Quartet-metric/multi-component gravity: scalar graviton as emergent dark substance

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Abstract. In the general frameworks of an earlier introduced quartet-metric/multi-component gravity, a theory of a massive scalar graviton supplementing the massless tensor one is consistently deduced. The peculiarities of the scalar-graviton field compared to the canonical scalar one are demonstrated. The (ultra-)light scalar graviton is treated as an emergent dark substance of the Universe: dark matter and/or dark energy depending on the solution. The case with scalar graviton as dark energy responsible for the late-time accelerated expansion of the Universe is studied in more detail. In particular, it is shown that due to an attractor solution for the light scalar graviton there naturally emerges at the classical level a tiny nonzero effective cosmological constant, even in the absence of the Lagrangian one. The prospects of going beyond LCDM model per scalar graviton are shortly indicated.

Keywords: modified gravity, dark energy theory, dark matter theory

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1 Introduction

The present-day scenario for the evolution of the Universe is described by the so-called Cosmological Standard Model or, more particularly, the ΛCDM model. The latter incorporates, in accord with its name, such new ingredients as the cosmological constant (CC) Λ and a (cold) dark matter (DM) of an unknown nature. Being extremely economic in the theoretical concepts the model shows an impressive phenomenological success in describing the observational data. Nevertheless, some arguments (though mainly of the theoretical nature) concerning CC may imply the necessity of going eventually beyond such a model, in particular, through a hypothetical dark energy (DE) superseding CC.\textsuperscript{1} With the ΛCDM model being based on General Relativity (GR) as a working tool, going beyond the model may, in particular, imply going beyond GR\textsuperscript{2} in looking for explanation of DM and DE.

In this vein, in refs. [6, 7] there was proposed an effective field theory (EFT) of the quartet-metric/multi-component gravity. The theory is based on the two physical concepts. (i) There exist in spacetime some distinct dynamical coordinates, given by a scalar quartet, playing the role of the Higgs-like fields for gravity. The number of the original gravitational components increases thus to fourteen: ten for metric and four for scalars (in the four spacetime dimensions). (ii) The diffeomorphism invariance of the quartet-metric GR gets (partially) spontaneously broken/hidden, with the gauge components contained in metric

\textsuperscript{1} For ΛCDM and beyond, see, e.g., [1, 2].
\textsuperscript{2} For the modified and extended theories of gravity beyond GR, see, e.g., [3]–[5].
becoming physical through absorbing the scalar quartet. In a general case, such a multi-
component theory describes in a fully dynamical and generally-covariant (GC) fashion the
(massive) scalar-vector-tensor gravity, with the additional physical gravity components a pri-
ori serving as DM and/or DE (and, conceivably, beyond) depending on the solution. It was
argued that the mere admixture to metric of the scalar quartet may result in the wide vari-
ety of the particular versions of the theory, with an extremely rich spectrum of the emergent
physical phenomena beyond GR. To systematically study the latter ones, with the aim of
picking-out the most relevant version of the theory (if any), presents a big challenge. Though,
in the spirit of the so-called Occam’s razor, the most likely version of the multi-component
gravity to serve as the next-to-GR one may be given by that just with the massive scalar
graviton supplementing the conventional (massless) tensor one of GR. So, in the present
paper we systematically adhere to such a line of reasoning.\(^3\)

In section 2, there is presented the genesis of a consistent scalar-graviton theory star-
ning from the first principles of the more general multi-component gravity. This allows to
clarify the nature of the (otherwise ad hoc) scalar graviton, as well as to open prospects
for its possible future modifications and generalizations. To this end, the multi-component
gravity is concisely exposed, with its scalar-graviton reduction consistently deduced. The
relation of the latter with and distinction from the Horndeski scalar-tensor theory [8, 9] is
exposed. In section 3, the scalar graviton is worked-out as a dark substance, with a set of
simplifications step-by-step imposed, and the peculiarities of the scalar graviton compared to
a canonical scalar field are demonstrated. In particular, the attributes of the scalar-graviton
solutions required for the scalar graviton to serve as DM or DE are shown. In section 4,
the scalar-graviton field is applied as DE filling-up homogeneously the whole Universe on the
late-time stage of its expansion. In particular, a mechanism of producing the tiny nonzero
effective CC through an attractor solution for the (ultra-)light scalar graviton is put forward.
In Conclusion, the necessity of further studying the scalar graviton to validate it as an
emergent dark substance of the Universe is stressed.

2 Multi-component gravity and scalar graviton

2.1 Multi-component gravity: generalities

Let us start with a concise exposition of EFT of the quartet-metric/multi-component grav-
ity [6, 7], the latter reducing in a limit to the metric GR. Such a theory is generically given
by a GC action

\[
I = \int L_G(g_{\mu\nu}, g, \partial_\mu \omega^a, \eta_{ab}) d^4x, \tag{2.1}
\]

with a Lagrangian scalar density \(L_G\) for the extended gravity dependent on the metric \(g_{\mu\nu}\)
\((g \equiv \det(g_{\mu\nu}) < 0)\) and a quartet of the scalar fields \(\omega^a, a = 0, \ldots, 3\). At that, \(a, b, \ldots\) are
the indices of the global Lorentz symmetry \(SO(1,3)\), with the invariant Minkowski symbol
\(\eta_{ab}\). By default, the signatures of \(g_{\mu\nu}\) and \(\eta_{ab}\) are chosen to coincide. The scalar fields \(\omega^a\) are
defined up to the (patch-wise) global Poincare transformations (independent of the space-
time) composed of the Lorentz ones and shifts \(\omega^a \rightarrow \omega^a + c^a\), with the arbitrary constant
parameters \(c^a\). The quartet \(\omega^a\) defines the (patch-wise) invertible coordinate transformations

\(^3\)Of course, this by no means deprives other GR modifications conceivable within the multi-component
gravity, such as, say, the pure-tensor gravity with the massive (tensor) graviton possessing, in particular, a
modified kinetic term, etc, [6, 7].
in spacetime from the arbitrary observer’s coordinates $x^\mu$ to some distinct dynamical world coordinates $\hat{x}^a$, the so-called, quasi-affine ones: $\hat{x}^a = \omega^a(x)$ (with the inverse $x^\mu = x^\mu(\hat{x})$). Physically, such coordinates may be considered as comoving with the vacuum treated ultimately as a dynamical system on par with the observable world. Due to GC and the global Poincare invariance, $\omega^a$ enters, in fact, through an auxiliary quasi-Lorentz metric

$$\omega_{\mu\nu} \equiv \partial_\mu \omega^a \partial_\nu \omega^b \eta_{ab}, \quad (2.2)$$

with

$$\omega \equiv \det(\omega_{\mu\nu}) = \det(\partial_\mu \omega^a)^2 \det(\eta_{ab}) < 0. \quad (2.3)$$

To ensure the (patch-wise) invertibility of the spacetime coordinate transformations $\hat{x}^a = \omega^a(x)$ one should have the Jacobian $\det(\partial_\mu \omega^a) \neq 0$ and thus $\omega \neq 0$, implying the non-degeneracy of the quasi-Lorentz metric $\omega_{\mu\nu}$, with an inverse $\omega^{-1}_{\mu\nu}$. \(^4\) In view of $\omega \neq 0$, the sign of $\sqrt{-\omega}$ is preserved and we can choose $\sqrt{-\omega} > 0$. In these terms, the Lagrangian of the multi-component gravity may most generally be rewritten in an entirely spacetime form as

$$L_G = L_G(g_{\mu\nu}, \omega_{\mu\nu}, g, \omega). \quad (2.4)$$

In particular, the kinetic terms beyond GR enter through the GC tensor given by the difference of the two Christoffel connections:

$$B^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}(g_{\mu\nu}) - \gamma^\lambda_{\mu\nu}(\omega_{\mu\nu}), \quad (2.5)$$

including, in particular,

$$B^\lambda_{\mu\lambda} = \partial_\mu \ln \sqrt{-g}/\sqrt{-\omega} \quad (2.6)$$

for the kinetic term of the scalar graviton. The potential terms enter through the GC scalars built from the powers of the GC tensor

$$\Omega^\mu_\nu = \omega_{\mu\lambda} g^{\lambda\nu} \quad (2.7)$$

and its inverse $\Omega^{-1}_\nu^\mu = g_{\rho\kappa} \omega^{-1}_{\mu\kappa}$, as well as $\det(\Omega^\mu_\nu) = \omega/g$. At that, the true dynamical variables remain still the metric $g_{\mu\nu}$ and the scalar quartet $\omega^a$. Ultimately, the latter serves as a Higgs-like field for gravity (partially) breaking the diffeomorphism symmetry of the quartet-metric/multi-component gravity and making the gauge components of $g_{\mu\nu}$ physical. Technically, the quasi-affine coordinates $\hat{x}^a$ are distinct by the fact that under using them the quasi-Lorentz metric gets Minkowskian form, $\omega_{ab}(\hat{x}) \equiv \eta_{ab}$ (respectively, $\omega^{-1}_{ab}(\hat{x}) \equiv \eta^{ab}$). As a result, the auxiliary affine connection $\gamma^\lambda_{\mu\nu}$ becomes in these coordinates zero, $\gamma^\lambda_{ab}(\hat{x}) = 0$. Yet, the Christoffel connection corresponding to the spacetime metric $g_{\mu\nu}$ in these coordinates, $\Gamma^\lambda_{\mu\nu}(\hat{x})$, remains, generally, nonzero.

The Lagrangian density $L_G$ may further be decomposed as

$$L_G = L_G(g_{\mu\nu}, \omega_{\mu\nu}, g/\omega) \mathcal{M}(g, \omega), \quad (2.8)$$

with $L_G$ a GC scalar Lagrangian and $\mathcal{M}$ a GC scalar density of the proper weight (a spacetime measure entering the spacetime volume element $dV = M d^4 x$). The measure may generally be chosen as

$$\mathcal{M} = \varphi(g/\omega) \sqrt{-g} \equiv \varphi(g/\omega)(g/\omega)^{1/2} \sqrt{-\omega}, \quad (2.9)$$

\(^4\) Not to mix $\omega^{-1}_{\mu\nu}$ with $\omega^{\mu\nu} \equiv g^{\mu\kappa} g^{\nu\lambda} \omega_{\kappa\lambda}$. 


with \( \varphi(g/\omega) \) an arbitrary function of the scalar \( g/\omega \). With the proper redefinition of \( L_G \), the measure may equivalently be chosen either as \( \sqrt{-g} \) or \( \sqrt{-\omega} \), depending on the context. So, prior fixing the Lagrangian we can without loss of generality put

\[
I = \int L_G(g_{\mu\nu}, \omega_{\mu\nu}, g/\omega) \sqrt{-g} \, d^4x,
\]

(2.10)

An \( L_G \) quadratic in the first derivatives of metric is considered in [6]. Generally, such an \( L_G \) describes the massive scalar (s), vector (v) and tensor (g) gravitons contained in the metric, with \( \omega^a \) serving ultimately as a gravity counterpart of the Higgs fields. Imposing on the parameters of \( L_G \) some “natural” (in a technical sense) restrictions, one can exclude in the linear approximation the vector graviton as the most “suspicious” theoretically and phenomenologically, leaving in this approximation in addition to the tensor graviton just the massive scalar one as the most “auspicious”. A more general multi-component gravity Lagrangian is discussed in [7]. Such a Lagrangian admits the two generic types of reductions significantly simplifying the theory: the scalar-graviton reduction and the second-derivative reduction. To result in as simple as possible version of the theory we impose step-by-step both types of reduction.

2.2 Scalar-graviton reduction

2.2.1 High-order derivatives

The multi-component gravity is significantly simplified (remaining still rather rich of the new content) under considering a reduced case given by the Lagrangian \( L_G \) dependent on \( \omega_{\mu\nu} \) exclusively through its determinant \( \omega \). In fact, due to GC \( \omega \) should enter the Lagrangian through the ratio \( \omega/g \). Without loss of generality this ratio may be substituted by

\[
\sigma \equiv \ln \sqrt{-g}/\sqrt{-\omega}.
\]

(2.11)

With \( \omega \) having the same weight as \( g \) under the general coordinate transformations, \( \sigma \) is a true GC scalar field normalized to zero in a flat limit (\( g = \omega = -1 \)). Stress that the scalar graviton \( \sigma \) has a combined nature, ultimately distinguishing \( \sigma \) from an elementary scalar field.

In the scalar-tensor gravity with a canonical scalar field, the most general Lagrangian of the arbitrary derivative order in the metric and scalar field in the four spacetime dimensions, resulting still in the second-order field equations (FEs) in both fields is given in [8, 9]. The most general Lagrangian with the derivable FEs which are quasi-linear in the second derivatives of both the metric and scalar field (in the sense that coefficients of the second derivatives contain no derivatives) is developed by Horndeski [9].

The respective Lagrangian for the scalar-reduced multi-component gravity

\[
L_{sg} = L_{sg}(g_{\mu\nu}, \sigma),
\]

(2.12)

may be obtained from the Horndeski one by imposing the restriction (2.11) on the scalar field or by adding to (2.12) a constraint Lagrangian, say, in the form

\[
L_\lambda = \lambda(e^{-\sigma} - \sqrt{-\omega}/\sqrt{-g}),
\]

(2.13)

This is to avoid potentially possible Ostrogradsky instabilities for the higher then second-order classical FEs, resulting in a theory with a ghost vacuum.

At that, all the terms present in the Horndeski theory may be shown to originate from the scalar-tensor terms having Galilean symmetry in the flat spacetime [10]. For a generalized Horndeski’s theory, see, [11].
with $\lambda$ an indefinite Lagrange multiplier. This is a sole but crucial difference compared to the original Horndeski scalar-tensor theory resulting in all the specifics of the scalar graviton. Eqs. (2.3) and (2.11), with $g_{\mu\nu}$ and $\omega^a$ as the independent gravity field variables, are the key ingredients of the dynamical theory of the scalar graviton. For completeness, the so obtained pure-gravity Lagrangian $L_{sg}$ is to be supplemented by a matter one $L_m$ dependent on some generic matter fields $\phi^I$ and, generally, on $\sigma$, too. The scalar-reduced multi-component gravity in terms of the Lagrangian (2.12) under constraint (2.11) or (2.13) may be proposed as the next-to-GR EFT of gravity describing the massless tensor graviton supplemented by the massive scalar one. The latter is assumed to serve as an emergent gravitational dark substance, in particular, DM and/or DE depending on the solution.

2.2.2 Second-order derivatives

Imposing additionally the second-derivative restriction we get the tensor-scalar gravity Lagrangian as

$$L = L_{sg} + L_m = L_g(\partial_{\lambda}g_{\mu\nu}, g_{\mu\nu}, \sigma) + L_s(\partial_{\lambda}\sigma, g_{\mu\nu}, \sigma) + L_m(\partial_{\lambda}\phi^I, \phi^I, g_{\mu\nu}, \sigma).$$

(2.14)

More particularly, we put in the second order

$$L_{sg} = \left[-\frac{1}{2} \kappa_g^2 \varphi_g(\sigma) R(g_{\mu\nu}) + \frac{1}{2} \kappa_s^2 \varphi_s(\sigma) g_{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V_s(\sigma)\right] \sqrt{-g},$$

(2.15)

were $R(g_{\mu\nu})$ is the Ricci scalar, $\varphi_g > 0$ and $\varphi_s > 0$ are some arbitrary scalar functions, and $V_s$ is a scalar-graviton potential, generally, including as a constant part a cosmological constant. The parameters $\kappa_g$ and $\kappa_s$ of the dimension of mass characterize the strength, respectively, of the tensor and scalar gravity, with $\kappa_g = \kappa_P \equiv 1/\sqrt{8\pi G_N}$ given by the reduced Planck mass. For the dominance of tensor gravity it is moreover assumed that $\kappa_s \ll \kappa_g$. The function $\varphi_g$ characterizes the type of modification of the tensor GR, while $\varphi_s$, in fact, corresponds to a redefinition of the scalar graviton compared to (2.11), the latter being taken by default as appearing naturally in the multi-component gravity.

Introducing the conformally rescaled metric $\hat{g}_{\mu\nu}$ through

$$\hat{g}_{\mu\nu} \equiv \varphi_g(\sigma) g_{\mu\nu},$$

(2.16)

with $(-\det(\hat{g}_{\mu\nu}))^{1/2} \equiv \sqrt{-\hat{g}} = \varphi_g^2 \sqrt{-g}$, we can present (2.15) equivalently as

$$L_{sg} = \left[-\frac{1}{2} \kappa_g^2 R(\hat{g}_{\mu\nu}) + \frac{1}{2} \kappa_s^2 \varphi_s(\sigma) \hat{g}^{-1}_{\mu\nu} \partial_\mu \hat{\sigma} \partial_\nu \hat{\sigma} - \hat{V}_s(\sigma)\right] \sqrt{-\hat{g}},$$

(2.17)

with $\hat{\varphi}_s$ and $\hat{V}_s$ properly redefined, and $\hat{g}^{-1}_{\mu\nu}$ being an inverse of $\hat{g}_{\mu\nu}$. In particular, putting $\varphi_g = e^{-\sigma/2}$ we get

$$\hat{g}_{\mu\nu} = (\omega/g)^{1/4} g_{\mu\nu},$$

(2.18)

implying $\hat{g} = \omega$. This case, supplemented by a properly modified matter Lagrangian, $\hat{L}_m(\phi^I, \hat{g}_{\mu\nu}, \sigma)$, may be referred to as the quasi-Weyl transverse gravity (WTDiff).\footnote{For WTDiff, see, e.g., [12]-[16]. Stress that in distinction with WTDiff, $\omega$ in qWTDiff is a dynamical variable, $\omega = \omega(\omega^a)$. Clearly, under neglecting by the explicit dependence on $\sigma$, qWTDiff reduces to GR with the redefined metric $\hat{g}_{\mu\nu}$. For matching with WTDiff (supplemented by a scalar graviton), $\omega$ in qWTDiff should effectively be “frozen” to an auxiliary non-dynamical/“absolute” scalar density by dropping-off the variation of the action with respect to $\omega^a$. This, in fact, means abandoning a proper FE (cf., section 3).}
As a paradigm, we choose in what follows $\varphi_g = \varphi_s = 1$ considering the scalar-graviton modification of GR with the canonical $\sigma$. Restricting ourselves by $L_s$ at energies less than $\kappa_s$, we retain only the leading term in the derivatives of $\sigma$. On the other hand, the scalar potential $V_s$ is still allowed to be an arbitrary function of $\sigma$. Put $V_s(\sigma) \equiv V_s|_{\min} + \Delta V_s(\sigma)$, where $V_s|_{\min} \equiv \kappa_s^2 \Lambda$, with $\Lambda \geq 0$ being a counterpart of the cosmological constant, and $\Delta V_s \geq 0$. Under $\Delta V_s \equiv 0$ the Lagrangian (2.17) gets at $\varphi_g = \varphi_s = 1$ moreover invariant under the global shifts $\sigma \rightarrow \sigma + \sigma_0$, with arbitrary constant $\sigma_0$. Thus, $\Delta V_s = 0$, as extending the symmetry of the Lagrangian, is natural in a technical sense, justifying the relative lightness of the scalar graviton. But this does not concern the constant part $V_s|_{\min}$ which requires additional arguments in the favor of its absence/smallness.

3 Scalar graviton as dark substance

3.1 General case

Varying the Lagrangian density with respect to $g_{\mu\nu}$, $\omega^\alpha$ and the generic matter fields $\phi^I$, and using, in particular, the relations

$$
\delta \sigma = \delta \sqrt{-g}/\sqrt{-g} - \delta \sqrt{-\omega}/\sqrt{-\omega}, \\
\delta \sqrt{-g} = -(1/2)\sqrt{-g}g_{\kappa\lambda}\delta g^{\kappa\lambda}, \\
\delta \sqrt{-\omega} = (1/2)\sqrt{-\omega}\omega^{-1\lambda\kappa}\delta \omega_{\kappa\lambda}, \\
\delta \omega_{\kappa\lambda} = \eta_{ab}(\omega^a_\lambda \delta \omega^b_\kappa + \omega^a_\kappa \delta \omega^b_\lambda),
$$

(3.1)

where $\omega^{-1\lambda\kappa} = \omega^{-1}_a \omega^{-1}_b \eta^{ab}$ is an inverse of $\omega_{\kappa\lambda}$, and $\omega^{-1}_a = \partial x^a/\partial \omega^a$ is a tetrad\(^8\) inverse of $\omega^a_\lambda \equiv \partial \omega^a_\lambda/\partial \omega^a$, we get the system of the coupled FEs for the metric, scalar quartet and matter fields in the conventional notation, respectively, as

$$
G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{\kappa^2 g} (T_{s\mu\nu} + T_{m\mu\nu}), \\
\frac{\delta}{\delta \omega^a}(L_g + L_s + L_m) \equiv \nabla_{\lambda} \left( \left( \frac{\delta L_g}{\delta \sigma} + \frac{\delta L_m}{\delta \sigma} \right) \omega^{-1\alpha}_a \right) = 0, \\
\frac{\delta L_m}{\delta \sigma^I} = \frac{\partial L_m}{\partial \sigma^I} - \nabla^a \frac{\partial L_m}{\partial \nabla_a \sigma^I} = 0,
$$

(3.2)

where $\delta/\delta$ is a total variational derivative and $\partial/\partial$ a partial one. The first and the last FEs in (3.2) are clearly the counterparts of the tensor gravity and matter FEs in GR. The second FE for $\omega^a$ is a generalization of the scalar-field one. This follows from the fact that this equation embodies, in particular, the ordinary one for $\sigma$ in a marginal case with $\delta(L_s + L_m)/\delta \sigma = 0$. But the latter should not fulfill in a general case. The specific form of the second FE, which looks like a continuity condition, is due to $L_G$ being dependent only on the derivatives of $\omega^a$ and thus invariant under the global shifts $\omega^a \rightarrow \omega^a + c^a$, with any constant $c^a$.

In the end, this implies a GC conserved current density $J^\alpha_\mu = \sqrt{-g}(\delta L_s/\delta \sigma + \delta L_m/\delta \sigma)\omega^{-1\mu}_a$. The r.h.s. of the first of FEs (3.2) may be treated as the total energy-momentum tensor, with $T_{s\mu\nu}$ and $T_{m\mu\nu}$ the canonical energy-momentum tensors, respectively, for the scalar graviton, as a kind of dark substance, and the matter obtained by means of varying the Lagrangian $L_f$ of the respective fraction $f = (s, m)$ through $g_{\mu\nu}$ as follows:

$$
T_{f\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g}L_f)}{\delta g^{\mu\nu}}.
$$

(3.3)

\(^8\)Not to mix $\omega^{-1\lambda}_a$ with $\omega^a_\lambda \equiv g^{\mu\nu} \eta_{ab} \omega^b_\nu$. 

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– 6 –
By this token, we get

\[ T_{\mu\nu} = \kappa_s^2 \nabla_\mu \sigma \nabla_\nu \sigma - \frac{1}{2} \kappa_s^2 \nabla^\lambda \sigma \nabla_\lambda \sigma g_{\mu\nu} + U_s g_{\mu\nu}, \]

\[ T_{m\mu\nu} = 2 \frac{\partial L_m}{\partial g^{\mu\nu}} - \left( L_m + \frac{\delta L_m}{\delta \sigma} \right) g_{\mu\nu}, \]  

(3.4)

where \( U_s \) is a generalized potential

\[ U_s \equiv V_s + W_s, \]  

(3.5)

with

\[ W_s \equiv -\delta L_s/\delta \sigma = \kappa_s^2 \nabla^\lambda \nabla_\lambda \sigma + V'_s \]  

(3.6)

being the rescaled wave operator for the scalar field \( \kappa_s \sigma \), and the prime-sign meaning the derivative with respect to \( \sigma \). The reduced Bianchi identity, \( \nabla_\nu G^{\nu \mu} = 0 \), results in

\[ \nabla_\mu T^{\mu}_\nu \equiv \nabla_\mu (T^{\mu}_s + T^{\mu}_m) = 0 \]  

(3.7)

representing the covariant conservation/continuity of the total energy-momentum tensor of matter supplemented by the scalar graviton. More particularly, (3.7) proves to read

\[ \partial_\mu W_s + W_s \partial_\mu \sigma = -\nabla_\nu T^{\nu \mu}_m. \]  

(3.8)

This is a consistency condition for \( \sigma \) (supplementing its FE (3.2)) being, with account for (3.6), of the third order what, in a general case, may encounter some instabilities.

### 3.2 Special case

#### 3.2.1 Matter conservation

The general solution to (3.8) may be looked-for in the form \( W_s = W_0(x) e^{-\sigma} \), with \( \partial_\mu W_0 = -e^\sigma \nabla_\mu T^{\mu}_m \). A crucial simplification occurs in the case if \( L_m \) is independent of \( \sigma \), \( \delta L_m/\delta \sigma = 0 \), so that \( T_{m\mu\nu} \) is covariantly conserved/continuous per se, \( \nabla_\nu T^{\nu \mu}_m = 0 \). In this case (or under the absence of matter), (3.8) possesses the first integral (playing the role a global degree of freedom) reducing the order of the equation for \( W_s \) to the second one:

\[ W_s = \kappa_s^2 \nabla^\lambda \nabla_\lambda \sigma + V'_s = \kappa_g^2 \Lambda_0 e^{-\sigma}, \]  

(3.9)

where there is put \( W_0 = \kappa_g^2 \Lambda_0 \), with \( \Lambda_0 \) an integration constant. With \( U_s = V_s + W_s \) becoming now the bona fide effective scalar potential \( U_s \equiv V_s + \kappa_g^2 \Lambda_0 e^{-\sigma} \), the scalar field satisfies the canonical second-order FE:

\[ \kappa_s^2 \nabla^\lambda \nabla_\lambda \sigma + U'_s = 0, \]  

(3.10)

At that, the Ostrogradsky instability potentially possible for a solution of the third-order eq. (3.8) is explicitly eliminated. At last, accounting for (3.9) and (2.11) one can in this case present FE (3.2) for \( \omega^a \) at \( W_0 \neq 0 \) as follows:

\[ \sqrt{-g} \nabla_\lambda (e^{-\sigma} \omega^{-1\lambda}_a) = \partial_\lambda (\sqrt{-g} e^{-\sigma} \omega^{-1\lambda}_a) = \partial_\lambda (\sqrt{-g} \omega^{-1\lambda}_a) = 0, \]  

(3.11)

\(^9\)This may in reality be an oversimplification, with a direct correlation of the scalar graviton and matter becoming, conceivably, in some cases even crucial, e.g., in the case of the scalar-graviton DM halos of galaxies to match with the well-known Tully-Fisher law.
which proves to be independent of $g_{\mu\nu}$. To put it differently, this FE expresses the conservation of the GC current density $J_\lambda^a = \sqrt{-\omega} \omega^{-1} \omega^{-1}$. Now, having found from FEs the metric and $\sigma$, and extracting hereof $\omega = ge^{-2\sigma}$, one can find the proper $\omega^a$ up to a residual freedom consistent with the required $\omega$. Such an ambiguity is insignificant and may, in principle, be eliminated afterwards in a more complete theory.

### 3.2.2 Scalar graviton: dark matter vs. dark energy

Altogether, depending on $\Lambda_0$ there are conceivable three generic cases with the principally different behavior for the scalar graviton as an emergent dark substance.\(^{10}\)

(i) $\Lambda_0 < 0$ arbitrary (varying). This case may be argued to be associated with the stationary scalar-graviton field in the closed spatial regions corresponding to the galaxy DM halos.\(^{11}\)

(ii) $\Lambda_0 = 0$. This intermediate case corresponds to the scalar graviton as a canonical scalar field.\(^{12}\)

(iii) $\Lambda_0 > 0$ arbitrary (fixed). We associate this case below with the homogeneous scalar-graviton field as DE filling-up the Universe as a whole.\(^{13}\)

In what follows, we concentrate on the scalar-graviton DE alone and do not dwell into the specific nature of DM, in particular, is it (and how) associated with the scalar graviton or not.

### 4 Scalar graviton as dark energy

#### 4.1 General case

Now we apply the results above to the Universe as a whole. Assuming the latter to be homogeneous and isotropic choose conventionally the FRW metric given in the standard cosmological coordinates $x^\mu = (t, r, \theta, \varphi)$ by the line element

$$ds^2 = dt^2 - a^2 \left( \frac{1}{1 - Kr^2} dr^2 + r^2 d\Omega^2 \right).$$ (4.1)

Here $t$ is the standard cosmological time, $r$ the radial distance from an (arbitrary chosen) spatial origin, $a(t)$ a scale factor, $K = k/R_0^2$, with $R_0$ an arbitrary fixed unit of length, and $k = 0, \pm 1$ for the zero, positive and negative spatial curvature of the spatially flat, closed and open Universe, respectively. Let the Universe be filled-up with a continuous medium/matter (taken for simplicity to be of one kind) possessing the energy-momentum tensor

$$T^\mu_\nu \equiv \left( \rho_m + p_m \right) u^\mu u^\nu - p_m g^\mu_\nu,$$ (4.2)

\(^{10}\)Note that in the case of $L_m$ being dependent on $\sigma$ the value and sign of $\Lambda_0$ for the different solutions (treated in the different spacetime regions as approximations to an exact one) may be allowed to vary, in distinction with the parameters of $V_s$ fixed ab initio for all the solutions.

\(^{11}\)In favor of such a possibility there was argued in [17], though in the frameworks with a non-dynamical auxiliary scalar density $\omega$. For a dynamical $\omega$, this case in the context of DM remains to be investigated.

\(^{12}\)For a canonical (ultra-)light scalar field in the context of the galaxy DM halos, see, e.g. [18, 19].

\(^{13}\)It may thus be said that the term $W_s$ in the effective scalar potential, being completely ad hoc for a canonical scalar field and drastically influencing the manifestations of the latter, is a kind of the “Black Swan” for the scalar graviton. The appearance of $W_s$ may, in turn, be traced back to the combined (dependent, in particular, on metric) nature of the scalar graviton, $\sigma = \sigma(g/\omega)$. 
where \( \rho_m(t) \) and \( p_m(t) \) are, respectively, the medium energy density and pressure, and \( u^\mu \) \( (u^\lambda u_\lambda = 1) \) the medium comoving four-velocity, with \( u^\mu = (1, 0, 0, 0) \) in the standard cosmological coordinates. The same, with \( \rho_{DM} \) and \( p_{DM} \), is assumed for a conceivable DM. Additionally, these substances are assumed to be characterized by some, given ab initio, indices of state, \( w_m = \rho_m/p_m \) and \( w_{DM} = \rho_{DM}/p_{DM} \), respectively \((w_{DM} = 0 \text{ for a cold DM (CDM)})\). In the spirit of \( \Lambda \text{CDM} \), the total energy density and pressure of the Universe, \( \rho \) and \( p \), in the presence of the scalar graviton \( s \) are given by the sum of the four fractions:

\[
\begin{align*}
\rho &= \rho_m + \rho_{DM} + \rho_\Lambda + \rho_s \equiv \rho_M + \rho_s^\Lambda \\
p &= p_m + p_{DM} + p_\Lambda + p_s \equiv p_M + p_s^\Lambda,
\end{align*}
\]

(4.3)

with \( M = (m, DM) \) referring to the total matter, incorporating the ordinary one and DM, and the the Lagrangian CC \( \Lambda \) \((w_\Lambda = -1)\). The latter will, in a general case, be included into a constant part of a redefined scalar-graviton potential, with the superscript \( \Lambda \) being in what follows omitted. In these terms, the Friedman-Lemaître gravity equations for the evolution of the Universe look like:

\[
\begin{align*}
\ddot{a}/a &= -(\rho + 3p)/6\kappa_g^2, \\
H^2 + K/a^2 &= \rho/3\kappa_g^2,
\end{align*}
\]

(4.4)

with a dot meaning a time derivative, and the Hubble parameter \( H \equiv \dot{a}/a \) giving the relative expansion rate of the Universe.

At that, the homogeneous scalar-graviton field, \( \sigma(t) \), is treated as the omnipresent DE spilled all over the Universe. More particularly, one gets\(^{14}\)

\[
\begin{align*}
\rho_s &= \frac{1}{2}\kappa_s^2 \dot{\sigma}^2 + U_s, \\
p_s &= \frac{1}{2}\kappa_s^2 \dot{\sigma}^2 - U_s.
\end{align*}
\]

(4.5)

In the above, \( U_s = V_s + W_s \) is the effective scalar-graviton potential, with \( V_s \) the Lagrangian scalar potential, and the scalar wave operator \( W_s \) as follows:

\[
W_s \equiv -\delta L_s/\delta \sigma = \kappa_s^2 (\ddot{\sigma} + 3H \dot{\sigma}) + \partial V_s/\partial \sigma.
\]

(4.6)

These expressions are valid at any \( k \) and correspond to the scalar-graviton DE with the variable effective index of state \( w_s(\sigma) \equiv \rho_s/\rho_s \). Under weakly changing \( \sigma \), \( \sigma \simeq 0 \), \((\text{though, generally, } \dot{\sigma} \neq 0)\) one has \( w_s = -1 \) mimicking thus the \( \Lambda \)-term. The second time-derivative of \( \sigma \) generally appears in \( \rho_s \) and \( p_s \), even under the simplest Lagrangian \( L_{sg} \), through the off-shell contribution \( W_s \) due to the intrinsic dependence of \( \sigma \) on metric.

The evolution equations (4.4) are to be supplemented by the covariant conservation/continuity condition for the scalar graviton \( s \) and the total matter \( M \):

\[
\dot{W}_s + W_s \dot{\sigma} = -(\dot{\rho}_M + 3H(\rho_M + p_M)),
\]

(4.7)

which follows from the reduced Bianchi identify. Generally, this is the third-order equation for \( \sigma \) due to a correlation of the scalar-graviton field as DE and the total matter.

\(^{14}\)The role of \( u^\mu \) here plays \( n^\mu = \nabla^\mu \sigma/(\nabla^\lambda \nabla_\lambda \sigma)^{1/2} \), which at \( \sigma = \sigma(t) \) is \( n^\mu = (1, 0, 0, 0) \) in the standard cosmological coordinates.
The equations above derivable in the multi-component gravity under the quadratic-scalar reduction describe the looked-for scenario for the evolution of the homogeneous isotropic Universe filled-up with the ordinary matter, DM and the scalar gravitons. Having found hereof $a(t)$ and $\sigma(t)$, and thus (at $k = 0$) $\sqrt{-g} = a^3$ and $\sqrt{-\omega} = e^{-\sigma}a^3$, one can then get from (3.11) the inverse tetrad $\omega^{-1\lambda}_\alpha$: 

$$\omega^{-10}_0 \sim 1/\sqrt{-\omega}, \quad \omega^{-1l}_\alpha = \delta^l_\alpha,$$  

(4.8)

where $a = (0, \alpha)$, $\lambda = (0, l)$; $\alpha, l = 1, 2, 3$. Inverting (4.8), so that $\omega^0_0 \equiv \partial_0 \omega^0 \sim \sqrt{-\omega}$ and $\omega^\alpha_l \equiv \partial_l \omega^\alpha = \delta^\alpha_l$, and choosing properly the coefficients one gets 

$$\omega^0 = \int \omega^0_0 dt + c^0 = \int \sqrt{-\omega} dt + c^0, \quad \omega^\alpha = \delta^\alpha_l x^l + c^\alpha$$  

(4.9)

defined up to some integration constants $c^0$ and $c^\alpha$ (to be put for simplicity zero), and $\det(\partial \omega^\alpha_l) = \sqrt{-\omega}$, as it should be. As for the quasi-affine coordinates $\hat{x}^\alpha = (\hat{t}, \hat{x}^\alpha)$, eq. (4.9) determines the quasi-affine time $\hat{t} = \omega^0_0(t)$, related with the cosmological one through $d\hat{t} = \sqrt{-\omega} dt$, and the spatial quasi-affine coordinates $\hat{x}^\alpha$, coinciding (at $k = 0$) with the cosmological ones.

### 4.2 Special case

#### 4.2.1 Matter conservation

A significant simplification of the preceding consideration occurs if each of the matter components $M = (m, DM)$ is independent of $\sigma$, and thus covariantly conserved/continuous, with the r.h.s. of (4.7) being zero\(^{15}\)

$$\dot{\rho}_M + 3H(\rho_M + p_M) = 0.$$  

(4.10)

Then it follows as before that $W_s = \kappa^2 g \Lambda_0 e^{-\sigma}$, with $\Lambda_0$ an integration constant taken to be positive. eq. (4.6) then implies, in turn, that the scalar-graviton FE is determined at any $k$ by the effective potential $U_s = V_s + \kappa^2 g \Lambda_0 e^{-\sigma}$ as

$$\ddot{\sigma} + 3H \dot{\sigma} + \kappa_s^2 \partial U_s/\partial \sigma = 0.$$  

(4.11)

Let now $\bar{\sigma}$ be the position of the minimum of the effective potential, $\partial U_s/\partial \sigma|_{\bar{\sigma}} = 0$. Under neglecting by $\dot{\sigma}$ and $\ddot{\sigma}$ this FE reduces to $\partial U_s/\partial \sigma = 0$, meaning $\sigma$ to be restricted by $\bar{\sigma}$. By this token, designating $U_s|\bar{\sigma} \equiv \kappa^2 g \bar{\Lambda}_s$ and replacing in $\rho_s$ and $p_s$ the $\sigma$-dependent $U_s$ by the constant $U_s|\bar{\sigma}$, one arrives (assuming $w_{DM} = 0$) at the standard $\Lambda$CDM model, corresponding to the effective CC $\bar{\Lambda}_s$ with

$$\bar{\rho}_s = -\bar{p}_s = \kappa^2 g \bar{\Lambda}_s,$$  

(4.12)

reproducing thus in such an approximation the standard $\Lambda$CDM model.

#### 4.2.2 Scalar-graviton dominance

To proceed explicitly further, consider the evolution of the Universe after a long time of its preceding expansion. Adopt at such a late-time stage the overwhelming dominance of DE by putting\(^{16}\)

$$\rho_M = p_M = 0.$$  

(4.13)

---

\(^{15}\)Though, this may, generally, be an oversimplification, especially what concerns DM.

\(^{16}\)Qualitatively, this may be not unrealistic due to the ratio of the energy of the total matter ($M = (m, DM)$) to DE at the present epoch being $0.3 : 0.7$. 

---
With account for the second part of FEs (4.4), eq. (4.11) acquires at \( k = 0 \) an autonomous form as follows:

\[
\ddot{\sigma} + \sqrt{3} v_s \left( \frac{1}{2} \dot{\sigma}^2 + U_s / \kappa_s^2 \right)^{1/2} \dot{\sigma} + \kappa_s^{-2} \partial U_s / \partial \sigma = 0,
\]

(4.14)

where

\[
v_s \equiv \kappa_s / \kappa_g \quad (4.15)
\]

(supposedly, \( v_s \ll 1 \)). This is the master equation for the evolution of the Universe due to the pure scalar-graviton DE. In particular, \( \sigma \equiv \bar{\sigma} \) is the exact solution to the equation. Having found from (4.14) \( \sigma \) one can then find from the second part of the Friedman-Lemaître equations (4.4) at \( k = 0 \) the Hubble parameter

\[
H \equiv \dot{a} / a = \rho_s^{1/2} / \sqrt{3} \kappa_g \geq 0.
\]

(4.16)

and the respective scale factor

\[
a = a_0 \exp \frac{1}{\sqrt{3} \kappa_g} \int \rho_s^{1/2} dt = a_0 \exp \frac{v_s}{\sqrt{3}} \int \left( \frac{1}{2} \dot{\sigma}^2 + U_s / \kappa_s^2 \right)^{1/2} dt,
\]

(4.17)

with \( a_0 \) an integration constant, To envisage the behavior of \( H(t) \) note that combining (4.4) at \( k = 0 \) one can find

\[
\dot{H} = -\frac{1}{2} (\rho_s + p_s) = -\frac{1}{2} v_s^2 \sigma^2 \leq 0,
\]

(4.18)

independently of \( U_s \). This means, in particular, that \( H \) always monotonically decays approaching atop a constant value \( \bar{H} \geq 0 \). At that, the Hubble horizon \( H^{-1} \) monotonically expands to \( \bar{H}^{-1} \).

### 4.2.3 Effective cosmological constant

More specifically, let us put for the Lagrangian CC \( \Lambda = 0 \) and choose the scalar potential quadratic in \( \sigma \):

\[
V_s = \frac{1}{2} m_s^2 (\kappa_s \sigma)^2 \equiv \frac{1}{2} \kappa_s^2 \mu_s^2 \sigma^2,
\]

(4.19)

with \( m_s \) the scalar-graviton mass and \( \mu_s \equiv v_s m_s \) its reduced mass, so that

\[
U_s = \kappa_g^2 \left( \frac{1}{2} \mu_s^2 \sigma^2 + \Lambda_0 e^{-\sigma} \right),
\]

(4.20)

Rewrite the scalar-graviton FE in this case as follows:

\[
\sigma'' + \sqrt{3} v_s \left( \frac{1}{2} \sigma^2 + \frac{1}{2} \sigma^2 + \alpha e^{-\sigma} \right)^{1/2} \sigma' + \sigma - \alpha e^{-\sigma} = 0,
\]

(4.21)

where

\[
\alpha \equiv \Lambda_0 / \mu_s^2 = \Lambda_0 / v_s^2 m_s^2
\]

(4.22)

and \( \sigma' \equiv d\sigma / d\tau \), with \( \tau = m_s t \). In these terms, the expansion rate at \( k = 0 \) reads

\[
H / m_s = a' / a = (v_s / \sqrt{3}) \left( \sigma'^2 / 2 + \sigma^2 / 2 + \alpha e^{-\sigma} \right)^{1/2}.
\]

(4.23)

The evolution of the Universe entirely due the scalar-graviton DE is given by solutions to (4.21) and (4.23) presented below. The behavior of the looked-for solution is mainly
\( u_s = 0.1, \alpha = 1. \)

**Figure 1.** The phase plot \((\sigma, \sigma')\) describing the (clock-wise) evolution of the scalar-graviton DE (thick line) depending on \(\tau = m_s t\) at the representative values of parameters \(u_s = \kappa_s / \kappa_g = 0.1\) and \(\alpha \equiv \Lambda_0 / u_s^2 m_s^2 = 1\). The thin lines are the flow ones. The initial point \((-5, 0)\) corresponds to the initial time \(\tau_0 = 1\). Independent of the initial point, all the solutions approach at \(\tau \to \infty\) the same attractor point \((\bar{\sigma}, 0)\) corresponding to an emergent effective CC \(\Lambda_s\). A similar behavior takes place for other values of \(\alpha\).

Figure 1. The phase plot \((\sigma, \sigma')\) describing the (clock-wise) evolution of the scalar-graviton DE (thick line) depending on \(\tau = m_s t\) at the representative values of parameters \(u_s = \kappa_s / \kappa_g = 0.1\) and \(\alpha \equiv \Lambda_0 / u_s^2 m_s^2 = 1\). The thin lines are the flow ones. The initial point \((-5, 0)\) corresponds to the initial time \(\tau_0 = 1\). Independent of the initial point, all the solutions approach at \(\tau \to \infty\) the same attractor point \((\bar{\sigma}, 0)\) corresponding to an emergent effective CC \(\Lambda_s\). A similar behavior takes place for other values of \(\alpha\).

The behavior of the respective solution to FEs on the phase plot \((\sigma, \sigma')\) for the case \(\alpha = 1\), starting from a freely chosen initial point \((\sigma_0 = -5, \sigma'_0 = 0)\), is shown in figure 1. It is seen that the solution tends to a point \((\bar{\sigma}, 0)\) serving as an attractor. The solution winds clock-wise around the attractor approaching the latter asymptotically at \(\tau \to \infty\). A similar behavior can be shown to tale place at other values of \(\alpha\) considered previously. The value \(u_s = 0.1\) is chosen just for illustration purposes. For the smaller \(u_s\) the tightness of winding
increases (under unchanged $\alpha$), with the picture becoming less prominent. The behavior of $H/m_s$ vs. $\tau = m_s t$ extracted numerically from figure 1 and the similar ones for other $\alpha$, with the initial time $\tau_0 = 1$ attributed to the initial point $(\sigma_0 = -0.5; 0)$, is shown in figure 2. As an independent test, eq. (4.18) rewritten as

$$H'/m_s v_s^2 \sigma'^2 = -1/2$$

proves to be numerically valid up to a proper accuracy for the numerical $\sigma(\tau)$ and $H(\tau)/m_s$. Clearly, $H$ approaches monotonically the asymptotic value $\bar{H}$ at $\tau = m_s t \rightarrow \infty$. Imposing for definiteness the asymptotic value of the Hubble parameter $\bar{H} = 75 \text{ km/s/Mpc}$, one can infer from figure 2 for the respective Compton wave lengths of the scalar graviton (in the units $c = h = 1$) the following values:

(a1) $\alpha = 0.01$, $1/m_s \simeq 3 \cdot 10^{-2} \text{ Gpc}$;

(a2) $\alpha = 0.1$, $1/m_s \simeq 10^{-1} \text{ Gpc}$;

(b) $\alpha = 1$, $1/m_s \simeq 3 \cdot 10^{-1} \text{ Gpc}$;

(c) $\alpha = 100$, $1/m_s \simeq 1 \text{ Gpc}$.

(4.26)

Stress that such values of the Compton wave length, less or of the order of the Universe size, are just the representative ones corresponding to the (rather loosely) chosen values of the parameters $v_s$ and $\alpha$.

It follows from (4.17) that at $k = 0$ the attractor produces the exponential expansion

$$\bar{a}(t) = a_0 \exp(\bar{A}_s/3)^{1/2} t \equiv a_0 \exp \bar{H} t.$$  

(4.27)

---

\textsuperscript{17}For comparison, $m_s = 1/\text{Gpc} \simeq 10^{-28} \text{ eV}/c^2$ in the conventional units.
In other words, there takes place the late-time inflation corresponding to the spontaneously emerging at the level of FEs the effective CC $\bar{\Lambda}$ even under the absence of the Lagrangian CC.\textsuperscript{18} Had any of $\Lambda_0$ or $\mu_s$ been zero the asymptotic inflation would not take place (under the assumed $V_s|_{\min} = \kappa^2_g \Lambda = 0$). By this token, having adopted the vanishing of the Lagrangian CC $\Lambda$, one could eventually explain the effective CC $\bar{\Lambda}$ to be tiny but nonzero, showing the way to partially solving one of the CC problems. Still, justifying the Lagrangian CC being zero and ensuring the quantum stability of the tiny classical effective CC is beyond the scope of the present paper.\textsuperscript{19}

As a final remark, at $\sigma = \bar{\sigma}$ due to $\sqrt{-\bar{\omega}} \sim \exp 3\bar{H}t$, the quasi-affine time $\bar{t}$ as defined by the attractor at $k = 0$ is $\bar{t} = t_0 \exp 3\bar{H}t$, or inversely $t = (3\bar{H})^{-1} \ln \bar{t}/t_0$, with $t_0$ an integration constant. eq. (4.27) implies then that $\ddot{a} = a_0 (\bar{t}/t_0)^{1/3}$, with the characteristic volume ($\bar{a}^3 = a_0^3 \bar{t}/t_0$) and the characteristic energy ($\kappa^2_g \bar{\Lambda} \bar{a}^3$) of the Universe increasing linearly in the quasi-affine time $\bar{t}$ (or, rather, v.v.), with the late-time inflation appearing just in disguise. Conceivably, this may present an alternative view on the evolution of the Universe and the meaning of time.

5 Conclusion

In conclusion, the scalar-graviton reduction of the multi-component gravity, with the massless tensor graviton supplemented by the massive scalar one, may present the viable modified gravity nearest-to-GR. At that, the scalar graviton proves to be quite plausible candidate on the role of a dark substance of the Universe: DM and/or DE depending on the solution. The scalar graviton as DE may, in a natural way, explain the appearance at the classical level of a nonzero but tiny effective CC due to the attractor solution for the (ultra-)light scalar graviton. Such a signature of the scalar-graviton DE makes going beyond the $\Lambda$CDM model per scalar graviton quite promising. Further studying the scalar graviton to validate it as an emergent dark substance of the Universe is urgent.

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\textsuperscript{18}This may to an extent be reminiscent of the Unimodular Relativity(UR)/Unimodular Gravity/TDiff, missing the scalar graviton effectively due to the restriction $g = \omega$, with a non-dynamical $\omega$. Here CC is also not a Lagrangian parameter $\Lambda$ but an integration constant $\Lambda_0$ appearing spontaneously at the level of FEs. For the (modified) UR, cf., e.g., [20], with the numerous references therein.

\textsuperscript{19}For the CC problems, see, e.g. [21], and for some recent (far from being exhaustive) attempts at solving them, cf., e.g., [14]–[16] and [22]–[25].
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