On symmetries in covariant Galilei mechanics

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Abstract

In the framework of covariant classical mechanics (i.e., generally relativistic classical mechanics on a spacetime with absolute time) developed by Jadczyk and Modugno, we analyse systematically the relations between symmetries of geometric objects. We show that the (holonomic) infinitesimal symmetries of the cosymplectic structure and of its horizontal potentials are also symmetries of spacelike metric, gravitational and electromagnetic fields, Euler-Lagrange morphism and Lagrangians. Then, we provide a definition for a covariant momentum map associated with a group of cosymplectic symmetries using a covariant lift of functions of phase space. In the case when the cosymplectic symmetries projects on spacetime we see that the components of this momentum map are quantisable functions in the sense of Jadczyk and Modugno. Finally, we illustrate the results by some examples.

Key words: Covariant classical mechanics, Covariant quantum mechanics, Lie groups and symmetry, Jet contact structure, Cosymplectic manifolds
1 Introduction

At the beginning of the 90’s M. Modugno and A. Jadczyn proposed a new geometric framework for a generally covariant classical and quantum mechanics on a curved spacetime with absolute time [15, 16, 31], based on jets, connections and cosymplectic forms. This approach was later developed in [4, 14, 30, 32, 38, 39, 41]. The theory will be referred to as ‘covariant classical Galilei theory’ (CCG) and ‘covariant quantum Galilei theory’ (CQG).

The theory was partially inspired by the wide literature on geometric formulations of classical and quantum mechanics. Main sources were symplectic and cosymplectic classical mechanics [1, 3, 6, 13, 24, 26, 27, 28], Newton or Galilei classical mechanics as general relativistic theories [2, 4, 8, 10, 21, 22, 36, 37], quantum theories of mechanics within a symplectic or cosymplectic framework (such as geometric quantisation) [5, 9, 13, 19, 23, 34, 35, 37, 42].

The models of CCG and CQG share nice ideas with the above literature, trying at the same time to avoid some typical problems. For instance, both theories are explicitly covariant with respect to changes of coordinates, even time-dependent ones. This feature partially comes from the cosymplectic structure of the phase space in the classical theory, which is also the classical background for the quantum theory, and overcomes the problem of explicit time independence of symplectic mechanics. On the other hand, the cosymplectic structure of CCG is of a special, physically reasonable, type. Therefore, it excludes those problems of the general cosymplectic formalism, which have no physical meaning. Additionally, both theories are covariant with respect to the choice
of units of measurement, because of the fact that the geometric framework incorporates ‘unit spaces’. This kind of covariance requires different geometric techniques compared to the standard literature. The theory provides a model for classical and quantum holonomically constrained systems of particles, supported by non trivial physically relevant examples, such as the quantised rigid body [32].

We think that these aspects are promising for the theory to be a mathematical framework for quantum mechanics.

The goal of this paper is to analyse systematically the symmetries of the structures involved in the covariant classical Galilei theory and to introduce an associated momentum map. In a subsequent paper we shall apply these results to the covariant quantum Galilei theory.

In the second section we summarise the basic aspects of the covariant classical Galilei theory [15, 16, 31, 30] emphasising the natural bijections between the fundamental geometric objects of the theory.

In the model, classical spacetime is an \((n+1)\)-dimensional manifold fibred over a 1-dimensional affine space. Spacetime is supposed to be equipped with further geometric objects. Namely, a scaled vertical metric of spacetime, called spacelike metric, a linear connection of the tangent bundle of spacetime which preserves the time form, called spacetime connection (gravitational field), and a scaled 2–form of spacetime (electromagnetic field). The phase space is taken to be the first jet space with respect to the spacetime fibring.

We shall see that there is a natural bijective correspondence between a spacetime connection, a distinguished affine connection of the phase space, called phase connection, and a distinguished (nonlinear) homogenous connection of the phase space, called dynamical connection. Moreover, there is a natural bijective correspondence between pairs of a phase connection and a spacelike metric and distinguished 2–forms of the phase space, called dynamical phase 2–forms. On the other hand, there is a natural bijective correspondence between pairs of a dynamical connection and a spacelike metric and distinguished 2–forms of the second jet space, called horizontal phase 2–forms.

By means of the natural correspondences, we can regard the gravitational field as a dynamical phase 2–form of phase space if a spacelike metric is chosen. Then, the electromagnetic field can be incorporated in the gravitational objects through a ‘minimal coupling’. This yields a total dynamical phase 2–form. Assuming the closure of the gravitational and the electromagnetic 2–forms as the dynamical equations for these fields, the postulated structure yields naturally a cosymplectic structure on phase space. The dynamical connection that corresponds to the total dynamical phase 2–form turns out to be the (scaled) Reeb vector field for this cosymplectic structure [1, 3, 6, 24, 26]. Consequently, it yields naturally the dynamics on spacetime, associated with the total dynamical phase 2–form, as the geodesic flow of the corresponding dynamical connection. This leads directly to the notion of conserved quantities.

Equivalently, the dynamics can be described using the contact structure of the second jet space which yields naturally the (intrinsic) Euler–Lagrange morphism as the
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horizontal phase 2–form.

On the other hand, any closed dynamical phase 2–form has local potentials. The spacetime-horizontal local potentials are said to be dynamical phase 1–forms and yield a time-horizontal part, called Lagrangian, according to the contact splitting induced by phase space. This turns out to be a (local) Lagrangian for the above (global and intrinsic) Euler–Lagrange morphism. For any Lagrangian, its associated dynamical phase 1–form turns out to be the Poincaré–Cartan form for this Lagrangian. Furthermore, given an observer, i.e. any connection of spacetime, a Poincaré–Cartan form splits into the observed Hamiltonian and the observed momentum.

Finally, we recall the notion of $\tau$-Hamiltonian lift [15, 16, 31], which is a covariant lift of functions of the phase space to vector fields of phase space motivated from the covariant quantum Galilei theory. We will see that this lift plays an important role in the definition of momentum map.

In the third section, we analyse systematically the infinitesimal symmetries of the geometric objects that we have introduced so far. Hereby, we are mainly interested in holonomic symmetries. Natural bijections are used to find the relations between the symmetries of the different objects.

After recalling the basic properties about symmetries and infinitesimal symmetries we apply this notion to the objects $CCG$. However, some of the geometric objects that we need for our analysis are not tensors. Therefore, we show first, how to get well defined expressions for the infinitesimal symmetries of these objects. We need these results to show in Theorem 3.1 that any holonomic infinitesimal symmetry of a spacetime connection is a symmetry of the corresponding phase connection, and vice-versa. Moreover, any holonomic infinitesimal symmetry of a spacetime connection turns out to be a symmetry of the corresponding dynamical connection, and vice-versa. On the other hand, Theorem 3.2 shows that any holonomic infinitesimal symmetry of a dynamical phase 2–form is also a symmetry of the corresponding spacelike metric and the corresponding affine connection of the phase space, and vice-versa. Similarly, we prove in Theorem 3.3 that any holonomic infinitesimal symmetry of the Euler–Lagrange morphism is also a symmetry of the corresponding pair of a spacelike metric and a dynamical connection, and vice-versa. This yields directly the equivalence of (holonomic) symmetries of the dynamical phase 2–form and the corresponding Euler-Lagrange morphism.

In subsection 3.3, we restrict our attention to closed dynamical 2–forms. We see, how conserved quantities are related to symmetries of such forms. This yields directly the Theorem of Noether 3.5 for symmetries of Poincaré–Cartan forms. Using Theorem 3.6 which is also suggested from a natural bijective correspondence and which shows that any holonomic infinitesimal symmetry of a Poincaré–Cartan form is a symmetry of the corresponding Lagrangian, and vice versa, we get another equivalent version of the Noether theorem in Corollary 3.1 which may be more popular to the physical reader.

In subsection 3.4, we provide a definition of covariant momentum map for an action of a group of symmetries of the cosymplectic structure in our model. This definition is similar to momentum map in presymplectic and cosymplectic literature [1, 8, 6, 24, 26].
However, the covariance of our theory requires the concept of \(\tau\)-Hamiltonian lifts. This leads us directly to the quantisable functions and thus to the quantum theory.

We see that we can associate to any infinitesimal generator, associated with the infinitesimal action, a pair, namely a conserved quantity and an element of the vector space of time units. Such a pair is determined up to an additive real constant of the conserved quantity. We say a map that associates to each element of the Lie algebra for the group of cosymplectic symmetries such a pair to be a momentum map for the (infinitesimal) action of the group. We prove that the \(\tau\)-Hamiltonian lift of any component of a momentum map is equal to the infinitesimal generator associated with the group of symmetries for which this component was defined. Moreover, we introduce, a bracket for pairs such that the map that associates to a pair its \(\tau\)-Hamiltonian turns out to be a morphism of Lie algebras.

For the case of an action that, additionally, projects on an action of the group on spacetime, we find that the components of a momentum map are of special type. Namely, they are ‘quantisable functions’, i.e. functions which are polynomials of second degree in the velocities, and whose second derivative (with respect to velocities) is proportional to the metric. Such functions are of fundamental importance in the quantum part of this model \([15, 16, 31]\). This feature is new with respect to standard symplectic momentum map formalism, and is mainly due to covariance requirements. Its importance is fundamental in quantum mechanics. In this case, a momentum map is determined by the first component of the pairs. Then, we consider an action of a group of symmetries that, additionally, is a group of symmetries of a Poincaré–Cartan form. Here, the momentum map turns out to be unique and it is determined by the Poincaré–Cartan form.

Finally, three very simple examples are provided in order to show the machinery at work. These examples show also, that in standard time–independent situations of particle mechanics we get the same results as standard symplectic mechanics \([27]\). However, we are able to treat time–dependent cases, as well. Therefore, we can relate time translations to the Hamiltonian in the standard (relativistic) way. For a less trivial application of the momentum map in our theory, see \([32]\).

We find our results promising also in view of a next research about symmetries of quantised systems.

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## 2 Covariant classical Galilei theory

In order to be rather self contained, we start by recalling the main points of the covariant classical Galilei theory. \([4, 16, 15, 31]\)
2.1 Unit spaces

Now, we are going to assume the fundamental spaces of units of measurement and coupling constants.

The theory of unit spaces has been developed in \cite{15,16} to make the independence of classical and quantum mechanics from scales explicit. Unit spaces are defined similarly to vector spaces, but using the abelian semigroup $\mathbb{R}^+$ instead of the field of real numbers $\mathbb{R}$. In particular, positive unit spaces are defined to be 1–dimensional (over $\mathbb{R}^+$) unit spaces. It is possible to define $n$–th tensorial powers and $n$–th roots of unit spaces.

Moreover, if $P$ is a positive unit space and $p \in P$, then we denote by $1/p \in P^*$ the dual element. Hence, we can set $P^{-1} := P^*$. In this way, we can introduce rational powers of unit spaces.

We assume the following unit spaces.

– $T$, the oriented one–dimensional semi-vector space of time intervals;

– $L$, the positive unit space of length units;

– $M$, the positive unit space of mass units.

We denote by $\mathbb{P} := P \otimes \mathbb{R}$ the associated vector space to the unit space $P$. An element $u_0 \in T$ (or $u^0 \in T^{-1}$) represents a time unit of measurement, a charge is represented by an element $q \in Q := T^{-1} \otimes L^{3/2} \otimes M^{1/2}$, and a particle is represented by a pair $(m, q)$, where $m$ is a mass and $q$ is a charge. A tensor field with values into mixed rational powers of $T$, $L$, $M$ is said to be scaled. We assume the Planck’s constant $\hbar \in T^{-1} \otimes L^2 \otimes M$.

We will often be involved with the Lie derivative of scaled tensor fields. It can be shown \cite{15,16} that the Lie derivative commutes with the scaling.

In the following, we assume all manifolds and maps to be $C^\infty$.

2.2 Spacetime and phase space

Assumption C.1. We assume spacetime to be an $(n + 1)$-dimensional oriented fibred manifold

\[ t : E \to T \]

over a 1-dimensional oriented affine space $T$ (time) associated with the vector space $\mathbb{T}$, where $n \in \mathbb{N}$ with $n \geq 2$.

We shall refer to spacetime charts $(x^0, x^i)$, which are adapted to the fibring, to a time unit of measurement $u_0 \in T$ and to the chosen orientation of $E$. The index 0 will refer to the base space, Latin indices $i, j, \cdots = 1, 2, 3$ will refer to the fibres, while Greek indices $\lambda, \mu, \cdots = 0, 1, 2, 3$ will refer both to the base space and the fibres.

A motion is defined to be a section $s : T \to E$. The coordinate expression of a motion $s$ is of the type

\[ s^i := x^i \circ s : T \to \mathbb{R}. \]

We shall be involved with the tangent bundle $\tau_E : TE \to E$ and the vertical tangent subspace $VE := \ker Tt \subset TE$. We denote the charts induced on $TE$ by $(x^\lambda, \dot{x}^\lambda)$;
2.3 Natural bijective correspondences

Moreover, we denote the induced local bases of vector fields, of forms and of vertical forms of $E$, respectively, by $(\partial_\lambda)$, $(d^\lambda)$ and $(\tilde{d}^\lambda)$.

The phase space is defined to be the first jet space of motions $t^1_0 : J_1E \to E$. We denote the charts induced on $J_1E$ by $(x^0, x^i, x^i_0)$. We will be involved with the second jet space of motions $t^2_0 : J_2E \to E$. We denote the charts induced on $J_2E$ by $(x^0, x^i, x^i_0, x^i_{00})$.

The velocity of a motion $s$ is the section $j_1s : T \to J_1E$, with coordinate expression $x^i_0 \circ j_1s = \partial_0 s^i$.

The phase space is equipped with the natural maps $d_1 : J_1E \to T^* \otimes TE$ and $\vartheta_1 : J_1E \to T^*E \otimes VE$, with coordinate expressions $d_1 = u^0 \otimes d_{1,0} = u^0 \otimes (\partial_0 + x^i_0 \partial_i)$ and $\vartheta_1 = \vartheta_1^i \otimes \partial_i = (d^i - x^i_1 d^0) \otimes \partial_i$. Analogously, the jet space $J_2E$ is equipped with the natural maps $d_2 : J_2E \to T^* \otimes T_1J_1E$ and $\vartheta_2 : J_2E \to T^*E \otimes V_1J_1E$, with coordinate expressions $d_2 = u^0 \otimes d_{2,0} = u^0 \otimes (\partial_0 + x^i_0 \partial_i + x^i_0 \partial^i_0)$ and $\vartheta_2 = \vartheta_2^i \otimes \partial_i = (d^i - x^i_0 d^0) \otimes \partial_i + (d^i - x^i_0 d^0) \otimes \partial_i$.

An observer is defined to be a section $o : E \to J_1E$. Its coordinate expression is of the type $o = u^0 \otimes (\partial_0 + o^i_0 \partial_i)$, where $o^i_0 : E \to \mathbb{R}$.

An observer $o$ can be regarded as a scaled vector field of $E$. The integral motions of an observer $o$ are defined to be the motions $s$ such that $j_1s = o s$. An observer $o$ yields locally a fibred splitting $E \to T \times P$, where $P$ is the manifold of integral motions of $o$. An observer is said to be complete if it yields a global splitting of $E$. A spacetime chart is said to be adapted to $o$ if it is adapted to the local splitting of $E$ induced by $o$, i.e. if $o^i_0 = 0$.

An observer $o$ can be regarded as a connection of the fibred manifold $E \to T$. Accordingly, it yields the translation fibred isomorphism $\nabla[o] : J_1E \to T^* \otimes VE$, given by $\nabla[o](e_1) := e_1 - o(t^1_0(e_1))$. We have the coordinate expression $\nabla[o] = (x^0_0 - o^1_0) d^0 \otimes \partial_i$.

2.3 Natural bijective correspondences

In the following we define distinguished objects living on spacetime or phase space and we investigate their relations. In a concrete model of a classical system these objects will be either determined by further assumptions or as a consequence of the assumptions.

Definition 2.1. A scaled vertical Riemannian metric

(2.1) \[ g : E \to \mathbb{L}^2 \otimes (V^*E \otimes V^*E) \]

is said to be a spacelike metric.

The coordinate expression of a spacelike metric $g$ is $g = g_{ij} \tilde{d}^i \otimes \tilde{d}^j$, where $g_{ij} : E \to \mathbb{L}^2$.

Given a mass $m \in \mathbb{M}$, it is convenient to introduce a “normalised” metric $G \equiv \frac{m}{\hbar} g$, with coordinate expression $G = G^0_{ij} u_0 \otimes \tilde{d}^i \otimes \tilde{d}^j$, where $G^0_{ij} : E \to \mathbb{R}$. 

A metric $G$ yields a family of Riemannian connections of the fibres of $E \to T$

$$\kappa : V E \to V^* E \otimes V V E$$

with coordinate expression $\kappa = d^k \otimes (\partial_k + \kappa_{k}^{i} \dot{x}^{i} \hat{\partial}_k)$, where $\kappa_{k}^{i} \dot{x}^{i}$ are the usual Christoffel symbols on the fibres of $E \to T$ related to $G$.

Next, we analyse distinguished connections that can be defined on spacetime or phase space. We recall the fact, that for any fibred manifold $p : F \to B$, there are natural bijective correspondences $c \leftrightarrow \nu \leftrightarrow \nabla$ between the connections $c : F \to T^* F \otimes TF$, the vertical projectors $\nu : F \to T^* F \otimes VF$ and the covariant derivatives $\nabla : J_1 F \to T^* F \otimes VF$ of $p$. In order to give their coordinate expressions, we take, only in this case, greek indices for the basis coordinates and latin indices for the fibres. Then, their expressions are $c = d^\lambda \otimes (\partial_\lambda + \kappa_{\lambda}^{i} \partial_i)$, $\nu[c] = (d^i - c_i^\lambda \partial^\lambda) \otimes \partial_i$ and $\nabla[c] = (x_i^\lambda - c_i^\lambda )d^\lambda \otimes \partial_i$.

**Definition 2.2.** A spacetime connection is defined to be a $dt$–preserving torsion free linear connection of the vector bundle $TE \to E$

$$K : TE \to T^* E \otimes TTE . \quad \Box$$

The coordinate expression of any spacetime connection $K$ is of the type $K = d^\lambda \otimes (\partial_\lambda + K_{\lambda}^{i} \partial_i)$, where $K_{\lambda}^{i} = K_{\lambda}^{i} \nu : E \to \mathbb{R}$. The compatibility with $dt$, i.e. the condition $\nabla[K]dt = 0$, is expressed by $K_{\mu}^0 \nu = 0$.

The restriction of a spacetime connection $K$ to the vertical tangent bundle is a linear connection $K' : VE \to V^* E \otimes TVE$.

A spacetime connection $K$ is said to be metric if $\nabla[K'] G = 0$. In this case, $K$ is partially determined by the metric according to the local formulas $K_{ihj} = -\frac{1}{2}(\partial_i G_{hj} + \partial_j G_{hi} - \partial_h G_{ij})$ and $K_{0ij} + K_{0ji} = -\partial_0 G_{ij}$ where indices have been raised or lowered by the metric $G$.

**Definition 2.3.** A phase connection is defined to be a torsion–free affine connection of the affine bundle $J_1 E \to E$

$$\Gamma : J_1 E \to T^* J_1 E . \quad \Box$$

The coordinate expression of $\Gamma$ is of the type $\Gamma = d^\lambda \otimes (\partial_\lambda + \Gamma_{\lambda}^i \partial^0_i)$, where $\Gamma_{\lambda}^i \equiv \Gamma_{\lambda 0^i}^0 x^0 + \Gamma_{\lambda 0^0}^i$ and $\Gamma_{\lambda 0^0}^i, \Gamma_{\lambda 0^i} : E \to \mathbb{R}$.

It can be easily seen in coordinates, that there is a natural bijective correspondence $K \leftrightarrow \Gamma[K]$.
between the pairs of a spacelike metric and a phase connection and the dynamical phase 2–forms.

A second order connection of spacetime is defined to be a (nonlinear) connection of the fibred manifold $J_1E \to T$  

\[(2.5) \quad \gamma : J_1E \to \mathbb{T}^* \otimes T J_1E,\]

which is projectable on the contact map $d_1$. The coordinate expression of $\gamma$ is of the type $\gamma = u^0 \otimes (\partial_0 + x_i^0 \partial_i + \gamma^{i}_0)$, where $\gamma^{0}_0 : J_1E \to \mathbb{R}$.

**Definition 2.4.** A second order connection is said to be a dynamical connection if it is “homogeneous” in the sense of [8], i.e. its coordinate expression is of the type $\gamma^{i}_0 \equiv \gamma^{i00}_0 x_0^0 x_0^0 + 2\gamma^{i0}_0 x_0^0 + \gamma^{0i}_0$ and $\gamma^{i00}_0$, $\gamma^{i0}_0$, $\gamma^{0i}_0 : E \to \mathbb{R}$.  

We will be involved with the covariant derivative of a dynamical connection, i.e. the morphism $\nabla[\gamma] : J_2E \to \mathbb{T}^* \otimes \mathbb{T}^* \otimes VE$ with coordinate expression $\nabla[\gamma] = u^0 \otimes (x_{00}^0 - \gamma^{00}_0) d^0 \otimes \partial_i$.

We can easily see in coordinates that the map $\Gamma \mapsto \gamma[\Gamma] := d_1 \circ \Gamma$ is a natural bijective correspondence between the phase connections and dynamical connections, namely such that $\gamma^{i00}_0 = \Gamma^{i00}_0$, $\gamma^{i0}_0 = \Gamma^{i0}_0$ and $\gamma^{0i}_0 = \Gamma^{0i}_0$.

Next, we define distinguished forms of $J_1E$ and $J_2E$ that can be defined through the above objects.

**Definition 2.5.** A 2–form $\Omega$ of $J_1E$ of the type

\[(2.6) \quad \Omega[G, \Gamma] = \nu[\Gamma] \wedge \partial_1 : J_1E \to \Lambda^2 T^* J_1E,\]

where $\nu[\Gamma]$ is the vertical projection associated with a phase connection $\Gamma$ and where the contracted wedge product is taken with respect to a spacelike metric $G$ a dynamical phase 2–form.

We have the coordinate expression $\Omega[G, \Gamma] = G_{ij}^0 (d^0_i - \Gamma_{ij}^0 d^\lambda) \wedge \partial^j_1 = G_{ij}^0 (d^0_i - \gamma_{ij}^0 - \Gamma_{ij}^0) \wedge \partial^j_1$.

We observe that the above form is the only natural 2–form which can be obtained from $\Gamma$ and $G$ [3], [16]. Moreover, the form $\Omega[G, \Gamma]$ is non degenerate in the sense that $dt \wedge \Omega[G, \Gamma]$ is a volume form of $J_1E$. We can easily prove that there is a unique scaled vector field $X : J_1E \to T J_1E$ such that $i_X dt = 1$ and $i_X \Omega[G, \Gamma] = 0$, namely, $X = \gamma[\Gamma]$, the dynamical connection. This yields a natural bijective correspondence

\[(G, \Gamma) \leftrightarrow \Omega[G, \Gamma]\]

between the pairs of a spacelike metric and a phase connection and the dynamical phase 2–forms.
Definition 2.6. A 2–form $\mathcal{E}$ of $J_2E$ of the type
\begin{equation}
\mathcal{E}[G, \gamma] = \nabla[\gamma] \wedge \vartheta_1 : J_2E \to \Lambda^2T^*E,
\end{equation}
where $\nabla[\gamma]$ is the covariant differential associated with a dynamical connection $\gamma$ and where the contracted wedge product is taken with respect to a spacelike metric $G$ is called a horizontal phase 2–form.

We have the coordinate expression $\mathcal{E}[G, \gamma] = G_{ij}^0(x^i_0-\gamma^i_0)d^0 \wedge (d^j - x^j_0d^0)$.

Analogously to the case of a dynamical phase 2–form, it turns out that there is a natural bijective correspondence
\[(G, \gamma) \leftrightarrow \mathcal{E}[G, \gamma]\]
between the pairs of a spacelike metric and a dynamical connection and the horizontal phase 2–forms.

Consequently, we obtain a natural bijective correspondence
\[\Omega[\Gamma, G] \leftrightarrow \mathcal{E}[\nabla[\gamma][\Gamma], G]\]
between the dynamical phase 2–forms and the horizontal phase 2–forms.

It turns out that the horizontal phase 2–form $\mathcal{E}$ corresponding to a dynamical phase 2–form $\Omega$ coincides with the horizontal part of $\Omega$ according to the contact splitting of forms of $J_1E$ induced by $J_2E$.

Let us consider the case of a closed dynamical phase 2–form $\Omega$. We can prove \[\nabla[\Gamma']G = 0\] that $d\Omega = 0$ is equivalent to the conditions that $\nabla[\Gamma']G = 0$ and that, in coordinates, the curvature $R[\Gamma] : E \to \Lambda^2T^*E \otimes V E \otimes T^*E$ fulfills $R^{ij}_{\lambda \mu} = R^{ji}_{\mu \lambda}$.

Given an observer $o$ we can define the 2–form of $E, \Phi[o] := 2o^*\Omega : E \to \Lambda^2T^*E$. It turns out that $d\Phi[o] = 0$.

Remark 2.1. If the phase space $J_1E$ is equipped with a closed dynamical phase 2–form $\Omega$, the triple $(J_1E, \Omega, dt)$ turns out to be a cosymplectic manifold. The corresponding dynamical connection turns out to be the (scaled) Reeb vector field for this cosymplectic structure.

Moreover, a closed dynamical phase 2–form $\Omega$ admits potential 1–forms of $J_1E$. In the following we introduce a special kind of such potential forms.

Definition 2.7. A horizontal 1–form $\Theta : J_1E \to T^*E$ such that $d\Theta = \Omega$ where $\Omega$ is a closed dynamical phase 2–form is said to be a dynamical phase 1–form associated with $\Omega$.

For any observer $o$, the expression of a dynamical phase 1–form associated with $\Omega$ in adapted coordinates is given by $\Theta = -(\frac{1}{2}G^0_{ij}x^i_0x^j_0-A_0)d^0 + (G^0_{ij}+A_i)d^j$ where $A_\lambda d^\lambda$ is a potential of the closed 2–form $\Phi[o] = 2o^*\Omega$. Clearly, a dynamical phase 1–form associated with $\Omega$ is determined up to a closed 1–form of $E$.

According to the contact splitting of $\Theta$ induced by $J_1E$ we define the following
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**Definition 2.8.** The Lagrangian $\mathcal{L}[\Theta]$ associated with a dynamical phase 1–form $\Theta$ is defined to be the time-horizontal 1–form

\[
\mathcal{L} := \partial_1 \Theta : J_1 E \to T^* T.
\]

The momentum $\mathcal{P}[\mathcal{L}]$ of the Lagrangian $\mathcal{L}$ is defined to be the vertical derivative of $\mathcal{L}$ with respect to the fibering $t^1_1 : J_1 E \to E$, i.e. $\mathcal{P} := V_E \mathcal{L}$. \[\square\]

We have the coordinate expressions $\mathcal{L}[\Theta] = (\frac{1}{2}G^0_{ij}x^i_0x^j_0 + A_i x^i_0 + A_0)d^0$ and $\mathcal{P}[\mathcal{L}] = (G^0_{ij}x^i_0 + A_i)(d^i - x^i_0d^0)$.

It turns out that $\Theta$ splits into $\Theta = \mathcal{L}[\Theta] + \mathcal{P}[\mathcal{L}[\Theta]]$. This splitting coincides with the contact splitting of $\Theta$ induced by $J_1 E$.

This yields directly the natural bijective correspondence

$$\Theta \leftrightarrow \mathcal{L}[\Theta]$$

between dynamical phase 1–forms and Lagrangians.

Moreover, $\Theta[\mathcal{L}]$ turns out to be the Poincaré–Cartan form associated with the Lagrangian $\mathcal{L}$. Hence, in the following, we say any dynamical phase 1–form to be a Poincaré–Cartan form. We stress the fact that the objects $\Theta$, $\mathcal{L}$ and $\mathcal{P}$ do not depend on an observer but on a chosen local gauge. It is for practical convenience that we have given the coordinate expressions with respect to an observer.

Given a closed dynamical phase 2–form $\Omega$ it can be proved that the horizontal phase 2–form $E[\Omega]$ coincides with the Euler-Lagrange morphism associated with any Lagrangian $\mathcal{L}[\Theta]$ where $\Theta$ is a dynamical phase 1–form associated with $\Omega$. Hence, in the following, we say a horizontal phase 2–form to be an Euler-Lagrange morphism, when we are dealing with closed dynamical phase 2–forms.

The above results concerning $\Theta$, $\Omega$, $\mathcal{L}$ and $E$ are described by the following commutative diagram

$$\begin{array}{ccc}
\Theta & \xrightarrow{d} & \Omega \\
\downarrow \scriptstyle{i_{\delta_1}} & & \downarrow \scriptstyle{i_{\delta_2} i_{\omega_2}} \\
\mathcal{L} & \xrightarrow{\epsilon} & E
\end{array}$$

This diagram can be regarded as a piece of a more comprehensive natural bicomplex, which accounts for the Lagrangian formalism via a cohomological scheme [20, 30, 40].

We summarise the above results in the following proposition

**Proposition 2.1.** The following natural bijective correspondences hold

\[
\begin{align*}
\text{(2.9)} & \quad K \leftrightarrow \Gamma \leftrightarrow \gamma \\
\text{(2.10)} & \quad \Omega \leftrightarrow (\Gamma, G) \leftrightarrow (\gamma, G) \leftrightarrow \mathcal{E} \\
\text{(2.11)} & \quad \Theta \leftrightarrow \mathcal{L}
\end{align*}
\]

It can be seen that there is another splitting of a Poincaré–Cartan form $\Theta$ which is observer dependent. Namely, given an observer $o$, each Poincaré–Cartan form splits, according to the splitting of $T^* E$ induced by $o$ into the morphism $-H[o, \Theta] := -o \downarrow \Theta : J_1 E \to T^* T$, called observed Hamiltonian, and the morphism $\mathcal{P}[o, \Theta] := \nu_1 \downarrow (\nu[o] \downarrow \Theta) : J_1 E \to T^* E$, called observed momentum.
2.4 Classical dynamics

We are involved with two different approaches to the classical dynamics in our context. The first seems to be more direct and consists in defining directly the law of particle motion, namely, the (generalised) Newton law.

Definition 2.9. Let $\gamma$ be a dynamical connection and $s$ a motion. Then, the condition on $s$,

$$\nabla[\gamma]j_1s := j_2s - \gamma \circ j_1s = 0.$$

is said to be the law of motion for the dynamical connection $\gamma$ [4, 16, 15, 31].

The law of motion has the coordinate expression

$$\partial_0^0 s_i - \gamma^0_0 \circ s = 0.$$

A dynamical connection $\gamma$ can be regarded (up to a time scale) as a vector field of $J_1E$, hence, a motion fulfilling the above equation is just an integral curve of $\gamma$.

It is easy to see that a motion $s$ fulfills the above law if and only if, for any $f : J_1E \to \mathbb{R}$, we have

$$d(f \circ j_1s) = (\gamma.f) \circ j_1s.$$

(2.12)

where $\gamma.f := df(\gamma)$. In the particular case when $\gamma.f = 0$ we call $f$ a conserved quantity.

On the other hand, if the metric $G$ is given by a concrete model, the natural bijective correspondence $E \leftrightarrow (G, \gamma)$ leads to an equivalent approach to the equations of motion, namely, the equation $E(L)j_2s = 0$ for any Lagrangian $L$.

2.5 Hamiltonian lift and quantisable functions

Let us consider a dynamical phase 2–form $\Omega$. Then, $\Omega$ yields in a natural way the Hamiltonian lift of functions $f : J_1E \to \mathbb{R}$ to vertical vector fields $H[f] : J_1E \to VJ_1E$. The musical morphism $\Omega^\flat : VJ_1E \to T^*J_1E$ turns out to be an isomorphism of vector spaces between vertical vector fields of $J_1E$ and forms of $J_1E$ that annihilate the corresponding $\gamma$. Clearly, in the case when $\Omega = \Omega[G, \Gamma]$, then $\gamma = \gamma[\Gamma]$.

More generally, taking into account the independence of units, the choice of a time scale $\tau : J_1E \to TT$ yields, in a natural (covariant) way, the $\tau$-Hamiltonian lift of functions $f : J_1E \to \mathbb{R}$ to vector fields

$$(2.13) \quad H_\tau[f] := < \tau, \gamma > + (\Omega^\flat)^{-1}(df - < \gamma.f, dt >) : J_1E \to TJ_1E$$

whose time component is $\tau$. We observe, that it is the scaling of the time form which requires such a (covariant) lift, rather than the standard lift in cosymplectic mechanics. Its coordinate expression is

$$H_\tau[f] = \tau^0(\partial_0 + x^0_0 \partial_h + \gamma^{0i}_0 \partial^0_h) + G^0_{hk}(-\partial^0_k f \partial_h + (\partial_k f + (\Gamma^i_{k0} - G^0_{kr}G^l_0 \Gamma^r_{s0}) \partial^0_l f) \partial^0_h)$$

where $\tau^0 := < \tau, u^0 >$.

In view of later developments in the quantum theory, it can be proved [16, 13] that $H_\tau[f]$ is projectable on a vector field $X[f] : E \to TE$ if and only if the following
conditions hold: i) the function $f$ is quadratic with respect to the affine fibres of $J_1 E \to E$ with second fibre derivative $f'' \otimes G$, where $f'' : E \to TT$ and ii) $\tau = f''$. A function of this kind is called special quadratic and is of the type

$$f = \frac{1}{2} f^0 G^0_{i,j} x^i_0 x^j_0 + f^0_i x^i_0 + \text{of},$$

with $f^0, f^0_i, f : E \to \mathbb{R}$.

Next, assume that $\Omega$ is closed. Then, the $\tau$-Hamiltonian lift yields a Poisson bracket for functions of $J_1 E$, namely, the bracket $\{f, g\} := i_{H_0[f]} i_{H_0[g]} \Omega$. We observe that if $\tau, \sigma$ are time scales, then $\{f, g\} = i_{H_0[f]} i_{H_0[g]} \Omega$. The vector space of special functions is not closed under the Poisson bracket, but it turns out to be an $\mathbb{R}$-Lie algebra through the natural special bracket $[f, g] = \{f, g\} + \gamma(f''), g - \gamma(g''), f$, where $\{f, g\}$ is the Poisson bracket of the functions $f$ and $g$. Of particular interest are such special functions whose time component is a constant. They are called quantisable functions with constant time component.

It is easy to see that we can apply this bracket also onto scaled functions. In particular, the Hamiltonian is a (scaled) quantisable function with constant time component.

### 3 Symmetries in covariant classical mechanics

In this section we want to introduce the notion of symmetry to the objects which we have defined on spacetime and phase space. We give theorems about the relation between the symmetries of these objects. The natural correspondences turn out to be an indicator for symmetry relations. We see that these results can be directly compared to standard results of Hamiltonian or Lagrangian mechanics. On the other hand, we think that our (non standard) approach to these results within the covariant framework is promising, especially in view of applications to quantum theory (CQG).

#### 3.1 Symmetries and infinitesimal symmetries

First, we want to recall the basic facts about symmetries, groups of symmetries and infinitesimal symmetries of manifolds, fibred manifolds and tensors.

Let $M$ be a manifold. Then, we define a symmetry of the manifold $M$ to be a diffeomorphism $f : M \to M$.

The diffeomorphisms $f : M \to M$ constitute a group, called the diffeomorphism group $Diff(M)$, which operates on the manifold $M$ via the natural left action $\Phi : Diff(M) \times M \to M : (f, m) \mapsto f(m)$. The group $Diff(M)$ is infinite dimensional, hence, it is difficult to deal with the full group. But, in practice, we are interested in finite dimensional subgroups with the structure of a Lie group.

On the other hand, we are often interested in an “abstract” Lie group $G$ which acts on the manifold $M$ through a left action $\Phi : G \times M \to M$. The map $\Phi$ yields a group morphism (which may still be denoted by $\Phi$) $\Phi : G \to Diff(M)$. This map needs not
to be injective. In the particular case when this map is injective we can identify the "abstract" group \( G \) with the corresponding subgroup of \( \text{Diff}(M) \).

By taking the tangent prolongation of the action \( \Phi \) with respect to \( G \), at the unit element \( e \in G \), we obtain the linear fibred morphism over \( M \), \( \partial \Phi : T_e G \times M \to TM \).

By considering the natural identification of \( T_e G \) with the Lie algebra \( \mathfrak{g} \) of left invariant vector fields of \( G \) we can write the above morphism as a map

\[
\partial \Phi : \mathfrak{g} \to \text{Sec}(TM) : \xi \mapsto X[\xi] := \partial \Phi(\xi),
\]

which turns out to be an antihomomorphism of Lie algebras. We call \( \partial \Phi : \mathfrak{g} \times M \to TM \) an infinitesimal left action of the Lie algebra \( \mathfrak{g} \) on \( M \) and the vector field \( X[\xi] \) of \( M \) infinitesimal generator of the infinitesimal action corresponding to \( \xi \).

Clearly, if \( \Phi : G \to \text{Diff}(M) \) is injective, then also \( \partial \Phi : \mathfrak{g} \to \text{Sec}(TM) \) is injective.

The above discussion suggests the following definition. We call a vector field \( X : M \to TM \) an infinitesimal symmetry of the manifold \( M \) (since its local flow is a local group of diffeomorphisms). Now, by considering a left action \( \Phi \) of a Lie group \( G \) on \( M \), the set of infinitesimal generators \( \{ X[\xi] : M \to TM, \forall \xi \in \mathfrak{g} \} \) turns out to be a subalgebra of the Lie algebra of infinitesimal symmetries of \( M \).

Now we extend the definition of symmetries to manifolds that are equipped with further structure.

Let \( p : E \to B \) be a fibred manifold. Then, we define a symmetry of the fibred manifold \( p \) to be a fibred diffeomorphism \( f \) of \( E \) over \( B \), i.e. a symmetry \( f : E \to E \) of the manifold \( E \) which projects on a symmetry \( \underline{f} : B \to B \) of the base space \( B \). Given a (Lie) group of symmetries of \( M \) any infinitesimal generator \( X[\xi] \) turns out to be a vector field \( X : E \to TE \) which projects on a vector field \( \underline{X} : B \to TB \). This suggests to call any vector field \( X \) of \( E \) that projects on a vector field \( \underline{X} \) of \( B \) an infinitesimal symmetry of the fibred manifold \( p \) (since its local flow is a local group of fibred diffeomorphisms).

Let \( f : M \to M \) be a symmetry of the manifold \( M \). Then, the tangent prolongation \( Tf : TM \to TM \) turns out to be a symmetry of the fibred manifold \( \tau_M : TM \to M \).

Moreover, for each left action \( \Phi : G \times M \to M \), the tangent prolongation \( T\Phi : G \times TM \to TM : (g, y) \mapsto T(\Phi_g)(y) \) turns out to be a left action, called the tangent prolongation of \( \Phi \).

Let \( \nu \) be a tensor field of \( M \), which is contravariant of order \( s \) and covariant of order \( r \). Then, we define a symmetry of the tensor field \( \nu \) to be a diffeomorphism \( f : M \to M \) such that \( \nu \circ \underline{\otimes} T f = \underline{\otimes} T f \circ \nu \).

As before, we can define groups of symmetries of \( \nu \). It turns out that each infinitesimal generator \( X[\xi] \) associated with a (Lie) group of symmetries of the tensor field \( \nu \) fulfills the equation \( L_X[\xi] \nu = 0 \). This suggests to call any vector field \( X \) of \( M \) such that \( L_X \nu = 0 \) an infinitesimal symmetry of the tensor field \( \nu \) (since its local flow is a local group of symmetries of the tensor field).

Now, let \( f : E \to E \) be a symmetry of the fibred manifold \( p : E \to B \). Then, for each \( 1 \leq k \), the \( k \)-jet prolongation \( J_k f : J_k E \to J_k E \) turns out to be a symmetry of the fibred manifolds \( p^h_k : J_k E \to J_h E \) and \( p^k : J_k E \to B \), for each \( 1 \leq h < k \).
Moreover, for each left action of symmetries \( \Phi : G \times E \to E \), the \( k \)-jet prolongation \( \mathcal{J}_k \Phi : G \times J_k E \to J_k E : (g, e_k) \mapsto J_k(\Phi_g)(e_k) \) turns out to be a left action, called the \( k \)-jet prolongation of \( \Phi \).

Let us recall the natural involution \( s : TTM \to TTM \) [12]. This map yields the natural prolongation of each vector field \( X : M \to TM \) to the vector field \( X(T) := s \circ TX : TM \to TTM \). If \( X = X^\lambda \partial_\lambda \), then \( X(T) = X^\lambda \partial_\lambda + \partial_\mu X^\lambda x^\mu \partial_\lambda \). The map \( X \mapsto X(T) \) turns out to be a morphism of Lie algebras. Let \( \Phi : G \times M \to M \) be a left action of \( G \) on \( M \). Then, we obtain \( s \circ \mathcal{T} \partial \Phi = \partial \mathcal{T} \Phi : g \times TM \to TTM \), hence, \( s \circ \mathcal{T} \partial \Phi \) turns out to coincide with the infinitesimal left action of the Lie algebra \( g \) on the manifold \( TM \).

We recall the natural map \( r^k : J_k TE \to T J_k E \) [25]. This map yields the natural prolongation of each vector field \( X : E \to TE \) to the vector field \( X(k) := r^k \circ J_k X : J_k E \to T J_k E \). In particular, if \( X = X^\lambda \partial_\lambda + X^i \partial_i \), then \( X(1) = X^\lambda \partial_\lambda + X^i \partial_i + (\partial_\mu X^i + \partial_i \partial_\mu x^\mu - \partial_\mu X^\lambda x^\mu) \partial_\lambda \). The map \( X \mapsto X(k) \) turns out to be a morphism of Lie algebras. Let \( \Phi : G \times E \to E \) be a left action of a group \( G \) such that \( \{ \Phi_g, g \in G \} \) is a group of symmetries of the fibred manifold \( p : E \to B \). Then, we obtain \( r^k \circ \mathcal{J} \partial \Phi = \partial \mathcal{J} \Phi : g \times J_k E \to T J_k E \), hence, \( r^k \circ \mathcal{J} \partial \Phi \) turns out to coincide with the infinitesimal left action of the Lie algebra \( g \) on the manifold \( J_k E \).

In general, we say all above natural prolongation of symmetries, infinitesimal symmetries and actions to be *holonomic*. By abuse of language we often call the group \( G \) a group of symmetries if the left action \( \Phi \) of \( G \) is given such that \( \Phi_g \) is a symmetry for all \( g \in G \).

### 3.2 Infinitesimal symmetries of spacetime structures

Now, we apply the above general definitions of infinitesimal symmetries to covariant classical mechanics. However, some of the morphisms (contact maps, spacelike metric) cannot be regarded as tensors, naturally. Consequently, a (direct) definition of their symmetries would require naturality techniques [17]. Instead, we show that it is possible in our case to give a meaning to their infinitesimal symmetries keeping the standard Lie derivative.

**Symmetries of spacetime**

An infinitesimal symmetry of spacetime is an infinitesimal symmetry of the fibred manifold \( t : E \to T \) which, additionally, preserves the affine structure of \( T \). More precisely, we define an *infinitesimal symmetry of spacetime* as a vector field \( X : E \to TE \) which is projectable on a vector field \( \mathcal{X} : T \to TT \) and such that \( \mathcal{X} \) is constant.

An easy calculation shows that

**Proposition 3.1.** A vector field \( X : E \to TE \) is an infinitesimal symmetry of spacetime if and only if \( L_X dt = 0 \).

If \( X = X^0 \partial_0 + X^i \partial_i \) is the coordinate expression, then the conditions are equivalent to \( \partial_\mu X^0 = 0 \).
Proof. In coordinates, any vector field $X$ of $E$ is of the type $X = X^0\partial_0 + X^i\partial_i$, where $X^0, X^i : E \to \mathbb{R}$ are functions. The time form writes as $dt = u_0d^0$. Hence, $L_X dt = u_0 \otimes (\partial_0 X^0 d^0 + \partial_i X^0 d^i)$. Therefore, $L_X dt = 0$ if and only if $\partial_0 X^0 = 0$ and $\partial_i X^0 = 0$. But the first condition on $X^0$ means that the vector field $X$ is projectable and, together with the second condition, that $X^0$ is even constant.

Symmetries of the contact maps

On any fibred manifold $p : E \to B$, the contact maps $\vartheta_1$ and $\vartheta$ are not tensors. Hence, their symmetries require naturality techniques. However, it is possible to use a standard Lie derivative for both objects by using the 1–dimensional affine structure of $T$ in the following way.

The affine structure of $p_1 ^1 : J_1 E \to E$ yields the following Lemma

Lemma 3.1. The map $\vartheta_1$ can be naturally regarded as a (scaled) tensor

$$\vartheta_1 : J_1 E \to T^* E \otimes V E \to \mathcal{T} \otimes (T^* J_1 E \otimes T J_1 E) .$$

An easy calculation gives the following lemma which relates infinitesimal spacetime symmetries and holonomic infinitesimal symmetries of $\vartheta_1$.

Proposition 3.2. Let $X$ be an infinitesimal spacetime symmetry. Then, $X_{(1)}$ is a holonomic infinitesimal symmetry of $\vartheta_1$, i.e. $L_{X_{(1)}} \vartheta_1 = 0$.

Proof. If $X = X^0\partial_0 + X^i\partial_i$ where $X^0 \in \mathbb{R}, X^i : E \to \mathbb{R}$, then, $L_{X_{(1)}} \vartheta_1 = u_0 \otimes (\partial_0 X^0 d^k \otimes \partial_k - \partial_i X^0 x^k_0 d^i \otimes \partial_k - \partial_0 X^0 x^k_0 d^i \otimes \partial_k) = 0$.

Lemma 3.2. Let $p : F \to B$ a fibred manifold. Let $X : F \to TF$ be a vector field which projects on a vector field $\tilde{X} : B \to TB$, and let $Y : F \to TF$ be a fibred morphism (over the identity on $B$). Let $\tilde{Y} : F \to TF$ be any extension of $Y$, i.e. a vector field projectable on $Y$. Then, $L_X Y := T p \circ L_X \tilde{Y}$ is well defined, i.e. it does not depend on the extension $\tilde{Y}$ of $Y$.

We obtain the following coordinate expression

$$L_X Y = (X^\mu \partial_\mu Y^\lambda - Y^\mu \partial_\mu X^\lambda + X^i \partial_i Y^\lambda) \partial_\lambda .$$

Proof. In coordinates, we have

$$L_X \tilde{Y} = (X^\mu \partial_\mu Y^\lambda - Y^\mu \partial_\mu X^\lambda + X^i \partial_i Y^\lambda - \tilde{Y}^i \partial_i X^\lambda) \partial_\lambda$$

$$+ (X^\mu \partial_\mu \tilde{Y}^j - Y^\mu \partial_\mu X^j + X^i \partial_i \tilde{Y}^j - \tilde{Y}^i \partial_i X^j) \partial_j .$$

Since $X$ is projectable, this yields $T p \circ L_X \tilde{Y} = (X^\mu \partial_\mu Y^\lambda - Y^\mu \partial_\mu X^\lambda + X^i \partial_i Y^\lambda) \partial_\lambda$.

This does not depend on the coefficients $Y_i$. The result turns out to be independent of the chart.
3.2 Infinitesimal symmetries of spacetime structures

In the following we use Lemma 3.2 to define symmetries of $d$.

**Proposition 3.3.** Let $X : E \rightarrow TE$ an infinitesimal spacetime symmetry. Then, $X_{(1)}$ is a holonomic infinitesimal symmetry of $d$, i.e. $L_{X_{(1)}} d_1 = 0$.

**Proof.** $d$ is a (scaled) fibred morphism over $E$. Thus, Lemma 3.2 yields the equality

$$L_{X_{(1)}} d_1 = u^0 \otimes (X^\lambda \partial_\lambda d_1^{\mu} + X^i_0 \partial^0_\mu d_1^{\mu} - d_1^\lambda_0 \partial_\lambda X^\mu) \partial_\mu = 0$$

where the components of $d_1$ were given by $d_1^{\mu} = (1, x^i_0)$.

Symmetries of spacelike metrics

Let $G$ be a spacelike metric. Clearly, $G$ is not a tensor of $E$. Hence, in order to define its infinitesimal symmetries, we use the following lemma.

**Lemma 3.3.** Let us consider a fibred manifold $p : F \rightarrow B$, a vector field $X$ of $F$ which is projectable on a vector field $X$ of $B$ and a vertical covariant tensor $\alpha : F \rightarrow \mathbb{R} \otimes V^* F$. Then, the vertical restriction $(L[X] \tilde{\alpha})^* : F \rightarrow \mathbb{R} \otimes V^* F$ of the Lie derivative $L[X] \tilde{\alpha}$, where $\tilde{\alpha} : F \rightarrow T^* F$ is an extension of $\alpha$, does not depend on the choice of the extension $\tilde{\alpha}$.

Hence, the Lie derivative $L[X] \alpha := (L[X] \tilde{\alpha})^* : F \rightarrow \mathbb{R} \otimes V^* F$ is well defined.

Its coordinate expression is

$$L_X \alpha = (X^\lambda \partial_\lambda \alpha_{j_1...j_r} + X^i \partial_i \alpha_{j_1...j_r} + \alpha_{i_1...r} \partial_{j_1} X^i + ... + \alpha_{j_1...j_r} \partial_{j_r} X^i) \partial^{j_1} \otimes ... \otimes \partial^{j_r}$$

where $(j_1, ... j_r)$ is any permutation of the fibre indices, and where we have used greek indices for the coordinates of the base space and latin indices for the fibres.

**Proof.** The coordinate expression shows that because of the projectability of $X$ the vertical restriction does not contain coefficients of the type $\alpha_{... \mu}$. The result turns out to be independent of the chart.

Thus, for each spacelike metric $G$, we call a projectable vector field $X$ of $E$ such that $L_X G = 0$ an infinitesimal symmetry of the spacelike metric $G$. An easy calculation shows

**Proposition 3.4.** Let $X$ be an infinitesimal spacetime symmetry and $G$ a spacelike metric. Then, we have the coordinate expression

$$L_X G = \{X^\lambda (\partial_\lambda G^0_{ij}) + G^0_{kj} (\partial_{j} X^k) + G^0_{ih} (\partial_{j} X^k)\} d^i \otimes d^j \, .$$
Symmetries of connections

Let \( K \) be a spacetime connection, \( \Gamma \) a phase connection and \( \gamma \) a dynamical connection. An easy calculation yields the following coordinate expressions.

**Proposition 3.5.** Let \( X \) be an infinitesimal spacetime symmetry. Then,

\[
L_X \Gamma = \{ \partial_\mu x_0^i - (\Gamma_{\lambda_00}^i + \Gamma_{\lambda_0k}^k)(\partial_\mu x_0^\lambda) - \partial_\lambda (\Gamma_{\mu_00}^i + \Gamma_{\mu_0k}^k x_0^k) X_\lambda \\
- \Gamma_{\mu_0k} x_0^k + (\Gamma_{\mu_00}^i + \Gamma_{\mu_0k}^k x_0^k)(\partial_\mu x_0^i) \} d^\mu \otimes \partial_0,
\]

\[
L_X \gamma = u^0 \{ X^0 (\partial_0 \gamma_0^i_0) + X^i (\partial_j \gamma_0^j_0) - (\partial_0 X_0^i) - (\partial_j X_0^i x_0^j) + X_0^j (\partial_0 \gamma_0^j_0) - \gamma_0^j_0 (\partial_0 X_0^j) \} \partial_0,
\]

\[
L_X (T) K = \{ (\partial_\lambda x^i) \Gamma_{i00}^k \dot{x}^\nu - \partial_\lambda \partial_\nu X^k \dot{x}^\nu + X^\alpha \partial_\alpha \Gamma_{\lambda_0k}^k \dot{x}^\mu \\
+ (\partial_\alpha x^i) \Gamma_{\lambda_0k}^k \dot{x}^\alpha - (\partial_j x^k) \Gamma_{\lambda_0k}^i \dot{x}^\mu \} \dot{\partial}_k \otimes d^\lambda.
\]

Now let us consider the case when \( \Gamma \) and \( \gamma \) are the corresponding connections for a spacetime connection \( K \). The natural bijective correspondences 2.9 suggest the following theorem.

**Theorem 3.1.** Let \( X \) be an infinitesimal spacetime symmetry. The following equivalences hold

1) \( L_X (T) K = 0 \) \( \iff \) 2) \( L_X (1) \Gamma = 0 \) \( \iff \) 3) \( L_X (1) \gamma = 0 \)

**Proof.** The proof of 1) \( \Rightarrow \) 2) \( \Rightarrow \) 3) follows easily in virtue of the Leibnitz rule and the fact that \( X (1) \) is a symmetry of the contact maps.

The proof of 3) \( \Rightarrow \) 2) \( \Rightarrow \) 1) can be obtained easily by considering the expressions of Lemma 3.5 which are polynomial in the coordinates \( x_0^i \) or \( \dot{x}^\mu \).

**Symmetries of phase 2–forms**

Let us consider a spacelike metric \( G \) and a phase connection \( \Gamma \). Moreover, let \( \Omega \) be the corresponding dynamical phase 2–form, \( \gamma \) the corresponding dynamical connection and \( E \) the corresponding Euler-Lagrange morphism. The natural correspondences 2.10 suggest the following theorem

**Theorem 3.2.** Let \( X \) be an infinitesimal symmetry of spacetime. The following equivalence holds

1) \( L_X (\Omega) = 0 \) \( \iff \) 2) \( L_X (1) \Gamma = 0 \), \( L_X G = 0 \).
3.2 Infinitesimal symmetries of spacetime structures

Proof. The definition (2.6) yields, \( L_{X(1)} \Omega = L_{X(1)} \nu[\Gamma] \land \partial_1 + \nu[\Gamma] L_{X(1)} \partial_1 + \nu[\Gamma] \partial_1 \). The proof of 2) \( \Rightarrow \) 1) can be seen directly because of the fact that \( X_{(1)} \) is a symmetry of \( \partial_1 \).

For the proof of 1) \( \Rightarrow \) 2) we consider the coordinate expressions \( L_{X(1)} \nu[\Gamma] \land \partial_1 = G_{ij}^0 \alpha^i_0 (d^\mu \land \partial - x_0^i d^\mu \land d^0) \) and \( \nu[\Gamma] \partial_1 = \beta^0_{ij} (d^0_0 \land \partial - x_0^i d^0 \land d^0 - \Gamma^0_0 d^\lambda \land d^0 + x_0^i \Gamma^0_0 d^\lambda \land d^0) \), where we have set \( \alpha^i_0 \) to be the coefficient of \( L_{X(1)} \Gamma \) in Proposition 3.5 and \( \beta^0_{ij} \) the coefficient of \( L_X G \) in Proposition 3.4. It can be easily seen that, if the sum of these expressions is zero, then, all \( \alpha^i_0 \) and all \( \beta^0_{ij} \) have to be zero.

Moreover, the correspondences 2.10 suggest the following

**Theorem 3.3.** Let \( X \) be an infinitesimal spacetime symmetry. The following equivalence holds.

1) \( L_{X(1)} \Omega = 0 \) \( \Leftrightarrow \) 2) \( L_{X(2)} \mathcal{E} = 0 \)

Proof. Expression 2.7 yields \( L_{X(1)} \mathcal{E} = L_{X(1)} \nabla[\gamma] \land \partial_1 + \nabla[\gamma] L_{X(1)} \partial_1 + \nabla[\gamma] \partial_1 \). In analogy to the proof of Theorem 3.2 this yields the equivalence between the condition \( L_{X(2)} \mathcal{E} = 0 \) and the two conditions \( L_X G = 0, \ L_{X(1)} \nabla[\gamma] = 0 \). Clearly, \( L_{X(1)} \nabla[\gamma] = 0 \) is equivalent to \( L_{X(1)} \gamma = 0 \). Thus, Theorem 3.2 and Theorem 3.3 yield the result.

Eventually, we add an example of a distinguished nonholonomic infinitesimal symmetry of \( \Omega \).

**Proposition 3.6.** The corresponding dynamical connection \( \gamma \) is a nonholonomic (scaled) infinitesimal symmetry of the cosymplectic structure \((J_1 \mathbf{E}, \Omega, dt)\), i.e. \( L_{\gamma} \Omega = 0 \) and \( L_{\gamma} dt = 0 \).

Proof. This follows directly from Cartan’s formula using \( i_{\gamma} \Omega = 0, i_{\gamma} dt = 1 \) and the closure of \( \Omega \).

We observe that this proposition is essentially the standard result for the Reeb vector field of any cosymplectic structure applied to our particular case.

But we can even say more about \( \gamma \).

**Theorem 3.4.** There is exactly one second order connection \( \tilde{\gamma} \) such that \( L_{\tilde{\gamma}} \Omega = 0 \). Namely, \( \tilde{\gamma} = \gamma \).

Proof. Let \( \tilde{\gamma} \) be a second order connection such that \( L_{\tilde{\gamma}} \Omega = 0 \). This implies that there exists a local function \( f : J_1 \mathbf{E} \to \mathbb{R} \) such that \( i_{\tilde{\gamma}} \Omega = df \). Using \( i_{\gamma} \Omega = 0 \), we get \( i_{\tilde{\gamma} - \gamma} \Omega = df \). Let us set \( c := \tilde{\gamma} - \gamma \); \( c \) is valued into \( T^* \otimes T^* \otimes V \mathbf{E} \), and has coordinate expression \( c = c_0^i u^0 \otimes \partial_i \), with \( c_0^i := \tilde{\gamma}_0^i - \gamma_0^i : J_1 \mathbf{E} \to \mathbb{R} \). We have to show that a local function \( f \) only exists if \( c = 0 \). Calculating in coordinates using \( c_0^i := G_{ij}^0 c_0^i \) one gets the following system of equalities \( \partial_0 f = -c_0^i x_0^i, \ \partial_i f = c_0^i, \ \partial_i f = 0 \). This systems implies that \( c = 0 \). Thus, \( \tilde{\gamma} = \gamma \).
3.3 Noether Symmetries

Let us consider a closed dynamical phase 2–form $\Omega$. The next proposition relates infinitesimal symmetries of $\Omega$ to conserved quantities.

**Lemma 3.4.** Let $\Omega$ be a closed dynamical phase 2–form and let $Y : J^1E \rightarrow TJ^1E$ be an infinitesimal symmetry of $\Omega$. Then, the 1–form $i_Y \Omega$ is closed, and any local potential function $f$ of $i_Y \Omega$ is a conserved quantity.

**Proof.** $L_Y \Omega = 0$ is locally equivalent to the closure of $i_Y \Omega$. Hence, there is a local function $f$ such that $df = i_Y \Omega$. Therefore, $\gamma(f) = df(\gamma) = i_Y \Omega(\gamma) = i_Y i_\gamma \Omega = 0$. \hfill QED

**Symmetries of Poincaré–Cartan forms**

Let us consider a local Poincaré–Cartan form $\Theta$ associated to $\Omega$.

Clearly, any infinitesimal symmetry $Y : J^1E \rightarrow TJ^1E$ of $\Theta$ is an infinitesimal symmetry of $\Omega$. In fact, if $L_Y \Theta = 0$, then $0 = dL_Y \Theta = L_Y d\Theta = L_Y \Omega$.

Now, we can formulate the following (Noether) theorem which relates holonomic infinitesimal symmetries of $\Theta$ to conserved quantities.

**Theorem 3.5.** Let $X$ be an infinitesimal spacetime symmetry which, additionally, is a holonomic infinitesimal symmetry of $\Theta$. Then, on the domain of $\Theta$, $i_{X(1)} \Omega$ is exact and $f := -i_X \Theta$ is a potential, hence, a conserved quantity.

**Proof.** $0 = L_X(1) \Theta = (d i_{X(1)} + i_{X(1)} d) \Theta = d i_{X(1)} \Theta + i_{X(1)} \Omega$. This is equivalent to the equation $i_{X(1)} \Omega = -d i_{X(1)} \Theta = -d i_X \Theta$. It follows directly from Lemma 3.4 that $-i_X \Theta$ is a conserved quantity. \hfill QED

**Remark 3.1.** In particular, if an observer $o$ is a (scaled) infinitesimal symmetry of $\Theta$, then the Hamiltonian $H[o]$ turns out to be the associated conserved quantity.

**Symmetries of Lagrangians**

Let $L$ be the Lagrangian corresponding to a Poincaré-Cartan form $\Theta$ and let $P$ be the corresponding momentum. The natural correspondence 2.11 indicates the following theorem

**Theorem 3.6.** Let $X$ be an infinitesimal spacetime symmetry. Then, the following equivalence holds

1) $L_{X(1)} \Theta = 0 \quad \Leftrightarrow \quad 2) L_{X(1)} L = 0$

**Proof.** Both directions can be proved in analogy to the proof of the equivalence $L_{X(1)} \Gamma = 0 \Leftrightarrow L_{X(1)} \gamma = 0$. \hfill QED
3.4 The momentum map in covariant classical mechanics

Theorem 3.6 yields immediately another formulation of the (Noether) Theorem 3.5. This version may be more popular to the physicist.

Corollary 3.1. Let $X$ be an infinitesimal spacetime symmetry which, additionally, is a holonomic infinitesimal symmetry of $\mathcal{L}$. Then, on the domain of $\Theta$, a conserved quantity is given by

$$f := -(X \lrcorner P + X \lrcorner L).$$

### 3.4 The momentum map in covariant classical mechanics

Let us suppose a closed dynamical phase $2$–form $\Omega$ and a left action $\hat{\Phi} : G \times J^1_1E \to J^1_1E$ of a group $G$ of symmetries of the cosymplectic structure $(J^1_1E, \Omega, dt)$. That is, $\hat{\Phi}_g^*\Omega = \Omega$ and $\hat{\Phi}_g^*dt = dt$. Let $\mathfrak{g}$ be the associated Lie algebra. Hence, $L_{\partial \hat{\Phi}(\xi)}\Omega = 0$ and $L_{\partial \hat{\Phi}(\xi)}dt = 0$ for all $\xi \in \mathfrak{g}$.

We would like to define a momentum map in our setting by analogy with the standard symplectic and cosymplectic literature [1, 6, 26, 27] and ref. therein.

Lemma 3.4 shows that any vector field $Y$ of $J^1_1E$ is an infinitesimal symmetry of $\Omega$ if and only if there exists a (local) function $f$ such that $i_Y \Omega = df$. Clearly, $f$ is determined up to an additive constant $c \in \mathbb{R}$. Each $f$ of this type is a conserved quantity.

Analogously, we can easily see that the following lemma holds

**Lemma 3.5.** Let $Y$ be any vector field of $J^1_1E$. Then, $Y$ is an infinitesimal symmetry of $dt$ if and only if $i_Y dt$ is a constant $c \in \mathbb{T}$.

Hence, by Lemma 3.4 and Lemma 3.5, we can locally associate with any infinitesimal symmetry $\partial \hat{\Phi}(\xi)$ of $\Omega$ and $dt$ locally a pair $(f_\xi, \tau_\xi)$, where $\tau_\xi$ is the constant $dt(\partial \hat{\Phi}(\xi))$ and $f_\xi$ is a potential function of $i_{\partial \hat{\Phi}(\xi)} \Omega$. In the following we denote by $Co(J^1_1E)$ the vector space of conserved quantities.

**Definition 3.1.** A (local) map $J$

$$J : \mathfrak{g} \to Co(J^1_1E) \times \mathbb{T} : \xi \mapsto (J_\xi, \tau_\xi),$$

where $J_\xi$ is a potential of $i_{\partial \hat{\Phi}(\xi)} \Omega$ and $\tau_\xi := i_{\partial \hat{\Phi}(\xi)} dt$ for all $\xi \in \mathfrak{g}$, is said to be a momentum map for the action $\hat{\Phi}$.

**Remark 3.2.** In general, a momentum map $J$ is defined locally. But if we assume suitable hypotheses on spacetime or on the Lie algebra $\mathfrak{g}$, then we can find a global momentum map. Of course, a global $J$ always exists if $H^1(E) = \{0\}$. A detailed list of other hypotheses under which $J$ is globally defined is given in [27]; they are the same as in our case.
On the other hand, given a time scale \( \tau \), it is possible to associate with any function \( f \) of \( J_1 \mathbf{E} \) a distinguished vector field of \( J_1 \mathbf{E} \), namely, the \( \tau \)-Hamiltonian lift of \( f \). The following theorem holds

**Theorem 3.7.** Let \( J \) be a momentum map for the action \( \hat{\Phi} \) and let \( H_\tau[J_\xi] \) be the \( \tau \)-Hamiltonian lift of \( J_\xi \) with respect to an arbitrary time scale \( \tau \).

Then, the following equivalence holds

\[
\partial \hat{\Phi}(\xi) = H_\tau[J_\xi] \iff \tau := \tau_\xi = i_{\partial \hat{\Phi}(\xi)} dt.
\]

**Proof.** By recalling that \( \gamma . J_\xi = 0 \) we obtain that \( i_{H_\tau[J_\xi]} \Omega = dJ_\xi - \langle \gamma . J_\xi, dt \rangle = dJ_\xi \). Hence, by observing that two vector fields \( X, Y : J_1 \mathbf{E} \to T J_1 \mathbf{E} \) are equal if and only if \( i_X dt = i_Y dt \) and \( i_X \Omega = i_Y \Omega \), we obtain the result. \( \square \)

This theorem shows, why we have included the time scale \( \tau_\xi \) in the definition of momentum map. Again, we stress the fact, that the function \( J_\xi \) is only determined up to a gauge, whereas the time scale \( \tau_\xi \) is determined uniquely by \( \partial \hat{\Phi}(\xi) \).

It is obvious that we want to know if the map that associates to a pair \((J_\xi, \tau_\xi)\) its \( \tau \)-Hamiltonian lift \( H(J_\xi, \tau_\xi) := H_{\tau_\xi}[J_\xi] \) is a homomorphism of Lie algebras. Therefore, we define the following bracket for pairs in \( \text{Co}(J_1 \mathbf{E}) \times \bar{T} \).

**Definition 3.2.** The bracket \( \{(f, \tau), (g, \sigma)\} := (0, \{f, g\}) \) is said to be the Poisson bracket of pairs.

Then we can prove the following theorem

**Theorem 3.8.** The map \( H \) is a homomorphism of Lie algebras between pairs \((f, \tau) \in \text{Co}(J_1 \mathbf{E}) \times \bar{T} \), with respect to the Poisson bracket of pairs, and vector fields \( H_\tau[f] \) of \( J_1 \mathbf{E} \), with respect to the standard Lie bracket.

**Proof.** The equalities \( i_{[H_\tau[f], H_\tau[g]]} dt = [L_{H_\tau[f]}, i_{H_\tau[g]}] dt = L_{H_\tau[f]} \sigma - i_{H_\tau[g]} d\tau = 0 \) and \( i_{[H_\tau[f], H_\tau[g]]} \Omega = [L_{H_\tau[f]}, i_{H_\tau[g]}] \Omega = di_{H_\tau[f]} i_{H_\tau[g]} \Omega + i_{H_\tau[f]} L_{H_\tau[g]} \Omega - i_{H_\tau[g]} L_{H_\tau[f]} \Omega = d\{f, g\} \). yield the result. \( \square \)

Let us recall that the \( \tau \)-Hamiltonian lift of a function \( f : J_1 \mathbf{E} \to \mathbb{R} \) is projectable on a vector field \( X \) of \( \mathbf{E} \) if and only if \( f \) is a special function and the second fibre derivative of \( f \) (with respect to the velocities) is equal to the time scale \( \tau \). Thus, Theorem 3.7 yields the following theorem that relates the function \( J_\xi \) to the time scale \( \tau_\xi \).

**Corollary 3.2.** Let \( \hat{\Phi} \) be projectable on a left action \( \Phi : G \times \mathbf{E} \to \mathbf{E} \). Then, any component \( J_\xi \) of a momentum map for \( \hat{\Phi} \) is a quantisable function and the second fibre derivative of \( J_\xi \) is equal to the time scale \( \tau_\xi = dt(\hat{\Phi}(\xi)) \).
Thus, in this case, each function $J_\xi$ encodes all information of the pair $(J_\xi, \tau_\xi)$. Hence, we call the map $J: g \to Co(J_1E): \xi \to J(\xi) := J_\xi$ momentum map, denoted by the same symbol $J$.

Now let us consider a Poincaré–Cartan form $\Theta$. Furthermore, let us suppose that $\hat{\Phi}$ is holonomic, i.e. $\hat{\Phi} = \Phi(1)$ where $\Phi$ is a left action of $G$ on $E$ and, additionally, we suppose that $\Phi$ preserves $\Theta$. Then the following holds

**Theorem 3.9.** There exists a momentum map on the domain of $\Theta$. Namely, the map

$$ J_\xi = \partial \Phi(\xi) \cdot \mathcal{P} + \partial \Phi(\xi) \cdot L. \tag{3.14} $$

Let $(e_p)$ be a basis of $g$, and $\xi = \xi^p e_p$. Then, the coordinate expression is

$$ J_\xi = \xi^p \left( (\partial_\rho \phi^i - x_0 \partial_\rho \phi^0) \partial^0 L + \partial_\rho \phi^0 L \right), $$

Given an observer $o$, the momentum map can be expressed in terms of the observed Hamiltonian $\mathcal{H}[o]$ and the observed momentum $\mathcal{P}[o]$ by

$$ J_\xi = \partial \Phi(\xi) \cdot \mathcal{P}[o] + \partial \mathcal{P}(\xi) \cdot \mathcal{H}[o]. \tag{3.15} $$

**Proof.** The first expression follows simply from the contact splitting of $\Theta$ and Theorem 3.5. The observer dependent expression of $J$ follows simply from the splitting of $\Theta$ through the observer.

The coordinate expression $\partial \phi(\xi) = \xi^p \partial_\rho \phi^0 \partial_0 + \xi^p \partial_\rho \phi^i \partial_i$ with respect to a basis $(e_p)$ yields the second expression.

**Remark 3.3.** There is a connection between the momentum of a Lagrangian and the momentum map. In fact, let $G$ be a group of vertical holonomic symmetries of $\Theta$, i.e. $i_{\partial \Phi(\xi)} dt = 0$. Then we have the expression

$$ J_\xi = \partial \phi(\xi) \cdot \mathcal{P} \equiv \mathcal{P}(\partial \phi(\xi)), $$

so the momentum map coincides with the momentum of the Lagrangian. \hfill Q.E.D.

### 3.5 General model of CCG

The general model of CCG is constituted by a spacetime $t: ET$, a spacelike metric $G$, a spacetime connection $K^2$ (gravitational field) and a scaled 2–form $f: E \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^* E$ of spacetime (electromagnetic field).

The gravitational connection $K^2$ yields the gravitational objects $\Gamma^i$, $\gamma^i$ and $\Omega^2$ through the correspondences discussed in the above section: $K \leftrightarrow \Gamma[K] \leftrightarrow \gamma[\Gamma]$ and $\Gamma \leftrightarrow \Omega[G, \Gamma]$.

On the other hand, given a charge $q \in \mathbb{Q}$, it is convenient to introduce the normalised form $F := \frac{\phi}{m} f: E \to \Lambda^2 T^* E$. This form can be regarded naturally as a dynamical phase 2–form.
The first field equations for the gravitational and the electromagnetic field are assumed to be \( d\Omega^2 = 0 \) and \( dF = 0 \).

There is a natural way to incorporate the electromagnetic field in the gravitational objects, by means of total objects of the type \( K = K^2 + K^e \), \( \Gamma = \Gamma^2 + \Gamma^e \), \( \gamma = \gamma^2 + \gamma^e \) and \( \Omega = \Omega^2 + \Omega^e \). In fact, we can start by considering a natural “minimal coupling” \[ \Omega := \Omega^2 + \frac{1}{2} F, \] (3.16)

where the factor \( \frac{1}{2} \) has been chosen just in order to obtain standard normalisation. Then the other total objects are obtained by means of the correspondences \( K \leftrightarrow \Gamma [K] \leftrightarrow \gamma [\Gamma] \) and \( \Gamma \leftrightarrow \Omega [G, \Gamma] \). In particular, \( -\gamma^e \) turns out to be the Lorentz force. The coefficients of \( K \) turn out to be \( K^{hik} = K^2 hik \), \( K^{0ik} = K^2 0ik + \frac{q}{2m} F^i k \) and \( K^{0i0} = K^2 0i0 + \frac{q}{m} F^i 0 \).

This yields that \( dt \wedge \Omega^n = dt \wedge (\Omega^2)^n : J_1 E \rightarrow T \otimes \Lambda^n T^* J_1 E \) is a (scaled) volume form of \( J_1 E \) and that \( d\Omega = 0 \). Therefore, the phase space \( J_1 E \) together with the total phase 2–form \( \Omega \) and the time form \( dt \) turns out to be a (scaled) cosymplectic manifold \[ \square \square \square \square \square \]. The total dynamical connection \( \gamma \) turns out to be the (scaled) Reeb vector field for this cosymplectic structure since \( i_\gamma \Omega = 0 \) and \( i_\gamma dt = 1 \).

### 3.6 Examples

Now, we apply the machinery developed in the above subsection to analyse three groups of symmetries acting in simple cases.

**Example 3.1.** We suppose the spacetime \( E \) to be an affine space with affine projection \( t \). In this case \( V E \simeq E \times S \), where \( S := \ker Dt \). So, we assume an Euclidean scaled metric \( g \) on \( S \).

Let us consider the vertical action

\[ S \times E \rightarrow E : (v, e_0) \mapsto (e_0 + v); \]

Let \( K^2 \) be the natural flat connection on \( E \) and \( F = 0 \). Then, any Poincaré–Cartan form exists globally and \( S_0 \) is a group of symmetries of a \( \Theta \). The momentum map \( J \) is just the standard linear momentum.

In fact, \( \Theta \) is invariant with respect to spacelike translations. Of course, the Lie algebra of \( S_0 \) is \( S_0 \) and we have the momentum map

\[ J : S_0 \rightarrow C^\infty (J_1 E, \mathbb{R}) : v \mapsto J(v) \equiv p_L(v). \]

We have the coordinate expression \( p_L(v) = v^i G_{ij} x^j_0 \) (see remark 3.3). \( \square \)

**Example 3.2.** Assume the same spacetime and fields as in the above example, and assume additionally that \( E \simeq T \times P \), i.e., assume a complete observer \( o \). Then, we can consider the natural action

\[ \mathbb{T} \times (T \times P) \rightarrow T \times P : (v, (\tau, p)) \mapsto (v + \tau, p). \]
It turns out that $\mathbb{T}$ is a group of symmetries of $\Theta$, i.e. $o$ is a (scaled) infinitesimal symmetry of $\Theta$, and the momentum map $J$ is just the (observed) kinetic energy $\mathcal{H}[o]$.

In fact, $\Theta$ is as in the above example, hence it is invariant with respect to time translations because the metric does not depend on time. Of course, the Lie algebra of $\mathbb{T}$ is $\mathbb{T}$ and we have the momentum map

$$J : \mathbb{T} \to \text{Co}(J_1 E) : \xi \mapsto J(\xi) \equiv \xi \downarrow (o \downarrow \Theta).$$

Obviously, $J = \mathcal{H}[o]$.

**Example 3.3.** Now, we suppose our spacetime to be $T \times SO(g)$, where $g$ is the metric of the above spacetime. The manifold $T \times SO(g)$ is interpreted as the configuration space for the relative configurations of a rigid body with respect to the center of mass (see [27, 32] for a more detailed account).

We assume the *inertia tensor* $I$ as the scaled vertical metric. Consider the action

$$SO(g) \times (T \times SO(g)) \to T \times SO(g) : (A, (\tau, B)) \mapsto (\tau, AB).$$

Let $K^2$ be the natural flat connection on $T \times SO(g)$ and $F = 0$. Then, $SO(g)$ is a group of symmetries of $\Theta$ and a momentum map $J$ is just the angular momentum.

In fact, as in the previous examples, $\Theta$ reduces to the kinetic energy of particles with respect to the center of mass. This is obviously invariant with respect orthogonal transformations [32]. We have the momentum map

$$J : \text{so}(g_a) \to \text{Co}(T \times T \otimes SO(g)) : \omega \mapsto J_\omega \equiv \omega^* \downarrow \mathcal{P},$$

where, by definition, $\mathcal{P} = V_\mathcal{E}L$, with the coordinate expression $\mathcal{P} = I_{ij}x^i_0d^j$. A simple computation shows that

- $\omega^* : SO(g) \to TSO(g) : r \mapsto \omega(r);$  
- $\omega^* \downarrow \mathcal{P}(v) = I(\omega(r), v) = \omega(r \times v).$

The Lie algebra of $SO(g_a)$ is $\text{so}(g_a)$, but the Hodge star isomorphism yields a natural Lie algebra isomorphism $\text{so}(g_a) \simeq \mathbb{L}^{-1} \otimes \mathcal{S}_a$. The isomorphism carries the Lie bracket of $\text{so}(g_a)$ into the cross product. In this way, if $\omega \in \text{so}(g_a)$ and $\bar{\omega} \in \mathbb{L}^{-1} \otimes \mathcal{S}_a$ is the corresponding element, we can equivalently write

$$J : \mathbb{L}^{-1} \otimes \mathcal{S}_a \to \text{Co}(T \times T \otimes SO(g)) : \bar{\omega} \mapsto J_\bar{\omega} \equiv I(r \times v, \omega),$$

where $v \in T^* \otimes TR_a \equiv J_1(T \times R_a)$. This proves the last part of the statement.

4 Conclusions

In this paper we used natural bijective correspondences between distinguished objects in the covariant classical Galilei theory as an indicator for relations between symmetries of these objects. In particular, it turned out that the holonomic infinitesimal symmetries of the objects spacelike metric, gravitational field and electromagnetic field are equivalent
to the holonomic infinitesimal symmetries of induced cosymplectic structure. Moreover, these symmetries are equivalent to the holonomic infinitesimal symmetries of the associated Euler-Lagrange morphism. Analogously, we saw that the holonomic infinitesimal symmetries of the horizontal potential 1-forms of the symplectic form are equivalent to the holonomic infinitesimal symmetries of the associated Lagrangians. These results allowed us to give two equivalent versions of the Noether theorem, namely, a ‘Poincaré-Cartan’ version and a ‘Lagrangian’ version. Thus, we saw that the holonomic infinitesimal symmetries of the cosymplectic structure yield the holonomic infinitesimal symmetries of all physical objects and of the dynamics.

Then, we introduced a covariant momentum map in the model, associated with cosymplectic symmetries. Here, the covariance with respect to units, lead us to introduce a momentum map whose components are pairs, namely, a constant time scale and a conserved quantity. A momentum map was determined only up to an additive constant to the conserved quantities. We saw that any pair determines uniquely a covariant lift (τ-Hamiltonian lift) of functions. We introduced a Poisson bracket for these pairs, such that the covariant lift turned out to be a morphism of Lie algebras. Then, we showed that for any action of cosymplectic symmetries which projects on an action on spacetime, the components of the momentum map turned out to be ‘quantisable functions’ and their second fibre derivative determined the constant time scale. This feature is new with respect to standard symplectic momentum map formalism, and is mainly due to covariance requirements. Its importance is fundamental in quantum mechanics. We saw that, if the group of symmetries , additionally, preserves a given Poincaré-Cartan form, then the momentum map is uniquely determined by the Poincaré-Cartan form.

Finally, we provided simple examples illustrating our results at work.

These results are promising in view of a next research about symmetries of quantised systems.

References

[1] C. Albert: *Le théorème de réduction de Marsden–Weinstein en géométrie cosymplectique et de contact*, J. Geom. Phys., p.627 ff., V.6, n.4, 1989.

[2] M. Le Bellac, J.–M. Lévy–Leblond: *Galilean electromagnetism*, Nuovo Cimento 14 B, 217 (1973).

[3] F. Cantrijn, M. de León, E. A. Lacomba: *Gradient vector fields on cosymplectic manifolds*, J. Phys. A 25 (1992) 175–188.

[4] D. Canarutto, A. Jadczyk, M. Modugno: *Quantum mechanics of a spin particle in a curved spacetime with absolute time*, Rep. on Math. Phys., 36, 1 (1995), 95–140.

[5] M. de Leon, J. C. Marrero, E. Padron: *On the geometric quantization of Jacobi manifolds*, p. 6185 ff., J. Math. Phys. 38 (12), 1997.
REFERENCES

[6] M. de Leon, M. Saralegi: Cosymplectic reduction for singular momentum maps, J. Phys. A 26 (1993) 5033–5043.

[7] C. Duval: On Galilean isometries, Clas. Quant. Grav. 10 (1993), 2217-2221.

[8] C. Duval, G. Burdet, H. P. Künzle, M. Perrin: Bargmann structures and Newton–Cartan theory, Phys. Rev. D, 31, n. 8 (1985), 1841–1853.

[9] C. Duval, H. P. Künzle: Minimal gravitational coupling in the Newtonian theory and the covariant Schrödinger equation, G.R.G., 16, n. 4 (1984), 333–347.

[10] J. Ehlers: The Newtonian limit of general relativity, in Fisica Matematica Classica e Relatività, Elba 9-13 giugno 1989, 95-106.

[11] M. Francaviglia, M. Palese, R. Vitolo: Symmetries and conservation laws in variational sequences, submitted to Czech Math. J. (1999).

[12] Godbillon: Géométrie différentielle et mécanique analytique, Hermann Paris (1969).

[13] M. J. Gotay: Constraints, reduction and quantization, J. Math. Phys. 27 n. 8 (1986), 2051–2066.

[14] J. Janyška: Remarks on symplectic and contact 2–forms in relativistic theories, Boll. U.M.I., 7 (1994), 587-616.

[15] A. Jadczyk, J. Janyška, M. Modugno: Galilei general relativistic quantum mechanics revisited, “Geometria, Física-Matemática e Outros Ensaios”, A. S. Alves, F. J. Craveiro de Carvalho and J. A. Pereira da Silva Eds., 1998, 253–313.

[16] A. Jadczyk, M. Modugno: A scheme for Galilei general relativistic quantum mechanics, in Proceedings of the 10th Italian Conference on General Relativity and Gravitational Physics, World Scientific, New York, 1993.

[17] I. Kolář, P. Michor, J. Slovák: Natural Operations in Differential Geometry, Springer-Verlag, 1993.

[18] I. Kolář, M. Modugno: On the algebraic structure on the jet prolongation of fibred manifolds, Czech. Math. Jour., 40 (115), 1990, 601–611.

[19] B. Kostant: Quantization and unitary representations, Lectures in Modern Analysis and Applications III, Springer–Verlag, 170 (1970), 87–207.

[20] D. Krupka: Variational sequences on finite order jet spaces, in Proc. Conf. on Diff. Geom. and its Appl., World Scientific, New York, 1990, 236-254.

[21] K. Kuchař: Gravitation, geometry and nonrelativistic quantum theory, Phys. Rev. D, 22, n. 6 (1980), 1285–1299.

[22] H. P. Künzle: General covariance and minimal gravitational coupling in Newtonian spacetime, in Geometrodynamics Proceedings (1983), A. Prastaro ed., Tecnoprint, Bologna 1984, 37–48.
REFERENCES

[23] N. P. Landsman, N. Linden: *The geometry of inequivalent quantizations*, Nucl. Phys. B 365 (1991), 121-160.

[24] P. Libermann, Ch. M. Marle: *Symplectic Geometry and Analytical Mechanics*, Reidel Publ., Dordrecht, 1987.

[25] L. Mangiarotti, M. Modugno: *Fibered Spaces, Jet Spaces and Connections for Field Theories*, in Proceed. of the Int. Meet. on Geometry and Physics, Pitagora Editrice, Bologna, 1983, 135–165.

[26] C.-M. Marle: *Lie group action on a canonical manifold*, Symplectic Geometry (Research notes in mathematics), Boston, Pitman, 1983.

[27] J. E. Marsden, T. Ratiu: *Introduction to Mechanics and Symmetry*, Texts in Appl. Math. 17, Springer, New York, 1995.

[28] E. Massa, E. Pagani: *Classical dynamics of non-holonomic systems: a geometric approach*, Ann. Inst. H. Poinc. 55, 1 (1991), 511-544.

[29] M. Modugno: *Torsion and Ricci tensor for non-linear connections*, Diff. Geom. and Appl. 1 No. 2 (1991), 177–192.

[30] M. Modugno, R. Vitolo: *Quantum connection and Poincaré–Cartan form*, Proc. of the Conf. in Honour of A. Lichnerowicz, Frascati, ottobre 1995; ed. G. Ferrarese, Pitagora, Bologna.

[31] M. Modugno: *Covariant quantum mechanics*, book pre-print, 1999

[32] M. Modugno, C. Tejero Prieto, R. Vitolo: *A covariant approach to classical and quantum mechanics of a rigid body*, preprint 1999.

[33] D. J. Saunders: *The geometry of jet bundles*, Cambridge University Press, 1989.

[34] J. Sniatnicki: *Geometric quantization and quantum mechanics*, Springer–Verlag, New York 1980.

[35] J.–M. Souriau: *Structures des systèmes dynamiques*, Dunod, Paris 1970.

[36] A. Trautman: *Sur la théorie Newtonienne de la gravitation*, C. R. Acad. Sc. Paris, t. 257 (1963), 617–620.

[37] W. M. Tulczyjew: *An intrinsic formulation of nonrelativistic analytical mechanics and wave mechanics*, J. Geom. Phys., 2, n. 3 (1985), 93–105.

[38] R. Vitolo: *Spherical symmetry in classical and quantum Galilei general relativity*, Annales de l’Institut Henri Poincaré, 64, n. 2 (1996).

[39] R. Vitolo: *Quantum structures in general relativistic theories*, Proc. of the XII It. Conf. on G.R. and Grav. Phys., Roma, 1996; World Scientific.

[40] R. Vitolo: *Finite order Lagrangian bicomplexes*, Math. Proc. of the Camb. Phil. Soc., 128 n. 3, 1998.
[41] R. Vitolo: *Quantum structures in Galilei general relativity*, Ann. Inst. 'H. Poinc.' 70, n.3, 1999.

[42] N. Woodhouse: *Geometric quantization*, Second Ed., Clarendon Press, Oxford 1992.