Gain and Phase Calibration of Uniform Rectangular Arrays Based on Convex Optimization and Neural Networks

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Abstract: A calibration method based on convex optimization (CVX) and neural networks is proposed for the large planar arrays of phased array three-dimensional imaging sonar systems. The method only needs an acoustic calibration source at an unknown position in the far field, and the direction of arrival (DOA) and gain and phase error are jointly estimated. The method uses a CVX algorithm to solve an optimization problem and initially estimates the DOA of the calibration source robustly. Subsequently, according to the estimation results, a neural network is used for fitting to obtain off-grid DOA estimation of the calibration source. Thereafter, spatial matched filtering is performed to obtain the gain and phase residual estimations. The root mean square error (RMSE) of the beam pattern calibrated by the method for uniform planar arrays can reach a value of 4.9542 × 10−5. The experimental results demonstrate the efficiency of the proposed method for gain and phase calibration.

Keywords: gain and phase calibration; sensor array; convex optimization

1. Introduction

In phased array three-dimensional (3D) sonar systems, the inconsistent sensor performances and sensor position deviation cause amplitude and phase errors in the array, leading to an increase in the sidelobe of the beam pattern and a shift in the focus direction. To solve this problem, amplitude and phase calibration of the sensor array is required for beamforming.

Numerous array calibration methods have been proposed, among which the simplest approach involves placing the sensor array and calibration acoustical source at fixed positions [1]. However, this is difficult to achieve in practical applications. Therefore, the array auto-calibration of an unknown calibration acoustical source is a major research direction. Conventionally, the direction of arrival (DOA) needs to be estimated initially through auto-calibration. DOA estimation algorithms include spectrum searching such as multiple signal classification (MUSIC) [2], the estimation of signal parameters via rotation invariance techniques (ESPRIT) [3], the Toeplitz-based (TB) algorithm [4], and the three-step-iterative (TSI) algorithm [5]. The above-mentioned methods utilize the orthogonality of signal subspace and noise subspace to estimate DOA through auto-covariance and eigenvalue decomposition. However, these algorithms require a huge amount of computation for large planar arrays. In recent years, compressed sensing (CS)-based [6] and deep learning [7–10] methods have been introduced to estimate the DOA and perform array calibration. However, owing to the 2π ambiguity of phase, the calibration method based on a neural network (NN) is prone to overfitting, which creates challenges in accurately estimating the correction source DOA. Moreover, in the CS-based method, the estimated DOA is constrained to initial discrete directions, and off-grid directions are difficult to estimate.

To address the above-mentioned shortcomings, a combined convex optimization (CVX) and NN method is proposed for auto-calibration of large planar arrays. The proposed method requires an acoustical source at an unknown location in the far-field. A dataset...
is generated with an acoustical source at a random location and a Gaussian distribution of errors and noises. A CVX algorithm is used to solve the planning problem to initially estimate the DOA robustly. Then, according to the DOA estimation result, a neural network is used to fit the DOA to obtain a more accurate DOA estimation in the off-grid direction. Finally, the spatial matched filter is used to calculate the amplitude and phase errors. The method uses the CVX algorithm to effectively avoid the phase 2π ambiguity problem and uses the neural network to estimate the off-grid direction. The experimental results demonstrate that the proposed method is efficient for gain and phase calibration.

This paper is organized as follows. In Section 2, an array calibration method based on CVX and an NN is proposed and analyzed. In Section 3, several methods of array calibration are employed to evaluate the efficiency of the proposed method. In Section 4, the experimental results are examined. In Section 5, the conclusions are drawn.

2. Materials and Methods

Consider a planar array with an \( M \times N \) sensor array. The array spacing between the sensors is \( d \), the acoustical wavelength is \( \lambda \), and the number of snapshots is \( T \). The single calibration acoustical source is placed from the direction \((\theta_a, \theta_b)\) in the far-field, as illustrated in Figure 1.

![Schematic of the planar array and calibration source.](image)

**Figure 1.** Schematic of the planar array and calibration source.

The sampled array signal can be expressed as

\[
x(m, n) = \xi_g(m, n) \exp\left(j \frac{2\pi}{\lambda} (u_0d_m + v_0d_n + \xi_p(m, n)) + \epsilon(m, n)\right),
\]

where \( u_0 = \sin\theta_a \), \( v_0 = \sin\theta_b \), \( (d_m, d_n) \) denotes the spacing between the sensor position \((m, n)\) and reference position, \( \xi_g \) and \( \xi_p \) denote the gain error \( \xi_g \sim N(1, \sigma_g^2) \) and phase error \( \xi_p \sim N(0, \sigma_p^2) \), respectively, and \( \epsilon(m, n) \) represents the noise. Moreover, \( N(\eta, \sigma^2) \) denotes the Gaussian distribution with expectation \( \eta \) and standard deviation \( \sigma \).

The gain and phase error \( \Gamma \) can be expressed as

\[
\Gamma = \xi_g \exp(j \xi_p)
\]

The propagation model can be expressed as

\[
x = \Gamma \cdot Ay + \epsilon,
\]

where \( x \) is the matrix of the sampled signal \( x \), \( A \) is the propagation matrix \( A \), and \( y \) is the acoustical signal matrix \( y \). The element of \( A \) is given by

\[
da_{m,n,p,q} = \exp\left(j \frac{2\pi}{\lambda} (d_m \sin \theta_p + d_n \sin \theta_q)\right),
\]
where \((\theta_p, \theta_q)\) denotes the steering direction, \(u = \sin \theta_p, v = \sin \theta_q\), and \((u, v)\) is restricted within \((-1 \text{ to } 1, -1 \text{ to } 1)\) divided into \(P \times Q\) directions. The DOA estimation problem can be transformed into the following optimization problem [6]:

\[
\hat{y} = \arg\min_y \lVert x - Ay \rVert_2^2 + \mu \lVert y \rVert_2^2
\]  

(5)

The problem can be solved by CVX [11]. The index of the maximum \(\hat{y} (\hat{u}_0, \hat{v}_0)\) is the CVX-estimated DOA.

CVX is a modelling framework for solving disciplined convex problems, including linear and quadratic programs, semidefinite programs, and \(l_1\)-norms. CVX is implemented in MATLAB to conveniently solve constrained norm minimization, entropy maximization, and several other CVX problems.

SDPT3 [12] is the default solver of CVX problems. SDPT3 is a primal-dual interior-point algorithm via a path-following paradigm. In each iteration of the algorithm, a predictor search direction is calculated to decrease the duality gap as much as possible. The solver uses two search directions: the HKM direction [13–15] and the NT direction [16]. Subsequently, the algorithm generates a Mehrotra-type corrector step [17] to approach the central path. The algorithm does not impose any neighbourhood restrictions and tries to simultaneously achieve feasibility and optimality.

However, the optimization problem of the DOA estimation \((\hat{u}_0, \hat{v}_0)\) is located on \(P \times Q\) discrete grids. If the calibration acoustical source \((u_0, v_0)\) is in the off-grid direction, the method cannot precisely estimate the DOA, and the directions surrounding \((\hat{u}_0, \hat{v}_0)\) have strong intensity. As indicated in Figure 2, the calibration source direction \((u_0, v_0)\) is \((0.4617, 0.4617)\), the CVX-estimated DOA \(O\) is \((0.46, 0.46)\), and the directions surrounding \(O\) have strong intensity.

To overcome this problem, the NN was trained to precisely estimate the off-grid DOA. The input is the CVX-estimated DOA \(O\) and \(5 \times 5\) surrounding directions, and the output is the direction of the calibration acoustical source \((\hat{u}_0, \hat{v}_0)\). The NN model includes three fully connected layers of 25 neural elements, as depicted in Figure 3. The loss function is mean square error (MSE). The activation function is the rectified linear unit (ReLU) expressed as follows:

\[
\text{ReLU}(x) = \begin{cases} 
  x & x \geq 0 \\
  0 & x < 0
\end{cases}
\]  

(6)

Figure 2. Estimated DOA of CVX method: (a) normalized beam pattern; (b) estimated DOA \(O\) and \(5 \times 5\) surrounding directions.
Once the estimated DOA \((\hat{u}_0, \hat{v}_0)\) is obtained via the CVX–NN method, the gain and phase error \(\Gamma\) can be estimated using a spatial matched filter [5].

The beam pattern in each snapshot for the estimated DOA \((\hat{u}_0, \hat{v}_0)\) is expressed as

\[ B(t) = a(\hat{u}_0, \hat{v}_0)Hx(t), \]

where \(a(\hat{u}_0, \hat{v}_0)\) denotes the ideal propagation vector corresponding to the steering direction \((\hat{u}_0, \hat{v}_0)\), \(a\)H represents the Hermitian transpose of \(a\), and \(t\) represents the snapshot index. The response vector of \((\hat{u}_0, \hat{v}_0)\) is expressed as

\[ R = \frac{\sum_{t=1}^{T} B(t)x(t)}{\sum_{t=1}^{T} |B(t)|^2} \]

The gain and phase errors can be obtained as follows:

\[ \hat{\Gamma} = \text{diag}(R \odot a(\hat{u}_0, \hat{v}_0)) \]

where \(\odot\) denotes elementwise division. The deviations of the DOA estimation are \(\Delta u\) and \(\Delta v\); \(\tilde{u}_0 = u_0 + \Delta u\) and \(\tilde{v}_0 = v_0 + \Delta v\). Ignoring the noise, \(a(\hat{u}_0, \hat{v}_0)\) can be expressed as

\[ a(u_0, v_0) = a(u_0 + \Delta u, v_0 + \Delta v) = \exp(i\frac{2\pi}{\lambda}(\Delta u d_m + \Delta v d_n)) \odot a(u_0, v_0) \]

where \(\odot\) denotes elementwise multiplication. For a larger \(\Delta u\), \(\Delta v\), \(d_m\), and \(d_n\), the deviation of the phase estimation observably increases. Based on the least squares method, the optimal reference position for the uniform planar array was the centre of the array, and the reference sensor index \((m_{\text{ref}}, n_{\text{ref}})\) was selected as the rounding of \(((M + 1)/2, (N + 1)/2)\). The normalised estimated gain and phase error is expressed as

\[ \hat{\Gamma} = \hat{\Gamma}/\hat{\Gamma}(m_{\text{ref}}, n_{\text{ref}}) \]

The method used the CVX algorithm to recover the signal according to the propagation model and obtained the direction of maximum strength as a robust DOA estimation. Subsequently, the beam strength was set as the input, and the NN was used for fitting to estimate the off-grid direction. This method effectively avoided the ambiguity of phase \(2\pi\), and because the implementation of the CVX algorithm to estimate DOA in the off-grid direction is complex, secondary estimation was employed to overcome this challenge.

3. Results

A planar array with \(50 \times 50\) sensors and sensor spacing of \(0.5\lambda\) was considered for the study. The calibration source was located in the direction \((27.5^\circ, 27.5^\circ)\). The gain and
phase errors were $\xi_g \sim \mathcal{N}(1, 0.2^2)$ and $\xi_p \sim \mathcal{N}(0, 0.6^2)$, respectively. The signal-to-noise ratio (SNR) was 25 dB, and the number of sample snapshots $T$ was 1000. The accuracy of the DOA, gain, and phase estimation was evaluated by the following root mean square error (RMSE) $E_d$, $E_g$, and $E_p$, respectively:

$$E_d = \sqrt{\frac{1}{2T} \sum_{t=1}^{T} (\hat{u}_t - u_0)^2 + (\hat{v}_t - v_0)^2},$$  

$$E_g = \sqrt{\frac{1}{MNT} \sum_{t=1}^{T} \|\hat{\xi}_g - \xi_g\|_2^2},$$  

$$E_p = \sqrt{\frac{1}{MNT} \sum_{t=1}^{T} \|\hat{\xi}_p - \xi_p\|_2^2}.$$  

According to the proposed CVX–NN method, the reference sensor index was $(25, 25)$, $E_d$ was $4.4583 \times 10^{-5}$, $E_g$ was 0.0027, and $E_p$ was 0.0017. The actual and estimated values of the gain and phase error are indicated in Figure 4. The partial actual and estimated values of the gain and phase errors are displayed in Table 1.

**Table 1.** Partial actual and estimated values of gain and phase errors.

| Sensor Index | Gain Error Value (Normalized) | Phase Error Value (Rad) |
|--------------|-------------------------------|-------------------------|
|              | Actual | Estimated | Actual | Estimated |
| (1, 1)       | 1.1380 | 1.1413    | -0.3866 | -0.3847    |
| (5, 5)       | 0.8205 | 0.8227    | -0.4081 | -0.4093    |
| (10, 10)     | 1.1534 | 1.1551    | -0.5294 | -0.5278    |
| (15, 15)     | 0.9586 | 0.9600    | -1.0722 | -1.0712    |
| (20, 20)     | 0.8177 | 0.8188    | -1.0351 | -1.0358    |
| (25, 25)     | 1.0000 | 1.0000    | 0.0000  | 0.0000     |
| (30, 30)     | 1.1298 | 1.1311    | -0.5637 | -0.5655    |
| (35, 35)     | 0.6254 | 0.6261    | -0.3143 | -0.3129    |
| (40, 40)     | 0.9555 | 0.9581    | 0.6357  | 0.6322     |
| (45, 45)     | 1.1451 | 1.1459    | -0.2079 | -0.2127    |
| (50, 50)     | 0.7934 | 0.7933    | 0.9309  | 0.9260     |

To verify the influence of the calibration results on the imaging quality, the beam pattern after calibration was compared with that before calibration and the ideal array. Figure 5 shows the comparison of beam patterns via calibration. The RMSE between the beam pattern after calibration and the ideal array was $4.9542 \times 10^{-5}$.
The results reveal that the proposed method was significantly superior to the TSI method. When using the proposed method versus the SNR when \( \sigma_p \) is set to a value of 0.2 throughout the experiments. A total of 100 independent experiments were carried out under each condition, and the results were averaged. Figure 6 presents the curve of the estimated DOA and phase error RMSE versus SNR when \( \sigma_p \) is set to a value of 0.2 throughout the experiments. A total of 100 independent experiments were carried out under each condition, and the results were averaged. Figure 6 presents the curve of the estimated DOA and phase error RMSE when using the proposed method versus the phase error standard deviation \( \sigma_p \) when the SNR is 25 dB. The results reveal that the proposed method was significantly superior to the TSI method when \( \sigma_p > 0.3 \). The RMSE of the TB method rapidly increases when \( \sigma_p > 0.3 \) because the method needs to calculate the sum/difference of phase, thus causing the ambiguity of phase \( 2\pi \) [5].

![Comparison of beam pattern via calibration: (a) ideal; (b) before calibration; (c) after calibration; (d) comparison of 2D beam pattern.](image1)

**Figure 5.** Comparison of beam pattern via calibration: (a) ideal; (b) before calibration; (c) after calibration; (d) comparison of 2D beam pattern.

To verify the performance of the proposed method under different conditions, the estimated results of the proposed method versus the gain and phase error standard deviations and SNR were compared with those of the TSI, TB and CVX methods. As the gain error had little effect on the DOA and phase error estimation, the gain error standard deviation \( \sigma_g \) was set to a value of 0.2 throughout the experiments. A total of 100 independent experiments were carried out under each condition, and the results were averaged.

![Table 1. Partial actual and estimated values of gain and phase errors.](image2)

**Table 1.** Partial actual and estimated values of gain and phase errors.

| Sensor Index | Gain Error Value (Normalized) | Phase Error Value (Rad) |
|--------------|-------------------------------|-------------------------|
| (1, 1)       | 1.1380                        | 1.1413                  |
| (5, 5)       | 0.8205                        | 0.8227                  |
| (5, 10)      | 1.0722                        | 1.0711                  |
| (10, 5)      | 1.0722                        | 1.0711                  |
| (10, 10)     | 1.1534                        | 1.1551                  |
| (15, 5)      | 1.0000                        | 1.0000                  |
| (15, 10)     | 1.0000                        | 1.0000                  |
| (20, 5)      | 0.8177                        | 0.8188                  |
| (20, 10)     | 0.8177                        | 0.8188                  |
| (25, 5)      | 0.9555                        | 0.9581                  |
| (25, 10)     | 0.9555                        | 0.9581                  |
| (30, 5)      | 1.1298                        | 1.1311                  |
| (30, 10)     | 1.1298                        | 1.1311                  |
| (35, 5)      | 0.6254                        | 0.6261                  |
| (35, 10)     | 0.6254                        | 0.6261                  |

**Figure 6.** Estimated DOA and phase error RMSE versus \( \sigma_p \): (a) DOA; (b) phase error.

![Estimated DOA and phase error RMSE versus \( \sigma_p \): (a) DOA; (b) phase error.](image3)
Figure 7 presents the curve of the estimated DOA and phase error RMSE obtained when using the proposed method versus the SNR when \( \sigma_p = 0.5 \). Under this condition, the RMSE of the TB method is significantly larger than that of the other methods. Thus, the results of the proposed method are significantly superior to those of the CVX and TSI methods.

![Figure 7. Estimated DOA and phase error RMSE versus SNR: (a) DOA; (b) phase error.](image)

Figure 8 presents the curve of the estimated gain error RMSE obtained when using the proposed method versus the SNR and \( \sigma_g \). As the estimation of gain error is independent of DOA, the results of each method are consistent.

![Figure 8. Estimated gain error RMSE versus SNR and \( \sigma_g \): (a) versus \( \sigma_g \); (b) versus SNR.](image)

4. Discussion

The proposed method uses CVX to robustly recover the signal and an NN to fit the beam intensity. This method could effectively avoid the phase \( 2\pi \) ambiguity problem and estimate the off-grid direction. Unlike alternative calibration methods such as TSI that are applied only for uniform arrays, the proposed method is not restricted by array configuration.

5. Conclusions

This paper presents a new gain and phase calibration method for the large planar arrays of 3D imaging sonar systems. A calibration acoustical source in the far-field is required. First, the CVX estimation method was adopted to obtain a preliminary estimation of the DOA of the calibration source. Thereafter, an NN was trained to accurately estimate the DOA. Ultimately, the DOA estimation and sampled signal covariance matrix were used to evaluate the gain and phase errors. Specifically, an RMSE of \( 4.9542 \times 10^{-3} \) was achieved for the beam pattern of the uniform planar array calibrated using the proposed method.
which was significantly superior to that of the TSI method for \( \sigma_p > 0.3 \). The experimental results demonstrate that the proposed CVX–NN method can accurately estimate and calibrate the gain and phase errors.

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**References**

1. Ng, B.P.; Lie, J.P.; Er, M.H.; Feng, A. A Practical Simple Geometry and Gain/Phase Calibration Technique for Antenna Array Processing. *IEEE Trans. Antennas Propag.* 2009, 57, 1963–1972. [CrossRef]

2. Schmidt, R. Multiple emitter location and signal parameter estimation. *IEEE Trans. Antennas Propag.* 1986, 34, 276–280. [CrossRef]

3. Roy, R.; Kailath, T. ESPRIT-estimation of signal parameters via rotational invariance techniques. *IEEE Trans. Acoust. Speech Signal Process.* 1989, 37, 984–995. [CrossRef]

4. Li, Y.; Er, M.H. Theoretical analyses of gain and phase error calibration with optimal implementation for linear equispaced array. *IEEE Trans. Signal Process.* 2006, 54, 712–723. [CrossRef]

5. Yuan, L.; Jiang, R.; Chen, Y. Gain and Phase Autocalibration of Large Uniform Rectangular Arrays for Underwater 3-D Sonar Imaging Systems. *IEEE J. Ocean. Eng.* 2014, 39, 458–471. [CrossRef]

6. Terada, T.; Nishimura, T.; Ogawa, Y.; Oghane, T.; Yamada, H. DOA Estimation for Multi-Band Signal Sources Using Compressed Sensing Techniques with Khatri-Rao Processing. *IEICE Trans. Commun.* 2014, E97.B, 2110–2117. [CrossRef]

7. Ogut, M.; Bosch-Lluis, X.; Reising, S.C. Deep Learning Calibration of the High-Frequency Airborne Microwave and Millimeter-Wave Radiometer (HAMMR) Instrument. *IEEE Trans. Geosci. Remote Sens.* 2020, 58, 3391–3399. [CrossRef]

8. Wang, Y.; Yang, A.; Chen, X.; Wang, P.; Wang, Y.; Yang, H. A Deep Learning Approach for Blind Drift Calibration of Sensor Networks. *IEEE Sens. J.* 2017, 17, 4158–4171. [CrossRef]

9. Goodman, J.; Salmond, D.; Davis, C.; Acosta, C. Ambiguity Resolution in Direction of Arrival Estimation using Mixed Integer Optimization and Deep Learning. In Proceedings of the 2019 IEEE National Aerospace and Electronics Conference (NAECON), Dayton, OH, USA, 15–19 July 2019; pp. 317–322. [CrossRef]

10. Ahmed, A.M.; Thanhtirge, U.S.K.P.M.; El Gamal, A.; Sezgin, A. Deep Learning for DOA Estimation in MIMO Radar Systems via Emulation of Large Antenna Arrays. *IEEE Commun. Lett.* 2021, 25, 1559–1563. [CrossRef]

11. CVX: MATLAB Software for Disciplined Convex Programming. Available online: http://cvxr.com/cvx/ (accessed on 8 August 2021).

12. Tütüncü, R.H.; Toh, K.C.; Todd, M.J. Solving semidefinite-quadratic-linear programs using SDPT3. *Math. Program.* 2003, 95, 189–207. [CrossRef]

13. Helmberg, C.; Rendl, F.; Vanderbei, R.J.; Wolkowicz, H. An Interior-Point Method for Semidefinite Programming. *SIAM J. Optim.* 1996, 6, 342–361. [CrossRef]

14. Kojima, M.; Shindoh, S.; Hara, S. Interior-Point Methods for the Monotone Semidefinite Linear Complementarity Problem in Symmetric Matrices. *SIAM J. Optim.* 1997, 7, 86–125. [CrossRef]

15. Monteiro, R.D.C. Primal-Dual Path-Following Algorithms for Semidefinite Programming. *SIAM J. Optim.* 1997, 7, 663–678. [CrossRef]

16. Nesterov, Y.E.; Todd, M.J. Self-Scaled Barriers and Interior-Point Methods for Convex Programming. *Math. Oper. Res.* 1997, 22, 1–42. [CrossRef]

17. Mehrotra, S. On the Implementation of a Primal-Dual Interior Point Method. *SIAM J. Optim.* 1992, 2, 575–601. [CrossRef]