Heckmann’s wormholes in Jordan-BransDicke gravity.

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Abstract

A simple Heckmann’s vacuum wormhole solution of Jordan-Brans-Dicke gravitation is presented and analysed. It is shown that in contrast with class I Brans solution where the throat radius becomes real when \( \omega < -4/3 \) here it becomes positive when \( \omega < -1 \).

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1 Introduction

Lately there has been renewed interest in the scalar-tensor theories of gravitation. Important arena where these theories have found immense applications is the field of wormhole physics. Several classes of solutions in the scalar-tensor theories support wormhole geometry. The most prominent example of scalar-tensor theories is perhaps the Jordan-Brans-Dicke (JBD) [1], [2]. Now a day, predictions of JBD theory to be consistent not only with the weak field solar system tests but also with the recent cosmological observations [3].

JBD theory describes gravitation through a metric tensor \( g_{\mu \nu} \) and a massless scalar field \( \phi \). In this theory, static wormhole solutions were found in vacuum, the source of gravity being the scalar field. Several static wormhole solutions in JBD theory have been widely investigated in the literature [4],[5], [6]. It was shown that three of the four Brans classes of vacuum solutions admit a wormholelike spacetime for convenient choices of their parameters.

In what follows, we shall present a static vacuum wormhole solution of the BD theory endowed with first exact solution of JBD field equations were obtained in parametric form by Heckmann [7], soon after Jordan proposed scalar tensor theory.

2 The Heckmann’s wormholes solutions.

JBD theory are described by the following action in the Jordan frame is:

\[
S = \int dx \sqrt{-g} (\phi R - \omega g^{\mu \nu} \nabla_\mu \phi \nabla_\nu \phi - ) + S_m.
\]  

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Here, $R$ is the Ricci scalar curvature with respect to the space-time metric $g_{\mu\nu}$ and $S_m$ denote action of matter fields. (We use units in which gravitational constant $G=1$ and speed of light $c=1$.)

Variation of (1) with respect to $g_{\mu\nu}$ and $\phi$ gives, respectively, the field equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{\phi} T^M_{\mu\nu} + T^{JBD}_{\mu\nu},$$

where

$$T^{JBD}_{\mu\nu} = \frac{\omega}{\phi^2} \left( \nabla_{\mu} \phi \nabla_{\nu} \phi - \frac{1}{2} g_{\mu\nu} \nabla_{\alpha} \phi \nabla^{\alpha} \phi \right) + \frac{1}{\phi} \left( \nabla_{\mu} \nabla_{\nu} \phi - g_{\mu\nu} \nabla^{\alpha} \phi \right),$$

and

$$\nabla_{\alpha} \nabla^{\alpha} \phi = \frac{T^M_{\alpha} \lambda}{3 + 2\omega}.$$  

and $T^M_{\alpha} \lambda$ is the energy momentum tensor of ordinary matter which obeys the conservation equation $T^M_{\mu\nu;\lambda} g^{\nu\lambda} = 0$.

One can choose the static spherically symmetric metric in curvature coordinates form

$$ds^2 = -e^\nu(r) dt^2 + e^\lambda(r) dr^2 + r^2 d\Omega^2.$$  

Then the solutions of the gravitational field equations in the vacuum take the form [7]

$$r = \frac{\alpha_0}{\sqrt{\tau (\tau^h - \tau^{-h})}},$$

$$e^\lambda = \frac{4h^2}{\left[\left(\frac{1}{2} + h\right) \tau^h - \left(\frac{1}{2} - h\right) \tau^{-h}\right]^2},$$

$$e^\nu = \tau^{\frac{\nu}{h}},$$

$$\phi = \phi_0 \tau^{\frac{\phi}{h}},$$

where $\tau$ parameter, arbitrary constant and

$$h^2 = \frac{1}{4} - \frac{A}{B^2}; A = \frac{\beta_0}{2} (1 - \beta_0 \omega); B = 1 + 2\beta_0.$$  

The constant $\phi_0$ are determined by an asymptotic flatness condition as $\phi_0 = 1$, while $\alpha_0$ and $\beta_0$ is determined by the requirement of having Schwarzschild geometry in the weak field limit [1]. Thereby

$$\alpha_0 = 4hM (1 + 2\beta_0)$$
is the function of central mass of the configuration and

\[ \beta_0 = -\frac{1}{3 + 2\omega}. \]  

This implies that the range of \( \beta_0 \) is dictated by the range of \( \omega \), which, in turn, is to be dictated by the requirements of wormhole geometry. In order to investigate whether a given solution represents a wormhole geometry, it is convenient to cast the metric into Morris-Thorne canonical form:

\[ ds^2 = -e^{2\chi(\hat{R})} dt^2 + \left[ 1 - \frac{b(\hat{R})}{\hat{R}} \right]^{-1} dr^2 + \hat{R}^2 d\Omega^2, \]  

where \( \chi(\hat{R}) \) and \( b(\hat{R}) \) are arbitrary functions of the radial coordinate, \( \hat{R} \). \( \chi(\hat{R}) \) is denoted as the redshift function, for it is related to the gravitational redshift; \( b(\hat{R}) \) is called the form function, because as can be shown by embedding diagrams, it determines the shape of the wormhole [8]. The radial coordinate has a range that increases from a minimum value at \( \hat{R}_0 \), corresponding to the wormhole throat, to infinity. The Heckmann solution can be cast to the form (11) by defining a radial coordinate \( \hat{R} \) which is related with \( r \) via the expression

\[ \hat{R} = \frac{\alpha_0}{\sqrt{\tau (\tau - h - \tau h)}}, \]  

The functions \( \chi(\hat{R}) \) and \( b(\hat{R}) \) are the given by

\[ \chi(\hat{R}) = \tau^{\frac{1}{h}}, \]  

\[ b(\hat{R}) = \hat{R} \left[ 1 - \frac{1}{4h} \right]^{\frac{1}{h} - \frac{1}{2h}} \]  

The axially symmetric embedded surface \( z = z(\hat{R}) \) shaping the wormhole’s spatial geometry is obtained from

\[ \frac{dz}{d\hat{R}} = \pm \left[ \frac{\hat{R}}{b(\hat{R})} - 1 \right]^{-\frac{1}{2}} \]  

By definition of wormhole at throat its embedded surface is vertical. The throat of the wormhole occurs at \( \hat{R} = \hat{R}_0 \) such that \( b(\hat{R}_0) = \hat{R}_0 \). This gives minimum allowed \( R \)-coordinate radii \( R_0^\pm \) as [7]

\[ R_0^\pm = \frac{\alpha_0}{\sqrt{\frac{1 - 4h^2}{4h} \left( \frac{1 + 2h}{1 - 2h} \right)^{\frac{1}{h}}}}, \]
The values $R^\pm_0$ can be obtained from (17) using the (10).

$$R^\pm_0 = \sqrt{-3 \left(1 + \omega\right)} \left(\frac{1 + 2\omega + \Omega}{1 + 2\omega - \Omega}\right)^{\frac{1 + 2\omega}{2}},$$  \hspace{1cm} (17)

where $\Omega = \sqrt{7 + 10\omega + 4\omega^2}$.

In contrast with class I Brans solution where the throat radius becomes real when $\omega < -4/3$ here it becomes positive when $\omega < -1$. This range thus gives rise viable wormhole geometry. The redshift function has a singularity at $\tau = 0$ which corresponds to the point $^*R = 0$.

3 Discussion

The field equations in JBD gravity theory are non-linear in nature. Moreover, the physical and the geometrical meaning of the radial coordinate $r$ are not defined by symmetry reasons and are unknown a priori. In this context, using the key assumption that the Heckmann [7] solution physically acceptable it was constructed the spherically symmetric wormholes.

Our analysis reveals that $\omega$ may take on arbitrary negative values less then -1. This result extends the scope for the feasibility of wormhole scenarios even to the regime of ordinary observations.

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