Shimura varieties and moduli

J.S. Milne

Abstract. Connected Shimura varieties are the quotients of hermitian symmetric domains by discrete groups defined by congruence conditions. We examine their relation with moduli varieties.

Contents

Notations 470
1 Elliptic modular curves 471
Definition of elliptic modular curves 471
Elliptic modular curves as moduli varieties 472
2 Hermitian symmetric domains 475
Preliminaries on Cartan involutions and polarizations 475
Definition of hermitian symmetric domains 477
Classification in terms of real groups 478
Classification in terms of root systems 479
Example: the Siegel upper half space 480
3 Discrete subgroups of Lie groups 481
Lattices in Lie groups 481
Arithmetic subgroups of algebraic groups 482
Arithmetic lattices in Lie groups 484
Congruence subgroups of algebraic groups 485
4 Locally symmetric varieties 486
Quotients of hermitian symmetric domains 486
The algebraic structure on the quotient 486
Locally symmetric varieties 488
Example: Siegel modular varieties 488
5 Variations of Hodge structures 489
The Deligne torus 490
Real Hodge structures 490
Rational Hodge structures 491
Polarizations 491
Local systems and vector sheaves with connection 492
Variations of Hodge structures 492
Introduction

The hermitian symmetric domains are the complex manifolds isomorphic to bounded symmetric domains. The Griffiths period domains are the parameter spaces for polarized rational Hodge structures. A period domain is a hermitian symmetric domain if the universal family of Hodge structures on it is a variation of Hodge
structures, i.e., satisfies Griffiths transversality. This rarely happens, but, as Deligne showed, every hermitian symmetric domain can be realized as the subdomain of a period domain on which certain tensors for the universal family are of type \((p, p)\) (i.e., are Hodge tensors).

In particular, every hermitian symmetric domain can be realized as a moduli space for Hodge structures plus tensors. This all takes place in the analytic realm, because hermitian symmetric domains are not algebraic varieties. To obtain an algebraic variety, we must pass to the quotient by an arithmetic group. In fact, in order to obtain a moduli variety, we should assume that the arithmetic group is defined by congruence conditions. The algebraic varieties obtained in this way are the connected Shimura varieties.

The arithmetic subgroup lives in a semisimple algebraic group over \(\mathbb{Q}\), and the variations of Hodge structures on the connected Shimura variety are classified in terms of auxiliary reductive algebraic groups. In order to realize the connected Shimura variety as a moduli variety, we must choose the additional data so that the variation of Hodge structures is of geometric origin.

The main result of the article classifies the connected Shimura varieties for which this is known to be possible. Briefly, in a small number of cases, the connected Shimura variety is a moduli variety for abelian varieties with polarization, endomorphism, and level structure (the PEL case); for a much larger class, the variety is a moduli variety for abelian varieties with polarization, Hodge class, and level structure (the PHL case); for all connected Shimura varieties except those of type \(E_6, E_7\), and certain types \(D\), the variety is a moduli variety for abelian motives with additional structure. In the remaining cases, the connected Shimura variety is not a moduli variety for abelian motives, and it is not known whether it is a moduli variety at all.

We now summarize the contents of the article.

§1. As an introduction to the general theory, we review the case of elliptic modular curves. In particular, we prove that the modular curve constructed analytically coincides with the modular curve constructed algebraically using geometric invariant theory.

§2. We briefly review the theory of hermitian symmetric domains. To give a hermitian symmetric domain amounts to giving a real semisimple Lie group \(H\) with trivial centre and a homomorphism \(u\) from the circle group to \(H\) satisfying certain conditions. This leads to a classification of hermitian symmetric domains in terms of Dynkin diagrams and special nodes.

§3. The group of holomorphic automorphisms of a hermitian symmetric domain is a real Lie group, and we are interested in quotients of the domain by certain discrete subgroups of this Lie group. In this section we review the fundamental theorems of Borel, Harish-Chandra, Margulis, Mostow, Selberg, Tamagawa, and others concerning discrete subgroups of Lie groups.
§4. The arithmetic locally symmetric varieties (resp. connected Shimura varieties) are the quotients of hermitian symmetric domains by arithmetic (resp. congruence) groups. We explain the fundamental theorems of Baily and Borel on the algebraicity of these varieties and of the maps into them.

§5. We review the definition of Hodge structures and of their variations, and state the fundamental theorem of Griffiths that motivated their definition.

§6. We define the Mumford-Tate group of a rational Hodge structure, and we prove the basic results concerning their behaviour in families.

§7. We review the theory of period domains, and explain Deligne’s interpretation of hermitian symmetric domains as period subdomains.

§8. We classify certain variations of Hodge structures on locally symmetric varieties in terms of group-theoretic data.

§9. In order to be able to realize all but a handful of locally symmetric varieties as moduli varieties, we shall need to replace algebraic varieties and algebraic classes by more general objects. In this section, we prove Deligne’s theorem that all Hodge classes on abelian varieties are absolutely Hodge, and have algebraic meaning, and we define abelian motives.

§10. Following Satake and Deligne, we classify the symplectic embeddings of an algebraic group that give rise to an embedding of the associated hermitian symmetric domain into a Siegel upper half space.

§11. We use the results of the preceding sections to determine which Shimura varieties can be realized as moduli varieties for abelian varieties (or abelian motives) plus additional structure.

Although the expert will find little that is new in this article, there is much that is not well explained in the literature. As far as possible, complete proofs have been included.

Notations

We use $k$ to denote the base field (always of characteristic zero), and $k^{\text{al}}$ to denote an algebraic closure of $k$. “Algebraic group” means “affine algebraic group scheme” and “algebraic variety” means “geometrically reduced scheme of finite type over a field”. For a smooth algebraic variety $X$ over $\mathbb{C}$, we let $X^{\text{an}}$ denote the set $X(\mathbb{C})$ endowed with its natural structure of a complex manifold. The tangent space at a point $p$ of space $X$ is denoted by $T_p(X)$.

Vector spaces and representations are finite dimensional unless indicated otherwise. The linear dual of a vector space $V$ is denoted by $V^\vee$. For a $k$-vector space $V$ and commutative $k$-algebra $R$, $V_R = R \otimes_k V$. For a topological space $S$, we let $V_S$ denote the constant local system of vector spaces on $S$ defined by $V$. By a lattice in a real vector space, we mean a full lattice, i.e., the $\mathbb{Z}$-module generated by a basis for the vector space.
A vector sheaf on a complex manifold (or scheme) $S$ is a locally free sheaf of $\mathcal{O}_S$-modules of finite rank. In order for $\mathcal{W}$ to be a vector subsheaf of a vector sheaf $\mathcal{V}$, we require that the maps on the fibres $\mathcal{W}_s \to \mathcal{V}_s$ be injective. With these definitions, vector sheaves correspond to vector bundles and vector subsheaves to vector subbundles.

The quotient of a Lie group or algebraic group $G$ by its centre $Z(G)$ is denoted by $G^{\text{ad}}$. A Lie group or algebraic group is said to be adjoint if it is semisimple (in particular, connected) with trivial centre. An algebraic group is simple (resp. almost simple) if it connected noncommutative and every proper normal subgroup is trivial (resp. finite). An isogeny of algebraic groups is a surjective homomorphism with finite kernel. An algebraic group $G$ is simply connected if it is semisimple and every isogeny $G' \to G$ with $G'$ connected is an isomorphism. The inner automorphism of $G$ defined by an element $g$ is denoted by $\text{inn}(g)$. Let $\text{ad}: G \to G^{\text{ad}}$ be the quotient map. There is an action of $G^{\text{ad}}$ on $G$ such that $\text{ad}(g)$ acts as $\text{inn}(g)$ for all $g \in G(k^{\text{ad}})$. For an algebraic group $G$ over $\mathbb{R}$, $G(\mathbb{R})^+$ is the identity component of $G(\mathbb{R})$ for the real topology. For a finite extension of fields $L/k$ and an algebraic group $G$ over $L$, we write $(G)_{L/k}$ for algebraic group over $k$ obtained by (Weil) restriction of scalars. As usual, $G_m = \text{GL}_1$ and $\mu_N$ is the kernel of $G_m \to \mathbb{G}_m$.

A prime of a number field $k$ is a prime ideal in $\mathcal{O}_k$ (a finite prime), an embedding of $k$ into $\mathbb{R}$ (a real prime), or a conjugate pair of embeddings of $k$ into $\mathbb{C}$ (a complex prime). The ring of finite adeles of $\mathbb{Q}$ is $\mathbb{A}_f = \mathbb{Q} \otimes \left( \prod_p \mathbb{Z}_p \right)$.

We use $\iota$ or $z \mapsto \bar{z}$ to denote complex conjugation on $\mathbb{C}$ or on a subfield of $\mathbb{C}$, and we use $X \simeq Y$ to mean that $X$ and $Y$ isomorphic with a specific isomorphism — which isomorphism should always be clear from the context.

For algebraic groups we use the language of modern algebraic geometry, not the more usual language, which is based on Weil’s Foundations. For example, if $G$ and $G'$ are algebraic groups over a field $k$, then by a homomorphism $G \to G'$ we mean a homomorphism defined over $k$, not over some universal domain. Similarly, a simple algebraic group over a field $k$ need not be geometrically (i.e., absolutely) simple.

1. Elliptic modular curves

The first Shimura varieties, and the first moduli varieties, were the elliptic modular curves. In this section, we review the theory of elliptic modular curves as an introduction to the general theory.

Definition of elliptic modular curves

Let $D$ be the complex upper half plane,

$$D = \{ z \in \mathbb{C} \mid \Im(z) > 0 \}.$$
The group $\text{SL}_2(\mathbb{R})$ acts transitively on $D$ by the rule

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.
$$

A subgroup $\Gamma$ of $\text{SL}_2(\mathbb{Z})$ is a congruence subgroup if, for some integer $N \geq 1$, $\Gamma$ contains the principal congruence subgroup of level $N$,

$$
\Gamma(N) \overset{\text{def}}{=} \{ \Lambda \in \text{SL}_2(\mathbb{Z}) | \Lambda \equiv I \mod N \}.
$$

An elliptic modular curve is the quotient $\Gamma \backslash D$ of $D$ by a congruence group $\Gamma$. Initially this is a one-dimensional complex manifold, but it can be compactified by adding a finite number of “cusps”, and so it has a unique structure of an algebraic curve compatible with its structure as a complex manifold. This curve can be realized as a moduli variety for elliptic curves with level structure, from which it is possible to deduce many beautiful properties of the curve, for example, that it has a canonical model over a specific number field, and that the coordinates of the special points on the model generate class fields.

**Elliptic modular curves as moduli varieties**

For an elliptic curve $E$ over $\mathbb{C}$, the exponential map defines an exact sequence

$$(1.1) \quad 0 \to \Lambda \to T_0(E^{an}) \overset{\text{exp}}{\longrightarrow} E^{an} \to 0$$

with

$$
\Lambda \simeq \pi_1(E^{an},0) \simeq H_1(E^{an},\mathbb{Z}).
$$

The functor $E \mapsto (T_0E,\Lambda)$ is an equivalence from the category of complex elliptic curves to the category of pairs consisting of a one-dimensional $\mathbb{C}$-vector space and a lattice. Thus, to give an elliptic curve over $\mathbb{C}$ amounts to giving a two-dimensional $\mathbb{R}$-vector space $V$, a complex structure on $V$, and a lattice in $V$. It is known that $D$ parametrizes elliptic curves plus additional data. Traditionally, to a point $\tau$ of $D$ one attaches the quotient of $\mathbb{C}$ by the lattice spanned by 1 and $\tau$. In other words, one fixes the real vector space and the complex structure, and varies the lattice. From the point of view of period domains and Shimura varieties, it is more natural to fix the real vector space and the lattice, and vary the complex structure.\(^1\)

Thus, let $V$ be a two-dimensional vector space over $\mathbb{R}$. A complex structure on $V$ is an endomorphism $J$ of $V$ such that $J^2 = -1$. From such a $J$, we get a decomposition $V = V^+_J \oplus V^-_J$ of $V$ into its $+i$ and $-i$ eigenspaces, and the isomorphism $V \to V^+_J/V^-_J$ carries the complex structure $J$ on $V$ to the natural complex structure on $V^+_J/V^-_J$. The map $J \mapsto V^+_J/V^-_J$ identifies the set of complex structures on $V$ with the

\(^1\)We are using that the functor $S \mapsto S^{an}$ from smooth algebraic varieties over $\mathbb{C}$ to complex manifolds defines an equivalence from the category of complete smooth algebraic curves to that of compact Riemann surfaces.

\(^2\)The choice of a trivialization of a variation of integral Hodge structures attaches to each point of the underlying space a fixed real vector space and lattice, but a varying Hodge structure — see below.
set of nonreal one-dimensional quotients of $V_C$, i.e., with $\mathbb{P}(V_C) \setminus \mathbb{P}(V)$. This space has two connected components.

Now choose a basis for $V$, and identify it with $\mathbb{R}^2$. Let $\psi: V \times V \to \mathbb{R}$ be the alternating form

$$\psi((\xi_0, \xi), (\xi_1)) = \det(\xi_0 \xi_1^t) = a d - b c.$$  

On one of the connected components, which we denote $D$, the symmetric bilinear form

$$(x, y) \mapsto \psi_J(x, y) \overset{\text{def}}{=} \psi(x, jy): V \times V \to \mathbb{R}$$

is positive definite and on the other it is negative definite. Thus $D$ is the set of complex structures on $V$ for which $+\psi$ (rather than $-\psi$) is a Riemann form. Our choice of a basis for $V$ identifies $\mathbb{P}(V_C) \setminus \mathbb{P}(V)$ with $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ and $D$ with the complex upper half plane.

Now let $\Lambda$ be the lattice $\mathbb{Z}^2$ in $V$. For each $J \in D$, the quotient $(V, J)/\Lambda$ is an elliptic curve $E$ with $H_1(E^{an}, \mathbb{Z}) \simeq \Lambda$. In this way, we obtain a one-to-one correspondence between the points of $D$ and the isomorphism classes of pairs consisting of an elliptic curve $E$ over $\mathbb{C}$ and an ordered basis for $H_1(E^{an}, \mathbb{Z})$.

Let $E_N$ denote the kernel of multiplication by $N$ on an elliptic curve $E$. Thus, for the curve $E = (V, J)/\Lambda$,

$$E_N(\mathbb{C}) = \frac{1}{N}\Lambda/\Lambda \simeq \Lambda/\Lambda_N \simeq (\mathbb{Z}/N\mathbb{Z})^2.$$  

A level-$N$ structure on $E$ is a pair of points $\eta = (t_1, t_2)$ in $E(\mathbb{C})$ that forms an ordered basis for $E_N(\mathbb{C})$.

For an elliptic curve $E$ over any field, there is an algebraically defined (Weil) pairing

$$\varepsilon_N: E_N \times E_N \to \mu_N.$$  

When the ground field is $\mathbb{C}$, this induces an isomorphism $\bigwedge^2(E_N(\mathbb{C})) \simeq \mu_N(\mathbb{C})$. In the following, we fix a primitive $N$th root $\zeta$ of 1 in $\mathbb{C}$, and we require that our level-$N$ structures satisfy the condition $\varepsilon_N(t_1, t_2) = \zeta$.

Identify $\Gamma(N)$ with the subgroup of $\text{SL}(V)$ whose elements preserve $\Lambda$ and act as the identity on $\Lambda/\Lambda_N$. On passing to the quotient by $\Gamma(N)$, we obtain a one-to-one correspondence between the points of $\Gamma(N) \setminus D$ and the isomorphism classes of pairs consisting of an elliptic curve $E$ over $\mathbb{C}$ and a level-$N$ structure $\eta$ on $E$. Let $Y_N$ denote the algebraic curve over $\mathbb{C}$ with $Y_N^{an} = \Gamma(N) \setminus D$.

Let $f: E \to S$ be a family of elliptic curves over a scheme $S$, i.e., a flat map of schemes together with a section whose fibres are elliptic curves. A level-$N$ structure on $E/S$ is an ordered pair of sections to $f$ that give a level-$N$ structure on $E_s$ for each closed point $s$ of $S$.

**Proposition 1.2.** Let $f: E \to S$ be a family of elliptic curves on a smooth algebraic curve $S$ over $\mathbb{C}$, and let $\eta$ be a level-$N$ structure on $E/S$. The map $\gamma: S(\mathbb{C}) \to Y_N(\mathbb{C})$ sending
s \in \mathcal{S}^{\mathbb{C}} \) to the point of \( \Gamma(N) \setminus D \) corresponding to \((E_s, \eta_s)\) is regular, i.e., defined by a morphism of algebraic curves.

**Proof.** We first show that \( \gamma \) is holomorphic. For this, we use that \( \mathbb{P}(V_{\mathbb{C}}) \) is the Grassmann manifold classifying the one-dimensional quotients of \( V_{\mathbb{C}} \). This means that, for any complex manifold \( M \) and surjective homomorphism \( \alpha : \mathcal{O}_M \otimes_{\mathbb{R}} V \to W \) of vector sheaves on \( M \) with \( W \) of rank 1, the map sending \( m \in M \) to the point of \( \mathbb{P}(V_{\mathbb{C}}) \) corresponding to the quotient \( \alpha_m : V_{\mathbb{C}} \to W_m \) of \( V_{\mathbb{C}} \) is holomorphic.

Let \( f : E \to S \) be a family of elliptic curves on a connected smooth algebraic variety \( S \). The exponential map defines an exact sequence of sheaves on \( S \)

\[
0 \rightarrow R_1 f_* \mathbb{Z} \rightarrow \mathcal{I}_0(E^{an}/S^{an}) \rightarrow E^{an} \rightarrow 0
\]

whose fibre at a point \( s \in S^{an} \) is the sequence (1.1) for \( E_s \). From the first map in the sequence we get a surjective map

\[
\mathcal{O}_{S^{an}} \otimes_{\mathbb{Z}} R_1 f_* \mathbb{Z} \twoheadrightarrow \mathcal{I}_0(E^{an}/S^{an}).
\]

Let \((t_1, t_2)\) be a level-\( N \) structure on \( E/S \). Each point of \( S^{an} \) has an open neighbourhood \( U \) such that \( t_1|_U \) and \( t_2|_U \) lift to sections \( \tilde{t}_1 \) and \( \tilde{t}_2 \) of \( \mathcal{I}_0(E^{an}/S^{an}) \) over \( U \); now \( N\tilde{t}_1 \) and \( N\tilde{t}_2 \) are sections of \( R_1 f_* \mathbb{Z} \) over \( U \), and they define an isomorphism

\[
\mathbb{Z}_U^2 \rightarrow R_1 f_* \mathbb{Z}|_U.
\]

On tensoring this with \( \mathcal{O}_{U^{an}} \),

\[
\mathcal{O}_{U^{an}} \otimes_{\mathbb{Z}} \mathbb{Z}_U^2 \rightarrow \mathcal{O}_{U^{an}} \otimes R_1 f_* \mathbb{Z}|_U
\]

and composing with (1.3), we get a surjective map

\[
\mathcal{O}_{U^{an}} \otimes_{\mathbb{R}} V \twoheadrightarrow \mathcal{I}_0(E^{an}/S^{an})|_U
\]

of vector sheaves on \( U \), which defines a holomorphic map \( U \to \mathbb{P}(V_{\mathbb{C}}) \). This maps into \( D \), and its composite with the quotient map \( D \to \Gamma(N) \setminus D \) is the map \( \gamma \). Therefore \( \gamma \) is holomorphic.

It remains to show that \( \gamma \) is algebraic. We now assume that \( S \) is a curve. After passing to a finite covering, we may suppose that \( N \) is even. Let \( \tilde{Y}_N \) (resp. \( \tilde{S} \)) be the completion of \( Y_N \) (resp. \( S \)) to a smooth complete algebraic curve. We have a holomorphic map

\[
S^{an} \rightarrow \tilde{Y}_N^{an} \subset \tilde{Y}_N^{an},
\]

to show that it is regular, it suffices to show that it extends to a holomorphic map of compact Riemann surfaces \( \tilde{S}^{an} \to \tilde{Y}_N^{an} \). The curve \( Y_2 \) is isomorphic to \( \mathbb{P}^1 \setminus \{0,1,\infty\} \). The composed map

\[
S^{an} \rightarrow \tilde{Y}_N^{an} \rightarrow Y_2^{an} \approx \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}
\]
does not have an essential singularity at any of the (finitely many) points of $\bar{S}_{\text{an}} \setminus S_{\text{an}}$ because this would violate the big Picard theorem.\(^3\) Therefore, it extends to a holomorphic map $\bar{S}_{\text{an}} \rightarrow \mathbb{P}^1(\mathbb{C})$, which implies that $\gamma$ extends to a holomorphic map $\bar{\gamma}: \bar{S}_{\text{an}} \rightarrow \bar{Y}_{\text{an}}$, as required. □

Let $\mathcal{F}$ be the functor sending a scheme $S$ of finite type over $\mathbb{C}$ to the set of isomorphism classes of pairs consisting of a family elliptic curves $f: E \rightarrow S$ over $S$ and a level-$N$ structure $\eta$ on $E$. According to Mumford [44], Chapter 7, the functor $\mathcal{F}$ is representable when $N \geq 3$. More precisely, when $N \geq 3$ there exists a smooth algebraic curve $S_N$ over $\mathbb{C}$ and a family of elliptic curves over $S_N$ endowed with a level $N$ structure that is universal in the sense that any similar pair on a scheme $S$ is isomorphic to the pullback of the universal pair by a unique morphism $\alpha: S \rightarrow S_N$.

**Theorem 1.4.** There is a canonical isomorphism $\gamma: S_N \rightarrow Y_N$.

Proof. According to Proposition 1.2, the universal family of elliptic curves with level-$N$ structure on $S_N$ defines a morphism of smooth algebraic curves $\gamma: S_N \rightarrow Y_N$. Both sets $S_N(\mathbb{C})$ and $Y_N(\mathbb{C})$ are in natural one-to-one correspondence with the set of isomorphism classes of complex elliptic curves with level-$N$ structure, and $\gamma$ sends the point in $S_N(\mathbb{C})$ corresponding to a pair $(E, \eta)$ to the point in $Y_N(\mathbb{C})$ corresponding to the same pair. Therefore, $\gamma(\mathbb{C})$ is bijective, which implies that $\gamma$ is an isomorphism. □

In particular, we have shown that the curve $S_N$, constructed by Mumford purely in terms of algebraic geometry, is isomorphic by the obvious map to the curve $Y_N$, constructed analytically. Of course, this is well known, but it is difficult to find a proof of it in the literature. For example, Brian Conrad has noted that it is used without reference in [30].

Theorem 1.4 says that there exists a single algebraic curve over $\mathbb{C}$ enjoying the good properties of both $S_N$ and $Y_N$.

2. **Hermitian symmetric domains**

The natural generalization of the complex upper half plane is a hermitian symmetric domain.

**Preliminaries on Cartan involutions and polarizations**

Let $G$ be a connected algebraic group over $\mathbb{R}$, and let $\sigma_0: g \mapsto \bar{g}$ denote complex conjugation on $G_{\mathbb{C}}$ with respect to $G$. A **Cartan involution** of $G$ is an involution $\theta$ of $G$ (as an algebraic group over $\mathbb{R}$) such that the group

$$G^{(\theta)}(\mathbb{R}) = \{ g \in G(\mathbb{C}) \mid g = \theta(\bar{g}) \}$$

---

\(^3\)Recall that this says that a holomorphic function on the punctured disk with an essential singularity at 0 omits at most one value in $\mathbb{C}$. Therefore a function on the punctured disk that omits two values has (at worst) a pole at 0, and so extends to a function from the whole disk to $\mathbb{P}^1(\mathbb{C})$. 

is compact. Then $G^{(\theta)}$ is a compact real form of $G_\mathcal{C}$, and $\theta$ acts on $G(\mathcal{C})$ as $\sigma_0 \sigma = \sigma \sigma_0$ where $\sigma$ denotes complex conjugation on $G_\mathcal{C}$ with respect to $G^{(\theta)}$.

Consider, for example, the algebraic group $GL_V$ attached to a real vector space $V$. The choice of a basis for $V$ determines a transpose operator $g \mapsto g^t$, and $\theta: g \mapsto (g^t)^{-1}$ is a Cartan involution of $GL_V$ because $GL_V^{(\theta)}(\mathbb{R})$ is the unitary group. The basis determines an isomorphism $GL_V \simeq GL_n$, and $\sigma_0(A) = \bar{A}$ and $\sigma(A) = (\bar{A}^t)^{-1}$ for $A \in GL_n(\mathbb{C})$.

A connected algebraic group $G$ has a Cartan involution if and only if it has a compact real form, which is the case if and only if $G$ is reductive. Any two Cartan involutions of $G$ are conjugate by an element of $G(\mathbb{R})$. In particular, all Cartan involutions of $GL_V$ arise, as in the last paragraph, from the choice of a basis for $V$. An algebraic subgroup $G$ of $GL_V$ is reductive if and only if it is stable under $g \mapsto g^t$ for some basis of $V$, in which case the restriction of $g \mapsto (g^t)^{-1}$ to $G$ is a Cartan involution. Every Cartan involution of $G$ is of this form. See [53], I, §4.

Let $C$ be an element of $G(\mathbb{R})$ whose square is central (so $\text{inn}(C)$ is an involution). A $C$-polarization on a real representation $V$ of $G$ is a $G$-invariant bilinear form $\varphi: V \times V \to \mathbb{R}$ such that the form $\varphi_C: (x, y) \mapsto \varphi(x, Cy)$ is symmetric and positive definite.

**Theorem 2.1.** If $\text{inn}(C)$ is a Cartan involution of $G$, then every finite dimensional real representation of $G$ carries a $C$-polarization; conversely, if one faithful finite dimensional real representation of $G$ carries a $C$-polarization, then $\text{inn}(C)$ is a Cartan involution.

**Proof.** An $\mathbb{R}$-bilinear form $\varphi$ on a real vector space $V$ defines a sesquilinear form $\varphi^t: (u, v) \mapsto \varphi_C(u, \bar{v})$ on $V(\mathbb{C})$, and $\varphi^t$ is hermitian (and positive definite) if and only if $\varphi$ is symmetric (and positive definite).

Let $G \to GL_V$ be a representation of $G$. If $\text{inn}(C)$ is a Cartan involution of $G$, then $G^{(\text{inn} C)}(\mathbb{R})$ is compact, and so there exists a $G^{(\text{inn} C)}$-invariant positive definite symmetric bilinear form $\varphi$ on $V$. Then $\varphi_C$ is $G(\mathbb{C})$-invariant, and so

$$\varphi^t(gu, (\sigma g)v) = \varphi^t(u, v), \quad \text{for all } g \in G(\mathbb{C}), u, v \in V_\mathbb{C},$$

where $\sigma$ is the complex conjugation on $G_\mathcal{C}$ with respect to $G^{(\text{inn} C)}$. Now $\sigma g = \text{inn}(C)(\bar{g}) = \text{inn}(C^{-1})(\bar{g})$, and so, on replacing $v$ with $C^{-1}v$ in the equality, we find that

$$\varphi^t(gu, (C^{-1}\bar{g}C^{-1}v) = \varphi^t(u, C^{-1}v), \quad \text{for all } g \in G(\mathbb{C}), u, v \in V_\mathbb{C}.$$

In particular, $\varphi(gu, C^{-1}gv) = \varphi(u, C^{-1}v)$ when $g \in G(\mathbb{R})$ and $u, v \in V$. Therefore, $\varphi_C^{-1}$ is $G$-invariant. As $(\varphi_C^{-1})_C = \varphi$, we see that $\varphi$ is a $C$-polarization.

For the converse, one shows that, if $\varphi$ is a $C$-polarization on a faithful representation, then $\varphi_C$ is invariant under $G^{(\text{inn} C)}(\mathbb{R})$, which is therefore compact. \hfill $\Box$

**2.2. Variant.** Let $G$ be an algebraic group over $\mathbb{Q}$, and let $C$ be an element of $G(\mathbb{R})$ whose square is central. A $C$-polarization on a $\mathbb{Q}$-representation $V$ of $G$ is a $G$-invariant bilinear form $\varphi: V \times V \to \mathbb{Q}$ such that $\varphi_\mathbb{R}$ is a $C$-polarization on $V_\mathbb{R}$. In order to
show that a $\mathbb{Q}$-representation $V$ of $G$ is polarizable, it suffices to check that $V_{\mathbb{R}}$ is polarizable. We prove this when $C^2$ acts as $+1$ or $-1$ on $V$, which are the only cases we shall need. Let $P(\mathbb{Q})$ (resp. $P(\mathbb{R})$) denote the space of $G$-invariant bilinear forms on $V$ (resp. on $V_{\mathbb{R}}$) that are symmetric when $C^2$ acts as $+1$ or skew-symmetric when it acts as $-1$. Then $P(\mathbb{R}) = \mathbb{R} \otimes_{\mathbb{Q}} P(\mathbb{Q})$. The $C$-polarizations of $V_{\mathbb{R}}$ form an open subset of $P(\mathbb{R})$, whose intersection with $P(\mathbb{Q})$ consists of the $C$-polarizations of $V$.

**Definition of hermitian symmetric domains**

Let $M$ be a complex manifold, and let $J_p: T_pM \to T_pM$ denote the action of $i = \sqrt{-1}$ on the tangent space at a point $p$ of $M$. A **hermitian metric** on $M$ is a riemannian metric $g$ on the underlying smooth manifold of $M$ such that $J_p$ is an isometry for all $p$. A **hermitian manifold** is a complex manifold equipped with a hermitian metric $g$, and a **hermitian symmetric space** is a connected hermitian manifold $M$ that admits a symmetry at each point $p$, i.e., an involution $s_p$ having $p$ as an isolated fixed point. The group $\text{Hol}(M)$ of holomorphic automorphisms of a hermitian symmetric space $M$ is a real Lie group whose identity component $\text{Hol}(M)^+$ acts transitively on $M$.

Every hermitian symmetric space $M$ is a product of hermitian symmetric spaces of the following types:

- **Noncompact type** — the curvature is negative and $\text{Hol}(M)^+$ is a noncompact adjoint Lie group; example, the complex upper half plane.
- **Compact type** — the curvature is positive and $\text{Hol}(M)^+$ is a compact adjoint Lie group; example, the Riemann sphere.
- **Euclidean type** — the curvature is zero; $M$ is isomorphic to a quotient of a space $\mathbb{C}^n$ by a discrete group of translations.

In the first two cases, the space is simply connected. A hermitian symmetric space is **indecomposable** if it is not a product of two hermitian symmetric spaces of lower dimension. For an indecomposable hermitian symmetric space $M$ of compact or noncompact type, the Lie group $\text{Hol}(M)^+$ is simple. See [27], Chapter VIII.

A **hermitian symmetric domain** is a connected complex manifold that admits a hermitian metric for which it is a hermitian symmetric space of noncompact type. The hermitian symmetric domains are exactly the complex manifolds isomorphic to bounded symmetric domains (via the Harish-Chandra embedding; [53], II §4). Thus a connected complex manifold $M$ is a hermitian symmetric domain if and only if

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$^4$Then $g_p$ is the real part of a unique hermitian form on the complex vector space $T_pM$, which explains the name.

$^5$This means that the sectional curvature $K(p, E)$ is $< 0$ for every $p \in M$ and every two-dimensional subspace $E$ of $T_pM$.

$^6$Usually a hermitian symmetric domain is defined to be a complex manifold *equipped* with a hermitian metric etc. However, a hermitian symmetric domain in our sense satisfies conditions (A.1) and (A.2) of [31], and so has a canonical Bergman metric, invariant under all holomorphic automorphisms.
478 Shimura varieties and moduli

(a) it is isomorphic to a bounded open subset of \( \mathbb{C}^n \) for some \( n \), and
(b) for each point \( p \) of \( M \), there exists a holomorphic involution of \( M \) (the symmetry at \( p \)) having \( p \) as an isolated fixed point.

For example, the bounded domain \( \{ z \in \mathbb{C} \mid |z| < 1 \} \) is a hermitian symmetric domain because it is homogeneous and admits a symmetry at the origin \( (z \mapsto -1/z) \). The map \( z \mapsto \frac{z-1}{z+1} \) is an isomorphism from the complex upper half plane \( D \) onto the open unit disk, and so \( D \) is also a hermitian symmetric domain. Its automorphism group is

\[ \text{Hol}(D) \simeq \text{SL}_2(\mathbb{R})/(\{\pm I\}) \simeq \text{PGL}_2(\mathbb{R})^+. \]

**Classification in terms of real groups**

2.3. Let \( U^1 \) be the circle group, \( U^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \). For each point \( o \) of a hermitian symmetric domain \( D \), there is a unique homomorphism \( u_o : U^1 \to \text{Hol}(D) \) such that \( u_o(z) \) fixes \( o \) and acts on \( T_o D \) as multiplication by \( z (z \in U^1) \).

**Example 2.4.** Let \( D \) be the complex upper half plane and let \( o = i \). Let \( h : U^1 \to \text{SL}_2(\mathbb{R}) \) be the homomorphism \( a + bi \mapsto \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) \). Then \( h(z) \) fixes \( o \), and it acts as \( z^2 \) on \( T_o D \). For \( z \in U^1 \), choose a square root \( \sqrt{z} \) in \( U^1 \), and let \( u_o(z) = h(\sqrt{z}) \mod \pm 1 \). Then \( u_o(z) \) is independent of the choice of \( \sqrt{z} \) because \( h(-1) = -I \). The homomorphism \( u_o : U^1 \to \text{SL}_2(\mathbb{Z})/\{\pm 1\} = \text{Hol}(D) \) has the correct properties.

Now let \( D \) be a hermitian symmetric domain. Because \( \text{Hol}(D) \) is an adjoint Lie group, there is a unique real algebraic group \( H \) such that \( H(\mathbb{R})^+ = \text{Hol}(D)^+ \). Similarly, \( U^1 \) is the group of \( \mathbb{R} \)-points of the algebraic torus \( S^1 \) defined by the equation \( X^2 + Y^2 = 1 \). A point \( o \in D \) defines a homomorphism \( u : S^1 \to H \) of real algebraic groups.

**Theorem 2.5.** The homomorphism \( u : S^1 \to H \) has the following properties:

- **SU1:** only the characters \( z, 1, z^{-1} \) occur in the representation of \( S^1 \) on \( \text{Lie}(H)_{\mathbb{C}} \) defined by \( u \);
- **SU2:** \( \text{inn}(u(-1)) \) is a Cartan involution.

Conversely, if \( H \) is a real adjoint algebraic group with no compact factor and \( u : S^1 \to H \) satisfies the conditions (SU1,2), then the set \( D \) of conjugates of \( u \) by elements of \( H(\mathbb{R})^+ \) has a natural structure of a hermitian symmetric domain for which \( u(z) \) acts on \( T_u D \) as multiplication by \( z \); moreover, \( H(\mathbb{R})^+ = \text{Hol}(D)^+ \).

**Proof.** The proof is sketched in [40], 1.21; see also [53], II, Proposition 3.2. 

\footnote{See, for example, [40], Theorem 1.9.}

\footnote{The maps \( S^1 \to H_{\mathbb{R}} \to \text{Aut}(\text{Lie}(H)) \) define an action of \( S^1 \) on \( \text{Lie}(H) \), and hence on \( \text{Lie}(H)_{\mathbb{C}} \). The condition means that \( \text{Lie}(H)_{\mathbb{C}} \) is a direct sum of subspaces on which \( u(z) \) acts as \( z, 1 \), or \( z^{-1} \).}
Thus, the pointed hermitian symmetric domains are classified by the pairs \((H, u)\) as in the theorem. Changing the point corresponds to conjugating \(u\) by an element of \(H(\mathbb{R})\).

**Classification in terms of root systems**

We now assume that the reader is familiar with the classification of semisimple algebraic groups over an algebraically closed field in terms of root systems (e.g., [29]).

Let \(D\) be an indecomposable hermitian symmetric domain. Then the corresponding group \(H\) is simple, and \(H_C\) is also simple because \(H\) is an inner form of its compact form (by SU2). Thus, from \(D\) and a point \(o\), we get a simple algebraic group \(H_C\) over \(\mathbb{C}\) and a nontrivial cocharacter \(\mu\) defined \(u_C: G_m \to H_C\) satisfying the condition:

\[(*) \quad G_m \text{ acts on } \text{Lie}(H_C) \text{ through the characters } z, 1, z^{-1}.\]

Changing \(o\) replaces \(\mu\) by a conjugate. Thus the next step is to classify the pairs \((G, M)\) consisting of a simple algebraic group over \(\mathbb{C}\) and a conjugacy class of nontrivial cocharacters of \(G\) satisfying \((*)\).

Fix a maximal torus \(T\) of \(G\) and a base \(S\) for the root system \(R = R(G, T)\), and let \(R^+\) be the corresponding set of positive roots. As each \(\mu\) in \(M\) factors through some maximal torus, and all maximal tori are conjugate, we may choose \(\mu \in M\) to factor through \(T\). Among the \(\mu\) in \(M\) factoring through \(T\), there is exactly one such that \(\langle \alpha, \mu \rangle \geq 0\) for all \(\alpha \in R^+\) (because the Weyl group acts simply transitively on the Weyl chambers). The condition \((*)\) says that \(\langle \alpha, \mu \rangle \in \langle 1, 0, -1 \rangle\) for all roots \(\alpha\). Since \(\mu\) is nontrivial, not all of the \(\langle \alpha, \mu \rangle\) can be zero, and so \(\langle \alpha, \mu \rangle = 1\) where \(\alpha\) is the highest root. Recall that the highest root \(\alpha = \sum_{\alpha \in S} n_\alpha \alpha\) has the property that \(n_\alpha \geq m_\alpha\) for any other root \(\sum_{\alpha \in S} m_\alpha \alpha\); in particular, \(n_\alpha \geq 1\). It follows that \(\langle \alpha, \mu \rangle = 0\) for all but one simple root \(\alpha\), and that for that simple root \(\langle \alpha, \mu \rangle = 1\) and \(n_\alpha = 1\). Thus, the pairs \((G, M)\) are classified by the simple roots \(\alpha\) for which \(n_\alpha = 1\) — these are called the special simple roots. On examining the tables, one finds that the special simple roots are as in the following table:

| type \(A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2\) | \(\alpha\) \(=\) \(\alpha_1 + \alpha_2 + \cdots + \alpha_n\) | special roots \(=\) \(\alpha_1, \ldots, \alpha_n\) | \(\#\) |
|---|---|---|---|
| \(A_n\) | \(\alpha_1 + \alpha_2 + \cdots + \alpha_n\) | \(\alpha_1, \ldots, \alpha_n\) | \(n\) |
| \(B_n\) | \(\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n\) | \(\alpha_1\) | \(1\) |
| \(C_n\) | \(2\alpha_1 + \cdots + 2\alpha_n\) | \(\alpha_n\) | \(1\) |
| \(D_n\) | \(\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n\) | \(\alpha_1, \alpha_n\) | \(3\) |
| \(E_6\) | \(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 2\alpha_6\) | \(\alpha_1, \alpha_6\) | \(2\) |
| \(E_7\) | \(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7\) | \(\alpha_7\) | \(1\) |
| \(E_8, F_4, G_2\) | none | 0 |

9If \(H_C\) is not simple, say, \(H_C = H_1 \times H_2\), then \(H = (H_1)_{C/\mathbb{R}}\), and every inner form of \(H\) is isomorphic to \(H\) itself (by Shapiro’s lemma), which is not compact because \(H(\mathbb{R}) = H_1(\mathbb{C})\).
Mnemonic: the number of special simple roots is one less than the connection index \((P(R): Q(R))\) of the root system.\(^{10}\)

To every indecomposable hermitian symmetric domain we have attached a special node, and we next show that every special node arises from a hermitian symmetric domain. Let \(G\) be a simple algebraic group over \(\mathbb{C}\) with a character \(\mu\) satisfying (*). Let \(U\) be the (unique) compact real form of \(G\), and let \(\sigma\) be the complex conjugation on \(G\) with respect to \(U\). Finally, let \(H\) be the real form of \(G\) such that \(\text{inn}(\mu(-1)) \circ \sigma\) is the complex conjugation on \(G\) with respect to \(H\). The restriction of \(\mu\) to \(U^1 \subset \mathbb{C}^x\) maps into \(H(\mathbb{R})\) and defines a homomorphism \(u\) satisfying the conditions \((SU1,2)\) of \((2.5)\). The hermitian symmetric domain corresponding to \((H, u)\) gives rise to \((G, \mu)\). Thus there are indecomposable hermitian symmetric domains of all possible types except \(E_8, F_4,\) and \(G_2\).

Let \(H\) be a real simple group such that there exists a homomorphism \(u: S^1 \to H\) satisfying \((SV1,2)\). The set of such \(u\)'s has two connected components, interchanged by \(u \leftrightarrow u^{-1}\), each of which is an \(H(\mathbb{R})^+\)-conjugacy class. The \(u\)'s form a single \(H(\mathbb{R})\)-conjugacy class except when \(S\) is moved by the opposition involution \(([19], 1.2.7, 1.2.8)\). This happens in the following cases: type \(A_n\) and \(s \neq \frac{n}{2}\), type \(D_n\) with \(n\) odd and \(s = \alpha_{n-1}\) or \(\alpha_n\); type \(E_6\) (see p. \(527\) below).

**Example: the Siegel upper half space**

A **symplectic space** \((V, \psi)\) over a field \(k\) is a finite dimensional vector space \(V\) over \(k\) together with a nondegenerate alternating form \(\psi\) on \(V\). The **symplectic group** \(S(\psi)\) is the algebraic subgroup of \(GL_V\) of elements fixing \(\psi\). It is an almost simple simply connected group of type \(C_{n-1}\) where \(n = \frac{1}{2} \dim_k V\).

Now let \(k = \mathbb{R}\), and let \(H = S(\psi)\). Let \(D\) be the space of complex structures \(J\) on \(V\) such that \((x, y) \mapsto \psi_J(x, y) \overset{\text{def}}{=} \psi(x, Jy)\) is symmetric and positive definite. The symmetry is equivalent to \(J\) lying in \(S(\psi)\). Therefore, \(D\) is the set of complex structures \(J\) on \(V\) for which \(J \in H(\mathbb{R})\) and \(\psi\) is a \(J\)-polarization for \(H\).

The action,

\[ g, J \mapsto gJg^{-1} : H(\mathbb{R}) \times D \to D, \]

of \(H(\mathbb{R})\) on \(D\) is transitive \(([40], \S 6)\). Each \(J \in D\) defines an action of \(C\) on \(V\), and \(2.6\)

\[ \psi(Jx, Jy) = \psi(x, y) \quad \text{all} \quad x, y \in V \implies \psi(zx, zy) = |z|^2\psi(x, y) \quad \text{all} \quad x, y \in V. \]

Let \(h_J : \mathbb{S} \to GL_V\) be the homomorphism such that \(h_J(z)\) acts on \(V\) as multiplication by \(z\), and let \(V_C = V^+ \oplus V^-\) be the decomposition of \(V_C\) into its \(\pm 1\) eigenspaces for \(J\). Then \(h_J(z)\) acts on \(V^+\) as \(z\) and on \(V^-\) as \(\bar{z}\), and so it acts on

\[ \text{Lie}(H)_C \subset \text{End}(V)_C \cong V_C^* \otimes V_C = (V^+ \oplus V^-)^* \otimes (V^+ \oplus V^-), \]

through the characters \(z^{-1}\bar{z}, 1, z\bar{z}^{-1}\).

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\(^{10}\)It is possible to prove this directly. Let \(S^+ = S \cup \{\alpha_0\}\) where \(\alpha_0\) is the negative of the highest root — the elements of \(S^+\) correspond to the nodes of the completed Dynkin diagram \(([7], \S 4, 3)\). The group \(P/Q\) acts on \(S^+\), and it acts simply transitively on the set \([\text{simple roots}] \cup \{\alpha_0\}\) \(([19], 1.2.5)\).
For $z \in \mathbb{U}^1$, (2.6) shows that $h_J(z) \in H$; choose a square root $\sqrt{z}$ of $z$ in $\mathbb{U}^1$, and let $u_J(z) = h_J(\sqrt{z}) \mod \pm 1$. Then $u_J$ is a well-defined homomorphism $\mathbb{U}^1 \to H^\mathrm{ad}(\mathbb{R})$, and it satisfies the conditions (SU1,2) of Theorem 2.5. Therefore, $D$ has a natural complex structure for which $z \in \mathbb{U}^1$ acts on $T_J(D)$ as multiplication by $z$ and $\mathrm{Hol}(D)^+ = H^\mathrm{ad}(\mathbb{R})^+$. With this structure, $D$ is the (unique) indecomposable hermitian symmetric domain of type $C_{n-1}$. It is called the Siegel upper half space (of degree, or genus, $n$).

3. Discrete subgroups of Lie groups

The algebraic varieties we are concerned with are quotients of hermitian symmetric domains by the action of discrete groups. In this section, we describe the discrete groups of interest to us.

Lattices in Lie groups

Let $H$ be a connected real Lie group. A lattice in $H$ is a discrete subgroup $\Gamma$ of finite covolume, i.e., such that $H/\Gamma$ has finite volume with respect to an $H$-invariant measure. For example, the lattices in $\mathbb{R}^n$ are exactly the $\mathbb{Z}$-submodules generated by bases for $\mathbb{R}^n$, and two such lattices are commensurable if and only if they generate the same $\mathbb{Q}$-vector space. Every discrete subgroup commensurable with a lattice is itself a lattice.

Now assume that $H$ is semisimple with finite centre. A lattice $\Gamma$ in $H$ is irreducible if $\Gamma \cdot N$ is dense in $H$ for every noncompact closed normal subgroup $N$ of $H$. For example, if $\Gamma_1$ and $\Gamma_2$ are lattices in $H_1$ and $H_2$, then the lattice $\Gamma_1 \times \Gamma_2$ in $H_1 \times H_2$ is not irreducible because $$(\Gamma_1 \times \Gamma_2) \cdot (1 \times H_2) = \Gamma_1 \times H_2$$ is not dense. On the other hand, $\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$ can be realized as an irreducible lattice in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ via the embeddings $\mathbb{Z}[\sqrt{2}] \to \mathbb{R}$ given by $\sqrt{2} \mapsto \sqrt{2}$ and $\sqrt{2} \mapsto -\sqrt{2}$.

**Theorem 3.1.** Let $H$ be a connected semisimple Lie group with no compact factors and trivial centre, and let $\Gamma$ be a lattice $H$. Then $H$ can be written (uniquely) as a direct product $H = H_1 \times \cdots \times H_r$ of Lie subgroups $H_i$ such that $\Gamma_i \overset{\text{def}}{=} \Gamma \cap H_i$ is an irreducible lattice in $H_i$ and $\Gamma_1 \times \cdots \times \Gamma_r$ has finite index in $\Gamma$.

**Proof.** See [42], 4.24. □

**Theorem 3.2.** Let $D$ be a hermitian symmetric domain, and let $H = \mathrm{Hol}(D)^+$. A discrete subgroup $\Gamma$ of $H$ is a lattice if and only if $\Gamma \setminus D$ has finite volume. Let $\Gamma$ be a lattice in $H$; then $D$ can be written (uniquely) as a product $D = D_1 \times \cdots \times D_r$ of hermitian symmetric domains such that $\Gamma_i \overset{\text{def}}{=} \Gamma \cap \mathrm{Hol}(D_i)^+$ is an irreducible lattice in $\mathrm{Hol}(D_i)^+$ and $\Gamma_1 \setminus D_1 \times \cdots \times \Gamma_r \setminus D_r$ is a finite covering of $\Gamma \setminus D$.

11Recall that two subgroup $S_1$ and $S_2$ of a group are commensurable if $S_1 \cap S_2$ has finite index in both $S_1$ and $S_2$. Commensurability is an equivalence relation.
Proof. Let $u_o$ be the homomorphism $S^1 \to H$ attached to a point $o \in D$ (see 2.3), and let $\theta$ be the Cartan involution $\text{inn}(u_o(-1))$. The centralizer of $u_o$ is contained in $H(R) \cap H^{(0)}(R)$, which is compact. Therefore $D$ is a quotient of $H(R)$ by a compact subgroup, from which the first statement follows. For the second statement, let $H = H_1 \times \cdots \times H_r$ be the decomposition of $H$ defined by $\Gamma$ (see 3.1). Then $u_o = (u_1, \ldots, u_r)$ where each $u_i$ is a homomorphism $S^1 \to H_i$ satisfying the conditions $SU_{1,2}$ of Theorem 2.5. Now $D = D_1 \times \cdots \times D_r$ with $D_i$ the hermitian symmetric domain corresponding to $(H_i, u_i)$. This is the required decomposition. \[\Box\]

Proposition 3.3. Let $\varphi : H \to H'$ be a surjective homomorphism of Lie groups with compact kernel. If $\Gamma$ is a lattice in $H$, then $\varphi(\Gamma)$ is a lattice in $H'$.

Proof. The proof is elementary (it requires only that $H$ and $H'$ be locally compact topological groups). \[\Box\]

Arithmetic subgroups of algebraic groups

Let $G$ be an algebraic group over $\mathbb{Q}$. When $r : G \to \text{GL}_n$ is an injective homomorphism, we let

$$G(\mathbb{Z})_r = \{g \in G(\mathbb{Q}) \mid r(g) \in \text{GL}_n(\mathbb{Z})\}.$$ 

Then $G(\mathbb{Z})_r$ is independent of $r$ up to commensurability ([4], 7.13), and we sometimes omit $r$ from the notation. A subgroup $\Gamma$ of $G(\mathbb{Q})$ is arithmetic if it is commensurable with $G(\mathbb{Z})_r$ for some $r$.

Theorem 3.4. Let $\varphi : G \to G'$ be a surjective homomorphism of algebraic groups over $\mathbb{Q}$. If $\Gamma$ is an arithmetic subgroup of $G(\mathbb{Q})$, then $\varphi(\Gamma)$ is an arithmetic subgroup of $G'(\mathbb{Q})$.

Proof. See [4], 8.11. \[\Box\]

An arithmetic subgroup $\Gamma$ of $G(\mathbb{Q})$ is obviously discrete in $G(R)$, but it need not be a lattice. For example, $G_m(\mathbb{Z}) = \{\pm 1\}$ is an arithmetic subgroup of $G_m(\mathbb{Q})$ of infinite covolume in $G_m(R) = \mathbb{R}^\times$.

Theorem 3.5. Let $G$ be a reductive algebraic group over $\mathbb{Q}$, and let $\Gamma$ be an arithmetic subgroup of $G(\mathbb{Q})$.

(a) The quotient $\Gamma \backslash G(R)$ has finite volume if and only if $\text{Hom}(G, G_m) = 0$; in particular, $\Gamma$ is a lattice if $G$ is semisimple.

(b) (Godement compactness criterion) The quotient $\Gamma \backslash G(R)$ is compact if and only if $\text{Hom}(G, G_m) = 0$ and $G(\mathbb{Q})$ contains no unipotent element other than 1.

Proof. See [4], 13.2, 8.4.\[12\]

\[12\]Statement (a) was proved in particular cases by Siegel and others, and in general by Borel and Harish-Chandra [6]. Statement (b) was conjectured by Godement, and proved independently by Mostow and Tamagawa [43] and by Borel and Harish-Chandra [6].
Let $k$ be a subfield of $\mathbb{C}$. An automorphism $\alpha$ of a $k$-vector space $V$ is said to be neat if its eigenvalues in $\mathbb{C}$ generate a torsion free subgroup of $\mathbb{C}^\times$. Let $G$ be an algebraic group over $\mathbb{Q}$. An element $g \in G(\mathbb{Q})$ is neat if $\rho(g)$ is neat for one faithful representation $G \hookrightarrow \text{GL}(V)$, in which case $\rho(g)$ is neat for every representation $\rho$ of $G$ defined over a subfield of $\mathbb{C}$. A subgroup of $G(\mathbb{Q})$ is neat if all its elements are. See [4], §17.

**Theorem 3.6.** Let $G$ be an algebraic group over $\mathbb{Q}$, and let $\Gamma$ be an arithmetic subgroup of $G(\mathbb{Q})$. Then, $\Gamma$ contains a neat subgroup of finite index. In particular, $\Gamma$ contains a torsion free subgroup of finite index.

**Proof.** In fact, the neat subgroup can be defined by congruence conditions. See [4], 17.4. □

**Definition 3.7.** A semisimple algebraic group $G$ over $\mathbb{Q}$ is said to be of compact type if $G(\mathbb{R})$ is compact, and it is said to be of noncompact type if it does not contain a nontrivial connected normal algebraic subgroup of compact type.

Thus a simply connected or adjoint group over $\mathbb{Q}$ is of compact type if all of its almost simple factors are of compact type, and it is of noncompact type if none of its almost simple factors is of compact type. In particular, an algebraic group may be of neither type.

**Theorem 3.8** (Borel density theorem). Let $G$ be a semisimple algebraic group over $\mathbb{Q}$. If $G$ is of noncompact type, then every arithmetic subgroup of $G(\mathbb{Q})$ is dense in the Zariski topology.

**Proof.** See [4], 15.12. □

**Proposition 3.9.** Let $G$ be a simply connected algebraic group over $\mathbb{Q}$ of noncompact type, and let $\Gamma$ be an arithmetic subgroup of $G(\mathbb{Q})$. Then $\Gamma$ is irreducible as a lattice in $G(\mathbb{R})$ if and only if $G$ is almost simple.

**Proof.** $\Rightarrow$: Suppose $G = G_1 \times G_2$, and let $\Gamma_1$ and $\Gamma_2$ be arithmetic subgroups in $G_1(\mathbb{Q})$ and $G_2(\mathbb{Q})$. Then $\Gamma_1 \times \Gamma_2$ is an arithmetic subgroup of $G(\mathbb{Q})$, and so $\Gamma$ is commensurable with it, but $\Gamma_1 \times \Gamma_2$ is not irreducible.

$\Leftarrow$: Let $G(\mathbb{R}) = H_1 \times \cdots \times H_r$ be a decomposition of the Lie group $G(\mathbb{R})$ such that $\Gamma_1 \overset{\text{def}}{=} \Gamma \cap H_1$ is an irreducible lattice in $H_1$ (cf. Theorem 3.1). There exists a finite Galois extension $F$ of $\mathbb{Q}$ in $\mathbb{R}$ and a decomposition $G_F = G_1 \times \cdots \times G_r$ of $G_F$ into a product of algebraic subgroups $G_i$ over $F$ such that $H_i = G_i(\mathbb{R})$ for all $i$. Because $\Gamma_1$ is Zariski dense in $G_1$ (Borel density theorem), this last decomposition is stable under the action of $\text{Gal}(F/\mathbb{Q})$, and hence arises from a decomposition over $\mathbb{Q}$. This contradicts the almost simplicity of $G$ unless $r = 1$. □

The rank, $\text{rank}(G)$, of a semisimple algebraic group over $\mathbb{R}$ is the dimension of a maximal split torus in $G$, i.e., $\text{rank}(G) = r$ if $G$ contains an algebraic subgroup isomorphic to $\mathbb{G}_m^r$ but not to $\mathbb{G}_m^{r+1}$.
Theorem 3.10 (Margulis superrigidity theorem). Let $G$ and $H$ be algebraic groups over $\mathbb{Q}$ with $G$ simply connected and almost simple. Let $\Gamma$ be an arithmetic subgroup of $G(\mathbb{Q})$, and let $\delta: \Gamma \rightarrow H(\mathbb{Q})$ be a homomorphism. If $\text{rank}(G_{\mathbb{R}}) \geq 2$, then the Zariski closure of $\delta(\Gamma)$ in $H$ is a semisimple algebraic group (possibly not connected), and there is a unique homomorphism $\varphi: G \rightarrow H$ of algebraic groups such that $\varphi(\gamma) = \delta(\gamma)$ for all $\gamma$ in a subgroup of finite index in $\Gamma$.

Proof. This the special case of [33], Chapter VIII, Theorem B, p. 258, in which $K = \mathbb{Q} = l$, $S = \{\infty\}$, $G = G$, $H = H$, and $\Lambda = \Gamma$. □

Arithmetic lattices in Lie groups

For an algebraic group $G$ over $\mathbb{Q}$, $G(\mathbb{R})$ has a natural structure of a real Lie group, which is connected if $G$ is simply connected (Theorem of Cartan).

Let $H$ be a connected semisimple real Lie group with no compact factors and trivial centre. A subgroup $\Gamma$ in $H$ is arithmetic if there exists a simply connected algebraic group $G$ over $\mathbb{Q}$ and a surjective homomorphism $\varphi: G(\mathbb{R}) \rightarrow H$ with compact kernel such that $\Gamma$ is commensurable with $\varphi(G(\mathbb{Z}))$. Such a subgroup is a lattice by Theorem 3.5(a) and Proposition 3.3.

Example 3.11. Let $H = \text{SL}_2(\mathbb{R})$, and let $B$ be a quaternion algebra over a totally real number field $F$ such that $H \otimes_F \mathbb{R} \approx M_2(\mathbb{R})$ for exactly one real prime $v$. Then $H \otimes_{\mathbb{Q}} \mathbb{R} \approx M_2(\mathbb{R}) \times \mathbb{H} \times \mathbb{H} \times \cdots$ where $\mathbb{H}$ is usual quaternion algebra, and so there exists a surjective homomorphism $\varphi: G(\mathbb{R}) \rightarrow \text{SL}_2(\mathbb{R})$ with compact kernel. The image under $\varphi$ of any arithmetic subgroup of $G(\mathbb{Q})$ is an arithmetic subgroup $\Gamma$ of $\text{SL}_2(\mathbb{R})$, and every arithmetic subgroup of $\text{SL}_2(\mathbb{R})$ is commensurable with one of this form. If $F = \mathbb{Q}$ and $B = M_2(\mathbb{Q})$, then $G = \text{SL}_2(\mathbb{Q})$ and $\Gamma \backslash \text{SL}_2(\mathbb{R})$ is noncompact (see §1); otherwise $B$ is a division algebra, and $\Gamma \backslash \text{SL}_2(\mathbb{R})$ is compact by Godement’s criterion (3.5b).

For almost a century, $\text{PSL}_2(\mathbb{R})$ was the only simple Lie group known to have non arithmetic lattices, and when further examples were discovered in the 1960s they involved only a few other Lie groups. This gave credence to the idea that, except in a few groups of low rank, all lattices are arithmetic (Selberg’s conjecture). This was proved by Margulis in a very precise form.

Theorem 3.12 (Margulis arithmeticity theorem). Every irreducible lattice in a semisimple Lie group is arithmetic unless the group is isogenous to $\text{SO}(1, n) \times \text{compact}$ or $\text{SU}(1, n) \times \text{compact}$.

Proof. For a discussion of the theorem, see [42], §5B. For proofs, see [33], Chapter IX, and [67], Chapter 6. □

Theorem 3.13. Let $H$ be the identity component of the group of automorphisms of a hermitian symmetric domain $D$, and let $\Gamma$ be a discrete subgroup of $H$ such that $\Gamma \backslash D$
has finite volume. If rank $H_i \geq 2$ for each factor $H_i$ in (3.1), then there exists a simply connected algebraic group $G$ of noncompact type over $\mathbb{Q}$ and a surjective homomorphism $\varphi: G(\mathbb{R}) \to H$ with compact kernel such that $\Gamma$ is commensurable with $\varphi(G(\mathbb{Z}))$. Moreover, the pair $(G, \varphi)$ is unique up to a unique isomorphism.

**Proof.** The group $\Gamma$ is a lattice in $H$ by Theorem 3.2. Each factor $H_i$ is again the identity component of the group of automorphisms of a hermitian symmetric domain (Theorem 3.2), and so we may suppose that $\Gamma$ is irreducible. The existence of the pair $(G, \varphi)$ just means that $\Gamma$ is arithmetic, which follows from the Margulis arithmeticity theorem (3.12).

Because $\Gamma$ is irreducible, $G$ is almost simple (see 3.9). As $G$ is simply connected, this implies that $G = (G^s)_{F/\mathbb{Q}}$ where $F$ is a number field and $G^s$ is a geometrically almost simple algebraic group over $F$. If $F$ had a complex prime, $G(\mathbb{R})$ would have a factor $(G')_{C/R}$ but $(G')_{C/R}$ has no inner form except itself (by Shapiro’s lemma), and so this is impossible. Therefore $F$ is totally real.

Let $(G_1, \varphi_1)$ be a second pair. Because the kernel of $\varphi_1$ is compact, its intersection with $G_1(\mathbb{Z})$ is finite, and so there exists an arithmetic subgroup $\Gamma_1$ of $G_1(\mathbb{Q})$ such $\varphi_1|\Gamma_1$ is injective. Because $\varphi(G(\mathbb{Z}))$ and $\varphi_1(\Gamma_1)$ are commensurable, there exists an arithmetic subgroup $\Gamma'$ of $G(\mathbb{Q})$ such that $\varphi(\Gamma') \subset \varphi_1(\Gamma_1)$. Now the Margulis superrigidity theorem 3.10 shows that there exists a homomorphism $\alpha: G \to G_1$ such that

$$(3.14) \quad \varphi_1(\alpha(\gamma)) = \varphi(\gamma)$$

for all $\gamma$ in a subgroup $\Gamma''$ of $\Gamma'$ of finite index. The subgroup $\Gamma''$ of $G(\mathbb{Q})$ is Zariski-dense in $G$ (Borel density theorem 3.8), and so (3.14) implies that

$$(3.15) \quad \varphi_1 \circ \alpha|\mathbb{R}) = \varphi.$$  

Because $G$ and $G_1$ are almost simple, (3.15) implies that $\alpha$ is an isogeny, and because $G_1$ is simply connected, this implies that $\alpha$ is an isomorphism. It is unique because it is uniquely determined on an arithmetic subgroup of $G$. □

**Congruence subgroups of algebraic groups**

As in the case of elliptic modular curves, we shall need to consider a special class of arithmetic subgroups, namely, the congruence subgroups.

Let $G$ be an algebraic group over $\mathbb{Q}$. Choose an embedding of $G$ into $GL_n$, and define

$$\Gamma(N) = G(\mathbb{Q}) \cap \{ A \in GL_n(\mathbb{Z}) \mid A \equiv 1 \mod N \}.$$  

A *congruence subgroup*\(^{13}\) of $G(\mathbb{Q})$ is any subgroup containing $\Gamma(N)$ as a subgroup of finite index. Although $\Gamma(N)$ depends on the choice of the embedding, this definition does not — in fact, the congruence subgroups are exactly those of the form $K \cap G(\mathbb{A}_f)$ for $K$ a compact open subgroup of $G(\mathbb{A}_f)$.

\(^{13}\)Subgroup defined by congruence conditions.
For a surjective homomorphism $G \to G'$ of algebraic groups over $\mathbb{Q}$, the homomorphism $G(\mathbb{Q}) \to G'(\mathbb{Q})$ need not send congruence subgroups to congruence subgroups. For example, the image in $\text{PGL}_2(\mathbb{Q})$ of a congruence subgroup of $\text{SL}_2(\mathbb{Q})$ is an arithmetic subgroup (see 3.4) but not necessarily a congruence subgroup.

Every congruence subgroup is an arithmetic subgroup, and for a simply connected group the converse is often, but not always, true. For a survey of what is known about the relation of congruence subgroups to arithmetic groups (the congruence subgroup problem), see [49].

Aside 3.16. Let $H$ be a connected adjoint real Lie group without compact factors. The pairs $(G, \varphi)$ consisting of a simply connected algebraic group over $\mathbb{Q}$ and a surjective homomorphism $\varphi: G(\mathbb{R}) \to H$ with compact kernel have been classified (this requires class field theory). Therefore the arithmetic subgroups of $H$ have been classified up to commensurability. When all arithmetic subgroups are congruence, there is even a classification of the groups themselves in terms of congruence conditions or, equivalently, in terms of compact open subgroups of $G(\mathbb{A}_f)$.

4. Locally symmetric varieties

To obtain an algebraic variety from a hermitian symmetric domain, we need to pass to the quotient by an arithmetic group.

Quotients of hermitian symmetric domains

Let $D$ be a hermitian symmetric domain, and let $\Gamma$ be a discrete subgroup of $\text{Hol}(D)^+$. If $\Gamma$ is torsion free, then $\Gamma$ acts freely on $D$, and there is a unique complex structure on $\Gamma \backslash D$ for which the quotient map $\pi: D \to \Gamma \backslash D$ is a local isomorphism. Relative to this structure, a map $\varphi$ from $\Gamma \backslash D$ to a second complex manifold is holomorphic if and only if $\varphi \circ \pi$ is holomorphic.

When $\Gamma$ is torsion free, we often write $D(\Gamma)$ for $\Gamma \backslash D$ regarded as a complex manifold. In this case, $D$ is the universal covering space of $D(\Gamma)$ and $\Gamma$ is the group of covering transformations. The choice of a point $p \in D$ determines an isomorphism of $\Gamma$ with the fundamental group $\pi_1(D(\Gamma), \pi p)$.

The complex manifold $D(\Gamma)$ is locally symmetric in the sense that, for each $p \in D(\Gamma)$, there is an involution $s_p$ defined on a neighbourhood of $p$ having $p$ as an isolated fixed point.

The algebraic structure on the quotient

Recall that $X^\text{an}$ denotes the complex manifold attached to a smooth complex algebraic variety $X$. The functor $X \rightsquigarrow X^\text{an}$ is faithful, but it is far from being surjective on arrows or on objects. For example, $(\mathbb{A}^1)^\text{an} = \mathbb{C}$ and the exponential function is a nonpolynomial holomorphic map $\mathbb{C} \to \mathbb{C}$. A Riemann surface arises from an

\footnote{Let $\gamma \in \Gamma$, and choose a path from $p$ to $\gamma p$; the image of this in $\Gamma \backslash D$ is a loop whose homotopy class does not depend on the choice of the path.}
algebraic curve if and only if it can be compactified by adding a finite number of points. In particular, if a Riemann surface is an algebraic curve, then every bounded function on it is constant, and so the complex upper half plane is not an algebraic curve (the function $\frac{z - i}{z + i}$ is bounded).

**Chow’s theorem**  An algebraic variety (resp. complex manifold) is *projective* if it can be realized as a closed subvariety of $\mathbb{P}^n$ for some $n$ (resp. closed submanifold of $(\mathbb{P}^n)^{\text{an}}$).

**Theorem 4.1** (Chow 1949 [11]). The functor $X \mapsto X^{\text{an}}$ from smooth projective complex algebraic varieties to projective complex manifolds is an equivalence of categories.

In other words, a projective complex manifold has a unique structure of a smooth projective algebraic variety, and every holomorphic map of projective complex manifolds is regular for these structures. See [63], 13.6, for the proof.

Chow’s theorem remains true when singularities are allowed and “complex manifold” is replaced by “complex space”.

**The Baily-Borel theorem**

**Theorem 4.2** (Baily-Borel 1966 [3]). Every quotient $D(\Gamma)$ of a hermitian symmetric domain $D$ by a torsion-free arithmetic subgroup $\Gamma$ of $\text{Hol}(D)^+$ has a canonical structure of an algebraic variety.

More precisely, let $G$ be the algebraic group over $\mathbb{Q}$ attached to $(D, \Gamma)$ in Theorem 3.13, and assume, for simplicity, that $G$ has no normal algebraic subgroup of dimension 3. Let $A_n$ be the vector space of automorphic forms on $D$ for the $n$th power of the canonical automorphy factor. Then $A = \bigoplus_{n \geq 0} A_n$ is a finitely generated graded $\mathbb{C}$-algebra, and the canonical map

$$D(\Gamma) \to D(\Gamma)^* \overset{\text{def}}{=} \text{Proj}(A)$$

realizes $D(\Gamma)$ as a Zariski-open subvariety of the projective algebraic variety $D(\Gamma)^*$ ([3], §10).

**Borel’s theorem**

**Theorem 4.3** (Borel 1972 [5]). Let $D(\Gamma)$ be the quotient $\Gamma \backslash D$ in (4.2) endowed with its canonical algebraic structure, and let $V$ be a smooth complex algebraic variety. Every holomorphic map $f: V^{\text{an}} \to D(\Gamma)^{\text{an}}$ is regular.

In the proof of Proposition 1.2, we saw that for curves this theorem follows from the big Picard theorem. Recall that this says that every holomorphic map from a punctured disk to $\mathbb{P}^1(\mathbb{C}) \setminus \{\text{three points}\}$ extends to a holomorphic map from the whole disk to $\mathbb{P}^1(\mathbb{C})$. Following earlier work of Kwack and others, Borel generalized the big Picard theorem in two respects: the punctured disk is replaced by a product
of punctured disks and disks, and the target space is allowed to be any quotient of a hermitian symmetric domain by a torsion-free arithmetic group.

Resolution of singularities ([28]) shows that every smooth quasi-projective algebraic variety \( V \) can be embedded in a smooth projective variety \( \bar{V} \) as the complement of a divisor with normal crossings. This condition means that \( \bar{V}^{\text{an}} \setminus V^{\text{an}} \) is locally a product of disks and punctured disks. Therefore \( f|V^{\text{an}} \) extends to a holomorphic map \( \bar{V}^{\text{an}} \to D(\Gamma)^* \) (by Borel) and so is a regular map (by Chow).

**Locally symmetric varieties**

A locally symmetric variety is a smooth algebraic variety \( X \) over \( \mathbb{C} \) such that \( X^{\text{an}} \) is isomorphic to \( \Gamma \setminus D \) for some hermitian symmetric domain \( D \) and torsion-free subgroup \( \Gamma \) of \( \text{Hol}(D) \).\(^{15}\) In other words, \( X \) is a locally symmetric variety if the universal covering space \( D \) of \( X^{\text{an}} \) is a hermitian symmetric domain and the group of covering transformations of \( D \) over \( X^{\text{an}} \) is a torsion-free subgroup \( \Gamma \) of \( \text{Hol}(D) \). When \( \Gamma \) is an arithmetic subgroup of \( \text{Hol}(D)^+ \), \( X \) is called an arithmetic locally symmetric variety. The group \( \Gamma \) is automatically a lattice, and so the Margulis arithmeticity theorem (3.12) shows that nonarithmetic locally symmetric varieties can occur only when there are factors of low dimension.

A nonsingular projective curve over \( \mathbb{C} \) has a model over \( \mathbb{Q}^{\text{al}} \) if and only if it contains an arithmetic locally symmetric curve as the complement of a finite set (Belyi; see [55], p. 71). This suggests that there are too many arithmetic locally symmetric varieties for us to be able to say much about their arithmetic.

Let \( D(\Gamma) \) be an arithmetic locally symmetric variety. Recall that \( \Gamma \) is arithmetic if there is a simply connected algebraic group \( G \) over \( \mathbb{Q} \) and a surjective homomorphism \( \varphi : G(\mathbb{R}) \to \text{Hol}(D)^+ \) with compact kernel such that \( \Gamma \) is commensurable with \( \varphi(G(\mathbb{Z})) \). If there exists a congruence subgroup \( \Gamma_0 \) of \( G(\mathbb{Z}) \) such that \( \Gamma \) contains \( \varphi(\Gamma_0) \) as a subgroup of finite index, then we call \( D(\Gamma) \) a connected Shimura variety. Only for Shimura varieties do we have a rich arithmetic theory (see [14], [19], and the many articles of Shimura, especially, [57, 58, 59, 60, 61]).

**Example: Siegel modular varieties**

For an abelian variety \( A \) over \( \mathbb{C} \), the exponential map defines an exact sequence

\[
0 \longrightarrow \Lambda \longrightarrow T_0(A^{\text{an}}) \overset{\exp}{\longrightarrow} A^{\text{an}} \longrightarrow 0
\]

with \( T_0(A^{\text{an}}) \) a complex vector space and \( \Lambda \) a lattice in \( T_0(A^{\text{an}}) \) canonically isomorphic to \( H_1(A^{\text{an}}, \mathbb{Z}) \).

---

\(^{15}\)As \( \text{Hol}(D) \) has only finitely many components, \( \Gamma \cap \text{Hol}(D)^+ \) has finite index in \( \Gamma \). Sometimes we only allow discrete subgroups of \( \text{Hol}(D) \) contained in \( \text{Hol}(D)^+ \). In the theory of Shimura varieties, we generally consider only "sufficiently small" discrete subgroups, and we regard the remainder as "noise". Algebraic geometers do the opposite.
Theorem 4.4 (Riemann’s Theorem). The functor \( A \mapsto (T_0(A), \Lambda) \) is an equivalence from the category of abelian varieties over \( \mathbb{C} \) to the category of pairs consisting of a \( \mathbb{C} \)-vector space \( V \) and a lattice \( \Lambda \) in \( V \) that admits a Riemann form.

Proof. See, for example, [47], Chapter I.

A Riemann form for a pair \((V, \Lambda)\) is an alternating form \( \psi : \Lambda \times \Lambda \rightarrow \mathbb{Z} \) such that the pairing \( (x, y) \mapsto \psi(x, \sqrt{-1}y) : V \times V \rightarrow \mathbb{R} \) is symmetric and positive definite. Here \( \psi_\mathbb{R} \) denotes the linear extension of \( \psi \) to \( \mathbb{R} \otimes \mathbb{Z} \Lambda \simeq V \). A principal polarization on an abelian variety \( A \) over \( \mathbb{C} \) is Riemann form for \((T_0(A), \Lambda)\) whose discriminant is \( \pm 1 \). A level-\( N \) structure on an abelian variety over \( \mathbb{C} \) is defined similarly to an elliptic curve (see §1; we require it to be compatible with the Weil pairing).

Let \( (V, \psi) \) be a symplectic space over \( \mathbb{R} \), and let \( \Lambda \) be a lattice in \( V \) such that \( \psi(\Lambda, \Lambda) \subset \mathbb{Z} \) and \( \psi|_{\Lambda \times \Lambda} \) has discriminant \( \pm 1 \). The points of the corresponding Siegel upper half space \( D \) are the complex structures \( J \) on \( V \) such that \( \psi_J \) is Riemann form (see §2). The map \( J \mapsto (V, J)/\Lambda \) is a bijection from \( D \) to the set of isomorphism classes of principally polarized abelian varieties over \( \mathbb{C} \) equipped with an isomorphism \( \Lambda \rightarrow H_1(A, \mathbb{Z}) \). On passing to the quotient by the principal congruence subgroup \( \Gamma(N) \), we get a bijection from \( D_N \overset{\text{def}}{=} \Gamma(N) \backslash D \) to the set of isomorphism classes of principally polarized abelian over \( \mathbb{C} \) equipped with a level-\( N \) structure.

Proposition 4.5. Let \( f : A \rightarrow S \) be a family of principally polarized abelian varieties on a smooth algebraic variety \( S \) over \( \mathbb{C} \), and let \( \eta \) be a level-\( N \) structure on \( A/S \). The map \( \gamma : S(\mathbb{C}) \rightarrow D_N(\mathbb{C}) \) sending \( s \in S(\mathbb{C}) \) to the point of \( \Gamma(N) \backslash D \) corresponding to \((A_s, \eta_s)\) is regular.

Proof. The holomorphicity of \( \gamma \) can be proved by the same argument as in the proof of Proposition 1.2. Its algebraicity then follows from Borel’s theorem 4.3.

Let \( \mathcal{F} \) be the functor sending a scheme \( S \) of finite type over \( \mathbb{C} \) to the set of isomorphism classes of pairs consisting of a family of principally polarized abelian varieties \( f : A \rightarrow S \) over \( S \) and a level-\( N \) structure on \( A \). When \( N \geq 3 \), \( \mathcal{F} \) is representable by a smooth algebraic variety \( S_N \) over \( \mathbb{C} \) ([44], Chapter 7). This means that there exists a (universal) family of principally polarized abelian varieties \( A/S_N \) and a level-\( N \) structure \( \eta \) on \( A/S_N \) such that, for any similar pair \((A'/S, \eta')\) over a scheme \( S \), there exists a unique morphism \( \alpha : S \rightarrow S_N \) for which \( \alpha^*(A/S_N, \eta) \simeq (A'/S', \eta') \).

Theorem 4.6. There is a canonical isomorphism \( \gamma : S_N \rightarrow D_N \).

Proof. The proof is the same as that of Theorem 1.4.

Corollary 4.7. The universal family of complex tori on \( D_N \) is algebraic.

5. Variations of Hodge structures

We review the definitions.
The Deligne torus

The Deligne torus is the algebraic torus $S$ over $\mathbb{R}$ obtained from $G_m$ over $\mathbb{C}$ by restriction of the base field; thus

$$S(\mathbb{R}) = \mathbb{C}^\times,$$

$$S_\mathbb{C} \simeq G_m \times G_m.$$ The map $S(\mathbb{R}) \to S(\mathbb{C})$ induced by $\mathbb{R} \to \mathbb{C}$ is $z \mapsto (z, \bar{z})$. There are homomorphisms

$$\mathbb{R} \times a \mapsto a^{-1} \to \mathbb{C}^\times \xrightarrow{z \mapsto z\bar{z}} \mathbb{R}^\times.$$ The kernel of $t$ is $S^1$. A homomorphism $h: S \to G$ of real algebraic groups gives rise to cocharacters $\mu_h: G_m \to G$, $z \mapsto h_G(z, 1)$, $z \in G_m(\mathbb{C}) = \mathbb{C}^\times$, $w_h: G_m \to G$, $w_h = h \circ w$ (weight homomorphism).

The following formulas are useful ($\mu = \mu_h$):

$$h_G(z_1, z_2) = \mu(z_1) \cdot \overline{\mu(z_2)}; \quad h(z) = \mu(z) \cdot \overline{\mu(z)} (5.1)$$

$$h(i) = \mu(-1) \cdot w_h(i). (5.2)$$

Real Hodge structures

A real Hodge structure is a representation $h: S \to \text{GL}_V$ of $S$ on a real vector space $V$. Equivalently, it is a real vector space $V$ together with a Hodge decomposition,

$$V_\mathbb{C} = \bigoplus_{p, q \in \mathbb{Z}} V^{p, q}$$

such that $V^{p, q} = V^{q, p}$ for all $p, q$. To pass from one description to the other, use the rule ([16, 19]):

$$v \in V^{p, q} \iff h(z)v = z^{-p} \bar{z}^{-q} v, \text{ all } z \in \mathbb{C}^\times.$$ The integers $h^{p, q} \stackrel{\text{def}}{=} \dim \mathbb{C} V^{p, q}$ are called the Hodge numbers of the Hodge structure. A real Hodge structure defines a (weight) gradation on $V$,

$$V = \bigoplus_{m \in \mathbb{Z}} V_m, \quad V_m = V \cap \left( \bigoplus_{p+q=m} V^{p, q} \right),$$

and a descending Hodge filtration,

$$V_\mathbb{C} \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots \supset 0, \quad F^p = \bigoplus_{p' \geq p} V^{p', q}.$$ The weight gradation and Hodge filtration together determine the Hodge structure because

$$V^{p, q} = (V_{p+q})_C \cap F^p \cap \overline{F^q}.$$ Note that the weight gradation is defined by $w_h$. A filtration $F$ on $V_\mathbb{C}$ arises from a Hodge structure of weight $m$ on $V$ if and only if

$$V = F^p \oplus \overline{F^q} \text{ whenever } p + q = m + 1.$$
The $\mathbb{R}$-linear map $C = h(i)$ is called the Weil operator. It acts as $i^{q-p}$ on $V^{p,q}$, and $C^2$ acts as $(-1)^m$ on $V_m$.

Thus a Hodge structure on a real vector space $V$ can be regarded as a homomorphism $h: S \to \text{GL}_V$, a Hodge decomposition of $V$, or a Hodge filtration together with a weight gradation of $V$. We use the three descriptions interchangeably.

5.3. Let $V$ be a real vector space. To give a Hodge structure $h$ on $V$ of type $\{(-1,0), (0,-1)\}$ is the same as giving a complex structure on $V$: given $h$, let $J$ act as $C = h(i)$; given a complex structure, let $h(z)$ act as multiplication by $z$. The Hodge decomposition $V_C = V^{-1,0} \oplus V^{0,-1}$ corresponds to the decomposition $V_C = V^+ \oplus V^-$ of $V_C$ into its $J$-eigenspaces.

Rational Hodge structures

A rational Hodge structure is a $\mathbb{Q}$-vector space $V$ together with a real Hodge structure on $V_R$ such that the weight gradation is defined over $\mathbb{Q}$. Thus to give a rational Hodge structure on $V$ is the same as giving

- a gradation $V = \bigoplus_m V_m$ on $V$ together with a real Hodge structure of weight $m$ on $V_{mR}$ for each $m$,
- a homomorphism $h: S \to \text{GL}_V$ such that $w_h: G_m \to \text{GL}_{V_m}$ is defined over $\mathbb{Q}$.

The Tate Hodge structure $\mathbb{Q}(m)$ is defined to be the $\mathbb{Q}$-subspace $(2\pi i)^m \mathbb{Q}$ of $\mathbb{C}$ with $h(z)$ acting as multiplication by $\text{Norm}_{\mathbb{C}/\mathbb{R}}(z)^m = (z \bar{z})^m$. It has weight $-2m$ and type $(-m,-m)$.

Polarizations

A polarization of a real Hodge structure $(V, h)$ of weight $m$ is a morphism of Hodge structures

$$\psi: V \otimes V \to \mathbb{R}(-m), \quad m \in \mathbb{Z},$$

such that

$$\langle x, y \rangle \mapsto (2\pi i)^m \psi(x, Cy): V \times V \to \mathbb{R}$$

is symmetric and positive definite. The condition (5.5) means that $\psi$ is symmetric if $m$ is even and skew-symmetric if it is odd, and that $(2\pi i)^m \cdot i^{p-q} \psi_C(x, \bar{x}) > 0$ for $x \in V^{p,q}$.

A polarization of a rational Hodge structure $(V, h)$ of weight $m$ is a morphism of rational Hodge structures $\psi: V \otimes V \to \mathbb{Q}(-m)$ such that $\psi_R$ is a polarization of $(V_{R}, h)$. A rational Hodge structure $(V, h)$ is polarizable if and only if $(V_{R}, h)$ is polarizable (cf. 2.2).
Local systems and vector sheaves with connection

Let $S$ be a complex manifold. A connection on a vector sheaf $\mathcal{V}$ on $S$ is a $\mathbb{C}$-linear homomorphism $\nabla: \mathcal{V} \to \Omega^1_S \otimes \mathcal{V}$ satisfying the Leibniz condition

$$\nabla(fv) = df \otimes v + f \cdot \nabla v$$

for all local sections $f$ of $\mathcal{O}_S$ and $v$ of $\mathcal{V}$. The curvature of $\nabla$ is the composite of $\nabla$ with the map

$$\nabla_1: \Omega^1_S \otimes \mathcal{V} \to \Omega^2_S \otimes \mathcal{V}$$

$$\omega \otimes v \mapsto d\omega \otimes v - \omega \wedge \nabla(v).$$

A connection $\nabla$ is said to be flat if its curvature is zero. In this case, the kernel $\nabla_1$ of $\nabla_1$ is a local system of complex vector spaces on $S$ such that $\mathcal{O}_S \otimes \nabla_1 \simeq \mathcal{V}$.

Conversely, let $\mathcal{V}$ be a local system of complex vector spaces on $S$. The vector sheaf $\mathcal{V} = \mathcal{O}_S \otimes \mathcal{V}$ has a canonical connection $\nabla$: on any open set where $\mathcal{V}$ is trivial, say $\mathcal{V} \simeq \mathbb{C}^n$, the connection is the map $(f_i) \mapsto (df_i): (\mathcal{O}_S)^n \to (\Omega^1_S)^n$. This connection is flat because $d \circ d = 0$. Obviously for this connection, $\mathcal{V} \simeq \nabla$.

In this way, we obtain an equivalence between the category of vector sheaves on $S$ equipped with a flat connection and the category of local systems of complex vector spaces.

Variations of Hodge structures

Let $S$ be a complex manifold. By a family of real Hodge structures on $S$ we mean a holomorphic family. For example, a family of real Hodge structures on $S$ of weight $m$ is a local system $\mathcal{V}$ of $\mathbb{R}$-vector spaces on $S$ together with a filtration $F$ on $\mathcal{V} \equiv \mathcal{O}_S \otimes \mathcal{V}$ by holomorphic vector subsheaves that gives a Hodge filtration at each point, i.e., such that

$$F^p \mathcal{V}_s \oplus \overline{F^{m+1-p} \mathcal{V}_s} \simeq \mathcal{V}_s, \quad \forall s \in S, p \in \mathbb{Z}. $$

For the notion of a family of rational Hodge structures, replace $\mathbb{R}$ with $\mathbb{Q}$.

A polarization of a family of real Hodge structures of weight $m$ is a bilinear pairing of local systems

$$\psi: \mathcal{V} \times \mathcal{V} \to \mathbb{R}(-m)$$

that gives a polarization at each point $s$ of $S$. For rational Hodge structures, replace $\mathbb{R}$ with $\mathbb{Q}$.

Let $\nabla$ be connection on a vector sheaf $\mathcal{V}$. A holomorphic vector field $Z$ on $S$ is a map $\Omega^1_S \to \mathcal{O}_S$, and it defines a map $\nabla_Z: \mathcal{V} \to \mathcal{V}$. A family of rational Hodge structures $\mathcal{V}$ on $S$ is a variation of rational Hodge structures on $S$ if it satisfies the following axiom (Griffiths transversality):

$$\nabla_Z(F^p \mathcal{V}) \subset F^{p-1} \mathcal{V} \text{ for all } p \text{ and } Z.$$

Equivalently,

$$\nabla(F^p \mathcal{V}) \subset \Omega^1_S \otimes F^{p-1} \mathcal{V} \text{ for all } p.$$
Here $\nabla$ is the flat connection on $V \overset{\text{def}}{=} \mathcal{O}_S \otimes_{\mathbb{Q}} V$ defined by $V$.

These definitions are motivated by the following theorem.

**Theorem 5.6** (Griffiths 1968 [24]). Let $f: X \rightarrow S$ be a smooth projective map of smooth algebraic varieties over $\mathbb{C}$. For each $m$, the local system $\mathbb{R}^m f_* \mathbb{Q}$ of $\mathbb{Q}$-vector spaces on $S^{an}$ together with the de Rham filtration on $\mathcal{O}_S \otimes_{\mathbb{Q}} \mathbb{R} f_* \mathbb{Q} \simeq \mathbb{R} f_* (\Omega^\bullet_{X/\mathbb{C}})$ is a polarizable variation of rational Hodge structures of weight $m$ on $S^{an}$.

This theorem suggests that the first step in realizing an algebraic variety as a moduli variety should be to show that it carries a polarized variation of rational Hodge structures.

### 6. Mumford-Tate groups and their variation in families

We define Mumford-Tate groups, and we study their variation in families. Throughout this section, “Hodge structure” means “rational Hodge structure”.

**The conditions (SV)**

We list some conditions on a homomorphism $h: S \rightarrow G$ of real connected algebraic groups:

- **SV1**: the Hodge structure on the Lie algebra of $G$ defined by $\text{Ad} \circ h: S \rightarrow \text{GL}_{\text{Lie}(G)}$ is of type $\{(1, -1), (0,0), (-1,1)\}$;
- **SV2**: $\text{inn}(h(i))$ is a Cartan involution of $G^{ad}$.

In particular, (SV2) says that the Cartan involutions of $G^{ad}$ are inner, and so $G^{ad}$ is an inner form of its compact form. This implies that the simple factors of $G^{ad}$ are geometrically simple (see footnote 9, p. 479).

Condition (SV1) implies that the Hodge structure on $\text{Lie}(G)$ defined by $h$ has weight 0, and so $w_h(G_m) \subset \mathbb{Z}(G)$. In the presence of this condition, we sometimes need to consider a stronger form of (SV2):

- **SV2\*:** $\text{inn}(h(i))$ is a Cartan involution of $G/w_h(G_m)$.

Note that (SV2\*) implies that $G$ is reductive.

Let $G$ be an algebraic group over $\mathbb{Q}$, and let $h$ be a homomorphism $S \rightarrow G_{\mathbb{R}}$. We say that $(G, h)$ satisfies the condition (SV1) or (SV2) if $(G_{\mathbb{R}}, h)$ does. When $w_h$ is defined over $\mathbb{Q}$, we say that $(G, h)$ satisfies (SV2\*) if $(G_{\mathbb{R}}, h)$ does. We shall also need to consider the condition:

- **SV3**: $G^{ad}$ has no $\mathbb{Q}$-factor on which the projection of $h$ is trivial.

In the presence of (SV1,2), the condition (SV3) is equivalent to $G^{ad}$ being of non-compact type (apply Lemma 4.7 of [40]).

Each condition holds for a homomorphism $h$ if and only if it holds for a conjugate of $h$ by an element of $G(\mathbb{R})$. 
Let $G$ be a reductive group over $\mathbb{Q}$. Let $h$ be a homomorphism $S \to G_{\mathbb{R}}$, and let $\tilde{h} : S \to G_{\mathbb{R}}^{ad}$ be $\text{ad} \circ h$. Then $(G, h)$ satisfies (SV1,2,3) if and only if $(G^{ad}, \tilde{h})$ satisfies the same conditions.\footnote{For (SV1), note that $\text{Ad}(h(z)) : \text{Lie}(G) \to \text{Lie}(G)$ is the derivative of $\text{ad}(h(z)) : G \to G$. The latter is trivial on $Z(G)$, and so the former is trivial on $\text{Lie}(Z(G))$.}

**Remark 6.1.** Let $H$ be a real algebraic group. The map $z \mapsto z/\bar{z}$ defines an isomorphism $S/w(G_m) \simeq S^1$, and so the formula

\[(6.2) \quad h(z) = u(z/\bar{z})\]

defines a one-to-one correspondence between the homomorphisms $h : S \to H$ trivial on $w(G_m)$ and the homomorphisms $u : S^1 \to H$. When $H$ has trivial centre, $h$ satisfies SV1 (resp. SV2) if and only if $u$ satisfies SU1 (resp. SU2).

**Notes.** Conditions (SV1), (SV2), and (SV3) are respectively the conditions (2.1.1.1), (2.1.1.2), and (2.1.1.3) of [19], and (SV2*) is the condition (2.1.1.5).

**Definition of Mumford-Tate groups**

Let $(V, h)$ be a rational Hodge structure. Following [15], 7.1, we define the *Mumford-Tate group* of $(V, h)$ to be the smallest algebraic subgroup $G$ of $GL_V$ such that $G_{\mathbb{R}} \supset h(S)$. It is also the smallest algebraic subgroup $G$ of $GL_V$ such that $G_{\mathbb{C}} \supset \mu_h(G_m)$ (apply (5.1), p. 490). We usually regard the Mumford-Tate group as a pair $(G, h)$, and we sometimes denote it by $MT_V$. Note that $G$ is connected, because otherwise we could replace it with its identity component. The weight map $w_h : G_m \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$ and maps into the centre of $G$.\footnote{Let $Z(w_h)$ be the centralizer of $w_h$ in $G$. For any $\alpha \in \mathbb{R}^*$, $w_h(\alpha) : V_{\mathbb{R}} \to V_{\mathbb{R}}$ is a morphism of real Hodge structures, and so it commutes with the action of $h(S)$. Hence $h(S) \subset Z(w_h)_{\mathbb{R}}$. As $h$ generates $G$, this implies that $Z(w_h) = G$.}

Let $(V, h)$ be a polarizable rational Hodge structure, and let $T^{m,n}$ denote the Hodge structure $V^{\otimes m} \otimes V^{\otimes n}$ ($m, n \in \mathbb{N}$). By a *Hodge class* of $V$, we mean an element of $V$ of type $(0,0)$, i.e., an element of $V \cap V^{0,0}$, and by a *Hodge tensor* of $V$, we mean a Hodge class of some $T^{m,n}$. The elements of $T^{m,n}$ fixed by the Mumford-Tate group of $V$ are exactly the Hodge tensors, and $MT_V$ is the largest algebraic subgroup of $GL_V$ fixing all the Hodge tensors of $V$ (cf. [20], 3.4).

The real Hodge structures form a semisimple tannakian category\footnote{For the theory of tannakian categories, we refer the reader to [21]. In fact, we shall only need to use the elementary part of the theory (ibid. §1.2).} over $\mathbb{R}$; the group attached to the category and the forgetful fibre functor is $S$. The rational Hodge structures form a tannakian category over $\mathbb{Q}$, and the polarizable rational Hodge structures form a semisimple tannakian category, which we denote $\text{Hdg}_{\mathbb{Q}}$. Let $(V, h)$ be a rational Hodge structure, and let $(V, h)\otimes$ be the tannakian subcategory generated by $(V, h)$. The Mumford-Tate group of $(V, h)$ is the algebraic group attached $(V, h)\otimes$ and the forgetful fibre functor.
Let \( G \) and \( G^e \) respectively denote the Mumford-Tate groups of \( V \) and \( V \oplus \mathbb{Q}(1) \). The action of \( G^e \) on \( V \) defines a homomorphism \( G^e \to G \), which is an isogeny unless \( V \) has weight 0, in which case \( G^e \simeq G \times \mathbb{G}_m \). The action of \( G^e \) on \( \mathbb{Q}(1) \) defines a homomorphism \( G^e \to \text{GL}_\mathbb{Q}(1) \) whose kernel we denote \( G^1 \) and call the special Mumford-Tate group of \( V \). Thus \( G^1 \subset \text{GL}_V \), and it is the smallest algebraic subgroup of \( \text{GL}_V \) such that \( G^1 \cap h(S^1) \). Clearly \( G^1 \subset G \) and \( G = G^1 \cdot w_h(G_m) \).

**Proposition 6.3.** Let \( G \) be a connected algebraic group over \( \mathbb{Q} \), and let \( h \) be a homomorphism \( S \to G_R \). The pair \( (G, h) \) is the Mumford-Tate group of a Hodge structure if and only if the weight homomorphism \( w_h : G_m \to G_R \) is defined over \( \mathbb{Q} \) and \( G \) is generated by \( h \) (i.e., any algebraic subgroup \( H \) of \( G \) such that \( h(S) \subset H_R \) equals \( G \)).

**Proof.** If \( (G, h) \) is the Mumford-Tate group of a Hodge structure \( (V, h) \), then certainly \( h \) generates \( G \). The weight homomorphism \( w_h \) is defined over \( \mathbb{Q} \) because \( (V, h) \) is a rational Hodge structure.

Conversely, suppose that \( (G, h) \) satisfy the conditions. For any faithful representation \( \rho : G \to \text{GL}_V \) of \( G \), the pair \( (V, h \circ \rho) \) is a rational Hodge structure, and \((G, h)\) is its Mumford-Tate group. \( \square \)

**Proposition 6.4.** Let \( (G, h) \) be the Mumford-Tate group of a Hodge structure \( (V, h) \). Then \( (V, h) \) is polarizable if and only if \( (G, h) \) satisfies \((SV2^*)\).

**Proof.** Let \( C = h(i) \). For notational convenience, assume that \( (V, h) \) has a single weight \( m \). Let \( G^1 \) be the special Mumford-Tate group of \( (V, h) \). Then \( C \in G^1(\mathbb{R}) \), and a pairing \( \psi : V \times V \to \mathbb{Q}(-m) \) is a polarization of the Hodge structure \((V, h)\) if and only if \((2\pi i)^m \psi \) is a \( C \)-polarization of \( V \) for \( G^1 \) in the sense of §2. It follows from (2.1) and (2.2) that a polarization \( \psi \) for \((V, h)\) exists if and only if inn\((C)\) is a Cartan involution of \( G^1_R \). Now \( G^1 \subset G \) and the quotient map \( G^1 \to G/w_h(G_m) \) is an isogeny, and so \( \text{inn}(C) \) is a Cartan involution of \( G^1 \) if and only if it is a Cartan involution of \( G/w_h(G_m) \). \( \square \)

**Corollary 6.5.** The Mumford-Tate group of a polarizable Hodge structure is reductive.

**Proof.** An algebraic group \( G \) over \( \mathbb{Q} \) is reductive if and only if \( G_R \) is reductive, and we have already observed that \((SV2^*)\) implies that \( G_R \) is reductive. Alternatively, polarizable Hodge structures are semisimple, and an algebraic group in characteristic zero is reductive if its representations are semisimple (e.g., [21], 2.23). \( \square \)

**Remark 6.6.** Note that (6.4) implies the following statement: let \( (V, h) \) be a Hodge structure; if there exists an algebraic group \( G \subset \text{GL}_V \) such that \( h(S) \subset G_R \) and \((G, h)\) satisfies \((SV2^*)\), then \( (V, h) \) is polarizable.

**Notes.** The Mumford-Tate group of a complex abelian variety \( A \) is defined to be the Mumford-Tate group of the Hodge structure \( H_1(A^{an}, \mathbb{Q}) \). In this context, special Mumford-Tate groups were first introduced in the talk of Mumford [45] (which is "partly joint work with J. Tate").
Special Hodge structures

A rational Hodge structure is special\(^{19}\) if its Mumford-Tate group satisfies (SV1, 2*) or, equivalently, if it is polarizable and its Mumford-Tate group satisfies (SV1).

**Proposition 6.7.** The special Hodge structures form a tannakian subcategory of $\text{Hdg}_{\mathbb{Q}}$.

*Proof.* Let $(V, h)$ be a special Hodge structure. The Mumford-Tate group of any object in the tannakian subcategory of $\text{Hdg}_{\mathbb{Q}}$ generated by $(V, h)$ is a quotient of $\text{MT}_V$, and hence satisfies (SV1,2*). □

Recall that the *level* of a Hodge structure $(V, h)$ is the maximum value of $|p - q|$ as $(p, q)$ runs over the pairs $(p, q)$ with $V^{p,q} \neq 0$. It has the same parity as the weight of $(V, h)$.

**Example 6.8.** Let $V_n(a_1, \ldots, a_d)$ denote a complete intersection of $d$ smooth hypersurfaces of degrees $a_1, \ldots, a_d$ in general position in $\mathbb{P}^{n+d}$ over $\mathbb{C}$. Then $H^n(V_n, \mathbb{Q})$ has level $\leq 1$ only for the varieties $V_n(2)$, $V_n(2,2)$, $V_2(3)$, $V_n(2,2,2)$ ($n$ odd), $V_3(3)$, $V_5(2,3)$, $V_5(3)$, $V_5(4)$ ([50]).

**Proposition 6.9.** Every polarizable Hodge structure of level $\leq 1$ is special.

*Proof.* A Hodge structure of level 0 is direct sum of copies of $\mathbb{Q}(m)$ for some $m$, and so its Mumford-Tate group is $\mathbb{G}_m$. A Hodge structure $(V, h)$ of level 1 is of type $\{(p,p+1), (p+1,p)\}$ for some $p$. Then

$$\text{Lie}(\text{MT}_V) \subset \text{End}(V) = V^\vee \otimes V,$$

which is of type $\{(-1,1), (0,0), (1,-1)\}$. □

**Example 6.10.** Let $A$ be an abelian variety over $\mathbb{C}$. The Hodge structures $H^n_B(A)$ are special for all $n$. To see this, note that $H^1_B(A)$ is of level 1, and hence is special by (6.9), and that

$$H^n_B(A) \simeq \bigwedge^n H^1_B(A) \subset H^1_B(A)^\otimes n,$$

and hence $H^n_B(A)$ is special by (6.7).

It follows that a nonspecial Hodge structure does not lie in the tannakian subcategory of $\text{Hdg}_{\mathbb{Q}}$ generated by the cohomology groups of abelian varieties.

**Proposition 6.11.** A pair $(G, h)$ is the Mumford-Tate group of a special Hodge structure if and only if $h$ satisfies (SV1,2*), the weight $w_h$ is defined over $\mathbb{Q}$, and $G$ is generated by $h$.

*Proof.* Immediate consequence of Proposition 6.3, and of the definition of a special Hodge structure. □

Note that, because $h$ generates $G$, it also satisfies (SV3).}

\(^{19}\)Poor choice of name, since "special" is overused and special points on Shimura varieties don’t correspond to special Hodge structures, but I can’t think of a better one. Perhaps an "SV Hodge structure"?
Example 6.12. Let $f : X \to S$ be the universal family of smooth hypersurfaces of a fixed degree $\delta$ and of a fixed odd dimension $n$. For $s$ outside a meagre subset of $S$, the Mumford-Tate group of $H^n(X_s, \mathbb{Q})$ is the full group of symplectic similitudes (see 6.23 below). This implies that $H^n(X_s, \mathbb{Q})$ is not special unless it has level $\leq 1$. According to (6.8), this rarely happens.

The generic Mumford-Tate group

Throughout this subsection, $(V, F)$ is a family of Hodge structures on a connected complex manifold $S$. Recall that “family” means “holomorphic family”.

Lemma 6.13. For any $t \in \Gamma(S, V)$, the set

$$Z(t) = \{ s \in S \mid t_s \text{ is of type } (0,0) \text{ in } V_s \}$$

is an analytic subset of $S$.

Proof. An element of $V_s$ is of type $(0,0)$ if and only if it lies in $F_0 V_s$. On $S$, we have an exact sequence

$$0 \to F^0 V \to V \to \mathcal{Q} \to 0$$

of locally free sheaves of $\mathcal{O}_S$-modules. Let $U$ be an open subset of $S$ such that $\mathcal{Q}$ is free over $U$. Choose an isomorphism $\mathcal{Q} \cong \mathcal{O}_U^r$, and let $t|U$ map to $(t_1, \ldots, t_r)$ in $\mathcal{O}_U^r$. Then

$$Z(t) \cap U = \{ s \in U \mid t_1(s) = \cdots = t_r(s) = 0 \}. \quad \square$$

Example 6.15. For a “general” abelian variety of dimension $g$ over $\mathbb{C}$, it is known that the $\mathbb{Q}$-algebra of Hodge classes is generated by the class of an ample divisor class ([12], [34]). It follows that the same is true for all abelian varieties in the subset $\hat{S}$ of the moduli space $S$. The Hodge conjecture obviously holds for these abelian varieties.

Let $t$ be a section of $T^{m,n}$ over an open subset $U$ of $\hat{S}$; if $t$ is a Hodge class in $T^{m,n}_s$ for some $s \in U$, then it is Hodge tensor for every $s \in U$. Thus, there exists a local system of $\mathbb{Q}$-subspaces $HT^{m,n}$ on $\hat{S}$ such that $(HT^{m,n})_s$ is the space of Hodge classes in $T^{m,n}_s$ for each $s$. Since the Mumford-Tate group of $(V_s, F_s)$ is the largest algebraic subgroup of $GL_{V_s}$ fixing the Hodge tensors in the spaces $T^{m,n}_s$, we have the following result.
Proposition 6.16. Let $G_s$ be the Mumford-Tate group of $(V_s, F_s)$. Then $G_s$ is locally constant on $\tilde{S}$.

More precisely:
Let $U$ be an open subset of $S$ on which $V$ is constant, say, $V = V_U$; identify the stalk $V_s$ ($s \in U$) with $V$, so that $G_s$ is a subgroup of $GL_V$; then $G_s$ is constant for $s \in U \cap \tilde{S}$, say $G_s = G$, and $G \supset G_s$ for all $s \in U \setminus (U \cap \tilde{S})$.

6.17. We say that $G_s$ is generic if $s \in \tilde{S}$. Suppose that $V$ is constant, say $V = V_S$, and let $G = G_{s_0} \subset GL_V$ be generic. By definition, $G$ is the smallest algebraic subgroup of $GL_V$ such that $G_R$ contains $h_{s_0}(S)$. As $G \supset G_s$ for all $s \in S$, the generic Mumford-Tate group of $(V, F)$ is the smallest algebraic subgroup $G$ of $GL_V$ such that $G_R$ contains $h_s(S)$ for all $s \in S$.

Let $\pi: \tilde{S} \to S$ be a universal covering of $S$, and fix a trivialization $\pi^*V \simeq V_S$ of $V$. Then, for each $s \in S$, there are given isomorphisms

\begin{equation}
V \simeq (\pi^*V)_s \simeq V_{\pi s}.
\end{equation}

There is an algebraic subgroup $G$ of $GL_V$ such that, for each $s \in \pi^{-1}(\tilde{S})$, $G$ maps isomorphically onto $G_s$ under the isomorphism $GL_V \simeq GL_{V_{\pi s}}$ defined by (6.18). It is the smallest algebraic subgroup of $GL_V$ such that $G_R$ contains the image of $h_s: S \to GL_{V_s}$ for all $s \in \tilde{S}$.

Aside 6.19. For a polarizable integral variation of Hodge structures on a smooth algebraic variety $S$, Cattani, Deligne, and Kaplan ([8], Corollary 1.3) show that the sets $\pi_*|Z(t)|$ in (6.14) are algebraic subvarieties of $S$. This answered a question of Weil [65].

Variation of Mumford-Tate groups in families

Definition 6.20. Let $(V, F)$ be a family of Hodge structures on a connected complex manifold $S$.

(a) An integral structure on $(V, F)$ is a local system of $\mathbb{Z}$-modules $\Lambda \subset V$ such that $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda \simeq V$.

(b) The family $(V, F)$ is said to satisfy the theorem of the fixed part if, for every finite covering $\alpha: S' \to S$ of $S$, there is a Hodge structure on the $\mathbb{Q}$-vector space $\Gamma(S', \alpha^*V)$ such that, for all $s \in S'$, the canonical map $\Gamma(S', \alpha^*V) \to \alpha^*V_s$ is a morphism of Hodge structures, or, in other words, if the largest constant local subsystem $V^f$ of $\alpha^*V$ is a constant family of Hodge substructures of $\alpha^*V$.

(c) The algebraic monodromy group at point $s \in S$ is the smallest algebraic subgroup of $GL_{V_s}$ containing the image of the monodromy homomorphism $\pi_1(S, s) \to GL(V_s)$. Its identity connected component is called the connected
monodromy group $M_s$ at $s$. In other words, $M_s$ is the smallest connected algebraic subgroup of $GL_{V_s}$ such that $M_s(\mathbb{Q})$ contains the image of a subgroup of $\pi_1(S, s)$ of finite index.

6.21. Let $\pi: \bar{S} \to S$ be the universal covering of $S$, and let $\Gamma$ be the group of covering transformations of $\bar{S}/S$. The choice of a point $s \in \bar{S}$ determines an isomorphism $\Gamma \cong \pi_1(S, s)$. Now choose a trivialization $\pi^* V \cong V_{\bar{S}}$. The choice of a point $s \in \bar{S}$ determines an isomorphism $V \cong V_{\pi(s)}$. There is an action of $\Gamma$ on $V$ such that, for each $s \in \bar{S}$, the diagram

$$
\begin{array}{ccc}
\Gamma & \times & V \\
\downarrow \cong & & \downarrow \cong \\
\pi_1(S, \pi s) & \times & V_s \\
\end{array}
$$

commutes. Let $M$ be the smallest connected algebraic subgroup of $GL_{V}$ such $M(\mathbb{Q})$ contains a subgroup of $\Gamma$ of finite index; in other words,

$$
M = \bigcap \{H \subset GL_{V} \mid H \text{ connected, } \langle \Gamma : H(\mathbb{Q}) \cap \Gamma \rangle < \infty \}.
$$

Under the isomorphism $V \cong V_{\pi s}$ defined by $s \in S$, $M$ maps isomorphically onto $M_s$.

**Theorem 6.22.** Let $(V, F)$ be a polarizable family of Hodge structures on a connected complex manifold $S$, and assume that $(V, F)$ admits an integral structure. Let $G_s$ (resp. $M_s$) denote the Mumford-Tate (resp. the connected monodromy group) at $s \in S$.

(a) For all $s \in \bar{S}$, $M_s \subset G_s^{der}$.

(b) If $T^{m,n}$ satisfies the theorem of the fixed part for all $m, n$, then $M_s$ is normal in $G_s^{der}$ for all $s \in \bar{S}$; moreover, if $G_{s'}$ is commutative for some $s' \in S$, then $M_s = G_s^{der}$ for all $s \in S$.

The theorem was proved by Deligne (see [15], 7.5; [66], 7.3) except for the second statement of (b), which is Proposition 2 of [1]. The proof of the theorem will occupy the rest of this subsection.

**Example 6.23.** Let $f: X \to \mathbb{P}^1$ be a Lefschetz pencil over $\mathbb{C}$ of hypersurfaces of fixed degree and odd dimension $n$, and let $S$ be the open subset of $\mathbb{P}^1$ where $X_s$ is smooth. Let $(V, F)$ be the variation of Hodge structures $R^n f_* \mathbb{Q}$ on $S$. The action of $\pi_1(S, s)$ on $V_s = H^n(X_s^{\text{an}}, \mathbb{Q})$ preserves the cup-product form on $V_s$, and a theorem of Kazhdan and Margulis ([17], 5.10) says that the image of $\pi_1(S, s)$ is Zariski-dense in the symplectic group. It follows that the generic Mumford-Tate group $G_s$ is the full group of symplectic similitudes. This implies that, for $s \in \bar{S}$, the Hodge structure $V_s$ is not special unless it has level $\leq 1$. 
Proof of (a) of Theorem 6.22. We first show that $M_s \subset G_s$ for $s \in \mathring{S}$. Recall that on $\mathring{S}$ there is a local system of $\mathbb{Q}$-vector spaces $HT^{m,n} \subset T^{m,n}$ such that $HT^{m,n}$ is the space of Hodge tensors in $T^{m,n}_s$. The fundamental group $\pi_1(S, s)$ acts on $HT^{m,n}_s$ through a discrete subgroup of $GL(HT^{m,n})$ (because it preserves a lattice in $T^{m,n}_s$), and it preserves a positive definite quadratic form on $HT^{m,n}_s$. It therefore acts on $HT^{m,n}_s$ through a finite quotient. As $G_s$ is the algebraic subgroup of $GL_V$, fixing the Hodge tensors in some finite direct sum of spaces $T^{m,n}_s$, this shows that the image of some finite index subgroup of $\pi_1(S, s)$ is contained in $G_s(\mathbb{Q})$. Hence $M_s \subset G_s$.

We next show that $M_s$ is contained in the special Mumford-Tate group $G^1_s$ at $s$. Consider the family of Hodge structures $V \oplus \mathbb{Q}(1)$, and let $G^\epsilon_s$ be its Mumford-Tate group at $s$. As $V \oplus \mathbb{Q}(1)$ is polarizable and admits an integral structure, its connected monodromy group $M^\epsilon_s$ at $s$ is contained in $G^\epsilon_s$. As $\mathbb{Q}(1)$ is a constant family, $M^\epsilon_s \subset \text{Ker}(G^\epsilon_s \to GL_{\mathbb{Q}(1)}) = G^1_s$. Therefore $M_s = M^\epsilon_s \subset G^1_s$.

There exists an object $W$ in $\text{Rep}_Q G_s \simeq (V_s)^{\oplus} \subset \text{Hdg}_Q$, such that $G^\text{der}_s \cdot \text{w}_{h_s}(G_m)$ is the kernel of $G_s \to GL_V$. The Hodge structure $W$ admits an integral structure, and its Mumford-Tate group is $G' \simeq G_s / (G^\text{der}_s \cdot \text{w}_{h_s}(G_m))$. As $W$ has weight 0 and $G'$ is commutative, we find from (6.4) that $G'(\mathbb{R})$ is compact. As the action of $\pi_1(S, s)$ on $W$ preserves a lattice, its image in $G'(\mathbb{R})$ must be discrete, and hence finite. This shows that

$$M_s \subset (G^\text{der}_s \cdot \text{w}_{h_s}(G_m)) \cap G^1_s = G^\text{der}_s.$$

Proof of the first statement of (b) of Theorem 6.22. We first prove two lemmas.

Lemma 6.24. Let $V$ be a $\mathbb{Q}$-vector space, and let $H \subset G$ be algebraic subgroups of $GL_V$. Assume:

(a) the action of $H$ on any $H$-stable line in a finite direct sum of spaces $T^{m,n}$ is trivial;
(b) $(T^{m,n})^H$ is $G$-stable for all $m, n \in \mathbb{N}$.

Then $H$ is normal in $G$.

Proof. There exists a line $L$ in some finite direct sum $T$ of spaces $T^{m,n}$ such that $H$ is the stabilizer of $L$ in $GL_V$ (Chevalley's theorem, [20], 3.1a,b). According to (a), $H$ acts trivially on $L$. Let $W$ be the intersection of the $G$-stable subspaces of $T$ containing $L$. Then $W \subset T^H$ because $T^H$ is $G$-stable by (b). Let $\varphi$ be the homomorphism $G \to GL_{V \otimes W}$ defined by the action of $G$ on $W$. As $H$ acts trivially on $W$, it is contained in the kernel of $\varphi$. On the other hand, the elements of the kernel of $\varphi$ act as scalars on $W$, and so stabilize $L$. Therefore $H = \text{Ker}(\varphi)$, which is normal in $G$. □

Lemma 6.25. Let $(V, F)$ be a polarizable family of Hodge structures on a connected complex manifold $S$. Let $L$ be a local system of $\mathbb{Q}$-vector spaces on $S$ contained in a finite direct sum of local systems $T^{m,n}$. If $(V, F)$ admits an integral structure and $L$ has dimension 1, then $M_s$ acts trivially on $L_s$. 
Proof. The hypotheses imply that $L$ also admits an integral structure, and so $\pi_1(S, s)$ acts through the finite subgroup $\{\pm 1\}$ of $GL_\ast$. This implies that $M_s$ acts trivially on $L_s$. \hfill \Box

We now prove the first part of (b) of the theorem. Let $s \in \hat{S}$; we shall apply Lemma 6.24 to $M_s \subset G_s \subset GL_{V_s}$. After passing to a finite covering of $S$, we may suppose that $\pi_1(S, s) \subset M_s(\mathbb{Q})$. Any $M_s$-stable line in $\bigoplus_{m,n} T_{s}^{m,n}$ is of the form $L_s$ for a local subsystem $L$ of $\bigoplus_{m,n} T_{s}^{m,n}$, and so hypothesis (a) of Lemma 6.24 follows from (6.25). It remains to show $(T_{s}^{m,n})^{M_s}$ is stable under $G_s$. Let $H$ be the stabilizer of $(T_{s}^{m,n})^{M_s}$ in $GL_{T_{s}^{m,n}}$. Because $T_{s}^{m,n}$ satisfies the theorem of the fixed part, $(T_{s}^{m,n})^{M_s}$ is a Hodge substructure of $T_{s}^{m,n}$, and so $(T_{s}^{m,n})^{M_s}$ is stable under $h(S)$. Therefore $h(S) \subset H_S$, and this implies that $G_s \subset H$.

**Proof of the second statement of (b) of Theorem 6.22** We first prove a lemma.

**Lemma 6.26.** Let $(V, F)$ be a variation of polarizable Hodge structures on a connected complex manifold $S$. Assume:

(a) $M_s$ is normal in $G_s$ for all $s \in \hat{S}$;
(b) $\pi_1(S, s) \subset M_s(\mathbb{Q})$ for one (hence every) $s \in S$;
(c) $(V, F)$ satisfies the theorem of the fixed part.

Then the subspace $\Gamma(S, V)$ of $V_s$ is stable under $G_s$, and the image of $G_s$ in $GL_{\Gamma(S,V)}$ is independent of $s \in S$.

In fact, (c) implies that $\Gamma(S, V)$ has a well-defined Hodge structure, and we shall show that the image of $G_s$ in $GL_{\Gamma(S,V)}$ is the Mumford-Tate group of $\Gamma(S, V)$.

Proof. We begin with observation: let $G$ be the affine group scheme attached to the tannakian category $\text{Hdg}_\mathbb{Q}$ and the forgetful fibre functor; for any $(V, h_V)$ in $\text{Hdg}_\mathbb{Q}$, $G$ acts on $V$ through a surjective homomorphism $G \to \text{MT}_V$; therefore, for any $(W, h_W)$ in $(V, h_V) \otimes \text{MT}_V$ acts on $W$ through a surjective homomorphism $\text{MT}_V \to \text{MT}_W$.

For every $s \in S$,

$$\Gamma(S, V) = \Gamma(S, V^f) = (V^f)_s = V_{s}^{\Gamma_{M_s}} \equiv (b) V_s^{M_s}.$$  

The subspace $V_{s}^{M_s}$ of $V_s$ is stable under $G_s$ when $s \in \hat{S}$ because then $M_s$ is normal in $G_s$, and it is stable under $G_s$ when $s \notin \hat{S}$ because then $G_s$ is contained in some generic Mumford-Tate group. Because $(V, F)$ satisfies the theorem of the fixed part, $\Gamma(S, V)$ has a Hodge structure (independent of $s$) for which the inclusion $\Gamma(S, V) \to V_s$ is a morphism of Hodge structures. From the observation, we see that the image of $G_s$ in $GL_{\Gamma(S,V)}$ is the Mumford-Tate group of $\Gamma(S, V)$, which does not depend on $s$. \hfill \Box

We now prove that $M_s = G_{s}^{\text{der}}$ when some Mumford-Tate group $G_{s}^{\text{der}}$ is commutative. We know that $M_s$ is a normal subgroup of $G_{s}^{\text{der}}$ for $s \in \hat{S}$, and so it remains to show that $G_s/M_s$ is commutative for $s \in \hat{S}$ under the hypothesis.
We begin with a remark. Let $N$ be a normal algebraic subgroup of an algebraic group $G$. The category of representations of $G/N$ can be identified with the category of representations of $G$ on which $N$ acts trivially. Therefore, to show that $G/N$ is commutative, it suffices to show that $G$ acts through a commutative quotient on every $V$ on which $N$ acts trivially. If $G$ is reductive and we are in characteristic zero, then it suffices to show that, for one faithful representation $V$ of $G$, the group $G$ acts through a commutative quotient on $(T^{m,n})^N$ for all $m, n \in \mathbb{N}$.

Let $T = T^{m,n}$. According to the remark, it suffices to show that, for $s \in \hat{S}$, $G_s$ acts on $T^{M_s}$ through a commutative quotient. This will follow from the hypothesis, once we check that $T$ satisfies the hypotheses of Lemma 6.26. Certainly, $M_s$ is a normal subgroup of $G_s$ for $s \in \hat{S}$, and $\pi_1(S, s)$ will be contained in $M_s$ once we have passed to a finite cover. Finally, we are assuming that $T$ satisfies the theorem of the fixed part.

Variation of Mumford-Tate groups in algebraic families

When the underlying manifold is an algebraic variety, we have the following theorem.

Theorem 6.27 (Griffiths, Schmid). A variation of Hodge structures on a smooth algebraic variety over $\mathbb{C}$ satisfies the theorem of the fixed part if it is polarizable and admits an integral structure.

Proof. When the variation of Hodge structures arises from a projective smooth map $X \rightarrow S$ of algebraic varieties and $S$ is complete, this is the original theorem of the fixed part ([25], §7). In the general case it is proved in [54], §.22. See also [13], 4.1.2 and the footnote on p. 45. □

Theorem 6.28. Let $(V, F)$ be a variation of Hodge structures on a connected smooth complex algebraic variety $S$. If $(V, F)$ is polarizable and admits an integral structure, then $M_s$ is a normal subgroup of $G_s^{\text{der}}$ for all $s \in \hat{S}$, and the two groups are equal if $G_s$ is commutative for some $s \in S$.

Proof. If $(V, F)$ is polarizable and admits an integral structure, then $T^{m,n}$ is polarizable and admits an integral structure, and so it satisfies the theorem of the fixed part (Theorem 6.27). Now the theorem follows from Theorem 6.22. □

7. Period subdomains

We define the notion of a period subdomain, and we show that the hermitian symmetric domains are exactly the period subdomains on which the universal family of Hodge structures is a variation of Hodge structures.
Flag manifolds

Let $V$ be a complex vector space and let $d = (d_1, \ldots, d_r)$ be a sequence of integers with $\dim V > d_1 > \cdots > d_r > 0$. The flag manifold $\text{Gr}_d(V)$ has as points the filtrations

$V \supset F^1 V \supset \cdots \supset F^r V \supset 0, \quad \dim F^i V = d_i.$

It is a projective complex manifold, and the tangent space to $\text{Gr}_d(V)$ at the point corresponding to a filtration $F$ is

$T_F(\text{Gr}_d(V)) \simeq \text{End}(V)/F^0 \text{End}(V)$

where

$F^j \text{End}(V) = \{ \alpha \in \text{End}(V) \mid \alpha(F^i V) \subset F^{i+j} V \text{ for all } i \}.$

**Theorem 7.1.** Let $V_S$ be the constant sheaf on a connected complex manifold $S$ defined by a real vector space $V$, and let $(V_S, F)$ be a family of Hodge structures on $S$. Let $d$ be the sequence of ranks of the subsheaves in $F$.

(a) The map $\varphi: S \to \text{Gr}_d(V_C)$ sending a point $s$ of $S$ to the point of $\text{Gr}_d(V_C)$ corresponding to the filtration $F_s$ on $V$ is holomorphic.

(b) The family $(V_S, F)$ satisfies Griffiths transversality if and only if the image of the map

$(d\varphi)_s: T_s S \to T_{\varphi(s)} \text{Gr}_d(V_C)$

lies in the subspace $F^{-1}_s \text{End}(V_C)/F^0_s \text{End}(V_C)$ of $\text{End}(V_C)/F^0_s \text{End}(V_C)$ for all $s \in S$.

**Proof.** Statement (a) simply says that the filtration is holomorphic, and (b) restates the definition of Griffiths transversality. \qed

Period domains

We now fix a real vector space $V$, a Hodge filtration $F_0$ on $V$ of weight $m$, and a polarization $t_0: V \times V \to \mathbb{R}(m)$ of the Hodge structure $(V, F_0)$.

Let $D = D(V, F_0, t_0)$ be the set of Hodge filtrations $F$ on $V$ of weight $m$ with the same Hodge numbers as $(V, F_0)$ for which $t_0$ is a polarization. Thus $D$ is the set of descending filtrations

$V_C \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots \supset 0$

on $V_C$ such that

(a) $\dim_C F^p = \dim_C F^0_0$ for all $p$,

(b) $V_C = F^p \oplus F^{p+1}_0$ whenever $p + q = m + 1$,

(c) $t_0(F^p, F^q) = 0$ whenever $p + q = m + 1$, and

(d) $(2\pi i)^m t_0(v, C\bar{v}) > 0$ for all nonzero elements $v$ of $V_C$. 
Condition (b) requires that $F$ be a Hodge filtration of weight $m$, condition (a) requires that $(V, F)$ have the same Hodge numbers as $(V, F_0)$, and the conditions (c) and (d) require that $t_0$ be a polarization.

Let $D^\vee = D^\vee (V, F_0, t_0)$ be the set of filtrations on $V_C$ satisfying (a) and (c).

**Theorem 7.2.** The set $D^\vee$ is a compact complex submanifold of $Gr_d(V)$, and $D$ is an open submanifold of $D^\vee$.

**Proof.** We first remark that, in the presence of (a), condition (c) requires that $F_{m+1-p}$ be the orthogonal complement of $F_p$ for all $p$. In particular, each of $F_p$ and $F_{m+1-p}$ determines the other.

When $m$ is odd, $t_0$ is alternating, and the remark shows that $D^\vee$ can be identified with the set of filtrations $V_C \supset F_{(m+1)/2} \supset F_{(m+3)/2} \supset \cdots \supset 0$ satisfying (a) and such that $F_{(m+1)/2}$ is totally isotropic for $t_0$. Let $S$ be the symplectic group for $t_0$. Then $S(C)$ acts transitively on these filtrations, and the stabilizer $P$ of the filtration $F_0$ is a parabolic subgroup of $S$. Therefore $S(C)/P(C)$ is a compact complex manifold, and the bijection $S(C)/P(C) \simeq D^\vee$ is holomorphic. The proof when $m$ is even is similar.

The submanifold $D$ of $D^\vee$ is open because the conditions (b) and (d) are open. □

The complex manifold $D = D(V, F_0, t_0)$ is the (Griffiths) *period domain* defined by $(V, F_0, t_0)$.

**Theorem 7.3.** Let $(V, F, t)$ be a polarized family of Hodge structures on a complex manifold $S$. Let $U$ be an open connected subset of $S$ on which the local system $V$ is trivial, and choose an isomorphism $V|U \simeq V_U$ and a point $o \in U$. The map $\mathcal{P}: U \to D(V, F_0, t_0)$ sending a point $s \in U$ to the point $(V_s, F_s, t_s)$ is holomorphic.

**Proof.** The map $s \mapsto F_s: U \to Gr_d(V)$ is holomorphic by (7.1) and it takes values in $D$. As $D$ is a complex submanifold of $Gr_d(V)$ this implies that the map $U \to D$ is holomorphic ([23], 4.3.3). □

The map $\mathcal{P}$ is called the *period map*.

The constant local system of real vector spaces $V_D$ on $D$ becomes a polarized family of Hodge structures on $D$ in an obvious way (called the *universal family*).

**Theorem 7.4.** If the universal family of Hodge structures on $D = D(V, F_0, t_0)$ satisfies Griffiths transversality, then $D$ is a hermitian symmetric domain.

**Proof.** Let $h_0: S \to GL_V$ be the homomorphism corresponding to the Hodge filtration $F_0$, and let $G$ be the algebraic subgroup of $GL_V$ whose elements fix $t_0$ up to scalar. Then $h_0$ maps into $G$, and $h_0 \circ \omega$ maps into its centre (recall that $V$ has a
single weight $m$). Therefore (see 6.1), there exists a homomorphism $u_0: \mathbb{S}^1 \to G^{ad}$ such that $h_0(z) = u_0(z/\bar{z}) \mod \mathbb{Z}(G)(\mathbb{R})$.

Let $0$ be the point $F_0$ of $D$, and let $g$ denote $G$ with the Hodge structure provided by $\text{Ad} \circ h_0$. Then

$$g_C/g_0 \cong T_0(D) \subset T_0(Gr_d(V)) \cong \text{End}(V)/F^0 \text{End}(V).$$

If the universal family of Hodge structures satisfies Griffiths transversality, then $g_C = F^{-1}g_C$ (by 7.1b). As $g$ is of weight 0, it must be of type $\{(-1,0), (0,0), (1,1)\}$, and so $h_0$ satisfies the condition SV1. Hence $u_0$ satisfies condition SU1 of Theorem 2.5.

Let $G^1$ be the subgroup of $G$ of elements fixing $t_0$. As $t_0$ is a polarization of the Hodge structure, $(2\pi i)^m t_0$ is a C-polarization of $V$ relative to $G^1$, and so $\text{inn}_1(C)$ is a Cartan involution of $G^1$ (Theorem 2.1). Now $C = h_0(i) = u_0(-1)$, and so $u_0$ satisfies condition SU2 of Theorem 2.5. The set $D$ is a connected component of the space of homomorphisms $u: \mathbb{S}^1 \to (G^1)^{ad}$, and so it is equal to the set of conjugates of $u_0$ by elements of $(G^1)^{ad}(\mathbb{R})^+$ (apply 7.6 below with $S$ replaced by $\mathbb{S}^1$). Any compact factors of $(G^1)^{ad}$ can be discarded, and so Theorem 2.5 shows that $D$ is a hermitian symmetric domain.

Remark 7.5. The universal family of Hodge structures on the period domain $D(V, h, t_0)$ satisfies Griffiths transversality only if (a) $(V, h)$ is of type $\{(-1,0), (0,1), (1,1)\}$, or (b) $(V, h)$ of type $\{(-1,0), (0,0), (1,1)\}$ and $h^{-1} \leq 1$, or (c) $(V, h)$ is a Tate twist of one of these Hodge structures.

Period subdomains

7.6. We shall need the following statement ([19], 1.1.12.). Let $G$ be a real algebraic group, and let $X$ be a (topological) connected component of the space of homomorphisms $S \to G$. Let $G_1$ be the smallest algebraic subgroup of $G$ through which all the $h \in X$ factor. Then $X$ is again a connected component of the space of homomorphisms of $S$ into $G_1$. Since $S$ is a torus, any two elements of $X$ are conjugate, and so the space $X$ is a $G_1(\mathbb{R})^+$-conjugacy class of morphisms from $S$ into $G$. It is also a $G(\mathbb{R})^+$-conjugacy class, and $G_1$ is a normal subgroup of the identity component of $G$.

Let $(V, F_0)$ be a real Hodge structure of weight $m$. A tensor $t: V^{\otimes 2r} \to \mathbb{R}(-mr)$ of $V$ is a Hodge tensor of $(V, F_0)$ if it is a morphism of Hodge structures. Concretely, this means that $t$ is of type $(0,0)$ for the natural Hodge structure on

$$\text{Hom}(V^{\otimes 2r}, \mathbb{R}(-mr)) \cong (V^\vee)^{\otimes 2r}(-mr),$$

or that it lies in $F^0(\text{Hom}(V^{\otimes 2r}, \mathbb{R}(-mr)))$.

We now fix a real Hodge structure $(V, F_0)$ of weight $m$ and a family $t = (t_i)_{i \in I}$ of Hodge tensors of $(V, F_0)$. We assume that $I$ contains an element $0$ such that $t_0$ is a polarization of $(V, F_0)$. Let $D(V, F_0, t)$ be a connected component of the set of Hodge filtrations $F$ in $D(V, F_0, t_0)$ for which every $t_i$ is a Hodge tensor. Thus, $D(V, F_0, t)$ is a
connected component of the space of Hodge structures on $V$ for which every $t_i$ is a Hodge tensor and $t_0$ is a polarization.

Let $G$ be the algebraic subgroup of $GL_V \times GL_{\mathbb{Q}(1)}$ fixing the $t_i$. Then $G(\mathbb{R})$ consists of the pairs $(g, c)$ such that

$$t_i(gv_1, \ldots, gv_{2r}) = c^{rm}t_i(v_1, \ldots, v_{2r})$$

for $i \in I$. Let $h$ be a homomorphism $S \to GL_V$. The $t_i$ are Hodge tensors for $(V, h)$ if and only if the homomorphism

$$z \mapsto (h(z), z\bar{z}): S \to GL_V \times G_m$$

factors through $G$. Thus, to give a Hodge structure on $V$ for which all the $t_i$ are Hodge tensors is the same as giving a homomorphism $h: S \to G$, and so $D$ is a connected component of the space of homomorphisms $S \to G$.

Let $G_1$ be the smallest algebraic subgroup of $G$ through which all the $h$ in $D$ factor. According to (7.6), $D$ is a $G_1(\mathbb{R})^+$-conjugacy class of homomorphisms $S \to G_1$. The group $G_1(\mathbb{C})$ acts on $D^\vee(V, F_0, t_0)$, and we let $D^\vee(V, F_0, t)$ denote the orbit of $F_0$.

**Theorem 7.7.** The set $D^\vee(V, F_0, t)$ is a compact complex submanifold of $D^\vee(V, F_0, t_0)$, and $D$ is an open complex submanifold of $D^\vee$.

**Proof.** In fact, $D^\vee(V, F_0, t_0)$ is a smooth projective algebraic variety. The stabilizer $P$ of $F_0$ in the algebraic group $G_{1C}$ is parabolic, and so the orbit of $F_0$ in the algebraic variety $D^\vee(V, F_0, t_0)$ is smooth projective variety. Thus, its complex points form a compact complex submanifold. As

$$D(V, h_0, t_0) = D(V, h_0, t_0) \cap D^\vee(V, h_0, t_0),$$

it is an open complex submanifold of $D^\vee(V, h_0, t_0)$.

We call $D = D(V, F_0, t)$ the period subdomain defined by $(V, F_0, t)$.

**Theorem 7.8.** Let $(V, F)$ be a family of Hodge structures on a complex manifold $S$, and let $t = (t_i)_{i \in I}$ be a family of Hodge tensors of $V$. Assume that $I$ contains an element $0$ such that $t_0$ is a polarization. Let $U$ be a connected open subset of $S$ on which the local system $V$ is trivial, and choose an isomorphism $V|U \xrightarrow{\cong} V_U$ and a point $o \in U$. The map $P: U \to D(V, F_0, t_0)$ sending a point $s \in U$ to the point $(V_s, F_s, t_s)$ is holomorphic.

**Proof.** Same as that of Theorem 7.3.

**Theorem 7.9.** If the universal family of Hodge structures on $D$ satisfies Griffiths transversality, then $D$ is a hermitian symmetric domain.

**Proof.** Essentially the same as that of Theorem 7.4.

**Theorem 7.10.** Every hermitian symmetric domain arises as a period subdomain.
Proof. Let $D$ be a hermitian symmetric domain, and let $o \in D$. Let $H$ be the real adjoint algebraic group such that $H(\mathbb{R})^+ = \text{Hol}(D)^+$, and let $u: \mathbb{S}^1 \to H$ be the homomorphism such that $u(z)$ fixes $o$ and acts on $T_0(D)$ as multiplication by $z$ (see §2). Let $h: S \to H$ be the homomorphism such that $h(z) = u_o(z/\bar{z})$ for $z \in \mathbb{C}^\times = S(\mathbb{R})$. Choose a faithful representation $\rho: H \to \text{GL}_V$ of $G$. Because $u$ satisfies (2.5, SU2), the Hodge structure $(V, \rho \circ h)$ is polarizable. Choose a polarization and include it in a family $t$ of tensors for $V$ such that $H$ is the subgroup of $\text{GL}_V \times \text{GL}_{\mathbb{Q}(1)}$ fixing the elements of $t$. Then $D \simeq D(V, h, t)$.

□

Notes. The interpretation of hermitian symmetric domains as moduli spaces for Hodge structures with tensors is taken from [19], 1.1.17.

Why moduli varieties are (sometimes) locally symmetric

Fix a base field $k$. A moduli problem over $k$ is a contravariant functor $\mathcal{F}$ from the category of (some class of) schemes over $k$ to the category of sets. A variety $S$ over $k$ together with a natural isomorphism $\phi: \mathcal{F} \to \text{Hom}_k(–, S)$ is called a fine solution to the moduli problem. A variety that arises in this way is called a moduli variety.

Clearly, this definition is too general: every variety $S$ represents the functor $h_S = \text{Hom}_k(–, S)$. In practice, we only consider functors for which $\mathcal{F}(T)$ is the set of isomorphism classes of some algebro-geometric objects over $T$, for example, families of algebraic varieties with additional structure.

If $S$ represents such a functor, then there is an object $\alpha \in \mathcal{F}(S)$ that is universal in the sense that, for any $\alpha' \in \mathcal{F}(T)$, there is a unique morphism $\alpha: T \to S$ such that $\mathcal{F}(\alpha)(\alpha) = \alpha'$. Suppose that $\alpha$ is, in fact, a smooth projective map $f: X \to S$ of smooth varieties over $\mathbb{C}$. Then $R^mf_*\mathbb{Q}$ is a polarizable variation of Hodge structures on $S$ admitting an integral structure (Theorem 5.6). A polarization of $X/S$ defines a polarization of $R^mf_*\mathbb{Q}$ and a family of algebraic classes on $X/S$ of codimension $m$ defines a family of global sections of $R^{2m}f_*\mathbb{Q}(m)$. Let $D$ be the universal covering space of $S^\text{an}$. The pull-back of $R^mf_*\mathbb{Q}$ to $D$ is a variation of Hodge structures whose underlying locally constant sheaf of $\mathbb{Q}$-vector spaces is constant, say, equal to $V_S$; thus we have a variation of Hodge structures $(V_S, F)$ on $D$. We suppose that the additional structure on $X/S$ defines a family $t = (t_i)_{i \in I}$ of Hodge tensors of $V_S$ with $t_0$ a polarization. We also suppose that the family of Hodge structures on $D$ is universal\(^\dagger\), i.e., that $D = D(V, F_0, t)$. Because $(V_S, F)$ is a variation of Hodge structures, $D$ is a hermitian symmetric domain (by 7.9). The Margulis arithmeticity theorem (3.12) shows that $\Gamma$ is an arithmetic subgroup of $G(D)$ except possibly when $G(D)$ has factors of small dimension. Thus, when looking at moduli varieties, we are naturally led to consider arithmetic locally symmetric varieties.

Remark 7.11. In fact it is unusual for a moduli problem to lead to a locally symmetric variety. The above argument will usually break down where we assumed that the

\(^\dagger\)This happens rarely!
variation of Hodge structures is universal. Essentially, this will happen only when a “general” member of the family has a Hodge structure that is special in the sense of §6. Even for smooth hypersurfaces of a fixed degree, this is rarely happens (see 6.8 and 6.12). Thus, in the whole universe of moduli varieties, locally symmetric varieties form only a small, but important, class.

Application: Riemann’s theorem in families

Let $A$ be an abelian variety over $\mathbb{C}$. The exponential map defines an exact sequence

$$0 \to H_1(A^{an}, \mathbb{Z}) \to T_0(A^{an}) \xrightarrow{\exp} A^{an} \to 0.$$  

From the first map in this sequence, we get an exact sequence

$$0 \to \ker(\alpha) \to H_1(A^{an}, \mathbb{Z})_C \xrightarrow{\alpha} T_0(A^{an}) \to 0.$$  

The $\mathbb{Z}$-module $H_1(A^{an}, \mathbb{Z})$ is an integral Hodge structure with Hodge filtration $F^{-1} = H_1(A^{an}, \mathbb{Z})_C \supset F^0 = \ker(\alpha) \supset 0$.

Let $\psi$ be a Riemann form for $A$. Then $2\pi i \psi$ is a polarization for the Hodge structure $H_1(A^{an}, \mathbb{Z})$.

**Theorem 7.12.** The functor $A \mapsto H_1(A^{an}, \mathbb{Z})$ is an equivalence from the category of abelian varieties over $\mathbb{C}$ to the category of polarizable integral Hodge structures of type $\{(-1,0), (0,-1)\}$.

**Proof.** In view of the correspondence between complex structures and Hodge structures of type $\{(-1,0), (0,-1)\}$ (see 5.3), this is simply a restatement of Theorem 4.4. □

**Theorem 7.13.** Let $S$ be a smooth algebraic variety over $\mathbb{C}$. The functor

$$(A \xrightarrow{f} S) \mapsto R_1f_*\mathbb{Z}$$

is an equivalence from the category of families of abelian varieties over $S$ to the category of polarizable integral variations of Hodge structures of type $\{(-1,0), (0,-1)\}$.

**Proof.** Let $f^A: A \to S$ be a family of abelian varieties over $S$. The exponential defines an exact sequence of sheaves on $S^{an}$,

$$0 \to R_1f^A_*\mathbb{Z} \to T_0(A^{an}) \to A^{an} \to 0.$$  

From this one sees that the map $\text{Hom}(A^{an}, B^{an}) \to \text{Hom}(R_1f^A_*\mathbb{Z}, R_1f^B_*\mathbb{Z})$ is an isomorphism. The $S$-scheme $\mathfrak{Hom}_S(A, B)$ is unramified over $S$, and so its algebraic sections coincide with its holomorphic sections (cf. [13], 4.4.3). Hence the functor is fully faithful. In particular, a family of abelian varieties is uniquely determined by its variation of Hodge structures up to a unique isomorphism. This allows us to construct the family of abelian varieties attached to a variation of Hodge structures locally. Thus, we may suppose that the underlying local system of $\mathbb{Z}$-modules is
trivial. Assume initially that the variation of Hodge structures on $S$ has a principal polarization, and endow it with a level-$N$ structure. According Proposition 4.5, the variation of Hodge structures on $S$ is the pull-back of the canonical variation of Hodge structures on $D_N$ by a regular map $\alpha: S \to D_N$. Since the latter variation arises from a family of abelian varieties (Theorem 4.6), so does the former.

In fact, the argument still applies when the variation of Hodge structures is not principally polarized, since [44], Chapter 7, hence Theorem 4.6, applies also to nonprincipally polarized abelian varieties. Alternatively, Zarhin’s trick (cf. [36], 16.12) can be used to show that (locally) the fourth multiple of the variation of Hodge structures is principally polarized. □

8. Variations of Hodge structures on locally symmetric varieties

In this section, we explain how to classify variations of Hodge structures on arithmetic locally symmetric varieties in terms of certain auxiliary reductive groups. Throughout, we write “family of integral Hodge structures” to mean “family of rational Hodge structures that admits an integral structure”.

Existence of Hodge structures of CM-type in a family

**Proposition 8.1.** Let $G$ be a reductive group over $\mathbb{Q}$, and let $h: S \to G_{\mathbb{R}}$ be a homomorphism. There exists a $G(\mathbb{R})^+$-conjugate $h_0$ of $h$ such that $h_0(S) \subset T_{0R}$ for some maximal torus $T_0$ of $G$.

**Proof.** (Mumford 1969 [46, p. 348]) Let $K$ be the centralizer of $h$ in $G_{\mathbb{R}}$, and let $T$ be the centralizer in $G_{\mathbb{R}}$ of some regular element of Lie $K$; it is a maximal torus in $K$. Because $h(S)$ centralizes $T$, $h(S) \cdot T$ is a torus in $K$, and so $h(S) \subset T$. If $T'$ is a torus in $G_{\mathbb{R}}$ containing $T$, then $T'$ centralizes $h$, and so $T' \subset K$; therefore $T = T'$, and so $T$ is maximal in $G_{\mathbb{R}}$. For a regular element $\lambda$ of Lie($T$), $T$ is the centralizer of $\lambda$. Choose a $\lambda_0 \in$ Lie($G$) that is close to $\lambda$ in Lie($G_{\mathbb{R}}$), and let $T_0$ be its centralizer in $G$. Then $T_0$ is a maximal torus of $G$ (over $\mathbb{Q}$). Because $T_{0R}$ and $T_R$ are close, they are conjugate: $T_{0R} = gTg^{-1}$ for some $g \in G(\mathbb{R})^+$. Now $h_0 \overset{\text{def}}{=} \text{inn}(g) \circ h$ factors through $T_{0R}$.

A rational Hodge structure is said to be of CM-type if it is polarizable and its Mumford-Tate group is commutative (hence a torus by 6.5).

**Proposition 8.2.** Let $(V, F_0)$ be a rational Hodge structure of some weight $m$, and let $t = (t_i)_{i \in I}$ be a family of tensors of $(V, F_0)$ including a polarization. Then the period subdomain defined by $(V, F_0, t)$ is the connected component containing $h_0$ of the space of homomorphisms $h: S \to G_{\mathbb{R}}$ (see §7).
This contains the $G(\mathbb{R})^+$-conjugacy class of $h_0$, and so the statement follows from Proposition 8.1. \hfill $\Box$

**Description of the variations of Hodge structures on $D(\Gamma)$**

Consider an arithmetic locally symmetric variety $D(\Gamma)$. Recall that this means that $D(\Gamma)$ is an algebraic variety whose universal covering space is a hermitian symmetric domain $D$ and that the group of covering transformations $\Gamma$ is an arithmetic subgroup of the real Lie group $\text{Hol}(D)^+$; moreover, $D(\Gamma)_{\text{an}} = \Gamma \backslash D$.

According to Theorem 3.2, $D$ decomposes into a product $D = D_1 \times \cdots \times D_r$ of hermitian symmetric domains with the property that each group $\Gamma_i \overset{\text{def}}{=} \Gamma \cap \text{Hol}(D_i)^+$ is an irreducible arithmetic subgroup of $\text{Hol}(D_i)^+$ and the map

$$D_1(\Gamma_1) \times \cdots \times D_r(\Gamma_r) \to D(\Gamma)$$

is finite covering. In order to be able to apply the theorems of Margulis we assume that

$$(8.3) \quad \text{rank}(\text{Hol}(D_i)) \geq 2 \text{ for each } i$$

in the remainder of this subsection. We also fix a point $o \in D$.

Recall (2.3) that there exists a unique homomorphism $u: U^1 \to \text{Hol}(D)$ such that $u(z)$ fixes $o$ and acts as multiplication by $z$ on $T_0(D)$. That $\Gamma$ is arithmetic means that there exists a simply connected algebraic group $H$ over $\mathbb{Q}$ and a surjective homomorphism $\varphi: H(\mathbb{R}) \to \text{Hol}(D)^+$ with compact kernel such that $\Gamma$ is commensurable with $\varphi(H(\mathbb{Z}))$. The Margulis superrigidity theorem implies that the pair $(H, \varphi)$ is unique up to a unique isomorphism (see 3.13).

Let

$$H_{\text{ad}}^{\mathbb{R}} = H_c \times H_{nc}$$

where $H_c$ (resp. $H_{nc}$) is the product of the compact (resp. noncompact) simple factors of $H_{\text{ad}}^{\mathbb{R}}$. The homomorphism $\varphi^{\mathbb{R}}(\mathbb{R}): H(\mathbb{R}) \to \text{Hol}(D)^+$ factors through $H_{nc}(\mathbb{R})^+$, and defines an isomorphism of Lie groups $H_{nc}(\mathbb{R})^+ \to \text{Hol}(D)^+$. Let $\tilde{h}$ denote the homomorphism $\mathbb{S}/\mathbb{G}_m \to H_{\text{ad}}^{\mathbb{R}}$ whose projection into $H_c$ is trivial and whose projection into $H_{nc}$ corresponds to $u$ as in (6.1). In other words,

$$(8.4) \quad \tilde{h}(z) = (h_c(z), h_{nc}(z)) \in H_c(\mathbb{R}) \times H_{nc}(\mathbb{R})$$

where $h_c(z) = 1$ and $h_{nc}(z) = u(z/\bar{z})$ in $H_{nc}(\mathbb{R})^+ \simeq \text{Hol}(D)^+$. The map $h \mapsto \tilde{h}$ identifies $D$ with the set of $H_{\text{ad}}^{\mathbb{R}}$-conjugates of $\tilde{h}$ (Theorem 2.5).

Let $(V, F)$ be a polarizable variation of integral Hodge structures on $D(\Gamma)$, and let $V = V_{\pi(o)}$. Then $\pi^*V \simeq V_D$ where $\pi: D \to \Gamma \backslash D$ is the quotient map. Let $G \subset \text{GL}_V$ be the generic Mumford-Tate group of $(V, F)$ (see p. 498), and let $t$ be a family of tensors of $V$ (in the sense of §7), including a polarization $t_0$, such that $G$ is the subgroup of $\text{GL}_V \times \text{GL}_{Q(1)}$ fixing the elements of $t$. As $G$ contains the Mumford-Tate group at each point of $D$, $t$ is a family of Hodge tensors of $(V_D, F)$. The period map $\mathcal{P}: D \to D(V, h_0, t)$ is holomorphic (Theorem 7.8).
We now assume that the monodromy map $\varphi' : \Gamma \to \mathbf{GL}(V)$ has finite kernel, and we pass to a finite covering, so that $\Gamma \subset G(\mathbb{Q})$. Now the elements of $t$ are Hodge tensors of $(V, F)$.

There exists an arithmetic subgroup $\Gamma'$ of $H(\mathbb{Q})$ such that $\varphi(\Gamma') \subset \Gamma$. The Margulis superrigidity theorem 3.10, shows that there is a (unique) homomorphism $\varphi'' : H \to G$ of algebraic groups that agrees with $\varphi' \circ \varphi$ on a subgroup of finite index in $\Gamma'$,

$$
\begin{array}{ccc}
H(\mathbb{Q})^+ & \xrightarrow{\varphi'} & \text{Hol}(D)^+ \\
\cup & \quad & \cup \\
\Gamma' & \xrightarrow{\varphi|\Gamma'} & \Gamma' \xrightarrow{\varphi'} \mathbb{G}(\mathbb{Q})
\end{array}
$$

It follows from the Borel density theorem 3.11 that $\varphi''(H)$ is the connected monodromy group at each point of $D(\Gamma)$. Hence $H \subset G^{\text{der}}$, and the two groups are equal if the Mumford-Tate group at some point of $D(\Gamma)$ is commutative (Theorem 6.22). When we assume that, the homomorphism $\varphi'' : H \to G$ induces an isogeny $H \to G^{\text{der}}$, and hence an isomorphism $H^{\text{ad}} \to G^{\text{ad}}$. Let $(V, h_o) = (V, F)_o$. Then $\text{ad} \circ h_o : S \to G^{\text{ad}}_R \cong H^{\text{ad}}$

equals $\check{h}$. Thus, we have a commutative diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\text{ad} \circ h_o} & G^{\text{ad}}_R \\
\downarrow & & \downarrow
\end{array}
$$

in which $G$ is a reductive group, the homomorphism $H \to G$ has image $G^{\text{der}}$, $w_h$ is defined over $\mathbb{Q}$, and $h$ satisfies (SV2*).

Conversely, suppose that we are given such a diagram (8.5). Choose a family $t$ of tensors for $V$, including a polarization, such that $G$ is the subgroup of $\mathbf{GL}_V \times G_{\mathbb{Q}(1)}$ fixing the tensors. Then we get a period subdomain $D(V, h, t)$ and a canonical variation of Hodge structures $(V, F)$ on it. Pull this back to $D$ using the period isomorphism, and descend it to a variation of Hodge structures on $D(\Gamma)$. The monodromy representation is injective, and some fibre is of CM-type by Proposition 8.2.

**Summary 8.6.** Let $D(\Gamma)$ be an arithmetic locally symmetric domain satisfying the condition (8.3) and fix a point $o \in D$. To give

---

21Let $G$ be a reductive group. The algebraic subgroup $Z(G) \cdot G^{\text{der}}$ is normal, and the quotient $G / (Z(G)^\circ \cdot G^{\text{der}})$ is both semisimple and commutative, and hence is trivial. Therefore $G = Z(G)^\circ \cdot G^{\text{der}}$, from which it follows that $Z(G^{\text{der}}) = Z(G) \cap G^{\text{der}}$. For any isogeny $H \to G^{\text{der}}$, the map $H^{\text{ad}} \to (G^{\text{der}})^{\text{ad}}$ is certainly an isomorphism, and we have just shown that $(G^{\text{der}})^{\text{ad}} \to G^{\text{ad}}$ is an isomorphism. Therefore $H^{\text{ad}} \to G^{\text{ad}}$ is an isomorphism.
a polarizable variation of integral Hodge structures on $D(\Gamma)$ such that some fibre is of CM-type and the monodromy representation has finite kernel

is the same as giving

a diagram (8.5) in which $G$ is a reductive group, the homomorphism $H \to G$ has image $G^\text{der}$, $w_h$ is defined over $\mathbb{Q}$, and $h$ satisfies (SV2*).

**Fundamental Question 8.7.** For which arithmetic locally symmetric varieties $D(\Gamma)$ is it possible to find a diagram (8.5) with the property that the corresponding variation of Hodge structures underlies a family of algebraic varieties? or, more generally, a family of motives?

In §§10,11, we shall answer Question 8.7 completely when “algebraic variety” and “motive” are replaced with “abelian variety” and “abelian motive”.

**Existence of variations of Hodge structures**

In this subsection, we show that, for every arithmetic locally symmetric variety, there exists a diagram (8.5), and hence a variation of polarizable integral Hodge structures on the variety.

**Proposition 8.8.** Let $H$ be a semisimple algebraic group over $\mathbb{Q}$, and let $\tilde{h}: S \to H^\text{ad}$ be a homomorphism satisfying (SV1,2,3). Then there exists a reductive algebraic group $G$ over $\mathbb{Q}$ and a homomorphism $h: S \to G^\mathbb{R}$ such that

- (a) $G^\text{der} = H$ and $\tilde{h} = \text{ad} \circ h$,
- (b) the weight $w_h$ is defined over $\mathbb{Q}$, and
- (c) the centre of $G$ is split by a CM field (i.e., a totally imaginary quadratic extension of a totally real number field).

**Proof.** We shall need the following statement:

Let $G$ be a reductive group over a field $k$ (of characteristic zero), and let $L$ be a finite Galois extension of $k$ splitting $G$. Let $G' \to G^\text{der}$ be a covering of the derived group of $G$. Then there exists a central extension

$$1 \to N \to G_1 \to G \to 1$$

such that $G_1$ is a reductive group, $N$ is a product of copies of $(G_m)_{L/k}$, and

$$(G_1^\text{der} \to G^\text{der}) = (G' \to G^\text{der}).$$

See [41], 3.1.

A number field $L$ is CM if and only if it admits a nontrivial involution $\iota_L$ such that $\sigma \circ \iota_L = \iota \circ \sigma$ for every homomorphism $\sigma: L \to \mathbb{C}$. We may replace $\tilde{h}$ with an $H^\text{ad}(\mathbb{R})^+$-conjugate, and so assume (by Proposition 8.1) that there exists a maximal torus $\tilde{T}$ of $H^\text{ad}$ such that $h$ factors through $\tilde{T}^\mathbb{R}$. Then $\tilde{T}^\mathbb{R}$ is anisotropic (by (SV2)), and so $\iota$ acts as $-1$ on $X^*(\tilde{T})$. It follows that, for any $\sigma \in \text{Aut}(\mathbb{C})$, $\iota \sigma$ and $\iota \sigma$ have the same
action on $X^*(\bar{T})$, and so $\bar{T}$ splits over a CM-field $L$, which can be chosen to be Galois over $\mathbb{Q}$. From the statement, there exists a reductive group $G$ and a central extension

$$1 \to N \to G \to H^{ad} \to 1$$

such that $G^{der} = H$ and $N$ is a product of copies of $(G_m)^n_{/\mathbb{Q}}$. The inverse image $T$ of $\bar{T}$ in $G$ is a maximal torus, and the kernel of $T \to \bar{T}$ is $N$. Because $N$ is connected, there exists a $\mu \in X_*(T)$ lifting $\mu_0 \in X_*(\bar{T})$.\footnote{The functor $X^*$ is exact, and so $0 \to X^*(\bar{T}) \to X^*(T) \to X^*(N) \to 0$ is exact. In fact, it is split-exact (as a sequence of $\mathbb{Z}$-modules) because $X^*(N)$ is torsion-free. On applying $\text{Hom}(-, \mathbb{Z})$ to it, we get the exact sequence $\cdots \to X_*(T) \to X_*(\bar{T}) \to 0$.}

The weight $w = -\mu - \mu_0$ of $\mu$ lies in $X_*(Z)$, where $Z = Z(G) = N$. Clearly $\iota w = w$ and so, as the Tate cohomology group\footnote{Let $g = \text{Gal}(\mathbb{C}/\mathbb{R})$. The $g$-module $X_*(Z)$ is induced, and so the Tate cohomology group $H^2_\ast(g, X_*(Z)) = 0$. By definition, $H^2_\ast(g, X_*(Z)) = X_*(Z)^g/(t + 1)X_*(Z)$.}

$H^0_\ast(\mathbb{R}, X_*(Z)) = 0$, there exists a $\mu_0 \in X_*(Z)$ such that $(t + 1)\mu_0 = w$. When we replace $\mu$ with $\mu - \mu_0$, we find that $w = 0$; in particular, $w$ is defined over $\mathbb{Q}$. Let $h : S \to G_{\mathbb{R}}$ correspond to $\mu$ as in (5.1), p. 490. Then $(G, h)$ fulfills the requirements.\hfill $\Box$

**Corollary 8.9.** For any semisimple algebraic group $H$ over $\mathbb{Q}$ and homomorphism $\bar{h} : S/G_m \to H^{ad}_{\mathbb{R}}$ satisfying (SV1,2,3), there exists a reductive group $G$ with $G^{der} = H$ and a homomorphism $h : S \to G_{\mathbb{R}}$ lifting $\bar{h}$ and satisfying (SV1,2*,3).

**Proof.** Let $(G, h)$ be as in the proposition. Then $G/G^{der}$ is a torus, and we let $T$ be the smallest subtorus of it such that $T_{\mathbb{R}}$ contains the image of $h$. Then $T_{\mathbb{R}}$ is anisotropic, and when we replace $G$ with the inverse image of $T$, we obtain a pair $(G, h)$ satisfying (SV1,2*,3).\hfill $\Box$

Let $G$ be a reductive group over $\mathbb{Q}$, and let $h : S \to G_{\mathbb{R}}$ be a homomorphism satisfying (SV1,2,3). The homomorphism $h$ is said to be **special** if $h(S) \subset T_{\mathbb{R}}$ for some torus $T \subset G$.\footnote{Of course, $h(S)$ is always contained in a subtorus of $G_{\mathbb{R}}$, even a maximal subtorus; the point is that there should exist such a torus defined over $\mathbb{Q}$.} In this case, there is a smallest such $T$, and when $(T, h)$ is the Mumford-Tate group of a CM Hodge structure we say that $h$ is **CM**.

**Proposition 8.10.** Let $h : S \to G_{\mathbb{R}}$ be CM. Then $h$ is CM if

(a) $w_h$ is defined over $\mathbb{Q}$, and

(b) the connected centre of $G$ is split by a $CM$-field.

**Proof.** It is known that a special $h$ is CM if and only if it satisfies the Serre condition:

$$(\tau - 1)(t + 1)\mu_h = 0 = (t + 1)(\tau - 1)\mu_h \quad \text{for all } \tau \in \text{Gal}(\mathbb{Q}^{al}/\mathbb{Q}).$$

As $w_h = (t + 1)\mu_h$, the first condition says that

$$\tau \mu_h = 0 \quad \text{for all } \tau \in \text{Aut}(\mathbb{C}).$$

and the second condition implies that

$$\tau \mu_h = \tau \mu_h \quad \text{for all } \tau \in \text{Aut}(\mathbb{C}).$$
Let $T \subset G$ be a maximal torus such that $h(S) \subset T_R$. The argument in the proof of (8.8) shows that $\tau \mu = \tau \mu$ for $\mu \in X_*(T)$, and since

$$X_*(T)_Q = X_*(Z)_Q \oplus X_*(T/Z)_Q$$

we see that the same equation holds for $\mu \in X_*(T)$. Therefore $(\tau + 1)(\tau - 1)\mu = (\tau - 1)(\tau + 1)\mu$, and we have already observed that this is zero. □

9. Absolute Hodge classes and motives

In order to be able to realize all but a handful of Shimura varieties as moduli varieties, we shall need to replace algebraic varieties and algebraic classes by more general objects, namely, by motives and absolute Hodge classes.

The standard cohomology theories

Let $X$ be a smooth complete25 algebraic variety over an algebraically closed field $k$ (of characteristic zero as always).

For each prime number $\ell$, the étale cohomology groups $H^r_\ell(X, \Omega^\bullet_X/k)(m)$ are finite dimensional $\mathbb{Q}_\ell$-vector spaces. For any homomorphism $\sigma: k \to k'$ of algebraically closed fields, there is a canonical base change isomorphism

(9.1) $H^r_\ell(X)(m) \to H^r_\ell(\sigma X)(m)$, $\sigma X \defeq X \otimes_k k'$.

When $k = \mathbb{C}$, there is a canonical comparison isomorphism

(9.2) $\mathbb{Q}_\ell \otimes \mathbb{Q} H^r_B(X)(m) \to H^r_\ell(X)(m)$.

Here $H^r_B(X)$ denotes the Betti cohomology group $H^r(X^{an}, \mathbb{Q})$.

The de Rham cohomology groups $H^r_{dR}(X, \Omega^\bullet_X/k)(m)$ are finite dimensional $k$-vector spaces. For any homomorphism $\sigma: k \to k'$ of fields, there is a canonical base change isomorphism

(9.3) $k' \otimes_k H^r_{dR}(X)(m) \to H^r_{dR}(\sigma X)(m)$.

When $k = \mathbb{C}$, there is a canonical comparison isomorphism

(9.4) $\mathbb{C} \otimes \mathbb{Q} H^r_B(X)(m) \to H^r_{dR}(X)(m)$.

We let $H^r_{k \times \mathbb{A}_f}(X)(m)$ denote the product of $H^r_{dR}(X)(m)$ with the restricted product of the topological spaces $H^r_B(X)(m)$ relative to their subspaces $H^r(X_{et}, \mathbb{Z}_\ell)(m)$. This is a finitely generated free module over the ring $k \times \mathbb{A}_f$. For any homomorphism

\begin{footnotesize}
25Many statements hold without this hypothesis, but we shall need to consider only this case.
26The “(m)” denotes a Tate twist. Specifically, for Betti cohomology it denotes the tensor product with the Tate Hodge structure $\mathbb{Q}(m)$, for de Rham cohomology it denotes a shift in the numbering of the filtration, and for étale cohomology it denotes a change in Galois action by a multiple of the cyclotomic character.
\end{footnotesize}
σ: k → k′ of algebraically closed fields, the maps (9.1) and (9.3) give a base change homomorphism

\[ H^2_{k \times \mathbb{A}_f}(X)(m) \xrightarrow{\sigma} H^2_{k' \times \mathbb{A}_f}(\sigma X)(m). \]

When k = C, the maps (9.2) and (9.4) give a comparison isomorphism

\[ (\mathbb{C} \times \mathbb{A}_f) \otimes_\mathbb{Q} H^2_B(X)(m) \rightarrow H^2_{\mathbb{C} \times \mathbb{A}_f}(X)(m). \]

**Notes.** For more details and references, see [20], §1.

### Absolute Hodge classes

Let X be a smooth complete algebraic variety over C. The cohomology group \( H^2_B(X)(r) \) has a Hodge structure of weight 0, and an element of type (0,0) in it is called a *Hodge class of codimension r* on X.\(^{27}\) We wish to extend this notion to all base fields of characteristic zero. Of course, given a variety X over a field k, we can choose a homomorphism \( \sigma: k \rightarrow \mathbb{C} \) and define a Hodge class on X to be a Hodge class on \( \sigma X \), but this notion depends on the choice of the embedding. Deligne’s idea for avoiding this problem is to use all embeddings ([18], 0.7).

Let X be a smooth complete algebraic variety over an algebraically closed field k of characteristic zero, and let \( \sigma \) be a homomorphism \( k \rightarrow \mathbb{C} \). An element \( \gamma \) of \( H^2_{k \times \mathbb{A}_f}(X)(r) \) is a \( \sigma \)-Hodge class of codimension r if \( \sigma \gamma \) lies in the subspace \( H^2_B(\sigma X)(r) \cap H^{0,0} \) of \( H^2_{\mathbb{C} \times \mathbb{A}_f}(\sigma X)(r) \). When k has finite transcendence degree over \( \mathbb{Q} \), an element \( \gamma \) of \( H^2_{k \times \mathbb{A}_f}(X)(r) \) is an *absolute Hodge class* if it is \( \sigma \)-Hodge for all homomorphisms \( \sigma: k \rightarrow \mathbb{C} \). The absolute Hodge classes of codimension r on X form a \( \mathbb{Q} \)-subspace \( AH^r(X) \) of \( H^2_{k \times \mathbb{A}_f}(X)(r) \).

We list the basic properties of absolute Hodge classes.

**9.7.** The inclusion \( AH^r(X) \subset H^2_{k \times \mathbb{A}_f}(X)(r) \) induces an injective map

\[ (k \times \mathbb{A}_f) \otimes_{\mathbb{Q}} AH^r(X) \rightarrow H^2_{k \times \mathbb{A}_f}(X)(r); \]

in particular \( AH^r(X) \) is a finite dimensional \( \mathbb{Q} \)-vector space.

This follows from (9.6) because \( AH^r(X) \) is isomorphic to a \( \mathbb{Q} \)-subspace of \( H^2_B(\sigma X)(r) \) (each \( \sigma \)).

**9.8.** For any homomorphism \( \sigma: k \rightarrow k' \) of algebraically closed fields of finite transcendence degree over \( \mathbb{Q} \), the map (9.5) induces an isomorphism \( AH^r(X) \rightarrow AH^r(\sigma X) \) ([20], 2.9a).

\(^{27}\)As \( H^2_B(X)(r) \simeq H^2_B(X) \otimes \mathbb{Q}(r) \), this is essentially the same as an element of \( H^2_B(X) \) of type \( (r, r) \).
This allows us to define $AH^r(X)$ for a smooth complete variety over an arbitrary algebraically closed field $k$ of characteristic zero: choose a model $X_0$ of $X$ over an algebraically closed subfield $k_0$ of $k$ of finite transcendence degree over $\mathbb{Q}$, and define $AH^r(X)$ to be the image of $AH^r(X_0)$ under the map $H^{2r}_{k_0 \times H_0}(X_0)(\tau) \to H^{2r}_{k \times H_0}(X)(\tau)$. With this definition, (9.8) holds for all homomorphisms of algebraically closed fields $k$ of characteristic zero. Moreover, if $k$ admits an embedding in $\mathbb{C}$, then a cohomology class is absolutely Hodge if and only if it is $\sigma$-Hodge for every such embedding.

9.9. The cohomology class of an algebraic cycle on $X$ is absolutely Hodge; thus, the algebraic cohomology classes of codimension $r$ on $X$ form a $\mathbb{Q}$-subspace $A^r(X)$ of $AH^r(X)$ ([20], 2.1a).

9.10. The Künneth components of the diagonal are absolute Hodge classes (ibid., 2.1b).

9.11. Let $X_0$ be a model of $X$ over a subfield $k_0$ of $k$ such that $k$ is algebraic over $k_0$; then $\text{Gal}(k/k_0)$ acts on $AH^r(X)$ through a finite discrete quotient (ibid. 2.9b).

9.12. Let

$$AH^r(X) = \bigoplus_{r \geq 0} AH^r(X);$$

then $AH^r(X)$ is a $\mathbb{Q}$-subalgebra of $\bigoplus H^{2r}_{k \times H_0}(X)(\tau)$. For any regular map $\alpha: Y \to X$ of complete smooth varieties, the maps $\alpha_*$ and $\alpha^*$ send absolute Hodge classes to absolute Hodge classes. (This follows easily from the definitions.)

**Theorem 9.13** (Deligne 1982 [20], 2.12, 2.14). Let $S$ be a smooth connected algebraic variety over $\mathbb{C}$, and let $\pi: X \to S$ be a smooth proper morphism. Let $\gamma \in \Gamma(S, R^{2r} \pi_*$ $Q(\tau))$, and let $\gamma_s$ be the image of $\gamma$ in $H^r_B(X_s)(\tau)$ ($s \in S(\mathbb{C})$).

(a) If $\gamma_s$ is a Hodge class for one $s \in S(\mathbb{C})$, then it is a Hodge class for every $s \in S(\mathbb{C})$.

(b) If $\gamma_s$ is an absolute Hodge class for one $s \in S(\mathbb{C})$, then it is an absolute Hodge class for every $s \in S(\mathbb{C})$.

**Proof.** Let $\bar{X}$ be a smooth compactification of $X$ whose boundary $\bar{X} \setminus X$ is a union of smooth divisors with normal crossings, and let $s \in S(\mathbb{C})$. According to [14], 4.1.1, 4.1.2, there are maps

$$H^r_B(\bar{X})(\tau) \xrightarrow{\text{onto}} \Gamma(S, R^{2r} \pi_* Q(\tau)) \xrightarrow{\text{injective}} H^r_B(X_s)(\tau)$$

whose composite $H^r_B(\bar{X})(\tau) \to H^r_B(X_s)(\tau)$ is defined by the inclusion $X_s \hookrightarrow \bar{X}$; moreover $\Gamma(S, R^{2r} \pi_* Q(\tau))$ has a Hodge structure (independent of $s$) for which the injective maps are morphisms of Hodge structures (theorem of the fixed part).

Let $\gamma \in \Gamma(S, R^{2r} \pi_* Q(\tau))$. If $\gamma_s$ is of type $(0,0)$ for one $s$, then so also is $\gamma$; then $\gamma_s$ is of type $(0,0)$ for all $s$. This proves (a).

Let $\sigma$ be an automorphism of $\mathbb{C}$ (as an abstract field). It suffices to prove (b) with “absolute Hodge” replaced with “$\sigma$-Hodge”. We shall use the commutative
Theorem 9.15

Therefore (9.5). The other maps \( \sigma \) are the base change isomorphisms and the remaining vertical maps are essential tensoring with \( \mathcal{A} \) (and are denoted \( \stackrel{?}{\rightarrow} \mathcal{A} \)).

Let \( \gamma \) be an element of \( \Gamma(S, R^{2r}\pi_s\mathbb{Q}(r)) \) such that \( \gamma_s \) is \( \sigma \)-Hodge for one \( s \). Recall that this means that there is a \( \gamma^s_\sigma \in H^{2r}_{\mathcal{B}}(\sigma X_s)(r) \) of type \((0,0)\) such that \( (\gamma^s_\sigma)_\mathcal{A} = \sigma(\gamma_s)_\mathcal{A} \) in \( H^{2r}_\mathcal{A}(\sigma X_s)(r) \). As \( \gamma_s \) is in the image of \( H^{2r}_{\mathcal{B}}(\mathcal{X})(r) \rightarrow H^{2r}_{\mathcal{B}}(X_s)(r) \),

\[ \sigma(\gamma_s)_\mathcal{A} \text{ is in the image of} \]

\[ H^{2r}_\mathcal{B}(\sigma \mathcal{X})(r) \rightarrow H^{2r}_\mathcal{A}(\sigma X_s)(r). \]

Therefore \( (\gamma^s_\sigma)_\mathcal{A} \) is also, which implies (by linear algebra\(^{28}\)) that \( \gamma^s_\sigma \) is in the image of \( H^{2r}_\mathcal{B}(\sigma \mathcal{X})(r) \rightarrow H^{2r}_\mathcal{B}(\sigma X_s)(r) \).

Let \( \tilde{\gamma}^s_\sigma \) be a pre-image of \( \gamma^s_\sigma \) in \( H^{2r}_\mathcal{B}(\sigma \mathcal{X})(r) \).

Let \( s' \) be a second point of \( S \), and let \( \tilde{\gamma}^s_\sigma \) be the image of \( \tilde{\gamma}^s_\sigma \) in \( H^{2r}_{\mathcal{B}}(\sigma X_{s'})(r) \). By construction, \( (\tilde{\gamma}^s_\sigma)_\mathcal{A} \) maps to \( \sigma(\gamma_s)_\mathcal{A} \) in \( \Gamma(\sigma S, R^{2r}(\sigma \pi)_s \mathcal{A}(r)) \), and so \( (\tilde{\gamma}^s_\sigma)_\mathcal{A} = \sigma(\gamma_s')_\mathcal{A} \) in \( H^{2r}_\mathcal{A}(\sigma X_{s'})(r) \), which demonstrates that \( \gamma_s' \) is \( \sigma \)-Hodge. \( \square \)

**Conjecture 9.14** (Deligne [18], 0.10). *Every \( \sigma \)-Hodge class on a smooth complete variety over an algebraically closed field of characteristic zero is absolutely Hodge, i.e.,

\[ \sigma \text{-Hodge (for one } \sigma \text{) } \implies \text{ absolutely Hodge.} \]

**Theorem 9.15** (Deligne 1982 [20], 2.11). *Conjecture 9.14 is true for abelian varieties.*

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\(^{28}\)Apply the following elementary statement:

Let \( E, W, \) and \( V \) be vector spaces, and let \( \alpha : W \rightarrow V \) be a linear map; let \( v \in V \); if \( e \otimes v \) is in the image of \( 1 \otimes \alpha : E \otimes W \rightarrow E \otimes V \) for some nonzero \( e \in E \), then \( v \) is in the image of \( \alpha \).

To prove the statement, choose an \( f \in E^\vee \) such that \( f(e) = 1 \). If \( \sum e_i \otimes \alpha(w_i) = e \otimes v \), then

\[ \sum f(e_i)w_i = v. \]
To prove the theorem, it suffices to show that every Hodge class on an abelian variety over \( \mathbb{C} \) is absolutely Hodge.\(^{29}\) We defer the proof of the theorem to the next subsection.

Aside 9.16. Let \( X_\mathcal{C} \) be a smooth complete algebraic variety over \( \mathbb{C} \). Then \( X_\mathcal{C} \) has a model \( X_0 \) over a subfield \( k_0 \) of \( \mathbb{C} \) finitely generated over \( \mathbb{Q} \). Let \( k \) be the algebraic closure of \( k_0 \) in \( \mathbb{C} \), and let \( X = X_0k \). For a prime number \( \ell \), let
\[
\mathcal{T}_\ell^r(X) = \bigcup_U \mathbb{H}_r^2(X)(\mathbb{Q}_\ell)^U \quad \text{(space of Tate classes)}
\]
where \( U \) runs over the open subgroups of \( \text{Gal}(k/k_0) \) — as the notation suggests, \( \mathcal{T}_\ell^r(X) \) depends only on \( X/k \). The Tate conjecture ([62], Conjecture 1) says that the \( \mathbb{Q}_\ell \)-vector space \( \mathcal{T}_\ell^r(X) \) is spanned by algebraic classes. Statement 9.11 implies that \( A^r(X) \) projects into \( \mathcal{T}_\ell^r(X) \), and (9.7) implies that the map \( \mathbb{Q}_\ell \otimes_{\mathbb{Q}} A^r(X) \to \mathcal{T}_\ell^r(X) \) is injective. Therefore the Tate conjecture implies that \( A^r(X) = A^r(X_0k) \), and so the Tate conjecture for \( X \) and one \( \ell \) implies that all absolute Hodge classes on \( X_\mathcal{C} \) are algebraic. Thus, in the presence of Conjecture 9.14, the Tate conjecture implies the Hodge conjecture. In particular, Theorem 9.15 shows that, for an abelian variety, the Tate conjecture implies the Hodge conjecture.

Proof of Deligne’s theorem

It is convenient to prove Theorem 9.15 in the following more abstract form.

Theorem 9.17. Suppose that for each abelian variety \( A \) over \( \mathbb{C} \) we have a \( \mathbb{Q} \)-subspace \( C^r(A) \) of the Hodge classes of codimension \( r \) on \( A \). Assume:

(a) \( C^r(A) \) contains all algebraic classes of codimension \( r \) on \( A \);
(b) pull-back by a homomorphism \( \alpha: A \to B \) of abelian varieties maps \( C^r(B) \) into \( C^r(A) \);
(c) let \( \pi: A \to S \) be an abelian scheme over a connected smooth complex algebraic variety \( S \), and let \( t \in \Gamma(S, R^2\pi_*\mathbb{Q}(r)) \); if \( t_s \) lies in \( C^r(A_s) \) for one \( s \in S(\mathbb{C}) \), then it lies in \( C^r(A_s) \) for all \( s \).

Then \( C^r(A) \) contains all the Hodge classes of codimension \( r \) on \( A \).

Corollary 9.18. If hypothesis (c) of the theorem holds for algebraic classes on abelian varieties, then the Hodge conjecture holds for abelian varieties. (In other words, for abelian varieties, the variational Hodge conjecture implies the Hodge conjecture.)

Proof. Immediate consequence of the theorem, because the algebraic classes satisfy (a) and (b).

The proof of Theorem 9.17 requires four steps.

\(^{29}\)Let \( A \) be an abelian variety over \( k \), and suppose that \( \gamma \) is \( \sigma_0 \)-Hodge for some homomorphism \( \sigma_0: k \to \mathbb{C} \). We have to show that it is \( \sigma \)-Hodge for every \( \sigma: k \to \mathbb{C} \). But, using the Zorn’s lemma, one can show that there exists a homomorphism \( \sigma': \mathbb{C} \to \mathbb{C} \) such that \( \sigma = \sigma' \circ \sigma_0 \). Now \( \gamma \) is \( \sigma \)-Hodge if and only if \( \sigma_0 \gamma \) is \( \sigma' \)-Hodge.
Step 1: The Hodge conjecture holds for powers of an elliptic curve. As Tate observed ([62], p. 19), the $\mathbb{Q}$-algebra of Hodge classes on a power of an elliptic curve is generated by those of type $(1,1)$. These are algebraic by a theorem of Lefschetz.

Step 2: Split Weil classes lie in $C$. Let $A$ be a complex abelian variety, and let $\nu$ be a homomorphism from a CM-field $E$ into $\text{End}(A)_{\mathbb{Q}}$. The pair $(A, \nu)$ is said to be of Weil type if the tangent space $T_0(A)$ is a free $E \otimes_{\mathbb{Q}} \mathbb{C}$-module. In this case, $d \defeq \dim_E H^1_B(A)$ is even and the subspace $\bigwedge^d H^1_B(A)(\frac{d}{2})$ of $H^d_B(A)(\frac{d}{2})$ consists of Hodge classes ([20], 4.4). When $E$ is quadratic over $\mathbb{Q}$, these Hodge classes were studied by Weil [65], and for this reason are called Weil classes. A polarization of $(A, \nu)$ is a polarization $\lambda$ of $A$ whose Rosati involution acts on $\nu(E)$ as complex conjugation. The Riemann form of such a polarization can be written

$$(x, y) \mapsto \text{Tr}_{E/\mathbb{Q}}(f \phi(x, y))$$

for some totally imaginary element $f$ of $E$ and $E$-hermitian form $\phi$ on $H^1(A, \mathbb{Q})$. If $\lambda$ can be chosen so that $\phi$ is split (i.e., admits a totally isotropic subspace of dimension $d/2$), then the Weil classes are said to be split.

Lemma 9.19. All split Weil classes of codimension $r$ on an abelian variety $A$ lie in $C^r(A)$.

Proof. Let $(A, \nu, \lambda)$ be a polarized abelian variety of split Weil type. Let $V = H^1(A, \mathbb{Q})$, and let $\psi$ be the Riemann form of $\lambda$. The Hodge structures on $V$ for which the elements of $E$ act as morphisms and $\psi$ is a polarization are parametrized by a period subdomain, which is hermitian symmetric domain (cf. 7.9). On dividing by a suitable arithmetic subgroup, we get a smooth proper map $\pi: \mathcal{A} \to S$ of smooth algebraic varieties whose fibres are abelian varieties with an action of $E$ (Theorem 7.13). There is a $\mathbb{Q}$-subspace $W$ of $\Gamma(S, R^d \pi_*(Q(\frac{d}{2})))$ whose fibre at every point $s$ is the space of Weil classes on $A_s$. One fibre of $\pi$ is $(A, \nu)$ and another is a power of an elliptic curve. Therefore the lemma follows from Step 1 and hypotheses (a,c). (See [20], 4.8, for more details.) \qed

Step 3: Theorem 9.17 for abelian varieties of CM-type. A simple abelian variety $A$ is of CM-type if $\text{End}(A)_{\mathbb{Q}}$ is a field of degree 2 $\dim A$ over $\mathbb{Q}$, and a general abelian variety is of CM-type if every simple isogeny factor of it is of CM-type. Equivalently, it is of CM-type if the Hodge structure $H^1(A)_{\mathbb{Q}}$ is of CM-type. According to [2]:

For any complex abelian variety $A$ of CM-type, there exist complex abelian varieties $B_j$ of CM-type and homomorphisms $A \to B_j$ such that every Hodge class on $A$ is a linear combination of the pull-backs of split Weil classes on the $B_j$.

Thus Theorem 9.17 for abelian varieties of CM-type follows from Step 2 and hypothesis (b). (See [20], §5, for the original proof of this step.)

---

30This is most conveniently proved by applying the criterion [39], 4.8.
Step 4: Completion of the proof of Theorem 9.17  Let $t$ be a Hodge class on a complex abelian variety $A$. Choose a polarization $\lambda$ for $A$. Let $V = H_1(A, \mathbb{Q})$ and let $h_A$ be the homomorphism defining the Hodge structure on $H_1(A, \mathbb{Q})$. Both $t$ and the Riemann form $t_0$ of $\lambda$ can be regarded as Hodge tensors for $V$. The period subdomain $D = D(V, h_A, \{t, t_0\})$ is a hermitian symmetric domain (see 7.9). On dividing by a suitable arithmetic subgroup, we get a smooth proper map $\pi: A \to S$ of smooth algebraic varieties whose fibres are abelian varieties (Theorem 7.13) and a section $t$ of $R^2 \pi_* \mathbb{Q}(r)$. For one $s \in S$, the fibre $(A, t)_s = (A, t)$, and another fibre is an abelian variety of CM-type (apply 8.1), and so the theorem follows from Step 3 and hypothesis (c). (See [20], §6, for more details.)

Motives for absolute Hodge classes

We fix a base field $k$ of characteristic zero; “variety” will mean “smooth projective variety over $k$”.

For varieties $X$ and $Y$ with $X$ connected, we let

$$C^r(X, Y) = \text{AH}^{\dim X + r}(X \times Y)$$

(correspondences of degree $r$ from $X$ to $Y$). When $X$ has connected components $X_i$, $i \in I$, we let

$$C^r(X, Y) = \bigoplus_{i \in I} C^r(X_i, Y).$$

For varieties $X, Y, Z$, there is a bilinear pairing

$$f, g \mapsto g \circ f: C^r(X, Y) \times C^s(Y, Z) \to C^{r+s}(X, Z)$$

with

$$g \circ f \overset{\text{def}}{=} (p_{XZ})_*(p_{XY}^* f \cdot p_{YZ}^* g).$$

Here the $p$’s are projection maps from $X \times Y \times Z$. These pairings are associative and so we get a “category of correspondences”, which has one object $hX$ for every variety over $k$, and whose Homs are defined by

$$\text{Hom}(hX, hY) = C^0(X, Y).$$

Let $f: Y \to X$ be a regular map of varieties. The transpose of the graph of $f$ is an element of $C^0(X, Y)$, and so $X \rightsquigarrow hX$ is a contravariant functor.

The category of correspondences is additive, but not abelian, and so we enlarge it by adding the images of idempotents. More precisely, we define a “category of effective motives”, which has one object $h(X, e)$ for each variety $X$ and idempotent $e$ in the ring $\text{End}(hX) = \text{AH}^{\dim X}(X \times X)$, and whose Homs are defined by

$$\text{Hom}(h(X, e), h(Y, f)) = f \circ C^0(X, Y) \circ e.$$ 

This contains the old category by $hX \leftrightarrow h(X, \text{id})$, and $h(X, \text{id})$ is the image of $hX \overset{\text{c}}{\to} hX$.

The category of effective motives is abelian, but objects need not have duals. In the enlarged category, the motive $h\mathbb{P}^1$ decomposes into $h\mathbb{P}^1 = h^0\mathbb{P}^1 \oplus h^2\mathbb{P}^1$, and it
turns out that, to obtain duals for all objects, we only have to “invert” the motive \( h^2 \mathbb{P}^1 \). This is most conveniently done by defining a “category of motives” which has one object \( h(X, e, m) \) for each pair \( (X, e) \) as before and integer \( m \), and whose Homs are defined by

\[
\text{Hom}(h(X, e, m), h(Y, f, n)) = f \circ C^{n-m}(X, Y) \circ e.
\]

This contains the old category by \( h(X, e) \leftrightarrow h(X, e, 0) \).

We now list some properties of the category \( \text{Mot}(k) \) of motives.

9.20. The Hom’s in \( \text{Mot}(k) \) are finite dimensional \( \mathbb{Q} \)-vector spaces, and \( \text{Mot}(k) \) is a semisimple abelian category.

9.21. Define a tensor product on \( \text{Mot}(k) \) by

\[
h(X, e, m) \otimes h(X, f, n) = h(X \times Y, e \times f, m + n).
\]

With the obvious associativity constraint and a suitable\(^{31}\) commutativity constraint, \( \text{Mot}(k) \) becomes a tannakian category.

9.22. The standard cohomology functors factor through \( \text{Mot}(k) \). For example, define

\[
\omega_\ell(h(X, e, m)) = e \left( \bigoplus_i H^i_\ell(X)(m) \right)
\]

(image of \( e \) acting on \( \bigoplus_i H^i_\ell(X)(m) \)). Then \( \omega_\ell \) is an exact faithful functor \( \text{Mot}(k) \rightarrow \text{Vec}_{\mathbb{Q}_\ell} \) commuting with tensor products. Similarly, de Rham cohomology defines an exact tensor functor \( \omega_{\text{dR}} : \text{Mot}(k) \rightarrow \text{Vec}_k \), and, when \( k = \mathbb{C} \), Betti cohomology defines an exact tensor functor \( \text{Mot}(k) \rightarrow \text{Vec}_{\mathbb{Q}} \). The functors \( \omega_\ell \), \( \omega_{\text{dR}} \), and \( \omega_B \) are called the \( \ell \)-adic, de Rham, and Betti fibre functors, and they send a motive to its \( \ell \)-adic, de Rham, or Betti realization.

The Betti fibre functor on \( \text{Mot}(\mathbb{C}) \) takes values in \( \text{Hdg}_{\mathbb{Q}} \) and is faithful (almost by definition). Deligne’s conjecture 9.14 is equivalent to saying that it is full.

**Abelian motives**

**Definition 9.23.** A motive is *abelian* if it lies in the tannakian subcategory \( \text{Mot}^{ab}(k) \) of \( \text{Mot}(k) \) generated by the motives of abelian varieties.

The Tate motive, being isomorphic to \( \bigwedge^2 h_1 E \) for any elliptic curve \( E \), is an abelian motive. It is known that \( h(X) \) is an abelian motive if \( X \) is a curve, a unirational variety of dimension \( \leq 3 \), a Fermat hypersurface, or a K3 surface.

Deligne’s theorem 9.15 implies that \( \omega_B : \text{Mot}^{ab} (\mathbb{C}) \rightarrow \text{Hdg}_{\mathbb{Q}} \) is fully faithful.

---

\(^{31}\)Not the obvious one! It is necessary to change some signs.
CM motives

**Definition 9.24.** A motive over $\mathbb{C}$ is of **CM-type** if its Hodge realization is of CM-type.

**Lemma 9.25.** Every Hodge structure of CM-type is the Betti realization of an abelian motive.

*Proof.* Elementary (see, for example, [37], 4.6).

Therefore $\omega_B$ defines an equivalence from the category of abelian motives of CM-type to the category of Hodge structures of CM-type.

**Proposition 9.26.** Let $G_{\text{Hdg}}$ (resp. $G_{\text{Mab}}$) be the affine group scheme attached to $\text{Hdg}_\mathbb{Q}$ and its forgetful fibre functor (resp. $\text{Mot}^{ab}(\mathbb{C})$ and its Betti fibre functor). The kernel of the homomorphism $G_{\text{Hdg}} \to G_{\text{Mab}}$ defined by the tensor functor $\omega_B : \text{Mot}^{ab}(\mathbb{C}) \to \text{Hdg}_\mathbb{Q}$ is contained in $(G_{\text{Hdg}})^{\text{der}}$.

*Proof.* Let $S$ be the affine group scheme attached to the category $\text{Hdg}^{\text{cm}}_\mathbb{Q}$ of Hodge structures of CM-type and its forgetful fibre functor. The lemma shows that the functor $\text{Hdg}^{\text{cm}}_\mathbb{Q} \hookrightarrow \text{Hdg}_\mathbb{Q}$ factors through $\text{Mot}^{ab}(\mathbb{C}) \hookrightarrow \text{Hdg}_\mathbb{Q}$, and so $G_{\text{Hdg}} \to S$ factors through $G_{\text{Hdg}} \to G_{\text{Mab}}$:

$$G_{\text{Hdg}} \to G_{\text{Mab}} \to S.$$  

Hence

$$\text{Ker}(G_{\text{Hdg}} \to G_{\text{Mab}}) \subset \text{Ker}(G_{\text{Hdg}} \to S) = (G_{\text{Hdg}})^{\text{der}}.$$  

□

Special motives

**Definition 9.27.** A motive over $\mathbb{C}$ is **special** if its Hodge realization is special (see p. 496).

It follows from (6.7) that the special motives form a tannakian subcategory of $\text{Mot}(k)$, which includes the abelian motives (see 6.10).

**Question 9.28.** Is every special Hodge structure the Betti realization of a motive? (Cf. [19], p. 248; [32], p. 216; [56], 8.7.)

More explicitly: for each simple special Hodge structure $(V, h)$, does there exist an algebraic variety $X$ over $\mathbb{C}$ and an integer $m$ such that $(V, h)$ is a direct factor of $\bigoplus_{r \geq 0} H^r_B(X)(m)$ and the projection $\bigoplus_{r \geq 0} H^r_B(X)(m) \to V \subset \bigoplus_{r \geq 0} H^r_B(X)(m)$ is an absolute Hodge class on $X$.

A positive answer to (9.28) would imply that all connected Shimura varieties are moduli varieties for motives (see §11). Apparently, no special motive is known that is not abelian.
Families of abelian motives  For an abelian variety $A$ over $k$, let

$$\omega_f(A) = \lim \Lambda_N(k^\text{al}), \quad \Lambda_N(k^\text{al}) = \text{Ker}(N: \Lambda(k^\text{al}) \to \Lambda(k^\text{al})).$$

This is a free $\Lambda_f$-module of rank $2 \dim A$ with a continuous action of $\text{Gal}(k^\text{al}/k)$.

Let $S$ be a smooth connected variety over $k$, and let $k(S)$ be its function field. Fix an algebraic closure $k(S)^\text{al}$ of $k(S)$, and let $k(S)^\text{un}$ be the union of the subfields $L$ of $k(S)^\text{al}$ such that the normalization of $S$ in $L$ is unramified over $S$. We say that an action of $\text{Gal}(k(S)^\text{al}/k(S))$ on a module is unramified if it factors through $\text{Gal}(k(S)^\text{un}/k(S))$.

**Theorem 9.29.** Let $S$ be a smooth connected variety over $k$. The functor $A \mapsto \Lambda_\tilde{S} \overset{\text{def}}{=} \Lambda_{k(S)}$ is a fully faithful functor from the category of families of abelian varieties over $S$ to the category of abelian varieties over $k(S)$, with essential image the abelian varieties $B$ over $k(S)$ such that $\omega_f(B)$ is unramified.

**Proof.** When $S$ has dimension $1$, this follows from the theory of Néron models. In general, this theory shows that an abelian variety (or a morphism of abelian varieties) extends to an open subvariety $U$ of $S$ such that $S \setminus U$ has codimension at least $2$. Now we can apply $^2$ [10], I 2.7, V 6.8.

The functor $\omega_f$ extends to a functor on abelian motives such that $\omega_f(h_1A) = \omega_f(A)$ if $A$ is an abelian variety.

**Definition 9.30.** Let $S$ be a smooth connected variety over $k$. A family $M$ of abelian motives over $S$ is an abelian motive $M_{\tilde{S}}$ over $k(S)$ such that $\omega_f(M_{\tilde{S}})$ is unramified.

Let $M$ be a family of motives over a smooth connected variety $S$, and let $\tilde{S} = \text{Spec}(k(S)^\text{al})$. The fundamental group $\pi_1(S, \tilde{S}) = \text{Gal}(k(S)^\text{un}/k(S))$, and so the representation of $\pi_1(S, \tilde{S})$ on $\omega_f(M_{\tilde{S}})$ defines a local system of $\Lambda_f$-modules $\omega_f(M)$. Less obvious is that, when the ground field is $\mathbb{C}$, $M$ defines a polarizable variation of Hodge structures on $S$, $\mathcal{H}^*_B(M/S)$. When $M$ can be represented in the form $(A, p, m)$ on $S$, this is obvious. However, $M$ can always be represented in this fashion on an open subset of $S$, and the underlying local system of $\mathbb{Q}$-vector spaces extends to the whole of $S$ because the monodromy representation is unramified. Now it is possible to show that the variation of Hodge structures itself extends (uniquely) to the whole of $S$, by using results from [54], [9], and [26]. See [38], 2.40, for the details.

**Theorem 9.31.** Let $S$ be a smooth connected variety over $\mathbb{C}$. The functor sending a family $M$ of abelian motives over $S$ to its associated polarizable Hodge structure is fully faithful, with essential image the variations of Hodge structures $(V, F)$ such that there exists a dense open subset $U$ of $S$, an integer $m$, and a family of abelian varieties $f: A \to S$ such that $(V, F)$ is a direct summand of $Rf_*(\Lambda_f)$. $\square$

**Proof.** This follows from the similar statement (7.13) for families of abelian varieties (see [38], 2.42). $\square$

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$^2$Recall that we are assuming that the base field has characteristic zero — the theorem is false without that condition.
10. Symplectic Representations

In this subsection, we classify the symplectic representations of groups. These were studied by Satake in a series of papers (see especially [51, 52, 53]). Our exposition follows that of Deligne [19].

In §8 we proved that there exists a correspondence between variations of Hodge structures on locally symmetric varieties and certain commutative diagrams

\[
\begin{align*}
&H \\
\downarrow &
\begin{array}{c}
(H^\text{ad}, \bar{h}) \\
\end{array}
\end{align*}
\begin{array}{c}
(G, h) \leftarrow (\rho) \rightarrow GL_V
\end{array}
\]

(10.1)

In this section, we study whether there exists such a diagram and a nondegenerate alternating form \( \psi \) on \( V \) such that \( \rho(G) \subset G(\psi) \) and \( \rho_R \circ h \in D(\psi) \). Here \( G(\psi) \) is the group of symplectic similitudes (algebraic subgroup of \( GL_V \) whose elements fix \( \psi \) up to a scalar) and \( D(\psi) \) is the Siegel upper half space (set of Hodge structures \( \bar{h} \) on \( V \) of type \( \{(-1,0), (0,-1)\} \) for which \( 2\pi i \psi \) is a polarization). Note that \( G(\psi) \) is a reductive group whose derived group is the symplectic group \( S(\psi) \).

Preliminaries

10.2. The universal covering torus \( \tilde{T} \) of a torus \( T \) is the projective system \( \{T_n, T_{nm} \rightarrow T_n\} \) in which \( T_n = T \) for all \( n \) and the indexing set is \( \mathbb{N} \setminus \{0\} \) ordered by divisibility. For any algebraic group \( G \),

\[
\text{Hom}(\tilde{T}, G) = \lim_{n \rightarrow \infty} \text{Hom}(T_n, G).
\]

Concretely, a homomorphism \( \tilde{T} \rightarrow G \) is represented by a pair \((f, n)\) in which \( f \) is a homomorphism \( T \rightarrow G \) and \( n \in \mathbb{N} \setminus \{0\} \); two pairs \((f, n)\) and \((g, m)\) represent the same homomorphism \( \tilde{T} \rightarrow G \) if and only if \( f \circ m = g \circ n \). A homomorphism \( f: \tilde{T} \rightarrow G \) factors through \( T \) if and only if it is represented by a pair \((f, 1)\). A homomorphism \( \tilde{G}_m \rightarrow GL_V \) represented by \((\mu, n)\) defines a gradation \( V = \bigoplus V_r, \ r \in \frac{1}{n} \mathbb{Z} \); here \( V_r = \{v \in V \mid \mu(t)v = t^r v\} \); the \( r \) for which \( V_r \neq 0 \) are called the weights the representation of \( \tilde{G}_m \) on \( V \). Similarly, a homomorphism \( \tilde{S} \rightarrow GL_V \) represented by \((h, n)\) defines a fractional Hodge decomposition \( V_C = \bigoplus V^{p.q} \) with \( p, q \in \frac{1}{n} \mathbb{Z} \).

The real case

Throughout this subsection, \( H \) is a simply connected real algebraic group without compact factors, and \( \bar{h} \) is a homomorphism \( \mathbb{S}/\mathbb{G}_m \rightarrow H^\text{ad} \) satisfying the conditions (SV1,2), p. 493, and whose projection on each simple factor of \( H^\text{ad} \) is nontrivial.

---

33This description agrees with that in §2 because of the correspondence in (5.3).
**Definition 10.3.** A homomorphism \( H \rightarrow \text{GL}_V \) with finite kernel is a **symplectic representation** of \( (H, \tilde{h}) \) if there exists a commutative diagram

\[
\begin{array}{ccc}
H & \rightarrow & \text{GL}_V \\
\downarrow & & \\
(H^{ad}, \tilde{h}) & \leftarrow & (G, h) \rightarrow (G(\psi), D(\psi)),
\end{array}
\]

in which \( \psi \) is a nondegenerate alternating form on \( V \), \( G \) is a reductive group, and \( h \) is a homomorphism \( S \rightarrow G \); the homomorphism \( H \rightarrow G \) is required to have image \( G^{\text{der}} \).

In other words, there exists a real reductive group \( G \), a nondegenerate alternating form \( \psi \) on \( V \), and a factorization

\[
H \xrightarrow{a} G \xrightarrow{b} \text{GL}_V
\]

of \( H \rightarrow \text{GL}_V \) such that \( a(H) = G^{\text{der}} \), \( b(G) \subset G(\psi) \), and \( b \circ h \in D(\psi) \); the isogeny \( H \rightarrow G^{\text{der}} \) induces an isomorphism \( H^{ad} \xrightarrow{c} G^{ad} \) (see footnote 21, p. 511), and it is required that \( \tilde{h} = c^{-1} \circ \text{ad} \circ h \).

We shall determine the complex representations of \( H \) that occur in the complexification of a symplectic representation (and we shall omit “the complexification of”).

**Proposition 10.4.** A homomorphism \( H \rightarrow \text{GL}_V \) with finite kernel is a symplectic representation of \( (H, \tilde{h}) \) if there exists a commutative diagram

\[
\begin{array}{ccc}
H & \rightarrow & \text{GL}_V \\
\downarrow & & \\
(H^{ad}, \tilde{h}) & \leftarrow & (G, h) \rightarrow \rho \text{GL}_V,
\end{array}
\]

in which \( G \) is a reductive group, the homomorphism \( H \rightarrow G \) has image \( G^{\text{der}} \), and \( (V, \rho \circ h) \) has type \( \{(-1,0), (0, -1)\} \).

**Proof.** Let \( G' \) be the algebraic subgroup of \( G \) generated by \( G^{\text{der}} \) and \( h(S) \). After replacing \( G \) with \( G' \), we may suppose that \( G \) itself is generated by \( G^{\text{der}} \) and \( h(S) \). Then \( (G, h) \) satisfies \( (SV2^*) \), and it follows from Theorem 2.1 that there exists a polarization \( \psi \) of \( (V, \rho \circ h) \) such that \( G \) maps into \( G(\psi) \) (cf. the proof of 6.4). \( \square \)
Let \((H, \tilde{h})\) be as before. The cocharacter \(\mu_{\tilde{h}}\) of \(H_{\text{ad}}\) lifts to a fractional cocharacter \(\tilde{\mu}\) of \(H_{\text{C}}\):

\[
\begin{array}{ccc}
\tilde{\mathbb{G}}_m & \xrightarrow{\tilde{\mu}} & H_{\text{C}} \\
\downarrow & & \downarrow \text{ad} \\
\mathbb{G}_m & \xrightarrow{\mu_{\tilde{h}}} & H_{\text{ad}}.
\end{array}
\]

**Lemma 10.5.** If an irreducible complex representation \(W\) of \(H\) occurs in a symplectic representation, then \(\tilde{\mu}\) has at most two weights \(a\) and \(a + 1\) on \(W\).

**Proof.** Let \(H \xrightarrow{\varphi} (G, h) \to \text{GL}_V\) be a symplectic representation of \((H, \tilde{h})\), and let \(W\) be an irreducible direct summand of \(V_{\text{C}}\). The homomorphisms \(\varphi_{\text{C}} \circ \tilde{\mu} : \tilde{\mathbb{G}}_m \to G_{\text{C}}\) coincides with \(\mu_{h}\) when composed with \(G_{\text{C}} \to G_{\text{ad}}^\text{C}\), and so \(\varphi_{\text{C}} \circ \tilde{\mu} = \mu_{h} \cdot \nu\) with \(\nu\) central. On \(V\), \(\mu_{h}\) has weights 0, 1. If \(a\) is the unique weight of \(\nu\) on \(W\), then the only weights of \(\tilde{\mu}\) on \(W\) are \(a\) and \(a + 1\). \(\square\)

**Lemma 10.6.** Assume that \(H\) is almost simple. A nontrivial irreducible complex representation \(W\) of \(H\) occurs in a symplectic representation if and only if \(\tilde{\mu}\) has exactly two weights \(a\) and \(a + 1\) on \(W\).

**Proof.** \(\Rightarrow:\) Let \((\mu, n)\) represent \(\tilde{\mu}\). As \(H_{\text{C}}\) is almost simple and \(W\) nontrivial, the homomorphism \(\mathbb{G}_m \to \text{GL}_W\) defined by \(\mu\) is nontrivial, therefore noncentral, and the two weights \(a\) and \(a + 1\) occur.

\(\Leftarrow:\) Let \((W, r)\) be an irreducible complex representation of \(H\) with weights \(a, a + 1\), and let \(V\) be the real vector space underlying \(W\). Define \(G\) to be the subgroup of \(\text{GL}_V\) generated by the image of \(H\) and the homotheties: \(G = r(H) \cdot \mathbb{G}_m\).

Let \(h\) be a fractional lifting of \(\tilde{h}\) to \(H\):

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\tilde{h}} & H_{\text{C}} \\
\downarrow & & \downarrow \text{ad} \\
S & \xrightarrow{h} & H_{\text{ad}}.
\end{array}
\]

Let \(W_a\) and \(W_{a+1}\) be the subspaces of weight \(a\) and \(a + 1\) of \(W\). Then \(\tilde{h}(z)\) acts on \(W_a\) as \((z/\bar{z})^a\) and on \(W_{a+1}\) as \((z/\bar{z})^{a+1}\), and so \(h(z) = \tilde{h}(z)z^{-a}z^{a+1}\) acts on these spaces as \(z\) and \(\bar{z}\) respectively. Therefore \(h\) is a true homomorphism \(S \to G\), projecting to \(\tilde{h}\) on \(H_{\text{ad}}\), and \(V\) is of type \(\{(-1,0), (0,-1)\}\) relative to \(h\). We may now apply Lemma 10.4. \(\square\)

We interpret the condition in Lemma 10.6 in terms of roots and weights. Let \(\tilde{\mu} = \mu_{\tilde{h}}\). Fix a maximal torus \(T\) in \(H_{\text{C}}\), and let \(R = R(H, T) \subset X^*(T)_{\mathbb{Q}}\) be the corresponding root system. Choose a base \(S\) for \(R\) such that \(\langle \alpha, \tilde{\mu} \rangle \geq 0\) for all \(\alpha \in S\) (cf. §2).
Recall that, for each $\alpha \in \mathbb{R}$, there exists a unique $\alpha^\vee \in X_*(T)_\mathbb{Q}$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and the symmetry $s_\alpha : x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$ preserves $\mathbb{R}$; moreover, for all $\alpha \in \mathbb{R}$, $\langle \mathbb{R}, \alpha^\vee \rangle \subset \mathbb{Z}$. The lattice of weights is
\[ P(\mathbb{R}) = \left\{ \varpi \in X^*(T)_\mathbb{Q} \mid \langle \varpi, \alpha^\vee \rangle \in \mathbb{Z} \text{ all } \alpha \in \mathbb{R} \right\}, \]
the fundamental weights are the elements of the dual basis $\{\varpi_1, \ldots, \varpi_n\}$ to $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$, and that the dominant weights are the elements $\sum n_i \varpi_i$, $n_i \in \mathbb{N}$. The quotient $P(\mathbb{R})/Q(\mathbb{R})$ of $P(\mathbb{R})$ by the lattice $Q(\mathbb{R})$ generated by $\mathbb{R}$ is the character group of $\mathbb{Z}(H)$:
\[ P(\mathbb{R})/Q(\mathbb{R}) \simeq X^*(\mathbb{Z}(H)). \]

The irreducible complex representations of $H$ are classified by the dominant weights. We shall determine the dominant weights of the irreducible complex representations such that $\tilde{\mu}$ has exactly two weights $\alpha$ and $\alpha + 1$.

There is a unique permutation $\tau$ of the simple roots, called the opposition involution, such that the $\tau^2 = 1$ and the map $\alpha \mapsto -\tau(\alpha)$ extends to an action of the Weyl group. Its action on the Dynkin diagram is determined by the following rules: it preserves each connected component; on a connected component of type $A_n$, $D_n$ ($n$ odd), or $E_6$, it acts as the unique nontrivial involution, and on all other connected components, it acts trivially ([64], 1.5.1). Thus:

![Dynkin diagrams](image)

**Proposition 10.7.** Let $W$ be an irreducible complex representation of $H$, and let $\varpi$ be its highest weight. The representation $W$ occurs in a symplectic representation if and only if
\[ \langle \varpi + \tau \varpi, \tilde{\mu} \rangle = 1. \]

**Proof.** The lowest weight of $W$ is $-\tau(\varpi)$. The weights $\beta$ of $W$ are of the form
\[ \beta = \varpi + \sum_{\alpha \in \mathbb{R}} m_\alpha \alpha, \quad m_\alpha \in \mathbb{Z}, \]
and
\[ \langle \beta, \tilde{\mu} \rangle \in \mathbb{Z}. \]
Thus, \( \langle \beta, \bar{\mu} \rangle \) takes only two values \( a, a + 1 \) if and only if
\[
-\langle \tau(\omega), \bar{\mu} \rangle = \langle \omega, \bar{\mu} \rangle - 1,
\]
i.e., if and only if (10.8) holds. \( \square \)

**Corollary 10.9.** If \( W \) is symplectic, then \( \omega \) is a fundamental weight. Therefore the representation factors through an almost simple quotient of \( H \).

**Proof.** For every dominant weight \( \omega \), \( \langle \omega + \tau\omega, \bar{\mu} \rangle \in \mathbb{Z} \) because \( \omega + \tau\omega \in \mathbb{Q}(R) \). If \( \omega \neq 0 \), then \( \langle \omega + \tau\omega, \bar{\mu} \rangle > 0 \) unless \( \bar{\mu} \) kills all the weights of the representation corresponding to \( \omega \). Hence a dominant weight satisfying (10.8) can not be a sum of two dominant weights. \( \square \)

The corollary allows us to assume that \( H \) is almost simple. Recall from §2 that there is a unique special simple root \( \alpha_s \) such that, for \( \alpha \in S \),
\[
\langle \alpha, \bar{\mu} \rangle = \begin{cases} 1 & \text{if } \alpha = \alpha_s \\ 0 & \text{otherwise.} \end{cases}
\]
When a weight \( \omega \) is expressed as a \( \mathbb{Q} \)-linear combination of the simple roots, \( \langle \omega, \bar{\mu} \rangle \) is the coefficient of \( \alpha_s \). For the fundamental weights, these coefficients can be found in the tables in [7], VI. A fundamental weight \( \omega \) satisfies (10.8) if and only if
\[
(10.10) \quad \text{(coefficient of } \alpha_s \text{ in } \omega + \tau\omega) = 1.
\]

In the following, we write \( \alpha_1, \ldots, \alpha_n \) for the simple roots and \( \omega_1, \ldots, \omega_n \) for the fundamental weights with the usual numbering. In the diagrams, the solid node is the special node corresponding to \( \alpha_s \), and the nodes \( \bullet \) correspond to symplectic representations (and we call them *symplectic nodes*).

**Type \( A_n \).** The opposition involution \( \tau \) switchi...
Let $\alpha_s = \alpha_j$, with $1 < j < n$. Then only the fundamental weights $\varpi_1$ and $\varpi_n$ satisfy (10.10):

As $P/Q$ is generated by $\varpi_1$, the symplectic representations form a faithful family.

**Type $B_n$.** In this case, $\alpha_s = \alpha_1$ and the opposition involution acts trivially on the Dynkin diagram, and so we seek a fundamental weight $\varpi_1$ such that $\varpi_1 = \cdots + \frac{1}{2} \alpha_n$.

According to the tables in Bourbaki,

$$\varpi_1 = \alpha_1 + 2 \alpha_2 + \cdots + (i - 1) \alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_n) \quad (1 \leq i < n)$$

$$\varpi_n = \frac{1}{2}(\alpha_1 + 2 \alpha_2 + \cdots + n \alpha_n),$$

and so only $\varpi_n$ satisfies (10.10):

As $P/Q$ is generated by $\varpi_n$, the symplectic representations form a faithful family.

**Type $C_n$.** In this case $\alpha_s = \alpha_n$ and the opposition involution acts trivially on the Dynkin diagram, and so we seek a fundamental weight $\varpi_1$ such that $\varpi_1 = \cdots + \frac{1}{2} \alpha_n$.

According to the tables in Bourbaki,

$$\varpi_1 = \alpha_1 + 2 \alpha_2 + \cdots + (i - 1) \alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-1} + \frac{1}{2} \alpha_n),$$

and so only $\varpi_1$ satisfies (10.10):

As $P/Q$ is generated by $\varpi_1$, the symplectic representations form a faithful family.

**Type $D_n$.** The opposition involution acts trivially if $n$ is even, and switches $\alpha_{n-1}$ and $\alpha_n$ if $n$ is odd. According to the tables in Bourbaki,

$$\varpi_i = \alpha_1 + 2 \alpha_2 + \cdots + (i - 1) \alpha_{i-1} + i(\alpha_i + \cdots + \alpha_{n-2}) + \frac{1}{2}(\alpha_{n-1} + \alpha_n),$$

$$\varpi_{n-1} = \frac{1}{2}(\alpha_1 + 2 \alpha_2 + \cdots + (n - 2) \alpha_{n-2} + \frac{n}{2} \alpha_{n-1} + \frac{1}{2}(n - 2) \alpha_n)$$

Let $\alpha_s = \alpha_1$. As $\alpha_1$ is fixed by the opposition involution, we seek a fundamental weight $\varpi_1$ such that $\varpi_1 = \frac{1}{2} \alpha_1 + \cdots$. Both $\varpi_{n-1}$ and $\varpi_n$ give rise to symplectic representations:
When $n$ is odd, $\omega_{n-1}$ and $\omega_n$ each generate $P/Q$, and when $n$ is even $\omega_{n-1}$ and $\omega_n$ together generate $P/Q$. Therefore, in both cases, the symplectic representations form a faithful family.

Let $\alpha_s = \alpha_{n-1}$ or $\alpha_n$ and let $n = 4$. The nodes $\alpha_1$, $\alpha_3$, and $\alpha_4$ are permuted by automorphisms of the Dynkin diagram (hence by outer automorphisms of the corresponding group), and so this case is the same as the case $\alpha_s = \alpha_1$:

The symplectic representations form a faithful family.

Let $\alpha_s = \alpha_{n-1}$ or $\alpha_n$ and let $n \geq 5$. When $n$ is odd, $\tau$ interchanges $\alpha_{n-1}$ and $\alpha_n$, and so we seek a fundamental weight $\omega_1$ such that $\omega_1 = \cdots + a\alpha_{n-1} + b\alpha_n$ with $a + b = 1$; when $n$ is even, $\tau$ is trivial, and we seek a fundamental weight $\omega_1$ such that $\omega_1 = \cdots + \frac{1}{2}\alpha_{n-1} + \cdots$ or $\cdots + \frac{1}{2}\alpha_n$. In each case, only $\omega_1$ gives rise to a symplectic representation:

The weight $\omega_1$ generates a subgroup of order 2 (and index 2) in $P/Q$. Let $C \subset \mathbb{Z}(H)$ be the kernel of $\omega_1$ regarded as a character of $\mathbb{Z}(H)$. Then every symplectic representation factors through $H/C$, and the symplectic representations form a faithful family of representations of $H/C$.

**Type $E_6$.** In this case, $\alpha_s = \alpha_1$ or $\alpha_6$, and the opposition involution interchanges $\alpha_1$ and $\alpha_6$. Therefore, we seek a fundamental weight $\omega_1$ such that $\omega_1 = a\alpha_1 + \cdots + b\alpha_6$ with $a + b = 1$. In the following diagram, we list the value $a + b$ for each fundamental weight $\omega_1$:

As no value equals 1, there are no symplectic representations.
**Type $E_7$.** In this case, $\alpha_s = \alpha_7$, and the opposition involution is trivial. Therefore, we seek a fundamental weight $\varpi_i$ such that $\varpi_i = \cdots + \frac{1}{2}\alpha_7$. In the following diagram, we list the coefficient of $\alpha_7$ for each fundamental weight $\varpi_i$:

As no value is $\frac{1}{2}$, there are no symplectic representations.

Following [19], 1.3.9, we write $D^R$ for the case $D_n(1)$ and $D^H$ for the cases $D_n(n-1)$ and $D_n(n)$.

**Summary 10.11.** Let $H$ be a simply connected almost simple group over $\mathbb{R}$, and let $\tilde{h}: S/G_m \to H^\text{ad}$ be a nontrivial homomorphism satisfying (SV1,2). There exists a symplectic representation of $(H, \tilde{h})$ if and only if it is of type $A$, $B$, $C$, or $D$. Except when $(H, \tilde{h})$ is of type $D_H^H$, $n \geq 5$, the symplectic representations form a faithful family of representations of $H$; when $(H, \tilde{h})$ is of type $D_H^H$, $n \geq 5$, they form a faithful family of representations of the quotient of the simply connected group by the kernel of $\varpi_1$.

**The rational case**

Now let $H$ be a semisimple algebraic group over $\mathbb{Q}$, and let $\tilde{h}: S/G_m \to H^\text{ad}$ satisfying (SV1,2) and generating $H^\text{ad}$.

**Definition 10.12.** A homomorphism $H \to \text{GL}_V$ with finite kernel is a symplectic representation of $(H, \tilde{h})$ if there exists a commutative diagram

$$
\begin{tikzcd}
H \\
(H^\text{ad}, \tilde{h}) \\
(G, h) \\
(G(\psi), D(\psi))
\end{tikzcd}
$$

in which $\psi$ is a nondegenerate alternating form on $V$, $G$ is a reductive group (over $\mathbb{Q}$), and $h$ is a homomorphism $S \to G_\mathbb{R}$; the homomorphism $H \to G$ is required to have image $G^\text{der}$.

Given a diagram (10.13), we may replace $G$ with its image in $\text{GL}_V$ and so assume that the representation $\rho$ is faithful.

We now assume that $H$ is simply connected and almost simple. Then $H = (H^s)_F/\mathbb{Q}$ for some geometrically almost simple algebraic group $H^s$ over a number field $F$. Because $H_\mathbb{R}$ is an inner form of its compact form, the field $F$ is totally real (see the proof of 3.13). Let $I = \text{Hom}(F, \mathbb{R})$. Then,

$$
H_\mathbb{R} = \prod_{v \in I} H_v, \quad H_v = H^s \otimes_{F_v} \mathbb{R}.
$$
The Dynkin diagram $D$ of $H_C$ is a disjoint union of the Dynkin diagrams $D_v$ of the group $H_{vC}$. The Galois group $\text{Gal}(\mathbb{Q}^{al}/\mathbb{Q})$ acts on it in a manner consistent with its projection to $I$. In particular, it acts transitively on $D$ and so all the factors $H_v$ of $H_\mathbb{R}$ are of the same type. We let $I_c$ (resp. $I_{nc}$) denote the subset of $I$ of $v$ for which $H_v$ is compact (resp. not compact), and we let $H_c = \prod_{v \in I_c} H_v$ and $H_{nc} = \prod_{v \in I_{nc}} H_v$. Because $\bar{h}$ generates $H^{ad}$, $I_{nc}$ is nonempty.

**Proposition 10.14.** Let $F$ be a totally real number field. Suppose that for each real prime $v$ of $F$, we are given a pair $(H_v, \bar{h}_v)$ in which $H_v$ is a simply connected algebraic group over $\mathbb{R}$ of a fixed type, and $\bar{h}_v$ is a homomorphism $S/\mathbb{G}_m \to H^{ad}_v$ satisfying (SV1,2) (possibly trivial). Then there exists an algebraic group $H$ over $\mathbb{Q}$ such that $H \otimes_{F,v} \mathbb{R} \approx H_v$ for all $v$.

**Proof.** There exists an algebraic group $H$ over $F$ such that $H \otimes_{F,v} \mathbb{R}$ is an inner form of its compact form for all real primes $v$ of $F$. For each such $v$, $H_v$ is an inner form of $H \otimes_{F,v} \mathbb{R}$, and so defines a cohomology class in $H^1(F_v, H^{ad})$. The proposition now follows from the surjectivity of the map

$$H^1(F, H^{ad}) \to \prod_{v \text{ real}} H^1(F_v, H^{ad})$$

([48], Proposition 1). □

**Pairs $(H, \bar{h})$ for which there do not exist symplectic representations**

$H$ is of exceptional type Assume that $H$ is of exceptional type. If there exists an $\bar{h}$ satisfying (SV1,2), then $H$ is of type $E_6$ or $E_7$ (see §2). A symplectic representation of $(H, \bar{h})$ over $\mathbb{Q}$ gives rise to a symplectic representation of $(H_\mathbb{R}, \bar{h})$ over $\mathbb{R}$, but we have seen (10.11) that no such representations exist.

$(H, \bar{h})$ is of mixed type $D$. By this we mean that $H$ is of type $D_n$ with $n \geq 5$ and that at least one factor $(H_v, \bar{h}_v)$ is of type $D_n^{\mathbb{R}}$ and one of type $D_n^{\mathbb{H}}$. Such pairs $(H, \bar{h})$ exist by Proposition 10.14. The Dynkin diagram of $H_\mathbb{R}$ contains connected components

$$D_n(1)$$

and

$$D_n(n)$$

or $D_n(n-1)$. To give a symplectic representation for $H_\mathbb{R}$, we have to choose a symplectic node for each real prime $v$ such that $H_v$ is noncompact. In order for the representation to be rational, the collection of symplectic nodes must be stable under $\text{Gal}(\mathbb{Q}^{al}/\mathbb{Q})$, but this is impossible, because there is no automorphism of the Dynkin diagram of type $D_n$, $n \geq 5$, carrying the node 1 into either the node $n-1$ or the node $n$.
Pairs \((H, \tilde{h})\) for which there exist symplectic representations

**Lemma 10.15.** Let \(G\) be a reductive group over \(\mathbb{Q}\) and let \(h\) be a homomorphism \(S \to G_\mathbb{R}\) satisfying \((SV1,2^*)\) and generating \(G\). For any representation \((V, \rho)\) of \(G\) such that \((V, \rho \circ h)\) is of type \(\{(-1,0), (0,-1)\}\), there exists an alternating form \(\psi\) on \(V\) such that \(\rho\) induces a homomorphism \((G, h) \to (G(\psi), D(\psi))\).

**Proof.** The pair \((\rho G, \rho \circ h)\) is the Mumford-Tate group of \((V, \rho \circ h)\) and satisfies \((SV2^*)\). The proof of Proposition 6.4 constructs a polarization \(\psi\) for \((V, \rho \circ h)\) such that \(\rho G \subset G(\psi)\). \(\square\)

**Proposition 10.16.** A homomorphism \(H \to \text{GL}_V\) is a symplectic representation of \((H, \tilde{h})\) if there exists a commutative diagram

\[
\begin{array}{ccc}
H & \longrightarrow & \text{GL}_V, \\
\downarrow & & \\
(G_{\text{ad}}, \tilde{h}) & \longleftarrow & (G, h)
\end{array}
\]

in which \(G\) is a reductive group whose connected centre splits over a CM-field, the homomorphism \(H \to G\) has image \(G^{\text{der}}\), the weight \(w_h\) is defined over \(\mathbb{Q}\), and the Hodge structure \((V, \rho \circ h)\) is of type \(\{(-1,0), (0,-1)\}\).

**Proof.** The hypothesis on the connected centre \(Z^o\) says that the largest compact subtorus of \(Z_\mathbb{R}^o\) is defined over \(\mathbb{Q}\). Take \(G'\) to be the subgroup of \(G\) generated by this torus, \(G^{\text{der}}\), and the image of \(w_h\). Now \((G', h)\) satisfies \((SV2^*)\), and we can apply 10.15. \(\square\)

We classify the symplectic representations of \((H, \tilde{h})\) with \(\rho\) faithful. Note that the quotient of \(H\) acting faithfully on \(V\) is isomorphic to \(G^{\text{der}}\).

Let \((V, \tau)\) be a symplectic representation of \((H, \tilde{h})\). The restriction of the representation to \(H_{\text{nc}}\) is a real symplectic representation of \(H_{\text{nc}}\), and so, according to Corollary 10.9, every nontrivial irreducible direct summand of \(\tau|_{H_{\text{nc}}}\) factors through \(H_v\) for some \(v \in I_{\text{nc}}\) and corresponds to a symplectic node of the Dynkin diagram \(D_v\) of \(H_v\).

Let \(W\) be an irreducible direct summand of \(V_{\mathbb{C}}\). Then

\[W \cong \bigotimes_{v \in T} W_v\]

for some irreducible symplectic representations \(W_v\) of \(H_{v,\mathbb{C}}\) indexed by a subset \(T\) of \(I\). The irreducible representation \(W_v\) corresponds to a symplectic node \(s(v)\) of \(D_v\). Because \(\tau\) is defined over \(\mathbb{Q}\), the set \(s(T)\) is stable under the action of \(\text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})\). For \(v \in I_{\text{nc}}\), the set \(s(T) \cap D_v\) consists of a single symplectic node.
Given a diagram (10.13), we let \( S(V) \) denote the set of subsets \( s(T) \) of the nodes of \( D \) as \( W \) runs over the irreducible direct summands of \( V \). The set \( S(V) \) satisfies the following conditions:

(10.17a) for \( S \in S(V) \), \( S \cap D_{nc} \) is either empty or consists of a single symplectic node of \( D_v \) for some \( v \in I_{nc} \);

(10.17b) \( S \) is stable under \( \text{Gal}(\mathbb{Q}^{al}/\mathbb{Q}) \) and contains a nonempty subset.

Given such a set \( S \), let \( H(S) \) be the quotient of \( H_C \) that acts faithfully on the representation defined by \( S \). The condition (10.17b) ensures that \( H(S) \) is defined over \( \mathbb{Q} \). According to Galois theory (in the sense of Grothendieck), there exists an étale \( \mathbb{Q} \)-algebra \( K_S \) such that

\[
\text{Hom}(K_S, \mathbb{Q}^{al}) \approx S \quad \text{(as sets with an action of } \text{Gal}(\mathbb{Q}^{al}/\mathbb{Q})\text{).}
\]

**Theorem 10.18.** For any set \( S \) satisfying the conditions (10.17), there exists a diagram (10.13) such that the quotient of \( H \) acting faithfully on \( V \) is \( H(S) \).

**Proof.** We prove this only in the case that \( S \) consists of one-point sets. For an \( S \) as in the theorem, the set \( S' \) of \( \{s\} \) for \( s \in S \in S \) satisfies (10.17) and \( H(S) \) is a quotient of \( H(S') \).

Recall that \( H = (H^s)_{F/\mathbb{Q}} \) for some totally real field \( F \). We choose a totally imaginary quadratic extension \( E \) of \( F \) and, for each real embedding \( v \) of \( F \) in \( I_c \), we choose an extension \( \sigma \) of \( v \) to a complex embedding of \( E \). Let \( T \) denote the set of \( \sigma \)'s. Thus

\[
E \xrightarrow{\sigma} \mathbb{C} \\
\cup \\
F \xrightarrow{v} \mathbb{R}
\]

We regard \( E \) as a \( \mathbb{Q} \)-vector space, and define a Hodge structure \( h_T \) on it as follows: \( E \otimes \mathbb{Q} \mathbb{C} \simeq \mathbb{C}^{\text{Hom}(E, \mathbb{C})} \) and the factor with index \( \sigma \) is of type \((-1,0)\) if \( \sigma \in T \), type \((0,-1)\) if \( \bar{\sigma} \in T \), and of type \((0,0)\) if \( \sigma \) lies above \( I_{nc} \). Thus \( (\mathbb{C}_\sigma = \mathbb{C}) \):

\[
E \otimes \mathbb{Q} \mathbb{C} = \bigoplus_{\sigma \in T} \mathbb{C}_\sigma \oplus \bigoplus_{\sigma \in T} \mathbb{C}_\sigma \oplus \bigoplus_{\sigma \in T \cup T} \mathbb{C}_\sigma.
\]

Because the elements of \( S \) are one-point subsets of \( D \), we can identify them with elements of \( D \), and so regard \( S \) as a subset of \( D \). It has the properties:

(a) if \( s \in S \cap D_{nc} \), then \( s \) is a symplectic node;

(b) \( S \) is stable under \( \text{Gal}(\mathbb{Q}^{al}/\mathbb{Q}) \) and is nonempty.

Let \( K_D \) be the smallest subfield of \( \mathbb{Q}^{al} \) such that \( \text{Gal}(\mathbb{Q}^{al}/K_D) \) acts trivially on \( D \). Then \( K_D \) is a Galois extension of \( \mathbb{Q} \) in \( \mathbb{Q}^{al} \) such that \( \text{Gal}(K_D/K) \) acts faithfully on \( D \). Complex conjugation acts as the opposition involution on \( D \), which lies in the centre of \( \text{Aut}(D) \); therefore \( K_D \) is either totally real or CM.
The $\mathbb{Q}$-algebra $K_S$ can be taken to be a product of subfields of $K_D$. In particular, $K_S$ is a product of totally real fields and CM fields. The projection $S \to I$ corresponds to a homomorphism $F \to K_S$.

For $s \in S$, let $V(s)$ be a complex representation of $H_C$ with dominant weight the fundamental weight corresponding to $s$. The isomorphism class of the representation $\bigoplus_{s \in S} V(s)$ is defined over $\mathbb{Q}$. The obstruction to the representation itself being defined over $\mathbb{Q}$ lies in the Brauer group of $\mathbb{Q}$, which is torsion, and so some multiple of the representation is defined over $\mathbb{Q}$. Let $V$ be a representation of $H$ over $\mathbb{Q}$ such that $V_C \approx \bigoplus_{s \in S} nV(s)$ for some integer $n$, and let $V_s$ denote the direct summand of $V_C$ isomorphic to $nV(s)$. These summands are permuted by $\text{Gal}(\mathbb{Q}^{al}/\mathbb{Q})$ in a fashion compatible with the action of $\text{Gal}(\mathbb{Q}^{al}/\mathbb{Q})$ on $S$, and the decomposition $V_C = \bigoplus_{s \in S} V_s$ corresponds therefore to a structure of a $K_S$-module on $V$: let $s' : K_S \to \mathbb{Q}^{al}$ be the homomorphism corresponding to $s \in S$; then $\alpha \in K_S$ acts on $V_s$ as multiplication by $s'(\alpha)$.

Let $H'$ denote the quotient of $H$ that acts faithfully on $V$. Then $H'_{\mathbb{R}}$ is the quotient of $H_{\mathbb{R}}$ described in (10.11).

A lifting of $\hat{h}$ to a fractional morphism of $S$ into $H'_{\mathbb{R}}$ defines a fractional Hodge structure on $V$ of weight 0, which can be described as follows. Let $s \in S$, and let $v$ be its image in $I_c$; if $v \in I_c$, then $V_s$ is of type $(0,0)$; if $v \in I_{nc}$, then $V_s$ is of type $\{(r, -r), (r - 1, 1 - r)\}$ where $r = \langle \omega, \mu \rangle$ (notations as in 10.7). We renumber this Hodge structure to obtain a new Hodge structure on $V$:

|          | old     | new     |
|----------|---------|---------|
| $V_s$, $v \in I_c$ | $(0,0)$ | $(0,0)$ |
| $V_s$, $v \in I_{nc}$ | $(r, -r)$ | $(0, -1)$ |
| $V_s$, $v \in I_{nc}$ | $(r - 1, 1 - r)$ | $(-1, 0)$ |

We endow the $\mathbb{Q}$-vector space $E \otimes F V$ with the tensor product Hodge structure. The decomposition

$$(E \otimes F V) \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_{v \in I_c} (E \otimes_{F,v} \mathbb{R}) \otimes_{\mathbb{R}} (V \otimes_{F,v} \mathbb{R}),$$

is compatible with the Hodge structures. The type of the Hodge structure on each direct summand is given by the following table:

|          | $E \otimes_{F,v} \mathbb{R}$ | $V \otimes_{F,v} \mathbb{R}$ |
|----------|-------------------------------|-------------------------------|
| $v \in I_c$ | $\{(-1,0), (0, -1)\}$ | $\{(0,0)\}$ |
| $v \in I_{nc}$ | $\{(0,0)\}$ | $\{(-1,0), (0, -1)\}$ |

Therefore, $E \otimes F V$ has type $\{(-1,0), (0, -1)\}$. Let $G$ be the algebraic subgroup of $\text{GL}_{E \otimes F V}$ generated by $E^x$ and $H'$. The homomorphism $h : S \to (\text{GL}_{E \otimes F V})_{\mathbb{R}}$ corresponding to the Hodge structure factors through $G_{\mathbb{R}}$, and the derived group of $G$ is $H'$. Now apply (10.16). □
Aside 10.19. The trick of using a quadratic imaginary extension $E$ of $F$ in order to obtain a Hodge structure of type $\{(-1,0), (0,-1)\}$ from one of type $\{(-1,0), (0,0), (0,-1)\}$ in essence goes back to Shimura (cf. [14], §6).

Conclusion. Now let $H$ be a semisimple algebraic group over $\mathbb{Q}$, and let $\tilde{h}$ be a homomorphism $S \to H_{\text{ad}}$ satisfying (SV1,2) and generating $H$.

Definition 10.20. The pair $(H, \tilde{h})$ is of Hodge type if it admits a faithful family of symplectic representations.

Theorem 10.21. A pair $(H, \tilde{h})$ is of Hodge type if it is a product of pairs $(H_i, \tilde{h}_i)$ such that either

(a) $(H_i, \tilde{h}_i)$ is of type $A, B, C,$ or $D$, and $H$ is simply connected, or
(b) $(H_i, \tilde{h}_i)$ is of type $D^H_n$ ($n \geq 5$) and equals $(H^s)_{F/\mathbb{Q}}$ for the quotient $H^s$ of the simply connected group of type $D^H_n$ by the kernel of $\omega_1$ (cf. 10.11).

Conversely, if $(H, \tilde{h})$ is a Hodge type, then it is a quotient of a product of pairs satisfying (a) or (b).

Proof. Suppose that $(H, \tilde{h})$ is a product of pairs satisfying (a) and (b), and let $(H', \tilde{h}')$ be one of these factors with $H'$ almost simple. Let $\tilde{H}'$ be the simply connected covering group of $H$. Then (10.11) allows us to choose a set $S$ satisfying (10.17) and such that $H' = H(S)$. Now Theorem 10.18 shows that $(H', \tilde{h}')$ admits a faithful symplectic representation. A product of pairs of Hodge type is clearly of Hodge type.

Conversely, suppose that $(H, \tilde{h})$ is of Hodge type, let $\tilde{H}$ be the simply connected covering group of $H$, and let $(H', \tilde{h}')$ be an almost simple factor of $(\tilde{H}, \tilde{h})$. Then $(H', \tilde{h}')$ admits a symplectic representation with finite kernel, and so $(H', \tilde{h}')$ is of type $E_6, E_7$, or mixed type $D$ (see p. 532). Moreover, if $(H', \tilde{h}')$ is of type $D^H_n, n \geq 5$, then (10.11) shows that it factors through the quotient described in (b). □

Notice that we haven’t completely classified the pairs $(H, \tilde{h})$ of Hodge type because we haven’t determined exactly which quotients of products of pairs satisfying (a) or (b) occur as $H(S)$ for some set $S$ satisfying (10.17).

11. Moduli

In this section, we determine (a) the pairs $(G, h)$ that arise as the Mumford-Tate group of an abelian variety (or an abelian motive); (b) the arithmetic locally symmetric varieties that carry a faithful family of abelian varieties (or abelian motives); (c) the Shimura varieties that arise as moduli varieties for polarized abelian varieties (or motives) with Hodge class and level structure.

Mumford-Tate groups

Theorem 11.1. Let $G$ be an algebraic group over $\mathbb{Q}$, and let $h: S \to G_{\text{ad}}$ be a homomorphism that generates $G$ and whose weight is rational. The pair $(G, h)$ is the Mumford-Tate
group of an abelian variety if and only if \( h \) satisfies (SV2\(^*\)) and there exists a faithful representation \( \rho: G \to \text{GL}_V \) such that \((V, \rho \circ h)\) is of type \( \{(-1,0), (0,-1)\}\).

**Proof.** The necessity is obvious (apply (6.4) to see that \((G, h)\) satisfies (SV2\(^*\))). For the sufficiency, note that \((G, h)\) is the Mumford-Tate group of \((V, \rho \circ h)\) because \( h \) generates \( G \). The Hodge structure is polarizable because \((G, h)\) satisfies (SV2\(^*\)) (apply 6.4), and so it is the Hodge structure \( H_1(Aan, \mathbb{Q}) \) of an abelian variety \( A \) by Riemann’s theorem 4.4. \( \square \)

The Mumford-Tate group of a motive is defined to be the Mumford-Tate group of its Betti realization.

**Theorem 11.2.** Let \((G, h)\) be an algebraic group over \( \mathbb{Q} \), and let \( h: S \to G_{\mathbb{R}} \) be a homomorphism satisfying (SV1,2\(^*\)) and generating \( G \). Assume that \( w_h \) is defined over \( \mathbb{Q} \). The pair \((G, h)\) is the Mumford-Tate group of an abelian motive if and only if \((G_{\text{der}}, \bar{h})\) is a quotient of a product of pairs satisfying (a) and (b) of (10.21).

The proof will occupy the rest of this subsection. Recall that \( G_{\text{Hdg}} \) is the affine group scheme attached to the tannakian category \( \text{Hdg}_{\mathbb{Q}} \) of polarizable rational Hodge structures and the forgetful fibre functor (see 9.26). It is equipped with a homomorphism \( h_{\text{Hdg}}: S \to (G_{\text{Hdg}})_{\mathbb{R}} \). If \((G, h)\) is the Mumford-Tate group of a polarizable Hodge structure, then there is a unique homomorphism \( \rho(h): G_{\text{Hdg}} \to G \) such that \( h = \rho(h)_{\mathbb{R}} \circ h_{\text{Hdg}} \). Moreover, \((G_{\text{Hdg}}, h_{\text{Hdg}}) = \lim \leftarrow (G, h)\).

**Lemma 11.3.** Let \( H \) be a semisimple algebraic group over \( \mathbb{Q} \), and let \( \bar{h}: S/G_m \to H_{\text{ad}}^{\mathbb{R}} \) be a homomorphism satisfying (SV1,2,3). There exists a unique homomorphism

\[
\rho(H, \bar{h}): (G_{\text{Hdg}})^{\text{der}} \to H
\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
(G_{\text{Hdg}})^{\text{der}} & \xrightarrow{\rho(H, \bar{h})} & H \\
\downarrow & & \downarrow \\
G_{\text{Hdg}} & \xrightarrow{\rho(h)} & H_{\text{ad}}^{\mathbb{R}}.
\end{array}
\]

**Proof.** Two such homomorphisms \( \rho(H, \bar{h}) \) would differ by a map into \( Z(H) \). Because \((G_{\text{Hdg}})^{\text{der}}\) is connected, any such map is constant, and so the homomorphisms are equal.

For the existence, choose a pair \((G, h)\) as in (8.9). Then \((G, h)\) is the Mumford-Tate group of a polarizable Hodge structure, and we can take \( \rho(H, \bar{h}) = \rho(h)(G_{\text{Hdg}})^{\text{der}} \).

**Lemma 11.4.** The assignment \((H, \bar{h}) \mapsto \rho(H, \bar{h})\) is functorial: if \( \alpha: H \to H' \) is a homomorphism taking \( Z(H) \) into \( Z(H') \) and carrying \( \bar{h} \) to \( \bar{h}' \), then \( \rho(H', \bar{h}') = \alpha \circ \rho(H, \bar{h}) \).
Proof. The homomorphism $\tilde{h}'$ generates $H^{\text{ad}}$ (by SV3), and so the homomorphism $\alpha$ is surjective. Choose a pair $(G, h)$ for $(H, \tilde{h})$ as in (8.9), and let $G' = G / \text{Ker}(\alpha)$. Write $\alpha$ again for the projection $G \to G'$ and let $h' = \alpha \circ h$. This equality implies that

$$\rho(h') = \alpha \circ \rho(h).$$

On restricting this to $(G_{\text{Hdg}})^{\text{der}}$, we obtain the equality

$$\rho(H', \tilde{h}') = \alpha \circ \rho(H, \tilde{h}).$$

Recall that $G_{\text{Mab}}$ is the affine group scheme attached to the category of abelian motives over $\mathbb{C}$ and the Betti fibre functor. The functor $\text{Mot}^{\text{ab}}(\mathbb{C}) \to \text{Hdg}_{\mathbb{Q}}$ is fully faithful by Deligne’s theorem (9.15), and so it induces a surjective map $G_{\text{Hdg}} \to G_{\text{Mab}}$.

Lemma 11.5. If $(H, h)$ is of Hodge type, then $\rho(H, \tilde{h})$ factors through $(G_{\text{Mab}})^{\text{der}}$.

Proof. Let $(G, h)$ be as in the definition (10.12), and replace $G$ with the algebraic subgroup generated by $h$. Then $(G, h)$ is the Mumford-Tate group of an abelian variety (Riemann’s theorem 4.4), and so $\rho(h) : G_{\text{Hdg}} \to G$ factors through $G_{\text{Hdg}} \to G_{\text{Mab}}$. Therefore $\rho(H, \tilde{h})$ maps the kernel of $(G_{\text{Hdg}})^{\text{der}} \to (G_{\text{Mab}})^{\text{der}}$ into the kernel of $H \to G$. By assumption, the intersection of these kernels is trivial.

Lemma 11.6. The homomorphism $\rho(H, \tilde{h})$ factors through $(G_{\text{Mab}})^{\text{der}}$ if and only if $(H, \tilde{h})$ has a finite covering by a pair of Hodge type.

Proof. Suppose that there is a finite covering $\alpha : H' \to H$ such that $(H', \tilde{h}')$ is of Hodge type. By Lemma 11.5, $\rho(H', \tilde{h}')$ factors through $(G_{\text{Mab}})^{\text{der}}$, and therefore so also does $\rho(H, \tilde{h}) = \alpha \circ \rho(H', \tilde{h}')$.

Conversely, suppose that $\rho(H, \tilde{h})$ factors through $(G_{\text{Mab}})^{\text{der}}$. There will be an algebraic quotient $(G, h)\text{ of } (G_{\text{Mab}}, h_{\text{Mab}})$ such that $(H, \tilde{h})$ is a quotient of $(G^{\text{der}}, \text{ad} \circ h)$. Consider the category of abelian motives $M$ such that the action of $G_{\text{Mab}}$ on $\omega_B(M)$ factors through $G$. By definition, this category is contained in the tensor category generated by $h_1(A)$ for some abelian variety $A$. We can replace $G$ with the Mumford-Tate group of $A$. Then $(G^{\text{der}}, \text{ad} \circ h)$ has a faithful symplectic embedding, and so it is of Hodge type.

Aside 11.7. Let $G$ be an algebraic group over $\mathbb{Q}$ and let $h$ be a homomorphism $S \to G_{\mathbb{R}}$. If $(G, h)$ is the Mumford-Tate group of a motive, then $h$ generates $G$, $w_h$ is defined over $\mathbb{Q}$, and $h$ satisfies (SV2*). Assume that $(G, h)$ satisfies these conditions. A positive answer to Question 9.28 would imply that $(G, h)$ is the Mumford-Tate
group of a motive if $h$ satisfies (SV1). If $G_{\text{der}}$ is of type $E_8$, $F_4$, or $G_2$, then there does not exist an $h$ satisfying (SV1) (apply §2 to $h|S^1$). Nevertheless, it has recently been shown that there exist motives whose Mumford-Tate group is of type $G_2$ ([22]).

Notes. This subsection follows §1 of [38].

Families of abelian varieties and motives

Let $S$ be a connected smooth algebraic variety over $\mathbb{C}$, and let $o \in S(\mathbb{C})$. A family $f: A \to S$ of abelian varieties over $S$ defines a local system $V = R^1f_*\mathbb{Z}$ of $\mathbb{Z}$-modules on $S^{\text{an}}$. We say that the family is faithful if the monodromy representation $\pi_1(S^{\text{an}}, o) \to GL(V_o)$ is injective.

Let $D(\Gamma) = \Gamma \backslash D$ be an arithmetic locally symmetric variety, and let $o \in D$. By definition, there exists a simply connected algebraic group $H$ over $\mathbb{Q}$ and a surjective homomorphism $\varphi: H(\mathbb{R}) \to \text{Hol}(D)^+$ with compact kernel such that $\varphi(H(\mathbb{Z}))$ is commensurable with $\Gamma$. Moreover, with a mild condition on the ranks, the pair $(H, \varphi)$ is uniquely determined up to a unique isomorphism (see 3.13). Let $\tilde{h}: S \to H^{\text{ad}}$ be the homomorphism whose projection into a compact factor of $H^{\text{ad}}$ is trivial and is such that $\varphi(\tilde{h}(z))$ fixes $o$ and acts on $T_o(D)$ as multiplication by $z/\bar{z}$ (cf. (8.4), p. 510).

Theorem 11.8. There exists a faithful family of abelian varieties on $D(\Gamma)$ having a fibre of CM-type if and only if $(H, \tilde{h})$ admits a symplectic representation (10.12).

Proof. Let $f: A \to D(\Gamma)$ be a faithful family of abelian varieties on $D(\Gamma)$, and let $(V, F)$ be the variation of Hodge structures $R^1f_*\mathbb{Q}$. Choose a trivialization $\pi^*V \cong V_D$, and let $G \subset GL_V$ be the generic Mumford-Tate group (see 6.17). As in (§8), we get a commutative diagram

$$
\begin{array}{ccc}
H & \rightarrow & (H^{\text{ad}}, \tilde{h}) \\
\downarrow & & \leftarrow \\
(G, h) & \leftarrow & GL_V
\end{array}
$$

(11.9)

in which the image of $H \to G$ is $G^{\text{der}}$. Because the family is faithful, the map $H \to G^{\text{der}}$ is an isogeny, and so $(H, \tilde{h})$ admits a symplectic representation.

Conversely, a symplectic representation of $(H, \tilde{h})$ defines a variation of Hodge structures (8.6), which arises from a family of abelian varieties by Theorem 7.13 (Riemann’s theorem in families).

Theorem 11.10. There exists a faithful family of abelian motives on $D(\Gamma)$ having a fibre of CM-type if and only if $(H, \tilde{h})$ has finite covering by a pair of Hodge type.

Proof. The proof is essentially the same as that of Theorem 11.8. The points are the determination of the Mumford-Tate groups of abelian motives in (11.2) and Theorem 9.31, which replaces Riemann’s theorem in families.
Shimura varieties

In the above, we have always considered connected varieties. As Deligne [14] observed, it is often more convenient to consider nonconnected varieties.

**Definition 11.11.** A Shimura datum is a pair \((G, X)\) consisting of a reductive group \(G\) over \(\mathbb{Q}\) and a \(G(\mathbb{R})^+\)-conjugacy class of homomorphisms \(S \to G_\mathbb{R}\) satisfying (SV1,2,3).\(^{35}\)

**Example 11.12.** Let \((V, \psi)\) be a symplectic space over \(\mathbb{Q}\). The group \(G(\psi)\) of symplectic similitudes together with the space \(X(\psi)\) of all complex structures \(J\) on \(V_\mathbb{R}\) such that \((x, y) \mapsto \psi(x, Jy)\) is positive definite is a Shimura datum.

Let \((G, X)\) be a Shimura datum. The map \(h \mapsto \tilde{h} \overset{\text{def}}{=} \text{ad} \circ h\) identifies \(X\) with a \(G^\text{ad}(\mathbb{R})^+\)-conjugacy class of homomorphisms \(\tilde{h}: S/\mathbb{G}_m \to G^\text{ad}_\mathbb{R}\) satisfying (SV1,2,3). Thus \(X\) is a hermitian symmetric domain (2.5, 6.1). More canonically, the set \(X\) has a unique structure of a complex manifold such that, for every representation \(\rho: G_\mathbb{R} \to GL_V\), \((V_X, \rho \circ h)_{h \in X}\) is a holomorphic family of Hodge structures. For this complex structure, \((V_X, \rho \circ h)_{h \in X}\) is a variation of Hodge structures, and so \(X\) is a hermitian symmetric domain.

The Shimura variety attached to \((G, X)\) and the choice of a compact open subgroup \(K\) of \(G(\mathbb{A}_f)\) is \(^{36}\)

\[
\text{Sh}_K(G, X) = G(\mathbb{Q})_+ \backslash X \times G(\mathbb{A}_f)/K
\]

where \(G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+\). In this quotient, \(G(\mathbb{Q})_+\) acts on both \(X\) (by conjugation) and \(G(\mathbb{A}_f)\), and \(K\) acts on \(G(\mathbb{A}_f)\). Let \(\mathcal{C}\) be a set of representatives for the (finite) double coset space \(G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K\); then

\[
G(\mathbb{Q})_+ \backslash X \times G(\mathbb{A}_f)/K \simeq \bigsqcup_{g \in \mathcal{C}} \Gamma_g \backslash X, \quad \Gamma_g = gKg^{-1} \cap G(\mathbb{Q})_+.
\]

Because \(\Gamma_g\) is a congruence subgroup of \(G(\mathbb{Q})\), its image in \(G^\text{ad}(\mathbb{Q})\) is arithmetic (3.4), and so \(\text{Sh}_K(G, X)\) is a finite disjoint union of connected Shimura varieties. It therefore has a unique structure of an algebraic variety. As \(K\) varies, these varieties form a projective system.

We make this more explicit in the case that \(G^\text{der}\) is simply connected. Let \(\nu: G \to T\) be the quotient of \(G\) by \(G^\text{der}\), and let \(Z\) be the centre of \(G\). Then \(\nu\) defines an isogeny \(Z \to T\), and we let

\[
T(\mathbb{R})^\dagger = \text{Im}(Z(\mathbb{R}) \to T(\mathbb{R})),
\]

\[
T(\mathbb{Q})^\dagger = T(\mathbb{Q}) \cap T(\mathbb{R})^\dagger.
\]

\(^{35}\)In the usual definition, \(X\) is taken to be a \(G(\mathbb{R})\)-conjugacy class. For our purposes, it is convenient to choose a connected component of \(X\).

\(^{36}\)This agrees with the usual definition because of [40], 5.11.
The set $T(Q)^\dagger \backslash T(A_f)/\nu(K)$ is finite and discrete. For $K$ sufficiently small, the map
\[(11.13) \quad [x, a] \mapsto [\nu(a)]: G(Q) \backslash X \times G(A_f)/K \to T(Q)^\dagger \backslash T(A_f)/\nu(K)\]
is surjective, and each fibre is isomorphic to $\Gamma \backslash X$ for some congruence subgroup $\Gamma$ of $G^\text{der}(Q)$. For the fibre over $[1]$, the congruence subgroup $\Gamma$ is contained in $K \cap G^\text{der}(Q)$, and equals it if $Z(G^\text{der})$ satisfies the Hasse principal for $H^j$, for example, if $G^\text{der}$ has no factors of type $A$.

**Example 11.14.** Let $G = \text{GL}_2$. Then
\[
(G \to T) = (\text{GL}_2 \xrightarrow{\text{det}} \mathbb{G}_m),
\]
and therefore
\[
T(Q)^\dagger \backslash T(A_f)/\nu(K) = Q^{>0} \backslash A_f^\times / \text{det}(K).
\]
Note that $A_f^\times = Q^{>0} \times \hat{\mathbb{Z}}^\times$ (direct product) where $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \simeq \prod\mathbb{Z}_l$. For
\[
K = K(N) \overset{\text{def}}{=} \{ a \in \hat{\mathbb{Z}}^\times \mid a \equiv 1, \text{mod}N \},
\]
we find that
\[
T(Q)^\dagger \backslash T(A_f)/\nu(K) \simeq (\mathbb{Z}/N\mathbb{Z})^\times.
\]

**Definition 11.15.** A Shimura datum $(G, X)$ is of *Hodge type* if there exists an injective homomorphism $G \to G(\psi)$ sending $X$ into $X(\psi)$ for some symplectic pair $(V, \psi)$ over $Q$.

**Definition 11.16.** A Shimura datum $(G, X)$ is of *abelian type* if, for one (hence all) $h \in X$, the pair $(G^\text{der}, \text{ad} \circ h)$ is a quotient of a product of pairs satisfying (a) or (b) of (10.21).

A Shimura variety $\text{Sh}(G, X)$ is said to be of Hodge or abelian type if $(G, X)$ is.

**Notes.** See [40], §5, for proofs of the statements in this subsection. For the structure of the Shimura variety when $G^\text{der}$ is not simply connected, see [19], 2.1.16.

**Shimura varieties as moduli varieties**

Throughout this subsection, $(G, X)$ is a Shimura datum such that

(a) $w_X$ is defined over $\mathbb{Q}$ and the connected centre of $G$ is split by a CM-field, and

(b) there exists a homomorphism $\nu: G \to G_m \simeq \text{GL}_{Q(1)}$ such that $\nu \circ w_X = -2$.

Fix a faithful representation $\rho: G \to \text{GL}_V$. Assume that there exists a pairing $t_0: V \times V \to \mathbb{Q}(m)$ such that (i) $gt_0 = \nu(g)^m t_0$ for all $g \in G$ and (ii) $t_0$ is a polarization of $(V, \rho \circ h)$ for all $h \in X$. Then there exist homomorphisms $t_i: V^{\otimes i} \to \mathbb{Q}(\frac{m^i}{2})$, $1 \leq i \leq n$, such that $G$ is the subgroup of $\text{GL}_V$ whose elements fix $t_0, t_1, \ldots, t_n$. When $(G, X)$ is of Hodge type, we choose $\rho$ to be a symplectic representation.
Let \( K \) be a compact open subgroup of \( G(\mathbb{A}_f) \). Define \( \mathcal{H}_K(\mathbb{C}) \) to be the set of triples
\[
(W, (s_i)_{0 \leq i \leq n}, \eta K)
\]
in which
- \( W = (W, h_W) \) is a rational Hodge structure,
- each \( s_i \) is a morphism of Hodge structures \( W^{\otimes r_i} \to \mathbb{Q}(\frac{m r_i}{2}) \) and \( s_0 \) is a polarization of \( W \),
- \( \eta K \) is a \( K \)-orbit of \( \mathbb{A}_f \)-linear isomorphisms \( V_{\mathbb{A}_f} \to W_{\mathbb{A}_f} \) sending each \( t_i \) to \( s_i \),

satisfying the following condition:
\[(*) \text{ there exists an isomorphism } \gamma: W \to V \text{ sending each } s_i \text{ to } t_i\]
and \( h_W \) onto an element of \( X \).

**Lemma 11.17.** For \((W, \ldots) \) in \( \mathcal{H}_K(\mathbb{C}) \), choose an isomorphism \( \gamma \) as in (*), let \( h \) be the image of \( h_W \) in \( X \), and let \( \alpha \in G(\mathbb{A}_f) \) be the composite \( V_{\mathbb{A}_f} \to W_{\mathbb{A}_f} \to V_{\mathbb{A}_f} \). The class \([h, \alpha]\) of the pair \((h, \alpha)\) in \( G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K \) is independent of all choices, and the map
\[
(W, \ldots) \mapsto [h, \alpha]: \mathcal{H}_K(\mathbb{C}) \to \text{Sh}_K(G, X)(\mathbb{C})
\]
is surjective with fibres equal to the isomorphism classes.

**Proof.** The proof involves only routine checking. \( \square \)

For a smooth algebraic variety \( S \) over \( \mathbb{C} \), let \( \mathcal{F}_K(S) \) be the set of isomorphism classes of triples \((A, (s_i)_{0 \leq i \leq n}, \eta K)\) in which
- \( A \) is a family of abelian motives over \( S \),
- each \( s_i \) is a morphism of abelian motives \( A^{\otimes r_i} \to \mathbb{Q}(\frac{m r_i}{2}) \), and
- \( \eta K \) is a \( K \)-orbit of \( \mathbb{A}_f \)-linear isomorphisms \( V_S \to \omega_f(A/S) \) sending each \( t_i \) to \( s_i \),

satisfying the following condition:
\[(**) \text{ for each } s \in S(\mathbb{C}), \text{ the Betti realization of } (A, (s_i), \eta K)_s\text{ lies in } \mathcal{H}_K(\mathbb{C}).\]

With the obvious notion of pullback, \( \mathcal{F}_K \) becomes a functor from smooth complex algebraic varieties to sets. There is a well-defined injective map \( \mathcal{F}_K(\mathbb{C}) \to \mathcal{H}_K(\mathbb{C})/\approx \), which is surjective when \((G, X)\) is of abelian type. Hence, in this case, we get an isomorphism \( \alpha: \mathcal{F}_K(\mathbb{C}) \to \text{Sh}_K(\mathbb{C}) \).

**Theorem 11.18.** Assume that \((G, X)\) is of abelian type. The map \( \alpha \) realizes \( \text{Sh}_K \) as a coarse moduli variety for \( \mathcal{F}_K \), and even a fine moduli variety when \( Z(\mathbb{Q}) \) is discrete in \( Z(\mathbb{R}) \) (here \( Z = Z(G) \)).

**Proof.** To say that \((\text{Sh}_K, \alpha)\) is coarse moduli variety means the following:

\[\text{The isomorphism } \eta \text{ is defined only on the universal covering space of } S^{\text{an}}, \text{ but the family } \eta K \text{ is stable under } \pi_1(S, o), \text{ and so is "defined" on } S.\]
(a) for any smooth algebraic variety $S$ over $\mathbb{C}$, and $\xi \in \mathcal{F}(S)$, the map $s \mapsto \alpha(\xi_s): S(\mathbb{C}) \to \text{Sh}_K(\mathbb{C})$ is regular;

(b) $(\text{Sh}_K, \alpha)$ is universal among pairs satisfying (a).

To prove (a), we use that $\xi$ defines a variation of Hodge structures on $S$ (see p. 523). Now the universal property of hermitian symmetric domains (7.8) shows that the map $s \mapsto \alpha(\xi_s)$ is holomorphic (on the universal covering space, and hence on the variety), and Borel’s theorem 4.3 shows that it is regular.

Next assume that $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{R})$. Then the representation $\rho$ defines a variation of Hodge structures on $\text{Sh}_K$ itself (not just its universal covering space), which arises from a family of abelian motives. This family is universal, and so $\text{Sh}_K$ is a fine moduli variety.

We now prove (b). Let $S'$ be a smooth algebraic variety over $\mathbb{C}$ and let $\alpha': \mathcal{F}_K(\mathbb{C}) \to S'(\mathbb{C})$ be a map with the following property: for any smooth algebraic variety $S$ over $\mathbb{C}$ and $\xi \in \mathcal{F}(S)$, the map $s \mapsto \alpha'(\xi_s): S(\mathbb{C}) \to S'(\mathbb{C})$ is regular. We have to show that the map $s \mapsto \alpha'\alpha^{-1}(s): \text{Sh}_K(\mathbb{C}) \to S'(\mathbb{C})$ is regular. When $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{R})$, the map is that defined by $\alpha'$ and the universal family of abelian motives on $\text{Sh}_K$, and so it is regular by definition. In the general case, we let $G'$ be the smallest algebraic subgroup of $G$ such that $h(S) \subset G'_R$ for all $h \in X$. Then $(G', X)$ is a Shimura datum (cf. 7.6), which now is such that $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{R})$; moreover, $\text{Sh}_{K \cap G'(A_f)}(G', X)$ consists of a certain number of connected components of $\text{Sh}_K(G, X)$. As the map is regular on $\text{Sh}_{K \cap G'(A_f)}(G', X)$, and $\text{Sh}_K(G, X)$ is a union of translates of $\text{Sh}_{K \cap G'(A_f)}(G', X)$, this shows that the map is regular on $\text{Sh}_K(G, X)$. □

Remarks

11.19. When $(G, X)$ is of Hodge type in Theorem 11.18, the Shimura variety is a moduli variety for abelian varieties with additional structure. In this case, the moduli problem can be defined for all schemes algebraic over $\mathbb{C}$ (not necessarily smooth), and Mumford’s theorem can be used to prove that the Shimura variety is moduli variety for the expanded functor.

11.20. It is possible to describe the structure $\eta K$ by passing only to a finite covering, rather than the full universal covering. This means that it can be described purely algebraically.

11.21. For certain compact open groups $K$, the structure $\eta K$ can be interpreted as a level-N structure in the usual sense.

11.22. Consider a pair $(\mathbb{H}, \tilde{h})$ having a finite covering of Hodge type. Then there exists a Shimura datum $(G, X)$ of abelian type such that $(G^\text{der}, \text{ad} \circ h) = (\mathbb{H}, \tilde{h})$ for some $h \in X$. The choice of a faithful representation $\rho$ for $G$ gives a realization of the connected Shimura variety defined by any (sufficiently small) congruence subgroup of $\mathbb{H}(\mathbb{Q})$ as a fine moduli variety for abelian motives with additional structure. For example, when $H$ is simply connected, there is a map $\mathcal{F}_K(\mathbb{C}) \to \mathbb{T}(\mathbb{Q}) \backslash \mathbb{T}(A_f)/\mathcal{V}(K)$.
Shimura varieties and moduli

(see (11.13), p. 541), and the moduli problem is obtained from $\mathcal{Y}_K$ by replacing $\mathcal{H}_K(\mathbb{C})$ with its fibre over $[1]$. Note that the realization involves many choices.

11.23. For each Shimura variety, there is a well-defined number field $E(G,X)$, called the reflex field. When the Shimura variety is a moduli variety, it is possible to choose the moduli problem so that it is defined over $E(G,X)$. Then an elementary descent argument shows that the Shimura variety itself has a model over $E(G,X)$. A priori, it may appear that this model depends on the choice of the moduli problem. However, the theory of complex multiplication shows that the model satisfies a certain reciprocity law at the special points, which characterizes it.

11.24. The (unique) model of a Shimura variety over the reflex field $E(G,X)$ satisfying (Shimura’s) reciprocity law at the special points is called the canonical model. As we have just noted, when a Shimura variety can be realized as a moduli variety, it has a canonical model. More generally, when the associated connected Shimura variety is a moduli variety, then $Sh(G,X)$ has a canonical model ([61], [19]). Otherwise, the Shimura variety can be embedded in a larger Shimura variety that contains many Shimura subvarieties of type $A_1$, and this can be used to prove that the Shimura variety has a canonical model ([35]).

Notes. For more details on this subsection, see [38].

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