New approach for solving Maxwell equations with strong singularity

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Abstract. In this note we introduce a notion of $R_\nu$-generalized solution to Maxwell equations with strong singularity in a 2D nonconvex polygonal domain. We develop a new weighted edge FEM. Results of numerical experiments prove the efficiency of this method.

1. Introduction
Maxwell equations are used in mathematical models of electromagnetic fields, for example, in plasma physics, electrodynamics and engineering of high-frequency devices. As a rule, in practical problems the computational domain is nonconvex with reentrant corners or edges on its boundary. Such geometry singularities lead to strong electromagnetic fields in their neighborhood, and a solution of Maxwell equations is strongly singular, i.e. it does not belong to the Sobolev space $H^1(W^{1,2})$.

For numerical solution of such problems P. Ciarlet Jr., F. Assous and others developed the singular complement method (SCM). This method consists in splitting of the space of solutions into a two-term direct sum, where the first term contains the regular part of solution and the second contains the singular part (see [1, 2, 3, 4]).

Another approach to solving Maxwell equations with strongly singular solution was developed by M. Costabel and M. Daug. This method is based on regularization of the initial equation with a weighted divergence term. The resulting elliptic problem considered in a special weighted space is equivalent to the original one (see [5, 6]).

For boundary value problems with strongly singular solutions we developed the theory of numerical methods based on the conception of $R_\nu$-generalized solution (see, for example, [7, 8, 9, 10, 11]). This conception allows us, depending on the singularity of input data (coefficients and right hand sides of equations and boundary conditions) and geometry of the boundary, to define a weighted space or a set containing the solution. It is also possible to build a regularizator which bounds the influence of singularity on the accuracy of numerical solution.

In the present paper we develop the weighted edge finite element method (FEM) based on the conception of $R_\nu$-generalized solution of the Maxwell equations with strong singularity due to a reentrant corner on the boundary. Numerical experiments of model problems showed that the rate of convergence of the numerical solution to the exact one is more than one and a half times better in comparison with the results established in [3, 6, 12]. Another advantage of this method is the simplicity of the solution determination which is an additional benefit for numerical experiments.
2. Notations and preliminaries
We denote two- and three-dimensional Euclidian spaces by \( R^2 \) and \( R^3 \), respectively, with the Cartesian coordinates \( x = (x_1, x_2), \) \( x = (x_1, x_2, x_3) \). Let \( \Omega \) be a bounded simply connected domain in these spaces with the boundary \( \partial \Omega \). Let \( \mathbf{n} \) be the unit outward normal to \( \partial \Omega \). In the case \( \Omega \subset R^2 \) \( \mathbf{t} \) means a unite tangent vector to \( \partial \Omega \). For vector-functions \( f : R^3 \rightarrow R^3 \) we use the notation \( f(x) = (E_1(x), E_2(x), E_3(x)) \). Vector-functions \( f : R^2 \rightarrow R^2 \) are denoted by boldface latin letters: \( \mathbf{E}(x) = (E_1(x), E_2(x)) \).

In the 2D case the notations \( \text{curl} \) and \( \text{curl} f \) distinguish between scalar and vector curl operators:

\[
\text{curl} \mathbf{E} = \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}, \quad \text{curl} f = \left( \frac{\partial f}{\partial x_2}, -\frac{\partial f}{\partial x_1} \right).
\]

Let us recall the definitions of some functional spaces associated with Maxwell equations:

\[
H_0(\text{curl}, \Omega) = \{ \mathbf{E} \in H(\text{curl}, \Omega) \mid \mathbf{E} \cdot \mathbf{t} = 0, x \in \partial \Omega \},
\]

\[
H(\text{div}, \Omega) = \{ \mathbf{E} \in L^2(\Omega) \mid \text{div} \mathbf{E} \in L^2(\Omega) \},
\]

\[
H(\text{div}0, \Omega) = \{ \mathbf{E} \in H(\text{div}, \Omega) \mid \text{div} \mathbf{E} = 0 \}, \quad V = H_0(\text{curl}, \Omega) \cap H(\text{div}0, \Omega).
\]

Let the domain \( \Omega \subset R^2 \) have one special point \( O(0, 0) \) on its boundary. Let \( B_\delta(O) \) be an open ball centered at point \( O(0, 0) \) with radius \( \delta \). We introduce a weight function \( \rho(x) \) which coincides with the distance to the point \( O(0, 0) \) in \( B_\delta(O) \) and it is equal to \( \delta \) outside \( B_\delta(O) \).

We introduce a weighted Sobolev space \( H^{k,\nu}_{2,\nu}(\Omega) \) with the norm

\[
\|f\|_{H^{k,\nu}_{2,\nu}(\Omega)} = \left( \sum_{|\lambda| \leq k} \left\| \rho^{2(\nu+|\lambda|-k)}(x) \left| D^\lambda f \right| \right\|^2 dx \right)^{1/2},
\]

where \( D^\lambda = \frac{\partial^{\lambda_1}}{\partial x_1^{\lambda_1}} \frac{\partial^{\lambda_2}}{\partial x_2^{\lambda_2}} \), \( \nu \) is nonnegative real, \( \lambda = (\lambda_1, \lambda_2), \) \( \lambda_1, \lambda_2 \) are nonnegative integers and \( |\lambda| = \lambda_1 + \lambda_2 \). When \( k = 0 \) \( H^0_{2,\nu}(\Omega) = L^2_{2,\nu}(\Omega) \). For the corresponding spaces of vector-functions we use notations \( H^{k,\nu}_{2,\nu}(\Omega), L^{2,\nu}_{2,\nu}(\Omega) \).

A subspace of \( H^k_{2,\nu}(\Omega) \), composed of the vector-functions with zero tangential component on \( \partial \Omega \), is denoted by \( H^k_{2,\nu}(\Omega) \):

\[
H^k_{2,\nu}(\Omega) = \{ \mathbf{E} \in H^k_{2,\nu}(\Omega) \mid \mathbf{E} \cdot \mathbf{t} = 0 \text{ for almost all } x \in \partial \Omega \}.
\]

3. Problem statement
Let \( \Omega \subset R^3 \). We suppose (see [1, 6]) that following conditions are satisfied:

- \( \partial \Omega \) is a perfect conductor;
- the physical environment is vacuum;
- free charges are absent;
- electromagnetic field and current density are monochromatic with circular frequency \( \omega \):

\[
\dot{E}(x, t) = \dot{E}(x) \exp(-i\omega t), \quad \dot{B}(x, t) = \dot{B}(x) \exp(-i\omega t), \quad \dot{J}(x, t) = \dot{J}(x) \exp(-i\omega t).
\]
Then the Maxwell equations with appropriate boundary conditions read:

\[
\begin{align*}
  i\omega \vec{E} + c^2 \text{curl} \vec{B} &= \frac{1}{\varepsilon_0} \vec{J}, \\
  -i\omega \vec{B} + \text{curl} \vec{E} &= 0, \\
  \text{div} \vec{E} &= 0, \quad x \in \Omega, \\
  \vec{E} \times \vec{n} &= 0, \\
  \vec{B} \cdot \vec{n} &= 0, \quad x \in \partial\Omega,
\end{align*}
\]

(1)

where \(\vec{E}\) and \(\vec{B}\) are, respectively, the electric and magnetic fields, \(\vec{J}\) is the current density, and \(c\) and \(\varepsilon_0\) are the speed of light and dielectric permittivity.

In what follows we assume that the data does not depend on variable \(x_3\). As noted in [1], we can decouple the initial system (1) in two systems of unknowns \(\vec{E} = (E_1, E_2, 0) = \vec{E}\), \(\vec{B} = (0, 0, B_3) = \vec{B}\) (so called TE mode) and \(\vec{E} = (0, 0, E_3) = \vec{E}\), \(\vec{B} = (B_1, B_2, 0) = \vec{B}\) (TM mode).

Let \(\Omega\) be a L-shaped domain with one reentrant corner on its boundary. Later in this paper we assume that \(\Omega = (-1, 1) \times (-1, 1) \setminus [0, 1] \times [-1, 0]\).

Using a standard technique (see [1]), we reduce the original problem for the TE mode to the first boundary value problem of the second order:

\[
\begin{align*}
  \text{curl} \text{curl} \vec{E} - k^2 \vec{E} &= \vec{f}, \\
  \text{div} \vec{E} &= 0, \quad x \in \Omega, \\
  \vec{E} \cdot \vec{n} &= 0, \quad x \in \partial\Omega.
\end{align*}
\]

(2)–(4)

It is well known that for a non-convex domain the solution of (2)–(4) belongs to \(V\) and does not belong to \(H^1(\Omega)\) ([1, 5, 6]). In other words, problem (2)–(4) has a strongly singular solution.

Let us introduce the notion of \(R_\nu\)-generalized solution for problem (2)–(4). Denote by

\[
a(\vec{E}, \vec{F}) = (\text{curl} \vec{E}, \text{curl}(\rho^{2\nu} \vec{F})) - k^2(\vec{E}, \rho^{2\nu} \vec{F}), \quad b(\vec{F}) = (\vec{f}, \rho^{2\nu} \vec{F})
\]

the bilinear and linear forms, respectively.

**Definition 3.1** A vector-function \(\vec{E} \in H^1_{2,\nu}(\Omega)\) is called \(R_\nu\)-generalized solution of problem (2)–(4) if it satisfies the condition (3) and for any \(\vec{F} \in H^1_{2,\nu}(\Omega)\) the identity \(a(\vec{E}, \vec{F}) = b(\vec{F})\) holds.

### 4. Construction of a scheme of the finite element method

We construct a scheme of the weighted edge FEM for the determination of an approximate \(R_\nu\)-generalized solution of the problem (2)–(4). This method is based on the \(H(\text{curl})\)-conforming edge FEM proposed by J.-C. Nédélec in 1980-s. Moreover, the finite element subspace is formed by solenoidal basis functions so that condition (3) holds automatically.

We divide \(\Omega\) into the set \(\{K\} = \{K_i\}_{i=1}^N\) of the closed squares \(K_i\) by means of vertical and horizontal straights \(x_1 = -1 + jh, x_2 = -1 + lh, j, l = 0, N\), where \(N\) is a positive even integer, \(h = \frac{2}{N}\) and \(N_i = \frac{3N^2}{4}\). Square \(K_i\) we name as element.

Let \(S_1, \ldots, S_{3N}\) be the set of sides of the squares \(K\) which do not belong to \(\partial\Omega\),

\[
S = \{S_i, \ i = 1, \ldots, 3N, \ S_i \notin \partial\Omega, \ S_{3N} = \frac{3N^2}{2} - 2N\}.
\]

We denote by \(M_i\) the center of the side \(S_i, i = 1, 3N\).

For every element \(K_i\) we denote by \(O_i^{K_i} = (o_1^{K_i}, o_2^{K_i})\) its center, \(S_{j}^{K_i}\) are its sides and \(M_{j}^{K_i} = (m_{1j}^{K_i}, m_{2j}^{K_i})\) are the centers of \(S_{j}^{K_i}, j = 1, 4\). The rule of the local numeration of the \(S_{j}^{K_i}, j = 1, 4\) is depicted on figure 1.
We introduce form functions of the element $K_i$ associated with its sides:

$$\psi^K_{1i} = \frac{1}{h^{\nu}}(m_{11}^{K_i}, x_2) \left( \frac{h}{2} + o_{2}^{K_i} - x_2 \right) \mathbf{i},$$

$$\psi^K_{2i} = \frac{1}{h^{\nu}}(m_{12}^{K_i}, x_2) \left( \frac{h}{2} + x_2 - o_{2}^{K_i} \right) \mathbf{i},$$

$$\psi^K_{3i} = \frac{1}{h^{\nu}}(m_{23}^{K_i}, x_1) \left( \frac{h}{2} + o_{1}^{K_i} - x_1 \right) \mathbf{j},$$

$$\psi^K_{4i} = \frac{1}{h^{\nu}}(m_{24}^{K_i}, x_1) \left( \frac{h}{2} + x_1 - o_{1}^{K_i} \right) \mathbf{j},$$

where $\mathbf{i}$ and $\mathbf{j}$ are the standard basis vectors of $R^2$.

For every side $S_i \in S$ we introduce the weighted basis function $\psi_i$ in the following way. Let $K_m$ and $K_n$ be two finite elements with common side $S_i$, and let $\psi_i^{K_m}$, $\psi_i^{K_n}$ be the form functions associated with side $S_i$. Then the corresponding basis function is given by

$$\psi_i = \begin{cases} 
\psi_i^{K_m}, & x \in K_m, \\
\psi_i^{K_n}, & x \in K_n, \\
0, & x \in \Omega \setminus (K_m \cup K_n). 
\end{cases}$$

Denote by $V_h$ the linear span of $\{\psi_i\}_{i=1}^{S_h}$. It is obvious that $V_h \subset H^1_{0}(\Omega)$. We shall approximate the $R_{\nu}$-generalized solution of problem (2)–(4) in this subspace.

**Definition 4.1** A function $E_h^{\nu} \in V_h$ satisfying the identity

$$a(E_h^{\nu}, v_h) = b(v_h) \quad \forall v_h \in V_h$$

is called the approximate $R_{\nu}$-generalized solution of problem (2)–(4).

An approximate solution $E_h^{\nu}$ will be found in the form

$$E_h^{\nu} = \sum_{i=1}^{S_h} d_i \psi_i,$$

where $d_i = \rho^{\nu}(M_i)g_i$, $g_i = \text{const.}$

The unknowns $d_i$ are defined from the system of equations

$$Ad = b,$$

where

$$d = (d_1, \ldots, d_{S_h})^T, \quad A = (A_{ij}), \quad i, j = 1, S_h,$$

$$A_{ij} = a(\psi_i, \psi_j), \quad b = (b_1, \ldots, b_{S_h})^T, \quad b_i = b(\psi_i).$$
5. Results of numerical experiments
We developed a program "Proba-III" for carrying out a set of numerical tests using our FEM described in section 4 and GMRES-method for solving system (5). The errors of numerical approximations to the $R_\nu$-generalized solution were computed as module between approximate and exact solutions in the points $M_i$ and in the norm of space $L_{2,\nu}(\Omega)$. We present here the results of numerical experiments for one model problem. To do this, introduce an auxiliary function $\varphi = (x_1 x_2 (x_1^2 - 1)(x_2^2 - 1))^2 \left( \sqrt{x_1^2 + x_2^2} \right)^{\beta+2}$. Define $E = \text{curl} \varphi$. Substituting $E$ in equation (2), we get the right hand side $f$ for the different coefficients (wave numbers) $k_1 = 220$ and $k_2 = 300$ (frequencies are 10.497 GHz and 14.314 GHz respectively). Numerical experiments were carried out on meshes with different step sizes $h$.

Table 1. Dependence of error $\|E - E^h_1\|_{L_{2,\nu}(\Omega)}$ on the number of segments of fragmentation $N$, $\beta = -6.5$, $\delta = 0.06$, $\nu = 3.5$.

| $N$ | 32  | 64  | 128 |
|-----|-----|-----|-----|
| $\|E - E^h_1\|_{L_{2,\nu}(\Omega)}$, $k = k_1$ | 0.0001249 | 0.0000650 | 0.0000333 |
| $\|E - E^h_1\|_{L_{2,\nu}(\Omega)}$, $k = k_2$ | 0.0001249 | 0.0000646 | 0.0000274 |

Results presented in table 1 are also visually depicted on figure 2.

Figure 2. Dependence of error $\|E - E^h_1\|_{L_{2,\nu}(\Omega)}$ on the number of segments of fragmentation $N$, $k = 220$ (a), $k = 300$ (b).

In table 2 for the determined approximate $R_\nu$-generalized solutions we present the numbers of points $n_1$ and $n_2$ (in percentage of their total number) where the errors $\delta_{1i} = |E_1(M_i) - E^h_{1\nu}(M_i)|$, $\delta_{2i} = |E_2(M_i) - E^h_{2\nu}(M_i)|$, $i = 1,2,\ldots, S_h$, are less than the given limit value $\bar{\Delta} = 0.001$.

On figure 3 we depict the distribution of the points $M_i$ with error $\delta_{1i}$ for the component $E^h_{1\nu}$ of the approximate $R_\nu$-generalized solution on different meshes. The figures for the component $E^h_{2\nu}$ are analogous.
Table 2. The numbers of points (in percentage of their total number) where the errors are less than the given limit value $\bar{\Delta} = 0.001$, $\beta = -6.5$, $\delta = 0.06$, $\nu = 3.5$.

| $N$  | 32  | 64  | 128 |
|------|-----|-----|-----|
| $n_1$, $k = k_1$ | 42.93% | 63.30% | 83.57% |
| $n_2$, $k = k_2$ | 42.93% | 69.28% | 83.40% |

Figure 3. Distribution of the points $M_i$ with error $\delta_{1i}$ for the component $E_{1\nu}^h$ of the approximate $R_{1\nu}$-generalized solution, $k = 220$ (a), $k = 300$ (b).

The series of numerical experiments showed that:

(i) approximate solution by a proposed FEM converges to an exact $R_{1\nu}$-generalized solution in the weight space $L_{2,\nu}(\Omega)$ with the first rate (see table 1 and figure 2), that is more than one and a half times better in comparison with the results obtained with some another methods (see [3], [6] and [12]);

(ii) the number of points where the errors are greater than the given limit value $\bar{\Delta}$ decreases with mesh refinement; the radius of the neighborhood containing these points decreases when $h$ decreases.
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