GEOMETRIC PROGRESSION-FREE SEQUENCES WITH SMALL GAPS

XIAOYU HE

Abstract. Various authors, including McNew, Nathanson and O’Bryant, have recently studied the maximal asymptotic density of a geometric progression free sequence of positive integers. In this paper we prove the existence of geometric progression free sequences with small gaps, partially answering a question posed originally by Beiglböck et al. Using probabilistic methods we prove the existence of a sequence $T$ not containing any 6-term geometric progressions such that for any $x \geq 1$ and $\varepsilon > 0$ the interval $[x, x + C_\varepsilon \exp((C_\varepsilon + \varepsilon) \log x / \log \log x)]$ contains an element of $T$, where $C = \frac{5}{6} \log 2$ and $C_\varepsilon > 0$ is a constant depending on $\varepsilon$. As an intermediate result we prove a bound on sums of functions of the form $f(n) = \exp(-d_k(n))$ in very short intervals, where $d_k(n)$ is the number of positive $k$-th powers dividing $n$, using methods similar to those that Filaseta and Trifonov used to prove bounds on the gaps between $k$-th power free integers.

Introduction

Let $k \geq 3$ be an integer, and $r$ be a positive rational number. A geometric progression of length $k$ with common ratio $r$ is a sequence $(a_0, \ldots, a_{k-1})$ of nonzero real numbers for which

$$a_i = ra_{i-1}, \quad i = 1, \ldots, k - 1.$$ 

A $k$-geometric progression, or $k$-GP, is a geometric progression of length $k$. Such a geometric progression is called trivial if $r = 1$ and henceforth we consider only nontrivial progressions.

Rankin [13] introduced the notion of a $k$-GP-free sequence, which for our purposes is a sequence of positive integers that contains no nontrivial $k$-geometric progressions. Whereas a theorem of Szemerédi [17] shows that $k$-term arithmetic-progression free sequences must have zero upper density in the naturals, there exist $3$-GP-free sequences with positive asymptotic density. For instance, the sequence of all squarefree positive integers avoids all geometric progressions of length 3 or more, and has asymptotic density $\zeta(2)^{-1} = \frac{6}{\pi^2}$.

Let $A$ be a $k$-GP-free sequence. The question of finding the maximal possible asymptotic density $d(A)$, or else the upper density $d_U(A)$, of $A$ has been a subject of recent study [2, 3, 12, 15, 15]. For an exposition of progress on this problem and the tightest known bounds on $d_U(A)$, along with constructions of $A$ with nearly optimal upper density, see the paper of McNew [12].

In this paper we are interested in a uniform version of this density problem. In particular, we would like to settle the existence of $k$-GP-free sequences with small gaps. Beiglböck, Bergelson, Hindman, and Strauss [2] proposed the following problem, which has implications in ergodic theory.

Problem. Does there exist a $c > 1$, a $k \geq 3$, and an increasing $k$-GP-free sequence $T = \{t_i\}_{i \in \mathbb{N}}$ of positive integers such that $t_{i+1} - t_i \leq c$ for all $i \in \mathbb{N}$? Such a sequence with bounded gaps is called syndetic.
We use the standard notations $f(x) = O(g(x))$ if there exists a constant $A > 0$ for which $f(x) < Ag(x)$ for all $x$, and $f(x) = o(g(x))$ if for any constant $A > 0$ the inequality $f(x) < Ag(x)$ holds for $x$ sufficiently large. Furthermore we write $f(x) = \Omega(g(x))$ if there exists a constant $B > 0$ for which $f(x) > Bg(x)$ for all $x$. In both cases the range of $x$ depends on context but is usually the natural numbers. We will often also use the shorthand $f(x) \ll g(x)$ for $f(x) = O(g(x))$, and similarly $f(x) \gg g(x)$ for $f(x) = \Omega(g(x))$.

Note that the sequence of squarefree integers fails to be syndetic, since there exist gaps as large as $\Omega(\log x / \log \log x)$ within the squarefree integers up to $x$. Erdős [5] knew this elementary result in 1951, though he was not the first. He showed that if $\{s_i\}_{i \in \mathbb{N}}$ is the sequence of squarefree integers in increasing order, then

$$s_{i+1} - s_i > (1 + o(1)) \frac{\pi^3 \log s_i}{6 \log \log s_i}$$

for infinitely many values of $i$. By roughly the same method, Rankin [14] proved that if $\{p_i\}_{i \in \mathbb{N}}$ is the sequence of primes in increasing order, then

$$p_{i+1} - p_i > (1 + o(1)) e^{\gamma} \frac{\log p_i \log \log p_i \log \log \log \log p_i}{(\log \log p_i)^2}$$

for infinitely many values of $i$, where $\gamma$ is the Euler-Mascheroni constant. The constant $e^{\gamma}$ in Rankin’s bound was improved several times; recently, it was replaced by a function growing to infinity by Ford, Green, Konyagin and Tao [8] concurrently with Maynard [11], settling a $10,000$ Erdős problem. Most recently, Ford, Green, Konyagin, Maynard, and Tao combined the previous methods and showed

$$p_{i+1} - p_i \gg \frac{\log p_i \log \log p_i \log \log \log \log p_i}{\log \log p_i}$$

for infinitely many $i$ and an effective implied constant, removing a $\log \log \log p_i$ factor from the denominator. For a history of these results and the current progress, see their paper [7].

It is a conjecture of Cramér [4] that

$$\limsup_{i \to \infty} \frac{p_{i+1} - p_i}{(\log p_i)^2} = 1,$$

but the best upper bound available is $p_{i+1} - p_i = O(p_i^{0.525})$ due to Baker, Harman, and Pintz [1]. For a discussion of Cramér’s model and its deficiencies see the paper of Pintz [16].

Large gaps between the squarefree integers up to $x$ are also poorly understood. In this direction the tightest bound is due to Filaseta and Trifonov [6], that the largest gaps between squarefree integers are at most $O(x^{5/7} \log x)$. Trifonov [18] established the generalization that the largest gaps between the $k$-th power free integers up to $x$ are at most $O(x^{1/(2k+1)} \log x)$, and assuming the $abc$ conjecture, Granville [9] was able to proved that the gaps are $O(x^{\varepsilon})$ for every $\varepsilon > 0$.

We are naturally interested in an unconditional construction of $k$-GP-free sequences with gaps of size $O(x^\varepsilon)$ for every $\varepsilon > 0$. Our main theorem proves the existence of such sequences using the probabilistic method.

**Theorem 1.** There exists a 6-GP-free sequence $T$ of positive integers $\{t_i\}_{i \in \mathbb{N}}$ such that for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ for which

$$t_{i+1} - t_i < C_\varepsilon \exp \left( \left( \frac{5}{6} \log 2 + \varepsilon \right) \frac{\log t_i}{\log \log t_i} \right),$$

$$A_{s + 1} - A_s > (1 + o(1)) \frac{\pi^3 \log A_s}{6 \log \log A_s}$$

for infinitely many values of $s$. By roughly the same method, Rankin [14] proved that if $\{p_i\}_{i \in \mathbb{N}}$ is the sequence of primes in increasing order, then

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**Theorem 1.** There exists a 6-GP-free sequence $T$ of positive integers $\{t_i\}_{i \in \mathbb{N}}$ such that for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ for which

$$t_{i+1} - t_i < C_\varepsilon \exp \left( \left( \frac{5}{6} \log 2 + \varepsilon \right) \frac{\log t_i}{\log \log t_i} \right).$$
holds for all \( i \in \mathbb{N} \).

Note that the previously best known unconditional result in this direction is the bound by Trifonov on the sequence of 5-th power free positive integers, with gaps of size at most \( O(x^{1/11} \log x) \).

Let \( d_i(n) \) denote the number of \( i \)-th powers dividing \( n \), and \( d_{i,j}(n) \) be the number of pairs \( (a, b) \in \mathbb{N}^2 \) with \( a^i b^j | n \). In the next section we prove a bound on short sums of the form

\[
S_{i,j}(x, h, D) = \sum_{x < n \leq x + h} \exp(-Dd_{i,j}(n)),
\]

where \( h \) is on the order of \( \exp \left( \frac{5}{6} \log 2 + \varepsilon \frac{\log x}{\log \log x} \right) \), and \( D \) is some constant. Once this computation is complete we will use the probabilistic method to show that a sequence randomly generated by removing terms from every 6-GP has gaps of the desired size with nonzero probability.

**Sums of Divisor Functions in Short Intervals**

We will be interested in a lower bound on the quantity

\[
S_{i,j}(x, h, D) = \sum_{x < n \leq x + h} \exp(-Dd_{i,j}(n)),
\]

where \( h \) is small and \( D \) is some fixed positive constant.

The following uniform bound was proven by Nair and Tenenbaum [14]. It is a generalization of a result of Shiu [16] to a larger class of functions. We state the relevant special case of their theorem here. Let \( M \) be the class of arithmetic functions \( f: \mathbb{N} \to \mathbb{N} \) satisfying

1. (Submultiplicativity) The function \( f \) satisfies \( f(mn) \leq f(m)f(n) \) for \( (m, n) = 1 \), and \( f(1) = 1 \).
2. There exists a fixed \( A > 0 \) such that for any prime \( p \), \( f(p^n) \leq A^n \) for all \( n \in \mathbb{N} \).
3. For all \( \varepsilon > 0 \), \( f(n) = O(n^{\varepsilon}) \).

**Theorem 2.** [14] Suppose that \( f \) is a function in the class \( M \). For any \( 0 < \beta < \frac{1}{2} \) and a function \( h(x) \gg x^\beta \), we have

\[
\sum_{x < n \leq x + h(x)} f(n) \ll \frac{h(x)}{\log x} \exp \left( \sum_{p \leq x} \frac{f(p)}{p} \right),
\]

where the implicit constant depends only on \( \beta \).

We first prove that Theorem 2 applies to the functions \( d_{i,j} \) we are interested in.

**Lemma 3.** For any \( i, j \in \mathbb{N} \), function \( d_{i,j}(n) \) which counts the number of pairs \( (a, b) \in \mathbb{N}^2 \) for which \( a^i b^j | n \) lies in the class \( M \).

**Proof.** Write

\[
D_{i,j}(n) = \{(a, b) \in \mathbb{N}^2 : a^i b^j | n\},
\]

so that \( d_{i,j}(n) = |D_{i,j}(n)| \). For the submultiplicativity of \( d_{i,j} \), it suffices to exhibit an injection \( \phi : D_{i,j}(mn) \to D_{i,j}(m) \times D_{i,j}(n) \), given \( (m, n) = 1 \). Given \( (a, b) \in D_{i,j}(mn) \), we let

\[
\phi((a, b)) = ((\gcd(a, m), \gcd(b, m)), (\gcd(a, n), \gcd(b, n))),(0, 0)),
\]

and submultiplicativity follows.
Next, for any prime \( p \) and positive integer \( n \), it is easy to see
\[
d_{i,j}(p^n) \leq (n+1)^2,
\]
since there are at most \( n+1 \) choices for \( a \) or \( b \). But \( (n+1)^2 \leq 4^n \) for all \( n \in \mathbb{N} \), so condition 2 is satisfied.

To show condition 3, we appeal to the well-known bound \( d(n) = O(n^\varepsilon) \), where \( d(n) \) is the divisor function. But \( d_{i,j}(n) \leq d(n)^2 \) since \( a, b \) are both chosen from the divisors of \( n \), and the growth condition \( d_{i,j}(n) = O(n^\varepsilon) \) for all \( \varepsilon > 0 \) holds. \[ \Box \]

With Theorem 2 in hand, it is easy to show the following bound.

**Corollary 4.** For \( i, j \geq 2 \), we have
\[
S_{i,j}(x, h, D) \gg h(x),
\]
if \( h(x) \gg x^\beta \) for some \( 0 < \beta < \frac{1}{2} \), where the implicit constant depends only on \( \beta \) and \( D \).

**Proof.** We use Jensen’s inequality and the convexity of the exponential function to get
\[
S_{i,j}(x, h, D) \geq h(x) \exp \left( -Dh(x)^{-1} \sum_{x<n \leq x+h} d_{i,j}(n) \right).
\]

Applying Theorem 2, we have then that
\[
S_{i,j}(x, h, D) \geq h(x) \exp \left( -D(\log x)^{-1} \exp \left( \sum_{p \leq x} \frac{d_{i,j}(p)}{p} \right) \right).
\]

Now for any prime \( p \), \( d_{i,j}(p) = 1 \), and so the inner sum is controlled by Mertens’ estimate \([13]\)
\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + O(1),
\]
from which the stated inequality follows. \[ \Box \]

Corollary 4 will already suffice to give gaps of size at most \( O(x^\varepsilon) \) for every \( \varepsilon > 0 \). However, we provide the following improvement to shorter intervals, using a basic idea of Filaseta and Trifonov \([6]\). To the best of our knowledge this bound, and the study of such sums in intervals shorter than \( x^\varepsilon \) for all \( \varepsilon > 0 \) is new.

**Lemma 5.** For any \( \varepsilon > 0 \) and \( i, j \geq 2 \), there \( E, N > 0 \) such that if \( x > N \) then
\[
S_{i,j}(x, h, D) \geq E \exp \left( (C_{i,j} + \varepsilon) \frac{\log x}{\log \log x} \right),
\]
where \( h(x) = \exp \left( (C_{i,j} + 2\varepsilon) \frac{\log x}{\log \log x} \right) \), \( C_{i,j} = \log 2 \left( \frac{1}{i} + \frac{1}{j} \right) \), and the constants \( E \) and \( N \) depend only on the choices of \( \varepsilon \) and \( D \).

**Proof.** First, we have that \( d_{i,j}(n) \leq d_i(n)d_j(n) \). Without loss of generality assume \( i \leq j \). Henceforth \( p \) always refers to a prime.

We enumerate the set \( A \) of \( n \in (x, x+h] \) divisible by some \( i \)-th prime power \( p^j \) where \( p \leq h \). Because \( i \geq 2 \), we have that
\[
|A| \leq \sum_{p \leq h} \left( \frac{h}{p^i} + 1 \right) \leq (\zeta(i) - 1)h + o(h),
\]
by the Chebyshev bound on the prime counting function, where $\zeta$ is the Riemann zeta function. Let $A^c$ denote the complement of $A$ in $(x, x+h]$. It follows that since $\zeta(i) < 2$ uniformly in $i > 2$, there exists a positive constant $0 < B < 1$ for which, whenever $x$ is sufficiently large, $|A^c| \geq Bh$. We restrict our attention to only $n \in A^c$.

Each such $n$ we can write as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\ell^{\alpha_\ell} m$ where each $p_k > h$, each $\alpha_k \geq i$, and $m$ is $i$-th power free, so that

$$d_i(n) = \prod_{k=1}^\ell \left( \left\lceil \frac{\alpha_k}{i} \right\rceil + 1 \right).$$

By a simple smoothing argument and the fact that if $n \in A^c$,

$$\sum_{k=1}^\ell \alpha_k \leq \frac{\log n}{\log h} \leq \frac{(1 + o(1)) \log \log x}{C_{i,j} + 2\varepsilon},$$

we find that for any $n \in A^c$, and $x$ sufficiently large,

$$d_i(n) \leq \exp \left( \frac{\log 2 \log \log x}{i C_{i,j} + \varepsilon} \right),$$

and similarly,

$$d_j(n) \leq \exp \left( \frac{\log 2 \log \log x}{j C_{i,j} + \varepsilon} \right).$$

Combining these two inequalities, we find

$$S_{i,j}(x, h, D) \geq \sum_{n \in A^c} \exp \left( -D d_i(n) d_j(n) \right) \geq Eh \exp \left( -D (\log x)^F \right),$$

where

$$F = \frac{\log 2}{C_{i,j} + \varepsilon} \left( \frac{1}{i} + \frac{1}{j} \right) < 1.$$

Plugging in the expression for $h$ the result follows, since

$$\frac{(C_{i,j} + 2\varepsilon) \log x}{\log \log x} - D (\log x)^F \geq \frac{(C_{i,j} + \varepsilon) \log x}{\log \log x}$$

for $x$ sufficiently large in terms of $D$ and $\varepsilon$, since $F < 1$. \hfill \Box

**The GP-Free Process**

For the proof of Theorem 1, we randomly generate a 6-GP-free sequence by removing at least one element from each 6-GP. For the technical details to work out, we only remove one of the middle two elements of each 6-GP.

Let $G_k$ denote the family of all nontrivial $k$-term geometric progressions of positive integers. Since each such family is countable, we can enumerate $G_k$

$$G_k = (G_{k,1}, G_{k,2}, \ldots),$$

where each $G_{k,i}$ is a nontrivial $k$-GP and they are ordered lexicographically as $k$-tuples of positive integers. To avoid double-counting we assume that each $G_{k,i}$ has a common ratio $r_{k,i} > 1$. 
Definition 6. The 6-GP-free process randomly generates a 6-GP-free sequence $T$ as follows. Generate a sequence $U = (u_1, u_2, \ldots)$ where $u_i \in G_{6,i}$ such that if

$$G_{6,i} = (a_i b_i^2, a_i b_i^4 c_i, a_i b_i^1 c_i^2, a_i b_i^2 c_i^2, a_i b_i^1 c_i^3, a_i b_i^4 c_i^3),$$

where $b_i < c_i$, then $u_i = a_i b_i^2 c_i^2$ or $u_i = a_i b_i^2 c_i^3$ with equal probability $\frac{1}{2}$. All of the $u_i$'s are chosen independently. Let $T$ be the random variable whose value is the sequence of all positive integers never appearing in $U$, sorted in increasing order.

It is clear that $T$ misses at least one term in each progression in $G_6$. In the next section we show that with nonzero probability $T$ has gaps smaller than $O\left(\exp\left((C_{2,3} + \varepsilon) \frac{\log x}{\log \log x}\right)\right)$ for every $\varepsilon > 0$, thereby proving Theorem 1. We rely extensively on Lemma 5.

**Proof of the Main Theorem**

Using Definition 6 it suffices to prove the following lemma to prove Theorem 1.

**Lemma 7.** For any $\varepsilon > 0$, there exists a constant $E > 0$ such that for every constant $C > 0$ and every $x \in \mathbb{N}$ sufficiently large, the probability that the sequence $T$ generated from the 6-GP-free process does not contain any element of $(x, x + Ch]$ is at most

$$P[T \cap (x, x + Ch] = \emptyset] \leq \exp\left(-CE \exp\left(\left(C_{2,3} + \varepsilon\right) \frac{\log x}{\log \log x}\right)\right),$$

where $C_{i,j}$, $h = h(x)$, and $E$ are as in Lemma 5, for the same $\varepsilon$ and $D = \log 2$.

**Proof.** Fix $C > 0$. The sequence $T$ contains no element of $(x, x + Ch]$ exactly when the sequence $U$ in Definition 6 contains every element of $(x, x + Ch]$. For any given positive integer $n \in \mathbb{N}$, the probability that $n \in U$ can be controlled as follows. It is easy to see that $n$ appears as the term $a_i b_i^2 c_i^2$ or $a_i b_i^2 c_i^3$ in some $G_{6,i}$ for at most $d_{3,2}(n)$ choices of $i$.

Furthermore, since the choices of $u_i \in G_{6,i}$ are independent, it follows that

$$P[n \notin U] \geq \left(\frac{1}{2}\right)^{d_{3,2}(n)}.$$

We can now bound the probability that every $n \in (x, x + Ch]$ appears in $U$. Given any fixed

$$G_{6,i} = (a_i b_i^2, a_i b_i^4 c_i, a_i b_i^1 c_i^2, a_i b_i^2 c_i^2, a_i b_i^1 c_i^3, a_i b_i^4 c_i^3),$$

note that the middle two elements $a_i b_i^2 c_i^2, a_i b_i^2 c_i^3$ differ by at least $a_i b_i^2 c_i^2$. Thus, if $n = a_i b_i^2 c_i^2$ lies in $(x, x + Ch]$, then

$$|a_i b_i^2 c_i^2 - n| \geq a_i b_i^2 c_i^2 \geq \sqrt{n},$$

and the same holds if $n = a_i b_i^2 c_i^3$. For $x$ sufficiently large we have $h(x) < \sqrt{x}$, so no two elements in $(x, x + Ch]$ lie in the same $G_{6,i}$ together. Thus each of the events $n \notin U$ are independent. It follows that we can multiply probabilities, getting

$$P[T \cap (x, x + Ch] = \emptyset] \leq \prod_{n \in (x, x + Ch]} \left(1 - \exp(-d_{3,2}(n) \log 2)\right),$$

to which we can apply $1 - t \leq e^{-t}$ to find that

$$P[T \cap (x, x + Ch] = \emptyset] \leq \exp\left(- \sum_{(x, x + Ch]} \exp(-d_{3,2}(n) \log 2)\right).$$
Now the inner sum can be bounded by Lemma 5 with $D = \log 2$, from which we get the desired inequality

$$P[T \cap (x, x + Ch] = \emptyset] \leq \exp \left( -CE \exp \left( (C_{2,3} + \varepsilon) \frac{\log x}{\log \log x} \right) \right)$$

for some constant $E > 0$ depending only on the choice of $\varepsilon$, for sufficiently large $x$. \qed

The proof of Theorem 1 is now dependent on choosing $C$ large enough in Lemma 7.

Proof. (Theorem 1) For any fixed $\varepsilon > 0$, we can pick $C_1 > 0$ such that

$$\sum_{x \in \mathbb{N}} \exp \left( -C_1 E \exp \left( (C_{2,3} + \varepsilon) \frac{\log x}{\log \log x} \right) \right) < 1.$$ 

In particular, by Lemma 7 and linearity of expectation, we see that there exists an $N > 0$ for which the expected number of intervals $(x, x + C_1 h]$ with $x > N$ that fail to intersect $T$ is less than 1. Thus, there exists a $T$ which intersects every $(x, x + C_1 h]$ for all $x > N$. Taking $C$ sufficiently large so that for every $x \in \mathbb{N}$, $(x, x + Ch(x)]$ contains an $(x', x' + C_1 h(x')]$ for some $x' > N$, we see that this $T$ intersects $(x, x + Ch]$ for all $x \in \mathbb{N}$. Thus no gap can be of size greater than $Ch(x)$, as desired. The value of $C_{2,3}$ is exactly \( \frac{5}{6} \log 2 \), as claimed. \qed

### Shorter Geometric Progressions

In this section we study the case when $k < 6$, and construct probabilistically a sequence $T$ with gaps $O(x^\varepsilon)$ which avoids 5-term geometric progressions. We also find such a $T$ avoiding 3-term geometric progressions with integer ratios. The proofs are very similar to that of Theorem 1, the only difference being we can now apply Theorem 2 in place of our Lemma 5.

**Proposition 8.** There exists a 5-GP-free sequence $T$ of positive integers $\{t_i\}_{i \in \mathbb{N}}$ such that for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ satisfying

$$t_{i+1} - t_i < C_\varepsilon t_i^\varepsilon$$

for all $i \in \mathbb{N}$.

Proof. Enumerate the nontrivial 5-GP’s of positive integers, and let the $n$-th one be

$$(a_n b_n^4, a_n b_n^3 c_n, a_n b_n^2 c_n^2, a_n b_n c_n^3, a_n c_n^4)$$

with ratio $c_n/b_n > 1$. We construct $T$ by removing from $\mathbb{N}$, randomly and independently, one of the two terms $a_n b_n^3 c_n$ and $a_n b_n^2 c_n^2$ from each such geometric progression, with probabilities $1 - p, p$ where $p = p(a_n b_n^2 c_n^2)$ depends only on the middle term of the progression, and is a nondecreasing function $p : \mathbb{N} \to (0, 1)$ we will choose to be

$$p(x) = 1 - \frac{1}{\log(x + 2)}.$$

For a given $x \in \mathbb{N}$, we compute the probability $P_x$ that $x$ is not removed in this process. By definition, $x$ is the second term $a_n b_n^3 c_n$ in at most $d_{3,1}(x)$ distinct progressions, and the third term $a_n b_n^2 c_n^2$ in at most $d_{2,2}(x)$ distinct progressions. Because $c_n > b_n$, we have

$$a_n b_n^2 c_n^2 \geq a_n b_n^3 c_n,$$

and so

$$P_x \geq p(x)^{d_{3,1}(x)} (1 - p(x))^{d_{2,2}(x)},$$
by the independence of all of these events. To finish the proof, pick \( h(x) = C x^\varepsilon \) to be the interval length. Then, by independence we get

\[
P[T \cap (x, x + h(x)] = \emptyset \leq \prod_{i=1}^{h(x)} (1 - p(x + i)^{d_{3,1}(x+i)}(1 - p(x + i))^{d_{2,2}(x+i)}),
\]

and applying \( 1 - t \leq e^{-t} \),

\[
P[T \cap (x, x + h(x)] = \emptyset \leq e^{-S},
\]

where

\[
S = \sum_{i=1}^{h(x)} p(x + i)^{d_{3,1}(x+i)}(1 - p(x + i))^{d_{2,2}(x+i)}.
\]

Applying Jensen’s inequality, we can further bound \( S \) by

\[
S \geq h \exp \left( \frac{1}{h} \sum_{i=1}^{h(x)} d_{3,1}(x+i) \log p(x + i) + d_{2,2}(x+i) \log(1 - p(x + i)) \right)
\]

\[
\gg h \exp \left( \log p(x) \log x + \log(1 - p(x)) \right),
\]

where we applied Theorem 2 on the two sums

\[
\sum_{i=1}^{h(x)} d_{3,1}(x+i) \ll h(x) \log x,
\]

since \( d_{3,1}(p) = 2 \) for all primes \( p \), and

\[
\sum_{i=1}^{h(x)} d_{2,2}(x+i) \ll h(x)
\]

since \( d_{2,2}(p) = 1 \) for all primes \( p \). We also used the fact that \( p \) is essentially constant on intervals of length \( h \). By choosing \( p(x) \sim 1 - \frac{1}{\log x} \) in the above, we find that

\[
\log p(x) \log x + \log(1 - p(x)) = O(\log \log x)
\]

and so we have shown \( S \gg h(x)/(\log x)^E \gg x^{e/2} \) for some constant \( E \), whence we can choose \( C \) sufficiently large in \( h(x) = C x^\varepsilon \) such that the sum \( \sum e^{-S} < 1 \), proving the existence of a \( T \) not missing any elements in intervals \( (x, x + h(x)] \).

If we further restrict to avoiding only geometric progressions with integer ratios, then we have by a similar argument the same bound for \( k = 3 \).

**Proposition 9.** There exists a sequence \( T \) of positive integers \( \{t_i\}_{i \in \mathbb{N}} \), avoiding all 3-GP’s with integer ratios, such that for every \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon > 0 \),

\[
t_{i+1} - t_i < C_\varepsilon t_i^\varepsilon
\]

for all \( i \in \mathbb{N} \).

**Proof.** Enumerate the integer-ratio 3-GP’s of positive integers, such that the \( n \)-th one is

\[
(a_n, a_n r_n, a_n r_n^2)
\]
with integer ratio $r_n \geq 2$. Construct $T$ by removing $a_n r_n$ or $a_n r_n^2$ from each such progression, randomly and independently, with probabilities $1 - p$, $p$ respectively, where $p = p(a_n r_n^2)$ is chosen as the nondecreasing function

$$p(x) = 1 - \frac{1}{\log(x + 2)}.$$  

Then if $P_x$ is the probability a given element $x \in \mathbb{N}$ remains in $T$, it is bounded by

$$P_x \geq p(x)^{d(x)} (1 - p(x))^{d_2(x)}.$$  

Following the exact same computation as in the proof of Proposition 8, we get

$$P[T \cap (x + h(x)) = \emptyset] \leq e^{-S},$$

and by two applications of Theorem 2, $S$ is bounded by

$$S \gg h \exp \left( \log p(x) \log x + \log(1 - p(x)) \right) \gg h/\log(x)^E$$

for some constant $E > 0$. As $h(x) = C x^\varepsilon$, this shows that $S \gg x^{\varepsilon/2}$ whereby we can pick $C$ sufficiently large that $\sum e^{-S} < 1$, proving the existence of a $T$ intersecting all $(x, x + h(x))$. \qed

Unfortunately, substantial improvement via shortening the intervals in the bound of Nair and Tenenbaum seems unlikely, due to the fact that individual values of $d_{i,j}(n)$ can grow quite large. In particular, it is a classical theorem of Wigert that

$$\limsup_{n \to \infty} \frac{\log d(n)}{\log n / \log \log n} = \log 2,$$

and the upper behavior of general $d_{i,j}$ is similar. Proceeding directly by convexity and the bound of Nair and Tenenbaum, one cannot hope to do better than Theorem 1, except possibly by improving the constant.

**Closing Remarks and Open Questions**

The large gaps problems for primes and squarefree integers remain poorly understood. Whereas the largest gaps in both sequences are expected to be $O((\log x)^2)$, and certainly $O(x^\varepsilon)$ for every $\varepsilon > 0$, the best that has been shown is $O(x^{0.325})$ due to Baker, Harman, and Pintz [1] for the primes and $O(x^{\frac{3}{10}} \log x)$ by Filaseta and Trifonov [6] for the squarefree integers. In light of these difficulties, a solution of the short gaps problem for GP-free sequences should sidestep our lack of understanding of the distribution of primes and squarefree integers.

Using the Chinese Remainder Theorem, it is easy to construct runs of consecutive integers of order approximately $\log x$ that avoid squarefree integers (and thus primes as well). We expect that a natural barrier of a similar type occurs for sequences generated by the 6-GP-free process, barring it from directly solving the original question of Beiglböck, Bergelson, Hindman and Strauss on syndetic sequences. Nevertheless improving Theorem 1 to the order of $\log x$ would already be significant. We make the following conjecture.

**Conjecture 10.** There exists a constant $C > 0$ such that with nonzero probability, the sequence $T$ generated by the 6-GP-free process has gaps of size $O((\log x)^C)$. 
It was originally a conjecture of Cramér [4] that the primes behave on a large scale as if they are randomly chosen out of the integers, with $n$ prime with probability $(\log n)^{-1}$. Of course this agrees with the Prime Number Theorem, but for short intervals of size $O((\log x)^C)$ a number of significant discrepancies between Cramér's predictions and the distribution of primes have been found by Maier [10] and Pintz [16]. Our method is based on the intuition that a truly randomized sequence generated by the 6-GP-free process or a similar probabilistic construction avoids these issues while still carrying the required multiplicative structure.

In the other direction, using a computer search we have found that any sequence of positive integers with gaps of size at most one contains 3-GP’s. In particular, an exhaustive search of all sequences containing one of every pair of consecutive integers in $[1, 640]$ showed that all such sequences contain at least one 3-GP. This is weak evidence that the original problem of Beiglböck, Bergelson, Hindman and Strauss about syndetic sequences is unlikely to be true, and in fact all syndetic sequences must contain arbitrarily long $k$-term geometric progressions.

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E-mail address: xiaoyuhe@college.harvard.edu

Eliot House, Harvard College, Cambridge, MA 02138.