Quantum phase transition in XXZ central spin model

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We investigate the quantum phase transition (QPT) in the XXZ central spin model, which can be described as a spin-$\frac{1}{2}$ particle coupled to $N$ bath spins. In general, the QPT is supposed to occur only in the thermodynamical limit. In contrast, we present that the central spin model exhibits a normal-to-superradiant phase transition in the limit where the ratio of the transition frequency of the central spin to that of the bath spins and the number of the bath spins tend to infinity. We give the low-energy effective Hamiltonian analytically in the normal phase and the superradiant phase, and we find that the longitudinal interaction $\Delta$ can significantly influence the excitation number and the coherence of the ground state. These two quantities are remarkably enhanced for the negative longitudinal interaction while suppressed for the positive longitudinal interaction. We also use the quantum Fisher information (QFI) to characterize the QPT and illustrate a measurement scheme that can be applied in practice. This work builds a novel connection between the qubit-spin systems and the qubit-field systems, which provides a possibility for the realization of criticality-enhanced quantum sensing in central spin systems.

I. INTRODUCTION

Quantum phase transition of many-body systems plays an important role in our understanding of physics [1]. While Landau’s symmetry-breaking theory gains great success in describing thermal phase transitions, quantum phase transitions, which is due to quantum fluctuations when temperature goes to zero beyond the symmetry-breaking paradigm, attracts a lot of interest in condensed matter physics [2–9] and quantum optics [10–18]. As a well-known model in quantum optics, the Dicke model [10] describes $N$ two-level atoms coupled to a single-mode cavity, and a quantum phase transition from the normal phase to the superradiant phase will occur in the thermodynamic limit $N \to \infty$. Recently, Hwang et al. [19, 20] presented that QPT can occur under the situation of a two-level atom coupled to a single-mode cavity, and they indicated that the ratio $\eta$ of the atomic transition frequency to the cavity field frequency plays the same role in the quantum Rabi model (QRM) and the Jaynes-Cummings model (JC) as the number of atoms in the Dicke model and the Tavis-Cummings model [21]. They also showed that the two-site JC lattice undergoes a Mott-insulating-superfluid QPT in the limit $\eta \to \infty$ [20].

Actually, there exist a similarity between the light-matter interaction in optical systems and the hyperfine interaction in spin systems. Like the phenomenon of superradiance in quantum optics mentioned earlier, the superradiant effect can occur in the nuclear spin environment [22–25]. Kessler et al. [22] showed that the superradiant effect can be realized in the systems that nuclear spin ensemble surrounding a quantum dot or an nitrogen-vacancy (NV) center, and some suitable optical pumping conditions are given. In Ref. [26], Dooley et al. exploited the spin coherent state as the initial state to discuss the collapse and revival phenomena in qubit-big spin model and revealed the similarities of the Hamiltonian between the qubit-spin systems and the qubit-field systems. In Ref. [27], He et al. gave the exact quantum dynamics of the XXZ central spin model and the analytic expression of quantum collapse and revival is also obtained. Moreover, they used the Holstein-Primakoff transformation to build a mapping between the central spin model and the JC model. Furthermore, the connection between the spin-$s$ central spin model and Tavis-Cummings model is discussed in Ref. [28].

However, the superradiant QPT has not been widely discussed in anisotropic central spin systems. Inspired by the work mentioned above, we follow the thoughts in Refs. [19, 20, 27] and analytically analyze the superradiant quantum phase transition in the XXZ central spin model. We first give the exact energy spectrum of the XXZ central spin model, and the asymptotic behavior between the central spin model and the JC model can be clearly observed in the case of a large number of the bath spins. The strength of longitudinal interaction $\Delta$ in the XXZ central spin model is different from that of the transverse interaction $A$, which can significantly influence the critical point of the phase transition. Thus, it is necessary to discuss the two following cases: (i) $\Delta = 0$, (ii) $|\Delta| < \omega$, where $\omega$ is frequency of the bath spins. It shows that the XXZ central spin model has a similar critical point with the JC model under the condition of $\Delta = 0$. However, for $|\Delta| < \omega$, the critical point is different from the one before and we give an analytical solution for this by means of the theory of low-energy effective Hamiltonian.

This article is organized as follows. In Sec. II, we give the exact energy spectrum of the XXZ central spin model and obtain the analytic expression of excitation number via the mean-field approximation. In Sec. III, we present the derivation of the low-energy effective Hamiltonian and exploit it to analyze the critical point and the ground state energy. In Sec. IV, we discuss the influence of the longitudinal interaction $\Delta$ on the excitation number and the coherence of the ground state. In Sec. V, we make use of the QFI to characterize the QPT of the XXZ central spin model, and give a measurement scheme. Finally, we give a conclusion in Sec. VI.

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II. MODEL

The central spin model can be described as a single spin-$\frac{1}{2}$ particle coupled to $N$ spin-$\frac{1}{2}$ particles (bath spins), which is also called qubit-big spin model in Ref. [26]. For different strengths of the longitudinal and the transverse interactions, the Hamiltonian of this model can be written as [27] (we set $\hbar = 1$)

$$H = \frac{\omega_0}{2} \sigma_0^z + \frac{\omega}{2} \sum_{k=1}^{N} \sigma_z^{(k)} + \sum_{k=1}^{N} \Delta_k \sigma_z^{(k)} \sigma_z^{(0)} + \sum_{k=1}^{N} \frac{A_k}{2} \left( \sigma_x^{(k)} \sigma_x^{(0)} + \sigma_y^{(k)} \sigma_y^{(0)} \right),$$

where $\omega_0$ and $\omega$ are respectively the transition frequency of the central spin and bath spins, $A_k$ is the strength of transverse interaction, and $\Delta_k$ is the longitudinal interaction. $\sigma_z^{(0)}$ denotes the Pauli operator of the central spin and $\sigma_z^{(k)}$ ($i = x, y, z$) denotes the Pauli operator of the bath spins. The central spin model is widely applied to solve the problem in quantum notes the Pauli operator of the bath spins. The central spin and bath spins, the Pauli operator of the central spin and bath spins, $\sigma_z^{(k)}$ are respectively the transition frequency of $\omega$, and its eigenvalue is $\Delta$. Now we introduce the Dicke states $|\uparrow\uparrow\uparrow\rangle$ for $|\downarrow\downarrow\downarrow\rangle$ as the eigenstates of $J_z$ and the $|\uparrow\rangle$ ($|\downarrow\rangle$) as the eigenstate of $\sigma_z^{(0)}$.

In this paper, we denote $|\uparrow\downarrow\rangle$ as $|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ as $|\uparrow\uparrow\rangle$ as the eigenstates of $J_z$ and the $|\downarrow\rangle$ ($|\uparrow\rangle$) of the JC model. Due to the $U(1)$ symmetry of the Hamiltonian in Eq. (2), it is easy to obtain the energy eigenvalues

$$E_{\pm} = \frac{1}{2} \left[ (2m+1) \omega - \Delta \pm \sqrt{(2m+1) \Delta - \omega + \omega_0}^2 + 4\Delta^2 k_n \right],$$

where $m = n - j - j$, $k_n = 2j - n + 1$. More detailed derivations are presented in Appendix A.

Up to now we have not discussed the value range of the parameter $\Delta$. Note that $|\uparrow\rangle$ is the ground state of Eq. (2) for $|\Delta| < \omega$, and its eigenvalue is $E_{\downarrow\downarrow\downarrow\downarrow} = -\frac{\omega_0}{4} - (\omega - \Delta)$. In the subsequent sections, we will prove $\omega < |\Delta|$ is a necessary condition for the superradiance QPT in the XXZ central spin model and we will also discuss the influence of different $\Delta$ on the XXZ central spin model. Now we consider the $E_{-}$ in the limit $\eta = \omega_0/\omega \to \infty$ and $g = A\sqrt{2j}/\sqrt{\omega_0 \omega} = \lambda/\sqrt{\omega_0 \omega}$. In order to make $g$ satisfies that $g \sim \mathcal{O}(1)$, we need to ensure that $\lambda/\omega \sim \sqrt{\eta}$. For $\Delta = 0$ and $\eta \gg 1$, $E_{-}$ in Eq. (3) can be expanded into

$$E_{-} = -\frac{\omega_0}{2} - \omega j + \left( 1 - g^2 + g^2 \frac{n - 1}{2j} \right) n \omega.$$

In the limit $N \to \infty$, the nonlinear term in Eq. (4) can be negligible and we obtain $E_{-} = (1 - g^2) n \omega + E_{-}\omega$, which has a similar harmonic spectrum presented in Ref. [20]. For $g < 1$, $E_{-}$ is minimum at $n = 0$ and the ground state energy in the normal phase is $-\omega_0/2 - \omega j$. For $g = 1$, there exist a degeneracy between $|\psi_{-}\rangle$ (Eq. (A4)) and $|\downarrow\rangle$, and the normal-to-superradiant phase transition occurs at this critical point. It is clear to see that for $g > 1$ the ground state is unstable and its energy can decrease infinitely as the excitation number increases, and the bath spins are macroscopically excited just like the behavior of the cavity field in the JC model [20].

Now we calculate the excitation number of the ground state. For $N \gg n$ and $\eta \gg 1$, $E_{-}$ in Eq. (3) can be written as

$$E_{-} = -\frac{\omega_0}{2} \sqrt{1 + 4g^2 n \eta^{-1}} + \omega_0 (n - j - 1) \eta^{-1},$$

and utilize $(\partial E_{-}/\partial n) / \omega_0 = 0$, we find that the excitation number of the ground state is $n_g = 0$ for $g < 1$. For $g > 1$, the excitation number of the ground state is given by

$$n_g = \frac{\eta}{4} (g^2 - g^{-2}),$$

which is consistent with the result in Ref. [20].

It is hard to acquire an analytical expression for $|\Delta| < \omega$ from Eq. (3), thus we use the mean-field approximation to get the mean-field energy, which is given by

$$E_{-} = \omega (n - j) - \bar{\omega}_0(n) \frac{n}{2},$$

where $\bar{\omega}_0(n) = \sqrt{4\lambda^2 n + 4n^2 \Delta^2 + 4n\Delta \omega_0 + \omega_0^2}$, $\bar{\omega}_0 = \omega_0 - N\Delta$, and $\bar{\omega} = \omega - \Delta$.

In the normal phase, the excitation number of the ground state is still $n_g = 0$. But for the superradiant phase, the excitation number is given by

$$n_g = -\frac{\lambda^2 + \Delta \omega_0}{2\Delta^2} + \frac{\lambda \omega}{2\Delta^2} \sqrt{\frac{\lambda^2 + 2\Delta \omega_0}{\omega^2 - \Delta^2}},$$

and the ground state energy is

$$E_{g}^{MF} = (n_g - j) \omega - \frac{1}{2} \bar{\omega}_0(n_g)$$

The detailed derivation is presented in Appendix A. This result is completely different from the previous situation since the existence of the nonlinear coupling term [45].
III. LOW-ENERGY EFFECTIVE HAMILTONIAN

In order to further understand the QPT in the XXZ central spin model, in this section we give the low-energy effective Hamiltonian both in the normal phase and the superradiant phase. Note that Eq. (2) can be mapped to the Hamiltonian of the JC model when $\Delta = 0$, which has been discussed in detail in Ref. [20], thus we focus on $|\Delta| < \omega$ in this paper.

For the normal phase, we apply a Holstein-Primakoff transformation and a Schrieffer-Wolff transformation $e^\delta$ with the anti-Hermitian operator $S = \lambda (a^\dagger S_- - a S_+) / \bar{\alpha}_0$ [19, 20] to Eq. (2), and we obtain the low-energy effective Hamiltonian which is

$$\hat{H}_{np} = -\frac{\bar{\alpha}_0}{2} - \omega j + \bar{\omega} (1 - \tilde{g}^2) a^\dagger a,$$

(10)

where $a$ ($a^\dagger$) is the bosonic annihilation (creation) operator, and $\tilde{g} = \lambda / \sqrt{\bar{\alpha}_0 \bar{\omega}}$. The detailed derivation of $\hat{H}_{np}$ is presented in Appendix B. Equation (9) shows a similar structure with Eq. (2) in the limit $N \to \infty$. For $|\Delta| > |\omega|$, we expect.

Now we prove that $|\Delta| < \omega$ is a necessary condition to acquire the QPT similar to the JC model. First of all, the comparison between Eq. (4) and Eq. (9) shows that $\bar{\omega}$ should satisfy that $\bar{\omega} > 0$, thus we have $\Delta < \omega$. Secondly, $n_g$ only makes sense if

$$\frac{\lambda^2 + 2\Delta \bar{\alpha}_0}{\omega^2 - \Delta^2} > 0,$$

(11)

and it is easy to verify that Eq. (10) can be always satisfied if $\Delta > -\omega$ in the range of $\tilde{g} > 1$. Finally, we get the necessary condition for the superradiance QPT in the XXZ central spin model is $|\Delta| < \omega$.

From Eq. (9) we see that the new critical point for $|\Delta| < \omega$ is $\lambda_c = \sqrt{\bar{\alpha}_0 \bar{\omega}}$. For $\tilde{g} < 1$, the ground state is $|\psi_{sp}^{np}\rangle = e^\delta |\downarrow, 0\rangle$ with energy $E_{np} = -\bar{\alpha}_0 / 2 - \bar{\omega} j$. Note that if $\Delta < \omega$ is not satisfied, then the ground state is $|\uparrow, 0\rangle$, which is not what we expect. However, the XXZ central spin model exhibits the instability when $\tilde{g} > 1$, and to solve this problem, we use the method proposed in Refs. [19, 20] to get the low-energy effective Hamiltonian in the superradiant phase.

Unlike the previous approach, we need to apply a displacement operator in addition to the two transformations mentioned before. Finally we get the low-energy effective Hamiltonian is

$$\hat{H}_{sp} = \omega \alpha^2 - \frac{\bar{\omega}_0}{2} + \kappa_0 \lambda^2 - \omega j,$$

(12)

where

$$\kappa_0 = \frac{\omega}{4} - \frac{2\alpha^2 \Delta^2 + \Delta \bar{\alpha}_0}{4\bar{\alpha}_0},$$

(13)

and

$$\lambda = \alpha^1 + a, \quad \alpha^2 = n_g \quad \text{and} \quad \bar{\omega}_0 = \sqrt{\frac{\lambda^2 + 2\Delta \bar{\alpha}_0}{\omega^2 - \Delta^2}}.$$
In the range of \(0 < \omega < \Delta< 0\), the excitation numbers are significantly enhanced in the superradiant phase. For \(\Delta < 0\), the ground state coherence of bath spins decreases appreciably. Therefore, compared with the situation of \(\Delta = 0\), the bath spins will be excited more quickly as \(|\Delta|\) approaches to \(\omega\).

But for \(0 < \Delta < \omega\), the corresponding results are different. Figure 2(b) shows that the excitation number decreases as \(\Delta\) increases, and when \(\Delta\) is close to \(\omega\), the variation of the excitation number \(n_g\) with respect to \(\bar{g}\) becomes discontinuous (shape of a stair). Similarly, we seek an explanation from Eq. (8). In the limit \(\bar{\omega} = \omega - \Delta \rightarrow 0\), Eq. (8) becomes

\[
 n_g = \frac{\bar{\omega}_0}{2\Delta^2 (\bar{g} \omega - \Delta)}.
\]  

(15)

In Eq. (15), for fixed \(\bar{g}\), the numerator decreases \((\bar{g} > 1\) and \(\bar{g} \omega > \Delta\)) and the denominator increases with \(\Delta\), whereas \(n_g\) decreases. Besides, the excitation number \(n_g\) is insensitive to \(\bar{g}\) when \(\Delta\) is close to \(\bar{\omega}\), therefore \(n_g\) will vary discretely (shown in Fig. 2(b)). In summary, we see that the macroscopic excitations of bath spins are significantly enhanced in the case of the negative longitudinal interaction \((-\omega < \Delta < 0\)) and the coherence increases significantly, whereas for \(-\omega < \Delta < 0\) the coherence decreases appreciably.

IV. INFLUENCE OF LONGITUDINAL INTERACTION

In this section, we will discuss the influence of the longitudinal interaction \(\Delta\) on the excitation number \(n_g\). The excitation number can be regarded as an order parameter since it keeps zero in the normal phase, and becomes non-zero in the superradiant phase. For \(\Delta = 0\), \(n_g\) can be described by Eq. (8), however, it is different for \(-\omega < \Delta < 0\) and \(0 < \Delta < \omega\), thus we must discuss it separately. In Fig. 2(a), the analytical results given by Eq. (8) agree well with the numerical results for different negative longitudinal interaction. Moreover, we find a significant increase in the number of excitations with the increase of the absolute value of the longitudinal interaction \(|\Delta|\) in the range of \(-\omega < \Delta < 0\). We can explain this phenomenon in terms of the analytical expression Eq. (8), which can be rewritten as

\[
 n_g = \frac{\lambda^2 (\lambda^2 + 2\Delta \bar{\omega}_0) - (\omega^2 - \Delta^2) \bar{\omega}_0^2}{2(\omega^2 - \Delta^2)(\lambda^2 + \Delta \bar{\omega}_0) + 2\lambda \omega \sqrt{(\omega^2 - \Delta^2)(\lambda^2 + 2\Delta \bar{\omega}_0)}}.
\]  

(14)

and we can find that Eq. (14) becomes Eq. (6) under the condition of \(\Delta = 0\). It is easy to see that the numerator remains finite when \(\Delta \simeq -\omega\), while the denominator tends to infinity. Therefore, compared with the situation of \(\Delta = 0\), the bath spins will be excited more quickly as \(|\Delta|\) approaches to \(\omega\).

In Eq. (16), for fixed \(\bar{g}\), the coherence decreases \((\bar{g} > 1\) and \(\bar{g} \omega > \Delta\)) and the denominator increases with \(\Delta\), whereas \(n_g\) decreases. Besides, the excitation number \(n_g\) is insensitive to \(\bar{g}\) when \(\Delta\) is close to \(\bar{\omega}\), therefore \(n_g\) will vary discretely (shown in Fig. 2(b)). In summary, we see that the macroscopic excitations of bath spins are significantly enhanced in the case of the negative longitudinal interaction \((-\omega < \Delta < 0\)) and the coherence decreases appreciably. Therefore, compared with the situation of \(\Delta = 0\), the bath spins will be excited more quickly as \(|\Delta|\) approaches to \(\omega\).

Similar to the excitation number of the ground state \(n_g\), the coherence is also affected by the longitudinal interaction \(\Delta\), which is shown in Fig. 3. In Fig. 3, we see that the results are similar to those of the excitation number discussed earlier. Furthermore, the coherence \(\langle a \rangle\) is also an order parameter of the QPT, which can be viewed as the result of the \(U(1)\) symmetry breaking [20].
V. PRACTICAL APPLICATION IN QUANTUM METROLOGY

Now we consider the application in quantum metrology with the XXZ central spin model. Quantum criticality which can be viewed as a quantum resource has been widely applied to quantum sensing [40, 42, 46–50], and quantum Fisher information (QFI) is a key concept in quantum metrology, which gives the lower bound for the variance of the parameter estimation. In addition, the QFI proportional to fidelity susceptibility is used to quantify the abrupt change of the ground state in the vicinity of a critical point, and can be viewed as a good indicator for quantum phase transitions [41, 51–53].

For a pure state $\psi(\bar{g})$ with a parameter $\bar{g}$, the fidelity is $f(\bar{g}, \delta \bar{g}) = |\langle \psi(\bar{g}) | \psi(\bar{g} + \delta \bar{g}) \rangle|$ and the QFI is given by

$$F_g = -4 \left. \frac{\partial^2 f(\bar{g}, \delta \bar{g})}{\partial (\delta \bar{g})^2} \right|_{\delta \bar{g} = 0}.$$  

(17)

where $\delta \bar{g}$ is a small perturbation.

In Fig. 4(a), we show the influence of different frequency ratios $\eta$ on the QFI. It is clear to see that the QFI has an abrupt change at the critical point $\bar{g}_c = 1$, which is due to a significant change in the ground state of the system. As the frequency ratio increases, the width of the QFI becomes narrower. Furthermore, we use the fidelity approach to verify the value range of the longitudinal interaction $\Delta$. In Fig. 4(b), we choose $\omega = 0.1$ and find that the abrupt changes of the QFI disappear when $|\Delta| > \omega$.

In practical measurements, one can obtain the parameter information via suitable observables and the error propagation formula. In this model, the observable of the central spin $\langle \sigma_z \rangle$ is able to measure, which is given by

$$\langle \sigma_z(t) \rangle = \langle \psi_{in} | e^{iHt} \sigma_z e^{-iHt} | \psi_{in} \rangle = 2 \text{Re} \left\{ b_\uparrow^* b_\downarrow \langle \varphi | e^{i[H(\varphi(\bar{g}^2 + \Delta))] \varphi} \right\}.$$  

(18)

where $|\psi_{in}\rangle = e^\alpha (b_\uparrow |\uparrow\rangle + b_\downarrow |\downarrow\rangle) \otimes |\varphi\rangle$ and $|\varphi\rangle = \sum_n d_n |n\rangle$ with $\sum_n |d_n|^2 = 1$. The inverse variance of the parameter $\bar{g}$ is given by [46]

$$\mathcal{F}_{\bar{g}} = \frac{(\partial_{\bar{g}} \langle \sigma_z \rangle)^2}{1 - \langle \sigma_z \rangle^2}.$$  

(19)

In Fig. 5, we show the inverse variance $\mathcal{F}_{\bar{g}}$ varies with $\bar{g}$. The initial state we choose is $b_\uparrow = b_\downarrow = 1/\sqrt{2}$ and $\varphi = |\alpha\rangle$. In spin systems, the spin coherent state can achieve the same effect with a large $N$ [26, 27]. It can be seen that $\mathcal{F}_{\bar{g}}$ changes periodically, and the amplitude of oscillation increases as the coupling strength $\bar{g}$ approaches the critical point. It is worth noting that choosing different longitudinal interaction $\Delta$ has a significant influence on the positions of the maximum value and minimum values of $\mathcal{F}_{\bar{g}}$. It means we can change the coupling strength $\bar{g}$ at which maximum precision is achieved by controlling the value of the longitudinal interaction $\Delta$, instead of being confined to the vicinity of the critical point.

VI. CONCLUSION

In this paper, we have investigated the quantum phase transition in the XXZ central spin model, and the exact energy spectrum can be given analytically due to the $U(1)$ symmetry. In addition, the similarity between the JC model and the central spin model is presented, and we have also demonstrated that the central spin model undergoes a superradiance QPT in the limit $\eta \to \infty$ and $N \to \infty$. To further explain the QPT in this model, we utilize the mean-field approximation to obtain the mean-field energy and the excitation number, which agree well with the numerical simulation.
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**Appendix A: Derivation of Exact Energy Spectrum**

Similar to the JC model, the dynamics of the XXZ central spin model is confined to the two-dimensional space spanned by $|\uparrow, n-1\rangle$ and $|\downarrow, n\rangle$ [44]. For a given $n$, the matrix elements of $H$ are

\[
\begin{align*}
\langle \uparrow, n-1 | H | \uparrow, n-1 \rangle &= \frac{\omega_0}{2} + (\omega + \Delta) m, \\
\langle \downarrow, n | H | \downarrow, n \rangle &= -\frac{\omega_0}{2} + (\omega - \Delta) (m+1), \\
\langle \uparrow, n-1 | H | \downarrow, n \rangle &= \langle \downarrow, n-1 | H | \uparrow, n-1 \rangle = A\sqrt{k_n},
\end{align*}
\]

(A1)

where $m = (n-1-j)$ and $k_n = (2j-n+1)n$. The matrix representation is

\[
H = \begin{pmatrix}
\frac{\omega_0}{2} + (\omega + \Delta) m & A\sqrt{k_n} \\
A\sqrt{k_n} & -\frac{\omega_0}{2} + (\omega - \Delta) (m+1)
\end{pmatrix}.
\]

(A2)

We set $\Omega_1 = 2\Lambda\sqrt{k_n}$, $\Omega_2 = (2m+1)\Delta - \omega + \omega_0$, $\Omega_3 = \Delta - (2m+1)\omega$. Then we can obtain the energy eigenvalues given by

\[
E_{\pm} = \frac{1}{2} \left( -\Omega_3 \pm \sqrt{\Omega_1^2 + \Omega_2^2} \right),
\]

(A3)

and the eigenstates given by

\[
\begin{align*}
|\psi_+ (n)\rangle &= \tilde{P}_{\uparrow,n-1}^+ |\uparrow, n-1\rangle + \tilde{P}_{\downarrow,n}^+ |\downarrow, n\rangle, \\
|\psi_- (n)\rangle &= \tilde{P}_{\uparrow,n-1}^- |\uparrow, n-1\rangle + \tilde{P}_{\downarrow,n}^- |\downarrow, n\rangle,
\end{align*}
\]

(A4)

where

\[
\begin{align*}
\tilde{P}_{\uparrow,n-1}^\pm &= \frac{\Omega \pm \sqrt{1 + \tilde{\Omega}^2}}{\sqrt{2\left(1 + \Omega_1^2\right) \pm 2\Omega\sqrt{1 + \Omega_2^2}}}, \\
\tilde{P}_{\downarrow,n}^\pm &= \frac{1}{\sqrt{2\left(1 + \Omega_1^2\right) \pm 2\Omega\sqrt{1 + \Omega_2^2}}},
\end{align*}
\]

(A5)

and $\tilde{\Omega} = \Omega_2/\Omega_1$. Now We use the mean-field approximation to get the mean-field energy. First, we apply the Holstein-Primakoff transformation to the Hamiltonian in Eq. (2), where

FIG. 5. Inverse variance $\mathcal{F}_\tilde{g}$ varies with $\tilde{g}$. (a) The evolution time is chosen as $\tau = 2\pi/\omega$, and different values of $\Delta$ change the peaks of $\mathcal{F}_\tilde{g}$. As the coupling strength approaches the critical point, the amplitude of oscillation increases. (b) For $\tau = 4\pi/\omega$, the number of the peaks are twice that of the previous one.
the angular momentum operators are represented by
\[ J_+ = \sqrt{N}a^\dagger \sqrt{1 - \frac{a^\dagger a}{N}}, \quad J_- = \sqrt{N} \sqrt{1 - \frac{a^\dagger a}{N}}, \quad J_z = a^\dagger a - \frac{N}{2}. \]
For large \( N \), the Hamiltonian becomes
\[ H_{hp} = \omega_0 S_z + \omega \left( a^\dagger a - j \right) + \lambda \left( a^\dagger S_- + aS_+ \right) + 2\Delta \left( a^\dagger a - \frac{N}{2} \right) S_z, \]
where \( \lambda = A\sqrt{2j} = A\sqrt{N} \). And the effective Hamiltonian under the mean-field approximation is given by
\[ H_{eff} = \langle \beta | H_{hp} | \beta \rangle = (\tilde{\omega}_0 + 2\Delta |\beta|^2) S_z + \omega (|\beta|^2 - j) + \lambda (\beta^\dagger S_- + \beta S_+), \]
and the mean-field energy is
\[ E_{mf} = \omega (|\beta|^2 - j) - \frac{1}{2} \tilde{\omega}_0, \]
where \( \tilde{\omega}_0(|\beta|^2) = \sqrt{4\lambda^2 |\beta|^2 + 4|\beta|^4 \Delta^2 + 4|\beta|^2 \Delta \tilde{\omega}_0 + \tilde{\omega}_0^2} \) is a function of \( |\beta|^2 \) and \( \tilde{\omega}_0 = \omega_0 - N\Delta \). Here we define \( \tilde{\omega} = \lambda / \sqrt{\tilde{\omega}_0 \omega_0} \) and replace \( |\beta|^2 \) with \( n \). Utilizing \( \partial E_{mf}/\partial n = 0 \), we find that for \( \tilde{\omega} < 1 \) the excitation number of the ground state is still \( n_g = 0 \) and \( E_g = -\tilde{\omega}_0/2 - \omega j \). However, for \( \tilde{\omega} > 1 \), we have
\[ n_g = -\frac{\lambda^2 + \Delta \tilde{\omega}_0}{2\Delta^2} + \frac{\lambda \omega}{2\Delta^2} \sqrt{\frac{\lambda^2 + 2\Delta \tilde{\omega}_0}{\omega^2 - \Delta^2}}, \]
and the energy of the ground state under the mean-field approximation is
\[ E_{g}^{MF} = (n_g - j) \omega - \frac{1}{2} \tilde{\omega}_0(n_g), \]

**Appendix B: Derivation of Low-Energy Effective Hamiltonian**

In this section, we give the derivation of the low-energy effective Hamiltonian in Eq. (9). We first consider the case of the normal phase. \( H_{hp} \) can be written as \( H_{hp} = H_0 + V \) where
\[ H_0 = \tilde{\omega}_0 S_z + \omega (a^\dagger a - j) + 2\Delta a^\dagger aS_z, \]
\[ V = \lambda (a^\dagger S_- + aS_+). \]

Now we use the method proposed in Refs. [19, 20]. First we apply a Schrieffer-Wolff transformation \( e^S \) to \( H_{hp} \), and the generator \( S \) is anti-Hermitian and block-off-diagonal. Then the Hamiltonian becomes
\[ \tilde{H} = e^{-S} H_{hp} e^S = \sum_{n=0}^{\infty} \frac{1}{n!} [H_{hp}, S]^{(n)}, \]
where \( [H, S]^{(n)} = [ [H, S]^{(n-1)}, S] \) and \( [H, S]^{(0)} = H \). Here we need the block-off-diagonal part of \( \tilde{H} \) to be zero up to the second order in \( \lambda \), thus \( S \) must satisfies that
\[ [H_0, S] = -\lambda (a^\dagger S_- + aS_+). \]
In the limit \( \eta \to \infty \), we find that \( S = \frac{\lambda}{\tilde{\omega}_0} (a^\dagger S_- - aS_+) \), it leads to
\[ \tilde{H} = H_0 + \frac{1}{2} [V, S] = H_0 + \lambda \left( a^\dagger S_- + aS_+ \right) \left( \frac{\lambda}{\tilde{\omega}_0} \left( a^\dagger S_- - aS_+ \right) \right) \]
\[ = H_0 + \frac{\lambda}{2 \tilde{\omega}_0} \left( 4S_z a^\dagger a + 2S_z + \sigma_0 \right), \]
and the low-energy effective Hamiltonian is expressed as
\[ \tilde{H}_{lp} = \langle \downarrow | \tilde{H} | \downarrow \rangle = -\frac{\tilde{\omega}_0}{2} - \omega j + \tilde{\omega} \left( 1 - \tilde{\omega}^2 \right) a^\dagger a, \]
where \( \tilde{\omega} = \omega - \Delta, \tilde{\omega} = \lambda / \sqrt{\omega_0 \tilde{\omega}_0} \).

For the superradiant phase, we need to apply a displacement operator \( D(\alpha) \) to \( H_{hp} \), which is given by
\[ \tilde{H} = D^\dagger (\alpha) H_{hp} D(\alpha) \]
\[ = \frac{\tilde{\omega}_0}{2} \sigma_z + \tilde{\omega} \sigma_z a^\dagger a + \omega \sigma_z \sigma_0 (a^\dagger a^\dagger) + \omega \alpha^2 + \lambda \left( a^\dagger S_- + aS_+ \right) + \Delta \left( a^\dagger a + a a^\dagger + \alpha \right) \]
\[ = \lambda \alpha \sigma_z + \Delta \left( a^\dagger a + a \sigma_0 a^\dagger a + \alpha \right) \sigma_z, \]
where \( D(\alpha) = e^{\alpha (a^\dagger - a)} \), \( \alpha^2 = n_g \), and here we make \( \alpha \) to be real for convenience. Now we get rid of the superscript of the Pauli operator and denote \( \sigma_i^{(0)} \equiv \sigma_i \) \( (i = x, y, z) \) for convenience.

We find that the part of the central spin in Eq. (B6) is \( \left( \tilde{\omega}_0 + 2\alpha^2 \right) \sigma_z / 2 + \lambda \alpha \sigma_z \), and its eigenstates are
\[ |\dagger\rangle = \cos \theta |\uparrow\rangle + \sin \theta |\downarrow\rangle, \quad |\downarrow\rangle = -\sin \theta |\uparrow\rangle + \cos \theta |\downarrow\rangle, \]
where \( \theta = \frac{1}{2} \arctan \left( \frac{2\alpha \Delta}{2\alpha^2 \Delta + \tilde{\omega}_0} \right) \). The corresponding eigenvalues are \( \pm \tilde{\omega}_0 |\alpha|^2 \) \( = \pm \sqrt{4\lambda^2 \alpha^2 + 4\alpha^4 \Delta^2 + 4\alpha^2 \Delta \tilde{\omega}_0 + \tilde{\omega}_0^2} \). Note that we have \( \alpha^2 = n_g \), thus utilizing Eq. (8) we have
\[ \tilde{\omega}_0(n_g) = \sqrt{\frac{\lambda^4 + 2\lambda^2 \tilde{\omega}_0 \alpha}{\omega^2 - \Delta^2}}. \]
Then we use the eigenstates $|↑\rangle(\downarrow)$ to rewrite Eq. (B6).

$$
\mathcal{H} = \left( \frac{\lambda \omega_0}{2\sigma_0} x - \frac{\lambda \Delta \sigma^2}{\sigma_0} x - 2\Delta \lambda \sigma \sigma_0 + \sigma_0 a^+ a \right) \tau_x
+ \left( \frac{\omega_0}{2} + \frac{\lambda^2 \sigma}{\sigma_0} x + 2\alpha^2 \Delta^2 + 2\Delta \lambda \sigma_0 + \sigma_0 a^+ a \right) \tau_z
- \frac{\lambda}{2} \rho \tau_x + \omega \alpha^2 + \omega \alpha_x a,
$$

(B9)

where $x = a^+ a$ and $p = i(a^+ - a)$.

The Hamiltonian in Eq. (B9) can be divided into diagonal and off-diagonal parts $\mathcal{H}_0$ and $\mathcal{V}$, where

$$
\mathcal{H}_0 = \omega a^+ a + \omega \alpha x + \omega \alpha^2 - \omega j
+ \left( \frac{\omega_0}{2} + \frac{\lambda^2 \sigma}{\sigma_0} x + 2\alpha^2 \Delta^2 + 2\Delta \lambda \sigma_0 + \sigma_0 a^+ a \right) \tau_x
+ \left( \frac{\lambda \omega_0}{2\sigma_0} x - \frac{\lambda \Delta \sigma^2}{\sigma_0} x - 2\Delta \lambda \sigma \sigma_0 + \sigma_0 a^+ a \right) \tau_z
- \frac{\lambda}{2} \rho \tau_x.
$$

Then we need to find the generator $\mathcal{S}$ satisfies that $[\mathcal{H}_0, \mathcal{S}] = -\mathcal{V}$. In the limit $\eta \to \infty$, $\mathcal{S}$ is given by

$$
\mathcal{S} = \left( \frac{\lambda \omega_0}{2\sigma_0} x - \frac{\lambda \Delta \sigma^2}{\sigma_0} x - 2\Delta \lambda \sigma \sigma_0 + \sigma_0 a^+ a \right) \tau_x + \frac{\lambda}{2} \rho \tau_x.
$$

(B10)

The transformed Hamiltonian is

$$
\mathcal{H}' = \mathcal{H}_0 + \frac{1}{2}[\mathcal{V}, \mathcal{S}]
= \omega a^+ a + \omega \alpha x + \omega \alpha^2 - \omega j
+ \left( \frac{\omega_0}{2} + \frac{\lambda^2 \sigma}{\sigma_0} x + 2\alpha^2 \Delta^2 + 2\Delta \lambda \sigma_0 + \sigma_0 a^+ a \right) \tau_x
+ \left( \frac{\lambda \omega_0}{2\sigma_0} x - \frac{\lambda \Delta \sigma^2}{\sigma_0} x - 2\Delta \lambda \sigma \sigma_0 + \sigma_0 a^+ a \right) \tau_z
+ \frac{\lambda^2 \sigma^2}{4\sigma_0^2} (a^+ a)^2 \tau_z
- \frac{\lambda \Delta \sigma^2}{\sigma_0^2} \frac{\lambda \omega_0}{2\sigma_0} x - \frac{\lambda \Delta \sigma^2}{\sigma_0^2} \frac{\lambda \omega_0}{2\sigma_0} \tau_x + \frac{\lambda^2 \sigma^2}{4\sigma_0^2} \rho^2 \tau_x,
$$

(B11)

Utilizing Eq. (8) and Eq. (B8), we find the relationship that $2\Delta^2 \sigma^2 + 2\Delta \sigma_0 = \omega \sigma_0$. It is easy to verify $\kappa_1$, $\kappa_2 = 0$ with the above relationship. Finally, we get Eq. (12) in main text.

$$
\mathcal{H}'_{sp} = \langle \downarrow \vert \mathcal{H}' \vert \downarrow \rangle
= \omega \alpha^2 - \frac{\sigma_0^2}{2} + \kappa_0 x^2 + \kappa_1 x + \kappa_2 p^2 - \omega j,
$$

(B12)

where

$$
\kappa_0 = \frac{2\alpha^2 \Delta^2 + 2\Delta \sigma_0 - \omega}{4\sigma_0},
$$

(B13)

$$
\kappa_1 = \omega \alpha - \frac{\lambda^2 \sigma^2 (2\alpha^2 \Delta^2 + 2\Delta \sigma_0)}{4\sigma_0},
$$

(B14)

$$
\kappa_2 = \frac{\omega}{4} - \frac{2\alpha^2 \Delta^2 + 2\Delta \sigma_0 + \lambda^2}{4\sigma_0}.
$$

(B15)

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