3-connected Reduction for Regular Graph Covers

Jiří Fiala · Pavel Klavík · Jan Kratochvíl · Roman Nedela

Abstract A graph $G$ covers a graph $H$ if there exists a locally bijective homomorphism from $G$ to $H$. We deal with regular covers in which this locally bijective homomorphism is prescribed by an action of a semiregular subgroup $\Gamma$ of $\text{Aut}(G)$; so $H \cong G/\Gamma$. Regular covers have many applications in constructions and studies of big objects all over mathematics and computer science.

In this paper, we study behaviour of regular graph covering with respect to 1-cuts and 2-cuts in $G$. An atom is an inclusion-minimal subgraph which is essentially 3-connected and cannot be further simplified. We use reductions which produce a series of graphs $G = G_0, \ldots, G_r$ such that $G_{i+1}$ is created from $G_i$ by replacing all atoms with colored edges. The primitive graph $G_r$ contains no atoms and it is either essentially 3-connected, or it is essentially a cycle, or $K_2$. An important property of our reductions is that a regular covering $G = G_0 \to H_0 = H$ induces a regular covering $G_1 \to H_1$ onto some graph $H_1$. Following the sequence of reductions we end with a covering $G_r \to H_r$, where $H_r = G_r/\Gamma_r$, for some semiregular $\Gamma_r \leq \text{Aut}(G_r)$.

Our aim is to revert the reduction and construct a quotient $H_0 = G_0/\Gamma_0$ from $H_r = G_r/\Gamma_r$. This process is called an expansion and it creates a series $H_r, H_{r-1}, \ldots, H_0$ with an extension series of semiregular subgroups $\Gamma_r, \Gamma_{r-1}, \ldots, \Gamma_0$. Inspired by Negami (1988), we show that an atom might have three types of quotients: the unique edge-quotient, the unique loop-quotient, and possibly several half-quotients.

The conference version of this paper appeared in ICALP 2014 [6]. For an interactive graphical map of the results, see [http://pavel.klavik.cz/orgpad/regular_covers.html](http://pavel.klavik.cz/orgpad/regular_covers.html) (supported for Google Chrome).

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quotients. Our main result states that all graphs \( H_i \) which can be created from \( H_{i+1} \) by replacing edges with the edge-quotients, loops with the loop-quotients and half-edges with some choices of half-quotients.

Applying the reductions and expansions to planar graphs, we obtain two structural results. First, we characterize the automorphism groups of planar graphs, similarly to Babai (1975). Second, we give a direct proof of Negami’s Theorem (1988) by describing all possible quotients of planar graphs. Also, this paper develops a theoretical background for an investigation of the complexity of testing whether a given input graph \( G \) regularly covers another input graph \( H \).

**Keywords** regular graph covers · 3-connected reduction · quotient expansion · half-quotients

1 Introduction

The notion of covering originates in topology to describe local similarity of two topological spaces. In this paper, we study coverings of graphs in a more restricting version called regular covering, for which the covering projection is described by a semiregular action of a group; see Section 2 for the formal definition. If \( G \) regularly covers \( H \), then \( H \) is called a quotient of \( G \). See Fig. 1 for an example.

Negami’s Conjecture. In 1988, Seiya Negami [11] formulated his famous conjecture stating that a connected graph \( H \) has a finite planar cover \( G \) if and only if \( H \) is projective planar. He was able to confirm this conjecture in the restricted case of regular coverings. We sketch his approach. If the graph \( G \) is 3-connected, the structure of automorphism groups of \( G \) is known and very simple, they are so-called spherical groups. Therefore the conjecture can be easily proved using geometry, since a quotient of the sphere is either the sphere, or the projective plane. The hard part of the proof is to deal with graphs \( G \) containing 1-cuts and 2-cuts.

Negami considered the minimal counterexample. In his proof, an essence of the crucial notion of an atom appears. A regular covering projection can behave on an atom in three different ways, and this understanding can be used to make the minimal counterexample smaller which forces a contradiction. In comparison, our work goes further and structurally describes all possible quotients \( H \) of a planar graph \( G \). Moreover, our results are independent of planarity in the following sense: If the behaviour of regular covering for 3-connected graphs in a considered class

![Fig. 1](image-url) A regular covering projection \( p \) from a graph \( G \) to one of its quotients \( H \). For every vertex \( v \in V(G) \), the image \( p(v) \) is written in the circle.
1.1 Our Results

In this paper, we study the behaviour of regular graph coverings with respect to 1-cuts and 2-cuts in \( G \). This problem is closely related to the problem on how 1- and 2-cuts behave under a semiregular action of a subgroup of the automorphism group \( \text{Aut}(G) \) of \( G \). For 1-cuts, it is quite simple since \( \text{Aut}(G) \) fixes the central block. But the behaviour of regular covering on 2-cuts is highly complex. We process the graph \( G \) by a series of reductions replacing some parts of \( G \), separated by 1- and 2-cuts, by edges. These reductions are inspired by the approach of Negami [11] and turn out to follow the same lines as the reductions introduced by Babai for studying the automorphism groups of planar graphs [12].

Atoms and Reductions. In Section 3, we introduce the important definition of an atom. Atoms are inclusion-minimal subgraphs which cannot be further simplified and are essentially 3-connected. The reduction constructs a series of graphs \( G = G_0, G_1, \ldots, G_r \). The reduction from \( G_i \) to \( G_{i+1} \) is done by replacing all the atoms of \( G_i \) by colored edges, where the colors code the isomorphism classes of atoms. In this process, we remove details from the graph but preserve its overall structure. Our definition of atoms is quite technical, dealing with many details necessary, so that it works nicely with respect to regular covering as it will become clear later.

The last (irreducible) graph in the sequence, denoted by \( G_r \), is called primitive. It is either very simple (\( K_2 \) or a cycle), or 3-connected. Therefore, we call this reduction process as the 3-connected reduction of \( G \).

When the graph \( G \) is not 3-connected, we consider its block-tree. The central block plays the key role in every regular covering projection. The reason is that a covering \( G \to H \) behaves non-trivially only on this central block; the remaining blocks are isomorphically preserved in \( H \). Therefore the atoms are defined with respect to the central block. We distinguish three types of atoms:

- **Proper atoms** are inclusion-minimal subgraphs separated by a 2-cut inside a block.
- **Dipoles** are formed by the sets of all parallel edges joining two vertices.
- **Block atoms** are blocks which are leaves of the block-tree, or stars of all pendant edges attached to a vertex. The central block is never a block atom.

The key properties of the automorphism groups are preserved by the reductions. More precisely, the reduction from \( G_i \) to \( G_{i+1} \) is defined in a way that an induced reduction epimorphism \( \Phi_i : \text{Aut}(G_i) \to \text{Aut}(G_{i+1}) \) possesses nice properties; see Proposition 4.1. Using it, we can describe the change of the automorphism group explicitly:

**Proposition 1.1** If \( G_i \) is reduced to \( G_{i+1} \), then

\[
\text{Aut}(G_{i+1}) = \text{Aut}(G_i)/\text{Ker}(\Phi_i).
\]

This shows that computing automorphism groups can be reduced to computing them for 3-connected graphs [11]. In [9,8], this is used to compute the automorphism groups of planar graphs since the automorphism groups of 3-connected graphs is understood, then we can extend this knowledge also to graphs with 1-cuts and 2-cuts in the class.
planar graphs are the automorphism groups of tilings of the sphere, and are well-understood. In Section 5, we use this to characterize the automorphism groups of planar graphs, similarly to Babai [1,2].

Expansions. We aim to investigate how the knowledge of regular quotients of \( G_{i+1} \) can be used to construct all regular quotients of \( G_i \). To do so, we introduce the reversal of the reduction called the expansion. If \( H_{i+1} = G_{i+1}/\Gamma_{i+1} \), then the expansion produces \( H_i \) by replacing colored edges back by atoms. To do this, we have to understand how regular covering behaves with respect to atoms. Inspired by Negami [11], we show that each proper atom/dipole has three possible types of quotients that we call an edge-quotient, a loop-quotient and a half-quotient. The edge-quotient and the loop-quotient are uniquely determined but an atom may have many non-isomorphic half-quotients.

The constructed quotients contain colored edges, loops and half-edges corresponding to atoms. Each half-edge in \( H_{i+1} \) is created from a halvable edge if an automorphism of \( \Gamma_{i+1} \) fixes this halvable edge and exchanges its endpoints. Roughly speaking it corresponds to cutting the edge in half. The following theorem is our main result and it describes every possible expansion of \( H_{i+1} \) to \( H_i \):

**Theorem 1.2** Let \( G_{i+1} \) be a reduction of \( G_i \). Every quotient \( H_i \) of \( G_i \) can be constructed from some quotient \( H_{i+1} \) of \( G_{i+1} \) by replacing each edge, loop and half-edge of \( H_{i+1} \) by the subgraph corresponding to the edge-, the loop-, or a half-quotient of an atom of \( G_i \), respectively.

Suppose that some regular quotient of the primitive graph \( G_r \) is chosen, so \( H_r = G_r/\Gamma_r \). The above theorem allows to describe all regular quotients \( H \) of \( G \) rising from \( H_r \), as depicted in the diagram in Fig. 2.

Algorithmic Applications. Our structural results have the following algorithmic implications, described in [6]. They allow to test for input graphs \( G \) and \( H \), where \( G \) is planar, whether \( G \) regularly covers \( H \) in time \( O(n^c \cdot 2^{|E(H)|}/2) \). By our description, there might be exponentially many quotients of \( G \), and so this algorithm is non-trivial since it has to test efficiently whether \( H \) is one of them. In particular, for every fixed graph \( H \), the constructed algorithm runs in polynomial time. This result contrasts with the complexity of general covering testing for which B́ılka et al. [4] prove that it is \( \mathsf{NP} \)-complete when \( G \) is planar and \( H \) is one of a few small fixed graph (e.g., \( K_4 \), \( K_5 \), or \( K_6 \)).
2 Definitions and Preliminaries

A multigraph $G$ is a pair $(V(G), E(G))$ where $V(G)$ is a set of vertices and $E(G)$ is a multiset of edges. We denote $|V(G)|$ by $v(G)$ and $|E(G)|$ by $e(G)$. The graph can possibly contain parallel edges and loops, and each loop at $u$ is incident twice with the vertex $u$. (So it contributes by two to the degree of $u$.) Each edge $e = uv$ gives rise to two half-edges, one attached to $u$ and the other to $v$. The collection of all half-edges is $H(G)$ and we denote $|H(G)|$ by $h(G)$; clearly $h(G) = 2e(G)$. In quotients, we sometime obtain graphs containing half-edges with free ends (missing the opposite half-edges).

We consider graphs with colored edges and also with three different edge types (directed edges, undirected edges and a special type called halvable edges). It might seem strange to consider such general objects. But when we apply reductions, we replace parts of the graph by edges and the colors encode isomorphism classes of replaced parts. Even if $G$ and $H$ are simple, the more general colored multigraphs are naturally constructed in the process of reductions.

2.1 Automorphisms and Groups

Automorphisms. We state the definitions in a very general setting of multigraphs and half-edges. An automorphism $\pi$ is fully described by a permutation $\pi_h : H(G) \to H(G)$ preserving edges and incidences between half-edges and vertices. Thus, $\pi_h$ induces two permutations $\pi_v : V(G) \to V(G)$ and $\pi_e : E(G) \to E(G)$ connected together by the very natural property $\pi_e(uv) = \pi_v(u)\pi_v(v)$ for every $uv \in E(G)$. In most of situations, we omit subscripts and simply use $\pi(u)$ or $\pi(uv)$. In addition, when we work with colored graphs, we require that an automorphism preserves the colors.

Groups. We assume that the reader is familiar with basic properties of groups; otherwise see [12]. We denote groups by Greek letters as for instance $\Psi$ or $\Gamma$. We use the following notation for some standard families of groups:

- $\mathfrak{S}_n$ for the symmetric group of all $n$-element permutations,
- $\mathfrak{C}_n$ for the cyclic group of integers modulo $n$,
- $\mathfrak{D}_n$ for the dihedral group of the symmetries of a regular $n$-gon, and
- $\mathfrak{A}_n$ for the alternating group of all even $n$-element permutations.

Automorphism Groups. Groups are quite often studied in the context of group actions, since their origin is in studying symmetries of mathematical objects. A group $\Psi$ acts on a set $S$ in the following way. Each $g \in \Psi$ permutes the elements of $S$, and the action is described by a mapping $\cdot : \Psi \times S \to S$ where $1 \cdot x = x$ and $(gh) \cdot x = g \cdot (h \cdot x)$. Usually, actions satisfy further properties that arise naturally from the structure of $S$.

In the language of graphs, an example of such an action is the group of all automorphisms of $G$, denoted by $\text{Aut}(G)$. Each element $\pi \in \text{Aut}(G)$ acts on $G$, permutes its vertices, edges and half-edges while it preserves edges and incidences between the half-edges and the vertices.

The orbit $[v]$ of a vertex $v \in V(G)$ is the set of all vertices $\{\pi(v) \mid \pi \in \Psi\}$, and the orbit $[e]$ of an edge $e \in E(G)$ is defined similarly as $\{\pi(e) \mid \pi \in \Psi\}$. The
Fig. 3 Two covers of $H$. The projections $p_v$ and $p'_v$ are written inside of the circles, and the projections $p_u$, $p'_u$, $p'_v$, and $p''_u$ are omitted. Notice that each loop is realized by having two neighbors labeled the same, and parallel edges are realized by having multiple neighbors labeled the same. Also covering projections preserve degrees.

The stabilizer of $x$ is the subgroup of all automorphisms which fix $x$. An action is called semiregular if it has no non-trivial (i.e., non-identity) stabilizers of both vertices and half-edges. Note that the stabilizer of an edge in a semiregular action may be non-trivial, since it may contain an involution transposing the two half-edges. We say that a group is semiregular if the associated action is semiregular. Thorough the paper, the letter $\Gamma$ is reserved for semiregular subgroups.

2.2 Coverings

A graph $G$ covers a graph $H$ (or $G$ is a cover of $H$) if there exists a locally bijective homomorphism $p$ called a covering projection. A homomorphism $p$ from $G$ to $H$ is given by a mapping $p_h : H(G) \rightarrow H(H)$ preserving edges and incidences between half-edges and vertices. It induces two mappings $p_v : V(G) \rightarrow V(H)$ and $p_e : E(G) \rightarrow E(H)$ such that $p_h(uv) = p_v(u)p_e(v)$ for every $uv \in E(G)$. The property to be local bijective states that for every vertex $u \in V(G)$ the mapping $p_h$ restricted to the half-edges incident with $u$ is a bijection. Figure 3 contains two examples of graph covers. Again, we mostly omit subscripts and just write $p(u)$ or $p(e)$. A fiber over a vertex $v \in V(H)$ is the set $p^{-1}(v)$, i.e., the set of all vertices $V(G)$ that are mapped to $v$, and similarly for fibers over edges and half-edges.

The Unique Walk Lifting Property. Let $uv \in E(H)$ be an edge which is not a loop. Then the set $p^{-1}(uv)$ corresponds to a perfect matching between the fibers $p^{-1}(u)$ and $p^{-1}(v)$. And if $uu \in E(H)$ is a loop, then the set $p^{-1}(uu)$ is a union of disjoint cycles which cover exactly $p^{-1}(u)$. Figure 4 shows examples. Further,
if \( h \in H(H) \) is a half-edge with free end based at \( u \in V(H) \), then the fiber \( p^{-1}(h) \) forms a perfect matching of \( p^{-1}(u) \). In general for a subgraph \( H' \) of \( H \), the preimages \( p^{-1}(H') \) in \( G \) are called lifts of \( H' \).

Let \( W \) be a walk \( u_0e_1u_1e_2\ldots e_nu_n \) in \( H \). Then \( p^{-1}(W) \) consists of \( k \) copies of \( W \). Suppose that we fix as the origin a vertex in the fiber \( p^{-1}(u_0) \). Due to the local bijectiveness of \( p \), there is exactly one edge incident with it mapped to \( e_1 \). Thus \( p^{-1}(u_1) \) is uniquely determined, and so on for \( p^{-1}(u_2) \) and for the other vertices of \( W \). When we proceed in this way, we conclude that the rest of the walk is uniquely determined. This important property of every covering is called the unique walk lifting property.

We adopt the standard assumption that both \( G \) and \( H \) are connected. Then as a simple corollary we get that all fibers of \( p \) are of the same size. To see that, observe that a path in \( H \) is lifted to a set of disjoint paths in \( G \). For \( u,v \in V(H) \), consider a path \( P \) between them. Then the paths in \( p^{-1}(P) \) define a bijection between the fibers \( p^{-1}(u) \) and \( p^{-1}(v) \). In other words, \(|G|=k|H|\) for some \( k \in \mathbb{N} \) which is the size of each fiber, and we say that \( G \) is a \( k \)-fold cover of \( H \).

Covering Transformation Groups. Every covering projection \( p \) defines a special subgroup of \( \text{Aut}(G) \) called the covering transformation group \( \text{CT}(p) \). It consists of all automorphisms \( \pi \) which preserve the fibers of \( p \), i.e., for every \( u \in V(G) \), the vertices \( u \) and \( \pi(u) \) belong to the same fiber, and similarly for edges and half-edges. Consider the graphs from Fig. 3. For the graph \( G \), we have \( \text{Aut}(G) = D_3 \) and \( \text{CT}(p) = \mathbb{C}_3 \). And \( G' \) has \( \text{Aut}(G') = \mathbb{C}_2 \) but \( \text{CT}(p') \) is trivial; and we note that it is often the case that a covering projection \( p \) has only one fiber-preserving automorphism, the trivial one.

Now suppose that \( \pi \in \text{CT}(p) \). Observe that a single choice of the image \( \pi(u) \) of one vertex \( u \in V(G) \) fully determines \( \pi \). This follows from the unique walk lifting property. Let \( v \in V(G) \) and consider some path \( P_{u,v} \) connecting \( u \) and \( v \) in \( G \). This path corresponds to a path \( P = p(P_{u,v}) \) in \( H \). Now we lift \( P \) and according to the unique walk lifting property, there exists a unique path \( P_{\pi(u),x} \) which starts in \( \pi(u) \). But since \( \pi \) is an automorphism and it maps \( P_{u,v} \) to \( P_{\pi(u),x} \), then \( x \) has to be equal \( \pi(v) \). In other words, we just proved that \( \text{CT}(p) \) is semiregular.

Regular Coverings. We aim to consider coverings which are highly symmetric. From the two examples from Fig. 3 the covering \( p \) is more symmetric than \( p' \). The size of \( \text{CT}(p) \) is a good measure of symmetricity of the covering \( p \). Since \( \text{CT}(p) \) is semiregular, it easily follows that \(|\text{CT}(p)| \leq k \) for any \( k \)-fold covering \( p \). A covering \( p \) is regular if \(|\text{CT}(p)| = k \). In Fig. 3 the covering \( p \) is regular since \(|\text{CT}(p)| = 3 \), and the covering \( p' \) is not regular since \(|\text{CT}(p')| = 1 \).

We use the following equivalent definition of regular covering. Let \( \Gamma \) be any semiregular subgroup of \( \text{Aut}(G) \). It defines a graph \( G/\Gamma \) called a regular quotient (or simply quotient) of \( G \) as follows: The vertices of \( G/\Gamma \) are the orbits of the action \( \Gamma \) on \( V(G) \), the half-edges of \( G/\Gamma \) are the orbits of the action \( \Gamma \) on \( H \). A vertex-orbit [\( v \)] is incident with a half-edge-orbit [\( h \)] if and only if the vertices of [\( v \)] are incident with the half-edges of [\( h \)]. (Because the action of \( \Gamma \) is semiregular, each vertex of [\( v \)] is incident with exactly one half-edge of [\( h \)], so this is well defined.) We naturally construct \( p : G \rightarrow G/\Gamma \) by mapping the vertices to its vertex-orbits and half-edges to its half-edge-orbits. Concerning an edge \( e \in E(G) \), it is mapped to an edge of \( G/\Gamma \) if the two half-edges belong to different half-edge-orbits of \( \Gamma \).
Fig. 5 The Hasse diagram of all quotients of the cube graph depicted in a geometric way. When semiregular actions fix edges, the quotients contain half-edges. The quotients connected by bold edges are obtained by 180 degree rotations. The quotients connected by dashed edges are obtained by reflections. The tetrahedron is obtained by the antipodal symmetry of the cube, and its quotient is obtained by a 180 degree rotation with the axis going through the centers of two non-incident edges of the tetrahedron.

If they belong to the same half-edge-orbits, it corresponds to a half-edge of $G/\Gamma$ with free end.

Since $\Gamma$ acts semiregularly on $G$, one can prove that $p$ is a $|\Gamma|$-fold regular covering with $\text{CT}(p) = \Gamma$. For the graphs $G$ and $H$ of Fig. 4, we get $H \cong G/\Gamma$ for $\Gamma \cong C_3$ which “rotates the cycle by three vertices”. As a further example, Fig. 5 geometrically depicts all quotients of the cube graph.

2.3 Block-trees and Their Automorphisms

The block-tree $T$ of $G$ is a tree defined as follows. Consider all articulations in $G$ and all maximal 2-connected subgraphs which we call blocks (with bridge-edges also counted as blocks). The block-tree $T$ is the incidence graph between the articulations and the blocks. For an example, see Fig. 6

There is the following well-known connection between $\text{Aut}(G)$ and $\text{Aut}(T)$:

**Lemma 2.1** Every automorphism $\pi \in \text{Aut}(G)$ induces an automorphism $\pi' \in \text{Aut}(T)$.

**Proof** First, observe that every automorphism $\pi$ of $G$ maps the articulations to the articulations and the blocks to the blocks which gives the induced mapping $\pi'$. It remains to show that $\pi'$ is an automorphism of $T$. Let $a$ be an articulation adjacent to a block $B$ in the tree. Then $a$ is contained in $B$. Therefore $\pi'(a)$ is
contained in $\pi'(B)$ and vice versa, which implies that $\pi'$ is an automorphisms of the block-tree $T$. \hfill \Box

We note that there is no direct relation between the structure of $\operatorname{Aut}(G)$ and $\operatorname{Aut}(T)$. First, $\operatorname{Aut}(T)$ may contain some additional automorphisms not induced by anything in $\operatorname{Aut}(G)$. Second, several distinct automorphisms of $\operatorname{Aut}(G)$ may induce the same automorphism of $\operatorname{Aut}(T)$. For example in Fig. 6, $\operatorname{Aut}(G) \cong D_3 \rtimes C_3^2$ and $\operatorname{Aut}(T) \cong S_6$.

The Central Block. Recall that for a tree, its center is either the central vertex or the central pair of vertices of a longest path, depending on the parity of its length. Every automorphism of a tree preserves its center.

Lemma 2.2 If $G$ has a non-trivial semiregular automorphism, then $G$ has a central block.

Proof For the block-tree $T$, all leaves are blocks, so each longest path is of an even length. Therefore $\operatorname{Aut}(T)$ preserves the central vertex. The central vertex can be either a central articulation, or a central block. If the central vertex is an articulation $u$, then every automorphism of $\operatorname{Aut}(T)$ fixes $u$. By Lemma 2.1, the same holds for every automorphism of $\operatorname{Aut}(G)$ which contradicts the assumptions. \hfill \Box

In the following, we shall assume that $T$ contains a central block. We orient the edges of the block-tree $T$ towards the central block; so the block-tree becomes rooted. A subtree of the block-tree is defined by any vertex different from the central block acting as root and by all its descendants.

Let $u$ be an articulation contained in the central block. By $T_u$ we denote the subtree attached to the central block at $u$.

Lemma 2.3 Let $\Gamma$ be a semiregular subgroup of $\operatorname{Aut}(G)$. If $u$ and $v$ are two articulations of the central block and of the same orbit of $\Gamma$, then $T_u \cong T_v$. Moreover there is a unique $\pi \in \Gamma$ which maps $T_u$ to $T_v$.

Proof Notice that either $T_u = T_v$, or $T_u \cap T_v = \emptyset$. Since $u$ and $v$ are in the same orbit of $\Gamma$, there exists $\pi \in \Gamma$ such that $\pi(u) = v$. Consequently $\pi(T_u) = T_v$. Suppose that there exist $\pi, \sigma \in \Gamma$ such that $\pi(T_u) = \sigma(T_u) = T_v$. Then $\pi \cdot \sigma^{-1}$ is an automorphism of $\Gamma$ fixing $u$. Since $\Gamma$ is semiregular, $\pi = \sigma$. \hfill \Box
3 Structural Properties of Atoms

In this section, we introduce special inclusion-minimal subgraphs of $G$ called atoms. We investigate their structural properties, in particular their behaviour with respect to regular covering projections.

3.1 Definition and Basic Properties of Atoms

Let $B$ be one block of $G$, so $B$ is a 2-connected graph. Two vertices $u$ and $v$ form a 2-cut $U = \{u, v\}$ if $B \setminus U$ is disconnected. We say that a 2-cut $U$ is non-trivial if $\deg(u) \geq 3$ and $\deg(v) \geq 3$.

Lemma 3.1 Let $U$ be a 2-cut and let $C$ be a component of $B \setminus U$. Then $U$ is uniquely determined by $C$.

Proof If $C$ is a component of $B \setminus U$, then $U$ has to be the set of all neighbors of $C$ in $B$. Otherwise $B$ would not be 2-connected, or $C$ would not be a component of $B \setminus U$.

The Definition. We first define a set $\mathcal{P}$ of subgraphs of $G$ called parts which are candidates for atoms:

- A block part is a subgraph non-isomorphic to $K_2$ induced by the blocks of a subtree of the block-tree. (Recall the definitions from Section 2.3)
- A proper part is a subgraph $S$ of $G$ defined by a non-trivial 2-cut $U$ of a block $B$. The subgraph $S$ consists of a connected component $C$ of $G \setminus U$ together with $u$ and $v$ and all edges between $\{u, v\}$ and $C$. In addition, we require that $S$ does not contain the central block; so it only contains some block of the subtree of the block-tree given by $B$.
- A dipole part is any dipole which is defined as follows. Let $u$ and $v$ be two distinct vertices of degree at least three joined by at least two parallel edges. Then the subgraph induced by $u$ and $v$ is called a dipole.

The inclusion-minimal elements of $\mathcal{P}$ are called atoms. We distinguish block atoms, proper atoms and dipoles according to the type of the defining part. Block atoms are either pendant stars called star block atoms, or pendant blocks possibly with single pendant edges attached to them called non-star block atoms. Also each proper atom is a subgraph of a block, together with some single pendant edges attached to it. Notice that a dipole part is by definition always inclusion-minimal, and therefore it is an atom. For an example, see Fig. 7.

We use the topological notation to denote the boundary $\partial A$ and the interior $\mathring{A}$ of an atom $A$. If $A$ is a dipole, we set $\partial A = V(A)$. If $A$ is a proper or block atom, we put $\partial A$ equal to the set of vertices of $A$ which are incident with an edge not contained in $A$. For the interior, we use the standard topological definition $\mathring{A} = A \setminus \partial A$ where we only remove the vertices $\partial A$, the edges adjacent to $\partial A$ are kept in $\mathring{A}$.

Note that $|\partial A| = 1$ for a block atom $A$, and $|\partial A| = 2$ for a proper atom or dipole $A$. The interior of a dipole is a set of free edges. We note that dipoles are exactly the atoms with no vertices in their interiors. Observe for a proper atom
An example of a graph with denoted atoms. The white vertices belong to the boundary of some atom, possibly several of them.

Let \( \partial A \) be the vertices of \( \partial A \) are exactly the vertices \( \{u, v\} \) of the non-trivial 2-cut used in the definition of proper parts. Also the vertices of \( \partial A \) of a proper atom are never adjacent in \( A \). Further, no block or proper atom contains parallel edges; otherwise a dipole would be its subgraph, so it would not be inclusion minimal.

**Non-overlapping Atoms.** Our goal is to replace atoms by edges, and so it is important to know that the atoms cannot overlap too much. The reader can see in Fig. 7 that the atoms only share their boundaries. This is true in general, and we are going to prove it in two steps.

**Lemma 3.2** The interiors of atoms are pairwise disjoint.

*Proof* For contradiction, let \( A \) and \( A' \) be two distinct atoms with non-empty intersections of \( \bar{A} \) and \( \bar{A}' \). First suppose that one of them, say \( A \), is a block item. Then \( A \) corresponds to a subtree of the block-tree which is attached by an articulation \( u \) to the rest of the graph. If \( A' \) is a block atom then it corresponds to some subtree, and we can derive that \( A \subseteq A' \) or \( A' \subseteq A \). And if \( A' \) is a dipole, then it is a subgraph of a block, and thus subgraph of \( A \). And if \( A' \) is a proper atom, it is defined with respect to some block \( B \). If \( B \) belongs to the subtree corresponding to \( A \), then \( A' \subseteq A \). Otherwise the subtree of \( A \) is attached to \( A' \), so \( A \subseteq A' \). In both cases, we get contradiction with minimality. Similarly, if one of the atoms is a dipole, we can easily argue contradiction with minimality.

Fig. 8 We depict the vertices of \( \partial A \) in black and the vertices of \( \partial A' \) in white. In both cases, we find a subset of \( A \) belonging to \( \mathcal{P} \) (its interior is highlighted in gray).
The last case is that both $A$ and $A'$ are proper atoms. Since the interiors are connected and the boundaries are defined as neighbors of the interiors, it follows that both $W' = A \cap \partial A'$ and $W = A' \cap \partial A$ are nonempty. We have two cases according to the sizes of these intersections depicted in Fig. 9.

If $|W| = |W'| = 1$, then $W \cup W'$ is a 2-cut separating $A \cap A'$ which contradicts minimality of $A$ and $A'$. And if, without loss of generality, $|W| = 2$, then there is no edge between $\bar{A} \setminus (A' \cup W')$ and the remainder of the graph $G \setminus (A \cup A')$. Therefore, $\bar{A} \setminus (A' \cup W')$ is separated by a 2-cut $W'$ which again contradicts minimality of $A$.

We note that in both cases the constructed 2-cut is non-trivial since it is formed by vertices of non-trivial cuts $\partial A$ and $\partial A'$.

Next we show a stronger version of the previous lemma which states that two atoms can intersect only in their boundaries.

**Lemma 3.3** Let $A$ and $A'$ be two atoms. Then $A \cap A' = \partial A \cap \partial A'$.

The proof is illustrated in Fig. 9. Suppose for contradiction that $\bar{A} \cap \partial A' \neq \emptyset$ and let $u' \in \bar{A}$. Since $u'$ has at least one neighbor in $\bar{A}'$, then without loss of generality $u \in \bar{A}'$ and $uu' \in E(G)$. Since $A$ is a proper atom, the set $\{u', v\}$ is not a 2-cut, so there is another neighbor of $u$ in $A$, which has to be equal $v'$. Symmetrically, $v'$ has another neighbor in $A'$ which is $v$. So $\partial A \subseteq A'$ and $\partial A' \subseteq A$. If $\partial A = A'$ and $\partial A' = A$, the graph is $K_4$ (since the minimal degree of cut-vertices is three) which contradicts existence of 2-cuts and atoms. And if for example $\bar{A} \neq \partial A'$, then $\partial A'$ does not cut a subset of $A$, so there has to a third neighbor $u'$ of $A'$, which contradicts that $\partial A'$ cuts $A'$ from the rest of the graph. \qed

**Connectivity of Atoms.** We call a graph *essentially 3-connected* if it is a 3-connected graphs with possibly single pendant edges attached to it. For instance, every non-star block atom is essentially 3-connected. A proper atom $A$ might not be essentially 3-connected. Let $\partial A = \{u, v\}$. We define $A^+$ as $A$ with the additional edge $uv$. It is easy to see that $A^+$ is an essentially 3-connected graph. Also, single pendant edges are always attached to $A$.

![Fig. 9](image.png) An illustration of the main steps of the proof of Lemma 3.3.
Lemma 3.4 Let $A$ be an essentially 3-connected graph, and we construct $B$ from $A$ by removing the single pendant edges of $A$. Then $\text{Aut}(A)$ is a subgroup of $\text{Aut}(B)$.

Proof These pendant single edges behave like markers, giving a 2-partition of $V(G)$ which $\text{Aut}(A)$ has to preserve.

\[ \square \]

3.2 Symmetry Types of Atoms

We distinguish three symmetry types of atoms which describe how symmetric each atom is. When such an atom is reduced, we replace it by an edge carrying the type. Therefore we work with multigraphs with three edge types: halvable edges, undirected edges and directed edges. We consider only the automorphisms which preserve these edge types and of course the orientation of directed edges.

Let $A$ be a proper atom or dipole with $\partial A = \{u, v\}$. Then we distinguish the following three symmetry types, see Fig. 10:

- The halvable atom. There exists a semiregular involutory automorphism $\tau \in \text{Aut}(A)$ which exchanges $u$ and $v$. More precisely, the automorphism $\tau$ fixes no vertices and no directed and undirected edges, but some halvable edges may be fixed.
- The symmetric atom. The atom is not halvable, but there exists an automorphism in $\text{Aut}(A)$ which exchanges $u$ and $v$.
- The asymmetric atom. The atom which is neither halvable, nor symmetric.

If $A$ is a block atom, then it is by definition symmetric.

Action of Automorphisms on Atoms. We show a simple lemma which states how automorphisms behave with respect to atoms.

Lemma 3.5 Let $A$ be an atom and let $\pi \in \text{Aut}(G)$. Then the following holds:

(a) The image $\pi(A)$ is an atom isomorphic to $A$. Further $\pi(\partial A) = \partial \pi(A)$ and $\pi(\bar{A}) = \bar{\pi}(A)$.
(b) If $\pi(A) \neq A$, then $\pi(A) \cap \bar{A} = \emptyset$.
(c) If $\pi(A) \neq A$, then $\pi(A) \cap A = \partial A \cap \partial \pi(A)$.

Proof (a) Every automorphism permutes the set of articulations and non-trivial 2-cuts. (Recall the definition from the first paragraph of Section 3.1) So $\pi(\partial A)$
separates $\pi(\bar{A})$ from the rest of the graph. It follows that $\pi(A)$ is an atom, since otherwise $A$ would not be an atom. And $\pi$ clearly preserves the boundaries and the interiors.

For the rest, (b) follows from Lemma 3.2 and (c) follows from Lemma 3.3. □

It follows that every automorphism $\pi \in \text{Aut}(G)$ gives a permutation of atoms and $\text{Aut}(G)$ induces an action on the set of all atoms.

3.3 Regular Projections and Quotients of Atoms.

Let $\Gamma$ be a semiregular subgroup of $\text{Aut}(G)$, which defines a regular covering projection $p : G \to G/\Gamma$. Negami [11, p. 166] investigated possible projections of proper atoms, and we investigate this question in more details. For a proper atom or a dipole $A$ with $\partial A = \{u, v\}$, we get one of the following three cases illustrated in Fig. 11.

(C1) The atom $A$ is preserved in $G/\Gamma$, meaning $p(A) \cong A$. Notice that $p(A)$ may just be a subgraph of $G/\Gamma$, not induced. For instance for a proper atom, it can happen that $p(u)p(v)$ is adjacent, even through $uv \notin E(G)$, as in Fig. 11.

(C2) The interior $\bar{A}$ is preserved and the vertices $u$ and $v$ are identified, i.e., $p(\bar{A}) \cong \bar{A}$ and $p(u) = p(v)$.

(C3) The covering projection $p$ is a $2k$-fold cover. There exists an involutory permutation $\pi$ in $\Gamma$ which exchanges $u$ and $v$ and preserves $A$. The projection $p(A)$ is a halved atom $\bar{A}$. This can happen only when $A$ is a halvable atom.

Fig. 11 The three cases for mapping of atoms (depicted in dots). Notice that for the third graph, a projection of the type (C1) could also be applied which would give a different quotient.
Lemma 3.6 For every atom $A$ and every semiregular subgroup $\Gamma$ defining the covering projection $p$, one of the cases (C1), (C2) and (C3) happens. Moreover, for a block atom we have exclusively the case (C1).

Proof For a block atom $A$, Lemma 2.3 implies that $p(A) \cong A$, so the case (C1) happens. It remains to deal with $A$ being a proper atom or a dipole, and let $\partial A = \{u, v\}$. According to Lemma 3.5b every automorphism $\pi$ either preserves $\hat{A}$, or $\hat{A}$ and $\pi(\hat{A})$ are disjoint. If there exists a unique non-trivial $\pi \in \Gamma$ which preserves $\hat{A}$, we get (C3); otherwise we get (C1) or (C2).

Let $\pi$ be the non-trivial automorphism of $\Gamma$ preserving $\hat{A}$. We know $\pi(\partial A) = \partial A$ and by semiregularity, $\pi$ has to exchange $u$ and $v$. Then the fiber containing $u$ and $v$ has to be of an even size, with $\pi$ being an involution reflecting $k$ copies of $A$, and therefore the covering $p$ is a $2k$-fold cover. This proves (C3).

Figure 12 shows how these projections $p(A)$ can look in $H_i$ depending on which of the three cases happens. So we get three types of quotients $p(A)$ of $A$. For (C1), we call this quotient an edge-quotient, for (C2) a loop-quotient and for (C3) a half-quotient. The following lemma allows to say “the” edge- and “the” loop-quotient of an atom.

Lemma 3.7 For every atom $A$, there is the unique edge-quotient and the unique loop-quotient up to isomorphism.

Proof For the cases (C1) and (C2), we have $\hat{A} \cong \hat{p}(A)$, so these quotients are uniquely determined.

For half-quotients, this uniqueness does not hold. First, an atom $A$ with $\partial A = \{u, v\}$ has to be halvable to admit a half-quotient. Then each half-quotient is determined by an involutory automorphism $\tau$ exchanging $u$ and $v$; here $\tau$ is the restriction of $\pi$ from (C3). There is a one-to-many relation between non-isomorphic half-quotients and automorphisms $\tau$, i.e., several different automorphisms $\tau$ may give the same half-quotient.

Lemma 3.8 A dipole $A$ has at most $\left\lfloor \frac{e(A)}{2} \right\rfloor + 1$ pairwise non-isomorphic half-quotients, and this bound is achieved.

![Fig. 12 How can $p(A)$ look in $G_i/\Gamma$, depending on the cases (C1), (C2) and (C3).]
Assuming that quotients can contain half-edges, the depicted dipole has four non-isomorphic half-quotients.

Proof Figure 13 shows a construction which achieves the bound. It remains to show that it is an upper bound. Without loss of generality, we can assume that all edges of this dipole are halvable. Let \( \tau \) be a semiregular involution. Edges which are fixed in \( \tau \) correspond to half-edges in the half-quotient \( A/\langle \tau \rangle \). Pairs of edges interchanged by \( \tau \) give rise to loops in \( A/\langle \tau \rangle \). In the quotient, we have \( \ell \) loops and \( h \) half-edges attached to a single vertex such that \( 2\ell + h = e(A) \). Since \( \ell \) is between 0 and \( \left\lfloor \frac{e(A)}{2} \right\rfloor \), the upper bound is established.

\[ \square \]

4 Graph Reductions and Quotient Expansions

We start with a quick overview. The reduction initiates with a graph \( G \) and produces a sequence of graphs \( G = G_0, G_1, \ldots, G_r \). To produce \( G_{i+1} \) from \( G_i \), we find the collection of all atoms \( A \) in \( G_i \) and replace each of them by an edge of the corresponding type. We stop after \( r \) steps when \( G_r \) contains no further atoms, and we call such a graph primitive. We call this sequence of graphs starting with \( G \) and ending with a primitive graph \( G_r \) as the reduction series of \( G \).

Suppose that \( H_r = G_r/\Gamma_r \) is a quotient of \( G_r \). The reductions applied to reach \( G_r \) are reverted on \( H_r \) and produce an expansion series \( H_r, H_{r-1}, \ldots, H_0 \) of \( H_r \). We obtain a series of semiregular subgroups \( \Gamma_r, \ldots, \Gamma_0 \) such that \( H_i = G_i/\Gamma_i \) and \( \Gamma_i \) extends \( \Gamma_{i+1} \). The entire process is depicted in the diagram in Fig. 3.

In this section, we describe structural properties of reductions and expansions. We study changes in automorphism groups done by reductions. Indeed, \( \text{Aut}(G_{i+1}) \) can differ from \( \text{Aut}(G_i) \). But the reduction is done right and important information of \( \text{Aut}(G_i) \) is preserved in \( \text{Aut}(G_{i+1}) \) which is key for expansions. The problem is that expansions are unlike reductions not uniquely determined. From \( H_{i+1} \), we can construct multiple \( H_i \). In this section, we characterize all possible ways how \( H_i \) can be constructed from \( H_{i+1} \), and thus establish Theorem 1.2.

4.1 Reducing Graphs Using Atoms

The reduction produces a series of graphs \( G = G_0, \ldots, G_r \). To construct \( G_{i+1} \) from \( G_i \), we find the collection of all atoms \( A \). We replace a block atom \( A \) by a pendant edge of some color based at \( u \) where \( \partial A = \{u\} \). We replace each proper atom or dipole \( A \) with \( \partial A = \{u, v\} \) by a new edge \( uv \) of some color and of one of the three edge types given by the symmetry type of \( A \):

- a halvable edge for a halvable atom \( A \),
- an undirected edge for a symmetric atom \( A \), and
- a directed edge for an asymmetric atom \( A \).
Fig. 14 On the left, we have a graph $G_0$ with three isomorphism classes of atoms, each having four atoms. The dipoles are halvable, the block atoms are symmetric and the proper atoms are asymmetric. We reduce $G_0$ to $G_1$ which is an eight cycle with single pendant edges, with four black halvable edges replacing the dipoles, four gray undirected edges replacing the block atoms, and four white directed edges replacing the proper atoms. The reduction series ends with $G_1$ since it is primitive. Notice the consistent orientation of the directed edges.

According to Lemma 3.3, the replaced the interiors of the atoms of $A$ are pairwise disjoint, so the reduction is well defined. We stop in the step $r$ when $G_r$ contains no atoms. We show in Lemma 4.6 that a primitive graph is either 3-connected, a cycle, or $K_2$ possibly with attached single pendant edges.

To be more precise, we consider graphs with colored edges and with three edge types: halvable, undirected and directed. We say that two graphs $G$ and $G'$ are isomorphic if there exists an isomorphism which preserves all colors and edge types, and we denote this by $G \cong G'$. We note that the results built in Section 3 transfers to colored graphs and colored atoms without any problems. Two atoms $A$ and $A'$ are isomorphic if there exists an isomorphism which maps $\partial A$ to $\partial A'$. We obtain isomorphism classes for the set of all atoms $A$ of $G_i$ such that $A$ and $A'$ belong to the same class if and only if $A \cong A'$.

To each isomorphism class, we assign one new color not yet used in the graph. When we replace atoms of $A$ by edges, we color these edges by the colors assigned to the isomorphism classes. It remains to say that for each isomorphism class of asymmetric atom, we consistently choose an arbitrary orientation of the directed edges replacing these atoms. For an example of the reduction, see Fig. 14.

The symmetry type of atoms depends on the types of edges the atom contains; see Fig. 13 for an example. Also, the figure depicts a quotient $G_2/\Gamma_2$ of $G_2$, and its expansions to $G_1/\Gamma_1$ and $G_0/\Gamma_0$. The resulting quotients $G_1/\Gamma_1$ and $G_2/\Gamma_2$ contain half-edges because $\Gamma_1$ and $\Gamma_2$ fix some halvable edges but $G_0/\Gamma_0$ contains no half-edges. This example shows that in reductions and expansions we need to consider half-edges even when the input graphs $G$ and $H$ are simple.

**Reduction Epimorphism.** We describe algebraic properties of the reductions, in particular how the groups $\text{Aut}(G_i)$ and $\text{Aut}(G_{i+1})$ are related. There exists a natural mapping $\Phi_i : \text{Aut}(G_i) \to \text{Aut}(G_{i+1})$ called *reduction epimorphism* which we define as follows. Let $\pi \in \text{Aut}(G_i)$. For the common vertices and edges of $G_i$ and $G_{i+1}$, we define $\Phi_i(\pi)$ exactly as in $\pi$. If $A$ is an atom of $G_i$, then according to Lemma 3.5a, $\pi(A)$ is an atom isomorphic to $A$. In $G_{i+1}$, we replace the interiors of
both $A$ and $\pi(A)$ by the edges $e_A$ and $e_{\pi(A)}$ of the same type and color. We define $\Phi_i(\pi)(e_A) = e_{\pi(A)}$. It is easy to see that each $\Phi_i(\pi) \in \text{Aut}(G_{i+1})$.

More precisely for purpose of Section 4.2, we define $\Phi_i$ on the half-edges. Let $e_A = uv$ and let $h_u$ and $h_v$ be the half-edges composing $e_A$, and similarly let $h_{\pi(u)}$ and $h_{\pi(v)}$ be the half-edges composing $e_{\pi(A)}$. Then we define $\Phi_i(\pi)(h_u) = h_{\pi(u)}$ and $\Phi_i(\pi)(h_v) = h_{\pi(v)}$.

**Proposition 4.1** The mapping $\Phi_i : \text{Aut}(G_i) \to \text{Aut}(G_{i+1})$ satisfies the following:

(a) The mapping $\Phi_i$ is a group homomorphism.

(b) The mapping $\Phi_i$ is an epimorphism, i.e., it is surjective.

(c) For a semiregular subgroup $\Gamma$ of $\text{Aut}(G_i)$, the mapping $\Phi_i|_\Gamma$ is an isomorphism.

Moreover, the subgroup $\Phi_i(\Gamma)$ remains semiregular.

**Proof** (a) Clearly, $\Phi_i(id) = id$. Let $\pi, \sigma \in \text{Aut}(G_i)$. We need to show that $\Phi_i(\sigma\pi) = \Phi_i(\sigma)\Phi_i(\pi)$. This is clearly true outside the interiors of the atoms. Let $A$ be an atom. By the definition, $\Phi_i(\sigma\pi)$ maps $e_A$ to $e_{\sigma(\pi(A))}$ while $\Phi_i(\pi)$ maps $e_A$ to $e_{\pi(A)}$ and $\Phi_i(\sigma)$ maps $e_{\sigma(A)}$ to $e_{\sigma(\pi(A))}$. So the equality holds everywhere and $\Phi_i$ is a group homomorphism.

(b) Let $\pi' \in \text{Aut}(G_{i+1})$, we want to extend $\pi'$ to $\pi \in \text{Aut}(G_i)$ such that $\Phi_i(\pi) = \pi'$. We just describe this extension on a single edge $e = uv$. If $e$ is an original edge of $G$, there is nothing to extend. Suppose that $e$ was created in $G_{i+1}$ from an atom $A$ in $G_i$. Then $\hat{e} = \pi'(e)$ is an edge of the same color and the same type as $e$, and therefore $\hat{e}$ is constructed from an isomorphic atom $\hat{A}$ of the same symmetry type. The automorphism $\pi'$ prescribes the action on the boundary $\partial A$. We need to show that it is possible to define an action on $\hat{A}$ consistently:

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Fig. 15 We reduce a part of a graph in two steps. In the first step, we replace five atoms by five edges of different types. As the result we obtain one halvable atom which we further reduce to one halvable edge. Notice that without considering edge types, the resulting atom would be just symmetric. In the bottom, we show a part of the corresponding quotient graphs when $\Gamma_i$ contains a semiregular involutory automorphism $\pi$ from (C3).
A is a block atom: The edges $e$ and $\bar{e}$ are pendant, attached by articulations $u$ and $u'$. We define $\pi$ by an isomorphism $\sigma$ from $A$ to $\hat{A}$ which takes $\partial A$ to $\partial \hat{A}$.

A is an asymmetric proper atom or dipole: By the definition, the orientation of $e$ and $\bar{e}$ is consistent with respect to $\pi'$. Since $\hat{A}$ is isomorphic to the interior of $A$, we define $\pi$ on $A$ according to one such isomorphism $\sigma$.

A is a symmetric or a halvable proper atom or a dipole: Let $\sigma$ be an isomorphism of $A$ and $\hat{A}$. Either $\sigma$ maps $\partial A$ exactly as $\pi'$, and then we can use $\sigma$ for defining $\pi$. Or we compose $\sigma$ with an automorphism of $A$ exchanging the two vertices of $\partial A$. (We know that such an automorphism exists since $A$ is not antisymmetric.)

So $\Phi_i$ is a surjective mapping.

(c) Recall that the kernel $\ker(\Phi_i)$ is the set of all $\pi$ such that $\Phi_i(\pi) = \text{id}$ and it is a normal subgroup of $\text{Aut}(G_i)$. It has the following structure: $\pi \in \ker(\Phi_i)$ if and only if it fixes everything except for the interiors of the atoms. Further, $\pi(\hat{A}) = \pi(A)$, so $\pi$ can non-trivially act only inside the interiors of the atoms.

For any subgroup $\Gamma$, the restricted mapping $\Phi_i|\Gamma$ is a group homomorphism with $\ker(\Phi_i|\Gamma) = \ker(\Phi_i) \cap \Gamma$. If $\Gamma$ is semiregular, then we show that $\ker(\Phi_i) \cap \Gamma$ is trivial. We know that $G_i$ contains at least one atom $A$. The boundary $\partial A$ is fixed by $\ker(\Phi_i)$, so by semiregularity of $\Gamma$ the intersection with $\ker(\Phi_i)$ is trivial. Hence $\Phi_i|\Gamma$ is an isomorphism.

For the semiregularity of $\Phi_i(\Gamma)$, let $\pi' \in \Phi_i(\Gamma)$. Since $\Phi_i|\Gamma$ is an isomorphism, there exists the unique $\pi \in \Gamma$ such that $\Phi_i(\pi) = \pi'$. If $\pi'$ fixes a vertex $u$, then $\pi$ fixes $u$ as well, so it is the identity, and $\pi' = \Phi_i(\text{id}) = \text{id}$. And if $\pi'$ only fixes an edge $e = uv$, then $\pi'$ exchanges $u$ and $v$. If $\pi$ also fixes $e$, this edge is halvable and so $\pi'$ can fix it as well. Otherwise there is an atom $A$ in $G_i$ replaced by $e$ in $G_{i+1}$. Then $\pi|A$ is an involutory semiregular automorphism exchanging $u$ and $v$, so $A$ is halvable. But then $e$ is a halvable edge, and thus $\pi'$ can fix it.

The above statement is an example of a phenomenon known in permutation group theory. Interiors of atoms behave as blocks of imprimitivity in the action of $\text{Aut}(G_i)$. It is well-known that the kernel of the action on the imprimitivity blocks is a normal subgroup of $\text{Aut}(G_i)$.

Now, we are ready to prove Proposition 4.1 which states that $\text{Aut}(G_{i+1}) \cong \text{Aut}(G_i)/\ker(\Phi_i)$.

Proof (Proposition 4.1) By Proposition 3.2, $\Phi_i$ is surjective, so by the well-known Homomorphism Theorem it follows that $\text{Aut}(G_{i+1}) \cong \text{Aut}(G_i)/\ker(\Phi_i)$. □

Corollary 4.2 We have $\text{Aut}(G_e) = \text{Aut}(G_0)/\ker(\Phi_{r-1} \circ \Phi_{r-2} \circ \cdots \circ \Phi_0)$.

Proof We have already proved that $\text{Aut}(G_{i+1}) = \text{Aut}(G_i)/\ker(\Phi_i)$. This equality easily follows from group theory. □

We can also describe the structure of $\ker(\Phi_i)$.

Lemma 4.3 The group $\ker(\Phi_i)$ is the direct product $\prod_{A \in A} \text{Fix}(A)$ where $\text{Fix}(A)$ is the point-wise stabilizer of $G_i \setminus \partial A$ in $\text{Aut}(G_i)$.

Proof According to Lemma 4.2 the interiors of the atoms are pairwise disjoint, so $\ker(\Phi_i)$ acts independently on each interior. Thus we get $\ker(\Phi_i)$ as the direct product of actions on each interior $A$ which is precisely $\text{Fix}(A)$. □
Alternatively, Fix(A) can be defined as the point-wise stabilizer of ∂A in Aut(A). Let $A_1, \ldots, A_s$ be pairwise non-isomorphic atoms in $G_i$, appearing with multiplicities $m_1, \ldots, m_s$. According to Lemma 4.3, we get

$$\text{Ker}(\Phi_i) \cong \text{Fix}(A_1)^{m_1} \times \cdots \times \text{Fix}(A_s)^{m_s}.$$ 

For the example of Fig. 14, we have $\text{Ker}(\Phi_0) \cong C_4^2 \times C_4^2 \times S_4^4$. 

### Semidirect Product

By Proposition 1.1, we know that $\text{Aut}(G_i)$ is an extension of $\text{Aut}(G_{i+1})$ by $\text{Ker}(\Phi_i)$. Our aim is to investigate when it is a semidirect product.

Let $A$ be an atom with $\partial A = \{u, v\}$. If $A$ is halvable, then by the definition there exists an involutory automorphism of $A$ exchanging $u$ and $v$ (which is even semiregular). On the other hand, if $A$ is symmetric, the definition states that there only exists some automorphism exchanging $u$ and $v$. If $A$ is a symmetric dipole, one can always find an involution exchanging $u$ and $v$. This is not true when $A$ is a symmetric proper atom, as illustrated in Fig. 16.

**Proposition 4.4** Suppose that every symmetric proper atom $A$ of $G_i$ with $\partial A = \{u, v\}$ has an involution automorphism $\tau$ exchanging $u$ and $v$. Then

(a) there exists $\Psi \leq \text{Aut}(G_i)$ such that $\Phi_i(\Psi) = \text{Aut}(G_{i+1})$ and $\Phi_i|\Psi$ is an isomorphism, and

(b) $\text{Aut}(G_i) \cong \text{Aut}(G_{i+1}) \rtimes \text{Ker}(\Phi_i)$.

**Proof** (a) We have already described in Proposition 4.1 how to extend $\pi' \in \text{Aut}(G_{i+1})$ on the interiors of the atoms to get an automorphism $\pi \in \text{Aut}(G_i)$. To establish (a), we need to do this consistently, in such a way that these extensions form a subgroup $\Psi$ which is isomorphic to $\text{Aut}(G_{i+1})$.

Let $e_1, \ldots, e_\ell$ be colored edges of one orbit of the action of $\text{Aut}(G_{i+1})$ such that these edges replace isomorphic atoms $A_1, \ldots, A_\ell$ in $G_i$; see Fig. 17 for an overview. As in Proposition 4.1, we divide the argument into three cases:

**Case 1:** The atom $A_1$ is a block atom: Let $u_1, \ldots, u_\ell$ be the articulations such that $\partial A_1 = \{u_i\}$. Choose arbitrarily isomorphisms $\sigma_{i,1}$ from $A_1$ to $A_i$ such that $\sigma_{i,1}(u_1) = u_i$, and put $\sigma_{1,1} = \text{id}$ and $\sigma_{i,j} = \sigma_{1,j}\sigma_{i,1}^{-1}$. If $\pi'(e_i) = e_j$, we set $\pi'_A = \sigma_{i,j}|_A$. Since

$$\sigma_{i,k} = \sigma_{j,k}\sigma_{i,j}, \quad \forall i, j, k,$$

the composition of the extensions $\pi_1$ and $\pi_2$ of $\pi'_1$ and $\pi'_2$ is defined on the interiors of $A_1, \ldots, A_\ell$ exactly as the extension of $\pi_2\pi_1$. Also, by (1), an identity $\pi'_k\pi'_{k-1} \cdots \pi'_1 = \text{id}$ is extended to an identity.

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Case 2: The atom $A_1$ is an asymmetric proper atom or dipole: Let $e_i = u_i v_i$. We approach it exactly as in Case 1, just we require that $\sigma_{1,i}(u_1) = u_i$ and $\sigma_{1,i}(v_1) = v_i$.

Case 3: The atom $A_1$ is a symmetric or a halvable proper atom or a dipole: For each $e_i$, we arbitrarily choose one endpoint as $u_i$ and one as $v_i$. Again, we arbitrarily choose isomorphisms $\sigma_{1,i}$ from $A_1$ to $A_i$ such that $\sigma_{1,i}(u_1) = u_i$ and $\sigma_{1,i}(v_1) = v_i$, and define $\sigma_{i,j} = \sigma_{1,j} \sigma_{1,i}^{-1}$.

We further consider an involution $\tau_1$ of $A_1$ which exchanges $u_1$ and $v_1$. (Such an involution exists for symmetric proper atoms by the assumptions, and for halvable atoms and symmetric dipoles by the definition.) Then $\tau_1$ defines an involution of $A_i$ by conjugation as $\tau_i = \sigma_{1,i} \tau_1 \sigma_{1,i}^{-1}$. It follows that

$$\tau_j = \sigma_{i,j} \tau_i \sigma_{i,j}^{-1}, \quad \text{and consequently} \quad \sigma_{i,j} \tau_i = \tau_j \sigma_{i,j}, \quad \forall i, j.$$  

We put $\hat{\sigma}_{i,j} = \sigma_{i,j} \tau_i$ which is an isomorphism mapping $A_i$ to $A_j$ such that $\hat{\sigma}_{i,j}(u_i) = v_j$ and $\hat{\sigma}_{i,j}(v_i) = u_j$. In the extension, we put $\pi|_{A_i} = \sigma_{i,j}|_{A_i}$ if $\pi'(u_i) = v_j$, and $\pi|_{A_i} = \sigma_{1,j}|_{A_i}$ if $\hat{\pi}(u_i) = v_j$.

Aside 4, we get the following additional identities:

$$\hat{\sigma}_{i,k} = \sigma_{j,k} \hat{\sigma}_{i,j}, \quad \hat{\sigma}_{i,k} = \hat{\sigma}_{j,k} \sigma_{i,j}, \quad \text{and} \quad \sigma_{i,k} = \hat{\sigma}_{j,k} \hat{\sigma}_{i,j}, \quad \forall i, j, k. \quad (2)$$

We just argue the last identity:

$$\hat{\sigma}_{j,k} \hat{\sigma}_{i,j} = \tau_k (\sigma_{j,k} \sigma_{i,j}) \tau_i = \tau_k \sigma_{i,k} \tau_i = \tau_k \tau_k \sigma_{i,k} = \sigma_{i,k}, \quad \text{where the last equality holds since } \tau_k \text{ is an involution. It follows that the composition } \tau_2 \tau_1 \text{ is correctly defined as above, and it maps identities to identities.}$$

We have described how to extend the elements of $\text{Aut}(G_{i+1})$ on one edge-orbit, and we apply this process repeatedly to all edge-orbits. The set $\Psi \leq \text{Aut}(G_i)$ consists of all these extensions $\pi$ from every $\pi' \in \text{Aut}(G_{i+1})$. It is a subgroup by (4) and (5), and since the extension $\pi' \mapsto \pi$ is injective, $\Psi \cong \text{Aut}(G_{i+1})$.

(b) By (a), we know that $\text{Ker}(\Phi_i) \subseteq \text{Aut}(G_i)$ has a complement $\Psi$ isomorphic to $\text{Aut}(G_{i+1})$. Actually, this already proves that $\text{Aut}(G_i)$ has the structure of the internal semidirect product.

Fig. 17 Case 1 is depicted on the left for three edges corresponding to isomorphic block atoms $A_1, A_2$ and $A_3$. The depicted isomorphisms are used to extend $\text{Aut}(G_{i+1})$ on the interiors of these atoms. Case 3 is on the right, with an additional semiregular involution $\tau_1$ which transposes $u_1$ and $v_1$. 

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We give more insight into its structure by describing it as an external semidirect product. Each element of Aut($G_i$) can be written as a pair $(\pi', \sigma)$ where $\pi' \in \text{Aut}(G_{i+1})$ and $\sigma \in \text{Ker}(\Phi_i)$. We first apply the extension $\pi \in \Psi$ of $\pi'$ and permute $G_i$, mapping interiors of the atoms as blocks. Then $\sigma$ permutes the interiors of the atoms, preserving the remainder of $G_i$.

It remains to understand how composition of two automorphisms $(\pi', \sigma)$ and $(\pi'', \tilde{\sigma})$ works. We get this as a composition of four automorphisms $\tilde{\sigma} \circ \pi \circ \sigma \circ \pi$, which we want to write as a pair $(\tau, \rho)$. Therefore, we need to swap $\tilde{\pi}$ with $\sigma$. This clearly preserves $\hat{A}$, since the action $\tilde{\sigma}$ on the interiors does not influence it; so we get $\tau = \tilde{\pi} \circ \sigma$.

But $\sigma$ is changed by this swapping. According to Lemma 4.3 we get $\sigma = (\sigma_1, \ldots, \sigma_s)$ where each $\sigma_i \in \text{Fix}(A_i)^{\mu_i}$. Since $\pi$ preserves the isomorphism classes of atoms, it acts on each $\sigma_i$ independently and permutes the isomorphic copies of $A_i$. Suppose that $A$ and $A'$ are two isomorphic copies of $A_i$ and $\pi(A) = A'$. Then the action of $\sigma_i$ on the interior of $A$ corresponds after the swapping to the same action on the interior of $A' = \pi(A)$. This can be described using the semidirect product, since each $\pi$ defines an automorphism of Ker($\Phi_i$) which permutes the coordinates of each $\text{Fix}(A_i)^{\mu_i}$, following the action of $\pi'$ on the colored edges of $G_{i+1}$. □

For the example in Fig. 13 $G_0$ has no symmetric proper atoms, so

$$\text{Aut}(G_1) \cong \mathbb{C}_2^2$$ and $$\text{Aut}(G_0) \cong \mathbb{C}_2^2 \ltimes (\mathbb{C}_2^4 \times \mathbb{C}_2^4 \times S_4^1).$$

**Primitive Graphs.** Recall that a graph is called primitive if it contains no atoms. We want to describe the structure of primitive graphs. First, we show that the central block is preserved:

**Lemma 4.5** Let $G$ admit a non-trivial semiregular automorphism $\pi$. Then each $G_{i+1}$ has a central block which is obtained from the central block of $G_i$ by replacing its atoms by colored edges.

**Proof** By Proposition 4.1, semiregular automorphisms are preserved during the reduction. By Lemma 2.2 each $G_i$ has a central block. Since we replace only proper atoms and dipoles in the central block, it remains to be a block after the reduction. We argue by induction that it remains central as well.

Let $B$ be the central block of $G_i$ and let $B'$ be this block in $G_{i+1}$. Consider the subtree $T_u'$ of the block tree $T'$ of $G_{i+1}$ attached to $B'$ in $u$ containing the longest path in $T'$ from $B'$. This subtree corresponds to $T_u$ in $G_i$. (See Section 2.3 for the definition of $T_u$.) Let $\pi$ be a non-trivial semiregular automorphism in $G_i$. Then $\pi(u) = v$, and by Lemma 2.3 we have $T_v \cong T_u$. Then $T_v'$ corresponds in $G_{i+1}$ to $T_v$ after reduction and $T_v' \cong T_v$. Therefore $B'$ is the central block of $G_{i+1}$. □

If $G$ has a non-trivial semiregular automorphism, then its central block is preserved in the primitive graph $G_r$. Therefore, we can assume that every primitive graph has a central block.

**Lemma 4.6** Let $G$ be a graph with a central block. Then $G$ is primitive if and only if it is a 3-connected graph, a cycle $C_n$ for $n \geq 2$, or $K_2$, or can be obtained from these graphs by attaching single pendant edges to at least two vertices.
Proof Clearly, the graphs mentioned in the statement are primitive; see Fig. 18.
For the other implication, the graph $G$ contains a central block. All blocks attached
to it have to be single pendant edges, otherwise $G$ would contain a block atom. By
removal of all pendant edges, we get the 2-connected graph $B$ consisting of only
the central block. We argue that $B$ is one of the stated graphs.

Now, let $u$ be a vertex of the minimum degree in $B$. If $\deg(u) = 1$, the graph
$B$ has to be $K_2$, otherwise it would not be 2-connected. If $\deg(u) = 2$, then either
the graph $B$ is a cycle $C_n$, or $u$ is an inner vertex of a path connecting two vertices
$x$ and $y$ of degree at least three such that all inner vertices are of degree two. But
then this path is an atom, a contradiction. Finally, if $\deg(u) \geq 3$, then every 2-cut
is non-trivial, and since $B$ contains no atoms, it has to be 3-connected. \qed

We note that if the existence of a central block is not required and we define
atoms with respect to the central articulation, then in addition the primitive graph
can be $K_1$.

4.2 Quotients and Their Expansion

Let $G_0, \ldots, G_r$ be the reduction series of $G$ and let $\Gamma_0$ be a semiregular subgroup
of $\text{Aut}(G_0)$. By repeated application of Proposition 4.1c, we get the uniquely
determined semiregular subgroups $\Gamma_1, \ldots, \Gamma_r$ of $\text{Aut}(G_1), \ldots, \text{Aut}(G_r)$ such that
$\Gamma_{i+1} = \Phi_i(\Gamma_i)$, each isomorphic to $\Gamma_0$. Let $H_i = G_i/\Gamma_i$ be the quotients with pre-
served colors of edges, and let $p_i$ be the corresponding covering projection from $G_i$
to $H_i$. Recall that $H_i$ can contain edges, loops and half-edges; depending on the
action of $\Gamma_i$ on the half-edges corresponding to the edges of $G_i$.

Lemma 4.7 Every semiregular subgroup $\Gamma_i$ of $\text{Aut}(G_i)$ corresponds to a unique semi-
regular subgroup $\Gamma_{i+1}$ of $\text{Aut}(G_{i+1})$ such that $\Gamma_{i+1} = \Phi_i(\Gamma_i)$. \qed

Quotients Reductions. Consider $H_i = G_i/\Gamma_i$ and $H_{i+1} = G_{i+1}/\Gamma_{i+1}$. We inves-
tigate relations between these quotients. Let $A$ be an atom of $G_i$ represented by
a colored edge $e$ in $G_{i+1}$. According to Lemma 4.7, we have three possible cases
(C1), (C2) and (C3) for the projection $p_i(A)$. It is easy to see that $\Phi_i$ is defined
exactly in the way that $p_{i+1}(e)$ corresponds to an edge in the case (C1), to a loop
in the case (C2) and to a half-edge in the case (C3). This explains the names of
the quotients $p_i(A)$ as the edge-quotient, the loop-quotient and a half-quotient. Figure 19 shows examples. We get the following commuting diagram:

\[
\begin{array}{ccc}
G_i & \xrightarrow{\text{red.}} & G_{i+1} \\
\downarrow \Phi_i & & \downarrow \Phi_{i+1} \\
H_i & \xrightarrow{\text{red.}} & H_{i+1}
\end{array}
\]

(3)

So we can construct the graph $H_{i+1}$ from $H_i$ by replacing the projections of atoms in $H_i$ by the corresponding projections of the edges replacing the atoms.

**Overview of Quotients Expansions.** Our goal is to reverse the horizontal edges in Diagram (3), i.e., to understand:

\[
\begin{array}{ccc}
G_i & \xleftarrow{\text{exp.}} & G_{i+1} \\
\downarrow \Phi_i & & \downarrow \Phi_{i+1} \\
H_i & \xleftarrow{\text{exp.}} & H_{i+1}
\end{array}
\]

(4)

Let $\Gamma_i$ and $\Gamma_{i+1}$ be semiregular groups such that $\Phi_i(\Gamma_i) = \Gamma_{i+1}$. Then we call $\Gamma_{i+1}$ a reduction of $\Gamma_i$, and $\Gamma_i$ an extension of $\Gamma_{i+1}$. There are two fundamental questions we address in this section in full details:

- **Question 1.** Given a group $\Gamma_{i+1}$, which semiregular groups $\Gamma_i$ are its extensions? Notice that all these groups $\Gamma_i$ are isomorphic to $\Gamma_{i+1}$ as abstract groups, but they correspond to different actions on $G_i$.

- **Question 2.** Let $\Gamma_i$ and $\Gamma_i'$ be two semiregular groups extending $\Gamma_{i+1}$. Under which conditions are the quotients $H_i = G_i/\Gamma_i$ and $H_i' = G_i/\Gamma_i'$ different?

**Extensions of Group Actions.** We first deal with Question 1.

**Lemma 4.8** For every semiregular group $\Gamma_{i+1}$, there exists an extension $\Gamma_i$.

**Proof** First notice that $\Gamma_{i+1}$ determines the action of $\Gamma_i$ everywhere on $G_i$ except for the interiors of the atoms of $G_i$, so we just need to define it there. Let $e = uv$
be one edge of \(G_{i+1}\) replacing an atom \(A\) in \(G_i\). First, we assume that \(A\) is not a block atom. Let \(|\Gamma_{i+1}| = k\). We distinguish two cases. Either the orbit \([e]\) contains exactly \(k\) edges, or it contains \(\frac{k}{2}\) edges. Our approach is similar to the proof of Proposition 4.4a.

Case 1: The orbit \([e]\) contains exactly \(k\) edges. Let \(e_1, \ldots, e_k\) be the orbit \([e]\) and let \(u_i = \pi'(u)\) and \(v_i = \pi'(v)\) for the unique \(\pi' \in \Gamma_{i+1}\) mapping \(e\) to \(e_i\). (We know that \(\pi'\) is unique because \(\Gamma_{i+1}\) is semiregular.) Let \(A_1, \ldots, A_k\) be the atoms of \(G_i\) corresponding to \(e_1, \ldots, e_k\) in \(G_{i+1}\). The edges \(e_1, \ldots, e_k\) have the same color and type, and thus the atoms \(A_i\) are pairwise isomorphic and of the same type.

We define the action of \(\Gamma_i\) on the interiors of \(A_1, \ldots, A_k\) exactly as in Proposition 4.4a for asymmetric atoms, using \(\sigma_i, j\). Let \(\pi' \in \Gamma_{i+1}\) and the extension \(\pi\) can be defined on the interiors as follows. If \(\pi'\) maps \(e_i\) to \(e_j\), we set \(\pi|_{A_i} = \sigma_{i,j}\). We note that for symmetric proper atoms, the existence of an involution \(\tau\) as in Proposition 4.4a is not required since \(\Gamma_{i+1}\) is semiregular.

Case 2: The orbit \([e]\) contains exactly \(\ell = \frac{k}{2}\) edges. Then we have \(k\) half-edges in one orbit, so in \(H_{i+1}\) we get one half-edge. Let \(e_1, \ldots, e_\ell\) be the edges of \([e]\). They have to be halvable, and consequently the corresponding atoms \(A_1, \ldots, A_\ell\) are halvable. Let \(u_i\) be an arbitrary endpoint of \(e_i\) and let \(v_i\) be the second endpoint of \(e_i\). Let \(\tau = \tau_1\) be any involutory semiregular automorphism of \(A_1\) which maps \(u_1\) to \(v_1\); we know that such \(\tau_1\) exists since \(A_1\) is a halvable atom.

Similarly as in Proposition 4.4a for symmetric and halvable atoms, we define \(\sigma_i, j, \tau_i\) and \(\sigma_i, j\). Let \(\pi' \in \Gamma_{i+1}\) and \(\pi'(e_i) = e_j\). To define the extension \(\pi\), we set \(\pi|_{A_i} = \sigma_{i,j}\) if \(\pi'(u_i) = u_j\), and \(\pi|_{A_i} = \sigma_{i,j}\) if \(\pi'(v_i) = v_j\).

We deal with block atoms in a similar manner as in Case 1, except the orbit \([u]\) consists of articulations, and the orbit \([v]\) consists of leaves. It is easy to observe that by semiregularity of \(\Gamma_{i+1}\) the constructed group \(\Gamma_i\) acts semiregularly on \(G_i\), as well.

\[\square\]

Corollary 4.9 The construction in the proof of Lemma 4.8 gives all possible extensions of \(\Gamma_{i+1}\).

Proof We get all possible choices for \(\Gamma_i\) in Case 1 by different choices of \(\sigma_{1,i}\), and in Case 2 by different choices of \(\sigma_{1,i}\) and \(\tau\). \[\square\]

Quotient Expansion. Recall the description of quotients of atoms from Section 4.3. We are ready to establish the main theorem of this paper. It states that every quotient \(H_i\) of \(G_i\) can be created from some quotient \(H_{i+1}\) of \(G_{i+1}\) by replacing edges, loops and half-edges of atoms replaced in the reduction from \(G_i\) to \(G_{i+1}\) with corresponding edge-, loop- and half-quotients.

Proof (Theorem 4.2) Let \(H_{i+1} = G_{i+1}/\Gamma_{i+1}\) and let \(H_i\) be constructed in the above way. We first argue that \(H_i\) is a quotient of \(G_i\), i.e., it is equal to \(G_i/\Gamma_i\) for some \(\Gamma_i\) extending \(\Gamma_{i+1}\). To see this, it is enough to construct \(\Gamma_i\) in the way described in the proof of Lemma 4.8. We choose \(\sigma_{1,i}\) arbitrarily, and the involutory permutations \(\tau\) are prescribed by chosen half-quotients replacing half-edges. It is easy to see that the resulting graph is the constructed \(H_i\). We note that only the choices of \(\tau\) matter, for arbitrary choices of \(\sigma_{1,i}\) we get isomorphic quotients \(H_i\).

On the other hand, if \(H_i\) is a quotient, it replaces the edges, loops and half-edges of \(H_{i+1}\) by some quotients, so we can generate \(H_i\) in this way. The reason...
is that according to Corollary 4.9 we can generate all $\Gamma_i$ extending $\Gamma_{i+1}$ by some choices $\sigma_{1,i}$ and $\tau$.

We say that two quotients $H_i$ and $H'_i$ extending $H_{i+1}$ are different if there exists no isomorphism of $H_i$ and $H'_i$ which fixes the vertices and edges common with $H_{i+1}$. (But $H_i$ and $H'_i$ still might be isomorphic.) According to Lemma 3.7 the edge and loop-quotients are uniquely determined, so we are only free in choosing half-quotients. For non-isomorphic choices of half-quotients, we get different graphs $H_i$. For instance suppose that $H_{i+1}$ contains a half-edge corresponding to the dipole from Fig. 20. Then in $H_i$ we can replace this half-edge by one of the four possible half-quotients of this dipole.

**Corollary 4.10** If $H_{i+1}$ contains no half-edge, then $H_i$ is uniquely determined. Thus, for an odd order of $\Gamma_r$, the quotient $H_r$ uniquely determines $H_0$.

**Proof** This is implied by Theorem 1.2 and Lemma 3.7 which states that edge- and loop-quotients are uniquely determined. If the order of $\Gamma_r$ is odd, no half-edges are constructed in $H_r$, so no half-quotients ever appear. $\square$

**Half-quotients of Dipoles.** In Lemma 3.8 we describe that a dipole $A$ without colored edges can have at most $\lfloor \frac{e(A)}{2} \rfloor + 1$ pairwise non-isomorphic half-quotients. This statement can be easily altered to dipoles with colored edges which admit a much larger number of half-quotients:

**Lemma 4.11** Let $A$ be a dipole with colored edges. Then the number of pairwise non-isomorphic half-quotients is bounded by $2^{\lfloor \frac{e(A)}{2} \rfloor}$ and this bound is achieved.

**Proof** Figure 20 shows a construction of dipoles achieving the upper bound. It remains to argue correctness of the upper bound.

First, we derive the structure of all involutory semiregular automorphisms $\tau$ acting on $A$. We have no freedom concerning the non-halvable edges of $A$: The undirected edges of each color class has to paired by $\tau$ together. Further, each directed edge has to be paired with a directed edges of the opposite direction and the same color. It remains to describe possible action of $\tau$ on the remaining at most $e(A)$ halvable edges of $A$. These edges belong to $c$ color classes having $m_1, \ldots, m_c$ edges. Each automorphism $\tau$ has to preserve the color classes, so it acts independently on each class.

We concentrate only on one color class having $m_i$ edges. We bound the number $f(m_i)$ of pairwise non-isomorphic quotients of this class. Then we get the upper bound $2^{\lfloor \frac{e(A)}{2} \rfloor}$.

![Fig. 20](image-url) An example of a dipole with a pair of black halvable edges and a pair of white halvable edges. There exist four pairwise non-isomorphic half-quotients. This example can easily be generalized to exponentially many pairwise non-isomorphic half-quotients by introducing more pairs of halvable edges of additional colors.
\[ \prod_{1 \leq i \leq c} f(m_i) \]  

for the number of non-isomorphic half-quotients of \( A \).

The rest of the proof is similar to the proof of Lemma 3.8. An edge \( e \) fixed in \( \tau \) is mapped into a half-edge of the given color in the half-quotient \( A/\langle \tau \rangle \). And if \( \tau \) maps \( e \) to \( e' \neq e \), then we get a loop in the half-quotient \( A/\langle \tau \rangle \). The resulting half-quotient only depends on the number of fixed edges and fixed two-cycles in the considered color class. We can construct at most \( f(m_i) = \lfloor \frac{m_i}{2} \rfloor + 1 \) pairwise non-isomorphic quotients, since we may have zero to \( \lfloor \frac{m_i}{2} \rfloor \) loops with the complementing number of half-edges.

The bound (5) is maximized when each class contains exactly two edges. (Except for one class containing either three edges, or one edge if \( e(A) \) is odd.) \( \Box \)

Assume that \( H_{i+1} \) contains a half-edge corresponding to a half-quotient of a dipole in \( H_i \). By Theorem 1.2, the number of non-isomorphic expansions \( H_i \) of \( H_{i+1} \) can be exponential in the size difference of \( H_i \) and \( H_{i+1} \).

The Block Structure of Quotients. We show how the block structure is preserved during expansions. A block atom \( A \) of \( G_i \) is always projected by (C1), and so it corresponds to a block atom of \( H_i \). Suppose that \( A \) is a proper atom or a dipole, and let \( \partial A = \{u, v\} \).

- For (C1), we get \( p(u) \neq p(v) \), and \( p(A) \) is isomorphic to an atom in \( H_i \).
- For (C2) and (C3), we get \( p(u) = p(v) \) and \( p(A) \) is an articulation of \( H_i \). If \( A \) is a dipole, then \( p(A) \) is a pendant star of half-edges and loops attached to \( p(u) \). If \( A \) is a proper atom, then \( p(A) \) is a pendant block (the fiber of an articulation in a double cover is a 2-cut) with attached single pendant edges and half-edges.

\[ \text{Lemma 4.12} \quad \text{The block structure of } H_{i+1} \text{ is preserved in } H_i \text{ with possible some new pendant blocks attached.} \]

\[ \text{Proof} \quad \text{Edges inside blocks are replaced using (C1) by edge-quotients of block atoms, proper atoms and dipoles which preserves 2-connectivity. The new pendant blocks in } H_i \text{ are created by replacing pendant edges with the block atoms, loops by loop-quotients, and half-edges by half-quotients.} \quad \Box \]

5 Planar Graphs

In this section, we show implication of our theory to planar graphs. Using the reduction, we describe the structure of the automorphism groups of planar graphs. We also characterize the quotients of planar graphs which results in a direct proof of Negami’s Theorem. The key point is that regular covering projections behave nicely on 3-connected planar graphs.

5.1 Automorphism Groups of 3-connected Planar Graphs

We review some well-known properties of planar graphs and their automorphism groups. These strong properties are based on Whitney’s Theorem \[14\] stating
that a 3-connected graph has a unique embedding into the sphere. This together with the well-known fact that polyhedral graphs are exactly 3-connected planar graphs implies that the automorphism groups of such graphs coincide with the automorphism group of the associated polyhedrons.

**Spherical Groups.** A group is spherical if it is the group of the symmetries of a tiling of the sphere. The first class of spherical groups are the subgroups of the automorphism groups of the platonic solids, i.e., $S_4$ for the tetrahedron, $C_2 \times S_4$ for the cube and the octahedron, and $C_2 \times A_5$ for the dodecahedron and the icosahedron; see Fig. 21. Table 1 shows the number of conjugacy classes of the subgroups of these three groups. Note that conjugate subgroups $\Gamma$ determine isomorphic quotients $G/\Gamma$. The second class of spherical groups is formed by the infinite families $C_n$, $D_n$, $C_n \times C_2$, and $D_n \times C_2$.

**Automorphisms of a Map.** A map $M$ is a 2-cell embedding of a graph $G$ onto a surface $S$. For purpose of this paper, $S$ is either the sphere or the projective plane. A rotation at a vertex is a cyclic ordering of the edges incident with the vertex. When working with abstract maps, they can be viewed as graphs endowed with rotations at every vertex. An angle is a triple $(v, e, e')$ where $v$ is a vertex, and $e$ and $e'$ are two incident edges which are consecutive in the rotation at $v$ or in the inverse rotation at $v$.

An automorphism of a map is an automorphism of the graph which preserves the angles; in other words the rotations are preserved. With the exception of paths and cycles, $\text{Aut}(M)$ is a subgroup of $\text{Aut}(G)$. In general these two groups might be very different. For instance, the star $S_n$ has $\text{Aut}(S_n) = S_n$, but for any map $M$

![Fig. 21 The five platonic solids together with their automorphism groups.](image)

| $S_4$ of the order 24 | $C_2 \times S_4$ of the order 48 | $C_2 \times A_5$ of the order 120 |
|----------------------|----------------------|----------------------|
| Order | Number | Order | Number | Order | Number |
|-------|--------|-------|--------|-------|--------|
| 1     | 1      | 1     | 1      | 1     | 1      |
| 2     | 2      | 8     | 1      | 9     | 4      |
| 3     | 1      | 12    | 1      | 6     | 3      |
| 4     | 3      |       |        |       |        |

![Table 1 The number of conjugacy classes of the subgroups of the groups of platonic solids.](chart)

| $S_4$ of the order 24 | $C_2 \times S_4$ of the order 48 | $C_2 \times A_5$ of the order 120 |
|----------------------|----------------------|----------------------|
| Order | Number | Order | Number | Order | Number |
|-------|--------|-------|--------|-------|--------|
| 1     | 1      | 8     | 1      | 1     | 1      |
| 2     | 3      | 10    | 3      | 9     | 4      |
| 3     | 1      | 12    | 2      | 6     | 3      |
| 4     | 3      | 20    | 1      | 5     | 1      |
| 5     | 1      | 24    | 1      |       |        |
| 6     | 3      | 60    | 1      |       |        |
of $S_n$ we just have $\text{Aut}(M) = D_n$. If $M$ is drawn on the sphere, then $\text{Aut}(M)$ is isomorphic to one of the spherical groups $\mathbb{S}_5$.

**Lemma 5.1** ([14]) If $G$ is a 3-connected planar graph, then $\text{Aut}(G)$ is isomorphic to one of the spherical groups.

**Proof** Since $G$ is a 3-connected planar graph, there exists the unique embedding of $G$ onto the sphere. Then for any map $M$ of $G$, we have $\text{Aut}(G) \cong \text{Aut}(M)$ [14]. $\square$

### 5.2 Automorphism Groups of Planar Graphs

In this section, we use the reductions to describe the automorphism groups of planar graphs. Unlike in the 3-connected case, their automorphism groups can be quite large and complicated. But we show that they can be described by a semidirect product series composed from few basic groups. Our approach is similar to Babai [11,12].

**Primitive Graphs.** We start with describing the automorphism groups of primitive graphs.

**Lemma 5.2** For a planar primitive graph $G$, the group $\text{Aut}(G)$ is a spherical group.

**Proof** Recall that a graph is essentially 3-connected if it is a 3-connected graph with attached single pendant edges to some of its vertices. If $G$ is essentially 3-connected, then $\text{Aut}(G)$ is a spherical group from Lemmas 3.4 and 5.1. If $G$ is $K_2$ or $C_n$ with attached single pendant edges, then it is a subgroup of $C_2$ or $D_n$. $\square$

**Atoms.** Next, we understand the automorphism groups of atoms. For an atom $A$, recall that $\text{Fix}(A)$ is the point-wise stabilizer of $\partial A$ in $\text{Aut}(A)$. Further, by $\text{Aut}_{\partial A}(A)$ we denote the set-wise stabilizer of $\partial A$ in $\text{Aut}(A)$. See Fig. 22 for examples. The groups $\text{Fix}(A)$ are used to describe automorphism groups of planar graphs, and $\text{Aut}_{\partial A}(A)$ is used in Section 5.3 to work with quotients of planar graphs.

**Lemma 5.3** Let $A$ be a planar atom.

(a) If $A$ is a star block atom, then $\text{Aut}_{\partial A}(A) = \text{Fix}(A)$ which is a direct product of symmetric groups.

(b) If $A$ is a non-star block atom, then $\text{Aut}_{\partial A}(A) = \text{Fix}(A)$ and it is a subgroup of a dihedral group.

\[
\begin{align*}
\text{Fix}(A) & \cong S_2 \times S_3 & \text{Fix}(A) & \cong D_6 & \text{Fix}(A) & \cong C_2 & \text{Fix}(A) & \cong S_2^2 \\
\text{Aut}_{\partial A}(A) & \cong S_2 \times S_3 & \text{Aut}_{\partial A}(A) & \cong D_6 & \text{Aut}_{\partial A}(A) & \cong C_2 & \text{Aut}_{\partial A}(A) & \cong S_2^2 \times C_2
\end{align*}
\]

**Fig. 22** An atom $A$ together with its groups $\text{Fix}(A)$ and $\text{Aut}_{\partial A}(A)$. From left to right, a star block atom, a non-star block atom, a proper atom, and a dipole.
(c) If $A$ is a proper atom, then $\text{Aut}_{\partial A}(A)$ is a subgroup of $C^2_2$ and $\text{Fix}(A)$ is a subgroup of $C_2$.

(d) If $A$ is a dipole, then $\text{Fix}(A)$ is a direct product of symmetric groups. If $A$ is symmetric, then $\text{Aut}_{\partial A}(A) = \text{Fix}(A) \rtimes C_2$. If $A$ is asymmetric, then $\text{Aut}_{\partial A}(A) = \text{Fix}(A)$.

Proof (a) The edges of each color class of the star block atom $A$ can be arbitrarily permuted, so $\text{Aut}_{\partial A} = \text{Fix}(A)$ which is a direct product of symmetric groups.

(b) For the non-star block atom $A$, $\partial A = \{u\}$ is preserved. We have one vertex in both $\text{Aut}_{\partial A}(A)$ and $\text{Fix}(A)$ fixed, thus the groups are the same. Since $A$ is essentially 3-connected, $\text{Aut}_{\partial A}(A)$ is a subgroup of $D_n$ where $n$ is the degree of $u$.

(c) Let $A$ be a proper atom with $\partial A = \{u, v\}$, and let $A^+$ be the essentially 3-connected graph created by adding the edge $uv$. Since $\text{Aut}_{\partial A}(A)$ preserves $\partial A$, we have $\text{Aut}_{\partial A}(A) = \text{Aut}_{\partial A}(A^+)$, and $\text{Aut}_{\partial A}(A^+)$ fixes in addition the edge $uv$. Because $A^+$ is essentially 3-connected, $\text{Aut}_{\partial A}(A^+)$ corresponds to the stabilizer of $uv$ in $\text{Aut}(M)$ for a map $M$ of $A^+$. But such a stabilizer has to be a subgroup of $C^2_2$. Since $\text{Fix}(A)$ stabilizes the vertices of $\partial A$, it is a subgroup of $C_2$.

(d) For an asymmetric dipole, we have $\text{Aut}_{\partial A} = \text{Fix}(A)$ which is a direct product of symmetric groups. For a symmetric dipole, we can permute the vertices in $\partial A$, so we get the semidirect product with $C_2$. ⊓ ⊔

The Characterization. We describe the automorphism groups of planar graphs. In comparison, a similar description was given by Babai [12] but it is less detailed and the language of his paper is difficult.

Lemma 5.4 For every planar symmetric proper atom $A$ with $\partial A = \{u, v\}$, there exists an involutory automorphism exchanging $u$ and $v$.

Proof Since $A$ is symmetric, there exists $\pi \in \text{Aut}_{\partial A}(A)$ which exchanges $u$ and $v$. Recall that $\text{Aut}_{\partial A}(A)$ is a subgroup of $C^2_2$, by Lemma 5.3. Therefore all elements of $\text{Aut}_{\partial A}(A)$ are involutions, so $\pi$ is an involution as well. ⊓ ⊔

As a corollary of Proposition 4.4b, we get the following description of the automorphism groups of planar graphs.

Corollary 5.5 Each automorphism group of a planar graph is obtained from a spherical group by repeated semidirect products with direct products of symmetric, dihedral and cyclic groups.

Proof The primitive graph $G_r$ has a spherical automorphism group by Lemma 5.2. By Lemma 5.4 we can apply Proposition 4.4b and $\text{Aut}(G_i) \cong \text{Aut}(G_{i+1}) \rtimes \text{Ker}(\Phi_i)$. By Lemma 4.3 the kernel $\text{Ker}(\Phi_i)$ is a direct product of the groups $\text{Fix}(A)$ for all atoms $A$ in $G_i$, which are by Lemma 5.3 subgroups of $C_2$ and $D_n$, and direct products of symmetric groups. ⊓ ⊔

On the other hand, not every abstract group obtained in this way is isomorphic to the automorphism group of some planar graph. The reason is that every orbit of $\text{Aut}(G_{i+1})$ has to swap isomorphic atoms which restricts their number in $G_i$. For instance if $\text{Aut}(G_r) \cong C_n$, then every orbit is of size 1, or $n$. Therefore the powers of $\text{Fix}(A)$ in the direct product are restricted. We can approach the characterization from the other side.
Fig. 23 The reduction tree for the reduction series in Fig. 14. The root is the primitive graph \( G_1 \) and each leaf corresponds to one atom of \( G_0 \).

For every graph \( G \), the reduction series corresponds to the reduction tree which is a rooted tree defined as follows. The root is the primitive graph \( G_r \), and the other nodes are the atoms obtained during the reductions. If a node contains a colored edge, it has the corresponding atom as a child. Therefore, the leaves are the atoms of \( G_0 \), after removing them, the new leaves are the atoms of \( G_1 \), and so on. For an example, see Fig. 23.

Proposition 4.4b constructs \( \text{Aut}(G) \) from the root to the leaves. Instead, we can approach it in the opposite direction. For an atom \( A \), let \( A^* \) denote the subgraph corresponding to the node of \( A \) and all its descendants in the reduction tree. In other words, \( A^* \) is the fully expanded atom \( A \). Let \( \text{Fix}(A^*) \) be the point-wise stabilizer of \( \partial A^* = \partial A \) in \( \text{Aut}(A^*) \). We define the following two classes of groups:

\[
\text{Aut(PLANAR)} = \{ \text{Aut}(G) : G \text{ is a connected planar graph} \},
\text{Fix(PLANAR)} = \{ \text{Fix}(A^*) : A \text{ is an atom of a reduction tree} \}.
\]

We note that the automorphism groups of disconnected planar graphs can be easily constructed from \( \text{Aut(PLANAR)} \) by the result of Jordan [10]. We first characterize \( \text{Fix(PLANAR)} \):

Lemma 5.6 The class \( \text{Fix(PLANAR)} \) is the class closed under:

(a) \( \{1\} \in \text{Fix(PLANAR)} \).
(b) If \( \psi_1, \psi_2 \in \text{Fix(PLANAR)} \), then \( \psi_1 \times \psi_2 \in \text{Fix(PLANAR)} \).
(c) If \( \psi \in \text{Fix(PLANAR)} \), then \( \psi^n \times \{S_n, C_n\} \in \text{Fix(PLANAR)} \) which denotes the semidirect product with one of the groups \( S_n \) and \( C_n \).
(d) If \( \psi_1, \psi_2 \in \text{Fix(PLANAR)} \), then \( (\psi_1^2 \times \psi_2^{2n}) \times D_n \in \text{Fix(PLANAR)} \).
Proof It is easy to observe that every abstract group from Fix(PLANAR) can be realized by a block atom, a proper atom or a dipole, and it can be realized in arbitrary many non-isomorphic ways. We argue that Fix(PLANAR) is closed under (a) to (d). It is clear for (a) and Fig. 24 shows the construction for (b) to (d).

It remains to show the opposite which we prove by induction according to the depth of a reduction tree. Let \( A \) be an atom. For each colored edge in \( A \) representing an atom \( \hat{A} \), by the induction hypothesis \( \text{Fix}(\hat{A}^*) \) can be generated using (a) to (d). The group \( \text{Fix}(A) \) consists of the automorphisms described in Lemma 5.3. Therefore \( \text{Fix}(A^*) \) consists of the groups of expanded atoms, permuted according to the action of \( \text{Fix}(A) \). We divide the argument according to the type of \( A \):

- Let \( A \) be a star block atom or a dipole. The edges of the same type and color can be arbitrarily permuted. Suppose that we have \( \ell \) types/colors of edges, with multiplicities \( m_1, \ldots, m_\ell \) and let \( A_1, \ldots, A_\ell \) be the corresponding atoms. Since the structure of the automorphisms is independent on each type/color class of atoms, each class contributes by one factor and \( \text{Fix}(A^*) \) is the direct product of these factors. Since each color class can be arbitrarily permuted, we get that the corresponding factor is isomorphic to \( \text{Fix}(A_i)^{m_i} \rtimes S_{m_i} \); the argument is similar to the proof of Proposition 4.4b. So \( \text{Fix}(A^*) \) can be generated using (b) and (c).

- Let \( A \) be a proper atom. We assume that \( \text{Fix}(A) \cong C_2 \), otherwise \( \text{Fix}(A^*) \) can be easily constructed only using (b). Let \( \text{Fix}(A) \) have \( \ell \) orbits of colored edges of size two, corresponding to atoms \( A_1, \ldots, A_\ell \). Each automorphism of \( \text{Fix}(A^*) \) either flips \( A \) which corresponds to swapping all these orbits, or preserves \( A \) which fixes all the orbits. Further, let \( \text{Fix}(A) \) have \( \ell' \) orbits of size one,

\[
\begin{align*}
\text{Fix}(A^*) &\cong \Psi_1 \times \Psi_2 \\
(b) & \\
\text{Fix}(A^*) &\cong \Psi_4 \rtimes S_4 \\
(c) & \\
\text{Fix}(A^*) &\cong \Psi_6 \rtimes S_6 \\
(d) & \\
\text{Fix}(A^*) &\cong (\Psi_1^6 \times \Psi_2^{12}) \rtimes D_6
\end{align*}
\]

**Fig. 24** Constructions for the operations (b) to (d), every colored edge corresponds to an atom \( \hat{A} \) with \( \text{Fix}(\hat{A}) \) isomorphic to the denoted group. Concerning (d), unlike in (c), there are two types of orbits of the action of \( D_n \). The gray edges (corresponding to \( \Psi_1 \)) are permuted as the edges of a regular \( n \)-gon, and the white edges (corresponding to \( \Psi_2 \)) are permuted as half-edges of this \( n \)-gon.
Suppose that extended atoms \( \text{Aut}(G) \). It follows that

\[
\text{Fix}(A^*) \cong (\text{Fix}(A^*_1) \times \cdots \times \text{Fix}(A^*_r))^2 \times C_2 \times \text{Fix}(\hat{A}^*_1) \times \cdots \times \text{Fix}(\hat{A}^*_r),
\]

so it can again be constructed using (b) and (c).

- **Let** \( A \) **be a non-star block atom.** Recall that \( A \) is essentially 3-connected, so it corresponds to a map. By Lemma 5.3, \( \text{Fix}(A) \) is either \( C_n \) or \( D_n \).

  - If \( \text{Fix}(A) \cong C_2 \), then the action of \( \text{Fix}(A) \) has orbits of sizes one or two. Exactly the same argument as in the case of proper atoms applies, so \( \text{Fix}(A^*) \) can be constructed using (b) and (c).

  - If \( \text{Fix}(A) \cong C_n \) for \( n \geq 3 \), then \( \text{Fix}(A) \) acts semiregularly on the edges and all edge-orbits are of size \( n \). Suppose the action of \( \text{Fix}(A) \) consists of \( \ell \) orbits of colored edges, corresponding to atoms \( A_1, \ldots, A_\ell \). Therefore

\[
\text{Fix}(A^*) \cong (\text{Fix}(A^*_1) \times \cdots \times \text{Fix}(A^*_\ell))^n \times C_n,
\]

so it can be constructed using (b) and (c).

  - If \( \text{Fix}(A) \cong D_n \), then \( \text{Fix}(A) \) acts semiregularly on the angles of the map and all edge-orbits are of size either \( n \) or \( 2n \). Suppose the action of \( \text{Fix}(A) \) consists of \( \ell \) orbits of colored edges of size \( n \), corresponding to atoms \( A_1, \ldots, A_\ell \), and \( \ell' \) orbits of size \( 2n \), corresponding to atoms \( A_1, \ldots, A_{\ell'} \). We get

\[
\text{Fix}(A^*) \cong (\text{Fix}(A^*_1) \times \cdots \times \text{Fix}(A^*_\ell) \times \text{Fix}^{2n}(\hat{A}^*_1) \times \cdots \times \text{Fix}^{2n}(\hat{A}^*_\ell)) \times D_n,
\]

where each element of \( D_n \) permutes the factors exactly as the colored edges in an automorphism of \( \text{Fix}(A) \). The orbits of the action of \( D_n \) are formed by the direct factors of the subgroups \( \text{Fix}^{2n}(A^*_1) \) and \( \text{Fix}^{2n}(A^*_\ell) \). In particular, \( D_n \) acts transitively on the first subgroups and regularly on the second subgroups. So \( \text{Fix}(A^*) \) can be constructed using (b) and (d).

This description is very similar to Jordan’s characterization of the automorphism groups of trees [10], which is the class of groups closed on (a), (b) and the part of (c) only with \( S_n \). Also, one can describe (c) and (d) by the group theoretic notation called the wreath product as \( \Psi \wr S_n \), \( (\Psi \wr S_n) \wr D_n \).

Now, we are ready to prove the characterization of \( \text{Aut}(\text{PLANAR}) \) in terms of the automorphism groups of 3-connected planar graphs.

**Theorem 5.7** The class \( \text{Aut}(\text{PLANAR}) \) consists of the groups constructed as follows. We take a planar graph \( G' \) with colored vertices and colored (possibly oriented) edges, which is either 3-connected, or \( K_2 \), or a cycle \( C_n \). Let \( m_1, \ldots, m_\ell \) be the sizes of the vertex- and edge-orbits of the action of \( \text{Aut}(G') \). Then for any choices \( \Psi_1, \ldots, \Psi_\ell \in \text{Fix}(\text{PLANAR}) \), we have

\[
(\Psi_1^{m_1} \times \cdots \times \Psi_\ell^{m_\ell}) \times \text{Aut}(G') \in \text{Aut}(\text{PLANAR}).
\]

On the other hand, every group of \( \text{Aut}(\text{PLANAR}) \) can be constructed in the above way.

**Proof** Suppose that \( G' \) is given. First, we replace colors of the vertices with colored single pendant edges attached to them. Then, we choose arbitrary non-isomorphic extended atoms \( A_1^*, \ldots, A_r^* \) such that \( \text{Fix}(A_i^*) \cong \Psi_i \), and we replace the corresponding colored edges with them. We denote this modified planar graph by \( G \) and we get that

\[
\text{Aut}(G) \cong (\Psi_1^{m_1} \times \cdots \times \Psi_\ell^{m_\ell}) \times \text{Aut}(G'),
\]

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in exactly the same way as in the proof of Proposition 4.4b, applied on $G_r$.

On the other hand, for a planar graph $G$, we apply the reduction series and obtain a primitive graph $G_r$. Each orbit of $\text{Aut}(G_r)$ consists of $m_i$ edges corresponding to one isomorphism class of fully expanded atoms $A_i^*$ and let $\Psi_i \cong \text{Fix}(A_i^*) \in \text{Fix(PLANAR)}$. By removing single pendant edges and replacing them with colors on the corresponding vertices, we get a graph $G'$ from the statement. It follows that

$$(\Psi_1^{m_1} \times \cdots \times \Psi_{\ell}^{m_{\ell}}) \rtimes \text{Aut}(G') \cong \text{Aut}(G).$$

We stop with the characterization of the automorphism groups of planar graphs here since it is just a biproduct of our structural theory and not the main focus of this paper. To describe precisely which abstract groups belong to $\text{Aut(PLANAR)}$, one only has to understand what are the possible restrictions on sizes of the orbits of $\text{Aut}(G')$.

5.3 Quotients of Planar Graphs and Negami’s Theorem

In this section, we describe quotients of planar graphs geometrically. Using Theorem 1.2, it only remains to understand the quotients of planar primitive graphs and the half-quotients of planar proper atoms. We also show that our structural theory gives a direct proof of Negami’s Theorem [11].

**Geometry and Quotients.** As we have already stated, automorphism groups of 3-connected planar graphs are isomorphic with automorphism groups of the corresponding maps, which allows to use geometry to study regular quotients. We first recall some basic definitions from geometry [13].

Let $G$ be a 3-connected planar graph. An automorphism of $G$ is called orientation preserving, if the respective map automorphism preserves the global orientation of the surface. It is called orientation reversing if it changes the global orientation of the surface. A subgroup of $\text{Aut}(G)$ is called orientation preserving if all its automorphisms are orientation preserving, and orientation reversing otherwise. We note that every orientation reversing subgroup contains an orientation preserving subgroup of index two. (The reason is that composition of two orientation reversing automorphisms is an orientation preserving automorphism.)

Let $\tau$ be an orientation reversing involution of an orientable surface. The involution $\tau$ is called antipodal if it is a semiregular automorphism of a closed orientable surface $S$ such that $S/\langle \tau \rangle$ is a non-orientable surface. Otherwise $\tau$ is called a reflection. A reflection of the sphere fixes a circle. An orientation reversing involution of a 3-connected planar graph is called antipodal if the respective map automorphism is antipodal and it is called a reflection if the respective map automorphism is a reflection. A reflection of a map on the sphere fixes always either an edge, or a vertex.

The quotient of the sphere by an orientation preserving group of automorphisms is again the sphere. The half-quotient of the sphere by a reflection is the disk and and the half-quotient by an antipodal involution is the projective plane. See Fig. 25.

**Quotients of Primitive Graphs.** By Lemma 4.6 we know that every primitive graph $G_r$ is either 3-connected with attached single pendant edges, or $K_2$ or $C_n$. 
with attached single pendant edges. By Lemma 5.7, these attached single pendant edges only make $\text{Aut}(G_r)$ smaller, which restricts the possible quotients. Therefore it is sufficient to understand how possible quotients can look for 3-connected planar graphs, $K_2$ and $C_n$.

**Lemma 5.8** (13) Let $G$ be a 3-connected planar graph and $\Gamma$ be a semiregular subgroup of $\text{Aut}(G)$. There are three types of quotients of $G$:

(a) Rotational quotients – The action of $\Gamma$ is orientation preserving and the quotient $G/\Gamma$ is planar.

(b) Reflectional quotients – The action of $\Gamma$ is orientation reversing but does not contain an antipodal involution. Then the quotient $G/\Gamma$ is planar and necessarily contains half-edges.

(c) Antipodal quotients – The action of $\Gamma$ is orientation reversing and contains an antipodal involution. Then $G/\Gamma$ is projective planar.

Figure 25 shows examples of these types of quotients. We note that an antipodal quotient can be planar, but not necessarily; for an example, see Fig. 1.

The quotients of $K_2$ are straightforward. Next, we characterize quotients of cycles, which completes the description of possible quotients of primitive graphs:

**Lemma 5.9** Let $\Gamma$ be a semiregular subgroup of $\text{Aut}(C_n)$. Then $C_n/\Gamma$ is either a cycle, or a path with two half-edges attached to its ends (only for $n$ even). □
Half-quotients of Proper Atoms. Next, we characterize the half-quotients of planar proper atoms. There are more restrictive than quotients of primitive graphs since the involution has to exchange the vertices of the boundary:

Lemma 5.10 Let $A$ be a planar proper atom and let $\partial A = \{u, v\}$. There are at most two half-quotients $A/\langle \tau \rangle$ where $\tau \in \text{Aut}_{\partial A}(A)$ is an involutory semiregular automorphism transposing $u$ and $v$:

(a) The rotational half-quotient – The involution $\tau$ is orientation preserving and $A/\langle \tau \rangle$ is planar with at most one half-edge.
(b) The reflectional half-quotient – The involution $\tau$ is a reflection and $A/\langle \tau \rangle$ is planar with at least two half-edges.

Proof The graph $A^+$ (obtained from $A$ by adding the edge $uv$) is an essentially 3-connected planar graph with a unique embedding into the sphere. By Lemma 5.3c, $\text{Aut}_{\partial A}(A)$ is a subgroup of $C_2^2$. An involution $\tau$ exchanging $u$ and $v$ corresponds to a map automorphism of $A^+$ fixing $uv$. Either $\tau$ is a $180^\circ$ rotation around the centre of $uv$ which gives the rotational half-quotient, or it is a reflection which gives the reflectional half-quotient; see Fig. 26. According to Lemma 5.8 both possible half-quotients are planar.

Direct Proof of Negami’s Theorem. Using the above statements, we give a direct proof of Negami’s Theorem. This theorem states that a graph $H$ has a finite planar
regular cover \( G \) (i.e., \( G/\Gamma \cong H \) for some semiregular \( \Gamma \leq \text{Aut}(G) \)), if and only if \( H \) is projective planar. For a given projective planar graph \( H \), the construction of a planar graph \( G \) is easy: by embedding \( H \) into the projective plane and taking the double cover of this embedding, we get the graph \( G \) embedded to the sphere. Below, we prove the harder implication:

**Theorem 5.11 (Negami [11])** Let \( G \) be a planar graph. Then every (regular) quotient of \( G \) is projective planar.

**Proof** We apply the reduction series on \( G \) which produces graphs \( G = G_0, G_1, \ldots, G_r \) such that \( G_r \) is primitive. If \( G_r \) is essentially 3-connected, then by Lemma 5.5 every quotient \( H_r = G_r/\Gamma_r \) is projective planar. If \( G_r \) is \( K_2 \) or \( C_n \) with single pendant edges attached, then by Lemma 5.5 every quotient \( H_r = G_r/\Gamma_r \) is even planar.

By Theorem 1.2, every quotient \( H = G/\Gamma \) can be constructed from some \( H_r \) by an expansion series in which we replace edges, loops and half-edges by edge-quotients, loop-quotients and half-quotients, respectively. All edge- and loop-quotients are clearly planar. By Lemma 5.10, every half-quotient of a proper atom is planar, and by Lemma 4.11 every half-quotient of a dipole is a set of loops and half-edges attached to a single vertex, which is also planar. Therefore, these replacements can be done in a way that the underlying surface of \( H_r \) is not changed, so \( H \) is also projective planar. \( \square \)

We note that deciding whether \( H = G/\Gamma \) is planar or non-planar projective is done on the primitive graph \( G_r \). It is non-planar if and only if \( \Gamma_r \) contains a semiregular antipodal involution and the resulting quotient \( H_r = G_r/\Gamma_r \) is non-planar.

6 Concluding Remarks

We recall the main points addressed in this paper:

- We describe the reduction series \( G = G_0, \ldots, G_r \) such that \( G_{i+1} \) is constructed from \( G_i \) by replacing the atoms of \( G_i \) with colored edges and the primitive graph \( G_r \) is either essentially 3-connected, or \( K_2 \), or a cycle (Lemma 4.6).
- We show that \( \text{Aut}(G_i) \) is an extension of \( \text{Aut}(G_{i+1}) \) (Proposition 1.1). By assuming existence of certain involutions, we can show that this extension can be described by the semidirect product (Proposition 4.4), which applies to planar graphs. Using this, we characterize the automorphism groups of planar graphs (Lemma 5.6, Theorem 5.7), similarly to Babai [1,2].
- For a prescribed quotient \( H_r = G_r/\Gamma_r \), we describe all possible expansions \( H_0 = G_0/\Gamma_0 \) which revert the reductions. Theorem 1.2 states that every quotient \( H \cong G/\Gamma \) can be obtained in this way, and different quotients \( H_0 \) are constructed by non-isomorphic quotients \( H_r \) and non-isomorphic choices of half-quotients in the expansions.
- Since the quotients of 3-connected planar graphs can be understood using geometry, we give a direct proof of Negami’s Theorem [11] (Theorem 5.11). The reason is that a quotient \( H_r = G_r/\Gamma_r \) is due to geometry always planar or projective planar. By Theorem 1.2 the expansions create \( H \) from \( H_r \) while preserving the underlying surface of \( H_r \).
Our results have as well algorithmic implications for regular covering testing, described in [6]. In particular, this allows to construct an algorithm for testing whether an input planar graph $G$ regularly covers an input graph $H$ can be constructed, running in time $O(n^c \cdot 2e(H)/2)$.

More General Graphs. Our structural results also work for more general graphs. We have assumed that the graphs $G$ and $H$ are without loops and free half-edges. We can work with loops and half-edges in $G$ in the same way as with pendant edges (of different colors). Since we assume that $H$ contains no half-edges, we set the reductions and expansions in the way that half-edges can appear in the expansion series but no expanded quotient $H_0$ contains half-edges. This is done by having all edges of $G_0$ as undirected edges. To admit quotients $H_0$ with half-edges, it is sufficient to change all edges of $G_0$ to halvable edges. Also, all the results can be used when $G$ and $H$ contain colored edges, vertices, some edges oriented, etc.

Harmonic Regular Covers. There is a generalization of regular graph covering for which it would be interesting to find out whether our techniques can be modified. Consider geometric regular covers of surfaces, like in Fig. 25 and 26. The orbits of the 180° rotations are of size two, with the exception of two points lying on the axis of the rotation. These exceptional points are called branch points. In general, a regular covering projection is locally homeomorphic around a branch point to the complex mapping $z \mapsto z^\ell$ for some integer $\ell \leq k$, and $\ell$ is called the order of the branch point.

Assume that $G$ is a 3-connected planar graph embedded onto the sphere, $\Gamma \leq \text{Aut}(G)$ is a semiregular subgroup of automorphisms of the sphere, and $p : G \to H = G/\Gamma$ is the regular covering projection. When $H$ is a standard graph (with no free half-edges), all branch points of $p$ belong to faces of the embedding. If a branch point (of order two) is placed in the center of an edge of $G$, this edge is projected to a half-edge in $H$. It is possible to consider covering projections between surfaces in which branch points can be placed in vertices of $G$ which gives harmonic regular covering [3]. If a branch point of order $\ell$ is placed in a vertex $v \in V(G)$, then the vertex $p(v) \in V(H)$ has the degree equal $\deg(v)/\ell$ and for an edge $e \in E(H)$ incident with $p(v)$, the fiber $p^{-1}(e)$ has exactly $\ell$ edges incident with $v$.

4-connected Reduction. We have described the way how to reduce a graph to a 3-connected one while preserving its essential structural information. This approach is highly efficient for planar graphs since many problems are much simpler for 3-connected planar graphs; for instance the considered regular graph covering problem. Suppose that we would like to push our results further, say to toroidal or projective planar graphs. The issue is that 3-connectivity does not restrict them much. Is it possible to apply some “4-connected reduction”, to reduce the input graphs even further? Suppose that one would generalize proper atoms to be inclusion minimal parts of the graph separated by a 3-cut. Would it be possible to replace them by triangles?

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