Topological classification of adiabatic processes

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Certain band insulators allow for the adiabatic pumping of quantized charge or spin for special
time-dependences of the Hamiltonian. These "topological pumps" are closely related to two di-
dimensional topological insulating phases of matter upon rolling the insulator up to a cylinder and
threading it with a time dependent flux. In this article we extend the classification of topological
pumps to the Wigner Dyson and chiral classes, coupled to multi-channel leads. The topological
index distinguishing different topological classes is formulated in terms of the scattering matrix
of the system. We argue that similar to topologically non-trivial insulators, topological pumps are
characterized by the appearance of protected gapless end states during the course of a pumping
cycle. We show that this property allows for the pumping of quantized charge or spin in the weak
coupling limit. Our results may also be applied to two dimensional topological insulators, where
they give a physically transparent interpretation of the topologically non-trivial phases in terms of
scattering matrices.

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Topological insulating states of matter differ from reg-
ular band insulators by the fact that they support pro-
tected gapless surface states. Holding promise for nu-
merous applications, this observation has considerably mo-
tivated the search for materials that realize such topo-
logical phases. The first experimental observation of a
topologically nontrivial insulator dates back to the dis-
covery of the quantum Hall effect 30 years ago [1, 2]. The
recent discovery of the quantum spin Hall effect [3-5], has
lead to a full classification of insulators with topological
order based on their underlying symmetries and spatial
dimensions [6].

The study of the quantum Hall effect has instigated nu-
merous theoretical works dedicated to the understanding
of its non-trivial topological nature. One particular ap-
ppealing argument was introduced by Laughlin [7], who
considered a pump formed by placing the two di-
sensional system on a cylinder and threading it with a time
dependent magnetic flux. As the flux is varied period-
ically in time, an integer number of charges are trans-
ferred across the pump upon completing one cycle. This
charge quantization is directly related to the quantized
Hall conductance of the underlying Hall insulator [8-11].

In this communication we extend Laughlin’s consider-
ations to pumps formed by two dimensional insulators
belonging to the Wigner Dyson and the chiral classes.
Based on their underlying symmetries, applying the
above construction imposes a symmetry constraint on the
pumping cycle. This allows for the classification of topo-
logical pumps in terms of invariants of their scattering
matrix and gives rise to a physically transparent interpre-
tation of the topologically non-trivial phases in terms of
quantized pumping properties. Similarly to topologically
non-trivial two-dimensional insulators, topologically non-
trivial pumps are characterized by the appearance of
gapless end states during the course of a pumping cy-

\[ S = 1 + 2iW^{-1} \left( H - i\pi WW^\dagger \right)^{-1} W, \]  

where W describes the coupling to the leads. Here we
assume the leads couple equally to both spin orienta-
tions at the edges of the insulator and that the coupling itself
does not break time reversal or sublattice symmetry.
This relation allows to obtain the symmetry restrictions on the reflection matrices, summarized in Table I.

![FIG. 1: A pump formed by rolling a two dimensional insulator on a cylinder and threading it with a time dependent flux $\phi$.](attachment:image1.png)

![FIG. 2: A pumping cycle can be visualized as a set of trajectories formed by the ring coordinates, shown for $N = 2$. Left hand side portrays a non-trivial winding of the center of mass coordinate, while the right hand side shows avoided level crossing corresponding to a non-trivial winding of the center of mass.](attachment:image2.png)

| class | $\Theta$ | $\mathcal{C}$ | symmetry restriction | index |
|-------|---------|-------------|----------------------|------|
| A     | 0       | 0           | $r(t) \in U(2N)$    | $\mathbb{Z}$ |
| AII   | 1       | 0           | $r(t) = r^{T}(-t)$  | 0    |
| AII   | -1      | 0           | $r(t) = \sigma_{y}r(-t)^{T}\sigma_{y}$ | $\mathbb{Z}_2$ |
| AIII  | 0       | 1           | $r(t) = r^{T}(t)$  | 0    |
| BDI   | 1       | 1           | $r(t) = r^{T}(t)$  | 0    |
| CII   | -1      | 1           | $r(t) = r^{T}(t)$  | 0    |

TABLE I: Symmetry restrictions on reflection matrices belonging to the Wigner Dyson (first three rows) and chiral (last three rows) classes. $\Theta$ and $\mathcal{C}$ are the time reversal and sublattice symmetry operations, respectively. Last column: Classes which allow for nontrivial topological $\mathbb{Z}$ or $\mathbb{Z}_2$ invariants.

Similarly to topologically non-trivial two-dimensional insulators, topologically non-trivial pumps are characterized by the appearance of gapless end states during the course of a pumping cycle. These emerge as resonances of the scattering matrix that introduce a $\pi$ phase shift of the incoming scattering states. (The presence of a bound state at the edge of the sample follows from Eq. 1 or from a Bohr-Sommerfeld quantization argument, when the scattering states acquire a $\pi$ phase shift at resonance.) In a topologically non-trivial pump, the appearance of resonances during a pumping cycle is topologically protected and is independent of smooth deformation of the Hamiltonian, although the specific moment of it’s appearance may vary. As we discuss below, these resonances manifest in the form of topologically protected pumping properties.

In order to relate the appearance of resonances during a pumping cycle to topological properties of the reflection matrix we note that the eigenvalues of the unitary reflection matrix are restricted to the unit circle, $\{z_{1} = e^{i\phi_{1}}, ..., z_{N} = e^{i\phi_{N}}\}$. The pumping cycle can be visualized as the set of trajectories formed by these coordinates $\{z_{1}(t), ..., z_{N}(t)\}$ on the unit circle, such that the original set of eigenvalues is recovered after a cycle is completed. From Eq. 1, the appearance of an edge state, i.e., a scattering resonance, corresponds to an eigenvalue $z_{i} = -1$ in the resonant channel $i$. As slight deformations of the Hamiltonian may shift the eigenvalues and lead to detuning away from the resonance, topologically protected resonances can only arise if the trajectory of the eigenvalue forms a non-contractable loop around the unit circle. Moreover, in the absence of any symmetries, any crossings of energy levels (and similarly of eigenvalues $z_{i}$) are generally avoided, or can be avoided in the presence of small perturbations, see Fig. 2. Hence, in the absence of any symmetries, topologically protected resonances can only arise if the center of mass of the ring coordinates, $\Phi = \sum_{i=1}^{N} \phi_{i}$, forms a non contractable loop within a pumping cycle. The sum of the eigenvalue phases is related to the determinant, $\det r = \Pi_{i} z_{i} = e^{i\Phi}$ and the winding of the phase during the course of a pumping cycle, gives rise to an integer index $n$:

$$n = \oint_{0}^{T} \frac{dt}{2\pi} \Phi = \oint_{0}^{T} \frac{dt}{2\pi} \frac{d}{dt} \ln \det r. \quad (2)$$

Symmetry constraints on the pumping cycle restrict the ‘dynamics’ of the eigenvalues during the cycle. In particular, a time reversal restriction relates the time-evolution of $r(t)$ during the second part of the cycle $T/2 \leq t < T$ to the evolution during the first part $0 \leq t < T/2$,

$$r(t) = \hat{O}r^{T}(T - t)\hat{O}^{-1}. \quad (3)$$

Here $\hat{O} = e^{i\pi \hat{S}_{y}/\hbar}$ is the unit matrix for spinless electrons and the Pauli matrix $\sigma_{y}$ for spinfull electrons. As a consequence, any winding of the phase $\Phi$ during the first part of the cycle is undone in the course of completing the cycle, $\Phi(t) = \Phi(T - t)$, resulting in

$$n = \int_{0}^{T/2} \frac{dt}{2\pi} \left( \Phi(t) + \dot{\Phi}(T/2 - t) \right) = 0,$$

which expresses the fact pumps with time reversal constraint cannot pump charge.
The time reversal constraint allows, however, for a more subtle classification, when the spin rotational symmetry is broken as a result of strong spin orbit scattering (e.g. in the symplectic class AII) \[14, 17\]. The time reversal restriction \[9\] in combination with the periodicity of the pump ensures the existence of two time reversal invariant moments (TRIM) \(t_1 = 0, T/2\) throughout the pumping cycle. At these moments, the reflection matrix is time-reversal symmetric, and its eigenvalues form Kramers degenerate pairs. While the time reversal restriction \[9\] inhibits any winding of \(\Phi\), topologically protected resonances may still arise if the sum of the phase differences acquired by the Kramer pairs during half a cycle has a topologically non-trivial winding. As we shall see below, the latter is defined up to multiples of 4\(\pi\), giving rise to a \(\mathbb{Z}_2\) index instead of a \(\mathbb{Z}\) index (A similar argument is made in Ref. \[13\] with respect to the "time-reversal polarization")

At the time reversal invariant moments, the eigenvalues of the reflection matrix occur in pairs, \((e^{i\phi_n}, e^{i\bar{\phi}_n})\) with \(n = 1, ..., N\). This introduces two possibilities. After half a cycle of the pump is completed, the pairs can either recombine, \(\phi_n(T/2) = \bar{\phi}_n(T/2)\) or exchange partners \(\phi_n(T/2) = \bar{\phi}_{n-1}(T/2)\). In the latter scenario, the sum of all phase differences acquired between former Kramers pairs, can pick up a phase of multiple of 2\(\pi\) when evolving into the two-fold degenerate configuration at \(t = T/2\), see Fig. 3a. We note that the crossing of the eigenvalues at the TRIM is protected by time reversal symmetry and therefore cannot be lifted by small deformations of the Hamiltonian. Moreover, higher winding of the relative phase, \(\delta \varphi = (\varphi - \bar{\varphi})\), corresponding to the Kramers pairs recombining with next to nearest neighbors eigenvalues, inevitably involve additional crossing points away from the TRIM. Here \(\varphi = \sum_{i=1}^N \phi_i\) and \(\bar{\varphi} = \sum_{i=1}^N \bar{\phi}_i\). As these are not protected by time reversal symmetry they are generically avoided or can be removed by small deformations of the Hamiltonian, see Fig. 3b. Hence, an odd number of winding of the relative phase by 2\(\pi\) is topologically protected, while an even number of winding is topologically trivial. The relative phase \(\delta \varphi\) acquired during half a cycle can be expressed in terms of the products of eigenvalues \(Z = e^{i\bar{\varphi}}\) and \(\bar{Z} = e^{i\varphi}\), as

\[
\theta = \int_0^{T/2} dt \delta \varphi = \ln \frac{Z(T/2)\bar{Z}(0)}{Z(0)\bar{Z}(T/2)}
\]

The presence of even or odd winding number of the relative phase is then expressed in terms of the \(\mathbb{Z}_2\) index

\[
e^{i\theta/2} = \frac{\sqrt{Z(T/2)\bar{Z}(0)}}{\sqrt{Z(0)\bar{Z}(T/2)}}
\]

which takes the values \(e^{i\theta/2} = \pm 1\). Using the relation between \(Z\) and the Pfaffian, \(Z = \text{Pf}(i r(\sigma_y))\), this result can be formulated in terms of the reflection matrix:

\[
e^{i\theta/2} = \frac{\text{Pf}(i r(T/2)\sigma_y)}{\text{Pf}(i r(0)\sigma_y)} \sqrt{\text{det}(0)}\sqrt{\text{det}(T/2)}
\]

where the same branch of the square root is chosen in the numerator and denominator. Eq. \[3\] is the \(N\)-channel generalization of the \(\mathbb{Z}_2\)-index in Ref. \[17\]. The same result appeared in Ref. \[19\] in the context of a classification of 2D topological insulators.

Finally, we consider a system with chiral, i.e. sublattice symmetry. Contrary to time-reversal symmetry, the sublattice symmetry is present at every instant in the cycle,

\[
r(t) = r^1(t).
\]

As the reflection matrix is hermitian at all times, its eigenvalues are restricted to the real axis, i.e. \(z_i = \pm 1\) throughout the cycle which prevents a non-trivial topology of the trajectory \(r(t)\) during a cycle. Consequently all pumps with a chiral symmetry are topologically trivial.

Following the discussion above, we may classify the pumps belonging to the Wigner Dyson and the chiral classes based on the presence of absence of time reversal, chiral and spin rotational symmetry. The result is summarized in the last column of Table I. The classification based on the reflection matrix of the one dimensional pump reproduces the corresponding table for two-dimensional insulators in the Wigner Dyson and chiral classes.

The appearance of protected gapless edge states during the course of a pumping cycle alters the pumping properties. In contrast to their trivial counterparts, topological nontrivial pumps allow for the pumping of quantized charge or spin. In the absence of time reversal or chiral symmetries (class A), the quantization of the charge is evident as the charge pumped through the insulator \[20, 21\].

FIG. 3: Fig. a) shows a non-trivial winding of the relative phase acquired between a Kramer’s pair after half a cycle corresponding to the exchange of partners. The higher winding of the relative phase shown in Fig. b) involves additional crossing points away from the TRIM which are generally avoided, leaving no net phase difference between the Kramers pairs.
is proportional to the topological index itself \[8, 9, 14, 22-27\]:

\[
Q = \frac{e}{2\pi i} \oint dt \, \text{tr} \left( \frac{d\hat{r}}{dt} \hat{r} \right) = en, \tag{7}
\]

where the trace is taken over the \( N \) channels.

Imposing a time reversal restriction on the pumping cycle inhibits the pumping of charge by restricting the winding of the phase of \( \text{det} r \). Nonetheless, topological pumps with a time reversal restriction allow for the pumping of quantized spin even in the presence of spin-orbit scattering. We notice that contrary to topological charge pumps, the spin \( \vec{S} \) pumped in a cycle

\[
\vec{S} = \frac{\hbar}{2\pi i} \oint dt \, \text{tr} \left( \frac{d\hat{r}}{dt} \hat{r} \right) \tag{8}
\]

is not directly related to the \( \mathbb{Z}_2 \) index \[5\]. Hence its quantization is in general not ascertained. Instead, quantization becomes asymptotically exact in the weak coupling limit when the coupling to the lead becomes arbitrary small \[28\].

Following the above discussion, a topological spin pump (class AII) crosses an odd number of resonance pairs during the course of a cycle. As multiple winding of the phase can be removed by small perturbations, a non-trivial pump generically crosses a single resonance pair during a cycle. In the weak coupling limit, the \( N \) channels decouple and only the resonant channel has a time-dependent phase shift. Since only one channel contributes to the pumped spin, the considerations of Ref. \[17\] for the single channel case, \( N = 1 \), can be applied: The typical time scale at which the resonance at time \( t_i \) is traversed vanishes in the weak coupling limit, as the broadening of the energy levels goes to zero. Contrarily, the time scale on which the spin quantization axis, \( \vec{e}_\phi(t) \), changes depends exclusively on microscopically details of the insulator and is independent of the coupling to the leads. Therefore, in the weak coupling limit, the reflection matrix of the resonant level describes a rotation around a fixed axis, and quantization \( \vec{S} = \hbar \vec{e}_\phi(t_i) \) readily follows \[17\].

These results can be extended to systems in which the gap arises due to electron electron interactions. The main difference arises as the many body nature of an interaction induced gap can change the Fermi sea of non-interacting electrons into \( d \) many-particle ground states. As a consequence, ground states may be interchanged in the course of a pumping cycle, and the pump may operate with an extended period, which is a multiple of the period \( T \) at which the Hamiltonian is varied see discussion in \[14, 29\]. In the weak coupling limit, depending on the symmetry group of the Hamiltonian, each extended periodicity can be characterized by a \( \mathbb{Z} \) (class A) or \( \mathbb{Z}_2 \) (class AII) topological index, corresponding for a transfer of an integer charge \( n \) or spin \( h \) during the extended cycle, respectively. Consequently, a topological pump with an extended pumping cycle, \( qT \), due to interactions, transfers on average a fractional charge \( en/q \) or spin \( h/q \) during a cycle \[14, 29\].

In conclusion, we have extended Laughlin’s construction of pumps formed by two dimensional insulators to the Wigner Dyson and chiral classes, coupled to multichannel leads. This mapping allows for a pedestrian derivation of the topological classification of insulators in terms of the reflection matrices of the corresponding pumps. We provide a physically transparent interpretation of the topologically non-trivial phases based on their quantized pumping properties.

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