NOTES ON GROUP DISTANCE MAGICNESS OF PRODUCT GRAPHS

A V PRAJEESH AND K PARAMASIVAM

ABSTRACT. If \( l \) is a bijection from the vertex set \( V(G) \) of a graph \( G \) to an additive abelian group \( \Gamma \) with \( |V(G)| \) elements in such a way that for any vertex \( u \) of \( G \), the weight \( w_G(u) = \sum_{v \in N_G(u)} l(v) \) is equal to the same element \( \mu_o \) of \( \Gamma \), then \( l \) is called a \( \Gamma \)-distance magic labeling of \( G \). A graph \( G \) that admits such a labeling is called \( \Gamma \)-distance magic and if \( G \) is \( \Gamma \)-distance magic for every additive abelian group \( \Gamma \) with \( |V(G)| \) elements, then \( G \) is called a group distance magic graph. In this paper, we provide few results on the group distance magic labeling of product graphs, namely lexicographic and direct product of two graphs. We also prove some necessary conditions for a graph to be group distance magic and provide a characterization for a tree to be group distance magic.

1. INTRODUCTION

In this paper, we consider only simple and finite graphs. We use \( V(G) \) for the vertex set and \( E(G) \) for the edge set of a graph \( G \). The neighborhood \( N_G(v) \), or shortly \( N(v) \) of a vertex \( v \) of \( G \) is the set of all vertices adjacent to \( v \), and the degree \( deg_G(v) \), or shortly \( deg(v) \) of \( v \) is the number of vertices in \( N_G(v) \). The distance \( d_G(u, v) \) between two vertices \( u \) and \( v \) of \( G \) is the length of shortest path connecting \( u \) and \( v \). For more standard graph theoretic notation and terminology, we refer Bondy and Murty [1] and Hammack et al. [2].

Recall two standard graph products (see [2]). Let \( G \) and \( H \) be two graphs. Both, the lexicographic product \( G \circ H \) and the direct product \( G \times H \) are graphs with the vertex set \( V(G) \times V(H) \). Two vertices \((g, h)\) and \((g', h')\) are adjacent in:

(i) \( G \circ H \) if and only if \( g \) is adjacent to \( g' \) in \( G \), or \( g = g' \) and \( h \) is adjacent to \( h' \) in \( H \).

(ii) \( G \times H \) if and only if \( g \) is adjacent to \( g' \) in \( G \) and \( h \) is adjacent to \( h' \) in \( H \).

A distance magic labeling of \( G \) is a bijection \( l : V(G) \rightarrow \{1, ..., |V(G)|\} \), such that for any \( u \) of \( G \), the weight of \( u \), \( w_G(u) = \sum_{v \in N_G(u)} l(v) \) is a constant \( \mu \). A graph \( G \) that admits such a labeling is called a distance magic graph [3].

The concept of distance magic labeling was studied by Vilfred [3] as sigma labeling. Later, Miller et al. [4] called it a 1-vertex magic vertex labeling and Sugeng et al. [5] referred the same as distance magic labeling.

The concept of distance magic labeling has been motivated by the construction of magic squares. It is worth to mention the motivation given by Froncek et al. [6] through an equalized incomplete tournament. An equalized incomplete tournament

\[\textit{Mathematics Subject Classification.} \textit{Primary 05C78, 05C25, 05C76.}\]
\[\textit{Key words and phrases.} \textit{Additive abelian group, group distance magic, lexicographic product, direct product.}\]
of \( n \) teams with \( r \) rounds, \( EIT(n, r) \) is a tournament which satisfies the following conditions:

(i) every team plays against exactly \( r \) opponents.
(ii) the total strength of the opponents, against which each team plays is a constant.

Therefore, finding a solution to an \( EIT(n, r) \) is equivalent to obtain a distance magic labeling of an \( r \)-regular graph with \( n \) vertices.

Most of important results and problems, which are more relevant and helpful in proving our results, are listed below.

**Theorem 1.1.** [3, 4, 7, 8] No \( r \)-regular graph with \( r \)-odd can be a distance magic graph.

**Lemma 1.1.** [4] If \( G \) contains two vertices \( u \) and \( v \) such that \( |N(u) \cap N(v)| = \deg(v) - 1 = \deg(u) - 1 \), then \( G \) is not distance magic.

**Theorem 1.2.** [9] Let \( r \geq 1 \) and \( n \geq 3 \). If \( G \) is an \( r \)-regular graph and \( C_n \) the cycle of length \( n \), then \( G \circ C_n \) admits a labeling if and only if \( n = 4 \).

In 2009, Shafiq et al. [9] posted a problem of the existence of distance magic labeling of the lexicographic product of a non-regular graph \( G \) with \( C_4 \).

**Problem 1.1.** [9] If \( G \) is a non-regular graph, determine if there is a distance magic labeling of \( G \circ C_4 \).

In 2018, Cichacz and Görlich [10], raised a similar question in the case of direct product of \( G \) with \( C_4 \).

**Problem 1.2.** [10] If \( G \) is non-regular graph, determine if there is a distance magic labeling of \( G \times C_4 \).

Anholcet al. [11] defined a distance magic graph \( G \) to be balanced if there exists a bijection \( l : V(G) \to \{1, ..., |V(G)|\} \) such that for any \( w \) of \( G \), the following holds: if \( u \in N(w) \) with \( l(u) = i \), then \( \exists v \in N(w), v \neq u \), with \( l(v) = |V(G)| + 1 - i \).

Further, we call \( u \), the twin vertex of \( v \) and vice versa.

From [11], it is clear that \( G \) is a balanced distance magic graph or shortly, balanced-\( dmg \) if and only if \( G \) is regular and the vertex set of \( G \) can be expressed as \( \{v_i, v'_i : 1 \leq i \leq |V(G)|\} \) such that for any \( i \), \( N(v_i) = N(v'_i) \), where \( v_i \) is the twin vertex of \( v'_i \). The graphs \( K_{2n, 2n} \) and \( K_{2n} - M, M \) any perfect matching of \( K_{2n} \) are examples of balanced-\( dmg \)’s.

The \( k^{th} \) power of a graph \( G \) is a graph \( G^k \) with the same set of vertices as \( G \) and any two vertices \( u \) and \( v \) are connected if and only if \( d_G(u, v) \leq k \).

In 2016, Arumugam et al. [12] proved the following result for the characterization of entire class of distance magic graphs \( G \) with \( \Delta(G) = |V(G)| - 1 \). It is interesting to see that the following class of graphs is derived from balanced-\( dmg \).

**Theorem 1.3.** [12] Let \( G \) be any graph of order \( n \) with \( \Delta(G) = n - 1 \). Then \( G \) is a distance magic graph if and only if \( n \) is odd and \( G \cong (K_{n-1} - M) + K_1 \) where \( M \) is a perfect matching in \( K_{n-1} \).

In 2004, Rao [8] proved the following result.

**Theorem 1.4.** [8] The graph \( C_k \square C_m \) is distance magic if and only if \( k = m \equiv 2 \mod 4 \).
Notes on group distance magicness of product graphs

Now, a natural question arises that for all graphs, which are not distance magic, whether one can introduce a new concept by replacing the existing co-domain, \{1, ..., |V(G)|\} of the distance magic labeling \(l\) by an another set or even by an algebraic structure such as group so that these graphs can admit such a magic-type labeling.

Motivated by this fact, in 2013, Froncek [13], introduced the notion of group distance magic labeling of graphs. He proved that \(C_k \square C_m\) is \(\mathbb{Z}_{km}\)-distance magic if and only if \(km\) is even, where \(\mathbb{Z}_{km}\) is a finite additive abelian group.

Throughout this paper, the algebraic structure \(\Gamma = (\Gamma, +)\) is a finite additive abelian group or shortly, abelian group, where ‘+’ is a binary operation on \(\Gamma\). The order and identity element of \(\Gamma\) are denoted by \(|\Gamma|\) and \(e\) respectively. Recall that any non-identity element \(g\) of \(\Gamma\) is an involution if \(g = -g\), where \(-g\) is the additive inverse of \(g\). If \(g\) is an involution of \(\Gamma\), then \(2g = e\). Also, any non-trivial finite group \(\Gamma\) has involution if and only if \(|\Gamma|\) is even. The fundamental theorem of abelian group states that a finite abelian group \(\Gamma\) can be expressed as a direct product of cyclic groups of prime power order, where the product is unique up to the order of subgroups. Moreover, the sum of all elements of \(\Gamma\), \(s(\Gamma) = \sum_{g \in \Gamma} g\) is equal to the sum of all involutions of \(\Gamma\).

**Lemma 1.2.** [14] Let \(\Gamma\) be an abelian group.

(i) If \(\Gamma\) has exactly one involution \(g'\), then \(s(\Gamma) = g'\).

(ii) If \(\Gamma\) has no involutions, or more than one involution, then \(s(\Gamma) = e\).

For more group theory related terminology and notation, we refer Herstein [15].

**Definition 1.1.** [13] If \(\Gamma\) is an abelian group and \(G\) is a graph such that \(|V(G)| = |\Gamma|\), then a bijection \(l : V(G) \rightarrow \Gamma\) is said to be a \(\Gamma\)-distance magic labeling of \(G\) if for any \(u\) of \(G\), the weight of \(u\), \(w_G(u) = \sum_{v \in N_G(u)} l(v)\) is equal to the same element \(\mu_o\) of \(\Gamma\). A graph \(G\) that admits such a labeling is called a \(\Gamma\)-distance magic graph and the element \(\mu_o\) is called the magic constant associated with the labeling \(l\) of \(G\).

Whenever \(l\) is a distance magic labeling of a graph \(G\) on \(n\) vertices with the magic constant \(\mu\), then consider a new labeling \(l^*\) on \(G\) as,

\[
l^*(v) = \begin{cases} l(v) & \text{if } l(v) < n \\ 0 & \text{if } l(v) = n, \end{cases}
\]

which is a \(\mathbb{Z}_n\)-distance magic labeling with magic constant \(\mu_0\), where \(\mu_0 \equiv \mu \mod n\).

On the other hand, it is observed from [13] that \(G\) is a \(\mathbb{Z}_n\)-distance magic does not imply that \(G\) is distance magic.

In 2014, Cichacz [16, 17] proved the following results.

**Theorem 1.5.** [17] Let \(G\) be a graph of order \(n\) and \(\Gamma\) be an arbitrary abelian group of order \(4n\) such that \(\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times B\) for some abelian group \(B\) on \(n\) vertices. Then there exists a \(\Gamma\)-distance magic labeling for the graph \(G \circ C_4\).

**Theorem 1.6.** [17] Let \(G\) be a graph of order \(n\) and \(\Gamma\) be an abelian group of order \(4n\). If \(n = 2^p(2k+1)\) for some natural numbers \(p, k\) and \(\deg(v) \equiv c \mod 2^{p+1}\) for some constant \(c\) for any \(v \in V(G)\), then there exists a \(\Gamma\)-distance magic labeling for the graph \(G \circ C_4\).
Theorem 1.7. \[16\] Let \( G \) be a graph of order \( n \) and if \( n = 2^p(2k+1) \) for some natural numbers \( p, k \) and \( \deg(v) \equiv c \mod 2^{p+2} \) for some constant \( c \) for any \( v \in V(G) \), then there exists a \( \Gamma \)-distance magic labeling for the graph \( G \times C_4 \).

Cichacz in \[18\] gave a complete characterization of group distance magicness of the complete bipartite graph.

Theorem 1.8. \[18\] The complete bipartite graph \( K_{m,n} \) is a group distance magic graph if and only if \( m + n \not\equiv 2 \mod 4 \).

Recently, Anholcer et al. \[19\] discussed the group distance magicness of the direct product of two graphs and obtained the following result.

Theorem 1.9. \[19\] If \( G \) is a balanced distance magic graph and \( H \) an \( r \)-regular graph for \( r \geq 1 \), then \( G \times H \) is a group distance magic graph.

In the following section, we provide some necessary conditions for a graph to be group distance magic and also characterize the group distance magic labeling of a tree \( T \). Further, we also discuss the group distance magic labeling of bi-regular graphs \( G \) with regularities \( r_1 = |V(G)| - 1 \) and \( r_2 = 1, 2, 3, |V(G)| - 2 \) or \( |V(G)| - 3 \).

Motivated by the results from \[16\] \[17\] \[19\], in third section, we discuss the group distance magic labeling of \( G \circ H \) and \( G \times H \), where \( G \) is a non-regular graph and \( H \) is a balanced-dmg.

2. GROUP DISTANCE MAGIC LABELING OF \((\overline{K_{n-1}} - M) + K_1\) AND TREES

Lemma 2.1. If \( \Gamma \) is any abelian group and if \( G \) is a graph with \( |\Gamma| \) vertices such that \( G \) has at least two distinct vertices of degree \( |\Gamma| - 1 \), then \( G \) is not \( \Gamma \)-distance magic.

\textit{Proof.} On the contrary, let \( l \) be a \( \Gamma \)-distance magic labeling of the graph \( G \) with magic constant \( \mu_0 \). Let \( u \) and \( v \) be two distinct vertices of \( G \) such that \( \deg_G(u) = \deg_G(v) = |\Gamma| - 1 = \deg_G(v) \). Then,

\[ s(\Gamma) = \mu_0 + l(u) = \mu_0 + l(v). \]

Since the left cancellation law holds in \( \Gamma \), we have \( l(u) = l(v) \), a contradiction. \( \square \)

Lemma 2.2. If \( \Gamma \) is any abelian group and if \( G \) is a graph with \( |\Gamma| \) vertices such that \( G \) has two distinct vertices \( u \) and \( v \) with \( |N_G(u) \cap N_G(v)| = \deg_G(u) = \deg_G(v) = 1 \), then \( G \) is not \( \Gamma \)-distance magic.

\textit{Proof.} On the contrary, let \( l \) be a \( \Gamma \)-distance magic labeling of the graph \( G \) with magic constant \( \mu_0 \). Choose two vertices \( u' \) and \( v' \) of \( G \) in such a way that the followings hold:

(i) \( u' \in N_G(u) \) and \( u' \not\in N_G(v) \), and
(ii) \( v' \in N_G(v) \) and \( v' \not\in N_G(u) \).

By comparing the weights of \( u \) and \( v \), we have

\[ \mu_0 = w_G(u) = g + l(u') = g + l(v') = w_G(v), \]

for some \( g \) in \( \Gamma \). By left cancellation law in \( \Gamma \), \( l(u') = l(v') \), a contradiction. \( \square \)

The following result characterizes the group distance magicness of a tree.
Theorem 2.1. A non-trivial tree $T$ is $\Gamma$-distance magic for an abelian group $\Gamma$ if and only if $T \cong K_{1,n}$, with $n \neq 1 \mod 4$.

Proof. If $\text{diam}(T) = 2$, the result is straightforward by Theorem 1.8. On the other hand if $\text{diam}(T) > 2$, then $T$ has two vertices $u$ and $v$ such that $d_G(u, v) = \text{diam}(T)$. Hence the result follows from Lemma 2.2.

Theorem 2.2. Let $\Gamma$ be any abelian group having at least one element $g^1$ such that $g^1$ is not an involution. If $l_1$ is a $\Gamma$-distance magic labeling of $G$ with magic constant $\mu_0$, then there exists a $\Gamma$-distance magic labeling $l_2$ of $G$ with magic constant $-\mu_0$.

Proof. Let $l_1$ be the distance magic labeling of a graph $G$ with magic constant $\mu_0$. Let $u$ be any vertex of $G$ with the neighbors $v_1, v_2, \ldots, v_t$, where $t = |N_G(u)|$. Then,

$$w_G(u) = \sum_{i=1}^{t} l_1(v_i) = \mu_0.$$

Consider a new function $l_2(u) = -(l_1(u))$. Since $l_1$ is a bijection, $l_2$ is also a bijection. Further, $l_1$ is not identically equal to $l_2$ because if $l_1(v) = l_2(v)$ for all $v$, in particular, if $l_1(v^1) = l_2(v^1) = g^1$, then $l_1(v^1) = -l_1(v^1) \implies 2l_1(v^1) = e$, that is $2g^1 = e$, a contradiction.

In an abelian group, the inverse of sum of elements is equal to the sum of inverses of each elements. Thus, the weight of $u$ of $G$ with respect to $l_2$ is given by

$$\sum_{i=1}^{t} l_2(v_i) = \sum_{i=1}^{t} -l_1(v_i) = -\left(\sum_{i=1}^{t} l_1(v_i)\right) = -\mu_0.$$

Since the choice of $u$ of $G$ is arbitrary, the result follows.

Theorem 2.3. Let $n > 1$ be an odd integer. Let $G$ be a graph isomorphic to $(K_{n-1} - M) + K_1$, where $M$ is any perfect matching of $K_{n-1}$. If $\Gamma$ is any abelian group with $|\Gamma| = n$, then $G$ admits a $\Gamma$-distance magic labeling $l$ if and only if $l(v_0) = e$, where $G[v_0] = K_1$.

Proof. Let $\Gamma$ be an abelian group with $|\Gamma| = n$. Define the vertex set of $G$ as,

$$\{v_0\} \cup \{v_i, v_i' : 1 \leq i \leq \frac{n-1}{2}\},$$

where $v_i$ and $v'_i$ are twin vertices of $K_{n-1} - M$ and $v_0$ is the vertex, which induces $K_1$ of $G$. Let $l$ be a $\Gamma$-distance magic labeling of $G$ with magic constant $\mu_0$. We know that $ng = e$ for any element $g$ of $\Gamma$, in particular, $n\mu_0 = e$. By using Lemma 1.2 and by comparing the total weights, we get

$$n\mu_0 = (n-1)l(v_0) + (n-2)(s(\Gamma) - l(v_0)),$$

which implies that $l(v_0)$ is $e$.

When $n$ is odd, for every element $g$ of $\Gamma$, there exists a unique $-g$ different from $g$ in $\Gamma$ such that $g + (-g) = e$. Consider a function $l$ from $V(G)$ to $\Gamma$ as, $l(v_0) = e$ and if $l(v_i) = g$, then $l(v'_i) = -g$, for all $i \neq 0$. It is not hard to verify that, the weight of each vertex of $G$ is $e$. Thus, $l$ is a $\Gamma$-distance magic labeling of $G$ with magic constant $\mu_0 = e$.

Theorem 1.8 and 2.1 confirm the fact that the number of group distance magic graphs on $n$ vertices with maximum degree $n - 1$ is comparatively higher than the number of distance magic graphs with maximum degree $n - 1$. Hence it is worth mentioning the group distance magic labeling of bi-regular graph with regularities $r_1 = n - 1$ and $r_2$, where $r_2$ is from the set $\{1, 2, 3, n - 3\}$. 

Notes on group distance magicness of product graphs
Let \( n > 3 \) be an odd integer and \( G \) be a bi-regular graph on \( n \) vertices with a unique vertex \( v \) of degree, \( r_1 = n - 1 \), and all other vertices of degree \( r_2 \), where \( r_2 = 1, 2, 3 \) or \( n - 3 \). It is observed that if \( l(v) \neq e \), then \( l \) is not an \( \Gamma \)-distance magic labeling of \( G \), for any abelian group \( \Gamma \) with \( |\Gamma| = n \).

Let \( v_e \) and \( v_{-g} \) be the vertices of \( G \) labeled with \( e \) and \( -g \) respectively. On contrary, if for a given abelian group \( \Gamma \) with \( |\Gamma| = n \), there exists a \( \Gamma \)-distance magic labeling \( l \) such that \( l(v) = g \neq e \). Then, \( w_G(v) = -g \), for every \( v \) in \( G \).

Case 1: If \( r_2 = 1 \), then comparing \( w_G(v_e) \) and \( w_G(v) \), we get \( g = -g \implies 2g = e \), a contradiction.

Case 2: If \( r_2 = 2 \), then there exists a unique vertex \( v^* \neq v, v_e \in N_G(v^*) \). Now, comparing the weights \( w_G(v^*) \) and \( w_G(v) \), we get \( 2g = e \), a contradiction.

Case 3: If \( r_2 = 3 \), then consider a vertex \( v^* \neq v, v^* \in N_G(v_{-g}) \). Further, comparing the weights \( w_G(v^*) = g + (-g) + l(v^{**}) = w_G(v) = -g \), we get \( l(v^{**}) = -g \), a contradiction for \( l \) being one-one.

Case 4: If \( r_2 = n - 3 \), then we get \( e = n(-g) = (n - 1 - (n - 3))g \) or \( 2g = e \), a contradiction.

When \( G \) is a regular graph and \( H \) is a balanced-\( \text{dmg} \), the distance magicness and the group distance magicness of \( G \circ H \) (only when \( H \cong C_4 \)) and \( G \times H \) are characterized by Theorem 1.2 and 1.9 respectively. In the case of a non-regular graph \( G \), analogous to Problem 1.1 and 1.2, natural questions arise on the existence of group distance magic labeling of \( G \circ H \) and \( G \times H \). The following section provides partial solutions to these problems.

3. Group Distance Magic Labeling of Lexicographic Product and Direct Product of Two Graphs

Throughout this section, we assume that \( H \) is a balanced-\( \text{dmg} \) on either \( 2^k \) or \( 4k + 2 \) vertices, \( \Gamma \) is an abelian group and \( \mathcal{A} \) is an abelian group with elements, \( a_0, a_1, ..., a_{|\mathcal{A}| - 1} \), where \( a_0 \) is the identity element in \( \mathcal{A} \). Observe that \( C_4 \cong K_4 - M \) is balanced-\( \text{dmg} \) of order \( 2^2 \) and \( C_{4k+2}^2 \cong K_{4k+2} - M \) is a balanced-\( \text{dmg} \) of order \( 4k + 2 \), where \( M \) is a perfect matching.

Theorem 3.1. Let \( G \) be a graph on \( n \) vertices and \( \Gamma \) be an abelian group with \( |\Gamma| = (4k + 2)n \) such that \( \Gamma \cong \mathbb{Z}_{4k+2} \times \mathcal{A} \), where \( k \geq 1 \) and \( \mathcal{A} \) an abelian group with \( |\mathcal{A}| = n \).

(i) If the degree of the vertices of \( G \) are either all even or all odd, then \( G \circ C_{4k+2}^2 \) is \( \Gamma \)-distance magic.

(ii) If there exists a constant \( m \in \mathbb{N} \) such that \( \text{deg}_G(u) \equiv m \mod 4k + 2 \), for all \( u \in V(G) \), then \( G \times C_{4k+2}^2 \) is \( \Gamma \)-distance magic.

Proof. Let \( G \) be a graph with the vertices \( u_0, ..., u_{n-1} \) and \( H \cong C_{4k+2}^2 \) be a balanced-\( \text{dmg} \) with the vertices \( x^0, x^0', ..., x^{2k}, x^{2k} \). For any \( i \in \{0, ..., n - 1\} \), let \( H_i = \{x^0_i, x^0'_i, ..., x^{2k}_i, x^{2k}_i'\} \) be the vertices of \( G \circ H \) and \( G \times H \) that replace \( u_i \) of \( G \).

Using the isomorphism \( \phi : \Gamma \rightarrow \mathbb{Z}_{4k+2} \times \mathcal{A} \), we identify \( g \in \Gamma \) with its image \( \phi(g) = (z, a_i) \), where \( z \in \mathbb{Z}_{4k+2} \) and \( a_i \in \mathcal{A}, i \) varies from 0 to \( n - 1 \).

For all \( i \) and for \( j \in \{0, ..., 2k\} \), define a function \( l \) as,
\[
l(x^j_i) = (j, a_i) \\
l(x^j_i') = (4k + 1, a_0) - l(x^j_i).
\]
Note that, the label sum of all the vertices of $H_i$ is $(2k+1, a_0)$, which is independent of $i$.

For all $i = 0, \ldots, n - 1$, if the degree of vertex $u_i$ is $2t_i$ for some $t_i \geq 1$, then for every vertex $v \in H_i$,

$$w_{G \circ H}(v) = \sum_{v^* \in N_{G \circ H}(v), v^* \notin N_{G \circ H}[H_i](v)} l(v^*) + \sum_{v^{**} \in N_{G \circ H}[H_i](v)} l(v^{**})$$

$$= 2t_i(2k + 1, a_0) + 2k(4k + 1, a_0) = (2k + 2, a_0),$$

and, if the degree of vertex $u_i$ is $2t_i + 1$, for some $t_i \geq 0$, then for every vertex $v \in H_i$,

$$w_{G \circ H}(v) = \sum_{v^* \in N_{G \circ H}(v), v^* \notin N_{G \circ H}[H_i](v)} l(v^*) + \sum_{v^{**} \in N_{G \circ H}[H_i](v)} l(v^{**})$$

$$= (2t_i + 1)(2k + 1, a_0) + 2k(4k + 1, a_0) = (1, a_0).$$

Further, the degree of each vertex $u$ of $G$ is congruent to $m$ modulo $(4k + 2)$. Then, for every vertex $v$ of $G \times H$,

$$w_{G \times H}(v) = \sum_{v^* \in N_{G \times H}(v)} l(v^*) = 2k((4k + 2)k' + m)(4k + 1, a_0) = (-2mk, a_0).$$

\[\square\]

In the following results, we assume $G$ is a graph with vertices $u_0, \ldots, u_{n-1}$ and $H$ is a balanced-dmg with the vertices $x^0, x^0', \ldots, x^{2k-1-1}, x^{(2k-1-1)'}$, in which $x^j$ and $x^j'$ are the twin vertices for all $j \in \{0, \ldots, 2k-1-1\}$. Moreover, for any $i \in \{0, \ldots, n-1\}$, we choose $H_i = \{x_i^0, x_i^0', \ldots, x_i^{2k-1-1}, x_i^{(2k-1-1)'}\}$ as the vertex set of $G \circ H$ and $G \times H$, that replaces $u_i$ of $G$ in which $x_i^j$ and $x_i^j'$ are the twin vertices.

**Lemma 3.1.** Let $G$ be a graph on $n$ vertices and $\Gamma$ be an abelian group with $|\Gamma| = 2^k n$, where $k \geq 2$ such that $\Gamma \cong \mathbb{Z}_{2^k} \times \mathcal{A}$ for $1 \leq s \leq k - 1$, $\mathcal{A}$ an abelian group with $|\mathcal{A}| = 2^{k-s} n$. Let $H$ be a balanced-dmg on $2^k$ vertices. Then,

(i) $G \circ H$ is $\Gamma$-distance magic.

(ii) If there exists a constant $m \in \mathbb{N}$ such that $\deg_G(u) \equiv m \mod 2^s$ for all $u \in V(G)$, then $G \times H$ is $\Gamma$-distance magic.

**Proof.** Using the isomorphism $\phi : \Gamma \to \mathbb{Z}_{2^k} \times \mathcal{A}$, we identify $g \in \Gamma$ with its image $\phi(g) = (z, a_i)$, where $z \in \mathbb{Z}_{2^k}$ and $a_i \in \mathcal{A}$, $i$ varies from 0 to $2^{k-s} n - 1$.

For $i \in \{0, \ldots, n - 1\}$ and $\alpha \in \{0, \ldots, 2^{s-1} - 1\}$, define a function $l$ as

$$l(x_i^j) = (\alpha, a_j \mod 2^{k-s} + 2^{k-s}),$$

where $\alpha 2^{k-s} \leq j \leq (\alpha + 1)2^{k-s} - 1$,

$$l(x_i^j) = (2^s - 1, a_0) - l(x_i^j).$$

Now for each $i = 0, \ldots, n - 1$, the label sum of all the vertices of $H_i$ is,

$$2^{k-1}(2^s - 1, a_0) = (-2^{k-1}, a_0) = (0, a_0),$$

which is the identity element of $\mathbb{Z}_{2^k} \times \mathcal{A}$ and label sum is independent of $i$. 

Note that, the degree of each vertex of $H$ is $2r$. For all $i = 0, \ldots, n - 1$, the vertex $v \in H_i$ has weight,

$$w_{G \circ H}(v) = \sum_{v^* \in N_{G \circ H}(v)} l(v^*) + \sum_{v^{**} \in N_{G \circ H_1}(v)} l(v^{**})$$

$$= \deg_{G_1}(v_i)(0, a_0) + r(2^s - 1, a_0) = (-r, a_0).$$

Moreover, if the degree of each vertex $u$ of $G$ is congruent to $m$ modulo $2^s$ then, for every $v$ of $G \times H$,

$$w_{G \times H}(v) = \sum_{v^* \in N_{G \times H}(v)} l(v^*) = r(2^s k' + m)(2^s - 1, a_0) = (-mr, a_0).$$

\[\Box\]

**Lemma 3.2.** Let $G$ be a graph on $n$ vertices and $\Gamma$ be an abelian group with $|\Gamma| = 2^k n$, such that $\Gamma \cong \mathbb{Z}_{2^k} \times A$, where $2 \leq k \leq s$ and $A$ is an abelian group with $|A| = 2^{k-s} n$. Let $H$ be a balanced-dmg on $2^k$ vertices.

(i) If there exists a constant $m \in \mathbb{N}$ such that $\deg_G(u) \equiv m \mod 2^s - 1$ for all $u \in V(G)$, then $G \circ H$ is $\Gamma$-distance magic.

(ii) If there exists a constant $m \in \mathbb{N}$ such that $\deg_G(u) \equiv m \mod 2^s$ for all $u \in V(G)$, then $G \times H$ is $\Gamma$-distance magic.

**Proof.** Using the isomorphism $\phi : \Gamma \to \mathbb{Z}_{2^k} \times A$, we identify $g \in \Gamma$ with its image $\phi(g) = (z, a_i)$, where $z \in \mathbb{Z}_{2^k}$, and $a_i \in A$, $i$ varies from 0 to $2^{k-s} n - 1$.

Consider the function $l$,

$$l(x_i) = ((2^{k-1}i + j) \mod 2^s - 1, a_{[2^{s-1}j]}),$$

$$l(x_i') = (2^s - 1, a_0) - l(x_i),$$

where $i \in \{0, \ldots, n - 1\}$ and $j \in \{0, \ldots, 2^{k-1} - 1\}$. For each $i = 0, \ldots, n - 1$, the label sum of all the vertices of $H_i$ is $(-2^{k-1}, a_0)$, which is independent of $i$. Recall that the degree of any vertex of $H_i$ is $2r$. Since the degree of any vertex $u$ of $G$ is congruent to $m$ modulo $2^s - 1$, for all $i = 0, \ldots, n - 1$, the vertex $v \in H_i$ has weight,

$$w_{G \circ H}(v) = \sum_{v^* \in N_{G \circ H}(v)} l(v^*) + \sum_{v^{**} \in N_{G \circ H_1}(v)} l(v^{**})$$

$$= (2^{s-1}k' + m)(-2^{k-1}, a_0) + r(2^s - 1, a_0) = (-r - 2^{k-1}m, a_0).$$

On the other hand, if the degree of any vertex $u$ of $G$ is congruent to $m$ modulo $2^s$ then for every $v$ of $G \times H$,

$$w_{G \times H}(v) = \sum_{v^* \in N_{G \times H}(v)} l(v^*) = r(2^s k' + m)(2^s - 1, a_0) = (-mr, a_0).$$

\[\Box\]

**Theorem 3.2.** Let $G$ be a graph on $n$ vertices and $\Gamma$ be an abelian group with $|\Gamma| = 2^k n$, where $k \geq 2$, $n = 2^s(2^t + 1)$, for some non-negative integers $s$ and $t$. Let $H$ be a balanced-dmg on $2^k$ vertices.

(i) If there exists a constant $m \in \mathbb{N}$ such that $\deg_G(u) \equiv m \mod 2^{k+s-1}$ for all $u \in V(G)$, then $G \circ H$ is $\Gamma$-distance magic.
(ii) If there exists a constant $m \in \mathbb{N}$ such that $\deg_G(u) \equiv m \mod 2^{k+s}$ for all $u \in V(G)$, then $G \times H$ is $\Gamma$-distance magic.

**Proof.** By the fundamental theorem of finite abelian groups, $\Gamma \cong \mathbb{Z}_{2^m} \times \mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_r^{n_r}}$, where $2^k n = 2^n \prod_{i=1}^{r} p_i^{n_i}$, $p_i$’s not necessarily distinct primes and $n_0 > 0$.

Note that if any vertex $u$ of $G$ is such that $\deg_G(u) \equiv m \mod 2^{k+s}$, then there exist unique integers $m_i$’s such that $\deg_G(u) \equiv m_i \mod 2^{n_0}$, where $n_0 \in \{1, \ldots, k+s-1\}$.

For each $\Gamma$ isomorphic to $\mathbb{Z}_{2^m} \times A$ with $1 \leq n_0 \leq k-1$, the result follows from Lemma 3.1. And for each $\Gamma$ isomorphic to $\mathbb{Z}_{2^m} \times A$ with $k \leq n_0 \leq k + s$, the result follows from Lemma 3.2 where $A$ is an abelian group with $|A| = \frac{n}{2^m}$. □

**Theorem 3.3.** Let $G$ be a graph on $n$ vertices and $\Gamma$ be an Abelian group with $|\Gamma| = 2^k n$, where $k \geq 2$. If all the vertices of $G$ are of even degree and $H$ is a balanced-dmg on $2^k$ vertices, then $G \circ H$ is a $\Gamma$-distance magic graph.

**Proof.** If $\Gamma$ is isomorphic to $\mathbb{Z}_{2^p} \times A$ for $p \in \{1, \ldots, k-1\}$, then the result follows from Lemma 3.1. Now, suppose that $\Gamma$ is isomorphic to $\mathbb{Z}_{2^k} \times A$, where $|A| = n$.

Using the isomorphism $\phi : \Gamma \to \mathbb{Z}_{2^k} \times A$, we identify $g \in \Gamma$ with its image $\phi(g) = (z, a_i)$, where $z \in \mathbb{Z}_{2^k}$ and $a_i \in A$, $i$ varies from 0 to $n-1$.

For all $i \in \{0, \ldots, n-1\}$ and $j \in \{0, \ldots, 2^{k-1} - 1\}$, define $l$ on $G \circ H$ as,

$$l(x_i^j) = (2j, a_i),$$

$$l(x_i^0) = (2^k - 1, a_0) - l(x_i^j).$$

Note that, the label sum of all the vertices of $H_i$ is,

$$(2^{k-1}(2^k - 1) \mod 2^k, a_0) = (-2^{k-1}, a_0),$$

which is independent of $i$.

Since for any $u_i$ of $G$, $\deg_G(u_i) = 2t_i$ with $t_i \geq 1$ and for any $x$ of $H$, $\deg_H(x) = 2r$, then the degree of $v$ in $G \circ H$ is $2(2^k t_i + r)$. Now, the weight of any vertex $v \in H_i$ is,

$$w_{G \circ H}(v) = \sum_{v^* \in N_{G \circ H}(v)} l(v^*) + \sum_{v^{**} \in N_{G \circ H[H_i]}(v)} l(v^{**})$$

$$= 2t_i(-2^{k-1}, a_0) + r(2^k - 1, a_0) = (-r, a_0).$$

□

**Corollary 3.1.** Let $t$ be an odd integer. Let $G \cong K_{m_1, m_2, \ldots, m_t}$ be a complete $t$-partite graph with, $m = \sum_{i=1}^{t} m_i$ and either all $m_i$’s are even or all $m_i$’s are odd. If $\Gamma$ is an abelian group with $|\Gamma| = 2^k m$, then $G \circ H$ is a $\Gamma$-distance magic graph, where $H$ is a balanced-dmg on $2^k$ vertices.

□

For an abelian group $\Gamma$, the following result discusses the $\Gamma$-distance magic labeling of $K_{m,n} \circ H$, where $m$ and $n$ are of different parity and $H$ is a balanced-dmg on $2^k$ vertices.

**Theorem 3.4.** Let $K_{m,n}$ be a complete bipartite graph with $m$ even and $n$ odd and let $\Gamma$ be an abelian group with $2^k (m+n)$ elements, where $k \geq 2$. If $H$ is a $2r$-regular balanced-dmg on $2^k$ vertices and $r$ is odd, then $K_{m,n} \circ H$ is $\Gamma$-distance magic.

□
Proof. If Γ is isomorphic to \( \mathbb{Z}_{2^k} \times A \) with \( p \in \{1, ..., k-1\} \), then the assertion follows from Lemma 3.1. Suppose that Γ is isomorphic to \( \mathbb{Z}_{2^k} \times A \), where \( |A| = m + n \).

Let \( G \cong K_{m,n} \) have the partition sets \( X = \{u_0, ..., u_{m-1}\} \) and \( Y = \{v_0, ..., v_{n-1}\} \). Then for each \( i \in \{0, ..., m-1\} \), let \( X_i = \{x_i^0, x_i^0, ..., x_i^{2^{k-1}-1}, x_i^{(2^{k-1}-1)'}\} \) be the vertex set of \( G \circ H \), that replace the vertex \( u_i \) of \( G \). Similarly, for each \( j \in \{0, ..., n-1\} \), let \( Y_j = \{y_j^0, y_j^0, ..., y_j^{2^{k-1}-1}, y_j^{(2^{k-1}-1)'}\} \) be the vertex set of \( G \circ H \), that replace the vertex \( v_j \) of \( G \).

Now, let the vertex set of \( G \circ H \) be \( X' \cup Y' \), where \( X' = \bigcup_{i=0}^{m-1} X_i \) and \( Y' = \bigcup_{j=0}^{n-1} Y_j \).

Using the isomorphism \( \phi: \Gamma \to \mathbb{Z}_{2^k} \times A \), we identify \( g \in \Gamma \) with its image \( \phi(g) = (z, a_i) \), where \( z \in \mathbb{Z}_{2^k} \) and \( a_i \in A \), \( i \) varies from 0 to \( m + n - 1 \).

For each \( q \in \{0, ..., 2^{k-1} - 1\} \), define \( l \) on \( X' \) as,

\[
l(x_i^q) = ((2^{k-1} + 1)q, a_i), \quad \text{and} \quad l(x_i^q) = (2^{k-1} - 1, a_0) - l(x_i^q), \quad \text{for all } i = 0, ..., m - 1.
\]

Again, for each \( q \in \{0, ..., 2^{k-1} - 1\} \), define \( l \) on \( Y' \) as,

\[
l(y_j^q) = (2q, a_{m+j}), \quad \text{and} \quad l(y_j^q) = (2^k - 1, a_0) - l(y_j^q), \quad \text{for all } j = 0, ..., n - 1.
\]

Then for all \( i \in \{0, ..., m - 1\} \), the label sum of all vertices of \( X_i \) is,

\[
(2^{k-1}(2^{k-1} - 1), a_0) = (2^{k-1}, a_0).
\]

Similarly, for all \( j \in \{0, ..., n - 1\} \), the label sum of all vertices of \( Y_j \) is,

\[
(2^{k-1}(2^{k-1} - 1), a_0) = (2^{k-1}, a_0).
\]

Further, since \( H \) is \( 2(2t + 1) \)-regular graph, for every \( x \) of \( X_i \),

\[
w_{G \circ H}(x) = \sum_{x^* \in N_{G \circ H}(x)} l(x^*) + \sum_{x^{**} \in N_{G \circ H}(X_i)(x)} l(x^{**})
= n(2^{k-1}, a_0) + (2t + 1)(2^{k-1} - 1, a_0) = (-2t + 1), a_0),
\]

and for every \( y \) of \( Y_j \),

\[
w_{G \circ H}(y) = \sum_{y^* \in N_{G \circ H}(y)} l(y^*) + \sum_{y^{**} \in N_{G \circ H}(Y_j)(y)} l(y^{**})
= m(2^{k-1}, a_0) + (2t + 1)(2^{k} - 1, a_0) = (-2t + 1), a_0),
\]

which completes the proof. \( \square \)

4. Conclusion

In this paper, we obtain few necessary conditions for a graph to be group distance magic and characterize the group distance magic labeling of a tree, few subclasses of bi-regular graphs and the lexicographic and direct product of a non-regular graph with a balanced distance magic graph.
Notes on group distance magicness of product graphs

REFERENCES

[1] J. A. Bondy and U. S. R. Murty, Graph Theory, Springer, New York, 2008.
[2] R. Hammack, W. Imrich and S. Klavžar, Handbook of Product Graphs, CRC Press, Boca Raton, FL, 2011.
[3] V. Vilfred, $\sum$-labelled graphs and circulant graphs, Ph.D. thesis, University of Kerala, Trivandrum, India, 1994.
[4] M. Miller, C. Rodger and R. Simanjuntak, Distance magic labelings of graphs, Australasian Journal of Combinatorics 28 (2003), 305–315.
[5] K. Sugeng, D. Froncek, M. Miller, T. Ryan and J. Walker, On distance magic labeling of graphs, Journal of Combinatorial Mathematics and Combinatorial Computing 71 (2009), 39–48.
[6] D. Froncek, P. Kovár and T. Kovárová, Fair incomplete tournaments, Bulletin of the Institute of Combinatorics and its Applications 48 (2006), 31–33.
[7] M. I. Jinnah, On $\sum$-labelled graphs, in: Technical Proceedings of Group Discussion on Graph Labeling Problems, (eds.) B.D. Acharya and S.M. Hedge (1999), 71–77.
[8] S. B. Rao, Sigma graphs—a survey, in: Labelings of Discrete Structures and Applications, (eds.) B.D. Acharya, S. Arumugam, A. Rosa, Narosa Publishing House, New Delhi (2008) 135–140.
[9] M. K. Shafiq, G. Ali and R. Simanjuntak, Distance magic labelings of a union of graphs, AKCE International Journal of Graphs and Combinatorics 6 (2009), 191–200.
[10] S. Cichacz and A. Görlich, Constant sum partition of sets of integers and distance magic graphs, Discussiones Mathematicae Graph Theory 38 (2018), 97–106.
[11] M. Anholcer, S. Cichacz, I. Peterin and A. Tepeh, Distance magic labeling and two products of graphs, Graphs and Combinatorics 31 (2015), 1125–1136.
[12] S. Arumugam, N. Kamatchi and P. Kovár, Distance magic graphs, Utilitas Mathematica 99 (2016), 131–142.
[13] D. Froncek, Group distance magic labeling of cartesian product of cycles, Australasian Journal of Combinatorics 55 (2013), 167–174.
[14] D. Combe, A. Nelson and W. Palmer, Magic labelings of graphs over finite abelian groups, Australasian Journal of Combinatorics 29 (2004), 259–272.
[15] I. N. Herstein, Topics in algebra, John Wiley & Sons, New York, 2006.
[16] S. Cichacz, Distance magic graphs $G \times C_n$, Discrete Applied Mathematics 177 (2014), 80–87.
[17] S. Cichacz, Note on group distance magic graphs $G[C_4]$, Graphs and Combinatorics 30 (2014), 565–571.
[18] S. Cichacz, Note on group distance magic complete bipartite graphs, Open Mathematics 12 (2014), 529–533.
[19] M. Anholcer, C. Sylwia, I. Peterin and A. Tepeh, Group distance magic labeling of direct product of graphs, Ars Mathematica Contemporanea 9 (2014), 93–107.

A V Prajeesh, Department of Mathematics, National Institute of Technology Calicut, Kozhikode 673601, India.
E-mail address: prajeesh_p150078ma@nitc.ac.in

Krishnan Paramasivam, Department of Mathematics, National Institute of Technology Calicut, Kozhikode 673601, India.
E-mail address: sivam@nitc.ac.in