CONTINUOUS SPINS IN 2D GRAVITY:
CHIRAL VERTEX OPERATORS AND LOCAL FIELDS

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Abstract

We construct the exponentials of the Liouville field with continuous powers within the operator approach. Their chiral decomposition is realized using the explicit Coulomb-gas operators we introduced earlier. From the quantum-group viewpoint, they are related to semi-infinite highest or lowest weight representations with continuous spins. The Liouville field itself is defined, and the canonical commutation relations verified, as well as the validity of the quantum Liouville field equations. In a second part, both screening charges are considered. The braiding of the chiral components is derived and shown to agree with the ansatz of a parallel paper of J.-L. G. and Roussel: for continuous spins the quantum group structure $U_q(sl(2)) \odot U\hat{q}(sl(2))$ is a non trivial extension of $U_q(sl(2))$ and $U\hat{q}(sl(2))$. We construct the corresponding generalized exponentials and the generalized Liouville field.

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1 Introduction

Until recent times, the progress in understanding the structure of two-dimensional gravity from the operator point of view[1]-[9], was based on the detailed study of the monodromy properties of the Virasoro null-vector equations, whose link with the quantum group $U_q(sl(2))$ is completely understood by now[8][9]. The degenerate fields correspond to standard representations of this quantum group, with positive half-integer spins[2]. The quantum group structure of the theory turns out to govern not only the chiral operator algebra[1][2][6][8][9], but also the reconstruction of (the exponential of) the Liouville field[7] $\exp(-J\alpha-\Phi)$, which is simply the $U_q(sl(2))$-singlet made out of two representations of spin $J$. It also opens the way towards understanding 2D gravity in the strong coupling regime[3]. However, the study of representations with half-integer spins does not by far answer all the physical questions we want to ask about 2D gravity. In particular, modular invariance in the strong coupling regime forces us to consider operators with quantum-group spins which are rational, but not halves of integers[13], and the possibility of defining the Liouville field itself, and not just some of its exponentials, is realized only if we can define the Liouville exponential with continuous $J$, so that we may let $\Phi = -\frac{d}{dJ}|_{J=0} \exp(-J\alpha-\Phi)/\alpha_-$. The basic difficulty in going away from half-integer spins is that one no longer deals with degenerate fields satisfying null-vector equations. In a recent letter we have shown[11] how to solve this problem, at least concerning the braiding, by using an operator Coulomb-gas realization, where the braiding matrix turned out to be computable from a simple quantum mechanical problem, which we could solve in closed form. The point of the present article is to go further in the same direction. Another line of attack has recently been followed in ref.[13], where the fusing and braiding matrices are generalized using the scheme[12] of Moore and Seiberg, and by requiring that the polynomial equations still hold for non-integer $2J$’s. As we will see the two methods agree.

This article is organized as follows. In section 2 we recall some background material. Section 3 is devoted to the case of a single screening charge, where the quantum group structure is the standard $U_q(sl(2))$. Besides completing the discussion of ref.[14] for the chiral algebra, we construct Liouville exponentials for arbitrary $J$ and give an expression for the Liouville field itself. It is shown that the canonical equal-time commutation relations as well as the quantum equations of motion are satisfied. We discuss the periodicity properties of our definition of the Liouville exponentials resp. of the Liouville field and their connection with the presence of singularities in the elliptic sector. The preservation on the quantum level of the symmetry under the exchange of the two equivalent Backlund free fields[15] is derived for half-integer spins. In section 4, we consider both screening charges together. The corresponding quantum group structure was noted $U_q(sl(2)) \odot U_q^\prime(sl(2))$ in ref.[14]. In the degenerate case, the primary fields of spins $J$ and $\hat{J}$ are of the type $(2\hat{J}+1,2J+1)$ in the BPZ classification. Then the braiding of a $(1,2J+1)$ field with a $(2\hat{J}+1,1)$ field is a simple phase, so that this $\odot$ symbol represents a sort of

\footnote{or of its suitable extension when the two types of screening charges are included.}
graded tensor-product. The situation is completely different for continuous spins, and this comes out very naturally from our approach where it is shown that, in this general situation, the relevant parameter is the quantity called effective spin given by
\[ J^e = J + \hat{J}_\pi/h \]
(\(h\) is defined by \(q = \exp(\text{i}h)\) as usual). Whereas, for half integer \(J\) \(\hat{J}\) and generic \(h\), it is equivalent to specify \(J\) \(\hat{J}\) or \(J^e\), the former loose their meaning for continuous spins. Thus the \(U_q(sl(2)) \circ U_q(sl(2))\) structure has novel features which we investigate in detail. In the parallel study of this general situation, extending the polynomial equations of the Moore Seiberg approach has allowed to write educated guesses for the fusion and braiding matrices that are remarkably simple in terms of these effective spins. Although it is unable, at present to directly deal with fusion, our approach allows us to actually derive the braiding matrices, which completely agree with the expression of ref.[13]. Assuming the standard relationship between braiding and fusion matrices, this fully determines the operator algebra of the chiral primary fields for continuous spins, and agrees with ref.[13]. It is also used to construct the Liouville exponential in this most general situation.

2 Trying to be self-contained

Let us rederive some background material about Liouville theory, in order to introduce the coming discussion. The solutions of the classical Liouville dynamics, which is described by the action
\[
S = \frac{1}{8\pi} \int d\tau d\sigma \left\{ (\partial_\tau \Phi)^2 - (\partial_\sigma \Phi)^2 - \mu^2 e^{2\sqrt{\gamma} \Phi} \right\},
\]
are given by
\[
2\sqrt{\gamma} \Phi = \ln \left[ \frac{8}{\mu^2 \sqrt{\gamma}} \left( \frac{A'(u)B'(v)}{(A(u) - B(v))^2} \right) \right], \quad u = \tau + \sigma, \quad v = \tau - \sigma
\]
with \(A\) and \(B\) arbitrary functions, and \(\sigma \in [0, 2\pi]\). The coupling constant is noted \(\gamma\). We have redefined \(\Phi \rightarrow 2\sqrt{\gamma} \Phi\) in order to agree with the classical limit of standard quantum normalizations, where \(2\sqrt{\gamma}\) is the limit of the screening charge \(\alpha_-\). Eq.2.2 is invariant under the projective transformations
\[
A \rightarrow \frac{aA + b}{cA + d}, \quad B \rightarrow \frac{aB + b}{cB + d},
\]
\(a, b, c, d\) complex, which on the quantum level gives rise to the \(U_q(sl(2))\) quantum group symmetry. Introducing the chiral fields\[4\]
\[
f_{J_m}^{(J)} = \sqrt{\binom{2J}{J+m}} (A^{1/2})^{J-m} (AA^{1/2})^{J+m},
\]
\[
\bar{f}_{J_m}^{(J)} = \sqrt{\binom{2J}{J+m}} (B^{1/2})^{J+m} (BB^{1/2})^{J-m},
\]
\[4\text{In this formula, contrary to the rest of the article, we of course use ordinary binomial coefficients.} \]
we can write the Liouville exponentials as
\[ e^{-2J\sqrt{\vartheta}} = \left(\frac{\mu^2 \sqrt{\gamma}}{8}\right)^J \sum_{m=-J}^J (-1)^{J+m} f_m^{(J)} \bar{f}_m^{(J)} \] (2.5)
for any positive half-integer J. It is easy to verify that, when A and B undergo the Möbius transformation Eq.2.3, the functions \( f_m^{(J)}, \bar{f}_m^{(J)} \) transform as standard (finite-dimensional) spin J representations of \( sl(2, \mathbb{C}) \), and Eq.2.3 is the corresponding singlet. For general J, Eq.2.3 is still valid formally, with the \( m \)-sum extending to \(+\infty\) or to \(-\infty\), depending on whether we work with the semi-infinite representations with \( J+m \) positive integer, or \( J-m \) positive integer. In practice, Eq.2.3 is an expansion in \((A/B)^m\), and highest-weight representations, with \(-\infty \leq m \leq J\) (resp. lowest-weight representations, with \(-J \leq m \leq \infty\)) will give a convergent expansion for \(|A| > |B|\) (resp. \(|B| > |A|\)). Both choices should represent the same function, since they are just related by the particular \( sl(2) \)-transformation
\[ A \rightarrow -1/A \quad B \rightarrow -1/B \] (2.6)
which sends \( m \rightarrow -m \) in \( f_m^{(J)}, \bar{f}_m^{(J)} \). Of course, in the case of positive half-integer J, this amounts only to a trivial permutation of terms in the sum Eq.2.5. For continuous J, however, the highest resp. lowest weight representations are representations only of the algebra but not of the group, due to the multivaluedness of the \( f_m^{(J)} \) under the group operations Eq.2.3. Consequently, the transformation Eq.2.6 exchanges highest and lowest weight representations. From the general point of view of Toda theory, it can be regarded as representing the Weyl group symmetry[14].

Periodicity of \( \Phi \) implies that A and B must be periodic up to a projective transformation, which is called the monodromy matrix. In the elliptic and hyperbolic sectors of the theory, we can always pick a representative of the equivalence class defined by Eq.2.3 such that the monodromy matrix is diagonal, i.e. such that A and B are periodic up to a multiplicative constant. In fact, there are precisely two such representatives, related by Eq.2.6 which can thus be viewed in this context as a kind of residual symmetry. It is then possible to define two equivalent sets of chiral free fields by
\begin{align*}
\sqrt{\gamma} \vartheta_1(u) & := \ln A'^{-1/2}(u), & \sqrt{\gamma} \vartheta_2(u) & := \ln AA'^{-1/2}(u), \\
\sqrt{\gamma} \bar{\vartheta}_1(v) & := \ln BB'^{-1/2}(v), & \sqrt{\gamma} \bar{\vartheta}_2(v) & := \ln B'^{-1/2}(v)
\end{align*}
(2.7)
Indeed, one may show that the canonical Poisson brackets of Liouville theory give the following free-field Poisson bracket relations
\[ \{ \vartheta'_1(\sigma_1), \vartheta'_2(\sigma_2) \}_{\text{P.B.}} = \{ \vartheta'_2(\sigma_1), \vartheta'_2(\sigma_2) \}_{\text{P.B.}} = 2\pi \delta'(\sigma_1 - \sigma_2) \] (2.8)
with similar relations for the bar components. The explicit mode expansions in terms of zero modes and oscillators are given by
\[ \vartheta_j(u) = q_0^{(j)} + p_0^{(j)} u + i \sum_{n \neq 0} e^{-\imath n u} p_n^{(j)} / n, \]
\[ \bar{\vartheta}_j(u) = -\bar{q}_0^{(j)} + \bar{p}_0^{(j)} u - i \sum_{n \neq 0} e^{-\imath n u} \bar{p}_n^{(j)} / n, \]
\[ -\vartheta_1 \rightarrow \vartheta_2, \quad -\vartheta_2 \rightarrow \vartheta_1. \]
\[ \vartheta_j(v) = q_0^{(j)} + p_0^{(j)} v + i \sum_{n \neq 0} e^{-inv} \bar{p}_n^{(j)}/n, \quad j = 1, 2 \]  

(2.9)

From Eq.2.3 we see that the periodicity properties of the A and B fields can be parametrized by the zero mode momenta e.g. of \( \vartheta_1 \):

\[ A(u + 2\pi) = e^{-4\pi p_0^{(1)}} \sqrt{\tau} A(u), \quad B(v + 2\pi) = e^{+4\pi \bar{p}_0^{(1)}} \sqrt{\tau} B(v) \]  

(2.10)

The complete symmetry of the treatment of the theory under the exchange of \( \vartheta_1, \bar{\vartheta}_1 \) and \( \vartheta_2, \bar{\vartheta}_2 \) even on the quantum level is the hallmark of Gervais-Neveu quantization and guarantees the preservation of the residual symmetry Eq.2.6. From Eqs.2.4, 2.7 we have that the fields \( f_m^{(j)} \) can be written as products of exponentials of the \( \vartheta_1 \) and \( \vartheta_2 \) fields:

\[ f_m^{(j)} = e^{(J-m)\sqrt{\tau} \vartheta_1} e^{(J+m)\sqrt{\tau} \vartheta_2} \]  

(2.11)

Though this form is, in principle, accessible directly to quantization for any \( J \) and \( m \) (cf. ref.10), for the purposes of the present paper it is more appropriate to work with an alternative Coulomb-gas-type representation in terms of one free field only. Let us consider the special cases \( m = \pm J \) of Eq.2.11 where we have

\[ f_{-J}^{(j)} = (\frac{1}{\sqrt{A}})^{2J} = e^{2J \sqrt{\tau} \vartheta_1}, \quad f_J^{(j)} = (\frac{A}{\sqrt{A}})^{2J} = e^{2J \sqrt{\tau} \vartheta_2}, \]  

(2.12)

Using the periodicity requirement Eq.2.10, one easily derives the relations

\[ A(u) = \left\{ e^{-4\pi p_0^{(1)}} \sqrt{\tau} \int_0^u f_1^{(-1)}(\rho) d\rho + \int_u^{2\pi} f_1^{(-1)}(\rho) d\rho \right\} / \left( e^{-4\pi p_0^{(1)}} \sqrt{\tau} - 1 \right) \]  

(2.13)

\[ -\frac{1}{A(u)} = \left\{ e^{-4\pi p_0^{(2)}} \sqrt{\tau} \int_0^u f_1^{(-1)}(\rho) d\rho + \int_u^{2\pi} f_1^{(-1)}(\rho) d\rho \right\} / \left( e^{-4\pi p_0^{(2)}} \sqrt{\tau} - 1 \right) \]  

(2.14)

Then we may rewrite Eq.2.3 as

\[ f_m^{(j)} = \sqrt{\frac{2J}{J+m}} f_{-J}^{(j)} A^{J+m} = \sqrt{\frac{2J}{J+m}} f_J^{(j)} (A^{-1})^{J-m} \]  

(2.15)

The starting point of the quantization is to replace Eqs.2.8 by their quantum counterparts, so that we now have

\[ \left[ \vartheta_1'(\sigma_1), \vartheta_1'(\sigma_2) \right] = \left[ \vartheta_2'(\sigma_1), \vartheta_2'(\sigma_2) \right] = 2\pi i \delta'(\sigma_1 - \sigma_2) \]  

(2.16)

It was shown in ref.15 that \( \vartheta_1 \) and \( \vartheta_2 \) are related by a complicated canonical transformation; however, the relation between the zero modes is simple:

\[ p_0^{(1)} = -p_0^{(2)}, \quad \bar{p}_0^{(1)} = -\bar{p}_0^{(2)}. \]  

(2.17)

Instead of \( p_0^{(1)} \) (or \( \bar{p}_0^{(2)} \)) it will be more convenient to work with the rescaled zero mode

\[ \varpi := i p_0^{(1)} \sqrt{\frac{2\pi}{h}} \]  

(2.18)

with \( h \) defined in terms of the central charge \( C \) by

\[ h = \frac{\pi}{12} (C - 13 - \sqrt{(C - 25)(C - 1)}) \]  

(2.19)

The parameter \( h \) which is the deformation parameter of \( sl(2) \), is also in effect the Planck constant of the quantum Liouville theory.
3 The case of a single screening charge

3.1 The braiding of the holomorphic components

Starting from the representation Eq.2.15, and following the method of ref. [11], we construct the quantum equivalents of the fields $f_{m}^{(J)}$. In the earlier papers they have been noted $V_{m}^{(J)}$, $\tilde{V}_{m}^{(J)}$, or $U_{m}^{(J)}$, depending upon the normalization chosen. They are periodic up to a multiplicative constant and thus can be considered as Bloch waves. On the other hand, there is also a second basis of chiral operators $\xi_{M}^{(J)}$, which are by construction explicitly covariant under the quantum group [2][9], and related to the Bloch wave fields by a linear transformation. In the present article, we will concentrate on the Bloch wave basis; the discussion of the $\xi_{M}^{(J)}$ fields for continuous spins will be carried out elsewhere. The construction of the Bloch wave vertex operators and their exchange algebra was essentially displayed in ref. [11], we go through it again as a preparation for the case of two screening charges, and to make some points which were left out before for brevity. As discussed above, we consider the semi-infinite families of Bloch wave operators with $J + m$ or $J - m$ a non-negative integer. It turns out that there exists a consistent operator algebra where the two types of families do not mix [13]. Thus we may concentrate on one type, say the case with $J + m = 0, 1, \ldots$. Then the quantum version of $f_{m}^{(J)}$ is most easily obtained from the quantum versions of (the left equality in) Eq.2.15, and of Eq.2.13. According to Eq.2.13 this leads to quantum expressions in terms of $\vartheta_{1}$ — note that the other case ($J - m$ integer) may be obtained by the replacement $\vartheta_{1} \leftrightarrow \vartheta_{2}$ everywhere (cf. also section 4.1.5).

To begin with, the factor $f_{-J}^{(J)}$ is replaced by the normal-ordered exponential

$$f_{-J}^{(J)}|_{\text{qu}} = U_{-J}^{(J)} = : e^{2J \sqrt{h/2\pi \vartheta_{1}}}. \quad (3.1)$$

The parameter $h$ is that of Eq.2.19. The change of the coefficient in the exponential is such that this field has conformal weight

$$\Delta_{J} = -J - \frac{h}{\pi} J(J + 1). \quad (3.2)$$

If $2J$ is a positive integer, this coincides with Kac's formula, and $U_{-J}^{(J)}$ is a $(1, 2J + 1)$ primary in the BPZ classification. Now let us turn to the second factor $A^{J+m}$ appearing on the left equation of Eq.2.15. As already used in [11], the classical expression Eq.2.13 for $A$ has a rather simple quantum generalization, which we will denote by $S$ to signify that it is is a primary field of dimension zero ("screening charge"), namely [17]

$$S(\sigma) = e^{2h(\varpi + 1)} \int_{0}^{\sigma} d\rho U_{1}^{(-1)}(\rho) + \int_{\sigma}^{2\pi} d\rho U_{1}^{(-1)}(\rho). \quad (3.3)$$

Apart from an overall change of normalization — removal of the denominator — and the introduction of normal orderings, the only change consists in the replacement
\[ \varpi \rightarrow \varpi + 1 \] in the prefactor of the first integral. The quantum formula is such that \( S \) is periodic up to a multiplicative factor

\[ S(\sigma + 2\pi) = e^{2ih(\varpi+1)}S(\sigma). \quad (3.4) \]

This is the quantum version of the left equation in Eq. 2.10. The basic primary field of the Coulomb gas picture is now defined as

\[ U_m^{(J)}(\sigma) = U_{-j}^{(J)}(\sigma)[S(\sigma)]^{J+m} \quad (3.5) \]

which is the quantum version of the first equality in Eq. 2.15. The product of operators at the same point implied in Eq. 3.5 exists for small enough \( h \) (more on this below). Since \( S \) is a screening operator, the conformal dimension of \( U_m^{(J)} \) agrees with Eq. 3.2. Furthermore, one easily verifies that

\[ U_m^{(J)}(\varpi) = (\varpi + 2m)U_m^{(J)} \quad (3.6) \]

Here we are assuming \( \varpi \) to be real, as is appropriate in the so-called elliptic sector of the theory (cf. section 4.1). Also in the rest of the paper we will concentrate on this case, if not indicated otherwise. It is the case which appears to be directly related to (tree level) amplitudes in \( c \leq 1 \) string theory\[7\]. The normalization of the \( U_m^{(J)} \) operators is given by

\[ <\varpi|U_m^{(J)}|\varpi + 2m> = I_m^{(J)}(\varpi) \quad (3.7) \]

and \( I_m^{(J)}(\varpi) \) is computed in appendix B to be\[7\]

\[ I_m^{(J)}(\varpi) = \left( 2\pi \Gamma(1 + \frac{h}{\pi}) \right)^{J+m} e^{ih(J+m)(\varpi-J+m)} \prod_{\ell=1}^{J+m} \frac{\Gamma[1 + (2J - \ell + 1)h/\pi]}{\Gamma[1 + \ell h/\pi] \Gamma[1 - (\varpi + 2m - \ell)h/\pi] \Gamma[1 + (\varpi + \ell)h/\pi]} \quad (3.8) \]

This formula illustrates an important point to be made about the integral representation Eq.3.3. For small enough \( h \), the arguments of the gamma functions are all positive, and this corresponds to the domain where the integral representation is convergent. When \( h \) increases, divergences appear. However, Eq.3.8 continues to make sense beyond the poles by the usual analytic continuation of the Gamma function. As is well known\[10\], the continuation of the ground state expectation value \( I_m^{(J)}(\varpi) \) defines the continuation of the operator \( U_m^{(J)} \) itself. Thus, \( U_m^{(J)} \) is related to the normalized operator \( V_m^{(J)} \) with \( <\varpi|V_m^{(J)}(\sigma = 0)|\varpi + 2m> = 1 \) (introduced already in ref.[8] for half-integer positive \( J \)) by

\[ U_m^{(J)} = I_m^{(J)}(\varpi)V_m^{(J)} \quad (3.9) \]

Note that \( U_{-j}^{(J)} \equiv V_{-j}^{(J)} \).

We now come to the braiding algebra of the fields \( U_m^{(J)} \). For half-integer positive spin — the case corresponding to Kac’s table — it is well known that the braiding

\[ \text{Eq.2.10} \]

\[ \text{Eq.3.15} \]

\[ \text{Eq.3.3} \]

\[ \text{Eq.3.2} \]

\[ \text{Eq.3.5} \]

\[ \text{Eq.3.7} \]

\[ \text{Eq.3.8} \]

\[ \text{Eq.3.9} \]
of the $U_n^{(J)}$ or $V_m^{(J)}$ is essentially given by a q-6j-symbol, and the explicit formulae were determined in ref. 3 (The general result is summarized in appendix A). In ref. 11, this result was extended to arbitrary $J$. We will recall some basic points of the derivation that will be useful later on. The braiding relation takes the form

$$U_m^{(J)}(\sigma)U_{m'}^{(J')}(\sigma') = \sum_{m_1,m_2} R_{U}(J,J';\varpi)m_{m_1}^{m_2}U_{m_2}^{(J')}m_{m_1}^{(J)}(\sigma)U_{m_1}^{(J)}(\sigma).$$

(3.10)

We only deal with the case $\pi > \sigma' > \sigma > 0$ explicitly. The other cases are deduced from the present one in the standard way. The sums extend over non-negative integer $J + m_1$ resp. $J' + m_2$ with the condition

$$m_1 + m_2 = m + m' =: m_{12}.$$  

(3.11)

Since one considers the braiding at equal $\tau$ one can let $\tau = 0$ once and for all. As there are no null-vector decoupling equations for continuous $J$, the derivation of Eq.(3.10) relies exclusively on the free field techniques summarized in the previous section. The basic point of our argument is that the exchange of two $U_m^{(J)}$ operators can be mapped into an equivalent problem in one-dimensional quantum mechanics, and becomes just finite-dimensional linear algebra. In view of Eqs.3, 3.4, the essential observation is that one only needs the braiding relations of $U_m^{(J)} \equiv V_{-J}$ operators which are normal ordered exponentials ("tachyon operators"). One has

$$V_{-J}^{(J)}(\sigma)V_{-J'}^{(J')}m_{m_1}^{(J)}(\sigma')V_{-J}^{(J)}(\sigma)$$

(3.12)

where $\epsilon(\sigma - \sigma')$ is the sign of $\sigma - \sigma'$. This means that when commuting the tachyon operators in $U_m^{(J)}(\sigma')$ through those of $U_m^{(J)}(\sigma)$, one only encounters phase factors of the form $e^{\pm 2\alpha\beta h}$ resp. $e^{\pm 6\alpha\beta h}$, with $\alpha$ equal to $J$ or $-1$, $\beta$ equal to $J'$ or $-1$, since we take $\sigma, \sigma' \in [0, \pi]$. Hence we are led to decompose the integrals defining the screening charges $S$ into pieces which commute with each other and with $V_{-J}^{(J)}(\sigma)$, $V_{-J'}^{(J')}(\sigma')$, up to one of the above phase factors. We consider explicitly only the case $0 < \sigma < \sigma' < \pi$ and write

$$S(\sigma) = S_{\sigma\sigma'} + S_{\Delta}, \quad S(\sigma') = S_{\sigma\sigma'} + k(\varpi)S_{\Delta} \equiv S_{\sigma\sigma'} + \tilde{S}_{\Delta},$$

$$S_{\sigma\sigma'} := k(\varpi) \int_{0}^{\sigma} V_1^{(\sigma)}(\rho)d\rho + \int_{\sigma}^{2\pi} V_1^{(\sigma)}(\rho)d\rho,$$

$$S_{\Delta} := \int_{\sigma}^{2\pi} V_1^{(\sigma)}(\rho)d\rho, \quad k(\varpi) := e^{2i\varpi(\varpi - 1)}$$

(3.13)

Using Eq.3.13, we then get the following simple algebra for $S_{\sigma\sigma'}, S_{\Delta}, \tilde{S}_{\Delta}$:

$$S_{\sigma\sigma'}S_{\Delta} = q^{-2}S_{\Delta}S_{\sigma\sigma'}, \quad S_{\sigma\sigma'}\tilde{S}_{\Delta} = q^2\tilde{S}_{\Delta}S_{\sigma\sigma'}, \quad S_{\Delta}\tilde{S}_{\Delta} = q^4\tilde{S}_{\Delta}S_{\Delta},$$

(3.14)

and their commutation properties with $V_{-J}^{(J)}(\sigma), V_{-J'}^{(J')} (\sigma')$ are given by

$$V_{-J}^{(J)}(\sigma)S_{\sigma\sigma'} = q^{-2J}S_{\sigma\sigma'}V_{-J}^{(J)}(\sigma), \quad V_{-J'}^{(J')} (\sigma')S_{\sigma\sigma'} = q^{-2J'}S_{\sigma\sigma'}V_{-J'}^{(J')} (\sigma'),$$

$$V_{-J}^{(J)}(\sigma)S_{\Delta} = q^{-2J}S_{\Delta}V_{-J}^{(J)}(\sigma), \quad V_{-J'}^{(J')} (\sigma')\tilde{S}_{\Delta} = q^{-6J}\tilde{S}_{\Delta}V_{-J'}^{(J')} (\sigma),$$

$$V_{-J'}^{(J')} (\sigma')S_{\Delta} = q^{2J}S_{\Delta}V_{-J'}^{(J')} (\sigma'), \quad V_{-J'}^{(J')} (\sigma')\tilde{S}_{\Delta} = q^{-2J}\tilde{S}_{\Delta}V_{-J'}^{(J')} (\sigma').$$

(3.15)
Finally, all three screening pieces obviously shift the zero mode in the same way:

\[
\begin{align*}
S_{\sigma}\sigma' \\
S_{\Delta} \\
\tilde{S}_{\Delta}
\end{align*}
\begin{array}{c}
\varpi = (\varpi + 2) \\
\frac{S_{\sigma'}}{S_{\Delta}} \\
\frac{\tilde{S}_{\Delta}}{S_{\Delta}}
\end{array}
\]  

(3.16)

Using Eqs. 3.13 we can commute \( \tilde{V}_{\sigma j}(\sigma) \) and \( \tilde{V}_{\sigma j'}(\sigma') \) to the left on both sides of Eq. 3.10, so that they can be cancelled. Then we are left with

\[
(q^{-2J'}S_{\Delta} + q^{2J'}S_{\sigma'})^{J + m}(\tilde{S}_{\Delta} + S_{\sigma'})^{J' + m'}q^{2JJ'} =
\sum_{m_1,m_2} R(J, J'; \varpi + 2(J + J'))^{m_2m_1} (q^{2J}S_{\sigma'} + q^{6J}\tilde{S}_{\Delta})^{J' + m_2}(S_{\sigma'} + S_{\Delta})^{J + m_1}
\]  

(3.17)

It is apparent from this equation that the braiding problem of the \( U_m^{(j)} \) operators is governed by the Heisenberg-like algebra Eq. 3.14, characteristic of one-dimensional quantum mechanics. We will proceed using the following simple representation of the algebra Eq. 3.14 in terms of one-dimensional quantum mechanics (\( y \) and \( y' \) are arbitrary complex numbers):

\[
S_{\sigma} = y'^2, \quad S_{\Delta} = ye^{2Q-P}, \quad \tilde{S}_{\Delta} = ye^{2Q+P}, \quad [Q, P] = ih.
\]  

(3.18)

The third relation in Eq. 3.18 follows from the second one in view of \( \tilde{S}_{\Delta} = k(\varpi)S_{\Delta} \) (cf. Eq. 3.13). This means we are identifying here \( P \equiv ih\varpi \) with the zero mode of the original problem. Using \( e^{2Q+P} = ec^{-2Q}q^2 \) we can commute all factors \( e^{2Q} \) to the right on both sides of Eq. 3.17 and then cancel them. This leaves us with

\[
q^{2JJ'} \prod_{s=1}^{J+m}(y'^2-q^{2J} + yq^{-(\varpi-2J+2s-1)}) \prod_{t=1}^{J'+m'} (y' + yq^{\varpi-2J'+2m+2t-1}) =
\sum_{m_1} R_U(J, J'; \varpi)^{m_2m_1} \prod_{t=1}^{J'+m_2}(y'^2q^{2J} + yq^{\varpi+4J-2J'+2t-1}) \prod_{s=1}^{J+m_1} (y' + yq^{-(\varpi-2J+2m_2+2s-1)})
\]  

(3.19)

where we have shifted back \( \varpi + 2(J + J') \rightarrow \varpi \) compared to Eq. 3.17. Since the overall scaling \( y \rightarrow \lambda y, y' \rightarrow \lambda y' \) only gives back Eq. 3.11, we can set \( y' = 1 \).

The solution of these equations, which was derived in ref. [1], will be cast under the convenient form

\[
R_U(J, J', \varpi)^{m_2m_1} = e^{-i\pi(\Delta_e+\Delta_k-\Delta_x-\Delta_f)k_{ab}^eK_{de}^f}\{a \ b \ | \ e \} \{d \ c \ | \ f \}
\]

(3.20)

where \( \{a \ b \ | \ e \} \) is the q-6j-symbol generalized to continuous spins, with arguments

\[
\begin{align*}
a &= J, \quad b = x + m + m', \quad c = x \equiv (\varpi - \varpi_0)/2 \\
d &= J', \quad e = x + m_2, \quad f = x + m
\end{align*}
\]

(3.21)
We have defined, as in the previous work along the same line, in Eq.3.23, the prefactor involves products of the type just recalled with indices of the square roots of the individual factors consisting of a single q-number. Eq.3.30). In particular, the square root in Eq.3.22 is to be understood as the product resp. quotient with \( \kappa \) ⌊.

Recall that we let

\[
\kappa := \frac{\sin(x)}{\sin h},
\]

where

\[
\kappa := \frac{\sin(x)}{\sin h}.
\]

This transformation formula is derived in ref.[13].

The method used to derive Eq.3.20 was to transform the hypergeometric function into another one such that the desired relations Eq.3.19 follow from the orthogonality relation of the associated Askey-Wilson polynomials. In this connection, let us note that a simple reshuffling of the parameters of the latter form allows to verify that the usual orthogonality relations of the 6-j symbols extend to our case. One has, in general,

\[
\sum_{J_{23}} \left\{ J_1 J_2 J_{123} \right\} \left\{ J_3 J_4 J_{123} \right\} K_{12} K_{23} = \delta_{J_{12} - K_{12}}.
\]

We follow the prescription of refs.[7],[13] for the definition of the square roots (cf. also below Eq.3.30). In particular, the square root in Eq.3.22 is to be understood as the product resp. quotient of the square roots of the individual factors consisting of a single q-number.

This transformation formula is derived in ref.[13].
where the $J$'s are arbitrary except for the constraint that the screening numbers

$$n_1 = J_1 + J_2 - J_{12}, \quad n_2 = J_3 + J_{12} - J_{13}, \quad n = J_1 + J_2 - J_{12}, \quad n' = J_2 + J_3 - J_{12},$$

$$\bar{n}_1 = J_1 + J_2 - K_{12}, \quad \bar{n}_2 = J_3 + K_{12} - J_{13},$$

are positive integers. These conditions fix the range of summation over $J_{12}$.

The basis $U_{m}^{(J)}$, apart from its manageability, has another practical virtue: Its braiding (and fusion) properties are given in a form which involves no square roots, but only $(q$-deformed) rational functions, and no phase ambiguities can arise. On the other hand, from the quantum group point of view it is more natural to consider a basis where the braiding (and fusion) is given exclusively in terms of the $6j$-symbol (the latter does however involve square roots). For this purpose, the authors of ref. [9] introduced the fields $\tilde{V}^{(J)}_m$, which are defined by

$$\tilde{V}^{(J)}_m = g^{x}_{J,x+m} V^{(J)}_m.$$

(3.29)

As before we let $x = (\varpi - \varpi_0)/2$. The coupling constants $g$ are defined by

$$g^{x}_{J,x+m} = \left( \frac{h}{\pi} \right)^{J+m} \prod_{k=1}^{J+m} \sqrt{F[1 + (2J - k + 1)h/\pi]} \sqrt{F[(\varpi + 2m - k)h/\pi]} \times$$

$$\prod_{k=1}^{J+m} \sqrt{F[-(\varpi + k)h/\pi]} \Big/ \sqrt{F[1 + kh/\pi]}$$

(3.30)

where, as usual, $F(z) := \Gamma(z)/\Gamma(1 - z)$. The treatment of the square roots requires some care. We follow the prescription of ref. [10] also used in ref. [13]. Eq. (3.30) immediately extends to the case of non integer $J$, as $J + m$ remains a positive integer. We then have ($m_{12} := m_1 + m_2 = m + m'$)

$$\tilde{V}^{(J)}_m (\sigma) \tilde{V}^{(J')}_{m'} (\sigma') = e^{-i\pi(\Delta_x + \Delta_{x+m_{12}} - \Delta_{x+m} - \Delta_{x+m'})} \times$$

$$\sum_{m_1,m_2} \left\{ J, \begin{array}{c} x+m_{12} \\ x \end{array} \bigg| \begin{array}{c} x+m_2 \\ x+m \end{array} \right\} \tilde{V}^{(J')}_{m_2} (\sigma') \tilde{V}^{(J)}_{m_1} (\sigma)$$

(3.31)

The relation with $U_{m}^{(J)}$ is given by

$$U_{m}^{(J)} = \frac{I_{m}(\varpi)}{g^{x}_{J,x+m}} \tilde{V}_{m}(\varpi) \equiv \frac{1}{\kappa^{x}_{J,x+m}} \tilde{V}_{m}(\varpi).$$

(3.32)

The $\kappa$ coefficient should, of course, be given by Eq. (3.22). This is checked in appendix B. Concerning the right-moving modes, the braiding algebra is given by

$$\tilde{V}^{(J)}_m (\sigma) \tilde{V}^{(J')}_{m'} (\sigma') = e^{i\pi(\Delta_x + \Delta_{x+m_{12}} - \Delta_{x+m} - \Delta_{x+m'})} \times$$

$$\sum_{m_1,m_2} \left\{ J_1, \begin{array}{c} \bar{x}+m_{12} \\ \bar{x} \end{array} \bigg| \begin{array}{c} \bar{x}+m_2 \\ \bar{x}+m \end{array} \right\} \tilde{V}^{(J')}_{m_2} (\sigma') \tilde{V}^{(J)}_{m_1} (\sigma)$$

(3.33)

where we let $\bar{x} = (\varpi - \varpi_0)/2$. The only difference with Eq. (3.31) is the change of sign of the phase factor. This may be verified by redoing the whole derivation. In ref. [7],
it was remarked that the right-mover braiding matrix is deduced from the left-mover one by changing $i = \sqrt{-1}$ into $-i$, since this correctly changes the orientation of the complex plane. This complex conjugation is most easily performed using the $U$ fields, since the braiding matrix Eq.3.20 is real apart from the first phase factor. For the $V$ fields, there is a slight subtlety related again to the appearance of the redundant square roots in Eq.3.22. The correct rule is to take the same definition for the square roots for left and right movers. Thus the right-moving coupling constant $\tilde{g}_{J,x+m}$ is given by the same expression Eq.3.30, not its complex conjugate. The same prescription should be followed for the roots appearing in $\overline{\kappa}_{J,x+m}$, while taking the usual complex conjugate for the phase factor appearing in front of the product in Eq.3.22. Note that $\varpi$ is always to be treated as real formally in this context, even in the hyperbolic sector where it is actually purely imaginary (cf. below).

4 Solving of the Liouville quantum dynamics

4.1 The Liouville exponential

First, let us note that, as $\hbar$ is real in the weak-coupling regime, the hermiticity of energy-momentum allows for $\varpi, \overline{\varpi}$ real or purely imaginary, corresponding to the elliptic resp. hyperbolic sector of the theory [18] (see also [16] for the case of open boundary conditions). In the former case, which we consider in this paper, we will see that the locality conditions are fulfilled if

$$\varpi - \overline{\varpi} = k\pi/\hbar, \quad k \in \mathbb{Z} \quad (4.1)$$

Eq.4.1 has an immediate interpretation as the natural generalization of the classical boundary conditions [17]. Moreover we will show that the appropriate definition of the Liouville exponential for arbitrary $J$ is

$$e^{-J\alpha_- \Phi(\sigma,\tau)} = \sum_{m=-J}^{\infty} \mu_0^{J+m} \tilde{V}_m(J)(u) \overline{V}_m(J)(v) \quad (4.2)$$

where $\alpha_- = 2\sqrt{\hbar/2\pi}$ is the screening charge. The constant $\mu_0^{J+m}$ will not be fixed by braiding or fusion. It will be determined below when we derive the field equations.

4.1.1 Locality

Let us now check locality. In the approach of refs. [17], one takes the zero modes of the left-moving and right-moving Liouville modes to commute, so that $U$ and $\overline{U}$ commute. However, operators involving both chiralities should be applied only to states fulfilling Eq.4.1, and conserve this condition. This is why we must have $\tilde{m} = m$ in Eq.4.2.\footnote{The situation is quite different in the strong coupling theory, see e.g. ref. [13].} Next we observe that if Eq.4.1 is valid,

$$\overline{\mathcal{R}}_{\tilde{V}}(J,J';\varpi)^{\tilde{m}_2\tilde{m}_1}_{\tilde{m}} = \mathcal{R}_{\tilde{V}}(J,J';\varpi)^{m_2m_1}_{m\tilde{m}} \quad (4.3)$$
as can be verified easily. The same is true for \( \overline{R}_\gamma \). Thus we have

\[
\overline{V}^{(J)}_{m}(\sigma)\overline{V}^{(J')}_{m'}(\sigma') = e^{+i\pi(\Delta_{x+m_1} - \Delta_{x+m_2} - \Delta_{x+m_3} - \Delta_{x+m_4})} \times \\
\sum_{m_1, m_2} \left\{ J_1 \right\}_{x+m_1} \left\{ J_2 \right\}_{x+m_2} \overline{V}^{(J')}_{m_2}(\sigma')\overline{V}^{(J)}_{m_1}(\sigma)
\]  

(4.4)

Then, according to Eqs. (3.31, 4.4) we get

\[
e^{-J\alpha_\Phi(\sigma, \tau)}e^{-J'\alpha_\Phi(\sigma', \tau)} = \\
\sum_{m, m', m_1, m_2} e^{i\theta(m_2-m_1)(m_2-m_2+\alpha)} \left\{ J \right\}_{x+m_1+m_2} \left\{ J' \right\}_{x+m_1} \overline{V}^{(J')}_{m_2}(u')\overline{V}^{(J')}_{m_1}(v')\overline{V}^{(J)}_{m_1}(v) \overline{V}^{(J)}_{m_2}(u) \times \\
\mu_0^{J+J'+m+m'} \overline{V}^{(J)}_{m_2}(u)\overline{V}^{(J')}_{m_1}(v) \overline{V}^{(J)}_{m_1}(v) \overline{V}^{(J)}_{m_2}(u) \mu_0^{J+J'+m+m'} \overline{V}^{(J')}_{m_2}(u') \overline{V}^{(J')}_{m_1}(v') \overline{V}^{(J)}_{m_1}(v) \overline{V}^{(J)}_{m_2}(u).
\]  

(4.5)

One first sums over \( m \), with fixed \( m_{12} = m + m' \). This precisely corresponds to the summation over \( J_{23} \) in Eq. (4.28). Thus only \( m_2 = \bar{m}_2 \) contributes. This gives immediately

\[
e^{-J_1\alpha_\Phi(\sigma_1, \tau)}e^{-J_2\alpha_\Phi(\sigma_2, \tau)} = e^{-J_2\alpha_\Phi(\sigma_2, \tau)}e^{-J_1\alpha_\Phi(\sigma_1, \tau)}.
\]  

(4.6)

and the Liouville exponential is local for arbitrary \( J \). We remark that Eq. (4.4) is not only sufficient, but also necessary for locality, as was observed in ref. [17] for the special case \( J = 1/2 \).

### 4.1.2 Closure by fusion

In the preceding analysis, we have discussed only the braiding properties of the chiral fields resp. the Liouville exponentials. However, according to the general Moore-Seiberg formalism [12], fusion (in the sense of the full operator product) and braiding are not independent. Assuming the validity of the Moore-Seiberg relation between fusion and braiding matrix, we then obtain immediately that the fusion of the \( \overline{V} \) fields should be given by (cf. also ref. [13])

\[
\overline{V}^{(J)}_{m_1}(z_1)\overline{V}^{(J')}_{m_2}(z_2) = \sum_{J_{12} = -m_1 - m_2} \left\{ J_1 \right\}_{x+m_1+m_2} \left\{ J_2 \right\}_{x+m_1} \times \\
\overline{V}^{(J_1)}_{m_1+m_2}(z_1)\overline{V}^{(J_2)}_{m_1+m_2}(z_2) < \varpi J_{12} \{ \nu_{12} \} |\overline{V}^{(J_1)}_{J_2-J_{12}}(z_1-z_2)|\varpi J_2 >
\]  

(4.7)

In Eq. (4.7) we have changed variables by letting \( z = e^{i(\tau+\sigma)} \), \( \bar{z} = e^{i(\tau-\sigma)} \) (recall that we are using Minkowski world-sheet variables). The only difference to the positive half-integer spin case, which was completely analyzed in ref. [8], is that the \( J_{12} \)-sum now extends to \(-m_1 - m_2 \) instead of \( |J_1 - J_2| \). Indeed, the positivity of the screening numbers appearing in the braiding matrix leads via the Moore-Seiberg relation to \footnote{11 with Euclidean variables, this would mean that we change to the sphere.}
the positivity of the screening numbers \( n_1 = J_1 + m_1, \ n_2 = J_2 + m_2, \ \rho_{12} = J_1 + J_2 - J_{12}, \ n = J_{12} + m_1 + m_2 \) of the fusion matrix. In ref. \[13\], it has been shown that the generalized 6j-symbol of Eq.3.20 together with the positivity condition for the screening charges, fulfills all the necessary identities for the polynomial equations to be valid with continuous spins. This provides a strong argument that the fusion matrix of Eq.4.7 is indeed the correct one, even though we have not attempted to derive it directly as we did for the braiding. Making use of the analogous equation for the bar components, one sees that the operator-product expansion of Liouville exponentials may be written as

\[
e^{-J_1 \alpha_\Phi(z_1, \bar{z}_1)} e^{-J_2 \alpha_\Phi(z_2, \bar{z}_2)} = 
\sum_{m_1, m_2} \sum_{J_{12}, J} \{ J_{J_1 x + m_1 + m_2} J_{J_2 x - m_1 + m_2} \} \times 
\mu_0^{J_{1 + m_1 + J_2 + m_2}} \sum_{\nu_{12}, \{ \nu \}} V^{(J_{12}, \{ \nu_{12} \})} (z_2) V^{(J_{12}, \{ \nu \})} (\bar{z}_2) 
\langle \omega J_{12}, \{ \nu_{12} \} | V_{J_{12} - J_{12}}^{(J_1)} (z_1 - z_2) | \omega J_{12} \rangle < \omega J_{12}, \{ \nu_{12} \} | V_{J_{12} - J_{12}}^{(J_1)} (\bar{z}_1 - \bar{z}_2) | \omega J_{12} \rangle. \tag{4.8}
\]

It follows from condition Eq.4.12 that \( \{ J_{J_1 x + m_1 + m_2} J_{J_2 x - m_1 + m_2} \} = \{ J_{J_1 x + m_1 + m_2} J_{J_2 x - m_1 + m_2} \} \). In the same way as for locality, the summation over \( m_1 \) with fixed \( m_1 + m_2 \) then reduces to the orthogonality relation for 6j-symbols so that only \( J_{12} = \bar{J}_{12} \) contributes, and one gets

\[
e^{-J_1 \alpha_\Phi(z_1, \bar{z}_1)} e^{-J_2 \alpha_\Phi(z_2, \bar{z}_2)} = 
\sum_{m, J_{12}} \sum_{\nu_{12}, \{ \nu \}} \mu_0^{J_{1 + m_1 + J_2 - J_{12}}} \times 
\langle \omega J_{12}, \{ \nu_{12} \} | V_{J_{12} - J_{12}}^{(J_1)} (z_1 - z_2) | \omega J_{12} \rangle < \omega J_{12}, \{ \nu_{12} \} | V_{J_{12} - J_{12}}^{(J_1)} (\bar{z}_1 - \bar{z}_2) | \omega J_{12} \rangle. \tag{4.9}
\]

The second line clearly involves the descendants of the Liouville exponentials which we denote by

\[
e^{-J_1 \alpha_\Phi(\nu, \{ \nu \}) (z, \bar{z})} \equiv \sum_m \mu_0^{J_{1 + m}} V_{m}^{(J_{12}, \{ \nu \})} (z) V_{m}^{(J_{12}, \{ \nu \})} (\bar{z}) \tag{4.10}
\]

As regards the last line, it is simply the corresponding matrix element of the Liouville exponential. One finally gets

\[
e^{-J_1 \alpha_\Phi(z_1, \bar{z}_1)} e^{-J_2 \alpha_\Phi(z_2, \bar{z}_2)} = 
\sum_{J_{12} = m_1 + m_2} \sum_{\nu, \{ \nu \}} e^{-J_{12} \alpha_\Phi(\nu, \{ \nu \}) (z_2, \bar{z}_2)} \times 
\langle \omega J_{12}, \{ \nu \} | V_{J_{12} - J_{12}}^{(J_1)} (z_1 - z_2, \bar{z}_1 - \bar{z}_2) | \omega J_{12}, \{ \nu \} \rangle \tag{4.11}
\]

The notation for the matrix element should be self-explanatory \[17\]. One sees that the Liouville exponential is closed by fusion for arbitrary \( J \) to all orders in the descendants.

\[\text{It is implied here that charge conservation should be used for the evaluation of the matrix element, such that only the term appearing in Eq.4.14 survives. According to ref.\[24\], charge conservation actually does not hold for the 3-point functions } \langle \omega | e^{-J_0 \Phi(z)} | \omega \rangle \text{ with continuous } J.\]

\[\text{From this point of view, the notation of Eq.4.11 is of course not rigorously appropriate.}\]
4.1.3 The cosmological constant revisited.

The braiding relation is invariant under the transformation

\[ \exp(-J\alpha_\Phi(z, \bar{z})) \rightarrow T \exp(-J\alpha_\Phi(z, \bar{z})) T^{-1}, \]  

where \( T \) is an arbitrary function of the zero modes \( \varpi \) and \( \bar{\varpi} \). This is why locality does not completely determine the Liouville exponentials. We have discussed this point in detail in ref.\[19\]. Concerning the fusion equation, the transformation just considered does not act on the last term on the right-hand side which is a c-number. The definition Eq.\[4.2\] we have chosen is such that this term — a compact bookkeeping device to handle all the descendants — is precisely given by the matrix elements of the Liouville exponential itself, without any additional normalization factor. It is thus quite natural. Note however that this choice of normalization differs from the previous one\[7\]; this point is discussed in appendix C and chapters 4.1.5, 4.1.6 below. The only remaining ambiguity is the arbitrariness in \( \mu_0 \). Changing this parameter is tantamount to changing the cosmological constant following ref.\[7\]. Indeed, the fusing and braiding relations of the \( \tilde{V} \) fields are invariant if we make the change \( \tilde{V}_m^{(J)} \rightarrow \mu_c^{(J+m)/2} \tilde{V}_m^{(J)} \). Any such change is generated by a combination of a field redefinition \( (\alpha_\Phi \rightarrow \alpha_\Phi - \ln \mu_c) \) and a similarity transformation of the form Eq.\[4.12\]. Thus the most general field satisfying Eqs.\[4.6\] and \[4.11\] is given by

\[ e^{-J\alpha_\Phi(\sigma, \tau)} = \sum_{m=-J}^{\infty} \left( \mu_0 \mu_c \right)^{J+m} \tilde{V}_m^{(J)}(u) \tilde{V}_m^{(J)}(v) = \mu_c^{1/2} e^{-J\alpha_\Phi(\sigma, \tau)} \mu_0^{\varpi/2} \]  

We will determine \( \mu_0 \) below so that it corresponds to a cosmological constant equal to one.

4.1.4 Expression in terms of Coulomb-gas fields

According to Eqs.\[3.35\] and \[3.32\] Eq.\[4.2\] may be rewritten as

\[ e^{-J\alpha_\Phi(\sigma, \tau)} = \sum_{m=-J}^{\infty} \mu_0^{J+m} \kappa_{J, x+m} \tilde{\kappa}_{J, \bar{x}+m} U_m^{(J)}(u) \bar{U}_m^{(J)}(v) \]

It is easy to see using condition Eq.\[4.1\] that

\[ \kappa_{J, \bar{x}+m} = \kappa_{J, \bar{x}+m} \]  

Thus the square roots combine pairwise and we are left with a rational expression.

\[ e^{-J\alpha_\Phi(\sigma, \tau)} = \sum_{m=-J}^{\infty} \tilde{\mu}_0^{J+m} (-1)^{J+m} \prod_{k=1}^{J+m} \frac{1 + 2J - k}{[k] [\varpi + 2m - k] [\varpi + k]} \times \]

\[ V_m^{(J)}(u) \bar{V}_m^{(J)}(v) S^{J+m} \tilde{S}^{J+m}, \]  

where we have let

\[ \tilde{\mu}_0 = \mu_0 \left( \frac{h}{2\pi \Gamma(1 + h/\pi) \sin h} \right)^2 \]  

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Using this Coulomb-gas expression, together with the mentioned orthogonality relations for Askey-Wilson polynomials, it is then possible to directly verify the locality of the Liouville exponential, without encountering any square root ambiguity. Note that Eq. 4.15 depends only on $\varphi$, not on $\overline{\varphi}$. On the other hand, the analysis of ref. [17] for $J = 1/2$ in the elliptic sector, when translated to the Coulomb gas basis, gives coefficients with an explicit dependence on $k$ of Eq. 4.1. Nevertheless, the two forms are equivalent, as they must, by means of a basis transformation Eq. 4.12, hence indistinguishable from the point of view of locality.

4.1.5 $\vartheta_1 \leftrightarrow \vartheta_2$ invariance

In section 2 we noted the existence of a symmetry of the theory under the exchange of the two free fields $\vartheta_1$ under $\vartheta_2$, the residual symmetry remaining after fixing the $SL_2(\mathbb{C})$ invariance. On the other hand, on the quantum level the expressions we have derived in the present paper for the Liouville field and its exponentials are not evidently symmetric under this exchange. However, we must remember here that the requirement of locality really fixed these operators only up to a similarity transformation Eq. 4.12 (the particular form Eq. 4.2 resp. Eq. 4.15 was only distinguished by its simplicity and its natural behaviour under fusion). Thus a priori we can expect $\vartheta_1 \leftrightarrow \vartheta_2$ invariance only to be valid up to a similarity transformation. As a matter of fact, we will show (for $J$ half-integer positive) that there exists a transformation $T(\varphi, \overline{\varphi})$ such that

$$T(\varphi, \overline{\varphi})e^{-Ja_\varphi T^{-1}}(\varphi, \overline{\varphi}) = T(-\varphi, -\overline{\varphi})e^{-Ja_{\overline{\varphi}} T^{-1}}(-\varphi, -\overline{\varphi}).$$

(4.17)

where the index (1) resp. (2) indicates the use of the $\vartheta_1$ resp. $\vartheta_2$ representation. This shows in addition that it is possible to choose particular representatives in the equivalence class of fields defined by Eq. 4.12 which are manifestly $\vartheta_1 \leftrightarrow \vartheta_2$ symmetric. To prove this we first observe that $\vartheta_1 \leftrightarrow \vartheta_2$ takes $\varphi$ into $-\varphi$ (cf. Eq. 2.17), whereas the normalized operators $V^{(J)}_m$ behave as

$$V^{(J)}_m \rightarrow V^{-(J)}_{-m}$$

(similarly for the right-movers). The latter follows by comparison of the conformal weights and zero mode shifts of the $V^{(J)}_m$ operators built from $\vartheta_1$ resp. $\vartheta_2$, as these two properties define normalized primary fields uniquely. For positive half-integer $J$, the summation range in Eq. 4.15 is $m = -J, \ldots, J$, hence symmetric under $m \rightarrow -m$, and the exchange $\vartheta_1 \leftrightarrow \vartheta_2$ essentially amounts only to a reorganization of terms. Then after commuting the $T$ operators to the left or to the right on both sides, Eq. 4.17 can be solved straightforwardly. A particular solution is

$$T(\varphi, \overline{\varphi}) = \sqrt{\frac{\Gamma(1 - \varphi h/\pi)\Gamma(1 - \overline{\varphi} h/\pi)}{\Gamma(1 + \varphi)\Gamma(1 + \overline{\varphi})}} \sqrt{\frac{|\varphi||\overline{\varphi}|}{\mu_0^{\varphi + \overline{\varphi}/4}}}(\varphi, \overline{\varphi}).$$

(4.19)
The last factor means effectively\(^\text{13}\) that we should put \(\mu_0 = 1\) in Eq.4.12 and Eq.4.15 (cf Eq.4.13), and so we will take \(\mu_0 = 1\) in the following. For \(\varpi = \overline{\varpi}\), Eq.4.19 reduces to the transformation written in appendix C to establish the connection between Eq.4.13 or 4.2 and the exponentials of ref.\([7]\). Thus the latter are also invariant under the \(\vartheta_1 \leftrightarrow \vartheta_2\) symmetry, and the same is true for the exponential of ref.\([17]\) which was constructed for arbitrary \(\varpi, \overline{\varpi}\). The solution Eq.4.19 is unique up to the replacement \(T(\varpi, \overline{\varpi}) \to T(\varpi, \overline{\varpi})T_1(\varpi, \overline{\varpi})\), with

\[
\frac{T_1(\varpi, \overline{\varpi})T_1(-\varpi - 2m, -\overline{\varpi} - 2m)}{T_1(\varpi + 2m, \overline{\varpi} + 2m)T_1(-\varpi, -\overline{\varpi})} = 1
\]  

(4.20)

Unfortunately the case of continuous \(J\), Eq.4.17 is not so easy to analyze, as the family of operators \(V^{(m)}_J\) with \(J + m = 0, 1, 2, \ldots\) is no longer invariant under the replacement \(m \to -m\), and Eq.4.17 becomes highly nontrivial. We leave this problem for a future publication and will restrict also in the next subsection to the case of positive half-integer \(J\).

4.1.6 Hermiticity

Another property of the Liouville exponentials that has not yet been discussed is hermiticity. As was worked out by Gervais and Neveu a long time ago\([10]\), the free fields possess the following behaviour under hermitian conjugation (for brevity of notation we write only the left-movers explicitly):

\[
\vartheta_1^\dagger = \vartheta_1, \quad \vartheta_2^\dagger = \vartheta_2 \quad (\varpi = -\overline{\varpi}^*)
\]

\[
\vartheta_1^\dagger = \vartheta_2, \quad \vartheta_2^\dagger = \vartheta_1 \quad (\varpi = \overline{\varpi}^*)
\]  

(4.21)

The first case corresponds to the hyperbolic sector of the theory, the second to the elliptic sector which we consider here. Consequently, one has for the vertex operators resp. screening charges:

\[
V_{-J}^{(j)} = V_{-J}^{(j)}, \quad V_{J}^{(j)} = V_{J}^{(j)}, \quad S_{(i)}^\dagger = S_{(i)} \quad (\varpi = -\overline{\varpi}^*)
\]

\[
V_{-J}^{(j)} = V_{J}^{(j)}, \quad V_{J}^{(j)} = V_{-J}^{(j)}, \quad S_{(i)}^\dagger = S_{(i)} \quad (\varpi = \overline{\varpi}^*)
\]  

(4.22)

where \(S_{(i)}\) denotes the screening charge constructed from \(\vartheta_i\). It is then immediate to show that in the elliptic sector,

\[
(e^{-(J\alpha - \Phi)}_{(1)})^\dagger = e^{-(J\alpha - \Phi)}_{(2)} \quad (\varpi = \overline{\varpi}^*)
\]  

(4.23)

Our exponentials can formally be interpreted also in the hyperbolic sector, and fulfill there

\[
(e^{-(\alpha - \Phi)}_{(i)})^\dagger = e^{-(\alpha - \Phi)}_{(i)} \quad (\varpi = -\overline{\varpi}^*)
\]  

(4.24)

\(^{13}\)Actually the last factor in Eq.4.13 removes only \(\mu_0^m\) in Eq.4.13, but the remaining normalization constant \(\mu_0^J\) plays no role here. It will become important, however, when we consider the equations of motion.
However, their locality properties are not entirely obvious in this sector—cf. below. Returning to the elliptic case, we note that Eq. 4.17 and Eq. 4.23 imply

\[(T e^{-J \alpha - \Phi} T^{-1})^\dagger = C (T e^{-J \alpha - \Phi} T^{-1}) C^{-1},\]

with

\[C(\varpi, \overline{\varpi}) = T^{-1\dagger}(\varpi, \overline{\varpi}) T^{-1}(-\varpi, -\overline{\varpi}) \quad (4.25)\]

Thus, hermiticity is realized only up to a similarity transformation. In fact, in the elliptic sector there exists no similarity transformation \(T\) at all such that \(C\) becomes trivial, even if Eq. 4.17 is not imposed. This fact was first observed in ref. [17] and later rediscovered in [23]. Nevertheless, the weaker hermiticity property Eq. 4.25 serves almost the same purpose as “true” hermiticity as far as correlators of the Liouville exponentials are concerned, as the similarity transformation \(C\) cancels out up to the contributions from the end points where \(C\) resp. \(C^{-1}\) hits the left resp. right vacuum. A more serious problem in the elliptic sector, also observed in ref. [17], is that the Liouville exponentials possess no natural restriction to the subspace of positive norm states, given by the condition \(|\varpi| < 1 + \pi/h, |\overline{\varpi}| < 1 + \pi/h\). For the coupling of \(c < 1\) matter to gravity, however, this problem is irrelevant, as all negative norm states become decoupled through the Virasoro conditions.

Let us add here some remarks on the hyperbolic sector. In this case, one should consider the hermitian zero mode \(P := i \varpi\). However, the chiral vertex operators shift \(P\) formally by imaginary amounts. In order to have a well-defined action of the zero mode shift operators \(e^{-m \alpha - q_0}\) in the hyperbolic sector, they should be applied to Gaussian wave packets rather than momentum eigenstates [21]. The former constitute a dense subset of the zero-mode Hilbert space. Then for any Gauss packet \((q_0 | \psi_G) = N e^{-\beta (q_0-b)^2}\) with \(\text{Re } \beta > 0, e^{-m \alpha - q_0} | \psi_G)\) is again a Gauss packet, possessing a Fourier decomposition in terms of real momenta \(P\). Matrix elements in the zero mode space can thus be evaluated by repeated Fourier transformation. Contour deformation considerations then show that in order for some function \(f(P)\) to fulfill

\[f(P)e^{-m \alpha - q_0} = e^{-m \alpha - q_0} f(P - 2im) \quad (4.26)\]

on the dense subspace, we need that \(f(P)\) be analytic on the strip \(\text{Im } P \in [-2m, 0]\) for \(m > 0\), resp. \(\text{Im } P \in [0, -2m]\) for \(m < 0\), with integrable singularities allowed on the real axis. Furthermore, \(f(P)\) needs to be exponentially bounded,

\[
\lim_{|P| \to \infty} \max_{-2m < \text{Im } P < 0} f(P) e^{-\beta P^2} = 0 \quad \forall \beta > 0 \quad \text{if} \quad m > 0, \quad \text{and analogously if} \quad m < 0.
\]

On the other hand, the coefficients of Eq. 4.13 contain poles at \(P + i(2m - k) = 0\) resp. \(P + ik = 0, k = 1, \ldots J + m\). The commutation of these coefficients with shift operators would then generically produce unwanted residue contributions in addition to the “naïve” formula Eq. 4.26, and this would prevent us from directly taking over the results of the locality analysis to the hyperbolic sector. It was observed in [21], however, that the problem is absent for \(J = 1/2\) (provided \(h < \pi\), which is

\[\text{More precisely, it was pointed out in the second of refs. [17] that } e^{-\alpha - \Phi/2} \text{ can be chosen hermitian resp. antimermitian in certain regions of } \varpi, \overline{\varpi} \text{ space, but } e^{-\alpha - \Phi/2} \text{ cannot be consistently restricted to these regions.}\]
assumed anyway in the weak coupling sector). One could think of defining the action of shift operators resp. functions \( f(P) \) through analytic continuation from the elliptic sector; however, it seems that such attempts lead immediately to problems with unitarity if the standard scalar product \( \langle P|P' \rangle = \delta(P - P') \) is kept.

### 4.2 The Liouville Field \( \Phi \)

#### 4.2.1 Definition

Having constructed Liouville exponentials with arbitrary continuous spins, we can now define the Liouville field \( \Phi \) itself by

\[
\alpha_\Phi := -\frac{d}{dJ} e^{-J\alpha_\Phi} \bigg|_{J=0} \tag{4.27}
\]

Though \( \Phi \) is not really a primary field — it is similar to the stress-energy tensor in this respect — it is needed to verify the validity of canonical commutation relations and the quantum equations of motion\(^{15}\). Thus we expand Eq.4.15 near \( J = 0 \). In this limit, the factor \( \prod_{k=1}^{J+m} [1 + 2J - k] \to \frac{2\hbar}{\sin h} \prod_{k=2}^{J+m} [1 + 2J - k] \) vanishes except for \( J + m = 0 \), and the exponential tends to one as it should. It then follows immediately that

\[
\Phi(\sigma, \tau) = -\left( \vartheta_1(u) + \bar{\vartheta}_1(v) \right) + \frac{2\hbar}{\alpha_\sin h} \times \\
\sum_{n=1}^{\infty} \mu_0^n \frac{1}{n} \prod_{k=1}^{n} \frac{1}{[\varpi + 2n - k][\varpi + k]} \mathcal{S}(u)^n \mathcal{S}(v)^n. \tag{4.28}
\]

#### 4.2.2 Periodicity properties and singularity structure

In the hyperbolic sector where \( \varpi = \varpi \), the Liouville field of Eq.4.28 is manifestly periodic. However, inspecting the periodicity behaviour of \( \Phi \) in the elliptic sector with \( \varpi \neq \varpi \), we find that

\[
\alpha_\Phi(\sigma + 2\pi, \tau) = \alpha_\Phi(\sigma, \tau) - 2\pi ik \tag{4.29}
\]

where \( k \) is the parameter appearing in Eq.4.1. The constant is entirely produced by the free field contribution to \( \Phi \), as the series in screening charges is periodic order by order (cf. Eq.3.4). Eq.4.29 obviously calls for some explanation, as at least classically the Liouville field should be periodic by definition. We will carry out the discussion classically, but this will suffice to obtain a qualitative understanding of the situation. The essential point is that the definitions Eq.2.2 and Eq.4.27, though seemingly equivalent classically, actually differ slightly in the elliptic sector. If \( 2\sqrt{\gamma} \Phi(\sigma, \tau) \) is regular everywhere in \([0, 2\pi]\), the two definitions clearly can differ only by a constant. But it is well known\(^{15}\) that in the elliptic sector (with \( k \neq 0 \)) there are \(|k| \) nonintersecting singularity lines, thus \(|k| \) singularities in \( \sigma \in [0, 2\pi] \). At a singular point, the constant connecting the two definitions may - and does

\(^{15}\)in their standard form; for a different approach, see ref.\(^{19}\).
- change, creating in this way a nontrivial periodicity behaviour of our field $\Phi$ of Eqs.4.27, 4.28. Indeed, classically the definition Eq.4.27 is equivalent to

$$2\sqrt{\gamma}\Phi = -2 \ln A^{-1/2} - 2 \ln BB'^{-1/2} - 2 \ln(1 - A/B) + \text{const.}$$  \hspace{1cm} (4.30)

with the series expansion representing the logarithm. It follows easily from the results of [18] that at each singular point, $2 \ln(1 - A/B)$ - and hence $\Phi$ of Eq.4.30 - jumps by an imaginary constant $-2\pi isgnk$, whereas there is no such jump, of course, in Eq.2.2 which is by definition real. Hence,

$$2\sqrt{\gamma}(\Phi_{\text{Eq.4.27}}(\sigma + 2\pi, \tau) - \Phi_{\text{Eq.4.27}}(\sigma, \tau))$$

$$- 2\sqrt{\gamma}(\Phi_{\text{Eq.2.2}}(\sigma + 2\pi, \tau) - \Phi_{\text{Eq.2.2}}(\sigma, \tau)) = -2\pi ik$$  \hspace{1cm} (4.31)

As the second difference is zero, we reproduce Eq.4.29. Thus the Liouville field we are using differs from the ”true” one only by a constant between any two singularity lines, but the constant changes at the singularities. In particular, our $\Phi$ cannot be real everywhere in $\sigma \in [0, 2\pi]$. (This has nothing to do with the nonhermiticity of the exponentials noted in section 4.1.6, as the latter is independent of $\sigma$; indeed, for the exponentials with half-integer $J$, the jumps play no role for the hermiticity behaviour). We remark that the periodicity behaviour of our Liouville field is actually quite natural, as the spectrum in the elliptic sector contains a winding number ($k$) and therefore looks like that of a compactified field. Pursuing this analogy further, we would be lead in the quantum case to impose the usual single-valuedness condition on the field operators. In the free string case, this enforces the quantization of the momenta, whereas here we obtain a discretization of the spin,

$$Jk \in \mathbb{Z}.$$ \hspace{1cm} (4.32)

Note that the periodicity behaviour of the (quantum) Liouville exponentials can be read off directly from Eq.4.29, though they are not actually naive exponentials of $\Phi$. If we admit also anti-periodic behaviour of the exponentials on the cylinder, then we can have half-integer spins as well for $k$ odd. One may speculate if the condition Eq.4.32 can be relaxed if we couple the theory to compactified matter, in such a way that the multivaluedness of the Liouville part is precisely cancelled by that of the matter, but we will not go into this here.

When using the free field $\vartheta_2$ instead of $\vartheta_1$, the periodicity behaviour of $\Phi$ of Eq.4.28 will be exactly opposite. One may think that the regions of convergence for the two (classical) expansions, $|A/B| < 1$ resp. $|B/A| < 1$ are complementary and therefore there is no contradiction. However, in contrast to the hyperbolic sector we have $|A/B| \equiv 1$ in the elliptic sector, such that both series expansions are exactly on their circle of convergence, and in fact converge there except for the singularities at $A = B$. Thus we see that $\vartheta_1 \leftrightarrow \vartheta_2$ invariance is broken by the singularities in the elliptic sector. However, for the exponentials with half-integer $J$, we get the same periodicity behaviour for the $\vartheta_1$ and the $\vartheta_2$ representation. Correspondingly, we were able even in the quantum case to construct these exponentials in a $\vartheta_1 \leftrightarrow \vartheta_2$ invariant way. For continuous $J$, however, it is not obvious how this invariance can be restored.
4.2.3 The case $k = 0$

A special consideration is required for the case where $\varpi$ is real (elliptic sector) but $k = 0$ in Eq.4.1. It is known from the work of [18] that in this situation, there is no real Liouville field even classically. However, from the point of view of the locality analysis, the case $k = 0$ is very natural and therefore we did not exclude it. It is not hard to show that indeed more generally one needs to have

$$\varpi \varpi < 0 \quad (4.33)$$

classically in order to obtain real solutions of the Liouville equation with positive cosmological constant. In the other case, one has $2\sqrt{\alpha} \Im \Phi = \pm i\pi$, and so the real part of $\Phi$ solves the Liouville equation with negative cosmological constant. As regards the singularity structure, $\sigma$ and $\tau$ essentially exchange their roles, and thus one obtains timelike instead of spacelike singularity lines[22]. The number of singularities can in general be greater than $|k|$, though $|k|$ continues to characterize the periodicity behaviour of our solution Eq.4.28. The explanation is that the additional singularities always come in pairs with opposite associated jumps $\pm 2\pi i$ of $2\sqrt{\alpha} \Phi$, so their effect is not seen in the overall periodicity behaviour of $\Phi$. In particular, for $k = 0$ we now understand why the Liouville field Eq.4.28 is periodic in spite of the possible presence of singularities. We stress also that in our analysis there is no restriction on the sign of $\varpi \varpi$, thus we describe solutions of the Liouville equation with both signs of the cosmological constant.

4.2.4 The field equations

For the considerations in this subsection and below, the similarity transformations $T$ discussed above play no role and so we return to the representation Eq.4.15 resp. Eq.4.12 of the Liouville exponentials. Our first task is to compute $\partial_u \Phi$ and $\partial_v \Phi$. For this we need an expression for $\partial_u S^n \equiv \partial_u U_n^{(0)}$. Since $U_n^{(0)}$ is a primary field with weight zero, its derivative is primary with weight one. It is easy to see, by looking at the shift properties in $\varpi$, that it must be proportional to $U(0)^{−1}$. Thus we have

$$\partial_u U_n^{(0)} = i c_n(\varpi) U_n^{(−1)}.$$

In order to determine $c_n(\varpi)$, one takes the matrix element between highest-weight states and uses $\partial_u U_n^{(0)} = i[L_0, U_n^{(0)}]$. With the help of Eq.3.8 for the normalization factor $I_m^{(f)}$ and the expression for the Virasoro weights $L_0|\varpi > = ((1 + \pi/h)^2 - \varpi^2)/h/4\pi|\varpi >$, this gives immediately

$$c_n(\varpi) = 2 \sin h q^{\varpi+1} |n| |\varpi + n| \quad (4.34)$$

Thus we obtain the following formulae:

$$\partial_u \Phi = -\partial_u \varpi + \frac{2i h}{\alpha - \sin h} \sum_{n=1}^{\infty} \frac{\bar{\mu}_n}{|n|} c_n(\varpi) \times$$

$$\prod_{k=1}^{n} \frac{1}{|\varpi + 2n - k||\varpi + k|} U_n^{(-1)} S_n,$$  \hspace{1cm} (4.35)
\[ \partial_v \Phi = -\partial_v \overline{r}_1 + \frac{2i\hbar}{\alpha_- \sin \hbar} \sum_{n=1}^{\infty} \frac{\bar{\mu}_0^n}{[n]} \overline{c}_n(\overline{\omega}) \times \prod_{k=1}^{n} \frac{1}{[\overline{\omega} + 2n - k][\overline{\omega} + k]} S^n U_n^{(-1)}. \]  

(4.36)

In these two equations we observe the appearance of the coefficients of the expansion Eq.4.13 with \( J = -1 \). Taking the crossed derivative we thus get

\[ \partial_u \partial_v \Phi = -\frac{\alpha_-}{8} e^{\alpha_- \Phi}, \]  

(4.37)

if we choose

\[ \bar{\mu}_0 = \frac{1}{32\pi \sin \hbar}. \]  

(4.38)

Eq.4.37 is the quantum Liouville field equation associated with an action given by Eq.2.1 with \( 2 \sqrt{\gamma} \) replaced by \( \alpha_- \), and with \( \mu = 1 \). In view of the recently shown equivalence of different frameworks\[19\], we can directly compare this result with the one obtained in an older analysis by Otto and Weigt\[20\], and find agreement\[16\].

### 4.2.5 Equal-time commutation relations

We now proceed to the canonical commutation relations. It is a trivial consequence of Eqs.4.6, 4.27 that

\[ [\Phi(\sigma, \tau), \Phi(\sigma', \tau)] = 0. \]  

(4.39)

Next, by differentiating Eq.4.39 twice with respect to time and using the equations of motion, we see that also

\[ [\Pi(\sigma, \tau), \Pi(\sigma', \tau)] = 0. \]  

(4.40)

where \( \Pi(\sigma, \tau) \) is the canonical momentum,

\[ \Pi(\sigma, \tau) = \frac{1}{4\pi} \partial_v \Phi(\sigma, \tau) \]  

(4.41)

Furthermore we note that \([\Pi(\sigma, \tau), \Phi(\sigma', \tau)]\) can be nonvanishing only at \( \sigma = \sigma' \), due to the fact the \( R \)-matrices for arbitrary spins depend on \( \tau, \sigma \) only via the step functions \( \theta(u - u') \) resp. \( \theta(v - v') \). On the other hand, the contribution of the free field parts of \( \Pi \) and \( \Phi \) gives precisely the expected result:

\[ [\Pi(\sigma, \tau), \Phi(\sigma', \tau)] \big|_{\text{free field}} = -i\delta(\sigma - \sigma') \]  

(4.42)

We will show now that the sum of the other contributions vanishes. First we observe that \([\Pi(\sigma, \tau), \Phi(\sigma', \tau)] = 0 \) order by order in powers of screening charges, for \( \sigma \neq \sigma' \) (the operators \( V_1^{(-1)}(u), \overline{V}_1^{(-1)}(v) \) appearing in \( \Pi(\sigma, \tau) \) count as screenings as well).

It suffices then to show that also at \( \sigma = \sigma' \),

\[ [\Pi(\sigma, \tau), \Phi(\sigma', \tau)]_N = 0 \quad \forall N \geq 1, \]  

(4.43)

\[ ^{16} \text{Note that the formula of Otto and Weigt quoted in ref.\[19\] needs to be multiplied by a factor } \frac{1}{(\sin \hbar)^{2N}} \text{ to be in accord with the equations of motion. (This was already noticed in \[20\].) } \]
where \([\Pi(\sigma, \tau), \Phi(\sigma', \tau)]\)_N denotes the \(N\)-screening contribution to the commutator. Let us represent the equal time commutator as a limit of time-ordered products,
\[
[\Pi(\sigma, \tau), \Phi(\sigma', \tau)]_N = \lim_{\Delta \to 0} \{ \Pi(\sigma, \tau + \Delta)\Phi(\sigma', \tau) - \Phi(\sigma', \tau)\Pi(\sigma, \tau - \Delta) \}_N \tag{4.44}
\]
as usual. Next we argue that for fixed \(N\) and \(\hbar\) small enough, the leading contributions to the operator product appearing in Eq.(4.44) simply are not singular enough to give a contribution to the commutator at \(\sigma = \sigma'\). Indeed, we have\(^{17}\)
\[
\partial_u \vartheta_1(u') \cdot U_N^{(0)}(u) \sim N\alpha \ln(z - z')U_N^{(0)}(u)
\]
with \(z := e^{iu}\), as well as
\[
\vartheta_1(u) \cdot U_N^{(-1)}(u') \sim \alpha \ln(z - z')U_N^{(-1)}(u') \tag{4.45}
\]
Eq.(4.45) is true even for arbitrary \(h\). Finally, there are also terms of the form \(U_n^{(0)}(u)U_n'^{(-1)}(u')\), \(n + n' = N\), which are finite for \(u \to u'\) if \(h\) is small enough. Similar contributions arise of course from the right-moving parts. As the singularities are only logarithmic, they cannot produce a nonvanishing distribution at \(\sigma = \sigma'\). Hence for fixed \(N\) and small enough \(\hbar\), Eq.(4.43) must be valid for all \(\sigma, \sigma'\).

On the other hand, using the \(R\)-matrix Eq.(3.20) and its right-moving counterpart and \((0 < \sigma < \sigma' < \pi, \tau = 0)\)
\[
\alpha_-[\vartheta_1(\sigma), V_1^{(-1)}(\sigma')] = -i\pi \alpha_-^2 V_1^{(-1)}(\sigma')
\]
\[
\alpha_-[\vartheta_1(\sigma), s(\sigma')] = i\pi \alpha_-^2 / (k(\varpi) - 1) (2k(\varpi)s(\sigma) + (1 - 3k(\varpi))s(\sigma'))
\]
\[
\alpha_-[\partial_v \vartheta_1(\sigma), s(\sigma')] = 2\pi i \alpha_-^2 k(\varpi)V_1^{(-1)}(\sigma) \tag{4.46}
\]
it is easy to see that \([\Pi(\sigma, \tau), \Phi(\sigma', \tau)]\)_N for general \(h\) can be written in the form
\[
[\Pi(\sigma, \tau), \Phi(\sigma', \tau)]_N = \sum_{l=0}^{N-1} \sum_{l'=0}^N \beta_{l'l} (\varpi)U_l^{(0)}(u')U_N^{(-1)}(u)\overline{U}_l^{(0)}(v')\overline{U}_{N-l}^{(0)}(v) + \sum_{l=0}^N \sum_{l'=0}^{N-1} \gamma_{l'\ell} (\varpi)U_l^{(0)}(u')U_N^{(-1)}(u)\overline{U}_l^{(0)}(v')\overline{U}_{N-l}^{(-1)}(v) \tag{4.47}
\]
Clearly the coefficients \(\beta_{l'l}\) and \(\gamma_{l'\ell}\) are analytic in \(h\), as the \(R\)-matrices and the coefficients on the r.h.s. of Eq.(4.46) have this property. Since \(\beta_{l'l}\) and \(\gamma_{l'\ell}\) vanish for small enough \(h\), they have to be zero everywhere. Thus we find for the full commutator,
\[
[\Pi(\sigma, \tau), \Phi(\sigma', \tau)] = -i\delta(\sigma - \sigma') \tag{4.48}
\]
as expected, showing that the quantization scheme is indeed canonical.

\(^{17}\)we use here the notation \(U_n^{(0)}\) instead of \(S^n\) to make it clear that these operators exist for arbitrary \(h\), according to the remarks below Eq.(3.8)
5  The case of two screening charges.

5.1  The braiding

In the above analysis, we have used only one quantum deformation parameter $h$, which tends to zero in the classical limit $C \to \infty$ according to Eq.2.10. However, as explained in refs.[8] one can consider also the operators $\tilde{U}_m^{(j)}$ which have the same form Eq.3.5 as the $U_m^{(j)}$ but involve the deformation parameter

$$\tilde{\hbar} \equiv \pi^2/h$$

(5.1)

Since $\tilde{U}_1^{(-1)}$ is also a dimension 1 operator, the $\tilde{U}_m^{(j)}$ have a Coulomb gas representation completely analogous to Eqs.2.1-2.3. Hence their exchange algebra (and also their fusion properties) are the same as that of the “unhatted” operators. More generally, one can combine the fields $\tilde{U}_m^{(j)}$ and $U_m^{(j)}$ by fusion to obtain the operators $U_{m\tilde{m}}^{(j\tilde{j})}$ with the properties (cf. Eqs.3.2, 3.3)

$$\Delta_{j,\tilde{j}} = \Delta_{j+\pi/h,\tilde{j}}$$

(5.2)

$$U_{m\tilde{m}}^{(j\tilde{j})}(\varpi)U_{m\tilde{m}}^{(j\tilde{j})}(\varpi') = e^{-2\pi i j\tilde{j}(\varpi-\varpi')}U_{m\tilde{m}}^{(j\tilde{j})}(\varpi')U_{m\tilde{m}}^{(j\tilde{j})}(\varpi)$$

(5.3)

where $\varpi = \varpi h/\pi$. Their normalization will be discussed below. For half-integer $J, \tilde{J}$ the braiding (and fusion) of the $U_{m\tilde{m}}^{(j\tilde{j})}$ follows immediately from that of the $U_m^{(j)}$ because the two sets of operators $U_m^{(j)}$ and $\tilde{U}_m^{(j)}$ commute up to a phase:

$$U_m^{(j)}(\sigma)\tilde{U}_m^{(j)}(\sigma') = e^{-2\pi i J\tilde{J}m(\sigma-\sigma')}U_m^{(j)}(\sigma')U_m^{(j)}(\sigma)$$

(5.4)

The natural expectation is that this will remain true even for noninteger $2J$. However, here we meet a surprise. The commutator of $V_{-j}^{(j)}(\sigma)$ and $\tilde{V}_{-\tilde{j}}^{(\tilde{j})}(\sigma')$ gives the factor in Eq.5.3, the screening charges $S(\sigma)$ and $\tilde{S}(\sigma')$ commute, but

$$S_{\sigma\sigma'}\tilde{V}_{-\tilde{j}}^{(\tilde{j})}(\sigma') = e^{2\pi i \tilde{\hbar} m(\sigma-\sigma')}S_{\sigma\sigma'}$$

(5.5)

whereas

$$S_{\Delta}\tilde{V}_{-\tilde{j}}^{(\tilde{j})}(\sigma') = e^{-2\pi i \tilde{\hbar} m(\sigma-\sigma')}S_{\Delta}$$

(5.6)

The phase factors agree only when $2J$ is integer, and thus the commutation of hatted and unhatted operators becomes nontrivial in general. We should therefore restart the machinery of section 2 with the operators $U_{m\tilde{m}}^{(j\tilde{j})}$, where

$$U_{m\tilde{m}}^{(j\tilde{j})} := V_{-j}^{(j)}\tilde{V}_{-\tilde{j}}^{(\tilde{j})}S_{J+m}\tilde{S}_{J+m}$$

with the product of the first two factors being defined by renormalizing the short-distance singularity as usual (cf. also ref.[11]). Following exactly the same steps, we arrive at the generalized version of Eq.3.2 (with shifted $\varpi$):
factor of the expression Eq.3.20 of Eq.5.7 (cf. Eq.3.19). At this point, it is convenient to combine it with the phase

\[ q \prod_{s=1}^{J+m_1} \left( y + yq^{(w+2J+2m_2)+2s-1} \right) \prod_{s=1}^{J+m_2} \left( y' + yq^{(w+2J'+2m_2)+2s-1} \right) \]

\[ \sum_{m_1m_2} R(J, J'; w + 2Je + 2J'e, \frac{m_1m_2}{m} \omega) q^{-J'J'e} q^{-\hat{\hat{J'}}\hat{\hat{J'}}'} \times \]

\[ \prod_{s=1}^{J'+m_2} \left( y'q^{2J'c} + yq^{(w+6J'c)+2s-1} \right) \prod_{s=1}^{J+m_1} \left( y + yq^{(w+2J+2m_2)+2s-1} \right) \]

\[ J' + m_2 \]

\[ J + m \]

\[ n = J + m, \quad n' = J' + m', \quad n_1 = J + m_1, \quad n_2 = J' + m_2, \]

\[ \hat{n} = \hat{J} + \hat{m}, \quad \hat{n'} = \hat{J'} + \hat{m'}, \quad \hat{n}_1 = \hat{J} + \hat{m}_1, \quad \hat{n}_2 = \hat{J'} + \hat{m}_2, \]

where

\[ J^e := J + \hat{J}\pi/h, \quad \hat{J}^e := \hat{J} + J\pi/h \]

At this point, it is useful to note that, since \( V^{(j)}(\hat{J}) \propto V^{(J^e)} \), and since the above braiding equations depend upon the \( m \)'s only through the screening numbers. In the continuous case 2\( J \) and \( 2\hat{J} \) loose meaning, since it is not possible to recover them from \( J^e \) (the same is of course also true for \( J'^e \)). The screening numbers are such that \( n + n' = n_1 + n_2, \quad \hat{n} + \hat{n'} = \hat{n}_1 + \hat{n}_2 \). Returning to our main line, we see that we may factorize the equation system into a product of one system of the type Eq.3.19 with another one of the same type with \( h \rightarrow \hat{h} \). This is realized if, in Eq.3.19, we change the spins from \( J \) to \( J^e \), and replace the \( m \)'s by the quantities

\[ m^e = m - \frac{\hat{J}\pi}{h}, \quad m'^e = m' - \frac{\hat{J}'\pi}{h}, \quad m^e_1 = m_1 - \frac{\hat{J}\pi}{h}, \quad m^e_2 = m_2 - \frac{\hat{J}'\pi}{h} \]

\[ \hat{m}^e = \hat{m} - \frac{\hat{J}\pi}{h}, \quad \hat{m}'^e = \hat{m}' - \frac{\hat{J}'\pi}{h}, \quad \hat{m}^e_1 = \hat{m}_1 - \frac{\hat{J}\pi}{h}, \quad \hat{m}^e_2 = \hat{m}_2 - \frac{\hat{J}'\pi}{h} \]

This last replacement is such that the screening numbers remain unchanged, that is \( J + m = J^e + m^e \), and so on. In view of the previous analysis, this tells us immediately what the solution of Eq.5.7 must be:

\[ R_U(J, J'; \omega) \frac{\hat{R}_U(J^e, J'^e; \omega)}{\hat{m}^e_1 \hat{m}^e_2} \frac{\hat{R}_U(J^e, J'^e; \omega)}{\hat{m}'^e_1 \hat{m}'^e_2} \]

\[ q^{J^eJ'^e} \hat{q}^{-\hat{J'}\hat{J'}} \]

\[ q^{2J^eJ'^e} \hat{q}^{2\hat{J'}\hat{J'}} \]

\[ q^{J^eJ'^e} \hat{q}^{2\hat{J'}\hat{J'}} \]

\[ q^{J^eJ'^e} \hat{q}^{2\hat{J'}\hat{J'}} \]

The \( R \)-matrices \( R_U \) and \( \hat{R}_U \) are given by the same expression Eq.3.20 written in terms of the deformation parameter \( h \) resp. \( \hat{h} \). The phase factor is due to the fact that it is \( q^{J^eJ'^e} \hat{q}^{-\hat{J'}\hat{J'}} \equiv q^{2J^eJ'^e} \hat{q}^{2\hat{J'}\hat{J'}} \) rather than \( q^{2J^eJ'^e} \hat{q}^{2\hat{J'}\hat{J'}} \equiv q^{4J^eJ'^e} \) which appears in Eq.5.9 (cf. Eq.3.19). At this point, it is convenient to combine it with the phase factor of the expression Eq.3.20 of \( R_U \) and the analogous one for \( \hat{R}_U \). Altogether one gets

\[ q^{-2J^eJ'^e} \hat{q}^{m^e(2m^e+m'^e+\omega)-m^e_1(2m^e_1+m'^e+\omega)-m^e_2(2m^e_2+m'^e+\omega)} \]

\[ q^{m'^e(2m^e+m'^e+\omega)-m'^e_1(2m^e_1+m'^e+\omega)-m'^e_2(2m^e_2+m'^e+\omega)} \]
Using the fact that $J + m$, $\hat{\mathcal{J}} + \hat{m}$, etc. are integers, this may may rewritten as
\[
e^{-i\pi(\Delta_x + \Delta_{x+1} + \Delta_{x+m_1} + \Delta_{x+m_2})} = e^{-i\pi(\Delta_x + \Delta_{x+m_1} + \Delta_{x+m_2})}
\]
where we introduced
\[
m^e \equiv m + \frac{\pi}{\hbar} \hat{m}, \quad m'^e \equiv m' + \frac{\pi}{\hbar} \hat{m}',
\]
\[
m'^1 \equiv m_1 + \frac{\pi}{\hbar} \hat{m}_1, \quad m'^2 \equiv m_2 + \frac{\pi}{\hbar} \hat{m}_2.
\]
(5.12)

This is precisely the factor expected from the general MS formalism, since it involves the same linear combination of Virasoro weights, as say Eq.3.13. We can write the complete $R$-matrix again in the form Eq.3.20, if we introduce "vectorial" quantities $\kappa_{m',m}$ and \{\fracs{}\fracs{}\} with $J = (J, \hat{J})$ etc. (cf. also ref.[13]). Then
\[
R_U(J, J'; \varpi)_{m, m'} = e^{-i\pi(\Delta_x + \Delta_{x+m_1} + \Delta_{x+m_2})}
\]
\[
\times \frac{\kappa^{x+m_1}_J \kappa^{x+m_2}_J}{\kappa^{x+m_1}_J \kappa^{x+m_2}_J} \frac{\kappa^{x+m_1}_J \kappa^{x+m_2}_J}{\kappa^{x+m_1}_J \kappa^{x+m_2}_J}
\]
where
\[
\kappa^{x+m} = \kappa^{x, x+m_0}_{J^e, J^e'} \kappa^{x, x+m_0}_{J^m, J^m'}
\]
and
\[
\{\fracs{}\fracs{}\} = \{\fracs{}\fracs{}\} \{\fracs{}\fracs{}\} \{\fracs{}\fracs{}\} \{\fracs{}\fracs{}\} \{\fracs{}\fracs{}\} \{\fracs{}\fracs{}\}
\]
where we have let $m'_1 = m_1 + m_0$, $\hat{m}'_2 = \hat{m}_1 + \hat{m}_2$. Finally, we make contact with the form proposed in ref.[13] by re-expressing the entries of the $6j$-symbols in terms of the effective quantum numbers defined by Eq.5.12. One easily finds, using the double-brace notation of ref.[13]
\[
\{\fracs{}\fracs{}\} = \{\fracs{}\fracs{}\} \{\fracs{}\fracs{}\} \{\fracs{}\fracs{}\} \{\fracs{}\fracs{}\} \{\fracs{}\fracs{}\} \{\fracs{}\fracs{}\}
\]
(5.14)

These expressions coincide with the ones introduced in ref.[13]. The complete $\kappa$ may also be written similarly as
\[
\kappa^{x+m} = \kappa^{x, x+m_0}_{J^e, x+m_0 - (\hat{J} + \hat{m})/h} \kappa^{x, x+m_0}_{J^m, x+m_0 - (J + m)/\hbar}
\]
(5.15)

Note that the hatted effective quantum numbers are simply equal to the unhatted one multiplied by $\hbar/\pi$. Thus everything has been expressed solely in terms of the effective quantum numbers where hatted and unhatted quantum numbers cannot be separated (clearly, the variable $x = (\varpi - \varpi_0)/2$ is also of this type), but which are such that the particular combinations $J^e + m^e$, $J^e' + m^e'$, $J^m + m^m_1$, $J^m + m^m_2$ may be written as sums of a positive integer plus another positive integer times

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is that, if we define $\tilde{\text{Eq.3.27}}$, and this completes the derivation. Each reduces to an orthogonality relation of the type appropriate for continuous spins, since $J^e + m^e = n + \bar{n}\pi/h$ instead of $\tilde{\text{Eq.3.27}}$. The outcome of the present discussion is that, if we define $\tilde{V}(J^e)$ by letting

$$
\tilde{V}(J^e) := \kappa J^e \frac{\pi}{2} 2 + \bar{m} V(J^e),
$$

(5.16)

the braiding of the $\tilde{V}$ fields is given by Eq.5.13 without the $\kappa$ factors, analogously to the case of a single screening charge, and thus coincides with the one written in ref.[13]. This completes the derivation of the braiding of the chiral vertex operators with both screening charges.

Finally let us check the orthogonality properties of the generalized 6-j symbols. Our aim is to prove that

$$
\sum_{\text{J}} \left\{ \frac{\hat{J}_1}{J_2, J_{123}} \mid \frac{\hat{J}_{12}}{J_{23}}, \frac{\hat{K}_{12}}{J_{12}} \right\} \left\{ \frac{\hat{J}_1}{J_2, J_{123}} \mid \frac{\hat{J}_{12}}{J_{23}}, \frac{\hat{K}_{12}}{J_{12}} \right\} = \delta_{J_{12}, K_{12}},
$$

(5.17)

What is the range of summation? In the generalized 6-j symbols $\left\{ \frac{\hat{J}_1}{J_2, J_{123}} \mid \frac{\hat{J}_{12}}{J_{23}}, \frac{\hat{K}_{12}}{J_{12}} \right\}$ there are four pairs of screening numbers

$$
p_{1,2} + \frac{\pi}{h} \hat{p}_{1,2} = J^e_1 + J^e_2 - J^e_{12}, \quad p_{1,2,3} + \frac{\pi}{h} \hat{p}_{1,2,3} = J^e_{12} + J^e_3 - J^e_{123},
$$

$$
p_{2,3} + \frac{\pi}{h} \hat{p}_{2,3} = J^e_2 + J^e_3 - J^e_{23}, \quad p_{2,3,1} + \frac{\pi}{h} \hat{p}_{2,3,1} = J^e_1 + J^e_{23} - J^e_{123}
$$

(5.18)

which are positive integers, such that $p_{1,2} + p_{1,2,3} = p_{2,3} + p_{2,3,1}$, and $\hat{p}_{1,2} + \hat{p}_{1,2,3} = \hat{p}_{2,3} + \hat{p}_{2,3,1}$. The two 6-j’s appearing in Eq.5.17 have the same $p_{2,3}, \hat{p}_{2,3}, p_{2,3,1}, \hat{p}_{2,3,1}$. The other screening numbers are modified when $\hat{J}_{12}$ is replaced by $\hat{K}_{12}$. We denote the result with the letter $q$ instead of $p$. Since the screening number are linearly related, only three of them are independent for each 6-j. We may choose to eliminate the screening numbers with index 12, 3. In the summation, only $J^e_3$ varies. Equivalently, we can then sum over the positive integers $p_{2,3}$ and $\hat{p}_{2,3}$ since $J^e_{23} = J^e_2 + J^e_3 + J^e_{23} - p_{2,3} - (\pi/h)\hat{p}_{2,3}$. Let us next consider the l.h.s. of Eq.5.17 making use of the explicit expression Eq.5.13. It becomes

$$
\sum_{p_{2,3}, \hat{p}_{2,3}} \left\{ \frac{\hat{J}_1}{J_2, -\hat{p}_{2,3} + p_{2,3}} \mid \frac{\hat{J}_{12}}{J_{123}, -p_{2,3} + \hat{p}_{1,2,3}} \right\} \left\{ \frac{\hat{J}_1}{J_2, -\hat{p}_{2,3} + p_{2,3}} \mid \frac{\hat{J}_{12}}{J_{123}, -p_{2,3} + \hat{p}_{1,2,3}} \right\} \times \left\{ \frac{\hat{J}_1}{J_2, -\hat{p}_{2,3} + p_{2,3}} \mid \frac{\hat{K}_{12}}{J_{123}, -p_{2,3} + \hat{p}_{1,2,3}} \right\} \times \left\{ \frac{\hat{J}_1}{J_2, -\hat{p}_{2,3} + p_{2,3}} \mid \frac{\hat{K}_{12}}{J_{123}, -p_{2,3} + \hat{p}_{1,2,3}} \right\}.
$$

(5.19)

The shifts of the entries of the 6-j symbols are not the same as in Eq.5.14. We used the liberty of changing them without changing the result — which was exhibited in ref.[13] — to go to a more convenient expression. Now, we recall[13] that these shifts are precisely such that the screening numbers of the first (resp. second) line only involve the $p$’s and $q$’s, (resp. the $\hat{p}$’s and $\hat{q}$’s), so that the two sums may be carried out independently. Each reduces to an orthogonality relation of the type Eq.3.27, and this completes the derivation.
5.2 The generalized Liouville field

The generalization of the Liouville exponential is given by

\[ e^{-J^e \alpha_- \Phi(\sigma, \tau)} = \sum_{n, \hat{n}=0}^{\infty} \mu_0^n \hat{\mu}_0^{\hat{n}} \tilde{V}^{(J_e)}_{n \hat{n}}(u) \tilde{V}^{(J_e)}_{n \hat{n}}(v) \]  

(5.20)

where \( \hat{\mu}_0 \) is given by equations similar to Eqs. 4.16 and 4.38 with \( h \rightarrow \hat{h} \). Of course, the preference of \( \alpha_- \) over \( \alpha_+ \) in Eq. 5.20 is purely notational as \( e^{-J^e \alpha_- \Phi} \equiv e^{-J^e \alpha_+ \hat{\Phi}} \).

Using Eq. 5.17, it is straightforward to verify that the generalized exponential is local and closed by fusion, provided \( \vartheta = \vartheta \). The passage to \( \vartheta_1 \leftrightarrow \vartheta_2 \) invariant exponentials (with \( J, \hat{J} \) positive half-integers) proceeds exactly as for the case of a single screening charge. The appropriate similarity transformation \( T \) can be read off directly from Eqs. C.6, C.7 of appendix C. It is equally straightforward to extend the hermiticity discussion. We now turn to the generalized Liouville field, which we define again by Eq. 4.27. Thus we obtain

\[ \Phi(\sigma, \tau) = -(\vartheta_1(u) + \hat{\vartheta}_1(v)) + \frac{2h}{\alpha_- \sin h} \sum_{n=1}^{\infty} \hat{\mu}_0^n \frac{1}{[n]} \prod_{k=1}^{n} \frac{1}{[\vartheta + 2n - k][\vartheta + k]} S(u)^n \hat{S}(v)^n. \]  

(5.21)

It differs from the previous one by the last line. Since \( \vartheta = \vartheta \), \( \Phi \) is periodic. As a result, the quantum field equation becomes

\[ \partial_u \partial_v \Phi = -\frac{\alpha_-}{8} e^{\alpha_- \Phi} - \frac{\alpha_+}{8} e^{\alpha_+ \hat{\Phi}}, \]  

(5.22)

involving both cosmological terms. Since \( \Phi = \Phi + \hat{\Phi} + \vartheta_1 + \hat{\vartheta}_1 \), the validity of Eq. 5.22 is a trivial consequence of the equations of motion with a single screening charge. The reason why \( \Phi \) must be shifted w.r.t. \( \Phi + \hat{\Phi} \) is that \( \Phi + \hat{\Phi} \) alone is not local. Indeed, using Eqs. 4.14 and its left-moving analog, and the fact that screening charges of different type commute, we see that the only nonzero contributions to \( [\Phi(\sigma, \tau) + \hat{\Phi}(\sigma, \tau), \Phi(\sigma', \tau) + \hat{\Phi}(\sigma', \tau)] = [\Phi(\sigma, \tau), \Phi(\sigma', \tau)] + [\hat{\Phi}(\sigma, \tau), \Phi(\sigma', \tau)] \) are of the form \( [\vartheta_1, \hat{S}] \) resp. \( [\hat{\vartheta}_1, S] \). These commutators are precisely cancelled by the free field shift. Finally, one may easily extend the previous discussion of the canonical commutation relations. The result is that Eqs. 4.39, 4.40 and 4.48 remain true for the generalized Liouville field without any modification.

6 Conclusions/Outlook

The operator approach to Liouville Theory, which originated more than ten years ago, has come a long way. Starting from the analysis of the simplest Liouville
field - the inverse square root of the metric - which corresponds to the \( J = 1/2 \)
representation, it has now progressed to the construction of the most general Liouville
operators in the standard (weak coupling) regime, corresponding to arbitrary
highest/lowest weight representations of the quantum group. The underlying chiral
algebra, either in its Bloch wave/Coulomb gas or its quantum group covariant guise,
has revealed beautiful structures which may find applications also in very different
contexts. Though the completion of the quantization program of Gervais and Neveu
thus finally comes into sight for the weak coupling sector, there remains an important
complex of questions yet to be addressed. While the Coulomb gas picture presented
here leads immediately to integral representations of arbitrary n-point functions in
the half-integer positive spin case\[17\], the correlators of operators with continuous
spins require more care. This is due to the fact that outside the half-integer positive
\( J \) region, the sums representing the Liouville exponentials become infinite, and their
evaluation within correlation functions is quite nontrivial even in the simplest case of
the three-point function\[23\]: the study of the latter (for arbitrary spins and central
charge) is the subject of ongoing investigations. Similarly, the hermiticity properties
of the Liouville operators and their invariance under the ”large” \( SL_2 \) transformation
which exchanges \( \vartheta_1 \) and \( \vartheta_2 \) still pose an open problem in the continuous spin case.

In ref.\[7\], where the computation of three-point functions relevant for minimal
matter coupled to gravity was discussed (here \( J \) is half-integer negative), these
difficulties were avoided in an interesting way by using instead of the ”canonical”
expression for the Liouville exponential as given by Eq.5.2 another operator with
the same conformal weight, which is however represented by a finite sum. This
approach appears to be closer in spirit to the analytic continuation procedure em-
ploied in the path integral framework, and its connection with the first-principle
approach along the lines of this paper certainly deserves a better understanding.
The Coulomb gas picture for general spins should however open up the possibility
of studying arbitrary local (i.e. not integrated) correlation functions, for any con-
tinuous value of the central charge, which is indispensible for a full understanding
of the integrability structure of Liouville theory as a local conformal field theory
in its own right. In contrast to this, the path integral or matrix model approaches
in their present form do not give any insight into the local structure of the theory.
In particular, the underlying quantum group symmetry and its implications for the
structure of the operator algebra have so far remained completely invisible in the
other approaches.

The techniques used in this paper are applicable not only to the standard weak
coupling \((C > 25)\) regime considered here, but also to the strong coupling theory
as developed in refs.\[1\] \[13\]. In this case, however, it turns out that one needs
inverse powers of the screening charges Eq.2.1. These can be immediately formulated
in the Gervais-Neveu framework, as the replacement \( A \rightarrow -1/A \) which inverts the
screening charge just corresponds to exchanging the free field \( \vartheta_1 \) with \( \vartheta_2 \).
Unfortunately, the corresponding \( V_m^{(J)} \) operators will then again involve both free fields
simultaneously, and the simple commutation technique of section3.1 cannot be used
directly. Nevertheless, it should be possible to relate the braiding of negative powers
of the screenings to that of positive powers making use of the known short-distance
product of $S$ with its inverse; we hope to address this question in a forthcoming publication.

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A Some previous results.

The conventions are the same as in previous papers so that we will not spell them out again. In the operator approach, the quantum group structure was shown[1, 2] to be of the type $U_q(sl(2)) \otimes U_{q^\alpha}(sl(2))$, where $\hbar$ is given by Eq.2.19, and

$$\hbar = \frac{\pi}{12} (C - 13 + \sqrt{(C - 25)(C - 1)}), \quad (A.1)$$

Each quantum group parameter is associated with a screening charge by the relations $h = \pi(\alpha_-)^2/2$, $\hbar = \pi(\alpha_+)^2/2$. The basic family of $(r, s)$ chiral operators in 2D gravity may be labelled by two quantum group spins $J$ and $\tilde{J}$, with $r = 2\tilde{J} + 1$, $s = 2J + 1$, so that the spectrum of Virasoro weights is given by

$$\Delta_{J, \tilde{J}} = \frac{C - 1}{24} - \frac{1}{24} ((J + \tilde{J} + 1)\sqrt{C - 1} - (J - \tilde{J})\sqrt{C - 25})^2, \quad (A.2)$$

in agreement with Kac’s formula. One outcome of ref.[3] was the fusion and braiding of the general chiral operators $V^{(J)}_m$, also denoted $V^{(J, \tilde{J})}_{m\tilde{m}}$, where underlined symbols denote double indices $\underline{J} \equiv (J, \tilde{J})$, $\underline{m} \equiv (m, \tilde{m})$, which were all taken to be half integers:

$$\mathcal{P}_{\underline{J}} V^{(\underline{J}_1)}_{\underline{L}_1 - \underline{L}} V^{(\underline{J}_2)}_{\underline{L}_2 - \underline{L}} = \sum_{\underline{J}_{12}} C_{\underline{J}_1 \underline{L}_1 \underline{L}}^{\underline{J}_2 \underline{L}_2 \underline{J}_{12}} \left| \underline{J}_1 \underline{L}_1 \underline{L}_{12} \right| \left| \underline{J}_2 \underline{L}_2 \underline{L}_{12} \right| \times$$

$$\mathcal{P}_{\underline{J}} \sum_{\{\nu\}} V^{(J_{12}; \nu)}_{\underline{L}_{12} - \underline{L}} \left< \varpi, \{\nu\} \right| V^{(J_{1})}_{\underline{L}_1 - \underline{L}} \left| \varpi \right>, \quad (A.3)$$

$$\mathcal{P}_{\underline{J}} V^{(\underline{J}_1)}_{\underline{L}_1 - \underline{L}} V^{(\underline{J}_2)}_{\underline{L}_2 - \underline{L}} = \sum_{\underline{J}_{13}} e^{i\pi(\Delta_{J} + \Delta_{J} - \Delta_{L} - \Delta_{L})} \times$$

$$\mathcal{P}_{\underline{J}} V^{(\underline{J}_1)}_{\underline{L}_1 - \underline{L}} V^{(\underline{J}_2)}_{\underline{L}_2 - \underline{L}} V^{(\underline{J}_3)}_{\underline{L}_3 - \underline{L}} \left| \varpi \right>, \quad (A.4)$$

In these formulae, world-sheet variables are omitted, and $\varpi$ is the rescaled zero-mode momentum of $\vartheta_1$ as in Eq.2.18. It characterizes the Verma modules $\mathcal{H}(\varpi)$, spanned by states noted $| \varpi, \{\nu\} >$, where $\{\nu\}$ is a multi-index. In the generic case, where the Verma module is trivial, $\mathcal{H}(\varpi)$ is a Fock space generated by the non-zero modes of the free field $\vartheta_1$ (or equivalently of $\vartheta_2$), with the ground state $| \varpi >$. The symbol $\varpi_{\underline{J}}$ stands for $\varpi_0 + 2J + 2\tilde{J}\pi/\hbar$ where $\varpi_0 = 1 + \pi/\hbar$, and $\mathcal{P}_{\underline{J}}$ is the projector.
on $\mathcal{H}(\varpi_J)$. The above formulae contain the recoupling coefficients for the quantum group structure $U_q(sl(2)) \otimes U_q(sl(2))$, which are defined by

$$\left\{ \frac{J_1}{J_2} | J_{23} \right\} = (-1)^{f_V(J_1, J_2, J_{23}, J_{12})} \left\{ \frac{J_1}{J_3} | J_{23} \right\} \left\{ \frac{J_1}{J_2} | J_{12} \right\}$$

where $\left\{ \frac{J_1}{J_2} | J_{23} \right\}$ is the 6j coefficient associated with $U_q(sl(2))$, while $\left\{ \frac{J_1}{J_2} | J_{23} \right\}$ stands for the 6j associated with $U_q(sl(2))$. $f_V(J_1, J_2, J_{23}, J_{12}, J_{13})$ is an integer given by

$$f_V(J_1, J_2, J_{23}, J_{12}, J_{13}) = 2\tilde{J}_2(J_{12} + J_{23} - J_2 - J_{123}) + 2J_2(\tilde{J}_{12} + \tilde{J}_{23} - \tilde{J}_2 - \tilde{J}_{123})$$

In addition to these group theoretic features there appear the coupling constants $g_{J_{12}}^{J_{23}}$, whose expression was given in ref. $[8]$. In order to connect with the general setting recalled in section 2, let us indicate that $V_{-J_{10}}^{(J_0)}$ is proportional to $f_{J_{10}}^{J_0}|_{\text{qu}}$, and that $V_{J_{10}}^{(J_0)}$ corresponds to the normal ordered exponential of $\vartheta_2$, that is to $f_{J_{10}}^{J_0}|_{\text{qu}}$. One may verify — this is left as an exercise to the dedicated reader — that the equations just written are such that the braiding of these fields is simply

$$V_{-J_{10}}^{(J_0)}(\sigma_1) V_{-J_{20}}^{(J_0)}(\sigma_2) = e^{-2ihJ_{12}(\sigma_1 - \sigma_2)} V_{-J_{20}}^{(J_0)}(\sigma_1) V_{-J_{10}}^{(J_0)}(\sigma_2), \quad (A.7)$$

$$V_{J_{10}}^{(J_0)}(\sigma_1) V_{J_{20}}^{(J_0)}(\sigma_2) = e^{-2ihJ_{12}(\sigma_1 - \sigma_2)} V_{J_{20}}^{(J_0)}(\sigma_2) V_{J_{10}}^{(J_0)}(\sigma_1). \quad (A.8)$$

where $\epsilon(\sigma_1 - \sigma_2)$ is the sign of $\sigma_1 - \sigma_2$ (for definiteness, we consider the interval $0 \leq \sigma_i \leq \pi$). This confirms that they are normal ordered exponentials of free fields, in agreement with the starting point of the GN quantization. Note that the braiding of $V_{-J_{10}}^{(J_0)}$ with $V_{J_{10}}^{(J_0)}$ involves the full complexity of the 6-j coefficients, so that the commutation relations of $\vartheta_1$ with $\vartheta_2$ are definitely not of free-field type.

### B Computation of the normalization integrals

#### B.1 The case of a single screening charge

To determine the normalization factors $I_m^{(J)}(\varpi)$ of Eq. $[38]$,

$$I_m^{(J)}(\varpi) = \langle \varpi | U_m^{(J)}(\sigma = 0) | \varpi + 2m \rangle \quad (B.1)$$

we will reduce the integrals involved to those computed by Fateev and Dotsenko $[25]$. Performing the Wick contractions in the standard fashion, one obtains $[17]$.

$$I_m^{(J)}(\varpi) = (-i)^n \oint dz_1 \cdots \oint dz_n \prod_{j=1}^n (1 - z_j)^\alpha z_j^\gamma \prod_{j<k} (z_j - z_k)^\beta$$

where

$$z_j = e^{i\sigma_j}, \quad \alpha = 2hJ/\pi, \quad \beta = -2h/\pi, \quad \gamma = (\varpi + 2m - 1)h/\pi - 1, \quad n = J + m. \quad (B.2)$$
The phase convention is that \((z_j - z_k)^\beta\) is real positive for \(0 < z_k < z_j < 1\), and otherwise defined by analytic continuation. As usual, this multiple integral is to be understood as the equal-time limit of a time-ordered integral with \(\sigma_j \to \sigma_j + i\tau_j\) and \(\tau_j \to 0^+\). Thus the contours are given by the drawing on the left of fig.1.

![Integration contours for Eq.A.2](image1)

fig1: Integration contours for Eq.A.2

These contours are quite different from those appearing in the Fateev-Dotsenko integrals[25]. To establish the connection, we first compress the contours towards the real axis as shown on the right of fig.1, the distance between the contours and the real axis being infinitesimally small. Then we split into contributions coming from the upper (UHP) resp. lower half plane (LHP). The LHP parts differ from the UHP ones only by phase factors; hence \(I_{(n)}^{(j)}\) can be reduced to a multiple integral over the UHP contours only as in fig.2.

![Equivalent set of integration contours](image2)

fig2: Equivalent set of integration contours

Indeed, suppose the \(z_n, \ldots, z_{n-p+1}\) integrations have already been brought into the form of fig.2. To continue with \(z_{n-p}\), notice first that \((z_j - z_{n-p})^\beta\) for \(j < n - p\) takes the same values for \(z_{n-p}\) infinitesimally above or below the real axis. Hence we need to consider only \((z_{n-p} - z_k)^\beta\), \(k > n - p\). The permutation symmetry of the multiple integral tell us in fact to consider together the set of \((p + 1)!\) configurations obtained
by performing all permutations of a given fixed configuration \( z_n, \ldots z_{n-p+1}, z_{n-p} \). They can be organized into groups \( A_q^{(P)} \) and \( B_q^{(P)} \) (\( P \) labelling the permutation), where \( z_{n-p} \) occurs at the \( q+1 \) th resp. \( p-q+1 \) th position (cf. fig.3),

with \( y_1 \cdots y_p = z_{P(n)} \cdots z_{P(n-p+1)} \). Thus \( B_q^{(P)} \) differs from \( A_q^{(P)} \) only by a permutation, except that \( z_{n-p} \) is now slightly below the real axis. It is now elementary to add up the phases \( \phi_{jk} \) of \( (z_j - z_k) \), \( j < k \), for \( A_q^{(P)} \) resp. \( B_q^{(P)} \). If \( j, k > n - p \) the phases generated by \( A_q^{(P)} \) and \( B_q^{(P)} \) agree since the relative ordering of the \( y_i \) is the same. For \( j = n - p \), we have

\[
\sum_{k>n-p} \phi_{n-p,k} \big|_{A_q^{(P)}} - \sum_{k>n-p} \phi_{n-p,k} \big|_{B_q^{(P)}} = \pi(p - q) - (\pi p + 2\pi(p - q)) = -\pi p, \quad (B.3)
\]

independent of \( q \). Taking into account also the phase dependence of \( z_k^\gamma \), we find by successively applying the above argument for \( p = 1, \ldots n - 1 \):

\[
I_m^{(J)}(\varpi) = (-i)^n \int_{C_1} dz_1 \cdots \int_{C_n} dz_n \prod_{j=1}^n (1 - z_j)^\alpha z_j^\gamma \prod_{j<k}(z_j - z_k)^\beta \prod_{l=0}^{n-1} (1 - e^{2\pi i\gamma + i\pi l\beta}) \quad (B.4)
\]

where the contours are as in fig.3. Reversing the direction of integration and
renaming \(z_1, \ldots, z_n \rightarrow z_n, \ldots, z_1\), we obtain finally, using \((z_k - z_j)^\beta = (z_j - z_k)^\beta e^{i\pi \beta}\),

\[
I_m^{(J)}(\varpi) = i^n \prod_{l=1}^n e^{i\pi \beta(l-1)}(1 - e^{2\pi i \gamma + i\pi \beta(l-1)})J_{n0}(\alpha', \beta'; \rho'),
\]

where

\[
J_{n0}(\alpha', \beta'; \rho') = \int_{\hat{C}_1} dz_1 \cdots \int_{\hat{C}_n} dz_n \prod_{j=1}^n (1 - z_j)^\alpha z_j^\gamma \prod_{j<k} (z_j - z_k)^\beta \tag{B.5}
\]

\(J_{n0}(\alpha', \beta'; \rho')\) is precisely the integral appearing in Eq. (A.4) of \cite{ref7}, with \(\alpha' = \gamma, \beta' = \alpha, 2\rho' = \beta\). The contours \(\hat{C}_i\) are the same as in fig. 3, except that all directions are reversed, as well as the order of \(z_1, \ldots, z_n\). Inserting the result of \cite{ref25} for \(J_{n0}\), one then obtains our formula Eq. \(\text{3.38}\).

**B.2 The relationship between \(g_{J,x+m}^x\), \(I_m^{(J)}\) and \(\kappa_{J,x+m}^x\)**

In this part, we verify the consistency of Eq. \(\text{3.32}\). Let us compute \(g_{J,x+m}^x / I_m^{(J)}(\varpi)\) from Eqs. \(\text{3.30}, \text{3.32}\). It is convenient to combine the results in three factors.

\[
\prod_{k=1}^n \frac{\sqrt{F[1 + (2J - k + 1)h/\pi]}}{\sqrt{F[1 + kh/\pi]}} \frac{\Gamma[1 + kh/\pi]}{\Gamma[1 + (2J - k + 1)h/\pi]} = \prod_{k=1}^n \sqrt{\frac{-[2J - k + 1]}{-|k|}}
\]

\[
\prod_{k=1}^n \sqrt{F[(\varpi + 2m - k)h/\pi] \Gamma[1 - (\varpi + 2m - k)h/\pi]} = \frac{\pi^{n/2}}{\prod_{k=1}^n \sqrt{\sin[h(\varpi + 2m - k)]}}
\]

\[
\prod_{k=1}^n \sqrt{F[-(\varpi + k)h/\pi] \Gamma[1 + (\varpi + k)h/\pi]} = \frac{\pi^{n/2}}{\prod_{k=1}^n \sqrt{-\sin[h(\varpi + k)]}} \tag{B.6}
\]

These equations easily lead to Eq. \(\text{3.22}^{\text{18}}\).

**B.3 The case of two screening charges**

Up to now we have been considering only the case of a single screening charge. However, the Fateev-Dotsenko formulae also cover the general case with both screening charges, corresponding to the normalizations \(I_m^{(J)}(\varpi)\). The above calculation is easily generalized and we skip the details. The final result can be written

\[
I_{\text{mm}}^{(J)}(\varpi) = I_{\text{m}^0}^{(J^e)}(\varpi + 2\hat{\pi}^/h) \hat{I}_{\text{m}^0}^{(J^e)}(\varpi + 2\hat{\pi}^/h)(i\pi/h)^{2\hat{\pi}^/h} \times \\
\prod_{l,l=1}^{n,\hat{n}} \{(l + \hat{l}^/h)(\varpi + 2m + 2\hat{\pi}^/h - l - \hat{l}^/h)(\varpi + l + \hat{l}^/h)(2J^e - (l - 1) - (\hat{l} - 1)/\pi/h)\}^{-1} \\
\times \prod_{l=1}^{2\hat{n}} \prod_{l=1}^n (l + (\varpi + l)/h) \prod_{l=1}^{2\hat{n}} \prod_{l=1}^n (l + (\varpi + l)/h) \tag{B.7}
\]

\(^{18}\)In order to define the square roots in the first of Eqs. B.6, we give a small negative imaginary part to \(h\) (cf. ref. \cite{ref25}). Then we can actually cancel the minus signs on the r.h.s. of this equation.
The earlier definition of the Liouville exponential

For completeness, let us connect Eq. 4.2 with the previous definition of ref. [7] for the half-integer case. In this article the following formula was introduced (we distinguish it by an index G.):

\[ e^{-J\alpha\phi(\sigma,\tau)_{G.}} = \tilde{c}_J \frac{1}{\sqrt{[\varpi]}} \sum_{m=-J}^{J} C_m^{(J)}(\varpi)\psi_m^{(J)}(z)\overline{\psi}_m^{(J)}(\bar{z})\sqrt{[\varpi]}, \]  

\( (C.1) \)

where \( \tilde{c}_J \) is an arbitrary constant, and

\[ C_m^{(J)}(\varpi) = (-1)^{J-m}(2i\sin \hbar)^{2J} e^{ihJ} \left( \frac{2J}{J-m} \right) \frac{|\varpi - J + m|_{2J+1}}{|\varpi + 2m|}. \]  

\( (C.2) \)

The fields \( \psi_m^{(J)} \) are similar to the \( \tilde{V}_m^{(J)} \)'s, but have a different normalization:

\[ \psi_m^{(J)} = E_m^{(J)} \tilde{V}_m^{(J)}, \]

with

\[ E_m^{(J)}(\varpi) = \frac{\prod_{r=1}^{J-m}(1 + r\hbar/\pi)}{\prod_{r=1}^{J+m}(1 + m\hbar/\pi)} \prod_{r=1}^{2m} \sqrt{\Gamma(\varpi + r - 1)\Gamma(\varpi + r + 1)} \times \]

\[ \prod_{r=1}^{J-m} \Gamma[\varpi - r\hbar/\pi] \prod_{r=1}^{J+m} \Gamma[-(\varpi + r)\hbar/\pi]. \]  

\( (C.3) \)

For \( \varpi = \varpi \), the case considered in ref. [7], elementary manipulations lead to the formula

\[ C_m^{(J)}(\varpi)E_m^{(J)}(\varpi)E_m^{(J)}(\varpi) = (2i\pi^2/\hbar)^{2J} e^{ihJ} \frac{\Gamma(\varpi + 2m + 1)\Gamma[(\varpi + 2m)\hbar/\pi]}{\Gamma(\varpi + 1)\Gamma(\varpi\hbar/\pi)}. \]  

\( (C.4) \)

Thus we can write \( e^{-J\alpha\phi(\sigma,\tau)_{G.}} \) in the form

\[ e^{-J\alpha\phi(\sigma,\tau)_{G.}} = c_J \rho^2(\varpi) \sum_{m=-J}^{J} \tilde{V}_m^{(J)}\overline{\tilde{V}}_m^{(J)} \sqrt{[\varpi]} / \rho^2(\varpi) \]

with

\[ \rho^2(\varpi) = \frac{\sqrt{\hbar/\pi}}{\Gamma(\varpi\hbar/\pi)\Gamma(\varpi + 1)} = \frac{1}{\Gamma(\varpi\hbar/\pi)\Gamma(\varpi/\hbar)\sqrt{\varpi\varpi}}. \]  

\( (C.5) \)

Thus the previous definition is related to Eq. 4.2 by a transformation of the type we have discussed in detail in ref. [19], that is a transformation of the Hilbert space that only involves the zero-mode \( \varpi \). We see that when written in the form Eq. C.5, the previous definition Eq. C.1 has meaning even for arbitrary \( J \). This was already
noticed in ref. [19], where the relation with other approaches [20] [21] was established by transformations of the same type.

Finally, let us deal with the case of two screening charges. In the half-integer case, the most general Liouville field was defined by

\[ e^{-(J\alpha_- + \hat{J}\alpha_+)}\Phi\bigg|_G \sim e^{-J\alpha_-}\Phi\bigg|_G e^{-\hat{J}\alpha_+}\Phi\bigg|_G \]  \hspace{1cm} (C.6)

where \( \sim \) means that one keeps the leading-order term in the fusion. According to ref. [8], the leading order fusion of the \( \tilde{V} \) fields is simply, for half-integer spins,

\[ \tilde{V}^{(J)}_{m}\tilde{V}^{(J)}_{\hat{m}} \sim \tilde{V}^{(J\hat{J})}_{m\hat{m}}. \]

Thus we get after commuting the intermediate \( \sqrt{[\varpi]} \) resp. \( \sqrt{[\hat{\varpi}]} \) factors to the right resp. to the left and redefining the normalization constant,

\[ e^{-(J\alpha_- + \hat{J}\alpha_+)}\Phi\bigg|_G = cJ\rho(\varpi)\sqrt{[\varpi]}\sqrt{[\hat{\varpi}]} \times \]

\[ \sum_{m,\hat{m}} (-1)^{J+m}(-1)^{\hat{J}+\hat{m}}\tilde{V}^{(J\hat{J})}_{mm\hat{m}}(u)\tilde{V}^{(J\hat{J})}_{mm\hat{m}}(v)\frac{\sqrt{[\varpi]}\sqrt{[\hat{\varpi}]}\rho(\varpi)^2}{\rho(\varpi)^2}, \]  \hspace{1cm} (C.7)

which is related to Eq.5.20 in a way similar to the case of a single screening charge. In contrast to Eq.C.5, we here have \( \mu_0 = \hat{\mu}_0 = -1 \), but it is also possible to have \( \mu_0 = \hat{\mu}_0 = 1 \) if we drop the \( \sqrt{[\varpi]} \) factors in the definition Eq.C.1 of the old Gervais exponentials, and its hatted counterpart. Note that it was crucial to define \( \rho(\varpi) \) to be symmetric between \( \hbar \) and \( \hat{\hbar} \), so that the \( \rho \) factors in between the two exponentials could cancel in Eq.C.6. For general continuous spins, \( \tilde{V}^{(J)}_{m}\tilde{V}^{(J)}_{\hat{m}} \) and \( \tilde{V}^{(J)}_{m}\tilde{V}^{(J)}_{\hat{m}} \) differ by a nontrivial normalization factor, and Eq.5.20 must be used instead of Eq.C.6.

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