Synthesis of Memory Gain-Scheduled Controllers for Discrete-Time LPV Systems

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Abstract: This paper considers synthesis of discrete-time gain-scheduled controllers for linear parameter varying systems based on linear matrix inequalities (LMIs). Unlike most of the previous results, gain-scheduled controllers that depend on memory of the scheduling parameters are investigated in this paper. Through the method of change-of-variables, parameter-dependent LMIs are obtained for synthesis of gain-scheduled controllers from extended LMIs for $H_{\infty}$ and $H_2$ performances. Numerical examples are provided to illustrate the proposed synthesis methods.

Key Words: gain-scheduled control, discrete-time systems, linear matrix inequalities, linear parameter varying systems, $H_{\infty}$ and $H_2$ control.

1. Introduction

Gain-scheduling is a method of control that utilizes linear time-varying controllers to compensate for the time-varying dynamics and nonlinearity of the plant. By employing linear parameter varying (LPV) systems to model a plant whose variation of the dynamics is represented by scheduling parameters, methods of analysis and synthesis of gain-scheduled control systems based on linear matrix inequalities (LMIs) enjoy guaranteed performances. In particular, extended LMIs with slack variables [1],[2] significantly improve resulting performances in gain-scheduled controller synthesis.

Several approaches have been proposed for synthesis of output feedback gain-scheduled controllers for discrete-time LPV plants via parameter-dependent LMIs. In [3] and [4], LMIs for synthesis have been proposed to obtain output-feedback gain-scheduled controllers. However, controllers of these papers may depend on the future value of the scheduling parameter $\theta(t+1)$ at time $t$, as well as $\theta(t)$. Such a controller cannot be implemented as it is. Extended LMIs have been applied to static output feedback gain-scheduled controller synthesis in [5], where a part of the variables of the LMI is restricted to zero to derive LMIs for synthesis via the methodology of linearizing change-of-variables. A dynamic output feedback controller synthesis method is proposed in [6]. Though the restriction on the variables is relaxed, a part of the variables is set to be constant of $\theta(t)$ again to apply linearizing change-of-variables. Notice that LMI approaches to discrete-time gain-scheduled controllers include synthesis of time-delay systems [7], 2-D systems [8], and systems with input saturation [9], and control via inexact measurement of the scheduling parameter [10], as well as recent results on state feedback synthesis [11]. There are several papers that apply Lyapunov functions that depend on the parameter for multiple instances to either state feedback or observer synthesis for T-S fuzzy systems (see, e.g., [12],[13]).

In view of previous results on output feedback synthesis via linearizing change-of-variables, we can see difficulty in applying parameter-dependent LMIs to synthesis of discrete-time gain-scheduled controllers as follows: If the Lyapunov matrix is a function of $\theta(t)$, the parameter-dependent LMI involves both $\theta(t)$ and $\theta(t+1)$, but the LPV controller considered in previous papers depend only on $\theta(t)$ or they are allowed to depend on $\theta(t)$ and $\theta(t+1)$. A controller depending on $\theta(t+1)$ is noncausal, while certain restrictions on the decision variables are needed to obtain a controller that only uses $\theta(t)$. From these observations, in this paper, we investigate discrete-time gain-scheduled controllers that can depend on $\theta(t−1)$ as well as $\theta(t)$, which we call a memory gain-scheduled controller. Also, we employ Lyapunov matrices that depend on both $\theta(t)$ and $\theta(t−1)$. We formulate a standard gain-scheduled control problem for discrete-time LPV systems as the plant and consider $H_{\infty}$ performance ($L_2$-gain) and $H_2$ performance of the closed-loop system. Then we propose parameter-dependent LMIs for synthesis of memory gain-scheduled controllers. Any solution to the proposed parameter-dependent LMIs gives a gain-scheduled controller that depends on $\theta(t)$ and $\theta(t−1)$, where a linearizing change-of-variables technique is applied with parameter-dependent congruent transformation. We also mention an extension of the proposed method for gain-scheduled controllers that utilize further past values, namely $\theta(t−k)$ with $k \geq 2$. Also, LMIs for memoryless gain-scheduled controller synthesis are provided with more degrees of freedom than previous results. The proposed LMIs for synthesis are verified by comparing computational results for a problem instance with those of previous results. Notice that the conference version of this paper presented as a position paper [14] only considered a part of $H_{\infty}$ performance of the present paper. At the same time, this paper generalizes the approach to exploiting more of past values, and also $H_2$ synthesis is considered in this paper.

The rest of the paper is organized as follows. Section 2 gives the problem formulation and preliminaries. Section 3 shows LMIs for $H_{\infty}$ and $H_2$ synthesis of discrete-time gain-scheduled controllers that depend on $\theta(t)$ and $\theta(t−1)$ at time $t$, and discusses relations to previous results and extensions to synthesis of controllers depending also on more past values of the
scheduling parameter, as well as an application of the main result to memoryless gain-scheduled controllers. In Section 4, numerical computations of upper bounds of $H_{\infty}$ and $H_2$ performances are provided to compare the previous and proposed methods. Lastly, Section 5 concludes the paper.

**Notation.** Let $\mathbf{Z}$ and $\mathbf{R}$ denote the set of integers and real numbers, respectively. For matrices $A$, $B$, $C$ with appropriate sizes, $\text{Tr} A$ is the trace of $A$, $A^T$ is the transpose of $A$, $\text{sym} A = A + A^T$. To represent a symmetric matrix divided into blocks, we use the abbreviation as

$$
\begin{bmatrix}
A & B \\
B^T & C
\end{bmatrix} = \begin{bmatrix} A & B \\
\ast & C
\end{bmatrix} = \begin{bmatrix} A & \ast \\
\ast & C
\end{bmatrix}.
$$

For a matrix or a matrix-valued function $M$, we mean by $M > 0$ that $M - \alpha I$ is positive semidefinite for some constant positive scalar $\alpha$, where $I$ is the identity matrix. Symbol $\| \cdot \|$ stands for the Euclidean norm. The block-diagonal matrix that consists of matrices $M_1, M_2, \ldots, M_k$ is denoted by $\text{diag}(M_1, M_2, \ldots, M_k)$.

### 2. Preliminaries

#### 2.1 Problem Formulation

Consider a gain-scheduled control system in a standard configuration shown in Fig. 1, where $w \in \mathbf{R}^{m_w}$ is the external input, $u \in \mathbf{R}^{m_u}$ is the control input, $z \in \mathbf{R}^{p_z}$ is the controlled output, and $\theta \in \mathbf{R}^r$ is the scheduling parameter.

**Definition 1** Let $\Theta, \Omega \subset \mathbf{R}^r$ be compact sets and define $\vartheta = [\theta : \mathbf{Z} \rightarrow \mathbf{R}^r | \theta(t) \in \Theta, \theta(t + 1) - \theta(t) \in \Omega \ \forall t \in \mathbf{Z}]$.

**Example 1** One of parameter sets frequently considered in the literature is given by $\Theta = \prod_{i=1}^{p_{\theta}} [\bar{\theta}_i, \tilde{\theta}_i]$ and $\Omega = \prod_{i=1}^{p_{\theta}} [\bar{\omega}_i, \tilde{\omega}_i]$ with $\bar{\theta}_i, \tilde{\theta}_i > 0$, $i = 1, 2, \ldots, r$. The region $R$ in Fig. 2 represents $\{((\theta(t), \theta(t + 1)) \in \mathbf{R}^r | (\theta(t)) \in \vartheta \}$, where

$$
\vartheta = [\theta : \mathbf{Z} \rightarrow \mathbf{R}^r | - \bar{\theta} \leq \theta(t) \leq \bar{\theta}, - \bar{\omega} \leq \theta(t + 1) - \theta(t) \leq \bar{\omega} \ \forall t \in \mathbf{Z}].
$$

Such parameter regions have been exploited in the literature of discrete-time gain-scheduled control synthesis in, e.g., [3]–[9].

Assume that the scheduling parameter $\theta(t)$ belongs to $\Theta$. Let the plant in Fig. 1 be given by the following discrete-time LPV system:

$$
\begin{aligned}
x(t + 1) &= \mathcal{A}_\theta x(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \\
z(t) &= \mathcal{C}_1 x(t) + \mathcal{D}_{11} w(t) + \mathcal{D}_{12} u(t), \\
y(t) &= \mathcal{C}_2 x(t) + \mathcal{D}_{21} w(t),
\end{aligned}
$$

where $x \in \mathbf{R}^n$ is the state. The coefficient matrices $\mathcal{A}, \mathcal{B}_1, \mathcal{C}_j, \mathcal{D}_{ij}, i, j = 1, 2$, depend on $\theta(t)$ at time $t$. We consider the following LPV system as the discrete-time gain-scheduled controller:

$$
\begin{aligned}
x_c(t + 1) &= \mathcal{A}_\theta x_c(t) + \mathcal{B}_j y(t), \\
u(t) &= \mathcal{C}_c x_c(t) + \mathcal{D}_c y(t),
\end{aligned}
$$

where $x_c \in \mathbf{R}^n$ is the state of the controller. The coefficient matrices $\mathcal{A}_\theta, \mathcal{B}_j, \mathcal{C}_c, \mathcal{D}_c$ are allowed to depend on $\theta(t)$ and $\theta(t - 1)$. We call (3) a memory gain-scheduled controller. The realization of the closed-loop system with state $x_{cl} = [x^T \ x_c^T]^T$ is given as

Fig. 1 Gain-scheduled control system.

**Fig. 2** Parameter box of $(\theta_a, \theta_b) = (\theta(t), \theta(t + 1))$ with $\Theta = [-\bar{\theta}, \tilde{\theta}]$ and $\Omega = [-\bar{\omega}, \tilde{\omega}]$.

$$
\begin{aligned}
x_c(t + 1) &= \mathcal{A}_c x_c(t) + \mathcal{B}_c \vartheta(t), \\
z(t) &= \mathcal{C}_c x_c(t) + \mathcal{D}_c \vartheta(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}_c &= \begin{bmatrix} \mathcal{A} + \mathcal{B}_2 \mathcal{D}_{12} \mathcal{C}_2 & \mathcal{B}_2 \mathcal{C}_1 \\
\mathcal{B}_2 \mathcal{C}_2 & \mathcal{A}_c
\end{bmatrix}, \\
\mathcal{B}_c &= \begin{bmatrix} \mathcal{B}_2 \mathcal{C}_2 \\
\mathcal{B}_2 \mathcal{D}_{21}
\end{bmatrix}, \\
\mathcal{C}_c &= \begin{bmatrix} \mathcal{C}_1 + \mathcal{D}_{12} \mathcal{D}_{21} \mathcal{C}_2 \\
\mathcal{D}_{12} \mathcal{C}_2
\end{bmatrix}, \\
\mathcal{D}_c &= \mathcal{D}_{11} + \mathcal{D}_{12} \mathcal{D}_{21}.
\end{aligned}
$$

We consider the $l_2$-gain ($H_{\infty}$-performance) and $H_2$-performance of the closed-loop system as the control specification under the exponential stability of the closed-loop system.

**Definition 2** (i) Suppose that $w(t) = 0$. Then the closed-loop system (4) is said to be exponentially stable if there exist $M > 0$ and $\lambda \in (0, 1)$ such that $\|x_{cl}(t)\| \leq M \lambda^t \|x_{cl}(0)\|$ holds for all integer $t \geq 0$. (ii) Suppose that $x_{cl}(0) = 0$. Then the $l_2$-gain $\gamma_{\infty}$ of closed-loop system (4) is defined as the infimum of $\gamma \geq 0$ for which $\sum_{t=0}^{\infty} \|z(t)\|^2 \leq \gamma^2 \sum_{t=0}^{\infty} \|w(t)\|^2$ for all integer $T \geq 0$. (iii) Suppose that $x_{cl}(0) = 0$, and $w(t)$ is a zero-mean white Gaussian process with covariance $\mathbf{E}[w(t)w(t)^T] = I$, where $\mathbf{E}$ stands for the expected value. Then the $H_2$-performance $\gamma_2 \geq 0$ of closed-loop system (4) is defined as

$$
\gamma_2^2 = \sup_{t \geq 0} \sup_{w(t) \neq 0} \mathbf{E}[\|z(t)\|^2].
$$

**2.2 Parameter-Dependent LMIs**

Consider an LPV system:

$$
\begin{aligned}
x(t + 1) &= \mathcal{A}_\theta x(t) + \mathcal{B}_1 w(t), \\
z(t) &= \mathcal{C}_\theta x(t) + \mathcal{D}_\theta w(t),
\end{aligned}
$$

where the coefficient matrices are functions of $\theta(t)$ at time $t$. It is widely known that the above LPV system is exponentially stable and its $l_2$-gain from $w$ to $z$ is less than $\gamma$ if there exists a
symmetric-matrix-valued function $Q = Q(\theta(t))$ that satisfies the following parameter-dependent LMI:

$$
\begin{bmatrix}
Q & 0 & ASQ & B \\
* & \gamma I & CQ & D \\
* & * & \gamma I & 0 \\
* & * & * & \gamma I
\end{bmatrix} > 0
$$

for all $\theta(\cdot) \in \mathcal{Q}$, where $Q_\gamma$ stands for $Q(\theta(t+1))$. The celebrated extended LMI which is equivalent to (5) is given as

$$
\begin{bmatrix}
Q & 0 & ASK & B \\
* & \gamma I & CK & D \\
* & * & \gamma I & 0 \\
* & * & * & \gamma I
\end{bmatrix} > 0,
$$

where $K$ is a function of $\theta(t)$. The equivalence between (5) and (6) is shown in [6] by generalizing the results of [1] and [2] for time-invariant systems to LPV systems with parameter-dependent variables $Q$ and $K$. Extended LMIs such as (6) often bring significantly superior results than standard LMIs like (5) in many problem formulations and in many problem instances in robustness analysis and synthesis (see, e.g., [15]).

However, a difficulty in synthesis is that these LMIs involve $Q_\gamma$, which depends on $\theta(t + 1)$. A simple application of the techniques for LMI-based synthesis, such as linearizing change-of-variables, results in a controller realization that might depend on $\theta(t + 1)$ [6,7]. Now recall that the LPV controller in (3) can depend on the values of the parameter $\theta(t − 1)$ as well as $\theta(t)$. This allows us to consider parameter-dependent LMIs for synthesis, as will be shown in the next section, with much more degrees-of-freedom in decision variables.

3. LMIs for Synthesis of Output Feedback Memory-Scheduled Controllers

3.1 $H_\infty$ Synthesis

Here we show a parameter-dependent LMI for synthesis, by which one can construct a memory-gain-scheduled controller that satisfies a guaranteed upper bound of the $l_2$-gain of the closed-loop system. In what follows, for a parameter-dependent matrix $M = M(\theta(t), \theta(t − 1), \ldots, \theta(t − k))$ with integer $k \geq 0$, by $M_+$ and $M_-$ we mean

$$
M_+ = M(\theta(t + 1), \theta(t), \ldots, \theta(t − k + 1)),
$$

$$
M_- = M(\theta(t − 1), \theta(t − 2), \ldots, \theta(t − k − 1)),
$$

respectively.

**Theorem 1** Suppose that there exist matrices $L_Q = L_Q(\theta), Y, Z, F, G, H, J$ that depend on $\theta(t)$ and $\theta(t − 1)$ at time $t$, and $X$ that depends on $\theta(t)$ at time $t$, for which

$$
\begin{bmatrix}
L_{Q+} & 0 & L_A & L_B \\
0 & \gamma I & L_C & L_D \\
* & * & \gamma I & 0 \\
* & * & * & \gamma I
\end{bmatrix} > 0 
$$

holds for all $\theta(\cdot) \in \mathcal{Q}$, where

$$
L_K = \begin{bmatrix} Y & I \\ Z & X \end{bmatrix}, 
$$

$$
L_A = \begin{bmatrix} [AF + B_2F & A + B_2Jc_2] \\ H & XR + GC_2 \end{bmatrix}, 
$$

$$
L_B = \begin{bmatrix} B_1 + B_2JD_{21} \\ XB_1 + GD_{21} \end{bmatrix}, 
$$

$$
L_C = \begin{bmatrix} C_1Y + D_{12}F & C_1 + D_{12}Jc_2 \end{bmatrix}, 
$$

$$
L_D = D_{11} + D_{12}JD_{21}.
$$

Then $X$ and $S := Y − X^T \gamma Z$ are nonsingular. Set the coefficient matrices of the gain-scheduled controller (3) as

$$
A_c = \begin{bmatrix} (AF + B_2F − X^T \gamma H) \\ −(B_2Jc_2 − X^T \gamma GC_2)Y \end{bmatrix}S^{-1}, 
$$

$$
B_c = B_2J − X^T \gamma G, 
$$

$$
C_c = (F − Jc_2Y)S^{-1}, 
$$

$$
D_c = J.
$$

Then the closed-loop system (4) is exponentially stable and its $l_2$-gain is less than $\gamma$.

**Remark 1** Theorem 1 involves two variables with a subscript $+ or −$, namely $L_Q$ and $X$. Since $L_Q$ depends on $(\theta(t), \theta(t − 1))$ (i.e., $k = 1$) and $X$ depends on $\theta(t)$ only (i.e., $k = 0$), we have $L_{Q+} = L_Q(\theta(t + 1), \theta(t))$ and $X_+ = X(\theta(t − 1))$ in accordance with the notation of the subscripts stated above Theorem 1.

**Proof.** Define

$$
U = \begin{bmatrix} I & 0 \\ I & −X \end{bmatrix}, \quad K = \begin{bmatrix} Y & F − X^T \gamma S \\ \mathbb{S} & \mathbb{S} \end{bmatrix},
$$

and $Q = U_LQU_L^T$. We see

$$
U_LK = \begin{bmatrix} Y & I \\ S & 0 \end{bmatrix}, \quad U_LKU_L^T = K,
$$

and

$$
U_L^T = (U_LK)^{-1}K = \begin{bmatrix} 0 & S^{-1} \\ I & −Y^T S^{-1} \end{bmatrix}K.
$$

Then, if ones sets $A_c, B_c, C_c$, and $D_c$ as in (8), it holds that

$$
\begin{bmatrix}
U & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix} \begin{bmatrix} U^T & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_cK & B_c \\ C_cK & D_c \end{bmatrix}.
$$

Multiplying $U_a = \text{diag}(U, I, U, I)$ and its transpose from the left and right of the inequality (7), respectively, we obtain

$$
\begin{bmatrix}
Q_+ & 0 & A_cK & B_c \\
* & \gamma I & C_cK & D_c \\
* & * & \gamma I & 0 \\
* & * & * & \gamma I
\end{bmatrix} > 0,
$$

which proves that the closed-loop system is exponentially stable and has $l_2$-gain less than $\gamma$ with the storage function

$$
V(t, x) = x^TQ^{-1}x.
$$

3.2 Discussion and Extension

The derivation of the LMI (7) for synthesis utilizes linearizing change-of-variables on output feedback synthesis, which has been proposed to handle standard LMIs [16,17] and generalized to extended LMIs [2,6]. A crucial point of the derivation in Theorem 1 is that, by allowing the controller to depend
on $\theta(t-1)$ as well as $\theta(t)$, we can apply a congruent transformation containing $\mathcal{U}$ and $\mathcal{U}_\circ$, which are functions of $\theta(t)$ and $\theta(t-1)$, respectively. With this formulation, the matrix $\mathbb{X}$ in (7) is able to depend on the scheduling parameter, while the counterpart of this matrix $\mathbb{X}_\circ$ is restricted to be constant of the scheduling parameter in [6]. We also notice that storage function $V(t,x) = x^T Q(t) x$ depends on $\theta(t)$ and $\theta(t-1)$ at time $t$. Such a storage function has never been considered before.

Theorem 1 extends [6] in the aspect of the dependency of the matrix variables on the scheduling parameter. More precisely, if we restrict $L_K$ to be independent of the scheduling parameter and the other variables to be functions of $\theta(t)$ only, then the LMI (14) of [6] is recovered from the LMI (7). Moreover, we can derive an LMI for synthesis of memoryless gain-scheduled controllers, namely controllers that depends only on $\theta(t)$, from (7) in view of the controller realization (8).

**Corollary 1** If LMI (7) holds for $L_Q = L_{Q_k}$, $Y$, $Z$, $F$, $G$, $H$, $J$ that depend on $\theta(t)$ at time $t$, and for $X$ that is constant of $\theta(t)$, then the closed-loop system (4) with the coefficients in (8) is exponentially stable and has $l_2$-gain less than $\gamma$. Moreover, the coefficient matrices of (8) depend only on $\theta(t)$ at time $t$.

In Corollary 1, variable $L_K$ is a function of $\theta(t)$ through $Y$ and $Z$, which are functions of $\theta(t)$. Thus Corollary 1 reveals that the matrix $\mathbb{X}$ can depend on the scheduling parameter even for synthesis of gain-scheduled controllers that depend only on $\theta(t)$, while the counterparts of $L_K$ (variables $S$, $X$, $Y$ contained in $T^T GT$) are chosen as constant matrices in the LMI (14) of [6]. In Section 4, we show numerical examples on a problem instance by which we compare upper bounds to performance indexes obtained by solving LMIs of [6], Theorem 1, Corollary 1, and LMIs to be shown in the following subsections.

Another direction for considering the dependency of the matrix variables on the scheduling parameter is utilizing more of past values of $\theta(\cdot)$. In the LMI (7), we restricted the dependency of the matrices to past values of the scheduling parameter up to $\theta(t-1)$. It is easy to extend Theorem 1 so that the matrix variables depend on $\theta(t)$, $\theta(t-1)$, ..., $\theta(t-k)$ with some finite positive integer $k$. This is summarized below, where, for the sake of generality, we let the coefficient matrices of the plant (2) also depend on past values of the scheduling parameter. For notational simplicity, define $\tilde{\theta}(t,k) = (\theta(t), \theta(t-1), \ldots, \theta(t-k+1))$ for integer $k \geq 1$ and $\tilde{\theta}(t;0) = 0$.

**Corollary 2** Let the coefficient matrices of the plant (2) depend on $\tilde{\theta}(t;N)$ at time $t$, where $N \geq 1$. Let $N_{i,1} \geq 1$, $N_{i,2} \geq 0$ and suppose that there exist matrices $L_Q = L_{Q_k}^T$, $Y$, $Z$, $F$, $G$, $H$, $J$, that depend on $\tilde{\theta}(t;N)$ at time $t$ and $X$ depending on $\tilde{\theta}(t;N_{i,2})$ at time $t$ for which the parameter-dependent LMI (7) holds for all $\tilde{\theta}(\cdot) \in \Theta$. Then the closed-loop system (4) with the coefficient matrices (8) is exponentially stable and its $l_2$-gain is less than $\gamma$, where the controller (3) depends at most on $\tilde{\theta}(t;N)$ at time $t$, where $N = \max(N, N_{i,1}, N_{i,2} + 1)$.

In Corollary 2, the left-hand side of the LMI (7) depends on $(\theta(t+1)) \cup (\theta(t;N))$. Theorem 1 and Corollary 1 correspond to Corollary 2 with $(N, N_{i,1}, N_{i,2}) = (1, 2, 1)$ and $(1, 1, 0)$, respectively.

### 3.3 Dual LMI for $H_\infty$ Synthesis

The synthesis method shown in Section 3.1 is based on the extended LMI (9) for the closed-loop system. As is widely known, there are dual LMIs that characterize stability, $H_\infty$ norm of linear systems. A dual formulation to the result of Section 3.1 is shown below.

**Corollary 3** (i) Suppose that there exist matrices $M_Q = M_Q^T$, $X$, $Z$, $F$, $G$, $H$, $J$ that depend on $\theta(t)$ and $\theta(t-1)$ at time $t$ and a matrix $Y$ that depends on $\theta(t)$ at time $t$, for which

\[
\begin{bmatrix}
M_P & 0 & * & * \\
0 & \gamma I & * & * \\
M_A & M_B & \text{sym} M_K - M_P & 0 \\
M_C & M_D & 0 & \gamma I
\end{bmatrix} > 0
\]  

holds for all $\theta(\cdot) \in \Theta$, where

\[
M_K = \begin{bmatrix} Y & I \\ Z & X \end{bmatrix},
\]

\[
M_A = \begin{bmatrix} AY & + & B_2 Z & A + B_2 J C_2 \\ H & XA & + & GC_2 \end{bmatrix},
\]

\[
M_B = \begin{bmatrix} B_1 & + & B_2 J D_2 \\ XB_1 & + & GD_2 \end{bmatrix},
\]

\[
M_C = \begin{bmatrix} C_1 Y & + & D_2 F & C_1 & + & D_1 J C_2 \\ C_1 & + & D_1 J C_2 & & \end{bmatrix},
\]

\[
M_D = D_1 & + & D_2 J D_2.
\]

Then $Y$ and $S := X - Z Y^{-1}$ are nonsingular. Set the coefficient matrices of the gain-scheduled controller (3) as

\[
A_c = S^{-1} \bigl( (XA + GC_2 - H Y^{-1}) - X (B_2 J C_2 - B_2 J F Y^{-1}) \bigr),
\]

\[
B_c = S^{-1} (G - XB_2 J),
\]

\[
C_c = IC_2 - FY^{-1},
\]

\[
D_c = J.
\]

Then the closed-loop system (4) is exponentially stable and its $l_2$-gain is less than $\gamma$. (ii) Let $N \geq 1$, $N_{i,1} \geq 1$, and $N_{i,2} \geq 0$ and suppose that the coefficient matrices of the plant (2) depends on $\tilde{\theta}(t;N)$ at time $t$ and that there exist matrices $M_P = M_P^T$, $X$, $Z$, $F$, $G$, $H$, $J$ that are functions of $\tilde{\theta}(t;N_{i,1})$ at time $t$ and $Y$ depending on $\tilde{\theta}(t;N_{i,2})$ at time $t$ for which the parameter-dependent LMI (7) holds for all $\tilde{\theta}(\cdot) \in \Theta$. Then the closed-loop system (4) with the coefficient matrices (11) is exponentially stable and its $l_2$-gain is less than $\gamma$, where the controller (3) depends at most on $\tilde{\theta}(t;N)$, where $N = \max(N, N_{i,1}, N_{i,2} + 1)$.

**Proof.** The proof of (i) is parallel to that of Theorem 1. Define

\[
\mathcal{U} = \begin{bmatrix} 0 & -Y^{-1} \\ I & I \end{bmatrix}, \quad \mathbb{K} = \begin{bmatrix} X & S \\ X - Y^{-1} & S \end{bmatrix},
\]

and $\mathbb{P} = \mathcal{U}^T L \mathcal{U}$, by which it holds that

\[
\begin{bmatrix}
\mathcal{U}^T & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix}
M_A & M_B \\
M_C & M_D
\end{bmatrix} \begin{bmatrix}
\mathcal{U} \\
0
\end{bmatrix} = \begin{bmatrix} XA_c & XB_c \\
C_c & D_c \end{bmatrix}
\]

and $\mathcal{U}^T L \mathcal{U} = \mathbb{K}$. Multiplying $U_d = \text{diag}(U_-, I, U, I)$ and its transpose from the right and left of the inequality (10), respectively, we obtain
Let the coe · γ imply (13) is equivalent to the generalized form of Corollary 2. For l depending on H 3.4 Setting k t = ∈ KA 2 = 2, suppose that there exist matrices ∗∗ ∈ Q RC cl, · γ depends on Ω 0 γ Θ (ii) The LMI (10) with variables as in (i) of Corollary 3 (iii) The LMI (7) with variables as in Corollary 1 (iv) The LMI (10) with variables as in (ii) of Corollary 3 where (N, Nc1, Nc2) = (1, 1, 0) (v) The LMI (7) with variables as in Corollary 1 but Y and Z are independent of θ(c) With settings (i) and (ii), we compute memory gain-scheduled controllers that depend on θ(t) and θ(t − 1) at time t. Settings (iii), (iv), and (v) provide memoryless gain-scheduled controllers, where (v) corresponds to the previous synthesis method of [6] in the sense of the dependency of the matrices that appear in the extended LMIs for synthesis.

To solve the parameter-dependent LMIs, we parameterized the decision variables of the LMIs as matrix-valued affine functions of θ(t) and θ(t − 1), where the latter is included for (i) and (ii). More specifically, for the case of (i), we set

\[ X = X_0 + \theta(t)X_1, \]
\[ Y = Y_0 + \theta(t)Y_1 + \theta(t − 1)Y_2, \]
\[ Z = Z_0 + \theta(t)Z_1 + \theta(t − 1)Z_2, \]
\[ J = W_0 + \theta(t)W_1 + \theta(t − 1)W_2, \]

where matrices V = \{X_0, X_1, Y_0, Y_1, Y_2, Z_0, Z_1, Z_2, W_0, W_1, W_2\} are constant of θ(c). Then one can represent the parameter-dependent LMI condition (7) as

\[
\begin{bmatrix}
\theta_1 I \\
\theta_2 I \\
\theta_3 I
\end{bmatrix}^T Q(V, \gamma) \begin{bmatrix}
\theta_1 I \\
\theta_2 I \\
\theta_3 I
\end{bmatrix} > 0
\]

for all (θ_1, θ_2, θ_3) ∈ Θ_v, where Q(V, γ) depends on (V, γ) and Θ_v = \{θ_1, θ_2, θ_3\} \in Θ | θ_1 − θ_2 ∈ Ω, θ_2 − θ_3 ∈ Ω \}

There are many methods to solve such parameter-dependent or robust LMIs (see, e.g., [18]). We use here the (D, G) scaling [19] to reduce parameter-dependent LMI (15) to a finite-dimensional LMI, by introducing D scalings as multipliers for the following six inequalities:

\[ \mathbb{P} * \mathbb{P} > 0, \]
\[ \mathbb{K} \mathbb{A}_{cl} + \mathbb{K} \mathbb{B}_{cl} \mathbb{C}_{cl} \mathbb{D}_{cl} \mathbb{C}_{cl} + \mathbb{D}_{cl} \mathbb{C}_{cl} > 0, \]

which proves that the closed-loop system is exponentially stable and has H 2-gain less than γ with the storage function \( V(x, t) = x^T P x \). The proof of (ii) can be seen easily. 

4. Numerical Examples

4.1 H∞ Synthesis

Consider the following coefficient matrices of LPV system (2):

\[
\begin{bmatrix}
0.46 & 0.23 & 0.05 & 0.101 & 0 & -5 \\
0.03 & 0.77 & -0.05 & 0.004 & 0.1 & -0.007 \\
-0.01 & 0.05 & 0.88 & 0.2 & 0.005 & 0.353
\end{bmatrix}
\]

Let the scheduling parameter θ(·) belong to Θ of (1) in Example 1 with Θ = [−θ̅, θ̄] = [−1, 1] and Ω = [−σ, σ] = [−0.5, 0.5]. We examined the LMIs (7) and (10) with the following settings:

(i) The LMI (7) with variables as in Theorem 1
(ii) The LMI (10) with variables as in (i) of Corollary 3
(iii) The LMI (7) with variables as in Corollary 1
(iv) The LMI (10) with variables as in (ii) of Corollary 3 where (N, Nc1, Nc2) = (1, 1, 0)
(v) The LMI (7) with variables as in Corollary 1 but Y and Z are independent of θ(c)
Since (16) is a sufficient condition even for (15) to hold, where

\[ D_t = D_2^T \geq 0, \quad G_i = -G_i^T, \quad i = 1, 2, \ldots, 6, \]

\[ R = \theta^T (D_t + D_2 + D_1) + \omega^2 (D_t + D_2) + (2 \omega)^2 D_6. \]

This type of use of \( G \) scalings can be found in [20]. The minimum of \( \gamma \) subject to (16) is denoted by \( \gamma_0^{(DG)} \). Also the other cases (ii) to (v) were handled similarly via a \((D,G)\) scaling.

Since (16) is a sufficient condition even for (15) to hold, the optimal \( \gamma \) subject to (16) is an upper bound. We provide a lower bound, denoted by \( \gamma_\infty^{(DG)} \), by minimizing \( \gamma \) with satisfying (15) but for grid points of \((\theta, \theta_0, \theta_c) \in \Theta_n\), in order to verify the quality of upper bounds. The minimum of \( \gamma \) subject to (15) is between \( \gamma_\infty^{(DG)} \) and \( \gamma_0^{(DG)} \).

The lower and upper bounds are computed via SeDuMi [21] with YALMIP [22]. The results are shown in Table 1, where the last column is the relative error \((\gamma_\infty^{(DG)} - \gamma_0^{(DG)}) / \gamma_0^{(DG)}\). We can estimate the true minimum of \( \gamma \) subject to (15) from the relative errors, which are at most 3\%. From Table 1, the results of (i), (ii) of memory gain-scheduled controllers enjoy smaller upper bounds than memoryless controllers of (iii)–(v). Thus these results support the usefulness of the memory gain-scheduled controllers and the synthesis methods. Also, with more freedom in the decision variables, (iii) and (iv) reduces the \( L_2 \)-gain upper bound of (v). We notice that there is no evident difference of superiority between dual LMI pairs.

### 4.2 \( H_2 \) Synthesis

Let us consider the same LPV plant as in the previous subsection and solve the LMI (13) in Theorem 2 with the following settings:

(i) \((N, N_1, N_2) = (1, 2, 1)\),

(ii) \((N, N_1, N_2) = (1, 1, 0)\),

(iii) \((N, N_1, N_2) = (1, 1, 0)\) but \( Y \) and \( Z \) are constant of \( \theta(t) \), where (i), (ii), and (iii) correspond to the memory gain-scheduled controllers depending on \( \theta(t) \) and \( \theta(t - 1) \), the memoryless gain-scheduled controller that uses \( \theta(t) \) only, and the memory gain-scheduled controller in [6], respectively.

We treated the parameter-dependent LMIs with affine parameterization of the decision variables and \((D,G)\) scaling similarly to the previous subsection for upper bounds, as well as lower bound computation via gridding. The upper bounds \( \gamma_2^{(DG)} \) and the lower bounds \( \gamma_2^{(DG)} \) are shown in Table 2. In this example, the improvement of the upper bound of the memory gain-scheduled control setting (i) is significant.

### 5. Conclusions

In this paper, we considered a memory gain-scheduled controller that utilizes past value \( \theta(t - 1) \) of the scheduling parameter as well as the current value \( \theta(t) \). We derived LMIs for synthesis via change-of-variables from extended LMIs for \( H_\infty \) and \( H_2 \) performances, where we applied the parameter-dependent congruence transformation. The insight into the structure of the LMIs allowed us to find more freedom on the decision variables even for memoryless gain scheduled controller than those considered in previous papers. The proposed formulation was extended to utilize more of past values of the scheduling parameter. We showed a problem instance for which the proposed methods attain smaller upper bounds for \( H_\infty \) and \( H_2 \) synthesis problems. Future work may include exactness issues of the proposed parameter-dependent LMIs when increasing the number of past values of the scheduling parameters \( N_1 \) and \( N_2 \).

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| Table 1 Results of \( \gamma_0^{(DG)} \) and \( \gamma_\infty^{(DG)} \). |
|---|---|---|
| setting | \( \gamma_0^{(DG)} \) | \( \gamma_\infty^{(DG)} \) | error (%) |
| (i) | 80.191 | 78.324 | 2.33 |
| (ii) | 85.168 | 82.623 | 2.99 |
| (iii) | 160.755 | 158.282 | 1.54 |
| (iv) | 125.106 | 124.047 | 0.85 |
| (v) | 192.763 | 191.643 | 0.58 |

| Table 2 Results of \( \gamma_2^{(DG)} \) and \( \gamma_\infty^{(DG)} \). |
|---|---|---|
| setting | \( \gamma_2^{(DG)} \) | \( \gamma_\infty^{(DG)} \) | error (%) |
| (i) | 55.221 | 54.627 | 2.84 |
| (ii) | 102.445 | 100.657 | 1.75 |
| (iii) | 116.694 | 116.091 | 0.52 |

[θ], [θ], [θ], [θ], [θ], [θ], [θ] along with six skew-symmetric matrices as \( G \) scalings, namely we solve the following LMI

\[
\begin{bmatrix}
-Q(V, \gamma) & V \\
V & \gamma
\end{bmatrix} > 0,
\]

which is sufficient for the inequality (15) to hold, where

\[ D_t = D_2^T \geq 0, \quad G_i = -G_i^T, \quad i = 1, 2, \ldots, 6, \]

\[ R = \theta^T (D_t + D_2 + D_1) + \omega^2 (D_t + D_2) + (2 \omega)^2 D_6. \]
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