Particle quantum states with indefinite mass and neutrino oscillations

A. E. Lobanov

Department of Theoretical Physics, Faculty of Physics, Moscow State University, 119991 Moscow, Russia

In this work we develop a mathematical formalism which allows obtaining oscillation formula for neutrino of any energy. We demonstrate that in the ultra-relativistic limit the results obtained in this new approach agree with the previously used phenomenological theory which is only applicable to ultra-relativistic neutrinos. To this end we do the following. The Hilbert spaces of particle states are constructed in such a way that the neutrinos are combined in a multiplet with its components being considered as different quantum states of a single particle. The same is done for the charged leptons and the down- and up-type quarks. In the theory based on the Lagrangian of the fermion sector of the Standard Model modified in accordance with this construction, the phenomenon of neutrino oscillations arises as a direct consequence of the general principles of quantum field theory. Using the example of the pion decay, when the resulting neutrino has to be ultra-relativistic, it is shown that the neutrino states produced in the decay process can be described by a superposition of states with different masses and identical canonical momenta with very high accuracy.

I. INTRODUCTION

The Standard Model of electroweak interactions based on the non-Abelian gauge symmetry of the interactions, the generation of particle masses due to the spontaneous symmetry breaking mechanism, and the philosophy of mixing of particle generations is universally recognized. Its predictions obtained in the framework of perturbation theory are in a very good agreement with the experimental data. There is no serious reason, at least at the energies available at present, for its main propositions to be revised.

However, in describing such an important and firmly experimentally established phenomenon as neutrino oscillations, an essentially phenomenological theory based on the pioneer works by B. Pontecorvo and Z. Maki et al. is used. Its statements are given in various review articles and monographs (see, e.g., [12–20]). The primary assumption of this theory is that the neutrinos are massive and there are three neutrino types with different masses. It is also postulated that the neutrinos produced in reactions are in the states which are superpositions of states with fixed masses, the so-called mass states. These superpositions build the so-called flavor basis. The transformation to this basis from the mass basis is given by a unitary mixing matrix. Initially, the mass basis elements are described by plane waves with the same (three-dimensional) momentum. The time evolution of the flavor states is described by the solution of the corresponding Cauchy problem.

The conclusions of this theory are valid when the neutrino energy is large compared to their masses. The neutrino masses are extremely small. Due to this fact, the neutrino energies, which satisfy this condition, are not too high. Therefore, there is an obvious logical contradiction. On the one hand, neutrino oscillations are observed at the energies that are typical for the Standard Model. On the other hand, oscillations suggest the possibility of transformations of free fermions with equal electroweak quantum numbers into each other, contrary to the Standard Model.

In the Standard Model of electroweak interactions, mass generation is due to spontaneous symmetry breaking. Mass matrices are diagonalized after spontaneous symmetry breaking. This transformation leads to the appearance of a mixing matrix in the terms of the interaction Lagrangian associated with charged currents. The mixing in the Lagrangian of free fields is absent and the fields of fermions with different masses are quantized independently. As the consequence, all fermions in this model are fundamental particles uncorrelated with each other.

Numerous efforts based on quantum field theory have been made to justify the conclusions of the phenomenological model. It is worth mentioning the description of the neutrino states in the form of wave packets (see, e.g., [21, 22]). Another approach involves a description of the neutrino production and detection as a single process (see, e.g., [23–26]). A review of the models based on the quantum theory of fields is given in paper [27]. The results obtained partially explain the oscillation mechanism, however, these models are not complete in the mathematical sense.

All the contradictions of the theory could be solved, if we could combine the neutrinos in a multiplet with their components being considered as different quantum states of a single particle. This could be done for the charged...
leptons and the down- and up-type quarks as well. To this purpose it is necessary to construct the Fock spaces of particle flavor states.

Attempts to construct the Fock space for neutrino flavor states by a mathematically-consistent way have been repeated many times \[28\] \[34\]. However, either the commutation relations for the creation and annihilation were differed from the canonical type \[28\] \[30\], or the vacuum of the theory was explicitly dependent on time \[31\] \[53\]. Thus it was concluded that it is likely impossible to construct the Fock space for the flavor states using the conventional approach \[34\], i.e. it is not possible to construct a unitary transformation between the mass and the flavor states using the mixing matrix only \[53\]. For a brief historical summary of the problem see, e. g., \[36\]. Note that a possibility of constructing the unitary transformation between the mass and the flavor states using the mixing matrix only was taken for granted in the phenomenological theory.

Let us consider this issue in detail. In relativistic quantum field theory a particle is usually associated with an irreducible representation of the Poincaré group \[37\] \[38\]. The eigenvalue of the Casimir operator constructed from the translation operators squared is identified with the observed mass of the particle. Therefore the existence of states that are superpositions of one-particle states with different masses contradicts the relativistic invariance of the theory, if the canonical momentum operator is identified with the translation operator. This obviously follows from the fact that the metric is defined on the hyperboloid in the momentum space determined by the value of the particle mass.

To overcome this difficulty it seems natural to associate some set of particles (a multiplet) with an irreducible representation of a wider group. A number of theorems \[39\] \[41\] indicate that the only reasonable extension of the symmetry group of the theory is the direct product of the Poincaré group and a group of internal symmetry. This immediately limits the set of the models under consideration. However, if we assume that the masses of particles in a multiplet can be set by hands, as it is done in the Standard Model, this circumstance is not essentially important. It is essential that in this point we encounter the Jost theorem \[42\] \[44\], \[52\]. It seems that due to this theorem the result of extending the symmetry group of the theory will be trivial. Since the irreducible unitary representations of the direct product of groups are unitary equivalent to the tensor product of the representations of these groups with the same values of the invariants \[53\], the masses of the components of the multiplets will be equal. It is this circumstance that is the most important reason of the failure of constructing the Fock space for flavor states.

In our opinion, it is possible to circumvent this obstacle \[47\]. The basic idea of the proposed approach is as follows. The conclusion mentioned above is not correct if there is more than one multiplet in the theory, and these multiplets interact with each other. As a matter of fact, in the general case the representation spaces for two multiplets cannot be simultaneously reduced to the direct sum of representation spaces of the Poincaré group. Even if the representation of the internal symmetry group is considered to be finite, which seems quite natural for a finite set of particle masses in the multiplet, the representations for both multiplets would be unitary equivalent, but not identical. The values of the invariant associated with the translation operators squared will be the same for all the components of the multiplets (for different multiplets the values may vary). However, there is no need to treat them as the observed squared masses of the particles. The masses will be determined by an operator from the enveloping algebra for the direct sum of the Lie algebras of the Poincaré group and the internal symmetry group. Therefore, the representations which were described above, being unitary equivalent are not “physically equivalent”. Such consideration is conformed to the spirit of the Standard Model, where the masses of all particles are generated due to the phenomenon of spontaneous symmetry breaking and are proportional to the vacuum expectation value of the Higgs field.

The purpose of this work is to implement these ideas to construct the Fock space for the flavor states and to develop a modification of the Standard Model, which is suitable to describe the particle oscillations. In such a model it will be possible to describe the behavior of not only ultra-relativistic neutrinos, but also low-energy neutrinos, for example, relic ones. To do this, in Section 11 following the results of the article \[45\], we construct wave functions of multiplets of fermions with spin 1/2, which form the spaces of irreducible representations of the direct product of the Poincaré group and the group \(SU(3)\). First, we consider the most simple version of such a construction that is the tensor product of the Dirac representation of the Poincaré group and the fundamental representation of \(SU(3)\), which we designate as \(\mathcal{H}_{m,1/2}\). In this space the wave functions describe multiplets with components possessing equal masses \(m\).

Then, using the example of neutrino multiplet, we find a unitary intertwining operator mapping \(\mathcal{H}_{m,1/2}\) to a new space \(\mathcal{H}^{(\nu)}_{m,1/2}\). In this space the wave functions describe a multiplet with components possessing different masses that are the eigenvalues of the operator, which is defined by a mass matrix of the multiplet. These components describe the so called mass states of the field. The mass states generate a basis in \(\mathcal{H}^{(\nu)}_{m,1/2}\), but the choice of this basis is not unique. We find all the plane-wave bases, which can describe the mass states. For any choice of the mass basis, linear combinations of its elements are bases in \(\mathcal{H}^{(\nu)}_{m,1/2}\). Due to the unitary equivalence of representations in \(\mathcal{H}_{m,1/2}\) and in \(\mathcal{H}^{(\nu)}_{m,1/2}\), the eigenvalues of the Casimir operator constructed from the translation operators squared on the elements of \(\mathcal{H}^{(\nu)}_{m,1/2}\) are equal to \(m^2\). However, the parameter \(m\) is not the observed mass, it sets the scale of the multiplet
masses. As a matter of fact, the parameter \( m \) has the meaning of the vacuum expectation value of the Higgs field.

In Section III, using the results of Section II we introduce a Lagrangian generalizing the fermion sector of the Standard Model in such a way that field functions of the individual particles are replaced by field functions of the multiplets. In the model under consideration the action defined by the Lagrangian of free fields is invariant with respect to the direct product of the Poincaré group and \( SU(3) \). Using Noether’s theorem we find the integrals of motion for the fields and carry out the quantization procedure. We make sure that the creation and annihilation operators satisfy the canonical commutation relations. However, these operators possess an additional discrete quantum number that is associated with the mass of the state. The results of the quantization make it possible to treat both the mass states of the multiplet and their superpositions as the quantum states of a single particle and to construct the appropriate Fock space. Therefore, the transition probabilities can be calculated in the framework of perturbation theory. In the theory based on the Lagrangian of the fermion sector of the Standard Model modified in accordance with this construction, the phenomenon of particle oscillations arises.

Note again that in the Standard Model non-diagonal matrices of Yukawa couplings are introduced to describe the interaction of fermions with the Higgs field. This procedure violates the \( SU(3) \)-symmetry of the theory. In the framework of the Kobayashi–Maskawa formalism, the diagonalization of these matrices is necessary to generate masses of fermions. This transformation leads to a mixing matrix in the charge-current interaction Lagrangian. This procedure presupposes that all the fermions described by the model (i.e. the neutrinos, the charged leptons and the down- and up-type quarks) are fundamental particles. Therefore, although the mixing of generations occurs, the direct transitions between free particles with the same electroweak quantum numbers, i.e. oscillations, are impossible.

In the proposed model the mixing occurs at the level of perturbation theory. So there is no need to carry out the diagonalization procedure and to introduce the mixing matrix into the interaction Lagrangian explicitly. It is obvious that the transition probabilities, obtained in this way, will oscillate. However, at least the transition probabilities obtained in the tree approximation will completely coincide with the predictions of the conventional approach at the distances from the source that are small compared with the oscillation lengths.

In the following sections we apply the results to the description of neutrino oscillations. In Section IV, we check that the formulas describing the neutral oscillations are independent of the type of the mass states, i.e. of the choice of the plane-wave basis in \( H^{(i)}_{m,1/2} \). In Section V, we calculate the probability of the pion decay and check that the probability of the production of one or another flavor state essentially depends on the types of the mass states of which the flavor states are constructed. A single flavor state is produced only when it is composed of the mass states with the same canonical momentum. Thus, we come to the conclusion that the phenomenological theory of high energy neutrino oscillations based on the ideas of Pontecorvo is a very good approximation for quantum field theory.

II. SPACES OF WAVE FUNCTIONS

Let us assume that the neutrinos, the charged leptons, the down- and up-type quarks are elements of different multiplets. For each multiplet we consider a space \( H_{m,1/2} \) which is the direct sum of the Dirac representation spaces for the Poincaré group with fixed (positive) frequency and equal values of the Casimir operator formed from the canonical momentum operators squared. Since the experimental data indicate that there are three generations of particles,

\[
H_{m,1/2} = H_{m,1/2}^{(1)} \oplus H_{m,1/2}^{(2)} \oplus H_{m,1/2}^{(3)},
\]

where the subspaces \( H^{(i)}_{m,1/2} \) correspond to the components of the multiplet.

In the coordinate representation a basis of \( H_{m,1/2} \) can be defined as

\[
\Psi_{\mu,\zeta}(x) = \psi_{\mu,\zeta}(x)e^{i(px)}.
\]

Here \( \psi_{\mu,\zeta}(x) \) are plane waves

\[
\psi_{\mu,\zeta}(x) = \sqrt{2m_p}u_{\mu,\zeta}e^{-i(px)},
\]

where \( p^0 = \sqrt{p^2 + m^2} \), and the spinors \( u_{\mu,\zeta} \) satisfy the equation

\[
(\gamma^\mu p_\mu - m)u_{\mu,\zeta} = 0.
\]

This spinors are normalized by the condition

\[
\bar{u}_{\mu,\zeta}u_{\nu,\zeta'} = 2m_\delta_{\zeta,\zeta'}, \quad \bar{u}_{\mu,\zeta} = u_{\mu,\zeta}^\dagger \gamma^0.
\]
The explicit formulas for the spinors $u_{\mu,\zeta,l}(x) = \psi_{-p,\zeta}(x)e^{(l)}$, $\psi_{-p,\zeta}(x) = \frac{1}{\sqrt{2p^0}}u_{-p,\zeta}e^{i(px)}$, \(\gamma^\mu p_\mu + m u_{-p,\zeta} = 0, \quad \bar{u}_{-p,\zeta}u_{-p,\zeta} = -2m\delta_{\zeta,\zeta'}\).

The explicit formulas for the spinors $u_{\mu,\zeta}$, $\bar{u}_{\mu,\zeta}$, $u_{-\mu,\zeta}$, $\bar{u}_{-\mu,\zeta}$, see, e.g., in [46]. In what follows we will work in the spaces of wave functions with the positive frequency. The conversions for the spaces of wave functions with the negative frequency are similar.

Consider the vectors $e^{(l)}$ which, for definiteness, can be chosen in the form

$$
e^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

as a basis of a three-dimensional vector space over the field of complex numbers. Introduce a scalar product in $\mathcal{H}_{m,1/2}$

$$(\Psi,\Phi) = \sum_{s=1}^{3} \int d\mathbf{x} \psi^\dagger_s(x,t)\Phi_s(x,t),$$

with summation over the coordinates of the vectors $e^{(l)}$. Then one can consider $\mathcal{H}_{m,1/2}$ as the space of an irreducible unitary representation of the direct product of the Poincaré and $SU(3)$ groups. This space is constructed as the direct product of the Dirac representation of the Poincaré group and the fundamental representation of the group $SU(3)$.

The explicit form of the Lie algebra elements in the present case is obvious. We have the standard realization for the generators of the Poincaré group

$$P_\mu = i\partial_\mu \mathbb{I}, \quad M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)/4 \mathbb{I},$$

where $\mathbb{I}$ is the $3 \times 3$ identity matrix. The Hermitian generators $X_k$ of the fundamental representation of the $SU(3)$ group are defined by the Gell-Mann matrices

$$X_1 = \begin{pmatrix} e^{(1)} \otimes e^{(2)} \\ e^{(2)} \otimes e^{(1)} \end{pmatrix},$$

$$X_2 = i \begin{pmatrix} e^{(2)} \otimes e^{(1)} \\ e^{(1)} \otimes e^{(2)} \end{pmatrix},$$

$$X_3 = \begin{pmatrix} e^{(1)} \otimes e^{(1)} \\ e^{(2)} \otimes e^{(2)} \end{pmatrix},$$

$$X_4 = \begin{pmatrix} e^{(3)} \otimes e^{(3)} \\ e^{(3)} \otimes e^{(3)} \end{pmatrix},$$

$$X_5 = i \begin{pmatrix} e^{(3)} \otimes e^{(1)} \\ e^{(1)} \otimes e^{(3)} \end{pmatrix},$$

$$X_6 = \begin{pmatrix} e^{(3)} \otimes e^{(1)} \\ e^{(3)} \otimes e^{(2)} \end{pmatrix},$$

$$X_7 = i \begin{pmatrix} e^{(3)} \otimes e^{(2)} \\ e^{(2)} \otimes e^{(3)} \end{pmatrix},$$

$$X_8 = \begin{pmatrix} e^{(1)} \otimes e^{(1)} \\ e^{(2)} \otimes e^{(2)} \end{pmatrix} - 2 \begin{pmatrix} e^{(3)} \otimes e^{(3)} \end{pmatrix}.$$

The irreducibility condition for this representation is the matrix Dirac equation (see, e.g., [47])

$$\left(i\gamma_\mu \partial_\mu - m\mathbb{I}\right) \Psi(x) = 0,$$

Obviously, in this representation all the components of the multiplet have equal masses.

Let us turn to another representation of the extended symmetry group of the theory for each multiplet. The idea of the proposed transformation is based on the fact that the derivation algebra of the Poincaré algebra contains not only the operators $P_\mu, M^{\mu\nu}$, but also the generator of dilatation $D$ (see, e.g., [17])

$$[P_\mu, D] = P_\mu, \quad [M^{\mu\nu}, D] = 0.$$

Because of this, it is possible to construct an outer automorphism of the direct product of the Poincaré group and $SU(3)$. For the representation in the space $\mathcal{H}_{m,1/2}$ the automorphism leads to scaling transformations of the
coordinates. The transformations can vary for different subspaces. As a consequence, it will allow to consider multiplets with different masses of the components.

To be specific, we assume that we work with the neutrino multiplet. Introduce new basis vectors \( n^{(l)} \) with the help of a unitary matrix \( V^{(\nu)} \) acting on \( e^{(l)} \)

\[
n^{(l)}_s = \sum_{r=1}^{3} V^{(\nu)}_{sr} e^{(l)}_r.
\]

(2.13)

The new representation space \( \mathcal{H}^{(\nu)}_{m,1/2} \) will also be defined as the direct sum of the spaces of the Dirac representations of the Poincaré group

\[
\mathcal{H}^{(\nu)}_{m,1/2} = \mathcal{H}^{(\nu_1)}_{m,1/2} \oplus \mathcal{H}^{(\nu_2)}_{m,1/2} \oplus \mathcal{H}^{(\nu_3)}_{m,1/2}.
\]

(2.14)

We define a basis in \( \mathcal{H}^{(\nu)}_{m,1/2} \) in the form

\[
\Psi^{(\nu)}_{q,\zeta,\mu_1}(x) = \psi^{(\nu)}_{q,\zeta,\mu_1}(x) n^{(l)},
\]

(2.15)

where \( \psi^{(\nu)}_{q,\zeta,\mu_1}(x) \) are the plane waves derived from (2.3) by a dilatation of the coordinates

\[
\Psi^{(\nu)}_{q,\zeta,\mu_1}(x) = \frac{\mu_1^{3/2}}{\sqrt{2q^0}} u_{q,\zeta} e^{-i\mu_1(qx)}.
\]

(2.16)

Here \( q^0 = \sqrt{q^2 + m^2} \), and \( \mu_1 \) are positive real numbers. The spinors \( u_{q,\zeta} \) satisfy the equation

\[
(\gamma^\mu q_\mu - m) u_{q,\zeta} = 0,
\]

(2.17)

and are normalized by the condition

\[
\bar{u}_{q,\zeta} u_{q,\zeta'} = 2m \delta_{\zeta,\zeta'}.
\]

(2.18)

Evidently, \( \Psi^{(\nu)}(x) \) (the elements of the space \( \mathcal{H}^{(\nu)}_{m,1/2} \)) and \( \Psi(x) \) (the elements of the space \( \mathcal{H}_{m,1/2} \)) are connected by the unitary (with respect to the scalar product (2.8)) transformation

\[
\Psi^{(\nu)}(x) = \mathcal{K}^{(\nu)} \Psi(x) = \int K(x,y) \Psi(y) \, dy.
\]

(2.19)

Its kernel is determined by the formula

\[
K(x,y) = \frac{1}{(2\pi)^3} \sum_{l=1}^{3} \sum_{\zeta = \pm 1} \int dq \Psi^{(\nu)}_{q,\zeta,\mu_1}(x) \otimes \Psi_{q,\zeta,\mu_1}^{(\nu)}(y).
\]

(2.20)

The explicit form of the dilatation operator is

\[
D = x_\alpha \partial^\alpha + 3/2.
\]

(2.21)

Therefore, the intertwining operator for the representations in the spaces \( \mathcal{H}^{(\nu)}_{m,1/2} \) and \( \mathcal{H}_{m,1/2} \) realizing the transformation (2.19) is

\[
\mathcal{K}^{(\nu)} = \sum_{l=1}^{3} D^{(l)} \left( n^{(l)} \otimes e^{(l)} \right).
\]

(2.22)

where

\[
D^{(l)} = \exp \left( \ln \mu_l (x_\alpha \partial^\alpha + 3/2) \right).
\]

(2.23)

A direct calculation yields the explicit form of the elements of the Lie algebra of the considered representation. The action of the generators \( X^{(\nu)}_k \) is reduced to obvious permutations of its elements

\[
X^{(\nu)}_1 \Psi^{(\nu)}_{q,\zeta,\mu_1}(x) = \Psi^{(\nu)}_{q,\zeta,\mu_2}(x), \quad X^{(\nu)}_1 \Psi^{(\nu)}_{q,\zeta,\mu_2}(x) = \Psi^{(\nu)}_{q,\zeta,\mu_1}(x), \quad X^{(\nu)}_1 \Psi^{(\nu)}_{q,\zeta,\mu_3}(x) = 0,
\]

(2.24)
and so on. The diagonal operators $X_3^{(\nu)}$, $X_8^{(\nu)}$ are determined by numerical matrices. The explicit form of these operators see in A.

The generators of the Lorentz group do not change the form

$$M^{(\nu)}_{\mu\nu} = \mathcal{K}^{(\nu)} M_{\mu\nu} \mathcal{K}^{(\nu)-1} = i \left( (x_\mu \partial_\nu - x_\nu \partial_\mu) + (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) / 4 \right) I,$$  

(2.25)

since the dilatation operator commute with the generators of rotations and boosts (see (2.12)). This fact is quite natural, because the results of observations, in particular, of the study of particle oscillations cannot depend on the choice of the inertial reference frame used for measurement.

For the translation generators, we have

$$P^{(\nu)}_\mu = \mathcal{K}^{(\nu)} P_\mu \mathcal{K}^{(\nu)-1} = i \partial_\mu \mathcal{R}^{(\nu)},$$  

(2.26)

where

$$\mathcal{R}^{(\nu)} = \sum_{l=1}^{3} \frac{1}{\mu_l} \left( n^{(l)} \otimes \hat{n}^{(l)} \right).$$  

(2.27)

That is $\mu_l^{-1}$ are the eigenvalues of the matrix $\mathcal{N}^{(\nu)}$, and $n^{(l)}$ are its eigenvectors normalized by the conditions (the asterisk denotes the complex conjugation)

$$\sum_{s=1}^{3} n^{(l) \ast}_{s} \hat{n}^{(k)} = \delta_{kl}, \quad \sum_{l=1}^{3} n^{(l) \ast}_{s} \hat{n}^{(l)} = \delta_{sr}. \tag{2.28}$$

The operators

$$P_{(l)}^{(\nu)} = n^{(l)} \otimes \hat{n}^{(l)}$$

(2.29)

are orthogonal projectors

$$P_{(l)}^{(\nu)} P_{(k)}^{(\nu)} = \delta_{kl} P_{(l)}^{(\nu)}, \quad \sum_{l=1,2,3} P_{(l)}^{(\nu)} = I.$$  

(2.30)

The Dirac equation, i.e. the irreducibility condition for this representation, is now written as

$$\left( i \gamma^\mu \partial_\mu \mathcal{N}^{(\nu)} - m I \right) \Psi^{(\nu)}(x) = 0.$$  

(2.31)

Since the parameters $\mu_l$ are chosen nonzero then $\mathcal{N}^{(\nu)}$ is non-degenerate and the inverse matrix exists

$$(\mathcal{N}^{(\nu)})^{-1} = \sum_{l=1}^{3} \mu_l P_{(l)}^{(\nu)}.$$  

(2.32)

If we multiply (2.31) by (2.32) then the Dirac equation takes a more familiar form

$$\left( i \gamma^\mu \partial_\mu - \mathcal{M}^{(\nu)} \right) \Psi^{(\nu)}(x) = 0,$$  

(2.33)

where

$$\mathcal{M}^{(\nu)} = m (\mathcal{N}^{(\nu)})^{-1} = \sum_{l=1}^{3} m_l P_{(l)}^{(\nu)}, \quad m_l = \mu_l m.$$  

(2.34)

The sets of solutions of equations (2.31) and (2.33) coincide. Further, we will refer only to equation (2.33).

We emphasize once again that the eigenvalues of the Casimir operators constructed from the generators (2.25), (2.26), and (A1) take the same values on the solutions of these equations and on the elements of the initial representation space $\mathcal{H}_{m,1/2}$. In particular, $P^{(\nu)\mu} P^{(\nu)} \Psi^{(\nu)}(x) = m^2 \Psi^{(\nu)}(x)$, but now the parameter $m$ is not the observed mass, it rather sets the scale of the multiplet masses. In other words, it represents the bare mass of the multiplet.
The observed masses are determined by the action of the canonical momentum operator squared on the functions \( \Psi(q, \zeta, m_l) \) and are equal to \( m_l = \mu m \) for the \( l \)-th component of the multiplet.

Let us discuss this point in detail. The general solution of (2.33) can be expanded in the basis functions that are the eigenfunctions of a complete set of operators commuting with the operator of the equation. Naturally, the choice of the complete set is ambiguous. For the standard Dirac equation, if we restrict ourselves to the case of plane-wave solutions, the complete set includes the space components of the momentum (i. e. the generators of the space translations) and the operator of the spin projection. This operator is constructed in the common way from the components of the Pauli–Lubanski–Bargmann vector. The ambiguity of the complete set composition in that case depends on the choice of the spin operator. In contrast to the standard Dirac equation, the complete set for Eq. (2.33) contains five operators. This circumstance makes the situation more complicated.

We now find the complete sets of operators for which the basis of the solution space of Eq. (2.33) consists of plane waves. Let \( \Lambda^{(l)} \) be a non-degenerate Hermitian matrix which commutes with \( M^{(l)} \). This matrix can be written as

\[
\Lambda^{(l)} = \sum_{l=1}^{3} \lambda_l \mathbf{p}^{(l)} , \quad \lambda_l \neq 0.
\]  

(2.35)

If we assume that the parameters \( \lambda_l \) are pairwise distinct, then, in the space of the solutions of Eq. (2.33), this matrix defines an action of the operator \( \mathcal{M}^{(l)} \), which is a linear combination of operators \( X_3^{(l)} \), \( X_8^{(l)} \), and the Casimir operators of \( SU(3) \). Since the representation is irreducible, then the action of the Casimir operators is given by the matrices that are multiple of the identity matrix. Obviously, \( \mathcal{M}^{(l)} \) can be chosen as one of the operators of the complete set. This operator is used to isolate the orthogonal subspaces determined by the projectors \( \mathbf{P}^{(l)} \). Therefore, the numerical values of the parameters \( \lambda_l \) can be arbitrary. It is natural to set \( \lambda_l = m_l \), that is, to assume that the action of the operator \( \mathcal{M}^{(l)} \) is defined by the matrix \( \mathbf{M}^{(l)} \).

For a basis to be a plane-wave one, the complete set must include three operators with continuous spectrum. If we choose the space translation generators, that is, the spatial components of the 4-vector \( \mathbf{p}^{(l)} \), as such operators, then any standard spin projector multiplied by \( \mathbf{N}^{(l)} \) will determine the spin projection. In this case the complete orthonormal system of solutions of Eq. (2.33) corresponding to a fixed (positive) frequency coincides with the basis \( \Psi^{(l)}_{q, \zeta, m_l}(x) \) (see Eq. (2.15)), which was considered previously.

However, we can take the spatial components of the 4-vector

\[
\Lambda^{(l)} \mathbf{p}^{(l)} = i \partial_\mu \Lambda^{(l)} \mathbf{N}^{(l)}
\]  

(2.36)

to play the role of the operators with continuous spectrum, where \( \Lambda^{(l)} \) is any matrix which satisfies (2.35). In particular, we can set \( \Lambda^{(l)} = (\mathbf{N}^{(l)})^{-1} \). Then the basis functions are the eigenfunctions of the spatial components of the canonical momentum operator \( i \partial_\mu \), and

\[
\psi^{(l)}_{p, \zeta, m_l}(x) = \psi^{(l)}_{p, \zeta, m_l}(x) n^{(l)},
\]  

(2.37)

where \( \psi^{(l)}_{p, \zeta, m_l}(x) \) are the plane waves describing particles with masses \( m_l \)

\[
\psi^{(l)}_{p, \zeta, m_l}(x) = \frac{1}{\sqrt{2p_l^0}} u_{p, \zeta}^{(m_l)} e^{-i(px)}.
\]  

(2.38)

Here \( p_l^0 = \sqrt{p^2 + m_l^2} \), the spinors \( u_{p, \zeta}^{(m_l)} \) satisfy the equation

\[
(\gamma^\mu p_\mu - m_l)u_{p, \zeta}^{(m)} = 0,
\]  

(2.39)

and are normalized by the condition

\[
\bar{u}_{p, \zeta}^{(m)} u_{p, \zeta'}^{(m)} = 2m_l \delta_{\zeta, \zeta'}.
\]  

(2.40)

Thus, the matrix \( \mathbf{M}^{(l)} \) can be interpreted as the mass matrix of the neutrino multiplet, and the parameters \( m_l = \mu m \) are the observed masses of the particles. The states that are described by the eigenfunctions of \( \mathcal{M}^{(l)} \), can be naturally called the mass states for any choice of the operators with continuous spectrum, that is, for any choice of the matrix \( \Lambda^{(l)} \) including the case \( \Lambda^{(l)} = I \).
However, we can choose a basis in the space $\mathcal{H}^{(\nu)}_{m,1/2}$ in the form of a superposition of the mass states. Consider an arbitrary unitary matrix $U$ with components $U_{\alpha l}$. The functions

$$
\Psi^{(\nu)}_{p,\zeta,\alpha}(x) = \sum_{l=1}^{3} U_{\alpha l} \Psi^{(\nu)}_{p,\zeta,l}(x),
$$

(2.41)

where $\Psi^{(\nu)}_{p,\zeta,l}(x)$ is the wave function of an arbitrary mass state, make up a complete orthonormal set in $\mathcal{H}^{(\nu)}_{m,1/2}$ with the scalar product (2.8). The basis (2.41) can be obtained as a result of the unitary transformation $\mathcal{U}$ of the space $\mathcal{H}^{(\nu)}_{m,1/2}$ onto itself

$$
\Psi^{(\nu)}_{p,\zeta,\alpha}(x) = \mathcal{U} \Psi^{(\nu)}_{p,\zeta,l}(x) = \int \tilde{K}(x, y) \Psi^{(\nu)}_{p,\zeta,l}(y) dy.
$$

(2.42)

The kernel of the transformation is determined by the formula

$$
\tilde{K}(x, y) = \frac{1}{(2\pi)^{3}} \sum_{\alpha=1}^{3} \sum_{k=1}^{3} \sum_{\zeta=\pm 1}^{3} \int d\mathbf{p} \delta_{\alpha k} \Psi^{(\nu)}_{p,\zeta,\alpha}(x) \otimes \Psi^{(\nu)*}_{p,\zeta,k}(y).
$$

(2.43)

The elements of the basis (2.41) are no longer eigenfunctions of the operator $\mathcal{M}^{(\nu)}$ determined by the mass matrix. The fifth operator of the complete set (denote it as $\mathcal{F}^{(\nu)}$) is defined now as

$$
\mathcal{F}^{(\nu)} = \mathcal{U} \mathcal{M}^{(\nu)} \mathcal{U}^{-1}.
$$

(2.44)

This operator can explicitly depend on the event space coordinates. It should be emphasized that the form of the causal Green function for Eq. (2.33) does not depend on the chosen basis

$$
S^{(\nu)}_{\gamma}(x) = \frac{1}{(2\pi)^{3}} \sum_{l=1}^{3} \mathbf{V}^{(\nu)}_{rl} \int \frac{(\gamma_i p^i + m_\gamma) e^{-i(px)}}{m_\gamma^2 - p^2 - i\epsilon} d^4 p.
$$

(2.45)

Similarly, we can construct the representation spaces for the multiplets of the charged leptons and the down- and up-type quarks, where the values $\mu_i$, and the matrices $V$, $\mathbf{N}$, $\mathbf{M}$, $\mathbf{L}$ for these multiplets may be different. Since we apply this approach to the problem of neutrino oscillations as an example, we introduce special notations for the multiplet of the charged leptons. The bare mass is denoted by $M$, without assuming in advance that $M = m$. According to the tradition in the theory of oscillations we denote the subspaces of the mass states by Greek letters. We will write the dilatation parameters as $\eta_\beta$ instead of $\mu_i$. Accordingly, the physical masses of the charged leptons will be $M_\beta = \eta_\beta M$, $\beta = e, \mu, \tau$. Applying a unitary matrix $\mathbf{V}^{(\nu)}$ to $\psi^{(l)}$, we obtain new basis vectors $a^{(\beta)}$

$$
a^{(\beta)}_s = \sum_{r=1}^{3} V^{(\nu)\gamma}_{sr} \psi^{(l)}_s, \quad \sum_{s=1}^{3} a^{(\alpha)}_s \eta^{(\beta)}_s = \delta_{\alpha \beta}, \quad \sum_{\beta=e,\mu,\tau} a^{(\beta)}_s \eta^{(\beta)}_s = \delta_{sr}.
$$

(2.46)

Therefore, the orthogonal projectors for the representation space of the charged leptons take the form

$$
\mathcal{P}^{(\beta)} = a^{(\beta)} \otimes a^{(\beta)*}.
$$

(2.47)

Hence,

$$
\mathbf{N}^{(\nu)} = \sum_{\beta=e,\mu,\tau} \frac{1}{\eta_\beta} \mathcal{P}^{(\beta)}, \quad \mathbf{M}^{(\nu)} = \sum_{\beta=e,\mu,\tau} M_\beta \mathcal{P}^{(\beta)}.
$$

(2.48)

We determine the basis elements of $\mathcal{H}^{(\nu)}_{m,1/2}$, corresponding to the basis (2.13), as

$$
\psi^{(\nu)}_{q,\zeta,\eta_\beta}(x) = \psi^{(\nu)}_{q,\zeta,\eta_\beta}(x) a^{(\beta)},
$$

(2.49)

where

$$
\psi^{(\nu)}_{q,\zeta,\eta_\beta}(x) = \frac{\eta_\beta^{3/2}}{\sqrt{2q^3}} u_{q,\xi} e^{-i\eta_\beta(qx)}.
$$

(2.50)
We determine the basis elements of $H^{(e)}_{m,1/2}$, corresponding to the basis $p_\pm, \zeta, M_\beta$, as

$$\Psi^{(e)}_{p_\pm, \zeta, M_\beta}(x) = \psi^{(e)}_{p_\pm, \zeta, M_\beta}(x) a^{(\beta)},$$

(2.51)

where $\psi^{(e)}_{p_\pm, \zeta, M_\beta}(x)$ are the plane waves describing particles with mass $M_\beta$

$$\psi^{(e)}_{p_\pm, \zeta, M_\beta}(x) = \frac{1}{\sqrt{2p_\pm^0}} u^{(M_\beta)}_{p_\pm, \zeta} e^{-i(px)}.$$

(2.52)

We do not introduce special notations for the quark multiplets, but use the appropriate indices only.

### III. MODIFIED MODEL OF ELECTROWEAK INTERACTIONS

Now we are able to write the Lagrangian of the modified theory describing the electroweak interactions. If, as usual, the interaction is carried by the gauge fields associated with the group $SU(2) \times U(1)$, then the Lagrangian of such a theory is the Lagrangian of the minimally extended Standard Model with changes in the fermion sector only.

The Lagrangian for the physical fermion fields in our model is written as follows:

$$\mathcal{L}_f = \mathcal{L}_0 + \mathcal{L}_{\text{int}},$$

(3.1)

where

$$\mathcal{L}_0 = \frac{i}{2} \sum_{i=\nu, e, u, d} \left[ \left( \Psi^{(i)} \gamma^\mu (\partial_\mu \Psi^{(i)}) \right) - (\partial_\mu \bar{\Psi}^{(i)} \gamma^\mu \Psi^{(i)}) \right] - \sum_{i=\nu, e, u, d} \bar{\Psi}^{(i)} M^{(i)} \Psi^{(i)}$$

(3.2)

is the Lagrangian of free fields and

$$\mathcal{L}_{\text{int}} = - \sum_{i=\nu, e, u, d} \bar{\Psi}^{(i)} M^{(i)} (H/v) \Psi^{(i)}$$

$$- \frac{g}{2\sqrt{2}} \left( \bar{\Psi}^{(e)} \gamma^\mu (1 + \gamma^5) \Psi^{(e)} W^-_\mu + \bar{\Psi}^{(u)} \gamma^\mu (1 + \gamma^5) \Psi^{(u)} W^+_\mu \right)$$

$$- \frac{g}{2\sqrt{2}} \left( \bar{\Psi}^{(d)} \gamma^\mu (1 + \gamma^5) \Psi^{(d)} W^-_\mu + \bar{\Psi}^{(u)} \gamma^\mu (1 + \gamma^5) \Psi^{(d)} W^+_\mu \right)$$

$$- \frac{g}{2\cos\theta_W} \sum_{i=\nu, e, u, d} \bar{\Psi}^{(i)} \gamma^\mu \left( T^{(i)} - 2Q^{(i)} \sin^2\theta_W + T^{(i)} \gamma^5 \right) \Psi^{(i)} Z_\mu$$

(3.3)

is the Lagrangian of the interaction between the fermion fields, the vector boson fields $W^\pm_\mu, Z_\mu, A_\mu$, and the Higgs field $H$. Here $\theta_W$ is the Weinberg angle, $e = g \sin\theta_W$ is the positron electric charge, $T^{(i)}$ is the weak isospin projection ($T^{(\nu)} = T^{(u)} = 1/2, T^{(e)} = T^{(d)} = -1/2$), $Q^{(i)}$ is the electric charge of the multiplet in the units of $e$, and $v$ is the vacuum expectation value of the Higgs field. Thus, (3.3) formally coincides with the Lagrangian of the Standard Model, supplemented with the singlets of the right-handed neutrinos (see, e.g., [9]). However, the field functions $\Psi^{(i)}$ describe not the individual particles, but the multiplet as a whole. So it is not necessary to introduce the mixing matrices into $\mathcal{L}_{\text{int}}$ explicitly.

The action defined by the Lagrangian of free fields (3.2) is explicitly invariant with respect to $SU(3)$ transformations generated by $X_k^{(u)}, X_k^{(e)}, X_k^{(u)}$, and $X_k^{(d)}$ (see (2.24)). Therefore, when quantizing the model, the multiplet can be considered as a single particle. The one-particle states in the Fock space are defined as usual, the creation and annihilation operators satisfy the canonical commutation relations. However, these operators carry an additional discrete quantum number that is associated with the mass of the state. The multiplet can be either in one of the three mass states, or in a pure quantum state that is a superposition of the states with fixed masses. In a certain sense we may say that there are only four fundamental fermions in our model.

We will discuss all this in detail, using the approach described in [49]. Consider the basis (2.15). We can write the
Accordingly, the angular momentum tensor is the same as in [49].

The current vector associated with the global gauge symmetry of the Lagrangian is defined by the relation

\[ J_\alpha = \sum_{s=1}^{3} \bar{\Psi}_s^{(\nu)}(x) \gamma_{\alpha} \Psi_s^{(\nu)}(x). \]  

It ensures the conservation of the total field charge

\[ Q = \sum_{l=1}^{3} \sum_{\zeta=\pm 1} \int dq [\bar{a}_{l,\zeta}^+(q)a_{l,\zeta}^-(q) + \bar{a}_{l,\zeta}^- a_{l,\zeta}^+(q)]. \]  

The tensor associated with the SU(3)-symmetry of the fields is defined by the relation

\[ S_{\kappa\alpha} = \frac{1}{2} \sum_{s=1}^{3} \bar{\Psi}_s^{(\nu)}(x) \gamma_{\alpha} (X_k^{(\nu)} \Psi_s^{(\nu)}(x))_s + (X_k^{(\nu)*} \bar{\Psi}_s^{(\nu)}(x))_s \gamma_{\alpha} \Psi_s^{(\nu)}(x), \quad k = 1...8. \]  

It ensures the existence of eight integrals of motion (see [A1] – [A11]). Using these integrals of motion and the total field charge it is possible to construct nine linear combinations

\[ \mathcal{X}_{lk} = \sum_{\zeta=\pm 1} \int dq [\bar{a}_{l,\zeta}^+(q)a_{k,\zeta}^-(q) + \bar{a}_{l,\zeta}^- a_{k,\zeta}^+(q)], \quad k, l = 1, 2, 3, \]  

and three of them are diagonal in the indices \( l, k \).

We do not consider the angular momentum tensor, because its analysis is not critical for what follows. Since the Lorentz group generators have the standard form, then the angular momentum tensor is the same as in [49].

The integrals of motion for the fields are associated with the generators of the Lie algebra of the symmetry group of the theory as follows:

\[ Z \Psi^{(\nu)}(x) = [\Psi^{(\nu)}(x), Z]. \]  

Here \( Z = \{ P^{(\nu)}_\beta, Q, X^{(\nu)}_k \ldots \} \) and \( Z = \{ P_\beta, Q, \mathcal{X}_k \ldots \} \). These relations enable one to interpret \( \bar{a}_{l,\zeta}^+(q) \) and \( a_{l,\zeta}^-(q) \) as the operators of neutrino creation and annihilation in the state \( l \) with the kinetic momentum \( q \) and the polarization \( \zeta \). Accordingly, \( a_{l,\zeta}^+(q) \) and \( \bar{a}_{l,\zeta}^-(q) \) are the operators of antineutrino creation and annihilation in the state \( l \) with the
kinetic momentum $q$ and the polarization $\zeta$. Eq. (3.11) and the invariance condition with respect to the change of particles to antiparticles for $P_\beta$ yield the canonical commutation relations

\[
\begin{align*}
\left[a_{\alpha,\zeta}^+(q), a_{\beta,\zeta'}^-(q')\right]_+ &= \delta_{\beta\zeta} \delta_{\zeta'\zeta} \delta(q - q'), \\
\left[a_{\alpha,\zeta}^-(q), a_{\beta,\zeta'}^+(q')\right]_+ &= \delta_{\beta\zeta} \delta_{\zeta'\zeta} \delta(q - q').
\end{align*}
\] (3.12)

Let us now consider linear combinations of these operators

\[
\tilde{a}_{\alpha,\zeta}^\pm(q) = \sum_{l=1}^{3} U_{al} a_{l,\zeta}^\pm(q), \quad \tilde{\alpha}_{\alpha,\zeta}^\pm(q) = \sum_{l=1}^{3} U_{al}^\ast \tilde{a}_{l,\zeta}^\pm(q),
\] (3.13)

where $U_{al}$ are components of an arbitrary unitary matrix $U$. The commutation relations for $a_{\alpha,\zeta}^\pm(q), \tilde{a}_{\alpha,\zeta}^\pm(q)$ are canonical. When expressed in terms of these operators, only $P_\beta$ and $Q$ are diagonal. However, using $Q$ and the remaining integrals of motion we can always construct three linear combinations of the form

\[
\lambda'_{\alpha} = \int dq \left[ \tilde{a}_{\alpha,\zeta}^+(q) a_{\alpha,\zeta}^-(q) + \tilde{a}_{\alpha,\zeta}^-(q) a_{\alpha,\zeta}^+(q) \right], \quad \alpha = 1, 2, 3.
\] (3.14)

Therefore, the operators $\tilde{a}_{\alpha,\zeta}^\pm(q), \tilde{\alpha}_{\alpha,\zeta}^\pm(q)$ lead to well-defined states in the Fock space. The difference between the number of particles and the number of antiparticles of each type $\alpha$ with the same kinetic momentum $q$ is an integral of motion.

Consider now the basis (2.37). We can write the components of the field functions as

\[
\Psi^{(\nu)}(x) = \frac{1}{(2\pi)^{3/2}} \sum_{l=1}^{3} \sum_{\zeta=\pm1} \int \frac{dp}{\sqrt{|2\mu|}} n_s(l) e^{-i(px)} a_{\alpha,\zeta}^-(p)
\]
\[
\tilde{\Psi}^{(\nu)}(x) = \frac{1}{(2\pi)^{3/2}} \sum_{l=1}^{3} \sum_{\zeta=\pm1} \int \frac{dp}{\sqrt{|2\mu|}} \tilde{n}_s(l) e^{-i(px)} \tilde{a}_{\alpha,\zeta}^+(p)
\]

The operators $a_{l,\zeta}^\pm(p), \tilde{a}_{l,\zeta}^\pm(p)$ arise as a result of the scaling transformation

\[
a_{l,\zeta}^\pm(p) = (\mu_l)^{-3/2} a_{l,\zeta}^\pm(q_l), \quad \tilde{a}_{l,\zeta}^\pm(p) = (\mu_l)^{-3/2} \tilde{a}_{l,\zeta}^\pm(q_l), \quad q_l = p/\mu_l.
\] (3.16)

Therefore, using Eq. (3.11) and the invariance condition with respect to the change of particles to antiparticles for $P_\beta$, we get that these operators as well as their linear combinations

\[
a_{\alpha,\zeta}^\pm(p) = \sum_{l=1}^{3} U_{al} a_{l,\zeta}^\pm(p), \quad \tilde{a}_{\alpha,\zeta}^\pm(p) = \sum_{l=1}^{3} U_{al}^\ast \tilde{a}_{l,\zeta}^\pm(p)
\] (3.17)

satisfy the canonical commutation relations (3.12).

A similar reasoning shows that the states, which are described by the operators $a_{l,\zeta}^\pm(p), \tilde{a}_{l,\zeta}^\pm(p)$ and $a_{\alpha,\zeta}^\pm(p), \tilde{a}_{\alpha,\zeta}^\pm(p)$ are well-defined in the Fock space, too. However, there is an important difference. The integral of motion $P_\beta$ is non-diagonal in terms of the operators $a_{\alpha,\zeta}^\pm(p), \tilde{a}_{\alpha,\zeta}^\pm(p)$. This situation is quite expected. The integral of motion $P_\beta$ is not the canonical momentum of the field, but the “kinetic momentum”. Since the superpositions of the mass states are non-stationary states, the non-diagonal form of $P_\beta$ reflects the fact of a possible energy transfer from one state to another.

If we admit that the matrices $V^{(i)}$ are equal, we come, at least in the framework of the perturbation theory, to a model with three independent generations of fermions. If the matrices $V^{(i)}$ are distinct, the terms of the Lagrangian that describe the interaction via the charged currents immediately cause a phenomenon, which is known as the mixing of generations. The matrix of the mixing coefficients for quarks is an analog of the Cabibbo–Kobayashi–Maskawa (CKM) matrix

\[
U^{\text{CKM}} = V^{(u)} V^{(d)}.
\] (3.18)
and for leptons it is an analog of the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix
\[ U_{PMNS} = V^{(c)} V^{(\nu)}. \]
(3.19)

The elements of these matrices can be expressed by the scalar products of the basis vectors. For example,
\[ \sum_{s=1}^{3} n^{(l)}_s \delta_s^{(\alpha)} = U_{PMNS}. \]
(3.20)

We shall use the notation \( U_{PMNS} \equiv P_{al} \) for the PMNS matrix elements to omit the lengthy indices.

In the presence of the mixing, the transition probabilities for the superpositions of the mass states cannot be reduced to the sum of the transition probabilities for the mass states. It should be emphasized that the total transition probabilities for these states can be different for the same unitary matrix \( U \), but various complete sets of the operators defining the mass state. The reason is that the operators \( 2.41 \) for such states can be different. Moreover, the measurement results can be different for different space-time localization points of the detector. In experiment, as it is well known, the generation mixing occurs for quarks, while the transfer of energy from one neutrino state to another is seen as the oscillation phenomenon.

IV. NEUTRINO OSCILLATIONS

Now we try to apply the developed formalism to the problem of neutrino oscillations. Eq. (2.44) is obviously translation invariant. Therefore, shifting the argument \( x^\mu \) in its solution by a constant 4-vector \( z^\mu \), we still obtain a solution. For the solutions describing the mass states, this transformation leads to a trivial phase multiplier. However, the form of the solutions that are determined by formula (2.44) will change. That is, an argument shift generates a unitary transformation in the solution space.

Let us examine this issue in detail. For clarity, we will describe the neutrino with the help of the density matrices of pure states. By definition, the density matrix of a pure state is \( \rho(x, y) = \Psi(x) \bar{\Psi}(y) \). We assume that the neutrino source and the detector are at a distance \( L \). In the area where a neutrino is produced, it is described by the density matrix \( \rho^{(v)}(x; y; q, \zeta, \alpha) \), and in the detection area it is described by the density matrix \( \rho^{(v)}(x - z, y - z; q, \zeta, \alpha) \). A mass state can be described, for example, by a density matrix
\[ \rho^{(v)}(x, y; q, \zeta, \mu) = \frac{1}{4q^0} e^{-i(q_z)(x - y)} \mu_5^\beta \left( n^{(l)} \otimes n^{(l)}(\gamma_\mu q^\mu + m)(1 - \zeta^5 \gamma_\mu S_0^\mu(q)), \right) \]
(4.1)
which is based on the solutions \( 2.15 \), or by a density matrix
\[ \rho^{(v)}(x, y; p, \zeta, m_l) = \frac{1}{4p^0} e^{-i(p_z)(x - y)} \left( n^{(l)} \otimes n^{(l)}(\gamma_\mu p_l^\mu + m_l)(1 - \zeta^5 \gamma_\mu S_0^\mu(p_l)), \right) \]
(4.2)
which is based on the solutions \( 2.37 \). In both cases there is no dependence on \( L \).

The situation changes in the case of superpositions of the mass states. If we take a state described by a superposition of the wave functions \( 2.15 \), then the group velocities of all the mass states of the neutrino will be the same: \( v = q^0 \). So we can suppose \( z^\mu = q^\mu L/|q| \). Therefore, the density matrix describing the neutrino state at the distance \( L \) from the source can be written as follows:
\[ \rho^{(v)}(x, y; q, \zeta, \mu_1, \mu_k; \alpha, L) = \frac{1}{4q^0} \sum_{k, l=1}^{3} e^{-i(qz)(\mu_1 + i(qy)\mu_k + 2izL/L^{(k)})} \times (\mu_1 \mu_k)^{3/2} \left( n^{(l)} \otimes n^{(l)}(\gamma_\mu q^\mu + m)(1 - \zeta^5 \gamma_\mu S_0^\mu(q)), \right) \]
(4.3)

Here we introduced the notation
\[ L^{(k)} = \frac{2\pi|q|}{m(m_1 - m_k)} = \frac{2\pi\beta}{(m_1 - m_k)\sqrt{1 - \beta^2}}, \quad \beta = |v|. \]
(4.4)

Recall that in formula \( 4.3 \), as well as in formulas \( 4.1 \), \( 4.2 \), the 4-vector \( q^\mu \) satisfies the condition \( q^2 = m^2 \). 4-vector \( S_0^\mu(q) \) determines the direction of the polarization of the particles, \( \zeta = \pm 1 \) is the sign of the spin projection on this
direction. If we consider the density matrix \( \rho^{(\nu)}(x, y; q, \mu_l) = \frac{1}{12q^0} \sum_{i=1}^{3} e^{-i(q(x-y))\mu_l^i} \left( n^{(l)} \otimes \bar{n}^{(l)} \right) (\gamma_{\mu} q^n + m) \). (4.5)

It is clear that the dependence of the density matrix on the distance from the source appears only when the source is coherent. This requires the dimensions of the source to be small compared with the parameters \( L^{(lk)} \).

If we take a state described by a superposition of wave functions \( \psi_l = p_l^0 p_l^0 \), then the group velocities of the mass states differ from each other, so for the state \( (4.3) \) the decoherence effect is absent and the state \( (4.7) \) spreads due to different group velocities. Due to the fact that the interaction conserves the energy, but not the group velocity, we can get the probability of the neutrino transition from one state to another. These states can be calculated if one uses the explicit form of the spinors \( \psi_{\alpha}^{(m)}(\mu) \) (see \( (4.6) \)). With the notation \( (4.10) \), we have

\[
\rho^{(\nu)}(x, y; p, \zeta, m_l, m_k; \alpha, L) = \frac{1}{8} \sum_{k,l=1}^{3} \frac{1}{\sqrt{p_l^0 p_k^0}} e^{-i(p(x-y)+i(\mu_k \gamma_l + \mu_l \gamma_k) + 2\pi i L / \tilde{L}^{(lk)}} \left( n^{(l)} \otimes \bar{n}^{(k)} \right) U_{\alpha k} U_{\alpha l}^* \\
\times (\gamma_{\mu} p^\mu_{k} + m_l)(1 - \zeta \gamma^5 \gamma_{\mu} S^\mu_{l}(p_l)) \left[ \sqrt{p_l^0 m_l p_l^0 + m_l} (1 + \gamma^0) + \sqrt{p_k^0 m_k - p_k^0} (1 - \gamma^0) \right] \] (4.7)

If we consider the density matrix \( (4.7) \), averaged over the parameters \( \alpha \) and \( \zeta \), we obtain

\[
\rho^{(\nu)}(x, y; p, m_l) = \frac{1}{12p_l^0} \sum_{l=1}^{3} e^{-i(p(x-y))} \left( n^{(l)} \otimes \bar{n}^{(l)} \right) (\gamma_{\mu} p^\mu_{l} + m_l). \] (4.8)

Using \( (4.3) \) and \( (4.7) \) we can get the probability of the neutrino transition from one state to another. These states differ from each other, so for the state \( (4.3) \) the decoherence effect is absent and the state \( (4.7) \) spreads due to different group velocities of its components. Due to the fact that the interaction conserves the energy, but not the group velocity, as will be shown later, the approximate equality \( (E_{\nu}^l \) is the average value of the neutrino energy)

\[
\tilde{L}^{(lk)} \approx L^{(lk)} \approx L^{(o)} = \frac{4\pi E_{\nu}}{m_l^2 - m_k^2} \] (4.9)

holds, and the condition of applicability of both formulas \( (4.3) \) and \( (4.7) \) is defined by the relation

\[
\frac{m_l^2 \xi L}{p^2 \tilde{L}^{(lk)}} \ll 1. \] (4.10)

In accordance with the fundamental principles of quantum mechanics the probability of observing the state \( \alpha \) at a distance \( L \) from the production point of the state \( \beta \) is

\[
\mathcal{P}_{\nu(\beta) \rightarrow \nu(\alpha)} = \text{Sp} \left( \rho(x, y; \alpha, L) \rho^\dagger(x, y; \beta) \right). \] (4.11)
Therefore, in both cases

\[
P_{\nu(\beta)\to\nu(\alpha)} = \sum_{k,l=1}^{3} U_{\beta k} U_{\alpha k}^* U_{\alpha l} \exp\left(\frac{2\pi i L_{\nu\beta\alpha}}{L}ight).
\]

The formula itself is quite trivial. Mathematically it gives the sum of the squared absolute values of the projections of the new basis unit vectors in the three-dimensional vector space over the field of complex numbers onto the original basis vectors. In our case, the unitary transformation of the basis is generated by the phase factors arising as a result of space-time translations. Regardless of the initial basis, i.e. the parameters \( U_{\beta l} \), the result is the same. Therefore, formula (4.12) describes the evolution of an arbitrary superposition of the mass states. In particular, we can consider the states that are conventionally called the flavor states. Their wave functions are defined by the PMNS matrix \( U^{PMNS}_{\alpha l} \).

In the phenomenological theory of neutrino oscillations only the transitions between the flavor states are considered. That is, it is postulated that the neutrino is produced in a flavor state. This postulate is quite natural, since the PMNS matrix is the only non-trivial unitary 3 \times 3 matrix that can be built using the elements of the representation space for leptons. However, as we have seen, the flavor states are not uniquely defined. Their properties are associated with the type of the mass states, from which they are composed. The only way to answer the question, in what state the neutrinos are produced, is to calculate the corresponding probabilities of the processes.

V. PION DECAY

In the processes involving neutrinos only the charged leptons (the electron, the muon and the tau-lepton) are detected directly. As the charged leptons have fixed masses \( M^{(\beta)} \), their description should be done using the density matrix of the mass states \( \varrho^{(\beta)}(x,y;p,\zeta,\eta) \) or \( \varrho^{(e)}(x,y;p,\zeta,M) \). These density matrices are quite similar to the density matrices of neutrinos that are given by (4.1), (4.2). The result of the total probability calculation does not depend on the type of the density matrix. It is quite obvious, since any mass state is defined by an eigenfunction of the operator which is determined by the mass matrix.

In the neutral currents processes the oscillations cannot be observed, so we can sum directly over all the discrete quantum numbers of the neutrino final states using either the density matrix (4.5) or (4.8) (multiplied by 6) in the calculations. Thus, the probability does not depend on \( L \).

Therefore, we will examine how neutrino oscillations affect the probability of the processes which occur via the charged currents only. As an example, consider the pion decay

\[
\pi^+ \to \nu_\mu^+ + \nu_e, \quad \mu^+ = \nu^+, e^+.
\]

Let the 4-momentum of the pion be \( k^\mu, k^2 = m^2 \), and 4-momentum of the lepton \( l^\mu \) be \( p^\mu, p^2 = M^2 \). We assume that the distance from the source of neutrinos is \( L \), and the linear size of the source is \( L_0 \). For clarity, we assume that the pion is at rest: \( k^0 = m_\pi, k = 0 \).

First, consider the probability of the process that produces a charged lepton with mass \( M_\beta \) and a neutrino with mass \( m_\alpha \). The probability of this process in the Fermi approximation is given by the formula

\[
W_{\beta l} = \frac{G_F^2 f_\pi^2}{4(2\pi)^3} \int d^4x d^4y \int d\mathbf{q} d\mathbf{p} \times \text{Sp}\left\{ \varrho^{(e)}(x,y;-p,\zeta,\beta)\gamma^\mu(1 + \gamma^5)\varrho^{(e)}(y,x;q,\zeta,\beta)\gamma^\mu(1 + \gamma^5)k_\mu k_\nu e^{-i(k(x-y))} \right\}.
\]

The density matrix of the neutrino \( \varrho^{(\nu)}(y,x;q,\zeta,\beta) \) can be taken either in the form (4.1) or (4.2). The density matrix of the charged antilepton \( \varrho^{(\ell)}(x,y;-p,\zeta,\beta) \) can be taken either in the form analogous to (4.1)

\[
\varrho^{(e)}(x,y;-p,\zeta,\eta) = \frac{1}{4p_\beta^\mu} e^{i(p(x-y))\eta_\beta^\mu} \left( a^{(\beta)} \otimes \bar{a}^{(\beta)} \right) (\gamma_\mu p^\mu - M)(1 - \zeta^5 \gamma_\mu S_0^\mu(p)),
\]

or in the form analogous to (4.2)

\[
\varrho^{(e)}(x,y;-p,\zeta,M_\beta) = \frac{1}{4p_\beta^\mu} e^{i(p(x-y))} \left( a^{(\beta)} \otimes \bar{a}^{(\beta)} \right) (\gamma_\mu p^\mu_{\beta} - M_\beta)(1 - \zeta^5 \gamma_\mu S_0^\mu(p_\beta)).
\]
Substituting these expressions into (5.1) and making elementary calculations, we obtain

\[
W_{\beta l} = \frac{G_\nu^2 f^2}{8\pi m_\nu^2} P_{\beta l} P_{\beta l}^* \sqrt{\left(m_\nu^2 - M_\beta^2 + m_\ell^2\right)^2 - 4m_\nu^2 m_\ell^2} 
\times \left[M_\beta^2(m_\nu^2 - M_\beta^2 + m_\ell^2) + m_\ell^2(m_\nu^2 - M_\beta^2 + m_\ell^2)\right].
\]

This expression shows that the total probability of the pion decay,

\[
W_\beta = \sum_{l=1}^3 W_{\beta l} = \frac{G_\nu^2 f^2}{8\pi m_\nu^2} M_\beta^2(m_\nu^2 - M_\beta^2)^2 \left(1 + \sum_{l=1}^3 \mathcal{O}(P_{\beta l} P_{\beta l}^* / \mathcal{E}_\nu^2)\right),
\]

weakly depends on the mixing parameters \(P_{\beta l}\) (here \(\mathcal{E}_\nu \approx (m_\nu^2 - M_\beta^2)/2m_\nu\)).

Now we calculate the probability of detecting a superposition of the neutrino mass states produced in the pion decay with the emission of a charged lepton with mass \(M_\beta\) at a distance \(L\) from the source. This probability is equal to

\[
W_{\beta \alpha}^L = \frac{G_\nu^2 f^2}{4(2\pi)^6} \int d^4x d^4y \int dq dp \times \text{Sp}\left\{\rho^{(e)}(x, y; -p, \zeta, \beta) \gamma^\mu (1 + \gamma^5) \rho^{(e)}(y, x; q, \zeta, \alpha; L) \gamma^\nu (1 + \gamma^5) k_\mu k_\nu e^{-i(k(x-y))}\right\}.
\]

First, perform the calculations taking the density matrix \(\rho^{(e)}(y, x; q, \zeta, \alpha; L)\) given by formula (4.3). That is, we assume that the group velocities of the mass states are the same. We change the integration variables

\[
z_\mu^0 = x^\mu - y^\mu, \quad z_\mu^r = (x^\mu + y^\mu)/2,
\]

and limit the range of integration over the variable \(z_\mu^r\) to the size of the area in which the reaction takes place [50]. Then, after dividing by the volume of the reaction area, we obtain (see [C])

\[
W_{\beta \alpha}^L = \frac{G_\nu^2 f^2}{8\pi m_\nu^2} M_\beta^2(m_\nu^2 - M_\beta^2)^2 \times \left[\sum_{k,l=1}^3 P_{\beta l} P_{\beta l}^* U_{\alpha k} U_{\alpha l}^* \frac{\sin(\pi L_0 / L_{\text{osc}})}{\pi L_0 / L_{\text{osc}}} \right] \left(1 + \mathcal{O}(m_{l,k}^2 / \mathcal{E}_\nu^2)\right).
\]

Here

\[
\Delta_{l k} = (m_l + m_k)/(2\sqrt{m_l m_k}) \geq 1
\]

is the ratio of the arithmetic mean to the geometric mean of the corresponding neutrino masses. The origin of the factor

\[
R = \frac{\sin(\pi L_0 / L_{\text{osc}})}{\pi L_0 / L_{\text{osc}}}
\]

is the incomplete coherence of the source that is due to its finite size \(L_0 \ll L\). Obviously, (see [C]), if \(l = k\) we have \(R \equiv 1\) for any \(L_0\). After summation over the parameter \(\alpha\) in formula (5.8), there appears an expression for the total probability of the pion decay [55].

Consider the probability of detecting the neutrino flavor states at a close range \(L \ll |L_{\text{osc}}|\) if the source is quite compact \(L_0 \ll L \ll |L_{\text{osc}}|\). Then, replacing \(U_{\alpha l}\) by \(P_{\alpha l}\) we obtain

\[
W_{\beta \alpha} = \frac{G_\nu^2 f^2}{8\pi m_\nu^2} M_\beta^2(m_\nu^2 - M_\beta^2)^2 \left[\sum_{k,l=1}^3 P_{\beta l} P_{\beta l}^* P_{\alpha k} P_{\alpha l}^* \frac{\sin(\pi L_0 / L_{\text{osc}})}{\pi L_0 / L_{\text{osc}}} \right] \left(1 + \mathcal{O}(m_{l,k}^2 / \mathcal{E}_\nu^2)\right).
\]

This formula shows that not only the probability of detecting the main neutrino flavor \(\beta\) is nonzero, but the probability of detecting the other flavors is nonzero as well. Thus, the state of the neutrino, that was produced in the decay is a superposition of the flavor states described by the density matrices [4.3]. It should be emphasized that we do not talk about a very small contribution \(\sim m_{l,k}^2 / \mathcal{E}_\nu^2\).
Now take the density matrix \((4.7)\) for \(q^{(\nu)}(y, x; q, \zeta, \alpha; L)\), that is, we assume that the mass states included in the superposition have the same canonical momenta. Using the same assumptions as in the previous case, in particular, the change of variables \((5.7)\) we obtain

\[
W^{L}_{\beta\alpha} = \frac{G_{F}^{2}f^{2}}{8\pi m_{\pi}^{2}} M_{\beta}^{2} \left( m_{\pi}^{2} - M_{\beta}^{2} \right)^{2} \left[ \sum_{k,l=1}^{3} P_{\beta l} P_{\beta k}^{*} U_{\alpha k} U_{\alpha l}^{*} \sin \left( \frac{\pi L_{0}/L_{osc}}{L_{0}/L_{osc}} \right) e^{2\pi i L/L_{osc}} \left( 1 + \mathcal{O}(m_{l,k}^{2}/E_{\nu}^{2}) \right) \right].
\]

(5.12)

When detecting the flavor states \((U_{\alpha l} = P_{\alpha l})\) at small distances from the compact source

\[
W_{\beta\alpha} = \frac{G_{F}^{2}f^{2}}{8\pi m_{\pi}^{2}} M_{\beta}^{2} \left( m_{\pi}^{2} - M_{\beta}^{2} \right)^{2} \left( \delta_{\alpha\beta} + \mathcal{O}(m_{l,k}^{2}/E_{\nu}^{2}) \right).
\]

(5.13)

Therefore, in the decay process the flavor state \(\beta\), which is described with a good accuracy by the density matrix \((4.7)\) is produced. This state can be called in the conventional sense. Naturally, its evolution is described by the standard formula

\[
P_{\nu(\beta)\rightarrow\nu(\alpha)} = \frac{3}{\sum_{k,l=1}^{3} P_{\beta l} P_{\beta k}^{*} P_{\alpha k} P_{\alpha l} e^{2\pi i L/L_{osc}}},
\]

(5.14)

which is analogous to \((4.12)\).

The question, what exact state is produced in the weak decays of particles, is still open. In our model, this state is pure in the quantum-mechanical sense, so that when building this state we need to vary, in general, all the quantum numbers of the final state of the neutrino. If this state could be expressed as a finite set of plane waves, then we would have to choose two appropriate matrices: a Hermitian matrix \(L^{(\nu)}\) (see \((2.30)\)) and a unitary matrix \(U^{(\nu)}\) (see \((2.41)\)). It is not improbable that the required state can be a wave packet, although not an abstract Gaussian wave packet, but a wave packet with characteristics explicitly depending on the type of the process in which it occurs. However, this is not essential, if the condition \((4.10)\) holds.

It should be recalled that all particles are produced in wave packet states. This circumstance helps to explain the phenomenon of the oscillations. However, there is no need to introduce wave packets explicitly to describe this phenomenon. The restriction of the space-time domain of integration with respect to the variable \(z_{\nu}^{0}\) (see \((5.7)\)) makes it possible to take into account the nonmonochromaticity of the produced particles. The oscillation lengths of charged leptons are very small due to their relatively large masses. The factor \(R\) (see Eq. \((5.10)\)) for such particles is negligible and the oscillations are non observable. Therefore, the technique used is essential only for the light particles, i.e. neutrinos.

When detecting the neutrino states from a very large \((L_{0} \gg |L_{osc}^{(k)}|)\) source we have

\[
W_{\beta\alpha} = \frac{G_{F}^{2}f^{2}}{8\pi m_{\pi}^{2}} M_{\beta}^{2} \left( m_{\pi}^{2} - M_{\beta}^{2} \right)^{2} \sum_{l=1}^{3} P_{\beta l} P_{\beta l}^{*} U_{\alpha l} U_{\alpha l}^{*} \left( 1 + \mathcal{O}(|L_{osc}^{(k)}/L_{0}|) \right) + \mathcal{O}(m_{l,k}^{2}/E_{\nu}^{2}).
\]

(5.15)

In this case oscillations are absent. If \(U_{\alpha l} = \delta_{\alpha l}\) we get the probability of the process that produces a charged lepton with mass \(M_{\beta}\) and a neutrino with mass \(m_{\alpha}\). And if \(U_{\alpha l} = P_{\alpha l}\) we get the probability of the process that produces a charged lepton with mass \(M_{\beta}\) and a neutrino flavor state \(\alpha\).

Similarly, we can consider the decay

\[
\pi^{-} \Rightarrow l_{\beta}^{-} + \bar{\nu}, \quad l_{\beta}^{-} = \mu^{-}, e^{-},
\]

using the density matrix for antineutrinos \((B1), (B2), (B3), (B4)\), the corresponding formulas can be obtained by replacing the sign of the oscillation length in the expression for the probability, that is, by replacing \(L_{osc}^{(k)}\) with \(-L_{osc}^{(k)}\). As is well known, this difference in the formulas for the probabilities of the processes involving particles and antiparticles can indicate CP violation in the theory.

Thus, our approach adequately describes the phenomenon of neutrino oscillations, which is, in fact, the phenomenon of the oscillation of the probabilities of the processes involving neutrinos. The flavor state, constructed as a superposition of the mass states with the same momentum, is distinguished by the fact that it is very close to the state which is formed in the decay process. The possibility to use such flavor states in order to describe neutrinos when the distance from the source is much larger than the oscillation length, directly follows from the smallness of the neutrino masses compared to their energies. Therefore, for ultra-relativistic neutrinos \((m_{l,k}^{2}/E_{\nu}^{2} \ll 1)\), which are really observed, the phenomenological theory of oscillations based on the ideas of Pontecorvo (see, e.g., \([12]\)), is a very good approximation for quantum field theory.
VI. CONCLUSIONS

In this paper we put forward a modification of the electroweak interaction theory, in which the fermions with the same electroweak quantum numbers are placed in fermion multiplets and are treated as different quantum states of a single particle. That is, in describing the electroweak interactions it is possible to use four fundamental fermions only. In this model, the mixing and oscillations of the particles arise as a direct consequence of the general principles of quantum field theory. This approach enables one to calculate the probabilities of the processes taking place in detectors at long distances from the source. Calculations of higher order processes including the computation of the contributions due to radiative corrections can be performed in the framework of perturbation theory using the regular diagram technique.

The developed approach is used to study neutrino oscillations. It is shown by the example of the pion decay that the states of the ultra-relativistic neutrino produced in the decay process can be described by a superposition of states with different masses and identical canonical momenta with very high accuracy.

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Appendix A: Explicit form for the generators of SU(3) transformations

A direct calculation yields the explicit form of the generators of SU(3) transformations \( X_{\nu}^{(\nu)} = \mathcal{K}^{(\nu)} X_{\nu} \mathcal{K}^{(\nu)-1} \)

\[
\begin{align*}
X_{1}^{(\nu)} & = \left( n^{(1)} \otimes \tilde{n}^{(2)} \right) D_{(1)} D_{(2)}^{-1} + \left( n^{(2)} \otimes \tilde{n}^{(1)} \right) D_{(2)} D_{(1)}^{-1}, \\
X_{2}^{(\nu)} & = i \left( n^{(2)} \otimes \tilde{n}^{(1)} \right) D_{(2)} D_{(1)}^{-1} - i \left( n^{(1)} \otimes \tilde{n}^{(2)} \right) D_{(1)} D_{(2)}^{-1}, \\
X_{3}^{(\nu)} & = \left( n^{(1)} \otimes \tilde{n}^{(1)} \right) - \left( n^{(2)} \otimes \tilde{n}^{(2)} \right), \\
X_{4}^{(\nu)} & = \left( n^{(1)} \otimes \tilde{n}^{(3)} \right) D_{(1)} D_{(3)}^{-1} + \left( n^{(3)} \otimes \tilde{n}^{(1)} \right) D_{(3)} D_{(1)}^{-1}, \\
X_{5}^{(\nu)} & = i \left( n^{(3)} \otimes \tilde{n}^{(1)} \right) D_{(3)} D_{(1)}^{-1} - i \left( n^{(1)} \otimes \tilde{n}^{(3)} \right) D_{(1)} D_{(3)}^{-1}, \\
X_{6}^{(\nu)} & = \left( n^{(2)} \otimes \tilde{n}^{(3)} \right) D_{(2)} D_{(3)}^{-1} + \left( n^{(3)} \otimes \tilde{n}^{(2)} \right) D_{(3)} D_{(2)}^{-1}, \\
X_{7}^{(\nu)} & = i \left( n^{(3)} \otimes \tilde{n}^{(2)} \right) D_{(3)} D_{(2)}^{-1} - i \left( n^{(2)} \otimes \tilde{n}^{(3)} \right) D_{(2)} D_{(3)}^{-1}, \\
X_{8}^{(\nu)} & = \left( n^{(1)} \otimes \tilde{n}^{(1)} \right) + \left( n^{(2)} \otimes \tilde{n}^{(2)} \right) - 2 \left( n^{(3)} \otimes \tilde{n}^{(3)} \right).
\end{align*}
\]

Operators of finite transformations are given by

\[
\begin{align*}
U_{1}^{(\nu)} (\alpha) & = \left( n^{(3)} \otimes \tilde{n}^{(3)} \right) + \left( n^{(1)} \otimes \tilde{n}^{(1)} \right) \left( n^{(2)} \otimes \tilde{n}^{(2)} \right) \cos \alpha - i \left( n^{(2)} \otimes \tilde{n}^{(2)} \right) \sin \alpha, \\
U_{2}^{(\nu)} (\alpha) & = \left( n^{(3)} \otimes \tilde{n}^{(3)} \right) + \left( n^{(1)} \otimes \tilde{n}^{(1)} \right) \left( n^{(2)} \otimes \tilde{n}^{(2)} \right) \cos \alpha - i \left( n^{(2)} \otimes \tilde{n}^{(2)} \right) \sin \alpha, \\
U_{3}^{(\nu)} (\alpha) & = \left( n^{(3)} \otimes \tilde{n}^{(3)} \right) + \left( n^{(1)} \otimes \tilde{n}^{(1)} \right) e^{i \alpha} + \left( n^{(2)} \otimes \tilde{n}^{(2)} \right) e^{-i \alpha}, \\
U_{4}^{(\nu)} (\alpha) & = \left( n^{(2)} \otimes \tilde{n}^{(2)} \right) + \left( n^{(1)} \otimes \tilde{n}^{(1)} \right) \left( n^{(3)} \otimes \tilde{n}^{(3)} \right) \cos \alpha - i \left( n^{(3)} \otimes \tilde{n}^{(3)} \right) \sin \alpha, \\
U_{5}^{(\nu)} (\alpha) & = \left( n^{(2)} \otimes \tilde{n}^{(2)} \right) + \left( n^{(1)} \otimes \tilde{n}^{(1)} \right) \left( n^{(3)} \otimes \tilde{n}^{(3)} \right) \cos \alpha - i \left( n^{(3)} \otimes \tilde{n}^{(3)} \right) \sin \alpha, \\
U_{6}^{(\nu)} (\alpha) & = \left( n^{(1)} \otimes \tilde{n}^{(1)} \right) + \left( n^{(2)} \otimes \tilde{n}^{(2)} \right) \left( n^{(3)} \otimes \tilde{n}^{(3)} \right) \cos \alpha - i \left( n^{(3)} \otimes \tilde{n}^{(3)} \right) \sin \alpha, \\
U_{7}^{(\nu)} (\alpha) & = \left( n^{(1)} \otimes \tilde{n}^{(1)} \right) + \left( n^{(2)} \otimes \tilde{n}^{(2)} \right) \left( n^{(3)} \otimes \tilde{n}^{(3)} \right) \cos \alpha - i \left( n^{(3)} \otimes \tilde{n}^{(3)} \right) \sin \alpha, \\
U_{8}^{(\nu)} (\alpha) & = \left( n^{(3)} \otimes \tilde{n}^{(3)} \right) e^{2 i \alpha} + \left( n^{(1)} \otimes \tilde{n}^{(1)} \right) \left( n^{(2)} \otimes \tilde{n}^{(2)} \right) \cos \alpha - i \left( n^{(2)} \otimes \tilde{n}^{(2)} \right) \sin \alpha.
\end{align*}
\]
where

\[
\left( n^{(1)} \otimes \bar{n}^{(1)} \right) = \frac{1}{6} (2I + X_s^{(\nu)} + 3X_s^{(\bar{\nu})}),
\]

\[
\left( n^{(2)} \otimes \bar{n}^{(2)} \right) = \frac{1}{6} (2I + X_s^{(\nu)} - 3X_s^{(\bar{\nu})}),
\]

\[
\left( n^{(3)} \otimes \bar{n}^{(3)} \right) = \frac{1}{3} (I - X_s^{(\bar{\nu})}).
\]

Integrals of motion

\[
X_3 = \sum_{\zeta = \pm 1} \int dq \left[ a_{1,\zeta}^+(q) a_{1,\zeta}^-(q) + a_{1,\zeta}^-(q) a_{1,\zeta}^+(q) - a_{2,\zeta}^+(q) a_{2,\zeta}^-(q) - a_{2,\zeta}^-(q) a_{2,\zeta}^+(q) \right],
\]

\[
X_5 = \sum_{\zeta = \pm 1} \int dq \left[ a_{2,\zeta}^+(q) a_{2,\zeta}^-(q) + a_{2,\zeta}^-(q) a_{2,\zeta}^+(q) + a_{3,\zeta}^+(q) a_{3,\zeta}^-(q) + a_{3,\zeta}^-(q) a_{3,\zeta}^+(q) \right],
\]

\[
X_6 = \sum_{\zeta = \pm 1} \int dq \left[ -a_{2,\zeta}^+(q) a_{2,\zeta}^-(q) + a_{2,\zeta}^-(q) a_{2,\zeta}^+(q) + a_{3,\zeta}^+(q) a_{3,\zeta}^-(q) + a_{3,\zeta}^-(q) a_{3,\zeta}^+(q) \right],
\]

Appendix B: Density matrix of antineutrino

The mass states of antineutrino are described either by the density matrix

\[
g^{(\nu)}(x, y; -q, \zeta, m) = \frac{1}{4q^3} e^{-i(q(x-y))} \rho_{\nu}^3 \left( n^{(l)} \otimes \bar{n}^{(l)} \right) (\gamma_iq^\mu - m)(1 - \zeta\gamma_5\gamma_iS_0^\nu(q)),
\]

based on the solutions similar to \((2.15)\), or the density matrix

\[
g^{(\nu)}(x, y; -p, \zeta, m) = \frac{1}{4p^3} e^{-i(p(x-y))} \rho_{\nu}^3 \left( n^{(l)} \otimes \bar{n}^{(l)} \right) (\gamma_iq^\mu - m)(1 - \zeta\gamma_5\gamma_iS_0^\nu(p)).
\]
based on the solutions similar to (2.37). In both cases, there is no dependence on $L$.

If we consider a state, which is described by a superposition of wave functions similar to (2.37), then the density matrix for the state of an antineutrino at a distance $L$ from the source can be written as

$$
\rho^{(\nu)}(x, y; -q, \zeta, \mu; \alpha, L) = \frac{1}{4q^0} \sum_{k, \ell=1}^{3} e^{i(qx)\mu_\ell-i(qy)\mu_k+2\pi i L/L^{(k)}} \times (\mu_k\mu_k)^{3/2} \left( n^{(l)} \otimes * n^{(k)} \right) U_{\alpha k} U_{\alpha l}^* \left( \gamma_\mu q^\mu - m \right)(1 - \zeta^5 \gamma_\mu S^\mu_\nu(q)).
$$

If we consider a state described by a superposition of wave functions similar to (2.37), then the density matrix for the state of an antineutrino at a distance $L$ from the source can be written as

$$
\rho^{(\nu)}(x, y; -p, \zeta, m_l, m_k; \alpha, L)
$$

$$
= \frac{1}{8} \sum_{k, \ell=1}^{3} \frac{1}{p_k^l p_k^\ell} e^{i(p_k x)-i(p_k y)+2\pi i L/L^{(k)}} \left( n^{(l)} \otimes * n^{(k)} \right) U_{\alpha k} U_{\alpha l}^* \times (\gamma_\mu p^\mu_l - m_l) (1 - \zeta^5 \gamma_\mu S^\mu_\nu(p_l)) \left[ \sqrt{p_k^l - m_k} \left( 1 + \gamma^0 \right) + \sqrt{p_k^\ell + m_l} \left( 1 - \gamma^0 \right) \right]
$$

$$
= \frac{1}{8} \sum_{k, \ell=1}^{3} \frac{1}{p_k^l p_k^\ell} e^{-i(p_k x)+i(p_k y)+2\pi i L/L^{(k)}} \left( n^{(l)} \otimes * n^{(k)} \right) U_{\alpha k} U_{\alpha l}^* \times \left[ \frac{p_k^l m_l}{p_k^l m_k} \left( 1 + \gamma^0 \right) + \frac{p_k^\ell + m_l}{p_k^\ell + m_k} \left( 1 - \gamma^0 \right) \right] (\gamma_\mu p^\mu_k - m_k) (1 - \zeta^5 \gamma_\mu S^\mu_\nu(p_k)).
$$

### Appendix C: Calculation of the pion decay probability

The expressions (5.6) after summation over the polarizations of the final particles can be easily reduced to the following form

$$
W^{L}_{\beta\alpha} = \frac{G^2_F f^2_\pi}{4(2\pi)^6 k^0} \int d^4x d^4y d^4p \delta(p^2 - M^2_\beta) \delta(q^2 - m^2) \times \text{Sp} \left[ (\gamma_\mu p^\mu + M_\beta) (1 + \gamma^5) (\gamma_\nu q^\nu + m) (1 + \gamma^5) k^\mu k^\nu \right] \times \sum_{k, \ell=1}^{3} e^{i(qx)\mu_k-i(qy)\mu_k+2\pi i L/L^{(k)}} \times \left( m_l/m \right)^{3/2} \left( m_k/m \right)^{3/2} P_{\beta k} P_{\beta k} U_{\alpha k} U_{\alpha l}^* \right].
$$

Having calculated the trace of $\gamma$-matrices and changed the integration variables

$$
q^\mu \rightarrow q^\mu (m/m_{kl}), \quad m_{kl} = (m_k + m_l)/2,
$$

we get

$$
W^{L}_{\beta\alpha} = \frac{2 G^2_F f^2_\pi}{(2\pi)^6 k^0} \left[ \sum_{k, \ell=1}^{3} \frac{P_{\beta k} P_{\beta k} U_{\alpha k} U_{\alpha l}^*}{\Delta^k_{l k}} \times \int d^4x d^4y d^4p \delta(p^2 - M^2_\beta) \delta(q^2 - m^2_{kl}) \left( 2(p_k)(q_k) - k^2(pq) \right) \times \int d^4x d^4y e^{i(qx)\mu_k-i(qy)\mu_k+2\pi i L/L^{(k)}} \right].
$$

Here

$$
L^{(k)} = \frac{4\pi|q|}{m_l^2 - m_k^2}, \quad \Delta_{lk} = \frac{m_k + m_l}{2\sqrt{m_lm_k}}.
$$
We change the integration variables

\[(x^0, x) \rightarrow \left( u^0 x^0 + (ux'), x' + u x^0 + \frac{u(ux')}{1 + u^0} \right), \]
\[(y^0, y) \rightarrow \left( u^0 y^0 + (uy'), y' + u y^0 + \frac{u(uy')}{1 + u^0} \right), \]

where

\[u^\mu = q^\mu / m_{kl}. \tag{C5}\]

The Jacobian of transformation \(J = 1\). This change of variables yields the following expression in the exponential

\[
\left( \frac{(k - p)q}{m_{kl}} - m_k \right) x^0 - \left( \frac{(k - p)q}{m_{kl}} - m_l \right) y^0
- \left( k - p - \frac{(k - p)q + m_{kl}(k^0 - p^0)}{m_{kl}(m_{kl} + q^0)} q \right) (x' - y'). \tag{C7}\]

We change the variables

\[
z'_- = x' - y', \ z'_+ = (x' + y')/2, \\
z'_0 = x'^0 - y'^0, \ z'^0 = (x'^0 + y'^0)/2, \tag{C8}\]

\[z'_- = \left( I - \frac{q \otimes q}{q^0(m_{kl} + q^0)} \right) z'^'. \tag{C9}\]

The Jacobian of transformation \(J' = m_{kl}/q^0\). As a result, the expression in the exponential takes the form

\[
\left( \frac{(k - p)q}{m_{kl}} - m_k \right) z'^0 - (m_k - m_l) z'^0 - \left( k - p - \frac{k^0 - p^0}{q^0} q \right) z'^'. \tag{C10}\]

To get the probability of the process in a unit volume per unit time we use the standard Fermi ansatz. We integrate \(C3\) over \(d^4z'_-\) in a finite four-dimensional region \(V'T'\) and divide by the invariant volume \(V'T' = VT\). As a result of this operation, a factor

\[R = \sin \left( \frac{(m_l - m_k)T'/2}{(m_l - m_k)T'/2} \right) \tag{C11}\]

arises. It follows from Eq. \(C3\) that

\[T' = \frac{q^0 T - (qL)}{m_{kl}}. \tag{C12}\]

Assuming \(L = vT, v = q/q^0\), we get

\[T' = \frac{m_{kl} T}{q^0} \equiv \frac{m_{kl}}{|q|} L_0, \tag{C13}\]

where \(L_0\) is the linear dimension of the region where the pion decays. Hence

\[R = \frac{\sin(\pi L_0/L^{(lz)})}{\pi L_0/L^{(lz)}}. \tag{C14}\]

The integration over \(dz'_-\) yields the \(\delta\)-functions associated with the momentum conservation

\[(2\pi)^3 \delta \left( k - p - \frac{k^0 - p^0}{q^0} q \right). \tag{C15}\]

It follows from Eq. \(C15\) that

\[p = k - \frac{k^0 - p^0}{q^0} q. \tag{C16}\]
and we have

\[
\frac{(k - p)q}{m_{kl}} - m_{kl} = \frac{m_{kl}}{q^0}(k^0 - p^0 - q^0).
\]  

(C17)

Therefore, using an integration variable

\[
\epsilon_0 = \epsilon_0 m_{kl}/q^0 = \epsilon_0 J'
\]  

(C18)

and integrating over it, we obtain the \( \delta \)-function associated with energy conservation

\[
2\pi \delta \left( k^0 - p^0 - q^0 \right).
\]  

(C19)

Thus,

\[
W^{L}_{\beta\alpha} = \frac{G_F^2 f_\tau^2}{(2\pi)^2 k^0} \left[ \sum_{k,l=1}^{3} \frac{P_{\beta l} P_{\delta k} U_{ak} U_{al}^*}{\Delta_{fk}^3} \int d^4 q d^4 p \frac{\sin(\pi L_0/L^{(lk)})}{\pi L_0/L^{(lk)}} e^{2\pi i L/L^{(lk)}} (2(pk)(qk) - k^2(pq)) \right.
\]

\[
\times \delta^4 (k - p - q) \delta^4 (p^2 - M_\beta^2) \delta^4 (q^2 - m_{kl}^2) \left. \right].
\]  

(C20)

The integral over the momentum variables coincides with the one that appears when calculating the probability of the mass state creation. Therefore,

\[
W^{L}_{\beta\alpha} = \frac{G_F^2 f_\tau^2}{(2\pi)^2 k^0} \left[ \sum_{k,l=1}^{3} \frac{P_{\beta l} P_{\delta k} U_{ak} U_{al}^*}{\Delta_{fk}^3} \int d^4 q d^4 p \frac{\sin(\pi L_0/L^{(lk)})}{\pi L_0/L^{(lk)}} e^{2\pi i L/L^{(lk)}} \right.
\]

\[
\times \left[ M_\beta^2 (m_{\pi}^2 - M_\beta^2 + m_{kl}^2) + m_{kl}^2 (m_{\pi}^2 + M_\beta^2 - m_{kl}^2) \right]
\]

\[
\times \delta^4 (q^2 - m_{kl}^2) \delta^4 (m_{\pi}^2 - M_\beta^2 + m_{kl}^2 - 2(kq)) \left. \right].
\]  

(C21)

Assuming \( k = 0 \) and neglecting \( m_{kl} \) in comparison with the neutrino energy we obtain the formula (5.8).

It should be noted that the change of variables (5.7) and limiting the integration domain to the size of the area

\[
\delta^4 (q^2 - m_{kl}^2) \delta^4 (m_{\pi}^2 - M_\beta^2 + m_{kl}^2 - 2(kq))
\]  

(C21)

is not only justified by physical considerations of Schwinger [50], but is also mathematically consistent [51].

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[52] Let $\Gamma$ be a continuous unitary representation of a finite-dimensional connected Lie group $G$ in a Hilbert space $\mathcal{H}$. Let $G$ contain the inhomogeneous Lorentz group as an analytic subgroup. Let finally the spectrum of the 4-momentum operator $P^\mu$ be contained in $\{0\} \cup V^+\cup V^+$ being the future cone in the Minkowski space $\mathbb{M}^4$. If $m_1 > 0$ is an isolated eigenvalue of the mass operator $M = (P^\mu P^\mu)^{1/2}$ then the corresponding eigenspace $\mathcal{H}_1$ is invariant under $\Gamma(G)$.
[53] As a matter of fact, the given statement is true for finite-dimensional representations. However, in what follows we shall consider only the constructions which are unitary equivalent to each other.
[54] This is true only for the Dirac equation describing free particles. The ambiguity of the choice of the operators with continuous spectrum arises, for example, in the problem of the neutrino spin evolution for a neutrino propagating in a dense matter and an electromagnetic field. [18].