Quantum theory of human communication
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Abstract

We use notions and techniques of Quantum Field Theory to formulate and investigate basic concepts and mechanisms of human communication. We start with attitudes which correspond to photons frequencies, then we introduce states-of-mind which correspond to wave functions. Finally, by way of the second quantization, we come to states-of-opinions which correspond to states of quantized radiation fields. In the present paper we shall only investigate superpositions of pairs of coherent states (e.g. the government and the opposition in a democratic country).

Contents

1 Introduction 2

2 The first quantization 4
   2.1 Profiles .............................. 4
   2.2 Attitudes ................................ 4
   2.3 The first quantization: from attitudes to states-of-mind .... 4
   2.4 Questions as observables .......................... 6

3 The second quantization 7
   3.1 Opinions and the Bose-Fock space for states-of-opinion ...... 7
   3.2 Occupation numbers and their statistics .......................... 9
   3.3 Coherent states .................................. 11
1 Introduction

Suppose we want to investigate the social behavior of a human community relative to a chosen subject. We expose the subject by producing a questionnaire with entries containing descriptions of the components of the subject in question. The questionnaire can be completed in several ways which we shall call attitudes. However, the process of filling in a questionnaire in a specific way is not a deterministic procedure. Usually, the mind of a respondent faces a number of preferences. Under casual influence from outside the preferences can change. Hence, the reaction of a respondent is not attached to a particular attitude but depends on the respondent’s state-of-mind.

A statement of a respondent selecting an attitude or a state-of-mind shall here be called a bit-of-information.

Fix a finite set of frequences and consider photons with frequencies from this set. Hence the momentum space for these photons consists of finitely many points and their wave-functions are superpositions of the frequencies. We create our social model by way of the following substitutions,

1) bits-of-information replace photons
2) attitudes replace frequencies
3) states-of-mind replace wave-functions.

As we know, the instance of passing from frequencies to wave-functions, and hence from attitudes to states-of-mind, constitutes the essence of the first quantization.

Totalities of states-of-mind produce new entities called opinions. Then states-of-opinion emerge as the result of the process of the second quantization which mathematically amounts to the repetition of the one used in constructing a free quantum radiation field (cf.[1], [2]). Hence

4) states-of-opinion replace quantized radiation fields.

To describe the second quantization we use the algebraic version of the concept of the Bose-Fock space, the so-called Bose algebra (cf.[4]). In [5] states-of-opinion were called information metabolisms.

The observables of the theory are just questions. Given a question and a state-of-opinion, the occupation number formalism provides the expectation
of the number of positive answers to the question in a poll performed under the given state-of-opinion (cf. [6]).

We treat bits-of-information circulating in human communities as bosons. The propagators of those bits-of-information are the individual respondents which, depending on their states-of-mind, provide answers yes or no to questions. Also organizations can get the status of respondents and quality of possessing a state-of-mind. Questions are coupled with orthogonal projections in the space of states-of-mind. Affirmative answers are weighted by the assigned number of energy-bits they carry: electing a Member of Parliament requires many energy-bits in the form of single votes whereas a shareholder’s single vote carries the number of energy-bits equal to the number of owned shares.

In what follows we shall mainly be interested in superpositions of coherent states-of-opinion. The coherent states considered in this paper are mathematically identical with those of quantum optics. They are defined within the polynomial representation of the Bose-Fock space (cf.[4]). In the present paper we restrict ourselves to analysis of states which are superpositions of two coherent states, e.g. the government and the opposition in a democratic country, the original inhabitants of a country and the immigrants, Christians and Moslems etc. Such states will be called bicoherent. We shall show that the bicoherent states depend on two parameters. The first, the interaction coefficient, is a number between 0 and 1 measuring the background for communication between respondents of interacting factions: a common language, traditions, religion, interest etc. The second one is called the superposition constant.

In a simple model constructed in Section 5, the superposition constant plays a double role - if it is greater or equal to one, it prevents interaction blocking the influence of the interaction coefficient. If negative, it controls regions of high and low frequencies of affirmation to questions asked in the superposition state-of-opinion. Moreover, under high interaction a sudden critical switch of opinion can occur in consequence of a minimal change of the superposition constant (cf. Remark 7). We also investigate change of opinion under temporary influence of some outside factors (as for instance election campaigns). It is shown that such a temporary influence diminishes high amplitudes (cf. Remark 8).

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his contribution.

2 The first quantization

2.1 Profiles

A selected sub-population characterized by a collection of attitudes will here be called a profile. For example the body of parliament members of a democratic country constitutes a profile. The attitudes will represent different political affiliations. The states-of-mind will then concern actual political problems. Also the government and its members can be considered as a profile. Here the set of attitudes will include different policies. For the profile of workers, the relevant attitudes will be concerned with the unions.

Hence, the same physical population consists of several different profiles. Profiles connected with a profession is easiest revealed by asking a question to which the answer “yes” selects the states-of-mind of the profession. For example, the question ”do you have a valid certificate qualifying you as a physician?” automatically extracts the profile of medical doctors. An examination will filter respondents of the profile of a particular profession. The whole population itself constitutes a profile as well.

2.2 Attitudes

Consider a community familiar with subjects which can be presented in a list of statements. The statements can be accepted or rejected by members of the community. In what follows we refer to this list of statements as a questionnaire. The term ”questionnaire” should not be taken literally. For instance, a questionnaire may consist of a set of examination questions but it can as well be an ordinary questionnaire prepared for a poll.

A copy of a completed questionnaire shall be called an attitude. The quantum mechanical counterpart of an attitude is a frequency. Hence the quantum mechanical equivalence of a space of attitudes is a momentum space consisting of finitely many frequencies.

2.3 The first quantization: from attitudes to states-of-mind

Take a space of attitudes consisting of \( n \) attitudes \( \{1, 2, ..., n\} \). Consider the real-valued functions \( x \) of \( n \) real variables \( t_1, t_2, ..., t_n \). To the attitude \( j \) we
attach the function $e_j$, which is the value of the variable $t_j$,

$$e_j(t_1, t_2, \ldots, t_n) = t_j.$$  

We shall consider the real vector space $\mathcal{F}$ of vectors

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_k e_k,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are arbitrary real numbers. Given another vector from $\mathcal{F}$,

$$y = \eta_1 e_1 + \eta_2 e_2 + \cdots + \eta_k e_k,$$

we define the inner product (Hermitian form) setting

$$\langle x, y \rangle = \lambda_1 \eta_1 + \lambda_2 \eta_2 + \cdots + \lambda_n \eta_n$$

so that $e_1, e_2, \ldots, e_n$ is an orthonormal basis in $\mathcal{F}$ and each vector $x$ from $\mathcal{F}$ can be written in the form

$$x = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 + \cdots + \langle x, e_n \rangle e_n.$$  

Then

$$\langle x, y \rangle = \langle x, e_1 \rangle \langle y, e_1 \rangle + \langle x, e_2 \rangle \langle y, e_2 \rangle + \cdots + \langle x, e_n \rangle \langle y, e_n \rangle.$$  

We shall write $|x|$ for the length of the vector $x$,

$$|x| = \langle x, x \rangle^{\frac{1}{2}}.$$  

A vector $x$ is called a state-of-mind if $|x| = 1$. We do not distinguish between states provided by $x$ and $-x$. Briefly we shall write

$$x_j \overset{\text{def}}{=} \frac{x}{|x|}$$

for the state-of-mind corresponding to the vector $x$.

If a respondent is in the state-of-mind $x$, his attitude will be $j$ with probability $\langle x, e_j \rangle^2$, i.e. to the question “which is your attitude?” he will name the attitude $j$ with probability $\langle x, e_j \rangle^2$.

The (real) vector space $\mathcal{F}$ shall be called the space of states-of-mind (as yet we have no interpretation for the process of multiplication by the imaginary unit $i$).

Continuing the analogy with photons, the space $\mathcal{F}$ of states-of-mind is the counterpart of the space of wave-functions (the state-space for photons)
depending on fixed finite number of frequencies. We just substitute attitudes for frequencies.

Given states-of-mind $x$ and $y$, the number $\langle x, y \rangle^2$ is called the *correlation* of $x$ and $y$. States for which the correlation is equal to zero shall be called *uncorrelated*.

Observe that the space $\mathcal{F}$ can be considered as the space of all real-valued functions $x$ on the set $\{1, 2, ..., n\}$, each such function assigning a real number $\lambda_j$ to $j$ from the set $\{1, 2, ..., n\}$.

### 2.4 Questions as observables

The process of assigning an attitude $j$ to a respondent can be "first quantized" to a question directed to a respondent, "are you fully accepting the attitude $j$?" The question itself then becomes an observable taking the form of the projection

$$Q_{e_j} = \langle e_j, \cdot \rangle e_j.$$ 

Now the procedure can be extended over arbitrary states-of-mind by attaching to a state-of-mind $x$ the projection

$$Q_x = \langle x, \cdot \rangle x$$

which directed to a respondent runs as follows

"are you in the state-of-mind $x$?"

We attach statistics to this question by way of the statement

$$\langle Q_x y, y \rangle^2 = \left\{ \begin{array}{ll}
\text{the probability of obtaining the} \\
\text{answer "yes" to the question $Q_x$} \\
\text{from a respondent in state $y$}
\end{array} \right.$$ 

i.e. the probability of the answer "yes" is equal to the correlation of $x$ and $y$.

More general questions $Q$ are linear combinations of questions of the form $Q_x$. Then

$$\langle Q_y, y \rangle = \left\{ \begin{array}{ll}
\text{the probability of obtaining the} \\
\text{answer "yes" to the question $Q$} \\
\text{from a respondent in state $y$}
\end{array} \right.$$ 

As an example we consider the projection

$$Qx = \langle e_i, x \rangle e_i + \langle e_j, x \rangle e_j,$$
where \( i \) and \( j \) are different attitudes. The question corresponding to this projection should read "do you favor precisely the attitudes \( i \) and \( j \) out of the collection of all possible attitudes?" Here we have \( Qx = x \), exactly for \( x = \langle e_i, x \rangle e_i + \langle e_j, x \rangle e_j \) which means that the answer "yes" comes with probability one from the states \( x = \lambda e_i + \eta e_j \), with \( \lambda^2 + \eta^2 = 1 \).

As explained in the Introduction, each affirmative answer to a question carries a number of energy-bits depending on the nature of the corresponding model.

3 The second quantization

The notions of attitude and state-of-mind concern individual respondents. The second quantization provides a formalism by use of which the parallel notions on the level of profiles can be defined (cf. [1], [4]). The counterpart of the notion of attitude attached to an individual member of a community will be the notion of opinion attached to a group of individuals. Similarly the counterpart of the notion of state-of-mind attached to a respondent will be the notion of state-of-opinion attached to a profile (which can as well be the whole community). As a state-of-mind assigns a number to every possible attitude, a state-of-opinion will assign a number to every possible opinion of a profile, i.e. the states-of-opinion are functions over the space of opinions. Given such a function, the square of its value on an opinion gives the probability that the profile shares this opinion.

3.1 Opinions and the Bose-Fock space for states-of-opinion

Suppose that from a poll we have gathered information about the actual distribution of attitudes of a profile. It means that we have a collection of attitudes, where the same attitude may appear many times, a single time or not at all. To obtain the precise definition we proceed as follows.

Let \( \{1, 2, ..., n\} \) be the set of all attitudes. Then a tuple of positive integers \( (k_1, k_2, ..., k_n) \) shall be called an opinion in which the attitude \( j \) appears \( k_j \) times for \( j = 1, 2, ..., n \). If a particular attitude, say \( i \), does not appear at all, we write \( k_i = 0 \). A poll assigns to each attitude the number of respondents sharing this attitude i.e. it provides the opinion of the community. We use the (real) Bargmann version of the Bose-Fock space construction. Write \( \tilde{F} \) for the algebra of all formal series

\[
f = \sum \lambda_{k_1,k_2,...,k_n} e_{k_1,k_2,...,k_n},
\]
where $e_{k_1,k_2,...,k_n}$ are products of variables $t_1, t_2, ..., t_n$:

$$e_{k_1,k_2,...,k_n}(t_1, t_2, ..., t_n) \equiv t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n}$$

and the sum runs through all the tuples $(k_1, k_2, ..., k_n)$ of non-negative integers.

We multiply the series in the standard way setting

$$e_{j_1,j_2,...,j_n} e_{k_1,k_2,...,k_n} = e_{j_1+k_1,j_2+k_2,...,j_n+k_n}.$$  

We postulate that a profile which consists of $k_1$ members carrying the state-of-mind $e_1$, $k_2$ members carrying the state-of-mind $e_2$ etc. up to $k_n$ members carrying the state-of-mind $e_n$, is in the state-of-opinion

$$\sqrt{k_1! k_2! \cdots k_n!}.$$  

The set \( \left\{ \frac{e_{k_1,k_2,...,k_n}}{\sqrt{k_1! k_2! \cdots k_n!}} \right\} \), where \((k_1, k_2, ..., k_n)\) runs through all possible different opinions, constitutes an orthonormal basis in the Bargmann Bose-Fock space representation

$$\Gamma F = \left\{ f \in \bar{F} : \sum_{k_1,k_2,...,k_n} \lambda_{k_1,k_2,...,k_n}^2 k_1! k_2! \cdots k_n! < \infty \right\},$$

where

$$\langle e_{j_1,j_2,...,j_n}, e_{k_1,k_2,...,k_n} \rangle = k_1! k_2! \cdots k_n! \delta_{j_1,k_1} \delta_{j_2,k_2} \cdots \delta_{j_n,k_n}.$$  

Hence, given

$$\sum_{j_1,j_2,...,j_n} \eta_{j_1,j_2,...,j_n} e_{j_1,j_2,...,j_n} \in \Gamma F,$$

$$\sum_{k_1,k_2,...,k_n} \lambda_{k_1,k_2,...,k_n} e_{k_1,k_2,...,k_n} \in \Gamma F,$$

we have

$$\left\langle \sum_{j_1,j_2,...,j_n} \eta_{j_1,j_2,...,j_n} e_{j_1,j_2,...,j_n}, \sum_{k_1,k_2,...,k_n} \lambda_{k_1,k_2,...,k_n} e_{k_1,k_2,...,k_n} \right\rangle$$

$$= \sum_{j_1,j_2,...,j_n} \sum_{k_1,k_2,...,k_n} \eta_{j_1,j_2,...,j_n} \lambda_{k_1,k_2,...,k_n} \langle e_{j_1,j_2,...,j_n}, e_{k_1,k_2,...,k_n} \rangle$$

$$= \sum_{j_1,j_2,...,j_n} \sum_{k_1,k_2,...,k_n} \eta_{j_1,j_2,...,j_n} \lambda_{k_1,k_2,...,k_n} \lambda_{k_1,k_2,...,k_n} k_1! k_2! \cdots k_n! \delta_{j_1,k_1} \delta_{j_2,k_2} \cdots \delta_{j_n,k_n}$$

$$= \sum_{k_1,k_2,...,k_n} \eta_{k_1,k_2,...,k_n} \lambda_{k_1,k_2,...,k_n} k_1! k_2! \cdots k_n!.$$  

8
For $x_1, x_2, \ldots, x_n, y$ from $F$ we have the following useful formula (cf.[4])

$$\langle x_1 x_2 \cdots x_n, y^m \rangle = \begin{cases} m! \langle x_1, y \rangle \langle x_2, y \rangle \cdots \langle x_m, y \rangle & \text{for } m = n \\ 0 & \text{otherwise} \end{cases}$$

A state-of-opinion will be a vector $f$ from $\Gamma F$ such that $\langle f, f \rangle = 1$. This way, for a state-of-opinion $f = \sum \lambda_{k_1,k_2,\ldots,k_n} e_{k_1,k_2,\ldots,k_n}$ we have $\sum \lambda_{k_1,k_2,\ldots,k_n}^2 = 1$, and for each opinion $(k_1, k_2, \ldots, k_n)$ the number $\lambda_{k_1,k_2,\ldots,k_n}$ represents the probability that the members of the concerned profile share the opinion $(k_1, k_2, \ldots, k_n)$. We identify states-of-opinion $f$ and $-f$. If all $k_j = 0$, then we get the vector $\phi$, $\phi(t_1, t_2, \ldots, t_n) = 1$, called the vacuum vector.

Notice that states-of-opinion can be interpreted as functions defined on the space of opinions, each such function assigning to an opinion $(k_1, k_2, \ldots, k_n)$ a real number $\lambda_{k_1,k_2,\ldots,k_n}$.

**Remark 1** In the present paper there is no need to take for $\lambda_{k_1,k_2,\ldots,k_n}$ the complex numbers. Should such a need occur in the future, the necessary adjustments are elementary.

We shall need the notion of the operator $w^*$ of annihilation by an element $w$ from $F$ (cf.[4]). We define $w^*$ first for the basis vectors $e_j$ of $F$ setting for $f \in -F$

$$(e_j^* f)(t_1, t_2, \ldots, t_n) = \frac{\partial}{\partial t_j} f(t_1, t_2, \ldots, t_n),$$

and then extend it linearly to include all $w$ from $F$.

The only infinite sums we will use are the elements of $\Gamma F$ called coherent vectors, which are the exponential functions

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

of $x \in F$. It is easy to verify that

$$\langle e^x, e^y \rangle = e^{\langle x, y \rangle}.$$

### 3.2 Occupation numbers and their statistics

To every orthogonal projection $Q$ in $F$ and every natural number $k$ we assign a projection $Q^{(k)}$ in $\Gamma F$ which we define as follows:

Take $x_1, \ldots, x_p, y_1, \ldots, y_q \in F$ such that $Q x_j = x_j$ for $j = 1, 2, \ldots, p$ and $Q y_i = 0$ for $i = 1, 2, \ldots, q$. Then we define
\[ Q^{(k)}(x_1 x_2 \cdots x_p y_1 y_2 \cdots y_q) \overset{\text{def}}{=} \begin{cases} x_1 x_2 \cdots x_p y_1 y_2 \cdots y_q & p = k \\ 0 & \text{otherwise} \end{cases} \]

It is easy to extend \( Q^{(k)} \) to an orthogonal projection in \( \Gamma \mathcal{F} \). The projection \( Q^{(k)} \) is an observable in the space of states-of-opinion and corresponds to the question:

Is there exactly \( k \) answers "yes" to the question \( Q \)?

Consequently, for a state-of-opinion \( f \) we have

\[ \langle Q^{(k)} f, f \rangle = \begin{cases} \text{the probability that in the state } f \text{ we get precisely } k \text{ answers "yes" to } Q \end{cases} \]

Let \( f \) be a state, i.e. let \(|f| = 1\). The numbers \( \langle Q^{(k)} f, f \rangle \) are called the occupation numbers of affirmation of \( Q \) in the state \( f \).

We extend \( Q \) to a derivation \( d\Gamma Q \), i.e. a transformation obeying the Leibniz rule,

\[ (d\Gamma Q) f g = (d\Gamma Q f) g + f (d\Gamma Q g) \]

This operation is often called the second quantization of \( Q \). It is easy to verify that the spectral decomposition of \( d\Gamma Q \) is

\[ d\Gamma Q = \sum_{k=0}^{\infty} k Q^{(k)}. \]

Hence, if \( f \) is a state-of-opinion, then

\[ \langle d\Gamma Q f, f \rangle = \sum_{k=0}^{\infty} k \langle Q^{(k)} f, f \rangle = \begin{cases} \text{the expected number of energy-bits} \\
\text{coming from the affirmative} \\
\text{answers to } Q \text{ in the state-of-opinion } f. \end{cases} \]

However, it is not the expected number of energy-bits coming from the affirmative answers which is measured by a poll but the expected percentage \( R(Q, f) \) of those energy-bits,

\[ R(Q, f) = \frac{\langle d\Gamma Q f, f \rangle}{\langle d\Gamma I f, f \rangle} = \begin{cases} \text{the expected percentage of energy-} \\
\text{bits coming from the affirmative} \\
\text{answers to } Q \text{ in the state-of-opinion } f. \end{cases} \]

We shall call \( R(Q, f) \) the relative expectation for the energy of affirmation of \( Q \) in the state \( f \). Here the identity operator \( I \) corresponds to the question: "How many energy-bits are available"?
Given a state-of-opinion $f$, we can produce a new one by making a superposition of $f$ with the vacuum

$$ (f + \alpha \phi) = \frac{f + \alpha \phi}{\sqrt{1 + \alpha^2 + 2 (\phi, f)}}. $$

Since

$$ \langle d\Gamma Q (f + \alpha \phi), f + \alpha \phi \rangle = \langle d\Gamma Q f, f \rangle, $$

we get

$$ R(Q, (f + \alpha \phi)) = \frac{\langle d\Gamma Q (f + \alpha \phi), (f + \alpha \phi) \rangle}{\langle d\Gamma I (f + \alpha \phi), (f + \alpha \phi) \rangle} = \frac{\langle d\Gamma Q f, f \rangle}{\langle d\Gamma I f, f \rangle} = R(Q, f) $$

which means that the superposition with the vacuum does not change the percentage of energy-bits coming from affirmation of $Q$.

### 3.3 Coherent states

A coherent state-of-opinion describes respondents with states-of-mind concentrated around a special state-of-mind called the mode of coherence, e.g. physicians with their professional curriculum as the mode, members of a political party with their party program as the mode, lawyers with their professional know-how as the mode etc.

Take a vector $x$ from the states-of-mind space $\mathcal{F}$. The coherent state $c(x)$ generated by $x$ is the normalized coherent vector $e^x$,

$$ c(x) = e^x = e^{-\frac{1}{2} (x,x) e^x} $$

$$ c(0) = \phi. $$

Observe that if the number $\langle x - y, x - y \rangle$ is very large, the correlation

$$ \langle c(x), c(y) \rangle = e^{-\frac{1}{2} |x - y|^2} $$

is almost 0, i.e. $c(x)$ and $c(y)$ are almost uncorrelated. Hence any experiment performed in one of those states has almost no probable relation to an experiment performed in the other state.

The coherent states are "almost" multiplicative; we have

$$ c(x + y) = e^{-\langle x, y \rangle} c(x) c(y). $$

The number $|x|^2$, the state-of-mind $x$, and the vector $x$ shall be respectively called the energy, the mode and the generating vector of the coherent
state $c(x)$. Hence in the background of a given coherent state lies the mode which is the state-of-mind that provides the right frequencies of occurrence of the attitudes from a fixed list. The mode for a given coherent state can be approximated as follows. We produce a ”super-questionnaire” out of all involved attitudes; then count the frequencies of the choice of particular attitudes in a poll and take their square roots as coefficients to the respective attitude.

We can easily compute the relative expectation $\mathcal{R}$ for $Q$ in a coherent state $c(x)$. Since $d\Gamma Q$ is a derivation, we have

$$d\Gamma Qc(x) = (Qx)c(x)$$

so that

$$\langle d\Gamma Qc(x), c(x) \rangle = \langle x, Qx \rangle = |Qx|^2$$

and we obtain the number

$$\mathcal{R}(Q, c(x)) = \langle x/, Qx/ \rangle = |Qx/|^2$$

which does not depend on the energy $|x|^2$ of $c(x)$.

## 4 Bicoherence

The concept of bicoherence concerns a community consisting of two coherent fractions, e.g. the government and the opposition in a democratic country, members of two different religious affiliations, a population consisting of natives and immigrants etc. In each of these cases the state-of-opinion of the whole population is a superposition of the states-of-opinion of two coherent sub-profiles. The state-of-opinion of the superposition is not any longer coherent and shall be called bicoherent.

One can easily quote important cases involving more than two coherent states but already in the case of three, the amount of necessary computation will double the size of this paper and hence must be postponed to a separate publication.

### 4.1 Bicoherent states

Take coherent states $c(u)$ and $c(v), u \neq v$, and a number $\lambda$. The number

$$\omega = \langle c(u), c(v) \rangle = e^{-\frac{1}{2}|u-v|^2}$$
shall be called the interaction coefficient. States of the form
\[ c_\lambda(u, v) = \frac{c(u) + \lambda c(v)}{\vartheta(\lambda, \omega)}, \tag{4} \]
where
\[ \vartheta(\lambda, \omega) = |c(u) + \lambda c(v)| = \sqrt{1 + \lambda^2 + 2\lambda \omega}, \tag{5} \]
shall be called bicoherent states. For \( \lambda \neq 0 \) we have
\[ c_\lambda(u, v) = c_{\lambda_1}(v, u) \]
so that for \( \lambda \) close to infinity, \( c_\lambda(u, v) \) behaves exactly as \( c_\lambda(v, u) \) behaves for \( \lambda \) close to zero. The coefficient \( \lambda \) will be called the superposition constant.

The closer to zero is \( \omega \), i.e. the greater is \( |u - v| \), the more the states \( c(u) \) and \( c(v) \) act as uncorrelated, and \( c_\lambda(u, v) \) describes a profile split into two groups which hardly communicate with each other.

With fixed \( u \) and \( v \), when \( \lambda \) increases to infinity, the state \( c_\lambda(u, v) \) converges to the state \( c(v) \), and when \( \lambda \) decreases to zero, it converges to the state \( c(u) \). Excluding the case of simultaneous \( \lambda = -1 \) and \( u = v \), we get from (1) the expected number of energy-bits of the affirmative answers to a question \( Q \):
\[ \langle c_\lambda(u, v), (d\Gamma Q) c_\lambda(u, v) \rangle = \frac{\kappa(Q; \lambda, u, v, \omega)}{\vartheta(\lambda, \omega)^2}, \]
where
\[ \kappa(Q; \lambda, u, v, \omega) = |Qu|^2 + 2\lambda \omega \langle Qu, v \rangle + \lambda^2 |Qv|^2. \tag{6} \]
Applying (2) we get
\[ R(Q, c_\lambda(u, v)) = \begin{cases} \text{the expected percentage of affirmations} \\ \text{of } Q \text{ in the state-of-opinion } c_\lambda(u, v) \end{cases} \]
\[ \frac{\kappa(Q; \lambda, u, v, \omega)}{\kappa(I; \lambda, u, v, \omega)}. \tag{7} \]
Define
\[ R_\omega(Q, \lambda, u, v) = \frac{|Qu|^2 + \lambda^2 |Qv|^2 + 2\lambda \langle Qu, v \rangle \omega}{|u|^2 + \lambda^2 |v|^2 + 2\lambda \langle u, v \rangle \omega} \]
\[ = \frac{|Qu/|^2 + \lambda^2 |Qv/|^2 + 2\lambda \langle Qu/, v/ \rangle \omega}{1 + \lambda^2 + 2\lambda \langle u/, v/ \rangle \omega}. \tag{8} \]
Then choosing \( \omega = e^{-\frac{1}{2}t^2|u-v|^2} \) we get

\[
\mathcal{R}_\omega (Q, \lambda, u, v) (9) = \frac{|Qu|^2 + \lambda^2 |Qv|^2 + 2\lambda \langle Qu, v \rangle e^{-\frac{1}{2}t^2|u-v|^2}}{|u|^2 + \lambda^2 |v|^2 + 2\lambda \langle u, v \rangle e^{-\frac{1}{2}t^2|u-v|^2}} = \mathcal{R} (Q, c_\lambda (tu, tv)),
\]

where \( \omega \) from the open interval \((0, 1)\) can now be treated as an independent variable modulo an adjustment of the amplitude of \( u - v \).

The interaction coefficient measures the ability for interaction (as for instance speaking the same everyday language, being a citizen of a democratic country, having the same cultural or religious background etc.).

The superposition constant plays two different roles. It measures the degrees of influence the participating coherent states have on their superposition. And it marks the existence of wish to enter the interaction at all: Catholics and Protestants of Northern Ireland are fully capable of interacting on an arbitrarily high social level but they will not enter the interaction due to some special reasons.

Suppose that some social forces alter the coherent state \( c(x) \) to another coherent state \( c(y) \). Then, writing \( z = y - x \), we can consider \( z \) as the vector altering the generating vector \( x \) of the given coherent state to a new generating vector \( x + z \) of the new coherent state \( c(x + z) = c(y) \). This reduces the process of changing \( c(x) \) into \( c(y) \) to the application of the transformation \( W_z \) dependent on a vector \( z \) from \( \mathcal{F} \). The transformation

\[
W_z c(x) = c(x + z)
\]

of \( c(x) \) into \( c(x + z) \) is called the Weyl transformation. Given \( z \), the Weyl transformation \( W_z \) is uniquely extendable to a linear isometry (states-of-mind preserving transformation) of \( \Gamma \mathcal{F} \) onto itself (cf. [4]). The Weyl transformation \( W_z \) is fully described by the coherent state \( c(z) \) which shall be called the generator of \( W_z \).

### 4.2 The mathematics of bicoherence

In this section we shall prove a series of results necessary for further development of the theory.

Let \( y, u \in \mathcal{F} \) and \( f, g \in \Gamma \mathcal{F} \) and let \( Q \) be an orthogonal projection. In the proofs below we shall freely use the following identities (cf.[4]):
\[ \langle yf, g \rangle = \langle f, y^*g \rangle \]
\[ y^* (fg) = (y^* f) g + f (y^* g) \]
\[ y^* c(u) = \langle y, u \rangle c(u) \]
\[ \langle d\Gamma Q f, g \rangle = \langle f, d\Gamma Q g \rangle \]
\[ d\Gamma Q (fg) = (d\Gamma Q f) g + f (d\Gamma Q g) \]
\[ d\Gamma Q c(u) = (Qu) c(u) \].

Given \( x, z \in F, |z| = 1 \), we briefly write
\[ z\# c(x) = \langle (x, z) \phi - z \rangle c(x) \].

\textbf{Lemma 2} Take \( u, z \in F \). If \( |z| = 1 \), then the vector \( z\# c(u) \) is a state-of-opinion.

\textbf{Proof.} We have
\[ \langle c(u), zc(u) \rangle = \langle z^* c(u), c(u) \rangle = \langle \langle z, u \rangle c(u), c(u) \rangle = \langle z, u \rangle \]
and
\[ \langle zc(u), zc(u) \rangle \\
\[ = \langle c(u), z^* (zc(u)) \rangle = |z|^2 + \langle z, u \rangle \langle c(u), zc(u) \rangle = |z|^2 + \langle z, u \rangle^2 \]
so that
\[ \langle \langle (u, z) \phi - z \rangle c(u), (u, z) \phi - z \rangle c(u) \rangle \\
\[ = \langle \langle u, z \rangle c(u) - zc(u), (u, z) c(u) - zc(u) \rangle \]
\[ = \langle u, z \rangle^2 - 2 \langle u, z \rangle \langle c(u), zc(u) \rangle + \langle zc(u), zc(u) \rangle \\
\[ = \langle u, z \rangle^2 - 2 \langle u, z \rangle^2 + |z|^2 + \langle z, u \rangle^2 = |z|^2. \]

Now we can verify the following

\textbf{Proposition 3} We have
\[ \lim_{\alpha \to 0} \left| \frac{c(u + \alpha z) - c(u)}{\sqrt{2} \sqrt{1 - e^{-\frac{1}{2}(\alpha |z|)^2}}} - z\# c(u) \right| = 0 \]
i.e. the bicoherent states \( c_{-1} (u + \alpha z, u) \) converge strongly to the state \( z\# c(u) \).
Proof. Take an arbitrary \( y \in \mathcal{F} \). Using l’Hospital Theorem, we get
\[
\lim_{\alpha \to 0} \left\langle \frac{c(u + \alpha z) - c(u)}{\sqrt{2} \sqrt{1 - e^{-\frac{1}{2}(\alpha z)^2}}} - \frac{\langle u, z_j \rangle \phi - z_j}{\partial c(u).c(y)} \right\rangle = 0.
\]
But \( z_j/\#c(u) \) lies on the unit sphere and \( \{ e^x : x \in \mathcal{F} \} \) is total so that the Proposition holds.

Proposition 4 We have
\[
\lim_{\alpha \to 0} \langle c_{-1} (u + \alpha z, u), (d\Gamma Q) c_{-1} (u + \alpha z, u) \rangle = \|Qu\|^2 + |Qz_j|^2.
\]

Proof. Indeed,
\[
\langle c(u + \alpha z) - c(u), d\Gamma Q (c(u + \alpha z) - c(u)) \rangle
\]
\[
= \langle c(u + \alpha z), (Q (u + \alpha z)) c(u + \alpha z) \rangle - \langle c(u + \alpha z), (Qu) c(u) \rangle
\]
\[
- \langle c(u), (Q (u + \alpha z)) c(u + \alpha z) \rangle + \langle c(u), (Qu) c(u) \rangle
\]
\[
= \langle Q (u + \alpha z), u + \alpha z \rangle - 2 \langle Qu, u + \alpha z \rangle e^{-\frac{1}{2} |\alpha z|^2} + \langle Qu, u \rangle,
\]
and using l’Hospital Theorem, we get
\[
\lim_{\alpha \to 0} \frac{|Q (u + \alpha z)|^2 - 2 \langle Qu, u + \alpha z \rangle e^{-\frac{1}{2} |\alpha z|^2} + |Qu|^2}{2 \left(1 - e^{-\frac{1}{2} |\alpha z|^2}\right)} = \|Qu\|^2 + |Qz_j|^2.
\]

Proposition 5 We have
\[
\lim_{\alpha \to 0} (c_{\lambda} (u, v) - W_{\alpha z} c_{\lambda} (u, v)) = (z_j/\#c(u) + \lambda z_j/\#c(v)).
\]

Proof. We have
\[
W_{\alpha z} c_{\lambda} (u, v) = \frac{c(u + \alpha z) + \lambda c(v + \alpha z)}{\vartheta (\lambda, \omega)},
\]
where \( \vartheta \) is given by (5). Let
\[
U_{\alpha} = c(u) - c(u + \alpha z)
\]
\[
V_{\alpha} = c(v) - c(v + \alpha z)
\]
\[
|U_{\alpha}|^2 = 2 \left(1 - e^{-\frac{1}{2} |\alpha z|^2}\right) = |V_{\alpha}|^2.
\]
Then

\[
\frac{U_\alpha}{|U_\alpha|} \rightarrow U = \left(\langle u, z \rangle \phi - z \right) c(u),
\]

\[
\frac{V_\alpha}{|V_\alpha|} \rightarrow V = \left(\langle v, z \rangle \phi - z \right) c(v)
\]

\[
c_\lambda (u, v) - W_{\alpha z} c_\lambda (u, v) = \frac{(c(u) - c(u + \alpha z)) + \lambda (c(v) - c(v + \alpha z))}{\vartheta(\lambda, \omega)}
\]

\[
= \frac{U_\alpha + \lambda V_\alpha}{\vartheta(\lambda, \omega)}
\]

\[
\frac{c_\lambda (u, v) - W_{\alpha z} c_\lambda (u, v)}{|c_\lambda (u, v) - W_{\alpha z} c_\lambda (u, v)|}
\]

\[
= \frac{U_\alpha + \lambda V_\alpha}{|U_\alpha + \lambda V_\alpha|} \rightarrow \frac{U + \lambda V}{|U + \lambda V|}
\]

\[
= \frac{\left(\langle u, z \rangle \phi - z \right) c(u) + \lambda \left(\langle v, z \rangle \phi - z \right) c(v)}{\left|\left(\langle u, z \rangle \phi - z \right) c(u) + \lambda \left(\langle v, z \rangle \phi - z \right) c(v)\right|}.
\]

Given a real number \(\lambda\) and \(u, v, z \in \mathcal{F}\), we define

\[\iota(Q, u, v, z) = \langle v - u, z \rangle \langle Qu, v \rangle \langle v - u, z \rangle + \langle (v - u), Q z \rangle.\]

Take \(u, v, z \in \mathcal{F}\), where \(|z| = 1\). Then

\[
\langle dQ (z \# c(u) + \lambda z \# c(v)), z \# c(u) + \lambda z \# c(v) \rangle = \vartheta^2(\lambda, \omega) |Q(z)|^2 + \kappa(Q; \lambda, u, v, \omega) - 2\lambda \omega c(Q, u, v, z).
\tag{10}
\]

**Proof.** By direct computations we verify the relation

\[
\frac{1}{\omega} \langle d\Gamma Q z \# c(u), z \# c(v) \rangle
\]

\[
= \langle z, Q z \rangle + \langle z, z \rangle \langle Qu, v \rangle + \iota(Q, u, v, z).
\]

which applied to the left side of (10) yields its right side. \(\blacksquare\)
4.3 Consequences of temporary external influence

Consider a profile in a bicoherent state $c_\lambda(u, v)$ and an element $z \in \mathcal{F}$. For $\alpha > 0$, let an external influence caused by $W_{\alpha z}$, 

$$c_\lambda(u, v) \rightarrow W_{\alpha z}c_\lambda(u, v),$$

induce a new state $c_\lambda(u + \alpha z, v + \alpha z)$. In consequence of the enforcement, the population falls into the superposition state 

$$(c_\lambda(u + \alpha z, v + \alpha z) - c_\lambda(u, v))/$$

of the original contra the enforced state-of-opinion $W_{\alpha z}c_\lambda(u, v)$. In Proposition 5 it is proved that when the enforcement fades away, i.e. when $\alpha \rightarrow 0$, the state-of-opinion tends to the limit state-of-opinion

$$(z#c(u) + \lambda z#c(v))/.$$

By (7) the expected percentage of affirmative answers to $Q$ in this state is

$$\mathcal{R}\left(Q, (z#c(u) + \lambda z#c(v))\right)$$

$$= \frac{\langle d\Gamma Q(z#c(u) + \lambda z#c(v)) , z#c(u) + \lambda z#c(v) \rangle}{\langle d\Gamma I(z#c(u) + \lambda z#c(v)) , z#c(u) + \lambda z#c(v) \rangle}$$

(11)

where $\omega = e^{-\frac{1}{2}|u-v|^2}$. Due to the lack of homogeneity relative to $u, v$, we cannot make $\omega$ in (11) an independent variable as in (9).

The term $\iota(Q, u, v, z)$ measures the balance of the influence of $c(z)$ on components $c(u)$ and $c(v)$ of $c_\lambda(u, v)$. If the influence of $c(z)$ on $c_\lambda(u, v)$ is equally distributed between $c(u)$ and $c(v)$, the term $\iota(Q, u, v, z)$ vanishes.

5 A model

Let us consider a special case. Suppose that in a community two complementary coherent profiles manifest. Fix two positive numbers $a$ and $b$, $a^2 + b^2 = 1$ and consider the states-of-mind

$$u = \mu\begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad v = \mu\begin{pmatrix} b \\ a \end{pmatrix}.$$

Suppose further that the community we investigate is polarized into two profiles - one in the state $c(u)$ and the other in the state $c(v)$. Let $100a^2\%$ of
the members of the profile in state \( c(u) \) support an attitude \( X \) while 100\( a^2 \)% of the members of the profile in state \( c(v) \) will reject \( X \). We consider the bicoherent state \( c_\lambda (\mu(u), \mu(v)) \).

Consider a question corresponding to the projection

\[
Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

where the eigenvector \((1, 0)\) implies the answer "yes" to the question "Do you support the attitude \( X \)?"

We shall now analyse the expected relative frequencies of affirmative answers to \( Q \) before and after the exertion of the influence which we choose equally divided between \( c(u) \) and \( c(v) \):

\[
z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

Hence

\[
\iota(Q,u,v,z) = \iota(I,u,v,z) = 0
\]

and

\[
\kappa(Q; \lambda, u, v, \omega) = (a^2 + 2\lambda ab + \lambda^2 b^2) \mu^2
\]

\[
\kappa(I; \lambda, u, v, \omega) = \mu^2 \left( (1 + \lambda^2) + 4\lambda ab \right).
\]

Using (8) we get

\[
R_\omega (Q, \lambda, u, v) = \frac{a^2 + 2\lambda ab \sqrt{1 - a^2} + \lambda^2 (1 - a^2)}{(1 + \lambda^2) + 4\lambda ab \sqrt{1 - a^2}}
\]

(12)

and using (11) we get

\[
R \left( Q, (z\# c(u) + \lambda z\# c(v)) \right)
\]

\[
\frac{1}{2} \left( \frac{(1 + \lambda^2 + 2\lambda \omega) \mu^{-2} + (a^2 + 2\lambda ab \sqrt{1 - a^2} + \lambda^2 (1 - a^2))}{(1 + \lambda^2 + 2\lambda \omega) \mu^{-2} + ((1 + \lambda^2) + 4\lambda ab \sqrt{1 - a^2})} \right),
\]

where \( \omega = e^{-\frac{1}{2}|u-v|^2} = e^{-\mu^2(a-b)^2} \) and \( \mu^2 = |u|^2 = |v|^2 \) are the energies of \( c(u) \) and \( c(v) \).

To illustrate the obtained results we shall draw the four pairs of graphs of (13) and (12) imposed on each other, for the choices of \( a^2 = \frac{19}{36}, \frac{22}{36}, \frac{26}{36}, \frac{28}{36} \), with respective interaction coefficients \( \omega = 0.99846, 0.97531, 0.90105, \)
When the energy $\mu$ grows, (13) converges to (12) so we can as well choose $\mu = 1$. In each of the four cases the graph corresponding to (13) is the one with smaller maximum and greater minimum than the graph corresponding to (12).

Let us look at the diagrams above. For $\lambda > 1$ there are no significant differences in the forms of the diagrams. In all cases maximum is not attained for $\lambda = 0$ but only for a negative $\lambda$. Movement of $\lambda$ from zero in the negative direction makes $R_\omega$ increase. Consequently we have

**Conclusion 6** *The increase of the influence of $c(v)$ acts as a buster for $c(u)$ providing more affirmative answers.*

Say the interaction coefficient $\omega$ is close to one. Starting at $\lambda = 0$ and moving in the negative direction, we observe a rapid increase of $R_\omega$. Then, continuing to move $\lambda$ in the same direction, the situation reverses - now the maximum decreases towards the minimum.

**Conclusion 7** *The closer to one is the interaction coefficient, the shorter is the interval within which $R_\omega$ attains first the maximum and then the minimum.*

Then the situation stabilizes and with further decrease of $\lambda$, $R_\omega$ converges to its limit $R(Q, c(v))$ in $-\infty$ so that the role of $c(u)$ is eliminated.
As an example we can take the government and the opposition of a democratic country each in a coherent state. Assume that there is an intensive interaction between $c(u)$ and $c(v)$. Say the government has majority: \( \mathcal{R}(Q, c(u)) > \mathcal{R}(Q, c(v)) \). A small negative weight $\lambda$ attached to $c(v)$ yields not much contribution itself but by way of the interaction it provokes the other fraction to vote. Similarly if \( \mathcal{R}(Q, c(u)) < \mathcal{R}(Q, c(v)) \), then the increase of negative answers is provoked. Still higher negative weight yields the reversed status - the respondents from $c(v)$ take over and in the first case cause a decrease and in the second case an increase of affirmation. These unusual variations can happen only in the presence of high interaction and in small intervals of negative $\lambda$ and hence will seldom occur in real life. However, such jumps in the distribution of votes have been observed in the past (cf.[3]).

It is clearly visible that equally distributed influence of $W_z$ makes the extreme values of $\mathcal{R}_\omega$ diminish.

**Conclusion 8** The influence of $W_z$ tempers the voters’ reactions.

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