A new technique for finding conservation laws of partial differential equations

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Abstract: We propose a new technique to construct conservation laws of partial differential equations (PDEs), where the whole computations can be fully implemented on a computer. The result is obtained by proving that any adjoint symmetry of PDEs is a differential substitution of nonlinear self-adjointness and vice versa. Consequently, each symmetry of PDEs corresponds to a conservation law via a formula if the PDEs are nonlinearly self-adjoint with differential substitution, which extends the results of Noether’s theorem. As a byproduct, we find that the set of differential substitutions includes the set of conservation law multipliers as a subset. With the help of the computer algebra system Mathematica, applications to the illustrated examples are performed and new conservation laws are constructed.

Keywords: Nonlinear self-adjointness with differential substitution, Adjoint symmetry, Conservation law, Multiplier

1 Introduction

It is well-known that Noether’s theorem established a close connection between symmetries and conservation laws for the PDEs possessing a variational structure [1,2]. However, the application of Noether’s approach relies on the following two conditions which heavily hinder the construction of conservation laws in such way:

(1). The PDEs under consideration must be derived from a variational principle, i.e. they are Euler-Lagrange equations.

(2). The used symmetries must leave the variational integral invariant, which means that not every symmetry of the PDEs can generate a conservation law via Noether’s theorem. Note that the symmetry stated here and below refers to the generalized symmetry of PDEs.

Thus many approaches are developed to get around the limitations of Noether’s theorem [3–8]. Recently, Ibragimov introduced the concept of nonlinear self-adjointness [9] which unified three subclasses stated earlier and provided a feasible method to construct conservation laws of PDEs [10–14]. The two required conditions of this approach are the symmetries admitted by the PDEs and the differential substitutions which convert...
nonlocal conservation laws to local ones. As for the first requirement, finding the symmetries of the PDEs, there exist a number of well-developed methods and computer algebra programs \[15\]–[18]. However, the way to obtain the required differential substitutions is only to use the equivalent identity of the definition which involves complicated computations \[9\]–[20].

Therefore a question naturally arise: would it be possible to derive the differential substitutions by other ways? The answer is positive. We show that each adjoint symmetry of PDEs is a differential substitution and vice versa. Consequently, each symmetry corresponds to a conservation law via a formula, where the formula only involves differential operation instead of integral operation and thus can be fully implemented on a computer.

It should be noted that there exists an approach, named by direct method, which does not require the symmetry information of PDEs but also connected with the symmetry and adjoint symmetry of PDEs. The direct method may be first used by Laplace to derive the well-known Laplace vector of the two-body Kepler problem \[3\]. Later, Steudel writes a conservation law in characteristic form \[4\], where the characteristics are also called multipliers and can be obtained by the fact that Euler operators annihilate the divergence expressions identically \[1\]. In particular, Anco and Bluman showed that, on the solution space of the given system of PDEs, multipliers are symmetries provided that its linearized system is self-adjoint, otherwise they are adjoint symmetries and can be obtained by choosing from the set of adjoint symmetries by virtue of the so-called adjoint invariance conditions \[6\]–[7]. Thus compare the computational process of the two methods, we find that the set of differential substitutions of nonlinear self-adjointness includes the one of multipliers as a subset.

The remainder of the paper is arranged as follows. In Section 2, some related notions and principles are reviewed and the main results are given. In Section 3, two illustrated examples are considered. The last section contains a conclusion of the results.

## 2 Main results

In this section, we first review some related notions and principles, and then give the main results of the paper.

### 2.1 Preliminaries

#### 2.1.1 Symmetry and adjoint symmetry

Consider a system of \(m\) PDEs with \(r\)th-order

\[
E^\alpha(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \alpha = 1, 2, \ldots, m,
\]

(1)

where \(x = (x^1, \ldots, x^n)\) is an independent variable set and \(u = (u^1, \ldots, u^m)\) is a dependent variable set, \(u_{(i)}\) denotes all \(i\)-th \(x\) derivatives of \(u\). Note that the summation convention for repeated indices will be used if no special explanations are added.
On the solution space of the given PDEs, a symmetry is determined by its linearized system while the adjoint symmetry is defined as the solution of the adjoint of the linearized system \([1, 2]\).

In particular, the determining system of a symmetry \(X_\eta = \eta^i(x, u, u_{(1)}, \ldots, u_{(s)}) \partial_{x^i}\) is the linearization of system (1) annihilating on its solution space, that is,

\[
(\mathcal{L}_E)^{\rho}_{\alpha} \eta^\rho = \frac{\partial E^\alpha}{\partial u^\rho} \eta^\rho + \frac{\partial E^\alpha}{\partial u_{i_1}^\rho} D_i \eta^\rho + \cdots + \frac{\partial E^\alpha}{\partial u_{i_1 \cdots i_r}^\rho} D_{i_1} \cdots D_{i_r} \eta^\rho = 0
\]

holds for all solutions of system (1). In (2) and below, \(D_i\) denotes the total derivative operator with respect to \(x^i\),

\[
D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots, \quad i = 1, 2, \ldots, n.
\]

The adjoint equations of system (2) are

\[
(\mathcal{L}_E^{\alpha})^{\rho}_{\beta} \omega^\rho = \omega^\rho \frac{\partial E^{\alpha}}{\partial u^{\beta}} - D_i \left( \omega^\rho \frac{\partial E^{\alpha}}{\partial u_{i_1}^{\beta}} \right) + \cdots + (-1)^r D_i \cdots D_{i_r} \left( \omega^\rho \frac{\partial E^{\alpha}}{\partial u_{i_1 \cdots i_r}^{\beta}} \right) = 0,
\]

which are the determining equations for an adjoint symmetry \(X_\omega = \omega_\rho(x, u, u_{(1)}, \ldots, u_{(r)}) \partial_{x^\rho}\) of system (1).

In general, solutions of the adjoint symmetry determining system (3) are not solutions of the symmetry determining system (2). However, if the linearized system (2) is self-adjoint, then adjoint symmetries are symmetries and system (1) has a variational principle and thus Noether’ approach is applicable in this case (1).

### 2.1.2 Nonlinear self-adjointness with differential substitution

We begin with nonlinear self-adjointness introduced by Ibragimov [9], whose main idea is first to turn the system of PDEs into Lagrangian equations by artificially adding new variables, and then to apply the theorem proved in [21] to construct local and nonlocal conservation laws.

Let \(\mathcal{L}\) be the formal Lagrangian of system (1) written as

\[
\mathcal{L} = v^\beta E^\beta(x, u, u_{(1)}, \ldots, u_{(r)}),
\]

where \(v^\beta\) is the new introduced dependent variable, then the adjoint equations of system (1) are defined by

\[
(E^{\alpha})^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(r)}, v_{(r)}) = \frac{\delta \mathcal{L}}{\delta u^\alpha} = 0,
\]

where \(v = (v^1, \ldots, v^m)\) and hereafter, \(\delta/\delta u^\alpha\) is the Euler operator defined as

\[
\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}^\alpha}.
\]
Then the definition of nonlinear self-adjointness of system (1) is given as follows.

**Definition 2.1 (Nonlinear self-adjointness)** The system (1) is said to be nonlinearly self-adjoint if the adjoint system (5) is satisfied for all solutions $u$ of system (1) upon a substitution $v = \varphi(x, u)$ such that $\varphi(x, u) \neq 0$.

Here, $\varphi(x, u) = (\varphi^1(x, u), \ldots, \varphi^m(x, u))$ and $v = \varphi(x, u)$ means $v^i = \varphi^i(x, u)$, $\varphi(x, u) \neq 0$ means that not all elements of $\varphi(x, u)$ equal zero and is called a nontrivial substitution. Definition 2.1 is equivalent to the following identities holding for the undetermined functions $\lambda_\alpha^\beta = \lambda_\alpha^\beta(x, u, u^{(1)}, \ldots, u^{(r)})$

$$(E^\alpha)^*(x, u, v, u^{(1)}, \ldots, u^{(r)})|_{v=\varphi} = \lambda_\alpha^\beta E^\beta,$$  

which is applicable in the proofs and computations.

Nonlinear self-adjointness contains three subclasses. In particular, if the substitution $v = \varphi(x, u)$ becomes $v = u$, then system (1) is called strict self-adjointness. If $v = \varphi(u)$, then it is named by quasi self-adjointness. If $v = \varphi(x, u)$ involving $x$ and $u$, then it is called weak self-adjointness.

As an extension of the substitution, if $v = \varphi(x, u, u^{(1)}, \ldots, u^{(s)})$, then it is called nonlinear self-adjointness with differential substitution [9, 19, 20].

**Definition 2.2 (Nonlinear self-adjointness with differential substitution)** The system (1) is said to be nonlinearly self-adjoint with differential substitution if the adjoint system (5) is satisfied for all solutions of system (1) upon a substitution $v = \varphi(x, u, u^{(1)}, \ldots, u^{(s)})$ such that $v \neq 0$.

Similarly, Definition 2.2 is equivalent to the following equality

$$(E^\alpha)^*(x, u, v, u^{(1)}, \ldots, u^{(r)})|_{v=\varphi(x, u, u^{(1)}, \ldots, u^{(s)})} = (\lambda_\alpha^\beta + \lambda_\alpha^\beta_{i_1} D_{i_1} + \cdots + \lambda_\alpha^\beta_{i_1 \cdots i_s} D_{i_1} \cdots D_{i_s}) E^\beta,$$  

where $\lambda_\alpha^\beta, \lambda_\alpha^\beta_{i_1}, \ldots, \lambda_\alpha^\beta_{i_1 \cdots i_s}$ are undetermined functions of arguments $x, u, u^{(1)}, \ldots, u^{(r+s)}$ respectively. Since the highest order derivatives in $\lambda_\alpha^\beta, \lambda_\alpha^\beta_{i_1}, \ldots, \lambda_\alpha^\beta_{i_1 \cdots i_s}$ maybe larger than the highest order derivative in $E^\alpha$, the right side of system (8) may not linear in $D_{i_1} \cdots D_{i_k} E^\alpha$, $k = 1, \ldots, m$ and thus application of equality (8) to find the differential substitutions is a difficult task.

For example, consider the Klein-Gordon equation in the form

$$G = u_{tt} - u_{xx} - g(u) = 0,$$  

where $g(u)$ is a nonlinear function of $u$. Following the main idea of nonlinear self-adjointness with differential substitution, set the formal Lagrangian $\mathcal{L} = \alpha(u_{tt} - u_{xx} - g(u))$ with a new introduced dependent variable $\alpha$, then the adjoint equation of Eq.(9) is

$$\frac{\delta \mathcal{L}}{\delta u} = D_t^2 \alpha - D_x^2 \alpha - g'(u) \alpha.$$  

(10)
Assume the differential substitution \( \alpha = \varphi(x, t, u, \partial_x u, \partial_t u, \ldots, \partial_x^p u, \partial_x^{p-1} \partial_t u) \) and use the equality (8), then Eq.(10) becomes

\[
\mathcal{D}_t^2 \varphi + \mathcal{L}_\varphi[u] G + \sum_{i=0}^{p-1} D_i^2 G(D_t + \mathcal{D}_t) \varphi_{\partial_x u} - D_x^2 \varphi - g'(u) \varphi = \sum_{j=0}^p \lambda_j D_j G + \sum_{k=0}^{p-1} \mu_k D_x^k G D_t G + \sum_{l,s=0}^{p-1} \nu_{ls} D_x^l G D_x^s G,
\]

where \( \lambda_j, \mu_k, \nu_{ls} \) (\( i, k, l, s = 1, \ldots, p - 1 \)) are arbitrary functions of \( x, t, u \) and up to \( p + 2 \) order derivatives of \( u \) without containing \( u_{tt} \) and its differential results, and

\[
\mathcal{D}_t = \partial_t + u_t \partial_u + u_x \partial_{u_x} + (u_{xx} + g(u)) \partial_{u_t} + \ldots
\]

is the total derivative operator which expresses \( u_{tt} \) and its derivatives through Eq.(9). The symbol \( \mathcal{L}_\varphi[u] \) stands for the linearization operator of function \( \varphi \) defined by

\[
\mathcal{L}_\varphi[u] G = \varphi_u G + \varphi_{\partial_x u} D_x G + \varphi_{\partial_t u} D_t G + \varphi_{\partial_x \partial_t u} D_x D_t G + \varphi_{\partial_x \partial_x u} D_x^2 G + \ldots.
\]

Obviously, the right side of Eq.(11) is not linear in \( D_x G \) since the last term contains \( D_x^2 G D_x^2 G \), thus the computation of differential substitution for Eq.(10) is not an easy task. For details of the example we refer to reference [22].

After finding the differential substitutions of nonlinear self-adjointness, we will use the following theorem to construct conservation laws of the system [21].

**Theorem 2.3** Any infinitesimal symmetry (Local and nonlocal)

\[
X = \xi^i(x, u, u(1), \ldots) \frac{\partial}{\partial x^i} + \eta^\sigma(x, u, u(1), \ldots) \frac{\partial}{\partial u^\sigma}
\]

of system (1) leads to a conservation law \( D_i(C^i) = 0 \) constructed by the formula

\[
C^i = \xi^i \mathcal{L} + W^\sigma \left[ \frac{\partial \mathcal{L}}{\partial u^\sigma_{ij}} - D_j \left( \frac{\partial \mathcal{L}}{\partial u^\sigma_{ij}} \right) + D_j D_k \left( \frac{\partial \mathcal{L}}{\partial u^\sigma_{ijk}} \right) - \ldots \right]
\]

\[
+ D_j(W^\sigma) \left[ \frac{\partial \mathcal{L}}{\partial u^{\sigma}_{ij}} - D_k \left( \frac{\partial \mathcal{L}}{\partial u^{\sigma}_{ijk}} \right) + \ldots \right] + D_j D_k(W^\sigma) \left[ \frac{\partial \mathcal{L}}{\partial u^{\sigma}_{ijk}} - \ldots \right] + \ldots,
\]

where \( W^\sigma = \eta^\sigma - \xi^i u^\sigma_{ij} \) and \( \mathcal{L} \) is the formal Lagrangian (4) which is written in the symmetric form about the mixed derivatives.

### 2.1.3 Multiplier

Multipliers are a set of functions which multiplies a system of PDEs in order to make the system get a divergence form, then for any solution of the equations this divergence will equal zero and one will get a conservation law. Explicitly, multiplier of conservation law for system (1) is defined as follows.
Definition 2.4  (Multiplier [7]) Multipliers for PDE system (1) are a set of expressions

\[ \Lambda = \{ \Lambda_1(x, u, u(1), \ldots, u(r)), \ldots, \Lambda_m(x, u, u(1), \ldots, u(r)) \} \]

satisfying

\[ \Lambda_\beta(x, u, u(1), \ldots, u(r)) E_\beta(x, u, u(1), \ldots, u(r)) = D_i(C^i) \]

with some expressions \( C^i \) for any function \( u \).

Since Euler operator \( \delta/\delta u^\sigma \) with \( \sigma = 1, 2, \ldots, m \) acting on the divergence expression \( D_i(C^i) \) yields zero identically, so the following theorem for computing multipliers is established [1, 2].

Theorem 2.5  A set of non-singular local multipliers (13) yields a local conservation law for the PDEs system (1) if and only if the set of identities

\[ \frac{\delta}{\delta u^\sigma} (\Lambda_\beta(x, u, u(1), \ldots, u(r)) E_\beta(x, u, u(1), \ldots, u(r))) = 0 \] (13)

holds for arbitrary functions \( u(x) \).

Since system (13) holds for arbitrary \( u \), one can treat each \( u^\sigma \) and its derivatives as independent variables, and consequently system (13) is separated into an over-determined linear PDEs system whose solutions are multipliers. When the calculation works on the solution space of the PDEs expressed in a Cauchy-Kovalevskaya form, the multipliers are selected from the set of adjoint symmetries using the adjoint invariance condition. For details we refer to references [6, 7] and therein.

2.2 Main results

We first give an equivalent definition of nonlinear self-adjointness with differential substitution. Definition 2.2 means that adjoint system (5), after inserted by the differential substitution \( v = \varphi(x, u, u(1), \ldots, u(s)) \), holds identically on the solution space of original system (1). This property can be used as the following alternative definition for nonlinear self-adjointness with differential substitution.

Definition 2.6  (Nonlinear self-adjointness with differential substitution) The system (1) is nonlinearly self-adjoint with differential substitution if the adjoint system (5) upon a nontrivial differential substitution \( v = \varphi(x, u, u(1), \ldots, u(s)) \) holds on the solution space of system (1).

In the sense of Definition 2.6 nonlinear self-adjointness with differential substitution is equivalent to the following equality

\[ (E^\alpha)^*(x, u, v, u(1), v(1), \ldots, u(r), v(r)) \big|_{v=\varphi(x, u, u(1), \ldots, u(s))} = 0, \text{ when } E^\alpha = 0 \] (14)

which is called the determining system of differential substitution.
On the other hand, since $v$ is a new introduced dependent variable set, then adjoint equations (15) can be explicitly expressed as

$$(E^\alpha)^* = \frac{\delta \mathcal{L}}{\delta u^\alpha} = v^\beta \frac{\partial E^\beta}{\partial u^\alpha} + \sum_{r=1}^{\infty} (-1)^r D_{i_1} \ldots D_{i_r} \left( v^\beta \frac{\partial E^\beta}{\partial u^\alpha_{i_1 \ldots i_r}} \right) = 0,$$

then compare system (15) with the adjoint symmetry determining equations (3), we find that they are identical if set $v = \omega$, where $\omega = (\omega_1, \ldots, \omega_m)$, thus we have the following result.

**Theorem 2.7** Any adjoint symmetry of system (1) is a differential substitution of nonlinear self-adjointness and vice versa.

Theorem 2.7 provides a new way to search for differential substitutions of nonlinear self-adjointness, which is identical to find the adjoint symmetries of PDEs. Furthermore, given a differential substitution of nonlinear self-adjointness, formula (12) can generate a conservation law with the symmetry of system (1), thus together with Theorem 2.3 and Theorem 2.7 we obtain the following result.

**Theorem 2.8** If system (1) is nonlinearly self-adjoint with differential substitution, then each symmetry corresponds to a conservation law via formula (12).

Theorem 2.8 builds a direct connection between symmetries and conservation laws by means of formula (12). In particular, if system (1) is strictly self-adjoint, then a symmetry is adjoint symmetry and thus is a differential substitutions of nonlinear self-adjointness.

In summary, we formulate the above procedure as the following algorithm of constructing conservation laws of PDEs.

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**Step 1:** Check whether the system of PDEs has a Lagrangian.
- If the system has a Lagrangian, then compute symmetries directly; Otherwise, compute symmetries and adjoint symmetries admitted by the system.

**Step 2:** Construct the formal Lagrangian $\mathcal{L}$ and find the differential substitution.
- By Theorem 2.7, the obtained adjoint symmetries are the differential substitutions of nonlinear self-adjointness.

**Step 3:** With the above known results, use formula (12) to construct conservation laws of the PDEs.

Since computing symmetry and adjoint symmetry is an algorithmic procedure, thus a variety of symbolic manipulation programs have been developed for many computer algebra systems (See [16, 17] and references therein). Furthermore, the general conservation law formula (12) only involve differential operation instead of integral operation. Hence, the proposed algorithm can be fully implemented on a computer.

To end this section, we compare nonlinear self-adjointness with differential substitution with multiplier method. Conservation law multiplier method for the PDEs admitting a
Cauchy-Kovalevskaya form is further studied in [6,7], which states that multipliers can be obtained by choosing from the set of adjoint symmetries with the so-called the adjoint invariance conditions, thus by Theorem 2.7 we have:

**Corollary 2.9** For the PDEs admitting a Cauchy-Kovalevskaya form, the set of differential substitutions contains the one of multipliers as a subset.

By Corollary 2.9, we find that there exist some adjoint symmetries which are differential substitutions but not multipliers of system (1), this case will be exemplified by a nonlinear wave equation in the next section.

## 3 Two illustrated examples

In this section, by means of the computer algebra system Mathematica, we consider two examples, where one is a nonlinear wave equation used to demonstrate the result of Corollary 2.9, the other is the Thomas equation used to illustrate the effectiveness of nonlinear self-adjointness with differential substitution.

### 3.1 A nonlinear wave equation

The first example is to consider a nonlinear wave equation [2,6]

\[ E = u_{tt} - u^2 u_{xx} - uu_x^2 = 0, \tag{16} \]

which has a variational principle given by the action integral \( S = \int (u_t^2 + u_x^2)/2 \, dtdx \) and so the adjoint symmetry and the symmetry are identical.

We first apply multiplier method to study conservation laws of Eq. (16). A function \( \Lambda = \Lambda(x,t,u,u_x,u_t) \) is a local multiplier of Eq. (16) if and only if Euler operator (6) acting on \( \Lambda \) yields zero for any \( u = u(x,t) \), i.e.,

\[
\frac{\delta (\Lambda E)}{\delta u} = D_t^2 \Lambda - u^2 D_x^2 \Lambda - 2uu_x D_x \Lambda \\
-(2uu_{xx} + u_x^2) \Lambda + E\Lambda_u - D_x (E\Lambda_{u_x}) - D_t (E\Lambda_{u_t}) = 0. \tag{17}
\]

Splitting Eq. (17) with respect to \( u_{tt} \) and its differential results, we find that the determining system for multiplier \( \Lambda \) consists of the symmetry determining system

\[ D_t^2 \Lambda - u^2 D_x^2 \Lambda - 2uu_x D_x \Lambda - (2uu_{xx} + u_x^2) \Lambda = 0, \tag{18} \]

where \( D_t = \partial_t + u_t \partial_u + u_{xt} \partial_{u_x} + (u^2 u_{xx} + uu_x^2) \partial_{u_t} + \ldots \) is the total derivative operator on the solution space of Eq. (16), and

\[ 2\Lambda_u - D_t \Lambda_{u_t} - D_x \Lambda_{u_x} = 0, \tag{19} \]

which is called the adjoint invariance condition.
It is known that Eq. (16) is invariant under the symmetry $X = (u - xu_x)\partial_u$, then function $\Lambda = u - xu_x$ is a solution of Eq. (18) but does not satisfy the adjoint invariance condition (19), thus it is not a multiplier. However, by Theorem 2.7, it is a differential substitution of nonlinear self-adjointness for Eq. (16), then set the formal lagrangian

$$L = (u - xu_x)(u_{tt} - u^2u_{xx} - uu_{xx}^2),$$

and by means of Theorem 2.3 we obtain a conservation law $D_tC^{t16} + D_xC^{x16} = 0$ given by the formulae

$$C^{t16} = (u - xu_x)D_t\eta - \eta(u_t - xu_{xt}),$$
$$C^{x16} = (xu_x - u)u^2D_x\eta - xu^2u_{xx}\eta,$$

where $X = \eta(x, t, u, u_x, u_t, \ldots)\partial_u$ is a symmetry of Eq. (16).

For example, let $\eta = -u_t$, then we obtain

$$C^{t16} = (u - xu_x)(u^2u_{xx} + uu_x^2) - u_t(u_t - xu_{xt}),$$
$$C^{x16} = u^2(xu_x - u)u_{xt} - xu^2u_{xx}u_t,$$

which is not reported in the literatures to the authors’ knowledge.

### 3.2 The Thomas equation

The Thomas equation is written as

$$G = u_{xt} + \alpha u_x + \beta u_t + \gamma u_xu_t = 0, \quad (20)$$

which arises in the study of chemical exchange process [23], where the constants $\alpha, \beta$ and $\gamma$ satisfy $\alpha > 0, \beta > 0, \gamma \neq 0$. The property of nonlinear self-adjointness had been studied in [9].

Following the infinitesimal symmetry criterion for PDEs [1], the determining equation of a symmetry in evolutionary vector field $X = \varphi(x, t, u, \partial_xu, \partial_tu, \ldots)\partial_u$ is

$$\mathcal{D}_tD_x\varphi + \alpha D_x\varphi + \beta \mathcal{D}_t\varphi + \gamma u_x\mathcal{D}_t\varphi + \gamma u_tD_x\varphi = 0, \quad (21)$$

where $\mathcal{D}_t = \partial_t + u_t\partial_u - (\alpha u_x + \beta u_t + \gamma u_xu_t)\partial_{ux} + u_{tt}\partial_{ut} + \ldots$ is the total derivative operator which expresses $u_{xt}$ and its derivatives through Eq. (20). The adjoint equation of Eq. (21) is

$$\mathcal{D}_tD_x\psi - \alpha D_x\psi - \beta \mathcal{D}_t\psi - \gamma u_x\mathcal{D}_t\psi - \gamma u_tD_x\psi + 2\gamma(\alpha u_x + \beta u_t + \gamma u_xu_t)\psi = 0, \quad (22)$$

which is the determining system of a adjoint symmetry $X = \psi(x, t, u, \partial_xu, \partial_tu, \ldots)\partial_u$. Then by Theorem 2.7 solutions of Eq. (22) are the differential substitutions of nonlinear self-adjointness.
Introduce the formal Lagrangian of Eq. (20) in the symmetric form
\[ L = v\left(\frac{1}{2}u_{xt} + \frac{1}{2}u_{tx} + \alpha u_x + \beta u_t + \gamma u_x u_t\right), \]
where \( v \) is a new dependent variable, then by formula (12), we obtain the following general conservation law formulae.

**Proposition 3.1** A conservation law \( D_t C^t + D_x C^x = 0 \) of Eq. (20) is given by
\[
C^t = \left(\gamma u_x v + \beta v - \frac{1}{2}D_x v\right)\eta + \frac{1}{2}vD_x \eta,
C^x = \left(\gamma u_t v + \alpha v - \frac{1}{2}D_t v\right)\eta + \frac{1}{2}vD_t \eta,
\]
where differential substitution \( v = \psi \) determined by Eq. (23) and \( X = \eta(x, t, u, u_x, u_t, \ldots)\partial_u \) is a symmetry of Eq. (20).

In what follows, we will use formulae (23) to construct conservation laws for some special cases of Eq. (20). Assume \( \psi = f(x, t, u)u_x + g(x, t, u)u_t + h(x, t, u) \), then after direct computations, we obtain
\[
\psi = B(x, t)e^{\gamma u} + c_1 e^{2(\gamma u + \alpha t + \beta x)} + e^{2(\gamma u + \alpha t + \beta x)}\left[(c_3 - c_2 t)u_t + (c_2 x + c_4)u_x + \frac{1}{\gamma}(c_2 \beta x - c_2 \alpha t + c_3 \alpha + c_4 \beta)\right],
\]
where \( c_1, \ldots, c_4 \) are arbitrary constants and \( B(x, t) \) satisfies \( B_{xt} - \alpha B_x - \beta B_t = 0 \) such that \( \alpha \neq 0 \).

Obviously, when \( c_2 = c_3 = c_4 = 0 \), adjoint symmetry (24) becomes the substitution of nonlinear self-adjointness, which is identical to the results in [9], while expression (24) with \( B(x, t) = c_1 = 0 \) is a new differential substitution and may generate new conservation laws.

**Example 1.** Let \( v = e^{2(\gamma u + \alpha t + \beta x)}(u_x + \alpha / \gamma) \) and \( \eta = -u_x \), then we have
\[
C^t = -e^{2(\gamma u + \alpha t + \beta x)}\left[\alpha \gamma u_x^2 + \alpha u_{xx} + \gamma(\beta u_x + \gamma u_x^2 + u_{xx})u_t\right],
C^x = e^{2(\gamma u + \alpha t + \beta x)}\left[\gamma u_x u_{tt} + \alpha^2 u_x + \alpha(2\gamma u_x + \beta)u_t + \gamma(\gamma u_x + \beta)u_t^2\right],
\]
which gives a second order conservation law \( D_t C^t + D_x C^x = 0 \).

**Example 2.** The second example is \( v = e^{2(\gamma u + \alpha t + \beta x)}(u_x + \beta / \gamma) \) and \( \eta = f(x, t)e^{-\gamma u} \), where \( f \) satisfies \( f_{xt} + \alpha f_x + \beta f_t = 0 \), then one has
\[
C^t = e^{\gamma u + 2\alpha t + 2\beta x}\left[f_t(\gamma u_x + \beta) - \gamma f(\beta u_x + \gamma u_x^2 + u_{xx})\right],
C^x = e^{\gamma u + 2\alpha t + 2\beta x}\left[f_t(\gamma u_x + \beta) + \alpha \gamma f u_x\right],
\]
which gives a second order conservation law in the form
\[ D_t C^t + D_x C^x = (f_{xt} + \alpha f_x + \beta f_t)(\gamma u_x + \beta)e^{\gamma u + 2\alpha t + 2\beta x} = 0. \]
Example 3. The last example is $v = e^{2(\gamma u + \alpha t + \beta x)}(xu_x - tu_t + (\beta t - \alpha x)/\gamma)$ and $\eta = -u_t$, then a second order conservation law $D_t C_{t0}^t + D_x C_{x0}^x = 0$ is given by

$$C_{t0}^t = -e^{2(\gamma u + \alpha t + \beta x)} \left[ \gamma xu_{xx} + \gamma^2 xu_x^2 + \alpha(\gamma xu_x - \alpha t + \beta x)u_x \right. + \gamma(2\beta x - \alpha t + 1)u_xu_t + \beta(\beta x - \alpha t + 1)u_t \right],$$

$$C_{x0}^x = e^{2(\gamma u + \alpha t + \beta x)} \left[ (\gamma xu_x - \alpha t + \beta x)u_{tt} + \alpha(\gamma xu_x + 1)u_t + \gamma(\gamma xu_x + \beta x + 1)u_t^2 \right].$$

4 Conclusion

We show that the set of adjoint symmetries admitted by the PDEs is identical to the one of differential substitutions of nonlinear self-adjointness, and then build a correspondence between symmetries and conservation laws via formula (12), which avoids integral operation by multiplier method and extends the results of Noether’ theorem. Furthermore, we demonstrate that the set of differential substitutions contains the one of conservation law multipliers as a subset. In addition, the presented results, after proper arrangements, can be applied to study approximate nonlinear self-adjointness of perturbed PDEs [9, 24, 25]. We will report these results in the future work.

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