RANDOM SECTIONS OF ELLIPSOIDS AND THE POWER OF RANDOM INFORMATION

AICKE HINRICHS, DAVID KRIEG, ERICH NOVAK, JOSCHA PROCHNO, AND MARIO ULLRICH

Abstract. We study the circumradius of the intersection of an $m$-dimensional ellipsoid $E$ with semi-axes $\sigma_1 \geq \cdots \geq \sigma_m$ with random subspaces of codimension $n$, where $n$ can be much smaller than $m$. We find that, under certain assumptions on $\sigma$, this random radius $R_n = R_n(\sigma)$ is of the same order as the minimal such radius $\sigma_{n+1}$ with high probability. In other situations $R_n$ is close to the maximum $\sigma_1$. The random variable $R_n$ naturally corresponds to the worst-case error of the best algorithm based on random information for $L_2$-approximation of functions from a compactly embedded Hilbert space $H$ with unit ball $E$. In particular, $\sigma_k$ is the $k$th largest singular value of the embedding $H \hookrightarrow L_2$. In this formulation, one can also consider the case $m = \infty$ and we prove that random information behaves very differently depending on whether $\sigma \notin \ell_2$ or not. For $\sigma \notin \ell_2$ we get $\mathbb{E}[R_n] = \sigma_1$ and random information is completely useless. For $\sigma \in \ell_2$ the expected radius tends to zero at least at rate $o(1/\sqrt{n})$ as $n \to \infty$. In the important case

$$\sigma_k \asymp k^{-\alpha} \ln^{-\beta}(k+1),$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ (which corresponds to various Sobolev embeddings), we prove

$$\mathbb{E}[R_n(\sigma)] \asymp \begin{cases} 
\sigma_1 & \text{if } \alpha < 1/2 \text{ or } \beta \leq \alpha = 1/2, \\
\sigma_{n+1} \sqrt{\ln(n+1)} & \text{if } \beta > \alpha = 1/2, \\
\sigma_{n+1} & \text{if } \alpha > 1/2.
\end{cases}$$

In the proofs we use a comparison result for Gaussian processes à la Gordon, exponential estimates for sums of chi-squared random variables, and estimates for the extreme singular values of (structured) Gaussian random matrices. The upper bound is constructive. It is proven for the worst case error of a least squares estimator.
1. Introduction

We are interested in the circumradius of the intersection of a centered ellipsoid $E$ in $\mathbb{R}^m$ with a random subspace $E_n$ of codimension $n$, where $n$ can be much smaller than $m$. While the maximal radius is the length of the largest semi-axis $\sigma_1$, the minimal radius is the length of the $(n+1)$-st largest semi-axis $\sigma_{n+1}$. But how large is the radius of a typical intersection? Is it comparable to the minimal or the maximal radius or does it behave completely different? We prove that the radius of a random intersection satisfies

$$\text{rad}(E \cap E_n) \leq \frac{c}{\sqrt{n}} \left( \sum_{j \geq n/4} \sigma_j^2 \right)^{1/2}$$

with overwhelming probability, where $c \in (0, \infty)$ is an absolute constant. For many sequences $\sigma$ of semi-axes, the right-hand side is of the same order as $\sigma_{n+1}$. This means that a typical intersection has radius comparable to the smallest one. One example are semi-axes of length $\sigma_j = j^{-\alpha}$ of polynomial decay $\alpha > 1/2$.

If the sequence $\sigma$ decays too slowly, this is no longer true and we find that a typical intersection often has radius comparable to the largest one. Indeed, if the ellipsoid is ‘fat’ in the sense that the semi-axes satisfy $\|\sigma\|_2 \geq c\sqrt{n}\sigma_1$, then we show that

$$\text{rad}(E \cap E_n) \geq \frac{\sigma_1}{2}$$

with overwhelming probability, where $c \in (0, \infty)$ is an absolute constant. An example are semi-axes of length $\sigma_j = j^{-\alpha}$ of polynomial decay $\alpha \leq 1/2$. Altogether, we obtain

$$\mathbb{E}[\text{rad}(E \cap E_n)] \asymp \begin{cases} \sigma_1 & \text{if } \alpha \leq 1/2, \\ \sigma_{n+1} & \text{if } \alpha > 1/2, \end{cases}$$

where $\asymp$ denotes equivalence up to positive constants not depending on $n$ and $m$.

The study of diameters of sections of symmetric convex bodies with a lower-dimensional subspace has been initiated by Giannopoulos and Milman [8, 10] and further advanced in the subsequent works of Litvak and Tomczak-Jaegermann [24], Giannopoulos, Milman, and Tsolomitis [9], or Litvak, Pajor, and Tomczak-Jaegermann [23]. However, as has already been pointed out in [8, 10], one cannot expect these bounds to be sharp for the whole class of symmetric convex bodies as is indicated by ellipsoids with highly incomparable semi-axes for which the diameter of sections of proportional dimension does not concentrate around some value [10, Example 2.2]. Moreover, the focus in these papers was on subspaces of proportional codimension, whereas we are mainly interested in subspaces with small codimension such as $m = n^2$ or $m = 2^n$ or even $m = \infty$.

Our motivation comes from the theory of information-based complexity (IBC). In IBC we often want to approximate the solution of a linear problem based on $n$ pieces of information about the unknown problem instance. We refer to [28, 29, 30] for a detailed exposition. It is usually assumed that some kind of oracle is available which grants us
this information at our request. We call this oracle $n$ times to get $n$ well-chosen pieces of information, trying to obtain optimal information about the problem instance. Often, however, this model is too idealistic. There might be no such oracle at our disposal and the information comes in randomly. We simply have to work with the information at hand. This is in fact a standard assumption in learning theory and uncertainty quantification, see [36]. It may also happen that an oracle is available but we simply do not know what to ask in order to obtain optimal information. In such a case, it seems natural to ask random questions. Both scenarios suggest the analysis of random information and the question how it compares to optimal information. For a survey of some classical results as well as new results see [13]. Here we study the case of $L_2$-approximation of vectors or functions from a Hilbert space.

More precisely, we consider the problem of recovering $x \in E$ from the data $N_n(x) \in \mathbb{R}^n$ which is obtained from an information mapping $N_n \in \mathbb{R}^{n \times m}$ and measure the error in the Euclidean norm. The power of the information mapping is given by its radius, which is the worst case error of the best recovery algorithm based on $N_n$, that is,

$$\text{rad}(N_n, E) = \inf_{\varphi: \mathbb{R}^n \to \mathbb{R}^m} \sup_{x \in E} \|\varphi(N_n(x)) - x\|_2.$$  

For problems of this type, it is known that the worst data is the zero data, resulting in

$$\text{rad}(N_n, E) = \sup_{x \in E \cap E_n} \|x\|_2,$$

where $E_n$ is the kernel of $N_n$, see [5, 28, 40]. Thus, if $N_n$ is a standard Gaussian matrix, we indeed arrive at the same problem as above. The radius of a random intersection is the worst case error of the best algorithm based on Gaussian random information, whereas the radius of the minimal intersection is the worst case error of the best algorithm based on optimal information. So the geometric questions above translate as follows: How good is random information? Is it comparable to the optimal information or is it much worse? The answers are the same. For instance, for polynomial decay $\sigma_j = j^{-\alpha}$, we have

$$\mathbb{E}[\text{rad}(N_n, E)] \asymp \begin{cases} 
\sigma_1 & \text{if } \alpha \leq 1/2, \\
\sigma_{n+1} & \text{if } \alpha > 1/2.
\end{cases}$$

As a matter of fact, the results for the radius of random information even hold when $m = \infty$, where our geometric interpretation fails. Namely, for any $\sigma \not\in \ell_2$, we obtain that $\mathbb{E}[\text{rad}(N_n, E)] = \sigma_1$ and random information is completely useless. For $\sigma \in \ell_2$ the expected radius of random information tends to zero with the same polynomial rate as the radius of optimal information. The proof of this upper bound is constructive. We present a least squares estimator based on random information that is almost as good as the optimal algorithm based on optimal information.

Remark 1. Using isomorphisms, our results can easily be transferred to any compact embedding $S$ of a Hilbert space $H$ into a separable $L_2$-space. That is, we may also consider
the problem of approximation an unknown function $f$ from the unit ball $E$ of $H$ in the $L_2$-norm. In this case, optimal information is given by the generalized Fourier coefficients

$$N_n^*(f) = \left( \langle f, b_i \rangle_2 \right)_{i=1 \ldots n}$$

where $b_i$ is the $L_2$-normalized eigenfunction belonging to the $i$th largest eigenvalue of the operator $S^*S$. The radius of optimal information $\sigma_{n+1}$ is the square-root of the $(n+1)$st largest eigenvalue. Random information, on the other hand, is given by

$$N_n(f) = \left( \sum_{j=1}^{\infty} g_{ij} \langle f, b_j \rangle_2 \right)_{i=1 \ldots n},$$

where the $g_{ij}$ are independent standard Gaussian variables. Equivalently, if $\sigma \in \ell_2$, we have

$$N_n(f) = \left( \langle f, h_i \rangle_H \right)_{i=1 \ldots n},$$

where the $h_i$ are iid Gaussian fields on $H$ whose correlation operator $C: H \to H$ is defined by $Cb_j = \sigma_j^2 b_j$. The results are the same. In particular, random information is (almost) as good as optimal information as long as $\sigma \in \ell_2$. An important case, which is often needed in approximation theory and complexity studies, are Sobolev embeddings, i.e., $H$ is a Sobolev space of functions that are defined on a bounded domain in $\mathbb{R}^d$. It is well known that then the singular values behave as $\sigma_k \approx k^{-\alpha} \ln^{-\beta}(k+1)$, where $\alpha$ and $\beta$ depend on the smoothness and the dimension $d$ and the condition $\sigma \in \ell_2$ means that the functions in $H$ are continuous, see also [4].

**Remark 2.** The phenomenon, that the results very much depend on whether $\sigma$ is square summable or not, is known from a related problem that was studied earlier in several papers. There $E$ is the unit ball of a reproducing kernel Hilbert space $H$. That is, $H \subseteq L_2(D)$ consists of functions on a common domain $D$ and function evaluation $f \mapsto f(x)$ is a continuous functional on $H$ for every $x \in D$. Again, the optimal linear information $N_n$ for the $L_2$-approximation problem is given by the generalized Fourier coefficients and has radius $\sigma_{n+1}$. This information might be hard to get and hence one might allow only function evaluations, i.e., information of the form

$$N_n(f) = \left( f(x_1), \ldots, f(x_n) \right), \quad x_i \in D.$$
The rest of the paper is organized as follows. In Section 2 we discuss the relation between the geometric problem and the IBC problem in more detail. We give general upper bounds (Theorem 3 and 4) and lower bounds (Theorem 5) for the radius of random information in terms of the sequence \( \sigma \) which hold with high probability. We derive the \( \ell_2 \)-dichotomy discussed above (Corollary 6) and apply the general theorems to sequences of polynomial decay (Corollary 7) and exponential decay (Corollary 9). The proofs are contained in Section 3. We add a final section about alternative approaches. We show an upper bound via the lower \( M^* \)-estimate and an elementary lower bound. These bounds are slightly weaker, but give a better insight into the geometric aspect of the problem.

2. Problem and results

We consider the ellipsoid

\[
\mathcal{E}_\sigma = \left\{ x \in \ell_2 : \sum_{j \in \mathbb{N}} \left( \frac{x_j}{\sigma_j} \right)^2 \leq 1 \right\}
\]

with semi-axes of lengths \( \sigma_1 \geq \sigma_2 \geq \cdots \geq 0 \).\(^1\) This is the unit ball of a Hilbert space, which we denote by \( H_\sigma \). We study the problem of recovering an unknown vector \( x \in \mathcal{E}_\sigma \) from \( n \) pieces of information, where we want to guarantee a small error in \( \ell_2 \). The information about \( x \in \mathcal{E}_\sigma \) is given by the outcome \( L_1(x), \ldots, L_n(x) \) of \( n \) linear functionals.\(^2\) The mapping \( N_n = (L_1, \ldots, L_n) \) is called the information mapping. A recovery algorithm is a mapping \( A_n : H_\sigma \to \ell_2 \) of the form \( A_n = \varphi \circ N_n \), where \( \varphi \) maps \( N_n(x) \) back to \( \ell_2 \). The worst case error of the algorithm is given by

\[
e(A_n) = \sup_{x \in \mathcal{E}_\sigma} \| A_n(x) - x \|_2.
\]

The quality of the information mapping is measured by its radius, which is the worst case error of the best recovery algorithm based on the information \( N_n \), i.e.,

\[
\text{rad}(N_n, \mathcal{E}_\sigma) := \inf_{\varphi: \text{im}(N_n) \to \ell_2} e(\varphi \circ N_n).
\]

Note that this is a linear problem over Hilbert spaces as described in [28, Section 4.2.3]. In particular, we have the relation

\[
\text{rad}(N_n, \mathcal{E}_\sigma) \geq \sup \{ \| x \|_2 : x \in \mathcal{E}_\sigma \text{ with } N_n(x) = 0 \}
\]

with equality for all bounded information mappings \( N_n \). We refer to [5, 28, 40]. It is easy to see that optimal information is given by the mapping

\[
N_n^* : H_\sigma \to \mathbb{R}^n, \quad N_n^*(x) = (x_1, \ldots, x_n),
\]

which satisfies

\[
\text{rad}(N_n^*, \mathcal{E}_\sigma) = \inf_{N_n \text{ linear}} \text{rad}(N_n, \mathcal{E}_\sigma) = \sigma_{n+1}.
\]

\(^1\)For convenience, we use the convention that \( a/0 = \infty \) for all \( a > 0 \) and \( 0/0 = 0 \).

\(^2\)Note that we also consider unbounded functionals, meaning that \( L_i(x) \in \mathbb{R} \cup \{ \text{NaN} \} \).
We want to compare this to the radius of random information, which is given by a random matrix $G_n \in \mathbb{R}^{n \times \infty}$ with independent standard Gaussian entries. That is, we study the random variable

$$R_n(\sigma) = \text{rad}(G_n, \mathcal{E}_\sigma).$$

Clearly, we always have $R_n(\sigma) \in [\sigma_{n+1}, \sigma_1]$. There are two alternative interpretations of the quantity $R_n(\sigma)$ if the sequence $\sigma$ is finite in the sense that $\sigma_j = 0$ for all $j > m$.

**Variant 1.** The quantity $R_n(\sigma)$ is the circumradius of the $(m - n)$-dimensional ellipsoid that is obtained by slicing the $m$-dimensional ellipsoid $\mathcal{E}_\sigma$ with a subspace $E_n$ that is uniformly distributed on the Grassmannian manifold $G_{m,m-n}$ of $n$-codimensional subspaces in $\mathbb{R}^m$ (equipped with the Haar probability measure). That is,

$$R_n(\sigma) = \text{rad}(\mathcal{E}_\sigma \cap E_n) = \sup \{ \|x\|_2 : x \in \mathcal{E}_\sigma \cap E_n \}.$$

This easily follows from the fact that the kernel of the matrix $G_n$ (when restricted to $\mathbb{R}^m$) is uniformly distributed on the Grassmannian.

**Variant 2.** Since the radius of information is invariant under a (component-wise) scaling of the information mapping, we also have

$$R_n(\sigma) = \text{rad}(U_n, \mathcal{E}_\sigma),$$

where $U_n$ is obtained from $G_n$ by erasing all but the first $m$ columns and then normalizing the rows in $\ell_2$. That is, in the finite-dimensional case, we may just as well study the quality of the information given by $n$ coordinates in random directions that are independent and uniformly distributed on the sphere $S^{m-1}$.

We want to give upper and lower bounds for $R_n(\sigma)$ which hold with high probability. Clearly, upper bounds are stronger if they are proven for the error $e(A_n)$ of a concrete algorithm $A_n = \varphi \circ G_n$. Here, we consider a least squares estimator. We define $\psi$ as the restriction of $G_n$ to $\mathbb{R}^k$, where $k \leq n$ is of order $n$ and will be specified later. Note that we identify $\mathbb{R}^k$ with the space of all $x \in \ell_2$ such that $x_j = 0$ for all $j > k$. We then take $\varphi = \psi^+$ where $\psi^+$ is the Moore-Penrose inverse of $\psi \in \mathbb{R}^{n \times k}$. This algorithm satisfies, almost surely, that $A_n(x) = x$ for all $x \in \mathbb{R}^k$. Moreover, one may write

$$A_n(x) = \arg \min_{y \in \mathbb{R}^k} \|G_n(x - y)\|_2, \quad x \in \mathbb{R}^m.$$

Let us now present the results. We note that it is not an essential assumption that the vector of semi-axes is non-increasing or even that the semi-axes are aligned with the standard basis of the Euclidean space. It simply eases the notation.

**Theorem 3.** Let $\sigma \in \ell_2$ be non-increasing and let $n \in \mathbb{N}$. Then the following estimate holds with probability at least $1 - 2 \exp(-n/100)$,

$$R_n(\sigma) \leq e(A_n) \leq \frac{221}{\sqrt{n}} \left( \sum_{j \geq \lfloor n/4 \rfloor} \sigma_j^2 \right)^{1/2}.$$
This estimate turns out to be useful for sequences $\sigma$ of polynomial decay. For sequences of exponential decay, we add a second upper bound. It is better suited for such sequences since the starting index $\lfloor n/4 \rfloor$ of the sum in the upper bound is replaced by $n$.

**Theorem 4.** Let $\sigma \in \ell_2$ be non-increasing. Then, for all $n \in \mathbb{N}$ and $c, s \in [1, \infty)$ we have

$$\mathbb{P} \left[ R_n(\sigma) \leq e(A_n) \leq 14sn \left( \sum_{j>n} \sigma_j^2 \right)^{1/2} \right] \geq 1 - e^{-c^2n} - \frac{c\sqrt{2e}}{s}.$$

On the other hand, we obtain the following lower bound for $R_n(\sigma)$. This lower bound is even satisfied for the smaller quantity $R_n(1) := \sup \{ x_1 : x \in E_\sigma, G_n(x) = 0 \}$, which corresponds to the difficulty of the easier problem of recovering just the first coordinate of $x \in E_\sigma$ from Gaussian information.

**Theorem 5.** Let $\sigma \in \ell_2$ be non-increasing, $\varepsilon \in (0, 1)$ and $n, k \in \mathbb{N}$ with

$$\sum_{j>k} \sigma_j^2 \geq \frac{3n\sigma_k^2}{\varepsilon^2}.$$

Then it holds with probability at least $1 - 5 \exp(-n/64)$ that

$$R_n(\sigma) \geq R_n^{(1)}(\sigma) \geq \sigma_k(1 - \varepsilon).$$

As a consequence of these theorems, we obtain that random information is useful if and only if $\sigma$ is square summable.

**Corollary 6.** If $\sigma \notin \ell_2$, then $R_n(\sigma) = \sigma_1$ holds almost surely for all $n \in \mathbb{N}$. On the other hand, if $\sigma \in \ell_2$, then

$$\lim_{n \to \infty} \sqrt{n} \mathbb{E}[R_n(\sigma)] = 0.$$

Before we present the proofs of our main results, let us provide some of the results on the expected radius that follow from our main results for special sequences. For sequences $(a_n)$ and $(b_n)$ we write $a_n \asymp b_n$ if there is a constant $C > 0$ such that $a_n \leq C b_n$ for all $n \in \mathbb{N}$. We write $a_n \asymp b_n$ in the case that both $a_n \leq b_n$ and $b_n \leq a_n$. We start with the case of polynomial decay.

**Corollary 7.** Let $\sigma$ be non-increasing with $\sigma_n \asymp n^{-\alpha} \ln^{-\beta}(n + 1)$ for some $\alpha \geq 0$ and $\beta \in \mathbb{R}$ (where $\beta \geq 0$ for $\alpha = 0$). Then

$$\mathbb{E}[R_n(\sigma)] \asymp \begin{cases} 
\sigma_1 & \text{if } \alpha < 1/2 \text{ or } \beta \leq \alpha = 1/2, \\
\sigma_{n+1}\sqrt{\ln(n+1)} & \text{if } \beta > \alpha = 1/2, \\
\sigma_{n+1} & \text{if } \alpha > 1/2.
\end{cases}$$

Similar results can be derived for the finite-dimensional case (i.e., $\sigma_j = 0$ for $j > m$) under the condition that $m$ is large enough in comparison to $n$. Details can be found in
the thesis [15, Corollaries 4.31–4.33]. This means that random information is just as good as optimal information if the singular values decay with a polynomial rate greater than 1/2. The size of a typical intersection ellipsoid is comparable to the size of the smallest intersection. On the other hand, if the singular values decay too slowly, random information is useless. A typical intersection ellipsoid is almost as large as the largest. There is also an intermediate case where random information is worse than optimal information, but only slightly.

Remark 8. The case $\sigma_n \asymp n^{-\alpha} \ln^{-\beta}(n+1)$ with $\alpha > 1/2$ can be extended to $\sigma_n \asymp n^{-\alpha}\varphi(n)$ for any slowly varying function $\varphi$. Also in this case, we get $\mathbb{E}[R_n(\sigma)] \asymp \sigma_{n+1}$.

We also discuss sequences of exponential decay. We have seen that $\mathbb{E}[R_n(\sigma)] \asymp \sigma_{n+1}$ holds for sequences with sufficiently fast polynomial decay. It remains open whether the same holds for sequences of exponential decay. With Theorem 4 we obtain the following.

**Corollary 9.** Assume that $\sigma_n \asymp a^n$ for some $a \in (0,1)$. Then

$$a^n \leq \mathbb{E}[R_n(\sigma)] \leq n^2 a^n.$$  

Despite the gap, this result is stronger than the result for polynomial decay if considered from the complexity point of view. Corollary 7 states that there is a constant $c \in (0, \infty)$ such that $cn$ pieces of random information are at least as good as $n$ pieces of optimal information. Corollary 9 states that there is a constant $c \in (0, \infty)$ such that $n + c \ln n$ pieces of random information are at least as good as $n$ pieces of optimal information.

3. The Proofs

In the proofs we will use the following tools:

- exponential estimates for sums of chi-squared random variables,
- Gordon’s min-max theorem for Gaussian processes,
- estimates for the extreme singular values of (structured) Gaussian matrices.

Before we enter the proofs, we recall and extend some of our notation. Let $\sigma = (\sigma_j)_{j=1}^{\infty}$ be a non-increasing sequence of non-negative numbers. We consider the Hilbert space

$$H_\sigma = \left\{ x \in \ell_2 : \sum_{j \in \mathbb{N}} \left( \frac{x_j}{\sigma_j} \right)^2 < \infty \right\},$$

using the convention that $a/0 = \infty$ for $a > 0$ and $0/0 = 0$, with inner product

$$\langle x, y \rangle_\sigma = \sum_{j \in \mathbb{N}} \frac{x_j y_j}{\sigma_j^2}.$$  

The unit ball of $H_\sigma$ is denoted by $E_\sigma$. The matrix $G_n = (g_{ij})_{1 \leq i \leq n,j \in \mathbb{N}}$ for $n \in \mathbb{N}$ has independent standard Gaussian entries. We want to study the distribution of the random variable

$$R_n(\sigma) = \sup \{ \|x\|_2 : x \in E_\sigma, G_n x = 0 \}.$$
Of course, the equation $G_n x = 0$ requires that the series $\sum_{j=1}^{\infty} g_{ij} x_j$ converges. For index sets $I \subseteq \mathbb{N}$ and $J \subseteq \mathbb{N}$, we consider the (structured) Gaussian $I \times J$-matrices

$$G_{I,J} = (g_{ij})_{i \in I, j \in J} \quad \text{and} \quad \Sigma_{I,J} = (\sigma_{ij})_{i \in I, j \in J}.$$  

Note that $G_n = G_{[n],[n]}$ and $G_{n,m} = G_{[n],[m]}$, where $[n]$ denotes the set of integers from 1 to $n$. We consider $H_J(\sigma) = \{ x \in H_\sigma : x_j = 0 \text{ for all } j \in \mathbb{N} \setminus J \}$ as a closed subspace of the Hilbert space $H_\sigma$ and denote its unit ball by $E^J_\sigma$. The projection of $x \in H_\sigma$ onto $H_J(\sigma)$ is denoted by $x_J$.

A crucial role in our proofs is played by estimates for the extreme singular values of random matrices. We recall some basic facts about singular values. Let $A$ be a real $r \times k$-matrix, where we allow that $r = \infty$ or $k = \infty$ provided that $A$ describes a compact operator from $\mathbb{R}^k$ to $\mathbb{R}^r$ (with Euclidean norm). For every $j \leq k$, the $j$th singular value $s_j(A)$ of this matrix can be defined as the square-root of the $j$th largest eigenvalue of the symmetric matrix $A^\top A$, which describes a positive operator on $\mathbb{R}^k$. Note that $s_j(A) = s_j(A^\top)$ if we have $j \leq \min\{r,k\}$. Our interest lies in the extreme singular values of $A$.

The largest singular value of $A$ is given by

$$s_1(A) = \sup_{x \in \mathbb{R}^k \setminus \{0\}} \frac{\|Ax\|_2}{\|x\|_2} = \|A : \mathbb{R}^k \to \mathbb{R}^r\|.$$  

This number is also called the spectral norm of $A$. The smallest singular value is given by

$$s_k(A) = \inf_{x \in \mathbb{R}^k \setminus \{0\}} \frac{\|Ax\|_2}{\|x\|_2}.$$  

Clearly, we have $s_k(A) = 0$ whenever $k > r$. If $r \leq k$, it also makes sense to talk about the $r$th singular value of $A$. This number equals the radius of the largest Euclidean ball that is contained in the image of the unit ball of $\mathbb{R}^k$ under $A$, that is

$$s_r(A) = \sup \{ \rho \geq 0 : \rho \mathbb{B}^k_2 \subseteq A(\mathbb{B}^k_2) \},$$  

where $\mathbb{B}^k_2$ denotes the unit ball in $k$-dimensional Euclidean space. These extreme singular values are also defined for noncompact operators $A$, where $A$ is restricted to its domain if necessary. We now turn to the proofs of our results.

3.1. The Upper Bound. We start with an almost sure upper bound for the worst case error of the least squares algorithm $A_n$ from (1). The upper bound is given in terms of the extreme singular values of the corresponding (structured) Gaussian matrices. The spectral statistics of random matrices, in particular the behavior of the least and largest singular value, attracted considerable attention over the years and we refer the reader to, e.g., [1, 3, 6, 22, 33, 34, 38, 41] and the references cited therein.
Proposition 10. Let $\sigma \in \ell_2$ be non-increasing and let $k \leq n$. If $G_{n,k} \in \mathbb{R}^{n \times k}$ has full rank, then

$$e(A_n) \leq \sigma_{k+1} + \frac{s_1(\Sigma_{[n],N\setminus[k]})}{s_k(G_{n,k})}.$$ 

Proof. We first note that $s_k(G_{n,k}) > 0$ if $G_{n,k}$ has full rank. Let $x \in H_\sigma$ with $\|x\|_\sigma \leq 1$. We recall that $A_n(x[k]) = x[k]$. This yields

$$\|x - A_n x\|_2 \leq \|x - x[k]\|_2 + \|A_n(x - x[k])\|_2 \leq \sigma_{k+1} + \|G_{n,k}^+ G_n(x - x[k])\|_2 \leq \sigma_{k+1} + \|G_{n,k}^+ : \mathbb{R}^n \rightarrow \mathbb{R}^k\| \cdot \|G_n : H_{N\setminus[k]}(\sigma) \rightarrow \mathbb{R}^n\|.$$ 

The norm of $G_{n,k}^+$ is the inverse of the $k$th largest (and therefore the smallest) singular value of the matrix $G_{n,k}$. The norm of $G_n$ is the largest singular value of the matrix

$$\Sigma = (\sigma_j g_{ij})_{1 \leq i \leq n, j > k} \in \mathbb{R}^{n \times \infty}.$$ 

To see this, note that $G_n = \Sigma \Delta$ on $H_{N\setminus[k]}(\sigma)$, where the mapping $\Delta : H_{N\setminus[k]}(\sigma) \rightarrow \ell_2$ with $\Delta y = (y_j/\sigma_j)_{j > k}$ is an isomorphism. This yields the stated inequality. \qed

The task now is to bound the $k$th singular value of the Gaussian matrix $G_{n,k}$ from below and the largest singular value of the structured Gaussian matrix $\Sigma_{[n],N\setminus[k]}$ from above. We start with the largest singular value of the latter. Let us remark that the question for the order of the expected value of the largest singular value of a structured Gaussian random matrix has recently been settled by Latała, Van Handel, and Youssef [19] (see also [3, 7, 12, 18, 41] for earlier work in this direction). The result we shall use here is due to Bandeira and Van Handel [3].

Lemma 11. Let $\sigma \in \ell_2$ be non-increasing. For every $c \in [1, \infty)$ and $n, k \in \mathbb{N}$, we have

$$\mathbb{P} \left[ s_1(\Sigma_{[n],N\setminus[k]}) \geq \frac{3}{2} \sqrt{\sum_{j > k} \sigma_j^2 + 11c \sigma_{k+1} \sqrt{n}} \right] \leq e^{-c^2 n}.$$ 

Proof. Without loss of generality, we may assume that $\sigma_{k+1} \neq 0$. Let us first consider the finite matrix

$$A_m = \Sigma_{[n],m+k\setminus[k]} \in \mathbb{R}^{n \times m} \quad \text{for} \quad m \in \mathbb{N},$$ 

and set

$$C_m = \frac{3}{2} \left( \sum_{j=k+1}^{k+m} \sigma_j^2 \right)^{1/2} + \frac{103c}{10} \sigma_{k+1} \sqrt{n},$$

where $A$ and $C$ denote their infinite dimensional variants. It is proven in [3, Corollary 3.11] that, for every $t \geq 0$ (and $\varepsilon = 1/2$), we have

$$\mathbb{P} \left[ s_1(A_m) \geq \frac{3}{2} \left( \left( \sum_{j=k+1}^{k+m} \sigma_j^2 \right)^{1/2} + \sigma_{k+1} \sqrt{n} + \frac{5 \sqrt{\ln(n)}}{ \sqrt{\ln(3/2)} \sigma_{k+1}} \right) + t \right] \leq e^{-t^2/2\sigma_{k+1}^2}.$$ 

By setting $t = \sqrt{2c} \sigma_{k+1} \sqrt{n}$, it follows that

$$\mathbb{P}[s_1(A_m) \geq C_m] \leq e^{-c^2 n}.$$
Turning to the infinite dimensional case, we note that we have $s_1(A) > C$ if and only if there is some $m \in \mathbb{N}$ such that $s_1(A_m) > C$. This yields

$$
\mathbb{P}[s_1(A) > C] = \mathbb{P}[\exists m \in \mathbb{N}: s_1(A_m) > C] = \lim_{m \to \infty} \mathbb{P}[s_1(A_m) > C] \leq e^{-c^2n}
$$

since $s_1(A_m)$ is increasing in $m$ and $C \geq C_m$. □

Together with Proposition 10 this yields that the estimate

$$
e(A_n) \leq \sigma_{k+1} + \frac{3}{2} \sqrt{\sum_{j > k} \sigma_j^2} + 11c \sigma_{k+1} \sqrt{n}
$$

holds with probability at least $1 - e^{-c^2n}$ for all $k \leq n$ and $c \geq 1$. It remains to bound the $k$-th singular value of the Gaussian matrix $G_{n,k}$ from below. It is known from [35, Theorem 1.1] that this number typically is of order $\sqrt{n} - \sqrt{k-1}$ for all $n \in \mathbb{N}$ and $k \leq n$. To exploit our upper bound to full extend, the number $k \leq n$ may be chosen such that the right-hand side of (2) becomes minimal. We realize that the term $1/s_k(G_{n,k})$ increases with $k$, whereas all remaining terms decrease with $k$. However, the inverse singular number achieves its minimal order $n^{-1/2}$ already for $k = cn$ with some $c \in (0, 1)$.

If $\sigma$ does not decay extremely fast, this does not lead to a loss regarding the other terms of (2). For instance, we may choose $k = \lfloor n/2 \rfloor$ and use the following special case of [6, Theorem II.13].

**Lemma 12.** Let $n \in \mathbb{N}$ and $k = \lfloor n/2 \rfloor$. Then

$$
\mathbb{P}\left[s_k(G_{n,k}) \leq \sqrt{n}/7\right] \leq e^{-n/100}.
$$

**Proof.** It is shown in [6, Theorem II.13] that, for all $k \leq n$ and $t > 0$, we have

$$
\mathbb{P}\left[s_k(G_{n,k}) \leq \sqrt{n} \left(1 - \sqrt{k/n} - t\right)\right] \leq e^{-n t^2/2}.
$$

The statement follows by putting $k = \lfloor n/2 \rfloor$ and $t^{-1} = \sqrt{50}$. □

If $\sigma$ decays very fast, $k = \lfloor n/2 \rfloor$ might not be the best choice. The term $\sigma_{k+1}$ in estimate (2) may be much smaller for $k = n$ than for $k = \lfloor n/2 \rfloor$. It is better to choose $k = n$. In this case, the inverse singular number is of order $\sqrt{n}$. We state a result of [37, Theorem 1.2].

**Lemma 13.** Let $n \in \mathbb{N}$ and $t \geq 0$. Then

$$
\mathbb{P}\left[s_n(G_{n,n}) \leq \frac{t}{\sqrt{n}}\right] \leq t \sqrt{2e}.
$$

This leads to the proof of Theorem 3 and 4 as presented in the introduction.

**Proof of Theorem 3 and 4.** To prove the first statement, let $k = \lfloor n/2 \rfloor$. We combine Lemma 12 and Lemma 11 for $c = 1$ with Proposition 10 and obtain that

$$
e(A_n) \leq 78 \sigma_{k+1} + \frac{21}{2\sqrt{n}} \left(\sum_{j > \lfloor n/2 \rfloor} \sigma_j^2\right)^{1/2}
$$
with probability at least \(1 - e^{-n} - e^{-n/100}\). The statement follows if we take into account that

\[
\sigma_{k+1}^2 \leq \frac{4}{n} \sum_{j=\lfloor n/2 \rfloor}^{\lfloor n/4 \rfloor} \sigma_j^2.
\]

To prove the second statement, we set \(t = c/s\). We combine Lemma 13 and Lemma 11 with Proposition 10 and obtain that

\[
e(A_n) \leq \sigma_{n+1} + \frac{1}{t} \left( \frac{3\sqrt{n}}{2} \left( \sum_{j>n} \sigma_j^2 \right)^{1/2} + 11 cn \sigma_{n+1} \right)
\]

with probability at least \(1 - e^{-c^2 n - t\sqrt{2e}}\). The rough estimates \(\sigma_{n+1}^2 \leq \sum_{j>n} \sigma_j^2\) and \(3\sqrt{n}/2 \leq 2cn\) and \(1 \leq sn\) yield the statement. □

3.2. The Lower Bound. We want to give lower bounds on the radius of information

\[
\mathcal{R}_n(\sigma) = \sup \left\{ \|x\|_2 : x \in \mathcal{E}_\sigma, G_n x = 0 \right\}
\]

which corresponds to the difficulty of recovering an unknown element \(x \in \mathcal{E}_\sigma\) from the information \(G_n x\) in \(\ell_2\). In fact, our lower bounds already hold for the smaller quantity

\[
\mathcal{R}_n^{(k)}(\sigma) = \sup \left\{ |x_k| : x \in \mathcal{E}_\sigma, G_n x = 0 \right\}
\]

which corresponds to the difficulty of recovering just the \(k\)th coordinate of \(x\). Again, we start with an almost sure estimate.

**Proposition 14.** Let \(\sigma \in \ell_2\) be non-increasing. For all \(n, k \in \mathbb{N}\) with \(\sigma_k \neq 0\) we have almost surely

\[
\mathcal{R}_n^{(k)}(\sigma) \geq \sigma_k \left( 1 - \frac{\|(g_{ik})_{i=1}^n\|_2}{\sigma_k^{-1} s_n (\Sigma_{\lfloor n/2 \rfloor}) + \|(g_{ik})_{i=1}^n\|} \right).\]

**Proof.** We may assume that the operator \(G_n : H_{\mathbb{N}\setminus\{k\}}(\sigma) \to \mathbb{R}^n\) is onto and that \(g = (g_{ik})_{i=1}^n\) is nonzero since these events occur with probability 1. Observe that

\[
G_n (\mathcal{E}_{\sigma_{\mathbb{N}\setminus\{k\}}}^n) = \Sigma_{\lfloor n/2 \rfloor} (\sigma_{\mathbb{N}\setminus\{k\}} (\mathbb{B}_2),
\]

where \(\mathbb{B}_2\) is the unit ball of \(\ell_2\). In particular, this implies

\[
s_n := s_n (\Sigma_{\lfloor n/2 \rfloor}) = \sup \left\{ \varrho \geq 0 : \varrho \mathbb{B}_2^n \subseteq \Sigma_{\lfloor n/2 \rfloor} (\mathbb{B}_2) \right\} > 0.
\]

Let \(e^{(k)}\) be the \(k\)-th standard unit vector in \(\ell_2\). Then we have

\[
\|e^{(k)}\|_2 = 1 \quad \text{and} \quad \|e^{(k)}\|_\sigma = \sigma_k^{-1}.
\]

Since the image of \(\mathcal{E}_{\sigma_{\mathbb{N}\setminus\{k\}}}^n\) under \(G_n\) contains a Euclidean ball of radius \(s_n\), we find an element \(\bar{y}\) of \(\mathcal{E}_{\sigma_{\mathbb{N}\setminus\{k\}}}^n\) such that

\[
G_n \bar{y} = \frac{s_n \cdot G_n e^{(k)}}{\|G_n e^{(k)}\|_2}.
\]
For $y = s_n^{-1} \| G_n e^{(k)} \|_2 \cdot \bar{y}$, we obtain $G_n y = G_n e^{(k)} = g$ and

$$ \| y \|_\sigma = s_n^{-1} \| G_n e^{(k)} \|_2 \| \bar{y} \|_\sigma \leq s_n^{-1} \| g \|_2. $$

Then the vector $z := e^{(k)} - y$ satisfies $G_n z = 0$ and $z_k = 1$ as well as

$$ \| z \|_\sigma \leq \| e^{(k)} \|_\sigma + \| y \|_\sigma \leq \sigma_k^{-1} + s_n^{-1} \| g \|_2. $$

The statement is obtained by

$$ \mathcal{R}_n^{(k)}(\sigma) \geq \frac{|z_k|}{\| z \|_\sigma} = \frac{1}{\| z \|_\sigma}. \quad \square $$

It remains to bound the $n$th singular value of $\Sigma_{[n] \backslash \{k\}}$ and the norm of the Gaussian vector $(g_{ik})_{i=1}^n$ with high probability. For both estimates, we use the following concentration result for chi-square random variables going back to Laurent and Massart [20, Lemma 1]. Alternatively, one could use the concentration of Gaussian random vectors in Banach spaces (see, e.g., [21, Proposition 2.18]).

**Lemma 15.** For $1 \leq j \leq m$, let $u_j$ be independent centered Gaussian variables with variance $a_j$. Then, for any $\delta \in (0, 1]$, we have

$$ \mathbb{P} \left[ \sum_{j=1}^m u_j^2 \leq (1 - \delta) \sum_{j=1}^m a_j \right] \leq \exp \left( -\frac{\delta^2 \| a \|_1}{4 \| a \|_\infty} \right), $$

$$ \mathbb{P} \left[ \sum_{j=1}^m u_j^2 \geq (1 + \delta) \sum_{j=1}^m a_j \right] \leq \exp \left( -\frac{\delta^2 \| a \|_1}{16 \| a \|_\infty} \right). $$

**Proof.** The lemma [20, Lemma 1] states that, for all $t > 0$, we have

$$ \mathbb{P} \left[ \sum_{j=1}^m u_j^2 \leq \| a \|_1 - 2 \| a \|_2 t \right] \leq e^{-t^2}, $$

$$ \mathbb{P} \left[ \sum_{j=1}^m u_j^2 \geq \| a \|_1 + 2 \| a \|_2 t + 2 \| a \|_\infty t^2 \right] \leq e^{-t^2}. $$

The formulation of Lemma 15 follows if we put

$$ t = \frac{\delta \| a \|_1}{2 \| a \|_2}, \quad \text{respectively} \quad t = \min \left\{ \frac{\delta \| a \|_1}{4 \| a \|_2}, \sqrt{\frac{\delta \| a \|_1}{4 \| a \|_\infty}} \right\}. $$

The desired probability estimate then follows by using $\| a \|_2^2 \leq \| a \|_1 \| a \|_\infty$. \quad \square

In particular, the norm of the Gaussian vector $(g_{ik})_{i=1}^n$ concentrates around $\sqrt{n}$. To bound the $n$th singular value of $\Sigma_{[n] \backslash \{k\}}$ we shall use Gordon’s min-max theorem. Let us state Gordon’s theorem [11, Lemma 3.1] in a form which can be found in [39].

**Lemma 16 (Gordon’s min-max theorem).** Let $n, m \in \mathbb{N}$ and let $S_1 \subseteq \mathbb{R}^n$, $S_2 \subseteq \mathbb{R}^m$ be compact sets. Assume that $\psi : S_1 \times S_2 \to \mathbb{R}$ is a continuous mapping. Let $G \in \mathbb{R}^{m \times n}$,
We want to apply Gordon’s theorem for the matrix $G$ in $\mathbb{R}^{m \times n}$. Note that the statement is trivial if $n = m$. We have the identity $A = DG$ where $G \in \mathbb{R}^{m \times n}$ is a random matrix with independent standard Gaussian entries and $D \in \mathbb{R}^{m \times m}$ is the diagonal matrix

$$D = \text{diag}(\sqrt{a_1}, \ldots, \sqrt{a_m}).$$

We want to apply Gordon’s theorem for the matrix $G$ and $\psi = 0$, where $S_1$ is the sphere in $\mathbb{R}^n$ and $S_2$ is the image of the sphere in $\mathbb{R}^m$ under $D$. Then we have

$$\Phi_1(G) := \min_{x \in S_1} \max_{y \in S_2} \left( \langle y, Gx \rangle + \psi(x, y) \right),$$

$$\Phi_2(u, v) := \min_{x \in S_1} \max_{y \in S_2} \left( \|x\|_2 \langle u, y \rangle + \|y\|_2 \langle v, x \rangle + \psi(x, y) \right).$$

Then, for all $c \in \mathbb{R}$, we have

$$\mathbb{P}[\Phi_1(G) < c] \leq 2 \mathbb{P}[\Phi_2(u, v) \leq c].$$

This yields the following lower bound on the smallest singular value of structured Gaussian matrices. Note that this is a generalization of Lemma 12.

**Lemma 17.** Let $A \in \mathbb{R}^{m \times n}$ be a random matrix whose entries $a_{ij}$ are centered Gaussian variables with variance $a_i$ for all $i \leq m$ and $j \leq n$. Then, for all $\delta \in (0, 1)$, we have

$$\mathbb{P} \left[ s_n(A) \leq \sqrt{(1 - \delta)\|a\|_1} - \sqrt{(1 + \delta)n \|a\|_\infty} \right] \leq 4 \exp \left( -\frac{\delta^2}{16} \min \left\{ n, \frac{\|a\|_1}{\|a\|_\infty} \right\} \right).$$

**Proof.** Note that the statement is trivial if $m \leq n$. We may assume that the $a_i$ are positive since an additional row of zeros does neither change $s_n(A)$ nor the norms of the vector $a$. We have the identity $A = DG$ where $G \in \mathbb{R}^{m \times n}$ is a random matrix with independent standard Gaussian entries and $D \in \mathbb{R}^{m \times m}$ is the diagonal matrix

$$D = \text{diag}(\sqrt{a_1}, \ldots, \sqrt{a_m}).$$

We may assume that the $a_i$ are positive since an additional row of zeros does neither change $s_n(A)$ nor the norms of the vector $a$. We have the identity $A = DG$ where $G \in \mathbb{R}^{m \times n}$ is a random matrix with independent standard Gaussian entries and $D \in \mathbb{R}^{m \times m}$ is the diagonal matrix

$$D = \text{diag}(\sqrt{a_1}, \ldots, \sqrt{a_m}).$$

We want to apply Gordon’s theorem for the matrix $G$ and $\psi = 0$, where $S_1$ is the sphere in $\mathbb{R}^n$ and $S_2$ is the image of the sphere in $\mathbb{R}^m$ under $D$. Then we have

$$\Phi_1(G) = \min_{x \in S_1} \max_{y \in S_2} \langle y, Gx \rangle = \min_{\|x\|_2 = 1} \max_{\|z\|_2 = 1} \langle Dz, Gx \rangle = \min_{\|x\|_2 = 1} \max_{\|z\|_2 = 1} \langle z, Ax \rangle = \min_{\|x\|_2 = 1} \|Ax\|_2 = s_n(A).$$

On the other hand, if $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ are standard Gaussian vectors, the choice of $z = Du/\|Du\|_2$ yields

$$\Phi_2(u, v) = \min_{x \in S_1} \max_{y \in S_2} \left( \langle u, y \rangle + \|y\|_2 \langle v, x \rangle \right) = \min_{\|x\|_2 = 1} \max_{\|z\|_2 = 1} \left( \langle u, Dz \rangle + \|Dz\|_2 \langle v, x \rangle \right) \geq \min_{\|x\|_2 = 1} \left( \|Du\|_2 + \frac{\|D^2 u\|_2}{\|Du\|_2} \langle v, x \rangle \right) = \|Du\|_2 - \frac{\|D^2 u\|_2}{\|Du\|_2} \|v\|_2 \geq \|Du\|_2 - \sqrt{\|a\|_\infty} \|v\|_2.$$
To obtain the statement of our lemma, we set \( c = \sqrt{(1 - \delta)\|a\|_1 - (1 + \delta)n\|a\|_\infty} \). By Lemma 15, we have
\[
P\left[ \|Du\|_2 \leq \sqrt{(1 - \delta)\|a\|_1} \right] \leq \exp\left( -\frac{\delta^2\|a\|_1}{4\|a\|_\infty} \right)
\]
and
\[
P\left[ \|v\|_2 \geq \sqrt{(1 + \delta)n} \right] \leq \exp\left( -\frac{\delta^2n}{16} \right).
\]
Now the statement is obtained from a union bound.

We need the statement of Lemma 17 for matrices with infinitely many rows, which is obtained from a simple limit argument.

**Lemma 18.** Formula (3) also holds for \( m = \infty \) provided that \( a \in \ell_1 \).

**Proof.** Again, we may assume that \( a \) is strictly positive. For \( m \in \mathbb{N} \) let \( A_m \) be the sub-matrix consisting of the first \( m \) rows of \( A \) and let \( a^{(m)} \) be the sub-vector consisting of the first \( m \) entries of \( a \). We use the notation
\[
c_m(\delta) = \sqrt{(1 - \delta)\|a^{(m)}\|_1} - \sqrt{(1 + \delta)n\|a^{(m)}\|_\infty},
\]
\[
p_m(\delta) = 4\exp\left( -\frac{\delta^2}{16} \min\left\{ n, \frac{\|a^{(m)}\|_1}{\|a^{(m)}\|_\infty} \right\} \right),
\]
where \( c(\delta) \) and \( p(\delta) \) correspond to the case \( m = \infty \). For any \( \varepsilon > 0 \) with \( \varepsilon < \delta/2 \) we can choose \( m \geq n \) such that \( c(\delta) \leq c_m(\delta - \varepsilon) \) and \( p_m(\delta - \varepsilon) \leq p(\delta - 2\varepsilon) \). Note that we have \( s_n(A) \geq s_n(A_m) \) and thus
\[
P \left[ s_n(A) \leq c(\delta) \right] \leq P \left[ s_n(A_m) \leq c(\delta) \right] \leq P \left[ s_n(A_m) \leq c_m(\delta - \varepsilon) \right] \leq p_m(\delta - \varepsilon) \leq p(\delta - 2\varepsilon).
\]
Letting \( \varepsilon \) tend to zero yields the statement.

We arrive at our main lower bound.

**Lemma 19.** Let \( \sigma \in \ell_2 \) be non-increasing and let \( n, k \in \mathbb{N} \) be such that \( \sigma_k \neq 0 \). Define
\[
C_k := C_k(\sigma) = \sigma_k^{-2} \sum_{j > k} \sigma_j^2.
\]
Then, for all \( \delta \in (0, 1) \), we have
\[
P \left[ \mathcal{R}^{(k)}_n(\sigma) \leq \sigma_k \left( 1 - \sqrt{(1 + \delta)n \over (1 - \delta)C_k} \right) \right] \leq 5 \exp\left( -(\delta/4)^2 \min\{n, C_k\}\right).
\]

**Proof.** First note that, in the setting of Proposition 14, the matrix \( \Sigma_{[n],N\setminus[k]}^T \) and the vector \( (g_{ik})_{i=1}^n \) are independent. Lemma 15 and Lemma 18 yield
\[
\| (g_{ik})_{i=1}^n \|_2 \leq \sqrt{1 + \delta} \sqrt{n} \quad \text{and} \quad \sqrt{1 - \delta} \sigma_k \sqrt{C_k} - \sqrt{1 + \delta} \sigma_{k+1} \sqrt{n}
\]
\[
s_n \left( \Sigma_{[n],N\setminus[k]}^T \right) \geq \sqrt{1 - \delta} \sigma_k \sqrt{C_k} - \sqrt{1 + \delta} \sigma_{k+1} \sqrt{n}
\]
with probability at least \(1 - 5\exp(-(\delta/4)^2 \min\{n, C_k\})\). Note that we have
\[
s_n \left(\Sigma_{[n],N\setminus\{k\}}\right) = s_n \left(\Sigma_{[n],N\setminus\{k\}}^T\right) \geq s_n \left(\Sigma_{[n],N\setminus\{k\}}\right)
\]
since erasing rows can only shrink the smallest singular value. In this case, we have
\[
\frac{\sigma_k^{-1} s_n \left(\Sigma_{[n],N\setminus\{k\}}\right)}{\|g_{ik}\|_2} + \|g_{ik}\|_2 \leq \frac{\sqrt{1 + \delta \sqrt{n}}}{\sqrt{1 - \delta \sqrt{C_k}} - (\sigma_{k+1}/\sigma_k)\sqrt{1 + \delta \sqrt{n}} + \sqrt{1 + \delta \sqrt{n}}}
\leq \frac{\sqrt{1 + \delta \sqrt{n}}}{1 - \delta \sqrt{C_k}}.
\]
Now the statement is obtained from Proposition 14. \(\square\)

This also proves Theorem 5 as stated in the previous section.

**Proof of Theorem 5.** We simply apply Lemma 19 and choose \(\delta = 1/2\) to obtain the desired lower bound for \(\mathcal{R}_n^{(k)}(\sigma)\). Since the lower bound is independent of \(\sigma_1, \ldots, \sigma_{k-1}\), we actually get the same lower bound for \(\mathcal{R}_n^{(k)}(\tilde{\sigma})\), where \(\tilde{\sigma}\) is obtained from \(\sigma\) by replacing the first \(k - 1\) coordinates with \(\sigma_k\). To see that the lower bound also holds for \(\mathcal{R}_n^{(1)}(\sigma)\) (as opposed to \(\mathcal{R}_n^{(k)}(\sigma)\)), we only need to realize that \(\mathcal{R}_n^{(1)}(\sigma) \geq \mathcal{R}_n^{(1)}(\tilde{\sigma})\), where \(\mathcal{R}_n^{(1)}(\tilde{\sigma})\) clearly has the same distribution as \(\mathcal{R}_n^{(k)}(\tilde{\sigma})\). \(\square\)

### 3.3. Corollaries

In order to optimize the lower bound of Theorem 5, we may choose \(k \in \mathbb{N}\) such that the right-hand side of our lower bound becomes maximal. If the Euclidean norm of \(\sigma\) is large, we simply choose \(k = 1\). Taking into account that \(\mathcal{R}_n(\sigma)\) is decreasing in \(n\), we immediately arrive at the following result.

**Lemma 20.** Let \(\sigma \in \ell_2\) be a nonincreasing sequence of nonnegative numbers and let
\[
n_0 = \left\lfloor \frac{\varepsilon^2}{3\sigma_1^2} \sum_{j=2}^{\infty} \sigma_j^2 \right\rfloor, \quad \varepsilon \in (0, 1).
\]
Then \(\mathcal{R}_n(\sigma) \geq \sigma_1 (1 - \varepsilon)\) for all \(n \leq n_0\) with probability at least \(1 - 5\varepsilon^{-n_0/64}\).

This leads to a proof of Corollary 6 which states that random information is useful if and only if \(\sigma \in \ell_2\).

**Proof of Corollary 6.** We first consider the case that \(\sigma \in \ell_2\). Since \(\mathcal{R}_n(\sigma) \leq \sigma_1\), Theorem 3 yields
\[
\mathbb{E}[\mathcal{R}_n(\sigma)] \leq 2e^{-n/100} \cdot \sigma_1 + \frac{156}{\sqrt{n}} \left(\sum_{j \geq \lceil n/4 \rceil} \sigma_j^2\right)^{1/2}.
\]
The statement is now implied by the fact that \(\sigma \in \ell_2\).

For the case that \(\sigma \notin \ell_2\), let \(0 < \varepsilon < 1\). For \(m \in \mathbb{N}\) let \(\sigma^{(m)}\) be the sequence obtained from \(\sigma\) by replacing the \(j\)th element with zero for all \(j > m\). For any \(N \geq n\), we can choose \(m \in \mathbb{N}\) such that
\[
\frac{\varepsilon^2}{3\sigma_1^2} \sum_{j=2}^{m} \sigma_j^2 \geq N
\]
since \( \sigma \not\in \ell_2 \). The first part of this corollary yields that
\[
P[\mathcal{R}_n(\sigma) \geq \sigma_1(1 - \varepsilon)] \geq P[\mathcal{R}_n(\sigma^{(m)}) \geq \sigma_1(1 - \varepsilon)] \\
\geq P[\mathcal{R}_N(\sigma^{(m)}) \geq \sigma_1(1 - \varepsilon)] \geq 1 - 5 \exp(-N/64).
\]
Since this holds for any \( N \geq n \), we get that the event \( \mathcal{R}_n(\sigma) \geq \sigma_1(1 - \varepsilon) \) happens with probability 1 for any \( \varepsilon \in (0, 1) \). This yields the statement since the event \( \mathcal{R}_n(\sigma) \geq \sigma_1 \) is the intersection of countably many such events.

We now apply our general estimates for \( \mathcal{R}_n(\sigma) \) to specific sequences \( \sigma \) to prove the statements of Corollaries 7 and 9.

**Proof of Corollary 7.** Part 1. The upper bound in the first equivalence is trivial since \( R_n(\sigma) \leq \sigma_1 \) almost surely. The lower bound follows immediately from Corollary 6.

Part 2. To prove the second equivalence of Corollary 7, it is enough to consider the sequence
\[
\sigma_j = j^{-1/2}(1 + \ln j)^{-\beta} \quad \text{for} \quad j \in \mathbb{N}
\]
with \( \beta > 1/2 \). Note that we have for any \( k \in \mathbb{N} \) that
\[
\sum_{j=k+1}^{\infty} \sigma_j^2 \asymp \ln^{1-2\beta}(k),
\]
where the implied constants depend only on \( \beta \). Now it follows from Theorem 3 and from Theorem 5 for \( k = \lceil c'_\beta n/(1 + \ln n) \rceil \) with some \( c'_\beta > 0 \) that
\[
\mathcal{R}_n(\sigma) \asymp n^{-1/2}(1 + \ln n)^{1/2-\beta}
\]
with probability at least \( 1 - 7e^{-n/100} \), where the implied constants depend only on \( \beta \). The statement for the expected value follows from \( 0 \leq R_n(\sigma) \leq 1 \).

Part 3. In the third equivalence of Corollary 7, the lower bound is trivial and even holds almost surely. To prove the upper bound, it is enough to consider the sequence
\[
\sigma_j = \min\{1, j^{-\alpha}(1 + \ln j)^{-\beta}\} \quad \text{for} \quad j \in \mathbb{N},
\]
where \( \alpha > 1/2 \) and \( \beta \in \mathbb{R} \). Theorem 3 yields for large \( n \) that
\[
\mathcal{R}_n(\sigma)^2 \lesssim \frac{1}{n} \sum_{j \geq [n/4]} \sigma_j^2 \lesssim n^{-2\alpha}(1 + \ln n)^{-2\beta}
\]
with probability at least \( 1 - 2e^{-n/100} \) and implied constants only depending on \( \alpha \) and \( \beta \). This yields the statement since \( \mathcal{R}_n(\sigma) \leq 1 \) almost surely.

**Proof of Corollary 9.** The lower bound follows from the trivial estimate \( \mathcal{R}_n(\sigma) \geq \sigma_{n+1} \). To prove the upper bound, we consider the case \( \sigma_j = a^{j-1} \) for all \( j \in \mathbb{N} \). The general case follows from the monotonicity and homogeneity of \( \mathcal{R}_n(\sigma) \) with respect to \( \sigma \). We use Theorem 4. We choose \( c \in [1, \infty) \) such that \( e^{-c^2} \leq a \). Note that there is some \( b \in (0, \infty) \).
such that
\[(\sum_{j>n} \sigma_j^2)^{1/2} = \frac{b a^n}{14}\]
for all \(n \in \mathbb{N}\). Theorem 4 yields for all \(t \geq bna^n\) that
\[P[R_n(\sigma) \geq t] \leq a^n + \frac{b n a^n c \sqrt{2e}}{t}.\]
This yields that
\[E[R_n(\sigma)] = \int_0^1 P[R_n(\sigma) \geq t] \, dt \leq a^n + bna^n + na^n \int_{bna^n}^1 \frac{b c \sqrt{2e}}{t} \, dt \leq n^2 a^n,
\]
as it was to be proven. \(\square\)

4. Alternative approaches

In this section we present alternative ways to estimate the radius of random information from above and below. We choose to do this because these approaches give a better insight into the geometric aspect of the problem. The results, however, are slightly weaker than those obtained in Section 3. The upper bound is weaker since it is not constructive and the lower bound is weaker since it requires a little more than \(\sigma \notin \ell_2\). For these geometric approaches, we restrict to the case of finite sequences \(\sigma\) with \(\sigma_j = 0\) for all \(j > m\). We write \(E_{\sigma}^m\) when we consider the ellipsoid \(E_{\sigma}\) as a subset of \(\mathbb{R}^m\).

4.1. Upper bound via the lower \(M^*\)-estimate. We present an alternative proof of our main upper bound. As already explained in the introduction, the radius of information can also be expressed as the radius of the ellipsoid that is obtained by slicing the \(m\)-dimensional ellipsoid \(E_{\sigma}^m\) with a random subspace of codimension \(n\). To estimate the radius from above, we use a result of Gordon from [11] on estimates of the Euclidean norm against a norm induced by a symmetric convex body \(K\) on large subsets of Grassmannians. Note that the first result in this direction had been established by V.D. Milman in [26].

We start with some notation and background information. Let \(K \subseteq \mathbb{R}^m\) be an origin symmetric convex body, i.e., a compact and convex set with non-empty interior such that \(x \in K\) implies \(-x \in K\). We define the quantity
\[M^*(K) = \int_{S^{m-1}} h_K(x) \mu(dx),\]
where \(S^{m-1}\) is the unit Euclidean sphere in \(\mathbb{R}^m\), integration is with respect to the normalized surface measure \(\mu = \mu_{m-1}\) on \(S^{m-1}\), and \(h_K : S^{m-1} \rightarrow \mathbb{R}\) is the support function of \(K\) given by
\[h_K(x) = \sup_{y \in K} \langle x, y \rangle.\]
Obviously, the support function is just the dual norm to the norm \(\| \cdot \|_K\) induced by \(K\), i.e., if \(K^o = \{ y \in \mathbb{R}^m : \langle y, x \rangle \leq 1 \ \forall x \in K\}\) is the so-called polar body of \(K\), then \(h_K(x) = \|x\|_{K^o}\). Since for \(x \in S^{m-1}\) the support function quantifies the distance from the
origin to the supporting hyperplane orthogonal to \( x \), the quantity \( M^*(K) \) is simply (half) the mean width of the body \( K \).

**Remark 21.** In the theory of asymptotic geometric analysis, the quantities \( M^*(K) \) together with

\[
M(K) := \int_{S^{m-1}} \|x\|_K \mu(dx)
\]

play an important rôle since the work of V.D. Milman on a quantitative version of Dvoretzky’s theorem on almost Euclidean subspaces of a Banach space. Using Jensen’s inequality together with polar integration and Urysohn’s inequality, it is not hard to see that

\[
M(K)^{-1} \leq \text{vrad}(K) \leq M^*(K) = M(K^o),
\]

where \( \text{vrad}(K) := (|K|/|B^m_2|)^{1/m} \) is the volume radius of \( K \) (here \( | \cdot | \) stands for the \( m \)-dimensional Lebesgue measure). For isotropic convex bodies in \( \mathbb{R}^m \) (i.e., convex bodies of volume 1 with centroid at the origin satisfying the isotropic condition – we refer to [2] for details), this immediately yields

\[
M^*(K) \geq \text{vrad}(K) \geq c\sqrt{m},
\]

for some absolute constant \( c \in (0, \infty) \). The question about upper bounds for \( M^*(K) \) with \( K \) in isotropic position has been essentially settled by E. Milman in [25, Theorem 1.1] who proved that

\[
M^*(K) \leq CL_K \sqrt{\log m},
\]

with absolute constant \( C \in (0, \infty) \). In fact, the \( \sqrt{m} \)-term is optimal and also the logarithmic part (up to the power). The optimality of the \( L_K \)-term is intimately related to the famous hyperplane conjecture. For a detailed exposition, we refer the reader to [2] and the references cited therein.

We continue with the so-called lower \( M^* \)-estimate. The first estimate of this type was proved by V.D. Milman in [26], see also [27]. We use the asymptotically optimal version obtained by Pajor and Tomczak-Jaegermann in [32] with improved constants from [11]. For the precise formulation, we refer to [2, Theorem 7.3.5].

**Proposition 22.** For \( n \in \mathbb{N} \), define

\[
a_n = \frac{\sqrt{2} \Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n}{2} \right)} = \sqrt{n} \left( 1 - \frac{1}{4n} + O(n^{-2}) \right).
\]

Let \( K \) be the unit ball of a norm \( \| \cdot \|_K \) on \( \mathbb{R}^m \). For any \( \gamma \in (0,1) \) and \( 1 \leq n < m \) there exists a subset \( \mathcal{B} \) in the Grassmannian \( \mathbb{G}_{m,m-n} \) of \( n \)-codimensional linear subspaces of \( \mathbb{R}^m \) with Haar measure at least

\[
1 - \frac{7}{2} \exp \left( -\frac{1}{18} (1 - \gamma)^2 a_n^2 \right).
\]
such that for any \( E_n \in B \) and all \( x \in E_n \) we have

\[
\frac{\gamma a_n}{d_n M^*(K)} \|x\|_2 \leq \|x\|_K.
\]

We should observe here that the distribution of the kernels of the Gaussian matrices \( G_n \) is the uniform distribution, i.e., the distribution of the Haar measure, on the Grassmann manifold \( \mathbb{G}_{m,m-n} \). This follows immediately from the rotational invariance of both measures on \( \mathbb{G}_{m,m-n} \). Hence, the probability estimate in the theorem is exactly with respect to the probability on the kernels we use elsewhere.

We want to apply the lower \( M^* \)-estimate to obtain an upper bound on the radius of information. For this note that the ellipsoid \( E_m \sigma \) satisfies

\[
(E_m \sigma)^o = \left\{ x \in \mathbb{R}^m : \sum_{j=1}^{m} \sigma_j^2 x_j^2 \leq 1 \right\}.
\]

This implies \( h_{E_m \sigma}^2(x) = \sum_{j=1}^{m} \sigma_j^2 x_j^2 \), and therefore,

\[
M^*(E_m \sigma)^2 \leq \int_{S_{m-1}} h_{E_m \sigma}(x)^2 \mu(dx) = \sum_{j=1}^{m} \sigma_j^2 \int_{S_{m-1}} x_j^2 \mu(dx) = \frac{1}{m} \sum_{j=1}^{m} \sigma_j^2.
\]

A direct application of Proposition 22 with \( K = E_m \sigma \) leads to the upper bound

\[
R_n(\sigma) \leq C_n \sum_{j=1}^{m} \sigma_j^2
\]

with an absolute constant \( C \in (0, \infty) \). This estimate is not very good if the semi-axes \( \sigma_j \) decay quickly. A better estimate can be obtained by switching from \( E_m \sigma \) to its intersection with a Euclidean ball of small radius, a renorming argument going back to Pajor and Tomczak-Jaegermann [31].

**Proposition 23.** There exists an absolute constant \( C \in (0, \infty) \) such that for any non-increasing finite sequence \( \sigma \in \ell_2 \) and all \( n \in \mathbb{N} \), we have

\[
R_n(\sigma) \leq C \frac{1}{n} \sum_{j=n/C}^{m} \sigma_j^2
\]

with probability at least \( 1 - \frac{7}{2} \exp(-n/32) \).

**Proof.** Let \( K_\varrho \) be the intersection of the ellipsoid \( E_m \sigma \) with a centered Euclidean ball of radius \( \varrho > 0 \). For all \( x \in \mathbb{R}^m \), \( y \in K_\varrho \) and \( k \leq m \), Cauchy-Schwarz inequality yields

\[
\langle x, y \rangle^2 \leq 2 \left( \sum_{j=1}^{k} x_j y_j \right)^2 + 2 \left( \sum_{j=k+1}^{m} x_j y_j \right)^2 \leq 2 \varrho^2 \sum_{j=1}^{k} x_j^2 + 2 \sum_{j=k+1}^{m} \sigma_j^2 x_j^2
\]

and thus the same upper bound holds for \( h_{K_\varrho}(x)^2 \). We obtain that

\[
M^*(K_\varrho)^2 = \left( \int_{S_{m-1}} h_{K_\varrho}(x)^2 \mu(dx) \right) \leq \int_{S_{m-1}} h_{K_\varrho}(x)^2 \mu(dx) \leq \frac{2}{m} \left( \varrho^2 + \sum_{j=k+1}^{m} \sigma_j^2 \right).
\]
Proposition 22 tells us that for any $\gamma \in (0, 1)$ there exists a subset $B$ of $G_{m,m-n}$ with measure at least $1 - \frac{7}{2} \exp(-(1-\gamma)^2 n/18)$ such that

$$\text{rad}(K_\varrho \cap E_n)^2 \leq \frac{\alpha_m^2 M_*(K_\varrho)^2}{\gamma^2 a_n^2} \leq \frac{c}{\gamma^2 n} \left( k \varrho^2 + \sum_{j=k+1}^m \sigma_j^2 \right)$$

for any $E_n \in B$ and an absolute constant $c \in (0, \infty)$. Choosing $\varrho$ such that $k \varrho^2 = \sum_{j=k+1}^m \sigma_j^2$ and $k = \lfloor \gamma^2 n/4c \rfloor$ yields

$$\text{rad}(K_\varrho \cap E_n)^2 \leq \frac{\varrho^2}{2}.$$ 

This clearly implies that also

$$\text{rad}(E_\varrho \cap E_n)^2 \leq \frac{\varrho^2}{2}.$$ 

For simplicity, we choose $(1-\gamma)^2 = 1/2$ and obtain the stated inequality. □

4.2. Elementary lower bound. In this section we prove the following lower bound.

**Proposition 24.** For any $\varepsilon \in (0, 1)$ and $c > 0$ there is a constant $c' > 0$ such that the following holds. If $\sigma_m \geq cm^{-\alpha}$ for some $m \in \mathbb{N}$ and $\alpha \in (0, 1/2)$ then $R_n(\sigma) \geq c'$ holds for all $n < m^{1-2\alpha}$ with probability at least $1 - \varepsilon$.

To obtain this lower bound, we first consider the problem of just recovering the first coordinate $x_1$ of $x$ in the unit ball $B_m^2$ of $\mathbb{R}^m$. The corresponding radius of information is given by

$$\tilde{\text{rad}}(G_n, B_m^2)^2 = \sup \{ x_1 : \|x\|_2 = 1 \text{ with } G_n x = 0 \}.$$ 

**Lemma 25.** For $n < m$, we have

$$\mathbb{E} \left[ \tilde{\text{rad}}(G_n, B_m^2)^2 \right] = \frac{m-n}{m}. \tag{4}$$

In particular, for any $\varepsilon \in (0, 1)$, we have

$$\tilde{\text{rad}}(G_n, B_m^2)^2 \geq 1 - \frac{n}{\varepsilon m} \tag{5}$$

with probability at least $1 - \varepsilon$.

**Proof.** Let $k = m-n$. To prove (4), we observe that we want to compute the expectation of the random variable

$$\tilde{\text{rad}}(G_n, B_m^2)^2 = \max \{ \langle x, y \rangle : x \in E, \|x\|_2 = 1 \},$$

where $E$ is uniformly distributed on $G_{m,k}$ and $y = e^{(1)}$ is fixed. Involving an orthogonal transformation of the coordinate system, we may also fix the subspace

$$E = \langle e^{(1)}, \ldots, e^{(k)} \rangle$$

and assume that $y$ is uniformly distributed on the sphere. This does not change the distribution of $\tilde{\text{rad}}(G_n, B_m^2)^2$. By the Cauchy-Schwarz inequality, the maximum is attained
for
\[ x = \frac{P_E(y)}{\|P_E(y)\|_2}, \]
where \( P_E \) denotes the orthogonal projection on \( E \). We obtain
\[ \widetilde{\text{rad}}(G_n, \mathbb{B}_2^m)^2 = \|P_E(y)\|_2^2 = \sum_{j=1}^k y_j^2. \]
We observe that \( \mathbb{E}(y_j^2) = 1/m \) for all \( j \leq m \), since these terms are equal and sum up to 1. This shows (4). Estimate (5) is a direct consequence of (4) taking into account that \( 0 \leq \widetilde{\text{rad}}(G_n, \mathbb{B}_2^m) \leq 1. \)

**Proof of Proposition 24.** If we choose \( G_n \) satisfying (5), by definition of \( \widetilde{\text{rad}}(G_n, \mathbb{B}_2^m) \) and compactness of the unit sphere of \( \mathbb{R}^m \), we find \( x \in \mathbb{R}^m \) with \( \|x\|_2 = 1 \) and \( G_n x = 0 \) satisfying
\[ x_1^2 \geq 1 - \frac{n}{\varepsilon m}. \]
Thus \( x_1 \) is already rather close to 1 which implies that the other coordinates can not be too big. Indeed we find
\[ \|x\|_\sigma^2 \leq x_1^2 + \frac{1}{\sigma_m^2} (1 - x_1^2) \leq 1 + \frac{1}{\sigma_m^2} \frac{n}{\varepsilon m} \leq 1 + \frac{1}{\varepsilon m^2 c^2} \leq 1 + \frac{1}{\varepsilon c^2}. \]
Rescaling \( x \) yields
\[ \text{rad}(G_n, \mathcal{E}_m^m)^2 \geq \frac{\varepsilon c^2}{1 + \varepsilon c^2} \]
and finishes the proof. \( \square \)

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(J. Prochno) Institut für Mathematik & Wissenschaftliches Rechnen, Karl-Franzens-Universität Graz, Heinrichstrasse 36, 8010 Graz, Austria