The elastic modulus reduction method for upper bound limit analysis of pressure vessels incorporating material strain hardening effect

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Abstract. Pressure vessel is a kind of special equipment which is involved in various industries. Once it is damaged, it will cause great loss to people's life and property. The traditional elastic modulus reduction method for upper bound limit analysis of pressure vessels adopts the elastic-perfectly plastic material model. This method will result in waste of materials without considering the material strain hardening effect. So the material strain hardening effect is introduced into upper bound limit analysis. Firstly, the equivalent yield strength considering material strain hardening effect is established. Then, the formula of limit load multiplier which is expressed by the equivalent yield strength is derived based on the upper bound theorem. Combined with the strategy for elastic modulus adjustment, the elastic modulus reduction method for upper bound limit analysis of pressure vessels incorporating material strain hardening effect is constructed. The limitation of the elastic-perfectly plastic model for the upper bound analysis method is eliminated by the method proposed in this paper. The result calculated by this method is closer to the true limit load of pressure vessels.

1. Introduction

Analytical method and numerical method are the main methods to analyze the ultimate bearing capacity of the structure considering the strain hardening effect of materials. In the analytical method[1], it is assumed that the material is incompressible and satisfies the power hardening constitutive relation; the ultimate bearing capacity of the structure is solved on the basis of the total theory of plasticity. This method is applicable to frame structures with the simple constitutive model. The numerical method mainly includes elastoplastic incremental analysis (EPIA) and elastic modulus adjustment procedure (EMAP). EPIA[2] simulates the strain hardening process of materials and the damage process of structures through the iterative analysis of the incremental load, but EPIA has such problems as great sensitivity of calculations to parameters when considering the strain hardening effect of materials. In order to overcome the shortcomings of EPIA[3], EMAP obtains the ultimate bearing capacity of the structure considering the strain hardening of materials by considering the strain hardening property of materials in the ultimate load multiplier[4]. According to the bound theorem of limit analysis, EMAP can be divided into lower bound method[5] and up bound method[6]. EMAP lower bound method is widely used[7]; EMAP higher bound method is suitable for plastic pressure machining and geotechnical engineering, and can be combined with the lower bound method to
estimate the value of true ultimate bearing capacity. However, the strain hardening effect of materials is mainly used in the lower bound limit analysis of EMAP[8], and has not been introduced into the upper bound limit analysis of structures.

In view of this, in this paper, the study on elastic modulus reduction method[9] (EMRM) for upper bound limit analysis of pressure vessels considering strain hardening effect of materials was carried out with EMRM as a typical EMAP. The element bearing ratio (EBR) expressed as a generalized yield function was defined. The threshold of EMAP was determined. The ultimate load multiplier considering the strain hardening effect was determined according to the upper bound theorem of plastic limit analysis. This method relieves the limitation of the upper bound analysis method by the ideal elastoplastic constitutive model.

2. Equivalent yield strength
For the problem that the EMRM for the upper bound limit analysis of pressure vessels fails to consider the strain hardening effect of materials, the equivalent yield strength considering the strain hardening of materials was given on the basis of the principle of strain energy equivalence; and the equivalent ideal elastoplastic model for bilinear, power hardening and Ramberg-Osgood materials was constructed.

2.1. Equivalent ideal elastoplastic model based on principle of strain energy equivalence
In order to consider the strain hardening effect of materials upon the upper bound limit analysis for shell structures by EMRM, the simplified elastoplastic constitutive model reflecting the real stress-strain relationship should be converted into an equivalent ideal elastoplastic model. In Figure 1, the equivalent yield strength corresponding to the equivalent ideal elastoplastic model is obtained according to the principle of strain energy reciprocity (that is, areas A1 and A2 are equivalent) [8].

\[ \sigma = g(\varepsilon, \sigma) \quad (1) \]

The equivalent yield strength under the equivalent ideal elastoplastic model \( \sigma^* \) is obtained by Formula (2):

\[ \int_{\varepsilon_i}^{\varepsilon_f} \varepsilon \, d\sigma - \frac{1}{2} (\sigma_i^* - \sigma_s) (\varepsilon_f - \varepsilon_i) = \varepsilon_f (\sigma_f - \sigma_s) - \int_{\varepsilon_i}^{\varepsilon_f} \varepsilon \, d\sigma \]

\[ \varepsilon_i = \frac{\sigma_i}{E}, \quad \varepsilon_f = \frac{\sigma_f}{E} \quad (2) \]

2.2. Equivalent yield strength based on complete stress-strain data
In this paper, the equivalent yield strength corresponding to the common elastoplastic stress-strain model was solved on the basis of complete stress-strain data. The complete stress-strain data can be
obtained by the static tensile test of materials. Different elastoplastic constitutive models can be fitted according to the stress-strain data. The common elastoplastic constitutive models are shown in Figure 2.

![Commonly used constitutive model](image)

The corresponding equivalent yield strength $\sigma^*$ can be obtained by substituting the basic parameters of the common elastoplastic constitutive models into Formula (2):

Bilinear: \[
\sigma^* = (1 - \frac{E}{E_i})\sigma_f + E \frac{E_i}{E} \sigma_j \pm (\sigma_f - \sigma_j) \frac{E}{E_i} (\frac{E}{E_i} - 1)
\] (3)

Power Hardening: \[
\sigma^* = E \left[ \left( \frac{\sigma_f}{A} \right)^{\frac{1}{m}} \pm \left( \frac{\sigma_f}{A} \right)^{\frac{1}{m}} - \frac{2}{EA^m(1 + \frac{1}{m})} \left( \frac{\sigma_f^{m+1}}{1 + \frac{1}{m}} \right) + \frac{2}{E^2} \right]
\] (4)

Ramberg-Osgood: \[
\sigma^* = \left( \frac{\sigma_f + a\sigma_f^n}{\sigma_f^{m+1}} \right) \pm \left( \frac{\sigma_f + a\sigma_f^n}{\sigma_f^{m+1}} \right)^n - \frac{2a^n\sigma_f^{m+1}}{(n+1)\sigma_f^{m+1}} - \sigma_j^2 = \frac{2a^n\sigma_f^{m+1}}{n+1}
\] (5)

Where, $\sigma_f$ is the tensile strength of material. The simplified elastoplastic constitutive model is converted into the equivalent ideal elastoplastic constitutive model to obtain the equivalent yield strength considering the strain hardening effect. The upper bound load multiplier based on the equivalent yield strength can be obtained accordingly.

### 3. EMRM for upper bound limit analysis of pressure vessels considering strain hardening effect of materials

The EMRM for the upper bound limit analysis of shell structures considering the strain hardening effect of materials proposed herein takes EBR as a basic parameter of EMAP and the reference bearing ratio as the threshold of modulus adjustment, determines the ultimate load multiplier considering the strain hardening effect based on equivalent yield strength, and is combined with the EMAP strategy to form the maneuvering allowable strain field and displacement field gradually approaching the limit state by reducing the elastic modulus of the high bearing element, so as to obtain the ultimate bearing capacity of shell structures.

#### 3.1. EBR

Element bearing ratio (EBR) is a control parameter that can simultaneously represent the internal force and resistance of the element in the EMRM modulus adjustment process. It is used to characterize the degree to which discrete elements approach plastic yield. For shell structures, the EBR $r^*_k$ can be expressed as:

\[
r^*_k = \sqrt{f}
\] (6)

where, the superscript $e$ is the element number; the subscript $k$ is an iterative step; $f$ is a generalized
yield function expressed as [10]:

\[
\begin{align*}
 f &= f_N + f_M + \frac{|f_{NM}|}{\sqrt{3}} \\
 f_N &= n_x^2 + n_y^2 - n_x n_y + 3n_{xy}^2 \\
 f_M &= m_x^2 + m_y^2 - m_x m_y + 3m_{xy}^2 \\
 f_{NM} &= n_x m_x - \frac{1}{2} n_y m_y - \frac{1}{2} n_x m_y + n_y m_x + 3n_{xy} m_{xy} \\
\end{align*}
\]

(7)

where, \(N_x\), \(N_y\) and \(N_{xy}\) are the internal forces of the film on the element section; \(M_x\), \(M_y\) and \(M_{xy}\) are bending moments and torques on the element section; \(N_{px}\), \(N_{py}\), \(N_{pdy}\), \(M_{px}\), \(M_{py}\), \(M_{pdy}\) are the section resistance under the action of each internal force alone. The diagram for internal forces of shell element with a thickness of \(T\) is shown in Figure 3.

![Figure 3. Diagram for internal force of plate and shell elements](image)

3.2. EMAP strategy

The bearing ratio uniformity \(d_k\) can reflect the approaching degree of bearing capacity of all elements in the structure. \(d_k\) can be expressed as[7]:

\[
d_k = \frac{\bar{r}_k + r_k^{\text{min}}}{\bar{r}_k + r_k^{\text{max}}} = \frac{1}{N} \sum_{e=1}^{N} r_k^e
\]

(9)

where, \(\bar{r}_k\), \(r_k^{\text{max}}\) and \(r_k^{\text{min}}\) are the mean, maximum and minimum of the EBR in the structure, respectively; \(N\) is the number of structural discrete elements.

As a reference value in the EMAP process, the reference bearing ratio \(r_k^0\) can be expressed as [11]:

\[
r_k^0 = r_k^{\text{max}} - d_k \times (r_k^{\text{max}} - r_k^{\text{min}})
\]

(10)

According to Hooke's law, the EMRM strategy can be expressed as [7]:

\[
E_{k+1}^e = \begin{cases} 
E_k^e \left( \frac{2(r_k^0)^2}{(r_k^e)^2 + (r_k^0)^2} \right), & r_k^e > r_k^0 \\
E_k^e, & r_k^e \leq r_k^0
\end{cases}
\]

(11)

where, \(E_{k+1}^e\) and \(E_k^e\) are the elasticity modulus of element \(e\) at the \(k+1\)th and \(k\)th iteration,
respectively; the elastic modulus of high bearing element with an EBR greater than \( r_k^0 \) will be reduced to simulate the damage evolution process of the structure.

3.3. Upper bound load multiplier considering strain hardening effect of materials

An optimization model for upper bound limit analysis can be constructed according to the condition that the work done by the external load is not less than the plastic dissipation work of the structure [10]:

\[
P^u_L = \min \left\{ \int_V \sigma \varepsilon^e dV \right\}
\]

s.t. \( u^* = 0 \) on \( S_u \)
\( u^*_i = 0 \) in \( V \)

where, \( P^u_L \) is the upper bound load multiplier; \( V \) is the volume of the structure; \( S_u \) and \( S_p \) are the boundary surface of force and displacement, respectively; \( \sigma \) is the yield strength of the material; \( \varepsilon^e \) is the equivalent strain field; \( u^*_i \) is the maneuvering allowable displacement field.

Combined with the EMRM strategy in Formula (11), this optimization model can simulate the process of structural plastic damage in the iterative analysis process to gradually obtain the failure mode approaching the plastic limit state of the structure, and then obtain the ultimate bearing capacity of the structure in combination of Formula (12) according to the principle of functional reciprocity:

\[
P^u_L = \min\left\{ P^u_{L,1}, P^u_{L,2}, \ldots, P^u_{L,k}, \ldots, P^u_{L,\rho} \right\}
\]

\[
P^u_{L,k} = \frac{\int_V D^e dV}{\int_V U^e dV} = \frac{\int_V \sigma^e \varepsilon^e_d dV}{\int_V \sigma^e \varepsilon^e_k dV}
\]

where, \( D^e \) is the plastic dissipation work of the structure at the \( k \)th iteration; \( U^e_k \) is the elastic strain energy at the \( k \)th iteration; \( \sigma^e_d \) and \( \varepsilon^e_d \) are the equivalent stress field and equivalent strain field at the \( k \)th iteration under the reference load.

The ultimate bearing capacity of the shell structure considering the strain hardening effect \( P^u_{L,k} \) can be obtained by considering structure discretization and substituting in the equivalent yield strength:

\[
P^u_{L,k} = \frac{\sum_{e=1}^{N} \sigma^e \varepsilon^e_{eq,k} V^e}{\sum_{e=1}^{N} \sigma^e \varepsilon^e_{eq,k} V^e}, \quad V = \sum_{e=1}^{N} V^e
\]

where, \( V^e \) is the volume of element \( e \). The mesh division of the shell structure is fine enough. Meanwhile, the thickness of the element is small. Therefore, the stress and strain at the centroid of the element can be approximated to represent the stress and strain of the whole element.

Formula (14) is solved iteratively until the ultimate bearing capacity of two adjacent iterative steps satisfies the convergence criterion:

\[
\left| \frac{P^u_{L,k} - P^u_{L,k-1}}{P^u_{L,k}} \right| \leq \varepsilon
\]

where, \( \varepsilon \) is the preset allowable error and takes 0.001 in the calculating example herein.

If the convergence of calculation results occurs after \( k \) iterations, the ultimate bearing capacity of the structure is:
Accordingly, the equivalent yield strength is solved according to complete stress-strain data; the ultimate load multiplier is defined on the basis of equivalent yield strength; the ultimate load multiplier of pressure vessels considering the strain hardening of materials are solved in combination with the EMAP process.

4. Examples

4.1. Pressure vessel

4.1.1. Basic parameters
In order to verify the accuracy of the upper bound method of EMRM considering the strain hardening of materials proposed herein, the test data in the relevant study of Zhejiang University of Technology (ZJUT)[11] were selected for comparison. Figure 4 shows the test model structure of the pressure vessel, the material of which is Q235B. See Table 1 for steel parameters. See Table 2 for geometrical parameters of the vessel.

![Figure 4. Schematic diagram of test model](image)

| Steel     | Elastic modulus $E$ / MPa | Yield strength $\sigma_s$ / MPa | Tensile strength $\sigma_f$ / MPa | Poisson's ration $\mu$ |
|-----------|---------------------------|-------------------------------|----------------------------------|------------------------|
| Q235B     | $2.00 \times 10^5$       | 324                           | 460                              | 0.3                    |

| Experiment object | D/ mm | L/ mm | T/ mm |
|-------------------|-------|-------|-------|
| Pressure vessel    | 160   | 380   | 1.5   |

4.1.2. Establishment of finite element model
The pressure vessel is under uniform internal pressure. The finite element analysis software ANSYS was needed for calculating the ultimate bearing capacity by the upper bound method of EMRM. The element SHELL181 was selected. See Figure 5 for the finite element mesh.

![Figure 5. Finite element mesh for the pressure vessel](image)
4.1.3. Calculation of equivalent yield strength

In order to compare the influence of different constitutive models on ultimate bearing capacity, three common elastoplastic constitutive models were selected. For the bilinear constitutive model, the tangent modulus is \( E_t = 0.83 \times 10^5 \text{MPa} \). For the power hardening constitutive model, the material parameter is \( A = 545.93 \) and the power hardening coefficient \( m = 0.096 \). For the Ramberg-Osgood constitutive model, the coefficient \( \alpha = 2.35 \); and \( n = 10.08 \). See Table 3 for the equivalent yield strength corresponding to the common elastoplastic constitutive models.

| Table 3. Equivalent yield strength / Mpa |
|----------------------------------------|
| Constitutive model                     |
| Bilinear                               |
| Power Hardening                        |
| Ramberg-Osgood                         |
| Equivalent yield strength based on     |
| complete stress-strain data            |
| 392.07                                 |
| 421.34                                 |
| 421.67                                 |

4.1.4. Influence of strain hardening effect of materials on ultimate bearing capacity

In this paper, the upper bound method of EMRM and the lower bound method of EMRM considering strain hardening effect were used to analyze the ultimate bearing capacity; and the results were compared with test results[11]. See Table 4 for calculations. The comparison of the ultimate bearing capacity calculated by the two numerical methods with the measured values obtained from the test is shown in Figure 6.

| Table 4. Ultimate load bearing capacity of the pressure vessel calculated by lower bound EMRM and up bound EMRM |
|-----------------------------------------------------------------------------------------------------------|
| Constitutive model | Measured values / MPa | Lower bound EMRM | Up bound EMRM |
|---------------------|------------------------|------------------|---------------|
|                     | Ultimate bearing       | Iterations /     | Ultimate      | Iterations /     | Error /%       |
|                     | capacity / MPa         | times            | capacity / MPa| times            |               |
| Elastic-perfectly   | 8.75                   | 6.370            | 3             | 27.20           | 6.961          | 8             | 20.45          |
| plastic             |                        |                  |               |                 |                |               |
| Bilinear            | 8.75                   | 7.602            | 3             | 15.13           | 8.424          | 8             | 3.73           |
| Power Hardening     | 8.75                   | 8.160            | 3             | 7.23            | 9.053          | 8             | 3.46           |
| Ramberg-Osgood      | 8.75                   | 8.160            | 3             | 7.23            | 9.060          | 8             | 3.54           |

Figure 6. Comparison of ultimate load bearing capacity
According to Table 4, the error between the ultimate bearing capacity of the vessel calculated by the upper bound method of EMRM and the lower bound method of EMRM and with ideal elastoplastic constitutive models and the measured values obtained from the test [16] exceeded 20%. The reason is that there are some differences between the ideal elastoplastic constitutive models and the real stress-strain relationship of the pressure vessel. Compared with the lower bound method of EMRM, the calculation of the ultimate bearing capacity of the pressure vessel by the upper bound method of EMRM is more accurate with an error with the measured values within 4%. It can also be seen from Figure 6 that the calculation results by the upper bound method of EMRM considering the strain hardening effect of materials are closer to the measured values. Although constitutive models are different, the numerical values of the ultimate bearing capacity considering the strain hardening effect of materials are close. The reason is that these constitutive models are based on a real stress-strain relationship and can reflect the important properties of the material after reaching yield.

This pressure vessel calculating example verified the effectiveness of the calculation of the ultimate bearing capacity of the shell structure by the upper bound method of EMRM by comparing the calculation results by this method with the test results. In this paper, the EPIA with recognized effectiveness was used for the limit analysis of shell structures with different constitutive models and the comparison with the calculation results by the upper bound method of EMRM to further verify the effectiveness and applicability of the upper bound method of EMRM in shell structures.

5. Conclusion
In this paper, the equivalent yield strength was given on the basis of the principle of strain energy equivalence; the ultimate load multiplier based on equivalent yield strength was determined according to the upper bound theorem of plastic limit analysis; the EMRM for the upper bound limit analysis of pressure vessels considering the strain hardening effect of materials was established in combination with EMAP strategy. This method relieves the limitation of the upper bound analysis method by the ideal elastoplastic constitutive model and provides an efficient way to calculate the ultimate bearing capacity of pressure vessels. Based on the method proposed in this paper, a fast method for safety evaluation of pressure vessels can be presented.

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References
[1] Galambos, T.V. (1998) Guide to stability design criteria for metal structures. John Wiley and Sons, New York.
[2] Tong, R.C., Wang, X.C. (1997) Simplified method based on the deformation theory for structural limit analysis—I. Theory and formulation. International Journal of Pressure Vessels and Piping, 70: 43-49.
[3] Adibi-Asl, R., Seshadri, R. (2007) Local limit-load analysis using mβ method. Journal of Pressure Vessel Technology, 129: 296-305.
[4] Mahmood, S.L., Adibi-Asl R., Daley C.G. (2013) Plastic response estimation in repeated elastic analyses for strain hardening material model. Journal of Pressure Vessel Technology, 135:051201.
[5] Mahmood, S., Haddara, M., Seshadri, R. (2010) Lower bound limit loads of ship structure components using the mo-tangent method based on single linear elastic analysis[J]. Ocean Engineering, 37: 1139-1148.
[6] Wu, W.L., Yang, L.F., Zhang, W. (2012) Elastic modulus reduction method for the upper bound limit analysis of beam structures. Chinese Journal of Applied Mechanics, 29: 687-691.
[7] Yu B., Yang L.F. (2010) Elastic modulus reduction method for limit analysis considering initial
constant and proportional loadings. Finite Elements in Analysis and Design, 46: 1086-1092.

[8] Zhang W., Zhang Y., Ye Z.Y. (2017) Elastic modulus reduction method for limit analysis of structures consideration of material strain hardening effect. Journal of Basic Science and Engineering, 4: 109-118.

[9] Yu B., Yang L. (2010) Elastic modulus reduction method for limit analysis of thin plate and shell structures. Thin-Walled Structures, 48: 291-298.

[10] Yang L., Zhang W., Yu B. (2012) Safety evaluation of branch pipe in hydropower station using elastic modulus reduction method. Journal of Pressure Vessel Technology, 134: 041202.

[11] Jiang, Y.Z. (2015) Numerical Simulation and Experimental Study of Burst Pressure of Pressure Vessel. Zhejiang University of Technology, Hangzhou.