The leading power Regge asymptotic behaviour of dimensionally regularized massless on-shell planar triple box

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Abstract

The leading power asymptotic behaviour of the dimensionally regularized massless on-shell planar triple box diagram in the Regge limit $t/s \to 0$ is analytically evaluated.
Systematical analytical evaluation of two-loop Feynman diagrams with four external lines within dimensional regularization \([1]\) began three years ago. In the pure massless case with all end-points on-shell, i.e. \(p_i^2 = 0, \ i = 1, 2, 3, 4\), the problem of analytical evaluation of two-loop four-point diagrams in expansion in \(\epsilon = (4 - d)/2\), where \(d\) is the space-time dimension, has been completely solved in \([2, 3, 4, 5, 6, 7]\). The corresponding analytical algorithms have been successfully applied to the evaluation of two-loop virtual corrections to various scattering processes \([8]\) in the zero-mass approximation.

In the case of massless two-loop four-point diagrams with one leg off-shell the problem of the evaluation has been solved in \([9, 10]\), with subsequent applications \([11]\) to the process \(e^+e^- \rightarrow 3\text{jets}\). (See \([12]\) for recent reviews of the present status of NNLO calculations. See \([13]\) for a brief review of results on the analytical evaluation of various double-box Feynman integrals and the corresponding methods of evaluation.) For another three-scale calculational problem, where all four legs are on-shell and there is a non-zero internal mass, a first analytical result was obtained in \([14]\) for the scalar master double box.

The purpose of this paper is to turn attention to three-loop on-shell massless four-point diagrams. As a first step, the leading power asymptotic behaviour of the dimensionally regularized massless on-shell planar triple box diagrams shown in Fig. 1 in the Regge limit \(t/s \rightarrow 0\) will be analytically evaluated. This calculation will demonstrate that a three-loop BFKL analysis \([15]\) (at least its virtual part which can be reduced to the evaluation of Regge asymptotics) is possible\(^2\).

The calculation will be based on the technique of alpha parameters and Mellin–Barnes (MB) representation which was successfully used in \([2, 4, 9, 14]\) and reduces, due to taking residues and shifting contours, to a decomposition of a given MB integral into pieces where a Laurent expansion of the integrand in \(\epsilon\) becomes possible. At a final stage, summation formulae for series of \(S_1(n)S_3(n)/n^2, \psi''(n + 1)S_2(n)/n\) etc., where \(S_k(n) = \sum_{j=1}^n j^{-k}\), are used. A table of such formulae is presented in Appendix.

\(^2\)In three loops, non-planar diagrams as well as higher terms of expansion of double boxes in \(\epsilon\) are also needed.
The general planar triple box Feynman integral without numerator takes the form

\[
T(a_1, \ldots, a_{10}; s, t; \epsilon) = \int \int \int \frac{d^d k \, d^d l \, d^d r}{\left( k^2 \right)^{a_1} \left[ (k + p_2)^2 \right]^{a_2} \left[ (k + p_1 + p_2)^2 \right]^{a_3}}
\times \frac{1}{\left[ (l + p_1 + p_2)^2 \right]^{a_4} \left[ (r - l)^2 \right]^{a_6} \left[ (k - l)^2 \right]^{a_7}}
\times \frac{1}{\left[ (r + p_1 + p_2)^2 \right]^{a_8} \left[ (r + p_1 + p_2 + p_3)^2 \right]^{a_9} \left( r^2 \right)^{a_{10}}}
\]  

(1)

where \( s = (p_1 + p_2)^2 \) and \( t = (p_2 + p_3)^2 \) are Mandelstam variables, and \( k, l \) and \( r \) are loop momenta. Usual prescriptions \( k^2 = k^2 + i0, \ s = s + i0, \) etc. are implied.

To evaluate the leading power asymptotic behaviour of the master triple box (1), i.e. for all \( a_i = 1 \), in the limit \( t/s \to 0 \) one can use the strategy of expansion by regions \([13, 17, 18]\). It shows that in the leading power only \( (1c-1c-1c) \) and \( (2c-2c-2c) \) regions contribute, with the leading power behaviour \( 1/t \). (See \([17]\) and Chapter 8 of \([18]\) for definitions of these contributions.) The leading power \( (2c-2c-2c) \) contribution for the general planar triple box takes the form

\[
T^{(2c-2c-2c)}(a_1, \ldots, a_{10}; s, t; \epsilon) = \int \int \int \frac{d^d k \, d^d l \, d^d r}{\left( k^2 \right)^{a_1} \left[ (k + p_2)^2 \right]^{a_2} \left( 2p_1 k + s \right)^{a_3}}
\times \frac{1}{\left[ (2p_1 l + s)^2 \right]^{a_4} \left[ (r - l)^2 \right]^{a_6} \left[ (k - l)^2 \right]^{a_7} \left( 2p_1 r + s \right)^{a_8} \left[ (r + p_2 + \bar{p})^2 \right]^{a_9} \left( r^2 \right)^{a_{10}}}
\]  

(2)

where \( \bar{p} \) is such that \( \bar{p}^2 = t, \ 2p_1 \bar{p} = 0, \ 2p_2 \bar{p} = -t \). The leading power \( (1c-1c-1c) \) contribution is obtained due to the symmetry \( \{ 1 \leftrightarrow 3, \ 4 \leftrightarrow 6, \ 8 \leftrightarrow 10 \} \).

To resolve the singularity structure of Feynman integrals in \( \epsilon \) it is very useful to apply the MB representation

\[
\frac{1}{(X + Y)^\nu} = \frac{1}{\Gamma(\nu)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz}{X^{\nu + z}} \Gamma(\nu + z) \Gamma(-z)
\]  

(3)

that makes it possible to replace sums of terms raised to some power by their products in some powers, at the cost of introducing extra integrations. In \([2, 4, 9]\) MB integrations were introduced directly in alpha/Feynman parametric integrals. It turns out more convenient to follow (as in \([2, 14]\)) the strategy of \([19]\) and introduce, in a suitable way, MB integrations, first, after integration over one of the loop momenta, \( r \), then after integration over \( l \), and complete this procedure after integration over the loop momentum, \( k \).

After appropriate changes of variables we arrive at the following fourfold MB representation of (2):

\[
T^{(2c-2c-2c)}(a_1, \ldots, a_{10}; s, t; \epsilon) = \frac{(i\pi)^{d/2}}{\Gamma(4 - a_{589(10)} - 2\epsilon)(-s)^{a_{348}}(-t)^{a_{125679(10)} - 6 + 3\epsilon}}
\]
\begin{align*}
\times \frac{\Gamma(6 - a_{15679(10)} - 3\epsilon)\Gamma(a_{125679(10)} - 6 + 3\epsilon)}{\prod_{j=2,5,7,8,9,10} \Gamma(a_j)} \frac{1}{(2\pi i)^4} \int_{-\infty}^{+\infty} \prod_{j=1}^{4} dz_j \, \Gamma(a_8 + z_3)\Gamma(-z_3) \\
\times \frac{\Gamma(4 - a_{569(10)} - 2\epsilon + z_1)\Gamma(-z_1)\Gamma(a_{569(10)} - 4 + 2\epsilon + z_2)\Gamma(6 - a_{125679(10)} - 3\epsilon - z_2)}{\Gamma(8 - a_{1325679(10)} - 4\epsilon + z_1)\Gamma(a_{15679(10)} - 4 + 2\epsilon + z_2)\Gamma(a_4 - z_3) \\
\times \frac{\Gamma(a_{15679(10)} - a_3 - 4 + 2\epsilon + z_1 + z_2)\Gamma(a_4 + z_1 - z_3)\Gamma(2 - a_{9(10)} - \epsilon + z_3)\Gamma(z_2 - z_4)}{\Gamma(6 - a_{45679(10)} - 3\epsilon + z_3)\Gamma(a_{569(10)} - 2 + \epsilon + z_4) \\
\times \Gamma(a_9 + z_4)\Gamma(a_{59(10)} - 2 + \epsilon + z_4) \\
\times \Gamma(2 - a_{47} - \epsilon - z_1 - z_2 + z_3 + z_4)\Gamma(2 - a_{589} - \epsilon - z_3 - z_4),
\end{align*}

where \( a = \sum_{i=1}^{10} a_i, a_{589(10)} = a_5 + a_8 + a_9 + a_{10}, a_{348} = a_3 + a_4 + a_8, \text{ etc.}, \) and integration contours are chosen in the standard way.

We then turn to the master triple box, i.e. set \( a_i = 1 \) and apply, after a preliminary analysis, a standard procedure of taking residues and shifting contours which results in contributions where one can expand in \( \epsilon \) in the integrand. In this way, it is not immediately clear how to perform integrations in some of 2-dimensional MB integrals obtained. One can however proceed numerically at this stage and arrive at a result in expansion in \( \epsilon \) where the simple pole and the finite part are numerically evaluated.

On the other hand, one can follow a straightforward method based on the MB representation and organize the evaluation of the leading power asymptotic behaviour in such a way that it will be a part of a future evaluation of the unexpanded triple box. Using the same way of introducing MB integrations as outlined above, after appropriate changes of variables, we arrive at the following (only!) sevenfold MB representation of the (unexpanded) triple box \( \mathbb{I} \):

\begin{align*}
T(a_1, \ldots, a_8; s, t; \epsilon) &= \frac{\pi^d/2}{3} (-1)^a \prod_{j=2,5,7,8,9,10} \Gamma(a_j)\Gamma(4 - a_{589(10)} - 2\epsilon)(-s)^{a-6+3\epsilon} \\
\times \frac{1}{(2\pi i)^7} \int_{-\infty}^{+\infty} \frac{dw}{w} \prod_{j=2}^{7} dz_j \left( \frac{t}{s} \right)^{w} \frac{\Gamma(a_2 + w)\Gamma(-w)\Gamma(z_2 + z_4)\Gamma(z_3 + z_4)}{\Gamma(a_1 + z_3 + z_4)\Gamma(a_3 + z_2 + z_4) \\
\times \frac{\Gamma(2 - a_{12} - \epsilon + z_2)\Gamma(2 - a_{23} - \epsilon + z_3)\Gamma(a_7 + w - z_4)\Gamma(-z_3)\Gamma(-z_6)}{\Gamma(4 - a_{123} - 2\epsilon + w - z_4)\Gamma(a_6 - z_5)\Gamma(a_4 - z_6) \\
\times \Gamma(+a_{123} - 2 + \epsilon + z_4)\Gamma(w + z_2 + z_3 + z_4 - z_7) \\
\times \Gamma(2 - a_{59(10)} - \epsilon - z_5 - z_7)\Gamma(2 - a_{589} - \epsilon - z_6 - z_7) \\
\times \Gamma(a_{467} - 2 + \epsilon + w - z_4 - z_5 - z_6 - z_7)\Gamma(a_9 + z_7)\Gamma(a_5 + z_5 + z_6 + z_7) \\
\times \Gamma(4 - a_{467} - 2\epsilon + z_5 + z_6 + z_7)\Gamma(a_{589(10)} - 2 + \epsilon + z_5 + z_6 + z_7) \\
\times \Gamma(2 - a_{67} - \epsilon - w - z_2 + z_5 + z_7)\Gamma(2 - a_{47} - \epsilon - w - z_3 + z_6 + z_7).}
\end{align*}

In the case of the master triple box, we set \( a_i = 1 \) for \( i = 1, 2, \ldots, 10 \) and obtain

\[ T^{(0)}(s, t; \epsilon) \equiv T(1, \ldots, 1; s, t; \epsilon) \]
Observe that, because of the presence of the factor $\Gamma(-2\epsilon)$ in the denominator, we are forced to take some residue in order to arrive at a non-zero result at $\epsilon = 0$, so that the integral is effectively sixfold.

The asymptotic expansion in the Regge limit is determined, in the language of the MB representation (3), by the poles in the variable $w$ that come from a $-w$ dependence of the gamma functions. The poles of $\Gamma(-w)$ correspond to the hard contribution to the asymptotic expansion (within expansion by regions) which turns out to be subleading (order $\epsilon^0$). The rest of the poles are not explicit and are generated due to integration over the other variables, $z_i$. An analysis of the integrand shows that the leading Regge asymptotic behaviour, which is effectively described by a gamma function of the type $\Gamma(-1 - 3\epsilon - w)$, is generated by the gamma functions

$$\Gamma(-\epsilon + z_2)\Gamma(-\epsilon - w - z_2 + z_6 + z_7)\Gamma(-1 - \epsilon - z_6 - z_7)$$

or, symmetrically, by

$$\Gamma(-\epsilon + z_3)\Gamma(-\epsilon - w - z_3 + z_5 + z_7)\Gamma(-1 - \epsilon - z_5 - z_7).$$

Then the standard procedure of shifting contours and taking residues is applied. It results again in a sum of MB integrals where a Laurent expansion in $\epsilon$ in the integrand becomes possible.

Following this second variant of evaluation we see that all integration over $z_i$ variables are then explicitly performed by means of the first and the second Barnes lemmas and their corollaries. The last integral, over $w$, is taken by closing the integration contour in the $w$ complex plane to the right and taking a series of residues at $w = 0, 1, 2, \ldots$ At the final stage, summation formulae for series involving $1/n^j$, $S_k(n)$, $S_{ik}(n) = \sum_{j=1}^{n} j^{-i} S_k(j)$, etc. are used. Some of them are presented in Appendix.

The final result for the Regge asymptotics of the planar triple box takes the following form:

$$T^{(0)}(s, t; \epsilon) = -\frac{(i\pi^{d/2}e^{-\gamma\epsilon})^3}{s^3(-t)^{1+3\epsilon}} \left\{ \frac{16}{9\epsilon^6} - \frac{5L}{3\epsilon^5} - \frac{3\pi^2}{2\epsilon^4} - \left[ \frac{11\pi^2}{12}L + \frac{131\zeta(3)}{9} \right] \frac{1}{\epsilon^3} \right\}.$$
\[
+ \left[ \frac{49\zeta(3)}{3} L - \frac{1411\pi^4}{1080} \right] \frac{1}{\epsilon^2} - \left[ \frac{503\pi^4}{1440} L - \frac{73\pi^2\zeta(3)}{4} + \frac{301\zeta(5)}{15} \right] \frac{1}{\epsilon} \\
+ \left[ \frac{223\pi^2\zeta(3)}{12} + 149\zeta(5) \right] \frac{L}{\epsilon} - \frac{624607\pi^6}{544320} + \frac{167\zeta(3)^2}{9} + O(\epsilon) \right] ,
\]

where \( L = \ln s/t \) and \( \zeta(z) \) is the Riemann zeta function.

A non-trivial check of this result comes from the first variant of the evaluation of the Regge asymptotic presented above where the coefficients at the poles \( 1/\epsilon^j, \ j = 2, \ldots, 6 \) agree exactly with (7) while the coefficient at \( 1/\epsilon \) and the finite part in \( \epsilon \) agree numerically.

The value 16/9 for the coefficient at the highest pole is also in agreement with the explicit result for a general \( L \)-loop ladder planar diagram (L-box) by Davydychev and Ussyukina [19] if we believe in a well-known empirical rule which corresponds \( 1/\epsilon \) to a logarithm of an expansion parameter. From their result one can find the leading power and logarithmic asymptotics: \((i\pi^2) L^4 4^L/(L!s^{L}t) \ln^{2L}(-s)\). This gives, in particular, for \( L = 1 \) and \( L = 2 \), and 16/9 for \( L = 3 \), in agreement with existing results at one-, two- and (now) at three-loop level. For higher poles, this correspondence hardly exists, even in some generalized form.

The coefficients at \( 1/\epsilon^6, \ldots, 1/\epsilon^3 \) in (3) have been also confirmed by a numerical check [20] with the help of a method [21] based on numerical integration in the space of alpha parameters.

The procedure described above can be applied, in a similar way, to the calculation of Regge asymptotics of any massless planar on-shell triple box.

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\section{Summation Formulae}

Summation formulae with at least \( 1/n^2 \) factor, which explicitly provides convergence, for example,

\[
\sum_{n=1}^{\infty} S_1(n-1) S_{12}(n-1) \frac{1}{n^2} = \frac{313\pi^6}{45360} - 2\zeta(3)^2 ,
\]

can be obtained\footnote{This result stays unique, starting from the two-loop level, for the class of massless four-point diagrams with all legs off-shell. For example, if one contracts some line (other than a rung) or puts a dot on some line, no analytical results are available for the moment.} using FORM [22] procedures called SUMMER and described in [23].

\footnote{I have derived these formulae myself and then was informed about existence of these procedures. For a person who has never used FORM, it looks simpler to derive (and check) these formulae than...}

5
Here is a table of summation formulae with the factor $1/n$. They are not explicitly present in SUMMER. The convergence is here provided by other factors, $\psi^{(k)}(n+1) = (-1)^k k! [S_{k+1}(n) - \zeta(k+1)]$, $k = 1, 2, \ldots$. We have

\[
\sum_{n=1}^{\infty} \frac{\psi'''(n+1)}{n} = -\frac{2\pi^6}{105} + 12\zeta(3)^2,
\]
\[
\sum_{n=1}^{\infty} \frac{\psi''(n+1)S_1(n)}{n} = \frac{\pi^6}{1512},
\]
\[
\sum_{n=1}^{\infty} \frac{\psi''(n+1)S_1(n)^2}{n} = \frac{\pi^6}{90} - 8\zeta(3)^2,
\]
\[
\sum_{n=1}^{\infty} \frac{\psi'(n+1)^2 S_1(n)}{n} = -\frac{\pi^6}{432} + 2\zeta(3)^2,
\]
\[
\sum_{n=1}^{\infty} \frac{\psi'(n+1)S_1(n)^3}{n} = \frac{269\pi^6}{22680},
\]
\[
\sum_{n=1}^{\infty} \frac{\psi'(n+1)\psi''(n+1)}{n} = \frac{61\pi^6}{22680} - 2\zeta(3)^2,
\]
\[
\sum_{n=1}^{\infty} \frac{\psi'''(n+1)}{n} = -\pi^2 \zeta(3) + 12\zeta(5),
\]
\[
\sum_{n=1}^{\infty} \frac{\psi'(n+1)^2 S_1(n)}{n} = \frac{\pi^2 \zeta(3)}{3},
\]
\[
\sum_{n=1}^{\infty} \frac{\psi''(n+1)S_1(n)}{n} = -\frac{2\pi^2 \zeta(3)}{3} + 7\zeta(5),
\]
\[
\sum_{n=1}^{\infty} \frac{\psi'(n+1)^3}{n} = \frac{5\pi^2 \zeta(3)}{6} - 9\zeta(5),
\]
\[
\sum_{n=1}^{\infty} \frac{\psi''(n+1)^2}{n} = -\frac{\pi^4}{180},
\]
\[
\sum_{n=1}^{\infty} \frac{\psi'(n+1)S_1(n)^2}{n} = \frac{7\pi^4}{360},
\]
\[
\sum_{n=1}^{\infty} \frac{\psi'(n+1)}{n} = \zeta(3).
\]

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