On transition of non-stationary waves

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Abstract

The probability of the events that the final states are detected with or interact with the nucleus in a finite time interval $T$ was found to be, $P = T\Gamma_0 + P^{(d)}$. $\Gamma_0$ is computed with Fermi’s golden rule, and does not depend on the nuclear wave functions. $P^{(d)}$ is not given by Fermi’s golden rule, and depends on the nuclear wave functions. In the electron mode of pion decays, $\Gamma_0$ is proportional to $m_{\pi}^2$ but $P^{(d)}$ for the event that the neutrino is detected is proportional to $m_{\nu}^{-2}$. $P^{(d)}$ does not hold the helicity suppression satisfied in $\Gamma_0$ and is inevitable in non-stationary quantum phenomena.
TRANSITIONS OF NON-STATIONARY WAVES

The transition rates are computed using the initial and final states at the asymptotic regions, which satisfy boundary conditions of the infinite time-interval $T = \infty$. Stationary phenomena are studied with methods for stationary waves. Now the non-stationary phenomena have non-trivial $T$-dependence and are studied with $T$-dependent transition probabilities, which can not be given from the stationary method. They are computed with a method of non-stationary waves, which satisfy boundary conditions different from those of stationary waves. The method is applied to a neutrino in pion decays in the present paper. We show, using the non-stationary method, that the transition probability at large $T$ has a constant term in addition to a $T$-linear term \[ P = \Gamma_0 T + P^{(d)}. \] \[ (1) \]

The rate $\Gamma_0$ is computed with Fermi’s golden rule \[2, 3\] and satisfies known properties. Now $P^{(d)}$ is not computed with the stationary waves and was obtained with the non-stationary method in Ref. \[4]. $P^{(d)}$ possesses unique properties that $\Gamma_0$ does not possess, and is important in various situations, especially in processes of $\Gamma_0 = 0$, $T \approx 0$. An example is presented.

In a microscopic system described by the Lorentz invariant Lagrangian,

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}, \] \[ (2) \]

the state vectors can be defined in Poincaré covariant manner. They are expressed by a many body wave function $|\Psi\rangle$ that follows a Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} |\Psi\rangle = (H_0 + H_{int}) |\Psi\rangle, \] \[ (3) \]

where the free part, $H_0$, and the interaction part, $H_{int}$, are derived from the previous Lagrangian. The transition processes are studied with Eq. \[3\] and classified to various cases depending on a situation of the system.

The transitions of lower excited states are studied with $S$-matrix $S[\infty]$ defined with an initial state at $t = -\infty$ and a final state at $t = \infty$. When the former states and the latter states are non-interacting each other and the overlap between them are negligible, the decay rates and cross section are considered to those of isolated particles in vacuum. The
asymptotic quantities at $T = \infty$ thus computed with $S[\infty]$ are Poincaré invariant \[4–6\]. The transition probability is proportional to $T$,
\begin{equation}
    P = T \Gamma_0,
\end{equation}
and $\Gamma_0$ is computed with Fermi’s golden rule. Waves of small spatial extensions follow Eq. (4). Their successive reactions occur separately, and the probability becomes in-coherent sum of those of each process. This region has been studied well and is called particle zone, here.

Transition of non-stationary waves reveals different probability. At a finite $T$, an overlap of the in-coming or out-going waves is not ignored and modifies the probabilities from those of the asymptotic region. The probability of successive processes becomes non-factorized to each process and the probability of the event that the final state interacts with others shows a different behavior from the particle zone. The detector in experiments is composed of many atoms and the outgoing particle in the event interacts with them. These scatterings occur at a finite $T$, instead of the infinite $T$. Accordingly, an $S$-matrix of the finite-time interval $T$, denoted as $S[T]$, was introduced \[1\]. $S[T]$ is defined in such way that satisfies the boundary condition at $T$ with Møller operators at a finite $T$, $\Omega_{\pm}(T)$, as $S[T] = \Omega_{\pm}(T)\Omega_{\pm}(T)$. $\Omega_{\pm}(T)$ are expressed by a free Hamiltonian $H_0$ and a total Hamiltonian $H$ by $\Omega_{\pm}(T) = \lim_{t \to \mp T/2} e^{iHt} e^{-iH_0t}$. From this expression, $S[T]$ satisfies
\begin{equation}
    [S[T], H_0] = i \left\{ \frac{\partial}{\partial T} \Omega_{\pm}(T) \right\} \Omega_{\pm}(T) - i \Omega_{\pm}^\dagger(T) \frac{\partial}{\partial T} \Omega_{\pm}(T).
\end{equation}

Due to Eq. (5), a matrix element of $S[T]$ between two eigenstates of $H_0$, $|\alpha\rangle$ and $|\beta\rangle$ of eigenvalues $E_\alpha$ and $E_\beta$, is decomposed into two components
\begin{equation}
    \langle \beta | S[T] | \alpha \rangle = \langle \beta | S^{(n)}[T] | \alpha \rangle + \langle \beta | S^{(d)}[T] | \alpha \rangle,
\end{equation}
where $\langle \beta | S^{(n)} | \alpha \rangle$ becomes finite for $E_\beta = E_\alpha$ and $\langle \beta | S^{(d)} | \alpha \rangle$ becomes finite for $E_\beta \neq E_\alpha$. The first term is equivalent to the asymptotic value and the second term is added to it. Thus the probability has a correction to Fermi’s golden rule. The fact that the correction is derived from the kinetic-energy non-conserving term and may modify dynamical properties was known to Peierls and Landau \[7\], and Landau concluded that the correction was negligible for processes of small energy transfer. We showed that the corrections are in fact important for processes of large energy transfer in the previous work \[1\], and the probability has a
-linear and constant, Eq. (1). \( \Gamma_0 \) is computed with \( S[\infty] \) and has been studied literature, but the \( P^{(d)} \) can not be computed with \( S[\infty] \) but by \( S[T] \).

The states of continuous spectrum of kinetic-energy couple with \( S^{(d)}[T] \). Among the infinite number of states of \( |\beta\rangle \) of \( E_\beta \neq E_\alpha \), certain states satisfy boundary conditions at \( t = \pm T/2 \). They are expressed for the scalar field \( \phi(x) \) with field operators [1, 8, 9]

\[
\lim_{t \to -T/2} \langle \alpha|\phi^f|\beta \rangle = \langle \alpha|\phi^f_{in}|\beta \rangle, \tag{7}
\]

\[
\lim_{t \to +T/2} \langle \phi^f|0 \rangle = \langle \alpha|\phi^f_{out}|\beta \rangle, \tag{8}
\]

where \( \phi_{in}(x) \) and \( \phi_{out}(x) \) satisfy the free wave equation, and \( \phi^f, \phi^f_{in} \) and \( \phi^f_{out} \) are the expansion coefficient of \( \phi(x) \), \( \phi_{in}(x) \) and \( \phi_{out}(x) \), with the normalized wave functions \( f(x) \) of the form

\[
\phi^f(t) = i \int d^3x f^* (\vec{x}, t) \overleftarrow{\partial_0} \phi (\vec{x}, t). \tag{9}
\]

For events that the neutrino is detected with the nucleon in the detector, the probability amplitude is expressed with the nucleon wave function in nucleus. Hence the nucleon wave function is used for \( f(x) \). \( S^{(d)}[T] \) thus defined depends on the base functions \( f(x) \), and is appropriate to write as \( S^{(d)}[T; f] \). The neutrino in the final state is expressed by the small wave function despite of its large mean free path. Accordingly, the probability of the events is expressed by this normalized wave function, called wave packet. Wave packets that satisfy free wave equations and are localized in space are important for rigorously defining scattering amplitude [8, 9].

At \( T \to \infty \), the right-hand side of Eq. (5) and the second term of Eq. (6) vanish, hence the energy defined by \( H_0 \) is conserved. Conversely, all the states \( |\beta\rangle \) of \( E_\beta = E_\alpha \) contribute and \( S[\infty] \) is uniquely defined. At a finite \( T \), \( S^{(n)}[T] \) satisfies the conservation of kinetic energy and is uniquely defined. \( S^{(d)}[T; f] \), on the other hand, does not satisfy the conservation of the kinetic energy and depends on \( f \). Furthermore, \( S^{(d)}[T; f] \) is not invariant under the Poincaré transformation defined by \( \mathcal{L}_0 \). The state \( |\beta\rangle \) of \( E_\beta \) is orthogonal to \( |\alpha\rangle \) of \( E_\alpha \neq E_\beta \) and the cross term of the first and second terms of Eq. (6) in a square of the modulus vanishes. Consequently the finite-size correction to the probability becomes positive semi-definite, and the probability \( P(T) \) is larger than \( P(\infty) \). Unitarity \( S[T]S^*[T] = 1 \) is satisfied and ensures the conservation of probability. For \( S[\infty] \), the states have constant kinetic energy, and reveal the particle properties. The states of continuous kinetic energy are different and are waves. It will be shown that they reveal features of wave properties.
FORBIDDEN PROCESS

For a process $\langle \beta | S^{(n)}[T] | \alpha \rangle = 0$, the rate vanishes, $\Gamma_0 = 0$, and $P$ is not proportional to $T$ but is constant, $P^{(d)}$. $P^{(d)}$ is computed from $\langle \beta | S^{(d)}[T; f] | \alpha \rangle$. Thus states $|\beta\rangle$ of $E_\beta \to \infty$ that satisfy the boundary conditions contribute to $P^{(d)}$. These waves propagate with the speed of light, hence necessarily give a finite contribution to the probability of the event that a light particle is detected.

Probabilities at finite $T$ using $S[\infty]$ were studied without wave packets in Refs. [10–15] and with wave packets in Refs. [16–23]. The probabilities from $S[\infty]$ do not show the finite-size corrections, because they were the asymptotic values. The probabilities from $S[T]$ are different, and show $T$-dependence. For the event that the neutrino is detected at $T$, $S[T]$ with the wave packet were studied in Refs. [24–27]. The $T$-dependent probabilities are derived from $S^{(d)}[T; f]$, and depend on $f$. The wave packets cannot be replaced with plane waves in $S[T]$.

$\pi \to \nu_\ell + \ell$

The probabilities of the events that the electron neutrino from the decay of pion interacts with the nearby nucleus at a finite distance satisfies $\Gamma_0 \approx 0$, $P^{(d)} \neq 0$ and is studied in this section. The system is described by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}},$$

$$\mathcal{L}_0 = \partial_\mu \varphi^* \partial^\mu \varphi - m_\pi^2 \varphi^* \varphi + i(\gamma \cdot p - m_\pi) l + \bar{\nu} (\gamma \cdot p - m_\nu) \nu,$$

$$\mathcal{L}_{\text{int}} = g J_{\text{hadron}}^{V-A} \times J_{\text{lepton}}^{V-A}, \quad g = G_F / \sqrt{2},$$

where $G_F$ is the Fermi coupling constant and $J_i^{V-A}$ is $V-A$ current. The amplitude for a neutrino of an average momentum $\vec{p}_\nu$ in the decay of a pion prepared at $t = T_\pi$ of a momentum $\vec{p}_\pi$ and a neutrino is expressed by a wave packet. These states are expressed as $|\pi\rangle = |\vec{p}_\pi, T_\pi\rangle$, $|l, \nu\rangle = |\mu, \vec{p}_l; \nu, \vec{p}_\nu, \vec{X}_\nu, T_\nu\rangle$. $\mathcal{M}$ is written with the hadronic matrix element of $V-A$ current and Dirac spinors

$$\mathcal{M} = \int d^4 x d\vec{k}_\nu N_1 \langle 0 | J_{i}^{V-A}(0) | \pi \rangle \bar{u}(\vec{p}_l) \gamma_\mu (1 - \gamma_5) \nu(\vec{k}_\nu)$$

$$\times \exp \left[ -i (p_\pi - p_l) \cdot x / h + i k_\nu \cdot (x - X_\nu) / h - \frac{\sigma^\mu}{2} (\vec{k}_\nu - \vec{p}_\nu)^2 \right].$$

(11)

where $N_1 = ig (\sigma_\nu / \pi)^{\frac{3}{2}} (m_l m_\nu / (2\pi)^3 E_\pi V E_{l, \nu})^{\frac{1}{2}}$ and $\langle 0 | J_{i}^{V-A}(0) | \pi \rangle = i f_{\pi} p_{\pi}^\mu$. The time $t$ is
integrated over the region $T_\pi \leq t \leq T_\nu$, in the region $T_\nu - T_\pi \ll \tau_\pi$, where $\tau_\pi$ is the pion life. $\sigma_\nu$ is the size of the nucleon wave function that the neutrino interacts with and is estimated from the size of a nucleus. For the sake of simplicity, we use the Gaussian form of the wave packet for the function $f(x)$ of Eq. (9). The result for the finite-size correction is the same in general wave packets.

The total probability is computed from the amplitude as

$$P = \int d\vec{X}_\nu \frac{d\vec{p}_\nu}{(2\pi)^3} \frac{d\vec{p}_\pi}{(2\pi)^3} \sum_{s_1,s_2} |M|^2,$$

(12)

where the unmeasured momentum of the final state is integrated over the whole positive energy region and that of the measured momentum and the position is integrated in the inside of the detector. Hereafter the natural unit, $c = \hbar = 1$, is taken in majority of places, but $c$ and $\hbar$ are written explicitly when it is necessary.

Equation (12) was computed in Ref. [1]. Integrating over the neutrino’s coordinate $\vec{X}_\nu$, we obtain the total volume, which is canceled by the factor $V^{-1}$ from the normalization of the initial pion state. The total probability is then expressed as the sum of the standard term, $\Gamma_0$, and the new term proportional to $\tilde{g}(\omega_\nu T)$:

$$P = T\Gamma_0 + P^{(d)}$$

(13)

$$\Gamma_0 = \tilde{N}_4 \int \frac{d^3p_\pi}{(2\pi)^3} \frac{p_\pi \cdot p_\nu (m_\pi^2 - 2p_\pi \cdot p_\nu)}{E_\nu} G_0,$$

$$P^{(d)} = \tilde{N}_4 \int \frac{d^3p_\pi}{(2\pi)^3} \frac{p_\pi \cdot p_\nu (m_\pi^2 - 2p_\pi \cdot p_\nu)}{E_\nu} T\tilde{g}(\omega_\nu T),$$

where $\tilde{N}_4 = 8g^2 f^2_\pi \sigma_\nu / E_\pi$ and $L = cT$ is the length of the decay region, and $\omega_\nu = m_\nu^2 / 2E_\nu$. $G_0$ comes from term of $p_\pi \approx p_e + p_\nu$, and

$$(m_\pi^2 - 2p_\pi \cdot p_\nu) = m_e^2,$$

(14)

which vanishes for $m_e = 0$. The $\Gamma_0$ becomes independent of $\sigma_\nu$ and is proportional to the square of charged lepton mass. The ratio between the electron and muon is $(m_e / m_\mu)^2 = 10^{-4}$ and $\Gamma_0$ for the electron is negligibly small, which is known as the helicity suppression.

$\tilde{g}(\omega_\nu T)$ comes from the kinetic-energy non-conserving term and satisfies $\tilde{g}(0) = \pi$, $\frac{\partial}{\partial T} \tilde{g}(\omega_\nu T)|_{T=0} = -\omega_\nu$ and $\tilde{g}(\omega_\nu T) = \frac{2}{\omega_\nu T}$, for $\omega_\nu T \rightarrow \infty$. In small $T$, $\tilde{g}(\omega T) = \tilde{g}(0) = \pi$, and $P^{(d)}$ is proportional to $T$, and the probability from $T\Gamma_0$ is ignorable. Hence

$$P = T \frac{8\pi g^2 f^2_\pi \sigma_\nu}{E_\pi} \int \frac{d^3p_\nu}{(2\pi)^3} \frac{p_\nu \cdot p_\nu (m_\pi^2 - 2p_\pi \cdot p_\nu)}{E_\nu} \theta(m_\pi^2 - m_e^2 - 2p_\pi \cdot p_\nu).$$

(15)
The kinetic-energy non-conserving term from the momentum region

\[ p_\pi - (p_e + p_\nu) \neq 0, \]
\[ (m_\pi^2 - 2p_\pi \cdot p_\nu) \gg m_e^2 \]
gives \( P^{(d)} \). Consequently the integral does not hold the helicity suppression, and \( P/T |_{T \to 0} \) is sizable of around 20 percent of the rate of muon-mode. At a large macroscopic \( T \), \( \tilde{g}(\omega_\nu T) \approx \frac{4E_\nu}{(m_\nu^2 T)} \), and we have,

\[ P = \frac{16g^2 f_\pi^2 \sigma_\nu \hbar}{E_\pi m_\nu^2 c^3} \int \frac{d^3 p_\nu}{(2\pi)^3} p_\pi \cdot p_\nu (m_\pi^2 - 2p_\pi \cdot p_\nu) \theta(m_\pi^2 - m_e^2 - 2p_\pi \cdot p_\nu). \]

\( P \) at a large \( T \) is independent of \( T \), which reveals the wave nature. The integral in Eq. (15) is Lorentz invariant but that in Eq. (17) is Lorentz non-invariant even in high-energy region. Thus \( P \) is governed by \( P^{(d)} \), Eq. (15) or Eq. (17) and has sizable magnitudes, which increase with \( \sigma_\nu \) because the number of kinetic-energy non-conserving states increases. Furthermore, the constant \( P^{(d)} \) of Eq. (17) is inversely proportional to the square of the neutrino mass. \( P^{(d)} \) is consistent with existing data on neutrino experiments \[28\] if \( \sigma_\nu \) is the size of nucleus, and a future precision measurement of \( P^{(d)} \) at \( T \approx 0 \) and at a macroscopic \( T \) will be able to supply the precise value of absolute neutrino mass. Implications of \( P^{(d)} \) and other processes of \( \Gamma_0 = 0, P^{(d)} \neq 0 \) will be studied in a subsequent paper.

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