THE EXISTENCE OF EXTREMAL FUNCTIONS FOR DISCRETE SOBOLEV INEQUALITIES ON LATTICE GRAPHS

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Abstract. In this paper, we study the existence of extremal functions (pairs) of the following discrete Sobolev inequality (0.1) and Hardy-Littlewood-Sobolev inequality (0.2) in the lattice $\mathbb{Z}^N$:

(0.1) $\|u\|_{L^q} \leq C_{p,q} \|u\|_{D^1,p}, \forall u \in D^1,p(\mathbb{Z}^N),$

where $N \geq 3, 1 \leq p < N, q > p^* = \frac{Np}{N-p}, C_{p,q}$ is a constant depending on $N, p$ and $q$;

(0.2) $\sum_{i,j \in \mathbb{Z}^N} f(i)g(j) |i-j|^\lambda \leq C_{r,s,\lambda} \|f\|_{\ell^r} \|g\|_{\ell^s}, \forall f \in \ell^r(\mathbb{Z}^N), g \in \ell^s(\mathbb{Z}^N),$

where $r, s > 1, 0 < \lambda < N, \frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} > 2, C_{r,s,\lambda}$ is a constant depending on $N, r, s$ and $\lambda$.

We introduce the discrete Concentration-Compactness principle, and prove the existence of extremal functions (pairs) for the best constants in the super-critical cases $q > p^*$ and $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} > 2$, respectively.

1. Introduction

For $N \geq 3, k \geq 1, p \geq 1, kp < N, \frac{1}{p'} = 1 - \frac{k}{N},$ we have the classical Sobolev inequality,

(1.1) $\|u\|_{p'}^{p'} \leq C_{p,q} \|u\|_{D^k,p}^{p'}, \forall u \in D^{k,p}(\mathbb{R}^N),$

where $C_{p,q}$ is a constant depending on $N, p$ and $q$, we omit the dependence of constants $N$ for convenience, and $D^{k,p}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ in the norm $\|u\|_{D^k,p} := \sum_{|\alpha| = k} \int |D^\alpha u|^p dx$.

Whether the best constant can be obtained by some $u \in D^{k,p}(\mathbb{R}^N)$, which is called the extremal function, has been intensively investigated in the literature. When $k = 1, p = 1$, H. Federer, W. Fleming [19] and W. Fleming, R. Rishel [20] proved that the best constant is the isoperimetric constant and the extremal function is the characteristic function of a ball. Using Schwarz symmetrization [40], best constants and extremal functions were obtained by Talenti [44], Rodemich [41] and Aubin [1] independently in the case of $k = 1, p > 1$. Moreover, Talenti’s paper [45] reveals the deep relation between isoperimetric inequalities and Sobolev inequalities, see also Cianchi [16]. For $k > 1$, P. L. Lions [33, 34, 35, 36] established the Concentration-Compactness method, which provided a new idea for proving the existence of extremal functions. The general idea is as follows. The best constant
in the Sobolev inequality (1.1) is given by

\[ S := \inf_{u \in D^{k,p}(\mathbb{R}^N)} \|u\|^{p}_{D^{k,p}} > 0. \]

Take a minimizing sequence \( \{u_n\} \) and regard \( \{|u_n|^p' \, dx\} \) as a sequence of probability measures. He proved in [33, 35] that there are three cases of the limit of the sequence: compactness, vanishing and dichotomy. Vanishing and dichotomy are ruled out by the rescaling trick and subadditivity inequality. Therefore, the extremal function exists by the compactness. Since the Concentration-Compactness principle requires weak convergence \( u_n \rightharpoonup u \) in \( D^{k,p}(\mathbb{R}^N) \), this method does not apply the case of \( p = 1 \).

For \( r, s > 1, 0 < \lambda < N, \frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2 \), we have the classical Hardy-Littlewood-Sobolev (HLS for abbreviation) inequality in \( \mathbb{R}^N \) [24, 25, 18],

\[
(1.2) \quad \iint f(x)g(y) \frac{|x - y|^{\lambda}}{|x - y|^N} \, dx \, dy \leq C_{r,s,\lambda} \|f\|_{L^r} \|g\|_{L^s}, \quad \forall f \in L^r(\mathbb{R}^N), g \in L^s(\mathbb{R}^N).
\]

In [31], Lieb proved the existence of the maximizing pair \((f, g)\), i.e. a pair that gives equality in (1.2). He also gave the explicit \((f, g)\) and best constant in the case \( r = s \). He’s method requires rearrangement inequalities to exclude the vanishing case and a compactness technique for maximizing sequences [31, Lemma 2.7], which is induced by the Brézis-Lieb lemma [4]. It also applies to Sobolev inequalities, doubly weighted HLS inequalities and weighted Young inequalities. Similar to Sobolev inequalities, Lions [36] also proved the existence of extreme functions for HLS inequalities by the Concentration-Compactness principle.

In recent years, people paid attention to the analysis on graphs. Since the Sobolev inequalities and HLS inequalities are useful analytical tools, they have been extended to the discrete setting [15, 29, 39]. For finite graphs, Sobolev inequalities and sharp constants have been obtained by [37, 38, 42, 43, 49, 50]. In this article, we study extremal functions of the discrete Sobolev inequality (0.1) in the lattice \( \mathbb{Z}^N \). Next, we consider the discrete HLS inequality (0.2). For \( N = 1, (0.2) \) is just the Hardy-Littlewood-Pólya inequality [20]. In [30, 14], the authors considered (0.2) in finite subgraphs of \( \mathbb{Z}^N \). For the supercritical case \( \frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} > 2 \), Huang, Li and Yin [29] proved the existence of the maximizing pair for (0.2) on \( \mathbb{Z}^N \) by analyzing the Euler-Lagrange equation on bounded subsets with Dirichlet boundary condition and taking the exhaustion. Inspired by this paper, we also consider (0.2) in the supercritical case and get the result by another proof.

A simple and undirected graph \( G = (V, E) \) consists of the set of vertices \( V \) and the set of edges \( E \). Two vertices \( x, y \) are called neighbours, denoted by \( x \sim y \), if there is an edge \( e \) connecting \( x \) and \( y \), i.e. \( e = \{x, y\} \in E \). In this paper, we mainly consider integer lattice graphs which serve as the discrete counterparts of \( \mathbb{R}^N \). The \( N \)-dimensional integer lattice graph, denoted by \( \mathbb{Z}^N \), is the graph consisting of the set of vertices \( V = \mathbb{Z}^N \) and the set of edges

\[
E = \left\{ \{x, y\} : x, y \in \mathbb{Z}^N, \sum_{i=1}^{N} |x_i - y_i| = 1 \right\}.
\]

We denote by \( \ell^p(\mathbb{Z}^N) \) the \( \ell^p \)-summable functions on \( \mathbb{Z}^N \) and by \( D^{1,p}(\mathbb{Z}^N) \) the completion of finitely supported functions in the \( D^{1,p} \) norm, see Section 2 for details.
The discrete Sobolev inequality (1.3) and HLS inequality (1.4) in $\mathbb{Z}^N$ are well-known, see [27, Theorem 3.6] and [29] for proofs:

(1.3) \[ \|u\|_{\ell^p} \leq C_p \|u\|_{D^1,p}, \forall u \in D^1,p(\mathbb{Z}^N), \]

where $N \geq 3, 1 \leq p < N$, $p^* = \frac{Np}{N-p}$;

(1.4) \[ \sum_{i,j \in \mathbb{Z}^N, i \neq j} \frac{f(i)g(j)}{|i-j|^\lambda} \leq C_{r,s,\lambda} \|f\|_{\ell^r}\|g\|_{\ell^s}, \forall f \in \ell^r(\mathbb{Z}^N), g \in \ell^s(\mathbb{Z}^N), \]

where $r, s > 1, 0 < \lambda < N, \frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$.

Since $\ell^p(\mathbb{Z}^N)$ embeds into $\ell^q(\mathbb{Z}^N)$ for any $q > p$, see Lemma 7, one verifies that the discrete Sobolev inequality (1.3) and HLS inequality (1.4) hold when $q \geq p^*$ and $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} \geq 2$ respectively. Recalling the continuous setting, it is called subcritical for $q < p^*$ (resp. $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} < 2$), critical for $q = p^*$ (resp. $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$), and supercritical for $q > p^*$ (resp. $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} > 2$) for the Sobolev inequality (resp. HLS inequality). Therefore, (1.3) and (1.4) hold in both critical and supercritical cases on $\mathbb{Z}^N$.

The optimal constant in the Sobolev inequality (1.1) is given by

(1.5) \[ S := \inf_{u \in D^1,p(\mathbb{Z}^N)} \|u\|_{D^1,p}^p \]

In order to prove that the infimum is achieved, we consider a minimizing sequence $\{u_n\} \subset D^1,p(\mathbb{Z}^N)$ satisfying

(1.6) \[ \|u_n\|_2 = 1, \|u_n\|_{D^1,p}^p \rightarrow S, n \rightarrow \infty. \]

We want to prove $u_n \rightarrow u$ strongly in $D^1,p(\mathbb{Z}^N)$, which will imply $u$ is a minimizer.

For the discrete HLS inequality, we first consider the following equivalent form of (1.2):

(1.7) \[ \|i|^{-\lambda} * f\|_r \leq C_{r,t} \|f\|_r, \]

where $|i|^{-\lambda} * f := \sum_{j \neq i} \frac{f(j)}{|i-j|^{\lambda+t}}, r, r^*, t > 1, 0 < \lambda < N, \frac{1}{r} + \frac{\lambda}{N} = 1 + \frac{1}{r^*}, r < r^*$.

The optimal constant in the inequality (1.7) is given by

(1.8) \[ K := \sup_{\|f\|_r = 1} \|i|^{-\lambda} * f\|_r. \]

We consider a maximizing sequence $\{f_n\}$ satisfying

(1.9) \[ \|f_n\|_r = 1, \|i|^{-\lambda} * f_n\|_r \rightarrow K, n \rightarrow \infty. \]

We want to prove $f_n \rightarrow f$ strongly in $\ell^r(\mathbb{Z}^N)$, and hence $f$ is a maximizer. Then by Lemma 3 for $\frac{1}{r} + \frac{1}{r^*} = 1$ there exists $g \in \ell^s(\mathbb{Z}^N)$ with $\|g\|_s = 1$ such that $(f, g)$ is a maximizing pair for (1.2).

We prove the following main results.

**Theorem 1.** For $N \geq 3, 1 \leq p < N$, $q > p^* = \frac{Np}{N-p}$, let $\{u_n\} \subset D^1,p(\mathbb{Z}^N)$ be a minimizing sequence satisfying (1.6). Then there exists a sequence $\{i_n\} \subset \mathbb{Z}^N$ such that the sequence after translation $\{u_n(i) := u_n(i + i_n)\}$ contains a convergent subsequence that converges to $v$ in $D^1,p(\mathbb{Z}^N)$. And $v$ is a minimizer for $S$. 

Remark. (1) We prove the case \( p = 1 \) on \( \mathbb{Z}^N \), while it is not true in the continuous case. In our case, \( \{ |\nabla u_n|_1 \} \) contains a \( v^\ast \)-convergent subsequence in \( \ell^1(\mathbb{Z}^N) \) by the discrete nature. This fails for the continuous case since \( L^1(\mathbb{R}^N) \) is not a dual space of any normed linear space. (2) The best constant can be obtained in the supercritical case.

Let \((\Gamma, S)\) be a Cayley graph of a discrete group \( G \) with a finite generating set \( S \). In particular, \( \mathbb{Z}^N \) is a Cayley graph of a free abelian group. For any \( r \in \mathbb{N} \), we denote by \( V(r) \) the number of group elements with word length at most \( r \). For a Cayley graph it is well known that if \( V(r) \geq C r^D, \quad \forall r \geq 1 \) for \( D \geq 3 \), then the Sobolev inequality holds,

\[
(1.9) \quad \|u\|_{\ell^p} \leq C_{p,q}\|u\|_{D^1,p},
\]

where \( 1 \leq p < D, q \geq \frac{Dp}{D-p} \). In fact, this follows from a standard trick and the isoperimetric estimate \textit{[47, Theorem 4.18]}. By the same argument, we can prove the following result.

**Theorem 2.** Let \((\Gamma, S)\) be a Cayley graph satisfying \( V(r) \geq C r^D, \quad \forall r \geq 1 \) for \( D \geq 3 \). For \( 1 \leq p < D, q \geq \frac{Dp}{D-p} \). Let \( \{u_n\} \subset D^{1,p} \) be a minimizing sequence in \((1.9)\) with \( \|u_n\|_q = 1 \). Then there exists a sequence \( \{g_n\} \subset \Gamma \) such that the sequence after translation \( \{v_n\} = \{u_n(g_n)\} \) contains a convergent subsequence that converges to \( v \) in \( D^{1,p} \) and \( v \) is a minimizer for \( S \).

Similarly, we have the following theorem for the discrete HLS inequality.

**Theorem 3.** For \( r, s, t > 1, 0 < \lambda < N, \quad \frac{1}{r} + \frac{1}{s} = \frac{1}{t} = \frac{1}{r} + \lambda + \frac{\lambda}{N} > 1 + \frac{1}{7} \). Let \( \{f_n\} \subset \ell^\lambda(\mathbb{Z}^N) \) be a maximizing sequence satisfying \textit{(18)}. Then there exists a sequence \( \{i_n\} \subset \mathbb{Z}^N \) such that the sequence after translation \( \{v_n(i) := f_n(i + i_n)\} \) contains a convergent subsequence that converges to \( f \) in \( \ell^\lambda(\mathbb{Z}^N) \). And \( f \) is a maximizer for \( K \). Moreover, there exists \( g \in \ell^\lambda(\mathbb{Z}^N) \) with \( \|g\|_s = 1 \) such that \( (f, g) \) is a maximizing pair for \textit{(12.3)} in the supercritical case \( \frac{1}{r} + \frac{1}{s} + \lambda > N > 2. \)

Remark. (1) Unlike Lieb’s proof in \textit{[31]}, our proof does not depend on the rearrangement trick and the special properties of \( I(i) = |i|^{-\lambda} \), namely \( I(i) \) is spherically symmetric and decreasing. This enables us to treat general classes of potentials \( I(i) \).

(2) The best constant can be obtained in the supercritical case. This has been proved by \textit{[29]}. Here we give an alternative proof.

We will provide two proofs for the main results. In the continuous setting, Lions proved the existence of extremal functions by Concentration-Compactness principle \textit{[35, Lemma 1.1]}. and a rescaling trick \textit{[35, Theorem 1.1, (17)]}. And Lieb in \textit{[31]} used a compactness technique and the rearrangement inequalities. Following Lions, the main idea of proof I is to prove a discrete analog of Concentration-Compactness principle, see Lemma \textit{[14]}. However, we don’t know proper notion of rescaling and rearrangement tricks on \( \mathbb{Z}^N \) to exclude the vanishing case of the limit function. Inspired by \textit{[29]}, for the supercritical case, we prove that the translation sequence has a uniform positive lower bound at the origin, see Lemma \textit{[14]} which excludes the vanishing case. The idea of proof II is based on a compactness technique by Lieb \textit{[31], Lemma 2.7} and the nontrivial nonvanishing of the limit of translation sequence.
According to [28, 5], we define the $p$-Laplacean of $u$ for $p > 1$,
\[
\Delta_p u(x) := \sum_{y \sim x} |u(y) - u(x)|^{p-2} (u(y) - u(x)),
\]
and for $p = 1$,
\[
\Delta_1 u(x) := \left\{ \sum_{y \sim x} f_{xy} : f_{xy} = -f_{yx}, f_{xy} \in \text{sign}(u(y) - u(x)) \right\},
\]
where $\text{sign}(x) = \begin{cases} 1, & x > 0, \\ [-1, 1], & x = 0, \\ -1, & x < 0. \end{cases}$

Similar to the continuous setting [1, 44], we have the following corollary.

**Corollary 4.** For $N \geq 3, 1 \leq p < N, q > p^*$, there is a positive solution of the equation
\[(1.10) \quad \Delta_p u + u^{q-1} = 0, \ x \in \mathbb{Z}^N.\]

By observing that any non-negative solution to (1.10) is bounded, see Lemma 9, we prove the following theorem using the results in Lin and Wu’s papers [32, 48].

**Theorem 5.** For $N \geq 3, p = 2, 2 < q \leq \frac{2+2N}{N}$, there does not exist a non-trivial non-negative solution for (1.10).

**Remark.** Compared with the continuous case [21, 22, 23], we conjecture that if $u$ is a non-negative solution of (1.10) in $\mathbb{Z}^N$ with $2 \leq q < p^*$, then $u \equiv 0$. According to Lions [35, Corollary I.1], we conjecture that (1.10) has a positive solution when $q = p^*$. For $p = 2$, we don’t know the existence of non-trivial non-negative solutions for (1.10) when $\frac{2+2N}{N} < q \leq 2^* = \frac{2N}{N-2}$.

By Theorem 3, we can get the Euler-Lagrange equation for (0.2) as follows, see [6, 7, 8, 9, 10, 11, 12] for continuous setting and [13, 29] for discrete setting.

**Corollary 6.** For $r, s > 1, 0 < \lambda < N, \frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} > 2$, there is a pair of positive solution $(f, g)$ of the following Euler-Lagrange equation for (0.2),
\[(1.11) \quad \begin{cases} K (f(i))^{r-1} = \sum_{j \neq i} \frac{g(j)}{|i-j|^r} \\ K (g(i))^{s-1} = \sum_{j \neq i} \frac{f(j)}{|i-j|^s}. \end{cases}\]

The paper is organized as follows. In Section 2, we recall some basic facts and prove some useful lemmas. In Section 3, we introduce the Brézis-Lieb lemma and prove the Concentration-Compactness principle in $\mathbb{Z}^N$. In Section 4, we prove a key lemma to exclude the vanishing case and give the proof I for Theorem 1. In Section 5, we give another proof II for Theorem 1 and prove Theorem 3 in a similar way.

## 2. Preliminary

Consider integer lattice graph $\mathbb{Z}^N$, which is the graph consisting of the set of vertices $V = \mathbb{Z}^N$ and the set of edges
\[ E = \left\{ \{x, y\} : x, y \in \mathbb{Z}^N, \sum_{i=1}^N |x_i - y_i| = 1 \right\}. \]
functions with finite support. For any \( u \in C(\mathbb{Z}^N) \), its support set is defined as \( \text{supp}(u) := \{ x \in \mathbb{Z}^N : u(x) \neq 0 \} \). Let \( C_0(\mathbb{Z}^N) \) be the set of all functions with finite support. For any \( u \in C(\mathbb{Z}^N) \), the \( \ell^p \) norm of \( u \) is defined as

\[
\|u\|_{\ell^p(\mathbb{Z}^N)} := \begin{cases} 
\left( \sum_{x \in \mathbb{Z}^N} |u(x)|^p \right)^{1/p} & 0 < p < \infty, \\
\sup_{x \in \mathbb{Z}^N} |u(x)| & p = \infty.
\end{cases}
\]

The \( \ell^p(\mathbb{Z}^N) \) space is defined as

\[
\ell^p(\mathbb{Z}^N) := \{ u \in C(\mathbb{Z}^N) : \|u\|_{\ell^p(\mathbb{Z}^N)} < \infty \}.
\]

In this paper, we shall write \( \|u\|_{\ell^p(\mathbb{Z}^N)} \) as \( \|u\|_p \) for convenience, when there is no confusion.

For any \( u \in C(\mathbb{Z}^N) \), we define difference operator for any \( x \sim y \) as

\[
\nabla_{xy} u = u(y) - u(x).
\]

Let

\[
|\nabla u(x)|_p := \left( \sum_{y \sim x} |\nabla_{xy} u|^p \right)^{1/p}
\]

be the \( p \)-norm of the gradient of \( u \) at \( x \).

The \( D^{1,p} \) norm of \( u \) is given by

\[
\|u\|_{D^{1,p}(\mathbb{Z}^N)} := \|\nabla u\|_{\ell^1(\mathbb{Z}^N)}^{1/p} = \left( \sum_{x \in \mathbb{Z}^N} \sum_{y \sim x} |\nabla_{xy} u|^p \right)^{1/p},
\]

and \( D^{1,p}(\mathbb{Z}^N) \) is the completion of \( C_0(\mathbb{Z}^N) \) in \( D^{1,p} \) norm.

The following lemma is well-known, see [29, Lemma 2.1].

**Lemma 7.** Suppose \( u \in L^p(\mathbb{Z}^N) \), then \( \|u\|_q \leq \|u\|_p, \forall q \geq p. \)

The combinatorial distance \( d \) is defined as \( d(x, y) = \inf \{ k : x = x_0 \sim \cdots \sim x_k = y \} \), i.e. the length of the shortest path connecting \( x \) and \( y \) by assigning each edge of length one.

Let \( \Omega \) be a subset of \( \mathbb{Z}^N \). We denoted by

\[
\delta \Omega := \{ x \in \mathbb{Z}^N \setminus \Omega : \exists y \in \Omega, \text{ s.t. } x \sim y \}
\]

the vertex boundary of \( \Omega \), possibly an empty set. We set \( \Omega := \Omega \cup \delta \Omega \).

We denoted by \( c_0(\mathbb{Z}^N) \) the completion of \( C_0(\mathbb{Z}^N) \) in \( \ell^\infty \) norm. Then it is well-known that \( \ell^1(\mathbb{Z}^N) = (c_0(\mathbb{Z}^N))^* \). We set

\[
\|\mu\| := \sup_{u \in c_0(\mathbb{Z}^N), \|u\|_{\ell^\infty} = 1} \langle \mu, u \rangle, \quad \forall \mu \in \ell^1(\mathbb{Z}^N).
\]

By definition,

\[
\mu_n \rightharpoonup^* \mu \text{ in } \ell^1(\mathbb{Z}^N) \text{ if and only if } \langle \mu_n, u \rangle \rightarrow \langle \mu, u \rangle, \forall u \in c_0(\mathbb{Z}^N).
\]

In the proof, we will use the following facts (see [17]).
Fact. (a) Every bounded sequence of $\ell^1(\mathbb{Z}^N)$ contains a $w^*$-convergent subsequence. (b) If $\mu_n \xrightarrow{w^*} \mu$ in $\ell^1(\mathbb{Z}^N)$, then $\mu_n$ is bounded and
$$\|\mu\| \leq \lim_{n \to \infty} \|\mu_n\|.$$ (c) If $\mu \in \ell^{1+}(\mathbb{Z}^N) := \{\mu \in \ell^1(\mathbb{Z}^N) : \mu \geq 0\}$, then
$$\|\mu\| = \langle \mu, 1 \rangle.$$

In the functional analysis, the following lemma is well-known, see [17] for a proof.

Lemma 8. For $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, let $0 \neq f \in \ell^p(\mathbb{Z}^N)$, then there exists unique $g \in \ell^q(\mathbb{Z}^N)$ with $\|g\|_q = 1$ satisfying
$$\|f\|_p = \sum_i f(i)g(i) = \max_{\|h\|_q \leq 1} \sum_i f(i)h(i).$$

$G = (V, E)$ is a locally finite unweighted graph, that is, the degree $d_x := \#\{y : y \sim x\}$ is finite for each $x \in V$. Then for any $u \in C(V)$, we define the normalized Laplace as
$$\Delta u(x) := \sum_{y \sim x} \frac{1}{d_x} (u(y) - u(x)).$$

Then we have the following lemma.

Lemma 9. For a locally finite graph $G = (V, E)$, if $u$ is a non-negative solution of equation
$$-\Delta u = u^a, \quad a > 1.$$ Then $u(x) \leq 1, \forall x \in V.$

Proof. For any $x \in V$, without loss of generality, we can assume that $u(x) > 0$. Then we get
$$u(x) \geq -\sum_{y \sim x} \frac{1}{d_x} (u(y) - u(x)) = u(x)^a.$$ Hence, $u(x) \leq 1$. $\square$

By Lemma 9 we can prove Theorem 5 in Section 1.

Proof of Theorem 5. If $u$ is a non-trivial non-negative solution for (1.10), we can define $v(t, x) := u(x)$, which satisfies the following heat equation
$$\begin{cases} v_t = \Delta^2 v + v^{a-1} & \text{in } (0, +\infty) \times \mathbb{Z}^N, \\ v(0, x) = u(x) & \text{in } \mathbb{Z}^N, \end{cases}$$ where $u(x)$ is bounded by Lemma 9. Using the results in Lin and Wu’s papers [32, 48], we know that for $0 < N(q - 2) < 2$, any non-trivial non-negative solution $v$ is not global, i.e. $v$ blows up in finite time, which yields a contradiction. This proves the theorem. $\square$
3. Concentration-Compactness Principle

In this section, we prove the discrete Concentration-Compactness principle. We first introduce a key lemma as follows [3] Theorem 1.

Consider a measure space \((\Omega, \Sigma, \mu)\), which consists of a set \(\Omega\) equipped with a \(\sigma\)-algebra \(\Sigma\) and a Borel measure \(\mu: \Sigma \rightarrow [0, \infty]\).

**Lemma 10.** (Brézis-Lieb lemma) Let \((\Omega, \Sigma, \mu)\) be a measure space, \(\{u_n\} \subset L^p(\Omega, \Sigma, \mu)\), and \(0 < p < \infty\). If
(a) \(\{u_n\}\) is uniformly bounded in \(L^p\),
(b) \(u_n \rightarrow u, n \rightarrow \infty \quad \mu\text{-almost everywhere in } \Omega\), then
(3.1) \(\lim_{n \rightarrow \infty} (\|u_n\|_{L^p}^p - \|u_n - u\|_{L^p}^p) = \|u\|_{L^p}^p\).

**Remark.** (1) The preceding lemma is a refinement of Fatou’s Lemma.
(2) Since \(\{u_n\}\) is uniformly bounded in \(L^p\), passing to a subsequence if necessary, we have
\(\lim_{n \rightarrow \infty} \|u_n\|_{L^p}^p = \lim_{n \rightarrow \infty} \|u_n - u\|_{L^p}^p + \|u\|_{L^p}^p\).
(3) If \(\Omega\) is countable and \(\mu\) is a positive measure defined on \(\Omega\), then we get a discrete version of Lemma 10.

**Corollary 11.** Let \(\Omega \subset \mathbb{Z}^N\), \(\{u_n\} \subset D^{1,p}(\mathbb{Z}^N)\), and \(1 \leq p < \infty\). If
(a) \(\{u_n\}\) is uniformly bounded in \(D^{1,p}(\mathbb{Z}^N)\),
(b) \(u_n \rightarrow u, n \rightarrow \infty \quad \text{pointwise in } \mathbb{Z}^N\), then
(3.2) \(\lim_{n \rightarrow \infty} \left( \sum_{i \in \Omega} |\nabla u_n(i)|_p^p - \sum_{i \in \Omega} |\nabla (u_n - u)(i)|_p^p \right) = \sum_{i \in \Omega} \left| \nabla u(i) \right|_p^p\).

**Proof.** We define two directed edge sets as follows
\(E_1 := \{e = (e_-, e_+) : e_-, e_+ \in \Omega, e_- \sim e_+\}\),
\(E_2 := \{e = (e_-, e_+) : (e_-, e_+) \in \Omega \times \partial \Omega, e_- \sim e_+\}\),
where \(E_1\) is the set of internal edges of \(\Omega\), \(E_2\) is the set of edges that cross the boundary of \(\Omega\), and \(e_-\) and \(e_+\) are the initial and terminal endpoints of \(e\).

Set \(\tilde{E} := E_1 \cup E_2\). We define \(\overline{\pi}, \mu: \tilde{E} \rightarrow \mathbb{R}, \overline{\pi}(e) = u(e_+) - u(e_-), \mu(e) = 1\).
Then we get
\(\sum_{\Omega} |\nabla u_n(i)|_p^p = \sum_{E_1} |\overline{\pi}_n(e)|_p^p + \sum_{E_2} |\overline{\pi}_n(e)|_p^p = \|\overline{\pi}_n\|_{\ell^p(\tilde{E}, \Sigma, \mu)} < \infty, \quad \overline{\pi}_n \rightarrow \overline{\pi} \text{ pointwise in } \tilde{E}\).

For the measure space \((\tilde{E}, \Sigma, \mu)\), by Lemma 10 we have
\(\lim_{n \rightarrow \infty} \left( \|\overline{\pi}_n\|_{\ell^p(\tilde{E}, \Sigma, \mu)} - \|u_n - u\|_{\ell^p(\tilde{E}, \Sigma, \mu)} \right) = \|\overline{\pi}\|_{\ell^p(\tilde{E}, \Sigma, \mu)}\),
which is equivalent to the equation (3.2). \(\square\)

In the continuous setting, P. L. Lions [35], Bianchi et al. [3] and Ben-Naoum et al. [2] proved that the limit of the minimizing sequence norm can be divided into three parts, i.e. the norm of the limit, the norm of the limit of the difference between the sequence and the limit, and the norm of the sequence at infinity. The corresponding parts still satisfy the Sobolev inequalities, also see [46, Lemma 1.40]. Next, we establish the Concentration-Compactness principle in the lattice \(\mathbb{Z}^N\).
Lemma 12. (Discrete Concentration-Compactness lemma) For \( N \geq 3, 1 \leq p < N, q \geq p^* \), if \( \{u_n\} \) is uniformly bounded in \( D^{1,p}(\mathbb{Z}^N) \). Then passing to a subsequence if necessary, still denoted as \( \{u_n\} \), we have

\[ u_n \rightharpoonup u \text{ pointwise in } \mathbb{Z}^N, \tag{3.3} \]

\[ \lim |\nabla u_n|^p_p \rightharpoonup |\nabla u|^p_p \text{ in } \ell^1(\mathbb{Z}^N). \tag{3.4} \]

And the following limits

\[ \lim_{R \to \infty} \lim_{n \to \infty} \sum_{d(i,0) > R} |\nabla u_n(i)|^p_p := \mu_\infty, \quad \lim_{R \to \infty} \lim_{n \to \infty} \sum_{d(i,0) > R} |u_n(i)|^q_q := \nu_\infty, \]

exist. For the above \( \{u_n\} \), we have

\[ |\nabla (u_n - u)|^p_p \rightharpoonup 0 \text{ in } \ell^1(\mathbb{Z}^N), \tag{3.5} \]

\[ |u_n - u|^q_q \rightharpoonup 0 \text{ in } \ell^1(\mathbb{Z}^N), \tag{3.6} \]

\[ \nu_\infty \leq S^{-1} \mu_\infty, \tag{3.7} \]

\[ \lim_{n \to \infty} \|u_n\|_{D^{1,p}}^p = \|u\|_{D^{1,p}}^p + \mu_\infty, \tag{3.8} \]

\[ \lim_{n \to \infty} \|u_n\|_q^q = \|u\|_q^q + \nu_\infty. \tag{3.9} \]

Proof. Since \( \{u_n\} \) is uniformly bounded in \( \ell^q(\mathbb{Z}^N) \), and hence in \( \ell^\infty(\mathbb{Z}^N) \). By diagonal principle, passing to a subsequence we get (3.3). Since \( \{\nabla u_n|^p_p\} \) is uniformly bounded in \( \ell^1(\mathbb{Z}^N) \), we get (3.4) by the Banach-Alaoglu theorem and (3.3). For every \( R \geq 1 \), passing to a subsequence if necessary,

\[ \lim_{n \to \infty} \sum_{d(i,0) > R} |\nabla u_n(i)|^p_p, \quad \lim_{n \to \infty} \sum_{d(i,0) > R} |u_n(i)|^q_q, \]

exist, where \( d \) is the combinatorial distance as defined in Section 2. Then we can define \( \mu_\infty, \nu_\infty \) by the monotonicity in \( R \).

Let \( v_n := u_n - u \), then \( v_n \rightharpoonup 0 \) pointwise in \( \mathbb{Z}^N \) and \( \{\nabla v_n|^p_p\} \) is uniformly bounded in \( \ell^1(\mathbb{Z}^N) \). Then any subsequence of \( \{\nabla v_n|^p_p\} \) contains a subsequence (still denoted as \( \{\nabla v_n|^p_p\} \)) that \( w^*- \)converges to 0 in \( \ell^1(\mathbb{Z}^N) \), which follows from

\[ \sum_h \|\nabla v_n|^p_p \rightharpoonup 0, \quad \forall h \in C_0(\mathbb{Z}^N). \]

Hence we get (3.5). Similarly, we get (3.6).

For \( R \geq 1 \), let \( \Psi_R \in C(\mathbb{Z}^N) \) such that \( \Psi_R(i) = 1 \) for \( d(i,0) \geq R + 1 \), \( \Psi_R(i) = 0 \) for \( d(i,0) \leq R \). By the discrete Sobolev inequality (0.1), we have

\[ \left( \sum |\Psi_R v_n|^q_q \right)^{p/q} \leq S^{-1} \sum |\nabla (\Psi_R v_n)|^p_p = S^{-1} \sum_{j \sim i} |\nabla_{ij} \Psi_R v_n(j) + \Psi_R(i) \nabla_{ij} v_n|)|^p. \]

And for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that

\[ |\nabla_{ij} \Psi_R v_n(j) + \Psi_R(i) \nabla_{ij} v_n|)|^p \leq C_\varepsilon |\nabla_{ij} \Psi_R^p| v_n(j)|^p + (1 + \varepsilon) |\nabla_{ij} v_n|^p |\Psi_R(i)|^p. \]

Since \( v_n \rightharpoonup 0 \) pointwise in \( \mathbb{Z}^N \), by \( \varepsilon \to 0^+ \) we obtain

\[ \lim_{n \to \infty} \left( \sum |\Psi_R v_n|^q_q \right)^{p/q} \leq S^{-1} \lim_{n \to \infty} \sum |\nabla v_n|^p_p \Psi_R. \tag{3.10} \]
From the definition of $\Psi_R$, we have

\[(3.11) \quad \lim_{R \to \infty} \lim_{n \to \infty} \sum_{d(i,0) > R} |\nabla v_n(i)|_p = \lim_{R \to \infty} \lim_{n \to \infty} \sum_{d(i,0) > R} |\nabla v_n(i)|^p \Psi^p_R.\]

\[(3.12) \quad \lim_{R \to \infty} \lim_{n \to \infty} \sum_{d(i,0) > R} |v_n(i)|^q = \lim_{R \to \infty} \lim_{n \to \infty} \sum_{d(i,0) > R} |v_n|^q \Psi^q_R.\]

By Lemma 10 and Corollary 11, we have

\[\lim_{n \to \infty} \left( \sum_{d(i,0) > R} |\nabla u_n(i)|_p^p - \sum_{d(i,0) > R} |\nabla v_n(i)|_p^p \right) = \sum_{d(i,0) > R} |\nabla u(i)|_p^p,\]

\[\lim_{n \to \infty} \left( \sum_{d(i,0) > R} |u_n(i)|^q - \sum_{d(i,0) > R} |v_n(i)|^q \right) = \sum_{d(i,0) > R} |u(i)|^q.\]

Hence by the above equalities, we get

\[(3.13) \quad \lim_{R \to \infty} \lim_{n \to \infty} \sum_{d(i,0) > R} |\nabla v_n(i)|_p^p = \mu_\infty,\]

\[(3.14) \quad \lim_{R \to \infty} \lim_{n \to \infty} \sum_{d(i,0) > R} |v_n(i)|^q = \nu_\infty.\]

Combining the equations (3.10), (3.11), (3.12), (3.13) and (3.14), we get

\[\nu^{p/q} \leq S^{-1} \mu_\infty.\]

Since $u_n \to u$ pointwise in $\mathbb{Z}^N$, then for every $R \geq 1$, we have

\[\lim_{n \to \infty} \sum_{d(i,0) > R} |\nabla u_n(i)|_p^p = \lim_{n \to \infty} \sum_{d(i,0) > R} \Psi_R |\nabla u_n(i)|_p^p = \sum_{d(i,0) > R} |\nabla u(i)|_p^p,\]

and

\[\lim_{n \to \infty} \sum_{d(i,0) > R} |u_n|^q = \lim_{n \to \infty} \sum_{d(i,0) > R} \Psi_R |u_n|^q = \sum_{d(i,0) > R} |u(i)|^q.\]

Letting $R \to \infty$, we obtain

\[\lim_{n \to \infty} \sum_{d(i,0) > R} |\nabla u(i)|_p^p = \mu_\infty + \sum_{d(i,0) > R} |\nabla u(i)|_p^p = \mu_\infty + \|u\|_{D^1,p}^p,\]

\[\lim_{n \to \infty} \sum_{d(i,0) > R} |u(i)|^q = \nu_\infty + \sum_{d(i,0) > R} |u(i)|^q = \nu_\infty + \|u\|_q^q.\]

**Remark.** (1) We prove the case $p = 1$ on $\mathbb{Z}^N$, while it is not true in the continuous case, since $L^1(\mathbb{R}^N)$ is not a dual space of any normed linear space.

(2) The difference between the sequence and the limit $u^*$-converges to 0 in $\ell^1(\mathbb{Z}^N)$, i.e. (3.35) and (3.36), which is not true in continuous setting. For example, consider the sequence of probability measures $\{\delta_n\}$ in $[0,1]$, where $\delta_n(x) := n \chi_{[0,\frac{1}{n}]}(x)$, then $\delta_n \to 0$ almost everywhere in $[0,1]$. However, $\delta_n \rightharpoonup^* \delta_0$ in $(C[0,1])^*$ and the Dirac measure $\delta_0$ is non-zero.
4. Proof I for Theorem 1

In this section, we will prove the existence of the extremal function for the discrete Sobolev inequality (1.1). Firstly, we prove that the minimizing sequence after translation has a uniform positive lower bound at the origin. This is crucial to rule out the vanishing case of the limit function.

**Lemma 13.** For $N \geq 3, 1 \leq p < N, q > p^*$, let $\{u_n\} \subset D^{1,p}(\mathbb{Z}^N)$ be a minimizing sequence satisfying (1.6). Then $\lim_{n \to \infty} \|u_n\|_{\ell^\infty} > 0$.

**Proof.** Choosing $q'$ such that $p^* < q' < q < \infty$, by interpolation inequality we have

$$1 = \|u_n\|_q \leq \|u_n\|_{q'} \|u_n\|_{\ell^{q'-q'}} \leq C_{q',p} q' \|u_n\|_{\ell^{q'-q'}} ,$$

where $C_{q',p}$ is the constant in the Sobolev inequality (1.4).

By taking the limit, we obtain

$$1 \leq C_{q',p} \lim_{n \to \infty} \|u_n\|_{\ell^{q'-q'}} .$$

This proves the lemma. \hfill \Box

**Remark.** The maximum of $|u_n|$ is attainable since $\|u_n\|_q = 1$. Define $v_n(i) := u_n(i + i_n)$, where $|u_n(i_n)| = \max |u_n(i)|$. Then the translation sequence $\{v_n\}$ is uniformly bounded in $D^{1,p}(\mathbb{Z}^N)$, $\|v_n\|_q = 1$ and $|v_n(0)| = \|u_n\|_{\ell^\infty}$. By Lemma 13 passing to a subsequence if necessary, we have

$$v_n \longrightarrow v \text{ pointwise in } \mathbb{Z}^N ,$$

(4.1) \hfill \lim_{n \to \infty} \|u_n\|_{\ell^\infty} > 0.

Similarly, we have the following corollary for the discrete HLS inequality (1.7).

**Corollary 14.** For $r, t > 1, 0 < \lambda < N, 1/\lambda + 1/r > 1 + 1/\lambda$, let $\{f_n\} \subset \ell^r(\mathbb{Z}^N)$ be a maximizing sequence satisfying (1.8). Then $\lim_{n \to \infty} \|f_n\|_{\ell^r} > 0$.

**Proof.** Choosing $r'$ such that $r < r' < r^* < \infty$, by interpolation inequality we have

$$C_{r',t} \|i^{-\lambda} \ast f_n\|_{r'} \leq \|f_n\|_{r'} \leq \|f_n\|_{r} \|f_n\|_{\ell^{r'-r}} = \|f_n\|_{\ell^{r'-r}} ,$$

where $C_{r',t}$ is the constant in the HLS inequality (1.7).

By taking the limit, we obtain

$$C_{r',t} \|f_n\|_{r'} \leq \lim_{n \to \infty} \|f_n\|_{\ell^{r'-r}} .$$

This proves the corollary. \hfill \Box

Next, we give the proof I of Theorem 1.

**Proof I of Theorem 1.** Let $\{u_n\} \subset D^{1,p}(\mathbb{Z}^N)$ be a minimizing sequence satisfying (1.4). And the translation sequence $\{v_n\}$ is defined in the remark after Lemma 10.

By equalities (4.1) and (4.8) in Lemma 12 passing to a subsequence if necessary, we get

$$S = \lim_{n \to \infty} \|v_n\|_{\ell^{p_1,p}} = \|v\|_{\ell^{p_1,p}} + \mu_{\ell^\infty} ,$$

$$1 = \lim_{n \to \infty} \|v_n\|_q = \|v\|_q + \nu_{\ell^\infty} .$$
From the Sobolev inequality, (3.7) and the inequality
\[(a^q + b^q)^{p/q} \leq a^p + b^p, \forall a, b \geq 0,\]
we get
\[S = \|v\|^p_{D^{1,p}} + \mu_{\infty} \]
\[\geq S\left(\|v\|^q_{q} + \nu_{\infty}\right) \]
\[\geq S\left(\|v\|^q_{q} + \nu_{\infty}\right)^{p/q} = S.\]
Since \((a^q + b^q)^{p/q} < a^p + b^p\) unless \(a = 0\) or \(b = 0\), we deduce from (4.1) that
\[\|v\|^q_{q} = 1.\]
By
\[\|v\|^p_{D^{1,p}} \geq S\|v\|^p_{q},\]
we get
\[\|v\|^p_{D^{1,p}} = S = \lim_{n \to \infty} \|v_n\|^p_{D^{1,p}}.\]
That is, \(v\) is a minimizer. \(\square\)

Now we are ready to prove Theorem 2.

Proof of Theorem 2. By the same argument as in Lemma 13, we can show that
\[\lim_{n \to \infty} \|u_n\|_{\ell_{\infty}} > 0.\]
Let \(g_n \in \Gamma\) such that \(|u_n(g_n)| = \max_{g \in \Gamma} |u_n(g)|\). Then for \(v_n := u_n(g_ng)\), we can prove the result verbatim as in Theorem 1. \(\square\)

5. PROOF II FOR THEOREM 1 AND THEOREM 3

In this section, we give another proof for Theorem 1 using the discrete Brézis-Lieb lemma. Then we prove Theorem 3 in a similar way.

Proof II of Theorem 1. Using Lemma 13 by the translation and taking a subsequence if necessary, we can get a minimizing sequence \(\{u_n\}\) satisfying (1.6), \(u_n \rightharpoonup u\) pointwise in \(\mathbb{Z}^N\), and \(|u(0)| > 0\).

By Lemma 10, the inequality (4.2) and the Sobolev inequality, then passing to a subsequence if necessary, we have
\[(5.1) \quad S = \lim_{n \to \infty} \|u_n\|^p_{D^{1,p}} = \lim_{n \to \infty} \frac{\|u_n\|^p_{D^{1,p}}}{\|u_n\|^q_{q}} = \lim_{n \to \infty} \frac{\|u_n - u\|^p_{D^{1,p}}}{\|u_n - u\|^q_{q} + \|u\|^q_{q}} \]
\[\geq \lim_{n \to \infty} \frac{\|u_n - u\|^p_{D^{1,p}}}{\|u_n - u\|^q_{q} + \|u\|^q_{q}} \]
\[\geq \lim_{n \to \infty} S\|u_n - u\|^p_{q} + \|u\|^p_{D^{1,p}}.\]
Since \(u \neq 0\), we have that
\[\|u\|^p_{D^{1,p}} \leq S\|u\|^p_{q},\]
which implies
\[\|u\|^p_{D^{1,p}} = S\|u\|^p_{q} .\]
By (5.1), passing to a subsequence, we get
\[\lim_{n \to \infty} \|u_n - u\|^p_{D^{1,p}} = S \lim_{n \to \infty} \|u_n - u\|^p_{q}.\]
Since $0 < \|u\|_q \leq \lim_{n \to \infty} \|u_n\|_q = 1$, it suffices to show that $\|u\|_q = 1$. Suppose that it is not true, i.e. $0 < \|u\|_q = D < 1$, then by Lemma 10
\[
\lim_{n \to \infty} \|u_n - u\|_q^2 = \lim_{n \to \infty} \|u_n\|_q^2 - \|u\|_q^2 = 1 - D^q > 0.
\]
However, $(a^q + b^q)^{p/q} < a^p + b^p$ if $a, b > 0$. This yields a contradiction by (5.1).

Thus, $\|u\|_q = 1$ and $u$ is a minimizer.

Taking measure spaces $(M, \Sigma, \mu)$ and $(M', \Sigma', \mu')$ in [31, Lemma 2.7] as $\mathbb{Z}^N$, we have the following lemma.

**Lemma 15.** Let $A$ be a bounded linear operator from $\ell^p(\mathbb{Z}^N)$ to $\ell^q(\mathbb{Z}^N)$ with $1 \leq p \leq q < \infty$. For $u \in \ell^p(\mathbb{Z}^N)$, $u \neq 0$, let
\[
R(u) := \frac{\|Au\|_q}{\|u\|_p}, \quad \text{and} \quad N := \sup \{ R(u) \mid u \in \ell^p(\mathbb{Z}^N), u \neq 0 \}.
\]

Let $\{u_n\}$ be a uniformly normed-bounded maximizing sequence for $N$. Suppose that $u_n \to u \neq 0$ pointwise,
\[
Au_n \to Au \text{ pointwise .}
\]
Then $u$ is a maximizer, i.e. $R(u) = N$.

Moreover, if $p < q$ and $\lim_{n \to \infty} \|u_n\|_p = C$ exists, then $\|u\|_p = C$ and $\lim_{n \to \infty} \|Au_n\|_q = \|Au\|_q$.

Next, we can prove Theorem 3 in a similar way.

**Proof of Theorem 3.** Let $A = |i|^{-\lambda} : \ell^r(\mathbb{Z}^N) \to \ell^t(\mathbb{Z}^N)$. By Corollary 14 and Lemma 15 we obtain a maximizer $f$ for $K$. And by Lemma 8 there exists a unique $g \in \ell^s(\mathbb{Z}^N)$ with $\|g\|_s = 1$ such that $(f, g)$ is a maximizing pair for (0.2).

Finally, we prove Corollary 4 in Section 1.

**Proof of Corollary 4.** By Theorem 1 there exists a minimizer $u$ for $S$. Replacing $u$ by $|u|$, we know that $|u|$ is still a minimizer. Therefore, we get a non-negative solution $u$. It follows from the Lagrange multiplier that $u$ is a solution of equation (1.10). The maximum principle yields that $u$ is positive.

We prove Corollary 6 in Section 1.

**Proof of Corollary 6.** Define the functional
\[
J(f, g) := \sum_{i,j \in \mathbb{Z}^N, i \neq j} \frac{f(i)g(j)}{|i-j|^{\lambda}}.
\]
From Theorem 3 we know that there is a maximizing pair $(f, g)$ for (0.2) under the constraint $\|f\|_r = \|g\|_s = 1$. A computation yields the Euler-Lagrange equation (1.11) for (0.2). Replace $(f, g)$ by $(|f|, |g|)$, which is still a maximizing pair. The pair is positive by (1.11).

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