C*-IRREducIBILITY OF COMMENSURATED SUBGROUPS

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ABSTRACT. Given a commensurated subgroup Λ of a group Γ, we completely characterize when the inclusion Λ ≤ Γ is C*-irreducible and provide new examples of such inclusions. In particular, we obtain that PSL(n, Z) ≤ PGL(n, Q) is C*-irreducible for any n ∈ N.

The main ingredient that we use is the fact that the action of a commensurated subgroup Λ ≤ Γ on its Furstenberg boundary ∂F Λ can be extended in a unique way to an action of Γ on ∂F Λ. Finally, we also investigate the counterpart of this kind of extension result for the universal minimal proximal space of a group.

1. Introduction

A group Γ is said to be C*-simple if its reduced C*-algebra C*_r(Γ) is simple. After the breakthrough characterizations of C*-simplicity in [KK17] and [BKKO17], several directions of research applying the new methods in different settings arose.

One of the recent interesting directions is investigating when inclusions of groups Λ ≤ Γ are C*-irreducible, in the sense that every intermediate C*-algebra B in C*_r(Λ) ⊂ B ⊂ C*_r(Γ) is simple. In [Rør21], Rørdam started a systematic study of this property and provided a dynamical criterion for an inclusion of groups to be C*-irreducible. Together with results in [Amr21], [Urs22] and [BO22], this has provided a complete characterization of C*-irreducibility of an inclusion in the case that Λ is a normal subgroup of Γ.

Recall that a subgroup Λ of a group Γ is said to be commensurated if, for any g ∈ Γ, Λ ∩ gΛg^{-1} has finite index in Λ. This is a much more flexible generalization of normal subgroups and finite-index subgroups. For example, for every n ∈ N, PSL(n, Z) is an infinite-index commensurated subgroup of the simple group PSL(n, Q).

In this work, we generalize the above characterization of C*-irreducibility to commensurated subgroups (see Theorem 3.5). The main ingredient in our proof is the fact that the action of Λ on its Furstenberg boundary ∂F Λ can be uniquely extended to an action of Γ on ∂F Λ if Λ is a commensurated subgroup in Γ (see Theorem 3.1).

As one of the applications, we show that, if Γ is a C*-simple group, then the inclusion of Γ in its abstract commensurator Comm(Γ) is C*-irreducible (see Corollary 3.14). To our best knowledge, this is also the first observation of the fact that, if Γ is a C*-simple group, then Comm(Γ) is C*-simple as well.

Given a subgroup Λ of a group Γ, Ursu introduced in [Urs22] a universal Λ-strongly proximal Γ-boundary B(Γ, Λ) and showed that, if Λ ≤ Γ, then B(Γ, Λ) =

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This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 817597).
In Section 4 we generalize this fact to commensurated subgroups and also observe that, in general, $B(\Gamma, \Lambda)$ is not extremally disconnected.

Finally, we also show that, given a commensurated subgroup $\Lambda$ of a group $\Gamma$, the action of $\Lambda$ on its universal minimal proximal space $\partial F \Lambda$ can also be extended in a unique way to an action of $\Gamma$ on $\partial F \Lambda$ (see Theorem 5.4), and use this fact for concluding that, for a certain locally finite commensurated subgroup $G$ of Thompson’s group $V$, the resulting action of $V$ on $\partial F G$ is free (see Corollary 5.4).

2. Preliminaries

Given a compact Hausdorff space $X$, we denote by $\text{Prob}(X)$ the space of regular probability measures on $X$. An action of a group $\Gamma$ on $X$ by homeomorphisms is said to be minimal if $X$ does not contain any non-trivial closed invariant subset, and to be topologically free if, for any $g \in \Gamma \setminus \{e\}$, the set $\{x \in X : gx = x\}$ has empty interior (if $\Gamma$ is countable, then $\Gamma \curvearrowright X$ is topologically free if and only if the set of points in $X$ which are not fixed by any non-trivial element of $\Gamma$ is dense in $X$). The action is said to be proximal if, given $x, y \in X$, there is a net $(g_i) \subset \Gamma$ such that the nets $(g_i x)$ and $(g_i y)$ converge and $\lim_{i} g_i x = \lim_{i} g_i y$. We say that the action is strongly proximal if the induced action $\Gamma \curvearrowright \text{Prob}(X)$ is proximal. The action is called a boundary action (or $X$ is a $\Gamma$-boundary) if it is both minimal and strongly proximal. We denote by $\partial F \Gamma$ the Furstenberg boundary of $\Gamma$, i.e., the universal $\Gamma$-boundary (see [Gla76, Section III.1]). The group $\Gamma$ is $C^*$-simple if and only if $\Gamma \curvearrowright \partial F \Gamma$ is free (see [BKKO17, Theorem 3.1]).

Given $\Gamma$-boundaries $X$ and $Y$, if there exists $\varphi : X \to Y$ a homeomorphism which is $\Gamma$-equivariant ($\Gamma$-isomorphism), then it follows from [Gla76, Lemma II.4.1] that $\varphi$ is the unique $\Gamma$-isomorphism between $X$ and $Y$.

Let $\Lambda \leq \Gamma$ be a finite-index subgroup. Then any strongly proximal $\Gamma$-action is also $\Lambda$-strongly proximal ([Gla76, Lemma II.3.1]) and any $\Gamma$-boundary is also a $\Lambda$-boundary ([Gla76, Lemma II.3.2]). Furthermore, by [Gla76, Theorem II.4.4], which is stated for the universal minimal proximal space but whose proof also works for the Furstenberg boundary, the action $\Lambda \curvearrowright \partial F \Lambda$ can be extended to $\Gamma \curvearrowright \partial F \Lambda$ and $\partial F \Lambda$ is $\Gamma$-isomorphic to $\partial F \Gamma$. In particular, $\partial F \Lambda$ and $\partial F \Gamma$ are also $\Lambda$-isomorphic.

Given a group isomorphism $\psi : \Gamma_1 \to \Gamma_2$, by universality there is a unique homeomorphism $\psi : \partial F \Gamma_1 \to \partial F \Gamma_2$ such that $\psi(gx) = \psi(g)\psi(x)$ for any $g \in \Gamma_1$ and $x \in \partial F \Gamma_1$.

Given a group $\Gamma$, let $\text{Sub}(\Gamma)$ be the space of subgroups of $\Gamma$ endowed with the pointwise convergence topology and with the $\Gamma$-action given by conjugation. Given a subgroup $\Lambda \leq \Gamma$, a $\Lambda$-uniformly recurrent subgroup (URS) is a non-empty closed $\Lambda$-invariant minimal set $U \subset \text{Sub}(\Gamma)$. Moreover, we say that $U$ is amenable if one of its elements is amenable. By [Ken20, Theorem 4.1], a group $\Gamma$ is $C^*$-simple if and only if it does not admit any non-trivial amenable $\Gamma$-uniformly recurrent subgroup.

An inclusion of groups $\Lambda \leq \Gamma$ is said to be $C^*$-irreducible if every intermediate $C^*$-algebra of $C^*_r(\Lambda) \subset C^*_r(\Gamma)$ is simple.

Given $\Lambda \leq \Gamma$ and $g \in \Lambda$, let $g^\Lambda := \{hgh^{-1} : h \in \Lambda\}$. We say that $\Gamma$ is icc relatively to $\Lambda$ if, for any $g \in \Gamma \setminus \{e\}$, $|g^\Lambda| < \infty$. The group $\Gamma$ is said to be icc if it is icc relatively to itself.
3. C*-irreducibility of commensurated subgroups

Let \( \Gamma \) be a group. Two subgroups \( \Lambda_1, \Lambda_2 \leq \Gamma \) are said to be commensurable if \([\Lambda_1 : \Lambda_1 \cap \Lambda_2] < \infty \) and \([\Lambda_2 : \Lambda_1 \cap \Lambda_2] < \infty \). Notice that this is an equivalence relation.

A subgroup \( \Lambda \leq \Gamma \) is said to be commensurated if, for any \( g \in \Gamma \), \( \Lambda \) is commensurable with \( g\Lambda g^{-1} \). Equivalently, for any \( g \in \Gamma \), \([\Lambda : \Lambda \cap g\Lambda g^{-1}] < \infty \). In this case, we write \( \Lambda \leq_c \Gamma \). In the literature, this notion is also referred to by saying that \( \Lambda \) is an almost normal subgroup of \( \Gamma \) or that \( (\Gamma, \Lambda) \) is a Hecke pair.

The following result generalizes [Gab76, Theorem II.4.4] and [Oza14, Lemma 20].

**Theorem 3.1.** Let \( \Lambda \leq_c \Gamma \). Then \( \Lambda \rtimes \partial_F \Lambda \) extends in a unique way to an action of \( \Gamma \) on \( \partial_F \Lambda \).

**Proof.** Given \( g \in \Gamma \), let \( \varphi_g : \partial_F \Lambda \to \partial_F (\Lambda \cap g\Lambda g^{-1}) \) be the \((\Lambda \cap g\Lambda g^{-1})\)-isomorphism. Also let \( \psi_g : \partial_F (\Lambda \cap g^{-1}\Lambda g) \to \partial_F (\Lambda \cap g\Lambda g^{-1}) \) be the homeomorphism such that for all \( h \in \Lambda \cap g^{-1}\Lambda g \), we have \( \psi_g(hx) = ghg^{-1}\psi_g(x) \). Let \( T_g := (\varphi_g)^{-1}\psi_g : \partial_F \Lambda \to \partial_F \Lambda \). We claim that \( g \mapsto T_g \) is a \( \Gamma \)-action which extends \( \Lambda \rtimes \partial_F \Lambda \).

Given \( h \in \Lambda \cap g^{-1}\Lambda g \) and \( x \in \partial_F \Lambda \), one can readily check that \( T_g(hx) = ghg^{-1}T_g(x) \).

Given \( g, h \in \Gamma \), we have that \([\Lambda : \Lambda \cap h^{-1}\Lambda h \cap (gh)^{-1}\Lambda(gh)] < \infty \). Furthermore, given \( k \in \Lambda \cap h^{-1}\Lambda h \cap (gh)^{-1}\Lambda(gh) \), we have \( T_{gh}(kx) = (gh)k(gh)^{-1}T_gT_h(x) \). On the other hand, \( T_gT_h(kx) = (gh)k(gh)^{-1}T_gT_h(x) \). In particular, \( T_gT_h^{-1}T_{gh} \) is a \((\Lambda \cap h^{-1}\Lambda h \cap (gh)^{-1}\Lambda(gh))\)-automorphism, hence \( T_{gh} = T_gT_h \).

Finally, given \( g \in \Lambda \), we have that \( x \mapsto g^{-1}T_g(x) \) is a \((\Lambda \cap g^{-1}\Lambda g)\)-automorphism, so that \( g^{-1}T_g = \text{Id}_{\partial_F \Lambda} \). \( \square \)

**Remark 3.2.** The existence part of Theorem 3.1 was shown by Dai and Glasner in [DGI19, Theorem 6.1] using a different method and assuming that \( \Gamma \) is countable.

Given a subset \( S \) of a group \( \Gamma \), let \( C_{\Gamma}(S) \) be the centralizer of \( S \) in \( \Gamma \). In the next result, we follow the argument of [BKKO17, Lemma 5.3].

**Lemma 3.3.** Let \( \Lambda \leq_c \Gamma \) and consider \( \Gamma \rtimes \partial_F \Lambda \). Given \( s \in \Gamma \), if \( s \in C_{\Gamma}(\Lambda \cap s^{-1}\Lambda s) \), then \( \text{Fix}(s) = \partial_F \Lambda \). Conversely, if \( \Lambda \rtimes \partial_F \Lambda \) is free and \( \text{Fix}(s) \neq \emptyset \), then \( s \in C_{\Gamma}(\Lambda \cap s^{-1}\Lambda s) \).

**Proof.** If \( s \in C_{\Gamma}(\Lambda \cap s^{-1}\Lambda s) \), then, given \( h \in \Lambda \cap s^{-1}\Lambda s \), we have \( s(hx) = hs(x) \). Since \([\Lambda : \Lambda \cap s^{-1}\Lambda s] < \infty \), we conclude that \( s \) acts trivially on \( \partial_F \Lambda \).

Suppose now that \( \Lambda \rtimes \partial_F \Lambda \) is free and \( \text{Fix}(s) \neq \emptyset \). Given \( t \in \Lambda : t \text{Fix}(s) \neq \emptyset \), we have that the action of \( st^{-1} \) and \( t \) coincide on \( \text{Fix}(s) \). Since \( st^{-1} \in \Lambda \) and \( \Lambda \rtimes \partial_F \Lambda \) is free, we obtain that \( t = st^{-1} \). Since, by [BKKO17, Lemma 5.1], \( \Lambda \) generates \( \Lambda \cap s^{-1}\Lambda s \), we conclude that \( s \in C_{\Gamma}(\Lambda \cap s^{-1}\Lambda s) \). \( \square \)

The proof of the following result is an adaptation of the argument in [Ken20, Remark 4.2] and its hypothesis is the same as in [Kor21, Theorem 5.3.(ii)].

**Proposition 3.4.** Let \( \Lambda \leq_c \Gamma \). Suppose that there exists a \( \Gamma \)-boundary \( X \) such that, for any \( \mu \in \text{Prob}(X) \), there exists a net \( (g_i) \in \Lambda \) such that \( g_i \mu \) converges to \( \delta_x \),
for some \( x \in X \), on which \( \Gamma \) acts freely. Then \( \Gamma \) does not admit any non-trivial amenable \( \Lambda \)-URS.

**Proof.** Suppose \( \mathcal{U} \) is a non-trivial amenable \( \Lambda \)-URS, and take \( K \in \mathcal{U} \). Since \( K \) is amenable, there exists \( \mu \in \text{Prob}(X) \) fixed by \( K \). Let \( (g_i) \subset \Lambda \) be a net such that \( g_i \mu \to \delta_x \), for some \( x \in X \), on which \( \Gamma \) acts freely. By taking a subnet, we may assume that \( g_i K g_i^{-1} \to L \in \text{Sub}(\Gamma) \). Take \( g \in L \setminus \{e\} \) and \( (k_i) \subset K \) such that \( g k_i g_i^{-1} = g \) for \( i \) sufficiently big. Then

\[
\delta_x = \lim g_i \mu = \lim g_i k_i \mu = \lim g_i k_i g_i^{-1} g_i \mu = g \delta_x,
\]

contradicting the fact that \( \Gamma \) acts freely on \( x \). \( \square \)

The following result generalizes [Urs22, Theorems 1.3 and 1.9] and [BO22, Theorem 6.4], as well as the claim about finite-index subgroups in [Rør21, Theorem 5.3].

**Theorem 3.5.** Let \( \Lambda \subseteq_c \Gamma \). The following conditions are equivalent:

1. \( \Lambda \) is \( C^\ast \)-irreducible;
2. \( \Lambda \) is \( C^\ast \)-simple and \( \Gamma \) is icc relatively to \( \Lambda \);
3. \( \Lambda \) is \( C^\ast \)-simple and, for any \( s \in \Gamma \setminus \{e\} \), we have that \( s \notin C_\Gamma(\Lambda \cap s^{-1} \Lambda s) \);
4. \( \Gamma \cap \partial_\ell \Lambda \) is free;
5. There is no non-trivial amenable \( \Lambda \)-URS of \( \Gamma \);
6. \( \Lambda \) is \( C^\ast \)-simple and \( \Gamma \cap \partial_\ell \Lambda \) is faithful.

**Proof.**

(1) \( \Rightarrow \) (2) follows from [Rør21, Remark 3.8 and Proposition 5.1].

(2) \( \Rightarrow \) (3). Suppose that there is \( s \in \Gamma \setminus \{e\} \) such that \( s \notin C_\Gamma(\Lambda \cap s^{-1} \Lambda s) \). Take \( g_1, \ldots, g_n \in \Lambda \) left coset representatives for \( \Lambda s^{-1} \Lambda s \). Then

\[
s^\Lambda = \{g_i k s k^{-1} k^{-1} g_i^{-1} : 1 \leq i \leq n, k \in \Lambda \setminus s^{-1} \Lambda s\} = \{g_i s g_i^{-1} : 1 \leq i \leq n\}
\]

is finite.

(3) \( \Rightarrow \) (4) follows from Lemma 3.3.

(4) \( \Rightarrow \) (1) follows from [Rør21, Theorem 5.3].

(5) \( \Rightarrow \) (2). If \( \Lambda \) is not \( C^\ast \)-simple, then it contains a non-trivial amenable \( \Lambda \)-uniformly recurrent subgroup. If \( \Gamma \) is not icc relatively to \( \Lambda \), there exists \( s \in \Gamma \setminus \{e\} \) such that \( s^\Lambda \) is finite. Hence the \( \Lambda \)-orbit of \( \langle s \rangle \) is a finite non-trivial amenable \( \Lambda \)-uniformly recurrent subgroup.

(4) \( \Rightarrow \) (5) follows from Proposition 3.4.

(3) \( \iff \) (6) follows from Lemma 3.3. \( \square \)

**Remark 3.6.** In [Rør21, Theorem 5.3], Rørdam showed that an inclusion \( \Lambda \leq \Gamma \) satisfying the hypothesis of Proposition 3.4 is \( C^\ast \)-irreducible, and asked whether the converse holds. We do not know whether the converse of Proposition 3.4 holds and whether the absence of non-trivial amenable \( \Lambda \)-URS is equivalent to \( \Lambda \leq \Gamma \) being \( C^\ast \)-irreducible in general.

**Corollary 3.7.** Given \( n \in \mathbb{N} \), the inclusion

\[
\text{PSL}(n, \mathbb{Z}) \leq \text{PGL}(n, \mathbb{Q})
\]

is \( C^\ast \)-irreducible.

**Proof.** It was shown in [BCdlH94] that \( \text{PSL}(n, \mathbb{Z}) \) is \( C^\ast \)-simple.

Let \( U(n, \mathbb{Z}) \) be the group of units of the ring \( M_n(\mathbb{Z}) \). By [Kri90, Corollary V.5.3], \( U(n, \mathbb{Z}) \leq_c \text{GL}(n, \mathbb{Q}) \). Since \( [U(n, \mathbb{Z}) : \text{SL}(n, \mathbb{Z})] = 2 \), we conclude that \( \text{SL}(n, \mathbb{Z}) \leq_c \)
GL(n, Q) as well. Since taking quotients preserves being commensurated, it follows that \( \text{PSL}(n, \mathbb{Z}) \leq \text{PGL}(n, \mathbb{Q}) \).

Let \((e_{ij})_{1 \leq i, j \leq n} \in M_n(\mathbb{Z})\) be the matrix units and fix \([a] \in \text{PGL}(n, \mathbb{Q}) \setminus \{[\text{Id}]\}\). By taking conjugates of \([a]\) by elements of the form \([\text{Id} + m \cdot e_{ij}] \in \text{PSL}(n, \mathbb{Z})\), \(m \in \mathbb{Z}\), \(1 \leq i \neq j \leq n\), it is easy to see that \([a]\text{PSL}(n, \mathbb{Z})\) is infinite, so that \(\text{PGL}(n, \mathbb{Q})\) is icc relatively to \(\text{PSL}(n, \mathbb{Z})\).

The conclusion then follows from Theorem 3.5. \(\square\)

**Remark 3.8.** Let us sketch a different proof of Corollary 3.7 which gives the stronger statement that \(\text{PSL}(n, \mathbb{Z}) \leq \text{PGL}(n, \mathbb{R})\) is \(C^*\)-irreducible, where \(\text{PGL}(n, \mathbb{R})\) is seen as a discrete group.

Clearly, it suffices to show that, for any countable group \(\Gamma\) such that \(\text{PSL}(n, \mathbb{Z}) \leq \Gamma \leq \text{PGL}(n, \mathbb{R})\), the inclusion \(\text{PSL}(n, \mathbb{Z}) \leq \Gamma\) is \(C^*\)-irreducible. By the argument in [Bry17, Example 3.4.3], the action of \(\text{PGL}(n, \mathbb{R})\) on the projective space \(\mathbb{P}^{n-1}(\mathbb{R})\) is topologically free. Since \(\text{PSL}(n, \mathbb{Z}) \cap \mathbb{P}^{n-1}(\mathbb{R})\) is a boundary action, the result follows from [Rør21, Theorem 5.3].

**Corollary 3.9.** Let \(\Lambda\) be a finite-index subgroup of a group \(\Gamma\). If \(\Gamma\) is \(C^*\)-simple, then \(\Lambda \leq \Gamma\) is \(C^*\)-irreducible. Conversely, if \(\Lambda\) is \(C^*\)-simple, then \(\Gamma\) is icc if and only if \(\Lambda \leq \Gamma\) is \(C^*\)-irreducible.

**Proof.** If \(\Gamma\) is \(C^*\)-simple, then \(\Gamma \cap \partial_F \Gamma\) is free. Since \(\partial_F \Gamma\) is \(\Gamma\)-isomorphic to \(\partial_F \Lambda\), it follows that \(\Lambda \leq \Gamma\) is \(C^*\)-irreducible.

If \(\Gamma\) is icc, then, since \([\Gamma : \Lambda] < \infty\), it is also icc relatively to \(\Lambda\), hence \(\Lambda \leq \Gamma\) is \(C^*\)-irreducible by Theorem 3.5. The last implication is immediate. \(\square\)

**Example 3.10.** The inclusion given by the Sanov subgroup \(F_2 \leq \text{PSL}(2, \mathbb{Z})\) is finite-index, hence it is \(C^*\)-irreducible by Corollary 3.8.

**Free groups.** Fix \(m, n \in \mathbb{N}\) such that \(2 \leq m < n\) and consider the free groups \(F_m = \langle a_1, \ldots, a_m \rangle \leq \langle a_1, \ldots, a_n \rangle = F_n\). In [Rør21, Example 5.4], Rørdam observed that \(F_m \leq F_n\) is \(C^*\)-irreducible. Notice that \(F_m\) is far from being commensurated in \(F_n\). In fact, given \(g \in F_n \setminus F_m\), we have that \(F_m \cap gF_mg^{-1} = \{e\}\) (i.e., \(F_m\) is malnormal in \(F_n\)). In particular, this example is not covered by Theorems 3.4 and 3.5. Nonetheless, there does exist an extension to \(F_n\) of the action \(F_m \rtimes \partial_F F_m\), but it is far from being unique, since the generators \(a_{m+1}, \ldots, a_n\) can be mapped into any homeomorphisms on \(\partial_F F_m\).

Furthermore, we claim that \(F_m \leq F_n\) satisfies condition (5) in Theorem 3.8 We will prove this by using Proposition 3.3. Let

\[
\partial F_n := \{(x_i) \in \prod_{i \in \mathbb{N}} \{a_1^{-1}, \ldots, a_n^{-1}\} : \forall i \in \mathbb{N}, x_{i+1} \neq x_i^{-1}\}
\]

be the Gromov boundary of \(F_n\), and consider the action of \(F_n\) on \(\partial F_n\) by left multiplication. Fix \(\mu \in \text{Prob}(\partial F_n)\) and we will show that there is \(w \in \partial F_n\) on which \(F_n\) acts freely and such that \(\delta_w \in \overline{F_m \mu}\).

Let \(z_+ := (a_1)_{i \in \mathbb{N}} \in \partial F_n\) and \(z_- := (a_1^{-1})_{i \in \mathbb{N}} \in \partial F_n\). Notice that, for all \(y \in \partial F_n \setminus \{z_-\}\), we have that, as \(k \to +\infty\), \(a_1^ky \to z_+\). Furthermore, \(a_1\) fixes \(z_-\).

It follows from the dominated convergence theorem that

\[a_1^k \mu \to \mu(\{z_-\})\delta_{z_-} + (1 - \mu(\{z_-\}))\delta_{z_+},\]
as $k \to +\infty$. In particular, $\nu := \mu(\{z_\pm\})\delta_{z_+} + (1 - \mu(\{z_\pm\}))\delta_{z_-} \in F_n\mu$.

Let $w := a_1a_2a_2^{-1}a_2a_3 \cdots a_1a_2a_2^{-1}a_2^{-1} \cdots \in \partial F_n$. Since $w$ is not eventually periodic, we have that $F_n$ acts freely on $w$. Given $k \in \mathbb{N}$, let $g_k := w_1 \cdots w_k a_2 \in F_m$. We have that $g_kz_\pm = w_1 \cdots w_k a_2z_\pm \to w$, as $k \to +\infty$. Therefore, $\delta_w \in F_m\nu \subset F_m\mu$, thus showing the claim.

**Abstract commensurator.** Let $\Gamma$ be a group and $\Omega$ be the set of isomorphisms between finite-index subgroups of $\Gamma$. Given $\alpha, \beta \in \Omega$, we say that $\alpha \sim \beta$ if there exists a finite-index subgroup $H \leq \text{dom}(\alpha) \cap \text{dom}(\beta)$ such that $\alpha|_H = \beta|_H$. Recall that the abstract commensurator of $\Gamma$, denoted by $\text{Comm}(\Gamma)$, is the group whose underlying set is $\Omega/\sim$, with product given by composition (defined up to finite-index subgroup).

Let $\Lambda$ be a commensurated subgroup of $\Gamma$. Given $g \in \Gamma$, let

$$
\beta_g : \Lambda \cap g^{-1}\Lambda \to \Lambda \cap g\Lambda g^{-1}
$$

and $j^\Lambda_\Gamma : \Gamma \to \text{Comm}(\Lambda)$ be the homomorphism given by $j^\Lambda_\Gamma(g) := [\beta_g]$. In order to ease the notation, we will sometimes denote $j^\Lambda_\Gamma$ simply by $j$, and it will always be clear from the context what are the involved groups. Let us now collect a few elementary facts about $j$.

**Lemma 3.11.** Let $\Gamma$ be a group. Then $j^\Gamma \Gamma(\Gamma) \leq_c \text{Comm}(\Gamma)$.

**Proof.** Fix $[\alpha] \in \text{Comm}(\Gamma)$. Given $g \in \text{dom}(\alpha)$, we have that $[\alpha]j(g)[\alpha]^{-1} = j(\alpha(g))$. In particular, $j(\Gamma) \cap [\alpha]j(\Gamma)[\alpha]^{-1} \supset j(\text{Im}(\alpha))$. Since $[\Gamma : \text{Im}(\alpha)] < \infty$, we conclude that $[j(\Gamma) : j(\Gamma) \cap [\alpha]j(\Gamma)[\alpha]^{-1}] < \infty$. \hfill $\square$

**Lemma 3.12.** Let $\Lambda \leq_c \Gamma$. Then $\ker j^\Lambda_\Gamma = \{g \in \Gamma : |g^\Lambda| < \infty\}$.

**Proof.** Given $g \in \ker j$, there exists a finite-index subgroup $H \leq \Lambda \cap g^{-1}\Lambda$ such that, for all $h \in H$, $ghg^{-1} = h$, which implies that $|g^\Lambda| < \infty$. Conversely, if $|g^\Lambda| < \infty$, then $H := \{k \in \Lambda : kg = gk\}$ is a finite-index subgroup of $\Lambda$ and $g \in \ker j$. \hfill $\square$

As a consequence of Lemma 3.12 if $\Gamma$ is an icc group, then $j : \Gamma \to \text{Comm}(\Gamma)$ is injective ([Kid11, Lemma 3.8.(i)]). The next result is known ([Kid11, Lemma 3.8.(iii)]). For the convenience of the reader, we provide the proof here.

**Lemma 3.13.** If $\Gamma$ is an icc group, then $\text{Comm}(\Gamma)$ is icc relatively to $\Gamma$.

**Proof.** Given $[\alpha] \in \text{Comm}(\Gamma)$ and $g \in \text{dom}(\alpha)$, we have

$$
j(g)[\alpha](g^{-1}) = j(\alpha(g^{-1}))[\alpha].
$$

If $[\alpha] \neq e$, then $H := \{g \in \text{dom}(\alpha) : g = \alpha(g)\}$ has infinite-index in $\text{dom}(\alpha)$. Given $g_1, g_2 \in \text{dom}(\alpha)$ such that $g_1H \neq g_2H$, one can readily check that $g_1\alpha(g_1)^{-1} \neq g_2\alpha(g_2)^{-1}$. From this, it follows immediately that $[\alpha]^H$ is infinite. \hfill $\square$

In [BO22, Corollary 6.6], Bédos and Omland showed that if $\Gamma$ is a $C^*$-simple group, then $\Gamma \leq \text{Aut}(\Gamma)$ is $C^*$-irreducible. The same conclusion holds when we consider the abstract commensurator:

**Corollary 3.14.** Given a $C^*$-simple group $\Gamma$, we have that $\Gamma \leq \text{Comm}(\Gamma)$ is $C^*$-irreducible.
Proof. Recall that any $C^*$-simple group is icc (this follows, e.g., from Theorem 3.5). The result is then a consequence of Theorem 3.5 and Lemma 3.13. □

Remark 3.15. Let $\mathbb{F}_n$ be a non-abelian free group of finite rank. Then Corollary 3.14 implies that $\text{Comm}(\mathbb{F}_n)$ is $C^*$-simple. Notice that it is an open problem whether $\text{Comm}(\mathbb{F}_n)$ is a simple group ([CMS20, Problem 7.2]).

4. Relative boundaries

Given groups $\Lambda \leq \Gamma$, Ursu introduced in [Urs22, Proposition 4.1] a $\Lambda$-strongly proximal $\Gamma$-boundary $B(\Gamma, \Lambda)$ which is universal with these properties.

Consider $\Gamma := \text{PSL}(2, \mathbb{Z})$ and the boundary action $\Gamma \acts \mathbb{R} \cup \{\infty\}$. The stabilizer $\Gamma_{\infty}$ of $\infty$ is isomorphic to $\mathbb{Z}$ and consists of the translations $g_n(x) := x + n, n \in \mathbb{Z}, x \in \mathbb{R}$.

Proposition 4.1. The action of $\Gamma = \text{PSL}(2, \mathbb{Z})$ on $B(\Gamma, \Gamma_{\infty})$ is topologically free but non-free. In particular, $B(\Gamma, \Gamma_{\infty})$ is not extremally disconnected.

Proof. For any $x \in \mathbb{R} \cup \{\infty\}$, we have $g_n(x) \to \infty$ as $n \to +\infty$. As a consequence of the dominated convergence theorem, it follows easily that $\Gamma_{\infty} \acts \mathbb{R} \cup \{\infty\}$ is strongly proximal. Hence, there is a $\Gamma$-equivariant map $B(\Gamma, \Gamma_{\infty}) \to \mathbb{R} \cup \{\infty\}$. Since $\Gamma_{\infty} \acts B(\Gamma, \Gamma_{\infty})$ is strongly proximal, it follows from amenability of $\Gamma_{\infty}$ that $\Gamma_{\infty}$ fixes some point in $B(\Gamma, \Gamma_{\infty})$. In particular, $\Gamma \acts B(\Gamma, \Gamma_{\infty})$ is not free. On the other hand, since $\Gamma \acts \mathbb{R} \cup \{\infty\}$ is topologically free, it follows from [BKKO17, Lemma 3.2] that $\Gamma \acts B(\Gamma, \Gamma_{\infty})$ is topologically free. As a consequence of [Pro71, Theorem 3.1], $B(\Gamma, \Gamma_{\infty})$ is not extremally disconnected. □

Remark 4.2. Let $\Gamma$ be a group. One of the key properties in the applications of $\partial_F \Gamma$ to $C^*$-simplicity of $\Gamma$ is the fact that $C(\partial_F \Gamma)$ is injective, shown in [KK17, Theorem 3.12]. Proposition 4.1 implies that $C(B(\Gamma, \Lambda))$ is not injective, in general. We believe that this is an evidence that $B(\Gamma, \Lambda)$ is not likely to play the same role of the Furstenberg boundary in $C^*$-algebraic applications.

Our next aim is to show that, given $\Lambda \leq_\epsilon \Gamma$, it holds that $B(\Gamma, \Lambda) = \partial_F \Lambda$. We start with a result which we believe has its own interest.

Theorem 4.3. Let $\Lambda \leq_\epsilon \Gamma$ and $\Gamma \acts X$ a minimal action on a compact space such that $\Lambda \acts X$ is proximal. Then $\Lambda \acts X$ is minimal as well.

Proof. Let $M \subset X$ be a closed non-empty $\Lambda$-invariant set. For any $g \in \Gamma$, we have that $gM$ is $g\Lambda g^{-1}$-invariant.

Fix $g_1, \ldots, g_n \in \Gamma$. We have that $H := \Lambda \cap g_1 \Lambda g_1^{-1} \cap \cdots \cap g_n \Lambda g_n^{-1}$ has finite index in $\Lambda$. In particular, $H \acts X$ is proximal and admits a unique minimal component $K$. Since each $g_i M$ is $g_i \Lambda g_i^{-1}$-invariant, we conclude that $K \subset \bigcap_{i=1}^n g_i M$.

By compactness of $X$, we obtain that $L := \bigcap_{g \in \Gamma} gM \neq \emptyset$. Since $L$ is $\Gamma$-invariant, we have $X = L \subset M$. □

The following is an immediate consequence of the previous theorem:

Corollary 4.4. Let $\Lambda \leq_\epsilon \Gamma$. If $X$ is a $\Gamma$-boundary which is also $\Lambda$-strongly proximal, then $X$ is a $\Lambda$-boundary.

By arguing as in [Urs22, Corollary 4.3], we conclude the following:

Corollary 4.5. If $\Lambda \leq_\epsilon \Gamma$, then $B(\Gamma, \Lambda) = \partial_F \Lambda$. 
Given a group $\Gamma$, there exists a universal minimal proximal $\Gamma$-space $\partial_\Gamma \Gamma$ (Gla76 Theorem II.4.2). It was shown in [PTVF19 Proposition 2.12] and [GTWZ21 Theorem 1.5] that a countable group $\Gamma$ is icc if and only if $\Gamma \actson \partial_\Gamma \Gamma$ is faithful, if and only if $\Gamma \actson \partial_\Gamma \Gamma$ is free.

One can easily check that the statements of Theorem 5.1 and Lemma 3.3 hold with $\partial_\Gamma \Lambda$ instead of $\partial_\Gamma \Gamma$, with the exact same proofs (in particular, [BKKO17 Lemma 5.1], which is needed in the proof of Lemma 3.3, uses only proximality).

For future reference, let us record this:

**Theorem 5.1.** Let $\Lambda \leq_c \Gamma$. Then $\Lambda \actson \partial_\Gamma \Lambda$ extends in a unique way to an action of $\Gamma$ on $\partial_\Gamma \Lambda$. Furthermore, given $s \in \Gamma$, if $s \in C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$, then $\text{Fix}(s) = \partial_\Gamma \Lambda$.

Conversely, if $\Lambda \actson \partial_\Gamma \Lambda$ is free and $\text{Fix}(s) \neq \emptyset$, then $s \in C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$.

As a consequence, we obtain the following:

**Theorem 5.2.** Let $\Lambda \leq_c \Gamma$ and assume that $\Lambda$ is countable. The following conditions are equivalent:

1. $\Gamma$ is icc relatively with $\Lambda$;
2. $\Lambda$ is icc and, for any $s \in \Gamma \setminus \{e\}$, we have that $s \notin C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$;
3. $\Gamma \actson \partial_\Gamma \Lambda$ is free;
4. $\Gamma \actson \partial_\Gamma \Lambda$ is faithful.

**Proof.** The implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) are proven as in Theorem 3.5. 
(4) $\implies$ (1). Suppose that there is $g \in \Gamma \setminus \{e\}$ such that $|g\Lambda| < \infty$. Then $H := \{h \in \Lambda : gh = hg\}$ is a finite-index subgroup of $\Lambda$, hence $H \actson \partial_\Gamma \Lambda$ is also minimal and proximal. Since the homeomorphism on $\partial_\Gamma \Lambda$ given by $g$ is $H$-equivariant, we conclude that $g$ acts trivially on $\partial_\Gamma \Lambda$. 

**Remark 5.3.** Given a group $\Gamma$, let $L(\Gamma)$ be its group von Neumann algebra. Given $\Lambda \leq \Gamma$, it follows from [Rør21 Proposition 5.1] that $\Gamma$ is icc relatively to $\Lambda$ if and only if any intermediate von Neumann algebra of $L(\Lambda) \subset L(\Gamma)$ is a factor.

Let us now apply Theorem 5.2 to a certain locally finite commensurated subgroup of Thompson’s group $V$.

Let $X := \{0, 1\}$ and, given $n \geq 0$, let $X^n$ be the set of words in $X$ of length $n$. Given $w \in X^n$, let $C(w) := \{(s_n) \in X^\mathbb{N} : s_{[1,n]} = w\}$. Recall that Thompson’s group $V$ is the group of homeomorphisms on $X^\mathbb{N}$ consisting of elements $g$ for which there exist two partitions $\{C(w_1), \ldots, C(w_m)\}$ and $\{C(z_1), \ldots, C(z_m)\}$ of $\{0, 1\}^\mathbb{N}$ such that $g(w_is) = z_is$ for every $1 \leq i \leq m$ and $s$ infinite binary sequence.

Let us define inductively groups $G_n$ acting by permutations on $X^n$.

Let $G_1 := \mathbb{Z}_2$ acting non-trivially on $X$ and, for $n \in \mathbb{N}$,

$$G_{n+1} := \bigoplus_{w \in X^n} \mathbb{Z}_2 \rtimes G_n,$$

where the action of $G_{n+1}$ on $X^{n+1}$ is defined as follows: given $v \in X^n$, $x \in X$, $\sigma \in G_n$ and $f \in \bigoplus_{X^n} \mathbb{Z}_2$,

$$(f, \sigma)(vx) := \sigma(v)f_{\sigma(v)}(x).$$

Let $G := \lim_{n \in \mathbb{N}} G_n$. Then $G$ acts faithfully on $X^\mathbb{N}$ and, as observed in [LB17 Proposition 7.11], $G \leq_c V$. 

5. Commensurated subgroups and proximal actions
Corollary 5.4. The extended action of $V$ on $\partial_p G$ is free.

Proof. It follows from [Gri11, Theorem 9.17] that $G$ is icc. Since $V$ is simple, the conclusion follows from Theorem 5.2. □

Remark 5.5. In [LBMB18, Theorem 1.5], Le Boudec and Matte Bon showed that Thompson’s group $V$ is $C^*$-simple, hence $V \acts \partial_p V$ is free. However, their proof is done by showing that $V$ does not admit non-trivial amenable URS, not by exhibiting a concrete topologically free $V$-boundary. It seems as an interesting problem to determine whether $V \acts \partial_p G$ is strongly proximal, thus providing an alternative proof for $C^*$-simplicity of $V$.

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