GEOMETRIC PRESENTATION FOR THE COHOMOLOGY RING OF POLYGON SPACES

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Abstract. We describe the cohomology ring of the moduli space of a flexible polygon in geometrically meaningful terms. We propose two presentations, both are computation friendly: there are simple rules for cup product.

1. Introduction

With an \( n \)-tuple of positive numbers \( L = (l_1, ..., l_n) \) we associate a flexible polygon, that is, \( n \) rigid bars of lengths \( l_i \) connected in a cyclic chain by revolving joints. A configuration of \( L \) is an \( n \)-tuple of points \( (q_1, ..., q_n) \in (\mathbb{R}^3)^n \) with \( |q_i q_{i+1}| = l_i \) and \( |q_n q_1| = l_n \).

Definition 1.1. The moduli space \( \mathcal{M}_3(L) \) is the set of all configurations of \( L \) lying in \( \mathbb{R}^3 \) modulo orientation preserving isometries of \( \mathbb{R}^3 \).

Definition 1.2. Equivalently, one defines

\[
\mathcal{M}_3(L) = \left\{ (u_1, ..., u_n) \in (S^2)^n : \sum_{i=1}^{n} l_i u_i = 0 \right\} / SO(3).
\]

The second definition shows that \( \mathcal{M}_3(L) \) does not depend on the ordering of \( \{l_1, ..., l_n\} \); however, it depends on the values of \( l_i \).

The space \( \mathcal{M}_3(L) \) is a \( 2n - 6 \)-dimensional complex-analytic manifold\(^1\). The cohomology ring \( H^*(\mathcal{M}_3(L), \mathbb{Z}) \) is explicitly described in [6]. Some of its generators and relations have a clear geometrical meaning, while the other do not. In [2] Kamiyama writes: "Although (R1) and (R2) are manageable, (R3) is complicated and mysterious. It is not easy to obtain information from (R3)."\(^2\)

In the cited paper he gives a reasonable answer by providing a combinatorial comprehension of the relation modulo \( \mathbb{Z}_2 \).

In turn, the present paper interprets generators and relations in entirely geometric terms. The first presentation (that is, the ring \( \tilde{\mathcal{N}} \)) is described in Section 3. Oversimplifying, its elements are \( \mathbb{Z} \)-linear combinations of (Poincaré

\(^1\)Moreover, A. Klyachko [3] showed that it is an algebraic variety.

\(^2\)In notation of the present paper, (R1), (R2), and (R3) are relations (2), (3), and (4) from Theorem 2.1 respectively.
duals of) nice submanifolds of $\mathcal{M}_3(L)$. The latter are characterized by codirected and oppositely directed pairs of edges, see Section 2.

The first and the second series of relations in the ring $\hat{\mathcal{N}}$ have a transparent geometrical meaning.

The third series of relations comes from computation of Chern classes of some natural bundles over the moduli space. The computation has in the background the (scholarly exercise!) way of finding the Chern class of tangent bundle $T\mathbb{S}^2$ to the 2-sphere $\mathbb{S}^2$, see comments to Proposition 2.6 and also [5] for more details.

These bundles are quite remarkable: on the one hand, they naturally appear in [6]. On the other hand, they are a counterpart of M. Kontsevich’s tautological bundles over the space $\mathcal{M}_{0,n}$ [4].

In the second presentation (the ring $\mathcal{P}$, see Section 4) the generators are (Poincaré duals of) perfect submanifolds of $\mathcal{M}_3(L)$. The latter are characterized by codirected edges only.

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2. Preliminaries

Moduli space $\mathcal{M}_3(L)$. Assume that an $n$-tuple of positive real numbers $L = (l_1, \ldots, l_n)$ is fixed. The moduli space $\mathcal{M}_3(L)$ is not empty if and only if $L$ satisfies the inequalities:

$$l_i < \frac{l_1 + \cdots + l_n}{2} \text{ for all } i \in [n] = \{1, 2, \ldots, n\}.$$

A subset $I \subset [n]$ is called long if

$$\sum_{i \in I} l_i > \frac{1}{2} \sum_{i=1}^{n} l_i.$$

Otherwise, $I$ is called short.

Throughout the paper we assume that no configuration fits in a straight line. This happens iff the moduli space $\mathcal{M}_3(L)$ is a smooth closed manifold. In more details, let us take all subsets $I \subset \{1, \ldots, n\}$. The associated hyperplanes

$$\sum_{i \in I} l_i = \sum_{i \notin I} l_i$$

called walls subdivide the parameter space $\mathbb{R}_+^n$ into a number of chambers. The diffeomorphic type of $\mathcal{M}_3(L)$ depends only on the chamber containing $L$; this becomes clear in view of the (coming below) stable configurations representations. For $L$ lying strictly inside a chamber, the space $\mathcal{M}_3(L)$ is a smooth manifold.

Let us make an additional assumption: the sum $\sum_{I \neq \emptyset} l_i$ never vanishes for all non-empty $I \subset \{1, \ldots, n\}$. This agreement does not restrict generality: one may perturb the edge lengths while staying in the same chamber.
We make use of yet another representation of \( M_3(L) \). Consider configurations of \( n \) (not necessarily all distinct) points \( p_i \) in the complex projective line. Each point \( p_i \) is assigned the weight \( l_i \). The configuration of (weighted) points is called \textit{stable} if sum of the weights of coinciding points is less than half the weight of all points. Denote by \( S_C(L) \) the space of stable configurations in the complex projective line. The group \( PSL(2, \mathbb{C}) \) acts naturally on this space. In this setting we have:

\[
M_3(L) = S_C(L)/PSL(2, \mathbb{C}).
\]

That is, for each \( n \) we have a finite series \( M_3(L) \) of compactifications of the space \( M_{0,n}(\mathbb{C}) \) depending on the particular choice of the lengths \( L \).

\textbf{Theorem 2.1 (\cite{6}).} The (co)homology groups of \( M_3(L) \) are free abelian. The cohomology ring \( H^*(M_3(L), \mathbb{Z}) \) is isomorphic to the quotient of the ring \( \mathbb{Z}[R, V_1, \ldots, V_{n-1}, U_1, \ldots, U_{n-1}] \) by the ideal \( \mathcal{I} \) generated by the four families of elements:

\begin{enumerate}
  \item \( U_i - V_i - R \) for \( i \in [n-1] \);
  \item \( U_i V_i \) for \( i \in [n-1] \);
  \item \( \prod_{i \in I} V_i \) for \( I \subset [n-1] \) such that \( I \cup \{n\} \) is long;
  \item \[
  \sum_{S \subseteq H} \left( \prod_{i \in S} V_i \right) \cdot R^{|H-S|-1},
  \]
  such that \( H \subset [n-1] \) is a long set, and \( S \subset H \) ranges over all the sets such that \( S \cup \{n\} \) is short.
\end{enumerate}

\textbf{Nice manifolds \cite{5}.} Nice manifolds are submanifolds of \( M_3(L) \). We define them as point sets, equip them with orientation, and then focus on their Poincaré dual cocycles.

\textit{Nice manifolds as point sets.} We start with an example: let \( i \neq j \in [n] \). Denote by \((ij)\) the image of the natural embedding of the space \( M_3(l_i + l_j, l_1, \ldots, \hat{l}_i, \ldots, \hat{l}_j, \ldots, l_n) \) into the space \( M_3(L) \).

That is, we think of the configurations of the new \( n-1 \)-gon as the configurations of \( L \) with parallel \textbf{codirected} edges \( i \) and \( j \) \textit{frozen} together to a single edge of the length \( l_i + l_j \). Since the moduli space does not depend on the ordering of the edges, it is convenient to think that \( i \) and \( j \) are consecutive indices.

Analogously, we can define the space \((i \bar{j})\) of configurations with edges \( i \) and \( j \) \textbf{parallel}, but \textbf{oppositely directed}. In other words, the space \((i \bar{j})\) is an image of the space \( M_3(|l_i - l_j|, l_1, \ldots, \hat{l}_i, \ldots, \hat{l}_j, \ldots, l_n) \) under natural embedding in the space \( M_3(L) \).

Generally, we can freeze several collections of edges either oppositely directed or codirected, and analogously define a nice submanifold labeled by the formal
product. All submanifolds arising this way are called *nice submanifolds of* \( \mathcal{M}_3(L) \), or just *nice manifolds* for short.

Putting the above more formally, each nice manifold is labeled by an unordered formal product

\[(I_1J_1) \cdot \ldots \cdot (I_kJ_k),\]

where \( I_1, \ldots, I_k, J_1, \ldots, J_k \) are some pairwise disjoint subsets of \([n]\) such that each set \( I_i \cup J_i \) has at least one element. The manifold \((I_1J_1) \cdot \ldots \cdot (I_kJ_k)\) is a subset of \( \mathcal{M}_3(L) \) defined by the conditions:

1. \( i, j \in I_k \) implies \( u_i = u_j \),
2. \( i, j \in J_k \) implies \( u_i = u_j \), and
3. \( i \in I_k, j \in J_k \) implies \( u_i = -u_j \).

**Orientation on nice manifolds.** By construction, each nice manifold is the moduli space \( \mathcal{M}_3(\tilde{L}) \) for some length vector \( \tilde{L} \). Therefore a nice manifold is a complex-analytic manifold and has a *canonical* orientation coming from the complex structure.

By definition, the *relative orientation* of a nice manifold \((IJ)\) coincides with its canonical orientation iff

\[\sum_i l_i > \sum_j l_i.\] (‡)

Further, the two orientations (canonical and relative ones) of a nice manifold

\[(I_1J_1) \cdot \ldots \cdot (I_kJ_k)\]

coincide iff the above inequality (‡) fails for even number of \((I_iJ_i)\).

From now on, by \((I_1J_1) \cdot \ldots \cdot (I_kJ_k) \in H^*(\mathcal{M}_3(L), \mathbb{Z})\) we mean (the Poincaré dual of) the nice manifold taken with its relative orientation. Some of the nice manifolds might be empty and thus represent the zero cocycles.

**Cup product rules for nice manifolds** [5]. The following rules are valid for nice submanifolds of \( \mathcal{M}_3(L) \).

1. The cup product is a commutative operation.
2. \((IJ) = -(JI)\).
3. If the factors have no common entries, the cup-product equals the formal product, e.g.:

\[(12) \cdot (34) \cdot (5\overline{6}) = (12) \cdot (34) \cdot (5\overline{6}).\]

4. If \( I_1 \cap I_2 = \{i\}, \ I_1 \cap J_2 = \emptyset, \ I_2 \cap J_1 = \emptyset, \) and \( J_1 \cap J_2 = \emptyset \), then

\[(I_1J_1) \cdot (I_2J_2) = (I_1 \cup I_2 J_1 \cup J_2).\]

Examples:

\[(123) \cdot (345) = (12345),\]
\[(123) \cdot (345) = (12345).\]

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\(^3\)For further computations is convenient to define also nice manifolds with \( I_i \cup J_i \) consisting of one element. That is, we set \( (1) = \mathcal{M}_3(L) \), \( (1) \cdot (23) = (23) \) etc.
(5) If \( J_1 \cap J_2 = \{i\} \), \( I_1 \cap J_2 = \emptyset \), \( I_2 \cap J_1 = \emptyset \), and \( I_1 \cap I_2 = \emptyset \), then
\[
(I_1 \overline{J}_1) \sim (I_2 \overline{J}_2) = -(I_1 \cup I_2 \overline{J}_1 \cup \overline{J}_2).
\]

Example:
\[
(12 \overline{3}) \sim (45 \overline{3}) = -(1245 \overline{3}).
\]

Let us comment shortly on the rules: (1) comes from the complex structure, (2) is valid just by definition, and (3)–(5) reflect the transversal intersection of nice manifolds.

**Natural bundles over** \( \mathcal{M}_3(L) \). Consider the following natural bundles over \( \mathcal{M}_3(L) \):

**Definition 2.2.** [6] For \( 1 \leq i \leq n \) let \( A_i = A_i(L) \) be the U(1)-bundle over \( \mathcal{M}_3(L) \) whose fiber over a point \( (u_1, \ldots, u_n) \) consists of all rotations of the configuration around the vector \( u_i \) (the direction of the rotation agrees with the right-hand rule).

**Definition 2.3.** [5] The line bundle \( E_i = E_i(L) \) is the complex line bundle over the space \( \mathcal{M}_3(L) \) whose fiber over a point \( (u_1, \ldots, u_n) \in (\mathbb{C}P^1)^n \) is the complex tangent line to the complex projective line \( \mathbb{C}P^1 \) at the point \( u_i \).

An easy observation is:

**Proposition 2.4.** The bundle \( A_i \) is a principal U(1)-bundle corresponding to the complex line bundle \( E_i \). \( \square \)

Denote by \( Ch(i) \) the first Chern class of \( A_i \) (or, equivalently, of \( E_i \)).

**Proposition 2.5.** [6] In notation of Theorem 2.1, one has
\[
Ch(i) = \begin{cases} 
2V_i + R & \text{for } i \neq n, \\
-R & \text{for } i = n. 
\end{cases} \quad \square
\]

On the other side, we proved the analogous result in terms of nice manifolds:

**Proposition 2.6.** [1, 5] For any \( j \neq i \in [n] \) we have
\[
Ch(i) = (ij) - (i \overline{j}). \quad \square
\]

Let us reveal the geometric meaning of this proposition. We are going to describe a section of \( E_i \) which is transversal to the zero section. Its zero locus (modulo orientation) is the desired Chern class \( Ch(i) \). Fix any \( j \neq i \), and consider the following (discontinuous) section of \( E_i \): at a point \( (u_1, \ldots, u_n) \in \mathcal{M}_3(L) \) we take the unit vector in the complex tangent line at \( u_i \) pointing in the direction of the shortest arc connecting \( u_i \) and \( u_j \). The section is well-defined except for the points of \( \mathcal{M}_3(L) \) with \( u_i = \pm u_j \), that is, except for nice manifolds \( (ij) \) and \( (i \overline{j}) \). One cooks a continuous section with zeroes at \( (ij) \) and \( (i \overline{j}) \). A detailed analysis (which we omit here; the reader is referred to [5] for details) shows that the signs are exactly as is stated in the proposition.
Proposition 2.7. For any $i \neq j \neq k \in [n]$ we have
\[ \text{Ch}(i) = (ij) + (ik) - (jk). \]

Proof. On the one hand, this relation can be derived from computation rules, see proof of Theorem 3.2. On the other hand, it is good to understand the geometrical meaning: Take a stable configuration $(x_1, ..., x_n) \in \mathcal{M}_3(L)$. Take the circle passing through $x_i, x_j,$ and $x_k$. It is oriented by the order $ijk$. Take the vector lying in the tangent complex line to $x_i$ which is tangent to the circle and points in the direction of $x_j$. It gives rise to a section of $E_i$ which is defined correctly whenever these three points are distinct. Therefore, $\text{Ch}(i) = A(ij) + B(ik) + C(jk)$ for some integer $A, B, C$. Detailed analysis specifies their values. □

Corollary 2.8. (The four-term relation) For any distinct $i, j, k, l$ we have
\[ (ij) + (kl) = (jk) + (il). \] □

The above rules do not exhaust all existing products of nice manifolds. The following technical proposition shows a way to compute cup product in two particular cases that are not covered in the rules.

Proposition 2.9. For any $i, j, k$, we have
\[ (ij) \mapsto (ij) = (ij) \mapsto \text{Ch}(i) = (ijk) - (ij\overline{k}), \]
\[ (i\overline{j}) \mapsto (i\overline{j}) = (i\overline{j}) \mapsto \text{Ch}(i) = -(i\overline{j}k) + (i\overline{j}k). \]

Proof of Proposition 2.9. Using $(ij) \mapsto (i\overline{j}) = 0$ and the rules (1) – (5), we get
\[ (ij) \mapsto (ij) = (ij) \mapsto [(ij) - (i\overline{j})] = (ij) \mapsto \text{Ch}(i) = (ij) \mapsto [(ik) - (i\overline{k})] = (ijk) - (ij\overline{k}), \]
and
\[ (i\overline{j}) \mapsto (i\overline{j}) = (i\overline{j}) \mapsto [(i\overline{j}) - (ij)] = -(i\overline{j}) \mapsto \text{Ch}(i) = -(i\overline{j}) \mapsto [(ik) - (i\overline{k})] = -(i\overline{k}k) + (i\overline{j}k). \] □

3. The ring of nice manifolds. The first presentation

Let us denote by $\mathcal{N}$ the $\mathbb{Z}$-linear closure of the set of all nice manifolds in the ring $H^*(\mathcal{M}_3(L), \mathbb{Z})$.

Theorem 3.1. (1) $\mathcal{N}$ coincides with the ring $H^*(\mathcal{M}_3(L), \mathbb{Z})$. In particular, $\mathcal{N}$ is closed under cup product.

(2) In notation of Theorem 2.1, we have:
\[ R = -(ni) + (n\overline{i}), \quad U_i = (ni), \quad V_i = (ni). \]

Proof. Prove first that the abelian group $\mathcal{N}$ is closed under cup product, so the set $\mathcal{N}$ is a subring of $H^*(\mathcal{M}_3(L), \mathbb{Z})$. It suffices to prove that each monomial in elementary nice manifolds (i.e. manifolds $(ij)$ or $(i\overline{j})$ for some $i, j$) is a nice manifold. So let us consider a monomial of the form
\[\pm \prod (i_p \bar{j}_p)^{a_p} \sim \prod (i_q \bar{j}_q)^{\beta_q}.\]

We may assume that \(\Sigma \alpha_p + \Sigma \beta_q \leq n - 3 = \dim \mathcal{M}_3(L)\), otherwise we get zero. Computation rules together with repeated applications of Proposition 2.9 do the job.

Here are two examples of such computation:

\[(123) \sim (124) = (12)^2 \sim (13) \sim (14) =
= [(125) - (12\bar{5})] \sim (13) \sim (14) =
= (12345) - (12\bar{3}\bar{4}\bar{5}).\]

\[(12)^4 \sim (56) = (12)^2 \sim [(123) - (12\bar{3})] \sim (56) =
= (12) \sim [(1234) - (12\bar{3}4) - (123\bar{4}) + (12\bar{3}\bar{4})] \sim (56) =
= (1234\bar{5}6) - (12\bar{3}4\bar{5}6) - (12\bar{3}\bar{4}\bar{5}6) - (123\bar{4}5\bar{6}) + (12\bar{3}45\bar{6}) - (123\bar{4}\bar{5}\bar{6}) + (12\bar{3}\bar{4}\bar{5}\bar{6}) - (123\bar{4}\bar{5}\bar{6}).\]

On the one hand, the ring \(\mathcal{N}\) is contained in the cohomology ring. On the other hand, the cohomology ring is generated by nice manifolds due to Theorem 2.1, Proposition 2.5, and Proposition 2.6.

Now we are ready to describe the first computation friendly presentation of the cohomology ring. Define the commutative graded ring \(\tilde{\mathcal{N}}\) with the ring operation 
"\(*\)" as follows:

\textbf{(Generators)} The generators of \(\tilde{\mathcal{N}}\) are the labels of all existing nice manifolds. That is, the generators bijectively correspond to all unordered formal products

\[(I_1 \bar{J}_1) \cdot \ldots \cdot (I_k \bar{J}_k),\]

where \(I_1, \ldots, I_k, J_1, \ldots, J_k\) are some pairwise disjoint subsets of \([n]\) such that each set \(I_i \cup J_i\) has at least one element.

\textbf{(Relations)} The relations in \(\tilde{\mathcal{N}}\) split in three sets:

1. Relations coming from "computation rules", see Section 2
   a. \((I \bar{J}) + (J \bar{I}) = 0\).
   b. If the factors have no common entries, the product equals the formal product, e.g.:
   \[(12) \ast (34) = (12) \cdot (34).\]
   c. If \(I_1 \cap I_2 = \{i\}, \ I_1 \cap J_2 = \emptyset, \ I_2 \cap J_1 = \emptyset, \text{ and } J_1 \cap J_2 = \emptyset, \) then
   \[(I_1 \bar{J}_1) \ast (I_2 \bar{J}_2) = (I_1 \cup I_2 \bar{J}_1 \cup \bar{J}_2).\]
   d. If \(J_1 \cap J_2 = \{i\}, \ I_1 \cap J_2 = \emptyset, \ I_2 \cap J_1 = \emptyset, \text{ and } I_1 \cap I_2 = \emptyset, \) then
   \[(I_1 \bar{J}_1) \ast (I_2 \bar{J}_2) = -(I_1 \cup I_2 \bar{J}_1 \cup \bar{J}_2).\]

\textbf{Remark.} Note that the relation (d) follows from (a) and (c).
(e) \((ij) \ast (i\overline{j}) = 0\). The geometrical meaning of this relation is transparent: the nice manifolds \((ij)\) and \((i\overline{j})\) are disjoint.

(2) A relation comes from length constraints:
\((IJ) = 0\) whenever one of the sets \(I\) or \(J\) is long.

(3) A relation coming from Chern class computation:
for any \(i,j,k \in [n]\) such that \(i \neq j, i \neq k\), we have
\[(ij) - (i\overline{j}) = (ik) - (i\overline{k}).\]

**Theorem 3.2.** The rings \(\tilde{\mathcal{N}}\) and \(H^*(\mathcal{M}_3(L), \mathbb{Z})\) are isomorphic.

**Proof.** Since all the defining relations of \(\tilde{\mathcal{N}}\) are valid in \(\mathcal{N} = H^*(\mathcal{M}_3(L), \mathbb{Z})\), we have an epimorphism
\[\tilde{\mathcal{N}} \xrightarrow{\phi} H^*(\mathcal{M}_3(L), \mathbb{Z}) = \mathcal{N},\]
where \(\phi\) maps each formal expression to the dual of the associated nice manifolds. To prove that \(\phi\) is bijective, it suffices to show that for any set \(H \subset [n - 1]\), we have
\[
\sum_{S \subseteq H} (Sn) \ast (-Ch(n))^{|H - S| - 1} = \sum_{S \subseteq H} (Sn) \ast (-Ch(n))^{|H - S| - 1} = (H),
\]
where in the first sum, as in Theorem 2.1, \(S \subsetneq H\) ranges over all the sets such that \(S \cup \{n\}\) is short, and by \(Ch(n) \in \tilde{\mathcal{N}}\) we mean \((nj) - (n\overline{j})\).

The first equality is trivial, so we prove the second equality.

The case \(|H| = 2\) gives a base of induction. Without loss of generality, assume that \(H = \{1, 2\}\). The left-hand side equals
\[
-Ch(n) + (1n) + (2n) = \frac{(1n) - (1n) + (2n) - (2n) + 2 \cdot (1n) + 2 \cdot (2n)}{2} =
\]
\[
\frac{Ch(1) + Ch(2)}{2} = \frac{(12) - (1\overline{2}) + (12) + (1\overline{2})}{2} = (12).
\]
(The division by 2 is eligible for degree one elements.)

In the general case, the proof goes by induction on \(|H|\). For \(H = \{1, 2, \ldots, h\}\), denote
\[
\Sigma(h) := \sum_{S \subseteq H} (Sn) \ast (-Ch(n))^{|H - S| - 1}.
\]
In this sum, we first group together the summands such with \(h \notin S\), and after those with \(h \in S\). Next, we apply the inductive assumption.
\[
\Sigma(h) = -Ch(n) \ast \Sigma(h - 1) + (12 \ldots h - 1 n) + (hn) \ast \Sigma(h - 1) =
\]
\[
((n\overline{h}) - (nh)) \ast (12 \ldots h - 1) + (nh) \ast (1 \ldots h - 1) = (12 \ldots h).
\]
Let us exemplify the induction step for \( |H| = 3 \):

\[
-Ch(n) \ast \left( \underbrace{-Ch(n) + (1n) + (2n)}_{= (12)} \right) + (12n) + (23n) + (13n) + (3n) \ast (-Ch(n)) =
\]

\[
= (12) + (23n) + (13n) + (3n) \ast (-Ch(n)) =
\]

\[
= (12) \ast (13) = (123). \quad \Box
\]

4. The ring of perfect manifolds. The second presentation

A nice manifold is perfect if its label has no overlines. That is, a perfect manifold is characterized by codirected collections of its edges.

**Proposition 4.1.** As a \( \mathbb{Z} \)-module, the ring \( H^*(\mathcal{M}_3(L), \mathbb{Z}) \) is generated by perfect manifolds.

**Proof.** The group \( H^2(\mathcal{M}_3(L), \mathbb{Z}) \) is generated by \((ij)\) and \((ij)^\ast\). Due to Propositions 2.6 and 2.7, each \((ij)^\ast\) is a linear combination of perfect manifolds. Indeed, \((ij)^\ast = (jk) - (ik)\) for any \( k \neq i, j \).

Now let us show that any product of perfect manifolds is a perfect manifold.

Let us call the perfect manifolds of type \((ij)\) elementary perfect manifolds. Clearly, each perfect manifold is a product of elementary ones.

Now, we prove that the product of two perfect manifolds is an integer linear combination of perfect manifolds. We may assume that the second factor is an elementary perfect manifold, say, \((12)\). Let the first factor be \((I_1) \cdot (I_2) \cdot (I_3) \cdot \ldots \cdot (I_k)\).

We need the following case analysis:

1. If at least one of 1, 2 does not belong to \( \bigcup I_i \), the product is a perfect manifold by the computation rule (4).
2. If 1 and 2 belong to different \( I_i \), we use the following:

   for any perfect manifold \((I_1) \cdot (I_2)\) with \( i \in I_1, j \in I_2 \), we have

   \[
   (I_1) \cdot (I_2) \sim (ij) = (I_1) \sim (ij) \sim (I_2) = (I_1j) \sim (I_2) = (I_1 \cup I_2).
   \]

3. Finally, assume that 1, 2 \( \in \bigcup I_i \). Choose \( i \notin I_1, j \notin I_1 \) such that \( i \) and \( j \) do not belong to one and the same \( I_k \). By Corollary 2.8

   \[
   (I_1) \cdot (I_2) \cdot (I_3) \cdot \ldots \cdot (I_k) \sim (12) = (I_1) \cdot (I_2) \cdot (I_3) \cdot \ldots \cdot (I_k) \sim ((1i) + (2j) - (ij)).
   \]

   After expanding the brackets, one reduces this to the above cases. \( \Box \)

We are ready to describe a presentation of the cohomology ring in terms of perfect manifolds. Define the commutative graded ring \( \mathcal{P} \) with the ring operation “*” as follows:
The ring $P$ is a free $\mathbb{Z}$–module generated by the labels of all existing perfect manifolds. That is, the generators bijectively correspond to all unordered formal products $$(I_1) \cdot \ldots \cdot (I_k),$$ where $I_1, \ldots, I_k$ are some non-empty pairwise disjoint subsets of $[n]$.

(Relations) The relations in $P$ are

1. Let $I$ and $J$ be disjoint subsets of $[n]$. Then
   
   (a) $(I \cdot (J) = (I) \cdot (J)$.
   (b) $(Ii) \cdot (Ji) = (IJi)$.

2. Length constraints: $(I) = 0$ whenever the set $I$ is long.

3. The four-term relations:
   
   $(ij) + (kl) = (jk) + (il)$ holds for any distinct $i, j, k, l$.
   
   This is equivalent to:
   
   The value of $(ij) + (ik) - (jk)$ does not depend on $j$ and $k$.

Remark. All these relations hold true in $\tilde{N}$.

Theorem 4.2. The rings $P$ and $H^*(\mathcal{M}_3(L), \mathbb{Z})$ are isomorphic.

Proof. We construct two ring homomorphisms $\phi : P \to \tilde{N}$ and $\psi : \tilde{N} \to P$

and prove that (a) $\phi$ is epimorphism, and (b) $\psi \circ \phi = id$.

By definition, $\phi$ sends each $(I_1) \cdot \ldots \cdot (I_k) \in P$ to an element of $\tilde{N}$ with the same label. The homomorphism $\phi$ is an epimorphism. The proof goes analogously to that of Proposition 4.1.

Now define $\psi$:

1. Set $\psi(\overline{ij}) := (jk) - (ik)$ for any $k \neq i$ and $k \neq j$. The target expression does not depend on $k$ due to the four-term relation.

2. Set $\psi(\overline{IJ}) := (I) \cdot (Jk) - (Ik) \cdot (J)$ for any $k \notin I$ and $k \notin J$. The target expression does not depend on $k$. Indeed, $(I) \cdot (Jk) - (Ik) \cdot (J) = (I) \cdot (J) \cdot [(jk) - (ik)]$ for $i \in I, j \in J$. The four-term relation
   
   $(jk) - (ik) = (jk') - (ik')$ completes the proof of correctness.

3. Set $\psi\left(\overline{(I_1) \cdot \ldots \cdot (I_m)J_m}\right)$ as the product $\psi(I_1J_1) \ast \ldots \ast \psi(I_mJ_m)$.

To prove the correctness, we need to show that $\psi$ sends all the relations in $\tilde{N}$ to zero.

1. $\psi\left(\overline{(IJ) + (JT)}\right) = 0$. This follows directly from the definition of the map $\psi$. 


(2) For \( I_1 \cap I_2 = \{i\}, \ I_1 \cap J_2 = \emptyset, \ I_2 \cap J_1 = \emptyset, \) and \( J_1 \cap J_2 = \emptyset, \) we need to prove that

\[
\psi(I_1 \overline{J}_1) \ast \psi(I_2 \overline{J}_2) = \psi(I_1 \cup I_2 \overline{J}_1 \cup \overline{J}_2).
\]

First, we prove that \( \psi(i \overline{j}) \ast \psi(i \overline{k}) = \psi(i \overline{j} \overline{k}) \) by a straightforward calculation:

\[
\psi(i \overline{j}) \ast \psi(i \overline{k}) = \left((ji) - (il)\right) \ast \left((jk) - (ij)\right) = (kj) \ast \left((ji) - (il)\right) = \psi(i \overline{j} \overline{k}).
\]

Then the general case follows from the calculation:

\[
\psi(I_1 \overline{J}_1) \ast \psi(I_2 \overline{J}_2) = (-1)^2 \cdot \psi(I_1) \ast \psi(i \overline{j}) \ast \psi(I_2) \ast \psi(i \overline{k}) \ast \psi(\overline{J}_2) = \psi(I_1 \cup I_2) \ast \psi(\overline{J}_1) \ast \psi(\overline{J}_2) \ast \psi(\overline{j} \overline{k}) = (-1)^3 \cdot \psi(I_1 \cup I_2) \ast \psi(J_1) \ast \psi(J_2) \ast \psi(\overline{j} \overline{k}) = (-1) \cdot \psi(I_1 \cup I_2) \ast \psi(J_1 \cup J_2) = (-1)^2 \cdot \psi(I_1 \cup I_2) \ast \psi(i \overline{J}_1 \cup \overline{J}_2) = \psi(I_1 \cup I_2 \overline{J}_1 \cup \overline{J}_2).
\]

(3) \( \psi(I \overline{J}) = 0 \) whenever one of the sets \( I \) or \( J \) is long. This follows from the definition.

(4) \( \psi\left((ij) \ast (i \overline{j})\right) = 0. \) Indeed, \( \psi\left((ij) \ast (i \overline{j})\right) = (ij) \ast \left((ju) - (iu)\right) = (iju) - (iju) = 0. \)

(5) For any \( i, j, k \in [n] \) such that \( i \neq j, i \neq k, \)

\[
\psi\left((ij) - (i \overline{j}) - (ik) + (i \overline{k})\right) = 0.
\]

Indeed, \( \psi\left((ij) - (i \overline{j}) - (ik) + (i \overline{k})\right) = (ij) - (jk) + (ik) - (ik) + (kj) - (ij) = 0. \)

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