Mean-Field Linear-Quadratic Stochastic Differential Games

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Abstract. The paper is concerned with two-person zero-sum mean-field linear-quadratic stochastic differential games over finite horizons. By a Hilbert space method, a necessary condition and a sufficient condition are derived for the existence of an open-loop saddle point. It is shown that under the sufficient condition, the associated two Riccati equations admit unique strongly regular solutions, in terms of which the open-loop saddle point can be represented as a linear feedback of the current state. When the game only satisfies the necessary condition, an approximate sequence turns out to be equivalent to the open-loop solvability of the game, and the limit is exactly an open-loop saddle point, provided that the game is open-loop solvable.

Keywords. linear-quadratic differential game, mean-field stochastic differential equation, two-person, zero-sum, open-loop saddle point, Riccati equation, closed-loop representation, perturbation approach.

AMS subject classifications. 91A15, 93E20, 49N10, 49N70.

1 Introduction

Let $\Omega, \mathcal{F}, \mathbb{P}$ be a complete probability space on which a standard one-dimensional Brownian motion $W = \{W(t) : 0 \leq t < \infty\}$ is defined. The augmented natural filtration of $W$ is denoted by $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$. Consider the following controlled linear mean-field stochastic differential equation (MF-SDE, for short) on a finite horizon $[0, T]$:

\[
\begin{align*}
\text{d}X(s) &= \left\{ A(s)X(s) + \bar{A}(s)\mathbb{E}[X(s)] + B_1(s)u_1(s) + \bar{B}_1(s)\mathbb{E}[u_1(s)] \\
&\quad + B_2(s)u_2(s) + \bar{B}_2(s)\mathbb{E}[u_2(s)] \right\} \text{d}s + \left\{ C(s)X(s) + \bar{C}(s)\mathbb{E}[X(s)] \\
&\quad + D_1(s)u_1(s) + \bar{D}_1(s)\mathbb{E}[u_1(s)] + D_2(s)u_2(s) + \bar{D}_2(s)\mathbb{E}[u_2(s)] \right\} \text{d}W(s),
\end{align*}
\]

(1.1)

where $A, \bar{A}, C, \bar{C} : [0, T] \to \mathbb{R}^{n \times n}$, $B_i, \bar{B}_i, D_i, \bar{D}_i : [0, T] \to \mathbb{R}^{n \times m_i}$ ($i = 1, 2$), called the coefficients of the state equation (1.1), are given deterministic functions. The solution $X$ of (1.1) is called a state process, and $u_i$ ($i = 1, 2$), belonging to the space

\[
U_i = \left\{ \varphi : [0, T] \times \Omega \to \mathbb{R}^{m_i} \mid \text{$\varphi$ is $\mathbb{F}$-progressively measurable, } \mathbb{E} \int_0^T |\varphi(s)|^2 \text{d}s < \infty \right\},
\]

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is called the control process of Player \(i\). To measure the performance of the controls \(u_1\) and \(u_2\), we introduce the following functional:

\[
J(x; u_1, u_2) = \mathbb{E}\left\{ (GX(T), X(T)) + (\bar{G}E[X(T)], E[X(T)]) \right\}
\]

\[
+ \int_0^T \begin{pmatrix} \bar{Q} & S_1^T & S_2^T \\ S_1 & R_{11} & R_{12} \\ S_2 & R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} X_1 \\ u_1 \\ u_2 \end{pmatrix} ds
\]

\[
+ \int_0^T \begin{pmatrix} \bar{Q} & \bar{S}_1^T & \bar{S}_2^T \\ \bar{S}_1 & \bar{R}_{11} & \bar{R}_{12} \\ \bar{S}_2 & \bar{R}_{21} & \bar{R}_{22} \end{pmatrix} \begin{pmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[u_1] \\ \mathbb{E}[u_2] \end{pmatrix} \begin{pmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[u_1] \\ \mathbb{E}[u_2] \end{pmatrix} ds
\}

(1.2)

where \(G\) and \(\bar{G}\) are \(n \times n\) symmetric matrices; \(Q, \bar{Q} : [0, T] \rightarrow \mathbb{R}^{n \times n}\), \(S_i : [0, T] \rightarrow \mathbb{R}^{m_i \times n}\), and \(R_{ij}, \bar{R}_{ij} : [0, T] \rightarrow \mathbb{R}^{m_i \times m_j}\) \((i, j = 1, 2)\) are deterministic functions with \(Q = Q^T\), \(\bar{Q} = \bar{Q}^T\), \(R_{ij} = R_{ji}^T\) and \(\bar{R}_{ij} = \bar{R}_{ji}^T\) \((i, j = 1, 2)\). In the Lebesgue integral on the right-hand side of (1.2), we have suppressed the argument \(s\), and we will do so in the sequel as long as no ambiguity arises.

The functional \(J(x; u_1, u_2)\) represents the cost of Player 1 and the payoff of Player 2 for using \(u_1\) and \(u_2\) to control the state process that starts from \(x\). Naturally, in this two-person zero-sum mean-field linear-quadratic stochastic differential game (Problem (MF-SG), for short), Player 1 wishes to minimize (1.2) by selecting his/her control from \(U_1\), and Player 2 wishes to maximize (1.2) by selecting his/her control from \(U_2\). The control pair \((u_1^*, u_2^*)\) acceptable to both players is called an open-loop saddle point of Problem (MF-SG), which is mathematically defined by the following inequalities:

\[
J(x; u_1^*, u_2^*) \leq J(x; u_1^*, u_2^*) \leq J(x; u_1, u_2^*), \quad \forall (u_1, u_2) \in U_1 \times U_2.
\]

(1.3)

From (1.3) one sees that if one of the players keeps his/her control \(u_s^*\) unchanged, the other cannot benefit by changing his/her control. In this sense, an open-loop saddle point (if exists) will be the best choice for both players.

When \(\bar{A}, \bar{B}_i, \bar{C}, \bar{D}_i, \bar{G}, \bar{S}, \bar{T}_{ij}\) and \(\bar{R}_{ij}\) \((i, j = 1, 2)\) all vanish, Problem (MF-SG) reduces to the classical two-person zero-sum linear-quadratic (LQ, for short) stochastic differential game (Problem (SG), for short), which has been studied for a long history and is widely applied in engineering, economy, and biology, etc. Since the purpose of the paper is not to make a lengthy survey on the literature, we only list here some closely related works (see, e.g., [6, 38, 21, 9, 10, 28, 37]) and refer the reader to the book [29] by Sun–Yong for more details and references cited therein. It is particularly worthy to mention that in a recent paper [24] by Sun, the strongly regular solvability of the Riccati equation associated with Problem (SG) is established under the so-called uniform convexity-concavity condition. This result brings new insights into the two-person zero-sum LQ stochastic differential game and serves as a foundation for our study on Problem (MF-SG).

Mean-field stochastic optimal control problems, which can be regarded as special cases of the mean-field stochastic differential game in the sense that one is interested in a single decision maker, have also attracted a lot of attention; see, for example, [1, 2, 7, 22, 19, 4]. The mean-field LQ stochastic optimal control problem was initially studied by Yong [33] and was later generalized by Huang–Li–Yong [13], Yong [34], Sun [23], Li–Sun–Xiong [17], and Sun–Wang [26] to various cases. Let us briefly recall the motivation for studying mean-field LQ stochastic optimal control problems proposed by Yong [33]. In some cases, one hopes that the optimal state process and/or control process could be not too sensitive with respect to the possible variation of the random events. To achieve this, one needs to keep the variances \(\text{var}[X]\) and \(\text{var}[u]\) small. Therefore, it is natural to take \(\text{var}[X]\) and \(\text{var}[u]\) into account and consider cost functionals of the form

\[
J(x; u) = \mathbb{E}\{ (GX(T), X(T)) + g\text{var}[X(T)] + \int_0^T [(Q(s)X(s), X(s))] ds \}
\]
In particular, in the mean-variance model (see [39], for example), the cost functional is simply 
\[ \mu \text{var} [X(T)] - \mathbb{E}[X(T)] \]. Note that 
\[ \text{var} [X(s)] = \mathbb{E}[[X(s)]^2] - (\mathbb{E}[X(s)])^2, \quad \text{var} [u(s)] = \mathbb{E}[|u(s)|^2] - (\mathbb{E}[u(s)])^2. \]

The control problem with functional (1.4) is actually a mean-field LQ control problem, due to the presence of \((\mathbb{E}|X(s)|^2)^2\) and \((\mathbb{E}|u(s)|^2)^2\). Another motivation for studying such type of problems is that the mean-field SDEs, also called McKean–Vlasov SDEs, can be used to describe particle systems at the mesoscopic level. Recently, the mean-field/McKean–Vlasov SDE has been wildly used in mean-field game theory. In general, it is the mean-square limit of the system of interacting particles (see [2, 7], for example). For more details of such type of motivations, we refer the reader to Huang–Malhâmé–Caines [14], Lasry–Lions [15], Bensoussan–Frehse–Yam [4], Carmona–Delarue [8], and the references cited therein. Along with the development of mean-field LQ optimal control problems, the LQ differential games for mean-field SDEs have also attracted extensive research, among which, we would like to mention Bensoussan–Sun–Yam–Yung [5], Graber [11], Barreiro-Gomez–Duncan–Tembine [3], Li–Shi–Yong [16], Moon [20], and Tian–Yu–Zhang [30].

For mean-field LQ control problems, two Riccati equations are derived by Yong [33] to construct an open-loop optimal control. The solvability of these two Riccati equations is established in [33] under certain positivity conditions and is further shown to be equivalent to the uniform convexity of the cost functional by Sun [23]. However, to our best knowledge, there are few significant results on the general solvability of the Riccati equations associated with Problem (MF-SG) so far. One of the main contributions of this paper is to fill up this gap. Compared with Sun [24], the additional difficulty mainly comes from the solvability of the second Riccati equation associated with Problem (MF-SG). To overcome this difficulty, besides establishing a technical Lemma 2.3, we also show that the solution of the first Riccati equation satisfies a comparison property (i.e., inequality (4.7) in Theorem 4.2). This property follows from the following observations: One control in the saddle point of Problem (SG) is optimal for a backward stochastic LQ control problem and the value function of this backward problem is given exactly in terms of the solution to the first Riccati equation. This observation is interesting in its own right and to our best knowledge, it is completely new in the literature.

In the literature, forward-backward stochastic differential equations (FBSDEs, for short) are usually used to characterize the open-loop solvability of Problem (MF-SG) (see, for example, [28, 29]). This method is suitable for deciding whether a control pair is an open-loop saddle point or not, but not very effective in constructing open-loop saddle points, because the associated Riccati equations might be not solvable and then the optimality system cannot be decoupled. This paper provides an alternative characterization (see Theorem 5.1) for the open-loop solvability of Problem (MF-SG) by a perturbation approach, which can be regarded as another important contribution. It is worthy to point out that the characterization is new even for Problem (SG) (in which there are no mean-field terms present).

The idea is to add two terms, \( \varepsilon \|u_1\|^2 \) and \( -\varepsilon \|u_2\|^2 \), to the original functional so that the Problem (MF-SG) with the new functional \( J_\varepsilon(x; u_1, u_2) \triangleq J(x; u_1, u_2) + \varepsilon \|u_1\|^2 - \varepsilon \|u_2\|^2 \) admits a unique open-loop point \((u_1^\varepsilon, u_2^\varepsilon)\) that can be represented explicitly in terms of the solutions to the associated Riccati equations. Then using the boundedness/convergence of the family \( \{(u_1^\varepsilon, u_2^\varepsilon)\}_{\varepsilon>0} \) to justify the open-loop solvability of the original game. The main difficulty here is that the value function \( V_\varepsilon(x) \) of the perturbed game is not monotone in \( \varepsilon \), due to which the technique used in the LQ control problem (see [25]) cannot be applied directly. The significant difference between the perturbation methods of controls and games is illustrated by presenting an elaborate example (see Example 5.3). To overcome the difficulty, we restate the perturbation approach by a Hilbert space method, which helps us to change the boundedness problem of \( \{(u_1^\varepsilon, u_2^\varepsilon)\}_{\varepsilon>0} \) into an equivalent
one: the norm estimate of some perturbed operators with special structures (see Proposition 2.4). Furthermore, it is found that the explicit upper bound estimate (2.5) in Proposition 2.4 also plays a crucial role in proving the strong convergence of \( \{(u^1_\varepsilon, u^2_\varepsilon)\}_{\varepsilon > 0} \), because in Theorem 5.1 we hope to show that \( \{(u^1_\varepsilon, u^2_\varepsilon)\}_{\varepsilon > 0} \) itself is strongly convergent when Problem (MF-SG) is open-loop solvable.

To summarize, we list the main contributions of the paper as follows.

1. The open-loop solvability of Problem (MF-SG) is studied by a Hilbert space method. A necessary and sufficient condition for the existence of an open-loop saddle point is derived (see Proposition 3.1).

2. Under the uniform convexity-concavity condition, the strongly regular solvability of the Riccati equations associated with Problem (MF-SG) is established (see Theorem 4.2 and Theorem 4.4). Further, in terms of the solutions to the Riccati equations, a closed-loop representation of the unique open-loop saddle point is obtained (see Theorem 4.6).

3. Under a necessary condition for the existence of an open-loop saddle point, an equivalent characterization of the open-loop solvability is established by a perturbation approach. This approach also provides an explicit procedure for finding open-loop saddle points (see Theorem 5.1).

In other words, under the uniform convexity-concavity condition, we first extend the results obtained in Sun [24] to the mean-field system. As explained before, to prove the solvability of the associated Riccati equations, we need to make some new observations and to overcome some new difficulties. Then under the weaker convexity-concavity condition, we develop a perturbation approach to characterize the open-loop solvability of Problem (MF-SG). This approach is first established for the game problem and can be regarded as the most technical part in the paper.

The rest of the paper is organized as follows. Section 2 collects some preliminary results. Section 3 is devoted to the study of the performance functional from a Hilbert space point of view. Section 4 establishes the solvability of the associated Riccati equations and provides a closed-loop representation of the open-loop saddle point. Section 5 investigates the open-loop solvability of Problem (MF-SG) by a perturbation method. An example is presented in Section 6 to illustrate the results obtained in previous sections.

### 2 Preliminaries

Throughout this paper, let \( \mathbb{R}^{n \times m} \) be the Euclidean space consisting of \( n \times m \) real matrices, endowed with the Frobenius inner product \((M, N) \triangleq \text{tr}[M^T N]\), where \( M^T \) and \( \text{tr}(M) \) stand for the transpose and the trace of \( M \), respectively. The norm of a matrix \( M \) induced by the Frobenius inner is denoted by \( |M| \) and the identity matrix of size \( n \) is denoted by \( I_n \). Let \( S^n \) be the subspace of \( \mathbb{R}^{n \times n} \) consisting of symmetric matrices and \( S^n_+ \) be the subset of \( S^n \) consisting of positive semidefinite matrices. For any Euclidean space \( \mathbb{H} \) (which could be \( \mathbb{R}^n, \mathbb{R}^{n \times m}, S^n, \) etc.), we introduce the following spaces:

- \( C([0, T]; \mathbb{H}) \) : the space of \( \mathbb{H} \)-valued, continuous functions on \([0, T]\);
- \( L^\infty(0, T; \mathbb{H}) \) : the space of \( \mathbb{H} \)-valued, essentially bounded functions on \([0, T]\);
- \( L^2_F(\Omega; \mathbb{H}) \) : the space of \( \mathcal{F}_T \)-measurable, \( \mathbb{H} \)-valued random variables \( \xi \) such that \( \mathbb{E}|\xi|^2 < \infty \);
- \( L^2(0, T; \mathbb{H}) \) : the space of \( \mathcal{F} \)-progressively measurable, \( \mathbb{H} \)-valued processes \( \varphi : [0, T] \times \Omega \to \mathbb{H} \) with \( \mathbb{E}\int_0^T |\varphi(s)|^2 ds < \infty \);
- \( L^2(\Omega; C([0, T]; \mathbb{H})) \) : the space of \( \mathcal{F} \)-adapted, continuous, \( \mathbb{H} \)-valued processes
We denote the norm of the Banach space $X$ by $\| \cdot \|_X$, which is often simply written as $\| \cdot \|$ when no confusion occurs. For $M, N \in \mathbb{S}^n$, we use the notation $M \succeq N$ (respectively, $M > N$) to indicate that $M - N$ is positive semidefinite (respectively, positive definite). For any $\mathbb{S}^n$-valued measurable function $F$ on $[0, T]$, we denote

$$\varphi : [0, T] \times \Omega \to \mathbb{H}$$

with

$$E \left[ \sup_{s \in [0, T]} |\varphi(s)|^2 \right] < \infty.$$  

For any

$$L$$

for the initial state $x$.

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$$E \left[ \sup_{s \in [0, T]} |\varphi(s)|^2 \right] < \infty.$$  

For any

$$L$$

for the initial state $x$.
The following result is concerned with bounded linear operators, by which we shall develop a perturbation approach for the open-loop solvability of Problem (MF-SG) in Section 5. Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two real Hilbert spaces. Let 

\[ \mathcal{M}_{ij} : \mathcal{H}_j \to \mathcal{H}_i, \quad i, j = 1, 2 \]

be linear bounded operators with \( \mathcal{M}_{ji} = \mathcal{M}_{ij}^* \), where \( \mathcal{M}_{ij}^* \) denotes the adjoint operator of \( \mathcal{M}_{ij} \). Then

\[ \mathcal{M} \triangleq \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix} \]

is a self-adjoint linear bounded operator on the product Hilbert space \( \mathcal{H}_1 \times \mathcal{H}_2 \) equipped with the inner product

\[ \langle (x_1, y_1), (x_2, y_2) \rangle \triangleq \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle, \quad \forall x_1, x_2 \in \mathcal{H}_1, \ y_1, y_2 \in \mathcal{H}_2. \]

**Proposition 2.4.** Suppose that \( \mathcal{M}_{11} \) and \( -\mathcal{M}_{22} \) are positive operators; that is \( \mathcal{M}_{11} \geq 0 \) and \( -\mathcal{M}_{22} \geq 0 \). Then for any \( \varepsilon > 0 \),

\[ \mathcal{M}_\varepsilon \triangleq \begin{pmatrix} \mathcal{M}_{11} + \varepsilon I & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} - \varepsilon I \end{pmatrix} \]

is invertible. Moreover,

\[ \| \mathcal{M}_{\varepsilon}^{-1} \mathcal{M} \| \leq 1, \quad \forall \varepsilon > 0. \]  \hspace{1cm} (2.5)

**Proof.** Since \( \mathcal{M}_{11} + \varepsilon I \geq \varepsilon I \), \( \mathcal{M}_{11,\varepsilon} \triangleq \mathcal{M}_{11} + \varepsilon I \) is invertible with \( \| \mathcal{M}_{11,\varepsilon}^{-1} \| \leq \varepsilon^{-1} \). Similarly, \( \mathcal{M}_{22,\varepsilon} \triangleq \mathcal{M}_{22} - \varepsilon I \) is invertible with \( \| \mathcal{M}_{22,\varepsilon}^{-1} \| \leq \varepsilon^{-1} \), and the self-adjoint operator

\[ \Phi_\varepsilon \triangleq \mathcal{M}_{22,\varepsilon} - \mathcal{M}_{21,\varepsilon}^* \mathcal{M}_{12} \equiv \mathcal{M}_{22,\varepsilon} - \mathcal{M}_{12}^* \mathcal{M}_{11,\varepsilon}^{-1} \mathcal{M}_{12} \]

is invertible with \( \| \Phi_\varepsilon^{-1} \| \leq \varepsilon^{-1} \). Now it is straightforward to verify that \( \mathcal{M}_\varepsilon \) is invertible with inverse

\[ \mathcal{M}_\varepsilon^{-1} = \begin{pmatrix} \mathcal{M}_{11,\varepsilon}^{-1} + \mathcal{M}_{11,\varepsilon}^{-1} \mathcal{M}_{12} \Phi_\varepsilon^{-1} (\mathcal{M}_{11,\varepsilon}^{-1} \mathcal{M}_{12})^* & -\mathcal{M}_{11,\varepsilon}^{-1} \mathcal{M}_{12} \Phi_\varepsilon^{-1} \\ -\Phi_\varepsilon^{-1} (\mathcal{M}_{11,\varepsilon}^{-1} \mathcal{M}_{12})^* \end{pmatrix}. \]

To prove (2.5), we write

\[ \mathcal{M}_\varepsilon^{-1} \mathcal{M} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \varepsilon \mathcal{M}_\varepsilon^{-1} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \varepsilon \begin{pmatrix} \mathcal{M}_{11,\varepsilon} & \mathcal{M}_{12} \\ -\mathcal{M}_{12}^* & -\mathcal{M}_{22,\varepsilon} \end{pmatrix}^{-1}. \]

Denote

\[ \hat{\mathcal{M}}_\varepsilon \triangleq \begin{pmatrix} \mathcal{M}_{11,\varepsilon} & \mathcal{M}_{12} \\ -\mathcal{M}_{12}^* & -\mathcal{M}_{22,\varepsilon} \end{pmatrix}. \]  \hspace{1cm} (2.6)

Then

\[ \mathcal{M}_\varepsilon^{-1} \mathcal{M} = I - \varepsilon \hat{\mathcal{M}}_\varepsilon^{-1}, \]  \hspace{1cm} (2.7)

and thus

\[ (\mathcal{M}_\varepsilon^{-1} \mathcal{M})^* \mathcal{M}_\varepsilon^{-1} \mathcal{M} = (I - \varepsilon \hat{\mathcal{M}}_\varepsilon^{-1})^* (I - \varepsilon \hat{\mathcal{M}}_\varepsilon^{-1}) \]

\[ = I - \varepsilon (\hat{\mathcal{M}}_\varepsilon^{-1})^* - \varepsilon \hat{\mathcal{M}}_\varepsilon^{-1} + \varepsilon^2 (\hat{\mathcal{M}}_\varepsilon^{-1})^* \hat{\mathcal{M}}_\varepsilon^{-1} \]

\[ = I - \varepsilon (\hat{\mathcal{M}}_\varepsilon^{-1})^* [\hat{\mathcal{M}}_\varepsilon + \hat{\mathcal{M}}_\varepsilon^* - \varepsilon I] \hat{\mathcal{M}}_\varepsilon^{-1}. \]  \hspace{1cm} (2.8)
Note that
\[ \hat{M}_\varepsilon + \hat{M}_\varepsilon^* - \varepsilon I = \begin{pmatrix} M_{11,\varepsilon} & M_{12} \\ -M_{12} & -M_{22,\varepsilon} \end{pmatrix} + \begin{pmatrix} M_{11,\varepsilon} & -M_{12} \\ M_{12} & -M_{22,\varepsilon} \end{pmatrix} - \varepsilon \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \]
\[ = \begin{pmatrix} 2M_{11} + \varepsilon I & 0 \\ 0 & -2M_{22} + \varepsilon I \end{pmatrix}. \]

Using the fact that \( M_{11} \) and \( -M_{22} \) are positive operators, we have
\[ \begin{pmatrix} 2M_{11} + \varepsilon I & 0 \\ 0 & -2M_{22} + \varepsilon I \end{pmatrix} \geq 0; \]
that is
\[ \hat{M}_\varepsilon + \hat{M}_\varepsilon^* - \varepsilon I \geq 0, \]
which implies that
\[ \varepsilon (\hat{M}_\varepsilon^{-1})^* [\hat{M}_\varepsilon + \hat{M}_\varepsilon^* - \varepsilon I] \hat{M}_\varepsilon^{-1} \geq 0. \]
Combining the above with (2.8) yields
\[ 0 \leq (\hat{M}_\varepsilon^{-1} M)^* \hat{M}_\varepsilon^{-1} M = I - \varepsilon (\hat{M}_\varepsilon^{-1})^* [\hat{M}_\varepsilon + \hat{M}_\varepsilon^* - \varepsilon I] \hat{M}_\varepsilon^{-1} \leq I. \]
Thus, we get
\[ \| \hat{M}_\varepsilon^{-1} M \|^2 = \| (\hat{M}_\varepsilon^{-1} M)^* \hat{M}_\varepsilon^{-1} M \| \leq 1. \]
The proof is complete.

3 Representation of the functional

In this section, we shall study the functional (1.2) from a Hilbert space point of view and represent it as a quadratic functional of the controls \((u_1, u_2)\), by which a necessary condition and a sufficient condition will be derived for the open-loop solvability. For any \( u_i \in \mathcal{U}_i \) \((i = 1, 2)\), consider the following MF-SDE:
\[
\left\{\begin{array}{l}
\displaystyle dX^{i,0}(s) = \{A(s)X^{i,0}(s) + A(s)\mathbb{E}[X^{i,0}(s)] + B_i(s)u_i(s) + B_i(s)\mathbb{E}[u_i(s)]\} ds \\
\quad + \{C(s)X^{i,0}(s) + C(s)\mathbb{E}[X^{i,0}(s)] + D_i(s)u_i(s) + D_i(s)\mathbb{E}[u_i(s)]\} dW(s), \quad s \in [0, T], \\
X^{i,0}(0) = 0.
\end{array}\right.
\]
Under (H1), the above MF-SDE admits a unique solution \( X^{i,0} \in \mathcal{L}^2_\mathbb{F}(\Omega; C([0, T]; \mathbb{R}^n)) \) satisfying
\[ \mathbb{E} \left[ \sup_{s \in [0, T]} |X^{i,0}(s)|^2 \right] \leq K \mathbb{E} \int_0^T |u_i(s)|^2 ds, \]
where the constant \( K > 0 \) is independent of \( u_i \). Thus we can define two bounded linear operators \( \mathcal{L}_i : \mathcal{U}_i \to \mathcal{L}^2_\mathbb{F}(\Omega; C([0, T]; \mathbb{R}^n)) \) and \( \hat{\mathcal{L}}_i : \mathcal{U}_i \to \mathcal{L}^2_{\mathbb{F}}(\Omega; \mathbb{R}^n) \) as follows:
\[ \mathcal{L}_i u_i = X^{i,0}, \quad \hat{\mathcal{L}}_i u_i = X^{i,0}(T), \quad \forall u_i \in \mathcal{U}_i; \quad i = 1, 2. \]
Also we can define the linear operators \( \mathcal{N} : \mathbb{R}^n \to \mathcal{L}^2_\mathbb{F}(\Omega; C([0, T]; \mathbb{R}^n)) \) and \( \hat{\mathcal{N}} : \mathbb{R}^n \to \mathcal{L}^2_{\mathbb{F}}(\Omega; \mathbb{R}^n) \) as follows:
\[ \mathcal{N} x = X^{0,x}, \quad \hat{\mathcal{N}} x = X^{0,x}(T), \quad \forall x \in \mathbb{R}^n, \]
with $X^{0,x}$ being the unique solution to the following MF-SDE:

$$
\begin{align*}
&dX^{0,x}(s) = \{A(s)X^{0,x}(s) + \dot{A}(s)E[X^{0,x}(s)]\}ds \\
&\quad + \{C(s)X^{0,x}(s) + \dot{C}(s)E[X^{0,x}(s)]\}dW(s), \quad s \in [0,T], \\
&X^{0,x}(0) = x.
\end{align*}
$$

(3.5)

For any given $(x, u_1, u_2) \in \mathbb{R}^n \times U_1 \times U_2$, it is easily checked that $X^{0,x} + X^{1,0} + X^{2,0}$ satisfies state equation (1.1). Thus, by the uniqueness of the solution to MF-SDE (1.1), we have

$$
X = X^{0,x} + X^{1,0} + X^{2,0} = \mathcal{N}x + \mathcal{L}_1u_1 + \mathcal{L}_2u_2, \quad \forall (x, u_1, u_2) \in \mathbb{R}^n \times U_1 \times U_2.
$$

(3.6)

In particular, the terminal value of $X$ can be represented by

$$
X(T) = X^{0,x}(T) + X^{1,0}(T) + X^{2,0}(T)
= \dot{\mathcal{N}}x + \dot{\mathcal{L}}_1u_1 + \dot{\mathcal{L}}_2u_2, \quad \forall (x, u_1, u_2) \in \mathbb{R}^n \times U_1 \times U_2.
$$

(3.7)

Then using (3.6)-(3.7), by the completion of squares technique it is straightforward to obtain the following representation of the functional (1.2):

$$
J(x; u_1, u_2) = \langle \mathcal{M}u, u \rangle + 2\langle \mathcal{K}x, u \rangle + \langle \mathcal{O}x, x \rangle, \\
\forall x \in \mathbb{R}^n, \, u = (u_1^T, u_2^T) \in U_1 \times U_2,
$$

(3.8)

where

$$
\mathcal{M} \triangleq \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix}, \quad \mathcal{K} \triangleq \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{pmatrix}, \quad \mathcal{O} \triangleq \dot{\mathcal{N}}^*(G + E^*\dot{G}E)\dot{N} + \mathcal{N}^*(Q + E^*\dot{Q}E)N,
$$

(3.9)

with

$$
\begin{align*}
\mathcal{M}_{ij} &\triangleq \hat{\mathcal{L}}_i^*(G + E^*\dot{G}E)\hat{\mathcal{L}}_j + \mathcal{L}_j^*(Q + E^*\dot{Q}E)\mathcal{L}_i + R_{ji} + E^*\dot{R}_{ji}E \\
&\quad + (S_{ji} + E^*\dot{S}_{ji}E)\mathcal{L}_i + \mathcal{L}_j^*(S_{ji}^T + E^*\dot{S}_{ji}^T)E), \quad i, j = 1, 2; \\
\mathcal{K}_i &\triangleq \hat{\mathcal{L}}_i^*(G + E^*\dot{G}E)\dot{N} + \mathcal{L}_i^*(Q + E^*\dot{Q}E)N + (S_i + E^*\dot{S}_iE)N, \quad i = 1, 2.
\end{align*}
$$

(3.10) (3.11)

By the above expression of $\mathcal{M}_{ij}; i, j = 1, 2$, we have $\mathcal{M}_{ji} = \mathcal{M}^*_j; i, j = 1, 2$, which implies that $\mathcal{M}$ is a self-adjoint operator. With the representation (3.8), we provide the following characterization for the open-loop saddle points of Problem (MF-MG).

**Proposition 3.1.** Let (H1)-(H2) hold. Let $x \in \mathbb{R}^n$ be any given initial state and $u^* = (u_1^T, u_2^T)^T \in U_1 \times U_2$. Then $u^*$ is an open-loop saddle point of Problem (MF-MG) for $x$ if and only if

$$
(-1)^{i+1}\mathcal{M}_{ii} \geq 0; \quad i = 1, 2 \quad \text{and} \quad \mathcal{M}u^* + \mathcal{K}x = 0.
$$

(3.12)

**Proof.** By Definition 2.1, $u^* = (u_1^T, u_2^T)^T$ is an open-loop saddle point of Problem (MF-MG) if and only if

$$
\begin{align*}
J(x; u_1^* + \lambda u_1, u_2^*) - J(x; u_1^*, u_2^*) &\geq 0, \quad \forall u_1 \in U_1, \, \lambda \in \mathbb{R}; \\
J(x; u_1^*, u_2^* + \lambda u_2) - J(x; u_1^*, u_2^*) &\leq 0, \quad \forall u_2 \in U_2, \, \lambda \in \mathbb{R}.
\end{align*}
$$

(3.13) (3.14)

For any $u_1 \in U_1$ and $\lambda \in \mathbb{R}$, by (3.8) we have

$$
\begin{align*}
J(x; u_1^* + \lambda u_1, u_2^*) - J(x; u_1^*, u_2^*) &= \lambda^2 \langle \mathcal{M}_{11}u_1, u_1 \rangle + 2\lambda \langle \langle \mathcal{M}_{11}u_1^*, u_1 \rangle + \langle \mathcal{M}_{12}u_2^*, u_1 \rangle + \langle \mathcal{K}_1x, u_1 \rangle \rangle.
\end{align*}
$$

(3.15)
Thus (3.13) holds if and only if
\[ M_{11} \geq 0 \quad \text{and} \quad M_{11}u_1^* + M_{12}u_2^* + K_1 x = 0. \]  
(3.16)

By the same argument as the above, we can show that (3.14) holds if and only if
\[ M_{22} \leq 0 \quad \text{and} \quad M_{22}u_2^* + M_{21}u_1^* + K_2 x = 0. \]  
(3.17)

Note that (3.12) is equivalent to (3.16) and (3.17). The proof is thus complete. \( \square \)

From Proposition 3.1, we see that the following convexity-concavity condition,
\[ \langle M_{11}u_1, u_1 \rangle = J(0; u_1, 0) \geq 0, \quad \forall u_1 \in U_1; \]
\[ \langle M_{22}u_2, u_2 \rangle = J(0; 0, u_2) \leq -\alpha\|u_2\|^2, \quad \forall u_2 \in U_2, \]  
(3.18)
is necessary for the existence of an open-loop saddle point. Next we introduce a condition slightly stronger than (3.18):

\textbf{(H3)} There exists a constant \( \alpha > 0 \) such that
\[ \langle M_{11}u_1, u_1 \rangle = J(0; u_1, 0) \geq \alpha\|u_1\|^2, \quad \forall u_1 \in U_1; \]
\[ \langle M_{22}u_2, u_2 \rangle = J(0; 0, u_2) \leq -\alpha\|u_2\|^2, \quad \forall u_2 \in U_2. \]  
(3.19)

If (H3) holds, for convenience we usually write (3.19) as follows:
\[ \langle M_{11}u_1, u_1 \rangle = J(0; u_1, 0) \geq 0, \quad \forall u_1 \in U_1; \]
\[ \langle M_{22}u_2, u_2 \rangle = J(0; 0, u_2) \leq 0, \quad \forall u_2 \in U_2. \]

The following result shows that the \textit{uniform convexity-concavity condition} (H3) is sufficient for the open-loop solvability of Problem (MF-MG).

\textbf{Proposition 3.2.} Let (H1)–(H3) hold, and the operators \( \mathcal{M} \) and \( K \) be defined by (3.9). Then \( \mathcal{M} \) is invertible and for any \( x \in \mathbb{R}^n \), Problem (MF-MG) admits a unique open-loop saddle point \( u^* = (u_1^*, u_2^*)^\top \) given by
\[ u^* = -\mathcal{M}^{-1}Kx. \]  
(3.20)

\textit{Proof.} By (3.19), we obtain that the operators \( M_{11}, M_{22} \) and \( \Phi \triangleq M_{22} - M_{21}\mathcal{M}^{-1}_{11}M_{12} \) are invertible. Then it is straightforward to verify that \( \mathcal{M} \) is invertible with inverse
\[ \mathcal{M}^{-1} = \begin{pmatrix} M_{11}^{-1} + (M_{11}^{-1}M_{12})\Phi^{-1}(M_{11}^{-1}M_{12})^\ast & -(M_{11}^{-1}M_{12})\Phi^{-1} \\ -\Phi^{-1}(M_{11}^{-1}M_{12})^\ast & \Phi^{-1} \end{pmatrix}. \]  
(3.21)
The remaining results follow from Proposition 3.1 directly. \( \square \)

\section{4 Open-loop saddle points and Riccati equations}

According to Proposition 3.2, under the uniform convexity-concavity condition (H3), the open-loop saddle point for the given \( x \in \mathbb{R}^n \) can be uniquely determined by (3.20). However, since \( \mathcal{M}^{-1}K \) is an abstract operator and very complicated, it is usually difficult to find the open-loop saddle point by computing (3.20) directly. Thus in this section, we shall give a more explicit form of the open-loop saddle point by introducing two associated Riccati equations. Furthermore, it will be shown that the unique open-loop saddle point admits a closed-loop representation.

Recall from [24] that the Riccati equation associated with Problem (SG) reads
\[
\begin{cases}
\dot{P} + PA + A^\top P + C^\top PC + Q \\
- (PB + C^\top PD + S^\top)(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S) = 0,
\end{cases}
\]  
(4.1)

\[ P(T) = G, \]
where

\[ B = (B_1, B_2), \quad D = (D_1, D_2), \quad S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \quad R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \]

\[ R + D^T P D = \begin{pmatrix} R_{11} + D_1^T P D_1 & R_{12} + D_1^T P D_2 \\ R_{21} + D_2^T P D_1 & R_{22} + D_2^T P D_2 \end{pmatrix}, \]

\[ B^T P + D^T P C + S = \begin{pmatrix} B_1^T P + D_1^T P C + S_1 \\ B_2^T P + D_2^T P C + S_2 \end{pmatrix}. \quad (4.2) \]

**Definition 4.1.** An absolutely continuous function \( P : [0, T] \rightarrow S^n \) is called a strongly regular solution of Riccati equation (4.1) if

(i) For \( i = 1, 2, \) \((-1)^{i+1} [R_{ii} + D_i^T P D_i] \gg 0, \) and

(ii) \( P \) satisfies (4.1) almost everywhere on \([0, T].\)

To establish the solvability of Riccati equation (4.1), we introduce the following two optimal control problems: For \( i = 1, 2, \) consider the state equation

\[ \begin{aligned}
  dX(s) &= \left\{ A(s)X(s) + \bar{A}(s)\mathbb{E}[X(s)] + B_i(s)u_i(s) + \bar{B}_i(s)\mathbb{E}[u_i(s)] \right\} ds \\
  &\quad + \left\{ C(s)X(s) + \bar{C}(s)\mathbb{E}[X(s)] + D_i(s)u_i(s) + \bar{D}_i(s)\mathbb{E}[u_i(s)] \right\} dW(s), \\
  X(0) &= x,
\end{aligned} \] (4.3)

and the cost functional

\[ J_i(x; u_i) = (-1)^{i+1} \mathbb{E} \left\{ \langle GX(T), X(T) \rangle + \langle \bar{G}\mathbb{E}[X(T)], \mathbb{E}[X(T)] \rangle \right\} \\
&\quad + \int_0^T \left\langle \begin{pmatrix} Q & S_i^\top \\ S_i & R_{ii} \end{pmatrix} \begin{pmatrix} X(u_i) \\ X_{u_i} \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X] \\ \mathbb{E}[u_i] \end{pmatrix} \right\} ds. \] (4.4)

If the mean-field terms in the above vanish, then (4.3) and (4.4) reduce to

\[ \begin{aligned}
  dX(s) &= \left\{ A(s)X(s) + B_i(s)u_i(s) \right\} ds + \left\{ C(s)X(s) + D_i(s)u_i(s) \right\} dW(s), \\
  X(0) &= x,
\end{aligned} \] (4.5)

and

\[ J_i(x; u_i) = (-1)^{i+1} \mathbb{E} \left\{ \langle GX(T), X(T) \rangle \right\} + \int_0^T \left\langle \begin{pmatrix} Q & S_i^\top \\ S_i & R_{ii} \end{pmatrix} \begin{pmatrix} X(u_i) \\ X_{u_i} \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X] \\ \mathbb{E}[u_i] \end{pmatrix} \right\} ds. \]

The Riccati equations associated with the above LQ control problems read

\[ \begin{aligned}
  \dot{P}_i + P_i A + A^T P_i + C^T P_i C + Q - (P_i B_i + C^T P_i D_i + S_i^T) \\
  \times (R_{ii} + D_i^T P_i D_i)^{-1} (B_i^T P_i C + S_i) = 0, \quad i = 1, 2, \\
  P_i(T) = G, \quad i = 1, 2,
\end{aligned} \] (4.6)

**Theorem 4.2.** Let (H1)--(H3) hold. Then Riccati equation (4.1) admits a unique strongly regular solution \( P \in C([0, T]; S^n) \). Moreover, the strongly regular solution \( P \) satisfies

\[ P_1(t) \leq P(t) \leq P_2(t), \quad \forall t \in [0, T], \] (4.7)

and

\[ (-1)^{i+1} [R_{ii} + \bar{R}_{ii} + (D_i + \bar{D}_i)^T P(D_i + \bar{D}_i)] \gg 0; \quad i = 1, 2, \] (4.8)

where \( P_i \in C([0, T]; S^n) \) \((i = 1, 2)\) is the unique solution of (4.6).
Proof. Note that the assumption (H3) implies that for any \((u_1, u_2) \in U_1 \times U_2,\)
\[
J_1(0; u_1) = J(0; u_1, 0) \gg 0 \quad \text{and} \quad J_2(0; u_2) = -J(0; 0, u_2) \gg 0.
\]
Thus for \(i = 1, 2,\) we obtain from [23, Theorems 4.2 and 4.4] that Riccati equation (4.6) admits a unique solution \(P_i \in C([0, T]; S^n)\) satisfying
\[
(-1)^{i+1}[R_{ii} + D_i^\top P_i D_i] \gg 0,
\] (4.9)
and
\[
(-1)^{i+1}[R_{ii} + \bar{R}_{ii} + (D_i + \bar{D}_i)^\top P_i (D_i + \bar{D}_i)] \gg 0.
\] (4.10)
According to [23, Theorem 5.2], (4.9) implies that the mapping \(u_i \mapsto J_i(0; u_i)\) is uniformly convex. Thus,
\[
J(0; u_1) = J_1(0; u_1) \gg 0, \quad \forall u_1 \in U_1;
\]
\[
J(0; u_2) = J_2(0; u_2) \ll 0, \quad \forall u_2 \in U_2,
\] (4.11)
where \(J\) denotes the functional of Problem (SG); that is \(J(x; u_1, u_2) \triangleq J(x; u_1, u_2)\) with \(A, \bar{A}, C, \bar{C}, \bar{D}_i, G, \bar{Q}, \bar{S}_1\) and \(\bar{R}_{ij} (i, j = 1, 2)\) all vanishing. Then from [24, Theorem 4.3], we obtain that Riccati equation (4.1) has a strongly regular solution \(P \in C([0, T]; S^n)\).

To prove the uniqueness, we suppose that \(\bar{P}, \tilde{P} \in C([0, T]; S^n)\) are two strongly regular solutions of (4.1). Then \(\bar{P} - \tilde{P}\) satisfies:
\[
(-1)^{i+1}[R_{ii} + D_i^\top \bar{P} D_i] \gg 0 \quad \text{and} \quad (-1)^{i+1}[R_{ii} + D_i^\top \tilde{P} D_i] \gg 0, \quad i = 1, 2.
\]
Similar to (3.21), \(R + D^\top \bar{P} D\) and \(R + D^\top \tilde{P} D\) are invertible with their inverses being bounded. Denote \(\Delta P = \bar{P} - \tilde{P}\). Then \(\Delta P\) satisfies the following linear ordinary differential equation:
\[
\begin{cases}
\Delta \ddot{P} + \Delta PA + A^\top \Delta P + C^\top \Delta PC - (\Delta PB + C^\top \Delta PD)(R + D^\top \bar{P} D)^{-1} \\
\times (B^\top \bar{P} + D^\top \bar{P} C + S) - (\bar{P} B^\top + C^\top \bar{P} D + S^\top)(R + D^\top \bar{P} D)^{-1} D^\top \Delta PD \\
\times (R + D^\top \bar{P} D)^{-1}(B^\top \bar{P} + D^\top \bar{P} C + S) - (\bar{P} B^\top + C^\top \bar{P} D + S^\top) \\
\times (R + D^\top \bar{P} D)^{-1}(B^\top \Delta P + D^\top \Delta PC) = 0,
\end{cases}
\] (4.12)
\[
\Delta P(T) = 0.
\]
Note that \(\bar{P}, \tilde{P}, (R + D^\top \bar{P} D)^{-1}\) and \((R + D^\top \tilde{P} D)^{-1}\) are bounded. Then by a standard argument using the Grönwall’s inequality, we get \(\Delta P \equiv 0\), which yields the uniqueness of the strongly regular solution to (4.1).

Next let us prove the unique strongly regular solution \(P \in C([0, T]; S^n)\) of (4.1) satisfies (4.7). To this end, for any fixed \(u_2 \in U_2\), we introduce the following LQ control problem: Consider the state equation
\[
\begin{cases}
\frac{dX(s)}{ds} = \{A(s)X(s) + B_1(s)u_1(s) + B_2(s)u_2(s)\} ds \\
+ \{C(s)X(s) + D_1(s)u_1(s) + D_2(s)u_2(s)\} dW(s),
\end{cases}
\] (4.13)
and the cost functional
\[
J_{u_2}(x; u_1) \triangleq J(x; u_1, u_2) = \mathbb{E}\left\{ \langle GX(T), X(T) \rangle + \int_0^T \left[ \langle QX, X \rangle + 2\langle S_1X, u_1 \rangle \\
+ \langle R_{11}u_1, u_1 \rangle + 2\langle X, S_2 u_2 \rangle + 2\langle u_1, R_{12}u_2 \rangle + \langle R_{22}u_2, u_2 \rangle \right] ds \right\}.
\] (4.14)
Note that (4.11) implies the mapping \( u_1 \mapsto J_{u_1}(x; u_1) = J(x; u_1, u_2) \) is uniformly convex, then by [25, Theorem 4.3] the unique optimal control \( \bar{u}_1(\cdot) \equiv \bar{u}_1(\cdot; u_2) \) of the above LQ control problem admits the following closed-loop representation:

\[
J_{u_1}(x; \bar{u}_1) = J_{u_2}(x; \bar{u}_1, u_2) = \mathcal{J}(x; \bar{u}_1, u_2) = \mathcal{J}(x; \bar{X} + v, u_2),
\]

where

\[
\bar{X}(s) = \mathcal{X}(s) + \mathcal{X}(s) + v(s) = \Theta(s)X(s) + v(s; u_2), \quad s \in [0, T],
\]

and \( \bar{X}(\cdot) \equiv \bar{X}(\cdot; u_2) \) is the solution of the closed-loop system:

\[
\begin{align*}
\dot{X}(s) &= A(s)X(s) + B_2(s)u_2(s) + [C(s)X(s) + D_2(s)u_2(s)]dW(s), \\
X(0) &= x.
\end{align*}
\]

Moreover,

\[
J_{u_1}(x; \bar{u}_1) = J_{u_2}(x; \bar{X} + v) = J(x; \bar{X} + v, u_2)
\]

\[
= \mathbb{E} \left\{ \langle P(0), x \rangle + 2 \langle Y(0), x \rangle + \int_0^T \left( \langle P_1D_2u_2, D_2u_2 \rangle + 2 \langle Y, B_2u_2 \rangle + 2 \langle Z, D_2u_2 \rangle + \langle (R_1 + D_1^T P_1 D_1)^{-1}(B_1^T Y + D_1^T Z + D_1^T P_1 D_2u_2 + R_1 u_1), D_2u_2 \rangle \right) ds \right\},
\]

Since the condition (4.11) holds, by [24, Theorem 4.4] Problem (SG) admits a unique open-loop saddle point \((u_1^*, u_2^*)\); that is

\[
\mathcal{J}(x; u_1^*, u_2^*) = \sup_{u_2 \in \mathcal{U}_2} \inf_{u_1 \in \mathcal{U}_1} \mathcal{J}(x; u_1, u_2) = \inf_{u_1 \in \mathcal{U}_1} \sup_{u_2 \in \mathcal{U}_2} \mathcal{J}(x; u_1, u_2) = \langle P(0), x \rangle,
\]

which implies that

\[
\inf_{u_1 \in \mathcal{U}_1} J_{u_2}(x; u_1) = J_{u_2}(x; u_1^*) \quad \text{and} \quad J(x; u_1^*, u_2^*) = \sup_{u_2 \in \mathcal{U}_2} J(x; u_1^*, u_2).
\]

Recall that the mapping \( u_1 \mapsto \mathcal{J}(x; u_1, u_2) \) is uniformly convex, which implies that \( u_1^\ast \) is the unique control satisfying the first equality in the above, thus we have

\[
\mathcal{J}(x; u_1^*, u_2^*) = J_{u_2}(x; u_1^*) = J_{u_2}(x; \bar{X} + v^*), \quad \mathcal{J}(x; \bar{X} + v^*, u_2^*) = \mathcal{J}(x; \bar{X} + v^* + v^*, u_2^*),
\]

with \( \bar{X}^*(\cdot) = \bar{X}(\cdot; u_2^*) \) and \( v^*(\cdot) = v(\cdot; u_2^*) \). On the other hand, by the second equality in (4.19) and the closed-loop representation (4.15) of the optimal control \( u_1^\ast \), we get

\[
\mathcal{J}(x; u_1^*, u_2^*) \geq \mathcal{J}(x; u_1^*, u_2) = \mathcal{J}_{u_2}(x; u_1^*) \geq J_{u_2}(x; u_1) = \mathcal{J}(x; \bar{X} + v, u_2), \quad \forall u_2 \in \mathcal{U}_2,
\]

with \( \bar{X}(\cdot) = \bar{X}(\cdot; u_2) \) and \( v(\cdot) = v(\cdot; u_2) \). Combining the above with (4.20) yields that

\[
\mathcal{J}(x; \bar{X} + v^*, u_2^*) \geq \mathcal{J}(x; \bar{X} + v, u_2^*) \equiv \mathcal{J}(x; \bar{X}(\cdot; u_2) + v(\cdot; u_2), u_2^*), \quad \forall u_2 \in \mathcal{U}_2.
\]
Taking \( u_2 = 0 \) in (4.21) and then making use of (4.18) and (4.16), we have

\[
(P(0)x, x) = \mathcal{J}(x; \Theta \hat{X} + v^*, u_2^*) \geq \mathcal{J}(x; \Theta \hat{X} + v, 0) = (P_1(0)x, x).
\]

Using the above arguments to the Problem (SG) with the initial pair replaced by \((t, x) \in [0, T] \times \mathbb{R}^n\), we obtain

\[
(P(t)x, x) \geq (P_1(t)x, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n.
\]

In a similar manner, we can also show that

\[
(P(t)x, x) \leq (P_2(t)x, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n.
\]

Thus, the unique strongly regular solution \( P \) of Riccati equation (4.1) satisfies (4.7). Finally, combining (4.7) with (4.10), we get (4.8) immediately. The proof is thus complete. \( \blacksquare \)

**Remark 4.3.** If we define the state processes \((Y, Z)\) by (4.16) and the cost functional \( \tilde{J}(x; u_2) \triangleq -\mathcal{J}(x; \Theta \hat{X} + v, u_2) \) by (4.18), then the corresponding control problem is a backward LQ problem. From (4.21), we see that under (H3), if \((u_1^*, u_2^*)\) is the unique open-loop saddle point of Problem (SG), then \(u_2^*\) is optimal for the above designed backward problem. For more results of backward LQ control problems, we refer the reader to \([18, 17, 27]\) and the references cited therein.

With the strongly regular solution \( P \) of (4.1), we now introduce the following deterministic two-person zero-sum LQ differential game problem (Problem (DG), for short): Consider the state equation

\[
\begin{cases}
\dot{y}(s) = [A(s) + \bar{A}(s)]y(s) + [B_1(s) + \bar{B}_1(s)]v_1(s) + [B_2(s) + \bar{B}_2(s)]v_2(s), \\
y(0) = x,
\end{cases}
\]

and the functional

\[
\tilde{J}(x; v_1, v_2) = \langle (G + \bar{G})y(T), y(T) \rangle + \int_0^T \left( \begin{array}{c}
\Upsilon \\
\Gamma_1 \\
\Sigma_{11} \\
\Sigma_{12}
\end{array} \right) \left( \begin{array}{c}
y \\
v_1 \\
v_2
\end{array} \right) \left( \begin{array}{c}
y \\
v_1 \\
v_2
\end{array} \right) ds,
\]

where

\[
\begin{cases}
\Upsilon = Q + \bar{Q} + (C + \bar{C})^TP(C + \bar{C}), \\
\Gamma_i = (D_i + \bar{D}_i)^TP(C + \bar{C}) + (S_i + \bar{S}_i), \quad i = 1, 2, \\
\Sigma_{ij} = R_{ij} + \bar{R}_{ij} + (D_{ij} + \bar{D}_{ij})^TP(D_{ij} + \bar{D}_{ij}), \quad i, j = 1, 2.
\end{cases}
\]

Since the strongly regular solution \( P \) also satisfies (4.8), the matrices \( \Sigma \) and \( \Sigma \), defined by

\[
\Sigma \equiv R + D^TPD, \quad \Sigma \equiv R + \bar{R} + (D + \bar{D})^TP(D + \bar{D}),
\]

are invertible. The Riccati equation associated with Problem (DG) is

\[
\begin{cases}
\dot{\Pi} + \Pi(A + \bar{A}) + (A + \bar{A})^T\Pi + Q + \bar{Q} + (C + \bar{C})^TP(C + \bar{C}) - \Pi(B + \bar{B}) \\
\quad + (C + \bar{C})^TP(D + \bar{D}) + (S + \bar{S})^T[R + \bar{R} + (D + \bar{D})^TP(D + \bar{D})]^{-1} \\
\quad \times [(B + \bar{B})^TP + (D + \bar{D})^TP(C + \bar{C}) + (S + \bar{S})] = 0,
\end{cases}
\]

where \( B, \bar{B}, D, \bar{D}, \) and \( R \) are defined in a similar way to (4.2). The following result shows that under (H3), Riccati equation (4.26) is also solvable.

**Theorem 4.4.** Let (H1)–(H3) hold. Then Riccati equation (4.26) is uniquely solvable.
Proof. By [24, Theorem 4.3], to prove the solvability of Riccati equation (4.26), it suffices to show that
\[ \bar{J}(0; v_1, 0) \gg 0 \quad \text{and} \quad \bar{J}(0; v_2, 0) \ll 0. \] (4.27)
Denote
\[ \bar{J}_1(x; v_1) = \langle (G + \bar{G})y(T), y(T) \rangle + \int_0^T \begin{pmatrix} \bar{Y}_1 & \bar{Y}_1^T \\ \bar{Y}_1 & \Sigma_{11} \end{pmatrix} \begin{pmatrix} y \\ v_1 \end{pmatrix} \begin{pmatrix} y \\ v_1 \end{pmatrix} ds, \] (4.28)
where \( y \) is the unique solution of (4.22) with \( v_2 \equiv 0 \), and
\[
\begin{aligned}
\bar{Y}_1 &= Q + \bar{Q} + (C + \bar{C})^T P_1(C + \bar{C}), \\
\bar{Y}_1 &= (D_1 + \bar{D}_1)^T P_1(C + \bar{C}) + (S_1 + \bar{S}_1), \\
\Sigma_{11} &= R_{11} + \bar{R}_{11} + (D_1 + \bar{D}_1)^T P_1(D_1 + \bar{D}_1),
\end{aligned}
\] (4.29)
with \( P_1 \) being the unique solution of (4.6) for \( i = 1 \). Recall the definition (4.4) of \( J_1 \) and note that \( J_1(0; u_1) = J(0; u_1, 0) \gg 0 \). By [23, Theorem 4.4], we have
\[ J_1(0; v_1) \gg 0. \] (4.30)
On the other hand, for any \( \delta > 0 \), by (4.23)–(4.28) we obtain
\[ \bar{J}(0; v_1, 0) - \bar{J}_1(0; v_1) + \delta \| v_1 \|^2 = \int_0^T \left\{ \langle (C + \bar{C})^T (P - P_1)(C + \bar{C}) \rangle, y, y \rangle + 2 \langle (D_1 + \bar{D}_1)^T (P - P_1)(C + \bar{C}) \rangle, v_1 \rangle \\
+ \langle (D_1 + \bar{D}_1)^T (P - P_1)(D_1 + \bar{D}_1) + \delta I_{m_1}, v_1, v_1 \rangle \right\} ds \\
\equiv \int_0^T \left[ \langle Qy, y \rangle + 2 \langle Sy, x \rangle + \langle Rv_1, v_1 \rangle \right] ds. \] (4.31)
Note that \( P \gg P_1 \) (recalling (4.7)), thus
\[
\mathcal{R} \equiv (D_1 + \bar{D}_1)^T (P - P_1)(D_1 + \bar{D}_1) + \delta I_{m_1} \geq \delta I_{m_1} \gg 0.
\]
Moreover, by Lemma 2.3 we have
\[ Q - \mathcal{S}^T \mathcal{R}^{-1} \mathcal{S} = (C + \bar{C})^T (P - P_1)(C + \bar{C}) - (C + \bar{C})^T (P - P_1)(D_1 + \bar{D}_1) \]
\[ \times [(D_1 + \bar{D}_1)^T (P - P_1)(D_1 + \bar{D}_1) + \delta I_{m_1}]^{-1}(D_1 + \bar{D}_1)^T (P - P_1)(C + \bar{C}) \geq 0. \]
Thus, the weighting matrices \( Q, \mathcal{S} \) and \( \mathcal{R} \) satisfy the so-called standard condition in the literature (see [35, Chapter 6], for example) of LQ optimal control problems. Then
\[
\bar{J}(0; v_1, 0) - \bar{J}_1(0; v_1) + \delta \| v_1 \|^2 = \int_0^T \left[ \langle (Q - \mathcal{S}^T \mathcal{R}^{-1} \mathcal{S})y, y \rangle + \langle \mathcal{R}(v_1 + \mathcal{R}^{-1} \mathcal{S}y), (v_1 + \mathcal{R}^{-1} \mathcal{S}y) \rangle \right] ds \\
\geq 0. \] (4.32)
Since \( \delta > 0 \) is arbitrary, we get
\[ \bar{J}(0; v_1, 0) - \bar{J}_1(0; v_1) \geq 0, \] (4.33)
which, together with (4.30), implies that
\[ \bar{J}(0; v_1, 0) \gg 0. \] (4.34)
Similarly, we can also prove that
\[ \bar{J}(0; 0, v_1) \ll 0. \] (4.35)
Combining (4.34) with (4.35), we get (4.27), which completes the proof of the existence. Then by the same arguments as in the proof of Theorem 4.2, we obtain the unique solvability of Riccati equation (4.26). \qed

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Remark 4.5. It is noteworthy that the comparison property (4.7) of \( P \) and \( P_i; i = 1, 2 \) serves as a crucial bridge to prove the solvability of Riccati equation (4.26) in Theorem 4.4. The technical Lemma 2.3 is used to show the weighting matrices \( Q, S \) and \( R \) defined by (4.31) exactly satisfy the so-called standard condition.

With the strongly regular solvability of Riccati equations (4.1)–(4.26) having been established, we present the closed-loop representation for the open-loop saddle point of Problem (MF-SG).

**Theorem 4.6.** Let (H1)–(H3) hold. Let \( P \in C([0, T]; \mathbb{S}^n) \) be the strongly regular solution to Riccati equation (4.1) satisfying (4.7)–(4.8) and \( \Pi \in C([0, T]; \mathbb{S}^n) \) be the solution to Riccati equation (4.26). Then with the notations

\[
\Sigma = R + D^\top PD, \quad \bar{\Sigma} = R + \bar{R} + (D + \bar{D})^\top P(D + \bar{D}),
\]

\[
\Theta = -\Sigma^{-1}(B^\top P + D^\top PC + S),
\]

\[
\bar{\Theta} = -\bar{\Sigma}^{-1}[(B + \bar{B})^\top \Pi + (D + \bar{D})^\top P(C + \bar{C}) + S + \bar{S}],
\]

the unique open-loop saddle point \( u^* = (u_1^*, u_2^*)^\top \) for the initial state \( x \) has the following closed-loop representation:

\[
u^* = \Theta \{ X^* - \mathbb{E}[X^*] \} + \bar{\Theta} \mathbb{E}[X^*],
\]

where \( X^* \) is the solution to the closed-loop system:

\[
\begin{align*}
dX^*(s) &= \{(A + B\Theta)(X^* - \mathbb{E}[X^*]) + [(A + \bar{A}) + (B + \bar{B})\bar{\Theta}]\mathbb{E}[X^*]\} ds \\
&\quad + \{(C + D\Theta)(X^* - \mathbb{E}[X^*]) + [(C + \bar{C}) + (D + \bar{D})\bar{\Theta}]\mathbb{E}[X^*]\} dW(s),
\end{align*}
\]

\[X^*(0) = x.\]

Moreover, the value function of Problem (MF-SG) is given by \( V(x) = \langle \Pi(0)x, x \rangle \).

**Remark 4.7.** By Theorem 4.6, we give an explicit representation for the unique open-loop saddle point of Problem (MF-SG). Indeed, the LQ problems occupied the center stage for research in control theory not only for its elegant solutions but also for its ability to approximate more general nonlinear problems, as pointed out by Wang–Zariphopoulou–Zhou in their recent work [32] of the stochastic control approach in reinforcement learning.

In order to prove Theorem 4.6, we need the following lemma, whose proof is standard and is similar to that of [28, Theorem 4.1].

**Lemma 4.8.** A pair \( (\bar{u}_1, \bar{u}_2) \in \mathcal{U}_1 \times \mathcal{U}_2 \) is an open-loop saddle point of Problem (MF-SG) if and only if (3.18) holds and \( \bar{u} = (\bar{u}_1^*, \bar{u}_2^*)^\top \) satisfies

\[
B(s)^\top \bar{Y}(s) + B(s)^\top \mathbb{E}[^{\top}\bar{Y}(s)] + D(s)^\top \bar{Z}(s) + D(s)^\top \mathbb{E}[Z(s)] + S(s)\bar{X}(s)
\]

\[+ \bar{S}(s)\mathbb{E}[\bar{X}(s)] + R(s)\bar{u}(s) + \bar{R}(s)\mathbb{E}[\bar{u}(s)] = 0, \quad s \in [0, T], \quad \text{a.s.}, \]

where \( (\bar{X}, \bar{Y}, \bar{Z}) \) is the unique solution to the following mean-field forward-backward SDE (MF-FBSDE, for short):

\[
\begin{align*}
d\bar{X}(s) &= \{AX + \bar{A}\mathbb{E}[\bar{X}] + B\bar{u} + B\mathbb{E}[\bar{u}] \} ds \\
&\quad + \{C\bar{X} + \bar{C}\mathbb{E}[\bar{X}] + D\bar{u} + D\mathbb{E}[\bar{u}] \} dW(s), \quad s \in [0, T],
\end{align*}
\]

\[
\begin{align*}
d\bar{Y}(s) &= -\{A^\top \bar{Y} + \bar{A}^\top \mathbb{E}[\bar{Y}] + C^\top \bar{Z} + \bar{C}^\top \mathbb{E}[\bar{Z}] + Q\bar{X} + Q\mathbb{E}[\bar{X}] \\
&\quad + S^\top \bar{u} + S^\top \mathbb{E}[\bar{u}] \} ds + \bar{Z}dW(s), \quad s \in [0, T],
\end{align*}
\]

\[
\bar{X}(0) = x, \quad \bar{Y}(T) = G\bar{X}(T) + G\mathbb{E}[\bar{X}(T)].
\]

**Proof of Theorem 4.6.** We first prove that the control \( u^* \) defined by (4.39) is the unique open-loop saddle point. Denote

\[Y^* = P(X^* - \mathbb{E}[X^*]) + \Pi \mathbb{E}[X^*],\]
\[ Z^* = P \{ C(X^* - E[X^*]) + (C + \bar{C})E[X^*] + D(u^* - E[u^*]) + (D + \bar{D})E[u^*] \}. \]

Note that

\[ E[Y^*] = \Pi E[X^*], \quad E[u^*] = \Theta E[X^*], \quad E[Z^*] = P \{ (C + \bar{C})E[X^*] + (D + \bar{D})E[u^*] \}. \]

By Itô's formula, we have

\[ dY^*(t) = \dot{P}(X^* - E[X^*])dt + P \{(A + B\Theta)(X^* - E[X^*])\} dt + P \{(C + D\Theta)(X^* - E[X^*]) + \[(C + \bar{C}) + (D + \bar{D})\Theta \] E[X^*] \} dW(t) \]

\[ + \Pi[(A + \bar{A}) + (B + \bar{B})\Theta] E[X^*] dt \]

\[ = -[A^T P + C^T PC + Q + (C^T PD + S^T)\Theta](X^* - E[X^*])dt - \{(A + \bar{A})^T \Pi + Q + \bar{Q} \]

\[ + (C + \bar{C})^T P(C + \bar{C}) + [(C + \bar{C})^T P(D + \bar{D}) + S^T + S^T)\Theta \] E[X^*] dt + Z^* dW(t) \]

\[ = -\{A^T Y^* + \bar{A}^T E[Y^*] + C^T P\{C(X^* - E[X^*]) + (C + \bar{C})E[X^*] + D(u^* - E[u^*]) \]

\[ + (D + \bar{D})E[u^*] \} + C^T P\{(C + \bar{C})E[X^*] + (D + \bar{D})E[u^*] \} + QT^* + \bar{Q}E[X^*] \]

\[ + S^T u^* + \bar{S}^T E[u^*] \} dt + Z^* dW(t) \]

\[ = -\{A^T Y^* + \bar{A}^T E[Y^*] + C^T Z^* + \bar{C}^T E[Z^*] + QX^* + \bar{Q}E[X^*] + S^T u^* \]

\[ + \bar{S}^T E[u^*] \} dt + Z^* dW(t). \]

Further, by the terminal values of \( P \) and \( \Pi \), we get

\[ Y^*(T) = P(T)\{X^*(T) - E[X^*(T)]\} + \Pi(T)E[X^*(T)] \]

\[ = GX^*(T) + GE[X^*(T)]. \]

Thus \((X^*, Y^*, Z^*)\) satisfies MF-FBSDE (4.42) with the control \( \bar{u} \equiv u^* \). Moreover, note that

\[ B^T Y^* + \bar{B}^T E[Y^*] + D^T Z^* + \bar{D}^T E[Z^*] + SX^* + \bar{S}E[X^*] + Ru^* + \bar{R}E[u^*] \]

\[ = B^T \{ P(X^* - E[X^*]) + \Pi E[X^*] \} + \bar{B}^T \Pi E[X^*] + SX^* + \bar{S}E[X^*] + Ru^* + \bar{R}E[u^*] \]

\[ + R(\Theta E[X^*] + \Theta E[X^*]) + \bar{D}^T P\{(C + \bar{C})E[X^*] + (D + \bar{D})E[X^*] \} \]

\[ = \{ B^T P + S + D^T PC + (D^T PD + R)\Theta \} X^* - E[X^*] \} + \{(B + \bar{B})^T \Pi \]

\[ + S + \bar{S} + (D + \bar{D})^T P(C + \bar{C}) + [(D + \bar{D})^T P(D + \bar{D}) + R + \bar{R}\Theta] E[X^*] \]

\[ = 0. \]

(4.46)

Then by Lemma 4.8, the control \( u^* \) defined by (4.39) is an open-loop saddle point. The uniqueness of open-loop saddle points follows from Proposition 3.2.

By integration by parts and (4.46), we get

\[ E(GX^*(T), X^*(T)) + \langle GE[X^*(T)]], E[X^*(T)]\rangle = E(Y^*(T), X^*(T)) \]

\[ = E(Y^*(0), X^*(0)) + \int_0^T \left[ (B^T Y^* + \bar{B}^T E[Y^*] + D^T Z^* + \bar{D}^T E[Z^*] - SX^* \]

\[ - \bar{S}E[X^*], u^* \rangle - \langle QX^*, X^* \rangle - \langle \bar{Q}E[X^*], X^* \rangle \right] ds \]

\[ = E(Y^*(0), X^*(0)) - \int_0^T \left[ \langle Rx^*, u^* \rangle + \langle \bar{R}E[u^*], E[u^*] \rangle + 2\langle SX^*, u^* \rangle \]

\[ + 2 \langle \bar{S}E[X^*], E[u^*] \rangle + \langle QX^*, X^* \rangle + \langle \bar{Q}E[X^*], E[X^*] \rangle \right] ds. \]
Noting that \( Y^*(0) = \Pi(0)x \), we have the following representation of the value function:

\[
V(x) = J(x; u^*) = \mathbb{E}\langle G X^*(T), X^*(T) \rangle + \langle G \mathbb{E}[X^*(T)], \mathbb{E}[X^*(T)] \rangle \\
+ \mathbb{E} \int_0^T \left[ \langle Ru^*, u^* \rangle + \langle R \mathbb{E}[u^*], \mathbb{E}[u^*] \rangle + 2 \langle SX^*, u^* \rangle + 2 \langle S \mathbb{E}[X^*], \mathbb{E}[u^*] \rangle \\
+ \langle Q X^*, X^* \rangle + \langle \bar{Q} \mathbb{E}[X^*], \mathbb{E}[X^*] \rangle \right] \, ds \\
= \mathbb{E}\langle Y^*(0), X^*(0) \rangle = (\Pi(0)x, x), \quad x \in \mathbb{R}^n.
\]

**5 Open-loop solvability: a perturbation approach**

In **Theorem 4.6**, it is shown that under the uniform convexity-concavity condition (H3), Problem (MF-SG) is uniquely open-loop solvable and the unique open-loop saddle point admits the closed-loop representation (4.39). In this section, we shall establish a characterization for the open-loop solvability of Problem (MF-SG) without assumption (H3). Recall from **Proposition 3.1** that the following convexity-concavity condition is necessary for the open-loop solvability of Problem (MF-SG):

\[
\langle M_{11} u_1, u_1 \rangle = J(0; u_1, 0) \geq 0, \quad \forall u_1 \in \mathcal{U}_1; \\
\langle M_{22} u_2, u_2 \rangle = J(0; 0, u_2) \leq 0, \quad \forall u_2 \in \mathcal{U}_2.
\]

Thus in this section, we always assume that (5.1) holds.

**5.1 The perturbation approach**

For each \( \varepsilon > 0 \), we introduce the following perturbed functional:

\[
J_\varepsilon(x; u_1, u_2) \triangleq J(x; u_1, u_2) + \varepsilon \mathbb{E} \int_0^T |u_1(s)|^2 \, ds - \varepsilon \mathbb{E} \int_0^T |u_2(s)|^2 \, ds \\
= \mathbb{E}\{ \langle GX(T), X(T) \rangle + \langle G \mathbb{E}[X(T)], \mathbb{E}[X(T)] \rangle \\
+ \int_0^T \left( \begin{pmatrix} Q & S_1 \\ S_2 & R_{11} + \varepsilon I_{m_1} \end{pmatrix} \begin{pmatrix} S_1^T \\ R_{21} \end{pmatrix} \right) \begin{pmatrix} X \\ u_1 \end{pmatrix}, \begin{pmatrix} X \\ u_2 \end{pmatrix} \right) ds \\
+ \int_0^T \left( \begin{pmatrix} Q & S_1 \\ S_2 & R_{11} \end{pmatrix} \begin{pmatrix} R_{12} \\ R_{22} - \varepsilon I_{m_2} \end{pmatrix} \right) \begin{pmatrix} \mathbb{E}[X] \\ \mathbb{E}[u_1] \end{pmatrix}, \begin{pmatrix} \mathbb{E}[X] \\ \mathbb{E}[u_2] \end{pmatrix} \right) ds \}. \tag{5.2}
\]

We denote the two-person zero-sum LQ stochastic differential game associated with (1.1)–(5.2) by Problem (MF-SG)_\varepsilon and the value function by \( V_\varepsilon \). Notice that

\[
J_\varepsilon(0; u_1, 0) = J(0; u_1, 0) + \varepsilon \mathbb{E} \int_0^T |u_1(s)|^2 \, ds \geq \varepsilon \| u_1 \|^2, \quad \forall u_1 \in \mathcal{U}_1; \\
J_\varepsilon(0; 0, u_2) = J(0; 0, u_2) - \varepsilon \mathbb{E} \int_0^T |u_2(s)|^2 \, ds \leq -\varepsilon \| u_2 \|^2, \quad \forall u_2 \in \mathcal{U}_2.
\]

Then with

\[
R_{\varepsilon} \triangleq \begin{pmatrix} R_{11} + \varepsilon I_{m_1} & R_{12} \\ R_{21} & R_{22} - \varepsilon I_{m_2} \end{pmatrix}, \tag{5.4}
\]

it follows from **Theorem 4.2** and **Theorem 4.4** that the Riccati equations

\[
\begin{cases}
\dot{P}_\varepsilon + P_\varepsilon A + A^T P_\varepsilon + C^T P_\varepsilon C + Q \\
- (P_\varepsilon B + C^T P_\varepsilon D + S^T)(R_{\varepsilon} + D^T P_\varepsilon D)^{-1}(B^T P_\varepsilon + D^T P_\varepsilon C + S) = 0,
\end{cases} \tag{5.5}
\]

\[
P_\varepsilon(T) = G
\]
and
\[
\begin{align*}
\dot{\Pi}_\varepsilon + \Pi_\varepsilon (A + \tilde{A}) + (A + \tilde{A})^T \Pi_\varepsilon + Q + \dot{Q} + (C + \tilde{C})^T P_\varepsilon (C + \tilde{C}) \\
- [\Pi_\varepsilon (B + \tilde{B}) + (C + \tilde{C})^T P_\varepsilon (D + \tilde{D}) + (S + \tilde{S})^T] \\
\times [R_\varepsilon + \dot{R} + (D + \tilde{D})^T P_\varepsilon (D + \tilde{D})]^{-1} \\
\times [(B + \tilde{B})^T \Pi_\varepsilon + (D + \tilde{D})^T P_\varepsilon (C + \tilde{C}) + (S + \tilde{S})] = 0,
\end{align*}
\]
\quad (5.6)

admit unique solutions \(P_\varepsilon \in C([0, T]; \mathbb{S}^n)\) and \(\Pi_\varepsilon \in C([0, T]; \mathbb{S}^n)\) satisfying
\[
(\varepsilon) + (\varepsilon)I_m_i_i \quad i = 1, 2,
\]
\quad (5.7)

\[
(\varepsilon) + (\varepsilon)I_m_i_i \quad i = 1, 2.
\]
\quad (5.8)

Denote
\[
\begin{align*}
\Sigma_\varepsilon &= R_\varepsilon + D^T P_\varepsilon D, \quad \Sigma_\varepsilon = R_\varepsilon + \tilde{R} + (D + \tilde{D})^T P_\varepsilon (D + \tilde{D}), \\
\Theta_\varepsilon &= -\Sigma_\varepsilon^{-1} (B^T P_\varepsilon + D^T P_\varepsilon C + S), \\
\tilde{\Theta}_\varepsilon &= -\Sigma_\varepsilon^{-1} [(B + \tilde{B})^T \Pi_\varepsilon + (D + \tilde{D})^T P_\varepsilon (C + \tilde{C}) + S + \tilde{S}].
\end{align*}
\]
\quad (5.9) (5.10) (5.11)

By Theorem 4.6, the unique open-loop saddle point \(u_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)^T\) of Problem (MF-SG) is given by:
\[
u_\varepsilon = \Theta_\varepsilon \{X_\varepsilon - E[X_\varepsilon]\} + \tilde{\Theta}_\varepsilon E[X_\varepsilon],
\]
\quad (5.12)

with \(X_\varepsilon\) solving the closed-loop system:
\[
\begin{align*}
\left\{ \begin{array}{l}
\dot{X}_\varepsilon(s) = \{(A + B\Theta_\varepsilon)(X_\varepsilon - E[X_\varepsilon]) + [(A + \tilde{A}) + (B + \tilde{B})\tilde{\Theta}_\varepsilon]E[X_\varepsilon]\}ds \\
\quad + \{(C + D\Theta_\varepsilon)(X_\varepsilon - E[X_\varepsilon]) + [(C + \tilde{C}) + (D + \tilde{D})\tilde{\Theta}_\varepsilon]E[X_\varepsilon]\}dW(s),
\end{array} \right.
\]
\quad (5.13)

\(X_\varepsilon(0) = x.\)

For any \(\varepsilon > 0\) and \(x \in \mathbb{R}^n\), the value of Problem (MF-SG) at \(x\) is given by
\[
V_\varepsilon(x) = J_\varepsilon(x; u_1^\varepsilon, u_2^\varepsilon),
\]

where \(u_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)\) is defined by (5.12). If the value \(V(x)\) of Problem (MF-SG) exists for some \(x \in \mathbb{R}^n\), then by the same arguments as in the proof of [24, Proposition 3.5] we have
\[
\lim_{\varepsilon \to 0} V_\varepsilon(x) = V(x).
\]

This means that the family \(\{u_\varepsilon\}_{\varepsilon > 0}\) defined by (5.12) is an approximate sequence of Problem (MF-SG). But as pointed out in [24], the existence of a value function does not imply that Problem (MF-SG) has an open-loop saddle point. Thus in terms of the family \(\{u_\varepsilon\}_{\varepsilon > 0}\), we present the following characterization for the open-loop solvability of Problem (MF-SG), which is the main result of this section.

**Theorem 5.1.** Let (H1)–(H2) and (5.1) hold. Let \(x \in \mathbb{R}^n\) be any given initial state and \(\{u_\varepsilon\}_{\varepsilon > 0}\) be the sequence defined by (5.12). Then the following statements are equivalent:

(a) Problem (MF-SG) has an open-loop saddle point at \(x\);

(b) the family \(\{u_\varepsilon\}_{\varepsilon > 0}\) is bounded in the Hilbert space \(L_2^2(0, T; \mathbb{R}^{m_1 + m_2}) \equiv U_1 \times U_2\), i.e.,
\[
\sup_{\varepsilon > 0} \mathbb{E} \int_0^T |u_\varepsilon(s)|^2 ds < \infty;
\]

(c) the family \(\{u_\varepsilon\}_{\varepsilon > 0}\) is strongly convergent in \(L_2^2(0, T; \mathbb{R}^{m_1 + m_2})\) as \(\varepsilon \to 0\).
Whenever (a), (b), or (c) is satisfied, the strong limit of \( \{u_\varepsilon\}_{\varepsilon>0} \) is an open-loop saddle point of Problem (MF-SG) for the initial state \( x \).

**Remark 5.2.** Since there is no coupled system in the above perturbation approach and all the equations involved can be solved by iteration method, it will be much more convenient for computational purposes.

The perturbation approach in stochastic LQ control problems was initially introduced by Sun–Li–Yong [25], and further sharpened by Wang–Sun–Yong [31] for finding the so-called weak closed-loop optimal strategies. However, compared with the control problems [25, 31], in the game problem there are some new difficulties, especially in proving the boundedness of \( \{u_\varepsilon\}_{\varepsilon>0} \). Before proving Theorem 5.1, we present the following tailormade example, from which we can perceive some essential differences between the perturbation approaches of LQ game and control problems.

**Example 5.3.** For any \( x \in \mathbb{R} \), consider the one-dimensional state equation

\[
\begin{cases}
\dot{X}(s) = su_1(s) + u_2(s), & s \in [0, 1], \\
X(0) = x,
\end{cases}
\]  
(5.13)

and the quadratic functional

\[ J(x; u_1, u_2) = -|X(1)|^2 + \int_0^1 s^2|u_1(s)|^2ds. \]  
(5.14)

In the example, we let \( U_1 = U_2 \) be the space of \( \mathbb{R} \)-valued square-integrable functions on \([0, 1]\). Note that

\[
J(1; 0, u_2) = -|X(1)|^2 = -\left|1 + \int_0^1 u_2(s)ds\right|^2 \leq 0 = J(1; 0, -1), \quad \forall u_2 \in U_2;
\]

\[
J(1; u_1, -1) = -\int_0^1 su_1(s)ds^2 + \int_0^1 |su_1(s)|^2ds \geq 0 = J(1; 0, -1), \quad \forall u_1 \in U_1.
\]

Thus the control pair \((0, -1)\) is an open-loop saddle point for \( x = 1 \) and the convexity-concavity condition (5.1) holds.

For any \( \varepsilon > 0 \) and \((u_1, u_2) \in U_1 \times U_2\), denote

\[ J_\varepsilon(x; u_1, u_2) = J(x; u_1, u_2) - \varepsilon \int_0^1 |u_2(s)|^2ds. \]  
(5.15)

Then

\[
J_\varepsilon(0; 0, u_2) = J(0; 0, u_2) - \varepsilon \int_0^1 |u_2(s)|^2ds \leq -\varepsilon \int_0^1 |u_2(s)|^2ds, \quad \forall u_2 \in U_2; \tag{5.16}
\]

\[
J_\varepsilon(0; u_1, 0) = J(0; u_1, 0) \geq 0, \quad \forall u_1 \in U_1. \tag{5.17}
\]

Roughly speaking, the above implies that the convexity of the mapping \( u_1 \mapsto J_\varepsilon(x; u_1, u_2) \) equals that of \( u_1 \mapsto J(x; u_1, u_2) \) and the concavity of \( u_2 \mapsto J_\varepsilon(x; u_1, u_2) \) is stronger than that of \( u_2 \mapsto J(x; u_1, u_2) \). However, we will show that the LQ game problem associated with (5.13)–(5.15) has no open-loop saddle point for \( x = 1 \).

We shall prove the above claim by contradiction. Suppose that the LQ game problem associated with (5.13)–(5.15) has an open-loop saddle point \((u_1^*, u_2^*)\). Then we must have

\[ \int_0^1 u_2^*(s)ds = -1. \]  
(5.18)
Otherwise, \( u_1^* \) is optimal for the LQ control problem associated with the state equation

\[
\dot{X}(s) = su_1(s), \quad s \in [0, T]; \quad X(0) = h \triangleq 1 + \int_0^1 u_2^*(s)ds \neq 0, \tag{5.19}
\]

and the cost functional \( J(u_1) = J_\varepsilon(1; u_1, u_2^*) \). The corresponding optimal state is denoted by \( X^* \). Then by [25, Corollary 3.3], \( u_1^* \) satisfies the following stationary condition:

\[
sY^*(s) + s^2 u_1^*(s) = 0, \quad s \in [0, 1], \tag{5.20}
\]

where \( Y^* \) is the solution to the following adjoint equation:

\[
\dot{Y}^*(s) = 0, \quad s \in [0, T]; \quad Y^*(1) = -X^*(1). \tag{5.21}
\]

By solving (5.21), the stationary condition (5.20) can be rewritten as

\[
-sX^*(1) + s^2 u_1^*(s) = 0, \quad s \in [0, 1]. \tag{5.22}
\]

Recalling that \( u_1^* \in \mathcal{U}_1 \) is a square-integrable function, we get

\[
X^*(1) = 0, \tag{5.23}
\]

which implies that

\[
u_1^*(s) = 0, \quad s \in [0, 1]. \tag{5.24}
\]

Thus,

\[
0 = X^*(1) = h + \int_0^1 su_1^*(s)ds = h. \tag{5.25}
\]

This contradicts (5.19) and thus (5.18) holds. Further, by (5.18), note that

\[
J_\varepsilon(1; u_1, u_2^*) = -\left| \int_0^1 su_1(s)ds \right|^2 + \int_0^1 |su_1(s)|^2 ds - \varepsilon \int_0^1 |u_2^*(s)|^2 ds,
\]

\[
J_\varepsilon(1; u_1^*, u_2^*) = \inf_{u_1 \in \mathcal{U}_1} J_\varepsilon(1; u_1, u_2^*), \tag{5.26}
\]

then we have

\[
u_1^*(s) = 0, \quad s \in [0, 1]. \tag{5.27}
\]

It follows that \( u_2^* \) is optimal for the control problem associated with the state equation

\[
\dot{X}(s) = u_2(s), \quad s \in [0, 1]; \quad X(0) = 1, \tag{5.28}
\]

and the cost functional \( J(u_2) = -J_\varepsilon(1; 0, u_2) \). Then, \( u_2^* \) satisfies the stationary condition:

\[
Y^*(s) + \varepsilon u_2^*(s) = 0, \quad s \in [0, 1], \tag{5.29}
\]

with \( Y^* \) solving the adjoint equation:

\[
\dot{Y}^*(s) = 0, \quad s \in [0, 1]; \quad Y^*(1) = X^*(1) = 0. \tag{5.30}
\]

It follows that

\[
u_2^*(s) = \frac{Y^*(s)}{\varepsilon} = 0, \quad s \in [0, 1], \tag{5.31}
\]

which contradicts (5.18). Therefore, the claim is proved; that is the LQ game problem associated with (5.13)–(5.15) has no open-loop saddle points for \( x = 1 \).
In the LQ control problems [25, 31], the perturbed cost functional \( J_\varepsilon(x; u) \) is defined by adding \( \varepsilon \|u\|^2 \) to the original one \( J(x; u) \), where \( u \) is the control process. By the monotonicity of the mapping \( \varepsilon \mapsto V_\varepsilon(x) \), [25] showed that \( \|u_\varepsilon\|^2 \) is bounded by \( \|u^*\|^2 \), provided the original problem has an optimal control \( u^* \).

From Example 5.3, we see that the new game associated with \( J(x; u_1, u_2) - \varepsilon\|u_2\|^2 \) possibly has no saddle point even if the original is open-loop solvable. Thus in the perturbation approach of games, adding both \( \varepsilon \|u_1\|^2 \) and \( -\varepsilon\|u_2\|^2 \) to the original functional turns out to be necessary, due to which the value function \( V_\varepsilon(x) \) is not monotone in \( \varepsilon \). Noticing this key point, a seemingly feasible approach is to introduce two parameters \( \varepsilon_1, \varepsilon_2 > 0 \) and consider the game problem with the functional:

\[
J_{\varepsilon_1, \varepsilon_2}(x; u_1, u_2) = J(x; u_1, u_2) + \varepsilon_1 \|u_1\|^2 - \varepsilon_2 \|u_2\|^2.
\]

Since \( J_{\varepsilon_1, \varepsilon_2} \) is monotone in each \( \varepsilon_i \), it seems that the convergence of \( \{u_{\varepsilon_1, \varepsilon_2}\}_{\varepsilon_1, \varepsilon_2>0} \) can be obtained by letting \( \varepsilon_1 \to 0 \) and \( \varepsilon_2 \to 0 \) separately. But in fact, after letting \( \varepsilon_1 \to 0 \), one cannot take the limit by letting \( \varepsilon_2 \to 0 \), because the game with the functional \( J_{0, \varepsilon_2} \) is possibly unsolvable (see Example 5.3).

In conclusion, the perturbation approaches of LQ games (i.e., Theorem 5.1) and controls (i.e., [25, 31]) are essentially different.

### 5.2 Proof of Theorem 5.1

With the preparations in Sections 2.3 and Subsection 5.1, now we are ready to prove Theorem 5.1 by a Hilbert space method, in which Proposition 2.4 and the Mazur’s theorem (see Yosida [36, p.120, Theorem 2]) play important roles.

**Proof of Theorem 5.1.** (i) We begin by proving the implication (a) \( \Rightarrow \) (b). Let \( v^* \) be an open-loop saddle point of Problem (MF-SG) for the initial state \( x \). Then by Proposition 3.1, \( v^* \) must satisfy

\[
\mathcal{M}v^* + \mathcal{K}x = 0,
\]

where the operators \( \mathcal{M} \) and \( \mathcal{K} \) are defined by (3.9). On the other hand, Proposition 3.2 shows that for any \( \varepsilon > 0 \), the unique open-loop saddle point \( u_\varepsilon \) of Problem (MF-SG)_\varepsilon for \( x \) can be also given by

\[
u_\varepsilon = -\mathcal{M}_\varepsilon^{-1} \mathcal{K}x,
\]

where

\[
\mathcal{M}_\varepsilon \triangleq \mathcal{M} + \begin{pmatrix} \varepsilon I_{m_1} & 0 \\ 0 & -\varepsilon I_{m_2} \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{11} + \varepsilon I_{m_1} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} - \varepsilon I_{m_2} \end{pmatrix}.
\]

Combining (5.32) with (5.33) yields that

\[
u_\varepsilon = -\mathcal{M}_\varepsilon^{-1} \mathcal{K}x = \mathcal{M}_\varepsilon^{-1} \mathcal{M}v^*.
\]

Then by Proposition 2.4, noting (5.1), we have

\[
\|u_\varepsilon\|^2 = \|\mathcal{M}_\varepsilon^{-1} \mathcal{M}v^*\|^2 \leq \|\mathcal{M}_\varepsilon^{-1} \mathcal{M}\|^2 \|v^*\|^2 \leq \|v^*\|^2.
\]

Thus, \( \{u_\varepsilon\}_{\varepsilon>0} \) is bounded in the Hilbert space \( L^2_T(0, T; \mathbb{R}^{m_1 + m_2}) \).

(ii) We show that (b) \( \Rightarrow \) (a). For convenience, we let

\[
\sup_{\varepsilon>0} \|u_\varepsilon\| \leq \|\bar{u}\|, \quad \text{for some } \bar{u} \in \mathcal{U}_1 \times \mathcal{U}_2.
\]

Since \( \{u_\varepsilon\}_{\varepsilon>0} \) is bounded in the Hilbert space \( L^2_T(0, T; \mathbb{R}^{m_1 + m_2}) \), it admits a weakly convergent subsequence. We denote this subsequence by \( \{u_{\varepsilon_j}\}_{j \geq 1} = \{(u_{1j}, u_{2j})\}_{j \geq 1} \) and its weak limit by
By Mazur’s theorem there exist \( \lambda_{kj} \in [0, 1], j = 1, 2, ..., N_k \) such that
\[
\sum_{j=1}^{N_k} \lambda_{kj} = 1, \quad k = 1, 2, ..., \tag{5.38}
\]
and
\[
\left\| \sum_{j=1}^{N_k} \lambda_{kj} u_{kj} - u^* \right\| \to 0, \quad \text{as} \quad k \to \infty. \tag{5.39}
\]
By the convexity of the mapping \( u_1 \mapsto J(x; u_1, u_2) \), we have
\[
J(x; \sum_{j=1}^{N_k} \lambda_{kj} u_{1kj}^{\varepsilon_{k+j}}, \sum_{j=1}^{N_k} \lambda_{kj} u_{2kj}^{\varepsilon_{k+j}}) \leq \sum_{j=1}^{N_k} \lambda_{kj} J(x; u_{1kj}^{\varepsilon_{k+j}}, \sum_{j=1}^{N_k} \lambda_{kj} u_{2kj}^{\varepsilon_{k+j}})
\]
\[
= \sum_{j=1}^{N_k} \lambda_{kj} J_{\varepsilon_{k+j}}(x; u_{1kj}^{\varepsilon_{k+j}}, \sum_{j=1}^{N_k} \lambda_{kj} u_{2kj}^{\varepsilon_{k+j}}) - \sum_{j=1}^{N_k} \lambda_{kj} \varepsilon_{k+j}\|u_{1kj}^{\varepsilon_{k+j}}\|^2
\]
\[
+ \sum_{j=1}^{N_k} \lambda_{kj} \varepsilon_{k+j}\left( \left\| \sum_{j=1}^{N_k} \lambda_{kj} u_{2kj}^{\varepsilon_{k+j}} \right\|^2
\]
\[
\leq \sum_{j=1}^{N_k} \lambda_{kj} J_{\varepsilon_{k+j}}(x; u_{1kj}^{\varepsilon_{k+j}}, \sum_{j=1}^{N_k} \lambda_{kj} u_{2kj}^{\varepsilon_{k+j}}) + \sum_{j=1}^{N_k} \lambda_{kj} \varepsilon_{k+j} \left\| \sum_{j=1}^{N_k} \lambda_{kj} u_{2kj}^{\varepsilon_{k+j}} \right\|^2. \tag{5.40}
\]
Moreover, note that \((u_{1kj}^{\varepsilon_{k+j}}, u_{2kj}^{\varepsilon_{k+j}})\) is an open-loop saddle of Problem (MF-SG)\(_{\varepsilon_{k+j}}\). Thus for any \( u_1 \in U_1 \),
\[
J_{\varepsilon_{k+j}}(x; u_{1kj}^{\varepsilon_{k+j}}, \sum_{j=1}^{N_k} \lambda_{kj} u_{2kj}^{\varepsilon_{k+j}}) \leq J_{\varepsilon_{k+j}}(x; u_{1kj}^{\varepsilon_{k+j}}, u_{2kj}^{\varepsilon_{k+j}}) \leq J_{\varepsilon_{k+j}}(x; u_1, u_2). \tag{5.41}
\]
Substituting the above into (5.40) and then by the concavity of the mapping \( u_2 \mapsto J(x; u_1, u_2) \), we have
\[
J(x; \sum_{j=1}^{N_k} \lambda_{kj} u_{1kj}^{\varepsilon_{k+j}}, \sum_{j=1}^{N_k} \lambda_{kj} u_{2kj}^{\varepsilon_{k+j}}) \leq \sum_{j=1}^{N_k} \lambda_{kj} J_{\varepsilon_{k+j}}(x; u_{1kj}^{\varepsilon_{k+j}}, u_{2kj}^{\varepsilon_{k+j}}) + \sum_{j=1}^{N_k} \lambda_{kj} \varepsilon_{k+j} \|u_1\|^2
\]
\[
\leq \sum_{j=1}^{N_k} \lambda_{kj} J(x; u_1, u_2) + \sum_{j=1}^{N_k} \lambda_{kj} \varepsilon_{k+j} \|u_1\|^2 + \sum_{j=1}^{N_k} \lambda_{kj} \varepsilon_{k+j} \|u_2\|^2
\]
\[
\leq J(x; u_1, \sum_{j=1}^{N_k} \lambda_{kj} u_{2kj}^{\varepsilon_{k+j}}) + \sum_{j=1}^{N_k} \lambda_{kj} \varepsilon_{k+j} \|u_1\|^2 + \sum_{j=1}^{N_k} \lambda_{kj} \varepsilon_{k+j} \|u_2\|^2. \tag{5.42}
\]
Thus by (5.38)-(5.39) and the (strong) continuity of the mapping \((u_1, u_2) \mapsto J(x; u_1, u_2)\), letting \(k \to \infty\) in (5.42) yields that

\[
J(x; u_1^*, u_2^*) \leq J(x; u_1, u_2^*), \quad \forall u_1 \in \mathcal{U}_1. \tag{5.43}
\]

By the same argument as the above, we also have

\[
J(x; u_1^*, u_2^*) \geq J(x; u_1^*, u_2), \quad \forall u_2 \in \mathcal{U}_2. \tag{5.44}
\]

The result then concludes from (5.43)-(5.44).

(iii) The implication (c) \(\Rightarrow\) (b) is trivially true. We next prove (b) \(\Rightarrow\) (c). We first claim: The family \(\{u_\varepsilon\}_{\varepsilon > 0}\) is weakly convergent as \(\varepsilon \to 0\) and the weak limit is an open-loop saddle point of Problem (MF-SG) for \(x\).

If the above claim holds, then the family \(\{u_\varepsilon\}_{\varepsilon > 0}\) converges weakly to an open-loop saddle point \(u^*\) of Problem (MF-SG) as \(\varepsilon \to 0\). Thus by the weakly lower semicontinuity of the mapping \(u \mapsto \|u\|^2\), we have

\[
\mathbb{E} \int_0^T |u^*(s)|^2 ds \leq \liminf_{\varepsilon \to 0} \mathbb{E} \int_0^T |u_\varepsilon(s)|^2 ds. \tag{5.45}
\]

On the other hand, by (5.36), with \(v^*\) replaced by \(u^*\), we obtain

\[
\mathbb{E} \int_0^T |u_\varepsilon(s)|^2 ds \leq \mathbb{E} \int_0^T |u^*(s)|^2 ds, \quad \forall \varepsilon > 0. \tag{5.46}
\]

Combining (5.45) with (5.46) yields that

\[
\lim_{\varepsilon \to 0} \mathbb{E} \int_0^T |u_\varepsilon(s)|^2 ds = \mathbb{E} \int_0^T |u^*(s)|^2 ds.
\]

Recall that \(u^*\) is the weak limit of \(\{u_\varepsilon\}_{\varepsilon > 0}\). Then the above implies that

\[
\lim_{\varepsilon \to 0} \mathbb{E} \int_0^T |u_\varepsilon(s) - u^*(s)|^2 ds = 0.
\]

It shows that \(\{u_\varepsilon\}_{\varepsilon > 0}\) converges strongly to \(u^*\) as \(\varepsilon \to 0\). Thus to show (b) \(\Rightarrow\) (c), we only need to prove the claim.

Noting that \(L^2_F(0, T; \mathbb{R}^{m_1+m_2})\) is a Hilbert space, to verify the claim, it suffices to show that every weakly convergent subsequence of \(\{u_\varepsilon\}_{\varepsilon > 0}\) has the same weak limit, which is an open-loop saddle point of Problem (MF-SG) for \(x\). Let \(u^*\) and \(\hat{u}\) be the weak limits of two different weakly convergent subsequences \(\{u_{\varepsilon_i}\}_{i=1}^\infty\) \((i = 1, 2)\) of \(\{u_\varepsilon\}_{\varepsilon > 0}\). Then by the same argument as in the proof of (b) \(\Rightarrow\) (a), we can show that both \(u^*\) and \(\hat{u}\) are open-loop saddle points. By the convexity of the mapping \(u_1 \mapsto J(x; u_1, u_2)\), we have

\[
J \left( x; \frac{u_1^* + \hat{u}_1}{2}, \frac{u_2^* + \hat{u}_2}{2} \right) \leq \frac{1}{2} J \left( x; u_1^*, \frac{u_2^* + \hat{u}_2}{2} \right) + \frac{1}{2} J \left( x; \hat{u}_1, \frac{u_2^* + \hat{u}_2}{2} \right). \tag{5.47}
\]

Noting that both \(u^*\) and \(\hat{u}\) are open-loop saddle points, and the mapping \(u_2 \mapsto J(x; u_1, u_2)\) is concave, we have

\[
\frac{1}{2} J \left( x; u_1^*, \frac{u_2^* + \hat{u}_2}{2} \right) \leq \frac{1}{2} J \left( x; u_1^*, u_2^* \right) + \frac{1}{2} J \left( x; \hat{u}_1, \frac{u_2^* + \hat{u}_2}{2} \right)
\]

\[
\leq \frac{1}{2} J \left( x; u_1^*, u_2^* \right) + \frac{1}{2} J \left( x; \hat{u}_1, \hat{u}_2 \right)
\]

\[
\leq \frac{1}{2} J \left( x; u_1, u_2^* \right) + \frac{1}{2} J \left( x; u_1, \hat{u}_2 \right)
\]

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Thus,
\[ J\left(x; \frac{u_1^* + \hat{u}_1}{2}, \frac{u_2^* + \hat{u}_2}{2}\right) \leq J\left(x; u_1, \frac{u_2^* + \hat{u}_2}{2}\right), \quad \forall u_1 \in \mathcal{U}_1. \]  
(5.48)

Similarly, we can prove
\[ J\left(x; \frac{u_1^* + \hat{u}_1}{2}, \frac{u_2 + \hat{u}_2}{2}\right) \geq J\left(x; \frac{u_1^* + \hat{u}_1}{2}, u_2\right), \quad \forall u_2 \in \mathcal{U}_2. \]  
(5.50)

Combining (5.49) with (5.50), we get that \( \frac{u_1^* + \hat{u}_1}{2} \) is also an open-loop saddle point of Problem (MF-SG) with respect to \( x \). Thus by (5.36), with \( v^* \) replaced by \( \frac{u_1^* + \hat{u}_1}{2} \), we obtain
\[ \mathbb{E} \int_0^T |u_{x_1}(s)|^2 ds \leq \mathbb{E} \int_0^T \left| \frac{u^*(s) + \hat{u}(s)}{2}\right|^2 ds, \quad i = 1, 2. \]  
(5.51)

By the weakly lower semicontinuity of the mapping \( u \mapsto ||u||^2 \) again, we have
\[ \mathbb{E} \int_0^T |u^*(s)|^2 ds \leq \liminf_{k \to \infty} \mathbb{E} \int_0^T |u_{x_1}(s)|^2 ds, \]
\[ \mathbb{E} \int_0^T |\hat{u}(s)|^2 ds \leq \liminf_{k \to \infty} \mathbb{E} \int_0^T |u_{x_2}(s)|^2 ds. \]

Thus by taking inferior limits on the both sides of (5.51), we get
\[ \mathbb{E} \int_0^T |u^*(s)|^2 ds \leq \mathbb{E} \int_0^T \left| \frac{u^*(s) + \hat{u}(s)}{2}\right|^2 ds, \]
\[ \mathbb{E} \int_0^T |\hat{u}(s)|^2 ds \leq \mathbb{E} \int_0^T \left| \frac{u^*(s) + \hat{u}(s)}{2}\right|^2 ds, \]
which implies that
\[ \mathbb{E} \int_0^T |u^*(s) - \hat{u}(s)|^2 ds \leq 0. \]

The claim is established. \( \blacksquare \)

Remark 5.4. The boundedness of \( ||M^{-1}_z M||^2 \) is sufficient for that of \( ||u_{x_1}||^2 \), which implies that \( \{u_x\}_{x > 0} \) admits a weakly convergent subsequence. With the help of Mazur’s theorem, the (strong) convergence of \( \{u_x\}_{x > 0} \) follows from the explicit upper bound (which is exactly 1) of \( ||M^{-1}_z M||^2 \).

Remark 5.5. If Problem (MF-SG) reduces to a (mean-field) LQ optimal control problem, then \( \mathcal{M} = \mathcal{M}_1 \) and (5.35) becomes
\[ u_x = M_1^{-1} M_1 v^* = (M_1 + \varepsilon I)^{-1} M_1 v^*. \]

Note that \( \mathcal{M}_1 \) is a positive operator, while \( \mathcal{M} \) is indefinite in general. Then the Hilbert space method brings the following new viewpoint: The perturbation approaches of LQ controls and games are the outcomes of the explicit norm estimates for perturbed positive operators and indefinite operators, respectively.

6 Example
In this section, we present a simple example to illustrate the procedure for finding the open-loop saddle points by Theorem 4.6 under the sufficient condition (3.19); and identifying the open-loop solvability by Theorem 5.1 under the necessary condition (3.18).
Example 6.1. Consider the one-dimensional state equation
\[
\begin{aligned}
dX(s) &= u_1(s)ds + u_2(s)dW(s), \quad s \in [0, 1], \\
X(0) &= x,
\end{aligned}
\] (6.1)
and the quadratic functional
\[
J(x; u_1, u_2) = \mathbb{E}\left\{ -|X(1)|^2 + \int_0^1 \left( |u_1(s)|^2 - |\mathbb{E}[u_2(s)]|^2 \right) ds \right\}.
\] (6.2)
It is straightforward to see that
\[
J(0; u_1, 0) \geq 0 \quad \text{and} \quad J(0; 0, u_2) \leq 0, \quad \forall (u_1, u_2) \in U_1 \times U_2.
\] (6.3)
Suppose that \((u_1^*, u_2^*)\) is an open-loop saddle point, then by Lemma 4.8, \((u_1^*, u_2^*)\) must satisfy
\[
u_1^* = -Y, \quad \mathbb{E}[u_2^*] = -Z,
\]
with
\[
\begin{aligned}
dX(s) &= -Y(s)ds - Z(s)dW(s), \\
dY(s) &= Z(s)dW(s), \\
X(0) &= x, \quad Y(T) = -X(T).
\end{aligned}
\] (6.4)
Note that the solution of the Riccati equation associated with (6.4) is \(-\frac{1}{s}\), which is not integrable over \([0, 1]\), thus the decoupling technique is not applicable. Moreover, FBSDE (6.4) does not satisfy the so-called monotone condition in Hu–Peng [12]. Thus we also cannot determine the solvability of (6.4) by [12] directly.

In the following, let us apply Theorem 5.1 to determine the open-loop solvability of the game. For any \(\varepsilon > 0\), we denote
\[
J_\varepsilon(x; u_1, u_2) = J(x; u_1, u_2) + \varepsilon \mathbb{E} \int_0^1 |u_1(s)|^2 ds - \varepsilon \mathbb{E} \int_0^1 |u_2(s)|^2 ds.
\] (6.5)
By (6.3), we have
\[
J_\varepsilon(0; u_1, 0) \geq \varepsilon \mathbb{E} \int_0^1 |u_1(s)|^2 ds, \quad \forall u_1 \in U_1,
\] (6.6)
\[
J_\varepsilon(0; 0, u_2) \leq -\varepsilon \mathbb{E} \int_0^1 |u_2(s)|^2 ds, \quad \forall u_2 \in U_2.
\]
Then we can apply Theorem 4.6 to find the unique open-loop saddle point of Problem (MF-SG)_\varepsilon with state equation (6.1) and functional (6.5). The corresponding Riccati equations (5.5)–(5.6) in the example read:
\[
\begin{aligned}
\dot{P}_\varepsilon(s) - \frac{P_\varepsilon(s)^2}{1 + \varepsilon} &= 0, \quad s \in [0, 1], \\
\dot{I}_\varepsilon(s) - \frac{I_\varepsilon(s)^2}{1 + \varepsilon} &= 0, \quad s \in [0, 1], \\
P_\varepsilon(1) &= -1, \quad I_\varepsilon(1) = -1.
\end{aligned}
\] (6.7)
Solving (6.7) by separating variables, we get
\[
P_\varepsilon(s) = -\frac{1 + \varepsilon}{s + \varepsilon}, \quad I_\varepsilon(s) = -\frac{1 + \varepsilon}{s + \varepsilon}, \quad s \in [0, 1].
\] (6.8)
Define the corresponding feedback operators \(\Theta_\varepsilon\) (5.10) and \(\bar{\Theta}_\varepsilon\) (5.11) by
\[
\Theta_\varepsilon(s) = \begin{pmatrix} \frac{1}{s + \varepsilon} \\ 0 \end{pmatrix}, \quad \bar{\Theta}_\varepsilon(s) = \begin{pmatrix} \frac{1}{s + \varepsilon} \\ 0 \end{pmatrix}, \quad s \in [0, 1].
\] (6.9)
Then the unique open-loop saddle point of Problem (MF-SG)$_\varepsilon$ is given by
\[
\begin{align*}
u_\varepsilon = \left( \begin{array}{c} u_1^\varepsilon \\ u_2^\varepsilon \end{array} \right) = \Theta \left( X_\varepsilon - \mathbb{E}[X_\varepsilon] \right) + \Theta \mathbb{E}[X_\varepsilon] = \left( \begin{array}{c} \frac{X_\varepsilon}{\varepsilon} \\ 0 \end{array} \right), \quad s \in [0, 1],
\end{align*}
\] (6.10)
with $X_\varepsilon$ being the unique solution to the following closed-loop system:
\[
\begin{align*}
dX_\varepsilon(s) &= \frac{X_\varepsilon(s)}{s + \varepsilon} ds, \quad s \in [0, 1],
X_\varepsilon(0) = x.
\end{align*}
\] (6.11)

By the variation of constants formula for ordinary differential equations, we get
\[
X_\varepsilon(s) = \frac{s + \varepsilon}{\varepsilon} x, \quad s \in [0, 1].
\] (6.12)

Combining the above with (6.10), we obtain the following explicit representation of $u_\varepsilon$:
\[
u_\varepsilon = \left( \begin{array}{c} u_1^\varepsilon \\ u_2^\varepsilon \end{array} \right) = \left( \begin{array}{c} \frac{X_\varepsilon(s)}{s} \\ 0 \end{array} \right), \quad s \in [0, 1].
\] (6.13)

Moreover, note that
\[
\sup_{\varepsilon > 0} \mathbb{E} \int_0^1 |u_\varepsilon(s)|^2 = \infty, \quad \text{if} \quad x \neq 0,
\] (6.14)
and
\[
\sup_{\varepsilon > 0} \mathbb{E} \int_0^1 |u_\varepsilon(s)|^2 = 0, \quad \text{if} \quad x = 0.
\] (6.15)

Thus according to Theorem 5.1, the problem has no open-loop saddle point for $x \neq 0$ and has an open-loop saddle point $(0, 0)$ for $x = 0$.

7 Appendix

Proof of Lemma 2.3. The proof is divided into four cases.

Case 1. If $K$ is invertible and $M$ is positive definite, then
\[
L^T ML - L^T MK(K^T MK + \delta I_m)^{-1} K^T ML
\geq L^T ML - L^T MK(K^T MK)^{-1} K^T ML = 0.
\] (7.1)

Case 2. If $m = n$, then there exist a sequence of invertible matrices $\{K_\varepsilon\}_{\varepsilon > 0}$ and a sequence of positive definite matrices $\{M_\varepsilon\}_{\varepsilon > 0}$ such that
\[
\lim_{\varepsilon \to 0} K_\varepsilon = K \quad \text{and} \quad \lim_{\varepsilon \to 0} M_\varepsilon = M.
\] (7.2)

Then from the facts $M \geq 0$, $\delta > 0$ and the result obtained in Case 1, we have
\[
L^T ML - L^T MK(K^T MK + \delta I_m)^{-1} K^T ML
= \lim_{\varepsilon \to 0} \left[ L^T M_\varepsilon L - L^T M_\varepsilon K_\varepsilon (K_\varepsilon^T M_\varepsilon K_\varepsilon + \delta I_m)^{-1} K_\varepsilon^T M_\varepsilon L \right] \geq 0.
\] (7.3)

Case 3. If $n > m$, set $\hat{K} = (K, 0)$ such that $\hat{K} \in \mathbb{R}^{n \times n}$, where $0$ is the zero matrix with an appropriate dimension. Then by the results obtained in Case 2, we have
\[
0 \leq L^T ML - L^T M\hat{K}(\hat{K}^T M\hat{K} + \delta I_n)^{-1} \hat{K}^T ML
\]
\[
= L^\top ML - \begin{pmatrix} L^\top MK & 0 \end{pmatrix} \begin{pmatrix} K^\top MK + \delta I_m & 0 \\ 0 & \delta I_{n-m} \end{pmatrix}^{-1} \begin{pmatrix} K^\top ML \end{pmatrix}
\]
\[
= L^\top ML - L^\top MK(K^\top MK + \delta I_m)^{-1}K^\top ML. \tag{7.4}
\]

**Case 4.** If \( n < m \), set \( \bar{L} = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix}, \bar{M} = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}, \bar{K} = \begin{pmatrix} K \\ 0 \end{pmatrix} \), such that \( \bar{L}, \bar{K} \in \mathbb{R}^{m \times m} \) and \( \bar{M} \in S_+^m \). By the results obtained in Case 2 again, we have
\[
0 \leq L^\top \bar{M} \bar{L} - L^\top \bar{M} \bar{K}(\bar{K}^\top \bar{M} \bar{K} + \delta I_m)^{-1}\bar{K}^\top \bar{M} \bar{L}
\]
\[
= \begin{pmatrix} L^\top ML & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} L^\top MK \\ 0 \end{pmatrix} (K^\top MK + \delta I_m)^{-1} \begin{pmatrix} K^\top ML \end{pmatrix}
\]
\[
= \begin{pmatrix} L^\top ML - L^\top MK(K^\top MK + \delta I_m)^{-1}K^\top ML & 0 \\ 0 & 0 \end{pmatrix}, \tag{7.5}
\]
which implies that
\[
L^\top ML - L^\top MK(K^\top MK + \delta I_m)^{-1}K^\top ML \geq 0.
\]

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