Enhanced Diffusion and the Continuous Spontaneous Localization Model

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Abstract

We find an analogy between turbulence and the dynamics of the continuous spontaneous localization model (CSL) of the wave function. The use of a standard white noise in the localization process gives Richardson’s $t^3$ law for the turbulent diffusion, while the introduction of an affine noise in the CSL allows us to obtain the intermittency corrections to this law.

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I. INTRODUCTION

In studying the phenomenon of enhanced diffusion, Shlesinger et al. [1] were led to introduce the concept of a Lévy walk as an extension of the more familiar Lévy flight [2]. The basic difference between a Lévy flight and a Lévy walk is that for the latter, although the walker visits all sites visited by the flight, the jumps do not occur instantaneously, but there may be a time delay before the next jump. Shlesinger et al. obtained an integral transport equation involving a scaled memory which is nonlocal in space and time. Contrary to the infinite mean square displacement obtained in a Lévy flight, the solution of such transport equation leads to a finite mean square displacement, which has the same time dependence as obtained by Richardson (<R²,t> ∼ t³) in his pioneering studies of turbulence [3]. The Mandelbrot intermittency corrections [4] are also considered by the authors of reference [1] and provide the necessary corrections to Richardson’s law, which are observed experimentally.

In this paper we show that similar results are obtained in the different physical context of the continuous spontaneous localization model of quantum mechanics (CSL) [5] such as introduced by Ghirardi, Pearle and Rimini (GPR) [6]. In this model the wave function is subjected to a stochastic process in Hilbert space. In one dimension the evolution equation in the Stratonovich form is [7]

\[
d\psi(x,t) = \left\{ -iH - \lambda \right\} dt + \sqrt{\gamma} \int dq dB(q,t) G(x-q) \psi(x,t),
\]

where dB is a white noise (<dB(t)> = 0 and <dB(t)dB(0)> = dt) and

\[
G(x-q) = \sqrt{\frac{\alpha}{2\pi}} \exp \left[ -\alpha \frac{(x-z)^2}{2} \right]
\]

is an indication of the localization of the wave function. The length parameter α and the frequency parameter λ are fundamental parameters of the spontaneous reduction model developed by Ghirardi, Rimini and Weber (GRW) [8] and are related to γ according to γ = λ(4π/α)² [5,6]. They are chosen in such a way that the new evolution equations do not give different results from the usual Schrödinger unitary evolution for microscopic systems with few degrees of freedom, but when a macroscopic system is described there is a fast decay of the macroscopic linear superpositions which are quickly transformed into statistical mixtures [6,8].

This analogy between the CSL process and turbulence allows us to obtain in section III the enhanced diffusion, the mean energy input into the turbulent medium, a Fokker-Planck equation for the probability density in phase space and Mandelbrot’s intermittency corrections with the introduction of an affine noise; all in the framework of the beable interpretation of the CSL model, which is presented in the next section.

II. A BEABLE INTERPRETATION OF THE CSL MODEL

The usual interpretation of quantum mechanics deals fundamentally with results of measurements and therefore presupposes, besides a system, an apparatus to perform the measurements. However what the apparatus is and how to distinguish it from the system are
questions with vague answers. In face of this problem, Bell [9] proposed an interpretation in
terms of ‘beables’ instead of observables. Beables correspond to things that exist independ-
ently of the observation, therefore they can be assigned well defined values. In this way
we avoid a cut between the microscopic (quantum) world and the macroscopic (classical)
world.

Vink [10] showed that two other well known interpretations of quantum mechanics -
the causal interpretation associated with Bohm [11] and the stochastic interpretation due to
Nelson [12] - are particular cases of the beable interpretation as developed by Bell. Moreover,
he proposed that all observables, even those that do not commute, can attain beable status
simultaneously.

Generalizing Vink’s results [10], we have recently extended the beable interpretation
to the GPR model for a free particle. We treated position and momentum as beables and
showed that in the continuum limit they satisfy the following stochastic differential equations

\[
\begin{align*}
    dx &= \frac{p_0}{M} dt + 2\nu \sqrt{\gamma} \left[ \int_0^t \int dq dB(q,t') \frac{\partial G(x - q)}{\partial x} \right] dt + (2\nu)^{\frac{1}{2}} dw, \\
    dp &= \hbar \sqrt{\frac{\alpha \lambda}{2}} dw.
\end{align*}
\]

In the next section we exploit equations (3) and (4) and obtain the main results of this
paper.

III. TURBULENCE RESULTS

In equation (3) the first term on the right hand side describes a single free particle
deterministic evolution as in the de Broglie-Bohm model. The two other terms describe the
stochastic processes, with \( dw \) and \( dB \) being two independent white noises, \(< dw(t) >= 0, < dw(t)dw(0) >= dt \) and \( \nu = \hbar / 2m \). The last stochastic term is a standard diffusion and
the second term, a non standard diffusion which exhibits the non-locality of the localization
process. This second term indicates that the particle position tracks the wave function. The
position increment induced by this term drives the particle to where the wave function is
increasing, and therefore localizing, according to the fluctuating term in equation (1). Notice
that \( dw \) and \( dB \) are two independent white noise, \(< dw(t) >= 0, < dw(t)dw(0) >= dt \) and \( \nu = \hbar / 2m \). The non standard diffusion term is responsible for the \( t^3 \) behavior for the mean
square displacement

\[
\begin{align*}
    < x^2(t) >= < x^2(t) >_S + \frac{\alpha \hbar^2}{6m^2} t^3,
\end{align*}
\]

where \(< x^2(t) >_S \) is the mean square displacement for the free Schrödinger evolution and
the last term corresponds to the enhanced diffusion typical of turbulence [1].

\( ^1 \)This \( t^3 \) behavior prompted us to investigate a possible analogy with turbulent diffusion.
With respect to the stochastic process for momentum we stress the fact that this is a consequence of the localization of the wave function, which vanishes when GRW parameters \((\alpha, \lambda)\) go to zero. As for the fluctuation in momentum, equation (4) gives

\[
< p^2(t) > = < p^2(t) >_S + \frac{\hbar^2 \alpha \lambda}{2} t. \tag{6}
\]

where \(< p^2(t) >_S\) is the mean square momentum for the Schrödinger evolution.

Working towards a more insightful physical picture of the above processes, we now consider the mean energy input in a turbulent medium and compare with the equivalent quantity in the GRW model.

Turbulence theories based on dimensional analysis give \[14\]

\[
< x^2(t) > \sim < \epsilon > t^3, \tag{7}
\]

where \(< \epsilon >\) is the mean energy input per unit time and per unit mass. Comparing (7) with our equation (5), we are led to identify

\[
< \epsilon > = \frac{\alpha \lambda \hbar^2}{6m^2}, \tag{8}
\]

which coincides with the term of energy non-conservation (per unit time per unit mass) of the GRW collapsing model ([8], eqs. 7.1 and 7.2).

A nice feature of our beable interpretation of the CSL model is the discontinuous nature of the velocity (eq. 4), which has its own analogy in turbulence. This point had already been noticed by Richardson [3], although his evolution equation does not take this into account [15]. In our case this comes about naturally as a consequence of giving beable status to both position and momentum. From the stochastic differential equations (3) and (4), we obtain the following Fokker-Planck equation for the probability density in phase space

\[
\frac{\partial P(x,p,t)}{\partial t} = -\frac{p_o}{m} \frac{\partial P(x,p,t)}{\partial x} \left( \frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} + \sqrt{\frac{\hbar^2 \alpha \lambda}{2m}} \frac{\partial^2}{\partial x \partial p} + \frac{\hbar^2 \alpha \lambda}{4} \frac{\partial^2}{\partial p^2} \right) P(x,p,t), \tag{9}
\]

which now has two diffusion coefficients for position \(\hbar/m\) and for momentum \(\hbar^2 \alpha \lambda/2\).

We have considered so far only white noise. If we wish to pursue the analogies with turbulence even further the corrections for intermittency should now be taken into account. In order to do so we consider an affine noise [16] called fractional Brownian noise

\[
< dB(t) dB(0) >= t^{4-\mu} dt. \tag{10}
\]

Noise \(dB\) when used in equation (3) gives for the anomalous diffusion term

\[
< x^2(t) > \sim t^{4-1+3}. \tag{11}
\]

This corresponds to one of the intermittency corrections obtained by Shlesinger et al provided we identify \(A - 1\) with \(3\mu/(4 - \mu)\), where \(\mu = E - df\), \(E\) being the Euclidean dimension and \(df\), the fractal dimension.
For the momentum variable the non-white noise gives

\[ <p^2> \sim t^A, \]

which leads to the scaling relation obtained by Shlesinger et al [1] for the root-mean-square velocity. Notice that contrary to reference [1], we obtain the intermittency correction (11) without having to use (12).

**IV. CONCLUSION**

We have exploited an analogy between turbulence and the beable interpretation of the spontaneous localization model in quantum mechanics thus providing an appealing physical picture for the localization process. The analogy with turbulence led us consider a non-white noise for the GPR process, which may be useful in the attempt to construct a more realistic model along the lines of GPR [7,17].

Within the beable interpretation of the CSL model, we have found a dynamics that has the character of a Lévy dynamics without having to use the Lévy stable laws. This point had already been noticed by Kusnezov et al [18].

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