A KMS-like state of Hadamard type on Robertson-Walker spacetimes and its time evolution

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Abstract

In this work we define a new state on the Weyl algebra of the free massive scalar Klein-Gordon field on a Robertson-Walker spacetime and prove that it is a Hadamard state. The state is supposed to approximate a thermal equilibrium state on a Robertson-Walker spacetime and we call it an adiabatic KMS state. This opens the possibility to do quantum statistical mechanics on Robertson-Walker spacetimes in the algebraic framework and the analysis of the free Bose gas on Robertson-Walker spacetimes. The state reduces to an adiabatic vacuum state if the temperature is zero and it reduces to the usual KMS state if the scaling factor in the metric of the Robertson-Walker spacetime is constant.

In the second part of our work we discuss the time evolution of adiabatic KMS states. The time evolution is described in terms of semigroups. We prove the existence of a propagator on the classical phase space. This defines a time evolution on the one-particle Hilbert space. We use this time evolution to analyze the evolution of the two-point function of the KMS state. The inverse temperature change is proportional to the scale factor in the metric of the Robertson-Walker spacetime, as one expects for a relativistic Bose gas.

1 Introduction

Beyond the standard model of cosmology the inflationary scenario, involving phase transitions and symmetry breaking, has been intensively discussed during the last years. For an investigation of such phenomena it is necessary to describe the thermal behavior of quantum fields and states. In this work we start a discussion of quantum statistical mechanics on Robertson-Walker spacetimes in the framework of algebraic quantum field theory. An analysis of the free Bose gas on Robertson-Walker spacetimes could serve as a model for

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more complicated quantum field theories. The system may show a phase transition, namely Bose-Einstein condensation.

The algebraic framework of quantum field theory started with the work of Haag and Kastler [HK64]; for an overview and basic results see the book of Haag [Haa92]. Dimock [Dim80] generalized the axioms to globally hyperbolic spacetimes. Basic object is a net of $C^*$-algebras arising from the assignment of a $C^*$-algebra $A(O)$ to each open, relatively compact subset $O$ of a manifold $M$. The algebra $A(O)$ is the algebra of local observables, i.e. the observables that can be measured in the region $O$. The quasilocal algebra $A$ is defined as the inductive limit of the $A(O)$, i.e. $A = \bigcup_{O \in M} A(O)$, where the bar denotes the norm closure and the union runs through all relatively compact open sets.

A state is a positive normalized linear functional on $A$. One of the major problems in algebraic quantum field theory on curved spacetimes is to pick out physically relevant states among all positive normalized linear forms. In quantum field theory on Minkowski spacetime there is the Poincaré group as the symmetry group and the spectrum condition which sort out the Minkowski vacuum. There is no symmetry group on a generic curved spacetime and therefore no way to distinguish a vacuum-like state.

In quantum field theory on curved spacetimes the class of Hadamard states is believed to be a class of physically relevant states. We mention two reasons supporting this opinion:

1. The class of Hadamard states allows the renormalization of the energy-momentum tensor $T_{\mu\nu}$. Quantum field theory on curved spacetimes is a semi-classical theory, where matter fields are quantized but not the metric, so one has to deal with the semiclassical Einstein equation

$$G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle_\omega.$$ 

The energy-momentum tensor contains products of fields and their derivatives at one point. To the right-hand side has to be given sense in a state $\omega$ by a renormalization procedure. For Hadamard states the renormalization by a point-splitting procedure is possible (see the book of Wald [Wal94, Chap. 4.6] and references therein).

2. Haag et al. [HNS84] formulated the “principle of local definiteness”, which contains requirements to physically relevant states, namely local quasi-equivalence, local primarity and local definiteness. It was shown by Verch [Ver94] that all Hadamard states on a globally hyperbolic spacetime are locally quasi-equivalent. For ultrastatic spacetimes he showed that the local von Neumann algebras arising from Hadamard states are factors (of type $\text{III}_1$), i.e. they are local primary and he also showed local definiteness. Recently he strengthened these results to arbitrary globally hyperbolic spacetimes [Ver97].

For these reasons it is reasonable to consider Hadamard states as good candidates for physically relevant states.

There are not many explicitly known Hadamard states although Junker [Jun96, Chap. 3.7] has given an explicit construction of Hadamard states. As explicitly known Hadamard
states we mention the ground state on ultrastatic spacetimes, KMS states on ultrastatic spacetimes with compact spacelike Cauchy surfaces, adiabatic vacuum on Robertson-Walker spacetimes and the Hartle-Hawking state on extended Schwarzschild spacetime.

The class of adiabatic vacuum states is a class of Hadamard states, defined on the Weyl algebra of the free massive Klein-Gordon field on Robertson-Walker spacetimes, which approximates a vacuum state. On the other hand it is possible to define KMS states, i.e. thermal equilibrium states, on ultrastatic spacetimes, e.g. the Einstein static universe, in a usual way. In this paper we combine these definitions to define a state which approximates a thermal equilibrium state on Robertson-Walker spacetimes, an adiabatic KMS state. We will prove that this state is a Hadamard state, i.e. belongs to the class of physically relevant states. We generalize our definition by introducing a chemical potential $\mu$. The state is a Hadamard state, if $\mu < m$, where $m$ is the mass parameter in the Klein-Gordon equation. We remark that Bose-Einstein condensation sets in, if the value of the chemical potential reaches the value of the mass parameter.

We also show that an adiabatic KMS state satisfies the KMS condition with respect to an automorphism group. This automorphism group does not generate the time translations of the system. This possibility was already mentioned in the fundamental work on KMS states by Haag, Hugenholtz and Winnink [HHW67].

The general idea of adiabatic KMS states and vacua is to maintain as many properties as possible of KMS states and ground states in the ultrastatic case. But it is known that a “naive” generalization does not lead to Hadamard states [Jim96, Chap. 3.6]. The “positive frequencies” cannot be fixed on a Cauchy surface, but must be determined dynamically off the Cauchy surface. We think of an adiabatic KMS state as a state which approximates a thermal equilibrium state in the sense that switching off the expansion of the Robertson-Walker spacetime would lead to a thermal equilibrium state of inverse temperature $\beta$ for $t \to \infty$. This can be seen with the methods of the second part of the work.

In the second part of the work we analyze the evolution of an adiabatic KMS state. We introduce new coordinates, so that the Klein-Gordon equation can be written as a first order system having only off-diagonal entries. This defines a semigroup for fixed $t$. We prove the existence of a propagator on the classical phase space. By a natural generalization of the notion of a one-particle Hilbert space structure we are able to define the time evolution on the one-particle Hilbert space. It is given by a unitary propagator. This unitary operator is not identical to the unitary operator coming from the group of automorphisms with respect to which an adiabatic KMS state satisfies the KMS condition. It can be seen that an adiabatic vacuum state is invariant under this time evolution in a certain sense. For an adiabatic KMS state the inverse temperature is proportional to the scale factor $R$ in the Robertson-Walker metric, as one expects for a relativistic Bose gas. Similarly one could show for a non-relativistic Bose gas the inverse temperature change to be proportional to $R^2$.

The work is organized as follows. In the next section the notion of adiabatic vacua is reviewed. After some preliminary remarks on one-particle Hilbert space structures and the definition of KMS states we give the definition of adiabatic KMS states in section 3. In section 4 we give a precise definition of Hadamard states and prove that adiabatic KMS
states are of this kind. A section on the KMS condition follows. In the following section we describe the time evolution in terms of semigroups. The existence of a propagator is proved. The time evolution on the one-particle Hilbert space is described in section 7. In the last section we compute the evolution of adiabatic KMS states. The necessary results on pseudodifferential operators and wave-front sets are summarized in the appendix.

2 Adiabatic vacuum states

In this section we briefly summarize the definition of adiabatic vacua. For a more detailed discussion see \cite{Jun96, LR90, Tru96}. Readers only interested in the definition of an adiabatic KMS state can skip over this section. Adiabatic vacua were originally introduced by Parker \cite{Par69}.

We consider the Klein-Gordon equation

\[(\Box_g - m^2)\varphi = 0,\]
on a Lorentz manifold \((\mathcal{M},g)\) topologically of the form \(\mathcal{M} = I \times S_\varepsilon, I \subset \mathbb{R}\), where \(\varepsilon = 1,0,-1\) corresponds to the spherical, the flat and the hyperbolic case, respectively and \(g\) is a Robertson-Walker metric

\[g = -dt^2 + R(t)^2\left[\sum_1^2 d\theta_i^2 + \sin^2 \theta_2 d\phi^2\right] = -dt^2 + h_{ij}(S_\varepsilon)dx^i dx^j, \quad i,j = 1,2,3, \tag{1}\]

with \(\Sigma_1 = \sin \theta_1, \Sigma_0 = \theta_1, \Sigma_{-1} = \sinh \theta_1\) and \(R(t) > 0\).

We construct the Weyl algebra \(CCR(D,\sigma)\) associated with the space of classical solutions of the Klein-Gordon operator on Robertson-Walker spacetimes, where \(D\) is a real vector space and \(\sigma\) a symplectic form on \(D\) (see e.g. \cite{BW92, 8.2}). We define the symplectic form by

\[\sigma(F,G) = \int_{S_\varepsilon} (f_1 g_2 - f_2 g_1) d\mu(S_\varepsilon), \quad F = f_1 \oplus f_2, \quad G = g_1 \oplus g_2,\]

where \(d\mu(S_\varepsilon) = \sqrt{\det h_{ij}(S_\varepsilon)} d^3x\) is the invariant measure on the spaces \(S_\varepsilon\). The algebras of local observables are

\[\mathcal{A}(\mathcal{O}) = C^*(W(F), F \in D, \text{ supp } (F) \subset \mathcal{O} \subset \mathcal{M}),\]
where we mean the $C^*$-algebra generated by these elements.

The Klein-Gordon operator on $(\mathcal{M}, g)$ has the form
\[ \Box_g - m^2 = -\partial_t^2 - 3H(t)\partial_t + R^{-2}(t)\Delta_\varepsilon - m^2, \]
where $\Delta_\varepsilon$ is the Laplace operator on the respective spatial parts, $H(t) = \dot{R}(t)/R(t)$ and $\partial_t \equiv \partial/\partial t$. The eigenvectors and eigenvalues of the Laplace operator are explicitly known in these three cases. This fact allows us to separate the time-dependent part of the equation. Denoting the eigenvectors by $Y_\vec{k}(x)$ and the eigenvalues of the Laplace operator by $-E(k)$, i.e. $\Delta Y_\vec{k}(x) = -E(k)Y_\vec{k}(x)$, we can express the elements of the set of Cauchy data $D$ on a Cauchy surface with the uniform notation
\[ F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \int d\vec{k} \begin{pmatrix} c(\vec{k}) \\ \hat{c}(\vec{k}) \end{pmatrix} Y_\vec{k}(x), \]
\[ G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \int d\vec{k} \begin{pmatrix} b(\vec{k}) \\ \hat{b}(\vec{k}) \end{pmatrix} Y_\vec{k}(x), \quad F, G \in D, \]
where the integral reduces to a sum in the spherical case (see Junker [Jun96, 3.5] for details). The main result concerning the structure of Fock states is summarized in the following

**Theorem 1** The Fock states for the Klein-Gordon field on a Robertson-Walker spacetime are given by a two-point function of the form
\[ \langle F | G \rangle_S = \int d\vec{k} \sqrt{c(\vec{k})b(\vec{k})} S_{00}(k) + \sqrt{c(\vec{k})b(\vec{k})} S_{01}(k) + b(\vec{k})\hat{c}(\vec{k}) S_{10}(k) + \hat{c}(\vec{k})b(\vec{k}) S_{11}(k). \]

The entries of the matrix $S$ can be expressed in the form
\[ S_{00}(k) = |p(k)|^2, \quad S_{11}(k) = R^6|q(k)|^2, \]
\[ S_{01}(k) = -R^3q(k)p(k), \quad S_{10} = \overline{S_{01}}, \]
where $p$ and $q$ are polynomially bounded measurable functions satisfying
\[ q(k)p(k) - p(k)q(k) = i. \]

Conversely every pair of polynomially bounded measurable functions satisfying equation (4) yields via (3) and (2) the two-point function of a Fock state.

For the proof see Lüders and Roberts [LR90, Thm. 2.3].

We consider the time-dependence of the Klein-Gordon equation
\[ (d_t^2 + 3H(t)d_t + m^2 + R^{-2}(t)E(k)) T_k(t) = 0, \quad \forall k. \]
This equation can be solved explicitly only in exceptional cases. In the general case, one tries to solve it by an iteration procedure. For finding the iteration, we consider

\[ T_k(t) = [2R^3(t)\Omega_k(t)]^{-1/2} \exp \left( i \int_{t_0}^{t} \Omega_k(t') dt' \right) , \quad \forall k , \]

where the functions \( \Omega_k \) have to be determined. Inserting this ansatz in equation (5) we find that the functions \( \Omega_k \) have to satisfy

\[ \Omega_k^2 = \omega_k^2 - \frac{3}{4} \left( \frac{\dot{R}}{R} \right)^2 - \frac{3}{2} \frac{\ddot{R}}{R} + \frac{3}{4} \left( \frac{\dot{\Omega}_k}{\Omega_k} \right)^2 - \frac{1}{2} \frac{\ddot{\Omega}_k}{\Omega_k} , \quad (6) \]

where \( \omega_k^2 = E(k)/R^2 + m^2 \). With

\[ (\Omega_k^{(0)})^2 := \omega_k^2 = E(k)/R^2 + m^2 \]

the iteration is given by

\[ (\Omega_k^{(n+1)})^2 = \omega_k^2 - \frac{3}{4} \left( \frac{\dot{R}}{R} \right)^2 - \frac{3}{2} \frac{\ddot{R}}{R} + \frac{3}{4} \left( \frac{\dot{\Omega}_k^{(n)}}{\Omega_k^{(n)}} \right)^2 - \frac{1}{2} \frac{\ddot{\Omega}_k^{(n)}}{\Omega_k^{(n)}} . \]

The functions \( T_k(t) \) and \( \dot{T}_k(t) \) are related to the functions \( q(k) \) and \( p(k) \), which constitute the matrix \( S \) by equation (3). On a Cauchy surface at time \( t \) these relations are

\[ T_k(t) = q(k) , \quad \dot{T}_k(t) = R^{-3}(t)p(k) . \quad (7) \]

An adiabatic vacuum state will now be defined by initial values at time \( t \):

**Definition 1** For \( t_0 , t \in \mathbb{R} \), let

\[ W_k^{(n)}(t) := [2R^3(t)\Omega_k^{(n)}(t)]^{-1/2} \exp \left( i \int_{t_0}^{t} \Omega_k^{(n)}(t') dt' \right) . \]

An adiabatic vacuum state of order \( n \) is a Fock state, obtained via equations (3) and (7), where the initial values at time \( t \) for equation (3) can be expressed by

\[ T_k(t) = W_k^{(n)}(t) , \quad \dot{T}_k(t) = \dot{W}_k^{(n)}(t) . \]

For later purposes we notice that for an adiabatic vacuum state of zeroth order we have

\[ S_{11}(k) = R^6 |q(k)|^2 = R^6 |T_k(t)|^2 = \frac{R^6}{2\omega_k} . \quad (8) \]
3 Adiabatic KMS states

We review the definitions of ground and KMS states on ultrastatic spacetimes by their one-particle Hilbert space structure. Then we describe an adiabatic vacuum state (of zeroth order) in a similar way. A short computation shows the coincidence of this definition with the one given by Lüders and Roberts [LR90]. Adiabatic KMS states are defined by imitating the connection of KMS states on ultrastatic spacetimes with the ground state on the same spacetime.

3.1 One-particle Hilbert space structures

Let $\omega_S$ be a state on the Weyl algebra $CCR(D,\sigma)$. A one-particle Hilbert space structure is a real-linear map $K : D \to \mathcal{H}$, $\mathcal{H}$ a Hilbert space, satisfying

1. $KD + iKD$ is dense in $\mathcal{H}$,
2. $\left[ S(F,G) + i\sigma(F,G) \right]/2 = \langle KF|KG \rangle$, $F, G \in D$,

where $S(\cdot,\cdot)$ is a real scalar product on $D$ and $\omega_S(W(F)) = \exp(-S(F,F)/4)$ is the generating functional of the state $\omega_S$. Usually it is also required that the map $K$ intertwines the time evolutions on the phase space $D$ and the Hilbert space $\mathcal{H}$. We come back to this point in section 7.

3.1.1 One-particle structure $K$ for a ground state on ultrastatic spacetimes

For a ground state on an ultrastatic spacetime, the one-particle Hilbert space structure is given by

$$K : D \to \mathcal{H} = L^2_C(S,d\mu), \quad f_1 \oplus f_2 \mapsto 2^{-1/2}(A^{1/4}f_1 + iA^{-1/4}f_2), \quad A := m^2 - \Delta,$$

where $\Delta$ is the Laplacian on the Cauchy surface $S$.

3.1.2 One-particle structure $K^\beta$ for a KMS state on ultrastatic spacetimes

For a KMS state of inverse temperature $\beta$, the one-particle Hilbert space structure is defined by doubling the Hilbert space:

$$K^\beta : D \to \mathcal{H} \oplus \mathcal{H}, \quad F \mapsto (\cosh Z^\beta)KF \oplus C(\sinh Z^\beta)KF,$$

where $Z^\beta$ is implicitly defined by $\tanh Z^\beta = \exp(-\beta A^{1/2})$, i.e.

$$\cosh^2 Z^\beta = \left[1 - \exp(-\beta A^{1/2})\right]^{-1}, \quad \sinh^2 Z^\beta = \left[1 - \exp(-\beta A^{1/2})\right]^{-1} \exp(-\beta A^{1/2}),$$

$C$ is a conjugation and $K$ is the map defined in subsubsection 3.1.1.
3.1.3 One-particle structure $K_t^\alpha$ for an adiabatic vacuum state

An adiabatic vacuum state (of zeroth order) can also be described by a one-particle Hilbert space structure, as we will show below. The mapping which defines an adiabatic vacuum state is given by

$$K_t^\alpha : D \rightarrow \mathcal{H} = L^2_C(S_\epsilon, d\mu), \quad f_1 \oplus f_2 \mapsto B_1(t)f_1 + iB_2(t)f_2,$$

where $A(t) = m^2 - \Delta/R^2(t)$. This one-particle Hilbert space structure leads to a two-point function

$$\langle K_t^\alpha(f_1 \oplus f_2)|K_t^\alpha(g_1 \oplus g_2)\rangle = \langle B_1f_1 + iB_2f_2|B_1g_1 + iB_2g_2 \rangle$$

We show this expression to be equivalent to the two-point function of an adiabatic vacuum state of zeroth order as defined by Lüders and Roberts [LR90]. For example for the fourth entry we have

$$2\langle f_2|B_2^*B_2g_2 \rangle = \langle f_2|A^{-1/2}g_2 \rangle = \int d\vec{k} \int d\vec{k}' \omega^{\omega^2} \langle \hat{b}(\vec{k})Y_{\vec{k}}(x)|Y_{\vec{k}'}(x) \rangle =$$

$$= \int d\vec{k} \int d\vec{k}' \omega^{\omega^2} \hat{b}(\vec{k})\omega^{-1}\delta(\vec{k} - \vec{k}') = R^3 \int d\vec{k} \hat{c}(\vec{k})\omega^{-1}\hat{b}(\vec{k}).$$

Comparing this with equation (3) gives $S_{11}(k) = R^3/(2\omega_k)$, which is the desired result of equation (3).

3.2 Definition of an adiabatic KMS state

If we look at the definition of KMS states on an ultrastatic spacetime, we find that it is connected with the ground state on this spacetime. In a similar way we connect an adiabatic KMS state with an adiabatic vacuum state:

**Definition 2** We define an **adiabatic KMS state** by a one-particle Hilbert space structure $K_t^{\alpha\beta}$ given by

$$K_t^{\alpha\beta} : D \rightarrow \mathcal{H} \oplus \mathcal{H},$$

$$F \mapsto (\cosh Z\beta)K_t^\alpha F \oplus C(\sinh Z\beta)K_t^\alpha F,$$
where $K^a_t$ is the one-particle structure of an adiabatic vacuum state, $C$ is a conjugation and $\tanh Z^\beta = \exp(-\beta A^{1/2}(t))$.

This definition leads to the following two-point function of an adiabatic KMS state

$$\langle K^a\beta(f_1 \mp f_2) \mid K^a\beta(g_1 \mp g_2) \rangle =$$

$$= \langle B_1 f_1 + iB_2 f_2 \mid \cosh^2 Z^\beta (B_1 g_1 + iB_2 g_2) \rangle + \langle B_1^* f_1 - iB_2^* f_2 \mid \sinh^2 Z^\beta (B_1^* g_1 - iB_2^* g_2) \rangle,$$

so that the two-point distribution $\Lambda$ has the form (we suppress the $t$-dependence of $A$):

$$\Lambda(f, g) =$$

$$= \frac{1}{2} \left[ (A^{1/2} + iH(1 + \frac{m^2}{2} A^{-1})) + i\partial_t \right] Ef \left[ A^{1/2} \cosh^2 Z^\beta [A^{1/2} + iH(1 + \frac{m^2}{2} A^{-1})] + i\partial_t \right] Eg \right],$$

for $f, g \in C_0^\infty(M)$ and $E$ is the causal propagator (see section 2). In section 4.2 we will prove that this two-point distribution is the two-point distribution of a Hadamard state.

## 4 Hadamard property of an adiabatic KMS state

In the next section we give a precise definition of a Hadamard state and prove in the following subsection that an adiabatic KMS state is a Hadamard state. The necessary results on pseudodifferential operators and wave-front sets are summarized in the appendix.

### 4.1 Hadamard states

Since the work of Radzikowski [Rad96] it is known that Hadamard states can be characterized by the wave-front set of its two-point distribution. Earlier definitions required a specific form of the two-point distribution (see Kay and Wald [KW91]). The characterization of a Hadamard state by its wave-front set is easier to handle and offers new possibilities to prove the Hadamard property of a state, but it requires some knowledge of pseudodifferential operators (PDO’s) and wave-front sets of distributions. We refer to the appendix for notation and some results used below.

**Definition 3** A quasifree state of a Klein-Gordon quantum field on a globally hyperbolic spacetime is a **Hadamard state** iff the wave-front set of its two-point distribution $\Lambda$ is of the form:

$$WF(\Lambda) = \{(x_1, \xi_1; x_2, -\xi_2) \in T^*(M \times M) \setminus \{0\} \mid (x_1, \xi_1) \sim (x_2, \xi_2), \xi_1^0 \geq 0\},$$

(10)

where the notation $(x_1, \xi_1) \sim (x_2, \xi_2)$ means that $x_1$ and $x_2$ can be joined by a null geodesic $\gamma$ and $\xi_1$ is tangent to $\gamma$ in $x_1$ and $\xi_2$ is the parallel transport of $\xi_1$ along $\gamma$ in $x_2$.

The proof that a state is a Hadamard state requires only the analysis of the wave-front set of its two-point distribution. We will do this in the next section for an adiabatic KMS state.
4.2 An adiabatic KMS state is a Hadamard state

In the proof that an adiabatic KMS state is a Hadamard state we use the following theorems due to Junker [Jun96, Thm.3.11 and 3.12].

**Theorem 2** Let \((\mathcal{M}, g)\) be a globally hyperbolic spacetime with Cauchy surface \(S\) and \((D, \sigma)\) be the phase space of initial data on \(S\) of the Klein-Gordon field. Let \(B, I, S, C\) be operators on \(L^2_{\mathbb{R}}(S, d\mu)\), such that \(I\) is symmetric, \(B\) is selfadjoint, positive and invertible and \(C^*C - S^*S = 1\).

Then, with \(H = L^2_{\mathbb{C}}(S, d\mu)\),

\[
K : D \to \tilde{H} = \mathcal{H} \oplus \mathcal{H},
\]

\[
(f_1, f_2) \mapsto C(2B)^{-1/2}[(B + iI)f_1 + if_2] \oplus S(2B)^{-1/2}[(B - iI)f_1 - if_2],
\]

is a one-particle Hilbert space structure.

For a proof see Junker [Jun96, Thm. 3.11] (where different conventions are used).

Under the assumption that the metric has the form of equation (1), the two-point distribution \(\Lambda\) resulting from this one-particle structure is given by

\[
\Lambda(f, g) = \frac{1}{2}\langle (B + iI + i\partial_t)Ef | B^{-1/2}C^*CB^{-1/2}((B + iI + i\partial_t)Eg) \rangle + \frac{1}{2}\langle (B - iI - i\partial_t)Ef | B^{-1/2}S^*SB^{-1/2}(B - iI - i\partial_t)Eg \rangle,
\]

where \(f, g \in C^\infty_0(M)\) and \(E\) is the causal propagator (see section 2).

Now let \((\mathcal{M}, g)\) be a globally hyperbolic spacetime, foliated in a neighborhood of \(S\) into \((-T, T) \times S\) with \(S_t := \{t\} \times S\) and \(S_0 = S\) and \(g\) of the form given in equation (1).

**Theorem 3** Let \(B(t), I(t), S(t), C(t)\) be PDO’s on \(S_t, t \in (-T, T)\), satisfying the properties stated in theorem 2, such that \(B\) is elliptic, \(S \in \text{OPS}^{-\infty}\), and such that there exists a PDO \(Q\) on \(\mathcal{M}\) with the property \(Q(B + iI + i\partial_t) = \Box_g - m^2\) which possesses a principal symbol \(q\) with

\[
q^{-1}(0) \setminus \{0\} \subset \{(x, \xi) \in T^*(\mathcal{M}) | \xi^0 \geq 0\}.
\]

Then the quasifree state given by the one-particle Hilbert space structure of theorem 2 is a Hadamard state, i.e. the wave-front set of the corresponding two-point distribution \(\Lambda\) has the form of equation (11).

For a proof see Junker [Jun96, Thm. 3.12].

This theorem will be used in the proof of the following theorem. The proof for closed Robertson-Walker spacetimes is in fact a generalization of Junker’s proof [Jun96, Chap. 3.4] that a KMS state on an ultrastatic spacetime with compact spacelike Cauchy surface is a Hadamard state.
Theorem 4  An adiabatic KMS state on the Weyl algebra of the free massive Klein-Gordon field on Robertson-Walker spacetimes as defined in definition 2 is a Hadamard state.

Proof: We have to show that the wave-front set of the two-point distribution (14) has the form of equation (15).

For \( F = f_1 \oplus f_2 \in D \), we have

\[
K_{t}^{a} F = (2A^{1/2})^{-1/2} \{ [A^{1/2} + iH(1 + m^2 A^{-1}/2)]f_1 + if_2 \}.
\]

We identify the operator \( B \) in theorem 2 respectively theorem 3 with \( A^{1/2} = (m^2 - \Delta/R^2)^{1/2} \), which is an elliptic, selfadjoint, positive PDO (of order 1). Furthermore the operator \( I \) is identified with \( H(1 + m^2 A^{-1}/2) \), which is a symmetric PDO. The operators \( S \) respectively \( C \) are identified with \( \sinh Z^{\beta} \) respectively \( \cosh Z^{\beta} \), so that \( C^*C - S^*S = 1 \).

For closed Robertson-Walker spacetimes we proceed as follows: Since

\[
S = \exp(-\beta A^{1/2}/2)(1 - \exp(-\beta A^{1/2}))^{-1/2}
\]

and \( A^{1/2} \) has the properties of theorem 8 in the appendix with the real-valued principal symbol given by \( a(x, \xi) = (\hbar^{ij}\xi_i\xi_j)^{1/2} \), we can apply this theorem, to conclude that \( S \) is a PDO with principal symbol

\[
p(a(x, \xi)) = \frac{\exp(-\beta a(x, \xi)/2)}{(1 - \exp(-\beta a(x, \xi)))^{1/2}}.
\]

This principal symbol falls off faster than any inverse power of \( \xi \), so that \( S \in OPS^{-\infty} \) and this also means that \( C \in OPS^{0} \).

For flat Robertson-Walker spacetimes we show directly that the involved operators are PDO’s:

\[
(cosh Z^{\beta} f)(t, \bar{x}) = (1 - \exp(-\beta A^{1/2}))^{-1/2} f(t, \bar{x}) = (2\pi)^{-3/2} \int (1 - \exp(-\beta \omega_k))^{-1/2} \tilde{f}(t, \bar{k})Y_{k}(\bar{x}) \, d\bar{k},
\]

which is a PDO of order zero, because

\[
a(k) = (1 - \exp(-\beta \sqrt{k^2 + m^2}))^{-1/2}
\]

is a symbol of order zero. Furthermore

\[
(sinh Z^{\beta} f)(t, \bar{x}) = (2\pi)^{-3/2} \int \frac{\exp(-\beta \omega_k/2)}{(1 - \exp(-\beta \omega_k))^{1/2}} \tilde{f}(t, \bar{k})Y_{k}(\bar{x}) \, d\bar{k},
\]

is a PDO of order \(-\infty\), because

\[
a(k) = \frac{\exp(-\beta \sqrt{k^2 + m^2}/2)}{(1 - \exp(-\beta \sqrt{k^2 + m^2}))^{1/2}}
\]

This completes the proof.

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and the derivatives of $a(k)$ tend to zero as $k \to \infty$. For hyperbolic Robertson-Walker spacetimes one has to express the eigenfunctions of the Laplace operator in terms of $\exp(i\mathbf{x} \cdot \mathbf{\xi})$ in the same way as it is done in Jun96, Proof of Lemma 3.26, to see that the operators are PDO’s of the desired type.

The operator $Q$ is given by

$$Q = 3iH/2 - A^{-1/2}\partial_tA^{1/2} - A^{1/2} + i\partial_t.$$  

This can be verified with the help of equation (3).

\[\square\]

### 4.3 Introduction of a chemical potential

We generalize the definition to the case of a non-vanishing chemical potential $\mu$. This can be done by changing the definition of $\tanh Z^\beta$: Let

$$\tanh Z^\beta = \exp(-\beta h(\mu)), \quad h(\mu) = A^{1/2}(t) - \mu.$$  

The operator $h(\mu)$ is selfadjoint on $\text{dom}(A^{1/2})$, positive if $\mu < m$ and elliptic. So we can generalize the proof in the case of a closed Robertson-Walker spacetimes to the case of a non-vanishing chemical potential $\mu$ under the restriction $\mu < m$. In the case of non-closed Robertson-Walker spacetimes we can again directly compute it.

**Theorem 5** An adiabatic KMS state on the Weyl algebra of the free massive Klein-Gordon field on Robertson-Walker spacetimes as defined in definition 3 generalized by equation (12) is a Hadamard state if $\mu < m$.

### 5 On the KMS condition

In this section we show that an adiabatic KMS state fulfills the KMS condition with respect to an automorphism group $\alpha_s$. It is not the automorphism group that generates the time translations of the system. It was already remarked in the fundamental paper on the KMS condition by Haag, Hugenholtz and Winnink [HHW67], that such a situation can occur. This means the system were in equilibrium if time evolution were given by the automorphism group $\alpha_s$, i.e. if $h(s) = (m^2 - \Delta/R^2(s))^{1/2}$ were independent of $s$.

A KMS state can be defined in the following way (see Kay and Wald [KW91]).

**Definition 4** Let $\alpha_s$ be an automorphism group on a $C^*$-algebra $\mathcal{A}$ and $\omega$ an $\alpha_s$-invariant state. $\omega$ is a KMS state at inverse temperature $\beta$ if its GNS triple $(\mathcal{F}, \pi_\beta, \Omega_\beta)$ satisfies the following properties:

1. The unique unitary group $U(s) : \mathcal{F} \to \mathcal{F}$ which implements $\alpha_s$ and leaves $\Omega_\beta$ invariant, is strongly continuous, so that $U(s) = \exp(-iHs)$ for some selfadjoint operator $H$. 

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2. $\pi_\beta(A)\Omega_\beta$ is contained in the domain of $\exp(-\beta H/2)$.

3. There exists a complex conjugation $J$ on $\mathcal{F}$ satisfying

$$[J, \exp(-iHs)] = 0, \forall s \in \mathbb{R}, \text{ and } \exp(-\beta H/2)\pi_\beta(A)\Omega_\beta = J\pi_\beta(A^*)\Omega_\beta, A \in \mathcal{A}.$$  

For quasifree states the definition can be reduced on the one-particle Hilbert space. $K^\alpha_\beta$ maps to $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$, so the representation Hilbert space can be chosen to be

$$\mathcal{F} = \mathcal{F}_s(\mathcal{H} \oplus \mathcal{H}) = \mathcal{F}_s(\mathcal{H}) \otimes \mathcal{F}_s(\mathcal{H}),$$

where $\mathcal{F}_s(\mathcal{H})$ is the symmetric Fock space over $\mathcal{H}$. The Weyl operator $W(f)$ on $\mathcal{F}$ is represented by

$$W(f) = W_F(\cosh Z^\beta K^\alpha_\beta f) \otimes W_F(C\sinh Z^\beta K^\alpha_\beta f),$$

where $W_F$ is the usual Weyl operator on $\mathcal{F}_s$. With $h = (m^2 - \Delta_s/R^2)^{1/2}$ we define on $\tilde{\mathcal{H}}$

$$e^{-ihs} = e^{-ihs} \oplus e^{ihs}, \quad e^{-\beta h/2} = e^{-\beta h/2} \oplus e^{\beta h/2}, \quad j(x \oplus y) = (-Cy) \oplus (-Cx)$$

and the operators on $\mathcal{F}$ by second quantization. The second condition in definition 4 can be reduced to Kay’s regularity condition $KD \subset \text{dom}(h^{-1/2})$ (see Kay [Kay85]). $h$ is a positive, selfadjoint operator, so that $h^{-1/2}$ is bounded and $\text{dom}(h^{-1/2}) = \mathcal{H}$. The condition $[J, e^{-iHs}] = 0$ reduces to $[j, e^{-ihs}] = 0$, which can easily be verified. The condition $\exp(-\beta H/2)\pi_\beta(A)\Omega_\beta = J\pi_\beta(A^*)\Omega_\beta$ reduces to $e^{-\beta h/2}(iK^\alpha_\beta f) = j(-iK^\alpha_\beta f)$ and can also be verified. One finds $e^{-\beta h/2}(x \oplus y) = Cy \oplus Cx$, so that the one-particle KMS condition

$$\langle e^{-ish}x | y \rangle_{\tilde{\mathcal{H}}} = \langle e^{-\beta h/2}y | e^{-ish}e^{-\beta h/2}x \rangle_{\tilde{\mathcal{H}}}$$

is valid for $x, y \in \tilde{\mathcal{H}}$ and all $s \in \mathbb{R}$.

6 Time evolution by semigroups

In this section we analyze the time evolution of adiabatic KMS states. It is well known that the inverse temperature of a relativistic Bose gas on Robertson-Walker spacetime is proportional to the scale parameter $R$ of the metric, while for a non-relativistic gas the inverse temperature is proportional to $R^2$. We will find the same behavior for a Bose gas on a Robertson-Walker spacetime described by an adiabatic KMS state.

We describe the time evolution on the classical phase space in terms of semigroups and prove the existence of a propagator.
6.1 The Klein-Gordon equation as a first order system

To analyze the time evolution of an adiabatic KMS state, we first have to describe the time evolution on the classical phase space $D$. This can be achieved by introducing different coordinates, so that the metric has the form

$$g = -R^6(t)dt^2 + R^2(t)[d\theta_1^2 + \Sigma_\varepsilon^4(R^2(t) + \sin^2 \theta_2 d\phi^2)], \quad R(t) > 0,$$

where again $\varepsilon = -1, 0, 1$ corresponds to the spherical, the flat and the hyperbolic spatial part respectively. We assume $R(t)$ and $\dot{R}(t)$ to be positive and continuous on any compact subset. In these coordinates the Klein-Gordon equation has the form

$$[-\partial_t^2 + R^4(t)\Delta_\varepsilon - m^2 R^6(t)]\phi = 0,$$

where $\Delta_\varepsilon$ is again the Laplace operator on the respective spatial spaces. The Klein-Gordon equation can be written as a first order system:

$$\partial_tF = -H(t)F$$

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad -H(t) = \begin{pmatrix} 0 & 1 \\ -B^2(t) & 0 \end{pmatrix}, \quad -B^2(t) = R^4(t)\Delta_\varepsilon - m^2 R^6(t),$$

We define the operator $H(t)$ on the real Hilbert space $\mathcal{H}_t = \text{dom}(B(t)) \oplus L^2(S_\varepsilon(t))$, where $S_\varepsilon$ are the respective spatial spaces, with scalar product

$$\langle f_1 \oplus f_2 | g_1 \oplus g_2 \rangle_B = \langle Bf_1 | Bg_1 \rangle + \langle f_2 | g_2 \rangle.$$ 

The phase space $D$ is of course a dense subspace of $\mathcal{H}_t$. For fixed $s$ the operator $H(s)$ is skew-adjoint on this space and therefore defines a contractive semigroup $T(t) = \exp[-tH(s)]$ on $\mathcal{H}$.

6.2 Existence of the propagator

We use the following theorem to prove the existence of the propagator. For each positive integer $k$, we define an approximate propagator $U_k(t, s)$ on $0 \leq s \leq t \leq 1$ by

$$U_k(t, s) = \exp \left( -(t - s)H \left( \left( i - \frac{1}{k} \right) \right) \right)$$

$$\text{and} \quad U_k(t, r) = U_k(t, s)U_k(s, r) \quad \text{if} \quad 0 \leq r \leq s \leq t \leq 1.$$

We also define $C(t, s) = H(t)H(s)^{-1} - 1$.

**Theorem 6** Let $X$ be a Banach space and let $I$ be an open interval in $\mathbb{R}$. For each $t \in I$, let $H(t)$ be the generator of a contraction semigroup on $X$ so that $0 \in \rho(H(t))$, the resolvent set of $H(t)$, and
1. The $H(t)$ have a common dense domain $D$.

2. For each $\varphi \in X$, $(t - s)^{-1}C(t, s)\varphi$ is uniformly strongly continuous and uniformly bounded in $s$ and $t$ for $t \neq s$ lying in any fixed compact subinterval of $I$.

3. For each $\varphi \in X$, $C(t)\varphi \equiv \lim_{s\rightarrow t}(t - s)^{-1}C(t, s)\varphi$ exists uniformly for $t$ in each compact subinterval and $C(t)$ is bounded and strongly continuous in $t$.

Then for all $s \leq t$ in any compact subinterval of $I$ and any $\varphi \in X$,

\[ U(t, s)\varphi = \lim_{k \rightarrow \infty} U_k(t, s)\varphi \]

exists uniformly in $s$ and $t$. Further, if $\psi \in D$, then $\varphi_s(t) = U(t, s)\psi$ is in $D$ for all $t$ and satisfies

\[ \frac{d}{dt}\varphi_s(t) = -H(t)\varphi_s(t), \quad \varphi_s(s) = \psi \]

and $\|\varphi_s(t)\| \leq \|\psi\|$ for all $t \geq s$.

For a proof see [RS75, Thm. X.70].

As a consequence of the positivity of $B^2(t)$, $0$ is in the resolvent set of $H(t)$.

We will verify condition 1, i.e. we have to show for all $t \in I$ the operators $H(t)$ have a common dense domain. For this it is sufficient to show that the spaces $L^2(S_x(t))$ are setwise equivalent. Let $h$ be the determinant of the spatial part of the metric $g$ and let $\mu_h(t) = \sqrt{h(t)}$ and $\mu_h(t') = \sqrt{h(t')}$ be the invariant measures induced by the metric $g$ on the Cauchy surfaces at time $t$ and $t'$ respectively. Then

\[ \int_{S_x} |f|^2 \sqrt{h(t)} \, d^3x = \int_{S_x} |f|^2 \frac{\sqrt{h(t)}}{\sqrt{h(t')}} \sqrt{h(t')} \, d^3x, \]

where $\sqrt{h(t)}/\sqrt{h(t')}$ is smooth, bounded and strictly positive, namely the Radon-Nykodim derivative of the measure $\mu_h(t)$ with respect to the measure $\mu_h(t')$. Therefore the measures $\{\mu_h(t)\}_{t \in I}$ are mutually absolutely continuous and because of the boundedness of $\sqrt{h(t)}/\sqrt{h(t')}$ we have $f \in L^2(S_x(t))$ iff $f \in L^2(S_x(t'))$.

We have to verify condition 2. The operator $C(t, s)$ is given by

\[ C(t, s) = H(t)H(s)^{-1} - 1 = \left( \begin{array}{cc} 0 & -1 \\ B^2(t) & 0 \end{array} \right) \left( \begin{array}{cc} 0 & B^{-2}(s) \\ -1 & 0 \end{array} \right) - \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ 0 & B^2(t)B^{-2}(s) - 1 \end{array} \right), \]

so we have

\[ (t - s)^{-1}C(t, s)F = (t - s)^{-1} \left( \frac{m^2R^6(t) - R^4(t)\Delta_x}{m^2R^6(s) - R^4(s)\Delta_x} - 1 \right) \pi, \quad F = \varphi \oplus \pi \in D. \]
We assumed $R(t)$ to be continuous and as a consequence $R(t)$ is jointly continuous on every compact subinterval. Therefore $C(t, s)$ is jointly continuous in $t$ and $s$. The operator is also jointly bounded on every compact subinterval (by using the eigenvalues of $\Delta_\varepsilon$) because $R(t)$ is bounded.

The last step is for $\varphi \in \mathcal{H}$ to show that $C(t)\varphi = \lim_{s\to t}(t - s)^{-1}(H(t)H(s)^{-1} - 1)\varphi$ exists uniformly for $t$ in each compact subinterval and $C(t)$ is bounded and strongly continuous in $t$. The existence of the limit can be shown with the rule of de l’Hospital. It is

$$C(t) = 4\frac{\dot{R}(t)}{R(t)} + 2m^2\dot{R}(t)R(t)(m^2R^2(t) - \Delta_\varepsilon)^{-1}.$$ 

The operator is bounded by the boundedness of $(m^2R^2(t) - \Delta_\varepsilon)^{-1}$ and because we assumed $R(t)$ and $\dot{R}(t)$ to be bounded functions and it is of course strongly continuous.

**Remark:** We have introduced new coordinates. In these coordinates the existence proof is most easy. It is also possible to proof the existence of the propagator $\tilde{U}(t, s)$ as the strong limit of the approximate propagators $\tilde{U}_k(t, s)$.

7 Time evolution on the one-particle Hilbert space

We will now describe the time evolution on the one-particle Hilbert space. In definition 3.1 we defined a one-particle Hilbert space structure. For static spacetimes it is also required that

$$U(t)K = K\mathcal{T}(t),$$

where $\mathcal{T}(t)$ describes the time evolution on $D$ and $U(t)$ the time evolution on $\mathcal{H}$. i.e. $K$ intertwines the time evolutions.

The time evolution on the classical phase space is given by the propagator of subsection 6.2. The propagator maps Cauchy data at time $s$ to Cauchy data at time $t$. It is therefore natural to require the following generalization of condition (13). For the one-particle Hilbert space structure $K^a_t$ of an adiabatic vacuum state we demand that $K^a_t$ intertwines the time evolutions such that

$$\tilde{U}(t, s)K^a_s = K^a_tU(t, s).$$

where $\tilde{U}(t, s)$ is time evolution on $\mathcal{H}$. This is a natural generalization. On the right hand side the evolution from a Cauchy surface at time $s$ to a Cauchy surface at time $t$ is given on phase space followed by the mapping to the one-particle Hilbert space at time $t$. On the left hand side we map to the one-particle Hilbert space at time $s$ and evolution is given on the Hilbert space to a Cauchy surface at time $t$.

The operator $K^a_s$ is injective: For $F = f_1 \oplus f_2, G = g_1 \oplus g_2 \in D$, we conclude from

$$\sqrt{2}K^a_s(F - G) = A^{1/4}(f_1 - g_1) + iA^{-1/4}H(1 + \frac{m^2}{2}A^{-1})(f_1 - g_1) + iA^{-1/4}(f_2 - g_2) = 0,$$
that $f_1 = g_1$, because $A^{1/4}$ maps real-valued functions to real-valued functions and this leads to $f_2 = g_2$, i.e. $F = G$. Since the kernel of $K^a_s$ contains only the zero vector, we can define the propagator $\tilde{U}(t, s)$ on the one-particle Hilbert space by

$$\tilde{U}(t, s) = K^a_t U(t, s)(K^a_s)^{-1},$$

on the range of $K^a_s$, which is dense in $\mathcal{H}$.

Furthermore we will show that $\tilde{U}(t, s)$ is isometric on the range of $K^a_s$ and can be extended to a unitary operator on $\mathcal{H}$. The propagator $U(t, s)$ leaves the symplectic form $\sigma$ invariant, because $\sigma$ is invariant under solutions of the Klein-Gordon equation. Since the real-scalar product $S$ on $D$ can be defined by a complexification $J$ ($J : D \to D$, $J^2 = -1$ see [BW92, 8.2.4]) via

$$S(F, G) = \sigma(F, JG),$$

we also have $S(U(t, s)F, U(t, s)G) = S(F, G)$ and $U(t, s)J_s = J_t U(t, s)$, where $J_s$ means the complexification, defining a state at time $s$. Now with $f = K^a_s F, g = K^a_s G, F, G \in D$, we have

$$\langle f | g \rangle = \langle K^a_s F | K^a_s G \rangle =$$

$$= [S(F, G) + i\sigma(F, G)]/2 =$$

$$= [S(U(t, s)F, U(t, s)G) + i\sigma(U(t, s)F, U(t, s)G)]/2 =$$

$$= [S(U(t, s)(K^a_s)^{-1}f, U(t, s)(K^a_s)^{-1}g) + i\sigma(U(t, s)(K^a_s)^{-1}f, U(t, s)(K^a_s)^{-1}g)]/2 =$$

$$= \langle K^a_t U(t, s)(K^a_s)^{-1}f | K^a_t U(t, s)(K^a_s)^{-1}g \rangle =$$

$$= \left\langle \tilde{U}(t, s)f \bigg| \tilde{U}(t, s)g \right\rangle,$$

which shows the that $\tilde{U}(t, s)$ is an isometry and because $\tilde{U}(t, s)$ is a bounded linear operator, densely defined on the range of $K^a_s$, it can be extended to a unitary operator on $\mathcal{H}$.

The propagator $U(t, s)$ leaves the symplectic form $\sigma$ invariant. Therefore it defines an automorphism $\alpha_{t,s}$ of the algebra given by $\alpha_{t,s}(W(F)) = W(U(t, s)F)$. For an adiabatic vacuum state $\omega$ we have

$$\omega_t(\alpha_{t,s}(W(F))) = \omega_t(W(U(t, s)F)) =$$

$$= \exp(-\|K^a_t U(t, s)F\|/4) =$$

$$= \exp(-\|\tilde{U}(t, s)K^a_s F\|/4) =$$

$$= \exp(-\|K^a_s F\|/4) =$$

$$= \omega_s(W(F)),$$

i.e. $\omega_t \circ \alpha_{t,s} = \omega_s$. 

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8 Time evolution of an adiabatic KMS state

In this section we will answer the question about the time evolution of an adiabatic KMS state.

We start with the two-point function of an adiabatic KMS state of inverse temperature $\beta_t$ on a Cauchy surface at time $t$ and compute the two-point function on a Cauchy surface $s$.

$$
\langle K^{\alpha \beta_t}(U(t, s) F)|K^{\alpha \beta_t}(U(t, s) F) \rangle = \\
= \left\langle K^\alpha_t(U(t, s) F) \left| \frac{1 + \exp[-\beta_t A^{1/2}(t)]}{1 - \exp[-\beta_t A^{1/2}(t)]} K^\alpha_t(U(t, s) F) \right. \right\rangle \\
= \left\langle \tilde{U}(t, s) K^\alpha_s(F) \left| \frac{1 + \exp[-\beta_t A^{1/2}(t)]}{1 - \exp[-\beta_t A^{1/2}(t)]} \tilde{U}(t, s) K^\alpha_s(F) \right. \right\rangle \\
= \left\langle K^\alpha_s(F) \left| \frac{1 + \exp[-\beta_t A^{1/2}(t)]}{1 - \exp[-\beta_t A^{1/2}(t)]} K^\alpha_s(F) \right. \right\rangle , \quad F \in D.
$$

We have

$$
-\beta_t A^{1/2}(t) = -\beta_t \left( m^2 - \frac{\Delta}{R^2(t)} \right)^{1/2} \\
= -\beta_t \frac{R(s)}{R(t)} \left( m^2 \frac{R^2(t)}{R^2(s)} - \frac{\Delta}{R^2(s)} \right)^{1/2} \\
= -\beta_s \left( m^2 \frac{R^2(t)}{R^2(s)} - \frac{\Delta}{R^2(s)} \right)^{1/2},
$$

so the state on the Cauchy surface $s$ can be interpreted as a state of inverse temperature $\beta_s = \beta_t R(s)/R(t)$. This means that the inverse temperature changes proportional to the scale parameter $R$.

Appendix: Pseudodifferential operators and wave-front sets

We shortly review in this appendix the necessary results on pseudodifferential operators and wave-front sets (see e.g. Taylor [Tay81]).

**Definition 5** Let $O$ be an open subset of $\mathbb{R}^n$. We define the symbol class $S^m(O)$, $m \in \mathbb{R}$, to consist of all functions $p \in C^\infty(O \times \mathbb{R}^n)$ with the property that, for any compact set $K \subset O$, any multi-indices $\alpha, \beta$, there exists a constant $C_{K,\alpha,\beta}$ such that

$$
|D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C_{K,\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}
$$

for all $x \in K, \xi \in \mathbb{R}^n$. 
Remark: It is possible to define more general symbol classes but it is not necessary for our work.

With each symbol $p(x, \xi)$ we associate the **pseudodifferential operator** $P$ by

$$(Pf)(x) = (2\pi)^{-n} \int p(x, \xi) \tilde{f}(\xi) e^{i\xi \cdot x} \, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where $\tilde{\cdot}$ denotes the Fourier transform and if $p(x, \xi) \in S^m$, we say $P \in OPS^m$. The operator $P$ is a continuous operator of $D(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$ and can be extended to a continuous operator of $E'(\mathbb{R}^n)$ to $D'(\mathbb{R}^n)$. By the Schwartz kernel theorem we can associate a distribution kernel $K_P \in D'(\mathbb{R}^n \times \mathbb{R}^n)$ with the map $P$ such that $\langle Pu | v \rangle = \langle K_P | u \otimes v \rangle$.

It is also possible to define a pseudodifferential operator (PDO) on a paracompact manifold.

**Definition 6** An operator $P : C_0^\infty(M) \to C^\infty(M)$ belongs to $OPS^m(M)$ if the kernel of $P$ is smooth off the diagonal in $M \times M$ and for any coordinate neighborhood $U \subset M$ there is a diffeomorphism $\chi : U \to O \subset \mathbb{R}^n$, such that the map of $C_0^\infty(O)$ into $C^\infty(O)$ given by $u \mapsto P(u \circ \chi) \circ \chi^{-1}$ belongs to $OPS^m(O)$.

If $P \in OPS^m(O)$, we define the principal symbol of $P$ to be the member of the equivalence class in $S^m(O)/S^{m-1}(O)$.

Now we define the wave-front set of a distribution. If $p \in S^m(O)$ and $p_m$ its principal symbol, the characteristic set $charP$ of the PDO $P$ associated with the symbol $p$ is given by

$$charP = \{(x, \xi) \in T^*(O) \setminus \{0\} | p_m(x, \xi) = 0\}.$$

The **wave-front set** $WF(u)$ of a distribution $u$ is defined by

$$WF(u) = \bigcap \{charP | P \in OPS^0, Pu \in C^\infty\}.$$

A useful characterization of the wave-front set is given in the next

**Theorem 7** The point $(x_0, \xi_0) \not\in WF(u)$ iff there is $\phi \in C_0^\infty(O)$, $\phi(x_0) \neq 0$, and a conic neighborhood $\Gamma$ of $\xi_0$, such that, for every $n$

$$|\tilde{\phi}u(\xi)| \leq C_n(1 + |\xi|)^{-n}, \quad \xi \in \Gamma.$$

For a proof see [Lay81], Chap. VI §1. We quote two further results, important for this work.

**Theorem 8** Let $A \in OPS^1(M)$ be an elliptic, selfadjoint, positive operator on a compact manifold $M$ with real valued principal symbol $a(x, \xi)$. Let $p(\lambda) \in S^m(\mathbb{R})$ be a Borel function. Then $p(A) \in OPS^m(M)$ with principal symbol $p(a(x, \xi))$. 

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For a proof see [Tay81, Chap. XII §1].

We remark that the square-root of the Laplace operator on a compact manifold is of this type, \((-\Delta)^{1/2} \in OPS^1(M)\) with principal symbol given by \(\sqrt{h_{ij}\xi^i\xi^j}\), where \(h_{ij}\) are the metric coefficients.

**Theorem 9**

1. If \(A \in OPS^m(O)\), then the associated kernel distribution \(K_A\) is smooth everywhere off the diagonal in \(O \times O\).

2. If \(A \in OPS^{-\infty}(O)\), then \(K_A\) is smooth everywhere in \(O \times O\).

For a proof see [Jun96, Lemma 2.6].

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