Beltrami operators and their application to constrained diffusion in Beltrami fields

N Sato

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail: sato@kurims.kyoto-u.ac.jp

Received 23 August 2018, revised 20 March 2019
Accepted for publication 26 April 2019
Published 13 May 2019

Abstract
Beltrami fields occur as stationary solutions of the Euler equations of fluid flow and as force free magnetic fields in magnetohydrodynamics. In this paper we discuss the role of Beltrami fields when considered as operators acting on a Hamiltonian function to generate particle dynamics. Beltrami operators, which include Poisson operators as a special subclass, arise in the description of topologically constrained diffusion in non-Hamiltonian systems. Extending previous results (Sato and Yoshida 2018 Phys. Rev. E 97 022145), we show that random motion generated by a Beltrami operator satisfies an H-theorem, leading to a generalized Boltzmann distribution on the coordinate system where the Beltrami condition holds. When the Beltrami condition is violated, random fluctuations do not work anymore to homogenize the particle distribution in the coordinate system where they are applied. The resulting distribution becomes heterogeneous. The heterogeneity depends on the ‘field charge’ measuring the departure of the operator from a Beltrami field. Examples of both Beltrami and non-Beltrami operators in three and four real dimensions together with the corresponding equilibrium distribution functions are given.

Keywords: Beltrami fields, almost Poisson brackets, diffusion, H-theorem

1. Introduction

A 3-dimensional Beltrami field is a vector field aligned with its own curl. Beltrami fields arise as stationary solutions of hydrodynamic [2, 3] and magnetohydrodynamic fluid equations [4, 5]. In these systems, a Beltrami field represents a physical field, such as fluid velocity or magnetic field. When the motion of a particle is considered, these fields behave as antisymmetric operators that generate particle dynamics by acting on the particle Hamiltonian.
Antisymmetric operators, mathematically represented by bivectors, generalize Poisson operators [6, 7] of noncanonical Hamiltonian mechanics by allowing violation of the Jacobi identity [8]. This generalization becomes necessary when non-integrable constraints affect a dynamical system [9–11]. An example pertaining to plasma physics is $\mathbf{E} \times \mathbf{B}$ drift motion [12], where the magnetic field plays the role of antisymmetric operator, and the electric field represents the gradient of the Hamiltonian, in this case given by the electrostatic potential (in this paper we refer to particle energy as Hamiltonian function, even if the Jacobi identity is violated). Object of the present study are Beltrami operators, i.e. antisymmetric operators that satisfy the Beltrami condition.

If we consider an ensemble of particles endowed with an antisymmetric operator, particle interactions that drive the system toward the equilibrium state can be represented by allowing a non-deterministic time-dependent part in the Hamiltonian function. Then the equation of motion is stochastic in nature, and it can be translated into a Fokker–Planck equation for a probability density (the distribution function of the ensemble, see [1, 13, 14]). The resulting diffusion operator, which is written in terms of the antisymmetric operator, is a second order non-elliptic partial differential operator. Violation of ellipticity (see [15, 16] for the definition of elliptic differential operator) occurs due to the null-space of the antisymmetric operator. Such null-space, which reflects constraints affecting particle motion, makes the coefficient matrix of the diffusion operator degenerate, thus breaking ellipticity.

Integrability of constraints (in the sense of the Frobenius theorem [17]) is essential in determining the geometrical properties of the antisymmetric operator: according to the Lie–Darboux theorem of differential geometry [6, 18, 19], the null-space of any constant rank Poisson operator is locally and completely integrable in terms of Casimir invariants. The level sets of the Casimir invariants define a symplectic submanifold where phase space coordinates are available. Hence, the outcome of diffusion in a noncanonical Hamiltonian system is a generalized Boltzmann distribution on the phase space metric of the symplectic submanifold weighted by the Casimir invariants [1, 20, 21].

When considering diffusion of an ensemble of particles in a given coordinate system and according to a prescribed antisymmetric operator, it has been shown in [1] that the Beltrami condition is the minimal requirement needed for the distribution function to homogenize (i.e. for the entropy to maximize) in that same reference frame.

Aim of the present paper is to elucidate the statistical properties of Beltrami operators. The manuscript is organized as follows. Section 2 reviews the mathematical formalism and the geometrical quantities characterizing antisymmetric operators. In section 3 we generalize the result of [1], which concerned pure diffusion processes, by allowing a deterministic part in the Hamiltonian function, as well as a friction term in the stochastic equation of motion, and prove an H-theorem for Beltrami operators. The result is a generalized Boltzmann distribution on the coordinate system where the Beltrami condition holds. We remark that the treatment of Beltrami operators of [1] was incomplete in that the proof of the H-theorem for the general case requires a new term in the Fokker–Planck equation, depending on a new constant $\kappa$, which cannot be predicted without physical considerations on the collective reaction of the particle ensemble to macroscopic ‘currents’ that self-consistently set up in the system. In section 4, we examine weak Beltrami operators. It is shown that generalized Boltzmann states are local entropy maxima. In section 5 we prove an H-theorem for a class of non-Beltrami operators. Both entropy measure and equilibrium distribution function are found to be ‘distorted’ by the non-vanishing field charge. In section 6 we derive the value of the physical constant $\kappa$, arising from the proof of the H-theorem of section 3, from a physical model describing electron diffusion in a magnetized plasma. In section 7 examples of Beltrami and non-Beltrami operators in $\mathbb{R}^3$ are given, and a systematic method to construct solenoidal Beltrami fields in
$\mathbb{R}^3$ is obtained. Examples of Beltrami and non-Beltrami operators in $\mathbb{R}^4$ are studied in section 8. Section 9 is for the conclusion.

2. Mathematical preliminaries

We consider a smooth bounded connected domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial \Omega$ and a coordinate system $x = (x^1, ..., x^n)$ with tangent basis $(\partial_1, ..., \partial_n)$. An antisymmetric operator is a bivector field $\mathcal{J} \in \bigwedge^2 T\Omega$ such that:

$$\mathcal{J} = \sum_{i<j} \mathcal{J}^i_j \partial_i \wedge \partial_j = \frac{1}{2} \mathcal{J}^{ij} \partial_i \wedge \partial_j, \quad \mathcal{J}^{ij} = -\mathcal{J}^{ji}.$$  

(1)

In the following we assume that $\mathcal{J}^{ij} \in C^\infty(\Omega)$, $i, j = 1, ..., n$. Given a Hamiltonian function $H \in C^\infty(\Omega)$, the equations of motion generated by $\mathcal{J}$ can be written as a vector field $X \in T\Omega$:

$$X = \mathcal{J}(dH) = \mathcal{J}^{ij} H_j \partial_i.$$  

(2)

Here, a lower index indicates partial differentiation, $H_j = \partial H / \partial x^j$. This convention will be used in the rest of the paper. Due to antisymmetry we have:

$$\dot{H} = i_X dH = \mathcal{J}^{ij} H_i H_j = 0.$$  

(3)

Here $i$ is the contraction operator. Since the Hamiltonian $H$ is preserved, $X$ is called a conservative vector field. Each $\mathcal{J}$ defines an antisymmetric bilinear form (bracket) $\{ \cdot, \cdot \}$. Given a pair of smooth functions $f, g \in C^\infty(\Omega)$, we have:

$$\{ f, g \} = \mathcal{J}(df, dg) = f_i \mathcal{J}^{ij} g_j.$$  

(4)

The evolution of a function $f$ with respect to the flow (2) can be written as:

$$\dot{f} = \{ f, H \}.$$  

(5)

$\mathcal{J}$ is called a Poisson operator whenever it satisfies the Jacobi identity:

$$h^{ijk} = \mathcal{J}^{im} \mathcal{J}^{jk}_m + \mathcal{J}^{jm} \mathcal{J}^{ki}_m + \mathcal{J}^{km} \mathcal{J}^{ij}_m = 0, \quad \forall i, j, k = 1, ..., n.$$  

(6)

The Jacobi identity is equivalent to demanding that the following trivector vanishes identically:

$$\Theta = \frac{1}{2} \mathcal{J}^{im} \mathcal{J}^{jk}_m \partial_i \wedge \partial_j \wedge \partial_k = \sum_{i<j<k} h^{ijk} \partial_i \wedge \partial_j \wedge \partial_k.$$  

(7)

We shall refer to $\Theta$ as the Jacobiator, and to the tensor $h^{ijk}$ as the helicity density of $\mathcal{J}$.

We introduce an auxiliary volume form:

$$\text{vol}_g = g \, dx^1 \wedge ... \wedge dx^n, \quad g \in C^\infty(\Omega), \quad g \neq 0.$$  

(8)

By contracting the bivector $\mathcal{J}$ with this volume form we define the covorticity n-2 form:

$$\mathcal{J}^{n-2} = i_{\mathcal{J}} \text{vol}_g = 2 \sum_{i<j} (-1)^{i+j-1} g \mathcal{J}^{ij} dx_g^{n-2}.$$  

(9)

In this notation $dx_g^{n-2} = dx^1 \wedge ... \wedge dx^{i-1} \wedge dx^{i+1} \wedge ... \wedge dx^{j-1} \wedge dx^{j+1} \wedge ... \wedge dx^n$. The cocurrent n-1 form is defined to be:

$$\mathcal{O}^{n-1} = d\mathcal{J}^{n-2} = 2 (-1)^j \frac{\partial (g \mathcal{J}^{ij})}{\partial x^i} \, dx_j^{n-1}.$$  

(10)
The flow (2) admits an invariant measure \( \text{vol}_g \) for some appropriate function \( g \) and for any choice of \( H \) provided that:

\[
\mathcal{L}_X \text{vol}_g = \frac{1}{g} \frac{\partial}{\partial x} (gJ^0) H_j \text{vol}_g = 0 \quad \forall H.
\]

This is equivalent to demanding that \( \mathcal{O}^{n-1} \) is identically zero. Therefore, an antisymmetric operator satisfying \( \mathcal{O}^{n-1} = 0 \) for some function \( g \) will be called measure preserving.

Due to the Lie–Darboux theorem, a constant rank Poisson operator locally defines a symplectic submanifold. This submanifold is endowed with the invariant measure provided by Liouville’s theorem. Hence, a constant rank Poisson operator is locally measure preserving. Note that however not all measure preserving operators are locally Poisson.

In the following we restrict our attention to a Cartesian coordinate system \( x = (x^1, \ldots, x^n) \) with the standard Euclidean metric of \( \mathbb{R}^n \). All quantities and operations, including the Hodge map \( * \), will be defined according to such metric. Consider the differential forms:

\[
b^{n-1} = J^{n-2} \wedge *dJ^{n-2},
\]

\[
\mathfrak{B} = *db^{n-1}.
\]

We call (12a) the field force of \( \mathcal{J} \) and (12b) its field charge. An antisymmetric operator \( \mathcal{J} \) will be called a Beltrami operator whenever \( b^{n-1} = 0 \). If \( \mathfrak{B} = 0 \), the operator will be a weak Beltrami operator. A Beltrami operator will be nontrivial if \( G \neq 0 \). Notice that, in principle, (12a) and (12b) can be defined with respect to other metrics.

When \( n = 3 \), \( (x^1, x^2, x^3) = (x, y, z) \) is a Cartesian coordinate system, and \( * \) is defined with respect to the Euclidean metric of \( \mathbb{R}^3 \), the field force reduces to \( *b^1 = 4 \left[ w \times (\nabla \times w) \right] \), where \( w_x = J^{yz}, \ w_y = J^{zx}, \ w_z = J^{xy} \), and \( w = (w_x, w_y, w_z) \). Thus, in \( \mathbb{R}^3 \) a Beltrami operator is nothing but a vector field satisfying the Beltrami condition \( w \times (\nabla \times w) = 0 \). It is also worth observing that, in \( \mathbb{R}^3 \), the Jacobi identity reduces to \( h = w \cdot \nabla \times w = 0 \). Hence, a Beltrami operator is ‘dual’ to a Poisson operator in the sense that while the former is aligned with its own curl, the latter is perpendicular to it. Table 1 summarizes the geometrical quantities introduced in this section. Note that violation of the summation convention may occur in formulas involving \( \mathcal{J} \). In particular, repeated indices that are both in the upper position should be summed (this notation is adopted in the rest of the paper). Such violation occurs because \( \mathcal{J} \) is a bivector, and its components \( J^0 \) have been used to define differential forms (covectors) that are useful for the later analysis.

### 3. H-theorem for Beltrami operators

We consider an ensemble of \( N \) particles, at first not interacting, and each obeying the equation of motion:

\[
X_0 = \mathcal{J} \left( dH_0 \right).
\]

Here \( H_0 \in C^\infty(\Omega) \) is a time-independent Hamiltonian function and \( \mathcal{J} \) a Beltrami operator. In this case, the Beltrami condition \( b^{n-1} = 0 \) reads:

\[
b^{n-1} = 4 (-1)^{i-1} J^0 \frac{\partial J^0}{\partial x^i} dx^{i-1} = 0.
\]

This implies:
Here, we let particles interact with each other. The energy of a single particle now has the form $H = H_0 + H_1$, where $H_0 = H_0(x,t)$ is a deterministic component, and $H_1 = H_1(x,t)$ a stochastic interaction term. $H_0$ may include self-induced potentials, such as an electric potential generated by electromagnetic interactions. In the following, we require $H_0$ to satisfy the condition:

$$
\langle \partial_t H_0 \rangle = \int_{\Omega} f \partial_t H_0 \, dV = 0.
$$

(16)

Next, we take into account the stochastic interaction term $H_1$, and we demand that:

$$
H_1 = D^{1/2} x^i \Gamma_i, \quad i, 1, \ldots, n.
$$

(17)

Here $\Gamma = (\Gamma_1, \ldots, \Gamma_n)$ is an n-dimensional Gaussian white noise random process, and $D > 0$ is a positive spatial constant (diffusion parameter). Equation (17) implies that the stochastic force causing relaxation, $-\nabla H_1 = -D^{1/2} \Gamma$, is homogeneous in the Cartesian coordinate system of $\mathbb{R}^n$. We further assume that such stochastic force is counterbalanced by a friction (viscous damping) force, $-\gamma x_0$, where $\gamma$ is a positive spatial constant (friction or damping parameter). In summary, the equation of motion of a particle in the ensemble now reads:

$$
X^i = J^{ij} \left( H_0 + \gamma J^{jk} H_0k + D^{1/2} \Gamma_j \right) - \kappa J^{ij} + \gamma J^{jk} J^{ij} H_0k + D^{1/2} J^{ij} \Gamma_j - \kappa J^{ij}.
$$

(18)

Table 1. List of geometrical quantities. AS stands for antisymmetric.

| Name           | Symbol | Definition | Expression |
|----------------|--------|------------|------------|
| Volume         | $\text{vol}_i$ | $g \, dx^1 \wedge \ldots \wedge dx^n$ |            |
| AS matrix      | $J$    | $J^{ij} = -J^{ji}$ |            |
| AS operator    | $\mathcal{J}$ | $\frac{1}{2} \mathcal{J}^{ij} \partial_i \wedge \partial_j$ |            |
| Helicity density | $h^{ij}$ | $J^{i\alpha} J^{j\beta} + J^{i\beta} J^{j\alpha} + J^{i\mu} J^{j\nu} \delta^{\mu\nu}$ |            |
| Jacobian       | $\mathcal{O}$ | $\frac{1}{2} J^{i\alpha} \partial_i \wedge \partial_j \wedge \partial_k$ |            |
| Covermix      | $\mathcal{C}^{\mu-1}$ | $d_\mathcal{C} \mathcal{J}^{\mu-2}$ | $2 \sum_{\mu<i<j} (-1)^{i+j-1} g J^{ij} \mathcal{d}^{\mu-2}$ |
| Field force    | $b^{\mu-1}$ | $\mathcal{J}^{\mu-2} \wedge \ast d_\mathcal{C} \mathcal{J}^{\mu-2}$ | $4 \sum_{\mu<i<j<k} (-1)^{i+j+k-1} \times g J^{ij} \frac{\partial (x^j \mathcal{J}^{ij})}{\partial x^k} \mathcal{d}^{\mu-2} \wedge \ast \mathcal{d}^{\mu-1}$ |

$\mathcal{J}^{ij} \partial_i \mathcal{J}^{ij} / \partial x^j = 0, \quad \forall i = 1, \ldots, n.$

(15)
\[
\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x^i}(J^i^f) \\
= \frac{\partial}{\partial x^i} \left[ -(J^i^f - \gamma J^i^k J^k f) H_{0j} f + \frac{1}{2} D J^i^k \frac{\partial}{\partial x^j}(J^k f) \right].
\]

(19)

Here we defined \( Z^i \) to be the Fokker–Planck velocity of the system:
\[
Z^i = (J^i_j - \gamma J^i_k J^k_j) H_{0j} - \kappa J^i_j - \frac{1}{2} D J^i_k \frac{\partial}{\partial x^j}(J^k_j f).
\]

(20)

If \( J \) satisfies the Beltrami condition (15), the Fokker–Planck equation (19) can be simplified to:
\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial x^i} \left\{ f \left[ -J^i_j H_{0j} + \kappa J^i_j + \frac{1}{2} D J^i_k J^k_j \left( \frac{\partial \log f}{\partial x^j} + 2\gamma D H_{0j} \right) \right] \right\}.
\]

(21)

The goal of the remaining part of this section is to show that the solution of the Fokker–Planck equation (21) is such that the entropy functional
\[
S = -\int_{\Omega} f \log f \, dV,
\]

(22)
does not decrease in time, i.e. we wish to show that for \( t \geq 0 \)
\[
\frac{dS}{dt} \geq 0.
\]

(23)

This statement represents an ‘H-theorem’, which describes the non-increasing nature of the functional \( H = -S \). Here, the quantity \( H \) is defined in analogy with the H-functional occurring in the classical result of statistical mechanics due to Boltzmann. Observe that the validity of equation (23) is not trivial in the present context because the system under consideration is not Hamiltonian. Therefore Liouville’s theorem does not hold, and there is no phase space measure that can be used to naturally induce an entropy functional.

We have the following:

**Theorem 3.1.** Let \( f \in C^\infty([0, +\infty) \times \Omega) \) be a smooth function in a smooth bounded domain \( \Omega \subset \mathbb{R}^n \) satisfying equation (19), with \( J \in \Lambda^2 T \Omega \) a Beltrami operator in \( \mathbb{R}^n \), \( J^i_j \in C^\infty(\Omega) \), \( H_0 \in C^\infty([0, +\infty) \times \Omega) \) the Hamiltonian function, and \( D \), \( \gamma = \beta D/2 \), and \( \kappa = \beta^{-1} \) positive real constants. Define \( X_0 = J^i_j H_{0j} \) and \( \xi^i = J^i_j \). Let \( Z \) be the Fokker–Planck velocity of (20) and \( N \) the unit outward normal to \( \partial \Omega \). Suppose that \( f > 0 \) in \( \Omega \), \( \langle \partial_t H_0 \rangle = 0 \), and \( Z \cdot N = 0 \) and \( (X_0 + \kappa \xi) \cdot N = 0 \) on \( \partial \Omega \). Then,
\[
\frac{dS}{dt} \geq 0,
\]

(24)

with \( S = -\int_{\Omega} f \log f \, dV \). Furthermore
\[
\lim_{t \to \infty} \int_{\Omega} (d \log f + \beta d H_0) = 0.
\]

(25)

Note that the boundary conditions of theorem 3.1 ensure the closure of the system, i.e. they prevent a net outflow of particles from the domain \( \Omega \).

**Proof.** Using the antisymmetry of \( J \), the Beltrami property (15), the Fokker–Planck equation (21), and the boundary conditions to eliminate surface integrals, the rate of change in \( S \) is:
\[
\frac{dS}{dr} = -\int_{\Omega} \partial_t f (1 + \log f) \, dV
= -\int_{\Omega} f Z \cdot \partial \log f \, dV
= \int_{\Omega} \left[ J^i_0 H_0 f + \frac{1}{2} D J^i_0 J^0_j \left( \frac{\partial \log f}{\partial x^i} \right) f_i \right] \, dV.
\] (26)

Set \( \beta = 2\gamma/D \). After some manipulations:

\[
\frac{dS}{dr} = (1 - \beta \kappa) \int_{\Omega} f J^i_0 H_0 \, dV + \beta \kappa \int_{\Omega} f J^i_0 H_0 \, dV
+ \frac{1}{2} D \int_{\Omega} f J^i_0 J^0_j \left( \beta H_0 + \frac{\partial \log f}{\partial x^i} \right) \left( \beta H_0 + \frac{\partial \log f}{\partial x^j} \right) \, dV
- \frac{1}{2} D \int_{\Omega} f J^i_0 J^0_j \left( \beta H_0 + \frac{\partial \log f}{\partial x^i} \right) \beta H_0 \, dV. \tag{27}
\]

On the other hand, conservation of total energy \( E = \int_{\Omega} H_0 \, dV \) implies that:

\[
\frac{dE}{dr} = \int_{\Omega} \partial_t H_0 + f \partial_x H_0 \, dV = \int_{\Omega} f Z^i H_0 \, dV = 0. \tag{28}
\]

Here we used equation (16). Substituting the Fokker–Planck velocity \( Z \) into the equation above, we arrive at the condition:

\[
\frac{1}{2} D \int_{\Omega} f J^i_0 J^0_j \left( \beta H_0 + \frac{\partial \log f}{\partial x^i} \right) H_0 \, dV = \beta \kappa \int_{\Omega} f J^i_0 H_0 \, dV. \tag{29}
\]

Substituting equation (29) into (27) gives:

\[
\frac{dS}{dr} = (1 - \beta \kappa) \int_{\Omega} f J^i_0 H_0 \, dV + \frac{1}{2} D \int_{\Omega} f |J_0 \left( d \log f + \beta d H_0 \right)|^2 \, dV. \tag{30}
\]

Hence, recalling that \( \kappa = \beta^{-1} \), we arrive at:

\[
\frac{dS}{dr} = \frac{1}{2} D \int_{\Omega} f |J_0 \left( d \log f + \beta d H_0 \right)|^2 \, dV \geq 0. \tag{31}
\]

At \( t \to \infty \), we must have \( dS/dt = 0 \). It follows that, if \( f > 0 \), the distribution function satisfies:

\[
\lim_{t \to \infty} J_0 \left( d \log f + \beta d H_0 \right) = 0. \tag{32}
\]

\( \square \)

Solutions of equation (32) are generalized Boltzmann distributions. Indeed, if the matrix \( J^i_0 \) is invertible, equation (32) reduces to the standard Boltzmann distribution \( f^\infty \propto \exp \{ -\beta H_0^\infty \} \), where \( f^\infty = \lim_{t \to \infty} f \) and \( \beta H_0^\infty = \lim_{t \to \infty} \beta H_0 \). When the matrix \( J^i_0 \) has a null-space that is at least partially integrable by some invariants \( C \), i.e. \( J \left( d C \right) = 0 \), one has \( f^\infty \propto \exp \left\{ -\beta H_0^\infty - \mu \left( C \right) \right\} \), with \( \mu \) a function of the invariants \( C \) depending on the initial configuration of the system. Remember that if \( J \) is a constant rank Poisson operator, its null-space is always completely locally integrable in terms of Casimir invariants (see [1]).
Finally, in theorem 3.1, as well as in theorems 4.1 and 5.1 of the next sections, it is assumed that the Fokker–Planck equation admits a global smooth solution \( f \in C^\infty ([0, \infty) \times \Omega) \). While stationary regular solutions of the Fokker–Planck equation can be written out explicitly (the Boltzmann state is an example), existence of smooth global solutions to the time-dependent case is nontrivial. Numerical simulations (see the examples discussed in [1]) suggest that such solutions exist. Nevertheless, rigorous results are not available at present. It is, however, worth observing that existence of global classical solutions has been shown for the standard Boltzmann equation [23, 24] under suitable assumptions on the interaction driving the ensemble toward equilibrium. Although Boltzmann’s collision operator is different, these results do suggest the existence of global classical solutions of the present problem, since the non-uniform ellipticity of the diffusion operator is expected to only affect the uniqueness of the solution.

4. Local entropy maximum for weak Beltrami operators

In this section we study the case of weak Beltrami operators in \( \mathbb{R}^n \), i.e. operators with vanishing field charge \( B = 0 \) with respect to the Euclidean metric of \( \mathbb{R}^n \). The components \( J^{ij} \) now satisfy the condition:

\[
B = \ast dN - 1 = 4 \frac{\partial}{\partial x^i} \left( J^{ij} \frac{\partial J^{kl}}{\partial x^l} \right) = 0. \tag{33}
\]

It is convenient to define the quantity

\[
\theta = \log f + \beta H_0, \tag{34}
\]

and the vector fields

\[
\xi = \frac{\partial J^{h}}{\partial x^i} \partial_i, \quad b = J^{ij} \frac{\partial J^{kl}}{\partial x^l} \partial_i = J \xi. \tag{35}
\]

\( \xi \) and \( b \) are the vector representations of cocurrent \( \mathcal{O}^{n-1} \) and field force \( b^{n-1} \) in \( \mathbb{R}^n \). Here, the action of the bivector \( J \) on a vector field \( \xi \) is defined as the standard matrix multiplication in \( \mathbb{R}^n 

\[ J \xi = J^{ij} \xi_j \frac{\partial}{\partial x^i}. \]

We wish to show that generalized Boltzmann states of the type \( J d\theta = 0 \) are local entropy maxima. More precisely, we have two cases.

**Theorem 4.1.** Let \( f \in C^\infty ([0, 1) \times \Omega) \) be a smooth function in a smooth bounded domain \( \Omega \subset \mathbb{R}^n \) satisfying equation (19), with \( \mathcal{J} \in \bigwedge^2 T\Omega \) a weak Beltrami operator in \( \mathbb{R}^n \) with components \( J^{ij} \in C^\infty (\Omega) \), \( H_0 \in C^\infty ([0, +\infty) \times \Omega) \) the Hamiltonian function, and \( D, \gamma = \beta D/2 \), and \( \kappa = \beta^{-1} \) positive real constants. Define \( X_0 = J^0 H_0, \xi^i = J^{ij}, \) and \( b^i = J^{ik} J^{jh} \). Let \( Z \) be the Fokker–Planck velocity of (20) and \( N \) the unit outward normal to \( \partial \Omega \).

1. Suppose that \( \nabla H_0 \cdot b = 0 \) and \( f > 0 \) in \( \Omega \), \( (\partial_t H_0) = 0 \), and \( Z \cdot N = 0 \) and \( (X_0 + \kappa \xi - \frac{1}{2} Db) \cdot N = 0 \) on \( \partial \Omega \). Then,

\[
\frac{dS}{dt} \geq 0, \tag{36}
\]

with \( S = -\int_{\Omega} f \log f \, dV \). Furthermore

\[
\lim_{t \to \infty} \mathcal{J} (d \log f + \beta dH_0) = 0. \tag{37}
\]
2. Suppose that $\nabla H_0 \cdot b \neq 0$ and $\partial_t H_0 = 0$ in $\Omega$, and $Z \cdot N = 0$ and $(X_0 + \kappa \xi) \cdot N = 0$ on $\partial \Omega$. Let $u \in C^\infty(\overline{\Omega})$ be a smooth function such that $u > 0$ in $\Omega$ and

$$J (d \log u + \beta \, dH_0) = 0. \tag{38}$$

Then, $u$ is a local entropy maximum, i.e.

$$\left( \frac{dS}{dt} \right)_{f=u} = 0, \quad \left( \frac{d^2 S}{dt^2} \right)_{f=u} < 0. \tag{39}$$

**Proof.** The proof of the first statement is analogous to that of theorem 3.1. In particular, the rate of change of the entropy $S$ has the form

$$\frac{dS}{dt} = \frac{1}{2} D \int_{\Omega} f |J d\theta|^2 \, d\nu + \frac{1}{2} D \beta \int_{\Omega} f \nabla H_0 \cdot b \, d\nu. \tag{40}$$

Since, by hypothesis, $\nabla H_0 \cdot b = 0$, the rate of change is non-negative:

$$\frac{dS}{dt} = \frac{1}{2} D \int_{\Omega} f |J d\theta|^2 \, d\nu \geq 0. \tag{41}$$

Equation (37) follows by considering the limit of the integrand when $t \to \infty$.

The proof of the second statement can be obtained as follows. By repeating the calculations of theorem 3.1 without imposing the condition $b = 0$, the rate of change in the entropy $S$ can be evaluated as

$$\frac{dS}{dt} = \frac{1}{2} D \int_{\Omega} f \left( |J d\theta|^2 + b \cdot \nabla \theta \right) \, d\nu = \frac{1}{2} D \int_{\Omega} f \left( |J d\theta|^2 - \xi \cdot J d\theta \right) \, d\nu. \tag{42}$$

If $f = u$, $J d\theta = 0$. Hence

$$\left( \frac{dS}{dt} \right)_{f=u} = 0. \tag{43}$$

Observe that, since the condition $B = 0$ has not yet been used, this shows that the generalized Boltzmann state (38) is always a stationary point of the entropy $S$. To see that the point $f = u$ is a local maximum when $B = 0$, we consider how the rate of change $\sigma = dS/dt$ is modified by a small perturbation around the state $f = u$ at time $t = 0$. We have

$$\delta \sigma = \frac{1}{2} D \int_{\Omega} \left[ \delta f \left( |J d\theta|^2 - \xi \cdot J d\theta \right) + f \left( 2 J d\theta \cdot J d\theta + b \cdot \nabla \delta \theta \right) \right] \, d\nu. \tag{44}$$

It follows that

$$\left( \delta \sigma \right)_{f=u} = \left( \frac{1}{2} D \int_{\Omega} f b \cdot \nabla \delta \theta \, d\nu \right)_{f=u} = - \left( \frac{1}{2} D \int_{\Omega} \delta \theta \nabla f \cdot b \right)_{f=u}. \tag{45}$$

In the last passage we used the fact that the boundary conditions $Z \cdot N = (X_0 + \kappa \xi) \cdot N = 0$ on $\partial \Omega$ imply $b \cdot N = 0$ on $\partial \Omega$ when $f = u$, and the assumption $B = 0$. Next, observe that,
since $\partial_t H_0 = 0$, one has
\[
\delta \theta = \frac{\delta f}{f} = t \frac{\partial f}{f} + o \left( t^2 \right) = - t \frac{\nabla \cdot (fZ)}{f} + o \left( t^2 \right).
\] (46)

On the other hand,
\[
-[\nabla \cdot (fZ)]_{f=u} = \left[ - H_0 J^i_i f + \kappa J^i_i f_i + \frac{1}{2} D \frac{\partial}{\partial \nu} \left( f J^i_k J^k_j \theta_j + f b_j \right) \right]_{f=u}
\]
\[
= \left[ - \kappa f \nabla \cdot (J \theta) + \frac{1}{2} D \nabla \cdot (f b) \right]_{f=u}
\]
\[
= \frac{1}{2} D \nabla u \cdot b.
\] (47)

Here, we imposed $\kappa = \beta^{-1}$ and $\mathcal{B} = 0$. Hence,
\[
\left( \frac{d^2 S}{dt^2} \right)_{f=u} = \left( \lim_{t \to 0} \frac{\delta \sigma}{t} \right)_{f=u} = - \frac{1}{4} D^2 \int_{\Omega} u (\nabla \log u \cdot b)^2 \, dV.
\] (48)

From equation (38) we see that $J \, du = - u \beta J \, dH_0$. Equation (48) can thus be rewritten as
\[
\left( \frac{d^2 S}{dt^2} \right)_{f=u} = - \frac{1}{4} \beta^2 D^2 \int_{\Omega} u (\nabla H_0 \cdot b)^2 \, dV < 0.
\] (49)

In the last passage we used the hypothesis $\nabla H_0 \cdot b \neq 0$. Thus, $f = u$ is a local maximum because $dS/dt = 0$ and $d^2 S/dt^2 < 0$ there. □

It is worth noticing that, under the hypothesis of the second part of theorem 4.1, the generalized Boltzmann state (38) does not represent a stationary solution to the Fokker–Planck equation (19). Therefore, a system initially in the state (38) will move to a different equilibrium state. During this process the entropy $S$ will decrease, at least for a short time, since (38) is a local entropy maximum. Physically, this means that the system undergoes self-organization.

5. H-Theorem for non-Beltrami operators

In this section, we prove an H-theorem for a class of non-Beltrami operators in $\mathbb{R}^n$. The form of the distribution function at equilibrium is obtained.

The analysis of the previous sections shows that weak Beltrami operators, characterized by the vanishing of the field charge $\mathcal{B} = 0$, represent the largest class for which the entropy $S = - \int_{\Omega} f \log f \, dV$ can be used as a predictor of the statistical behavior of the system. Therefore, in order to deal with operators in the class $\mathcal{B} \neq 0$ it is necessary to modify the entropy measure to a form that is compatible with entropy growth. In particular, we consider functionals defined as
\[
\Sigma_g = - \int_{\Omega} f \log \left( \frac{f}{g} \right) \, dV, \quad g \in C^\infty (\Omega), \quad g > 0.
\] (50)

The natural interpretation of the function $g$ is that of the Jacobian of the coordinate change $x \to y$ such that
\[ dV = dx^1 \wedge \ldots \wedge dx^n = g^{-1} dy^1 \wedge \ldots \wedge dy^n = g^{-1} dV', \]  
(51)

where \( dV' = dy^1 \wedge \ldots \wedge dy^n \). If \( \Sigma_\phi \) is maximized by the statistical process (19), the equilibrium distribution function should be attainable in terms of the following variational principle:

\[ \delta (\Sigma_\phi - \alpha N - \beta E - \eta C) = 0, \]  
(52)

where \( N = \int_\Omega f \ dV, \) \( E = \int_\Omega \mathcal{H}_0 \ dV, \) and \( C = \int_\Omega fC' \ dV \) are constraints representing conservation of particle number, total energy, and total Casimir invariants (if they exist) respectively, and \( \alpha, \beta, \) and \( \eta \) real constants (Lagrange multipliers). Hence, at equilibrium we expect

\[ f^\infty = g e^{-\alpha - \beta \mu - \eta C}. \]  
(53)

The remaining task is to relate \( g \) with the field charge \( \mathfrak{B} \). In the context of three-dimensional diffusion in a magnetic field \( B \) of uniform strength \( |B| = \) constant, the field charge reduces to the divergence of the Lorentz force \( B \times (\nabla \times B) \). In analogy with Poisson’s equation, where the divergence of the electric field gives the electric charge density, the quantity \( \mathfrak{B} \) can thus be interpreted as an effective charge density (this analogy is the reason of the naming ‘field charge’). Hence, the potential energy \( \phi \) associated to the charge \( \mathfrak{B} \) should be given as \( \mathcal{D}\phi = \mathfrak{B} \), with \( \mathcal{D} \) a second order partial differential operator to be determined. Then, \( g = e^{-\phi} \).

In theorem 5.1 it is shown that this is what happens for a certain class of non-Beltrami operators. Before proving theorem 5.1, we need to elucidate some properties of the vector fields \( \xi \) and \( b \).

**Proposition 5.1.** There exist smooth vector fields \( u, k \in T\Omega \) with \( k \in \ker J \) such that

\[ \xi = J u + k, \ b = J^2 u. \]  
(54)

**Proof.** The vector field \( \xi \) can be decomposed as \( \xi = J u + k \) with \( J u \in \text{Im} J \), where \( \text{Im} J \) is the image of \( J \), and \( k \perp \text{Im} J \). It follows that

\[ 0 = k \cdot J v = -J k \cdot v \ \forall v \in T\Omega. \]  
(55)

Hence \( k \in \ker J \). Furthermore, \( b = J \xi = J^2 u \).

The following H-theorem pertains to operators such that \( u = -\nabla \phi \) for some scalar function \( \phi \in C^\infty (\Omega) \), implying \( \xi = -J d\phi + k \) and \( b = -J^2 d\phi \). Recalling that \( \mathfrak{B} = 4\nabla \cdot b \), one has \(-4\nabla \cdot (J^2 d\phi) = \mathfrak{B} \), giving the second-order degenerate elliptic partial differential operator \( \mathcal{D} = -4\nabla \cdot (J^2 d) \).

**Theorem 5.1.** Let \( f \in C^\infty ([0, +\infty) \times \overline{\Omega}) \) be a smooth function in a smooth bounded domain \( \Omega \subset \mathbb{R}^n \) satisfying equation (19), with \( J \in \Lambda^1 T\Omega, J^0 J = C^\infty (\overline{\Omega}), H_0 \in C^\infty ([0, +\infty) \times \overline{\Omega}) \) the Hamiltonian function, and \( \gamma = \beta D/2 \), and \( \kappa = \beta^{-1} \) positive real constants. Define \( X_0 = J^0 H_0 \) and \( \xi = J^2 \phi \). Let \( Z \) be the Fokker–Planck velocity of (20) and \( N \) the unit outward normal to \( \partial \Omega \). Suppose that \( \xi = -J d\phi + k, \ \phi \in C^\infty (\Omega), \ k \in \ker J, \ f > 0 \) and \( \nabla \phi \cdot (X_0 + \kappa \xi) = 0 \) in \( \Omega \), \( \langle \partial \phi, H_0 \rangle = 0 \), and \( Z \cdot N = 0 \) and \( (X_0 + \kappa \xi) \cdot N = 0 \) on \( \partial \Omega \). Then,

\[ \frac{d\Sigma_\phi}{dt} \geq 0, \]  
(56)

with \( \Sigma_\phi = -\int_{\Omega} (f \log f + \phi) \ dV \). Furthermore

\[ \lim_{t \to \infty} J (d \log f + \beta dH_0 + d\phi) = 0. \]  
(57)
Proof. The rate of change of the entropy $\Sigma_{\phi}$ can be evaluated in a similar way to theorem 3.1. The result is

\[
\frac{d\Sigma_{\phi}}{dt} = \int_{\Omega} \mathcal{J} (d \log f + \beta dH_0 + d\phi) \cdot dV - \int_{\Omega} f \nabla \cdot (X_0 + \kappa \mathbf{\xi}) \ dV. 
\] (58)

Using the hypothesis $\nabla \phi \cdot (X_0 + \kappa \mathbf{\xi}) = 0$, it follows that

\[
\frac{d\Sigma_{\phi}}{dt} = \int_{\Omega} f \mathcal{J} (d \log f + \beta dH_0 + d\phi) \cdot dV \geq 0. 
\] (59)

Equation (57) follows by considering the limit $t \to \infty$ of the integrand. □

When $u = -\nabla \phi$ and $k = 0$ the vector field $\mathbf{\xi}$ is given by $\mathbf{\xi} = -\mathcal{J} \phi$. Then, one has

\[
\frac{\partial}{\partial x^i} (e^\phi J_{ij}) = e^\phi (-J_{ji} \phi_i + J_{ij} \phi_i) = 0. 
\] (60)

Recalling the definition of measure preserving operator, equation (11), we see that such $\mathcal{J}$ is measure preserving with respect to the volume form $e^\phi dx^1 \wedge ... \wedge dx^n$. This observation characterizes the class of non-Beltrami operators for which theorem 5.1 applies: any such operator departs from a measure preserving one (defined by $u = -\nabla \phi$ and $k = 0$) only up to an element of the kernel $k \in \ker \mathcal{J}$.

6. Charged particle diffusion in a magnetized plasma

The purpose of this section is to show that the assumption $\kappa = \beta^{-1}$ in theorem 3.1 is satisfied in the physical scenario of the diffusion of electrons in an electromagnetic field. Under the conditions to be described below, the antisymmetric operator $\mathcal{J}$ associated to this system is defined in $\Omega \subset \mathbb{R}^3$.

In $\mathbb{R}^3$ the action of $\mathcal{J}$ on a function $H$ can be represented as a cross product:

\[
\mathcal{J} (dH) = w \times \nabla H. 
\] (61)

Here $w \in C^\infty (\Omega)$ is the smooth vector field encountered in section 2. When written in terms of $w$, the Fokker–Planck equation (19) takes the form:

\[
\frac{\partial f}{\partial t} = \nabla \left\{ w \times \left[ (-\nabla H_0 + \gamma \nabla H_0 \times w) f + \frac{1}{2} D \nabla \times (wf) \right] - \kappa f \nabla \times w \right\}. 
\]

Observe that the Fokker–Planck velocity $Z$ associated to (62) is

\[
Z = w \times \nabla H_0 + \gamma w \times (w \times \nabla H_0) - \frac{1}{2f} D w \times [\nabla \times (wf)] + \kappa \nabla \times w. 
\] (63)

The first three terms represent single particle dynamics, friction, and diffusion respectively. To understand the physical meaning of the fourth term, $\kappa \nabla \times w$, we consider an ensemble of electrons, each with mass $m$ and charge $e$. In the absence of interaction, and denoting $v = X$, the equation of motion of an electron is

\[
m \frac{dv}{dt} = e (v \times B + E). 
\] (64)
Here, \( \mathbf{B} \neq 0 \) is the time-independent background magnetic field, and \( \mathbf{E} = -\nabla \phi \) the electric field with electrostatic potential \( \phi \). Equation (64) is a 6D canonical Hamiltonian system. Under the assumption of a small electron mass, \( m \ll 1 \), equation (64) can be reduced to the force balance equation
\[
0 = \mathbf{v} \times \mathbf{B} + \mathbf{E}.
\] (65)

Next, observe that from equation (65) \( \mathbf{B} \cdot \mathbf{E} = 0 \). Hence the particle is not accelerated along the magnetic field. It follows that the component of the velocity along the magnetic field \( v_\parallel = \mathbf{v} \cdot \mathbf{B} / B \) of a particle initially at rest will satisfy \( v_\parallel = 0 \) at all times. Then, \( \mathbf{v} = \mathbf{v}_\perp \), with \( \mathbf{v}_\perp \) the component of the velocity perpendicular to \( \mathbf{B} \). Under these conditions, equation (65) can be solved for \( \mathbf{v} \),
\[
\mathbf{v} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} = \frac{\mathbf{B} \times \nabla \phi}{B^2}.
\] (66)

The dynamics described by equation (66) is called \( \mathbf{E} \times \mathbf{B} \) drift [12]. Observe that \( H_0 = e\phi \) is a constant of motion representing conservation of energy (inertial terms do not appear in the particle energy due to the small mass assumption). With the identification \( \mathbf{w} = \mathbf{B} / eB^2 \), one can verify that equation (66) is not a Hamiltonian system whenever the magnetic field \( \mathbf{B} \) has finite helicity density. Indeed, the Jacobiator for the antisymmetric operator \( \mathbf{w} \) is
\[
\mathcal{G} = h^{123} \partial_1 \wedge \partial_2 \wedge \partial_3 = - (\mathbf{w} \cdot \nabla \times \mathbf{w}) \partial_1 \wedge \partial_2 \wedge \partial_3
\] (67)

Consider now the term \( \kappa \nabla \times \mathbf{w} \) when \( \kappa = \beta^{-1} = k_B T \), with \( k_B \) the Boltzmann constant and \( T \) the temperature. If \( \mathbf{w} = \mathbf{B} / eB^2 \), we have
\[
\kappa \nabla \times \mathbf{w} = \frac{k_B T}{e} \left( \frac{\nabla \times \mathbf{B}}{B^2} + \frac{2 \mathbf{B} \times \nabla B}{B^3} \right) = \frac{\mu_0 k_B T}{e} \mathbf{J} + \frac{2 k_B T}{e} \mathbf{B} \times \nabla B/B^3.
\] (68)

Here \( \mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B} \) is the electric current and \( \mu_0 \) the vacuum permeability. Let us show that the two terms on the right-hand side of equation (68) represent charged particle drifts predicted by guiding center theory. According to the guiding center equations, the drift \( \mathbf{v}_d \) experienced by a charged particle across the magnetic field is (see equation (1) of [22])
\[
\mathbf{v}_d = \mathbf{B} / eB^2 \times \left( \mu \nabla B + e \nabla \phi + m v_\parallel^2 \hat{b} \cdot \nabla \hat{b} \right).
\] (69)

Here, \( \mu \) is the magnetic moment of the charged particle and \( \hat{b} = \mathbf{B} / B \) is the unit vector along the magnetic field. The second term on the right-hand side is nothing but the \( \mathbf{E} \times \mathbf{B} \) equation of motion (66). After some manipulations, equation (69) can be rearranged as
\[
\mathbf{v}_d = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{m v_\parallel^2}{e} \frac{\mathbf{J}}{B^2} + \frac{\mu B + m v_\parallel^2}{e} \frac{\mathbf{B} \times \nabla B}{B^3}.
\] (70)

Next, recall that \( \mu B = \frac{2}{e} v_\parallel^2 = K_\perp \) is the kinetic energy of the cyclotron gyration around the magnetic field, while \( \frac{2}{e} v^2 = K_\parallel \) is the kinetic energy associated to motion along the magnetic field. Hence, (70) has the form
\[
\mathbf{v}_d = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{2 K_\perp}{e} \frac{\mathbf{J}}{B^2} + \frac{K_\parallel + 2 K_\parallel}{e} \frac{\mathbf{B} \times \nabla B}{B^3}.
\] (71)
Let $\mathcal{P}$ denote the distribution function of the electron plasma in 6D canonical phase space $(p, q)$. At thermal equilibrium, we expect $\mathcal{P}$ to be the Boltzmann distribution $\mathcal{P} \propto e^{-\beta H}$, with $H = \frac{q^2}{2} + e\phi$. For such distribution, denoting with $\langle \cdot \rangle$ ensemble averaging, the ensemble averaged kinetic energies are $\langle K_L \rangle = 2\langle K_B \rangle = k_B T = \beta^{-1}$. We thus recover equation (68). In physical terms, we have thus shown that $\beta^{-1}\nabla \times \mathbf{w}$ accounts for the tendency of the charged particle to follow the current $\mathbf{J}$, and for the drift caused by the inhomogeneity and curvature of the magnetic field (the last term on the right-hand side of equation (71) is the sum of the so-called curvature and gradient-B drifts [12]). Observe that these effects become negligible at low temperatures.

Finally, it is worth considering how equation (62) is reduced if $\mathbf{w}$ is a Beltrami field. Assume that $\mathbf{w} \neq 0$ in $\Omega$. Then, if $\mathbf{w}$ is Beltrami field, $\nabla \times \mathbf{w} = \mathbf{h}$ with $\mathbf{h} = (\mathbf{w} \cdot \nabla \times \mathbf{w})/\mathbf{w}^2$. Recalling that $\kappa = \beta^{-1} = D/2\gamma$, equation (62) can be written as

\[
\frac{\partial f}{\partial t} = -\frac{\hbar}{\beta} \nabla (\beta H_0 + \log f) \cdot \mathbf{w} + \nabla H_0 \cdot \mathbf{w} \times \nabla f + \frac{1}{2} D \nabla \cdot \{[\mathbf{w} \times \nabla (\beta H_0 + \log f) \times \mathbf{w}] f\}. \tag{72}
\]

### 7. Examples in three real dimensions

The purpose of the present section is to provide examples of Beltrami operators in domains $\Omega \subseteq \mathbb{R}^3$. In particular, we describe a rigorous method to construct three dimensional solenoidal Beltrami fields. We remark that the solenoidal Beltrami fields obtained by application of this method represent exact solutions of the constant density stationary ideal Euler equations with Beltrami fields. We remark that the solenoidal Beltrami fields obtained by application of this method represent exact solutions of the constant density stationary ideal Euler equations with pressure $P = -w^2/2$ and of the ideal MHD equilibrium equations at constant pressure.

Before discussing the examples, let us consider what we expect for the three dimensional equilibrium states in light of theorems 3.1 and 4.1. The consequences of the maximum entropy criterion become especially clear if we restrict our attention to stationary solutions of the purely diffusive form of (62), which is obtained by setting $H_0 = \kappa = 0$. This is because in such case it is sufficient to consider the conditions on the operator $\mathbf{w}$ that allow a flat equilibrium state $\nabla f = 0$. In the limit $t \to \infty$ we have $\partial f = 0$. Therefore:

\[
0 = \nabla \cdot \{\mathbf{w} \times [\nabla \times (\mathbf{w} f)]\}. \tag{73}
\]

Here we have written $f$ in place of $f^\infty$ to simplify the notation. This convention will be used in the rest of the paper.

The field force associated to $\mathbf{w}$ can be expressed in vector form as $\mathbf{b} = \mathbf{w} \times (\nabla \times \mathbf{w})$. Then the field charge reads $\mathfrak{B} = 4\nabla \cdot \mathbf{b}$. In terms of $\mathbf{b}$ and $\mathfrak{B}$ equation (73) becomes:

\[
0 = \frac{1}{4} [\mathfrak{B} f + \mathbf{b} \cdot \nabla f + \nabla \cdot [\mathbf{w} \times (\nabla f \times \mathbf{w})]]. \tag{74}
\]

From equation (74) it is clear that any weak Beltrami operator ($\mathfrak{B} = 0$) admits the stationary solution $\nabla f = 0$. This solution is also the outcome of the dynamical diffusion process for any nontrivial Beltrami operator, as follows from the proof of the H-theorem. Indeed if the operator is nontrivial, $\mathfrak{C} \neq 0$, implying $\mathbf{h} = \mathbf{w} \cdot \nabla \times \mathbf{w} \neq 0$. This means that the Frobenius integrability condition for the vector field $\mathbf{w}$ is violated, and there is no function $C$ such that $\mathbf{w} \times \nabla C = 0$. Hence the only solution to (32) is $\nabla f = 0$. Conversely, if $\mathfrak{B} \neq 0$ such solution is not admissible.

Let us now study some explicit examples of Beltrami operators in three dimensions.
71. Beltrami operators in $\mathbb{R}^3$

The classical Beltrami field in $\mathbb{R}^3$ has the form:

$$\mathbf{w} = \sin z \nabla x + \cos z \nabla y.$$  \hfill (75)

This vector field satisfies $\nabla \times \mathbf{w} = \mathbf{w}$, $\nabla \cdot \mathbf{w} = 0$, and $\mathbf{w}^2 = 1$. Let $\sigma = \sigma(z)$ be a smooth function of the variable $z$. We can slightly generalize equation (75) as

$$\mathbf{w} = \sin \sigma \nabla x + \cos \sigma \nabla y,$$  \hfill (76)

which satisfies $\nabla \times \mathbf{w} = \sigma \mathbf{w}$, $\nabla \cdot \mathbf{w} = 0$, and $\mathbf{w}^2 = 1$. Notice that the vector fields (75) and (76) are solenoidal.

More generally, suppose that $(\ell, \psi, \theta)$ is an orthogonal coordinate system such that $|\nabla \ell| = |\nabla \psi|$. Furthermore, suppose that the orientation of the coordinate system is such that $\nabla \ell \cdot \nabla \psi \times \nabla \theta / |\nabla \ell \cdot \nabla \psi \times \nabla \theta| = 1$. Consider the vector field:

$$\mathbf{w} = \cos u \nabla \psi + \sin u \nabla \ell,$$  \hfill (77)

where $u = u(\theta)$ is a smooth function of the variable $\theta$. We have:

$$\nabla \times \mathbf{w} = u_0 \nabla \ell \cdot \nabla \psi \times \nabla \theta \left( \sin u \frac{\nabla \ell}{|\nabla \ell|^2} + \cos u \frac{\nabla \psi}{|\nabla \psi|^2} \right) = u_0 |\nabla \theta| \mathbf{w}. $$  \hfill (78)

Hence, a vector field $\mathbf{w}$ constructed in this way is a Beltrami field. If $u_0 \neq 0$, this equation also implies that:

$$\mathbf{w} = u_0^{-1} |\nabla \theta|^{-1} \nabla \times \nabla (\ell \cos \theta - \psi \sin \theta).$$  \hfill (79)

If we interpret $\mathbf{w}$ as a flow, we see that it has two integral invariants $\theta$ and $\ell \cos \theta - \psi \sin \theta$.

For example, take $(\ell, \psi, \theta) = (\sqrt{\rho + z}, \sqrt{\rho - z}, \arctan(y/x))$ to be a parabolic coordinate system with $\rho^2 = x^2 + y^2 + z^2$. This coordinate system is orthogonal. Furthermore $|\nabla \ell|^2 = |\nabla \psi|^2 = 1/2 \rho$ and $|\nabla \theta| = 1/\sqrt{x^2 + y^2}$. Hence, the vector field:

$$\mathbf{w} = \cos \left[ \arctan \left( \frac{y}{x} \right) \right] \nabla \sqrt{\rho - z} + \sin \left[ \arctan \left( \frac{y}{x} \right) \right] \nabla \sqrt{\rho + z},$$  \hfill (80)

is a Beltrami field with proportionality factor $\hat{h} = 1/\sqrt{x^2 + y^2}$, i.e. $\nabla \times \mathbf{w} = \hat{h} \mathbf{w}$.

With the aid of the procedure described above, we can construct arbitrarily complex Beltrami fields by finding appropriate orthogonal coordinate systems. More precisely, suppose that we want a Beltrami field with a given proportionality coefficient $\hat{h}$ such that $\nabla \times \mathbf{w} = \hat{h} \mathbf{w}$. This can be accomplished by first solving the eikonal equation $|\nabla \theta| = |\hat{h}|$ and then by trying to determine the orthogonal coordinate system $(\ell, \psi, \theta)$ with respect to the obtained $\theta$ (it is essential that $|\nabla \ell| = |\nabla \psi|$). Notice that however, while the eikonal equation for the variable $\theta$ can be solved within the framework of the method of characteristics for first order partial differential equations, the existence of the coordinate system $(\ell, \psi, \theta)$ is not guaranteed in general.

As an example, take $\hat{h} = \exp(x + y)$. Applying the procedure described above, one can construct the Beltrami field:

$$\mathbf{w} = \frac{1}{\sqrt{2}} \cos \left[ \frac{\exp(x + y)}{\sqrt{2}} \right] \nabla (x - y) + \sin \left[ \frac{\exp(x + y)}{\sqrt{2}} \right] \nabla z.$$  \hfill (81)

Observe that this vector field is solenoidal. If one wants a Beltrami field with opposite proportionality factor $-\hat{h}$, it is sufficient to consider the dual vector field $\mathbf{w}^* = \sin \theta \nabla \psi + \cos \theta \nabla \ell$. 

15
In physical applications involving the Euler and MHD equations, one is often interested in solenoidal Beltrami fields. Therefore, the orthogonal coordinates \((\ell, \psi, \theta)\) must satisfy the additional requirement

\[
\nabla \cdot \mathbf{w} = \cos \theta \Delta \psi + \sin \theta \Delta \ell = 0. \tag{82}
\]

The following result gives a systematic method to obtain an orthogonal coordinate system \((\ell, \psi, \theta)\) such that the vector field \(\mathbf{w} = \cos \theta \nabla \psi + \sin \theta \nabla \ell\) is a solenoidal Beltrami field.

**Proposition 7.1.** Let \(D \subset \mathbb{R}^2\) be a smooth bounded simply connected domain. Then, there exist smooth functions \((\ell, \psi, \theta)\) with \(\ell, \psi \in C^\infty (D)\) and \(\theta \in C^\infty (\mathbb{R})\) such that the vector fields

\[
\mathbf{w} = \cos [\sigma (\theta)] \nabla \psi + \sin [\sigma (\theta)] \nabla \ell, \tag{83}
\]

with \(\sigma = \sigma (\theta)\) an arbitrary smooth function of the coordinate \(\theta\), are a family of smooth solenoidal Beltrami fields in \(\Omega = D \times \mathbb{R}\).

**Proof.** The variable \(\theta\) can be chosen to be one of the three Cartesian coordinates \((x, y, z)\). We set \(\theta = z\). With this choice, the remaining variables are functions of the coordinates \(x\) and \(y\), i.e. \(\ell = \ell (x, y)\) and \(\psi = \psi (x, y)\). Observe that \(\nabla \psi \cdot \nabla \ell = \nabla \psi \cdot \nabla \psi = 0\) in \(\Omega\). We define the function \(\ell\) to be a solution of Laplace’s equation

\[
\Delta \ell = 0 \text{ in } D. \tag{84}
\]

Laplace’s equation is elliptic and can be solved in terms of harmonic functions. The solution \(\ell\) is also smooth in the domain \(D\). On the other hand, since \(D\) is simply connected, the harmonic function \(\ell\) admits an harmonic conjugate \(\psi\) which satisfies the Cauchy–Riemann equations

\[
\ell_x = \psi_y, \quad \ell_y = -\psi_x. \tag{85}
\]

Observe that these equations imply

\[
|\nabla \ell| = |\nabla \psi|, \quad \nabla \ell \cdot \nabla \psi = 0 \text{ in } D. \tag{86}
\]

Hence, we have shown that the gradients of the functions \((\ell, \psi, \theta)\) are orthogonal to each other, the scale factors \(\nabla \ell\) and \(\nabla \psi\) are equal, and the functions \(\ell\) and \(\psi\) are harmonic. Therefore, from equations (78) and (82), we see that the vector fields (83) are solenoidal Beltrami fields. These Beltrami fields are nontrivial whenever \(|\nabla \ell| \neq 0\) and \(\sigma \theta \neq 0\).

The following is an example of solenoidal Beltrami field constructed in terms of harmonic conjugate functions.

\[
\mathbf{w} = \cos z \nabla (\cosh y \cos x) + \sin z \nabla (\sinh y \sin x). \tag{87}
\]

Here, the orthogonal coordinate system is \((\ell, \psi, \theta) = (\sinh y \sin x, \cosh y \cos x, z)\), with \(\ell\) and \(\psi\) harmonic conjugates. Observe that \(\nabla \cdot \mathbf{w} = 0\), \(h = 1\), and \(|\nabla \ell|^2 = |\nabla \psi|^2 = |\nabla \ell|^2 = |\nabla \psi|^2 = \cosh (2y) - \cos (2x)\)/2.

Finally, we conclude this section with an example of weak Beltrami operator, i.e. a vector field such that \(\mathbf{b} \neq 0\) but \(\mathcal{B} = 0\). Consider the vector field:

\[
\mathbf{w} = \sqrt{\gamma^2 - 2z^2} \nabla x + z \nabla y. \tag{88}
\]

This vector field satisfies \(\mathbf{b} = \frac{\sqrt{2}}{2} \nabla (y^2 - z^2) - \frac{2z}{\sqrt{\gamma^2 - 2z^2}} \nabla x\) and \(\mathcal{B} = 0\) as desired.
7.2. Non-Beltrami operators in $\mathbb{R}^3$

Suppose that we can find a smooth function $g \in C^\infty(\Omega)$, $g \neq 0$, such that the Beltrami condition is satisfied by the vector field $gw$, i.e.:

$$gw \times [\nabla \times (gw)] = 0.$$  

(89)

Then $g$ is a stationary solution to (73). Observe that now $b = -w \times (\nabla \log g \times w)$. Hence $w$ is not a Beltrami operator in $\mathbb{R}^3$. Condition (89) can be obtained by using the auxiliary volume element $\text{vol}_g = g \, dx \wedge dy \wedge dz$ while keeping the Hodge star on $\mathbb{R}^3$ in the definition of $b^{n-1}$, and by setting $b^{n-1} = 0$. It follows that the proper entropy measure for the diffusion process is $S = -\int_\Omega f \log \left( \frac{f}{g} \right) \, dV$. Then, by the H-theorem, $w \times \nabla (f / g) \to 0$ in the limit $t \to \infty$. If $G \neq 0$ this implies $f \to g$.

There is a class of vector fields that always satisfy equation (89) for some appropriate choice of the function $g$. Let $(\ell, \psi, \theta)$ be the orthogonal coordinate system introduced in the previous example. Let $u = u(\theta)$ be a smooth function of the variable $\theta$. Consider the vector fields:

$$w = \nabla \psi + u \nabla \ell.$$  

(90)

It can be verified that, by setting $g = 1/\sqrt{1 + u^2}$, condition (89) is satisfied. Indeed, if we define a new variable $\sigma = \arctan u$ we obtain $gw = \cos \sigma \nabla \psi + \sin \sigma \nabla \ell$, which has the same form of the Beltrami field (77).

A simple example of (90) is:

$$w = \nabla x + y \nabla z.$$  

(91)

This vector field satisfies $b = y \nabla y$ and $\mathfrak{B} = 1$. The outcome of the corresponding diffusion process is the equilibrium distribution function $f = 1/\sqrt{1 + y^2}$.

8. Examples in four real dimensions

Aim of the present section is to provide examples of Beltrami and non-Beltrami operators in $\mathbb{R}^4$. Systems with four dynamical variables may arise, for example, when the motion of a particle is subject to a pair of constraints that reduce the six-dimensional phase space to a four-dimensional manifold. An example is guiding-center motion [12], where the dynamical variables are $(x, y, z, v_\parallel)$, with $v_\parallel$ the particle velocity along the magnetic field, and the constraints are the conservation of magnetic moment $\mu$ and the redundancy of the cyclotron phase $\theta_c$.

8.1. Beltrami operators in $\mathbb{R}^4$

In $\mathbb{R}^4$, an antisymmetric operator $J$ cannot be represented in terms of a vector field $w$ as in the case of $\mathbb{R}^3$. Hence, we return to the general bivector representation, equation (1). Furthermore, in four dimensions we must distinguish two types of nontrivial Beltrami operators: the measure preserving class, satisfying $O^{n-1} = 0$ and $G \neq 0$, and the general Beltrami class, satisfying $O^{n-1} \neq 0$, $G \neq 0$, and $b^{n-1} = 0$. The measure preserving class does not arise in three dimensions because in such case $O^{n-1} = 0$ always implies $G = 0$, and vice versa by appropriate choice of the Jacobian $g$. Remember that the H-theorem derived in section 3 applies to Beltrami operators. Hence, statistical equilibrium will realize the generalized Boltzmann state, equation (32).
For $x \in \mathbb{R}^4$, we denote the fourth coordinate by $s$, so that $x = (x, y, z, s)$. Consider the following bivector field
\[ \mathcal{J} = z \partial_y \wedge \partial_z + y \partial_z \wedge \partial_x + x \partial_x \wedge \partial_y + c \partial_x \wedge \partial_s, \quad c \in \mathbb{R}. \] (92)

When $c = 0$, this operator reduces to the Poisson operator of the Euler rotation equations for a rigid body with $(x, y, z)$ as components of the angular momentum. Let us verify that (92) falls in the nontrivial measure preserving class when $c \neq 0$. We have
\[ \mathcal{G} = \frac{1}{2} \mathcal{J}^i_{jk} \mathcal{J}^j_{im} \partial_i \wedge \partial_j \wedge \partial_k \]
\[ = h^{123} \partial_z \wedge \partial_x \wedge \partial_y + h^{124} \partial_y \wedge \partial_z \wedge \partial_x + h^{134} \partial_z \wedge \partial_x \wedge \partial_y + h^{234} \partial_z \wedge \partial_y \wedge \partial_x + h^{234} \partial_x \wedge \partial_z \wedge \partial_y - c \partial_x \wedge \partial_z \wedge \partial_s \neq 0. \] (93)

In the last passage we used the fact that $h^{123} = h^{124} = h^{134} = 0$ and $h^{234} = -c$. Hence, (92) does not satisfy the Jacobi identity. Next, we evaluate the covorticity (n − 2) form with respect to the Euclidean metric of $\mathbb{R}^4$:
\[ \mathcal{J}^{n-2} = 2 \left( x \, ds \wedge dx + y \, ds \wedge dy + z \, ds \wedge dz + c \, dz \wedge dy \right) \]
\[ = ds \wedge d \left( x^2 + y^2 + z^2 \right) + 2c \, dz \wedge dy. \] (94)

It follows that the corresponding covorticity $n - 1$ form is
\[ \mathcal{O}^{n-1} = d \mathcal{J}^{n-2} = 0, \] (95)
which is the desired result.

Consider the bivector field
\[ \mathcal{J} = u \partial_x \wedge \partial_y + v \partial_y \wedge \partial_z, \] (96)
with $u = u(z, s)$ a smooth function of the coordinates $z$ and $s$, and $v = v(x, y)$ a smooth function of the coordinates $x$ and $y$. Observe that this operator reduces to the symplectic matrix of $\mathbb{R}^4$ for $u = v = 1$. Equation (96) is a nontrivial measure preserving operator when $\nabla (uv) \neq 0$. Indeed,
\[ \mathcal{G} = \frac{1}{2} \mathcal{J}^i_{jk} \mathcal{J}^j_{im} \partial_i \wedge \partial_j \wedge \partial_k \]
\[ = h^{123} \partial_z \wedge \partial_x \wedge \partial_y + h^{124} \partial_x \wedge \partial_z \wedge \partial_y + h^{134} \partial_x \wedge \partial_z \wedge \partial_y + h^{234} \partial_y \wedge \partial_x \wedge \partial_z + u \partial_y \wedge \partial_z \wedge \partial_x - u \partial_z \wedge \partial_y \wedge \partial_x \neq 0. \] (97)

In the last passage we used the fact that $h^{123} = v u_r, h^{124} = -v u_s, h^{134} = u v_y$, and $h^{234} = -u v_x$. Furthermore, the covorticity $n - 2$ form is
\[ \mathcal{J}^{n-2} = 2 \left( u \, dz \wedge dx + v \, dx \wedge dy \right). \] (98)

Hence, $\mathcal{J}$ is an exact form, and the covortient identically vanishes,
\[ \mathcal{O}^{n-1} = d \mathcal{J}^{n-2} = 0. \] (99)
Consider the bivector field
\[ \mathcal{F} = \cos \theta \partial_x \wedge \partial_t + \sin \theta \partial_x \wedge \partial_t + \sin \phi \partial_x \wedge \partial_t. \]
(100)
If \( \partial_x = 0 \), this operator corresponds to the Beltrami field \( w = \cos \theta \nabla y + \sin \theta \nabla x \) of \( \mathbb{R}^3 \). Let us verify that (100) is a nontrivial Beltrami operator in \( \mathbb{R}^4 \). The Jacobiator is
\[ \mathfrak{G} = \frac{1}{2} \mathcal{F}^{ij} \mathcal{F}^{jk} \partial_i \wedge \partial_j \wedge \partial_k \]
\[ = h^{123} \partial_x \wedge \partial_x \wedge \partial_z + h^{124} \partial_x \wedge \partial_y \wedge \partial_z + h^{134} \partial_x \wedge \partial_z \wedge \partial_z \]
\[ + h^{234} \partial_z \wedge \partial_z \wedge \partial_z - \sin x \sin z \partial_x \wedge \partial_z \wedge \partial_z - \sin x \cos z \partial_x \wedge \partial_x \wedge \partial_z \neq 0. \]
(101)
In the last passage, we used the fact that \( h^{123} = -1 \), \( h^{124} = 0 \), \( h^{134} = -\sin x \sin z \), and \( h^{234} = -\sin x \cos z \). The covorticity \( n - 1 \) form is
\[ \mathcal{J}^{n-2} = 2 (\sin z \, dx \wedge dy + \cos z \, dx \wedge dz + \sin x \, dy \wedge dx) \]
\[ = 2 [ds \wedge (\cos x \, dy \wedge dz) + d \cos x \wedge dy]. \]
(102)
The cocurrent \( n - 1 \) form is
\[ \mathcal{O}^{n-1} = d\mathcal{J}^{n-2} = 2 (\cos z \, dz \wedge dx - \sin z \, dz \wedge dy) \neq 0. \]
(103)
Hence, such operator is not measure preserving with respect to the Euclidean metric of \( \mathbb{R}^4 \). Finally, the field force \( n - 1 \) form is
\[ b^{n-1} = \mathcal{J}^{n-2} \wedge d\mathcal{J}^{n-2} = 2 \mathcal{J}^{n-2} \wedge (\cos z \, dy \wedge dz + \sin x \, dx) = 0, \]
(104)
which is the desired result. It is worth observing that the kernel of the operator (100) is spanned by the linearly independent covectors
\[ \theta = \tan z \, dx + dy, \quad \eta = \sin x \, dx - \cos z \, ds. \]
(105)
These covectors do not satisfy the Frobenius integrability conditions \( \theta \wedge \eta \wedge d\theta = 0 \) and \( \theta \wedge \eta \wedge d\eta = 0 \). Therefore, dynamics \( X \) is subject to the pair of nonholonomic constraints
\[ i_x \theta = \dot{x} \tan z + \dot{y} = 0, \quad i_x \eta = \dot{x} \sin x - \dot{z} \cos z = 0. \]
(106)
More generally, one can verify that the families of bivector fields in \( \mathbb{R}^4 \),
\[ \mathcal{J} = \epsilon_{ijkl} \left[ \sin (c \, x') \partial_i \wedge \partial_j + \cos (c \, x') \partial_k \wedge \partial_l \right] + u (x') \partial_i \wedge \partial_l \]
(107)
and
\[ \mathcal{J} = \epsilon_{ijkl} \left[ \cos (c \, x') \partial_i \wedge \partial_j - \sin (c \, x') \partial_k \wedge \partial_l \right] - u (x') \partial_i \wedge \partial_l \]
(108)
have a non-vanishing Jacobiator, \( \mathfrak{G} \neq 0 \), a non-vanishing cocurrent, \( \mathcal{O}^{n-1} \neq 0 \), and a vanishing field force, \( b^{n-1} = 0 \). Here \( u = u (x') \) is a smooth function of the coordinate \( x' \). Therefore, they represent families of nontrivial Beltrami operators in \( \mathbb{R}^4 \).
Finally, consider the following bivector field
\[ \mathcal{J} = c \partial_x \wedge \partial_t + \cos x \partial_x \wedge \partial_t + \sin s \partial_x \wedge \partial_t + \sin x \partial_x \wedge \partial_t, \quad c \in \mathbb{R}. \]
(109)
When \( c \neq 0 \), this operator is a nontrivial weak Beltrami operator with respect to the Euclidean metric of \( \mathbb{R}^4 \). Indeed, one can verify that
\[
\mathcal{B} = \partial_t \wedge \partial_y \wedge \partial_z - \sin x \sin s \partial_y \wedge \partial_z \partial_z - (c \cos x - \cos s \sin x) \partial_y \wedge \partial_z \partial_z,
\]
\[
\mathcal{J}^{n-2} = 2 [dy \wedge d\cos x + (\sin s \, dx - \cos s \, dy) \wedge dz + c \, dz \wedge ds],
\]
\[
\mathcal{O}^{n-1} = 2 (\sin s \, dy + \cos s \, dx) \wedge dz \wedge dx \neq 0,
\]
\[
\mathcal{B} = s \, dB^{n-1} = 0.
\]
As discussed in part 1 of theorem 4.1, for the weak Beltrami class the generalized Boltzmann state (32) is obtained when \( b \cdot \nabla H_0 = 0 \). When this condition is not satisfied, the results of the second part of theorem 4.1 apply. We also remark again that the violation of the Beltrami condition with respect to a given metric does not preclude the possibility of satisfying the same condition in a different coordinate system by appropriate choice of the Jacobian \( g \). In such case, the equilibrium state (32) will be obtained in the transformed metric.

### 8.2. Non-Beltrami operators in \( \mathbb{R}^4 \)

In this section we discuss a four dimensional operator that does not belong to the Beltrami class of \( \mathbb{R}^4 \). Consider the bivector field
\[
\mathcal{J} = c \, \partial_t \wedge \partial_y + (x + cz) \, \partial_y \wedge \partial_z, \quad c \in \mathbb{R}.
\]
When \( c = 0 \), this operator reduces to the Poisson operator of the Heisenberg algebra, i.e. the class A type II algebra of the Bianchi classification of three dimensional Lie–Poisson algebras [25]. With respect to the Euclidean metric of \( \mathbb{R}^4 \), one finds that
\[
\mathcal{B} = - c \, \partial_t \wedge \partial_y \wedge \partial_z,
\]
\[
\mathcal{J}^{n-2} = 2 [c \, dy \wedge dz + (x + cs) \, dx \wedge ds],
\]
\[
\mathcal{O}^{n-1} = -2c \, dx \wedge dz \wedge ds,
\]
\[
\mathcal{B} = 4c^2.
\]
Hence, this operator does not satisfy the Jacobi identity and does not belong to the Beltrami class when \( c \neq 0 \).

According to equation (19), the Fokker–Planck equation associated with (111) for the case of pure diffusion, \( \nabla H_0 = 0 \), is:
\[
\frac{\partial f}{\partial t} = \kappa \mathcal{J}^{ij} f_i + \frac{1}{8} D (\mathcal{B} f + 4b f_i) + \frac{1}{2} D \frac{\partial}{\partial x} (\mathcal{J}^{ik} J^k f_j)
\]
\[
= \kappa c f_y + \frac{1}{2} D (c^2 f + c (x + cz) f_z)
\]
\[
+ \frac{1}{2} D \left[ c^2 (f_x + f_y) + (x + cz)^2 (f_y + f_z) + 2c (x + cz) f_z \right].
\]
Set \( \nu = 2\kappa / D \). At equilibrium we thus have
\[
0 = c^2 f + \nu c f_y + 3c (x + cz) f_z + c^2 (f_x + f_y) + (x + cz)^2 (f_y + f_z).
\]
If $c = 0$, any function $f = f(x,s)$ of the coordinates $x$ and $s$ is a stationary solution. This is consistent with the fact that in such case both $x$ and $s$ become Casimir invariants of the operator (111). It is clear that the solution $\nabla f = 0$ is not admissible when $c \neq 0$. Stationary solutions for the case $c \neq 0$ can be obtained by solving the eigenvalue problem $f_{xx} + f_{ss} = -f$ with appropriate boundary conditions and by setting $f_x = f_s = 0$.

9. Concluding remarks

Beltrami operators are bivectors that satisfy the Beltrami condition. This class of operators allows a formulation of statistical mechanics even in the absence of canonical phase space, i.e. even if the underlying dynamics is not Hamiltonian. In this paper we studied the properties of Beltrami operators in the context of statistical mechanics of topologically constrained mechanical systems. We proved an H-theorem for Beltrami operators in $\mathbb{R}^n$, and obtained the equilibrium distribution function, a generalized Boltzmann distribution. Beyond Beltrami operators, homogeneous perturbations in a given reference system do not result in the flattening of the corresponding distribution function: by proving an H-theorem for a class of non-Beltrami operators, we found that entropy measure and equilibrium distribution function are distorted by a non-vanishing field charge. Examples of both Beltrami and non-Beltrami operators were given together with the resulting equilibrium distribution function for the case of pure diffusion processes in $\mathbb{R}^3$ and $\mathbb{R}^4$.

Acknowledgments

The research of NS was supported by JSPS KAKENHI Grant No. 18J01729. The author would like to acknowledge useful discussion with Professor Z Yoshida and Professor M Yamada.

ORCID iDs

N Sato ø https://orcid.org/0000-0002-2973-0635

References

[1] Sato N and Yoshida Z 2018 Phys. Rev. E 97 022145
[2] Moffatt H K 2014 Proc. Natl Acad. Sci. 111 10
[3] Enciso A and Peralta-Salas D 2016 Arch. Ration. Mech. Anal. 220 243–60
[4] Yoshida Z and Mahajan S M 2002 Phys. Rev. Lett. 88 9
[5] Mahajan S M and Yoshida Z Phys. Rev. Lett. 81 4863
[6] Littlejohn R 1982 AIP Conf. Proc. 88 47–66
[7] Morrison P J 1998 Rev. Mod. Phys. 70 467
[8] Caligari C E and Chandre C 2016 Chaos 26 053101
[9] Bloch A M, Marsden J E and Zenkov D V 2005 Not. AMS 52 320–9
[10] Bates L and Sniatycki J 1993 Rep. Math. Phys. 32 1
[11] van der Schaft A J and Maschke B M 1994 Rep. Math. Phys. 34 2
[12] Cary J R and Brizard A J 2009 Rev. Mod. Phys. 81 693
[13] Gardiner C W 1985 Handbook of Stochastic Methods (Berlin: Springer) pp 80–102
[14] Risken H 1989 The Fokker–Planck Equation (Berlin: Springer) p 63
[15] Gilbarg D and Trudinger N S 2001 Elliptic Partial Differential Equations of Second Order (Berlin: Springer) p 31
[16] Evans L C 2010 Partial Differential Equations (Providence, RI: American Mathematical Society) p 314
[17] Frankel T 2012 The Geometry of Physics, an Introduction (Cambridge: Cambridge University Press) pp 165–78
[18] de León M 1989 Methods of Differential Geometry in Analytical Mechanics (New York: Elsevier) pp 250–3
[19] Arnold V I 1989 Mathematical Methods of Classical Mechanics (New York: Springer) pp 230–2
[20] Yoshida Z and Mahajan S M 2014 Prog. Theor. Exp. Phys. 2014 073J01
[21] Sato N and Yoshida Z 2016 Phys. Rev. E 93 062140
[22] Boozer A H 1980 Phys. Fluids 23 5
[23] Gressman P T and Strain R M 2010 Proc. Natl Acad. Sci. 107 5744–9
[24] Di Perna R J and Lions P L 1989 Ann. Math. 130 321–66
[25] Yoshida Z, Tokieda T and Morrison P J 2017 Phys. Lett. A 381 2772–7