Elastic flow of networks: short-time existence result

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Abstract. In this paper we study the $L^2$-gradient flow of the penalized elastic energy on networks of $q$-curves in $\mathbb{R}^n$ for $q \geq 3$. Each curve is fixed at one end-point and at the other is joint to the other curves at a movable $q$-junction. For this geometric evolution problem with natural boundary condition we show the existence of smooth solutions for a (possibly) short interval of time. Since the geometric problem is not well-posed, due to the freedom in reparametrization of curves, we consider a fourth-order non-degenerate parabolic quasilinear system, called the analytic problem, and show first a short-time existence result for this parabolic system. The proof relies on applying Solonnikov’s theory on linear parabolic systems and Banach fixed point theorem in proper Hölder spaces. Then the original geometric problem is solved by establishing the relation between the analytical solutions and the solutions to the geometrical problem.

1. Introduction

The elastic energy of a smooth regular curve immersed in $\mathbb{R}^n$, $f : \bar{I} \rightarrow \mathbb{R}^n$, $n \geq 2$, $I = (0, 1)$, is given by

$$\mathcal{E}(f) = \frac{1}{2} \int_I |\vec{\kappa}|^2 dx,$$

where $ds = |\partial_x f| dx$ is the arc-length element and $\vec{\kappa}$ is the curvature vector of the curve. The latter is given by $\vec{\kappa} = \partial_x^2 f$ where $\partial_x = |\partial_x f|^{-1} \partial_x$ denotes the differentiation with respect to the arclength parameter. The elastic energy in (1.1) is also called bending energy of curves. It was proposed by Jacob Bernoulli in 1691 for studying the equilibrium shape of curves, called elasticae or elastic curves, [21]. Besides being used as a simple model in mechanics, the elastic energy has also been used for defining and studying the so-called nonlinear splines in computer graphics, see e.g., [15] and the references therein.

Since the elastic energy of a curve can be made arbitrarily small by enlarging the curve, in minimization problems one usually penalizes the length or consider curves with fixed length. In the first case one is led to consider the energy

$$\mathcal{E}_\lambda(f) = \mathcal{E}(f) + \lambda \mathcal{L}(f), \quad \lambda > 0,$$
where
\[ L(f) = \int_I ds \]
is the length of the curve. The term \( \lambda L(f) \), when \( \lambda > 0 \), in (1.2) is a natural term to be considered, since it could be viewed as the energy naively responsible for the stretching of curves in elasticity.

In both cases, critical points of the energy satisfy the equation
\[ \nabla_{L^2} E_\lambda(f) = \nabla_s^2 \vec{\kappa} + \frac{1}{2} |\vec{\kappa}|^2 \vec{\kappa} - \lambda \vec{\kappa} = 0, \]
where in the case of fixed length \( \lambda \) is a Lagrange multiplier (see for instance [9], [18]). Here \( \nabla_s \) is an operator that on a smooth vector field \( \phi \) acts as follows \( \nabla_s \phi = \partial_s \phi - \langle \partial_s \phi, \partial_s f \rangle \partial_s f \), i.e., it is the normal projection of \( \partial_s \phi \). It may also be understood as a covariant differentiation.

The attempt to associate the elastic energy to networks appears in many disciplines, for examples, mechanical studies of polymer gels, fiber and protein networks in material sciences (e.g. [3,14]), and interpolation of surfaces by meshes formed by curve networks minimizing elastic energy in Computer Aided Geometric Design (e.g. [20]). These attempts motivate the study of the elastic energy on networks in particular in 3-dimensional spaces in mathematical analysis and numerical analysis.

The mathematical treatment of networks with elastic energy has only started quite recently. In [5,8] the authors provide first results concerning the existence of minimizers in special classes of networks (in particular an angle condition is imposed at the junction). Here we look at the steepest descent flow of the elastic energy on networks of \( q \) curves, \( q \geq 3 \), starting from a point (the \( q \)-junction) and ending at \( q \) fixed points in \( \mathbb{R}^n \). In this setting we assign orientation to each curve, although the energy is independent of the orientation of the curves. In other words, the network \( f = \{f_1, f_2, \ldots, f_q\} \), where \( f_i : \bar{I} \to \mathbb{R}^n, \bar{I} = (0, 1), i \in \{1, \ldots, q\} \), are \( q \) regular curves, satisfies the followings:

1. The end-points are fixed:
\[ f_i(1) = P_i \text{ for } i \in \{1, \ldots, q\}, \quad (1.3) \]

with given points \( P_i, i \in \{1, \ldots, q\}, \) in \( \mathbb{R}^n \).

2. The curves start at the same point
\[ f_i(0) = f_j(0) \text{ for all } i, j \in \{1, \ldots, q\} \quad (\text{concurrence condition}). \quad (1.4) \]

We write \( \Gamma = \{f_1, f_2, \ldots, f_q\} \) when we think of the network as a geometrical object, that is when the parametrization chosen for each curve plays no role. The energy of the network \( \Gamma = \{f_1, \ldots, f_q\} \) is given by
\[ E_\lambda(\Gamma) = \sum_{i=1}^q E_\lambda(f_i), \quad (1.5) \]
where $\lambda = (\lambda_1, \ldots, \lambda_q)$, $\lambda_i \geq 0$ (the penalization of the length is obviously not necessary for a short time existence result).

We call the above configuration a $q$-network. The aim of this work is to complete our work undertaken in [4], where we analyse the long time behaviour for the elastic flow of triods (3-networks). More precisely, we give here full details on the short-time existence result exploited in [4]. At the same time we generalize the needed short-time existence statement to the case of $q$-networks for $q \geq 3$. A short-time existence result for the elastic evolution of networks appeared first in [11]: there the planar case for triods is discussed and the existence is demonstrated in $C^{4+\alpha,4+\alpha}$ spaces. For our arguments in [4] to be complete we need however a statement for networks with curves in $\mathbb{R}^n$ whose parametrization is smooth is space and time up to time zero. As it turns out, we are able to demonstrate what is needed, independently of the number ($q \geq 3$) of curves meeting at the junction. Our approach to the problem is inspired by the method used by Bronsard and Reitich in their seminal work [2] for the short time existence of the motion by curvature of a planar network (see the more recent monumental work [16] on the subject). New ideas are needed since the elastic flow is of fourth order and the ambient space is $\mathbb{R}^n$.

For an overview on the current research undertaken on the elastic flow of networks, we refer the interested reader to [4, 12, 17] and the references given there. In particular, this problem was proposed and studied numerically in [1].

**Main results** The aim of this work is to establish a short time existence result for the $L^2$-gradient flow of the energy $E_\lambda$ of a network as described above. In other words, given an initial $q$-network $\Gamma_0 = \{f_{0,1}, \ldots, f_{0,q}\}$ of sufficiently smooth regular curves satisfying (1.3) and (1.4), we look for the existence of $\forall T > 0$ and $f_i : [0,T] \times [0,1] \to \mathbb{R}^n$, $f_i \in C^{4+\alpha,4+\alpha}([0,T] \times [0,1])$, $k \in \mathbb{N}$, $k \geq 4$, $\alpha \in (0,1)$ (resp. $f_i \in C^\infty([0,T] \times [0,1])$, see “Appendix B” for the definition of the parabolic Hölder spaces) for $i \in \{1, \ldots, q\}$, regular curves and solution to

$$(\partial_t f_i)^+ = -\nabla_s^2 \vec{k}_i - \frac{1}{2} |\vec{k}_i|^2 \vec{k}_i + \lambda_i \vec{k}_i, \quad i \in \{1, \ldots, q\},$$

with initial datum $f_i(t=0) = f_{0,i}$ and boundary conditions

$$\begin{align*}
\begin{cases}
    f_i(t,1) = P_i, & \forall t \in [0,T], i \in \{1, \ldots, q\}, \\
    \vec{k}_i(t,1) = 0 = \vec{k}_i(t,0) & \forall t \in [0,T], i \in \{1, \ldots, q\}, \\
    f_i(t,0) = f_j(t,0) & \forall t \in [0,T], i, j \in \{1, \ldots, q\}, \\
    \sum_{i=1}^q (\nabla_s \vec{k}_i(t,0) - \lambda_i \partial_s f_i(t,0)) = 0 & \forall t \in [0,T].
\end{cases}
\end{align*}$$

(1.6)

As usual $(\partial_t f)^+$ denotes the normal part of the velocity, i.e. $(\partial_t f)^+ = \partial_t f - \langle \partial_t f, \partial_{s} f \rangle \partial_{s} f$. The first and third line in (1.6) ensure that during the flow the network satisfies (1.3) and (1.4), while the other boundary conditions are the so called natural ones, derived by imposing that the first variation of the energy is zero. For the
derivation of the first variation and the natural boundary conditions in the case $q = 3$, the readers are referred to Section 2 of [4] (see also “Appendix A” below). The case of general $q$ goes similarly.

For the initial datum $\Gamma_0 = \{f_{0,1}, \ldots, f_{0,q}\}$, we assume that $f_{0,i} \in C^{k,\alpha}([0,1], \mathbb{R}^n)$, $k \geq 4$, $\alpha \in (0,1)$ (resp. $f_{0,i} \in C^\infty([0,1], \mathbb{R}^n)$, $i \in \{1, \ldots, q\}$, are regular curves satisfying
\[
\begin{align*}
    f_{0,i}(1) &= P_i, && \text{for all } i \in \{1, \ldots, q\}, \\
    \kappa_{0,i}(1) &= 0 = \kappa_{0,i}(0), && \text{for all } i \in \{1, \ldots, q\}, \\
    f_{0,i}(0) &= f_{0,j}(0) && \text{for all } i, j \in \{1, \ldots, q\}, \\
    \text{and } \sum_{i=1}^q (\nabla_s \kappa_{0,i}(0) - \lambda_i \partial_s f_{0,i}(0)) &= 0,
\end{align*}
\]
(1.7)
(where $\kappa_{0,i}$ denotes the curvature of $f_{0,i}$) and with further compatibility conditions (specified in the statements below). Furthermore, the initial datum has to satisfy the following non-collinearity condition.

**Definition 1.1.** (Non-collinearity condition (NC)) We say that the initial datum satisfies the non-collinearity condition if
\[
\dim \text{span}\{\partial_s f_{0,1}, \ldots, \partial_s f_{0,q}\}_{|x=0} \geq 2.
\]
Similarly, a family of regular curves $f_i : [0, T] \times [0, 1] \to \mathbb{R}^n$, $i \in \{1, \ldots, q\}$, satisfies the non-collinearity condition if
\[
\dim \text{span}\{\partial_s f_i(t, x), \ldots, \partial_s f_q(t, x)\}_{|x=0} \geq 2 \text{ for all } t \in [0, T].
\]

**Remark 1.1.** The non-collinearity condition (NC) establishes that the $q$ unit tangent vectors at the $q$-junction should not span a one-dimensional subspace. Analytically and equivalently, we can express this fact by considering the (geometric) expression $nc : [0, T] \times [0, 1] \to \mathbb{R}$,
\[
nc(t, x) = 1 - \prod_{1 \leq i < j \leq q} |\langle \partial_s f_i(t, x), \partial_s f_j(t, x) \rangle|,
\]
and asking that $nc$ is strictly positive at the junction point $x = 0$.

As we will see below, the non-collinearity condition is necessary in our analysis in order to guarantee the short time existence of a solution. Moreover it has been used also in [4] to prove long-time existence (in the case $q = 3$). Note that in case $q = 3$, then $nc$ is simply given by
\[
nc = 1 - \langle \partial_s f_1(t, x), \partial_s f_2(t, x) \rangle \langle \partial_s f_1(t, x), \partial_s f_3(t, x) \rangle \langle \partial_s f_2(t, x), \partial_s f_3(t, x) \rangle,
\]
which has also appeared in [11]. In [4, § 5], it is shown that the non-collinearity condition arises naturally when imposing $\partial_t f_i = \partial_t f_j$ at the junction: in particular if
nc > 0 holds, which is equivalent to the invertibility of the matrix Q in (1.12), then at the boundary the tangential components of the velocity vectors (that is \(\langle \partial_t f_i, \partial_s f_i \rangle\)) can be expressed in purely geometric terms by solving the algebraic equation (1.12). The reader is referred to Remark 1.6 below for the arguments and the generalization to the case of \(q\) curves.

Observe that the formulation of the problem given so far involves purely geometric quantities and hence it is invariant under reparametrizations.

In order to treat the problem analytically and to keep the topology of the network with movable junction point during the evolution, we need to allow some tangential components in the flow equations. Hence, we rewrite the flow equations as

\[
\partial_t f_i = -\nabla_s^2 \vec{k}_i - \frac{1}{2} |\vec{k}_i|^2 \vec{k}_i + \lambda_i \vec{k}_i + \varphi_i \partial_s f_i \quad \text{on} \quad (0, T) \times I \quad \text{for} \quad i \in \{1, \ldots, q\},
\]

for some (sufficiently smooth, that is \(\varphi_i \in C^{\frac{k+\alpha-4}{4}+\alpha}([0, T] \times [0, 1])\) resp. \(\varphi_i \in C^\infty([0, T] \times [0, 1])\)) tangential components \(\varphi_i = \langle \partial_t f_i, \partial_s f_i \rangle\), which are part of the problem.

Our main result reads as follows.

**Theorem 1.2.** (Geometric existence Theorem) Let \(n \geq 2, q \geq 3, \alpha \in (0, 1)\) and \(P_i, i \in \{1, \ldots, q\}\), be given points in \(\mathbb{R}^n\). Given \(f_{0,i} : [0, 1] \rightarrow \mathbb{R}^n, f_{0,i} \in C^{4,\alpha}([0, 1]), i \in \{1, \ldots, q\}\), regular curves, satisfying the non-collinearity condition (NC), (1.7), and

\[
\nabla_s^2 \vec{k}_{0,i} = 0 \quad \text{at} \quad x = 1, \quad i \in \{1, \ldots, q\},
\]

\[
-\nabla_s^2 \vec{k}_{0,i} + \varphi_{0,i} \partial_s f_{0,i} = -\nabla_s^2 \vec{k}_{0,j} + \varphi_{0,j} \partial_s f_{0,j} \quad \text{at} \quad x = 0 \quad \text{for} \quad i, j \in \{1, \ldots, q\},
\]

(1.9)

with \(\varphi_{0,i}(0) = \varphi_i(t = 0, 0), \varphi_i\) given in (1.13) below, then there exist \(T > 0\) and regular curves \(f_i \in C^{4,\alpha,4+\alpha}([0, T] \times I; \mathbb{R}^n), i \in \{1, \ldots, q\}\), such that

\[
(\partial_t f_i) = -\nabla_s^2 \vec{k}_i - \frac{1}{2} |\vec{k}_i|^2 \vec{k}_i + \lambda_i \vec{k}_i, \quad i \in \{1, \ldots, q\},
\]

(1.10)

together with the boundary conditions (1.6) and the initial condition \(\Gamma = \{f_1, \ldots, f_q\}|_{t=0} = \Gamma_0 = \{f_{0,1}, \ldots, f_{0,q}\}\) that is

\[
f_i(t = 0) = f_{0,i} \circ \psi_i, \quad i \in \{1, \ldots, q\},
\]

(1.11)

with \(\psi_i \in C^{4,\alpha}([0, 1], [0, 1])\), orientation preserving diffeomorphisms. Moreover, we have instant parabolic smoothing, that is \(f_i \in C^\infty((0, T] \times [0, 1])\) for any \(i \in \{1, \ldots, q\}\), and the non-collinearity condition holds at the triple junction for any time \(t \in [0, T]\).

It turns out that also (1.10) is a fully geometric condition, as discussed in detail in Remark 1.6 (cf. also Remark 1.1 above). Since the problem and formulation are fully
Upon imposing higher regularity and stronger compatibility condition for the initial data, we can obtain a smooth solution. Precisely

**Theorem 1.3. (Smooth Geometric existence Theorem)** Let \( n \geq 2, q \geq 3 \), and \( P_i, i \in \{1, \ldots, q\} \), be given points in \( \mathbb{R}^n \). Given \( f_{0,i} : [0, 1] \to \mathbb{R}^n, f_{0,i} \in C^\infty([0, 1]), i \in \{1, \ldots, q\} \), regular curves which, when appropriately reparamerized, satisfy the compatibility conditions of any order (as stated in Remark 3.5 below) and the non-collinearity condition (NC), then there exist \( T > 0 \) and regular curves \( f_i \in C^\infty([0, T] \times [0, 1]; \mathbb{R}^n), i \in \{1, \ldots, q\}, \) such that

\[
(\partial_t f_i)^\perp = -\nabla_s^2 \kappa_i - \frac{1}{2} |\kappa_i|^2 \kappa_i + \lambda_i \kappa_i, \quad i \in \{1, \ldots, q\},
\]

together with the boundary conditions (1.6) and the initial condition \( \Gamma = \{f_1, \ldots, f_q\} \) equal to \( \Gamma_0 = \{f_{0,1}, \ldots, f_{0,q}\} \) (in the sense of (1.11) for smooth orientation preserving diffeomorphisms). Moreover, the non-collinearity condition (NC) holds at the \( q \)-junction for any time \( t \in [0, T] \).

**Remark 1.4.** Note that Theorems 1.2 and 1.3 are said to be geometric since the result does not depend on the specific parametrization of curves/networks. Moreover, in order to find a solution we will consider another system of equations where, in particular, the tangential component of the evolution is specified. This second problem we call, as usually done in the literature, the analytic problem, see Theorem 2.3 below.

**Remark 1.5.** In order to be able to use the expression “network” to describe our geometrical setting we restrict ourself to the case of \( q \) curves with \( q \geq 3 \), but as the analysis below shows, all arguments used are still valid also in the case of \( q = 2 \). Moreover, the analysis performed below can be easily generalized to \( q \)-networks with curves meeting in two \( q \)-junctions (the so called theta-networks when \( q = 3 \)), therefore the previous results hold also for this configuration.

The problem of geometric uniqueness is briefly treated in Lemma 4.1.

**Remark 1.6.** In the statement of Theorem 1.2 we ask that the initial datum satisfies (1.10). This condition is geometrical (i.e. independent of the choice of parametrization) since the non-collinearity condition (NC) holds along the flow. This has been observed and exploited already in [4, Rem.5.1] in the case \( q = 3 \). For general \( q \) the argument goes as follows. If \( f_i, \varphi_i, i = 1, 2, \ldots, q \), solve (1.8), (1.6), (1.7) in the sense of Theorem 1.2 (in particular also the compatibility conditions of order zero are satisfied), then at \( x = 0 \) we have that for any \( t \in [0, T] \), \( \partial_t f_i = \partial_t f_j \), that is

\[
-A_i + \varphi_i T_i = -A_j + \varphi_j T_j
\]

for any \( i, j \in \{1, \ldots, q\} \), where for brevity of notation we write

\[
A_i = A_i(t) := \nabla_s^2 \kappa_i \bigg|_{x=0}, \quad T_i = T_i(t) = \partial_s f_i(t, 0),
\]
and where we have used the fact that the curvature vanishes at the boundary. Taking the scalar product with $T_i$ gives $\varphi_i = -\langle A_j, T_i \rangle + \varphi_j \langle T_j, T_i \rangle$, and summing up yields

$$(q - 1)\varphi_i = \sum_{j \neq i, j=1}^{q} \varphi_i = \sum_{j \neq i, j=1}^{q} (-\langle A_j, T_i \rangle + \varphi_j \langle T_j, T_i \rangle).$$

In other words, for any $i \in \{1, \ldots, q\}$ we have

$$(q - 1)\varphi_i - \sum_{j \neq i, j=1}^{q} \varphi_j \langle T_j, T_i \rangle = \sum_{j \neq i, j=1}^{q} \langle A_j, T_i \rangle,$$

which can be written as

$$Q \cdot \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_q \end{pmatrix} = \begin{pmatrix} \langle \sum_{j=1}^{q} A_j, T_1 \rangle \\ \langle \sum_{j=2}^{q} A_j, T_2 \rangle \\ \vdots \\ \langle \sum_{j=q}^{q} A_j, T_q \rangle \end{pmatrix},$$

where

$$Q = \begin{pmatrix} (q - 1) & -\langle T_1, T_2 \rangle & -\langle T_1, T_3 \rangle & \cdots & -\langle T_1, T_q \rangle \\ -\langle T_2, T_1 \rangle & (q - 1) & -\langle T_2, T_3 \rangle & \cdots & -\langle T_2, T_q \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\langle T_q, T_1 \rangle & -\langle T_q, T_2 \rangle & -\langle T_q, T_3 \rangle & \cdots & (q - 1) \end{pmatrix}.$$  

We see that the submatrix $(Q_{ij})_{i,j=1}^{q-1} \in \mathbb{R}^{(q-1) \times (q-1)}$ composed out of the first $(q - 1)$ rows and $(q - 1)$ columns is strictly diagonal dominant and hence invertible. This in turns implies that the first $(q - 1)$ columns of $Q$ are linearly independent and hence $\text{rank}(Q) \geq q - 1$. The rank of $Q$ is precisely $q - 1$ if we can write the last column as a linear combination of the first $(q - 1)$, that is if $Q$ has zero as an eigenvalue. Hence, suppose there exists $v \in \mathbb{R}^q$, $v \neq 0$, such that $Qv = 0$. Let $w = \frac{v}{\|v\|_\infty}$. Then $Qw = 0$ and $|w_i| \leq 1$ for any $i = 1, \ldots, q$. Possibly multiplying $w$ with $-1$ we obtain the existence of an entry $w_j$ such that $w_j = 1$. From $(Qw)_j = 0$ it follows then

$$(q - 1) = \sum_{i \neq j, i=1}^{q} w_i \langle T_i, T_j \rangle$$

Since $|w_i \langle T_i, T_j \rangle| \leq |\langle T_i, T_j \rangle| \leq 1$ this can not be realized if $\text{dim}(T_1, \ldots, T_q) \geq 2$. Hence it becomes clear that the validity of the non-collinearity condition (NC) at time $t \in [0, T)$ ensures the invertibility of $Q$ and thus also the fact that $\varphi_i(t, 0)$, $i = 1, \ldots, q, t \in [0, T)$ can be expressed in geometrical terms at the junction point, namely

$$\varphi_i(t, x = 0) = \varphi_i(\partial_s f_j(t, 0), \nabla_s^2 \kappa_j(t, 0), j = 1, 2, \ldots, q)$$

(1.13)

where the exact expression can be immediately deduced from (1.12).
Structure of the article

In the next section we give the analytical problem which we are going to solve. We consider (1.8) with a specific choice of the tangential component $\varphi_i$ (cf. (2.1) below) and with boundary conditions a bit stronger than (1.6). These choices are dictated by the need to obtain a (fourth order) non-degenerate system of quasilinear PDEs. In Sect. 3, we give the proof of the short-time existence for the non-degenerate parabolic system of fourth-order, solving the analytic problem, by applying Solonnikov’s theory on linear parabolic systems and Banach fixed point theorem in proper Hölder spaces. Our approach to the problem is inspired by [2] and similar ideas are used in [11] for the case of planar networks of three curves. The approach we chose is typical for this kind of problems. We aim in giving a clear and complete exposition and at the same time considering an arbitrary number of curves and any codimension. In Sect. 4 we discuss the relation between the analytical solutions obtained in Sect. 2 and the solutions to the geometrical problem we are interested in. In the “Appendix” we collect some useful results.

2. The analytical problem

As already observed, the geometrical problem we want to study is not well posed due to the freedom given by the invariance with respect to reparametrizations. This is why, we consider now a fourth order (non-degenerate) system of quasilinear PDEs for which we prove existence of a solution.

In the flow equations (1.8) only the normal components of derivatives with respect to arc-length appear and hence the operator is not uniformly elliptic. From formula (A4) we see that by choosing tangential components

$$\varphi_i^* = -\left(\frac{\partial f_i}{\partial x f_i}^4, \frac{\partial f_i}{\partial x f_i}^5, \frac{\partial f_i}{\partial x f_i}^6\right) + 10 \frac{\partial f_i}{\partial x f_i}^3 \frac{\partial f_i}{\partial x f_i}^4 \frac{\partial f_i}{\partial x f_i}^5 + 5 \frac{\partial f_i}{\partial x f_i}^2 \frac{\partial f_i}{\partial x f_i}^3 \frac{\partial f_i}{\partial x f_i}^4 \frac{\partial f_i}{\partial x f_i}^5 \frac{\partial f_i}{\partial x f_i}^6$$

$$- \frac{35 \langle \partial f_i^2 \partial f_i, \partial f_i \rangle^3}{2} + \frac{\lambda_i}{\partial x f_i^3}$$

we get the parabolic equations

$$\partial_t f_i = -\frac{1}{\partial x f_i^4} \partial_x^4 f_i + h(f_i),$$

for $i = 1, \ldots, q$, with

$$h(f_i) = 6 \langle \partial f_i^2 \partial f_i, \partial f_i \rangle \frac{\partial f_i^3}{\partial x f_i^6}$$

$$+ \frac{\partial f_i^2}{\partial x f_i^2} \left(\frac{5 \langle \partial f_i^2 \partial f_i, \partial f_i \rangle^2}{2 \langle \partial f_i^2 \partial f_i, \partial f_i \rangle^4} + 4 \frac{\partial f_i^3}{\partial x f_i^4} \frac{\partial f_i^4}{\partial x f_i^6} \right) - \frac{35 \langle \partial f_i^2 \partial f_i, \partial f_i \rangle^2}{2} \frac{\lambda_i}{\partial x f_i^6}.$$  

This choice of tangential component is the most natural one to get parabolicity of the equation and has been used for instance also in [11]. Further, we observe that the
boundary condition $\vec{\kappa} = 0$ is not well posed. Indeed, the curvature of a curve $f$ can be written as

$$\vec{\kappa} = \frac{1}{|\partial_x f|^2} \left( \text{Id}_{n \times n} - \partial_s f \otimes \partial_s f \right) \partial_x^2 f.$$

(Here and in the following, for vectors $v, w \in \mathbb{R}^n$ we write $v \otimes w$ to denote the $n \times n$ matrix $vw^T$.) Clearly, the matrix given by the terms between the brackets has 0 as an eigenvalue and this creates problems in the analytical treatment of the problem. If one instead imposes the boundary condition $\partial_x^2 f = 0$ this in particular implies that the curvature is zero and it also gives a well posed problem. For this reason the problem we construct a solution to is in fact (2.2) with boundary conditions

$$\begin{cases} f_i(t, 1) = P_i, & \text{for all } t \in [0, T), i \in \{1, \ldots, q\}, \\
\partial_x^2 f_i(t, 1) = 0 = \partial_s^2 f_i(t, 0) & \text{for all } t \in [0, T), i \in \{1, \ldots, q\}, \\
f_i(t, 0) = f_j(t, 0) & \text{for all } t \in [0, T), i, j \in \{1, \ldots, q\}, \\
and \sum_{i=1}^q (\nabla_s \vec{\kappa}_i(t, 0) - \lambda_i \partial_s f_i(t, 0)) = 0 & \text{for all } t \in [0, T), \\
\end{cases}$$

instead of (1.6). The case $q = 3$ and $n = 2$ appears in [11].

In order to find solutions that are $C^{4+\alpha}([0, T] \times [0, 1])$, $\alpha \in (0, 1)$, for some $T > 0$ the initial datum has to satisfy some compatibility conditions. Following the notation of [19, page 98] (see also [10, page 217]) we need to impose compatibility conditions of order zero.

**Compatibility conditions 2.1.** We assume that $f_{0, i} : [0, 1] \rightarrow \mathbb{R}^n$, $f_{0, i} \in C^{4,\alpha}([0, 1])$, $i \in \{1, \ldots, q\}$, regular curves, satisfy compatibility condition of order zero for the problem (2.2), (2.4). That is, $f_{0, i}$ satisfy the boundary conditions

$$\begin{cases} f_{0, i}(1) = P_i, & \partial_x^2 f_{0, i}(1) = 0 = \partial_s^2 f_{0, i}(0), i \in \{1, \ldots, q\}, \\
f_{0, i}(0) = f_{0, j}(0), i \in \{1, \ldots, q\}, \\
and \sum_{i=1}^q (\nabla_s \vec{\kappa}_0(i, 0) - \lambda_i \partial_s f_{0, i}(0)) = 0, \\
\end{cases}$$

and at the fixed boundary point $x = 1$ the curves satisfy

$$\frac{1}{|\partial_x f_{0, i}|^4} \partial_x^4 f_{0, i} \big|_{x=1} = 0,$$

(i.e. $\partial_t f_i = 0$ at $t = 0, x = 1$) while at the junction point $x = 0$

$$\frac{1}{|\partial_x f_{0, i}|^4} \partial_x^4 f_{0, i} \big|_{x=0} = \frac{1}{|\partial_x f_{0, j}|^4} \partial_x^4 f_{0, j} \big|_{x=0},$$

for $i \neq j \in \{1, \ldots, q\}$ (i.e. $\partial_t f_i = \partial_t f_j$ at $t = 0, x = 0$).

**Remark 2.2.** In the formulas above we have used that $h(f_i)$ is zero at the boundary due to the boundary condition $\partial_x^2 f_i = 0$. 

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Theorem 2.3. Let \( n \geq 2, q \geq 3, \alpha \in (0, 1) \) and \( P_i, i \in \{1, \ldots, q\} \), be points in \( \mathbb{R}^n \). Given \( f_{0,i} : [0, 1] \to \mathbb{R}^n, f_{0,i} \in C^{4,\alpha}([0, 1]), i \in \{1, \ldots, q\} \), regular curves satisfying the Compatibility conditions 2.1 and the non-collinearity condition (NC), then there exist \( T > 0 \) and regular curves \( f_i \in C^{\delta + 4 + \alpha}([0, T] \times [0, 1]; \mathbb{R}^n), i \in \{1, \ldots, q\} \), such that \( f = (f_1, \ldots, f_q) \) is the unique solution of (2.2) together with the boundary conditions (2.4) and the initial condition

\[
f_i(t = 0) = f_{0,i}. 
\]   

Moreover, we have instant parabolic smoothing, that is \( f_i \in C^\infty((0, T] \times [0, 1]) \) for any \( i \in \{1, \ldots, q\} \) and the non-collinearity condition (NC) holds at the triple junction for any time \( t \in [0, T] \).

3. Proof of Theorem 2.3

We start by fixing some notation. Given \( q \) time dependent curves: \( f_i : [0, T] \times [0, 1] \to \mathbb{R}^n, i \in \{1, \ldots, q\} \), we denote their components by

\[
f_i^j \text{ with } j = 1, \ldots, n \text{ for each curve } f_i, \ i \in \{1, \ldots, q\}. \tag{3.1}
\]

In the following, when it does not create confusion, we will not write the dependence in \( t \) and \( x \) to keep the notation as slender as possible. As we will see the arguments are independent of \( q \) the number of curves. When reading the arguments for the first time it might be useful to consider simply the case \( q = 3 \).

Here \( f_{0,i} : [0, 1] \to \mathbb{R}^n, i \in \{1, \ldots, q\} \), denote the initial data as given in Theorem 2.3 and their components are denoted by \( f_i^j \), according to (3.1). Let us recall that these are regular curves and satisfy the Compatibility Conditions 2.1 as well as the non-collinearity condition (NC). Set \( \delta > 0 \) as

\[
\delta := \min\{|\partial_x f_{0,i}(x)| : x \in [0, 1] \text{ and } i \in \{1, \ldots, q\}\}. \tag{3.2}
\]

Set for \( \alpha \in (0, 1) \), and for some \( 0 < T < 1 \) and \( M > 0 \) both to be chosen later

\[
X_i = \left\{ u \in C^{\frac{4+\alpha}{\delta} + 4 + \alpha}([0, T] \times [0, 1]; \mathbb{R}^n) : \|u\|_{C^{\frac{4+\alpha}{\delta} + 4 + \alpha}} \leq M, u(0, x) = f_{0,i} \right\}, \tag{3.3}
\]

for \( i \in \{1, \ldots, q\} \) (recall “Appendix B”, where definition of parabolic Hölder spaces and useful properties are collected).

We proceed now as follows. We first associate a linear system to (2.2), (2.4) with (2.8) for each \( \tilde{f} \in \prod_{i=1}^q X_i \) by computing the coefficients at the initial datum and choosing the right-hand side depending also on \( \tilde{f} \) in such a way that a fixed point of the associated solution operator solves the original non-linear problem. Thanks to the
non-collinearity condition we show that the linear parabolic system is well-posed and hence we have the solution operator

\[ \mathcal{R} : \prod_{i=1}^{q} X_i \rightarrow \prod_{i=1}^{q} C^{\frac{4+\alpha}{4},4+\alpha}([0,T] \times [0,1] : \mathbb{R}^n) \]

\[ \bar{f} \mapsto f = \mathcal{R} \bar{f}, \quad (3.4) \]

with \( \bar{f} = (\bar{f}_1, \ldots, \bar{f}_q) \). Then a fixed point of this map is a solution of (2.2), (2.4) with (2.8).

Below we show in detail how this operator \( \mathcal{R} \) is constructed, but first of all let us prove with the following lemma that, by choosing first \( M \) and then \( T \), we can guarantee that the maps \( \bar{f}_i \in X_i \) are regular on the whole considered time interval. The choice of \( M \) is specified in (3.51) below.

Here we consider \( f_{0,i} \in C^{\frac{4+\alpha}{4},4+\alpha}([0,T] \times [0,1]) \) by extending it as a constant function in time.

**Lemma 3.1.** Let \( \bar{f}_i \in X_i \) for some \( i \in \{1, \ldots, q\} \). Then for any \( T < 1 \) we have

\[ \| \partial_x f_{0,i} - \partial_x \bar{f}_i \|_{C^{\frac{4+\alpha}{4},4+\alpha}([0,T] \times [0,1])} \leq C T^{\frac{\alpha}{4}} \left( \| \bar{f}_i \|_{C^{\frac{4+\alpha}{4},4+\alpha}([0,T] \times [0,1])} + \| f_{0,i} \|_{C^{4,\alpha}([0,1])} \right), \quad (3.5) \]

for some universal constant \( C \). Moreover, there exists \( 0 < T_1 < 1 \), such that

\[ |\partial_x \bar{f}_i(t,x)| \geq \frac{1}{2} \delta > 0 \text{ for all } (t,x) \in [0, T_1] \times [0,1], \quad (3.6) \]

with \( \delta \) defined as in (3.2). Here \( T_1 = T_1(M, \delta, f_0) \) and is chosen independently of \( i \).

**Proof.** Since \( \bar{f}_i \equiv f_{0,i} \) at \( t = 0 \) the first inequality follows directly from Lemma B.5 with \( m = 0 \) and \( l = 1 \) and the fact that we have extended \( f_{0,i} \) as a constant in time.

For (3.6) we observe that using the first part of the claim and the definition of \( \delta \) in (3.2) we have

\[ |\partial_x \bar{f}_i(t,x)| \geq |\partial_x f_{0,i}(t,x)| - \| \partial_x f_{0,i} - \partial_x \bar{f}_i \|_{C^0([0,T] \times [0,1])} \]

\[ \geq \delta - C T^{\frac{\alpha}{4}} \left( \| \bar{f}_i \|_{C^{\frac{4+\alpha}{4},4+\alpha}} + \| f_{0,i} \|_{C^{4,\alpha}([0,1])} \right) \]

\[ \geq \delta - C T^{\frac{\alpha}{4}} \left( M + \| f_{0,i} \|_{C^{4,\alpha}([0,1])} \right) \geq \frac{1}{2} \delta, \]

by choosing \( T_1 < 1 \) such that \( T_1^{\frac{\alpha}{4}} C (M + \| f_{0,i} \|_{C^{4,\alpha}([0,1])}) \leq \frac{1}{2} \delta \) for any \( i \in \{1, \ldots, q\} \). \( \square \)

Next, let us construct the operator \( \mathcal{R} \).

3.1. The linear system

**The linear PDEs** Define for \( i \in \{1, \ldots, q\} \) the coefficients \( D_i = D_i(x) \) where

\[ 0 < D_i := \frac{1}{|\partial_x f_{0,i}|}. \quad (3.7) \]
In particular, since the initial curves are sufficiently smooth, there exists $\tilde{\delta} > 0$ such that
\begin{equation}
\tilde{\delta} = \min\{D_i(x) : x \in [0, 1] \text{ and } i \in \{1, \ldots, q\}\}. \tag{3.8}
\end{equation}
Moreover, for $\tilde{f}_i \in X_i$ with $i \in \{1, \ldots, q\}$ fixed, set $R^j_i = R^j_i(t, x)$, $j \in \{1, \ldots, n\}$, where
\begin{equation}
R^j_i := \left(\frac{1}{|\partial_x f_0, i|} - \frac{1}{|\partial_x \tilde{f}_i|}\right) \partial_x^4 \tilde{f}^j_i. \tag{3.9}
\end{equation}
Then, for $f_0, i, i \in \{1, \ldots, q\}$, as in Theorem 2.3 the linear system we consider is
\begin{equation}
\begin{aligned}
\partial_t f^j_i + (D_i)^4 \partial_x^4 f^j_i &= R^j_i + h^j(\tilde{f}_i) \text{ in } (0, T) \times (0, 1), \\
f^j_i(t = 0, x) &= f^j_{0, i}(x), \quad x \in [0, 1],
\end{aligned} \tag{3.10}
\end{equation}
for $j$ and $i$ as before and with appropriate linear boundary condition that we now derive. Notice that $h^j(\tilde{f}_i)$ denotes the $j$-th component of the vector $h(\tilde{f}_i)$ defined in (2.3).

**The linear boundary conditions** We have two boundary points for each curve. One is fixed while the other is the junction point and hence moving. At the fixed boundary points we already have linear boundary conditions (recall (2.4)) and hence we can concentrate on the junction point.

At the junction point we have the boundary conditions
\begin{equation}
\begin{aligned}
\partial_x^2 f_i(t, 0) &= 0 \quad \text{ for all } t \in (0, T), \ i \in \{1, \ldots, q\}, \\
f_i(t, 0) &= f_j(t, 0) \quad \text{ for all } t \in (0, T), \ i, j \in \{1, \ldots, q\},
\end{aligned}
\end{equation}
and
\begin{equation}
\sum_{i=1}^q (\nabla_x \tilde{k}_i(t, 0) - \lambda_i \partial_x f_i(t, 0)) = 0 \quad \text{ for all } t \in (0, T),
\end{equation}
and hence only the last one needs to be linearized. By (A3) and since $\partial_x^2 f_i = 0$ at $x = 0$ we find
\begin{equation}
\begin{aligned}
\nabla_x \tilde{k}_i &= \partial_x \tilde{k}_i = \frac{\partial_x^3 f_i}{|\partial_x f_i|^3} - \left(\frac{\partial_x^3 f_i}{|\partial_x f_i|^5}, \partial_x f_i\right) \partial_x f_i \\
&= \frac{1}{|\partial_x f_i|^3} \left(\text{Id}_{n \times n} - \partial_x f_i \otimes \partial_x f_i\right) \partial_x^3 f_i. \tag{3.11}
\end{aligned}
\end{equation}
We consider a linear boundary condition using the initial datum and a given vector $\tilde{f} \in \prod_{i=1}^q X_i$ as follows
\begin{equation}
\sum_{i=1}^q E_i \partial_x^3 f_i = b \in \mathbb{R}^n,
\end{equation}
where the $n \times n$ matrices $E_i, i \in \{1, \ldots, q\}$, are given by
\begin{equation}
E_i = (D_i)^3 (\text{Id}_{n \times n} - d_i \otimes d_i), \tag{3.12}
\end{equation}
where the \( d_i \)'s are the normalized tangential vectors of the initial data, that is
\[
d_i := D_i \partial_x f_{0,i} = \frac{\partial_x f_{0,i}}{|\partial_x f_{0,i}|}.
\]

The vector field \( b \) is given by
\[
b = b(\vec{f}) = \sum_{i=1}^{q} (E_i - \vec{E}_i) \partial_x^3 \vec{f}_i + \lambda_i \frac{\partial_x \vec{f}_i}{|\partial_x \vec{f}_i|},
\]
where
\[
\vec{E}_i = \frac{1}{|\partial_x \vec{f}_i|^3} (\text{Id}_{n \times n} - \partial_s \vec{f}_i \otimes \partial_s \vec{f}_i).
\]

Let us notice that each matrix \( E_i \) has determinant zero, but as we will see below, since we consider the sum \( \sum_i E_i \) the boundary condition is still well posed under the assumption of non-collinearity (NC).

Summing up the linear boundary conditions we consider are
\[
\begin{align*}
  f_i(t, 1) &= P_i, & \text{for all } t \in (0, T), \ i \in \{1, \ldots, q\}, \\
  \partial_x^2 f_i(t, 1) &= 0 = \partial_x^2 f_j(t, 0) & \text{for all } t \in (0, T), \ i \in \{1, \ldots, q\}, \\
  f_i(t, 0) &= f_j(t, 0) & \text{for all } t \in (0, T), \ i, j \in \{1, \ldots, q\}, \\
  and \sum_{i=1}^{q} E_i \partial_x^3 f_i(t, 0) &= b & \text{for all } t \in (0, T),
\end{align*}
\]
with \( b \) defined in (3.14). This choice of \( b \) ensures that a fixed point of the associated solution operator will satisfy the boundary conditions (2.4).

3.2. Existence of solutions to the linear problem

The operator \( \mathcal{R} \) is defined as follows: given \( \vec{f} \in \prod_{i=1}^{q} X_i \) we set \( \mathcal{R} \vec{f} \) to be the unique solution \( f \) of the linear parabolic system (3.10), (3.15). This can be done according to the next theorem:

**Theorem 3.2.** Let the assumptions of Theorem 2.3 hold. Let \( M > 0 \) and let \( T > 0 \) be such that the curves belonging to \( X_i \) are regular. Then for any \( \vec{f} \in \prod_{i=1}^{q} X_i \) there exists \( f = (f_1, \ldots, f_q), f_i \in C^{4+\alpha}([0, T] \times [0, 1]; \mathbb{R}^n), i \in \{1, \ldots, q\}, \) unique solution of the linear parabolic system (3.10) together with the boundary conditions (3.15).

Moreover, there exists a constant \( C_0 > 0 \) such that
\[
\sum_{i=1}^{q} \| f_i \|_{C^{4+\alpha}([0, T] \times [0, 1])}^q \leq C_0 \left( \sum_{i=1}^{q} (\| R_i + h(\vec{f}_i) \|_{C^{4+\alpha}([0, T] \times [0, 1])} + \| f_{0,i} \|_{C^{4+\alpha}([0, 1])} + |P_i|) + \| b \|_{C^{4+\alpha}([0, T])} \right),
\]
(3.16)
The constant $C_0$ depends on $n$, $q$, $\delta$ and $\tilde{\delta}$.

The theorem above gives us the solution operator $R$ described in (3.4) above.

### 3.2.1. Well-posedness of the linear problem

Here we check using [19] that the linear parabolic problem (3.10) with boundary conditions (3.15) is well posed.

First of all, observe that the left-hand side of our system (3.10) can be written as $L(x, t, \partial_x, \partial_t) f$ with $f \in \prod_{i=1}^{q} X_i$ (i.e. $f = (f_1, \ldots, f_q)$) and

$$L(x, t, \partial_x, \partial_t) = \text{diag}(\ell_{kk})_{k=1}^{qn}$$

(3.17)

where

$$\ell_{kk}(x, t, \partial_x, \partial_t) = \partial_t + (D_i)^4 \partial_x^4$$

if $k = (i-1)n + j$ for some $j \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, q\}$, with $D_i$ defined in (3.7). Notice that in [19, page 8] also $L_0$ the principal part of $L$ is used. Since here $L$ coincide with its principal part, for simplicity we work only with $L$ avoiding $L_0$ altogether.

As usual, we associate to these differential operators polynomials with coefficients depending (possibly) on $(t, x)$ by replacing $\partial_x$ by $i\xi$, $\xi \in \mathbb{R}$, and $\partial_t$ by $p$, $p \in \mathbb{C}$. Then,

$$\ell_{kk}(x, t, i\xi, p) = p + (D_i)^4 (i\xi)^4,$$

if $k = (i-1)n + j$ for some $j \in \{1, \ldots, n\}$ and $i \in \{1, \ldots, q\}$. In particular, for $\lambda \in \mathbb{R}$

$$\ell_{kk}(x, t, i\lambda, p\lambda^4) = p\lambda^4 + (D_i)^4 (i\lambda)^4 = \lambda^4 \ell_{kk}(x, t, i\xi, p).$$

In the following,

$$L(x, t, i\xi, p) := \det L(x, t, i\xi, p) = \prod_{i=1}^{q} (p + (D_i)^4 (i\xi)^4)^n,$$

(3.18)

and $L(x, t, i\xi\lambda, p\lambda^4) = \lambda^4 q^n L(x, t, i\xi, p)$, see [19, Eq. (1.2)].

Let

$$\hat{L}(x, t, i\xi, p) := L(x, t, i\xi, p) L^{-1}(x, t, i\xi, p)$$

$$\hat{L}(x, t, i\xi, p) = \prod_{i=1}^{q} (p + (D_i)^4 (i\xi)^4)^n \text{diag}((\ell_{kk})^{-1})_{k=1}^{qn} = \text{diag}(A_{kk})_{k=1}^{qn}$$

(3.19)

with

$$A_{kk} = A_{kk}(x, t, i\xi, p) = \prod_{i=1}^{q} (p + (D_i)^4 (i\xi)^4)^n (p + (D_i)^4 (i\xi)^4)^{-n}.$$
if \( k = (l - 1)n + j \) for \( l \in \{1, \ldots, q\} \) and \( j \in \{1, \ldots, n\} \). Since most of the terms are equal, let \( A_1 := A_{11} \), \( A_2 := A_{n+1,n+1} \) and so on, i.e. let

\[
A_i := A_{(i-1)n+1,(i-1)n+1} \quad \text{for} \quad i \in \{1, \ldots, q\}.
\]

(3.20)

Notice that for \( k \in \{1, \ldots, n\} \) we have \( A_{(i-1)n+k,(i-1)n+k} = A_i \) for \( i \in \{1, \ldots, q\} \).

**Parabolicity condition** For \( \xi \in \mathbb{R} \) and by (3.18) we see that the roots (in the variable \( p \)) of the polynomial \( L(x, t, i\xi, p) \) are given by

\[
p = -(D_i)^4 \xi^4 \quad \forall \ i \in \{1, \ldots, q\} \quad \text{and each one with multiplicity } n
\]

and satisfy

\[
\text{Re } p = -(D_i)^4 \xi^4 \leq -\tilde{\delta}^4 \xi^{2b} \quad \text{with } b = 2 \quad \forall \ \xi \in \mathbb{R}, \ \forall \ (t, x),
\]

and with \( \tilde{\delta} = \min\{D_i : i \in \{1, \ldots, q\}\} \) (see (3.8)). So the parabolicity condition [19, Page 8] is satisfied and we even have uniform parabolicity.

**Complementary conditions on the initial datum** Since our initial conditions are

\[
f^i_j(t, x) \mid_{t=0} = f^i_0(x), \quad i \in \{1, \ldots, q\}, \quad j \in \{1, \ldots, n\},
\]

the associated matrix is

\[
C_0(\partial_x, \partial_t) = Id_{qn \times qn}.
\]

(3.21)

According to [19, Page 12] we need to verify that the rows of the matrix \( \mathcal{D}(x, p) = C_0(x, 0, p) \cdot \hat{L}(x, 0, 0, p) \) are linearly independent modulo \( p^{qn} (r = qn \text{ in [19, page 12], since we have } q \text{ curves}) \). With (3.19) and (3.20) we find that \( \mathcal{D}(x, p) = \text{diag}(p^{qn-1}) \in \mathbb{R}^{qn \times qn} \) from which the linear independency of the rows immediately follows.

**The polynomial** \( M^+ \): Next, consider the polynomial \( L = L(x, t, i\tau, p) \) given in (3.18) (with \( \tau \) instead \( \xi \) as in [19] to stress that we are now working at the boundary points), that is

\[
L(x, t, i\tau, p) = \prod_{i=1}^{q} \left( p + (D_i)^4 \tau^4 \right)^n.
\]

As a function of \( \tau \), the polynomial \( L \) has \( 2qn \) roots with positive real part and \( 2qn \) roots with negative real part provided \( \text{Re } p \geq 0 \) and \( p \neq 0 \) (see [19, Page 11]). Indeed, due to the assumptions on \( p \) we may write \( p = |p|e^{i\theta_p} \) with \(-\frac{1}{2}\pi \leq \theta_p \leq \frac{1}{2}\pi \) and \(|p| \neq 0 \). Then, the roots have to satisfy for some \( i \in \{1, \ldots, q\} \)

\[
\tau^4 = -\frac{p}{(D_i)^4} \quad \Rightarrow \quad \tau^4 = \left| p \right| \frac{e^{i(\pi + \theta_p)}}{(D_i)^4}.
\]

\[1\]In our case \( \xi \) is a vector in \( \mathbb{R} \) but in the general setting of [19] \( \xi \in \mathbb{R}^d \). Now working at the boundary one uses the tangential \( \zeta \) and normal \( \tau \) directions of the vector \( \xi \). Since we are in dimension one there is no tangential direction, i.e. \( \zeta = 0 \).
The (distinct) roots with positive imaginary part are for \( i \in \{1, \ldots, q\} \)

\[
\tau_{i,1}(x, p) = r_i e^{i \frac{1}{4}(\theta_p + \pi)} \quad \text{and} \quad \tau_{i,2}(x, p) = r_i e^{i \frac{1}{4}(\theta_p + 3\pi)}, \quad \text{with} \quad r_i := \frac{\sqrt{|p|}}{D_i},
\]

each with multiplicity \( n \). With these roots we define the polynomial

\[
M^+(x, \tau, p) = \prod_{i=1}^{q} (\tau - \tau_{i,1})^n (\tau - \tau_{i,2})^n.
\]

For later let us write also the (distinct) roots with negative imaginary part. These are

\[
\tau_{i,3}(x, p) = r_i e^{i \frac{1}{4}(\theta_p + 5\pi)} \quad \text{and} \quad \tau_{i,4}(x, p) = r_i e^{i \frac{1}{4}(\theta_p + 7\pi)}, \quad i \in \{1, \ldots, q\},
\]

each with multiplicity \( n \).

**Complementary conditions at the fixed boundary points** By (3.15), at \( x = 1 \) the boundary condition system reads: \( B f = \tilde{b} \) where \( f = (f_1, \ldots, f_q)^T \in \mathbb{R}^{qn} \) with

\[
B(x = 1, t, \partial_x, \partial_t) = \begin{pmatrix}
B & 0 & \ldots & 0 \\
0 & B & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & B
\end{pmatrix}
\]

a \( 2qn \times qn \) matrix where \( B \) is a \( 2n \times n \) matrix given by

\[
B = B(x = 1, t, \partial_x, \partial_t) = \begin{pmatrix}
Id_{n \times n} \\
Id_{n \times n} \partial_x^2
\end{pmatrix}
\]

so that

\[
B = B(x = 1, t, i\tau, p) = \begin{pmatrix}
Id_{n \times n} \\
-IId_{n \times n} \tau_2
\end{pmatrix},
\]

and

\[
\tilde{b} = (P_1, 0, P_2, 0, \ldots, P_q, 0)^T \in \mathbb{R}^{2qn}.
\]

According to [19, Page 11] we need to check that at \( x = 1 \) the rows of the matrix

\[
A(x = 1, t, i\tau, p) := B(x = 1, t, i\tau, p) \hat{L}(x = 1, t, i\tau, p)
\]

are linearly independent modulo \( M^+(x = 1, \tau, p) \) for \( \text{Re } p \geq 0 \) and \( p \neq 0 \). In the following we simply write \( M^+(\tau) \) since \( x \) and \( p \) are fixed and there is no explicit dependence on time.

---

2In the notation of Solonnikov \( L = L(\tau) \) has \( b \cdot r = 2 \cdot qn = 2qn \) roots with negative imaginary parts and \( 2qn \) with negative imaginary parts.
Since $B$ is a block-matrix and $\hat{L}$ is a diagonal matrix (see (3.19)), $A$ is also a block matrix. Hence for the linear independence of the rows it is sufficient to consider the different blocks separately, i.e. each curve separately. For simplicity we consider the first curve only, that is the first $2n$ rows. We do not need to consider the columns that are identically zero and hence we simply have to consider the rows of the $2n \times n$ matrix

$$B = B(x = 1, t, i\tau, p) = \begin{pmatrix} A_1 I_{d_{2n} \times n} \\ -A_1 I_{d_{2n} \times n} \tau^2 \end{pmatrix},$$

since $A_{kk} = A_1$ for $k \in \{1, \ldots, n\}$ by (3.20) with

$$A_1(x = 1, t, i\tau, p) = (p + (D_1)^4 \tau^4)^n \prod_{i=2}^{q} (p + (D_i)^4 \tau^4)^n.$$

Now to check the linear independence of the rows modulo $M^+$ we have to verify that if there exists $\omega \in \mathbb{R}^{2n}$ such that

$$\omega^T \begin{pmatrix} A_1 I_{d_{2n} \times n} \\ -A_1 I_{d_{2n} \times n} \tau^2 \end{pmatrix} = (0, \ldots, 0) \mod M^+(\tau),$$

then necessarily $\omega = 0$.

Now let us recall that $M^+$ is the polynomial whose roots are exactly the roots with positive imaginary part of $\prod_{i=1}^{q} (p + (D_i)^4 \tau^4)^n = A_1(p + (D_1)^4 \tau^4)$. As a consequence, $A_1$ and $M^+$ have many factors in common, which we can factor out. More precisely, looking at the first equation of the system above and denoting by $\omega^j$ the $j$-th component of $\omega$, we observe that

$$A_1(\omega^1 - \omega^{n+1} \tau^2) = 0 \mod M^+(\tau)$$

if and only if

$$a_1(\tau)(\omega^1 - \omega^{n+1} \tau^2) = 0 \mod s_1(\tau)$$

where

$$a_1(\tau) = (\tau - \tau_{1,3}(p))^{n-1}(\tau - \tau_{1,4}(p))^{n-1} \prod_{i=2}^{q} (\tau - \tau_{i,3}(p))^{n}(\tau - \tau_{i,4}(p))^n,$$

$$s_1(\tau) = (\tau - \tau_{1,1}(p))(\tau - \tau_{1,2}(p)).$$

(3.23) \hspace{1cm} (3.24)

Since $s_1(\tau)$ can not divide $a_1(\tau)$ then it has to divide $\omega^1 - \omega^{n+1} \tau^2$ that is also a polynomial of degree two. This polynomial has the same zeroes as $s_1(\tau)$ iff

$$\begin{cases}
\omega^1 - \omega^{n+1} \tau^2 e^{\frac{\theta p}{\tau}} = 0 \\
\omega^1 + \omega^{n+1} \tau^2 e^{\frac{\theta p}{\tau}} = 0
\end{cases}$$
which implies $\omega^1 = \omega^{n+1} = 0$. Similarly one consider the other components and also the other curves.

**Complementary conditions at the junction** At $x = 0$, using (3.15), the boundary condition of the linearized system reads: $B \tilde{f} = \tilde{b}$ where $f = (f_1, f_2, \ldots, f_q)^T \in \mathbb{R}^{nq}$ with $B$ a $(2nq) \times qn$ matrix given by

$$B(x = 0, t, \partial_x, \partial_t) = \begin{pmatrix}
1d_{n \times n} & -1d_{n \times n} & 0 & \ldots & 0 \\
1d_{n \times n} & 0 & -1d_{n \times n} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1d_{n \times n} & 0 & 0 & \ldots & -1d_{n \times n} \\
id_{n \times n} & 0 & 0 & \ldots & 0 \\
0 & id_{n \times n} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
E_1 \partial_x^3 & E_2 \partial_x^3 & E_3 \partial_x^3 & \ldots & E_q \partial_x^3
\end{pmatrix}$$

with $E_i, i = 1, \ldots, q, n \times n$ matrices defined in (3.12) and $\tilde{b} = (0, 0, \ldots, 0, b)^T \in \mathbb{R}^{2nq}$ with $b \in \mathbb{R}^n$ defined in (3.14). The first $(q-1)n$ rows describe the concurrency condition, the last $n$ rows give the third order boundary condition while the others correspond to the second order boundary condition.

As before, we need to check that at $x = 0$ the rows of the matrix

$$A(x = 0, t, \imath \tau, p) := B(x = 0, t, \imath \tau, p) \hat{L}(x = 0, t, \imath \tau, p)$$

are linearly independent modulo $M^+(x = 0, \tau, p)$ for $\text{Re} p \geq 0$ and $p \neq 0$. By (3.19) we have to study the rows of the matrix $A_0(\tau) := A(x = 0, t, \imath \tau, p)$ with

$$A(x = 0, t, \imath \tau, p) = \begin{pmatrix}
A_1 id_{n \times n} & -A_2 id_{n \times n} & 0 & \ldots & 0 \\
A_1 id_{n \times n} & 0 & -A_3 id_{n \times n} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
A_1 id_{n \times n} & 0 & 0 & \ldots & -A_q id_{n \times n} \\
-A_1 \tau^2 id_{n \times n} & 0 & 0 & \ldots & 0 \\
0 & -A_2 \tau^2 id_{n \times n} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
-iA_1 \tau^3 E_1 & -iA_2 \tau^3 E_2 & -iA_3 \tau^3 E_3 & \ldots & -iA_q \tau^3 E_q
\end{pmatrix}$$

where the coefficients $A_i, i = 1, 2, \ldots, q$, are defined in (3.20).

Let us assume that there exists $\omega \in \mathbb{R}^{2nq}$ such that

$$\omega^T A_0(\tau) = (0, \ldots, 0), \mod M^+(\tau).$$

This is a system of $qn$ equations in $2qn$ variables.
The factors $A_1, A_2, \ldots, A_q$ have many factors in common with the polynomial $M^+$ so that we can rewrite the system in the following way. The first $n$ equations can be written for $k = 1, \ldots, n$ as

$$a_1(\tau)(\omega^k + \omega^{n+k} + \cdots + \omega^{n(q-2)+k} - \omega^{(q-1)n+k} + \tau^2 - i \sum_{j=1}^{n} E_{1,jk} \omega^{(2q-1)n+j} \tau^3) = 0, \mod s_1(\tau),$$

with $a_1, s_1$ defined in (3.23) and (3.24) respectively and $E_{1,jk}$ denoting the $j, k$ entry of the matrix $E_1$. Defining $a_2, a_3, \ldots, a_q, s_2, s_3, \ldots, s_q$ accordingly we find that the $(n+k)$-th equation, $k = 1, \ldots, n$, can be simplified to

$$a_2(\tau)(-\omega^k - \omega^{n+k} \tau^2 - i \sum_{j=1}^{n} E_{2,jk} \omega^{(2q-1)n+j} \tau^3) = 0, \mod s_2(\tau).$$

Likewise

$$a_3(\tau)(-\omega^{n+k} - \omega^{(q+1)n+k} \tau^2 - i \sum_{j=1}^{n} E_{3,jk} \omega^{(2q-1)n+j} \tau^3) = 0, \mod s_3(\tau),$$

while the $((q-1)n+k)$-th equation for $k = 1, \ldots, n$ can be written as

$$a_q(\tau)(-\omega^{n(q-2)+k} - \omega^{2(q-1)n+k} \tau^2 - i \sum_{j=1}^{n} E_{q,jk} \omega^{(2q-1)n+j} \tau^3) = 0, \mod s_q(\tau).$$

Since the polynomial $a_i$ does not vanish on the zeros of the polynomial $s_i$ we do not need to consider further the factors $a_i$’s in each one of the equations above. Hence, each algebraic equation above is reduced to the form

$$a - bx^2 + cx^3 = 0 \mod (x - x_1)(x - x_2),$$

with $x_1 \neq \pm x_2$. Our idea is now to plug in the zeroes $x_1$ and $x_2$ as done before for the other boundary conditions. We get then the conditions

$$\begin{cases}
a - bx_1^2 + cx_1^3 = 0, \\
a - bx_2^2 + cx_2^3 = 0.
\end{cases}$$

Subtracting the two equations we get first the relation

$$b = \frac{x_1^2 + x_1x_2 + x_2^2}{x_1 + x_2}c.$$

and then, from the first equation,

$$a = \frac{x_1^2x_2}{x_1 + x_2}c.$$
The roots $\tau_{i,1}, \tau_{i,2}$ of $s_i(\tau)$ are given in (3.22) and we compute (taking $x_k = \tau_{i,k}$, $k = 1, 2$)

$$\frac{x_1^2 + x_1x_2 + x_2^2}{x_1 + x_2} = ir_i \frac{1}{\sqrt{2}} e^{i\frac{3}{4}\theta_p} \quad \text{and} \quad \frac{x_1^2 + x_2^2}{x_1 + x_2} = -ir_i^3 \frac{1}{\sqrt{2}} e^{i\frac{3}{4}\theta_p}.$$ 

We now write the $qn$ equations obtained by imposing the algebraic equation (3.28) to the equations (3.25), (3.26), ..., up to (3.27). In (3.25) we have $b = \omega^{(q-1)n+k}$ and $c = -i \sum_{j=1}^n E_{1,jk}\omega^{(2q-1)n+j}$, for $k = 1, \ldots, n$, so that from (3.28) we get the $n$ equations

$$\omega^{(q-1)n+k} = r_1 \frac{1}{\sqrt{2}} e^{i\frac{1}{4}\theta_p} \sum_{j=1}^n E_{1,jk}\omega^{(2q-1)n+j}, \quad k = 1, \ldots, n. \quad (3.30)$$

Similarly, in (3.26), (and then analogously up to (3.27)), we have $b = \omega^{qn+k}$ and $c = -i \sum_{j=1}^n E_{2,jk}\omega^{(2q-1)n+j}$ and (3.28) implies

$$\omega^{qn+k} = r_2 \frac{1}{\sqrt{2}} e^{i\frac{1}{4}\theta_p} \sum_{j=1}^n E_{2,jk}\omega^{(2q-1)n+j}, \quad k = 1, \ldots, n, \quad (3.31)$$

$$\vdots$$

$$\omega^{2(q-1)n+k} = r_q \frac{1}{\sqrt{2}} e^{i\frac{1}{4}\theta_p} \sum_{j=1}^n E_{q,jk}\omega^{(2q-1)n+j}, \quad k = 1, \ldots, n. \quad (3.32)$$

From these equations we immediately see that the components $\omega^m$ for $(q-1)n+1 \leq m \leq 2(q-1)n$ are determined by the components $\omega^m$ for $(2q-1)n+1 \leq m \leq 2qn$.

The $qn$ equations obtained by imposing the algebraic equation (3.29) to (3.25), (3.26), ..., up to (3.27) are given by, for $k = 1, \ldots, n$

$$\omega^k + \omega^{n+k} + \ldots + \omega^{n(q-2)+k} = -r_1^3 \frac{1}{\sqrt{2}} e^{i\frac{3}{4}\theta_p} \sum_{j=1}^n E_{1,jk}\omega^{(2q-1)n+j}$$

$$-\omega^k = -r_2^3 \frac{1}{\sqrt{2}} e^{i\frac{3}{4}\theta_p} \sum_{j=1}^n E_{2,jk}\omega^{(2q-1)n+j}$$

$$-\omega^{k+n} = -r_3^3 \frac{1}{\sqrt{2}} e^{i\frac{3}{4}\theta_p} \sum_{j=1}^n E_{3,jk}\omega^{(2q-1)n+j}$$

$$\vdots$$

$$-\omega^{k+n(q-2)} = -r_q^3 \frac{1}{\sqrt{2}} e^{i\frac{3}{4}\theta_p} \sum_{j=1}^n E_{q,jk}\omega^{(2q-1)n+j}. \quad (3.33)$$
This system of $qn$ equations implies the system of $n$ equations
\[
0 = r_1^3 \sum_{j=1}^{n} E_{1,j,k} \omega^{(2q-1)n+j} + r_2^3 \sum_{j=1}^{n} E_{2,j,k} \omega^{(2q-1)n+j} + \ldots
+ r_q^3 \sum_{j=1}^{n} E_{q,j,k} \omega^{(2q-1)n+j}, \quad k = 1, \ldots, n,
\]
that by definition of $r_i$, in (3.22), of $E_i$ in (3.12) and since $|p| \neq 0$ can be rewritten as
\[
0 = (\tilde{E}_1 + \tilde{E}_2 + \ldots + \tilde{E}_q)v \quad \text{where} \quad \tilde{E}_i = Id_{n \times n} - \pi_i,
\]
with $\pi_i$ the projection onto the space spanned by $d_i = D_i \partial_x f_{0,i}$ and $v \in \mathbb{R}^n$ with
\[
v^k = \omega^{(2q-1)n+k}, \quad k = 1, 2, \ldots, n.
\]
Even better, this can be rewritten as
\[
qv - \sum_{i=1}^{q} \pi_i v = 0. \quad (3.34)
\]
Since $|\pi_i w| \leq |w|$ for all $w \in \mathbb{R}^n$ and $|\pi_i w| = |w|$ iff $w$ and $d_i$ are linearly dependent, we see that the system has a non-trivial solution iff there exists a vector $v \in \mathbb{R}^n$ such that $\pi_i v = v$ for $i = 1, 2, \ldots, q$. This is the case iff $\pi_1 = \pi_2 = \cdots = \pi_q$, i.e., the vectors $d_i$'s are linearly dependent.

By the non-collinearity condition (NC), the vectors fulfill $\dim(\text{span}\{d_1, d_2, \ldots, d_q\}) \geq 2$ and hence the only solution to (3.34) is the zero-vector, that is $\omega^{(2q-1)n+k} = 0$ for $k = 1, \ldots, n$. In turn, $\omega^m$ for $(q-1)n + 1 \leq m \leq 2qn$ are zero (by (3.30), (3.31) to (3.32)). Then by (3.33) also $\omega^m$ for $1 \leq m \leq (q-1)n$ are zero. The rows are linearly independent and hence the complementary conditions are satisfied also at the junction.

3.2.2. Proof of Theorem 3.2

By the regularity assumptions of the initial data $f_{0,i}$ and $\tilde{f}_i \in X_i$ and using the properties of the parabolic spaces collected in “Appendix B” one can readily check that the regularity assumptions required by [19, Thm.4.9, page 121] for the coefficients of the elliptic operator $L$ in (3.17), and those for the coefficients of the boundary operators $B$ and $C_0$ are satisfied.

Next, thanks to the choice of the space $X_i$ in (3.3) (precisely, being each function $\tilde{f}_i$ equal to $f_{0,i}$ at time $t = 0$) and having linearized at the initial datum one sees that since $f_{0,i}, i \in \{1, \ldots, q\}$, satisfy the Compatibility conditions 2.1 (i.e. the compatibility conditions of order zero of the non-linear problem) than $f_{0,i}, i \in \{1, \ldots, q\}$, satisfies also the compatibility condition of order zero of the linear problem. Precisely, the boundary conditions
\[
\begin{cases}
  f_{0,i}(1) = P_i, \quad \partial_x^2 f_{0,i}(1) = 0 = \partial_x^2 f_{0,i}(0), \quad i \in \{1, \ldots, q\}, \\
  f_{0,i}(0) = f_{0,j}(0), \quad i, j \in \{1, \ldots, q\}, \\
  \text{and} \sum_{i=1}^{q} E_i \partial_x^3 f_{0,i} = b,
\end{cases}
\]
are satisfied since at $t = 0$, $\tilde{E}_i = E_i$. Similarly, since $R_i = 0$ at time $t = 0$ (recall (3.9)) we see that (2.6) and (2.7) give also the remaining compatibility condition of order zero.

Since linear problem is well posed by the considerations in the previous section, the claim follows by [19, Thm.4.9, page 121] or [10, Thm. VI.21].

3.3. Existence by Banach fixed point theorem

Let us recall that $f_{0,i} \in C^{4+a}(0, 1]$ and $\tilde{f}_i, \tilde{g}_i \in X_i$ and $\delta$ as defined in (3.2). Then, for $T < 1$ we have

$$\|\partial_x f_{0,i} - \partial_x \tilde{f}_i\|_{C^{4+a}_{\delta}(0, T] \times [0, 1)} \leq CT^\frac{a}{2} \left(\|\tilde{f}_i\|_{C^{4+a}(0, T] \times [0, 1)} + \|f_{0,i}\|_{C^{4,a}(0,1)}\right)^3,$$

(3.35)

with $C = C(n, \delta)$. Moreover, for $m \in \mathbb{N}$ and any $T \leq T_1$ (with $T_1$ as defined in Lemma 3.1) we have

$$\left\|\frac{1}{|\partial_x f_{0,i}|^m} - \frac{1}{|\partial_x \tilde{f}_i|^m}\right\|_{C^{4+a}_{\delta}(0, T] \times [0, 1)} \leq CT^\frac{a}{2}$$

for any $x \in [0, 1]$ and with $C = C(n, m, \delta, \|\tilde{f}_i\|_{C^{4+a}(0, T] \times [0, 1)}, \|f_{0,i}\|_{C^{4,a}(0,1)})$ as well as

$$\left\|\frac{1}{|\partial_x \tilde{f}_i|^m} - \frac{1}{|\partial_x \tilde{g}_i|^m}\right\|_{C^{4+a}_{\delta}(0, T] \times [0, 1)} \leq CT^\frac{a}{2} \left\|\tilde{f}_i - \tilde{g}_i\right\|_{C^{4+a}(0, T] \times [0, 1)};$$

$$\left\|\frac{1}{|\partial_x \tilde{f}_i|^{\frac{m}{2}} - \frac{1}{|\partial_x \tilde{g}_i|^{\frac{m}{2}}}}\right\|_{C^{4+a}_{\delta}(0, T] \times [0, 1)} \leq CT^\frac{a}{2} \left\|\tilde{f}_i - \tilde{g}_i\right\|_{C^{4+a}(0, T] \times [0, 1)};$$

again for $x \in [0, 1]$ and with $C = C(n, m, \delta, \|\tilde{f}_i\|_{C^{4+a}(0, T] \times [0, 1)}, \|\tilde{g}_i\|_{C^{4+a}(0, T] \times [0, 1)}$.

Proof. The first inequality is a direct consequence of Lemma B.3, Remark B.1, (3.5) and the definition of $\delta$ in (3.2). Indeed,
\[ \| \partial_x f_{0,i} - \partial_x \tilde{f}_i \|_{C^{\frac{m}{4}, \alpha}} \leq C(n) \left( \frac{1}{\| \partial_x f_{0,i} \|_{C^0(0,1)}} \right)^2 \left( \| \tilde{f}_i \|_{C^{\frac{m}{4}, \alpha}}^2 + \| f_{0,i} \|_{C^4,\alpha(0,1)}^2 \right) \times \| \partial_x f_{0,i} - \partial_x \tilde{f}_i \|_{C^{\frac{m}{4}, \alpha}} \]

\[ \leq C(n) T^\frac{m}{4} \left( \frac{1}{\| \partial_x f_{0,i} \|_{C^0(0,1)}} \right)^2 \left( \| \tilde{f}_i \|_{C^{\frac{m}{4}, \alpha}}^2 + \| f_{0,i} \|_{C^4,\alpha(0,1)}^2 \right)^3. \]

Next, let us denote with \( p_k(\cdots) \) a polynomial of degree at most \( k \) in its given variables. For the second and fourth inequalities using Remark 3.3, Lemmas 3.1, B.2, B.3 and (3.35) we find for \( T \leq T_1 \)

\[ \| \partial_x f_{0,i} - \partial_x \tilde{f}_i \|_{C^{\frac{m}{4}, \alpha}((0,T) \times [0,1])} \leq \| \partial_x f_{0,i} - \partial_x \tilde{f}_i \|_{C^{\frac{m}{4}, \alpha}((0,T) \times [0,1])} \leq \]

\[ \leq CT^\frac{m}{4} \left( \| \tilde{f}_i \|_{C^{\frac{m}{4}, \alpha}((0,T) \times [0,1])} + \| f_{0,i} \|_{C^4,\alpha((0,T) \times [0,1])} \right)^3 p_{3m-1} \left( \| \partial_x f_{0,i} \|_{C^0,\alpha((0,T) \times [0,1])}, \| \partial_x \tilde{f}_i \|_{C^{\frac{m}{4}, \alpha}((0,T) \times [0,1])} \right). \]

with \( C = C(n, m, \delta) \). The same way we obtain

\[ \| \partial_x f_{0,i}(\cdot, x) \|_{C^{\frac{m}{4}, \alpha}((0,T))} \leq \| \partial_x f_{0,i}(\cdot, x) \|_{C^{\frac{m}{4}, \alpha}((0,T))} \leq \]

\[ \leq C p_{3m-1} \left( \| \tilde{f}_i \|_{C^{\frac{m}{4}, \alpha}((0,T) \times [0,1])}, \| f_{0,i} \|_{C^4,\alpha((0,T) \times [0,1])} \right). \]
where once again the constant depends on $n, m$ and $\delta$.

\textbf{Strict contraction} Consider the solution operator $R$ given in (3.4). Our aim is to show that $R$ is a (strict) contraction, that is with $f = R(\tilde{f}), g = R(\tilde{g})$ the contraction estimate,

$$
\|f - g\|_{C^{\frac{n+4}{4}, \frac{n+4}{4}}([0,T] \times [0,1])} \leq C_1 T^\frac{\nu}{2} \|\tilde{f} - \tilde{g}\|_{C^{\frac{n+4}{4}, \frac{n+4}{4}}([0,T] \times [0,1])}
$$

(3.36)

holds for some constant $C_1 = C_1(n, q, \delta, \tilde{\delta}, \|f_0\|_{C^{4,4}([0,1])}, M)$ and $T \leq T_1$ with $T_1$ defined in Lemma 3.1. Here $M$ is the constant in (3.3).

Observe that $f - g$ fulfills the parabolic linear system,

$$
\partial_t (f_i - g_i) + D_i^4 \partial_x^4 (f_i - g_i) = R_i(\tilde{f}_i) + h(\tilde{f}_i) - R_i(\tilde{g}_i) - h(\tilde{g}_i),
$$

(3.37)

for all $(t, x) \in (0, T) \times (0, 1)$, $i \in \{1, \ldots, q\}$, with $R_i, h$ defined in (3.9) and (2.3) respectively, together with the initial condition $(f_i - g_i)(t = 0) = 0$ and the boundary conditions

$$
\begin{aligned}
(f_i - g_i)(t, 1) &= 0, & \forall t \in [0, T], i \in \{1, \ldots, q\}, \\
(f_i - g_i)(t, 0) &= (f_j - g_j)(t, 0), & \forall t \in [0, T], i, j \in \{1, \ldots, q\}, \\
\partial_x^2 (f_i - g_i)(t, 0) &= 0 = \partial_x^2 (f_i - g_i)(t, 1), & \forall t \in [0, T], i \in \{1, \ldots, q\}, \\
\sum_{i=1}^{q} E_i \partial_x^3 (f_i - g_i)(t, 0) &= b(\tilde{f}) - b(\tilde{g}), & \forall t \in [0, T],
\end{aligned}
$$

where the boundary term $b$ is defined in (3.14).

By the same arguments as in Sect. 3.2.2 the linear problem is well posed and the regularity assumptions on the coefficients are satisfied. Moreover, since $\tilde{f} = \tilde{g}$ at $t = 0$ we see that the zero initial datum satisfies the compatibility conditions of order zero and hence $f - g$ is the solution given by [19, Thm.4.9, page 121] or [10, Thm. VI.21] of (3.37) with the boundary conditions given above. Moreover, the same theorems give the following estimate

$$
\sum_{i=1}^{q} \|f_i - g_i\|_{C^{\frac{n+4}{4}, \frac{n+4}{4}}([0,T] \times [0,1])} 
\leq C_0 \left( \sum_{i=1}^{q} (\|R_i(\tilde{f}_i) + h(\tilde{f}_i) - R_i(\tilde{g}_i) - h(\tilde{g}_i)\|_{C^{\frac{n+4}{4}, \frac{n+4}{4}}([0,T] \times [0,1])}) 
+ \|b(\tilde{f}) - b(\tilde{g})\|_{C^{0, \frac{n+4}{4}}([0,T])} \right),
$$

(3.38)

where $C_0 = C_0(n, q, \delta, \tilde{\delta})$ is the constant in Theorem 3.2. To obtain inequality (3.36), we need to estimate the terms on the right-hand side of (3.38). First of all, for any $i \in \{1, \ldots, q\}$, by applying triangle inequality of Hölder norms, Remark B.1 and
Lemmas B.2, 3.4 we have for $T \leq T_1$ with $T_1$ from Lemma 3.1

$$\| R_i(\tilde{f}_t) - R_i(\bar{g}_t) \|_{C^{\frac{q}{2}, \alpha}([0,T] \times [0,1])} \leq C \left( \left\| \frac{1}{|\partial_x f_0|} \right\|_{C^{\frac{q}{2}, \alpha}([0,T] \times [0,1])} \right) \left( \left\| \frac{1}{|\partial_x \bar{g}_i|} \right\|_{C^{\frac{q}{2}, \alpha}([0,T] \times [0,1])} \right)$$

$$\leq C T^{\frac{q}{2}} \| \tilde{f}_t - \bar{g}_t \|_{C^{\frac{q}{4} + \alpha + 4}([0,T] \times [0,1])},$$

(3.39)

where $C = C(n, \delta, \| f_0 \|_{C^{4, \alpha}([0,1])}, \| \tilde{f}_t \|_{C^{\frac{q}{4} + \alpha, \alpha+4}}, \| \bar{g}_i \|_{C^{\frac{q}{4} + \alpha, \alpha+4})}.$

With the same ideas, since $\tilde{f}_t = \bar{g}_t$ at $t = 0$, using Lemmas B.2, B.5, 3.4, 3.1 we have for $T \leq T_1$

$$\| h(\tilde{f}_t) - h(\bar{g}_t) \|_{C^{\frac{q}{2}, \alpha}([0,T] \times [0,1])} \leq C \left( \sum_{k=1}^{3} \| \partial^k_x \tilde{f}_t - \partial^k_x \bar{g}_t \|_{C^{\frac{q}{2}, \alpha}([0,T] \times [0,1])} + \sum_{k=1}^{4} \left\| \frac{1}{|\partial_x \bar{g}_i|} \right\|_{C^{\frac{q}{2}, \alpha}([0,T] \times [0,1])} \right)$$

$$\leq C T^{\frac{q}{2}} \| \tilde{f}_t - \bar{g}_t \|_{C^{\frac{q}{4} + \alpha + 4}([0,T] \times [0,1])},$$

(3.40)

where $C = C(n, \delta, \lambda_i, \| \tilde{f}_t \|_{C^{\frac{q}{4} + \alpha, \alpha+4}}, \| \bar{g}_i \|_{C^{\frac{q}{4} + \alpha, \alpha+4}).}$

For the estimates of the boundary terms, $\| b(\tilde{f}) - b(\bar{g}) \|_{C^{0, \frac{q}{4} + 1}([0,T])}$, we start from the term multiplied by $\lambda_i$, see (3.14). With help from Lemmas B.2, 3.1, B.3, 3.4 and Remark B.1, we have

$$\| \frac{\partial_x \tilde{f}_t}{|\partial_x \tilde{f}_t|} - \frac{\partial_x \bar{g}_t}{|\partial_x \bar{g}_t|} \|_{C^{0, \frac{q}{4} + 1}([0,T])} \leq C \left( \left\| \frac{1}{|\partial_x \tilde{f}_t|} \right\|_{C^{0, \frac{q}{4} + 1}([0,T])} \| \partial_x \tilde{f}_t - \partial_x \bar{g}_t \|_{C^{0, \frac{q}{4} + 1}([0,T])} \right)$$

(3.41)

$$\leq C T^{\frac{q}{2}} \| \tilde{f}_t - \bar{g}_t \|_{C^{\frac{q}{4} + \alpha + 4}([0,T] \times [0,1])},$$

where $C = C(n, \delta, \| \tilde{f}_t \|_{C^{\frac{q}{4} + \alpha, \alpha+4}}, \| \bar{g}_i \|_{C^{\frac{q}{4} + \alpha, \alpha+4}).}$

For the highest-order terms at the boundary we compute

$$\| \sum_{i=1}^{q} (E_i - \tilde{E}_i(\tilde{f}_i)) \partial^3_x \tilde{f}_t - (E_i - \tilde{E}_i(\bar{g}_i)) \partial^3_x \bar{g}_t \|_{C^{0, \frac{q}{4} + 1}([0,T])} \leq \sum_{i=1}^{q} \| (E_i - \tilde{E}_i(\tilde{f}_i)) (\partial^3_x \tilde{f}_t - \partial^3_x \bar{g}_t) \|_{C^{0, \frac{q}{4} + 1}([0,T])}$$

(3.42)

$$\leq \sum_{i=1}^{q} \| (E_i - \tilde{E}_i(\bar{g}_i)) \partial^3_x \bar{g}_t \|_{C^{0, \frac{q}{4} + 1}([0,T])}. $$
Note that we may split the matrix term as
\[
E_i - \bar{E}_i(\bar{f}_i) = \left( \frac{1}{|\partial_x f_{0,i}|^3} - \frac{1}{|\partial_x \bar{f}_i|^3} \right) I_{d_n \times n} - \left( \frac{1}{|\partial_x f_{0,i}|^3} d_i \otimes d_i - \frac{1}{|\partial_x \bar{f}_i|^3} \partial_x \bar{f}_i \otimes \partial_x \bar{f}_i \right)
\]
\[
= \left( \frac{1}{|\partial_x f_{0,i}|^3} - \frac{1}{|\partial_x \bar{f}_i|^3} \right) I_{d_n \times n} - \left( \frac{1}{|\partial_x f_{0,i}|^5} - \frac{1}{|\partial_x \bar{f}_i|^5} \right) \partial_x f_{0,i} \otimes \partial_x f_{0,i}
\]
\[
+ \frac{1}{|\partial_x \bar{f}_i|^5} \partial_x f_{0,i} \otimes (\partial_x f_{0,i} - \partial_x \bar{f}_i)
\]
\[
+ \frac{1}{|\partial_x \bar{f}_i|^5} (\partial_x f_{0,i} - \partial_x \bar{f}_i) \otimes \partial_x \bar{f}_i.
\]
(3.43)

By applying the linear algebra, \((\bar{u} \otimes \bar{v})\bar{w} = \bar{u}(\bar{v}, \bar{w})\), from (3.43) we have using Lemmas B.2, 3.1, 3.4, B.5 and Remark B.1 (for simplicity here we mostly write \(C^{0, \frac{\alpha+1}{4}}\) instead of \(C^{0, \frac{\alpha+1}{4}}([0, T])\))
\[
\|(E_i - \bar{E}_i(\bar{f}_i)) \cdot (\partial_x^3 \bar{f}_i - \partial_x^3 \bar{g}_i)\|_{C^{0, \frac{\alpha+1}{4}}([0, T])}
\]
\[
\leq C \left\| \frac{1}{|\partial_x f_{0,i}|^3} - \frac{1}{|\partial_x \bar{f}_i|^3} \right\|_{C^{0, \frac{\alpha+1}{4}}} \| \partial_x^3 \bar{f}_i - \partial_x^3 \bar{g}_i \|_{C^{0, \frac{\alpha+1}{4}}}
\]
\[
+ C \left\| \frac{1}{|\partial_x f_{0,i}|^5} - \frac{1}{|\partial_x \bar{f}_i|^5} \right\|_{C^{0, \frac{\alpha+1}{4}}} \| \partial_x f_{0,i} \|_{C^{0, \frac{\alpha+1}{4}}} \| \partial_x^3 \bar{f}_i - \partial_x^3 \bar{g}_i \|_{C^{0, \frac{\alpha+1}{4}}}
\]
\[
+ C \left\| \frac{1}{|\partial_x \bar{f}_i|^5} \right\|_{C^{0, \frac{\alpha+1}{4}}} \| \partial_x f_{0,i} \|_{C^{0, \frac{\alpha+1}{4}}} \| \partial_x \bar{f}_i \|_{C^{0, \frac{\alpha+1}{4}}} \| \partial_x f_{0,i} - \partial_x \bar{f}_i \|_{C^{0, \frac{\alpha+1}{4}}} \| \partial_x^3 \bar{f}_i - \partial_x^3 \bar{g}_i \|_{C^{0, \frac{\alpha+1}{4}}}
\]
\[
\leq CT^{\frac{\alpha}{4}} \| \bar{f}_i - \bar{g}_i \|_{C^{\frac{\alpha+1}{4}}([0, T] \times \{0, 1\})},
\]
with \(C = C(n, \delta, \| \bar{f} \|_{C^{\frac{\alpha+1}{4}}([0, 1])}, \| \bar{g} \|_{C^{\frac{\alpha+1}{4}}([0, 1])})\).

Similarly, we may apply the same trick of estimates to the second term in (3.42).

More precisely, writing
\[
(\bar{E}_i(\bar{f}_i) - \bar{E}_i(\bar{g}_i)) \partial_x^3 \bar{g}_i = \left( \frac{1}{|\partial_x \bar{f}_i|^3} - \frac{1}{|\partial_x \bar{g}_i|^3} \right) \partial_x^3 \bar{g}_i + \left( \frac{1}{|\partial_x \bar{f}_i|^5} - \frac{1}{|\partial_x \bar{g}_i|^5} \right) \partial_x \bar{f}_i \otimes \partial_x \bar{f}_i \partial_x^3 \bar{g}_i
\]
\[
+ \frac{1}{|\partial_x \bar{g}_i|^5} (\partial_x \bar{f}_i - \partial_x \bar{g}_i) \otimes \partial_x \bar{f}_i + \partial_x \bar{g}_i \otimes (\partial_x \bar{f}_i - \partial_x \bar{g}_i) \partial_x^3 \bar{g}_i
\]

using Remark B.1, Lemmas 3.1, 3.4, B.2 and (B2) we obtain
\[
\|(\bar{E}_i(\bar{f}_i) - \bar{E}_i(\bar{g}_i)) \partial_x^3 \bar{g}_i\|_{C^{0, \frac{\alpha+1}{4}}([0, T])} \leq CT^{\frac{\alpha}{4}} \| \bar{f}_i - \bar{g}_i \|_{C^{\frac{\alpha+1}{4},\frac{\alpha+1}{4}}([0, T] \times [0, 1])},
\]
with \(C = C(n, \delta, \| \bar{f} \|_{C^{\frac{\alpha+1}{4}}([0, 1])}, \| \bar{g} \|_{C^{\frac{\alpha+1}{4}}([0, 1])})\).

Combining the previous estimates we therefore infer
\[
\|b(\bar{f}) - b(\bar{g})\|_{C^{0, \frac{\alpha+1}{4}}([0, T])} \leq CT^{\frac{\alpha}{4}} \sum_{i=1}^{q} \| \bar{f}_i - \bar{g}_i \|_{C^{\frac{\alpha+1}{4},\frac{\alpha+1}{4}}([0, T] \times [0, 1])}
\]
(3.44)
with \( C = C(\lambda, n, \delta, \| \tilde{f} \|_{C^{4+\alpha},4+\alpha([0,T] \times [0,1])}, \| f_0 \|_{C^{4,\alpha}([0,1])}, \| \tilde{g} \|_{C^{4+\alpha},4+\alpha([0,T] \times [0,1])}) \).

From (3.38) together with (3.44), (3.40) and (3.39), we obtain (3.36) since \( \| f_j \|_{C^{4+\alpha},4+\alpha} \)
and \( \| \tilde{g} \|_{C^{4+\alpha},4+\alpha} \) are bounded by \( M \) by definition of \( X_i \).

**Self-map** We show now that \( \mathcal{R} \) defined in (3.4) indeed maps \( \prod_{i=1}^q X_i \) into itself by choosing first \( M \) and then \( T \) sufficiently small. Given \( f \in \prod_{i=1}^q X_i \) by Theorem 3.2
the solution \( f \) of (3.10), (3.15) satisfies estimate (3.16) so that for each \( j \in \{1, \ldots, q\} \),

\[
\| f_j \|_{C^{4+\alpha},4+\alpha([0,T] \times [0,1])} \leq \sum_{i=1}^q \| f_i \|_{C^{4+\alpha},4+\alpha([0,T] \times [0,1])}
\]

\[
\leq C_0 \left( \sum_{i=1}^q (\| R_i + h(\tilde{f}_i) \|_{C^{q,\alpha}([0,T] \times [0,1])} + \| f_0,i \|_{C^{4,\alpha}([0,1])} + | P_i | + \| b \|_{C^{0,\alpha}([0,1])} \right),
\]

with \( C_0 = C_0(n, q, \delta, \tilde{\delta}) \). It then follows from applying triangle inequalities of Hölder-norms and (3.14) that for each \( j \in \{1, \ldots, q\} \),

\[
\| f_j \|_{C^{4+\alpha},4+\alpha([0,T] \times [0,1])}
\]

\[
\leq C_0 \sum_{i=1}^q \left( \| R_i \|_{C^{\frac{q}{4},\alpha}([0,T] \times [0,1])} + \| h(\tilde{f}_i) - h(f_0,i) \|_{C^{\frac{q}{4},\alpha}([0,T] \times [0,1])}
\]

\[
+ \| (E_i - \tilde{E}_i) \partial_3^3 \tilde{f}_i \|_{C^{0,\alpha}([0,T] \times [0,1])} + | \lambda | \left( \| \partial_x f_i \|_{C^{0,\alpha}([0,T] \times [0,1])} + \| f_0,i \|_{C^{4,\alpha}([0,1])} + | P_i | + n|\lambda| \right). \quad (3.45)
\]

The last term on the right-hand side of (3.45) depends only on the initial data and \( P_i, \lambda \), and will dictate the choice of the constant \( M \). From the other terms on the right-hand side we are able to gain a power of \( T \) and hence to bound them choosing \( T \) sufficiently small. Indeed, from (3.9) using Lemmas B.2, 3.4 and Remark B.1 we find for \( T \leq T_1 \)

\[
\| R_i \|_{C^{\frac{q}{4},\alpha}([0,T] \times [0,1])} \leq C \left( \frac{1}{| \partial_x f_{0,i} |^4} - \frac{1}{| \partial_x \tilde{f}_i |^4} \right) \| \partial_3^3 \tilde{f}_i \|_{C^{\frac{q}{4},\alpha}([0,T] \times [0,1])}
\]

\[
\leq CT^{\frac{q}{4}}, \quad (3.46)
\]

with \( C = C(n, \delta, \| \tilde{f} \|_{C^{4+\alpha},4+\alpha}, \| f_0 \|_{C^{4,\alpha}([0,1])}) \). Furthermore, from (2.3) and using Lemmas B.2, B.5, 3.1, 3.4, Remark B.1 again for \( T \leq T_1 \), we find

\[
\| h(\tilde{f}_i) - h(f_0,i) \|_{C^{\frac{q}{4},\alpha}([0,T] \times [0,1])}
\]

\[
\leq C \sum_{k=1}^3 \| \partial_x^k f_i - \partial_x^k f_0,i \|_{C^{\frac{q}{4},\alpha}([0,T] \times [0,1])} + C \sum_{k=1}^4 \left( \frac{1}{| \partial_x f_{0,i} |^{2k}} - \frac{1}{| \partial_x \tilde{f}_i |^{2k}} \right) \| \tilde{f}_i \|_{C^{\frac{q}{4},\alpha}([0,T] \times [0,1])}
\]
with $C = C(n, \delta, \|f_i\|_{C^{4+\alpha}}, \|f_0,i\|_{C^{4,\alpha}([0,1])})$. Similarly, for the boundary terms with Lemmas B.2, 3.1, B.3, B.5, 3.4, Remark B.1 and (B2) when $T \leq T_1$ we find

$$
\left\| \frac{\partial_x f_i}{\partial x f_i} \right\|_{C^{0,1+\alpha}([0,T])} \leq C \left\| \frac{1}{\partial_x f_i} \right\|_{C^{0,1+\alpha}([0,T])} + C \left\| \frac{f_0,i}{\partial_x f_i} \right\|_{C^{0,1+\alpha}([0,T])} 
$$

and finally by (3.43)

$$
\left\| (E_i - \tilde{E}_i) \partial_x^3 f_i \right\|_{C^{0,1+\alpha}([0,T])} \leq CT^{\frac{\alpha}{2}},
$$

with $C = C(n, \delta, \|f_i\|_{C^{4+\alpha}}, \|f_0,i\|_{C^{4,\alpha}([0,1])})$.

From (3.45) together with (3.46), (3.47), (3.48) and (3.49) we obtain

$$
\| f_i \|_{C^{4+\alpha}([0,T] \times [0,1])} \leq C_2 T^{\frac{\alpha}{2}} + C_0 \sum_{i=1}^{q} \left( \| h(f_0,i) \|_{C^{0,\alpha}([0,1])} + \| f_0,i \|_{C^{4,\alpha}([0,1])} + |P_i| + n|\lambda_i| \right),
$$

with $C_2 = C_2(n, q, \delta, \tilde{\delta}, \|f_0\|_{C^{4,\alpha}([0,1])}, M)$ since $\|f_i\|_{C^{4+\alpha}} \leq M$ by definition of $X_i$ for each $i \in \{1, 2, \ldots, q\}$.

**Proof of Theorem 2.3**

**Proof of Theorem 2.3. Existence of a solution** We start by fixing $M$ and $T$. Let

$$
M := 2C_0 \sum_{i=1}^{q} \left( \| h(f_0,i) \|_{C^{0,\alpha}([0,1])} + \| f_0,i \|_{C^{4,\alpha}([0,1])} + |P_i| + n|\lambda_i| \right)
$$

see the last term in (3.50). Now fix $T \leq \min\{T_1, 1\}$, with $T_1$ defined in Lemma 3.1, such that

$$
C_1 T^{\frac{\alpha}{2}} < 1 \quad \text{and} \quad C_2 T^{\frac{\alpha}{2}} < \frac{M}{2},
$$

with $C_1, C_2$ the constants depending on $n, q, \delta$, $\tilde{\delta}$, $\|f_0\|_{C^{4,\alpha}([0,1])}$ and $M$ appearing in (3.36) and (3.50) respectively.

Since the $\bigcup_{i=1}^{q} X_i$ is a closed set of the Banach space $C^{4+\alpha}([0, T] \times [0, 1]; \mathbb{R}^q)$ and, by the choice of $M$ and $T$, the map $\mathcal{R}$ is a self-map and a strict contraction, by applying Banach’s fixed point theorem we get a unique fixed point of $\mathcal{R}$ and hence, by construction, a solution to (2.2) with (2.4) and (2.8) in $C^{4+\alpha}([0, T] \times [0, 1]; \mathbb{R}^q)$. Moreover, $f_i$ is a regular curve for each $i \in \{1, \ldots, q\}$ and $t \in [0, T]$ by Lemma 3.1.
Non-collinearity Consider the function
\[ nc : [0, T] \times [0, 1] \rightarrow \mathbb{R}, \quad nc(t, x) = 1 - \prod_{1 \leq i < j \leq q} |\langle \partial_x f_i, \partial_x f_j \rangle|, \]
for \( f = (f_1, \ldots, f_q) \) the solution provided so far. The non-collinearity condition (NC) yields that at time \( t = 0 \) and at \( x = 0 \) this function is strictly positive. Moreover, the regularity and smoothness of the solution ensures that \( nc \in C^0([0, T] \times [0, 1]) \) (see Remark B.1) and hence, by possibly choosing \( T \) smaller, \( nc > 0 \) on \([0, T] \times [0] \). In other words, the non-collinearity condition remains satisfied on \([0, T] \).

Uniqueness of the solution Let \( f, \tilde{f} \) be two different solutions on \([0, T] \times [0, 1]\) of (2.2) with (2.4) and (2.8). Let
\[ \bar{t} = \sup\{t \in [0, T] | f(\tau, x) = \tilde{f}(\tau, x) \quad \forall x \in [0, 1] \text{ and } \forall \tau \leq t\}. \]
Obviously \( 0 \leq \bar{t} < T \). Now consider the problem (2.2), (2.4), with initial data \( f(\bar{t}, \cdot) = \tilde{f}(\bar{t}, \cdot) \). Note that for this new initial data (NC) is satisfied as well as all necessary compatibility conditions. By repeating the arguments provided so far, this problem has a unique solution on some small time interval \([\bar{t}, \bar{t} + \epsilon] \). Since \( f \) and \( \tilde{f} \) are also solutions, this yields that \( f = \tilde{f} \) for some time after \( \bar{t} \) giving a contradiction to the definition of \( \bar{t} \).

Parabolic-Smoothing for positive time The smoothness of the solution in \([0, T] \times [0, 1]\) follows with similar arguments as presented in [7, App.B.2.3]. For completeness we report the main ideas here. Given \( 0 < \epsilon < T \), we consider the network \( \gamma := f \eta = (f_1 \eta, \ldots, f_q \eta) \), where \( \eta : [0, T] \rightarrow [0, 1] \) is some smooth cut-off function with \( \eta(t) = 0 \) for \( 0 < t < \epsilon / 4 \) and \( \eta(t) = 1 \) for \( t \in [\epsilon / 2, T] \). By the regularity of \( f \) it follows that \( \gamma \in C^{\frac{4+\alpha}{4},\frac{4+\alpha}{4}}([0, T] \times [0, 1]) \). Moreover, upon recalling (2.2), (2.4), (2.8), and (3.11), we infer that \( \gamma \) satisfies the linear parabolic boundary value problem
\[ \partial_t \gamma_i = -\frac{1}{|\partial_x f_i|^4} \partial_x^4 \gamma_i + \eta h(f_i) + f_i \frac{d}{dt} \eta \quad i \in \{1, \ldots, q\}, \tag{3.52} \]
with boundary conditions
\[
\begin{align*}
\gamma_i(t, 1) &= \eta(t) \gamma_i, & \text{for all } t \in (0, T), i \in \{1, \ldots, q\}, \\
\partial_x^2 \gamma_i(t, 1) &= 0 = \partial_x^2 \gamma_i(t, 0), & \text{for all } t \in (0, T), i \in \{1, \ldots, q\}, \\
\gamma_i(t, 0) &= \gamma_j(t, 0), & \text{for all } t \in (0, T), i, j \in \{1, \ldots, q\}, \\
\sum_{i=1}^{q} (E_i(f_i) \partial_x^3 \gamma_i(t, 0) - \lambda_i \frac{1}{|\partial_x f_i|^4} \partial_x \gamma_i(t, 0)) &= 0 & \text{for all } t \in (0, T),
\end{align*} \tag{3.53}
\]
with \( E_i(f_i) = \frac{1}{|\partial_x f_i|^4} (Id - \partial_x f_i \otimes \partial_x f_i) \) and initial datum \( \gamma_0, i = 0, i \in \{1, \ldots, q\} \). Note that the system is linear and parabolic by regularity of \( f_i, i \in \{1, \ldots, q\} \). The compatibility conditions of any order are satisfied (thanks to \( \gamma \) being identically zero close to the origin) and the complementary conditions are also satisfied (this is done
in a similar way as in the previous section and exploiting the fact that (NC) holds for all times in $[0, T])$. The coefficients of the elliptic and boundary operators belong to $C^{\frac{3+\alpha}{2}, \frac{3+\alpha}{2}}([0, T] \times [0, 1])$ and to $C^{0, \frac{3+\alpha}{2}}([0, T])$ respectively, whereas the inhomogeneity in $(3.52)$ is in $C^{1,1+\alpha}([0, T] \times [0, 1])$. Application of [19, Thm.4.9, page 121] yields $\gamma \in C^{\frac{3}{5+\alpha}, 5+\alpha}([0, T] \times [0, 1])$ and therefore $f \in C^{\frac{3+\alpha}{2}, \frac{3+\alpha}{2}}([\varepsilon/2, T] \times [0, 1])$.

To apply a bootstrapping argument we now repeat the same procedure, but since the higher regularity of $f$ is guaranteed only for $t \geq \varepsilon/2$ the next cutting function must be zero, say on $[0, \frac{2}{5}\varepsilon]$ and equal one on $[\frac{2}{5}\varepsilon, T]$, i.e. we have to “shift and reduce” progressively the interval where $\eta \in (0, 1)$. More details in this respect can be found in [7, App.B.2.3]. Eventually we attain $f \in C^\infty([\varepsilon, T])$ and since $\varepsilon$ was arbitrarily chosen the claim follows.

3.4. The case of a smooth initial datum

If the initial data $f_{0,i}$, $i \in \{1, \ldots, q\}$, are in $C^{k,\alpha}([0, 1])$, $k \geq 4$, and higher order compatibility conditions are satisfied we get a solution with higher regularity. Let us first state the compatibility conditions of general order (see Remark 2.1 for compatibility conditions of order zero).

**Remark 3.5.** (Compatibility conditions analytical problem) Following [19, page 98] and [10, page 217, Example 6.12], for the problem $(2.2)$, $(2.4)$, with initial datum $f_0$, we say that compatibility conditions of order $\mu \in \mathbb{N} \cup \{0\}$ are satisfied if the following hold:

- we have
  $$f_{0,i}(1) = P_i, \quad \text{and} \quad f_{0,i}(0) - f_{0,j}(0) = 0, \quad i, j \in \{1, \ldots, q\},$$

- for any $i_q \in \mathbb{N}$ such that $4i_q - 4 \leq \mu$ we have
  $$\partial_t^{i_q} f_i \bigg|_{(t,x) = (0,1)} = 0, \quad \text{and} \quad (\partial_t^{i_q} f_i - \partial_t^{i_q} f_j) \bigg|_{(t,x) = (0,0)} = 0, \quad i, j \in \{1, \ldots, q\},$$

- for any $i_q \in \mathbb{N} \cup \{0\}$ such that $4i_q - 2 \leq \mu$ we have
  $$\partial_t^{i_q} \left( \partial_x^2 f_i \right) \bigg|_{(t,x) = (0,1)} = 0, \quad \text{and} \quad \partial_t^{i_q} \left( \partial_x^2 f_i \right) \bigg|_{(t,x) = (0,0)} = 0, \quad i \in \{1, \ldots, q\},$$

- for any $i_q \in \mathbb{N} \cup \{0\}$ such that $4i_q - 1 \leq \mu$ we have
  $$\partial_t^{i_q} \left( \sum_{i=1}^q (\nabla_s \kappa_i(t,0) - \lambda_i \partial_s f_i(t,0)) \right) \bigg|_{t=0} = 0.$$

The above conditions should be understood as follows: upon recalling $(2.2)$, $(2.3)$, and $(2.1)$, let $L_i^*$, $i \in \{1, \ldots, q\}$, be the differential operator such that

$$L_i^* f_i = \partial_t f_i = -\frac{1}{|\partial_x f_i|^4} \partial_x^4 f_i + h(f_i)$$
\[ \begin{align*}
&= -\nabla_s^2 \tilde{\kappa}_i - \frac{1}{2} |\tilde{\kappa}_i|^2 \tilde{\kappa}_i + \lambda_i \tilde{\kappa}_i + \varphi_i^* \partial_s f_i \\
\text{and let } L_i^{* (i_q)} f_i = \partial_t^{i_q} f_i,
\end{align*} \]

where one can use [4, Lemmas 3.1, 3.5, 3.6] to derive an explicit expression for \( \partial_t^{i_q} f_i \) free of time derivatives. Then the first condition can be rephrased as

\[ L_i^{* (i_q)} f_{0,i} = 0 \text{ at } x = 1 \text{ for } i \in \{1, \ldots, q\}, \text{ and} \]

\[ L_i^{* (i_q)} f_{0,i} = L_j^{* (i_q)} f_{0,j} \text{ at } x = 0 \text{ for } i \neq j. \]

The other conditions are understood in a similar way. For instance the second set of conditions can be rephrased as

\[ \partial_x^2 L_i^{* (i_q)} f_{0,i} = 0 \text{ at } x \in \{0, 1\} \text{ for } i \in \{1, \ldots, q\}. \]

\textbf{Theorem 3.6.} \textit{Let } \( n \geq 2, \ q \geq 3, \ \alpha \in (0, 1), \ k \in \mathbb{N}, \ k \geq 4 \text{ and } P_i, \ i \in \{1, \ldots, q\}, \text{ be points in } \mathbb{R}^n. \ \text{Given } f_{0,i} : [0, 1] \to \mathbb{R}^n, \ f_{0,i} \in C^{k,\alpha}([0, 1]), \ i \in \{1, \ldots, q\}, \text{ regular maps satisfying the compatibility conditions of order } (k - 4) \text{ (as stated in Remark 3.5) and the non-collinearity condition (NC), then there exist } T > 0 \text{ and regular curves } f_i \in C^{k+\alpha,k+\alpha}([0, T] \times I; \mathbb{R}^n), \ i \in \{1, \ldots, q\}, \text{ such that } f = (f_1, \ldots, f_q) \text{ is the unique solution of (2.2) together with the boundary conditions (2.4) and the initial condition } f_i(t = 0) = f_{0,i}. \}

\textit{Moreover, we have instant parabolic smoothing, that is } f_i \in C^{\infty}((0, T] \times [0, 1]) \text{ for any } i \in \{1, \ldots, q\} \text{ and the non-collinearity condition (NC) holds at the triple junction for any time } t \in [0, T]. \}

\textbf{Proof.} \textit{Since the assumption of Theorem 2.3 are satisfied, there exist } T > 0 \text{ and regular curves } f_i \in C^{k+\alpha,k+\alpha}([0, T] \times I; \mathbb{R}^n) \cap C^{\infty}((0, T] \times [0, 1]), \ i \in \{1, \ldots, q\}, \text{ such that } f = (f_1, \ldots, f_q) \text{ is the unique solution of (2.2) satisfying the boundary conditions (2.4) and the initial condition. Moreover the solution satisfies the non-collinearity condition on } [0, T]. \ \text{It remains to show that, in case } k \geq 5, \text{ the solution is actually more regular. We observe that } f_i \text{ for } i \in \{1, \ldots, q\} \text{ solve the linear PDE system }

\[ \partial_t f_i = -a_i \partial_x^4 f_i + b_i \text{ in } (0, T] \times (0, 1), \ i \in \{1, \ldots, q\}, \]

\textit{with boundary conditions }

\[ \begin{align*}
&f_i(t, x = 1) = P_i, \quad f_i(t, x = 0) = f_j(t, x = 0) \quad \forall t \in [0, T], \ i, j \in \{1, \ldots, q\} \\
&\partial_x^2 f_i = 0 \quad \forall t \in [0, T], \ x = 0, 1, \ i \in \{1, \ldots, q\} \\
&\sum_{i=1}^q c_i \partial_x^3 f_i = \sum_{i=1}^q q_i \quad \forall t, \ x = 0.
\end{align*} \]
and initial condition \( f_i(t = 0) = f_{i,0} \) on \([0, 1], i \in \{1, \ldots, q\}\), by looking at the non-linear initial boundary value problem satisfied by \( f_i \) as a linear problem for \( f_i \) with given coefficients (since we already have a solution: recall (2.2), (2.3), (3.11)). The coefficients satisfy \( a_i \in C^{\frac{3}{4}+\alpha, 3+\alpha}([0, T] \times [0, 1]), b_i \in C^{\frac{1}{4}+\alpha, 1+\alpha}([0, T] \times [0, 1]) \) and \( c_i \in C^{\frac{3}{4}+\alpha, 3+\alpha}([0, T] \times [0, 1]) \) and the boundary data satisfies \( q_i \in C^{\frac{3}{4}+\alpha, 3+\alpha}([0, T] \times [0, 1]) \). The system is parabolic by the regularity of \( f_i, i \in \{1, \ldots, q\} \), and by the assumptions on the initial datum the compatibility conditions of order zero and one are satisfied. Proceeding similarly as in the previous Sect. 3.2.1 one shows that the complementary conditions are satisfied since the non-collinearity condition is satisfied on \([0, T]\). By the regularity of the initial datum and [19, Thm.4.9, page 121] we find \( f_i \in C^{\frac{3}{4}+\alpha, 5+\alpha}([0, T] \times [0, 1]), i \in \{1, \ldots, q\} \). Being the solution more regular, we can repeat the argument as long as the smoothness of the initial datum and the order of the compatibility condition allow. 

By the previous result we immediately infer an existence result in \( C^\infty \).

**Corollary 3.7.** Let \( n \geq 2, q \geq 3, \) and \( P_i, i \in \{1, \ldots, q\} \), be points in \( \mathbb{R}^n \). Given \( f_{0,i} : [0, 1] \to \mathbb{R}^n, \) \( f_{0,i} \in C^\infty([0, 1]), i \in \{1, \ldots, q\}\), regular maps satisfying the compatibility conditions of any order (as stated in Remark 3.5) and the non-collinearity condition (NC), then there exist \( T > 0 \) and regular curves \( f_i \in C^\infty([0, T] \times [0, 1]; \mathbb{R}^n), i \in \{1, \ldots, q\}\), such that \( f = (f_1, \ldots, f_q) \) is the unique solution of (2.2) together with the boundary conditions (2.4) and the initial condition \( f_i(t = 0) = f_{0,i}\). Moreover, the non-collinearity condition (NC) holds at the triple junction for any time \( t \in [0, T] \).

**Proof.** By the arguments in the proof of Theorem 3.6 one sees that the time interval of existence of the solution is independent of \( k \). This immediately yields the result. 

**4. Solutions to the geometrical problem**

We are now ready to prove Theorem 1.2, i.e. the existence of a geometric solution. We show that initial curves \( f_{0,i} : [0, 1] \to \mathbb{R}^n, i \in \{1, \ldots, q\}\), satisfying the assumptions of Theorem 1.2 can be reparametrized in such a way that the reparametrizations are admissible initial data for Theorem 2.3. Then, the solution of the analytical problem is also solution of the geometric problem. Here the fact that in Theorem 1.2 the initial datum is attained up to reparametrizations plays a crucial role.

**Proof of Theorem 1.2.** Let \( f_0 = (f_{0,1}, f_{0,2}, \ldots, f_{0,q}) \) be as in the statement. We look for a reparametrization of each of these curves such that the reparametrizations satisfy the Compatibility Conditions 2.1. As a first step we reparametrize each \( f_{0,i}, i \in \{1, \ldots, q\}\), by constant speed. Set

\[
v_i : [0, 1] \to [0, 1], \quad v_i(x) = \frac{1}{\mathcal{L}(f_{0,i})} \int_0^x |\partial_x f_{0,i}| \, dx. \tag{4.1}
\]
Then \( \tilde{f}_{0,i}(x) := f_{0,i}(v_i^{-1}(x)), \) \( i = 1, 2, \ldots, q, \) are parametrized by constant speed. By Remark B.4 we have \( \tilde{f}_{0,i} \in C^{4,\alpha}([0, 1], \mathbb{R}^n). \) In order to keep the notation simple, let us denote \( \tilde{f}_{0,i} \) again by \( f_{0,i}. \) Then (1.7), (1.9), (1.10) and Remark A.1 imply that \( f_{0,i}, i \in \{1, \ldots, q\}, \) satisfy (NC),

\[
\begin{cases}
    f_{0,i}(1) = P_i, \\
    \partial_x^2 f_{0,i}(1) = 0 = \partial_x^2 f_{0,i}(0), \\
    f_{0,i}(0) = f_{0,j}(0) \\
    \text{for all } i \in \{1, \ldots, q\}, \\
    \text{for all } i, j \in \{1, \ldots, q\},
\end{cases}
\tag{4.2}
\]

and

\[
\sum_{i=1}^{q} (\nabla_x \tilde{\kappa}_{0,i}(0) - \lambda_i \partial_x f_{0,i}(0)) = 0,
\]

\[
\frac{1}{|\partial_x f_{0,i}(0)|^4} \partial_x^4 f_{0,i} = 0 \quad \text{at } x = 1, \quad i \in \{1, \ldots, q\},
\tag{4.3}
\]

\[
- \frac{1}{|\partial_x f_{0,i}(0)|^4} \partial_x^4 f_{0,i} + \varphi_0 \partial_x f_{0,i} = - \frac{1}{|\partial_x f_{0,j}(0)|^4} \partial_x^4 f_{0,j} + \varphi_0 \partial_x f_{0,j} \quad \text{at } x = 0,
\tag{4.4}
\]

for \( i, j \in \{1, \ldots, q\}. \) As one sees, in order to apply Theorem 2.3, the only difference to the Compatibility Conditions 2.1 appears in (4.4). For \( i \in \{1, \ldots, q\} \) fixed, consider now the map \( \psi_i \) (which is a perturbation near 0 of the identity on the interval \([0, 1])

\[
\psi_i : [0, 1] \to [0, 1], \quad \psi_i(y) = y - a_i \eta_i(y)y^4,
\]

where \( a_i := \frac{1}{\pi} \varphi_0 \eta_i(0)|\partial_x f_{0,i}(0)|^3 \) and \( \eta_i : [0, 1] \to [0, 1] \) is a smooth function fulfilling the following conditions in case \( a_i \neq 0: \)

(i) \( \eta_i(y) \equiv 1 \) on \([0, \epsilon], \) \( \eta_i(y) \equiv 0, \) on \([1 - \epsilon, 1]\) with \( \epsilon = \min\{\frac{1}{4}, |a_i|^{-\frac{1}{3}}\}, \)

(ii) \( \eta_i(y) < \frac{1}{2}|a_i|^{-1}y^{-3} \) for all \( y \in [0, 1]. \)

In case \( a_i = 0 \) we set \( \eta_i \equiv 0. \) It is easy to see that such smooth function \( \eta_i \) exists. The assumptions on \( \eta_i \) guarantee that \( \partial_y \psi_i(y) > 0 \) holds, \( \forall y \in [0, 1], \) since \( \frac{d}{dy}(a_i \eta_i(y)y^4) < \frac{1}{2}. \) Hence, the map \( \psi_i \) is a diffeomorphism. Consider now \( \tilde{f}_{0,i}(y) := f_{0,i}(\psi_i(y)) \) for each \( i \in \{1, \ldots, q\}. \) Since \( \partial_y^2 \psi_i(y) = 0 \) at \( y = 0, 1 \) and by (4.2) a direct computation gives that \( \tilde{f}_{0,i}, i \in \{1, \ldots, q\}, \) satisfy (2.4). By (4.3) and since \( \psi_i(y) = y \) near \( y = 1, \) also (2.6) is satisfied. It remains to discuss (2.7). At \( y = 0 \) we have

\[
\frac{1}{|\partial_y \tilde{f}_{0,i}(0)|^4} \partial_y^4 \tilde{f}_{0,i}(0) = \frac{1}{|\partial_x f_{0,i}(0)|^4} \left( \partial_x^4 f_{0,i}(0) + \partial_x f_{0,i}(0) \partial_y^4 \psi_i(0) \right) = \frac{1}{|\partial_x f_{0,i}(0)|^4} \partial_x^4 f_{0,i}(0) - \partial_x f_{0,i}(0) \varphi_0 \eta_i(0),
\]

and hence by (4.4) and repeating the same computations for \( \tilde{f}_{0,j} \) we find

\[
\frac{1}{|\partial_y \tilde{f}_{0,i}(0)|^4} \partial_y^4 \tilde{f}_{0,i}(0) = \frac{1}{|\partial_y \tilde{f}_{0,j}(0)|^4} \partial_y^4 \tilde{f}_{0,j}(0),
\]
Proof of Theorem 1.3. The statement is a direct consequence of Corollary 3.7.

Lemma 4.1. (Geometric uniqueness) Let \( f, \bar{f} \) be two solutions according to Theorem 1.2 defined on \([0, T] \times [0, 1]\) and on \([0, \bar{T}] \times [0, 1]\) respectively. Moreover, let \( f, \bar{f} \) have initial data \( f_0 \) satisfying (1.9) and (1.10). Then the sets \( \Gamma(t) = \{ (f_1(t, x), \ldots, f_q(t, x)) : x \in [0, 1] \} \) and \( \bar{\Gamma}(t) = \{ (\bar{f}_1(t, x), \ldots, \bar{f}_q(t, x)) : x \in [0, 1] \} \) coincide on \([0, T']\) for some \( T' > 0 \).

Proof. Let \( f, \bar{f} \) be as in the statement. Then, according to (1.11), there exist orientation preserving diffeomorphisms \( \psi_i, \bar{\psi}_i : [0, 1] \to [0, 1] \) such that
\[
fi(t = 0) = f_{0,i} \circ \psi_i \quad \text{and} \quad \bar{fi}(t = 0) = f_{0,i} \circ \bar{\psi}_i \quad \text{for} \ i = 1, \ldots, q.
\]
Consider \( g_i(t, x) = fi(t, \psi_i^{-1}(x)) \) and \( \bar{g}_i(t, x) = \bar{fi}(t, \bar{\psi}_i^{-1}(x)) \) defined on \([0, T] \times [0, 1]\) and on \([0, \bar{T}] \times [0, 1]\) respectively. Note that the regularity is preserved (recall Remark B.4) and the sets \( \Gamma(t) \) and \( \bar{\Gamma}(t) \) do not change. Now \( g_i \) and \( \bar{g}_i \) are solutions of (1.8) with possibly different tangential components but the same initial parametrization \( f_{0,i} \). In particular, the tangential components of the speed at the boundary at \( t = 0 \) coincide, i.e., \( \phi_{0,i} = \bar{\phi}_{0,i} \) at the boundary. Of course, also \( \mathcal{L}(g_i(0)) = \mathcal{L}({\bar{g}_i}(0)) \).

We argue now that the sets \( \Gamma(t) = \{ (g_1(t, x), \ldots, g_q(t, x)) : x \in [0, 1] \} \) and \( \bar{\Gamma}(t) = \{ (\bar{g}_1(t, x), \ldots, \bar{g}_q(t, x)) : x \in [0, 1] \} \) coincide for some small time using the uniqueness of the solution of the analytic problem. By Lemma C.1 there exist \( T' \in (0, \min\{T, \bar{T}\}) \) and diffeomorphisms \( \phi_i(t, \cdot), \bar{\phi}_i(t, \cdot) \) with \( \phi_i, \bar{\phi}_i \in \mathcal{C}^{4+\alpha,4+\alpha}([0, T'] \times [0, 1], \mathbb{R}), i = 1, \ldots, q \), such that
\[
\bar{g}_i(t, y) := g_i(t, \phi_i(t, y)) \quad \text{and} \quad \bar{g}_i(t, y) := \bar{g}_i(t, \bar{\phi}_i(t, y)),
\]
for \( i = 1, \ldots, q \), are in \( \mathcal{C}^{4+\alpha,4+\alpha}([0, T'] \times [0, 1], \mathbb{R}^n) \) and solve the analytical problem (2.2) together with (2.4) and with initial datum \( g_i(t = 0) \circ \phi_i(t = 0, \cdot) \) and \( \bar{g}_i(t = 0) \circ \bar{\phi}_i(t = 0, \cdot) \) respectively. The initial diffeomorphisms \( \phi_i(t = 0, \cdot) \) and \( \bar{\phi}_i(t = 0, \cdot) \) have to satisfy (C1) and, since \( g_i, \bar{g}_i \) have the same initial datum, can be chosen equal.

By uniqueness of the solution for Problem (2.2) (recall Theorem 2.3),
\[
\bar{g}_i(t, \cdot) = g_i(t, \phi_i(t, \cdot)) = \bar{g}_i(t, \cdot) := \bar{g}_i(t, \bar{\phi}_i(t, \cdot)),
\]
on \([0, T']\) as \( C^{4,\alpha}\)-maps. In particular, the claim follows.
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A Supporting materials

Here we collect some useful formulas. For \( f : I \to \mathbb{R}^n \) a regular parametrization of a curve and for sufficiently smooth \( \phi : I \to \mathbb{R}^n \) the first variation of the length is given by

\[
\frac{d}{d\varepsilon} \mathcal{L}(f + \varepsilon \phi) \bigg|_{\varepsilon = 0} = \frac{d}{d\varepsilon} \int_I \|\partial_x (f + \varepsilon \phi)\| \, dx \bigg|_{\varepsilon = 0} = \langle \partial_s f, \phi \rangle \bigg|_{\partial I} - \int_I \langle \kappa, \phi \rangle \, ds, \tag{A1}
\]

while the first variation of elastic energy (see [6, Proof of Lemma A1]) is

\[
\frac{d}{d\varepsilon} \mathcal{E}(f + \varepsilon \phi) \bigg|_{\varepsilon = 0} = \frac{d}{d\varepsilon} \int_I \|\partial_x (f + \varepsilon \phi)\|^2 \, dx \bigg|_{\varepsilon = 0} = \langle \partial_s f, \kappa \rangle \bigg|_{\partial I} + \langle \phi, \nabla_s \kappa + \frac{1}{2} \|\kappa\|^2 \partial_s f \rangle \bigg|_{\partial I} + \int_I \langle \nabla_s^2 \kappa + \frac{1}{2} \|\kappa\|^2 \kappa, \phi \rangle \, ds. \tag{A2}
\]

Moreover,

\[
\kappa = \partial_s^2 f = \frac{\partial_x^2 f}{\|\partial_x f\|^4} - (\partial_x^2 f, \partial_x f) \frac{\partial_x f}{\|\partial_x f\|^4} = \frac{\partial_x^2 f}{\|\partial_x f\|^4} - (\frac{\partial_x^2 f}{\|\partial_x f\|^4}, \partial_s f) \partial_s f,
\]

\[
|\kappa|^2 = \frac{|\partial_x^2 f|^2}{\|\partial_x f\|^4} - \left(\frac{\langle \partial_x^2 f, \partial_x f \rangle \rangle}{\|\partial_x f\|^6}\right)^2,
\]

\[
\partial_s \kappa = \partial_s^3 f = \frac{\partial_x^3 f}{\|\partial_x f\|^3} - \left(\frac{\partial_x^3 f}{\|\partial_x f\|^3}, \partial_s f \right) \partial_s f
\]

\[
- 3\kappa \langle \frac{\partial_x^2 f}{\|\partial_x f\|^2}, \partial_s f \rangle + \left(\frac{\langle \partial_x^2 f, \partial_x f \rangle \rangle}{\|\partial_x f\|^6}\right)^2 \partial_s f - \frac{|\partial_x^2 f|^2}{\|\partial_x f\|^4} \partial_s f,
\]

\[
\langle \kappa, \partial_s \kappa \rangle = \frac{\langle \partial_x^3 f, \partial_x^2 f \rangle}{\|\partial_x f\|^5} - \frac{\langle \partial_x^2 f, \partial_x f \rangle}{\|\partial_x f\|^7} \langle \partial_s f, \partial_x^2 f \rangle - 3\frac{|\partial_x^2 f|^2}{\|\partial_x f\|^7} \langle \partial_x f, \partial_x^3 f \rangle
\]

\[
+ 3 \frac{(\langle \partial_x f, \partial_x^2 f \rangle)^3}{\|\partial_x f\|^9},
\]

\[
\partial_x^2 \kappa = \partial_s^4 f = \frac{\partial_x^4 f}{\|\partial_x f\|^4} - 6\langle \partial_x^2 f, \partial_x f \rangle \frac{\partial_x^3 f}{\|\partial_x f\|^6} - 4 \frac{|\partial_x^2 f|^2}{\|\partial_x f\|^6} \partial_s^2 f - 4 \frac{|\partial_x^2 f|^2}{\|\partial_x f\|^6} \langle \partial_x^3 f, \partial_x f \rangle
\]
\[ + 19 \partial_x^2 f \left( \frac{(\partial_x^2 f, \partial_x f)^2}{|\partial_x f|^8} \right) + \frac{\partial_x f}{|\partial_x f|} \left[ - \frac{\partial_x^2 f}{|\partial_x f|^5}, \partial_x f \right] - 3 \langle \frac{\partial_x^3 f}{|\partial_x f|^5}, \partial_x f \rangle \\
+ 13 \langle \frac{\partial_x^3 f}{|\partial_x f|^7}, \partial_x f \rangle (\partial_x^2 f, \partial_x f) + 13 \langle \frac{\partial_x^2 f}{|\partial_x f|^7}, \partial_x f \rangle |\partial_x f|^2 \\
- \frac{28}{|\partial_x f|^9} \left( \frac{(\partial_x^2 f, \partial_x f)^3}{|\partial_x f|^3} \right) \right]. \tag{A3} \]

In particular it follows for the velocity in (1.8)

\[
\partial_t f = -\nabla_s^2 \vec{k} - \frac{1}{2} |\vec{k}|^2 \vec{k} + \lambda \vec{k} + \phi \partial_s f = -\partial_t^2 \vec{k} - 3 (\partial_t \vec{k}, \vec{k}) \partial_t f - \frac{3}{2} |\vec{k}|^2 \vec{k} + \lambda \vec{k} + \phi \partial_s f
\]

\[
= - \frac{\partial_t^4 f}{|\partial_t f|^4} + 6 \langle \frac{\partial_t^2 f}{|\partial_t f|^4}, \partial_t f \rangle \frac{\partial_t^3 f}{|\partial_t f|^6} \\
+ \frac{\partial_t^2 f}{|\partial_t f|^2} \left( \frac{5}{2} \frac{|\partial_t^2 f|^2}{|\partial_t f|^4} + 4 \frac{(\partial_t^3 f, \partial_t f)}{|\partial_t f|^4} - \frac{35}{2} \frac{(\partial_t^2 f, \partial_t f)^2}{|\partial_t f|^6} + \lambda \right) \\
- \frac{\partial_s f}{|\partial_s f|^4} \left[ 1 + \frac{2}{3} \frac{(\partial_s^2 f, \partial_s f)}{|\partial_s f|^4} + 10 \frac{(\partial_s^3 f, \partial_s f)}{|\partial_s f|^6} (\partial_s^2 f, \partial_s f) + \frac{5}{2} \frac{(\partial_s^2 f, \partial_s f)}{|\partial_s f|^4} \right] \\
- \frac{35}{2} \frac{(\partial_s^2 f, \partial_s f)^3}{|\partial_s f|^9} + \lambda \frac{2}{3} \frac{(\partial_s^2 f, \partial_s f)^2}{|\partial_s f|^8} \phi. \tag{A4}\]

**Remark A.1.** Let \( f : [0, 1] \to \mathbb{R}^n \) be a regular sufficiently smooth curve parametrized by constant speed (equal to its length), that is \( |\partial_t f| = L(f) \) on \([0, 1]\). Then \( \partial_t f, \partial_s^2 f \)

\[ = 0, \quad |\partial_s^2 f|^2 + |\partial_s^2 f, \partial_s^3 f| = 0, \quad \vec{k} = \frac{\partial_s^2 f}{|\partial_s f|^2}, \quad \text{and} \quad 3 \langle \partial_s^2 f, \partial_s^3 f \rangle + \langle \partial_s f, \partial_s^4 f \rangle = 0 \]

so that (with similar calculations as in (A4)) we immediately obtain

\[
\nabla_s^2 \vec{k} = \frac{1}{|\partial_s f|^4} \partial_s^2 f - \frac{1}{|\partial_s f|^6} (\partial_s^3 f, \partial_s f) \partial_s f + \frac{1}{|\partial_s f|^6} |\partial_s^2 f|^2 \partial_s^2 f
\]

\[
= \frac{1}{|\partial_s f|^4} \partial_s^2 f + \frac{3}{|\partial_s f|^6} (\partial_s^3 f, \partial_s^3 f) \partial_s f + \frac{1}{|\partial_s f|^6} |\partial_s^2 f|^2 \partial_s^2 f. \]

**B Function spaces**

Our short-time existence theory uses parabolic Hölder spaces, which are defined as follows. Following [19, page 66], for a function \( v : [0, T] \times [0, 1] \to \mathbb{R} \) and \( \rho \in (0, 1) \) let

\[
[v]_{\rho, x} := \sup_{(t, x), (t', y) \in [0, T] \times [0, 1]} \frac{|v(t, x) - v(t, y)|}{|x - y|^\rho},
\]

\[
[v]_{\rho, t} := \sup_{(t, x), (t', x') \in [0, T] \times [0, 1]} \frac{|v(t, x) - v(t', x)|}{|t - t'|^\rho}.
\]

As in [19, pages 91 and 66] we define

\[
C^{k+\alpha, k+\alpha} ([0, T] \times [0, 1]) \quad \text{for } \alpha \in (0, 1) \text{ and } k \in \mathbb{N}_0
\]
to be the space of all maps \( v : [0, T] \times [0, 1] \to \mathbb{R} \) with continuous derivatives \( \partial_t^j \partial_x^i v \) for \( i, j \in \mathbb{N} \cup \{0\} \) with \( 4i + j \leq k \) and such that the norm

\[
\|v\|_{C^{\frac{k+m}{4}, m+\alpha}([0,T]\times[0,1])} := \sum_{4i+j=0}^{k} \sup_{(t,x) \in [0,T]\times[0,1]} |\partial_t^j \partial_x^i v(t,x)| + \sum_{4i+j=k} |\partial_t^j \partial_x^i v|_{\alpha,x} + \sum_{0 < k+\alpha - 4i - j < 4} |\partial_t^j \partial_x^i v|_{\frac{k+\alpha - 4i - j}{4},t}
\]

is finite. Notice that in the last term we sum over \( i, j \)'s satisfying the inequality. In the proofs, in order to avoid lengthy notation, we do not write the set when considering the parabolic Hölder spaces. That is we write simply \( \|v\|_{C^{\frac{k+m}{4}, m+\alpha}} \) instead of \( \|v\|_{C^{\frac{k+m}{4}, m+\alpha}([0,T]\times[0,1])} \). When considering the Hölder norms in only one variable we always write the set, for instance in \( C^{4,0}([0,1]) \) or \( C^{0,\frac{4}{5}}([0,T]) \).

When dealing with vector-valued maps we use the convention that the \( C^{\frac{k+m}{4}, m+\alpha} \)-norm of the vector is the sum of the norms of its components.

Remark B.1. From the definition it follows that for \( m \leq k, m, k \in \mathbb{N}_0 \)

\[
C^{\frac{m+k}{4}, m+\alpha}([0,T]\times[0,1]) \subset C^{\frac{k+m}{4}, k+\alpha}([0,T]\times[0,1]),
\]

and if \( v \in C^{\frac{k+l}{4}, k+\alpha}([0,T]\times[0,1]) \), then \( \partial_t^l v \in C^{\frac{k-l+\alpha}{4}, k-l+\alpha}([0,T]\times[0,1]) \) for all \( 0 \leq l \leq k \) so that

\[
\|\partial_t^l v\|_{C^{\frac{k-l+\alpha}{4}, k-l+\alpha}([0,T]\times[0,1])} \leq \|v\|_{C^{\frac{k+l}{4}, k+\alpha}([0,T]\times[0,1])}.
\]

In particular at each fixed \( x \in [0,1] \) we have \( \partial_t^l v(.,x) \in C^{s,\beta}([0,T]) \) with \( s = \frac{k-l+\alpha}{4} \) and \( \beta = \frac{k-l+\alpha}{4} - s \).

We will use often the following properties of the Hölder norms.

Lemma B.2. For \( k \in \mathbb{N}_0 \), \( \alpha, \beta \in (0,1) \) and \( T > 0 \) we have

1. if \( v, w \in C^{\frac{k+l}{4}, k+\alpha}([0,T]\times[0,1]) \), then

\[
\|vw\|_{C^{\frac{k+l}{4}, k+\alpha}} \leq C \|v\|_{C^{\frac{k+l}{4}, k+\alpha}} \|w\|_{C^{\frac{k+l}{4}, k+\alpha}},
\]

with \( C = C(k) > 0 \);

2. if \( v \in C^{\frac{2}{5}, \alpha}([0,T]\times[0,1]) \), \( v(t,x) \neq 0 \) for all \((t,x)\), then

\[
\frac{1}{2} \|v\|_{C^{\frac{2}{5}, \alpha}} \leq \left( \int_{[0,T]\times[0,1]} |v|^2 \right)^{\frac{1}{2}} \|v\|_{C^{\frac{2}{5}, \alpha}}.
\]

Similar statements are true for functions in \( C^{k,\beta}([0,T]) \) and \( C^{k,\beta}([0,1]) \).

 Proof. It follows by the definition of the norms and direct computation. \( \square \)

Lemma B.3. For \( n \in \mathbb{N}, k \in \mathbb{N}_0, \alpha, \beta \in (0,1) \) and \( T > 0 \) we have
1. if a vector-field $v \in C^{\frac{\alpha}{4},\alpha}([0, T] \times [0, 1]; \mathbb{R}^n)$, then

$$\| |v| \|_{C^{\frac{\alpha}{4},\alpha}} \leq C \|v\|_{C^{\frac{\alpha}{4},\alpha}},$$

with $C = C(n)$. 

2. for $v, w \in C^{\frac{\alpha}{4},\alpha}([0, T] \times [0, 1]; \mathbb{R}^n)$ we have

$$\| |v| - |w| \|_{C^{\frac{\alpha}{4},\alpha}} \leq C \frac{1}{\|v| + |w| \|_{C^0([0,T] \times [0,1])}} (\|v\|_{C^{\frac{\alpha}{4},\alpha}} + \|w\|_{C^{\frac{\alpha}{4},\alpha}})^2 \|v - w\|_{C^{\frac{\alpha}{4},\alpha}}$$

with $C = C(n)$. Similar statements are true for functions in $C^{k,\beta}([0, T])$ and $C^{k,\beta}([0, 1])$.

**Proof.** The main observation is that one has to be careful about the treatment of the Hölder seminorms. The first statement relies on the equivalence of the $l_2$-norm and $l_1$-norm in $\mathbb{R}^n$, which gives

$$| |v(x)| - |v(y)| | \leq |v(x) - v(y)| \leq C(n) \sum_{j=1}^{n} |v^j(x) - v^j(y)|$$

for any vector valued map $v$. The last inequality is needed because of our convention for the Hölder norm of a vector valued function. For the second statement, the aim is to manipulate the considered map in such a way that it is written as a product of functions and we can apply the previous lemma. We can write

$$\| |v| - |w| \|_{C^{\frac{\alpha}{4},\alpha}} = \left\| \frac{|v|^2 - |w|^2}{|v| + |w|} \right\|_{C^{\frac{\alpha}{4},\alpha}} \leq C \| |v|^2 - |w|^2 \|_{C^{\frac{\alpha}{4},\alpha}} \frac{1}{|v| + |w|} \|_{C^{\frac{\alpha}{4},\alpha}}$$

$$\leq C \frac{1}{|v| + |w|} \|_{C^0} \left\| |v| + |w| \right\|_{C^{\frac{\alpha}{4},\alpha}} \left\| \sum_{j=1}^{n} ((v^j)^2 - (w^j)^2) \right\|_{C^{\frac{\alpha}{4},\alpha}}$$

$$\leq C \sum_{j=1}^{n} \frac{1}{|v| + |w|} \|_{C^0} \left\| |v| + |w| \right\|_{C^{\frac{\alpha}{4},\alpha}} \left\| v^j + w^j \right\|_{C^{\frac{\alpha}{4},\alpha}} \left\| v^j - w^j \right\|_{C^{\frac{\alpha}{4},\alpha}}$$

and the claim follows. 

We will also need that the composition of Hölder functions is again Hölder. Here one needs to pay attention since, in general, if $f \in C^{0,\alpha}(I)$, $g \in C^{0,\beta}(I)$ then $f \circ g \in C^{0,\alpha\beta}(I)$, i.e. the Hölder exponent of the composition is given by the product of the Hölder exponents. Therefore, in order not to lose in regularity by applying directly this rule, we need to look carefully at the terms we are working with. In particular we exploit that we always consider the convolution of a Hölder map with a diffeomorphism and hence we do not lose in the Hölder power.

**Remark B.4.** If $f_0 \in C^{k,\alpha}([0, 1])$, $k \geq 4$, then the diffeomorphism $v$ defined as in (4.1) is also in $C^{k,\alpha}([0, 1])$ by the previous lemmata. By direct verification we find that
also $\phi := v^{-1}$ belongs to $C^{k,\alpha}([0, 1])$. Then $(\partial_x^i f_0) \circ \phi \in C^{0,\alpha}([0, 1])$ for $0 \leq i \leq k$ thanks to the fact that $\phi$ is a diffeomorphism and hence in particular in $C^{0,1}([0, 1])$. Since $\partial_x^k (f_0 \circ \phi)$ is a polynomial in the maps $(\partial_x^i f_0) \circ \phi$ and (several products of) $\partial_x^j \phi$, for $1 \leq i \leq k$, $1 \leq j \leq k$, we see that $f_0 \circ \phi \in C^{k,\alpha}([0, 1])$.

**Lemma B.5.** Let $T < 1$ and $v \in C^{4+\alpha,4+\alpha}([0, T] \times [0, 1])$ such that $v(0, x) = 0$, for any $x \in [0, 1]$ then

$$\|\partial_x^l v\|_{C^{m+\beta, m+\beta}} \leq C(m) T^\beta \|v\|_{C^{4+\alpha,4+\alpha}}$$

for all $l, m \in \mathbb{N}_0$ such that $l + m < 4$. Here $\beta = \max\{\frac{1-\alpha}{4}, \frac{\alpha}{4}\} \in (0, 1)$; more precisely for $l \geq 1$ then $\beta = \frac{\alpha}{4}$.

In particular, for each $x \in [0, 1]$ fixed

$$\|\partial_x^l v(\cdot, x)\|_{C^{m+\beta, m+\beta}([0,T])} \leq C(m) T^\beta \|v\|_{C^{4+\alpha,4+\alpha}}$$

for all $l, m \in \mathbb{N}_0$ such that $l + m < 4$.

**Proof.** Since by definition of the norm we have (recall $m, l < 4$)

$$\|\partial_x^l v\|_{C^{m+\beta, m+\beta}} = \sum_{j=0}^m \sup_{(t,x) \in [0,T] \times [0,1]} |\partial_x^{l+j} v(t,x)| + [\partial_x^m v]_{\alpha,x} + \sum_{j=0}^m [\partial_x^l \partial_x^j v]_{m+\beta-l,t}$$

we observe that for $j \in \{0, \ldots, m\}$, $0 < l + j \leq l + m < 4$, and we have

$$\sup_{(t,x) \in [0,T] \times [0,1]} |\partial_x^{l+j} v(t,x)| = \sup_{(t,x) \in [0,T] \times [0,1]} \frac{|\partial_x^{l+j} v(t,x) - \partial_x^{l+j} v(0,x)|}{|t - 0|^{4+\alpha-(l+j)}} T^{\frac{4+\alpha-(l+j)}{4} - \frac{\alpha}{4}} \leq \sup_{(t,x) \in [0,T] \times [0,1]} [\partial_x^{l+j} v]_{\frac{4+\alpha-(l+j)}{4},t} T^\frac{\alpha}{4}.$$  

Instead, for the case $j = l = 0$ we compute

$$\sup_{(t,x) \in [0,T] \times [0,1]} |v(t,x)| = \sup_{(t,x) \in [0,T] \times [0,1]} \frac{|v(t,x) - v(0,x)|}{|t - 0|} \leq T \left( \sup_{(0,T) \times [0,1]} |\partial_t v| \right).$$

Next, with similar ideas, using the fact that $|x - y| \leq 1$ and $l + m + 1 \leq 4$ we compute

$$[\partial_x^{m+l} v]_{\alpha,x} = \sup_{(t,x), (t,y) \in [0,T] \times [0,1]} \frac{|\partial_x^{m+l} v(t,x) - \partial_x^{m+l} v(t,y)|}{|x - y|^{\alpha}}$$

$$= \sup_{(t,x), (t,y) \in [0,T] \times [0,1]} \frac{|\partial_x^{m+l} v(t,x) - \partial_x^{m+l} v(t,y)|}{|x - y|} |x - y|^{1-\alpha}$$

$$\leq \sup_{(t,x) \in [0,T] \times [0,1]} |\partial_x^{m+l} v(t,x)|$$

$$\leq [\partial_x^{m+l+1} v]_{\frac{4+\alpha-(m+l+1)}{4},t} T^{\frac{4+\alpha-(m+l+1)}{4} - \frac{\alpha}{4}} \leq [\partial_x^{m+l+1} v]_{\frac{4+\alpha-(m+l+1)}{4},t} T^\frac{\alpha}{4}$$

and for $j \in \{0, \ldots, m\}$.
\[
\left[ \partial_x^j \partial_x^l v \right]_{m+\alpha-j,l}^{m+\alpha} = \sup_{(t,x), (t',x) \in [0,T] \times [0,1]} \frac{|\partial_x^{j+l} v(t, x) - \partial_x^{j+l} v(t', x)|}{|t - t'|^{\frac{1}{4}}} T^\frac{j}{4}
\]

The above computation makes sense except for the case \( m = 3, j = 0 \) (and hence \( l = 0 \)), for which \( \frac{m+\alpha+1+j}{4} > 1 \). The case \( m = 3, l = j = 0 \) is treated as follows

\[
\left[ v \right]^{3+\alpha}_{4},t = \sup_{(t,x), (t',x) \in [0,T] \times [0,1]} \frac{|v(t) - v(t')|}{|t - t'|^{\frac{1}{4}}} T^{\frac{1}{4}}
\]

Finally putting all estimates together and recalling that \( l + j \leq l + m \leq 3, T < 1 \) and \( \alpha < 1 \) we obtain (here for \( m < 3 \), for \( m = 3 \) the arguments are similar)

\[
\| \partial_x^l v \|_{C^{\frac{m+\alpha}{4},m+\alpha}} \leq \left( \sum_{l \neq 0, j = 0}^m \left[ \partial_x^{l+j} v \right]_{\frac{m+\alpha+1}{4},j}^{\frac{m+\alpha+1}{4}} + \sum_{j = 0}^m \left[ \partial_x^{j+l} v \right]_{\frac{m+\alpha+1}{4},l}^{\frac{m+\alpha+1}{4}} \right) T^\frac{j}{4} + C \left( \sup_{[0,T] \times [0,1]} |\partial_x^l v| \right) T^{\frac{1}{4}}
\]

and the first estimate follows. The second part of the claim follows from the observation that

\[
\| \partial_x^l v (\cdot, x) \|_{C^0, \frac{m+\alpha}{4}([0,T])} \leq \| \partial_x^l v \|_{C^{\frac{m+\alpha}{4},m+\alpha}([0,T] \times [0,1])}, \tag{B2}
\]

due to (B1).

**C Diffeomorphisms**

With the following lemma we show the existence of a diffeomorphism, which transforms a solution of the geometric problem into a solution of the analytical one.

**Lemma C.1.** Let \( \{ f_1, \ldots, f_q \} \), with \( f_i : [0, T] \times [0, 1] \rightarrow \mathbb{R}^n \), be a \( C^{\frac{4+\alpha}{4},4+\alpha}([0, T] \times [0, 1]; \mathbb{R}^n) \) solution of (1.8) and (1.6) with initial datum \( \{ f_{0,1}, \ldots, f_{0,q} \} \) that satisfies (1.7), (1.9) and (1.10), i.e. a geometric solution in the sense of Theorem 1.2. Then there exists \( T' \in (0, T) \) and orientation preserving diffeomorphisms \( \phi_i(t, \cdot) \) with \( \phi_i \in C^{\frac{4+\alpha}{4},4+\alpha}([0, T'] \times [0, 1], \mathbb{R}) \), such that \( \tilde{f}_i(t, y) := f_i(t, \phi_i(t, y)), i = 1, \ldots, q \), are in \( C^{\frac{4+\alpha}{4},4+\alpha}([0, T'] \times [0, 1], \mathbb{R}^n) \) and solve the analytical problem (2.2) together with (2.4) (with the obvious notation changes) and with initial datum \( f_{0,i} \circ \phi_i(t = 0, \cdot) \), that is \( \{ \tilde{f}_1, \ldots, \tilde{f}_q \} \) is the unique solution of the analytical problem with initial datum \( f_{0,i} \) opportuneily reparametrized. \( \square \)
Proof. Let \( \{f_1, f_2, \ldots, f_q\} \) be as in the statement. Let \( \varphi_i \) be the tangential component of the velocity of \( f_i \), that is

\[
\varphi_i(t, x) := (\partial_t f_i(t, x), \partial_x f_i(t, x)) \partial_x f_i(t, x)
\]

for \((t, x) \in [0, T] \times [0, 1]\). By (1.7), (1.9), (1.10) the initial datum \( f_{0,i}, i \in \{1, \ldots, q\} \), satisfies (NC) and at the boundary points \( x = 0, 1 \)

\[
\partial_x f_{0,i}(x) = \langle \partial_x^2 f_{0,i}, \partial_x f_{0,i} \rangle \partial_x f_{0,i}(x),
\]

\[
\nabla_x^2 \tilde{\kappa}_{0,i}(x) = \frac{\partial_x^4 f_{0,i}}{|\partial_x f_{0,i}|^4}(x) - \langle \partial_x^4 f_{0,i}, \partial_x f_{0,i} \rangle \frac{\partial_x f_{0,i}}{|\partial_x f_{0,i}|^6}(x)
\]

\[
-6 \frac{\partial_x^3 f_{0,i}}{|\partial_x f_{0,i}|^6} \langle \partial_x^2 f_{0,i}, \partial_x f_{0,i} \rangle + 6 \langle \partial_x^3 f_{0,i}, \partial_x f_{0,i} \rangle \langle \partial_x^2 f_{0,i}, \partial_x f_{0,i} \rangle \frac{\partial_x f_{0,i}}{|\partial_x f_{0,i}|^8}(x).
\]

For each \( i \in \{1, \ldots, q\} \), fix a smooth diffeomorphism \( \psi_i : [0, 1] \rightarrow [0, 1] \) such that \( \partial_y \psi_i > 0 \) on \([0, 1]\) and

\[
\psi_i(0) = 0, \quad \psi_i(1) = 1,
\]

\[
\partial_y^2 \psi_i(y) + \frac{\langle \partial_x^2 f_{0,i}, \partial_x f_{0,i} \rangle}{|\partial_x f_{0,i}|^2}(y)(\partial_y \psi_i)^2(y) = 0 \quad \text{at } y = 0, 1,
\]

as well as

\[
\left( -\frac{1}{|\partial_x f_{0,i}|^6} \langle \partial_x^4 f_{0,i}, \partial_x f_{0,i} \rangle + \frac{6}{|\partial_x f_{0,i}|^8} \langle \partial_x^3 f_{0,i}, \partial_x f_{0,i} \rangle \langle \partial_x^2 f_{0,i}, \partial_x f_{0,i} \rangle 
\right)
\]

\[
-\frac{3}{|\partial_x f_{0,i}|^{10}} ((\partial_x^2 f_{0,i}, \partial_x f_{0,i}))^3 - \frac{4}{|\partial_x f_{0,i}|^6} (\partial_y \psi_i)^2 \langle \partial_x^2 f_{0,i}, \partial_x f_{0,i} \rangle \partial_y^3 \psi_i
\]

\[\left( -\frac{1}{|\partial_x f_{0,i}|^4} (\partial_y \psi_i)^4 \partial_y^4 \psi_i - \frac{1}{|\partial_x f_{0,i}|^4} \phi_{0,i} \right)(y) = 0, \quad \text{at } y = 0, 1
\]

whereby recall that \( \phi_{0,i}(1) = 0 \). There are many ways to give a diffeomorphism with the above properties. To simplify the construction, let us further impose that

\[
\partial_y \psi_i(y) = 1, \quad \text{and } \partial_y^3 \psi_i(y) = 0 \quad \text{at } y = 0, 1.
\]

Then \( \psi_i \) can be given as a perturbation of the identity map, with a similar construction as performed in the proof of Theorem 1.2, the main difference being that now we have two smooth cut-off functions \( \tilde{\eta}_0, \tilde{\eta}_1 : [0, 1] \rightarrow [0, 1] \) such that \( \tilde{\eta}_0 \) is equal to one near the origin, \( \tilde{\eta}_1 \) is equal to one near to \( y = 1 \), \( \text{supp}(\tilde{\eta}_0) \) and \( \text{supp}(\tilde{\eta}_1) \) are small and well separated, and with \( \tilde{\eta}_0, \tilde{\eta}_1 \) dropping to zero sufficiently fast to guarantee that \( \partial_y \psi_i > 0 \) everywhere. More precisely \( \psi_i : [0, 1] \rightarrow [0, 1] \) is given by

\[
\psi_i(y) = y + \tilde{\eta}_0(y)(b_0 y^2 + c_0 y^4) + \tilde{\eta}_1(y)(b_1 (y - 1)^2 + c_1 (y - 1)^4)
\]

with appropriately chosen coefficients \( b_j, c_j, j = 0, 1 \), that depend on the values of \( f_{0,i} \) and \( \phi_{0,i} \) at the boundary.
For later purposes let us observe that the inverse $\psi_i^{-1}$ satisfies $\partial_x(\psi_i^{-1}) > 0$ and
\[
\psi_i^{-1}(0) = 0, \quad \psi_i^{-1}(1) = 1,
\]
\[
\partial_x^2(\psi_i^{-1})(x) - \frac{\partial^2 f_{0,i}}{|\partial_x f_{0,i}|^2}(x) \partial_x(\psi_i^{-1})(x) = 0 \quad \text{at } x = 0, 1
\]
\[
0 = -\frac{1}{|\partial_x f_{0,i}|^2} \partial_x^4 \psi_i^{-1}(x) + 6 \frac{\partial^2 f_{0,i}}{|\partial_x f_{0,i}|^6} \partial_x^3(\psi_i^{-1})(x)
\]
\[
+ \left( \frac{\partial^4 f_{0,i}}{|\partial_x f_{0,i}|^6} - 6 \frac{\partial^3 f_{0,i}}{|\partial_x f_{0,i}|^8} \partial_x^2(\psi_i^{-1})(x) + \frac{\psi_{0,i}}{|\partial_x f_{0,i}|} \right) \partial_x(\psi_i^{-1})(x), \quad \text{at } x = 0, 1.
\]

Consider now for fixed $i \in \{1, \ldots, g\}$ the evolution equation
\[
\partial_t \phi_i(t, y) = -\frac{1}{|\partial_x f_i|^6} (\partial^4 f_i, \partial_x f_i) + \frac{4}{|\partial_x f_i|^6} (\partial^3 f_i, \partial_x f_i) \frac{\partial^2 \phi_i}{(\partial_x \phi_i)^2} - \frac{18}{|\partial_x f_i|^6} (\partial\phi_i) \frac{(\partial^2 f_i, \partial_x f_i)}{(\partial_x \phi_i)^2} + \frac{6}{|\partial_x f_i|^6} (\partial^3 f_i, \partial_x f_i) \partial_x^3 \phi_i - \frac{1}{|\partial_x f_i|^4} \frac{1}{(\partial_y \phi_i)^2} \partial_x^4 \phi_i
\]
\[
+ 10 \frac{1}{|\partial_x f_i|^8} (\partial^3 f_i, \partial_x f_i) (\partial^2 f_i, \partial_x f_i) - \frac{35}{2} \frac{1}{|\partial_x f_i|^8} (\partial_x \phi_i)^2 ((\partial^2 f_i, \partial_x f_i))^2 \partial_x^2 \phi_i
\]
\[
+ 10 \frac{1}{|\partial_x f_i|^4} \frac{1}{(\partial_y \phi_i)^2} \partial_x^3 \phi_i \partial_x^2 \phi_i + \frac{5}{2} \frac{1}{|\partial_x f_i|^8} |(\partial_x f_i)|^2 (\partial_x^2 f_i, \partial_x f_i)
\]
\[
- \frac{35}{2} \frac{1}{|\partial_x f_i|^10} ((\partial_x^2 f_i, \partial_x f_i))^3 + \lambda_j \frac{1}{|\partial_x f_i|^4} (\partial_x^2 f_i, \partial_x f_i) + \lambda_j \frac{1}{|\partial_x f_i|^2} \frac{1}{(\partial_y \phi_i)^2} \partial_x^2 \phi_i
\]
\[
- \frac{1}{|\partial_x f_i|} \phi_i(t, \phi_i(t, y)), \quad \text{(C3)}
\]

with boundary conditions
\[
\phi_i(t, 0) = 0, \quad \phi_i(t, 1) = 1, \quad \partial_x^2 \phi_i(t, y) + \frac{\langle 2 f_i, \partial_x f_i \rangle}{|\partial_x f_i|^2} (\partial_y \phi_i)^2 (t, y) = 0 \quad \text{at } y = 0, 1,
\]
\[
\text{for all } t \text{ and initial condition}
\]
\[
\phi_i(0, y) = \psi_i(y), \quad y \in [0, 1]. \quad \text{(C5)}
\]

Since at the boundary points
\[
\partial_x^2 f_i = \langle 2 f_i, \partial_x f_i \rangle \frac{\partial_x f_i}{|\partial_x f_i|^2}, \quad \text{(C6)}
\]
using (C4) we infer that the right-hand side of (C3) is given by
\[- \frac{1}{|\partial_x f_i|^6} (\partial_x^4 f_i, \partial_x f_i) + \frac{6}{|\partial_x f_i|^8} (\partial_x^3 f_i, \partial_x f_i)(\partial_x^2 f_i, \partial_x f_i) - \frac{3}{|\partial_x f_i|^{10}} ((\partial_x^2 f_i, \partial_x f_i))^3\]
\[- \frac{4}{|\partial_x f_i|^6} \frac{1}{(\partial_y \phi_i)^2} (\partial_x^2 f_i, \partial_x f_i) \partial_x \phi_i - \frac{1}{|\partial_x f_i|^4} (\partial_y \phi_i)^4 \partial_y \phi_i - \frac{1}{|\partial_x f_i|} \phi_t(t, y),\]
therefore, by (C2), our initial data \(\psi_i\) satisfies compatibility conditions of order zero.

The idea is that \(\tilde{f}_i := f_i \circ \phi_i\) is a solution of the analytical problem with initial datum \(f_{i,0} \circ \psi_i, i \in \{1, \ldots, q\}\). As a first step one should prove existence of solutions \(\phi_i\) to (C3), (C4) with (C5). One could proceed as in Sect. 3 to prove existence of a solution. By doing this one notices that, for instance, the term \(\tilde{f}_i \circ \phi_i\) creates a problem in the contraction argument since the tangential component \(\phi_i\) is only \(C^{2,\alpha}\) (since it depends also on \(\partial_x^4 f_i\)). To avoid this problem we follow an idea presented in [13] and look instead for the evolution equation satisfied by the inverse of the diffeomorphisms \(y \mapsto \phi_i(t, y)\). Let \(\eta_i = \eta_i(t, x)\) be such that \(\eta_i(t, \phi_i(t, y)) = y\) for \(y \in [0, 1]\). Since
\[
\partial_y \phi_i = \frac{1}{\partial_x \eta_i}, \quad \partial_x^2 \phi_i = -\frac{\partial_x^2 \eta_i}{(\partial_x \eta_i)^3}, \quad \partial_x^3 \phi_i = -\frac{\partial_x^3 \eta_i}{(\partial_x \eta_i)^4} + \frac{3(\partial_x^2 \eta_i)^2}{(\partial_x \eta_i)^5},
\]
we find that the evolution equation for the inverse reads
\[
\partial_t \eta_i(t, x) = -\partial_x \eta_i \partial_x \phi_i = \frac{6}{|\partial_x f_i|^6} (\partial_x^4 f_i, \partial_x f_i) \partial_x \eta_i + \frac{4}{|\partial_x f_i|^6} (\partial_x^3 f_i, \partial_x f_i) \partial_x^2 \eta_i
\]
\[
+ \frac{10}{|\partial_x f_i|^6} (\partial_x^2 f_i, \partial_x f_i) \partial_x \eta_i - \frac{1}{|\partial_x f_i|^4} \partial_x \eta_i
\]
\[
- \frac{35}{|\partial_x f_i|^8} ((\partial_x^2 f_i, \partial_x f_i))^3 \partial_x \eta_i
\]
\[
- \frac{5}{2} \frac{1}{|\partial_x f_i|^6} (\partial_x^2 f_i, \partial_x f_i)^2 \partial_x \eta_i + \frac{5}{2} \frac{1}{|\partial_x f_i|^6} (\partial_x^2 f_i, \partial_x f_i)^2 \partial_x \eta_i
\]
\[
+ \frac{35}{2} \frac{1}{|\partial_x f_i|^10} ((\partial_x^2 f_i, \partial_x f_i))^3 \partial_x \eta_i - \lambda_i \frac{1}{|\partial_x f_i|^4} (\partial_x^2 f_i, \partial_x f_i) \partial_x \eta_i + \lambda_i \frac{1}{|\partial_x f_i|^2} \partial_x \eta_i \]
\[
+ \frac{1}{|\partial_x f_i|} \phi_i(t, x) \partial_x \eta_i
\]
(C7)

with boundary conditions
\[
\eta_i(t, 0) = 0, \quad \eta_i(t, 1) = 1,
\]
\[
\partial_x^2 \eta_i(t, x) - \frac{(\partial_x^2 f_i, \partial_x f_i)}{|\partial_x f_i|^2} \partial_x \eta_i(t, x) = 0, \quad \forall x \in [0, 1),
\]
and initial condition
\[
\eta_i(t = 0, x) = \psi_i^{-1}(x), \quad x \in [0, 1].
\]
Note that by (C6) and (C8) the right hand side of (C7) at the boundary points is given by
\[
\begin{align*}
&- \frac{1}{|\partial_x f_1|^4} \partial_x^4 \eta_i(t, x) + 6 \frac{\partial^2 f_1}{|\partial_x f_1|^6} \partial_x^3 \eta_i(t, x) \\
&+ \left( \frac{\partial^4 f_1}{|\partial_x f_1|^6} - 6 \frac{\partial^3 f_1}{|\partial_x f_1|^8} \right) \partial_x \eta_i(t, x).
\end{align*}
\]
We see that by the conditions satisfied by \( \psi_i^{-1} \) at the boundary also for this equation the compatibility condition of order zero are satisfied. Observe that \( \eta_i \) satisfies a linear parabolic problem of fourth order with linear boundary conditions (as opposed to its inverse \( \phi_i \)). Standard linear parabolic theory (see [19]) applies so that we obtain the existence of a solution \( \eta_i \in C^{4+\alpha,4+\alpha}([0, T] \times [0, 1], \mathbb{R}) \). Since \( \eta_i(0, \cdot) \) is a diffeomorphism, by possibly reducing the time interval we can ensure that \( \eta_i(t, \cdot) \) is a family of diffeomorphism on \([0, T']\) for some \( 0 < T' \leq T \). The existence of its inverse \( \phi_i \) yields the existence of its inverse \( \eta_i \).

Finally, by a direct computation one sees that \( \tilde{f}_i := f_i \circ \phi_i \) is a solution of the analytical problem with initial datum \( f_{0,i} \circ \psi_i, i \in \{1, \ldots, q\} \). Indeed,
\[
\partial_t \tilde{f}_i(t, y) = (\partial_t f_i)(t, \phi_i(t, y)) + \partial_x f_i(t, \phi_i(t, y)) \partial_y \phi_i(t, y),
\] (C9)

and the computations performed in (A4), or the formula (2.2) and (2.3), yield
\[
(\partial_t f_i)(t, \phi_i(t, y))) = \left[ - \nabla_x^2 \tilde{f}_i - \frac{1}{2} \tilde{\kappa}_i \tilde{\kappa}_i + \lambda_i \kappa_i \right](t, y) + \phi_i(t, \phi_i(t, y)) \partial_x f_i(t, \phi_i(t, y))
\]
\[
= \left[ - \frac{\partial_y^4 \tilde{f}_i}{|\partial_y \tilde{f}_i|^4} + h(\tilde{f}_i, \partial_y \tilde{f}_i, \partial_x^2 \tilde{f}_i, \partial_x^3 \tilde{f}_i) \right](t, y)
\]
\[
- \left[ \mathcal{R}_i(t, y) - \phi_i(t, \phi_i(t, y)) \right] \frac{1}{|\partial_x \tilde{f}_i(t, \phi_i(t, y))|} \partial_x \tilde{f}_i(t, \phi_i(t, y))
\] (C10)

where
\[
\mathcal{R}_i(t, y) = \left[ - \left( \frac{\partial_y^4 \tilde{f}_i}{|\partial_y \tilde{f}_i|^5}, \partial_y \tilde{f}_i \right) + 10 \left( \frac{\partial_y^2 \tilde{f}_i, \partial_y \tilde{f}_i}{|\partial_y \tilde{f}_i|^7} \right) \left( \partial_y^3 \tilde{f}_i, \partial_y \tilde{f}_i \right) + \frac{5}{2} \left( \partial_y^2 \tilde{f}_i, \partial_y \tilde{f}_i \right) \frac{|\partial_y^2 \tilde{f}_i|^2}{|\partial_y \tilde{f}_i|^7} \right]
\]
\[
- \frac{35}{2} \left( \frac{\partial_y^2 \tilde{f}_i, \partial_y \tilde{f}_i}{|\partial_y \tilde{f}_i|^3} \right)^3 + \frac{\lambda_i}{|\partial_y \tilde{f}_i|^3} \left( \frac{\partial_y^2 \tilde{f}_i, \partial_y \tilde{f}_i}{|\partial_y \tilde{f}_i|^3} \right)^2 \right](t, y).
\]

Note that \( \partial_y \tilde{f}_i(t, y) = \partial_x f_i(t, \phi_i(t, y)) \partial_y \phi_i(t, y) \) and
\[
\begin{align*}
\partial_y^2 \tilde{f}_i &= \partial_x^2 f_i(\partial_y \phi_i)^2 + \partial_x f_i \partial_y^2 \phi_i, \\
\partial_y^3 \tilde{f}_i &= \partial_x^3 f_i(\partial_y \phi_i)^3 + 3 \partial_x^2 f_i \partial_y \phi_i \partial_y \phi_i + \partial_x f_i \partial_y^3 \phi_i \\
\partial_y^4 \tilde{f}_i &= \partial_x^4 f_i(\partial_y \phi_i)^4 + 6 \partial_x^3 f_i(\partial_y \phi_i)^2 \partial_y^2 \phi_i + 3 \partial_x^2 f_i (\partial_y^3 \phi_i)^2 + 4 \partial_x^2 f_i \partial_y \phi_i \partial_y^3 \phi_i + \partial_x f_i \partial_y^4 \phi_i.
\end{align*}
\]
Using that $\vec{k}_i = 0$ at the boundary, we can immediately verify that $\partial^2_{\tau} f_i(t, y) = 0$ at $y = 0, 1$ for all times. Moreover with (C4) and (C6) we obtain at the boundary

$$
\frac{\partial^4_{y} \vec{f}_i}{|\partial_x f_0, i|^4}(t, y) = \left( \frac{\partial^4_{\tau} f_i}{|\partial_x f_0, i|^4} - 6\partial^3_{\tau} f_i \frac{1}{|\partial_x f_0, i|^6}(\partial^2_{\tau} f_i, \partial_x f_i) \right)(t, y) + \left( 3(\partial^2_{\tau} f_i, \partial_x f_i)^3 \frac{1}{|\partial_x f_0, i|^4} + 4(\partial^2_{\tau} f_i, \partial_x f_i) \frac{1}{|\partial_x f_0, i|^6} \partial^3_x \phi_i + \frac{1}{|\partial_x f_0, i|^4} \partial^4_{\phi} \phi_i \right) \partial_x f_0, i(t, y).
$$

Therefore at time $t = 0$, using the compatibility conditions for $\phi_i$ (i.e. our choice of $\psi_i$) and the expression for $\nabla^2_x \vec{k}_0, i$ we obtain at $y = 0, 1$

$$
\frac{\partial^4_{y} \vec{f}_0, i}{|\partial_x f_0, i|^4} = \nabla^2_x \vec{k}_0, i + \left( \frac{\langle \partial^4_{y} f_0, i, \partial_x f_0, i \rangle}{|\partial_x f_0, i|^6} \frac{1}{|\partial_x f_0, i|^8} \partial_x f_0, i \right) - 6(\partial^3_{y} f_0, i, \partial_x f_0, i) \langle \partial^2_{y} f_0, i, \partial_x f_0, i \rangle \frac{1}{|\partial_x f_0, i|^8} \partial_x f_0, i + \left( 3(\partial^2_{y} f_0, i, \partial_x f_0, i)^3 \frac{1}{|\partial_x f_0, i|^4} + 4(\partial^2_{y} f_0, i, \partial_x f_0, i) \frac{1}{|\partial_x f_0, i|^6} \partial^3_x \phi_i + \frac{1}{|\partial_x f_0, i|^4} \partial^4_{y} \phi_i \right) \partial_x f_0, i
$$

so that we immediately derive that the Compatibility conditions 2.1 for the analitical problem are verified. Next, since $\partial_x \phi_i > 0$ we verify that $\mathcal{R}_i = \phi_i + |\partial_x f_i| \partial_y \phi_i$. This, together with (C9) and (C10) shows that indeed, $\vec{f}_i := f_i \circ \phi_i$ solves (2.2) and (2.4). \hfill \Box

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