W GRAVITY, $N = 2$ STRINGS AND $2 + 2 \, SU^*(\infty)$ YANG-MILLS INSTANTONS

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(August 1993)

Abstract

We conjecture that $W$ gravity can be interpreted as the gauge theory of $\phi$ diffeomorphisms in the space of dimensionally-reduced $D = 2 + 2 \, SU^*(\infty)$ Yang-Mills instantons. These $\phi$ diffeomorphisms preserve a volume-three form and are those which furnish the correspondence between the dimensionally-reduced Plebanski equation and the KP equation in $(1+2)$ dimensions. A supersymmetric extension furnishes super-$W$ gravity. The Super-Plebanski equation generates self-dual complexified super gravitational backgrounds (SDSG) in terms of the super-Plebanski second heavenly form. Since the latter equation yields $N = 1 \, D = 4$ SDSG complexified backgrounds associated with the complexified-cotangent space of the Riemannian surface, $(T^*\Sigma)^c$, required in the formulation of $SU^*(\infty)$ complexified Self-Dual Yang-Mills theory (SDYM); it naturally follows that the recently constructed $D = 2 + 2 \, N = 4$ SDSYM theory-as the consistent background of the open $N = 2$ superstring- can be embedded into the $N = 1 \, SU^*(\infty)$ complexified Self-Dual-Super-Yang-Mills (SDSYM) in $D = 3 + 3$ dimensions. This is achieved after using a generalization of self-duality for $D > 4$. We finally comment on the the plausible relationship between the geometry of $N = 2$ strings and the moduli of $SU^*(\infty)$ complexified SDSYM in $3 + 3$ dimensions.

1 Introduction

Recently [1,2], starting from a self-dual $SU^*(\infty)$ (complexified) Yang-Mills theory in $(2+2)$ dimensions, the Plebanski second heavenly equation was obtained after a suitable dimensional reduction. The self-dual gravitational background was the complexified cotangent space of the internal two-dimensional Riemannian surface required in the formulation of $SU^*(\infty)$ Yang-Mills theory [22]. A subsequent dimensional reduction leads to the KP equation in $(1+2)$ dimensions after the relationship from the Plebanski second heavenly function, $\Omega$, to the KP function, $u$, was obtained. Also a complexified KP equation was found when
a different dimensional reduction scheme was performed. Such relationship between $\Omega$ and $u$ is based on the correspondence between the $SL(2, R)$ self-duality conditions in $(3 + 3)$ dimensions of Das, Khviengia, Sezgin (DKS) [3] and the ones of $SU(\infty)$ in $(2 + 2)$ dimensions. The generalization to the Supersymmetric KP equation is straightforward by extending the construction of the bosonic case to the previous Super-Plebanski equation, found by us in [1], yielding self-dual supergravity backgrounds in terms of the light-cone chiral superfield, $\Theta$, which is the supersymmetric analog of $\Omega$. The most important consequence of this Plebanski-KP correspondence is that $W$ gravity can be seen as the gauge theory of $\phi$-diffeomorphisms in the space of dimensionally-reduced $D = 2 + 2$, $SU^*(\infty)$ Yang-Mills instantons. These $\phi$ diffeomorphisms preserve a volume-three-form and are, precisely, the ones which provide the Plebanski-KP correspondence. This was the main result of [2].

As a byproduct, we can generalize now our results to the supersymmetric case where the super KP equation can be obtained from the Super-Plebanski one using the results of [1]. The supersymmetric case is very relevant because there has been a lot of work recently [4] pertaining to the crucial role that $N = 4$ SDSYM in $D = 2 + 2$ has as the consistent $N = 2$ open superstring background [5]. The relevance of $N = 8$ (gauged) Self-Dual-Supergravity (SDSG) for the consistent background of the closed $N = 2$ (heterotic) superstring was noticed by Siegel [5]. Lately Nishino has shown [4] that the $N = 4$ SDSYM in $D = 2 + 2$ generates Witten’s topological field theory after appropriate twisted dimensional reduction/truncations. Therefore, the supersymmetric case is very relevant to current research.

There has been many derivations of the (super) KP equation in the literature. In [4] Supersymmetric KP systems were embedded in a SDSYM theory in $D = 2 + 2$. DKS [3] derived the KP equation from a self-duality condition in $(3 + 3)$ dimensions. Barcelos-Neto et al [7] derived it from the vanishing curvature equation in $2 + 1$ dimensions for an $SL(2, R)$ valued potential. A derivation of the KP equation based on the asymptotic $h \rightarrow 0$ limit of the continual $sl(N + 1, C)$ Toda molecule equation was given earlier by Chakravarty and Ablowitz [8]. The continual Toda molecule equation was obtained, first, by a suitable ansatz, dimensional reduction and continuous version of the the Cartan basis for $sl(N + 1, C)$. And many others. Six reasons why our results in [2] differed from the previous authors were outlined in [2]. The seven, and most important difference, was that we were able to provide with the geometrical origin of $W$ (super) gravity! The former can be interpreted as a gauge theory in the space of $D = 2 + 2$ $SU^*(\infty)$ (super) Yang-Mills instantons!

The outline of this letter goes as follows: In section II we review the derivation in [2] and show why $W$ gravity can be interpreted as the gauge theory of $\phi$ diffeomorphisms in the space of dimensionally reduced $D = 2 + 2$ $SU^*(\infty)$ Yang-Mills instantons. The $W$ metric is the gauge field that gauges these $\phi$ volume-preserving diffs. We discuss a very natural and plausible procedure to truncate the theory and yield $W_N$ gravity. A way how to obtain $W_\infty(\lambda)$ algebras is also discussed briefly. Finally, in III we present a discussion about the supersymmetric case and point out why the importance of $N = 4$ SDSYM in $D = 2 + 2$ as a consistent $N = 2$ open superstring background- really originates from a dimensional reduction of the $N = 1$ $SU^*(\infty)$ SDSYM in $D = 3 + 3$. (Of course, for those Lie algebras that can be embedded in $SU^*(\infty)$). i.e; the geometry (moduli) of the $N = 2$ open superstring stems from the moduli space of $N = 1$ $D = 3 + 3$ $SU^*(\infty)$ SDSYM theory. This is not surprising in view of the ubiquitous presence of Calabi-Yau manifolds in the
string literature. Generalized self-duality conditions for spaces with $D > 4$ are currently under investigation by many people [23].

2 On $W$ gravity and the KP equation from Plebanski

Let us choose complex coordinates for the complexified- spacetime, $C^4$: $y = (1/\sqrt{2})(x_1 + ix_2); \tilde{y} = (1/\sqrt{2})(x_1 - ix_2); \tilde{z} = (1/\sqrt{2})(x_3 - ix_4)$ and $z = (1/\sqrt{2})(x_3 + ix_4)$. The metric of signature $(4,0)$ and $(2+2)$ is, respectively, $ds^2 = dyd\tilde{y} + (-)dzd\tilde{z}$ and the complexified-spacetime SDYM equations are $F_{yz} = F_{\tilde{y}\tilde{z}} = 0$ and $F_{y\tilde{y}} + (-)F_{z\tilde{z}} = 0$. Such equations require an explicit signature-dependent definition of the Hodge * operation which is consistent with the double-Wick rotation of the Euclidean SDYM equations and with the fact that $**F = F$.

For our choice of coordinates: $\epsilon_{y\tilde{y}z\tilde{z}} = 1$. The * operation in spaces where $t = 0, 2$ is taken to be such that one of the SDYM eqs is: $F_{y\tilde{y}} = i^2/2 \sqrt{|g|} \epsilon_{y\tilde{y}z\tilde{z}} F^{z\tilde{z}}$; where $F^{z\tilde{z}} = -F_{z\tilde{z}}$.

The internal coordinates, $q, p$ can be incorporated into a pair of complex-valued, canonical-conjugate variables: $\hat{q} = Q(q,p)$, $\hat{p} = P(p,q)$ such as $\{\hat{q}, \hat{p}\}_{qp} = 1$. $Q, P$ are independent maps from a sphere ($S^2 \sim CP^1$), per example, to $C^1$, such as $Q \neq \lambda P$; $\lambda = constant$. This is in agreement with the fact that the true symmetry algebra of Plebanski’s equation is the $CP^1$ extension of the $sdiff \Sigma$ Lie-algebra as discussed by Park [10].

It was the suitable dimensional reduction in [1]: $\partial_y = \partial_{\hat{q}}; -\partial_{\tilde{y}} = \partial_{\hat{p}}$ and the ansatz (where for convenience we drop the “hats” over the $q, p$ variables):

$$\partial_z A_z = (1/2\kappa^2)\Omega_{zq}; \partial_z A_{\tilde{z}} = (1/2\kappa^2)\Omega_{z\tilde{p}};$$ (1a)
$$\partial_p A_z = (1/2\kappa^2)\Omega_{zp}; \partial_q A_{\tilde{z}} = (1/2\kappa^2)\Omega_{q\tilde{p}};$$ (1b)
$$\partial_q A_z = (1/2\kappa^2)\Omega_{zq}; \partial_p A_{\tilde{z}} = (1/2\kappa^2)\Omega_{zp};$$ (1c)
$$A_y = (1/2\kappa^2)\Omega_{zq}; A_{\tilde{y}} = -(1/2\kappa^2)\Omega_{z\tilde{p}}.$$ (1d)

that yields the Plebanski equation in [1]. $\kappa$ is a constant that has dimensions of length and can be set to unity. The semicolon stands for partial derivatives and $\Omega(z, \tilde{z}; \hat{q}, \hat{p})$ is the Plebanski’s second heavenly function. Upon such an anstaz and dimensional reduction, Plebanski’s second heavenly equation was obtained in [1]:

$$(\Omega_{zq})^2 - \Omega_{z\tilde{q}} \Omega_{\tilde{q}q} + \Omega_{z\tilde{z}} - \Omega_{\tilde{z}q} = 0.$$ (2)

Eq(2) yields self-dual solutions to the complexified-Einstein’s equations, and gives rise to hyper-Kahler metrics on the complexification of $T^*\Sigma$, through a continuous self-dual deformation, represented by $\Omega$, of the flat metric in $(T^*\Sigma)^c$ [1].

One of the plausible first steps in the dimensional-reduction of Plebanski’s equation is to take a real-slice. A natural real slice can be taken by setting: $\tilde{z} = \tilde{z}$. $\tilde{y} = \tilde{y}$ which implies, after using: $\partial_y = \partial_{\tilde{y}}$; $-\partial_{\tilde{y}} = \partial_{\tilde{p}}$, that $-(\partial_{\tilde{y}})^* = \partial_{\tilde{p}}$ and, hence, the Poisson-bracket degenerates to zero; i.e. it “collapses”: The quantity $\{Q, P\}_{qp} = \{Q, -Q^*\}_{q,p}$, if real, cannot be equal to 1 but is zero as one can verify by taking complex-conjugates on both sides of the equation. Therefore, since the Poisson brackets between any two potentials, $\{A_1, A_2\}_{qp} = \{A, B\}_{Q, P}\{Q, P\}_{q,p} = 0$, the $CP^1$-extension of the $sdiff \Sigma$ Lie-algebra is Abelianized (no commutators) in the process. DKS [3] already made the remark that their
derivation was also valid for $U(1)$. To sum up, taking a real slice reduces $C^4$-dependent solutions to $C^2$-dependent ones “killing”, in the process, the Poisson-brackets.

The reader might feel unhappy with this fact. Another option is not to take a real slice but, instead, one imposes the $C^1$-valued dimensional-reduction condition (the complexification of eq-3c, below): $\partial x_1 - \partial x_3 = 0$; where $x_1, x_3$ are complex coordinates. Since in this case $Q^*$ is no longer equal to $-P$, the Poisson-bracket is well defined, one ends up still having the $CP^1$-extended $sdiff \Sigma$ Lie-algebra untarnished and with a $C^3$-dependent theory (since the Plebanski equation was dependent of $C^4$). Complex-valued gauge potentials is precisely what is needed in order to have the $CP^1$-extended $sdiff \Sigma$ to be locally isomorphic to $su^*(\infty)$. Following the same step by step procedure as the one outline below, yields a complexification of the KP equation [2].

The second step of the dimensional-reduction (in the real case) is to take $\partial x_1 - \partial x_3 = \partial x_- = 0$; $x_- = x_1 - x_3$. and, hence, we end up with an effective real three-dimensional theory.

Now we are ready to establish the correspondence with the dimensionally-reduced $sl(2, R)$ SDYM equations in $(3 + 3)$ dimensions by DKS [3]. Set:

$$x_1 \to X_6 = Y; \quad x_2 \to x_2(X, Y; T); \quad x_3 \to X_3 = X; \quad x_4 \to X_1 = T. \quad (3a)$$

$$\partial x_2 = 0. \quad \partial x_3 = \partial x_4. \quad \partial x_5 = 0. \quad (3b)$$

$$\partial x_- = 0 \Rightarrow u(X; Y; T) \to \Omega(x_1 + x_3; x_2; x_4) = \Omega(x_+; x_2; x_+). \quad (3c)$$

(Notice the variables in eq-(3c); $\Omega$ is a function of a spatial, timelike and null variable. Compare this with the variables in the KP function; two temporal and one spacelike. For us, $X_2, X_4, X_6$ are spacelike and $X_1, X_3, X_5$ timelike.) the Jacobian:

$$J = \frac{\partial(X, Y, T)}{\partial(x_+, x_2, x_4)} = 1 \quad (3c)$$

Eq-(3a) defines a class of maps from $\mathcal{M}^{1+2} \to \mathcal{N}^{1+1+1}$; i.e. $\phi : P(X, Y, T) \to P'(x_+, x_2, x_4)$. The Jacobian $J$ should not vanish and without loss of generality can be set to one: $J = J^{-1} = 1$. Hence, eq-(3a) defines a class of volume-preserving-diffs.

Using eqs-(3a-3c) in equations (25,37,39,40,41) of DKS, we learn, from the correspondence given in eqs-(1,2) of [2], respectively, that a one to one correspondence with the $SU(\infty)$ SDYM equations, is possible iff we take for an Ansatz (see eq-46 in DKS) $A_{x_1} = A_{x_3}$. (eq-3d in [2]). The DKS-Plebanski ’dictionary’ reads:

$$F_{36} = 0 \to F_{x_1 x_3} = F_{y z} + F_{\bar{y} z} = 0. \quad (4a)$$

$$F_{13} = 0 \to F_{x_3 x_4} = iF_{x z} = 0. \quad (4b)$$

$$F_{16} = 0 \to F_{x_1 x_4} = F_{y z} - F_{\bar{y} z} = 0. \quad (4c)$$

The reason the right hand side of (4a) is zero is a result of the eqs-(3c,3d). Exactly the same happens to eq-(4b) which is nothing but one of the SDYM equations. $F_{y z} = \{\Omega_y, \Omega_{\bar{z}}\} = 0$ is a result of the ansatz in eqs-(1a-1d); the dim-reduction conditions in [1] and all of the 2 + 2 SDYM equations. (This is not the case in the Euclidean regime. After all there must be subtle differences between the Euclidean versus the 2 + 2 version; this is one of them.
besides the fact that two timelike directions provide with two Hamiltonian structures which are essential in many integrable systems). Eq-(4b) becomes after setting \( \{\Omega_q, \Omega_p\} = 0 \) in (2): \( \Omega_{zq} - \Omega_{zp} = 0 \). Equation-(4c) is zero iff (i). The ansatz of eq-(3d) in [2] is used. (ii). The dim-reduction conditions in eq-(3c) are taken and, (iii). The eqs-(2,4b) are satisfied. If conditions (i-iii) are met it is straightforward to verify that eq-(3c) \( = (\partial_{x_1} - \partial_{x_3})A_{x_4} = 0 \).

The crux of [2] was to obtain the desired relationship between \( u \) and \( \Omega \) in order to have self-consistent loop arguments and equations; to render the right handsides of eq-(4a,4b,4c) to zero; and, to finally, obtain the desired KP equation from the dimensional reduction of the Plebanski equation (DRPE).

Having established the suitable correspondence and the assurance that the right-hand sides of eqs-(4a,4b,4c) are in fact zero, we can now claim, by construction, that equations (39,40,41,47,53) of DKS [3] are the equivalent, in the \( u \) variable language, to eqs-(4a,4b,4c) above, in the \( \Omega \) language. Therefore the equivalence for eq-(4c) reads:

\[
\beta u_X - u_Y = 1/2(-\partial^2_{x_+} + \partial_{x_2} \partial_{x_4} + 2i \partial_{x_+} \partial_{x_2}) \Omega = 0. \tag{5b}
\]

and we include the DRPE:

\[
1/2(\partial^2_{x_+} + \partial_{x_2} \partial_{x_4}) \Omega(x_+;x_2;x_4) = 0 \rightarrow (\text{DRPE}) \tag{5c}
\]

The function, \( u(X,Y,T) \), satisfies the KP equation:

\[
\partial_X (u_T - 1/4 u_{XXX} - 3/2 uu_X + (\lambda^2 + \alpha \beta)u_X = \)

\[
1/2(\partial_{x_+}^2 - \partial_{x_2}^2 - i \partial_{x_+} \partial_{x_4} - i \partial_{x_+} \partial_{x_2}) \Omega(x_+;x_2;x_4) = 0. \tag{5a}
\]

And, similarly, eq-(47) of DKS is the equivalent of eq-(4a) above:

\[
\beta u_X - u_Y = 1/2(-\partial^2_{x_+} + \partial_{x_2} \partial_{x_4} + 2i \partial_{x_+} \partial_{x_2}) \Omega = 0. \tag{5b}
\]

The function, \( u(X,Y,T) \), satisfies the KP equation:

\[
\partial_X (u_T - 1/4 u_{XXX} - 3/2 uu_X) = -(\lambda^2 + \alpha \beta)\beta^{-2} u_{YY}. \tag{6}
\]

after using the relation, \( u_Y = \beta u_X \rightarrow u_{YY} = \beta^2 u_{XX} \) and differentiating the l.h.s. of (5a). One has to make a suitable scaling of the variables because the KP equation obtained by DKS was given in terms of dimensionless quantities. We must emphasize that \( \Omega \) is not constrained, in any way whatsoever, to satisfy two independent differential equations. The l.h.s of (5a) is zero as a consequence of (4a,4b). Viceversa, if (4a,4c) are satisfied then (4b) follows. We have seven equations to deal with. These are:

(i). The lhs and rhs of eqs (5a); (ii). The lhs and rhs of eqs (5b); (iii). The KP equation, (6); and, (v). Equation (3e), nonvanishing Jacobian.

This is all what is needed in order to arrive finally to our main result of [2]: For every solution \( \Omega \) of the DRPE (5c) we set: \( \Omega[x_+ (X,Y,T); x_2 (X,Y,T); x_4(X,Y,T)] = u(X,Y,T) \) and plugging \( \Omega \) into the l.h.s of (5a,5b) we get three partial differential equations, once we include (3e): \( \mathcal{J} = \mathcal{J}^{-1} = 1 \), for the volume-preserving diffs, \( \phi: P(X,Y,T) \rightarrow P'(x_+,x_2,x_4) \).

Once a solution for the three diffeomorphisms that comprise \( \phi \) is found, then we have an explicit expression for \( u(X,Y,T) \) that solves the KP equation by construction.

And, viceversa, once a solution for the KP equation is found we set \( u[X(x_+,x_2,x_4); Y(); T()] \) equal to \( \Omega(x_+,x_2,x_4) \) and plugging \( u \) into the r.h.s of (5a,5b) we get three partial differential equations, once we include (3e) (nonvanishing Jacobian), for the inverse volume-preserving-diffs, \( \phi^{-1}: P'(x_+,x_2,x_4) \rightarrow P(X,Y,T). \) Once a solution is found, we then have an explicit
expression for $\Omega(x_+, x_2, x_4)$ that solves the DRPE by construction because eqs-(4a,4c) $\Rightarrow$ eq-(4b).

Now, we arrive at the most important result of [2]. This construction that relates the KP to Plebanski is precisely what furnishes the geometrical meaning of $W$ gravity. Remember what we said earlier about the fact that the DRPE provides a solution-space, $\mathcal{S}$, of dim-reduced hyper-Kahler metrics in the complexified cotangent space of the Riemannian surface, $(T^*\Sigma)^\mathbb{C}$, required in the formulation of $SU^*(\infty)2 + 2$ SDYM theory. Since the volume-preserving diffs, $\phi; \phi^{-1}$, yield the dim-reduced- Plebanski- KP correspondence; it is natural that the $W_\infty$ symmetry algebra associated with the KP equation is the $\phi$ transform of the dimensional reduced $CP^1$-extended $sdiff\Sigma$- Lie algebra.

Because these $\phi$, volume-preserving- diffs, act on the solution space, $\mathcal{S}$, alluded earlier, it becomes clear that the $W$ metric can be interpreted as the gauge field which gauges these $\phi$ diffeomorphisms acting on the space $\mathcal{S}$!! In [1] we already made the remark that one could generalized matters evenfurther by starting with a $SU(\infty)$ SDYM theory on a self dual curved $2 + 2$ background. Upon imposing the ansatz of eqs-(1a-1d) one gets a generalization of equation (2) where an extra field, $\Omega^1$, the Plebanski first heavenly form, appears in addition to $\Omega$. $\Omega^1$ is the field which generates the self-dual metric for the $D = 2 + 2$ background. The generalization of (2) encodes the interplay between “two” self-dual “gravities”; one stemming from $\Omega$, the other from $\Omega^1$. In any case, in we wish to gauge the volume-preserving-diffs, $\phi$, in order to get $W$ gravity where the $W$ metric is the gauge field, we need to come up with an extra field, which is pressumably the role played precisely by $\Omega^1$.

Upon attaching fibers at each “point”, $\Omega$ of $\mathcal{S}$, these $\phi$ volume-preserving diffs act as gauge transformations (automorphisms) along the fiber; since $\Omega$ remains fixed as we vary $\phi$ along the fiber, one generates a continuous family of functions $u, u', u''$, ..... obeying the KP equation. Therefore, the $\phi$ transform of the dim-reduced $CP^1$-extended $sdiff\Sigma$-Lie algebra is indeed the symmetry algebra of the KP equation; i.e. A classical $W_\infty$ algebra which rotates solutions of the KP equation into eachother. We speak loosely when we say $W_\infty$ algebra; i.e; we also could be referring to the $W_{1+\infty}$ algebra. This all depends on the Topology of the two-surface, $\Sigma$. The project for the future is to elucidate what types of $W_\infty$-algebras one generates in this fashion. Our aim is to have a universal $W_\infty$ algebra from which all $W_\infty$ algebras can be obtained by a judicious truncation. Then one can pursue their quantization. All this is, of course, a non-trivial task.

What happens when we set the Jacobian, $J$ to an arbitrary constant, $\lambda$? In this case one gets a class of volume-preserving-diffs depending on $\lambda$, $\phi_\lambda$ which furnishes the $W_\infty(\lambda)$ algebras discussed in the literature, see references in [11]. A further and detailed discussion of this will be presented in a future publication.

In the case that one imposes a truncation the algebra should reduce to $W_N$. A very natural truncation can be done by a filtering procedure : $SU(N-1) \subset SU(N) \subset SU(N + 1) \subset$ ..... It was in this way that Hoppe [6] showed that a basis dependent limit of $SU(N)$ when $N \rightarrow \infty$ is isomorphic to the $sdiffS^2$-Lie algebra. A 'similar' construction based on Jet Bundles over Riemannian surfaces and a Drinfeld-Sokolov reduction was given by Bilal, Fock and Kogan to explain the origins of $W$ algebras [12]. A thorough discussion on area-preserving diffs and (classical) $w_\infty$ algebras was given by Sezgin [6]. The Topology of $\Sigma$ is crucial as to what type of $w_\infty, w_{1+\infty}$ algebras one gets. For higher genus surfaces, $g > 1$
one can start with Siegel’s upper-half of the complex-plane (a disc, whose \( sdiff \) Lie algebra is isomorphic to \( SL(\infty) \)) and take suitable quotients by finite/discrete groups. This won’t concern us at the moment. There is a large mathematical literature on this subject. What mostly concerns us is what are the physical implications to string theory and how to select its true vacuum state. This will see shortly.

We’ve been working with a dim-reduction procedure; is it possible to incorporate other types of reduction schemes, like Killing symmetry types? Park [10] showed that a Killing symmetry reduction of the Plebanski first heavenly equation yields the \( sl(\infty) \) continual Toda equation whose asymptotic expansion by Ablowitz and Chakravarty [8] led to the KP equation. The (classical) \( W_\infty \) algebra was obtained as a Killing symmetry reduction of the \( CP^1 \)-extended \( sdiff \) \( \Sigma \)-Lie algebra. Since the physics of self dual gravity should be independent of which formulation one is using, either in terms of the first or second heavenly form, this fact corroborates, once more, that our findings should be correct because \( sl(\infty, C) \) can be embedded into \( sl(\infty, H) \sim s\nu^*(\infty) \). Gervais and Matsuo [13] proposed viewing \( W \) gravity as holomorphic embeddings of \( \Sigma \) into higher-dim Kahler manifolds, like \( CP^n \) or some suitable coset space, \( G/H \), where \( W \) transformations look like particular diffs in the target space. The \( W \) surfaces in \( CP^n \) were shown to be instantons of the corresponding non-linear \( \sigma \)-models. A geometrical meaning of the associated conformally-reduced WZNW models was given in terms of the Frenet-Serret equations for these \( CP^n \) embeddings. The relevance of these WZNW models was already pointed out by Park [10] who showed that WZNW models valued in \( sdiff \) \( \Sigma \) were equivalent to \( D = 4 \) SD gravity. In [14] we found that the \( N = 2 \) SWZNW model valued in \( sdiff \) \( \Sigma \) was SDSG in four dimensions. We see once again the crucial role that SDG and SDSG has on the nature of the (super) \( W \) geometry. All these findings would fit into our picture when a Killing-symmetry reduction scheme is chosen.

Another clue that we are on the right track was provided by Boer and Goeree [15] who studied arbitrary \( W \) algebras related to the embeddings of \( sl(2) \) in a Lie algebra \( G \). A simple general formula for all \( W \) transformations enabled them to construct covariant actions for \( W \) gravity and this action was nothing but a Fourier transform of the WZNW action. It was precisely such general formula that provided a ’geometrical’ interpretation of \( W \) transformations as a homotopy contractions of ordinary gauge transformations. These homotopy-contractions should correspond to the volume-preserving \( \phi \)-diffs alluded earlier. For further geometrical interpretations of \( W \) transformations see Figueroa-O’Farril et al [16]; Yi and Soloviev [17]; Sotkov et al [18]. Giveon and Shapere [19] noticed the importance of volume-preserving diffs in the gauge symmetries of the \( N = 2 \) string. Since it is their physics we are mainly concerned with, it is the geometry of \( N = 2 \) strings that we turn our attention to in the conclusion of this letter.

3 Conclusion: The SKP Equation and the Geometry of \( N = 2 \) Strings

Using the results by us in [1] we can automatically extend the construction here to the supersymmetric case. The Lorentzian version of the Plebanski equation was derived for \((3 + 1)\) superspace:

\[
(\Theta_{\bar{q}\bar{p}})^2 - \Theta_{\bar{q}\bar{q}} \Theta_{\bar{p}\bar{p}} + \Theta_{\bar{q}\bar{z}} - \Theta_{\bar{p}\bar{z}} = 0.
\]
where \( \Theta(z, \tilde{z}, \hat{q}, \hat{p}; \theta^+) \) is a light-cone chiral superfield described by Gilson et al [20]. The derivatives in (7) are taken with respect to the left-handed variables: \( x^m \equiv x^m + i \theta \sigma^m \bar{\theta} \). For details and self-duality conditions in Euclidean and Atiyah-Ward spacetimes, see [1,4,14,20]. A double-Wick rotation from 2 + 2 (or a single one from 3 + 1) SDSL into Euclidean SG can only be performed iff \( N = \text{even} \) because there are no Majorana spinors in Euclidean space. A simple Wick-rotation of 3 + 1 SDSL into 2 + 2 SDSL should be possible, after a suitable truncation, since SDSL in 2 + 2-dim is based on the existence of Majorana-Weyl spinors. In any case it is not difficult to see due to the fact that we were working in complexified superspaces [21] (these authors also discussed quaternionic superspaces), that a \( N = 1, 2 + 2SU^*(\infty) \) complexified SDSL theory accomodates a doubling of the number of real degrees of freedom. Therefore, one should get the \( N = 2 \) SKP equation and its associated \( N = 2 \) super-\( W_\infty \) algebra. Nishino has conjectured recently the existence of \( N = 2 \) SKP equations in [4]. If we had started initially by taking a real slice, a \( N = 1 \) SKP equation would have been obtained. It is not difficult to see that in order to embed the \( N = 4 \) 2 + 2 SDSL theory [4], one must start from a \( N = 1 3 + 3 SU^*(\infty) \) complexified SDSL theory instead; a complexified-fiber-bundle over a complexified six-dim superspace. (we are not concerned with the mathe matical subtleties for the time being, like the non uniqueness of the \( N \rightarrow \infty \) limits; the existence of complex structures in infinite-dimensional spaces;.....). A dim-reduction to 2 + 2 dimensions yields a doubling, which in turn, implies a fourth-folding in terms of the number of real degrees of freedom (supersymmetries). Therefore, for those algebras which can be embedded into \( su^*(\infty) \), we have that six dimensional \( N = 1 3 + 3 SU^*(\infty) \) complexified SDSL is the "master" theory of lower-dim supersymmetric integrable systems! One could have started from an \( N = 1 D = 5 + 5 SYM \) coupled to \( N = 1 D = 5 + 5 SG \) and after suitable dimensional reductions/ truncations obtain different types of \( N \)-extended lower-dim supersymmetric theories. A further constraint would extract the Self-Dual pieces from the latter [4]. However, this is not necessary! We have now at our disposal ways to generalize the notion of self-duality to higher than-four dimensions [23]. It is the moduli of these higher dimensional \( 3 + 3 SU^*(\infty) \) complexified generalized-self-dual super symmetric theories that encode the geometry of \( N = 2 \) strings. This is the main result of this letter.

In view of this we can see that the geometry of \( N = 2 \) open superstrings is tightly connected to that of the moduli space of \( N = 1 3 + 3 SU^*(\infty) \) complexified SDSL theory. The super \( W_\infty \) symmetry would follow by extending our bosonic-construction in [2] to the SKP equation. Surely enough, the latter moduli space (and the associated with the closed (heterotic) \( N = 2 \) superstring) must be related to the infinite-dimensional super-Grassmannian where super-Riemannian surfaces (SRS) of arbitrary genus are represented as 'points'. We learnt from the Manin-Radul SKP hierarchy [24] that the super-\( \tau \) function satisfies the super-Hirota equations of the SKP hierarchy which is the generating function for the super-conformal field theory on the SRS of arbitrary genus. An infinite perturbation expansion, in principle, should allow us to extract nonperturbative information to our superstring theory. And where, if any, should this nonperturbative information come from ?? From the moduli space of \( D = 2 + 2 SU^*(\infty) \) Super-Yang-Mills instantons. We should not be worried about using complex-valued gauge-potentials as long as we know how to extract real-valued information from the theory. We must not forget that the chiral counterparts, \( \hat{W} \) algebras, are always there!: the \( CP^1 \)-extended \( sdiff \Sigma \)-Lie algebra does contain both the
W, \bar{W} \text{ algebras} [10]. Therefore, we expect to see a connection between the large } N \text{ limit of } SU(N) \text{ complexified Gauge Theories and the infinite genus-limit of Riemann surfaces in the sum over world-sheets approach to string theory. Based on the counting above and on the results in [1,2], starting with a } N = 4 \text{ } SU^*(\infty) \text{ complexified SDSYM theory coupled to a } N = 4 2+2 \text{ } SDSG \text{ complexified-background should establish a link with Siegel's observation that } N = 8 \text{ (gauged) SDSG in } 2+2 \text{ is the underlying theory of the (heterotic) closed } N = 2 \text{ superstring.}

Since } W_\infty \text{ is a symmetry of string-field theory, [25], the quest for the true vacuum of string theory might, in principle, not be so far away because, at least, we know where to look: It ought to lie behind the moduli space of } SU^*(\infty) \text{ SDSYM in } 3+3 \text{ dimensions. This might also shed some light as to how extract } D = 4 \text{ QCD from string theory in four dimensions. All this has to be proven, of course. What we know for certain is that the construction of [2] did not require any laborious nor sophisticated algebra; it is right there in section II! Supersymmetric theories in } D = 2+2 \text{ have been constructed in [4]; generalized-self-dual theories in [23]; therefore, the tools appear to be available to tackle and prove/disprove our conjecture.}

REFERENCES

1. C. Castro : Journ. Math. Phys. \textbf{34}, 2 (1993) 681.
2. C. Castro :” The KP equation from Plebanski and } SU(\infty) \text{ SDYM theory” I.A.E.C-7-93 preprint. Submitted to the Jour. Math. Physics.
3. A.Das, Z. Khviengia, E. Sezgin : Phys. Letters B \textbf{289} (1992) 347.
4. H. Nishino : UMDPP-93-preprint ;” Supersymmetric-KP systems Embedded in Supersymmetric Self-Dual Yang-Mills Theory. UMDEPP-93-145.
“SDSYM Generates Witten’s Topological Field Theory” UMDEPP-93-14.
S.J.Ketov, S.J.Gates Jr. and H. Nishino : Nucl. Phys. B \textbf{393} (1993) 149.
5. W. Siegel : Stony Brook ITP-SB-92-24 (May 1992). H. Ooguri and C. Vafa. Nucl. Phys. B \textbf{361} (1991) 469; \textbf{ibid}. \textbf{367} (1991) 83.
6. Jens Hoppe : Int. Journal Mod. Phys. A\textbf{4} (1989) 5235. E. Sezgin : “Area-Preserving Diffeomorphisms , } w_\infty \text{ Algeb and } w_\infty \text{ Gravity. CTP-TAMU-13-92.
7. J.Barcelos-Neto, A. Das, S.Panda and S. Roy. Phys. Lett. B \textbf{282} (1992) 365.
Cambridge University Press. (1991).
8. M.J. Ablowitz, P.A. Clarkson :“\textbf{Solitons, Nonlinear Evolution Equations and Inverse Scattering}” London Math. Soc. Lecture Notes \textbf{149};
Cambridge University Press (1991).
9. Edward Witten :”Surprises with Topological Field Theories “ Proceedings of Strings’ 90 at College Station, Texas, USA.
10. Q.H. Park : Phys. Letters. B \textbf{236} (1990) 429; \textbf{ibid} B \textbf{238} (1990) 287.
Int.Jornal of Modern Phys A \textbf{7}; page 1415, (1991).
11. P. Bouwknegt, K. Schouetens : “W symmetry in Conformal Field Theories “ CERN-TH-6583/92. To appear in Physics Reports.
12. A.Bilal, V.V. Fock and I.I. Kogan : “On the Origins of } W \text{ Algebras.}
CERN-TH-5965-90 preprint.
13. J.L.Gervais, Y. Matsuo :” Classical $A_n$ $W$ geometry” L.P.T.E.N.S 91-35 preprint. Phys. Lett. B 274 (1992) 309-316.
14. C. Castro : “The $N = 2$ SWZNW model valued in $sdiff$ $\Sigma$ is Self Dual Supergravity in four Dimensions “. I.A.E.C-11-92 preprint; submitted to the Journ. Math. Phys.
15. J. de Boer and J. Goeree : “Covariant $W$ Gravity and its Moduli Space from Gauge Theory” THU-92-14 preprint. Univ. of Utrecht.
16. J. Figueroa-O’Farril,E. Ramos and S. Stanciu : “A Geometrical Interpretation of Classical $W$ Transformations. BONN-HE-92-27
17. W.S.Yi and O.A. Soloviev : “ $W_3$ gravity from affine maximal hypersurfaces”. UMDEPP-92-146.
18. G. Sotkov, M. Stanishkov. Nucl. Phys. B 356 (1991) 439.
19. A. Giveon and A. Shapere : “Gauge Symmetries of the $N = 2$ String” IASSNS-HEP-92-14 preprint.
20. G.R.Gilson, I.Martin, A. Restuccia and J.G. Taylor : Comm. Math. Physics 107 (1986) 377.
21. J. Lukierski and A. Nowicki; Ann. Phys. 166 (1986) 164.
22. E.G.Floratos, J. Iliopoulos, G.Tiktopoulos : Phys. Lett. B 217 (1989) 285.
23. A.D. Popov; JINR Rapid communications, no.6 57 (1992) 57.
24. Y. Manin and A.O. Radul; Comm. Math. Physics. 98 (1985) 65.
25. M. Kaku; Introduction to Superstrings. Springer-Verlag, 1988.
M. Kaku; Strings, Conformal Field Theory and Topology, An Introduction. Springer-Verlag, 1990.