Lie algebraic noncommuting structures from reparametrisation symmetry

Sunandan Gangopadhyay *
S.N. Bose National Centre for Basic Sciences,
JD Block, Sector III, Salt Lake, Kolkata-700098, India

Abstract

We extend our earlier work of revealing both space-space and space-time noncommuting structures in various models in particle mechanics exhibiting reparametrisation symmetry. We show explicitly (in contrast to the earlier results in our paper [9]) that for some special choices of the reparametrisation parameter $\epsilon$, one can obtain space-space noncommuting structures which are Lie-algebraic in form even in the case of the relativistic free particle. The connection of these structures with the existing models in the literature is also briefly discussed. Further, there exists some values of $\epsilon$ for which the noncommutativity in the space-space sector can be made to vanish. As a matter of internal consistency of our approach, we also study the angular momentum algebra in details.

Keywords: Noncommutativity, Reparametrisation Symmetry

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1 Introduction

The interest in noncommutative theories has been motivated by the results in string theory. The early results in this subject [1], have been followed by a vast number of papers dealing with the problem of formulating a noncommutative (NC) quantum mechanics [2], [3], [4] and field theory. In this context, it has been observed that an important role is played by redefinitions or change of variables providing a map between the commutative and noncommutative structures [5], [6], [7]. In a recent paper [9], in contrast to the earlier approaches, we have shown that noncommuting structures can be obtained for models in particle mechanics with reparametrisation symmetry. In general, the associative algebraic structure $A_x$ which defines a noncommutative space can be defined in terms of a set of generators $x^i$ and relations $R$. Some important explicit cases are of the form of a canonical structure

$$\{x^i, x^j\} = \theta^{ij}; \quad \theta^{ij} \in \mathcal{C},$$

(1)

a Lie algebraic structure

$$\{x^i, x^j\} = C^{ij}_k x^k; \quad C^{ij}_k \in \mathcal{C},$$

(2)

and a quantum space structure

$$x^i x^j = q^{-1} R^{ij}_{kl} x^k x^l; \quad R^{ij}_{kl} \in \mathcal{C}.$$  

(3)

*sunandan@bose.res.in

1In all these cases the index $i$ representing the spatial coordinates takes values from 1 to $d$. 

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Gauge theories have been formulated on each of these NC spaces based on the notion of covariant coordinates and Seiberg-Witten (SW) maps have been established in all cases in [10]. Hence, a thorough understanding of the above types of NC spaces and their emergence is important in its own right, if not essential. Generation of NC phase-space with Lie algebraic forms of noncommutativity have also appeared in [11, 12, 13, 14, 15]. In [9], we exploited the reparametrisation symmetry of the problem to find a nonstandard gauge in which the space-time and space-space coordinates become noncommuting. There we presented a definite method of finding this gauge and also showed that the change of variables relating the nonstandard and standard gauges is a gauge transformation. The structures obtained were Lie-algebraic in the case of a nonrelativistic free particle, but not so in its relativistic counterpart. In this paper, we make use of the change of variables derived in our earlier paper [9] to show explicitly that for some special choice of the reparametrisation parameter, one can obtain noncommuting space-space structures falling in the Lie-algebraic category even in the relativistic case. We emphasize that these Lie-algebraic structures may be useful in giving explicit forms of the star products and SW maps (discussed in [10]) by reading off the structure constants of the algebra.

Moreover, there exists solutions of $\epsilon$ for which the noncommutativity between spatial coordinates vanish, but the space-time algebra still remains noncommutative. However, we do not get solutions for which the space-time algebra vanishes while the space-space algebra remains noncommuting. Finally, in order to show the internal consistency of our analysis, we study the angular momentum algebra in details.

2 Lie algebraic noncommuting structures : relativistic free particle

In this section, we first give a brief review of reparametrisation symmetry exhibited by the free relativistic particle [9]. The standard reparametrisation invariant action of a relativistic free particle which propagates in $d + 1$-dimensional “target spacetime” reads:

$$S_0 = -m \int d\tau \sqrt{-\dot{x}^2}. \quad (4)$$

The canonically conjugate momenta to $x^\mu$ are given by

$$p_\mu = \frac{m \dot{x}_\mu}{\sqrt{-\dot{x}^2}}. \quad (5)$$

These are subject to the Einstein constraint

$$\phi_1 = p^2 + m^2 \approx 0 \quad (6)$$

and satisfy the standard PB relations

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu; \quad \{x^\mu, x^{\nu}\} = \{p^\mu, p^\nu\} = 0. \quad (7)$$

Now using the reparametrisation symmetry of the problem (under which the action is invariant) and the fact that $x^\mu(\tau)$ transforms as a scalar under world-line reparametrisation, $x^\mu$ are the space-time coordinates, $\mu = 0, 1, \ldots, d$, the dot here denotes differentiation with respect to the evolution parameter $\tau$, and the Minkowski metric is $\eta = \text{diag}(-1, 1, \ldots, 1)$.

$^3$The generator of the above reparametrisation invariance is proportional to the Einstein constraint $\phi_1$. This has been discussed in [9] and we shall not elaborate on this aspect here.
\[ \tau \to \tau' = \tau'(\tau) \]
\[ x^\mu(\tau) \to x'^\mu(\tau') = x^\mu(\tau) \]  
leads to the following infinitesimal transformation of the space-time coordinate
\[ \delta x^\mu(\tau) = x'^\mu(\tau) - x^\mu(\tau) = \epsilon \frac{dx^\mu}{d\tau}. \]  
The simplest gauge condition to get rid of the gauge freedom generated by \( \phi_1(\tau) \), is obtained by identifying the time coordinate \( x^0 \) with the evolution parameter \( \tau \),
\[ \phi_2 = x^0 - \tau \approx 0. \]  
The constraints form a second class set with
\[ \{ \phi_a, \phi_b \} = 2p_0 \epsilon_{ab}; \quad (a, b = 1, 2). \]  
The resulting non-vanishing Dirac brackets (DB) are
\[ \{ x^i, p_0 \}_{DB} = \frac{p_i}{p_0} \quad \{ x^i, p_j \}_{DB} = \delta^i_j \]  
which imposes the constraints \( \phi_1 \) and \( \phi_2 \) strongly.
Using the transformations that relates the primed coordinates in terms of the unprimed coordinates can be written down in terms of phase-space variables as:
\[ x'^0 = x^0 + \epsilon; \quad x'^i = x^i - \epsilon \frac{p_i}{p_0} \]  
where we have used the relation \( \frac{dx^i}{d\tau} = -\frac{p_i}{p_0} \). With the above change of variables (derived from reparametrisation symmetry) at our disposal, enables us to choose some special values of the reparametrisation parameter \( \epsilon \) which leads to noncommuting structures falling in the Lie-algebraic category as we shall see subsequently.
Setting
\[ \epsilon = -\theta^0k p_k \frac{p_0}{m} \]  
and using \( \theta^i \) we obtain the following algebra between the primed coordinates:
\[ \{ x'^i, x'^j \}_{DB} = \frac{1}{m} \left( \theta^0j p^j - \theta^0i p^i \right) \]  
\[ \{ x'^0, x'^i \}_{DB} = \frac{1}{m} \left( \theta^0i p_0 + \theta^0k p_k \frac{p^i}{p_0} \right) \]  
\[ \{ x'^i, p^i_0 \}_{DB} = \frac{p^i}{p_0} \quad ; \quad \{ x'^i, p^j \}_{DB} = \delta^i_j \]

\[ \text{The Dirac brackets are defined as} \quad \{ A, B \}_{DB} = \{ A, B \} - \{ A, \phi_a \}(\phi^{-1})_{ab} \{ \phi_b, B \}, \quad \text{where} \ A, B \ \text{are any pair of phase-space variables.} \]
\[ \text{The parameter} \ \epsilon \ \text{of the reparametrisation transformation in (14) is not a Lorentz scalar (or four-vector). However, this problem is not serious and we shall present a detailed discussion of this issue later in the paper.} \]
\[ \text{Note that} \ p^i_0(\text{s}) \ \text{are gauge invariant objects as} \ \{ p^i_0, \phi \} = 0, \ \text{hence} \ p^i_0 = p^i_0. \]
It is now important to observe that the noncommutativity in the space-space coordinates \((11)\) has a Lie-algebraic structure in phase-space (with the inclusion of identity) and not in space-time\(^7\). This is in contrast to the results derived in \((9)\) (for the relativistic free particle) where space-space noncommutativity was not Lie-algebraic in form because of the presence of \(p_0\) in the denominator which did not have a vanishing bracket with all other phase-space variables. However, the algebra between the space-time coordinates is not Lie-algebraic in form.

Alternatively, one may demand that the space-space algebra between the primed coordinates is of the Lie-algebraic form \((11)\). A simple inspection (after the substitution of \((10)\) in the left hand side of \((11)\)) gives the solution \((10a)\) for the reparametrisation parameter \(\epsilon\). The change of variables relating the primed coordinates with the unprimed ones hence read:

\[
\begin{align*}
x'_{0} &= x_{0} - \theta^{0k} p_k \frac{p_0}{m} \\
x'_{i} &= x_{i} + \theta^{0k} p_k \frac{p^i}{m}
\end{align*}
\](18)

Note that the above change of variables is different to that derived in \((9)\). This is because \(\epsilon\) in \((9)\) is of the form \(\epsilon = -\theta^{0k} p_k\).

The above solution of \(\epsilon\) \((10a)\) shows that the desired gauge fixing condition is given by \(8\):

\[
\phi_3 = x^0 + \theta^{0k} p_k \frac{p_0}{m} - \tau \approx 0, \quad k = 1, 2, \ldots d.
\](19)

It is easy to check that the constraints \((6, 14)\) form a second class pair as

\[
\{\phi_a, \phi_b\} = 2 p_0 \epsilon_{ab}; \quad (a, b = 1, 3).
\](20)

The set of non-vanishing DB(s) consistent with the strong imposition of the constraints \((6, 14)\) reproduces the results \((11, 15, 12, 16, 13, 17)\).

We now investigate the algebra of the Lorentz generators (rotations and boosts) for the above choice of the reparametrisation parameter \(\epsilon\) to illuminate the internal consistency of our analysis. As we have pointed out in \((9)\), the definition of the Lorentz generators remains unchanged in our approach, because these are gauge invariant objects. The Lorentz generators (rotations and boosts) are defined as

\[
M_{ij} = x_i p_j - x_j p_i
\]

\[
M_{0i} = x_{0} p_i - x_i p_0
\]

They satisfy the usual algebra in both the unprimed and the primed coordinates as \(M_{\mu\nu}\) and \(p_{\mu}\) are both gauge invariant \([13]\):

\[
\{M_{ij}, p_k\}_{DB} = \delta_{ik} p_j - \delta_{jk} p_i
\]

\[
\{M_{ij}, M_{kl}\}_{DB} = \delta_{ik} M_{jl} - \delta_{jk} M_{il} + \delta_{jl} M_{ik} - \delta_{il} M_{jk}
\]

\[
\{M_{ij}, M_{0k}\}_{DB} = \delta_{ik} M_{0j} - \delta_{jk} M_{0i}
\]

\[
\{M_{0i}, M_{0j}\}_{DB} = M_{ji}
\]

\(^7\)Note that following \([10]\), one can therefore associate an appropriate “diamond star product” for this in order to compose any pair of phase-space functions.

\(^8\)Note that we have dropped the prime from \(x'_{0}\) for convenience in \((14)\).
\{ M_{0i}, p_k \}_DB = -\delta_{ik} p_0 \quad (27)  

On the other hand, the algebra between the space coordinates and those of Lorentz generators (rotations and boosts) are different in the two gauges \[\text{(10, 14y19)}\] since \(x^k\) is not gauge invariant under gauge transformation. We find:

\{ M_{ij}, x^k \}_DB = \delta_{ik} x^j - \delta_{j k} x^i \quad (28) \quad A3

\{ M_{0i}, x^j \}_DB = x_i \frac{p^j}{p_0} - x_0 \delta_{i j} \quad (29) \quad A4

\{ M_{ij}, x^{'k} \}_DB = \delta_{ik} x^{'j} - \delta_{j k} x^{'i} + \frac{p^k}{m} \left( \theta^0_i p_j - \theta^0_j p_i \right) \quad (30) \quad A5

\{ M_{0i}, x^{'j} \}_DB = x^{'i} \frac{p^j}{p_0} - x^{'0} \delta_{i j} - \theta^0_i p^j \frac{p_0}{m} - \theta^0_k p_k \frac{p^0 p^j}{mp_0} \quad (31) \quad A6

Note at this stage that the gauge choice \[\text{(14y19)}\] is not Lorentz invariant. However, the Dirac bracket procedure forces this constraint equation to be strongly valid in all Lorentz frames \[\text{hanson19}\].

This can be made consistent if and only if an infinitesimal Lorentz boost to a new frame

\[ p^\mu \rightarrow p'^\mu = p^\mu + \omega^{\mu\nu} p_\nu; \quad \omega^{\mu\nu} = -\omega^{\nu\mu} \quad (32) \quad A7a \]

is accompanied by a compensating infinitesimal gauge transformation

\[ \tau \rightarrow \tau' = \tau + \Delta \tau \quad (33) \quad A8 \]

The change in \(x^\mu\), upto first order in \(\omega\), is therefore

\[ x^{'\mu}(\tau) = x^\mu(\tau') + \omega^{\mu\nu} x_\nu(\tau) \]

\[ = x^\mu(\tau) + \Delta \tau \frac{dx^\mu}{d\tau} + \omega^{\mu\nu} x_\nu \quad (34) \quad A9 \]

In particular, the zero-th component is given by,

\[ x^{'0}(\tau) = x^0(\tau) + \Delta \tau \frac{dx^0}{d\tau} + \omega^0 i \quad (35) \quad A10 \]

Since the gauge condition \[\text{(14y19)}\] is \(x^0(\tau) \approx \tau - \theta^{0k} p_k p_0/m\), \(x^0(\tau)\) also must satisfy \(x^0(\tau) = (\tau - \theta^{0k} p_k p_0/m)\) in the boosted frame, which can now be written, using \[\text{(A7a, A24)}\] as:

\[ x^{'0}(\tau) = \tau - \frac{\theta^{0k}}{m} (p_k + \omega^0 k p_0)(p_0 + \omega^0 l p_l) \quad (36) \quad A11 \]

Comparing the left hand side of the above equation with \[A10\] and using the gauge condition \[\text{(14y19)}\], one can now solve for \(\Delta \tau\) up to terms linear in \(\omega\) to get,

\[ \Delta \tau = \frac{\theta^{0k}}{m} \left( \omega^0 k p_0^2 - \omega^0 l p_k p_l \right) \quad (37) \quad A12u \]
Therefore, for a pure boost, the spatial components of \((A9)\) satisfy
\[
\delta x^j(\tau) = x'^j(\tau) - x^j(\tau) = \Delta \tau \frac{dx^j}{d\tau} + \omega^0 x_0
\]
\[
= \omega^0 \left( x_i \frac{p^j}{p_0} - x_0 \delta^j_i - \theta^0 p^j p_0 m - \theta^0 p_k p_k p^j \frac{p_j}{m p_0} \right) + O(\omega^2)
\]
(38) \hspace{1cm} A12a

Hence we find that \((A6)\) and \((A12a)\) are consistent with each other up to first order in \(\omega\).

Next we observe that there is another interesting choice of \(\epsilon\) which reads the following:
\[
\epsilon = -d_k \theta^{kl} p^l p_0
\]
(39) \hspace{1cm} 15aaa

where; \(d_k\) are arbitrary dimensionless constants.

This yields (using \((9.3)\) and \((10)\)) the following algebra between the primed coordinates:
\[
\{x'^i, x'^j\}_{DB} = \frac{d_k}{m} \left( \theta^{kl} p^l - \theta^{kj} p^l \right)
\]
(40) \hspace{1cm} 16aaa

\[
\{x'^0, x'^i\}_{DB} = \frac{d_k}{m} \left( \theta^{0l} p^l + \theta^{kl} p^l \frac{p_i}{p_0} \right)
\]
(41) \hspace{1cm} 12aaa

\[
\{x'^i, p'_0\}_{DB} = \frac{p_i}{p_0} ; \quad \{x'^i, p'_j\}_{DB} = \delta^i_j
\]
(42) \hspace{1cm} 13aaa

Once again we obtain a Lie-algebraic noncommutative structure in the space-space sector. However, note that \((16aaa)\) is different from \((11aaa)\) because the noncommutative parameter \(\theta\) in \((16aaa)\) has space indices in contrast to the space-time indices appearing in \((11aaa)\). The space-time algebra is once again not Lie-algebraic in form.

The desired gauge fixing condition which leads to the above Dirac brackets read:
\[
\phi_4 = x^0 + d_k \theta^{kl} p^l p_0 - \tau \approx 0, \quad k = 1, 2, \ldots, d.
\]
(43) \hspace{1cm} 14

As before the algebra between the Lorentz generators \(M_{\mu\nu}\) and \(p_\mu\) remains the same \hspace{1cm} 23, 24, 25, 26, 27. Also the algebra between \(M_{\mu\nu}\) and \(x'^\mu\) in the unprimed coordinates remain the same \hspace{1cm} 23, 24, 25, 26. However, the algebra between \(M_{\mu\nu}\) and \(x'^\mu\) in the primed coordinates are different and read:
\[
\{M_{ij}, x'^k\}_{DB} = \delta_i^k x'_j - \delta_j^k x'_i + \frac{d_i}{m} \left( \theta^i_j p_j - \theta^j_i p_i \right) p^k
\]
(44) \hspace{1cm} A5aa

\[
\{M_{0i}, x'^j\}_{DB} = x'_i \frac{p^j}{p_0} - x'_0 \delta^j_i - d_i \theta^i_j p^j \frac{p_0}{m} - d_i \theta^{0k} p_k p_j \frac{p_j}{m p_0}
\]
(45) \hspace{1cm} A6aa

Rerunning our earlier analysis of enforcing the constraint equation \((14)\) to be strongly valid in all Lorentz frames leads to the following solution for \(\Delta \tau\) in \((A9)\) up to linear in \(\omega\):
\[
\Delta \tau = \frac{d_k}{m} \theta^{kl} \left( \omega^0 p_0^2 + \omega^0 p_1 p_0 \right) x_i
\]
(46) \hspace{1cm} A12

6
Therefore, for a pure boost, the spatial components of $A$ satisfy
\begin{align*}
\delta x^j(\tau) &= x^j(\tau) - \tau x^j(\tau) = \frac{dx^j}{d\tau} + \omega^0 x_0 \\
&= \omega^0 \left( x_i \frac{p^j}{p_0} - x_0 \delta_i^j - d_k \theta^k_i p^j \frac{P_0}{m} - d_k \theta^k_l p_l \frac{p^j}{m p_0} \right) + O(\omega^2)
\end{align*}

Hence we find that (A6aa) and (A12aa) are consistent with each other up to first order in $\omega$.

We make certain observations now. Although, the relations (11), (16) have a close resemblance to Snyder’s algebra [17], there is a subtle difference. Note that the right hand side of these relations do not have the structure of an angular momentum operator in their differential representation (obtained by replacing $p_j$ by $(-i\partial_j)$ in contrast to the Snyder’s algebra. Further, the relations are not reminiscent of $\kappa$-Minkowski algebra (that has been studied extensively in the literature recently [11, 12, 13, 14]) but has a similar structure to the commutation relations describing the Lie-algebraic deformation of the Minkowski space [15], the only difference being that momentum operators appear at the right hand side of the relations instead of the position operators. Interestingly, the values of the reparametrisation parameter $\epsilon$ (10a, 11) that leads to the noncommuting Lie-algebraic structures [11, 13] are the only choices possible as can be easily seen from purely dimensional considerations.

Finally, there exists choices of $\epsilon$ for which the space-space noncommutativity can be made to vanish. The choices are:
\begin{align*}
\epsilon &= e_k \theta^0 k \frac{p_j}{m} \\
\epsilon &= -f_{kl} \theta^{kl} p_0
\end{align*}

where, $e_k$ and $f_{kl}$ are arbitrary dimensionless constants.

The space-time algebras however do not vanish for the above values of $\epsilon$ and are as follows:
\begin{align*}
\{x^0, x^i\} &= \frac{2e_k \theta^0 k \frac{p_j}{m}}{m} \\
\{x^0, x^i\} &= f_{kl} \theta^{kl} \frac{p_j}{p_0}
\end{align*}

Another interesting question that can be asked is the following. Can we get canonical NC space-space structures from reparametrisation symmetry. To see this we ask whether there exists a solution of $\epsilon$ for which the following equation holds between the primed coordinates:
\begin{align*}
\{x^0, x^i\} = \theta^{ij}
\end{align*}

Substituting (12) in the left hand side of the above equation in terms of the unprimed coordinates, it is easy to note that there does not exist an $\epsilon$ for which the above equation is satisfied. Thus, one cannot obtain canonical NC space-space structures from reparametrisation symmetry.

3 Conclusions

In this paper, we have extended our earlier work on noncommutativity and reparametrisation symmetry [9] to obtain Lie-algebraic NC structures in case of a free relativistic particle. This is
in contrast to the results obtained in [9], since the NC structures although Lie-algebraic in form in case of the non-relativistic free particle were not so in its relativistic counterpart. The change of variables derived in this paper are different than those appearing in [9]. This is related to the fact that the choice of the reparametrisation parameter $\epsilon$ is different in the two cases. We also find solutions (for $\epsilon$) for which the algebra between space-space coordinates in the primed sector vanishes while the space-time algebra still survives. However, we do not get canonical [11] or quantum space structures [12] from reparametrisation symmetry.

As a matter of internal consistency of our analysis, we study the angular momentum algebra in details. As in [9], the angular momentum remains gauge invariant since the change of variables is just a gauge transformation. Hence, we feel that our approach is more elegant than those [6] where such change of variables are found by inspection and leads to ambiguities in the definition of physical variables like the angular momentum.

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