Families of Almost Complex Structures and Transverse $(p, p)$-Forms

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Abstract
An almost $p$-Kähler manifold is a triple $(M, J, \Omega)$, where $(M, J)$ is an almost complex manifold of real dimension $2n$ and $\Omega$ is a closed real transverse $(p, p)$-form on $(M, J)$, where $1 \leq p \leq n$. When $J$ is integrable, almost $p$-Kähler manifolds are called $p$-Kähler manifolds. We produce families of almost $p$-Kähler structures $(J_t, \Omega_t)$ on $\mathbb{C}^3$, $\mathbb{C}^4$, and on the real torus $\mathbb{T}^6$, arising as deformations of Kähler structures $(J_0, g_0, \omega_0)$, such that the almost complex structures $J_t$ cannot be locally compatible with any symplectic form for $t \neq 0$. Furthermore, examples of special compact nilmanifolds with and without almost $p$-Kähler structures are presented.

Keywords Almost $p$-Kähler manifold · Almost complex deformation · Semi-Kähler metric

Mathematics Subject Classification 53C55 · 53C25

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1 Introduction

Let \((M, J, g, \omega)\) be a \(2n\)-dimensional compact Kähler manifold, that is \(M\) is a compact \(2n\)-dimensional smooth manifold endowed with an integrable almost complex structure \(J\) and \(J\)-Hermitian metric \(g\) whose fundamental form \(\omega\) is closed, namely \((J, g, \omega)\) is a Kähler structure on \(M\). Then, the celebrated theorem of Kodaira and Spencer [9, Theorem 15] states that the Kähler condition is stable under small \(C^\infty\)-deformations of the complex structure \(J\). As a consequence of complex Hodge Theory, the existence of a Kähler structure on a compact manifold imposes strong restrictions on the topology of \(M\), e.g., the odd index Betti numbers of \(M\) are even, the even index Betti numbers are greater than zero and the de Rham complex of \(M\) is a formal differential graded algebra in the sense of Sullivan. In [8] Harvey and Lawson give an intrinsic characterization in terms of currents of compact complex manifolds admitting a Kähler metric. However there are many examples of compact complex manifolds without any Kähler structures; examples are easily constructed by taking compact quotients of simply connected nilpotent Lie groups. The underlying differentiable manifolds of such complex manifolds may carry further structures, e.g., symplectic structures, and balanced, SKT, Astheno Kähler metrics, that is Hermitian metrics whose fundamental form \(\omega\) satisfies \(d\omega^{n-1} = 0\), respectively \(\bar{\partial}\bar{\partial}\omega = 0\), respectively \(\partial_\omega^{n-2} = 0\), where \(\dim_{\mathbb{C}} M = n\).

An almost Kähler manifold is an almost complex manifold equipped with a Hermitian metric whose fundamental form is closed. In the almost Kähler setting stability properties are drastically different. First of all, a dense subset of almost complex structures \(J\) on \(\mathbb{R}^{2n}\), with \(n > 2\), are not compatible with any symplectic form, that is, there are no symplectic forms \(\omega\), such that \(g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)\) is a \(J\)-Hermitian metric, or equivalently, \(\omega\) is a closed positive \((1, 1)\)-form. Such almost complex structures can be extended to any Kähler manifold \((M, J)\) of dimension bigger than 2, showing that there exists a curve \(\{J_t\}_{t \in (-\varepsilon, \varepsilon)}\) such that \(J_0 = J\) and \(J_t\), for \(t \neq 0\), is a (non-integrable) almost complex structure on \(M\), which is not even locally compatible with respect to any symplectic form. This is a direct consequence of e.g., [11, Theorem 2.4, Corollary 2.5] which deals with the local case.

In the present paper, starting with a complex manifold \((M, J)\), we are interested in studying the stability properties of transversely closed \((p, p)\)-forms, namely the stability properties of \(p\)-Kähler manifolds in the terminology of Alessandrini and Andreata [2] under possible non integrable small deformations of the complex structure. The notion of transversality was firstly introduced by Sullivan in [19], in the context of cone structures, namely a continuous field of cones of \(p\)-vectors on a manifold, and then the \(p\)-Kähler condition was studied by several authors (see e.g., [2, 3] and the references therein). In particular, 1-Kähler manifolds correspond to Kähler manifolds and \((n - 1)\)-Kähler manifolds correspond to balanced manifolds in the terminology of Michelsohn (see [10]): for the proofs of these results see [2, Proposition 1.15] or [17, Corollary 4.6]. Kähler manifolds are \(p\)-Kähler for all \(p\) (by taking the \(p\)-th power of the form) but \(p\)-Kähler manifolds may not be Kähler. Note that there is a difference between the case \(p = n - 1\) and \(p < n - 1\). Indeed, according to [10, p.279], if \(\Omega\) is an \((n - 1)\)-Kähler structure, then \(\Omega = \omega^{n-1}\) for a suitable fundamental form \(\omega\) of a Hermitian metric on \(M\). Hence \(d\omega^{n-1} = 0\) and so \(M\) admits a balanced metric.
the other hand if $p < n - 1$ and $\omega$ is the fundamental form of an almost Hermitian metric on $M$, then $d\omega^p = 0$ implies that $d\omega = 0$, so in fact $M$ is almost Kähler (see [6, Theorem 3.2]).

In contrast to the Kähler case, the $p$-Kähler condition on a compact complex manifold is not stable under small deformation of the complex structure. This was proved in [3] by constructing a non balanced deformation of the natural complex structure for the Iwasawa manifold, which carries a balanced metric.

Recently, in [16], Rao, Wan and Zhao further studied the stability of $p$-Kähler compact manifolds under small integrable deformations of the complex structure. Assuming that the $(p, p + 1)$-th mild $\partial\bar{\partial}$-lemma holds, it is shown that $p$-Kähler structures are stable for all $1 \leq p \leq n - 1$. Here the $(p, p + 1)$-th mild $\partial\bar{\partial}$-lemma for a complex manifold means that each $\partial\bar{\partial}$-exact and $\partial\bar{\partial}$-closed $(p, p + 1)$-form on this manifold is $\partial\bar{\partial}$-exact. Note that such a condition does not hold in the Iwasawa example in [3, p.1062]. For other recent results on families of compact balanced manifolds see [18].

In the terminology by Gray and Hervella [6, p. 40], a $J$-Hermitian metric $g$ on an almost complex manifold $(M, J)$ of complex dimension $n$ is said to be semi-Kähler if $d\omega^{n-1} = 0$.

The aim of this paper is to produce families of almost $p$-Kähler structures $(J_t, \Omega_t)$ arising as deformations of Kähler structures $(J_0, g_0, \omega_0)$, such that the almost complex structures $J_t$ cannot be locally compatible with any symplectic form for $t \neq 0$.

Our first main result is the following, see Theorem 5.1

**Theorem** Let $(J, \omega)$ be the standard Kähler structure on the standard torus $\mathbb{T}^6 = \mathbb{R}^6/\mathbb{Z}^6$, with coordinates $(z_1, z_2, z_3)$, $z_j = x_j + iy_j$, $j = 1, 2, 3$. Let $f = f(z_2, \bar{z}_2)$ be a $\mathbb{Z}^6$-periodic smooth complex valued function on $\mathbb{R}^6$ and set $f = u + iv$, where $u = u(x_2, y_2)$, $v = v(x_2, y_2)$.

Let $I = (-\varepsilon, \varepsilon)$. Assume that $u, v$ satisfy the following condition

$$\left. \left( \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial y_2} \right) \right|_{(x, y) = 0} \neq 0$$

Then, for $\varepsilon > 0$ small enough, there exists a 1-parameter complex family of almost complex structures $\{J_t\}_{t \in I}$ on $\mathbb{T}^6$, such that

i) $J_0 = J$;

ii) $J_t$ admits a semi-Kähler metric for all $t \in I$;

iii) For any given $0 \neq t \in I$, the almost complex structure $J_t$ is not locally compatible with respect to any symplectic form on $\mathbb{T}^6$.

Then, starting with a balanced structure on $(J, g, \omega)$ on $\mathbb{C}^n$, we provide necessary conditions in order that a curve $(J_t, g_t, \omega_t)$ give rise to semi-Kähler structures on $\mathbb{C}^n$ (Theorem 4.1). Special results are obtained for $n = 3, p = 2$ (Theorem 4.3, Corollaries 4.4 and 4.5).

Finally, we have a result for almost 2-Kähler structures on $\mathbb{C}^4$, see Theorem 5.7 and the example following.
Theorem There exists a family of almost-complex structures $J_t$ on $\mathbb{C}^4$ such that $J_0 = i$ is the standard integrable complex structure and $J_t$ is almost 2-Kähler for all $t \neq 0$, but $J_t$ is not locally Kähler for all $t \neq 0$.

It is interesting to contrast this with the linear case, where we see in Proposition 3.1 that a complex structure preserving a product $\omega^p$ automatically preserves $\omega$ (up to sign). Hence, for $p < n - 1$, if the almost $p$-Kähler forms in a deformation are not given by almost Kähler structures then they cannot be powers of 2-forms $\omega_t$ (since by [6, Theorem 3.2] the $\omega_t$ would necessarily be closed and hence almost Kähler).

In particular it is impossible to find a deformation as in the theorem above where the almost 2-Kähler forms remain equal to $\omega_0^2$ (where $\omega_0$ is the standard Kähler form on $\mathbb{C}^4$). This differs from the Kähler case, where by Moser’s theorem we may apply a family of diffeomorphisms $\Phi_t$ to any deformation $(J_t, \omega_t)$, with $[\omega_t]$ constant, so that the form is unchanged, that is $(\Phi_t^* J_t, \Phi_t^* \omega_t) = (J_t', \omega_0)$.

The paper is organized as follows: in Sect. 2 we start by fixing notation and recalling some basic facts on $p$-Kähler and almost $p$-Kähler structures, giving a simple example of a compact $(2n - 1)$-dimensional complex manifold without any $p$-Kähler structure, for $1 \leq p \leq (n - 1)$. In Sect. 3 we prove Proposition 3.1. In Proposition 3.3 we provide an example of a deformation of a non-Kähler integrable complex structure into nonintegrable structures such that none of the structures are almost Kähler, in fact they cannot be tamed by any symplectic form, but they are all almost 2-Kähler, and in fact all compatible with the same $(2, 2)$ form. Finally Sects. 4 and 5 are devoted to the proofs of the main results and to the constructions of the almost $p$-Kähler families. In particular, in Theorem 5.7 we obtain the family of almost 2-Kähler structures in $\mathbb{C}^4$ which are not almost Kähler, and in Proposition 5.4 we construct a curve of almost complex structures $\{J_t\}_{t \in \mathbb{R}}$ (non integrable for $t \neq 0$) on the Iwasawa manifold such that $J_0$ admits a balanced metric and $J_t$ does not admit any semi-Kähler metric for $t \neq 0$.

2 Almost $p$-Kähler Structures

Let $V$ be a real $2n$-dimensional vector space endowed with a complex structure $J$, that is an automorphism $J$ of $V$ satisfying $J^2 = -\text{id}_V$. Let $V^*$ be the dual space of $V$ and denote by the same symbol the complex structure on $V^*$ naturally induced by $J$ on $V$. Then the complexified $V^{*\mathbb{C}}$ decomposes as the direct sum of the $\pm i$-eigenspaces, $V^{1,0}$, $V^{0,1}$ of the extension of $J$ to $V^{*\mathbb{C}}$, given by

$$
V^{1,0} = \{ \varphi \in V^{*\mathbb{C}} \mid J\varphi = i\varphi \} = \{ \alpha - iJ\alpha \mid \alpha \in V^* \},
$$

$$
V^{0,1} = \{ \psi \in V^{*\mathbb{C}} \mid J\psi = -i\psi \} = \{ \beta + iJ\beta \mid \beta \in V^* \},
$$

that is

$$
V^{*\mathbb{C}} = V^{1,0} \oplus V^{0,1}.
$$

According to the above decomposition, the space $\Lambda^r(V^{*\mathbb{C}})$ of complex $r$-covectors on $V^{\mathbb{C}}$ decomposes as

\[ \text{Springer} \]
\[ \Lambda^r(V^C) = \bigoplus_{p+q=r} \Lambda^{p,q}(V^*C), \]

where

\[ \Lambda^{p,q}(V^*C) = \Lambda^p(V^{1,0}) \otimes \Lambda^q(V^{0,1}). \]

If \( \{\varphi^1, \ldots, \varphi^n\} \) is a basis of \( V^{1,0} \), then

\[
\{\varphi^{i_1} \wedge \cdots \wedge \varphi^{i_p} \wedge \overline{\varphi^{j_1}} \wedge \cdots \wedge \overline{\varphi^{j_q}} \mid 1 \leq i_1 < \cdots < i_p \\
\leq n, \ 1 \leq j_1 < \cdots < j_q \leq n\}
\]

is a basis of \( \Lambda^{p,q}(V^*C) \). Set \( \sigma = i^{p^2}2^{-p} \). Then, given any \( \varphi \in \Lambda^{p,0}(V^*C) \) we have that

\[ \overline{\sigma p \varphi} \wedge \overline{\varphi} = \sigma p \varphi \wedge \overline{\varphi}, \]

that is \( \sigma p \varphi \wedge \overline{\varphi} \) is a \((p, p)\)-real form. Consequently, denoting by

\[ \Lambda^{p,p}_{\mathbb{R}}(V^*C) = \{\psi \in \Lambda^{p,p}(V^*C) \mid \psi = \overline{\psi}\}, \]

we get that

\[
\{\sigma p \varphi^{i_1} \wedge \cdots \wedge \varphi^{i_p} \wedge \overline{\varphi^{j_1}} \wedge \cdots \wedge \overline{\varphi^{j_p}} \mid 1 \leq i_1 < \cdots < i_p \leq n\}
\]

is a basis of \( \Lambda^{p,p}_{\mathbb{R}}(V^*C) \).

**Remark 2.1** The complex structure \( J \) acts on the space of real \( k \)-covectors \( \Lambda^k(V^*) \) by setting, for any given \( \alpha \in \Lambda^k(V^*) \),

\[ J\alpha(V_1, \ldots, V_k) = \alpha(JV_1, \ldots, JV_k). \]

then it is immediate to check that if \( \psi \in \Lambda^{p,p}_{\mathbb{R}}(V^*C) \) then \( J\psi = \psi \). For \( k = 2 \), the converse holds.

Denoting by

\[ \text{Vol} = (\frac{i}{2}\varphi^1 \wedge \overline{\varphi^1}) \wedge \cdots \wedge (\frac{i}{2}\varphi^n \wedge \overline{\varphi^n}), \]

we obtain that

\[ \text{Vol} = \sigma_n \varphi^1 \wedge \cdots \wedge \varphi^n \wedge \overline{\varphi^1} \wedge \cdots \wedge \overline{\varphi^n}, \]

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that is Vol is a volume form on $V$. A real $(n, n)$-form $\psi$ is said to be positive respectively strictly positive if

$$\psi = a \text{Vol},$$

where $a \geq 0$, respectively $a > 0$. By definition, $\psi \in \Lambda^{p,0}(V^*\mathbb{C})$ is said to be simple or decomposable if

$$\psi = \eta^1 \wedge \cdots \wedge \eta^p,$$

for suitable $\eta^1, \ldots, \eta^p \in V^{1,0}$. Let $\Omega \in \Lambda^{p,p}_\mathbb{R}(V^*\mathbb{C})$. Then $\Omega$ is said to be transverse if, given any non-zero simple $(n-p)$-covector $\psi$, the real $(n, n)$-form

$$\Omega \wedge \sigma_{n-p} \psi \wedge \overline{\psi}$$

is strictly positive.

The notion of positivity on complex vector spaces can be transferred pointwise to almost complex manifolds. Let $(M, J)$ be an almost complex manifold of real dimension $2n$; we will denote by $A^{p,q}(M)$ the space of complex $(p, q)$-forms, that is the space of smooth sections of the bundle $\Lambda^{p,q}(M)$ and by $A^{p,p}_\mathbb{R}(M)$ the space of real $(p, p)$-forms.

**Definition 2.2** Let $(M, J)$ be an almost complex manifold of real dimension $2n$ and let $1 \leq p \leq n$. A $p$-Kähler form is a closed real transverse $(p, p)$-form $\Omega$, that is $\Omega$ is $d$-closed and, at every $x \in M$, $\Omega_x \in \Lambda^{p,p}_\mathbb{R}(T^*_x M)$ is transverse. The triple $(M, J, \Omega)$ is said to be an almost $p$-Kähler manifold.

### 3 Curves of Almost Complex Structures

Let $I = (-\varepsilon, \varepsilon)$ and $\{J_t\}_{t \in I}$ be a smooth curve of almost complex structures on $M$, such that $J_0 = J$. Then, for small $\varepsilon$, there exists a unique $L_t : TM \to TM$, with $L_t J + JL_t = 0$, for every $t \in I$, such that

$$J_t = (\text{Id} + L_t)J(\text{Id} + L_t)^{-1},$$

for every $t$ (see e.g., [4], [5, Sec. 3]). We can write $L_t = tL + o(t)$. Assume that $J$ is compatible with respect to a symplectic form $\omega$ on $M$, that is, at any given $x \in M$,

$$g_x(\cdot, \cdot) := \omega_x(\cdot, J \cdot)$$

is a positive definite Hermitian metric on $M$. Equivalently, $\omega$ is a positive $(1, 1)$-form with respect to $J$. Then $\omega$ is a positive $(1, 1)$-form with respect to $J_t$ if and only if $L_t$ is $g_J$-symmetric and $\|L_t\| < 1$.

We can show the following.
Proposition 3.1 Let $\omega$ be a positive $(1, 1)$-form on the complex vector space $\mathbb{C}^n$. Let $\{J_t\}$ be a curve of linear complex structures on $\mathbb{C}^n$. If $J_t$ preserves $\omega^p$ for all $t$, $1 \leq p < n$, then $J_t$ preserves $\omega$.

Proof We will show that a complex structure on $\mathbb{C}^n$ preserving $\omega^p$ preserves $\omega$ up to sign. Hence in a 1-parameter family with $J_0 = i$ all complex structures must preserve $\omega$ itself.

First note that a subspace $W \subset \mathbb{C}^n$ of real dimension at least $2p$ is symplectic if and only if $\omega^p$ is nondegenerate. Also, for a symplectic subspace of dimension at least $2p$ the symplectic complement can be defined either in the usual way as

$$W^\perp := \{v \in \mathbb{C}^n | \omega(v, w) = 0 \text{ for all } w \in W\}$$

or equivalently as

$$W^\perp := \{v \in \mathbb{C}^n | \omega^p(v, w_1, \ldots, w_{2p-1}) = 0 \text{ for all } w_1, \ldots, w_{2p-1} \in W\}.$$

Suppose then that $J$ is a complex structure preserving $\omega^p$ and $V$ is a 2-dimensional symplectic plane. Then $V^\perp = W$ is symplectic and hence $\omega^p|_W$ is nondegenerate. As $J$ preserves $\omega^p$ we have that $\omega^p|_{JW}$ is also nondegenerate and therefore also symplectic, and using the definition of $(JW)^\perp$ only in terms of $\omega^p$ we see that $(JW)^\perp = JV$. Hence $JV$ is also a symplectic plane. Moreover, if $U$ is a symplectic plane in $V^\perp$ then $JU \subset (JW)^\perp = JW = (JV)^\perp$.

Let $x_1, y_1, \ldots, x_n, y_n$ be a basis of $\mathbb{C}^n$ with $\omega(x_k, y_k) = 1$ for all $k$ and such that the symplectic planes $V_k = \text{Span}(x_k, y_k)$ are orthogonal (for example we can take the standard symplectic basis). Then by the remark at the end of the previous paragraph the planes $JV_k$ are also symplectically orthogonal. Set $\lambda_k = \omega(Jx_k, Jy_k)$. Since $J$ preserves $\omega^p$ we have that $\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_p} = 1$ for all $1 \leq i_1 < \cdots < i_p \leq n$. Hence either all $\lambda_k = 1$ or all $\lambda_k = -1$. (The second case is only possible when $p$ is even.) It follows that $J$ is either symplectic or anti-symplectic.

Notice that there exist exact Kähler structures on 3-dimensional compact complex manifolds, that is balanced metrics $g$, such that $\omega^2$ is $d$-exact, where $\omega$ denotes the fundamental form of $g$. This is in contrast with the almost Kähler case, in view of Stokes Theorem.

Example 3.2 Let $G = SL(2, \mathbb{C})$. Then $G$ admits compact quotients by uniform discrete subgroups $\Gamma$, so that

$$M = \Gamma \backslash G$$

is a 3-dimensional compact complex manifold. Denote by

$$Z_1 = \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad Z_2 = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Z_3 = \frac{1}{2} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix};$$

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then \( \{Z_1, Z_2, Z_3\} \) is a basis of the Lie algebra of \( G \). We have

\[
[Z_1, Z_2] = -Z_3, \quad [Z_1, Z_3] = Z_2, \quad [Z_2, Z_3] = -Z_1.
\]

Accordingly, the dual left-invariant coframe \( \{\psi^1, \psi^2, \psi^3\} \) satisfies the Maurer-Cartan equations

\[
d\psi^1 = \psi^2 \wedge \psi^3, \quad d\psi^2 = -\psi^1 \wedge \psi^3, \quad d\psi^3 = \psi^1 \wedge \psi^2.
\]

Let

\[
\Omega = \frac{1}{4}(\psi^{12\bar{1}} + \psi^{13\bar{1}} + \psi^{23\bar{3}}),
\]

where we indicated \( \psi^{r\bar{s}} = \psi^r \wedge \bar{\psi}^s \). Then,

\[
\Omega = \frac{1}{8}d(\psi^{12\bar{3}} + \psi^{1\bar{2}3} - \psi^{13\bar{2}} - \psi^{\bar{1}32} + \psi^{23\bar{1}} + \psi^{2\bar{3}1}) := dy
\]

In [14, p.467] it is proved that every left-invariant \((2, 2)\)-form is \(d\)-exact (see also [15] for cohomological computations). Let us define a complex curve of almost complex structures on \( \Gamma \setminus G \) through a basis of \((1, 0)\) forms by setting

\[
\psi^1_t = \psi^1, \quad \psi^2_t = \psi^2, \quad \psi^3_t = \psi^3 - t\bar{\psi}^3,
\]

for \( t \in \mathbb{B}(0, \varepsilon) \). A direct computation gives

\[
\begin{align*}
\frac{d\psi^1_t}{\psi^1_t} &= \frac{1}{1-|t|^2}(\psi^{12\bar{3}} + t\psi^{23\bar{3}}) \\
\frac{d\psi^2_t}{\psi^2_t} &= -\frac{1}{1-|t|^2}(\psi^{13\bar{2}} + t\psi^{\bar{1}32}) \\
\frac{d\psi^3_t}{\psi^3_t} &= \psi^{12\bar{3}} - t\psi^{23\bar{1}}.
\end{align*}
\]

In particular, the last equation shows that \( J_t \) is not integrable for \( t \neq 0 \). Now we compute the action of \( J_t \) on the forms \( \psi^1, \psi^2, \psi^3 \). A straightforward calculation yields to

\[
J_t\psi^1 = i\psi^1, \quad J_t\psi^2 = i\psi^2, \quad J_t\psi^3 = \frac{i}{1 - |t|^2}(\psi^3 - t\bar{\psi}^3)
\]

and

\[
J_t\bar{\psi}^1 = -i\bar{\psi}^1, \quad J_t\bar{\psi}^2 = -i\bar{\psi}^2, \quad J_t\bar{\psi}^3 = \frac{i}{1 - |t|^2}(-\psi^3 + 2t\psi^3)
\]

Therefore, it immediate to check that

\[
J_t\Omega = \Omega,
\]
so that $\Omega$ is $(2, 2)$-with respect to $J_t$ for every $t$.

Finally, there are no symplectic structures taming $J_t$, for every given $t \in \mathbb{B}(0, \varepsilon)$. By contradiction: assume that there exist a symplectic structure $\omega_t$ on $\Gamma \setminus SL(2, \mathbb{C})$ taming $J_t$. Then, since for every $t$ the almost complex structure is left invariant, by an average process, we may produce a left invariant symplectic structure $\hat{\omega}$ on $\Gamma \setminus SL(2, \mathbb{C})$, taming $J_t$. Let $\hat{\omega}$ be given as

$$2\hat{\omega} = i A \psi_t^{1\bar{1}} + i B \psi_t^{2\bar{2}} + i C \psi_t^{3\bar{3}} + i u \psi_t^{1\bar{2}} - \bar{u} \psi_t^{2\bar{1}} + v \psi_t^{1\bar{3}} - \bar{v} \psi_t^{3\bar{1}} + w \psi_t^{2\bar{3}} - \bar{w} \psi_t^{3\bar{2}},$$

where $A, B, C, u, v, w \in \mathbb{C}$. Then, a direct calculation using $(3)$ gives that, if $\hat{\omega}$ is closed, then $C = 0$. This is absurd.

Therefore, we have proved the following.

**Proposition 3.3** For any given $t \in \mathbb{B}(0, \varepsilon) \subset \mathbb{C}$, $(\Gamma \setminus SL(2, \mathbb{C}), J_t, \Omega)$ is a compact almost $2$-Kähler manifold of complex dimension $3$, such that:

i) the almost complex structure $J_t$ is integrable if and only if $t = 0$;

ii) the almost $2$-Kähler structure $\Omega$ is $d$-exact;

iii) $J_t$ has no tamed symplectic structures for every given $t \in \mathbb{B}(0, \varepsilon)$.

We end this Section proving a result of non-existence of almost $p$-Kähler structures.

**Proposition 3.4** Let $(M, J)$ be a closed almost complex manifold of (complex) dimension $n$.

Suppose $\alpha$ is a non closed $1$-form such that the $(1, 1)$ part

$$(d\alpha)^{1,1} = \sum_k c_k \psi_k \wedge \bar{\psi}_k$$

where the $\psi_k$ are $(1, 0)$-covectors and the $c_k$ have the same sign. Then $(M, J)$ does not have a balanced metric.

More generally, suppose there exists a non closed $(2n - 2p - 1)$-form $\beta$ such that

$$(d\beta)^{n-p,n-p} = \sum_k c_k \psi_k \wedge \bar{\psi}_k$$

where the $\psi_k$ are simple $(n - p, 0)$-covectors and the $c_k$ have the same sign. Then $(M, J)$ does not admit an almost $p$-Kähler form.

**Proof** It suffices to prove the second statement. Without loss of generality, we may assume that all the $c_k$ are positive. Arguing by contradiction, suppose that $\Omega$ is a $p$-Kähler form and $\beta$ is a non-zero $(2n - 2p - 1)$-form as above. Then since $M$ is closed we have

$$0 = \int_M \sigma_{n-p} d(\Omega \wedge \beta) = \sum_k c_k \int_M \Omega \wedge \sigma_{n-p} \psi_k \wedge \bar{\psi}_k > 0$$

as all integrals on the right are strictly positive. This gives a contradiction. $\square$
Remark 3.5 All Riemann surfaces are almost Kähler. Therefore if a 1-form $\alpha$ as in Proposition 3.4 exists, it must restrict to a closed form on all 1-dimensional subvarieties of $M$.

As an application of Proposition 3.4, we provide a family of $n$-dimensional compact complex manifolds which are not $p$-Kähler, for any given $1 \leq p \leq (n - 1)$.

Example 3.6 Let

$$\mathbb{H}_{2n-1}(\mathbb{R}) := \left\{ A = \begin{bmatrix} 1 & X & v \\ 0 & I_{n-1} & Y \\ 0 & 0 & 1 \end{bmatrix} \mid X^t, Y \in \mathbb{R}^{n-1}, \ v \in \mathbb{R} \right\}$$

be the $(2n - 1)$-dimensional real Heisenberg group, where $I_{n-1}$ denotes the identity matrix of order $n - 1$. Then $\mathbb{H}_{2n-1}(\mathbb{R})$ is $(2n - 1)$-dimensional nilpotent Lie group and the subset $\Gamma \subset \mathbb{H}_{2n-1}(\mathbb{R})$, formed by matrices having integers entries, is a uniform discrete subgroup of $\mathbb{H}_{2n-1}(\mathbb{R})$, so that $\Gamma \backslash \mathbb{H}_{2n-1}(\mathbb{R})$ is a compact $(2n - 1)$-dimensional nilmanifold. Then

$$M = \Gamma \backslash \mathbb{H}_{2n-1}(\mathbb{R}) \times \mathbb{R}/\mathbb{Z}$$

is a $2n$-dimensional compact nilmanifold having a global coframe $\{e^1, \ldots, e^n, f^1, \ldots, f^n\}$, defined as

$$e^\alpha = dx^\alpha, \ 1 \leq \alpha \leq n - 1, \ e^n = du$$

$$f^\alpha = dy^\alpha, \ 1 \leq \alpha \leq n - 1, \ f^n = dv - \sum_{\beta=1}^{n-1} x^\beta dy^\beta,$$

where $u$ denotes the natural coordinate on $\mathbb{R}$. It is immediate to check that

$$\begin{cases} 
  de^\alpha = 0 & 1 \leq \alpha \leq n \\
  df^\beta = 0 & 1 \leq \beta \leq n - 1 \\
  df^n = -\sum_{\gamma=1}^{n-1} e^\gamma \wedge f^\gamma 
\end{cases} \quad (4)$$

Then

$$\varphi^\alpha = e^\alpha + if^\alpha, \ 1 \leq \alpha \leq n$$

give rise to a complex coframe of $(1, 0)$-forms on $M$ such that

$$\begin{cases} 
  d\varphi^\alpha = 0 & 1 \leq \alpha \leq n - 1 \\
  d\varphi^n = \frac{1}{2} \sum_{\beta=1}^{n-1} \varphi^\beta \wedge \overline{\varphi^\beta} 
\end{cases}$$
so that induced almost complex structure $J$ is integrable. Fix any $1 \leq p \leq (n - 1)$. We show that $(M, J)$ is not $p$-Kähler. Define

$$\beta = \varphi^n \wedge \varphi^1 \cdots (n-p-1)(n-p-1).$$

Then,

$$d\beta = (d\beta)^{n-p,n-p} = \left( \sum_{\beta=1}^{n-1} \varphi^\beta \beta \right) \wedge \varphi^1 \cdots (n-p-1)(n-p-1)$$

and the result follows from Proposition 3.4.

Note that the smooth manifold $M$ does not carry any Kähler structure $(J, g, \omega)$, since it is a non toral nilmanifold.

4 Semi-Kähler Deformations of Balanced Metrics

Let $M = \mathbb{C}^n$ endowed with a balanced structure $(J, g, \omega)$, that is $J$ is a complex structure on $\mathbb{C}^n$, $g$ is a Hermitian metric such that the fundamental form $\omega$ of $g$ satisfies $d\omega^{n-1} = 0$. Denote by $\{\varphi^1, \ldots, \varphi^n\}$ be a complex $(1, 0)$-coframe on $(\mathbb{C}^n, J)$, and let

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} \omega_{jk}(z) \varphi^j \wedge \overline{\varphi^k},$$

be the fundamental form of $g$, where $(\omega_{jk}(z))$ is Hermitian and positive definite. Then, assuming $\omega$ is a balanced metric,

$$\Omega := \frac{1}{(n-1)!} \omega^{n-1}$$

is $d$-closed. By assumption $d\Omega = 0$. Denote by $\{\zeta_1, \ldots, \zeta_n\}$ the dual $(1, 0)$-frame of $\{\varphi^1, \ldots, \varphi^n\}$. Let $I = (-\epsilon, \epsilon)$ and let $\{J_t\}_{t \in I}$ be a smooth curve of almost complex structures on $\mathbb{C}^n$ such that $J_0 = J$. Then, as already recalled in Sect. 3, there exists a unique $L_t \in \text{End}(T \mathbb{C}^n)$ such that $L_t J + J L_t = 0$ and

$$J_t = (I + L_t)J(I + L_t)^{-1}.$$
In other words, the curve of almost complex structures \( \{ J_t \}_{t \in I} \) is encoded by such a \( \Phi(t) \in \Gamma(\mathbb{C}^n, \Lambda^{0,1} \mathbb{C}^n \otimes T^{1,0} \mathbb{C}^n) \) so that, if the expression of \( \Phi(t) \) is

\[
\Phi(t) = \sum_{h,k=1}^{n} \sigma^j_k(z,t) \bar{\varphi}^k \otimes \xi_j,
\]

with \( \sigma^j_k = \sigma^j_k(z,t) \) smooth on \((z,t)\), then a complex \((1,0)\)-coframe on \((\mathbb{C}^n, J_t)\) is given by

\[
\varphi^j_t = \varphi^j - \langle \Phi(t), \xi_j \rangle, \quad j = 1, \ldots, n,
\]

where

\[
\langle \Phi(t), \xi_j \rangle = \sum_{k=1}^{n} \sigma^j_k \bar{\varphi}^k
\]

Then, explicitly

\[
\varphi^j_t = \varphi^j - \sum_{k=1}^{n} \sigma^j_k \bar{\varphi}^k, \quad j = 1, \ldots, n. \quad (5)
\]

According to the Kodaira and Spencer theory of small deformations of complex structures (see [12], [7]) \( J_t \) is integrable if and only if the Maurer-Cartan equation holds, that is

\[
\bar{\partial} \Phi(t) + \frac{1}{2} [[\Phi(t), \Phi(t)]] = 0.
\]

Here, we are not assuming that \( J_t \) is integrable. Let \((J_t, g_t, \omega_t)\) be a curve of almost Hermitian metrics on \( \mathbb{C}^n \) such that \((J_0, g_0, \omega_0) = (J, g, \omega)\). Then,

\[
\omega_t := \frac{i}{2} \sum_{j,k=1}^{n} \omega_{jk}(z,t) \varphi^j_t \wedge \bar{\varphi}^k_t; \quad (6)
\]

where \( \omega_{jk}(z,t) \) are smooth and \( \omega_{jk}(z,0) = \omega_{jk}(z), \ j, k = 1, \ldots, n. \)

In the sequel we will use the symbol \( \dot{\cdot} \) to denote the derivative with respect to \( t \), e.g., we will use the following notation

\[
\dot{\varphi}^j_0 = \frac{d}{dt} \varphi^j_t \big|_{t=0}
\]
and we will drop 0 for the \( t \) derivative of the functions \( \sigma^j_k(z_1, \ldots, z_n, t) \) evaluated at \( t = 0 \), that is

\[
\sigma^j_k = \frac{d}{dt} \sigma^j_k(z_1, \ldots, z_n, t)|_{t=0}.
\]

Set

\[
\Omega_t := \frac{1}{(n-1)!} \omega_t^{n-1}.
\]  \hspace{1cm} (7)

Let us compute the \( t \) derivative of \( \Omega_t \) at \( t = 0 \). In view of (5), we easily compute

\[
\dot{\psi}^j_0 = - \sum_{k=1}^n \sigma^j_k \psi^k,
\]

by the definition of \( \Omega_t \), we obtain that

\[
\dot{\Omega}_0 \in A^{n,n-2,\mathbb{C}^n} \oplus A^{n-1,n-1,\mathbb{C}^n} \oplus A^{n-2,n,\mathbb{C}^n}, \quad \dot{\Omega}_0 = \overline{\Omega}_0.
\]

Consequently, we can define the \((n - 2, n)\)-form \( \eta \) and the \((n - 1, n - 1)\)-form \( \lambda \) on \((\mathbb{C}^n, J_t)\) respectively by

\[
\eta := (\dot{\Omega}_0)^{n-2,n}
\]  \hspace{1cm} (8)

and

\[
\lambda := (\dot{\Omega}_0)^{n-1,n-1}.
\]  \hspace{1cm} (9)

We have

\[
\dot{\Omega}_0 = \eta + \overline{\eta} + \lambda.
\]  \hspace{1cm} (10)

We are ready to state the following.

**Theorem 4.1** Let \((J, g, \omega)\) be a balanced structure on \(\mathbb{C}^n\). Let \((J_t, g_t, \omega_t)\), for \( t \in I \), be a curve of almost Hermitian metrics on \(\mathbb{C}^n\) such that \((J_0, g_0, \omega_0) = (J, g, \omega)\). If \((J_t, g_t, \omega_t)\) is a curve of semi-Kähler structures on \(\mathbb{C}^n\), then

\[
\partial \eta + \overline{\partial} \lambda = 0.
\]  \hspace{1cm} (11)

**Proof** By definition, \(\Omega_t := \frac{1}{(n-1)!} \omega_t^{n-1}\) and by assumption,

\[
d\Omega_t = 0.
\]  \hspace{1cm} (12)
Thus, by taking the derivative of (12) with respect to \( t \) evaluated at \( t = 0 \) and taking into account (10), we obtain

\[
0 = d\Omega_0 = d(\eta + \bar{\eta} + \lambda) = (\partial + \bar{\partial})(\eta + \bar{\eta} + \lambda) = \partial\eta + \bar{\partial}\bar{\eta} + \partial\lambda + \bar{\partial}\lambda,
\]

where we have used that \( J_t \) is integrable for \( t = 0 \) and that \( \eta \in A^{n-2,n}\mathbb{C}^n \). Therefore, the above equation

\[
\partial\eta + \bar{\partial}\bar{\eta} + \partial\lambda + \bar{\partial}\lambda = 0
\]

turns to be equivalent, by type reasons, to

\[
\partial\eta + \bar{\partial}\lambda = 0,
\]

that is, if \( \Omega_t \) is d-closed, then (11) holds. The Theorem is proved. \( \square \)

Let \( M \) be a compact holomorphically parallelizable complex manifold. Then, by a Theorem of Wang (see [20]), there exists a simply-connected, connected complex Lie group \( G \) and a lattice \( \Gamma \subset G \) such that \( M = \Gamma \setminus G \). Assume that \( M \) is a solvmanifold, that is \( G \) is solvable. Then, according to Nakamura [13, Prop. 1.4], the universal covering of \( M \) is biholomorphically equivalent to \( \mathbb{C}^N \). Due to Abbena and Grassi [1, Theorem 3.5], the natural complex structure \( J \) on \( M \) admits a balanced metric \( g \). As a direct consequence of Theorem 4.1, we get the following (see also Sferruzza [18])

**Corollary 4.2** Let \( M = \Gamma \setminus \mathbb{C}^N \) be a compact complex solvmanifold endowed with the natural balanced structure \((J, g, \omega)\). If \((J_t, g_t, \omega_t)\) is a curve of semi-Kähler structures on \( M \) such that \((J_0, g_0, \omega_0) = (J, g, \omega)\), then

\[
0 = [\partial\eta]_{\bar{\partial}} \in H^{N-1,N}_{\bar{\partial}}(M).
\]  

(13)

Coming back to \( \mathbb{C}^n \), in the particular case that

\[
\omega_t := \frac{i}{2} \sum_{j=1}^{n} \varphi_t^j \wedge \overline{\varphi_t^j},
\]  

(14)

we obtain the following.

**Proposition 4.3** Let \((J, g, \omega)\) be a balanced structure on \( \mathbb{C}^n \). Let \( \{J_t\}_{t \in I} \) be a smooth curve of almost complex structures on \( \mathbb{C}^n \) such that \( J_0 = J \). Let \( \omega_t \) be the real \( J_t \)-positive \((1, 1)\)-form defined by (14) and \( g_t \) be the associated \( J_t \)-Hermitian metric on \( \mathbb{C}^n \). If \((J_t, g_t, \omega_t)\) is a curve of semi-Kähler structures on \( \mathbb{C}^n \), then

\[
\partial\eta = 0.
\]  

(15)

The proof of the above Proposition follows at once from Theorem 4.1 by noting that in such a case \( \lambda = 0 \).

Finally, under the same assumptions as in the last Proposition 4.3, for \( n = 3 \), we derive the following.
Corollary 4.4 Let \((J, g, \omega)\) be a balanced structure on \(\mathbb{C}^3\). Let \(\{J_t\}_{t \in I}\) be a smooth curve of almost complex structures on \(\mathbb{C}^3\) such that \(J_0 = J\). Let \(\omega_t\) be the real \(J_t\)-positive \((1, 1)\)-form defined by (14) and \(g_t\) be the associated \(J_t\)-Hermitian metric on \(\mathbb{C}^3\). If \((J_t, g_t, \omega_t)\) is a curve of semi-Kähler structures on \(\mathbb{C}^3\), then

\[
\partial \left( \left( (\dot{\sigma}_2^2 - \dot{\sigma}_1^2) \varphi^1 + (\dot{\sigma}_1^3 - \dot{\sigma}_3^1) \varphi^2 + (\dot{\sigma}_2^1 - \dot{\sigma}_1^2) \varphi^3 \right) \wedge \varphi^{133} \right) = 0. \tag{16}
\]

Corollary 4.5 In the same assumptions as in the Corollary 4.4, under the additional assumption that \(\varphi^j = dz_j, j = 1, 2, 3\), if \((J_t, g_t, \omega_t)\) is a curve of semi-Kähler structures on \(\mathbb{C}^3\), then the following equations hold

\[
\begin{aligned}
\frac{\partial}{\partial z_1} (\dot{\sigma}_1^3 - \dot{\sigma}_3^1) - \frac{\partial}{\partial z_3} (\dot{\sigma}_3^2 - \dot{\sigma}_3^2) &= 0 \\
\frac{\partial}{\partial y_1} (\dot{\sigma}_2^1 - \dot{\sigma}_1^2) - \frac{\partial}{\partial y_2} (\dot{\sigma}_2^2 - \dot{\sigma}_2^3) &= 0 \\
\frac{\partial}{\partial z_2} (\dot{\sigma}_1^1 - \dot{\sigma}_1^2) - \frac{\partial}{\partial z_3} (\dot{\sigma}_1^3 - \dot{\sigma}_2^2) &= 0
\end{aligned} \tag{17}
\]

Remark 4.6 The previous constructions can be easily adapted replacing the parameter space \(I = (-\varepsilon, \varepsilon)\), with

\[
\mathcal{B}(0, \varepsilon) = \{ t = (t_1, \ldots, t_k) \in \mathbb{R}^k : |t| < \varepsilon. \}
\]

In particular, Theorem 4.1 can be generalized by considering

\[
\eta_j = \left( \frac{\partial}{\partial t_j} \Omega \big|_{t=0} \right)^{n-2,n}, \quad \lambda_j = \left( \frac{\partial}{\partial t_j} \Omega \big|_{t=0} \right)^{n-1,n-1}.
\]

5 Applications and Examples

First, we construct a family of semi-Kähler structures on the 6-dimensional torus \(\mathbb{T}^6\), obtained as a deformation of the standard Kähler structure on \(\mathbb{T}^6\), which cannot be locally compatible with any symplectic form. More precisely, we start by showing the following.

Theorem 5.1 Let \((J, \omega)\) be the standard Kähler structure on the standard torus \(\mathbb{T}^6 = \mathbb{R}^6 / \mathbb{Z}^6\), with coordinates \((z_1, z_2, z_3)\), \(z_j = x_j + iy_j, j = 1, 2, 3\). Let \(f = f(z_2, \bar{z}_2)\) be a \(\mathbb{Z}^6\)-periodic smooth complex valued function on \(\mathbb{R}^6\) and set \(f = u + iv, \) where \(u = u(x_2, y_2), v = v(x_2, y_2).\)

There is an almost complex structure \(J = J(f)\) on \(\mathbb{T}^6\) such that

i) If \(f = 0\) then \(J = J(0)\) is the standard complex structure;

ii) \(J\) admits a semi-Kähler metric provided \(f\) is sufficiently small (in the uniform norm);

iii) Suppose that \(u, v\) satisfy the following condition

\[
\left( \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial y_2} \right)_{(x,y)=0} \neq 0. \tag{18}
\]
Then, $J$ is not locally compatible with respect to any symplectic form on $\mathbb{T}^6$.

**Proof**

i) Define

\[
\begin{align*}
J \partial x_1 &= 2v \partial x_3 + \partial y_1 - 2u \partial y_3 \\
J \partial x_2 &= \partial y_2 \\
J \partial x_3 &= \partial y_3 \\
J \partial y_1 &= -\partial x_1 - 2u \partial x_3 - 2v \partial y_3 \\
J \partial y_2 &= -\partial x_1 \\
J \partial y_3 &= -\partial x_3
\end{align*}
\]

By definition of $J$, in view of the $\mathbb{Z}^6$-periodicity of the functions $u, v$ we immediately get that $J$ gives rise to an almost complex structure on $\mathbb{T}^6$, such that $J(0)$ coincides with the standard complex structure on $\mathbb{T}^6$.

ii) The almost complex structure defined by (19) induces an almost complex structure on the cotangent bundle of $\mathbb{T}^6$, still denoted by $J$ and expressed as

\[
\begin{align*}
Jdx_1 &= -dy_1 \\
Jdx_2 &= -dy_2 \\
Jdx_3 &= -dy_3 + 2vdx_1 + \partial y_1 - 2udy_1 \\
Jdy_1 &= dx_1 \\
Jdy_2 &= dx_2 \\
Jdy_3 &= dx_3 - 2udx_1 - 2vdy_1
\end{align*}
\]

Accordingly, a $(1, 0)$-coframe on $\mathbb{T}^6$ with respect to $J$ is given by

\[
\varphi^1 = dz_1, \quad \varphi^2 = dz_2, \quad \varphi^3 = dz_3 - fd\bar{z}_1.
\]

Thus,

\[
\sigma^3_1 = f(z_2, \bar{z}_2),
\]

with the other $\sigma^i_k$ vanishing. Then, for $f$ small enough, $J$ admits a semi-Kähler metric $g$, whose fundamental form is provided by

\[
\omega = \frac{i}{2} \sum_{j=1}^{3} \varphi^j \wedge \overline{\varphi^j}
\]

iii) To show that the almost complex structure admits no locally compatible symplectic forms when (18) holds, we need to recall the following result.

Let $P$ be an almost complex structure on $\mathbb{R}^6$, with coordinates $(x_1, \ldots, x_6)$. Then, according to [11, Theorem 2.4], if $P$ is locally compatible with respect to a symplectic
form, then the following necessary conditions hold
\[
\begin{aligned}
-\frac{\partial}{\partial x_1}(P_{26} - P_{62}) - \frac{\partial}{\partial x_2}(P_{16} - P_{61}) - \frac{\partial}{\partial x_3}(P_{15} - P_{51}) \\
-\frac{\partial}{\partial x_4}(P_{23} - P_{32}) + \frac{\partial}{\partial x_5}(P_{13} - P_{31}) - \frac{\partial}{\partial x_6}(P_{12} - P_{21}) = 0 \\
-\frac{\partial}{\partial x_4}(P_{26} - P_{62}) + \frac{\partial}{\partial x_5}(P_{16} - P_{61}) - \frac{\partial}{\partial x_6}(P_{15} - P_{51}) = 0,
\end{aligned}
\]
where all the derivatives are computed at \( x = 0 \). In our notation, \( x_4 = y_1, x_5 = y_2, \)
\( x_6 = y_3 \).

Now, by the assumption (18)
\[
\left( \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial y_2} \right)\bigg|_{(x,y)=0} \neq 0
\]
on the partial derivatives at \((x,y) = 0\) of \( f = u + iv = u(x_2, y_2) + iv(x_2, y_2) \), so we immediately see that the first equation of (21) is not satisfied. Therefore, \( J(f) \) cannot be locally compatible with any symplectic form. \( \square \)

**Remark 5.2** We can generate 1-parameter families of almost complex structures by setting \( J_t = J(tf) \). These show that the necessary condition (17) of Corollary 4.5 is also sufficient in this case. Indeed, in such a case the natural \( \omega_t = i\sum_{j=1}^3 \varphi^j_t \wedge \bar{\varphi}^j_t \) satisfies \( d\omega_t^2 = 0 \).

As a consequence of iii), for any given \( t \neq 0 \), \( J_t \) is not integrable.

**Example 5.3** (Iwasawa manifold) Let \( \mathbb{C}^3 \) be endowed with the product \(*\) defined by
\[
(w_1, w_2, w_3) * (w_1', w_2', w_3') = (w_1 + z_1, w_2 + z_2, w_3 + w_1z_2 + z_3).
\]
Then \((\mathbb{C}^3, *)\) is a complex nilpotent Lie group which admits a lattice \( \Gamma = \mathbb{Z}[i]^3 \) and accordingly it turns out that \( \mathbb{I}_3 = \Gamma \backslash \mathbb{C}^3 \) is a compact complex 3-dimensional manifold, the Iwasawa manifold. It is immediate to check that
\[
\{ \varphi^1 = dz_1, \varphi^2 = dz_2, \varphi^3 = dz_3 - z_1dz_2 \}
\]
is a complex \((1,0)\)-coframe for the standard complex structure naturally induced by \( \mathbb{C}^3 \), whose dual frame is
\[
\{ \xi_1 = \frac{\partial}{\partial z_1}, \xi_2 = \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_3}, \xi_3 = \frac{\partial}{\partial z_3} \}.
\]
The following
\[
g = \sum_{j=1}^3 \varphi^j \circ \bar{\varphi}^j
\]
is a balanced metric on \( \mathbb{I}_3 \). Indeed, the fundamental form of \( g \) is
\[
\omega = \frac{i}{2} (dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 - z_1 dz_2 \wedge d\overline{z}_3 - \overline{z}_1 dz_3 \wedge d\overline{z}_2 + dz_3 \wedge d\overline{z}_3),
\]
which satisfies \( d\omega^2 = 0 \).

We will provide a smooth curve of almost complex structures \( \{J_t\}_{t \in I} \) such that \( J_0 \) coincides with the complex structure on \( \mathbb{I}_3 \) and such that, for any \( t \neq 0 \), \( J_t \) admits no semi-Kähler metric. To this purpose let us define \( \{J_t\}_{t \in I} \) on \( \mathbb{I}_3 \) by assigning
\[
\Phi_1(t) = t \varphi_1^1 \otimes \zeta_2 - t \varphi_2^2 \otimes \xi_1.
\]
Then, accordingly,
\[
\{\varphi_t^1 = dz_1 + td\overline{z}_2, \quad \varphi_t^2 = dz_2 - td\overline{z}_1, \quad \varphi_t^3 = dz_3 - z_1 dz_2\}
\]
is a complex \((1, 0)\)-coframe on \( (\mathbb{I}_3, J_t) \). A simple calculation yields to the following structure equations
\[
d\varphi_t^1 = 0, \quad d\varphi_t^2 = 0, \quad d\varphi_t^3 = -\frac{1}{1 + t^2} \left[ \varphi_t^{12} + t(\varphi_t^{1\overline{1}} + \varphi_t^{2\overline{2}}) + t^2 \varphi_t^{1\overline{2}} \right]
\]
(22)
The last equation implies that \( J_t \) is not integrable for \( t \neq 0 \). First we observe that, by the definition of \( \Phi(t) \),
\[
\sigma_1^2 = t, \quad \sigma_2^1 = -t, \quad \sigma_j^k = 0, \text{ otherwise.}
\]
Hence
\[
\partial \left( \left( (\dot{\sigma}_3^2 - \dot{\sigma}_2^3)\varphi^1 + (\dot{\sigma}_1^3 - \dot{\sigma}_3^1)\varphi^2 + (\dot{\sigma}_2^1 - \dot{\sigma}_1^2)\varphi^3 \right) \wedge \varphi^{123} \right) = 2\partial\varphi^{1\overline{2}\overline{3}} = -2\varphi^{1\overline{2}\overline{3}}
\]
A simple calculation shows that
\[
0 \neq [-2\varphi^{1\overline{2}\overline{3}}]_{\overline{\sigma}} \in H^2_{\overline{\sigma}}(\mathbb{I}_3),
\]
that is the necessary condition
\[
0 \neq [\partial \eta]_{\overline{\sigma}} \in H^2_{\overline{\sigma}}(\mathbb{I}_3)
\]
of Corollary 4.2 is not satisfied. Indeed, the form
\[
-2\varphi^{1\overline{2}\overline{3}}
\]

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is $\overline{\partial}$-harmonic with respect to the Hermitian metric

$$g = \sum_{j=1}^{3} \phi^j \otimes \overline{\phi^j}$$

on $\mathbb{I}_3$. In fact a stronger statement is true. We show that $J_t$ does not admit any semi-Kähler metric for $t \neq 0$. By the third equation of (22), we immediately get that

$$\left( d\phi_t^3 \right)^{1,1} = -\frac{t}{1+t^2} \left( \phi_t^{1\overline{1}} + \phi_t^{2\overline{2}} \right)$$

Then, Proposition 3.4 applies, proving the following.

**Proposition 5.4** The curve $\{J_t\}_{t \in I}$ of almost complex structures on $\mathbb{I}_3$ satisfies the following:

i) $J_0$ coincides with the natural complex structure on $\mathbb{I}_3$ and admits a balanced metric.

ii) For any given $t \neq 0$, $J_t$ is not integrable and it has no semi-Kähler metrics.

**Example 5.5** (A family of almost 2-Kähler structures on $\mathbb{C}^4$) Let $M = \mathbb{C}^4$ with real coordinates $(x_1, \ldots, x_4, y_1, \ldots, y_4)$ and $g$ be a smooth real valued function on $\mathbb{C}^4$. Define an almost complex structure $\mathcal{J} = \mathcal{J}_g$ on $\mathbb{C}^4$ by setting

$$\begin{align*}
\mathcal{J} \partial x_1 &= g \partial x_3 + \partial y_1 \\
\mathcal{J} \partial x_2 &= \partial y_2 \\
\mathcal{J} \partial x_3 &= \partial y_3 \\
\mathcal{J} \partial x_4 &= \partial y_4 \\
\mathcal{J} \partial y_1 &= -\partial x_1 - g \partial y_3 \\
\mathcal{J} \partial y_2 &= -\partial x_2 \\
\mathcal{J} \partial y_3 &= -\partial x_3 \\
\mathcal{J} \partial y_4 &= -\partial x_4
\end{align*}$$

Then, a straightforward calculation shows that

$$\begin{align*}
\Phi^1 &= dx_1 + idy_1, \\
\Phi^2 &= dx_2 + idy_2, \\
\Phi^3 &= dx_3 + i(-gdx_1 + dy_3), \\
\Phi^4 &= dx_4 + idy_4
\end{align*}$$

is a complex $(1, 0)$-coframe on $(\mathbb{C}^4, \mathcal{J})$. Define

$$\Omega = \frac{1}{4} \sum_{j<k} \Phi^{ijk\overline{k}}$$
and set

$$\Omega_\tau = \Omega + \tau \text{Re} \Phi^{234\bar{4}}$$

We have the following.

**Lemma 5.6** The form $\Omega_\tau$ is $(2, 2)$ with respect to $\mathcal{J}$ and positive for every $\tau \in (-\varepsilon, \varepsilon)$, for $\varepsilon$ small enough (independently of $g$). Furthermore, assume that $g = g(x_2, x_3)$. Then, for any given $\tau$, we have that

$$d\Omega_\tau = 0 \iff \frac{\partial g}{\partial x_2} - 2\tau \frac{\partial g}{\partial x_3} = 0$$

**Proof** By (24) we obtain

$$\Phi^{112\bar{2}} = -4dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2, \quad \Phi^{113\bar{3}} = -4dx_1 \wedge dy_1 \wedge dx_3 \wedge dy_3$$

$$\Phi^{114\bar{4}} = -4dx_1 \wedge dy_1 \wedge dx_4 \wedge dy_4, \quad \Phi^{234\bar{4}} = -4dx_2 \wedge dy_2 \wedge dx_4 \wedge dy_4$$

$$\Phi^{233\bar{3}} = 4 (gdx_1 \wedge dx_2 \wedge dx_3 \wedge dy_2 - dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3)$$

$$\Phi^{334\bar{4}} = -4 (gdx_1 \wedge dx_3 \wedge dx_4 \wedge dy_4 + dx_3 \wedge dy_3 \wedge dx_4 \wedge dy_4)$$

In view of the above formulae, since $g = g(x_2, x_3)$, we get

$$d\Omega = \frac{1}{4} d \sum_{j < k} \Phi^{j\bar{j}k\bar{k}} = \frac{\partial g}{\partial x_2} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_4 \quad (25)$$

Expanding $\Phi^{234\bar{4}}$, we obtain

$$\text{Re} \Phi^{234\bar{4}} = 2 (gdx_1 \wedge dx_2 \wedge dx_4 \wedge dy_4 - dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3 +$$

$$dx_3 \wedge dx_4 \wedge dy_4)$$

Consequently,

$$d\text{Re} \Phi^{234\bar{4}} = -2 \frac{\partial g}{\partial x_3} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_4 \quad (26)$$

Taking into account the definition of

$$\Omega_\tau = \Omega + \tau \text{Re} \Phi^{234\bar{4}},$$

the Lemma follows immediately from (25) and (26). \qed

We are ready to state and prove the following.
Theorem 5.7 Let $g$ be a smooth real valued function on $\mathbb{C}^4$ such that $g = g(x_2, x_3)$. Assume that, for some $\tau \in (-\varepsilon, \varepsilon)$ the function $g$ satisfies the partial differential equation

$$\frac{\partial g}{\partial x_2} - 2\tau \frac{\partial g}{\partial x_3} = 0.$$ 

Let $\mathcal{J}$ be the almost complex structure on $\mathbb{C}^4$ associated with $g$. Then,

i) $(\mathcal{J}, \Omega_\tau)$ is an almost 2-Kähler structure on $\mathbb{C}^4$.

ii) If $\frac{\partial g}{\partial x_2}|_{(x, y)} \neq 0$, then $\mathcal{J}$ is not locally compatible with respect to any symplectic form on $\mathbb{C}^4$.

Proof i) By Lemma 5.6 and by assumption on $g$, we immediately obtain that $(\mathcal{J}, \Omega_\tau)$ is an almost 2-Kähler structure on $\mathbb{C}^4$.

ii) We will apply again [11, Theorem 2.4].

Let $\mathbb{R}^6 \cong \mathbb{C}^3_{z_1, z_2, z_3}$, where $z_j = x_j + iy_j, \ j = 1, 2, 3$. Then, the restriction $\mathcal{J}|_{\mathbb{R}^6}$ of $\mathcal{J}$ to $\mathbb{R}^6$ gives rise to an almost complex structure on $\mathbb{R}^6$. We have that

$$\begin{align*}
\mathcal{J} \partial x_1 &= g \partial x_3 + \partial y_1 \\
\mathcal{J} \partial x_2 &= \partial y_2 \\
\mathcal{J} \partial x_3 &= \partial y_3 \\
\mathcal{J} \partial y_1 &= -\partial x_1 - g \partial y_3 \\
\mathcal{J} \partial y_2 &= -\partial x_1 \\
\mathcal{J} \partial y_3 &= -\partial x_3
\end{align*}$$

where $g = g(x_2, x_3)$ satisfies

$$\frac{\partial g}{\partial x_2} - 2\tau \frac{\partial g}{\partial x_3} = 0.$$ 

By assumption, $\frac{\partial g}{\partial x_2}|_{(x, y)} \neq 0$. Therefore, in view of [11, Theorem 2.4], the second equation of (21) is not satisfied. Consequently, $\mathcal{J}|_{\mathbb{R}^6}$ cannot be locally compatible with any symplectic form on $\mathbb{R}^6 \cong \mathbb{C}^3_{z_1, z_2, z_3}$.

By contradiction: assume that there exists a symplectic form on $\mathbb{C}^4$ locally compatible with $\mathcal{J}$. Thus $(\mathcal{J}, \omega)$ gives rise to an almost Kähler metric $\mathcal{G}$ on $\mathbb{C}^4$. Hence, $(\mathcal{J}|_{\mathbb{R}^6}, \mathcal{G}|_{\mathbb{R}^6})$ would be an almost Kähler structure on the submanifold $\mathbb{R}^6 \subset \mathbb{C}^4$. This is absurd. \hfill $\square$

As an explicit example, we may take $g$ defined as

$$g(x_2, x_3) = 2\tau^2 x_2 + \tau x_3.$$ 

Then,
a) $g$ verifies the partial differential equation
\[
\frac{\partial g}{\partial x_2} - 2\tau \frac{\partial g}{\partial x_3} = 0.
\]

b) For every $\tau \in (-\varepsilon, \varepsilon)$, $\tau \neq 0$, we have $\frac{\partial g}{\partial x_2}(x, y) \neq 0$.

c) For $\tau = 0$ the almost complex structure is standard.

Therefore, for any $\tau \in (-\varepsilon, \varepsilon)$, $\tau \neq 0$, $(\mathcal{J}, \Omega_\tau)$ is a almost 2 Kähler structure on $\mathbb{C}^4$, such that $\mathcal{J}$ admits no compatible symplectic structures.

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