GENERALISED HYPERBOLICITY: HYPERSURFACE SINGULARITIES

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Abstract. Sufficient conditions for the well-posedness of the initial value problem for the scalar wave equation are obtained in space-times with hypersurface singularities

1. Introduction

A desirable property of any space-time used to model a physically plausible scenario is that the evolution of the Einstein’s equations is well posed; that is the initial value problem admits a unique solution. Space-times whose metrics are at least $C^{2-}$, which guarantees the existence of unique geodesics, fall within the context of the Cosmic Censorship Hypothesis of Penrose (1979). This hypothesis states that the space-time will be globally hyperbolic, i.e. strong causality is satisfied and $J^+(p) \cap J^-(q)$ is compact $\forall p, q \in M$, and hence the evolution of Einstein’s equations is well defined.

There are however a number of space-times which violate cosmic censorship in this sense, but for which there may be a well posed initial value theorem for test fields. This suggests that a weaker notion of hyperbolicity should be defined (Clarke, 1996). Space times which may fall into this category include those with weak singularities such as as thin cosmic strings (Vickers, 1987), impulsive gravitational waves (Penrose, 1972) and dust caustic space-times (Clarke and O’Donnell, 1992), all of which model physically plausible scenarios. Such space-times typically admit a locally bounded metric whose differentiability level is lower than $C^{2-}$ and a curvature tensor that is well defined as a distribution, often with its support on a proper submanifold.

A concept of hyperbolicity for such space-times was proposed by Clarke (1998). This was based on the extent to which singularities disrupted the local evolution of the initial value problem for the scalar wave equation.

$\Box \phi = f$
$\phi|_S = \phi_0$
$n^a\phi_a|_S = \phi_1$

Clarke reformulated the initial value problem, on an open region $\Omega$ with a compact closure admitting a space-like hypersurface $S$, which partitions $\Omega$ into two disjoint sets $\Omega^+$ and $\Omega^-$, in a distributional form, obtained by multiplying by a test field $\omega$ and integrating by parts once to give

$$\int_{\Omega^+} \phi_{,\omega} h g^{ab} (-g)^{1/2} d^4x = - \int_{\Omega^+} f \omega (-g)^{1/2} d^4x - \int_S \phi_1 \omega dS \quad \forall \omega \in D(\Omega)$$

$\phi|_S = \phi_0$

and then defined a point $p \in M$ as being $\Box$-regular if it admitted such a neighbourhood $\Omega$ for which the above equation had a unique solution for each set of Cauchy data $(\phi_0, \phi_1) \in H^1(S) \times H^0(S)$. A space-time which was $\Box$-regular everywhere could then be said to be $\Box$-globally hyperbolic. It was shown if a space-time satisfied the following curve integrability conditions at a given point $p \in M$, then that point was $\Box$-regular:

1. $g_{ab}$ and $g^{ab}$ are continuous
2. $g_{ab}$ is $C^1$ on $M - J^+(p)$
3. $g_{ab,c}$ exists as a distribution and is locally square integrable.
4. The distributional Riemann tensor components $R^a_{bcd}$ may be interpreted as locally integrable functions.
5. There exists a non empty open set $C \subset \mathbb{R}^4$ and functions $M, N : \mathbb{R}^+ \to \mathbb{R}^+$, with $M(\varepsilon), N(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that if $\gamma : [0,1] \to M$ is a curve with $\dot{\gamma} \in C$ then $\gamma$ is future time-like and

$$\int_0^\varepsilon |\Gamma^a_{bc}(\gamma(s))|^2 ds < M(\varepsilon)$$

$$\int_0^\varepsilon |R^a_{bcd}(\gamma(s))| ds < N(\varepsilon)$$
The proof involved the construction of a congruence of time-like geodesics, whose tangent admitted an essentially bounded weak derivative, and a suitable energy inequality from which uniqueness and existence could be deduced. In particular it was shown that these results were applicable to the dust caustic space-times.

There is an important class of space-times with weak singularities that do not satisfy these curve integrability conditions; space-times whose curvature has support on a proper submanifold. This class includes both impulsive gravitational waves (Co-dimension 1) and thin cosmic strings (Co-dimension 2).

This paper will consider the question of hyperbolicity in space-times with a singular hypersurface. For hyperbolicity in conical space-times see Wilson (2000) and also Vickers and Wilson (2000).

2. Existence and uniqueness

In order to prove hyperbolicity, we must show that in each open region \( \Omega \) with a compact boundary \( \partial \Omega \), that a unique solution \( \phi \in H^1(\Omega) \) exists to the initial value problem;

\[
\Box \phi = f \\
\phi |_{S} = \phi_0 \\
n^a \phi_a |_{S} = \phi_1
\]

where \( f \in H^1(\Omega) \) and the initial data \( \phi_0 \) and \( \phi_1 \) lie in suitable function spaces. We shall assume that \( (\phi_0, \phi_1) \in H^1(S) \times H^0(S) \) at least; this is what was required for hyperbolicity in curve integrable space-times (Clarke, 1998). However we shall see that, in the given circumstances, that one requires some what stronger conditions on \( (\phi_0, \phi_1) \).

In an open region \( \Omega \) with a closed boundary, we shall assume that we have the following scenario

1. A local coordinate system \((t, x^a)\) exists for \( \Omega \) such that the initial surface \( S \) is described by \( t = 0 \) and \( g_{ab} \) and \( g^{ab} \) are essentially bounded (i.e. bounded almost everywhere)
2. There exists a hypersurface \( \Lambda \) such that \( g_{ab} \) is \( C^2 \), on \( \Omega - \Lambda \). This characterises \( \Lambda \) as a singular hypersurface in a regular space-time.
3. \( g_{ab} \) is \( C^1 \) on \( \Omega \)
4. A congruence of time-like geodesics exists whose tangent field \( T^a \) has an essentially bounded covariant derivative. Without loss of generality we shall assume that \( t \) is proper time along each of these geodesics; this means that \( T^a = \delta^a_t \) and \( g_{tt} = -1 \).

We shall also assume without loss of generality that \( \Omega \) is generated by a foliation of space-like hypersurfaces \((S_\tau)_{a<\tau<b}\) (with \( a < 0 < b \)) having a common boundary, with \( S_0 \) coinciding with the initial surface \( S \) (see figure 1).

Figure 1. The foliation of region \( \Omega \)
2.1 The energy inequality.

We shall follow the approach of Hawking and Ellis (1973), Clarke (1998) and Wilson (2000) in discussing existence and uniqueness of solutions to (1) in terms of an energy inequality. The energy inequality gives a bound to the solution $\phi$ at a given time in terms of the initial data $(\phi_0, \phi_1)$ and the source term $f$.

We define the following energy integral

$$E(\tau) = \int_{S_\tau} S^{ab} T_b n_a \, dS$$

where $n^a$ is the future pointing normal vector to the surface $S_\tau$ and

$$S^{ab} = (g^{ac} g^{bd} - \frac{1}{2} g^{ab} g^{cd}) \phi_c \phi_d - \frac{1}{2} g^{ab} \phi^2$$

It will turn out that estimating this energy integral easier than working directly with the classical Sobolev norm $\|\phi\|^{1}_{S_\tau}$. However $E(\tau)$ and $(\|\phi\|^{1}_{S_\tau})^2$ are equivalent in the sense that there exist positive constants $B_1$ and $B_2$ such that for $a \leq \tau \leq b$

$$B_1 (\|\phi\|^{1}_{S_\tau})^2 \leq E(\tau) \leq B_2 (\|\phi\|^{1}_{S_\tau})^2$$

(2)

See Wilson (2000) for a proof of this result.

**Lemma 1.** Solutions $\phi \in H^1(\Omega^+)\,$ of the initial value problem (1) satisfy an energy inequality.

**Proof.** In order to obtain an energy inequality we must apply Stokes’ theorem to the vector $S^{ab} T_b$ on the region

$$\Omega_\tau = \bigcup_{0 < \tau' < \tau} S_{\tau'},$$

however caution must be exercised because of the possible lack of differentiability of $g_{ab}$ and $\phi$ in a neighbourhood of $\Lambda$. We shall therefore apply Stokes’ theorem to a sequence of regions $\Omega_{\tau'}$ which converges to $\Omega_\tau$. Without loss of generality we shall illustrate the procedure for the case where $\Lambda$ intersects each $S_{\tau'}$ exactly once, partitioning $\Omega$ exactly into two regions (Figure 2).

On applying Stokes’ theorem to both regions we have

$$\int_{\Omega_{\tau'}^+ \cup \Omega_{\tau'}^-} (S^{ab} T_a)_b \, dV = \int_{S_{\tau'}^+ \cup S_{\tau'}^-} S^{ab} T_a n_b \, dS - \int_{S_{\tau'}^+ \cup S_{\tau'}^-} S^{ab} T_a n_b \, dS$$

$$+ \int_{\Lambda_{\tau'}^-} S^{ab} T_a \lambda_b \, d\lambda - \int_{\Lambda_{\tau'}^+} S^{ab} T_a \lambda_b \, d\lambda$$

The integrand on the left hand side may be written as

$$(S^{ab} T_b)_a = T^a \phi_a (f - \phi) + T^c d \phi_c \phi_d - \frac{1}{2} T^a \phi_a (g^{cd} \phi_c \phi_d + \phi^2)$$

Now $\phi \in H^1(\Omega)$ implies that

$$\lim_{\varepsilon \to 0} \int_{\Omega_{\tau'}^+ \cup \Omega_{\tau'}^-} (S^{ab} T_a)_b \, dV = \int_{\Omega_\tau} T^a \phi_a (f - \phi) \, dV + \int_{\Omega_\tau} (T^c d \phi_c \phi_d - \frac{1}{2} T^a \phi_a (g^{cd} \phi_c \phi_d + \phi^2)) \, dV$$

![Figure 2. Applying Stokes’ theorem to the region $\Omega$](image)
and similarly $\phi_{|S_x} \in H^1(S_x)$ implies
\[
\lim_{\varepsilon \to 0} \int_{S^2_{\varepsilon} \cup S^2_{-\varepsilon}} S^{ab} T_n dS = E(\tau)
\]
Finally we define the energy flux across $\Lambda$, which will exist for $\phi \in H^1(\Omega)$:
\[
F(\tau) = \lim_{\varepsilon \to 0} \left( \int_{\Lambda_{\varepsilon}^+} S^{ab} T_n dS - \int_{\Lambda_{\varepsilon}^-} S^{ab} T_n dS \right)
\]
The energy equation, in the limit is therefore
\[
E(\tau) = E(0) - F(\tau) + \int_{\Omega_{\varepsilon}} \phi_t (f - \phi) dV + \int_{\Omega_{\varepsilon}} (T^{cd} \phi_{,c} \phi_{,d} - \frac{1}{2} T^{ab} \phi_{,a} (g^{cd} \phi_{,c} \phi_{,d} + \phi^2)) dV
\]
The fact that $g_{ab} \in C^{1,-}(\Omega)$, $T^a_{\bullet} \in L^\infty(\Omega)$ and $\phi \in H^1(\Omega)$ implies that for all $0 < \tau < b$, there exist constants $C_1, C_2 > 0$ such that
\[
E(\tau) \leq E(0) + |F(\tau)| + C_1 \int_{\Omega_{\varepsilon}} |\phi_t| (|f| + |\phi|) d\tau + C_2 \int_{\Omega_{\varepsilon}} \left( \sum_{a,b=0}^4 |\phi_{,a}||\phi_{,b}| + |\phi|^2 \right) d\tau
\]
On applying the Cauchy-Schwarz inequality
\[
E(\tau) \leq E(0) + |F(\tau)| + \frac{1}{2} C_1 (|f|_{\Omega_{\varepsilon}}^0)^2 + (C_1 + 4C_2)(|\phi|_{\Omega_{\varepsilon}}^1)^2
\]
By applying (2)
\[
E(\tau) \leq E(0) + |F(\tau)| + \frac{1}{2} C_1 (|f|_{\Omega_{\varepsilon}}^0)^2 + \frac{1}{B_1} (C_1 + 4C_2) \int_{\tau=0}^\tau E(\tau') d\tau'
\]
Finally Gronwall’s inequality gives
\[
E(\tau) \leq \left( E(0) + |F(\tau)| + \frac{1}{2} C_1 (|f|_{\Omega_{\varepsilon}}^0)^2 \right) \exp \left( \frac{C_1 + 4C_2}{B_1} \tau \right)
\]
Equivalently (3) may be written in terms of the Sobolev norms as
\[
(|\phi|_{S_x}^1)^2 \leq \frac{1}{B_1} (B_2 (|\phi|_{S_0}^0)^2 + |F(\tau)| + \frac{1}{2} C_1 (|f|_{\Omega_{\varepsilon}}^0)^2) \exp \left( \frac{C_1 + 4C_2}{B_1} b \right)
\]
This differs from the standard energy inequality by the presence of an additional flux term $F(\tau)$. However if $F(\tau)$ vanishes then we are able to establish that a solution is unique.

**Proposition 2.** Solutions with a vanishing flux $F(\tau)$ are unique.

**Proof.** Suppose that $\gamma$ is the difference of two such solutions of (1), then it must be a solution of the initial value problem
\[
\square \gamma = 0
\]
\[
\gamma_{|S} = 0
\]
\[
n^a \gamma_{,a} |_{S} = 0
\]
and so the corresponding energy inequality is
\[
(|\gamma|_{S_x}^1)^2 \leq \frac{1}{B_1} \exp \left( \frac{C_1 + 4C_2}{B_1} b \right) |F(\tau)|
\]
Therefore it follows that if $F(\tau) = 0$ then $\gamma = 0$. □

A sufficient, but not necessary condition for $F(\tau)$ to vanish is that the solution $\phi$ is $C^1.$
2.2 Existence of solutions in $H^1(\Omega)$.

In order to prove that an $H^1$ solution $\phi$ to (1) exists, we shall prove the existence of $\psi \in H^1(\Omega)$ to the following initial value problem with zero initial data

\[
\begin{align*}
\Box \psi &= f - \Box q \\
\psi|_{S^0} &= 0 \\
n^a \psi|_{a|S^0} &= 0
\end{align*}
\]

(5)

where

\[
q(t, x^\alpha) = \phi_0(x^\alpha) + t\phi_1(x^\alpha)
\]

The choice of $q$ ensures that it satisfies the conditions

\[
q|_{S^0} = \phi_0, \quad n^a q|_{a|S^0} = \phi_1
\]

so that $\phi = \psi + q$ will be a solution of (1).

It will be assumed that $\phi_0$ and $\phi_1$ are at least in $H^1(S)$, which guarantees that $q \in H^1(\Omega)$; however the energy inequality (3) will only be applicable to (5) if the source term $f - \Box q$ of (5) is also in $H^1(\Omega)$. As the connection is essentially bounded, sufficient conditions to achieve $\Box q \in H^1(\Omega)$ are $\phi_0, \phi_1 \in H^3(S)$.

Our construction of a solution $\psi \in H^1(\Omega)$ to (5) will follow the methods of Egorov and Shubin (1992) and Clarke (1998). We shall work exclusively in the space $H^1(\Omega)$ as a Hilbert space with the inner product

\[
\langle \psi, \omega \rangle = \int_{\Omega} (e^{ab} \psi_\alpha \omega_b + \psi \omega) \, dV
\]

where $e^{ab} = g^{ab} + 2T^aT^b$ is a positive definite metric. It should be noted that in general we do not have $e^{ab}_{;c} = 0$.

The induced norm is equivalent to the Sobolev norm $\|\psi\|_{1,\Omega}$ in that there exist constants $B_3, B_4 > 0$ such that for all $\psi, \omega \in H^1(\Omega)$

\[
B_3(\|\psi\|_{1,\Omega})^2 \leq \langle \psi, \psi \rangle \leq B_4(\|\psi\|_{1,\Omega})^2
\]

(6)

We shall define two subspaces of $H^1(\Omega)$

\[
V_0 = \{ \psi \in H^3(\Omega) \cap C^2(\Omega) \mid \psi|_{S^0} = n^a \psi|_{a|S^0} = \psi_{,b}|_{S^0} = n^a \psi|_{a,\beta|S^0} = 0 \}
\]

\[
V_1 = \{ \omega \in H^1(\Omega) \cap C^2(\Omega) \mid \omega|_{S_{r_2}} = n^a \omega|_{a|S_{r_2}} = \omega_{,\beta}|_{S^0} = n^a \omega|_{,\alpha,\beta|S^0} = 0 \}
\]

The condition that $V_0, V_1 \subset H^3(\Omega)$ ensures that $\Box V_0, \Box V_1 \subset H^1(\Omega)$ whereas the $C^2$ differentiability is sufficient to force the flux term for both $\psi$ and $\psi_\alpha$ in the energy inequality (3) to vanish. The energy inequality for $\psi \in V_0$ then becomes

\[
(\|\psi\|_{1,\Omega}^2)^2 \leq \frac{A}{2B_1} e^{Ar_2/B_1} (\|\Box \psi\|_{1,\Omega}^2)
\]

In particular we apply this to the region $\Omega^+$, integrate over the time variable $\tau$ and apply (6) to give a bound for $\langle \psi, \psi \rangle$

\[
\langle \psi, \psi \rangle \leq C_4 (\|\Box \psi\|_{1,\Omega}^2) \quad \forall \psi \in V_0 \quad C_3 > 0 \text{ constant}
\]

We may obtain a similar inequality for $\omega \in V_1$, by regarding $S_{r_2}$ as an initial surface, with zero initial data, evolving back in time and constructing an analogous energy inequality; thus we obtain

\[
\langle \omega, \omega \rangle \leq C_4 (\|\Box \omega\|_{1,\Omega}^2) \quad \forall \omega \in V_1 \quad C_4 > 0 \text{ constant}
\]

(7)

The construction of $V_0, V_1$ and $\langle \psi, \omega \rangle$ are motivated by the need to have $\Box$ behaving as a self-adjoint operator, however in general this cannot be guaranteed and extra conditions on the space-time are needed to achieve this.
Lemma 3. If the following conditions are satisfied

\( T^a_{\ b} = 0 \)
\( R_{ab}T^b = 0 \)

then

\[
\langle \Box \psi, \omega \rangle = \langle \psi, \Box \omega \rangle \quad \forall \psi \in V_0, \omega \in V_1
\]

Proof. Let \( \psi \in V_0 \) and \( \omega \in V_1 \). We apply Stokes' theorem to the region \( \Omega^1_\epsilon \cup \Omega^2_\epsilon \) and take the limit \( \epsilon \to 0 \)

\[
\int_{\Omega^+} (e^{ab}\psi_{;a}^c + e^{abc}\psi_{;ab}^c + e^{abc}\psi_{;abc}^c) dV = \int_{\Omega^+} e^{ab}\psi_{;a}^c \omega_{;bc} dV + \int_{\Omega^+} e^{abc}\psi_{;abc}^c dV
\]

On combining the above equations

\[
\int_{\Omega^+} (\Box \omega + e^{ab}\Box(\psi_{;a}^c) \omega_{;bc}) dV + \int_{\Omega^+} e^{abc}\psi_{;abc}^c dV
\]

and on applying the contracted Ricci identity

\[
\Box(\psi_{;a}^c) = (\Box \psi)_{;a} + R^b_{\ da} \psi_{;b}
\]

we obtain

\[
\int_{\Omega^+} (\Box \omega + e^{ab}(\Box \psi)_{;a} \omega_{;bc}) dV = \int_{\Omega^+} (\psi \Box \omega + e^{ab}\psi_{;a} (\Box \omega)_{;b}) dV
\]

Using the fact that \( e^{ab} = g^{ab} + 2T^a T^b \) and applying the hypotheses to this equation, it is now evident that

\[
\langle \Box \psi, \omega \rangle = \langle \psi, \Box \omega \rangle.
\]

As a consequence we may define a linear functional \( k : \Box V_1 \to \mathbb{R} \) by

\[
k(\Box \omega) = (f - \Box q, \omega)
\]
Proposition 4. \( k \) defines an function \( \psi \in H^1(\Omega^+) \) with \( \psi \) and \( \psi, t \) vanishing on \( S \).

Proof. This result is a consequence of the Hahn-Banach theorem, in conjunction with the Riesz representation theorem. In order to apply these theorems, we must show that

1. \( k \) is bounded by a semi-norm on \( \square V_1 \)
2. \( \square V_1 \) is a dense subspace of \( H^1(\Omega^+) \)

A bound for \( k \) may be constructed by applying (7):

\[
|k(\square \omega)| = |\langle f - \square q, \omega \rangle| \\
\leq (f - \square q, f - \square q)^{1/2} \langle \omega, \omega \rangle^{1/2} \\
\leq C_2^{1/2} B_1^{1/2} \| f - \square q \|_{1+} \| \square \omega \|_{1+}
\]

To prove that \( \square V_1 \) is dense in \( H^1(\Omega^+) \), It is sufficient to show that any function in \( (\square V_1)^{\perp} \) is necessarily zero. Suppose that \( \lambda \in (\square V_1)^{\perp} \) then, because \( V_0 \) is dense, there exists a sequence \( (\psi_n) \) in \( V_0 \) converging to \( \lambda \).

By using Lemma 3 and taking the limit \( n \to \infty \), we have for all \( \omega \in V_1 \)

\[
\langle \square \psi_n, \omega \rangle = \langle \psi_n, \square \omega \rangle \to \langle \lambda, \square \omega \rangle = 0
\]

Since \( V_1 \) is dense, this implies that \( \square \psi_n \to 0 \) almost everywhere and, by (7), we have \( \psi_n \to 0 \) almost everywhere. Therefore \( \lambda = 0 \) in \( H^1(\Omega^+) \). \( \square \)

We have therefore proved the following

Theorem 5.

Let \((M, g)\) be a space-time containing a singular hypersurface \( \Lambda \) with \( g_{ab} \) being \( C^{1-} \) across \( \Lambda \). Moreover suppose there exists a congruence of time-like geodesics with a tangent field \( T^a \) satisfying

1. \( T^a_{;b} = 0 \)
2. \( R_{ab}T^b = 0 \)

then solution \( \phi \in H^1(\Omega^+) \) exists to the initial value problem

\[
\square \phi = f \\
\phi|_{\partial S} = \phi_0 \\
n^a \phi, a|_{\partial S} = \phi_1
\]

where

\[
f \in H^1(\Omega^+) \\
\phi_0, \phi_1 \in H^3(S)
\]

Moreover if a solution \( \phi \) satisfies the vanishing flux condition \( F(\tau) = 0 \), then it is unique.

If we replace conditions (1) and (2) by the weaker condition that \( T^a_{;b} \) is essentially bounded then \( \square \) is only a self adjoint operator in \( H^0(\Omega) \) with the usual \( L^2 \) inner product. This is sufficient to prove existence of solutions in \( H^0(\Omega) \).

3. Examples

Thin shells of matter.

An elementary example of a space-time containing a hypersurface singularity may be constructed using the following metric

\[
ds^2 = -dt^2 + F(z)^2 dx^2 + G(z)^2 dy^2 + dz^2
\]

where

\[
F(z) = 1 - az H(z) \\
G(z) = 1 + az (1 - H(z))
\]

It may be shown that the non zero components of the curvature tensor are

\[
R_{zzzz} = a \delta(z) \\
R_{yyzz} = a \delta(z)
\]
and the non-zero components of the energy-momentum tensor are

\[ T_{tt} = 2a \delta(z) \]
\[ T_{xx} = a \delta(z) \]
\[ T_{yy} = a \delta(z) \]

and with \( a > 0 \) this space-time represents a positive energy matter distribution whose support is the hypersurface \( z = 0 \).

This space-time possesses a metric which is \( C^{1-} \) across the singular hypersurface and moreover the integral curves of \( \partial/\partial t \) forms a congruence of time-like geodesics whose tangent satisfies the conditions \( T^a_{\;;b} = 0 \) and \( R_{ab}T^b = 0 \). Therefore Theorem 5 is applicable and we are able to establish that hyperbolicity is applicable.

**Impulsive gravitational waves.**

Another example of a space-time with a singular hypersurface that satisfies similar differentiability conditions is one which contains impulsive gravitational waves. In double null coordinates

\[ u = \frac{1}{\sqrt{2}}(t - z), \quad v = \frac{1}{\sqrt{2}}(t + z) \]

the metric may be written as

\[ ds^2 = -2du \, dv + F(u)^2 \, dx^2 + G(u)^2 \, dy^2 \]

where

\[ F(z) = 1 + auH(u) \]
\[ G(z) = 1 - auH(u) \]

The space-time is vacuum, but has a Weyl tensor with delta function components

\[ C_{uxux} = -a \delta(u) \]
\[ C_{uyuy} = a \delta(u) \]

A congruence of time-like geodesics whose tangent field is

\[ T^a = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0) \]

may be constructed; however \( T^a_{\;b} \) is essentially bounded but not vanishing, so theorem 5 is not applicable to this space-time in its current form.

The proof of Theorem 5 breaks down at Lemma 3 in that self-adjointness of \( \Box \) cannot be established in \( H^1 \). At most we are able to show existence in \( H^0 \) by defining the subspaces \( V_0 \) and \( V_1 \) to be in \( H^0 \) and work with the \( H^0 \) inner product, with which \( \Box \) will be self-adjoint without the need for \( T^a_{\;b} \) to vanish completely.

4. **Conclusion**

Theorem 5 gives us sufficient conditions on the initial data for the existence of an \( H^1 \) solution in space-times with a singular hypersurface; provided certain conditions on the existence of a certain time-like geodesic congruence have been met.

It was found that stronger conditions on the initial data are required, than what is required in Minkowski space. However the fact that the connection components were essentially bounded avoided the need for imposing extra conditions involving the rate at which spatial derivatives vanished as were needed for the conical case (Wilson 2000).

The behaviour of \( T^a_{\;b} \) was found to be more of an issue here as although \( T^a_{\;b} \) being essentially bounded is sufficient for establishing energy inequalities, it was found that the vanishing of \( T^a_{\;b} \) was required \( \Box \) to be self adjoint in \( H^1 \) and therefore to enable existence to be proved in this space. The conical space-time (Wilson, 2000) has vanishing \( T^a_{\;b} \).
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