Some spaces are more equal than others

Boudewijn F. Roukema

1 Toruń Centre for Astronomy, Nicolaus Copernicus University, ul. Gagarina 11, 87-100 Toruń, Poland

Key words cosmology, cosmic topology
PACS 98.80.-k, 98.80.Bp, 98.80.Es, 98.80.Jk, 95.30.Sf

It has generally been thought that in perturbed Friedmann-Lemaître-Robertson-Walker models of the Universe, global topology should not have any feedback effects on dynamics. However, a weak-field limit heuristical argument, assuming a finite particle horizon for the transmission of gravitational signals, shows that a residual acceleration effect can occur. The nature of this effect differs algebraically between different constant curvature 3-manifolds. This potentially provides a selection mechanism for the 3-manifold of comoving space.

1 Cosmic topology and dynamics

The geometrical and global topological freedom allowed by constant curvature Riemannian 3-manifolds as a model of the comoving spatial section of a relativistic universe [5, 6, 7, 13, 21] imply that a Friedmann-Lemaître-Robertson-Walker (FLRW) model may be finite in total comoving spatial volume for any of the three curvatures: negative, zero or positive. This resolves the historical dilemma in cosmology according to which two physically desirable qualities of space seemed to be contradictory: a finite universe without boundaries seemed to be a self-contradiction. Riemannian geometry gave a solid mathematical foundation to the resolution of this apparent conflict. Even before the relativistic era, Riemannian 3-manifolds of curvature and/or topology different to that of simply-connected Euclidean space were proposed as models of space by Karl Schwarzschild [27, 32]. Skipping forward, the start of the “modern” epoch of active cosmic topology research can probably be dated back to the two 1993 papers by Starobinsky [30] and Stevens et al. [31]. It is a great pleasure to have Alexei Starobinsky here with us in this meeting. For reviews of the historical and modern theoretical and observational aspects of cosmic topology, see refs [12, 17, 29, 15, 3, 20].

Although, in general, these 3-manifolds are not vector spaces, Grassmann’s pioneering work in linear algebra [8, 9] nevertheless provides useful tools (as it does in nearly all of modern science), since working in the covering spaces $\mathbb{H}^3$, $\mathbb{R}^3$, and $S^3$, for negative, zero, and positive curvature, respectively, is useful for many observational and theoretical purposes. The covering space $\mathbb{R}^3$ is a vector space, and calculations made when the covering space is the hypersphere $S^3$ are very straightforward to program when $S^3$ is embedded in $\mathbb{R}^4$. Both the analytical and numerical calculations referred to below for the spherical well-proportioned spaces specifically use this latter technique—thanks to Hermann Grassmann.

What was long taken as self-evident in cosmic topology research was the inference that since the Einstein field equations are local, there is no way that the global topology of the spatial section of an FLRW universe could have an effect on that universe’s dynamics. The only known link between topology and dynamics was through curvature, since the three different curvatures allow three different families of 3-manifolds.
2 Residual gravity

However, it was shown heuristically in ref \cite{22} that in a multiply-connected universe containing a density perturbation, the effects of the distant copies of the perturbation in the covering space are not exactly symmetrical on a massless test particle, leading to a non-zero residual acceleration effect. Consider Fig. 3 in ref \cite{22}. In a flat, multiply-connected model of comoving in-diameter \( L \), the perturbation is modelled as a point-sized massive object, and the gravitational pull on a massless test particle at a small distance \( x \) is estimated in the weak-field limit, i.e. Newtonian gravity within a finite particle horizon is used. In the covering space \( \mathbb{R}^3 \) along a given holonomy transformation axis, one of the two adjacent copies (at \( L - x \) and \( L + x \)) of the massive object is slightly closer to the test particle than the other. Hence, at the position of the massless test particle, after removing the gravitational pull towards the “original” copy of the massive object at distance \( x \), there is a residual gravitational force that pulls the massless test particle towards the slightly closer of the two topological images of the “original” massive object. A Taylor expansion in \( x/L \)

\[
\ddot{x} = \frac{4Gm}{L^3}x 
\]

where \( G \) is the gravitational constant and \( m \) is the mass of the massive object.

This example does not constitute a full relativistic calculation, but it is difficult to see how the effect could be avoided in a perturbed FLRW model, or in an hypothetical exact solution for an almost FLRW model. A universe which is small and homogeneous except for one positive density fluctuation is seen by an observer to be slightly anisotropic unless the observer is located at the centre of the fluctuation, which can be assumed to be spherically symmetric. This anisotropy concerns the gravitational potential seen from different directions, as it is transmitted to the observer by gravitational waves. It is difficult to avoid the conclusion that the dynamics of a perturbed FLRW model can be affected by global topology, even if the effect at the present epoch is likely to be small.

2.1 Stabilisation towards equal fundamental lengths

In a right-angled 3-torus model, i.e. \( T^3 \equiv \mathbb{R}^3/\mathbb{Z}^3 \) where the fundamental domain is a rectangular prism, if the three side-lengths of the prism \( L_i \) are very unequal to one another, then the \( L_i^{-3} \) factor will cause the shortest length to induce the strongest residual acceleration. As suggested in Sect. 3.2.3 of \cite{22}, this may cause the shorter length of the fundamental domain to expand faster than the other two directions, tending towards equality of the three side-lengths. The effects would equalise when the side-lengths equalise. If the model initially has an isotropic scale factor, then it will become anisotropic in the sense that the scale factors in different directions will become slightly unequal, and will remain so until equal side-lengths are obtained.

2.2 \( T^3 \) is both well-proportioned and well-balanced

However, at the equilibrium state of equal side-lengths of a \( T^3 \) model, there residual acceleration is different to that given in Eq. (1). As shown algebraically and numerically in Sect. 3.1 of ref \cite{26}, the residual accelerations induced by a massive object in an exactly regular \( T^3 \) model cancel down to the third order in \( \epsilon_x, \epsilon_y, \epsilon_z \), the distances to the test particle as fractions of the respective fundamental lengths. Hence, not only is a regular \( T^3 \) model an equilibrium state in the sense that the residual accelerations in the three directions will tend to equalise, but the residual acceleration as a vector (\( T^3 \) is a flat space) will also drop sharply in amplitude as this equalisation is approached.

Spaces that have approximately equal fundamental lengths have been termed “well-proportioned” \cite{33}. It is now clear that (regular) \( T^3 \) is not only well-proportioned: it is also well-balanced.

\footnote{Physically, there is no difference between an “original” and a “copy” of the object—these are two images in the covering space. See the review papers cited above for a fuller introduction.}
3 Well-proportioned spherical FLRW models with a perturbation

Well-proportioned spherical FLRW models also exist. While well-proportionality might seem to be a subjective, aesthetic criterion for a preferred model of the Universe, the lack of structure on scales above \( \sim 10h^{-1} \) Gpc in the Wilkinson Microwave Anisotropy Probe (WMAP) sky maps, predicted as a sign of cosmic topology based on the low-resolution COBE maps [30, 31] is best explained by a well-proportioned model [33]. Among these, the Poincaré dodecahedral space model, \( S^3/I^* \), has become a particularly good (though disputed) candidate [28, 16, 25, 1, 2, 10, 11, 18, 4, 14, 24, 23].

Are the well-proportioned spherical spaces, \( S^3/T^* \), \( S^3/O^* \), and \( S^3/I^* \), also well-balanced in the sense of the residual gravity effect? As shown by Grassmann [8] over a century and a half ago, \( \mathbb{R}^4 \) exists as a self-consistent mathematical object—a four-dimensional vector space—and on a modern computer, calculations in \( \mathbb{R}^4 \) require only a minor change in computer code compared to those in \( \mathbb{R}^3 \). By embedding \( S^3 \) in \( \mathbb{R}^4 \), both algebraic and numerical calculations of the dominant terms of the residual gravity effect are rendered tractable for these spaces, as presented in ref [26].

The result is that all three of these spaces are indeed well-balanced. The linear term in the Taylor expansion of the residual gravity effect cancels in all three cases. However, the Poincaré space \( S^3/I^* \), the space that has been selected by empirical arguments, is even better balanced than the other spaces. Not only does the linear term cancel, but the third-order term also cancels, leaving an expression dominated by the fifth order. This fifth-order term can be written as a vector in \( \mathbb{R}^4 \)

\[
\hat{r} = \frac{12\sqrt{5} (297\sqrt{5} + 655)}{125\sqrt{5} - \sqrt{5}} \left( \frac{r}{R_C} \right)^5 \left\{ \begin{array}{l}
[70 y^4 + (42 \sqrt{5} + 70) x^2 y^2 - (14 \sqrt{5} + 70) y^2 + (21 \sqrt{5} - 7) x^4 - 28 \sqrt{5} x^2 \\
+ 7 \sqrt{5} + 5 \end{array} x,
\right.
\left( \begin{array}{l}
[70 z^4 + (42 \sqrt{5} + 70) y^2 z^2 - (14 \sqrt{5} + 70) z^2 + (21 \sqrt{5} - 7) y^4 - 28 \sqrt{5} y^2 \\
+ 7 \sqrt{5} + 5 \end{array} y,
\right.
\left( \begin{array}{l}
[70 x^4 + (42 \sqrt{5} + 70) x^2 z^2 - (14 \sqrt{5} + 70) z^2 + (21 \sqrt{5} - 7) x^4 - 28 \sqrt{5} z^2 \\
+ 7 \sqrt{5} + 5 \end{array} z,
\right.
\left(0\right),
\right.
\right.
\right.
\right.
\end{array}
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
\right.
expands. While only very elementary calculations have been performed so far, the initial results are tantalising. Not only does the effect seem to be a stabilising effect towards equal side-lengths in a $T^3$ model, but it also seems that the effect selects out the space that is preferred empirically by several groups based on the WMAP data—the Poincaré dodecahedral space, $S^3/I^*$—as being better balanced than other spaces. Could this effect have provided a selection criterion during the quantum epoch of the Universe?

**Acknowledgements** The author thanks the organisers for a meeting that was very enjoyable and scientifically productive.

**References**

[1] Aurich R., Lustig S., Steiner F., 2005a, CQG, 22, 3443, [arXiv:astro-ph/0504656]
[2] Aurich R., Lustig S., Steiner F., 2005b, CQG, 22, 2061, [arXiv:astro-ph/0412569]
[3] Blanlœil V., Roukema B. F., eds, 2000, “Cosmological Topology in Paris 1998” Paris: Blanlœil & Roukema, [arXiv:astro-ph/0010170]
[4] Caillerie S., Lachièze-Rey M., Luminet J. , Lehoucq R., Riazuelo A., Weeks J., 2007, A&A, 476, 691, [arXiv:0705.0217v2]
[5] de Sitter W., 1917, MNRAS, 78, 3.
[6] Friedmann A., 1923, Mir kak prostranstvo i vremya (The Universe as Space and Time). Leningrad: Academia.
[7] Friedmann A., 1924, Zeitschr. f ü r Phys., 21, 326.
[8] Grassmann H. G., 1844, Die lineare Ausdehnungslehre. Leipzig: Wiegand.
[9] Grassmann H. G., 1862, Die Ausdehnungslehre, vollständig und in strenger Form bearbeitet. Berlin: Enslin.
[10] Gundermann J., 2005, ArXiv e-prints, [arXiv:astro-ph/0503014]
[11] Key J. S., Cornish N. J., Spergel D. N., Starkman G. D., 2007, Phys. Rev. D, 75, 084034, [arXiv:astro-ph/0504656]
[12] Lachièze-Rey M., Luminet J., 1995, Phys. Rep., 254, 135, [arXiv:gr-qc/9605010]
[13] Lemaître G., 1931, MNRAS, 91, 490.
[14] Lew B., Roukema B. F., 2008, A&A, 482, 747, [arXiv:0801.1358]
[15] Luminet J., Roukema B. F, 1999, in NATO ASIC Proc. 541: Theoretical and Observational Cosmology, Publisher: Dordrecht: Kluwer, Topology of the Universe: Theory and Observation. p. 117, [arXiv:astro-ph/9901364]
[16] Luminet J., Weeks J. R., Riazuelo A., Lehoucq R., Uzan J., 2003, Nature, 425, 593, [arXiv:astro-ph/0302535]
[17] Luminet J.-P., 1998, Acta Cosmologica, XXIV-1, 105, [arXiv:gr-qc/9804006]
[18] Niarcho A., Jaffe A., 2007, Physical Review Letters, 99, 081302, [arXiv:astro-ph/0702436]
[19] Orwell G., 1945, Animal Farm: A Fairy Story. London: Secker and Warburg.
[20] Reboücas M. J., Gomero G. I., 2004, Braz. J. Phys., 34, 1358, [arXiv:astro-ph/0402324]
[21] Robertson H. P., 1935, ApJ, 82, 284.
[22] Roukema B. F., Bajtlik S., Biesiada M., Szaniewska A., Jurkiewicz H., 2007, A&A, 463, 861, [arXiv:astro-ph/0602159]
[23] Roukema B. F., Buliński Z., Gaudin N. E., 2008, A&A, 492, 673, [arXiv:0807.4260]
[24] Roukema B. F., Buliński Z., Szaniewska A., Gaudin N. E., 2008, A&A, 486, 55, [arXiv:0801.0006]
[25] Roukema B. F., Lew B., Cechowska M., Marecki A., Bajtlik S., 2004, A&A, 423, 821, [arXiv:astro-ph/0402608]
[26] Roukema B. F., Rózański P. T., 2009, A&A, 502, 27, [arXiv:0902.3402]
[27] Schwarzschild K., 1900, Vier.d.Astr.Gess, 35, 337.
[28] Spergel D. N., Verde L., Peiris H. V., Nolta M. R., Bennett C. L., Halpern M., Hinshaw G., Jarosik N., Kogut A., Limon M., Meyer S. S., Page L., Tucker G. S., Weiland J. L., Wollack. E., Wright E. L., 2003, ApJSupp, 148, 175, [arXiv:astro-ph/0302209]
[29] Starkman G. D., 1998, CQG, 15, 2529.
[30] Starobinsky A. A., 1993, Journal of Experimental and Theoretical Physics Letters, 57, 622.
[31] Stevens D., Scott D., Silk J., 1993, Physical Review Letters, 71, 20.
[32] Stewart J. M., Stewart M. E., Schwarzschild K., 1998, CQG, 15, 2539.
[33] Weeks J., Luminet J.-P., Riazuelo A., Lehoucq R., 2004, MNRAS, 352, 258, [arXiv:astro-ph/0312312]