ON THE VELOCITY AVERAGING FOR EQUATIONS WITH OPTIMAL HETEROGENEOUS ROUGH COEFFICIENTS

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ABSTRACT. Assume that \((u_n)\) is a sequence of solutions to heterogeneous equations with rough coefficients and fractional derivatives, weakly converging to zero in \(L^p(\mathbb{R}^{d+m})\), with \(p > 1\). We prove that the sequence of averaged quantities \(\left(\int \rho(y)u_n(x,y)dy\right)\) is strongly precompact in \(L^1_{\text{loc}}(\mathbb{R}^d)\) for any \(\rho \in C_c(\mathbb{R}^m)\), provided that restrictive non-degeneracy conditions are satisfied. These are fulfilled for elliptic, parabolic, fractional convection-diffusion equations, as well as for parabolic equations with a fractional time derivative. The main tool that we are using is an adapted version of H-distributions. As a consequence of the introduced methods, we obtain an optimal velocity averaging result in the \(L^p\), \(p \geq 2\), framework under the standard non-degeneracy conditions, as well as a connection between the H-measures and the H-distributions.

1. INTRODUCTION

In the paper we extend results from [15] concerning the velocity averaging for a general transport-type equations to an \(L^p\) setting with an arbitrary \(p > 1\). A simplified version of some results presented here, has been outlined, mostly without proof, in [16].

Accordingly, we consider a sequence of functions \((u_n)\) weakly converging to zero in \(L^p(\mathbb{R}_x^d \times \mathbb{R}_y^m)\) for some \(p > 1\), and satisfying the following sequence of equations

\[ \mathcal{P}u_n(x,y) = \sum_{k \in I} \partial^{\alpha_k}_x \left( a_k(x,y)u_n(x,y) \right) = \partial^\kappa_y G_n(x,y), \]

where \(I\) is a finite set of indices, \(\partial^{\alpha_k} = \partial^{\alpha_k}_{x_1} \cdots \partial^{\alpha_k}_{x_d}\) for a multi-index \(\alpha_k = (\alpha_{k_1}, \ldots, \alpha_{kd}) \in (\mathbb{R}^+)^d\), and similarly for \(\kappa = (\kappa_1, \ldots, \kappa_m) \in \mathbb{N}_0^m\). The fractional derivative \(\partial^{\alpha_k}_x u\) is defined by

\[ \partial^{\alpha_k}_x u = \bar{F}((2\pi i \xi_k)^{\alpha_k} \hat{u}), \]

where \(\hat{u} = \mathcal{F}(u)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} u(x)dx\) is the Fourier transform while \(\bar{F}\) (or \(\check{\cdot}\)) is the inverse Fourier transform. Remark that in our setting, the fractional derivative is actually the Fourier multiplier operator with the symbol \((2\pi i \xi_k)^{\alpha_k}\) (see Definition 2).

Denote by \(A\) the principal symbol of the (pseudo-)differential operator \(\mathcal{P}\), which is of the form

\[ A(x,y,\xi) = \sum_{k \in I} a_k(x,y)(2\pi i \xi)^{\alpha_k}. \]
The sum given above is taken over all terms from (1) whose order of derivative \( \alpha_k \) is not dominated by any other multi-index from \( I \).

For the principal symbol we assume that there exists a multi-index \( \beta = (\beta_1, \cdots, \beta_d) \in \mathbb{R}^d_+ \) such that for any positive \( \lambda \in \mathbb{R} \) the following generalised homogeneity assumption holds

\[
A(x, y, \lambda^{1/\beta_1} \xi_1, \ldots, \lambda^{1/\beta_d} \xi_d) = \lambda A(x, y, \xi),
\]

implying that

\[
\sum_j (\alpha_{kj} / \beta_j) = 1 \quad \text{for every } k \in I'.
\]

In addition we assume that the order of derivatives \( \alpha_{kj} \) entering the principal symbol are either integers, or larger or equal to the space dimension \( d \).

As for the coefficients from (1), we assume that

\( a_k \in L^p(R^m; L^{\bar{p}}(R^d)), \quad k = 1, \ldots, d, \quad \frac{1}{p} + \frac{1}{p'} + \frac{1}{q} = \frac{1}{p} + \frac{1}{p'} = 1, \)

where \( q = p\bar{p}/(p - \bar{p}) \), while \( p' \) stands for a dual index of \( p \).

\textbf{b)} The sequence \((G_n)\) is strongly precompact in the anisotropic space \( L^1(R^m; W^{(-\beta_1, \ldots, -\beta_d), d}(R^d)) \).

We see that the coefficients \( a_k \) are chosen in such a way that the sequences \((a_k u_n)\) are bounded in \( L^1(R^m; L^{1+\varepsilon}(R^d)) \) for some \( \varepsilon > 0 \). We are not able to prove that the velocity averaging result holds if \((a_k u_n)\) are merely bounded in \( L^1(R^d + m) \), but the results from [21] hint that it is not possible to propose better assumptions than those given in \textbf{a)} (unless having additional requirements on \((u_n)\) as in [13]).

Let us now introduce a definition of the weak solution to (1). Assume for the moment that the subindex \( n \) is removed from (1).

\textbf{Definition 1.} We say that a function \( u \in L^p(R^{d+m}) \) is a weak solution to (1) if for every \( g \in C_c^\infty(R^m; W^{\beta, q}(R^d)) \) it holds

\[
\int_{R^{m+d}} \sum_{k \in I} a_k(x, y) \rho^{1/\alpha_k} u(x, y) \partial_x^{\alpha_k} g(x, y) dx dy = (-1)^{|\kappa|} \int_{R^m} \left< G(\cdot, y), \partial_\xi^{\kappa} g(\cdot, y) \right> dy,
\]

where in the last term duality on \( W^{\beta, q}(R^d) \) is considered.

It has been noticed since long time ago that even in the case of homogeneous coefficients [1] one cannot expect the very sequence \((u_n)\), but only the associated sequence of its averages with respect to the velocity variable to be strongly precompact in \( L^p_{\text{loc}}(R^d) \). More precisely, it was proved in [15] for \( p \geq 2 \) (or in [12] in the hyperbolic case) that an averaged quantity

\[
\left( \int_{R^m} \rho(y) u_n(x, y) dy \right), \quad \rho \in C_c(R^m),
\]

where \((u_n)\) are solutions to (1), will be strongly \( L^2_{\text{loc}} \) precompact provided the following non-degeneracy condition is fulfilled

\[
(\forall (x, \xi) \in D \times P) \quad A(x, y, \xi) \neq 0 \quad (ae \ y \in R^m),
\]
where $D \subseteq \mathbb{R}^d$ is a full measure set, while $P$ stands for an appropriate $d-1$ dimensional compact manifold in $\mathbb{R}^d$.

A result of this type is usually called a velocity averaging lemma.

Its importance is demonstrated in many works, but we shall mention only very famous [18] and [10]. Concerning the averaging lemma itself, there are also indeed interesting works [9, 13, 21, 23], but almost all of them were given for homogeneous equations (i.e. the ones where coefficients do not depend on $x \in \mathbb{R}^d$), exclusively with integer-order derivatives. The reason for this one can search in the fact that, in the homogeneous situation, one can separate the solutions $u_n$ from the coefficients (e.g. by applying the Fourier transform with respect to $x$), and this is basis of most of the methods (see e.g. [23] and references therein). We remark that more detailed observations on this issue one can find in the introduction of our recent work [15].

In order to attack the heterogeneous situation a different tool is required, and it was provided independently by P. Gerard [12] and L. Tartar [24] through the concept of microlocal defect measures (in the terminology of the former), or H-measures (as named by the latter).¹ After their work, different variants of the concept appeared, adapted to a problem under consideration [3, 15, 19, 20].

In [12] one can find a velocity averaging lemma in a heterogeneous setting proved by using H-measures. In [15] we proved a general version of the velocity averaging lemma. However, in both papers, the sequence $(u_n)$ was bounded in $L^p(\mathbb{R}^{d+m})$ for $p \geq 2$ since the notion of H-measures is defined in the $L^2$ framework (i.e. they describe a loss of compactness for such sequences). Such $L^2$ character of the H-measures provides their non-negativity, which enables the authors to conclude that given object is actually a Radon measure defined on $\mathbb{R}^d \times P$, for an appropriate $d-1$ dimensional manifold $P$ (Gerard and Tartar worked with $P = S^{d-1}$, where $S^{d-1}$ is the unit sphere in $\mathbb{R}^d$). If the $\mathbb{R}^d$ projection of the H-measure associated to the sequence $(u_n)$ equals zero, then the sequence $(u_n)$ is strongly precompact in $L^2_{\text{loc}}(\mathbb{R}^d)$. Remark that the $\mathbb{R}^d$ projection is actually the standard defect measure [17].

In order to override the mentioned $p \geq 2$ confinement we must invent a more sophisticated tool. It will be based on a generalisation of the H-distributions concept from [2]. The H-distributions were introduced in order to describe a defect of strong convergence for $L^p(\mathbb{R}^d; \mathbb{R}^n)$ sequences (i.e. for $n$-dimensional $L^p$ sequences). It is not difficult to generalise such a concept when the sequences have countable dimension, and even when functions assume values in a separable Hilbert space $H$, i.e. $u_n \in L^p(\mathbb{R}^d; H)$ (Proposition 19). For a further clarification, one can compare works [12] and [24], and also to consult [15, Proposition 12].

Thus in the case of $L^p$ sequences $(u_n)$ for $p < 2$, we have merely a distribution (instead of a measure) describing eventual loss of strong precompactness. Therefore, in order to use the H-distributions on the velocity averaging problem for (1), we must increase assumptions on the principal symbol, and we require the following restrictive non-degeneracy condition

$$|A| \neq 0, \forall \xi \in P.$$ (8)

The condition implies (actually it is equivalent to) the following strong convergence

$$\frac{|A|^2}{|A|^2 + \delta} \rightarrow 1 \text{ in } L^p_{\text{loc}}(\mathbb{R}^m; L^p_{\text{loc}}(\mathbb{R}^d; C^d(P))),$$ (9)

¹In the sequel we shall use the terminology of H-measures.
as $\delta \to 0$, which is needed for the proof of the main theorem.

Indeed, if (8) holds, then for almost every $(x, y) \in \mathbb{R}^{d+m}$ we have that

$$1 - \frac{|A|^2}{|A|^2 + \delta} \in C^4(P),$$

which goes to zero as $\min_{\xi} |A(x, y, \xi)| > 0$. Note that the fractions above are smooth enough due to the assumptions on order of derivatives entering the principal symbol $A$.

Physically relevant equations satisfying the restrictive non-degeneracy conditions are elliptic and parabolic equations, but also fractional convection-diffusion equations [7, 8], and parabolic equations with a fractional time derivative [4, 5, 6] which degenerate on a set of measure zero.

The paper is organised as follows.

In Section 2, we introduce auxiliary notions and notations.

In Section 3, we introduce a variant of the H-distributions required to prove the main result of the paper – the velocity averaging lemma for (1) under the assumption (9). Unlike the proof sketched in [16], here we propose a different approach which gives rise to the connection between the H-measures and the H-distributions (given in the Appendix), but also incorporates methods that were previously applied to elliptic problems with singular data [11].

In [15] we have proved that in the case $p \geq 2$ the velocity averaging result holds under the classical non-degeneracy condition (7) merely. Due to the H-measures techniques used there, the coefficients $a_k$ in (1) are restricted to $L^2(\mathbb{R}^m; L^r(\mathbb{R}^d))$ where $2/p + 1/r = 1$. In Section 4, we combine methods developed in previous sections with the H-measures to improve the velocity averaging result from [15] in the sense that the coefficients to (1) belong to $L^p(\mathbb{R}^m; L^{\bar{p}'}(\mathbb{R}^d))$ where $1/p + 1/\bar{p} < 1$.

2. Notions and notations

We start with the notion of the Fourier multiplier which forms the basis of the current contribution.

**Definition 2.** A (Fourier) multiplier operator $A_\psi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ associated to a bounded function $\psi$ (see e.g. [14]), is a mapping defined by

$$A_\psi(u) = \tilde{F}(\psi \hat{u}),$$

where $\hat{u}$ is the Fourier transform while $\tilde{F}$ (or $\check{}$) is the inverse Fourier transform.

If, for a given $p \in [1, \infty)$, the multiplier operator $A_\psi$ satisfies

$$\|A_\psi(u)\|_{L^p} \leq C\|u\|_{L^p}, \quad u \in S,$$

where $C$ is a positive constant, while $S$ stands for a Schwartz space, then its symbol $\psi$ is called an $L^p$ (Fourier) multiplier.

We shall analyse multipliers defined on the manifold $P$ determined by the order of the derivatives entering the principal symbol (2):

$$P = \{\xi \in \mathbb{R}^d : \sum_{i=1}^{d} |\xi_i|^{l_\beta_i} = 1\},$$

where $\beta$ is the homogeneity index from (3), while $l$ is a minimal number such that either $l_\beta_i > d$ or $l_\beta_i$ is an even integer for each $i$. These assumptions ensure...
that the introduced manifold is of class $C^d$ which enables us to analyse associated multipliers, as well as to define appropriate variant of the H-distributions on them (see Theorem 9).

In order to associate an $L^p$ multiplier to a function defined on $P$ we extend it to $\mathbb{R}^d \setminus \{0\}$ by means of the projection
\[
(\pi_P(\xi))_i = \xi_i \left( |\xi_1|^{\beta_1} + \cdots + |\xi_d|^{\beta_d} \right)^{-1/\beta_i}, \quad i = 1, \ldots, d, \quad \xi \in \mathbb{R}^d \setminus \{0\},
\]
where here and in the sequel we use abbreviation $|\xi|_\beta = \left( \sum |\xi_i|^{\beta_i} \right)^{1/\beta}$.

There are many criteria on a symbol $\psi$ providing it to be an $L^p$ multiplier. In the paper, we shall need the Marcinkiewicz multiplier theorem \cite[Theorem 5.2.4.]{14}, more precisely its corollary which we provide here:

**Corollary 3.** Suppose that $\psi \in C^d(\mathbb{R}^d \cup \{\xi_j = 0\})$ is a bounded function such that for some constant $C > 0$ it holds
\[
|\xi^{\alpha} \partial^\beta \psi(\xi)| \leq C, \quad \xi \in \mathbb{R}^d \cup \{\xi_j = 0\}
\]
for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ such that $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d \leq d$. Then, the function $\psi$ is an $L^p$-multiplier for $p \in (1, \infty)$, and the operator norm of $A_\psi$ depends only on $C$, $p$ and $d$.

**Remark 4.** Using this corollary, we have proved that for a bounded function $\psi$ defined on the manifold $P$ and smooth outside coordinate hyperplanes, its extension $\psi_P = \psi \circ \pi_P$ is an $L^p$ multiplier (see \cite[Lemma 5]{15}). If in addition we assume that $\psi$ is smooth on the whole manifold, i.e. $\psi \in C^d(P)$, then the corresponding operator satisfies
\[
\|A_\psi\|_{L^p \to L^p} \leq C\|\psi\|_{C^d(P)},
\]
with a constant $C$ depending only on $p \in (1, \infty)$ and $d$.

Here, we shall need a similar statement.

**Lemma 5.** Let $\beta \in \mathbb{R}_+^d$ and let $\theta : \mathbb{R}^d \to \mathbb{R}$ be a smooth compactly supported function equal to one on the unit ball centered at origin.

Then for any $\gamma > 0$ the multiplier operator $T^\gamma$ with the symbol
\[
T^\gamma(\xi)(1 - \theta(\xi)) = \frac{1}{|\xi|_\beta} (1 - \theta(\xi))
\]
is a continuous $L^p(\mathbb{R}^d) \to W^{\gamma, \beta, p}(\mathbb{R}^d)$ operator for any $p \in (1, \infty)$. Specially, due to the Rellich theorem it is a compact $L^p(\mathbb{R}^d) \to L^p_{\text{loc}}(\mathbb{R}^d)$ operator.

**Proof:** We shall first prove that the operator $T^\gamma$ is a continuous operator on $L^p(\mathbb{R}^d)$. To this effect, remark that it is enough to prove that $T^\gamma$ satisfies condition of Theorem 3 away from the origin. Around the origin, the operator $T^\gamma$ is controlled by the term $(1 - \theta)$ (which is equal to zero on $B(0, 1)$ and obviously satisfies conditions of Theorem 3). We use the induction argument with respect to the order of derivative in (11).

- $n = 1$

  In this case, we compute
\[
\partial_k T^\gamma(\xi) = C_k \frac{1}{\xi_k} T^\gamma(\xi)(\pi_P(\xi))^{\beta_k}_k
\]
for some constant $C_k$. From here, it obviously follows $|\xi_k \partial_k T^\gamma(\xi)| \leq C$ for $\xi \in \mathbb{R}^d$ away from the origin.

• $n = m$

Our inductive hypothesis is that a $\alpha$-order derivatives of $T^\gamma(\xi)$ can be represented in the following way

$$\partial^\alpha T^\gamma(\xi) = \frac{1}{\xi^\alpha} T^\gamma(\xi) P_\alpha(\xi), \quad (13)$$

where $P_\alpha$ is a bounded function satisfying (11) for $|\alpha| \leq d - |\alpha|$.

• $n = m + 1$

To prove that (13) holds for $|\alpha| = m + 1$ it is enough to notice that $\alpha = e_k + \alpha'$, where $|\alpha'| = m$, and that according to the induction hypothesis we have

$$\partial^\alpha T^\gamma = \partial_k \partial^{\alpha'} T^\gamma = \partial_k \left( \frac{1}{\xi^{\alpha'}} T^\gamma(\xi) P_{\alpha'}(\xi) \right) = \frac{1}{\xi^{\alpha'}} T^\gamma(\xi) P_{\alpha'}(\xi),$$

where

$$P_{\alpha}(\xi) = (P_{\alpha_k} P_{\alpha'} + \xi_k \partial_k P_{\alpha'} - \alpha_k P_{\alpha'})(\xi),$$

thus satisfying conditions (11) as well.

From here, (11) immediately follows for $T^\gamma$ away from the origin, thus proving that the operator $T^\gamma$ is a continuous operator on $L^p(\mathbb{R}^d)$.

It remains to prove that for any $\beta_j$ from the $d$-tuple $\beta = (\beta_1, \ldots, \beta_d)$, the multiplier operator $\partial^{\beta_j} T^\gamma$ is a continuous $L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$ operator. To accomplish this, notice that its symbol is

$$\frac{(1 - \theta(\xi))(2\pi i \xi)^{\gamma \beta_j}}{|\xi|^{\beta_j}} = (1 - \theta(\xi)) (\pi_\beta(\xi))^{\gamma \beta_j}. \quad (14)$$

Thus, away from the origin, it is a composition of a function which is smooth outside coordinate hyperplanes and the projection $\pi_\beta$, and by Remark 4 satisfies conditions of Theorem 3.

In the paper we shall need the following generalisation of Tartar’s commutation lemma [25, Lemma 28.2] to $L^p, p \neq 2$ sequences.

**Lemma 6.** Let $B$ be the operator of multiplication by a continuous function $b \in C_0(\mathbb{R}^d)$. Let $(v_n)$ be a bounded sequence in $L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d), p \in [1, \infty]$ such that $v_n \rightarrow 0$ in the sense of distributions, and let $\psi \in C^d(\mathbb{P})$. Then for the commutator $C = A_{\psi B} - B A_{\psi}$, the sequence $(C v_n)$ converges strongly to zero in $L^q(\mathbb{R}^d)$ for any $q \in [2, p \setminus \{\infty\}]$ if $p \geq 2$, and any $q \in [p, 2 \setminus \{1\}]$ if $p < 2$.

**Proof:** In [15, Theorem 6] we have proved that for an arbitrary $\psi \in C^d(\mathbb{P})$ the extension $\psi_\mathbb{P}$ satisfies the conditions of Tartar’s commutation lemma, thus ensuring that $C$ is a compact operator on $L^2(\mathbb{R}^d)$.

According to the interpolation inequality for any $r$ between 2 and $p$, and $\alpha \in \langle 0, 1 \rangle$ we have

$$|C v_n|_q \leq |C v_n|_2^2 |C v_n|_r^{1-\alpha}, \quad (15)$$

where $1/q = \alpha/2 + (1-\alpha)/r$. As $C$ is a compact operator on $L^2(\mathbb{R}^d)$, while $C$ is bounded on $L^r(\mathbb{R}^d)$ for $r \in \langle 1, \infty \rangle$, we get the claim. \qed
Next, we shall need the following truncation operator

$$T_l(u) = \begin{cases} 0, & |u| > l \\ u, & u \in [-l, l], \end{cases}, \quad l \in \mathbb{N}. \quad (16)$$

The operator (more precisely its variant) was introduced in [11] where it was noticed that convergence of \((T_l(u_n))\) for every \(l \in \mathbb{N}\) in \(L^1_{\text{loc}}(\mathbb{R}^d)\) implies the strong convergence of \((u_n)\) in \(L^1(\mathbb{R}^d)\) (Lemma 8 below). This property will be used in order to prove the strong precompactness of the averaged family (6). Moreover, the truncation operator will enable us to obtain a relation between the H-measures and the H-distributions (see the Appendix).

The following statements ensure the above mentioned property of the truncation operator.

**Lemma 7.** Let \((u_n)\) be a bounded sequence in \(L^p(\Omega)\) for some \(p > 1\), where \(\Omega\) is an open set in \(\mathbb{R}^d\). Then for the sequence of truncated functions it holds

$$\limsup_n \|T_l(u_n) - u_n\|_{L^1(\Omega)} \to 0. \quad (17)$$

**Proof:** Denote by

$$\Omega^l_n = \{x \in \Omega : u_n(x) > l\}.$$

Since \((u_n)\) is bounded in \(L^p(\Omega)\) we have

$$\sup_{k \in \mathbb{N}} \int_\Omega |u_n(x)|^p dx \geq \sup_{k \in \mathbb{N}} \int_{\Omega^l_n} l^p dx,$$

implying that

$$\lim_{l \to \infty} \sup_{k \in \mathbb{N}} \text{meas}(\Omega^l_n) = 0. \quad (18)$$

Now, we use the Hölder inequality

$$\int_\Omega |u_n - T_l(u_n)| dx = \int_{\Omega^l_n} |u_n| dx \leq \text{meas}(\Omega^l_n)^{1/p'} \|u_n\|_{L^p(\Omega)}$$

which tends to zero uniformly with respect to \(n\) according to (18). Thus, (17) is proved. \(\square\)

**Lemma 8.** Let \((u_n)\) be a bounded sequence in \(L^p(\Omega)\) for some \(p > 1\), where \(\Omega\) is an open set in \(\mathbb{R}^d\). Suppose that for each \(l \in \mathbb{N}\) the sequence of truncated functions \((T_l(u_n))\) is precompact in \(L^1(\Omega)\). Then there exists a subsequence \((u_{n_k})\) and function \(u \in L^p(\Omega)\) such that

$$u_{n_k} \to u \quad \text{in} \quad L^1(\Omega).$$

**Proof:** Due to the strong precompactness assumptions on truncated sequences, there exists a subsequence \((u_{n_k})\) such that for every \(l \in \mathbb{N}\) the sequence \((T_l(u_{n_k}))\) is convergent in \(L^1(\Omega)\), with a limit denoted by \(u^l\). We prove that the obtained sequence \((u^l)\) strongly converges in \(L^1(\Omega)\) as well.

To this end, note that

$$\|u^{l_1} - u^{l_2}\|_{L^1(\Omega)} \leq \|u^{l_1} - T_{l_1}(u_{n_k})\|_{L^1(\Omega)} + \|T_{l_1}(u_{n_k}) - u_{n_k}\|_{L^1(\Omega)}$$

$$+ \|T_{l_2}(u_{n_k}) - u_{n_k}\|_{L^1(\Omega)} + \|T_{l_2}(u_{n_k}) - u^{l_2}\|_{L^1(\Omega)},$$

which together with Lemma 7 implies that \((u^l)\) is a Cauchy sequence. Thus, there exists \(u \in L^1(\Omega)\) such that

$$u^l \to u \quad \text{in} \quad L^1(\Omega). \quad (19)$$
Now it is not difficult to see that entire \((u_{n_k})\) converges toward \(u\) in \(L^1(\Omega)\) as well. Namely, it holds

\[
\|u_{n_k} - u\|_{L^1(\Omega)} \leq \|u_{n_k} - T_i(u_{n_k})\|_{L^1(\Omega)} + \|T_i(u_{n_k}) - u^i\|_{L^1(\Omega)} + \|u^i - u\|_{L^1(\Omega)},
\]

which by the definition of functions \(u^i\), and convergences (17) and (19) imply the statement.

3. H-distributions and velocity averaging

We start the section with description of the variant of H-distributions that we use in the proof of the main theorem. It has been recently introduced in [16] in an isotropic case, and it is an extension of the concept proposed in [2].

**Theorem 9.** Let \((u_n)\) be a bounded sequence in \(L^p(\mathbb{R}^{d+m}), p > 1\), and let \((v_n)\) be a sequence converging weakly to zero in \(L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)\) for some (finite) \(q \geq p'\). Let \(\bar{p} \in [1, p]\) be such that \(\frac{1}{\bar{p}} + \frac{1}{p} + \frac{1}{q} = 1\). Then, after passing to a subsequence (not relabeled), there exists a continuous bilinear functional \(B\) on \(L^p(\mathbb{R}^m; L^\bar{p}(\mathbb{R}^d)) \otimes C^d(\mathbb{P})\) (with \(L^\infty(\mathbb{R}^d)\) being replaced by \(C_0(\mathbb{R}^d)\) if \(p = p'\)) such that for every \(\phi \in L^p(\mathbb{R}^m; L^\bar{p}(\mathbb{R}^d)), \phi_2 \in C_0(\mathbb{R}^d)\), and \(\psi \in C^d(\mathbb{P})\), it holds

\[
B(\phi \phi_2, \psi) = \lim_{n \to \infty} \int_{\mathbb{R}^{d+m}} \phi_1(x, y)u_n(x, y)A_{\psi}^{-}(\phi_2 v_n)(x)dx dy,
\]

where \(A_{\psi}^{-}\) is the (Fourier) multiplier operator on \(\mathbb{R}^d\) associated to \(\psi \circ \pi_\mathbb{P}\).

The functional \(B\) we call the H-distribution corresponding to (sub)sequences \((u_n)\) and \((v_n)\).

**Remark 10.** In the case \(p = q = 2\), the H-distribution defined above is actually the (generalised) H-measure corresponding to sequences \((u_n)\) and \((v_n)\) (see Theorem 15).

**Proof:** First, remark that according to the commutation lemma (Lemma 6), it holds

\[
\lim_{n \to \infty} \int_{\mathbb{R}^{d+m}} \phi_1(x, y)u_n(x, y)A_{\psi}^{-}(\phi_2 v_n)(x)dx dy
\]

\[
= \lim_{n \to \infty} \int_{\mathbb{R}^{d+m}} \phi_1(x, y)\phi_2(x)u_n(x, y)A_{\psi}^{-}(v_n)(x)dx dy.
\]

Thus the limit in (20) depends only on the product \(\phi_1 \phi_2 \in L^p(\mathbb{R}^m; L^\bar{p}(\mathbb{R}^d))\).

Next, consider the bilinear mapping \(B_n\) defined for every \(\phi \in L^p(\mathbb{R}^m; L^\bar{p}(\mathbb{R}^d))\) and \(\psi \in C^d(\mathbb{P})\) by

\[
B_n(\phi, \psi) = \int_{\mathbb{R}^{d+m}} \phi(x, y)u_n(x, y)A_{\psi}(v_n)(x)dx dy.
\]

According to the Hölder inequality and the Marcinkiewicz theorem, it holds

\[
|B_n(\phi, \psi)| \leq C\|\psi\|_{C^d(\mathbb{P})}\|v_n\|_{L^p(\mathbb{R}^d)}\int_{\mathbb{R}^m} \|\phi(\cdot, y)\|_{L^\bar{p}(\mathbb{R}^d)}\|u_n(\cdot, y)\|_{L^p(\mathbb{R}^d)}dy,
\]

where \(C\) is the constant from relation (12) depending on \(d\) and \(q\). By using the Hölder inequality again (now applied in the variable \(y\)), we get that
where $\bar{C}$ depends on $C$, and bounds on $\|u_n\|_{L^{p'}(\mathbb{R}^{d+m})}$ and $\|v_n\|_{L^q(\mathbb{R}^d)}$.

Thus it follows that $(B_n)$ is an equibounded sequence of bilinear functionals, and by [2, Lemma 3.2] and (21), there exists a functional $B$ for which (20) holds. \hfill $\square$

It is not difficult to see that the H-distributions given in the previous theorem exhibit similar properties as the H-measures, in the sense that trivial H-distribution functions $b$ strongly precompact in $L^{p'}(\mathbb{R}^d)$ imply a velocity averaging result. Indeed, for a fixed $l \in \mathbb{N}$ and $\chi \in L^\infty(\mathbb{R}^{d+m})$, take $\langle \psi, u \rangle = \langle T_l(\int_{\mathbb{R}^{d+m}} \chi(x,y) u_n(x,y) dy), \psi \rangle$, where $(u_n)$ is a sequence from the last theorem. Suppose that the projection on $\mathbb{R}^d$ of the H-distribution $(B_l)$ corresponding to the sequences $(u_n)$ and $(v_n)$ is equal to zero for every $l \in \mathbb{N}$, i.e.

$$B_l(\phi, 1) = 0, \ \phi \in L^{p'}(\mathbb{R}^{d+m}; L^{p'}(\mathbb{R}^d)), \ l \in \mathbb{N}. $$

By choosing $\phi_1 \bar{\phi}_2 = \chi$ in (20), we conclude that

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} T_l \left( \int_{\mathbb{R}^m} \chi(x,y) u_n(x,y) dy \right)^2 dx = 0. $$

From here and Lemma 8, we get that $(\int_{\mathbb{R}^m} \chi(x,y) u_n(x,y) dy)$ converges to zero strongly in $L^1_{\text{loc}}(\mathbb{R}^d)$. From the interpolation inequalities, the sequence is also strongly precompact in $L^p_{\text{loc}}(\mathbb{R}^d)$ for any $\tilde{p} < p$.

More information concerning the connection between the H-measures and the H-distributions one can find in the Appendix.

Now, we go back to the main subject of the paper – the velocity averaging result and tools required for its proof. Remark that according to the Schwartz kernel theorem [22], one can extend functional $B$ to a distribution on $D'(\mathbb{R}^{d+m} \times \mathbb{P})$. An improved result can be obtained by means of the next theorem. It is a generalisation of [16, Theorem 2.1] to Lebesgue spaces with mixed norms, with the proof going along the same lines. The considered anisotropic Lebesgue spaces $L^p(\mathbb{R}^d)$ are Banach spaces with the norm given by

$$\|f\|_p = \left( \int \cdots \left( \int |f(x_1, x_2, \ldots, x_d)|^{p_1} \, dx_1 \right)^{p_2/p_1} \, dx_2 \right)^{p_3/p_2} \cdots \, dx_d \right)^{1/p_d}. $$

Theorem 11. Let $B$ be a continuous bilinear functional on $L^p(\mathbb{R}^d) \otimes E$, where $E$ is a separable Banach space, and $p \in (1, \infty)^d$. Then $B$ can be extended as a continuous functional on $L^p(\mathbb{R}^d; E)$ if and only if there exists a (nonnegative) function $b \in L^p(\mathbb{R}^d)$ such that for every $\psi \in E$ and almost every $x \in \mathbb{R}^d$, it holds

$$|\hat{B}\psi(x)| \leq b(x)\|\psi\|_E, $$

where $\hat{B}$ is a bounded linear operator $E \to L^p(\mathbb{R}^d)$ defined by $\langle \hat{B}\psi, \phi \rangle = B(\phi, \psi)$, $\psi \in L^p(\mathbb{R}^d)$. \hfill (22)

The proof in [16] is presented for real functionals, but the result holds for complex ones as well, with the proof going along the same lines (just by considering the real and imaginary part of $B$ separately).

We use the theorem in order to get the following result for the functional $B$ from Theorem 9.
Corollary 12. If $\bar{p} > 1$ the bilinear functional $B$ defined in Theorem 9 can be extended as a continuous functional on $L^p\left(\mathbb{R}^m; L^\bar{p}(\mathbb{R}^d; C^d(P))\right)$.

Proof: We shall prove that the functional $B$ satisfies conditions of Theorem 11.

As the first step, choose a dense countable set $E$ of functions $\psi_j$ on the unit sphere in $C^d(P)$. In addition, we assume that for each such function, $-\psi_j$ belongs to the same set as well and is indexed by $-j, j \in \mathbb{N}$.

Let $\tilde{B}$ be an operator defined in Theorem 11, which in this setting is a bounded linear operator $C^d(P) \to L^p(\mathbb{R}^m; L^\bar{p}(\mathbb{R}^d))$. For each function $\tilde{B}\psi_j$ denote by $D_j$ the corresponding set of Lebesgue points. The set $D_j$ is of full measure, and thus the set $D = \bigcap_j D_j$ as well.

For any $(x, y) \in D$ and $k \in \mathbb{N}$ denote

$$b^R_k(x, y) := \max_{|j| \leq k} |\Re(\tilde{B}\psi_j(x, y))| = \sum_{|j| = 1}^k |\Re(\tilde{B}\psi_j(x, y))| \chi^R_k(x, y)$$

(23)

$$b^I_k(x, y) := \max_{|j| \leq k} |\Im(\tilde{B}\psi_j(x, y))| = \sum_{|j| = 1}^k |\Im(\tilde{B}\psi_j(x, y))| \tilde{\chi}^I_k(x, y),$$

where $\chi^R_k, \chi^I_k, |j| = 1, \ldots, k$ are characteristic functions of the sets of all points for which the above maxima are achieved for $\psi_j$.

We shall prove that the functions $b^R_k, b^I_k$ given by (23) are uniformly bounded in $L^p(\mathbb{R}^m; L^\bar{p}(\mathbb{R}^d))$. Since $(b^R_k), (b^I_k)$ are increasing sequence of positive functions, denoting by $b^R, b^I$ their (pointwise) limits, the function $b = b^R + b^I$ will satisfy the conditions of Theorem 11. Indeed, according to (23), we see that (22) will hold for every $\psi_j \in E$, and by continuity, the statement can be generalised to an arbitrary $\psi \in C^d(P)$.

Thus it remains to prove the boundedness of the sequence $(b^R_k), (b^I_k)$. We write down the proof just for the first one, as for the second one is performed in completely the same way. To this effect, take an arbitrary $\phi \in C_c(\mathbb{R}^{d+m})$, and denote $K = \text{supp} \phi$. Let $\chi^{k,e}_j \in C_c(\mathbb{R}^{d+m})$ be smooth approximations of characteristic functions from (23) on $K$ such that

$$\|\chi^{k,e}_j - \chi^k_j\|_{L^r(K)} \leq \frac{\varepsilon}{K},$$

where $r > 1$ is chosen such that $q = r'p'$. Denote by $C_u$ an $L^p$ bound of $(u_n)$ and by $C_v$ an $L^q$ bound of $(v_n)$. 


According to (23) and the definition of the operator $\tilde{B}$, we have

$$\langle b_k^R, \phi \rangle = \lim_{n \to \infty} \frac{1}{\mathbb{R}} \left( \int_{\mathbb{R}^{d+m}} \sum_{|j|=1}^k |\phi u_n A^k_j(x,y)(A v_n)(x)dx dy \right)$$

$$\leq \limsup_{n \to \infty} \int_{\mathbb{R}^{d+m}} \left( \sum_{|j|=1}^k |u_n A^k_j (x,y)|^{1/p} \left( \sum_{|j|=1}^k \chi_j^k [\phi A v_n (x,y)]^{1/p'} \right)dxdy \right)$$

$$\leq \limsup_{n \to \infty} \|u_n\|_{L^p(\mathbb{R}^{d+m})} \left( \left\{ \sum_{|j|=1}^k (\chi_j^k - \chi_j^{k,\varepsilon}) \right\} \|\phi A v_n\|_{L^{p'}(\mathbb{R}^{d+m})} \right)$$

$$\leq C_u \limsup_{n \to \infty} \left( \sum_{|j|=1}^k \|\chi_j^k - \chi_j^{k,\varepsilon}\|_{L^r(K)} \|A v_n\|_{L^p(\mathbb{R}^{d+m})} \right)^{1/p'}$$

$$\leq C_u C_p' \limsup_{n \to \infty} \left( \varepsilon C_\phi \|v_n\|_{L^p(\mathbb{R}^{d+m})} + \sum_{|j|=1}^k \|\chi_j^{k,\varepsilon} A v_n\|_{L^{p'}(\mathbb{R}^{d+m})} \right)^{1/p'}$$

where $C_p'$ is the constant from the corollary of the Marcinkiewicz theorem (more precisely from (12)), while $C_\phi = \|\phi\|_{L^q(\mathbb{R}^{d+m}; L^\infty(\mathbb{R}^{d+m}))}$. By letting $\varepsilon \to 0$, we conclude

$$\langle b_k^R, \phi \rangle \leq C_u C_p' \limsup_{n \to \infty} \left( \sum_{|j|=1}^k \|\chi_j^k A v_n\|_{L^{p'}(\mathbb{R}^{d+m})} \right)^{1/p'}$$

since $\chi_j^{k,\varepsilon} \to \chi_j^k$ in $L^r(K)$ for any $r \in [1, \infty)$. Thus, as

$$\sum_{|j|=1}^k \|\chi_j^k A v_n\|_{L^{p'}(\mathbb{R}^{d+m})} = \|\phi v_n\|_{L^{p'}(\mathbb{R}^{d+m})} \leq \left( \|\phi\|_{L^{p'}(\mathbb{R}^{d+m}; L^{p'}(\mathbb{R}^{d}))} \|v_n\|_{L^p(\mathbb{R}^{d})} \right)^{1/p'}$$

it follows

$$\lim_{k \to \infty} \left| \langle b_k^R, \phi \rangle \right| \leq C_u C_p' C_v \|\phi\|_{L^{p'}(\mathbb{R}^{d+m}; L^{p'}(\mathbb{R}^{d}))}$$

Since $C_c(\mathbb{R}^{d+m})$ is dense in $L^{p'}(\mathbb{R}^{d+m}; L^{p'}(\mathbb{R}^{d}))$ we conclude that the sequence $(b_k^R)$ is bounded in $L^{p'}(\mathbb{R}^{d+m}; L^{p'}(\mathbb{R}^{d}))$. Concluding the same for $(b_k^2)$, and denoting respectively the limits by $b^R, b^2$, one gets that $b = b^R + b^2$ satisfies (22). The result now follows from Theorem 11.

Now, we can prove the main result of the paper.
Theorem 13. For the sequence of equations (1) we assume

- the coefficients satisfy conditions a), b);
- the principal symbol (2) satisfies the homogeneity assumption (3) and the restrictive non-degeneracy condition (8);
- \( u_n \to 0 \) in \( L^p(\mathbb{R}^{d+m}) \), for some \( p > 1 \).

Then for any \( \rho \in \mathcal{C}_c(\mathbb{R}^m) \) the sequence of averaged quantities \( \left( \int_{\mathbb{R}^m} \rho(y)u_n(x,y)dy \right) \) converges to \( 0 \) strongly in \( L^1_{loc}(\mathbb{R}^d) \).

Proof: Fix \( \rho \in \mathcal{C}_c(\mathbb{R}^m) \), \( \varphi \in L^\infty(\mathbb{R}^d) \), and \( l \in \mathbb{N} \). Denote by \( V_l \) a weak \( * \) \( L^\infty(\mathbb{R}^{d+m}) \) limit along a subsequence of truncated averages defined by \( V'_n = \varphi T_l(\int_{\mathbb{R}^m} \rho(y)u_n(x,y)dy) \), where \( T_l \) is the truncation operator introduced in (16).

Denote \( v'_n = V'_n - V_l \) and remark that \( v'_n \to 0 \) in \( L^\infty(\mathbb{R}^m) \) with respect to \( n \).

Next, let \( B_l \) be the H-distribution defined in Theorem 9 corresponding to (sub)sequences (of) \( u_n \) and \( v'_n \).

Take a dual product of (1) with the test function \( g_n \) of the form \( (T^1 \) below is defined in Lemma 5)

\[ g_n(x,y) = \rho_1(y)(T^1 \circ A_{\psi^1})(\varphi_1 v_n)(x), \]

where \( \psi \in \mathcal{C}_c^d(P) \), \( \varphi_1 \in \mathcal{C}_c^\infty(\mathbb{R}^d) \), and \( \rho_1 \in \mathcal{C}_c^{|\alpha|}(\mathbb{R}^m) \) are arbitrary test functions, while \( \alpha \) is the multi-index appearing in (1) (see (5)). We get

\[ \sum_{k \in I} \int_{\mathbb{R}^{d+m}} a_k(x,y)u_n(x,y)\varphi_1(y)A_{\psi^1} \circ A_{(1-\theta(\xi))(-2\pi i \xi)^{\alpha_k}/|\xi|^\beta} \varphi_1 v_n(x)d\mathbf{x} = o_n(1), \]

where \( |\xi|_\beta \) is defined in (10). The right-hand side of the last expression tends to zero as \( n \to \infty \) as, by assumption b), the sequence \( (G_n) \) of functions on the right hand side of (1) converges strongly to zero in \( L^2(\mathbb{R}^m; W^\beta,q(\mathbb{R}^d)) \), while, according to Lemma 5, the multiplier operator \( T^1 \circ A_{\psi^1} : L^2(\mathbb{R}^d) \to W^\beta,q(\mathbb{R}^d) \) is bounded.

Rewriting the last relation and passing to the limit we get

\[ \lim_{n} \sum_{k \in I} \int_{\mathbb{R}^{d+m}} a_k(x,y)u_n(x,y)\varphi_1(y) \times \]

\[ \times \left( A_{\psi^1} \circ A_{k_1} \circ \cdots \circ A_{k_d} \circ T - \sum_{j=1}^{d} \alpha_j \right)^{\alpha_{kj}} \left( \varphi_1 v_n(x)d\mathbf{x} \right) = 0, \]

where \( A_{kj} \), is the multiplier operator with the symbol

\[ (1-\theta(\xi))(-2\pi i \xi^j)^{\alpha_{kj}/|\xi|^\beta}. \]

Since the powers \( \alpha_{kj}, j = 1, \ldots, d, \) are either greater than \( d \) or natural numbers, the above symbol is the composition of the projection \( \pi_{\varphi} \) and a smooth function (of class \( \mathcal{C}_c^d \)), thus satisfying conditions of Theorem 3. Thus, the corresponding operators \( A_{-\pi_{\varphi}}^{\alpha_{kj}}, j = 1, \ldots, d \) are \( L^p \) continuous and satisfy bound (12).

According to (4) and the definition of the main symbol, we conclude that for every \( k \notin I' \) it must be \( \sum_{j=1}^{d} \alpha_{kj}/|\xi|^\beta < 1 \). By means of Lemma 5 we conclude that for such
an index \( k \) the limit of the integral in (24) vanishes, and, due to the arbitrariness of test functions, the relation takes the form

\[
A B_l = 0, \tag{25}
\]

where \( A \) is the principal symbol given by (2).

According to Corollary 12, we can test (25) on the function

\[
\phi(x, y) \psi(\xi) \frac{A(x, y, \xi)}{|A(x, y, \xi)|^2 + \delta},
\]

for an arbitrary \( \phi \in C_c^\infty(\mathbb{R}^{d+m}), \psi \in C^d(\mathbb{P}) \). Thus, we obtain

\[
\left\langle B_l, \phi(x, y) \psi(\xi) \frac{|A(x, y, \xi)|^2}{|A(x, y, \xi)|^2 + \delta} \right\rangle = 0,
\]

and by letting \( \delta \to 0 \), using (9) and the continuity of the functional \( B_l \), we conclude

\[
B_l = 0, \quad \forall l \in \mathbb{N}.
\]

From the definitions of the H-distributions and the truncation operator \( T_l \), we conclude by taking in (20) test functions \( \psi = 1 \) and \( \phi_1 \phi_2 = \varphi \times \rho \) for the previously chosen \( \varphi \) and \( \rho \) (see the beginning of the proof):

\[
0 = \lim_{n \to \infty} \int_{\mathbb{R}^d} \varphi^2(x) |T_l \int_{\mathbb{R}^m} u_n(x, y) \rho(y) dy|^2 dx, \quad l \in \mathbb{N}. \tag{26}
\]

Now, using Lemma 8, we conclude that

\[
\left( \int_{\mathbb{R}^m} \rho(y) u_n(\cdot, y) dy \right) \text{ is strongly precompact in } L^1_{\text{loc}}(\mathbb{R}^d).
\]

\[\square\]

4. OPTIMAL VELOCITY AVERAGING IN \( L^p, \ p \geq 2 \), FRAMEWORK

Using the method from Theorem 13 we are able to optimize the velocity averaging results when the sequence of solutions to (1) are bounded in \( L^p(\mathbb{R}^{d+m}) \) for some \( p \geq 2 \), under the classical non-degeneracy conditions given by (7). We shall need the extension of the H-measures introduced in [15] whose existence and properties are restated in the next theorem.

**Theorem 14.** Assume that a sequence \( (u_n) \) converges weakly to zero in \( L^2(\mathbb{R}^{d+m}) \cap L^2(\mathbb{R}^m; L^p(\mathbb{R}^d)), p \geq 2 \). Then there exists a measure \( \mu \in L^2_{\text{loc}}(\mathbb{R}^{2m}, M(\mathbb{R}^d \times \mathbb{P})) \) such that for all \( \phi \in L^2(\mathbb{R}^m; L^2(\mathbb{R}^d)), x^\frac{1}{p} + \frac{p}{2} = 1 \) (with \( L^\infty(\mathbb{R}^d) \) being replaced by \( C_0(\mathbb{R}^d) \) if \( p = 2 \), \( \phi_2 \in L^2(\mathbb{R}^m; C_0(\mathbb{R}^d)) \), and \( \psi \in C^d(\mathbb{P}) \) it holds

\[
\lim_{n \to \infty} \int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^d} (\phi_1 u_{n'})(x, y) \left( A_{\psi, \phi_2} u_{n'}(\cdot, \tilde{y}) \right)(x) dxdy \tilde{y} = \int_{\mathbb{R}^{2m}} \langle \mu(y, \tilde{y}, \cdot, \cdot), \phi_1(\cdot, y) \overline{\phi_2(\cdot, \tilde{y})} \otimes \overline{\psi} \rangle dxdy \tilde{y},
\]

where \( A_{\psi, \phi} \) is the (Fourier) multiplier operator on \( \mathbb{R}^d \) associated to \( \psi \circ \pi_p \).

Furthermore, the operator \( \mu \) has the form

\[
\mu(y, \tilde{y}, x, \xi) = f(y, \tilde{y}, x, \xi) \nu(x, \xi) dy d\tilde{y}, \tag{27}
\]
where \( \nu \in \mathcal{M}_b(\mathbb{R}^d \times P) \) is a non-negative scalar Radon measure whose \( \mathbb{R}^d \) projection \( \int \rho \, dv(x, \xi) \) can be extended to a bounded functional on \( L^{p'}(\mathbb{R}^d) \) in the case \( p > 2 \), while \( f \) is a function from \( L^2(\mathbb{R}^{2m}; L^1(\mathbb{R}^d \times P : \nu)) \).

By \( L^2_{\text{w}}(\mathbb{R}^{2m}; \mathcal{M}_b(\mathbb{R}^d \times P)) \) we have denoted the Banach space of weakly * measurable functions \( \mu : \mathbb{R}^{2m} \to \mathcal{M}_b(\mathbb{R}^d \times P) \) such that \( \int_{\mathbb{R}^{2m}} \| \mu(y, \tilde{y}) \|^2 dyd\tilde{y} < \infty \).

An H-measure defined above is an object associated to a single \( L^2 \) sequence. However, there are no obstacles to adjoin a similar object to different sequences as in the case of the H-distributions (Theorem 9). This can be done by forming a vector sequence, and consider non-diagonal elements of corresponding (matrix) H-measure. Another way is to joint two sequences in a single one by means of a dummy variable, as it is done in the next theorem.

**Theorem 15.** Let \((u_n)\) be a bounded sequence in \( L^2(\mathbb{R}^{d+m}) \cap L^2(\mathbb{R}^m; L^p(\mathbb{R}^d)) \), for some \( p \geq 2 \), and let \((v_n)\) be a sequence weakly converging to zero in \( L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \) where \( 1/q + 1/p < 1 \). Then, after passing to a subsequence (not relabeled), there exists a measure \( \mu \in L^2_{\text{w}}(\mathbb{R}^{m}; \mathcal{M}_b(\mathbb{R}^d \times P)) \) such that for all \( \phi_1 \in L^2(\mathbb{R}^m); C_0(\mathbb{R}^d) \), \( \phi_2 \in C_0(\mathbb{R}^d) \), \( \psi \in C^2(\mathbb{R}^d) \), we have

\[
\langle \mu, \phi_1 \phi_2 \otimes \psi \rangle = \lim_{n \to \infty} \int_{\mathbb{R}^{d+m+k}} \phi_1(x, y)u_n(x, y)\overline{A_{\psi_\nu}^c(\phi_2v_n)}(x)dxdy.
\]

Furthermore, the measure \( \mu \) is of the form

\[
\mu(y, x, \xi) = f(y, x, \xi)dv(x, \xi)dy,
\]

where \( \nu \in \mathcal{M}_b(\mathbb{R}^d \times P) \) is a non-negative, bounded, scalar Radon measure, while \( f \in L^2(\mathbb{R}^m; L^1(\mathbb{R}^d \times P : \nu)) \). We call it the generalised H-measure corresponding to (sub)sequences of \((u_n)\) and \((v_n)\).

**Proof:** Denote by \( u \) an \( L^2 \) weak limit of the sequence \((u_n)\) along a (non-relabeled) subsequence. Fix an arbitrary non-negative compactly supported \( \rho \in C_c(\mathbb{R}^m) \) with the total mass equal to one. Let

\[
W_n(x, y, \lambda) = \begin{cases} (u_n - u)(x, y), & \lambda \in (0, 1) \\ \rho(y)v_n(x), & \lambda \in (-1, 0) \\ 0, & \text{else}. \end{cases}
\]

Clearly, we have that \( W_n \rightharpoonup 0 \) in \( L^2(\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}) \), and by Theorem 14 it admits a measure \( \tilde{\mu} \in L^2_{\text{w}}(\mathbb{R}^{2(m+1)}; \mathcal{M}_b(\mathbb{R}^d \times P)) \) such that for any \( \tilde{\phi}_1 \in L^2(\mathbb{R}^{m+1}; C_0(\mathbb{R}^d)), \tilde{\phi}_2 \in L^2(\mathbb{R}^{m+1}; C_0(\mathbb{R}^d)), \) and \( \psi \in C(\mathbb{R}) \) it holds

\[
\langle \tilde{\mu}, \tilde{\phi}_1 \tilde{\phi}_2 \otimes \psi \rangle = \lim_{n \to \infty} \int_{\mathbb{R}^{d+2(m+1)}} \overline{\tilde{A}_{\psi_{\nu}}^c(\tilde{\phi}_2W_n)}(x, y, \lambda)(x)dxdw,
\]

where \( w = (y, \tilde{y}, \lambda) \in \mathbb{R}^{2m+2} \).

According to the representation (27) the measure \( \tilde{\mu} \) is of the form

\[
\tilde{\mu} = \tilde{f}(y, \tilde{y}, \lambda, \tilde{\lambda}, x, \xi)dv(x, \xi)dyd\tilde{y}d\lambda d\tilde{\lambda}, \quad y, \tilde{y} \in \mathbb{R}^m, \lambda, \tilde{\lambda} \in \mathbb{R},
\]

where \( \nu \in \mathcal{M}_b(\mathbb{R}^d \times P) \) is a non-negative scalar Radon measure, while \( \tilde{f} \) is a function from \( L^2(\mathbb{R}^{2(m+1)}; L^1(\mathbb{R}^d \times P : \nu)) \).

By taking in (30) \( \tilde{\phi}_1(x, y, \lambda) = \phi_1(x, y) \otimes \theta_1(\lambda) \), and \( \tilde{\phi}_2(x, y, \lambda) = \phi_2(x) \otimes \rho_2(y) \otimes \theta_2(\lambda) \), where \( \phi_1 \in L^2(\mathbb{R}^m; C_0(\mathbb{R}^d)) \) and \( \phi_2 \in C_0(\mathbb{R}^d) \) are arbitrary test functions,
while \( \theta_1 = \chi_{[0,1]}, \theta_2 = \chi_{[-1,0]} \), and \( \rho_2(\mathbf{\hat{y}}) = 1 \) for \( \mathbf{\hat{y}} \in \text{supp}\rho \), we see that the measure
\[
d\mu(y, x, \xi) = \left( \int_{\mathbb{R}^m} f(y, \mathbf{\hat{y}}, \lambda, \mathbf{\hat{\lambda}}, x, \xi) d\mathbf{\hat{y}}d\lambda d\mathbf{\hat{\lambda}} \right) d\nu(x, \xi)dy,
\]
satisfies (28).

**Remark 16.** The last theorem is stated for sequence of functions \( u_n \) being in \( L^2 \) space with respect to the velocity variable \( y \), as this was the setting in which generalised \( H \)-measures have been defined in [15]. However, if in addition one assumes that \( (u_n) \) is bounded in \( L^p(\mathbb{R}^m; L^s(\mathbb{R}^d)) \) for some \( p \in (1, \infty) \), then a test function can be taken merely from \( L^{p'}(\mathbb{R}^m; C_0(\mathbb{R}^d)) \).

By using the above characterisation of \( H \)-measures we are able to improve the main result of the paper, namely Theorem 13, in the case \( p \geq 2 \) by assuming merely the classical non-degeneracy condition (7) instead of the restrictive one given by (8). Note that due to the lower regularity assumptions on the coefficients the following theorem also generalises the velocity averaging results provided in [15].

**Theorem 17.** Assume that \( u_n \rightharpoonup 0 \) weakly in \( L^p(\mathbb{R}^{d+m}) \cap L^2(\mathbb{R}^{d+m}) \), \( p \geq 2 \), where \( u_n \) represent weak solutions to (1) in the sense of Definition 1 (with conditions a) and b) together with the homogeneity assumption (3) being fulfilled). Furthermore, assume that the classical non-degeneracy conditions (7) are satisfied.

Then, for any \( \rho \in L^2(\mathbb{R}^m) \),
\[
\int_{\mathbb{R}^m} u_n(x, y)\rho(y)dy \to 0 \quad \text{strongly in } L^1_{\text{loc}}(\mathbb{R}^d).
\]

**Proof:** We stick to the notations from Theorem 13.

Denote by \( B_t \) the \( H \)-distribution corresponding to the sequences \( (u_n) \) and \( (v^c_n) \).

Thus, from the proof of Theorem 13, we conclude that \( B_t \) satisfies localization principle given by (25). We wish to prove that from here, under condition (7), it follows that \( B_t \equiv 0 \).

Since the sequence \( (u_n) \) is bounded in \( L^2(\mathbb{R}^{d+m}) \) then, according to Theorem 15, together with the sequence \( (v^c_n) \), it forms the \( H \)-measure which coincides with the \( H \)-distribution \( B_t \) at least on the space \( C_c(\mathbb{R}^{d+m}; C^d(\mathbb{P})) \) where therefore it admits the representation given by (29). By using the density arguments and the continuity of the \( H \)-distribution on \( L^{p'}(\mathbb{R}^m; L^p(\mathbb{R}^d); C^d(\mathbb{P})) \) (provided by Corollary 12), for an arbitrary test function \( g \in L^{p'}(\mathbb{R}^m; L^p(\mathbb{R}^d); C^d(\mathbb{P})) \) it holds
\[
\langle B_t, g \rangle = \int_{\mathbb{R}^m} \int_{\mathbb{R}^d \times \mathbb{P}} g(y, x, \xi) f_t(y, x, \xi) d\nu_t(x, \xi) dy,
\]
for some \( \nu_t \in \mathcal{M}_b(\mathbb{R}^d \times \mathbb{P}) \) and \( f_t \in L^2(\mathbb{R}^m; L^1(\mathbb{R}^d \times \mathbb{P}; \nu)) \).

Now, take an arbitrary \( \delta > 0 \), and for a \( \rho \in L^2(\mathbb{R}^m) \) and \( \phi \in C_c(\mathbb{R}^d; C^d(\mathbb{P})) \) consider the test function
\[
\rho(y)\phi(x, \xi) \frac{A(x, \xi, y)}{|A(x, \xi, y)|^2 + \delta}.
\]

The localisation principle (25) implies
\[
\left\langle B_t, \rho(y)\phi(x, \xi) \frac{|A(x, y, \xi)|^2}{|A(x, y, \xi)|^2 + \delta} \right\rangle = 0,
\]
which by means of representation (29) and Fubini’s theorem takes the form

$$\int_{\mathbb{R}^d \times P} \int_{\mathbb{R}^m} \frac{\rho(y)\phi(x, \xi)|A(x, \xi, y)|^2}{|A(x, \xi, y)|^2 + \delta} f_l(y, x, \xi) dy dv_l(x, \xi) = 0. \quad (32)$$

Let us denote

$$I_\delta(x, \xi) = \int_{\mathbb{R}^m} \rho(y) \frac{|A(x, \xi, y)|^2}{|A(x, \xi, y)|^2 + \delta} f_l(y, x, \xi) dy.$$

According to the non-degeneracy condition (7), we have

$$I_\delta(x, \xi) \rightarrow \int_{\mathbb{R}^m} \rho(y) f_l(y, x, \xi) dy,$$

as $\delta \to 0$ for $\nu$–a.e. $(x, \xi) \in \mathbb{R}^d \times P$. By using the Lebesgue dominated convergence theorem, it follows from (32) after letting $\delta \to 0$:

$$\langle B_l, \rho \otimes \phi \rangle = \int_{\mathbb{R}^d \times P} \int_{\mathbb{R}^m} \rho(y) \phi(x, \xi) f_l(y, x, \xi) dy dv_l(x, \xi) = 0,$$

i.e. $B_l = 0$ for every $l$. Now, as in the proof of Theorem 13, we conclude that the sequence of truncated averages $T_l \int_{\mathbb{R}^m} u_n(x, y) \rho(y) dy$ is strongly precompact in $L^1_{\text{loc}}(\mathbb{R}^d)$, which together with Lemma 8 concludes the theorem.

5. Appendix

We are able to use previously introduced techniques to point out a connection between the H-distributions that we have introduced and the H-measures that we used in [15] (Theorem 14). We shall need a kind of truncation function again:

$$u^l = \begin{cases} 
  u, & l < |u| \leq l + 1 \\
  0, & \text{else.}
\end{cases}$$

The following theorem holds.

**Theorem 18.** Let $(u_n)$ be a sequence bounded in $L^p(\mathbb{R}^{d+m})$, $p > 1$. Let $(v_n)$ be a sequence weakly converging to zero in $L^2(\mathbb{R}^d) \cap L^s(\mathbb{R}^d)$ for every finite $s \geq p'$. Then, the following representation holds for the H-distribution $B$ corresponding to (sub)sequences of $(u_n)$ and $(v_n)$

$$B = \sum_{l=1}^{\infty} f^l(y, x, \xi) dv^l(x, \xi) dy,$$

where $f^l(y, x, \xi) dv^l(x, \xi) dy$ are generalised H-measures corresponding to $(u_n^l)$ and $(v_n)$, $l \in \mathbb{N}$.

**Proof:** First remark that we can write

$$u_n = \sum_{l=1}^{\infty} u_n^l,$$

and that $(u_n^l)$ is a sequence of functions with $L^\infty$ norms bounded by $l$. Denote by $(u_n)$ a non-relabeled subsequence of $(u_n)$ such that for each $l$ $(u_n^l)$ converges weakly * in $L^\infty(\mathbb{R}^d \times \mathbb{R}^m)$ toward a limit $u^l$. 

\[\square\]
By means of the last relation the limit from (20) can be express as
\[
\lim_{n \to \infty} \int_{\mathbb{R}^{d+m}} \phi_1(x, y) \sum_{l=1}^\infty u^l_n(x, y)A_{\psi^l}(\phi_2 v_n)(x) dx dy.
\] (33)

The test function \( \phi_1 \) is taken from the space \( L^p(\mathbb{R}^m; L^p(\mathbb{R}^d)) \) for some \( p \in (1, p) \), and the integral is well defined since by assumption \( (v_n) \) is specially bounded in \( L^q(\mathbb{R}^d) \), with \( q \) given in a).

The result of the theorem will follow easily if we show the summation sign in (33) can be put in front of the limit.

To this effect notice that \( \sum_{l=L}^\infty u^l_n \) is supported within the set
\[
\Omega^L_n = \{ x \in \Omega : u_n(x) > l \}
\]
for which we have shown in Lemma 7 that
\[
\lim_{L \to \infty} \sup_{n \in \mathbb{N}} \text{meas}(\Omega^L_n) = 0. \tag{34}
\]

For some \( \phi^m_1 \in C^\infty_c(\mathbb{R}^{d+m}) \) such that \( \| \phi_1 - \phi^m_1 \|_{L^p(\mathbb{R}^m; L^p(\mathbb{R}^d))} < \varepsilon \) we have
\[
\left| \lim_{n \to \infty} \int_{\mathbb{R}^{d+m}} \phi_1(x, y) \sum_{l=L}^\infty u^l_n(x, y)A_{\psi^l}(\phi_2 v_n)(x) dx dy \right|
\leq C \lim_{n \to \infty} \sup_{n \in \mathbb{N}} \| u_n \|_{L^p(\Omega^L_n)} \| \phi_2 v_n \|_{L^q(\mathbb{R}^d)} \left( \varepsilon + \| \phi^m_1 \|_{L^p(\mathbb{R}^d)} \right),
\]
where we have used that \( L^p(K) \hookrightarrow L^p(K) \) for a compact set \( K \subset \mathbb{R}^d \). As \( \varepsilon \) is arbitrary, by using (34) it follows that the above expression goes to zero as \( L \) goes to infinity. Thus we can shift the summation sign in (33), and we get
\[
\langle B, \phi_1 \overline{\phi_2} \otimes \overline{\psi} \rangle = \sum_{l=1}^\infty \lim_{n \to \infty} \int_{\mathbb{R}^{d+m}} \phi_1(x, y) u^l_n(x, y)A_{\psi^l}(\phi_2 v_n)(x) dx dy.
\]

Now it is enough to rely on (28) to conclude the proof.

If the sequence \( (u_n) \) from Theorem 9 is bounded in \( L^2 \) with respect to the velocity variable, then the corresponding H-distribution can be represented as an infinite (weighted) sum of the H-distributions \( \mu_i, \ i \in \mathbb{N} \), corresponding to the sequences \( (\int_{\mathbb{R}^m} u_n(\cdot, y)e_i(y)dy) \) and \( (v_n) \), where \( \{e_i\}_i \in \mathbb{N} \) is an orthonormal basis in \( L^2(\mathbb{R}^m) \). A similar representation holds for the H-measures (see the proof of [15, Proposition 12]), but in that case, by using the positivity property, it can be further simplified to the form given in (27).

**Proposition 19.** Denote by \( \mu \) the generalised H-distribution corresponding to (sub)-sequences \( (u_n) \), taken to be bounded in \( L^p(\mathbb{R}^d; L^q(\mathbb{R}^m)) \) \( \cap L^p(\mathbb{R}^{d+m}) \), \( p \in (1, 2) \), and \( (v_n) \), weakly converging to zero in \( L^2(\mathbb{R}^d) \) \( \cap L^q(\mathbb{R}^d) \), for some \( q \geq 2 \). Denoting by \( \mu_i \) H-distributions corresponding to \( (\int_{\mathbb{R}^m} u_n(\cdot, y)e_i(y)dy) \) and \( (v_n) \), the following representation holds
\[
\langle \mu, \phi_1 \overline{\phi_2} \otimes \overline{\psi} \rangle = \sum_{i=1}^\infty \langle \mu_i, \int_{\mathbb{R}^m} \phi_1(\cdot, y)e_i(y)dy \overline{\phi_2} \otimes \overline{\psi} \rangle, \tag{35}
\]
with test functions $\phi_1 \in L^p(R^m; L^{p'}(R^d)) \cap L^{p'}(R^d; L^{m}(R^m))$, $\phi_2 \in C_0(\mathbb{R}^d)$, and $\psi \in C^d(\mathbb{P})$.

**Proof:** Rewrite an arbitrary test function $\phi_1$ as

$$\phi_1(x, y) = \sum_{i=1}^{\infty} c_i(x)e_i(y),$$

where $c_i(x) = \int_{\mathbb{R}^m} \phi_1(x, y)e_i(y)dy$ and $(\sum_i |c_i(x)|^2)^{1/2}$ belongs to $L^{p'}(\mathbb{R}^d)$.

According to Theorem 9, for $\phi_1$ from above, and $\phi_2 \in C_0(\mathbb{R}^d)$, $\psi \in C^d(\mathbb{P})$ we have that

$$\langle \mu, \phi_1 \phi_2 \otimes \psi \rangle = \lim_{n \to \infty} \int_{\mathbb{R}^{d+m}} (\phi_1 u_n)(x, y) \mathcal{A}_\psi(\phi_2 v_n)(x) dx dy$$

$$= \lim_{n \to \infty} \int_{\mathbb{R}^d} \sum_{i=1}^{\infty} c_i(x) \int_{\mathbb{R}^m} u_n(x, y)e_i(y)dy \mathcal{A}_\psi(\phi_2 v_n)(x) dx,$$

(36)

By taking into account properties of the coefficients $c_i$, a procedure similar to the one applied in the preceding theorem enables us to estimate the limit of

$$\int_{\mathbb{R}^d} \sum_{i=1}^{\infty} c_i(x) \int_{\mathbb{R}^m} u_n(x, y)e_i(y)dy \mathcal{A}_\psi(\phi_2 v_n)(x),$$

which goes to zero as $L$ approaches infinity, uniformly with respect to $n$. Thus we can relocate the summation sign in (36) in order to get

$$\langle \mu, \phi_1 \phi_2 \otimes \psi \rangle = \sum_{i=1}^{\infty} \langle \mu_i(x, \xi), c_i \phi_2 \otimes \psi \rangle = \sum_{i=1}^{\infty} \langle \mu_i(x, \xi), \int_{\mathbb{R}^m} \phi_1(\cdot, y)e_i(y)dy \phi_2 \otimes \psi \rangle,$$

which completes the proof of (35). \qed

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