Dynamics and thermodynamics of systems with long-range interactions: interpretation of the different functionals

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Abstract. We discuss the dynamics and thermodynamics of systems with weak long-range interactions. Generically, these systems experience a violent collisionless relaxation in the Vlasov regime leading to a (usually) non-Boltzmannian quasi stationary state (QSS), followed by a slow collisional relaxation leading to the Boltzmann statistical equilibrium state. These two regimes can be explained by a kinetic theory, using an expansion of the BBGKY hierarchy in powers of $1/N$, where $N$ is the number of particles. We discuss the physical meaning of the different functionals appearing in the analysis: the Boltzmann entropy, the Lynden-Bell entropy, the “generalized” entropies arising in the reduced space of coarse-grained distribution functions, the Tsallis entropy, the generalized $H$-functions increasing during violent relaxation (not necessarily monotonically) and the convex Casimir functionals used to settle the formal nonlinear dynamical stability of steady states of the Vlasov equation. We show the connection between the different variational problems associated with these functionals. We also introduce a general class of nonlinear mean field Fokker-Planck (NFP) equations that can be used as numerical algorithms to solve these constrained optimization problems.

1. INTRODUCTION

Recently there was a renewed interest for the dynamics and thermodynamics of systems with long-range interactions \[1\]. These systems are numerous in Nature and appear in different domains of physics, astrophysics, fluid mechanics and biology. Systems with long-range interactions have very particular properties that contrast from those of more familiar systems with short-range interactions like gases and neutral plasmas. First of all, they can organize spontaneously into “coherent structures” in physical space. This corresponds to stars, galaxies, clusters of galaxies... in astrophysics, jets and vortices (like the gulf stream or jovian vortices) in two-dimensional geophysical turbulence, clumps and filaments (interpreted as the initiation of a vasculature) in the chemotaxis of biological populations, clusters in the HMF model... These systems have a very special thermodynamic limit corresponding to $N \to +\infty$ in such a way that the coupling constant tends to zero ($k \sim 1/N \to 0$) while the volume remains unity ($V \sim 1$). This limit describes systems with weak long-range interactions. For these systems, the mean field approximation becomes exact for $N \to +\infty$. Generically, these systems display two successive types of relaxation: a violent collisionless relaxation leading to a Quasi Stationary State (QSS) and a slow collisional relaxation tending to the Boltzmann distribution of statistical equilibrium. The QSS is a stationary solution of the Vlasov equation on the coarse-grained scale. It is usually described by a non-Boltzmannian
distribution and its lifetime diverges with \( N \). Mathematically, the Bolzmann equilibrium is obtained when the \( t \to +\infty \) limit is taken before the \( N \to +\infty \) limit and the QSS is obtained when the \( N \to +\infty \) limit is taken before the \( t \to +\infty \) limit. These two regimes can be explained by a kinetic theory, using an expansion of the BBGKY hierarchy in powers of \( 1/N \) with \( N \to +\infty \) \[2, 3, 4, 5\]. Other peculiar behaviors in the dynamics and thermodynamics of systems with long-range interactions have been evidenced: negative specific heats in the microcanonical ensemble \[6, 7, 8, 9\], ensemble inequivalence \[10, 11, 12, 13\], numerous types of phase transitions persisting at the thermodynamic limit \[14, 15\], non-trivial dependence of the collisional relaxation time with the number \( N \) of particles \[16, 17\], algebraic decay of the correlation functions \[18, 19, 20, 21\], front structure and slow relaxation of the velocity tails \[22, 21\], metastable states whose lifetimes diverge exponentially with \( N \) \[23, 24\], out-of-equilibrium phase transitions \[25, 26, 27, 28, 29\], kinetic blocking due to the absence of resonances \[21, 30\], curious effects due to spatial inhomogeneity \[31\], non-ergodic behaviors \[32, 33\], glassy dynamics \[34\] etc... Certainly, the physical richness of these systems (Pandora’s box) will lead to further investigations and applications.

In this paper, we discuss the meaning of the different functionals appearing in the study of the dynamics and thermodynamics of systems with weak long-range interactions. The Boltzmann entropy describes the statistical equilibrium state resulting from a collisional relaxation. The Lynden-Bell entropy \[35\] and the “generalized” entropic functionals arising in the reduced space of coarse-grained distributions \[36\] describe the QSS resulting from a complete violent collisionless relaxation. The maximization of these entropic functionals rely on an assumption of ergodicity (efficient mixing). Tsallis entropies \[37\] have been proposed as an alternative when the system has a non-ergodic behaviour so that violent relaxation is incomplete. Incomplete relaxation may also be understood by developing a kinetic theory of the process of violent relaxation \[38, 39, 40\]. Generalized \( H \)-functions \[40\] are functionals of the coarse-grained distribution function that increase during violent relaxation (not necessarily monotonically). Finally, convex Casimir functionals are used to settle the formal nonlinear dynamical stability of steady states of the Vlasov equation \[41, 42\]. Although the above-mentioned functionals may have similar mathematical forms, they have very different physical interpretations.

### 2. SYSTEMS WITH WEAK LONG-RANGE INTERACTIONS

#### 2.1. The \( N \)-body Hamiltonian system

We consider an isolated system of particles in interaction whose dynamics is fully described by the Hamilton equations

\[
\frac{m}{\frac{d\mathbf{r}_i}{dt}} = \frac{\partial H}{\partial \mathbf{v}_i}, \quad \frac{m}{\frac{d\mathbf{v}_i}{dt}} = -\frac{\partial H}{\partial \mathbf{r}_i},
\]

\[
H = \frac{1}{2} \sum_{i=1}^{N} m\mathbf{v}_i^2 + m^2 \sum_{i<j} u(\mathbf{r}_i - \mathbf{r}_j),
\]

(1)
where \( u_{ij} = u(r_i - r_j) \) is a binary potential of interaction depending only on the absolute distance \(|r_i - r_j|\) between the particles. Basically, the evolution of the \( N \)-body distribution function \( P_N(r_1, v_1, ..., r_N, v_N, t) \) is governed by the Liouville equation

\[
\frac{\partial P_N}{\partial t} + \sum_{i=1}^{N} \left( v_i \cdot \frac{\partial P_N}{\partial r_i} + F_i \cdot \frac{\partial P_N}{\partial v_i} \right) = 0,
\]

(2)

where

\[
F_i = -\frac{\partial \Phi}{\partial r_i} = -m \sum_{j \neq i} \frac{\partial u_{ij}}{\partial r_i} = \sum_{j \neq i} F(j \rightarrow i),
\]

(3)
is the force by unit of mass experienced by particle \( i \) due to the interaction with the other particles. The Liouville equation is equivalent to the \( N \)-body Hamiltonian system (1).

Now, the basic postulate of statistical mechanics states that: *At statistical equilibrium, all the microscopic configurations that are accessible (i.e. that have the correct value of the energy) are equiprobable*. There is no guarantee that the dynamics will lead the system to that “uniform” state. This relies upon a hypothesis of ergodicity which may not always be fulfilled for complex systems (some regions of the \( \Gamma \)-phase space could be more probable than others). This is why it is important to develop a kinetic theory in order to vindicate (or not!) this result. In the sequel, following the wisdom of ordinary statistical mechanics, we shall assume that all the accessible microscopic configurations are equiprobable at equilibrium (for \( t \to +\infty \)) but we must keep in mind the importance of this postulate.

At statistical equilibrium, the \( N \)-body distribution is given by the microcanonical distribution expressing the equiprobability of the accessible configurations

\[
P_N(r_1, v_1, ..., r_N, v_N) = \frac{1}{g(E)} \delta(E - H(r_1, v_1, ..., r_N, v_N)).
\]

(4)

Since \( \int P_N \prod_i dr_i dv_i = 1 \), the density of states with energy \( E \) is given by

\[
g(E) = \int \delta(E - H(r_1, v_1, ..., r_N, v_N)) \prod_i dr_i dv_i.
\]

(5)

This is the normalization factor of the \( N \)-body distribution function \( P_N \). The microcanonical entropy of the system is defined by \( S(E) = \ln g(E) \) and the microcanonical temperature by \( \beta(E) = 1/T(E) = \partial S/\partial E \) (we take the Boltzmann constant \( k_B = 1 \)). We introduce the reduced probability distributions

\[
P_j(x_1, ..., x_j) = \int P_N(x_1, ..., x_N) dx_{j+1} ... dx_N,
\]

(6)

where \( x = (r, v) \). For identical particles, the average density in \( \mu \)-space is related to the one-body distribution function by

\[
f(r, v) = \left\langle \sum_{i=1}^{N} m \delta(r - r_i) \delta(v - v_i) \right\rangle = N m P_1(r, v),
\]

(7)
and the average value of the energy is

\[ E = \langle H \rangle = Nm \int P_1(r,v) \frac{v^2}{2} drdv + \frac{1}{2}N(N-1)m^2 \int u(r-r')P_2(r,v,r',v')drdvdr'dv'. \quad (8) \]

By differentiating the defining relation for \( P_j \) and using Eq. (4), we can obtain an equilibrium BBGKY-like hierarchy for the reduced distributions [2]:

\[ \frac{\partial P_j}{\partial r_1} = -\frac{1}{g(E)} \frac{\partial}{\partial E} \left[ g(E)P_j \right] \sum_{i=2}^{j} m^2 \frac{\partial u_{i,j}}{\partial r_1} - (N-j)m^2 \int \frac{\partial u_{1,j+1}}{\partial r_1} \frac{1}{g(E)} \frac{\partial}{\partial E} \left[ g(E)P_{j+1} \right] d\varepsilon_{j+1}, \quad (9) \]

\[ \frac{\partial P_j}{\partial v_1} = -\frac{1}{g(E)} \frac{\partial}{\partial E} \left[ g(E)P_j \right] mv_1. \quad (10) \]

This is the counterpart of the equilibrium hierarchy in plasma physics. It is however more complex in the present situation because it has been derived in the microcanonical ensemble. Since statistical ensembles are generically inequivalent for systems with long-range interactions, we must formulate the problem in the microcanonical ensemble which is the fundamental one. We note that the terms involving the derivative of the density of states can be split in two parts according to

\[ \frac{1}{g(E)} \frac{\partial}{\partial E} \left[ g(E)P_j \right] = \beta P_j + \frac{\partial P_j}{\partial E}. \quad (11) \]

The terms with the \( E \) derivative would not have emerged if we had started from the Gibbs canonical distribution [2].

### 2.2. Thermodynamic limit and mean-field approximation

We define the thermodynamic limit as \( N \to +\infty \) in such a way that the normalized energy \( \varepsilon = E/(u_s N^2 m^2) \) and the normalized temperature \( \eta = \beta N m^2 u_s \) are fixed, where \( u_s \) represents the typical value of the potential of interaction. Since the normalized coupling constant \( \beta m^2 u_s = \eta/N \sim 1/N \) goes to zero for \( N \to +\infty \), we are studying systems with weak long-range interactions. For 3D self-gravitating systems \( u = -G/|r-r'| \), \( u_s = G/R, \eta = \beta GMm/R, \varepsilon = ER/GM^2 \), for 2D point vortices \( u = -\frac{1}{2\pi} \ln |r-r'| \), \( u_s = 1, \eta = \beta N \gamma^2, \varepsilon = E/\Gamma^2 \) and for the HMF model \( u = -\frac{k}{2\pi} \cos(\theta - \theta'), u_s = k, \eta = \beta kN, \varepsilon = E/kN^2 \) we recover the dimensionless parameters already introduced in the literature. In general, the potential of interaction is written as \( u(r_{ij}) = k\tilde{u}(r_{ij}) \) where \( k \) is the coupling constant (e.g., \( G \) for self-gravitating systems or \( k \) for the HMF

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1 Two-dimensional point vortices (with circulation \( \gamma \)) are special because they have no inertia. Their kinetic theory, which must take into account this particularity, is developed in [30].
model). The dynamical time is \( t_D \sim R/v_{typ} \sim R/\sqrt{Nmu_s} \) where the typical velocity \( v_{typ} \sim (Nmu_s)^{1/2} \) has been obtained by equating the kinetic energy \( \sim Nmv^2 \) and the potential energy \( \sim N^2m^2u_s \). For a long-range interaction, we have \( u_s = k/R^\alpha \) where \( \alpha \) is less than the dimension \( d \) of the system. Then, we get \( t_D \sim R^{(2+\alpha-d)/2}/\sqrt{k\rho} \) where \( \rho \sim M/V \) is the average density. By a suitable normalization of the parameters, the proper thermodynamic limit for systems with long-range interactions is such that the coupling constant behaves like \( k \sim u_s \sim 1/N \), while \( m \sim 1, \beta \sim 1, E/N \sim 1 \) and \( V \sim 1 \). This implies that \( |r| \sim 1, |v| \sim 1, u \sim 1/N, |F(j \rightarrow i)| \sim 1/N \) and \( t_D \sim 1 \). For example, in the case of self-gravitating systems, this scaling corresponds to \( G \sim 1/N \) while \( m \sim 1, \beta \sim 1, E/N \sim 1, V \sim 1, t_D \sim 1/\sqrt{G\rho} \sim 1 \) and in the case of the HMF model, it corresponds to \( k \sim 1/N \) while \( \beta \sim 1, E/N \sim 1, V \sim 1, t_D \sim 1/\sqrt{G\rho} \sim 1 \) (for self-gravitating systems, we could also consider the dilute limit \( N \rightarrow +\infty \) with \( N/R \sim 1 \) while \( G \sim 1, m \sim 1, \beta \sim 1, E/N \sim 1 \) \([43]\) but in that case we notice that \( t_D \sim N \)). Using another normalization of the parameters, the thermodynamic limit is such that the mass of the particles behaves like \( m \sim 1/N \) while \( k \sim u_s \sim 1, \beta \sim N, E \sim 1, V \sim 1 \) and \( t_D \sim 1 \). In this scaling, the total mass \( M \sim Nm \) is of order unity. For example, in the case of self-gravitating systems, this scaling corresponds to \( m \sim 1/N \) while \( G \sim 1, \beta \sim N, E \sim 1, V \sim 1, t_D \sim 1/\omega \sim R^2/\Gamma \sim 1 \) (where \( \Gamma = N\gamma \) is the total circulation).

The first two equations of the equilibrium BBGKY-like hierarchy are

\[
\frac{\partial P_1}{\partial r_1} (x_1) = -(N-1)m^2 \int \frac{\partial u_{12}}{\partial r_1} \left( \beta P_2 + \frac{\partial P_2}{\partial E} \right) dx_2, \tag{12}
\]

\[
\frac{\partial P_2}{\partial r_1} (x_1, x_2) = -m^2 \left( \beta P_2 + \frac{\partial P_2}{\partial E} \right) \frac{\partial u_{12}}{\partial r_1} - (N-2)m^2 \int \frac{\partial u_{13}}{\partial r_1} \left( \beta P_3 + \frac{\partial P_3}{\partial E} \right) dx_3. \tag{13}
\]

We note that the ratio of \( \partial P_1/\partial E \) on \( \beta P_1 \) is of order \( 1/(E\beta) = 1/(\epsilon \eta N) \). Therefore, in the thermodynamic limit \( N \rightarrow +\infty \) with \( \epsilon, \eta \) fixed, the second term in Eq. (11) is always negligible with respect to the first. We now decompose the two- and three-body distribution functions in the suggestive form

\[
P_2(x_1, x_2) = P_1(x_1)P_1(x_2) + P'_2(x_1, x_2), \tag{14}
\]

\[
P_3(x_1, x_2, x_3) = P_1(x_1)P_1(x_2)P_1(x_3) + P'_2(x_1, x_2)P_1(x_3)
+ P'_2(x_1, x_3)P_1(x_2) + P'_2(x_2, x_3)P_1(x_1) + P'_3(x_1, x_2, x_3). \tag{15}
\]

This decomposition corresponds to the first terms of the Mayer expansion in plasma physics. The non-trivial correlations \( P'_j \) are called the cumulants. We can now substitute these decompositions in the equilibrium BBGKY-like hierarchy. The first equation \([12]\) of the hierarchy can be written

\[
\frac{\partial P_1}{\partial r_1} (x_1) = -\beta (N-1)m^2 P_1(x_1) \int P_1(x_2) \frac{\partial u_{12}}{\partial r_1} dx_2

- \beta (N-1)m^2 \int P'_2(x_1, x_2) \frac{\partial u_{12}}{\partial r_1} dx_2 - (N-1)m^2 \int \frac{\partial u_{12}}{\partial r_1} \frac{\partial P_2}{\partial E} (x_1, x_2) dx_2. \tag{16}
\]
In the thermodynamic limit defined previously, it can be shown that the cumulants \( P'_n \) are of order \( N^{-(n-1)} \) \([2]\). In particular, we have

\[
P_2(x_1, x_2) = P_1(x_1)P_1(x_2) + O(1/N).
\]

Therefore, the mean field approximation is exact for \( N \to +\infty \). In the thermodynamic limit, the two-body correlation function is the product of two one-body distribution functions: \( P_2(x_1, x_2) \simeq P_1(x_1)P_1(x_2) \).

### 2.3. The mean field equilibrium distribution

Taking the limit \( N \to +\infty \) and using Eq. (17), the first equation (16) of the equilibrium BBGKY-like hierarchy becomes

\[
\nabla \rho (r) = -\beta m \rho (r) \nabla \int \rho (r')u(r - r')dr',
\]

where \( \rho (r) = NmP_1(r) \) is the spatial density. After integration, this can be written in the form of a meanfield Boltzmann distribution

\[
\rho (r) = A'e^{-\beta m \Phi(r)},
\]

where

\[
\Phi(r) = \int \rho (r')u(r - r')dr',
\]

is the potential produced self-consistently by the smooth distribution of particles. Therefore, the equilibrium density profile of the particles is determined by an integrodi
differential equation (18) or (19)-(20). On the other hand, for \( N \to +\infty \), Eq. (10) leads to

\[
\frac{\partial P_1}{\partial v_1} = -\beta m P_1 v_1,
\]

so that the velocity distribution \( \phi (v) = NmP_1(v) \) is maxwellian

\[
\phi (v) = A''e^{-\beta m \frac{v^2}{2}}.
\]

Combining Eqs. (19) and (22) and introducing the distribution function \( f(r, v) = NmP_1(r, v) \), we get the mean-field Maxwell-Boltzmann distribution function

\[
f(r, v) = Ae^{-\beta m \frac{v^2}{2} + \Phi(r)},
\]

where \( \Phi(r) \) is given by Eq. (20) and \( \rho = \int f dv \). If we define the entropy by

\[
S_N = -\int P_N \ln P_N dx_1...dx_N,
\]
and use the mean-field approximation

\[ P_N(x_1, \ldots, x_N) = P_1(x_1) \cdots P_1(x_N), \quad (25) \]

valid at the thermodynamic limit \( N \to +\infty \) defined previously, we obtain

\[ S = -N \int P_1(x) \ln P_1(x) dx. \quad (26) \]

In terms of the distribution function \( f = NmP_1 \), we get

\[ S = -\int \frac{f}{m} \ln \left( \frac{f}{Nm} \right) dr dv. \quad (27) \]

On the other hand, in the mean-field approximation, the average energy (8) is given by

\[ E = Nm \int P_1 v^2 dr dv + \frac{1}{2} N^2 m^2 \int P_1(r) u(r - r') P_1(r') dr dr'. \quad (28) \]

This can also be written

\[ E = \int f v^2 dr dv + \frac{1}{2} \int \rho(r) \Phi(r) dr. \quad (29) \]

The mass is given by

\[ M = \int f dr dv. \quad (30) \]

Using these relations, we can relate the inverse temperature \( \beta \) and the constant \( A \) in the mean field Maxwell-Boltzmann distribution (23) to the energy \( E \) and the mass \( M \).

### 2.4. Statistical equilibrium state: the Boltzmann entropy

We wish to determine the most probable distribution of particles at statistical equilibrium by using a combinatorial analysis, assuming that all accessible microstates (with given energy \( E \) and mass \( M \)) are equiprobable. To that purpose, we divide the \( \mu \)-space \( \{r, v\} \) into a very large number of microcells with size \( h \). We do not put any exclusion, so that a microcell can be occupied by an arbitrary number of particles. We shall now group these microcells into macrocells each of which contains many microcells but remains nevertheless small compared to the phase-space extension of the whole system. We call \( \nu \) the number of microcells in a macrocell. Consider the configuration \( \{n_i\} \) where \( n_i \) is the number of particles in the macrocell \( i \). Using the standard combinatorial procedure introduced by Boltzmann, the probability of the state \( \{n_i\} \), i.e. the number of microstates corresponding to the macrostate \( \{n_i\} \), is given by

\[ W(\{n_i\}) = \prod_i \frac{N!}{n_i!} \nu^{n_i}. \quad (31) \]
This is the Maxwell-Boltzmann statistics. As is customary, we define the entropy of the state \( \{ n_i \} \) by
\[
S(\{ n_i \}) = \ln W(\{ n_i \}).
\] (32)

It is convenient here to return to a representation in terms of the distribution function giving the phase-space density in the \( i \)-th macrocell: \( f_i = f(r_i, v_i) = n_i m / \nu h^d \) (where \( d \) is the dimension of space). Using the Stirling formula \( \ln n! \simeq n \ln n - n \) for \( n \gg 1 \), we have
\[
\ln W(\{ n_i \}) = - \sum_i n_i \ln n_i = - \sum_i \nu h^d f_i / m \ln f_i / m.
\] (33)

Passing to the continuum limit \( \nu \to 0 \), we obtain the usual expression of the Boltzmann entropy
\[
S_B[f] = - \int \frac{f}{m} \ln \frac{f}{m} d\mathbf{r} d\mathbf{v},
\] (34)
up to some unimportant additive constant. Assuming ergodicity, the statistical equilibrium state, corresponding to the most probable distribution of particles (i.e. the macrostate that is the most represented at the microscopic level), is obtained by maximizing the Boltzmann entropy (34) while conserving the total mass (30) and the total energy (29). Introducing Lagrange multipliers and writing the variational principle in the form
\[
\delta S_B - \beta \delta E - \alpha \delta M = 0,
\] (35)
we obtain the meanfield Maxwell-Boltzmann distribution
\[
f = A e^{- \beta m (\frac{\mathbf{v}^2}{2} + \Phi)}. \tag{36}
\]
This returns the equations obtained in Sec. 2.3. The potential \( \Phi \) is obtained by solving the integrodifferential equation (19)-(20) and the Lagrange multipliers \( A \) and \( \beta \) must be related to the constraints \( M \) and \( E \) using Eqs. (30)-(29). Then, we have to make sure that the distribution is a maximum of \( S_B \) at fixed mass and energy, not a minimum or a saddle point. Therefore, this method gives a condition of thermodynamical stability which was not obtained in Sec. 2.3 (to get a similar relation, we need to consider the next order term in the expansion in \( 1/N \) of the equilibrium BBGKY-like hierarchy).

The maximization problem
\[
\max_f \{ S_B[f] \mid E[f] = E, M[f] = M \}, \tag{37}
\]
corresponds to a condition of microcanonical stability. Alternatively, the minimization problem
\[
\min_f \{ F_B[f] = E[f] - TS_B[f] \mid M[f] = M \}, \tag{38}
\]
corresponds to a condition of canonical stability, where \( F_B[f] \) is the Boltzmann free energy. The optimization problems (37) and (38) have the same critical points (cancelling the first order variations of the thermodynamical potential). On the other hand, canonical stability (38) implies microcanonical stability (37): if \( f(\mathbf{r}, \mathbf{v}) \) is a minimum of \( F_B[f] \)
at fixed mass, then it is a maximum of $S_B[f]$ at fixed mass and energy. However, the converse is wrong in case of ensemble inequivalence. This means that the ensemble of solutions of (38) is included in the ensemble of solutions of (37) but the two ensembles may not coincide for systems with long-range interactions $^2$. Ensemble inequivalence was first encountered in astrophysics $^6, 7$ where it was realized that configurations with negative specific heats are allowed in the microcanonical ensemble but not in the canonical ensemble (see reviews in $^10, 15$). This notion of ensemble inequivalence has been developed and formalized recently at a general level $^11, 13, 14$. The condition of ensemble equivalence or ensemble inequivalence, related to the concavity of the entropy $S(E)$, is discussed in detail in these papers. Ensemble equivalence/inequivalence can also be deduced from the study of the series of equilibria $\beta(E)$ using the Poincaré criterion $^44, 15$.

3. KINETIC THEORY FROM THE BBGKY HIERARCHY

In this section, we derive a general kinetic equation (53) for Hamiltonian systems with weak long-range interactions. We start from the BBGKY hierarchy and use a systematic expansion of the solutions of the equations of this hierarchy in powers of $1/N$ in a proper thermodynamic limit $N \to +\infty$. The kinetic equation (53) is valid at order $O(1/N)$.

3.1. The $1/N$ expansion

From the Liouville equation (2) we can construct the complete BBGKY hierarchy for the reduced distribution functions $P_j$. It reads

$$\frac{\partial P_j}{\partial t} + \sum_{i=1}^j v_i \frac{\partial P_j}{\partial r_i} + \sum_{i=k=1, k \neq i}^j \mathbf{F}(k \to i) \frac{\partial P_j}{\partial v_i} + (N-j) \sum_{i=1}^j \int d\mathbf{x}_{j+1} \mathbf{F}(j+1 \to i) \frac{\partial P_{j+1}}{\partial v_i} = 0.$$  (39)

This hierarchy of equations is not closed since the equation for the one-body distribution $P_1(x_1, t)$ involves the two-body distribution $P_2(x_1, x_2, t)$, the equation for the two-body distribution $P_2(x_1, x_2, t)$ involves the three-body distribution $P_3(x_1, x_2, x_3, t)$, and so on. The idea is to close the hierarchy by using a systematic expansion of the solutions in powers of $1/N$ in the thermodynamic limit $N \to +\infty$. Considering the scaling of the terms in each equation of the hierarchy, we argue that there exists solutions of the whole BBGKY hierarchy such that the correlation functions $P'_j$ scale like $1/N^{j-1}$ at any time

$^2$ Since energy is non-additive for systems with long-range interactions, the canonical ensemble cannot be used to study a subpart of the system $^10, 11$. Therefore, the problem (38) has no physical justification for an isolated Hamiltonian system interacting via purely long-range forces. However, the canonical ensemble is justified for dissipative systems, such as Brownian systems with long-range interactions, that experience a friction force and a stochastic force (modelling short-range interactions) in addition to long-range forces $^2$. Here, we shall only consider isolated Hamiltonian systems so that the fundamental ensemble is the microcanonical ensemble. However, the minimization problem (38) can be useful mathematically since it offers a sufficient condition of microcanonical stability (since (38) $\Rightarrow$ (37)) as discussed in the text.
This implicitly assumes that the initial condition has no correlation, or that the initial correlations respect this scaling (if there are strong correlations in the initial state, like “binaries”, the kinetic theory will be different from the one developed in the sequel). If this scaling is satisfied, we can consider an expansion of the solutions of the equations of the hierarchy in terms of the small parameter $1/N$. This is similar to the expansion in terms of the plasma parameter made in plasma physics. However, in plasma physics the systems are spatially homogeneous while, in the present case, we shall take into account spatial inhomogeneity. This brings additional terms in the kinetic equations that are absent in plasma physics. Therefore, strictly speaking, the hierarchy that we consider is different from the ordinary BBGKY hierarchy. If we introduce the notations $f = N m P_1$ (distribution function) and $g = N^2 P'_2$ (two-body correlation function), we get at order $1/N$:

$$\frac{\partial f_1}{\partial t} + v_1 \frac{\partial f_1}{\partial r_1} + \frac{N-1}{N} \langle F \rangle_1 \frac{\partial f_1}{\partial v_1} = -m \frac{\partial}{\partial v_1} \int F(2 \to 1) g(x_1, x_2) dx_2, \quad (40)$$

$$\frac{\partial g}{\partial t} + v_1 \frac{\partial g}{\partial r_1} + \langle F \rangle_1 \frac{\partial g}{\partial v_1} + \frac{1}{m^2} \mathcal{F}(2 \to 1) f_2 \frac{\partial f_1}{\partial v_1} + \frac{1}{m} \int F(3 \to 1) g(x_2, x_3, t) \frac{f_1}{m} dx_3 + (1 \leftrightarrow 2) = 0, \quad (41)$$

where we have introduced the abbreviations $f_1 = f(r_1, v_1, t)$ and $f_2 = f(r_2, v_2, t)$. We have also introduced the mean force (by unit of mass) created in $r_1$ by all the particles

$$\langle F \rangle_1 = \int F(2 \to 1) \frac{f_2}{m} dr_2 dv_2 = -\nabla \Phi_1, \quad (42)$$

and the fluctuating force (by unit of mass) created by particle 2 on particle 1:

$$\mathcal{F}(2 \to 1) = F(2 \to 1) - \frac{1}{N} \langle F \rangle_1. \quad (43)$$

These equations are exact at the order $O(1/N)$ where the three-body correlation function can be neglected. They form therefore the right basis to develop a kinetic theory for Hamiltonian systems with weak long-range interactions. We note that these equations are similar to the BBGKY hierarchy of plasma physics but not identical. One difference is the $(N-1)/N$ term in Eq. (40). The other difference is the presence of the fluctuating force $\mathcal{F}(2 \to 1)$ instead of $F(2 \to 1)$ due to the spatial inhomogeneity of the system. In plasma physics, the system is homogeneous over distances of the order of the Debye length so the mean force $\langle F \rangle$ vanishes.

**3.2. The limit $N \to +\infty$: the Vlasov equation**

Recalling that $P'_2 \sim 1/N$, we note that

$$P_2(x_1, x_2, t) = P_1(x_1, t) P_1(x_2, t) + O(1/N). \quad (44)$$
If we consider the limit $N \to +\infty$ (for a fixed time $t$), we see that the correlations between particles can be neglected so the two-body distribution function factorizes in two one-body distribution functions i.e. $P_2(x_1, x_2, t) = P_1(x_1, t)P_1(x_2, t)$. Therefore the mean field approximation is exact in the limit $N \to +\infty$. Substituting this result in Eq. (39), we obtain the Vlasov equation

$$\frac{\partial f_1}{\partial t} + v_1 \frac{\partial f_1}{\partial r_1} + \langle F \rangle_1 \frac{\partial f_1}{\partial v_1} = 0.$$  \hspace{1cm} (45)

This equation also results from Eq. (40) if we neglect the correlation function $g = N^2 P'_2$ in the r.h.s. The Vlasov equation describes the collisionless evolution of the system up to a time at least of order $Nt_D$ (where $t_D$ is the dynamical time). In practice, $N \gg 1$ so that the domain of validity of the Vlasov equation is huge (for example, in stellar systems $N \sim 10^6 - 10^{12}$ stars). When the Vlasov equation is coupled to a long-range potential of interaction it can develop a process of phase mixing and violent relaxation leading to a quasi-stationary state (QSS) on a very short timescale, of the order of the dynamical time $t_D$ [35]. This process will be discussed in Sec. 4.

### 3.3. The order $O(1/N)$: a general kinetic equation

If we want to describe the collisional evolution of the system, we need to consider finite $N$ effects. Equations (40) and (41) describe the evolution of the system on a timescale of order $Nt_D$. The equation for the evolution of the smooth distribution function is of the form

$$\frac{\partial f_1}{\partial t} + v_1 \frac{\partial f_1}{\partial r_1} + \frac{N-1}{N} \langle F \rangle_1 \frac{\partial f_1}{\partial v_1} = C_N[f_1],$$  \hspace{1cm} (46)

where $C_N$ is a “collision” term analogous to the one arising in the Boltzmann equation. In the present context, there are not real collisions between particles. The term on the right hand side of Eq. (46) is due to the development of correlations between particles as time goes on. It is related to the two-body correlation function $g(x_1, x_2, t)$ which is itself related to the distribution function $f(x_1, t)$ by Eq. (41). Our aim is to obtain an expression for the collision term $C_N[f]$ at the order $1/N$. The difficulty with Eq. (41) for the two-body correlation function is that it is an integrodifferential equation. The second and third terms are advective terms, the fourth term is the source of the correlation and the fifth term takes into account collective effects. If we neglect the contribution of the integral in Eq. (41), we get the simpler coupled system

$$\frac{\partial f_1}{\partial t} + v_1 \frac{\partial f_1}{\partial r_1} + \frac{N-1}{N} \langle F \rangle_1 \frac{\partial f_1}{\partial v_1} = -m \frac{\partial}{\partial v_1} \int F(2 \to 1)g(x_1, x_2)dx_2,$$  \hspace{1cm} (47)

$$\frac{\partial g}{\partial t} + \left[ v_1 \frac{\partial}{\partial r_1} + v_2 \frac{\partial}{\partial r_2} + \langle F \rangle_1 \frac{\partial}{\partial v_1} + \langle F \rangle_2 \frac{\partial}{\partial v_2} \right] g$$

$$+ \left[ \mathcal{F}(2 \to 1) \frac{\partial}{\partial v_1} + \mathcal{F}(1 \to 2) \frac{\partial}{\partial v_2} \right] \frac{f_1 f_2}{m m} = 0.$$  \hspace{1cm} (48)
The integral that we have neglected contains “collective effects” that describe the polarization of the medium. In plasma physics, they are responsible for the Debye shielding. These collective effects are taken into account in the Lenard-Balescu equation through the dielectric function \[45\]. However, this equation is restricted to spatially homogeneous systems and based on a Markovian approximation (see, e.g., \[3\]). These assumptions are necessary to use Laplace-Fourier transforms in order to solve the integro-differential equation \[44\]. Here, we want to describe more general situations where the interaction is not shielded so that the system can be spatially inhomogeneous. If we neglect collective effects, we can obtain a general kinetic equation in a closed form \[53\] that is valid for systems that are not necessarily homogeneous and that can take into account memory effects. This equation has interest in its own right (despite its limitations) because its structure bears a lot of physical significance.

The equation \[43\] for the correlation function can be written

\[
\frac{dg}{dt} + \mathcal{L}g = -\left[\mathcal{F}(2 \to 1) \frac{\partial}{\partial v_1} + \mathcal{F}(1 \to 2) \frac{\partial}{\partial v_2}\right] f_m(x_1,t) \frac{f_m(x_2,t)}{m},
\]

where we have denoted the advective term by $\mathcal{L}$ (Liouvillian operator). Solving formally this equation with the Green function

\[
G(t,t') = \exp\left\{-\int_{t'}^t \mathcal{L}(\tau)d\tau\right\},
\]

we obtain

\[
g(x_1,x_2,t) = -\int_0^t d\tau G(t,t-\tau) \left[\mathcal{F}(2 \to 1) \frac{\partial}{\partial v_1} + \mathcal{F}(1 \to 2) \frac{\partial}{\partial v_2}\right] f_m(x_1,t-\tau) \frac{f_m(x_2,t-\tau)}{m}.
\]

The Green function constructed with the smooth field $\langle F \rangle$ means that, in order to evaluate the time integral in Eq. \[51\], we must move the coordinates $r_i(t-\tau)$ and $v_i(t-\tau)$ of the particles with the mean field flow in phase space, adopting a Lagrangian point of view. Thus, in evaluating the time integral, the coordinates $r_i$ and $v_i$ placed after the Greenian must be viewed as $r_i(t-\tau)$ and $v_i(t-\tau)$ where

\[
r_i(t-\tau) = r_i(t) - \int_0^\tau v_i(t-s)ds, \quad v_i(t-\tau) = v_i(t) - \int_0^\tau \langle F \rangle (r_i(t-s),t-s)ds. \tag{52}
\]

Substituting Eq. \[51\] in Eq. \[47\], we get

\[
\frac{df_i}{dt} + v_i \frac{\partial f_i}{\partial v_1} + \frac{N - 1}{N} \langle F \rangle_i \frac{\partial f_i}{\partial v_1} = \frac{\partial}{\partial v_1^\mu} \int_0^t d\tau \int dr_2 d\nu_2 F^{\mu}(2 \to 1,t)G(t,t-\tau)
\]

\[
\times \left[\mathcal{F}^{\nu}(2 \to 1) \frac{\partial}{\partial v_1^\nu} + \mathcal{F}^{\nu}(1 \to 2) \frac{\partial}{\partial v_2^\nu}\right] f(r_1,v_1,t-\tau) \frac{f_m(r_2,v_2,t-\tau)}{m}. \tag{53}
\]

This kinetic equation can also be obtained from a more abstract projection operator formalism \[46\] or from a quasilinear theory based on the Klimontovich equation \[5\].
(note that we can replace $F^\mu(2 \to 1, t)$ by $\mathcal{P}^\mu(2 \to 1, t)$ in the first term of the r.h.s. of the equation since the fluctuations vanish in average). This kinetic equation (53) is valid at order $1/N$ so that it describes the “collisional” evolution of the system on a timescale of order $N t_D$ (ignoring collective effects). Equation (53) is a non-Markovian integrodifferential equation. If we implement a Markovian approximation valid at order 1 of the equation since the fluctuations vanish in average). This kinetic equation (53) is exact at order $O(1/N)$, the energy must be conserved. Indeed, the integral constraints of the Hamiltonian system must be conserved at any order of the $1/N$ expansion (47). On the other hand, we cannot establish the $H$-theorem for an equation of the form (53). It is only when additional approximations are implemented (markovian approximation) that the $H$-theorem is obtained. To be more precise, let us compute the rate of change of the Boltzmann entropy $S_B = -\int f_1 \ln \frac{f_1}{\mu} d\mathbf{r}_1 d\mathbf{v}_1$

3 As we have already indicated, Eq. (53) ignores collective effects. However, we do not think that this approximation alters the conservation of the energy. When we make a Markovian approximation (valid for $N \to +\infty$) and extend the time integral to infinity, leading to Eq. (54), the conservation of the energy is shown in Sec. 3.5 for homogeneous systems and in [48] for inhomogeneous systems (using angle-action variables).
with respect to the general kinetic equation (53). After straightforward manipulations obtained by interchanging the indices 1 and 2, it can be put in the form

\[
\dot{S}_B = \frac{1}{2m^2} \int dx_1 dx_2 \frac{1}{f_1 f_2} \int_0^\infty d\tau \left[ \mathcal{P}^\mu (2 \rightarrow 1) f_2 \frac{\partial f_1}{\partial v_1^\mu} + \mathcal{P}^\mu (1 \rightarrow 2) f_1 \frac{\partial f_2}{\partial v_2^\mu} \right]_t 
\times G(t, t - \tau) \left[ \mathcal{P}^\nu (2 \rightarrow 1) f_2 \frac{\partial f_1}{\partial v_1^\nu} + \mathcal{P}^\nu (1 \rightarrow 2) f_1 \frac{\partial f_2}{\partial v_2^\nu} \right]_{t - \tau}.
\] (56)

We note that its sign is not necessarily positive. This depends on the importance of memory effects. In general, the Markovian approximation is justified for \( N \rightarrow +\infty \) because the correlations decay on a timescale \( \tau_{corr} \) that is much smaller than the time \( \sim Nt_D \) on which the distribution changes (with, again, the exception of gravity). In that case, the entropy increases monotonically as shown in Sec. 3.5 for homogeneous systems and in [48] for inhomogeneous systems (using angle-action variables). Now, even if the energy is conserved and the entropy increases monotonically, it is not completely clear whether the general kinetic equation (53) will relax towards the mean field Maxwell-Boltzmann distribution (36) of statistical equilibrium. It could be trapped in a steady state that is not the state of maximal entropy because there is no resonance anymore to drive the relaxation (see the discussion in Secs. 3.5 and 3.7 and in [48]). Indeed, the kinetic equation (53) can admit several stationary solutions, not only the Boltzmann distribution. It could also undergo everlasting oscillations without reaching a steady state. The kinetic equation (53) may have a rich variety of behaviors and its complete study is of great complexity.

### 3.5. Spatially homogeneous systems: the Landau and Lenard-Balescu equations

If we consider spatially homogeneous systems, make a Markovian approximation and extend the time integration to infinity, the kinetic equation (53) becomes

\[
\frac{\partial f_1}{\partial t} = \frac{\partial}{\partial v_1^\mu} \int_0^{+\infty} d\tau \int d\mathbf{r}_2 d\mathbf{v}_2 F^\mu (2 \rightarrow 1, t) F^\nu (2 \rightarrow 1, t - \tau) 
\times \left( \frac{\partial}{\partial v_1^\nu} - \frac{\partial}{\partial v_2^\nu} \right) f(\mathbf{v}_1, t) \frac{f(\mathbf{v}_2, t)}{m},
\] (57)

where we have used \( F(2 \rightarrow 1) = -F(1 \rightarrow 2) \). Since \( \langle F \rangle = 0 \), the particles follow linear trajectories with constant velocity to leading order in \( N \). Therefore, \( \mathbf{v}_i(t - \tau) = \mathbf{v}_i - \mathbf{v}_i \tau \) where \( \mathbf{r}_i \) and \( \mathbf{v}_i \) denote their position and velocity at time \( t \). In that case, the integrals on time and position in Eq. (57) can be easily performed in Fourier space [3] and we obtain the Landau equation

\[
\frac{\partial f_1}{\partial t} = \pi (2\pi)^d m \frac{\partial}{\partial v_1^\mu} \int d\mathbf{v}_2 d\mathbf{k} k^\mu k^\nu \hat{u}(k)^2 \delta(\mathbf{k} \cdot \mathbf{w}) \left( f_2 \frac{\partial f_1}{\partial v_1^\nu} - f_1 \frac{\partial f_2}{\partial v_2^\nu} \right),
\] (58)
where \( w = v_1 - v_2 \). Other equivalent expressions of the Landau equation are given in [3]. The Landau equation ignores collective effects. Collective effects can be taken into account by keeping the contribution of the last integral in Eq. (41). For spatially homogeneous systems, the calculations can be carried out explicitly in the complex plane [51] and lead to the Lenard-Balescu equation

\[
\frac{\partial f_1}{\partial t} = \pi (2\pi)^d m \frac{\partial}{\partial v_1} \int dv_2 dk k^d k^v \frac{\hat{u}(k)^2}{|\epsilon(k, k \cdot v)|^2} \delta(k \cdot w) \left( f_2 \frac{\partial f_1}{\partial v_1} - f_1 \frac{\partial f_2}{\partial v_2} \right),
\]

where \( \epsilon(k, \omega) \) is the dielectric function

\[
\epsilon(k, \omega) = 1 - (2\pi)^d \hat{u}(k) \int \frac{k \cdot \partial f / \partial v}{k \cdot v - \omega} dv.
\]

We note that the Lenard-Balescu equation is obtained from the Landau equation by replacing the potential \( \hat{u}(k) \) by the “screened” potential \( \hat{u}(k) / |\epsilon(k, k \cdot v)| \). The Landau and Lenard-Balescu equations conserve mass and energy (reducing to the kinetic energy for a spatially homogenous system) and monotonically increase the Boltzmann entropy [45]. Thus, \( \dot{M} = \dot{E} = 0 \) and \( \dot{S}_R \geq 0 \) (H-theorem). The collisional evolution is due to a condition of resonance between the particles’ orbits. For homogeneous systems, the condition of resonance encapsulated in the \( \delta \)-function appearing in the Landau and Lenard-Balescu equations corresponds to \( k \cdot v_1 = k \cdot v_2 \) with \( v_1 \neq v_2 \). For \( d > 1 \) the only stationary solution is the Maxwell distribution. Therefore, the Landau and Lenard-Balescu equations relax towards the Maxwell distribution. Since the collision term in Eq. (53) is valid at order \( O(1/N) \), the relaxation time scales like

\[
t_R \sim N t_D, \quad (d > 1)
\]

as can be seen directly from Eq. (58) by dimensional analysis (comparing the l.h.s. and the r.h.s., we have \( 1/t_R \sim u^2 N \sim 1/N \) while \( t_D \sim R / v_{typ} \sim 1 \) with the scalings introduced in Sec. 2.2). This scaling, predicted in [3], has been observed for 2D Coulombian plasmas [49, 50].

For one-dimensional systems, like the HMF model, the situation is different. For \( d = 1 \), the kinetic equation (58) reduces to

\[
\frac{\partial f_1}{\partial t} = 2\pi^2 m \frac{\partial}{\partial v_1} \int dv_2 dk k^2 \frac{\hat{u}(k)^2}{|k| |\epsilon(k, kv_2)|^2} \delta(v_1 - v_2) \left( f_2 \frac{\partial f_1}{\partial v_1} - f_1 \frac{\partial f_2}{\partial v_2} \right) = 0.
\]

Therefore, the collision term \( C_N[f] \) vanishes at the order \( 1/N \) because there is no resonance. The kinetic equation reduces to \( \partial f / \partial t = 0 \) so that the distribution function does not evolve at all on a timescale \( \sim N t_D \). This implies that, for one-dimensional homogeneous systems, the relaxation time towards statistical equilibrium is larger than \( N t_D \). Thus, we expect that

\[
t_R > N t_D, \quad (d = 1).
\]

For the HMF model, when the system remains spatially homogeneous, it is found that the relaxation time scales like \( t_R \sim e^N \) [17]. The fact that the Lenard-Balescu collision term vanishes in 1D is known for a long time in plasma physics (see, e.g., the last paragraph in [52]) and has been rediscovered recently in the context of the HMF model [20, 53].
3.6. The Vlasov-Landau equation for stellar systems

Self-gravitating systems are spatially inhomogeneous, but the collisional current can be calculated by making a local approximation [54]. Therefore, the evolution of the distribution function is governed by the Vlasov-Landau equation

$$\frac{\partial f_1}{\partial t} + v_1 \cdot \frac{\partial f_1}{\partial r_1} + \langle F \rangle_1 \cdot \frac{\partial f_1}{\partial v_1} = \frac{\partial}{\partial v_1} \left( f_2 \frac{\partial f_1}{\partial v_1} - f_1 \frac{\partial f_2}{\partial v_2} \right) dv_2,$$

(64)

where

$$K^{\mu \nu} = 2\pi mG^2 \frac{1}{w} \ln \Lambda \left( \delta^{\nu \nu} - \frac{w^{\mu} w^{\nu}}{w^2} \right),$$

(65)

and we have set $f_1 = f(r_1, v_1, t)$ and $f_2 = f(r_1, v_2, t)$ assuming that the encounters can be treated as local (see, e.g., [46, 55, 56, 4] for a more complete discussion). The Vlasov-Landau-Poisson system conserves the total mass and the total energy (kinetic + potential) of the system. It also increases the Boltzmann entropy (34) monotonically: $\dot{S}_B \geq 0$ (H-theorem). The mean field Maxwell-Boltzmann distribution (36) is the only stationary solution of this equation (cancelling both the advective term and the collision term individually). Therefore, the system tends to reach the Boltzmann distribution. However, there are two reasons why it cannot attain it: (i) Evaporation: when coupled to the gravitational Poisson equation, the mean field Maxwell-Boltzmann distribution (36) yields a density profile $\rho \sim r^{-2}$ (for $r \to +\infty$) with infinite mass so there is no physical distribution of the form (36) in an infinite domain [10, 15]. The system can increase the Boltzmann entropy indefinitely by evaporating. Therefore, the Vlasov-Landau-Poisson system has no steady state with finite mass and the density profile tends to spread indefinitely. (ii) Gravothermal catastrophe: if the energy of the system is lower than the Antonov threshold $E_c = -0.335GM^2/R$ (where $R$ is the system size), it will undergo core collapse. This is called gravothermal catastrophe [6] because the system can increase the Boltzmann entropy indefinitely by contracting and overheating. This process usually dominates over evaporation and leads to the formation of binary stars [54, 57, 15]. If the system is confined within a box so as to prevent evaporation, the Vlasov-Landau-Poisson system admits stationary solutions for sufficiently large energies (above the Antonov threshold). They correspond to the mean field Boltzmann distributions (36). There can exist several Boltzmann distributions with the same values of mass and energy having different density contrasts $R = \rho(0)/\rho(R)$. Since the Boltzmann entropy is the Lyapunov functional of the Vlasov-Landau equation, a Boltzmann distribution (steady state) is linearly dynamically stable iff it is a (local) maximum of entropy at fixed mass and energy (thermodynamical stability). Therefore, only the local entropy maxima at fixed mass and energy can be reached in practice. This corresponds to configurations with density contrast $R < 709$. Minima and saddle points of entropy ($R > 709$) are unstable. The relaxation time to the Boltzmann distribution is

$$t_R \sim \frac{N}{\ln N} t_D,$$

(66)

where the logarithmic correction comes from the divergence of the Coulombian factor $\ln \Lambda \sim \ln N$. Since there is no global entropy maximum at fixed mass and energy for a
self-gravitating system confined within a box, the equilibrium configurations are only metastable (local entropy maxima). However, the lifetime of the metastable states is considerable, scaling like $e^{N}$, so that these states are stable in practice. Depending on the initial condition and on the structure of the basin of attraction, the system can either relax towards a long-lived metastable mean field Boltzmann distribution or undergo gravitational collapse (gravothermal catastrophe) and form a binary star surrounded by a hot halo (leading to a structure with unbounded entropy).

3.7. Orbit-averaged kinetic equation

We have seen in Sec. 3.5 that one dimensional systems that are spatially homogeneous do not evolve at all on a timescale $\sim Nt$ or larger because of the absence of resonances. However, if the system is spatially inhomogeneous, new resonances can appear as described in [48] so that an evolution is possible on a timescale $Nt$. Then, we can expect that one dimensional inhomogeneous systems will tend to approach the Boltzmann distribution on the timescale $Nt$. To be more precise, let us consider the orbit-averaged-Fokker-Planck equation derived in [48]. Exploiting the timescale separation between the dynamical time and the relaxation time, we can average Eq. (54) over the orbits, assuming that at any stage of its evolution the system reaches a mechanical equilibrium on a short dynamical time. Therefore, the distribution function is a quasi-stationary solution of the Vlasov equation

$$f \approx f(\varepsilon, t)[\text{where } \varepsilon = v^2/2 + \Phi \text{ is the individual energy}]$$

slowly evolving in time under the effect of “collisions” (= correlations due to finite $N$ effects). Introducing angle-action variables, we get an equation of the form [48]:

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial}{\partial J} \sum_{m,m'} \int \left| A_{mm'}(J, J') \right|^2 \delta(m\Omega(J) - m'\Omega(J')) \left\{ f(J')m \frac{\partial f}{\partial J} - f(J)m' \frac{\partial f}{\partial J'} \right\} dJ'. \tag{67}$$

The important point to notice is that the evolution of the system is due to a condition of resonance between the pulsations $\Omega(J)$ of the particles’ orbits (this property extends to $d$ dimensions). Only particles whose pulsations satisfy $m\Omega(J) = m'\Omega(J')$ with $(m, J) \neq (m', J')$ participate to the diffusion current. This is similar to the collisional relaxation of two dimensional point vortices [21, 30]. It can be shown that Eq. (67) conserves mass and energy and monotonically increases the Boltzmann entropy so that the system tends to approach the Boltzmann distribution of statistical equilibrium on a timescale $\sim Nt$ (the Boltzmann distribution is always a stationary solution of Eq. (67)) [48]. However, it may happen that there is not enough resonances so that the system can be trapped in a quasi stationary state different from the Boltzmann distribution. This happens when the condition of resonance cannot be satisfied so that $m\Omega(J) \neq m'\Omega(J')$ for all $(m, J) \neq (m', J')$ for which $|A_{mm'}(J, J')|^2 \neq 0$. In that case, the system is in a steady state of Eq. (67) that is not the Boltzmann distribution. Indeed, Eq. (67) may admit other stationary solutions than the Boltzmann distribution [48]. This is what happens to point vortices in 2D hydrodynamics: the collisional relaxation stops when the profile of angular velocity becomes monotonic even if the system has not reached the Boltzmann distribution [21]. In that case, the system will relax towards the Boltzmann distribution (if it truly does!) on a timescale larger than $Nt$ (this requires to develop the kinetic
theory at higher order in $1/N$, taking into account three body, four body,... correlations). We may wonder whether the same situation can happen to systems described by a kinetic equation of the form \(^{(53)}\).

4. STATISTICAL THEORY OF VIOLENT RELAXATION: THE LYNDEN-BELL ENTROPY

For $t \ll N t_D$, the evolution of the system is governed by the Vlasov equation

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} + F \cdot \frac{\partial f}{\partial v} = 0,$$

where $F = -\nabla \Phi$ is the force by unit of mass experienced by a particle. It is produced self-consistently by the other particles through the mean field

$$\Phi(r,t) = \int u(r-r')\rho(r',t)dr'.$$

Mathematically, the Vlasov equation is obtained when the $N \to +\infty$ limit is taken before the $t \to +\infty$ limit. Indeed, the collision term in Eq. \((53)\) scales as $1/N$ so that it vanishes for $N \to +\infty$. The Vlasov equation, or collisionless Boltzmann equation, simply states that, in the absence of “collisions” (more properly correlations), the distribution function $f$ is conserved by the flow in phase space. This can be written $df/dt = 0$ where $d/dt = \partial/\partial t + v \cdot \partial/\partial r - \nabla \Phi \cdot \partial/\partial v$ is the material derivative. The Vlasov equation conserves the usual constraints: the mass $M$ and the energy $E$, but also an infinite number of invariants called the Casimirs. They are defined by $I_h = \int h(f) dr dv$ for any continuous function $h(f)$. The conservation of the Casimirs is equivalent to the conservation of the moments of the distribution function denoted

$$M_n = \int f^n dr dv.$$

The Vlasov equation admits an infinite number of stationary solutions whose general form is given by the Jeans theorem \([54]\). Starting from an unstable initial condition $f_0(r,v)$, the Vlasov equation develops very complex filaments as a result of a mixing process in phase space (collisionless mixing). In this sense, the fine-grained distribution function $f(r,v,t)$ will never reach a stationary state but will rather produce intermingled filaments at smaller and smaller scales. However, if we introduce a coarse-graining procedure, the coarse-grained distribution function $\overline{f}(r,v,t)$ will reach a metaequilibrium state $\overline{f}(r,v)$ on a very short timescale, of the order of the dynamical time $t_D$. This is because the evolution continues at scales smaller than the scale of observation (coarse-grained). This process is known as “phase mixing” and “violent relaxation” (or collisionless relaxation) \([54]\). Then, on a longer timescale, collisions (correlations) will come into play and drive the system towards the Boltzmann statistical equilibrium state. Lynden-Bell \([35]\) has tried to predict the metaequilibrium state, or quasi stationary state (QSS), achieved by the system through violent relaxation in terms of statistical mechanics. This approach is of course quite distinct from the statistical mechanics of the $N$-body system.
(exposed in Sec. 2.4) which describes the statistical equilibrium state reached by a discrete $N$-body Hamiltonian system for $t \to +\infty$. In Lynden-Bell’s approach, we make the statistical mechanics of a field, the distribution function $f(r, v, t)$ whose evolution is governed by the Vlasov equation (68) while in Sec. 2.4 we made the statistical mechanics of a system of point particles described by the Hamilton equations (1).

Let $f_0(r, v)$ denote the initial (fine-grained) distribution function (DF). We discretize $f_0(r, v)$ in a series of levels $\eta$ on which $f_0(r, v)$ $\approx \eta$ is approximately constant. Thus, the levels $\{\eta\}$ represent all the values taken by the fine-grained DF. If the initial condition is unstable, the distribution function $f(r, v, t)$ will be stirred in phase space (phase mixing) but will conserve its values $\eta$ and the corresponding hypervolumes $\gamma(\eta) = \int \delta(f(r, v, t) - \eta) \, dr \, dv$ as a property of the Vlasov equation (this is equivalent to the conservation of the Casimirs). Let us introduce the probability density $\rho(r, v, \eta)$ of finding the level of phase density $\eta$ in a small neighborhood of the position $r, v$ in phase space. This probability density can be viewed as the local area proportion occupied by the phase level $\eta$ and it must satisfy at each point the normalization condition

$$\int \rho(r, v, \eta) \, d\eta = 1. \quad (71)$$

The locally averaged (coarse-grained) DF is then expressed in terms of the probability density as

$$\overline{f}(r, v) = \int \rho(r, v, \eta) \eta \, d\eta, \quad (72)$$

and the associated macroscopic potential satisfies

$$\overline{\Phi}(r, t) = \int u(r - r') \overline{f}(r', v', t) \, dr' \, dv'. \quad (73)$$

Since the potential is expressed by space integrals of the density, it smoothes out the fluctuations of the distribution function, supposed at very fine scale, so $\Phi$ has negligible fluctuations (we thus drop the bar on $\Phi$). The conserved quantities of the Vlasov equation can be decomposed in two groups. The mass and energy will be called robust integrals because they are conserved by the coarse-grained distribution function: $M[f] = M[\overline{f}]$ and $E[f] = E[\overline{f}]$. Hence

$$M = \int \overline{f} \, dr \, dv, \quad (74)$$

$$E = \int \frac{1}{2} \overline{v^2} \, dr \, dv + \frac{1}{2} \int \overline{\Phi} \, dr \, dv. \quad (75)$$

As discussed above, the potential can be considered as smooth, so we have expressed the energy in terms of the coarse-grained fields $\overline{f}$ and $\overline{\Phi}$ neglecting the internal energy of the fluctuations $\Phi$. Therefore, the mass and the energy can be calculated at any time of the evolution from the coarse-grained field $\overline{f}$. By contrast, the moments $M_n$ with $n \geq 2$ will be called fragile integrals because they are altered on the coarse-grained scale since $\overline{f^n} \neq \overline{f^n}$ (the local moments of the distribution are defined by $\int \rho(r, v, t) = \int \rho \eta^n \, d\eta$). Therefore, only the moments of the fine-grained field $M_n^{fg} = M_n[f] = \int \rho \eta^n \, d\eta$ are
conserved, i.e.

$$M^f_{m} = \int \rho(r, v, \eta) \eta^n dr dv d\eta. \quad (76)$$

The moments of the coarse-grained field $M^c_{m} = \int f^n dr dv$ are not conserved during the evolution since $M^f_{m} \neq M^c_{m}$. In a sense, the moments $M^f_{m}$ are “hidden constraints” because they are expressed in terms of the fine-grained distribution $\rho(r, v, \eta)$ and they cannot be measured from the (observed) coarse-grained field. They can be only computed from the initial conditions before the system has mixed or from the fine-grained field. Since in many cases we do not know the initial conditions nor the fine-grained field, they often appear as “hidden”. Note that instead of conserving the fine-grained moments, we can equivalently conserve the total hypervolume $\gamma(\eta) = \int \rho dr dv$ of each level $\eta$.

After a complex evolution, we may expect the system to be in the most probable, i.e. most mixed state, consistent with all the constraints imposed by the dynamics. This, however, relies on a hypothesis of ergodicity (efficient mixing) whose validity will be discussed in Sec. 6. We define the mixing entropy as the logarithm of the number of microscopic configurations associated with the same macroscopic state characterized by the probability density $\rho(r, v, \eta)$. To get this number, we divide the macrocells $(r, r + dr; v, v + dv)$ into $v$ microcells of size $h$ and denote by $n_{ij}$ the number of microcells occupied by the level $\eta_j$ in the $i$-th macrocell. Note that a microcell can be occupied only by one level $\eta_j$. This is due to the fact that we make the statistical mechanics of a continuous field $f(r, v, t)$ instead of point particles. Therefore, we cannot “compress” that field, unlike point particles. A simple combinatorial analysis indicates that the number of microstates associated with the macrostate $\{n_{ij}\}$ is

$$W(\{n_{ij}\}) = \prod_j N_j! \prod_i \frac{v!}{n_{ij}!}, \quad (77)$$

where $N_j = \sum_i n_{ij}$ is the total number of microcells occupied by $\eta_j$ (this is a conserved quantity equivalent to $\gamma(\eta)$). We have to add the normalization condition $\sum_j n_{ij} = v$, equivalent to Eq. (71), which prevents overlapping of different levels (we note that we treat here the level $\eta = 0$ on the same footing as the others). This constraint plays a role similar to the Pauli exclusion principle in quantum mechanics. Morphologically, the Lynden-Bell statistics (77) corresponds to a 4th type of statistics since the particles are distinguishable but subject to an exclusion principle [35]. There is no such exclusion for the statistical equilibrium of point particles since they are free a priori to approach each other, so we can put several particles in the same microcell (see Sec. 2.4).

Taking the logarithm of $W$ and using the Stirling formula, we get

$$\ln W(\{n_{ij}\}) = -\sum_{i, j} n_{ij} \ln n_{ij} = -\sum_{i, j} v h^{2d} \rho_{ij} \ln \rho_{ij}, \quad (78)$$

where $\rho_{ij} = \rho(r_i, v_i, \eta_j) = n_{ij} / v h^{2d}$. Passing to the continuum limit $v \to 0$, we obtain the mixing entropy introduced by Lynden-Bell [35]:

$$S_{LB}[\rho] = -\int \rho(r, v, \eta) \ln \rho(r, v, \eta) dr dv d\eta. \quad (79)$$
Note that the Lynden-Bell entropy can be interpreted as the Boltzmann entropy for a distribution of levels \( \eta \) (including \( \eta = 0 \)). Equation (79) is sometimes called a collisionless entropy to emphasize the distinction with the collisional entropy (34). Assuming ergodicity or "efficient mixing", the statistical equilibrium state is obtained by maximizing the Lynden-Bell entropy \( S_{LB}[\rho] \) while conserving the mass \( M \), the energy \( E \) and all the Casimirs (or moments \( M_n \)). We need also to account for the local normalization condition (71). This problem is treated by introducing Lagrange multipliers, so that the first order variations satisfy
\[
\delta S_{LB} - \beta \delta E - \sum_{n\geq 1} \alpha_n \delta M_n - \int \zeta(\mathbf{r}, \mathbf{v}) \delta \left( \int \rho(\mathbf{r}, \mathbf{v}, \eta) d\eta \right) d\mathbf{r} d\mathbf{v} = 0, \tag{80}
\]
where \( \beta \) is the inverse temperature and \( \alpha_n \) the "chemical potential" associated with \( M_n \). The resulting optimal probability density is a Gibbs state which has the form
\[
\rho(\mathbf{r}, \mathbf{v}, \eta) = \frac{1}{Z} \chi(\eta) e^{- (\beta \varepsilon + \alpha) \eta}, \tag{81}
\]
where \( \varepsilon = \frac{v^2}{2} + \Phi(\mathbf{r}) \) is the energy of a particle by unit of mass. In writing Eq. (81), we have distinguished the Lagrange multipliers \( \alpha \) and \( \beta \) associated with the robust integrals \( M \) and \( E \) from the Lagrange multipliers \( \alpha_n > 1 \), associated with the conservation of the fragile moments \( M_n > 1 = \int \rho \eta^n d\eta d\mathbf{r} d\mathbf{v} \), which have been regrouped in the function \( \chi(\eta) \equiv \exp(- \sum_{n>1} \alpha_n \eta^n) \). This distinction will make sense in the following. Under this form, we see that the equilibrium distribution of phase levels is a product of a universal Boltzmann factor \( e^{- (\beta \varepsilon + \alpha) \eta} \) by a non-universal function \( \chi(\eta) \) depending on the initial conditions. The partition function \( Z \) is determined by the local normalization condition \( \int \rho d\eta = 1 \) leading to
\[
Z = \int \chi(\eta) e^{- \eta (\beta \varepsilon + \alpha) } d\eta. \tag{82}
\]
Finally, the equilibrium coarse-grained distribution function defined by \( \mathcal{F} = \int \rho \eta d\eta \) can be written
\[
\mathcal{F} = \frac{\int \chi(\eta) \eta e^{- \eta (\beta \varepsilon + \alpha) } d\eta}{\int \chi(\eta) e^{- \eta (\beta \varepsilon + \alpha) } d\eta}, \tag{83}
\]
or, equivalently,
\[
\mathcal{F} = -\frac{1}{\beta} \frac{\partial \ln Z}{\partial \varepsilon} = F(\beta \varepsilon + \alpha) = \mathcal{F}(\varepsilon). \tag{84}
\]
The function \( F \) is given by
\[
F(x) = -(\ln \hat{\chi})'(x), \tag{85}
\]
where we have defined \( \hat{\chi}(x) = \int_0^{\infty} \chi(\eta) e^{- \eta x} d\eta \). It is straightforward to check that a distribution function \( f = f(\varepsilon) \) depending only on the energy \( \varepsilon \) is a stationary solution of the Vlasov equation [54]. Therefore, for a given initial condition, the statistical theory of Lynden-Bell selects a particular stationary solution of the Vlasov equation (most mixed)
among all possible ones (an infinity!) Specifically, the equilibrium state is obtained by solving the integro-differential equation

$$\Phi(r) = \int u(r-r') f_{\alpha_n, \beta} [v'^2/2 + \Phi(r')]dr'dv', \quad (86)$$

and relating the Lagrange multipliers $\alpha_n$, $\beta$ to the constraints $M_n$, $E$. Then, we have to make sure that the distribution is a maximum of $S_{LB}$, not a minimum or a saddle point. We note that the coarse-grained distribution function $\overline{f}(\varepsilon)$ predicted by Lynden-Bell can take a wide diversity of forms depending on the function $\chi(\eta)$ determined by the fragile moments ("hidden constraints") fixed by the initial condition. The coarse-grained distribution function $f(\varepsilon)$ can be viewed as a sort of superstatistics as it is expressed as a superposition of Boltzmann factors (on the fine-grained scale) weighted by a non-universal function $\chi(\eta)$ $^{[53]}$. Some examples will be given in the sequel. In the present context, the function $\chi(\eta)$ is determined from the constraints $a posteriori$. Indeed, we have to solve the full problem in order to get the expression of $\chi(\eta)$. In this sense, the constraints associated with the conservation of the fine-grained moments are treated microcanonically $^{[5]}$. We emphasize that the function $\overline{f}(\varepsilon)$ determining the metaequilibrium state depends on the details of the initial conditions. This differs from the ordinary statistical equilibrium state which only depends on the mass $M$ and the energy $E$. Here, we need to know the value of the fine-grained moments $M_{n>1}^{f,g}$ which are accessible only in the initial condition (or from the fine-grained field) since the observed moments are altered for $t > 0$ by the coarse-graining as the system undergoes a mixing process ($M_{n>1}^{f,g} \neq M_{n>1}^{f,g}$). This makes the practical prediction of $\overline{f}(\varepsilon)$ very complicated, or even impossible, since, in practice, we often do not know the initial conditions in detail (e.g., the initial state giving rise to a galaxy or to a large scale vortex). In addition, in many cases, we cannot be sure that the initial condition is not already mixed (coarse-grained). If it has a fine-grained structure, this would change a priori the prediction of the metaequilibrium state (see p. 284 of $^{[25]}$ and $^{[60]}$).

We note that the coarse-grained distribution function predicted by Lynden-Bell depends only on the individual energy $\varepsilon$ of the particles. According to the Jeans theorem $^{[54]}$, such distribution functions form just a particular class of stationary solutions of the Vlasov equation. We also note that $\overline{f}(\varepsilon)$ is a monotonically decreasing function of energy. Indeed, from Eqs. (81) and (84), it is easy to establish that

$$\overline{f}'(\varepsilon) = -\beta f_2, \quad f_2 \equiv \int \rho(\eta - \bar{f})^2 d\eta \geq 0, \quad (87)$$

---

$^4$ Incidentally, the fact that the coarse-grained distribution function should be a stationary solution of the Vlasov equation is not obvious. This depends on the definition of coarse-graining $^{[58]}$.

$^5$ In the context of two-dimensional turbulence, some authors $^{[41, 59]}$ have proposed that the vorticity distribution $\chi(\sigma)$ could be imposed by a small-scale forcing, in which case it should be treated canonically (as a prior distribution). It is not clear whether there exists a similar interpretation for other systems with long-range interactions (note, however, that a small-scale forcing is present in certain models of dark matter in astrophysics). Here, we exclusively consider isolated systems where the distribution $\chi(\eta)$ is determined implicitly by the initial conditions.
where \( f_2 \equiv \bar{r}^2 - \bar{r}'^2 \) is the centered local variance of the distribution \( \rho(r,v,\eta) \). Therefore, \( \bar{r}(\eta) \leq 0 \) since \( \beta \geq 0 \) is required to make the velocity profile normalizable. We also have \( F'(x) = -f_2(x) \) so that \( F \) is a monotonically decreasing function. Finally, the coarse-grained distribution function satisfies \( \bar{f}(r,v) \leq f_0^{\max} \) where \( f_0^{\max} = f_{\text{max}}(r,v) = 0 \) is the maximum value of the initial (fine-grained) distribution function. This inequality can be obtained from Eq. (83) by taking the limit \( \varepsilon \to -\infty \) for which \( \bar{f}(\varepsilon) \to \eta_{\text{max}} = f_0^{\max} \) and using the fact that \( f(\varepsilon) \) is a decreasing function. Of course, the inequality \( 0 \leq \bar{f} \leq f_0^{\max} \) is clear from physical considerations since the coarse-grained distribution function is always intermediate between the minimum and the maximum values of \( f_0 \). We also note [61] that the most probable level \( \eta_*(r,v) \) of the distribution (81) is given by \( \eta_* = \left[ \langle \ln \chi \rangle \right]^{-1}(\beta \varepsilon + \alpha) \), provided that \( \langle \ln \chi \rangle''(\eta_*) < 0 \). The function \( \eta_*(r,v) \) is a stationary solution of the Vlasov equation of the form \( \eta_0 = \eta_0(\varepsilon) \) which monotonically decreases with the energy since \( \eta_0'(\varepsilon) = \beta \langle \ln \chi \rangle''(\eta_*) < 0 \). It usually differs from the average value \( \bar{f}(r,v) \) of the distribution (81).

If the initial DF takes only two values \( f_0 = 0 \) and \( f_0 = \eta_0 \), the Lynden-Bell entropy becomes

\[
S_{LB} = -\int \left\{ \frac{\bar{T}}{\eta_0} \ln \frac{\bar{T}}{\eta_0} + \left( 1 - \frac{\bar{T}}{\eta_0} \right) \ln \left( 1 - \frac{\bar{T}}{\eta_0} \right) \right\} d\tau d\psi,
\]

which is similar to the Fermi-Dirac entropy (but with, of course, a completely different interpretation). In that case, the statistical prediction of Lynden-Bell for the metaequilibrium state resulting from a “collisionless” violent relaxation is [35, 38, 62]:

\[
\bar{T} = \frac{\eta_0}{1 + e^{\eta_0(\beta \varepsilon + \alpha)}},
\]

which is similar to the Fermi-Dirac distribution [6]. This has to be contrasted from the statistical equilibrium state resulting from a “collisional” relaxation which is the Maxwell-Boltzmann distribution [36]. In the dilute limit of Lynden-Bell’s theory \( \bar{T} \ll \eta_0 \), the DF (89) becomes

\[
\bar{T} = A e^{-\beta \eta_0 e},
\]

This is similar to the statistical equilibrium state [36]. Therefore, in this approximation, collisional and collisionless relaxation lead to the same distribution function (the mean field Maxwell-Boltzmann distribution) but with a completely different interpretation, corresponding to very different timescales. To emphasize the difference, note in particular the bar on \( f \) in Eq. (90) and the fact that the mass of the individual stars \( m \) in Eq. [36] is replaced by the value \( \eta_0 \) of the fine-grained distribution function in Eq. (90).

In conclusion, assuming ergodicity, the Lynden-Bell statistical equilibrium state \( \rho(r,v,\eta) \) is obtained by solving the maximization problem

\[
\max_{\rho} \{ S_{LB}[\rho] \mid E[\rho] = E, \ M[\rho] = M, \ M_{n>1}^{\text{f,g}}[\rho] = M_{n>1}^{\text{f,g}}, \int \rho d\eta = 1 \}.
\]

\(^6\) Degeneracy effects associated with the Lynden-Bell statistics [85] have been observed in 2D turbulence [63], self-gravitating systems [64] and for the HMF model [27].
This corresponds to a condition of microcanonical stability. Let us introduce the Lynden-Bell free energy \( F_{LB}[\rho] = E[\rho] - TS_{LB}[\rho] \). The condition of canonical stability is
\[
\min_{\rho} \quad \{ F_{LB}[\rho] \mid M[\rho] = M, \ M^{f,g}_{n>1}[\rho] = M^{f,g}_{n>1}, \int \rho \, d\eta = 1 \}. \tag{92}
\]

The optimization problems (91) and (92) have the same critical points. Furthermore, if \( \rho(r, v, \eta) \) solves the minimization problem (92) then it solves the maximization problem (91) and represents therefore a maximum entropy state. However, the reciprocal is wrong in case of ensemble inequivalence. A solution of (91) is not necessarily a solution of (92): the minimization problem (92) just determines a subclass of the solutions of the maximization problem (91). Therefore, the condition of canonical stability (92) provides just a sufficient condition of microcanonical stability: (92) \( \Rightarrow \) (91).

**Numerical algorithms:** A relaxation equation has been proposed to solve the maximization problem (91). It is obtained from a Maximum Entropy Production Principle (MEPP) and has the form of a generalized mean field Fokker-Planck equation \([38]\). It can serve as a numerical algorithm to determine the Lynden-Bell maximum entropy state for given \( E, M \) and \( M^{f,g}_{n>1} \) specified by the initial conditions. The relaxation equation solving the microcanonical problem (91) is
\[
\frac{\partial \rho}{\partial t} + v \cdot \frac{\partial \rho}{\partial r} - \nabla \Phi \cdot \frac{\partial \rho}{\partial v} = \frac{\partial}{\partial v} \left\{ D \frac{\partial}{\partial v} + \beta(t)(\eta - \bar{\eta}) \rho v \right\}, \tag{93}
\]
with
\[
\beta(t) = -\frac{\int D\frac{\partial f}{\partial v} \cdot v \, dr \, dv}{\int D f^2 \, dr \, dv}. \tag{94}
\]
This equation conserves the mass \( M \), the energy \( E \) (through the time dependent Lagrange multiplier \( \beta(t) \)) and the fine-grained moments \( M^{f,g}_{n>1} \), and monotonically increases the Lynden-Bell entropy: \( \dot{S}_{LB} \geq 0 \) (H-theorem). Furthermore, \( \dot{S}_{LB} = 0 \) iff \( \rho \) is a steady solution of Eq. (93). Therefore, \( S_{LB}[\rho] \) is the Lyapunov functional of the relaxation equation (93). It results that \( \rho(r, v, \eta) \) is a linearly dynamically stable stationary solution of Eq. (93) iff it is a (local) maximum of \( S_{LB} \) at fixed \( E, M \) and \( M^{f,g}_{n>1} \). Minima or saddle points of entropy are dynamically unstable. Furthermore, if \( S_{LB} \) is bounded from above (within the previous constraints) we can conclude for Lyapunov’s theory that \( \rho(r, v, \eta, t) \) will relax towards a (local) maximum of \( S_{LB} \) at fixed \( E, M \) and \( M^{f,g}_{n>1} \) for \( t \rightarrow +\infty \) (if there exists several local entropy maxima, the selection will depend on a complicated notion of basin of attraction). Therefore, Eq. (93) with Eq. (94) can serve as a numerical algorithm to solve the microcanonical problem (91) and compute the Lynden-Bell statistical equilibrium state. It can also provide a phenomenological description of the out-of-equilibrium dynamics towards Lynden-Bell’s statistical equilibrium \([38]\).

If we fix the inverse temperature \( \beta \) instead of the energy in Eq. (93), we get a relaxation equation solving the canonical problem (92). This equation conserves the mass \( M \) and the fine-grained moments \( M^{f,g}_{n>1} \), and monotonically decreases the Lynden-Bell free energy: \( \dot{F}_{LB} \leq 0 \). Furthermore, \( F_{LB} = 0 \) iff \( \rho \) is a steady solution of (93) with fixed \( \beta \). Therefore, \( F_{LB}[\rho] \) is the Lyapunov functional of this relaxation equation. It
results that $\rho(r, v, \eta)$ is a linearly dynamically stable stationary solution of Eq. (93) with fixed $\beta$ iff it is a (local) minimum of $F_{LB}$ at fixed $M$ and $M_{n>1}^{f,g}$. Furthermore, if $F_{LB}$ is bounded from below, $\rho(r, v, \eta, t)$ will relax towards a (local) minimum of $F_{LB}$ at fixed $M$ and $M_{n>1}^{f,g}$. Therefore, Eq. (93) with fixed $\beta$ can serve as a numerical algorithm to solve the canonical problem (92) and determine a subclass of Lynden-Bell statistical equilibrium states since (92) $\Rightarrow$ (91).

5. GENERALIZED ENTROPIES IN THE REDUCED SPACE OF COARSE-GRAINED DISTRIBUTIONS

Assuming ergodicity, we have seen that the most probable local distribution of phase levels $\rho(r, v, \eta)$ maximizes the Lynden-Bell mixing entropy (79) while conserving mass, energy and all the fine-grained moments. This functional of $\rho$ is the proper form of Boltzmann entropy in the context of the violent relaxation. It is obtained by a combinatorial analysis taking into account the specificities of the collisionless (Vlasov) evolution. We shall first show that the equilibrium coarse-grained distribution function $\overline{f}(r, v)$ (which is the function directly accessible to the observations) extremizes a certain functional $S[f]$ at fixed mass $M$ and energy $E$ [65, 36]. This functional is non-universal and depends on the initial conditions. It is determined indirectly by the statistical theory of Lynden-Bell and cannot be obtained from a combinatorial analysis, unlike $S[\rho]$. Such functionals arise because they encapsulate the influence of fine-grained constraints (Casimirs) that are not accessible on the coarse-grained scale. They play the role of “hidden constraints”. Now, if the distribution of phase levels $\chi(\eta)$ is treated canonically, it can be shown that the equilibrium coarse-grained distribution function $\overline{f}(r, v)$ maximizes $S[f]$ at fixed mass $M$ and energy $E$. In this sense, $S[f]$ will be called a “generalized entropy” in the reduced space of coarse-grained distributions.[7] A coarse-grained DF that maximizes a “generalized entropy” $S[f]$ at fixed mass and energy is a Lynden-Bell statistical equilibrium state (but the reciprocal is wrong in case of ensemble inequivalence). We note that the entropic functionals $S[\rho]$ and $S[f]$ are defined on two different spaces. The $\rho$-space is the relevant one to make the statistical mechanics of violent relaxation [35]. The $f$-space is a sort of projection of the $\rho$-space in the space of directly observable (coarse-grained) distributions.

Since the coarse-grained distribution function $\overline{f}(\varepsilon)$ predicted by the statistical theory of Lynden-Bell depends only on the individual energy and is monotonically decreasing (see Sec. 4), it extremizes a functional of the form [65, 36]:

$$S[f] = - \int C(f) d\varepsilon dv,$$

(95)

---

7 We emphasize that this notion of “generalized entropies” [65, 36] is completely different from the notion of generalized entropies introduced by Tsallis (see Sec. 7). These entropies take into account the influence of the Casimir constraints for an ergodic evolution while the Tsallis entropies aim at describing non-ergodic behaviours.
at fixed mass $M$ and energy $E$, where $C(\mathcal{F})$ is a convex function, i.e. $C'' > 0$. Indeed, introducing Lagrange multipliers and writing the variational principle as

$$\delta S - \beta \delta E - \alpha \delta M = 0,$$

we find that

$$C'(\mathcal{F}) = -\beta \epsilon - \alpha.$$  \hspace{1cm} (97)

Since $C'$ is a monotonically increasing function of $\mathcal{F}$, we can inverse this relation to obtain

$$\mathcal{F} = F(\beta \epsilon + \alpha) = \mathcal{F}(\epsilon),$$  \hspace{1cm} (98)

where

$$F(x) = (C')^{-1}(-x).$$  \hspace{1cm} (99)

From the identity

$$\mathcal{F}'(\epsilon) = -\beta / C''(\mathcal{F}),$$  \hspace{1cm} (100)

resulting from Eq. (97), $\mathcal{F}(\epsilon)$ is a monotonically decreasing function of energy (if $\beta > 0$). Thus, Eq. (84) is compatible with Eq. (98) provided that we identify (99) with (85). Therefore, to any function $F(x)$ determined by the function $\chi(\eta)$ in the statistical theory according to Eq. (85), we can associate to the metaequilibrium state (84) a functional (95) where $C(\mathcal{F})$ is given by Eq. (99) or equivalently by

$$C(\mathcal{F}) = -\int \mathcal{F}^{-1}(x)dx.$$  \hspace{1cm} (101)

Using Eq. (85), this function is explicitly given in terms of $\dot{\chi}$ by

$$C(\mathcal{F}) = -\int \mathcal{F}[(\ln \dot{\chi})']^{-1}(-x)dx.$$  \hspace{1cm} (102)

On the other hand, comparing Eqs. (87) and (100), we find that

$$f_2 = 1/C''(\mathcal{F}).$$  \hspace{1cm} (103)

This equation relates the variance of the equilibrium distribution $f_2$ to the coarse-grained distribution function $\mathcal{F}$ through the function $C$. We note that $C(\mathcal{F})$ is a non-universal function which depends on the initial conditions. Indeed, it is determined by the function $\chi(\eta)$ which depends indirectly on the initial conditions through the complicated procedures discussed in Sec. 4. In general, $S[\mathcal{F}]$ is not the Boltzmann functional $S_B[\mathcal{F}] = -\int \mathcal{F} \ln \mathcal{F} d\mathbf{r} d\mathbf{v}$ (except in the dilute limit of the Lynden-Bell theory) due to fine-grained constraints (Casimirs) that modify the form of entropy that we would naively expect. This is why the quasi-stationary state is described by non-standard distributions (even for an assumed ergodic evolution). The existence of “hidden constraints” (here the Casimir invariants that are not accessible on the coarse-grained scale) is the physical reason for the occurrence of non-standard distributions and “generalized entropies” in our problem. In fact, the distribution is standard (Boltzmann) at the level of the local distribution of fluctuations $\rho(\mathbf{r}, \mathbf{v}, \eta)$ ($\rho$-space) and non-standard at the level of the macroscopic coarse-grained field $\mathcal{F}(\mathbf{r}, \mathbf{v})$ ($\mathcal{F}$-space).
So far, we have shown that the coarse-grained DF extremizes the functional at fixed mass and energy. The conditions under which it maximizes this functional at fixed mass and energy have been discussed recently by Bouchet [66]. In the following, we give a complementary discussion, adopting a presentation similar to the one used in [42] in a different context (see also Sec. 11). We introduce the grand entropy [36]:

\[ S_{\chi}[\rho] = S_{LB}[\rho] - \sum_{n>1} \alpha_n M_n^{f.g.}[\rho], \quad (104) \]

which is the Legendre transform of \( S_{LB}[\rho] \) with respect to the fragile moments \( M_n^{f.g.}[\rho] \). Expliciting these fine-grained moments, it can be rewritten

\[ S_{\chi}[\rho] = - \int \rho \ln \left( \frac{\rho}{\chi(\eta)} \right) d\mathbf{r} d\mathbf{v} d\eta, \quad (105) \]

where we have noted

\[ \chi(\eta) \equiv \exp \left( - \sum_{n>1} \alpha_n \eta^n \right). \quad (106) \]

We now consider the grand microcanonical ensemble defined by [36]:

\[
\max_{\rho} \{ S_{\chi}[\rho] \mid E[\rho] = E, M[\rho] = M, \int \rho d\eta = 1 \}. \quad (107)
\]

If \( \rho(\mathbf{r}, \mathbf{v}, \eta) \) solves the maximization problem (107), then it solves the maximization problem (91) and represents therefore a Lynden-Bell maximum entropy state. Note that the reciprocal is wrong in case of ensemble inequivalence [11, 14]: a solution of (91) is not necessarily a solution of (107). In other words, the maximization problem (107) determines only a subclass of solutions of the maximization problem (91). Therefore, it only provides a sufficient condition of Lynden-Bell thermodynamical stability. Note that we can also consider the grand canonical ensemble

\[
\min_{\rho} \{ F_{\chi}[\rho] = E - T S_{\chi}[\rho] \mid M[\rho] = M, \int \rho d\eta = 1 \}, \quad (108)
\]

where \( F_{\chi}[\rho] \) is a grand free energy. We have the implications: (108) \( \Rightarrow \) (107) \( \Rightarrow \) (91) and (108) \( \Rightarrow \) (92) \( \Rightarrow \) (91) so that (108) gives a sufficient condition of Lynden-Bell thermodynamical stability (less refined than (107)). In the following, we shall consider the maximization problem (107). To determine the distribution \( \rho_+(\mathbf{r}, \mathbf{v}, \eta) \) which maximizes \( S_{\chi}[\rho] \) with the robust constraints \( E[\overline{f}] = E, M[\overline{f}] = M, \) and the normalization condition \( \int \rho d\eta = 1 \), we can proceed in two steps (we consider here the global maximization problem. The local maximization problem is treated in the Appendix). First step: we determine the distribution \( \rho_1(\mathbf{r}, \mathbf{v}, \eta) \) which maximizes \( S_{\chi}[\rho] \) with the constraint \( \int \rho d\eta = 1 \) and a fixed coarse-grained distribution \( \int \rho \eta d\eta = \overline{f}(\mathbf{r}, \mathbf{v}) \) (note that fixing the coarse-grained distribution function automatically determines the robust constraints \( E \) and \( M \)). This gives a distribution \( \rho_1[\overline{f}(\mathbf{r}, \mathbf{v}), \eta] \) depending on \( \overline{f}(\mathbf{r}, \mathbf{v}) \) and \( \eta \). Substituting this distribution in the functional \( S_{\chi}[\rho] \), we obtain a functional \( S[\overline{f}] \equiv S_{\chi}[\rho_1] \) of the coarse-grained distribution \( \overline{f} \). Second step: we determine the distribution function
\( \mathcal{T}_* (r,v) \) which maximizes \( S[\mathcal{T}] \) with the constraints \( E[\mathcal{T}] = E \) and \( M[\mathcal{T}] = M \). Finally, we have \( \rho_1 (r,v,\eta) = \rho_1 [\mathcal{T}_*(r,v),\eta] \). Let us now show that \( S[\mathcal{T}] \equiv S_\chi [\rho_1] \) is the functional \( (102) \). The distribution \( \rho_1 (r,v,\eta) \) that extremizes \( S_\chi [\rho] \) with the constraints \( \int \rho \, d\eta = 1 \) and \( \int \rho \eta \, d\eta = \mathcal{T}(r,v) \) satisfies the first order variations

\[
\delta S_\chi - \int \Phi (r,v) \delta \left( \int \rho \eta \, d\eta \right) \, dr \, dv \, d\eta = 0,
\]

where \( \Phi (r,v) \) and \( \zeta (r,v) \) are Lagrange multipliers. This yields

\[
\rho_1 (r,v,\eta) = \frac{1}{Z(r,v)} \chi (\eta) e^{-\eta \Phi(r,v)},
\]

where \( Z(r,v) \) and \( \Phi (r,v) \) are determined by

\[
Z(r,v) = \int \chi (\eta) e^{-\eta \Phi(r,v)} \, d\eta = \hat{\chi} (\Phi),
\]

\[
\mathcal{T}(r,v) = \frac{1}{Z(r,v)} \int \chi (\eta) \eta e^{-\eta \Phi(r,v)} \, d\eta = - (\ln \hat{\chi})' (\Phi).
\]

The critical point \( (110) \) is a maximum of \( S_\chi [\rho] \) with the above-mentioned constraints since \( \delta^2 S_\chi = - \int [2(\delta \rho)^2 / \rho] \, dr \, dv \, d\eta \leq 0 \). Then, we compute

\[
S_\chi [\rho_1] = - \int \rho_1 (\eta \Phi - \ln \hat{\chi}) \, dr \, dv \, d\eta = \int (\mathcal{T} \Phi + \ln \hat{\chi} (\Phi)) \, dr \, dv.
\]

Therefore, the functional of the coarse-grained DF is

\[
S[\mathcal{T}] = - \int C(\mathcal{T}) \, dr \, dv,
\]

with

\[
C(\mathcal{T}) = - \mathcal{T} \Phi - \ln \hat{\chi} (\Phi).
\]

Now \( \Phi (r,v) \) is related to \( \mathcal{T}(r,v) \) by Eq. \( (112) \). This implies that

\[
C' (\mathcal{T}) = - \Phi = - [((\ln \hat{\chi})')^{-1} (- \mathcal{T})],
\]

so that

\[
C(\mathcal{T}) = - \int [((\ln \hat{\chi})')^{-1} (-x)] \, dx.
\]

This returns the functional \( (102) \). We now conclude that \( \rho (r,v,\eta) \) solves the maximization problem \( (107) \) if, and only if, \( \mathcal{T}(r,v) \) solves the maximization problem

\[
\max \{ S[\mathcal{T}] \mid E[\mathcal{T}] = E, M[\mathcal{T}] = M \},
\]
where $S[f]$ is given by Eqs. (114) and (117). The optimal distributions are related to each other by $\rho^*(r, v, \eta) = \rho_1[f^*(r, v)]$, according to the procedure described previously. We have the implications: (118) $\Leftrightarrow$ (107) $\Rightarrow$ (91). Therefore, if $f(r, v)$ solves the maximization problem (118), the corresponding distribution $\rho(r, v, \eta)$ solves the maximization problem (91) and is therefore a Lynden-Bell statistical equilibrium state. However, the reciprocal is wrong in case of ensemble inequivalence. The maximum entropy state satisfying (91) may not satisfy (107) or (118). Therefore, although the coarse-grained DF always extremizes $S[f]$ at fixed $E, M$, it is not a maximum of $S[f]$ at fixed $E, M$ in the region of ensemble inequivalence. These maximization problems (107) or (118) provide only sufficient conditions of Lynden-Bell thermodynamical stability. Returning to the maximization problem (118), we introduce the generalized free energy $F[f] = E[f] - T S[f]$. Now, if $f(r, v)$ solves the minimization problem

$$\min_f \{F[f] \mid M[f] = M\}, \quad (119)$$

then it solves the maximization problem (118). However, the reciprocal is wrong in case of ensemble inequivalence: the solutions of (118) are not necessarily solutions of (119). Therefore, the generalized canonical problem (119) determines only a subclass of solutions of the generalized microcanonical problem (118). We have the implications: (119) $\Rightarrow$ (118) $\Leftrightarrow$ (107) $\Rightarrow$ (91) and (119) $\Leftrightarrow$ (108) $\Rightarrow$ (107) $\Rightarrow$ (91).

**Numerical algorithms**: The numerical algorithms solving the optimization problems (118) and (119) will be presented in Sec. 10 in connection with the formal nonlinear dynamical stability with respect to the Vlasov equation.

### 6. INCOMPLETE VIOLENT RELAXATION

The statistical approach presented previously rests on the assumption that the collisionless mixing is efficient so that the ergodic hypothesis which sustains the statistical theory of Lynden-Bell is fulfilled. There are situations where the Lynden-Bell prediction works relatively well. However, there are other situations where the Lynden-Bell prediction fails. It has been understood since the beginning [35] that violent relaxation may be incomplete in certain cases so that the Lynden-Bell mixing entropy is not maximized in the whole available phase space. Incomplete relaxation [67] can lead to more or less severe deviations from the Lynden-Bell prediction. Physically, the system tends to reach

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8 Note that the maximization problem (118) determines a state that is both Lynden-Bell thermodynamically stable with respect to fine grained perturbations $\delta \rho(r, v, \eta)$ (according to criterion (91) $\Leftrightarrow$ (118)) and formally nonlinearly dynamically Vlasov stable with respect to coarse-grained perturbations $\delta f(r, v)$ (according to criterion (136)). On the other hand, since (71) does not imply (118) in case of ensemble inequivalence, we conclude that there may exist stable Lynden-Bell statistical equilibrium states that do not satisfy the criterion of formal nonlinear dynamical stability (136). Therefore, either: (i) the ensembles are always equivalent with respect to the conjugate variables $(M_{n>1}^{LR}, \alpha_{n>1})$ (ii) Lynden-Bell thermodynamical stability does not imply formal nonlinear dynamical stability (iii) there exists a stronger condition of nonlinear dynamical stability than (136). This intriguing point needs further development. In particular, it would be interesting to exhibit a situation of ensemble inequivalence with respect to the conjugate variables $(M_{n>1}^{LR}, \alpha_{n>1})$. 
the Lynden-Bell maximum entropy state during violent relaxation but, in some cases, it cannot attain it because the variations of the potential, that are the engine of the evolution, die away before the relaxation process is complete (there may be other reasons for incomplete relaxation). Since the Vlasov equation admits an infinite number of stationary solutions, the coarse-grained distribution \(\bar{f}(r,v,t)\) can be trapped in one of them and remain frozen in that quasi stationary state \(\bar{f}_{QSS}(r,v)\) until collisional effects finally come into play (on longer timescales). This steady solution is not always the most mixed state (it can be only partially mixed) so it may differ from Lynden-Bell’s statistical prediction. Thus, for dynamical reasons, the system does not always explore the whole phase space ergodically. In general, the statistical theory of Lynden-Bell gives a relatively good first order prediction of the QSS without fitting parameter and is able to explain out-of-equilibrium phase transitions between different types of structures, depending on the values of the control parameters fixed by the initial condition. However, there are cases where the prediction does not work well (it can sometimes be very bad) because of incomplete relaxation. Other stationary solutions of the Vlasov equation, differing from Lynden-Bell’s prediction, can arise in case of incomplete violent relaxation. The difficulty is that we do not know a priori whether the prediction of Lynden-Bell will work or fail because this depends on the dynamics and it is difficult to know in advance if the system will mix well or not. Let us give some examples of complete and incomplete violent relaxation in stellar systems, 2D turbulence and for the HMF model (for more discussion, see [67, 4]).

**Stellar systems:** The concept of violent relaxation was first introduced by Lynden-Bell [35] to explain the apparent regularity of elliptical galaxies in astrophysics. However, for 3D stellar systems the prediction of Lynden-Bell leads to density profiles whose mass is infinite (the density decreases as \(r^{-2}\) at large distances). In other words, there is no maximum entropy state at fixed mass and energy in an unbounded domain. Furthermore, it is known that the distribution functions (DF) of galaxies do not only depend on the energy \(\varepsilon = v^2/2 + \Phi(r)\) contrary to what is predicted by the Lynden-Bell statistical theory. This means that other ingredients are necessary to understand their structure. However, the approach of Lynden-Bell is able to explain why elliptical galaxies have an almost isothermal core. Indeed, it is able to justify a Boltzmannian distribution \(\bar{f} \sim e^{-\beta \varepsilon}\) in the core without recourse to collisions which operate on a much longer timescale \(t_R \sim (N/\ln N)t_D\). However, violent relaxation is incomplete in the halo. The concept of incomplete violent relaxation explains why galaxies are more confined than predicted by statistical mechanics (the density profile of elliptical galaxies decreases as \(r^{-4}\) instead of \(r^{-2}\) [54]).

For one dimensional self-gravitating systems, the Lynden-Bell entropy has a global maximum at fixed mass and energy in an unbounded domain. Early simulations of the 1D Vlasov-Poisson system starting from a water-bag initial condition have shown a relatively good agreement with the Lynden-Bell prediction [64]. In other cases, the Vlasov equation (and the corresponding N-body system) can have a very complicated, non-ergodic, dynamics. For example, starting from an annulus in phase space, Mineau et al. [68] have observed the formation of phase-space holes which block the relaxation towards the Lynden-Bell distribution. In that case, the system does not even relax towards a stationary state of the Vlasov equation but develops everlasting oscillations.
Two-dimensional vortices: In the context of two-dimensional turbulence, Miller [69] and Robert & Sommeria [70] have developed a statistical mechanics of the 2D Euler equation which is similar to the Lynden-Bell theory (see [38] for a description of this analogy). This theory works relatively well to describe vortex merging [71] or the nonlinear development of the Kelvin-Helmholtz instability in a shear layer [63]. It can account for the numerous bifurcations observed between different types of vortices (monopoles, dipoles, tripoles,...) [25, 72] and is able to reproduce the structure of geophysical and jovian vortices like Jupiter’s great red spot [73, 74, 59].

However, some cases of incomplete relaxation have been reported. For example, in the plasma experiment of Huang & Driscoll [75], the Miller-Robert-Sommeria (MRS) statistical theory gives a reasonable prediction of the QSS without fit but the agreement is not perfect [76]. The observed central density is larger than predicted by theory and the tail decreases more rapidly than predicted by theory, i.e. the vortex is more confined. This is related to the fact that mixing is not very efficient in the core and in the tail of the distribution (this can be explained by invoking the concept of “maximum entropy bubbles” to delimitate the region where the vorticity mixes well [72] or by developing a kinetic theory of violent relaxation, see [30] and Sec. 8). As noted by Boghosian [77], in this case the QSS can be fitted by a Tsallis distribution where the density drops to zero at a finite distance.

The HMF model: Recently, Antoniazzi et al. [27] have performed numerical simulations of the HMF model, starting from a water bag initial condition, to test the prediction of the Lynden-Bell theory. For a given value of the energy $U = 0.69$, this theory predicts an out-of-equilibrium phase transition between a spatially homogeneous phase and a spatially inhomogeneous phase above a critical magnetization $M_{\text{crit}} = 0.897$ discovered in [26, 27]. Numerical simulations show a relatively good agreement with the Lynden-Bell theory when the initial magnetization $M_0 < M_{\text{crit}}$. However, for $M_0 > M_{\text{crit}}$ several authors report important deviations to the Lynden-Bell theory. In particular, for $M_0 = 1$, Latora et al. [32] find that the QSS is spatially homogeneous ($M_{\text{QSS}} = 0$) and the velocity distribution is non-gaussian, while the Lynden-Bell theory predicts a spatially inhomogeneous state with gaussian tails (the same as the Boltzmann equilibrium state, since we are in the non-degenerate limit [4]). Using isotropic water-bag initial conditions, Campa et al. [17] show that the QSS is well-fitted by a semi-elliptical distribution. As remarked in [4], this corresponds to a Tsallis distribution with a compact support (with index $q = 3$) so that the distribution function drops to zero above a finite velocity. This confinement is a typical result of incomplete relaxation and can be explained by a kinetic theory of violent relaxation (see [5] and Sec. 8).

On the other hand, using different initial conditions, Morita & Kaneko [78] find that the system does not relax to a QSS but exhibits oscillations whose duration diverges with $N$. Therefore, the Lynden-Bell prediction clearly fails. This long-lasting periodic or quasi periodic collective motion appears through Hopf bifurcation and is due to the presence of clumps (high density regions) in phase space. We remark that this behaviour is relatively similar to the one reported by Mineau et al. [68] for self-gravitating systems, except that they observe phase space holes instead of phase space clumps.
7. GENERALIZED THERMODYNAMICS: THE TSALLIS ENTROPY

To account for the non-Boltzmannian nature of the QSS, some authors [77, 32, 79] have proposed to change the form of the entropy. In particular, it has been proposed that the QSS could be in the “most probable” state that maximizes the Tsallis entropy

\[ S_q[f] = -\frac{1}{q-1} \int (f^q(r,v) - f(r,v)) \, dr \, dv, \tag{120} \]

at fixed mass and energy. This leads to a distribution function of the form

\[ f_q(r,v) = \left[ \mu - \beta (q-1) \epsilon / q \right]^{1/(q-1)} \]

where \( \mu \) and \( \beta \) are Lagrange multipliers determined by the mass and the energy. Therefore, the Tsallis entropy (120) can be viewed as a generalization of the Boltzmann entropy [37]. However, it was remarked by Brands et al. [76] that the entropy \( S_q[f] \) can lead to inconsistencies when applied to the context of the violent relaxation (in order to describe the QSS) where the system’s dynamics is governed by the Vlasov equation. Indeed, the Tsallis distribution \( f_q(r,v) \) may not respect the properties of the Vlasov equation, in particular the fact that the DF is bounded by the maximum value of the initial condition:

\[ f(r,v) \leq f_{\text{max}}(r,v,t=0). \]

In our point of view [76, 36], the proper form of Tsallis entropy in the context of the violent relaxation is

\[ S_q[\rho] = -\frac{1}{q-1} \int (\rho^q(r,v,\eta) - \rho(r,v,\eta)) \, dr \, dv \, d\eta. \tag{121} \]

This is a functional of the probability density \( \rho(r,v,\eta) \) which takes into account the specificities of the collisionless (Vlasov) dynamics. For \( q \to 1 \), it returns the Lynden-Bell entropy (79). For \( q \neq 1 \), it could take into account incomplete mixing and non-ergodicity. In that context, the \( q \) parameter could be interpreted as a measure of mixing (\( q = 1 \) if the system mixes efficiently) and Tsallis entropy (121) could be interpreted as a functional attempting to take into account non-ergodicity in the process of incomplete violent relaxation. Maximizing \( S_q[\rho] \) at fixed mass, energy and Casimirs leads to a \( q \)-generalization of the equilibrium state. This maximization principle is a condition of thermodynamical stability (in Tsallis’ generalized sense) in the context of violent relaxation. Then, we can obtain a \( q \)-generalization of the equilibrium coarse-grained distribution function. In the case of two levels \( f \in \{0, \eta_0\} \), and in the dilute limit of the theory \( f \ll \eta_0 \), \( S_q[\rho] \) can be written in terms of the coarse-grained distribution

\[ A \text{ striking property of the Tsallis entropy is to yield distribution functions that have a compact support (for } q > 1 \text{) so that the density vanishes above a maximum energy. This leads to a confinement in phase space that is qualitatively (and sometimes quantitatively) similar to what is observed in situations of incomplete relaxation (see discussion in Sec. 6). Indeed, since mixing is never complete, the high energy tail of the distribution is less populated than what is predicted by the Boltzmann statistical mechanics (note that this notion of confinement is similar to the notion of maximum entropy “bubble” introduced in [72]). However, as discussed in [67], it is not clear why this confinement should necessarily (universally) be described by a Tsallis } q \text{-distribution. Other distributions with a compact support could also result from an incomplete relaxation.} \]
\( f = \rho \eta_0 \) in the form

\[
S_q[f] = -\frac{1}{q-1} \int [(\bar{f}/\eta_0)^q - (\bar{f}/\eta_0)] d\mathbf{r} d\mathbf{v},
\]

(122)

similar to Eq. (120), but with a different interpretation than the one given in [77, 32, 79]. However, as stated in [76], it is not clear why complicated effects of non-ergodicity (incomplete mixing) could be encapsulated in a simple functional such as (121). Other functionals of the form

\[
S = -\int C(\rho) d\mathbf{r} d\mathbf{v} d\eta
\]

(123)

where \( C \) is a convex function could be considered as well. Tsallis generalized entropy \( S_q \) and Tsallis \( q \) distributions can probably describe a certain type of non-ergodic behaviour but not all of them: they are not “universal attractors” of the (coarse-grained) Vlasov equation. For example, elliptical galaxies are not stellar polytropes [54] so their DF cannot be fitted by the Tsallis distribution [12, 67]. An improved model is a composite model that is isothermal in the core (justified by Lynden-Bell’s theory of violent relaxation) and polytropic in the halo (due to incomplete relaxation) with an index \( n = 4 \) corresponding to \( q = 7/5 \) [80, 12]. This observation suggests that the achieved distributions could be described by entropies of the form (121) where \( q(q, \mathbf{r}, \mathbf{v}) \) depends on the position (in phase space) so as to take into account the degree of mixing [12]. Unfortunately, this idea is not very useful in practice if we do not give an a priori prescription to relate the value of the index \( q(q, \mathbf{r}, \mathbf{v}) \) to the phase-space region. Finally, we note that the Tsallis distribution with a time dependent index \( q(t) \) can provide a good fit of the distribution of the system during its slow collisional evolution towards the Boltzmann distribution (\( q = 1 \)). This idea has been developed by Taruya & Sakagami [81] in stellar dynamics and proposed by Chavanis [4] for the HMF model.

8. KINETIC THEORY OF VIOLENT RELAXATION

An alternative idea to understand the problem of incomplete violent relaxation is to develop a kinetic theory of the Vlasov equation on the coarse-grained scale in order to understand dynamically what limits the convergence towards the Lynden-Bell distribution [38]. Kinetic theories of collisionless relaxation have been developed in [52, 82, 38, 39, 58, 5] with the aim to determine the dynamical equation satisfied by the coarse-grained DF \( \bar{f}(\mathbf{r}, \mathbf{v}, t) \). An interesting approach is based on a quasilinear theory of the Vlasov equation [52, 82, 39]. In the two levels approximation, the quasilinear theory leads to a kinetic equation for the coarse-grained DF of the form [5]:

\[
\frac{\partial \bar{f}}{\partial t} + \mathbf{v} \frac{\partial \bar{f}}{\partial \mathbf{r}} - \nabla \Phi \frac{\partial \bar{f}}{\partial \mathbf{v}} = \epsilon^d \epsilon^v \frac{\partial}{\partial \mathbf{v}^\mu} \int_0^t d\tau \int d\mathbf{r}_1 d\mathbf{v}_1 \frac{F^\mu}{m} (1 \rightarrow 0) G(t, t - \tau) \times \frac{F^v}{m} (1 \rightarrow 0) \left\{ \bar{f}_1 (\eta_0 - \bar{f}_1) \frac{\partial \bar{f}}{\partial \mathbf{v}^\nu} - \bar{f}_1 (\eta_0 - \bar{f}) \frac{\partial \bar{f}_1}{\partial \mathbf{v}_1^\nu} \right\} \right|_{t - \tau},
\]

(123)

We stress, however, that the Tsallis entropy is the simplest generalization of the Boltzmann entropy because, on an axiomatic point of view, the Tsallis entropy possesses almost all the properties of the Boltzmann entropy. This is not the case for other types of “generalized entropies”.
where $\varepsilon_r, \varepsilon_v$ are the coarse-graining mesh sizes in position and velocity spaces, $\Phi$ is the smooth field produced by the coarse-grained DF and $f = f(r, v, t), f_1 = f(r_1, v_1, t)$. This equation is expected to describe the late quiescent stages of the relaxation process when the fluctuations have weaken so that the quasilinear approximation can be implemented. It does not describe the early, very chaotic, process of violent relaxation driven by the strong fluctuations of the potential. The quasilinear theory of the Vlasov equation is therefore a theory of “quiescent” collisionless relaxation.

Equation (123) is very similar, in structure, to Eq. (53) for the collisional evolution, with nevertheless three important differences: (i) the fluctuating force $F(1 \to 0)$ is replaced by the direct force $F_1$ because the fluctuations have a different nature in the two problems. (ii) The distribution function $f$ in the collisional term of Eq. (53) is replaced by the product $f(\eta_0 - \bar{f})$ in Eq. (123). This nonlinear term arises from the effective “exclusion principle”, discovered by Lynden-Bell, accounting for the non-overlapping of phase levels in the collisionless regime. This is consistent with the Fermi-Dirac-like entropy (88) and Fermi-Dirac-like distribution (89) at statistical equilibrium. (iii) Considering the dilute limit $f \ll \eta_0$ to fix the ideas, we see that the equations (123) and (53) have the same mathematical form differing only in the prefactors: the mass $m$ of a particle in Eq. (53) is replaced by the mass $\eta_0 \varepsilon_r \varepsilon_v$ of a completely filled macrocell in Eq. (123). This implies that the timescales of collisional and collisionless “relaxation” are in the ratio

$$\frac{t_{\text{ncoll}}}{t_{\text{coll}}} \sim \frac{m}{\eta_0 \varepsilon_r \varepsilon_v}.$$  

Since $\eta_0 \varepsilon_r \varepsilon_v \gg m$, this ratio is in general quite small implying that the collisionless relaxation is much more rapid than the collisional relaxation. Typically, $t_{\text{ncoll}}$ is of the order of a few dynamical times $t_D$ (its precise value depends on the size of the mesh) while $t_{\text{coll}}$ is of order $\sim Nt_D$ or larger. The kinetic equation (123) conserves the mass and, presumably, the energy. By contrast, we cannot prove an $H$-theorem for the Lynden-Bell entropy (88). Indeed, the time variation of the Lynden-Bell entropy is of the form

$$\dot{S}_{\text{LB}} = \frac{1}{2} \varepsilon_r \varepsilon_v \int d r d v \int_{r_1, v_1} d r_1 d v_1 \frac{1}{\bar{f}(\eta_0 - \bar{f}) \bar{f}_1(\eta_0 - \bar{f}_1)} \int_0^t d \tau Q(t) G(t, t - \tau) Q(t - \tau),$$

and its sign is not necessarily positive. This depends on the importance of memory effects. Now, even if Eq. (123) conserves the energy and the mass and increases the Lynden-Bell entropy (88) monotonically, this does not necessarily imply that the system will converge towards the Lynden-Bell distribution (89). Indeed, it has been observed in several experiments and numerical simulations that the QSS does not exactly coincide with the strict statistical equilibrium state predicted by Lynden-Bell because of the complicated problem of incomplete relaxation (see Sec. 6). This is usually explained by a lack of ergodicity or “incomplete mixing”. Here, we try to be a little more precise by using the kinetic theory. There can be several reasons of incomplete relaxation:
(i) Absence of resonances: Very few is known concerning kinetic equations of the form of Eq. (123) and it is not clear if the Lynden-Bell distribution (89) is a stationary solution of that equation (and if it is the only one). As explained in Sec. 3.4 for the kinetic equation (53) describing the collisional relaxation, the relaxation may stop because the current $J$ vanishes due to the absence of resonances. This argument may also apply to Eq. (123) which has a similar structure and can be a cause for incomplete relaxation. The system tries to approach the statistical equilibrium state (as indicated by the increase of the entropy) but may be trapped in a QSS that is different from the statistical prediction (89). This QSS is a steady solution of Eq. (123) which cancels individually the advective term (l.h.s.) and the effective collision term (r.h.s.). This determines a subclass of steady states of the Vlasov equation (cancellation of the l.h.s.) such that the complicated “turbulent” current $J$ in the r.h.s. vanishes. This offers a large class of possible steady state solutions that can explain the deviation between the QSS and the Lynden-Bell statistical equilibrium state (89) observed, in certain cases, in simulations and experiments of violent relaxation.

(ii) Incomplete relaxation in phase space: The turbulent current $J$ in Eq. (123) is driven by the fluctuations $f_2 \equiv \tilde{f}^2 = f_2^2 - \bar{f}^2$ of the distribution function generating the fluctuations $\delta \Phi$ of the potential (see $[5]$ for more details). In the “mixing region” of phase space where the fluctuations are strong, the DF tends to reach the Lynden-Bell distribution (89). As we depart from the “mixing region”, the fluctuations decay ($f_2 \to 0$) and the mixing is less and less efficient $||J|| \to 0$. In these regions, the system takes a long time to reach the Lynden-Bell distribution (89) and, in practice, cannot attain it in the time available (see (iii)). In the two levels case, we have $f_2 = \bar{f}(\eta_0 - \bar{f})$. Therefore, the phase space regions where $\bar{f} \to 0$ or $\bar{f} \to \eta_0$ do not mix well (the diffusion current $J$ is weak) and the observed DF can be sensibly different from the Lynden-Bell distribution in these regions of phase space. This concerns essentially the core ($\bar{f} \to \eta_0$) and the tail ($\bar{f} \to 0$) of the distribution. Note that the confinement of the tail ($\bar{f} \to 0$) is consistent with the notion of maximum entropy “bubble” $[72]$. However, mixing can also be incomplete inside the “bubble”, in particular in the core ($\bar{f} \to \eta_0$) of the distribution.

(iii) Incomplete relaxation in time: during violent relaxation, the system tends to approach the statistical equilibrium state (89). However, as it approaches equilibrium, the fluctuations of the potential, which are the engine of the evolution, become less and less effective to drive the relaxation. This is because the scale of the fluctuations becomes smaller and smaller as time goes on. This effect can be taken into account in the kinetic theory by considering that the correlation lengths $\varepsilon_r(t)$ and $\varepsilon_v(t)$ decrease with time so that, in the kinetic equation (123), the prefactor $\varepsilon_r(t)\varepsilon_v(t) \to 0$ for $t \to +\infty$. As a result, the “turbulent” current $J$ in Eq. (123) can vanish before the system has reached the statistical equilibrium state (89). In that case, the system can be trapped in a QSS that is a steady solution of the Vlasov equation different from the statistical prediction (89).

Similar arguments have been given in $[83, 38]$ on the basis of a more phenomenological kinetic theory of violent relaxation based on the MEPP. The idea is to keep the Lynden-Bell entropy (79) unchanged (as being the fundamental entropy for the process of violent relaxation) but describe the dynamical evolution of $\rho (\mathbf{r}, \mathbf{v}, \eta, t)$ by a relaxation
equation of the form

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \frac{\partial \rho}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \rho}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \left\{ D(\mathbf{r},\mathbf{v},t) \left[ \frac{\partial \rho}{\partial \mathbf{v}} + \beta(t)(\eta - \mathbf{r})\rho \mathbf{v} \right] \right\}, \quad (127)$$

with a diffusion coefficient $D(\mathbf{r},\mathbf{v},t)$ going to zero for large time (as the variations of the potential $\Phi$ decay) and in regions of phase-space where the fluctuations $\delta \Phi$ are not strong enough to provide efficient mixing (so that $f_2 \to 0$). The vanishing of the diffusion coefficient can “freeze” the system in a subdomain of phase space and account for incomplete relaxation and non-ergodicity. In general, the resulting state, although incompletely mixed, is not a Tsallis $q$-distribution. This approach is interesting because it is not based on a generalized entropy, so there is no free parameter like $q$. However, it demands to solve a dynamical equation (127) to predict the QSS. The idea is that, in case of incomplete relaxation (non-ergodicity), the prediction of the QSS is impossible without considering the dynamics [36].

9. GENERALIZED $H$-FUNCTIONS IN $\mathcal{f}$-SPACE

In order to quantify the importance of mixing, Tremaine et al. [40] have introduced the notion of generalized $H$-functions. A generalized $H$-function is a functional of the coarse-grained DF of the form

$$H[\mathcal{f}] = -\int C(\mathcal{f}) d\mathbf{r} d\mathbf{v}, \quad (128)$$

where $C$ is any convex function ($C'' > 0$). We assume that the initial condition at $t = 0$ has been prepared without small-scale structure so that the fine-grained and coarse-grained DF are equal: $f(\mathbf{r},\mathbf{v},0) = \mathcal{f}(\mathbf{r},\mathbf{v},0)$. For $t > 0$, the system will mix in a complicated manner and develop intermingled filaments so that these two fields will not be equal anymore. We have

$$H(t) - H(0) = \int \left\{ C[\mathcal{f}(\mathbf{r},\mathbf{v},0)] - C[\mathcal{f}(\mathbf{r},\mathbf{v},t)] \right\} d\mathbf{r} d\mathbf{v}$$

$$= \int \left\{ C[f(\mathbf{r},\mathbf{v},0)] - C[\mathcal{f}(\mathbf{r},\mathbf{v},t)] \right\} d\mathbf{r} d\mathbf{v}. \quad (129)$$

The fine-grained DF is solution of the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \quad (130)$$

Thus

$$\frac{d}{dt} \int C(f) d\mathbf{r} d\mathbf{v} = \int C'(f) \frac{\partial f}{\partial \mathbf{r}} d\mathbf{r} d\mathbf{v} = -\int C'(f) \left( \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \frac{\partial f}{\partial \mathbf{v}} \right) d\mathbf{r} d\mathbf{v}$$

$$= -\int \left( \mathbf{v} \frac{\partial}{\partial \mathbf{r}} - \nabla \Phi \frac{\partial}{\partial \mathbf{v}} \right) C(f) d\mathbf{r} d\mathbf{v} = 0. \quad (131)$$
This shows that the functional $H[f]$ calculated with the fine-grained DF is independent on time (it is a particular Casimir) so Eq. (129) becomes

$$H(t) - H(0) = \int \{ C[f(r,v,t)] - C[\mathcal{F}(r,v,t)] \} dr dv. \tag{132}$$

Let us now divide a macrocell of surface $\Delta$ into $\nu$ microcells. We call $f_i$ the value of the DF in a microcell. The contribution of a macrocell to $H(t) - H(0)$ is

$$\Delta \left\{ \frac{1}{\nu} \sum_i C(f_i) - C \left( \frac{1}{\nu} \sum_i f_i \right) \right\}, \tag{133}$$

which is positive since $C$ is convex. This implies that $H(t) - H(0) \geq 0$. We conclude that the generalized $H$-functions $H[\mathcal{F}]$ calculated with the coarse-grained DF increase in the sense that $H(t) \geq H(0)$ for any $t \geq 0 \,[40]$. This is similar to the Boltzmann $H$-theorem in kinetic theory. However, contrary to the Boltzmann equation, the Vlasov equation does not single out a unique functional (the above inequality is true for all $H$-functions) and the time evolution of the $H$-functions is not necessarily monotonic (nothing is implied concerning the relative values of $H(t)$ and $H(t')$ for $t, t' > 0$).

**Example**: the Tsallis functional expressed in terms of the coarse-grained DF

$$H_q[\mathcal{F}] = -\frac{1}{q-1} \int (\mathcal{F}^q - \mathcal{F}) dr dv, \tag{134}$$

is a particular generalized $H$-function. This interpretation of Tsallis functional as a generalized $H$-function in the sense of [40] was given by Plastino & Plastino [84] and Chavanis [65, 36, 12]. The Boltzmann functional $H_1[\mathcal{F}] = -\int \mathcal{F} \ln \mathcal{F} dr dv$, corresponding to $q = 1$ is also a particular $H$-function in the sense of [40].

### 10. FORMAL NONLINEAR DYNAMICAL STABILITY FOR THE VLASOV EQUATION

In this section, we discuss the formal nonlinear dynamical stability of stationary solutions of the Vlasov equation. We consider a special class of Casimir functionals of the form

$$S[f] = -\int C(f) dr dv, \tag{135}$$

where $C$ is any convex function ($C'' > 0$). These functionals, as well as the energy and the mass, are conserved by the Vlasov equation. It results that the maximization problem

$$\max_f \{ S[f] \mid E[f] = E, M[f] = M \} \tag{136}$$

determines a steady state of the Vlasov equation that is formally nonlinearily dynamically stable. Writing the first variations as $\delta S - \beta \delta E - \alpha \delta M = 0$, the steady state satisfies $f = F(\beta \epsilon + \alpha)$ where $F(x) = (C')^{-1}(-x)$. Therefore, this distribution function is a monotonically decreasing function of the energy (for $\beta > 0$): $f = f(\epsilon)$ with $f'(\epsilon) < 0$. 


Formally, the maximization problem (136) is similar to a condition of “microcanonical stability” in thermodynamics where $S$ plays the role of an entropy. Due to this thermodynamical analogy [65], we can use the methods of thermodynamics to study the nonlinear dynamical stability problem (136). We also note that if $f(r,v)$ solves the minimization problem

$$\min_f \{ F[f] = E[f] - TS[f] \mid M[f] = M \},$$

(137)

then it solves the maximization problem (136) and is therefore nonlinearly dynamically stable. However, the reciprocal is wrong in case of ensemble inequivalence. A solution of (136) is not necessarily a solution of (137). Therefore, the minimization problem (137) provides just a sufficient condition of nonlinear dynamical stability. The stability criterion (136) is more refined than the stability criterion (137): if a distribution $f = f(\varepsilon)$ with $f'(\varepsilon) < 0$ satisfies (136) or (137), then it is nonlinearly dynamically stable; however, if it does not satisfy (137), it can be nonlinearly dynamically stable provided that it satisfies (136). This means that we can “miss” stable solutions if we just use the optimization problem (137) to construct stable stationary solutions of the Vlasov equation of the form $f = f(\varepsilon)$ with $f'(\varepsilon) < 0$. The criterion (137) determines a subclass of solutions of the maximization problem (136). The criterion (136) is richer and allows to construct a larger class of nonlinearly dynamically stable stationary solutions. This is similar to a situation of “ensemble inequivalence” in thermodynamics [11, 14]. Indeed, (137) is similar to a criterion of “canonical stability” in a thermodynamical analogy [65], where $F$ is similar to a “free energy”. Canonical stability implies microcanonical stability but the converse is wrong in case of ensemble inequivalence.

For a long time, most studies related to the formal nonlinear dynamical stability of the Vlasov equation considered the minimization of an Energy-Casimir functional $E - S$ at fixed mass (problem with one constraint) [85]. This Energy-Casimir method was first introduced by Arnold [86] in two-dimensional hydrodynamics for the Euler equation. This corresponds to the “canonical” criterion (137). It only provides a sufficient condition of formal nonlinear dynamical stability. Recently, Ellis et al. [41] introduced a refined stability criterion for the 2D Euler equation in the form the “microcanonical” criterion (136) (problem with two constraints). In [42], we introduced a similar criterion to study the formal nonlinear dynamical stability of stellar systems with respect to the Vlasov-Poisson system. Explicit examples of “ensemble inequivalence” in the context of the nonlinear dynamical stability for the 2D Euler and Vlasov equations, where the criteria (136) and (137) do not coincide, were explicitly constructed by Ellis et al. [41] for geophysical flows and by Chavanis [42] for stellar polytropes. In the astrophysical context, it was shown [42] that “ensemble inequivalence” leads to a nonlinear version of the Antonov first law (see Sec. 11).

**Example:** the extremization of the Tsallis functional

$$S_q[f] = -\frac{1}{q-1} \int (f^q - f) dr dv,$$

(138)

at fixed mass and energy leads to distribution functions of the form

$$f(r,v) = \left[ \mu - \frac{\beta(q-1)}{q} \varepsilon \right]^\frac{1}{q-1},$$

(139)
where $\mu$ and $\beta$ are Lagrange multipliers determined by $M$ and $E$ \cite{87}. These DF are particular stationary solutions of the Vlasov equation called stellar polytropes in astrophysics \cite{54}. Furthermore, if the DF maximizes the Tsallis functional at fixed mass and energy, then it is formally nonlinearly dynamically stable with respect to the Vlasov equation according to \cite{136}. In this context, the Tsallis functional \cite{138} is a particular Casimir functional of the form \cite{135}, not a generalized entropy. Its maximization at fixed mass and energy forms a criterion of nonlinear dynamical stability for the Vlasov equation, not a criterion of generalized thermodynamical stability in the microcanonical ensemble. This dynamical interpretation of the Tsallis functional related to the Vlasov equation was given by Chavanis & Sire \cite{88}. It differs from the thermodynamical interpretation given by Taruya & Sakagami \cite{79} in terms of generalized thermodynamics. On the other hand, the minimization of the Tsallis functional $F_q = E - T S_q$ at fixed mass forms a sufficient criterion of nonlinear dynamical stability for the Vlasov equation, not a criterion of generalized thermodynamical stability in the canonical ensemble. In the present context, the resemblance with a generalized thermodynamical formalism is effective and is the mark of a thermodynamical analogy \cite{65}. Note that the preceding discussion also applies to the Boltzmann distribution, corresponding to $q = 1$, which is also a particular solution of the Vlasov equation.

**Numerical algorithms:** A relaxation equation has been proposed to solve the maximization problem \cite{136}. It is obtained from a Maximum Entropy Production Principle (MEPP), exploiting the thermodynamical analogy, and has the form of a generalized mean field Kramers equation \cite{65}. It can serve as a numerical algorithm to determine a nonlinearly dynamically stable stationary solution of the Vlasov equation, specified by the convex function $C(f)$, for given $E$ and $M$. The relaxation equation solving the “microcanonical” problem \cite{136} is

$$
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left\{ D \left[ \frac{\partial f}{\partial v} + \frac{\beta(t)}{C''(f)} v \right] \right\},
$$

with

$$
\beta(t) = - \frac{\int D \frac{\partial f}{\partial v} : v d\mathbf{r} d\mathbf{v}}{\int D \frac{\partial^2 f}{\partial v^2} d\mathbf{r} d\mathbf{v}} \tag{141}
$$

This equation conserves the mass $M$, the energy $E$ (through the time dependent Lagrange multiplier $\beta(t)$) and monotonically increases the $S$-functional: $\dot{S} \geq 0$ (H-theorem). Furthermore, $\dot{S} = 0$ iff $f$ is a steady solution of Eq. \cite{140}. Therefore, $S[f]$ is the Lyapunov functional of the relaxation equation \cite{140}-\cite{141}. It results that $f(r,v)$ is a linearly dynamically stable stationary solution of Eq. \cite{140}-\cite{141} iff it is a (local) maximum of $S$ at fixed $E, M$. Minima or saddle points of $S$ are dynamically unstable. Furthermore, if $S$ is bounded from above (within the previous constraints), $f(r,v,t)$ will relax towards a (local) maximum of $S$ at fixed $E, M$ for $t \to +\infty$ (if there exists several local maxima of $S$, the selection will depend on a complicated notion of basin of attraction). Therefore, Eq. \cite{136}-\cite{141} can serve as a numerical algorithm to solve the “microcanonical” problem \cite{136} and compute a nonlinearly dynamically stable stationary solution of the Vlasov equation, specified by $C$, for given $E$ and $M$. Note that the generalized Vlasov-Landau...
(GVL) equation

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = \Lambda \frac{\partial}{\partial \nu} \int \frac{\delta^{\mu \nu} w^{2} - w^{\mu} w^{\nu}}{w^{3}} \left( \frac{1}{C''(f_1)} \frac{\partial f}{\partial v^{\nu}} - \frac{1}{C''(f)} \frac{\partial f_{1}}{\partial v^{\nu}} \right) d_{v_1},
\]

(142)

introduced in [89] has the same properties as the relaxation equation (140)-(141). In particular, \( f(r,v) \) is a linearly dynamically stable stationary solution of the GVL equation iff it is a (local) maximum of \( S \) at fixed \( E, M \). Minima or saddle points of \( S \) are dynamically unstable. Therefore, the GVL equation can also provide a numerical algorithm to compute nonlinearly dynamically stable stationary solutions of the Vlasov equation, specified by \( C \), for given \( E \) and \( M \). We note that \( f(r,v) \) is a linearly dynamically stable stationary solution of the generalized Vlasov-Landau equation iff it is a formally nonlinearly dynamically stable stationary solution of the Vlasov equation.

If we fix the inverse temperature \( \beta \) instead of the energy in Eq. (140), we get a relaxation equation

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = \frac{\partial}{\partial \nu} \left( D \left[ \frac{\partial f}{\partial v} + \frac{\beta}{C''(f)} v \right] \right),
\]

(143)

solving the “canonical” problem (137). It is called the generalized Vlasov-Kramers (GVK) equation. This equation conserves the mass \( M \) and monotonically decreases the \( F \)-functional: \( \dot{F} \leq 0 \). Furthermore, \( \dot{F} = 0 \) iff \( f \) is a steady solution of Eq. (143). Therefore, \( F[f] \) is the Lyapunov functional of the GVK equation. It results that \( f(r,v) \) is a linearly dynamically stable stationary solution of the GVK equation iff it is a (local) minimum of \( F \) at fixed \( M \). Maxima or saddle points of \( F \) are dynamically unstable. Furthermore, if \( F \) is bounded from below (within the previous constraints), \( f(r,v,t) \) will relax towards a (local) minimum of \( F \) at fixed \( M \). Therefore, Eq. (143) can serve as a numerical algorithm to solve the “canonical problem” (137) and determine a subclass of nonlinearly dynamically stable stationary solutions of the Vlasov equation, specified by \( C \), since (137) \( \Rightarrow \) (136). We note that a linearly dynamically stable stationary solution of the generalized Vlasov-Kramers equation is (i) a formally nonlinearly dynamically stable stationary solution of the Vlasov equation (ii) a linearly dynamically stable stationary solution of the generalized Vlasov-Landau equation (but the reciprocal is wrong in case of ensemble inequivalence).

We can also use the preceding relaxation equations to solve the optimization problems (118) and (119) of Sec. 5. If we view \( f \) as the coarse-grained DF \( \overline{f} \) and if we take \( C(f) \) of the form (117), a linearly dynamically stable stationary solution of the relaxation equation (140)-(141) is a Lynden-Bell statistical equilibrium state according to the sufficient condition (118) \( \Rightarrow \) (91). Similarly, a linearly dynamically stable stationary solution of the generalized Vlasov-Landau equation is a Lynden-Bell statistical equilibrium state (but the reciprocal is wrong in case of ensemble inequivalence). On the other hand, a linearly dynamically stable stationary solution of the relaxation equation (143) is a Lynden-Bell statistical equilibrium state according to the sufficient condition (119) \( \Rightarrow \) (91). Therefore, a linearly dynamically stable stationary solution of the generalized Vlasov-Kramers equation is a Lynden-Bell statistical equilibrium state (but the reciprocal is wrong in case of ensemble inequivalence).
11. THE NONLINEAR ANTONOV FIRST LAW

It was first realized by Antonov [90] in astrophysics, that there exists deep connections between the dynamical stability of steady solutions of the Vlasov equation (68) of the form \( f = f(\varepsilon) \) with \( f'(\varepsilon) < 0 \) and the dynamical stability of steady solutions of the Euler equations

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \tag{144}
\]

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p - \nabla \Phi, \tag{145}
\]

with a barotropic equation of state \( p = p(\rho) \) (as usual \( \Phi(r, t) = \int u(|r - r'|) \rho(r', t) dr' \) is the mean field potential). The Euler equations conserve the mass \( M[\rho] = \int \rho d r \) and the energy

\[
\mathcal{W}[\rho, u] = \int \rho \int^{\rho} \frac{p'(\rho')}{\rho'^{2}} d\rho' dr + \frac{1}{2} \int \rho \Phi dr + \frac{1}{2} \int \rho u^{2} dr, \tag{146}
\]

including the internal energy, the potential energy and the kinetic energy of the mean motion. It results that the minimization problem

\[
\min_{\rho, u} \{ \mathcal{W}[\rho, u] \mid M[\rho] = M \}, \tag{147}
\]

determines a steady state of the Euler equation that is formally nonlinearly dynamically stable. Writing the first order variations as \( \delta \mathcal{W} - \alpha \delta M = 0 \), we get \( u = 0 \) and

\[
\int^{\rho} \frac{p'(\rho')}{\rho'} d\rho' = -\Phi. \tag{148}
\]

This leads to the condition of hydrostatic equilibrium

\[
\nabla p + \rho \nabla \Phi = 0. \tag{149}
\]

We also note that \( \rho = \rho(\Phi) \) and that \( p'(\rho)/\rho = -1/\rho'(\Phi) \) so that \( \rho(\Phi) \) is monotonically decreasing (since \( p'(\rho) > 0 \) in cases of physical interest).

Let us now return to the Vlasov equation. We have seen that the minimization problem (137) provides a sufficient condition of formal nonlinear dynamical stability. In order to solve this minimization problem, we can proceed in two steps (we consider here the global minimization problem. The local minimization problem is treated in the Appendix). First step: we determine the distribution \( f_{1}(r, v) \) which minimizes \( F[f] \) at fixed density profile \( \rho(r) = \int f dv \). This gives a distribution \( f_{1}[\rho(r), v] \) depending on \( \rho(r) \) and \( v \). Substituting this distribution in the functional \( F[f] \), we obtain a functional \( F[\rho] \equiv F[f_{1}] \) of the density. Second step: we determine the density \( \rho_{*}(r) \) which minimizes \( F[\rho] \) at fixed mass \( M[\rho] = M \). Finally, we have \( f_{*}(r, v) = f_{1}[\rho_{*}(r), v] \). Let us be more explicit. If we fix the density profile \( \rho(r) \), the potential energy \( W[\rho] \) is automatically determined. Therefore, minimizing \( F[f] = E[f] - TS[f] \) at fixed density profile
is equivalent to minimizing $\tilde{F}[f] = K[f] - TS[f]$ at fixed density profile, where $K[f] = \frac{1}{2} \int f v^2 dv$ is the kinetic energy. The distribution $f_1(r, v)$ that extremizes $\tilde{F}[f]$ with the constraint $\int f dv = \rho(r)$ satisfies the first order variations $\delta F + \int \lambda(r) \delta \int f dv dr = 0$, where $\lambda(r)$ is a Lagrange multiplier. This leads to

$$
    f_1 = F \left\{ \beta \left[ \frac{v^2}{2} + \lambda(r) \right] \right\},
$$

where $F(x) = (C')^{-1}(-x)$ and $\lambda(r)$ is determined by $\rho(r)$ by writing $\rho = \int f_1 dv$. Since

$$
    \delta^2 F = -T \delta^2 S = \frac{1}{2} \int C''(f_1)(\delta f)^2 dv dr \geq 0,
$$

this is a minimum of $F[f]$ at fixed density profile. According to Eq. (150), the density $\rho = \int f_1 dv = \rho[\lambda(r)]$ and the pressure $p = \frac{1}{2} \int f_1 v^2 dv = p[\lambda(r)]$ are functions of $\lambda(r)$. Eliminating $\lambda(r)$ between these two expressions we obtain a barotropic equation of state $p = p(\rho)$. Then, after simple calculations (see Appendix B of [42]), we can show that the functional $F[\rho] \equiv F[f_1]$ is given by

$$
    F[\rho] = \int \rho \int \rho \frac{p(\rho')}{\rho^2} d\rho' dr + \frac{1}{2} \int \rho \Phi dr.
$$

Therefore, we conclude that $f_*(r, v) = f_1[\rho_*(r), v]$ solves the minimization problem (137) iff $\rho_*(r)$ solves the minimization problem (153) if $\rho_*(r)$ solves the minimization problem (137).

Now we observe that the solutions of the minimization problems (147) and (153) coincide. Therefore, we conclude that $f_*(r, v) = f_1[\rho_*(r), v]$ solves the minimization problem (137) iff $\rho_*(r)$ solves the minimization problem (147). On the other hand, since the ensemble of solutions of (137) is included in the ensemble of solutions of (136) we conclude that $f_*(r, v) = f_1[\rho_*(r), v]$ solves the maximization problem (147), but the reciprocal is wrong in case of ensemble inequivalence. In conclusion, we find that a DF $f = f(\epsilon)$ with $f'(\epsilon) < 0$ is a nonlinerae dynamically stable steady state of the Vlasov equation (kinetic system) if the corresponding density profile $\rho(r)$ is a nonlinerarly dynamically stable steady state of the Euler equation (fluid system), but the reciprocal is wrong. In astrophysics, this forms the nonlinear Antonov first law [42]: “a stellar system with $f = f(\epsilon)$ and $f'(\epsilon) < 0$ is a nonlinerarly dynamically stable steady state of the Vlasov-Poisson system if the corresponding barotropic star is a nonlinerarly dynamically stable steady state of the Euler-Poisson system (but the reciprocal is wrong).”[11] This can be seen as a manifestation of “ensemble inequivalence” in a thermodynamical analogy [42]. For example, complete polytropes

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[11] This law was proven by Antonov [90] for the problem of linear dynamical stability and extended by Chavanis [42] to the problem of formal nonlinear dynamical stability.
with index $3 < n < 5$ satisfy the “microcanonical” criterion (136) but not the “canonical” criterion (137) ⇔ (153) ⇔ (147). Thus, stellar polytropes with index $3 < n < 5$ are nonlinearly dynamically stable with respect to the Vlasov-Poisson system while the corresponding gaseous polytropes are unstable with respect to the Euler-Poisson system [42].

**Example:** When $S[f]$ is the Tsallis functional (138), the corresponding barotropic gas is a gaseous polytrope with an equation of state [88]:

$$p = K \rho^{\gamma}, \quad \gamma = 1 + \frac{1}{n}, \quad n = \frac{1}{q-1} + \frac{d}{2}.$$  \hspace{1cm} (154)

The functional (152) is of the form

$$F[\rho] = \frac{K}{\gamma-1} \int (\rho^{\gamma} - \rho) d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r}.$$  \hspace{1cm} (155)

This can be viewed as a Tsallis “free energy” functional $F_\gamma[\rho] = W[\rho] - KS_\gamma[\rho]$ with index $\gamma$ where $W$ is the (potential) energy and $K$ plays the role of a “polytropic temperature” (recall again that these analogies with thermodynamics are purely formal in the present dynamical context). It can be compared with the Tsallis “free energy” functional $F_q[f] = E[f] - TS_q[f]$ in phase space (see Sec. 10). The steady density distribution (148) is of the form

$$\rho = \left[ \lambda - \frac{\gamma - 1}{\gamma K \Phi} \right]^{\gamma}.$$  \hspace{1cm} (156)

which is morphologically similar to Eq. (139). Therefore, a Tsallis distribution $f_q(\varepsilon)$ in phase space yields a Tsallis distribution $\rho_\gamma(\Phi)$ in position space [88].

**Numerical algorithm:** A relaxation equation has been proposed to solve the minimization problem (153). It is called the generalized Smoluchowski (GS) equation [65]. It can serve as a numerical algorithm to determine nonlinearly dynamically stable stationary solutions of the Euler equations, specified by $p(\rho)$, for given $M$. It is written as

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\epsilon} (\nabla \rho + \rho \nabla \Phi) \right].$$  \hspace{1cm} (157)

This equation conserves the mass $M$ and monotonically increases the $F$-functional: $\dot{F} \geq 0$ (H-theorem). Furthermore, $\dot{F} = 0$ iff $\rho$ is a steady solution of (157). Therefore, $F[\rho]$ is the Lyapunov functional of the GS equation (157). It results that $\rho(\mathbf{r})$ is a linearly dynamically stable stationary solution of the generalized Smoluchowski equation iff it is a (local) minimum of $F$ at fixed $M$. Maxima or saddle points of $F$ are dynamically unstable. Furthermore, if $F$ is bounded from below (within the previous constraint), $\rho(\mathbf{r}, t)$ will relax towards a (local) minimum of $F$ at fixed $M$ for $t \rightarrow +\infty$. Therefore, Eq. (157) can serve as a numerical algorithm to solve the “canonical” problem (153) and compute a nonlinearly dynamically stable stationary solution of the Euler equation, specified by $p(\rho)$, for given $M$ since (153) ⇔ (147). We note that $\rho(\mathbf{r})$ is a linearly dynamically stable stationary solution of the generalized Smoluchowski equation iff it
is a formally nonlinearly dynamically stable stationary solution of the barotropic Euler equation. There are other corollaries to that result. Since (137) ⇔ (153) we conclude that \( f(\mathbf{r}, \mathbf{v}) \) is linearly dynamically stable with respect to the generalized Vlasov-Kramers equation (143) iff the corresponding density \( \rho(\mathbf{r}) \) is linearly dynamically stable with respect to the generalized Smoluchowski equation (157). On the other hand, according to the implications (153) ⇔ (147) ⇔ (137) ⇒ (136) and (153) ⇔ (137) ⇔ (108) ⇒ (107) ⇒ (97), we conclude that a linearly dynamically stable stationary solution of the generalized Smoluchowski equation determines (i) a nonlinearly dynamically stable stationary solution of the Vlasov equation (ii) a linearly dynamically stable stationary solution of the generalized Vlasov-Landau equation (142) (iii) a Lynden-Bell statistical equilibrium state (but the reciprocal is wrong in case of ensemble inequivalence).

12. SELECTIVE DECAY PRINCIPLE

Let us conclude this paper by some phenomenological considerations. The results of Sec. 9 suggest a notion of generalized selective decay principle [36, 67]: among all invariants of the collisionless dynamics, the generalized \( H \)-functions (fragile constraints) increase \((-H \text{ decrease})\) on the coarse-grained scale while the mass and the energy (robust constraints) are approximately conserved\(^{12}\). According to this principle, we may expect (phenomenologically) that the QSS reached by the system as a result of violent relaxation will maximize a certain \( H \)-function called \( H^*[\mathcal{F}] \) (non-universal) at fixed mass \( M \) and energy \( E \). Therefore, it is expected to satisfy a problem of the form

\[
\max_{\mathcal{F}} \{H^*[\mathcal{F}] \mid E[\mathcal{F}] = E, M[\mathcal{F}] = M\}. \quad \text{(Phenomenological)} \quad (158)
\]

As we have seen, this maximization problem determines a distribution function \( \mathcal{F} = \mathcal{F}(\varepsilon) \) with \( \mathcal{F}(\varepsilon) < 0 \) which is a stationary solution of the Vlasov equation (recall that our argument applies to the coarse-grained distribution). In general, the \( H \)-function \( H^*[\mathcal{F}] \) that is effectively maximized by the system as a result of violent relaxation (if it really maximizes an \( H \)-function!) is difficult to predict. It depends on the initial conditions and on the efficiency of mixing. If mixing is complete (ergodicity), the statistical theory of Lynden-Bell shows that the coarse-grained DF (84) extremizes a functional of the form (95)-(102) at fixed mass and energy. Although we cannot conclude that this extremum is always a maximum because of the problem of ensemble inequivalence reported in Sec. 5, it is likely however that in many cases the coarse-grained DF predicted by the statistical theory of Lynden-Bell maximizes the generalized entropy (114)-(117) at fixed mass and energy. Therefore, we conclude that when mixing is complete \( H^* = S \), where \( S \) is given

\(^{12}\) A selective decay principle was first introduced in two-dimensional turbulence, for the 2D Navier-Stokes equation, based on different arguments [91]. It states that, in the presence of a small viscosity \( \nu \rightarrow 0 \), the energy of the flow is approximately conserved while the enstrophy decays. This leads to a phenomenological minimum enstrophy principle where the "equilibrium" flow minimizes the enstrophy at fixed circulation and energy. Our selective decay principle, which also applies to the (inviscid) 2D Euler equation [36], is more general (the enstrophy is a particular \( H \)-function) and uses the notion of coarse-graining rather than viscosity or other source of dissipation.
by Eqs. (114)-(117). In other words, apart from the problem of ensemble inequivalence which demands further investigation, the statistical theory of Lynden-Bell justifies the generalized selective decay principle (158) and gives the expression of $H^*[\mathcal{F}]$, depending on the initial condition. We again emphasize the importance of the notion of coarse-graining: the functionals $H[\mathcal{F}]$ calculated with the fine-grained DF are conserved while the $H$-functions $\mathcal{H}[\mathcal{F}]$ calculated with the coarse-grained DF increase. Furthermore, the particular $H$-function $H^*[\mathcal{F}] = \delta[\mathcal{F}]$ (whose form depends on the initial conditions) is maximum at statistical equilibrium while $E[\mathcal{F}]$ and $M[\mathcal{F}]$ are approximately conserved. Therefore, although the Lynden-Bell theory takes into account the conservation of the Casimirs $H[\mathcal{F}]$, it leads to a coarse-grained distribution which maximizes a functional $H^*[\mathcal{F}]$ at fixed $E$ and $M$. On the other hand, if mixing is incomplete (see Sec. 6), the coarse-grained DF may still maximize a certain generalized $H$-function at fixed mass and energy but $H^*[\mathcal{F}]$ and $\mathcal{F}(\mathbf{r})$ can take forms that are not compatible with the expressions (114)-(117) and (83) derived in the statistical approach. In case of incomplete relaxation, the prediction of this functional $H^*[\mathcal{F}]$ seems out-of-reach because its form depends on the efficiency of mixing which is not known a priori. In particular, there is no reason why $H^*[\mathcal{F}]$ should always (universally) be the Tsallis functional. However, it has been observed in several occasions [77, 17, 4] that the Tsallis distributions are “attractors” of the solutions of the (coarse-grained) Vlasov equation for some initial conditions.

We now observe that the criterion of formal nonlinear dynamical stability (136) is remarkably consistent with the phenomenological selective decay principle of violent relaxation (158) provided that we interpret $\mathcal{F}$ as the coarse-grained DF. This coincidence looks surprising at first sight. A priori, the optimization problems (136) and (158) are completely different: the problem (136) explicitly uses the fact that the functional $H[\mathcal{F}]$ is conserved by the Vlasov equation while the problem (158) uses the fact that the functional $\mathcal{H}[\mathcal{F}]$ calculated with the coarse-grained DF increases during violent relaxation. In fact, the problem (136) indicates that if a DF is a maximum of $H$ at fixed mass and energy, then it is nonlinearly dynamically stable with respect to the Vlasov equation while the phenomenology of violent relaxation leading to the problem (158) explains how, starting from an unstable initial condition, a DF can possibly reach a maximum of $H$ at fixed mass and energy (through mixing) although $H$ is rigorously conserved by the Vlasov equation. Indeed, during mixing the coarse-grained DF is not conserved by the Vlasov equation ($D\mathcal{F}/Dt \neq 0$ where $D/Dt$ is the material derivative in phase space) and the $H$-functions $H[\mathcal{F}]$ increase. Once it has mixed, the coarse-grained DF satisfies the Vlasov equation $D\mathcal{F}/Dt = 0$ so that the Casimirs calculated with the coarse-grained DF are now conserved $\mathcal{H}[\mathcal{F}] = 0$, as well as the mass and the energy. Indeed, after mixing the coarse-grained field must be viewed as a new fine-grained field in a possibly further evolution. Now, the phenomenology of violent relaxation (158) suggests that $\mathcal{F}(\mathbf{r}, \mathbf{v}, t)$ will be brought to a maximum $\mathcal{F}_* (\mathbf{r}, \mathbf{v})$ of a certain $H$-function $H^*$. This is proven by the theory of Lynden-Bell in case of efficient mixing (up to the problem of ensemble inequivalence) and this may remain true even in case of incomplete relaxation. Since $\mathcal{H}[\mathcal{F}]$, $E[\mathcal{F}]$ and $M[\mathcal{F}]$ are conserved by the Vlasov equation after mixing, and since $\mathcal{F}_*(\mathbf{r}, \mathbf{v})$ is a maximum of $H^*[\mathcal{F}]$ at fixed mass and energy, then it is a formally nonlinearly dynamically stable steady state of the Vlasov equation according to the criterion (136).

Let us make a final comment. The maximization (under appropriate constraints) of the Lynden-Bell entropy [79], of the Tsallis entropy [121], of an $H$-function [128] or
of a convex Casimir \([135]\) leads to a distribution function of the form \(\mathcal{f} = \mathcal{f}(\varepsilon)\) with \(\mathcal{f}'(\varepsilon) < 0\) depending only on the energy. In astrophysics, these DF can only describe spherical stellar systems (and even a sub-class of them) \([54]\). In reality, stellar systems are not spherical and their distribution functions are not function of the energy alone \([13]\). Indeed, according to the Jeans theorem \([54]\), there exists more general stationary solutions of the Vlasov equation which depend on other integrals of motion. This indicates that the structure of the final state of a collisionless stellar system depends on its dynamical evolution in a complicated manner. An important problem in astrophysics is therefore to find the form of distribution function appropriate to real galaxies. Simple concepts based on entropies, \(H\)-functions or formal nonlinear dynamical stability are not sufficient to understand the structure of real galaxies. This is particularly deceptive. However, conceptually, the theory of violent relaxation is important to explain how a collisionless stellar system reaches a steady state (QSS). This is due to phase mixing in phase space. The coarse-grained DF \(\mathcal{f}(r,v,t)\) reaches a steady state \(\mathcal{f}_{QSS}(r,v)\) in a few dynamical times while the fine-grained distribution function \(f(r,v,t)\) develops filaments at smaller and smaller scales and is never steady (presumably). Since this mixing process is very complex, the resulting structure \(\mathcal{f}_{QSS}(r,v)\) should be extremely robust and should be therefore a nonlinearly dynamically stable stationary solution of the Vlasov equation. Thus, the theory of incomplete violent relaxation explains how collisionless stellar systems can be trapped in nonlinearly dynamically stable (robust) stationary solutions of the Vlasov equation on the coarse-grained scale.

13. SUMMARY: THE DIFFERENT FUNCTIONALS

We would like to again emphasize the distinction between entropies and \(H\)-functions \([36]\). An entropy is a quantity which is proportional to the logarithm of the disorder, where the disorder is equal to the number of microstates consistent with a given macrostate. This is how the Lynden-Bell entropy \([79]\) has been defined. Tsallis entropy \([121]\) could be considered as a generalization of this definition in the case where the phase-space has a complex structure so that the evolution is non-ergodic and the potentially accessible microstates are not equiprobable. In each case, the entropy is a functional of the probability \(\rho(r,v,\eta)\) and the maximization of these entropies at fixed mass, energy and Casimirs is a condition of thermodynamical stability. The “generalized entropies” \([114]-[117]\) defined in Sec. 5 can be regarded as entropies which are proportional to the logarithm of the number of microstates consistent both with a given macrostate and with the constraints imposed by the Vlasov equation (Casimirs). Their functional form depends on the initial condition. They are defined on a projection space (\(\mathcal{f}\)-space) where a macrostate is defined by the specification of \(\mathcal{f}(r,v)\) instead of \(\rho(r,v,\eta)\). On the other hand, the \(H\)-functions do not have a statistical origin. They are just arbitrary functionals of the coarse-grained distribution \(\mathcal{f}(r,v,t)\) of the form \([128]\).

\[^{13}\text{This shows that the phenomenological principle} \([155]\) \text{is not always valid. Indeed, real (non-spherical) galaxies do not maximize an} H \text{ function at fixed mass and energy although they probably result from a process of violent relaxation.}\]
They increase during violent relaxation and they are useful to characterize the degree of mixing of a collisionless system [40]. According to the phenomenological selective decay principle (158), the QSS reached after violent relaxation is expected to maximize a certain $H$-function (non-universal) at fixed mass and energy. According to criterion (136), this principle ensures that the QSS is non-linearly dynamically stable with respect to the Vlasov equation. This is to be expected since the QSS results from a turbulent mixing which makes it very robust. In case of complete violent relaxation, the selective decay principle (158) is justified by the statistical theory of Lynden-Bell and $H^* = S$ where $S$ is given by Eqs. (114)-(117). In case of incomplete relaxation, the $H$-function $H^*$ and the distribution function $f_{QSS}$ of the QSS are different from the Lynden-Bell prediction. In some cases of incomplete relaxation [77, 76, 17, 4], the QSS is close to a Tsallis distribution and $H^*$ is close to the Tsallis functional, but this is not universally true.

Examples: Tsallis functional $S_q[\rho]$ expressed in terms of $\rho(r, v, \eta)$ is a generalized entropy. Tsallis functional $S_q[f]$ expressed in terms of $f(r, v)$ is (i) a generalized $H$-function (ii) a particular case of generalized entropy $S_q[\rho]$ for two levels $f \in \{0, \eta_0\}$ in the dilute limit $\bar{f} \ll \eta_0$. Tsallis functional $S_q[f]$ expressed in terms of $f(r, v)$ is a particular convex Casimir whose maximization at fixed mass and energy provides a condition of formal nonlinear dynamical stability for the Vlasov equation (this leads to stellar polytropes in astrophysics).

14. CONCLUSION

In this paper, we have studied the dynamics and thermodynamics of systems with weak long-range interactions. In particular, we have stressed the physical interpretation of the different functionals appearing in the analysis. In our discussion, we have exclusively considered the case of isolated Hamiltonian systems described by the microcanonical ensemble. In this concluding section, we would like to briefly discuss the physics of related systems.

It is interesting to compare the microcanonical evolution of Hamiltonian systems with long-range interactions to the canonical evolution of Brownian systems with long-range interactions [2]. In the case of Brownian particles in interaction, the microscopic equations of motion consist in $N$ coupled Langevin equations including a friction force and a stochastic force in addition to the long-range coupling. These additional terms can take into account short-range interactions with an external medium playing the role of a thermal bath. These Brownian systems are dissipative and they are described by the canonical ensemble. In the thermodynamic limit $N \to +\infty$ defined in Sec. 2.2., the mean field approximation is exact and the evolution of the distribution function $f(r, v, t)$ of the Brownian system is governed by the mean field Kramers equation (decreasing monotonically the Boltzmann free energy) [3]. This is the canonical counterpart of the Vlasov equation for Hamiltonian systems. This shows that, for $N \to +\infty$, the Brownian system tends to relax towards the mean field Boltzmann distribution on a typical timescale $\xi^{-1}$ where $\xi$ is the friction coefficient. In the strong friction limit $\xi \to +\infty$, the velocity distribution is close to Maxwellian and the evolution of the spatial density $\rho(r, t)$ is governed by the mean field Smoluchowski equation [3]. So far, two types of Brownian systems
with weak long-range interactions have been studied: (i) self-gravitating Brownian particles (or bacterial populations) where the particles interact via a Newtonian potential [92] (ii) the Brownian Mean Field (BMF) model where the particles interact via a cosine potential [53, 93].

The previously described Hamiltonian and Brownian systems [2] are “simple” systems (this does not mean that their study is trivial!) because everything is contained in the \( N \)-body Hamiltonian or Langevin equations. Recently, some authors have considered the case of “complex” systems where the microscopic dynamics is not perfectly known or can be biased with respect to the ordinary one because of some microscopic constraints whose influence is difficult to formalize. It has been proposed that such systems could be described by an effective generalized thermodynamics (E.G.T.). The Tsallis [37] entropy has become very popular to describe complex systems, but other forms of generalized entropies can also be considered [65, 94]. In that context, generalized kinetic equations, associated with generalized forms of entropic functionals, have been introduced either in the microcanonical ensemble or in the canonical ensemble. In the first case, they have the form of generalized Boltzmann or Landau equations [95, 96] and they conserve the energy and the mass. In the second case, they have the form of generalized Fokker-Planck equations, like the generalized Kramers and Smoluchowski equations [95, 97, 94]. If the particles interact via weak long-range forces, we obtain generalized mean field kinetic equations [65, 94]. The general study of these equations, which combine both long-range interactions and generalized thermodynamics, is very rich and can lead to a wide diversity of phase transitions and blow-up phenomena. For example, generalized mean field Fokker-Planck equations appear in the physics of bacterial populations driven by chemotaxis (Keller-Segel models) [98, 94]. More generally, long-range interactions and generalized thermodynamics can have applications in several domains of physics, astrophysics and biology [99].

**APPENDIX: SECOND VARIATIONS AND LOCAL STABILITY EQUIVALENCE**

The results of Sec. 5 show that the variational problems (107) and (118) are equivalent for global maximization. In this Appendix, adapting the procedure developed by Bouchet [66] for the 2D Euler equation, we show that they are also equivalent for local maximization. A critical point of (107) is a local maximum of \( S_\chi[\rho] \) at fixed mass, energy and normalization iff

\[
\delta^2 F[\delta \rho] = T \int \frac{(\delta \rho)^2}{2\rho} \, d\rho \, d\nu \, d\eta + \frac{1}{2} \int \delta \chi \delta \Phi \, d\rho \, d\nu \geq 0,
\]

for all perturbations \( \delta \rho \) that conserve mass, energy and normalization at first order. On the other hand, a critical point of (118) is a local maximum of \( S[f] \) at fixed mass and energy iff

\[
\delta^2 F[f] = \frac{1}{2} T \int C''(f) (\delta f)^2 \, d\rho \, d\nu + \frac{1}{2} \int \delta f \delta \Phi \, d\rho \, d\nu \geq 0,
\]
for all perturbations \( \delta \overline{f} \) that conserve mass and energy at first order. To make the connection between the second order variations (159) and (160), the idea is to project the perturbation \( \delta \rho \) on a suitable space (to be defined) and write \( \delta \rho = \delta \rho_\parallel + \delta \rho_\perp \) where \( \delta \rho_\perp \) is the orthogonal perturbation [66]. The perturbation \( \delta \rho \) must satisfy \( \int \delta \rho d\eta = 0 \) and \( \int \delta \rho \eta d\eta = \overline{f} \). We impose the same constraints on \( \delta \rho_\parallel \), i.e. \( \int \delta \rho_\parallel d\eta = 0 \) and \( \int \delta \rho_\parallel \eta d\eta = \overline{f} \) (we shall see that this leads naturally to the orthogonality of \( \delta \rho_\parallel \) and \( \delta \rho_\perp \)). Then, writing \( \delta \rho_\parallel = \overline{f} g \rho \) where \( g \) is a function to be determined, we must have \( \int \rho g d\eta = 0 \) and \( \int \rho \eta g d\eta = 1 \). We look for a solution of the form \( g = a + b\eta \) and find that \( a + b\overline{f} = 0 \) and \( a\overline{f} + b\overline{f}^2 = 1 \) leading to \( a = -\overline{f}/f_2 \) and \( b = 1/f_2 \) where \( f_2 \) is the centered local variance defined in Eq. (87). Therefore, we write

\[
\delta \rho = \frac{\overline{f}}{f_2} (\eta - \overline{f}) \rho + \delta \rho_\perp,
\]

(161)

where \( \delta \rho_\perp \) ensures that all the perturbations are considered. By construction, we have \( \int \delta \rho_\perp d\eta = 0 \) and \( \int \delta \rho_\perp \eta d\eta = 0 \). Therefore, \( \delta \rho_\perp \) is orthogonal to \( \delta \rho_\parallel \) in the sense that

\[
\int \frac{\delta \rho_\parallel}{\rho} \delta \rho_\perp d\eta = 0.
\]

(162)

Then, we readily obtain

\[
\int \frac{(\delta \rho)^2}{\rho} d\eta = \int \frac{(\delta \rho_\perp)^2}{\rho} d\eta + \int \frac{(\delta \rho_\parallel)^2}{\rho} d\eta = \int \frac{(\delta \rho_\perp)^2}{\rho} d\eta + \frac{(\overline{f})^2}{f_2} = \int \frac{(\delta \rho_\perp)^2}{\rho} d\eta + C''(\overline{f})(\delta \overline{f})^2,
\]

(163)

where we have used identity (103). Finally, the second variations (159) can be rewritten

\[
\delta^2 F[\delta \rho] = T \int \frac{(\delta \rho_\perp)^2}{2\rho} d\mathbf{r} d\mathbf{v} d\eta + \delta^2 F[\delta \overline{f}].
\]

(164)

If \( \delta^2 F[\delta \overline{f}] \geq 0 \) for all perturbations \( \delta \overline{f} \) that conserve mass and energy at first order, then \( \delta^2 F[\delta \rho] \geq 0 \) for all perturbations \( \delta \rho \) that conserve mass, energy and normalization at first order. Alternatively, if there exists a perturbation \( \delta \overline{f}_* \) such that \( \delta^2 F[\delta \overline{f}_*] < 0 \), by taking \( \delta \rho_* \) in the form (161) with \( \delta \overline{f} = \delta \overline{f}_* \) and \( \delta \rho_\perp = 0 \), we get \( \delta^2 F[\delta \rho_*] < 0 \). We conclude that \( \rho(\mathbf{r}, \mathbf{v}, \eta) \) is a local maximum of \( S_\chi[\rho] \) at fixed \( E, M \) and normalization iff \( \overline{f}(\mathbf{r}, \mathbf{v}, \eta) \) is a local maximum of \( S[\overline{f}] \) at fixed \( E, M \). Thus: (107) \( \leftrightarrow \) (118) locally.

We now adapt the same procedure to show that the variational problems (137) and (153) are equivalent for local minimization (we have already shown in Sec. 11 that they are equivalent for global minimization). A critical point of (137) is a local minimum of \( F[f] \) at fixed mass iff

\[
\delta^2 F[f] = \frac{1}{2} T \int C''(f)(\delta f)^2 d\mathbf{r} d\mathbf{v} + \frac{1}{2} \int \delta \rho \delta \Phi d\mathbf{r} \geq 0,
\]

(165)
for all perturbations $\delta f$ that conserve mass at first order. On the other hand, a critical point of \((153)\) is a local minimum of $F[\rho]$ at fixed mass iff
\[
\delta^2 F[\rho] = \int \frac{p'(\rho)}{2 \rho} (\delta \rho)^2 d\mathbf{r} + \frac{1}{2} \int \delta \rho \delta \Phi d\mathbf{r} \geq 0, \tag{166}
\]
for all perturbations $\delta \rho$ that conserve mass. We can always write the perturbation in the form
\[
\delta f = \frac{\delta \rho}{\int \frac{dx}{C''(f)}} C''(f) + \delta f_{\perp}, \tag{167}
\]
where $\delta f_{\perp}$ ensures that all the perturbations are considered. By construction, we have $\int \delta f d\mathbf{v} = \int \delta f_{\parallel} d\mathbf{v} = \delta \rho$ so that $\int \delta f_{\perp} d\mathbf{v} = 0$. Therefore, $\delta f_{\perp}$ is orthogonal to $\delta f_{\parallel}$ in the sense that
\[
\int C''(f) \delta f_{\parallel} \delta f_{\perp} d\mathbf{v} \propto \int \delta f_{\perp} d\mathbf{v} = 0. \tag{168}
\]
Then, we readily obtain
\[
\int C''(f)(\delta f)^2 d\mathbf{v} = \int C''(f)(\delta f_{\parallel})^2 d\mathbf{v} + \int C''(f)(\delta f_{\perp})^2 d\mathbf{v} + \frac{1}{\int \frac{dx}{C''(f)}} \frac{(\delta \rho)^2}{\int \frac{dx}{C''(f)}}. \tag{169}
\]
Now, a critical point of \((137)\) is of the form $f = F(\beta \varepsilon + \alpha) = f(\varepsilon)$ with $\varepsilon = \nu^2 / 2 + \Phi(\mathbf{r})$. If we introduce the density $\rho = \int f d\mathbf{v}$ and the pressure $p = \frac{1}{d} \int f v^2 d\mathbf{v}$, we find that $\rho = \rho(\Phi)$ and $p = p(\Phi)$. Eliminating $\Phi(\mathbf{r})$ between these relations, we obtain a barotropic equation of state $p = p(\rho)$, the same as in Sec. 11. We easily establish that
\[
\nabla p = \frac{1}{d} \int \frac{df}{d\mathbf{r}} v^2 d\mathbf{v} = \frac{1}{d} \nabla \Phi \int f'(\varepsilon) v^2 d\mathbf{v} = \frac{1}{d} \nabla \Phi \int \frac{df}{d\varepsilon} \cdot v d\mathbf{v} = -\nabla \Phi \int f d\mathbf{v} = -\rho \nabla \Phi, \tag{170}
\]
so that the condition $f = f(\varepsilon)$ is equivalent to the condition of hydrostatic equilibrium $\nabla p = -\rho \nabla \Phi$. This implies $p'(\rho) = -\rho(\Phi)$ and
\[
\frac{p'(\rho)}{\rho} = -\frac{1}{\rho'(\Phi)} = -\frac{1}{\int f'(\varepsilon) d\mathbf{v}} = \frac{T}{\int \frac{dv}{C''(f)}}, \tag{171}
\]
where we have used the identity $C''(f) = -\beta / f'(\varepsilon)$ resulting from $C'(f) = -\beta \varepsilon - \alpha$. Combining Eqs. \((169)\) and \((171)\) we finally obtain
\[
\delta^2 F[\delta f] = \frac{1}{2} T \int C''(f)(\delta f_{\perp})^2 d\mathbf{v} + \delta^2 F[\delta \rho]. \tag{172}
\]
From this equality, we conclude that $f(\mathbf{r}, \mathbf{v})$ is a local minimum of $F[f]$ at fixed mass iff $\rho(\mathbf{r})$ is a local minimum of $F[\rho]$ at fixed mass. Thus: \((137) \iff (153)\) locally.
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