"Centrifugal force: A gedanken experiment”—new surprises

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Abstract

A recently proposed gedanken experiment [G.Z. Machabeli and A.D. Rogava. Phys. Rev. A 50, 98 (1994)], exhibiting surprising behavior, is reexamined. A description of this behavior in terms of the laboratory inertial frame is presented, avoiding uncertainties arising due to a definition of a centrifugal force in relativity. The surprising analogy with the radial geodesic motion in Schwarzschild geometry is found. The definition of the centrifugal force, suggested by J.C. Miller and M.A. Abramowicz, is discussed.

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I. INTRODUCTION

Recently, in [1], we described a simple *gedanken experiment*, revealing the strange dynamics of rotational motion in special relativity. The experimental layout consisted of a straight, long and narrow pipe rotating around an axle normal to its symmetry axis and a small bead, which could move inside the pipe without friction. The pipe rotated with constant angular velocity $\omega = \text{const}$ and was assumed to be massless and absolutely rigid. At $t = 0$ the bead was just above the pivot ($r_0 = 0$) and had an initial velocity $v_0$. In this particular case the bead, contrary to common intuitive expectations, appeared to move in a quite unusual way (see, for details, [1]). The problem was considered in the rotating non-inertial frame (RNF) of reference of the pipe.

In [1] we have interpreted the surprising behavior of the bead in terms of reversal in direction of a centrifugal force. The approach evoked some comments and criticism [2,3]. Miller & Abramowicz in their Comment [2] pointed at the relationship of this behavior with the relativistic dependence of the bead mass on its velocity. They recommend to define the relativistic centrifugal force in the way as forces are defined in general. Such definition excludes confusing "reversal" of the force, while the *actual deceleration* of the bead is ascribed to the relativistic mass variation. These interesting comments throw a new light on the subject and we definitely greet their appearance. However, we would like to reply on the comments in order to clarify our approach and to point out at some additional aspects of the problem, which we found out after [1] had appeared in press.

It is, certainly, indisputable that a centrifugal force is a seeming (or, "apparent" [2]) force, which arises only in non-inertial frames of reference. However, the main results of [1] were not connected with the peculiarities of the frame in which the problem was considered. In this context it seems interesting to reconsider briefly our experiment in the laboratory inertial frame (LIF), relative to which the pipe rotates. The purpose is to study the bead dynamics in this frame, *without invoking the centrifugal force conception*. In the next section such a consideration is presented. The main physically significant result of [1] (decelarative
character of the bead motion) is verified and obtained again in terms of the LIF.

In the third section we describe an interesting analogy found between the results of [1] and the radial geodesic motion of the test particle in Schwarzschild geometry.

In the concluding section of this letter we discuss the definition of the centrifugal force suggested by Miller & Abramowicz and indicate at the advantages and disadvantages which, in our opinion, it has.

II. BEAD MOTION: LIF TREATMENT

Let us consider the motion of the bead in the LIF, where the spacetime is just Minkowskian. Owing to the polar symmetry of the experimental set-up it is convenient for the forthcoming purposes to write the spatial part of the metric in polar \( (g_{\varphi\varphi} = r^2, \quad g_{rr} = 1) \) coordinates:

\[
ds^2 = -d\tau^2 = -dt^2 + r^2 d\varphi^2 + dr^2,
\]

where \( \tau \) is the proper time of the bead. We use units in which \( c = 1 \).

The motion of the bead in LIF is characterised by the three-velocity \( \vec{v} \) with the nonzero physical components: \( v \equiv v_r = \frac{dr}{dt} \) and \( u \equiv v_\varphi = r \omega \). Note that \( v_\varphi \) and \( v_r \) are connected with their contravariant and covariant components as \( v_\varphi = rv^r = v_\varphi/r \) and \( v_r = v^r = v_r \).

The equation of the bead motion in LIF may be written simply as:

\[
\frac{d\vec{p}}{dt} = \vec{f},
\]

where \( \vec{p} = m_0 \gamma \vec{v} \) is a three-momentum of the bead, \( m_0 \) is its rest mass, and \( \gamma \)–its Lorentz factor as measured in LIF:

\[
\gamma = \left(1 - \omega^2 r^2 - v^2\right)^{-1/2},
\]

while \( \vec{f} \) is a real three-force acting on the bead (pipe reaction force). This force has only one, azimuthal, nonzero component: \( f_\varphi \neq 0 \). It means that the orthogonal radial component of the bead three-acceleration \( (d\vec{p}/dt)_r \), specified by the left hand side of (2), is equal to
zero. Note that \( (d\vec{p}/dt)_i = dp_i/dt + v^k p_{i;k} \neq dp_i/dt \) since the spatial part of the metric (1) is curved. In particular, radial component of the equation of the bead motion leads to

\[
\frac{d}{dt}(mv) - m\omega^2 r = 0, \tag{4}
\]

where \( m(t) \equiv m_0 \gamma \) is the relativistic (inertial) mass of the bead, which varies with time and measures the beads' variable resistance to acceleration. Note that \( m(t) \) depends not only on the radial velocity of the bead \( v(t) \), but also on its radial coordinate \( r(t) \). Taking into account (3) we find

\[
\frac{dm}{dt} = m v \gamma^2 \left( \frac{\omega^2 r}{1} + \frac{dv}{dt} \right), \tag{5}
\]

and (4) yields:

\[
\frac{d^2 r}{dt^2} = \frac{\omega^2 r}{1 - \omega^2 r^2} \left( 1 - \omega^2 r^2 - 2v^2 \right). \tag{6}
\]

This is exactly the same equation, which we get in [1] for the radial acceleration of the bead as measured in RNF. This equation also may be written in the following surprisingly elegant form:

\[
\frac{d^2 r}{dt^2} = \frac{(1 - \gamma^2 v^2)}{(1 + \gamma^2 v^2)} \omega^2 r. \tag{7}
\]

This equation distinctly represents peculiarities of the bead motion: when the motion is nonrelativistic \( (\gamma v \ll 1) \) it reduces to the usual classic equation for centrifugal acceleration: \( d^2r/dt^2 = \omega^2 r \), in the ultrarelativistic limit \( (v_0 \rightarrow 1) \) the sign of the right hand side is just the opposite: \( d^2r/dt^2 = -\omega^2 r \). When \( \gamma_0 v_0 = 1 \) \( (v_0 = \sqrt{2}/2) \) the sign reversal occurs from the very beginning of the motion.

Another interesting point, which also slipped off our attention in [1], is that if we introduce new variables: \( \phi \equiv 2\arccos(\omega r), \lambda \equiv \omega t, \) and \( \Omega^2 \equiv 1 - v_0^2 \), we can reduce (6) to the following, remarkably simple equation:

\[
\frac{d^2 \phi}{d\lambda^2} + \Omega^2 \sin \phi = 0. \tag{8}
\]

This is well-known pendulum equation, describing nonlinear oscillations of a free mathematical pendulum. The easiest way for getting (8) is to write the equation for the radial
velocity of the bead [1]:

\[
\frac{dr}{dt} = \sqrt{(1 - \omega^2 r^2)[1 - (1 - v_0^2)(1 - \omega^2 r^2)]},
\]

to rewrite it in above introduced notations as \(d\phi/d\lambda = -2\sqrt{1 - \Omega^2 \sin^2(\phi/2)}\), and to take one more derivative by \(\lambda\). The striking resemblance of our solutions with mathematical pendulum motion, noticed already in [1], becomes, now, more clear and appreciable. If we introduce the concept of an analogous pendulum, governed by (8), then our initial conditions \((r_0 = 0, \frac{dr}{dt})_{t=0} = v_0\) for this pendulum are replaced by \(\phi_0 = \pi\) and \((d\phi/d\lambda)_{\lambda=0} = -2v_0\).

This pendulum rotates in the vertical plane, performing periodic motion with the effective frequency \(\Omega\). The time interval, needed by the bead to reach \(\omega r = 1\) "light cylinder" point [1] corresponds, now, to the time needed by the analogous pendulum to reach its stable equilibrium \((\phi = 0)\) point.

The LIF treatment has one more advantage: it allows to find out the pipe reaction force \(f \equiv f_\phi\), which acts on the bead and forces it to corotate with the rigidly rotating pipe. This force is explicitly expressed by the azimuthal component of Eq. (3). The result is:

\[
f = m_0\omega\left(r \frac{d\gamma}{dt} + 2\gamma v\right) = \frac{2m_0\omega v}{1 - \omega^2 r^2}.
\]

(9)

Note that in RNF this is also the expression for a relativistic Coriolis (inertial) force acting on the moving bead.

**III. ANALOGY WITH THE MOTION IN SCHWARZSCHILD GEOMETRY**

As it appears, this problem is also in a rather unexpected analogy with the certain kind of geodesic motion in Schwarzschild geometry. In particular, let us consider a radial geodesic "fall" of a test particle onto a Schwarzschild black hole with \(M\) mass. Let a radial velocity of the particle at infinity be \(V_\infty\) and pointed inwards. If one denotes by \(E = \gamma_\infty \equiv (1 - V_\infty^2)^{-1/2} > 1\) the specific energy of the particle per its rest mass, then for its radial velocity relative to the observer at infinity \(V_r \equiv \sqrt{g_{rr}} \frac{dr}{dt}\) one gets

\[
V_r^2 = \frac{E^2}{4} - \left[\frac{E}{2} - \frac{1}{E} \left(1 - \frac{2M}{r}\right)\right]^2,
\]

(10)
(see e.g., Mc.Vittie’s book [4], where this equation is derived in different notations). As for the quantity \( dV_r/dt \), called by McVittie nontensor radial acceleration of the particle, one gets:

\[
\frac{dV_r}{dt} = \frac{2M}{Er^2} \left[ \frac{E}{2} - \frac{1}{E} \left( \frac{2M}{r} \right) \right] \sqrt{1 - \frac{2M}{r}}. \tag{11}
\]

We see that the acceleration of the particle is negative (i.e., the modulus of the particle infall velocity should be increasing), until the particle reaches \( r_1 = 4M/(2 - \gamma_{\infty}^2) \) radius, where the acceleration changes its sign, and \( V_r \) reaches its maximum velocity \( V_{\text{max}} = \gamma_\infty/2 \).

More specifically, we have the following kinds of the motion: (a) \( V_\infty \ll 1 \) (\( \gamma_\infty \approx 1 \)): in this case the particle begins to move with increasing speed, at \( r_1 \approx 4M \) reaches its maximum value (\( V_{\text{max}} \approx 1/2 \)) and decelerates, afterwards, down to zero radial velocity, when the particle approaches the black hole horizon \( r \to 2M \); (b) \( V_\infty = \sqrt{2}/2 \) (\( \gamma_\infty = \sqrt{2} \)): in this “threshold” case at the initial moment \( dV_r/dt = 0 \) (\( V_{\text{max}} = V_\infty, r_1 \to \infty \)), and the particle decelerates smoothly down to the horizon; (c) \( V_\infty > \sqrt{2}/2 \): the acceleration is positive during the whole course of the motion—the motion is decelerative.

It is easy to notice that this picture impressively resembles (even quantitatively) the situation in our gedanken experiment. This likeness is settled due to the remarkable similarity between (10) and the corresponding equation for \( dr/dt \) from [1], which may be written as:

\[
\left( \frac{dr}{dt} \right)^2 = \frac{E^2}{4} - \frac{1}{E} \left( \frac{2M}{r} \right) \left( 1 - \omega^2 r^2 \right)^2. \tag{12}
\]

The resemblance is, as it seems likely, a manifestation of some likeness of the spacetime in the rotating pipe [1]

\[
ds_p^2 = -(1 - \omega^2 r^2)dt^2 + dr^2 \tag{13}
\]

with the Schwarzschild spacetime along a radial geodesic (\( \theta = \text{const}, \ \varphi = \text{const} \))

\[
ds_s^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2. \tag{14}
\]

Comparing, for instance, lapse functions for these two metrics (\( \alpha_p = \sqrt{-g_{tt}} = \sqrt{1 - \omega^2 r^2} \), and \( \alpha_s = \sqrt{-g_{tt}} = \sqrt{1 - 2M/r} \)) we see that \( \alpha_p \) and \( \alpha_s \) become infinite at the light cylinder.
and horizon, respectively. However, the likeness is not complete: spatial part of the pipe metric is flat, while the spatial part of (14) metric is curved \((g_{rr} \neq 1)\). The latter difference is also essential: it ensures finiteness of the time \(t^*\) \([1]\), needed by the bead to reach the light cylinder \(r = \omega^{-1}\). Remember, that an analogous time to reach the Schwarzschild black hole horizon, as measured by the distant observer, goes to infinity.

The extensive examination of this analogy needs separate consideration, which is beyond the scope of this reply and will be presented elsewhere.

### IV. CENTRIFUGAL FORCE DEFINITION

It is well known that a generalization of Newton’s second law

\[
\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt},
\]

is the most useful and convenient definition of the (three-) force \(\vec{F}\) in special relativity. According to Rindler ([5], p.88) "This definition has no physical content until other properties of force are specified, and the suitability of the definition will depend on these other properties."

For real applied forces, arising in relativistic dynamics, this definition is physically justified and is proved to be the most appropriate. But, how one should define inertial forces in relativity? Miller & Abramowicz suggest to use the same general method for these forces too. We would have nothing against such extrapolation. However, we should like to point at one problem arising in the framework of this approach.

This definition contains a quantity \(m(t) = m_0 \gamma(t)\), which has, clearly, the meaning of the relativistic mass in the LIF. However, note that this quantity is used for the definition of another physical quantity (centrifugal force), which exists in another—non-inertial—frame of reference. Such a procedure (definition of a quantity in one frame, through some other quantity, defined in another frame) seems to us quite unusual for the spirit of special relativity.
Certainly, the variable mass may be defined also in RNF, but it would not be equal to \( m(t) \). The point is that \( \gamma(t) \), having in LIF a meaning of Lorentz factor, has not the same meaning in RNF, because Lorentz factor of a moving particle is not invariant between frames [5]. This circumstance appears mostly obvious in the ”1 + 1” formulation of the same problem. In fact, for the two-dimensional curved metric (13) in the RNF \( V \equiv v/\alpha_p \), and \( \Gamma = [(1 - \omega^2 r^2)/(1 - \omega^2 r^2 - v^2)]^{1/2} \). Now, it is possible to define relativistic mass in the RNF as \( M(t) \equiv m_0 \Gamma \) and write, instead of Eq.(15), the following equation:

\[
F^* = \frac{d}{dt} (MV) = \frac{M \omega^2 r}{\alpha_p}.
\] (16)

This definition is already made by means of the true Lorentz factor of the bead as measured in RNF; like (15) it, also, gives ”not reversible” centrifugal force, but lacks the attractive simplicity of (15). However, we should always remember that the mass, which the RNF observer actually measures, is \( M(t) \) and not \( m(t) \). It seems, therefore, more consistent to express the physical quantity existing in a particular frame (centrifugal force \( f_c \), which exists in RNF) through the other physical quantity \( M(t) \) defined and measured in the same frame.

Despite this uncertainty, we should like to note that an importance of the mass variation effect, noticed by Miller & Abramowicz, is, indeed, a very remarkable feature of this problem. As it appears, the capability of mass variation drastically affects the dynamics of the motion. A relativistic body is able to ”absorb” in itself an energy, which in nonrelativistic case is expended only on the increase of its acceleration. But, is it appropriate (and possible) to describe this secondary dynamical effect as an action of some ”negative self-thrust” force? Saying ”secondary” we do not mean its significance but, rather, its status in the causal order of true physical reasons. If one introduces such force it, certainly, will be, like the centrifugal force, an ”apparent force”. The actual dependence of the bead mass on time is governed explicitly by the concrete kind of its motion, which is entirely determined by the outer real force (pipe reaction force \( f \) (12), in our case) applied to this body.

We would like to finish our reply with modest questions and not with bold statements. Is it necessary to define seeming, or, ”apparent” [2] force by the rule known for real forces?
If, as it was above cited, the definition is physically aimless until it is not suitable for describing the properties of a dynamical problem, perhaps it is still more appropriate to define the centrifugal force in its own way. Maybe, it is more appropriate to relate the sign reversal of the acceleration with the corresponding reversal of the properly defined centrifugal force—the sole inertial force, acting on the moving bead in RNF, which has the clear nonrelativistic analogy? We hope that this interesting and intricate problem will attract further more attention of the wide physical audience.

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REFERENCES

[1] G. Z. Machabeli, and A. D. Rogava, Phys.Rev. A 50, 98 (1994)

[2] J. C. Miller and M. A. Abramowicz, SISSA ref. 178/94/A (1994)

[3] F. de Felice, Phys. Rev. A 52, 3452 (1995)

[4] G. C. McVittie, General Relativity and Cosmology, (Chapman and Hall, London, 1956)

[5] W. Rindler, Essential Relativity, (Springer-Verlag, New-York 1980)