The metric in the superspace of Riemannian metrics and its relation to gravity*

Hans-Jürgen Schmidt

Universität Potsdam, Institut für Mathematik, Am Neuen Palais 10
D-14469 Potsdam, Germany, E-mail: hjschmi@rz.uni-potsdam.de

Abstract

The space of all Riemannian metrics is infinite-dimensional. Nevertheless a great deal of usual Riemannian geometry can be carried over. The superspace of all Riemannian metrics shall be endowed with a class of Riemannian metrics; their curvature and invariance properties are discussed. Just one of this class has the property to bring the lagrangian of General Relativity into the form of a classical particle's motion. The signature of the superspace metric depends in a non-trivial manner on the signature of the original metric, we derive the corresponding formula. Our approach is a local one: the essence is a metric in the space of all symmetric rank-two tensors, and then the space becomes a warped product of the real line with an Einstein space.

Key words: Quantum gravity, Wheeler-DeWitt equation, superspace metric

MS classification: 53C20

*This paper is in final form and no version of it will be submitted for publication elsewhere.
1 THE SUPERSPACE

Let \( n \geq 2 \), \( n \) be the dimension of the basic Riemannian spaces. Let \( M \) be an \( n \)-dimensional differentiable manifold with an atlas \( x \) of coordinates \( x^i, i = 1, \ldots, n \). The signature \( s \) ( = number of negative eigenvalues) shall be fixed; let \( V \) be the space of all Riemannian metrics \( g_{ij}(x) \) in \( M \) with signature \( s \), related to the coordinates \( x^i \). This implies that isometrical metrics in \( M \) are different points in \( V \) in general. The \( V \) is called superspace, its points are the Riemannian metrics. The tangent space in \( V \) is the vector space

\[
T = \{ h_{ij}(x) | x \in M, \ h_{ij} = h_{ji} \}
\]

the space of all symmetric tensor fields of rank 2. All considerations are local ones, so we may have in mind one single fixed coordinate system in \( M \).

2 COORDINATES IN SUPERSPACE

Coordinates should possess one contravariant index, so we need a transformation of the type

\[
y^A = \mu^{Aij} g_{ij}(x)
\]

such that the \( y^A \) are the coordinates for \( V \). To have a defined one-to-one correspondence between the index pairs \((i, j)\) and the index \( A \) we require

\[
A = 1, \ldots N = n(n+1)/2,
\]

and \( A = 1, \ldots N \) corresponds to the pairs

\[
(1,1), (2,2), \ldots (n,n), (1,2), (2,3), \ldots (n-1,n), (1,3), \ldots (n-2,n), \ldots (1,n)
\]

consecutively. \( (i, j) \) and \( (j, i) \) correspond to the same \( A \). We make the ansatz

\[
\mu^{Aij} = \mu_{Aij} = b \text{ for } i \neq j, \ c \text{ for } i = j
\]

\[
0 \text{ if } (i,j) \text{ does not correspond to } A
\]
and require the usual inversion relations

\[ \mu^{Aij} \mu_{Bij} = \delta^A_B \quad \text{and} \quad \mu^{Aij} \mu_{Akl} = \delta^i_k \delta^j_l. \]  

(5)

Bracketed indices are to be symmetrized, which is necessary because of symmetry of the metric \( g_{ij} \). Inserting ansatz (4) into (5) gives \( c^2 = 1 \), \( b^2 = 1/2 \).

Changing the sign of \( b \) or \( c \) only changes the sign of the coordinates, so we may put

\[ c = 1, \quad b = 1/\sqrt{2}. \]  

(6)

The object \( \mu^{Aij} \) is analogous to the Pauli spin matrices relating two spinorial indices to one vector index.

3 METRIC IN SUPERSPACE

The metric in the superspace shall be denoted by \( H_{AB} \), it holds

\[ H_{AB} = H_{BA} \]  

(7)

and the transformed metric is

\[ G^{ijkl} = H_{AB} \mu^{Aij} \mu^{Bkl}, \quad H_{AB} = \mu_{Aij} \mu_{Bkl} G^{ijkl}. \]  

(8)

From (4) and (7) it follows that

\[ G^{ijkl} = G^{jikl} = G^{klij}. \]  

(9)

The inverse to \( H_{AB} \) is \( H^{AB} \), and we define

\[ G_{ijkl} = H^{AB} \mu_{Aij} \mu_{Bkl} \]  

(10)

which has the same symmetries as (9). We require \( G^{ijkl} \) to be a tensor and use only the metric \( g_{ij}(x) \) to define it. Then the ansatz

\[ G^{ijkl} = z g^{i(k} g^{l)j} + \alpha g^{ij} g^{kl} \]  

(11)

\[ G_{ijkl} = v g_{i(k} g_{l)j} + \beta g_{ij} g_{kl} \]  

(12)
where $v, z, \alpha$ and $\beta$ are constants, is the most general one to fulfil the symmetries (9). One should mention that also curvature-dependent constants could have been introduced. The requirement that $H_{AB}$ is the inverse to $H^{AB}$ leads via (8,10) to

$$G_{ijkl}G^{kmp} = \delta^{(m}_{i} \delta^{p)}_{j}. \tag{13}$$

The requirement that $G^{ijkl}$ is a tensor can be justified as follows: Let a curve $y^{A}(t), 0 < t < 1$ in $V$ be given, then its length is

$$\sigma = \int_{0}^{1} \left( H_{AB} \frac{dy^{A}}{dt} \frac{dy^{B}}{dt} \right)^{1/2} dt$$

i.e., with (2) and (8)

$$\sigma = \int_{0}^{1} \left( G_{ijkl} \frac{dg_{ij}}{dt} \frac{dg_{kl}}{dt} \right)^{1/2} dt. \tag{14}$$

A coordinate transformation in $M$: $x^{i} \rightarrow \epsilon x^{i}$ changes $g_{ij} \rightarrow \epsilon^{-2} g_{ij}$. We now require that $\sigma$ shall not be changed by such a transformation. Then $\alpha$ and $z$ are constant real numbers. Inserting (11,12) into (13) gives $v z = 1$, hence $z \neq 0$. By a constant rescaling we get

$$v = z = 1 \tag{15}$$

and then (11,12,13) yield

$$\alpha \neq - \frac{1}{n}, \quad \beta = \frac{-\alpha}{1 + \alpha n}. \tag{16}$$

So we have got a one-parameter set of metrics in $V$. Eq. (16) fulfils the following duality relation: with $f(\alpha) = -\alpha/(1 + \alpha n)$, $f(f(\alpha)) = \alpha$ holds for all $\alpha \neq -1/n$.

$$G^{ijkl} = g^{i(k} g^{l)j} + \alpha g^{ij} g^{kl}$$

$$G_{ijkl} = g_{i(k} g_{l)j} + \beta g_{ij} g_{kl}. \tag{17}$$

It holds: For $\alpha = -1/n$, the metric $H_{AB}$ is not invertible.

**Indirect proof:** $G^{ijkl}$ depends continuously on $\alpha$, so it must be the case with the inverse. But

$$\lim_{\alpha \to -1/n}$$

applied to $G_{ijkl}$ gives no finite result. Contradiction.
4 Signature of the superspace metric

Let $S$ be the signature of the superspace metric $H_{AB}$. $S$ depends on $\alpha$ and $s$. For convenience we define

$$\Theta = 0, \quad (\alpha > -1/n), \quad 1, \quad (\alpha < -1/n).$$  \hfill (18)

From continuity reasons it follows that $S$ is a function of $\Theta$ and $s$: $S = S(\Theta, s)$. If we transform $g_{ij} \rightarrow -g_{ij}$ i.e., $s \rightarrow n - s$, then $H_{AB}$ is not changed, i.e.,

$$S(\Theta, s) = S(\Theta, n - s).$$  \hfill (19)

We transform $g_{ij}$ to diagonal form as follows

$$g_{11} = g_{22} = \ldots = g_{ss} = -1, \quad g_{ij} = \delta_{ij} \quad \text{otherwise}.$$  \hfill (20)

4.1 Signature for $\Theta = 0$

To calculate $S(0, s)$ we may put $\alpha = 0$ and get with (8,11,15)

$$H_{AB} = \mu_{Ai j} \mu_{B kl} g^{ik} g^{jl}.$$  \hfill (21)

which is a diagonal matrix. It holds $H_{11} = \ldots = H_{nn} = 1$ and the other diagonal components are $\pm 1$. A full estimate gives in agreement with (19)

$$S(0, s) = s(n - s).$$  \hfill (22)

4.2 Signature for $\Theta = 1$

To calculate $S(1, s)$ we may put $\alpha = -1$ and get

$$H_{AB} = \mu_{Ai j} \mu_{B kl} \left( g^{ik} g^{jl} - g^{ij} g^{kl} \right).$$  \hfill (23)

For $A \leq n < B$, $H_{AB} = 0$, i.e., the matrix $H_{AB}$ is composed of two blocks. For $A, B \leq n$ we get

$$H_{AB} = 0 \quad \text{for} \quad A = B, \quad 1 \quad \text{for} \quad A \neq B$$

a matrix which has the $(n - 1)$-fold eigenvalue $1$ and the single eigenvalue $1 - n$. For $A, B > n$ we have the same result as for the case $\alpha = 0$, i.e., we get $S(1, s) = 1 + s(n - s)$. 

5
4.3 Result

The signature of the superspace metric is

\[ S = \Theta + s(n - s) . \]  

(24)

5 SUPERCURVATURE

We use exactly the same formulae as for finite-dimensional Riemannian geometry to define Christoffel affinities \( \Gamma^A_{BC} \) and Riemann tensor \( R^A_{BCD} \). Using (4) we write all equations with indices \( i,j = 1,...,n \). Then each pair of covariant indices \( i,j \) corresponds to one contravariant index \( A \). The following formulae appear:

\[ \frac{\partial g^{ij}}{\partial g_{km}} = -g^{i(k} g^{m)j} \]  

(25)

\[ \Gamma^{ijklmp} = -\frac{1}{2} g^{i(k} g^{l(m} g^{p)j} - \alpha g^{ij} g^{k(l} g^{p)m} - \frac{1}{2} g^{j(k} g^{i(l} g^{p)m} \]  

(26)

and, surprisingly independent of \( \alpha \) we get

\[ \Gamma^{kmp} = -\delta^{(k}_{(i} g^{l(m} \delta^{p)}_{j)} \]  

(27)

Consequently, also Riemann- and Ricci tensor do not depend on \( \alpha \):

\[ R^{kmpij}_{rs} = \frac{1}{2} \left( \delta^{(k}_{(r} g^{l(m} g^{p)i} \delta^{j)}_{s)} - \delta^{(k}_{(r} g^{l(i} \delta^{p)}_{s)} \right) . \]  

(28)

Summing over \( r = m \) and \( s = p \) we get

\[ R^{klij} = \frac{1}{4} \left( g^{ij} g^{kl} - n g^{k(i} g^{l)j} \right) . \]  

(29)

The Ricci tensor has one eigenvalue 0. Proof: It is not invertible because it is proportional to the metric for the degenerated case \( \alpha = -1/n \), cf. sct. 3.

The co–contravariant Ricci tensor reads

\[ R^{ij}_{kl} = G^{kmpij} R^{mpij} = \frac{1}{4} \left( g^{ij} g^{kl} - n \delta^{(i}_{k} \delta^{j)}_{l} \right) , \]  

(30)
and the curvature scalar is

$$R = -\frac{1}{8}n(n-1)(n+2).$$  \hspace{1cm} (31)$$

The eigenvector to the eigenvalue 0 of the Ricci tensor is $g_{ij}$. All other eigenvalues equal $-n/4$, and the corresponding eigenvectors can be parametrized by the symmetric traceless metrics, i.e. the multiplicity of the eigenvalue $-n/4$ is $(n-1)(n+2)/2$.

## 6 Superdeterminant

We define the superdeterminant

$$H = \det H_{AB}.$$  \hspace{1cm} (32)$$

$H$ is a function of $g, \alpha$ and $n$ which becomes zero for $\alpha = -1/n$, cf. sct. 3. We use eqs. (8) and (17) to look in more details for the explicit value of $H$. The formal calculation for $n = 1$ leads to

$$H = H_{11} = G^{1111} = g^{11}g^{11} + \alpha g^{11}g^{11} = (1 + \alpha)g^{-2}.$$  \hspace{1cm} (33)$$

Multiplication of $g_{ij}$ with $\epsilon$ gives $g \rightarrow \epsilon^n g$, $H_{AB} \rightarrow \epsilon^{-2}H_{AB}$ and $H \rightarrow \epsilon^{-n(n+1)}H$. So we get in an intermediate step

$$H = H_1 g^{-n-1}$$

where $H_1$ is the value of $H$ for $g = 1$. $H_1$ depends on $\alpha$ and $n$ only. To calculate $H_1$ we put $g_{ij} = \delta_{ij}$ and get via $H_{ij} = \delta_{ij} + \alpha$, $H_{Ai} = 0$ for $A > n$, and $H_{AB} = \delta_{AB}$ for $A, B > n$ finally

$$H_1 = 1 + \alpha n.$$  \hspace{1cm} (34)$$

This is in agreement with the $n = 1$-calculation.
7 GRAVITY

Now, we come to the main application: The action for gravity shall be expressed by the metric of superspace. We start from the metric

\[ ds^2 = dt^2 - g_{ij} \, dx^i \, dx^j \]  

(35)

\[ i,j = 1, \ldots n \] with positive definite \( g_{ij} \) and \( x^0 = t \). We define the second fundamental form \( K_{ij} \) by

\[ K_{ij} = \frac{1}{2} g_{ij,0} . \]

(36)

The Einstein action for (35) is

\[ I = - \int *R \frac{1}{2} \sqrt{g} \, d^{n+1}x \]

(37)

where \( g = \det g_{ij} \) and \( *R \) is the \((n+1)\)-dimensional curvature scalar for (35). Indices at \( K_{ij} \) will be shifted with \( g_{ij} \), and \( K = K^i_i \). With (36) we get

\[ (K \sqrt{g}),0 = (K,0 + K^2) \sqrt{g} . \]

(38)

This divergence can be added to the integrand of (37) without changing the field equations. It serves to cancel the term \( K_0 \) of \( I \). So we get

\[ I = \int \frac{1}{2} \left( K^{ij} K_{ij} - K^2 + R \right) \sqrt{g} \, d^{n+1}x \]

(39)

where \( R \) is the \( n \)-dimensional curvature scalar for \( g_{ij} \). We make now the ansatz for the kinetic energy

\[ W = \frac{1}{2} G^{ij mp} K_{ij} K_{mp} = \frac{1}{2} \left( K^{ij} K_{ij} + \alpha K^2 \right) . \]

(40)

Comparing (40) with (39) we see that for \( \alpha = -1 \) (surprisingly, this value does not depend on \( n \))

\[ I = \int \left( W + \frac{R}{2} \right) \sqrt{g} \, d^{n+1}x \]

(41)

holds. Because of \( n \geq 2 \) this value \( \alpha \) gives a regular superspace metric. (For \( n = 1 \), eq. (37) is a divergence, and \( \alpha = -1 \) gives not an invertible superspace-metric.)
Using the $\mu^{Aij}$ and the notations $z^A = \mu^{Aij} g_{ij} / 2$ and $v^A = dz^A / dt$ we get from (40,41)

$$I = \int \frac{1}{2} \left( H_{AB} v^A v^B + R(z^A) \right) \sqrt{g} d^{n+1}x$$

(42)

i.e., the action has the classical form of kinetic plus potential energy. The signature of the metric $H_{AB}$ is $S = 1$. This can be seen from eqs. (18,24).

8 CONCLUSION

In eq. (42), Einstein gravity is given in a form to allow canonical quantization: The momentum $v^A$ is replaced by $-i \partial / \partial z^A \ (\hbar = 1)$, and then the Wheeler - DeWitt equation for the world function $\psi(z^A)$ appears as Hamiltonian constraint in form of a wave equation:

$$\left( \Box - R(z^A) \right) \psi = 0.$$  (43)

After early attempts in [1], the Wheeler - DeWitt equation has often been discussed, especially for cosmology, see e.g. [2-5]. Besides curvature, matter fields can be inserted as potential, too. It is remarkable that exactly for Lorentz and for Euclidean signatures in (35) (positive and negative definite $g_{ij}$ resp.) the usual D’Alembert operator ($S = 1$) in (43) appears. For other signatures in (35), (43) has at least two timelike axes.

The author gratefully acknowledges stimulating discussions with U. Bleyer, D.-E. Liebscher and A. I. Zhuk before and with J. Kijowski and P. Michor during the conference.

REFERENCES

[1] R. Arnowitt, S. Deser, C. Misner, 1962 in: E. Witten, Gravitation, An Introduction to current research, New York.
[2] U. Bleyer, D.-E. Liebscher, H.-J. Schmidt, A.I. Zhuk, PRE-ZIAP 89-11.
[3] G. Gibbons, S. Hawking, J. Stewart, Nucl. Phys. B 281 (1987), 736.
[4] L. P. Grishchuk, Yu. V. Sidorov, p. 700 in: Proc. 4. Sem. Quantum Gravity Moscow, WSPC Singapore 1988, Ed. M. A. Markov.
[5] J. Halliwell, S. Hawking, p. 509 in: Proc. 3. Sem. Quantum Gravity Moscow, WSPC Singapore 1985, Ed. M. A. Markov.

Received September 22, 1989

In this reprint we removed only obvious misprints of the original, which was published in Proc. Conf. Brno August 27 - September 2, 1989: Differential Geometry and its Applications, Eds.: J. Janyska, D. Krupka, World Scientific PC Singapore 1990, pages 405-411.

Author’s address that time: H.-J. Schmidt, Zentralinstitut für Astrophysik der Akademie der Wissenschaften, DDR-1591 Potsdam, R.-Luxemburg-Str. 17a