Moduli of vector bundles on projective surfaces: some basic results.

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March 28 1994

Vector bundles on algebraic surfaces have been studied since the 1960’s. Moduli spaces for stable bundles were constructed in the 70’s. Subsequently, Gieseker and Maruyama constructed moduli spaces for semistable torsion-free sheaves: these provide natural compactifications of the moduli spaces of vector bundles. Many detailed and interesting results have been proved regarding these moduli spaces when the surface belongs to some particular class, for example if it is $\mathbb{P}^2$ or a $K3$, but our knowledge decreases as the Kodaira dimension of the surface increases, and in particular very little is known if the surface is of general type. In this paper we address two basic questions:

1. When is the moduli space reduced and of the expected dimension?
2. When is the moduli space irreducible?

In order to present our results we need to introduce some notation. Let $S$ be a smooth irreducible projective surface over $\mathbb{C}$, and let $H$ be an ample divisor on $S$. To define a moduli space of sheaves on the polarized surface $(S,H)$ we need a set of sheaf data $\xi$, i.e. a triple

$$\xi = (r_\xi, \det_\xi, c_2(\xi)),$$

where $r_\xi$ is a positive integer, $\det_\xi$ is a line bundle on $S$, and $c_2(\xi) \in H^4(S; \mathbb{Z}) \cong \mathbb{Z}$. We let $M_\xi$ be the moduli space of semistable (with respect to $H$) torsion-free sheaves, $F$, on $S$ with

$$r_F = r(F) = r_\xi, \quad \det F \cong \det_\xi, \quad c_2(F) = c_2(\xi). \quad (0.1)$$

A fundamental theorem of Gieseker and Maruyama [G1, Ma] asserts that $M_\xi$ is projective. If $F$ is a semistable sheaf satisfying (0.1), we let $[F]$ be the point in $M_\xi$ corresponding to the equivalence class of $F$. We recall some known facts concerning the local structure of $M_\xi$. First let’s define the discriminant of a torsion-free sheaf $F$ on $S$ as

$$\Delta_F := c_2(F) - \frac{r(F) - 1}{2r(F)} c_1(F)^2$$

(warning: our normalization differs from that of [DL]). If $\xi$ is a set of sheaf data, the discriminant $\Delta_\xi$ is defined in the obvious way: if $[F] \in M_\xi$, then $\Delta_\xi = \Delta_F$. The expected dimension of $M_\xi$ is given by

$$\expdim (M_\xi) := 2r_\xi \Delta_\xi - (r_\xi^2 - 1) \chi(O_S).$$
Now assume that \([F] \in \mathcal{M}_\xi\) and that \(F\) is stable. Then deformation theory \([F]\) gives

\[
\dim_{[F]} \mathcal{M}_\xi \geq 2r_\xi \Delta_\xi - (r_\xi^2 - 1) \chi(\mathcal{O}_S), \tag{0.2}
\]
\[
\dim T_{[F]} (\mathcal{M}_\xi) = 2r_\xi \Delta_\xi - (r_\xi^2 - 1) \chi(\mathcal{O}_S) + h^0(F, F \otimes K)^0, \tag{0.3}
\]

where, for a line bundle \(L\) on \(S\), we set

\[
h^0(F, F \otimes L)^0 := \dim \{ \varphi \in \text{Hom}(F, F \otimes L) \mid \text{tr}\varphi = 0 \}.
\]

Let \([F] \in \mathcal{M}_\xi\). Following Friedman we say that \(\mathcal{M}_\xi\) is good at \([F]\) if \(F\) is stable and \(h^0(F, F \otimes K)^0\) vanishes, where \(K\) is the canonical line bundle. In this case (0.2) is an equality and the moduli space is smooth near \([F]\). We say that \(\mathcal{M}_\xi\) is good if it is good at the generic point of every one of its irreducible components: this means that \(\mathcal{M}_\xi\) is reduced and its dimension equals the expected dimension. Now we can go back to Questions (1) and (2). First of all notice that if \(r_\xi = 1\) then \(\mathcal{M}_\xi\) is good (at each point) for trivial reasons, and furthermore, as is well-known, it is always irreducible. Thus we will only be concerned with the case \(r_\xi \geq 2\). Our main result is that if \(\Delta_\xi \gg 0\) then \(\mathcal{M}_\xi\) is good and irreducible, i.e. both questions have a positive answer. The significance of the condition \(\Delta_\xi \gg 0\) is the following: if \(\Delta_\xi < 0\) then \(\mathcal{M}_\xi\) is empty by Bogomolov’s Inequality, and on the other hand \(\mathcal{M}_\xi \neq \emptyset\) if \(\Delta_\xi \gg 0\) (see [HL,LQ]). Actually we will prove more than the simple statement that \(\mathcal{M}_\xi\) is asymptotically good. To explain this, let \(L\) be a line bundle on \(S\) and set

\[
W^L_\xi = \{ [F] \in \mathcal{M}_\xi \mid h^0(F, F \otimes L)^0 > 0 \}.
\]

Thus if \(F\) is stable then \(\mathcal{M}_\xi\) is good at \([F]\) if and only if \([F] \notin W^K_\xi\). We prove that (if \(L\) is fixed) the growth of \(\dim W^L_\xi\) (for fixed rank and increasing \(\Delta_\xi\)) is smaller than that of the expected dimension of \(\mathcal{M}_\xi\). The theorem about \(\mathcal{M}_\xi\) being good for \(\Delta_\xi \gg 0\) follows at once from this result (setting \(L = K\)), together with some dimension counts to take care of properly semistable sheaves. To see that it is interesting to bound the dimension of \(W^L_\xi\) for arbitrary \(L\) consider the case \(L = \mathcal{O}_S(K + C)\), where \(C\) is a smooth curve on \(S\). In this case, if \(F\) is locally-free and stable, the geometric significance of \([F] \notin W^L_\xi\) is the following: the natural morphism from a neighborhood of \([F]\) in \(\mathcal{M}_\xi\) to the deformation space of \([F]|_C\) surjects onto the subspace of deformations fixing the isomorphism class of \(\text{det}(F|_C)\).

The precise statements of the results we have described are given in Theorems B, C, D, E and their corollaries. One feature of these theorems is that they are for the most part effective. Thus we give an explicit upper bound for \(\dim W^L_\xi\). From this one can compute an explicit lower bound for \(\Delta_\xi\) guaranteeing that \(\mathcal{M}_\xi\) is good. This lower bound for arbitrary rank might not be very practical, however it appears to depend on the ”correct” quantities. If the rank is two our methods are somewhat stronger: in this case we have computed the lower bound explicitly, and we will show that it can not be too far off from the optimal one. Regarding irreducibility our results are less explicit, but we do give a lower bound for \(\Delta_\xi\) guaranteeing irreducibility of the moduli space for rank-two bundles with trivial determinant on a complete intersection.

All of the above results spring from Propositions (1.1)-(1.2). To explain the content of these propositions let the boundary of \(\mathcal{M}_\xi\), denoted by \(\partial \mathcal{M}_\xi\), be the subset of \(\mathcal{M}_\xi\) parametrizing sheaves which are singular, i.e. not locally-free. Furthermore, if \(X \subset \mathcal{M}_\xi\) let the boundary of \(X\) be the intersection

\[
\partial X := X \cap \partial \mathcal{M}_\xi.
\]

Propositions (1.1)-(1.2) assert that if \(X \subset \mathcal{M}_\xi\) is a closed subvariety whose dimension satisfies certain conditions then \(\partial X \neq \emptyset\). These propositions are proved by further developing the ideas
in the proof of Theorem (1.0.3) of [O]. (This theorem states that, in rank two, any irreducible component intersects the boundary, if $\Delta_\xi \gg 0$.) From Propositions (1.1)-(1.2) one obtains Theorem A, which bounds the maximum dimension of complete subvarieties of $M_\xi$ not intersecting the boundary. Given Theorem A one can easily bound $\dim W^L_\xi$, this is the content of Theorem B, and in particular prove that $M_\xi$ is asymptotically good. Theorem C gives an explicit lower bound for $\Delta_\xi$ guaranteeing that $M_\xi$ is good, in rank two: this is obtained by arguments similar to those that give Theorem B. Asymptotic irreducibility (Theorem D) follows from Theorems A and B: the argument is due to Gieseker and Li. We reproduce their proof because we will then apply it to complete intersections in order to obtain an explicit result (Theorem E). Our proofs depend on certain estimates: in particular we need an upper bound for the dimension of the loci in $M_\xi$ parametrizing sheaves with subsheaves with (relatively) large slope. This and other estimates are proved in the last section of the paper.

The first to prove that $M_\xi$ is asymptotically good, if the rank is two, was Donaldson [D] (see also [F,Z]). He proved that $\dim W^L_\xi$ is bounded, up to lower order terms, by $3\Delta_\xi$ (the same proof works also for $W^L_\xi$); by (0.2) this implies that $M_\xi$ is good for $\Delta_\xi \gg 0$. Donaldson’s bound for $\dim W^L_\xi$ is asymptotically better than that given by Theorem B, but since the lower order terms in his formula have eluded computation, it does not give an effective result. Recently Gieseker and Li [GL2] have proved that $M_\xi$ is asymptotically good. They proved that the codimension of $W^L_\xi$ in $M_\xi$ goes to infinity for $\Delta_\xi \gg 0$, but they did not make their result effective. Finally Gieseker and Li [GL1] were the first to prove asymptotic irreducibility in rank two.

**Statement of results.** Throughout the paper surface means a smooth irreducible projective surface: we will always denote it by $S$. We let $K$ be its canonical divisor, and $H$ be an ample divisor on $S$. We will often make the following assumption:

$$|H| \text{ is base-point-free and } \dim |H| \geq 2. \quad (0.4).$$

When considering a moduli space $M_\xi$ we always tacitly assume that $r_\xi \geq 2$. If $r > 2$ is an integer, set

$$\rho(r) := 8(16r^3 - 39r^2 + 36r - 12)^{-1},$$

$$\Delta_0(r, S, H) := \rho^{-1}H^2,$$

$$\lambda_2(r) := 2r - \frac{r - 1}{2}\rho,$$

$$\lambda_1(r, S, H) := \sqrt{\rho} \left[ \frac{r^3}{2} \frac{|K \cdot H|}{\sqrt{H^2}} + r^5 \sqrt{H^2} \right],$$

$$\lambda_0(r, S, H) := \frac{r^7}{2}H^2 + \frac{r^4}{5} \frac{(K \cdot H)^2}{H^2} + \frac{r^2}{2} \frac{(K \cdot H + (r^2 + 1)H^2 + 1)^2}{H^2} + r^2 |\chi(O_S)| + \frac{r^3}{8} |K^2|.$$  

When $r = 2$, we set

$$\Delta_0(2, S, H) := \begin{cases} 3H^2, & \text{if } K \cdot H < 0, \\ 3H^2 \left( 1 + \frac{K \cdot H}{H^2} \right)^2, & \text{if } K \cdot H \geq 0, \end{cases}$$

$$\lambda_2(2) := \frac{23}{6},$$

$$\lambda_1(2, S, H) := \frac{1}{2\sqrt{3H^2}} \left( 4H^2 + 3K \cdot H + 4 \right),$$

$$\lambda_0(2, S, H) := \begin{cases} \frac{3(K \cdot H + H^2 + 1)^2}{4H^2} - \frac{K^2}{4} + 4 - 3|\chi(O_S)|, & \text{if } K \cdot H < 0, \\ \frac{3(K \cdot H)^2}{4H^2} + 6K \cdot H + \frac{3H^2}{2} - \frac{K^2}{4} + 8 - 3|\chi(O_S)|, & \text{if } K \cdot H \geq 0. \end{cases}$$
**Theorem A.** Let \((S, H)\) be a polarized surface. Assume that \(H\) satisfies (0.4). Let \(\xi\) be a set of sheaf data such that \(\Delta_\xi > \Delta_0(r_\xi, S, H)\). If \(X \subset \mathcal{M}_\xi\) is a closed subvariety such that

\[
\dim X > \lambda_2(r_\xi) \Delta_\xi + \lambda_1(r_\xi, S, H) \sqrt{\Delta_\xi} + \lambda_0(r_\xi, S, H),
\]

(0.5)

then the boundary of \(X\) is non-empty.

Notice that the above result is meaningful because \(\lambda_2(r) < 2r\) (see (0.2)). Let

\[
\lambda'_0(r, S, H) := \max \{\lambda_0(r, S, H), \epsilon(r, S, H) + r\} ,
\]

\[
\Delta_1(r, S, H) := \max \left\{ (\lambda_2 - (2r - 1))^{-1} \cdot (\Delta_0 + \epsilon_K - \lambda_0'(r^2 - 1)\chi(O_S)), \Delta_0 \right\}
\]

if \((r, S, H) \neq (2, \mathbb{P}^2, O_{\mathbb{P}^2}(1))\),

\[
\Delta_1(2, \mathbb{P}^2, O_{\mathbb{P}^2}(1)) := 3.
\]

Here \(\epsilon\) and \(\epsilon_K\) are given by (5.39) and (5.6) respectively.

**Theorem B.** Let \((S, H)\) be a polarized surface, with \(H\) satisfying (0.4). Let \(L\) be a line bundle on \(S\). Let \(\xi\) be a set of sheaf data such that \(\Delta_\xi > \Delta_1(r_\xi, S, H)\). Then

\[
\dim W^L_\xi \leq \lambda_2(r_\xi) \Delta_\xi + \lambda_1(r_\xi, S, H) \sqrt{\Delta_\xi} + \lambda'_0(r_\xi, S, H) + \epsilon_L(r_\xi, S, H),
\]

(0.6)

where \(\epsilon_L\) is given by (5.6).

Applying the above result with \(L = K\), one gets the following

**Corollary B’.** There exists a function \(\Delta'_1(r, S, H)\) (depending only on \(r, K^2, K \cdot H, H^2\), and \(\chi(O_S)\)) such that the following holds. Let \((S, H)\) be as above, and \(\xi\) be a set of sheaf data such that \(\Delta_\xi > \Delta'_1(r_\xi, S, H)\). Then \(\mathcal{M}_\xi\) is good. Furthermore \(\mathcal{M}_\xi\) is the closure of the open subset parametrizing \(\mu\)-stable vector bundles.

In the rank-two case we will carry out the computations necessary to determine explicitly the lower bound of this corollary. The result is the following

**Theorem C.** Let \((S, H)\) be a polarized surface. Assume that \(H\) is effective and that there exists a smooth curve in \([n_0H]\), where \(n_0\) is given by (2.13). Let \(\Delta_2(S, H)\) be the function defined by (2.12). Let \(\xi\) be a set of sheaf data with \(r_\xi = 2\). If

\[
\Delta_\xi > \Delta_2(S, H) + 2h^0(2K),
\]

then \(\mathcal{M}_\xi\) is good. Furthermore \(\mathcal{M}_\xi\) is the closure of the open subset parametrizing \(\mu\)-stable vector bundles.

To exemplify the possible uses of this theorem we give the following two results.

**Corollary C’.** Let \(S\) be a surface with ample canonical divisor \(K\). Assume that \(p_g(S) > 0\), and that \(K^2 \gg 0\) (\(K^2 > 100\) will do). Let \(\xi\) be a set of sheaf data with \(r_\xi = 2\), and let \(\mathcal{M}_\xi\) be the corresponding moduli space for semistable sheaves on the polarized surface \((S, K)\). If

\[
\Delta_\xi \geq 42K^2 + 15\chi(O_S),
\]
then $M_\xi$ is good, and furthermore the subset parametrizing $\mu$-stable vector bundles is dense in $M_\xi$.

**Corollary C''.** Let $S$ be a surface with ample canonical divisor $K$. Assume that $H$ is a divisor satisfying (0.4), and that $c_1(K) = kc_1(H)$ for $k \gg 0$ (say $k > 100$). Let $\xi$ be a set of sheaf data with $r_\xi = 2$, and let $M_\xi$ be the corresponding moduli space for semistable sheaves on $(S,K)$. If

$$\Delta_\xi \geq 17K^2 + 10\chi(O_S),$$

then the same conclusions as in the previous corollary hold.

These two corollaries only apply to minimal surfaces of general type with no $(-2)$-curves. Of course Theorem C applies to any surface, in particular we can apply it to general-type surfaces with $(-2)$-curves, but the lower bound one gets is not as nice as that of Corollaries C'-C''. However we believe that bounds similar to those of these corollaries hold for any surface of general type, if $K^2$ is replaced by $\omega_S^{\text{can}}$, where $S_\text{can}$ is the canonical model of $S$, and the polarization is close enough to $\omega_{S_\text{can}}$ (how "close" will depend on $\Delta_\xi$). When $S_\text{can}$ is smooth this should follow from the results in Section 2 of [MO]; the case when $S$ contains $(-2)$-curves should be analyzable by similar methods.

Finally we come to irreducibility.

**Theorem D.** There exists a function $\Delta_3(r,S,H)$ such that the following holds. If $(S,H)$ is a polarized surface, and if $\xi$ is set of sheaf data such that $\Delta_\xi > \Delta_3(r_\xi,S,H)$, then $M_\xi$ is irreducible (and is the closure of the locus parametrizing $\mu$-stable vector bundles).

We give an explicit value for $\Delta_3$ valid for complete intersections of large degree, if the rank is two and the determinant is trivial.

**Theorem E.** Let $S$ be a complete intersection in a projective space, and let $H = O_S(1)$. Suppose also that the integer $k$ such that $K \sim kH$ is very large ($k > 100$ suffices). Let

$$\xi = (2, O_S, c_2(\xi)).$$

If

$$\Delta_\xi > 95K^2 + 11\chi(O_S) + 1,$$

then $M_\xi$ is irreducible.

**Notation and conventions.**

We work throughout over the complex numbers. Sheaves are always coherent. We let $\pi_X: X \times Y \to X$ be the projection. A family of sheaves on $X$ parametrized by $B$ consists of a sheaf $F$ on $X \times B$, flat over $B$. If $b \in B$ we set $F_b := F|_{X \times \{b\}}$. Let $\xi$ be a set of sheaf data for $S$, and let $U \subset M_\xi$ be an algebraic subset all of whose points parametrize stable sheaves. A tautological family parametrized by $U$ consists of a family of stable sheaves $F$ on $S$, parametrized by $U$, such that for all $b \in U$ the isomorphism class of $F_b$ is represented by $b$. If $F$ is a sheaf on a smooth curve or surface, we let $Def(F)$ be the versal deformation space of $F$, and $Def^0(F)$ be the subscheme parametrizing deformations which fix the isomorphism class of $\det F$. (See [F].)

If $F$ is a semistable torsion-free sheaf on $S$, we denote by $Gr(F)$ the direct sum of the successive quotients of any Jordan-Hölder filtration of $F$. Recall that the (closed) points of $M_\xi$ are in one-to-one correspondence with equivalence classes of torsion-free semistable sheaves, $\tilde{F}$, satisfying (0.1): two semistable sheaves $F_1, F_2$ are equivalent if $Gr(F_1) \cong Gr(F_2)$.
Let $X$ be a projective irreducible variety, and $D$ be an ample divisor on $X$. The \textit{slope} of a torsion-free sheaf $F$ on $X$ with respect to $D$ is given by

$$
\mu_F = \mu(F) := \frac{1}{\tau_F} c_1(F) \cdot D^{n-1},
$$

where $n := \dim X$. We recall that $F$ is $\mu$-semistable (equivalently slope-semistable) if, for all subsheaves $E \subset F$ we have

$$
\mu_E \leq \mu_F.
$$

If the above inequality is strict whenever $r(E) < r(F)$, then $F$ is $\mu$-stable (slope-stable). If $F$ is semistable then it is $\mu$-semistable, and $\mu$-stability implies stability. Let $\alpha \in \mathbb{R}$. A torsion-free sheaf $F$ on $S$ is $\alpha$-stable if, for every subsheaf $E \subset F$ with $0 < r(E) < r(F)$, one has

$$
\mu_E < \mu_F - \frac{\alpha}{r_E} \sqrt{H^2}.
$$

Thus $\mu$-stability is equivalent to 0-stability. As is immediately verified $F$ is $\alpha$-stable if and only if, for every non-trivial torsion-free quotient $F \rightarrow Q$, one has

$$
\mu_Q > \mu_F + \frac{\alpha}{r(Q)} \sqrt{H^2}.
$$

Furthermore the notion of $\alpha$-stability only depends on the ray spanned by $c_1(H)$, and $F$ is $\alpha$-stable if and only if so is $F^*$. We let

$$
\mathcal{M}_\xi(\alpha) := \{ [F] \in \mathcal{M}_\xi \mid F \text{ is not } \alpha\text{-stable} \}.
$$

We will show (Proposition (5.9)) that $\mathcal{M}_\xi(\alpha)$ is a constructible subset of $\mathcal{M}_\xi$. In case $r_\xi = 2$ we will also consider a (constructible) subset of $\mathcal{M}_\xi(\alpha)$, defined as follows.

\textbf{(0.7) Definition.} Let $\xi$ be a set of sheaf data with $r_\xi = 2$. Let $C \subset S$ be a smooth irreducible curve. For $\alpha \in \mathbb{R}$ we let $\mathcal{M}_\xi^C(\alpha) \subset \mathcal{M}_\xi$ be the subset of points $[F]$ such that $F|_C$ is locally-free, and such that there exists a rank-one subsheaf $A \subset F$ with:

1. $\mu_A \geq \mu_F - \alpha \sqrt{H^2}$ (so $F$ is not $\alpha$-stable).
2. The restriction $A|_C$ spans a destabilizing subline bundle of $F|_C$.

\section{A criterion for non-emptiness of the boundary.}

In this section we will prove the two following propositions.

\textbf{(1.1) Proposition.} Let $(S, H)$ be a polarized surface, with $H$ satisfying (0.4). Let $\xi$ be a set of sheaf data, and let $X \subset \mathcal{M}_\xi$ be a closed subvariety. Assume that there exists a positive integer $n$ such that:

1. $\dim X > \frac{1}{2} (r_\xi^2 - 1)(H^2 n^2 + K \cdot Hn)$,
2. $\dim X > \frac{1}{4} r_\xi^2 (H^2 n^2 + K \cdot Hn) + \frac{1}{4} r_\xi^2 + \frac{1}{2} + (2r_\xi - 1) \Delta_\xi + \epsilon(r_\xi, S, H)$,
3. $\dim X > \frac{1}{4} r_\xi^2 (H^2 n^2 + K \cdot Hn) + \frac{1}{4} r_\xi^2 + \frac{1}{2} + \dim \mathcal{M}_\xi \left( (r_\xi - 1) \sqrt{H^2 n} \right)$,
4. $\dim X > 2r_\xi \Delta_\xi - (r_\xi^2 - 1) \chi(\mathcal{O}_S) + \epsilon_K(r_\xi, S, H) + \frac{1}{4} r_\xi^2 - \frac{1}{2} (r_\xi - 1) (H^2 n^2 - K \cdot Hn)$,
where \([\ast]\) denotes integer part, and \(e_K(r_\xi, S, H), e(r_\xi, S, H)\) are as in (5.6) and (5.39) respectively. Then \(\partial X \neq \emptyset\).

(1.2) Proposition. Let \((S, H)\) be a polarized surface. Assume that \(H\) is effective. Let \(\xi\) be a set of sheaf data with \(r_\xi = 2\). Let \(X \subset M_\xi\) be a closed subvariety. Suppose that there exist a positive integer \(n\) and a smooth curve \(C \in |nH|\) such that Items (1) and (2) of Proposition (1.1) are satisfied, and furthermore

\[
\dim X > \frac{1}{2}(H^2n^2 + K \cdot Hn) + 1 + \dim \mathcal{M}_\xi^C \left( \frac{\sqrt{H^2}}{n} \right),
\]

(1.3)

\[
\dim X > \frac{1}{2}(H^2n^2 + K \cdot Hn) + 1 + \dim \mathcal{M}_\xi \left( \frac{K \cdot H}{2\sqrt{H^2}} \right),
\]

(1.4)

\[
\dim X > 4\Delta_\xi - 3\chi(O_S) + h^0(2K) + 1 - \frac{1}{2}(H^2n^2 - K \cdot Hn).
\]

(1.5)

Then \(\partial X \neq \emptyset\).

The above propositions will be proved at the end of the section.

Certain families of elementary modifications.

In this subsection we will prove:

(1.6) Proposition. Let \((S, H)\) be a polarized surface. Let \(\xi\) be a set of sheaf data. Let \([F] \in M_\xi\), and assume that \(F\) is locally-free and \(\mu\)-stable. Let \(C \subset S\) be a smooth irreducible curve. Set \(\alpha_C := (r_\xi - 1)(C \cdot H)/\sqrt{H^2}\). Assume that:

1. \(F|_C\) is not stable,
2. \([F] \notin \mathcal{M}_\xi(\alpha_C)\).

Let \(X \subset M_\xi\) be a closed subvariety containing \([F]\), and such that

\[
\dim X > 2r_\xi\Delta_\xi - (r_\xi^2 - 1)\chi(O_S) + h^0(F, F \otimes K_S)^0 + \frac{r_\xi^2}{4} - \frac{1}{2}(r_\xi - 1)C^2 + \frac{1}{2}(r_\xi - 1)C \cdot K.
\]

(1.7)

Then the boundary of \(X\) is non-empty. If \(r_\xi = 2\), the same conclusion holds if Item(2) is replaced by the condition

\([F] \notin \mathcal{M}_\xi^C(\alpha_C)\).

We will prove this proposition at the end of the subsection. The key ingredient in its proof is provided by a certain family of elementary modifications. We now proceed to introduce this family.

Let \(C \subset S\) be as above. Let \([F] \in M_\xi\), and assume that \(F|_C\) satisfies Item (1) of Proposition (1.6). Choose a destabilizing sequence

\[
0 \to \mathcal{L}_0 \to F|_C \to \mathcal{Q}_0 \to 0.
\]

(1.8)

By definition we have:

\(\mathcal{L}_0\) and \(\mathcal{Q}_0\) are locally-free,

\[\mu(\mathcal{L}_0) - \mu(\mathcal{Q}_0) \geq 0.\]

(1.9)
Let $E$ be the elementary modification of $F$ associated to the destabilizing quotient of (1.8), i.e. the sheaf on $S$ fitting into the exact sequence

$$0 \to E \to F \xrightarrow{φ} \iota_∗ Q_0 \to 0,$$

where $ι : C \hookrightarrow S$ is the inclusion. Restricting the above sequence to $C$, one gets an exact sequence

$$0 \to Q_0 \otimes O_C(−C) \to E|_C \xrightarrow{f_0} L_0 \to 0.$$  \hspace{1cm} (1.12)

Hence by (1.9) $E|_C$ is locally-free. Since $E$ and $F$ are isomorphic outside of $C$ we conclude that $E$ is locally-free. Let

$$Y_F := Quot(E|_C; L_0)$$

be the Grothendieck Quot-scheme parametrizing quotients of $E|_C$ which have the same Hilbert polynomial as $L_0$. Notice that the notation is slightly imprecise, since a destabilizing sequence for $F|_C$ is not necessarily unique. However this will not create confusion because we will always fix a sequence (1.8) once and for all. We denote by $0$ the point of $Y_F$ corresponding to $f_0$ (see (1.12)). We are now ready to define the promised family of elementary modifications. The parameter space will be $Y_F$. Let

$$\pi^*_C(E|_C) \xrightarrow{f} L$$

be the tautological quotient sheaf on $C \times Y_F$, and let $G$ be the sheaf on $S \times Y_F$ fitting into the exact sequence

$$0 \to G \to \pi^*_S E \xrightarrow{φ} (ι \times id_{Y_F})_* L \to 0,$$

where $φ$ is the composition of restriction to $C \times Y_F$ and $f$.

(1.13) Lemma. The sheaf $G$ is flat over $Y_F$, and thus we can regard it as a family of sheaves on $S$ parametrized by $Y_F$. Let $y \in Y_F$. Then $G_y$ fits into the exact sequence

$$0 \to G_y \to E \xrightarrow{φ_y} ι_∗L_y \to 0,$$

where $φ_y := φ|_{S \times \{y\}}$. In particular $G_y$ is torsion-free. Finally, $G_y$ is singular if and only if so is $L_y$.

Proof. The sheaf $G$ is $Y_F$-flat because so are $π^*_S E$ (obvious) and $(ι \times id_{Y_F})_* L$ (by definition of the Quot-scheme). Flatness of the latter implies that (1.14) is exact. For $y \in Y_F$, let $R_y := ker f_y$; since $E|_C$ is locally-free, so is $R_y$. The exact sequence

$$0 \to L_y(−C) \to G_y|_C \to R_y \to 0$$

shows that $G_y$ is singular at a point $P \in C$ if and only if $L_y$ is singular at $P$. Since

$$G_y|_{(S−C)} \cong E|_{(S−C)},$$

and $E$ is locally-free, we conclude that $G_y$ is singular if and only if so is $L_y$. q.e.d.

Let $F := G \otimes π^*_S O_S(C)$. By the above lemma, we can regard $F$ as a family of torsion-free sheaves on $S$ parametrized by $Y_F$. Let $∂Y_F \subset Y_F$ be the subset parametrizing singular sheaves.
(1.15) Lemma. Let notation be as above. Then:
1. \( F_0 \cong F \).
2. For \( y \in Y_F \) we have \( \det F_y \cong \det \xi \) and \( c_2(F_y) = c_2(\xi) \).
3. Let \( \Sigma \subset Y_F \) be a closed subvariety. If \( \dim \Sigma > r^2/4 \), then \( \Sigma \cap \partial Y_F \neq \emptyset \).

Proof. Clearly the subsheaf \( F(-C) \to F \) is in the kernel of the map \( g \) of (1.11). Hence \( F(-C) \) is actually a subsheaf of \( E \); let \( \lambda: F(-C) \to E \) be the inclusion map. As is easily checked

\[
\Im(\lambda|_C) = \ker f_0.
\]

Since \( \lambda \) is an isomorphism outside of \( C \), we conclude that \( F(-C) \) fits into the exact sequence

\[
0 \to F(-C) \to E \xrightarrow{\phi_0} \ell_* L_0 \to 0.
\]

By Lemma (1.13) the sheaf \( G_0 \) fits into the same exact sequence, and thus \( G_0 \cong F(-C) \). This proves Item (1). Let’s consider the second Item. It follows from Exact sequences (1.11) and (1.14) that \( \det E \cong \det F(-rQ_0/C) \) and that \( \det G_0 \cong \det F(-rC_0 Y) \). This gives \( \det F_y \cong \det F \cong \det \xi \). Since \( c(t_*(L_0)) \) is independent of \( y \in Y_F \), so is \( c(G_y) \), and hence also \( c(F_c) \). Since \( c_2(F_0) = c_2(F) = c_2(\xi) \), we conclude that \( c_2(F_y) = c_2(\xi) \) for all \( y \in Y_F \). Now let’s prove Item (3). Assume that \( \Sigma \cap \partial Y_F = \emptyset \); we will arrive at a contradiction. Fix \( P \in C \). Since we are assuming that \( F_0 \) is locally-free for all \( y \in \Sigma \), it follows from the last sentence of Lemma (1.13) that

\[
\mathcal{L}|_{P \times \Sigma}
\]

is a vector bundle. Thus the kernel of the map

\[
f|_{P \times \Sigma}: E_P \otimes \mathcal{O}_\Sigma \to \mathcal{L}|_{P \times \Sigma},
\]

is a rank-\( r(Q_0) \) subbundle of the trivial bundle on \( \Sigma \) with fiber \( E_P \) (here \( E_P \) is the fiber of \( E \) at \( P \)). Let

\[
\rho: \Sigma \to \text{Gr} := \text{Gr}(r(Q_0), E_P)
\]

be the morphism defined by setting \( \rho(y) := \ker(f|_{(P,y)}) \). Now fix \( [V] \in \text{Gr} \). By hypothesis \( \dim \Sigma > \dim \text{Gr} \), and hence \( \dim \rho^{-1}([V]) \geq 1 \). (\( * \))

Set

\[
\Omega := \bigcup_{y \in \rho^{-1}([V])} \mathbb{P}(\ker f_y).
\]

Since the map \( \Sigma \to \mathbb{P}(E) \) defined by \( y \to \mathbb{P}(\ker f_y) \) is injective, we conclude by (\( * \)) that

\[
\dim \Omega \geq (1 + r(Q_0)) \cdot (\dagger)
\]

Since \( Y_F \) is complete so is \( \Sigma \), and hence \( \Omega \) is a closed subvariety of \( \mathbb{P}(E) \). By (\( \dagger \)) we conclude that

\[
\dim \Omega \cap \mathbb{P}(E_P) \geq r(Q_0).
\]

This is absurd because the above intersection is \( \mathbb{P}(V) \), and \( \dim \mathbb{P}(V) = (rQ_0 - 1) \). q.e.d.

In order to use the above lemma, we need to ensure that the dimension of \( Y_F \) is large, and that \( F \) is a family of semistable sheaves. (Notice that by Item (1) the sheaves \( F_y \) are stable for \( y \) varying in an open non-empty subset of \( Y_F \); however this will not be good enough.)
(1.16) Lemma. Keep notation as above. Then
\[ \dim Y_F \geq \frac{1}{2}(r_\xi - 1)C^2 - \frac{1}{2}(r_\xi - 1)C \cdot K. \]

Proof. By (1.12) we have the lower bound
\[ \dim Y_F \geq \chi(Q_0^* \otimes O_C(C) \otimes L_0). \]
Let \( g \) be the genus of \( C \). Riemann-Roch gives
\[ \chi(Q_0^* \otimes O_C(C) \otimes L_0) = rL_0rQ_0[\mu(O_C(C)) + \mu(L_0) - \mu(Q_0) + 1 - g], \]
where the slopes are as bundles on \( C \). By Inequality (1.10) we conclude that
\[ \dim Y_F \geq rL_0rQ_0\left(C^2 + 1 - g\right). \]
Using adjunction one gets the lemma. (Notice that if \((C^2 - C \cdot K) < 0\) then the lemma is trivially verified.)

Regarding stability we have the following

(1.17) Lemma. Keep notation as above, and let \( \alpha_C \) be as in the statement of Proposition (1.6). If \([F] \notin M_\xi(\alpha_C)\), then \( F \) is a family of stable sheaves. In the case \( r_\xi = 2 \) the same conclusion holds if:

1. \( F \) is \( \mu \)-stable, and
2. \([F] \notin M_\xi^C(\alpha_C)\).

Proof. Let \( y \in Y_F \). We will show that \( G_y \) is \( \mu \)-stable; this will prove the lemma. First notice that, by Item (2) of Lemma (1.15), we have
\[ \mu(G_y) = \mu_F - C \cdot H. \]  
Now let \( A \hookrightarrow G_y \) be a subsheaf with \( 0 < r(A) < r(G_y) \). Let \( \lambda: A \rightarrow F \) be the composition (see (1.14) and (1.11))
\[ A \hookrightarrow G_y \rightarrow E \rightarrow F. \]
Since \( \lambda \) is injective outside of \( C \), and since \( A \) is torsion-free, we conclude that \( \lambda \) is an injection. If \([F] \notin M_\xi(\alpha_C)\) then
\[ \mu_A < \mu_F - \frac{\alpha_C}{r_A}\sqrt{H^2} \leq \mu_F - C \cdot H = \mu(G_y), \]
and hence \( G_y \) is \( \mu \)-stable. Now assume \( r_\xi = 2 \) and \([F] \notin M_\xi^C(\alpha_C)\). If \( \lambda \) is zero at the generic point of \( C \), then we get an injection \( A(C) \hookrightarrow F \). By hypothesis \( F \) is \( \mu \)-stable, and hence
\[ \mu_A + C \cdot H < \mu_F. \]
By (*) we get \( \mu_A < \mu(G_y) \). Now assume \( \lambda \) is not zero at the generic point of \( C \). Then
\[ \text{Im } (\lambda|_C) = \text{Im } (E|_C \rightarrow F|_C) \quad \text{at the generic point of } C. \]
Since the right-hand side is a destabilizing subline bundle of \( F|_C \), and since \([F] \notin M_\xi^C(\alpha_C)\) we conclude that (†) holds. Thus \( G_y \) is \( \mu \)-stable. q.e.d.
Proof of Proposition (1.6). Since $F$ satisfies Item (1), we can construct $Y_F$ and $\mathcal{F}$ as above. By Lemma (1.17), $\mathcal{F}$ is a family of stable sheaves on $S$ parametrized by $Y_F$. Hence, by Item (2) of Lemma (1.15), $\mathcal{F}$ induces a classifying morphism

$$\varphi: Y_F \to \mathcal{M}_\xi.$$  

By Item (1) of (1.15), we have $F_0 \cong F$. Hence $[F] \in X$ and the inverse image $\varphi^{-1}X$ is a closed subvariety of $Y_F$ containing the point 0. We have

$$\dim \varphi^{-1}X \geq \dim Y_F - (\dim T[F], M_\xi - \dim X).$$  

To be precise: the right-hand side is a lower bound for the dimensions of all irreducible components of $\varphi^{-1}X$ containing 0. By (0.3), (1.7) and Lemma (1.16) we conclude that $\dim \varphi^{-1}X > \frac{r_\xi}{4}$. By Item (3) of Lemma (1.15) there exists $y \in \varphi^{-1}X$ such that $F_y$ is singular. Then $\varphi(y) \in \partial X$, and hence $\partial X \neq 0$. q.e.d.

Determinant bundles.

In this subsection $C$ will be a smooth irreducible curve in the linear system $|nH|$. Let $\mathcal{M}(C; \xi)$ be the moduli space of rank-$r_\xi$ semistable bundles on $C$ with determinant isomorphic to $\det_\xi|C$. Let $X \subset \mathcal{M}_\xi$ be a subvariety such that, for all $[F] \in X$, the restriction $F|_C$ is a stable locally-free bundle. Since $C \in |nH|$ this implies that $F$ is $\mu$-stable for all $[F] \in X$. Then, as is easily verified there exists a morphism $$\rho: X \to \mathcal{M}(C; \xi)$$ given by restriction, i.e. $\rho([F]) = [F|_C]$. Our goal will be to prove the following

(1.18) Proposition. Let $X, C, \rho$ be as above. Assume also that $X$ is closed and irreducible, and that all sheaves parametrized by points of $X$ are locally-free. Let $\Theta$ be the theta-divisor on $\mathcal{M}(C; \xi)$ (see DN]). Then

$$(\rho^* \Theta)^{\dim X} > 0.$$  

We will first prove a series of preliminary results. We start with a weak version of closedness of non-stability for a family of vector bundles on a variable degenerating curve. More precisely:

Let $B$ be a smooth curve, $0 \in B$ be a base point, and $B_0 := (B - 0)$. Let $\pi: C \to B$ be a family of curves; for $b \in B$ set $C_b := \pi^{-1}(b)$. We assume that:

1. $C$ is smooth outside a finite set of points in $C_0$.
2. all fibers $C_b$ are reduced and connected, and for $b \neq 0$ they are smooth.

Let $D_1, \ldots, D_s$ be the irreducible components of the central fiber $C_0$.

(1.19) Lemma. Keep notation as above. Let $\mathcal{F}$ be a vector bundle on $C$, and set $\mathcal{F}_b := \mathcal{F}|_{C_b}$. Assume that for all $b \neq 0$ the bundle $\mathcal{F}_b$ is non-stable. There exists $1 \leq i \leq s$ such that, letting $\lambda_i: D_i \to D_i$ be the normalization, the bundle $\lambda_i^* (\mathcal{F}|_{D_i})$ is not stable.

Proof. For generic $b \in B^0$ the Harder-Narasimhan filtration of $\mathcal{F}_b$ has constant type (i.e. length, and rank and slope of the successive quotients). Thus, shrinking $B^0$ if necessary, we can assume that there exists a vector bundle $Q^0$ on $C_0 := \pi^{-1}(B^0)$, and an exact sequence

$$\mathcal{F}|_{C_0} \xrightarrow{\alpha} Q^0 \to 0,$$  

(*)
whose restriction to \( C_0 \) is a destabilizing sequence, for all \( b \in B^0 \). By properness of the relative Quot-scheme parametrizing quotients of \( F_b \), there is a \( B \)-flat sheaf \( Q \) on \( C \) extending \( Q^0 \), and an exact sequence
\[
F \to Q \to 0.
\]

By flatness \( Q \) is torsion-free. In particular it is locally-free outside a finite set of points in \( C_0 \). Let \( f: \tilde{C} \to C \) be a desingularization such that, for all \( 1 \leq i \leq s \), the proper tranform of \( D_i \) is smooth. We denote this proper transform by \( \tilde{D}_i \). Let \( \text{Tor}(f^*Q) \) be the torsion subsheaf of \( f^*Q \): since \( Q \) is torsion-free \( \text{Tor}(f^*Q) \) is supported on the exceptional divisors of \( f \). Pulling back \((*)\) we get an exact sequence
\[
0 \to K \to f^*F \to f^*Q/\text{Tor}(f^*Q) \to 0.
\]

Since \( \tilde{C} \) is a smooth surface, and since \( f^*F \) is locally-free, \( K \) is also locally-free. Let \( \varphi := \pi \circ f \), and set \( \tilde{C}_b := \varphi^{-1}(b) \). If \( b \neq 0 \), then the restriction of \((\dagger)\) to \( \tilde{C}_b = C_b \) is the destabilizing sequence associated to \( \alpha_b \) (see \((*)\)), and hence
\[
\frac{1}{r(K)}c_1(K) \cdot \tilde{C}_b \geq \frac{1}{r(F)}c_1(f^*F) \cdot \tilde{C}_b.
\]

The same inequality holds also when \( b = 0 \). We have
\[
\tilde{C}_0 = \sum_{i=1}^s \tilde{D}_i + \sum_{j=1}^p n_j E_j,
\]

where \( E_1, \ldots, E_p \) are the exceptional divisors of \( f \), and \( n_j > 0 \) for all \( j \). The map \( K \to f^*F \) has isolated zeroes, and hence the restriction of \((\dagger)\) to \( \tilde{D}_i \) and \( E_j \) is exact. Thus, since \( f^*F|_{E_j} \) is trivial,
\[
\frac{1}{r(K)}c_1(K) \cdot E_j \leq \frac{1}{r(F)}c_1(f^*F) \cdot E_j.
\]

By \((\sharp)\) and \((b)\) we conclude that there exists \( 1 \leq i \leq s \) such that
\[
\frac{1}{r(K)}c_1(K) \cdot \tilde{D}_i \geq \frac{1}{r(F)}c_1(f^*F) \cdot \tilde{D}_i.
\]

Since \( 0 < r(K) < r(f^*F) \), we conclude that \( f^*F|_{\tilde{D}_i} \) is not stable. \( \text{q.e.d.} \)

\textbf{(1.20) Proposition.} Let \( C \in |nH| \) be a smooth curve. Let \( X \subset M_\xi \) be a closed subset such that \( F|_C \) is locally-free and stable for all \( \lbrack F \rbrack \in X \). Let \( k \) be a positive integer such that there exist smooth curves in \( |k(nH)| \) (e.g. \( k \gg 0 \)). Then there exists a smooth \( D_k \in |k(nH)| \) such that \( F|_{D_k} \) is stable for all \( \lbrack F \rbrack \in X \).

\textbf{Proof.} Since \( C \in |nH| \) it follows from our hypothesis that \( F \) is \( \mu \)-stable for all \( \lbrack F \rbrack \in X \). Thus we can cover \( X \) by subsets \( X_i, \) open in the analytic topology, so that there exists a tautological sheaf on each \( S \times X_i \). For convenience of exposition we will assume that these local tautological sheaves fit together to give a tautological sheaf on \( S \times X \); however, as the reader will readily check, the proof works in general. Now let \( U_k \subset |k(nH)| \) be the open dense subset parametrizing smooth curves. Let
\[
\tilde{Z}_k := \{ ([D], [F]) \in U_k \times X | F|_D \text{ is not stable} \}.
\]

By openness of stability, \( \tilde{Z}_k \) is closed in \( U_k \times X \). Let \( Z_k \) be the image of \( \tilde{Z}_k \) under the projection \( U_k \times X \to U_k \). Since \( X \) is closed in \( M_\xi \), it is proper, and hence \( Z_k \) is closed in \( U_k \). We must show
that $Z_k \neq U_k$. (Of course, if $k = 1$ this is true by hypothesis.) The proof is by contradiction, so we assume $Z_k = U_k$. Let

$$R_1, \ldots, R_k \in U_1 - Z_1$$

be $k$ distinct curves. Set

$$C_0 := R_1 + \cdots + R_k.$$

Since $|knH| \times X \to |knH|$ is proper, and since $Z_k = U_k$, there exists a smooth connected curve $B$, a point $0 \in B$ and a map

$$g = (g_1, g_2): B \to |knH| \times X,$$

with the following properties:

1. $g_1(B^0) \subset U_k$, where $B^0 := (B - 0)$.
2. $g_1^{-1}([C_0]) = 0$.
3. The family $\pi: C \to B$, obtained pulling back by $g_1$ the tautological family of curves parametrized by $|knH|$, satisfies Items (1) and (2) preceding Lemma (1.19).
4. Let $\psi: C \to S \times X$ be given by $(\psi_1, \psi_2)$, where $\psi_1: C \to S$ is the natural map, and $\psi_2 := g_2 \circ \pi$. Let $F := \psi^*E$, where $E$ is a tautological family on $S \times X$. Then the restriction $F|_{\pi^{-1}(b)}$ is non-stable for all $b \neq 0$.

Thus we can apply Lemma (1.19) to the bundle $F$ on $C$. We conclude that there exist $1 \leq i \leq k$ and $[F] \in X$ such that $F|_{R_i}$ is not stable. This is absurd because $[R_i] \in (U_1 - Z_1)$. \textbf{q.e.d.}

Assume that $C$, $X$ are as in the hypotheses of Proposition (1.20). Let $D_k \in |knH|$ be a smooth curve as in the proposition. Let

$$\rho_k: X \to M(D_k; \xi)$$

be the morphism given by restriction. Let $\Theta_k$ be the theta-divisor on $M(D_k; \xi)$.

(1.21) \textbf{Lemma.} Let notation be as above. Then there exists a positive rational $\lambda_k$ such that

$$c_1(\rho_k^*\Theta_k) = \lambda_k c_1(\rho_1^*\Theta_1).$$

\textbf{Proof.} First let’s assume that there exists a tautological sheaf $\mathcal{E}$ on $S \times X$. Let $\mathcal{E}_k$ be its restriction to $D_k \times X$; by our hypotheses it is a vector bundle. Let $M_k$ be a vector bundle on $D_k$ such that

$$\chi(\mathcal{E}|_{D_k \times \{x\}} \otimes M_k) = 0 \quad (\ast)$$

for $x \in X$. (As is easily verified, such an $M_k$ always exists.) Then Grothendieck-Riemann-Roch (see [O] for the computation in the rank-two case) gives

$$c(\pi_X! (\mathcal{E}_k \otimes \pi_C^* M_k)) = \pi_{X,*} \left[ \text{ch} (\mathcal{E}_k \otimes \pi_C^* M_k) TdC \right].$$

Considering the degree-one components of both sides of the above equality, and using $(\ast)$, one gets

$$-c_1(\det \pi_X! (\mathcal{E}_k \otimes \pi_C^* M_k)) = r(M_k) \cdot \pi_{X,*} \left( c_2(\mathcal{E}_k) - \frac{r(\mathcal{E}_k) - 1}{2r(\mathcal{E}_k)} c_1(\mathcal{E}_k)^2 \right).$$

Now assume that the rank of $M_k$ is minimal. Then the left-hand side of the above equality is identified with the first Chern class of $\rho_k^*\Theta_k$ (see [DN]), while the right-hand side equals the slant product

$$r(M_k) \cdot \left( c_2(\mathcal{E}) - \frac{r(\mathcal{E}) - 1}{2r(\mathcal{E})} c_1(\mathcal{E})^2 \right) /[knH].$$
This proves the lemma (with $\lambda_k = k \cdot rk(M_k)$) under the assumption that there is a tautological sheaf on $S \times X$. In general, by Theorem (A.5) in [Mu] there exists a quasi-tautological sheaf $\mathcal{F}$ on $S \times X$, i.e. such that for $[F] \in X$, the restriction $\mathcal{F}|_{S \times \{[F]\}}$ is isomorphic to $F^\oplus \sigma$ for some positive integer $\sigma$. Then one can repeat the proof above with $\mathcal{E}$ replaced by $\mathcal{F}$.

q.e.d.

**Proof of Proposition (1.18).** By Serre’s vanishing Theorem, if $k \gg 0$ then for all $[F_1], [F_2] \in X$ we have

$$H^1(F_1^* \otimes F_2(-knH)) = 0. \quad (*)$$

By Proposition (1.20) there exists a smooth $D_k \in \lceil knH \rceil$ such that $F|_{D_k}$ is stable for all $[F] \in X$. By (†) the restriction map $\rho_k$ is an injection. Since $\Theta_k$ is ample [DN], we conclude that

$$(\rho_k^* \Theta_k)^{\dim X} > 0.$$ 

Proposition (1.18) follows at once from the above inequality and Lemma (1.21).

**Proof of Propositions (1.1)-(1.2).**

The proof is by contradiction, so we assume that $F$ is locally-free for all $[F] \in X$. Let $C \in \lceil nH \rceil$ be a smooth curve. We start by showing that there exists $[E] \in X$ such that $E|_C$ is not stable. This again we prove by reductio ad absurum. Clearly we can assume $X$ is irreducible. If $E|_C$ is stable for all $[E] \in X$, then we have the restriction morphism

$$\rho: X \to \mathcal{M}(C; \xi).$$

Let $g$ be the genus of $C$. By adjunction

$$g - 1 = \frac{1}{2} H^2 n^2 + \frac{1}{2} K \cdot H n, \quad (*)$$

and hence

$$\dim \mathcal{M}(C; \xi) = (r_\xi^2 - 1) (g - 1) = \frac{1}{2} (r_\xi^2 - 1) (H^2 n^2 + K \cdot H n).$$

By Item (1) of Proposition (1.1) we see that $\dim X > \dim \mathcal{M}(C; \xi)$, and thus

$$(\rho^* \Theta)^{\dim X} = 0.$$ 

This contradicts Proposition (1.18). Hence we conclude that there exists $[E] \in X$ such that $E|_C$ is not stable. Let $X_C \subset X$ be the (closed) subset parametrizing sheaves whose restriction to $C$ is not-stable; we have just proved that $X_C \neq \emptyset$. Let $X^\mu_C \subset X_C$ be the subset parametrizing $\mu$-stable sheaves. We claim that $X^\mu_C \neq \emptyset$. Suppose that the “original” $E$ (with $E|_C$ not stable) is not $\mu$-stable. Let $\mathcal{E}$ be the family of sheaves on $S$ parametrized by $Def^0(E)$. By Luna’s étale slice Theorem the map

$$\lambda: Def^0(GrE) \to \mathcal{M}_\xi$$

induced by $\mathcal{E}$ is surjective onto a neighborhood of $[E]$. Hence

$$\dim (\lambda^{-1} X) \geq \dim X. \quad (*)$$

Obviously

$$\lambda^{-1} X_C = \{ x \in \lambda^{-1} X \mid \mathcal{E}_x|_C \text{ is not stable} \}.$$
By Proposition (5.47) we have

\[ \dim (\lambda^{-1}X_C) \geq \dim \lambda^{-1}X - \frac{r_\xi^2}{4} \cdot q. \]

By (\ast), Inequality (\ast) and Item (2) of Proposition (1.1) we conclude that

\[ \dim (\lambda^{-1}X_C) > (2r_\xi - 1)\Delta_\xi + \epsilon(r_\xi, S, H). \]

Hence by Proposition (5.40) there exists \( x \in \lambda^{-1}X_C \) such that \( E_x \) is \( \mu \)-stable. Then \( \lambda(x) \in X^\mu_C \), and hence \( X^\mu_C \neq \emptyset \). Let \([E'] \in X^\mu_C \); since \( E' \) is stable there is a neighborhood (in the analytic topology) of \([E'] \) in \( M_\xi \) parametrizing a tautological family \( F \). Applying again Proposition (5.47), this time to \( F \), we get

\[ \dim X^\mu_C \geq \dim X - \frac{r_\xi^2}{8} (K^2 n^2 + K \cdot H n) - \frac{r_\xi^2}{4}. \]

This implies that, under the hypotheses of (1.1), there exists \([F] \in X^\mu_C \) such that \([F] \notin M_\xi(\alpha_C) \) (by Item (3)) and, that under the hypotheses of (1.2), there exists \([F] \in X^\mu_C \) such that \([F] \notin M^\prime_\xi(\alpha_C) \) and \([F] \notin M_\xi \left( K \cdot H / 2\sqrt{\Delta} \right) \) (by (1.3) and (1.4)). (Here \( \alpha_C \) is as in Proposition (1.6).) Hence in the latter case Corollary (5.8) gives

\[ h^0(F, F \otimes K) \leq h^0(2K). \]

The above properties of \( F \) together with Item (4) of (1.1) or Inequality (1.5) of (1.2) show that the hypotheses of (1.6) are satisfied by \( F \) and \( X \). By Proposition (1.6) we conclude that \( \partial X \neq \emptyset \). This contradicts the assumption \( \partial X = \emptyset \); thus we have proved Propositions (1.1)-(1.2).

2. Closed subsets not intersecting the boundary.

In this section we will prove Theorem A, by applying Propositions (1.1) and (1.2). As a further application of this last proposition we will give explicit conditions ensuring that every irreducible component of \( M_\xi \) has non-empty boundary (Proposition (2.11)), when \( r_\xi = 2 \). This last result will be the key ingredient in the proof of Theorem C. To simplify notation we will set \( \epsilon = \epsilon(r_\xi, S, H), e_K = e_K(r_\xi, S, H), \) and \( \chi = \chi(\mathcal{O}_S). \)

**Proof of Theorem A for** \( r_\xi > 2 \).

Let

\[ \psi_1(\xi, n) := \frac{1}{2}(r_\xi^2 - 1)H^2 n^2 + \frac{1}{2}(r_\xi^2 - 1)K \cdot H n, \]

\[ \psi_2(\xi, n) := \frac{1}{8}r_\xi^2 H^2 n^2 + \frac{1}{8}r_\xi^2 K \cdot H n + \frac{1}{4}r_\xi^2 + (2r_\xi - 1)\Delta_\xi + \epsilon, \]

\[ \psi_3(\xi, n) := (2r_\xi - 1)\Delta_\xi + (2r_\xi^3 - \frac{39}{8} r_\xi^2 + 4r_\xi - 1)H^2 n^2 + \left[ \frac{r_\xi^3}{2} K \cdot H + \frac{r_\xi^3 H^2}{2} \right] n + \frac{r_\xi^4}{5} \frac{(K \cdot H)^2}{H^2} + \frac{r_\xi^7}{2} \frac{H^2 (K \cdot H + r_\xi^2 H^2)^2}{H^2} + r_\xi^2 |\chi| + \frac{r_\xi^3}{8}|K^2|, \]

\[ \psi_4(\xi, n) := 2r_\xi \Delta_\xi - \frac{1}{2}(r_\xi - 1)H^2 n^2 + \frac{1}{2}(r_\xi - 1)K \cdot H n - (r_\xi^2 - 1)\chi + e_K + \frac{r_\xi^2}{4}. \]
If \( i = 1, 4 \) then \( \psi_i(\xi, n) \) equals the right-hand side of the inequality in Item (i) of Proposition (1.1). Clearly \( \psi_2(\xi, n) \) bounds from above the right-hand side of Item (2) of (1.1). Finally Proposition (5.9) and easy estimates show that \( \psi_3(\xi, n) \) is an upper bound for the right-hand side of Item (3) in the same proposition. Thus if

\[
\dim X > \max\{\psi_1(\xi, n), \psi_2(\xi, n), \psi_3(\xi, n), \psi_4(\xi, n)\},
\]

for some integer \( n \geq 1 \), then we conclude that \( \partial X \neq \emptyset \). As is easily checked

\[
\max\{\psi_1(\xi, n), \psi_2(\xi, n)\} \leq \psi_3(\xi, n)
\]

for \( n \geq 1 \). Hence we have

**Lemma.** Let \( X \subset M_\xi \) be a closed irreducible subset such that

\[
\dim X > \max\{\psi_3(\xi, x_0), \psi_4(\xi, x_0 - 1)\}.
\]

Then \( \partial X \neq \emptyset \).

**Proof.** First notice that, since \( \Delta_\xi \geq \Delta_0 \), we have \( x_0 \geq 1 \), and hence \( n_0 \) is a positive integer. If \( \xi \) is fixed, the function \( \psi_3(\xi, n) \) is increasing for positive \( n \). Thus by (2.3) we have

\[
\dim X > \psi_3(\xi, n_0).
\]

Now let’s show that

\[
\dim X > \psi_4(\xi, n_0).
\]

First we will prove that \( \dim X > \psi_4(\xi, x_0 - 1) \) implies that

\[
x_0 - 1 \geq \frac{K \cdot H}{2H^2},
\]

obtained replacing the two sides of (†) by their dominant terms (that is dominant for \( \Delta_\xi \) and \( n \) large). Thus

\[
x_0 = \sqrt{\frac{\rho(r_\xi)}{H^2}} \sqrt{\Delta_\xi}.
\]

Set \( n_0 := \lfloor x_0 \rfloor \). We will prove Theorem A by applying (2.2) with \( n = n_0 \). By the discussion above this choice of \( n \) is almost optimal if \( \Delta_\xi \) is large (and with this choice the computations are relatively simple).

**Lemma.** Let \( X \subset M_\xi \) be a closed irreducible subset such that

\[
\dim X > \max\{\psi_3(\xi, x_0), \psi_4(\xi, x_0 - 1)\}.
\]

Then \( \partial X \neq \emptyset \).

**Proof.** First notice that, since \( \Delta_\xi \geq \Delta_0 \), we have \( x_0 \geq 1 \), and hence \( n_0 \) is a positive integer. If \( \xi \) is fixed, the function \( \psi_3(\xi, n) \) is increasing for positive \( n \). Thus by (2.3) we have

\[
\dim X > \psi_3(\xi, n_0).
\]

First we will prove that \( \dim X > \psi_4(\xi, x_0 - 1) \) implies that

\[
x_0 - 1 \geq \frac{K \cdot H}{2H^2},
\]
or, in other words, if the hypotheses of Theorem A are satisfied by some $X \subset \mathcal{M}_\xi$, then

$$\Delta_\xi \geq \rho^{-1}H^2 \left( 1 + \frac{K \cdot H}{2H^2} \right)^2.$$ 

For this observe that, if $\xi$ is fixed, then the unique critical point of the concave-down quadratic polynomial $\psi_4(\xi, n)$ is given by $(K \cdot H)/2H^2$. Hence if $(\ast)$ does not hold then, since $0 \leq (x_0 - 1)$, we have

$$\psi_4(\xi, x_0 - 1) \geq \psi_4 \left( \xi, \frac{K \cdot H}{H^2} \right) > 2r_\xi \Delta_\xi - (r_\xi^2 - 1)\chi + e_K.$$ 

By Inequality (0.2) we conclude that all points of $X$ parametrize non-stable sheaves. On the other hand, by $(\ast)$ and (2.1) we have $\dim X > \psi_2(\xi, n_0)$. As is easily checked this implies that $X$ satisfies the hypotheses of Corollary (5.45); thus by this same corollary we get a contradiction. We conclude that $(\ast)$ holds. Now $(\dagger)$ follows at once from (2.3), $(\ast)$ and the fact that $\psi_4(\xi, n)$ is decreasing for $n \geq (K \cdot H)/2H^2$. \textbf{q.e.d.}

Now we can finish the proof of Theorem A for $r_\xi > 2$. A straightforward computation gives

$$\psi_3(\xi, x_0) = \lambda_2 \Delta_\xi + \sqrt{\rho} \left[ \frac{r_\xi^3 |K \cdot H|}{2 \sqrt{H^2}} + r_\xi^5 \sqrt{H^2} \right] \sqrt{\Delta_\xi}$$

$$+ \frac{r_\xi^4 (K \cdot H)^2}{5H^2} + \frac{r_\xi^7}{2} H^2 + \frac{r_\xi^2 (K \cdot H + r_\xi^2 H^2)^2}{2H^2} + r_\xi^5 |\chi| + \frac{r_\xi^3}{8} |K^2|,$$

$$\psi_4(\xi, x_0 - 1) = \lambda_2 \Delta_\xi + \sqrt{\rho} \left[ \frac{r_\xi - 1}{2} \frac{K \cdot H}{\sqrt{H^2}} + (r_\xi - 1)\sqrt{H^2} \right] \sqrt{\Delta_\xi}$$

$$- \frac{r_\xi - 1}{2} H^2 - \frac{r_\xi - 1}{2} K \cdot H - (r_\xi^2 - 1)\chi + e_K + \frac{r_\xi^2}{4}.$$ 

As is easily checked, if $\dim X$ satisfies (0.5) then it is greater than both these quantities. By the previous lemma we conclude that $\partial X \neq \emptyset$.

**Proof of Theorem A when $r_\xi = 2$.**

The proof will be similar to the one given above, with the difference that instead of Proposition (1.1) we will be using (1.2). We set $P_2 := h^0(2K)$.

(2.4) **Lemma.** Assume that $H$ is effective. Let $\xi$ be a set of sheaf data, with $r_\xi = 2$. Let $C \in |nH|$ be a smooth curve. Then

$$\dim \mathcal{M}_\xi^C (n\sqrt{H^2}) \leq 3\Delta_\xi + 2H^2 n^2 + (K \cdot H + 2H^2 + 2)n$$

$$+ \frac{3(K \cdot H + H^2 + 1)^2}{2H^2} - \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 3 - 3\chi - q_S.$$ 

**Proof.** By Propositions (5.10) and (5.11) we have

$$\dim \mathcal{M}_\xi^C \left( n\sqrt{H^2} \right) \leq \max \left\{ 3\alpha_0^2 + \left( (K \cdot H + 2H^2 + 2)(H^2)^{-1/2} - n\sqrt{H^2} \right) \alpha_0 \right\}_{0 \leq \alpha_0 \leq \alpha}$$

$$+ \frac{3(K \cdot H + H^2 + 1)^2}{2H^2} + \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 3 - 3\chi(\mathcal{O}_S) - q_S.$$ 

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The above expression is a concave-up function of $\alpha_0$, and hence its maximum is achieved at one of its end-points. A computation shows that its maximum is achieved for $\alpha_0 = \alpha$, and equals the right-hand side of the inequality in the lemma. \[ q.e.d. \]

Set
\[
\phi_1(\xi, n) := \frac{3}{2}H^2n^2 + \frac{3}{2}K \cdot Hn,
\]
\[
\phi_2(\xi, n) := \frac{1}{2}H^2n^2 + \frac{1}{2}K \cdot Hn + 3\Delta_\xi + \epsilon + \frac{3}{2},
\]
\[
\phi_3(\xi, n) := 3\Delta_\xi + \frac{5}{2}H^2n^2 + \left(\frac{3}{2}K \cdot H + 2H^2 + 2\right)n
\]
\[
+ \frac{3(K \cdot H + H^2 + 1)^2}{2H^2} + \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 4 - 3\chi - q_S,
\]
\[
\phi_4(\xi, n) := 3\Delta_\xi + \frac{1}{2}H^2n^2 + \frac{1}{2}K \cdot Hn + 3\Delta_\xi + \epsilon + \frac{3}{2},
\]
\[
\phi_5(\xi, n) := 4\Delta_\xi - \frac{1}{2}H^2n^2 + \frac{1}{2}K \cdot Hn + P_2 - 3\chi + 1,
\]

where
\[
\tau(S, H) := \frac{5(K \cdot H)^2}{4H^2} + K \cdot H + \frac{K \cdot H}{H^2} + \frac{3(K \cdot H + H^2 + 1)^2}{2H^2}
\]
\[
- \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 3 - 3\chi - q_S \quad \text{if } K \cdot H \geq 0
\]
\[
\tau(S, H) := 0 \quad \text{if } K \cdot H < 0.
\]

For $i = 1, 2, 5$ the value of $\phi_i(\xi, n)$ equals the right-hand side of the inequality in Items (1)-(2) of Proposition (1.1) (with $r_\xi = 2$), and of Inequality (1.5), respectively. By Lemma (2.4) the right-hand side of (1.3) is bounded above by $\phi_3(\xi, n)$, and by Proposition (5.10) $\phi_4(\xi, n)$ is an upper bound for the right-hand side of (1.4). Thus by Proposition (1.2) it suffices to show that, for some integer $n \geq 1$, we have
\[
\dim X > \max \{\phi_1(\xi, n), \phi_2(\xi, n), \phi_3(\xi, n), \phi_4(\xi, n), \phi_5(\xi, n)\}.
\]

As is easily checked,
\[
\max \{\phi_1(\xi, n), \phi_2(\xi, n)\} \leq \phi_3(\xi, n) \quad \text{for } n \geq 0,
\]

and thus we have
\[
\text{(2.5). Keep notation as above. If}
\]
\[
\dim X > \max \{\phi_3(\xi, n), \phi_4(\xi, n), \phi_5(\xi, n)\} \quad (2.6)
\]

for some integer $n \geq 1$, then $\partial X \neq \emptyset$.

Proceeding as in the previous case, we let $w_0$ be the positive root of the equation in $n$
\[
\frac{5}{2}H^2n^2 + 3\Delta_\xi = -\frac{1}{2}H^2n^2 + 4\Delta_\xi,
\]

obtained by equating the two dominant terms of $\phi_3(\xi, n)$ and $\phi_5(\xi, n)$. Explicitely
\[
w_0 = \frac{\sqrt{\Delta_\xi}}{\sqrt{3H^2}}.
\]
**Lemma.** Keep notation as above. If

$$
\dim X > \max \{ \phi_3(\xi, w_0), \phi_4(\xi, w_0), \phi_5(\xi, w_0 - 1) \},
$$

(2.8)

then \( \partial X \neq \emptyset \).

**Proof.** Let \( n_0 := [w_0] \). Since \( \Delta_\xi > \Delta_0(2, S, H) \), we have \( n_0 \geq 1 \). We claim that (2.6) holds with \( n = n_0 \); by (2.5) this will imply the lemma. If \( \xi \) is fixed, then

1. \( \phi_3(\xi, n) \) is increasing for \( n \geq 1 \),
2. \( \phi_4(\xi, n) \) is increasing for \( n \geq -(K \cdot H)/2H^2 \), and
3. \( \phi_5(\xi, n) \) is decreasing for \( n \geq (K \cdot H)/2H^2 \).

Since \( n_0 \geq 1 \), \( \phi_3(\xi, n) \) is increasing for \( n \geq n_0 \), and since \( \Delta_\xi > \Delta_0 \), \( \phi_5(\xi, n) \) is decreasing for \( n \geq (w_0 - 1) \). Hence (2.8) implies that

$$
\dim X > \max \{ \phi_3(\xi, n_0), \phi_5(\xi, n_0) \}.
$$

If \( K \cdot H \geq 0 \) then, by Item (2) above, Inequality (2.8) also implies \( \dim X > \phi_4(\xi, n_0) \), so we are done. If \( K \cdot H < 0 \), then as is easily checked \( \phi_4(\xi, n) \leq \phi_3(\xi, n) \) for all \( n \geq 1 \) (use (5.5) to take care of the contribution of \( \chi \) in \( \phi_3 \)). Hence also in this case \( \dim X > \phi_4(\xi, n_0) \). **q.e.d.**

Now one finishes the proof of Theorem A in the rank-two case by checking that if (0.5) holds, then the hypotheses of the above lemma are satisfied.

**Another application of Proposition (1.2).**

Our goal in this subsection is to determine an effective \( \Delta_1 \) with the property that, if \( \Delta_\xi > \Delta_1 \), then every closed subset of \( M_\xi \) whose dimension is at least the expected dimension of \( M_\xi \) has non-empty boundary. One such lower bound can be obtained by applying Theorem A. However, while Theorem A provides the best “asymptotic” result of its kind obtainable from Proposition (1.1), it is not sharp for \( \Delta_\xi \) small; hence we proceed differently. We will limit ourselves to rank-two sheaves. Let \( z_0 \) be the positive root of the equation in \( n \)

$$
4\Delta_\xi - 3\chi = \phi_5(\xi, n),
$$

(2.9)

i.e.

$$
z_0 = \frac{1}{2H^2} \left[ K \cdot H + \sqrt{(K \cdot H)^2 + 8H^2(P_2 + 1)} \right].
$$

(2.10)

**Proposition.** Assume that \( H \) is effective. Keeping notation as above, let

$$
\Delta_2(S, H) := (4K \cdot H + 7H^2 + 2)z_0 + 6H^2 + \frac{9}{2}K \cdot H + 3\frac{K \cdot H}{H^2} + \frac{7(K \cdot H)^2}{4H^2} + \frac{3}{2H^2} + 14 + 5P_2(S) - \frac{K^2}{4} - q_S,
$$

(2.12)

\( n_0 := \) the least positive integer such that \( n_0 > z_0 \).

(2.13)

Let \( \xi \) be a set of sheaf data, with \( r_\xi = 2 \). Assume that \( \Delta_\xi > \Delta_2(S, H) \) and that the linear system \( |n_0H| \) contains a smooth curve. Then the following hold:

1. If \( Y \) is an irreducible component of \( M_\xi \), then the generic point of \( Y \) parametrizes a \( \mu \)-stable sheaf, and hence

$$
\dim Y \geq 4\Delta_\xi - 3\chi(O_S).
$$
2. If $X \subset \mathcal{M}_\xi$ is a closed irreducible subset such that

$$\dim X \geq 4\Delta_\xi - 3\chi(O_S),$$  \hspace{1em} (2.14)

then $\partial X \neq \emptyset$.

**Proof.** We start by proving Item (2). We will show that

$$4\Delta_\xi - 3\chi > \max \{\phi_3(\xi, n_0), \phi_4(\xi, n_0), \phi_5(\xi, n_0)\}. \hspace{1em} (*)$$

Item (2) will then follow from (2.14) and (2.5). Since $z_0$ is the positive root of (2.9), and since $\phi_5(\xi, n)$ is a concave-down function of $n$, we have

$$4\Delta_\xi - 3\chi > \phi_5(\xi, n_0),$$

for any $\Delta_\xi$ (i.e. even if $\Delta_\xi \leq \Delta_2$). Now let’s first examine the case $K \cdot H \geq 0$. In this case both $\phi_3(\xi, n)$ and $\phi_4(\xi, n)$ are increasing functions for $n \geq 1$ (see the proof of Lemma (2.7)). Thus it suffices to show that

$$4\Delta_\xi - 3\chi > \phi_3(\xi, z_0 + 1), \hspace{1em} (\dagger)$$

$$4\Delta_\xi - 3\chi > \phi_4(\xi, z_0 + 1).$$

Computing, one gets that the two above inequalities are equivalent to

$$\Delta_\xi > \Delta_2(S, H) \hspace{1em} (\ast)$$

$$\Delta_\xi > (K \cdot H + H^2)z_0 + 2H^2 + \frac{9}{2}K \cdot H + 4\frac{K \cdot H}{H^2} + 3\frac{(K \cdot H)^2}{H^2} + \frac{3}{2H^2} + 8 + P_2 - \frac{K^2}{4} - qS,$$

respectively. As is easily checked using (2.10), the first inequality implies the second. This proves $(\ast)$ if $K \cdot H \geq 0$. Now let’s assume $K \cdot H < 0$. In this case $\phi_3(\xi, n)$ is again increasing for $n \geq 1$ and $\phi_4(\xi, n_0) \leq \phi_3(\xi, n_0)$ (see the proof of Lemma (2.7)). Hence it suffices to show that $(\dagger)$ holds. Since this inequality is equivalent to $(\ast)$ we are done. Now let’s prove Item (1). As is easily checked one has

$$\phi_3(\xi, n_0) > 3\Delta_\xi + \epsilon,$$

and hence

$$4\Delta_\xi - 3\chi > 3\Delta_\xi + \epsilon.$$  

Item (1) follows from the above inequality and Corollary (5.46).

We wish to rewrite $\Delta_2(S, H)$ when

$$c_1(K) = kc_1(H)$$

for some rational positive $k$. Let

$$N_2(S, H) := \frac{17}{2} + 6\sqrt{1 + \frac{8(\chi + 1)}{9K^2}} + \left[8 + \frac{4}{H^2} + \left(\frac{21}{2} + \frac{3}{H^2}\right)\sqrt{1 + \frac{8(\chi + 1)}{9K^2}}\right]k^{-1} + \left(6 + \frac{14}{H^2} + \frac{3}{2(H^2)^2}\right)k^{-2}. \hspace{1em} (2.15)$$

A straightforward computation gives
(2.16) Let \((S, H)\) be a polarized surface. Assume that \(c_1(K) = kc_1(H)\) for a rational positive \(k\). (In particular \(K\) is ample.) Then
\[
\Delta_2(S, H) := N_2(S, H)K^2 + 5\chi - q_S.
\]

**Comments.**

Theorem A naturally raises the question: what is the maximum dimension of closed subsets of \(\mathcal{M}_\xi\) which do not intersect \(\partial \mathcal{M}_\xi\)? The following proposition gives a lower bound for this maximum which can be contrasted with the upper bound provided by Theorem A.

**Proposition.** Let \(S\) be a surface such that \(H^{1,1}_Z(S) = \mathbb{Z}c_1(C)\), where \(C\) is a curve of (arithmetic) genus \(g\). Fix an integer \(r \geq 2\) and, for \(d \in \mathbb{Z}\), let
\[
\xi(d) = (r, -[C], d).
\]
If \(d\) is sufficiently large there exists a closed subvariety \(X_d \subset \mathcal{M}_{\xi(d)}\) such that
\[
\partial X_d = \emptyset, \\
\dim X_d \geq (r - 2)d - (r - 2)(r + g - 1) - \dim \text{Aut}(C).
\]

**Proof.** Let \(L\) be a non-special line bundle on \(C\). Set \(d = \deg(L)\), \(n := h^0(L) - 1\). We assume that:

1. The complete linear system \(|L|\) defines an embedding
\[
C \hookrightarrow \mathbb{P}^n = \mathbb{P}(H^0(L)^*)
\]
2. \(n + 1 \geq r\) and \(d > C \cdot C\).

Given an \(r\)-dimensional subspace \(V \subset H^0(L)\) with no base-points, let \(F_V\) be the sheaf on \(S\) fitting into the exact sequence
\[
0 \to F_V \to V \otimes \mathcal{O}_S \xrightarrow{e_V} \iota_* \mathcal{O}_C(L) \to 0,
\]
where \(e_V\) is the evaluation map, and \(\iota: C \hookrightarrow S\) is the inclusion. Clearly \(F_V\) is locally-free. The Chern classes of \(F_V\) are given by
\[
c_1(F_V) = -[C], \\
c_2(F) = d.
\]

As is easily checked, it follows from \(H^{1,1}_Z(S) = \mathbb{Z}c_1(C)\) that \(F_V\) is \(\mu\)-stable. One can identify the set of base-point free \(r\)-dimensional linear subspaces of \(H^0(L)\) with the set \(U_C\) of \((n - r)\)-dimensional linear subspaces of \(\mathbb{P}^n\) not intersecting \(C\) (embedded by \(|L|\)). Let \(\mathbb{P}^{n-2} \subset \mathbb{P}^n\) be disjoint from \(C\), and set
\[
B_d := \text{Gr}(n - r, \mathbb{P}^{n-2})
\]
Then \(B_d\) is a projective subset of \(U_C\), and
\[
\dim B_d = (r - 2)(d - g - r + 1).
\]
Clearly one can construct a family \(\mathcal{F}\) of vector bundles on \(S\) parametrized by \(B_d\), with the property that if \([V] \in B_d\), then
\[
\mathcal{F}|_{S \times [V]} \cong F_V.
\]
Since the Chern classes of $F_V$ are given by (†), and since $F_V$ is stable for all $V$, the family $\mathcal{F}$ defines a morphism

$$\varphi: B_d \to M_{\xi(d)}.$$ 

Let $X_d := \varphi(B_d)$. Clearly $X_d$ is closed and $\partial X_d = \emptyset$. Now let’s check that (2.17) holds. Dualizing (*) one gets

$$0 \to V^* \otimes O_S \to F_V^* \to O_C(C) \otimes L^* \to 0.$$ 

By Item (2) above we have $V^* \cong H^0(F_V^*)$, and hence the isomorphism class of $F_V$ determines $V$ up to isomorphism. Formula (2.17) follows at once from this and Equation (*). \hfill \text{q.e.d.}

### 3. Moduli of bundles with twisted endomorphisms.

In this section we will prove Theorems B, C, and their corollaries. First some notation. If $X \subset M_{\xi}$, we let $X \supset X$ be the closure of $X$ in $M_{\xi}$, and $X^\mu \subset X$ be the subset parametrizing $\mu$-stable sheaves.

To simplify notation we will set

$$\partial X = \partial (X) \quad \partial X^\mu := (\partial X)^\mu \quad \overline{\partial X}^\mu := (\overline{\partial X})^\mu.$$ 

**The double-dual construction.**

The basic idea in the proof of Theorems B and C is to consider the double-duals of sheaves parametrized by $\overline{\partial W}_L^{\xi}$, or by a non-good irreducible component of $M_{\xi}$ respectively. In this subsection we will establish some results concerning this “double-dual construction”. Assume that $X \subset M_{\xi}$ and that $\partial X^\mu \neq \emptyset$. Let $[F] \in \partial X^\mu$; we have the canonical exact sequence

$$0 \to F \to F^{**} \xrightarrow{\psi_F} Q_F \to 0,$$  

(3.1)

where $Q_F$ is an Artinian sheaf of finite length

$$\ell(Q_F) = h^0(Q_F) > 0.$$ 

Since $F$ is $\mu$-stable, so is $F^{**}$, and hence it determines a point $[F^{**}] \in M_{\xi'}$, where

$$\xi' = (r_\xi, \det_\xi, c_2(\xi) - \ell(Q_F)).$$  

(3.2)

The sheaves $F^{**}$, for $[F]$ varying in $\partial X^\mu$, do not fit together to give a family of sheaves (their second Chern classes might vary). However there is a (maximal) stratification of $\partial X^\mu$ by locally closed subsets with the property that the double duals of sheaves parametrized by points of the same stratum fit together to give (locally) a family of vector bundles. We call this the double-dual stratification of $\partial X^\mu$. We will be interested in the open strata. First we give a lower bound for their dimension.

**Proposition.** Let $\mathcal{F}$ be a family of rank-$r$ torsion-free sheaves on $S$ parametrized by an equidimensional variety $B$. Let $\partial B \subset B$ be the subset parametrizing singular sheaves. Then $\partial B$ is closed, and

$$\text{cod} (\partial B, B) \leq r - 1.$$ 

**Proof.** Since $\mathcal{F}$ is a family of rank-$r$ torsion-free sheaves on a smooth surface it has a short locally-free resolution

$$0 \to \mathcal{E}_1 \xrightarrow{\phi} \mathcal{E}_0 \to \mathcal{F} \to 0.$$ 

Let $D(\phi) \subset S \times B$ be the degeneracy locus of $\phi$ (i.e. the locus where $\phi$ drops rank). Clearly $D(\phi)$ is closed, and since $r(\mathcal{E}_0) - r(\mathcal{E}_1) = r$ we have

$$\text{cod} (D(\phi), S \times B) \leq r + 1.$$ 

The result follows because $\partial B = \pi_B(D(\phi))$. \hfill \text{q.e.d.}
(3.4) Corollary. Let \( X \subset \mathcal{M}_\xi \) be a locally closed equidimensional subset. Assume that \( \partial X^\mu \neq \emptyset \). Let \( Y \subset \partial X^\mu \) be an irreducible component of an open stratum of the double-dual stratification. Then
\[
\dim Y \geq \dim X - (r_\xi - 1).
\]
Let \( Y \) be as in the above corollary, and set
\[
Y^{**} := \{ [F^{**}] \mid [F] \in Y \}.
\]
Thus \( Y^{**} \subset \mathcal{M}_{\xi'} \), where \( \xi' \) is given by (3.2). We will relate the dimensions of \( X \) and \( Y^{**} \). For this we need to consider certain Quot-schemes. If \( E \) is a vector bundle on \( S \), and \( \ell \) is a positive integer, let \( \text{Quot}(E; \ell) \) be the parameter space for quotients of \( E \) of finite length equal to \( \ell \). Let \( \text{Quot}_0(E; \ell) \) be the (open) subset parametrizing quotients
\[
E \to \bigoplus_{i=1}^\ell C_{P_i},
\]
where \( C_{P_i} \) is the skyscraper sheaf at \( P_i \). The following result is due to Li \([Li]\) (the proof there is given for \( r(E) = 2 \), but in fact it carries over to any rank, see \([GL2]\)):

(3.5) Theorem (Li). Let notation be as above. Then \( \text{Quot}_0(E; \ell) \) is dense in \( \text{Quot}(E; \ell) \). In particular \( \dim \text{Quot}(E; \ell) = (r(E) + 1)\ell \).

(3.6) Corollary. Let \( X \subset \mathcal{M}_\xi \), and assume that \( \partial X^\mu \neq \emptyset \). Let \( Y \subset \partial X^\mu \) be an irreducible component of an open stratum of the double-dual stratification. Let \( \ell := \ell(Q_F) \), where \([F] \in Y \). Then
\[
\dim Y^{**} \geq \dim X - (r_\xi + 1)\ell - r_\xi + 1. \tag{3.7}
\]
If (3.7) is an equality, the following holds: Let \([E] \in Y^{**} \), and let \( \phi \in \text{Quot}_0(E; \ell) \) be a generic quotient. Then the sheaf \( F_\phi \) fitting into the exact sequence
\[
0 \to F_\phi \to E \xrightarrow{\phi} \bigoplus_{i=1}^\ell C_{P_i} \to 0
\]
is parametrized by a point of \( Y \).

Proof. Let \( \text{Quot}(Y^{**}; \ell) \to Y^{**} \) be the relative \( \text{Quot} \)-scheme, with fiber \( \text{Quot}(E; \ell) \) over \([E] \in Y^{**} \), and let \( \text{Quot}_0(Y^{**}; \ell) \) be the open subset with fiber \( \text{Quot}_0(E; \ell) \) over \([E] \). We have an injection
\[
f : Y \hookrightarrow \text{Quot}(Y^{**}; \ell),
\]
mapping \([F] \) to the canonical quotient \( \psi_F \) (see (3.1)). By Theorem (3.5) we conclude that
\[
\dim Y \leq \dim Y^{**} + (r_\xi + 1)\ell. \tag{*}
\]
Inequality (3.7) follows from the above inequality and Corollary (3.4). Now suppose that (3.7) is an equality then we must have equality also in (*). By Theorem (3.5) we conclude that \( f(Y) \cap \text{Quot}_0(Y^{**}; \ell) \) is dense in \( \text{Quot}_0(Y^{**}; \ell) \); this proves the second statement of the corollary. \textbf{q.e.d.}

Now let \( L \) be a line bundle on \( S \). If \( X \subset \mathcal{M}_\xi \) is an irreducible locally closed subset with \( X^\mu \neq \emptyset \), we set
\[
h_L(X) := \min \{ h^0(F, F \otimes L) \mid [F] \in X^\mu \}.
\]
By semicontinuity of cohomology dimension, if \([F] \in X^\mu \) is a generic point then \( h^0(F, F \otimes L) = h_L(X) \). The following proposition contains the observation that will allow us to deduce Theorems B and C from Theorem A.
**Proposition (3.8).** Let notation be as above. Let $X \subset \mathcal{M}_\xi$ be a locally closed irreducible subset such that $\partial X^\mu \neq \emptyset$. Let $Y \subset \partial X^\mu$ be an irreducible component of an open stratum of the double-dual stratification, and let $\ell := \ell(Q_F)$ for $[F] \in Y$. Then:

1. $h_L(Y^{**}) \geq h_L(X)$.
2. $\dim Y^{**} \geq \dim X - (2\mathbf{r}_\xi - 1)\ell - 1$.
3. If $h_L(X) > 0$ then one at least of the inequalities in Items (1)–(2) is strict.

**Proof.** Let $[E] \in Y^{**}$ be such that $h^0(E, E \otimes L)^0 = h_L(Y^{**})$. Let $[F] \in Y$ be a sheaf such that $F^{**} \cong E$. There exists a natural injection

$$\delta: H^0(F, F \otimes L)^0 \hookrightarrow H^0(E, E \otimes L)^0.$$  

(1)

Item (1) follows at once from this. Item (2) follows from Corollary (3.6). Now let’s prove Item (3). So assume that we have equality in Item (2): we will prove that the inequality of Item (1) is strict. Let $[E] \in Y^{**}$ be as above. Let $F_\phi$ be as in the statement of Corollary (3.6) where $\phi$ is an arbitrary quotient. Letting $\delta_\phi$ be as in (1), with $F$ replaced by $F_\phi$, we have

$$\delta_\phi (H^0(F_\phi, F_\phi \otimes L)) = \{ f \in H^0(E, E \otimes L)^0 \mid f^* \phi_i = \lambda_i \phi_i \},$$

where $\phi_i$ is the restriction of $\phi$ to $P_i$, and $\lambda_i \in \mathbb{C}$. Now suppose that $f \neq 0$ and that $\phi$ is generic. Then, since $f$ is not a dilation, it is not contained in the image of $\delta_\phi$. Hence

$$h^0(F_\phi, F_\phi \otimes L)^0 < h_L(Y^{**}) \text{ for } \phi \text{ generic.}$$

(1)

On the other hand, if Item (2) is an equality then so must be Inequality (3.7). Hence by Corollary (3.6) $[F_\phi] \in Y$ for $\phi$ generic. By (1) we conclude that the inequality in Item (1) is strict. q.e.d.

**Proof of Theorem B.**

The proof will be by contradiction. So we assume that $\Delta_\xi > \Delta_1$, and that Inequality (0.6) is violated, for some line bundle $L$. Let $X_0 \subset W_\xi^L$ be an irreducible component of maximum dimension. By hypothesis

$$\dim X_0 > \lambda_2 \Delta_\xi + \lambda_1 \sqrt{\Delta_\xi} + \lambda_0 + \epsilon_L.$$  

(3.9)

By the above inequality and Theorem A we have $\partial \overline{X}_0 \neq \emptyset$. The following lemma will show that in fact $\partial \overline{X}_0 \neq \emptyset$.

**Lemma (3.10).** Let $X \subset \mathcal{M}_\xi$ be an equidimensional locally closed subset such that:

1. $\partial X \neq \emptyset$, and
2. $\dim X > (2\mathbf{r}_\xi - 1)\Delta_\xi + \epsilon(\mathbf{r}_\xi, S, H) + r_\xi - 1$.

Then $\partial X^\mu \neq \emptyset$.

**Proof.** Let $[F] \in \partial X$. If $F$ is $\mu$-stable there is nothing to prove, so assume $F$ is not $\mu$-stable. Let $\mathcal{F}$ be the family of sheaves on $S$ parametrized by $Def^0(Gr F)$. By Luna’s étale slice Theorem, the map

$$\lambda: Def^0(Gr F) \rightarrow \mathcal{M}_\xi$$

induced by $\mathcal{F}$ is surjective onto a neighborhood of $[F]$, and hence

$$\dim (\lambda^{-1} X) \geq \dim X.$$  

(*)
Let $\partial (\lambda^{-1}X) \subset \lambda^{-1}X$ be the subset parametrizing singular sheaves. By Proposition (3.3) and by Item (2) we conclude that
\[
\dim \partial (\lambda^{-1}X) > (2r_{\xi} - 1)\Delta_{\xi} + \epsilon(r_{\xi}, S, H).
\]
By (5.40) the subset of $\partial (\lambda^{-1}X)$ parametrizing $\mu$-stable sheaves is non-empty. Since this subset is open, and since its image is contained in $\partial X^\mu$, we see that $\partial X^\mu \neq \emptyset$.

An easy computation shows that (3.9) together with $\Delta_{\xi} > \Delta_1$ give
\[
\dim X_0 > (2r_{\xi} - 1)\Delta_{\xi} + \epsilon(r_{\xi}, S, H) + r_{\xi} - 1.
\]

Hence by Lemma (3.10) we have $\partial X^\mu_0 \neq \emptyset$. Let $Y_0 \subset \partial X^\mu_0$ be an irreducible component of an open stratum of the double-dual stratification. Let
\[
\xi_1 := (r_{\xi}, \det_{\xi}, c_2(\xi) - \ell_0),
\]
where $\ell_0 := \ell(Q_{F_0})$ for $[F_0] \in Y_0$. Set $X_1 := Y_0^{**}$. Thus
\[
X_1 \subset M_{\xi_1}.
\]

Now consider $\overline{X}_1$ and, if $\partial \overline{X}_1^{\mu} \neq \emptyset$ continue in the same fashion. We will get a sequence
\[
X_i \subset M_{\xi_i}, \quad Y_i \subset \partial \overline{X}_i^{\mu}, \quad X_{i+1} = Y_i^{**}, \quad (3.11)
\]
for $i = 0, \ldots, n$ (with $\xi_0 := \xi$), until we reach a point when $\partial \overline{X}_n^{\mu} = \emptyset$. We will show that the dimension of $X_n$ is “too big”, and thus get a contradiction. Let $\ell_i := \ell(Q_{F_i})$, where $[F_i] \in X_i$, and set
\[
\ell := \ell_0 + \cdots + \ell_{n-1}.
\]

**Lemma.** Keeping notation as above, we have
\[
\dim X_n \geq \dim X_0 - (2r_{\xi} - 1)\ell - e_L.
\]

**Proof.** By Item (1) of (3.8) we have
\[
h_L(X_0) \leq h_L(X_1) \leq \cdots \leq h_L(X_n).
\]

In particular, since $h_L(X_0) > 0$, we have $h_L(X_i) > 0$ for all $i$. Since by (5.7) $h_L(X_i) \leq e_L$ there can be at most $(e_L - 1)$ strict inequalities in ($\ast$). Thus by Proposition (3.8) again we have
\[
\dim X_{i+1} \geq \dim X_i - (2r_{\xi} - 1)\ell_i - \delta_i,
\]
where $\delta_i = 0$ or $\delta_i = 1$, and $\delta_i$ is equal to 1 for at most $(e_L - 1)$ values of $i$. The result follows at once from this.

The above Lemma together with (3.9) gives
\[
\dim X_n > \lambda_2\Delta_{\xi} + \lambda_1\sqrt{\Delta_{\xi}} + \lambda_0 - (2r_{\xi} - 1)\ell
\]
\[
= (2r_{\xi} - 1)\Delta_{\xi} + [\lambda_2 - (2r_{\xi} - 1)]\Delta_{\xi} + \lambda_1\sqrt{\Delta_{\xi}} + \lambda_0
\]
\[
> \max \left\{ \lambda_2\Delta_{\xi} + \lambda_1\sqrt{\Delta_{\xi}} + \lambda_0, (2r_{\xi} - 1)\Delta_{\xi} + \epsilon + r_{\xi} - 1 \right\}.
\]
Lemma. Keeping notation as above, we have
\[ \Delta_{\xi_n} \leq \Delta_0. \] (3.14)

Proof. The proof is by contradiction, so we assume that (3.14) is violated. Then Inequality (3.13) and Theorem A give that \( \partial X_n \neq \emptyset \). By (3.13) and Lemma (3.10) we conclude that \( \partial X_n \neq \emptyset \). This contradicts the definition of \( X_n \).

q.e.d.

Now we can finish the proof of Theorem B. We will show that
\[ \dim X_n > 2r_\xi \Delta_\xi - (r_\xi^2 - 1)\chi + e_K. \] (\*)

From this one concludes as follows: Since \( X_n \subset M_{\xi_n} \) the above inequality together with (0.3) implies that all sheaves parametrized by \( X_n \) are non-stable. But this is absurd by Inequality (3.13) and by (5.45). Now let’s prove (\*). By (3.12) it suffices to show that
\[ [\lambda_2 - (2r_\xi - 1)] \Delta_\xi + \lambda_1 \sqrt{\Delta_\xi} + \lambda_0' > \Delta_{\xi_n} - (r_\xi^2 - 1)\chi + e_K. \]

By Inequality (3.14) it suffices to check that
\[ [\lambda_2 - (2r_\xi - 1)] \Delta_\xi + \lambda_1 \sqrt{\Delta_\xi} + \lambda_0' > \Delta_0 - (r_\xi^2 - 1)\chi + e_K. \]

This follows at once from \( \Delta_\xi > \Delta_1 \).

q.e.d.

Proof of Corollary B’.

Let \( \tilde{\Delta}_1(r_\xi, S, H) \) be the smallest number such that:
\begin{itemize}
  \item \( \tilde{\Delta}_1 \geq \Delta_1 \), and
  \item if \( \Delta_\xi > \Delta_1 \) then
\end{itemize}
\[ 2r_\xi \Delta_\xi - (r_\xi^2 - 1)\chi(O_S) > (2r_\xi - 1)\Delta_\xi + \epsilon(r_\xi, S, H) + r_\xi, \] (3.15)
\[ 2r_\xi \Delta_\xi - (r_\xi^2 - 1)\chi(O_S) > \lambda_2 \Delta_\xi + \lambda_1 \Delta_\xi + \lambda_0' + e_K. \] (3.16)

Set
\[ \Delta'_1(r_\xi, S, H) := \tilde{\Delta}_1(r_\xi, S, H) + (r_\xi - 1)^{-1}e_K(r_\xi, S, H). \]

As is easily checked \( \Delta'_1 \) depends only on \( r_\xi, K^2, K \cdot H, H^2 \) and \( \chi(O_S) \).

(3.17) Claim. Let \((S, H)\) be a polarized surface, with \( H \) satisfying (0.4). Let \( \xi \) be a set of sheaf data such that \( \Delta_\xi > \tilde{\Delta}_1(r_\xi, S, H) \). Then \( M_{\xi} \) is good, and the generic point of any of its irreducible components parametrizes a \( \mu \)-stable sheaf.

Proof. Let \( X \) be an irreducible component of \( M_{\xi} \). Inequality (3.15) and Corollary (5.46) give that the generic point of \( X \) parametrizes a \( \mu \)-stable sheaf. Since \( \Delta_\xi > \Delta_1 \), we conclude by Theorem B and Inequality (3.16) that
\[ \dim W^K_\xi < 2r_\xi \Delta_\xi - (r_\xi^2 - 1)\chi(O_S). \]

Thus by Inequality (0.2) we have
\[ \dim X > W^K_\xi, \]
i.e. \( h^0(F, F \otimes K)^0 = 0 \) for a (stable) sheaf \( F \) parametrized by the generic point of \( X \). Thus \( M_{\xi} \) is good.

q.e.d.
Now assume that $\Delta_\xi > \Delta'_1$. Let $X$ be an irreducible component of $\mathcal{M}_\xi$. By (3.17) we have $X^\mu \neq \emptyset$. We will show that if $[F] \in X^\mu$ is generic then $F$ is locally-free. Assume the contrary, i.e. $X^\mu \subset \partial \mathcal{M}_\xi$. Let $Y \subset X^\mu$ be an irreducible component of an open stratum of the double-dual stratification. Then

$$Y^{**} \subset \mathcal{M}_{\xi'} \quad c_2(\xi) - c_2(\xi') > 0.$$ 

It follows from Theorem (3.5) that

$$\dim X^\mu \leq \dim Y^{**} + (r_\xi + 1) (c_2(\xi) - c_2(\xi')). \quad (*)$$

We distinguish between two cases:

1. If $\Delta_{\xi'} > \Delta_1$, then $\dim Y^{**} \leq 2r_\xi \Delta_{\xi'} - (r_\xi^2 - 1)\chi(\mathcal{O}_S)$ by Claim (3.17).
2. If $\Delta_{\xi'} \leq \Delta_1$, then $\dim Y^{**} \leq 2r_\xi \Delta_{\xi'} - (r_\xi^2 - 1)\chi(\mathcal{O}_S) + e_K$ by (0.3).

In both cases, Inequality (*) gives

$$\dim X^\mu < 2r_\xi \Delta_\xi - (r_\xi^2 - 1)\chi(\mathcal{O}_S).$$

This is absurd because $\mathcal{M}_\xi$ is good. Hence the generic point of $X^\mu$ parametrizes a locally-free sheaf. This finishes the proof of Corollary $C'$.

**Proof of Theorem C.**

The essential step is provided by the following

(3.18) **Proposition.** Let $(S,H)$ be a polarized surface satisfying the hypotheses of Theorem C. Let $\xi$ be a set of sheaf data with $r_\xi = 2$. Assume that

$$\Delta_\xi > \Delta_2(S,H) + 2h^0(2K).$$

If $X_0 \subset \mathcal{M}_\xi$ is an irreducible component then there exists $[F] \in \partial X_0^\mu$ such that

$$h^0(F^{**}, F^{**} \otimes K)^0 = 0. \quad (3.19)$$

**Proof.** By Proposition (2.11) we have $\partial X_0 \neq \emptyset$. As is easily checked we have

$$4\Delta_2 - 3\chi > 3\Delta_2 + \epsilon + 1. \quad (3.20)$$

Thus by Lemma (3.10) we conclude that $\partial X_0^\mu \neq \emptyset$. Let $Y_0 \subset \partial X_0^\mu$ be an irreducible component of an open stratum of the double-dual stratification, and set $X_1 := Y_0^{**}$. Assume that

$$h_K(X_1) > 0. \quad (3.21)$$

We will arrive at a contradiction, and thus we will conclude that $h_K(X_1) = 0$; this will prove the proposition. Proceeding as in the proof of Theorem B, we construct a series of locally closed irreducible subsets as in (3.11); the only difference is that in the present case we define $n$ by requiring that:

1. $\Delta_{\xi_{n-1}} > \Delta_2$, and
2. either $\partial \overline{X_n^\mu} = \emptyset$, or $\Delta_{\xi_n} \leq \Delta_2$. 

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Claim. Keeping notation as above, we have

$$\Delta_{\xi_n} \leq \Delta_2,$$

$$h_K(X_i) \leq h^0(2K) \text{ for } 0 \leq i \leq n - 1.$$  \hspace{1cm} (3.22)

Proof of the claim. Since $\Delta_{\xi} > \Delta_2$ we have $X_0 \geq 4\Delta_{\xi} - 3\chi$ (Proposition (2.11)). Thus, by Item (2) of Proposition (3.8) we have

$$\dim X_i \geq 4\Delta_{\xi_i} - 3\chi \text{ for all } 0 \leq i \leq n.$$  \hspace{1cm} (3.23)

Now assume that $\Delta_{\xi_n} > \Delta_2$. Then by the above inequality and by Proposition (2.11) we have $\partial^X_n \neq \emptyset$. By Inequality (3.20) and Lemma (3.10) we conclude that $\partial^X_n \neq \emptyset$. This contradicts the definition of $n$, and thus we conclude that (3.22) holds. In order to prove (3.23) we need the following easily checked inequality:

$$4\Delta_2 - 3\chi > 3\Delta_2 + 3\frac{(K \cdot H)^2}{4H^2} + \frac{(K \cdot H)(K \cdot H + 2H^2 + 2)}{2H^2} \frac{2H^2}{2H^2} + \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 3 - 3\chi(O_S) - q_S.$$  \hspace{1cm} (3.25)

The right-hand side of (3.25) equals the right-hand side of the inequality in Proposition (5.10) with $\alpha := (K \cdot H)/2\sqrt{H^2}$, and $\Delta_{\xi}$ replaced by $\Delta_2$. The above inequality, together with (3.24) and Item (1) preceding the claim, gives

$$\dim X_i > \dim M_{\xi_i} \left( \frac{K \cdot H}{2\sqrt{H^2}} \right) \text{ for } 0 \leq i \leq n - 1.$$  \hspace{1cm} q.e.d.

By Corollary (5.8) we conclude that (3.23) holds.

Now we will use the full strength of Proposition (3.8) to give a lower bound for $\dim X_n$ which is larger than (3.24). By (3.21) and by Item (1) of (3.8) we have

$$0 < h_K(X_1) \leq h_K(X_{n-1}) \leq h_K(X_n).$$

By (3.23) there are at most $h^0(2K)$ strict inequalities. Thus applying Proposition (3.8) repeatedly to $X_0$, $X_1$, ... one gets

$$\dim X_n \geq \dim X_0 - 3(c_2(\xi) - c_2(\xi_n)) - h^0(2K) \geq 3\Delta_{\xi_n} + \Delta_{\xi} - 3\chi - h^0(2K).$$

Since $\Delta_{\xi} > \Delta_2 + 2h^0(2K)$ we conclude that

$$\dim X_n > 3\Delta_{\xi_n} - 3\chi + \Delta_2 + h^0(2K).$$  \hspace{1cm} (*)

By construction the generic point of $X_n$ parametrizes a $\mu$-stable vector bundle. Hence:

1. If $e_K(X_n) \leq h^0(2K)$ then by (0.3) we have $\dim X_n \leq 4\Delta_{\xi_n} - 3\chi + h^0(2K)$, and

2. if $e_K(X_n) > h^0(2K)$, then by Corollary (5.8) $\dim X_n \leq \dim M_{\xi_n} \left( \frac{K \cdot H}{2\sqrt{H^2}} \right)$.

By (3.22) and by (*) we see that Item (1) is impossible. On the other hand also Item (2) is impossible, by (5.10), (*), (3.22) and (3.25). Thus we have a contradiction. This proves that (3.21) cannot hold, and hence proves the proposition.  \hspace{1cm} q.e.d.
Now we can prove Theorem C. Let \( X_0 \subset \mathcal{M}_\xi \) be an irreducible component. Let \([F] \in X_0\) be as in the statement of Proposition (3.18). By openness of \( \mu \)-stability the generic point of \( X_0 \) parametrizes a \( \mu \)-stable sheaf. Furthermore, since we have an injection

\[
H^0(F,F \otimes K)^0 \hookrightarrow H^0(F^{**},F^{**} \otimes K)^0,
\]

we conclude by (3.19) that \( \mathcal{M}_\xi \) is good at \([F]\). This proves that \( \mathcal{M}_\xi \) is good. The only thing to prove is that the generic point of \( X \) parametrizes a locally-free sheaf. Since \( F \) is stable, a neighborhood of \([F]\) in \( \mathcal{M}_\xi \) is isomorphic to \( \text{Def}^0(F) \), which is smooth because \( \mathcal{M}_\xi \) is good at \([F]\). Thus it suffices to show that there exists \( x \in \text{Def}^0(F) \) parametrizing a locally-free sheaf. This is equivalent to the existence of \( x \in \text{Def}(F) \) parametrizing a locally-free sheaf. As is well-known this follows from (3.19); we will recall the proof. Let

\[
\text{Supp}(F^{**}/F) = \{P_1,\ldots,P_\ell\}.
\]

The versal deformation space of \( F_{P_i} \) (the localization of \( F \) at \( P_i \)) is smooth, and \( F_{P_i} \) deforms to a free \( \mathcal{O} \)-module \([F]\). Hence, since \( \text{Def}(F) \) is smooth, it suffices to show that the map of tangent spaces

\[
T_0\text{Def}(F) \overset{\rho}{\longrightarrow} \bigoplus_{i=1}^\ell \text{Def}(F_{P_i})
\]

induced by a versal sheaf on \( S \times \text{Def}(F) \), is surjective. The map \( \rho \) is part of the local-to-global exact sequence coming from the spectral sequence abutting to \( \text{Ext}^1(F,F) \). The piece of interest to us is

\[
\text{Ext}^1(F,F) \overset{\rho}{\longrightarrow} H^0\left( \bigoplus_{i=1}^\ell \text{Ext}^1(F_{P_i},F_{P_i}) \right) \rightarrow H^2(\text{Hom}(F,F)) . \tag{*}
\]

We have an exact sequence

\[
0 \rightarrow \text{Hom}(F,F) \rightarrow \text{Hom}(F^{**},F^{**}) \rightarrow R \rightarrow 0 ,
\]

where \( R \) is an Artinian sheaf (supported at the \( P_i \)'s). Hence (3.19) implies that the last term of (\( * \)) is zero, thus \( \rho \) is surjective. This completes the proof of Theorem C.

**Surfaces with ample canonical bundle.**

Theorem C together with (D2def) gives the following

**3.26 Proposition.** Let \( S \) be a surface with \( K \) ample. Assume that there exists an effective divisor \( H \) on \( S \) such that \( c_1(K) = kc_1(H) \) for some rational positive \( k \), and such that \( |n_0H| \) contains a smooth curve, where \( n_0 \) is given by (2.13). Let \( N_2(S,H) \) be as in (2.15). Let \( \xi \) be a set of sheaf data with \( r_\xi = 2 \), and let \( \mathcal{M}_\xi \) be the corresponding moduli space of sheaves on \( S \), polarized by \( K \). If

\[
\Delta_\xi > N_2(S,H)K^2 + 2K^2 + 7\chi(\mathcal{O}_S) ,
\]

then \( \mathcal{M}_\xi \) is good.

**Proof of Corollaries C' and C''.** For Corollary C' notice that the hypotheses of Proposition (2.11) are satisfied by \( H = K \). In fact \( K \) is effective by hypothesis. Furthermore an easy computation gives \( z_0 > 2 \), hence \( n_0 \geq 3 \). Since \( K^2 \) is large (\( K^2 \geq 6 \) suffices), \( n_0K \) is very ample by [Bo], in particular there exists a smooth curve in the linear system \( |n_0K| \). The result follows from Proposition (3.26) and the easy estimate

\[
N_2(S,K) < 40 + \frac{22(\chi + 1)}{3K^2} ,
\]

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valid for $K^2 > 100$. Similarly Corollary $C''$ follows from Proposition (3.26) and the estimate

$$N_2(S, H) < 15 + \frac{8(\chi + 1)}{3K^2},$$

valid for $k > 100$.

**Examples of non-good moduli spaces.** We will show that the lower bound given in Corollaries $C'$-$C''$ is, if not sharp, at least of the right form. Assume $H$ is effective. We will consider non-trivial extensions

$$0 \to \mathcal{O}_S \to F \to I_Z(H) \to 0,$$

where $Z$ is a zero-dimensional subscheme of $S$ of length $\ell$. These extensions are parametrized by

$$\text{Ext}^1(I_Z(H), \mathcal{O}_S) \cong H^1(I_Z(K + H))^*.$$

From this one easily gets

(3.29). Keep notation as above. Assume that

$$\ell > h^0(K + H) = \chi(\mathcal{O}_S) + \frac{1}{2}K \cdot H + \frac{1}{2}H^2.$$

Then if $Z$ is generic we have

$$\dim \text{Ext}^1(I_Z(H), \mathcal{O}_S) = \ell - \chi(\mathcal{O}_S) - \frac{1}{2}K \cdot H - \frac{1}{2}H^2,$$

and the generic non-trivial extension (3.28) is locally-free.

The following is also an easy exercise.

(3.31). Keeping notation as above, assume that (3.30) is satisfied and that

$$\ell > \frac{9}{8}H^2 + 2 + q_S.$$

Then if $Z$ is generic the generic non-trivial extension (3.28) is $\mu$-stable, and furthermore its space of global sections is one-dimensional.

Let

$$\xi(\ell) := (2, \mathcal{O}_S(H), \ell).$$

By (3.29)-(3.31) if

$$\ell > \max \left\{ \chi(\mathcal{O}_S) + \frac{1}{2}K \cdot H + \frac{1}{2}H^2, \frac{9}{8}H^2 + 2 + q_S \right\}$$

then there is an irreducible subset $\Sigma_\ell \subset \mathcal{M}_{\xi(\ell)}$ parametrizing extensions (3.28) with $Z$ generic, and we have

$$\dim \Sigma_\ell = 3\ell - \chi(\mathcal{O}_S) - \frac{1}{2}K \cdot H - \frac{1}{2}H^2.$$

Since $\Delta_{\xi(\ell)} = \ell - H^2/4$ we conclude that
Let notation be as above. Assume that \( \ell \) satisfies (3.32) and that
\[
\ell < 2\chi(\mathcal{O}_S) - \frac{1}{2}K \cdot H + \frac{1}{2}H^2.
\]
Then \( \mathcal{M}_{\xi(\ell)} \) is not good. More precisely it contains a subset \( \Sigma_{\ell} \) whose dimension is greater than the expected dimension of \( \mathcal{M}_{\xi(\ell)} \). Furthermore the generic point of \( \Sigma_{\ell} \) parametrizes a \( \mu \)-stable vector bundle.

One can easily check that if \( q_S = 0 \) and if \( c_1(K) = kc_1(H) \) for \( k \gg 0 \), then the hypotheses of (3.33) are satisfied. Hence letting
\[
\ell_0 := 2\chi(\mathcal{O}_S) - \frac{1}{2}K \cdot H + \frac{1}{2}H^2 - 1,
\]
one gets

**Corollary (3.34).** Let notation be as above. Assume that \( c_1(K) = kc_1(H) \) for \( k \gg 0 \). Then

1. The moduli space \( \mathcal{M}_{\xi(\ell_0)} \) is not good, and
2. \( \Delta_{\xi(\ell_0)} = 2\chi(\mathcal{O}_S) - \frac{1}{2}K \cdot H + \frac{1}{4}H^2 - 1. \)

By considering surfaces with arbitrarily large \( k \), we conclude that a bound of the form given in Corollaries C'-C" is the best we can hope for.

### 4. Irreducibility.

We will first derive Theorem D from Theorem A and Corollary B'; the argument is due to Gieseker-Li [GL1]. Then we will obtain Theorem E by making explicit Gieseker-Li’s proof in the case considered by the theorem.

**Proof of Theorem D.**

We will prove the following

(4.1) **Proposition.** Let \((S, H)\) be a polarized surface. For any integer \( r \geq 2 \) and any line bundle \( M \) on \( S \), there exists a number \( \Delta_3(r, M, S, H) \) such that the following holds. Let \( \xi \) be set of sheaf data with \( \det\xi \cong M \), and such that
\[
\Delta_\xi > \Delta_3(r, M, S, H).
\]

Then \( \mathcal{M}_\xi \) is irreducible (and is the closure of the subset parametrizing \( \mu \)-stable vector bundle).

This result implies Theorem D. In fact, since tensorization by a line bundle \( N \) identifies \( \mathcal{M}_\xi \) with \( \mathcal{M}_{\xi \otimes N} \) (with the obvious notation), Theorem D will hold if we set
\[
\Delta_3(r, S, H) := \max_{M \in \mathcal{S}} \{ \Delta_3(r, M, S, H) \},
\]
where \( \mathcal{S} \) is any set of line bundles whose first Chern classes are a set of representatives for the finite group \( H^1_{\mathbb{Z}}(S)/r_\xi H^1_{\mathbb{Z}}(S) \). The following lemma is the key ingredient in the proof of Proposition (4.1).
Lemma. Let \((S, H)\) be a polarized surface. For any integer \(r \geq 2\) there exists a number \(\hat{\Delta}_1(r, S, H)\) (with \(\hat{\Delta}_1 \geq \Delta'_1\), where \(\Delta'_1\) is as in Corollary B') such that the following holds. Let \(\xi, \ell\) be a set of sheaf data and a positive integer respectively, such that

\[\Delta_\xi \geq \hat{\Delta}_1(r_\xi, S, H) + \ell.\]

Set

\[\xi' := (r_\xi, \det_\xi, c^2_2(\xi) - \ell),\]

Let \(X\) be an irreducible component of \(M_\xi\). Then there exist a locally closed non-empty subset \(Y \subset \partial X\) and an open subset \(\tilde{V}\) of an irreducible component of \(M_\xi'\), with the following properties. If \([E] \in \tilde{V}\) then \(E\) is locally-free, \(\mu\)-stable, and

\[h^0(E, E \otimes K)^0 = 0.\]

Furthermore \([F] \in Y\) if and only if \(F\) fits into an exact sequence

\[0 \rightarrow F \rightarrow E \xrightarrow{\phi} \bigoplus_{i=1}^\ell \mathbb{C}P_i \rightarrow 0,\]

for some \([E] \in \tilde{V}\) (here \(\{P_1, \ldots, P_\ell\}\) is a set of distinct points of \(S\)).

Proof. First one proves the lemma in the case \(\ell = 1\), then the general case follows easily from this. The case \(\ell = 1\) follows from Theorem A, Corollary B' and dimension counts (use Proposition (3.3) and Theorem (3.5)). The argument is similar to that used in the proof of Corollary B'; we leave the details to the reader.

Now we are ready to prove Proposition (4.1). Set \(r = r_\xi\). Choose a set of sheaf data

\[\xi_0 = (r, M, c_2(\xi_0)),\]

such that

\[\Delta_{\xi_0} \geq \hat{\Delta}_1(r_\xi, S, H) \text{ with } c^2_2(\xi_0) \text{ minimal.}\]

Notice that, since by [HL,LQ] the moduli space \(M_\xi\) is non-empty for \(\Delta_\xi \gg 0\), Lemma (4.3) shows that also \(M_{\xi_0}\) is non-empty. Let \(U_{\xi_0} \subset M_{\xi_0}\) be the open subset parametrizing \(\mu\)-stable locally-free sheaves \(E\) such that

\[h^0(E, E \otimes K)^0 = 0.\]

Since \(\hat{\Delta}_1 \geq \Delta'_1\), the subset \(U_{\xi_0}\) is dense in \(M_{\xi_0}\), and hence non-empty. There exists an integer \(n\) such that for all \([E] \in U_{\xi_0}\) the bundle \(E(nH)\) has \((r - 1)\) independent sections, and such that the degeneracy locus of the corresponding map

\[O^{(r-1)}_S \rightarrow E(nH)\]

is (at most) zero-dimensional. (For example if \(E(nH)\) is generated by global sections.) Hence if \([E] \in U_{\xi_0}\) then \(E\) fits into an exact sequence

\[0 \rightarrow O_S(-nH)^{(r-1)} \rightarrow E \rightarrow I_Z \otimes M \otimes [(r - 1)nH] \rightarrow 0,\]

where \(I_Z\) is the ideal sheaf of a zero-dimensional subscheme \(Z\). We can, and will, assume that

\[h^1(M \otimes [rnH + K]) = 0.\]
Set
\[ \Delta_3(r, M, S, H) := \Delta_1(r, S, H) + h^0(M \otimes [rnH + K]) + 1. \] (4.8)

We will prove that Proposition (4.1) holds with this value of \( \Delta_3 \). Thus we assume that \( \Delta_\xi \) satisfies Inequality (4.2). Set
\[ \ell := \Delta_\xi - \Delta_1. \] (\#)

Let \( X \) be an irreducible component of \( M_\xi \). By Lemma (4.3) there exists \( [E] \in U_{\xi_0} \) such that \( X \) contains all the isomorphism classes of sheaves \( F \) fitting into (4.4), with \( E \) the chosen vector bundle, and \( \ell \) given by (\#). Since \( E \) fits into Exact Sequence (4.6), if we choose \( \phi \) appropriately we can arrange that \( F := \ker \phi \) fit into the exact sequence
\[ 0 \to O_S(-nH)^{(r-1)} \to F \to I_W \otimes M \otimes [(r - 1)nH] \to 0, \] (*)

with \( W = Z \cup \{P_1, \ldots, P_s\} \), for \( s = (\Delta_\xi - \Delta_{\xi_0}) \). Furthermore, since by hypotheses
\[ \Delta_\xi - \Delta_{\xi_0} > h^0(M(rnH + K)), \]
we can also assume (using (4.7)) that
\[ h^1(I_W \otimes M \otimes [rnH + K]) = \ell(W) - h^0(M \otimes [rnH + K]), \]
and hence by Serre duality
\[ \dim \text{Ext}^1 \left( I_W \otimes M \otimes [(r - 1)nH], O_S(-nH)^{(r-1)} \right) = (r-1) \cdot (\ell(W) - h^0(M \otimes [rnH + K])). \] (\dagger)

Now let \( \Sigma_\ell \) be the space parametrizing non-trivial extensions (*), such that (\dagger) holds. Then \( \Sigma_\ell \) fibres over a non-empty open subset of \( \text{Hilb}^b(S) \), with projective spaces as fibres. In particular \( \Sigma_\ell \) is irreducible. Since \( E \) is \( \mu \)-stable we conclude that the subset \( \Sigma^\mu_\ell \subset \Sigma_\ell \) parametrizing \( \mu \)-stable extensions is non-empty. Let \( \Omega_\ell \subset M_\ell \) be the image of \( \Sigma^\mu_\ell \) under the classifying map; since \( \Sigma_\ell \) is irreducible, so is \( \Omega_\ell \). We have proved that \( X \) contains a point \( [F] \in \Omega_\ell \). Since \( h^0(F^{**}, F^{**} \otimes K)^0 \) vanishes, \( M_\xi \) is smooth at \( [F] \). Hence we conclude that \( X \) contains all of \( \Omega_\ell \). To sum up: every irreducible component of \( M_\xi \) contains the irreducible (non-empty) subset \( \Omega_\ell \), and \( M_\xi \) is smooth at the generic point of \( \Omega_\ell \). This implies that \( M_\xi \) is irreducible.

(4.9) Remark. The above proof works also if we only assume that for the generic \( [E] \) in any irreducible component of \( U_{\xi_0} \), \( E \) fits into (4.6).

Complete intersections with Picard number one.

The goal of this subsection is to prove the following

(4.10) Proposition. Let \( (S, H) \) be as in the statement of Theorem E. Assume also that \( \text{Pic}(S) = \mathbb{Z}[H] \). Then the conclusion of Theorem E holds for \( (S, H) \).

We begin by giving an explicit value for \( \Delta_1 \) for rank-two sheaves.
(4.11) Proposition. Let \((S, H)\) be a polarized surface. Assume that \(H\) is effective and that the linear system \([n_0 H]\) contains a smooth curve, where \(n_0\) is defined by (2.13). Then Lemma (4.3) holds (in rank two) with 
\[
\tilde{\Delta}_1(2, S, H) := \Delta_2(S, H) + 2h^0(2K),
\]
where \(\Delta_2(S, H)\) is defined by (2.12).

Proof. First we prove that (4.3) holds for \(\ell = 1\). Thus we assume that \(\Delta_\xi \geq (\tilde{\Delta}_1 + 1)\); in particular \(S\), \(H\) and \(\xi\) satisfy the hypotheses of Proposition (3.18). Let \(X\) be an irreducible component of \(\mathcal{M}_\xi\). Let \([F_0] \in \partial X^\mu\) be such that Proposition (3.18) holds with \(F = F_0\). Let \(\tilde{Y}_1 \subset \partial X^\mu\) be the irreducible component of the stratum of the double dual stratification which contains \([F_0]\). Let \(Y_1 \subset \tilde{Y}_1\) be the open subset parametrizing sheaves \(F\) such that 
\[
h^0(F^{**}, F^{**} \otimes K)^0 = 0.
\]
The subset \(Y_1\) is non-empty because \([F_0] \in Y_1\). We have \(Y_1^{**} \subset \mathcal{M}_\xi'\), where 
\[
\xi' = (2, \det_\xi, c_2(\xi')), \quad c_2(\xi') < c_2(\xi).
\]
By Proposition (3.3) and Theorem (3.5) 
\[
\dim Y_1^{**} + 3(c_2(\xi) - c_2(\xi')) \geq \dim Y_1 \geq \dim X - 1.
\]
By (3.19) both \(\mathcal{M}_\xi\) and \(\mathcal{M}_\xi'\) are good at \([F_0]\) and \([F_0^{**}]\) respectively. Hence (\*\) gives 
\[
\Delta_{\xi'} \geq \Delta_\xi - 1.
\]
By (\#\) we conclude that \(c_2(\xi') = c_2(\xi) - 1\). Thus (\dag\) is an equality, and hence also all the inequalities in (\*\). The result is that \(\mathcal{M}_{\xi'}\) is good (by Proposition (3.18)) and furthermore 
\[
\dim Y_1^{**} = \dim \mathcal{M}_{\xi'}.
\]
Now set \(V := Y_1^{**}\). By (\*) \(V\) is an open subset of \(\mathcal{M}_{\xi'}\). By construction, if \([F] \in Y_1^{**}\) then \(F\) fits into an exact sequence (4.4) for some \([E] \in V\) (with \(\ell = 1\)). Conversely, since the inequalities of (\*) are equalities, if \([F]\) is the generic sheaf fitting into (4.4) for \([E] \in V\) (with \(\ell = 1\)) then \([F] \in Y_1^{**}\).

We conclude that 
\[
Y := \{[F] \in \mathcal{M}_\xi | \text{\(F\) fits into (4.4) for some \([E] \in V, \text{ with } \ell = 1\)}\}
\]
is contained in the closure of \(Y_1\), and hence \(Y \subset X\). This proves Lemma (4.3) if \(\ell = 1\). When \(\ell > 1\) one iterates this construction. Define \(Y_1 \subset \partial X\) as above. Let \(X_2 := Y_1^{**}\), and define \(X_2 \subset \partial X_2\) in the same way as we defined \(Y_1 \subset \partial X^\mu\). We continue this process up to \(Y_\ell\) (that this is possible is guaranteed by Proposition (3.18)). Set \(V := Y_\ell^{**}\), and let \(Y \subset \mathcal{M}_\xi\) be the parameter space for all sheaves \(F\) fitting into (4.4) for \([E] \in V\). One checks easily that \(Y \subset \partial X\) and hence that Lemma (4.3) holds. q.e.d.

(4.12) Corollary. Let \((S, H)\) be a polarized surface satisfying the hypotheses of Theorem E. Then Lemma (4.3) holds with 
\[
\tilde{\Delta}_1(2, S, H) = 17K^2 + 10\chi(O_S).
\]

Proof. Immediate from Proposition (4.11) together with (2.16) and (3.27). q.e.d.

Set 
\[
\xi_0 := (2, O_S, 17K^2 + 10\chi(O_S)).
\]
Then \(\Delta_{\xi_0}\) satisfies (4.5).
(4.13) **Lemma.** Keep notation as above. Let \((S, H)\) be a polarized surface satisfying the hypotheses of Proposition (4.10). Let \(X \subset M_{\xi_0}\) be an irreducible component. There is an open dense subset \(U \subset X\), parametrizing locally-free sheaves, such that if \([E] \in U\) then \(E\) fits into an exact sequence
\[
0 \to \mathcal{O}_S(-6K) \to E \to I_Z \otimes [6K] \to 0,
\]
where \(Z\) is a zero-dimensional subscheme of \(S\) of length
\[
\ell(Z) = 53K^2 + 10\chi(O_S).
\]

**Proof.** We will prove that \(E\) fits into (4.14); once this is done the length of \(Z\) is obtained by computing \(c_2(E)\). We begin by showing that if \([E] \in M_{\xi_0}\) and \(E\) is locally-free, then \(E \otimes [5K]\) fits into an exact sequence
\[
0 \to \mathcal{O}_S(nH) \to E \otimes [5K] \to I_W \otimes [10K - nH] \to 0,
\]
where \(W\) is a zero-dimensional subscheme of \(S\), and \(n \geq 0\). In fact by Serre duality and \(\mu\)-semistability we have
\[
h^2(E \otimes [5K]) = h^0(E \otimes [-4K]) = 0,
\]
and hence \(h^0(E \otimes [5K]) \geq \chi(E \otimes [5K])\). Applying the Hirzebruch-Riemann-Roch Theorem one gets
\[
\chi(E \otimes [5K]) = 3K^2 - 8\chi(O_S).
\]
The right-hand side is positive for \(k \gg 0\), e.g. if \(k > 100\), and hence \(h^0(E \otimes [5K]) > 0\). (Here \(k\) is as in the statement of Theorem E.) That \(E\) fits into (4.14) follows from this and the hypothesis that \(\text{Pic}(S) = \mathbb{Z}[H]\) (in fact this is the only place where we use this assumption). We need to bound \(n\). Of course by semistability we have \(n \leq 5k\); we will show that if \([E] \in X\) is generic there is a better bound.

**Claim.** Assume that \([E] \in X\) is generic. Then \(n < 4k\).

**Proof of the claim.** Assume that for generic \([E] \in X\) the sheaf \(E \otimes [5K]\) fits into (4.14) with \(n \geq 4k\). Then
\[
\dim M_{\xi_0}(K \cdot H) = 4\Delta_{\xi_0} - 3\chi(O_S).
\]
In fact this follows by writing
\[
\mu(nH) = 5K \cdot H - \alpha\sqrt{H^2} \quad \text{for} \quad \alpha \leq \frac{K \cdot H}{\sqrt{H^2}}.
\]
Applying Proposition (5.10) we get
\[
3\frac{(K \cdot H)^2}{H^2} + \frac{(K \cdot H)^2 + 2(K \cdot H)H^2 + 2K \cdot H}{H^2} + \frac{3((K \cdot H) + H^2 + 1)^2}{2H^2} + 3 \geq 17K^2 + 10\chi(O_S).
\]
As is easily verified this is false as soon as \(k \geq 0\), and hence we conclude that \(n < 4k\) for generic \([E] \in X\) (with \(E\) locally-free).

Now we are ready to show that if \([E] \in X\) is generic and \(E\) is locally-free, then \(E \otimes [6K]\) has a section with isolated zeroes. By the above claim the vector-bundle \(E \otimes [6K]\) fits into an exact sequence
\[
0 \to \mathcal{O}_S(K + nH) \to E \otimes [6K] \to I_W \otimes [11K - nH] \to 0,
\]

\[\text{ }(\ast)\]
with $n < 4k$. First we show that

$$H^0(I_W \otimes [11K - nH]) \neq 0.$$  

For this it suffices to prove that

$$h^0(O_S(11K - nH)) > \ell(W). \quad (\dagger)$$

The left-hand side equals $\chi(O_S(11K - nH))$, hence is computed by Hirzebruch-Riemann-Roch. An easy computation then gives that for $(\dagger)$ to hold we need that

$$(13k^2 - 2k - 2)H^2 > 9\chi(O_S).$$

This inequality is satisfied as soon as $k > 2$. Let

$$\sigma \in H^0(I_W \otimes [11K - nH])$$

be a non-zero section; it lifts to section $\tilde{\sigma}$ of $E \otimes [6K]$ because $h^1(K + nH) = 0$. If $\tau$ is any section of $O_S(K + nH)$, then

$$(\tau + \tilde{\sigma}) \subset (\sigma),$$

where $(\cdot)$ denotes "zero-locus". Since $O_S(K + nH)$ is very ample one easily concludes that there exists $\tau$ such that $\theta := (\tau + \tilde{\sigma})$ is section with isolated zeroes. Thus $E$ fits into

$$0 \rightarrow O_S \xrightarrow{\theta} E \otimes [6K] \rightarrow I_Z \otimes [12K] \rightarrow 0,$$

where $Z = (\theta)$. Tensoring the above exact sequence with $O_S(-6K)$ one obtains (4.14). \textbf{q.e.d.}

\textbf{Proof of Proposition (4.10).} By Remark (4.9), Formula (4.8), Lemma (4.13) and Corollary (4.12), we can set

$$\Delta_3(2, O_S, S, H) = \hat{\Delta}_1 + h^0(13K) + 1,$$

where the value of $\hat{\Delta}_1$ is given by Corollary (4.12). Proposition (4.10) follows at once.

\textbf{Proof of Theorem E.}

Let $S$ be a surface satisfying the hypotheses of Theorem E. Then

$$S = V_1 \cap \cdots \cap V_n \subset \mathbb{P}^{n+2},$$

where $V_i$ is a degree-$d_i$ hypersurface. Let

$$\rho: S \rightarrow B$$

be the family of smooth complete intersections of $n$ hypersurfaces in $\mathbb{P}^{n+2}$ of degrees $d_1, \ldots, d_n$. Thus $B$ is an open subset of a Grassmannian, and $S = \rho^{-1}(b_0)$ for a certain $b_0 \in B$. If $b \in B$ we set $S_b := \rho^{-1}(b)$. We let $B_1 \subset B$ be the subset parametrizing surfaces whose Picard group is generated the hyperplane class. We recall the following
(4.15) Noether-Lefschetz Theorem. Keep notation as above. Assume that \( p_g(S_b) > 0 \) for \( b \in B \). Then \( B_1 \) is dense in \( B \).

For \( c \in \mathbb{Z} \) let \( \mathcal{M}_c(S_b) \) be the moduli space of torsion-free sheaves \( F \) on \( S_b \), semistable with respect to \( \mathcal{O}_{S_b}(1) \), with

\[
r(F) = 2, \quad \det F \cong \mathcal{O}_{S_b}, \quad c_2(F) = c.
\]

By Maruyama [Ma] there exists a relative moduli space

\[
\pi: \mathcal{M}_c(S) \to B
\]

proper over \( B \), such that \( \pi^{-1}(b) \cong \mathcal{M}_c(S_b) \) for all \( b \in B \).

(4.16) Proposition. Keep notation as above. Assume that the integer \( k \) such that \( K_{S_b} \sim kH \) is large (e.g. \( k > 100 \)). If

\[
c > 95K_{S_b}^2 + 11\chi(\mathcal{O}_{S_b}) + 1
\]

then \( \mathcal{M}_c(S) \) is irreducible.

Proof. For \( b \in B \) let \( \mathcal{M}_c^0(S_b) \subset \mathcal{M}_c(S_b) \) be the (open) subset parametrizing sheaves \( F \) such that

1. \( F \) is locally-free and stable,
2. \( h^0(F, F \otimes K_{S_b}) = 0 \).

Let \( \mathcal{M}_c^0(S) := \bigcup_{b \in B} \mathcal{M}_c^0(S_b) \). By Corollary C" \( \mathcal{M}_c^0(S_b) \) is dense in \( \mathcal{M}_c(S_b) \) for all \( b \), and hence \( \mathcal{M}_c^0(S) \) is dense in \( \mathcal{M}_c(S) \). Thus it suffices to show that \( \mathcal{M}_c^0(S) \) is irreducible. Let \( X \) be anyone of its irreducible components. Let \( X^* \subset X \) be the complement of the intersection with all other irreducible components. We claim that \( \pi(X^*) \) contains an open non-empty subset of \( B \). Since \( \pi(X^*) \) is constructible it suffices to prove that it is not contained in any proper subvariety of \( B \). Let \( b \in \pi(X^*) \), and let \( v \in T_b(B) \). By deformation theory (see for example Proposition 2.1 in [G2]) there exist a curve \( \iota: \Lambda \to X^* \) and a point \( P \in \Lambda \) such that \( \iota(P) = [F] \) and

\[
v \in \text{Im}D(\pi \circ \iota)(P),
\]

where \( D \) is the differential. This proves that \( \pi(X^*) \) is not contained in any proper subvariety of \( B \), and hence it contains an open non-empty subset. Now let \( Y \) be another irreducible component of \( \mathcal{M}_c^0(S) \). Then

\[
X^* \cap Y^* = \emptyset.
\]

(*)

Since \( \pi(X^*) \) and \( \pi(Y^*) \) both contain a Zariski-open non-empty subset of the irreducible variety \( B \), we conclude by (4.15) that there exists

\[
b_0 \in \pi(X^*) \cap \pi(Y^*)
\]

such that \( \text{Pic}(S_{b_0}) \cong \mathbb{Z} \mathcal{O}_{S_{b_0}} \). By Proposition (4.10) the moduli space \( \mathcal{M}_c(S_{b_0}) \) is irreducible. Since both \( X^* \) and \( Y^* \) must contain an open non-empty subset of \( \mathcal{M}_c(S_{b_0}) \), we conclude that

\[
X^* \cap Y^* \neq \emptyset.
\]

This contradicts (*), and hence \( \mathcal{M}_c^0(S) \) is irreducible.

q.e.d.
**Corollary.** Keep assumptions as in Proposition (4.16). Then \( M_c(S_b) \) is connected for all \( b \in B \).

**Proof.** Let \( b \in B_1 \). By Proposition (4.10), \( \pi^{-1}(b) = M_c(S_b) \) is irreducible, hence connected. Since \( B_1 \) is dense in \( B \) (by (4.15)), and since \( M_c(S) \) is irreducible (Proposition (4.16)) and proper over \( B \), we conclude that \( M_c(S_b) \) is connected for all \( b \in B \). \( \text{q.e.d.} \)

Now we are ready to prove Theorem E. Let \( S \) and \( H \) be as in the statement of the theorem, and let \( c \) be an integer such that
\[
c > 95K^2 + \chi(O_S) + 1. \quad (cgr)
\]
We must prove that \( M_c(S) \) is irreducible.

**Claim.** Keep notation and assumptions as above. Suppose also that \( M_c(S) \) is reducible. Then there exist two irreducible components \( X_1, X_2 \) such that \( X_1 \cap X_2 \) contains a point parametrizing a stable sheaf.

**Proof.** By Corollary (4.17) there exist two irreducible components \( X_1, X_2 \) such that their intersection is non-empty. Let \([F] \in X_1 \cap X_2\). If \( F \) is stable there is nothing to prove, so assume \( F \) is non-stable. Let \( E := Gr(F) \). By Luna’s étale slice Theorem the natural map
\[
\lambda: Def(E) \to M_c(S)
\]
surjects onto a neighborhood of \([F]\). (Since \( S \) is regular, \( Def^0(E) = Def(E) \).) Hence \( \lambda^{-1}X_i \) is a closed non-empty subset of \( Def(E) \), and
\[
dim \lambda^{-1}X_i \geq \dim X_i = 4c - 3\chi(O_S). \quad (*)
\]
On the other hand the dimension of the tangent space to \( Def(E) \) at the origin is given by
\[
dim T_0(Def^0(E)) = h^1(E, E) = -\chi(E, E) + h^0(E, E) + h^0(E, E \otimes K).
\]
Here \( \chi(E, E) \) and the other terms are defined as in (5.1). By Lemma (5.41) and by (5.7) we conclude that
\[
dim T_0 Def(E) \leq 4c - 3\chi(O_S) + 3 + \frac{2}{H^2}(K \cdot H + 5H^2 + 1)^2.
\]
Thus, by Inequality (*), we have
\[
dim (\lambda^{-1}X_1 \cap \lambda^{-1}X_2) \geq 4c - 3\chi(O_S) - 3 - \frac{2}{H^2}(K \cdot H + 5H^2 + 1)^2.
\]
As is easily checked
\[
4c - 3\chi(O_S) - 3 - \frac{2}{H^2}(K \cdot H + 5H^2 + 1)^2 > 3c + \epsilon(2, S, H),
\]
if \( k > 0 \). Hence by Proposition (5.40) there exists a point in \( x \in \lambda^{-1}X_1 \cap \lambda^{-1}X_2 \) parametrizing a \( \mu \)-stable sheaf. Then \( \lambda(x) \in X_1 \cap X_2 \) parametrizes a stable sheaf. \( \text{q.e.d.} \)

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So let’s assume that $\mathcal{M}_c(S)$ is reducible. By the above claim there exist two irreducible components $X, Y$, of $\mathcal{M}_c(S)$ and a point $[F]$ in their intersection such that $F$ is stable. By (0.3) we conclude that
\[
dim_{[F]} X \cap Y \geq 4c - 3\chi(\mathcal{O}_S) - h^0(F, F \otimes K)^0.
\]
Applying (5.7) we get
\[
dim_{[F]} X \cap Y \geq 4c - 3\chi(\mathcal{O}_S) - \frac{2}{H^2}(K \cdot H + 5H^2 + 1)^2. \tag{\dagger}
\]
Since $X \cap Y$ is in the singular locus of $\mathcal{M}_c(S)$ and since $F$ is stable, we have
\[
dim_{[F]} X \cap Y \leq W^K_\xi,
\]
where $\xi = (2, \mathcal{O}_S, c)$. By Theorem B and by (\dagger) we get
\[
4c - 3\chi(\mathcal{O}_S) - \frac{2}{H^2}(K \cdot H + 5H^2 + 1)^2 \leq \lambda_2c + \lambda_1 \sqrt{c} + \lambda'_0 + e_K.
\]
A straightforward computation shows that this is impossible if $c$ satisfies (cgr) and $k$ is large, for example $k > 100$. This proves that $\mathcal{M}_c(S)$ is irreducible.

5. Estimates.

In this section we prove various technical results which have been used in the course of the paper. We let $H$ be an ample divisor on the surface $S$. Unless otherwise stated stability and $\mu$-stability of sheaves on $S$ is with respect to $H$. If $F, G$ are sheaves on the same scheme we set
\[
h^i(F, G) := \dim Ext^i(F, G), \quad \chi(F, G) := \sum (-1)^i h^i(F, G). \tag{5.1}
\]
A bound for the number of sections of a semistable sheaf on $S$.

The bound provided by the following proposition is due to Simpson and Le Potier [S,Le].

(5.2) Proposition (Simpson-Le Potier). Keeping notation as above, assume that $H$ satisfies (0.4). Let $G$ be a $\mu$-semistable torsion-free sheaf on $S$. Then
\[
h^0(G) \leq \frac{r_G}{2H^2}(\mu_G + (r_G + 1)H^2 + 1)^2.
\]

The following corollary is what we use.

(5.3) Corollary. Assume that $H$ satisfies (0.4). Let $F, G$ be $\mu$-semistable torsion-free sheaves on $S$. Then
\[
h^0(F, G) \leq \frac{r_F \cdot r_G}{2H^2}(\mu_G - \mu_F + (r_F \cdot r_G + 1)H^2 + 1)^2.
\]

Proof. Since $F$ and $G$ are both torsion-free $\mu$-semistable sheaves on $S$, so is $Hom(F, G)$. The corollary follows by applying Proposition (5.2) to $Hom(F, G)$.

In the rank-one case there is a slightly sharper version of Proposition (5.2). If $A$ is an effective divisor on $S$, let
\[
A_f := \text{fixed part of } |A|, \quad A_m := A - A_f = \text{moving part of } A.
\]
(5.4) Lemma. Let notation be as above. Assume that $H$ is effective. Let $C \in |H|$. If $A$ is a divisor on $S$, then
\[ h^0(\mathcal{O}_C(A_m)) \leq [C \cdot A + 1]_+ , \]
where $[x]_+ := \max\{x, 0\}$.

Proof. We might as well assume that $A_m$ is effective non-zero. Since $|A_m|$ has no fixed part, there exists $D \in |A_m|$ such that the scheme-theoretic intersection of $C$ and $D$, call it $Z$, is zero-dimensional. Then the exact sequence
\[ 0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(D) \rightarrow \mathcal{O}_Z \rightarrow 0 \]
gives
\[ h^0(\mathcal{O}_C(A_m)) \leq h^0(\mathcal{O}_C) + C \cdot D \leq h^0(\mathcal{O}_C) + C \cdot A. \]
As is easily checked $C$ is 1-connected, hence $h^0(\mathcal{O}_C) = 1$ (see [BPV]). Hence the result follows from the above inequality. \textbf{q.e.d.}

(5.5) Proposition. Assume that $H$ is effective. Let $A$ be a rank-one torsion-free sheaf on $S$. Then
\[ h^0(A) \leq \frac{1}{2H^2} \left( \mu_A + H^2 + 1 \right)^2 . \]

Proof. Let $C \in |H|$. By considering the exact sequence
\[ 0 \rightarrow \mathcal{O}_S(A - (i + 1)H) \rightarrow \mathcal{O}_S(A - iH) \rightarrow \mathcal{O}_C(A - iH) \rightarrow 0 \]
for all non-negative integers $i$, one easily concludes that
\[ h^0(\mathcal{O}_S(A)) \leq \sum_{i=0}^{N} h^0(\mathcal{O}_C((A - iH)_m)) , \]
where $N$ is the maximum value of $i$ such that
\[ h^0(\mathcal{O}_S(A - iH)) > 0. \]
Thus by Lemma (5.4)
\[ h^0(\mathcal{O}_S(A)) \leq \sum_{i=0}^{N} [\mu_A - iH^2 + 1]_+ . \]
An easy estimate of the right-hand side gives the proposition. \textbf{q.e.d.}

Twisted endomorphisms.

If $L$ is a line bundle on $S$, and $r$ is a positive integer we set
\[ e_L(r, S, H) := \begin{cases} \frac{r^2}{2H^2} (\mu_L + (r^2 + 1)H^2 + 1)^2 - h^0(L), & \text{if } L \cdot H \geq 0, \\ 0, & \text{if } L \cdot H < 0. \end{cases} \quad (5.6) \]
By applying Corollary (5.3) with $G = F \otimes L$ one gets
Assume that $H$ satisfies (0.4). Let $F$ be a $\mu$-semistable rank-$r$ torsion-free sheaf on $S$. Then

$$\dim \text{Hom}(F, F \otimes L)^0 \leq e_L(r, S, H).$$

The following is a useful observation:

**Proposition.** Let $L$ be a line bundle on $S$. Let $F$ be a rank-two torsion-free sheaf on $S$ such that

$$h^0(F, F \otimes L)^0 > h^0(L^2).$$

Then $F$ is not $(L \cdot H/2\sqrt{H^2})$-stable, i.e. there exists a rank-one subsheaf $A \subset F$ such that

$$\mu_A \geq \mu_F - \frac{L \cdot H}{2}.$$

**Proof.** Replacing $F$ by $F^{**}$ we can assume that $F$ is locally-free. Let

$$\phi: H^0(F, F \otimes L)^0 \to H^0(L^2)$$

be the map defined by $\phi(\sigma) := \det \sigma$. Since $0 \in \phi^{-1}(0)$, $\phi^{-1}(0)$ is non-empty, and hence by our hypothesis $\dim \phi^{-1}(0) > 0$. Thus there exists a non-zero map

$$f: F \to F \otimes L$$

with $\det f = \text{tr} f = 0$, i.e. $f \circ f = 0$. The kernel of $f$ is a rank-one torsion-free subsheaf of $F$, which we denote by $A$. We have

$$0 \to A \to F \xrightarrow{g} B \to 0.$$

The sheaf $B$ is also torsion-free, because it is isomorphic to $\text{Im} f \subset F \otimes L$. Since $f \circ f = 0$, the map $f$ is obtained as the composition

$$F \xrightarrow{g} B \xrightarrow{g} A \otimes L,$$

for some non-zero map $g$. Thus

$$\mu_B - \mu_A \leq \mu_L.$$

This implies the proposition. \(\text{q.e.d.}\)

**Corollary.** Let $\xi$ be a set of sheaf data for $S$, with $r_\xi = 2$. Let $\Sigma_\xi \subset \mathcal{M}_\xi$ be the subset parametrizing sheaves $F$ such that

$$h^0(F, F \otimes K)^0 > h^0(2K).$$

Then $\Sigma_\xi \subset \mathcal{M}_\xi \left((K \cdot H)/2\sqrt{H^2}\right)$.

**A bound for the dimension of $\mathcal{M}_\xi(\alpha)$.**

Let

$$s(r) := ((r - 1)^2 + 1) H^2 + 1,$$

$$M(r, S, H) := \begin{cases} \frac{s^2}{s^2} |K^2 - \frac{1}{2} r(r + 1) \chi(O_S)| & \text{if } \chi(O_S) \geq 0 \text{ and } K^2 \geq 0, \\ \frac{s^2}{s^2} |\chi(O_S)| + \frac{s^2}{s^2} |K^2| & \text{otherwise.} \end{cases}$$

The goal of this subsection is to prove the following propositions.
(5.9) Proposition. Assume that $H$ satisfies (0.4). Let $\xi$ be a set of sheaf data, and $\alpha \in \mathbb{R}$. Then $\mathcal{M}_\xi(\alpha)$ is a constructible subset of $\mathcal{M}_\xi$. If $K \cdot H \geq 0$, then

$$
\dim \mathcal{M}_\xi(\alpha) \leq (2r_{\xi} - 1) \Delta_{\xi} + (2r_{\xi} - 1)\alpha^2 + (r_{\xi}^2 - 2r_{\xi} + 2) \frac{K \cdot H + 2s(r_{\xi})}{2\sqrt{H^2}} \alpha
$$

$$
+ (r_{\xi} - 1)(r_{\xi}^3 - 4r_{\xi}^2 + 6r_{\xi} - 2) \frac{(K \cdot H)^2}{8H^2} + [(r_{\xi} - 1)^3 + r_{\xi}] \frac{s(r_{\xi})^2}{2H^2}
$$

$$
+ (r_{\xi}^2 - r_{\xi} + 1) \frac{(K \cdot H + s(r_{\xi}))^2}{2H^2} - q_{\xi} + M(r_{\xi}, S, H).
$$

If $K \cdot H < 0$,

$$
\dim \mathcal{M}_\xi(\alpha) \leq (2r_{\xi} - 1) \Delta_{\xi} + (2r_{\xi} - 1)\alpha^2 + \left[ (r_{\xi}^2 - 2r_{\xi} + 2) \frac{s(r_{\xi})}{\sqrt{H^2}} - (r_{\xi}^2 - 2r_{\xi} - 2) \frac{K \cdot H}{2\sqrt{H^2}} \right] \alpha
$$

$$
+ \frac{r_{\xi}^3}{16} (17r_{\xi} + 4) \frac{(K \cdot H)^2}{8H^2} + [(r_{\xi} - 1)^3 + r_{\xi}] \frac{s(r_{\xi})^2}{2H^2}
$$

$$
+ (r_{\xi}^2 - r_{\xi} + 1) \frac{(K \cdot H + s(r_{\xi}))^2}{2H^2} + \frac{r_{\xi}^2}{2} (1 - 2\chi(O_S)) + \frac{r_{\xi}^3}{8} |K^2| - q_{\xi}.
$$

The next proposition provides a better bound for the case of rank two.

(5.10) Proposition. Assume that $H$ is effective. Let $\xi$ be a set of sheaf data with $r_{\xi} = 2$, and let $\alpha \in \mathbb{R}$. Then

$$
\dim \mathcal{M}_\xi(\alpha) \leq 3\Delta_{\xi} + 3\alpha^2 + \frac{K \cdot H + 2H^2 + 2}{\sqrt{H^2}} \alpha
$$

$$
+ \frac{3(K \cdot H + H^2 + 1)^2}{2H^2} + \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 3 - 3\chi(O_S) - q_{\xi}.
$$

The proposition below gives a bound for the dimension of the subset $\mathcal{M}_\xi^C(\alpha) \subset \mathcal{M}_\xi(\alpha)$ defined in (0.7).

(5.11) Proposition. Assume $H$ is effective. Let $C \in |nH|$ be a smooth curve, and let $\xi$ be a set of sheaf data with $r_{\xi} = 2$. Let $\varphi(\xi, \alpha)$ be the right-hand side of the inequality in the previous proposition. Then $\mathcal{M}_\xi^C(\alpha)$ is a constructible subset of $\mathcal{M}_\xi$, and

$$
\dim \mathcal{M}_\xi^C(\alpha) \leq \max \{ \varphi(\xi, \alpha_0) - n\alpha_0 \sqrt{H^2} \}_{0 \leq \alpha_0 \leq \alpha}.
$$

First we will show that the above propositions follow from a bound for the dimension of certain extension groups (Propositions (5.15) and (5.16)). Then we will obtain these bounds.

Filtrations. We will prove Propositions (5.9)-(5.10) by bounding the number of moduli of certain filtrations. Let $F$ be a torsion-free sheaf on $S$. Let

$$
0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n+1} = F
$$

(5.12)

be a filtration with $n \geq 1$. We set

$$
Q_i := F_i/F_{i-1}, \ r_i := r(Q_i), \ \mu_i := \mu(Q_i), \ \Delta_i := \Delta(Q_i).
$$

(5.13)

We also set $A := F_1$, $B := F/A$. We will make the following assumption
(5.14).

1. $Q_i$ is torsion-free and $\mu$-semistable, and
2. $\mu_2 > \mu_3 > \cdots > \mu_{n+1}$.

The significance of the following two propositions is that they give an upper bound on the number of moduli of such filtrations.

(5.15) Proposition. Keep notation as above. Assume that $H$ satisfies (0.4). Suppose that $F$ is $\mu$-semistable and that (5.14) holds. Define $\alpha$ by setting

$$\mu_A = \frac{\alpha}{r_A} \sqrt{H^2}.$$ 

If $K \cdot H \geq 0$, then

$$\sum_{i \leq j} h^1(Q_j, Q_i) \leq (2r_F - 1)\Delta_F + \frac{1}{2}(1 + \frac{1}{r_A})(2r_F - r_A)\alpha^2 + [r_B(r_B - 1) + r_F] \frac{K \cdot H + 2s(r_F)}{2\sqrt{H^2}} \alpha$$

$$+ r_B(r_B^3 - r_B^2 - r_F r_B + r_F^2) \frac{(K \cdot H)^2}{8H^2} + (r_B^3 + r_A r_F) \frac{s(r_F)^2}{2H^2}$$

$$+ (r_F^2 - r_A r_B) \frac{(K \cdot H + s(r_F))^2}{2H^2} + M(r_F, S, H).$$

If $K \cdot H < 0$, then

$$\sum_{i \leq j} h^1(Q_j, Q_i) \leq (2r_F - 1)\Delta_F + \frac{1}{2}(1 + \frac{1}{r_A})(2r_F - r_A)\alpha^2$$

$$+ \left\{ [r_F + r_B(r_B - 1)] \frac{K \cdot H + 2s(r_F)}{2\sqrt{H^2}} - r_F(r_F - 1) \frac{K \cdot H}{\sqrt{H^2}} \right\} \alpha$$

$$+ [r_F r_B(r_B - 1)(r_F + 2r_A) + r_A r_B^2(r_A + 1)] \frac{(K \cdot H)^2}{8H^2} + (r_B^3 + r_A r_F) \frac{s(r_F)^2}{2H^2}$$

$$+ (r_F^2 - r_A r_B) \frac{(K \cdot H + s(r_F))^2}{2H^2} + \frac{r_F^2}{2}(1 - 2\chi(O_S)) + \frac{r_B^3}{8} |K^2|.$$ 

(5.16) Proposition. Keeping notation as above, assume that $H$ is effective. Suppose that $F$ is a rank-two $\mu$-semi-stable sheaf with a filtration (5.12), and that (5.14) is satisfied. Define $\alpha$ by setting

$$\mu_A = \frac{\alpha}{r_A} \sqrt{H^2}.$$ 

Then

$$\sum_{i \leq j} h^1(Q_j, Q_i) \leq 3\Delta_F + 3\alpha^2 + \frac{K \cdot H + 2H^2 + 2}{\sqrt{H^2}} \alpha$$

$$+ \frac{3(K \cdot H + H^2 + 1)^2}{2H^2} + \frac{(K \cdot H)^2}{4H^2} - \frac{K^2}{4} + 3 - 3\chi(O_S).$$

Proof of (5.9)-(5.10)-(5.11) assuming (5.15)-(5.16). First we prove Propositions (5.9) and (5.10). Let $F$ be a family of sheaves on $S$ parametrized by a scheme $B$. Let $\chi := (\chi_1, \ldots, \chi_{n+1})$ be a sequence of one-variable polynomials. Drezet-Le Potier [DL] have constructed a scheme

$$\text{Drap}(F; \chi) \to B.$$
proper over $B$, parametrizing all filtrations (5.12) with $F = F_x$ for some $x \in B$, and where
\[ \chi(Q_i(nH)) = \chi_i \quad i = 1, \ldots, (n + 1). \] (\textcircled{z})

Now let $P_\xi$ be a parameter space for semistable sheaves of which $M_\xi$ is the geometric invariant theory quotient. Thus there is a family of semistable sheaves $F_\xi$ on $S$ parametrized by $P_\xi$ inducing a surjection $P_\xi \to M_\xi$. For $\alpha_0 \in \mathbb{R}$ let $I(\alpha_0)$ be the set of sequences $\chi$ such that if $F$ fits into (5.12) and (\textcircled{z}) holds, then
\[ \mu(A) \geq \mu_F - \frac{\alpha_0}{r(A)} \sqrt{H^2}. \] (5.17)

(The definition of $I(\alpha_0)$ makes sense because the slope of a sheaf is determined by its Hilbert polynomial.) If $\chi$ is as above, let $\text{Drap}_0(\mathcal{F}_\xi; \chi)$ be the subset of $\text{Drap}(\mathcal{F}_\xi; \chi)$ parametrizing filtrations such that (5.14) is satisfied. Since each of the properties of (5.14) is open, $\text{Drap}_0(\mathcal{F}_\xi; \chi)$ is an open subset of $\text{Drap}(\mathcal{F}_\xi; \chi)$. Set
\[ \text{Drap}_0(\mathcal{F}_\xi, \alpha_0) := \bigcup_{\chi \in I(\alpha_0)} \text{Drap}_0(\mathcal{F}_\xi; \chi). \]

Claim. Let notation be as above. Then $\text{Drap}_0(\mathcal{F}_\xi, \alpha_0)$ is a scheme of finite type.

Proof. Since $\text{Drap}_0(\mathcal{F}_\xi; \chi)$ is of finite type for every $\chi$, all we have to show is that
\[ I^*(\alpha_0) := \{ \chi \in I(\alpha_0) \mid \text{Drap}_0(\mathcal{F}_\xi; \chi) \neq \emptyset \} \]
is a finite set. So let $F$ be a sheaf satisfying the hypotheses of Proposition (5.15), and let $Q_i$ be as in (5.13). Since the set of slopes of subsheaves $Q_i \subset (\mathcal{F}_\xi)_x$, for $x \in P_\xi$, satisfying (5.17) is finite, it suffices to check that there is a universal bound for the size of the coefficients of the Hilbert polynomials of the $Q_i$, if $(S, H)$, $\alpha$, $r_F$, $c_1(F)$ and $c_2(F)$ are fixed. This in turn amounts to bounding the size of
\[ c_1(Q_i) \cdot H, \quad c_1(Q_i)^2, \quad c_1(Q_i) \cdot K, \quad c_2(Q_i). \]

That $c_1(Q_i) \cdot H$ is bounded follows from (5.17) and from Item (2) of (5.14). Thus, by Hodge index, we also get that $c_1(Q_i)^2$ is bounded above. Let’s show that it is also bounded below. By Item (1) of (5.14) and Bogomolov’s Inequality we have
\[ \frac{1}{2r_i} c_1(Q_i)^2 \geq \frac{1}{2} c_1(Q_i)^2 - c_2(Q_i). \]

Thus
\[ \sum_{i=1}^{n+1} \frac{1}{2r_i} c_1(Q_i)^2 \geq \sum_{i=1}^{n+1} \frac{1}{2} c_1(Q_i)^2 - c_2(Q_i) = \frac{1}{2} c_1(F)^2 - c_2(F). \] (\textcircled{*})

Since the values of $c_1(Q_i)^2$ are bounded above one concludes that they are also bounded below. Since $c_1(Q_i) \cdot H$ is bounded we conclude by the Hodge index theorem that $c_1(Q_i) \cdot K$ is bounded. Finally boundedness of $c_2(Q_i)$ follows from boundedness of $c_1(Q_i)^2$, Bogomolov’s Inequality and the equality in (\textcircled{*}). q.e.d.

Let $\pi$ be the composition of the projection $\text{Drap}_0(\mathcal{F}_\xi, \alpha_0) \to P_\xi$ and the quotient map $P_\xi \to M_\xi$. By construction we have
\[ M_\xi(\alpha_0) = \pi(\text{Drap}_0(\mathcal{F}_\xi, \alpha_0)). \] (\textcircled{b})
Hence, by the claim above, $\mathcal{M}_\xi(\alpha_0)$ is a constructible set. Now let’s consider $\dim \mathcal{M}_\xi(\alpha_0)$. For convenience of exposition we will assume that there is a tautological family of sheaves $\mathcal{G}_\xi$ parametrized by $\mathcal{M}_\xi$; it will be clear how to modify the argument if this is not the case. Let $x \in \text{Drap}(\mathcal{G}_\xi; \bar{\chi})$ correspond to (5.12). An easy inductive argument with extensions shows that

$$\dim \text{Drap}(\mathcal{G}_\xi; \bar{\chi})_x \leq \sum_{i \leq j} h^1(Q_j, Q_i).$$

By (b) we conclude that $\dim \mathcal{M}_\xi(\alpha_0)$ is bounded above by the maximum of the values of the right-hand side of the inequality in (5.15) (or in (5.16)) for $r_F = r_\xi$, $\Delta_F = \Delta_\xi$, $0 \leq \alpha \leq \alpha_0$, and $1 \leq r_A \leq r_\xi$. This immediately gives (5.10). It also gives a slightly weaker version of (5.9).

In order to get (5.9) one argues that if $[F] \in \mathcal{M}_\xi(\alpha_0)$ then at least one of $F$, $F^{**}$ fits into a filtration (5.12) such that (5.14) is satisfied and furthermore $r(A) \leq r_\xi/2$. The arguments above together with Theorem (3.5) will show then that $\dim \mathcal{M}_\xi(\alpha_0)$ is bounded above by the maximum of the values of the right-hand side of the inequality in (5.15) for $r_F = r_\xi$, $\Delta_F = \Delta_\xi$, $0 \leq \alpha \leq \alpha_0$, and $1 \leq r_A \leq r_\xi/2$. Proposition (5.9) follows from this together with easy estimates.

Now let’s prove Proposition (5.11). We keep the notation introduced in the previous proof. For simplicity of exposition we assume that there exists a tautological family $\mathcal{G}_\xi$ of sheaves parametrized by $\mathcal{M}_\xi$. Let $\text{Drap}_0^C(\mathcal{G}_\xi; \alpha_0)$ be the subset of $\text{Drap}_0(\mathcal{G}_\xi; \alpha_0)$ parametrizing filtrations

$$0 \to A \xrightarrow{f} F \to B \to 0,$$  \hspace{1cm} (5.18)

such that the restriction $A|_C$ is locally-free and spans a destabilizing subline bundle of $F|_C$. As is easily checked $\text{Drap}_0^C(\mathcal{G}_\xi; \alpha_0)$ is closed. Since

$$\pi(\text{Drap}_0^C(\mathcal{G}_\xi; \alpha_0)) = \mathcal{M}_\xi^C(\alpha_0),$$

we conclude that $\mathcal{M}_\xi^C(\alpha_0)$ is constructible. Now let’s prove the upper bound for its dimension. We start by examining the map of vector bundles $f|_C$. Let $\Omega$ be its zero-locus. Since $A|_C$ spans a destabilizing subline bundle of $F|_C$, we have

$$\mu(A|_C) + \deg \Omega \geq \mu(F|_C).$$  \hspace{1cm} (*)

Define $\alpha$ by setting

$$\mu_A = \mu_F - \alpha \sqrt{H^2}. $$

(Thus $0 \leq \alpha \leq \alpha_0$.) Since $C \in |nH|$ we conclude by (*) that

$$\deg \Omega \geq n\alpha \sqrt{H^2}. $$  \hspace{1cm} (5.19)

Since $B$ is a rank-one torsion-free sheaf we have $B = I_Z \otimes B^{**}$, for a zero-dimensional subscheme $Z$ of $S$. We claim that for each point $P \in \Omega$ we have

$$\text{mult}_P(Z) \geq \text{mult}_P(\Omega). $$  \hspace{1cm} (5.20)

In fact, let $f = (f_1, f_2)$ be an expression for $f$ in a neighborhood of $P$, so that $I_Z$ is locally generated by $f_1, f_2$. Let $y$ be a local equation for $C$. Then

$$\text{mult}_P(Z) = \dim_C \mathcal{O}/(f_1, f_2), \quad \text{mult}_P(\Omega) = \dim_C \mathcal{O}/(f_1, f_2, y). $$
where \(\mathcal{O}\) is the local ring at \(P\); Inequality (5.20) follows at once from this. Putting together (5.20) and (5.19) we get

\[
\sum_{P \in \mathcal{C}} \text{mult}_P(Z) \geq n\alpha \sqrt{H^2}.
\] (5.21)

To conclude the proof of Proposition (5.11) we go back to the proof that (5.16) implies (5.10). Let \(x \in \text{Drap}(\mathcal{G}_{\xi}; \alpha_0)\) correspond to the filtration (5.18). We argued that

\[
\dim \text{Drap}(\mathcal{G}_{\xi}; \alpha_0)_x \leq h^1(A, A) + h^1(B, A) + h^1(B, B).
\] (\star)

Here

\[
h^1(B, B) = \dim \text{Pic}(S) + \dim \text{Hilb}^C(S),
\]
is the number of moduli of rank-one torsion-free sheaves with the same Chern classes as \(B\) (\(\ell := \ell(Z) = c_2(B)\)). For \(m \in \mathbb{R}\) let \(I_m(C) \subset \text{Hilb}^C(S)\) be the locus parametrizing subschemes \(Z\) such that \(\sum_{P \in \mathcal{C}} \text{mult}_P(Z) \geq m\). By (5.21), Inequality (\star) is replaced in the present case by

\[
\dim \text{Drap}^C(\mathcal{G}_{\xi}; \alpha_0)_x \leq h^1(A, A) + h^1(B, A) + \dim \text{Pic}(S) + \dim I_{n\alpha \sqrt{H^2}}.
\]

An easy application of a theorem of Iarrobino [I] shows that

\[
\text{cod}(I_m(C), \text{Hilb}^C(S)) \geq m,
\]
and hence

\[
\dim \text{Drap}^C(\mathcal{G}_{\xi}; \alpha_0)_x \leq h^1(A, A) + h^1(B, A) + h^1(B, B) - n\alpha \sqrt{H^2}.
\]

Since the right-hand side of the inequality in (5.10) is an upper bound for the above sum of \(h^1\)'s for \(0 \leq \alpha \leq \alpha_0\) we conclude that (5.11) holds.

**Proof of Proposition (5.15).** By additivity of the Euler characteristic, and by Serre duality, we have

\[
\sum_{i \leq j} h^1(Q_j, Q_i) = -\chi(F, F) + \sum_{i \leq j} \chi(Q_i, Q_j) + \sum_{i \leq j} h^0(Q_j, Q_i) + \sum_{i \leq j} h^0(Q_i, Q_j \otimes K_S).
\]

For \(x \in \text{Pic}(S) \otimes \mathbb{Q}\), set \(\chi(x) := \chi(\mathcal{O}_S) + (x^2 - x \cdot K)/2\). Applying the formula

\[
\chi(F, G) = r_F \cdot r_G \left[ \chi \left( \frac{c_1(G)}{r_G} - \frac{c_1(F)}{r_F} \right) - \frac{\Delta_F}{r_F} - \frac{\Delta_G}{r_G} \right],
\]
valid for any couple of sheaves \(F, G\) (of positive rank) on \(S\), we get:

\[
\sum_{i \leq j} h^1(Q_j, Q_i) = 2r_F \Delta_F - r_F^2 \chi(\mathcal{O}_S) + \sum_{i \leq j} r_i r_j \left[ \chi \left( \frac{c_1(Q_j)}{r_j} - \frac{c_1(Q_i)}{r_i} \right) - \frac{\Delta_i}{r_i} - \frac{\Delta_j}{r_j} \right]
+ \sum_{1 < j} h^0(A, Q_j \otimes K_S) + \sum_{2 \leq j \leq i} h^0(Q_j, Q_i) + h^0(A, A) + h^0(A, A \otimes K_S) + \sum_{1 < j} h^0(Q_j, A) + \sum_{2 \leq j \leq i} h^0(Q_i, Q_j \otimes K_S).
\]

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Set

\[ \Theta := \sum_{i<j} r_i r_j \left[ \chi \left( \frac{c_i(Q_j)}{r_j} - \frac{c_i(Q_i)}{r_i} \right) - \frac{\Delta_i}{r_i} - \frac{\Delta_j}{r_j} \right], \tag{5.22} \]

\[ \Lambda := \sum_{1<j} h^0(A, Q_j \otimes K_S), \]

\[ \Gamma := \sum_{2 \leq i \leq j} h^0(Q_j, Q_i), \]

\[ \Omega := h^0(A, A) + h^0(A, A \otimes K_S) + \sum_{1 < j} h^0(Q_j, A) + \sum_{2 \leq i < j} h^0(Q_i, Q_j \otimes K_S). \]

Then the previous equation becomes

\[ \sum_{i \leq j} h^1(Q_j, Q_i) = 2r_F \Delta_F - r_F^2 \chi(O_S) + \Theta + \Lambda + \Gamma + \Omega. \]

Proposition (5.15) follows from the bounds for \( \Theta, \Lambda, \Gamma \) and \( \Omega \) given below, together with a straightforward computation.

(5.23). Let notation be as above. If \( K \cdot H \geq 0 \), then

\[ \Theta \leq - \Delta_F + \left( \frac{r_A + 1}{2r_A} \right)^2 + [r_F(r_F - 1) - r_F] \frac{K \cdot H}{2 \sqrt{H^2}} + r_F [r_F(r_F - 1)(r_F - 2r_A) + r_A r_F^2(r_A + 1)] \frac{(K \cdot H)^2}{8H^2} + M(r_F, S, H) + r_F^2 \chi(O_S). \]

(5.24). With notation as above, assume \( K \cdot H < 0 \). Then

\[ \Theta \leq - \Delta_F + \left( \frac{r_A + 1}{2r_A} \right)^2 - [r_F(r_F - 1)(2r_F - r_B) + r_F] \frac{K \cdot H}{2 \sqrt{H^2}} + r_F [r_F r_B(r_B - 1)(r_F + 2r_A) + r_A r_B^2(r_A + 1)] \frac{(K \cdot H)^2}{8H^2} + \frac{r_F^2}{2} + \frac{r_F^3}{8} |K^2|. \]

In both cases (i.e. regardless of the sign of \( K \cdot H \)), we have

\[ \Lambda \leq \frac{1}{2r_A} (r_A^2 + r_A + r_B) \alpha^2 + \frac{r_F}{\sqrt{H^2}} (K \cdot H + s(r_F)) \alpha + \frac{r_A r_B}{2H^2} (K \cdot H + s(r_F))^2, \tag{5.25} \]

\[ \Gamma \leq \frac{r_B - 1}{2} \alpha^2 + \frac{r_B(r_B - 1) s(r_F)}{\sqrt{H^2}} \alpha + \frac{r_B^3}{2H^2} s(r_F)^2, \tag{5.26} \]

\[ \Omega \leq \frac{r_A r_F}{2H^2} s(r_F)^2 + \frac{r_A^2 + r_B^2}{2H^2} (K \cdot H + s(r_F))^2. \tag{5.27} \]

Proof of (5.23)-(5.24). Let

\[ r_\ell := \max \{r_i \mid 1 \leq i \leq n + 1\}, \]

\[ \sigma_i := - r_1 - \ldots - r_{i-1} + r_{i+1} + \ldots r_{n+1}. \tag{5.28} \]
For future reference we notice that
\[ \sum_{i=1}^{n+1} r_i \sigma_i = 0. \] (5.29)

\((5.30)\) Lemma. Keeping notation as above,
\[ \Theta \leq -(r_F - r_\ell) \Delta_F + \frac{1}{2} \left( r_F^2 - \sum_{i=1}^{n+1} r_i^2 \right) \chi(O_S) + r_\ell \sum_{i=1}^{n+1} c_1(Q_i)^2 - r_\ell \frac{c_1(F)^2}{2r_F} + \frac{1}{2} \sum_{i=1}^{n+1} \sigma_i c_1(Q_i) \cdot K_S. \]

Proof. We brake the sum defining \( \Theta \) (Equation (5.22)) into two pieces. First consider
\[ \Theta' := \sum_{i<j} r_i r_j \chi \left( \frac{c_1(Q_j)}{r_j} - \frac{c_1(Q_i)}{r_i} \right). \]
A straightforward computation gives
\[ \Theta' = r_F \sum_{i=1}^{n+1} \frac{c_1(Q_i)^2}{2r_i} - \frac{1}{2} c_1(F)^2 + \frac{1}{2} \sum_{i=1}^{n+1} \sigma_i c_1(Q_i) \cdot K_S + \frac{1}{2} \left( r_F^2 - \sum_{i=1}^{n+1} r_i^2 \right) \chi(O_S). \]

Now consider
\[ \Theta'' := \sum_{i<j} (r_j \Delta_i + r_i \Delta_j). \]
Then
\[ \Theta'' = \sum_{i=1}^{n+1} (r_F - r_i) \Delta_i. \]
Since \((r_F - r_i) \geq (r_F - r_\ell)\), and since \(\Delta_i \geq 0\) (Bogomolov’s theorem), we conclude that
\[ \Theta'' \geq (r_F - r_\ell) \sum_{i=1}^{n+1} \Delta_i = (r_F - r_\ell) \sum_{i=1}^{n+1} \left[ c_2(Q_i) - \frac{1}{2} c_1(Q_i)^2 \right] + (r_F - r_\ell) \sum_{i=1}^{n+1} \frac{1}{2r_i} c_1(Q_i)^2. \]
Additivity of the Chern character gives
\[ \sum_{i=1}^{n+1} \left[ c_2(Q_i) - \frac{1}{2} c_1(Q_i)^2 \right] = c_2(F) - \frac{1}{2} c_1(F)^2 = \Delta_F - \frac{1}{2r_F} c_1(F)^2, \]
and thus we have
\[ -\Theta'' \leq -(r_F - r_\ell) \Delta_F + (r_F - r_\ell) \frac{1}{2r_F} c_1(F)^2 - (r_F - r_\ell) \sum_{i=1}^{n+1} \frac{1}{2r_i} c_1(Q_i)^2. \]

Since \( \Theta = (\Theta' - \Theta'') \), the lemma follows. q.e.d.

Set
\[ \Xi := r_\ell \sum_{i=1}^{n+1} \frac{c_1(Q_i)^2}{2r_i} - r_\ell \frac{c_1(F)^2}{2r_F} + \frac{1}{2} \sum_{i=1}^{n+1} \sigma_i c_1(Q_i) \cdot K_S. \] (5.31)
Lemma (5.32). Let notation be as above. Then

\[
\Xi \leq \frac{r_A + r_AR_B}{2} \left( \frac{r_B}{r_A} \alpha^2 + \frac{1}{2} \left\{ [r_B(r_B - 1) - r_B] (K_S \cdot H) + r_F(r_B - 1) [||K_S \cdot H| - (K_S \cdot H)] \right\} \right)
\]

Proof. First of all notice that \( \Xi \) is left invariant if \( F \) and the \( Q_i \) are tensored by a line bundle \( \zeta \), or even if they are formally tensored by \( \zeta \in \text{Pic}(S) \otimes Q \). (Use Equation (5.29).) Choosing \( \zeta \) such that \( c_1(F \otimes \zeta) = 0 \), we can assume that \( c_1(F) = 0 \). Of course by doing this the classes \( c_1(F) \) and \( c_1(Q_i) \) become elements of \( NS(S) \otimes Q \). Now rewrite the right-hand side of (5.31) to get

\[
\Xi = \frac{r_A}{2r_A} \left[ \frac{r_A}{r_A} c_1(A) + \frac{r_B}{2} K \right]^2 + \frac{1}{2} \sum_{i=1}^{n+1} r_i \left[ \frac{r_A}{r_A} c_1(Q_i) + \frac{\sigma_i}{2} K \right]^2 - \frac{K^2}{8r_F} \sum_{i=1}^{n+1} r_i \sigma_i^2. 
\]

In what follows we will use the inequality

\[
L^2 \leq \frac{(L \cdot H)^2}{H^2},
\]

which, by the Hodge index theorem, holds for all \( L \in NS(S) \otimes Q \). It gives

\[
\frac{r_A}{2r_A} \left[ \frac{r_A}{r_A} c_1(A) + \frac{r_B}{2} K \right]^2 \leq \frac{r_B}{2r_A} \left( \frac{r_B}{2} \alpha^2 - \frac{r_B(K \cdot H)}{2\sqrt{H^2}} \right) + \frac{r_B^2}{8r_F} (K \cdot H)^2,
\]

and

\[
\frac{1}{2} \sum_{i=2}^{n+1} r_i \left[ \frac{r_A}{r_A} c_1(Q_i) + \frac{\sigma_i}{2} K \right]^2 \leq \frac{1}{2} \sum_{i=2}^{n+1} r_i \left[ r_i \mu_i + \frac{\sigma_i}{2} \mu_K \right]^2.
\]

We will bound the right-hand side of the above inequality by applying the following lemma. Its (easy) proof is left to the reader.

Lemma (5.36). Let \( x_1, \ldots, x_n \) be real numbers and \( r_1, \ldots, r_n \) be positive integers. Let \( N := \sum_{i=1}^n r_i \). If \( x_i \geq a \) for \( i = 1, \ldots, n \), then

\[
\sum_{i=1}^n r_i x_i^2 \leq \sum_{i=1}^n r_i x_i - (N-1)a^2 + (N-1)a^2.
\]

We apply the lemma to the sum on the right-hand side of (5.35). Set \( x_i := r_i \mu_i + \sigma_i \mu_K/2 \). Then \( N = r_B \), and

\[
\sum_{i=2}^{n+1} r_i x_i = r_B \sqrt{H^2} \alpha - \frac{1}{2} r_A r_B \mu_K.
\]

Since \( \mu_2 \geq \cdots \geq \mu_{n+1} \geq 0 \), we can set \( a := -\frac{1}{2} r_F |\mu_K| \). Lemma (5.36) gives

\[
\frac{1}{2} \sum_{i=2}^{n+1} r_i \left[ r_i \mu_i + \frac{\sigma_i}{2} \mu_K \right]^2 \leq \frac{1}{2} r_B \alpha^2 + \frac{1}{2} \left[ r_F (r_B - 1) |\mu_K| - r_A r_B \mu_K \right] \alpha
\]

\[
+ \frac{1}{8r_B H^2} \left\{ (r_F (r_B - 1) |\mu_K| - r_A r_B \mu_K)^2 + r_F (r_B - 1) \mu_K^2 \right\}.
\]

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Lemma (5.32) now follows from (5.33), (5.34), (5.35), and the above inequality, together with a straightforward computation.

q.e.d.

Inequalities (5.23)-(5.24) follow from Lemmas (5.30)-(5.32) and some easy estimates. The only estimate which is not completely trivial is provided by the following

**Lemma.** Keeping notation as above, we have

\[ r_F \leq \frac{1}{r_\ell} \sum_{i=1}^{n+1} r_i \sigma_i^2. \]

**Proof.** One checks easily that, if \( n \) is replaced by \((n+1)\), and \( r_{n+1} \) by \( t_{n+1}, t_{n+2} \) (with \( t_{n+1}+t_{n+2} = r_{n+1} \)), then the right-hand side of the above inequality increases. Thus the minimum, with a fixed \( r_1 = r_A \), is given by \( n = 1 \). The lemma then follows by direct computation.

q.e.d.

**Proof of Inequality (5.25).** By Corollary (5.3), we have

\[ \Lambda \leq \frac{r_A}{2H^2} \sum_{1<j} r_j (\mu_j + \mu_K - \mu_A + s(r_F))^2. \]

Lemma (5.36) together with an easy computation gives (5.25).

**Proof of Inequality (5.26).** To simplify notation let \( s := s(r_F) \). By Corollary (5.3) we have

\[ \Gamma \leq \frac{1}{2H^2} \sum_{1<i<j} r_i r_j (\mu_i - \mu_j + s)^2 + \sum_{1<i} r_i^2 s^2. \]

Expanding the squares in the first sum on the right-hand side, we write

\[ \Gamma = \frac{1}{2H^2} \left( \Gamma_1 + \Gamma_2 + \Gamma_3 \right), \quad (5.37) \]

where

\[ \Gamma_1 := s^2 \sum_{1<i} r_i^2 + s^2 \sum_{1<i<j} r_i r_j, \]

\[ \Gamma_2 := \sum_{1<i<j} \left( r_i r_j \mu_i^2 + r_i r_j \mu_j^2 \right) - 2 \sum_{1<i<j} r_i r_j \mu_i \mu_j, \]

\[ \Gamma_3 := 2s \sum_{1<i<j} (r_i r_j \mu_i - r_i r_j \mu_j). \]

We rewrite \( \Gamma_2 \) as

\[ \Gamma_2 = \sum_{1<i} r_i (r_B - r_i) \mu_i^2 - 2 \sum_{1<i<j} r_i r_j \mu_i \mu_j = r_B \sum_{1<i} r_i \mu_i^2 - r_B^2 \mu_B^2. \]

For the second equality we have used the relation \( r_B \mu_B = \sum_{1<i} r_i \mu_i \). We also write

\[ \Gamma_3 = 2s \sum_{1<i} r_i \tau_i \mu_i, \]

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where \( \tau_i := (r_2 - \cdots - r_{i-1} + r_{i+1} + \cdots + r_{n+1}) \). (See (5.28).) Now write

\[
\begin{align*}
\sum_{1 < i} r_i \left( \mu_i + \frac{\tau_i s}{r_B} \right)^2 & \leq r_B \sum_{1 < i} r_i \left( \mu_i + \frac{\tau_i s}{r_B} \right)^2 - r_B^2 \mu_B^2 \\
& \leq r_B \sum_{1 < i} r_i \left( \mu_i + \frac{\tau_i s}{r_B} \right)^2 - r_B^2 \mu_B^2.
\end{align*}
\]

(5.38)

We bound the sum of squares on the last line by applying Lemma (5.36). We let \( a := (\mu_F - s) \).

Clearly \( N = r_B \). Furthermore

\[
\sum_{1 < i} r_i \left( \mu_i + \frac{\tau_i s}{r_B} \right) = r_B \mu_B + \frac{s}{r_B} \sum_{1 < i} r_i \tau_i = r_B \mu_B.
\]

Lemma (5.36) gives

\[
\sum_{1 < i} r_i \left( \mu_i + \frac{\tau_i s}{r_B} \right)^2 \leq r_B^2 \mu_B^2 - 2r_B(r_B - 1)(\mu_F - s) + r_B(r_B - 1)(\mu_F - s)^2.
\]

Putting this together with Inequality (5.38) we conclude that

\[
\Gamma_2 + \Gamma_3 \leq r_B^2(r_B - 1)(\mu_F - \mu_F + s)^2 = (r_B - 1)H^2\alpha^2 + 2r_B(r_B - 1)s \sqrt{H^2\alpha} + r_B^2(r_B - 1)s^2.
\]

Adding \( \Gamma_1 \leq r_B^2 s^2 \), and using Equation (5.37), one gets (5.26).

**Proof of Inequality (5.27).** It follows from Corollary (5.3) and some simple estimates.

**Proof of Proposition (5.16).** The proof follows that of Proposition (5.15), with the difference that we replace Corollary (5.3) by Proposition (5.5).

**Properly semistable sheaves.**

Let \( F \) be a torsion-free sheaf on \( S \) which is properly \( \mu \)-semistable, i.e. \( \mu \)-semistable but not \( \mu \)-stable.

We will bound the dimension of the locus \( V^0(F) \subset Def^0(F) \) parametrizing properly \( \mu \)-semistable sheaves.

(5.39) For \( r \geq 2 \) an integer, set

\[
\epsilon(r, S, H) := \begin{cases} 
\frac{r^2}{2H^2} [K \cdot H + s(r)]^2 + \frac{r^2}{16} \left[ \frac{(K \cdot H)^2}{H^2} - K^2 \right] + r^2 + pr^2 |\chi(O_S)| - q_S & \text{if } r > 2, \\
\frac{3}{2H^2} (K \cdot H + H^2 + 1)^2 + \frac{1}{8} \left[ \frac{(K \cdot H)^2}{H^2} - K^2 \right] + 4 - 3\chi(O_S) - q_S & \text{if } r = 2,
\end{cases}
\]

where \( p := -3/4 \) if \( \chi(O_S) \geq 0 \), and \( p := 1 \) if \( \chi(O_S) < 0 \).

(5.40) Proposition. Let \( (S, H) \) be a polarized surface. Let \( F \) be a torsion-free sheaf on \( S \) of rank \( r_F \geq 2 \), which is \( \mu \)-semistable but not \( \mu \)-stable (i.e. properly \( \mu \)-semistable). If \( r_F > 2 \) assume that \( H \) satisfies (0.4), if \( r_F = 2 \) assume only that \( H \) is effective. Then, letting \( V^0(F) \subset Def^0(F) \) be as above, we have

\[
\dim V^0(F) \leq (2r_F - 1) \Delta_F + \epsilon(r_F, S, H).
\]

Notice that if \( F \) is simple, then the proof that Proposition (5.15) implies Proposition (5.9) gives also a bound for \( \dim V^0(F) \). If however \( F \) is not simple, then we need to argue differently. Before proving the above bound, we state a lemma.
(5.41) Lemma. Let $A$, $B$ be torsion-free sheaves on a projective irreducible variety. Suppose that $A$, $B$ are both properly $\mu$-semistable, and that $\mu_A = \mu_B$. Then

$$h^0(A, B) \leq r_A r_B.$$ 

Proof. The proof is by double induction on the lengths of $\mu$-Jordan-Hölder filtrations for $A$ and $B$. If the lengths are both one, then $A$ and $B$ are stable, hence $h^0(A, B) \leq 1$, and the result is true. To prove the inductive step, let $A_1 \subset A$ and $B_1 \subset B$ be the first terms of Jordan-Hölder filtrations. Then

$$h^0(A, B) \leq 1 + r_{A_1}(r_B - r_{B_1}) + r_{B_1}(r_A - r_{A_1}) + (r_A - r_{A_1})(r_B - r_{B_1}).$$

The lemma follows by simplifying the right-hand side. q.e.d.

Proof of Proposition (5.40). Let $V(F) \subset \text{Def}(F)$ be the locus parametrizing properly $\mu$-semistable sheaves. We will prove that

$$\dim V(F) \leq (2r_F - 1) + \epsilon(r_F, S, H) + q_S.$$ (5.42)

Clearly this is equivalent to Proposition (5.40). Let $F$ be the family of sheaves parametrized by $\text{Def}(F)$. Let $\text{Quot}_0(F)$ be the quot-scheme parametrizing torsion-free quotients of $F_x$, for $x$ varying in $\text{Def}(F)$, with slope equal to that of $F$ (i.e. destabilizing quotients). By a theorem of Grothendieck [G], $\text{Quot}_0(F)$ is of finite type. Let

$$\pi: \text{Quot}_0(F) \to \text{Def}(F),$$

be the projection. Clearly

$$V(F) = \pi(\text{Quot}_0(F)),$$ (5.43)

and hence $\dim V(F) \leq \dim \text{Quot}_0(F)$. Now let $y \in \text{Quot}_0(F; P)$ correspond to the $\mu$-detabilizing sequence

$$0 \to A \to F_x \to B \to 0.$$ (*)&

Then there is an exact sequence [DL]

$$0 \to \text{Hom}(A, B) \to T_y \text{Quot}_0(F_x; P) \xrightarrow{\pi_y} \text{Ext}^1(F_x, F_x) \xrightarrow{\omega_+} \text{Ext}^1(A, B),$$ (5.44)

where $\omega_+$ is induced by the inclusion $A \subset F_x$ and the quotient $F_x \to B$. By applying the functors $\text{Hom}(\cdot, F_x)$ and $\text{Hom}(A, \cdot)$ to (*) one concludes that

$$\dim \text{Quot}_0(F_x; P)_y \leq h^0(A, B) + h^1(A, A) + h^1(B, B) + h^1(B, A).$$

Now we estimate the sum of the $h^1$'s by repeating the proof of Proposition (5.15). (We will get sharper results because the filtration consists of only two terms.) Adopting the notation used in that proof, we have

$$\Theta \leq -r_A \Delta_F + \frac{r_A r_F}{8} \left[ \frac{(K \cdot H)^2}{H^2} - K^2 \right].$$

Then, by applying (5.3), (5.5), or (5.41) to bound $\Lambda$, $\Gamma$ and $\Omega$, we get

$$\dim \text{Quot}_0(F; P)_y \leq (2r_F - 1) + \epsilon(r_F, S, H) + q_S.$$ (5.42)

By (5.43) this proves Inequality (5.42), and hence the proposition.
(5.45) Corollary. Let $(S, H)$ be a polarized surface, and $\xi$ be set of sheaf data for $S$. If $r_\xi > 2$ we assume that $H$ satisfies (0.4), if $r_\xi = 2$ we only assume that $H$ is effective. Let $X \subset M_\xi$ be a subset such that

$$\dim X > (2r_\xi - 1)\Delta_\xi + \epsilon(r_\xi, S, H).$$

Then there exists a point of $X$ parametrizing a $\mu$-stable sheaf.

Proof. Let $[F] \in X$; we can assume $F$ is properly $\mu$-semistable. Let $F$ be the family of sheaves on $S$ parametrized by $Def^0(Gr F)$. By Luna’s étale slice Theorem the map

$$\lambda: Def^0(Gr F) \to M_\xi,$$

induced by $F$ is surjective onto a neighborhood of $[F]$. Thus $\dim \lambda^{-1}X$ satisfies the same inequality as $\dim X$. By (5.40) there exists $x \in \lambda^{-1}X$ parametrizing a $\mu$-stable sheaf. Then $\lambda(x) \in X$ is a point parametrizing a $\mu$-stable sheaf. q.e.d.

(5.46) Corollary. Let $(S, H)$ and $\xi$ be as in the previous corollary. If

$$2r_\xi\Delta_\xi - (r_\xi^2 - 1)\chi(\mathcal{O}_S) > (2r_\xi - 1)\Delta_\xi + \epsilon(r_\xi, S, H),$$

then the generic point of any irreducible component of $M_\xi$ parametrizes a $\mu$-stable sheaf.

Proof. Let $X \subset M_\xi$ be an irreducible component. Let $[F] \in X$ be a point not belonging to any other component of $M_\xi$. If $F$ is $\mu$-stable there is nothing to prove, so assume $F$ is not $\mu$-stable. By deformation theory $[F]$ we have

$$\dim Def^0(F) \geq 4\Delta_\xi - 3\chi(\mathcal{O}_S).$$

Hence by Proposition (5.40) we conclude that the generic point $x \in Def^0(F)$ parametrizes a $\mu$-stable sheaf. This implies that the generic point of $X$ parametrizes a $\mu$-stable sheaf. q.e.d.

Non-stable vector bundles on curves.

Let $C$ be a smooth irreducible curve of genus $g$. Let $\mathcal{F}$ be a family of rank-$r$ vector bundles on $C$, parametrized by an equidimensional variety $B$. Let $B^{ns} \subset B$ be the subset parametrizing bundles which are not stable (i.e. either unstable or properly semistable). The goal of this subsection is to prove the following

(5.47) Proposition. Keeping notation as above, assume that $B^{ns}$ is not empty. Then

$$\text{cod}(B^{ns}, B) \leq \frac{r^2}{4}g.$$

If $g = 0$, there are no stable vector bundles, and hence the proposition is trivially verified. Thus we can assume that $g > 0$. Let $F$ be a non-stable vector bundle on $C$. Let $V(F) \subset Def(F)$ be the subset parametrizing non-stable bundles. Since $Def(F)$ is smooth, it suffices to show that, if $V_i$ is an irreducible component of $V(F)$, then

$$\text{cod}(V_i, Def(F)) \leq \frac{r^2}{4}g.$$

This is what we will prove. One can stratify $V(F)$ according to the ranks and slopes of the successive quotients of the Harder-Narasimhan filtration. Thus the stratum corresponding to the type

$$t := ((r_1, \mu_1), \ldots, (r_n, \mu_n)),$$
where \( \mu_1 > \cdots > \mu_n \), consists of the points \( x \in \text{Def}(F) \) such that there is a filtration
\[
F_1 \subset F_2 \subset \cdots \subset F_n = F_x,
\]
with \( F_i/F_{i-1} \) a rank-\( r_i \) semistable bundle with slope \( \mu_i \). As is well-known \cite{AB, Le}, each \( V_i \) is smooth, and
\[
\text{cod}(V_\ell, \text{Def}(F)) = \sum_{i<j} r_ir_j(\mu_i - \mu_j + g - 1). \tag{5.49}
\]

In order to prove Inequality (5.48) we need the following

**Lemma.** Keep notation as above. Assume that the genus \( g \) of \( C \) is positive. Let \( F \) be a non-stable vector bundle on \( C \). Then every irreducible component of \( V(F) \) contains an open dense subset parametrizing minimally non-stable bundles, i.e. bundles \( F \) fitting into an exact sequence
\[
0 \to A \to F \to B \to 0, \tag{5.50}
\]
where \( A, B \) are semistable vector bundles such that
\[
\mu_A \geq \mu_B, \tag{5.51}
\mu_A - \frac{1}{r_A} < \mu_B + \frac{1}{r_B}. \tag{5.52}
\]

**Proof.** Fix a component \( V_\ell \) of \( V(F) \). If the generic point of \( V_\ell \) parametrizes a semistable bundle, then there is nothing to prove. Thus we can assume that the bundles parametrized by \( V_\ell \) are unstable. Let \( t \) be the type of the Harder-Narasimhan filtration for the generic bundle parametrized by \( V_\ell \). Since versality is an open condition, we can replace \( F \) by \( F_x \) for a generic point \( x \in V_\ell \), and thus we can assume that \( V(F) \) is irreducible and that \( V(F) = V_\ell \). First let’s show that the Harder-Narasimhan filtration corresponding to \( t \) consists of only two terms. Let \( A \) be the first term of the H.-N. filtration for \( F \), and let
\[
0 \to A \to F \to B \to 0, \tag{*}
\]
be the corresponding exact sequence. Let \( P \) be the Hilbert polynomial of \( B \), and let \( \text{Quot}(F; P) \) be the \( \text{Quot} \)-scheme parametrizing quotients of \( F_x \) for \( x \in \text{Def}(F) \), with Hilbert polynomial equal to \( P \). If \( y \in \text{Quot}(F; P) \), then there is an exact sequence (5.44). The last map is surjective, because \( C \) is smooth of dimension one. By a criterion of Drezet-LePotier \cite{DL}, \( \text{Quot}(F; P) \) is smooth. Hence its dimension can be computed from (5.44). Furthermore, since \( h^0(A, B) = 0 \), the projection \( \pi: \text{Quot}(F; P) \to \text{Def}(F) \) is an embedding; let \( W \) be its image. An easy computation gives
\[
\text{cod}(W, \text{Def}(F)) = r_Ar_B(\mu_A - \mu_B + g - 1).
\]

Clearly \( W \subset V(F) \), and hence \( \text{cod}(W, \text{Def}(F)) \geq (V(F), \text{Def}(F)) \). Since \( V(F) = V_\ell \), one easily concludes from Equation (5.49) that \( t = ((r_A, \mu_A), (r_B, \mu_B)) \). Now let’s show that (\( * \)) is minimally destabilizing, i.e. that (5.52) is satisfied. We argue by contradiction. Assume that (5.52) is violated. Let \( A' \) be a sheaf fitting into an exact sequence
\[
0 \to A' \to A \overset{\phi}{\to} k_P \to 0, \tag{5.53}
\]
where \( k_P \) is the skyscraper sheaf at some point \( P \in C \). Let \( B' := B \oplus k_P \). Then there is an exact sequence
\[
0 \to A' \to F \to B' \to 0. \tag{\dagger}
\]
Let $P'$ be the Hilbert polynomial of $B'$, and let $\text{Quot}(\mathcal{F}; P')$ be the Quot-scheme of quotients of $\mathcal{F}_x$ with Hilbert polynomial equal to $P'$, for $x \in \text{Def}(F)$. As in the previous case, $\text{Quot}(\mathcal{F}; P')$ is smooth, and (5.44) gives

$$\dim \text{Quot}(\mathcal{F}; P') = h^1(F, F) - r_A r_B (\mu_A - \mu_B + g - 1) + r_F. \quad (\sharp)$$

Let $\pi: \text{Quot}(\mathcal{F}; P') \to \text{Def}(F)$ be the projection. Since (5.52) is violated,

$$\pi(\text{Quot}(\mathcal{F}; P')) \subset V(F). \quad (\star)$$

As is easily checked, $\pi^{-1}(o)$ consists of all sequences $A'$, where $A'$ fits into an exact sequence (5.53). (Here $o$ is the point parametrizing $F$.) Letting $P$ and $\phi$ vary, we get

$$\dim \pi^{-1}(o) = r_A.$$ 

Since $r_A < r_F$, we conclude by (\sharp) that

$$\dim \pi(\text{Quot}(\mathcal{F}; P')) > h^1(F, F) - r_A r_B (\mu_A - \mu_B + g - 1).$$

This inequality, together with (\star), contradicts Formula (5.49). This proves that (5.52) is satisfied, and hence $F$ is minimally non-stable. \textbf{q.e.d.}

\textbf{Proof of (5.48).} By the previous lemma we can assume that every sheaf parametrized by $V_i$ fits into Exact sequence (5.50), with (5.51)-(5.52) satisfied. We distinguish two cases, according to whether $V_i$ parametrizes unstable or semistable sheaves. In the first case Formula (5.49) gives

$$\text{cod}(V_i, \text{Def}(F)) = r_A r_B (\mu_A - \mu_B + g - 1).$$

By (5.51)-(5.52) we conclude that (5.48) is satisfied. Now assume that $F$ is semistable. Let $\text{Quot}_0(\mathcal{F})$ be the Quot-scheme parametrizing quotients of $\mathcal{F}_x$ whose slope is equal to that of $\mathcal{F}_x$, for $x \in \text{Def}(F)$. Let $\pi: \text{Quot}_0(\mathcal{F}) \to \text{Def}(F)$ be the projection. Exact sequence (5.44) and the smoothness of $\text{Quot}_0(\mathcal{F})$ give

$$\dim \pi(\text{Quot}_0(\mathcal{F})) \geq h^1(F, F) + \chi(A, B) - h^0(A, B).$$

Since $\pi(\text{Quot}_0(\mathcal{F})) = V_i$ one concludes, by Lemma (5.41), that (5.48) is satisfied.

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