An epsilon-delta bound for plane algebraic curves and its use for certified homotopy continuation of systems of plane algebraic curves

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Abstract

We explain how, given a plane algebraic curve \( C : f(x,y) = 0, x_1 \in C \) not a singularity of \( y \) w.r.t. \( x \), and \( \varepsilon > 0 \), we can compute \( \delta > 0 \) such that \( |y_j(x_1) - y_j(x_2)| < \varepsilon \) for all holomorphic functions \( y_j(x) \) which satisfy \( f(x, y_j(x)) = 0 \) in a neighbourhood of \( x_1 \) and for all \( x_2 \) with \( |x_1 - x_2| < \delta \). Consequently, we obtain an algorithm for reliable homotopy continuation of plane algebraic curves. As an example application, we study continuous deformation of closed discrete Darboux transforms.

Moreover, we discuss a scheme for reliable homotopy continuation of triangular polynomial systems. A general implementation has remained elusive so far. However, the epsilon-delta bound enables us to handle the special case of systems of plane algebraic curves. The bound helps us to determine a feasible step size and paths, which are equivalent w.r.t. analytic continuation to the actual paths of the variables but along which we can proceed more easily.

1 Motivation

In many geometric problems, variables depend analytically on some parameter. If we want to analyze and experiment with these problems using interactive software, whenever the user continuously modifies the parameter, we must update the dependent variables accordingly. For many applications, in doing so, the analytical relationship between variables and parameter should be preserved at all times. Therefore we need reliable algorithms for analytic continuation.

Consider for example the following problem of discrete differential geometry (Hoffmann, 2009, Section 2.6). Let there be a regular discrete curve \( \gamma \) in \( \mathbb{CP}^1 \), i.e. a polygonal chain with distinct vertices \( \gamma_0, \gamma_1, \ldots, \gamma_n \in \mathbb{CP}^1 \). We define the discrete Darboux transform \( \tilde{\gamma} \) of \( \gamma \) with initial point \( \tilde{\gamma}_0 \in \mathbb{CP}^1 \) and parameter \( \mu \in \mathbb{C} \) as follows: for all \( j = 1, 2, \ldots, n \), let \( \tilde{\gamma}_j \in \mathbb{CP}^1 \) be the unique point for which the cross-ratio

\[
(\gamma_{j-1}, \gamma_j; \tilde{\gamma}_j, \tilde{\gamma}_{j-1}) := \frac{(\gamma_{j-1} - \tilde{\gamma}_j)(\gamma_j - \tilde{\gamma}_{j-1})}{(\gamma_{j-1} - \tilde{\gamma}_{j-1})(\gamma_j - \tilde{\gamma}_j)} = \mu.
\]

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It can be shown that $\tilde{\gamma}_{j-1}$ is mapped to $\tilde{\gamma}_j$ by a unique M"{o}bius transformation, which depends only on $\gamma_{j-1}$, $\gamma_j$, and $\mu$, but not on $\tilde{\gamma}_j$. Hence, there exists a unique M"{o}bius transformation $M$ depending on $\gamma_0$, $\gamma_1$, ..., $\gamma_n$, and $\mu$, which maps an initial point $\tilde{\gamma}_0$ to the corresponding last point $\tilde{\gamma}_n$ of $\tilde{\gamma}$. Consequently, for every choice of $\mu \in \mathbb{C}$, there are two choices of initial point $\tilde{\gamma}_0$ (counted with multiplicity) such that $\tilde{\gamma}$ is a closed polygonal chain. These are exactly the fixed points of $M$ or, in other words, the roots of the characteristic polynomial of $M$. The vanishing of the characteristic polynomial establishes an algebraic (particularly analytical) relationship between $\mu$ and $\tilde{\gamma}_0$.

If we want to study closed Darboux transforms of a discrete curve $\gamma$ for varying parameter $\mu$ using interactive software, then we must analytically continue $\tilde{\gamma}_0$. Otherwise we may observe sudden jumps of $\tilde{\gamma}_0$ under continuous movement of $\mu$, which have no mathematical justification.

In practice, of course, we cannot modify a parameter continuously. Instead, we obtain a series of parameter values at a series of discrete points in time. We do not know how the parameter moves between sample points. A natural approach would be to interpolate linearly between consecutive parameter values (using a time parameter in the unit interval). However, the segment between parameter values may contain singularities beyond which analytic continuation becomes impossible. Thus it seems reasonable to analytically continue along the polygonal chain of parameter values as long as this is possible, and to deviate from that path otherwise. Such a deviation can still be interpreted as a linear interpolation between consecutive parameter values if we let the time parameter run from 0 to 1 on an arbitrary path through the complex plane instead of restricting it to the unit interval.

This is the paradigm of ‘complex detours’ invented by Kortenkamp and Richter-Gebert for their interactive geometry software Cinderella (Kortenkamp and Richter-Gebert, 2006). It is described in more detail in (Kortenkamp, 1999, esp. Chapter 7; Kortenkamp and Richter-Gebert, 2002). Essentially the same concept was conceived in the context of homotopy continuation by Morgan and Sommese (1987), who later named it the ‘gamma trick’ (Sommese and Wampler, 2005, Lemma 7.1.3 on p. 94).

Once we have chosen a path for the parameter, we must determine the right value of the dependent variable at consecutive sample points. How this can be achieved may in fact be relatively easy to see for us—just determine values in a way such that there are no jumps—but hard to see for an algorithm. The tracing problem of dynamic geometry, i.e. tracing the positions of dependent elements of a geometric construction under movement of a free element, is NP-complete already for constructions that only involve points, lines through two points, intersection of lines, and angle bisectors (Kortenkamp and Richter-Gebert, 2002).

The interactive geometry software Cinderella currently uses a heuristic for path following. Most homotopy continuation methods use a predictor-corrector approach, which is generally also heuristic. For an overview of homotopy continuation methods, consider the books by Allgower and Georg (1990) or Sommese and Wampler (2005). Lately, certified homotopy continuation methods have emerged (Beltrán and Leykin, 2012; Hauenstein and Sottile, 2012; Hauenstein et al., 2014). They are based on Smale’s alpha theory (Smale, 1986).

In what follows, we derive a certified algorithm for analytic continuation
of plane algebraic curves based on the following simple observation: Due to
continuity, if the parameter moves little, so does the dependent variable. Hence,
if we take small enough steps along the parameter path, we can choose the right
value of the dependent variable based on proximity. As an application, we return
to the example of continuous deformation of closed discrete Darboux transforms.
Moreover, we show how the algorithm generalizes to systems of plane algebraic
curves. A comparison with other approaches demonstrates the practicability of
our algorithms.

2 Computing an epsilon-delta bound for plane
algebraic curves

Theorem 2.1. Let \( C : f(x, y) = 0 \) be a complex plane algebraic curve, where
\[
f(x, y) = \sum_{k=0}^{n} a_k(x) y^{n-k}
\]
is a polynomial of degree \( n \) in \( y \) whose coefficients \( a_k(x) \) are polynomials in
\( x \). Let \( x_1 \in \mathbb{C} \) be a point in the complex plane at which neither the leading
coefficient \( a_0(x) \) nor the discriminant of \( f(x, y) \) w.r.t. \( y \) vanish. Then for every
\( \varepsilon > 0 \), we can algorithmically compute \( \delta > 0 \) such that
\[
|y_j(x_1) - y_j(x_2)| < \varepsilon
\]
for all holomorphic functions \( y_j(x) \), \( j = 1, 2, \ldots, n \), that satisfy \( f(x, y_j(x)) = 0 \)
in a neighbourhood of \( x_1 \) and for all \( x_2 \) with \( |x_1 - x_2| < \delta \).

Remark 2.2. How does Theorem 2.1 help us to perform analytic continuation?
Let \( \varepsilon \) be half the minimal distance between the \( y \)-values at \( x_1 \). Then for any \( x_2 \)
less than \( \delta \) away from \( x_1 \) the following holds: The \( y \)-value \( y_j(x_2) \), which results
from analytic continuation of \( y_j(x) \) along the segment from \( x_1 \) to \( x_2 \), is closer
to \( y_j(x_1) \) than to any other \( y \)-value at \( x_1 \). In other words, \( \delta \) provides an upper
bound for the step width of parameter \( x \) such that we may match \( y \)-values on
the same branch based on proximity.

Our plan for the proof of Theorem 2.1 is as follows: We will see that there is an
upper bound of \( \delta \) depending on

1. the radius of convergence of the Taylor expansion of \( y_j(x) \) at \( x_1 \),
2. the modulus of the derivative of \( y_j(x) \) at \( x_1 \),
3. the maximum modulus of \( y_j(x) \) on a circle centred at \( x_1 \),

for \( j = 1, 2, \ldots, n \), respectively. We derive a formula for that upper bound and
then compute bounds for its ingredients. To this end, we need the following
lemmas.

Lemma 2.3. Let \( U \subset \mathbb{C} \) be an open subset of the complex plane, and let
\[
y_j : U \to \mathbb{C}
\]
be holomorphic. Taylor expansion of \( y_j \) around \( x_1 \in U \) yields
\[
y_j(x_2) = y_j(x_1) + (x_2 - x_1)y_j'(x_1) + (x_2 - x_1)^2 R(x_2),
\]
for all \( x_2 \in \mathbb{C} \) such that \(|x_2 - x_1| < \rho \) and sufficiently small \( \rho > 0 \). The remainder \( R(x_2) \) satisfies
\[
|R(x_2)| \leq \frac{M}{\rho(|x_2 - x_1|)}
\]
where
\[
M = \max_{t \in [0,2\pi]} |y_j(x_1 + \rho e^{it})|.
\]
\textbf{Lemma 2.3} is a standard result of complex analysis ([Ahlfors, 1979], p. 124–126), which we therefore do not prove here.

\textbf{Lemma 2.4} (implicit differentiation). Let \( f(x,y) \) be a complex polynomial. Let \( U \subset \mathbb{C} \) be an open subset of the complex plane. Let \( y_j: U \to \mathbb{C} \) be a holomorphic function that satisfies \( f(x,y_j(x)) = 0 \) for all \( x \in U \). Then for all \( x_1 \in U \) with \( f_y(x_1,y_j(x_1)) \neq 0 \) it follows that
\[
y_j'(x_1) = -\frac{f_x(x_1,y_j(x_1))}{f_y(x_1,y_j(x_1))}.
\]

\textbf{Proof.} By the chain rule, the total differential of \( f(x,y_j(x)) = 0 \) w.r.t. \( x \) is
\[
Df(x,y_j(x)) = f_x(x,y_j(x)) + f_y(x,y_j(x)) \cdot y_j'(x) = 0.
\]
Therefore
\[
y_j'(x_1) = -\frac{f_x(x_1,y_j(x_1))}{f_y(x_1,y_j(x_1))}.
\]

\textbf{Lemma 2.5} ([Fujwara (1916) Inequality 3 on p. 168]). Consider a polynomial
\[
p(x) = \sum_{k=0}^{n} a_k x^{n-k}
\]
of degree \( n \) with complex coefficients \( a_k \in \mathbb{C}, k = 0, 1, \ldots, n \). Then all \( \bar{x} \in \mathbb{C} \) with \( p(\bar{x}) = 0 \) satisfy
\[
|\bar{x}| < 2 \max \left\{ \frac{1}{|a_0|} \frac{a_k}{a_0^{\frac{1}{k}}} : k = 1, \ldots, n \right\}.
\]

\textbf{Proof.} Consider the inequality
\[
|p(x)| \geq |a_0||x|^n - \sum_{k=1}^{n} |a_k||x|^{n-k}. \tag{1}
\]
The RHS of \( (1) \) is positive if
\[
|a_0||x|^n \geq 2^k|a_k||x|^{n-k}, \quad k = 1, 2, \ldots, n,
\]
because then
\[ |a_0||x|^n > (1 - 2^{-n})|a_0||x|^n = \sum_{k=1}^{n} 2^{-k}|a_0||x|^n \geq \sum_{k=1}^{n} |a_k||x|^{n-k}. \]

Hence, \(|p(x)| > 0\) if
\[ |x| > \max \left\{ 2^k \frac{|a_k|}{|a_0|} \right\} \]
and thus
\[ |\bar{x}| < 2 \max \left\{ |a_k|^{\frac{1}{k}} : k = 1, \ldots, n \right\} \]
for all zeros \(\bar{x} \in \mathbb{C}\) of \(p(x)\).

**Lemma 2.6** (bounds for trigonometric polynomials). Consider a trigonometric polynomial of degree \(n\) of the form
\[ p(x_1 + \rho e^{it}) = \sum_{k=0}^{n} a_k (x_1 + \rho e^{it})^{n-k}. \]

Then
\[ |p(x_1 + \rho e^{it})| \leq \sum_{k=0}^{n} |a_k|(|x_1| + |\rho|)^{n-k}. \]

Moreover, if the zeros \(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\) of \(p(x)\) satisfy \(|\bar{x}_k - x_1| > \rho\) then
\[ |p(x_1 + \rho e^{it})| \geq |a_0| \prod_{k=0}^{n} (|\bar{x}_k - x_1| - \rho) > 0. \]

**Proof.** The upper bound follows from the triangle inequality. The lower bound follows from the factorization
\[ p(x_1 + \rho e^{it}) = a_0 \prod_{k=0}^{n} (x_1 + \rho e^{it} - \bar{x}_k) \]
and the fact that \(|x_1 + \rho e^{it} - \bar{x}_k| \geq |\rho - |\bar{x}_k - x_1|||\. Note that the lower bound is positive by the assumptions that \(|\bar{x}_k - x_1| > \rho\) and that \(p\) has degree \(n\), i.e. \(a_0 \neq 0\).

**Proof of Theorem 2.1** Let \(y_j(x)\), \(j = 1, \ldots, n\), denote the holomorphic functions that satisfy \(f(x, y_j(x)) = 0\) in a neighbourhood of \(x_1\). By **Lemma 2.3**
\[ y_j(x_2) = y_j(x_1) + (x_2 - x_1)y'_j(x_1) + (x_2 - x_1)^2 R_j(x_2) \quad (2) \]
for all \(x_2 \in \mathbb{C}\) such that \(|x_2 - x_1| < \rho\) and sufficiently small \(\rho > 0\). If we bring \(y_j(x_1)\) to the LHS of (2), take the absolute value on both sides, and apply the triangle inequality, we see that
\[ |y_j(x_1) - y_j(x_2)| = |x_2 - x_1||y'_j(x_1) + (x_2 - x_1)R_j(x_2)| \leq |x_2 - x_1|(|y'_j(x_1)| + |x_2 - x_1||R_j(x_2)|) = |R_j(x_2)||x_2 - x_1|^2 + |y'_j(x_1)||x_2 - x_1|. \]
Hence, under the above assumptions,

\[ |R_j(x_2)||x_2 - x_1|^2 + |y_j'(x_1)||x_2 - x_1| - \varepsilon < 0. \] (4)

is a sufficient condition for \(|y_j(x_1) - y_j(x_2)| < \varepsilon\).

The LHS of (4) is strictly increasing in \(|y_j'(x_1)|\) and \(|R_j(x_2)|\). Therefore, if we plug in the bounds

\[ |y_j'(x_1)| \leq \max_j |y_j'(x_1)| =: Y \] (5)

and

\[ |R_j(x_2)| \leq \frac{M}{\rho(\rho - |x_2 - x_1|)} \]

(see Lemma 2.3) into (4), we obtain a stronger sufficient condition for

\[ |y_j(x_1) - y_j(x_2)| < \varepsilon, \]

namely

\[
\frac{M}{\rho(\rho - |x_2 - x_1|)}|x_2 - x_1|^2 + Y|x_2 - x_1| - \varepsilon < 0 \\
\iff M|x_2 - x_1|^2 + \rho(\rho - |x_2 - x_1|)(Y|x_2 - x_1| - \varepsilon) < 0 \\
\iff (M - \rho Y)|x_2 - x_1|^2 + \rho(\rho + \varepsilon)|x_2 - x_1| - \varepsilon \rho^2 < 0. \] (6)

How we can transform (6) into a sufficient bound on \(|x_2 - x_1|\) depends on the sign of \(M - \rho Y\).

First case: \(M - \rho Y > 0\). The LHS of (6) describes a smile parabola in \(|x_2 - x_1|\) with a positive and a negative root. Since \(|x_2 - x_1| \geq 0\), we need only bound \(|x_2 - x_1|\) from above by the positive root, i.e.

\[
|x_2 - x_1| < \frac{-\rho(\rho Y + \varepsilon) + \sqrt{\rho^2(\rho Y + \varepsilon)^2 + 4(M - \rho Y)\varepsilon^2}}{2(M - \rho Y)} \\
= \frac{\rho \left(\sqrt{(\rho Y - \varepsilon)^2 + 4\varepsilon M - (\rho Y + \varepsilon)}\right)}{2(M - \rho Y)}. 
\]

Second case: \(M - \rho Y < 0\). The LHS of (6) describes a frown parabola in \(|x_2 - x_1|\) with one root greater than \(\rho\) and one root between 0 and \(\rho\). Since \(|x_2 - x_1| < \rho\) by definition, we need only bound \(|x_2 - x_1|\) from above by the smaller root, i.e.

\[
|x_2 - x_1| < \frac{\rho(\rho Y + \varepsilon) - \rho\sqrt{(\rho Y + \varepsilon)^2 - 4(\rho Y - M)\varepsilon}}{2(\rho Y - M)} \\
= \frac{\rho \left(\sqrt{(\rho Y - \varepsilon)^2 + 4\varepsilon M - (\rho Y + \varepsilon)}\right)}{2(\rho Y - M)}. 
\]

Third case: \(M - \rho Y = 0\). The LHS of (6) reduces to

\[
\rho(\rho Y + \varepsilon)|x_2 - x_1| - \varepsilon \rho^2 < 0 \iff |x_2 - x_1| < \frac{\varepsilon \rho}{\rho Y + \varepsilon}.
\]
This bound is asymptotically equivalent to the previous bounds as \( M \) approaches \( \rho Y \). Altogether, we thus arrive at the sufficient bound
\[
|x_2 - x_1| < \frac{\rho \sqrt{(\rho Y - \varepsilon)^2 + 4\varepsilon M - (\rho Y + \varepsilon)}}{2(M - \rho Y)}.
\] (7)

The RHS of (7) has the expected qualitative behaviour: It is strictly increasing in \( \varepsilon \) and \( \rho \), and strictly decreasing in \( M \) and \( Y \).

It remains to be shown that we can compute bounds for the ingredients \( \rho \), \( Y \), and \( M \) of (7).

Lemma 2.3 (and thus our argument) is valid if and only if \( \rho \) is smaller than the radius of convergence of the Taylor expansion of \( y_j(x) \). Therefore, we must choose \( \rho \) smaller than the distance between \( x_1 \) and the singularities of \( y_j(x) \), \( j = 1, 2, \ldots, n \). Recall that \( y_j(x) \) satisfies \( f(x, y_j(x)) = 0 \) in a neighbourhood of \( x_1 \), where
\[
f(x, y) = \sum_{k=0}^{n} a_k(x) y^{n-k}.
\]
In particular, \( \rho \) must be smaller than the distance between \( x_1 \) and the zeros of \( a_0(x) \). The zeros of \( a_0(x) \) are exactly the poles of \( y_j(x) \). The remaining finite singularities of \( y_j(x) \) are exactly the finite ramification points of \( y_j(x) \). These are zeros of the discriminant of \( f(x,y) \) w.r.t. \( y \). Hence, we may choose any
\[
\rho < \min\{|x_1 - x|: a_0(x) \cdot \Delta_y(f(x,y))(x) = 0\},
\]
where \( \Delta_y(f(x,y))(x) \) denotes the discriminant of \( f \) w.r.t. \( y \).

We can compute
\[
Y = \max_j |y_j'(x_1)| = \max_j \left| \frac{f_x(x_1, y_j(x_1))}{f_y(x_1, y_j(x_1))} \right|
\]
by Lemma 2.4. Note that the denominator does not vanish by the assumption that \( x_1 \) is not a zero of the discriminant of \( f(x,y) \) w.r.t. \( y \).

Therefore, \( M \) remains to be computed or bounded from above. To that end, we can apply Lemma 2.5 to
\[
f(x, y_j(x)) = \sum_{k=0}^{n} a_k(x) y_j(x)^k,
\]
interpreted as a polynomial in \( y_j(x) \). By our choice of \( \rho \), the leading coefficient \( a_0(x) \) does not vanish for all \( x \) with \( |x - x_1| \leq \rho \). For those \( x \) and for all \( j = 1, 2, \ldots, n \), Lemma 2.5 yields
\[
|y_j(x)| < 2 \max \left\{ \frac{|a_k(x)|}{|a_0(x)|} \right\}^{\frac{1}{k}} |k = 1, \ldots, n\}
\]
Consequently,
\[
M < 2 \max_{t \in [0,2\pi]} \left\{ \frac{|a_k(x_1 + \rho e^{it})|}{|a_0(x_1 + \rho e^{it})|} \right\}^{\frac{1}{k}} |k = 1, \ldots, n\}
\]
By Lemma 2.6, we can compute upper bounds \( \tilde{a}_k \) of \( \max_{t \in [0, 2\pi]} |a_k(x + \rho e^{it})| \) and a lower bound \( \tilde{a}_0 > 0 \) of \( \min_{t \in [0, 2\pi]} |a_0(x + \rho e^{it})| \), which are much easier to compute than these extreme values.

The zeros of \( a_0(x) \) and of \( \Delta_y(f(x, y))(x) \) can be computed (at least to arbitrary precision) using a root-finding algorithm. Similarly, the values \( y_j(x_1) \), \( j = 1, 2, \ldots, n \), can be computed (at least to arbitrary precision) by solving

\[
f(x_1, y_j(x_1)) = 0
\]

for \( y_j(x_1) \).

Let us summarize our argument: We may choose

\[
\delta = \frac{\rho \left( \sqrt{(\rho Y - \varepsilon)^2 + 4\varepsilon M - (\rho Y + \varepsilon)} \right)}{2(M - \rho Y)},
\]

where

\[
\rho < \min \{|x_1 - x| : a_0(x) \cdot \Delta_y(f(x, y))(x) = 0\},
\]

\[
Y := \max_j \left| \frac{f_x(x_1, y_j(x_1))}{f_y(x_1, y_j(x_1))} \right|, \quad M := 2 \max_k \left( \frac{\tilde{a}_k}{\tilde{a}_0} \right)^{\frac{1}{2}}.
\]

Remark 2.7. For Theorem 2.1 to hold, \( f(x, y) \) needs neither be irreducible nor square-free. However, if \( f(x, y) \) is not square-free, the discriminant may vanish identically and the epsilon-delta bound is no longer useful. If \( f(x, y) \) is square-free but not irreducible, the epsilon-delta bound for \( y \)-values on one irreducible component may be smaller than necessary due to the influence of zeros of the discriminant of other irreducible components.

3 Certified homotopy continuation of plane algebraic curves

Theorem 2.1 enables us to solve the following problem:

Problem 3.1. Consider a plane algebraic curve

\[
C: f(x, y) = 0.
\]

Let \( x: [0, 1] \to \mathbb{C}, t \mapsto x(t) \) be a monotonic (distance non-decreasing) path, i.e.

\[
|x(0) - x(t_1)| \leq |x(0) - x(t_2)| \text{ for } 0 \leq t_1 \leq t_2 \leq 1.
\]

Let \( y(0) \in \mathbb{C} \) satisfy \( f(x(0), y(0)) = 0 \). If analytic continuation of \( y \) along \( x(t) \) is possible, determine the value \( y(1) \) that results from initial value \( y(0) \) under analytic continuation of \( y \) along \( x(t) \).

The algorithm for Problem 3.1 follows from Remark 2.2

Algorithm 3.2. Let \( f(x, y), x(t), \) and \( y(0) \) be defined as in Problem 3.1

1. Let \( T = 0 \).
2. While $T < 1$,
   
   (a) Let $\varepsilon$ be half the minimum distance between the $y$ with $f(x(T), y) = 0$.
   
   (b) Compute $\delta$ by the epsilon-delta bound of Theorem 2.1.
   
   (c) Use bisection to maximize $T^* \in [T, 1]$ such that $|x(T) - x(T^*)| < \delta$.
   
   (d) Let $y(T^*)$ be the $y$ with $f(x(T^*), y) = 0$ closest to $y(T)$.
   
   (e) Let $T = T^*$.

3. Output $y(1)$ and stop.

4 Case study: continuous deformation of closed discrete Darboux transforms

Algorithm 3.2 shows how the epsilon-delta bound can be used for certified homotopy continuation of plane algebraic curves. In this section, as an example application, let us return to the closed discrete Darboux transform introduced in section 1.

We generally follow the exposition of Hoffmann (2009, Section 2.6) but use a slightly different definition of cross-ratio. (A value $\mu$ of our cross-ratio corresponds to a value $1 - \mu$ of the cross-ratio in Hoffmann, 2009, Section 2.6 and vice versa.)

Recall the definition of discrete Darboux transform:

**Definition 4.1** (discrete Darboux transform). Let $\gamma$ be a regular discrete curve in $\mathbb{CP}^1$ with vertices $\gamma_0, \gamma_1, \ldots, \gamma_n \in \mathbb{CP}^1$. We choose an initial point $\tilde{\gamma}_0 \in \mathbb{CP}^1$ and prescribe a cross-ratio $\mu \in \mathbb{C}$. The **discrete Darboux transform of $\gamma$ with initial point $\tilde{\gamma}_0$ and parameter $\mu$** is the unique discrete curve $\tilde{\gamma}$ whose vertices $\tilde{\gamma}_j, j = 1, 2, \ldots, n,$ satisfy

\[
(\gamma_{j-1}, \gamma_j; \tilde{\gamma}_j, \tilde{\gamma}_{j-1}) := \frac{(\gamma_{j-1} - \tilde{\gamma}_j)(\gamma_j - \tilde{\gamma}_{j-1})}{(\gamma_{j-1} - \tilde{\gamma}_j)(\gamma_j - \tilde{\gamma}_{j-1})} = \mu.
\]

**Lemma 4.2.** Let $a, b, d \in \mathbb{CP}^1$ be in general position. For every $\mu \in \mathbb{C}$, there exists a Möbius transformation depending on $a, b$, and $\mu$ that maps $d$ to $c \in \mathbb{CP}^1$ such that $(a, b; c, d) = \mu$.

**Proof.** Consider the Möbius transformation

\[
M: x \mapsto \frac{x - a}{x - b},
\]

which maps $a$, $b$, and $d$ to $0$, $\infty$, and $d'$ respectively. The cross-ratio is invariant under Möbius transformations. Hence, if we denote the image of $c$ under $M$ as $c'$, we want that

\[
(0, \infty; c', d') = \frac{(0 - c')(\infty - d')}{(0 - d')(\infty - c')} = \frac{c'}{d'} = \mu.
\]
We define the Möbius transformations
\[N: \, d' \mapsto e' = \mu d', \quad M^{-1}: \, x' \mapsto \frac{bx' - a}{x' - 1}.
\]
Then the Möbius transformation
\[M^{-1} \circ N \circ M: \, d \mapsto \frac{(\mu b - a)d - (\mu - 1)ab}{(\mu - 1)d + b - \mu a},\]
maps \(d \in \mathbb{CP}^1\) to \(c \in \mathbb{CP}^1\) such that \((a, b; c, d) = \mu\).

Note that \((M^{-1} \circ N \circ M)(a) = a\) and \((M^{-1} \circ N \circ M)(b) = b\), independent of \(\mu\).

**Lemma 4.3.** There exists a Möbius transformation depending on \(\gamma_0, \gamma_1, \ldots, \gamma_n\), and \(\mu\) that maps an initial point \(\tilde{\gamma}_0\) of a discrete Darboux transform of \(\gamma\) with parameter \(\mu\) to the corresponding end point \(\tilde{\gamma}_n\).

**Proof.** By Lemma 4.2, there exist Möbius transformations \(M_j\), \(j = 1, 2, \ldots, n\), depending on \(\gamma_{j-1}, \gamma_j, \) and \(\mu\) that map \(\tilde{\gamma}_{j-1}\) to \(\tilde{\gamma}_j\). Therefore their composition \(M_n \circ M_{n-1} \circ \cdots \circ M_1\) is a Möbius transformation depending on \(\gamma_0, \gamma_1, \ldots, \gamma_n\), and \(\mu\) that maps \(\tilde{\gamma}_0\) to \(\tilde{\gamma}_n\).

**Remark 4.4.** A discrete Darboux transform \(\tilde{\gamma}\) is closed if and only if its initial point \(\tilde{\gamma}_0\) is a fixed point of the Möbius transformation of Lemma 4.3. The Möbius transformation of Lemma 4.3 is of the form
\[x \mapsto \frac{ax + b}{cx + d},\]
where \(a, b, c,\) and \(d\) are polynomials in \(\mu\) with complex coefficients depending on \(\gamma_0, \gamma_1, \ldots, \gamma_n\). Its fixed points are the roots of the equation
\[(cx + d)x - (ax + b) = cx^2 + (d - a)x - b = 0.
\]
This equation is quadratic in \(x\). Its degree in \(\mu\) increases with the number of points of \(\gamma\). Equivalently, in homogeneous coordinates, the fixed points are the eigenvectors of matrix
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.
\]

**Example 4.5.** As a simple but interesting enough example, consider the closed discrete curve \(\gamma\) spanned by the fifth roots of unity,
\[\gamma_j = e^{2\pi ij/5}, \quad j = 0, 1, \ldots, 5.\]
The relationship between \(\mu\) and the initial point \(\tilde{\gamma}_0\) of a closed discrete Darboux transform \(\tilde{\gamma}\) of \(\gamma\) is governed by the equation
\[
\left[\left((-3 + \sqrt{5})\mu^2 + 6\mu - 3 - \sqrt{5}\right)\tilde{\gamma}_0^2 + \left((-2 - 4\sqrt{5})\mu + 1 + \sqrt{5}\right)\tilde{\gamma}_0 + \left(-3 + \sqrt{5}\right)\mu^2 + 6\mu - 3 - \sqrt{5}\right](1 - \mu) = 0. \tag{9}
\]
Equation (9) is quadratic in \(\tilde{\gamma}_0\), cubic in \(\mu\), and has total degree 5. For almost every value of \(\mu\), exactly two values of \(\tilde{\gamma}_0\) satisfy the equation. The only exceptions are \(\mu = 1\), where all values of \(\tilde{\gamma}_0\) satisfy the equation, and \(\mu = \frac{3 + \sqrt{5}}{8}\) and
$\mu = \infty$, which are ramification points of $\tilde{\gamma}_0$, i.e. points where there is only one value of $\tilde{\gamma}_0$, which is a root of multiplicity 2 of (9).

The discrete Darboux transform $\tilde{\gamma}$ of $\gamma$ with initial point $\tilde{\gamma}_0 = \gamma_1$ and parameter $\mu = 0$ is identical to $\gamma$ up to a rotation by $2\pi/5$, i.e.

$$\tilde{\gamma}_{j-1} = e^{2\pi i/5} \cdot \gamma_{j-1} = \gamma_j$$

for all $j = 0, 1, \ldots, n$. Particularly, the discrete curve $\tilde{\gamma}$ is closed.

We would like to examine how the closed Darboux transform $\tilde{\gamma}$ behaves when $\mu$ makes two full anticlockwise turns around the ramification point $\frac{3+\sqrt{5}}{8} + \frac{1}{1000}$.

Figure 4.6 attempts to illustrate the behaviour of the closed Darboux transform $\tilde{\gamma}$ under the aforementioned motion of $\mu$. (It is notoriously difficult to reproduce dynamic behaviour on paper. A video of the experiment is available in the supplementary material for this article.) We can read Figure 4.6 in two different ways:

Firstly, the left image shows the movement of $\tilde{\gamma}_0$ (grey points) as $\mu$ (white points) completes one full circle. Then the right image shows the movement of $\tilde{\gamma}_0$ (grey points) as $\mu$ (white points) completes another full circle. The position of the Darboux transform (black) after one turn of $\mu$ is identical to the initial position up to rotation. The final position of the Darboux transform after the second turn is absolutely identical to the initial position.

Secondly, the left image shows the movement of one choice of $\tilde{\gamma}_0$ such that $\tilde{\gamma}$ is closed as $\mu$ completes one full circle. The right image shows how the other choice of $\tilde{\gamma}_0$ such that $\tilde{\gamma}$ is closed moves at the same time. After one turn of $\mu$ we reach the initial position up to interchanged choices of $\tilde{\gamma}_0$. (In the left image, $\tilde{\gamma}_0$ moves from $\gamma_1$ to $\gamma_4$ while in the right image, $\tilde{\gamma}_0$ moves from $\gamma_4$ to $\gamma_1$.) After another turn of $\mu$, the two choices of $\tilde{\gamma}_0$ reach their initial positions again.

Following Remark 2.2, the steps of $\mu$ are chosen according to the epsilon-delta bound of Theorem 2.1 for (9) with $\varepsilon$ half the distance between the two choices of $\tilde{\gamma}_0$. As we expect, the closer $\mu$ approaches the singularity the smaller the steps become. At its rightmost position, $\mu$ is only $\frac{1}{1000}$ away from the singularity. It takes 127 steps until $\mu$ completes one full circle.
Remark 4.7. If we want to prevent jumps, $\mu$ and $\tilde{\gamma}_0$ cannot both be freely movable, i.e. we cannot let $\mu$ and $\tilde{\gamma}_0$ interchange their roles as movable and dependent point. Otherwise, we can force a jump as follows: We move parameter $\mu$ to $\mu = 1$. At the same time, according to (9), the two possible initial points of $\tilde{\gamma}$ move to the origin and the point at infinity, respectively. Without loss of generality, we assume that $\tilde{\gamma}_0$ moved to the origin. Note that $\mu = 1$ describes an irreducible component of the plane algebraic curve (9). This means that if we remove $\tilde{\gamma}_0$ from the origin, $\mu$ will simply rest at $\mu = 1$. Then we cannot move $\mu$ without a jump of $\tilde{\gamma}_0$ because in order to move continuously, $\tilde{\gamma}_0$ would initially have to be arbitrarily close to the origin (or the point at infinity).

Remark 4.8. Floating point arithmetic introduces rounding errors into the computation of $\tilde{\gamma}_0$. This has a peculiar effect: If $\tilde{\gamma}_0$ is closed, we know that $\tilde{\gamma}_0$ must be one of the two fixed points of the M"obius transformation $M$ that maps the initial point $\tilde{\gamma}_0$ of $\tilde{\gamma}$ to its last point $\tilde{\gamma}_n$. In general, a M"obius transformation has one attracting and one repelling fixed point. When the fixed point is repelling, any numerical error in its position is amplified by M"obius transformation $M$. Therefore, the closed Darboux transform may (numerically) no longer be closed when computed naively. We have observed (see Figure 4.6) that we can move from one choice of $\tilde{\gamma}_0$ to the other by moving $\mu$ around a ramification point. The natural domain for the map $c: \mu \mapsto \tilde{\gamma}_0$ is a Riemann surface. Different choices of $\tilde{\gamma}_0$ correspond to different branches of the Riemann surface. When we compute the vertices of $\tilde{\gamma}$, we step by step compute $M \circ c = M_n \circ M_{n-1} \circ \cdots \circ M_1 \circ c$. Note that function $M \circ c$ is an example of a function on a Riemann surface that is numerically stable on one branch and numerically unstable on the other.

Luckily, we can stabilize the computation by considering the inverse M"obius transformation $M^{-1}$. A repelling fixed point of a M"obius transformation is an attracting fixed point of its inverse. We can step by step compute $M^{-1} \circ c = M_1^{-1} \circ M_2^{-1} \circ \cdots \circ M_n^{-1} \circ c$ to obtain the vertices of $\tilde{\gamma}$ in reversed order. Since the cross-ratio of $A, B, C, D$ satisfies $(A, B; C, D) = (B, A; D, C)$, we need only change our algorithm very little in order to obtain $M^{-1} = M_1 \circ M_2 \circ \cdots \circ M_n$ instead of $M = M_n \circ M_{n-1} \circ \cdots \circ M_1$; we only need to reverse the order of the vertices of $\tilde{\gamma}$ before we compute the M"obius transformations.

Besides, we can determine whether $\tilde{\gamma}_0$ approximates an attracting fixed point of $M$ by considering the derivative of $M$ at $\tilde{\gamma}_0$. The fixed point near $\tilde{\gamma}_0$ is attracting if the absolute value of the derivative is smaller than 1.

5 Towards homotopy continuation of triangular systems of polynomials

In this section, we discuss a scheme for certified homotopy continuation of triangular systems of polynomials. A general implementation has remained elusive so far. However, we later follow the same scheme when we derive an algorithm for certified homotopy continuation of systems of plane algebraic curves (Algorithm 6.4).

Problem 5.1. Consider a triangular system of polynomials, without loss of
Let \( x_0(0), x_1(0), \ldots, x_n(0) \) be initial values that satisfy the system of equations and let \( x_0(1) \) be a target value for variable \( x_0 \), i.e. a value to which \( x_0 \) should move continuously. We define function \( x_0(t) \) as a parameterization of the segment between \( x_0(0) \) and \( x_0(1) \),

\[
x_0(t) = (1 - t)x_0(0) + tx_0(1).
\]

By analytic continuation w.r.t. \( t \in [0, 1] \) we can (unless there are singularities on the curves along which we perform analytic continuation) step by step define holomorphic functions \( x_1(t), x_2(t), \ldots, x_n(t) \). For example, we obtain \( x_1(t) \) from \( p_1(x_0(t), x_1(t)) = 0 \), then \( x_2(t) \) from \( p_2(x_0(t), x_1(t), x_2(t)) = 0 \), etc.

Compute the target values \( x_1(1), x_2(1), \ldots, x_n(1) \) from the given polynomial system, all initial values and the first target value.

Remark 5.2. Any algorithm for this problem has to face the following fundamental difficulty: Among all paths \( x_j(t) \) along which we perform analytic continuation to define the next path \( x_k(t) \), generally only \( x_0(t) \) is linear. The other paths \( x_1(t), x_2(t), \ldots, x_n(t) \) are almost always curvilinear—and unknown. We can at best evaluate \( x_1(t), x_2(t), \ldots, x_n(t) \) at finitely many discrete points in time and interpolate between the sample points. However, we must make sure that the approximate paths we obtain by discretization remain close enough to the actual paths such that they yield the same result w.r.t. analytic continuation. In particular, we must make sure that in every step no singularities lie between approximate and actual path. To make things worse, this includes singularities of variables that occur only in later equations, whose position in time may change depending on how we approximate the current step.

One way to attack this difficulty is to eliminate \( x_1, x_2, \ldots, x_{n-1} \) from the polynomial system \( (1) \), e.g. using resultants. However, this approach is expensive and suffers from exponential expression swell. The resulting polynomial equation in \( x_0, x_n \) very likely has a high total degree, huge coefficients, and many (artificial) critical points. This means that we can in principle apply the method for analytic continuation of plane algebraic curves of Algorithm 3.2 but that in practice it will often be too expensive (see Example 7.4). If elimination introduces artificial critical points on the path of \( x_0 \), Algorithm 3.2 does not even terminate.

Instead we pursue the following idea:

Remark 5.3 (General scheme for homotopy continuation of triangular systems). We perform homotopy continuation of one equation after another, interpolating linearly between sample points (using a time parameter in the unit interval). In order to obtain sample points on the actual paths of the variables, we synchronize the time step. This means that we let all variables make time steps of the same size. We determine a step width such that analytic continuation by proximity is possible (as in Remark 2.2), and such that the linearly interpolated
paths between consecutive sample points are equivalent to the actual paths of the variables w.r.t. analytic continuation. To fulfil the latter requirement, the step width must be small enough such that there are no singularities between linearly interpolated paths and actual paths. We cannot foresee whether linear paths and actual paths enclose singularities of variables that occur only in later equations. We must determine whether this is the case when we later analytically continue the respective variable. Should we find that we have ‘caught’ a singularity, we start over with a smaller step width. Unless there are singularities on the actual paths of variables, there is a small neighbourhood around the actual paths that is free of singularities. Eventually, after finitely many reductions of step size, the linear paths approximate the actual paths of the variables well enough such that we do not encounter singularities anymore. Then we can make one synchronized time step with all variables. We proceed until we reach time $t = 1$.

6 Certified homotopy continuation of systems of plane algebraic curves

In full generality, it may be very difficult to decide whether or not there are singularities between linearly interpolated paths and actual paths. (Among other things, we may want to ensure that the $(k-1)$-dimensional discriminant locus of variable $x_k$ w.r.t. equation $p_k(x_0, x_1, \ldots, x_k) = 0$ does not intersect the polydisc around the last sample point with radii lengths of the linear paths.) Therefore, we restrict ourselves to systems of plane algebraic curves, a special case of Problem 5.1.

Problem 6.1. Consider a system of bivariate polynomials, without loss of generality

\begin{align*}
p_1(x_0, x_1) &= 0, \\
p_2(x_1, x_2) &= 0, \\
&\vdots \\
p_n(x_{n-1}, x_n) &= 0.
\end{align*}

(11)

Let $x_0(0), x_1(0), \ldots, x_n(0)$ be initial values that satisfy the system of equations and let $x_0(1)$ be a target value. We define function $x_0(t)$ as a parameterization of the segment between $x_0(0)$ and $x_0(1)$,

$$x_0(t) = (1-t)x_0(0) + tx_0(1).$$

By analytic continuation w.r.t. $t \in [0,1]$ we can (unless there are singularities on the curves along which we perform analytic continuation) step by step define holomorphic functions $x_1(t), x_2(t), \ldots, x_n(t)$. For example, we obtain $x_1(t)$ from $p_1(x_0(t), x_1(t)) = 0$, then $x_2(t)$ from $p_2(x_1(t), x_2(t)) = 0$, etc.

Compute the target values $x_1(1), x_2(1), \ldots, x_n(1)$ from the given polynomial system, all initial values and the first target value.

Before we describe an algorithm for Problem 6.1 we need the following lemma.
Lemma 6.2. Let $C: f(x, y) = 0$, $x_1 \in \mathbb{C}$ be defined as in Theorem 2.1. Let $\varepsilon > 0$. Suppose that we have determined $\delta > 0$ by the epsilon-delta bound of Theorem 2.1 such that

$$|y_j(x_1) - y_j(x_2)| < \varepsilon$$

for all holomorphic functions $y_j(x)$ that satisfy $f(x, y_j(x)) = 0$ in a neighbourhood of $x_1$ and for all $x_2$ with $|x_1 - x_2| < \delta$.

Then for all $x_2$ with $\delta' = |x_1 - x_2| < \delta$,

$$\varepsilon' = \frac{\delta'}{\delta} \cdot \varepsilon < \varepsilon$$

satisfies

$$|y_j(x_1) - y_j(x_2)| < \varepsilon'.$$

This means that we can find a better estimate for the range of $y_j(x)$ w.r.t. an actual feasible movement of $x$ from $x_1$ to $x_2$.

**Proof.** Under the assumptions of Lemma 6.2,

$$f(x) = \frac{y_j(\delta x + x_1) - y_j(x_1)}{\varepsilon}$$

is a holomorphic function from the open unit disk to the open unit disk. By the maximum modulus principle, we know that there exists a point on the boundary of the disk of radius $\delta'$ around $x_1$ where $|y_j(x) - y_j(x_1)|$ is greater or equal than at any point $x$ with $|x - x_1| < \delta'$. Hence, there exists a point on the boundary of the disk of radius $\frac{\delta'}{\pi}$ around the origin where $|f(x)|$ is greater or equal than at any point $x$ with $|x| < \frac{\delta'}{\pi}$. Schwarz lemma states that

$$|f(x)| \leq |x|$$

for all $x$ in the open unit disk. Therefore

$$\frac{\varepsilon'}{\varepsilon} = \max_{t \in [0, 1]} |f(\delta' \cdot e^{2\pi i t})| \leq \max_{t \in [0, 1]} \left| \frac{\delta'}{\delta} \cdot e^{2\pi i t} \right| = \frac{\delta'}{\delta},$$

and thus

$$\varepsilon' \leq \frac{\delta'}{\delta} \cdot \varepsilon$$

for all $x_2$ with $|x_1 - x_2| < \delta' < \delta$. \qed

Remark 6.3. Alternatively, if we plug in $\delta = \delta'$ and $\varepsilon = \varepsilon'$ into (8) and solve for $\varepsilon'$, we obtain

$$\varepsilon' = \delta' \left( \tilde{y} + \frac{M \delta'}{\rho(\rho - \delta')} \right) < \varepsilon,$$

with $M, \rho, \tilde{y}$ as in the proof of Theorem 2.1. This yields another better estimate for the range of $y_j(x)$ w.r.t. an actual feasible movement of $x$ from $x_1$ to $x_2$.

**Algorithm 6.4.** Consider the system of bivariate polynomials of Problem 6.1 with initial values $x_0(0), x_1(0), \ldots, x_n(0)$ and a target value $x_0(1)$.

1. Define $x_0(t) = (1 - t)x_0(0) + tx_0(1)$.  

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2. Let \( T = 1 \).

3. Let \( \varepsilon'_k = |x_0(0) - x_0(T)| \).

4. For all \( k = 1, 2, \ldots, n \):
   
   (a) Let \( \varepsilon_k \) be half the minimum distance between the \( x_k \) with \( p_k(x_{k-1}(0), x_k) = 0 \).

   (b) Compute \( \delta_k \) according to the epsilon-delta bound of Theorem 2.1 for \( f(x, y) = p_k(x_{k-1}, x_k), x_1 = x_{k-1}(0) \) and \( \varepsilon = \varepsilon_k \).

   (c) If \( \delta_k < \varepsilon'_k \) then let \( T = T/2 \) and go to 3.

   (d) Let \( x_k(T) \) be the \( x_k \) with \( p_k(x_{k-1}(T), x_k) = 0 \) closest to \( x_k(0) \).

   (e) Let \( \delta'_k = |x_{k-1}(0) - x_{k-1}(T)| \).

   (f) Let \( \varepsilon'_k \) be \( (\delta'_k + \varepsilon)/\delta \cdot \varepsilon_k \) with \( \epsilon > 0 \).

5. If \( T = 1 \) then output \( x_1(T), x_2(T), \ldots, x_n(T) \) and stop.

6. Let \( x_0(0) = x_0(T), x_1(0) = x_1(T), \ldots, x_n(0) = x_n(T) \) and go to 1.

**Theorem 6.5.** If the target values \( x_1(1), x_2(1), \ldots, x_n(1) \) of Problem 6.1 are well-defined, Algorithm 6.4 computes them in finitely many steps.

**Proof.** The first two steps of Algorithm 6.4 are initialization steps. In step 1, we define a linear homotopy between initial value \( x_0(0) \) and final value \( x_0(1) \) of \( x_0 \). We first want to test whether we can perform analytic continuation of the system in a single time step. Therefore, in step 2, we set target time \( T = 1 \).

Steps 3–6 form the main loop of our algorithm. They are repeated until we reach time \( T = 1 \), in which case step 5 terminates the algorithm.

In step 3, we estimate the range of \( x_0 \) as it runs from its initial position \( x_0(0) \) to its target position \( x_0(T) \). Since \( x_0(t) \) is linear by definition (step 1), our estimate \( \varepsilon'_0 = |x_0(0) - x_0(T)| \) is exact.

Step 4 is the inner loop of our algorithm, in which we try to perform analytic continuation equation by equation of our system. Run variable \( k \) denotes the index of the equation \( p_k(x_{k-1}, x_k) \) under consideration.

In steps 4a–4b, we use the epsilon-delta bound of Theorem 2.1 and Remark 2.2 to compute a feasible step width \( \delta_k \) for variable \( x_{k-1} \). If \( x_{k-1} \) moves at most \( \delta_k \) then we can perform analytic continuation of \( x_k \) w.r.t. \( p_k(x_{k-1}, x_k) = 0 \) by selecting as \( x_k(T) \) the value of \( x_k \) with \( p_k(x_{k-1}(T), x_k) = 0 \) closest to \( x_k(0) \).

Hence, in step 4c, we test whether feasible step \( \delta_k \) is smaller than an upper bound \( \varepsilon'_{k-1} \) of the range of \( x_{k-1} \) as it runs from \( x_{k-1}(0) \) to \( x_{k-1}(T) \).

If \( \delta_k < \varepsilon'_{k-1}, \) we cannot be sure that there are no singularities between the actual path of \( x_{k-1} \) and the interpolated path, i.e. the segment from \( x_{k-1}(0) \) to \( x_{k-1}(T) \). Our attempt to reach target time \( T \) in one step has failed. Therefore, we halve target time \( T \) and go back to step 3.

Otherwise, if \( \delta_k \geq \varepsilon'_{k-1}, \) the epsilon-delta bound of Theorem 2.1 guarantees that actual path and interpolated path of \( x_{k-1} \) are equivalent w.r.t. analytic continuation of \( x_k \). Then in step 4d, we determine target value \( x_k(T) \). By construction, \( x_k(T) \) is a point on the actual path of \( x_k \).
In steps 4e–4f, we use [Lemma 6.2] to compute an upper bound for the range of $x_k$ as it runs from $x_k(0)$ to $x_k(T)$. The computation is independent of whether $x_{k-1}$ runs along actual or interpolated path. The bound $c'_k$ holds for both paths, particularly also for analytic continuation of $x_k$ along the actual path of $x_{k-1}$.

We then proceed with analytic continuation of the next variable, if any. When we leave the inner loop (step 4), we obtain valid positions for $x_1, x_2, \ldots, x_n$ at target time $T$. If $T = 1$, we output the solution and stop (step 5). Otherwise, we use $x_0(T), x_1(T), \ldots, x_n(T)$ as a valid initial position from which we again try to reach target time $T = 1$ (step 6).

By the assumption that $x_1(1), x_2(1), \ldots, x_n(1)$ are well-defined, there are only finitely many singularities in a neighbourhood of the actual paths of $x_0, x_1, \ldots, x_n$. The algorithm terminates after finitely many steps as eventually the interpolated paths of $x_1, x_2, \ldots, x_n$ approximate the actual paths well enough such that we do not encounter singularities anymore.

7 Comparison with other approaches

Let us discuss more examples, which allow us to compare the performance of our algorithm with that of other approaches. (The number of steps needed by Algorithm 3.2 and Algorithm 6.4 stated below relate to an experimental implementation in Haskell that is available in the supplementary material for this article.)

Example 7.1 ([Hauenstein et al. (2014], Section 7.1]). Consider the Newton homotopy

$$H(x, t) = f(x) + vt$$

where $f(x) = x^2 - 1 - m$ and $v = m$ for various values of $m > -1$. The goal is to analytically continue $x$ as $t$ moves from 1 to 0.

In Table 7.2 and Table 7.3, we compare the performance of Algorithm 3.2 with that of the algorithms of Beltrán and Leykin (2013) and Hauenstein et al. (2014), for various values of $m$. The data for the latter algorithms is quoted from Hauenstein et al. (2014, Table 1 and Table 2). Both Beltrán and Leykin (2013) and Hauenstein et al. (2014) present algorithms designed for certified homotopy continuation of arbitrary polynomial systems whereas Algorithm 3.2 can only deal with plane algebraic curves. However, the example indicates that in the univariate case Algorithm 3.2 may perform much better than those more general algorithms, which do not exploit the special structure of the univariate case.

Furthermore, let us elaborate on Remark 5.2. The following example shows that it may be better to apply Algorithm 6.4 to a system of plane algebraic curves than to eliminate variables and apply Algorithm 3.2 to the resultant.

Example 7.4. Consider the system of bivariate polynomials

$$p_1(x_0, x_1) = -4 + 2x_0 + x_1 + 2x_0x_1 + x_1^2 = 0,$$

$$p_2(x_1, x_2) = x_1^2 + x_2^2 = 0,$$

with initial values

$$x_0(0) = 0, \quad x_1(0) = \frac{-1 - \sqrt{17}}{2}, \quad x_2(0) = \left(\frac{-9 - \sqrt{17}}{2}\right)^\frac{1}{4},$$

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| $m$ | Number of steps of Algorithm 3.2 | Number of a priori steps of Beltrán and Leykin (2013) | Number of a posteriori certified intervals of Hauenstein et al. (2014) |
|-----|---------------------------------|--------------------------------------------------|--------------------------------------------------|
| 10  | 9                               | 184                                             | 51                                               |
| 20  | 12                              | 217                                             | 67                                               |
| 30  | 14                              | 237                                             | 78                                               |
| 40  | 16                              | 250                                             | 82                                               |
| 50  | 17                              | 260                                             | 88                                               |
| 60  | 18                              | 269                                             | 92                                               |
| 70  | 19                              | 276                                             | 96                                               |
| 80  | 20                              | 282                                             | 99                                               |
| 90  | 21                              | 288                                             | 103                                              |
| 100 | 21                              | 292                                             | 105                                              |
| 1000| 41                              | 395                                             | 162                                              |
| 2000| 49                              | 426                                             | 180                                              |
| 3000| 54                              | 446                                             | 191                                              |
| 4000| 58                              | 457                                             | 197                                              |
| 5000| 62                              | 468                                             | 204                                              |
| 6000| 73                              | 499                                             | 220                                              |
| 7000| 87                              | 530                                             | 238                                              |
| 8000| 96                              | 547                                             | 250                                              |

Table 7.2: Performance of Algorithm 3.2 in comparison with the algorithms of Beltrán and Leykin (2013) and Hauenstein et al. (2014), for various values of $m$. The data in the last two columns is quoted from Hauenstein et al. (2014, Table 1).

| $k$ | Number of steps of Algorithm 3.2 | Number of a priori steps of Beltrán and Leykin (2013) | Number of a posteriori certified intervals of Hauenstein et al. (2014) |
|-----|---------------------------------|--------------------------------------------------|--------------------------------------------------|
| 1   | 5                               | 176                                             | 64                                               |
| 2   | 9                               | 287                                             | 68                                               |
| 3   | 14                              | 390                                             | 70                                               |
| 4   | 18                              | 492                                             | 71                                               |
| 5   | 22                              | 593                                             | 71                                               |
| 6   | 27                              | 695                                             | 71                                               |
| 7   | 31                              | 798                                             | 71                                               |
| 8   | 36                              | 901                                             | 71                                               |
| 9   | 40                              | 1003                                            | 71                                               |
| 10  | 44                              | 1108                                            | 71                                               |

Table 7.3: Performance of Algorithm 3.2 in comparison with the algorithms of Beltrán and Leykin (2013) and Hauenstein et al. (2014), for various values of $m = -1 + 10^{-k}$. The data in the last two columns is quoted from Hauenstein et al. (2014, Table 2).
and target value \( x_0(1) = 1 \). The \( x_1 \)-resultant of \( p_1(x_0, x_1) \) and \( p_2(x_1, x_2) \) is
\[
q(x_0, x_2) = 16 - 16x_0 + 4x_0^2 + 9x_2 + 4x_0^2x_2 + x_2^2 = 0. \tag{13}
\]

Let us compare the performance of Algorithm 6.4 for (12) with the performance of Algorithm 3.2 for (13) as \( x_0 \) moves linearly (in the unit interval) from 0 to 1. We find that Algorithm 6.4 subdivides once, i.e. it needs two steps. In contrast, Algorithm 3.2 needs six steps. One possible explanation is that \( x_0 = -\frac{1}{2} \) is a singularity of (13) but not of (12). For \( x_0 = -\frac{1}{2} \), (13) has three zeros of multiplicity two, whereas (12) has six simple roots. Each zero of multiplicity two of (13) corresponds to two simple zeros of (12) with differing signs of \( x_1 \).

Generally, elimination introduces artificial singularities. Due to an artificial singularity it can even happen that we cannot analytically continue the resultant: For example, the \( x_1 \)-resultant of
\[
\tilde{p}_1(x_0, x_1) = -4 + 2x_0 + x_2 - 2x_0x_1 + x_1^2 = 0,
\]
\[
p_2(x_1, x_2) = x_1^2 + x_2^2 = 0,
\]
has an artificial singularity at \( x_0 = \frac{1}{2} \). In this case, Algorithm 3.2 does not terminate whereas Algorithm 6.4 produces the desired result.

8 Conclusion

From an epsilon-delta bound for plane algebraic curves [Theorem 2.1], we have derived algorithms for certified homotopy continuation of plane algebraic curves (Algorithm 3.2) and systems of plane algebraic curves (Algorithm 6.4). Our certificate is rigorous for exact real arithmetic. For floating point arithmetic, [Theorem 2.1] can be considered a soft certificate. Several examples demonstrate the practicability of our approach.

A generalization of [Algorithm 6.4] to arbitrary systems of polynomials might be an interesting challenge for further research.

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