ON EMBEDDINGS BETWEEN SPACES OF FUNCTIONS OF GENERALIZED BOUNDED VARIATION

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ABSTRACT. In this note, we aim to establish a number of embeddings between various function spaces that are frequently considered in the theory of Fourier series. More specifically, we give sufficient conditions for the embeddings \( \Phi V[h] \subseteq ABV(h\uparrow p) \), \( ABV[h_1] \subseteq TV[h_2] \), and \( ABV^{(p,\gamma)} \subseteq \Gamma BV^{(\gamma,\tau)} \). Our results are new even for the well-known spaces that have been studied in the literature. In particular, a number of results due to M. Avdispahić, that describe relationships between the classes \( \Lambda BV \) and \( V[h] \), are derived as special cases.

1. Introduction and preliminaries

The Jordan class \( BV \) of functions of bounded variation has been generalized by many authors in various ways (see [1]). In particular, Schembari and Schramm introduced the space \( \Phi V[h] \) in [7] to encompass previous generalizations:

Let \( \Phi = \{\phi_j\}_{j=1}^{\infty} \) be a sequence of increasing convex functions on \([0, \infty)\) such that \( \phi_j(0) = 0 \) for all \( j \), and \( 0 < \phi_{j+1}(x) \leq \phi_j(x) \) for all \( x > 0 \). If \( h \) is a nondecreasing sequence of positive reals, we say that \( \Phi \) is a Schramm sequence (with respect to \( h \)) provided that for each \( x > 0 \), \( \sum_{j=1}^{\infty} \phi_j(x)/h(n) = \infty \) as \( n \to \infty \). A real-valued function \( f \) on \([a, b] \) is said to be of bounded \( \Phi \)-variation if

\[
V_{\Phi,h}(f) = \sup_{1 \leq n < \infty} \frac{v(n, \Phi, f)}{h(n)} < \infty,
\]

where \( v(n, \Phi, f) = v(n, \Phi, f, [a, b]) \) is the \( \Phi \)-modulus of variation of \( f \), that is, the supremum of the sums \( \sum_{j=1}^{n} \phi_j(|f(I_j)|) \), taken over all finite collections \( \{I_j\}_{j=1}^{\infty} \) of nonoverlapping subintervals of \([a, b] \) and \( f(I_j) = f(\sup I_j) - f(\inf I_j) \). We denote by \( \Phi V[h] \) the linear space of all functions \( f \) on \([a, b] \) such that \( V_{\Phi,h}(cf) < \infty \) for some constant \( c > 0 \).

It is shown in [7] that \( \Phi V[h] \) is indeed a Banach space with respect to the norm

\[
\|f\|_{\Phi,h} := |f(a)| + \inf\{c > 0 : V_{\Phi,h}(f - f(a)/c) \leq 1\}.
\]

This space has many applications in Fourier analysis as well as in treating topics such as integration, convergence, summability, etc. (see e.g. [12, 8, 7]).

If \( \phi \) is a strictly increasing convex function on \([0, \infty)\) with \( \phi(0) = 0 \), and if \( \Lambda = \{\lambda_j\}_{j=1}^{\infty} \) is a Waterman sequence (i.e., \( \Lambda \) is a nondecreasing sequence of positive numbers such that \( \sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty \)), by taking \( \phi_j(x) = \phi(x)/\lambda_j \) for all \( j \), we get the class \( \phi\Lambda BV \) of functions of \( \phi\Lambda \)-bounded variation. This class was introduced by Schramm and Waterman in [9] (see also [?]). More specifically, if \( \phi(x) = x^p \) (\( p \geq 1 \)), we get the Waterman-Shiba class \( ABV(p) \), which was introduced by Shiba in [10]. When \( p = 1 \), we obtain the well-known Waterman class \( ABV \). Also, if \( h \) is a modulus of variation (i.e., a nondecreasing and concave sequence of positive reals) and \( \phi_j(x) = x \) for all \( j \), the Chanturiya class \( V[h] \) is obtained as a special subclass of \( \Phi V[h] \).

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In the literature, much attention has been devoted to the study of relationships between the above-mentioned classes; see [12], [6], [2], [5], [3] and the references therein for some results in this direction. In particular, a characterization of embeddings between $\Lambda BV$ classes was obtained by Perlman and Waterman [6]. Ge and Wang characterized the embeddings $\Lambda BV \subseteq \phi BV$ and $\phi BV \subseteq \Lambda BV$. Kita and Yoneda showed in [5] that the embedding $BV_p \subseteq BV^{(p, 1)}$ is both automatic and strict for all $1 \leq p < \infty$. Furthermore, Goginava characterized the embedding $\Lambda BV \subseteq BV^{(q_n, \infty)}$, and a characterization of the embedding $ABV^{(p) \subseteq BV^{(q_n, \infty)}}$ was given by Hormozi, Prus-Wiśniowski and Rosengren in [7]. More recently, the embeddings $ABV^{(p) \subseteq GV^{(q_n, \infty)}}$ and $ABV \subseteq BV^{(q_n, \infty)}$ $(1 \leq q < \infty)$ were investigated by Goodarzi, Hormozi and Memić (see [3]).

2. Results

Our first main result presents a sufficient condition for the embedding $\Phi V[h] \subseteq \Lambda ABV^{(p, \infty)}$ (see Theorem 2.2 below). Before that, we need a lemma.

If $\Phi = \{\phi_j\}_{j=1}^\infty$ is a Schramm sequence, we define $\Phi_k(x) := \sum_{j=1}^k \phi_j(x)$ for $x \geq 0$. Then $\Phi_k(x)$ is clearly an increasing convex function on $[0, \infty)$ such that $\Phi_k(0) = 0$ and $\Phi_k(x) > 0$ for $x > 0$. Without loss of generality we assume that $\Phi_k(x)$ is strictly increasing on $[0, \infty)$. We denote by $\Phi_k^{-1}(x)$ the inverse function of $\Phi_k(x)$. If $\lambda = \{\lambda_j\}$ and $\Gamma = \{\gamma_j\}$ are Waterman sequences, for each $n$ we define $\Lambda(n) := \sum_{j=1}^n \frac{1}{\lambda_j}$ and $\Gamma(n) := \sum_{j=1}^n \frac{1}{\gamma_j}$.

**Lemma 2.1.** Let $1 < q < \infty$ and $k \in \mathbb{N}$. If $f \in \Phi V[h]$ and $x_1, x_2, ..., x_k$ are nonnegative real numbers such that

$$\sum_{j=1}^k \phi_j(x_{\tau(j)}) \leq v(k, \Phi, f)$$

for any permutation $\tau$ of $k$ letters, then

$$\left( \sum_{j=1}^k x_j^q \right)^{\frac{1}{q}} \leq 16 (1 + V_{\Phi, h}(f)) \max_{1 \leq m \leq k} m^{\frac{1}{q}} \Phi_m^{-1}(h(k)).$$

**Proof.** Note first that following the arguments in the proof of [13, Theorem 2.1] one can verify that

$$\left( \sum_{j=1}^k x_j^q \right)^{\frac{1}{q}} \leq 16 \max_{1 \leq m \leq k} m^{\frac{1}{q}} \Phi_m^{-1}(v(k, \Phi, f)).$$

On the other hand, since the $\Phi_m^{-1}$ are strictly increasing concave functions with $\Phi_m^{-1}(0) = 0$, we get

$$\Phi_m^{-1}(at) \leq (1 + a)\Phi_m^{-1}(t), \quad \text{for any} \quad a, t > 0.$$ 

Now, applying the latter inequality with $a := V_{\Phi, h}(f)$ and $t := h(k)$ yields (2.1), as desired. \qed

**Theorem 2.2.** The embedding $\Phi V[h] \subseteq \Lambda ABV^{(p)}$ holds whenever

$$\sum_{k=1}^{\infty} \Delta \left( \frac{1}{\lambda_k} \right) \max_{1 \leq m \leq k} m \left( \Phi_m^{-1}(h(k)) \right)^p < \infty,$$

where $\Delta(a_k) = a_k - a_{k+1}$.
**Proof.** Let \( f \in \Phi V[h] \), so there exists some \( c > 0 \) such that \( V_{\Phi,h}(cf) < \infty \). Without loss of generality we may assume that \( c = 1 \). Let \( \{I_j\}_{j=1}^s \) be a nonoverlapping collection of subintervals of \([0,1]\). When \( q \geq 1 \) we may use Lemma (2.1) with \( x_j = |f(I_j)| \) to get

\[
(2.2) \quad \left( \sum_{j=1}^s |f(I_j)|^q \right)^{\frac{1}{q}} \leq 16 \left(1 + V_{\Phi,h}(f)\right) \max_{1 \leq m \leq s} \frac{1}{m} \Phi^{-1}_m(h(s)).
\]

In order to prove that \( V_A(f) < \infty \), we need to estimate the sum \( \sum_{k=1}^s \frac{|f(I_k)|^p}{\lambda_k} \). Taking \( x_k := \frac{1}{\lambda_k} \) and \( y_k := |f(I_k)|^p \) in Abel’s partial summation formula

\[
\sum_{k=1}^s x_k y_k = \sum_{k=1}^{s-1} \Delta(x_k) \sum_{j=1}^k y_j + s \sum_{j=1}^s y_j,
\]

one can write

\[
\sum_{k=1}^s \frac{|f(I_k)|^p}{\lambda_k} = \sum_{k=1}^{s-1} \Delta \left( \frac{1}{\lambda_k} \right) \sum_{j=1}^k |f(I_j)|^p + \frac{1}{\lambda_s} \sum_{j=1}^s |f(I_j)|^p.
\]

Then, applying (2.2) with \( q = p \) to estimate the right-hand side of the preceding equality, it follows that

\[
\sum_{k=1}^s \frac{|f(I_k)|^p}{\lambda_k} \leq \sum_{k=1}^{s-1} \Delta \left( \frac{1}{\lambda_k} \right) C^p \max_{1 \leq m \leq k} m \left( \Phi^{-1}_m(h(k)) \right)^p + \frac{1}{\lambda_s} C^p \max_{1 \leq m \leq s} m \left( \Phi^{-1}_m(h(s)) \right)^p
\]

\[
\leq \sum_{k=1}^{s-1} \Delta \left( \frac{1}{\lambda_k} \right) C^p \max_{1 \leq m \leq k} m \left( \Phi^{-1}_m(h(k)) \right)^p + \sum_{k=s}^\infty \Delta \left( \frac{1}{\lambda_k} \right) C^p \max_{1 \leq m \leq k} m \left( \Phi^{-1}_m(h(k)) \right)^p
\]

\[
\leq C^p \sum_{k=1}^\infty \Delta \left( \frac{1}{\lambda_k} \right) \max_{1 \leq m \leq k} m \left( \Phi^{-1}_m(h(k)) \right)^p < \infty,
\]

where \( C = 16 \left(1 + V_{\Phi,h}(f)\right) \) and the penultimate inequality is due to the fact that

\[
\frac{1}{\lambda_s} \max_{1 \leq m \leq s} m \left( \Phi^{-1}_m(h(s)) \right)^p \leq \sum_{k=s}^\infty \Delta \left( \frac{1}{\lambda_k} \right) \max_{1 \leq m \leq k} m \left( \Phi^{-1}_m(h(k)) \right)^p.
\]

This means that \( f \in \Lambda BV^{(p)} \), as desired. \( \square \)

**Corollary 2.3.** The embedding \( \Phi BV \subseteq ABV \) holds whenever

\[
\sum_{n=1}^\infty \Delta \left( \frac{1}{\lambda_n} \right) n \Phi^{-1}_n(1) < \infty.
\]

In particular, the embedding \( \phi ABV \subseteq \Gamma BV \) holds whenever

\[
\sum_{n=1}^\infty \Delta \left( \frac{1}{\gamma_n} \right) n \phi^{-1}(\Lambda(n)^{-1}) < \infty.
\]

**Corollary 2.4.** ([2, Theorem 2]) The embedding \( V[h] \subseteq \Lambda BV \) holds whenever

\[
\sum_{n=1}^\infty \Delta \left( \frac{1}{\lambda_n} \right) h(n) < \infty.
\]
Recently the second author et al. [3] obtained the following inequality and used it to characterize the embedding $\Lambda BV^{(p)} \subseteq \Gamma BV^{(q, q)}$:

$$\left( \sum_{j=1}^{n} x_{j}^{q} z_{j} \right)^{\frac{1}{q}} \leq \max_{1 \leq k \leq n} \left( \sum_{j=1}^{k} z_{j} \right)^{\frac{1}{q}} \left( \sum_{j=1}^{k} y_{j} \right)^{-1},$$

where $1 \leq q < \infty$, and $\{x_{j}\}$, $\{y_{j}\}$ and $\{z_{j}\}$ are positive nonincreasing sequences. In the sequel, we will further exploit (2.3) to prove the forthcoming results.

**Theorem 2.5.** Let $1 \leq p \leq q < \infty$. Let either $\left\{ \Gamma(n)^{\frac{2}{q}}/\Lambda(n)^{\frac{1}{p}} \right\}$ or $\left\{ h_{2}(n)^{\frac{2}{q}}/h_{1}(n)^{\frac{1}{p}} \right\}$ be nondecreasing. Then the embedding $\Lambda BV[h_{1}]^{(p)} \subseteq \Gamma BV[h_{2}]^{(q)}$ holds whenever

$$\sup_{1 \leq n < \infty} \left( \frac{\Gamma(n)}{h_{2}(n)} \right)^{\frac{2}{q}} \left( \frac{h_{1}(n)}{\Lambda(n)} \right)^{\frac{1}{p}} < \infty.$$

**Proof.** Let $f \in \Lambda BV[h_{1}]^{(p)}$ and consider a fixed $n$. Let $\{I_{j}\}_{j=1}^{n}$ be a nonoverlapping collection of subintervals of $[0, 1]$. Set $x_{j} := |f(I_{j})|^{p}$, $y_{j} := 1/\lambda_{j}$ and $z_{j} := 1/\gamma_{j}$. In view of the equimonotonic sequences inequality [4, Theorem 368] we can, and do, assume that the $x_{j}$ are arranged in descending order. Now, applying (2.3) with $q/p \geq 1$ in place of $q$ we obtain

$$\left( \sum_{j=1}^{n} \frac{|f(I_{j})|^{q}}{\gamma_{j}} \right)^{\frac{1}{q}} \leq \sum_{j=1}^{n} \frac{|f(I_{j})|^{p}}{\lambda_{j}} \max_{1 \leq k \leq n} \frac{\Gamma(k)^{\frac{2}{q}}}{\Lambda(k)^{\frac{1}{p}}}.$$

Therefore, we get

$$\left( \sum_{j=1}^{n} \frac{|f(I_{j})|^{q}}{\gamma_{j}} \right)^{\frac{1}{q}} \leq \left( \sum_{j=1}^{n} \frac{|f(I_{j})|^{p}}{\lambda_{j}} \right)^{\frac{1}{p}} \max_{1 \leq k \leq n} \frac{\Gamma(k)^{\frac{2}{q}}}{\Lambda(k)^{\frac{1}{p}}}$$

$$\leq \left( v(n; \Lambda, p, f) \right)^{\frac{1}{p}} \max_{1 \leq k \leq n} \frac{\Gamma(k)^{\frac{2}{q}}}{\Lambda(k)^{\frac{1}{p}}}$$

$$\leq C h_{1}(n)^{\frac{1}{p}} \frac{h_{2}(n)^{\frac{1}{q}}}{h_{1}(n)^{\frac{1}{p}}} = C h_{2}(n)^{\frac{1}{q}}.$$

for some positive constant $C$, depending solely on $f$. As a result, taking supremum over all collections $\{I_{j}\}_{j=1}^{n}$ as above, it follows that

$$v(n; \Gamma, q, f) \leq C^{q} h_{2}(n),$$

which means that $f \in \Gamma BV[h_{2}]^{(q)}$. \hfill \Box

An important consequence of the preceding theorem is the following result which provides a sufficient condition for the embedding $\Lambda BV \subseteq V[h]$ (see Remark (2.7)).

**Corollary 2.6.** The embedding $\Lambda BV \subseteq V[h]$ holds whenever

$$\sup_{1 \leq n < \infty} \frac{n}{\Lambda(n) h(n)} < \infty.$$

**Proof.** Note that $\{n \Lambda(n)^{\frac{1}{p}}\}$ is nondecreasing and apply Theorem (2.5) with $p = q = 1$, $h_{2} = h$, $h_{1}(n) = 1$ for all $n$, and $\gamma_{j} = 1$ for all $j$. \hfill \Box

**Remark 2.7.** It is worth noting that the existence of a condition that characterizes when $\Lambda BV$ can be embedded into $V[h]$ seems to have been unknown for a long time. We conjecture that (2.5) is a necessary condition as well.
As an application of Corollary (2.6), we deduce the following result by taking $h(n) = \frac{n}{\lambda(n)}$.

**Corollary 2.8.** ([2, Theorem 1]) The following embedding holds:

$$\Lambda BV \subseteq V[n\Lambda(n)^{-1}].$$

**Corollary 2.9.** Let $1 \leq p \leq q < \infty$. Then the embedding $\Lambda BV^{(p)} \subseteq \Gamma BV^{(q)}$ holds whenever

$$\sup_{1 \leq n < \infty} \frac{\sum_{j=1}^{s} |f(I_j)|^p}{\Lambda(n)^{p}} < \infty.$$

**Corollary 2.10.** The embedding $V[h_1] \subseteq V[h_2]$ holds whenever

$$\sup_{1 \leq n < \infty} \frac{h_1(n)}{h_2(n)} < \infty.$$

Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $1 \leq p_n \uparrow p \leq \infty$. A real-valued function $f$ on $[a, b]$ is said to be of $p_n$-a-bounded variation if

$$V_{\Lambda}(f) = V_{\Lambda}(f; p_n \uparrow p) := \sup_{n \geq 1} \sup_{I \subseteq [a, b]} \left( \sum_{j=1}^{s} \frac{|f(I_j)|^{p_n}}{\lambda_j} \right)^{\frac{1}{p_n}} < \infty,$$

where the $\{I_j\}_{j=1}^{s}$ are collections of nonoverlapping subintervals of $[a, b]$ such that $\inf_{j} |I_j| \geq \frac{b-a}{2n}$. The class of functions of $p_n$-a-bounded variation is denoted by $\Lambda BV^{(p_n \uparrow p)}$. This class was introduced by Vyas in [11]. When $\lambda_j = 1$ for all $j$, we obtain the class $BV^{(p_n \uparrow p)}$—introduced by Kita and Yoneda [5]—which is a generalization of the well-known Wiener class $BV_p$.

The mutual relationship between the generalized Wiener classes $\Lambda BV^{(p_n \uparrow p)}$ is rather chaotic even in the special case where $p_n = q_n = 1$ for all $n$; see [?] for a nice and detailed discussion on this. Besides, in order to determine when $BV^{(p_n \uparrow p)} \subseteq BV^{(q_n \uparrow q)}$ and $\Lambda BV^{(p)} \subseteq \Gamma BV^{(q_n \uparrow q)}$ ([5, Theorem 3.1] and [3, Theorem 1.4]), fairly significant restrictions have been imposed. So, it would be highly desirable to find a condition that implies the embedding $\Lambda BV^{(p_n \uparrow p)} \subseteq \Gamma BV^{(q_n \uparrow q)}$ without any additional restrictions on the $p_n$, $q_n$, $\Lambda$ and $\Gamma$. Theorem (2.12) provides such a condition.

Next lemma supplements (2.3) and is used in the proof of Theorem (2.12).

**Lemma 2.11.** If $0 < q < 1$, then (2.3) holds whenever the sequence $\left\{ \frac{\sum_{i=1}^{k} z_i}{\sum_{i=1}^{k} y_i} \right\}_{k}$ is nondecreasing.

**Proof.** First, we apply (2.3) with $q = 1$ to obtain

$$\sum_{j=1}^{n} x_j z_j \leq \sum_{j=1}^{n} x_j y_j \max_{1 \leq k \leq n} \left( \sum_{i=1}^{k} z_i \right) \left( \sum_{i=1}^{k} y_i \right)^{-1}. \quad (2.6)$$

Then an application of the Hölder inequality yields

$$\sum_{j=1}^{n} x_j^q z_j = \sum_{j=1}^{n} (x_j z_j)^{q} z_j^{1-q} \leq \left( \sum_{j=1}^{n} x_j z_j \right)^{q} \left( \sum_{j=1}^{n} z_j \right)^{1-q} \leq \left( \sum_{j=1}^{n} x_j y_j \right)^{q} \left( \sum_{j=1}^{n} z_j \right)^{1-q} \max_{1 \leq k \leq n} \left( \sum_{i=1}^{k} z_i \right)^{q} \left( \sum_{i=1}^{k} y_i \right)^{-q}.$$
where the last two inequalities are due, respectively, to (2.6) and the fact that \( \left\{ \sum_{i=1}^{k} z_i / \sum_{i=1}^{k} y_i \right\}_k \) is nondecreasing.

**Theorem 2.12.** The embedding \( ABV^{(p_n, q_n)} \subseteq \Gamma BV^{(q_n, q)} \) holds whenever

\[
\sup_{1 \leq n < \infty} \sum_{k=1}^{\infty} \Delta \left( \frac{1}{\gamma_k} \right) \max_{1 \leq m \leq k} m \Lambda(m)^{-\frac{q_n}{p_n}} < \infty.
\]

**Proof.** Assume that \( f \in ABV^{(p_n, q_n)} \). For an arbitrary but fixed \( n \), let \( \{I_j\}_{j=1}^{s} \) be a nonoverlapping collection of subintervals of \([0, 1]\) with \( \inf |I_j| \geq \frac{1}{m} \), and put \( q = q_n/p_n \), \( x_j = |f(I_j)|^{p_n} \), \( y_j = 1/\lambda_j \), \( z_j = 1/\gamma_j \). Without loss of generality, we may also assume that the \( x_j \) are arranged in descending order. Now, by Abel's transformation and applying (2.3) we obtain

\[
\sum_{k=1}^{s} \left| f(I_k) \right|^{q_n} \gamma_k^{-1} = \sum_{k=1}^{s-1} \Delta \left( \frac{1}{\gamma_k} \right) \sum_{j=1}^{k} \left| f(I_j) \right|^{p_n} \gamma_k + \frac{1}{\gamma_s} \sum_{j=1}^{s} \left| f(I_j) \right|^{q_n} \gamma_s
\]

\[
\leq \sum_{k=1}^{s-1} \Delta \left( \frac{1}{\gamma_k} \right) \sum_{j=1}^{k} \left| f(I_j) \right|^{p_n} \lambda_j \gamma_k \gamma_s \max_{1 \leq m \leq k} m \Lambda(m)^{-\frac{q_n}{p_n}} + \frac{1}{\gamma_s} \sum_{j=1}^{s} \left| f(I_j) \right|^{p_n} \lambda_j \gamma_s \max_{1 \leq m \leq s} m \Lambda(m)^{-\frac{q_n}{p_n}}
\]

\[
\leq \sum_{k=1}^{s-1} \Delta \left( \frac{1}{\gamma_k} \right) V_{\Lambda}(f)^{q_n} \max_{1 \leq m \leq k} m \Lambda(m)^{-\frac{q_n}{p_n}} + \frac{1}{\gamma_s} V_{\Lambda}(f)^{q_n} \max_{1 \leq m \leq s} m \Lambda(m)^{-\frac{q_n}{p_n}}
\]

\[
\leq \sum_{k=1}^{s-1} \Delta \left( \frac{1}{\gamma_k} \right) V_{\Lambda}(f)^{q_n} \max_{1 \leq m \leq k} m \Lambda(m)^{-\frac{q_n}{p_n}} + \sum_{k=s}^{\infty} \Delta \left( \frac{1}{\gamma_k} \right) V_{\Lambda}(f)^{q_n} \max_{1 \leq m \leq k} m \Lambda(m)^{-\frac{q_n}{p_n}}
\]

\[
= V_{\Lambda}(f)^{q_n} \sum_{k=1}^{\infty} \Delta \left( \frac{1}{\gamma_k} \right) \max_{1 \leq m \leq k} m \Lambda(m)^{-\frac{q_n}{p_n}} < \infty,
\]

where we have used the fact that

\[
\frac{1}{\gamma_s} \max_{1 \leq m \leq s} m \Lambda(m)^{-\frac{q_n}{p_n}} \leq \sum_{k=s}^{\infty} \Delta \left( \frac{1}{\gamma_k} \right) \max_{1 \leq m \leq k} m \Lambda(m)^{-\frac{q_n}{p_n}}.
\]

Taking suprema over all collections \( \{I_j\}_{j=1}^{s} \) as above, and over all \( n \) yields \( V_{\Gamma}(f) < \infty \). That is, \( f \in \Gamma BV^{(q_n, q)} \). \( \square \)

**Corollary 2.13.** The embedding \( ABV^{(p)} \subseteq \Gamma BV^{(q)} \) holds whenever

\[
\sum_{n=1}^{\infty} \Delta \left( \frac{1}{\gamma_n} \right) \max_{1 \leq k \leq n} \frac{k}{\Lambda(k)^{p}} < \infty.
\]

In particular, the embedding \( ABV \subseteq \Gamma BV \) holds whenever

\[
\sum_{n=1}^{\infty} \Delta \left( \frac{1}{\gamma_n} \right) \frac{n}{\Lambda(n)} < \infty.
\]
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