Statistical properties of the low-temperature conductance peak-heights for Corbino discs in the quantum Hall regime

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A recent theory has provided a possible explanation for the “non-universal scaling” of the low-temperature conductance (and conductivity) peak-heights of two-dimensional electron systems in the integer and fractional quantum Hall regimes. This explanation is based on the hypothesis that samples which show this behavior contain density inhomogeneities. Theory then relates the non-universal conductance peak-heights to the “number of alternating percolation clusters” of a continuum percolation model defined on the spatially-varying local carrier density. We discuss the statistical properties of the number of alternating percolation clusters for Corbino disc samples characterized by random density fluctuations which have a correlation length small compared to the sample size. This allows a determination of the statistical properties of the low-temperature conductance peak-heights of such samples. We focus on a range of filling fraction at the center of the plateau transition for which the percolation model may be considered to be critical. We appeal to conformal invariance of critical percolation and argue that the properties of interest are directly related to the corresponding quantities calculated numerically for bond-percolation on a cylinder. Our results allow a lower bound to be placed on the non-universal conductance peak-heights, and we compare these results with recent experimental measurements.

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I. INTRODUCTION

Experimental studies of the dissipative conductivity \(\sigma_{xx}\) of two-dimensional electron systems in the integer and fractional quantum Hall regimes have shown this to obey an unusual “non-universal scaling” at very low temperatures \[\text{[1]}. \] “Scaling” refers to the observation that the height of the peak in \(\sigma_{xx}\) associated with a transition between a plateau with quantized Hall conductivity \(\sigma_1\) and one with quantized Hall conductivity \(\sigma_2\) is proportional to \(|\sigma_1 - \sigma_2|\) for all well-developed peaks in a given sample. The constant of proportionality is found to fluctuate between samples, so this scaling is described as “non-universal”. This behavior is in contrast to theoretical predictions of scaling with a universal prefactor of 0.5 \[\text{[2]–[4]}\]. Non-universal scaling was most clearly demonstrated in recent experiments \[\text{[2]–[4]}\] on two Corbino disc samples for which prefactors of approximately 0.2 and 0.3 were found.

In Ref. \[\text{[2]}\] an explanation for non-universal scaling was proposed. This explanation is based on the hypothesis that in samples exhibiting this phenomenon there exist fluctuations in the electron density on scales large compared to microscopic length scales. From a classical calculation of the transport through the resulting inhomogeneous conductor, it was shown that, in samples with such inhomogeneities, non-universal scaling appears for all plateau transitions that have reached their low-temperature limiting forms (we shall refer to such transitions as “well-developed”). We will concentrate on the conclusions that were obtained for Corbino disc samples (analogous results hold for Hall bars \[\text{[2]}\]). For a well-developed transition between two quantized Hall plateaus with Hall conductivities \(\sigma_1\) and \(\sigma_2\) the two-terminal conductance of a Corbino disc sample takes the form \[\text{[2]}\]

\[ G = M|\sigma_1 - \sigma_2|, \tag{1} \]

where \(M\) is an integer related to the geometrical properties of a classical percolation problem defined on the spatially-varying filling fraction \(\nu(r)\) (the local filling fraction \(\nu(r)\) is defined as the ratio of the local electron number density \(n(r)\) to the density of flux quanta \(eB/h\)). We refer to \(M\) as the “number of alternating percolation clusters”; this quantity will be defined below, and the reason for this name will then become clear. As the magnetic field is varied, such that the sample sweeps through the plateau transition, the number of alternating percolation clusters passes through a sequence of integer values, starting and finishing with \(M = 0\) (in the plateau regions). Significantly, although the specific sequence of values that \(M\) passes through will in general differ for different configurations of the density fluctuations, the same sequence is predicted to occur for all well-developed transitions within a given sample. In particular, the maximum value \(M^{\max}\) will be the same for all such transitions. Thus, within the theory of Ref. \[\text{[2]}\], the maximum heights of all well-developed conductance peaks of a Corbino disc sample show a scaling with a sample-
dependent, but integer, prefactor $M^{\text{max}}$. If the measurements of the conductance $G$ are used to define an observed conductivity $\sigma_{xx}$ via the usual relation $\sigma_{xx} \equiv A_0 G$ where $A_0 \equiv (1/2\pi)\ln(r_2/r_1)$ is the geometrical aspect ratio of the Corbino disc determined by its inner $r_1$ and outer $r_2$ radii, then the peak-heights of the conductivity will also exhibit non-universal scaling. In this case the non-universal prefactor is $A_0 M^{\text{max}}$, which can vary between samples due to both fluctuations in the integer $M^{\text{max}}$ and variations in the aspect ratio $A_0$. In Ref. [5] it was shown that the non-universal prefactors for the conductivity peak-heights of the two Corbino discs studied in Ref. [5] are both consistent with the form $A_0 M^{\text{max}}$ if $M^{\text{max}} = 1$ in each case.

It is of interest to understand the statistical properties of the integer $M^{\text{max}}$ which determines the non-universal scaling prefactor. This is the issue that we address in the present paper. We study the probability distribution of the number of alternating percolation quantities $M$ for a Corbino disc sample as a function of the aspect ratio $A_0$, denoting the probabilities $P_M(A_0)$. As discussed in Section II, we study these probabilities for an ensemble of Corbino disc samples characterized by density fluctuations with a scale much less than the sample dimensions, and within a narrow range of filling fraction at the center of the quantum Hall transition, in which case the percolation model determining $M$ can be considered to be critical. By appealing to conformal invariance of critical percolation, we argue that the analogous probabilities which have recently been calculated numerically for bond-percolation in the cylindrical geometry [7] can be directly related to $P_M(A_0)$. For some disorder configurations, the maximum value $M^{\text{max}}$ could lie outside the region of filling fraction that we address, so our results provide information on a lower bound to $M^{\text{max}}$. We show that, if the two samples studied in Ref. [5] are drawn from the statistical ensemble that we have assumed, the combined probability for $M^{\text{max}} = 1$ in both of these samples is less than 15%. We view this small probability as evidence that short-range isotropic density inhomogeneities may not well represent the experimental samples, and suggest that density fluctuations on a scale comparable to the sample size could be present.

II. RELATION TO CONTINUUM PERCOLATION

We begin by explaining how the integer $M$ appearing in Eq. (1) is related to the geometrical properties of a continuum percolation model (the reader is referred to Ref. [6] for a detailed discussion). The connection to percolation arises when one considers the form of the spatial distribution of the local conductivity in the sample. In the theory of Ref. [6], the conductivity tensor at a position $\mathbf{r}$ is assumed to be specified by the local value of the filling fraction $\nu(\mathbf{r})$, and equal to the conductivity of a homogeneous system with this filling fraction. The inhomogeneous samples are assumed to contain short-range disorder on length scales much smaller than those of the density inhomogeneities, so the appropriate homogeneous system to consider is one which retains this impurity scattering. Consequently, the dependence of the local conductivity on the local filling fraction $\sigma(\nu)$ exhibits a quantized Hall effect, with the conductivity tensor taking quantized values over finite ranges of filling fraction. At low temperatures (which we consider in detail below), the transition regions between quantized Hall plateaus are very narrow in filling fraction and the conductivity is quantized at almost all values of the filling fraction. These quantized plateau regions lead to spatially-extended regions in the inhomogeneous sample within which the local conductivity takes the same, quantized value (in these regions the local filling fraction remains within a quantized Hall plateau). In particular, consider a situation in which the average filling fraction of the inhomogeneous sample $\sigma$ is close to the narrow range of filling fractions within which the homogeneous sample displays a transition between plateaus with quantized Hall conductivities $\sigma_1$ and $\sigma_2$. We denote the filling fraction at which the quantum Hall transition of the homogeneous sample is centered by $\nu_c$, and its width in filling fraction by $\delta\nu$. In the inhomogeneous sample, regions in which the local filling fraction is less than $\nu_c - \delta\nu/2$ will have a local conductivity that is quantized with Hall conductivity $\sigma_1$ (we shall refer to these as “$\sigma_1$-regions”), and regions where the local filling fraction is larger than $\nu_c + \delta\nu/2$ will have a local conductivity tensor that is quantized according to $\sigma_2$ (“$\sigma_2$-regions”). As temperature decreases, the regions of intermediate filling fraction that spatially separate the $\sigma_1$- and $\sigma_2$-regions, and inside of which the local conductivity tensor is not quantized, become progressively narrower, due to the progressive decrease in the width $\delta\nu$ of the plateau transition for an infinite homogeneous sample. In the limit of low temperatures, for which the plateau transition of the finite inhomogeneous sample is “well-developed” and Eq. (1) applies, these intermediate regions may be considered to be infinitesimally narrow. The sample then divides cleanly into $\sigma_1$- and $\sigma_2$-regions separated by sharp boundaries along lines on which the filling fraction has the threshold value $\nu_c$ (this is the filling fraction at which the plateau transition occurs for an infinite homogeneous sample in the limit of zero temperature). The spatial distribution of the conductivity is determined by a continuum percolation model defined on the spatially-varying filling fraction $\nu(\mathbf{r})$ with the threshold $\nu_c$: in a region where the filling fraction is below (above) $\nu_c$ the local conductivity is quantized with a Hall conductivity $\sigma_1$ ($\sigma_2$). Figure [6] illustrates a particular configuration of the local conductivity, for an average filling fraction close to $\nu_c$ such that the two regions occupy almost equal area.
As discussed in detail in Ref. [3], in the limit of low temperatures when the local conductivity tensor can be assumed to be purely off-diagonal, dissipation does not occur in the bulk of the sample but only at junctions formed between the $\sigma_1$- and $\sigma_2$-clusters and the contacts. The net conductance of the sample depends only on the topology of the conductivity distribution. Specifically, the two-terminal conductance of the sample is found to be given by Eq. (3), with the “number of alternating percolation clusters” $M$ defined by

$$M \equiv \min(n_1, n_2),$$

where $n_i$ is the number of clusters of type $\sigma_i$ that connect between inner and outer contacts of the Corbino disc. The configuration illustrated in Figure 1 is a case in which two $\sigma_1$-clusters and two $\sigma_2$-clusters connect the contacts, so $n_1 = n_2 = 2$ and $M = 2$.

We will briefly discuss the evolution of $M$ as the sample is swept through a transition between the $\sigma_1$ and $\sigma_2$ quantum Hall plateaus. Consider first a situation in which the average filling fraction $\bar{\nu}$ is sufficiently below $\nu_c$ that the sample is in the $\sigma_1$ quantized Hall plateau. The conductivity at almost all points in the sample is quantized at $\sigma_1$ so the inner and outer contacts are connected by a single cluster of this kind. Therefore $n_1 = 1$ and $n_2 = 0$, and, by Eqs. (2) and (1), $M = 0$ and $G = 0$. This correctly attributes a vanishingly small conductance to the sample in this quantized Hall plateau. When $\bar{\nu}$ is sufficiently larger than $\nu_c$, the sample is in the $\sigma_2$ quantized Hall plateau, and a similar picture emerges (with the roles of $\sigma_1$ and $\sigma_2$ reversed): $n_2 = 1$ and $n_1 = 0$ and the conductance vanishes $G = 0$.

As the average filling fraction is swept through $\nu_c$, such that the sample passes between the two quantized Hall plateaus, there appear distributions of the two phases that are quite different from those in the plateau regions. It is straightforward to convince oneself that the topology of the Corbino disc constrains the only other possible values of the pair $(n_1, n_2)$ to the cases $(0, 0)$, $(1, 1)$, $(2, 2)$, . . . , for which $M = 0, 1, 2, . . .$. In the course of the plateau transition, the distribution of the conductivity will pass through one or more of these different topological configurations, so $M$ will pass through a sequence of integer values and a step-like peak in the conductance $G$ will appear (as noted in Ref. [3], it is possible that no peak appears, since the sequence $(1, 0) \rightarrow (0, 0) \rightarrow (0, 1)$ is permitted). The sequence of topological configurations and consequently the shape and height of the conductance peak depend on the specific spatial distribution of the filling fraction $\nu(r)$ on which the percolation model is defined. The statistical properties of the peak heights and shapes therefore depend on the statistical form of the density inhomogeneities in the sample.

We will assume that the spatial fluctuations in the filling fraction $\nu(r)$ are isotropic and homogeneous and have a correlation length $R_c$ that is small compared to the sample size $L$. The form of the long-range density fluctuations in typical quantum Hall samples is not known, so it is not clear if the form we study is generally appropriate. However, in the absence of any prior knowledge, it is natural to consider the fluctuations to be homogeneous and isotropic. Furthermore, provided the correlation length $R_c$ is small compared to the sample size, universality of percolation causes the statistical properties at large length scales to be insensitive to the details of the density fluctuations at a scale $R_c$. The large-scale properties of the resulting percolation model are controlled by the correlation length $\xi$, which diverges at the percolation threshold according to

$$\xi \simeq R_c|\bar{\nu} - \nu_c|^{-4/3}.$$  

(The percolation threshold of an infinite inhomogeneous sample $\bar{\nu}_c$ is equal to the critical filling fraction of an infinite homogeneous sample $\nu_c$ if the density fluctuations are statistically symmetric about $\bar{\nu}$.) In particular, the transition region between the two quantized Hall plateaus occurs within the range of average filling fraction $\bar{\nu}$ close to the percolation threshold $\bar{\nu}_c$, for which the correlation length $\xi$ becomes comparable to the system size, $\xi \simeq L$.

We will restrict attention to the narrow range at the center of the transition for which $\xi \gg L$. In this central range, the distribution of the two phases is equivalent to the distribution exactly at the critical point $\bar{\nu} = \bar{\nu}_c$: we will refer to this range of filling fraction as the “critical region” of the transition. In the following, we discuss how the properties in the critical region of the continuum percolation model may be related to numerical studies of a critical lattice model.

III. UNIVERSALITY AND CONFORMAL INVARIANCE OF CRITICAL PERCOLATION

It is well-established that there exist “universal” properties of percolation models that are the same for all
models of percolation within a given “universality class” (specified by the symmetry and dimensionality of the model). In particular, it is believed that at their respective critical points all percolation models within the same universality class give rise to the same probability distribution of configurations coarse-grained on a length scale \( \Lambda \) in the limit \( \Lambda \gg a \), where \( a \) is the largest microscopic length scale of the model. Any property that is a function of this coarse-grained probability distribution will also be a universal property of critical percolation. We anticipate (as we discuss further below) that the probabilities \( P_M(A_0) \) in a finite system are universal properties of two-dimensional isotropic percolation at criticality, since these depend on the structure of clusters on large scales (of order the system size \( L \)) and can be expected to be insensitive to the introduction of coarse-graining over small length scales \( \Lambda \ll L \).

If this is the case, these probabilities may be determined by studying bond-percolation on a lattice, which falls within the same universality class as the continuum percolation in which we are primarily interested. One can achieve a discretization of the Corbino disc that is suitable for studying bond-percolation by introducing a finite square lattice of the form illustrated in Fig. 2(a) (one could equally well use a triangular, honeycomb or random lattice, all of which lead to isotropic behavior on large length scales). However, motivated by the fact that a percolation model is defined by local rules, it is natural to extend the concept of universality to include percolation models whose properties vary slowly in space, provided that locally these models are suitable representations of critical percolation (within the appropriate universality class). This hypothesis forms the basis for the application of conformal field theory to percolation and other critical systems [10]. In particular, bond-percolation on any conformal transformation of a square lattice presents a good model of the critical percolation in which we are interested, since this transformation generates a lattice that is locally square. A discretization of the Corbino disc by a lattice of this kind is illustrated in Fig. 2(b).

![FIG. 2. Two possible discretizations of the Corbino disc. (a) Uniform square lattice with boundaries chosen to best reproduce the circular contacts (dashed). (b) Conformal transformation of a square lattice defined on a cylinder: the lattice constant varies continuously with position, but the lattice is everywhere locally square.](image)

The work that we present in this paper and the conclusions that we will draw are based on the hypothesis that a critical bond-percolation model defined on the discretization of the Corbino disc illustrated in Fig. 2(b) may be used to determine the probabilities \( P_M(A_0) \) for a continuum percolation model at criticality in this geometry. This is equivalent to assuming that these probabilities are conformally-invariant properties of critical percolation.

While the statements of universality and conformal invariance of critical percolation do not have rigorous foundations, they are supported by a great deal of numerical evidence [8,9]. Of particular relevance to the present work are numerical studies that have shown the probabilities for any number of clusters of one kind to connect between the free edges of rectangles [11] and cylinders [12,13] to be universal quantities. Furthermore, it has been shown that these probabilities can be expressed as conformally-invariant properties of the standard field-theoretical formulation of percolation [14], and this conformal-invariance has recently been demonstrated numerically [12,13]. It has also been demonstrated that the probabilities for \( n \) clusters of one kind to span the free edges of cylinders are universal quantities [13]. Since the probabilities \( P_M(A_0) \) and the spanning probabilities studied in Refs. [12,13] have similar definitions in terms of the properties of large-scale clusters, we believe that these works provide strong motivation for our hypothesis that \( P_M(A_0) \) are conformally-invariant quantities.

We note for reference that conformal invariance is also a property of the continuum conduction problem in two dimensions. For example, suppose that \( S \) and \( S' \) are two regions of annular topology which can be mapped onto each other by a conformal mapping \( r' = f(r) \). Let the local conductivity tensor at each point \( r \) in \( S \) be identical to the conductivity tensor at its image point \( r' \). Then the conductance between the inner and outer edges of \( S' \) will be identical to the conductance between the inner and outer edges of \( S \). As a particular case of this theorem, one may use the conformal mapping described below [Eq. (4)] to show that the two-terminal conductance of a Corbino disc sample with a constant local conductivity and with an aspect ratio \( A_0 = (1/\pi) \ln(r_2/r_1) \) is equal to that of a cylinder with the same constant local conductivity and with a length that is \( A_0 \) times its circumference.

IV. CONNECTION TO NUMERICAL STUDIES ON A CYLINDER

We now show in detail how a lattice of the form illustrated in Fig. 2(b) can be constructed. We start from a \( N_x \times N_y \) square lattice whose nodes have the Cartesian co-ordinates \( (x, y) = (n_x, n_y) \), where \( n_x = 0, \ldots, N_x - 1 \) and \( n_y = 0, \ldots, N_y - 1 \). Consider the set of points defined by the images \( \{(u, v)\} \) of each node \( \{(x, y)\} \) under the transformation

\[
 w = e^{-2\pi iz/N_x}, \tag{4}
\]

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\[
 w = e^{-2\pi iz/N_x}, \tag{4}
\]
The set of points \((u, v)\) lie in a region the shape of a Corbino disc with inner and outer radii \(r_1 = 1, r_2 = \exp[2\pi(N_y - 1)/N_x]\). The aspect ratio of the disc is therefore \(A_0 = N_y/N_x\) when \(N_y \gg 1\) which we assume to be the case. The resulting lattice is locally square (in the limit \(N_x, N_y \gg 1\)) as is guaranteed by the properties of analytic functions in the complex plane \(z = x + iy\), and therefore represents a suitable discretization of critical percolation (within the assumptions of conformal invariance). Thus, through the use of the above transformation, a model of bond-percolation defined on the lattice \((u, v)\) covering a Corbino disc with aspect ratio \(A_0 = N_y/N_x\) may be related to a model of bond-percolation on a square lattice \((x, y)\) of size \(N_x \times N_y\). To reproduce the topology of the Corbino disc, periodic boundary conditions must be imposed on the square lattice and a row of bonds introduced between the edges at \(n_x = 0\) and \(n_x = N_x - 1\): this therefore represents a bond-percolation model on a cylinder.

A recent paper by two of the present authors \([7]\) contains a discussion of bond-percolation in the cylindrical space discretized by the \(N_x \times N_y\) square lattice described above. Numerical calculations of the probability distribution of the number of alternating percolation clusters are reported (the definition of this number in terms of the distribution of clusters on a cylinder is analogous to that on a Corbino disc, with the ends of the cylinder playing the role of the contacts of the Corbino disc). At the critical point, these probabilities are found to depend only on the ratio \(N_x/N_y\) in the large-system limit \((N_x, N_y \gg 1)\). The resulting values in the large system limit are presented in Fig. 3 as a function of \(N_x/N_y\).

![Fig. 3](image)

**FIG. 3.** Probabilities for the number of alternating percolation clusters \(M\) at the critical point of a bond-percolation model on a cylinder with circumference \(N_x\) and length \(N_y\) (from the calculations reported in Refs. \([8, 9]\)). By conformal invariance, the same values describe continuum percolation on a Corbino disc with aspect ratio \(A_0 = N_y/N_x\).

In view of the discussion of Sec. \([11]\) and the use of the conformal transformation \([4]\), we claim that, at the critical point, the probability distribution of the number of alternating percolation clusters found numerically for bond-percolation on a cylinder with aspect ratio \(A_0 = N_y/N_x\) is equal to the corresponding quantity \(P_M(A_0)\) for continuum percolation on a Corbino disc with aspect ratio \(A_0 = N_y/N_x\) (in the limit \(N_x, N_y \gg 1\)). The equivalence between the quantities \(P_M(A_0)\) and the results of Ref. \([8]\) is made explicit by the relation \(P_M(A_0) = F^*_M(L_1/L_2 = 1/A_0, 0)\), where \(F^*_M(L_1/L_2, x)\) is the scaling function discussed in Ref. \([8]\) (note that the aspect ratio \(R = L_1/L_2\) defined in Refs. \([10, 11]\) is the inverse of the aspect ratio we have defined \(A_0 = N_y/N_x\)).

To conclude our discussion of the numerical studies reported in Ref. \([7]\), we note that this work also presents results for the number of alternating percolation clusters away from the critical point. The corresponding probability distributions are found to satisfy a scaling form, with only the ratio of the correlation length to the system size appearing as an additional variable. These results cannot be directly transformed to analogous quantities for the Corbino disc, since under the transformation \(z = x + iy\) the local value of the correlation length will vary with position (due to the fact that the local lattice constant varies with position). To recover a suitable model for percolation on the Corbino disc (in which the local correlation length is uniform), one should study a cylindrical system in which the deviation of the local bonding probability \(p(r)\) from the critical value \(p_c\) varies as \(|p(r) - p_c|^\nu \propto \exp(2\pi y/N_x),\) where \(y\) is the distance of the center of the bond along the cylinder and \(\nu = 4/3\) is the correlation length exponent \([3]\).

**V. RELATION TO EXPERIMENTAL MEASUREMENTS**

Combining the results shown in Fig. 3 with Eq. \([3]\) leads to the probability distribution for the low-temperature dissipative conductance of a Corbino disc under the conditions for which our theory applies: for a sample containing homogeneous and isotropic density fluctuations with a correlation length that is small compared to the sample size, and within the “critical region” of the transition. Since the maximum value of \(M\) could occur outside of the critical region, these results place a lower bound on the maximum peak height. Work is under way to follow the evolution of the number of spanning clusters as the system sweeps through the percolation threshold \([12]\): this will allow the distribution of peak-shapes (and therefore maximum peak heights) to be determined.

Before we compare our results with experimental observations, we will discuss how they would be affected if the samples were to contain an additional uniform density gradient.

Consider a sample that contains a weak uniform gradient in density (and no random component). In this case, the evolution of \(M\) as a function of the average filling fraction \(\bar{\nu}\) is very simple. The evolution of the spatial distribution of the local conductivity is illustrated in Fig. 4.
FIG. 4. Schematic diagram of the evolution of the distribution of local conductivity for a Corbino disc with a uniform density gradient. During the transition between quantized Hall plateaus, the distribution passes through a configuration with $M = 1$.

For small $\bar{\nu}$ the Hall conductivity at all points in the sample is quantized at $\sigma_1$. As $\bar{\nu}$ increases, a $\sigma_2$-region appears on one side of the sample and the (straight) boundary separating the $\sigma_1$- and $\sigma_2$-regions sweeps across the sample. Over the range of filling fraction for which this boundary touches the inner contact, the two contacts are connected by one $\sigma_1$-cluster and one $\sigma_3$-cluster so $M = 1$. Once the boundary has passed the inner contact, the contacts are connected by a single $\sigma_2$-region and $M = 0$ again. Thus, during the plateau transition $M$ passes through a single peak with $M_{\text{max}} = 1$. This is true for any value of the aspect ratio of the Corbino disc.

For samples drawn from an ensemble characterized by both a uniform density gradient and homogeneous and isotropic inhomogeneities, the probability for observing $M_{\text{max}} = 1$ is increased above the value one would calculate in the absence of the uniform gradient. If the uniform gradient is sufficiently large, then the probability for $M_{\text{max}} = 1$ becomes 100%. To determine the threshold value of the uniform gradient above which the random fluctuations may be neglected, consider a sample of characteristic size $L$ containing a uniform density gradient of size $\nabla \sigma$, and isotropic and homogeneous fluctuations with amplitude $\delta \sigma$ and length scale $R_c = L$. The uniform gradient introduces a spatial variation in the deviation of the local percolation model from criticality. Within the transition region between quantum Hall plateaus, the local model will be critical along a straight line passing through the sample in a direction perpendicular to $\nabla \sigma$. However, at a distance $L$ from this line there will be a fractional deviation from criticality of order $L \nabla \sigma/\delta \sigma$, so the local value of the correlation length will be reduced to a size $\xi \simeq R_c (\delta \sigma/L \nabla \sigma)^{4/3}$. If this local correlation length is small compared to the sample size, then, on each side of the line at which the local model is critical, the conductivity distribution contains a single large cluster (on one side $\sigma_1$, on the other $\sigma_3$) and the form of the distribution is the same as in the absence of the random fluctuations. The condition under which the uniform density gradient dominates and the random fluctuations can be neglected is therefore $R_c (\delta \sigma/L \nabla \sigma)^{4/3} \ll L$. One may rewrite this condition in the form

$$\Delta \sigma \equiv L \nabla \sigma \gg \delta \sigma (R_c/L)^{3/4};$$

which shows that, for small $R_c/L$, the total change in the density across the sample caused by the uniform gradient $\Delta \sigma$ need only be large compared to a small multiple of the typical random density fluctuation $\delta \sigma$. Thus even a small uniform gradient in the density is sufficient to be the dominant density inhomogeneity and cause $M_{\text{max}} = 1$ for almost all samples in the ensemble.

We will now compare our results with the measurements on the two samples studied in Ref. [5]. It was shown in Ref. [5] that the low-temperature conductance peak heights for these two samples are consistent with Eq. (1) if both samples have $M_{\text{max}} = 1$ (this consistency holds to an accuracy of approximately 10%). Let us calculate the probability for this to occur if the samples were drawn at random from the ensemble of samples discussed in previous sections (characterized by homogeneous and isotropic density fluctuations with short-range correlations). The probability that $M_{\text{max}} = 1$ may be expressed as the probability that $M = 1$ in the critical region times the conditional probability that in this case a larger value of $M$ does not appear outside this region, plus the probability that $M = 0$ in the critical region times the conditional probability that in this case a value of $M = 1$ (but no larger) occurs outside of the critical region. The conditional probabilities must be less than or equal to unity, leading to the inequality that the probability for $M_{\text{max}} = 1$ in a Corbino disc with aspect ratio $A_0$ is less than or equal to $P_1(A_0) + P_0(A_0)$. The aspect ratios of the samples studied in Ref. [5] are 0.21 and 0.32, so the above inequalities together with the values for $P_M(A_0)$ (estimated by linear interpolation of the results shown in Fig. 1) lead to the conclusion that the probability for $M_{\text{max}} = 1$ in these two samples is less than or equal to 0.25 and 0.61, respectively. Thus the combined probability for both samples to have $M_{\text{max}} = 1$ is less than or equal to 0.15.

The availability of only two experimental data points prevents strong conclusions to be drawn from the comparison of our results with experiments. However, the reasonably small chance ($\leq 15\%$) for both samples to exhibit $M_{\text{max}} = 1$ if drawn at random from the ensemble of samples we have assumed suggests a discrepancy between these observations and our model. We suggest that this small probability may indicate the presence of long-range density inhomogeneities in these samples. As shown above, even a weak uniform density gradient is sufficient to significantly increase the probability of $M_{\text{max}} = 1$. Clearly further experiments are required to fully determine the statistical properties of the low-temperature non-universal conductance as a function of the aspect ratio of the Corbino disc, both to test the theory of Ref. [5] and to determine the form of the density inhomogeneities. To this end, it would be particularly interesting to perform transport measurements on Corbino disc samples with split outer electrodes (provided the fabrication of such a structure does not introduce significant additional density inhomogeneities). If the voltage
of each outer contact is held at the same value but the current through each is measured separately, then the theory of Ref. [6] predicts that, at low temperatures, the conductance of each contact will be \( M_i |\sigma_1 - \sigma_2| \), where \( M_i \) is a non-negative integer associated with the contact \( i \). For instance, if the samples studied in Ref. [5] for which \( M_i = \sum M_i = 1 \) had split outer contacts, this prediction means that \( M_i = 1 \) for only one outer contact, while \( M_i = 0 \) for all others. The statistical properties of the conductances \( M_i \) measured for a large number of samples would provide valuable information on the form of the density inhomogeneities.

VI. SUMMARY

An explanation for the observed “non-universal scaling” of the conductance (and conductivity) peak-heights in two-dimensional electron systems in the quantized Hall regime has recently been proposed [6]. Within this theory, the heights of the conductance peaks are related to the number of alternating percolation clusters for a percolation model defined on the spatially-varying local filling fraction in the sample. Motivated by this work, we have studied the statistical properties of the number of alternating percolation clusters for Corbino disc samples. We considered the samples to contain random density fluctuations that are isotropic and homogeneous and have a correlation length that is small compared to the sample size. By appealing to conformal invariance of critical percolation, we have argued that in the “critical region” of the quantum Hall transition the probability distribution of the number of alternating percolation clusters can be found from numerical calculations of the same quantity for a bond-percolation model defined on a cylinder [7]. We used this identification and the results of Ref. [6] to obtain a lower bound to the low-temperature conductance peak heights, and compared our results with recent experimental measurements [5]. The experimental observations clearly show non-universal scaling for two Corbino disc samples, with prefactors that are consistent with the theory of Ref. [6] if \( M_f = 1 \). We concluded that, within the assumptions we had made, the combined probability for \( M_f = 1 \) in these two samples is less than 15%. We suggested that this small probability could indicate a failure of our assumptions concerning the form of the density inhomogeneities in these samples, and that additional long-wavelength components could be present.

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