A proposal for realizing a 3-qubit controlled-phase gate with superconducting qubit systems coupled to a cavity

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We present a way to realize a 3-qubit quantum controlled-phase gate with superconducting qubit systems coupled to a cavity. This proposal does not require adjustment of the qubit level spacings or identical qubit-cavity coupling constants. Moreover, since only a resonant interaction is applied, the gate can be performed fast, within ~ 10 nanosecond. This proposal is quite general, which can be applied to various types of superconducting qubits, atoms trapped in a cavity, or quantum dots coupled to a resonator.

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I. INTRODUCTION

Superconducting qubit systems, including Cooper pair boxes, Josephson junctions, and superconducting quantum interference devices (SQUIDs), are among the most promising candidates for scalable quantum computing [1]. In the past decade, many theoretical methods for realizing a single-qubit gate and a two-qubit gate with superconducting qubit systems have been proposed [2-7]. Moreover, a two-qubit gate was experimentally realized using superconducting qubit systems coupled through capacitors [8-10], mutual inducance [11], or cavities [12,13]. However, the experimental realization of a 3-qubit quantum gate with superconducting devices has not been reported so far. Experimentally, a 3-qubit controlled-phase gate has been realized in NMR quantum system and a 3-qubit controlled NOT gate has been demonstrated with trapped ions [14,15].

A 3-qubit controlled-phase (CP) gate is of significance, which has applications in quantum information processing such as quantum network circuit construction, error correction and quantum algorithms [16-19]. A 3-qubit CP gate can in principle be constructed using basic two-qubit gates and single-qubit gates only (i.e., the conventional gate-decomposing protocol). However, when using the conventional gate-decomposing protocol, the gate operation is complicated because at least 25 steps of operations will be required, assuming that the realization of a single-qubit gate or a two-qubit controlled phase gate requires a one-step operation only [16] (see Fig. 1).

In this work, we focus on the physical realization of a 3-qubit CP gate with superconducting qubit systems based on cavity QED technique. Recently, cavity QED with superconducting qubits has attracted considerable attention [20]. A cavity or resonator acts as a quantum bus which can mediate long-distance, fast interaction between distant superconducting qubit systems [4,12,21-26]. Based on cavity QED technique, 2-qubit quantum gates, 2-qubit quantum algorithms, 3-qubit quantum entanglement, and quantum state transfer have been experimentally demonstrated with superconducting qubits coupled to a cavity or resonator [12,13,27].

In the following, we propose a way for realizing a 3-qubit CP gate with superconducting qubit systems coupled to a cavity or resonator (hereafter, we use the term cavity and resonator interchangeably). As shown below, this proposal has the following features: (a) there is no need for adjusting the qubit level spacings during the gate operation, thus decoherence caused by tuning the qubit level spacings is avoided; (b) no photon detection is needed during the entire gate operation, thus the effect of the photon-detection imperfection on the gate performance is avoided; (c) identical qubit-cavity coupling constants are not required, thus this proposal is tolerable to the inevitable nonuniformity in device parameters; (d) since only a resonant interaction is applied, the gate can be performed fast, within ~ 10 nanosecond.; and (e) this proposal only requires five steps of operations. This proposal is quite general, which can be applied to various types of superconducting qubits, atoms trapped in a cavity, or quantum dots coupled to a resonator.

This paper is organized as follows. In Sec. II, we briefly review the basic theory of resonant qubit-cavity and qubit-pulse interactions. In Sec. III, we show how to realize a 3-qubit CP gate with superconducting qubit systems coupled to a cavity or resonator. In Sec. IV, we briefly discuss possible experimental implementation with superconducting qubit systems coupled to a one-dimensional transmission line resonator. A concluding summary is presented in Sec. V.

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FIG. 1: (a) Schematic circuit of a 3-qubit CP gate with two control qubits (linked to the filled circles) and a target qubit (at the bottom). Here, Z represents a controlled-phase flip on the target qubit. The circuit in (a) is equivalent to the circuit in (b), which consists of six 2-qubit controlled NOT (CNOT) gates and seven single-qubit phase shift gates (i.e., the elements each containing a $T$ or $T^+$). (c) Schematic circuit for a 2-qubit CNOT gate with a control qubit (linked to the filled circle) and a target qubit (at the bottom). The symbol $\oplus$ represents a controlled NOT on the target qubit. The circuit in (c) is equivalent to the circuit in (d), where the part enclosed in a dash-line box represents a 2-qubit CP gate and each element containing $H$ stands for a single-qubit Hadamard gate. The combination of circuits in (a), (b), (c), and (d) indicates that constructing a 3-qubit CP gate requires six 2-qubit CP gates, twelve single-qubit Hadamard gates, and seven single-qubit phase shift gates. Namely, a total of 25 basic gates are needed to build up a 3-qubit CP gate, by using the conventional gate-decomposing protocol (for details, see reference [16]). Therefore, at least 25 steps of operations will be required, assuming that the realization of a single-qubit gate or a two-qubit controlled phase gate requires a one-step operation only.

| Charge-qubit system | Phase-qubit system | Flux-qubit system |
|---------------------|-------------------|------------------|
| $|3\rangle$         | $|3\rangle$       | $|3\rangle$     |
| $|2\rangle$         | $|2\rangle$       | $|2\rangle$     |
| $|1\rangle$         | $|1\rangle$       | $|1\rangle$     |
| $|0\rangle$         | $|0\rangle$       | $|0\rangle$     |
| atom, quantum dot  |                   |                  |

FIG. 2: (Color online) Illustration of four-level qubit systems. The energy eigenvalues for the four levels $|0\rangle$, $|1\rangle$, $|2\rangle$, and $|3\rangle$ are denoted by $E_0$, $E_1$, $E_2$, and $E_3$, respectively. In (a), the level spacings satisfy $E_2 - E_1 > E_1 - E_0$, $E_3 - E_2$; and $E_3 - E_2 < E_1 - E_0$. In (b), the level spacings satisfy $E_1 - E_0 > E_2 - E_1 > E_3 - E_2$. In (c), the level spacings meet $E_2 - E_1 > E_1 - E_0$, $E_3 - E_2$; and $E_3 - E_2 > E_1 - E_0$. For the availability of the four levels in superconducting systems, also see discussion in [33].
II. BASIC THEORY

The superconducting qubit systems considered in this work have four levels shown in Fig. 2. Note that the four-level structure in Fig. 2(a) applies to superconducting charge-qubit systems [28], the one in Fig. 2(b) applies to phase-qubit systems [29,30], and the one in Fig. 2(c) applies to flux-qubit systems [28,31]. In addition, the four-level structure in Fig. 2(b) is also available in natural atoms and quantum dots.

A. System-cavity resonant interaction. Consider a system with four levels as shown in Fig. 2. Suppose that the transition between the two levels [2] and [3] is resonant with the cavity mode while the transition between any other two levels is highly detuned with (decoupled from) the cavity mode. In the interaction picture, the interaction Hamiltonian of the system and the cavity mode is given by (after the rotating-wave approximation)

\[ H = \hbar g (a^{+} \sigma_{Z3} + \text{H.c.}), \]  

where \( a^{+} \) and \( a \) are the photon creation and annihilation operators of the cavity mode, \( g \) is the coupling constant between the cavity mode and the [2] ↔ [3] transition of the system, and \( \sigma_{Z3} = [2] \langle 3 \rangle \).

It is straightforward to show that the initial states \([3] |0\rangle_{c} \) and \([2] |1\rangle_{c} \) of the system and the cavity mode evolve as follows

\[ |3\rangle |0\rangle_{c} \rightarrow -i \sin (gt) |2\rangle |1\rangle_{c} + \cos (gt) |3\rangle |0\rangle_{c}, \]
\[ |2\rangle |1\rangle_{c} \rightarrow \cos (gt) |2\rangle |1\rangle_{c} - i \sin (gt) |3\rangle |0\rangle_{c}. \]

On the other hand, the state \([0] |0\rangle_{c} \) remains unchanged under the Hamiltonian (1).

The coupling strength \( g \) may vary with different systems due to different level structures, non-uniform device parameters, and/or non-exact placement of systems in the cavity. Therefore, in the gate operation below, we will replace \( g \) by \( g_{1} \), \( g_{2} \) and \( g_{3} \) for systems 1, 2, and 3, respectively.

B. System-pulse resonant interaction. Consider a system with four levels as depicted in Fig. 2. Assume that the pulse is resonant with the transition between the two levels \([i] \leftrightarrow [j] \) of the system. Here, the level \([i] \) is a lower-energy level. The interaction Hamiltonian in the interaction picture is given by

\[ H_{I} = \hbar \left( \Omega_{ij} e^{i\phi} |i\rangle \langle j| + \text{H.c.} \right), \]

where \( \Omega_{ij} \) and \( \phi \) are the Rabi frequency and the initial phase of the pulse, respectively. Based on the Hamiltonian (3), it is easy to show that a pulse of duration \( t \) results in the following rotation

\[ |i\rangle \rightarrow \cos \Omega_{ij} t |i\rangle - ie^{-i\phi} \sin \Omega_{ij} t |j\rangle, \]
\[ |j\rangle \rightarrow \cos \Omega_{ij} t |j\rangle - ie^{i\phi} \sin \Omega_{ij} t |i\rangle. \]

Note that the state transformation (4) can be completed within a very short time, by increasing the pulse Rabi frequency \( \Omega_{ij} \) (i.e., by increasing the intensity of the pulse).

III. REALIZING A THREE-QUBIT CP GATE WITH SUPERCONDUCTING QUBIT SYSTEMS COUPLED TO A CAVITY

For three qubits, there are a total of eight computational basis states, denoted by \([000], [001], ..., [111]\), respectively. A 3-qubit CP gate results in the transformation \([111] \rightarrow -[111] \) but nothing to the remaining seven computational basis states. Namely, when the two control qubits (the first two qubits) are in the state \([1]\), a phase flip (i.e., \([1] \rightarrow -[1]\)) happens to the state \([1]\) of the target qubit (the last qubit). To realize this gate, let us consider three superconducting qubit systems 1, 2, and 3. Each system has four levels as depicted in Fig. 2. The two lowest levels \([0]\) and \([1]\) of each qubit system are used to represent the two logic states of a qubit while the two higher-energy lowest levels \([0]\) and \([1]\) of each qubit system are employed for the coherent manipulation of the quantum states. For simplicity, we consider the four-level structure shown in Fig. 2(b) or Fig. 3 in our following discussion. For the purpose of the gate, we denote the first (second) lowest level of system 1 or 2 as the level \([0]\) \((|0\rangle\)) while the first (second) level of system 3 as the level \([1]\) \((|0\rangle\)) (see Fig 3). We should mention that the gate operation procedure presented below is applicable to the gate configuration using the four-level structure as depicted in Fig. 2(a) or Fig. 2(c).

The implementation of our gate below requires the resonant interaction between the cavity mode and the \([2] \leftrightarrow [3]\) transition of each qubit system. This condition can be achieved by setting the level spacing between the two levels \([2]\) and \([3]\) to be the same for each qubit system. Note that for superconducting qubit systems, by designing the qubit systems appropriately, one can easily make the level spacing between certain two levels (the two levels \([2]\) and \([3]\) here) to be identical [32], though it is difficult to have the level spacing between any two levels to be identical for each qubit system due to nonuniformity of the system parameters. In addition, as shown below, our gate realization requires
that the cavity mode is highly detuned (decoupled) from the transition between any other two levels of each system. This condition can be achieved via adjustment of the qubit level spacings before the gate operation. Note that for superconducting qubit systems, the level spacings can be readily adjusted by changing the external parameters (e.g., the external magnetic flux and gate voltage for superconducting charge-qubit systems, the current bias or flux bias in the case of superconducting phase-qubit systems and flux-qubit systems, see, e.g., [28,29,33]).

We now discuss how to implement our gate. The cavity mode is assumed to be initially in the vacuum state $|0\rangle_c$.

The procedure for realizing the three-qubit CP gate is as follows:

Step (i): The operation of this step is: (a) Apply a pulse (with a frequency $\omega = \omega_{31}$, a phase $\phi = -\frac{\pi}{2}$, and a duration $t_{1,a} = \frac{\pi}{2\omega_{31}}$) to system 1 [Fig. 3(a)], to transform the state $|1\rangle_1$ to $|3\rangle_1$ as described by Eq. (4); (b) Wait for a time $t_{1,b} = \frac{\pi}{2\omega_{31}}$ to have the cavity mode resonantly interacting with the $|2\rangle \leftrightarrow |3\rangle$ transition of system 1 [Fig. 3(a')] such that the state $|3\rangle_1 |0\rangle_c$ is transformed to $-i|2\rangle_1 |1\rangle_c$ as described by Eq. (2) while the state $|0\rangle_1 |0\rangle_c$ remains unchanged; then (c) Apply a pulse (with a frequency $\omega = \omega_{20}$, a phase $\phi = -\frac{\pi}{2}$, and a duration $t_{1,c} = \frac{\pi}{2\omega_{20}}$) to system 1 [Fig. 3(a'')], to transform the state $|0\rangle_1$ to $|2\rangle_1$ and the state $|2\rangle_1$ to $-|0\rangle_1$.

The pulse sequence for this step of operation is shown in Fig. 4(a).

It can be seen that after the operation of this step, the following transformation is obtained:

$$
\begin{align*}
|1\rangle_1 |0\rangle_c & \rightarrow (a) |3\rangle_1 |0\rangle_c \\
|0\rangle_1 |0\rangle_c & \rightarrow (b) -i|2\rangle_1 |1\rangle_c \\
|0\rangle_1 |0\rangle_c & \rightarrow (c) i|0\rangle_1 |1\rangle_c.
\end{align*}
$$

Step (ii): The operation for this step is: (a) Apply a pulse (with a frequency $\omega = \omega_{20}$, a phase $\phi = -\frac{\pi}{2}$, and a duration $t_{2,a} = \frac{\pi}{\omega_{20}}$) to system 2 [Fig. 3(b)], to transform the state $|0\rangle_2$ to $|2\rangle_2$; (b) Wait for a time $t_{2,b} = \frac{\pi}{2\omega_{20}}$ to have the cavity mode resonantly interacting with the $|2\rangle \leftrightarrow |3\rangle$ transition of system 2 [Fig. 3(b')], such that the state $|2\rangle_2 |1\rangle_c$ is transformed to $-i|3\rangle_2 |0\rangle_c$ while the states $|1\rangle_2 |0\rangle_c$, $|2\rangle_2 |0\rangle_c$, and $|1\rangle_2 |1\rangle_c$ remain unchanged; then (c) Apply a pulse (with a frequency $\omega = \omega_{30}$, a phase $\phi = \pi$, and a duration $t_{2,c} = \frac{\pi}{\omega_{30}}$) to system 2 [Fig. 3(b'')], to transform the state $|3\rangle_2$ to $i|0\rangle_2$. The pulse sequence for this step of operation is shown in Fig. 4(b).
The operations for these two steps are as follows: 

\[ (v) \]

The purpose of the last two steps presented below is to obtain the reverse transformations described by Eqs. (5) and (6). The operations for these two steps are as follows:

\[ (v) \]

One can see that after the operation of this step, the following transformation is achieved:

\[ (6) \]

Step (iii): The operation for this step is: (a) Apply a pulse (with a frequency \( \omega = \omega_{21} \), a phase \( \phi = -\pi \), and a duration \( t_{3,a} = \frac{\pi}{\Omega_{21}} \)) to system 3 [Fig. 3(c)], to transform the state \( |1\rangle_3 \) to \( |2\rangle_3 \); (b) Wait for a time \( t_{3,b} = \frac{\pi}{\Omega_2} \) to have the cavity mode resonantly interacting with the \( |2\rangle \leftrightarrow |3\rangle \) transition of system 3 [Fig. 3(c')], such that the state \( |2\rangle_3 |1\rangle_c \) changes to \( -|2\rangle_3 |1\rangle_c \), while the states \( |2\rangle_3 |0\rangle_c, |0\rangle_3 |0\rangle_c \), and \( |0\rangle_3 |1\rangle_c \) remain unchanged; then (c) Apply a pulse (with a frequency \( \omega = \omega_{21} \), a phase \( \phi = \frac{\pi}{2} \), and a duration \( t_{3,c} = \frac{\pi}{\Omega_{21}} \)) to system 3 [Fig. 3(c'')], to transform the state \( |2\rangle_3 \) back to \( |1\rangle_3 \). The pulse sequence for this step of operation is shown in Fig. 4(c).

After the operation of this step, we obtain the following transformation:

\[ (7) \]
Step (iv): (a) Apply a pulse (with a frequency \(\omega = \omega_{30}\), a phase \(\phi = \pi\), and a duration \(t_{4,a} = \frac{\pi}{2\Omega}\)) to system 2 [Fig. 3(b’)], to transform the state \(|0\rangle_2\) to \(|i\rangle_3\); (b) Wait for a time \(t_{4,b} = \frac{\pi}{2\Omega}\) to have the cavity mode resonant interact with the \(|2\rangle \leftrightarrow |3\rangle\) transition of system 2 [Fig. 3(b’)], such that the state \(|3\rangle_2|0\rangle_c\) is transformed to \(-i|2\rangle_2|1\rangle_c\), while the states \(|1\rangle_2|0\rangle_c\), \(|2\rangle_2|0\rangle_c\) and \(|1\rangle_2|1\rangle_c\) remain unchanged; then (c) Apply a pulse (with a frequency \(\omega = \omega_{30}\), a phase \(\phi = \frac{\pi}{2}\), and a duration \(t_{4,c} = \frac{\pi}{2\Omega}\)) to system 2 [Fig. 3(b’)], to transform the state \(|2\rangle_2|0\rangle_c\) to \(|0\rangle_2\). The pulse sequence for this step of operation is shown in Fig. 4(d).

One can see that after the operation of this step, the following transformation is achieved:

\[
\begin{align*}
|0\rangle_2|0\rangle_c & \quad \rightarrow \quad |i\rangle_3|0\rangle_c \\
|1\rangle_2|1\rangle_c & \quad \rightarrow \quad -|2\rangle_2|1\rangle_c \\
|2\rangle_2|0\rangle_c & \quad \rightarrow \quad \frac{1}{\sqrt{2}}|2\rangle_2|0\rangle_c + \frac{1}{\sqrt{2}}|2\rangle_2|1\rangle_c \\
|1\rangle_2|0\rangle_c & \quad \rightarrow \quad -i|2\rangle_2|0\rangle_c \\
|0\rangle_1|0\rangle_c & \quad \rightarrow \quad |0\rangle_1|0\rangle_c \\
|0\rangle_1|1\rangle_c & \quad \rightarrow \quad \frac{1}{\sqrt{2}}|0\rangle_1|0\rangle_c + \frac{1}{\sqrt{2}}|0\rangle_1|1\rangle_c \\
|0\rangle_1|1\rangle_c & \quad \rightarrow \quad |0\rangle_1|0\rangle_c \\
|0\rangle_2|0\rangle_c & \quad \rightarrow \quad |0\rangle_2|0\rangle_c \\
\end{align*}
\]

Step (v): (a) Apply a pulse (with a frequency \(\omega = \omega_{31}\), a phase \(\phi = \frac{\pi}{2}\), and a duration \(t_{5,a} = \frac{\pi}{2\Omega}\)) to system 1 [Fig. 3(a’)], to transform \(|0\rangle_1\) to \(- |2\rangle_1\) and \(|2\rangle_1\) to \(|0\rangle_1\); (b) Wait for a time \(t_{5,b} = \frac{\pi}{2\Omega}\) to have the cavity mode resonantly interacting with the \(|2\rangle \leftrightarrow |3\rangle\) transition of system 1 [Fig. 3(a’)], such that the state \(|2\rangle_1|1\rangle_c\) is transformed to \(-i|3\rangle_1|0\rangle_c\) as described by Eq. (2) while the state \(|0\rangle_1|0\rangle_c\) remains unchanged; then (c) Apply a pulse (with a frequency \(\omega = \omega_{31}\), a phase \(\phi = \frac{\pi}{2}\), and a duration \(t_{5,c} = \frac{\pi}{2\Omega}\)) to system 1 [Fig. 3(a)], to transform the state \(|3\rangle_1\) to \(- |1\rangle_1\). The pulse sequence for this step of operation is shown in Fig. 4(e).

It can be seen that after the operation of this step, the following transformation is obtained:

\[
\begin{align*}
|0\rangle_1|1\rangle_c & \quad \rightarrow \quad -|2\rangle_1|1\rangle_c \\
|2\rangle_1|0\rangle_c & \quad \rightarrow \quad |0\rangle_1|0\rangle_c \\
|1\rangle_2|0\rangle_c & \quad \rightarrow \quad |0\rangle_1|0\rangle_c \\
|0\rangle_1|1\rangle_c & \quad \rightarrow \quad |0\rangle_1|0\rangle_c \\
|0\rangle_2|0\rangle_c & \quad \rightarrow \quad |0\rangle_2|0\rangle_c \\
\end{align*}
\]

Based on the results (5-9) obtained above, we can find the following state evolution of the whole system after each step of the above operations:

\[
\begin{align*}
|000\rangle_0|0\rangle_c & \quad \rightarrow \quad |200\rangle_0|0\rangle_c \\
|001\rangle_0|0\rangle_c & \quad \rightarrow \quad |201\rangle_0|0\rangle_c \\
|010\rangle_0|0\rangle_c & \quad \rightarrow \quad |210\rangle_0|0\rangle_c \\
|011\rangle_0|0\rangle_c & \quad \rightarrow \quad |211\rangle_0|0\rangle_c \\
|000\rangle_0|0\rangle_c & \quad \rightarrow \quad |200\rangle_0|0\rangle_c \\
|001\rangle_0|0\rangle_c & \quad \rightarrow \quad |201\rangle_0|0\rangle_c \\
|010\rangle_0|0\rangle_c & \quad \rightarrow \quad |210\rangle_0|0\rangle_c \\
|011\rangle_0|0\rangle_c & \quad \rightarrow \quad |211\rangle_0|0\rangle_c \\
\end{align*}
\]

Here and below, \(|i\rangle_1|j\rangle_2|k\rangle_3\) is abbreviation of the state \(|i\rangle_1|j\rangle_2\) of systems \(1, 2, 3\) with \(i, j, k \in \{0, 1, 2, 3, 4\}\). This result (10) demonstrates that a phase flip happens to the state \(|111\rangle\) after the operations above.

Eq. (10) shows that during each step of operations, the two states \(|0\rangle_0\) and \(|1\rangle_0\) of each of two irrelevant qubit systems remain unchanged. For instance, one can see from Eq. (10) that during the operation of step (i) on qubit 1, the two states \(|0\rangle_0\) and \(|1\rangle_0\) of qubit system 2 or qubit system 3 remain unchanged. The reason for this is that the cavity mode was assumed to be resonant with the \(|2\rangle \leftrightarrow |3\rangle\) transition of each qubit but highly detuned (decoupled) from the transition between any other two levels. Thus, when a qubit is in the state \(|0\rangle_0\) or \(|1\rangle_0\), the qubit is decoupled from the cavity mode and thus the states \(|0\rangle_0\) and \(|1\rangle_0\) of this qubit are not affected by the cavity mode though a photon is populated in the cavity.

On the other hand, based on the results (5-9) given above, it is easy to see that for the other four initial states \(|000\rangle_0\), \(|001\rangle_0\), \(|010\rangle_0\), and \(|011\rangle_0\) of the qubit systems, the states of the whole system after each step of the above operations are listed below:
FIG. 5: (color online) (a) Setup for three superconducting qubit systems (red dots) and a (grey) standing-wave one-dimensional coplanar waveguide resonator. \( \lambda \) is the wavelength of the resonator mode, and \( L \) is the length of the resonator. The two (blue) curved lines represent the standing wave magnetic field in the \( z \)-direction. Each qubit system (a red dot) could be a superconducting charge-qubit system as depicted in (b), flux-biased phase-qubit system in (c), and flux-qubit system in (d). \( E_J \) is the Josephson junction energy \((0.6 < \alpha < 0.8)\) and \( V_g \) is the gate voltage. The qubit systems are placed at locations where the magnetic fields are the same to achieve an identical coupling strength for each qubit system. The superconducting loop of each qubit system, which is a large square for (b) and (d) while a large circle for (c), is located in the plane of the resonator between the two lateral ground planes (i.e., the \( x-y \) plane). For each qubit system, the external magnetic flux \( \Phi_c \) through the superconducting loop for each qubit system is created by the magnetic field threading the superconducting loop. A classical magnetic pulse is applied to each qubit system through an \( ac \) flux \( \Phi_e \) threading the qubit superconducting loop, which is created by an \( ac \) current loop (i.e., the red dashed-line loop) placed on the qubit loop. The pulse frequency and intensity can be adjusted by changing the frequency and intensity of the \( ac \) loop current.

which shows that the cavity mode remains in the vacuum state \( |0 \rangle_c \) during the entire operation, the four states \( |000 \rangle, |001 \rangle, |010 \rangle \) and \( |011 \rangle \) of the qubit systems undergo time evolution (induced by the applied pulses only) but return to their original states after the last step of operation. Note that the state \( |2 \rangle \) of qubit system 1, 2, or 3 does not change when no photon is populated in the cavity, due to the energy conservation. From Eqs. (10) and (11), it can be concluded that after the above process, a 3-qubit CP gate was implemented with three systems (i.e., the controlled systems 1 and 2, as well as the target system 3) while the cavity mode returns to its original vacuum state.

Several points need to be addressed as follows:

(i) From the description of operations above, it can be seen that due to the use of the four levels, the states \( |0 \rangle \) and \( |1 \rangle \) of qubit systems not involved in the operation are not affected by the cavity mode, no matter whether or not a photon is populated in the cavity. We note that when systems with two or three energy levels are used, the decoupling of other systems from the cavity mode can not be made during the operation performed on any one of the systems.

(ii) For certain kinds of superconducting qubit systems (e.g., flux qubit systems), the decay of the level \( |1 \rangle \) of each system is avoided when the transition between the two lowest levels is forbidden due to the optical selection rules \([31]\), or it can be suppressed by increasing the potential barrier between the two lowest levels \(|0 \rangle \) and \(|1 \rangle \) \([4,29,30,34]\)

(iii) For simplicity, we considered the identical Rabi frequency \( \Omega \) for each pulse during the gate operation above. Note that this requirement is unnecessary. The Rabi frequency for each pulse can be different and thus the pulse durations for each step of operations above can be adjusted accordingly.

(iv) To have the effect of the system-cavity resonant interaction on the state transformation induced by the pulse negligible, the pulse Rabi frequency \( \Omega \) needs to be set such that \( \Omega \gg g_1, g_2, g_3 \).

IV. POSSIBLE EXPERIMENTAL IMPLEMENTATION

As shown above, it can be found that the total operation time is given by

\[
\tau = \sum_{i=1}^{5} (t_{i,a} + t_{i,b} + t_{i,c}) = \frac{\pi}{g_1} + \frac{\pi}{g_2} + \frac{\pi}{g_3} + \frac{5\pi}{\Omega}.
\]

(12)
The $\tau$ should be much shorter than the energy relaxation time $\gamma_{3p}^{-1}$ and dephasing time $\gamma_{3p}^{-1}$ of the level $|3\rangle$ (note that the levels $|1\rangle$ and $|2\rangle$ have a longer decoherence time than the level $|3\rangle$), such that decoherence, caused due to spontaneous decay and dephasing process of the qubit systems, is negligible during the operation. And, the $\tau$ needs to be much shorter than the lifetime of the cavity photon, which is given by $\kappa^{-1} = Q/2\pi\nu_c$, such that the decay of the cavity photon can be neglected during the operation. Here, $Q$ is the (loaded) quality factor of the cavity and $\nu_c$ is the cavity field frequency. To obtain these requirements, one can design the qubit systems to have sufficiently long energy relaxation time and dephasing time, such that $\tau \ll \gamma_{3p}^{-1}, \gamma_{3p}^{-1}$; and choose a high-$Q$ cavity such that $\tau \ll \kappa^{-1}$.

For the sake of definitiveness, let us consider the experimental possibility of realizing the 3-qubit CP gate, using three identical superconducting qubit systems coupled to a resonator [Fig. 5(a)]. Each qubit system could be a superconducting charge-qubit system [Fig. 5(b)], flux-qubit system [Fig. 5(c)], or flux-biased phase-qubit system [Fig. 5(d)]. As a rough estimate, assume $g_1 \approx g_2 \approx g_3 = g$, and $g/2\pi \approx 220$ MHz, which could be reached for a superconducting qubit system coupled to a one-dimensional standing-wave CPW (coplanar waveguide) transmission resonator [27]. With the choice of $\Omega \sim 10g$, one has $\tau \sim 8$ ns, much shorter than $\min\{\gamma_{3p}^{-1}, \gamma_{3p}^{-1}\} \sim 1$ $\mu$s [29,35]. In addition, consider a resonator with frequency $\nu_c \sim 5$ GHz (e.g., Ref. [13]) and $Q \sim 5 \times 10^4$, we have $\kappa^{-1} \sim 1.6$ $\mu$s, which is much longer than the operation time $\tau$ here. Note that superconducting coplanar waveguide resonators with a (loaded) quality factor $Q \sim 10^6$ have been experimentally demonstrated [36,37]. We remark that further investigation is needed for each particular experimental setup. However, this requires a rather lengthy and complex analysis, which is beyond the scope of this theoretical work.

V. CONCLUSION

Before conclusion, we should mention two previous proposals on multiqubit-controlled phase gates [38,39], which are relevant to this work. However, we note that the present proposal is quite different from the ones in [38,39] as follows:

(i) The previous proposal in [38] requires using five-level qubits while our present proposal employs four-level qubits only. Since one more level is used, the former is more challenging in experiments when compared with the latter. Furthermore, the proposal in [38] is based on the adiabatic passage technique while ours is based on the resonant interaction only. As is well known, the adiabatic passage requires slowly changing the Rabi frequencies of the pulses applied. Hence, the gate speed for the proposal in [38] is far slower than that using our present proposal.

(ii) The previous proposal in [39] requires adjusting the qubit level spacings during the gate operation while as shown above our present proposal does not need adjustment of the qubit level spacings during the gate operation. Thus, decoherence caused due to adjustment of the qubit level spacings is avoided by our present proposal. Moreover, adjusting the qubit level spacings during the gate operation is undesirable in experiments, which however does not apply to our present proposal. Hence, the present proposal is much improved when compared with the proposal in [39].

In summary, we have presented a way to realize a 3-qubit controlled-phase gate with four-level superconducting qubit systems in cavity QED. As shown above, this proposal has the following advantages: (i) No adjustment of the level spacings of qubit systems during the entire operation is needed, thus decoherence caused due to the adjustment of the level spacings is avoided in this proposal; (ii) The coupling constants of each system with the cavity are not required to be identical, which makes neither identical qubits nor exact placement of qubits to be required by this proposal; (iii) No photon detection is needed during the entire gate operation, and thus the effect of the photon-detection imperfection on the gate performance is avoided; (iv) Because only resonant interactions are used, the gate can be performed fast within $\sim 10$ nanosecond; and (v) The gate realization requires five steps of operations only. Finally, it is noted that this proposal is quite general, which can be applied to other physical systems, such as atoms trapped in a cavity or quantum dots coupled to a resonator.

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