Fractional Darboux Transformations

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Abstract

In this paper we utilize the covariance of Riccati equation with respect to linear fractional transformations to define classes of conformally equivalent second order differential equations. This motivates then the introduction of fractional Darboux transformations which can be recognized also as generalized Cole-Hopf transformations. We apply these transformations to find Schrodinger equations with isospectral potentials and to the linearization of some new classes of nonlinear partial differential equations.
1 Introduction

Darboux transformations [1,2,3] of the form

$$\psi = \left[ A(x) + B(x) \frac{\partial}{\partial x} \right] \phi(x) \quad (1.1)$$

have many different applications in mathematical physics [9,10,11,12,13] and the theory of differential equations. Among these one can find the Factorization method (raising and lowering operators for one or several coupled differential equations [1,4]) separation of coupled Schrodinger equations [6] and applications to nonlinear integrable equations. Furthermore with proper generalization this transformation has been applied to multidimensional problems [6,10,14,16] and discrete systems [15,17].

In its simplest context the transformation (1.1) “connects” the solutions of two Schrodinger equations with different potentials $u(x), v(x)$ i.e.

$$\phi'' = (u(x) + \lambda) \phi \quad (1.2)$$

$$\psi'' = (v(x) + \lambda) \psi. \quad (1.3)$$

Letting $B(x) = 1$ one can easily show that in order for eqs (1.2), (1.3) to be related by the transformation (1.1) $A(x), u(x), v(x)$ must satisfy;

$$A'' + u' + A(u - v) = 0 \quad (1.4)$$

$$2A' + u - v = 0 \quad (1.5)$$

Eliminating $(u - v)$ between these equations and integration yields

$$A' - A^2 + u = -\nu \quad (1.6)$$

where $\nu$ is an integration constant. Eq. (1.6) is a Riccati equation which can be linearized by the transformation $A = -\zeta'/\zeta$ which leads to

$$\zeta'' = (u(x) + \nu)\zeta. \quad (1.7)$$
Thus ζ is an eigenfunction of the original eq. (1.2) with λ = ν. From (1.5) we now infer that

\[ v = u - 2(lnζ)'' \]  

(1.8)

i.e. a Darboux transformation changes the potential function \( u(x) \) by \( Δu = -2(lnζ)'' \) where ζ is an arbitrary eigenfunction of (1.2).

Our objective in this paper is to introduce fractional Darboux transformations which are defined as transformations of the form

\[ ψ = \frac{A(x)φ(x) + B(x)φ'}{C(x)φ(x) + D(x)φ'} \]  

(1.9)

and elaborate on some of their applications to linear and nonlinear differential equations. To motivate the introduction of these transformations we use the covariance of the Riccati equation with respect to linear fractional transformations. We then show that this covariance induces an equivalence relation between different second order differential equations whose solutions are related by a transformation of the form (1.9) (Sec. 2). In Section 3 we consider the classification problem for these “conformally equivalent” equations. In Section 4 and 5 we explore then the application of the transformation (1.2) to Schrödinger equations and nonlinear equations. We end with some observations and conclusions in Section 6.

2 Conformally Equivalent Equations

Consider a linear second order differential equation

\[ p(x)w'' + q(x)w' + r(x)w = 0 \]  

(2.1)

(with smooth enough coefficients).

Introducing

\[ y = \frac{w'}{w} \]  

(2.2)

eq. (2.1) is transformed into a Riccati equation

\[ y' = -y^2 - \frac{q(x)}{p(x)} y - \frac{r(x)}{p(x)}. \]  

(2.3)
However it is well known that Riccati equation is “covariant” with respect to the linear fractional transformation \([5,7]\)

\[
y = \frac{\alpha(x)z(x) + \gamma(x)}{\beta(x)z(x) + \delta(x)}
\]

\[\text{(2.4)}\]

i.e. such a transformation takes a Riccati equation into another one in its class

\[
z' = F(x)z^2 + G(x)z + H(x)
\]

\[\text{(2.5)}\]

In particular for eq. \((2.3)\) we have;

\[
F = -[\alpha^2 + \alpha \beta \left( \frac{q(x)}{p(x)} \right) + \beta^2 \left( \frac{r(x)}{p(x)} \right) + (\alpha' \beta - \alpha \beta')]/\Delta
\]

\[\text{(2.6)}\]

\[
G = -[2\alpha \gamma + (\alpha \delta + \beta \gamma) \left( \frac{q(x)}{p(x)} \right) + 2\beta \delta \left( \frac{r(x)}{p(x)} \right) + (\alpha' \delta - \alpha \delta')]\Delta
\]

\[\text{(2.7)}\]

\[
H = -[\gamma^2 + \gamma \delta \left( \frac{q(x)}{p(x)} \right) + \delta^2 \left( \frac{r(x)}{p(x)} \right) + (\gamma' \delta - \gamma \delta')]\Delta
\]

\[\text{(2.8)}\]

where \(\Delta = \alpha \delta - \beta \gamma \neq 0\).

However we can transform eq. \((2.3)\) back to a second order linear differential equation by the transformation

\[
z = -\frac{u'(x)}{F(x)u(x)}
\]

\[\text{(2.9)}\]

to obtain

\[
u'' - \left( G(x) + \frac{F'(x)}{F(x)} \right) u' + H(x)F(x)u = 0.
\]

\[\text{(2.10)}\]

We conclude then that the sequence of transformations \((2.2), (2.4), (2.9)\) establishes an equivalence relation between linear second order differential equations and we shall say that such equations are “conformally equivalent”.

To see how the solutions of eqs \((2.2), (2.10)\) are related we note that from \((2.2), (2.4)\) we have

\[
z = \frac{\delta w' - \gamma w}{\alpha w' - \beta w'}
\]

\[\text{(2.11)}\]

and hence using \((2.9)\)

\[
u(x) = \exp \left( \int F(x) \frac{\gamma w - \delta w'}{\alpha w' - \beta w'}dx \right).
\]

\[\text{(2.12)}\]
We conclude then that up to exponentiation and integration the relationship between the solutions is given by a Fractional Darboux transformation.

\[ \tilde{w}(x) = \frac{A(x)w + B(x)w'}{C(x)w + D(x)w'} \] (2.13)

We now explore some properties of this relationship when \( \alpha, \beta, \gamma, \delta \) is eq. (2.4) are constants. Under this restriction it is well known that the transformation (2.4) is equivalent to a sequence of three transformations

\[
\begin{align*}
y &= a_1z_1 + b_1 \\
z_1 &= \frac{1}{z_2} \\
z_2 &= a_2z_3 + b_2
\end{align*}
\] (2.14a, 2.14b, 2.14c)

Since affine transformations (2.14a, 2.14c) induce only a minor change in the original equation (they just add constant to \( r(x) \)) it is interesting to find those equations which remain invariant with respect to the transformation \( y = \frac{1}{z} \). To this end we let \( p(x), q(x), r(x) \) be polynomials in \( x \)

\[
p(x) = \sum_{k=0}^{n} p_k x^k, \quad q(x) = \sum_{k=0}^{n} q_k x^k, \quad r(x) = \sum_{k=0}^{n} r_k x^k
\] (2.15)

and solve the equations

\[
\frac{q(x)}{p(x)} = -\left( G(x) + \frac{F'(x)}{F(x)} \right) \quad (2.16)
\]

\[
\frac{r(x)}{p(x)} = H(x)F(x) \quad (2.17)
\]

for the coefficients \( p_k, q_k, h_k \). When \( n = 2 \) we find that up to translations in \( x \) (i.e. \( x = x + a \)) there are only three equations with this property

(1) \( w'' - n^2 w = 0 \) (2.18)

(2) Chebychev equation

\[
(p_0 - x^2)w'' - xw' + h_0 w = 0
\] (2.19)
\[ x \left( x - 2 \frac{q_0}{p_0} \right) w'' + \frac{q_0}{p_0} w' + \frac{h_0}{p_0} x^2 w = 0 \quad (2.20) \]

Letting \( p_0 = 1, \ q_0 = \frac{1}{2} \) in (2.20) we have

\[ x(1 - x)w'' - \frac{1}{2} w' - h_0 x^2 w = 0 \]

This equation does not appear in Kamke’s book [5]. However it belongs to the class of Hill-Mathieu equations through the transformation \( x = \cos \theta \).

In Table 1 we present the effect of the transformation \( y = \frac{1}{x} \) on several classes of equations whose solutions are the Special Functions of Mathematical Physics. We would like to emphasize however that in general \( \alpha, \beta, \gamma, \delta \) in eq. (2.4) can be functions of \( x \) rather than just constants and consequently the conformal equivalence class of each of these equations can be very large indeed.

Finally for the Schrodinger equation \((1.2)\) we find that when \( \alpha, \beta, \gamma, \delta \) are constant with \( \alpha \delta - \beta \gamma = 1 \), \( Q(x), \ R(x) \) in eq. \((2.10)\) take the following form;

\[
Q(x) = 2\alpha\gamma - 2\beta\delta(u(x) + \lambda) - \frac{\beta^2 u'(x)}{\beta^2(u + \lambda) - \alpha^2} \quad (2.21)
\]

\[
R(x) = [\delta^2(u + \lambda) - \gamma^2][\beta^2(u + \lambda) - \alpha^2] \quad (2.22)
\]

### 3 The Classification Problem

To use the algorithm described in the previous section in a practical manner one must be able to determine when a differential equation

\[ w_1'' + q_1(x)w_1' + r_1(x)w_1 = 0 \quad (3.1) \]

is conformally equivalent to eq. \((2.1)\). To this end one must solve the equations

\[ q_1(x) = G(x) + \frac{F'(x)}{F(x)} \quad (3.2) \]

\[ r_1(x) = H(x)F(x) \quad (3.3) \]
for appropriate $\alpha(x), \beta(x), \gamma(x), \delta(x)$ subject to the condition $\Delta \neq 0$.

To make progress towards the solution of this classification problem we shall consider two separate cases; $\beta(x) \neq 0$ and $\beta(x) = 0$.

A. $\beta(x) \neq 0$.

In this case we can assume without loss of generality that $\beta(x) = 1$. The resulting expression for $q_1(x)$ can be used to solve for $\gamma(x)$ in terms of $q_1(x), \alpha(x), \delta(x)$ and the coefficients of eq. (2.1). Substituting this expression for $\gamma(x)$ in eq. (4.3) (with $\beta(x) = 1$) we obtain;

$$r_1(x) = \frac{q_1(x)^2}{4} + \frac{1}{2} \frac{dq_1(x)}{dx} + R_1(x)$$

(3.4)

where $R_1(x)$ which depends only on $\alpha(x)$ and the coefficients of eq. (2.1) is given by eq. (A.1). (To simplify this equation we let $p(x) = 1$).

Eq. (3.4) is a single ODE for $\alpha(x)$ and if a solution to this equation exists then equations (2.1), (3.1) are conformally equivalent. However in general eq. (3.4) is nonlinear (and intractable) equation for $\alpha(x)$. Due to this circumstance it is appropriate to use eq. (3.4) in a “reverse” manner viz. assume a fixed functional form for $\alpha(x)$ and determine the corresponding form of $R_1(x)$. This expression can be evaluated for various classes of 2nd order ODEs and an appropriate table can provide a quick reference to determine if a given equation is conformally equivalent to any of the classes that appear in the table (under the restrictions imposed on $\alpha(x)$).

Table 2 presents the different $R_1(x)$ that correspond to some classes of special functions when $\alpha(x) = 0$.

From this table we gather the following:

**Corollary 3.1**

1. $\omega'' + \omega = 0$ is conformally equivalent to Bessel equation of order $1/2$

2. Bessel equations of order $0, 1$ are conformally equivalent to each other.

B. $\beta(x) = 0$

As can be expected this is a simple case to treat.
To begin with we can assume without loss of generality that $\delta(x) = 1(\alpha(x) \neq 0)$ and obtain for $q_1(x), r_1(x)$ the following expressions (we let $p(x) = 1$)

$$q_1(x) = 2\alpha(x) + q(x) \quad (3.5)$$

$$r_1(x) = \alpha^2(x) + \alpha(x)q(x) + r(x) + \frac{\partial \alpha(x)}{\partial x}. \quad (3.6)$$

Solving (3.3) for $\alpha(x)$ and substituting in (3.4) we obtain eq. (3.4) with $R_1(x)$ given by

$$R_1(x) = -\frac{q(x)^2}{4} - \frac{1}{2} \frac{dq(x)}{dx} + r(x) \quad (3.7)$$

(observe that $R(x)$ is independent of $\alpha(x)$).

To recast this algorithm in perspective we note that the Forbenius method expresses the solution of (2.1) in terms of powers of $x$. The algorithms described above enable us to approach this problem differently by asking whether the solution of (3.1) can be expressed naturally in terms of other classes of function (e.g. special functions) i.e. when (3.1) is conformally equivalent to an equation whose solution is already known.

### 4 Application to Schrodinger Equation

In this section we consider the application of fractional Darboux transformations in the form

$$\psi(x) = \frac{A(x)\phi(x) + \phi'(x)}{B(x)\phi(x) + \phi'(x)} \quad (4.1)$$

to relate the solutions of equations (1.2), (1.3). Differentiating (4.1) twice and substituting in (1.3) using (1.2) to eliminate the higher order derivatives of $\phi(x)$ we obtain a polynomial which contains the monomials $\phi^3, \phi^2\phi', \phi(\phi')^2$ and $(\phi')^3$. To annul this polynomial we let the coefficient of each of these monomials be zero. By using the last three equations to express $\lambda, v(x)$ and $\frac{du}{dx}$ and simplifying we obtain from the first equation (coefficient of $\phi^3$)

$$[(B - A)' - (B^2 - A^2)'] - 2B[(B - A)' - (B^2 - A^2)] = 0. \quad (4.2)$$
This can be solved by the Ansatz

\[(B - A)' - (B^2 - A^2) = 0\]  \hspace{1cm} (4.3)

substituting this relation in the equations for \(\lambda, v\) and \(\frac{du}{dx}\) yield the following equations

\[u(x) = A^2 - A' - c\]  \hspace{1cm} (4.4)

\[v(x) = 2A(A - B) - c\]  \hspace{1cm} (4.5)

\[\lambda = c\]  \hspace{1cm} (4.6)

where \(c\) is a constant of integration. Thus

\[\Delta u = v(x) - u(x) = A(A - 2B) + A'.\]  \hspace{1cm} (4.7)

To use these equations in practical applications we start with eq. (4.4) to find \(A\). This Riccati equation can be converted to

\[\zeta_1'' = (u(x) + c)\zeta_1\]  \hspace{1cm} (4.8)

by the transformation

\[A = -\frac{\zeta_1'}{\zeta_1}.\]  \hspace{1cm} (4.9)

It follows then that \(\zeta_1\) is a solution of eq. (1.2) with \(\lambda = c\). Similarly if we substitute for \(A\) in (1.3) using (4.4) we obtain

\[B' - B^2 = -u - c.\]  \hspace{1cm} (4.10)

Thus \(\zeta_2\) \((\text{where } B = -\frac{\zeta_2'}{\zeta_2})\) is also a solution of (1.2) with the same eigenvalue. It follows then that

\[\Delta u = [(ln\zeta_1)']^2 - 2(ln\zeta_1)'(ln\zeta_2)' - (ln\zeta_1)''\]  \hspace{1cm} (4.11)

which is completely different from (1.8) for Darboux transformation (1.1).
5 Generalization of Cole-Hopf Transformation

It is well known that Cole-Hopf transformation [8]

\[
\psi(x,t) = -2\nu \frac{\phi_x(x,t)}{\phi(x,t)}
\]  \hspace{1cm} (5.1)

can be used to linearize the Burger’s equation

\[
\psi_t + \psi\psi_x = \nu \psi_{xx}
\]  \hspace{1cm} (5.2)

and transform it to the heat equation

\[
\phi_t = \nu \phi_{xx}.
\]  \hspace{1cm} (5.3)

It is now easy to recognize that the fractional Darboux transformation

\[
\psi(x,t) = A(x)\phi(x,t) + C(x)\phi_x(x,t) + B(x)\phi(x,t) + D(x)\phi_x(x,t) = N
\]  \hspace{1cm} (5.4)

is actually a generalization of (5.1). It is natural therefore to apply it to the linearization of nonlinear partial differential equations.

In the following we restrict ourselves to the classification of nonlinear equations which can be transformed to the heat equation through the transformation (5.4).

To this end we compute

\[
\frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial x^2}
\]

using (5.4) and simplify the expressions thus obtained by replacing \(\phi_t\) by \(\phi_{xx}\) at each step (we set \(\nu = 1\) in eq. (5.3)).

After some algebra we obtain

\[
\psi_t - \psi_{xx} = \frac{\left[ A'' + (2A' + C'')(\frac{\phi_x}{\phi}) + C'' \left( \frac{\phi_{xx}}{\phi} \right) \right]}{B + D \left( \frac{\phi_x}{\phi} \right)} + \psi \left[ \frac{B'' + (2B' + D'')(\frac{\phi_x}{\phi}) + 2D' \left( \frac{\phi_{xx}}{\phi} \right)}{B + D \left( \frac{\phi_x}{\phi} \right)} \right] + 2\psi_x \frac{E_x}{E}.
\]  \hspace{1cm} (5.5)

However from (5.4)

\[
\frac{\phi_x}{\phi} = \frac{B\psi - A}{C - D\psi}.
\]  \hspace{1cm} (5.6)
Substituting (5.6) and using the identity

\[
\frac{\phi_{xx}}{\phi} = \left( \frac{\phi_x}{\phi} \right)_x + \left( \frac{\phi_x}{\phi} \right)^2.
\] (5.7)

in eq. (5.5) we obtain for the right hand side an expression which contains monomials of \( \psi, \psi_x \) up to the fourth and second degree respectively. However if we restrict ourselves to second order nonlinearity it is enough to let \( D = 0 \) and set \( B(x) = 1 \). The simplified form of eq. (5.5) in this case is

\[
C^2(\psi_t - \psi_{xx}) = -C'\psi^2 + [C'(2A + C') - C(C' - 2A)'] + 2C\psi_x]\psi
- C(C' + 2A)\psi_x + AC(C' + 2A)' - A(C'' + C'A)
+ C(C'A' - A''C).
\] (5.8)

In particular if \( A, B, C, D \) are constants eq. (5.8) simplifies and we obtain

\[
\psi_t - \psi_{xx} = 2\psi_x \frac{A - B\psi - D\psi_x}{D\psi - C}
\] (5.9)

which is a generalization of Burger's equation.

Similarly we can use the transformation (5.1) (with \( A, B, C, D \) being functions of \( x \) only) to relate a linear second order equation of the form (2.1) (with \( w \) replaced by \( \phi \)) to second order nonlinear ODEs. If we restrict ourselves to nonlinearities up to the third order we find that we must set \( D(x) = 0 \) (and without loss generality let \( B(x) = 1 \)). The general form of the resulting nonlinear second order equation is

\[
C^2\psi'' = 2\psi^3 - (3qC + 6A + 2C')\psi^2
+ [(-2r + q' + q^2)C^2 + (C'' + 2qC' + 6AQ)C
+ 6A^2 + 4C'A]\psi + (r' + q r)C^3
+ [-q^2 - 2r]A + A'' + 2C' r]C^2
+ [-3qA^2 - (C'' + 2qC')A]C - 2A^3 - 2C'A^2
\] (5.10)

Similar algorithm can be used to find other nonlinear equations that can be transformed to a given linear equation via a transformation of the form (5.4).
6 Conclusions

We introduced in this paper Fractional Darboux transformations and elaborated on some of their applications. In view of the wide range of applications that regular Darboux transformations have in the literature we expect that this new transformation will generalize many known results. In particular the systematic adaption of this new transformation to the computation of Ladder operators for new potentials in the spirit of [1,4] has yet to be worked out. We hope to elaborate on these subjects in subsequent publications.
Appendix

In this appendix we give the general form of $R_1(x)$ in eq. (3.4).

$$ R_1(x) = \frac{N_1(x)}{D_1(x)} \quad (A.1) $$

where

$$ N_1 = 2(\alpha^2 + \alpha q + \alpha' + r)\alpha''' - 3(\alpha'')^2 - 6(2\alpha + (q\alpha)' + r')\alpha''
+ 12(\alpha')^3 + 6(4r + q' - q^2)(\alpha')^2 + [4(4r - 2q' - q^2)\alpha^2
+ 2(q'' + 8qr - 8r' - 2q^3)\alpha + 8r(2r + q') - 4q(qr + 2r') + 2r''\alpha']
+ (4r - 2q' - q^2)\alpha^4 + 2(q'' - 2r' - q^3 + 4qr - q'q)\alpha^3
+ [8r^2 - q^4 + 2q^2r + (2q'' - 6r')q - 3q'' + 2r'']\alpha^2 + [8qr^2
+ 2(q'' + q'q - 2r' - q^3)r + 2r''q - 2q^2r' - 6r'q]\alpha
+ 4r^3 + (2q' - q^2)r^2 + (2r'' - 2qr')r - 3(r')^2
$$

$$ D_1 = 4[\alpha(\alpha + q) + \alpha' + r] \quad (A.3) $$

For brevity we suppressed the dependence on $x$ in eqs. (A.2), (A.3).
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Table 1: The effect of the sequence of transformations (2.2), (2.4), (2.9) on the equations in the second column when (2.4) is given by $y = \frac{1}{z}$

| Class of equations | Original equation | Transformed equation |
|--------------------|-------------------|----------------------|
| Constant Coefficient | $a\omega'' + b\omega' + c\omega = 0$ | Same |
| Legendre           | $(1 - x^2)w'' - 2xw' + n(n + 1)w = 0$ | $(1 - x^2)u'' + n(n + 1)u = 0$ |
| Hermite            | $w'' - 2xw' + 2nw = 0$ | $u'' + 2xu' + 2nu = 0$ |
| Bessel             | $x^2w'' + xw' + (x^2 - n^2)w = 0$ | $u'' - \frac{x^2 + n^2}{x(x^2 - n^2)} u' + \frac{x^2 - n^2}{x^2} u = 0$ |
| Leguerre           | $xw'' + (1 - x)w' + nw = 0$ | $u'' + u' + \frac{n}{x} u = 0$ |
| Chebyshev          | $(1 - x^2)\omega'' - x\omega' + h^2\omega = 0$ | Same |
| Hypergeometric     | $x(1 - x)w'' + [\gamma - (\alpha + \beta + 1)x]w'$ | $x(1 - x)u'' + [1 - \gamma + (\alpha + \beta - 1)x]u$ |

$$-\alpha \beta w = 0$$

$$-\alpha \beta u = 0$$
Table 2: Form of $R_1(x)$ in eq. (3.4) for some classes of special functions when $\alpha(x) = 0$

| Function Type       | $R_1(x)$                                                                 |
|---------------------|--------------------------------------------------------------------------|
| Constant coefficients| $c - \frac{1}{4}b^2$                                                     |
| Legendre            | $\frac{n(n + 1)}{1 - x^2}$                                               |
| Hermite             | $-x^2 + 2n - 1$                                                          |
| Bessel              | $\frac{x^2[4x^4 - 3x^2(4n^2 + 1) + 2n^2(6n^2 - 5)] + n^4(4n^2 + 1)}{4x^2(x^2 - n^2)^2}$ |
| Leguerre            | $\frac{n}{x} - \frac{1}{4}$                                             |
| Chebyshev           | $\frac{-(4n^2 - 1)x^2 - 2(1 + 2n^2)}{4(x^2 - 1)^2n^4}$                   |
| Hypergeometric      | $\frac{[1 - (\alpha - \beta)^2]x^2 + 2[(\alpha + \beta + 1)\gamma - (2\alpha + 1)\beta - (1 + \alpha)]x + \gamma^2 - 1}{x^2(x - 1)^2}$ |