THE CLASSICAL BERNOULLI-EULER ELASTIC CURVE IN A MANIFOLD

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Abstract. In this study, we describe the classical Bernoulli-Euler elastic curve in a manifold by the property that the velocity vector field of the curve is harmonic. Then, a condition is obtained for the elastic curve in a manifold. Finally, we give an example which provides the condition mentioned in this paper and illustrate it with a figure.

Keywords: Energy; energy of a unit vector field; elastic curve.

1. Introduction

The history of the elastica or the elastic curve is very old and many researchers have worked on this issue, for example [6, 11]. One can study a bent thin rod and consider the energy it stores. The classical Euler-Bernoulli model assigns a numerical value to this energy, which is proportional to \( \int_{s}^{a} k^2(u)du \). The elastica is the critical point for this total squared curvature functional on regular curves with given boundary conditions [8].

In [1] the author calculated the energy of the Frenet vector fields in \( \mathbb{R}^n \), showing that the energy of the velocity vector field was \( E(V_1(s)) = \frac{1}{2} \int_{s}^{a} k_1^2(u)du \). By means of this result, we have seen that the speed vector field of the Bernoulli-Euler elastic curve is harmonic.

In this paper, using the above result, we give a condition for elastica on a manifold.

Definition 1.1. Let \((M, g)\) be a Riemann manifold and \(\alpha : I \to M\), be a unit speed curve.

If \(\{E_i\}_{i=1}^r\) is an orthonormal frame along \(\alpha\) and

\[
E_1 = \frac{d\alpha}{ds},
\]

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\[\nabla_0^n E_1 = k_1 E_2,\]
\[\nabla_0^n E_i = -k_{i-1} E_{i-1} + k_i E_{i+1}, \quad \forall i = 2, ..., r - 1\]
\[\nabla_0^n E_r = -k_{r-1} E_{r-1},\]

where \(k_1, ..., k_{r-1}\) are positive functions with a real value on \(I\), then \(\alpha\) is said to be an \(r\)-th order Frenet curve. These functions are called the curvature functions of the curve \(\alpha\).

**Proposition 1.1.** The connection map \(K : T(T^1M) \to T^1M\) verifies the following conditions.

1) \(\pi \circ K = \pi \circ d\pi\) and \(\pi \circ K = \pi \circ \tilde{\pi}\), where \(\tilde{\pi} : T(T^1M) \to T^1M\) is the tangent bundle projection.

2) For \(\omega \in T_x M\) and a section \(\xi : M \to T^1M\), we have
\[K(d\xi(\omega)) = \nabla_\omega \xi\]

where \(T^1M\) is the unit tangent bundle and \(\nabla\) is the Levi-Civita covariant derivative [3].

**Definition 1.2.** For \(\eta_1, \eta_2 \in T_\xi(T^1M)\), we define
\[(1.1) \quad g_S(\eta_1, \eta_2) = \langle d\pi(\eta_1), d\pi(\eta_2) \rangle + \langle K(\eta_1), K(\eta_2) \rangle.\]

This gives a Riemannian metric on tangent bundle \(TM\). As mentioned, \(g_S\) is called the Sasaki metric. The metric \(g_s\) makes the projection \(\pi : T^1M \to M\) a Riemannian submersion [3, 10].

**Definition 1.3.** Let \(f : (M, <, >) \to (N, h)\) be a differentiable map between Riemannian manifolds. The energy of \(f\) is given by

\[(1.2) \quad \mathcal{E}(f) = \frac{1}{2} \int_M (\sum_{a=1}^n h(df(e_a), df(e_a)))v\]

where \(v\) is the canonical volume form in \(M\) and \(\{e_a\}\) is a local basis of the tangent space (see [12, 4], for example).

By a (smooth) variation of \(f\) we mean a smooth map \(f : M \times (-\epsilon, \epsilon) \to N, \ (x, t) \to f_t(x) \ (\epsilon > 0)\) such that \(f_0 = f\). We can think of \(\{f_t\}\) as a family of smooth mappings which depend ‘smoothly’ on a parameter \(t \in (-\epsilon, \epsilon)\).

**Definition 1.4.** A smooth map \(f : (M, g) \to (N, h)\) is said to be harmonic if
\[\left.\frac{d}{dt}\mathcal{E}(f_t; D)\right|_{t=0} = 0\]

where \(\mathcal{E}(f; D) = \frac{1}{2} \int_D (\sum_{a=1}^n h(df(e_a), df(e_a)))v_g\), for all compact domains \(D\) and all smooth variations \(f_t\) of \(f\) supported in \(D\), [2].
Definition 1.5. Let $\alpha : [a, b] \to \mathbb{R}^n$ be a regular curve. Elastica is defined for the curve $\alpha$ over the each point on a fixed interval $[a, b]$ as a minimizer of the bending energy:

$$E_B = \frac{1}{2} \int_a^b k_1^2(s) ds,$$

with some boundary conditions [5, 7].

The right side of Equation (1.3) is the energy of the velocity vector field according to [1]. By combining this resultant with the definition 1.4 we can give the following definition

2. Elastica in a Manifold

Definition 2.1. A curve on a manifold is called a classical Bernoulli-Euler elastic curve if the velocity vector field of the curve is harmonic.

Theorem 2.1. Let $M$ be a Riemann manifold, $\alpha$ be r-th order Frenet curve in $M$ and $\alpha(a) = p$, $\alpha(b) = q$. If $\alpha$ is classical elastic curve, then the following equation is satisfied,

$$\int_a^b \lambda(s) k_1(s) k_1'(s) ds = 0$$

where $k_1$ is the 1st curvature function and $\lambda$ is the real-valued function on $[a, b]$.

Proof. Let $\alpha : I \to M$ be the r-th order Frenet curve $C$ on $\varphi(U) \subset M$ and $\alpha = \varphi \circ \gamma$, $\gamma = (\gamma_1, ..., \gamma_m), \gamma : I \to U \subset \mathbb{R}^m; \varphi : U \to M$. Let $\{E_i\}_{i=1}^m$ be the Frenet frame field on $\alpha$.

We define the $\lambda$ and $v_i$ functions to create a curve family between two fixed points on the manifold. The functions are: $\lambda : [a, b] \subset I \to R$, $\lambda(s) = (s-a)(b-s)$, $\lambda(a) = 0$, $\lambda(b) = 0$ and $\lambda(s) \neq 0$ for all $s \in (a, b)$, of class $C^2$ and

$$\lambda(s) E_1(s) = (v_1(s), v_2(s), ..., v_n(s)).$$

Since $\{\varphi_1(\gamma(s)), ..., \varphi_m(\gamma(s))\}$ is a local basis of the tangent space, where $\varphi_1, ..., \varphi_m$ are first-order partial derivatives, we have

$$\lambda(s) E_1(s) = \sum_{i=1}^m v_i(s) \varphi_i(\gamma(s)); \text{ where } v_i : [a, b] \to R.$$

Let the collection of the curve be

$$\alpha'(s) = \varphi(\gamma_1(s) + tv_1(s), ..., \gamma_m(s) + tv_m(s)),$$
for \( t = 0, \) \( \alpha^0(s) = \alpha(s) \) and
\[
(\varphi^{-1} \circ \alpha')(s) = \gamma'(s) = (\gamma_1(s) + tv_1(s), ..., \gamma_m(s) + tv_m(s)).
\]

From (2.2) we get \( \lambda(a)E_1(a) = \sum_{i=1}^m v_i(a)\phi_i(\gamma(a)). \) Since \( \lambda(a) = 0 \) we have \( v_i(a) = 0 \) and
\[
\gamma'(a) = (\gamma_1(a) + tv_1(a), ..., \gamma_m(a) + tv_m(a) = (\gamma_1(a), ..., \gamma_m(a)) = \gamma(a).
\]

Similarly, we get \( \gamma'(b) = \gamma(b) \). Using these results in (2.3) we obtain
\[
\alpha'(a) = (\varphi \circ \gamma')(a) = \alpha(a) = p \quad \text{and} \quad \alpha'(b) = (\varphi \circ \gamma')(b) = \alpha(b) = q.
\]

These results show that \( \alpha' \) is a curve segment from \( p \) to \( q \) on \( M \). Take this collection \( \alpha'(s) = \alpha(s, t) \) for all curves. The expression for the energy of the velocity vector field \( E_1 \) of \( \alpha' \) from \( p \) to \( q \) on \( M \) becomes \( \mathcal{E}(E_1) \).

Let \( TC_t \) be the tangent bundle. So we have \( E_1_t : C_t \to TC_t \), where \( TC_t = \bigcup_{j \in J} T \alpha'(j)C_1 \), \( C_t = \alpha'(I) \) and \( T \alpha'(j)C_t \) is the straight line through the point \( \alpha'(j) \) in the \( E_1 \) direction. Let \( \pi : TC_t \to C_t \) be the bundle projection. By using Equation (1.2) we calculate the energy of \( E_1 \) as
\[
(2.4) \quad \mathcal{E}(E_1) = \frac{1}{2} \int_a^b g_S(dE_1(E_1, \alpha(s, t)), dE_1(E_1, \alpha(s, t)))ds
\]
where \( ds \) is the element arc length. From (1.1) we have
\[
g_S(dE_1(E_1, dE_1(E_1))) = < d\pi(dE_1(E_1)), d\pi(dE_1(E_1)) > + < K(dE_1(E_1)), K(dE_1(E_1)) >.
\]

Since \( E_1 \) is a section, we have \( d(\pi) \circ d(E_1) = d(\pi \circ E_1) = d(id_{C_1}) = id_{TC_1} \). By Proposition 1.1, we also have that
\[
K(dE_1(E_1)) = \nabla^2_{E_1} E_1 = E_1' = \frac{\partial E_1}{\partial s}.
\]

Using these results in (2.4) we get
\[
(2.5) \quad \mathcal{E}(E_1) = \frac{1}{2} \int_a^b (< E_1, E_1 > + < E_1', E_1' >)ds
\]

By Definition 1.4, if \( E_1 \) is a harmonic, then \( t = 0 \) should be the critical point of \( \mathcal{E}(E_1) \). Supposing that \( \frac{\partial \mathcal{E}(E_1)}{\partial t} \big|_{t=0} = 0 \), from (2.5) we obtain:
\[
\frac{\partial \mathcal{E}(E_1)}{\partial t} = \frac{1}{2} \int_a^b ( < E_1, E_1 > + < E_1', E_1' > )ds \\
= \frac{1}{2} \int_a^b \frac{\partial}{\partial t} [ ( < E_1, E_1 > + < E_1', E_1' > )ds].
\]
Since $<E_1,E_1>=1$ we have $\frac{\partial}{\partial t} <E_1,E_1>=0$ and we get

$$
(2.6) \frac{\partial\mathcal{E}(E_{1t})}{\partial t} = \frac{1}{2} \int_a^b \frac{\partial}{\partial t} <\frac{\partial E_{1t}}{\partial s}, \frac{\partial E_{1t}}{\partial s}> ds = \int_a^b <\frac{\partial^2 E_{1t}}{\partial s \partial t}, \frac{\partial E_{1t}}{\partial s}> ds.
$$

We can write

$$
\frac{\partial}{\partial s} <\frac{\partial E_{1t}}{\partial t}, \frac{\partial E_{1t}}{\partial s}> = <\frac{\partial^2 E_{1t}}{\partial s \partial t}, \frac{\partial E_{1t}}{\partial s}>.
$$

Thus, we can deduce,

$$
(2.7) <\frac{\partial^2 E_{1t}}{\partial s \partial t}, \frac{\partial E_{1t}}{\partial s}> = \frac{\partial}{\partial s} <\frac{\partial E_{1t}}{\partial t}, \frac{\partial E_{1t}}{\partial s}> - \frac{\partial E_{1t}}{\partial t} \frac{\partial^2 E_{1t}}{\partial s^2}.
$$

Substituting (2.7) in (2.6), for $t=0$, we have

$$
\frac{\partial\mathcal{E}(E_{1t})}{\partial t} \bigg|_{t=0} = \int_a^b \frac{\partial}{\partial s} <\frac{\partial E_{1t}}{\partial s}(s,0), \frac{\partial E_{1t}}{\partial s}(s,0)> - <\frac{\partial E_{1t}}{\partial t}(s,0), \frac{\partial^2 E_{1t}}{\partial s^2}(s,0)> ds
$$

and

$$
(2.8) \frac{\partial\mathcal{E}(E_{1t})}{\partial t} \bigg|_{t=0} = <\frac{\partial E_{1t}}{\partial t}(s,0), \frac{\partial E_{1t}}{\partial s}(s,0)>|_a^b
$$

$$
- \int_a^b <\frac{\partial E_{1t}}{\partial t}(s,0), \frac{\partial^2 E_{1t}}{\partial s^2}(s,0)> ds.
$$

From (2.2) and (2.3), we obtain,

$$
(2.9) \quad \frac{\partial}{\partial t}(s,t) = \lambda(s)E_1(s).
$$

and

$$
(2.10) \quad \frac{\partial}{\partial s}(s,t)|_{t=0} = \alpha'(s) = E_1(s).
$$

Now we calculate the partial derivatives of (2.10) with respect to $s$ and $t$; using Frenet formulas, we get

$$
(2.11) \quad \frac{\partial E_{1t}}{\partial s}(s) = \frac{\partial^2}{\partial s^2}(s,t)|_{t=0} = \alpha''(s) = E'_1(s) = k_1(s)E_2(s)
$$

and

$$
\frac{\partial E_{1t}}{\partial t}(s,t) = \frac{\partial^2}{\partial s \partial t}(s,t) = \frac{\partial^2}{\partial t \partial s}(s,t).
$$

From (2.9), we have

$$
(2.12) \quad \frac{\partial E_{1t}}{\partial t}(s,t)|_{t=0} = \frac{\partial E_{1t}}{\partial t}(s,0) = \lambda'(s)E_1(s) + \lambda(s)k_1(s)E_2(s).
$$
It follows from (2.11) and (2.12) that
\[ < \frac{\partial E_1(t,s)}{\partial t}, \frac{\partial E_1(s,t)}{\partial s} > = \lambda(s)k_1^2(s). \]

Considering the candidate function \( \lambda(a) = \lambda(b) = 0 \), we get:

\[ < \frac{\partial E_1}{\partial t}(s,0), \frac{\partial E_1}{\partial s}(s,0) > |_{a}^{b} = \lambda(b)k_1^2(b) - \lambda(a)k_1^2(a) = 0. \]

From (2.11), we get
\[ \frac{\partial^2 E_1}{\partial s^2}(s,0) = -k_1^2(s)E_1(s) + k_1(s)E_2(s) + k_1(s)k_2(s)E_3(s) \]

Therefore, (2.12) and (2.14) gives
\[ < \frac{\partial E_1}{\partial t}(s,0), \frac{\partial^2 E_1}{\partial s^2}(s,0) >= [-\lambda(s)k_1^2(s)]' + 3\lambda(s)k_1(s)k_1'(s) \]

Substituting (2.13) and (2.15) in (2.8) yields
\[ \left. \frac{\partial \mathcal{E}(E_1)}{\partial t} \right|_{t=0} = - \int_{a}^{b} \left( [-\lambda(s)k_1^2(s)]' + 3\lambda(s)k_1(s)k_1'(s) \right) ds = 0 \]

and
\[ \left. \frac{\partial \mathcal{E}(E_1)}{\partial t} \right|_{t=0} = [\lambda(s)k_1^2(s)] |_{a}^{b} - 3 \int_{a}^{b} \lambda(s)k_1(s)k_1'(s) ds = 0 \]

We are looking the candidate function \( \lambda(a) = \lambda(b) = 0 \),
which given \( [\lambda(s)k_1^2(s)] |_{a}^{b} = 0 \) and
\[ \left. \frac{\partial \mathcal{E}(E_1)}{\partial t} \right|_{t=0} = -3 \int_{a}^{b} \lambda(s)k_1(s)k_1'(s) ds = 0 \]

This completes the proof of the theorem. ■

**Example 1.** Let \( \varphi : R^2 \rightarrow R^3, \varphi = (x, y, \frac{1}{2}xy) \), \( \varphi(R^2) = M \) and \( \alpha(s) = (3s, s^2, s^3) \).
If we can choose \( \lambda : [-10, 10] \rightarrow R, \lambda(s) = 10^2 - s^2 \) then \( \lambda(-10) = 0\lambda(10) = 0 \) and \( \lambda(s) \neq 0 \) for all \( s \in (-10, 10) \). We calculate
\[ k_1(s) = \frac{6\sqrt{s^4 + 9s^2 + 1}}{(\sqrt{9s^4 + 4s^2 + 9})^3}, \]
\[ k_1'(s) = 6\frac{2s^3 + 9s}{(\sqrt{9s^4 + 4s^2 + 9})^3} - 3\sqrt{s^4 + 9s^2 + 1}(\sqrt{9s^4 + 4s^2 + 9})^2(35s^3 + 8s) \]
\[ \frac{(9s^4 + 4s^2 + 9)^3}{(9s^4 + 4s^2 + 9)^3}, \]
and

\[
\frac{\partial \mathcal{E}(T_k)}{\partial k} \bigg|_{k=0} = -\int_{-10}^{10} (10^2 - s^2) k_1(s) k_1'(s) ds = 0.
\]

Thus \( \alpha \) is an elastica on \( M \), Figure 2.1.

**Conclusion.** In this paper, we have determined the classical Bernoulli-Euler elastic curve that is the harmonic of the velocity vector field of the curve on a manifold. We have obtained the collection of curves passing through \( p \) and \( q \) points using \( \lambda \) and \( v_i \) functions on the manifold. We have also proposed a novel condition to be the classical Bernoulli-Euler elastic curve in the collection of curves. In the end, we have given an example of the elastic curve satisfying the novel condition on a two-dimensional manifold and shown the graphs of both the manifold and the elastic curve.

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