Generalized hypergeometric series for Racah matrices in rectangular representations

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ABSTRACT

One of spectacular results in mathematical physics is the expression of Racah matrices for symmetric representations of the quantum group SU_q(2) through the Askey-Wilson polynomials, associated with the q-hypergeometric functions 4φ3. Recently it was shown that this is in fact the general property of symmetric representations, valid for arbitrary SU_q(N) – at least for exclusive Racah matrices $\bar{S}$. The natural question then is what substitutes the conventional q-hypergeometric polynomials when representations are more general? New advances in the theory of matrices $\bar{S}$, provided by the study of differential expansions of knot polynomials, suggest that these are multiple sums over Young sub-diagrams of the one, which describes the original representation of SU_q(N).

A less trivial fact is that the entries of the sum are not just the factorized combinations of quantum dimensions, as in the ordinary hypergeometric series, but involve non-factorized quantities, like the skew characters and their further generalizations – as well as associated additional summations with the Littlewood-Richardson weights.

1 Introduction

Racah matrices [1] describe the deviation from associativity in the product of representations and they play a prominent and increasing role in modern quantum field and string theory. Despite in many cases we need these quantities for sophisticated representations of infinite-dimensional algebras, they are still far from being well known even for the simplest quantum algebras SU_q(N). In these simple cases Racah matrices are responsible for at least two subjects of primary importance: modular transformations of conformal blocks [2] in 2d conformal field theory [3] and for calculation of knot invariants [4] (knot polynomials) in 3d Chern-Simons theory [5]. Part of the problem is that Racah matrices are actually maps, and explicit formulas are available in particular bases – what often makes these formulas non-invariant (this is often referred to as the multiplicity problem), and thus not-very-interesting for pure mathematicians. Thus the progress in the field is largely due to physical methods and inspirations – while the rigorous presentation awaits the completion of the phenomenological part of the story.

In the present paper we open one more new chapter of this exciting mystery-book: the relation between Racah matrices and hypergeometric orthogonal polynomials of the Askey-Wilson type [6]. Recently we reviewed the subject from the point of view of orthogonal polynomials [7], and now we approach it again from the Racah matrix side – with the natural question: what generalizes hypergeometric series if we switch from symmetric to generic representations of SU_q(N).

As the first step we demonstrated in [8] that exclusive Racah matrices $\bar{S}^R_{\mu\nu\tau}$,

$$(R \otimes \bar{R}) \otimes R \longrightarrow \bar{R} \quad \bar{S} \quad \left( R \otimes (\bar{R} \otimes R) \longrightarrow R \right)$$

needed for arborescent knot calculus of [9], in the case of symmetric representations $R = [r]$ are expressed through the hypergeometric Racah/Askey-Wilson polynomials for arbitrary algebra SU_q(N) (for $N = 2$ this is the classical result).

Not surprisingly, similar formulas follow from the general $\bar{S}$-calculus of [10]–[16]. Since this calculus is applicable to arbitrary representations $R$, and is especially well understood for rectangular $R = [r^*]$, this opens a possibility to generalize the hypergeometric realization. Relevant substitute of the hypergeometric series in this case are sums over Young diagrams with the entries made from the skew characters of su_q(N).

We begin from reminding the general ideas of $\bar{S}$-calculus in sec.2 and then describe in sec.3 what is currently known about the structure of the underlying $F$-functions and their relation [13] to skew Schur polynomials. After
that in secs. 4 and 5 we explain what happens in symmetric representations \( R = [r] \) and how the Askey-Wilson realization arises for them in this context. We conclude in sec. 6 with the suggestion that the formulas in sec. 2 are exactly the ones, which provide the extension of Askey-Wilson realization from symmetric to rectangular representations. Non-rectangular case is also straightforward, but there are still some technical difficulties to be resolved to handle non-trivial multiplicities – the most interesting part of both Racah and arborescent calculi. We refer to [15, 16] for explanations and leave non-rectangular case for the future considerations.

2 Factorization of differential expansion for double braids

According to [10, 11] and [12, 13] the rectangularly-colored HOMFLY for a double braid \((m, n)\)

\[
\frac{\chi^\ast_{\lambda}(N) = \text{Schur}\left\{ p_k = \frac{[Nk]}{[k]} \right\}}{h_\lambda = \prod_{(\alpha,\beta) \in \lambda} [\text{hook length}(\alpha,\beta)] = \prod_{(\alpha,\beta) \in \lambda} [\text{leg}(\alpha,\beta) + \text{arm}(\alpha,\beta) + 1]}
\]

is given by a factorized differential expansion [17]-[19]:

\[
H_{R}^{(m,n)} = \sum_{\mu,\nu \subseteq R} \frac{\sqrt{D_{\mu}D_{\nu}}}{d_{\mu}} S_{\mu \nu}^R \Lambda^m_{\mu} \Lambda^n_{\nu} = \sum_{\lambda \subseteq R} \chi^\ast_{\lambda}(r)\chi^\ast_{\lambda}(s) \cdot \{q\}^{2|\lambda|} h^2_\lambda \chi^\ast_{\lambda}(N + r)\chi^\ast_{\lambda}(N - s) \cdot \frac{F_{\lambda}^{(m)}F_{\lambda}^{(n)}}{F_{\lambda}^{(1)}F_{\lambda}^{(1-1)}}
\]

(1)

Here \( \{x\} = x - x^{-1} \), quantum numbers are \( [n] = \{q^n\} \) and \( h_\lambda \) is denominator in

\[
\chi^\ast_{\lambda}(N) = \text{Schur}\left\{ p_k = \frac{[Nk]}{[k]} \right\} = \frac{1}{h_\lambda} \prod_{(\alpha,\beta) \in \lambda} [N + \alpha - \beta]
\]

(2)

given by the hook formula:

For the figure-eight knot 4_1 parameters are \((m, n) = (1, -1)\), and the last factor (the ratio of four \( F \)) is absent. Generic twist knots correspond to \( n = 1 \), and, since actually \( F_{\lambda}^{(-1)} = 1 \) the last ratio turns into just \( F_{\lambda}^{(m)} \).

The difference between knot and Racah calculi is that for the latter one we need not the \( F_{\lambda}^{(m)} \) as a total for a given \( m \), but its decomposition into particular eigenvalues, dictated by the evolution method of [20] and [18]. Among other things this eliminates the factor \( \{q\}^{2|\lambda|}h_\lambda \) from the expression for \( S \), making it from just the characters \( \chi^\ast \) at the topological locus [20] (quantum dimensions) with a small admixture of additional quantum-number factors.

\( F_{\lambda}^{(m)} \) are actually sums over \( m \)-the powers of the squared ”eigenvalues” \( \Lambda^m_{\mu} \), where \( \mu \) are Young sub-diagrams of \( \lambda \) – and according to [11] one can read from this eigenvalue expansion the Racah matrices \( S_{\mu \nu}^R \). According
to [13] the coefficients of the F expansion are essentially the skew characters \( \chi_{\lambda/\mu} = \sum \nu C_{\mu\nu}^{\lambda} \chi_\nu \) with the Littlewood-Richardson coefficients \( C_{\mu\nu}^{\lambda} \), defined from \( \chi_\mu \chi_\nu = \sum \lambda C_{\mu\nu}^{\lambda} \chi_\lambda \).

Eq.(1) and the general formula for F-expansion in [13] implies for the case of rectangular representation \( R = [r^s] \) with \( s \leq 2 \) that

\[
\sum_{\mu,\nu \subseteq \lambda \subseteq r} \frac{(-1)^{|\lambda|} \chi_\lambda^{(s)}(r) \chi_\lambda^{(s)}(s) \chi_\lambda^{(s)}(N + r) \chi_\lambda^{(s)}(N - s)}{\chi_\lambda^{(s)}(N)^2} \frac{C_{\mu/\nu}^{\lambda} \chi_{\mu/\nu}^{(s)}(N_{\mu/\nu}) \chi_{\mu/\nu}^{(s)}(N_{\mu/\nu}) \chi_{\mu/\nu}^{(s)}(N_{\mu/\nu}) \chi_{\mu/\nu}^{(s)}(N_{\mu/\nu})}{\chi_\lambda^{(s)}(N)} \chi_\lambda^{(s)}(N)
\]

(4)

where \( N_{\mu/\nu} = N + \sum i - \sum j - \lambda \)-dependent shift for the hook-parametrization [11] of \( \mu = (i_1, j_1 | i_2, j_2 | \ldots) \) and \( C_{\mu/\nu}^{\lambda} \) are some factorized quantities: ratios of quantum numbers \( [N + u] \) with various shifts \( u \), also described in terms of the hook parameters \( \{i, j\} \) for \( \mu \) and \( \{a, b\} \) for \( \lambda \). As to \( D_{\mu} \) at the l.h.s., they are dimensions of representations, appearing in decomposition of the product \( R \otimes \bar{R} \) – for which rectangular \( R \) are in one-to-one correspondence with the Young sub-diagrams of \( R \) [12].

3 F-functions and G-factors

According to [11] and [13] the F-functions are best described in a peculiar hook parametrization of Young diagrams:

![Diagram of a 3-hook Young diagram](image)

\( a_1, a_2, a_3 \)
\( b_1, b_2, b_3 \)

The main drawback of this parametrization is that it changes discontinuously with the number of hooks: the empty diagram \( \emptyset \) is not a particular case of any 1-hook diagram \( (a_1, b_1) \), of which the minimal is \( [1] = (0, 0) \) and so on. Formally one could associate \( \emptyset \) with \( a_1 + b_1 = -1 \), but this is not quite respected by the formulas. Because of this one needs to write

\[
F^{(m)}(A, q) = \sum \frac{c_\lambda}{\{q\} |\lambda| h_\lambda \cdot \chi_\lambda(N)} \sum_{\mu \subseteq \lambda} f_\mu(N, q) \cdot \Lambda_\mu^m
\]

(5)

with different expressions for different hook numbers \( \#^h \). The F-functions depend explicitly on \( A = q^N \) and \( q \), but mostly are made from the quantum numbers, involving \( N \). The only exceptions are the squared eigenvalues

\[
\Lambda_\mu = \Lambda_{(i_1, j_1 | i_2, j_2 | \ldots)} = \prod_{k=1}^{\#^h} (A \cdot q^{i_k - j_k})^{2(i_k + j_k + 1)}
\]

(6)

and the overall coefficients

\[
c_\lambda = c_{(a_1, b_1 | a_2, b_2 | \ldots)} = \prod_{k=1}^{\#^h} (A \cdot q^{a_k - b_k})^{(a_k + b_k + 1)}
\]

(7)

Both, however, drop away from the expression (1) for the Racah matrix \( \tilde{S}^R - \Lambda_\mu \) because \( \tilde{S}_{\mu\nu}^R \) are coefficients of the A-expansion and \( c_\lambda \) because of cancellations, dictated by the properties:

\[
F^{(-1)}_\lambda = 1, \quad F^{(0)}_\lambda = \delta_{\lambda, \emptyset}, \quad F^{(1)}_\lambda = (-)^{\#^h (a_k + b_k + 1)} c_\lambda^2
\]

(8)

which are responsible for the simplicity of the differential expansion [17]-[19] at the r.h.s. of (1) for respectively the figure-eight knot \( 4_1 \), unknot and the trefoil \( 3_1 \). As already mentioned after (1), the factors \( \{q\} \) and \( h_\lambda \) also drop away from the expressions for HOMFLY polynomials and \( \tilde{S} \).
The sum rules (8) are non-trivial analogues of the elementary identity
\[
\sum_{\mu \subseteq \lambda} (-1)^{|\mu|} \cdot \chi_{\lambda/\mu} \cdot \chi_{\mu^{tr}} = \delta_{\lambda,\emptyset}
\]
(9)
which follows from the defining property of skew characters,
\[
\sum_{\mu \subseteq \lambda} \chi_{\lambda/\mu} \{p'_k\} \cdot \chi_{\mu^{tr}} \{p''_k\} = \chi_{\lambda} \{p'_k + p''_k\}
\]
(10)
and the transposition law
\[
\chi_{\mu} \{-p_k\} = (-1)^{|\mu|} \chi_{\mu^{tr}} \{p_k\}
\]
(11)
While (9) holds beyond the topological locus (i.e. for all values of time variables), it does not survive introduction of weights $A^{\pm 1}$ even on the locus, i.e. there is no analogue of the other two identities in (8).
The difficult part of the story is to describe $f^0_{\lambda}$ which satisfy all the three. Currently they are fully known for $\lambda = (a_1, b_1(a_2, 0)$ what is enough to get the Racah matrices $S$ for the case $R = [r, r]$ (actually, for this purpose $b_1 = 0, 1$ is sufficient). After (9) it is not such a big surprise that they involve skew characters, but exact formulas [11,13] are still not very easy to interpret and understand.
• For the empty diagram $\mu$ always
\[
f^0_{\emptyset} = 1
\]
(12)
• Since $\mu \subseteq \lambda$ the number of hooks $#^h_{\mu} \leq #^h_{\lambda}$. Thus for the single-hook $\lambda$ it remains to describe only the contributions of the single-hook $\mu$. These are relatively simple factorized expressions [11]:
\[
f^{i,j}_{(a,b)} = g^{i,j}_{(a,b)} \cdot K^{i,j}_{(a,b)} = (-)^{i+j+1} \cdot \frac{[a]!}{[a-i][i]!} \cdot \frac{[b]!}{[b-j][j]!} \cdot \frac{[a + b + 1]}{[i + j + 1]} \cdot \frac{D_{a}!D_{a+i+1}!}{D_{a+i+j+1}!} \cdot \frac{\tilde{D}_{j}!\tilde{D}_{j+i+1}!}{D_{b+j+1}!D_{b-j-1}!} \frac{D_{2a+1}D_{2a+1-j}}{D_{0}D_{1-j}}
\]
(13)
with
\[
g^{i,j}_{(a,b)} = (-)^{i+j+1} \frac{D_{2a+1}D_{2a+1-j}}{D_{0}D_{1-j}} \frac{(D_{a})^2}{D_{a+i+j+1}!D_{a-i-1}!} \frac{(\tilde{D}_{j})^2}{D_{b+j+1}!D_{b-j-1}!}
\]
and
\[
K^{i,j}_{(a,b)} = \frac{\chi^{a,b}_{\lambda/\mu}(N)}{\chi^a_{\lambda}(N)}
\]
(15)
Note that this combination involves $\chi_{\mu}$ rather than $\chi_{\mu^{tr}}$, thus $\sum_{\mu \subseteq \lambda} (-1)^{|\mu|} K^{i,j}_{\lambda} \neq 0$ (in fact, it vanishes, but only for diagrams $\lambda$ of odd size $|\lambda| = odd$, because $(-)^{|\mu|} \chi_{\mu}(p_k) = \chi_{\mu}((-)^k p_k)$, i.e. only odd times change sign).
Notation in (14) is: $D_{a} = [N + a]$, $D_{b} = [N - b]$ and $D_{a}! = \prod_{k=0}^{a} D_{k} = \frac{[N+a]!}{[N-a]!}$ and $D_{b}! = \prod_{k=0}^{b} \tilde{D}_{k} = \frac{[N]!}{[N-b-1]!}$ (note that these products start from $k = 0$ and include respectively $a + 1$ and $b + 1$ factors).
• For two-hook $\lambda = (a_1, b_1(a_2, b_2)$ the formulas are far more involved, and they are different for different number of hooks in $\mu$:
\[
f^{i_1,j_1}_{(a_1,b_1)} \cdot g^{i_1,j_1}_{(a_1,b_1)} = g^{i_1,j_1}_{(a_1,b_1)} \cdot K^{i_1,j_1}_{(a_1,b_1)}(N) \cdot \xi^{i_1,j_1}_{(a_1,b_1)}(N)
\]
(16)
\[
f^{i_2,j_2}_{(a_2,b_2)} = \frac{[N + i_1 + i_2 + 1][N - j_1 - j_2 - 1]}{[N + i_1 - j_2][N + i_2 - j_1]} \cdot g^{i_2,j_2}_{(a_2,b_2)} \cdot K^{i_2,j_2}_{(a_2,b_2)}(N) \cdot \xi^{i_2,j_2}_{(a_2,b_2)}(N)
\]
(17)
Non-trivial are the correction factors:
\[
\xi^{i_1,j_1}_{(a_1,b_1)}(N) = \left[ \frac{[N + a_2 - j_1][N - b_2 + i_1]}{[N + a_2 + i_1 + 1][N - b_2 - j_1 - 1]} \cdot \frac{K^{i_1,j_1}_{(a_1,b_1)}(N + i_1 - j_1)}{K^{i_1,j_1}_{(a_1,b_1)}(N)} \cdot \delta_{i_1,j_1-0} \right] + \frac{K^{i_1,j_1}_{(a_1,b_1)}(N + i_1 + 1, b_2)}{K^{i_1,j_1}_{(a_1,b_1)}(N + i_1 + 1, b_2)} \cdot \left[ \frac{(1 - \delta_{i_1,0})(1 - \delta_{j_1,0})}{(1 - \delta_{a_2,0})(1 - \delta_{b_2,0})} \right] + \frac{K^{i_1,j_1}_{(a_1,b_1)}(N + i_1 + 1, b_2)}{K^{i_1,j_1}_{(a_1,b_1)}(N + i_1 + 1, b_2)} \cdot \left[ \frac{(1 - \delta_{i_1,0})(1 - \delta_{j_1,0})}{(1 - \delta_{a_2,0})(1 - \delta_{b_2,0})} \right]
\]
(18)
and
\[
\xi^{i_2,j_2}_{(a_2,b_2)}(N) = \left[ \frac{[N + i_2 + 2][N - j_2 + 2]}{[N + i_2 - 1][N + j_2 - 1]} \cdot \frac{K^{i_2,j_2}_{(a_2,b_2)}(N + i_2 + 2, b_2)}{K^{i_2,j_2}_{(a_2,b_2)}(N + i_2 + 2, b_2)} \cdot \delta_{b_2,0} \right] + \frac{K^{i_2,j_2}_{(a_2,b_2)}(N + i_2 + 2, b_2)}{K^{i_2,j_2}_{(a_2,b_2)}(N + i_2 + 2, b_2)} \cdot \left[ \frac{(1 - \delta_{b_2,0})(1 - \delta_{a_2,0})}{(1 - \delta_{a_2,0})(1 - \delta_{b_2,0})} \right]
\]
(19)
Note that we provide expressions only for the case when \( a_2 \cdot b_2 = 0 \) (i.e. when either \( b_2 = 0 \) or \( a_2 = 0 \)), what is emphasized by boxes in above formulas. Sufficient for all the simplest non-symmetric rectangular representations \( R = [r, r] \) and \( R = [2r] \) are respectively \( b_2 = 0 \) and \( a_2 = 0 \). Note also, that \( K \) factorizes nicely for the single-hook diagrams \( \lambda \):

\[
K^{(i,j)}_{(a,b)}(N) = \frac{\chi^{(i,j)}_{(a,b)}(N) \cdot \chi^{(i,j)}_{(a,b)}(N)}{\chi^{(a,b)}_{(a,b)}(N)} = \frac{[a]! \cdot [b]! \cdot [a+b+1] \cdot \ldots \cdot [N]}{\ldots \cdot [N+j+1] \cdot [a+b-i] \cdot \ldots \cdot [N-i]} \cdot \frac{[N+i]!}{[N-j-1]!} \cdot \frac{[N+a-i-1]!}{[N+a]!} \cdot \frac{[N-b-1]!}{[N-b+j]!}.
\]

(19)

but does not do so for the two-hook \( \lambda \), even if \( \mu \) is still a single-hook, like in (18), e.g.

\[
K^{(1,1)}_{(3,1,1,0)}(N) \sim \chi^{(1,1)}_{(3,1,1,0)}(N) = \chi^{(1,1)}_{(3,1,1)}(N) + \chi^{(1,1)}_{(2,2)}(N) \sim A^2 q^8 + 2 A^2 q^6 + A^2 q^4 + A^2 q^2 - q^6 - q^4 - 2 q^2 - 1 \sim [3][2][N+2] + [4][N] \sim [2][N+2] + [N-2]
\]

(20)

Therefore there seems to be no freedom to change the somewhat mysterious shifts of \( N \) in these formulas.

In the case of \( a_2 \cdot b_2 \neq 0 \) skew characters are further deformed into still more complicated quantities [14], which still lack a proper identification.

4 Symmetric representations

For symmetric representations \( R = [r] \) with \( s = 1 \) we obtain from (1), by recursively substituting (5), (13), (14) and (19):

\[
\bar{a}_{\mu}^{[r]} = d_r \cdot \frac{[\mu]![\nu]![N-1]![N-2]!}{[N+\mu-2]![N+\nu-2]!} \cdot \sum_{\mu, \nu \leq \lambda \leq r} (-)^\lambda \frac{[\lambda]!}{[\lambda-\mu]!}[\lambda-\nu]! \cdot \frac{[N+r+\lambda-1]![N+\lambda-2]!}{[N+\lambda+\mu-1]![N+\lambda+\nu-1]!} = \frac{[r]!}{[\mu]![\nu]!} \cdot \chi^r_\mu(N)^2 \cdot \chi^r_\nu(N)^2 \cdot \sum_{\lambda = \max(\mu, \nu)} r \frac{(-)^\lambda [\lambda]!}{[\lambda-\mu]!}[\lambda-\nu]! \chi^r_\lambda(N+r) \chi^r_\lambda(N-1) \cdot \frac{\chi^r_\lambda(N+\mu) \chi^r_\lambda(N+\nu)}{\chi^r_\lambda(N+\mu) \chi^r_\lambda(N+\nu)}
\]

(21)

where used are also explicit expression for dimensions

\[
D_\mu = [N+2\mu-1][N-1] \left( \frac{[N+\mu-2]!}{[\mu]![N-1]!} \right)^2
\]

(22)

of representations in \( R \otimes \bar{R} = [r] \otimes [r]^{N-1} = \oplus_{\mu=0}^r [r+\mu, r^{N-2}, r-\mu] = \oplus_{\mu=0}^r [2\mu, \mu^{N-2}] \) to convert the original \( \sqrt{D_\mu D_\nu} \bar{S}_{\mu}^{[r]} \) into \( \bar{a}_{\mu}^{[r]} = (-)^{\mu+\nu} \frac{d}{\sqrt{D_\mu D_\nu}} \bar{S}_{\mu}^{[r]} \).

This expression is partly in terms of quantum dimensions \( \chi^r_\lambda \) and does contains neither skew characters, nor shifts. The only thing which reminds that the skew characters are somehow behind the scene is the presence of factorials \( [\lambda-\mu]! [\lambda-\nu]! \) in denominator. Of course, it is possible to make this explicit, by rewriting (21) in the form (4):

\[
\bar{a}_{\mu}^{[r]} = \frac{1}{\chi^r_\mu(N-1) \chi^r_\mu(N-1)} \cdot \sum_{\lambda = \max(\mu, \nu)} r \frac{(-)^\lambda [\lambda]!}{[\lambda-\mu]!}[\lambda-\nu]! \cdot \frac{[N-1]!}{[\lambda-\mu]!}[\lambda-\nu]! \cdot \frac{\chi^r_{\lambda-\mu}(N+\mu) \chi^r_{\lambda-\mu}(N+\nu)}{\chi^r_\lambda(N+\mu) \chi^r_\lambda(N+\nu)}
\]

with relatively simple shifts \( N^{[\lambda]}_{[\mu]} = N+\mu-1 \), which are independent of \( \lambda \), and \( G \)-factors \( G^{[r]}_{[\lambda]} = (-)^\mu \frac{[N+\lambda-1]!(N+\mu-1)}{[N+\lambda+\mu-1](N+\mu-1)} \).

Example of \( R = [1] \):

In we denote the combination in the box in the last formula through \( B_\lambda \), then

\[
\bar{a}^{[1]} = \begin{pmatrix}
\frac{1}{\chi_0(N-1) \chi_0(N-1)} \cdot \frac{B_0 \chi_0(N) \chi_0(N) \chi_0(N) \chi_0(N) \chi_0(N)}{\chi_0(N-1) \chi_0(N-1)} & \frac{1}{\chi_0(N-1) \chi_0(N-1)} \cdot \frac{B_1 \chi_0(N+1) \chi_0(N+1) \chi_0(N+1) \chi_0(N+1) \chi_0(N+1)}{\chi_0(N-1) \chi_0(N-1)}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\chi_1(N-1) \chi_1(N-1)} \cdot \frac{B_0 + B_1}{\chi_1(N-1) \chi_1(N-1)} & \frac{1}{\chi_1(N-1) \chi_1(N-1)} \cdot \frac{B_1}{\chi_1(N-1) \chi_1(N-1)}
\end{pmatrix}
\]

(23)
with \( B_1 = -[N + 1][N - 1] \) and \( B_0 = [N]^2 \).

After multiplication by \( \frac{(-)^{r+\ell}}{d_{[\ell]}} \sqrt{D_0 D_1} \) with \( D_0 = 1, D_1 = [N + 1][N - 1] \) and \( d_{[\ell]} = [N] \) this gives the unitary symmetric Racah matrix

\[
\tilde{S}^{[\ell]}_1 = \frac{1}{[N]}
\begin{pmatrix}
1 & \sqrt{[N + 1][N - 1]} \\
\sqrt{[N + 1][N - 1]} & -1
\end{pmatrix}
\]

(24)

**Example of \( R = [2] \):**

Similarly, in this case

\[
\begin{pmatrix}
\frac{B_0 x_0(N)(N)(N - 1)}{x_0(N)(N)(N - 1)} + \frac{B_1 x_0(N)(N)(N - 1)}{x_1(N)(N + 1)(N + 2)} + \frac{B_2 x_0(N)(N)(N - 1)}{x_2(N)(N + 2)(N + 3)} & x_0(N)(N + 1)(N + 1)\chi_1(N+1,N+1) + \frac{B_2 x_0(N)(N)(N - 1)}{x_2(N)(N + 2)(N + 3)} \\
x_1(N)(N + 1)(N + 1)\chi_1(N+1,N+1) + \frac{B_2 x_0(N)(N)(N - 1)}{x_2(N)(N + 2)(N + 3)} & x_0(N)(N)(N) + \frac{B_2 x_0(N)(N)(N - 1)}{x_2(N)(N + 2)(N + 3)}
\end{pmatrix}
\]

with \( B_0 = [N]^2, B_1 = -[N + 2][N - 1][N - 1][N - 1] \) and \( B_2 = [N - 1][N - 2][N - 1][N - 1] \).

Multiplication by \( \frac{(-)^{r+\ell}}{d_{[\ell]}} \sqrt{D_0 D_1} \) with \( D_0 = 1, D_1 = [N + 1][N - 1], D_2 = [N + 2][N - 1][N - 1] \) and \( d_{[2]} = [N][N - 1][N - 1][N - 1][N - 1] \) provides the unitary symmetric Racah matrix

\[
\tilde{S}^{[2]}_2 = \frac{[2]}{[N + 1][N]}
\begin{pmatrix}
\frac{1}{\sqrt{[N + 1][N - 1]}} & \frac{\sqrt{[N + 1][N - 1]}}{[N + 1][N - 1][N - 1][N - 1][N - 1]} \\
\frac{\sqrt{[N + 1][N - 1]}}{[N + 1][N - 1][N - 1][N - 1][N - 1]} & \frac{[N][N + 1][N - 1]}{[N + 3][N + 1][N + 1]}
\end{pmatrix}
\]

(26)

## 5 Hypergeometric series

Coming back to (21), one can make a change of summation variable \( \lambda = r - k \) to get a factorial \([k]!\) in the denominator. Then the sum turns into

\[
\sigma_{\mu \nu}^{[r]} = \frac{1}{\chi_\mu(N - 1)\chi_\nu(N - 1)} \sum_{k=0}^{\min(r - \mu, r - \nu)} (-)^{r - k} \frac{[r - k]!}{[r - \mu - k]!(r - \nu - k)!} \frac{[N + 2r - 1 - k]!(N + r - 2 - k)!}{[N + r + \mu - 1 - k]!(N + r + \nu - 1 - k)!} \sum_{k=\max(\mu, \nu)}^{\min(r, \mu + \nu)} (-)^{k} \frac{[k - \mu]!(k - \nu)!}{[r + \mu + N - 1]!(r + \nu + N - 1)!} \frac{[k + N + r - 1]!}{[k + N + r - 1]!(k + N + r - 1)!(k + N + r - 1)!}
\]

(27)

what is proportional to the \( q \)-hypergeometric polynomial

\[
\phi_{3} \left( \begin{array}{c}
\mu - r, \nu - r, 1 - N - r - \mu, 1 - N - r - \nu \\
-\mu - r, 2 - r - N, 2 - r - N
\end{array} \middle| z = q^2 \right)
\]

(28)

This looks different but is actually equivalent to the result of [8] for the same \( \dot{\sigma}_{\mu \nu}^{[r]} \):
\[ 4 \phi_3 \left( 1 - 2r - N, \mu + r - \nu - r, \mu - r, \nu - r \mid z = q^2 \right) \] (29)

Note that (21) and (27) contain just five factorials in denominator, instead of seven in the first line of (29), which are usual in the standard formulas for SU_q(2) Racah matrices.

We remind [7] that in the balanced case, i.e. for \( \alpha_1 + \ldots + \alpha_{p+1} + 1 = \beta_1 + \ldots + \beta_p \), hypergeometric series are expressed through quantum numbers and

\[ \sum_{n} \frac{[\alpha + n - 1]! \ldots [\alpha_{p+1} + n - 1]! \beta_1 - 1]! \ldots [\beta_p - 1]!}{[\alpha_1 - 1]! \ldots [\alpha_{p+1} - 1]! [\beta_1 + n - 1]! \ldots [\beta_p + n - 1]! [n]!} \sim \sum_{k} \frac{[-\beta_1 - n]! \ldots [-\beta_p - k]!}{[-\alpha_1 - k]! \ldots [-\alpha_{p+1} - k]! [k]!} \] (30)

6 Conclusion

In this sense (4) can be considered as the generalization of the q-hypergeometric polynomials, which is relevant for description of generic Racah matrices, at least in rectangular representations. In non-rectangular case some sub-diagrams \( \lambda \in R \) appear with non-trivial multiplicities and contribute additional terms into this expansion, see [15,16] for more details.

In general the elements of Racah matrix \( S^{[\mu,\nu]}_{[\lambda,\rho]} \) are expressed through quantum numbers, but are not factorized – for two reasons: because of the sum over sub-diagrams \( \lambda \in R \) and because the items in the sum are made not just from the nicely-factorized quantum dimensions \( \chi^s_{\lambda}, \chi^s_{\mu}, \chi^s_{\nu} \), but also from the skew characters \( \chi^s_{\lambda/\mu} \) and \( \chi^s_{\lambda/\nu} \) and their further generalizations, like (23) and (31) in [14], which do not have this factorization property. For \( R = [r,r] \) one can alternatively represent \( S \) as a triple sum with the Littlewood-Richardson weights

\[ S^{[\mu,\nu]}_{[\lambda,\rho]} = \sum_{\mu', \nu', \rho, \lambda \in [r,r]} C^{\lambda}_{\mu \mu'} C^{\lambda}_{\nu \nu'} B_{\mu \nu'}^{\lambda \rho} \] (31)

then \( B_{\mu \nu'}^{\lambda \rho} \) will be factorized combinations of quantum numbers, but the number of sums seem to grow further for \( R = [r^s] \) with \( s \geq 3 \) [14].

Specifics of symmetric representations is that for them the skew characters \( \chi^s_{\lambda/[\mu]} = \chi^s_{\lambda/\mu} \) do factorize, \( C^{[\lambda]}_{[\mu], [\nu]} = \delta_{\mu + \nu', \lambda} \) and also the sum over \( \lambda \) is a one-fold sum – what reduces the generic triple sum (31) over Young diagrams to the ordinary q-hypergeometric polynomial, though of a rather complicated Askey-Wilson type \( 4 \phi_3 \) with the fixed hypergeometric argument \( z = q^2 \).

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