Abstract
In this work, a new class of stochastic gradient algorithm is developed based on $q$-calculus. Unlike the existing $q$-LMS algorithm, the proposed approach fully utilizes the concept of $q$-calculus by incorporating a time-varying $q$ parameter. The proposed enhanced $q$-LMS (Eq-LMS) algorithm utilizes a novel, parameterless concept of error-correlation energy and normalization of signal to ensure high convergence, stability and low steady-state error. The proposed algorithm automatically adapts the learning rate with respect to the error. For evaluation purposes the system identification problem is considered. The necessary condition of convergence for the proposed algorithm is analyzed, and the validation of analytical findings and simulation results is discussed. Extensive experiments show better performance of the proposed Eq-LMS algorithm compared to the standard $q$-LMS approach.

Keywords Adaptive algorithms · Least mean squares algorithm · $q$-calculus · Jackson derivative · System identification · $q$-LMS

1 Introduction
The least square method is considered to be a widely used optimization technique. It has been applied in diversified applications such as plant identification [17], detection of elastic inclusions [1], noise cancelation [43], echo cancelation [14], ECG signal analysis [50], elasticity imaging [10], and time series prediction [54]. The least mean square (LMS) is one of the most popular least square algorithms for adaptive filtering due to its low computational complexity; however, it has a slow convergence rate due to the dependency on the eigenvalue spread of the input correlation matrix [21]. Extensive research has been carried out toward the optimization of the LMS algorithm [8,18,23,41,52]. One of the disadvantages of the LMS is that it is sensitive to the scaling of its input. In [18,21], the normalized LMS (NLMS) and its variants were proposed to solve...
this problem through normalization. To improve the convergence rate and steady-state performance, variable step size frameworks were devised in [2,35]. In [4,25,26,53], different solutions for complex signal processing were proposed. Similarly, to deal with nonlinear signal processing problem, the concept of kernel function-based LMS algorithms was proposed in [16,40,44].

Besides these variants, various definitions of gradient have also been used to derive improved LMS algorithms; for instance in [30], a fractional-order calculus (FOC)-based least mean square algorithm, named the robust variable step size fractional least mean square (RVSS-FLMS), is proposed. The algorithm is derived using a modified Riemann–Liouville fractional derivative for high convergence performance. In [3,31], some adaptive schemes were proposed for maintaining stability through adaptive variable fractional power. The FOC variants are, however, not stable and diverge if the weights are negative or the input signal is complex. The detailed comments on fractional-order variants and suggested improvements are discussed in [15,33,34,51].

Recently, the $q$-LMS algorithm is proposed which utilizes the $q$-gradient from the Jackson’s derivative so that the secant of the cost function is computed instead of the tangent [8]. The algorithm takes larger steps toward the optimum solution and, therefore, achieves a higher convergence rate. The $q$-LMS algorithm has also been used for various applications, such as adaptive noise cancelation [12], system identification, and designing of whitening filter [7]. In [6] the $q$-normalized LMS algorithm is proposed and its convergence performance is analyzed. In [5], using the same definition of $q$-calculus, variants of the steady-state least mean algorithms are derived.

All the aforementioned variants of the $q$-LMS algorithm enhance convergence speed at the cost of increased computational complexity and steady-state error. In order to improve convergence rate without compromising the steady-state performance, a time-varying $q$-LMS is proposed in [11]. However, it requires the tuning of two additional parameters ($\beta$ and $\gamma$), and the performance of the time-varying $q$-LMS [11] is very sensitive to the selection of the tuning parameters. In this paper, we propose a new variant of the $q$-LMS by making the $q$-parameter time varying. The proposed enhanced $q$-LMS (Eq-LMS) utilizes a novel, parameterless concept of error-correlation energy and normalization to ensure rapid convergence without compromising stability and low steady-state error. The proposed algorithm automatically adapts the learning rate with respect to error. It takes larger steps in case of larger error and reduces the learning rate with decreased error. Unlike the contemporary methods [2,8,35], the proposed method is a parameterless technique and does not require manual tuning of any parameter. The proposed algorithm is evaluated for the system identification problem, and the results are demonstrated for both the steady-state performance and the convergence rate. Extensive experiments are performed to show the superiority of the proposed Eq-LMS algorithm over a variety of contemporary methods.

The rest of the paper is structured as follows. An overview of the $q$-calculus and $q$-least mean square algorithm is provided in Sect. 2. The details of proposed algorithm are discussed in Sect. 3, followed by the experimental results in Sect. 4. The paper is finally concluded in Sect. 5.
2 Overview of \(q\)-least Mean Square Algorithm

The conventional LMS algorithm is derived using the concept of steepest descent with the weight-update rule

\[ \mathbf{w}(i + 1) = \mathbf{w}(i) - \frac{\mu}{2} \nabla_{\mathbf{w}} J(\mathbf{w}), \tag{1} \]

where \(J(\mathbf{w})\) is the cost function for the LMS algorithm and is defined as

\[ J(\mathbf{w}) = E[2e^2(i)], \tag{2} \]

where \(E[\cdot]\) is the expectation operator and \(e(i)\) is the estimation error between the desired response \(d(i)\) and the output signal at the \(i\)th instant, i.e.,

\[ e(i) = d(i) - \mathbf{x}^\top(i) \mathbf{w}(i), \tag{3} \]

Here, \(\mathbf{x}(i)\) is the input signal vector defined as

\[ \mathbf{x}(i) = [x_1(i), x_2(i), \ldots x_M(i)]^\top, \tag{4} \]

and \(\mathbf{w}(i)\) is the weight vector defined as:

\[ \mathbf{w}(i) = [w_1(i), w_2(i), \ldots w_M(i)] \tag{5} \]

where \(M\) is the length of the filter.

2.1 Overview of \(q\)-Calculus

The quantum calculus or \(q\)-calculus is sometimes referred to as the calculus without a limit [22]. It has been successfully used in various areas such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions and other sciences quantum theory, operational theory, mechanics, and the theory of relativity [9,13,47,48].

In \(q\)-calculus, the differential of a function is defined as (See, [24])

\[ dq(f(x)) = f(qx) - f(x). \tag{6} \]

The derivative therefore takes the form

\[ D_q(f(x)) = \frac{dq(f(x))}{dq(x)} = \frac{f(qx) - f(x)}{(q - 1)x}. \tag{7} \]
When $q \to 1$, the expression becomes the derivative in the classical sense. The $q$-derivative of a function of the form $x^n$ is

$$D_{q,x}x^n = \begin{cases} q^n - 1, & q \neq 1, \\ q - 1, & q = 1. \end{cases}$$  \hfill (8)

For a function $f(x)$ of $n$ number of variables, $x = [x_1, x_2, \ldots, x_n]^T$, the $q$-gradient is defined as \cite{9,22,24}

$$\nabla_{q,w} f(x) \triangleq [D_{q_1,x_1} f(x), D_{q_2,x_2} f(x), \ldots, D_{q_n,x_n} f(x)]^T,$$  \hfill (9)

where $q = [q_1, q_2, \ldots, q_N]^T$. The product rule takes two equivalent forms \cite{9,22,24}:

$$D_q (f(x)g(x)) = g(x)D_q f(x) + f(qx)D_q g(x) = g(qx)D_q f(x) + f(x)D_q g(x).$$  \hfill (10)

The definition in (9) also satisfies the quotient rule \cite{9,22,24}:

$$D_q \frac{f(x)}{g(x)} = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(qx)g(x)}, \quad g(x)g(qx) \neq 0.$$  \hfill (11)

Also for $g(x) = cx^k$

$$D_q f(g(x)) = D_q^k(f)(g(x))D_q(g(x)).$$  \hfill (12)

### 2.2 $q$-least Mean Square (q-LMS) Algorithm

The performance of the LMS algorithm depends on the eigenvalue spread of the input correlation matrix. The LMS is therefore regarded as an inherently slowly converging approach \cite{35}. In order to resolve this issue, the $q$-LMS has been proposed in \cite{8}. Instead of the conventional gradient, the $q$-LMS is derived using the $q$-calculus and utilizes the Jackson derivative method \cite{8}, and it takes larger steps (for $q > 1$) in the search direction as it evaluates the secant of the cost function rather than the tangent \cite{8}. By replacing the conventional gradient in (1) with the q-gradient, we get

$$w(i + 1) = w(i) - \frac{\mu}{2} \nabla_{q,w} J(w).$$  \hfill (13)

The $q$-gradient of the cost function $J(w)$ for the $k$th weight is defined as

$$\nabla_{q,w_k} J(w) = \frac{\partial_{q_k} e}{\partial_{q_k}} J(w) \frac{\partial_{q_k} e(i)}{\partial_{q_k} y} \frac{\partial_{q_k} y(i)}{\partial_{q_k} w_k(i)} y(i).$$  \hfill (14)
Solving partial derivatives in (14) using the Jackson derivative defined in Sect. (2) gives

\[
\frac{\partial q}{\partial q} e J(w) = \frac{\partial q}{\partial q} E[2e^2(i)] = E \left[ \frac{q_k^2 - 1}{q_k - 1} e(i) \right] = 2E[(q_k + 1)e(i)],
\]

where \( J(w) = E[2e^2(i)] \), \( E[\cdot] \) is the expectation operator and \( e(i) = d(i) - y(i) \).

Similarly

\[
\frac{\partial q_k}{\partial q_k} w_k(i) y = x_k(i),
\]

and

\[
\frac{\partial q_k}{\partial q_k} e = -1,
\]

Substituting Eqs. (15), (16), and (17) in (14) gives

\[
\nabla_{q,w}(i) J(w) = -2E[(q_k + 1)e(i)x_k(i)].
\]

Similarly, for \( k = 1, 2, \ldots, M \),

\[
\nabla_{q,w} J(w) = -2E[(q_1 + 1)e(i)x_1(i), (q_2 + 1)e(i)x_2(i), \ldots (q_M + 1)e(i)x_M(i)].
\]

Consequently, Eq. (19) can be written as

\[
\nabla_{q,w} J(w) = -4E[Gx(i)e(i)],
\]

where \( \mu \) is the learning rate (step size) and \( G \) is a diagonal matrix

\[
\text{diag}(G) = \left[ \left( \frac{q_1 + 1}{2} \right), \left( \frac{q_2 + 1}{2} \right), \ldots \left( \frac{q_M + 1}{2} \right) \right]^T.
\]

Dropping the expectation from the q-gradient in (20) results in

\[
\nabla_{q,w} J(w) \approx -4Gx(i)e(i).
\]

Substituting (22) in (1) renders the weight-update rule of the q-LMS algorithm by

\[
w(i + 1) = w(i) + 2\mu Gx(i)e(i).
\]
2.2.1 $q$-LMS as a Whitening Filter-$q$-Normalized LMS

The $q$-normalized least mean square ($q$-NLMS) algorithm is defined [6]1:

$$w(i + 1) = w(i) + \mu \frac{Gx(i)e(i)}{\zeta + ||x(i)||_G^2},$$

where $\zeta$ is a small value added in the denominator to avoid the indeterminate form, and $||x(i)||_G^2$ is the weighted norm of the input vector. By selecting the $q$ parameter in (23) as $q = 1/\lambda_{\text{max}}$, we can design a whitening filter and hence the dependency on the input correlation can be avoided [6].

2.2.2 Time-Varying $q$-LMS

The time-varying $q$-LMS algorithm is based on the variable step size (VSS) method [35] and is given as [11]

$$\Psi(i + 1) = \beta\Psi(i) + \gamma e(i)^2, \quad (0 < \beta < 1, \gamma > 0),$$

$$q(i + 1) = \begin{cases} q_{\text{upper}}, & \Psi(i + 1) > q_{\text{upper}}, \\ 1, & \Psi(i + 1) < 1, \\ \Psi & \text{otherwise}. \end{cases}$$

where $q_{\text{upper}}$ is so chosen to satisfy the stability bound [8]

$$q_{\text{upper}} = \frac{2}{\mu \lambda_{\text{max}}}.\quad (27)$$

The $q(i + 1)$ is updated according to the square of the estimation error. When the estimation error is large, $q(i)$ will approach its upper bound denoted by $q_{\text{upper}}$, while for the smaller values, $q(i)$ tends to unity.

3 The Proposed Enhanced $q$-least Mean Square ($Eq$-LMS) Algorithm

The $q$-LMS algorithm has an extra degree of freedom to control the performance via the diagonal matrix $G$, which comprises the $q$-dependent entries. The weight-update rule of the $q$-LMS algorithm can be written as

$$w(i + 1) = w(i) + 2\mu \hat{x}(i)e(i),$$

where $\hat{x}(i) = Gx(i)$. For the special case of $G=I$ (identity matrix), the $q$-LMS algorithm will be transformed into the conventional LMS. Based on the above discussion, we make the following important observations.

- We argue that the $q$-gradient with $q > 1$ enhances the speed of convergence as it takes the secant of function rather than the tangent [8]. The larger the value of the
The $q$ parameter, the faster the convergence of the algorithm. But this improvement in the rate of convergence comes at the cost of a degradation in the steady-state performance.

- The time-varying $q$-LMS technique [11] is based on variable step size method [35], which requires the tuning of additional parameters such as $\beta$ and $\gamma$.
- By selecting the $q$ parameter in (23) as $q = 1/\lambda_{\text{max}}$, we can design a whitening filter and hence it can remove the dependency on the input correlation [6]. However, with a large step size the $q$-NLMS converges rapidly with a compromised steady-state performance. Similarly, a smaller step size results in better steady-state performance but with slow convergence. As such, the two important performance parameters cannot be optimized simultaneously.

### 3.1 Proposed Improvements

To overcome the aforementioned issues, we propose the $Eq$-LMS algorithm with the following improvements.

- To achieve higher convergence rate with lower steady-state error, we propose to incorporate the instantaneous error energy to adapt the $q$-parameter. The proposed algorithm automatically takes large steps when the error is large and reduces the step size with the decreasing error. Note that, unlike the time-varying $q$-LMS [11], no additional tuning parameters are introduced and the proposed approach is completely automatic.
- The whitening factor ($q = 1/\lambda_{\text{max}}$) is also utilized to set the limits of adaptive $q$-vector, and this allows the algorithm to operate at a higher convergence rate without worrying about the divergence issues.
- To update each $q$-parameter, the proposed $Eq$-LMS utilizes a responsible error (error at each tap of the filter). With this improvement, the $q$-variable for each tap will be updated accordingly; hence, both the steady-state error and convergence performance can be improved significantly.

### 3.2 Formulation of the $Eq$-LMS Algorithm

By replacing the fixed $G$ in (23) with its time-varying form $G(i)$, the weight-update rule of the proposed $Eq$-LMS is given as

$$w(i + 1) = w(i) + 2\mu e(i)x(i)G(i), \quad (29)$$

where $\mu$ is the learning rate, and $e(i)$ is the error at the $i$th instant defined by

$$e(i) = d(i) - y(i), \quad (30)$$

where $d(i)$ and $y(i)$ are the desired and estimated output at the $i$th instant, respectively. Here, $G(i)$ is a diagonal matrix with time-varying diagonal elements and is defined as
\[
G(i) = \begin{bmatrix}
q_1(i) \\
q_2(i) \\
\vdots \\
q_M(i)
\end{bmatrix}.
\] (31)

It can also be written as
\[
G(i) = \text{diag}(q_1(i), \ldots, q_M(i)) = \text{diag}(q(i)),
\] (32)

where
\[
q(i) = \{q_1(i), q_2(i), \ldots, q_M(i)\}.
\] (33)

We propose an update rule for vector \( q \) defined in the following steps.

- **Step 1** Initialize vector \( q \) with any positive random values.
- **Step 2** Use instantaneous error to update first entry \( q_1 \) of \( q \) vector, which is associated with weight of the instant input tap, i.e.,

\[
q_1(i + 1) = \frac{1}{M + 1} \left\{ |e(i)| + \sum_{j=1}^{M} q_j(i) \right\},
\] (34)

where \( M \) is the length of the filter.
- **Step 3** To avoid divergence while maintaining the higher convergence rate, the following standard conditions will be evaluated: [20,45,49]

\[
q(i + 1) = \begin{cases} 
\frac{1}{\lambda_{\text{max}}} i f & |q_1(i + 1)| > \frac{1}{\lambda_{\text{max}}}, \\
q_1(i + 1) & \text{otherwise},
\end{cases}
\] (35)

where \( \lambda_{\text{max}} \) is the maximum eigenvalue of the input autocorrelation matrix.
- **Step 4** Update all entries of vector \( a \) except for the first entry, simply by shifting:

\[
q_{k+1}(i + 1) = q_k(i),
\] (36)

where \( 1 < k < M - 1 \)
- **Step 5** For next iterations, repeat steps 2 to 5.

Finally, the weight-update equation of the proposed Eq-LMS can be written as:

\[
w(i + 1) = w(i) + 2\mu e(i)x(i) \odot q(i),
\] (37)

where \( \odot \) indicates the elementwise multiplication.

### 3.3 Optimal Solution of the Eq-LMS

The optimal solution for the proposed Eq-LMS is derived by equating \( \nabla_{q,w} J(w) \) in Eq. (20) to zero:

\[
- E[4Ge(i)x(i)] \approx 0
\] (38)
Taking the expectation and substituting the value of $e(n)$ in Eq. (38) gives:

$$4G[P - Rw_{opt}] \approx 0 \quad (39)$$

where $R$ is the input autocorrelation matrix, and it is defined as:

$$R = E[x(i)x(i)^\top] \quad (40)$$

and $P$ is the cross-correlation vector of the desired output and input, and it is given as:

$$R = E[d(i)x(i)] \quad (41)$$

and the optimal weights are therefore found to be:

$$w_{opt} \approx R^{-1}P \quad (42)$$

### 3.4 Necessary Condition for Convergence

The proposed Eq-LMS algorithm is analyzed below for the necessary condition of convergence. For the convenience of analysis the following assumptions are made:

- input vectors $x(i)$, at different time instants, are statistically independent of each others. This implies that $E[x(i)x(i - 1)] = 0$
- the input signal is statistically independent of all the previous samples of the desired output, i.e., $x(i)$ is independent of $d(i - k)$, where $k = \pm 1, \pm 2, \pm 3 \ldots$
- $w(i)$ is statistically independent of $x(i)$ and $d(i)$
- the error is independent of the step size $\mu(i)$ \[36\]
- the error signal, at the optimal solution, is orthogonal to the elements of the input signal vector.
- the input vector is independent of the weight error vector $E[x(i)\Delta(i)] = 0$

The weight error vector is the error between the optimal weights $w_{opt}$ and the estimated weights $w(i)$ and is given as:

$$\Delta(i) = w(i) - w_{opt} \quad (43)$$

With this definition, the weight error recursion can be defined using Eq. (37):

$$E[\Delta(i + 1)] = E[\Delta(i)] + 2E[\mu(i)]E[x(i)e(n)] \quad (44)$$

The $\mu(i)$ is the time-varying step size, which varies with the value of $q(i)$:

$$\mu(i) = \mu q(i) \quad (45)$$

where $\mu$ is the fixed step size and $q(i)$ contains the time-varying diagonal elements of $G$ and is updated according to the rule defined in Eq. (35).
The mean value of the weight error vector can be shown to be governed by the following recursion:

\[
E[\Delta(i + 1)] = [I - 2E[\mu(i)]R]E[\Delta(i)]
\]  

(46)

where I is the identity matrix.

The \((I - 2E[\mu(i)]D)\) is a diagonal matrix.

\[
(I - 2E[\mu(i)]D) = \begin{bmatrix}
1 - 2\mu(1)\lambda_0 & 0 & \cdots & 0 \\
0 & 1 - 2\mu(2)\lambda_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 - 2\mu(N)\lambda_N
\end{bmatrix}
\]  

(47)

and \(E[\Delta'(i + 1)] =

\[
\begin{bmatrix}
(1 - 2\mu(1)\lambda_0)^n + 1 & 0 & \cdots & 0 \\
0 & (1 - 2\mu(2)\lambda_1)^n + 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (1 - 2\mu(N)\lambda_N)^n + 1
\end{bmatrix}
\]  

(48)

where \(E[\Delta'(i + 1)] = T^T\Delta(n + 1)\).

It shows that in order to guarantee the convergence of the Eq-LMS algorithm, each diagonal element of Eq. (48) must have an absolute value that is less than one.

A sufficient condition for the time-varying step size is:

\[
0 < E[\mu(i)] < \frac{1}{\lambda_{\text{max}}}
\]  

(49)

where \(\lambda_{\text{max}}\) is the maximum eigenvalue of \(R\). The value of \(\mu(i)\) in this range guarantees that all the elements of the diagonal matrix in Eq. (48) tend to zero as \(i \to \infty\); as a result, \(E[\Delta'(i + 1)]\) tends to zero for large \(i\).

### 3.4.1 Experimental Validation of the Analytically Derived Necessary Condition of Convergence

In this section, we make the comparison of the derived analytical findings with the simulation results. Equation (49) provides the necessary condition of convergence for the proposed Eq-LMS algorithm to converge. Namely that the expected value of step size \(\mu(i)\) should be less than the maximum eigenvalue of the input correlation matrix. The input signal \(x(n)\) is chosen to consist of \(1 \times 10^2\) randomly generated samples obtained from the Gaussian distribution of mean zero and a variance of 1. The system noise is a zero mean i.i.d. sequence with a variance of 0.01 which sets the SNR to 20 dB. A system of order 5 is randomly generated. The algorithm took 100 iterations to converge, and the results are averaged over 1000 independent runs. Figure 1 clearly demonstrates a good match between the theoretical findings and the simulation results.
Fig. 1 Analytical and simulation MAE behavior of the Eq-LMS algorithm

Table 1 Computational complexities of various algorithms in terms of the number of unknown weights (M)

| Method         | Multiplications | Additions |
|----------------|-----------------|-----------|
| LMS            | 2M + 1          | 2M        |
| q-LMS          | 3M + 1          | 2M        |
| Time-varying q-LMS | 2M + 4      | 2M + 1    |
| NLMS           | 3M + 2          | 3M + 1    |
| Eq-LMS         | 2M + 2          | 3M + 1    |

3.5 Computational Complexity of the Proposed Eq-LMS Algorithm

The complexity analysis of various variants of the LMS algorithm has been conducted in terms of the number of operations required for adaptation process, and the results are shown in Table 1. The M denotes the number of unknown weight parameters, the LMS algorithm requires 2M + 1 multiplications and 2M additions, while q-LMS requires 3M + 1 multiplications due to the presence of diagonal matrix G and 2M additions. Time-varying q-LMS requires 2M + 4 multiplications and 2M + 1 additions, while NLMS requires 3M + 2 multiplications and 3M + 1 additions. In comparison with LMS, the multiplications required by NLMS and Eq-LMS are, respectively, M+1 and 1 more than that required by LMS. Table 1 shows that the proposed Eq-LMS is M times less expensive than the NLMS while maintaining a better performance index.

4 Experiments

For the evaluation of the proposed algorithm, the problem of system identification is used. Adaptive learning methods have been successfully used to identify the unknown
system, with numerous applications, for example, in control engineering, communication systems [18,19,27–29,32,37–39,42,46]. Channel estimation, for instance, is a widely used method in communication systems to estimate the characteristics of an unknown channel. Consider a linear channel shown in Fig. 2.

\[ y(t) = h_1 x(t) + h_2 x(t - 1) + h_3 x(t - 2) + h_4 x(t - 3) + h_5 x(t - 4). \]  

Equation (50) shows the mathematical model of the system, where \( x(t) \) and \( y(t) \) are the input and output of the system, respectively, and \( d(t) \) is the disturbance which is taken to be white Gaussian noise in this case. For this experiment, \( x(t) \) consists of \( 1 \times 10^6 \) randomly generated samples obtained from Gaussian distribution of mean zero and variance of 1. In (50), the system is defined by its impulse response \( h(t) \), while \( \hat{y}(t) \), \( \hat{h}(t) \), and \( e(t) \) are the estimated output, estimated impulse response, and the error of estimation, respectively. The simulation parameters selected are as follows: coefficient values of \( h_1 = -2, h_2 = -1, h_3 = 0, h_4 = 1, \) and \( h_5 = 2 \) are selected for the channel, and the experiments are performed on three noise levels with the SNR values of 10dB, 20dB, and 30dB. The weights are initialized to zero for all algorithms. Specifically, the objective of these simulations is to compare the performance of the proposed enhanced \( q \)-LMS (Eq-LMS) algorithm with the contemporary counterparts, i.e., least mean square (LMS)/\( q \)-LMS [8] at \( q = 1 \), \( q \)-LMS [8] at \( q = 2 \), time-varying \( q \)-LMS [11], and the normalized LMS (NLMS), for a given convergence rate and a given steady-state error in three different scenarios. The idea behind taking two different values of \( q \) is to show that the higher value of \( q \) results in a faster convergence rate with a higher steady-state error, while a smaller steady-state error is produced with a smaller value of \( q \).

For the performance evaluation, the normalized weight deviation (NWD) in the actual and the obtained weights is compared. Specifically, we define

\[ \text{NWD} = \frac{\| h - w \|}{\| h \|}. \]
Table 2 Evaluation protocol 1: configuration of learning rates of different approaches for an equal convergence rate of $1 \times 10^{-1}$

| Algorithm           | Learning rate $\mu$ |
|---------------------|---------------------|
|                     | 10 dB SNR           | 20 dB SNR           | 30 dB SNR           |
| LMS/q-LMS           | $4.4 \times 10^{-2}$ | $2.45 \times 10^{-2}$ | $2.7 \times 10^{-5}$ |
| Time-varying $q$-LMS| $3.5 \times 10^{-1}$ | $1.3 \times 10^{-1}$ | $3.2 \times 10^{-5}$ |
| Normalized LMS      | $2.45 \times 10^{-1}$ | $1.2 \times 10^{-1}$ | $8.7 \times 10^{-5}$ |
| Proposed $E_q$-LMS  | $1 \times 10^{-1}$ | $1 \times 10^{-1}$ | $1 \times 10^{-1}$ |

where $h$ is the actual impulse response of the channel and $w$ is the estimated weight vector. Simulations are repeated for 1000 independent runs, and mean results are reported. Simulations are performed primarily to evaluate the steady-state and convergence performances of the proposed algorithm for various learning rates. Since $E_q$-LMS is a time-varying algorithm, we manually configured the values of $\mu$ for different scenarios to achieve similar steady-state or convergence performances. Accordingly, three evaluation protocols are designed:

1. **Evaluation protocol 1** learning rate $= 1 \times 10^{-1}$, SNR $= \{10, 20, 30\}$ dB.
2. **Evaluation protocol 2** learning rate $= 1 \times 10^{-2}$, SNR $= \{10, 20, 30\}$ dB.
3. **Evaluation protocol 3** learning rate $= 1 \times 10^{-3}$, SNR $= \{10, 20, 30\}$ dB.

4.1 Experiments to Evaluate the Steady-State Performance

4.1.1 Evaluation Protocol 1: Fast Convergence

For the comparison of the steady-state performance, all algorithms were set up for equal convergence, and after 10,000 iterations, the steady-state value of the NWD is examined. The learning rate (step size) configurations for equal convergence rate (Evaluation protocol 1) are shown in Table 2.

The relevant normalized weight difference (NWD) curves with three different SNR values are depicted in Fig. 3. From Fig. 3a–c, it can be seen that the proposed $E_q$-LMS produced the best performance under all three conditions: (1) for the SNR value of 10 dB, it outperformed the LMS/q-LMS at $q = 1$, the $q$-LMS at $q = 2$, the time-varying $q$-LMS, and the NLMS by the NWD value of $-1.03$ dB, $-2.95$ dB, $-8.06$ dB, and $-2.14$ dB, respectively, (2) for the SNR value of 20 dB, it surpassed the listed algorithms by the NWD value of $-2.36$ dB, $-4.05$ dB, $-6.96$ dB, and $-3.29$ dB, respectively, and (3) for the SNR value of 30 dB, the above-mentioned algorithms were outperformed by a margin of $-2.29$ dB, $-3.86$ dB, $-7.18$, and $-3.47$ dB, respectively. Note that the proposed $E_q$-LMS algorithm showed the lowest steady-state error in all conditions, while the LMS, NLMS, and the $q$-LMS at $q = 2$ showed faster convergence than the time-varying $q$-LMS but with a greater steady-state error than the proposed $E_q$-LMS. With the above-discussed settings, results for the channel estimation problem are summarized in Table 3.
Fig. 3 NWD curves for the LMS/q-LMS at $q = 1$, $q$-LMS at $q = 2$, time-varying $q$-LMS, NLMS, and the $Eqq$-LMS. Normalized weight deviation with learning rate and SNR of $a$ $10^{-1}$, $10$ dB, $b$ $10^{-1}$, $20$ dB, $c$ $10^{-1}$, $30$ dB, $d$ $e^{-2}$, $10$ dB, $e$ $e^{-2}$, $20$ dB, $f$ $e^{-2}$, $30$ dB, $g$ $10^{-3}$, $10$ dB, $h$ $10^{-3}$, $20$ dB, and $i$ $10^{-3}$, $30$ dB

Table 3 Evaluation protocol 1: results of various approaches for an equal convergence rate

| Algorithm               | 10 dB NWD (dB) | 20 dB NWD (dB) | 30 dB NWD (dB) |
|-------------------------|---------------|---------------|---------------|
| LMS/q-LMS ($q = 1$)     | $-14.63$      | $-21.06$      | $-28.77$      |
| $q$-LMS ($q = 2$)       | $-12.71$      | $-19.37$      | $-27.02$      |
| Time-varying $q$-LMS   | $-7.60$       | $-16.46$      | $-23.88$      |
| Normalized LMS          | $-13.52$      | $-20.13$      | $-27.59$      |
| Proposed $Eqq$-LMS      | $-15.66$      | $-23.42$      | $-31.06$      |

Bold indicates the best performance result

4.1.2 Evaluation Protocol 2: Medium Convergence

The learning rate (step size) configurations for an equal convergence rate (Evaluation protocol 2) are shown in Table 4.
Table 4: Evaluation protocol 2: configuration of learning rates of different approaches for an equal convergence rate of $1 \times 10^{-2}$

| Algorithm          | Learning rate $\mu$ |
|--------------------|---------------------|
|                    | 10 dB SNR           | 20 dB SNR           | 30 dB SNR           |
| LMS/q-LMS          | $4 \times 10^{-3}$  | $1.5 \times 10^{-3}$| $5.2 \times 10^{-4}$|
| Time-varying $q$-LMS | $2 \times 10^{-2}$  | $1 \times 10^{-3}$  | $1.8 \times 10^{-3}$|
| Normalized LMS     | $2 \times 10^{-2}$  | $9 \times 10^{-3}$  | $2.5 \times 10^{-3}$|
| Proposed Eq-LMS    | $1 \times 10^{-2}$  | $1 \times 10^{-2}$  | $1 \times 10^{-2}$  |

Table 5: Evaluation protocol 2: results of various approaches for an equal convergence rate

| Algorithm          | Steady-state NWD (dB) |
|--------------------|-----------------------|
|                    | 10 dB SNR           | 20 dB SNR           | 30 dB SNR           |
| LMS/q-LMS (q = 1)  | $-25.42$            | $-32.70$            | $-40.00$            |
| $q$-LMS (q = 2)    | $-23.62$            | $-30.86$            | $-38.19$            |
| Time-varying $q$-LMS | $-22.40$            | $-29.40$            | $-37.45$            |
| Normalized LMS     | $-24.18$            | $-31.45$            | $-38.80$            |
| Proposed Eq-LMS    | $-25.85$            | $-33.60$            | $-41.13$            |

Bold indicates the best performance result.

The relevant normalized weight difference (NWD) curves with three different SNR values are depicted in Fig. 3. From Fig. 3d–f, it can be seen that the proposed Eq-LMS produced the best performance under all three conditions: (1) for the SNR value of 10 dB, it outperformed the LMS/q-LMS at $q = 1$, $q$-LMS at $q = 2$, time-varying $q$-LMS and the NLMS by the NWD value of $-0.68$ dB, $-2.23$ dB, $-4.08$ dB, and $-1.74$ dB, respectively, (2) for the SNR value of 20 dB, it surpassed the above-mentioned algorithms by the NWD value of $-1.22$ dB, $-2.74$ dB, $-5.34$ dB, and $-2.45$ dB, respectively, and (3) for the SNR value of 30 dB, the above-mentioned algorithms were outperformed by a margin of $-1.44$ dB, $-2.94$ dB, $-4.07$ and $-2.47$ dB, respectively. Note that the proposed Eq-LMS algorithm showed the lowest steady-state error in all conditions, while the LMS, NLMS and the $q$-LMS at $q = 2$ showed faster convergence than time-varying $q$-LMS but with a greater steady-state error than the proposed Eq-LMS. With the above-discussed settings, results for the channel estimation problem are summarized in Table 5.

4.1.3 Evaluation Protocol 3: Slow Convergence

The learning rate (step size) configurations for equal convergence rate (Evaluation protocol 3) is shown in Table 6.

The relevant NWD curves with three different SNR values are delineated in Fig. 3. From Fig. 3g–i, it can be seen that the proposed Eq-LMS produced the best performance under all three conditions: (1) for the SNR value of 10 dB, it outperformed the LMS/q-LMS at $q = 1$, $q$-LMS at $q = 2$, time-varying $q$-LMS, and the NLMS...
Table 6 Evaluation protocol 3: configuration of learning rates of different approaches for the fix convergence rate of $1 \times 10^{-3}$

| Algorithm              | Learning rate $\mu$ |
|------------------------|----------------------|
|                        | 10 dB SNR | 20 dB SNR | 30 dB SNR |
| LMS/q-LMS              | $3.6 \times 10^{-4}$ | $1.3 \times 10^{-4}$ | $4.5 \times 10^{-3}$ |
| Time-varying $q$-LMS  | $1.6 \times 10^{-3}$ | $6 \times 10^{-4}$ | $1.5 \times 10^{-4}$ |
| Normalized LMS         | $1.9 \times 10^{-3}$ | $7 \times 10^{-4}$ | $2.3 \times 10^{-4}$ |
| Proposed $Eq$-LMS      | $1 \times 10^{-3}$   | $1 \times 10^{-3}$ | $1 \times 10^{-3}$ |

Table 7 Evaluation protocol 3: results of various approaches for an equal convergence rate

| Algorithm (q = 1)      | Steady-state NWD (dB) |
|------------------------|------------------------|
|                        | 10 dB SNR | 20 dB SNR | 30 dB SNR |
| LMS/q-LMS              | $-20.16$   | $-27.34$  | $-34.67$  |
| $q$-LMS $(q = 2)$      | $-18.92$   | $-26.13$  | $-33.45$  |
| Time-varying $q$-LMS  | $-16.76$   | $-23.22$  | $-32.04$  |
| Normalized LMS         | $-19.10$   | $-26.11$  | $-33.64$  |
| Proposed $Eq$-LMS      | **$-20.84$** | **$-28.56$** | **$-36.11$** |

Bold indicates the best performance result

by the NWD value of $-0.43$ dB, $-1.92$ dB, $-3.45$ dB, and $-1.67$ dB, respectively, (2) for the SNR value of 20 dB, it surpassed the above-mentioned algorithms by the NWD value of $-0.9$ dB, $-2.43$ dB, $-4.2$ dB, and $-2.15$ dB, respectively, and (3) for the SNR value of 30 dB, the above-mentioned algorithms were outperformed by a margin of $-1.13$ dB, $-2.66$ dB, $-3.68$, and $-2.33$ dB, respectively. Note that the proposed $Eq$-LMS algorithm showed the lowest steady-state error in all conditions, while the $q$-LMS at $q = 2$ showed faster convergence with a greater steady-state error than the proposed $Eq$-LMS. With the above-discussed settings, results for the channel estimation problem are summarized in Table 7.

4.2 Experiments to Evaluate the Convergence Performance

4.2.1 Evaluation Protocol 1: Fast Convergence

For the comparison of convergence performances, all algorithms were set up for equal steady-state error. The learning rate (step size) configurations for equal steady-state (Evaluation protocol 1) are shown in Table 8. The learning rate for the proposed $Eq$-LMS has been set according to the three evaluation protocols.

The relevant normalized weight difference (NWD) curves with three different SNR values are depicted in Fig. 4. From Fig. 4a–c, it can be seen that the proposed $Eq$-LMS algorithm produced the best results under all three conditions: (1) for the SNR value of 10 dB, algorithms are run for 1000 iterations. The convergence point of the proposed $Eq$-LMS is reached at 120th iteration, $q$-LMS at ($q = 2$) converged on the 80th
Table 8 Evaluation protocol 1: configuration of learning rates of different approaches for an equal steady-state error

| Algorithm          | Learning rate $\mu$ |
|--------------------|---------------------|
|                    | 10 dB SNR | 20 dB SNR | 30 dB SNR |
| LMS/q-LMS          | $3 \times 10^{-2}$ | $8.9 \times 10^{-3}$ | $2.7 \times 10^{-3}$ |
| Time-varying q-LMS | $3.3 \times 10^{-2}$ | $8.8 \times 10^{-3}$ | $3.1 \times 10^{-3}$ |
| Normalized LMS     | $1 \times 10^{-1}$ | $2.8 \times 10^{-2}$ | $8.5 \times 10^{-3}$ |
| Proposed Eq-LMS    | $1 \times 10^{-1}$ | $1 \times 10^{-1}$ | $1 \times 10^{-1}$ |

Fig. 4 NWD curves for the LMS/q-LMS at $q = 1$, q-LMS at $q = 2$, time-varying q-LMS, NLMS, and the Eq-LMS. Normalized weight deviation with learning rate and SNR of $a 1e^{-1}$, 10 dB, $b 1e^{-1}$, 20 dB, $c 1e^{-1}$, 30 dB, $d 1e^{-2}$, 10 dB, $e 1e^{-2}$, 20 dB, $f 1e^{-2}$, 30 dB, $g 1e^{-3}$, 10 dB, $h 1e^{-3}$, 20 dB, and $i 1e^{-3}$, 30 dB iteration, but its steady-state error is much larger compared to the proposed Eq-LMS, and (2) for the SNR value of 20 dB, algorithms are run for 5000 iterations. Note that the proposed Eq-LMS algorithm outperformed all competing approaches by converging in only 400 iterations. The q-LMS ($q = 2$) was unable to reach the given error floor and took 400 iterations to reach a much higher error, and (3) for the SNR value of
Table 9 Evaluation protocol 1: results of various approaches for an equal steady-state error

| Algorithm                  | Convergence point (number of iterations $\times 1000$) |
|----------------------------|-------------------------------------------------------|
|                            | 10 dB SNR | 20 dB SNR | 30 dB SNR |
| LMS/$q$-LMS at ($q = 1$)   | 0.20      | 0.70      | 2.70      |
| $q$-LMS at ($q = 2$)       | 0.08      | 0.40      | 1.55      |
| Time-varying $q$-LMS       | 0.64      | 2.80      | 7.20      |
| Normalized LMS             | 0.23      | 1.10      | 4.40      |
| Proposed Eq-$L$-MS         | 0.12      | 0.40      | 1.60      |

Table 10 Evaluation protocol 2: configuration of learning rates of different approaches for an equal steady-state error

| Algorithm         | Learning rate $\mu$ |
|-------------------|----------------------|
|                   | 10 dB SNR | 20 dB SNR | 30 dB SNR |
| LMS/$q$-LMS       | $3 \times 10^{-3}$  | $8.8 \times 10^{-4}$ | $2.7 \times 10^{-4}$ |
| Time-varying $q$-LMS | $3.3 \times 10^{-3}$  | $9.3 \times 10^{-4}$ | $3.1 \times 10^{-4}$ |
| Normalized LMS    | $9 \times 10^{-2}$  | $2.72 \times 10^{-3}$ | $8.5 \times 10^{-4}$ |
| Proposed Eq-$L$-MS | $1 \times 10^{-2}$  | $1 \times 10^{-2}$  | $1 \times 10^{-2}$  |

30 dB, algorithms are run for 10,000 iterations, and the proposed Eq-$L$-LMS algorithm took the least number of iterations by converging at the 1600th iteration. The proposed Eq-$L$-LMS shows the best performance in terms of steady-state error and convergence rate, thus showing the best overall performance. With the above-discussed settings, results for the channel estimation problem are summarized in Table 9.

4.2.2 Evaluation Protocol 2: Medium Convergence

The learning rate (step size) configuration for equal steady-state (Evaluation protocol 2) is shown in configuration Table 10. The learning rate for the proposed Eq-$L$-LMS has been set according to the three evaluation protocols.

The relevant NWD curves with three different SNR values are depicted in Fig. 4. From Fig. 4d–f, it can be seen that the proposed Eq-$L$-LMS algorithm produced the best results under all three conditions: (1) for the SNR value of 10 dB, algorithms are run for 10,000 iterations, the convergence point of the proposed Eq-$L$-LMS is reached at the 1500th iteration, the $q$-LMS at ($q = 2$) converged on the 1100th iteration, but its steady-state error is much larger than the proposed Eq-$L$-LMS, (2) for the SNR value of 20 dB, algorithms are run for 50,000 iterations, the proposed Eq-$L$-LMS outperformed all competing approaches in terms of convergence point with least steady-state error, and (3) for the SNR value of 30 dB, algorithms are run for 10,000 iterations, the proposed Eq-$L$-LMS algorithm converged at 19000th iteration, and it showed the best performance in terms of steady-state error and convergence rate, thus showing the best overall performance. With the above-discussed settings, results for the channel estimation problem are summarized in Table 11.
Table 11  Evaluation protocol 2: results of various approaches for an equal steady-state error

| Algorithm                        | Convergence point (number of iterations × 1000) |
|----------------------------------|-------------------------------------------------|
|                                  | 10 dB SNR | 20 dB SNR | 30 dB SNR |
| LMS/q-LMS at (q = 1)             | 2         | 9         | 32        |
| q-LMS at (q = 2)                 | 1.1       | 5.9       | 19        |
| Time-varying q-LMS              | 6.2       | 28        | 80        |
| Normalized LMS                  | 3.1       | 14        | 52        |
| Proposed Eq-LMS                 | 1.5       | 6         | 19        |

Table 12  Evaluation protocol 3: configuration of learning rates of different approaches for an equal steady-state error

| Algorithm       | Learning rate μ |
|-----------------|-----------------|
|                 | 10 dB SNR | 20 dB SNR | 30 dB SNR |
| LMS/q-LMS       | 3 × 10^{-4} | 8.9 × 10^{-4} | 2.7 × 10^{-5} |
| Time-varying q-LMS | 3.3 × 10^{-4} | 9 × 10^{-5} | 3.2 × 10^{-5} |
| Normalized LMS  | 1 × 10^{-3}  | 2.8 × 10^{-4} | 8.5 × 10^{-5} |
| Proposed Eq-LMS | 1 × 10^{-3}  | 1 × 10^{-3}  | 1 × 10^{-3}  |

4.2.3 Evaluation Protocol 3: Slow Convergence

The learning rate (step size) configuration for equal steady-state (Evaluation protocol 3) is shown in configuration Table 12. The learning rate for the proposed Eq-LMS has been set according to the three evaluation protocols.

The relevant NWD curves with three different SNR values are depicted in Fig. 4. From Fig. 4g–i, it can be seen that the proposed Eq-LMS algorithm produced the best results under all three conditions: (1) for the SNR value of 10 dB, algorithms are run for 10,000 iterations, the convergence point of the proposed Eq-LMS is reached at the 21,000th iteration, the q-LMS at (q = 2) converged at the 12,000th iteration, but its steady-state error is much larger than the proposed Eq-LMS, (2) for the SNR value of 20 dB, the algorithms are run for 50,000 iterations, the proposed Eq-LMS algorithm outperformed all competing approaches in terms of convergence point with least steady-state error, and (3) for the SNR value of 30 dB, the algorithms are run for 1,000,000 iterations, and the proposed Eq-LMS algorithm converged at the 200000th iteration. The proposed Eq-LMS showed the best performance in terms of steady-state error and convergence rate. With the above-discussed settings, results for the channel estimation problem are summarized in Table 13.

5 Conclusion

In this work, we proposed a quantum calculus-based steepest descent algorithm called enhanced q-least mean square algorithm (Eq-LMS) using a novel concept of error-
**Table 13** Evaluation protocol 3: results of various approaches for an equal steady-state error

| Algorithm          | Convergence point (number of iterations × 1000)          |
|--------------------|----------------------------------------------------------|
|                    | 10 dB SNR | 20 dB SNR | 30 dB SNR |
| LMS/q-LMS at (q = 1) | 23        | 100       | 370       |
| q-LMS at (q = 2)     | 12        | 60        | 200       |
| Time-varying q-LMS   | 72        | 28        | 840       |
| Normalized LMS       | 36        | 320       | 600       |
| Proposed Eq-LMS      | 21        | 70        | 240       |

Correlation energy. The proposed algorithm is a parameterless method, and unlike the contemporary time-varying q-LMS, it does not require additional tuning. The proposed Eq-LMS algorithm is analyzed for the necessary condition of convergence, and the validation of the analytical and simulation results is presented. The proposed Eq-LMS was compared with the LMS, q-LMS, time-varying q-LMS, and the NLMS algorithms for the problem of linear channel estimation. Extensive simulation tests were conducted to analyze the convergence and the steady-state performance at three different SNR levels. For all scenarios, the proposed Eq-LMS algorithm comprehensively outperformed the contemporary approaches achieving the best performance in terms of steady-state error and convergence.

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Affiliations

Alishba Sadiq1 · Shujaat Khan2 · Imran Naseem1,3 · Roberto Togneri3 · Mohammed Bennamoun4

Alishba Sadiq
alishba.sadiq@pafkiet.edu.pk

Shujaat Khan
shujaat@iqra.edu.pk

Roberto Togneri
roberto.togneri@uwa.edu.au
Mohammed Bennamoun
mohammed.bennamoun@uwa.edu.au

1 College of Engineering, Karachi Institute of Economics and Technology, Korangi Creek, Karachi 75190, Pakistan

2 Faculty of Engineering Science and Technology, Iqra University, Defence View, Shaheed-e-Millat Road (Ext.), Karachi 75500, Pakistan

3 School of Electrical, Electronic and Computer Engineering, The University of Western Australia, 35 Stirling Highway, Crawley, WA 6009, Australia

4 School of Computer Science and Software Engineering, The University of Western Australia, 35 Stirling Highway, Crawley, WA 6009, Australia