A Semidefinite Programming Method for Integer Convex Quadratic Minimization

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Abstract

We consider the NP-hard problem of minimizing a convex quadratic function over the integer lattice \( \mathbb{Z}^n \). We present a semidefinite programming (SDP) method for obtaining a nontrivial lower bound on the optimal value of the problem. By interpreting the solution to the SDP relaxation probabilistically, we obtain a randomized algorithm for finding good suboptimal solutions. The effectiveness of the method is shown for numerical problem instances of various sizes. Finally, we introduce several extensions to the idea, including how to reduce the search space of existing branch-and-bound type enumeration algorithms for solving the problem globally, by using the tighter lower and upper bounds.

1 Introduction

We consider the problem

\[
\text{minimize} \quad f(x) = x^T P x + 2q^T x \\
\text{subject to} \quad x \in \mathbb{Z}^n,
\]

with variable \( x \), where \( P \in \mathbb{R}^{n \times n} \) is symmetric and positive definite, and \( q \in \mathbb{R}^n \).

A number of other problems can be reduced to the form of (1). The integer least squares problem,

\[
\text{minimize} \quad \|Ax - b\|_2^2 \\
\text{subject to} \quad x \in \mathbb{Z}^n,
\]

with variable \( x \) and data \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \), is easily reduced to the form of (1) by expanding out the objective function, provided that \( A^T A \) is positive definite. The mixed integer version of the problem, where some components of \( x \) are allowed to be real numbers, also reduces to an equivalent problem with integer variables only. This transformation uses Schur complement to explicitly minimize over the noninteger variables [BV04 §A.5.5].

Another equivalent formulation of (1) is the closest vector problem,

\[
\text{minimize} \quad \|v - z\|_2^2 \\
\text{subject to} \quad z \in \{Bx \mid x \in \mathbb{Z}^n\},
\]
in the variable $z \in \mathbb{R}^n$. Although not equivalent to (1), the **shortest vector problem** is also a closely related problem, which in fact, is reducible to solving the closest vector problem:

$$
\begin{align*}
\text{minimize} & \quad \|z\|_2^2 \\
\text{subject to} & \quad z \in \{Bx \mid x \in \mathbb{Z}^n\} \\
& \quad z \neq 0.
\end{align*}
$$

Problem (1) arises in several applications. For example, in position estimation using the Global Positioning System (GPS), resolving the integer ambiguities of the phase data is posed as a mixed integer least squares problem [HB98]. In multiple-input multiple-output (MIMO) wireless communication systems, maximum likelihood detection of (vector) Boolean messages involves solving an integer least squares problem [JO05]. The mixed integer version of the least squares problem appears in data fitting applications, where some parameters are integer-valued. (See, e.g., [UR15].) The closest vector problem and shortest vector problem have numerous application areas in cryptanalysis of public key cryptosystem such as RSA [NS01]. The spectral test, which is used to check the quality of linear congruential random number generators, is an application of the shortest vector problem [Knu97, §3.3.4].

Several hardness results are known for the integer least squares problem (2). Given an instance of the integer least squares problem, define the approximation factor of a point $x$ to be $\|Ax - b\|_2^2 / \|Ax^\ast - b\|_2^2$, where $x^\ast$ is the global (integer) solution of (2). Finding a constant factor approximation is an NP-hard problem [ABSS93]. In fact, finding an approximation still remains NP-hard even when the approximation factor is $n^{c/\log \log n}$, where $c > 0$ is some constant [DKRS03].

Standard methods for obtaining the optimal solution to (1) work by enumerating all integer points within a suitably chosen box or ellipsoid [FP85]. The worst case running time of these methods is exponential in $n$, making it impractical for problems of large size. Algorithms such as Lenstra–Lenstra–Lovász lattice reduction algorithm [LLL82] can be used to find an approximate solution in polynomial time, but the approximation factor guarantee is exponential in $n$ [GLS, §5.3].

A simple lower bound on $f^\ast$, the optimal value of (1), can be obtained by removing the integer constraint. This continuous relaxation has the solution $x^{cts} = -P^{-1}q$, with objective value $f^{cts} = -q^TP^{-1}q$. A method of obtaining a better lower bound than $f^{cts}$ was developed in [BHS15, BCL12]. This lower bound is computed by finding another quadratic function $\tilde{f}$ that is a global underestimator of $f$, and has an additional property that the integer point minimizing $\tilde{f}$ can be found efficiently, e.g., separable or well-conditioned quadratic functions.

A simple upper bound on $f^\ast$ can also be obtained by observing some properties of the problem. First of all, $x = 0$ gives a trivial upper bound of $f(0) = 0$, which immediately gives $f^\ast \leq 0$. Another simple approximate solution can be obtained by rounding each entry of $x^{cts}$ to the nearest integer point, $x^{rnd}$. Let $f^{rnd} = f(x^{rnd})$. To get a bound on $f^{rnd}$, we start by rewriting the objective function as

$$
f(x) = (x - x^{cts})^T P(x - x^{cts}) + f^{cts}.
$$
Since rounding changes each coordinate by at most $1/2$, we have

$$
\|x^{\text{rnd}} - x^{\text{cts}}\|_2^2 = \sum_{i=1}^{n} (x^{\text{rnd}}_i - x^{\text{cts}}_i)^2 \leq n/4.
$$

It follows that

$$
f^{\text{rnd}} - f^{\text{cts}} = (x^{\text{rnd}} - x^{\text{cts}})^T P (x^{\text{rnd}} - x^{\text{cts}}) \leq \sup_{\|v\|_2 \leq \sqrt{n}/2} v^T P v = (n/4) \omega_{\text{max}},
$$

where $\omega_{\text{max}}$ is the largest eigenvalue of $P$. Since $f^{\text{cts}}$ is a lower bound on $f^*$, this inequality bounds the suboptimality of $x^{\text{cts}}$. We note that in the special case of diagonal $P$, the objective function is separable, and thus the rounded solution is optimal. However, in general, $x^{\text{rnd}}$ is not optimal, and in fact, $f^{\text{rnd}}$ can be positive, which is even worse than the trivial upper bound $f(0) = 0$.

### 2 Lagrange duality

In this section, we discuss a Lagrangian relaxation for obtaining a nontrivial lower bound to (1). We make two assumptions without loss of generality. Firstly, we assume that $x^{\text{cts}} \notin \mathbb{Z}^n$, for otherwise $x^{\text{cts}}$ is already the global solution. Secondly, we assume that $x^{\text{cts}}$ is in the box $[0, 1]^n$. For any arbitrary problem instance, we can translate the coordinates in the following way to satisfy this assumption. Note that for any $v \in \mathbb{Z}^n$, the problem below is equivalent to (1):

$$
\begin{align*}
\text{minimize} & \quad (x - v)^T P (x - v) + 2 (P v + q)^T (x - v) + f(v) \\
\text{subject to} & \quad x \in \mathbb{Z}^n.
\end{align*}
$$

By renaming $x - v$ to $x$ and ignoring the constant term $f(v)$, the problem can be rewritten in the form of (1). Clearly, this has different optimal solution as well as different objective value from the original problem, but the two problems are related by a simple change of coordinates: any point $x$ in the new problem corresponds to $x + v$ in the original problem. To translate the coordinates, find $x^{\text{cts}} = -P^{-1} q$, and take elementwise floor to $x^{\text{cts}}$ to get $x^{\text{flr}}$. Then, substitute $x^{\text{flr}}$ in place of $v$ above.

We note a simple fact that every integer point $x$ satisfies either $x_i \leq 0$ or $x_i \geq 1$ for all $i$. Equivalently, this condition can be written as $x_i (x_i - 1) \geq 0$ for all $i$. Using this, we relax the integer constraint $x \in \mathbb{Z}^n$ into a set of nonconvex quadratic constraints: $x_i (x_i - 1) \geq 0$ for all $i$. The following nonconvex problem is then a relaxation of (1):

$$
\begin{align*}
\text{minimize} & \quad x^T P x + 2 q^T x \\
\text{subject to} & \quad x_i (x_i - 1) \geq 0, \quad i = 1, \ldots, n.
\end{align*}
$$

It is easy to see that the optimal value of (4) is greater than or equal to $f^{\text{cts}}$, because $x^{\text{cts}}$ is not a feasible point, due to the two assumptions that $x^{\text{cts}} \notin \mathbb{Z}^n$ and $x^{\text{cts}} \in [0, 1]^n$. Note that
the second assumption was necessary, for otherwise \( x^{\text{cts}} \) is the global optimum of (4), and the Lagrangian relaxation described below would not produce a lower bound that is better than \( f^{\text{cts}} \).

The Lagrangian of (4) is given by

\[
L(x, \lambda) = x^T P x + 2q^T x - \sum_{i=1}^n \lambda_i x_i (x_i - 1) = x^T (P - \text{diag}(\lambda)) x + 2(q + (1/2)\lambda)^T x,
\]

where \( \lambda \in \mathbb{R}^n \) is the vector of dual variables. Define \( \bar{q}(\lambda) = q + (1/2)\lambda \). By minimizing the Lagrangian over \( x \), we get the Lagrangian dual function

\[
g(\lambda) = \begin{cases} 
-\bar{q}(\lambda)^T (P - \text{diag}(\lambda))^{\dagger} \bar{q}(\lambda) & P - \text{diag}(\lambda) \succeq 0, \quad \bar{q}(\lambda) \in \mathcal{R}(P - \text{diag}(\lambda)) \\
-\infty & \text{otherwise},
\end{cases}
\]

where \( \mathcal{R}(A) \) denotes the range of \( A \), and the inequality \( \succeq \) is with respect to the positive semidefinite cone. The Lagrangian dual problem is then

\[
\text{maximize} \quad g(\lambda) \\
\text{subject to} \quad \lambda \geq 0,
\]

in the variable \( \lambda \in \mathbb{R}^n \), or equivalently,

\[
\text{maximize} \quad -\bar{q}(\lambda)^T (P - \text{diag}(\lambda))^{\dagger} \bar{q}(\lambda) \\
\text{subject to} \quad P - \text{diag}(\lambda) \succeq 0 \\
\quad \bar{q}(\lambda) \in \mathcal{R}(P - \text{diag}(\lambda)) \\
\quad \lambda \geq 0.
\]

Also, by using Schur complements, the problem can be reformulated into a semidefinite program (SDP):

\[
\text{maximize} \quad -\gamma \\
\text{subject to} \quad \begin{bmatrix} P - \text{diag}(\lambda) & q + (1/2)\lambda \\ (q + (1/2)\lambda)^T & \gamma \end{bmatrix} \succeq 0 \\
\quad \lambda \geq 0,
\]

in the variables \( \lambda \in \mathbb{R}^n \) and \( \gamma \in \mathbb{R} \).

We note that any feasible \( \lambda \) for (6) yields a lower bound on \( f^* \). To show this property, let \( \tilde{x} \) be any feasible point to (4), and let \( \lambda \geq 0 \). Then we have

\[
\sum_{i=1}^n \lambda_i \tilde{x}_i (\tilde{x}_i - 1) \geq 0,
\]

and thus

\[
L(\tilde{x}, \lambda) \leq f(\tilde{x}).
\]

It follows that

\[
g(\lambda) = \inf_x L(x, \lambda) \leq L(\tilde{x}, \lambda) \leq f(\tilde{x}).
\]
Since \( \tilde{x} \) was an arbitrary feasible point to (4), and because (4) is a relaxation to the original problem (1) which has \( f^* \) as the optimal value, we have

\[
g(\lambda) \leq f^*,
\]

for any \( \lambda \geq 0 \).

### 2.1 Comparison to simple lower bound

We observed in the previous section that \( g(\lambda) \leq f^* \) for any \( \lambda \geq 0 \), where \( g(\lambda) \) is defined by (5). Using this property, we show a provable bound on the Lagrangian lower bound. Let \( f^{\text{cts}} = -q^T P^{-1} q \) be the simple lower bound on \( f^* \), and \( f^{\text{sdp}} = \sup_{\lambda \geq 0} g(\lambda) \) be the lower bound obtained by solving the Lagrangian dual. Also, let \( \omega_1 \geq \cdots \geq \omega_n \) be the eigenvalues of \( P \). For clarity of notation, we use \( \omega_{\max} \) and \( \omega_{\min} \) to denote the largest and smallest eigenvalues of \( P \), namely \( \omega_1 \) and \( \omega_n \).

**Theorem 1.** The lower bounds satisfy

\[
f^{\text{sdp}} - f^{\text{cts}} \geq \frac{n\omega_{\min}^2}{4\omega_{\max}} \left( 1 - \frac{\|x^{\text{cts}} - (1/2) \mathbf{1}\|_2}{n/4} \right)^2.
\]  

(8)

**Proof.** Let \( P = Q \text{diag}(\omega) Q^T \) be the eigenvalue decomposition of \( P \), where \( \omega = (\omega_1, \ldots, \omega_n) \).

We consider \( \lambda \) of the form \( \lambda = \alpha \mathbf{1} \), and rewrite the dual function in terms of \( \alpha \), where \( \alpha \) is restricted to the range \( \alpha \in [0, \omega_{\min}) \):

\[
g(\alpha) = - (q + (1/2)\alpha \mathbf{1})^T (P - \alpha I)^{-1} (q + (1/2)\alpha \mathbf{1}).
\]

We note that \( g(0) = -q^T P^{-1} q = f^{\text{cts}} \), so it is enough to show the same lower bound on \( g(\alpha) - g(0) \) for any particular value of \( \alpha \).

Let \( s = Q^T \mathbf{1} \), and \( \tilde{q} = Q^T q \). By expanding out \( g(\alpha) \) in terms of \( s, \tilde{q}, \) and \( \omega \), we get

\[
g(\alpha) = - \sum_{i=1}^n \frac{\tilde{q}_i^2 + \alpha s_i \tilde{q}_i^2 + (1/4) \alpha^2 s_i^2}{\omega_i - \alpha}.
\]
Then,
\[
g(\alpha) - g(0) = \sum_{i=1}^{n} \frac{\tilde{q}_i^2}{\omega_i} - \sum_{i=1}^{n} \frac{\tilde{q}_i^2 + \alpha s_i \tilde{q}_i^2 + (1/4)\alpha^2 s_i^2}{\omega_i - \alpha}
\]
\[
= -\sum_{i=1}^{n} \frac{(\alpha/\omega_i)(\tilde{q}_i + (1/2)\omega_i s_i)^2 - (1/4)\alpha s_i^2(\omega_i - \alpha)}{\omega_i - \alpha}
\]
\[
= \alpha \sum_{i=1}^{n} s_i^2 - \sum_{i=1}^{n} \frac{\alpha(\tilde{q}_i + (1/2)\omega_i s_i)^2}{\omega_i(\omega_i - \alpha)}
\]
\[
= \frac{\alpha n}{4} - \alpha \sum_{i=1}^{n} \left(1 - \frac{\alpha}{\omega_i} \right) \left(\frac{\tilde{q}_i + (1/2)\omega_i s_i}{\omega_i - \alpha} \right)^2.
\]

By differentiating the expression above with respect to \(\alpha\), we get
\[
\frac{dg}{d\alpha} = \frac{n}{4} - \frac{\alpha n}{4} \sum_{i=1}^{n} \left(1 - \frac{\alpha}{\omega_i} \right) \left(\frac{\tilde{q}_i + (1/2)\omega_i s_i}{\omega_i - \alpha} \right)^2.
\]

We note that \(dg/d\alpha\) is a decreasing function in \(\alpha\) in the interval \([0, \omega_{\min})\). Also, at \(\alpha = 0\), we have
\[
\frac{dg(0)}{d\alpha} = \frac{n}{4} - \sum_{i=1}^{n} \left(\frac{\tilde{q}_i}{\omega_i} + (1/2)s_i\right)^2
\]
\[
= \frac{n}{4} - \left\|\text{diag}(\omega)^{-1}Q^Tq + (1/2)QQ^T1\right\|_2^2
\]
\[
= \frac{n}{4} - \left\|-Q\text{diag}(\omega)^{-1}Q^Tq - (1/2)QQ^T1\right\|_2^2
\]
\[
= \frac{n}{4} - \left\|-P^{-1}q - (1/2)1\right\|_2^2
\]
\[
= \frac{n}{4} - \left\|x_{cts} - (1/2)1\right\|_2^2
\]
\[
\geq 0.
\]

The last line used the fact that \(x_{cts}\) is in the box \([0, 1]^n\).

Suppose that \(dg/d\alpha \geq 0\) on the range \([0, \omega_{\min})\). That is, for every \(\alpha \in [0, \omega_{\min})\),
\[
\sum_{i=1}^{n} \left(\frac{\tilde{q}_i + (1/2)\omega_i s_i}{\omega_i - \alpha}\right)^2 \leq \frac{n}{4}.
\]
Then, for all \( \alpha \in [0, \omega_{\text{min}}) \),
\[
fsdp - g(0) \geq g(\alpha) - g(0) = \frac{\alpha n}{4} - \alpha \sum_{i=1}^{n} \left(1 - \frac{\alpha}{\omega_i}\right) \left(\frac{\tilde{q}_i + (1/2)\omega_i s_i}{\omega_i - \alpha}\right)^2 \\
\geq \frac{\alpha n}{4} - \alpha \left(1 - \frac{\alpha}{\omega_{\text{max}}}\right) \sum_{i=1}^{n} \left(\frac{\tilde{q}_i + (1/2)\omega_i s_i}{\omega_i - \alpha}\right)^2 \\
\geq \frac{\alpha n}{4} - \alpha \left(1 - \frac{\alpha}{\omega_{\text{max}}}\right) n \frac{n}{4} \\
= \frac{n\alpha^2}{4\omega_{\text{max}}}.
\]
and thus,
\[
fsdp - g(0) \geq \lim_{\alpha \to \omega_{\text{min}}} \frac{n\alpha^2}{4\omega_{\text{max}}} = \frac{n\omega_{\text{min}}^2}{4\omega_{\text{max}}}.\]

Now, suppose that \( dg/d\alpha = 0 \) is achieved at some \( \alpha^* \in [0, \omega_{\text{min}}) \). Then, we have
\[
g(\alpha^*) - g(0) = \frac{\alpha^* n}{4} - \alpha^* \sum_{i=1}^{n} \left(1 - \frac{\alpha^*}{\omega_i}\right) \left(\frac{\tilde{q}_i + (1/2)\omega_i s_i}{\omega_i - \alpha^*}\right)^2 \\
= \alpha^* \left(\frac{n}{4} - \sum_{i=1}^{n} \left(\frac{\tilde{q}_i + (1/2)\omega_i s_i}{\omega_i - \alpha^*}\right)^2\right) + \sum_{i=1}^{n} \frac{\alpha^*^2}{\omega_i} \left(\frac{\tilde{q}_i + (1/2)\omega_i s_i}{\omega_i - \alpha^*}\right)^2 \\
= \alpha^* \sum_{i=1}^{n} \frac{\omega_i}{\omega_{\text{max}}} \left(\frac{\tilde{q}_i + (1/2)\omega_i s_i}{\omega_i - \alpha^*}\right)^2 \\
\geq \alpha^* \sum_{i=1}^{n} \left(\frac{\tilde{q}_i + (1/2)\omega_i s_i}{\omega_i - \alpha^*}\right)^2 \\
= \frac{n\alpha^*^2}{4\omega_{\text{max}}}.
\]
Using this, we go back to the equation \( dg(\alpha^*)/d\alpha = 0 \) and establish a lower bound on \( \alpha^* \):
\[
\frac{n}{4} = \sum_{i=1}^{n} \left(\frac{\tilde{q}_i + (1/2)\omega_i s_i}{\omega_i - \alpha^*}\right)^2 \\
= \sum_{i=1}^{n} \frac{\omega_i}{\omega_{\text{min}} - \alpha^*} \left(\frac{\tilde{q}_i}{\omega_i} + (1/2)s_i\right)^2 \\
\leq \frac{\omega_{\text{min}}}{\omega_{\text{min}} - \alpha^*} \sum_{i=1}^{n} \left(\frac{\tilde{q}_i}{\omega_i} + (1/2)s_i\right)^2 \\
= \frac{\omega_{\text{min}}}{\omega_{\text{min}} - \alpha^*} \|x^{\text{cts}} - (1/2)1\|_2^2.
\]
From this inequality, we have

$$\alpha^* \geq \omega_{\min} \left(1 - \frac{\|x^{\text{cts}} - (1/2)1\|_2^2}{n/4}\right).$$

Plugging in this lower bound gives

$$f^{\text{sdp}} - g(0) \geq g(\alpha^*) - g(0) \geq \frac{n\omega_{\min}^2}{4\omega_{\max}} \left(1 - \frac{\|x^{\text{cts}} - (1/2)1\|_2^2}{n/4}\right)^2.$$

Therefore, regardless of whether $dg/d\alpha = 0$ is achieved or not, the increase in the lower bound is guaranteed to be at least

$$\frac{n\omega_{\min}^2}{4\omega_{\max}} \left(1 - \frac{\|x^{\text{cts}} - (1/2)1\|_2^2}{n/4}\right)^2,$$

as claimed.

Now we discuss several implications of Theorem 1. First, we note that the right-hand side of (8) is always nonnegative, and is monotonically decreasing in $\|x^{\text{cts}} - (1/2)1\|_2^2$. In particular, when $x^{\text{cts}}$ is an integer point, then we must have $f^{\text{cts}} = f^{\text{sdp}} = f^*$. Indeed, for $x^{\text{cts}} \in \{0, 1\}^n$, we have $\|x^{\text{cts}} - (1/2)1\|_2^2 = n/4$, and the right-hand side of (8) is zero.

Let $\delta$ be the vector defined by $\delta_i = |x^{\text{rnd}}_i - x^{\text{cts}}_i|$. For each $i$, we have $\delta_i + |x^{\text{cts}}_i - 1/2| = 1/2$. It follows that

$$\|x^{\text{cts}} - (1/2)1\|_2^2 = \sum_{i=1}^n (x^{\text{cts}}_i - 1/2)^2 = \sum_{i=1}^n (1/2 - \delta_i)^2 = \|\delta\|_2^2 - \|\delta\|_1 + n/4.$$

Plugging this into the statement of the theorem, we get

$$f^{\text{sdp}} - f^{\text{cts}} \geq \frac{4\omega_{\min}^2}{n\omega_{\max}} (\|\delta\|_1 - \|\delta\|_2^2)^2.$$

On the other hand, using $\delta$, we can a tighter upper bound on $f^{\text{rnd}}$ than (3):

$$f^{\text{rnd}} \leq f^{\text{cts}} + \omega_{\max}\|\delta\|_2^2.$$

Combining with the result of Theorem 1 gives

$$f^{\text{rnd}} - f^{\text{sdp}} \leq \omega_{\max}\|\delta\|_2^2 - \frac{4\omega_{\min}^2}{n\omega_{\max}} (\|\delta\|_1 - \|\delta\|_2^2)^2.$$

Since $0 \leq \delta_i \leq 1/2$, we have $\delta_i - \delta_i^2 \geq (1/2)\delta_i$ for all $i$. Then, $\|\delta\|_1 - \|\delta\|_2^2 \geq (1/2)\|\delta\|_1$. This gives a looser, but simpler expression for the suboptimality of $x^{\text{rnd}}$:

$$f^{\text{rnd}} - f^* \leq f^{\text{rnd}} - f^{\text{sdp}} \leq \omega_{\max}\|\delta\|_2^2 - \frac{\omega_{\min}^2}{\omega_{\max}} \frac{\|\delta\|_1^2}{n}.$$
3 Semidefinite relaxation

In this section, we show another convex relaxation of (4) for finding a nontrivial lower bound on \( f^\star \). By introducing a new variable \( X = xx^T \), we can reformulate (4) as:

\[
\begin{align*}
\text{minimize} & \quad \text{Tr}(PX) + 2q^Tx \\
\text{subject to} & \quad \text{diag}(X) \geq x \\
& \quad X = xx^T,
\end{align*}
\]

in the variables \( X \in \mathbb{R}^{n \times n} \) and \( x \in \mathbb{R}^n \).

Then, we relax the nonconvex constraint \( X = xx^T \) into a convex constraint \( X \succeq xx^T \), and write it using a Schur complement to obtain a convex relaxation:

\[
\begin{align*}
\text{minimize} & \quad \text{Tr}(PX) + 2q^Tx \\
\text{subject to} & \quad \text{diag}(X) \geq x \\
& \quad \begin{bmatrix}
X & x \\
x^T & 1
\end{bmatrix} \succeq 0.
\end{align*}
\]

(9)

The optimal value of problem (9) is a lower bound on \( f^\star \), just as the Lagrangian relaxation (7) gives a lower bound \( f^\text{sdp} \) on \( f^\star \). In fact, problems (7) and (9) are duals of each other, and they yield the same lower bound \( f^\text{sdp} \) [VB96].

Using the dual relationship between (7) and (9), we derive the Karush-Kuhn-Tucker (KKT) optimality condition for the SDP (9). Let \( X, x, \lambda, \) and \( \gamma \) be any points that satisfy the following conditions:

\[
\begin{align*}
\text{diag}(X) & \geq x \\
X & \succeq xx^T \\
\lambda & \geq 0 \\
\begin{bmatrix}
P - \text{diag}(\lambda) & q + (1/2)\lambda \\
(q + (1/2)\lambda)^T & \gamma
\end{bmatrix} & \succeq 0 \\
\lambda_i(X_{ii} - x_i) & = 0, \quad i = 1, \ldots, n
\end{align*}
\]

\[
\text{Tr} \left( \begin{bmatrix}
X & x \\
x^T & 1
\end{bmatrix} \begin{bmatrix}
P - \text{diag}(\lambda) & q + (1/2)\lambda \\
(q + (1/2)\lambda)^T & \gamma
\end{bmatrix} \right) = 0.
\]

Then, \((X, x)\) are optimal in (9), and \((\lambda, \gamma)\) are optimal in (7). The converse also holds, \(i.e.,\) if \((X^\star, x^\star)\) and \((\lambda^\star, \gamma^\star)\) are optimal solutions of (9) and (7), then they satisfy the KKT conditions above.

In terms of the asymptotic running time, there is no benefit to solve one relaxation over the other, as both relaxations require \(O(n^3)\) flops to solve using an interior point method, assuming that a constant number of iterations is needed (which is the case in practice).

3.1 Randomized algorithm

The semidefinite relaxation (9) has a natural probabilistic interpretation, which can be used to construct a simple randomized algorithm for obtaining good suboptimal solutions, \(i.e.,\)
feasible points with low objective value. Let \((x^*, X^*)\) be any optimal solution to (1). Suppose \(z \in \mathbb{R}^n\) is a Gaussian random variable with mean \(\mu\) and covariance matrix \(\Sigma\). Then, \(\mu = x^*\) and \(\Sigma = X^* - x^*x^T\) solve the following problem of minimizing the expected value of a quadratic form, subject to quadratic inequalities:

\[
\begin{align*}
\text{minimize} & \quad \mathbf{E}(z^T P z + 2q^T z) \\
\text{subject to} & \quad \mathbf{E}(z_i(z_i - 1)) \geq 0, \quad i = 1, \ldots, n,
\end{align*}
\]

in variables \(\mu \in \mathbb{R}^n\) and \(\Sigma \in \mathbb{R}^{n \times n}\). Intuitively, this distribution \(\mathcal{N}(\mu, \Sigma)\) has mean close to \(x^\text{cts}\) so that the expected objective value is low, but each diagonal entry of \(\Sigma\) is large enough so that when \(z\) is sampled from the distribution, \(z_i(z_i - 1) \geq 0\) holds in expectation. While sampling \(z\) from \(\mathcal{N}(\mu, \Sigma)\) does not give a feasible point to (1) immediately, we can simply round it to the nearest integer point to get a feasible point. Using these observations, we construct the following randomized algorithm.

\[
\begin{align*}
\text{Algorithm 3.1} & \quad \text{Randomized algorithm for suboptimal solution to (1).} \\
\text{given} & \quad \text{number of iterations } K. \\
1. & \quad \text{Solve SDP. Solve (9) to get } X^* \text{ and } x^*. \\
2. & \quad \text{Form covariance matrix. } \Sigma := X^* - x^*x^T, \text{ and its Cholesky factorization } LL^T = \Sigma. \\
\text{for } k = 1, 2, \ldots, K & \quad 3. \text{Random sampling. } z^{(k)} := x^* + Lw, \text{ where } w \sim \mathcal{N}(0, I). \text{ (Same as } z^{(k)} \sim \mathcal{N}(x^*, \Sigma).) \\
& \quad 4. \text{Round to nearest integer. } x^{(k)} := \text{round}(z^{(k)}). \\
& \quad 5. \text{Update best point. } x^\text{best} := x^{(k)} \text{ and } f^\text{best} := f(x^{(k)}) \text{ if } f(x^\text{best}) > f(x^{(k)}).
\end{align*}
\]

The SDP in step 1 takes \(O(n^3)\) time to solve, assuming that the number of iterations required by an interior point method is constant. Step 2 is dominated by the computation of Cholesky factorization, which uses roughly \(n^3/3\) flops. Steps 3 through 5 can be done in \(O(n^2)\) time. The overall time complexity of the method is then \(O(n^2(K + n))\). By choosing \(K = O(n)\), the time complexity can be made \(O(n^3)\).

4 Greedy algorithm for obtaining a 1-opt solution

Here we discuss a simple greedy descent algorithm that starts from an integer point, and iteratively moves to another integer point that has a lower objective value. This method can be applied to the simple suboptimal point \(x^\text{rand}\), or every \(x^{(k)}\) found in Algorithm 3.1 to yield better suboptimal points.

We call \(x \in \mathbb{Z}^n\) is 1-opt if the objective value at \(x\) doesn’t improve by changing a single coordinate, i.e., \(f(x + ce_i) \geq f(x)\) for all index \(i\) and integer \(c\). The difference in the function values at \(x\) and \(x + ce_i\) can be written as

\[
f(x + ce_i) - f(x) = c^2 P_{ii} + cg_i = P_{ii}(c + g_i/(2P_{ii}))^2 - g_i^2/(4P_{ii}),
\]
where \( g = 2(Px + q) \) is the gradient of \( f \) at \( x \). It is easily seen that given \( i \), the expression above is minimized when \( c = \text{round}\left(-g_i/(2P_{ii})\right) \). For \( x \) to be optimal with respect to \( x_i \), then \( c \) must be 0, which is the case if and only if

\[
-\frac{1}{2} \leq -\frac{g_i}{2P_{ii}} \leq \frac{1}{2},
\]

or equivalently, \( P_{ii} \geq |g_i| \). Thus, \( x \) is 1-opt if and only if \( \text{diag}(P) \geq |g| \), where the absolute value on the righthand side is taken elementwise.

Also, observe that

\[
P(x + ce_i) + q = (Px + q) + cP_i,
\]

where \( P_i \) is the \( i \)th column of \( P \). Thus, when \( x \) changes by a single coordinate, the value of \( g \) can be updated just by referencing a single column of \( P \). These observations suggest a simple and quick greedy algorithm for finding a 1-opt point from any given integer point \( x \).

**Algorithm 4.1** Greedy descent algorithm for obtaining 1-opt point.

given an initial point \( x \in \mathbb{Z}^n \).

1. **Compute initial gradient.** \( g = 2(Px + q) \).
2. **Stopping criterion.** \textbf{quit} if \( \text{diag}(P) \geq |g| \).
3. **Find descent direction.** Find index \( i \) and integer \( c \) minimizing \( c^2P_{ii} + cg_i \).
4. **Update \( x \).** \( x_i := x_i + c \).
5. **Update gradient.** \( g := g + 2cP_i \).

Initializing \( g \) takes \( O(n^2) \) flops, but each subsequent iteration only takes \( O(n) \) flops. This is because steps 2 and 3 only refer to the diagonal elements of \( P \), and step 5 only uses a single column of \( P \). Though we do not give an upper bound on the total number of iterations, we show, using numerical examples, that the average number of iterations until convergence is roughly \( 0.14n \), when the initial points are sampled according to the probability distribution given in §3.1. The overall time complexity of Algorithm 4.1 is then \( O(n^2) \) on average. Thus, we can run the greedy 1-opt descent on every \( x^{(k)} \) in Algorithm 3.1 without changing its overall time complexity \( O(n^2(K + n)) \).

A similar approach as above yields a 2-opt point, which cannot be improved by changing at most two coordinates at the same time. However, every iteration uses \( O(n^2) \) flops, which makes the algorithm significantly slower for larger problems.

## 5 Examples

In this section, we consider numerical examples to show the performance of the SDP-based lower bound and randomized algorithm, developed in previous sections.
5.1 Method

We combine the techniques developed in previous sections to find lower and upper bounds, as well as suboptimal solutions to the problem. By solving the simple relaxation and rounding the solution, we immediately get a lower bound $f^{\text{cts}}$ and an upper bound $f^{\text{rnd}} = f(x^{\text{rnd}})$. We also run Algorithm 4.1 on $x^{\text{rnd}}$ to get a 1-opt point, namely $\hat{x}^{\text{rnd}}$. This gives another upper bound $\hat{f}^{\text{rnd}} = f(\hat{x}^{\text{rnd}})$.

Then, we solve the semidefinite relaxation (9) to get a lower bound $f^{\text{sdp}}$. Using the solution to the SDP, we run Algorithm 3.1 to obtain suboptimal solution $s$, and keep the best suboptimal solution $x^{\text{best}}$. In addition, we run Algorithm 4.1 on every feasible point considered in step 4 of Algorithm 3.1 and find the best 1-opt suboptimal solution $\hat{x}^{\text{best}}$. The randomized algorithm thus yields two additional upper bounds to (1), namely $f^{\text{best}} = f(x^{\text{best}})$ and $\hat{f}^{\text{best}} = f(\hat{x}^{\text{best}})$.

The total number of iterations $K$ in Algorithm 3.1 is set to $K = 3n$, so that the overall time complexity of the algorithm, not counting the running time of the 1-opt greedy descent algorithm, is $O(n^3)$. We note that the process of sampling points and running Algorithm 4.1 trivially parallelizes.

5.2 Numerical examples

We consider random instances of integer least squares problem (2), where the entries of $A \in \mathbb{R}^{m \times n}$ are sampled independently from $\mathcal{N}(0, 1)$. The dimensions are set as $m = 2n$, which makes the condition number of $P = A^T A$ close to 30, regardless of $n$. We set $q = -P x^{\text{cts}}$, where $x^{\text{cts}}$ is randomly drawn from the box $[0, 1]^n$. The problem is then normalized so that the simple lower bound is $-1$, i.e., $f^{\text{cts}} = -q^T P^{-1} q = -1$.

There are other ways to generate random problem instances. For example, the eigen-spectrum of $P$ is controlled by the magnitude of $m$ relative to $n$. We note that $P$ becomes a near-diagonal matrix as $m$ diverges to infinity, because the columns of $A$ are uncorrelated. This makes the integer least squares problem easier to solve. On the contrary, smaller $m$ makes the problem harder to solve. Another way of generating random problem instances is to construct $P$ from a predetermined eigenspectrum $\omega_1, \ldots, \omega_n$, as $P = Q \text{diag}(\omega) Q^T$, where $Q$ is a random rotation matrix. This makes it easy to generate a matrix with a desired condition number. Our method showed the same qualitative behavior on data generated in these different ways, for larger or smaller $m$, and also for different eigenspectra.

The SDP (9) was solved using CVX \cite{GB14, GB08} with the MOSEK 7.1 solver \texttt{MOS}, on a 3.40 GHz Intel Xeon machine. For problems of relatively small size $n \leq 60$, we found the optimal solution using MILES \cite{CZ07}, a MATLAB package for mixed integer least squares problems. MILES solves (1) by enumerating lattice points in a suitably chosen ellipsoid. The enumeration method is based on various algorithms developed in \cite{CYZ05, AEVZ02, SE94, FP85}.
Table 1: Average lower bound by number of variables.

| n  | SDP  | BHS  |
|----|------|------|
| 40 | −0.912 | −0.941 |
| 50 | −0.918 | −0.946 |
| 60 | −0.924 | −0.950 |
| 70 | −0.924 | −0.975 |
| 80 | −0.923 | −0.974 |
| 100 | −0.928 | −0.957 |
| 500 | −0.941 | −0.967 |
| 1000 | −0.943 | −0.978 |

5.3 Results

In Table 1 we compare the SDP-based lower bound, denoted by SDP, against the lower bound shown in [BHS15], denoted by BHS. For each problem size, the average lower bound was computed from 100 random problem instances. Note that in all instances, the simple lower bound was $f_{cts} = -1$. We observe not only that our method found a tighter lower bound on average, but also that our method performed consistently better, i.e., in all problem instances, the SDP based lower bound was higher than the lower bound shown by [BHS15].

Algorithm 3.1 gives a better suboptimal solution as the number of samples $K$ grows. To test the relationship between the number of samples and the quality of suboptimal solutions, we consider a specific problem instance of size $n = 500$, sample $K = 50n$ points. Figure 1 shows convergence of the best upper bounds $f_{best}$ and $f_{sdp}^\star$. Other random problem instances showed the same qualitative results. The experiments suggest that $K = 3n$ is enough number of samples for most problems, as many more samples are necessary for $f_{best}^\star$ to decrease further. All subsequent experiments discussed below used $K = 3n$ as the number of sample points.

Next, we consider another set of 100 random problem instances of small size $n = 60$, where we compare the lower and upper bounds obtained from our method with the optimal value. Figure 2 shows histograms of the gap between the optimal solution and simple lower bound, and the gap between the optimal solution and SDP-based lower bound, namely $f^\star - f_{cts}$ and $f^\star - f_{sdp}$. In every problem instance, $f_{sdp} \geq f_{cts}$ held, as shown in Theorem 1. The mean increase in the lower bound was 0.0797, and the mean gap between the lower bound and optimal value was 0.1567 for the simple bound, and 0.0770 for the SDP-based lower bound. Roughly speaking, then, the SDP-based lower bound as twice as good as the simple lower bound.

Figure 3 shows histograms of the suboptimality of $x_{rnd}^\star$, $\hat{x}_{rnd}^\star$, $x_{best}^\star$, and $\hat{x}_{best}^\star$. The mean suboptimality of $x_{rnd}$ was 0.1063, and simply finding a 1-opt point from $x_{rnd}$ improved the mean suboptimality to 0.0185. Algorithm 3.1 itself, without 1-opt refinement, produced suboptimal points of mean suboptimality 0.0142, and running Algorithm 4.1 on top of it reduced the suboptimality to 0.0003. It is noteworthy that in 87% of the problem instances,
Figure 1: Best upper bounds $f_{\text{best}}$ and $\hat{f}_{\text{best}}$ found in Algorithm 3.1 and lower bounds $f_{\text{cts}}$ and $f_{\text{sdp}}$, versus iteration $k$.

Figure 2: Histograms of the gap between the optimal value $f^*$ and the two lower bounds $f_{\text{cts}}$ and $f_{\text{sdp}}$, for 100 random problem instances of size $n = 60$. 
Algorithm 3.1 terminated with $\hat{f}_{\text{best}} = f^*$, i.e., $\hat{x}_{\text{best}}$ was the optimal solution.

Now, we consider 100 problems of larger size $n = 1000$, where all existing methods for solving the problem exactly run too slowly. As the optimal value is unobtainable, we consider the gap given by the difference between the upper and lower bounds. Figure 4 shows histograms of the four optimality gaps obtained from our method, namely $f^{\text{rnd}} - f^{\text{cts}}$, $\hat{f}^{\text{rnd}} - f^{\text{cts}}$, $f^{\text{best}} - f^{\text{sdp}}$, and $\hat{f}^{\text{best}} - f^{\text{sdp}}$. The mean value of these quantities were: 0.2512, 0.1630, 0.1509, and 0.0990. As seen from the experiment with smaller problems, we believe that the best upper bound $\hat{f}^\text{best}$ is very close to the optimal value, whereas the lower bound $f^{\text{sdp}}$ is farther away from the optimal value.

In Table 2, we compare the running time of our method and that of MILES for problems of various sizes. It should be noted that MILES always terminates with the optimal solution, whereas our method does not have such a guarantee, even though the experiment results suggest that the best suboptimal point found is close to optimal. A failed run means that the algorithm did not produce an output within 10 minutes. The average running time was computed only using successful runs. For problems of size $n \geq 100$, MILES failed to solve any problem instance within 10 minutes. On the other hand, our method had no failed runs, and terminated within 4 minutes for every problem instance of the biggest size $n = 1000$.

In Table 3, we show the breakdown of the running time of our method into three parts: running time of solving the SDP (9), running time of sampling and evaluating $K = 3n$...
Figure 4: Histograms of the four optimality gaps for 100 random problem instances of large size $n = 1000$.

| $n$ | MILES | Our Method |
|-----|-------|------------|
|     | Time | Failure | Time |
| 40  | 0.663 | 0%       | 0.247 |
| 50  | 3.977 | 0%       | 0.315 |
| 60  | 37.10 | 0%       | 0.390 |
| 70  | 276.1 | 24%      | 0.469 |
| 80  | 409.7 | 96%      | 0.597 |
| 100 | N/A  | 100%     | 1.245 |
| 500 | N/A  | 100%     | 31.21 |
| 1000| N/A  | 100%     | 210.3 |

Table 2: Average running time of MILES and our method in seconds, and percentage of failed runs.
Table 3: Breakdown of the running time of our method.

| n    | SDP  | Random Sampling | Greedy 1-opt |
|------|------|-----------------|--------------|
| 40   | 0.126| 0.078           | 0.043        |
| 50   | 0.141| 0.108           | 0.066        |
| 60   | 0.161| 0.139           | 0.090        |
| 70   | 0.187| 0.171           | 0.111        |
| 80   | 0.230| 0.217           | 0.150        |
| 100  | 0.622| 0.361           | 0.262        |
| 500  | 11.86| 10.23           | 9.125        |
| 1000 | 80.81| 71.18           | 58.33        |

Table 4: Average number of iterations of Algorithm 4.1.

| n    | Iterations |
|------|------------|
| 40   | 5.17       |
| 50   | 6.24       |
| 60   | 7.33       |
| 70   | 8.46       |
| 80   | 9.57       |
| 100  | 12.02      |
| 500  | 66.02      |
| 1000 | 141.3      |

sample points, and the total running time of the 1-opt refinement Algorithm 4.1 on all of the K sample points. We found that none of the three parts of our method clearly dominates the total running time. We also note that the total running time grows subcubically, despite the theoretical running time of \(O(n^3)\).

There is no simple bound for the number of iterations that Algorithm 4.1 takes, as it depends on both \(P\) and \(q\), as well as the initial point. In Table 4, we give the average number of iterations for Algorithm 4.1 to terminate at a 1-opt point, when the initial points are sampled from the distribution found in \(3.1\). We see that the number of iterations is roughly \(0.14n\). The asymptotic growth in the number of iterations appears to be slightly superlinear. However, since it is computationally unreasonable to solve problems of much larger size (e.g., \(n = 10^4\)), we claim that the number of iterations is effectively linear, as far as practical applications are concerned.
6 Extensions

6.1 Constrained problems

The technique developed in §2 can be directly applied to problems with quadratic inequality and equality constraints, even when the constraints are nonconvex. The convex relaxation of a quadratically constrained quadratic program is an SDP, which is easily solved to give a lower bound.

Algorithm 3.1 however, will not produce an upper bound in general, because after rounding, the resulting point may not satisfy the given constraints. For example, in the case of equality constrained problems, rounding a point to the nearest integer point will almost certainly produce an infeasible point. Nevertheless, simple constraints such as box constraints, or sum constraints $1^T x = m$ with $m \in \mathbb{Z}$ can be handled, since it is easy to produce a feasible integer point closest to any given real-valued vector.

6.2 Nonquadratic objective

Our method can be applied to problems with nonquadratic objective function to get suboptimal solutions. Consider the problem

$$\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in \mathbb{Z}^n,
\end{align*}$$

in the variable $x$, where $f : \mathbb{R}^n \to \mathbb{R}$ is convex but not necessarily quadratic. A simple convex relaxation can be obtained by removing the integer constraint. Let $x^{cts}$ be the solution to the convex relaxation. Then, form a quadratic approximation $\hat{f}$ of $f$ around $x^{cts}$ to get the following problem:

$$\begin{align*}
\text{minimize} & \quad \hat{f}(x) = (x - x^{cts})^T H(x - x^{cts}) + f(x^{cts}) \\
\text{subject to} & \quad x \in \mathbb{Z}^n,
\end{align*}$$

in the variable $x$. Here, $H \in \mathbb{R}^{n \times n}$ is the Hessian of $f$ evaluated at $x^{cts}$. Then, our method can be applied to this quadratic approximation to yield suboptimal solutions. In particular, if $\hat{f}(x) \leq f(x)$ for all $x \in \mathbb{R}^n$, then we not only obtain suboptimal solutions from Algorithm 3.1 but also get a lower bound to the optimal value that is no worse than $f(x^{cts})$.

6.3 Unimodular transformation

An integer matrix $T$ is called unimodular if $|\det T| = 1$. It has a property that for every integer point $x$, both $Tx$ and $T^{-1}x$ are integer points. Therefore, given problem (1), one can apply the change of coordinates $z = T^{-1}x$, and solve an equivalent problem

$$\begin{align*}
\text{minimize} & \quad z^T (T^T PT) z + 2(T^T q)^T z \\
\text{subject to} & \quad z \in \mathbb{Z}^n,
\end{align*}$$
in the variable $z$, where $T \in \mathbb{Z}^{n \times n}$ is a unimodular matrix.

The techniques developed in previous sections are not invariant under linear transformations. That is, applying a unimodular transformation to a problem changes both the SDP-based lower bound and the expected performance of the randomized algorithm. For example, running the Lenstra–Lenstra–Lovász lattice reduction algorithm on the square root matrix $P^{1/2}$ of $P$ induces a unimodular transformation, and applying this transformation to the problem changes the SDP-based lower bound. The change in lower bound, however, is unpredictable, i.e., it is not guaranteed that the new lower bound no worse than the previous lower bound. It is an open problem to find an appropriate $T$ that improves the performance of our method.

### 6.4 Branch-and-bound method

The SDP-based lower bound can be applied to enumeration algorithms finding the global optimum to achieve improved running time. Algorithms such as the Fincke–Pohst algorithm [FPS5], its variants, and the standard branch-and-bound method all rely on lower bounds to the subproblems to determine the order of enumeration, and prune the search tree. As these subproblems are in the form of (1), an SDP-based lower bound can be found by solving either (6) or (9), with no further modification. We note that (6) needs not be solved optimally, as any feasible solution to it yields a lower bound on $f^\star$.

Enumeration methods that solve (1) globally can also utilize Algorithm 3.1 by taking the suboptimal solution $\hat{x}_{\text{best}}$ and using its objective value $\hat{f}_{\text{best}}$ as the initial bound on the optimal value, reducing the search space of the enumeration algorithm. This reduction in search space can be applied not only to the original problem, but also to every subproblem considered in the enumeration algorithm.

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