The authors find some new inequalities of Jordan type for the sine function. These newly established inequalities are of new form and are applied to deduce some known results.

1. Introduction

For \( x \in (0, \pi/2] \), we have

\[
\frac{\sin x}{x} \geq \frac{2}{\pi}.
\] (1)

The inequality is sharp with equality if and only if \( x = \pi/2 \).

This inequality is known in the literature as Jordan's sine inequality for the sine function. See [1, page 33] and other references cited in the first page of [2].

Motivated by [3], it was established in [5] that

\[
\frac{1}{\pi^3} \left( \pi^2 - 4x^2 \right) + \frac{2}{\pi} \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^3} \left( \pi^2 - 4x^2 \right),
\] (2)

for \( x \in (0, \pi/2] \). The equalities hold if and only if \( x = \pi/2 \).

This refines Jordan's inequality (1).

Motivated by [3], it was established in [5] that

\[
\frac{1}{2\pi^5} \left( \pi^4 - 16x^4 \right) + \frac{2}{\pi} \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^5} \left( \pi^4 - 16x^4 \right),
\] (3)

for \( x \in (0, \pi/2] \). The equalities are valid if and only if \( x = \pi/2 \).

This also refines Jordan's inequality (1). Also, see the double inequality (3.10) in the survey article [2, page 17].

In recent years, the above inequalities have been refined, extended, generalized, and applied by many mathematicians in a large amount of papers. See, for example, [3–19].

For a systematic review on this topic, please refer to the expository paper [2].

The aim of this paper is to further refine and generalize these inequalities of Jordan type for the sine function.

Our main results may be stated as in the following theorems.

**Theorem 1.** If \( n \geq 0 \) and \( m \geq 2 \) are integers, then

\[
\frac{2^{m+2}}{(2m+n\pi)^{m+1}} \left\{ \left( \frac{\pi}{2} \right)^m - x^m \exp \left[ n \left( x - \frac{\pi}{2} \right) \right] \right\}
+ \frac{2}{\pi} \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{\pi - 2}{\pi^m} \left( \left( \frac{\pi}{2} \right)^m - x^m \exp \left[ n \left( x - \frac{\pi}{2} \right) \right] \right),
\] (4)

on \((0, \pi/2]\).
on $(0, \pi/2]$, then
\[
\frac{4}{\pi^2 g'(\pi/2)} \left[ g \left( \frac{\pi}{2} \right) - g(x) \right] + \frac{2}{\pi} \sin x \\
\leq \frac{2 + 2(\pi - 2)}{\pi \left[ g(\pi/2) - g(0) \right]} \left[ g \left( \frac{\pi}{2} \right) - g(x) \right].
\]
(6)

Remark 3. Taking $n = 0$ in Theorem 1 yields
\[
\frac{2^{m+1}}{mn^{m+1}} \left( \left( \frac{n}{2} \right)^m - x^m \right) + \frac{2}{\pi} \sin x \\
\leq \frac{2 + 2(\pi - 2)}{\pi} \left[ \left( \frac{n}{2} \right)^m - x^m \right],
\]
(7)
on $(0, \pi/2]$ for $m \geq 2$. The equalities in (7) are valid if and only if $x = \pi/2$.

Putting $m = 2, 4$ in (7) results in (2) and (3), respectively. This means that Theorem 1 generalizes the inequalities (2) and (3).

Remark 4. Let the function $g(x)$ in Theorem 2 be $x^m$ for $m \geq 2$. A straightforward computation gives
\[
g \left( \frac{\pi}{2} \right) = \left( \frac{n}{2} \right)^m \neq 0 = g(0),
\]
\[
g'(x) = mx^{m-1} > 0,
\]
\[3g''(x) + xg'''(x)
= 3m(m-1)x^{m-2} + mx(m-1)(m-2)x^{m-2} > 0,
\]
\[2g'(x) - 2xg''(x) + x^2g'''(x) = -m(m-1)(m-2)
\leq 0,
\]
\[
\frac{4}{\pi^2 g'(\pi/2)} \left[ g \left( \frac{\pi}{2} \right) - g(x) \right]
= \frac{2^{m+1}}{mn^{m+1}} \left[ \left( \frac{n}{2} \right)^m - x^m \right] + \frac{2}{\pi},
\]
\[
\frac{\pi - 2}{\pi \left[ g(\pi/2) - g(0) \right]} \left[ g \left( \frac{\pi}{2} \right) - g(x) \right]
= \frac{2 + 2^{m}(\pi - 2)}{\pi^{n+1}} \left[ \left( \frac{n}{2} \right)^m - x^m \right].
\]
(8)

This implies inequality (7). Hence, Theorem 2 generalizes Theorem 1.

In the final section of this paper, we will apply Theorem 1 to refine and generalize Yang’s inequality and construct some integral inequalities.

2. A Lemma

In order to prove Theorems 1 and 2, the following lemma is necessary.

Lemma 5. Let $f, g : [a, b] \to \mathbb{R}$ be differentiable on $(a, b)$. If $g' \neq 0$ and $f'/g'$ are decreasing on $(a, b)$, then the functions
\[
\frac{f(x) - f(b)}{g(x) - g(b)}, \quad \frac{f(x) - f(a)}{g(x) - g(a)}
\]
are also decreasing on $(a, b)$.

Remark 6. Lemma 5 can be found in many papers such as [10, 12, 13, 20].

3. Proofs of Theorems

We are now in a position to prove our theorems.

Proof of Theorem 1. Let
\[
f_1(x) = \frac{\sin x}{x}, \quad f_2(x) = -x^m e^{nx},
\]
\[f_3(x) = \sin x - x \cos x, \quad f_4(x) = (mx^{m+1} + nx^{m+2}) e^{nx},
\]
on $(0, \pi/2]$. A direct calculation gives
\[
\frac{f_1'(x)}{f_1(x)} = \frac{\sin x - x \cos x}{(mx^{m+1} + nx^{m+2}) e^{nx}} = \frac{f_3(x)}{f_4(x)},
\]
\[
\frac{f_3(x)}{f_4'(x)}
= \frac{\sin x}{[m(m+1)x^{m-1} + 2n(m+1)x^m + n^2x^{m+1}] e^{nx}},
\]
\[
\frac{f_3'(x)}{f_4'(x)}
= \frac{h_m(x)}{\sec x[m(m+1)x^{m-1} + 2n(m+1)x^m + n^2x^{m+1}]^2 e^{nx}},
\]
(11)
where
\[
h_m(x) = m(m+1)x^{m-1} + 2n(m+1)x^m + n^2x^{m+1}
\]
\[+ 3nm(m+1)x^{m-1} + 3n^2(m+1)x^m
\]
\[+ n^3x^{m+1}\tan x.
\]

\[
- \frac{\sin x - x \cos x}{(mx^{m+1} + nx^{m+2}) e^{nx}} = \frac{f_3(x)}{f_4(x)},
\]
(9)
Utilizing \( \tan x > x \) on \((0, \pi/2)\) leads to
\[
h'_m(x) \leq m(m + 1)x^{m-1} + 2n(m + 1)x^m
+ n^2x^{m+1} - x[m(m + 1)(m - 1)x^{m-2}
+ 3nm(m + 1)x^{m-1}
+ 3n^2(m + 1)x^m + n^3x^{m+1}]
= -[m(m + 1)(m - 2)x^{m+1} + n(m + 1)(3m - 2)x^m
+ n^2(3m - 2)x^{m+1} + n^3x^{m+2}]
\leq 0, \tag{13}
\]
on \((0, \pi/2)\) for \( m \geq 2 \) and \( n \geq 0 \). As a result, the function \( \frac{f_3(x)}{f_4(x)} \) is decreasing on \((0, \pi/2)\). In virtue of Lemma 5, it follows that the functions
\[
\frac{f_3(x)}{f_4(x)} = \frac{f_3(x) - f_3(0)}{f_4(x) - f_4(0)}, \quad \frac{f_3'(x)}{f_4'(x)}
H(x) = \frac{f_1(x) - f_1(\pi/2)}{f_2(x) - f_2(\pi/2)}
\]
are all decreasing on \((0, \pi/2)\). Since
\[
\lim_{x \to 0^+} H(x) = \frac{2^m(\pi - 2)}{\pi^{m+1}}e^{-\pi n/2},
\lim_{x \to (\pi/2)^-} H(x) = \frac{2^m(\pi - 2)}{\pi^{m+1}}e^{-\pi n/2},
\]
we have
\[
\frac{2^m e^{-\pi n/2}}{\pi^{m+1}} \leq H(x) \leq \frac{2^m(\pi - 2)}{\pi^{m+1}}e^{-\pi n/2}, \tag{16}
\]
which can be reformulated as the inequality (4). Theorem 1 is thus proved.

**Proof of Theorem 2.** Let
\[
f_1(x) = \frac{\sin x}{x}, \quad f_2(x) = -g(x),
f_3(x) = \sin x - x \cos x, \quad f_4(x) = x^2g'(x),
on \((0, \pi/2)\). It is easy to see that
\[
f'_2(x) = -g'(x) < 0, \quad f'_2(x) \neq 0,
f'_4(x) \neq 0,
f'_4(x) = 2xg'(x) + x^2g''(x)
= x(2g'(x) + xg''(x)) > 0,
\]
on \((0, \pi/2)\). Furthermore, we have
\[
\frac{f'_3(x)}{f'_4(x)} = \frac{\sin x}{x^2g'(x)} = \frac{f_3(x)}{f_4(x)}, \tag{19}
\frac{f'_3(x)}{f'_4(x)} = \frac{\sin x}{2g'(x) + xg''(x)}, \tag{20}
\]
Employing \( \tan x > x \) and the conditions in (5), it is not difficult to show that the numerator of \( \frac{f'_3(x)}{f'_4(x)} \) is negative on \((0, \pi/2)\). This means that the function \( \frac{f_3(x)}{f_4(x)} \) is decreasing on \((0, \pi/2)\). Consequently, making use of Lemma 5 consecutively, it is revealed that the functions
\[
\frac{f_3(x)}{f_4(x)} = \frac{f_3(x) - f_3(0)}{f_4(x) - f_4(0)}, \quad \frac{f_3'(x)}{f_4'(x)}
H(x) = \frac{f_1(x) - f_1(\pi/2)}{f_2(x) - f_2(\pi/2)} = \frac{(\sin x) - (2/\pi)}{g(\pi/2) - g(x)}
\]
are all decreasing on \((0, \pi/2)\). Since
\[
\lim_{x \to 0^+} H(x) = \lim_{x \to 0^+} \frac{f_1(x) - f_1(\pi/2)}{f_2(x) - f_2(\pi/2)} = \frac{1 - (2/\pi)}{g(\pi/2) - g(0)}
= \frac{\pi - 2}{\pi (g(\pi/2) - g(0))},
\lim_{x \to (\pi/2)^-} H(x) = \lim_{x \to (\pi/2)^-} \frac{(\sin x) - (2/\pi)}{g(\pi/2) - g(x)}
= \frac{\sin x - (2/\pi)x}{g(\pi/2) - xg'(x)}
= \frac{-xg'(x)}{g(\pi/2) - xg'(x)} = \frac{4}{\pi^2g'(\pi/2)}, \tag{22}
\]
from \( H(\pi/2) \leq H(x) \leq H(0) \), the inequality (6) follows. The proof of Theorem 2 is complete.

**4. Applications of Theorem 1**

After proving Theorems 1 and 2, we now start off to apply them to construct some new inequalities.

Let \( 0 \leq \lambda \leq 1 \) and \( A, B > 0 \) with \( A + B \leq \pi \). Then,
\[
\cos^2(AA) + \cos^2(AB) - 2 \cos(AA) \cos(AB) \cos(\lambda \pi)
\geq \sin^2(\lambda \pi). \tag{23}
\]
This inequality is known in the literature as Yang’s inequality. Since paper [16], many mathematicians mistakenly referred this inequality to [21, pages 116–118]. Indeed, the paper we should refer to is [22] or even an earlier paper in Chinese.

The first application of Theorem 1 is to refine and generalize Yang’s inequality (23) as follows.

**Theorem 7.** For \( k \geq 2 \), let \( A_i > 0 \) and \( \sum_{i=1}^{k} A_i \leq \pi \). If \( 0 \leq \lambda \leq 1 \), then
\[
R(\lambda) \leq \sum_{1 \leq i < j \leq k} H_{ij} \leq T(\lambda),
\]
where
\[
H_{ij} = \cos^2(\lambda A_i) + \cos^2(\lambda A_j) - 2 \cos(\lambda A_i) \cos(\lambda A_j) \cos(\lambda \pi),
\]
\[
T(\lambda) = 4 \left( \frac{k}{2} \right) \left( \lambda + \frac{(\pi - 2) \lambda}{2} \left[ 1 - \lambda^m \right] \right)^2 \cos^2 \frac{\lambda \pi}{2},
\]
and \( n \geq 0 \) and \( m \geq 2 \) are integers.

**Proof.** Substituting \( x = \lambda \pi / 2 \) in the inequality (4) reveals that
\[
\sin \frac{\lambda \pi}{2} \geq \lambda + \frac{2 \lambda}{2m + n \pi} \left[ 1 - \lambda^m \right],
\]
\[
\sin \frac{\lambda \pi}{2} \leq \lambda + \frac{(\pi - 2) \lambda}{2} \left[ 1 - \lambda^m \right].
\]
Using
\[
\sin^2(\lambda \pi) = 4 \sin^2 \frac{\lambda \pi}{2} \cos^2 \frac{\lambda \pi}{2},
\]
the inequality
\[
\sin^2(\lambda \pi) \leq H_{ij} \leq 4 \sin^2 \frac{\lambda \pi}{2},
\]
see either [22], [16, (2.13)], or [2, page 17, (3.4)], becomes
\[
4 \left\{ \lambda + \frac{2 \lambda}{2m + n \pi} \left[ 1 - \lambda^m \right] \right\}^2 \cos^2 \frac{\lambda \pi}{2} \leq H_{ij}
\]
\[
\leq 4 \left\{ \lambda + \frac{(\pi - 2) \lambda}{2} \left[ 1 - \lambda^m \right] \right\}^2.
\]
Finally, taking the sum of the above inequality for all \( 1 \leq i < j \leq n \) results in (24). The required proof is complete.

**Corollary 8.** Under the conditions of Theorem 7, one has
\[
R_1(\lambda) \leq \sum_{1 \leq i < j \leq k} H_{ij} \leq T_1(\lambda),
\]
where
\[
T_1(\lambda) = \left( \frac{k}{2} \right) \pi^2 \lambda^2, \quad R_1(\lambda) = 4 \left( \frac{k}{2} \right) \lambda^2 \cos^2 \frac{\lambda \pi}{2}.
\]

**Theorem 9.** For \( x \in (0, \pi/2] \), if \( n \geq 0 \) and \( m \geq 2 \) are integers, then
\[
\int_0^{\pi/2} \sin \frac{x}{x} dx + \frac{2^m \lambda}{2m + n \pi} \lambda \int_0^{\pi/2} x^m e^{ix} dx \geq 1 + \frac{2}{(2m + n \pi)},
\]
\[
\int_0^{\pi/2} \sin \frac{x}{x} dx + \frac{2m \pi}{2m + n \pi} \int_0^{\pi/2} x^m e^{ix} dx \leq \frac{\pi}{2}.
\]

**Proof.** This follows from integrating on all sides of the double inequality (4).

**Remark 10.** Applying Theorem 9 to \( n = 0 \) gives
\[
\frac{1}{m + 1} + 1 \leq \int_0^{\pi/2} \sin \frac{x}{x} dx \leq 1 + \frac{(\pi - 2)}{2} \frac{m}{m + 1}.
\]

Applying Theorem 9 to \( n = 0 \) and \( m = 2 \) yields
\[
\frac{4}{3} \leq \int_0^{\pi/2} \sin \frac{x}{x} dx \leq \frac{\pi + 1}{3}.
\]
This is a recovery of an inequality established in [3, page 101]. It was also collected in [2, (2.14)]. Such a kind of inequalities can be found in [23].

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**References**

[1] D. S. Mitrinović, *Analytic Inequalities*, Springer, 1970.
[2] F. Qi, D.-W. Niu, and B.-N. Guo, “Refinements, generalizations, and applications of jordan’s inequality and related problems,” *Journal of Inequalities and Applications*, vol. 2009, Article ID 271923, 52 pages, 2009.
[3] F. Qi, “Extensions and sharpenings of Jordan’s and Kober’s inequality,” *Journal of Mathematics for Technology*, vol. 12, no. 4, pp. 98–102, 1996 (Chinese).
[4] F. Qi and Q. D. Hao, “Refinements and sharpenings of Jordan’s and Kober’s inequality,” *Mathematics and Informatics Quarterly*, vol. 8, no. 3, pp. 116–120, 1998.
[5] D. J. Wai and Y. Hua, "Sharpening of Jordan's inequality and its applications," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 3, article 102, 2006.

[6] R. P. Agarwal, Y.-H. Kim, and S. K. Sen, "A new refined Jordan's inequality and its application," *Mathematical Inequalities and Applications*, vol. 12, no. 2, pp. 255–264, 2009.

[7] Á. Baricz, "Jordan-type inequalities for generalized Bessel functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 2, article 39, 2008.

[8] Á. Baricz, "Some inequalities involving generalized bessel functions," *Mathematical Inequalities and Applications*, vol. 10, no. 4, pp. 827–842, 2007.

[9] Á. Baricz and S.-H. Wu, "Sharp Jordan-type inequalities for Bessel functions," *Publicationes Mathematicae*, vol. 74, no. 1–2, pp. 107–126, 2009.

[10] Z.-H. Huo, D.-W. Niu, J. Cao, and F. Qi, "A generalization of Jordan's inequality and an equality," *Hacettepe Journal of Mathematics and Statistics*, vol. 40, no. 1, pp. 53–61, 2011.

[11] R. Klén, M. Visuri, and M. Vuorinen, "On Jordan type inequalities for hyperbolic functions," *Journal of Inequalities and Applications*, vol. 2010, Article ID 362548, 14 pages, 2010.

[12] D.-W. Niu, J. Cao, and F. Qi, "Generalizations of Jordan's inequality and concerned relations," *Politehnica University of Bucharest. Scientific Bulletin A*, vol. 72, no. 3, pp. 83–98, 2010.

[13] D.-W. Niu, Z.-H. Huo, J. Cao, and F. Qi, "A general refinement of Jordan's inequality and a refinement of L. Yang's inequality," *Integral Transforms and Special Functions*, vol. 19, no. 3, pp. 157–164, 2008.

[14] A. Y. Özban, "A new refined form of Jordan's inequality and its applications," *Applied Mathematics Letters*, vol. 19, no. 2, pp. 155–160, 2006.

[15] S.-H. Wu, "On generalizations and refinements of Jordan type inequality," *Octogon Mathematical Magazine*, vol. 12, no. 1, pp. 267–272, 2004.

[16] C.-J. Zhao and L. Debnath, "On generalizations of L. Yang's inequality," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 4, article 56, 2002.

[17] L. Zhu, "A general form of Jordan's inequalities and its applications," *Mathematical Inequalities and Applications*, vol. 11, no. 4, pp. 655–665, 2008.

[18] L. Zhu, "A general form of Jordan-type double inequality for the generalized and normalized Bessel functions," *Applied Mathematics and Computation*, vol. 215, no. 11, pp. 3802–3810, 2010.

[19] L. Zhu, "Jordan type inequalities involving the Bessel and modified Bessel functions," *Computers and Mathematics with Applications*, vol. 59, no. 2, pp. 724–736, 2010.

[20] G. D. Anderson, S.-L. Qiu, M. K. Vamanamurthy, and M. Vuorinen, "Generalized elliptic integrals and modular equations," *Pacific Journal of Mathematics*, vol. 192, no. 1, pp. 1–37, 2000.

[21] L. Yang, *Theory of Value Distribution of Functions and New Research*, Science Press, Beijing, China, 1982, (Chinese).

[22] C.-J. Zhao, "The extension and strength of Yang Le inequality," *Mathematics in Practice and Theory*, vol. 30, no. 4, pp. 493–497, 2000 (Chinese).

[23] F. Qi, L.-H. Cui, and S.-L. Xu, "Some inequalities constructed by Tchebysheff's integral inequality," *Mathematical Inequalities and Applications*, vol. 2, no. 4, pp. 517–528, 1999.