Abstract

We describe in detail how one can extract space-time geometry in an exactly solvable model of quantum dilaton gravity of the type proposed by Callan, Giddings, Harvey and Strominger (CGHS). Based on our previous work, in which a model with 24 massless matter scalars was quantized rigorously in BRST operator formalism, we compute, without approximation, mean values of the matter stress-energy tensor, the inverse metric and some related quantities in a class of coherent physical states constructed in a specific gauge within the conformal gauge. Our states are so designed as to describe a variety of space-time in which in-falling matter energy distribution produces a black hole with or without naked singularity. In particular, we have been able to produce the prototypical configuration first discovered by CGHS, in which a (smeared) matter shock wave produces a black hole without naked singularity.

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1 Introduction

Many of the perplexing difficulties in quantum gravity are intimately associated with its physical interpretation. This stems from the ironic circumstance that while geometry is the deep key concept that captures the essence of gravity, our actual perception inherently hinges upon local measurements and hence cannot truly be geometrical. As a matter of fact this dilemma already exists in classical general relativity; no one knows how to describe physics in terms of a set of coordinate-independent numbers alone. Of course in classical case we know a way to circumvent this difficulty: We set up a suitable coordinate system, which we know how to interpret in relation to physical measurements made in our vicinity, and relying on this intuition we can extend our understanding to the whole of the space-time manifold. In quantum theory, however, the situation is much more non-trivial for various reasons. Let us list a few which will be relevant. First, the notion of a quantum state is global. It describes a state of the whole system at once and no “local” information is stored in itself. The second problem, related to the first, is that in the most popular formulation, where quantum gravity is treated as a constrained system \[1\], the wave functions are functionals of the fields and together with the lack of probability interpretation no shadow of space-time physics is recognized in them.

As long as one stays within the approximation where one deals only with small quantum fluctuations around a prescribed background geometry (possibly with some back reaction incorporated), these problems essentially do not present themselves. However, with the recent developments of quantum gravity, especially in two dimensions where one now has models which are exactly solvable, this problem of physical interpretation has become one of the central issues to be faced squarely. The purpose of this article is to discuss this problem in concrete and exact terms in a model of quantum dilaton gravity of the type proposed by Callan, Giddings, Harvey and Strominger \[2\]-\[15\], \[16\]-\[22\].

In our previous work, hereafter referred to as I \[23\], we have rigorously quantized a version of such a class of models with 24 massless matter scalars by developing a non-linear and non-local quantum canonical mapping of interacting fields into a set of free fields. Furthermore, all the physical states and operators of the model have been obtained as BRST cohomology classes. Technically this constitutes the exact solution of the model. However, as emphasized above, solvability and understandability are two different concepts in quantum gravity. Physical states obtained in I are expressed in terms of Fourier
mode operators of the auxiliary free fields and as they stand they do not yield to physical interpretations.

In order to extract the physical meaning of these abstract states, one must act on them by appropriate operators of physical significance and see the response. Indeed this is what we must do for as simple a theory as that of a single quantum harmonic oscillator: A Fock state by itself carries no physical meaning. Only by looking at its response to the action of the energy operator and by computing the expectation values of the coordinate and/or the momentum operators, can we understand the physical content of such an abstract state.

In gauge theories, these physics-probing operators should preferably be gauge invariant. The problem in the case of gravity, however, is that except for such an operator as the volume of the universe, there are few simple gauge invariant operators which we know how to interpret. Conceptually, one may imagine introducing gauge invariant “clocks and rulers” and try to describe the motion of particles and fields in relation to these quantities. In practice, however, it is extremely difficult, if not impossible, to construct such measuring apparatus out of the fields in a given model: One is free to pick certain gauge invariant quantities and declare them as one’s reference entities, but there is no guarantee that they will allow us to extract intuitively understandable physics. Although an attempt in this direction has recently been made [24, 25], it is not clear how classical space-time picture can be reconstructed from the first principle in this approach.

This brings us to the remaining alternative, i.e., to the use of more familiar operators, such as the metric, the curvature and the energy-momentum tensor of the matter fields, as our probe. These operators are obviously gauge dependent and hence in order to get definite responses we must fix the gauge completely. In the BRST formalism we are adopting, this corresponds to making a definite choice of a representative for each non-trivial cohomology class. What is the suitable criterion for making such a choice? It is connected to another fundamental issue, namely which matrix elements we should compute and how to interpret them. Our point of view is the following: Even with a lack of probabilistic interpretation of the wave functions, mean values of the operators listed above in a chosen state should be related to what we actually observe in a universe specified by that state. In particular, if we arrange a suitable state, classical geometry (with quantum corrections) should be recognizable in such averages. Among the classical
solutions of the CGHS model, the most interesting is the one in which a matter shock-wave produces a black hole configuration. Thus we shall try to choose a cohomology class and a particular representative thereof so that such a configuration is reproduced. For technical reasons, we shall be able to deal only with a few of the desired operators, including the matter stress-energy tensor and the inverse metric $g^{\alpha\beta}$. Nevertheless, we shall be able to compute, without approximation, the mean values of these operators in a certain class of coherent physical states and see that black holes with and without naked singularities can be formed by smeared shock-wave-like in-falling matter distributions. Although an attempt has recently been made [26], this is to our knowledge the first time that one can explicitly see the emergence of space-time geometry in an exactly solvable model of quantum dilaton gravity containing matter fields.

In the course of our calculation, we face the question of the choice of the inner product between states, especially in the space of zero-modes of the dilaton-Liouville sector which is generated by hermitian operators with continuous spectra. As was analyzed some time ago in [27, 28], essentially two types of inner product can be consistently implemented in such a case. One of them involves indefinite metric and was later shown [28] to be relevant for the prescription of the “conformal rotation” [29] in the Euclidean path integral formulation of four dimensional Einstein quantum gravity. In the present case, however, we find that such a choice is in conflict with the requirement of reality of various mean values. Instead the correct choice turned out to be of the remaining type in the classification of [27]. More details will be provided later.

We organize the rest of this article as follows: In Sect. 2, we provide a brief review of the results obtained in I, in preparation for the subsequent sections. Expressions of the physical states obtained there in the BRST formalism, however, are not quite useful for our purposes. Therefore in Sect. 3 we construct two different DDF-type representations [30], which will be used in the actual calculations. Some technical details concerning this construction are relegated to Appendix A. Sect. 4 and 5 constitute the main part of our work. In Sect.4, we first give some motivations for the class of physical states we shall consider and write down their explicit forms. Then, following a discussion of the choice of the inner product, we describe the essence of the actual computation of the mean values for the operators mentioned previously. Details of the exact results, which are rather involved, are listed in Appendix B. In Sect. 5, we focus on a class of particularly interesting cases and analyze them in the limit where the (parameter) size of the universe
becomes large. We shall be able to show explicitly how the presence of the matter energy flux produces black hole configurations of various sorts. The properties of the integrals that appear in the analysis are given in Appendix C. Finally, in Sect. 6, we discuss the remaining problems, including the difficult question of how to define and compute the S-matrix. The essence of our work has been reported in [31].

2 Brief Review of the Model

We begin by giving a brief review of the model and the results previously obtained in I in order to make this article reasonably self-contained. This will at the same time serve to define various quantities to be used in the subsequent sections.

The classical action of our model is taken to be of CGHS form [4], given by

\[ S = \frac{1}{\gamma^2} \int d^2 \xi \sqrt{-g} \left\{ e^{-2\phi} \left( -4g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - R_g + 4\lambda^2 \right) + \sum_{i=1}^{N} \frac{1}{2} g^{\alpha\beta} \partial_\alpha f_i \partial_\beta f_i \right\}, \] (2.1)

where \( \phi \) is the dilaton field and \( f_i \) (\( i = 1, \ldots, N \)) are N massless scalar fields representing matter degrees of freedom. We shall stay throughout in Minkowski space and use the metric convention such that for flat space \( \eta_{\alpha\beta} = \text{diag}(1, -1) \). (Compared with the form we adopted in I, the signs of the terms in the bracket \[ \] are reversed to conform to the original CGHS action. This leads to a minor sign change, to be indicated later, for the results obtained in I.)

In I, we adopted the usual convention of setting both the speed of light and \( \hbar \) to be unity. In this work, in order to critically examine the notion of “quantum corrections”, we shall explicitly retain \( \hbar \) dependence after quantization. This in turn requires us to properly keep track of dimensions of various quantities. In two dimensions, all the fields appearing in the action are dimensionless and the only dimensionful quantities at the classical level are \( \lambda \) (the dilatonic cosmological constant) and \( 1/\gamma^2 \) factor in front. They set the fundamental length and the mass scale, \( L_0 \) and \( M_0 \) respectively, as

\[ L_0 = \frac{1}{\lambda}, \quad M_0 = \frac{\lambda}{\gamma^2}. \] (2.2)

Note that \( \gamma \) has the dimension of \( 1/\sqrt{\hbar} \).

In order to define all the quantities unambiguously, we take our universe to be spatially periodic with period \( 2\pi L \). It is then convenient to introduce the dimensionless coordinates
\[ x^\mu = (t, \sigma) = \xi^\mu / L \] and require that all the fields in the action be invariant under \( \sigma \to \sigma + 2\pi \). When the action is rewritten in terms of \( x^\mu \), it retains its form except with the replacement \( \lambda \to \mu \equiv \lambda L \), where \( \mu \) is dimensionless. Later when we come to the physical interpretation of the results, we will get back to the original variables \( \xi^\mu \) and \( \lambda \).

Quantization of this model enforcing conformal invariance was proposed by several authors \[16, 19, 21, 22\] and we adapted the approach of Ref \[22\]. In their scheme, one first makes a classical transformation of fields

\[ \Phi \equiv e^{-2\phi}, \quad h_{\alpha\beta} \equiv e^{2\omega} g_{\alpha\beta}, \quad (2.3) \]

where

\[ \omega = \frac{1}{2} (\ln \Phi - \Phi). \quad (2.4) \]

The action thereby takes the form proposed by Russo and Tseytlin \[32\]

\[ S = \frac{1}{\gamma^2} \int d^2 x \sqrt{-h} \left[ -h^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi - R_h \Phi + 4\mu^2 e^{-\Phi} + \sum_i \frac{1}{2} h^{\alpha\beta} \partial_\alpha \vec{f}_i \cdot \partial_\beta \vec{f}_i \right], \quad (2.5) \]

where the curvature scalar \( R_h \) refers to the conformally transformed “metric” \( h_{\alpha\beta} \). By choosing a measure appropriately and going through an analysis similar to the one performed by David and Distler and Kawai \[33\] for non-critical string theory, one arrives at a quantum model. For the special case with 24 matter scalars, the model simplifies considerably and in the “conformal gauge” \( h_{\alpha\beta} = e^{2\rho} \eta_{\alpha\beta} \) the action takes the form

\[ S = S^{cl} + S^{gh}, \quad (2.6) \]

\[ S^{cl} = \frac{1}{\gamma^2} \int d^2 x \left( -\partial_\alpha \Phi \partial^\alpha \Phi - 2\partial_\alpha \Phi \partial^\alpha \rho + 4\mu^2 e^{\Phi + 2\rho} + \frac{1}{2} \partial_\alpha \vec{f} \cdot \partial^\alpha \vec{f} \right). \quad (2.7) \]

where \( S^{gh} \) is the usual \( b\)-\( c \) ghost action. This is the model which we solved exactly in our previous work by means of a quantum canonical mapping into free fields.

From the equations of motion, the dilaton field \( \Phi \) and the Liouville field \( \rho \) can be expressed in terms of periodic free fields \( \psi \) and \( \chi \) as

\[ \Phi = -\chi - AB, \quad (2.8) \]

\[ \rho = \frac{1}{2} (\psi - \Phi), \quad (2.9) \]

where the functions \( A(x^+) \) and \( B(x^-) \) are defined by

\[ \partial_+ A(x^+) = \mu e^{\psi^+/2}(x^+), \quad (2.10) \]

\[ \partial_- B(x^-) = \mu e^{\psi^-/2}(x^-). \quad (2.11) \]
(In Eq. (2.8) the sign of $\chi$ is reversed compared with $I$. It is not difficult to check that this is the only change necessary to be consistent with the original CGHS action we adopt in this article.) The light-cone coordinates are defined as usual by $x^\pm = t \pm \sigma$ and $\psi^{\pm/2}(x^\pm)$ are the left- and right-going components of the free field $\psi(x)$. We write the Fourier mode expansions of $\psi$ and $\chi$ as

$$\psi = \tilde{\gamma} \left\{ q^+ + p^+(x^+ + x^-) + i \sum_{n \neq 0} \left( \frac{\alpha_n^+ e^{-inx^+}}{n} + \frac{\tilde{\alpha}_n^+ e^{-inx^-}}{n} \right) \right\}, \quad (2.12)$$

$$\chi = \tilde{\gamma} \left\{ q^- + p^-(x^+ + x^-) + i \sum_{n \neq 0} \left( \frac{\alpha_n^+ e^{-inx^+}}{n} + \frac{\tilde{\alpha}_n^- e^{-inx^-}}{n} \right) \right\}, \quad (2.13)$$

where $\tilde{\gamma}$, to be often used hereafter, is defined as

$$\tilde{\gamma} \equiv \frac{\gamma}{\sqrt{4\pi}}. \quad (2.14)$$

Then $\psi^{+/2}$ for example takes the form

$$\psi^{+/2} = \tilde{\gamma} \left\{ \frac{q^+}{2} + p^+ x^+ + i \sum_{n \neq 0} \frac{\alpha_n^+ e^{-inx^+}}{n} \right\}. \quad (2.15)$$

A somewhat peculiar superscript $\pm/2$ on $\psi$ is designed to remind us that its zero-mode part contains $q^+/2$, i.e. half the corresponding part in the full $\psi$. On the other hand, we will need chiral free fields with full zero-mode structure, which possess better conformal properties. These will be denoted with the usual superscript $\pm$. For instance, we define

$$\psi^+ \equiv \tilde{\gamma} \left\{ q^+ + p^+ x^+ + i \sum_{n \neq 0} \frac{\alpha_n^+ e^{-inx^+}}{n} \right\}. \quad (2.16)$$

As $\psi^{\pm/2}$ each experiences a constant shift under $\sigma \to \sigma + 2\pi$, $A(x^+)$ and $B(x^-)$ are not periodic and satisfy the boundary conditions

$$A(x^+ + 2\pi) = \alpha A(x^+), \quad (2.17)$$

$$B(x^- - 2\pi) = \frac{1}{\alpha} B(x^-), \quad (2.18)$$

where $\alpha$ is related to the zero mode $p^+$ by

$$\alpha = e^{i\sqrt{\pi} p^+}. \quad (2.19)$$

Solutions for $A$ and $B$ which satisfy the proper boundary conditions are (suppressing the $t$ dependence)

$$A(\sigma) = \mu C(\alpha) \int_0^{2\pi} d\sigma' E_\alpha(\sigma - \sigma') e^{\psi^{+/2}(\sigma')}, \quad (2.20)$$

$$B(\sigma) = \mu C(\alpha) \int_0^{2\pi} d\sigma'' E_{1/\alpha}(\sigma - \sigma'') e^{\psi^{+/2}(\sigma'')}, \quad (2.21)$$

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where \( C(\alpha) = 1/\left(\alpha^{1/2} - \alpha^{-1/2}\right) \) and the functions \( E_\alpha(\sigma) \) and \( E_{1/\alpha}(\sigma) \) are defined by
\[
E_\alpha(\sigma) \equiv \exp\left(\frac{1}{2} \epsilon(\sigma) \ln \alpha\right), \quad E_{1/\alpha}(\sigma) \equiv \exp\left(-\frac{1}{2} \epsilon(\sigma) \ln \alpha\right).
\] (2.22)
\( \epsilon(\sigma) \) is a stair-step function with the property \( \epsilon(\sigma + 2\pi) = 2 + \epsilon(\sigma) \) and coincides with the usual \( \epsilon \)-function in the interval \([-2\pi, 2\pi]\). Note that we must require \( p^+ \) not to vanish since otherwise \( C(\alpha) \) blows up.

The left-going and the right-going energy-momentum tensor \( T_{\pm\pm} \) take simple forms in terms of the free fields. With a convenient normalization, they take the form
\[
T_{\pm\pm} = \frac{1}{\tilde{\gamma}^2} \left( \partial_\pm \chi \partial_\pm \psi - \partial_\pm^2 \chi + \frac{1}{2} (\partial_\pm \tilde{f})^2 \right).
\] (2.23)
This can be diagonalized by introducing canonically normalized scalar fields \( \phi_1, \phi_2 \) and \( \phi_f \):
\[
\psi = \frac{\tilde{\gamma}}{\sqrt{2}} (\phi_1 + \phi_2), \quad \chi = \frac{\tilde{\gamma}}{\sqrt{2}} (\phi_1 - \phi_2), \quad f^i = \tilde{\gamma} \phi_f^i,
\] (2.24)
\[
T_{\pm\pm} = \frac{1}{2} (\partial_\pm \phi_1)^2 - Q \partial_\pm^2 \phi_1 - \frac{1}{2} (\partial_\pm \phi_2)^2 + Q \partial_\pm^2 \phi_1 + \sum_{i=1}^{24} \frac{1}{2} (\partial_\pm \phi_f^i)^2,
\] (2.25)
where the background charge \( Q \) is given by
\[
Q = \frac{\sqrt{2\pi}}{\gamma} = \frac{1}{\sqrt{2\tilde{\gamma}}}.
\] (2.26)

Fourier mode expansion for \( f^i \) is just like for \( \psi \) with the replacements \((q^+, p^+, \alpha_+^m, \tilde{\alpha}^+_n) \rightarrow (q_f^i, p_f^i, \alpha_f^i, \tilde{\alpha}_f^i)\).

As was fully described in I, one can show that the mapping from the original fields \( \{\Phi, \rho\} \) into the free fields \( \{\psi, \chi\} \) is a quantum as well as classical canonical transformation. Namely, the canonical equal time commutation relations
\[
[\Phi(\sigma, t), \Pi_\Phi(\sigma', t)] = [\rho(\sigma, t), \Pi_\rho(\sigma', t)] = i\hbar \delta(\sigma - \sigma')
\] are reproduced if we assume the commutators among the modes of \( \psi \) and \( \chi \) to be
\[
[q^\pm, p^\mp] = i\hbar, \quad [\alpha_+^m, \alpha_-^n] = \left[\tilde{\alpha}_+^m, \tilde{\alpha}_-^n\right] = m\hbar \delta_{m+n,0}, \quad \text{Rest} = 0.
\] (2.27) (2.28) (2.29)
(Commutators between the modes of $f^i$ are of the usual form.) To establish this result quantum mechanically, it was important that the non-local operators $A(x^+)$ and $B(x^-)$ are well-defined without the need of normal-ordering due to the commutativity of the modes of $\psi$.

The quantized model continues to enjoy conformal invariance. The Fourier modes $L_m$ and $\bar{L}_m$ of the left- and right-going energy-momentum tensors in the dilaton-Liouville (dL) and the matter (f) sector satisfy the usual Virasoro algebra with the central charges $c_{dL} = 2$ and $c_f = 24$ respectively. It was shown in I that $\chi$ and the product $A(x^+)B(x^-)$ transform as genuine dimension zero primary fields, while due to the presence of the background charge $\psi$ transforms anomalously as

$$[L_{m+}, \psi(x)] = e^{imx^+} \left( \frac{1}{i} \partial_+ \psi + \frac{Q\gamma}{\sqrt{2\pi} m} \right).$$

(2.30)

In the subsequent sections, chiral primary fields with dimension 0 will play important roles. $\chi^+$ (not $\chi^{+}/2$) is one such field. Another one is a slight modification of $A(x^+)$ defined by

$$A(x^+) \equiv e^{q_+/2} A(x^+).$$

(2.31)

The additional zero-mode factor makes this field transform as a genuine chiral primary field.

As far as the mathematical structure is concerned, our model is a hybrid of critical and non-critical bosonic string theories. Hence the physical states are readily obtained by using the BRST analysis developed for these theories with appropriate modifications \[34\]-\[37\], \[23\]. Let us briefly summarize the results obtained in I. (Only the left-going sector will be treated explicitly.)

The nilpotent BRST operator is given by

$$d = \sum c_{-n} (L_{n}^{dL} + L_{n}^{f}) - \frac{1}{2} \sum (m - n)c_{-m}c_{-n}b_{m+n} : ,$$

(2.32)

where $sl(2)$ invariant normal-ordering for the ghosts is assumed. The physical ghost vacuum is defined as usual by $|0>_{\downarrow} = c_1 |0>_{inv}$. The operator $d$ is decomposed with respect to the ghost zero mode in the form $d = c_0 L_{0}^{tot} - Mb_0 + \hat{d}$, where $L_{0}^{tot}$ is the total Virasoro operator including the ghosts. It is well-known that the non-trivial $d$-cohomology...
must be in the sector satisfying \( L_0^{\text{tot}} \psi = 0 \). By assigning the degree to the mode operators
\[
\begin{align*}
\deg(\alpha^+_n) &= \deg(c_n) = 1, \\
\deg(\alpha^-_n) &= \deg(b_n) = -1, \\
\deg(\text{Rest}) &= 0,
\end{align*}
\] (2.33)
the BRST operator for the relative cohomology \( \hat{d} \) is decomposed as
\[
\begin{align*}
\hat{d} &= \hat{d}_0 + \hat{d}_1 + \hat{d}_2, \\
\hat{d}_0 &= \sum_{n \neq 0} P^+(n)c_{-n}\alpha^-_n, \\
\hat{d}_1 &= \sum_{n \neq 0} c_{-n}(\alpha^-_{-m-n} + \frac{1}{2} (m-n)c_{-n}b_{m+n} + L_0^f) , \\
\hat{d}_2 &= \sum_{n \neq 0} P^-(n)c_{-n}\alpha^+_n,
\end{align*}
\] (2.34)
where \( P^\pm(n) \) are given by
\[
P^+(n) = p^+ + i\sqrt{2}Qn, \quad P^-(n) = p^-.
\] (2.38)
One then studies \( \hat{d}_0 \)- or \( \hat{d}_2 \)-cohomology depending on the conditions on \( P^\pm(n) \), and upon them all the \( \hat{d} \)- and \( d \)-cohomologies can be constructed.

As our purpose in this article is to extract space-time geometry of states in which the matter fields carry finite energy in the limit of large \( L \), we only record the relevant \( d \)-cohomologies, namely the ones with arbitrarily high matter excitations without ghosts.

Let \( \psi_0 \) be a state of the form
\[
\begin{align*}
\psi_0 &= \hat{F} | \vec{P} >_{\uparrow}, \\
\vec{P} &= (p^+, p^-, \vec{p}_f), \\
L_0^{\text{tot}} | \vec{P} >_{\uparrow} &= \left( p^+p^- + \frac{1}{2}\vec{p}_f^2 - \hbar \right) | \vec{P} >_{\uparrow} = 0,
\end{align*}
\] (2.39)
where \( \hat{F} \) is an operator composed of arbitrary number of matter creation operators. Such a state simultaneously belongs to \( \hat{d}_0 \)- and \( \hat{d}_2 \)-cohomologies. If \( P^+(n) \neq 0 \) for all non-zero integer \( n \), the corresponding physical state \( \psi \) satisfying \( L_0^{\text{tot}} \psi = 0 \) can be constructed in the form
\[
\begin{align*}
\psi &= \sum_{n=0}^{\infty} (-1)^n (T^+)^n \psi_0, \\
T^+ &\equiv \hat{N}_{dL^2}^{-1} K^+ \hat{d}_1, \\
K^+ &\equiv \sum_{n \neq 0} \frac{1}{P^+(n)} \alpha^+_n b_n,
\end{align*}
\] (2.42)
where $\hat{N}_{dLg}$ is the level counting operator for the dilaton-Liouville-ghost sector. Similarly, if $p^- \neq 0$, the expression for $\psi$ becomes

$$\psi = \sum_{n=0}^{\infty} (-1)^n (T^-)^n \psi_0,$$  \hspace{1cm} (2.45)

$$T^- \equiv \hat{N}_{dLg}^{-1} K^- \hat{d}_1,$$  \hspace{1cm} (2.46)

$$K^- \equiv \frac{1}{p^-} \sum_{n \neq 0} \alpha_-^- b_n.$$  \hspace{1cm} (2.47)

In the next section, we shall give more useful representations for these somewhat formal expressions.

## 3 DDF Representations of Physical States

As we have seen, the structure of physical states of our model is formally extremely similar to that of bosonic string theories. However, the physical interpretation of them is quite different. In string theory, Virasoro levels specify the invariant masses of various fields, while in the present model they refer to the discretized energy levels of a field: The energy carried by a state at level $n$ is proportional to $n/L$. As we will be most interested in configurations where the matter fields carry finite energy in the limit of large $L$, we need to be able to deal with physical states at arbitrary high Virasoro levels, in marked contrast to the case of string theory.

For this purpose, the formal expressions (2.42) and (2.43) of physical states obtained through BRST analysis are not readily tractable. Fortunately, for states not involving ghosts, more useful expressions are available, the so called DDF states [30], developed long ago in the context of string theory. Let us briefly describe the essence of the construction in a manner suitable for our model.

Let $\phi^i_f(x^+)$ be canonically normalized left-going matter fields and $\varphi(x^+)$ be a dimension 0 primary field with the following properties:

(i) $e^{im\varphi}$ is periodic for $m \in \mathbb{Z},$

(ii) modes of $\varphi$ all commute with themselves and with the matter fields $\phi^i_f,$

(iii) $\int_0^{2\pi} \frac{dx^+}{2\pi} \partial_+ \varphi(x^+) = 1.$

Then the set of operators $B^i_m$ defined by

$$B^i_m = \int_0^{2\pi} \frac{dx^+}{2\pi} e^{im\varphi} \partial_+ \phi^i_f,$$  \hspace{1cm} (3.1)
are BRST invariant and satisfy the oscillator commutation relations

\[ [B^i_m, B^n_j] = \hbar m \delta_{m+n,0}. \]  

(3.2)

BRST invariance is trivial since the integrand of \( B^i_m \) is a periodic dimension 1 primary field and the commutation relations also follow easily using the properties \((i) \sim (iii)\) listed above. In particular, the periodicity requirement is crucial in order to perform the integration by parts during the course of the calculation.

For our model, the simplest candidate for \( \varphi \) (when \( p^- \neq 0 \)) is \( \varphi(x^+) = \chi^+(x^+)/\langle \tilde{\gamma} p^- \rangle \) and the corresponding \( B^i_m \), which we denote by \( A^i_m \), is given by

\[ A^i_m = \int_0^{2\pi} \frac{dx^+}{2\pi} e^{im\chi^+(x^+)/\langle \tilde{\gamma} p^- \rangle} \partial_x \phi^i_f. \]  

(3.3)

The factor of \( 1/p^- \) is necessary to assure the periodicity. Physical states can be built up using these oscillators as

\[ \sum_{m_1, n_2, \ldots, n_k} C_{i_1 i_2 \ldots i_k}^{n_1 n_2 \ldots n_k} A^i_{-n_1} A^{i_2}_{-n_2} \cdots A^{i_k}_{-n_k} | \vec{P} \rangle_{\downarrow} = \sum_{n=0}^{\infty} (-T^-)^n \psi_0, \]

where \( | \vec{P} \rangle_{\downarrow} \) is the zero-mode vacuum satisfying the condition \( p^+ p^- + (1/2)\tilde{p}_f^2 - \hbar = 0 \). A characteristic feature of this type of physical states is that they are composed solely of the oscillators \( \alpha_{-n} \) and \( \alpha_{-m}^\dagger \) and do not contain \( \alpha_{+n}^\dagger \)'s. In the BRST formalism, this property is precisely the one enjoyed by the states of the form in (2.47), namely,

\[ \sum_{n=0}^{\infty} \psi_0, \]

where \( \psi_0 \) is a state representing \( \tilde{d}_2 \)-cohomology with matter excitations only. Indeed one can show that they are identical. The precise identification is

\[ a^i_1 \cdot \vec{A}_{-k_1} \alpha_{i_2}^\dagger \cdot \vec{A}_{-k_2} \cdots a^i_q \cdot \vec{A}_{-k_q} | \vec{P} \rangle_{\downarrow} = \sum_{n=0}^{\infty} (-T^-)^n a^i_1 \cdot \vec{\alpha}_{-k_1} \alpha_{i_2}^\dagger \cdot \vec{\alpha}_{-k_2} \cdots a^i_q \cdot \vec{\alpha}_{-k_q} | \vec{P} \rangle_{\downarrow}, \]

(3.4)

where

\[ p'_+ p'_- + \frac{1}{2} \tilde{p}_f^2 - \hbar = 0, \quad p^+ p^- + \frac{1}{2} \tilde{p}_f^2 + \hat{N}_f - \hbar = 0, \]

\[ p'_+ = p'_- = p_-, \quad \tilde{p}_f = \tilde{p}_f, \]
and $\bar{\alpha}_i$’s are arbitrary constant vectors. ( The difference in the zero-mode sector is simply due to the fact that $A^\perp_{-m}$ contains the factor $\exp(-imq^-/p^-)$ which shifts $p^+$ by the amount $-m\hbar/p^-$, while $T^-$ does not contain such zero-mode operator. )

In fact the proof of this formula is already implicit in our previous work, namely in the proof of Eq.(4.24) (with technical details in Appendix B) in I, with the replacement of $T^+$ by $T^-$ and some associated changes. We now make its relevance more explicit.

Let $\psi_0$ be a state made up solely of matter oscillators as stated above. Hence it is of degree 0 and is annihilated by both $\hat{d}_0$ and $\hat{d}_2$. The problem is to construct a representative $\psi$ of $\hat{d}$-cohomology, which contains $\psi_0$. Since we are interested in $\psi$ composed only of $\alpha^-_n$ and $\alpha^\perp_{-m}$, the degree of $\psi$ must be non-positive and we can expand it as

$$\psi = \sum_{n \geq 0} \psi_{-n}, \quad (3.5)$$

where $\deg(\psi_{-n}) = -n$. Then the $\hat{d}$-closedness condition for $\psi$ reads

$$\hat{d}\psi = (\hat{d}_0 + \hat{d}_1 + \hat{d}_2) \sum_{n \geq 0} \psi_{-n}$$

$$= \sum_n \left( \hat{d}_0 \psi_{-n} + \hat{d}_1 \psi_{-(n+1)} + \hat{d}_2 \psi_{-(n+2)} \right) \quad (3.6)$$

which leads to the recursion relations

$$\hat{d}_1 \psi_0 + \hat{d}_2 \psi_{-1} = 0, \quad (3.7)$$

$$\hat{d}_0 \psi_{-n} + \hat{d}_1 \psi_{-(n+1)} + \hat{d}_2 \psi_{-(n+2)} = 0 \quad \text{(for} \ n \geq 0) \quad (3.8)$$

By an argument parallel to that given in Appendix B of I, provided $p^- \neq 0$, one can show recursively that $K^-\psi_{-n} = 0$ and $\hat{d}_0 \psi_{-n} = 0$. This latter statement means that indeed $\psi$ does not contain $\alpha^\perp_{-n}$ oscillators. Then the recursion relation simplifies to

$$\hat{d}_2 \psi_{-(n+1)} = -\hat{d}_1 \psi_{-n}, \quad (3.9)$$

and the rest of the argument in I amounted to showing that, within the space of states without the excitations of ghosts and $\alpha^\perp_{-n}$ oscillators, $\hat{d}_2$ has the inverse given by $\hat{N}^{-1}_{d\perp}K^-$ and hence the recursion relation is uniquely solved starting from $\psi_0$. Therefore to prove the validity of Eq.(3.4), all one has to do is to check that the degree zero part of the both sides are identical, but this is trivial.
From the argument just presented, it is clear that when \( P^+(n) \neq 0 \) physical states \((2.42)\) constructed in terms of \( T^+ \) must also have DDF type representation. To find it, one must look for a candidate for the periodic dimension 0 primary \( \varphi \), which consists only of the modes of \( \psi^+(x^+) \). \( \psi^+(x^+) \) itself, however, is not appropriate since it does not transform as a primary field due to the presence of the background charge (cf. Eq.\((2.30)\)). The correct choice of \( \varphi \) turns out to be

\[
\varphi = \frac{1}{\gamma p^+} \eta^+, \quad (3.10)
\]

\[
\eta^+ \equiv \ln \left( \frac{A(x^+)/\mu}{\mu} \right), \quad (3.11)
\]

where \( A(x^+) \) is a genuine primary field of dimension 0 defined in Eq.\((2.31)\). Notice that \( p^+ \) must not vanish for this construction, but this condition is already needed in defining \( A(x^+) \) and \( B(x^-) \). A useful explicit form of \( \eta^+ \) is derived in Appendix A, together with its conjugate denoted by \( \zeta^+ \). The fields \( \eta^+ \) and \( \zeta^+ \) are intimately related to the ones employed in \([24, 25]\).

Thus physical states can be built up by the BRST invariant oscillators

\[
\tilde{A}^i_{-n} \equiv e^{-i(n/\tilde{\gamma} p^+) \ln(\tilde{\gamma} p^+)} \int_0^{2\pi} dy^+ \frac{d}{2\pi} e^{-in\eta^+/\tilde{\gamma} p^+} \partial_\phi f(y^+). \quad (3.12)
\]

The extra phase factor in front, which commutes with the BRST operator, is added to remove the corresponding phase in the integrand so that the physical states built with these oscillators agree with the ones constructed with \( T^+ \) operators. In the next section, we shall make use of this type of oscillators to construct interesting physical states.

### 4 Extraction of Space-Time Geometry

As was already pointed out in the introduction, physical meaning of an abstract state can only be extracted by looking at its response to the action of appropriate operators of physical significance. In quantum gravity, each physical state corresponds to a possible choice of the universe and all the events which “take place” in that universe must already be encoded in a chosen state. This means that there is no meaning to a “transition” between different physical states and consequently we will be interested only in the average values of suitable operators in a particular physical state.
4.1 Choice of Probing Operators

The first question then is which operators are suitable for probing the content of a physical state. Preferably we wish to use an appropriate set of BRST invariant operators since their expectation values are independent of the choice of the representative of the physical state. They are essentially the integrals of dimension 1 vertex operators familiar in string theory. As they are manifestly coordinate-independent, their expectation values are simply a set of numbers. In string theory, these set of numbers have immediate physical significance; they are functions of the momenta of the particles which propagate in the target space. In quantum gravity context, however, they are very hard to interpret. One might try to draw an analogy to the description of a charge distribution in terms of a set of integrals, namely the multipole moments. A crucial distinction is that in that case a definite physical picture is already attached to the functions forming the basis of the expansion and with the knowledge of the values of the moments we can immediately reconstruct the physical distribution. Here we do not have such an underlying expansion. Thus although one cannot deny a possibility that in the future, with enough experience and ingenuity, we may be able to understand physics directly from an infinite set of gauge invariant numbers, but at present it is obviously not productive to pursue such a route.

We shall then try to deal with operators of more direct physical significance, such as the stress-energy tensor of the matter fields, the metric, and the curvature. As for the latter two entities, there are some ambiguities: First of all, it is not clear which of the two conformally related quantities, $g_{\alpha\beta}$ and $h_{\alpha\beta}$, should be regarded as the metric. This question of the choice of “conformal-frame” often occurs in dilaton gravity and the principle of general coordinate invariance alone cannot dictate the correct choice. We shall therefore keep both possibilities open. Classically, from the definitions (2.3), (2.4) and the canonical transformation (2.8),(2.9), the metric and the curvature in these two schemes can be expressed in terms of the free fields (in the original coordinate $\xi^\mu$) as

$$g_{\alpha\beta} = \Phi^{-1} e^{\psi} \eta_{\alpha\beta}, \quad g^{\alpha\beta} = \Phi e^{-\psi} \eta^{\alpha\beta},$$

$$R^g_{\alpha\beta} = \frac{1}{2} \Box \ln \Phi \eta_{\alpha\beta}, \quad R^g = e^{-\psi} \Phi \Box \ln \Phi,$$  

$$h_{\alpha\beta} = e^{\psi - \Phi} \eta_{\alpha\beta}, \quad h^{\alpha\beta} = e^{\Phi - \psi} \eta^{\alpha\beta},$$

$$R^h_{\alpha\beta} = -2\lambda^2 e^{\psi} \eta_{\alpha\beta}, \quad R^h = -4\lambda^2 e^{\Phi}.$$  

Due to the composite nature and the presence of the complicated expression $\Phi = -\chi - AB$,
it is not an easy task to give proper quantum definitions for these operators. In this article, we shall treat two of the relatively simple ones, namely $g^{\alpha\beta}$ and $R_h^{\alpha\beta}$. Recalling that all the modes of $\psi$ commute with each other, $R_h^{\alpha\beta}$ and $-ABe^{-\psi}$ part of $g^{\alpha\beta}$ is already well-defined. On the other hand, the remaining part of the latter operator, namely $\chi e^{-\psi}$, needs regularization. For this purpose, let us decompose $\chi$ and $\psi$ into the zero-mode, the annihilation, and the creation parts:

\[
\begin{align*}
\chi &= \chi_0 + \chi_a + \chi_c, \\
\psi &= \psi_0 + \psi_a + \psi_c = \psi_0 + \tilde{\psi},
\end{align*}
\]

(4.5) (4.6)

where $\tilde{\psi}$ denotes the non-zero-mode part of $\psi$. We can then define the operator $\chi e^{-\psi}$ by the normal-ordering:

\[
\begin{align*}
: \chi e^{-\psi} : &= \chi_0 e^{-\psi_0} e^{-\tilde{\psi}} + \chi_c e^{-\psi} + e^{-\psi} \chi_a .
\end{align*}
\]

(4.7)

For the first term, we have written out the zero-mode part explicitly. It is easy to check that $\chi_0$ and $\psi_0$ commute with each other and hence $\chi e^{-\psi}$ defined above is properly hermitian. Therefore, we can actually write

\[
\begin{align*}
: \chi e^{-\psi} : &= \chi e^{-\psi} - \left[ \chi_a, e^{-\psi} \right] .
\end{align*}
\]

(4.8)

Conformal property of $\chi e^{-\psi}$ is easily worked out to be

\[
\begin{align*}
\left[ L_n, : \chi e^{-\psi} : \right] &= \hbar e^{inz^+} \left( \frac{1}{\ell} \partial_+ - n \right) : \chi e^{-\psi} : \\
&\quad + \hbar^2 \gamma^2 ne^{inz^+} e^{-\psi} .
\end{align*}
\]

(4.9)

This shows that the conformal transformation property of regularized $g^{\alpha\beta}$ is slightly modified by a higher order contribution and it is no longer a conformal primary. However, as we shall fix the gauge completely, this will not cause any problems. Together with the matter part of the energy-momentum tensor, these operators will give interesting physical information.

4.2 Choice of States

Clearly the operators we have chosen to work with are gauge dependent and hence their expectation values inevitably depend on the choice of the representative of the physical state, which amounts to a choice of gauge within the conformal gauge. If we
denote by $| \Psi_0 >$ a special representative of a non-trivial cohomology class satisfying $L_0^{\text{tot}} | \Psi_0 > = \bar{L}_0^{\text{tot}} | \Psi_0 > = 0$, any other representative $| \Psi >$ of the same class is expressed as

$$
| \Psi > = | \Psi_0 > + | \Lambda >, \tag{4.10}
$$

$$
| \Lambda > = d | \Psi_{-1} > + \bar{d} | \bar{\Psi}_{-1} >, \tag{4.11}
$$

where $| \Psi_{-1} >$ and $| \bar{\Psi}_{-1} >$ are arbitrary states with left- and right- ghost number $-1$ respectively. The question is how we should choose $| \Psi_0 >$, $| \Psi_{-1} >$ and $| \bar{\Psi}_{-1} >$ so that the average value $< \Psi | O(x) | \Psi >$ for the operator indicated above exhibits an interesting physically interpretable behavior. A hint is provided by a simple fact about the coordinate dependence of a matrix element $< a | O(x) | b >$, where the states $| a >$, $| b >$ and the operator $O(x)$ carry definite global left-right dimensions i.e.

$$
L_0^{\text{tot}} | a > = \hbar \Delta_a | a >, \quad \bar{L}_0^{\text{tot}} | a > = \hbar \bar{\Delta}_a | a > \tag{4.12}
$$

$$
L_0^{\text{tot}} | b > = \hbar \Delta_b | b >, \quad \bar{L}_0^{\text{tot}} | b > = \hbar \bar{\Delta}_b | b > \tag{4.13}
$$

$$
\left[ L_0^{\text{tot}}, O(x) \right] = \hbar \frac{i}{\hbar} \partial^+ O(x), \quad \left[ \bar{L}_0^{\text{tot}}, O(x) \right] = \hbar \frac{i}{\hbar} \partial^- O(x). \tag{4.14}
$$

By evaluating the matrix element $< a | [L_0^{\text{tot}}, O(x)] | b >$ and a similar one with $\bar{L}_0^{\text{tot}}$, we easily deduce

$$
< a | O(x) | b > = \text{const.} \ e^{i(\Delta_a - \Delta_b) x^+} \cdot e^{i(\Delta_a - \Delta_b) x^-}, \tag{4.15}
$$

which expresses nothing but the conservation of energy and momentum. Let us apply this to the case of interest, namely to the expectation value

$$
< \Psi | O(x) | \Psi > = < \Psi_0 | O(x) | \Psi_0 >
$$

$$
+ < \Psi_0 | O(x) | \Lambda > + < \Lambda | O(x) | \Psi_0 > \tag{4.16}
$$

$$
+ < \Lambda | O(x) | \Lambda >.
$$

Then we immediately learn the following: First, $< \Psi_0 | O(x) | \Psi_0 >$ part can only be a constant. Second, the remaining part can produce non-trivial coordinate dependence if $| \Lambda >$ carries non-vanishing weights. In particular, by arranging $| \Lambda >$ to be a suitable superposition of states with various weights, it should be possible to generate a wide variety of coordinate dependence.

To further narrow down the appropriate choice of $| \Psi_0 >$ and $| \Lambda >$, let us note that the second line of (4.16) contains the information of the non-trivial part of the physical
state while the last line depends only on the gauge part $| \Lambda >$. Thus it is natural to try to choose $| \Lambda >$ such that the interesting coordinate dependence comes predominantly from the cross terms in the second line. As for $| \Psi_0 >$, various choices can be possible. It would however be most interesting if we can produce a shock-wave like configuration for the matter energy-momentum tensor since then we should see the formation of a black hole in the mean value of the metric. We expect that such a macroscopic configuration can be constructed if $| \Psi_0 >$ is taken to be a suitable coherent state.

Guided by the reasoning above, we have chosen to work with the following class of states. First, $| \Psi_0 >$ is taken to be a coherent state built up with the BRST invariant oscillators $\tilde{\Lambda}_{-n}$ introduced in the previous section:

$$
| \Psi_0 > \equiv e^G | \tilde{P} >, \hspace{1cm} \text{(4.17)}
$$

$$
G \equiv \frac{1}{\hbar} \sum_{n \geq 1} \tilde{\nu}_n \tilde{\Lambda}_{-n}, \hspace{1cm} \text{(4.18)}
$$

$$
\tilde{\nu}_n = \nu_n e^{inx_0^+}, \hspace{1cm} (\nu_n, x_0^+: \text{real constants}), \hspace{1cm} \text{(4.19)}
$$

$$
| \tilde{P} > \equiv e^{-icp^+/(\hbar \gamma)^2} \int_{-\infty}^{\infty} dp^+ \int_{-\infty}^{\infty} dp^- W(p^+) | p^+, p^-, \tilde{p}_f > \downarrow \times \delta(p^- - \frac{1}{p^+}(h - \frac{1}{2}p^2_f)), \hspace{1cm} \text{(4.20)}
$$

$$
c = \text{a real constant}, \hspace{1cm} \text{(4.21)}
$$

$$
\tilde{p}_f^2 \equiv \tilde{p}_f \cdot \tilde{p}_f. \hspace{1cm} \text{(4.22)}
$$

Some explanations are in order: For simplicity, we consider a coherent state in which only one kind of matter field is excited. Thus we omit the superscript $i$ on $\tilde{\Lambda}_{-n}$. $| \tilde{P} >$ is a zero-mode vacuum smeared with a real weight $W(p^+)$ and it clearly satisfies $L^{tot}_0 | \tilde{P} > = \tilde{L}^{tot} | \tilde{P} >= 0$. This smearing is necessary to make the mean value of the operator $q^-$ well-defined, which will appear in $< g^{\alpha\beta} >$. An appropriate choice of $W(p^+)$ will be given in Sect.5. The phase factor in front is a BRST invariant and will be seen to produce a coordinate-independent contribution in the mean value $< g^{\alpha\beta} >$ and the constant $c$ will be adjusted to cancel certain unwanted terms. The phase factor in the definition of $\tilde{\nu}_n$ will produce a shift $x^+ \rightarrow x^+ - x_0^+$ in certain terms and will eventually specify where a matter shock wave will traverse. Next, the reason for employing $\tilde{\Lambda}_{-n}$, rather than the apparently simpler $A_{-n}$, is two-fold. First, $A_{-n}$ consists of $\alpha^-_m$ oscillators, which have non-vanishing commutators with $e^{\pm \psi_i}$ contained in $g^{\alpha\beta}$ and $R^{\hbar}_{\alpha\beta}$. Consequently, when $A_{-n}$ is exponentiated to make a coherent state, calculations will become practically
intractable. The second and more strategic reason is that with this choice the energy balance between the matter sector and the dilaton-Liouville sector will be predominantly between \((1/2)(\partial_+ f)^2\) and \(\partial_+^2 \chi\), which is precisely the situation that prevails in the gauge \(\psi = 0\) often used in (semi-)classical discussions \footnote{In terms of the original variables, this means \(\phi = \rho_g\), where \(\phi\) and \(\rho_g\) are, respectively, the dilaton and the Liouville mode of the metric \(g_{\alpha\beta}\).}. These remarks will be substantiated when we display the result of our calculation. In any case, \(| \Psi_0 \rangle\) so constructed clearly satisfies an equation characteristic of a coherent state, namely

\[
\tilde{A}_m | \Psi_0 \rangle = \tilde{\nu}_m | \Psi_0 \rangle \quad (m \geq 1).
\]

(4.23)

We also wish to call the attention of the reader that \(| \Psi_0 \rangle\) contains, apart from the zero-modes, only the left-going oscillators. This is due to our intention to produce a left-going matter shock wave as treated in CGHS.

Let us now come to the choice of the gauge part \(| \Lambda \rangle\). We have chosen it to realize all the required features discussed above in the simplest possible form. It is written in the form

\[
| \Lambda \rangle = \frac{1}{\kappa} (d b_{-M} + \bar{d} \bar{b}_{-M}) | \Omega \rangle,
\]

(4.24)

where \(b_{-M}\) and \(\bar{b}_{-M}\) are, respectively, the left- and right-going anti-ghost oscillator at level \(M\) and \(\kappa\) is a constant carrying the dimension of \(\hbar^2\). As long as it is finite, the choice of \(M\) does not make any qualitative difference. \(| \Omega \rangle\) is chosen to be a superposition of zero-mode states of the form

\[
| \Omega \rangle \equiv \sum_{k=-\infty}^{\infty} \omega_k \sum_{l=\pm 1,0} | \tilde{P}(k,l) >_\downarrow,
\]

(4.25)

\[
| \tilde{P}(k,l) > \equiv e^{-inq^+/\hbar \gamma} \gamma^2 \int_{-\infty}^{\infty} dp^+ \int_{-\infty}^{\infty} dp^- W(p^+) | p^+, p^-(k,l), \vec{p}_f >_\downarrow
\]

\[
\times \delta (p^- - \frac{1}{p^+} (h - \frac{1}{2} p^2_f)),
\]

(4.26)

\[
p^-(k,l) \equiv p^- - \frac{\hbar}{p^+} k - i \hbar \gamma l.
\]

(4.27)

\(\omega_k\) are set of real coefficients. Notice that we sum over states with shifted \(p^-\) zero-modes. This is necessary for the following reasons: First since \(\tilde{A}_{-n}\) contains a factor \(\exp(-inq^+/p^+)\) which shifts \(p^-\) by the amount \(-n\hbar/p^+\), the shift of the form \(-\hbar k/p^+\) is needed in order to yield non-trivial overlap with \(| \Psi_0 \rangle\). Likewise, the second shift \(-i \gamma \hbar \ell\)
for \( \ell = \pm 1, 0 \) is required to produce non-vanishing results when we deal with the operators \( g^{\alpha\beta} \) and \( R_{\alpha\beta} \), which contain \( e^{\pm\psi} \) and hence carry imaginary \( p^- \). It is not difficult to see that our choice of \( | \Lambda > \) has the desired property that it is a superposition of states with various Virasoro weights and at the same time the pure gauge part of the mean value is rather insensitive to the matter content.

4.3 Specification of Inner Product

Before we start our calculation of the expectation values, we must settle one more important issue, namely the choice of the inner product. In our previous work, we have argued that the correct hermiticity assignments for the Fourier modes of the free fields should be the usual one, namely

\[
\alpha_n^\dagger = \alpha_{-n} \quad (n \neq 0),
\]

\[
p^\dagger = p, \quad q^\dagger = q
\]

for each field. As we emphasized there, this requirement does not yet fix the inner product completely. Some time ago, it was pointed out in \cite{27,28} that for hermitian operators with continuous spectra their eigenvalues need not be real and in fact there are two distinct classes of choices for the inner product which are compatible with their hermiticity. Their analysis applies to our zero-mode sector and especially we must take due caution for the dilaton-Liouville zero-modes \( p^\pm \). If we denote them generically by \( p \), their argument shows that one can either choose \( p \) to take values along a “real-like” path characterized by \( \text{Re} \, p > \text{Im} \, p \) at infinity or along an “imaginary-like” path defined by \( \text{Re} \, p < \text{Im} \, p \) at infinity. The former is a generalization of the usual choice along the real axis and the latter actually signifies the presence of indefinite metric structure of the Hilbert space in question. This latter scheme was later shown \cite{28} to correspond to the “conformal rotation” in four dimensional Euclidean quantum gravity proposed by Gibbons, Hawking and Perry \cite{29}.

In the present case, we are dealing with a Minkowski theory with an indefinite metric structure and occurrence of complex \( p^- \) zero-modes. (Recall the sign of the kinetic term for \( \phi_2 \) in \( (2.23) \) and the imaginary shift present in \( p^-(k,l) \)). Hence it is not obvious which of the two schemes mentioned above should be taken. As we describe below, the correct choice is dictated by the requirement of reality of the mean values of various operators.
In the general formulation of [27] which allows complex values for \( p \) (for both of the two schemes), the expectation value of an operator \( \mathcal{O} \) must be defined as \(< p^* | \mathcal{O} | p >\), where \( p^* \) is the complex conjugate of \( p \). This means that even when \( \mathcal{O} \) is hermitian its mean value need not be real. Indeed we have

\[
(\langle \Psi(P^*) | \mathcal{O} | \Psi(P) >)^* = \langle \Psi(P) | \mathcal{O} | \Psi(P^*) > \\
\neq \langle \Psi(P^*) | \mathcal{O} | \Psi(P) >,
\]

where we have displayed the \( P \) dependence of \( | \Psi > \) explicitly. An obvious way of making it real is to require

\[
| \Psi(P^*) > = | \Psi(P) >.
\]

It turns out that this condition is satisfied provided that we (i) take \( p^+ \) and \( p^- \) to be real, (ii) sum up the imaginary shift of \( p^- \) symmetrically with respect to \( l \) as in (4.25), and (iii) perform the smearing over \( p^+ \) with \( W(p^+) \). (The last of these procedures, which is already necessary to make the mean value of \( q^- \) in \( < g^{\alpha \beta} > \) well-defined, can be seen to effect cancellation of a number of imaginary contributions which otherwise remain.)

If this prescription is not followed, one can check that \( < g^{\alpha \beta} > \) and \( < R_{h\pm} > \) become complex due to the presence of \( p^+ \) or \( p^- \) in them. Thus the conclusion is rather simple: we should take the usual scheme \( i.e. \) quantization along a “real-like” path. We will explicitly show that all the mean values will be real with such an inner product.

Finally, a word should be added that in the ghost sector inclusion of the usual ghost zero-modes \( c_0 \bar{c}_0 \) will be implicitly assumed throughout.

### 4.4 Calculation of Mean Values

We are now ready to perform the computations of the mean values. Most of the calculations are tedious but straightforward. Thus, we shall sketch how we organize the calculation, explain some of the non-trivial manipulations, and then jump to the results.

Up to a certain point, we only need to assume that the operator \( \mathcal{O}(x) \) is hermitian and does not contain ghosts. Remembering that the inner product already contains \( c_0 \bar{c}_0 \),

\[
| \Psi > = | \Psi_0 > + \frac{\hbar}{\kappa} L_- | \Omega >,
\]

\[\text{The imaginary shift contained in } p^- (k, l) \text{ is consistent with this choice since they occur only in the finite domain in the complex plane.}\]
\[ < \Psi | = < \Psi_0 | + \frac{\hbar}{\kappa} < \Omega | \mathcal{L}_+, \] (4.33)
\[ \mathcal{L}_- = L_{-M}^{tot} + \bar{L}_{-M}^{tot}, \] (4.34)
\[ \mathcal{L}_+ = L_{+M}^{tot} + \bar{L}_{+M}^{tot}. \] (4.35)

Note that both \( | \Psi_0 > \) and \( | \Omega > \) are annihilated by \( L_{M}^{tot} \) and \( \bar{L}_{M}^{tot} \) and hence \( \mathcal{L}_+ | \Omega > = < \Omega | \mathcal{L}_- = 0 \) holds. After a simple calculation, we can then organize the mean value of an operator \( \mathcal{O} \) in the following way:

\[ < \Psi | \mathcal{O} | \Psi > = \mathcal{M}_0 + \mathcal{M}_1 + \mathcal{M}_2, \] (4.36)
\[ \mathcal{M}_0 = < \Psi_0 | \mathcal{O} | \Psi_0 >, \] (4.37)
\[ \mathcal{M}_1 = \mathcal{M}_{1+} + \mathcal{M}_{1-}, \] (4.38)
\[ \mathcal{M}_{1+} = \frac{\hbar}{\kappa} < \Omega | [\mathcal{L}_+, \mathcal{O}] | \Psi_0 >, \] (4.39)
\[ \mathcal{M}_{1-} = \frac{\hbar}{\kappa} < \Psi_0 | [\mathcal{O}, \mathcal{L}_-] | \Omega > = \mathcal{M}^*_{1+}, \] (4.40)
\[ \mathcal{M}_2 = \mathcal{M}_{20} + \mathcal{M}_{21}, \] (4.41)
\[ \mathcal{M}_{20} = \frac{\hbar^2}{\kappa^2} < \Omega | \mathcal{O} [\mathcal{L}_+, \mathcal{L}_-] | \Omega >, \] (4.42)
\[ \mathcal{M}_{21} = \frac{\hbar^2}{\kappa^2} < \Omega | [[\mathcal{L}_+, \mathcal{O}], \mathcal{L}_-] | \Omega >. \] (4.43)

The most important and somewhat non-trivial part of the calculation is that of \( \mathcal{M}_{1+} \) for \( \mathcal{O} =: \chi e^{-\psi} : \) and \( T^f \). Below let us give some details of this calculation.

First consider the case \( \mathcal{O} =: \chi e^{-\psi} : \). Since we have already worked out the conformal property of this operator in (4.3), the commutator in \( \mathcal{M}_{1+} \) is easily obtained to be

\[ [\mathcal{L}_+, : \chi e^{-\psi} :] = \hbar \hat{D}(M, x) : \chi e^{-\psi} : + \hbar^2 f(M, x) e^{-\psi}, \] (4.44)

where the differential operator \( \hat{D}(M, x) \) and the function \( f(M, x) \) are defined by

\[ \hat{D}(M, x) \equiv e^{iMx^+} \left( \frac{1}{i} \partial_+ - M \right) + e^{iMx^-} \left( \frac{1}{i} \partial_- - M \right), \] (4.45)
\[ f(M, x) \equiv \hat{\gamma}^2 M \left( e^{iMx^+} + e^{iMx^-} \right). \] (4.46)

Thus our calculation reduces to that of \( < \Omega | : \chi e^{-\psi} : | \Psi_0 > \) and \( < \Omega | e^{-\psi} | \Psi_0 > \). Calculation of the latter is rather trivial since the operator \( e^{-\psi} \) contains \( \alpha^+_n \) oscillators only and they act trivially on the coherent state \( | \Psi_0 > \). So we shall concentrate on the former. Recalling the form of the coherent state and using the fact that \( < \Omega | \) consists
only of zero-modes, we easily get
\[< \Omega | \chi e^{-\psi} | \Psi_0> = < \Omega | \chi_0 e^{-\psi_0} | \tilde{P} > + < \Omega | e^{-\psi_0} \chi_a e^G | \tilde{P} > . \] (4.47)

There is no problem evaluating the first term on the right hand side, but for the second we need to develop a formula for moving \( \chi_a \) through \( e^G \) to the right. By a standard formula,
\[\chi_a e^G = e^G \left\{ \chi_a - [G, \chi_a] + \frac{1}{2} [G, [G, \chi_a]] + \cdots \right\} . \] (4.48)

The series actually terminates after the double commutator for the following reason: \([G, \chi_a]\) no longer contains \( \alpha_m^- \)'s and also it is linear in \( \alpha^f_n \). Thus the double commutator can only contain \( \alpha_n^+ \)'s and hence the rest of the series vanishes.

Recalling \( G = \sum_{n \geq 1} (\bar{\nu}_n/\hbar n) \tilde{A}_{-n} \), we first need to compute \([\alpha^-_m, \tilde{A}_{-n}]\), where \( \tilde{A}_{-n} \) is given in (3.12). Since \( \alpha^-_m \) has a non-vanishing commutator with \( \alpha^+_m \) and the result is a c-number, we only need to compute \([\alpha^-_m, \eta^+]\). From the expression of \( \eta^+ \) obtained in Appendix A, we get
\[\left[ \alpha^-_m, \eta^+ \right] = \frac{1}{A} \mu \frac{1}{\gamma} \sum_k \frac{1}{P^+(k)} e^{i k y^+} \left[ \alpha^-_m, C_{-k} \right] . \] (4.49)

From the definition of \( C_{-k} \), we get a compact result for the commutator
\[\left[ \alpha^-_m, C_{-k} \right] = \frac{\bar{\gamma} h}{i} C_{m-k} . \] (4.50)

With this result back in (4.49), the commutator \([\alpha^-_m, \eta^+]\) is still quite involved due to the presence of the inverse of \( A \). What saves the day is the fact that we only need to evaluate this operator between \(< \Omega | \) and \( | \tilde{P} > \) both of which consist only of zero-modes. Thus effectively we can set all the non-zero modes of \( \psi \) (denoted by \( \tilde{\psi} \)) to zero at this stage. This greatly simplifies the rest of the calculation. We easily check
\[C_{k|\tilde{\psi}=0} = \delta_{k,0} . \] (4.51)
\[A|\tilde{\psi}=0 = \mu e^{i\psi_0} \frac{1}{\gamma P^+} , \] (4.52)
and \([\alpha^-_m, \eta^+]\) takes an extremely simple form:
\[\left[ \alpha^-_m, \eta^+ \right]_{\tilde{\psi}=0} = \frac{\bar{\gamma} h}{i} \frac{P^+}{P^+(m)} e^{i m y^+} . \] (4.53)

Using this result, we get
\[\left[ \alpha^-_m, \tilde{A}_{-n} \right]_{\tilde{\psi}=0} = - \frac{n h}{P^+(m)} e^{-i n q^+ / P^+} \alpha^f_{m-n} . \] (4.54)
This leads to
\[
\left[ G, \alpha_m^- \right]_{\tilde{\psi}=0} = \frac{1}{P^+(m)} \sum_{n \geq 1} \tilde{\nu}_n e^{-inq^+/p^+} \alpha_m^f_{m-n}. \tag{4.56}
\]

The double commutator \([G, \left[ G, \alpha^-_m \right]]\) can then be computed with the aid of the formula
\[
\left[ \alpha^f_k, G \right]_{\tilde{\psi}=0} = \tilde{\nu}_k e^{-ikq^+/p^+} \quad (\text{for } k \geq 1) \tag{4.57}
= 0 \quad (\text{otherwise}) \tag{4.58}
\]

Putting together the results obtained so far, we get the re-ordering formula
\[
\alpha_m^- e^G|_{\tilde{\psi}=0} = e^G \left( \alpha^-_m - \frac{1}{P^+(m)} \sum_{n \geq 1} \tilde{\nu}_n e^{-inq^+/p^+} \alpha_m^f_{m-n} \right. \\
- \frac{1}{2} \frac{1}{P^+(m)} e^{-imq^+/p^+} \sum_{n=1}^{m-1} \tilde{\nu}_n \tilde{\nu}_{m-n} \left. \right|_{\tilde{\psi}=0}. \tag{4.59}
\]

To obtain \(\langle \Omega \mid e^{-\psi_0 \chi_a e^G} \mid \tilde{\tilde{P}} >\), we multiply the above formula by \((\tilde{\gamma}/im) \exp(-imx^+)\) and sum over \(m\) for \(m \geq 1\). Further, we can now set \(\text{all}\) the non-zero modes to zero. The final result can be expressed in the form
\[
\langle \Omega \mid e^{-\psi_0 \chi_a e^G} \mid \tilde{\tilde{P}} > = i\tilde{\gamma} \langle \Omega \mid e^{-\psi_0} \left( \frac{1}{2} \sum_{m \geq 1} \frac{e^{-imq^+/p^+}}{mP^+(m)} \sum_{n=0}^{m} \nu_n \nu_{m-n} e^{-im(x^+ - x_0^+)} \right) \mid \tilde{\tilde{P}} >, \tag{4.60}
\]

where we have defined \(\nu_0 \equiv p_f\) and extended the range of the sum over \(n\) from 0 to \(m\) so as to incorporate the term linear in \(\nu_m\). Note that the phase factor included in \(\tilde{\nu}_m\) has produced a shift in \(x^+\).

Calculation of \(\mathcal{M}_{1+}\) for \(O = T^f\) proceeds in a similar manner. First the commutator is given by
\[
\left[ \mathcal{L}^+, T^f \right] = \left[ L^f_M, T^f \right] \\
= \hbar e^{iMx^+} \left\{ \left( \frac{1}{i} \partial_+ + 2M \right) T^f + \hbar 2M(M^2 - 1) \right\}. \tag{4.62}
\]

Thus we only need to compute \(\langle \Omega \mid T^f e^G \mid \tilde{\tilde{P}} >\). The only non-trivial part is for \(L^f_m (m \geq 1)\) in \(T^f\). Since \(\tilde{\tilde{P}} >\) contains only zero-modes, we can replace \(G\) by \(G_f\) given by
\[
G_f \equiv G|_{\tilde{\psi}=0} = \sum_{n \geq 1} \frac{\tilde{\nu}_n}{\hbar n} e^{-inq^+/p^+} \alpha_{-n}^f. \tag{4.63}
\]
Therefore,
\[
< \Omega | L_m^f e^G | \tilde{P} > = < \Omega | \left[ L_m^f, G_f \right] + \frac{1}{2} \left[ \left[ L_m^f, G_f \right], G_f \right] | \tilde{P} > .
\]  
(4.64)

Commutators are easily evaluated. First using \( \left[ L_m^f, \alpha_{-n}^f \right] = n\hbar \alpha_{m-n}^f \), we get
\[
\left[ L_m^f, G_f \right] = \sum_{n \geq 1} \tilde{\nu}_n e^{-imq^+/p} \alpha_{m-n}^f .
\]  
(4.65)

Notice that this is very similar to \( [G, \alpha_{-m}] \) with \( \tilde{\psi} = 0 \). The double commutator becomes
\[
\left[ \left[ L_m^f, G_f \right], G_f \right] = e^{-imq^+/p} \sum_{n=1}^{m-1} \tilde{\nu}_n \tilde{\nu}_{m-n} .
\]  
(4.66)

Combining them we get
\[
< \Omega | L_m^f e^G | \tilde{P} > = < \Omega | \frac{1}{2} e^{-imq^+/p} \sum_{n=0}^{m} \tilde{\nu}_n \tilde{\nu}_{m-n} | \tilde{P} > .
\]  
(4.67)

With this formula we can easily compute the desired matrix element as
\[
< \Omega | \left[ \mathcal{L}_+, T_f(x^+) \right] | \Psi_0 >
= -\hbar e^{iMx^+} < \Omega | \sum_{m \geq 0} (m - 2M) \frac{1}{2} e^{-imq^+/p} \sum_{n=0}^{m} \nu_n \nu_{m-n} e^{-im(x^+-x_0^+)} | \tilde{P} >
+ \hbar^2 e^{iMx^+} 2M(M^2 - 1) < \Omega | \tilde{P} > .
\]  
(4.68)

The rest of the calculations are simpler than the ones sketched above and we have computed, without approximation, the mean values for the operators \( \partial_{x^+} f, T_f (\xi^+), \)
\( R^\hbar_{++} (\xi) = -\lambda^2 e^{\psi} \) and \( g^{\alpha\beta} = - (\chi + AB)e^{-\psi} \eta^{\alpha\beta} \). The results, which are rather involved, are listed in Appendix B.

5 Black Hole Geometry in the Large \( L \) Limit

In the previous section, we have performed a rigorous computation of the mean values for a number of operators of physical interest in a class of physical states. The results obtained are, however, still quite involved containing some infinite sums and are hard to interpret. In this section we shall evaluate these sums in the most interesting limit where the (parameter) size of the universe, \( L \), becomes very large. By choosing various parameters specifying the state appropriately, we will be able to produce shock-wave-like energy-momentum distributions for the matter field and see that in response black hole configurations will be formed.
5.1 Preliminary Remarks

Before we begin our calculation, we must make several important remarks.

(i) The first remark is concerned with the precise meaning of the large $L$ limit to be adopted in this article. Recall that to make our analysis rigorous we have imposed spatially periodic boundary conditions for the fields. This does not mean of course that our universe is necessarily homogeneous. In fact we can arrange the parameters so that the bulk of the matter energy-momentum density will be concentrated along a line $\xi^+ \sim \xi_0^+$ where $\xi_0^+ = Lx_0^+$. What we will look at is what happens in the finite region around this line as $L \to \infty$. In other words, our large $L$ limit is such that $x^\pm = \xi^\pm/L$ tend to vanish. (This implies that we restrict ourselves to a finite interval in the “time variable” $\xi_0^+$ as well. As the speed of light is finite, \textit{i.e.} unity in our convention, this is causally reasonable.) In this limit many of the terms in our expressions of the mean values are easily seen to vanish. However, we must be very careful in taking this limit for the terms involving the infinite sums.

(ii) The second remark has to do with the phase factor $e^{-icp^+/\hbar\gamma}$ introduced in the definitions of $|\Psi_0>$ and $|\Omega>$. (See Eq.(4.20) and Eq.(4.26).) This factor plays an important role when the zero-mode $q^-$ occurs in the mean value, as in $<\chi e^{-\psi}:>$. It produces a constant shift through $e^{icp^+/\hbar\gamma}q^-e^{-icp^+/\hbar\gamma} = q^- + c/\gamma$ and by varying $c$ we can adjust the coordinate independent contributions in $<\chi e^{-\psi}:>$ freely. Thus in what follows the coordinate-independent part of $<g^{\alpha\beta}>$ will be denoted simply as an adjustable constant.

(iii) Next we make a comment on the positivity of $<T_f>$ and the overall energy balance. The composite operator $T_f$ is defined by a subtraction of an infinite positive constant and hence is not necessarily a positive definite operator. One notices however that the coordinate-independent part of $<T_f>$ proportional to the zero-mode $p_f^2$ is positive and by a suitable choice of $p_f$ one can always make $<T_f>$ positive throughout. On the other hand, the coordinate-dependence of the metric is not affected by such a choice. This has a natural explanation coming from the energy-momentum constraint $<T_{tot}> = 0$. It is easily checked that, for our choice of physical states, any change of $p_f$ in $<T_f>$ is precisely compensated by the corresponding change in the $<(1/\gamma^2)\partial_+\chi\partial_+\psi>$ and consequently the $<(1/\gamma^2)\partial_+^2\chi>$ part is left unchanged. This is responsible for the observed $p_f$-independence of the coordinate-dependent part of $<\chi e^{-\psi}:>$ (see Appendix B), which is the only part of the metric that can possibly depend on $p_f$. In physical terms,
this phenomenon means that the positive uniform matter energy density is “neutralized” by the negative uniform energy density of the dilaton-Liouville sector. As the metric couples to both of them, there is no net effect.

(iv) Finally, we make an important remark on the notion of “quantum corrections”. A glance at the results listed in Appendix B shows that all the coordinate dependence comes either with $\hbar^2/\kappa$ or with $\hbar^4/\kappa^2$ (except for a few terms which occur with an extra $\hbar$). Since $\kappa$ has the dimension of $\hbar^2$, we shall henceforth set $\kappa = \hbar^2$. Then, according to the usual terminology, bulk of the contributions will become “classical” and we have only a few minor “quantum corrections”. Is this a correct statement? The answer is interestingly ambiguous since in an exact quantum treatment like the one we are pursuing the notion of “quantum corrections” becomes rather meaningless. We have chosen the state $|\Psi_0>$ to be a coherent state so that we can expect to produce “classical” configurations. This is indeed achieved by the above assignment of $\kappa$. However, from the point of view of the exact quantum theory, a coherent state is a highly non-perturbative quantum state and precisely through its quantum coherence “classical objects” are formed. This is conceptually quite different from the semi-classical treatment where strictly classical objects are provided from the beginning.

5.2 Large $L$ Limit

With these remarks understood, we shall now take the large $L$ limit of the expressions listed in Appendix B. The infinite sums appearing for the coordinate-dependent parts are of the generic form

\[
S = a(L) \sum_{n \geq 1} b(n) c((n/L)\xi^+),
\]

\[
= a(L)L \sum_{u=1/L,2/L,...} b(Lu)c(u\xi^+) \frac{1}{L},
\]

where we have introduced a variable $u \equiv n/L$ proportional to the energy-momentum density. As $L$ becomes large, this can be replaced by the integral

\[
S \underset{L \to \infty}{\sim} a(L)L \int_{1/L}^{\infty} du b(Lu)c(u\xi^+),
\]

provided that the integral so obtained converges at both ends.
To examine this we now need to specify our parameters. With the purpose of producing shock-wave-black-hole configurations in mind, we have chosen them to be as follows:

\[ \nu_n = \nu(Lu) = \nu u^d e^{-au^2}, \]  
\[ \omega_n = \omega(Lu) = \frac{-\omega}{Lu} \quad (n \neq 0), \]  
\[ \omega = \text{a positive constant}, \]  
\[ \omega_0 = \text{a constant to be adjusted}, \]  
\[ W(p^+) = p^+ e^{-a(p^+-p_0^+)^2/2}, \]

where \( \nu, \omega \) and \( p_0^+ \) are constants and we study the cases for \( d = -1/2, 0, 1/2, 1 \). A factor of \( p^+ \) in \( W(p^+) \) is to suppress the contribution from \( p^+ = 0 \) where various expressions become singular.

Because of the Gaussian factor in \( \nu(Lu) \), the integrals are all convergent at the upper end. As for the lower end, one has to perform a somewhat tedious power counting analysis. The result of this analysis shows that we can write the large \( L \) limit of the mean values as

\[ < \partial_{\xi^+} f > \xrightarrow{L \to \infty} \left( \frac{\gamma p_f}{L} \mathcal{N} + 2\gamma \omega \nu I_f(\xi^+ - \xi_0^+) \right) < \tilde{P} | \tilde{P} >, \]

\[ < T^f(\xi^+) > \xrightarrow{L \to \infty} \left\{ \frac{p_f^2}{2L^2} (\mathcal{N} + 6M^2(\omega^2\zeta(2) + \omega_0^2)) + \omega^2 \nu^2 I_T(\xi^+ - \xi_0^+) \right\} < \tilde{P} | \tilde{P} >, \]

\[ < R^h_{+-}(\xi) > \xrightarrow{L \to \infty} -4\gamma^2 \left[ (M \omega_0 + M^2(\omega^2\zeta(2) + \omega_0^2)) < \tilde{P} | \tilde{P} > + 2\gamma^2 < \tilde{P} | p_+^2 | \tilde{P} > \right] = \text{constant}, \]

\[ < g^{-1} > \equiv -< : (\chi + AB) e^{-\psi} : > \]

\[ \xrightarrow{L \to \infty} \hat{\gamma}^2 \omega^2 \nu^2 I_\chi(\xi^+ - \xi_0^+) < \tilde{P} | \tilde{P} > -c_1 \xi^+ \delta_{d-1/2} < \tilde{P} | p^+ | \tilde{P} > -c_2 < \tilde{P} | \tilde{P} > -6 \left( \frac{\lambda}{\gamma} \right)^2 M^4 (\omega^2\zeta(2) + \omega_0^2) \xi^+ \xi^- < \tilde{P} | \frac{1}{p_+^2} | \tilde{P} >, \]

where \( I_f(\xi), I_T(\xi) \) and \( I_\chi(\xi) \) are integrals of the form

\[ I_f(\xi) = \int_{1/L}^{\infty} du u^d e^{-au^2} \cos u \xi, \]

\[ I_T(\xi) = \int_{1/L}^{\infty} du \cos u \xi \int_{0}^{u} dv [v(u - v)]^d e^{-a(v^2+(u-v)^2)}, \]

\[ I_\chi(\xi) = \int_{1/L}^{\infty} du \frac{\cos u \xi}{u^2} \int_{0}^{u} dv [v(u - v)]^d e^{-a(v^2+(u-v)^2)}, \]

\[ 28 \]
\[\mathcal{N} = e^{\sum_{n \geq 1} |\nu_n|^2 / \hbar n}, \quad \zeta(2) = \text{Riemann’s } \zeta \text{ function}, \]

\[
\langle \tilde{P} | \tilde{P} \rangle = \left\{ (p_0^+)^2 + \frac{1}{2\alpha} \right\} \sqrt{\frac{\pi}{\alpha}} \gamma^4 \delta^{1+N_f}(0), \quad (N_f = 24), \quad (5.15)
\]

\[
\langle \tilde{P} \mid p^+ \mid \tilde{P} \rangle = p_0^+ \left\{ (p_0^+)^2 + \frac{3}{2\alpha} \right\} \sqrt{\frac{\pi}{\alpha}} \gamma^4 \delta^{1+N_f}(0) \quad (5.16)
\]

\[
\langle \tilde{P} \mid p_+^2 \mid \tilde{P} \rangle = \left\{ (p_0^+)^4 + \frac{3}{\alpha} (p_0^+)^2 + \frac{3}{4\alpha^2} \right\} \sqrt{\frac{\pi}{\alpha}} \gamma^4 \delta^{1+N_f}(0), \quad (5.17)
\]

\[
\langle \tilde{P} \mid \frac{1}{p_+^2} \mid \tilde{P} \rangle = \sqrt{\frac{\pi}{\alpha}} \gamma^4 \delta^{1+N_f}(0). \quad (5.18)
\]

We must supply some explanations: First, as it is an overall common factor, we did not bother to regularize \( \delta^{1+N_f}(0) \) by additional smearing. Hence in the following, we shall consider quantities with \( \langle \tilde{P} \mid \tilde{P} \rangle \) removed. Secondly, in bringing \( \langle g^{-1} \rangle \) to the above form, we have adjusted the constant \( \omega_0 \) in the following way. The large \( L \) limit of \( \langle AB e^{-\psi} \rangle \) originally contained a term of the form \( \xi^+ \xi^- \) with a coefficient proportional to \( \omega_0 - 3M(\omega^2 \zeta(2) + \omega_0^2) \). We have chosen \( \omega_0 \) to make this coefficient vanish. This is a part of our gauge choice. Note that the term of the form \( -\lambda g^2 \xi^+ \xi^- \) arises entirely from the pure gauge part of \( \langle AB e^{-\psi} \rangle \), and it describes the so called linear dilaton vacuum when the matter field vanishes. The fact that our simple choice of \( | \Lambda \rangle \) produces precisely such a vacuum configuration is quite remarkable. Third comment is concerned with the third line of \( \langle g^{-1} \rangle \). The term linear in \( \xi^+ \) with some constant \( c_1 \) is an extra contribution present only for \( d = -1/2 \) and it is obtained by a careful examination of the process of replacing the infinite sums by integrals. The magnitude of this term however can be made arbitrary by changing \( p_0^+ \) and/or the exponent \( \alpha \) in the smearing function \( W(p^+) \). The coordinate independent term written as \( c_2 \langle \tilde{P} \mid \tilde{P} \rangle \), where \( c_2 \) depends on \( p_0^2 \), arises from several sources. But this term can also be adjusted to any value following the remark (ii) above.

### 5.3 Evaluation of the Integrals

We now briefly describe how one can evaluate the integrals \( I_f, I_T \) and \( I_\chi \).

First, for \( d > -1 \) the integral \( I_f \) can be expressed in terms of the Kummer’s confluent hypergeometric function \( _1F_1(a; b; z) \) as follows:

\[
I_f = \frac{1}{2} \Gamma \left( \frac{d+1}{2} \right) a^{-(d+1)/2} _1F_1 \left( \frac{d+1}{2}; -\frac{\xi^2}{4a} \right) \quad (5.20)
\]
\[ \xi \to 0 \quad \frac{1}{2} \Gamma \left( \frac{d+1}{2} \right) a^{-(d+1)/2} \left( 1 - \frac{d+1}{4a} \xi^2 + O(\xi^4) \right) \quad (5.21) \]

\[ \xi \to \infty \quad 2^d \sqrt{\pi} \frac{\Gamma \left( \frac{d+1}{2} \right)}{\Gamma \left( -\frac{d}{2} \right)} |\xi|^{-(d+1)} \quad (d \neq 0, 2, \ldots) \quad (5.22) \]

For \( d = 0 \), we have a simple Gaussian:

\[ I_f(d = 0) = \frac{1}{2} \sqrt{\pi} a^{-1/2} e^{-\xi^2/(4a)}. \quad (5.23) \]

\[ I_T \text{ and } I_\chi \text{ are of the similar type and can be treated together as follows. First consider the integration over } v. \text{ By making a change of variable from } v \text{ to } x \text{ of the form } x = \frac{2(v - (u/2))}{u}, \text{ it can be transformed into} \]

\[ \int_0^u dv \left[ v(u - v) \right]^d e^{-a(u^2 + (u-v)^2)} \]

\[ = 2^{-2d} e^{-au^2/2} u^{2d+1} \int_0^1 dx \left( 1 - x^2 \right)^d e^{-a/2) x^2 u^2}. \quad (5.25) \]

Put this back into the integral \( I_T \) or \( I_\chi \) and interchange the order of \( u \)- and \( x \)-integration. The result is

\[ I(d, \delta) \equiv 2^{-2d} \int_0^1 dx (1 - x^2)^d \int_{1/L}^{\infty} du u^{2(d-\delta)+1} e^{\alpha(x)u^2} \cos \xi u, \quad (5.26) \]

\[ \alpha(x) \equiv \frac{a}{2} (1 + x^2), \quad (5.27) \]

where \( I_T = I(d, \delta = 0) \) and \( I_\chi = I(d, \delta = 1) \). For \( d - \delta > -1 \), \( 1/L \) can put to zero and the \( u \)-integration can be performed just like for \( I_f \). In this way we obtain

\[ I(d, \delta) = 2^{-(d+\delta)} a^{-(d-\delta+1)} \Gamma(d - \delta + 1) \]

\[ \times \int_0^1 dx (1 - x^2)^d (1 + x^2)^{-(d-\delta+1)} \frac{\xi^2}{2a(1+x^2)} \frac{\xi^2}{2a(1+x^2)} \quad (5.28) \]

Although the remaining \( x \)-integral cannot be performed in a closed form, this is a very convenient form for numerical evaluation since we no longer have oscillatory integral.

For \( I_\chi \) for \( d = -1/2 \) and 0, the formula above does not apply since the integral diverges at the lower end as \( 1/L \) tends to zero. One can easily show, however, that the divergent piece is a constant and hence inessential. (Recall the remark (ii) again.) After removing these pieces, the \( u \)-integral can again be expressed in terms of confluent hypergeometric functions.
In Appendix C, we list the results of all the relevant integrals together with their asymptotic behavior for large $\xi$. (For small $\xi$ they all behave like $\sim A_0 - A_2(\xi^2/a) + O(\xi^4)$, where $A_0$ and $A_2$ are positive constants.) Utilizing these expressions and numerical analysis thereof, we shall now be able to give the physical interpretation of the final outcome of our long calculations.

### 5.4 Physical Interpretation

A good starting point of our discussion is the examination of the simplest of our results, namely that of the Ricci tensor $< R^h_{+-}(\xi) > = < -\lambda^2 e^{-\psi} > = \text{constant}$. Although it says that with respect to $h_{\alpha\beta}$ our configurations are rather trivial, it reveals an important nature of our choice of states. In the original work of CGHS and in many others which followed, the gauge in which $\psi = 0$ was recognized to be particularly convenient for discussing various (semi-)classical black hole configurations. In such a gauge, one obviously has $R^h_{+-} = -\lambda^2 = \text{constant}$, essentially of the same feature as our $< R^h_{+-}(\xi) >$. In our quantum calculation, this comes about because our $|\Psi\rangle$ was designed to contain no non-zero modes of $\chi$ field: Only the zero-mode part of $e^\psi$ is active and hence no coordinate dependence arises in $< R^h_{+-}(\xi) >$. Thus our choice essentially corresponds to the familiar gauge described above and this allows us to compare our results with the classical ones with ease.

Let us now describe the results for the matter energy-momentum tensor $< T^f(\xi) >$ and the inverse metric $< g^{-1} >$, obtained with the aid of numerical calculations. It is convenient to denote the expression (5.11) for $< g^{-1} >$ in the following form:

$$
< g^{-1} > = -K_g(\xi^+) - \lambda_g^2 \xi^+ \xi^- ,
$$

where

$$
K_g(\xi^+) = c_K \xi^+ + d_K - \tilde{\gamma}^2 \omega \nu^2 I_\chi(\xi^+ - \xi_0^+) ,
$$

$$
c_K, d_K, \lambda_g = \text{constants}.
$$

The constants can be adjusted as was discussed before, but we should remember that $c_k$ can only be non-vanishing for $d = -1/2$ case. The curvature scalar is then given by

$$
R^g = -4\lambda_g^2 \frac{K_g(\xi^+) - \xi^+ \partial_+ K_g(\xi^+)}{K_g(\xi^+) + \lambda_g^2 \xi^+ \xi^-} .
$$
Although we have performed numerical analysis for the four values of $d$ which control the behavior of the relevant integrals, we shall only discuss $d = -1/2$ and $1/2$ cases in some detail since the qualitative features for the remaining cases are not drastically different from these cases.

Let us begin with the $d = -1/2$ case. With suitable choice of parameters, it describes precisely the space-time in which an in-falling (smeared) shock wave of matter energy produces a black hole without naked singularity, the prototypical configuration discovered in [2]. From the expression given in (C.1), we see that the integral $I_T(\xi^+ - \xi^+_0)$ gives very nearly a Gaussian peaked around $\xi^+_0$ and as the parameter $a$ approaches 0 it becomes a $\delta$-function. As it is obviously positive by itself, we set $p_f = 0$ in the expression of $\langle T^f \rangle$. (This is also convenient since for this value of $d$, the quantity $\mathcal{N}$ diverges as $L$ becomes large.) In Fig.1a we plot the typical behavior of $\langle T^f \rangle$ and in Fig.1b the integral $I_x$ up to a constant. It is important to note that the asymptotic behavior for large $|\xi^+ - \xi^+_0|$ is linear as is seen in (C.3). Thus we can make $\langle g^{-1} \rangle$ behave very much like the CGHS case by adjusting the term $c_K\xi^+$ in $K_g$, present for this value of $d$, and the constant $d_K$, to cancel this linear portion for the range $\xi^+ < \xi^+_0$. $K_g(\xi^+)$ then becomes

$$K_g(\xi^+) = \tilde{\gamma}^2 \omega \nu^2 \left( \frac{\pi^2}{2} (\xi^+ - \xi^+_0) - I_x(\xi^+ - \xi^+_0) \right),$$  \hspace{1cm} (5.33)

which behaves like $(\pi^2/2)\tilde{\gamma}^2 \omega \nu^2 \theta(\xi^+ - \xi^+_0)$ for $|\xi^+ - \xi^+_0| >> \sqrt{a}$. In this way we obtain a smeared version of the CGHS black hole. In Fig.1c, we show the line of curvature singularity for this configuration. We clearly see that a black hole without a naked singularity is formed and to the left of the line $\xi^+ = \xi^+_0$ the space-time quickly becomes the linear dilaton vacuum configuration for $\sqrt{a} << \xi^+_0$.

The total matter energy sent in is expressed in this “Kruskal” coordinate system as [3]

$$E_f = \lambda g \int_0^\infty d\xi^+ \xi^+ < T^f(\xi^+) >$$

$$= \lambda g \omega \nu^2 \int_0^\infty d\xi^+ \xi^+ I_T(\xi^+ - \xi^+_0).$$  \hspace{1cm} (5.34)

We may now use the important relation $I_T = -\partial^2_\xi I_x$, evident from the definitions (5.13) and (5.14), which expresses the overall energy balance. Together with (5.33) above, this leads to

$$E_f = \frac{\lambda g}{\tilde{\gamma}^2} \left( \xi^+ \partial_\xi K_g - K_g \right) |^\infty_0.$$  \hspace{1cm} (5.35)
For sharply peaked matter distribution, this gives the familiar result proportional to $\xi_0^+$:

\[
E_f = \frac{\pi^2}{2} \lambda_g \nu^2 \xi_0^+ .
\]  
(5.36)

The same expression can also be obtained from the point of view of the energy stored in the dilaton-Liouville system. As our configuration is effectively a classical solution in the $\psi = 0$ gauge, with $\lambda_g$ as the dilatonic cosmological constant, it should be meaningful to look at the classical expression of the energy density $t_{00}$ of the dilaton-Liouville system.

It is straightforward to show that for $\psi = 0$ it takes the form

\[
t_{00} = \partial_{\xi_1}^2 \Phi - 2\lambda_g^2 ,
\]  
(5.37)

which upon substituting $\Phi = -\lambda_g^2 \xi^+ \xi^- - K_g$ becomes

\[
t_{00} = -\partial_{\xi_1}^2 K_g (\xi^+) = -\partial_{\xi_1}^2 K_g (\xi^+) = t_{++} .
\]  
(5.38)

Therefore the energy of the dilaton-Liouville system is obtained as

\[
E_{dL} = \lambda_g \int_0^\infty d\xi^+ \xi^+ t_{++}(\xi^+)
\]
\[
= \lambda_g \left( K_g - \xi^+ \partial_+ K_g \right) |^\infty_0 .
\]  
(5.39)

This is opposite in sign to the matter energy pumped in as expected.

One might think that all these results indicate that we have simply reproduced a classical configuration. This is not quite so: Comparison of $I_f$ with $I_T$ given in (C.4) and (C.1) shows that $<T_f>$ is not exactly equal to the square of $<\partial_+ f>$. It is due to the fact that $T_f$ is a composite operator and quantum interference has made the difference. In fact if all the expectation values behaved as classical, that would mean that only a single state, which is a coherent state with respect to all the operators, must dominate the intermediate sum. Such a situation is extremely difficult to arrange. In the present context, the energy density, not $\partial_+ f$, is the operator of prime importance which directly influences the form of the metric and hence we have arranged to make it behave as classically expected.

Next let us consider the case for $d = 1/2$. In this case, $I_T$ is given by an integral over a Gaussian times a polynomial as shown in (C.11). After the $x$-integral, it behaves like a Gaussian in the vicinity of $\xi^+ \sim \xi_0^+$, but develops a negative minimum and tends to vanish exponentially for large $|\xi^+ - \xi_0^+|$. Thus if we require the positivity of $<T_f(\xi^+)>
we have to take \( p_f \) in (5.9) to be non-zero and of order \( L \). As for \( I_\chi \), it is almost a Gaussian as seen in (C.13) and we can easily find the behavior of \( \langle g^{-1} \rangle \) by using (5.29). In Fig.2, we plot the line of curvature singularity for this configuration, where we have taken \( d_K \) to be an appropriate positive number and have chosen the width of the smeared shock wave to be rather broad. One sees that while a space-like singularity is formed near \( \xi^+ \sim \xi^+_0 \), as \( \xi^+ \) becomes large a time-like naked singularity develops and approaches the \( \xi^+ \) axis. This geometry is very similar to the one that appears in the model of \[5\].

With non-vanishing \( p_f \), the energy carried in by the matter field becomes

\[
E_f = \frac{\lambda_g}{\gamma^2} (\xi^+ \partial_\xi K_g - K_g)|_0^\infty + \lambda_g \frac{p_f^2}{2L^2} \left( N + 6M^2(\omega^2\zeta(2) + \omega_0^2) \right) \int_0^\infty d\xi^+ \xi^+
= \text{negative constant} + \lambda_g \frac{p_f^2}{2L^2} \left( N + 6M^2(\omega^2\zeta(2) + \omega_0^2) \right) \int_0^\infty d\xi^+ \xi^+ . \tag{5.40}
\]

Evidently the second term is divergent. This was to be expected since non-vanishing \( p_f^2 \) in \( \langle T^- \rangle \) represents a constant energy density permeating the whole universe \[8, 3\].

Although the behavior of the line of curvature singularity resembles that of \[3\], there are several differences. First, we do not impose additional boundary conditions on the naked singularity: The physical content of our universe is already completely determined by the specification of the physical state. Second, from (5.32) we observe that the scalar curvature \( R^g \) goes to zero for large \( \xi^+ \), not to \(-\infty\) as in \[3\]. This means that the black hole fades away in this region. Finally, we comment on the right-going part of the energy-momentum tensor of the matter fields, \( \langle T^- \rangle \), the existence of which is often regarded as a signal of Hawking radiation. As our physical states contain only the left-movers in the non-zero mode sector and, due to conformal invariance, they do not couple to the right-going counterparts, only the zero-mode \( p_f \) contributes to \( \langle T^- \rangle \). After a simple calculation we get

\[
\langle T^-(\xi^-) \rangle \sim \frac{p_f^2}{L^2} \left( N + 6M^2(\omega^2\zeta(2) + \omega_0^2) \right) \langle \hat{P} | \hat{P} \rangle . \tag{5.41}
\]

When \( p_f \) is of order \( L \), as in this case, \( \langle T^- \rangle \) is finite, positive and coordinate-independent. There exists no negative energy flow. This may be considered as a rather general phenomenon: In the case of an unstable black hole geometry, \( p_f \) should be non-zero from the requirement of the positivity of \( \langle T^{++} \rangle \), and this in turn leads to finite and positive \( \langle T^- \rangle \). Further discussion on the relation between Hawking radiation and \( \langle T^- \rangle \), however, is unfortunately beyond the scope of this paper.
6 Discussions

By using an exactly solvable model of CGHS type, we have shown explicitly how one can extract space-time geometry from an exact yet abstract physical state in quantum theory of gravity. Although the model employed is but a toy model in 1 + 1 dimensions, we believe that it is quite significant to be able to discuss a variety of important issues in quantum gravity in a concrete and unambiguous manner. Furthermore, the point of view and the procedures developed in this work should find wide applications in other models of quantum gravity and possibly in quantum cosmology.

In spite of the progress made, a number of important problems still remain to be understood. The first and the foremost is the question of how to define and compute the S-matrix: Without its understanding, we cannot even formulate the most interesting problem concerning the Hawking radiation, quantum coherence, and the fate of a black hole.

There appear to be two substantial obstacles we must overcome. In semi-classical treatment, a reference background geometry as well as a coordinate system are already available, and one has a space-time picture before one starts discussing the S-matrix for particle excitations around such a macroscopic background. On the other hand, in exact analysis geometry can emerge only after we compute some expectation values of appropriate operators, as we have seen in this work. In other words, it is extremely difficult to separate out the bulk geometry and the particle excitations for which to define the scattering matrix.

The second difficulty has to do with the very nature of the usual definition of the S-matrix. S-matrix elements describe the overlap between the “states” prepared in the “far past” and the ones defined in the “far future”, where the interactions are supposed to be negligible. Thus the notion of S-matrix inherently hinges upon our ability to separate out appropriate sub-regions or sub-systems. This is again very hard to do in advance before we obtain a space-time picture, in particular before the notion of “time” becomes available. To avoid any confusion, we emphasize that it is not the question of “boundary conditions” as sometimes argued. Such a terminology implicitly assumes that one already has a space-time picture, which we do not. Besides, boundary conditions are already used in obtaining the physical states and are not to be imposed again after the extraction of a space-time picture. In any case, the pressing task is to give a concrete procedural
definition of the S-matrix in quantum gravity to which everyone can agree.

Finally, as for the Hawking radiation and the problem of loss of quantum coherence, we do not have much to say since one cannot make a proper discussion of this issue without a satisfactory definition of the S-matrix. Nevertheless it may be worth remarking that an essentially similar situation is expected to occur in a many-body system, even without gravity, where the system can be approximately divided into a macroscopic classical sub-system and a microscopic quantum sub-system. If one is able to treat the whole system exactly, there should not be any quantum incoherence. On the other hand, if one makes an approximation as stated above, the macroscopic part would act as a germ of incoherence for the quantum sub-system. It would be quite interesting if one can set up a simple idealized model in which to study the emergence of quantum incoherence in an explicit manner.

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Appendix A

In this appendix, we shall derive an explicit form of the new field $\eta^+$ introduced in Eq.(3.11). Furthermore, we define its conjugate field $\zeta^+$ such that at the classical level the transformation from $(\chi^+, \psi^+)$ into $(\eta^+, \zeta^+)$ is a canonical transformation. Despite the non-local nature of this transformation, the energy-momentum tensor in the dilaton-Liouville sector will be seen to take a simple local form in terms of the new pair of fields.
Explicit Expression of \( \eta^+ \)

We begin by deriving a useful alternative representation of \( A(x^+) \). First it is clear from Eq.\((2.31)\) that \( q^+ \)-dependence of \( A(x^+) \) is simply an overall factor \( \exp(\tilde{\gamma}q^+) \). Next due to the boundary condition \( A(x^+ + 2\pi) = \alpha A(x^+) \) with \( \alpha = \exp(\gamma\sqrt{\pi}p^+) \), \( A(x^+) \) must contain a factor \( \exp(\tilde{\gamma}p^+x^+) \) and the rest must be periodic. This means that \( A(x^+) \) can be written as

\[
A(x^+) = \mu A_0(x^+) \tilde{A}(x^+) ,
\]

\[
A_0(x^+) = e^{\tilde{\gamma}(q^+ + p^+ x^+)} = e^{\psi_0^+} ,
\]

\[
\tilde{A}(x^+ + 2\pi) = \tilde{A}(x^+) .
\]

By applying \( \partial_+ \) on \( A(x^+) \) above and comparing it with the original definition of \( A(x^+) \) i.e. \( \partial_+A = \mu e^{\psi^+} \), we easily deduce

\[
e^{\tilde{\psi}^+} = \left( \partial_+ + \frac{p^+}{\sqrt{2Q}} \right) \tilde{A} = \hat{\mathcal{P}}_+ \tilde{A}, \tag{A.4}
\]

where \( Q = \sqrt{2\pi}/\gamma \) is the background charge. In the space of periodic functions the differential operator \( \hat{\mathcal{P}}_+ \) can be inverted. Indeed by Fourier analysis, we can easily obtain the Green’s function \( g(x^+ - y^+) \) for it as

\[
\hat{\mathcal{P}}_+ g(x^+ - y^+) = 2\pi \delta(x^+ - y^+) \tag{A.6}
\]

\[
g(x^+ - y^+) = \sqrt{2Q} \sum_{m \in \mathbb{Z}} \frac{1}{P^+(m)} e^{im(x^+ - y^+)} , \tag{A.7}
\]

where \( \delta(x^+ - y^+) \) is the periodic \( \delta \) function and \( P^+(m) \) is precisely the quantity that appeared in the construction of the BRST cohomology by means of the operator \( T^+ \) (see Eq. \((2.44)\)). Thus \( \tilde{A} \) can be solved to be

\[
\tilde{A} = \int_0^{2\pi} \frac{dy^+}{2\pi} g(x^+ - y^+) e^{\tilde{\psi}^+(y^+)} \tag{A.8}
\]

\[
= \sqrt{2Q} \sum_{m \in \mathbb{Z}} \frac{1}{P^+(m)} e^{imx^+} C_{-m} , \tag{A.9}
\]

\[
C_{-m} = \int_0^{2\pi} \frac{dy^+}{2\pi} e^{-imy^+} e^{\tilde{\psi}^+(y^+)} . \tag{A.10}
\]

Thus we obtain

\[
A(x^+)/\mu = e^{\psi_0^+} \cdot \sqrt{2Q} \sum_{m \in \mathbb{Z}} \frac{1}{P^+(m)} e^{imx^+} C_{-m} . \tag{A.11}
\]
Taking the logarithm of this expression and separating the result into the zero-mode and the non-zero-mode parts, we get

\[ \eta^+ = \eta_0^+ + \tilde{\eta}^+, \quad \eta_0^+ = \psi_0^+ + \ln \left( \frac{C_0}{\gamma p^+} \right), \quad \tilde{\eta}^+ = \ln \left( 1 + \sum_{n \neq 0} \frac{p^+ e^{-i n x}}{P^+(n) C_0} e^{i n x^+} \right). \]  

Note that the zero-mode \( \eta_0^+ \) contains, apart from the zero-mode part of \( \psi^+ \), the expression \( \ln \left( \frac{C_0}{\gamma p^+} \right) \) which is a complicated combination of non-zero modes of \( \psi^+ \).

The inverse relation, i.e. the expression of \( \psi^+ \) in terms of \( \eta^+ \), is immediately obtained from the defining relation \( \partial_+ e^{\eta^+} = e^{\psi^+} \) as

\[ \psi^+ = \eta^+ + \ln(\partial_+ \eta^+). \]  

**Classical Canonical Transformation**

We shall now try to find a field \( \zeta^+ \) the modes of which are conjugate to those of \( \eta^+ \), and express the energy-momentum tensor in terms of new fields \( \eta^+ \) and \( \zeta^+ \).

To find \( \zeta^+ \), we shall perform a classical canonical transformation from the “old” set of canonical pair \( (q, p) \sim (\chi^+, \psi^+) \) to the “new” set \( (Q, P) \sim (\eta^+, \zeta^+) \). To be rigorous, let us look at the individual modes and identify the canonical pair:

\[ (q, p) \sim (q^-, p^+), (p^-, -q^+), (\alpha_n^- / n, i \alpha_n^+) \]
\[ (Q, P) \sim (q_\eta, p_\zeta), (p_\eta, -q_\zeta), (-\beta_n^+ / n, i \beta_n^-) \]

where we denote the non-zero modes of \( \eta^+ \) and \( \zeta^+ \) by \( \beta_n^+ \) and \( \beta_n^- \) respectively. Our convention for the Poisson bracket is

\[ \{q, p\} = 1, \quad \{\alpha_n^+, \alpha_n^-\} = \frac{1}{i} n. \]

We take the generating function to be of type \( F(q, Q) \). Then, as is well known,

\[ p = \frac{\partial F}{\partial q}, \quad P = -\frac{\partial F}{\partial Q}. \]  

To realize the first of these relations, we must have

\[ F = q^- p^+ - p^- q^+ + \sum_{n \neq 0} \frac{i}{n} \alpha_n^- \alpha_n^+. \]
The rest of the procedure is to express \( p^+, q^+, \alpha_n^\pm \) in terms of \( q_\eta, p_\eta, \beta_n^\pm \) and then vary \( q_\eta, p_\eta, \beta_n^\pm \) to get the modes of \( \zeta^+ \).

Making use of the explicit form of \( \eta^+ \), we obtain, after some amount of work, the following expressions:

\[
q_\zeta = q^- - p^- \int_0^{2\pi} \frac{dx^+}{2\pi} \frac{1}{\partial_+ \eta^+}, \quad (A.18)
\]
\[
p_\zeta = p^-, \quad (A.19)
\]
\[
\beta_n^- = \imath p^- \int_0^{2\pi} \frac{dy^+}{2\pi} \frac{1}{\partial_+ \eta^+} \text{e}^{\imath y^+} + \alpha_n^- + \imath n \frac{\sqrt{4\pi}}{\gamma} \int_0^{2\pi} \frac{dy^+}{2\pi} \frac{\partial_+ \tilde{\chi}^+}{\partial_+ \eta^+} \text{e}^{\imath y^+}. \quad (A.20)
\]

We can now form the field \( \zeta^+ \) in the usual way:

\[
\zeta^+(x^+) = \frac{\gamma}{\sqrt{4\pi}} \left( q_\zeta + p_\zeta x^+ + \imath \sum_{n \neq 0} \frac{\beta_n^-}{n} \text{e}^{-\imath nx^+} \right). \quad (A.21)
\]

Using the identity \( \sum_{n \neq 0} \exp(\imath n(y^+ - x^+)) = 2\pi \delta(x^+ - y^+) - 1 \), we get

\[
\zeta^+ = \chi^+ - \frac{1}{\partial_+ \eta^+} \int_0^{2\pi} \frac{dy^+}{2\pi} \frac{\partial_+ \tilde{\chi}^+}{\partial_+ \eta^+}. \quad (A.22)
\]

Since the last integral is independent of \( x^+ \), the derivative \( \partial_+ \zeta^+ \) takes the form

\[
\partial_+ \zeta^+ = \partial_+ \chi^+ - \partial_+ \left( \frac{\chi^+}{\partial_+ \eta^+} \right)
= \frac{1}{\partial_+ \eta^+} \left[ \partial_+ \chi^+ \left( \partial_+ \eta^+ + \frac{\partial_+^2 \eta^+}{\partial_+ \eta^+} \right) - \partial_+^2 \chi^+ \right]. \quad (A.23)
\]

If we write the last equation in the form

\[
\partial_+ \zeta^+ \partial_+ \eta^+ = \partial_+ \chi^+ \left( \partial_+ \eta^+ + \frac{\partial_+^2 \eta^+}{\partial_+ \eta^+} \right) - \partial_+^2 \chi^+,
\]

and use the relation

\[
\partial_+ \psi^+ = \partial_+ \eta^+ + \frac{\partial_+^2 \eta^+}{\partial_+ \eta^+}, \quad (A.25)
\]

which follow from (A.15), we recognize that the above expression is nothing but the energy-momentum tensor in the dilaton-Liouville sector. Thus we obtain

\[
\tilde{\gamma}^2 T^{dL}(x^+) = \partial_+ \psi^+ \partial_+ \chi^+ - \partial_+^2 \chi^+,
\]

\[
= \partial_+ \zeta^+ \partial_+ \eta^+. \quad (A.26)
\]

\[
= \partial_+ \zeta^+ \partial_+ \eta^+. \quad (A.27)
\]

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It is remarkable that we have obtained a simple local expression of free-field type without a background charge despite the complicated non-linear and non-local transformations performed. One recognizes that $\eta^+$ and $\zeta^+$ are essentially the fields employed in the work of [24, 25]. However, as the form of $\zeta^+$ (Eq.(A.22)) shows, the canonical relation between $(\chi^+, \psi^+)$ and $(\eta^+, \zeta^+)$ cannot be easily extended to the quantum domain.
Appendix B

In this appendix, we list the exact results of the calculations of the mean values for the operators $\partial_\xi f$, $T^f(\xi^+)$, $R^h_{+-}(\xi) = -\lambda^2 e^\psi$ and $g^{\alpha\beta} = -(\chi + AB)e^{-\psi}\eta^{\alpha\beta}$ before we take the large $L$ limit.

$$\langle \partial_\xi f \rangle = (F_1 + F_2 + F_3) < \tilde{P} | \tilde{P} >$$ \hspace{1cm} (B.1)

$$F_1 = \frac{\gamma p_f}{L} \exp \left( \sum_{n \geq 1} \frac{|\nu_n|^2}{\hbar n} \right), \hspace{1cm} (B.2)$$

$$F_2 = -2\frac{\gamma h^2}{\kappa L} \sum_{n \geq 0} \nu_n \omega_n (n - M) \cos(n(x^+ - x^+_0) - Mx^+), \hspace{1cm} (B.3)$$

$$F_3 = -12\frac{\gamma h^4}{\kappa^2 L} M p_f \sum_k k\omega_k^2 \hspace{1cm} (B.4)$$

$$\langle T^f(\xi^+) \rangle = (T_1 + T_2 + T_3) < \tilde{P} | \tilde{P} >,$$ \hspace{1cm} (B.5)

$$T_1 = \frac{p_f^2}{2L^2} \exp \left( \sum_{n \geq 1} \frac{|\nu_n|^2}{\hbar n} \right), \hspace{1cm} (B.6)$$

$$T_2 = 2\frac{h^2}{\kappa L^2} \left\{ \cos Mx^+ \{Mp_f^2 + \frac{\hbar}{12} c_f M(M^2 - 1)\} \omega_0 
+ \sum_{k \geq 1} \omega_k \{p_f \nu_k + \frac{1}{2} \sum_{n=1}^{k-1} \nu_n \nu_{k-n} \} (2M - k) \cos(k(x^+ - x^+_0) - Mx^+) \right\}, \hspace{1cm} (B.7)$$

$$T_3 = \frac{h^4}{\kappa^2 L^2} \left\{ -6M p_f^2 \sum_k k\omega_k^2 
+ 3 \left[ M^2 p_f^2 + \frac{\hbar}{12} c_f M^2(M^2 - 1) \right] \sum_k \omega_k^2 \right\}. \hspace{1cm} (B.8)$$

$$\langle R^h_{+-}(\xi) \rangle = \langle -\lambda^2 e^\psi \rangle = R_1 + R_2 + R_3,$$ \hspace{1cm} (B.9)

$$R_1 = -2\lambda^2 \frac{h^2}{\kappa} \omega_0 < \tilde{P} | \left( \gamma p^+ (\sin Mx^+ + \sin Mx^-) 
+ M (\cos Mx^+ + \cos Mx^-) \right) e^{2\gamma p^+ t} | \tilde{P} >, \hspace{1cm} (B.10)$$

$$R_2 = 8\lambda^2 M \frac{h^4}{\kappa^2} \sum_k k\omega_k^2 < \tilde{P} | e^{2\gamma p^+ t} | \tilde{P} >.$$
\[-4\lambda^2 \frac{\hbar^4}{\kappa^2} \sum_k \omega_k^2 < \tilde{P} \mid (\tilde{\gamma}p^+)^2 e^{2\tilde{\gamma}p^+ t} \mid \tilde{P} > , \]  
(B.11)

\[ R_3 = -4 \lambda^2 \frac{\hbar^4}{\kappa^2} (\cos Mx^+ \cos Mx^- + \sin Mx^+ \sin Mx^-) \times \sum_k \omega_k^2 < \tilde{P} \mid (\tilde{\gamma}p^+)^2 + M^2 \mid \tilde{P} > . \]  
(B.12)

\[
< AB e^{-\psi} > = ((AB)_1 + (AB)_2 + (AB)_3 + (AB)_4) < \tilde{P} \mid \frac{1}{p^+_2} \mid \tilde{P} > 
\]  
(B.13)

\[ (AB)_1 = \frac{\lambda^2 L^2}{\tilde{\gamma}^2} \exp \left( \sum_{n \geq 1} \frac{|\nu_n|^2}{\hbar n} \right), \]  
(B.14)

\[ (AB)_2 = -2 \frac{\lambda^2 L^2}{\tilde{\gamma}^2} \frac{\hbar^2}{\kappa} M \omega_0 (\cos Mx^+ + \cos Mx^-) , \]  
(B.15)

\[ (AB)_3 = 12 \frac{\lambda^2 L^2}{\tilde{\gamma}^2} \frac{\hbar^4}{\kappa^2} M \sum_k (M - k) \omega_k^2 , \]  
(B.16)

\[ (AB)_4 = 6M^2 \frac{\lambda^2 L^2}{\tilde{\gamma}^2} \frac{\hbar^4}{\kappa^2} (\cos Mx^+ \cos Mx^- + \sin Mx^+ \sin Mx^-) . \]  
(B.17)

\[
< : \chi e^{-\psi} : > = C^+_{11} + C^-_{11} + C^+_{12} + C^-_{12} + C^+_{21} + C^-_{21} + C^+_{22} + C^-_{22} + C_3, \]  
(B.18)

\[ C^+_{11} = -2 \frac{\hbar^2}{\kappa} \left\{ -\tilde{\gamma}^2 hM \cos Mx^+ < \Psi_0 \mid e^{-\tilde{\gamma}(2p^+t+q^+)} \mid \Omega > 
+ (M \cos Mx^+ - \sin Mx^+ \partial_+ ) \times < \Psi_0 \mid \tilde{\gamma}(2p^- t + (c/\gamma)) e^{-\tilde{\gamma}(2p^+t+q^+)} \mid \Omega > 
- (M \sin Mx^+ + \cos Mx^+ \partial_+ ) \times < \Psi_0 \mid \tilde{\gamma} iq^- e^{-\tilde{\gamma}(2p^+t+q^+)} \mid \Omega > \right\} , \]  
(B.19)

\[ C^-_{11} = C^+_{11}(x^+ \rightarrow x^-, \partial_+ \rightarrow \partial_-) , \]  
(B.20)

\[ C^+_{12} = -2 \frac{\tilde{\gamma}^2}{\kappa} < \tilde{P} \mid e^{-2\tilde{\gamma}p^+ t} \left[ p_f \sum_{n \geq 1} \frac{\nu_n \omega_n}{n(p^+_2 + 2Q^2 n^2)} \cdot \left\{ - \cos Mx^+ \left( p^+ \sin n(x^+ - x^+_0) + \sqrt{2} Qn \cos n(x^+ - x^+_0) \right) 
+ \sin Mx^+ \left( p^+ \cos n(x^+ - x^+_0) - \sqrt{2} Qn \sin n(x^+ - x^+_0) \right) \right\} 
+ \frac{1}{2} \sum_{m,n \geq 1} \frac{\nu_m \nu_n \omega_{m+n}}{(m+n)(p^+_2 + 2Q^2 (m+n)^2)} \cdot \left\{ - \cos Mx^+ \left( p^+ \sin (m+n)(x^+ - x^+_0) + \sqrt{2} Q(m+n) \cos (m+n)(x^+ - x^+_0) \right) 
+ \sin Mx^+ \left( p^+ \cos (m+n)(x^+ - x^+_0) \right) \right\} \right] \]  
(B.21)
\[-\sqrt{2}Q(m+n)\sin(m+n)(x^+-x_0^-)\right\}] \mid \tilde{P} > , \text{ (B.22)}

\[C_{12}^- = -2\gamma \frac{\hbar^2}{\kappa} < \tilde{P} \mid e^{-2\gamma p^+ t} \left[ p_f M \sum_{n \geq 1} \frac{\nu_n \omega_n}{n(p_+^2 + 2Q^2 n^2)} \cdot \left\{ -\cos M x^+ (p^+ \sin n(x^+-x_0^-) + \sqrt{2}Q n \cos n(x^+-x_0^-)) \\
+ \sin M x^+ (p^+ \cos n(x^+-x_0^-) - \sqrt{2}Q n \sin n(x^+-x_0^-)) \right\} \\
+ \frac{1}{2} M \sum_{m,n \geq 1} \frac{\nu_m \nu_n \omega_{m+n}}{(m+n)(p_+^2 + 2Q^2 (m+n)^2)} \cdot \left\{ -\cos M x^+ (p^+ \sin(m+n)(x^+-x_0^-) + \sqrt{2}Q(m+n) \cos(m+n)(x^+-x_0^-)) \\
+ \sin M x^+ (p^+ \cos(m+n)(x^+-x_0^-) \\
- \sqrt{2}Q(m+n) \sin(m+n)(x^+-x_0^-)) \right\} \right] \mid \tilde{P} > , \text{ (B.23)}

\[C_{21} = \text{Re} \left\{ 4 M \frac{\hbar^4}{\kappa^2} < \Omega \mid \tilde{\gamma}(2p^- t + q^-) e^{-\tilde{\gamma}(2p^+ t + q^+)} (L_0/\hbar) \mid \Omega > \right\} \text{ (B.24)}

\[C_{22} = \text{Re} \left\{ 2 \frac{\hbar^4}{\kappa^2} \left[ < \Omega \mid \left\{ (2\tilde{\gamma} p^+ + iM)\tilde{\gamma} p^- + 3h\tilde{\gamma}^2 M^2 \right\} e^{-\tilde{\gamma}(2p^+ t + q^+)} \mid \Omega > \right. \\
+ < \Omega \mid \tilde{\gamma}(p^- t + q^-) (\tilde{\gamma} p^+)^2 + iM\tilde{\gamma} p^+ + 2 M^2) e^{-\tilde{\gamma}(2p^+ t + q^+)} \mid \Omega > \left. \right\} \text{ (B.25)}

\[C_3 = \text{Re} \left\{ 2 \frac{\hbar^4}{\kappa^2} e^{iMx^+} e^{-iMx^-} \cdot < \Omega \mid \left[ -2\tilde{\gamma}^2(\hbar M^2 + p^- p^+) + \tilde{\gamma}(2p^- t + q^-) \left(M^2 + (\tilde{\gamma} p^+)^2 \right) \right] \\
\cdot e^{-\tilde{\gamma}(2p^+ t + q^+)} \mid \Omega > \right\} \text{ (B.26)}

In the expressions above, some of the quantities in the final stage of the calculation are left unevaluated. This is because of the following two reasons: These quantities depend on how we choose the smearing function \(W(p^+)\) and should better be evaluated after we specify \(W(p^+)\). Moreover, many of them actually vanish as \(L \to \infty\), the limit we are most interested in.
Appendix C

In this appendix, we provide a list of relevant integrals that occur in the calculation of the mean values and their large $\xi$ asymptotic behavior. For certain special cases, the confluent hypergeometric function reduces to (a polynomial times) a Gaussian and we shall use such a simplified form whenever it occurs. For such essentially Gaussian cases, large $\xi$ form will not be listed.

$d = -1/2$ case

\[
I_T = \sqrt{\frac{2\pi}{a}} \int_0^1 dx (1 - x^2)^{-1/2}(1 + x^2)^{-1/2}e^{\frac{-\xi^2}{(2a(1+x^2))}} \tag{C.1}
\]

\[
I_x = L\pi - \sqrt{2\pi a} \int_0^1 dx (1 - x^2)^{-1/2}(1 + x^2)^{1/2}e^{\frac{-\xi^2}{(2a(1+x^2))}}
- \sqrt{\frac{2\pi}{a}} \xi^2 \int_0^1 dx (1 - x^2)^{-1/2}(1 + x^2)^{-1/2} F_1 \left( \frac{1}{2}; \frac{3}{2}; -\frac{\xi^2}{2a(1+x^2)} \right) \tag{C.2}
\]

\[
\xi \rightarrow \infty \quad L\pi - \pi^2 |\xi| \tag{C.3}
\]

\[
I_f = \frac{1}{2}\Gamma \left( \frac{1}{4} \right) a^{-1/4} F_1 \left( \frac{1}{4}; \frac{1}{2}; -\frac{\xi^2}{4a} \right) \tag{C.4}
\]

\[
\xi \rightarrow \infty \quad \sqrt{\frac{\pi}{2}} |\xi|^{-1/2} \tag{C.5}
\]

$d = 0$ case

\[
I_T = \frac{1}{a} \int_0^1 dx \frac{1}{1 + x^2} F_1 \left( 1; \frac{1}{2}; -\frac{\xi^2}{2a(1+x^2)} \right) \tag{C.6}
\]

\[
\xi \rightarrow \infty \quad -\xi^2 \tag{C.7}
\]

\[
I_x = \int_0^1 dx \int_{1/L}^{\infty} du e^{-\alpha(x)u^2}
- \frac{1}{a} \int_0^1 d\zeta \zeta \int_0^1 dx \frac{1}{1 + x^2} F_1 \left( 1; \frac{3}{2}; -\frac{\zeta^2}{2a(1+x^2)} \right) \tag{C.8}
\]

\[
\xi \rightarrow \infty \quad \text{const.} - \ln \xi \tag{C.9}
\]

\[
I_f = \frac{1}{2} \sqrt{\pi a}^{-1/2} e^{-\xi^2/(4a)} \tag{C.10}
\]
\( d = 1/2 \) case

\[
I_T = 2^{-3/2} \sqrt{\pi a^{-3/2}} \int_0^1 dx (1 - x^2)^{1/2}(1 + x^2)^{-3/2} \left( 1 - \frac{\xi^2}{a(1 + x^2)} \right) e^{-\xi^2/(2a(1+x^2))} \tag{C.11}
\]

\[
\xi \to \infty \sim -2^{-3/2} \sqrt{\pi a^{-5/2}} \xi^2 \int_0^1 dx (1 - x^2)^{1/2}(1 + x^2)^{-5/2} e^{-\xi^2/(2a(1+x^2))} \tag{C.12}
\]

\[
I_x = 2^{-3/2} \sqrt{\pi a^{-1/2}} \int_0^1 dx (1 - x^2)^{1/2}(1 + x^2)^{-1/2} e^{-\xi^2/(2a(1+x^2))} \tag{C.13}
\]

\[
I_f = \frac{1}{2} \Gamma \left( \frac{3}{4} \right) a^{-3/4} F_1 \left( \frac{3}{4}; \frac{1}{2}; -\frac{\xi^2}{4a} \right) \tag{C.14}
\]

\[
\xi \to \infty \sim -\frac{\sqrt{\pi}}{2\sqrt{2}} |\xi|^{-3/2} \tag{C.15}
\]

\( d = 1 \) case

\[
I_T = \frac{1}{2a} \int_0^1 dx (1 - x^2)(1 + x^2)^{-1} F_1 \left( 2; \frac{1}{2}; -\frac{\xi^2}{2a(1 + x^2)} \right) \tag{C.16}
\]

\[
\xi \to \infty \sim \frac{52}{35} |\xi|^{-4} \tag{C.17}
\]

\[
I_x = \frac{1}{4a} \int_0^1 dx (1 - x^2)(1 + x^2)^{-1} F_1 \left( 1; \frac{1}{2}; -\frac{\xi^2}{2a(1 + x^2)} \right) \tag{C.18}
\]

\[
\xi \to \infty \sim -\frac{1}{5} |\xi|^{-2} \tag{C.19}
\]

\[
I_f = \frac{1}{2a} F_1 \left( 1; \frac{1}{2}; -\frac{\xi^2}{4a} \right) \tag{C.20}
\]

\[
\xi \to \infty \sim -|\xi|^{-2} \tag{C.21}
\]
References

[1] B. S. DeWitt, Phys. Rev. 160, 113 (1967); J. A. Wheeler in Batelle Recontres, edited by C. DeWitt and J. A. Wheeler (Benjamin, New York, 1968).

[2] C.G. Callan, S.B. Giddings, J.A. Harvey and A. Strominger, Phys. Rev. D 45, R1005 (1992).

[3] J.G. Russo, L. Susskind and L. Thorlacius, Phys. Lett. B 292, 13 (1992).

[4] T. Banks, A. Dabholkar, M. Douglas and M. O’Loughlin, Phys. Rev. D 45, 3607 (1992).

[5] J. Russo, L. Susskind and L. Thorlacius, Phys. Rev. D 46, 3444 (1992).

[6] L. Susskind, L. Thorlacius, Nucl. Phys. B382, 123 (1992).

[7] S. Hawking, Phys. Rev. Lett. 69, 406 (1992).

[8] B. Birnir, S.B. Giddings, J. Harvey and A. Strominger, Phys. Rev. D 46, 638 (1992).

[9] J. Russo, L. Susskind and L. Thorlacius, Phys. Rev. D 47, 533 (1993).

[10] D. Lowe, Phys. Rev. D 47, 2446 (1993).

[11] S. W. Hawking and J. M. Stewart, Nucl. Phys. B400, 393 (1993).

[12] T. Piran and A. Strominger, ITP preprint NSF-ITP-93-36, hepth@xxx/9304148 (unpublished).

[13] L. Susskind, L. Thorlacius and J. Uglum, Stanford preprint, SU-ITP-93-15, hepth@xxx/9306069 (unpublished).

[14] S. W. Hawking and Hayward, University of Cambridge and CalTech preprint, DAMTP-R93-12, CALT-68-1861 (unpublished).

[15] J. G. Russo, University of Texas Report No. UTTG-22-93 (unpublished).

[16] S.P. de Alwis, Phys. Lett. B 289, 278 (1992); S.P. de Alwis, Phys. Rev. D 46, 5429 (1992); Phys. Lett. B 300, 330 (1993).

[17] A. Strominger, Phys. Rev. D 46, 4396 (1992).

[18] S. Giddings and A. Strominger, Phys. Rev. D 47, 2454 (1993).

[19] A. Bilal and C.G. Callan, Nucl. Phys. B394, 73 (1993).

[20] A. Miković, Phys. Lett. B 291, 19 (1992); 304, 70 (1993).
[21] K. Hamada, Phys. Lett. B 300, 322 (1993); University of Tokyo, Komaba Report No. UT-Komaba 92-9 (unpublished).

[22] K. Hamada and A. Tsuchiya, University of Tokyo, Komaba Report No. UT-Komaba 92-14 to appear in Int. J. Mod. Phys. A.

[23] S. Hirano, Y. Kazama and Y. Satoh, Phys. Rev. D 48, 1687 (1993).

[24] E. Verlinde and H. Verlinde, Nucl. Phys. B406, 43 (1993).

[25] K. Schoutens, E. Verlinde and H. Verlinde, Phys. Rev. D 48, 2690 (1993).

[26] S.P. de Alwis, University Of Cololado Report No. COLO-HEP-318, hepth@xxx/9307140 (unpublished).

[27] H. Arisue, T. Fujiwara, T. Inoue, and K. Ogawa, J. Math. Phys. 9, 2055 (1981).

[28] H. Arisue, T. Fujiwara, M. Kato, and K. Ogawa, Phys. Rev. D 35, 2309 (1987).

[29] G. W. Gibbons, S. W. Hawking and M. J. Perry, Nucl. Phys. B138, 141 (1978).

[30] E. Del Guidice, P. Di Vecchia and S. Fubini, Ann. Phys. 70, 378 (1972).

[31] Y. Kazama and Y. Satoh, University of Tokyo, Komaba Report No. UT-Komaba 93-13.

[32] J. Russo and A. Tseytlin, Nucl. Phys. B382, 259 (1992); S. Odintsov and I. Shapiro, Phys. Lett. 263B, 183 (1991).

[33] F. David, Mod. Phys. Lett. A3, 1651 (1988); J. Distler and H. Kawai, Nucl. Phys. B321, 509 (1989).

[34] B.H. Lian and G.J. Zuckerman, Phys. Lett. B 266, 21 (1991); Phys. Lett. B 254, 417 (1991); Commun. Math. Phys. 145, 561 (1992).

[35] P. Bouwknegt, J. McCarthy, and K. Pilch, Commun. Math. Phys. 145, 541 (1992).

[36] A. Bilal, Phys. Lett. 282B, 309 (1992).

[37] Y. Matsumura, N. Sakai, Y. Tani, and T. Uchino, Mod. Phys. Lett. A 8, 1507 (1993).
Figure Captions

Fig.1a: A plot of $I_T(\xi)$ for $d = -1/2$ and $a = 0.005$. The shape is very nearly a steeply peaked Gaussian.

Fig.1b: A plot of $I_\chi(\xi)$ up to a constant for $d = -1/2$ and $a = 0.005$. The asymptotic behavior for large $|\xi|$ is linear.

Fig.1c: The line of curvature singularity (solid line) for $d = -1/2$ produced by a left-going smeared shock wave along $\xi^+ = \xi_0^+$ (dotted line). The dot-dashed line represents the event horizon and the space-time quickly approaches the linear dilaton vacuum to the left of $\xi^+ = \xi_0^+$.

Fig.2: The line of curvature singularity (solid line) for $d = 1/2$ produced by a left-going smeared shock wave along $\xi^+ = \xi_0^+$ (dotted line). Note the appearance of naked singularity.
