complete toric variety $X$, being rational, is a very special kind of variety. But as an ambient space, $X$ can be home to some interesting subvarieties (we will see some Calabi-Yau examples in §8). In particular, using the homogeneous coordinates of §2 or the torus coordinates of §5, it is easy to describe hypersurfaces and complete intersections in $X$. In this section, we will study hypersurfaces $Y \subset X$ and the associated affine hypersurfaces $Y \cap T \subset T$, where $T$ is the torus of $X$. We will end the section with some remarks about complete intersections.

Affine and projective hypersurfaces. Given a Laurent polynomial $f$ contained in $\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, the equation $f = 0$ defines an affine hypersurface

$$Z_f \subset T = (\mathbb{C}^*)^n.$$ 

We can assume that $f \in L(\Delta)$ for some convex integral polytope $\Delta \subset \mathbb{R}^n$ (for example, let $\Delta$ be the Newton polytope of $f$). We will also assume that $\Delta$ has dimension $n$ (it is easy to reduce to this case). Then we have a toric variety $X_\Delta$ containing $T$, and $f \in L(\Delta)$ defines a projective hypersurface

$$Y_f \subset X_\Delta$$

since $f$ can be regarded as a global section of an ample line bundle on $X_\Delta$. Note that $Z_f = Y_f \cap T$, though $Y_f$ need not be the Zariski closure of $Z_f$. The latter happens, for instance, if $f$ is a single monomial $t^m$.

When $\Delta$ is the Newton polytope of $f$, there is a nice relation between the topology of $Z_f$ and the vertices of $\Delta$. Using the map

$$\text{(6.1)} \quad \log : (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (t_1, \ldots, t_n) \mapsto (\log |t_1|, \ldots, \log |t_n|)$$

one can show that unbounded connected components of $\mathbb{R}^n - \log(Z_f)$ correspond to vertices of $\Delta$ and in each such component contains a translate of the corresponding cone in the normal fan of $\Delta$. There is also a version of this (using the moment map) for $Y_f$—see [GKZ, Sect. 1.B and 1.C of Ch. 6].

Since $Z_f$ or $Y_f$ could be very singular for an arbitrary $f \in L(\Delta)$, we need some sort of genericity condition on $f$. There are two conditions which are used in
In this case, $F \in \text{notion of quasi-smooth}$ was introduced by Danilov (see [Dan2, Sect. 14]).

(6.2) $L(\Delta) \simeq S_\beta$
described in (5.3). Thus, $f \in L(\Delta)$ corresponds to a homogeneous polynomial $F \in S_\beta$. The equation $F = 0$ gives a hypersurface in $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ (where $\Sigma$ is the fan of $X_\Delta$). By Theorem 2.1, this descends to a hypersurface $Y_f \subset X_\Delta$, and one can check that $Y_f = Y_f$. Furthermore, $Y_f$ is quasi-smooth if and only if the partial derivatives $\partial F/\partial x_i$ have no common zeros in $\mathbb{C}^{\Sigma(1)} - Z(\Sigma)$ (see [BC, Sect. 3]).

When the defining polynomial $f$ or $F$ is clear from context, we will write $Z \subset T$ in the affine case and $Y \subset X_\Delta$ in the projective case.

**Cohomology of affine hypersurfaces.** For a nondegenerate affine hypersurface $Z = Z_f \subset T = (\mathbb{C}^*)^n$, the cohomology groups $H^i(Z)$ (always with complex coefficients) carry natural mixed Hodge structures. In [DK], the dimension $h^{p,q}(H^i(Z))$ of the $(p,q)$ Hodge component of $\text{Gr}^W_{p+i,H^i(Z)}$ (where $W$ is the weight filtration) is computed using the combinatorics of the Newton polytope $\Delta$ of $f$.

It suffices to compute $h^{p,q}(H^i_c(Z))$ since $H^i_c(Z)$ is dual (as a mixed Hodge structure) to $H^{2n-2-i}(Z)$. In fact, it is sufficient to compute

$$e^{p,q}(Z) = \sum_i (-1)^i h^{p,q}(H^i_c(Z))$$
since the Gysin map

$$H^i_c(Z) \rightarrow H^{i+2}(T)$$
is an isomorphism of mixed Hodge structures (suitably shifted) for $i > n - 1$ and $H^i_c(Z) = 0$ for $i < n - 1$ (recall that $Z$ is smooth and affine).

In the special case when $\Delta$ is simplicial, computing $e^{p,q}(Z)$ is fairly easy. First, for a face $\Gamma \subset \Delta$, define $\phi_\Gamma(\Gamma)$ for $0 < i \leq \dim \Gamma$ by the formulas

$$\phi_\Gamma(\Gamma) = \sum_{i=1}^k (-1)^{i-k} \binom{\dim \Gamma + 1}{i-k} I^*(k\Gamma)$$

$$= (-1)^{\dim \Gamma + 1} \sum_{k=0}^{\dim \Gamma + 1} (-1)^{i+k} \binom{\dim \Gamma + 1}{i+k} I(k\Gamma)$$

(6.3)
where \( l(k\Gamma) \) (resp. \( l^*(k\Gamma) \)) is the number of integer points in \( k\Gamma \) (resp. in the relative interior of \( k\Gamma \)). The two representations of \( \phi_i(\Gamma) \) are related to the remarkable properties of the Ehrhart polynomial from \( \S 5 \) (see [Bat4, Sect. 2] for more details).

When \( p > q \), [DK, 5.7] gives the formula

\[
e^{p,q}(Z) = (-1)^{n+p+q} \sum_{\dim \Gamma > p} (-1)^{\dim \Gamma} \left( \frac{n - \dim \Gamma}{n - 1 - p - q} \right) \phi_{\dim \Gamma - p}(\Gamma),
\]

where the sum is over all faces \( \Gamma \) of \( \Delta \) of dimension \( > p \). Since \( e^{p,q}(Z) = e^{q,p}(Z) \), this gives us everything except \( e^{p,p}(Z) \). However, we also have the identity

\[
(6.4) \quad (-1)^{n-1} \sum_q e^{p,q}(Z) = (-1)^p \left( \frac{n}{p+1} \right) + \phi_{n-p}(\Delta),
\]

from [DK, 4.4], which now enables us to compute \( e^{p,p}(Z) \).

The paper [DK] also describes an algorithm for computing the full mixed Hodge structure of \( Z_f \subset X_{\Delta} \) for any polytope \( \Delta \). There are tables giving explicit formulas when \( 1 \leq \dim(\Delta) \leq 4 \) (see [DK, 5.11]).

One way to represent the numbers \( e^{p,q}(Z) \) is via the \( E \)-polynomial

\[
E(Z; u, v) = \sum_{p,q} e^{p,q}(Z) u^p v^q.
\]

Then (6.4) describes \( E(Z; u, 1) \), which tells us about the Hodge filtration. The weight filtration, on the other hand, concerns \( E(Z; u, u) \). This polynomial is studied in [BB2] and [DL]. We should also mention that explicit formulas for \( E(Z; u, v) \) (for \( \Delta \) arbitrary, not just simplicial) can be found in [BB2].

We next describe some work of Batyrev [Bat4] on representing cohomology classes of \( Z \subset T \) as residues of forms. Here, we will focus on the mixed Hodge structure of \( H^i(Z) \). The dual of the above Gysin map is the natural map \( H^k(T) \to H^k(Z) \), which is an isomorphism for \( k < n-1 \) and injective for \( i = n-1 \). Because of this, we define the primitive cohomology of \( Z \) by the exact sequence

\[
(6.5) \quad 0 \to H^{n-1}(T) \to H^{n-1}(Z) \to H_0^{n-1}(Z) \to 0.
\]

It follows that \( H_0^{n-1}(Z) \) has a mixed Hodge structure. The graded pieces of the Hodge filtration will be denoted \( \text{Gr}^p H_0^{n-1}(Z) \). In [Bat4, Cor. 3.14], Batyrev observes that (6.4) can be reformulated as

\[
(6.6) \quad \dim \text{Gr}^p H_0^{n-1}(Z) = \phi_{p+1}(\Delta).
\]

Example. Let \( \Delta \) be a convex integer polygon in \( \mathbb{R}^2 \). Then \( Z \subset (\mathbb{C}^*)^2 \) is a curve which can be obtained by removing \( m \) points from a smooth complete curve of genus \( g \). As is well-known, this determines the Hodge numbers of \( H^1(Z) \), and using (6.5), we obtain \( h^{0,1}(H_0^1(Z)) = h^{0,1}(H_0^1(Z)) = g \) and \( h^{1,1}(H_0^1(Z)) = m - 3 \). However, (6.6) and (6.3) imply that

\[
h^{0,1}(H_0^1(Z)) = \phi_1(\Delta) = l^*(\Delta)
\]

\[
h^{1,0}(H_0^1(Z)) + h^{1,1}(H_0^1(Z)) = \phi_2(\Delta) = l(\Delta) - 3.
\]
It follows that \( g = l^*(\Delta) \) and \( m = l(\Delta) - l^*(\Delta) \) is the number of integer points on the boundary of \( \Delta \). Thus we can see the mixed Hodge structure of \( Z \) geometrically. This example can also be done using the formulas of [DK].

The next step is to represent cohomology classes in \( H^{n-1}_0(Z) \) algebraically. For this purpose, let \( g = t_0 f(t_1, \ldots, t_n) - 1 \), which is in the ring \( S_\Delta \) introduced in §5. We also set
\[
g_i = t_i \frac{\partial g}{\partial t_i}, \quad 0 \leq i \leq n.
\]
Then \( g_0 = t_0 f \) and \( g_i \in S_\Delta \) has degree 1 in \( S_\Delta \) because \( t_0 \) appears to the first power. These polynomials generate the graded ideal \( J_{f, \Delta} = \langle g_0, \ldots , g_n \rangle \subset S_\Delta \).

The following result is proved in [Bat4, Sect. 6].

**Theorem 6.1.** There is a natural isomorphism
\[
\text{Gr}_T^p H^{n-1}_0(Z) \simeq (S_\Delta / J_{f, \Delta})_{n-p}.
\]

The idea behind this isomorphism is that a polynomial \( t_0^{-p} g(t_1, \ldots, t_n) \in (S_\Delta)_{n-p} \) gives a \( n \)-form
\[
g \frac{dt_1}{f^{n-p} t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}
\]
on \( T - Z \) whose residue lies in \( \text{Gr}_T^p H^{n-1}_0(Z) \).

**Cohomology of projective hypersurfaces.** We now turn our attention to the projective hypersurface \( Y = Y_f \subset X = X_\Delta \), where as usual \( f \in L(\Delta) \) is non-degenerate. In addition, we will assume that \( X \) is simplicial. Then the natural map
\[
H^i(X) \rightarrow H^i(Y)
\]
is an isomorphism for \( i < n - 1 \) and is injective for \( i = n - 1 \). In analogy with the affine case, we define the \textit{primitive cohomology} by the exact sequence
\[
0 \rightarrow H^{n-1}(X) \rightarrow H^{n-1}(Y) \rightarrow H^{n-1}_0(Y) \rightarrow 0.
\]
Since \( X \) is simplicial, \( Y \) is quasi-smooth and hence \( H^{n-1}_0(Y) \) has a pure Hodge structure. Letting \( h^{p,n-1}_0 \) denote the dimension of the appropriate Hodge component, [DK, 5.5] can be restated as
\[
h^{p,n-1}_0 = (-1)^n \sum_{\dim \Gamma > p} (-1)^{\dim \Gamma} \phi_{\dim \Gamma - p}(\Gamma),
\]
where \( \phi(\Gamma) \) is as in (6.3). This formula implies that \( h^{n-1,0}_0 = \phi_1(\Delta) = l^*(\Delta) \).

We have already seen that \( Y \subset X \) can be defined in two ways, either using \( f \in L(\Delta) \) or \( F \in S_\beta \), where \( f \) and \( F \) are related via (6.2). Then we have the following results from [Bat4] and [BC].

**Theorem 6.2.** Let \( X = X_\Delta \) be simplicial and \( Y = Y_f \) be nondegenerate. If \( I^{(1)} \subset S_\Delta \) be the ideal generated by those \( t_i^m \in S_\Delta \) for which \( m \) is an interior point of \( k \Delta \), and \( H_f \) is its image in \( S_\Delta / J_{f, \Delta} \), then there are natural isomorphisms
\[
H^{p,n-1}_0(Y) \simeq \text{Gr}_T^p \mathcal{W}_{n-1} H^{n-1}_0(Z) \simeq (H_f)_{n-p},
\]
where \( Z = Z_f \) is the corresponding affine hypersurface.
**Theorem 6.3.** Let \( X = X_\Delta \) be simplicial and \( Y = Y_F \) be quasi-smooth. If \( J(F) = \langle \frac{\partial F}{\partial x_\rho} \rangle \subset S \) be the Jacobian ideal of \( F \), and \( \beta_0 = [\sum_\rho D_\rho] \) be the anticanonical class, then there is a natural isomorphism

\[
H_0^{p,n-1-p}(Y) \simeq (S/J(F))_{(n-p)\beta-\beta_0}
\]

when \( p \neq n/2 - 1 \), and for \( p = n/2 - 1 \), we have an exact sequence

\[
0 \to H^{n-2}(X) \xrightarrow{\cup [Y]} H^n(X) \to (S/J(F))_{(n/2+1)\beta-\beta_0} \to H_0^{n/2,n/2}(Y) \to 0.
\]

Theorem 6.3 generalizes a classic result of Griffiths on the cohomology of projective hypersurfaces (see [Gri] and [PS]). The basic idea is similar to Theorem 6.1: a homogeneous polynomial \( G \in S_{(n-p)\beta-\beta_0} \) gives a \( n \)-form on \( X - Y \) whose residue lies in \( H_0^{n-1-p}(Y) \) (see [BC] for a precise description). The complication in the case \( p = n/2 - 1 \) comes from the exact sequence

\[
0 \to H^{n-2}(X) \xrightarrow{\cup [Y]} H^n(X) \to H^n(X - Y) \to H_0^{n-1}(Y) \to 0.
\]

When \( X \) is projective space (or even a weighted projective space), the sequence implies \( H^n(X - Y) \simeq H_0^{n-1}(Y) \), but this can fail in the toric case. Fortunately, this only affects \( H_0^{n/2-1,n/2}(Y) \).

**Remarks.** 1. While Theorem 6.1 uses the vanishing of \( H^i(X, O_X(Y)) \) for \( Y \) ample and \( i > 0 \), Theorem 6.3 uses the Bott-Steenbrink-Danilov vanishing theorem:

\[
(6.7) \quad H^i(X, \Omega^p_X(Y)) = 0, \quad Y \text{ ample, } i > 0.
\]

This result is stated in [Dan2] and [Oda2] without proof. In the simplicial case, a proof appeared in [BC, Sect. 7], and a general proof (using characteristic \( p > 0 \)) can be found in [BFLM]. A generalization of (6.7) is the vanishing theorem:

\[
(6.8) \quad H^i(X, \mathcal{W}_k \Omega^p_X(log(-K))(Y)) = 0, \quad Y \text{ ample, } i > 0.
\]

Here, \( \mathcal{W}_k \Omega^p_X(log(-K)) = \hat{\Omega}^{p-k}_X \wedge \hat{\Omega}^k_X (log(-K)) \) is the usual weight filtration and \(-K = \sum_\rho D_\rho \) is the anticanonical divisor. When \( X \) is simplicial, (6.8) was proved in [BC], but the general case is still open.

2. Using the isomorphism (5.3), one can relate the ideal \( H_f \subset S_\Delta/J_{f,\Delta} \) to the ideal generated by \( \Pi_\rho x_\rho \) in \( S/(x_\rho \partial F/\partial x_\rho) \). This leads to a natural isomorphism

\[
H_0^{p,n-1-p}(Y) \simeq (S/J_1(F))_{(n-p)\beta-\beta_0},
\]

where \( J_1(F) \) is the ideal quotient

\[
J_1(F) = \langle x_\rho \partial F/\partial x_\rho \rangle : \Pi_\rho x_\rho
\]

(see [BC, Sect. 11]). For a weighted projective space, \( J(F) \) equals \( J_1(F) \), but in general the relation between these ideals is not well understood. The ideal \( J_1(F) \) arises naturally in certain mirror symmetry contexts (see [MP, 5.36]).
It can also happen that one is interested in a hypersurface $Y \subset X = X_\Delta$ which is big and nef but not ample. In the toric context, big and nef mean that $Y$ corresponds to a $n$-dimensional integer polytope $\Delta'$ which is a Minkowski summand of $\Delta$ (i.e., $\Delta' + \Delta'' = \mu \Delta$ for some integer polytope $\Delta''$ and $\mu \in \mathbb{Z}$—see [BB3, Sect. 2]). In this case, the map $H^i(X) \to H^i(Y)$ need not be an isomorphism for $i < n-1$, and Theorems 6.2 and 6.3 can also fail. An example of the latter is given by the proper transform of a degree 8 hypersurface in a resolution of $\mathbb{P}(1,1,2,2,2)$.

We will say more about this when we study Calabi-Yau hypersurfaces in §8.

Besides the cohomology of toric hypersurfaces, one can also study their moduli (see [AGM2], [Bat4] and [BC]). The resulting variations of Hodge structure are closely connected with hypergeometric functions, which will be discussed in §9.

**Complete intersections.** Complete intersections in toric varieties can be studied from several points of view. In the affine case, suppose we have $f_i \in L(\Delta_i)$ as in Proposition 5.1. Then we get the affine complete intersection

$$Z_{f_1} \cap \cdots \cap Z_{f_s} \subset T.$$  

To compute the cohomology of this variety, we use the *Cayley trick*. Consider the toric variety $T \times \mathbb{C}^s$ with variables $t_1, \ldots, t_n, \lambda_1, \ldots, \lambda_s$, and let

$$\mathcal{F} = \lambda_1 f_1 + \cdots + \lambda_s f_s - 1.$$  

This gives the affine hypersurface $Z_{\mathcal{F}} \subset T \times \mathbb{C}^s$, and one obtains

$$E(Z_{f_1} \cap \cdots \cap Z_{f_s}; u, v) = (uv - 1)^s - (uv)^{1-s} E(Z_{\mathcal{F}}; u, v)$$

under suitable nondegeneracy hypotheses. This follows by considering the projection $Z_{\mathcal{F}} \to T$ (see [DK, 6.2]). Hence we can reduce to the hypersurface case (note, however, that $Z_{\mathcal{F}}$ is a hypersurface in $T \times \mathbb{C}^s$ instead of a torus).

Turning to the projective case, let $\Delta = \Delta_1 + \cdots + \Delta_s$ be as in (5.4), and consider the complete intersection

$$Y_{f_1} \cap \cdots \cap Y_{f_s} \subset X = X_\Delta,$$

which we assume to be nondegenerate. Now consider the projective bundle

$$\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{O}_X(D_1) \otimes \cdots \otimes \mathcal{O}_X(D_s)),$$

where $D_i$ is the divisor corresponding to $\Delta_i$ in Proposition 5.1. If $\pi : \mathbb{P}(\mathcal{E}) \to X$ is the natural projection, then $\pi_* (\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = \mathcal{O}_X(D_1) \otimes \cdots \otimes \mathcal{O}_X(D_s)$. Hence there is a unique section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ corresponding to $(f_1, \ldots, f_s)$. This section defines a hypersurface $\mathcal{Y} \subset \mathbb{P}(\mathcal{E})$, and one can show that the natural map

$$\mathbb{P}(\mathcal{E}) - \mathcal{Y} \to X - Y_{f_1} \cap \cdots \cap Y_{f_s}$$

is a $\mathbb{C}^{s-1}$ bundle in the Zariski topology. Hence

$$H^i_c(X - Y_{f_1} \cap \cdots \cap Y_{f_s}) \simeq H^{i+2(s-1)}_c(\mathbb{P}(\mathcal{E}) - \mathcal{Y}),$$

$$H^i_c(X - Y_{f_1} \cap \cdots \cap Y_{f_s}) \simeq H^{i+2(s-1)}_c(\mathbb{P}(\mathcal{E}) - \mathcal{Y}),$$
which is an isomorphism of mixed Hodge structures of degree \((s - 1, s - 1)\). This is discussed in more detail in [BB1].

From the point of view of the homogeneous coordinate ring \(S\) of \(X\), each \(f_i\) corresponds to a homogeneous polynomial \(F_i \in S\). If we assume that each \(O_X(D_i)\) is ample, then it follows from [CCD] that the homogeneous coordinate ring of \(\mathbb{P}(\mathcal{E})\) is \(S \otimes \mathbb{C}[y_1, \ldots, y_s]\), where the variables \(y_i\) have the property that

\[
F = y_1 F_1 + \cdots + y_s F_s
\]

is homogeneous and defines the hypersurface \(Y \subset \mathbb{P}(\mathcal{E})\). This explains why the above construction can be regarded as the projective version of the Cayley trick.

It should also possible to represent primitive cohomology classes of \(Y\) using the Jacobian ideal of \(F \in S \otimes \mathbb{C}[y_1, \ldots, y_s]\). This has not been done in general, but the case \(X = \mathbb{P}^n\) has been studied in [ENV], [Konn], [LT] and [Ter]. Furthermore, [Dim] and [Nag] treat this case from the toric point of view, which we now explain. Let \(F_1, \ldots, F_s\) be homogeneous polynomials (in the usual sense) in \(\mathbb{C}[x_0, \ldots, x_n]\) of degrees \(d_1, \ldots, d_s\) at least 2. This gives the complete intersection \(Y_{F_1} \cap \cdots \cap Y_{F_s} \subset \mathbb{P}^n\) of dimension \(n - s\). The homogeneous coordinate ring of \(\mathbb{P}(\mathcal{E})\) is \(R = \mathbb{C}[x_0, \ldots, x_n, y_1, \ldots, y_s]\) and is graded by \(\mathbb{Z}^2\), where

\[
\deg(x_i) = (1, 0), \quad \deg(y_j) = (-d_j, 1).
\]

The polynomial \(F\) of (6.9) has degree \(\beta = (0, 1)\) in \(R\) and, defining primitive cohomology in the usual way, one obtains a natural isomorphism

\[
H^n_{p,n-s-p}(Y_{F_1} \cap \cdots \cap Y_{F_s}) \simeq (R/J(F))_{(n-p)\beta-\beta_0},
\]

where \(\beta = \deg(F) = (0, 1), \ \beta_0 = \deg(x_0 \cdots x_n y_1 \cdots y_s) = (n + 1 - \sum_{j=1}^s d_j, s)\), and \(J(F)\) is the Jacobian ideal. Note the similarity with the second part of Theorem 6.3. It should be possible to prove a version of (6.10) for complete intersections of ample hypersurfaces in an arbitrary complete simplicial toric variety.

§7. Secondary fans and polytopes

The secondary polytope was discovered in the year 1988 by Gelfand, Kapranov and Zelevinsky [GKZ1], with further developments by Billera, Filliman and Sturmfels [BFS] and Oda and Park [OP]. We will discuss the secondary polytope and its normal fan, which is called the secondary polytope or GKZ decomposition. At the end of the section we will also mention the Gröbner fan of a toric ideal.

The secondary fan. We begin with the secondary fan, following [OP]. As noted in §3, different toric varieties can have closely related Kähler cones. This happens because different fans in \(N_R\) can share the same 1-dimensional cones. To study the general case, fix a finite set of primitive vectors \(\mathcal{B} \subset N\), where as usual \(N \simeq \mathbb{Z}^n\) and \(M\) is its dual, and assume that \(\text{Cone}(\mathcal{B}) = N_R\). We will consider all complete fans \(\Sigma\) in \(N_R\) whose 1-dimensional cone generators \(\Sigma(1)\) lie in \(\mathcal{B}\).

Given \(\mathcal{B}\), we define \(A_B \simeq \mathbb{R}^{[B]-n}\) by the exact sequence

\[
0 \rightarrow M_R \xrightarrow{\alpha} \bigoplus_{\rho \in \mathcal{B}} \mathbb{R} \cdot e_\rho \xrightarrow{\beta} A_B \rightarrow 0,
\]
where \( \alpha(m) = \sum_{\rho \in \mathcal{B}} \langle m, \rho \rangle e_\rho \). Also let \( \mathcal{B}^0 = \{ \beta(e_\rho) : \rho \in \mathcal{B} \} \subset A_\mathcal{B} \). Then the pair \( (\mathcal{B}^0, A_\mathcal{B}) \) is the linear Gale transform of \( (\mathcal{B}, N_\mathcal{B}) \). The cone
\[
A_\mathcal{B}^+ = \text{Cone}(\mathcal{B}^0) = \{ \sum_{\rho \in \mathcal{B}} a_\rho \beta(e_\rho) : a_\rho \geq 0 \} \subset A_\mathcal{B},
\]
is strongly convex since \( \text{Cone}(\mathcal{B}) = N_\mathcal{B} \). From (2.2), we see that if \( \Sigma \) is a fan with \( \Sigma(1) = \mathcal{B} \), then \( A_\mathcal{B} = A_{n-1}(X_\Sigma) \otimes \mathbb{R} \) and \( A_\mathcal{B}^+ \) is the cone generated by effective divisors on \( X_\Sigma \).

Now suppose that \( \Sigma \) is a simplicial projective fan with \( \Sigma(1) = \mathcal{B} \). From §3, we have the cone \( \text{cpl}(\Sigma) \subset A_\mathcal{B} \) of convex classes, and it is easy to see that \( \text{cpl}(\Sigma) \subset A_\mathcal{B}^+ \).

Furthermore, the interior of \( \text{cpl}(\Sigma) \) is the Kähler cone of \( X_\Sigma \). In the example from §3, these cones filled up \( A_\mathcal{B}^+ \) as we varied \( \Sigma \). In general, this doesn’t always happen, which is why we must allow fans \( \Sigma \) where \( \Sigma(1) = \mathcal{B} \) when \( \Sigma \) is projective. According to [OP, Cor. 3.6], the cones \( \text{cpl}(\Sigma) \) fit together as follows.

**Theorem 7.1.** If \( \text{Cone}(\mathcal{B}) = N_\mathcal{B} \), then, as \( \Sigma \) ranges over all simplicial projective fans in \( N_\mathcal{B} \) with \( \Sigma(1) \subset \mathcal{B} \), the cones \( \text{cpl}(\Sigma) \) and their faces form a fan in \( A_\mathcal{B} \) whose support is \( A_\mathcal{B}^+ \).

The fan of this theorem is the **secondary fan** or **GKZ decomposition** of \( \mathcal{B} \). An example with two maximal cones was given in §3. In general, the structure of the secondary fan is quite interesting. For example, two cones \( \text{cpl}(\Sigma) \) and \( \text{cpl}(\Sigma') \) with a common codimension 1 face are related by a “flop” (as in the example from §3) or by adding or subtracting a single 1-dimensional cone to \( \Sigma \). Also, faces on the boundary of \( A_\mathcal{B}^+ \) correspond to certain “degenerate” fans. This is all explained in [OP, Sect. 3].

The secondary fan has a nice relation to the moment map from §4. If you look back at its construction, you’ll see that the group \( G \) and the moment map \( \mu_\Sigma \) of \( X_\Sigma \) depend only on \( \Sigma(1) \). Thus, given \( \mathcal{B} \subset N_\mathcal{B} \) as above, we get the moment map
\[
\mu_\mathcal{B} : \mathbb{C}^\mathcal{B} \longrightarrow A_\mathcal{B}^+.
\]
By Theorem 4.1, if \( a \) is in the interior of \( \text{cpl}(\Sigma) \), then
\[
\mu_\mathcal{B}^{-1}(a)/G_\mathcal{B} \simeq X_\Sigma
\]
when \( \Sigma \) is projective, simplicial and satisfies \( \Sigma(1) = \mathcal{B} \). In [CK], it is shown that this holds under the weaker hypothesis \( \Sigma(1) \subset \mathcal{B} \) (with \( \Sigma \) still projective and simplicial). Hence, the secondary fan gives a complete picture of the toric varieties we can build via symplectic reduction from a given moment map.

**The secondary polytope.** Given a finite subset \( \mathcal{A} \subset N_\mathcal{B} \simeq \mathbb{R}^n \), we can describe a very interesting polytope using certain triangulations of its convex hull \( \Delta^\mathcal{A} = \text{Conv}(\mathcal{A}) \). We will assume that \( \Delta^\mathcal{A} \) is \( n \)-dimensional. Then a **triangulation** of \( \mathcal{A} \) is a triangulation \( T \) of \( \Delta^\mathcal{A} \) such that the vertices of each simplex in \( T \) lie in \( \mathcal{A} \) (though not every element of \( \mathcal{A} \) need be used). Furthermore, the triangulation \( T \) is **regular**
or coherent if there is a function $\psi : \Delta^* \to \mathbb{R}$ which is affine on each simplex of $T$ and strictly convex. See [GKZ1] or [BFS] for precise definitions and proofs of the results stated below.

If $T$ is a triangulation of $\mathcal{A}$, we get the point

$$
\phi_T = \sum_{\sigma \in T} \sum_{u \in \sigma} \text{Vol}(\sigma) e_u \in \bigoplus_{u \in \mathcal{A}} \mathbb{R} \cdot e_u.
$$

The convex hull of these points is the secondary polytope of $\mathcal{A}$, denoted $\Sigma(\mathcal{A})$. One can show that the dimension of $\Sigma(\mathcal{A})$ is $|\mathcal{A}| - n - 1$ and that its vertices are precisely the points $\phi_T$ for which $T$ is a regular triangulation of $\mathcal{A}$. This polytope has some very interesting combinatorial properties [GKZ1], [Stu2].

The secondary polytope also has a normal fan $\mathcal{N}((\Sigma(\mathcal{A})))$, which is described using an affine version of the Gale transform. From $\mathcal{A}$, we get the set $\mathcal{A} \times \{1\} \subset \mathbb{R} \otimes \mathbb{R}$, and as in (7.1), we can construct an exact sequence

$$
0 \longrightarrow \mathbb{M}_\mathbb{R} \oplus \mathbb{R} \xrightarrow{\alpha} \bigoplus_{u \in \mathcal{A}} \mathbb{R} \cdot e_u \xrightarrow{\beta} \mathcal{A}_\mathbb{A} \longrightarrow 0,
$$

where $\alpha(m, \lambda) = \sum_{u \in \mathcal{A}} (\langle m, u \rangle + \lambda) e_u$. Also set $\mathcal{A}^0 = \{\beta(e_u) : u \in \mathcal{A}\}$. Then the pair $(\mathcal{A}^0, \mathcal{A}_\mathbb{A})$ is the affine Gale transform of $(\mathcal{A}, \mathcal{N}_\mathbb{R})$. Note that $\sum_{u \in \mathcal{A}} \beta(e_u) = 0$ and that $\mathcal{A}_\mathbb{A}$ has dimension $|\mathcal{A}| - n - 1$.

Using the dual map $\beta^* : \mathcal{A}^*_\mathbb{A} \to \bigoplus_{u \in \mathcal{A}} \mathbb{R} \cdot e_u$, one can show that the image of $\beta^*$ is parallel to the affine span of the secondary polytope $\Sigma(\mathcal{A})$. It follows that the normal fan of $\Sigma(\mathcal{A})$ lives naturally in $\mathcal{A}_\mathbb{A}$. This is the secondary fan of $\mathcal{A}$, denoted $\mathcal{N}(\Sigma(\mathcal{A}))$. The maximal cones of the secondary fan correspond to vertices of $\Sigma(\mathcal{A})$ and hence to regular triangulations of $\mathcal{A}$. Note also that $\mathcal{N}(\Sigma(\mathcal{A}))$ is a complete fan in $\mathcal{A}_\mathbb{A}$ since it comes from a polytope.

The enlarged secondary fan. The secondary fan $\mathcal{N}(\Sigma(\mathcal{A}))$ might seem rather different from the secondary fan defined earlier. To see the relation, let $\mathcal{B} = \Sigma(1)$, where $\Sigma$ is a projective fan, not necessarily simplicial. Then, as before, we get the secondary fan of $\mathcal{B}$, whose support is the strongly convex cone $A_\mathbb{B}^+ \subset A_\mathbb{B}$. Following [AGM2], we can enlarge this to a complete fan as follows. Fix an ample divisor $D = \sum_{\rho \in \mathcal{B}} a_{\rho} D_{\rho}$ with $a_\rho > 0$ for all $\rho$. This implies that 0 is an interior point of the corresponding polytope $\Delta \subset \mathbb{M}_\mathbb{R}$. Then the dual polytope $\Delta^\circ \subset \mathbb{N}_\mathbb{R}$ is defined by

$$
\Delta^\circ = \{u \in \mathbb{N}_\mathbb{R} : \langle m, u \rangle \geq -1 \text{ for all } m \in \Delta\}.
$$

Note that 0 is an interior point of $\Delta^\circ$, though $\Delta^\circ$ need not be integral. Also, the normal fan $\Sigma$ of $\Delta$ is obtained by taking cones over proper faces of $\Delta^\circ$. In particular, the vertices of $\Delta^\circ$ are $(1/a_{\rho}) \rho$ for $\rho \in \mathcal{B} = \Sigma(1)$. Now let

$$
\mathcal{A} = \text{Vert}(\Delta^\circ) \cup \{0\} = \{(1/a_{\rho}) \rho : \rho \in \Sigma(1)\} \cup \{0\}.
$$

We leave it to the reader to show that there is a natural isomorphism $A_{\mathcal{A}} \simeq A_{\mathcal{B}}$ which carries $\mathcal{A}^0$ to $\mathcal{B}^0 \cup \{- \sum_{\rho \in \mathcal{B}} a_{\rho} \beta(e_\rho)\}$. Under this isomorphism, we can compare the secondary fans of $\mathcal{A}$ and $\mathcal{B}$ as follows.
Proposition 7.2. Let $\mathcal{A}$ and $\mathcal{B}$ be as above. Then:

1. Under the isomorphism $A_\mathcal{A} \simeq A_\mathcal{B}$, the cone of $\mathcal{N}(\Sigma(\mathcal{A}))$ given by a regular triangulation $T$ of $\mathcal{A}$ lies in $A_\mathcal{B}^+$ if and only if $0$ is contained in every maximal simplex of $T$.

2. For such a triangulation $T$, let $\Sigma'$ be the fan obtained by taking cones over simplices of $T$. Then $\Sigma'$ is a projective simplicial fan with $\Sigma'(1) \subset \mathcal{B}$, and the cone $\text{cpl}(\Sigma')$ is the cone of $\mathcal{N}(\Sigma(\mathcal{A}))$ given by to $T$.

3. Conversely, every maximal cone $\text{cpl}(\Sigma') \subset A_\mathcal{B}^+$ comes from such a triangulation $T$. We get $T$ by intersecting $\Delta^\circ$ with the cones in $\Delta^\circ$.

In this situation, we call $\mathcal{N}(\Sigma(\mathcal{A}))$ the enlarged secondary fan of $\mathcal{B}$. A consequence of this proposition is that there is a bijective correspondence between maximal cones of the secondary fan of $\mathcal{B}$ and regular triangulations of $\mathcal{A}$ whose maximal cones all contain $0$.

Example. The toric variety $X = X_\Sigma$ from §3 has cone generators

$$e_0 = (0, 0, -2), \quad e_1 = (1, 1, 1), \quad e_2 = (1, -1, 1), \quad e_3 = (-1, -1, 1), \quad e_4 = (-1, 1, 1)$$

and maximal cones $\sigma_{1234}, \sigma_{012}, \sigma_{023}, \sigma_{034}$ and $\sigma_{041}$. Let $\Delta$ be polytope associated to the anticanonical divisor $D_0 + D_1 + D_2 + D_3 + D_4$, which is ample. One can compute that the dual polytope of $\Delta$ is

$$\Delta^\circ = \text{Conv}(e_0, e_1, e_2, e_3, e_4).$$

In §8, we will see that $\Delta$ and $\Delta^\circ$ are examples of reflexive polytopes.

Using $\mathcal{A} = \text{Vert}(\Delta^\circ) \cup \{0\} = \{e_0, e_1, e_2, e_3, e_4, 0\}$, one sees that there are exactly four triangulations of $\mathcal{A}$, which means that the enlarged secondary fan has four maximal cones. Two of the triangulations have $0$ as a vertex, and the corresponding cones were described in §3. We leave it to the reader to determine the other two cones in the fan and the corresponding triangulations of $\mathcal{A}$.

One can also describe the enlarged secondary fan using the total space of the line bundle over $X_\Delta$ given by $-\sum_\rho a_\rho D_\rho$ (see [AGM2]). This is closely related to an alternate approach to the secondary fan $\mathcal{N}(\Sigma(\mathcal{A}))$ which appears in [OP]. We will see in §10 that the enlarged secondary fan and the secondary polytope are used in mirror symmetry.

Other references for secondary polytopes and the GKZ-decomposition are [Loc] and [Par]. A significant generalization of the secondary polytope, called the fiber polytope, is described in [BS] and [Zie].

The Gröbner fan. If our finite set $\mathcal{A}$ is integral, i.e., $\mathcal{A} = \{u_1, \ldots, u_\ell\} \subset N$, then we can use term orders on a toric ideal to create a fan closely related to the secondary fan of $\mathcal{A}$. The basic idea is that $\mathcal{A} \times \{1\} \subset N \oplus \mathbb{Z}$ determines the toric ideal $I_{\mathcal{A} \times \{1\}} \subset \mathbb{C}[x_1, \ldots, x_\ell]$. This ideal defines the affine toric variety $\mathbb{A}^{\mathcal{A} \times \{1\}}$, which by §5 is the affine cone over $X_\mathcal{A} \subset \mathbb{P}^{\ell-1}$. For this reason, we write the toric ideal as $I_{\mathcal{A}}$ instead of $I_{\mathcal{A} \times \{1\}}$. By Lemma 5.5, we can express this ideal as

$$I_{\mathcal{A}} = (x^a^+ - x^a^- : a = (a_1, \ldots, a_\ell) \in \mathbb{Z}_\ell, \sum_{i=1}^\ell a_im_i = 0, \sum_{i=1}^\ell a_i = 0).$$
We can use elements \( \omega \in \mathbb{R}^\ell \) to create initial ideals of \( I_A \) (in the sense of Gröbner theory) as follows: if \( f = \sum a_c x^a \), define \( \text{LT}_\omega(f) = \sum \omega \cdot \text{maximal } c_a x^a \), and then set
\[
\text{LT}_\omega(I_A) = \langle \text{LT}_\omega(f) : f \in I_A \rangle.
\]
For generic \( \omega \), this is the initial ideal of \( I_A \) for some term order, and one can show that every initial ideal arises in this way [Stu2]. The interesting aspect is that different \( \omega \)'s can give the same initial ideal. We define
\[
\omega \sim \omega' \iff \text{LT}_\omega(I_A) = \text{LT}_{\omega'}(I_A).
\]
One can prove that each equivalence class is a relatively open convex polyhedral cone. Furthermore, if \( \alpha \) is the map from (7.3), then \( \omega + \alpha(m, \lambda) \sim \omega \) for all \( (m, \lambda) \in M_\mathbb{R} \oplus \mathbb{R} \). Hence it makes sense to take the quotient by the image of \( \alpha \). This leads to the following result.

**Proposition 7.3.** Under the map \( \beta : \mathbb{R}^\ell \to A_A \) from (7.3), we have:

1. The closures of the images of the equivalence classes form a complete fan in \( A_A \) called the Gröbner fan.
2. The Gröbner fan of \( A \) refines the secondary fan \( \mathcal{N}(\Sigma(A)) \).

This is proved in [Stu2]. One can also show that the Gröbner fan is the normal fan of a polytope called the state polytope. Algorithms for computing the state polytope and Gröbner fan are given in [Stu2], and the reader may also wish to consult [MR]. Although the Gröbner fan of \( I_A \) may be strictly finer than the secondary fan of \( A \), there are situations where the Gröbner fan is very useful. An example from mirror symmetry can be found in [HLY1].

### §8. Reflexive polytopes and Calabi-Yau hypersurfaces

In this section, we will create some interesting families of Calabi-Yau varieties using toric geometry. The key idea will be Batyrev’s notion of reflexive polytopes. At the end of the section we will also consider complete intersections and nef-partitions.

**Singular Calabi-Yau varieties.** The quintic threefold in \( \mathbb{P}^4 \) is one of the best-known examples of a Calabi-Yau manifold. However, there are many contexts (including toric geometry) where singular Calabi-Yau varieties arise naturally. Hence we define Calabi-Yau as follows. Let \( Y \) be a \( d \)-dimensional normal projective complex variety. Assume \( O_Y(K_Y) = \hat{\Omega}_Y^d \simeq O_Y \) and
\[
H^1(Y, O_Y) \simeq \cdots \simeq H^{d-1}(Y, O_Y) \simeq \{0\}.
\]
Then \( Y \) is *canonical Calabi-Yau* if it has at worst canonical singularities. Furthermore, \( Y \) is *minimal Calabi-Yau* if it has at worst \( \mathbb{Q} \)-factorial terminal singularities.

Since canonical singularities are Cohen-Macaulay and a Calabi-Yau has trivial canonical bundle, a singular Calabi-Yau is always Gorenstein. Thus its singularities are either Gorenstein canonical or \( \mathbb{Q} \)-factorial Gorenstein terminal.

To understand why these singularities are appropriate for Calabi-Yau varieties, recall that a singularity is *canonical* if there is a local resolution of singularities \( f : \tilde{Y} \to Y \) such that
\[
K_{\tilde{Y}} = f^*(K_Y) + \sum a_i E_i,
\]
where the sum is over the exceptional divisors \( E_i \) of \( f \) and \( a_i \geq 0 \). If in addition we have \( a_i > 0 \) for all \( i \), then the singularity is \textit{terminal}. Thus having terminal singularities means that any resolution must change the canonical class. In particular, if \( Y \) is a singular minimal Calabi-Yau, then its resolutions cease to be Calabi-Yau since the canonical class is no longer trivial.

As for canonical singularities, there are many situations (including threefolds and the Calabi-Yaus to be constructed below) where having canonical singularities implies the existence of a partial resolution \( \hat{f} : \hat{Y} \to Y \) such that \( K_{\hat{Y}} = f^*(K_Y) \) (so the canonical class doesn’t change) and \( \hat{Y} \) has \( \mathbb{Q} \)-factorial terminal singularities (so any further resolution changes the canonical class). Thus, if we start from a singular canonical Calabi-Yau \( Y \), then \( \hat{Y} \) (if it exists) is a minimal Calabi-Yau which is as close as possible to being smooth while remaining Calabi-Yau. In Batyrev’s terminology, \( \hat{Y} \) is called a \textit{maximal projective crepant partial desingularization} of \( Y \) (a MPCP-desingularization for short).

For more background on canonical and terminal singularities (including some nice examples), see [Rei5].

**Reflexive polytopes and Fano toric varieties.** Following Batyrev [Bat1], we say that a \( n \)-dimensional integral convex polytope \( \Delta \subset \mathbb{M} \) is \textit{reflexive} if it contains the origin as an interior point and if its dual polytope

\[
\Delta^\circ = \{ u \in \mathbb{N} : \langle m, u \rangle \geq -1 \text{ for all } m \in \Delta \} \subset \mathbb{N}
\]

is also integral. Since \((\Delta^\circ)^\circ = \Delta\), reflexive polytopes always come in pairs. It also follows that each facet \( F \) of a reflexive polytope \( \Delta \) is defined by an equation \( \langle u_F, m \rangle = -1 \) for some \( u_F \in \mathbb{N} \). This easily implies that 0 is the only point of \( \text{M} \) in the interior of \( \Delta \). The polytopes \( \Delta \) and \( \Delta^\circ \) from (7.4) are an example of a reflexive polytope and its dual.

It is known (see [Bat1]) that in each dimension, there are only finitely many reflexive polytopes up to unimodular equivalence, and work is underway to classify \textit{all} reflexive polytopes in dimension 4 (see [Ska]).

Given a reflexive polytope \( \Delta \), we get a toric variety \( X_\Delta \). Since the facets of \( \Delta \) are given by \( \langle u_F, m \rangle = -1 \), it follows easily that the divisor on \( X_\Delta \) determined by \( \Delta \) is precisely the anticanonical divisor \( -K = \sum \rho D_\rho \). Also, the anticanonical divisor is ample, which tells us that \( X_\Delta \) is a Fano variety, and it is Gorenstein since \( K \) is obviously Cartier. Conversely, one can show that every Gorenstein Fano toric variety arises from a reflexive polytope (see [Bat1]).

**Calabi-Yau hypersurfaces.** Now suppose that \( Y \subset X_\Delta \) is a general anticanonical hypersurface in a Gorenstein Fano toric variety coming from a reflexive polytope \( \Delta \). Then the adjunction formula

\[
\Omega_Y^{n-1} \simeq \Omega_{X_\Delta}(-K) \otimes \mathcal{O}_Y
\]

shows that \( \Omega_Y^{n-1} \simeq \mathcal{O}_Y \). Since \( \mathcal{O}_{X_\Delta}(-Y) \simeq \mathcal{O}_{X_\Delta}(K) = \Omega_{X_\Delta}^n \), we also get an exact sequence

\[
0 \longrightarrow \Omega_{X_\Delta}^n \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0,
\]

which makes it easy to show that \( H^i(Y, \mathcal{O}_Y) = 0 \) for \( 0 < i < n-1 \) (this uses Serre-Grothendieck duality and the vanishing of \( H^i(X_\Delta, \mathcal{O}_{X_\Delta}) \) for \( i > 0 \)). Finally,
Bertini theorems show that \( Y \) has at most Gorenstein toric singularities, which are known to be canonical. It follows that \( Y \) is a canonical Calabi-Yau variety.

When \( Y \) is singular, we would like to desingularize it as much as possible while remaining Calabi-Yau. As one might expect, toric geometry tells us exactly what to do. We’ve seen that the cone generators of the normal fan are the vertices of \( \Delta^o \). We will consider projective simplicial fans \( \Sigma \) which refine the normal fan of \( \Delta \) and whose cone generators satisfy

\[
\Sigma(1) = N \cap \Delta^o - \{0\}.
\]

To see why this condition is relevant, remember that one desingularizes \( X_\Delta \) by subdividing cones into simplicial ones and then adding new cone generators until we get smooth cones. For example, adding a new cone generator \( \rho \) gives a birational map of toric varieties \( f : X \to X_\Delta \), and if \( \rho \) lies in the cone over the facet \( \langle u, m_F \rangle = -1 \) of \( \Delta^o \), then using the techniques of [Rei5], one obtains

\[
K_X = f^*(K_{X_\Delta}) - (\langle \rho, m_F \rangle + 1)D,
\]

where \( D \) is the exceptional divisor (and is the divisor of \( X \) corresponding to \( \rho \)). Thus, as long as we add new cone generators coming from \( N \cap \Delta^o \), we don’t change the canonical class. But once we’ve used up all of \( N \cap \Delta^o - \{0\} \), as \( \Sigma \) does, then any further \( \rho \)’s must lie outside of \( \Delta^o \) (since reflexive implies 0 is the only integer interior point). Hence \( \langle \rho, m_F \rangle < -1 \), and from (8.2), it follows that we have terminal singularities. Since \( \Sigma \) is simplicial, we see that \( X_\Sigma \) is \( \mathbb{Q} \)-factorial. Thus we have the following theorem (see [Bat1] for details).

**Theorem 8.1.** Let \( \Delta \subset M_\mathbb{R} \) be a reflexive polytope, and let \( \Sigma \) be a fan in \( N_\mathbb{R} \) which refines the normal fan of \( \Delta \) and satisfies (8.1). Then:

1. The general anticanonical hypersurface \( Y \subset X_\Delta \) is a canonical Calabi-Yau variety.
2. The general anticanonical hypersurface \( \hat{Y} \subset X_\Sigma \) is a minimal Calabi-Yau variety. Furthermore, \( \hat{Y} \) is a MPCP-desingularization of its image \( Y \subset X_\Delta \) under the map \( X_\Sigma \to X_\Delta \).

When \( \Delta \) is a 4-dimensional reflexive polytope, one can show that the MPCP-desingularization \( \hat{Y} \subset X_\Sigma \) is smooth Calabi-Yau threefold (see [Bat1]).

It follows that for each fan \( \Sigma \) as in the statement of the theorem, we get a family of minimal Calabi-Yau varieties. In terms of what we considered in §7, these fans correspond to certain maximal cones in the secondary fan of the set \( B = N \cap \Delta^o - \{0\} \subset N \).

Of course, since \( \Delta \) gives the family \( \hat{Y} \subset X_\Sigma \) of minimal Calabi-Yaus, we can use the reflexive polytope \( \Delta^o \) to construct a “dual” family \( \hat{Y}^o \subset X_{\Sigma^o} \) of minimal Calabi-Yau varieties, where \( \Sigma^o(1) = M \cap \Delta - \{0\} \). We will see in §10 that \( \hat{Y}^o \subset X_{\Sigma^o} \) is conjectured to be the mirror family of \( \hat{Y} \subset X_\Sigma \). Because of the previous paragraph, the situation is complicated by the multiple choices for \( \Sigma \) and \( \Sigma^o \). This is related to the idea of multiple mirrors, also to be studied in §10.

There are also some nice formulas for Hodge numbers. The MPCP-desingularization \( \hat{Y} \subset X_\Sigma \) is simplicial, so that \( H^*(\hat{Y}) \) has a pure Hodge structure. But we can’t apply the results of §6 directly since \( \hat{Y} \) may fail to be ample (this happens, for example, when \( X_\Delta = \mathbb{P}(1,1,2,2,2) \)). However, we have the following results from [Bat1].
Theorem 8.2. Let $\Delta$ be a reflexive polytope of dimension $n \geq 4$ and let $\Sigma$ be a fan as in Theorem 8.1. If $\hat{Y} \subset X_{\Sigma}$ is a general anticanonical hypersurface, then

$$h^{n-2,1}(\hat{Y}) = l(\Delta) - n - 1 - \sum_{\text{codim } \Gamma = 1} l^*(\Gamma) + \sum_{\text{codim } \Gamma = 2} l^*(\Gamma)l^*(\Gamma^\circ),$$

where $\Gamma$ is a face of $\Delta$, $\Gamma^\circ$ is the corresponding dual face of $\Delta^\circ$ and, as in §6, $l(\Gamma)$ (resp. $l^*(\Gamma)$) is the number of integer (resp. interior integer) points in $\Gamma$. Also,

$$h^{1,1}(\hat{Y}) = l(\Delta^\circ) - n - 1 - \sum_{\text{codim } \Gamma^\circ = 1} l^*(\Gamma^\circ) + \sum_{\text{codim } \Gamma^\circ = 2} l^*(\Gamma^\circ)l^*(\Gamma),$$

where now $\Gamma^\circ$ is a face of $\Delta^\circ$ and $\Gamma$ is the dual face.

The remarkable symmetry evident between the formulas for $h^{1,1}$ and $h^{n-2,1}$ will be important when we discuss mirror symmetry in §10. A different approach to the study of $h^{1,1}$ can be found in [Roa1].

Although the above construction seems completely natural, it can take some thought to find the reflexive polytope. For example, a degree 7 hypersurface in $\mathbb{P}(1,1,1,2,2)$ is Calabi-Yau, yet the simplex giving $\mathbb{P}(1,1,1,2,2)$ in not reflexive, nor is the hypersurface Cartier. Here, the reflexive polytope $\Delta$ is (up to translation) the Newton polytope of all monomials of degree 7, and the toric variety $X_\Delta$ is a blow-up of $\mathbb{P}(1,1,1,2,2)$. Some early evidence for mirror symmetry came from a list of 7555 Calabi-Yau hypersurfaces in weighted projective spaces. The list exhibited an incomplete duality which was only fully understood after the definition of reflexive polytope (see [CdK]).

Calabi-Yau complete intersections. We next generalize the above construction to create families of Calabi-Yau complete intersections. In §6, we discussed some basic facts about complete intersections in toric varieties, though the special features of the Calabi-Yau case are due to [LBor], with subsequent work by [BB1–3]. Details of what follows can be found in these references.

The basic way to describe a Calabi-Yau complete intersection in a toric variety is through the idea of a nef-partition. Suppose that $\Delta$ is a $n$-dimensional reflexive polytope, so that $\text{Vert}(\Delta^\circ)$ is the set of cone generators of the fan of $X = X_\Delta$. Then a nef-partition is a disjoint union

$$\text{Vert}(\Delta^\circ) = E_1 \cup \cdots \cup E_r$$

such that the divisors $D_j = \sum_{\rho \in E_j} D_\rho$ are Cartier and nef (i.e., generated by global sections). Equivalently, a nef-partition is a partition of the anticanonical divisor $-K = \sum_{\rho \in E_j} D_\rho$ into a sum of nef Cartier divisors. Furthermore, if $\Delta_j$ is the polytope associated to the divisor $D_j$, then it follows that

$$\Delta = \Delta_1 + \cdots + \Delta_r.$$ 

We will always assume that $n - r \geq 1$.

In this situation, let $Y_j \subset |D_j|$ be generic. Then the complete intersection

$$V = Y_1 \cap \cdots \cap Y_r \subset X_\Delta$$
is a canonical Calabi-Yau. Furthermore, as in Theorem 8.1, suppose that $\Sigma$ is a fan refining the normal fan of $\Delta$ with the property that $\Sigma(1) = N \cap \Delta^\circ - \{0\}$. We can regard $D_j$ as divisors on $X_\Sigma$, and if $\tilde{Y}_j$ is a general member of $|D_j|$, then, as shown in [BB3], the complete intersection

$$ (8.5) \quad \tilde{V} = \tilde{Y}_1 \cap \cdots \cap \tilde{Y}_r \subset X_\Sigma $$

is a minimal Calabi-Yau variety which is a MPCM-desingularization of the corresponding $V \subset X_\Delta$. There are formulas similar to those of Theorem 8.2 for the Hodge numbers $h^{n-r-1,1}(\tilde{V})$ and $h^{1,1}(\tilde{V})$.

In §10, we will see that the nef-partition giving the family of Calabi-Yau complete intersections $\tilde{V} \subset X_\Sigma$ naturally determines a “dual” family of Calabi-Yau complete intersections. This construction is used in mirror symmetry.

§9. Resultants, Discriminants and Hypergeometric Functions

This section will discuss briefly those aspects of resultants, discriminants and hypergeometric functions which pertain to toric varieties. Basic references are the book [GKZ1] and, for hypergeometric functions, the papers [GKZ2] and [GKZ3].

$A$-resultants. Given a finite set of exponents $A \subset \mathbb{Z}^n$, we get the vector space $L(A)$ of Laurent polynomials

$$ f = \sum_{m \in A} c_m t^m. $$

For simplicity, we will assume that $A$ is affinely independent in $\mathbb{Z}^n$. Then let $\nabla_A \subset L(A)^n+1$ be the Zariski closure of those $(n+1)$-tuples of polynomials $(f_0, \ldots, f_n) \in L(A)^n+1$ where there is $t \in (\mathbb{C}^*)^n$ such that $f_0(t) = \cdots = f_n(t) = 0$. One can show that $\nabla_A$ is a hypersurface in $L(A)^n+1$ and hence is defined by a polynomial $R_A = 0$. Thus $R_A$ is a polynomial in the coefficients $c_{i,m}$ of $f_i$, and in fact $R_A \in \mathbb{Z}[c_{i,m}]$. This polynomial is the $A$-resultant.

For example, if $A = \{0, 1, \ldots, d\} \subset \mathbb{Z}$, then $R_A$ is the usual resultant of two polynomials $f_0, f_1$ of degree $d$ in one variable. Many other examples can be found in [GKZ3] and [Stu4]. In general, the $A$-resultant is sometimes called a sparse resultant since only the exponents in $A$ are used. It is also possible to allow the polynomials $f_0, \ldots, f_n$ to have different exponents—this leads to what is known as the $(A_0, \ldots, A_n)$-resultant or mixed sparse resultant. Algorithms for computing sparse resultants are described in [CE] and [Stu4].

Chow forms. We can think of the $A$-resultant in toric terms as follows. If $A$ has $\ell$ elements, then we get the (possibly non-normal) toric variety $X_A \subset \mathbb{P}^{\ell-1}$ as in (5.6). This projective variety has a Chow form $R_{X_A}$, which is a polynomial in the Plücker coordinates $[m_0, \ldots, m_n]$ that vanishes precisely on those subspaces $L \in G(\ell, n-\ell-1)$ where $\mathbb{P}(L) \cap X_A \neq \emptyset$.

It is customary to rewrite the Chow form as follows. If we regard $L$ as being defined by $n+1$ linear forms, then the coefficients of these forms give a $(n+1) \times \ell$ matrix $(c_{i,m})$, where we use elements $m \in A$ to label the coordinates of $\mathbb{P}^{\ell-1}$. Then, replacing the Plücker coordinate $[m_0, \ldots, m_n]$ by the bracket polynomial $[m_0, \ldots, m_n] = \det(c_{i,m})$, we get a polynomial in the $c_{i,m}$ which is easily seen to coincide with the $A$-resultant $R_A$. 
Chow polytopes and secondary polytopes. Using the Chow form, we can create the Chow polytope as follows. If we express $R_{X_A}$ as a polynomial in the brackets $|\sigma| = [m_0, \ldots, m_n]$, then a term $T$ in $R_{X_A}$ is of the form $T = c\Pi_{a|\sigma}^{|\sigma|}$, $c \neq 0$, and we assign $T$ the weight

$$
\phi_T = \sum_{\sigma} \sum_{m \in |\sigma|} a_\sigma c_m \in \bigoplus_{m \in \Delta \mathcal{A}} \mathbb{R} \cdot e_m.
$$

The Chow polytope of $X_A$ is defined to be the convex hull of these points $\phi_T$.

The remarkable fact is that the Chow polytope of $X_A$ is exactly the secondary polytope $\Sigma(\mathcal{A})$ from §7 (which is the convex hull of the points $\phi_T$ defined in (7.2)). This is proved in [GKZ1, Ch. 9]. We know that the vertices of $\Sigma(\mathcal{A})$ correspond to regular triangulations of $\mathcal{A}$, and given such a triangulation, there is a precise formula for the corresponding term of $R_{X_A}$. One can also formulate this in terms of certain toric degenerations of $X_A$ (see [KSZ1]). Some explicit examples can be found in [Stu4], and [Stu3] has similar results concerning the $(\mathcal{A}_0, \ldots, \mathcal{A}_n)$-resultant.

$\mathcal{A}$-Discriminants. Given $X_A \subset \mathbb{P}^{\ell-1}$ as above, let $\nabla_{\mathcal{A}} \subset L(\mathcal{A})$ be the Zariski closure of those Laurent polynomials $f$ for which the affine hypersurface $Z_f \subset (\mathbb{C}^*)^n$ is singular. Then $\nabla_{\mathcal{A}}$ is the affine cone over the dual variety $X_A^\vee$. If $\nabla_{\mathcal{A}}$ has codimension 1, we define the $\mathcal{A}$-discriminant $\Delta_{\mathcal{A}}$ to be the defining equation of $\nabla_{\mathcal{A}}$ (or, equivalently, of $X_A^\vee$). We set $\Delta_{\mathcal{A}} = 1$ in all other cases. Note that $\Delta_{\mathcal{A}}$ is a polynomial in the coefficients $c_m$ of $f = \sum_{m \in \Delta \mathcal{A}} c_m t^m$, and, in fact, $\Delta_{\mathcal{A}} \in \mathbb{Z}[c_m]$.

Examples. 1. When $\mathcal{A}$ consists of all non-negative exponent vectors of degree at most $d$, $\Delta_{\mathcal{A}}$ is the usual discriminant of a homogeneous polynomial of degree $d$. More precisely, if $F$ is the homogenization of $f \in L(\mathcal{A})$, then $\Delta_{\mathcal{A}}(f) = \text{Disc}(F)$.

2. Consider all linear combinations of the monomials $1, x, \ldots, x^p, y, xy, \ldots, yx^q$. Then $\Delta_{\mathcal{A}}$ is the usual resultant $R(f, g)$ of polynomials of degrees $p$ and $q$ respectively. This is an example of the Cayley trick and can be used to express any $(\mathcal{A}_0, \ldots, \mathcal{A}_n)$-resultant as a discriminant (see [GKZ, Prop. 1.7 of Ch. 9]).

3. An example where $\Delta_{\mathcal{A}} = 1$ is given by the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^3$. More generally, same is true for the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ whenever $n \neq m$.

Principal $\mathcal{A}$-determinants. Given a Laurent polynomial $f \in L(\mathcal{A})$, the principal $\mathcal{A}$-determinant $E_{\mathcal{A}}(f)$ is defined to be the resultant

$$
E_{\mathcal{A}}(f) = R_{\mathcal{A}}(f, t_1 \partial f / \partial t_1, \ldots, t_n \partial f / \partial t_n).
$$

Note that the resultant makes sense since $t_i \partial f / \partial t_i \in L(\mathcal{A})$.

The nicest fact about $E_{\mathcal{A}}$ is that its Newton polytope is precisely the secondary polytope $\Sigma(\mathcal{A})$ (see [GKZ, Thm. 1.4 of Ch. 10]). One way to understand this result is to observe that $E_{\mathcal{A}}$ is the polynomial obtained from the Chow form $R_{X_A}$ under the specialization which sends the bracket $|\sigma|$ to $\pm \text{Vol}(\sigma) \Pi_{m \in |\sigma|} c_m$ (where $\pm \text{Vol}(\sigma)$ is the signed volume of the simplex spanned by $m \in |\sigma|$).

When $X_A$ is smooth, there is an especially nice relation between $\Delta_{\mathcal{A}}$ and $E_{\mathcal{A}}$.

Theorem 9.1. If $X_A \subset \mathbb{P}^{\ell-1}$ is smooth and $Q = \text{Conv}(\mathcal{A})$, then

$$
E_{\mathcal{A}}(f) = \pm \prod_{\Gamma \subset Q} \Delta_{\mathcal{A} \cap \Gamma}(f|_{\Gamma}),
$$

where $\Delta_{\mathcal{A} \cap \Gamma}(f|_{\Gamma})$ is the $\mathcal{A} \cap \Gamma$-discriminant of $f|_{\Gamma}$.
where \( \Gamma \subset Q \) are the nonempty faces and \( f|_\Gamma = \sum_{m \in A \cap \Gamma} c_mt^m \).

When \( X_A \) is singular, there is a more complicated but very elegant formula relating \( \Delta_A \) and \( E_A \)—see [GKZ, Ch. 10] for the details.

Another way to see the relation between \( \Delta_A \) and \( E_A \) is to study the projective hypersurface \( Y_f \subset X_A \) defined by \( f \in L(A) \). Then we have the following result.

**Proposition 9.2.** Given \( X_A \subset \mathbb{P}^{\ell-1} \), we have:

1. If \( X_A \) is smooth and \( \Delta_A \) is nonconstant, then \( Y_f \subset X_A \) is smooth if and only if \( \Delta_A(f) \neq 0 \).
2. If \( E_A \) is nonconstant, then \( Y_f \subset X_A \) is nondegenerate (as defined in §6) if and only if \( E_A(f) \neq 0 \).

It is likely that the first part of this proposition remains true in the simplicial case, so that \( Y_f \) should be quasi-smooth in the sense of §6 if and only if \( \Delta_A(f) \neq 0 \). However, no proof has appeared in print.

As a final comment, note that resultants can be computed in terms of discriminants (using the Cayley trick) and vice versa (using \( E_A \) and Theorem 9.1 in the smooth case). Historically, \( A \)-discriminants were first discovered in the context of \( A \)-hypergeometric functions and led to the definition of \( A \)-resultants. In practice, resultants are easier to compute (see, for example, [Stu4]).

**\( A \)-Hypergeometric functions.** To define the \( A \)-hypergeometric equations, consider \( A = \{m_1, \ldots, m_\ell\} \subset \mathbb{Z}^n \). For simplicity, we will assume that \( \mathbb{Z}^n \) is affinely generated by \( A \), which means that \( \mathbb{Z}^{n+1} \) is generated by \( A \times \{1\} \). Then, for each \( \mathbf{a} = (a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell \) satisfying the conditions

\[
\sum_{i=1}^\ell a_im_i = 0, \quad \sum_{i=1}^\ell a_i = 0,
\]

consider the differential operator

\[
\Box_{\mathbf{a}} = \prod_{a_j > 0} \left( \frac{\partial}{\partial c_j} \right)^{a_j} - \prod_{a_j < 0} \left( \frac{\partial}{\partial c_j} \right)^{-a_j}
\]

where we regard \( c_1, \ldots, c_\ell \) as variables on \( L(A) = \mathbb{C}^\ell \). We will also consider the differential operators

\[
Z_0 = \sum_{j=1}^\ell c_j \frac{\partial}{\partial c_j}, \quad Z_i = \sum_{j=1}^\ell m_{ji}c_j \frac{\partial}{\partial c_j}, \quad i = 1, \ldots, n,
\]

where \( m_j = (m_{j1}, \ldots, m_{jn}) \in \mathbb{Z}^n \) are the elements of \( A \). Then, for a function \( \Phi(c_1, \ldots, c_\ell) \) on \( \mathbb{C}^\ell \), the **\( A \)-hypergeometric system with exponents** \( \beta_0, \ldots, \beta_n \in \mathbb{C} \) is the system of differential equations

\[
\Box_{\mathbf{a}} \Phi = 0, \quad \text{for all } \mathbf{a} \text{ satisfying (9.1)}
\]

\[
Z_i \Phi = \beta_i \Phi, \quad \text{for all } i = 0, \ldots, n.
\]

This system is holonomic, so that its solutions form a locally constant sheaf outside a hypersurface in \( L(A) = \mathbb{C}^\ell \). Two especially nice facts are first, that the
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generic number of linearly independent solutions of (9.3) is the normalized volume of the polytope \( \text{Conv}(A) \), and second, that \((c_1,\ldots,c_\ell)\) is generic if and only if the principal \( A \)-determinant is nonvanishing, i.e., if and only if \( E_A(f) \neq 0 \) for \( f = \sum_{i=0} c_i t^{m_i} \). Proofs can be found in [GKZ3].

A more direct connection with toric varieties can be seen by considering the symbols of the operators \( \square_a \). Using \( x_1,\ldots,x_\ell \) as variables on the dual of \( L(A) = \mathbb{C}^\ell \), the symbol of \( \square_a \) is its Fourier transform \( \hat{\square}_a \), which is obtained by replacing \( \partial/\partial c_j \) in (9.2) with \( x_j \). If we write \( a = a^+ - a^- \) as in Lemma 5.5, then \( \hat{\square}_a \) becomes \( x^{a^+} - x^{a^-} \). Since we do this for all \( a \) satisfying (9.1), it follows from (7.5) that we get the toric ideal \( I_A \). Hence the Fourier transform of (9.3) is supported on the toric variety \( X_A \). This toric variety plays an important role in the proofs of many results about \( A \)-hypergeometric functions. Also, although (9.3) involves infinitely many equations \( \square_a \Phi = 0 \), one can always reduce to a finite number using a Gröbner basis for the toric ideal \( I_A \) (see [HLY1]).

There are several ways to write down solutions to the \( A \)-hypergeometric system. For us, the most interesting method involves the periods of the affine hypersurface \( Z_f \subset (\mathbb{C}^*)^n \) defined by \( f = \sum_{i=0} c_i t^{m_i} \). We will assume that \( A = \Delta \cap \mathbb{Z}^n \), where \( \Delta \subset \mathbb{R}^n \) is an integer polytope. As in §5, we have the ring \( S_\Delta \subset \mathbb{C}[t_0,t_1,\ldots,t_n] \). In [Bat4, Thm. 14.2], the following result is proved.

**Proposition 9.3.** Let \( A = \Delta \cap \mathbb{Z}^n \) and \( t_0^k t^m \in S_\Delta \) (so that \( m \in k\Delta \cap \mathbb{Z}^n \)). Then, for a \( n \)-cycle \( \gamma \in H_n((\mathbb{C}^*)^n - Z_f) \) and \( f = \sum_{i=0} c_i t^{m_i} \in L(A) \) nondegenerate, the period integral

\[
\Pi(c_1,\ldots,c_\ell) = \Pi(f) = \int_{\gamma} \frac{t^m}{f^{k}} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}
\]

satisfies the \( A \)-hypergeometric system (9.3) for exponents \((\beta_0,\ldots,\beta_n) = (-k,-m)\).

By Proposition 9.2, \( Z_f \) is nondegenerate if and only if \( E_A(f) \neq 0 \). This helps explain why the singular set of the \( A \)-hypergeometric system is defined by \( E_A = 0 \). The integral in Proposition 9.3 is an example of an Euler integral, which is the main object of study in [GKZ2].

It is also possible to write down series solutions of the \( A \)-hypergeometric system. Formally, these solutions look like

\[
\Phi_\gamma(c_1,\ldots,c_\ell) = \sum_{a} \frac{c^{\gamma + a}}{\Pi_{j=1}^{\ell} \Gamma(\gamma_j + a_j + 1)},
\]

where \( \gamma \in \mathbb{C}^\ell \), \( \Gamma \) is the usual \( \Gamma \)-function, and the sum is over all \( a \) satisfying (9.1). By restricting to certain choices of \( \gamma \) and \( \beta_0,\ldots,\beta_n \), one can show that the above series converge locally when \((c_1,\ldots,c_\ell)\) lies in certain regions of \( \mathbb{C}^\ell \) determined by a regular triangulation \( T \) of \( \text{Conv}(A) \). This is described as follows. We saw in §7 that a regular triangulation gives a maximal cone of \( A_\Delta \) in the secondary fan of \( A \). Using the exact sequence (7.3), we get a cone \( C(T) \subset \bigoplus_{m \in A} \mathbb{R} \cdot e_m \simeq \mathbb{R}^\ell \). Then one can prove that the series (9.4) converges locally for those \((c_1,\ldots,c_\ell)\) such that \(- \log(c_1,\ldots,c_\ell)\) (as defined in (6.1)) lies in a suitable translate of \( C(T) \). Furthermore, one gets all solutions of (9.3) in this way. See [GKZ3] for details.

As mentioned earlier, the Newton polytope of the principal \( A \)-determinant \( E_A \) is the secondary polytope (with vertices corresponding to regular triangulations).
In terms of (6.1), the regions of nice convergence of the series (9.4) correspond (up to sign) to unbounded components of $R^l - \log(\{E_A = 0\})$. This again illustrates why the singular set of $A$-hypergeometric system is given by $E_A = 0$.

Finally, Proposition 9.3 shows that the system (9.3) is closely tied to the torus. Roughly speaking, the equations $Z_i \Pi = \beta_i \Pi$ express the invariance of the period integral $\Pi$ for $Z_f \subset \mathbb{C}^n$ under the infinitesimal automorphisms of $(\mathbb{C}^*)^n$. If we were to formulate a similar period integral for $Y_f \subset X_\Delta$, we would need to add further equations to (9.3) to account for automorphisms of $X_\Delta$. This extended $A$-hypergeometric system is described in [HLY1].

§10. Mirror symmetry

In 1991, a group of physicists made some startling enumerative predictions concerning rational curves on a quintic threefold (see [CdGP]). The basic idea was that a hard computation on the quintic threefold became easier by working on its “mirror”. The first mirror constructions involved finite quotients of weighted projective spaces which, as Roan pointed out in 1992, can be described naturally using toric methods (see [Roa2]). After Batyrev’s 1993 introduction of reflexive polytopes (see [Bat1]), toric geometry has become a basic tool of mirror symmetry.

We will not discuss mirror symmetry in general—the reader should consult [Yau], [Mor3] or [CK] for an introduction to this fascinating topic. Rather, we will concentrate on describing how toric geometry is used in mirror symmetry. This is a very active field, and the references given below are far from complete.

**Complex and Kähler moduli.** The complex moduli of varieties have been studied for many years. The infinitesimal deformations of a compact complex manifold $Y$ of dimension $d$ are given by $H^1(Y, \Theta_Y)$, which in nice cases is the tangent space to the complex moduli space. When $Y$ is Calabi-Yau, the isomorphism $\Omega^d_Y \simeq \mathcal{O}_Y$ implies $\Theta_Y \simeq \Omega^{d-1}_Y$, so that $H^1(Y, \Theta_Y) \simeq H^{d-1,1}(Y)$.

The intense study of Kähler moduli is more recent. The basic definition is as follows: given a Kähler manifold $Y$, the **complexified Kähler cone** is the cone

$$K_C(Y) = \{ B + iJ \in H^2(Y, \mathbb{C}) : J \text{ is Kähler} \}/\text{Im } H^2(Y, \mathbb{Z}),$$

and the **complexified Kähler moduli space** is the quotient $K_C(Y)/\text{Aut}(Y)$. The exact structure of this quotient space depends on how $\text{Aut}(Y)$ acts on the ordinary Kähler cone of $Y$ (as described in §3). In the Calabi-Yau case, it is conjectured that the Kähler cone is polyhedral modulo the action of $\text{Aut}(Y)$ (see [Mor1] for a precise statement). This would imply the existence of a semi-toric compactification of the complexified Kähler moduli space. Assuming that $\text{Aut}(Y)$ acts discretely, the tangent space to the Kähler moduli space (we will usually drop the adjective “complexified”) is $H^{1,1}(Y)$.

So far, we’ve assumed that $Y$ is smooth, and in dimension 3, this is sufficient. Higher dimensional generalizations of mirror symmetry lead naturally to singular varieties, and one can define both complex and Kähler moduli when $Y$ is a minimal Calabi-Yau variety as in §8.

**The naive idea of mirror symmetry.** A Calabi-Yau threefold $Y$ together with a complexified Kähler class $\omega = B + iJ$ determines an $N = 2$ superconformal field theory (SCFT), and mirror symmetry suggests that there should be another Calabi-Yau $Y^\circ$ with class $\omega^\circ$ which in some sense interchanges the complex and Kähler structures of $Y$ and $Y^\circ$ but still gives the same $N = 2$ SCFT.
This interchange of complex and Kähler moduli in particular gives isomorphisms of the corresponding moduli spaces (the “mirror map”) and hence induces isomorphisms

\[(10.2) \quad H^{d-1,1}(Y) \simeq H^{1,1}(Y^{\circ}), \quad H^{1,1}(Y) \simeq H^{d-1,1}(Y^{\circ})\]

between their tangent spaces. But mirror symmetry is much more than these isomorphisms, for an isomorphism of $N = 2$ SCFTs also implies that certain trilinear functions on $H^{1,1}(Y)$ and $H^{d-1,1}(Y^{\circ})$ should agree after a change of variables given by the mirror map. These trilinear functions are called 3-point functions or correlation functions in the physics literature and are related to enumerative geometry and quantum cohomology (for $H^{1,1}(Y)$) and Hodge theory (for $H^{d-1,1}(Y^{\circ})$). Unfortunately, these topics are beyond the scope of this survey.

The physical theories involved in mirror symmetry have yet to be defined rigorously, but there are “mathematical mirror symmetry conjectures” which capture the mathematically interesting consequences. This can be done in all dimensions, and versions of these conjectures are in [Mor1–3], [Giv2], [Kont] and [Ver]. Mirror symmetry for holomorphically symplectic manifolds is proved in [Ver2], and versions of these conjectures are in [Mor1–3], [Giv2], [Kont] and [Ver]. Mirror symmetry for Calabi-Yau toric hypersurfaces is conjectural. Hence, in what follows, our term “mirror” really means “conjectural mirror”.

**Mirror symmetry for toric hypersurfaces.** In [Bat1], Batyrev used reflexive polytopes to construct mirrors for Calabi-Yau toric hypersurfaces. As in §8, the process starts with an $n$-dimensional reflexive polytope $\Delta \subset M_{\mathbb{R}}$, which by definition determines the anticanonical divisor on $X_{\Delta}$. This gives canonical Calabi-Yau hypersurfaces $Y \subset X_{\Delta}$. Then, if $\Sigma$ is a projective simplicial fan refining the normal fan of $\Delta$ and satisfying $\Sigma(1) = N \cap \Delta^\circ - \{0\}$, we obtain a family of minimal Calabi-Yau hypersurfaces $\hat{Y} \subset X_{\Sigma}$. To construct the mirror of this family, we repeat the above procedure using the dual polytope $\Delta^\circ$. Thus, for a suitable fan $\Sigma^\circ$ in $M_{\mathbb{R}}$ with $\Sigma^\circ(1) = M \cap \Delta - \{0\}$, the anticanonical divisor determines a family $\hat{Y}^\circ \subset X_{\Sigma^\circ}$ of minimal Calabi-Yaus. This is conjectured to be the mirror of $\hat{Y} \subset X_{\Sigma}$.

Evidence for the mirror relation between $\hat{Y}$ and $\hat{Y}^\circ$ comes from Theorem 8.2, which shows that

\[H^{n-2,1}(\hat{Y}) \simeq H^{1,1}(\hat{Y}^\circ), \quad H^{1,1}(\hat{Y}) \simeq H^{n-2,1}(\hat{Y}^\circ),\]

as predicted by (10.2) (since $\hat{Y}$ and $\hat{Y}^\circ$ have dimension $n - 1$). Also, $H^{n-2,1}(\hat{Y})$ has a subspace $H^{n-2,1}_{\text{poly}}(\hat{Y})$ consisting of deformations of $\hat{Y}$ obtained by varying its defining equation in $X_{\Sigma}$, and similarly, $H^{1,1}(\hat{Y})$ has a subspace $H^{1,1}_{\text{toric}}(\hat{Y})$ consisting of restrictions of $(1,1)$-classes on $X_{\Sigma}$. The monomial-divisor mirror map of [AGM2] gives natural isomorphisms

\[H^{n-2,1}_{\text{poly}}(\hat{Y}) \simeq H^{1,1}_{\text{toric}}(\hat{Y}^\circ), \quad H^{1,1}_{\text{toric}}(\hat{Y}) \simeq H^{n-2,1}_{\text{poly}}(\hat{Y}^\circ).\]

**Example.** To compute the mirror of the quintic threefold $Y \subset \mathbb{P}^4$, note that $Y$ is an anticanonical hypersurface, and the corresponding polytope $\Delta \subset M_{\mathbb{R}} \simeq \mathbb{Z}^4$ is reflexive. Hence the mirror should be determined by the dual polytope $\Delta^\circ \subset N_{\mathbb{R}}$. 
The toric variety $X_{\Delta^0}$ comes from the normal fan of $\Delta^0$. The cone generators of this fan are the vertices of $\Delta$, which are

$(-1, -1, -1, -1), (1, -1, -1, -1), (-1, 1, -1, -1), (-1, 1, 1, -1), (-1, -1, 1, 4), (-1, -1, -1, 4)$.

These generate a sublattice $M' \subset M$ of index 125, and the quotient $M/M'$ is

$$H = \{(a_0, a_1, a_2, a_3, a_4) \in (\mathbb{Z}/5)^5 : \sum_{i=0}^4 a_i \equiv 0 \mod 5\}/\mathbb{Z}_5$$

where $\mathbb{Z}_5 \subset (\mathbb{Z}/5)^5$ is the diagonal subgroup. Using the lattice $M'$, the normal fan gives $\mathbb{P}^4$, so by [Oda1, Cor. 1.16], $X_{\Delta^0}$ is the quotient of $\mathbb{P}^4$ by $M/M' \simeq H$, where $[a_0, a_1, a_2, a_3, a_4] \in H$ acts on $\mathbb{P}^4$ by the map $(x_0, x_1, x_2, x_3, x_4) \mapsto (\zeta^{a_0} x_0, \zeta^{a_1} x_1, \zeta^{a_2} x_2, \zeta^{a_3} x_3, \zeta^{a_4} x_4)$ for $\zeta = \exp(2\pi i / 5)$.

The homogeneous coordinate ring $S$ of $X_{\Delta^0}$ from §2 is the polynomial ring $S = \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$, which is graded by the Chow group $A_3(X_{\Delta^0})$. From (2.2), we get

$$0 \longrightarrow \mathbb{Z}^4 \overset{\alpha}{\longrightarrow} \mathbb{Z}^5 \overset{\beta}{\longrightarrow} \mathbb{Z} \oplus H \longrightarrow 0,$$

where $\alpha$ is as usual and $\beta$ is given by

$$\beta(a_0, a_1, a_2, a_3, a_4) = (\sum_{i=0}^4 a_i, [-a_0 - a_1 - a_3 - a_4, a_1, a_2, a_3, a_4]) \in \mathbb{Z} \oplus H.$$

Thus $A_3(X_{\Delta^0}) \simeq \mathbb{Z} \oplus H$ and the grading on $S$ is obtained by letting a monomial $x_0^{a_0} x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}$ have degree $\beta(a_0, a_1, a_2, a_3, a_4) \in \mathbb{Z} \oplus H$.

It follows that the anticanonical class has degree $\beta(1, 1, 1, 1, 1) = (5, 0)$, and the only monomials in $S$ of this degree are $x_5^5$ and $x_0 x_1 x_2 x_3 x_4$ (this can be seen directly or using the isomorphism $S_{(5,0)} \simeq L(\Delta^0)$ from (6.2)). Furthermore, the automorphisms of $X_{\Delta^0}$ given by its torus show that any anticanonical hypersurface is isomorphic to one defined by an equation of the form

$$x_5^5 + x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + 5\psi x_0 x_1 x_2 x_3 x_4 = 0.$$

This gives a 1-dimensional family of hypersurfaces $Y^0 \subset X_{\Delta^0}$. In concrete terms, $Y^0$ is the quotient by $H$ of the hypersurface in $\mathbb{P}^4$ defined by the above equation.

Finally, to get the mirror, we pick a projective simplicial fan $\Sigma^5$ in $M_\mathbb{R}$ refining the normal fan of $\Delta^0$ such that $\Sigma^5(1) = M \cap \Delta - \{0\}$. Then the mirror family of the quintic threefold $Y$ is given by the hypersurfaces $\hat{Y}^0 \subset X_{\Sigma^5}$ which are the proper transforms of $Y^0 \subset X_{\Delta^0}$. Since $Y \subset \mathbb{P}^4$ has 1-dimensional Kähler moduli (i.e., $h^{1,1} = 1$), we expect its mirror $\hat{Y}^0$ to have 1-dimensional complex moduli. Note that $M \cap \Delta - \{0\}$ has lots of points besides the vertices, and hence there are many choices for $\Sigma^0$. A specific choice for $\Sigma^0$ is described in the appendix to [Mor3].

In general, once the mirror family has been found, one gets enumerative predictions on $\hat{Y}$ by computing the mirror map and the Yukawa coupling on $\hat{Y}^0$. Without getting into precise definitions, these objects are closely related to period integrals and Picard-Fuchs equations on a toric hypersurface, and the computations involve series expansions about certain “special” boundary points in the complex moduli space. These “special” points have maximal unipotent monodromy and their existence is suggested by the structure of the Kähler moduli space of the mirror. See [Mor2-3] for further details.

In many cases, the period integrals and Picard-Fuchs equations can be computed directly (see [Mor4] for some examples), though Proposition 9.3 suggests a connection with $\mathcal{A}$-hypergeometric equations. This has been studied carefully in [HLY1-2]. Further references for both of these methods can be found in [Mor2].
Multiple mirrors and global moduli. The theory described so far is only local. Given a family of minimal Calabi-Yau hypersurfaces $\hat{Y} \subset X$ and its mirror $\hat{Y}^\circ \subset X^\circ$, the complex moduli of $\hat{Y}$ should correspond to the Kähler moduli of $\hat{Y}^\circ$. The problem is that complex moduli typically form a quasi-projective variety, while Kähler moduli are often the quotient of a bounded domain by a finite group. These are very different types of mathematical objects.

Hence, the symmetry $\hat{Y} \leftrightarrow \hat{Y}^\circ$ only gives a local isomorphism between complex and Kähler moduli. To get a global version of mirror symmetry for toric hypersurfaces, we need to examine the role of the many fans $\Sigma$ and $\Sigma^\circ$ which can occur. This is the multiple mirror phenomenon. For complex moduli, we expect the $\hat{Y}$'s for different $\Sigma$'s to be related by a series of flops, and hence they should have the same complex moduli (although the varieties themselves are not isomorphic).

The picture on the Kähler side is more interesting, for here, we’ve seen in §7 that the various Kähler cones for $\Sigma^\circ(1) = M \cap \Delta - \{0\}$ fit together to form the secondary fan of $B = M \cap \Delta - \{0\}$. The support of this fan is a strongly convex cone in $A_{n_1}(X_\Delta) \otimes \mathbb{R}$. Gluing together the corresponding complexified Kähler cones (10.1) gives the partially enlarged Kähler moduli space from [AGM1]. (Strictly speaking, we are only dealing with the toric part of the moduli space, but we will ignore this detail.)

The partially enlarged moduli space is still too small to be quasi-projective. To get something bigger, we use the enlarged secondary fan from §7. This is determined by an ample divisor on a toric variety, which here is the anticanonical divisor. The corresponding fan is the secondary fan $\mathcal{N}(\Sigma(\mathcal{A}))$ for $\mathcal{A} = M \cap \Delta$. When we complexify the cones in this fan and glue them together, we get the enlarged Kähler moduli space from [AGM1], which corresponds to the whole complex moduli of the mirror.

This isomorphism between moduli spaces extends to certain natural compactifications. For simplicity, let’s restrict to polynomial moduli, which come from those $f \in L(\Delta)$ which are nondegenerate (i.e., $E_A(f) = 0$ for $A$ as above) modulo automorphisms of the toric variety. If we instead use only automorphisms coming from the torus, we can compactify using the Newton polytope of the principal $A$-determinant $E_A$. The resulting compactification has a natural toric structure where the fixed points correspond to the vertices of the Newton polytope. But the Newton polytope of $E_A$ is the secondary polytope $\Sigma(\mathcal{A})$, whose vertices correspond to cones in the enlarged Kähler moduli space. Furthermore, the fixed points for vertices corresponding to cones in the partially enlarged Kähler moduli space are precisely the “special” points mentioned earlier. For more of the mathematics behind this picture, see [AGM2].

The physics interpretation is also quite interesting: the cones coming from the partially enlarged Kähler moduli space correspond to different “Calabi-Yau phases” of the same SCFT, while the other cones in the enlarged moduli space correspond to “non-geometric phases”. See [AGM1] and [MP] for more details.

Nef-partitions and Gorenstein cones. In §8, we described how a nef-partition $\text{Vert}(\Delta^\circ) = E_1 \cup \cdots \cup E_r$ of a reflexive polytope $\Delta$ gave a Minkowski sum $\Delta = \Delta_1 + \cdots + \Delta_r$ in $M_\mathbb{R}$ and a canonical Calabi-Yau complete intersection $V = Y_1 \cap \cdots \cap Y_r \subset X_\Delta$. Furthermore, a projective simplicial fan $\Sigma$ refining the normal fan of $\Delta$ and satisfying $\Sigma(1) = N \cap \Delta^\circ - \{0\}$ gives the MPCP-desingularization $\hat{V} = \hat{Y}_1 \cap \cdots \cap \hat{Y}_r \subset X_\Sigma$ as in (8.5). This is a family of minimal Calabi-Yau complete
intersections.
To get the mirror family, we follow [LBor] and consider the polytopes

\[ \nabla_i = \text{Conv}(\{0\} \cup E_i) \subset \mathbb{N}_\mathbb{R}, \quad i = 1, \ldots, r. \]

Then \( \nabla = \nabla_1 + \cdots + \nabla_r \) is a reflexive polytope in \( \mathbb{N}_\mathbb{R} \) with a natural nef-partition coming from (10.3). It is interesting to note that \( \nabla^o \subset M_\mathbb{R} \) is different from our original \( \Delta \). In fact, [LBor] shows that

\[ \nabla^o = \text{Conv}(\Delta_1, \ldots, \Delta_r) \subset \Delta_1 + \cdots + \Delta_r = \Delta \subset M_\mathbb{R}, \]

with a similar relation between \( \Delta^o \) and \( \nabla \) in \( \mathbb{N}_\mathbb{R} \).

Then the nef-partition for \( \nabla \) gives a family of canonical Calabi-Yau complete intersections \( V^o \subset X_{\nabla^o} \), and picking a projective simplicial fan \( \Sigma^o \) refining the normal fan of \( \nabla^o \) and satisfying \( \Sigma^o(1) = M \cap \nabla^o - \{0\} \) gives a family \( \tilde{V}^o \subset X_{\Sigma^o} \) of minimal Calabi-Yau complete intersections which is conjectured in [LBor] to be the mirror of \( \tilde{V} \subset X_{\Delta^o} \). Evidence for this is presented in [BB2-3], [Bv] and [LT] (see also the references in [BB3]).

A significant generalization of this construction, which uses Gorenstein cones, appeared in [BB1]. We will not go into the details, but Gorenstein cones can explain all of the mirror constructions we’ve given so far, as well as describing mirrors for certain rigid Calabi-Yau manifolds (where the mirror may have a different dimension). Details can be found in [BB1].

**Reid’s fantasy.** If a Calabi-Yau threefold \( Y_1 \) degenerates to a variety with only nodes as singularities, then one can resolve the singularities to obtain another Calabi-Yau \( Y_2 \) (which could fail to be Kähler). Locally, a vanishing cycle \( S^3 \subset Y_1 \) collapses to a node and is then resolved to give \( \mathbb{P}^1 \simeq S^2 \subset Y_2 \). Hence \( Y_1 \) and \( Y_2 \) can have quite different Betti numbers. In [Rei3], Reid speculates that the moduli of all Calabi-Yau threefolds many be connected in this (possibly non-Kähler) way.

In the physics literature, the singular Calabi-Yau between \( Y_1 \) and \( Y_2 \) is a conifold and going from \( Y_1 \) to \( Y_2 \) is a conifold transition. Such transitions were long thought to produce unacceptable singularities in the physical theories, but recently (see [GMS]), these difficulties were resolved by allowing certain non-perturbative string states (electrically charged black holes) on \( Y_1 \) to become massless on the conifold and to be interpreted on \( Y_2 \) as elementary perturbative states (elementary particles).

It follows that a Kähler version of Reid’s fantasy would enable any two Calabi-Yau threefolds to be connected by a single physical theory. This implies that when \( \mathbb{R}^4 \times \text{(Calabi-Yau threefold)} \) is used to model the vacuum state of the universe, we don’t need to worry about which Calabi-Yau to use, since all can occur.

Although Reid’s fantasy is still conjectural, it has been verified that for the 7555 Calabi-Yau threefolds mentioned in §8, their moduli are connected through Kähler varieties, though the singularities may be more complicated than nodes (see [ACJM] and [CGGK]). The basic idea is as follows. Suppose we have reflexive polytopes \( \Delta_2 \subset \Delta_1 \), which implies \( \text{Vert}(\Delta_2) \subset \text{Vert}(\Delta_1) \). Then, for \( Y_1 \subset X_{\Delta_1} \) defined by \( f \in L(\Delta_1) \), we can degenerate \( Y_1 \) by letting the coefficients of \( f \) corresponding to vertices not in \( \Delta_2 \) become zero. This gives a singular variety \( \tilde{Y} \subset X_{\Delta_2} \), which, when resolved, corresponds to a Calabi-Yau hypersurface \( Y_2 \subset X_{\Delta_2} \). Once the MPCP-desingularizations are taken into account, we can link the corresponding moduli spaces. As already mentioned, the singularities of \( \tilde{Y} \) may be more complicated than
just simple nodes, and at present there is no physical explanation of the transition from \( Y_1 \) to \( Y_2 \). But this method is sufficient to link up the 7555 Calabi-Yaus on the list, and as noted in [Ska], it may be sufficient to connect the moduli of all 3-dimensional Calabi-Yau toric hypersurfaces.

**Further remarks.** Although our discussion of mirror symmetry has been rather superficial (and has omitted some important ideas), it should be clear that toric geometry has a prominent role to play, if for no other reason than providing a rich supply of examples. Notice also that virtually everything in the earlier sections of the paper has been used. Symplectic geometry seems to be an exception, but this is only because we didn’t describe Witten’s *linear sigma models* [Wit], which are physical theories where toric varieties enter by means of symplectic reduction. See [MP] for more details and for some other interesting uses of toric geometry in mirror symmetry.

We should also mention that it is possible to compute the quantum cohomology of toric varieties (see [Bat3]). In addition, certain mirror symmetry calculations suggest that in some situations, the usual Hodge numbers \( h^{p,q} \) need to be replaced by *string-theoretic Hodge numbers* \( h^{p,q}_{st} \). For Calabi-Yau complete intersections in a toric variety, these numbers are computed in [BB2].

§11. Other developments

Besides the topics reported on so far, the last few years have seen a lot of interesting work on other aspects of toric geometry. Here is a selection, with apologies for the many fine papers not mentioned.

**Very ample divisors.** As is well-known, an ample divisor \( D \) on a complete toric variety \( X \) is very ample if \( X \) is smooth or has dimension \( \leq 2 \). In general, \( D \) may fail to be very ample, but [EW] proves that \((n - 1)D\) is always very ample, where \( n \geq 2 \) is the dimension of \( X \). Examples show that this result is sharp.

There is also a notion of *\( k \)-very ampleness* which measures the behavior of \( D \) relative to 0-dimensional subschemes \( Z \subset X \) with \( h^0(O_Z) = k + 1 \) (so that 0-ample means spanned by global sections and 1-ample means very ample). For smooth toric surfaces, [DiR] shows that \( k \)-very ampleness can be interpreted in terms of convexity properties of the support function of \( D \) relative to the integer lattice.

**Embeddings into toric varieties.** Another nice fact about toric varieties concerns embeddings of a complete variety \( Y \). The Chevalley criterion (proved by Kleiman) states that \( Y \) can be embedded into a projective space if and only if every finite subset of \( Y \) is contained in an affine open. When \( Y \) is normal, [Wlo] proves that we can embed \( Y \) into a complete toric variety if and only if every *two element* subset of \( Y \) lies in an affine open.

**Classifying toric varieties.** There are several ways one can try to classify toric varieties. For example, one can work one dimension at a time, which is the approach taken in §8 when we discussed reflexive polytopes. Another strategy is to classify smooth complete toric varieties according to their Picard number \( \rho \). For \( \rho = 2 \), this is done in [Kle], which such varieties are shown to be projective (and [ESch] finds projective embeddings for which Conjecture 5.6 is satisfied.) Similarly, [KS] shows that smooth toric varieties with Picard number 3 are projective. Using this and the primitive collections defined in §2, a classification for smooth complete toric
varieties with $\rho = 3$ is given in [Bat2]. The papers mentioned here also contain references to earlier work on classification.

**Invariants of toric varieties.** When a toric variety $X$ is smooth (resp. simplicial), its cohomology over $\mathbb{Z}$ (resp. $\mathbb{Q}$) is well-known. However, it is also possible to compute the rational intersection cohomology of a compact toric variety (see [Fie]), and since a toric variety has a natural torus action, there is also equivariant cohomology to consider, which is computed in [Bif].

Turning to less topological invariants, one can study the K-theory (Grothendieck groups of vector bundles or coherent sheaves) of a toric variety, which coincide in this case and are computed in [More1]. The Brauer group of a toric variety is discussed in [DFM] in the case of an algebraically closed field, and the split case over arbitrary fields is studied in [For]. It would be interesting to see what results could be obtained for non-split tori (to be defined below). The paper [For] also considers certain invariants of $X_{\Sigma}$ which depend only on the combinatorial type of the fan $\Sigma$. Similar questions about $\text{Pic}(X_{\Sigma})$ are studied in [Eik].

Another invariant known in the smooth (resp. simplicial) toric case is the Chow ring $A^*(X)$ (resp. $A^*(X) \otimes \mathbb{Q}$). For a general toric variety, the Chow groups $A_k(X)$ are computed in [Dan2, Sect. 10], and when $X$ is a complete toric variety, one can relate $A_k(X) \otimes \mathbb{Q}$ to a certain weight filtration of the Borel-Moore homology of $X$ (see [Tot]). Besides these classical Chow groups, there are also the operational Chow groups $A^k(X)$ of Fulton and MacPherson, which give a Chow cohomology ring $A^*(X)$. When $X$ is a complete toric variety, this ring is computed in [FS].

**Intersection theory on toric varieties.** As mentioned in the introduction, Hard Lefschetz for simplicial toric varieties was used in Stanley’s proof of McMullen’s conjectures about convex simplicial polytopes. A nice discussion of this may be found in [Ful, Sect. 5.6]. Since Hard Lefschetz for intersection cohomology is a very deep result, it is reasonable to ask if a simpler proof exists in the toric case. This led to the papers [Oda1] and [Oda4] on the de Rham cohomology of toric varieties, although the question of finding a toric proof of Hard Lefschetz is still open. (We should also mention the paper [Dan1], which studies the de Rham cohomology of toroidal varieties.)

Subsequently, a proof McMullen’s conjectures which avoided toric varieties and Hard Lefschetz was found by McMullen [McM]. His proof used a certain polytope algebra. In [FS], it is shown that in dimension $n$, the polytope algebra is the inverse limit of the Chow cohomology rings $A^*(X) \otimes \mathbb{Q}$ over the directed system of all toric compactifications of the torus $(\mathbb{C}^*)^n$.

**Counting lattice points.** If $X_\Delta$ is the toric variety determined by an $n$-dimensional integer polytope $\Delta \subset M_\mathbb{R} \simeq \mathbb{R}^n$, then its Todd class can be written

$$Td(X_\Delta) = \sum_{\sigma \in \Sigma} r_{\sigma} [V(\sigma)] \in \bigoplus_{k=0}^{n} A_k(X_\Delta),$$

where $\Sigma$ is the normal fan of $\Delta$, $[V(\sigma)]$ is the class of the orbit closure corresponding to $\sigma$, and $r_{\sigma} \in \mathbb{Q}$. By [Dan2], the number of integer points in $\Delta$ is given by

$$l(\Delta) = \sum_{\sigma \in \Sigma} r_{\sigma} \text{Vol}(F(\sigma)), $$

where $F(\sigma)$ is the orbit closure corresponding to $\sigma$.
where $\text{Vol}(F(\sigma))$ is the normalized volume of the face of $\Delta$ corresponding to $\sigma$. There is a similar formula for the Ehrhart polynomial $E_\Delta(k) = l(k\Delta)$ for $k \geq 1$. See [Ful, Sect. 5.3] or [More2, Sect. 1.1] for a nice introduction to this topic.

These formulas reduce the problem of counting lattice points to finding an explicit expression for the Todd class. While this can be done for any given toric variety (see [Ful]), general formulas weren’t available until recently, when several different solutions were found. In [More2], a formula for $r_\sigma$ is given which depends only on the cone $\sigma$ and not the fan in which it sits. For simplicial toric varieties, [Pom] introduces the idea of a “mock” Todd class, denoted $TD(X, \Delta)$, which is built using formulas from the smooth case. The difference $Td(X, \Delta) - TD(X, \Delta)$ is then described using functions which involve Dedekind sums and lead to explicit formulas for the component $Td^i(X, \Delta)$. (For the weighted projective space $\mathbb{P}(q_0, q_1, q_2, q_3)$, another approach to this computation can be found in [Lat].) A slightly different definition of “mock” Todd class is given in [CS] and is used to obtain a third expression for $Td(X, \Delta)$. A corollary is an explicit formula for the number of lattice points in an integer simplex.

We should also mention that the Todd class is related to the total Chern class. For a singular toric variety, the formula $ch(X, \Delta) = \sum_{\sigma \in \Sigma} [V(\sigma)]$ is shown to hold in homology with closed supports (see [BBF]).

The quite different approach to the study of lattice points appears in [Bri1]. Here, the object of interest is the Laurent polynomial

$$l_\Delta(t) = \sum_{m \in \Delta \cap M} t^m,$$

where we still assume that $X, \Delta$ is simplicial. Using the Lefschetz-Riemann-Roch theorem from equivariant K-theory, $l_\Delta(t)$ can be written as a sum of rational functions in $t$ determined by the cones at the vertices of $\Delta$. Hence we get a formula which not only counts lattice points (by setting $t = (1, \ldots, 1)$) but also describes the lattice points themselves. Simpler proofs of this result can be found in [Ish] and [SI], and a weighted version is in [Bri2]. We should also mention that these questions can be studied from a purely “polytope” point of view, which uses a combinatorial Riemann-Roch theorem and avoids toric methods. See [KK] for details.

**Rational points on toric varieties over number fields.** To define a toric variety over a number field $K$, first observe that a torus $T$ over $K$ is determined by a lattice $M \cong \mathbb{Z}^n$ with an action of $\text{Gal}(E/K)$, where $K \subset E$ is a finite Galois extension over which the torus splits, i.e., becomes isomorphic to $(\mathbb{G}_m)^n$. Then a toric variety $X$ over $K$ containing $T$ is determined by a fan in $M_\mathbb{R}$ which is invariant under $\text{Gal}(E/K)$. See [BT] for details and references.

In this situation, one can study $K$-rational points on $X$ using the height function coming from a metrized ample line bundle $\mathcal{L}$ on $X$. The basic question concerns the asymptotics of $N(T, \mathcal{L}, B)$, which is the number of $K$-rational points in the torus $T \subset X$ with height bounded by $B$. When $X$ is a smooth projective Fano toric variety over $K$, the natural line bundle to use is the anticanonical line bundle $\mathcal{L} = \mathcal{O}(-K_X)$. In [BT], the asymptotic formula

$$N(T, \mathcal{L}, B) = c B (\log B)^{r-1}(1 + o(1)), \quad B \gg 0$$

is proved, where $r$ is the rank of the Picard group $\text{Pic}(X)$ over $K$ and $c$ is a nonzero constant. This verifies the toric case of a conjecture of Manin for Fano varieties.
over number fields (see [FMT]). The constant $c$ can be explicitly computed in terms of the geometry of the cone of effective divisors in $\text{Pic}(X)_\mathbb{R}$, the order of the nontrivial part of the Brauer group of $X$, and a certain Tamagawa number associated with the metrized bundle $\mathcal{L}$ on $X$.

**Quotients of toric varieties.** In §2, we constructed a toric variety $X$ as the quotient $\left(\mathbb{C}^\Delta(1) - Z(\Sigma)\right)/G$, which is a geometric quotient when $X$ is simplicial. Oda observed that $\mathbb{C}^\Delta(1) - Z(\Sigma)$ has a natural structure of a toric variety, so that we are taking the quotient of a toric variety by a subgroup of its torus. One can study this problem in general, and various types of quotients are possible, including *combinatorial quotients*, *GIT quotients* and *Chow quotients*. For GIT quotients, there can be several quotients in any given situation, and the different quotients are related by a chamber structure where quotients for chambers sharing a common wall are (in good cases) related by a flip. (This is similar to the secondary fan described in §7.) Furthermore, the Chow quotient can be described using a certain fiber polytope and is the inverse limit of the GIT quotients. See [B-BS], [Hu], [KSZ2] and [Tha] for details.

**Residues on toric varieties.** There is a huge literature on residues. For multidimensional residues, toric methods can be used in defining local residues (see [PT]), and Gröbner methods, including the Gröbner fan from §7, can be used in computing global residues (see [CDS]). A version of global residues specific to toric varieties was introduced in [Cox3], and various properties of these *toric residues* have been studied in [CCD] and [CD]. Toric residues have also been used in mirror symmetry (see [MP]).

**Singularities of toric varieties.** Singularities of affine toric varieties have been the subject of several recent papers. For example, basic tools in deformation theory are the spaces $T^1_X$ and $T^2_X$ which describe infinitesimal deformations and obstructions, and for an affine toric variety $X$, these are computed in [Alt1]. One can also study deformations which are themselves toric, and in the case of isolated 3-dimensional toric Gorenstein singularities, the versal deformation can be constructed entirely by toric means (see [Alt3]). Furthermore, the irreducible components of the deformation space correspond to certain decompositions of a lattice polytope into Minkowski sums of other lattice polytopes. The role of Minkowski sums in the deformation theory of affine toric varieties is also explored in [Alt2]. Another reference for Gorenstein toric singularities is [Nom].

In studying the resolutions of singularities of an affine toric variety $X$, a divisor in a resolution of $X$ is *essential* if a birational copy of the divisor appears in every resolution of the variety, and it is *equivariant essential* if it appears (again up to birational equivalence) in every toric resolution. In [BG-S], it is shown that if $X$ comes from a cone $\sigma \subset N_{\mathbb{R}}$, then equivariant essential divisors correspond bijectively to minimal generators of the semigroup $\sigma \cap N - \{0\}$. Furthermore, in dimension 3, the same is true for essential divisors. Other papers dealing with the resolution of 3-dimensional toric singularities are [ESpa] and [Pou].

In another context, a conjecture of Shokurov on the minimal discrepancies of log-terminal singularities was verified for toric singularities in [ABor]. Hypersurface sections of toric singularities are considered in [Tsu]. It is also possible to discuss toric singularities without reference to a base field (or even a base scheme). This topic is studied in [Kat] and may have applications to arithmetic algebraic geometry.
Resolution of singularities. A recent development is the discovery in [Ad] and [BP] of a simple proof of a weak form of resolution of singularities. The precise result is that if $X$ is a normal projective variety in characteristic 0 and $Y \subset X$ is a proper subvariety, then there is a birational morphism $f : \hat{X} \to X$ such that $\hat{X}$ is a smooth projective variety and $f^{-1}(Y)$ is a strict normal-crossings divisor. However, $f : \hat{X} - f^{-1}(Y) \to X - Y$ might not be the identity, so we don’t get a resolution of singularities in the usual sense. The proofs in [Ad] and [BP] are slightly different, but both make essential use of toric methods.

Conclusion

In this survey, we have attempted to convey the richness of the recent work in toric geometry. An unexpected consequence of all this activity is that it is less clear where to learn about the subject. One could start with the standard approach to toric varieties, as in [Dan2], [Ful], [Oda2]. (Other introductions include [Ewa], [Oda3] and [Rei4].) Alternatively, one could begin with the quotient construction of §2, where [Cox2] is one of many references. This is closely tied to the symplectic approach, as described in [Aud] and [Gui]. Yet another starting point would be the theory of non-normal toric varieties, as in [GKZ1] or [Stu2].

Of course, there is no “best” approach to toric varieties. The multiplicity of entry points is actually a virtue, for it enables people from different areas of mathematics to learn about and contribute to this fascinating and accessible part of algebraic geometry.

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