Nonlinear Wigner solid transport over superfluid helium under AC conditions

Yuriy P. Monarkha¹,² and Kimitoshi Kono¹

¹ Low Temperature Physics Laboratory, RIKEN, Hirosawa 2-1, Wako 351-0198, Japan
² B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine
47 Lenin Ave., Kharkov 61103, Ukraine
E-mail: monarkha@ilt.kharkov.ua

Received December 18, 2008

Nonlinear transport properties of the two-dimensional Wigner solid of surface electrons on superfluid helium are studied for alternating current conditions. For time-averaged quantities like Fourier coefficients, the field-velocity characteristics are shown to be qualitatively different as compared to that found in the DC theory. For a spatially uniform current we found a general solution for the field-velocity relationship which appears to be strongly dependent on the current frequency. If the current frequency is much lower than the ripplon damping parameter, the Bragg–Cherenkov resonances which appear at high enough drift velocities acquire a distinctive saw-tooth shape with long right-side tails independent of small damping. For current frequencies which are close or higher than the ripplon damping coefficient, the interference of ripplons excited at different time intervals results in a new oscillatory (in drift velocity) regime of Bragg–Cherenkov scattering.

PACS: 67.90.+z Other topics in quantum fluids and solids;
73.20.–r Electron states at surfaces and interfaces;
73.25.+i Surface conductivity and carrier phenomena.

Keywords: liquid helium, surface electrons, Wigner solid, nonlinear transport, Bragg–Cherenkov scattering.

Introduction

Electrons trapped on the surface of liquid helium form a clean two-dimensional electron system of which the average Coulomb interaction energy can greatly exceed the average kinetic energy (for a review, see [1,2]). For typical electron densities realized experimentally \( n_s \lesssim 10^9 \text{ cm}^{-2} \), the Fermi-energy of surface electrons (SEs) is much smaller than temperature. Therefore at a low enough temperature, depending on the surface electron density \( T_n \sim n_s \), the electron system undergoes a transition to the Wigner solid (WS) state. This was first observed by Grimes and Adams [3] from the onset of resonances induced by electron interaction with capillary-waves (ripplons) whose wave-vector \( q \) is close to electron reciprocal lattice vectors \( g \). Since the electron lattice determines a specific set of frequencies \( n g \sim \sqrt{n_s} \), the WS system is expected to be strongly affected by ripplons.

Nonlinear transport properties of the two-dimensional Wigner solid of surface electrons on superfluid helium are studied for alternating current conditions. For time-averaged quantities like Fourier coefficients, the field-velocity characteristics are shown to be qualitatively different as compared to that found in the DC theory. For a spatially uniform current we found a general solution for the field-velocity relationship which appears to be strongly dependent on the current frequency. If the current frequency is much lower than the ripplon damping parameter, the Bragg–Cherenkov resonances which appear at high enough drift velocities acquire a distinctive saw-tooth shape with long right-side tails independent of small damping. For current frequencies which are close or higher than the ripplon damping coefficient, the interference of ripplons excited at different time intervals results in a new oscillatory (in drift velocity) regime of Bragg–Cherenkov scattering.

The physics of this phenomenon can be explained as follows. The pressure at the interface induced by the WS moving with a constant drift velocity \( v \) can be represented as a series of terms proportional to \( \exp[-i(q \cdot r - vt)] \), where \( r \) is the in-plane coordinate vector. As a function of \( t \), it can be considered as a series of harmonic perturbations with frequencies \( \omega_g=\sqrt{\alpha / \rho q^2} \) is the ripplon spectrum, \( \alpha \) is the surface tension, and \( \rho \) is the mass density of liquid helium), experimental evidence for a triangular electron lattice on a liquid-He surface was given. Electron interaction with such ripplons appeared to be very strong, leading to a huge reconstruction of the WS phonon spectrum in the low frequency range [4].

The resonances of Grimes and Adams occur when the frequency of the input signal \( \omega_0 \) is close to \( \omega_n \). Transport properties of the WS of SEs are usually studied under much lower frequencies \( \omega_0 \ll \omega_n \). Nevertheless, even under low frequency conditions the resonant interaction with ripplons of frequencies which are close to \( \omega_n \) appears to be possible as a nonlinear conductivity effect [5]. The physics of this phenomenon can be explained as follows. The pressure at the interface induced by the WS moving with a constant drift velocity \( v \) can be represented as a series of terms proportional to \( \exp[i(q \cdot r - vt)] \), where \( r \) is the in-plane coordinate vector. As a function of \( t \), it can be considered as a series of harmonic perturbations with frequencies \( \omega_g \), and one can expect a resonance response of the system when \( \omega_g \) is close to \( \omega_n \). It should be noted that the corresponding velocity \( v_1 \sim \omega_1 / g_1 \) is rather low (typically about 1–10 m/s) while the thermal ve-
Locality of electrons in the liquid state is usually much higher (about 3·10⁻³ m/s). Therefore the single-electron Cherenkov emission of ripplons is a quite usual phenomenon contributing to resistivity of SEs on liquid helium. The important point is that for a moving WS with \( g \mathbf{v} \rightarrow \omega_q \), the response of the system can be considered as a coherent Bragg–Cherenkov (BC) scattering. This effect was first described by the usual perturbation treatment \([6]\) which leads to symmetrical peaks of the electron collision rate with non-Lorentzian tails.

Besides direct BC scattering effects limiting the WS velocity \([5]\), there are other interesting nonlinear conductivity (\( \sigma \)) phenomena observed for SE transport over superfluid helium. In the presence of the magnetic field oriented normally to the surface, \( \sigma^{-1} \) as a function of the input voltage \( V \) has a remarkable N-type anomaly \([7]\). The decreasing part of \( \sigma^{-1}(V) \) was attributed to the BC scattering. When studying WS confined in the channel geometry, periodic conductance oscillations with varying the drift velocity were observed \([8]\). These oscillations were attributed to anisotropic spatial order with lines of electrons along the channel edges. Complicated nonlinear conductivity of the WS was observed for current frequencies which are close to typical frequencies of plasmon–ripplon coupled modes \([9]\). Interesting WS velocity jumps caused by the decoupling of electrons from the surface deformation were recently observed for the WS in a channel \([10]\).

Unfortunately, for \( q \rightarrow g_1 \) the electron–ripplon coupling is strong which leads to a huge increase of the electron effective mass at low frequencies due to surface dimples \([4]\). Under such conditions the perturbation treatment is doubtful and coupled WS phonon–ripplon modes are usually treated in a self-consistent way \([11,12]\). In this treatment the most of the interaction Hamiltonian of the WS with ripplons of \( q \rightarrow g_1 \) is included in the description of the coupled modes. Therefore, a simple classical model of coherent BC scattering of capillary waves for the WS moving with a constant velocity \([13]\) seems to be more appropriate than a perturbation treatment. This model was introduced in order to analyze a complicated nonlinear magnetoconductivity observed previously \([7]\). An extension of this model applied to liquid \( ^3 \)He allows to explain the nonlinear field-velocity characteristics of the WS under strong ripplon damping conditions \([14,15]\).

Application of the models of coherent BC scattering to the nonlinear WS transport on liquid \( ^4 \)He is difficult for several reasons. First, experimental geometries usually imply that the driving electric field and electron current density are not spatially uniform. Secondly, measurements are done under AC conditions when the electron velocity and driving electric field are periodic functions of time with the period \( 2\pi / \omega \) which is much longer than the typical ripplon oscillation period \( 2\pi / \omega_1 \). Moreover, the damping of ripplons \( \gamma_q \) in superfluid \( ^4 \)He is anomalously small \([16]\), which means that BC peaks of the classical model are extremely narrow and some other effects not included in the model can significantly affect its main results.

In this work we study the surface-displacement profile and field-velocity relationship for spatially uniform alternating motion of the WS over superfluid \( ^3 \)He and \( ^4 \)He under small ripplon damping conditions: \( \gamma_q \ll \omega_1 \). For arbitrary frequency of the current \( \omega \), the exact expression for the field-velocity relationship can be found. This solution appears to be strongly dependent on the ratio \( \omega / \gamma_g \). Therefore, we separate two frequency regions: \( \omega \ll \gamma_g \) and \( \omega \gg \gamma_g \). In both regions, the nonlinear field-velocity characteristics obtained here for time averaged quantities differ significantly from those obtained in the DC model. In the high frequency region \( \gamma_g \leq \omega \ll \omega_1 \), we expect the appearance of a new BC scattering regime of the WS transport caused by interference of ripplons excited at different time intervals. This frequency region is usually realized in experiments on nonlinear WS transport over superfluid \( ^4 \)He, and, therefore, we expect that our new results will help to understand the nonlinear electronic response observed.

**Dimple profile evolution induced by the drift velocity**

The analysis of the classical BC scattering given in Ref. 13 was restricted to a simplified one-dimensional DC model. The damping effects were considered in a phenomenological way. Here we consider more realistic 2D model of alternating motion of the WS with a particular ripplon damping defined for both \( ^3 \)He and \( ^4 \)He. We investigate shape variations induced by the WS velocity and ripplon damping which are very important for understanding the nonlinear WS transport.

We assume spatially uniform motion of the WS, which means that all electron lattice sites have the same displacement vector \( \mathbf{s}(t) \) in external fields. In this case the electron pressure at the interface induced by the WS moving with an arbitrary velocity can be presented in the following form:

\[
P^{(el)}(r,t) = n_s \sum_g \tilde{V}_g \exp \left[ g \cdot (r - \mathbf{s}(t)) \right],
\]

where \( \tilde{V}_g = V_g \exp(-q^2 \langle u_q^2 \rangle / 4) \), the electron–ripplon coupling \( V_g \) depends on the holding electric field \( E_\perp \) directed normally to the surface and on the wavenumber \( q \) \([2]\), and \( \langle u_q^2 \rangle \) is the mean-square displacement of electrons from lattice sites due to fast coupled phonon–ripplon modes whose frequencies are limited by \( \omega_f \gg \omega_1 \). Actually, \( \omega_f \) is the frequency of electron oscillations in the potential of a steady dimple. A simple self-consistent treatment \([12]\) gives

\[\text{Fizika Nizkikh Temperatur, 2009, v. 35, No. 5}\]
The ripplon damping of superfluid 3 He decreases with $k_F$, where $k_F$ is the Fermi momentum of quasiparticles in liquid 3 He, $\Delta$ is the excitation gap, and $f(x) = (e^x + 1)^{-1}$. The ripplon damping of superfluid 3 He decreases with cooling at an exponential rate. Still, in experiments on WS it can be just reasonably small ($\gamma_g / \omega_g \sim 0.1$ or 0.01).

In contrast, the ripplon damping in superfluid 4 He is anomalously small. In the ballistic regime it is given by [16]

$$\gamma_q = \frac{\pi^2 \hbar}{60 \rho} \left( \frac{T}{\hbar v_{4He}} \right)^4 q,$$

where $v_{4He}$ is the first sound velocity. For $n_s = 10^9$ cm$^{-2}$ and $T = 0.5$ K, a simple estimate gives $\gamma_g / \omega_g \sim 10^{-4}$.

Thus, for WS transport on superfluid 4 He the damping coefficient of ripplons is extremely small. It is remarkable that this damping coefficient has the same dependence on the wave-vector $q$ as that found for the ballistic regime of liquid 3 He [Eq. (6)].

In the reference frame moving along with the WS, the dimple profile given by Eq. (5) can be evaluated as

$$\xi(r) = -\sum_{g} \frac{n_g \vec{V}_g}{\rho |\vec{D}(\vec{v})|^2 -(\vec{g} \cdot \vec{v})^2 | \cos (\vec{g} \cdot \vec{x}) - 2 \vec{g} \vec{v} \sin (\vec{g} \cdot \vec{x}) |},$$

where $\vec{g} = s \rho \cos (\omega t)$. This equation represents dimple sublattice moving in-phase with the WS. In the limit $\vec{g} \rightarrow 0$, Eq. (5) surely gives the well known shape of steady dimples which is independent of damping. If $\vec{g} \neq \vec{0}$, the shape of dimples is affected by the WS velocity and damping. Under the condition $\omega << \gamma_g$, the dimple shape changes with time continuously in such a way that it is always adjusted to a given velocity $\vec{v}(t)$. In other words, for any fixed value of $\vec{v}(t)$ the dimple shape is the same as that defined by the DC theory with the corresponding WS velocity.

For liquid 3 He, the weak ripplon-damping regime can be realized only for superfluid phase at $T < 0.3$ mK. In this case, $\gamma_g$ is determined by ballistic bulk quasiparticle scattering from an uneven interface [15]:

$$\gamma_g = \frac{\hbar k_F}{8 \pi^2 \rho} f(\Delta / T) q,$$

where $k_F$ is the Fermi momentum of quasiparticles in liquid 3 He, $\Delta$ is the excitation gap, and $f(x) = (e^x + 1)^{-1}$. The ripplon damping of superfluid 3 He decreases with cooling at an exponential rate. Still, in experiments on WS it can be just reasonably small ($\gamma_g / \omega_g \sim 0.1$ or 0.01).
friction acting on the dimple sublattice by the environment to the electron crystal.

For significantly smaller damping coefficient \( \gamma_1/\omega_1 = 0.1 \), which is expected for superfluid \( ^3 \text{He} \) at \( T = 0.2 \) mK, substantial shape changes appear only if the WS velocity approaches the first BC resonance or exceeds it. Dimple shape variations which occur near the vicinity of the first resonance are shown in Fig. 2. The important point is that the velocity-induced displacements found for \( u = u_c = 2/\sqrt{\pi} \) are much larger (about 15 times) than surface displacements in the initial surface dimple. This means that dynamic decoupling of the WS from surface dimples which shall be discussed below for a fixed field condition is accompanied by creation of huge displacement waves moving in the same direction.

At \( u < u_c \), in spite of huge changes in the dimple profile the average position of an electron remains the same being fixed to the potential minimum formed by the dimple potential and the driving electric field. For experimen-

![Fig. 1. Variations of the dimple sublattice profile \( \xi(x,0) \) induced by the WS velocity for two typical velocity orientations: NN direction \((a)\), and SN direction \((b)\). Steady dimples are shown by the solid line. The dimensionless velocity increases from \( u = 0.6 \) to higher values by steps equal 0.1. Calculations are performed for superfluid \( ^3 \text{He} \), \( n_s = 10^8 \) cm\(^{-2}\), \( E_\perp = 189 \) V/cm, and \( T = 0.277 \) mK (\( \gamma_1/\omega_1 = 0.1 \)).](image1)

![Fig. 2. Variations of the dimple sublattice profile \( \xi(x,0) \) induced by the WS velocity near the first BC resonance for very small ripplon damping \( \gamma_1/\omega_1 = 0.01 \). The drift velocity is oriented along the NN direction. Calculations are performed for superfluid \( ^3 \text{He} \), \( n_s = 10^8 \) cm\(^{-2}\), \( E_\perp = 189 \) V/cm, and \( T = 0.2 \) mK.](image2)
because the dimple lattice is rearranged in a raw of valleys (terms with $v g = 0, d$ do not contribute in $M_d$ because of the proportionality factor $g^2$).

**Field-velocity characteristics**

The asymmetry of the dimple shape with regard to the average electron position in a lattice site causes a force $F^{(D)}$ acting on the electron solid. In equilibrium this force is balanced by an external driving field. By definition, $F^{(D)}$ is the sum of forces acting on each electron $-\nabla V_{\text{int}}/\partial x_e$ averaged over electron distribution within the dimple [here $V_{\text{int}}(r)$ is the electron-ripplon interaction Hamiltonian whose Fourier transform $V_g$ was used in Eq. (1)]. Electron distribution caused by long wave-length fluctuations with $\omega < \omega_0$ occurs together with surface dimples and therefore it should be excluded from the averaging. Then, the average of the electron density operator $n_e = \sum_q \exp(i q \cdot r_e)$ can be found as

$$
n_e = N_e \exp[-q^2 \langle u_a^2 \rangle / 4 + i q \cdot s(t)] \delta_{q, g}. \quad (10)
$$

Using this equation and Eq. (4) in the general expression $-\langle \nabla V_{\text{int}}/\partial r_e \rangle f$, the force acting on the WS can be written as

$$
F^{(D)}(t) = -N_e \sum_g \frac{n_g V^2_g}{\rho_0 g} \int_0^{\infty} \sin(\omega_g \tau) e^{-\gamma_g \tau} \times \sin \left[ g [s(t) - s(t-\tau)] \right] d\tau. \quad (11)
$$

For any spatially uniform displacement $s(t)$ given, this equation defines the in-plane force induced by the dimple sublattice.

If the time scale of the WS displacement vector $s(t)$ is much longer than $\tau = 1/\gamma_g$, then $g [s(t) - s(t-\tau)]$ can be approximated by $g v(t) \tau$. In this limit $F^{(D)}$ has the same form at that given by the DC treatment with the constant velocity replaced by $v(t)$. In equilibrium, $F^{(D)}(t)$ as well as the kinetic friction of the electron lattice $F^{(s)}(t) = -N_e v_e v$ caused by electron scattering of other kinds are balanced by the external force $N_e e E$ (here we assume that magnetic field is zero). The solution of the balance equation can be represented as a field-velocity characteristic $E(v)$:

$$
E(v) = \frac{m_e v_e v}{e g} \sum_g \frac{g^2 V^2_g}{\omega_0^2 - g^2 v^2} + 4 \gamma_g^2 g^2 v^2 \left( \frac{m_e v_e v}{e g} \right)^2 + m_e v_e v. \quad (12)
$$

Using this equation, $E(v)$ can be calculated numerically for any given damping coefficient and the collision frequency $v_e$ caused by electron scattering with thermal ripplons, vapor atoms, or even walls if WS is formed in a channel geometry.

For the DC case, Eq. (12) is a two-dimensional extension of the classical one-dimensional model of BC scattering reported previously [13] with the real ripplon damping parameter and with more accurate electron-ripplon coupling. The driving field found from the balance equation has sharp maxima in the vicinity of BC resonance conditions $g^2 v^2 \rightarrow \omega_0^2$. If the driving field is given and the WS velocity is adjusted to the field, then regions with $dE/dv < 0$ are unstable. This means that for driving fields exceeding the major maximum of $E(v)$ the balance of forces is not possible and the WS decouples from surface dimples. According to Fig. 2, decoupling of the WS is accompanied by creation of huge surface waves moving with the group velocity $u_e$.

In experiments on WS transport, usually it is the current which is given, while the driving field is adjusted to the current by electron redistribution which screens the external potential variations. This is supported by the fact that regions with $dE/dv < 0$ are experimentally observed [8, 14]. Therefore, field-velocity characteristics of electrons moving ultra-fast with $u > u_e$ are very important for understanding the nonlinear WS transport on superfluid helium.

For liquid $^4$He with $\gamma_g / \omega_g \sim 10^{-3}$, Eq. (12) applied to the DC case would give just a set of extremely sharp peaks. At the same time, beyond these peaks $E(v)$ is close to zero. For liquid $^3$He, the parameter $\gamma_g / \omega_g$ can be much larger than it is for liquid $^4$He (even larger than unity) and BC peaks of $E(v)$ can substantially overlap. It is worth noting that $E(v)$ depends strongly on the velocity vector orientation with regard to the 2D electron lattice. For example, in the NN direction of motion there is only one major peak with $|g| = g_1$ at $u = 2/\sqrt{2} \omega$, while for the SN direction there are two equivalent peaks with $|g| = g_1$ at $u = 1$ and $u = 2$.
complication: DC measurements are practically impossible and by now all data are obtained under AC conditions. This means that WS velocity and the driving electric field are periodic functions of time with the period $2\pi/\omega_0$. The frequency of experimental signal usually varies from $10^4$ s$^{-1}$ to about $6 \cdot 10^5$ s$^{-1}$ which is much lower than the typical ripplon frequency $\omega_1$. Therefore, it is conventionally expected that the DC model of WS transport should give qualitatively correct description of data obtained in such experiments. We shall see that this is not true for two major reasons.

First, we note that the condition $\omega < \omega_1$ is not sufficient for adiabatic adjustment of surface displacements to WS velocity variations. For example at the BC resonance condition ($u = u_c$) we have a huge wave which follows the WS and for a change to much smaller displacements a shape variation (without changing the amplitude) is not sufficient. At the BC resonance the capillary wave accumulates great energy which should be transferred to the environment in order to make the transition from the surface displacements calculated for $u = u_c \approx 1.155$ (see Fig. 2) to the surface displacements calculated for $u = 1.3$ or even for $u = 1.1$. This means that an adiabatic AC extension of the DC model requires an additional restriction: the frequency of the current should be much lower than the corresponding ripplon damping ($\omega < \gamma_1$). This is actually the condition which allowed us to transform Eq. (11) into Eq. (12). For WS transport over superfluid $^3$He this condition is satisfied even at $\gamma_1 / \omega_0 \approx 10^{-2}$. Regarding liquid $^4$He, the condition $\omega < \gamma_1$ requires to use an AC frequency which is much lower than $10^4$ s$^{-1}$.

Secondly, even if the above noted condition is fulfilled, Eq. (12) cannot be used directly for plotting field-velocity characteristics. Under AC conditions, it is important which quantities are actually measured and presented in the field-velocity characteristics. If time averaged quantities are considered, in the nonlinear regime the outcome can be qualitatively different for different kinds of averaging. For example, even for harmonic velocity $u_0 \cos \omega t$, the mean-square time averaging of the driving field $\sqrt{\langle E^2 \rangle}$ and averaging of the absolute value $\langle |E| \rangle$ give qualitatively different results for the function $E(u_0 \cos \omega t)$ defined by Eq. (12). This could be easily proven by considering the right-side tail of the BC resonance in the limiting case $\gamma_g \to 0$. For averaging $\langle |E| \rangle$, in this limit a part of the corresponding integrand can be rearranged as the $\delta$-function [see Eq. (12)], and the final result will not depend on $\gamma_g$. In contrast, the integrand of the quantity $\sqrt{\langle E^2 \rangle}$ is squared, and, therefore, the resonance tail of $\sqrt{\langle E^2 \rangle}$ increases with reducing $\gamma_g$.

As a measure of the alternating field one can choose the main term of the Fourier series representing $E(t)$:

$$E_\omega = \frac{\alpha_0}{\pi} \int \frac{E(t) \cos(\omega t)}{-\nu/\omega} dt .$$

(13)

It is also a kind of time averaging, and similar to $\langle |E| \rangle$ it has a finite resonance tail in the limiting case $\gamma_g \to 0$. In the following we shall consider only quantities $E_\omega$ and $\langle |E| \rangle$ which will be used for presenting the field-velocity relationship.

Assume that the condition of given current is realized and $u(t) = u_0 \cos \omega t$. Using the adiabatic treatment discussed above, we insert $\gamma(t) = \gamma_1 u(t)$ into Eq. (12) and evaluate the time integral of Eq. (13). It is convenient to introduce two integer variables $m$ and $n$ to describe the reciprocal lattice vectors $g_{m,n} = mg^{(1)} + ng^{(2)}$ (here $g^{(1)}$ and $g^{(2)}$ are primitive vectors of this lattice). Then, after some algebra, the Fourier transform $E_\omega$ can be represented as a function of the velocity amplitude

$$E_\omega(u_0) = v_1 \frac{n_1}{e\alpha^2} \sum_{g_{1,m,n}} \frac{P_{m,n}}{S_{m,n}} \frac{\gamma_{g_{m,n}}}{\omega_{g_{m,n}}} Q(p_{m,n}u_0, \beta_{g_{m,n}}) +$$

$$+ m_n v_e \nu c_1 u_0 .$$

(14)

Here we use the following dimensionless notations

$$p_{m,n} = \left( \frac{(g_{m,n})^2}{g_1^2} \right)^{1/2}, \quad \beta_{g_{m,n}} = \frac{\gamma_{g_{m,n}}}{\omega_{g_{m,n}}} .$$

(15)

$$Q(w, \beta) = \frac{4 w_0^2}{\pi} \int_0^\infty dy \left[ (1 + y^2 - w_0^2)^{1/2} + w_0^2 \beta^2 (1 + y^2)^{1/2} \right] .$$

(16)

The integral $Q(w, \beta)$ can be evaluated in an analytical form:

$$Q(w, \beta) = \frac{2 \beta}{wG} \frac{2}{\sqrt{w^2 - 1 - \beta^2 / 2 - i w^2 G}}$$

(17)

where $G(\beta) = \beta \sqrt{1 - \beta^2 / 4}$. Remakrably, in the limit $\gamma_g \to 0$ ($\beta \to 0$) for $w > 1$ there is a finite asymptote

$$Q(w, \beta) \to \frac{20(w-1)}{w\sqrt{w^2 - 1}} ,$$

(18)

where $\theta(x)$ is the unit step-function.

For $\langle |E| \rangle$ as a function of the dimensionless velocity amplitude $u_0$, an equation similar to Eq. (14) is found. The only difference which appears for such averaging is that instead of $Q(w, \beta)$ we should use another function $I(w, \beta)$ defined by
\[ I(w, \beta) = \frac{2\beta}{\pi G} \Re \left[ \frac{\arctan \left( \frac{iw}{\sqrt{w^2 - 1 + \beta^2 / 2 + iG}} \right)}{\sqrt{w^2 - 1 + \beta^2 / 2 + iG}} \right]. \quad (19) \]

As expected in the limiting case \( \gamma_g \to 0 \) (\( \beta \to 0 \)), it also has a finite asymptote

\[ I(w, \beta) \to \frac{\theta(w-1)}{\sqrt{w^2-1}}, \quad (20) \]

which means that the right-side tails of BC resonances are independent of small damping.

Already from the analysis of the integrals \( Q(w, \beta) \) and \( I(w, \beta) \) given above one can conclude that BC resonance tails of the AC treatment differ (even qualitatively) from that found for the DC models. The left-side tail becomes even steeper for both \( E_\alpha(u_0) \) and \( \langle |E| \rangle \), while the right-side tails extend far beyond the resonance and are independent of ripplon damping in the limit \( \gamma_g \to 0 \). This behavior is illustrated in Fig. 4 where \( E_\alpha(u_0) \) is plotted for two typical directions of the WS velocity [NN direction (solid line), and SN direction (dashed line)] assuming \( \gamma_g \to 0 \). The other parameters are taken for the liquid \(^4\text{He}\) case. Thus, instead of \( \delta \)-peaks of the classical BC scattering model here we have saw-tooth shaped peaks with long right-side tails. As noted above, field-velocity characteristics depend strongly on the direction of the WS velocity. It is interesting that for the SN direction there are two major BC peaks and the second one (at \( u_0 \to 2 \)) becomes even more prominent than the first one because in the AC cause at \( u_0 > 2 \) the velocity sweeps through the both resonances. Of course, considering the limiting case \( \gamma_g / \omega_1 \to 0 \) we should keep in mind that \( \omega \) should be much smaller than \( \gamma_g \). Therefore, real ripplon damping should be taken into account for consistent analysis of field-velocity relationships induced by BC scattering.

For nonlinear WS transport over superfluid \(^4\text{He}\), the ripplon damping parameter is extremely small and the condition \( \gamma_g < \omega \ll \omega_g \) is realized in most of known experiments. This case requires a special treatment because large surface displacements excited at a first instance of \( u(t) = u_e \) cannot be relaxed back within the period of current oscillations. Therefore a new steady regime will be developed for each AC frequency, which can be far away from the solutions found above for the condition \( \omega \ll \gamma_g \).

We have to return back to the exact solution of Eq. (11). In this equation now we insert \( s_x(t) = (v_0/\omega) \sin (\omega t) \) and then perform the time averaging defined by Eq. (13). Here we disregard the unimportant correction induced by \( v_e \).

Then quite generally, \( E_\omega(v_0) \) can be found as

\[ E_\omega(v_0) = \sum_g g_\omega u_n g_r^2 \int_0^\infty \sin (\omega g \tau) e^{-\gamma g T} \cos \left( \frac{\omega \tau}{2} \right) \times \]

\[ \times J_1 \left( 2 g_x v_0 \sin \left( \frac{\omega \tau}{2} \right) \right) d\tau, \quad (21) \]

where \( J_1(z) \) is the Bessel function of the first kind. For \( E_\omega \) as a function of the dimensionless velocity amplitude \( u_0 \), an equation similar to Eq. (14) can be found. In the general case, the integral \( Q(w, \beta) \) of Eq. (14) should be replaced by \( Q(w, \beta, \omega') \) defined as

\[ Q(w, \beta, \omega') = 4 \int_0^\infty \sin (2y) e^{-\beta y} \cos (\omega' y) \times \]

\[ \times J_1 \left( \frac{2w}{\omega'} \sin (\omega' y) \right) dy, \quad (22) \]

where \( \omega' = \omega / \gamma_g \). The dimensionless function \( Q(w, \beta, \omega') \) describes a shape of a single BC resonance of \( E_\omega(u_0) \) for arbitrary current frequency and ripplon damping (here \( w \propto u_0 \) and \( \beta \propto \gamma_g \)). It is easy to see that in the limiting case \( \omega' \to 0 \) and \( \beta \to 0 \) analyzed above, Eq. (22) provides the correct asymptote shown in Eq. (18). For conditions \( \gamma_g < \omega < \omega_g \), and \( g_r v_0 > 0 \), the argument of the Bessel function entering the integrand of Eq. (21) attains huge numbers because \( \sin (\omega \tau / 2) \sim 1 \). This can lead to remarkable field-velocity relationships with side-oscillations which we shall discuss in the following.

**Results and discussions**

Consider briefly the velocity-field relationship \( E_\omega(u) \) for WS transport over superfluid \(^3\text{He}\) at \( T = 0.25 \text{ mK} \). At such a temperature the damping coefficient \( \gamma_g \) is \( \simeq 1.54 \times 10^6 \text{ s}^{-1} \) which is much lower than the BC resonance

![Fig. 4.](image-url)
frequency \( \omega_0 \) and is still much higher than the typical current frequency used in experiments. The later condition makes the adiabatic AC extension of the BC scattering model introduced here applicable. The results of numerical evaluations of Eq. (14) are shown in Fig. 5. The velocity-field characteristic given by the DC classical model is shown by the dotted line. It consists of series of BC peaks with nearly symmetric tails. For the first Fourier coefficient as a function of the drift velocity amplitude, the AC theory gives lower peaks which are asymmetric with regard to the maximum positions. It is important that time averaging used in evaluation of \( E_{0a} \) do not smooth out the BC resonances completely. Another important feature of the AC treatment discussed here is the appearance of long right-side tails of the BC resonances. The left-side tails \( (u_0 < u) \) become even more steeper, because under AC conditions electrons spend only a little time near the BC resonance.

The most interesting experimental results on nonlinear WS transport were obtained employing liquid \( ^{4}\text{He} \) [5, 7]. In this case even for \( T \approx 0.5 \text{ K} \) the ripplon damping parameter given by Eq. (7) is extremely small. First, we assume that \( \omega_0 \) is low enough to make Eq. (14) applicable. The comparison of results obtained for the two kinds of averaging of the electric field \( \langle E_{0a} \rangle \) and \( \langle \| E \| \rangle \) is given in Fig. 6, assuming that the velocity is along the NN direction and \( v_e = 0 \). Here the sharp peaks of the DC model (dotted line) are strongly smoothed by the time averaging of the AC model \( \langle E_{0a} \rangle \) is shown by the solid line, and \( \langle \| E \| \rangle \) — by the dashed line). Additionally, the maximum values of the BC peaks are greatly reduced as compared to the results calculated for the DC case.

The influence of a finite electron collision frequency \( v_e \) due to scattering with thermal ripplons and walls is analyzed for the SN direction and shown in Fig. 7. A reasonable estimate for the electron collision frequency \( v_e \approx 2.4 \times 10^{-9} \text{ s}^{-1} \) (for chosen \( n_e, E_{\perp} \) and \( T \)) is found considering electron scattering with thermal ripplons in the usual way [2] and taking into account that at low temperatures the average kinetic energy of an electron in the WS state differs substantially from \( T \). For experiments with WS in the channel geometry, \( v_e \) can be even higher because of the WS friction at the channel walls. In order to illustrate this effect is Fig. 7 we considered also a larger value \( v_e \approx 7.5 \times 10^{-9} \text{ s}^{-1} \). According to this figure, the

**Fig. 5.** Field velocity relationship for the DC case (dashed line), and the first Fourier coefficient of the driving field \( E_{0a} \) vs the drift velocity amplitude for the AC case (solid line). Drift velocity is oriented along the NN direction. Calculations where performed for superfluid \( ^{3}\text{He} \), \( n_s = 10^8 \text{ cm}^{-2}, T = 0.25 \text{ mK} \).

**Fig. 6.** Field velocity relationship for two kinds of averaging of the alternative driving field: \( E_{0a} \) (solid line) and \( \| E \| / 2 \) (dashed line). Drift velocity is oriented along the SN direction. Calculations where performed for superfluid \( ^{4}\text{He} \), \( n_s = 10^8 \text{ cm}^{-2}, T = 0.5 \text{ K} \) with the ripplon damping parameter defined by Eq. (7).

**Fig. 7.** The main Fourier coefficient of the driving field \( E_{0a} \) vs the drift velocity amplitude for different values of the electron collision frequency \( v_e \): 0 (solid line), \( 2.4 \times 10^{-9} \text{ s}^{-1} \) (dashed line), and \( 7.5 \times 10^{-9} \text{ s}^{-1} \) (short dashed line). Drift velocity is oriented along the NN direction. The DC case results are shown by the dotted line. Calculations where performed for superfluid \( ^{4}\text{He} \), \( n_s = 10^8 \text{ cm}^{-2}, T = 0.5 \text{ K} \).
electron collision frequency $v_e$ affects the both tails of the field velocity characteristic. At the left-side $v_e$ and time averaging of Eq. (13) act in the opposite ways. The left-side tail ($u_0 < u_c$) becomes less steep for a finite $v_e$. At the right-side $v_e$ acts in the same way as the time averaging, increasing the right-side tail and making $E_{\omega}(u_0)$ more flatter in the region $u_0 > u_c$. In general, due to the both these effects the field-velocity characteristic of the WS acquires a distinctive «nose» shape.

The numerical calculations presented in Figs. 6 and 7 were done assuming $\omega << \gamma_g$. In experiments on WS transport over superfluid $^4$He this condition was not realized. The influence of the condition $\omega > \gamma_g$ on the field-velocity characteristics can be understood using the general expressions for $E_{\omega}(v_0)$ and $Q(w, \beta, \omega')$ given in Eqs. (21) and (22). The main features of BC scattering under AC conditions can be revealed from the dimensionless function $Q(w, \beta, \omega')$ which describes the shape of a single BC resonance [see Eq. (14)]. Consider the main BC resonance when we can set $w = u_0$, $\beta = 2\gamma_g/\omega_0$ and $\omega' = \omega_0/\omega_1$, and for simplicity assume $\beta = 0.1$. In the limiting case $\omega' << \beta$, the function $Q(u_0, \beta, 0)$ coincides with $Q(u_0, \beta)$ obtained in Eq. (17). For example, it is practically impossible to distinguish $Q(u_0, \beta, 0.0001)$ shown in Fig. 8 by the solid line from $Q(u_0, \beta)$ given by Eq. (17). As a function of the dimensionless velocity, $Q(u_0, \beta, 0.0001)$ has a typical saw-tooth shape discussed above.

Remarkable shape transformations of $Q(u_0, \beta, \omega')$ as a function of $u_0$ occur when $\omega'$ approaches and exceeds the value of the parameter $\beta$ which is proportional to ripplon damping. A sharp (from the left-side) saw-tooth peak of a single BC resonance is developed into distinctive smooth oscillations which [according to Eq. (14)] result in similar oscillations of $E_{\omega}(u_0)$. The amplitude and the period of oscillations are gradually increase with $\omega'$ in the range considered. These oscillations represent a new regime of BC scattering of the WS which occur under the AC condition, when the current frequency becomes comparable or larger than the ripplon damping.

The period of new conductivity oscillations depends on the relation between the frequency of the current $\omega$ and the frequency of ripplons excited $\omega_g$. According to Fig. 8, it increases with the ratio $\omega / \omega_g$. The ripplon damping just decreases the amplitude of oscillations. The later is illustrated in Fig. 9 where $Q(u_0, \beta, \omega')$ is plotted vs $u_0$ for a fixed value of the ratio $\omega / \omega_1 = 0.05$ and different values of the damping parameter.

There are other important points which follow from Figs. 8 and 9. For low damping the first maximum of the field-velocity characteristic can be substantially larger than the BC peak value found in the limiting case $\omega << \gamma_g$. Secondly, due to the finite frequency $\omega$, even for a very small ripplon damping the left-side tail is not steep as it was for $\omega << \gamma_g$. Additionally, the maximum position is substantially shifted to higher drift velocities.

The physics of side-oscillations in the field-velocity relationship can be explained as follows. If $u_0 > u_c$, then even during a period of current oscillations the WS passes through the BC scattering point four times. The phase difference between surface waves of the same $\mathbf{q} = \mathbf{g}$ excited at different times increases with the velocity amplitude. Thus, depending on the velocity amplitude $u_0$ the excited waves can interfere constructively or destructively, which is the reason for the side-oscillations. The higher frequency of the current, the larger amplitude is necessary to produce the same phase-difference.
Most of experiments on the nonlinear WS transport are performed using a Corbino geometry of which the alternating current is spatially nonuniform, and only a limited area of the WS can satisfy the BC scattering conditions. This area changes with time because of the AC conditions. This experimental situation is very difficult to analyze. One may conclude that spatial variations of the current would additionally smooth out the BC resonances. It should be noted that the field-velocity characteristics of the WS with a «nose» shape without the BC peaks where observed in the experiment on WS transport in a channel geometry [8]. It is interesting that conductance oscillations similar to the BC oscillations shown in Fig. 8 were also reported in this experiment. Therefore, our theoretical results give an alternative explanation for oscillations in electronic response observed for SEs on superfluid helium $^4$He.

Conclusions

In summary, we have analyzed the nonlinear WS transport over superfluid $^3$He and $^4$He under AC conditions. The theory developed for a spatially uniform alternating current indicates that the field-velocity relationship obtained previously in the classical DC model of BC scattering is not applicable for time averaged quantities such as the first Fourier coefficient. The detailed analysis is given for two important limiting cases of low and high frequency of the electron current. For frequencies which are much lower than the ripplon damping coefficient, calculations based on the new theory lead to asymmetric BC peaks of a saw-tooth shape which are strongly broadened at the right side. The broadening of the right-side tails do not depend on small ripplon damping. The left-side tails of the BC resonances become even steeper which preserves the main BC anomaly in the field-velocity characteristic.

For current frequencies which are comparable with the ripplon damping or even higher, the new nonlinear regime of BC scattering of the WS is reported. In this regime each BC peak is transformed into an oscillatory field-velocity relationship due to interference of ripplons multiply excited at different times. The evolution of surface displacements of the dimple sublattice with increasing the current amplitude calculated in this work, as well as the new field-velocity relationships obtained for alternating current, help to understand the nonlinear conductivity of the WS on superfluid helium observed in different experiments.

The work is partly supported by the Grant-in-Aids for Scientific Research from Monka-sho.

1. Electrons on Helium and Other Cryogenic Substrates, E.Y. Andrei (ed.), Kluwer Academic Pub., Dordrecht (1997).
2. Yu.P. Monarkha and K. Kono, Two-Dimensional Coulomb Liquids and Solids, Springer-Verlag, Berlin, Heidelberg (2004).
3. C.C. Grimes and G. Adams, Phys. Rev. Lett. 42, 795 (1979).
4. D.S. Fisher, B.I. Halperin, and P.M. Platzman, Phys. Rev. Lett. 42, 798 (1979).
5. A. Kristensen, K. Djerfi, P. Fozooni, M.J. Lea, P.J. Richardson, A. Santrich-Badal, A. Blackburn, and R.W. van der Heijden, Phys. Rev. Lett. 77, 1350 (1996).
6. M.I. Dykman and Yu.G. Rubo, Phys. Rev. Lett. 78, 4813 (1997).
7. K. Shirahama and K. Kono, Phys. Rev. Lett. 74, 781 (1995).
8. P. Glasson, V. Dotsenko, P. Fozooni, M.J. Lea, W. Bailey, and G. Papageorgiou, Phys. Rev. Lett. 87, 176802 (2001).
9. V.E. Syvokon, K.A. Nasedkin, and A.S. Neoneta, Fiz. Nizk. Temp. 34, 761 (2008) [Low Temp. Phys. 34, 600 (2008)].
10. H. Ikekami, H. Akimoto, and K. Kono, accepted for publication in: Phys. Rev. Lett.
11. H. Namaizawa, Solid State Commun. 34, 607 (1980).
12. Yu.P. Monarkha and V.B. Shikin, Fiz. Nizk. Temp. 9, 913 (1983) [Sov. J. Low Temp. Phys. 9, 471 (1983)].
13. W.F. Vinen, J. Phys.: Condens. Matter 11, 7079 (1999).
14. K. Shirahama, Yu.P. Monarkha, and K. Kono, Phys. Rev. Lett. 93, 176805 (2004).
15. Yu.P. Monarkha and K. Kono, J. Phys. Soc. Jpn. 74, 960 (2005).
16. P. Roche, M. Roger, and F.I.B. Williams, Phys. Rev. B53, 2225 (1996).