Algorithmic Fair Allocation of Indivisible Items: A Survey and New Questions

Haris Aziz¹, Bo Li², Hervé Moulin³ and Xiaowei Wu⁴
¹UNSW Sydney and Data61 CSIRO, Sydney, Australia
²Department of Computing, The Hong Kong Polytechnic University, Hong Kong, China
³University of Glasgow, Glasgow, UK and Higher School of Economics, St. Petersburg, Russia
⁴IOTSC, University of Macau, Macau, China
haziz@cse.unsw.edu.au, comp-bo.li@polyu.edu.hk, herve.moulin@glasgow.ac.uk, xiaoweiwu@um.edu.mo

Abstract
The theory of algorithmic fair allocation is within the center of multi-agent systems and economics in the last decade due to its industrial and social importance. At a high level, the problem is to assign a set of items that are either goods or chores to a set of agents so that every agent is happy with what she obtains. Particularly, in this survey, we focus on indivisible items, for which absolute fairness such as envy-freeness and proportionality cannot be guaranteed. One main theme in the recent research agenda is about designing algorithms that approximately achieve the fairness criteria. We aim at presenting a comprehensive survey of recent progresses through the prism of algorithms, highlighting the ways to relax fairness notions and common techniques to design algorithms, as well as the most interesting questions for future research.

1 Introduction
While fair allocation is an age-old problem and the widely known Divide-and-Choose algorithm can trace back to the Bible, modern research on fair allocation is regarded to be initiated by Steinhaus at a meeting of the Econometric Society in Washington D.C. in 1947 [Steinhaus, 1948]. Since then, a large body of works in economics and mathematics have been directed towards understanding the theory of allocating resources among agents in a fair manner [Moulin, 2003]. The recent focus on indivisible items is motivated, in part, by the applications that inherently entail allocation of items that cannot be fractionally allocated, such as assigning computational resources in a cloud computing environment and courses to teachers in a school. In the last decade, computer science offered a fresh and practical angle to the research agenda – algorithmic fair allocation. Besides designing algorithms, computer science has brought many more ideas, such as computational and communication complexity, and informational assumptions, which do not align with the main theme of the current survey. Interested readers can refer to the surveys by Walsh [2020] and Aziz [2020a] for detailed discussion. Fair allocation algorithms have been implemented in the real-world; for instance, Course Match is employed for course allocation at the Wharton School in the University of Pennsylvania, and the websites Spliddit (spliddit.org) and Fair Outcomes (fairoutcomes.com) provide online access to fair allocation algorithms.

Although this survey mainly focuses on indivisible items, the study of fair allocation was classically centered around allocating a divisible resource, which is also known as the cake-cutting problem [Brams and Taylor, 1996; Robertson and Webb, 1998]. Fairness is mostly captured by envy-freeness and proportionality in the literature. An envy-free allocation (which is also proportional) of a divisible cake always exists and can be found in bounded steps [Aziz and Mackenzie, 2016]. Moreover, a competitive equilibrium from equal incomes guarantees envy-freeness and Pareto optimality simultaneously [Varian, 1973]. A recent line of research extends the study to chores, such as the computation of envy-free allocations [Delgabiani et al., 2018] and competitive equilibria [Boodaghians et al., 2021; Chaudhury et al., 2021]. Unlike divisible items, when items are indivisible, absolutely fair allocations rarely exist. For example, when allocating a single item to two agents, every allocation is not envy-free or proportional. Accordingly, an extensively studied subject is to investigate the extent to which these fairness notions or their relaxations can be approximately satisfied.

There are several surveys highlighting different perspectives of fair allocation theory. Moulin [2018] reviewed the theory through the prism of economics. Aleksandrov and Walsh [2020] and Sukosompong [2021] respectively focused on the online and constrained settings. Lang and Rothe [2016], Walsh [2020] and Aziz [2020] reviewed the problem in the perspective of broad computer science. Instead, the angle of the current survey is algorithmic and the focus is particularly on the introduction of common techniques to design (approximation) algorithms. Moreover, we will discuss more sophisticated settings that were introduced in the last couple of years which uncovered new challenges and open problems in the field of fair allocation.

Roadmap In the remaining of the survey, we define the model of fair allocation in Section 2 and introduce the widely adopted solution concepts in Sections 3 and 4. In Section 5, we review the commonly used techniques to design fair allocation algorithms. In Section 6, we introduce the more sophisticated settings that are proposed recently. Finally, we discuss two more properties, efficiency and truthfulness that are desired to be satisfied together with fairness, in Section 7.
2 Model

In a fair allocation instance, we allocate a set of \( m \) indivisible items \( M = \{1, \ldots, m\} \) to a group of \( n \) agents \( N = \{1, \ldots, n\} \). An allocation is represented by an \( n \)-partition \( X = (X_1, \ldots, X_n) \) of \( M \), where \( X_i \subseteq M \) is the bundle allocated to agent \( i \). It is required that each item is allocated to exactly one agent, i.e., \( X_i \cap X_j = \emptyset \) for all \( i \neq j \) and \( \bigcup_{i \in N} X_i = M \). If \( \bigcup_{i \in N} X_i \neq M \), \( X \) is a partial allocation. We sometimes consider fractional allocations, denoted by \( x = (x_{ie})_{i \in N, e \in M} \), where \( 0 \leq x_{ie} \leq 1 \) denotes the fraction of item \( e \) allocated to agent \( i \), and \( \sum_{i \in N} x_{ie} = 1 \) for all \( e \in M \). Each agent \( i \) has a valuation function \( v_i : 2^M \to \mathbb{R} \) that assigns a value to each bundle of items. When \( v_i(S) > 0 \) for all \( i \in N \) and \( S \subseteq M \), the items are goods; when \( v_i(S) \leq 0 \) for all \( i \) and \( S \), the items are chores. For ease of exposition, we mainly discuss the case when the valuations are additive and leave the discussion on more general valuations to Section 6. That is, for any \( i \in N \) and \( S \subseteq M \), we have \( v_i(S) = \sum_{e \in S} v_i(e) \). When there is no confusion, we use \( v_i(e) \) to denote \( v_i(\{e\}) \). Further, for any \( S \subseteq M \) and \( e \in M \), we use \( S + e \) and \( S - e \) to denote \( S \cup \{e\} \) and \( S \setminus \{e\} \), respectively. Let \( I = (N, M, v) \) be a fair allocation instance where \( v = (v_1, \ldots, v_n) \). When all agents agree on the same ordering of all items in values (e.g., \( v_1(1) \geq \cdots \geq v_i(m) \) for all \( i \)), the instance is called identical ordering (IDO).

Before the extensive study of fairness, efficiency was at the centre of the theory of resource allocation. The utilitarian welfare of an allocation \( X \) is \( \sum_{i \in N} v_i(X_i) \), by maximizing which the total happiness of the agents is maximized. The egalitarian welfare is \( \min_{i \in N} \{v_i(X_i)\} \), by maximizing which the smallest happiness is maximized. A compromise between utilitarian and egalitarian welfare is Nash welfare, i.e., \( \Pi_{i \in N} v_i(X_i) \). We say an allocation \( X \) Pareto dominates another allocation \( X' \) if \( v_i(X_i) \geq v_i(X'_i) \) for all \( i \in N \) and \( v_i(X_i) > v_i(X'_i) \) for some \( i \). An allocation is Pareto optimal (PO) if it is not Pareto dominated by any other allocation.

Naturally, the fairness of an allocation can be evaluated by its egalitarian welfare, such Santa Claus problem [Bansal and Sviridenko, 2006] and load balancing problem [Lenstra et al., 1990]. However, in practice, since agents may have heterogeneous valuations, the max-min objective is not enough to satisfy all of them. Thus, various notions were proposed to characterize the fairness of allocations, including envy-freeness (EF) [Foley, 1966], proportionality (PROP) [Steinhaus, 1948] and equity (EQ) [Dubins and Spanier, 1961]. The relationships among these notions are discussed by Amanatidis et al. [2018], Sun et al. [2021] and Chakraborty et al. [2021]. Due to space limit, this survey only focuses on two of the most widely studied, namely EF and PROP.

3 Envy-freeness

We first consider envy-freeness, whose study dates back to Foley [1966] and Tinbergen [1930], and its relaxations.

**Definition 3.1 (EF)** For the allocation of items (goods or chores), an allocation \( X \) is envy-free (EF) if for any two agents \( i, j \in N \), we have \( v_i(X_i) \geq v_j(X_j) \).

The problem of checking whether a given instance admits an EF allocation is NP-complete even for \( \{0, 1\} \)- or \( \{0, -1\} \)-valued instances [Aziz et al., 2015; Bhaskar et al., 2021]. Moreover, the example of allocating a single item between two agents defies any bounded multiplicative approximation of EF, and thus researchers turn their attention to additive approximations. Two of the most popular ones are envy-free up to one item (EF1) and envy-free up to any item (EFX).

**EF1** The notion of EF1 was first studied for the allocation of goods by Lipton et al. [2004], which allows an agent to envy another agent but requires that the envy can be eliminated by removing an item from the envied agent’s bundle. This notion naturally extends to chores by removing an item from the envious agent’s bundle. For both goods and chores, EF1 allocations always exist and can be efficiently computed by the Round Robin algorithm; see Section 5.

**Definition 3.2 (\( \alpha \)-EF1)** For any \( \alpha \geq 0 \), an allocation \( X \) is \( \alpha \)-approximate envy-free up to one item (\( \alpha \)-EF1) if for any \( i, j \in N \), there exists \( e \in X_i \cup X_j \) such that \( v_i(X_i - e) \geq \alpha v_i(X_j - e) \). When \( \alpha = 1 \), the allocation is EF1.

**EFX** Same as EF1, the EFX relaxation was proposed for the allocation of goods, by Caragiannis et al. [2019b]. Informally speaking, the notion of EFX strengthens the fairness by requiring that the envy between two agents can be eliminated by removing any item owned by these two agents.

**Definition 3.3 (\( \alpha \)-EFX)** For any \( \alpha \geq 0 \), an allocation \( X \) for goods (resp. chores) is \( \alpha \)-approximate envy-free up to any item (\( \alpha \)-EFX) if for any \( i, j \in N \) and any \( e \in X_i \) (resp. \( e \in X_j \)), \( v_i(X_i) \geq \alpha v_i(X_j - e) \) (resp. \( v_i(X_i) \geq \alpha v_i(X_j) \)). When \( \alpha = 1 \), the allocation is EFX.

Unlike the case of EF1 allocations, the existence of EFX allocations remains unknown. For the case of goods, it was shown by Plaut and Roughgarden [2020] that EFX allocations exist in some special cases: (1) identical (combinatorial) valuations, (2) IDO additive valuations, and (3) \( n = 2 \). Chaudhury et al. [2020] and Amanatidis et al. [2021] further extended the existence of EFX allocations to the cases when \( n = 3 \) and \( 5 \) bi-valued valuations. While the existence of EFX allocations remains unknown for the general case, there are fruitful results regarding EFX partial allocations (where unallocated items are assumed to be donated to a charity) and approximation of EFX allocations. Since allocating nothing to the agents is trivially EFX, researchers are interested in finding EFX partial allocations with high efficiency. Caragiannis et al. [2019a] showed that there exists an EFX partial allocation achieving half of the maximum Nash welfare. Chaudhury et al. [2021] proposed a pseudo-polynomial time algorithm that computes an EFX partial allocation with at most \( n - 1 \) unallocated items under which no agent envies the charity. This result was improved by Berger et al. [2021], who showed that there is an EFX allocation with at most a single unallocated item for \( n = 4 \), and \( n - 2 \) unallocated items for \( n \geq 5 \). There are also results that aim at computing approximately EFX allocations. Plaut and Roughgarden [2020] showed that every instance (even with subadditive valuations) admits a 0.5-EFX allocation. The approximation ratio was improved to 0.618 under additive valuations by a polynomial time algorithm proposed by Amanatidis et al. [2020].
Chaudhury et al. [2021b] proposed a polynomial time algorithm that computes a \((1 - \epsilon)\)-EFX allocation with \(o(n)\) unallocated items and high Nash welfare. In contrast to the case of goods, the chores counterpart is much less well studied. EFX allocations are known to exist only for a few special cases, e.g., IDO instances [Li et al., 2022] and leveled preference instances [Gafni et al., 2021]. For general instances, only \(O(n^2)\) approximation of EFX is known to exist [Zhou and Wu, 2021]. The existence of EFX allocations remains unknown even for \(n = 3\) agents or bi-valued instance.

4 Proportionality and Maximin Share

Proportionality (PROP) was proposed by Steinhaus [1948], which is the most widely studied threshold-based solution concept. PROP is weaker than EF under additive valuations.

**Definition 4.1 (PROP)** An allocation \(X\) is proportional (PROP) if for every agent \(i \in N\), we have \(v_i(X_i) \geq \text{PROP}_i\), where \(\text{PROP}_i = (1/n) \cdot v_i(M)\).

For divisible goods and normalized valuations, the items can be allocated such that every agent has value at least \(1/n\), which is not true for indivisible items. Hill [1987] studied the worst case guarantee that an agent can have as a function of \(n\) and \(\max_{x \in N, e \in M} \{v_i(e)\}\). With two agents, the chores version is equivalent to the goods one; but with three or more agents, the equivalence is far from clear, and may not hold. One drawback of this guarantee is that the value of the function decreases quickly and goes to 0 as \(\max_{x \in N, e \in M} \{v_i(e)\}\) becomes large.

Maximin Share Fairness Besides the worst case guarantee studied by Hill [1987], one popular relaxation of PROP is the maximin share fairness, motivated by the following imaginary experiment. If agent \(i\) is the mediator and divides all items into \(n\) bundles, the best way to approximate PROP for \(i\) is to maximize the smallest bundle according to her own valuation. Formally, define the maximin share (MMS) of \(i\) as

\[
\text{MMS}_i(M, n) = \max_{X \in \Pi_n(M)} \min_{j} v_i(X_j),
\]

where \(\Pi_n(M)\) denotes the set of all \(n\)-partitions of \(M\). When \(M\) and \(n\) are clear from the context, we write \(\text{MMS}_i\) for short. Note that \(\text{MMS}_i \leq \text{PROP}_i\), and the computation of \(\text{MMS}_i\) is NP-complete.

**Definition 4.2 \((\alpha, \text{MMS})\)** For any \(\alpha \geq 0\), an allocation \(X\) is \(\alpha\)-approximate maximin share fair \((\alpha, \text{MMS})\) if for any \(i \in N\), \(v_i(X_i) \geq \alpha \cdot \text{MMS}_i\). When \(\alpha = 1\), the allocation is MMS.

Note that the approximation ratio \(\alpha \leq 1\) for goods and \(\alpha \geq 1\) for chores. The definition of MMS fairness was first introduced by Budish [2011], based on the concept of [Moulin, 1990]. Unfortunately, it is shown that there exist instances for which no allocation can ensure MMS value for every agent for the case of goods [Kurokawa et al., 2018] and chores [Aziz et al., 2017b]. The best known approximation results are \((3/4 + 1/(12n))\)-MMS for goods [Huang and Taki, 2021] and 11/9-MMS for chores [Huang and Lu, 2021]. The best known negative results are that \(\alpha \leq 39/40\) for goods and \(\alpha \geq 44/43\) for chores [Feige et al., 2021].

**More Solution Concepts** Motivated by the definition of MMS, Caragiannis et al. [2019b] proposed pairwise MMS (PMMS) and Barman et al. [2018a] proposed groupwise MMS (GMMS). Informally, PMMS is similar to MMS, but instead requires that the allocation is MMS for the instance induced by any two agents. GMMS generalizes both MMS and PMMS and requires that the allocation is MMS for the instance induced by any subset of agents. We refer the readers to, e.g., [Caragiannis et al., 2019b; Barman et al., 2018a; Amanatidis et al., 2020], for more detailed discussions.

Finally, similar to EF1 and EFX, we can relax PROP to PROP1 and PROPX. It is known that a PROP1 allocation always exists and can be found in polynomial time when the items are goods [Conitzer et al., 2017; Barman and Krishnamurthy, 2019], chores [Brânzei and Sandomirsky, 2019] or mixture of goods and chores [Aziz et al., 2020b]. Regarding PROPX, when items are goods, PROPX allocations may not exist [Moulin, 2018; Aziz et al., 2020b]. However, when items are chores, PROPX allocations exist and can be found efficiently [Moulin, 2018; Li et al., 2022].

5 Algorithms and Common Techniques

In this section, we introduce the techniques to design fair allocation algorithms. Unfortunately, due to space limit, we were not able to include all of them. Instead, we choose some of the most commonly used and powerful ones that are also the basis of more complicated algorithms.

**Divide-and-Choose** Divide-and-Choose is one of the most classic allocation algorithms. The algorithm is very useful and intuitive when there are only two agents. The idea is to let the first agent partition the items into two bundles and the other agent choose her preferred bundle. The remaining bundle is allocated to the first agent, and thus her best strategy is to maximize the value of the smaller bundle, i.e.,

\[
(X_1, X_2) \in \arg\max_{(S_1, S_2) \in \Pi_2(M)} \min\{v_1(S_1), v_1(S_2)\}.
\]

Plaut and Roughgarden [2020] proved that \((X_1, X_2)\) is always MMS and EFX to the first agent if we break tie by maximizing the size of the smaller bundle in \((X_1, X_2)\), i.e., leximin+ allocation. Actually, this result holds for any number of agents which implies the existence of EFX allocations for the case of identical valuations. Since the second agent obtains her preferred bundle in \((X_1, X_2)\), the allocation is EF to her. Therefore, with two agents, Divide-and-Choose algorithm returns an allocation that is MMS and EFX.

**Adjusted-Winner** Adjusted-Winner is another widely used algorithm for the two-agent case [Brams and Taylor, 1996]. The idea is to sort the items according to the ratios between the utilities that they yield for the two agents, i.e.,

\[
\frac{v_1(1)}{v_2(1)} \geq \frac{v_1(2)}{v_2(2)} \geq \cdots \geq \frac{v_1(m)}{v_2(m)},
\]

and let agent 1 choose a minimal set of consecutive items for which she is EF1 starting from left (the remainings are given to agent 2). The advantage is that it ensures high social welfare between two agents [Bei et al., 2021c]. This allocation is EF1 but not necessarily MMS or EFX.
Sequential Allocation and Round-Robin A general class of algorithms that is also suitable for a distributed implementation is that of sequential-picking allocation [Brams and Taylor, 2000], which was formally studied in a general and systematic way by Bouveret and Lang [2011]. Under these methods, agents have a sequence of turns to pick their most preferred item that is still available. A popular sequence protocol is the Round-Robin, where the picking sequence repeats the pattern 1, . . . , n. The Round-Robin algorithm produces allocations that are EFX (but not necessarily EFX) for both goods and chores; however, not for mixture of them. For this, Aziz et al. [2020b] proposed the double Round-Robin method that computes EFX allocations for mixture of goods and chores. Amanatidis et al. [2016] and Aziz et al. [2020a] designed more involved picking sequences to approximate MMS fairness, for goods and chores respectively.

Envy-cycle Elimination The Envy-cycle Elimination algorithm is inherently a greedy algorithm that in each round a new item is assigned to the agent who is in a disadvantage for goods or advantage for chores [Lipton et al., 2004]. The main technique of the algorithm is to ensure the existence of an agent that is not envied (for goods) or not envious (for chores) by trading items among agents. We use goods as an illustration. The algorithm is based on an envy graph, where the nodes correspond to agents and there is an edge from agent i to agent j if i is envious of j’s bundle. The algorithm works by assigning, at each step, an unassigned item to an agent who is not envied by any other agents, i.e., a node with in-degree 0 in the envy graph. If no such agent exists, the graph must contain a directed cycle. Then the cycle can be resolved by exchanging the bundles of items along the cycle, i.e., an agent in the cycle gets the bundle of the agent she points to. The algorithm terminates when all items are allocated and outputs an EFX allocation for arbitrary monotone combinatorial valuation functions [Lipton et al., 2004].

The algorithm and its adaptations are very widely studied, combining with which stronger fairness notions can also be satisfied. For example, the algorithm itself ensures EFX [Plaut and Roughgarden, 2020] and 2/3-MMS [Barman and Krishnamurthy, 2020] for IDO instances. With more involved preprocessing procedures, it can ensure 0.618-EFX, 0.553-GMMS, 0.667-PMMS and EFX simultaneously [Amanatidis et al., 2020]. For chores, it is shown in [Barman and Krishnamurthy, 2020] that the returned allocation is 4/3-MMS. However, noted by Bhaskar et al. [2021], this allocation may not be EFX if the cycle is resolved arbitrarily. Instead, they used the top-trading technique (in which each agent only points to the agent she envies the most) to preserve EFX. Later, Li et al. [2022] further showed that with this technique, the returned allocation is PROPX. We can also observe the shadow of the algorithm in more complicated techniques, such as the (group) champion graphs and rainbow cycle number [Chaudhury et al., 2020; Chaudhury et al., 2021b] which enable stronger existence and approximation results of EFX.

Bag-filling Algorithms Bag-filling Algorithms are particularly helpful for the threshold-based fairness like MMS. The idea is to maintain a bag and keep adding items to it until some agent thinks the bag is good enough (for goods) or about to be too bad to all agents (for chores). Then the bag is taken away by some satisfied agent and the algorithm repeats the procedure with the remaining items. The difficulty is to select a proper threshold for the bag so that the approximation for the agent who takes away a bag is good and there remains sufficiently many (or few) items for the remaining agents. With a more careful design and analysis, the approximation ratio can be improved to 2/3 [Garg et al., 2019] and further better than 3/4 [Garg and Taki, 2021] for goods. For chores the approximation ratio can be improved to 11/9 [Huang and Lu, 2021]. There are several nice properties regarding MMS fairness [Amanatidis et al., 2017b; Garg et al., 2019], e.g., the scale invariant and the reduction to IDO instances. Interestingly, the second property shows that any algorithm for approximating MMS allocations for IDO instances applies to general instances with the same approximation ratio preserved. Making use of these properties the design of algorithms can be significantly simplified.

Rounding Fractional Solutions Although competitive equilibria may not exist for indivisible items, we can first compute a market equilibrium by assuming the items being divisible and then carefully round the fractional allocation to an integral one [Barman and Krishnamurthy, 2019; Brânzei and Sandomirsky, 2019; Garg et al., 2021a]. This approach is especially helpful when efficiency is desired along with fairness, e.g., for the computation of EFX+PO or PROPX+PO allocations for goods [Barman et al., 2018b; Barman and Krishnamurthy, 2019], EFX+PO allocations for bi-valued chores [Barman et al., 2019; Ebadian et al., 2021], PROPX+PO [Aziz et al., 2020b] and approximately MMS+PO allocations for mixture of goods and chores [Kulkarni et al., 2021].

Eating Algorithms The Probabilistic-Serial (PS) algorithm of [Bogomolnaia and Moulin, 2001] is a randomized algorithm for allocating indivisible items in an ex-ante EF manner. Agents eat their most preferred items at a uniform rate and move on to the next item when the previous one is consumed. The probability share of an agent for an item is the fraction of the item eaten by the agent. In recent works [Freeman et al., 2020; Aziz, 2020b], researchers have sought allocation algorithms that simultaneously satisfy ex-ante EF and ex-post EF1 for the allocation of indivisible items that are goods or chores. In particular, the PS-lottery method was proposed that provides an explicit lottery over a set of EF1 allocations. Aziz and Brandl [2020] presented an eating algorithm that is suitable for any type of feasibility constraint and allocation problem with ordinal preferences.

6 More Sophisticated Settings

The past several years have witnessed the emergence of more sophisticated settings that brought new challenges to the design of fair allocation algorithms, including mixture of goods and chores, weighted agents, partial information and general
valuations. In the following, we review their models, as well as the corresponding results and open problems.

**Mixture of Goods and Chores** The general case when items are mixture of goods and chores has recently been studied in [Bogomolnaia et al., 2017; Bogomolnaia et al., 2019; Aziz et al., 2020b; Aziz et al., 2022]. This model is particularly interesting because it includes the typical setting when the valuations are not monotone. Aziz et al. [2022] proved that a double Round Robin algorithm is able to compute an EF1 allocation for any number of agents, and a generalized adjusted winner algorithm can find an EF1 and PO allocation for two agents. A natural open question is whether PO and EF1 allocations exist for arbitrary number of agents. Recently, Aziz et al. [2020b] and Kulkarni et al. [2021] designed algorithms that compute PROPI+PO or approximately MMS+PO allocations, respectively. More generally, it is an intriguing future research direction to study the fair allocation problem under arbitrary non-monotonic valuations.

**Asymmetric Agents** For most of the aforementioned research works, the agents are assumed to be symmetric in the sense of taking the same share in the system. Motivated by the real-world scenarios where people at leadership positions take more responsibilities, some recent works took the asymmetry into consideration and studied the fair treatment of non-equals. The definitions of envy-freeness and maximin share fairness have been adapted to the weighted settings by Farhadi et al. [2019], Aziz et al. [2019a] and Chakraborty et al. [2020]. Regarding goods, it is shown by Farhadi et al. [2019] that the best approximation ratio for weighted MMS is $\Theta(n)$ and by Chakraborty et al. [2020] that weighted EF1 allocations exist. Regarding chores, although weighted MMS was studied in [Aziz et al., 2019a], the best approximation ratio and the existence of weighted EF1 allocations are still unknown. Novel fairness notions, such as AnyPrice share and l-out-of-d maximin share, were proposed and studied in [Babaioff et al., 2021a; Babaioff et al., 2021b] which highlight different perspectives of the weighted setting.

**With Monetary Transfers** Since fairness notion like envy-freeness cannot be satisfied exactly, there are works studying how to use payments or subsidies to compensate agents and achieve fairness accordingly [Halpern and Shah, 2019; Brustle et al., 2020]. The problem has been extensively considered in the economics literature under the context of rent division problem [Edward Su, 1999]. Halpern and Shah [2019] aimed at bounding the amount of external subsidies when the marginal value of each item is at most one for every agent, and Brustle et al. [2020] proved that at most one unit of subsidies per agent is sufficient to guarantee the existence of an envy-free allocation [Brustle et al., 2020]. Caragiannis and Ioannidis [2021] studied the optimization problem of computing allocations that are envy-freeable using minimum amount of subsidies, and designed a fully polynomial time approximation scheme for instances with a constant number of agents. A more general problem is the fair allocation of mixture of divisible and indivisible items, where the divisible item can be viewed as heterogeneous subsidies [Bei et al., 2021a; Bei et al., 2021b].

**Partial Information** Researchers also care about fair allocation with partial information, and particularly the ordinal preference setting, where the algorithm only knows each agent’s ranking over all items without the cardinal values. For goods, the best possible approximation ratio of MMS allocations using only ordinal preferences is $\Omega(\log n)$ by Amanatidis et al. [2016] and Halpern and Shah [2021] for chores, constant upper and lower bounds are proved by Aziz et al. [2020a]. Recently, Hosseini et al. [2021a] proposed the ordinal MMS fairness, which is more robust to cardinal values. Another interesting question is to investigate the query complexity of unknown valuations. In this model the algorithms can access the valuations by making queries to an oracle. Oh et al. [2021] proved that $\Theta(\log m)$ queries suffices to define an algorithm that returns EF1 allocations. In general, it is an important research direction to explore how much knowledge is sufficient to design a fair allocation algorithm.

**General Valuations** Besides additive valuations, we may have more complex and combinatorial preferences that involve substitutabilities and complementarities in the items, including submodular, XOS, and subadditive valuations. Formal definitions and discussions of these valuations can be found in [Nisan, 2000]. Some of the results we have discussed in previous sections also apply to general valuations. For example, the envy-cycle elimination algorithm returns an EF1 allocation for monotone combinatorial valuations [Lipton et al., 2004]. Plaut and Roughgarden [2020] proved the existence of 0.5-EFX allocations for subadditive valuations. Regarding MMS, Barman and Krishnamurthy [2020] and Ghodsi et al. [2018] designed polynomial time algorithms to compute constant-approximate allocations for submodular and XOS valuations, and $O(\log n)$-approximate for subadditive valuations. Chaudhury et al. [2021c] further designed algorithms to compute allocations that are approximately EFX and simultaneously achieve $O(n)$-approximation to the maximum Nash welfare for subadditive valuations.

Besides the above settings, there are more in the literature, such as constrained resources [Sukamongkolkij et al., 2021], public resources [Conitzer et al., 2017; Fain et al., 2018], group fairness [Sukamongkolkij, 2018; Conitzer et al., 2019], and dynamic settings [Aleksandrov and Walsh, 2020]. Due to space limit, we were not able to discuss all of them in details.

7 Beyond Fairness: Efficiency and Incentives

Beyond achieving fairness alone, more and more attention is paid to investigating the extent to which we can design algorithms to compute allocations that are fair and simultaneously satisfy other properties, such as efficiency and truthfulness.

7.1 Computing Fair and Efficient Allocations

Although finding an allocation that maximizes the utilitarian welfare is straightforward (by allocating each item to the agent who has highest value), finding such an allocation within fair allocations is NP-hard [Barman et al., 2019]. One interesting research question here is to bound the utilitarian welfare loss by enforcing the allocations to be fair, i.e., the price of fairness [Bei et al., 2021c;
Barman et al., 2020. Besides utilitarian welfare, a large body of works studied the compatibility between fairness notions and the weaker efficiency notion of PO. For the case of goods, Caragiannis et al. [2019b] proved that the allocation that maximizes Nash welfare is EF1 and PO. Later, Barman et al. [2018b] designed a pseudopolynomial time algorithm for computing EF1+PO allocations. Truly polynomial time algorithms for the problem remain unknown. Barman and Krishnamurthy [2019] designed a polynomial time algorithm for computing PROPI+PO allocations. Regarding the stronger fairness notion of EFX, Amanatidis et al. [2021a] proved that for bi-valued valuations, the allocation that maximizes Nash welfare is EFX+PO, and Garg and Murhekar [2021] improved this result by giving a polynomial time algorithm. Further, Garg and Murhekar [2021] proved that if the valuations have three different values, EFX+PO allocations may not exist. In contrast, Hosseini et al. [2021b] proved that if the valuations are lexicographic, EFX+PO allocations exist and can be found in polynomial time.

For chores, most of the problems are still open. The good news is that PROPI+PO allocations can be computed in polynomial time, even if the items are mixture of goods and chores [Aziz et al., 2020b]. However, for EF1 and PROPX, their compatibility with PO are still unknown. Ebadian et al. [2021] and Garg et al. [2021b] proved that for bi-valued instances, EF1+PO allocations always exist and can be found efficiently, which are the only exceptions so far.

7.2 Being Fair for Strategic Agents

Fair allocation problems are often faced by strategic agents in real-life scenarios, where an agent may intentionally misreport her values for the items to manipulate the outcome of the algorithm and obtain a bundle of higher value. The goal is to design truthful algorithms in which agents maximize their utilities by reporting true preferences. For two agents, Amanatidis et al. [2017a] gave a complete characterization of truthful algorithms, using which we have the tight approximation bounds for solution concepts such as EF1 and MMS. For an arbitrary number of agents, Amanatidis et al. [2016] and Aziz et al. [2019b] designed truthful approximation algorithms but the tight bounds are still unknown. The aforementioned works consider the social environment where monetary transfers are not allowed. With monetary transfers, polynomial time truthful mechanisms were designed in Barman et al., 2019) for single-parameter valuations, which maximize the social welfare and approximately satisfy fairness notions such as MMS and EF1.

Another game-theoretic research agenda is to investigate the agents’ strategic behaviours in the algorithms that may not be truthful. For example, Amanatidis et al. [2021b] proved that in the Round-Robin algorithm, the allocations induced by pure Nash equilibria are always EF1 (regarding the true values). Bouveret and Lang [2014] and Aziz et al. [2017a] studied the strategic setting in general sequential allocation algorithms. It is interesting to study the agents’ behaviours in other algorithms and with other fairness notions.

References

[Aleksandrov and Walsh, 2020] Martin Aleksandrov and Toby Walsh. Online fair division: A survey. In AAAI, 2020.

[Amanatidis et al., 2016] Georgios Amanatidis, Georgios Birmpas, and Evangelos Markakis. On truthful mechanisms for maximin share allocations. In IJCAI, pages 31–37, 2016.

[Amanatidis et al., 2017a] Georgios Amanatidis, Georgios Birmpas, George Christodoulou, and Evangelos Markakis. Truthful allocation mechanisms without payments: Characterization and implications on fairness. In EC, pages 545–562, 2017.

[Amanatidis et al., 2017b] Georgios Amanatidis, Evangelos Markakis, Ashin Nikzad, and Amin Saberi. Approximation algorithms for computing maximin share allocations. ACM Trans. Algorithms, 13(4):52:1–52:28, 2017.

[Amanatidis et al., 2018] Georgios Amanatidis, Georgios Birmpas, and Vangelis Markakis. Comparing approximate relaxations of envy-freeness. In IJCAI, pages 42–48, 2018.

[Amanatidis et al., 2020] Georgios Amanatidis, Evangelos Markakis, and Apostolos Ntanos. Multiple birds with one stone: Beating 1/2 for EFX and GMMS via envy cycle elimination. Theor. Comput. Sci., 841:94–109, 2020.

[Amanatidis et al., 2021a] Georgios Amanatidis, Georgios Birmpas, Aris Filos-Ratsikas, Alexandros Hollender, and Alexandros A. Voudouris. Maximum nash welfare and other stories about EFX. Theor. Comput. Sci., 563:69–85, 2021.

[Amanatidis et al., 2021b] Georgios Amanatidis, Georgios Birmpas, Federico Fusco, Philip Lazos, Stefano Leonardi, and Rebecca Reiffenhaus. Allocating indivisible goods to strategic agents: Pure nash equilibria and fairness. In WINE, volume 13112, pages 149–166, 2021.

[Aziz and Brandl, 2020] Haris Aziz and Florian Brandl. The vignetting rule: A general approach for probabilistic economic design with constraints. CoRR, abs/2008.08991, 2020.

[Aziz and Mackenzie, 2016] Haris Aziz and Simon Mackenzie. A discrete and bounded envy-free cake cutting protocol for any number of agents. In FOCS, pages 416–427, 2016.

[Aziz et al., 2015] Haris Aziz, Serge Gaspers, Simon Mackenzie, and Toby Walsh. Fair assignment of indivisible objects under ordinal preferences. Artif. Intell., 227:71–92, 2015.

[Aziz et al., 2017a] Haris Aziz, Sylvain Bouveret, Jérôme Lang, and Simon Mackenzie. Complexity of manipulating sequential allocation. In AAAI, pages 328–334, 2017.

[Aziz et al., 2017b] Haris Aziz, Gerhard Rauchchecker, Guido Schreyn, and Toby Walsh. Algorithms for max-min share fair allocation of indivisible chores. In AAAI, pages 335–341, 2017.

[Aziz et al., 2019a] Haris Aziz, Hau Chan, and Bo Li. Weighted maxmin fair share allocation of indivisible chores. In IJCAI, pages 46–52, 2019.

[Aziz et al., 2019b] Haris Aziz, Bo Li, and Xiaowei Wu. Strategyproof and approximately maxmin fair share allocation of chores. In IJCAI, pages 60–66, 2019.

[Aziz et al., 2020a] Haris Aziz, Bo Li, and Xiaowei Wu. Approximate and strategyproof maximin share allocation of chores with ordinal preferences. CoRR, abs/2012.13884, 2020.

[Aziz et al., 2020b] Haris Aziz, Hervé Moulin, and Fedor Sandomirsky. A polynomial-time algorithm for computing a pareto optimal and almost proportional allocation. Oper. Res. Lett., 48(5):573–578, 2020.
Simultaneously achieving ex-ante and ex-post fairness. In WINE, volume 12495, pages 341–355, 2020.

Bansal and Sviridenko, 2006] Nikhil Bansal and Maxim Sviridenko. Competitive equilibrium with indivisible goods and generic budgets. Math. Oper. Res., 46(1):382–403, 2021.

Barman and Krishnamurthy, 2019] Siddharth Barman and Sanath Kumar Krishnamurthy. On the proximity of markets with integral equilibria. In AAAI, pages 1748–1755, 2019.

Barman and Krishnamurthy, 2020] Siddharth Barman and Sanath Kumar Krishnamurthy. Approximation algorithms for maximin fair division. ACM Trans. Economics and Comput., 8(1):5:1–5:28, 2020.

Barman et al., 2018a] Siddharth Barman, Arpita Biswas, Sanath Kumar Krishna Murthy, and Yadati Narahari. Groupwise maximin fair allocation of indivisible goods. In AAAI, 2018.

Barman et al., 2018b] Siddharth Barman, Sanath Kumar Krishnamurthy, and Rohit Vaish. Finding fair and efficient allocations. In EC, pages 557–574. ACM, 2018.

Barman et al., 2019] Siddharth Barman, Ganesh Ghalme, Shweta Jain, Pooja Kulkarni, and Shivika Narang. Fair division of indivisible goods among strategic agents. In AAMAS, 2019.

Barman et al., 2020] Siddharth Barman, Umang Bhaskar, and Nisarg Shah. Optimal bounds on the price of fairness for indivisible goods. In WINE, pages 356–369, 2020.

Bei et al., 2021a] Xiaohui Bei, Zihao Li, Jinyan Liu, Shengxin Liu, and Xinhang Lu. Fair division of mixed divisible and indivisible goods. Artif. Intell., 293:103436, 2021.

Bei et al., 2021b] Xiaohui Bei, Shengxin Liu, Xinhang Lu, and Hongao Wang. Maximin fairness with mixed divisible and indivisible goods. Auton. Agents Multi Agent Syst., 35(2):34, 2021.

Bei et al., 2021c] Xiaohui Bei, Xinhang Lu, Pasin Manurangsi, and Warut Suksompong. The price of fairness for indivisible goods. Theory Comput. Syst., 65(7):1069–1093, 2021.

Berger et al., 2021] Ben Berger, Avi Cohen, Michal Feldman, and Amos Fiat. (almost full) EXF exists for four agents (and beyond). CoRR, abs/2102.10654, 2021.

Bhaskar et al., 2021] Umang Bhaskar, A. R. Sricharan, and Rohit Vaish. On approximate envy-freeness for indivisible chores and mixed resources. In APPROX-RANDOM, 2021.

Bogomolnaia and Moulin, 2001] Anna Bogomolnaia and Hervé Moulin. A new solution to the random assignment problem. J. Econ. Theory, 100(2):295–328, 2001.

Bogomolnaia et al., 2017] Anna Bogomolnaia, Hervé Moulin, Fedor Sandomirskiy, and Elena Yanovskaya. Competitive Division of a Mixed Manna. Econométrica, 85(6):1847–1871, 2017.

Bogomolnaia et al., 2019] Anna Bogomolnaia, Hervé Moulin, Fedor Sandomirskiy, and Elena Yanovskaya. Dividing goods and bads under additive utilities. Soc Choice Welf, 52:395–417, 2019.

Boudaghians et al., 2021] Shant Boudaghians, Bhaskar Ray Chaudhury, and Ruta Mehta. Polynomial time algorithms to find an approximate competitive equilibrium for chores. CoRR, abs/2107.06649, 2021.

Bouvier et al., and Lang, 2018] Sylvain Bouvier et al. and Jérôme Lang. A general elicitation-free protocol for allocating indivisible goods. In IJCAI, pages 73–78, 2011.

Bouvier and Lang, 2014] Sylvain Bouvier and Jérôme Lang. Manipulating picking sequences. In EC, pages 141–146, 2014.

Brams and Taylor, 1996] Steven J. Brans and Alan D. Taylor. Fair division - from cake-cutting to dispute resolution. 1996.

Brans and Taylor, 2000] Steven J. Brans and Alan D. Taylor. The win-win solution - guaranteeing fair shares to everybody. 2000.

Brânzei and Sandomirskiy, 2019] Simina Brânzei and Fedor Sandomirskiy. Algorithms for competitive division of chores. CoRR, abs/1907.01766, 2019.

Brustle et al., 2020] Johannes Brustle, Jack Dippel, Vishnu V. Narayan, Mashbat Suzuki, and Adrian Vetta. One dollar each eliminates envy. In EC, pages 23–39, 2020.

Budish, 2011] Eric Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. Journal of Political Economy, 119(6):1061–1103, 2011.

Caragiannis and Ioannidis, 2021] Ioannis Caragiannis and Stavros Ioannidis. Computing envy-free allocations with limited subsidies. In WINE, volume 13112, pages 522–539, 2021.

Caragiannis et al., 2019a] Ioannis Caragiannis, Nick Gravin, and Xin Huang. Envy-freeness up to any item with high nash welfare: The virtue of donating items. In EC, pages 527–545, 2019.

Caragiannis et al., 2019b] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. The unreasonable fairness of maximum nash welfare. ACM Trans. Economics and Comput., 7(3):12:1–12:32, 2019.

Chakraborty et al., 2020] Mithun Chakraborty, Ayumi Igarashi, Warut Suksompong, and Yair Zick. Weighted envy-freeness in indivisible item allocation. In AAMAS, pages 231–239, 2020.

Chakraborty et al., 2021] Mithun Chakraborty, Erel Segal-Halevi, and Warut Suksompong. Weighted fairness notions for indivisible items revisited. CoRR, abs/2112.04166, 2021.

Chaudhury et al., 2020] Bhaskar Ray Chaudhury, Jugal Garg, and Kurt Mehlhorn. EXF exists for three agents. In EC, 2020.

Chaudhury et al., 2021a] Bhaskar Ray Chaudhury, Jugal Garg, Peter McGlaughlin, and Rutv Mehta. Competitive allocation of a mixed manna. In SODA, pages 1405–1424, 2021.

Chaudhury et al., 2021b] Bhaskar Ray Chaudhury, Jugal Garg, Kurt Mehlhorn, Ruta Mehta, and Pranabendu Misra. Improving EXF guarantees through rainbow cycle number. In EC, 2021.

Chaudhury et al., 2021c] Bhaskar Ray Chaudhury, Jugal Garg, and Ruta Mehta. Fair and efficient allocations under subadditive valuations. In AAAI, pages 5269–5276, 2021.

Chaudhury et al., 2021d] Bhaskar Ray Chaudhury, Telikepalli Kavitha, Kurt Mehlhorn, and Alkmini SGouritsa. A little charity guarantees almost envy-freeness. SIAM J. Comput., 2021.

Conitzer et al., 2017] Vincent Conitzer, Rupert Freeman, and Nisarg Shah. Fair public decision making. In EC, 2017.

Conitzer et al., 2019] Vincent Conitzer, Rupert Freeman, Nisarg Shah, and Jennifer Wortman Vaughan. Group fairness for the allocation of indivisible goods. In AAAI, pages 1853–1860, 2019.
[Dehghani et al., 2018] Sina Dehghani, Alireza Farhadi, Mohammad Taghi Hajiaghayi, and Hadi Yami. Envy-free chore division for an arbitrary number of agents. In SODA, 2018.

[Dubins and Spanier, 1961] Lester E Dubins and Edwin H Spanier. How to cut a cake fairly. The American Mathematical Monthly, 68(1P1):1–17, 1961.

[Ebadian et al., 2021] Soroush Ebadian, Dominik Peters, and Nisarg Shah. How to fairly allocate easy and difficult chores. CoRR, abs/2110.11285, 2021.

[Edward Su, 1999] Francis Edward Su. Rental harmony: Sperner’s lemma in fair division. The American mathematical monthly, 106(10):930–942, 1999.

[Fain et al., 2018] Brandon Fain, Kamesh Munagala, and Nisarg Shah. Fair allocation of indivisible public goods. In EC, 2018.

[Farhadi et al., 2019] Alireza Farhadi, Mohammad Ghodsi, Mohammad Taghi Hajiaghayi, Sébastien Lahaie, David M. Pennock, Masoud Seddighin, Saeed Seddighin, and Hadi Yami. Fair allocation of indivisible goods to asymmetric agents. J. Artif. Intell. Res., 64:1–20, 2019.

[Feige et al., 2021] Uriel Feige, Ariel Sapir, and Laliv Tauber. A tight negative example for MMS fair allocations. In WINE, pages 355–372, 2021.

[Foley, 1966] Duncan Karl Foley. Resource allocation and the public sector. Yale University, 1966.

[Freeman et al., 2020] Rupert Freeman, Nisarg Shah, and Rohit Vaish. Best of both worlds: Ex-ante and ex-post fairness in resource allocation. In EC, pages 21–22, 2020.

[Gafni et al., 2021] Yotam Gafni, Xin Huang, Ron Lavi, and Inbal Talgam-Cohen. Unified fair allocation of goods and chores via copies. CoRR, abs/2109.08671, 2021.

[Garg and Murhekar, 2021] Jugal Garg and Aniket Murhekar. Computing fair and efficient allocations with few utility values. In SAGT, volume 12885 of Lecture Notes in Computer Science, pages 345–359. Springer, 2021.

[Garg and Taki, 2021] Jugal Garg and Setareh Taki. An improved approximation algorithm for maximin shares. Artif. Intell., 300:103547, 2021.

[Garg et al., 2019] Jugal Garg, Peter McGlaughlin, and Setareh Taki. Approximating maximin share allocations. In SOA, volume 69, pages 20:1–20:11, 2019.

[Garg et al., 2021a] Jugal Garg, Martin Hoefer, Peter McGlaughlin, and Marco Schmalhofer. When dividing mixed manna is easier than dividing goods: Competitive equilibria with a constant number of chores. In SAGT, volume 12885, pages 329–344, 2021.

[Garg et al., 2021b] Jugal Garg, Aniket Murhekar, and John Qin. Fair and efficient allocations of chores under bivalued preferences. CoRR, abs/2110.09601, 2021.

[Ghodsi et al., 2018] Mohammad Ghodsi, Mohammad Taghi Hajiaghayi, Masoud Seddighin, Saeed Seddighin, and Hadi Yami. Fair allocation of indivisible goods: Improvements and generalizations. In EC, pages 539–556, 2018.

[Halpern and Shah, 2019] Daniel Halpern and Nisarg Shah. Fair division with subsidy. In SAGT, pages 374–389, 2019.

[Halpern and Shah, 2021] Daniel Halpern and Nisarg Shah. Fair and efficient resource allocation with partial information. In IJCAI, pages 224–230, 2021.

[Hill, 1987] Theodore P Hill. Partitioning general probability measures. The Annals of Probability, pages 804–813, 1987.

[Hosseini et al., 2021a] Hadi Hosseini, Sujoy Sikdar, Rohit Vaish, and Lirong Xia. Fair and efficient allocations under lexicographic preferences. In AAAI, pages 5472–5480, 2021.

[Hosseini et al., 2021b] Hadi Hosseini, Sujoy Sikdar, Rohit Vaish, and Lirong Xia. Fair and efficient allocations under lexicographic preferences. In AAAI, pages 5472–5480, 2021.

[Huang and Lu, 2021] Xin Huang and Pinyan Lu. An algorithmic framework for approximating maximin share allocation of chores. In EC, pages 630–631, 2021.

[Kulkarni et al., 2021] Rucha Kulkarni, Ruta Mehta, and Setareh Taki. Indivisible mixed manna: On the computability of MMS+PO allocations. In EC, pages 683–684, 2021.

[Kurokawa et al., 2018] David Kurokawa, Ariel D. Procaccia, and Junxing Wang. Fair enough: Guaranteeing approximate maximin shares. J. ACM, 65(2):8:1–8:27, 2018.

[Lang and Rothe, 2016] Jérôme Lang and Jörg Rothe. Fair division of indivisible goods. In Economics and Computation, pages 493–550, 2016.

[Lenstra et al., 1990] Jan Karel Lenstra, David B. Shmoys, and Éva Tardos. Approximation algorithms for scheduling unrelated parallel machines. Math. Program., 46:259–271, 1990.

[Li et al., 2022] Bo Li, Yingkai Li, and Xiaowei Wu. Almost proportional allocations for indivisible chores. TheWebConf, 2022.

[Lipton et al., 2004] Richard J. Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. On approximately fair allocations of indivisible goods. In EC, pages 125–131, 2004.

[Moulin, 1990] Hervé Moulin. Uniform externalities: Two axioms for fair allocation. J. Public Econ., 43(3):305–326, 1990.

[Moulin, 2003] Hervé Moulin. Fair division and collective welfare. MIT Press, 2003.

[Moulin, 2018] Hervé Moulin. Fair division in the age of internet. Ann. Rev. Econ., 2018.

[Nisan, 2000] Noam Nisan. Bidding and allocation in combinatorial auctions. In EC, pages 1–12, 2000.

[Oh et al., 2021] Hoon Oh, Ariel D. Procaccia, and Warut Suksompong. Fairly allocating many goods with few queries. SIAM J. Discret. Math., 35(2):788–813, 2021.

[Plaut and Roughgarden, 2020] Benjamin Plaut and Tim Roughgarden. Almost envy-freeness with general valuations. SIAM J. Discret. Math., 34(2):1039–1068, 2020.

[Robertson and Webb, 1998] Jack M. Robertson and William A. Webb. Cake-cutting algorithms - be fair if you can. 1998.

[Steinhaus, 1948] Hugo Steinhaus. The problem of fair division. Econometrika, 16:101–104, 1948.

[Suksompong, 2018] Warut Suksompong. Approximate maximin shares for groups of agents. Math. Soc. Sci., 92:40–47, 2018.

[Suksompong, 2021] Warut Suksompong. Constraints in fair division. ACM SIGecom Exchanges, 19(2):46–61, 2021.

[Sun et al., 2021] Ankang Sun, Bo Chen, and Xuan Vinh Doan. Connections between fairness criteria and efficiency for allocating indivisible chores. In AAMAS, pages 1281–1289, 2021.

[Tinbergen, 1930] Jan Tinbergen. Mathematische psychologie. Mens en Maatschappij, 6(4):342–352, 1930.

[Varian, 1973] Hal R Varian. Equity, envy, and efficiency. 1973.

[Walsh, 2020] Toby Walsh. Fair division: The computer scientist’s perspective. In IJCAI, pages 4966–4972, 2020.

[Zhou and Wu, 2021] Shengwei Zhou and Xiaowei Wu. Approximately EFX allocations for indivisible chores. CoRR, abs/2109.07313, 2021.