ZERO CYCLES, MENNICKE SYMBOLS AND $K_1$-STABILITY OF CERTAIN REAL AFFINE ALGEBRAS

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ABSTRACT. Let $R$ be a reduced real affine algebra of (Krull) dimension $d \geq 2$ such that either $R$ has no real maximal ideals, or the intersection of all real maximal ideals in $R$ has a height of at least one. In this article, we prove the following: (1) the $d$-th Euler class group $E^d(R)$, defined by Bhatwadekar-R. Sridharan, is canonically isomorphic to the Levine-Weibel Chow group of zero cycles $\text{CH}_0(\text{Spec}(R))$; (2) the universal Mennicke symbol $\text{MS}_{d+1}(R)$ is canonically isomorphic to the universal weak Mennicke symbol $\text{WMS}_{d+1}(R)$; and (3) additionally, if $R$ is a regular domain, then the Whitehead group $\text{SK}_1(R)$ is canonically isomorphic to $\text{SK}_{d+1}(R)_{\text{w}}$. As an application, we investigate some Eisenbud-Evans type theorems.

1. INTRODUCTION

This article investigates a question of M. P. Murthy on zero cycles [35, 2.12], the canonical map between the top-length universal weak Mennicke symbol and universal Mennicke symbol, and the stabilization problem for the groups $K_1$ and $K_1\text{Sp}$ over certain (mostly singular) real affine algebras. The real affine algebras that are primarily focused on in this article, denoted by the symbol $R$ (throughout the article), satisfy one of the following conditions.

P-1 It has no real maximal ideals, or

P-2 the intersection of all real maximal ideals has a height of at least 1.

A large chunk of examples of such real affine algebras are motivated from algebraic geometry. Consider a real affine algebra $A$ such that the closure of the set of all $\mathbb{R}$-rational points in $\text{Spec}(A)$ has dimension $< \dim(\text{Spec}(A))$, then $A$ satisfies P-2. For a concrete example consider the variety $\text{Spec}(A)$, where $A = \mathbb{R}[X_0, \ldots, X_d]/(\sum_{i=0}^d X_i^2)$. Then the only real maximal ideal in $A$ is the maximal ideal corresponding to the origin. This class of real affine algebras is considered in [37], [36], and [35]. One of the primary objectives of this article is to extend their study into the realm of algebra. The main results of this article have three distinct themes, deeply interconnected by the divisibility property of some lower $K$-groups of certain curves. We discuss each of them separately below.

1.1. Zero cycles and projective modules. In geometry, the question of determining the precise obstruction for a vector bundle $\mathcal{E}$ on a (connected) affine variety $X$, which can determine whether $\mathcal{E}$ admits a nowhere vanishing section, still remains open. Whenever $\text{rank}(\mathcal{E}) > \dim(X)$, J-P. Serre proved in [41] that such a nowhere vanishing section will always exist. However, literature provides examples in which J-P. Serre’s theorem fails when considering the case $\text{rank}(\mathcal{E}) \leq \dim(X)$. It is evident that, among

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the remaining cases, the only practically feasible scenario tackled so far, to some extent, is the top case where \( \text{rank}(E) = \dim(X) \).

In simple algebraic terms, for a commutative noetherian ring \( A \) of dimension \( d \), the study of a finitely generated projective \( A \)-module \( P \) of rank \( d \), along with an obstruction class that can detect the splitting behavior for \( P \), remains absent to date, without any further assumption. That being said, a tremendous amount of research has been conducted on this problem since the late 1980s. In [35], M. P. Murthy settled this question for smooth affine variety \( A \) over algebraically closed fields by showing that the top Chern class \( c_d(P) \) in \( \text{CH}_0(\text{Spec}(A)) \) governs the splitting behavior for \( P \). Here \( \text{CH}_0(\text{Spec}(A)) \) is the Chow group of zero cycles on \( \text{Spec}(A) \) modulo rational equivalence, as defined by M. Levine and C. Weibel in [31]. Recently, it has been established in [29] that the hypothesis ‘smoothness’ in [35] can be removed. In [16], J.-L. Colliot-Thélène and C. Scheiderer studied the Chow groups of a real affine variety \( A \). Although over an arbitrary base field, the vanishing of the top Chern class is a necessary condition for \( P \) to split into a free factor, it is not sufficient, even for smooth real varieties. The study of a sufficient condition on a smooth real affine variety for the splitting problem of top rank projective modules via its top Chern class was initiated in [10]. It took a deep understanding of two other groups, namely the Euler class group \( E^d(A) \) and the weak Euler class group \( \tilde{E}^d(A) \) defined in [11] to completely solve the problem in [8]. A diligent reader might point out that while dealing with smooth real varieties the following theorem plays a crucial step in this development.

Theorem 1.1. [10] Theorem 5.5] Let \( \text{Spec}(A) \) be a smooth affine variety of dimension \( d \geq 2 \) over \( \mathbb{R} \). Then the canonical surjective map \( E^d_0(A) \to \text{CH}_0(\text{Spec}(A)) \) is an isomorphism.

Therefore, the following weaker question naturally arises when dealing with singular real affine algebras. Later in this subsection, we briefly provide some hints on how the following question plays a crucial role in understanding many open questions in literature.

**Question 1.2.** Let \( A \) be a reduced real affine algebra of dimension \( d \geq 2 \). Is the canonical surjective map \( E^d_0(A) \to \text{CH}_0(\text{Spec}(A)) \) an isomorphism?

The above question remains open, except in the smooth case. Recently, A. Asok and J. Fasel proved in [1] that the aforementioned question has an affirmative answer for a smooth affine scheme over an infinite perfect field of characteristic \( \neq 2 \). In Theorem 3.7 we prove the following.

**Theorem 1.3.** The canonical surjective map \( E^d_0(R) \to \text{CH}_0(\text{Spec}(R)) \) is an isomorphism, where \( \dim(R) = d \geq 2 \).

We give a sketch of our approach. The first step of the proof is to show that the group \( E^d(R) \) is divisible [Proposition 3.3]. To prove this, we utilize another variant of Euler class groups \( E^d(R) \) and \( \tilde{E}^d(R) \) [see 2.5], which are closely related to [10] Definition 2.1] and [29 7.2]. The notation \( \mu(-) \) stands for the minimal number of generators. In the second step, we employ the divisibility of \( E^d(R) \) to establish that any reduced ideal \( J \subset R \) such that \( \text{ht}(J) = \mu(J/J^2) = d \) is a surjective image of a projective \( R \)-module \( P \) of rank \( d \). Furthermore, we compute its Chern class with respect to an ideal \( K \) supported by only smooth complex maximal ideals [for details we refer to Proposition 3.6]. In the
third step, with the help of [16] and [29] we show that the cycle \([ K] \) associated to \( R/K \) in \( \text{CH}_0(\text{Spec}(R)) \) is not a torsion element. Then applying the second step it follows that the canonical map is an isomorphism.

As two of the interesting consequences, we (i) prove an analogy to A. A. Rojtman’s theorem [40], and (ii) provide an affirmative answer to [17, Question 2]. In the following, we describe a few of these consequences [for details we refer to [8,11]].

**Theorem 1.4.** Let \( \dim(R) = d \geq 2 \), and let \( P \) be a projective \( R \)-module of rank \( d \). Let \( I \subset R \) be an ideal of height \( d \) such that \( \mu(I/I^2) = d \). Then the following are true.

1. \( \text{CH}_0(\text{Spec}(R)) \) is torsion free.
2. Suppose that \( f : P/I \to I/I^2 \) is a surjective map. Then there exists a surjective lift \( f : P \to I \) of \( f \) if and only if \( e(P) = (I) \) in \( \text{E}^0_d(R,L) \), where \( \wedge^d P = L \).
3. Let \( J \subset R[T] \) be an ideal such that \( \mu(J/J^2) = \text{ht}(J) = d \). Then there exists a projective \( R[T] \)-module \( P \) of rank \( d \) such that \( J \) is a surjective image of \( P \).
4. Let \( (R) \) be the ring obtained from \( R[T] \) by inverting all monic polynomials in \( R[T] \). Then for all \( d \geq 3 \) the canonical map \( \Gamma : E^d(R[T]) \to E^d(R[T]) \) is injective.

**1.2. Mennicke Symbols.** The study of the elementary orbit space of unimodular rows \( \text{Um}_{d+1}(A) \) is well-documented in the literature, where \( A \) is a ring of dimension \( d \geq 2 \). L. N. Vaserstein, for \( d = 2 \) [51, Section 5] and W. van der Kallen, for \( d \geq 3 \) [46], have shown that the orbit space \( \text{Um}_{d+1}(A) \) possesses an abelian group structure. Furthermore, it is proven in [47] that the group \( \text{Um}_{d+1}(A) \) coincides with the universal weak Mennicke symbol \( \text{WMS}_{d+1}(A) \) of length \( d + 1 \). From the definition, one may observe that there exists a canonical surjection \( f : \text{WMS}_{d+1}(A) \to \text{MS}_{d+1}(A) \), where \( \text{MS}_{d+1}(A) \) denotes the universal Mennicke symbol of length \( d + 1 \). The existence of Bass-Kubota theorem in one-dimensional rings [46 Theorem 2.12], and the inductive approach of W. van der Kallen, as used in establishing the group structure in \( \text{Um}_{d+1}(A) \), naturally leads to the following question: Is \( f \) an isomorphism? An example over real affine algebras provides evidence that this is not the case in general (cf. [46 4.17] and [53 Example 2.2(c)]).

The group \( \text{WMS}_{d+1}(A) \) is said to have a *nice* group structure if for any two unimodular rows \( (a, a_1, \ldots, a_d) \) and \( (b, a_1, \ldots, a_d) \) of length \( d + 1 \) the following holds.

\[
[(a, a_1, \ldots, a_d)] \star [(b, a_1, \ldots, a_d)] = [(ab, a_1, \ldots, a_d)]
\]

Here \([ - ]\) denotes elementary orbit space of \([ - ]\). One may observe that \( f \) is an isomorphism if and only if \( \text{WMS}_{d+1}(A) \) has a nice group structure (cf. [22 Theorem 2.1]).

The injectivity of the map \( f \) has been studied across various algebras in a series of articles, including [22], [19], [24], [27] and [28]. In Theorem 1.1 we prove that the group \( \text{WMS}_{d+1}(R) \) has a nice group structure, where \( \dim(R) = d \geq 2 \). As a consequence, in Corollary 1.3 we show that \( \text{WMS}_{d+1}(R) \) is a divisible group.

**1.3. Stability for \( K_1 \) and \( K_1 \text{Sp} \) groups.** We begin this part by recalling the stabilization problem for the Whitehead group \( \text{SK}_1(A) \), where \( A \) is a ring. It follows from the definition of \( \text{SK}_1 \) that there exists a canonical map \( \Gamma_n : \text{SK}_1(A) \to \text{SK}_1(A) \), for all \( n \geq 2 \). The stabilization problem for the Whitehead group \( \text{SK}_1(A) \) asks the following.
**Question 1.5.** Given a ring $A$ of dimension $d \geq 2$, what is the least positive integer $n$ for which the maps $\Gamma_{n+i}$ become injective (similarly surjective) for all $i \geq 0$?

A similar question can be asked for the symplectic group $K_1\text{Sp}$ with the appropriate adjustments. For an arbitrary commutative noetherian ring, the stability for both groups $K_1$ and $K_1\text{Sp}$ were established by L. N. Vaser˘stein in [48], [49] and [50]. One can construct examples of smooth real varieties, which can establish that L. N. Vaser˘stein’s bounds are optimal in general (cf. [47, Proposition 7.10], [38], [20]). However, it was proved in [39], [5] and [4] that L. N. Vaser˘stein’s injective stability bound for both $K_1$ and $K_1\text{Sp}$ can be improved for regular affine algebras over a field $k$ of characteristic zero such that the cohomological dimension of $k$ is at most one (the condition on the base field $k$ is more general there, see the condition in [39, Proposition 3.1]). We prove analogies of their theorems in the following form [for details we refer to Section 5].

**Theorem 1.6.** Additionally, if we assume that $R$ is a regular domain, and let $I \subset R$ be a principal ideal, then the following are true.

1. $\text{SK}_1(R, I) \simeq \frac{\text{SL}_{n+1}(R, I)}{\text{E}_n(R, I)}$, for all $n \geq d + 1$.
2. The canonical map $\frac{\text{Sp}_{2n}(R, I)}{\text{ESp}_{2n}(R, I)} \to K_1\text{Sp}(R, I)$ is injective, provided if $d$ is even then $4$ divides $d$, for all $n \geq \lceil \frac{d+1}{2} \rceil$.

1.4. **Some Eisenbud-Evans type theorems.** Section [6] is an extension of Section [3]. Here we investigate relations between some Eisenbud-Evans type theorems as studied in [34]. Let $A$ be a ring. Recall that the group $\text{F}^dK_0(A)$, which is the subgroup of $K_0(A)$ generated by the images of all $[A/I]$, where $I$ is a locally complete intersection ideal of height $d$. One of the main results in Section [6] is Theorem [6.2] where we prove that $\text{F}^dK_0(R)$ is canonically isomorphic with $\text{E}^d(R)$. This provides an affirmative answer to [35, 2.12] for $R$.

1.5. **Notations.** Unless otherwise stated, all rings considered in this article are assumed to be commutative noetherian with $1(\neq 0)$, and all modules are assumed to be finitely generated. Any projective module is assumed to have a constant rank. We denote the vector $(1, \cdots, 0)$ as $e_1$. For a ring $A$, the set MaxSpec$(A)$ is the collection of all maximal ideals in $A$. Throughout the article the symbol $R$ will always represent a real affine algebra of dimension $d \geq 2$ satisfying one of the following conditions:

P-1 it has no real maximal ideals;

P-2 the intersection of all real maximal ideals has a height of at least 1.

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2. Preliminaries

This section is devoted to recollecting several results and definitions from the literature that serve as prerequisites for the main theorems in this article. In some cases, we modify or restate well-known results slightly to better suit our requirements.

2.1. Stably free modules. The purpose of this subsection is to give the proof of Theorem 2.8 We believe Theorem 2.8 must be well known due to A. A. Suslin [43] and others (see [56]). However, we could not find a complete reference. We begin with the following definition.

Definition 2.1. Let $A$ be a ring and $M$ be an $A$-module. The order ideal of an element $m \in M$ is defined by $O_M(m) := \{ \alpha(m) \in A : \alpha \in M^* = \text{Hom}_R(M,A) \}$. An element $m \in M$ is called a unimodular element if $O_M(m) = A$. The set of all unimodular elements of $M$ is denoted by $\text{Um}(M)$. If $M = A^n$, then we write $\text{Um}_n(A)$ instead of $\text{Um}(A^n)$.

We recall Swan’s version of Bertini theorem from [44, Theorem 1.3].

**Theorem 2.2.** Let $V = \text{Spec}(A)$ be a smooth affine variety over an infinite field $k$. Let $Q$ be a projective $A$-module of rank $r$ and $(q,a) \in \text{Um}(Q \oplus A)$. Then there is a $y \in Q$ such that the ideal $I := O_Q(q + ay)$ has the following properties.

1. The subscheme $U = \text{Spec}(A/I)$ of $V$ is smooth over $k$ and $\dim(U) = \dim(V) - r$, unless $U = \phi$;
2. If $\dim(U) \neq 0$ then $U$ is a variety.

The next two results follow from a theorem due to A. A. Suslin [43] and their proofs can be found in [56] Propositions 3 and 4).

**Proposition 2.3.** Let $C$ be a smooth real curve having no real maximal ideal. Then $\text{SK}_1(C)$ is a divisible group.

**Proposition 2.4.** Let $C$ be as in Theorem 2.3 Then the natural map $K_1\text{Sp}(C) \rightarrow \text{SK}_1(C)$ is an isomorphism.

Definition 2.5. Let $A$ be a ring. The elementary group $E_n(A)$ is the subgroup of $\text{GL}_n(A)$ generated by the matrices $E_{ij}(\lambda) = I_n + e_{ij}(\lambda)$, where $i \neq j$, and $e_{ij}(\lambda)$ is the matrix with only possible non-zero entry $\lambda$ is at the $(i,j)$-th position.

The proof of the following lemma can be found in [43, Corollary 2.3].

**Lemma 2.6.** Let $A$ be a ring and $n \geq 2$. Suppose $(a_0, \ldots, a_n) \in \text{Um}_{n+1}(A)$ satisfies $\dim(A/(a_2, \ldots, a_n)) \leq 1$ and $\dim(A/(a_3, \ldots, a_n)) \leq 2$. Furthermore, assume that there exists $\alpha \in \text{SL}_2(A/(a_2, \ldots, a_n)) \cap \text{ESp}(A/(a_2, \ldots, a_n))$ such that $(\overline{a_0}, \overline{a_1})\alpha = (\overline{b_0}, \overline{b_1})$. Then, there exists $\gamma \in E_{n+1}(A)$ such that $(a_0, \ldots, a_n)\gamma = (b_0, b_1, a_3, \ldots, a_n)$.

The next proposition is based on a clever observation of P. Raman, that one may avoid singularities on A. A. Suslin’s proof of [43, Theorem 2.4]. This proposition plays a crucial role in this article. Hence, we give a sketch of the proof.

**Proposition 2.7.** Let $A$ be a reduced affine algebra over a perfect field $k$ of dimension $n$, and let $v = (v_0, \ldots, v_n) \in \text{Um}_{n+1}(A)$. Assume that $S \subset \text{MaxSpec}(A)$ such that the ideal $I := \bigcap_{m \in S} m$, has height $\geq 1$. Then there exists a matrix $\epsilon \in E_{n+1}(A)$ such that if we take $((u_0, \cdots, u_n) = )u = v \epsilon$ then
(1) $A/\langle u_0, \cdots, u_{i-1} \rangle$ is a smooth affine $k$-algebra of dimension $n - i$, and

(2) $\text{MaxSpec}(A/\langle u_0, \cdots, u_{i-1} \rangle) \cap S = \emptyset$, for all for any $1 \leq i \leq n$.

Additionally, we may assume that $A/\langle u_0, \cdots, u_{i-1} \rangle$ is a smooth affine domain if $i < n$.

Proof. Since $A$ is reduced, and $k$ is a perfect field, the ideal defining the singular locus of $A$, say $J$, has height $\geq 1$. Let $I = T \cdot J$, then by our hypothesis we obtain that $ht(I) \geq 1$. Hence, going modulo $I$ and using standard stability arguments (such as the Prime avoidance lemma), one can find an $\omega \in E_n^{n+1}(A/I)$ such that $v \cdot \omega \equiv e_1 \mod I$.

We choose a lift $\Omega \in E_n^{n+1}(A)$ of $\omega$. Then we have $v \cdot \Omega = w = (w_0, \cdots, w_n)$, where $1 - w_0 \in I$ and $w_i \in I$ for all $i \geq 1$. Therefore, it is enough to prove the theorem for $w$.

So without loss of generality, we may begin with the assumption that $1 - v_0 \in I$ and $v_i \in I$ for $i \geq 1$.

Also, we observe that it is enough to prove the theorem for $i = 1$, as then we can repeat the same steps on $A/\langle v_0 \rangle$ to get the result on $A/\langle v_0, \cdots, v_{i-1} \rangle$. Moreover, since the completion is elementary we can always come back to our initial ring $A$.

Furthermore, the fact $1 - v_0 \in I \subset I$ will ensure that $S \cap \text{MaxSpec}(A/\langle v_0, \cdots, v_{i-1} \rangle) = \emptyset$.

Hence, in the remaining part, we give the proof for $i = 1$ only.

Applying Theorem 2.2, we get $\lambda_j \in A$, for $j = 1, \cdots, n$, such that if we replace $v_0$ by $v'_0 = v_0 + \sum \lambda_j v_j$, then $\text{Spec}(A/\langle v'_0 \rangle)$ is a smooth variety outside the singularities of $A$ such that $\dim(A/\langle v'_0 \rangle) = n - 1$. As $v_i \in I$ for $i \geq 1$, we have $v'_0 - 1 \in I$. Since $A/\langle v'_0 \rangle$ is a domain (in particular reduced), the set of all singular points in $\text{Spec}(A/\langle v'_0 \rangle)$ forms a closed set. Let $J$ be the ideal defining the singular locus of $A/\langle v'_0 \rangle$. Since $A/\langle v'_0 \rangle$ is smooth outside the singularities $A$ we have $T \subset J$, where ‘bar’ denotes going modulo $\langle v'_0 \rangle$. But $v'_0 - 1 \in I$ gives us the fact that $\overline{I} = A/\langle v'_0 \rangle$. This implies $A/\langle v'_0 \rangle$ is smooth. So by taking $u = (v'_0, v_1, \cdots, v_n)$ the proof completes.

Now we are ready to prove the main theorem of the subsection, which follows as a consequence of previous results.

**Theorem 2.8.** Any unimodular rows in $R$ of length $d + 1$ can be completable to the first row of an invertible matrix. As a consequence, any stably free $R$-modules of rank $d$ are free.

Proof. First we note that if $d < 2$, then there is nothing to prove. Hence, we may assume that $d \geq 2$. Note that, if $R$ satisfies P-2, then this is done in [37, Theorem 3.2].

Therefore, it is enough to assume that $R$ is a real affine algebra having no real maximal ideal. Let $v = (v_0, \cdots, v_d) \in \text{Um}_{d+1}(R)$. Using Lemma 2.7 we may assume that $C := R/\langle v_0, \cdots, v_{d-2} \rangle$ is a smooth curve. Let ‘bar’ denote going modulo $\langle v_0, \cdots, v_{d-2} \rangle$.

By Theorem 2.3 the group $\text{SK}_1(C)$ is divisible, there exists a $\sigma \in \text{SL}_2(C) \cap E_3(C)$ such that $\sigma(\overline{v}_0, \overline{v}_1) = (\overline{v}^d, b)$. Applying Theorem 2.4 we get $\sigma \in \text{SL}_{d}(C) \cap \text{ESp}(C)$.

Now one may use Lemma 2.6 to find an $\epsilon \in E_{d+1}(R)$ such that $\bar{v} = (a^d, b, v_3, \cdots, v_d)$. It follows from [42] that the unimodular row $(a^d, b, v_3, \cdots, v_d)$ is completable to the first row of an invertible matrix.

**Remark 2.9.** For any unimodular row $w = (w_0, \cdots, w_d) \in \text{Um}_{d+1}(A)$, we will call $(w_0, \cdots, w_{r-1}, w^d)$ as the factorial row of $w$ and will be denoted as $\chi_{r!}(w)$. It follows from the proof of Theorem 2.8 that, for any $v \in \text{Um}_{d+1}(R)$ there exist an $\epsilon \in E_{d+1}(R)$ and $w \in \text{Um}_{d+1}(R)$ such that $\bar{w} = \chi_{r!}(w)$.

2.2. Cancellation of projective modules. Let $A$ be a ring. Recall that a projective $A$-module $P$ of rank $n$ is said to be cancellative if $P \oplus A^r \cong P' \oplus A^r$ implies $P \cong P'$,
where $P'$ is another $A$-module. After Theorem 2.8, a natural question arises: whether a projective $R$-module of rank $d$ is cancellative. It turns out that this question has an affirmative answer. Again we believe this result is well-known. However, we were unable to find a suitable reference for the exact following version. When $R$ satisfies condition P-2, then the result follows from [32, Theorem 3.2]. In the following theorem, we show that if $R$ satisfies P-1, then one can still follow the proof of Theorem 2.8 with some suitable adjustments. For the sake of completeness, we give a sketch. Before that, we recall the following definition.

**Definition 2.10.** Let $A$ be a ring. Let $P$ be a projective $A$-module such that $\text{Um}(P) \neq \emptyset$. We choose $\phi \in P^*$ and $p \in P$ such that $\phi(p) = 0$. We define an endomorphism $\phi_p$ as the composite $\phi_p : P \to A \to P$, where $A \to P$ is the map sending $1 \to p$. Then by a transvection we mean an automorphism of $P$, of the form $1 + \phi_p$, where either $\phi \in \text{Um}(P^*)$ or $p \in \text{Um}(P)$. By $E(P)$ we denote the subgroup of $\text{Aut}(P)$ generated by all transvections. 

**Remark 2.11.** If $P$ is a free $A$-module of rank $n \geq 3$, then $E(P)$ coincides with $E_n(A)$ [2, Lemma 2.20].

**Theorem 2.12.** Let $P$ be a projective $R$-module of rank $d$. Then $P$ is cancellative.

Proof. It is sufficient to assume that $R$ satisfies P-1. Let us choose $(a, p) \in \text{Um}(R \oplus P)$. We will show that there exists $\sigma \in \text{Aut}(A \oplus P)$ such that $\sigma(a, p) = (1, 0)$. Furthermore, without loss of generality, we may assume that $R$ is reduced. Let $J$ be the ideal defining the singular locus of $R$, then $\text{ht}(J) \geq 1$. There exists a non-zero divisor $t \in J$ such that $P_t$ is free. Let $s = t^l$ be such that $sP \subset R^d$, where $l \in \mathbb{N}$.

Since $s$ is a non-zero divisor, applying a classical result due to Bass ([3] or see [6, Proposition 2.13]), we obtain that the canonical map $\text{Um}(R \oplus P) \to \text{Um}(\frac{R}{t^l} \oplus \frac{P}{t^l})$ is surjective. Therefore, without loss of generality, we may assume that $a - 1 \in \langle s \rangle$ and $p \in sP \subset R^d$. Hence, we may take $p = (a_1, \ldots, a_d) \in R^d$.

Using Lemma 2.7 we may further assume that $B = A/\langle a_1, \ldots, a_d \rangle$ is a smooth affine domain of dimension one. Since $P_t$ is free and $a - 1 \in \langle t \rangle$ we obtain that the module $\frac{P}{\langle a_1, \ldots, a_d \rangle}P_t$ is a free $B$-module. Now we may follow the arguments given in the proof of Theorem 2.8 to obtain a $\gamma \in E_d(R/\langle a \rangle) = E(P/ap)$ (since $P/ap$ is free) such that $\gamma(\tilde{a}_1, \ldots, \tilde{a}_d) = (\tilde{a}_1, \ldots, \tilde{a}_{d-2}, \tilde{b}_{d-1}, \tilde{b}_d^d)$, where ‘tilde’ denotes going modulo $\langle a \rangle$. Since $\langle a, s \rangle = R$ and $sP \subset R^d$, we obtain that $\tilde{P} = (\frac{a}{a})^d$. In particular, we get the equality $\text{Um}(\tilde{P}) = \text{Um}_d(R/\langle a \rangle)$. Applying [6, Proposition 2.12] we can lift $\gamma \in E_d(R/\langle a \rangle)$ to an $\alpha \in \text{Aut}(P)$. Therefore, we obtain that $\alpha p \equiv (a_1, \ldots, a_{d-2}, b_{d-1}, b_d^d)$ mod $aP$.

We define $\sigma := \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$. Moreover, as $p \in sP \subset R^d$, we get the following.

$$(a, p) \equiv (a, \alpha p) \equiv (a, a_1, \ldots, a_{d-2}, b_{d-1}, b_d^d) \mod E(R \oplus P).$$

Since factorial rows are completable by [42], one may conclude the proof (cf. [6, Theorem 4.1]).
2.3. Excision ring and relative cases. The purpose of this section is to state and prove a relative version of Theorem 2.8, which will be needed to improve the injective stability of $SK_1$ in Section 5. We begin by recalling some interesting facts about the excision ring (for the definition, we refer to Definition 2.5]). Let $A$ be a ring. For any ideal $I \subset A$, the excision ring $A \oplus I$ can be viewed as a fiber product of $A$ with respect to the ideal $I$. Therefore, if $A$ is an affine algebra over a field $k$, then $A \oplus I$ is also an affine algebra over $k$ [25 Proposition 3.1]. We denote $\pi_2 : A \oplus I \rightarrow A$ by $(r, i) \mapsto r + i$. The following lemma ensures that the excision ring $R \oplus I$ enjoys similar properties as $R$.

Lemma 2.13. Let $I \subset R$ be an ideal. The excision algebra $R \oplus I$ satisfies one of the following:

(i) If $R$ satisfies P-1, then $R \oplus I$ is also a real affine algebra of dimension $d$ satisfies P-1.
(ii) If $R$ satisfies P-2, then $R \oplus I$ is also a real affine algebra of dimension $d$ satisfies P-2.

Proof. Firstly, we point out that $R \oplus I$ is also a real affine algebra of dimension $d$, and the extension $R \rightarrow R \oplus I$ is an integral extension, for a proof one may see [25 Proposition 3.1]. Therefore, only need to show that if $R$ satisfies P-i, then so does $R \oplus I$, for $i=1,2$. We show this in separate cases.

Let $R$ satisfies P-1, and let $\mathfrak{M} \subset R \oplus I$ be a maximal ideal. Then $\frac{R}{\mathfrak{M}} \cong \mathfrak{M} \oplus I$ is an integral extension. This implies that $R \cap \mathfrak{M}$ is a maximal ideal in $R$. Since $R$ has no real maximal ideal, we have $\frac{R}{\mathfrak{M}} \cong \mathbb{C}$. Hence, $\frac{R}{\mathfrak{M}} \oplus I$ is also isomorphic to $\mathbb{C}$.

Let $R$ satisfy P-2, and let $\mathfrak{J}$ be the intersection of all real maximal ideals in $R \oplus I$. Let $\mathfrak{M}$ be a real maximal ideal in $R \oplus I$. Since $R \rightarrow R \oplus I$ is an integral extension, we have $\frac{R}{\mathfrak{M}} \cong \mathbb{R}$. This fact combined with the fact that the intersection of all real maximal ideals of $R$ has height $\geq 1$ implies that $ht(\mathfrak{J} \cap R) \geq 1$. Let $a \in \mathfrak{J} \cap A$ be a non-zero divisor. Then the element $(a,0) \in \mathfrak{J}$ is a non-zero divisor in $R \oplus I$. This concludes the proof.

Let us recall some definitions in the relative case.

Definition 2.14. Let $A$ be a ring $A$, and let $I \subset A$ be an ideal. We define the following.

(1) $Um_n(A, I) := \{v \in Um_n(A) : v \equiv e_1 \mod I\}$. We shall call $Um_n(A, I)$ the relative unimodular rows of length $n$ with respect to the ideal $I$.
(2) $SL_n(A, I) := \{\alpha \in SL_n(A) : \alpha \equiv I_n \mod I\}$.
(3) For all $n \geq 2$, the group $E_n(A, I)$ is the smallest normal subgroup of $E_n(A)$ containing the elements $E_{2n}(x)$, where $x \in I$ (see [46 2.1]).

Now we are ready to state and prove the main result of this subsection. The proof essentially uses the idea of [39 Proposition 3.3].

Proposition 2.15. Let $v \in Um_{d+1}(R, I)$, where $I = \langle a \rangle$ for some non-zero divisor $a \in R$. Then $v$ can be completed to a matrix $\sigma \in SL_{d+1}(R, I)$.

Proof. Let $v = (1 + av_1, av_2, \ldots, av_{d+1})$. Since $v \equiv e_1 \mod \langle a \rangle$, there exists $u \in Um_{d+1}(R)$ such that $vu^t = 1$ and $u \equiv e_1 \mod \langle a \rangle$. Let $u = (1 + av_1, av_2, \ldots, av_{d+1})$. Since $a$ is a non-zero divisor we obtain that $u_1 + v_1 = -a \sum_{i=1}^{d+1} u_i v_i$. Let $B = R[T]/\langle T^2 - aT \rangle$, and let ‘bar’ denote going modulo $\langle T^2 - aT \rangle$. We take $v(T) = (1 + v_1 T, v_2 T, \cdots, v_{d+1} T)$ and $u(T) = (1 + u_1 T, u_2 T, \cdots, u_{d+1} T)$. Then we get the following.

$$v(T)u(T)^t = 1 + T(u_1 + v_1) + T^2 \left(\sum_{i=1}^{d+1} u_i v_i\right) = 1 + T(T - a) \left(\sum_{i=1}^{d+1} u_i v_i\right) \equiv 1 \mod \langle T^2 - aT \rangle$$
This shows that \( v(T) \in Um_{d+1}(B) \). We observe that \( B \cong R \oplus (a) \) \cite[Corollary 3.2]{25}. Hence, by Lemma \ref{lemma2} it follows that \( B \) also satisfies one of the conditions D-I or D-2. Therefore, applying Theorem \ref{thm2} there exists \( \alpha(T) \in SL_{d+1}(B) \), such that \( v(T) = e_1 \alpha(T) \). Let \( \sigma = \alpha(0)^{-1} \alpha(a) \). Then \( e_1 \sigma = v \), and \( \sigma \equiv I_{d+1} \mod \langle a \rangle \).

The following lemma can be proven using exactly the same arguments given in \cite[Theorem 4.4]{23}. One just needs to use Remark \ref{remark2} in place of \cite[Lemma 4.1]{23} in their proof. Hence, we skip the proof.

**Lemma 2.16.** Let \( a \in R \) be a non-zero divisor and let \( I = \langle a \rangle \). Let \( v \in Um_{d+1}(R, I) \). Then there exist \( e \in Um_{d+1}(R, I) \) and \( e \in E_{d+1}(R, I) \) such that \( we = \chi_{d}(w) \).

**2.4. Mennicke symbols.** In this subsection, we briefly recall the Mennicke symbols and the weak Mennicke symbols. We begin with the following definitions.

**Definition 2.17.** Let \( A \) be a ring. A Mennicke symbol of length \( n + 1 \geq 3 \), is a pair \((\psi, G)\), where \( G \) is a group and \( \psi : Um_{n+1}(A) \to G \) is a map such that:

1. \( \psi((0, ..., 0, 1)) = 1 \) and \( \psi(v) = \psi(ve) \) for any \( e \in E_{n+1}(A) \);
2. \( \psi((b_1, ..., b_n, x)\psi((b_1, ..., b_n, y)) = \psi((b_1, ..., b_n, xy)) \) for any two unimodular rows \( (b_1, ..., b_n, x) \) and \( (b_1, ..., b_n, y) \).

**Definition 2.18.** Let \( A \) be a ring. A weak Mennicke symbol of length \( n + 1 \geq 3 \) is a pair \((\psi, G)\), where \( G \) is a group and \( \phi : Um_{n+1}(A) \to G \) is a map such that:

1. \( \phi(1, 0, ..., 0) = 1 \) and \( \psi(v) = \phi(ve) \) if \( e \in E_{n+1}(A) \);
2. \( \phi(a, a_1, ..., a_n)\phi(1 - a, a_1, ..., a_n) = \phi(a(1 - a), a_1, ..., a_n) \) for any unimodular row \((a, a_1, ..., a_n)\) such that \((1 - a, a_1, ..., a_n)\) is also unimodular.

Let \( A \) be a ring of dimension \( n \). Clearly, a universal Mennicke symbol \((ms, MS_{n+1}(A))\) and a universal weak Mennicke symbol \((wms, MS_{n+1}(A))\) exist. With an abuse of notation, we only use the notations \( MS_{n+1}(A) \) and \( WMS_{n+1}(A) \). W. van der Kallen defined an abelian group structure in \cite{46} on the elementary orbit space of unimodular rows \( Um_{n+1}(A) \). Moreover, in \cite{47} it was shown that the group \( Um_{n+1}(A) / wms(A) \) coincides with the group \( WMS_{n+1}(A) \). Hence, we shall stick to the notation \( WMS_{n+1}(A) \) only.

**2.5. Euler class groups.** In this subsection, we give a slightly alternative definition of top Euler class groups for \( R \). Our definition is motivated from \cite[Definition 2.1]{10} and \cite[7.2]{29}. In Section 3, we show that our definition matches with the Euler class groups defined in \cite{11} for \( R \).

Additionally, let \( R \) be reduced. Let \( I \subset R \) be an ideal such that \( \mu(I/I^2) = \text{ht}(I) = d \). Let \( L \) be a rank one projective \( R \)-module. Let \( \alpha, \beta : (L/IL) \oplus (R/I)^{d-1} \to I/I^2 \) be two surjections. We say \( \alpha \) and \( \beta \) are related if there exists \( \sigma \in E((L/IL) \oplus (R/I)^{d-1}) \) such that \( \alpha \sigma = \beta \). This defines an equivalence relation on the set of all surjections \( (L/IL) \oplus (R/I)^{d-1} \to I/I^2 \). Let \([\alpha] \) denote the equivalence class of \( \alpha \). We will call \([\alpha] \) a “local \( L \)-orientation” of \( I \). With abuse of notation sometimes we will call \( \alpha \) a local \( L \)-orientation of \( I \) instead of \([\alpha] \). The local \( L \)-orientation \([\alpha] \) is said to be a “global \( L \)-orientation” of \( I \), if there exists a surjective map \( \Gamma : L \oplus R^{d-1} \to I \) such that \( \Gamma \circ R/I = \alpha \). In this situation, we call \( \Gamma \) a surjective lift of \( \alpha \). Notice that, since the canonical map \( E(L \oplus R^{d-1}) \to E((L/IL) \oplus (R/I)^{d-1}) \) is surjective, if \( \alpha \) has a surjective lift, then so has any \( \beta \) equivalent to \( \alpha \). Whenever \( L \cong R \), we will say a local orientation (respectively global orientation) of \( I \) instead of an \( R \)-local orientation (respectively \( R \)-global orientation) of \( I \).
Let \( \tilde{G} \) be the free abelian group on the set \( B \) of pairs \((m, \omega_m)\), where:

1. \( m \subset R \) is a smooth complex maximal ideal of height \( d \);
2. \( \omega_m : (L/mL) \oplus (R/m)\)(\(d-1\) \(\rightarrow m/m^2\)) is a local L-orientations of \( m \).

We shall say that an ideal \( J \subset R \) of height \( d \) is regular if it is of the form \( I = \cap_i m_i \), where each \( m_i \) is a smooth complex maximal ideal of height \( d \) with \( m_i \neq m_j \) for \( i \neq j \). Given a regular ideal \( J = \cap_i m_i \), we observe that there always exists a local L-orientation. Let \( \omega_J \) be a local L-orientation of \( J \). Then \( \omega_J \) gives rise, in a natural way, to a local L-orientation \( \omega_{m_i} \) of \( m_i \). We associate to the pair \((J, \omega_J)\), to the element \( \sum_i(m_i, \omega_{m_i}) \) of \( \tilde{G} \). By abuse of notation, we denote the element \( \sum_i(m_i, \omega_{m_i}) \) by \((J, \omega_J)\).

Let \( \tilde{H} \) be the subgroup of \( \tilde{G} \) generated by the set \( S \) of pairs \((J, \omega_J)\), where \( \omega_J \) is a global L-orientation of \( J \). Then the quotient group \( \tilde{G}/\tilde{H} \) is the “\( d \)-th Euler class group” of \( R \) with respect to \( L \), denoted as \( \tilde{E}^d(R, L) \). Whenever \( L \cong R \), we shall write \( \tilde{E}^d(R) \) instead of \( \tilde{E}^d(R, R) \).

Similarly, one can define the \( d \)-th weak Euler class group \( \tilde{E}^d_0(R, L) \) as \( \tilde{F}/\tilde{K} \), where \( \tilde{F} \) and \( \tilde{K} \) are as follows.

Let \( \tilde{F} \) be the free abelian group on the set of all smooth complex maximal ideals \( m \) of height \( d \). Let \( J = \cap_i m_i \) be a regular ideal. Then we associate \((J)\) to the element \( \sum_i(m_i) \) of \( F \). By abuse of notation, we denote the element \( \sum_i(m_i) \) by \((J)\).

Let \( \tilde{K} \) be the subgroup of \( \tilde{F} \) generated by \((J)\) such that \((J)\) has a global L-orientation. Whenever \( L \cong R \), we shall write \( E^d_0(R) \) instead of \( \tilde{E}^d_0(R, R) \).

We denote the Euler class groups and weak Euler class groups defined in [11] by \( E^d(R, L) \) and \( E^d_0(R, L) \) respectively. One may observe that from the universal property of quotient group there exist canonical homomorphisms \( i_R : \tilde{E}^d(R, L) \rightarrow F^d(R, L) \) and \( \phi_R : \tilde{E}^d_0(R, L) \rightarrow E^d_0(R, L) \).

3. ZERO CYCLES AND PROJECTIVE MODULES

Let \( R \) be reduced and let \( P \) be a projective \( R \)-module of top rank. One of the primary objectives of this section is to show that the vanishing of the top Chern class of \( P \) in the group \( CH_0(\text{Spec}(R)) \) dictates the splitting problem for \( P \). As a preparation, we prove various results that are interesting in their own right. We begin with a rephrased version of a moving lemma within our framework. This arises as a direct consequence of the Bertini-Murthy-Swan theorem [35 Corollary 2.6]. We will employ this lemma repeatedly throughout this section. Hence, we give a sketch.

**Lemma 3.1.** Additionally, let \( R \) be reduced. Let \( J \subset R \) be an ideal containing a non-zero divisor \( s \) such that \( \mu(J/J^2) = d \). Then there exists an ideal \( I \subset R \) such that

1. \( I \) is a product of finitely many distinct smooth complex maximal ideals of height \( d \),
2. \( I + JK = R \), for any ideal \( K \) such that \( \text{ht}(K) \geq 1 \), and
3. \( \mu(I \cap J) = d \).

Moreover, if \( \text{ht}(J) = d \), then one can choose \( I \) with the following additional properties.

4. Given any local L-orientation \( \omega_J \) of \( J \), we get a local L-orientation \( \omega_I \) of \( I \) and a local L-orientation \( \omega_{IJ} \) of \( IJ \) such that \((I, \omega_I) = (I, \omega_I) + (J, \omega_J) = 0 \) in \( E^d(R, L) \), where \( L \) is line bundle. Consequently, we get \((IJ) = (I) + (J) = 0 \) in \( E^d_0(R, L) \).
Proof. Let \( J = \langle f_1, \cdots, f_d \rangle + J^2 \). Let \( \mathfrak{J} \) be the ideal defining the singular locus of \( A \).

Let \( \mathfrak{J} \) be the intersection of all real maximal ideals in \( R \) if \( R \) satisfies P-2, else we take \( \mathfrak{J} = A \). Let \( \mathfrak{R} := \mathfrak{J} \cap \mathcal{K} \), then \( \text{ht}(\mathfrak{R}) \geq 1 \). Now we apply Bertini-Murthy-Swan theorem \cite[Corollary 2.6]{35} to get \( h_i \in \mathfrak{J} \) and an ideal \( I \), which is a product of distinct smooth maximal ideals of height \( d \) such that \( IJ = \langle h_i, \cdots, h_d \rangle \), \( I + J\mathfrak{R} = R \), and \( f_i - h_i \in J^2 \).

Moreover, since \( I + \mathfrak{J} = R \) the ideal \( I \) is only supported by smooth complex maximal ideals. Hence this proves (1), (2) and (3).

Now let us assume that \( \text{ht}(J) = d \). Let the local \( L \)-orientation \( \omega_J \) induce a map \( f : L \oplus R^{d-1} \to J \), such that \( f(L \oplus R^d) + J^2 = J \). Now, we repeat the arguments given in the first paragraph and obtain an ideal \( I \) which is a product of finitely many distinct smooth complex maximal ideals of height \( d \) and a surjection \( g : L \oplus R^{d-1} \to IJ \) with \( f \otimes R/J \equiv g \otimes R/J \) such that \( I + J\mathfrak{R} = R \). Let \( \omega_{IJ} \) be the global \( L \)-orientation of \( IJ \) induce by \( g \). Now since \( I + J = R \), by Chinese remainder theorem \( g \) will induce local \( L \)-orientations \( \omega_I \) of \( I \) and \( \omega'_J \) of \( J \). Moreover, as \( f \otimes R/J \equiv g \otimes R/J \), we obtain that \( (J, \omega_J) = (J, \omega'_J) \) in \( E^d(R, L) \). This in particular give us that \( (IJ, \omega_{IJ}) = (I, \omega_I) + (J, \omega'_J) = 0 \) in \( E^d(A, L) \). Now the remaining part of (4) follows from using the canonical group homomorphisms \( E^d(R, L) \to E^d_0(R, L) \). This concludes the proof.

The next proposition is on the divisibility of the Euler class group \( E^d(R) \). Before that, we recall the following computation from \cite[Lemma 4.1]{18}.

**Lemma 3.2.** Let \( I, J \subseteq R \) be two ideals of height \( d \), and let \( n \in \mathbb{N} \). Suppose that, there exist \( a_i \in R \) such that \( I = \langle a_1, \cdots, a_d \rangle + I^2 \) and \( J = \langle a_1, \cdots, a_{d-1} \rangle + I^n \). Let \( L \) be a projective \( R \)-module of rank one. Then \( (J) = n(I) \) in \( E^d_0(R, L) \).

**Proposition 3.3.** The Euler class group \( E^d(R) \) is divisible.

Proof. Note that the group \( E^d(R) \) is canonically isomorphic to the group \( E^d(R_{\text{red}}) \) \cite[Corollary 4.6]{11}. Hence, without loss of generality, we may assume that \( R \) is reduced.

Let \( L \) be a projective \( R \)-module of rank one. We divide the proof into the following steps.

**Step - 1.** In this step we show that the canonical map \( i_R : \tilde{E}^d(R, L) \to E^d(R, L) \) is an isomorphism.

Proof of Step - 1. We note that if \( (I, \omega_I) \in \tilde{E}^d(A, L) \) such that its image vanishes in \( E^d(A, L) \), then by \cite[Theorem 4.2]{11} it follows that \( \omega_I \) is a global \( L \)-orientation of \( I \). Implies that \( (I, \omega_I) = 0 \) in \( E^d(A, L) \). This shows that \( i_R \) is injective.

Let \( (J, \omega_J) \in E^d(R, L) \), where \( J \subseteq R \) is an ideal such that \( \mu(J/J^2) = \text{ht}(J) = d \). Then by Lemma 3.1 there exists an ideal \( I \) which is a product of finitely many distinct smooth complex maximal ideals of height \( d \) and a local \( L \)-orientation \( \omega_I \) of \( I \) such that \( (J, \omega_J) = -(I, \omega_I) \) in \( E^d(R, L) \). Since \( I \) is a product of distinct smooth complex maximal ideals we get \( (I, \omega_I) \in E^d_0(R, L) \). Implies that \( (J, \omega_J) \in E^d(R, L) \).

**Step - 2.** In this step we show that the canonical map \( \psi_R : \tilde{E}^d(R, L) \to E^d_0(R, L) \) sending \( (m, \omega_m) \mapsto (m) \) is an isomorphism, where \( m \subseteq R \) is a smooth complex maximal ideal of height \( d \).

Proof of Step - 2. We observe that by definition the canonical map \( \tau_R : E^d(R, L) \to E^d_0(R, L) \) is surjective. Moreover, again from the definitions it follows that we have the following commutative triangle.
\[ E^d(R, L) \xrightarrow{\tau_R} E^d_0(R, L) \]
\[ i_R \quad \psi_R \]
\[ \tilde{E}^d(R, L) \]

Since \( i_R \) is an isomorphism by Step - 1, we get \( \psi_R \) is surjective. Hence, we only prove the injective part.

Let \((I, \omega_I) \in \tilde{E}^d(R, L)\) be such that there exists a global \(L\)-orientation \(\omega'_I\) of \(I\). First, we show that \((I, \omega_I) = 0\) in \(\tilde{E}^d(R, L)\). Now two local \(L\)-orientations of \(I\) must differ by a unit in \(R/I\). Let \(\omega_I\) and \(\omega'_I\) differ by \(u \in (R/I)^*\). Since \(I\) is a product of finitely many distinct smooth complex maximal ideals, there exists \(v \in (R/I)^*\) such that \(u = v^2\). Since \(\omega'_I\) is a global \(L\)-orientation, applying [11, Lemma 5.3] we conclude that \((I, \omega_I) = 0\) in \(E^d(R, L)\). Therefore, by [11, Theorem 4.2] it follows that \(\omega_I\) is a global \(L\)-orientation of \(I\). Hence, \((I, \omega_I) = 0\) in \(\tilde{E}^d(R, L)\).

Now let \((J, \omega_J)\) be in the kernel of the map \(\tilde{E}^d(R, L) \to E^d_0(R, L)\). Then the same proof of [6, Lemma 3.3] yields the following

\[ (J, \omega_J) + \sum_{i=1}^r (J_i, \omega_i) = \sum_{k=r+1}^s (J_k, \omega_k) \text{ in } \tilde{E}^d(R, L), \]

where \(J_i, J_k\) are regular ideals in \(R\) of height \(d\) such that there exist surjections \(R^{d-1} \oplus L \to J_i\) for each \(i\). However, then we have already proved that \((J_i, \omega_i) = 0\), for each \(i\). This completes the proof of Step - 2.

**Step - 3.** In this step we show that the canonical group homomorphism \(\phi_R : \tilde{E}^d_0(R) \to E^d_0(R)\) is an isomorphism.

Proof of Step - 3. Let \(I \subset R\) be a regular ideal of height \(d\) such that \(\phi_R(I) = 0\) in \(E^d_0(R)\). It follows from Step - 1, 2 and [11, Theorem 4.2] that any set of generators of \(I/\tau^2\) can be lifted to a set of generators of \(I\). In particular, the ideal \(I\) is a complete intersection ideal of height \(d\). Therefore, we get \((I) = 0\) in \(\tilde{E}^d_0(R)\).

Let \((J) \in E^d_0(R)\) be an element, where \(J \subset R\) is an ideal such that \(\mu(J/J^2) = \text{ht}(J) = d\). Now applying Lemma [31] there exists an ideal \(K\) of height \(d\) in \(R\) such that

1. \(\mu(K \cap J) = d\);
2. \(K + J = R\);
3. \(K\) is a product of distinct smooth maximal ideals in \(R\) of height \(d\).

The conditions (1) and (2) imply that \((J) + (K) = 0\) in \(E^d_0(R)\). From (3) it follows that \((K) \in \tilde{E}^d_0(R)\). This concludes the proof of Step - 3.

**Step - 4.** In this step we show that the group \(E^d_0(R)\) is divisible.

Proof of Step - 4. Let \(m \subset R\) be a smooth complex maximal ideal of height \(d\), and let \(n \in \mathbb{N}\). It follows from the previous steps that it is enough to find an ideal \(I \subset R\) such that \(\mu(I/I^2) = \text{ht}(I) = d\), and \((m) = n(I)\) in \(E^d_0(R)\). We now follow the argument of M. P. Murthy [35] with some suitable modifications.

Let \(J_1\) be the ideal defining the closed set which is the union of the singular locus of \(\text{Spec}(R)\) and all the irreducible components of \(\text{Spec}(R)\) of dimension less than \(d\). Let \(J_2\) be the intersection of all real maximal ideals in \(R\), if \(R\) satisfies P-2, else \(J_2 = R\). We set \(J := J_1 \cap J_2\). By the choice of \(m\) it follows that \(m + J = R\). Let \(m = (f_1, \cdots, f_d) + m^2\).
Then applying [35 Corollary 2.4] there exist \( g_i \in m^2, i = 1, \ldots, d - 1 \) such that if we set \( h_i = f_i + g_i \), then

1. \( \langle h_1, \ldots, h_{d-1} \rangle + J = R, \) and
2. \( C := \text{Spec}(R/\langle h_1, \ldots, h_{d-1} \rangle) \) is a smooth curve.

We note, it follows from (1) that the curve \( C \) does not have any real maximal ideals. Implying that \( \text{Pic}(C) \) is a divisible group. Furthermore, we still have the following.

\[
m = \langle h_1, \ldots, h_{d-1}, f_d \rangle + m^2
\]

Let ‘bar’ denote going modulo \( \langle h_1, \ldots, h_{d-1} \rangle \). As \( \mu(\overline{m}/m^2) = 1 \), the ideal \( \overline{m} \) is an invertible ideal in \( C \). Since \( \text{Pic}(C) \) is a divisible group, there exists an ideal \( \overline{K} \subset C \) such that \( (\overline{m})^{-1} = (\overline{K})^n \) in \( \text{Pic}(C) \). Moreover, using moving arguments without loss of generality we may assume that \( \overline{m} + \overline{K} = C \), and \( \overline{K} \) is a product of distinct maximal ideals in \( C \). Then we can find a non-zero divisor \( \overline{h} \in C \) such that the following hold.

\[
(*) \quad \overline{m}\overline{K}^n = \langle \overline{h} \rangle
\]

Let \( K \subset R \) be a lift of \( \overline{K} \). Then \( K \) is a product of distinct smooth complex maximal ideals in \( R \) of height \( d \). Implying that \( \mu(K/K^2) = \text{ht}(K) = d \). Moreover, as \( \overline{K} \) is an invertible ideal we have \( \mu(K/K^2) = 1 \). Let \( \overline{K} = \langle \overline{h}_d \rangle + \overline{K}^d \), and let \( h_d \in R \) be a lift of \( \overline{h}_d \). Then \( K = \langle h_1, \ldots, h_d \rangle + K^2 \). We take \( K' = \langle h_1, \ldots, h_{d-1} \rangle + K^n \). Then from Lemma 3.2 we get \( (K')^n = n(K) \) in \( E^n_0(R) \). It follows from \( (*) \) that \( \mu(m \cap K') = d \). Implying that \( m = -(K') = -n(K) \) in \( E^n_0(R) \). Therefore, we take \( (I) = -(K) \) to conclude the proof. \( \square \)

Remark 3.4. We would also like to point out two obvious observations of [Proposition 3.3 Step - 1, 2]. Let \( P \) be a projective \( R \)-module of rank \( d \), and \( I \subset R \) be an ideal such that \( \mu(I/I^2) = d = \text{ht}(I) \). Applying [11 Corollary 4.4] we get that \( (1) \ (I) = 0 \) in \( E^n_0(R, L) \) if and only if \( \mu(I) = d \), and \( (2) \ e(P) = 0 \) in \( E^n_0(R, L) \) if and only if \( P \) has a unimodular element, where \( L = \wedge^d P \).

Remark 3.5. It follows from [Proposition 3.3 Step - 1, 2] and [11 Theorem 6.8] that the groups \( E^d(R) \) and \( E^d(R, L) \) are isomorphic, for any rank one projective \( R \)-module \( L \).

The next proposition is due to M. P. Murthy [35 Theorem 3.3]. For smooth real affine varieties this was proved in [10 Lemma 5.1].

Proposition 3.6. Let \( R \) be reduced, and let \( J \subset R \) be a reduced ideal such that \( \mu(J/J^2) = \text{ht}(J) = d \). Let \( L \) be a rank one projective \( R \)-module. Then there exist (a) an ideal \( K \subset R \) of height \( d \), which is a product of distinct smooth complex maximal ideals, and (b) a projective \( R \)-module \( P \) of rank \( d \) (which is unique up to isomorphism if we fix \( K \)) with determinant \( L \) such that the following hold.

1. \( J + K = R, \)
2. \( J \) is a surjective image of \( P, \)
3. \( [P] - [L \oplus R^{d-1}] = [R/K] \in K_0(R), \)
4. \( e(P) = (J) \) in \( E^n_0(R, L), \)
5. \( (J) + (d - 1)! (K) = 0 \) in \( E^n_0(R, L), \) and additionally, if \( J \) is a product of distinct smooth maximal ideals in \( R \) of height \( d \), then \( [J] + (d - 1)! [K] = 0 \) in \( \text{CH}_0(\text{Spec}(R)). \)
Proof. Applying Lemma 3.1 we can find an ideal \( I \subset R \), comaximal with \( J \) such that \( I \) is a product of finitely many distinct smooth complex maximal ideals of height \( d \), and \( (IJ) = (I) + (J) = 0 \) in \( E^d_0(R, L) \).

Since \( E^d_0(R, L) \) is a divisible group [by Proposition 3.3 we get an ideal \( K \subset R \) with \( \text{ht}(K) = \mu(K/K^2) = d \) such that \( (I) = (d-1)! (K) \) in \( E^d_0(R, L) \). Moreover, using Lemma 3.1 twice we may choose \( K \) such that \( K + JJ = R \), and \( K \) is a product of distinct smooth complex maximal ideals of height \( d \). Let \( K = \langle a_1, \ldots, a_d \rangle + K^2 \), and let \( K' = \langle a_1, \ldots, a_{d-1} \rangle + K^{(d-1)!} \) be the Boratynski ideal of \( K \). Since \( K \) is comaximal with \( JI \), it follows from the definition of \( K' \) that \( K' + JJ = R \). Then by Lemma 3.2 we obtain that \( (K') = (d-1)! (K) \) in \( E^d_0(R, L) \). Now as \( (I) + (J) = 0 \), we get the following.

\[(*) \quad (J) + (d-1)! (K) = 0 \text{ in } E^d_0(R, L)\]

Implies that \( (JK') = (J) + (K') = 0 \) in \( E^d_0(R, L) \). Hence, we get \( (I) = (K') \) in \( E^d_0(R, L) \). By applying [35, Theorem 2.2] we get a (unique up to isomorphism if we fix \( K \)) projective \( R \)-module \( Q \) of rank \( d \) with trivial determinant such that (i) \( K' \) is a surjective image of \( Q \) and (ii) \( [Q] - [R^d] = [-R/K] \) in \( K_0(R) \). We point out that the uniqueness of \( Q \) such that \( Q \) satisfies (i) and (ii) follows from Theorem 2.12. Note that, since \( (JK') = 0 \in E^d_0(R, L) \), there is a surjection \( L \oplus R^{d-1} \to JK' \). Now applying [35, Theorem 1.3 and Remark 1.4] there exists a (unique up to isomorphism if we fix \( K \)) projective \( R \)-module \( P \) of rank \( d \) and a surjection \( P \to J \) such that \( P \oplus Q \cong R^d \oplus (L \oplus R^{d-1}) \). One may observe that \( \det(P) \cong L \). Now let \( z = [P] - [L \oplus R^{d-1}] \in K_0(R) \). Then we obtain the following.

\[z = [P] - [L \oplus R^{d-1}] = [R^d] - [Q] = [R/K] \in K_0(R)\]

Since \( J \) is a surjective image of \( P \), by definition [11, §6] we get (4).

Now assume that \( J \) is a product of distinct smooth maximal ideals in \( R \) of height \( d \). Since there exists a canonical homomorphism \( E^d_0(R, L) \to \text{CH}_0(\text{Spec}(R)) \) sending \( (m) \mapsto [m] \),

for any smooth maximal ideals in \( R \), (5) follows from \((*)\). This concludes the proof. \( \square \)

Now we are ready to prove the main theorem of this section.

**Theorem 3.7.** Let \( R \) be reduced and let \( J \subset R \) be an ideal which is a product of distinct smooth maximal ideals in \( R \) of height \( d \). If the cycle \( [J] \) associated to \( R/J \) is 0 in \( \text{CH}_0(\text{Spec}(R)) \), then \( \mu(J) = d \). Furthermore, the canonical map \( \theta_R : E^d_0(R) \to \text{CH}_0(\text{Spec}(R)) \) is an isomorphism.

Proof. Let \( X = \text{Spec}(R) \). Applying Proposition 3.6 there exists a projective \( R \)-module \( P \) of rank \( d \) with trivial determinant and a reduced ideal \( K \) of height \( d \), which is a product of distinct smooth complex maximal ideals such that the following hold.

1. \( J + K = R \),
2. \( J \) is a surjective image of \( P \),
3. \( [P] - [R^d] = [R/K] \in K_0(R) \),
4. \( e(P) = (J) \) in \( E^d_0(R) \),
5. \( [J] + (d-1)! [K] = 0 \) in \( \text{CH}_0(X) \).

Since \([J] = 0 \) in \( \text{CH}_0(X) \), from (5) we get the following.

\[(*) \quad (d-1)! [K] = 0 \in \text{CH}_0(X)\]
Let us define $R_C := \frac{R[T]}{(T^{d+1})} \cong R \otimes_R \mathbb{C}$, and let $X_C = \text{Spec}(R_C)$. As $K$ is supported by only smooth complex maximal ideals, the cycle $[K] \in \text{CH}_0(X)$ sits inside the image of the canonical homomorphism $\text{CH}_0(X_C) \rightarrow \text{CH}_0(X)$. Then, since $\text{CH}_0(X_C)$ is divisible by [29 Theorem 6.7], it follows that $[K] \in D(X)$, where $D(X)$ is the maximal divisible subgroup of $\text{CH}_0(X)$. Moreover, from (\ref{mu}) we get that $[K]$ is a torsion element in $D(X)$. It follows from [29 Theorem 1.3] that $\text{CH}_0(X_C)$ is torsion free. Therefore, applying [16 Proposition 1.4 (b)] we obtain that the torsion subgroup $(D(X))_{\text{tor}}$ is 0. In particular, implying that $[K] = 0$ in $\text{CH}_0(X)$. We recall that there is a natural homomorphism $\text{CH}_0(X) \rightarrow K_0(R)$ sending $[K] \mapsto [R/K]$ [31 \S 2]. Therefore, from (3) we get that $P$ is a stably free module of rank $d$. Hence, using Theorem [2.8] one may obtain that $P$ is free. Implied that $\mu(J) = d$.

Now we shall show that the canonical map $\theta_R : E^d_0(R) \xrightarrow{\sim} \text{CH}_0(X)$ is an isomorphism. Note that we have already proved the injective part. To elaborate this let $I \subset R$ be an ideal such that $\mu(I/I^2) = \text{ht}(I) = d$ and $\theta_R(I) = 0$ in $\text{CH}_0(X)$. Applying Lemma [3.1] twice we may assume that $I$ is a product of distinct smooth maximal ideals in $R$ of height $d$. Then $\mu(I) = d$ follows from the previous paragraph.

Since any smooth maximal ideal $m \subset R$ of height $d$ satisfies the condition $\mu(m/m^2) = d = \text{ht}(m)$, the surjectivity of the map $\theta_R$ follows from the fact that $(m) \in E^d_0(R)$. This concludes the proof. 

3.1. Consequences. Now we obtain the following series of results as a consequence of the results proved in this section until now. We commence with an analogy to A. A. Rojtman’s theorem [40].

Corollary 3.8. The $d$-th Euler class group $E^d(R)$ is torsion free.

Proof. By [11 Corollary 4.6] it is enough to assume that $R$ is reduced. Let $I \subset R$ be an ideal with $\mu(I/I^2) = \text{ht}(I) = d$ such that $n(I) = 0$ in $E^0_0(R)$, for some $n \geq 1$. Using Lemma [3.1] twice we may assume that $I$ is a product of finitely many distinct smooth complex maximal ideals of height $d$. It follows from [Proposition 3.3 Step 1, 2 and Theorem 3.7] that it is enough to show $[I] = 0$ in $\text{CH}_0(\text{Spec}(R))$. Since $I$ is a product of smooth complex maximal ideals, $[I]$ sits inside the image of the canonical map $\text{CH}_0(\text{Spec}(R_C)) \rightarrow \text{CH}_0(\text{Spec}(R))$. Hence, by [16 Proposition 1.4 (b)] and [29 Theorem 1.3] it follows that $[I] = 0$ in $\text{CH}_0(\text{Spec}(R))$. 

Remark 3.9. Since $E^0_0(R, L)$ is torsion free, in Proposition [3.6] the choice of the pair $(K, P)$ is unique in the sense that if there exists another pair $(K'', P')$ satisfying all the conditions of [3.6] then $(K) = (K'')$ in $E^d_0(R)$ and $P \cong P''$.

Since $R \hookrightarrow R_C$ is an integral extension, it will induce a natural homomorphism $\chi_R : E^d(R) \rightarrow E^d(R_C)$ (see [54 Definition 3.2]).

Corollary 3.10. The natural map $\chi_R : E^d(R) \xrightarrow{\sim} E^d(R_C)$ is an isomorphism.

Proof. Let $m \subset R_C$ be a smooth maximal ideal of height $d$. Since smoothness is a geometric property and $R \hookrightarrow R_C$ is an integral extension, the ideal $m \cap R$ is also smooth. Now let us assume that $\mu(m) = d$. Implied that $[R/m \cap R] = 0$ in $K_0(R)$. Applying Riemann-Roch theorem [18 Theorem 1.2] one may observe that $(d-1)(m \cap R) = 0$ in $E^d_0(R)$. Hence by Corollary 3.8 we get $(m \cap R) = 0$ in $E^d_0(R)$. Therefore, we obtain that $\mu(m \cap R) = d$. 

By [29] Lemma 7.4, the group $E_d^0(R_C)$ is generated the class of all such maximal ideals $m$ in $R_C$. This, together with the fact discussed in the previous paragraph, ensures that the existence of a natural group homomorphism $\Gamma_R : E_d^0(R_C) \to E_d^0(R)$ sending $(m) \mapsto (m \cap R)$. Now, it follows from the definitions of $\chi_R$ and $\Gamma_R$ that $\chi_R \circ \Gamma_R = \Gamma_R \circ \chi_R = \text{Id}$.

Corollary 3.11. Let $P$ be as in Proposition 3.6. Then the weak Euler class $e(P) = 0$ in $E_d^0(R, L)$ if and only if $P \cong L \oplus R^{d-1}$.

Proof. First, we assume that $e(P) = 0$ in $E_d^0(R, L)$. Then by Proposition 3.6 (4) we get that $J$ is a complete intersection ideal of height $d$. Applying Proposition 3.6 (5) and Corollary 3.8 we obtain that $[K] = 0$ in $\text{CH}_0(\text{Spec}(R))$. Hence, using Theorem 3.7 it follows that $(K) = 0$ in $E_d^0(R, L)$. Then using (Proposition 3.3, Step - 1, 2 and [11] Theorem 4.2) it follows that $K$ is a complete intersection ideal of height $d$. Imposing that $[R/K] = 0$ in $K_0(R)$. Therefore, from Proposition 3.6 (3) we get that $P$ is a stably isomorphic with $L \oplus R^{d-1}$. Now it follows from Theorem 2.12 that $P \cong L \oplus R^{d-1}$. The converse part is trivial.

One may compare the following result with [33] Theorem 2.1.

Corollary 3.12. Let $I \subset R$ be an ideal such that $\mu(I/I^2) = \text{ht}(I) = d$. Let $P$ be a projective $R$-module of rank $d$ with determinant $L$. Suppose that $\overline{\mathcal{T}} : P/IP \to I/I^2$ be a surjective map. Then there exists a surjective lift $f : P \to I$ of $\overline{\mathcal{T}}$ if and only if $e(P) = (I)$ in $E_d^0(R, L)$.

Proof. First we prove the theorem with the assumption that $R$ is reduced. We fix an isomorphism $\chi : \wedge^d P \cong L$. Note that if $f : P \to I$ then from the definition (see [11] Section 6, paragraph 7) it follows that $e(P) = (I)$ in $E_d^0(R, L)$. Therefore, we assume that $e(P) = (I)$. Let $\alpha : P \to I$ be any lift (might not be surjective) of $\overline{\mathcal{T}}$ (i.e. $\alpha \otimes R/I = \overline{\mathcal{T}}$). Then we get $I = \alpha(P) + I^2$. By [11] Lemma 2.11 there exists an ideal $J \subset R$ co-maximal with $I$ of height $d$ such that $I \cap J = \alpha(P)$. Therefore, in $E_d^0(R, L)$ we have $(I) = e(P) = (I \cap J) = (I) + (J)$. That is $(J) = 0$ in $E_d^0(R, L)$. Since $E_d^0(R, L) \cong E_d^0(R, L)$, we obtain that any local $L$-orientation $\omega_J$ of $J$ is a global $L$-orientation.

Let ‘bar’ denote going modulo $J$. We can find an isomorphism $\delta^{-1} : P/IP \cong (L/JL) \oplus (R/J)^{d-1}$ such that $\overline{\mathcal{T}} \delta = \overline{\mathcal{T}}$. Let us define $\overline{\mathcal{T}} \delta : (L/JL) \oplus (R/J)^{d-1} \to J/J^2$. Since $(J) = 0$ in $E_d^0(R, L)$ by [11] Corollary 4.4 and [Proposition 3.3, Step - 1, 2] we get a surjection $\beta : L \oplus R^{d-1} \to J$ such that $\beta \otimes R/J = \overline{\mathcal{T}}$. Then we note that $(\beta \otimes R/J)^{d-1} = \overline{\mathcal{T}}$. Hence, by Subtraction principal [11] Theorem 3.3) there exists $f : P \to I$ such that $f \otimes R/I = \alpha \otimes R/I = \overline{\mathcal{T}}$. This concludes the proof when $R$ is reduced.

Now let $\eta$ be the nilradical of $R$. Let $A = R/\eta$ and ‘tilde’ denote going modulo $\eta$. Let us assume that $e(P) = (I)$ in $E_d^0(R, L)$. From the previous step, there exists a surjective lift $f' : P \to \tilde{I}$ of $\overline{\mathcal{T}}$. We choose a lift (might not be surjective) $g : P \to \tilde{I}$ of $f'$. Then we get that $g(P) + \eta = I$. Now one can obtain the required surjection $f : P \to I$ using the arguments given in [11] Corollary 4.6] or see [26] Corollary 4.13] for a detailed version.

The next result is an analogy of [34] Theorem 1] in our set-up.

Corollary 3.13. Let $I \subset R$ be an ideal of height such that $\mu(I/I^2) = \text{ht}(I) = d$. Let $P$ be a projective $R$-module of rank $d$ such that $f : P \to I$ is a surjection. Then $P$ has a unimodular element if and only if $\mu(I) = d$. 
Proof. Let L be the determinant of P, and let $\chi : \wedge^d P \cong L$ be an isomorphism. By Theorem [3.12] we obtain that $e(P) = (I)$ in $E_0^d(R, L)$. Let $\{\chi, f\}$ induce the local L-orientation $\omega_I$ of I. Since the groups $E_0^d(R, L)$ and $E_0^d(R)$ are canonically isomorphic [11, Theorem 6.8] we obtain that $\mu(I) = d$ if and only if $(I)$ is in $E_0^d(R, L)$. Applying [Proposition 3.3, Step - 1, 2] we obtain that $(I) = 0$ in $E_0^d(R, L)$ if and only if $(I, \omega_I) = 0$ in $E^d(R, L)$. By [11, Corollary 4.4] we obtain that $(I, \omega_I) = 0$ in $E^d(R, L)$ if and only if P has a unimodular element. This concludes the proof.

We now move towards the polynomial extension of $R$ and prove some results as a corollary of our main theorems in this section. We begin with a theorem on projective generation of a locally complete intersection in $R[T]$ of height d. For affine algebras over an algebraically closed field of characteristic 0 a similar result was proved in [17].

**Corollary 3.14.** Let $I \subset R[T]$ be a locally complete intersection ideal such that $ht(I) = \mu(I/T^2) = d \geq 3$. Let $\omega_I$ be a local orientation of I. Then there exists a projective $R[T]$-module $P$ and an isomorphism $\chi : \wedge^d P \cong R[T]$ such that $e(P, \chi) = (I, \omega_I)$ in $E^d(R[T])$. As a consequence, we obtain a surjection $\alpha : P \twoheadrightarrow I$ such that $(I, \omega_I)$ is obtained from $(\alpha, \chi)$.

Proof. Since $\mathbb{Q} \subset R$ we may assume that there exists an element $\lambda \in R$ such that either $I(\lambda) = R$ or $ht(I(\lambda)) = d$. Moreover, taking the transformation $T \mapsto T - \lambda$ we may further assume that either $I(0) = R$ or $ht(I(0)) = d$. Then we note that $(I(0), \omega_I(0))$ is in $E^d(R)$. Applying [Proposition 3.6 (3) and Proposition 3.3, Step - 1, 2] there exists projective $R$-module $Q$ together with an isomorphism $\chi' : R \cong \wedge^d Q$ such that $e(Q, \chi') = (I(0), \omega_I(0))$ in $E^d(R)$. Now using [17, Theorem 3.5] there exists a projective $R[T]$-module $P$ and an isomorphism $\chi : \wedge^d P \cong R[T]$ such that $e(P, \chi) = (I, \omega_I)$ in $E^d(R[T])$. It follows from [17, Corollary 4.10] that there exists a surjection $\alpha : P \twoheadrightarrow I$ such that $(I, \omega_I)$ is obtained from $(\alpha, \chi)$.

The following corollary gives an affirmative answer to a question asked in [17] Question 2 in our set-up.

**Corollary 3.15.** Let $R(T)$ be the ring obtained from $R[T]$ by inverting all monic polynomials in $R[T]$. Then for all $d \geq 3$ the canonical map

$$\Gamma : E^d(R[T]) \rightarrow E^d(R(T))$$

is injective.

Proof. Let $(I, \omega_I)$ in $E^d(R[T])$ be such that $(IR(T), \omega_I \otimes R(T)) = 0$ in $E^d(R(T))$. Applying Proposition [3.14] there exists a projective $R[T]$-module $P$ together with an isomorphism $\chi : \wedge^d P \cong R[T]$, and a surjection $\alpha : P \twoheadrightarrow I$ such that (1) $e(P, \chi) = (I, \omega_I)$ in $E^d(R[T])$, and (2) $(I, \omega_I)$ is obtained from $(\alpha, \chi)$. Since $(IR(T), \omega_I \otimes R(T)) = 0$ by [11, Corollary 4.4] we obtain that $P \otimes R(T)$ has a unimodular element. Hence, applying [12, Theorem 3.4] we get that $P$ has a unimodular element. Therefore, using [17, Corollary 4.11] it follows that $(I, \omega_I) = 0$ in $E^d(R[T])$. This concludes the proof.

## 4. A nice group structure of WMS$_{d+1}(R)$

In this section, we prove that WMS$_{d+1}(R)$ has a nice group structure. Hence, it follows from [42] that WMS$_{d+1}(R)$ is canonically isomorphic to the universal Menicke symbol MS$_{d+1}(R)$. The outline of the proof follows the arguments given in [22, Theorem 3.9].
Theorem 4.1. The abelian group $WMS_{d+1}(R)$ has a nice group structure. That is for any $(a, a_1, \cdots, a_d)$ and $(b, a_1, \cdots, a_d) \in Um_{d+1}(R)$ we have
\[ [(a, a_1, \cdots, a_d)] \ast [(b, a_1, \cdots, a_d)] = [(ab, a_1, \cdots, a_d)]. \]
In particular $WMS_{d+1}(R) \cong MS_{d+1}(R)$.

Proof. Without loss of generality we may assume that $R$ is a reduced ring (see [22, Lemma 3.5]). Moreover, if $R$ satisfies condition P-2, then by Lemma 2.7 taking $S$ to be the collection of all real maximal ideals we may further assume that for any $2 \leq i \leq d$, the ring $R/\langle a_i, a_{i+1}, \cdots, a_d \rangle$ is a smooth real affine algebra of dimension $i-1$, having no real maximal ideal. On the other hand, if $R$ satisfies P-1, then using Theorem 2.2 one can achieve the same. Applying the product formula as given in [46] we get
\[ [(a, a_1, \cdots, a_d)] \ast [(b, a_1, \cdots, a_d)] = [(a(b + p) - 1, (b + p)a_1, a_2, \cdots, a_d)], \]
where $p$ is chosen such that $ap - 1 \in \langle a_2, \cdots, a_d \rangle$. Let $B = R/\langle a_2, a_3, \cdots, a_d \rangle$ and let ‘bar’ denote going modulo $\langle a_2, a_3, \cdots, a_d \rangle$. Then by Bass-Kubota theorem [46, Theorem 2.12] we have $SK_1(B) \cong MS_2(B)$. Therefore, in the group $MS_2(B)$ we get the following.
\[ [(\bar{a}(\bar{b} + \bar{p}) - \bar{1}, (\bar{b} + \bar{p})\bar{a}_1)] = [(\bar{a}(\bar{b} + \bar{p}) - \bar{1}, \bar{a}_1)] \]
Hence there exists a $\sigma \in SL_2(B) \cap E_3(B)$ such that
\[ (\bar{a}(\bar{b} + \bar{p}) - \bar{1}, (\bar{b} + \bar{p})\bar{a}_1)\sigma = (\bar{a}(\bar{b} + \bar{p}) - \bar{1}, \bar{a}_1). \]
By Theorem 2.4 we obtain that $\sigma \in SL_2(B) \cap ESP(B)$. Then applying [43, Corollary 2.3] we can find $\alpha \in E_{d+1}(R)$ such that
\[ (a(b + p) - 1, (b + p)a_1, a_2, \cdots, a_d)\alpha = (a(b + p) - 1, a_1, a_2, \cdots, a_d). \]
Now as $ap - 1 \in \langle a_2, \cdots, a_d \rangle$ we get
\[ [(a(b + p) - 1, a_1, a_2, \cdots, a_d)] = [(ab, a_1, a_2, \cdots, a_d)]. \]
This concludes the proof. \hfill \Box

The remaining part of the section is devoted to establishing some consequences of the above theorem.

Corollary 4.2. Let $I \subset R$ be an ideal. Then the abelian group $Um_{d+1}(R/I)$ has a nice group structure. That is
\[ [(a, a_1, \cdots, a_d)] \ast [(b, a_1, \cdots, a_d)] = [(ab, a_1, \cdots, a_d)], \]
where $[-]$ denote the class in the relative elementary orbit space of relative unimodular rows of length $d + 1$.

Proof. Applying Theorem 4.1 and Lemma 2.13 it follows that the group $WMS_{d+1}(R \oplus I)$ has a nice group structure. Hence, by [24, Lemma 3.6] the proof concludes. \hfill \Box

We end this section with an extension of a result due to J. Fasel [19, Theorem 2.2].

Corollary 4.3. The group $WMS_{d+1}(R)$ is divisible.
Proof. Let \( v = (v_0, \ldots, v_d) \in \text{Um}_{d+1}(R) \) and let \( n \in \mathbb{N} \). By Proposition \[2,7\] (taking \( S \) to be the collection of all real maximal ideals, whenever \( R \) satisfies P-2) or Theorem \[2,2\] we may assume that \( R/\langle v_2, \ldots, v_d \rangle \) is a smooth curve having no real maximal ideal. Let \( C = R/\langle v_2, \ldots, v_d \rangle \), and let ‘bar’ denote going modulo \( \langle v_2, \ldots, v_d \rangle \). Then by Bass-
Kubota theorem \[46\, \text{Theorem } 2.12\], we have \( \text{MS}_2(C) = \text{SK}_1(C) \). The latter one is a
divisible group by Proposition \[2,4\]. Now following the argument given in [Theorem \[2,8\] paragraph 2], one can get a \( \gamma \in \text{E}_{d+1}(R) \) such that \( v = \gamma (u_0^n, u_1, v_2, \ldots, v_d) \). Therefore, applying Theorem \[4,1\] we obtain the following.

\[
[v] = [(u_0^n, u_1, v_2, \ldots, v_d)] = [(u_0, u_1, v_2, \ldots, v_d)]^n
\]

This concludes the proof. \( \square \)

5. IMPROVED STABILITY FOR \( \text{K}_1 \) AND \( \text{K}_1 \text{Sp} \) GROUPS

Additionally, let \( R \) be a regular domain and let \( I \subset R \) be a principal ideal. In this
set-up, we prove that the injective stability of the groups \( \text{SK}_1(R, I) \) and \( \text{K}_1 \text{Sp}(R, I) \)
decreases by one. This improves L. N. Vaserstein’s general stability bounds \[48, 49\], and \[50\]. The arguments used in this section are motivated from \[39\].

**Notation.** Let \( A \) be a ring. For any two square matrices \( M \in M_m(A) \) and \( N \in M_n(A) \),
by \( M \perp N \) we denote the matrix

\[
\begin{pmatrix}
M & 0 \\
0 & N
\end{pmatrix} \in M_{m+n}(A).
\]

**Theorem 5.1.** Additionally, let \( R \) be an integral domain, and let \( I = \langle a \rangle \subset R \) be a principal ideal. Let \( \sigma \in \text{SL}_{d+1}(R, I) \cap \text{E}(R, I) \). Then \( \sigma \) is isotopic to identity. Moreover, if we assume that \( R \) is regular, then \( \text{SK}_1(R, I) \simeq \text{SL}_{d+1}(R, I)/\text{E}_{d+1}(R, I) \).

Proof. The surjective of the canonical map \( \Gamma_{d+1} : \text{SL}_{d+1}(R, I)/\text{E}_{d+1}(R, I) \to \text{SK}_1(R, I) \) follows from
\[48\]. To prove the injectivity, first, we note that the only non-trivial part is to prove the following.

\[ \text{E}(R, I) \cap \text{SL}_{d+1}(R, I) \subset \text{E}_{d+1}(R, I) \]

Let \( \sigma \in \text{E}(R, I) \cap \text{SL}_{d+1}(R, I) \). By stability theorem due to L. N. Vaserstein \[48\] we get \( \sigma \in \text{E}_{d+2}(R, I) \cap \text{SL}_{d+1}(R, I) \). It follows from \[51\, \text{Chapter } 1, \$$ 2 \] that \( 1 \perp \sigma \) is of the form

\[
\prod E_{ij}(b)E_{ji}(a \lambda_k)E_{ij}(-b),
\]

where the product runs for some \( i \neq j \) and \( b, \lambda_k \in R \). We define

\[
\tau(T) := \prod E_{ij}(b)E_{ji}(\lambda_k T)E_{ij}(-b).
\]

Then \( \tau(T) \in \text{E}_{d+2}(R[T], \langle T \rangle) \) such that \( \tau(a) = 1 \perp \sigma \). Let \( t = T^2 - T a \in R[T] \) be a
canonical divisor, and let \( \nu = c_T \tau(T) \). Then \( \nu \in \text{Um}_{d+2}(R[T], \langle t \rangle) \). Since \( R \) is an
affine domain, any maximal ideal in \( R[T] \) is of height \( d + 1 \). Hence, with the help of Suslin’s
monic polynomial theorem (see \[30\, \text{Chapter III, } \$$ 3,3, \text{page no } 108\] ) one may observe that \( R/\mathfrak{m} \cap R \rightarrow R[T]/\mathfrak{m} \) is an integral extension, for any maximal ideal \( \mathfrak{m} \subset R[T] \).
This in particular gives us that \( R[T] \) satisfies one of the conditions P-i [cf. the proof of
Lemma 2.13. Therefore, by Proposition 2.15, there exists $\chi(T) \in \text{SL}_{d+2}(R[T], \langle t \rangle)$, such that $v = e_1 \chi(T)$. Since $e_1 T \chi(T)^{-1} = e_1$, the matrix $\tau(T) \chi(T)^{-1}$ is of the form

$$
(1 - \rho(T)) \prod_{i=1}^{d+2} E_{i,1}(\lambda_i),
$$

where $\lambda_i \in \langle t \rangle$, $\rho(T) \in \text{SL}_{d+1}(R[T], \langle t \rangle)$ and $\rho(a) = \sigma$. Now since $\chi(T) \equiv I_{d+2}$ modulo $\langle t \rangle$, we have $\chi(0) = \chi(a) = I_{d+2}$. Hence, $\rho(0) = I_{d+1}$. In other words we get $\rho(T) \in \text{SL}_{d+1}(R[T], \langle t \rangle)$ is an isometry of $\sigma$.

Now, if we assume that $R$ is regular, then by [52, Theorem 3.3] we get $\rho(T) \in E_{d+1}(R[T], \langle t \rangle)$. Hence, $\sigma = \rho(a) \in \text{ESp}_{d+1}(R, \langle a \rangle)$. This concludes the proof.

For the rest of this section, we shift towards the symplectic matrices (for definitions we refer to [51, Chapter 1, § 4]), and prove an analogous result of Theorem 5.1 for the symplectic group $K_1 \text{Sp}(R, I)$.

**Proposition 5.2.** Let $d \equiv 1 \mod (4)$, and let $I = \langle a \rangle \subset R$ be a principal ideal, where $a \in R$ is a non-zero divisor. Then

$$
\text{Um}_{d+1}(R, I) = e_1 \text{Sp}_{d+1}(R, I).
$$

Proof. First, we note that for $d = 1$, we have $\text{Sp}_2(R, I) = \text{SL}_2(R, I)$. Hence, there is nothing to prove when $d = 1$, so we may assume that $d \geq 5$. Let $v \in \text{Um}_{d+1}(R, I)$. Recall that, for any unimodular row $w = (w_0, \ldots, w_r)$, we denote $\chi_r(w)$ by the unimodular row $(w_0, \ldots, w_{r-1}, w_r'^t)$. It follows from Lemma 2.16 that

$$
v \equiv \chi_{d!}(v') \mod E_{d+1}(R, I),
$$

for some $v' \in \text{Um}_{d+1}(R, I)$. Then applying [14, Theorem 5.5] we obtain the following.

(1) $v \equiv \chi_{d!}(v') \mod \text{ESp}_{d+1}(R, I)$

By Corollary 4.2 it follows that the group $\dfrac{\text{Um}_{d+1}(R, I)}{E_{d+1}(R, I)}$ has a nice group structure. Hence, using [41, Proposition 3.1] we get

(2) $\chi_{d!}(v') \equiv e_1 \mod \text{Sp}_{d+1}(R, I)$.

Combing (1) and (2) the proof concludes. \qed

**Definition 5.3.** Let $\sigma$ denote the permutation of the natural numbers given by $\sigma(2i) = 2i - 1$ and $\sigma(2i - 1) = 2i$. Let $x \in A$. Then $sE_{ij}(x)$ is defined as follows: $sE_{ij}(x) := I_{2n} + e_{ij}(x)$ if $i = \sigma(j)$; and $sE_{ij}(x) := I_{2n} + e_{ij}(x) - (-1)^{i+j} e_{\sigma(j)\sigma(i)}(x)$ if $i \neq \sigma(j)$ and $i < j$.

**Theorem 5.4.** Additionally, let $R$ be a regular domain of dimension $d \geq 4$, and let $I = \langle a \rangle \subset R$ be a principal ideal. Moreover, assume that if $d$ is even then 4 divides $d$. Let $n = \left\lceil \frac{d+1}{2} \right\rceil$, where $\lceil - \rceil$ denotes the smallest integer less than or equals to $-$. Then $\text{Sp}_n(R, I) \cap \text{ESp}(R, I) = \text{ESp}_n(R, I)$. Consequently, the canonical map $\text{Sp}_n(R, I) \cap \text{ESp}(R, I) \to K_1 \text{Sp}(R, I)$ is injective.

Proof. First, we observe that applying L. N. Vaseršteǐn’s stability theorem [50] it is enough to show that

$$
\text{Sp}_n(R, I) \cap \text{ESp}_{n+2}(R, I) \subset \text{ESp}_n(R, I).
$$
Let $\sigma \in \text{Sp}_n(R, I) \cap \text{ESP}_{n+2}(R, I)$. Then $I_2 \perp \sigma$ is of the form

$$\prod g_\sigma E_{ij}(a\lambda_k)g^{-1},$$

where the product runs for some $i \neq j$, $g \in \text{ESP}_{n+2}(R)$ and $\lambda_k \in R$ (cf. [15] Lemma 3.5, last paragraph). We define

$$\rho(T) := \prod g_\sigma E_{ij}(\lambda_k T)g^{-1}.$$ 

Then $\rho(T) \in \text{ESP}_{n+2}(R[T], \langle T \rangle)$ such that $\rho(a) = I_2 \perp \sigma$. Let us take $v(T) = e_1 \rho(T)$. Then we note that $v(T) \in \text{Um}_{n+2}(R[T], \langle T^2 - aT \rangle)$. Since $R$ is an affine domain, any maximal ideal in $R[T]$ is of height $d + 1$. Hence, with the help of Suslin’s monic polynomial theorem, one may observe that $R/m \cap R \hookrightarrow R[T]/m$ is an integral extension, for any maximal ideal $m \subset R[T]$. This in particular gives us that $R[T]$ satisfies one of the conditions $P_i$, $i=1,2$.

If $d$ is odd, then we have $n = d + 1$. Then applying [14] Lemma 6.3 and Theorem 5.5 we obtain the following.

$$v(T) \in \text{Um}_{d+3}(R[T], \langle T^2 - aT \rangle) = e_1 E_{d+3}(R[T], \langle T^2 - aT \rangle) = e_1 \text{ESP}_{d+3}(R[T], \langle T^2 - aT \rangle)$$

Hence there exists an $\alpha(T) \in \text{ESP}_{d+3}(R[T], \langle T^2 - aT \rangle)$ such that $v(T) = e_1 \alpha(T)$.

Now if 4 divides $d$, then we note that $n = d$. Applying Proposition 5.2 one can find an $\alpha(T) \in \text{Sp}_{d+2}(R[T], \langle T^2 - aT \rangle)$ such that $v(T) = e_1 \alpha(T)$.

Therefore, in either case there exists an $\alpha(T) \in \text{Sp}_{n+2}(R[T], \langle T^2 - aT \rangle)$ such that $v(T) = e_1 \alpha(T)$. Then the matrix $\rho(T)\alpha(T)^{-1}$ is of the form

$$\begin{pmatrix}
1 & 0 & 0 \\
* & 1 & * \\
* & 0 & \eta(T)
\end{pmatrix},$$

for some $\eta(T) \in \text{Sp}_n(R[T], \langle T \rangle)$. Since $R$ is regular applying [23] Theorem 5.3 we obtain that $\eta(T) \in \text{ESP}_n(R[T], \langle T \rangle)$. Since $\alpha(a) = I_{n+2}$, the matrix $\eta(T) \in \text{ESP}_n(R[T], \langle T \rangle)$ must satisfy the equality $\eta(a) = \sigma$. In particular, this implies $\sigma \in \text{ESP}_n(R, I)$. This concludes the proof.

Remark 5.5. Additionally, let $R$ be a regular domain. When $d \equiv 3 \mod (4)$, one can employ Theorem 5.1 and follow the argument in [45] Theorem 3.6 to conclude that $\text{Um}_{d+1}(R) = e_1 \text{Sp}_{d+1}(R)$. However, whether an improvement of the existing symplectic injective stability for $\text{K}_1\text{Sp}(R)$ is possible still remains unknown for the case $d \equiv 2 \mod (4)$.

6. A SUFFICIENT CONDITION FOR EFFICIENT GENERATION OF MODULES

This section is motivated from [24] and [35, §4]. Here we investigate relations between some Eisenbud-Evans type theorems as studied in [24]. To do so we need some preparation. We begin with the following stronger version of Proposition 3.6. The proof essentially uses Proposition 3.6 and a subtraction principle due to N. Mohan Kumar and M. P. Murthy [35, Theorem 1.3].

Proposition 6.1. Let $A$ be a reduced real affine algebra of dimension $d \geq 2$, and let $J \subset A$ be an ideal which is not contained in any minimal prime ideals in $A$ such that $\mu(J/J^2) = d$. Additionally, assume that $A$ satisfies one of the following conditions.

(i) $A$ satisfies $P$-I;
(ii) the intersection of all real maximal ideals in \( A \) has a height of at least 2.

Then there exist (a) a reduced ideal \( K \subset A \) of height \( d \), which is a product of distinct smooth complex maximal ideals, and (b) a projective \( A \)-module \( P \) of rank \( d \) with trivial determinant such that the following hold.

1. \( J \) is a surjective image of \( P \),
2. \([P] - [A^d] = -[A/K] \in K_0(A)\).

Proof. By Lemma 3.1 one can obtain an ideal \( I \subset A \) such that

(a) \( I \) is a product of distinct smooth complex maximal ideals of height \( d \),
(b) \( I + J = A \), and
(c) \( \mu(I \cap J) = d \).

Now applying Proposition 3.6 there exist a reduced ideal \( K \subset A \) of height \( d \), which is a product of distinct smooth complex maximal ideals and a projective \( A \)-module \( Q \) of rank \( d \) with trivial determinant and such that the following hold.

(I) \( I + K = A \),
(II) \( I \) is a surjective image of \( Q \),
(III) \([Q] - [A^d] = [A/K] \in K_0(A)\),
(IV) \( e(Q) = (I) \) in \( E_0^d(A) \),
(V) \([I] + (d - 1)! [K] = 0 \) in \( CH_0(\text{Spec}(A)) \).

We now show that the hypothesis of [35, Theorem 1.3] are satisfied so that we can come back to the ideal \( J \). We take \( F \) as the closure of \( \mathbb{R} \)-rational points of \( \text{Spec}(A) \) when \( A \) satisfies (ii) and empty set otherwise. Let \( a \subset A \) be an ideal such that \( \dim(A/a) \leq 1 \) and \( \dim(a \cap F) \leq d - 2 \). We take \( F = V(J) \).

Now if \( A \) satisfies (i) then \( A/a \) is a real affine algebra of dimension \( \leq d - 1 \) such that \( A/a \) does not have any real maximal ideal. In this case, it follows from Theorem 2.12 that all projective \( A/a \)-modules of rank \( \geq d - 1 \) are cancellative. In this case \( J = A \).

Implying that \( \dim(V(J) \cap F) = \dim(V(A)) \leq d - 2 \). The other hypotheses of [35, Theorem 1.3] follow from (a), (b), (c), and (III).

Now we assume that \( A \) satisfies (ii). In this case, we have \( \text{ht}(J) \geq 2 \). In this case if \( \text{ht}(a) \geq 2 \) then by Bass cancellation [3] all projective \( A/a \)-module of rank \( \geq d - 1 \) are cancellative. So we assume that \( \text{ht}(a) = 1 \). Since \( A \) satisfies (ii) it follows that \( A/a \) is a real affine algebra of dimension \( d \) such that either \( A/a \) does not have any real maximal ideals or the intersection of all real maximal ideals in \( A/a \) has a height of at least 1.

Therefore, again applying Theorem 2.12 it follows that that all projective \( A/a \)-modules of rank \( \geq d - 1 \) are cancellative. As \( \text{ht}(J) \geq 2 \), we get \( \dim(V(J) \cap F) = \dim(V(J + J)) \leq d - 2 \). The other hypotheses of [35, Theorem 1.3] follow from (a), (b), (c), and (III).

Therefore, applying [35, Theorem 1.3] there exist (A) a projective \( A \)-module \( P \) of rank \( d \) such that \( J \) is a surjective image of \( P \), and (B) \( P \oplus Q \cong A^d \). Now from (B), we get the following.

\[ [P] - [A^d] = [A^d] - [Q] = -[A/K] \in K_0(R) \]

This concludes the proof. \( \square \)

Let \( A \) be a ring (not necessarily containing \( \mathbb{Q} \)). Then one can still define the \( d \)-th Euler class group \( E^d(A) \) and prove that \( (K, \omega_K) = 0 \) if and only if \( \omega_K \) is a global orientation of \( K \) (see [13, Section 4]). We recall that the group \( F^dK_0(A) \), which is the
subgroup of $K_0(A)$ generated by the images of all $[A/I]$, where $I$ is a locally complete intersection ideal of height $d$ (for details we refer to [32, Section 1]). Let $J \subset A$ be an ideal such that $\mu(J/J^2) = \text{ht}(J) = d$. Since in a local ring $A_p$, one can always lift generators of $J_p/J^2_p$ (where $p \in V(J)$) to a generator of $J_p$, the ideal $J$ is in fact a locally complete intersection ideal of height $d$. Moreover, the group $F^dK_0(A)$ kills all the complete intersection ideals of height $d$. This evident that there exists a canonical group homomorphism $\kappa_A : E^d(A) \to F^dK_0(A)$ sending $(I, \omega_I) \mapsto [A/I]$, where $\omega_I$ is a local orientation of $I$. In the next theorem, we prove that $\kappa_R$ is an isomorphism.

**Theorem 6.2.** The canonical map $\kappa_R : E^d(R) \sim F^dK_0(R)$ is an isomorphism.

Proof. Without loss of generality we may assume that $R$ is reduced. Moreover, by [Proposition 3.3 Step 1, 2, 3] it is enough to show that the composition map $E^d_0(R) \sim F^dK_0(R)$ is an isomorphism. With an abuse of notation, we denote the composition map $E^d_0(R) \sim F^dK_0(R)$ by the same notation $\kappa_R$.

In our set-up, the injectivity of $\kappa_R$ is a consequence of results due to A. A. Suslin, M. Boratyński and M. P. Murthy. We elaborate: let $J \subset R$ be an ideal such that $\text{ht}(J) = \mu(J/J^2) = d$, and $(J) \in \ker(\kappa_R)$. Let $J = \langle a_1, \ldots, a_d \rangle + J^2$. We consider the Boratyński ideal $I := \langle a_1, \ldots, a_d \rangle + J^{(d-1)!}$ of $J$. Then by [35, Theorem 2.2] there exists a projective $R$-module $P$ of rank $d$ such that (1) $I$ is a surjective image of $P$, and (2) $(P) - (R^d) = [R/J] \in F^dK_0(R)$. Moreover, from Lemma 3.2 we get $(I) = (d-1)!(J)$ in $E^d_0(R)$. Now since $(J) \in \ker(\kappa_R)$ from (2) one can observe that $P$ is stably free. Hence, $P$ is free by Theorem 2.8. Implying that $(d-1)!(J) = (I) = 0$ in $E^d_0(R)$. Since $E^d_0(R)$ is torsion free by Corollary 3.8 it follows that $(J) = 0$ in $E^d_0(R)$.

Now we prove the surjectivity of the map $\kappa_R$. Applying [32, Theorem 1.5] we obtain that $F^dK_0(R)$ is generated by the classes $[R/m]$, where $m$ runs through all the smooth maximal ideals in $R$ of height $d$. Since any smooth maximal ideal $m$ of height $d$ will satisfy $\mu(m/m^2) = d = \text{ht}(m)$. Therefore, the ideal $m$ has a preimage in $E^d_0(R)$. □

**Remark 6.3.** Let $R$ be reduced, and let $P$ be a projective $R$-module of rank $d$. Corollary 3.8 and Theorem 6.2 immediately establish that the top Chern class $c_{d}(P)$ of $P$, as defined in [35], governs the splitting behavior for $P$ (see [35, Theorem 3.7]).

Let $A$ be a ring, and let $M$ be an $A$-module. We fix the following notations for the remaining part of this section.

- $\text{supp}(M) = \{ p \in \text{Spec}(A) : M_p \neq 0 \}$
- $\nu(p, M) = \mu(M_p) + \dim(A/p)$
- $\eta(M) = \sup \{ \nu(p, M) : p \in \text{supp}(M) \}$
- $\delta(M) = \sup \{ \nu(p, M) : p \in \text{supp}(M), \dim(A/p) < \dim(A) \}$

Now we state a theorem which is due to M. P. Murthy. We comment that with the results developed in this article so far, the exact same proof of [35, Theorem 4.1] will work in our set-up as well. To avoid repeating the same arguments we omit the proof.

**Proposition 6.4.** Let $A$ be as in Proposition 6.7 and let $M$ be an $A$-module. Then there exists is a projective $A$-module $P$ (with trivial determinant) of rank $\delta(M)$ such that (1) the module $M$ is a surjective image of $P$, and (2) $[P] - [A^{\delta(M)}] \in F^dK_0(A)$.

We now state and prove the main theorem of this section.

**Theorem 6.5.** Let $A$ be as in Proposition 6.7. Then the following are equivalent.
(1) Every $A$-module $M$ is generated by at-most $\delta(M)$ element.
(2) Every locally complete intersection ideal in $A$ of height $d$ is complete intersection.
(3) Every smooth maximal ideal in $A$ of height $d$ is complete intersection.
(4) Every projective $A$-module of rank $d$ splits into a free factor of rank one.

Proof. (1) $\implies$ (2) If $I \subset A$ is locally complete intersection ideal in $A$ of height $d$, then $\delta(I) = d$. Hence, (2) follows.
(2) $\implies$ (3) Since every smooth maximal ideal in $A$ of height $d$ is a locally complete intersection ideal of height $d$. Hence, (3) follows.
(3) $\implies$ (4) Statement (3) will imply that $E^d_0(R) = 0$. From [Proposition 5.3 Step - 1, 2, 3] it follows that $E^d(R, L) = 0$ for any line bundle $L$. Therefore, applying [11 Corollary 4.4] statement (4) follows.
(4) $\implies$ (1) Let $I \subset A$ be an ideal such that $\mu(I/I^2) = \text{ht}(I) = d$. Applying [Proposition 5.6 (2)] there exists a projective $A$-module $P$ of rank $d$ such that $I$ is a surjective image of $P$. Since $P$ has a unimodular element, applying Corollary 3.13 we get $\mu(I) = d$. This in particular proves that $E^d_0(R)$, and hence $E^d(R)$ vanishes. Therefore, by Theorem 6.2 we obtain that $F^dK_0(A) = 0$. Applying Proposition 6.4 one can find a projective $A$-module $Q$ of rank $\delta(M)$ such that (a) the module $M$ is a surjective image of $Q$, and (b) $|Q| - [A^{\delta(M)}] \in F^dK_0(A)$. If $\eta(M) \leq \delta(M)$ then $M$ is generated by $\delta(M)$ elements [21]. Therefore, we assume that $\eta(M) > \delta(M)$. In this case, one has $\delta(M) \geq d$, for details we refer to [35 Proof of Theorem 4.1, paragraph 1]. Since $F^dK_0(A) = 0$ we get $Q$ is stably free. Hence, by Theorem 2.8 the module $Q$ is free. This concludes the proof.

\[\square\]

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