Quantum Geometry of the Universal Hypermultiplet

Sergei V. Ketov

Caltech-USC Center for Theoretical Physics
Department of Physics and Astronomy
University of Southern California
Los Angeles, CA 90089-2535, USA
e-mail ketov@citusc.usc.edu

Abstract: The universal hypermultiplet moduli space metric in the type-IIA superstring theory compactified on a Calabi-Yau threefold is related to integrable systems. The instanton corrections in four dimensions arise due to multiple wrapping of BPS membranes and fivebranes around certain (supersymmetric) cycles of Calabi-Yau. The exact (non-perturbative) metrics can be calculated in the special cases of (i) the D-instantons (or the wrapped D2-branes) in the absence of fivebranes, and (ii) the fivebrane instantons with vanishing charges, in the absence of D-instantons. The solutions of the first type are governed by the three-dimensional Toda equation, whereas the solutions of the second type are governed by the particular Painlevé VI equation.

1 Introduction

Non-perturbative contributions to the effective supergravity theory, originating from the type IIA string compactification on a Calabi-Yau (CY) threefold $\mathcal{Y}$, are known to be due to the solitonic five-branes wrapped about the entire CY space and the supermembranes (D2-branes) wrapped about special Lagrangian (supersymmetric) three-cycles $C_3$ of $\mathcal{Y}$. The supersymmetric cycles minimize volume in their homology class, while the corresponding wrapped brane configurations lead to the BPS states. Being solitonic (BPS) classical solutions to the higher dimensional (Euclidean) equations of motion, these wrapped branes are localized in the uncompactified (four) dimensions and thus can be identified with 4d instantons. The instanton actions are essentially given by the volumes of the cycles on which the branes are wrapped.

The compactification of the type-IIA superstring theory on $\mathcal{Y}$ gives rise to the four-dimensional (4d) N=2 superstrings whose Low-Energy Effective Action (LEEA) is given by the 4d, N=2 supergravity coupled to N=2 vector supermultiplets and hypermultiplets. The hypermultiplet LEEA is most naturally described by the Non-Linear Sigma-Model (NLSM), whose scalar fields parametrize the quaternionic target space $M_H$. The instanton corrections to the LEEA due to the wrapped fivebranes and membranes can be easily identified and distinguished from each other in the semi-classical limit, since the

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fivebrane instanton corrections are organized by powers of \( e^{-1/g_{\text{string}}^2} \), whereas the membrane instanton corrections are given by powers of \( e^{-1/g_{\text{string}}} \), where \( g_{\text{string}} \) is the type-IIA superstring coupling constant \([3]\). The vacuum expectation value of the four-dimensional dilaton field \( \langle \phi \rangle \) in the compactified type-IIA superstring is simply related to the CY volume \( V_{\text{CY}} \) in M-theory, \( V_{\text{CY}} = e^{-2\langle \phi \rangle} \), so that the type-IIA superstring loop expansion amounts to the derivative expansion of the M-theory action \([3]\). Any CY compactification has the co-called Universal Hypermultiplet (UH) containing a dilaton, an axion, a complex RR-type pseudo-scalar and a Dirac dilatino. The target space of the universal hypermultiplet NLSM has to be an Einstein space with the (Anti)Self-Dual (ASD) Weyl tensor \([2]\). We restrict ourselves to a calculation of the instanton corrections to the universal hypermultiplet NLSM metric by analyzing generic quaternionic deformations of the classical UH metric. We use the simple fact that the (anti)self-dual Weyl tensor already implies the integrable system of partial differential equations on the components of the UH moduli space metric. Additional simplifications arise due to the Einstein condition and the physically motivated isometries. The exact UH metric is supposed to be regular and complete (cf. Seiberg-Witten theory — see, e.g., ref. \([5]\) for a review).

## 2 UH metric in string perturbation theory

The LEET of (tree) type-IIA superstrings in ten dimensions is given by the IIA supergravity. The universal (UH) sector of the 10d type-IIA supergravity compactified down to four dimensions is obtained by using the following Ansatz for the 10d metric:

\[
ds_{10}^2 = g_{mn} dx^m dx^n = e^{-\phi/2} ds_{\text{CY}}^2 + e^{3\phi/2} g_{\mu\nu} dx^\mu dx^\nu ,
\]

while keeping only \( SU(3) \) singlets in the internal CY indices and ignoring all CY complex moduli. In eq. (1) \( \phi(x) \) stands for the 4d dilaton, \( g_{\mu\nu}(x) \) is the spacetime metric in four uncompactified dimensions, \( \mu, \nu = 0, 1, 2, 3 \), and \( ds_{\text{CY}}^2 \) is the (Kähler and Ricci-flat) metric of the internal CY threefold \( Y \) in complex coordinates,

\[
ds_{\text{CY}}^2 = g_{ij}(y, \bar{y}) dy^i dy^j ,
\]

where \( i, j = 1, 2, 3 \). By definition, the CY threefold \( Y \) possesses the \((1, 1)\) Kähler form \( J \) and the holomorphic \((3, 0)\) form \( \Omega \). The universal hypermultipet (UH) unites the dilaton \( \phi \), the axion \( D \) coming from dualizing the three-form field strength \( H_3 = dB_2 \) of the NS-NS two-form \( B_2 \) in 4d, and the complex scalar \( C \) representing the RR three-form \( A_3 \) with \( A_{ijk}(x, y) = \sqrt{2}C(x)\Omega_{ijk}(y) \). When using a flat (or rigid) CY with

\[
g_{ij} = \delta_{ij} \quad \text{and} \quad \Omega_{ijk} = \varepsilon_{ijk} ,
\]

this yields the (Ferrara-Sabharwal) NLSM action in 4d \([3]\),

\[
S_4 = -\frac{1}{\kappa_4^2} \int d^4x \sqrt{-g} \left[ -\frac{1}{2} R + (\partial_m \phi)^2 + e^{2\phi} |\partial_\mu C|^2 + e^{4\phi} \left( \partial_\mu D + \frac{i}{2} \bar{C} \gamma_\mu C \right)^2 \right] .
\]

where \( H_3 \) has been traded for the pseudoscalar \( D \) via the Legendre transform.

The perturbative (one-loop) string corrections to the UH metric originate from the \((Riemann)^4\) terms in M-theory compactified on a CY three-fold \( Y \) \([3]\). These quantum corrections are known to be proportional to the CY Euler number \( \chi = 2(h_{1,1} - h_{1,2}) \). In fact, the corrected metric is related to the classical UH metric by a local field redefinition \([4]\), so that the local UH geometry is unchanged in superstring perturbation theory.
3 D-instantons and UH metric

The classical (Ferrara-Sabharwal) NLSM metric describes the symmetric quaternionic space $SU(2, 1)/SU(2) \times U(1)$. In particular, the $U(1)$ subgroup of the $SU(2)$ symmetry is given by the duality rotations $U_C(1)$ of the complex R-R pseudo-scalar $C$,

$$C \rightarrow e^{i\alpha}C.$$ 

These duality rotations are believed to be exact in quantum theory [1], as we assume too.

As regards generic four-dimensional quaternionic manifolds (relevant for UH), they all have Einstein-Weyl geometry of negative scalar curvature [2],

$$W_{abcd} = 0, \quad R_{ab} = \frac{\Lambda}{2} g_{ab},$$

where $W_{abcd}$ is the Weyl tensor and $R_{ab}$ is the Ricci tensor for the metric $g_{ab}$. When using the Ansatz [7]

$$ds^2_Q = \frac{P}{\omega^2} \left[ e^u(dx^2 + dy^2) + d\omega^2 \right] + \frac{1}{P^2 \omega^2} (dt + \Theta_1)^2$$

for a generic quaternionic metric with an abelian isometry, it is straightforward to prove that the restrictions (6) on the metric (7) precisely amount to the 3d Toda equation

$$u_{xx} + u_{yy} + (e^u)_{\omega\omega} = 0.$$ 

The second potential $P$ of eq. (7) is then given by [7]

$$P = \frac{1}{2\Lambda} (\omega u_{\omega} - 2),$$

whereas the remaining one-form $\Theta_1$ obeys the linear equation [7]

$$d\Theta_1 = -P_x dy \wedge d\omega - P_y d\omega \wedge dx - e^u (P_\omega + \frac{2}{\omega} P + \frac{2\Lambda}{\omega} P_2) dx \wedge dy.$$ 

In terms of the complex coordinate $\zeta = x + iy$, the 3d Toda equation (8) takes the form

$$4u_{\zeta\bar{\zeta}} + (e^u)_{\omega\omega} = 0.$$ 

Separable solutions to the 3d Toda equation, having the form

$$u(\zeta, \bar{\zeta}, \omega) = F(\zeta, \bar{\zeta}) + G(\omega),$$

are easily found to be [8]

$$e^u = \frac{4e^2(\omega^2 + 2\omega b \cos \alpha + b^2)}{(1 + e^2 |\zeta|^2)^2},$$

where $(\alpha, b, c)$ are all constants. Eq. (13) automatically possesses the rigid $U_C(1)$ symmetry with respect to the duality rotations $\zeta \rightarrow e^{i\alpha} \zeta$ of the complex RR-field $\zeta$.

The classical approximation corresponds to the conformal limit $\omega \rightarrow \infty$ and $|\zeta| \rightarrow \infty$, while keeping the ratio $|\zeta|^2 / \omega$ finite. Then one easily finds that $P \rightarrow -\Lambda^{-1} = \text{const.} > 0$, whereas the metric based on eq. (13) takes the form

$$ds^2 = \frac{1}{\lambda^2} \left( |dC|^2 + d\lambda^2 \right) + \frac{1}{\lambda^4} (dD + \Theta)^2,$$

in terms of the new variables $C = 1/\zeta$ and $\lambda^2 = \omega$, after a few rescalings. The metric (14) reduces to that of eq. (4) when using $\lambda^{-2} = e^{2\phi}$.  

4 Fivebrane instantons

As was demonstrated in ref. [9], the BPS condition on the fivebrane instanton solution with the vanishing charges defines a gradient flow in the hypermultiplet moduli space. The flow implies the SU(2) isometry of the UH metric since the non-degenerate action of this isometry in the four-dimensional UH moduli space gives rise to the well defined three-dimensional orbits that can be parametrized by the ‘radial’ coordinate to be identified with the flow parameter.

Let’s consider a generic SU(2)-invariant metric in four Euclidean dimensions. In the Bianchi IX formalism, where the SU(2) symmetry is manifest, the general Ansatz for such metrics reads [10]

\[ ds^2 = w_1 w_2 w_3 dt^2 + \frac{w_2 w_3}{w_1} \sigma_1^2 + \frac{w_3 w_1}{w_2} \sigma_2^2 + \frac{w_1 w_2}{w_3} \sigma_3^2 \]  

in terms of the su(2) (left)-invariant (Cartan) one-forms \( \sigma_i \) and the radial coordinate \( t \). The metric (15) is dependent upon three functions \( w_i(t), i = 1, 2, 3 \).

Being applied to the metric (15), the ASD Weyl condition gives rise to a (Halphen) system of Ordinary Differential Equations (ODE) [10, 11],

\[ \begin{align*}
    \dot{A}_1 &= -A_2 A_3 + A_1 (A_2 + A_3), \\
    \dot{A}_2 &= -A_3 A_1 + A_2 (A_3 + A_1), \\
    \dot{A}_3 &= -A_1 A_2 + A_3 (A_1 + A_2),
\end{align*} \]  

(16)

where the dots denote differentiation with respect to \( t \), and the functions \( A_i(t) \) are defined by the auxiliary ODE system,

\[ \begin{align*}
    \dot{w}_1 &= -w_2 w_3 + w_1 (A_2 + A_3), \\
    \dot{w}_2 &= -w_3 w_1 + w_2 (A_3 + A_1), \\
    \dot{w}_3 &= -w_1 w_2 + w_3 (A_1 + A_2). 
\end{align*} \]  

(17)

The Halphen system (16) has a long history. Perhaps, its most natural (manifestly integrable) derivation is provided via a reduction of the SL(2, C) anti-self-dual Yang-Mills equations from four Euclidean dimensions to one. The Painlevé VI equation is known to be behind the ASD-Weyl geometries having the SU(2) symmetry [10, 11]. In fact, all quaternionic metrics with the SU(2) symmetry are governed by the particular Painlevé VI equation:

\[ y'' = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) (y')^2 - \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) y' \]

\[ + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[ \frac{1}{8} - \frac{x}{8y^2} + \frac{x-1}{8(y-1)^2} + \frac{3x(x-1)}{8(y-x)^2} \right], \]

(18)

where \( y = y(x) \), and the primes denote differentiation with respect to \( x \) [11].

The equivalence between eqs. (16) and (18) is well known to mathematicians [10, 11]. An exact solution to the Painlevé VI equation (18), which leads to a regular (and complete) quaternionic metric (15), is unique [11]. The regular solution can be written down in terms of the standard theta-functions \( \vartheta_\alpha(z|\tau) \), where \( \alpha = 1, 2, 3, 4 \), and the arguments are related.
as $z = \frac{1}{2}(\tau - k)$, where $k$ is an arbitrary (real and positive) parameter. The variable $\tau$ is related to the variable $x$ of eq. (18) via the relation
\[
x = \vartheta^4_3(0)/\vartheta^4_1(0) ,
\]
where the value of the theta-function variable $z$ is explicitly indicated, as usual. The explicit solution to eq. (18) reads [12]
\[
y(x) = \frac{\vartheta''''(0)}{3\pi^2\vartheta''_1(0)\vartheta'_1(0)} + \frac{1}{3} \left[ 1 + \frac{\vartheta''_1(0)}{\vartheta'_1(0)} \right] \\
+ \frac{\vartheta''''(z)\vartheta'_1(0) - 2\vartheta''''(z)\vartheta'_1(z) + 2\pi i(\vartheta''''(z)\vartheta'_1(z) - \vartheta''''(z))}{2\pi^2\vartheta''_1(0)\vartheta'_1(z) + \pi i\vartheta'_1(z)} .
\]

The parameter $k > 0$ describes the monodromy of this solution around its essential singularities (branch points) $x = 0, 1, \infty$. This (non-abelian) monodromy is generated by the matrices (with the purely imaginary eigenvalues $\pm i$)
\[
M_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & i^{-1-k} \\ i^{1+k} & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & i^{-k} \\ -i^k & 0 \end{pmatrix} .
\]

The function (20) is meromorphic outside its essential singularities at $x = 0, 1, \infty$, while is also has simple poles at $\tilde{x}_1, \tilde{x}_2, \ldots$, where $\tilde{x}_n \in (x_n, x_{n+1})$ and $x_n = x(ik/(2n - 1))$ for each positive integer $n$. Accordingly, the metric is well-defined (complete) for $x = \tilde{x}_n$ [11]. Near the boundary the metric has the following asymptotical behaviour [13]
\[
ds^2 \propto \frac{dx^2}{(1 - x)^2} + \frac{4}{(1 - x)\cosh^2(\pi k/2)}\sigma^2_1 + \frac{16}{(1 - x)^2\sinh^2(\pi k/2)\cosh^2(\pi k/2)}\sigma^2_2 \\
+ \frac{4}{(1 - x)\sinh^2(\pi k/2)}\sigma^2_3 + \text{regular terms} .
\]

As is clear from eq. (22), the real parameter $k$ can be identified with the five-brane instanton action that is proportional to the CY volume and $1/g^2_{\text{string}}$ as well. The semiclassical regime thus arises near the boundary $x \to 1^-$ at $k \to +\infty$. In this limit, one gets back the Ferrara-Sabharwal metric out of that in eq. (22) after rescaling $(1 - x) \to 2^6e^{\pi k}(1 - x)$ and redefining $x = r^2$.

A few comments are in order.

The very notions of a ‘wrapped brane’, an ‘instanton’ and a ‘dilaton’ are essentially semiclassical, and they do not exist non-perturbatively. We consider the full UH theory as the NLSM, i.e. modulo field reparametrizations (or diffeomorphisms in the NLSM target space). The physical interpretation of the exact quaternionic solutions to the UH metric is, however, possible in the semiclassical regime. Hence, first, we identify the semilasccal region and, second, we rewrite a given exact solution as a sum of the known classical solution and the exponentially small corrections with respect to the well defined real parameter (or modulus). Those corrections are finally identified with the instanton contributions, whose origin (due to the wrapped BPS branes) we already know in the context of the CY compactified type-IIA superstrings.

The supersymmetric 3-cycles $C_3$ are defined by two conditions: (i) the pullback of the CY Kähler form $J$ on $C_3$ should vanish, $J|_{C_3} = 0$, and (ii) the pullback of the imaginary
part of the holomorphic CY 3-form $\Omega$ should vanish too, $\Im \Omega|_{C_3} = 0$. In the terminology of ref. [14], the supersymmetric 3-cycles $C_3$ are of the A-type, whereas the Calabi-Yau threefold itself is of the B-type. In our case, the only relevant modulus of a wrapped 5-brane is its volume, i.e. the CY ‘size’ parameter (a Kähler moduli). The semiclassical description can be valid only for large CY volumes. In the opposite limit of small CY volumes, an exact N=2 superconformal field theory (Landau-Ginzburg) description can apply [14]. Mirror symmetry may allow us to relate these two different descriptions.

The coefficient at $\sigma_2^2$ in eq. (22) vanishes faster than the coefficients at $\sigma_1^2$ and $\sigma_3^2$ when approaching the boundary, $x \to 1^-$. On the two-dimensional boundary annihilated by $\sigma_2$ one has the natural conformal structure

$$\sinh^2(\pi k/2)\sigma_1^2 + \cosh^2(\pi k/2)\sigma_3^2.$$  \hspace{1cm} (23)

The only relevant parameter $\tanh^2(\pi k/2)$ in eq. (23) represents the central charge (or the conformal anomaly) of the 2d conformal field theory on the boundary. Our results are, therefore, consistent with the holographic principle [15].

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