Macroscopic Quantum Dynamics of a Free Domain Wall in a Ferromagnet

Junya Shibata† and Shin Takagi††
Department of Physics, Tohoku University, Sendai 980-8578, Japan
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Abstract

We study macroscopic quantum dynamics of a free domain wall in a quasi-one-dimensional ferromagnet by use of the spin-coherent-state path integral in discrete-time formalism. Transition amplitudes between typical states are quantitatively discussed by use of stationary-action approximation with respect to collective degrees of freedom representing the center position and the chirality of the domain wall. It is shown that the chirality may be loosely said to be canonically conjugate to the center position; the latter moves with a speed depending on the former. It is clarified under what condition the center position can be regarded as an effective free-particle position, which exhibits the phenomenon of wave-packet spreading. We demonstrate, however, that in some case the non-linear character of the spin leads to such a dramatic phenomenon of a non-spreading wave packet as to completely invalidate the free-particle analogy. In the course of the discussion, we also point out various difficulties associated with the continuous-time formalism.

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I. INTRODUCTION

Recent developments in low-temperature measurement techniques and the so-called nanostructure technology enable us to study low-dimensional magnetism in mesoscopic magnetic systems. Among others, dynamics of a domain wall in a ferromagnet has attracted much attention both theoretically and experimentally, because it is expected to exhibit quantum-mechanical aspects at sufficiently low temperatures. A domain wall contains a (semi-)macroscopic number of spins, its width being typically $10 \sim 1000 \text{ Å}$. Hence, if its quantum-mechanical behavior was found, it would be an evidence of macroscopic quantum phenomena (MQP). To list just a few of the theoretical works about possible MQP involving such a domain wall: depinning of a domain wall via macroscopic quantum tunneling (MQT), coherent tunneling through a periodic pinning potential, macroscopic quantum coherence (MQC) of the chirality.

One of the standard procedures to discuss MQP begins by deriving an effective action in terms of those collective degrees of freedom which directly describe the tunneling in question. In the case of the magnetic domain wall, the relevant collective degrees of freedom are the center position and/or the chirality of the wall. Existing works in the literature then treat the effective action in the Caldeira-Leggett scheme to evaluate the tunneling rate. However, as emphasized by Leggett, one should probe a quantum-mechanical time evolution to check whether MQP (especially, MQC) have occurred. Hence, what is needed on the theoretical side is to evaluate not only tunneling rates but also relevant transition amplitudes.

As a technique to evaluate a transition amplitude, the spin-coherent-state path integral in continuous-time formalism is frequently used. However, as noted by some workers, this formalism has some fundamental difficulties even at the level of a single-spin system. The nature and implication of the difficulties have recently been examined in detail where it has been pointed out, among others, that the information on the initial and the final states fails to be retained in the transition amplitude in question, and that, when combined with the stationary-action approximation, the stationary value of the action is not always given correctly and the fluctuation integral diverges. Hence, at present, there are no reliable results for transition amplitudes. One of the interesting predictions made by the continuous-time formalism is an interference effect (the so-called spin-parity effect); the behavior of the domain wall is predicted to depend dramatically on whether the magnitude $S$ of each of the individual spins is an integer or a half-integer. The origin of the effect has been ascribed to the so-called Berry-phase term appearing in the effective action. However, the information of the initial and the final states are essential for interference effects. Hence, the purported interference effect need be re-considered.

In this paper, we present a first step to clarify these problems by use of the discrete-time formalism of the spin-coherent-state path integral. We focus upon a free domain wall and evaluate real-time transition amplitudes. This is the first case of an unambiguous evaluation of such amplitudes via the spin-coherent-state path integral as applied to an interacting many-spin system.

The paper is organized as follows. In Sec. II we present the model Hamiltonian to be treated, namely that consisting of the Heisenberg exchange and the anisotropy energies, and formulate transition amplitudes between spin coherent states by use of the spin-coherent-
state path integral in the discrete-time formalism. In Sec. III we introduce a domain wall together with the collective degrees of freedom for its center position and the chirality, respectively. This section is also devoted to the derivation of the effective action of the domain wall. We also point out some problems associated with the continuous-time treatment of the effective action. Sec. IV evaluates transition amplitudes for a free domain wall in the stationary-action approximation including the effects of fluctuations. In the course of the evaluation, we note the conjugate relation between the center position and the chirality. We also locate those terms which can induce interference effects. In Sec. V we compute transition probabilities between typical states and compare the quantum dynamics of the domain wall with that of a free particle. This comparison allows us, among others, to identify the "effective mass" of the domain wall. In Sec. VI it is explicitly shown that the continuous-time formalism leads to a wrong transition probability for a free domain wall. We conclude with a speculation on the possibility of MQT and/or MQC involving a domain wall.

II. MODEL

We consider a ferromagnet consisting of a spin $S$ of magnitude $S$ at each site in a quasi-one-dimensional cubic crystal (a linear chain) of lattice constant $a$. The magnet is assumed to have an easy axis and a hard axis in the $z$ and the $x$ directions, respectively. Accordingly, we adopt the Hamiltonian

$$
\hat{H} = -\tilde{J} \sum_{<i,j>} \hat{S}_i \cdot \hat{S}_j - \frac{1}{2} \sum_j (K \hat{S}_{j,z}^2 - K_\perp \hat{S}_{j,x}^2),
$$

(2.1)

where the index $i$ or $j$ represents a lattice point, $<i,j>$ denotes a nearest-neighbor pair, $N_L$ is the total number of lattice points, and $\tilde{J}$ is the exchange coupling constant, and $K$ and $K_\perp$ are longitudinal and transverse anisotropy constants; $\tilde{J}$, $K$, and $K_\perp$ are all positive.

Since we are interested in those transition amplitudes which are appropriate to describe quantum mechanical motion of a domain wall, we introduce a spin-coherent state at each site, which is suited for a vector picture of spin. By use of the eigenstate $|S\rangle$ of $\hat{S}_z$ associated with the eigenvalue $S$, the spin coherent state is defined by

$$
|n\rangle := (1 + |\xi|^2)^{-S} \exp(\xi \hat{S}_-)|S\rangle,
$$

(2.2)

where $n$ is a unit vector ($n_x = \sin \theta \cos \phi, n_y = \sin \theta \sin \phi, n_z = \cos \theta$) with the complex number $\xi$ being its Riemann projection:

$$
\xi = e^{i\phi} \tan \frac{\theta}{2}, \quad \xi^* = e^{-i\phi} \tan \frac{\theta}{2}.
$$

(2.3)

These states form an overcomplete set and possess, among others, the following properties:

$$
\langle n|\hat{S}|n\rangle = Sn,
$$

(2.4a)

$$
\langle \langle n'|(e \cdot \hat{S})^2|n\rangle \rangle := \frac{\langle n'| (e \cdot \hat{S})^2 |n\rangle}{\langle n'|n\rangle} = \left(1 - \frac{1}{2S}\right) \left(\langle \langle n'|e \cdot \hat{S}|n\rangle\rangle\right)^2 + \frac{S}{2},
$$

(2.4b)
where $\mathbf{e}$ is an arbitrary unit vector. Hereafter we work with the $\xi$-representation, and denote a state of the system as

\[
|\xi\rangle \equiv |\xi_1, \xi_2, \cdots, \xi_{N_L}\rangle := \bigotimes_j^N |\xi_j\rangle,
\]

(2.5)

where $|\xi_j\rangle (\equiv |n_j\rangle)$ is a spin coherent state at the site $j$. The transition amplitude between the initial state $|\xi_1\rangle$ and the final state $|\xi_F\rangle$ can be expressed as a spin-coherent-state path integral in the real discrete-time formalism by the standard procedure of the repeated use of the resolution of unity (see, e.g., Ref. 17 on which the present notation is based):

\[
\langle \xi_F | e^{-i\hat{H}T/\hbar} | \xi_1 \rangle = \lim_{N \to \infty} \int \prod_{n=1}^{N-1} \prod_j^{N_L} d\mu(\xi_j(n), \xi_j^*(n)) \exp \left( \frac{i}{\hbar} S[\xi^*, \xi] \right),
\]

(2.6)

where $N \equiv T/\epsilon$, $\epsilon$ is an infinitesimal time interval, $n$ represents discrete time, and the integration measure is

\[
d\mu(\xi_j(n), \xi_j^*(n)) := \frac{2S + 1}{(1 + |\xi_j(n)|^2)^2} \frac{d\xi_j(n)d\xi_j^*(n)}{2\pi i}, \quad \frac{d\xi_j(n)d\xi_j^*(n)}{2\pi i} \equiv \frac{d\Re\xi_j(n) d\Im\xi_j(n)}{\pi}.
\]

(2.7)

The action $S[\xi^*, \xi]$ consists of two parts, $S^c[\xi^*, \xi]$ and $S^d[\xi^*, \xi]$, which are to be called the canonical term and the dynamical term, respectively. They take the following forms:

\[
S[\xi^*, \xi] = S^c[\xi^*, \xi] + S^d[\xi^*, \xi],
\]

(2.8a)

\[
i \frac{\hbar}{h} S^c[\xi^*, \xi] := \sum_{n=1}^{N} \sum_j^{N_L} \ln \langle \xi_j(n) | \xi_j(n-1) \rangle = S \sum_{n=1}^{N} \sum_j^{N_L} \ln \frac{(1 + \xi_j^*(n) \xi_j(n-1))^2}{(1 + |\xi_j(n)|^2)(1 + |\xi_j(n-1)|^2)},
\]

(2.8b)

\[
i \frac{\hbar}{h} S^d[\xi^*, \xi] := -i \frac{\hbar}{h} \sum_{n=1}^{N} \epsilon H(\xi^*(n), \xi(n-1)),
\]

(2.8c)

\[
H(\xi^*, \eta) := \langle \{ \xi | \hat{H} | \eta \} \rangle, \quad \langle \xi(0) \langle N \equiv \xi_F.
\]

(2.8d)

Here, we emphasize that the integration variables are $\{\xi^*(n), \xi(n)|n = 1, 2, \ldots, N-1\}; \xi(0)$ and $\xi(N)$ are fixed complex numbers. In passing, note that

\[
\langle \xi_j(n) | \xi_j(n-1) \rangle = \left( \cos^2 \frac{\theta_j(n)}{2} + \frac{\sin^2 \theta_j(n-1)}{2} + \sin^2 \frac{\theta_j(n)}{2} \sin^2 \frac{\theta_j(n-1)}{2} e^{-i(\phi_j(n) - \phi_j(n-1))} \right)^{2S},
\]

(2.9)

which is a $2\pi$-periodic function of the phase difference $\phi_j(n) - \phi_j(n-1)$.

At this stage, if one regarded all the differences $|\xi_j(n) - \xi_j(n-1)|$ as small in some sense, expanded the action $S[\xi^*, \xi]$ and went over to the continuous-time formalism, one would obtain the following form which had been used in most of the literature including Refs. 8 and 10:
\[ \langle \xi_F | e^{-i\hat{H}T/\hbar} | \xi_i \rangle \sim \int \mathcal{D}\xi \mathcal{D}\xi^* \exp \left\{ \frac{i}{\hbar} \left( \mathcal{S}_\text{con}[\xi^*, \xi] + \mathcal{S}_\text{d}[\xi^*, \xi] \right) \right\}, \] (2.10a)

\[ \frac{i}{\hbar}\mathcal{S}_\text{con}[\xi^*, \xi] := S \sum_{j} \int_{0}^{T} dt \frac{\dot{\xi}_j(t)\xi_j(t) - \xi_j^*(t)\dot{\xi}_j(t)}{1 + \xi_j^*(t)\xi_j(t)}, \] (2.10b)

\[ \frac{i}{\hbar}\mathcal{S}_\text{d}[\xi^*, \xi] := -\frac{i}{\hbar} \int_{0}^{T} dt H(\xi^*(t), \xi(t)), \] (2.10c)

where \( \mathcal{D}\xi \mathcal{D}\xi^* \) represents a symbolic measure in the continuous-time formalism. However, as pointed out in our previous paper, this formalism has various difficulties (see also the next section). For this reason, we proceed to consider the transition amplitude in the discrete-time formalism (2.8).

We shall be interested in those spin configurations whose scale of spatial variation is much larger than the lattice constant \( a \). Accordingly, we take the spatial continuum limit in (2.8b) and (2.8c):

\[ \frac{i}{\hbar}\mathcal{S}_\text{c}[\xi^*, \xi] = S \sum_{n=1}^{N} \int_{-L/2}^{L/2} dx \ln \left( \frac{(1 + \xi^*(x, n)\xi(x, n - 1))^2}{(1 + |\xi(x, n)|^2)(1 + |\xi(x, n - 1)|^2)} \right), \] (2.11a)

\[ \frac{i}{\hbar}\mathcal{S}_\text{d}[\xi^*, \xi] = \frac{i}{\hbar} \sum_{n=1}^{N} \epsilon \int_{-L/2}^{L/2} dx \mathcal{H}(\xi^*(x, n), \xi(x, n - 1)), \] (2.11b)

\[ \mathcal{H}(\xi^*(x), \eta(x)) := \frac{S}{(1 + \xi^*(x)\eta(x))^2} \left[ 2JS\partial_x \xi^*(x)\partial_x \eta(x) - \frac{K}{2} \left\{ \left( S - \frac{1}{2} \right) (1 - \xi^*(x)\eta(x))^2 + \frac{1}{2} \right\} \right. \]
\[ \left. + \frac{\alpha K}{2} \left\{ \left( S - \frac{1}{2} \right) (\xi^*(x) + \eta(x))^2 + \frac{1}{2} \right\} \right], \] (2.11c)

where \( L \) is the length of the linear chain, \( J \equiv \tilde{J}a^2 \), and \( \alpha \equiv K_\perp/K \). In this paper we consider the case of a weak transverse anisotropy \( \alpha \ll 1 \), and study the dynamics of a domain wall to the first order in \( \alpha \).

### III. EFFECTIVE ACTION FOR A DOMAIN WALL

#### A. Kink configuration

We begin by finding a domain wall configuration. It is determined by one of the static solutions \( \{ \xi^*(x), \bar{\xi}(x) \} \) of the action \( \mathcal{S}[\xi^*, \bar{\xi}] \). They satisfy the following equations up to \( \mathcal{O}(\alpha^0) \):

\[ \lambda^2 \left\{ \partial_x^2 \xi^*(x) - \frac{2\bar{\xi}(x)(\partial_x \xi^*(x))^2}{1 + \xi^*(x)\xi^*(x)} \right\} - \frac{1 - \bar{\xi}(x)\xi^*(x)}{1 + \xi^*(x)\xi^*(x)} \xi^*(x) = 0, \] (3.1a)

\[ \lambda^2 \left\{ \partial_x^2 \bar{\xi}(x) - \frac{2\bar{\xi}(x)(\partial_x \bar{\xi}(x))^2}{1 + \xi^*(x)\xi^*(x)} \right\} - \frac{1 - \bar{\xi}(x)\xi^*(x)}{1 + \xi^*(x)\xi^*(x)} \bar{\xi}(x) = 0, \] (3.1b)
where $\lambda^2 \equiv JS/K(S - 1/2)$. An obvious solution is the ”vacuum” solution representing the uniform configuration in which the spins are either all parallel or all anti-parallel to the $z$ direction. The other solution is the ”kink” solution representing a domain-wall configuration in which the spins at $x \sim +\infty$ are parallel to the $z$ direction, the spins at $x \sim -\infty$ are anti-parallel to the $z$ direction, and there is a transition region (i.e., a domain wall) of width $\lambda$;

$$\xi^s(x) = \exp \left( -\frac{x - Q}{\lambda} + i\phi_0 \right), \quad \bar{\xi}^s(x) = \exp \left( -\frac{x - Q}{\lambda} - i\phi_0 \right),$$

(3.2)

where $Q$ and $\phi_0$ are arbitrary real constants. $Q$ is the center position of the domain wall, and $\phi_0$ is a quantitative measure of the chirality of the domain wall with respect to the $x$ axis (Fig. 1); the wall is maximally right-handed if $\phi_0 = \pi/2$ and maximally left-handed if $\phi_0 = -\pi/2$, while it has no chirality if $\phi_0 = 0$. The range of $\phi_0$ is chosen as $-\pi \leq \phi_0 \leq \pi$, with $\phi_0 = \pi$ and $\phi_0 = -\pi$ representing the same situation.

Hereafter we consider transition amplitudes between the following domain-wall states:

$$|\xi_\beta\rangle = |z_\beta\rangle := \bigotimes_j |\xi^s(ja; z_\beta)\rangle, \quad \beta = I, F,$$

(3.6)
where \( z_1 \) represents the center position \( q_1 \) and the chirality \( \phi_1 \) of the domain wall in the initial state, and \( z_F \) those in the final state. At both ends of the discrete time \((n = 0 \text{ or } n = N)\), we define

\[
z(0) \equiv z_1 := q_1 + i\phi_1, \quad z(N) \equiv z_F := q_F + i\phi_F, \quad \eta(x, 0; z(0)) = \eta^*(x, N; z^*(N)) = 0. \tag{3.7a, b}
\]

Putting Eqs. (3.4) into the action (2.11), we obtain up to \( O(\alpha) \)

\[
S[\xi^*, \xi] = S^S[z^*, z] + \text{terms involving the environment } \eta, \quad \tag{3.8a}
\]

\[
S^S[z^*, z] = S^{sc}[z^*, z] + S^{sd}[z^*, z], \quad \tag{3.8b}
\]

\[
\frac{i}{\hbar} S^{sc}[z^*, z] := \frac{N_{DW}}{2} \sum_{n=1}^{N} \int_{-L/\lambda}^{L/\lambda} dx \ln \left( \frac{1 + e^{-x+z^*(n)+z(n-1)}}{1 + e^{-x+z^*(n)+z(n)}}(1 + e^{-x+z^*(n-1)+z(n-1)}) \right), \quad \tag{3.8c}
\]

\[
\frac{i}{\hbar} S^{sd}[z^*, z] := -\frac{i}{\hbar} E_{DW} \sum_{n=1}^{N} \epsilon \left[ 1 + \frac{\alpha}{4} \{1 + \cosh(z^*(n) - z(n-1)) \} \right], \quad \tag{3.8d}
\]

where \( N_{DW} \equiv \lambda/a \) is the number of spins in the domain wall, and \( E_{DW} \equiv 2N_{DW} KS(S-1/2) \) is the kink energy. The zero point of energy has been adjusted in (3.8d). In this paper, we consider only \( S^S[z^*, z] \) which is expected to make the most dominant contribution to transition amplitudes. The influence of the environment shall be discussed in a separate paper.

Expression (3.8d) can be reduced to a simpler form (see Appendix A for the details of the derivation by use of dilogarithm [2]):

\[
\frac{i}{\hbar} S^{sc}[z^*, z]/N_{DW}S = \frac{1}{2} \sum_{n=1}^{N} \left[ - (q(n) - q(n-1))^2 - R(\phi_0(n) - \phi_0(n-1)) \\
- i \{2(q(n) + q(n-1)) + 2L/\lambda\} I(\phi_0(n) - \phi_0(n-1)) \right], \tag{3.9}
\]

where both \( R(\phi) \) and \( I(\phi) \) are \( 2\pi \)-periodic functions (see, Fig. 2) such that

\[
R(\phi) = \phi^2 \quad : |\phi| \leq \pi, \quad \tag{3.10a}
\]

\[
I(\phi) = \phi \quad : -\pi \leq \phi < \pi. \quad \tag{3.10b}
\]

This periodicity follows inevitably from the periodicity of (3.8c). In general this periodicity need be respected in performing the integration with respect to \( \{\phi_0(n)\} \). However, in special circumstances when the range of \( \phi_0(n) - \phi_0(n-1) \) can be restricted to \([-\pi, \pi]\) for all \( n \), the above action may be rewritten as

\[
\frac{i}{\hbar} S^{sc}[z^*, z]/N_{DW}S = \sum_{n=1}^{N} \left( - \frac{1}{2} (q(n) - q(n-1))^2 + (\phi_0(n) - \phi_0(n-1))^2 \right) - i(q(n)\phi_0(n-1) - \phi_0(n)q(n-1))
\]
\[-i \left\{ (2\phi_F - \phi_I) L/\lambda + (q_F \phi_F - q_I \phi_I) \right\} \]
\[= \sum_{n=1}^{N} \left\{ -\frac{1}{2} \left( z^* (n) z(n) + z^* (n-1) z(n-1) \right) + z^* (n) z(n-1) \right\} \]
\[-i \left\{ (2\phi_F - \phi_I) L/\lambda + (q_F \phi_F - q_I \phi_I) \right\}. \quad (3.11)\]

This expression, except for the last constant term, formally coincides with the corresponding action appearing in the (boson-)coherent-state path integral with a single degree of freedom. The last term can be neglected because it is a constant phase, which does not affect any physical quantity.

C. Effective action for a domain wall in continuous-time formalism

Let us comment on the continuous-time treatment of a domain wall and the associated problems.

If one started from (2.10), one would obtain the continuous-time counterpart of (3.8) as

\[
S_{\text{con}}[z^*, z] = \int_0^T dt \mathcal{L}_{\text{con}},
\]

\[
\mathcal{L}_{\text{con}} := \frac{\hbar}{i} N_{\text{DW}} S \left( \frac{1}{2} (\dot{z}(t) z(t) - z^*(t) \dot{z}(t)) - E_{\text{DW}} \left[ 1 + \frac{\alpha}{4} \left( 1 + \cosh(z^*(t) - z(t)) \right) \right] \right)
\]

\[
= 2 \frac{N_{\text{DW}} \hbar S}{\lambda} \dot{\phi}_0(t) - 2 N_{\text{DW}} \hbar S \Omega \cos^2 \phi_0(t),
\]

\[
\Omega \equiv \frac{K}{2 \hbar} \left( S - \frac{1}{2} \right) \alpha. \quad (3.12a)
\]

In the last expression, constant terms including those appearing as a result of partial integration have been neglected. If the transverse anisotropy is strong in the sense that \( N_{\text{DW}} S \Omega \gg 1 \), \( \phi_0(t) \) may be restricted to a region near \( \pm \pi/2 \);

\[
\phi_0(t) = C \pi/2 + \varphi(t), \quad C \equiv \pm 1, \quad |\varphi(t)| \ll 1. \quad (3.13)
\]

Substituting this into (3.12a), one would find

\[
\mathcal{L}_{\text{con}} \simeq CA \dot{\varphi}(t) + \frac{2}{\pi} A \dot{\varphi}(t) \varphi(t) - 2 N_{\text{DW}} \hbar S \Omega \varphi^2(t), \quad (3.14)
\]

where \( A \equiv N_{\text{DW}} \hbar S \pi/\lambda \). Provided that the path-integration measure is independent of \( \varphi \), Gaussian integration with respect to \( \varphi \) would then lead to the following effective action for \( Q \):

\[
\frac{i}{\hbar} S_{\text{eff}}^{\text{con}}[Q] = \frac{i}{\hbar} \int_0^T dt \left\{ C A \dot{Q}(t) + \frac{M_D}{2} \dot{Q}^2(t) \right\}, \quad (3.15)
\]

where \( M_D \) is the Döring mass:

\[
M_D \equiv \frac{2 \hbar^2}{a^2} \sqrt{\frac{KS}{J(S - 1/2) K_\perp}}. \quad (3.16)
\]
Accordingly, one might expect the following expression for the transition amplitude:

$$\langle z_F | e^{-i\hat{H}T/\hbar} | z_I \rangle \sim \int DQ \exp \left( \frac{i}{\hbar} S_{\text{eff}}^{\text{con}}[Q] \right). \quad (3.17)$$

This corresponds to the result of Braun and Loss. At this stage, it has been concluded that the center position of the domain wall behaves as a free particle with a possible modification due to interference effects induced by the first term $CA\dot{Q}(t)$ of the effective action, which is often called the "Berry-phase term".

If one is interested in the quantum depinning of the domain wall, a pinning potential is to be added to Lagrangian (3.14). If the effect of the transverse anisotropy is relatively larger than that of the pinning potential, the effective action (3.15) is simply augmented by the pinning potential. MQT of the center position has been discussed on the basis of this action. On the other hand, if the pinning effect is the larger, one could carry out Gaussian integration with respect to $Q(t)$ and obtain an effective action governing $\phi_0(t)$, with which MQC of the chirality has been discussed.

However, the whole series of the above-quoted arguments, which are based on the form of the derived effective actions regardless of the path-integration measure, are at best heuristic and their validity is rather dubious. In the literature it has been tacitly assumed that the right-hand side of (3.17) is a Feynman kernel with respect to $Q$. (Recall the following point: In order to obtain a transition amplitude between physical (i.e., normalizable) states from a Feynman kernel, the latter has to be multiplied by the initial and the final wave functions and integrated over $Q_F$ and $Q_I$.) The starting point of the whole arguments, on the other hand, is the left-hand side of (3.17), which is a transition amplitude between physical states. Its path-integral structure is different from that for a Feynman kernel. One should not be misled by the apparent form of the effective action (3.15). Though integration over $\varphi(t)$ may be carried out in principle, resulting action can not be a Feynman kernel. Indeed, as we shall illustrate in Sec.VI, the right-hand side of (3.17) as interpreted as a Feynman kernel does not give a correct transition amplitude. By the same token, interference effects predicted on the basis of the "Berry-phase term" $CA\dot{Q}(t)$ need be re-examined: as shown in the following section, since the continuous-time treatment neglects many other terms which can contribute to interference effects, there is no reason that only the "Berry-phase term" should be kept.

IV. TRANSITION AMPLITUDE IN STATIONARY-ACTION APPROXIMATION

In this section, we evaluate transition amplitudes by means of the stationary-action approximation, discuss the conjugate relation between the center position and the chirality of the domain wall, and locate those terms which can induce interference effects.

A. Stationary-action path

Let $\{\tilde{z}^s(n), z^s(n)\}$ be the stationary-action path, namely, the stationary point of the action $S^s[z^s, z]$: 
where the symbol \( \mid \) indicates the replacement \((z^*, z) \to (\bar{z}^*, z^*)\) after differentiation. It is convenient to define

\[
\begin{align*}
\bar{z}^s(0) &:= z_1 = q_1 + i\phi_1, \\
\bar{z}^s(N) &:= z^*_F = q_F - i\phi_F.
\end{align*}
\]

Let us work with (3.11) instead of (3.9). This procedure will be justified \textit{a posteriori}. Then, the above set of equations take the form:

\[
\begin{align*}
z^s(n) - z^s(n - 1) &= -i\epsilon \Omega \sinh \{\bar{z}^s(n) - z^s(n - 1)\}, \\
\bar{z}^s(n) - \bar{z}^s(n - 1) &= -i\epsilon \Omega \sinh \{\bar{z}^s(n) - z^s(n - 1)\},
\end{align*}
\]

By use of \(z^s(n)\) and \(\bar{z}^s(n)\), we define the stationary-action path for the center position and the chirality as

\[
\begin{align*}
q^s(n) &:= (z^s(n) + \bar{z}^s(n))/2, \\
\phi^s(n) &:= (z^s(n) - \bar{z}^s(n))/2i,
\end{align*}
\]

which are not real in general.

The left-hand side of (4.3) are \(\mathcal{O}(\epsilon)\) because of the factor \(\epsilon\) on the right-hand side. Hence, as far as the equations for the stationary-action path is concerned, we can go over to the continuous time:

\[
\begin{align*}
\frac{d\bar{z}^s(t)}{dt} &= -i\Omega \sinh(\bar{z}^s(t) - z^s(t)), \\
\frac{d\bar{z}^s(t)}{dt} &= -i\Omega \sinh(\bar{z}^s(t) - z^s(t)).
\end{align*}
\]

Note that the boundary condition is dictated by (4.2) as

\[
\begin{align*}
z^s(0) &= z_1, \\
\bar{z}^s(T) &= z^*_F.
\end{align*}
\]

It follows from Eqs. (4.5) that

\[
\frac{d}{dt}(\bar{z}^s(t) - z^s(t)) = 0.
\]

Hence,

\[
\bar{z}^s(t) - z^s(t) = -2i\phi,
\]

where \(\phi\) is a complex constant \((\phi \equiv \phi' + i\phi''); \phi', \phi'' \in \mathbb{R}\). Substituting (4.8) into (4.3), and taking account of the boundary condition (4.6), we obtain

\[
\begin{align*}
z^s(t) &= -\Omega t \sin 2\phi + z_1, \\
\bar{z}^s(t) &= -\Omega(t - T) \sin 2\phi + z^*_F.
\end{align*}
\]
Putting this back into (4.8), we find that the constant $\phi$ is determined by the following algebraic equation:

$$\Omega T \sin 2\phi + z_F^* - z_1 = -2i\phi, \quad (4.10)$$

or equivalently,

$$2\phi'' - \Omega T \sin 2\phi' \cosh 2\phi'' = q \equiv q_F - q_I, \quad (4.11a)$$

$$2\phi' + \Omega T \cos 2\phi' \sinh 2\phi'' = \phi_F + \phi_I. \quad (4.11b)$$

The stationary-action path can be expressed in terms of $q^s(t)$ and $\phi^s(t)$ as defined by (4.4) as

$$q^s(t) = -\Omega t \sin 2\phi + z_1 - i\phi, \quad (4.12a)$$

$$\phi^s(t) = \phi. \quad (4.12b)$$

Note that both of these are complex. Eq. (4.12b) shows that $\phi^s(n) - \phi^s(n-1) = 0$ for all $n$. This justifies our procedure of working with (3.11) instead of (3.9). Incidentally it follows from (4.12) that

$$\frac{dq^s(t)}{dt} = -\Omega \sin 2\phi^s(t). \quad (4.13)$$

Thus, the velocity of the center position depends on the chirality and is proportional to the transverse anisotropy (recall that $\Omega \propto \alpha$). In the special circumstance that $\phi^s(t)$ happens to be close to $\pm \pi/2$, we can put $\phi^s(t) = \pm \pi/2 + \varphi^s(t)$ to find

$$\frac{dq^s(t)}{dt} \propto \varphi^s(t). \quad (4.14)$$

This shows that in such a circumstance the center position and the chirality are mutually canonically conjugate, thereby confirming the claim made in Ref. 10. In any case it is clear that $q^s(t)$ and $\phi^s(t)$ are closely coupled; they should be treated on an equal footing.

**B. Stationary action**

The stationary action $S^s[\bar{z}^s, z^s]$ may be arranged as

$$S^{ss} := S^s[\bar{z}^s, z^s] = S^{ssc} + S^{ssd}, \quad (4.15a)$$

$$i\hbar S^{ssc} := N_{DW} \sum_{n=1}^{N} \left[ -\frac{1}{2}(|z_F|^2 + |z_I|^2) + \frac{1}{2}(z_F^*z^s(N - 1) + z^*(1)z_I) 
+ \frac{1}{2} \sum_{n=1}^{N-1} \{(\bar{z}^s(n + 1) - \bar{z}^s(n))z^s(n) - \bar{z}^s(n)(z^s(n) - z^s(n - 1)) \right], \quad (4.15b)$$

$$i\hbar S^{ssd} := -\frac{i}{\hbar} E_{DW} \sum_{n=1}^{N} \epsilon \left[ 1 + \frac{\alpha}{4} \{1 + \cosh(\bar{z}^s(n) - \bar{z}^s(n - 1)) \right]. \quad (4.15c)$$
The first and the second terms on the right-hand side of (4.15b) depend on the initial and the final state. The second term depends also on \( T \) through the stationary-action path. These terms, which have been neglected in the continuous-time formalism, turn out to be crucial for a correct evaluation of the transition amplitude. The third term corresponds to the "Berry-phase term" in the continuous-time formalism. In the latter formalism, interference effects have been ascribed to this term alone. However, the second term can also contribute to interference effects. This is another remarkable difference from the continuous-time formalism. Of course, interference effects in question can arise only if there exist two or more stationary-action paths. In the case of a free domain wall under consideration, there is no question of interference because the stationary-action path is unique.

Substituting (4.19) into (4.15), we obtain

\[
\begin{align*}
\frac{i}{\hbar} S^{sc} & = - \frac{N_{DW} S}{2} \left[ |z_F|^2 + |z_I|^2 - 2 z_F^* z_I - (\Omega T \sin 2\phi)^2 \right] + O(\epsilon), \\
\frac{i}{\hbar} S^{sd} & = - \frac{i}{\hbar} E_{DW} T \left( 1 + \frac{\alpha}{4} \right) - iN_{DW} S \Omega T \cos 2\phi + O(\epsilon).
\end{align*}
\]

(4.16a)

(4.16b)

Putting these together, we finally arrive at

\[
\begin{align*}
\frac{i}{\hbar} S^{ss} & = - \frac{N_{DW} S}{2} \left[ |z_F|^2 + |z_I|^2 - 2 z_F^* z_I - (\Omega T \sin 2\phi)^2 + 2i\Omega T \cos 2\phi \right] \\
& \quad - \frac{i}{\hbar} E_{DW} T \left( 1 + \frac{\alpha}{4} \right).
\end{align*}
\]

(4.17)

This stationary action is complex in general.

C. Fluctuations

As noted previously, well-defined evaluation of the fluctuation integral is possible only in the discrete-time formalism. Accordingly we separate the integration variables as

\[
\begin{align*}
z(n) & = z^s(n) + \zeta(n), \\
z^*(n) & = \bar{z}^s(n) + \bar{\zeta}(n) : n = 1, 2, \ldots, N - 1.
\end{align*}
\]

(4.18a)

It is convenient to define

\[
\zeta(0) = \zeta^*(N) = 0.
\]

(4.18b)

Substituting (4.18a) into the action (3.8d) and (3.11), and expanding up to the second order in the fluctuation, we get

\[
\begin{align*}
S^s[z^s, \bar{z}] & = S^{ss} + S_2[\zeta^*, \zeta], \\
S_2[\zeta^*, \zeta] & = S_2^s[\zeta^*, \zeta] + S_2^d[\zeta^*, \zeta],
\end{align*}
\]

(4.19a)

(4.19b)

where
\[
\frac{i}{\hbar} \mathcal{S}^{\text{ss}}[\zeta^*, \zeta] := -\frac{N_{DW} S}{2} \sum_{n=1}^{N} \left\{ 2\zeta^*(n)\zeta(n) - 2\zeta^*(n)\zeta(n-1) \right\},
\]
(4.20a)

\[
\frac{i}{\hbar} \mathcal{S}^{d}[\zeta^*, \zeta] := -i\frac{N_{DW} S}{2} \Omega \cos 2\phi \sum_{n=1}^{N} \epsilon \left\{ (\zeta^*(n))^2 + (\zeta(n-1))^2 - 2\zeta^*(n)\zeta(n-1) \right\}.
\]
(4.20b)

Accordingly, the transition amplitude reduces to
\[
\langle z_F | e^{-i\hat{H}T/\hbar} | z_I \rangle \simeq \exp \left[ \frac{i}{\hbar} \mathcal{S}^{\text{ss}} \right] K_2(T),
\]
(4.21a)

\[
K_2(T) := \lim_{N \to \infty} \int \frac{d\zeta(n)d\zeta^*(n)}{2\pi i} \exp \left( \frac{i}{\hbar} \mathcal{S}^{d}[\zeta^*, \zeta] \right),
\]
(4.21b)

where \( M \) is a constant, whose value as well as the detailed evaluation of \( K_2(T) \) are given in Appendix B. The result is
\[
K_2(T) = \frac{e^{i\Omega_{\phi} T/2}}{\sqrt{1 - i\Omega_{\phi} T}}, \quad \Omega_{\phi} \equiv \Omega \cos 2\phi.
\]
(4.22)

This completes a microscopic evaluation of transition amplitudes for a free domain wall.

V. TRANSITION PROBABILITY

We can now compute transition probabilities between various initial and final domain-wall states:
\[
P(q, \phi_F, \phi_I; T) := |\langle z_F | e^{-i\hat{H}T/\hbar} | z_I \rangle|^2
\]
\[
\simeq \frac{e^{3\Omega_{\phi} T}}{\sqrt{1 + 2i\Omega_{\phi} T + (\Omega_{\phi})^2 T^2}} \exp \left[ -\frac{2\hbar}{\hbar} \mathcal{S}^{\text{ss}} \right],
\]
(5.1a)

\[
\frac{2}{\hbar} \mathcal{S}^{\text{ss}} = N_{DW} S \left[ q^2 + (\phi_F - \phi_I)^2 - 2(\Omega T)^2 \Re(\sin 2\phi)^2 - 2\Omega T \Im \cos 2\phi \right].
\]
(5.1b)

The right-hand side does not depend on \( q_F \) and \( q_I \) separately but only on \( q(\equiv q_F - q_I) \) as expected from the translation invariance. This justifies the notation \( P(q, \phi_F, \phi_I; T) \).

A. Case of \( \alpha = 0 \)

In the absence of transverse anisotropy (\( \Omega \propto \alpha = 0 \)), the transition probability is independent of \( T \):
\[
P(q, \phi_F, \phi_I; T) = \exp \left[ -N_{DW} S \left\{ q^2 + (\phi_F - \phi_I)^2 \right\} \right] = |\langle z_F | z_I \rangle|^2.
\]
(5.2)

Thus, the transition probability coincides with the overlap between the initial and the final states and is a function of the differences \( q_F - q_I \) and \( \phi_F - \phi_I \). It is shown in Fig. 3.

The factor \( N_{DW} S \) in the exponent of \( (5.2) \) is large (typically of order of \( 10^3 \)), reflecting the semi-macroscopic character of a typical domain wall of interest. Hence, if the final state is even slightly different from the initial state, the transition is forbidden.
B. Case of $\alpha \neq 0$

In the presence of the transverse anisotropy, the transition probability depends on $T$:

\[
P(q, \phi_F, \phi_I; T) = \frac{\exp(-\tau \sin 2\phi' \sinh 2\phi'')}{\sqrt{1 - 2\tau \sin 2\phi' \sinh 2\phi' + \tau^2\{\cos 2\phi' \cosh 2\phi'' + \sin 2\phi' \sinh 2\phi''\}}}
\times \exp \left[-N_{DW}S\left\{q^2 + (\phi_F - \phi_I)^2\right.ight.
\left.-\tau^2\left(\sin 2\phi' \cosh 2\phi'' - \cos 2\phi' \sinh 2\phi''\right) + 2\tau \sin 2\phi' \sinh 2\phi''\right\}\right],
\]

(5.3)

where $\tau \equiv \Omega T$.

1. analytical evaluation in linear approximation

In order to find $\phi(\equiv \phi' + i\phi'')$, we need to solve the algebraic equation (4.11). On inspection we see that it has a solution $\phi = (\phi_F + \phi_I)/2$ if $q = -\tau \sin(\phi_F + \phi_I)$. This motivates us to look for a more general class of solutions by linearizing (4.11) under the following condition to be justified \textit{a posteriori}:

\[
\phi' = \frac{\phi_F + \phi_I}{2} + \varphi', \quad \phi'' = \varphi'', \quad |\varphi'|, |\varphi''| \ll 1.
\]

(5.4)

Then, we can write down the linearized version of (4.11) as

\[
2\varphi'' - \tau C_{FI} 2\varphi' = q', \quad 2\varphi' + \tau C_{FI} 2\varphi'' = 0,
\]

(5.5)

where

\[
q' \equiv q + \tau S_{FI}, \quad C_{FI} \equiv \cos(\phi_F + \phi_I), \quad S_{FI} \equiv \sin(\phi_F + \phi_I).
\]

(5.6)

Hence

\[
2\varphi' = -\frac{\tau C_{FI}}{1 + (\tau C_{FI})^2} q', \quad 2\varphi'' = \frac{1}{1 + (\tau C_{FI})^2} q'.
\]

(5.7)

Since the factors multiplying $q'$ are at most of order unity, the assumed condition (5.4) is satisfied if $|q'| \ll 1$. Substituting (5.7) into (5.14) we find

\[
-\frac{2}{\hbar} \Im S_{ss} = -N_{DW}S\left\{\frac{q^2}{1 + (\tau C_{FI})^2} + (\phi_F - \phi_I)^2\right\},
\]

(5.8)

while the prefactor of (5.14) is obtained, up to $O(q^2)$, as

\[
\frac{1}{\sqrt{1 + (\tau C_{FI})^2}} \left\{1 - \left(\frac{\tau C_{FI}}{1 + (\tau C_{FI})^2}\right)^2 \left\{2\tau S_{FI} q' + \left(\frac{1}{2} \left(\frac{S_{FI}}{C_{FI}}\right)^2 - \frac{(\tau C_{FI})^2}{1 + (\tau C_{FI})^2}\right) q'^2\right\}\right\}.
\]

(5.9)
Because of the large factor \( N_{DW} S \) in the exponent of (5.8), the \( q' \)-dependence of the prefactor is negligible. Hence,

\[
P(q, \phi_F, \phi_I; T) \simeq \frac{1}{\sqrt{1 + (\tau \cos(\phi_F + \phi_I))^2}} 
\times \exp\left[-N_{DW} S \left\{ \frac{(q + \tau \sin(\phi_F + \phi_I))^2}{1 + (\tau \cos(\phi_F + \phi_I))^2} + (\phi_F - \phi_I)^2 \right\}\right],
\]

which is valid provided that \(|q + \tau \sin(\phi_F + \phi_I)| \ll 1\). This approximate formula shows that transitions are suppressed if \( \phi_F \neq \phi_I \). It also suggests the following picture: for a given \((\phi_F, \phi_I)\), the peak of the wave packet representing the center position \( q \) moves with the velocity (in units of \( \lambda \Omega \))

\[
u_{packet} \equiv -\sin(\phi_F + \phi_I),
\]

while the width (in units of \( \lambda \)) of the wave packet increases as

\[
w_{packet}(\tau) \equiv \left\{ 1 + (\tau \cos(\phi_F + \phi_I))^2 \right\}^{1/2}.
\]

The minimal velocity and maximal spreading occurs at \(|\phi_F + \phi_I| = 0 \) (mod \( \pi \)), while the maximal velocity and minimal spreading occurs at \(|\phi_F + \phi_I| = \pi/2 \) (mod \( \pi \)). We choose these cases as well as the intermediate case of \(|\phi_F + \phi_I| = 2\pi/3 \), and depict (5.10) in Figs. 4–6 with solid curves. These figures show the case of minimal suppression \((\phi_F = \phi_I)\). We take \( N_{DW} S = 100 \) throughout (and also in Fig. 7 to be mentioned below).

Fig. 4 shows the case of \( \phi_F = \phi_I = \pi/2 \), where \( \nu_{packet} = 0 \) and \( w_{packet}(\tau) = (1 + \tau^2)^{1/2} \). The curve for \( q = 0 \), where the formula (5.10) is exact as mentioned at the beginning of this subsection, shows that the probability of remaining in the initial state decreases with time. Thus the domain wall exhibits a quantum phenomenon analogous to the wave-packet spreading for a free particle. The long-time tail originates from the prefactor of (5.3) coming from the fluctuation. If the final state corresponds to the mere displacement of the initial domain wall by the distance \( q \neq 0 \), the transition probability exhibits an initial increase, which may be interpreted as the appearance of an overlap between the final wave packet and the wave packet evolving from the initial state. For a small \( q \), the overlap initially suppressed by the large factor \( N_{DW} S \) is rapidly recovered resulting in a sharp initial increase of the transition probability. The origin of the long-time tail is the same in the case of \( q = 0 \). The case of \( \phi_F = \phi_I = -\pi/3 \), where \( \nu_{packet} = \sqrt{3}/2 \) and \( w_{packet}(\tau) = (1 + \tau^2/4)^{1/2} \), is shown in Fig. 5, which exhibits a moving and spreading wave packet. So far, the quantum dynamics of the domain wall resembles that of a free particle. However, Fig. 6, where \( \phi_F = \phi_I = -\pi/4 \), reveals quite a different feature; it may be interpreted as showing a non-spreading wave packet like a solitary wave with \( \nu_{packet} = 1 \) and \( w_{packet}(\tau) = 1 \). Finally, Fig. 7 shows that the transition probabilities between states with the common center position \((q = 0)\) and different chiralities are rather small. This is in contrast to the situation in Fig. 4, where there is an appreciable transition probability between states with the common chirality \((\phi_F = \phi_I)\) and \( q \neq 0 \).
2. numerical evaluation

To check the accuracy of the approximate formula (5.10), we have numerically solved (4.11) for $\phi'$ and $\phi''$, and put them into (5.3). The results are depicted in Figs. 4–7. It is confirmed that (5.10) remains valid even for $|q + \tau \sin(\phi_F + \phi_1)| \sim 1$.

C. Comparison with quantum dynamics of a free particle

Let us compare the above-found quantum behavior of the domain wall with that of a free particle.

The transition amplitude for a free particle with mass $m$ between the initial coherent state $|Z_I\rangle (Z_I = Q_I + iP_I)$ and the final coherent state $|Z_F\rangle (Z_I = Q_F + iP_F)$ is given by

$$
\langle Z_F | e^{-i\hat{H}T/\hbar} | Z_I \rangle = \lim_{N \to \infty} \int \prod_{n=1}^{N-1} dZ(n)dZ^*(n) \exp \left( \frac{i}{\hbar} S[Z^*, Z] \right),
$$

(5.13a)

where

$$
i\frac{\hbar}{\hbar} S[Z^*, Z] := \sum_{n=1}^{N} \left[ - \frac{1}{2} (|Z(n)|^2 + |Z(n-1)|^2 + Z^*(n)Z(n-1)
-i \frac{\epsilon}{8} \right] (1 - (Z^*(n) - Z(n-1))^2) \right].
$$

(5.13b)

This action resembles that of a free domain wall (3.8d) and (3.11). Indeed, if we expand the non-linear part in (3.8) as

$$
\cosh(z^*(n) - z(n - 1)) \approx 1 - \frac{(z^*(n) - z(n - 1))^2}{2},
$$

(5.14)

then the action (3.8) for a free domain wall reduces to (5.13b). In this sense, a free domain wall may be said to be a non-linear version of a free particle.

The transition probability for the free particle is exactly calculated as

$$
|\langle Z_F | e^{-i\hat{H}T/\hbar} | Z_I \rangle|^2 = \left( 1 + \left( \frac{\hbar T}{2m\delta^2} \right)^2 \right)^{-1/2}
$$

$$
\times \exp \left[ - \frac{1}{2\delta^2} \left( \frac{1}{1 + \left( \frac{\hbar T}{2m\delta^2} \right)^2} \left( (Q_F - Q_I) - \frac{T}{2m}(P_F + P_I) \right) \right)^2 - \frac{\delta^2}{2\hbar^2} (P_F - P_I)^2 \right],
$$

(5.15)

where $Q_I$ and $Q_F$ are the center of the initial and the final Gaussian wave packet, respectively, and likewise $P_I$ and $P_F$ are the initial and the final mean momentum. The width of the wave packet has been chosen to be $\delta$ both for the initial and the final state. The form of (5.10) differs from that of (5.13) only in the non-linear factors $\cos(\phi_F + \phi_1)$ and $\sin(\phi_F + \phi_1)$. If these are linearized as $\cos(\phi_F + \phi_1) \approx 1$ and $\sin(\phi_F + \phi_1) \approx \phi_F + \phi_1$, then (5.10) reduces to the same form as (5.13); in this case the behavior of a free domain wall is the same as that of a free particle. However, the linearization is not always allowed. In particular, when $\phi_F + \phi_1 = \pm \pi/2$, the behavior of the domain wall is completely different from that of the free particle; the wave packet does not spread! This is a manifestation of the non-linear character of spin.
D. Effective mass of a free domain wall

We can estimate the effective domain-wall mass from the correspondence of the domain wall and the particle as noted in the previous subsection.

Comparison of (5.10) and (5.15) reveals the following correspondence. First, the coefficient of $T$ in the prefactor suggests

$$\Omega |\cos(\phi_F + \phi_I)| \leftrightarrow \frac{\hbar}{2m\delta^2}.$$  \hspace{1cm} (5.16)

Second, the coefficient of $q^2$ in the exponent at $T = 0$ suggests

$$\delta_{DW} := \frac{\lambda}{\sqrt{2N_{DW}S}} \leftrightarrow \delta.$$  \hspace{1cm} (5.17)

$\delta_{DW}$ can thus be interpreted as the width of the wave packet describing the initial domain wall. These two correspondences suggest to associate the domain wall with the "effective mass" $M_{DW}$ given by

$$M_{DW} = \frac{\hbar}{2\Omega |\cos(\phi_F + \phi_I)|\delta_{DW}^2} = \frac{M_D}{|\cos(\phi_F + \phi_I)|}.$$  \hspace{1cm} (5.18)

It is to be noted, however, that this "effective mass" depends on the initial and the final chirality; as such, it can not be viewed as an effective mass of an ordinary dynamical entity. It coincides with the Döring mass if $\phi_F + \phi_I = 0$ or $\pm \pi$. On the other hand, it is infinite if $\phi_F + \phi_I = \pm \pi/2$, which is just another way of expressing the non-spreading of the wave packet as noted in the previous subsection.

VI. DISCUSSION

If we calculated the transition probability by use of the continuous-time action (3.12a), what result would have been obtained? Though the stationary-action path is the same as (4.9), the stationary action would be

$$i\frac{\hbar}{\mathcal{S}_{con}} = i\frac{\hbar}{\mathcal{S}_{con}[z^s, z^s]} = -iN_{DW}S\tau(\phi \sin 2\phi + \cos 2\phi) - \frac{i}{\hbar}E_{DW}T \left(1 + \frac{\alpha}{4}\right).$$  \hspace{1cm} (6.1)

Hence

$$|\langle z_F | e^{-i\hat{H}T/\hbar} | z_I \rangle|^2 \sim |K_{2con}(T)|^2 \exp \left(-\frac{2}{\hbar}\mathcal{S}_{con}^{ss}\right),$$  \hspace{1cm} (6.2)

$$-\frac{2}{\hbar}\mathcal{S}_{con}^{ss} = N_{DW}S\tau(\phi'' \sin 2\phi' \cosh 2\phi'' + \phi' \cos 2\phi' \sinh 2\phi'' - \sin 2\phi' \sinh 2\phi'').$$  \hspace{1cm} (6.3)

Let $T = 0$. Assuming that the fluctuation integral $K_{2con}(T)$ was somehow evaluated and that it agreed with the correct value (1.22), we would have found $|\langle z_F | e^{-i\hat{H}T/\hbar} | z_I \rangle|^2 |_{T = 0} \sim 1$. However, this is a completely meaningless result, since the left-hand side should be equal to $|\langle z_F | z_I \rangle|^2$. (If one started from the effective action (3.15), one would also find a similarly meaningless result.) Furthermore one can not rationally calculate the fluctuation integral from this formalism. [7]
VII. CONCLUDING REMARKS

We have considered the macroscopic quantum dynamics of a free domain wall in a quasi-one-dimensional ferromagnet by use of the spin-coherent-state path integral in the discrete-time formalism. The center position and the chirality, which have been chosen as the collective degrees of freedom, are noted to be mutually canonically conjugate in a loose sense. The quantum behavior of the domain wall is the same as that of the free particle and its effective mass is the Döring mass if $\phi_F + \phi_I = 0$ or $\pm \pi$, but in general it differs from the latter in some non-linear effects. We have also pointed out some grave difficulties associated with the continuous-time formalism. It cannot correctly evaluate transition amplitudes. Its assertion on interference effects on the basis of the "Berry-phase term" alone is also questionable.

Let us speculate on MQT and MQC involving a domain wall. Since a free domain wall with a fixed chirality ($\phi_F = \phi_I$) has been shown to behave roughly like a free particle unless $\phi_F = \phi_I = \pm \pi/4$, we expect that a quantum depinning (MQT) will occur in the case of a weak pinning and a strong transverse anisotropy as mentioned by many workers; a strong transverse anisotropy tends to fix the chirality at $\phi_F = \phi_I = \pm \pi/2$. However, if a transverse anisotropy energy is comparable to a pinning potential, the dependence of the domain-wall mass on the chirality can be important. This may somewhat affect the MQT. Such a possibility has been overlooked in the literature. The MQC has been suggested to occur in the case of a strong pinning and a weak transverse anisotropy, namely for a fixed center position ($q = 0$). However, Fig. 7 shows that the transition probability for $q = 0$ and $|\phi_F - \phi_I| \sim \pi$ is negligible. This is due to the large factor $N_{DW}S$ in the exponent. Hence, for the MQC to occur, it may be necessary to invoke a mechanism (e.g., a magnetic field) to decrease $|\phi_F - \phi_I|$. At any rate, a careful consideration is needed to make a conclusion on the possibility of the MQC.

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APPENDIX A: DERIVATION OF EQ. (3.9)

The numerator in the logarithm of Eq. (3.8c) can be rewritten in two forms;

$$
1 + e^{-x+z^*(n)+z(n-1)} = (1 + e^{-x+z^*(n)+z(n-1)})(1 + \frac{e^{-x+z^*(n)+z(n)}}{1+e^{-x+z^*(n)+z(n)}}(e^{-z(n)-z(n-1)} - 1)),
$$

(A1a)

$$
= (1 + e^{-x+z^*(n-1)+z(n-1)})(1 + \frac{e^{-x+z^*(n-1)+z(n-1)}}{1+e^{-x+z^*(n-1)+z(n-1)}}(e^{z^*(n)-z^*(n-1)} - 1)).
$$

(A1b)

In order to perform the $x$ integration, we pay attention to the following formula;
\[
\int dx \ln \left( 1 + \frac{A}{1 + e^{x+B}} \right) = Di \left( 1 + \frac{A}{1 + e^{x+B}} \right) - Di \left( \frac{1 + \frac{A}{1 + e^{x+B}}}{1 + A} \right) + \ln(1 + A) \ln \left( \frac{-Ae^{x+B}}{1 + e^{x+B}} \right), \tag{A2}
\]

where \( Di(z) \) is the dilogarithm defined by
\[
Di(z) := \int_1^z \frac{\ln t}{1 - t} \, dt. \tag{A3}
\]

Thus, the integral in (3.8c) can be evaluated as
\[
I \equiv \int_{-L/\lambda}^{L/\lambda} dx \ln \frac{(1 + e^{-x+z*(n)-z(n-1)})^2}{(1 + e^{-z*(n)+z(n)})(1 + e^{-z*(n-1)+z(n-1)})} = \sum_{\alpha=1}^2 \left[ Di \left( 1 + \frac{A_\alpha}{1 + e^{x+B_\alpha}} \right) - Di \left( \frac{1 + \frac{A_\alpha}{1 + e^{x+B_\alpha}}}{1 + A_\alpha} \right) + \ln(1 + A_\alpha) \ln \left( \frac{-A_\alpha e^{x+B_\alpha}}{1 + e^{x+B_\alpha}} \right) \right]_{-L/\lambda}^{L/\lambda}, \tag{A4}
\]

where
\[
A_1 \equiv e^{-(z(n)-z(n-1))} - 1 = e^{-(q(n)-q(n-1))-i(\phi_0(n)-\phi_0(n-1))} - 1, \tag{A5a}
B_1 \equiv -(z^*(n) - z(n)) = -2q(n), \tag{A5b}
A_2 \equiv e^{z^*(n)-z*(n-1)} - 1 = e^{q(n)-q(n-1)-i(\phi_0(n)-\phi_0(n-1))} - 1, \tag{A5c}
B_2 \equiv -(z^*(n-1) - z(n-1)) = -2q(n-1). \tag{A5d}
\]

Since \( L/\lambda \gg 1 \), (A4) may be simplified as
\[
I = \sum_{\alpha=1}^2 \left[ - \left( Di(1 + A_\alpha) + Di \left( \frac{1}{1 + A_\alpha} \right) \right) + \ln(1 + A_\alpha) \left( \frac{L}{\lambda} + \ln(\exp(-B_\alpha)) \right) \right]. \tag{A6}
\]

Next, we use the dilogarithm identity
\[
Di(A) + Di \left( \frac{1}{A} \right) = -\frac{1}{2}(\ln A)^2 \tag{A7}
\]
to find
\[
I = \sum_{\alpha=1}^2 \left[ \frac{1}{2}(\ln(1 + A_\alpha))^2 + \ln(1 + A_\alpha) \left( \frac{L}{\lambda} + \ln(\exp(-B_\alpha)) \right) \right] = - \left[ (q(n) - q(n-1))^2 + R(\Delta\phi_0(n)) + i \{ 2(q(n) + q(n-1)) + 2L/\lambda \} I(\Delta\phi_0(n)) \right], \tag{A8}
\]

where
\[
\Delta\phi_0(n) \equiv \phi_0(n) - \phi_0(n-1), \tag{A9a}
R(\phi) := - (\ln(\exp(-i\phi))^2 : |\phi| \leq \pi, \tag{A9b}
I(\phi) := i \ln(\exp(-i\phi)) : -\pi \leq \phi < \pi. \tag{A9c}
\]

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$R(\phi)$ and $I(\phi)$, which are $2\pi$-periodic functions, can be cast into the form (3.11). This periodicity follows from (3.8c), which originates from the overlap of between spin-coherent states (2.9) whose real and imaginary parts are even and odd periodic, respectively. The discontinuity or non-smoothness are the consequence of the spatial continuum approximation.

**APPENDIX B: CALCULATION OF FLUCTUATION INTEGRAL**

The fluctuation action (4.20) may be written as

$$\frac{i}{\hbar}S^s_2(N-1) := -i\frac{N_{DW}S}{2} \sum_{n=1}^{N-1} \left\{ A(n)(\zeta^*(n))^2 + B(n)(\zeta(n))^2 + 2C(n)\zeta^*(n)\zeta(n) + 2D(n)\zeta^*(n)\zeta(n-1) \right\}, \quad (B1)$$

where

$$A(n) = \epsilon\Omega \cos 2\phi, \quad (B2a)$$
$$B(n) = \epsilon\Omega \cos 2\phi, \quad (B2b)$$
$$C(n) = -i, \quad (B2c)$$
$$D(n) = i - \epsilon\Omega \cos 2\phi. \quad (B2d)$$

This can be expressed in the matrix form:

$$\frac{i}{\hbar}S^s_2(N-1) = -i\frac{N_{DW}S}{2} \zeta M(N-1)\zeta, \quad (B3)$$

where

$$\zeta := (\zeta^*(N-1), \zeta(N-1), \zeta^*(N-2), \ldots, \zeta^*(1), \zeta(1)) \quad (B4a)$$

$$M(N-1) := \begin{pmatrix}
A(N-1) & C(N-1) & 0 & \\
C(N-1) & B(N-1) & D(N-1) & \\
0 & D(N-1) & A(N-2) & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
D(2) & A(1) & 0 & \\
0 & C(1) & B(1)
\end{pmatrix}. \quad (B4b)$$

As to the integration measure, which has not been explicitly derived, we may make the following two assumptions:

(i) Its structure is the same as that of the (boson-)coherent-state path integral:

$$\frac{1}{\mathcal{M}} \frac{d\zeta(n)d\zeta^*(n)}{2\pi i}, \quad (B5)$$

where $\mathcal{M}$ is a constant.

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(ii) The value of $\mathcal{M}$ can be inferred from the overlap between the initial and the final states.

Assumption (ii) is reasonable since $\exp[i\mathcal{S}(T = 0)/\hbar]$ should coincide with $\langle z_f|z_i \rangle$. On the basis of these assumptions, the fluctuation integral can be cast into the form

$$K_2(T) = \lim_{N \to \infty} \int \frac{1}{\mathcal{M}} \frac{d\zeta(n) d\zeta^*(n)}{2\pi i} \exp \left[ -i \frac{N_{\text{DW}} S}{2} t \zeta \mathcal{M}(N - 1) \zeta \right]$$

$$= \lim_{N \to \infty} \left[ (\mathcal{M}N_{\text{DW}} S)^{2(N-1)} (-1)^{N-1} \det(\mathcal{M}(N - 1)) \right]^{-1/2}$$

$$= \lim_{N \to \infty} \left[ (\mathcal{M}N_{\text{DW}} S)^{2(N-1)} \det \mathcal{M}(N - 1) \right]^{-1/2}.$$  \hspace{1cm} (B6)

The above determinant can be evaluated as follows. Let

$$M(n) := \det \mathcal{M}(n) = \begin{vmatrix} A(n) & -i & 0 & \\
- & B(n) & D(n) & \\
& 0 & D(n) & A(n-1) \\
& & & & \ddots \end{vmatrix}.$$  \hspace{1cm} (B7)

This can be expanded in terms of the cofactor as

$$M(n) = A(n)M'(n) + M(n - 1),$$  \hspace{1cm} (B8)

where

$$M'(n) := \begin{vmatrix} B(n) & D(n) & 0 & \\
D(n) & A(n-1) & -i & \\
0 & -i & B(n-1) & \\
& & & \ddots \end{vmatrix}.$$  \hspace{1cm} (B9)

$M'(n)$ can in turn be expanded as

$$M'(n) = B(n)M(n - 1) - D^2(n)M'(n - 1).$$  \hspace{1cm} (B10)

The recursion relations (B8) and (B10) should be solved with the initial condition

$$M(0) = 1, \quad M'(0) = 0.$$  \hspace{1cm} (B11)

In the limit of $\epsilon \to 0$, they reduce to a set of coupled first-order differential equations:

$$\frac{dM(t)}{dt} = \Omega_{\phi} M'(t),$$  \hspace{1cm} (B12a)

$$\frac{dM'(t)}{dt} = \Omega_{\phi} \left\{ M(t) + 2iM'(t) \right\},$$  \hspace{1cm} (B12b)

where $\Omega_{\phi}$ is given by (4.22). This gives
\[ M(t) = (1 - i\Omega_\theta t)e^{i\Omega_\theta t}, \quad M'(t) = \Omega_\theta te^{i\Omega_\theta t}. \quad \text{(B13)} \]

Hence,

\[
K_2(T) = \lim_{N \to \infty} \left[ (\mathcal{M}N_{DW}S)^2(N-1)M(N-1) \right]^{-1/2} \\
= \frac{e^{-i\Omega_\theta T/2}}{\sqrt{1 - i\Omega_\theta T}} \lim_{N \to \infty} \frac{1}{(\mathcal{M}N_{DW}S)^{N-1}}. \quad \text{(B14)}
\]

Since \( K_2(0) \) should be unity, we conclude from assumption (ii) that

\[
\mathcal{M} = \frac{1}{N_{DW}S}. \quad \text{(B15)}
\]
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† email: shibata@cmpt01.phys.tohoku.ac.jp
‡ email: takagi@cmpt01.phys.tohoku.ac.jp
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FIGURES

FIG. 1. Domain walls with three chiralities (quoted from Ref. 10); (a) right-handed wall ($\phi_0 = \pi/2$), (b) left-handed wall ($\phi_0 = -\pi/2$), and (c) wall with no chirality ($\phi_0 = 0$). Circles in (a) and (b) drawn to guide the eye lie in the $yz$ plane, while the spins lie in the $zx$ plane in (c). The quasi-one-dimensional direction of the crystal is here aligned with the spin hard axis for ease of visualization. A different alignment, which may be the case for a real magnet, does not affect the content of the text; for instance, one could rotate all the spins by $\pi/2$ around the $y$ axis if the dominant anisotropy originates from the demagnetizing field.

FIG. 2. Functions $R(\phi)$ and $I(\phi)$.

FIG. 3. The transition probability in the absence of transverse anisotropy as a function of $q \equiv q_F - q_I$ and $\phi_F - \phi_I$ with $N_{DW}S = 100$. It is independent of $T$.

FIG. 4. Time-dependence of the transition probabilities for $\phi_F = \phi_I = \pi/2$. Solid lines represent analytical results in the linear approximation (the curve for $q = 0$ is exact). Squares, open circles, and triangles represent numerical results.

FIG. 5. The transition probability as a function of $q$ and $T$ in the case of $\phi_F = \phi_I = -\pi/3$. Numerical results are indistinguishable from the analytical ones.

FIG. 6. The transition probability as a function of $q$ and $T$ in the case of $\phi_F = \phi_I = -\pi/4$. Numerical results are indistinguishable from the analytical ones.

FIG. 7. Time-dependence of the transition probabilities for $q_F = q_I$ and $\phi_I = \pi/2$. Solid lines represent analytical results in the linear approximation. Squares represent numerical results.
Fig. 1

(c)

Fig. 2
Fig. 3

Fig. 4
Fig. 5

Fig. 6
Fig. 7