The longest increasing subsequence in involutions avoiding 3412 and another pattern

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Abstract

In this note, we study the mean length of the longest increasing subsequence of a uniformly sampled involution that avoids the pattern 3412 and another pattern.

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MSC2010: Primary 05A05, 05A16; Secondary 05A15.

1 Introduction

In this paper we study the longest increasing subsequence of involutions avoiding 3412 and another pattern. A permutation σ = σ₁σ₂ · · · σₙ of length n is defined as an arrangement of the elements of the set [n] := {1, 2, · · · , n}. A permutation σ is called an involution if σ = σ⁻¹, where σ⁻¹ᵢ = j if and only if σⱼ = i. We use notations Sₙ and Iₙ to denote, respectively, the set of all permutations and the set of all involutions of length n. A subsequence of σ ∈ Sₙ is defined as a sequence σᵢ₁σᵢ₂ · · · σᵢₖ, where 1 ≤ i₁ < i₂ < · · · < iₖ ≤ n. The subsequence is called an increasing subsequence if σᵢ₁ < σᵢ₂ < · · · < σᵢₖ.

For any permutation σ, there is at least one longest increasing subsequence. We denote the length of this subsequence by Lₙ(σ). The celebrated Ulam’s problem is concerned with the asymptotic behavior, as n tends to infinity, of the expectation of Lₙ(σ) when σ is chosen uniformly from Sₙ [1, 11]. The classical Ulam’s problem has been extended and generalized in various directions [13, 14]. In particular, asymptotic behavior of the distribution of the longest increasing subsequence of random involutions is the topic of [2, 7].

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Variations of Ulam’s problem have been considered also for permutations in $S_n$ avoiding certain patterns [3, 8, 9, 10]. For permutations $\pi = \pi_1 \pi_2 \cdots \pi_k \in S_k$ and $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$, we say that $\sigma$ contains pattern $\pi$ if there exist $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that

$$\sigma_{i_s} < \sigma_{i_t} \quad \text{if and only if} \quad \pi_s < \pi_t \quad \text{for all} \quad 1 \leq s, t \leq k.$$ 

For instance, the permutation 15243 contains 321 as a pattern because it has the subsequences $5*43$, and 543 matches the pattern 321. If $\sigma$ does not contain $\pi$ as a pattern, then we say that $\sigma$ avoids $\pi$ or $\sigma$ is a $\pi$-avoiding permutation. We denote by $S_n(\pi)$ and $I_n(\pi)$, respectively, the sets of $\pi$-avoiding permutations and $\pi$-avoiding involutions of $[n]$.

The goal of this paper is to study Ulam’s problem in the context of involutions in $I_n$ avoiding 3412 and another pattern. In [4] Egge connected generating functions for various subsets of $I_n(3412)$ with continued fractions and Chebyshev polynomials of the second kind, and gave a recursive formula for computing them. The formula exploits a bijection between $I_n(3412)$ and Motzkin paths established in [6]. Many of the results in [4] are concerned with statistics of decreasing subsequences of involutions in $I_n(3412)$. Later, Egge and Mansour [5] extended the results in [4] to certain bivariate generating functions involving statistics of two-cycles in involutions. In this paper we extend the method of [4, 5] to certain bivariate generating functions involving the statistic $L_n(\sigma)$, and use it as a tool for studying the Ulam’s problem for such pattern-restricted involutions.

For a given set of patterns $T$, let $I_n(T) = \bigcap_{\tau \in T} I_n(\tau)$ and denote by $P_{n,T}$ the uniform distribution on $I_n(T)$. Thus, the probability of choosing any $\sigma \in I_n(T)$ under $P_{n,T}$ is $\frac{1}{|I_n(T)|}$, where $|\cdot|$ is the size of the set. We use the notations $E_{n,T}(\cdot)$ and $\text{Var}_{n,T}(\cdot)$ to denote, respectively, the expectation and the variance operators under $P_{n,T}$. We use the shortcut $L_n$ to denote the random variable $L_n(\sigma)$, where $\sigma \in S_n$ is a random permutation sampled uniformly from $I_n(T)$.

Throughout the paper, we write $a_n \sim b_n$ to indicate that $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$. We have:

**Theorem 1.1.** Consider $L_n$ on $I_n(T)$ under the uniform probability measure. Then we have the following:

(i) If $T = \{3412\}$, then $E_{n,T}(L_n) = \frac{4n}{9}$.

(ii) If $T = \{3412, 123\}$, then $E_{n,T}(L_n) = \frac{n^{5/2} + 3/4 + (-1)^n/4}{n^2 / 4 + 7/8 + (-1)^n / 8} \sim 2$.

(iii) If $T = \{3412, 213\}$ or $T = \{3412, 132\}$, then $E_{n,T}(L_n) \sim \frac{n}{\sqrt{3}}$.

(iv) If $T = \{3412, 321\}$, then $E_{n,T}(L_n) \sim \frac{3 + \sqrt{5}}{5 + \sqrt{5}} n$.

(v) If $T = \{3412, 123 \cdots k\}$ for some $k \geq 1$, then $E_{n,T}(L_n) \sim k - 1$.

(vi) If $T = \{3412, 4123\}$, then,

$$E_{n,T}(L_n) \sim \frac{1}{457}(198\alpha^3 - 246\alpha^2 - 131\alpha + 299)n \approx 0.454689799955 \cdots n.$$ 

Here $\alpha$ is the complex root of smallest absolute value of the polynomial $3x^4 - 3x^3 - x^2 + 3x - 1$. 

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| $\tau$ | $H_n(x, q) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{I}_n(3412, \tau)} x^n q^{l_n(\sigma)}$ | $E_{n,T} = E_{n,T}(L_n), V_n,T = \text{Var}_{n,T}(L_n)$ for $T = \{3412, \tau\}$ |
|-------|----------------------------------|-------------------------------------------------|
| 1234  | $1 + \frac{x}{(1-x)^2} + \frac{x^2}{(1-x)^3} \alpha^2 + \frac{x^3(x^2+1)}{(1-x)^4(\alpha+1)} \tau^3$ | $E_{n,T} \sim 3, V_{n,T} \sim \frac{12}{7\tau}$ |
| 1243, 2134, 1324 | $1 + \frac{q\alpha x^2(1+\alpha(q-2)x^3)[1-xq]}{1-\alpha q x^3(1-xq)}$ | $E_{n,T} \sim \frac{3}{\sqrt{q}}, V_{n,T} \sim \frac{1}{12} n$ |
| 1342, 1423, 2314, 3124 | $\frac{(q-1)x^3+2x^2+x-1}{2x^2-(1+q)x+1}$ | $E_{n,T} \sim \frac{(3-2\alpha)(\alpha+1)}{7}, V_{n,T} \sim \frac{-7\alpha^2+5\alpha+10}{49} n$, where $\alpha^3 - \alpha^2 - 2\alpha + 1 = 0, \alpha \approx 0.45004$ |
| 1432, 3214, 2143, 4231 | $\frac{1-x}{x-q}$ | $E_{n,T} \sim \frac{2}{3}, V_{n,T} \sim \frac{4}{3} n$ |
| 2341, 4123 | $1 - \frac{x^2}{x^2+q} \frac{1}{1-\alpha^2x(1+x)}$ | $E_{n,T} \sim \frac{18\alpha^3 - 246\alpha^2 - 131\alpha + 299}{457} n$, $V_{n,T} \sim \frac{2880\alpha^3 - 7157\alpha^2 - 8959\alpha + 4723}{209849} n$, where $3\alpha^4 - 3\alpha^3 - \alpha^2 + 3\alpha - 1 = 0, \alpha \approx 0.45209$ |
| 2413, 3142 | $\frac{1-x^2}{x^2+q} \frac{x}{1-(q+1)x(1-xq)}$ | $E_{n,T} \sim \frac{49}{4}, V_{n,T} \sim \frac{49}{27} n$ |
| 2431, 3241, 4132, 4213 | $\frac{\sqrt{x^2+q(x-1)x^2}-2x}{2x^2}$ | $E_{n,T} \sim \frac{2(\alpha+1)(\alpha+2)}{7}, V_{n,T} \sim \frac{-7\alpha^2+4\alpha+13}{49} n$, where $\alpha^3 - \alpha^2 - 2\alpha + 1 = 0, \alpha \approx 0.44504$ |
| 3421, 4312 | $\frac{1-x}{x(1-x)(q+1)x^2}\frac{1}{1-\alpha^2x(1-xq)}$ | $E_{n,T} \sim \frac{x}{x} n, V_{n,T} \sim \frac{x}{x} n$ |
| 4321 | $\frac{1-x}{x(1-x)^2+q(x-1)x^2-2x^2}$ | $E_{n,T} \sim \frac{1}{n}, V_{n,T} \sim \frac{1}{n} n$ |

Table 1: The list of the generating functions and asymptotic values of the mean and variance of the length of the longest increasing subsequence for uniformly random involutions from $I_n(3412, \tau)$ with $\tau \in S_4$.

(vii) If $T = \{3412, 4321\}$, then $E_{n,T}(L_n) \sim \frac{5n}{8}$. Since 3412 contains the patterns 231 and 312, we have

$$I_n(3412, 231) = I_n(231) \quad \text{and} \quad I_n(3412, 312) = I_n(312).$$

As shown in section 3.2 of [8], $E_{n,T}(L_n) = \frac{n+1}{2}$ for $T = \{3412, 231\}$ and $T = \{3412, 312\}$. Thus, Theorem 1.1 covers all possible cases for $I_n(3412, \tau)$ with $\tau \in S_4$.

Using similar arguments we also obtained the asymptotic of $E_{n,T}(L_n)$ and $\text{Var}_{n,T}(L_n)$ for all possible cases $I_n(3412, \tau)$ with $\tau \in S_4$. We summarize these results in Table 1, without explicit calculations for the sake of space.

The rest of the paper is organized as follows. In Section 2 we consider $I_n(3412)$ and prove part (i) of Theorem 1.1. In Section 3 we consider $I_n(3412, \tau)$ with various patterns $\tau$ and prove the rest of Theorem 1.1.
2 Longest increasing subsequences in $I_n(3412)$

For $\rho \in S_k$ and $\sigma \in S_m$, we denote by $\rho \oplus \sigma$ their direct sum, which is a permutation in $S_{k+m}$ given by $\rho_1 \cdots \rho_k (\sigma_1 + k) \cdots (\sigma_m + k)$. Similarly, we denote by $\rho \ominus \sigma$ the skew sum of $\rho$ and $\sigma$, which is an element of $S_{k+m}$ given by $(\rho_1 + m) \cdots (\rho_k + m) \sigma_1 \cdots \sigma_m$.

Our proofs make use of the following recursive structure of the involutions in $I_n(3412)$, for the details see [6, Remark 4.28] and [4, Proposition 2.9]:

**Proposition 2.1.** Let $\rho \in I_n(3412)$. Then either

(i) $\rho = 1 \oplus \rho'$ and $\rho' \in I_{n-1}(3412)$, or

(ii) $\rho = (1 \ominus \rho'' \ominus 1) \oplus \rho'$, where $\rho'' \in I_{m-2}(3412)$ and $\rho' \in I_{n-m}(3412)$ for some $m \geq 2$.

**Proof of Theorem 1.1-(i).** Let $H(x, q)$ be the generating function for the number of involutions in $I_n(3412)$ according to the length of the longest increasing subsequence. More precisely,

$$H(x, q) = \sum_{n \geq 0} \sum_{\sigma \in I_n(3412)} x^n q^{L_n(\sigma)}. \quad (1)$$

To obtain a closed form for $H(x, q)$, we partition $I_n(3412)$ as a union of the following four non-overlapping subsets, by virtue of Proposition 2.1:

(i) $I_{n,1}$ - the set of the empty involution;
(ii) $I_{n,2}$ - the set of the involutions in $I_n(3412)$ that start with 1;
(iii) $I_{n,3}$ - the set of the involutions in $I_n(3412)$ that start with 21;
(iv) $I_{n,4}$ - the set of the involutions in $I_n(3412)$ that can be written as $(1 \ominus \sigma'' \ominus 1) \oplus \sigma'$, where $\sigma''$ is a nonempty 3412-avoiding involution and $\sigma'$ is any 3412-avoiding involution.

Adding together contributions of all the four sets, we obtain:

$$H(x, q) = \underbrace{1 + xqH(x, q)}_{I_{n,1}} + x^2qH(x, q) + x^2(H(x, q) - 1)H(x, q) \quad \underbrace{+ x^2}_{I_{n,2}} \underbrace{+ x^2}_{I_{n,3}} \underbrace{+ x^2}_{I_{n,4}}.$$ 

Hence,

$$H(x, q) = \frac{1 - xq - x^2(q - 1) - \sqrt{(1 - xq - x^2(q - 1))^2 - 4x^2}}{2x^2}.$$ 

Note that $H(x, 1) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$, which is the generating function for Motzkin numbers [4, 6]. Furthermore,

$$\frac{\partial}{\partial q} H(x, q) \bigg|_{q=1} = -\frac{x+1}{2x} + \frac{1+x^2}{2x\sqrt{1-2x-3x^2}}.$$ 

Hence,

$$E_n,3412(L_n) = \frac{[x^n]\frac{\partial}{\partial q} H(x, q) \bigg|_{q=1}}{[x^n]H(x, 1)} \sim \frac{2n\sqrt{2}}{9\pi nn^{n+1}} \frac{3^{n+1}}{2\sqrt{3}} \sim 4n \frac{3^{n+1}}{9},$$

which completes the proof of Theorem 1.1-(i).
3 Longest increasing subsequences in $I_n(3412, \tau)$

In this section, we extend our arguments from $I_n(3412)$ to $I_n(3412, \tau)$ for various patterns $\tau$. Toward this end, similar to (1), we define

$$H_\tau(x, q) = \sum_{n \geq 0} \sum_{\sigma \in I_n(3412, \tau)} x^n q^{L_n(\sigma)}.$$ 

More generally, for a collection of patterns $T$, we set

$$H_T(x, q) = \sum_{n \geq 0} \sum_{\sigma \in I_n(3412) \cap I_n(T)} x^n q^{L_n(\sigma)}.$$ 

When $T = \{\tau, \tau'\}$, for simplicity, we write $H_{\tau, \tau'}(x, q)$. We also set $H_{\emptyset}(x, q) := 0$ and let $H_{\tau', \tau'}(x, q) := H_{\tau}(x, q) - H_{\tau', \tau}(x, q)$ denote the corresponding generating function for the involutions in $I_n(3412, \tau)$ that contain the pattern $\tau'$. We call a permutation irreducible if it cannot be represented as a direct sum of two nonempty permutations. It is easy to show that every permutation $\rho$ can be written as a direct sum

$$\rho = \rho^{(1)} \oplus \rho^{(2)} \oplus \cdots \oplus \rho^{(k)},$$

where $\rho^{(1)}, \ldots, \rho^{(k)}$ are nonempty irreducible permutations, uniquely determined by $\rho$. We next introduce a bar operator for permutations following [4].

**Definition 3.1.** For $\rho \in S_m$, define $\overline{\rho}$ as follows:

1. $\overline{\emptyset} = \emptyset$ and $\overline{T} = \emptyset$.

2. If $m \geq 2$ and there exists a permutation $\sigma$ such that $\rho = 1 \ominus \sigma \ominus 1$, then $\overline{\rho} = \sigma$.

3. If $m \geq 2$ and there exists a permutation $\sigma$ such that $\rho = 1 \ominus \sigma$, and $\sigma$ does not end with 1, then $\overline{\rho} = \sigma$.

4. If $m \geq 2$ and there exists a permutation $\sigma$ such that $\rho = \sigma \ominus 1$, and $\rho$ does not begin with $m$, then $\overline{\rho} = \sigma$.

5. If $m \geq 2$ and $\rho$ does not begin with $m$, and it does not end with 1, then $\overline{\rho} = \rho$.

Our main technical tool for calculating the corresponding generating functions for the classes $I_n(3412, \tau)$ is the following extension of a result for $I_n(3412)$ given by Corollary 5.6 in [4].

**Proposition 3.2.** Suppose that $\tau = \tau^{(1)} \oplus \tau^{(2)} \oplus \cdots \oplus \tau^{(s)}$ is a direct sum of nonempty irreducible permutations $\tau^{(1)}, \ldots, \tau^{(s)}$ such that $\tau^{(1)}$ is not a decreasing sequence. For $i \in [s]$, define $\theta^{(i)} := \overline{\tau^{(1)} \oplus \cdots \oplus \tau^{(i)}}$ and $\theta^{<i>} := \tau^{(i)} \oplus \cdots \oplus \tau^{(s)}$.

Then we have:
(i) If \( \tau^{(1)} = 1 \), then
\[
H_\tau(x, q) = 1 + \frac{xq}{1-x} H_{\theta^{(1)}(x, q)} + x^2 \sum_{i=2}^{s} \left\{ H_{\theta^{(1)}(i)/12}(x, q) - H_{\theta^{(i-1)/12}}(x, q) \right\} H_{\theta^{(i)}(x, q)}. 
\]

(ii) If \( \tau^{(1)} = 21 \), then
\[
H_\tau(x, q) = 1 + xq H_{\rho}(x, q) + \frac{x^2 q}{1-x} H_{\theta^{2}}(x, q) + x^2 \sum_{i=2}^{s} \left\{ H_{\theta^{(1)}(i)/12}(x, q) - \delta_{i>2} H_{\theta^{(i-1)/12}}(x, q) \right\} H_{\theta^{(i)}(x, q)}, 
\]
where \( \delta_A \) is one if \( A \) is true, and is zero otherwise.

(iii) If \( \tau^{(1)} = m(m-1) \cdots 1 \) with \( m \geq 3 \), then
\[
H_\tau(x, q) = 1 + \sum_{i=1}^{s} \left\{ H_{\theta^{(1)}(i)/12}(x, q) - H_{\theta^{(i-1)/12}}(x, q) \right\} H_{\theta^{(i)}(x, q)}. 
\]

(iv) If \( \tau^{(1)} \neq m(m-1) \cdots 1 \) and \( \rho^{(1)} \in S_m \) with \( m \geq 3 \), then
\[
H_\tau(x, q) = 1 + \frac{xq}{1-x} H_{\rho}(x, q) + x^2 \sum_{i=1}^{s} \left\{ H_{\theta^{(1)}(i)/12}(x, q) - H_{\theta^{(i-1)/12}}(x, q) \right\} H_{\theta^{(i)}(x, q)}. 
\]

We will only prove parts (i) and (iv) of the proposition. The proofs of the other two cases are very similar, and therefore are omitted.

**Proof of Proposition 3.2-(i).** Assume first that \( \tau^{(1)} = 1 \). We partition the set \( I_n(3412, \tau) \) into three non-overlapping subsets:

(i) \( J_{n,1} \) - the set of the empty involution;
(ii) \( J_{n,2} \) - the set of those involutions of the form \( r(r+1) \cdots 1 \oplus \sigma' \) for some \( r \geq 2 \);
(iii) \( J_{n,3} \) - the set of those involutions which do not begin with a decreasing sequence.

It is easy to see that the involutions in the sets \( J_{n,1} \) and \( J_{n,2} \) contribute 1 and \( xq H_\rho(x, q) \), respectively, to \( H_\tau(x, y) \). To obtain the contribution of the involutions in the set \( J_{n,3} \), we first observe that in view of Proposition 2.1, all involutions in \( J_{n,3} \) can be written in the form \( \sigma = (1 \oplus \sigma'' \ominus 1) \oplus \sigma' \) with \( \sigma'' \) that contains 12. Thus, the involutions in \( J_{n,3} \) that avoid \( \tau^{(1)} \) contribute \( x^2 H_{\theta^{(i)/12}}(x, q) H_\tau(x, q) = 0 \). Furthermore, any involution in \( J_{n,3} \) that
contains $\tau^{(1)}$, avoids $\theta^{(i)}$ and contains $\theta^{(i-1)}$ for some $i = 2, 3, \ldots, s$. The total contribution of such involutions into $H_\tau(x, q)$ is equal to

$$x^2 \sum_{i=2}^s (H_{\theta^{(i)}/12}(x, q) - H_{\theta^{(i-1)}/12}(x, q)) H_{\theta^{<i>}}(x, q).$$

Adding together the contributions of $J_{n,1}$, $J_{n,2}$, and $J_{n,3}$, we obtain the desired result. \qed

**Proof of Proposition 3.2-(iv).** Suppose now that $\tau^{(1)} \neq m(m-1)\cdots1$ and $\tau^{(1)} \in S_m$ with $m \geq 3$. We will consider again the partition $I_n(3412, \tau) = \bigcup_{k=1}^3 J_{n,k}$ defined in the course of the proof of part (i) of the proposition. It is easy to verify that in this case, $\mathcal{J}_{n,1}$ contributes 1 to $H_\tau(x, q)$, while permutations in the set $\mathcal{J}_{n,2}$ contribute $\frac{x^2}{1-x} H_\tau(x, y)$. To obtain the contribution of $\mathcal{J}_{n,3}$, recall that by Proposition 2.1, all involutions in this set have the form $\sigma = (1 \oplus \sigma'' \oplus 1) \oplus \sigma'$ where $\sigma''$ contains 12. Thus, the involutions in $\mathcal{J}_{n,3}$ that avoid $\tau^{(1)}$ contribute $x^2 H_{\theta^{(1)}/12}(x, q) H_\tau(x, q)$, while the involutions in $\mathcal{J}_{n,3}$ that contain $\tau^{(1)}$ contribute

$$x^2 \sum_{i=2}^s (H_{\theta^{(i)}/12}(x, q) - H_{\theta^{(i-1)}/12}(x, q)) H_{\theta^{<i>}}(x, q).$$

Adding up all the contributing terms listed above, yields the desired result. \qed

The rest of this section is divided into five subsections, each one is concerned with $I_n(3412, \tau)$ for a particular type of pattern $\tau$ and presents the proof of the corresponding part in Theorem 1.1.

### 3.1 $E_{n,T}(L_n)$ on $I_n(3412, \tau)$ with $\tau \in S_2$

Note that the only involution in $I_n(3412, 12)$ is $n(n-1)\cdots1$. Thus,

$$H_{12}(x, q) = 1 + \frac{xq}{1-x}. \quad (2)$$

Similarly, the only involution in $I_n(3412, 21)$ is $12\cdots n$. Thus,

$$H_{21}(x, q) = \frac{1}{1-xq}.$$

### 3.2 $E_{n,T}(L_n)$ on $I_n(3412, \tau)$ with $\tau \in S_3$

**Proof of Theorem 1.1-(ii).** An application of Proposition 3.2-(i) with $\tau = 1 \oplus 1 \oplus 1 = 123$ gives

$$H_{123}(x, q) = 1 + \frac{xq}{1-x} H_{12}(x, q) + x^2(H_{12/12}(x, q) - H_{1/12}(x, q)) H_{12}(x, q)$$

$$+ x^2(H_{123/12}(x, q) - H_{12/12}(x, q)) H_1(x, q).$$

It follows from (2) and the decomposition

$$H_{123/12}(x, q) = H_{123}(x, q) - H_{12}(x, q) \quad (3)$$
that
\[ H_{123}(x, q) = 1 + \frac{xq}{1-x} \left( 1 + \frac{xq}{1-x} \right) + x^2 H_{123}(x, q) - x^2 \left( 1 + \frac{xq}{1-x} \right). \]

Therefore,
\[ H_{123}(x, q) = 1 + \frac{xq(1-x(1-q) - x^2 + x^3)}{(1-x)^3(1+x)}. \]

Hence, for \( T = \{3412, 123\} \) we have:
\[
E_{n,T}(L_n) = \frac{[x^n] \frac{\partial}{\partial q} H_{123}(x, q) \big|_{q=1}}{[x^n] H_{123}(x, 1)} = \frac{n^2/2 + 3/4 + (-1)^n/4}{n^2/4 + 7/8 + (-1)^n/8} \sim 2. \]

\( \square \)

**Proof of Theorem 1.1-(iii).** Proposition 3.2-(i) implies that for \( \tau = 1 \oplus 21 = 132, \)
\[ H_{132}(x, q) = 1 + \frac{xq}{1-x} H_{212}(x, q) + x^2(H_{132/12}(x, q) - H_{1/12}(x, q))H_{21}(x, q). \]

Using (3) and the fact that \( H_{1/12}(x, q) = 0, \) we get
\[ H_{132}(x, q) = \frac{1-x^2(1-q)}{1-xq-x^2}. \]

Therefore, for \( T = \{3412, 132\} \) we have:
\[
E_{n,T}(L_n) \sim \frac{n}{\sqrt{5}}. \]

We next apply Proposition 3.2-(ii) to \( \tau = 21 \oplus 1 = 213, \) to get
\[ H_{213}(x, q) = 1 + xqH_{213}(x, q) + \frac{x^2q}{1-x} + x^2(H_{213}(x, q) - H_{12}(x, q))H_{1}(x, q). \]

It follows then from (2) that
\[ H_{213}(x, q) = \frac{1-x^2(1-q)}{1-xq-x^2}. \]

Hence, for \( T = \{3412, 213\} \) we have: \( E_{n,T}(L_n) \sim \frac{n}{\sqrt{5}}. \)

\( \square \)

**Proof of Theorem 1.1-(iv).** An application of Proposition 3.2-(iii) to \( \tau = 321 \) yields:
\[ H_{321}(x, q) = 1 + (x + x^2)qH_{321}(x, q) + x^2(H_{1}(x, q) - H_{12}(x, q))H_{321}(x, q). \]

Since \( H_{1}(x, q) = H_{1,12}(x, q) = 1, \) this implies that
\[ H_{321}(x, q) = \frac{1}{1-qx-qx^2}. \]

Thus, for \( T = \{3412, 321\} \) we have \( E_{n,T}(L_n) \sim \frac{3+\sqrt{5}}{5+\sqrt{5}}n. \)

\( \square \)
3.3 $E_{n,T}(L_n)$ on $I_n(3412, \tau)$ with $\tau = 12 \cdots k$

Proof of Theorem 1.1-(v). Let $F_k(x, q) := H_{12-k}(x, q)$. Applying Proposition 3.2 to the permutation $\tau = 12 \cdots k$ with $k \geq 1$, we obtain:

$$F_k(x, q) = 1 + \frac{xq}{1-x} F_{k-1}(x, q) + x^2 \sum_{i=3}^{k} F_i(x, q) - F_{i-1}(x, q) F_{k-i+1}(x, q).$$

Let $F(x, q; y) := \sum_{k \geq 1} F_k(x, q) y^k$. Multiplying both sides of the above recurrence equation by $y^k$, summing over $k \geq 1$, and using the fact that $F_0(x, q) = 0$ and $F_1(x, q) = 1$, we obtain:

$$F(x, q; y) = \frac{y}{1-y} + \frac{xqy}{1-x} F(x, q; y) + \frac{x^2}{y} F(x, q; y) F(x, q; y) - y^2 H_{12}(x, q) F(x, q; y) - x^2 F(x, q; y) - y^2 F(x, q; y) + x^2 y F(x, q; y).$$

Taking (2) into account and solving for $F(x, y; q)$, we obtain:

$$F(x, y; q) = \frac{1}{1-y} + \frac{y}{1-y} \frac{(1-qxy - (1+qy)x^2 - \sqrt{(1-qyx - (qy+1)x^2)^2 - 4q(1+x)x^3y})}{(1-qyx)(1-x-qyx)^2} C \left( \frac{qxy^2}{(1-x)(1-x-qyx)^2} \right),$$

where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers $c_n = \frac{1}{n+1} \binom{2n}{n}$.

Substituting a series representation of the generating function from A001263 in [12], we obtain that

$$F(x, y; q) = \frac{y}{1-y} + \frac{1}{1-y} \sum_{j=0}^{k-2} \sum_{i=1}^{j+1} \frac{1}{i!} (\frac{1}{i-1}) x^{2i+j-1} (1-x)^{2j+1} (1+x)^j q^{j+1} y^{j+2}.$$ 

Therefore, for $k \geq 2$ we have:

$$[y^k] F(x, y; q) = 1 + \sum_{j=0}^{k-2} \sum_{i=1}^{j+1} \frac{1}{i!} (\frac{1}{i-1}) x^{2i+j-1} (1-x)^{2j+1} (1+x)^j q^{j+1} [y^{j+1}].$$

Hence, for all $k \geq 2$, using the usual bracket notation for coefficient extraction,

$$[x^n y^k] F(x, 1; y) \sim \frac{1}{(k-1)^2(2k-4)!} \binom{2k-4}{k-2} n^{2k-4}$$

and

$$[x^n y^k] \frac{\partial}{\partial q} F(x, q; y) \bigg|_{q=1} \sim \frac{1}{2^{k-2}(2k-4)!} \binom{2k-4}{k-2} n^{2k-4},$$

which yields the result in Theorem 1.1-(v). \qed
3.4 \( E_{n,T}(L_n) \) on \( I_n(3412, \tau) \) with \( \tau = k12 \cdots (k - 1) \)

**Proof of Theorem 1.1-(vi).** Let \( G_k(x, q) := H_{k12 \cdots (k-1)}(x, q) \). Applying Proposition 3.2-(iv) to \( \tau = k12 \cdots (k - 1) \) with \( k \geq 3 \), we obtain that

\[
G_k(x, q) = 1 + \frac{xq}{1-x} G_k(x, q) + x^2(F_{k-1}(x, q) - F_2(x, q))G_k(x, q),
\]

which in view of (2) leads to

\[
G_k(x, q) = \frac{1}{1 - \frac{xq}{1-x} - x^2(F_{k-1}(x, q) - 1 - \frac{xq}{1-x})}.
\]

Taking (4) into account, we arrive to the following result:

**Lemma 3.3.** For \( k \geq 3, \)

\[
H_{k12 \cdots (k-1)}(x, q) = \frac{1}{1 - \frac{xq}{1-x} - x^2 \sum_{j=1}^{k-3} \sum_{i=1}^{j} \sum_{j=1}^{i+1} \frac{1}{1-x} \frac{(j-1)(j+1)}{(1-x)^{2i+1} (1+x)} q^{j+1}}.
\]

For example, \( H_{4123}(x, q) = \frac{1}{1-\frac{xq}{1-x} - x^2 q^2/((1-x)^3(1+x))} \). Let \( \alpha \) be the root of smallest absolute value of the polynomial \( 3x^4 - 3x^3 - x^2 + 3x - 1 \). Thus \( \alpha \approx 0.45208778430 \), and for \( T = \{3412, 4123\} \) we have:

\[
E_{n,T}(L_n) = \left[ \frac{[x^n] \frac{\partial}{\partial q} H_{4123}(x, q)}{[x^n] H_{4123}(x, 1)} \right]_{q=1}
\]

\[
\sim \frac{1}{457} (198\alpha^3 - 246\alpha^2 - 131\alpha + 299)n \approx 0.454689799955 \cdots n.
\]

This completes the proof of Theorem 1.1-(vi).

\[\Box\]

3.5 \( E_{n,T}(L_n) \) on \( I_n(3412, \tau) \) with \( \tau = k(k-1) \cdots 1 \)

**Proof of Theorem 1.1-(viii).** Let \( F_k(x, q) := H_{k(k-1) \cdots 1}(x, q) \). Applying Proposition 3.2 to the permutation \( \tau = k(k-1) \cdots 1 \) with \( k \geq 3 \), we see that

\[
F_k(x, q) = 1 + (x + x^2 + \cdots + x^{k-1})q F_k(x, q) + x^2(F_{k-2}(x, q) - 1 - (x + x^2 + \cdots + x^{k-3})q) F_k(x, q).
\]

Thus,

\[
F_k(x, q) = \frac{1}{1 - qx - (q-1)x^2 - x^2 F_{k-2}(x, q)}
\]

with \( F_1(x, q) = 1 \) and \( F_2(x, q) = \frac{1}{1-x} \). Iterating this equation, one can obtain an expression for \( F_k(x, q) \) in the form of finite continued fractions. Alternatively, \( F_k(x, q) \) can be expressed in terms of Chebyshev polynomials.

Recall that Chebyshev polynomials of the second kind can be defined as the solution to the recursion

\[
U_n(t) = 2t U_{n-1}(t) - U_{n-2}(t)
\]

with initial conditions \( U_0(t) = 1 \) and \( U_1(t) = 2t \). Using this recursion and induction, one can derive the following result from (5).

\[10\]
Lemma 3.4. For all \( k \geq 1 \),

\[
H_{(2k+1)(2k)-1}(x, q) = \frac{U_{k-1} \left( \frac{1-qx-(q-1)x^2}{2x} \right) - xU_{k-2} \left( \frac{1-qx-(q-1)x^2}{2x} \right)}{x \left( U_k \left( \frac{1-qx-(q-1)x^2}{2x} \right) - xU_{k-1} \left( \frac{1-qx-(q-1)x^2}{2x} \right) \right)}
\]

and

\[
H_{(2k+2)(2k+1)-1}(x, q) = \frac{1-xqU_{k-1} \left( \frac{1-qx-(q-1)x^2}{2x} \right) - U_{k-2} \left( \frac{1-qx-(q-1)x^2}{2x} \right)}{x \left( 1-xqU_k \left( \frac{1-qx-(q-1)x^2}{2x} \right) - U_{k-1} \left( \frac{1-qx-(q-1)x^2}{2x} \right) \right)}.
\]

We remark that the results in Lemma 3.4 with \( q = 1 \) recover formulas (7) and (8) in [4] for ordinary generating functions for the number of involutions avoiding 3412 and \( k(k-1)\cdots1 \).

An application of the lemma with \( k = 1 \) yields for \( T = \{3412, 4321\} \):

\[
E_{n,T}(L_n) = \left[ x^n \right] \frac{\partial}{\partial q} H_{4321}(x,q) \bigg|_{q=1} \sim \frac{5}{8} n,
\]

which completes the proof of Theorem 1.1-(vii).

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