LOCALLY SASAKIAN MANIFOLDS

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Abstract

We show that every Sasakian manifold in dimension $2k + 1$ is locally generated by a free real function of $2k$ variables. This function is a Sasakian analogue of the Kähler potential for Kähler geometry. It is also shown that every locally Sasakian-Einstein manifold in $2k + 1$ dimensions is generated by a locally Kähler-Einstein manifold in dimension $2k$.

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1 Introduction

The Sasakian structure, which is defined on an odd dimensional manifold is, in a sense, the closest possible analog of the Kähler geometry of even dimension. It was introduced by S. Sasaki [12] in 1960, who considered it as a special kind of contact geometries. Sasakian structure consists, in particular, of the contact 1-form $\eta$ and the Riemannian metric $g$. The differential of $\eta$ defines a 2-form, which constitutes an analog of the fundamental form of Kähler geometry.

Sasakian geometry was primarily studied as a substructure within the category of contact structures. A review of this approach can be found in [1, 14]. In this letter we exploit the analogy between Sasakian and Kähler geometry. We show that a well known fact that a Kähler geometry can be locally generated by a Kähler potential has its Sasakian counterpart. This result may be of some use in constructing a vast family of examples of Sasakian and Sasakian-Einstein structures.

The Sasakian and Sasakian-Einstein structures appear in physics in the context of string theory. It turns out that a metric cone $(C=S^+\times S, \tilde{g}=dr^2+r^2g)$ over a Sasakian-Einstein manifold $(S, g)$ is Kähler and Ricci flat, i.e. it constitutes a Calabi-Yau manifold. Moreover, the Sasaki-Einstein manifolds in dimensions $2k+1$ and Sasakian manifolds with three Sasakian structures in dimension $4k+3$ are related to the Maldacena conjecture [3, 4, 6, 13]. It turns out that they are one of very few structures which can serve as a compact factor in (anti-de-Sitter) background for classical field theories which, via the Maldacena conjecture, correspond to the large $N$ limit of certain quantum conformal field theories.

A formal definition of a Sasaki manifold is as follows.

**Definition 1**

Let $S$ be a $(2k+1)$-dimensional manifold equipped with a structure $(\phi, \xi, \eta, g)$ such that:

(i) $\phi$ is a $(1,1)$ tensor field,
(ii) $\xi$ is a vector field,
(iii) $\eta$ is a field of an 1-form,
(iv) $g$ is a Riemannian metric.

Assume, in addition, that for any vector fields $X$ and $Y$ on $S$, $(\phi, \xi, \eta, g)$ satisfy the following algebraic conditions:

1. $\phi^2X = -X + \eta(X)\xi$,
2. $\eta(\xi) = 1$,
3. $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$,
4. $g(\xi, X) = \eta(X)$,

and the following differential conditions:

5. $N_\phi + d\eta \otimes \xi = 0$,

where $N_\phi(X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$ is the Nijenhuis tensor for $\phi$,

6. $d\eta(X, Y) = g(\phi X, Y)$.

Then $S$ is called a Sasakian manifold.

**Example 1**

A standard example of a Sasaki manifold is an odd dimensional sphere

$$S^{2k+1} = \{ C^{k+1} \ni (z^1, ..., z^{k+1}) : |z^1|^2 + ... + |z^{k+1}|^2 = 1 \} \subset C^{k+1},$$

viewed as a submanifold of $C^{k+1}$. Let $J$ be the standard complex structure on $C^{k+1}$, $\tilde{g}$ the standard flat metric on $C^{k+1} \equiv R^{2k+2}$, and $n$ be the unit normal to the sphere. The vector field $\xi$ on $S^{2k+1}$ is defined by $\xi = -Jn$. If $X$ is tangent vector to the sphere then $JX$ uniquely decomposes onto the part parallel to $n$ and the part tangent to the sphere. Denote this decomposition by $JX = \eta(X)n + \phi X$. This defines 1-form $\eta$ and tensor field $\phi$ on $S^{2k+1}$. Denoting the restriction of $\tilde{g}$ to $S^{2k+1}$ by $g$ we obtain $(\phi, \xi, \eta, g)$ structure on $S^{2k+1}$. It is a matter of checking that this structure equips $S^{2k+1}$ with a structure of a Sasakian-Einstein manifold. This construction is, in a certain sense, a Sasakian counterpart of the Fubini-Study Kähler structure on $CP^k$. 

2
Notation
We adapt the following notation:
\( K, l \) denotes the partial derivative of a function \( K \) with respect to the coordinate \( z^l \).
Complex conjugate of an indexed quantity, e.g. \( a^i_j \), is usually denoted by a bar over it, i.e. \( \overline{a^{i_j}} \). Our notation is: \( \overline{a^{i_j}} = \overline{a}^{\overline{i_j}} \).
Symmetrized tensor products of 1-forms \( \eta \) and \( \lambda \) is denoted by \( \eta \lambda = \frac{1}{2}(\eta \otimes \lambda + \lambda \otimes \eta) \).

The main result
The purpose of this letter is to prove the following theorem, which locally characterizes all Sasakian and Sasakian-Einstein manifolds.

Theorem.
Let \( U \) be an open set of \( C^k \times \mathbb{R} \) and let \((z^1, z^2, ..., z^k, x)\) be Cartesian coordinates in \( U \). Consider:
- a vector field \( \xi = \partial_x \)
- a real-valued function \( K \) on \( U \) such that \( \xi(K) = 0 \)
- an 1-form \( \eta = dx + i \sum_{m=1}^{k} (K_m dz^m) - i \sum_{\overline{m}=1}^{k} (\overline{K}_{\overline{m}} d\overline{z}^{\overline{m}}) \)
- a bilinear form \( g = \eta^2 + 2 \sum_{m,k=1}^{k} K_{m\overline{k}} dz^m d\overline{z}^{\overline{k}} \)
- a tensor field \( \phi = -i \sum_{m=1}^{k} [ (\partial_m - iK_m \partial_x) \otimes dz^m] + i \sum_{\overline{m}=1}^{k} (\partial_{\overline{m}} + i\overline{K}_{\overline{m}} \partial_x) \otimes d\overline{z}^{\overline{m}} \).

I) If the function \( K \) is chosen in such a way that the bilinear form \( g \) has positive definite signature then \( U \) equipped with the structure \((\phi, \xi, \eta, g)\) is a Sasakian manifold. Moreover, every Sasakian manifold can locally be generated by such a function \( K \).

II) The above Sasakian structure satisfies Einstein equation \( \text{Ric}(g) = \lambda g \) if and only if \( \lambda = 2k \) and the function \( K \) satisfies \(-[\log \det(K_{ij})]_{m\overline{n}} = 2(k+1)K_{m\overline{n}} \).

2 Almost contact versus Sasakian manifolds

Definition 2
Consider \((2k+1)\)-dimensional manifold \( S \) equipped with a structure consisting of a \((1,1)\) tensor field \( \phi \), a vector field \( \xi \) and a field of an 1-form \( \eta \). Assume, in addition, that for every vector field \( X \) on \( S \) \( (\xi, \eta, \phi) \) satisfy the following algebraic conditions:

1. \( \phi^2 X = -X + \eta(X)\xi \),
2. \( \eta(\xi) = 1 \).

Then \( S \) is called almost contact manifold. If, in addition, an almost contact manifold \((S, (\xi, \eta, \phi))\) is equipped with Riemannian metric \( g \) such that for every vector fields \( X \) and \( Y \) on \( S \) we have

3. \( g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \),
4. \( g(\xi, X) = \eta(X) \),

then the almost contact manifold is called almost contact metric manifold.

Note that every Sasakian manifold is an almost contact metric manifold.
Let $T^C S$ be the complexification of the tangent bundle of an almost contact manifold $S$. The almost contact structure $(\xi, \eta, \phi)$ on $S$ defines the decomposition

$$T^C S = C \otimes \xi \oplus N \oplus \bar{N},$$

where $C \otimes \xi$, $N$ and $\bar{N}$ are eigenspaces of $\phi$ with eigenvalues 0, $-i$ and $i$, respectively.

We say that a vector subbundle $Z$ of $T^C S$ is involutive if and only if $[\Gamma(Z), \Gamma(Z)] \subset \Gamma(Z)$, where $\Gamma(Z)$ denotes the set of all sections of $Z$.

**Lemma**

For an almost contact structure the condition $N \phi + d\eta \otimes \xi = 0$ is satisfied if and only if the bundle $N$ is involutive, $[\Gamma(N), \Gamma(N)] \subset \Gamma(N)$, and $[\xi, \Gamma(N)] \subset \Gamma(N)$.

**Proof.**

Let $X, Y \in \Gamma(N)$. Making use of the eigenvalue property of $\phi$ and of property (1) of Definition 2, we get the following expressions for the Nijenhuis tensor of $\phi$:

$$N_\phi(X, Y) = -2[X, Y] + 2i\phi([X, Y]) + \eta([X, Y])\xi,$$

$$N_\phi(X, \bar{Y}) = \eta([X, \bar{Y}])\xi,$$

$$N_\phi(X, \xi) = -[X, \xi] + i\phi([X, \xi]) + \eta([X, \xi])\xi.$$

Observe that the last component of the above formulae is the action of $-d\eta \otimes \xi$ on $(X, Y)$, $(X, \bar{Y})$ and $(X, \xi)$, respectively. Therefore $N_\phi + d\eta \otimes \xi = 0$ if and only if

$$\phi([X, Y]) = -i[X, Y]$$

and

$$\phi([X, \xi]) = -i[X, \xi].$$

This finishes the proof.

**Corollary**

For an almost contact structure satisfying $N_\phi + d\eta \otimes \xi = 0$ (in particular for a Sasaki structure) the bundle $C \otimes \xi \oplus N$ is involutive.

### 3 Sasakian geometry in a null frame

Let $(S, (\xi, \eta, \phi, g))$ be a Sasakian manifold of dimension $2k + 1$. The algebraic conditions (1)-(4) of Definition 1 imply an existence of a local basis $(\xi, m_i, \bar{m}_i)$, $i, \bar{i} = 1, 2, ..., k$, of complex-valued vector fields on $S$, with a cobasis $(\eta, \mu_i, \bar{\mu}_i)$, such that

$$g = \eta^2 + 2 \sum_{i=1, \bar{i}=1}^{k} \mu_i \bar{\mu}_{\bar{i}},$$

$$\phi = i \sum_{j=1}^{k} (\bar{m}_j \otimes \bar{\mu}_j) - i \sum_{j=1}^{k} (m_j \otimes \mu_j).$$

Since $(S, (\xi, \eta, \phi, g))$ is Sasakian then its bundle $C \otimes \xi \oplus N$ is involutive. This is equivalent to the condition that the forms $\mu^1, \mu^2, ..., \mu^k$ generate a closed differential ideal i.e.

$$d\mu^i \wedge \mu^1 \wedge \mu^2 \wedge ... \wedge \mu^k = 0 \quad \forall i = 1, 2, ..., k.$$
Condition (6) of Definition 1 of a Sasakian manifold in this basis reads

\[ d\eta = -2i \sum_{l=1}^{k} \mu^l \wedge \bar{\mu}^l. \quad (4) \]

Thus, the fact that the manifold is Sasakian necessarily implies the existence of a local basis \((\xi, m_i, \bar{m}_i)\) with a dual basis \((\eta, \mu^i, \bar{\mu}^i)\) such that \((\xi, m_i, \bar{m}_i)\) and \((\eta, \mu^i, \bar{\mu}^i)\) form a mutually dual basis for \(T^C S\) and \(T^* CS\), respectively, satisfying (3)-(4), then the structure \((\xi, \eta, \phi, g)\) defined by \(\xi, \eta\) and \(\phi\) of (3) is a Sasakian manifold. This fact can be seen by noting that condition (3) is equivalent to the existence of complex-valued functions \(a_{ijk}\), \(b_{ij\bar{k}}\) and \(c_{ij}\) such that

\[ d\mu^i = \sum_{j,n=1}^{k} a_{ijn} \mu^j \wedge \mu^k + \sum_{j,n=1}^{k} b_{ijn} \bar{\mu}^j \wedge \mu^k + \sum_{j}^{k} c_{ij} \wedge \eta. \quad (5) \]

The dual conditions to conditions (4)-(5) imply that \(N\) is involutive and that \([\xi, \Gamma(N)] \subset \Gamma(N)\).

These, when compared with Lemma of Corollary 1, imply condition (7), which is the only condition from Definition 1 which, a'priori, was not assumed for \((\xi, \eta, \phi, g)\). This proves the following Proposition.

**Proposition 1**

(I) (Local version)

Let \((\xi, \eta, \phi, g)\) be a Sasakian structure on a manifold \(S\) of dimension \(2k + 1\). Then there exists a local basis \((\xi, m_i, \bar{m}_i), i, \bar{i} = 1, 2, ... k\) of \(T^C S\) with dual basis \((\eta, \mu^i, \bar{\mu}^i)\) such that

\[ g = \eta^2 + 2 \sum_{l=1}^{k} \mu^l \bar{\mu}^l \]

\[ \phi = i \sum_{j=1}^{k} (\bar{m}_j \otimes \bar{\mu}^j) - i \sum_{j=1}^{k} (m_j \otimes \mu^j), \]

\[ d\mu^i \wedge \mu^1 \wedge \mu^2 \wedge ... \wedge \mu^k = 0 \quad \forall i = 1, 2, ..., k, \]

\[ d\eta = -2i \sum_{l=1}^{k} \mu^l \wedge \bar{\mu}^l. \]

(II) (Global version)

Every almost contact metric structure which satisfies condition (6) of Definition 1 is Sasakian if and only if its canonical decomposition \(T^C S = C \otimes \xi \oplus N \oplus \bar{N}\), consists of involutive \(C \otimes \xi \oplus N\) part.

We close this section with a quick application of part (I) of Proposition 1. It is well known that a vector field \(\xi\) on a Sasakian manifold \((S, (\xi, \eta, \phi, g))\) is a Killing vector field. This in particular means that the Lie derivatives \(L_\xi\) of \(g\) and \(\eta\) vanish. The second of these facts is an immediate consequence of (4). To calculate \(L_\xi g\) one uses (4) and (5). After some work one shows that vanishing of \(L_\xi g\) is equivalent to \(c_{ij} + c_{ij} = 0\) where \(c_{ij}\) are functions appearing in (5). On the other hand, these equations are automatically implied by application of \(d\) on both sides of equation (4).
4 Analogue of the Kähler potential

We pass to a construction of local coordinates on a Sasakian manifold \((S, (\xi, \eta, \phi, g))\). We assume that all the fields defining the Sasakian structure are smooth on \(S\).

In a considered region of \(S\), we chose a local frame \((\xi, m, \bar{m})\) of Proposition 1. Now, the fact that \(\xi\) is a Killing vector field on \(S\) together with the complex version of the Fröbenius theorem, assures that condition (3) is equivalent to an existence of complex-valued functions \(f^i\) and \(z^i\), \(i, j = 1, 2, \ldots k\) such that

\[
\mu^i = f^i_j dz^j. \tag{6}
\]

Since the forms \((\mu^i, \bar{\mu}^i)\) form a part of the basis on the considered region of \(S\) then we also have

\[
dz^1 \wedge dz^2 \wedge \ldots dz^k \wedge d\bar{z}^1 \wedge d\bar{z}^2 \wedge \ldots \wedge d\bar{z}^k \neq 0. \tag{7}
\]

For the basis \((\xi, \partial z^1, \ldots, \partial z^k, \partial \bar{z}^1, \ldots, \partial \bar{z}^k)\) and its dual \((\eta, dz^1, \ldots, dz^k, d\bar{z}^1, \ldots, d\bar{z}^k)\) the Maurer-Cartan relations for \(d(dz^i) = 0 = d(d\bar{z}^i)\) readilly show that \([\xi, \partial/\partial z^i] = 0 = [\xi, \partial/\partial \bar{z}^i]\). Therefore, there exists a real coordinate \(x\) complementary to \(z^1, \ldots, z^k, \bar{z}^1, \ldots, \bar{z}^k\) such that

\[
\xi = \partial_x \tag{8}
\]

and the form \(\eta\) reads

\[
\eta = dx + p^i dz^i + \bar{p}^i d\bar{z}^i. \tag{9}
\]

Comparing this with the fact that \(\xi\) preserves \(\eta\) leads to the conclusion that the functions \(p_i\) are independent of coordinate \(x\), \(\partial p_i/\partial x = 0\).

Condition (4) is now equivalent to the following two conditions for the differentials of functions \(p_i\):

\[
p_{i,j} - p_{j,i} = 0 \tag{10}
\]

and

\[
p_{j,i} - \bar{p}_{i,j} = 2i \sum_{l=l,l=1}^k f^l_j f^{i}_{\bar{l}}, \tag{11}
\]

In a simply connected region of \(S\) equation (10) guarantees an existence of a complex-valued function \(V\) such that

\[
p_i = \partial V/\partial z^i. \tag{12}
\]

Since \(p_i\) is independent of \(x\) it is enough to consider functions \(V\) such that \(\partial V/\partial x = 0\). Inserting so determined \(p_i\) to equation (11) we show that now (11) is equivalent to

\[
K_{j\bar{i}} = \sum_{l=l,l=1}^k f^l_j f^{\bar{l}}_{\bar{i}}, \tag{13}
\]

where we have introduce \(\text{Im} V = K\) and \(\text{Re} V = L\). Finally we note that now

\[
\eta = d(x + L) + i \sum_{j=1}^k K_{j} dz^j - i \sum_{j=1}^k K_{j} d\bar{z}^j,
\]

so redefining the \(x\) coordinate by \(x \rightarrow x + L\) we simplify \(\eta\) to the form \(\eta = dx + i \sum_{j=1}^k K_{j} dz^j - i \sum_{j=1}^k K_{j} d\bar{z}^j\). Using (13) we can eliminate functions \(f^i_j\) from formulae defining our Sasakian structure. Indeed,

\[
g = \eta^2 + 2 \sum_{l=l,l=1}^k \mu^l \bar{\mu}^l =
\]
In this way we obtain the following theorem.

**Theorem 1.**

Let $\mathcal{U}$ be an open set of $\mathbb{C}^k \times \mathbb{R}$ and let $(z^1, z^2, \ldots, z^k, x)$ be Cartesian coordinates in $\mathcal{U}$. Consider:

(i) a vector field $\xi = \partial_x$

(ii) a real-valued function $K$ on $\mathcal{U}$ such that $\xi(K) = 0$

(iii) an 1-form $\eta = dx + i \sum_{m=1}^{k} (K_m dz^m) - i \sum_{\overline{m}=1}^{k} (K_{\overline{m}} d\overline{z}^{\overline{m}})$

(iv) a bilinear form $g = \eta^2 + 2 \sum_{m,k=1}^{k} K_{m,k} dz^m d\overline{z}^{\overline{k}}$

(v) a tensor field

$$\phi = -i \sum_{m=1}^{k} [(\partial_m - iK_m \partial_x) \otimes dz^m] + i \sum_{m=1}^{k} (\partial_{\overline{m}} + iK_{\overline{m}} \partial_x) \otimes d\overline{z}^{\overline{m}}.$$

If the function $K$ is chosen in such a way that the bilinear form $g$ has positive definite signature then $\mathcal{U}$ equipped with the structure $(\phi, \xi, \eta, g)$ is a Sasakian manifold. Moreover, every Sasakian manifold can locally be generated by such a function $K$.

The function $K$ appearing in the above theorem is a Sasakian analogue of the Kähler potential generating Kähler geometries. We call it a Sasakian potential.

We close this section with a remark that several Sasakian potentials may generate the same Sasakian structure. This is evident if one notes that the following transformations

$$K \rightarrow K + f(z^j) + \overline{f}^{\overline{j}} \quad \text{and} \quad x \rightarrow x + i\overline{f}^{\overline{j}} - if(z^j), \quad (14)$$

with $f$ being a holomorphic function of $z^j$s, do not change the Sasakian structure of Theorem 1. Thus, transformations (14) are the gauge transformations for the Sasakian potential.

### 5 Locally Sasakian-Einstein structures

In this section we calculate the Ricci tensor for the Sasakian metric $g$ generated in a region $\mathcal{U}$ by the Sasakian potential $K$ of Theorem 1. We also derive the equation which the Sasakian potential has to obey for Sasakian metric to satisfy Einstein equations $Ric(g) = \lambda g$. In this section we use the Einstein summation convention.

Let $\mathcal{U}$ be a simply connected region of $\mathbb{C}^k \times \mathbb{R}$ as in Theorem 1. Consider a Sasakian structure defined in this Theorem by the Sasakian potential $K$. In the holonomic cobasis

$$(dy^\nu) = (dx, dz^i, d\overline{z}^{\overline{i}})$$
the covariant components of the Sasakian metric read

\[ g_{\mu\nu} = \begin{pmatrix} 1 & iK_j & -iK_j & 0 \\ iK_i & -K_iK_j & K_{ij} + K_iK_j & K_{ij} \\ -iK_i & K_{ij} + K_iK_j & -K_jK_i & 0 \\ 0 & K_{ij} & K_{ji} & 1 + 2K_iK_jK_{ij} \end{pmatrix}. \] (15)

The contravariant components of the metric read

\[ g^{\mu\nu} = \begin{pmatrix} 1 + 2K_iK_jK_{ij} & iK_jK_{ij} & -iK_jK_{ij} & K_{ij} \\ iK_jK_{ij} & 0 & K_{ij} & 0 \\ -iK_jK_{ij} & K_{ij} & 0 & K_{ij} \\ K_{ij} & K_{ij} & K_{ij} & 1 \end{pmatrix}, \] (16)

where

\[ K_{ij} = \delta^i_j \quad K_{ji} = \delta^i_j \quad K^{ij} = K^{ji}. \] (17)

The connection 1-forms \( \Gamma_{\mu\nu} = \frac{1}{2}(g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\nu\rho,\mu})dy^\rho \) read

\[ \Gamma_{xx} = 0 \quad \Gamma_{xi} = iK_{ij}dz^j \quad \Gamma_{xj} = -K_iK_{ij}d\bar{z}^j \quad \Gamma_{ix} = iK_{ij}d\bar{z}^j \quad \Gamma_{ij} = \Gamma_{ji} \] (18)

It is convenient to introduce the following functions:

\[ C_{jm}^i = K_{ij} \quad B_{jm}^i = C_{jm}^i + \delta^i_mK_j + \delta^i_jK_m \quad A_{jm} = C_{jm}^iK_i + 2K_jK_{jm} - K_{jm} \]

Then the connection 1-forms \( \Gamma_{\nu}^{\mu} \) read

\[ \Gamma^x_{x} = -dK \quad \Gamma^x_{i} = -K_i dx - iA_{jm}dz^m \quad \Gamma^i_{x} = -idz^i \quad \Gamma^x_{j} = \Gamma^x_{j} \quad \Gamma^j_{x} = \Gamma^j_{x} \]

\[ \Gamma^i_{j} = -i\delta^i_j dx - \delta^i_j K_j dz^i + B_{ji} dz^i \quad \Gamma^i_{j} = -K_j dz^i \quad \Gamma^j_{i} = \Gamma^j_{i} \]

The curvature 2-forms \( \Omega_{\nu}^{\mu} = \frac{1}{2}R_{\nu\rho\sigma}dy^\rho \wedge dy^\sigma = d\Gamma_{\nu}^{\mu} + \Gamma_{\nu}^{\rho} \wedge \Gamma_{\rho}^{\mu} \) read

\[ \Omega^x_{x} = iK_{i} dx \wedge dz^i \quad \Omega^x_{j} = -K_j dz^i \] \( \wedge dz^i \)

\[ \Omega^i_{x} = -\delta^i_j dx \wedge dz^j + i\delta^i_j K_j dz^j \wedge dz^i \] \( + iK_j dz^i \wedge dz^i \)

\[ \Omega^i_{j} = -i\delta^i_j K_j dx \wedge dz^i + (K_{ji} + K_{ij} K_{li}) dx \wedge dz^i + iA_{jm} dz^i \wedge dz^m \]

\[ \Omega^i_{j} = i\delta^i_j K_j dx \wedge dz^i + (K_{ji} + K_{ij} K_{li}) \delta^i_{n} dz^n \wedge dz^i \] \( + \delta^i_{n} K_j dz^i \wedge dz^i \)

\[ \Omega^j_{i} = \Omega^i_{j} \quad \Omega^i_{j} = \Omega^j_{i} \quad \Omega^i_{j} = \Omega^i_{j} \]

The Ricci tensor \( R_{\nu\sigma} = R_{\nu\rho\sigma} \) components read

\[ R_{xx} = 2k \quad R_{xj} = 2ikK_j \quad R_{ij} = -2kK_jK_i \quad R_{ij} = 2kK_jK_i - 2K_{ij} - C_{mi}^{m} \]

\[ R_{xj} = R_{xj} \quad R_{ij} = R_{ij}. \]
Now, the Einstein equations $\text{Ric}(g) = \lambda g$, which are nontrivial only for the components $R_{xx}$ and $R_{ij}$ become

$$\lambda = 2k - (\kappa^{\bar{m}l} K_{,\bar{m}l j}), j = 2(k + 1)K_{,ij}.$$ 

Since the matrix $(\kappa^{\bar{m}l})$ is the inverse of $(K_{,ij})$ then the left hand side of the second equations above is

$$-(\kappa^{\bar{m}l} K_{,\bar{m}l i}), j = -\log(\det(K_{,mn}))_{,ij},$$

see e.g. [7].

Thus we arrive to the following theorem.

**Theorem 2**

Any Sasakian manifold of dimension $(2k + 1)$ can be locally represented by the Sasakian potential $K$ of Theorem 1. In the region where the potential is well defined the manifold satisfies Einstein equations $\text{Ric}(g) = \lambda g$ if and only if the cosmological constant

$$\lambda = 2k$$

and the potential satisfies

$$-\log(\det(K_{,mn}))_{,ij} = 2(k + 1)K_{,ij}.$$ (19)

Surprisingly equation (19) is the same as the Einstein condition $\text{Ric}(h) = 2k + 1)h$ for the Kähler metric $h = 2K_{,ij} d\bar{z}^i dz^j$ in dimension $2k$. Thus we have the following Corollary.

**Corollary**

Every Sasakian-Einstein manifold in dimension $(2k + 1)$ is locally in one to one correspondence with a Kähler-Einstein manifold in dimension $2k$ whose cosmological constant $\lambda = 2k + 1$. The correspondence is obtained by identifying the Kähler potential for the Kähler-Einstein manifold with the Sasaki potential for the Sasaki-Einstein manifold.

**Examples**

1). Sasakian potential for the sphere $S^{2k+1}$.

Consider a function

$$K = \frac{1}{2} \log(z^1 \bar{z}^1 + ... + z^{k+1} \bar{z}^{k+1})$$

defined on $\mathbb{C}^{k+1} - \{0\}$. Let

$$N = i(K_{,j} dz^j - K_{,j} d\bar{z}^j),$$

$$H = 2K_{,ij} dz^i d\bar{z}^j$$

and

$$G = N^2 + H.$$

The tensor fields $N$ and $G$ restrict to a sphere

$$S^{2k+1} = \{(z^1, ..., z^{k+1}) \in \mathbb{C}^{k+1} - \{0\} \mid z^1 \bar{z}^1 + ... + z^{k+1} \bar{z}^{k+1} = 1\}.$$ 

Denote these restrictions by $\eta$ and $g$, respectively. Then the 1-form $\eta$ and the Riemannian metric $g$ define a Sasakian-Einstein structure on $S^{2k+1}$. This structure coincides with the one defined in Example 1 of Section 1.

To see this, recall the Hopf fibration $U(1) \to S^{2k+1} \to \mathbb{C}P^k$ with the action of $e^{i\phi} \in U(1)$ on $(z^1, ..., z^{k+1}) \in S^{2k+1}$ defined by $e^{i\phi}(z^1, ..., z^{k+1}) = (e^{i\phi} z^1, ..., e^{i\phi} z^{k+1})$. The canonical projection is given by $S^{2k+1} \ni (z^1, ..., z^{k+1}) \to \text{dir}(z^1, ..., z^{k+1}) \in \mathbb{C}P^k$. The sphere is covered by $k + 1$ charts

$$U_j = V_j \times U(1),$$

where $V_j$ is the annular subset $\{|z^1| < 1\} \times U(1)$.
where
\[ V_j = \{ \text{dir}(z^1, ..., z^{k+1}) \mid (z^1, ..., z^{k+1}) \in S^{2k+1} \text{ and } z^j \neq 0 \}. \]

The local coordinates on each \( U_j \) are
\[ (\xi^i = \frac{z^i}{z^j}, \phi_j = \frac{i}{2} \log \frac{z^j}{z^j}), \quad i = 1, ..., k + 1, \ i \neq j. \]

Then on each chart \( U_j \) the form \( \eta_j = \eta|_{U_j} \) reads
\[ \eta_j = d\phi_j + \frac{i}{2} \sum_{i=1, i \neq j}^{k+1} (\xi^i d\xi^j - \xi^j d\xi^i). \]

The metric \( g \) restricted to \( U_j \) is
\[ g_j = (\eta_j)^2 + \frac{(1 + \sum_{i=1, i \neq j}^{k+1} |\xi^i|^2)(\sum_{i=1, i \neq j}^{k+1} |d\xi^i|^2) - |\sum_{i=1, i \neq j}^{k+1} (\xi^i d\xi^j)|^2}{1 + \sum_{i=1, i \neq j}^{k+1} |\xi^i|^2}. \]

Now, observe that on each \( U_j \) the structure \((g_j, \eta_j)\) may be obtained by means of Theorems 1 and 2 choosing a Sasakian potential
\[ K^j = \frac{1}{2} \log(1 + \sum_{i=1, i \neq j}^{k+1} |\xi^i|^2) \]

on the corresponding \( V_j \). It is easy to check that \( K^j \) satisfies equation \( [13] \) on \( V_j \). Thus, Theorem 2 assures that the Sasakian structure generated by \((g_j, \eta_j)\) is Sasakian-Einstein. Easy, but lengthy, calculation shows that the Weyl tensor of \( g_j \) vanishes identically on \( U_j \). This proves that \((U_j, g_j)\) is locally isometric to a standard Riemannian structure on \( S^{2k+1} \). Since \((g_j, \eta_j)\) originate from the global structure \((g, \eta)\) then this global Sasakian structure must coincide with the standard Sasakian structure of Example 1. Note also that \( h_j = g_j - (\eta_j)^2 \) projects to \( V_j \) and patched together defines the Fubini-Study metric on \( CP^k \). In this sense the standard Sasakian structure on \( S^{2k+1} \) described in Example 1 is the analogue of the Fubini-Study Kähler structure on \( CP^k \).

2.) Sasakian-Einstein structure on \( C^q \times C^n \times R \).

Consider a function
\[ K = \frac{1}{q + n + 1} \left( \sum_{i=1}^{q} \log(1 + |v^i|^2) \right) + \frac{n + 1}{2(q + n + 1)} \log(1 + \sum_{i=1}^{n} |w^i|^2) \]
defined on \( C^q \times C^n \), with coordinates \((z^\mu) = (v^i, w^i)\). It is easy to check that
\[ \log \det(K_{\mu\nu})|_{\mu\rho} = -2(q + n + 1)K_{\rho\sigma}. \]

Thus, via Theorems 1 and 2, such \( K \) generates a Sasakian-Einstein structure on \( C^q \times C^n \times R \).

3) Locally Sasakian-Einstein structures in dimension 5.

If \( k = 2 \) then, modulo the gauge \([14]\), equation \([13]\) may be integrated to the form
\[ K_{i1i2}K_{j2i1} - K_{i1j2}K_{j2i1} = e^{-6K}. \]

This is a well known equation describing the gravitational instantons in four dimensions \([2, 3, 4, 11]\). Examples of the Kähler-Einstein metrics generated by its solutions can be found e.g. in \([2, 3, 11]\). Via Theorems 1 and 2, each of these Kähler-Einstein structures defines a nontrivial Sasakian-Einstein manifold in dimension 5.
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