SUBALGEBRAS OF ÉTALE ALGEBRAS IN BRAIDED FUSION CATEGORIES

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Abstract. In [7, Rem. 3.4] the authors asked the question if any étale subalgebra of an étale algebra in a braided fusion category is also étale. We give a positive answer to this question if the braided fusion category $\mathcal{C}$ is pseudo-unitary and non-degenerate. In the case of a pseudo-unitary fusion category we also give a new description of the lattice correspondence from [6, Theorem 4.10]. This new description enables us to describe the two binary operations on the lattice of fusion subcategories.

1. Introduction

In [7, Rem. 3.4] the authors asked the question if any étale subalgebra of an étale algebra in a braided fusion category $\mathcal{C}$ is also étale. It seems that an answer to this question does not appear in the literature at this moment. We give a positive answer to this question if the braided fusion category $\mathcal{C}$ is pseudo-unitary and non-degenerate.

For any fusion category $\mathcal{C}$ it was shown in [6, Lemma 3.5] that $A = R(1)$ is a connected étale algebra in $\mathcal{Z}(\mathcal{C})$ where $R$ is a right adjoint of the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. Moreover, in [6, Theorem 4.10] it was shown that the lattice of fusion subcategories of $\mathcal{C}$ is in a reverse one to one correspondence with the lattice of étale subalgebras of $A$. $A$ is sometimes called the adjoint algebra of $\mathcal{C}$ in [18].

Recall that given a connected étale subalgebra $L$ of $A$ in [6] it was associated a fusion subcategory $\beta(L)$ of $\mathcal{C}$. Under the tensor equivalence $R : \mathcal{C} \to \mathcal{Z}(\mathcal{C})_A$ the fusion subcategory $\beta(L)$ consists of those $A$-modules in $\mathcal{Z}(\mathcal{C})$ that are dyslectic as right $L$-modules.

For a pivotal tensor categories, Shimizu has developed in [18] a character theory for fusion categories, similar to the classical theory of representations of finite groups and semisimple Hopf algebras. Moreover

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he showed that the adjoint algebra $A$ associated to a fusion category has a natural action on any object of $\mathcal{C}$.

The main goal of this paper is to show that in the case of a pseudo-unitary fusion category $\mathcal{C}$ all unitary subalgebras of $A$ in $\mathcal{Z}(\mathcal{C})$ are connected étale. We also obtain a new characterization of the fusion subcategories from [6] associated to étale subalgebras of $A$.

In Section 3, in the case $\mathcal{C}$ is a pseudo-unitary fusion category, we associate to any unitary subalgebra $L$ of $A$ in $\mathcal{Z}(\mathcal{C})$ a fusion subcategory $\mathcal{S}_L$ of $\mathcal{C}$, consisting of those simple objects of $\mathcal{C}$ whose characters (as defined in [18]) have trivial restriction to $L$. We also denote by $\mathcal{C}_L^{\text{triv}}$ the full abelian subcategory of $\mathcal{C}$ consisting of those objects that receive a trivial action from the subalgebra $L$, see Definition 3.4.

It is shown that for any pivotal fusion category $\mathcal{C}$ one has a chain of inclusions of abelian full subcategories:

$$\beta(L) \subseteq \mathcal{C}_L^{\text{triv}} \subseteq \mathcal{S}_L. \tag{1.1}$$

On the other hand, we show in Theorem 3.2 that if $\mathcal{C}$ is a pseudo-unitary fusion category then $\text{FPdim}(\mathcal{S}_L) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(L)}$. Since by [6, Theorem 4.10] we also have $\text{FPdim}(\beta(L)) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(L)}$, this shows that the above chain inclusions of fusion subcategories are in fact equalities. This chain of equalities implies that $\mathcal{C}_L^{\text{triv}}$ is also a fusion subcategory and that any unitary connected subalgebra of $A$ is also an étale subalgebra of $A$. These facts can be stated as follows:

**Theorem 1.1.** Any fusion subcategory of a pseudo-unitary fusion category $\mathcal{C}$ is of the form $\mathcal{C}_L^{\text{triv}}$ for some unitary subalgebra $L$ of the algebra $A := R(1)$. Moreover, every such unitary subalgebra $L$ of $A$ is étale.

From Theorem 1.1, using some results concerning étale algebras developed in [6] we deduce that in a non-degenerate braided pseudo-unitary fusion category, any subalgebra of an étale algebra is also étale. More precisely we prove the following:

**Theorem 1.2.** Let $\mathcal{C}$ be a pseudo-unitary non-degenerate braided fusion category and $A$ an étale algebra of $\mathcal{B}$. Then any unitary subalgebra $L$ of $A$ is also étale.

For two subalgebras of the adjoint algebra $A$ we denote by $LM$ the image of $L \otimes M$ under the multiplication $m : A \otimes A \to A$. Since $A$ is commutative it follows that $LM = ML$ and moreover $ML$ is a subalgebra of $A$. Concerning the two binary operations on the lattice of fusion subcategories we prove the following.
Theorem 1.3. Let $C$ be a pseudo-unitary fusion category and $L, M$ two unitary subalgebras of the adjoint algebra $A$ of $C$. With the above notations one has

$$S_L \cap S_M = S_{LM}, \quad S_L \vee S_M = S_{L \oplus M}.$$

Shortly, the organization of the paper is as follows. In Section 2 we recall the basics on fusion categories and also the character theory developed by Shimizu in [18] for pivotal fusion categories. In Section 3 we construct the fusion subcategory $S_L$ mentioned above in the pivotal case. We also define for any fusion category the full abelian subcategory $C_{triv}$ and prove the inclusion $S_L \supseteq C_{triv}$ from Equation (1.1). In Section 4 we prove the inclusion $C_{triv} \supseteq \beta(L)$ from Equation (1.1). Also in this section we give the proofs of the above three theorems. In Subsection 4.6 we also consider as an example the case of fusion categories coming from representation categories semisimple Hopf algebras.

We work over an algebraically closed field $k$ of characteristic zero. All the Hopf algebras and fusion categories notations follow [9].

2. Preliminaries

In this section we review the basic properties of fusion categories that are needed through the paper. For the definition and standard theory of monoidal categories, we refer the reader to [14] and [13]. Given a monoidal category we define by $\mathcal{O}(C)$ the class of all its objects. Recall that a left dual object of $X \in \mathcal{O}(C)$ is a triple $(X^*, ev_X, coev_X)$ consisting of an object $X^* \in \mathcal{O}(C)$ and morphisms $ev_X : X^* \otimes X \to 1$ and $coev_X : 1 \to X \otimes X^*$ such that the following equalities are satisfied:

$$\begin{align*}
(ev_X \otimes id_{X^*}) \circ (id_X \otimes d) &= id_{X^*}, \\
(id_X \otimes ev_X) \circ (coev_X \otimes id_X) &= id_X.
\end{align*}$$

Similarly, one can define a right dual of $X$ (which in fact is a left dual of $X$ in $C^{rev}$). A monoidal category $C$ is said to be rigid if every object of $C$ has both a left and a right dual object.

Recall that a finite tensor category [11] over a field $k$ is a rigid monoidal category $C$ such that $C$ is a finite abelian category, the tensor product $\otimes : C \times C \to C$ is $k$-linear in each variable, and $\text{End}_C(1) \simeq k$ as algebras. A fusion category [10] is a semisimple finite tensor category.

For a monoidal category $C$, the left monoidal center (or the Drinfeld center) of $C$ is a category $\mathcal{Z}(C)$ defined as follows: An object of $\mathcal{Z}(C)$ is a pair $(V, \sigma_V)$ consisting of an object $V \in C$ and a natural isomorphism

$$\sigma_{V,X} : V \otimes X \to X \otimes V$$

for all $X \in \mathcal{O}(C)$, satisfying a part of the hexagon axiom. A morphism $f : (V, \sigma_V) \to (W, \sigma_W)$ in $\mathcal{Z}(C)$ is a morphism in $C$ such that $(id_X \otimes f) \circ$
\[ \sigma_{V, X} = \sigma_{W, X} \circ (f \otimes \text{id}_X) \] for all \( X \in \mathcal{C} \). The composition of morphisms is defined in an obvious way. The category \( \mathcal{Z}(\mathcal{C}) \) is in fact a braided monoidal category, see, e.g., [13] Chapt. XIII.3 for details.

Let \( \mathcal{C} \) be any finite tensor category and \( A \) be an algebra in \( \mathcal{C} \). Recall that the algebra \( A \) is called connected if \( \text{Hom}_\mathcal{C}(1, A) = k \). A subalgebra \( B \xrightarrow{\iota} A \) is called unitary if \( u_B \circ \iota = u_A \) where \( u_B \) and \( u_A \) are the units of \( B \) and \( A \). Recall [6] that an algebra \( A \) of a braided fusion category \( \mathcal{B} \) is called étale if it is separable and connected. An algebra \( A \) is called separable if its multiplication \( A \otimes A \xrightarrow{\rho} A \) has a section as \( A \)-bimodules in \( \mathcal{C} \). This is equivalent by [6, Theorem 3.2] to the fact that the category of right (or left) \( A \)-modules \( \mathcal{B}_A \) is a semisimple category.

Let \( A \) be an algebra in a braided fusion category \( \mathcal{B} \) and \( M \) be a right \( A \)-module in \( \mathcal{B} \). Then \( M \) is called dyslectic (local) if
\[
\rho_M \circ \pi_M \circ \rho_M = \pi_M
\]
where \( \rho_M : M \otimes A \to M \) is the module structure of \( M \).

Dyslectic modules form a full subcategory of \( \mathcal{B}_A \) which is usually denoted by \( \mathcal{B}_A^0 \). This subcategory is closed under \( \otimes \) and the braiding in \( \mathcal{B} \) induces a natural braided structure in \( \mathcal{B}_A^0 \), see [15, Section 2]. Thus, \( \mathcal{B}_A^0 \) is a braided fusion category.

2.1. The central Hopf comonad of a finite tensor category. Let \( \mathcal{C} \) be a fusion category. The forgetful functor \( F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C} \) admits a right adjoint functor \( R : \mathcal{C} \to \mathcal{Z}(\mathcal{C}) \) and \( Z := FR : \mathcal{C} \to \mathcal{C} \) is a Hopf comonad called the central Hopf comonad associated to \( \mathcal{C} \). Moreover, one has that
\[
Z(V) \simeq \int_{X \in \mathcal{C}} X \otimes V \otimes X^*. 
\]

see [18] Section 2.6]. We denote by \( \pi_{V; X} : Z(V) \to X \otimes V \otimes X^* \) the universal dinatural transformation associated to the end \( Z(V) \). The Hopf comonad structure of \( Z \) can be described in terms of the dinatural transformation \( \pi \). The comultiplication \( \delta : Z \to Z^2 \) is the unique natural transformation such that
\[
(\text{id}_X \otimes \pi_{V; Y} \otimes \text{id}_{X^*}) \circ \pi_{Z(V); X} \circ \delta_V = \pi_{V; X \otimes Y}
\]
The counit of \( \epsilon : Z \to \text{id}_\mathcal{C} \) is given by \( \epsilon_V := \pi_{V; 1} \). There is a lax monoidal structure defined on \( Z \) by
\[
Z_2(M, N) : Z(M) \otimes Z(N) \to Z(M \otimes N)
\]
as the unique map making the following diagram commutative:
\( Z(M) \otimes Z(N) \xrightarrow{\pi_{M, X} \otimes \pi_{N, X}} Z(M \otimes N) \)

\[
(X \otimes M \otimes X^*) \otimes (X \otimes N \otimes X^*) \xrightarrow{id_{X \otimes M} \otimes ev_X \otimes id_{N \otimes X^*}} X \otimes M \otimes N \otimes X^*
\]

which can be written as

\( \pi_{M\otimes N, X} \circ Z_2(M, N) = (id_{X \otimes M} \otimes ev_X \otimes id_{N \otimes X^*}) \circ (\pi_M \otimes \pi_N \otimes X) \)

2.2. On the adjoint algebra of a finite tensor category. It is known that \( A := R(1) \) has the structure of central commutative algebra in \( Z(\mathcal{C}) \). The half braiding of \( A \), denoted by \( c_{A,X} \), is defined by

\[
A \otimes X \xrightarrow{\delta_1 \otimes id_X} Z(A) \otimes X \xrightarrow{\pi_{Z(1),X} \otimes id_X} X \otimes A \otimes X^* \otimes X \xrightarrow{id_X \otimes id_A \otimes ev_X} X \otimes A.
\]

The multiplication \( m : A \otimes A \to A \) and the unit \( u_A : 1 \to A \) of the adjoint algebra \( A \) are uniquely determined by the universal property of the end \( A = Z(1) \) as:

\[
\pi_{1,X} \circ u_A = coev_X,
\]

\[
\pi_{1,X} \circ m = (id_X \otimes ev_X \otimes id_{X^*}) \circ (\pi_{1,X} \otimes \pi_{1,X}).
\]

Moreover \( \epsilon_1 : A \to 1 \) is a morphism of algebras, see \cite{18}.

Any object \( X \in \mathcal{C} \) is canonically an \( A \)-module in \( \mathcal{C} \), via the morphism:

\[
\alpha_X : A \otimes X \xrightarrow{\pi_{1,X} \otimes id_X} X \otimes X^* \otimes X \xrightarrow{id_X \otimes ev_X} X.
\]

We denote by \( c_{R(M),X} : R(M) \otimes X \to X \otimes R(M) \) the half braiding of \( R(M) \in Z(\mathcal{C}) \). Recall that with the above braiding \( A = R(1) \) becomes a commutative algebra in the center \( Z(\mathcal{C}) \). Moreover, by \cite{18} Equation (3.12) one has that

\[
\alpha_X = (A \otimes X \xrightarrow{c_{A,X}} X \otimes A \xrightarrow{id_X \otimes \epsilon_1} X).
\]

The vector space \( CE(\mathcal{C}) := \text{Hom}_C(1, A) \) is called the set of central elements. For \( a, b \in CE(\mathcal{C}) \) we set \( ab := m \circ (a \otimes b) \). Then the set \( CE(\mathcal{C}) \) is a monoid with respect to this operation. Note that for the algebra unit \( u_A : 1 \to A \) of the algebra \( A \) to the unit of the monoid \( CE(\mathcal{C}) \) is \( F(u_A) \).

2.3. Internal characters for pivotal structures. Recall that a pivotal structure \( j \) on a tensor category \( \mathcal{C} \) is a tensor isomorphism \( j : id_{\mathcal{C}} \to (-)^{**} \). Using the pivotal structure one can construct a right evaluation as follows:

\[
\tilde{ev}_X : X \otimes X^* \xrightarrow{j \otimes id} X^{**} \otimes X^* \xrightarrow{ev_{**}} 1.
\]
Then the right partial pivotal trace of $f : A \otimes X \to B \otimes X$ is defined as follows:

\[
(2.9) \quad \text{tr}_{A,B}^{X} : A = A \otimes 1 \xrightarrow{id_A \otimes \text{coev}_X} A \otimes X \otimes X^* \xrightarrow{f \otimes id} B \otimes X \otimes X^* \xrightarrow{id_B \otimes \tilde{\text{ev}}_X} B.
\]

The usual right pivotal trace of an endomorphism $f : X \to X$ is obtained as a particular case for $A = B = 1$. In particular, the right pivotal dimension $\dim_r(X)$ of $X$ is defined as the right trace of the identity of $X$. A pivotal structure $a$ on a tensor category $\mathcal{C}$ is called spherical if $\dim(V) = \dim(V^*)$ for any object $V \in \mathcal{O}(\mathcal{C})$. Given an object $X \in \mathcal{O}(\mathcal{C})$ the internal character $\text{ch}(X)$ is defined as the morphism $\text{ch}(X) := \text{tr}_{a,1}^{X}(\alpha_X) : A \to 1$.

Therefore using Equation (2.9) one can write that

\[
(2.10) \quad \text{ch}(X) = A \otimes 1 \xrightarrow{id_A \otimes \text{coev}_X} A \otimes X \otimes X^* \xrightarrow{\alpha_X \otimes id} X \otimes X^* \xrightarrow{\tilde{\text{ev}}_X} 1.
\]

Then the space $\text{CF}(\mathcal{C}) := \text{Hom}_\mathcal{C}(A,1)$ is called the space of class functions of $\mathcal{C}$. For two class functions $f, g \in \text{CF}(\mathcal{C})$ one can define a multiplication by

\[
f \ast g := f \circ Z(g) \circ \delta_1.
\]

Here $\delta : Z \to Z^2$ is the comultiplication structure of $Z$ defined in the Equation (2.3). By [18, Theorem 3.10] one has that $\text{ch}(X \otimes Y) = \text{ch}(X)\text{ch}(Y)$ for any two objects $X$ and $Y$ of $\mathcal{C}$. For a finite tensor category $\mathcal{C}$ the space of class functions $\text{CF}(\mathcal{C})$ is a finite-dimensional algebra.

2.4. The cointegral and integral of a fusion category. Let $\mathcal{C}$ be a fusion category and $A = Z(1)$ be its adjoint algebra as defined above.

An integral in $\mathcal{C}$ is a morphism $\Lambda : 1 \to A$ in $\mathcal{C}$ such that

\[
m \circ (id_A \otimes \Lambda) = e_1 \otimes \Lambda.
\]

A cointegral in $\mathcal{C}$ is a morphism $\lambda : A \to 1$ such that

\[
Z(\lambda) \circ \delta_1 = u \otimes \lambda
\]

where $u : 1 \to A$ is the unit of the algebra $A$. It is well known that the integral and cointegral of a finite unimodular tensor category are unique up to a scalar, see [18].

2.5. Fourier transform for finite tensor categories. Let $\lambda \in \text{CF}(\mathcal{C})$ be a non-zero integral of $\mathcal{C}$. The Fourier transform for finite tensor categories of $\mathcal{C}$ associated to $\lambda$ is the linear map

\[
(2.11) \quad \mathcal{F}_\lambda : \text{CE}(\mathcal{C}) \to \text{CF}(\mathcal{C}) \quad \text{given by} \quad a \mapsto \lambda \leftarrow S(a)
\]

where $S : \text{CE}(\mathcal{C}) \to \text{CE}(\mathcal{C})$ is the antipodal operator on $\text{CE}(\mathcal{C})$, see [18, Definition 3.6]. The Fourier transform is a bijective $k$-linear map whose inverse is given in [18, Equation (5.17)]. Here, the right action denoted by $\leftarrow$ of $\text{CE}(\mathcal{C})$ on $\text{CF}(\mathcal{C})$ given by $f \leftarrow b = f \circ m \circ (b \otimes id_A)$ for all
There is also a non-degenerate evaluation pairing \( \langle \cdot, \cdot \rangle_A \) given by

\[
\langle \cdot, \cdot \rangle_A : \text{CF}(\mathcal{C}) \otimes \text{CE}(\mathcal{C}) \to k, \quad \langle \chi, z \rangle \mapsto \chi \circ z.
\]

**Fourier transform for fusion categories.** For the rest of this section, suppose that \( \mathcal{C} \) is a pivotal fusion category over an algebraically closed field \( k \). Furthermore, let \( \text{Irr}(\mathcal{C}) := \{V_0, \ldots, V_m\} \) be a complete set of representatives of isomorphism classes of simple objects with \( V_0 = 1 \), the unit object. For \( i \in \{0, \ldots, m\} \), we define \( i^* \in \{0, \ldots, m\} \) by \( V_i^* \cong V_{i^*} \). Then \( i \mapsto i^* \) is an involution on \( \{0, \ldots, m\} \). As an object of \( \mathcal{C} \), the adjoint algebra decomposes as

\[
A \cong \bigoplus_{i=0}^{m} V_i \otimes V_i^*.
\]

Shimizu has defined in [18] the elements

\[
E_i : 1 \xrightarrow{\text{coev}_{V_i}} V_i \otimes V_i^* \hookrightarrow A, \quad \chi_i : A \xrightarrow{\pi_i} V_i \otimes V_i^* \xrightarrow{\text{ev}_{V_i}} 1.
\]

It is easy to see that \( \{E_i\}_{i=0,\ldots,m} \) and \( \{\chi_i\}_{i=0,\ldots,m} \) are bases for \( \text{CE}(\mathcal{C}) \) and \( \text{CF}(\mathcal{C}) \) respectively, such that

\[
\langle \chi_i, E_j \rangle = d_i \delta_{i,j}.
\]

where \( d_i := \text{dim}(V_i) \). Moreover, \( E_i E_j = \delta_{i,j} \) and \( S(E_i) = E_i^* \), where \( S : \text{CE}(\mathcal{C}) \to \text{CE}(\mathcal{C}) \) is the antipodal mentioned above. The elements \( \chi_i \) are called the irreducible characters of the simple objects \( V_i \) and \( E_i \in \text{CE}(\mathcal{C}) \) their corresponding primitive idempotents. Note that \( E_0 \), the idempotent associated to the unit object \( 1 \) of \( \mathcal{C} \), is the idempotent integral \( \Lambda \in \text{CE}(\mathcal{C}) \), see [18, Lemma 6.1]. By [18, Equation (6.8)] one has that the idempotent cointegral of \( \mathcal{C} \) can be written as:

\[
\lambda_\mathcal{C} = \frac{1}{\text{dim}(\mathcal{C})} \left( \sum_{[V_i] \in \text{Irr}(\mathcal{C})} d_i \chi_i \right).
\]

Since \( \Lambda = E_0 \) it follows by Equation (2.15) that \( \langle \lambda_\mathcal{C}, \Lambda \rangle = \frac{1}{\text{dim}(\mathcal{C})} \). It also follows that

\[
\mathcal{F}_\chi^{-1}(\chi_i) = \frac{\text{dim}(\mathcal{C})}{d_i} E_i^*.
\]

This equation was proven in [18, Equation (6.10)] under the hypothesis that the Grothendieck ring of \( \mathcal{C} \) is commutative but the proof works word by word also in the case of an arbitrary pivotal fusion category, with the not necessarily commutative Grothendieck ring.
For a spherical fusion category $\mathcal{C}$ one has $\dim(V_{\ast}) = \dim(V_{i})$ and it follows from \cite[Equation (4.7)]{5} that:

\begin{equation}
(\chi, F_{\chi}^{-1}(\mu))_{\lambda} = \dim(\mathcal{C}) \tau(\chi \mu).
\end{equation}

for any $\chi, \mu \in \text{CF}(\mathcal{C})$.

2.6. On the inclusion of class functions of a fusion subcategory.

Let $\mathcal{D}$ be a fusion subcategory of a given fusion category $\mathcal{C}$. As in \cite[Sect. 4.3]{18}, for any object $V$ of $\mathcal{C}$ the end

\[ \bar{\mathcal{Z}}(V) : \mathcal{D}^{\text{op}} \times \mathcal{D} \to \mathcal{C}, \quad \bar{\mathcal{Z}}(V) := \int_{X \in \mathcal{D}} X \otimes V \otimes X^{*} \]

exists and we denote by $\bar{\pi}_{1, X} : \bar{\mathcal{Z}}(1) \to X \otimes X^{*}$ the universal dinatural maps defining this end. From the universal property of $\bar{\mathcal{Z}}(1)$ there is a unique canonical map $\bar{Z}(V) : \mathcal{D}^{\text{op}} \times \mathcal{D} \to \mathcal{C}$, such that $\bar{\pi}_{V, X} \circ q = \pi_{V, X}$ for any object $X$ of $\mathcal{D}$. In \cite[Appendix]{5} it is shown that $q : \mathcal{Z}(1) \to \bar{\mathcal{Z}}(1)$ induces a map $q_{1}^{*} : \text{CF}(\mathcal{D}) \to \text{CF}(\mathcal{C})$, $\chi \mapsto \chi \circ q_{1}$ that is a monomorphism of $k$-algebras.

2.7. The adjunction isomorphisms for $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. Let $\mathcal{C}$ be any fusion category and $R : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ be a right adjoint of the forgetful functor as in the previous section. It is known that $Z := FR$ is a Hopf comonad in $\mathcal{C}$. One has that the category of $Z$-comodules in $\mathcal{C}$ is equivalent to the center $\mathcal{Z}(\mathcal{C})$, see \cite{8} and \cite{2}. Moreover, in this case, the comodule forgetful functor $C^{Z} \to \mathcal{C}$ identifies to $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. This shows that the adjunction isomorphisms are given by

\begin{equation}
\psi_{V, 1} : \text{Hom}_{\mathcal{C}}(F(V), 1) \to \text{Hom}_{\mathcal{Z}(\mathcal{C})}(V, A), \quad f \mapsto R(f) \circ \eta_{V},
\end{equation}

and its inverse is given by

\begin{equation}
\psi_{1, V}^{-1} : \text{Hom}_{\mathcal{Z}(\mathcal{C})}(V, A) \to \text{Hom}_{\mathcal{C}}(F(V), 1), \quad g \mapsto \epsilon_{1} \circ F(g).
\end{equation}

for any object $V \in \mathcal{Z}(\mathcal{C})$. Here $\eta_{V} : V \to R(F(V))$ is the unit of the adjunction which coincides to the $Z$-comodule structure of $V \in \mathcal{Z}(\mathcal{C})$.

2.8. Frobenius-Perron dimensions and pseudo-unitarity. For a simple object $X$ of a fusion category $\mathcal{C}$ the Frobenius-Perron eigenvalue of the left multiplication by $[X]$ on the Grothendieck ring $K_{0}(\mathcal{C})$ is denoted by $\text{FPdim}(X)$ and it is called the Frobenius-Perron dimension of $X$. Recall that $\text{FPdim}(X)$ is a positive real algebraic number and the Frobenius-Perron dimension extends linearly to an algebra morphism $\text{FPdim} : K_{0}(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{C}$. The Frobenius-Perron dimension $\text{FPdim}(\mathcal{C})$ of $\mathcal{C}$ is defined as

\[ \text{FPdim}(\mathcal{C}) := \sum_{X \in \text{Irr}(\mathcal{C})} \text{FPdim}(X)^{2}. \]
A fusion category $\mathcal{C}$ is called pseudo-unitary if $\text{FPdim}(\mathcal{C}) = \dim(\mathcal{C})$. In this case, $\mathcal{C}$ admits a unique (canonical) spherical structure with respect to which the categorical dimensions of simple objects are all positive, see [10, Proposition 8.23]. With respect to this spherical structure, the categorical dimension of any object coincides with its Frobenius-Perron dimension, i.e. $\text{FPdim}(X) = \dim(X)$ for any object $X \in \mathcal{O}(\mathcal{C})$. Moreover, every full fusion subcategory of $\mathcal{C}$ is pseudo-unitary.

3. ON THE SUBCATEGORY $\mathcal{S}_L$ VIA CHARACTER THEORY

Let as above $\mathcal{C}$ be a pivotal fusion category and $F : \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ be the forgetful functor with right adjoint $R : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$. Let also $A := R(1)$ be its adjoint algebra and $\iota : L \hookrightarrow A$ be a unitary subalgebra of $A$. We also denote by $L$ and $A$ the images of $L$ and $A$ under the forgetful functor $F$. Since $A$ is connected it follows that $L$ is also connected. Note also that $L$ is a commutative algebra in $\mathcal{Z}(\mathcal{C})$ as a subalgebra of $A$. Since $\mathcal{Z}(\mathcal{C})$ is a fusion category there is also a projection $A \twoheadrightarrow L$ in $\mathcal{Z}(\mathcal{C})$ of $A$ onto the object $L$ such that $\pi \circ \iota = \text{id}_L$. Since $\mathcal{C}$ is a fusion category, the left adjoint of $F$ is also isomorphic to $R$ (see [17]), and therefore $\text{Hom}_{\mathcal{Z}(\mathcal{C})}(1, A) = \text{Hom}_{\mathcal{C}}(1, 1) \cong \mathbb{K}$. Thus the unit $u_L$ of the subalgebra $L$ can be written as $\pi \circ u_A$, where $A \twoheadrightarrow L$ is the projection in $\mathcal{Z}(\mathcal{C})$ of $A$ into $L$.

Define also a character space of $L$ by $\text{CF}(L) := \text{Hom}_{\mathcal{C}}(L, 1)$. Then there are well-defined restriction maps:

\begin{equation}
\text{Res} := \text{Hom}_{\mathcal{C}}(F(\iota_L), -) : \text{CF}(\mathcal{C}) \rightarrow \text{CF}(L),
\end{equation}

\begin{equation}
\text{Res}^A_L := \text{Hom}_{\mathcal{Z}(\mathcal{C})}(\iota_L, -) : \text{Hom}_{\mathcal{Z}(\mathcal{C})}(A, A) \rightarrow \text{Hom}_{\mathcal{Z}(\mathcal{C})}(L, A).
\end{equation}

Since $Z = FR$ is a comonad it follows that the adjunction isomorphisms $F \dashv R$ are given in this case by

\begin{equation}
\psi_{L, 1} : \text{Hom}_{\mathcal{C}}(L, 1) \rightarrow \text{Hom}_{\mathcal{Z}(\mathcal{C})}(L, A), \quad \mu \mapsto R(\mu) \circ \eta_L.
\end{equation}

where $\eta_L$ is the unit of the adjunction $L \xrightarrow{\eta_L} R(F(L))$.

Moreover, its inverse is given by

\begin{equation}
\psi^{-1}_{L, 1} : \text{Hom}_{\mathcal{Z}(\mathcal{C})}(L, A) \rightarrow \text{Hom}_{\mathcal{C}}(L, 1), \quad g \mapsto \epsilon_1 \circ g.
\end{equation}
Naturality in the first variable of the adjunction isomorphisms $\psi_{(-)}^1$ of $F \dashv R$ implies that the following diagram is commutative:

\[
\begin{array}{ccc}
CF(L) = \text{Hom}_C(L, 1) & \xrightarrow{\psi_{L,1}} & \text{Hom}_Z(C)(L, A) \\
\text{Res} & & \text{Res}^A_1 \downarrow \\
CF(C) = \text{Hom}_C(A, 1) & \xrightarrow{\psi_{A,1}} & \text{Hom}_Z(C)(A, A).
\end{array}
\]

Note that $\text{End}_Z(C)(A)$ is an algebra with $f \ast g := f \circ g$. Moreover, $\text{Hom}_Z(C)(L, A)$ is a left $\text{End}_Z(C)(A)$-module via $f.\mu = f \circ \mu$ for any $f \in \text{End}_Z(C)(A)$ and $\mu \in \text{Hom}_Z(C)(L, A)$. Then the commutativity of diagram (3.5) also gives that $CF(L)$ is a left $CF(C)$-module via

\[
(3.6) \quad \chi.\mu = \psi_{A,1}^{-1}(\psi_{L,1}(\chi) \circ \psi_{L,1}(\mu)), \quad \text{for any } \chi \in CF(C), \mu \in CF(L).
\]

On the other hand, since the restriction map $\text{Res}_A^L: \text{Hom}_Z(C)(A, A) \to \text{Hom}_Z(C)(L, A)$ is an $\text{End}_Z(C)(A)$-module homomorphism it follows that $\text{Res}$ is also a $CF(C)$-module homomorphism. This can be written as:

\[
(3.7) \quad \chi.\text{Res}(\chi') = \text{Res}(\chi \ast \chi'),
\]

for any two $\chi, \chi' \in CF(C)$. In particular, for $\chi' = \epsilon_1$ the unit of $CF(C)$ it follows that for any $\chi \in CF(C)$ one has

\[
(3.8) \quad \chi.\text{Res}(\epsilon_1) = \text{Res}(\chi).
\]

Define also the *central subspace* of $CE(C)$ associated to $L$ as the vector space $CE(L) := \text{Hom}_C(1, L)$. Denote by

\[
(3.9) \quad \iota^C : CE(L) = \text{Hom}_C(1, L) \hookrightarrow \text{Hom}_C(1, A) = CE(C)
\]

the canonical inclusion given by $\iota^C(z) = F(\iota_L) \circ z$ for any $z \in CE(L)$.

For $z, z' \in CE(L)$, one can set $z z' := m_L \circ (z \otimes z')$. Then as in the case of $CE(C)$ it is easy to see that $CE(L)$ is a monoid with respect to this operation. Moreover, it is easy to see that $\iota^C$ is a unitary algebra embedding. Since $CE(C)$ is a semisimple commutative algebra over an algebraically closed field, see [18], it follows that $CE(L)$ is also a semisimple commutative algebra.

Let $\{\ell_s\}_{s \in \{0, \ldots, r\}}$ be the (central) primitive idempotents of $CE(L)$. One can write $\iota^C_\ell(\ell_s) = \sum_{i \in B_s} E_i$ for the decomposition of the primitive idempotents of $CE(L)$ inside $CE(C)$. In this way we get a partition $\{0, \ldots, m\} = \mathcal{B}_0 \sqcup \cdots \sqcup \mathcal{B}_r$. Through the rest of the paper, by abuse of notations we also denote the corresponding partition of the irreducible characters, $\text{Irr}(C) = \mathcal{B}_0 \sqcup \cdots \sqcup \mathcal{B}_r$ with the same symbols $\mathcal{B}_s$, with $0 \leq s \leq r$. 

s \leq r$. Without loss of generality we may suppose that the irreducible character $\chi_0 = \epsilon_1$ of the unit object of $\mathcal{C}$ satisfies $\chi_0 \in \mathcal{B}_0$ i.e.

$$\iota^c(\ell_0) = \Lambda + \sum_{\{i \in \mathcal{B}_0, i \neq 0\}} E_i$$

where $\Lambda$ is the idempotent integral of $\mathcal{C}$.

**Remark 3.11.** Note that the map $\pi : A \to L$ induces also two maps at the level of characters and centers. One is a canonical embedding of vector spaces $\pi^c : \text{CF}(L) \hookrightarrow \text{CF}(\mathcal{C})$, $\mu \mapsto \mu \circ F(\pi)$. The other one is a surjective linear map $\pi^e : \text{CE}(\mathcal{C}) \to \text{CE}(L)$, $z \mapsto \pi \circ z$. Clearly one has $\text{Res} \circ \pi^c = \text{id}_{\text{CF}(L)}$ and $\pi^e \circ \iota^c = \text{id}_{\text{CE}(L)}$.

There is also a nondegenerate canonical paring given by

$$\langle \ , \rangle_L : \text{CF}(L) \otimes \text{CE}(L) \to \mathbb{k}, \langle \alpha, z \rangle \mapsto \alpha \circ z.$$

Recall also the non-degenerate evaluation pairing $\langle \ , \rangle_A$ given by

$$\langle \ , \rangle_A : \text{CF}(\mathcal{C}) \otimes \text{CE}(\mathcal{C}) \to \mathbb{k}, \langle \chi, z \rangle \mapsto \chi \circ z.$$

**Lemma 3.14.** One has the following compatibility properties between pairings:

$$\langle \text{Res}(\chi), \bar{z} \rangle_L = \langle \chi, \iota^c(\bar{z}) \rangle_A.$$

$$\langle \mu, \pi^e(z) \rangle_L = \langle \pi^e(\mu), z \rangle_A$$

for any $\bar{z} \in \text{CE}(L)$, $z \in \text{CE}(\mathcal{C})$, $\chi \in \text{CF}(\mathcal{C})$ and any $\mu \in \text{CF}(L)$

**Proof.** Straightforward.

**Proposition 3.17.** For any two irreducible characters $\chi_i, \chi_j \in \text{Irr}(\mathcal{C})$ one has that $\chi_i, \chi_j \in \mathcal{B}_s$ for some $0 \leq s \leq r$ if and only if

$$\frac{\text{Res}(\chi_i)}{d_i} = \frac{\text{Res}(\chi_j)}{d_j}.$$ 

**Proof.** Let $\eta_s \in \text{CF}(L)$ be the dual basis of $\ell_s$ with respect to the non-degenerate pairing $\langle \ , \rangle_L$. Thus one has $\langle \eta_t, \ell_s \rangle_L = \delta_{s,t}$, for any $0 \leq s, t \leq r$. We will show that if $\chi_i \in \mathcal{B}_s$ then

$$\frac{\text{Res}(\chi_i)}{d_i} = \eta_s.$$

This would finish the proof of the proposition.

Suppose now that $\chi_i \in \mathcal{B}_s$. Then by Equation (3.15) one has

$$\langle \text{Res}(\chi_i), \ell_s \rangle_L = \langle \chi_i, \iota^c(\ell_s) \rangle_A = \langle \chi_i, \sum_{j \in \mathcal{B}_s} E_j \rangle_A = d_i.$$
hand, \( \langle \text{Res}(\chi_i), \ell_t \rangle_L = \langle \chi_i, \iota^e(\ell_t) \rangle_A = 0 \) if \( t \neq s \). Therefore \( \frac{1}{d_t} \text{Res}(\chi_i) = \eta_s \) for any irreducible character \( \chi_i \in \mathcal{B}_s \).

Recall that for the character \( \chi_0 \) of the unit object \( 1 \) of \( \mathcal{C} \) we assumed that \( \chi_0 \in \mathcal{B}_0 \). We denote by \( \underline{\epsilon}_L := \text{Res}(\chi_0) \) its restriction to \( L \). Note that the above proposition implies \( \underline{\epsilon}_L = \eta_0 \). Moreover the proof shows that \( \chi_i \in \mathcal{B}_0 \) if and only if

\[
(3.19) \quad \text{Res}(\chi_i) = d_i \underline{\epsilon}_L.
\]

**Remark 3.20.** We have chosen to add the underline bar in the notation \( \underline{\epsilon}_L : L \to 1 \) of the restriction \( \text{Res}(\epsilon_1) \) in order to distinguished it from the counit \( \epsilon_L : Z(L) \to L \) of the comonad \( Z \) evaluated at \( L \).

### 3.1. On the right \( L \)-module maps in \( \text{CF}(L) \).

Since \( \epsilon_1 : A \to 1 \) is an algebra morphism in \( \mathcal{C} \) it follows that \( 1 \) can be considered a right \( A \)-module in \( \mathcal{C} \) via the morphism \( \epsilon_1 \). As explained in [18, Subsection 5.2] the integral \( \Lambda : 1 \to A \) is the unique (up to scalar) morphism of right \( A \)-modules in \( \mathcal{C} \).

The next Lemma is an analogue of [18, Lemma 6.1] and its proof follows the same lines.

**Lemma 3.21.** Let \( \mathcal{C} \) be a fusion category and \( L \) be a unitary subalgebra of \( A \) in \( Z(\mathcal{C}) \).

1. An arrow \( a : 1 \to L \) in \( \mathcal{C} \) is a morphism of right \( L \)-modules in \( \mathcal{C} \) if and only if for any \( z \in \text{CE}(L) \) one has:

\[
(3.22) \quad za = \langle \underline{\epsilon}_L, z \rangle_L a.
\]

2. \( \ell_0 \) is a morphism of right \( L \)-modules in \( \mathcal{C} \).

**Proof.** The proof of the first item is straightforward, see also [18, Lemma 6.1].

Since \( \text{CE}(L) \) is a product of copies of the field \( k \) an element \( \phi \in \text{CE}(L) \) satisfying Equation (3.22) is unique up to a scalar. Moreover, note that the idempotent \( \ell_0 \) satisfies this equation. Indeed, since \( \text{CE}(L) \) is a semisimple algebra there is \( \mu \in \overline{\text{CE}(L)} \) such that \( z\ell_0 = \mu(z)\ell_0 \) for all \( z \in \text{CE}(L) \). Since \( \langle \underline{\epsilon}_L, \ell_0 \rangle_L = \langle \epsilon_1, \iota^e(\ell_0) \rangle_A = 1 \) it follows that

\[
\mu(z) = \langle \underline{\epsilon}_L, \mu(z)\ell_0 \rangle_L = \langle \underline{\epsilon}_L, z\ell_0 \rangle_A = \langle \epsilon_1, \iota^e(z\ell_0) \rangle_A \\
= \langle \epsilon_1, \iota^e(z) \rangle_A \langle \epsilon_1, \iota^e(\ell_0) \rangle_A = \langle \epsilon_1, \iota^e(z) \rangle_A \\
= \langle \underline{\epsilon}_L, z \rangle_L.
\]

\[ \square \]
On the full subcategory $\mathcal{S}_L$. We denote by $\mathcal{S}_L$ the full abelian subcategory of $\mathcal{C}$ generated by the simple objects $V_i$ of $\mathcal{C}$ whose irreducible characters $\chi_i$ satisfy $\chi_i \in \mathcal{B}_0$.

**Theorem 3.1.** Let $\mathcal{C}$ be a pseudo-unitary fusion category and $L$ be a unitary subalgebra of the adjoint algebra $A$ of $\mathcal{C}$. With the above notations one has that $\mathcal{S}_L$ is a fusion subcategory of $\mathcal{C}$.

**Proof.** Suppose that $M, M' \in \text{Irr}(\mathcal{S}_L)$ are two simple objects with characters $\chi_M = \chi, \chi_{M'} = \chi'$. Then $\text{Res}(\chi) = d(\chi)\xi_L$ and $\text{Res}(\chi') = d(\chi')\xi_L$. It follows that

$$\text{Res}(\chi \star \chi') = \chi \cdot \text{Res}(\chi') = d(\chi')\chi \cdot \xi_L = d(\chi)\text{Res}(\chi') = d(\chi')d(\chi)\xi_L.$$  \hfill (3.18)

On the other hand suppose that $\chi \star \chi' = \sum_{\chi_u \in \text{Irr}(\mathcal{C})} N_{M, M'}^u \chi_u$ with some $N_{M, M'}^u \geq 0$. It follows that

$$\text{Res}(\chi \star \chi') = \sum_{\chi_u \in \text{Irr}(\mathcal{C})} N_{M, M'}^u \text{Res}(\chi_u) = \sum_{s=0}^{r} \sum_{u \in \mathcal{B}_s} N_{M, M'}^u d_u \text{Res}(\frac{\chi_u}{d_u})$$  \hfill (3.18)\hfill

where $\{\eta_s\}$ is the dual basis of $\ell_s$ with respect to the pairing $(\, , \,)_L$. Since $\eta_s$ are linearly independent it follows that for any $s \neq 0$ one has

$$\sum_{u \in \mathcal{B}_s} N_{M, M'}^u d_u = 0$$

Since $\mathcal{C}$ is pseudo-unitary one has $d_u > 0$ and therefore in this case $N_{M, M'}^u = 0$ for any $u \in \mathcal{B}_s$. Thus all the simple constituents of $M \otimes M'$ are in full abelian subcategory $\mathcal{S}_L$.  \hfill $\square$

**On the** $\text{FPdim}(\mathcal{S}_L)$. In this subsection we prove the following:

**Theorem 3.2.** Let $\mathcal{C}$ be a pseudo-unitary fusion category and $L$ be a unitary subalgebra of the adjoint algebra $A$ of $\mathcal{C}$. With the above notations one has that

$$\text{FPdim}(\mathcal{S}_L) = \frac{\dim(\mathcal{C})}{\dim(L)}.$$

First we need to fix several notations. Let $\mathcal{C}$ be a fusion category and $F : Z(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor. As above, let $R$ be a right adjoint of $F$. We may suppose that $A = \bigoplus_{j=0}^{r} C^{(j)}$ is the decomposition of $A$ in homogenous components in $Z(\mathcal{C})$. Define $\mathcal{J} := \{0, \ldots, r\}$ the set of indices of homogenous components of $A$. 
We may write each homogenous component as \( C^{(j)} = \bigoplus_{s=1}^{m_j} C_s^{(j)} \) where \( C_s^{(j)} \) are the simple sub-objects of \( A \) entering in the homogenous component \( C^{(j)} \). Therefore as an object of \( \mathcal{Z}(\mathcal{C}) \) one has a decomposition in simple objects

\[
A = \bigoplus_{j \in J} \bigoplus_{s \in \mathcal{M}_j} C_s^{(j)}
\]

where \( C_s^{(j)} \) and \( C_t^{(j)} \) are isomorphic simple \( \mathcal{Z}(\mathcal{C}) \)-submodules of \( A \) and \( \mathcal{M}_j := \{1, \ldots, m_j\} \). One has \( \text{End}_{\mathcal{Z}(\mathcal{C})}(C^{(j)}) \approx M_{m_j}(k) \) as algebras and therefore \( \text{End}_{\mathcal{Z}(\mathcal{C})}(A) \approx \bigoplus_{j \in J} M_{m_j}(k) \). By the natural isomorphism \( \psi_{A,1} \) from Equation (3.4) one also has that \( \text{CF}(\mathcal{C}) \approx \bigoplus_{j \in J} M_{m_j}(k) \) as a semisimple algebra.

Without loss of generality we may also suppose that \( C^{(0)} = 1_{\mathcal{Z}(\mathcal{C})} \) is the simple trivial object of \( A \). Since \( \text{Hom}_{\mathcal{Z}(\mathcal{C})}(A, 1) \approx \text{Hom}_\mathcal{C}(1, 1) = k \) it follows that \( m_0 = 1 \).

Define \( F_{st}^j \in \text{End}_{\mathcal{Z}(\mathcal{C})}(A) \) as the unique endomorphisms of \( A \) that send (as identity) \( C_t^{(j)} \) into \( C_s^{(j)} \) and that are zero on the other conjugacy classes \( C_{t'}^{(j)} \) with \( t' \neq t \). Then these endomorphisms form a linear basis \( \text{End}_{\mathcal{Z}(\mathcal{C})}(A) \) such as

\[
\tilde{F}_{st}^j \circ \tilde{F}_{s't'}^j = \delta_{j,j'} \delta_{s',s} \tilde{F}_{st'}^j
\]

Denote \( F_{st}^j := \psi_{A,1}^{-1}(\tilde{F}_{st}^j) \in \text{CF}(\mathcal{C}) \). It follows that \( F_{st}^j \) are the standard matrix entries in the matrix-block \( M_{m_j}(k) \). Thus

\[
F_{st}^j F_{s't'}^j = \delta_{j,j'} \delta_{s',s} F_{st'}^j,
\]

and the corresponding primitive central idempotent corresponding to the unit of the block is \( F^J = \sum_{s=0}^{m_j} F_{ss}^j \). Note that the decompositions of \( A \) from Equation (3.23) are in bijection with the matrix bases \( \{F_{st}^j\} \) of \( \text{CF}(\mathcal{C}) \).

It is well known that the block \( k = M_{m_0}(k) \) from the above decomposition corresponds to the central primitive idempotent \( F^0 = \lambda \), the cointegral of \( \mathcal{C} \).

By [18, Lemma 6.2] the Grothendieck ring \( \text{Gr}_k(\mathcal{C}) \) is a symmetric Frobenius algebra with non-degenerate trace \( \tau : \text{Gr}_k(\mathcal{C}) \rightarrow \mathbb{C} \) given by \( [X] \mapsto \dim_k \text{Hom}_\mathcal{C}(1, X) \). For a pivotal fusion category one has \( \tau([X]) = \langle \chi(X), \Lambda \rangle \) for any object \( X \in \mathcal{O}(\mathcal{C}) \), where \( \Lambda \) is an idempotent cointegral associated to \( \mathcal{C} \).

Then the corresponding associative non-degenerate bilinear form on \( \text{CF}(\mathcal{C}) \) is given by \( \beta_\tau(\chi, \mu) := \langle \chi \mu, \Lambda \rangle \). A pair of dual bases for \( \beta_\tau \) is given by \( \{\chi_i, \chi_{i^*}\} \).
Suppose that on the matrix-block decomposition of $\text{CF}(\mathcal{C})$ one has

$$\beta_\tau(\chi, \mu) = \sum_j \frac{1}{n_j} \text{tr}_j(\chi \mu).$$

where $\text{tr}_j$ is the usual trace on the matrix algebra $M_{m_j}(k)$. Since $\{F_{st}^j, n_j F_{ts}^j\}$ is another pair of dual bases for $\beta_\tau(-, -)$ one can write:

$$\sum_{j=0}^m \sum_{s,t \in M_j} n_j F_{st}^j \otimes F_{ts}^j = \sum_{i=0}^m \chi_i \otimes \chi_i^*.$$  

Lemma 3.27. For a pivotal fusion category, with the above notations one has

$$n_j = \frac{\dim(\mathcal{C})}{\dim(\mathcal{C}_s^{(j)})}.$$  

Proof. By [18, Proposition 5.17] one has that for any $f \in \text{CF}(\mathcal{C})$

$$\text{tr}(\psi^{-1}_{-1}(f)) = \langle f, \Lambda \rangle \langle \lambda, u \rangle$$

where $(\Lambda, \lambda)$ is any pair of an integral and cointegral of $\mathcal{C}$ such that $\langle \lambda, \Lambda \rangle = 1$. One can chose such a pair with $\langle \lambda, u \rangle = \dim(\mathcal{C})$ and $\Lambda$ an idempotent integral, i.e $\epsilon_1(\Lambda) = 1$.

In particular, for $f = F_{jj}$ one has that $\psi^{-1}_{-1}(F_{jj})$ is the projection on the homogenous component $\mathcal{C}^{(j)}$. Therefore the above trace formula can be written as

$m_j \dim(\mathcal{C}_s^{(j)}) = \text{tr}(\psi^{-1}_{-1}(F_{jj})) = \langle F_{jj}, \Lambda \rangle \dim(\mathcal{C})$.

On the other hand, from Equation (3.25) one has

$$\tau(F_{ss}^j) = \langle F_{ss}^j, \Lambda \rangle = \frac{1}{n_j} \text{tr}_j(F_{ss}^j) = \frac{1}{n_j}$$

Thus $\langle F_{jj}, \Lambda \rangle = \tau(F_{jj}) = \sum_{s=0}^{m_j} \tau(F_{ss}^j) = \frac{m_j}{n_j}$ and the result follows. \qed

Definition 3.3. Denote by $C_{st}^{ij} := F_{ij}^{-1}(F_{st}^j) \in \text{CE}(\mathcal{C})$ and call this element the conjugacy class sum corresponding to $F_{st}^j$. Here we use the cointegral $\lambda$ with $\langle \lambda, u \rangle = 1$.

Lemma 3.30. Let $\mathcal{C}$ be a spherical fusion category. With the above notations one has that:

$$\langle F_{st}^j, C_{uv}^i \rangle_\Lambda = \delta_{i,j} \delta_{v,s} \delta_{u,t} \dim(\mathcal{C}_s^{(j)}).$$
Proof. In particular, for $\chi = F^j_{st}$ and $\mu = F^j_{ts}$ in Equation (2.17) one has that
\[
\langle F^j_{st}, C^i_{uv} \rangle_A = \langle F^j_{st}, F^{-1}_\lambda(F^i_{uv}) \rangle_A = \dim(C)\tau(F^j_{st}F^i_{uv}) = \dim(C)\tau(F^i_{uv}) = \delta_{i,j}\delta_{u,v}\delta_{u,t}\frac{\dim(C)}{n_j}.
\]
Then use the formula (3.29).

\[\square\]

Lemma 3.32. Let $C$ be a spherical non-degenerate fusion category. With the above notations one has

(3.33) \[\langle \epsilon_1, C^j_{st} \rangle_A = \delta_{s,t} \dim(C^{(j)})\]

Proof. Note that $\epsilon_1 = \chi_0 = \sum_j F^j$ is the unit of $\text{CF}(C)$. Equation (2.17) for $\chi = \epsilon_1$ and $\mu = F^j_{ts}$ gives:

\[
\langle \epsilon_1, C^j_{st} \rangle_A = \dim(C)\tau(F^j_{st}) = \delta_{s,t} \dim(C^{(j)})\]

\[\square\]

Lemma 3.34. With the above notations one has

(1) $F^j_{st} \in \text{Hom}_C(C^{(j)}_t, 1)$

(2) $C^j_{st} \in \text{Hom}_C(1, C^{(j)})$.

(3) On the homogenous components the Fourier transform sends $\text{Hom}_C(1, F(C^{(j)}))$ into $\text{Hom}_C(F(C^{(j)}), 1)$.

Proof. By the adjunction isomorphism $\psi^{-1}_{\lambda,1}$ from Equation (2.19) one has $F^j_{st} = \epsilon_1 \circ \widetilde{F}^j_{st}$ which gives the proof for the first item.

On the other hand Equation (2.17) shows that also $\{F^j_{st}, \frac{n_t}{\dim(C)}C^j_{ts}\}$ are also dual bases for $\langle , \rangle_A$. Since for $s \neq t$ the pairing $\langle , \rangle_A$ is zero on $\text{Hom}_C(C^{(j)}_t, 1) \times \text{Hom}_C(1, F(C^{(j)}))$ it follows that $C^j_{ts} \in \text{Hom}_C(C^{(j)}_t, 1)$.

\[\square\]

Recall the embedding $q^* : \text{CF}(D) \to \text{CF}(C)$ from Subsection 2.6. By abuse of notations we denote by the same symbol $\lambda_D$, the image of $\lambda_D \in \text{CF}(D)$ under $q^*$.

Proposition 3.35. Let $C$ be a non-degenerate spherical fusion category and $D \subseteq C$ a fusion subcategory of $C$. Suppose that

\[
\lambda_D = \sum_{(j,s,t) \in \mathcal{L}_D} \beta^j_{st} F^j_{st}
\]
where all $\beta^i_j \in \mathbb{k}$ are non-zero scalars and $\mathcal{L}_D \subseteq \sqcup_{j \in J} \{j\} \times \mathcal{M}_j \times \mathcal{M}_j$ is a subset of indices. Then

$$
\frac{\dim(C)}{\dim(D)} = \sum_{(j,s,s) \in \mathcal{L}_D} \beta^i_{ss} \dim(C_{ss}^j).
$$

Proof. We show that both terms of Equation (3.36) equal $\langle \epsilon_1, \ell_D \rangle_A$ where $\ell_D := F^{-1}_\lambda(\lambda_D)$. Here the cointegral $\lambda$ is chosen such that $\langle \lambda, u \rangle_A = 1$, where $u = F(\eta_A)$ is the unit of $\text{CE}(C)$. Applying $F^{-1}_\lambda$ to the above formula for $\lambda_D$ it follows that the element $\ell_D$ has the formula

$$
\ell_D = \sum_{(j,s,t) \in \mathcal{L}_D} \beta^i_{st} C_{st}^j.
$$

Equation (3.37) gives that

$$
\langle \epsilon_1, \ell_D \rangle_A = \sum_{(j,s,t) \in \mathcal{L}_D} \beta^i_{st} \langle \epsilon_1, C_{st}^j \rangle_A = \sum_{(j,s,s) \in \mathcal{L}_D} \beta^i_{ss} \dim(C_{ss}^j).
$$

On the other hand from Equation (2.15) one has

$$
\ell_D = F^{-1}_\lambda(\lambda_D) = \frac{1}{\dim(D)} \left( \sum_{\chi_i \in \text{Irr}(D)} d_i^* F^{-1}_\lambda(\chi_i) \right)
$$

$$
= \frac{1}{\dim(D)} \left( \sum_{\chi_i \in \text{Irr}(D)} d_i^* \frac{\dim(C)}{d_i} E_{i^*} \right).
$$

Since $C$ is spherical it follows that

$$
\ell_D = \frac{\dim(C)}{\dim(D)} \left( \sum_{\chi_i \in \text{Irr}(D)} E_i \right).
$$

By applying $\epsilon_1$ to Equation (3.38) it follows that $\langle \epsilon_1, \ell_D \rangle_A = \frac{\dim(C)}{\dim(D)}$. Indeed, note that $\langle \epsilon_1, E_j \rangle = \delta_{j,0}$ since we assumed that $\chi_0 = \epsilon_1$. □

Remark 3.39. Note that the above result holds for any decomposition of $A$ from Equation (3.23), thus basically for any matrix basis $F_{st}^j$ of $\text{CF}(C)$.

Remark 3.40. In [5] it was studied the central element $\ell_D := F^{-1}_\lambda(\lambda_D) \in \text{CE}(C)$ in the case of a fusion category with commutative Grothendieck ring.

To simplify the notations, through the rest of the paper, we denote $C_{st} := C_{st}^j$ for the class sum $C_{st}^j$ whenever it is specified that $s, t \in \mathcal{M}_j$, case in which the index $j$ is implicitly understood. The same notation $F_{st} := F_{st}^j$ will be used in the same circumstances.
Note that the central subspace of $\text{CE}(\mathcal{C}) := \text{Hom}_\mathcal{C}(1, A)$ associated to $L$ can be described as $\text{CE}(\mathcal{C}) := \bigoplus_{j \in \mathcal{J}} \bigoplus_{s \in \mathcal{M}_j} \text{Hom}_\mathcal{C}(1, F(\mathcal{C}_s^{(j)}))$. Therefore, a linear basis for $\text{CE}(\mathcal{C})$ is given by $C_s^{(j)}$ with $j \in \mathcal{J}$ and $s, t \in \mathcal{M}_j$.

### 3.2. On unitary subalgebras of $A$.

Let $L$ be a subalgebra of $A$ in $\mathcal{Z}(\mathcal{C})$. Since $A$ is a semisimple object of $\mathcal{Z}(\mathcal{C})$ we may choose the simple direct summands $C_s^{(j)}$ from Equation (3.23) such that

$$L = \bigoplus_{j \in \mathcal{J}_L} \bigoplus_{s \in \mathcal{L}_j} C_s^{(j)}$$

for some subset $\mathcal{J}_L \subseteq \mathcal{J}$ and a subset $\mathcal{L}_j \subseteq \mathcal{M}_j := \{1, \ldots, m_j\}$. Denote by $\widetilde{\mathcal{J}}_L := \bigcup_{j \in \mathcal{J}_L} \{j\} \times \mathcal{L}_j$.

By the naturality of the adjunction isomorphisms $\psi_{\mathcal{C}_s^{(j)}, 1}$ it follows that via the inclusion $\iota^e : \text{CE}(L) \rightarrow \text{CE}(\mathcal{C})$ one has

$$\text{CE}(L) := \text{Hom}_\mathcal{C}(1, L) = \bigoplus_{j \in \mathcal{J}_L} \bigoplus_{s \in \mathcal{L}_j} \text{Hom}_\mathcal{C}(1, F(\mathcal{C}_s^{(j)})).$$

Therefore we may denote by the same symbol $C_{st}^{(j)} \in \text{Hom}_\mathcal{C}(1, F(\mathcal{C}_s^{(j)}))$ the elements of the linear basis of $\text{CE}(\mathcal{C})$ that belong to $\text{CE}(L) \subseteq \text{CE}(\mathcal{C})$ if $s \in \mathcal{L}_j$. In other words, $\iota^e(C_{st}^{(j)}) = C_{st}^{(j)}$ if $j \in \mathcal{J}_L$ and $s \in \mathcal{L}_j$.

Thus, a linear basis for $\text{CE}(L)$ is given by $C_{st}^{(j)} := F^{-1}(F_{st}^{(j)})$ with $j \in \mathcal{J}_L$, $s \in \mathcal{L}_j$ and $t \in \mathcal{M}_j$.

Note also that $\text{CF}(\mathcal{C}) = \bigoplus_{j \in \mathcal{J}, s \in \mathcal{M}_j} \text{Hom}_\mathcal{C}(F(\mathcal{C}_s^{(j)}), 1)$. Recall that there is also a projection $\pi : A \rightarrow L$ in $\mathcal{Z}(\mathcal{C})$ since this is a semisimple category. Then, as above, the induced linear morphism $\pi^e := F(\pi)^* : \text{Hom}_\mathcal{C}(F(L), 1) \rightarrow \text{Hom}_\mathcal{C}(F(A), 1)$ gives an embedding $\text{CF}(L) \hookrightarrow \text{CF}(\mathcal{C})$.

Via this embedding one has $\text{CF}(L) = \bigoplus_{(j, s) \in \widetilde{\mathcal{J}}_L} \text{Hom}_\mathcal{C}(F(\mathcal{C}_s^{(j)}), 1)$. We may denote by the same symbol $F_{st}^{(j)}$ the same elements corresponding to $\text{CF}(L)$ when $(j, t) \in \widetilde{\mathcal{J}}_L$.

The following three lemmas are easy to verify.

**Lemma 3.43.** A basis for $\text{CF}(L)$ is given by $F_{st}^{(j)}$ with $(j, t) \in \widetilde{\mathcal{J}}_L$ and $s \in \mathcal{M}_j$.

**Lemma 3.44.** With the above identifications one has

$$\text{Res}(F_{st}^{(j)}) = F_{st}^{(j)}, \text{ if } (j, t) \in \widetilde{\mathcal{J}}_L, \text{ for any, } 0 \leq s \leq m_j$$

and

$$\text{Res}(F_{st}^{(j)}) = 0, \text{ if } (j, t) \notin \widetilde{\mathcal{J}}_L, \text{ for any, } 0 \leq s \leq m_j.$$
Lemma 3.47. For any $j \in J_L$ one has $\pi_i(C_{st}^j) = C_{st}^j$ if $j \in J_L$, $s \in L_j$ and, $\pi_i(C_{st}^j) = 0$ otherwise.

For any irreducible character $\chi_i$ write
\[
\chi_i = \sum_{j=0}^{m} \sum_{s,t \in M_j} \alpha(i)_{st}^j F_{st}^j
\]
for some scalars $\alpha(i)_{st}^j \in k$. Applying $F_{-1} \lambda$ to this Equation and using Equation (2.16) one has that
\[
\dim(C) \frac{d_i}{E_i} = \sum_{j=0}^{m} \sum_{s,t \in M_j} \alpha(i)_{st}^j C_{st}^j
\]
which gives that
\[
E_i = \frac{1}{\dim(C)} \sum_{j=0}^{m} \sum_{s,t \in M_j} d_i \alpha(i)_{st}^j C_{st}^j.
\]

Denote by
\[
\beta_{st}^j := \sum_{i \in B_0} d_i \alpha(i)_{st}^j.
\]

Lemma 3.52. Let $C$ be a fusion category and $L$ be a unitary subalgebra of $A$. With the above notations one has
\[
\beta_{st}^j = 0, \text{ for any } (j,s) \notin \tilde{J}_L, \text{ and any } 0 \leq t \leq m_j.
\]

Proof. Recall the linear basis $\ell_k$ of $CE(L)$, with $0 \leq k \leq r$. Applying Equation (3.49) one obtains that for any $0 \leq k \leq r$ one has
\[
\ell^i(\ell_k) = \sum_{\chi_i \in \text{Irr}(B_0)} E_{i} \overset{3.49}{=} \frac{1}{\dim(C)} \sum_{j=0}^{m} \sum_{s,t \in M_j} \left( \sum_{\chi_i \in \text{Irr}(B_0)} d_i \alpha(i)_{st}^j \right) C_{st}^j.
\]
Since $\{C_{st}^j\}_{j \in J_L, s \in L_j, t \in M_j}$ is a basis of $CE(L)$ it follows that
\[
\sum_{\chi_i \in \text{Irr}(B_0)} d_i \alpha(i)_{st}^j = 0
\]
if $j \notin J_L$ or if $j \in J_L$ but $s \notin L_j$. Note that for $k = 0$ this can also be written as $\beta_{st}^j = 0$, for any $(j,s) \notin \tilde{J}_L$, and any $0 \leq t \leq m_j$ since $S_L = B_0$ is a fusion subcategory. \qed

Lemma 3.55. With the above notations one has
\[
\varepsilon_{L} = \sum_{(j,t) \in \tilde{J}_L} F_{jt}^i.
\]
For any $\chi_i \in B_0$ one has

\begin{equation}
\alpha(i)_s^j = \delta_{s,t} d_i, \text{ for any } (j,t) \in \widetilde{J}_L, \text{ and any, } 1 \leq s \leq m_j.
\end{equation}

**Proof.** Since $\varepsilon_1 = \sum_{j=0}^{m_j} F^j$ is the unit of $\text{CF}(\mathcal{C})$ it follows by Lemma 3.44 that $\varepsilon_L := \text{Res}(\varepsilon_1) = \sum_{(j,t) \in \widetilde{J}_L} F^j_{tt}$. If $\chi_i \in B_0$ then by Equation (3.19) one has $\text{Res}(\chi_i) = d_i \varepsilon_L = d_i \left( \sum_{(j,t) \in \widetilde{J}_L} F^j_{tt} \right)$. On the other hand

\[
\text{Res}(\chi_i) = \sum_{j \in J} \sum_{s \in M_j} \alpha(i)_s^j \text{Res}(F^j_{st})
\]

Comparing the two expressions for $\text{Res}(\chi_i)$ it follows that for $(j,t) \in \widetilde{J}_L$ one has $\alpha(i)_s^j = d_i \delta_{s,t}$. \qed

**Proposition 3.58.** Suppose that $\mathcal{C}$ is a spherical fusion category and $L$ be a unitary subalgebra of $A$. With the above notations, for the idempotent integral of $S_L$ one has that

\begin{equation}
\lambda_{S_L} = \sum_{j \in J_L} \sum_{s \in L_j} F^j_{ss} + \sum_{j \in J_L} \sum_{s \in L_j, t \notin L_j} \beta^j_{st} F^j_{st}
\end{equation}

for some scalars $\beta^j_{st} \in \mathbb{K}$. 
Proof. Recall that \( \dim(S_L) = \sum_{i \in \Irr(S_L)} d_i \cdot d_i \). Then since \( C \) is spherical one has that
\[
\lambda_{S_L} = \frac{1}{\dim(S_L)} \left( \sum_{\chi_i \in \Irr(S_L)} d_i \cdot \chi_i \right)
\]
where the scalars \( \beta_{st}^j := \left( \sum_{\chi_i \in \Irr(S_L)} d_i^* \alpha(i)_{st}^j \right) \) are defined as in Equation (3.51). \( \square \)

Proof of Theorem 3.2.

Proof. Note that Proposition 3.58 shows that the set of indices \( L_{S_L} \) of \( S_L \) coincides to \( \bigsqcup_{j \in J_L} \{j\} \times M_j \times L_j \). Then by Equation (3.36) one has that
\[
\frac{\dim(C)}{\dim(S_L)} = \sum_{(j,s) \in \mathcal{J}_L} \dim(C_s^{(j)}) = \dim(L).
\]
which finishes the proof of Theorem 3.2 since $\mathcal{C}$ is a pseudo-unitary fusion category.

**Corollary 3.60.** Let $\mathcal{C}$ be a pseudo-unitary fusion category and $M, L$ be two unitary subalgebras of $A$. Then $\mathcal{S}_M = \mathcal{S}_L$ if and only if $M = L$.

**Proof.** Note that since $A$ is a semisimple object one can choose the simple direct summands $\mathcal{C}_s^{(j)}$ such that Equation (3.61) is satisfied for both $M$ and $L$, i.e.

\[
L = \bigoplus_{j \in \mathcal{J}_L} \bigoplus_{s \in \mathcal{L}_j^1} \mathcal{C}_s^{(j)}, \quad M = \bigoplus_{j \in \mathcal{J}_M} \bigoplus_{s \in \mathcal{L}_j^2} \mathcal{C}_s^{(j)}.
\]

for some subsets $\mathcal{J}_L, \mathcal{J}_M \subseteq \mathcal{J}$ and $\mathcal{L}_j^1, \mathcal{L}_j^2 \subseteq \mathcal{M}_j$. One has $\mathcal{S}_L = \mathcal{S}_M$ if and only if $\lambda_{\mathcal{S}_L} = \lambda_{\mathcal{S}_M}$. Then the result follows from Equation (3.59).

### 3.3. On the abelian full subcategories $\mathcal{C}^\text{triv}_L$

Let $M$ be any object of $\mathcal{C}$. We denote by $\alpha_M|_L$ the restriction to the subalgebra $L$ of the action $\alpha_M$ of $A$ on $M$ defined in Equation (2.7). We say that an object $M$ of $\mathcal{C}$ receives a trivial $L$-action if

\[
\alpha_M|_L = \mathcal{L}_L \otimes \text{id}_M.
\]

**Definition 3.4.** Let $\mathcal{C}$ be any finite tensor category and $L$ be a unitary subalgebra of $A$. We define $\mathcal{C}^\text{triv}_L$ as the full abelian subcategory of $\mathcal{C}$ whose objects $M$ receive a trivial action $\alpha_M$.

Next proposition holds for any finite tensor category, not necessarily fusion.

**Proposition 3.63.** Let $\mathcal{C}$ be a pivotal finite tensor category and $L$ be a unitary subalgebra of $A$. Then as abelian subcategories of $\mathcal{C}$ one has $\mathcal{C}^\text{triv}_L \subseteq \mathcal{S}_L$.

**Proof.** Note that by definition of the dimension one has

\[
d(M) \text{id}_1 = (1 \xrightarrow{\text{coev}_M} M \otimes M^* \xrightarrow{\overline{ev}_M} 1)
\]

where $\overline{ev}_M = (M \otimes M^* \xrightarrow{j_M} M^{**} \otimes M^* \xrightarrow{\text{ev}_M} 1)$ is defined using the pivotal structure $j_M$. This implies

\[
d(M)\mathcal{L}_L = (L \simeq L \otimes 1 \xrightarrow{j_M \otimes \text{id}_1} 1 \xrightarrow{\text{coev}_M} M \otimes M^* \xrightarrow{\overline{ev}_M} 1).
\]

Suppose that $M \in \mathcal{O}(\mathcal{C})$ satisfies Equation (3.62). We will show that $M \in \mathcal{O}(\mathcal{S}_L)$. Since for any two morphisms in $\mathcal{C}$ one has $f \otimes g = (f \otimes 1) \circ (1 \otimes g)$, it follows that

\[
(id_A \otimes \text{coev}_M) \circ F(\iota_L) = (F(\iota_L) \otimes id_M \otimes id_{M^*}) \circ (id_L \otimes \text{coev}_M).
\]
Then the restriction \( \text{Res}(\chi(M)) \) can be written as:

\[
\text{Res}(\chi(M)) = (L \overset{F_{(1)}^{(1)}}{\longrightarrow} A \overset{\chi(M)}{\longrightarrow} 1) = \tag{2.10}
\]

\[
\overset{\alpha_M \otimes \text{id}_{M^*}}{\longrightarrow} M \otimes M^* \overset{\text{ev}_M}{\longrightarrow} 1
\]

\[
\text{(L \simeq L \otimes 1 \overset{\text{id}_L \otimes \text{coev}_M}{\longrightarrow} L \otimes M \otimes M^* \overset{\alpha_M}{\longrightarrow} M \otimes M^* \overset{\text{ev}_M}{\longrightarrow} 1)}.
\]

Replacing from Equation \((3.62)\) \(\alpha_M \mid_L \) by \(\xi_L \otimes \text{id}_M\) one obtains that \(\text{Res}(\chi(M))\) can be written as

\[
(L \simeq L \otimes 1 \overset{\text{id}_L \otimes \text{coev}_M}{\longrightarrow} L \otimes M \otimes M^* \overset{\xi_L \otimes \text{id}_M \otimes \text{id}_{M^*}}{\longrightarrow} M \otimes M^* \overset{\text{ev}_M}{\longrightarrow} 1).
\]

Using Equation \((3.64)\) this shows that \(\text{Res}(\chi(M)) = d(M)\xi_L\), i.e \(M \in \mathcal{O}(S_L)\).

4. **Proof of Theorem 1.1**

Let \( \mathcal{C} \) be a finite tensor category and \( F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C} \) be the forgetful functor. As above we denote by \( R \) a right adjoint of \( F \). The monoidal structure of the adjoint functor \( R : \mathcal{C} \to \mathcal{Z}(\mathcal{C}) \) gives two actions of the algebra \( A = R(1) \) on the object \( R(M) \in \mathcal{Z}(\mathcal{C}) \). The left action is defined by \( \rho^l_{R(M)} = R_2(M, 1) : R(M) \otimes R(1) \to R(M \otimes 1) \simeq R(M) \) and similarly the right action is defined by \( \rho^r_{R(M)} = R_2(1, M) : R(1) \otimes R(M) \to R(1 \otimes M) \simeq R(M) \). By [16] Lemma 6.4] for any finite tensor category one has

\[
(4.1) \quad \rho^r_{R(M)} = \rho^l_{R(M)} \circ c_{R(M), A}
\]

where \( c_{R(M), -} \) is the braiding of \( R(M) \in \mathcal{Z}(\mathcal{C}) \). The author also has warned that in general \( \rho^l_{R(M)} \neq \rho^r_{R(M)} \circ c_{A, R(M)} \).

**Lemma 4.2.** Let \( \mathcal{C} \) be a finite tensor category. For any object \( M \) of \( \mathcal{C} \) with the above notations one has

\[
(4.3) \quad \epsilon_M \circ \rho^l_{R(M)} = \epsilon_1 \otimes \epsilon_M, \quad \epsilon_M \circ \rho^r_{R(M)} = \epsilon_1 \otimes \epsilon_M
\]

**Proof.** It follows from the definition of the actions \( \rho^l_{R(M)}; \rho^r_{R(M)} \) and Equation \((2.4)\). \(\square\)

By [13] Lemma 3.5] \( A \) is an étale algebra in \( \mathcal{Z}(\mathcal{C}) \) and the right adjoint functor \( R \) induces a tensor equivalence \( \mathcal{C} \xrightarrow{R} \mathcal{Z}(\mathcal{C})_A \). Recall also that [6] Theorem 4.10] gives an (inclusion reversing) bijection between the lattice of étale subalgebras of \( A \) and that of fusion subcategories of \( \mathcal{C} \). This bijection can be described as follows:
Given an étale subalgebra $L$ of the adjoint algebra $A$ one considers the full subcategory $\beta(L)$ of $C$ as the category generated by those right $A$-modules in $Z(C)$ that are dyslectic as right $L$-modules with respect to $L$. Its inverse associates to a fusion subcategory $D$ of $C$ the subalgebra $L_D = J_D(1)$ where $J_D$ is the right adjoint of the forgetful functor $F_D : Z(C) \to Z_C(D)$. Recall that $Z_C(D)$ is the relative center of $D$ as a fusion subcategory of $C$, see [6]. By the proof of [6, Theorem 4.10] one has

\begin{equation}
FPdim(\beta(L)) = \frac{FPdim(C)}{FPdim(L)}
\end{equation}

for any connected étale subalgebra $L$ of $A$. Using Equation (4.1), since $L \in Z(C)$ is a sub-object of $A$ we note that for any $M \in O(C)$, the dyslecticity of $R(M)$ as a right $L$-module is equivalent to

\begin{equation}
\rho^l_{R(M)} \mid_L = \rho^r_{R(M)} \mid_L \circ c_{L,R(M)}
\end{equation}

For any left object $M$ of $C$ we denote by $\alpha_M \mid_L$ the restriction of the action $\alpha_M : A \otimes M \to M$ of $A$ from Equation (2.7) to $L$. Recall that by Equation (2.8) one has $\alpha_M = (id_M \otimes \epsilon_1) \circ c_{A,M}$. Since $L$ is a subalgebra of $A$ in $Z(C)$ this formula can be restricted to $L$ as:

\begin{equation}
\alpha_M \mid_L = (id_M \otimes \epsilon_L) \circ c_{L,M},
\end{equation}

where as above $\epsilon_L$ denotes the restriction of $\epsilon_1 : A \to 1$ to $L$.

**Proposition 4.7.** Let $L$ be a unitary subalgebra of $A$ in $Z(C)$. With the above notations if $M$ is an object of $C$ such that $R(M)$ is dyslectic with respect to $L$ then the following diagram is commutative:

\[
\begin{array}{ccc}
L \otimes Z(M) & \xrightarrow{\alpha_{Z(M)} \mid_L} & Z(M) \\
\downarrow{\epsilon_L \otimes M} & & \downarrow{\epsilon_M} \\
M & \xleftarrow{\epsilon_M \circ \alpha_{Z(M)} \mid_L} & L \otimes Z(M)
\end{array}
\]

which can be written as:

\begin{equation}
\epsilon_M \circ \alpha_{Z(M)} \mid_L = \epsilon_L \otimes \epsilon_M.
\end{equation}

**Proof.** Assume that $R(M)$ is a dyslectic right $L$-module in $Z(C)$. From Equation (4.5) it follows that

\begin{equation}
\epsilon_M \circ \rho^l_{R(M)} \mid_L = \epsilon_M \circ \rho^r_{Z(M)} \circ \rho^r_{R(M)} \mid_L \circ c_{L,R(M)}.
\end{equation}

By Equation (4.3) the above equality can be written as

\begin{equation}
\epsilon_L \otimes \epsilon_M = (\epsilon_L \otimes \epsilon_M) \circ c_{L,R(M)}.
\end{equation}
On the other hand by Equation (4.6) one has

\[(\epsilon_M \otimes \epsilon_L) \circ c_{L,R}(M) = (\epsilon_M \otimes \text{id}_1) \circ (\text{id}_{R(M)} \otimes \epsilon_L) \circ c_{L,Z}(M)] = (\epsilon_M \otimes \text{id}_1) \circ \alpha_{Z(M)} \big|_L \]

Thus Equation (4.10) can be written as \(\epsilon_L \otimes \epsilon_M = \epsilon_M \circ \alpha_{Z(M)} \big|_L \). \(\square\)

4.1. **Proof of the inclusion** \(\beta(L) \subseteq \mathcal{C}_{L}^{\text{triv}}\). For any unitary subalgebra \(L\) of \(A\) we denote by \(\beta(L)\) the full subcategory of \(\mathcal{C}\) which by the tensor equivalence \(R : \mathcal{C} \rightarrow \mathcal{Z}^{(\mathcal{C})}\) correspond to the right \(A\)-modules that are dyslectic with respect to \(L\). Note that as mentioned above, if \(L\) is an étale subalgebra then \(\beta(L)\) is a fusion subcategory of \(\mathcal{C}\) by [6, Theorem 4.10].

**Proposition 4.11.** Let \(\mathcal{C}\) be a fusion category and \(L\) be a unitary subalgebra of \(A\). With the above notations one has \(\beta(L) \subseteq \mathcal{C}_{L}^{\text{triv}}\).

**Proof.** Suppose that \(M\) is an object of \(\beta(L)\), i.e. a dyslectic right \(L\)-module. By the previous Proposition the action of \(L\) on \(Z(M)\) satisfies \(\epsilon_M \circ \alpha_{Z(M)} \big|_L = \epsilon_L \otimes \epsilon_M\).

Since \(\epsilon_M : Z(M) \rightarrow M\) is a morphism in \(\mathcal{C}\) it follows that it is also an \(A\)-linear morphism, see [18], and by restriction it is also \(L\)-linear. Therefore we have

\[(\epsilon_M \circ \alpha_{Z(M)} \big|_L = \alpha_M \big|_L \circ (\text{id}_L \otimes \epsilon_M) \]

Then Equation (4.12) can be written as

\[(L \otimes Z(M) \xrightarrow{\epsilon_L \otimes \epsilon_M} M) = (L \otimes Z(M) \xrightarrow{\text{id}_L \otimes \epsilon_M} L \otimes M \xrightarrow{\alpha_M} M).\]

Then [11 Proposition 5.1] applied to \(\mathcal{C}^{\text{op}}\) implies that the counit \(\epsilon_M\) is an epimorphism. Since \(\mathcal{C}\) is semisimple it follows that \(\epsilon_M : Z(M) \rightarrow M\) has a section \(r_M : M \rightarrow Z(M)\) in \(\mathcal{C}\), i.e. a morphism satisfying \(\epsilon_M \circ r_M = \text{id}_M\). Composing Equation (4.13) with \((\text{id}_L \otimes r_M)\) one obtains that

\[(L \otimes M \xrightarrow{\text{id}_L \otimes r_M} L \otimes Z(M) \xrightarrow{\epsilon_L \otimes \epsilon_M} M) = (L \otimes M \xrightarrow{\text{id}_L \otimes r_M} L \otimes Z(M) \xrightarrow{\text{id}_L \otimes \epsilon_M} L \otimes M \xrightarrow{\alpha_M} M)\]

which can be written as

\[(L \otimes M \xrightarrow{\epsilon_M \otimes \epsilon_M} M) = (L \otimes M \xrightarrow{\alpha_M} M).\]

This shows that \(M \in \mathcal{C}_{L}^{\text{triv}}\). \(\square\)
4.2. Proof of Theorem 1.1

Proof. First, we show that $\beta(M) = C^\text{triv}_M = S_M$ for any étale subalgebra $M$ of $A$. Indeed, Proposition 3.63 and Proposition 4.11 imply that $\beta(M) \subseteq C^\text{triv}_M \subseteq S_M$. Moreover by Theorem 3.2 the two fusion subcategories $\beta(M)$ and $S_M$ have the same Frobenius-Perron dimensions, namely $\frac{FPdim(C)}{FPdim(M)}$.

On the other hand for any unitary subalgebra $L$ of $A$ we know that $S_L$ is a fusion subcategory of $C$ with dimension $\frac{FPdim(C)}{FPdim(L)}$. The bijective correspondence $\beta$ from [6] gives that $S_L = \beta(M)$ for some étale subalgebra $L$ of $A$. The first part of the proof implies that $S_L = S_M$ and Corollary 3.60 implies that $M = L$. □

4.3. Proof of Theorem 1.2 Let $C$ be a braided fusion category $F: C \to D$ a tensor functor to an arbitrary fusion category $D$. Recall that a central structure for $F$ is a braided functor $C \xrightarrow{F'} Z(D)$ such that the composition $C \xrightarrow{F'} Z(D) \xrightarrow{\text{Forg}} D$ coincides to $F$. A functor $F$ admitting such a structure is called central. It follows by [6, Lemma 3.5] that $R(1)$ is an étale algebra in $C$ where $R$ is a right adjoint of the central functor $F$.

Proof. By [6, Lemma 3.29] one has that the free functor $F_A: C \to C_A$ is a central functor and moreover, $A = R_A(1)$ where $R_A$ is the right adjoint of $F_A$. Since $C$ is non-degenerate, there is an equivalence of categories $Z(C) \cong C \boxtimes C^\text{rev}$. By [6, Corollary 3.30] there is also an equivalence of categories $Z(C_A) \cong C \boxtimes (C_A^0)^\text{rev}$. Moreover, under this equivalence, the free functor $F_A$ identifies as

$$(F_A: C \to C_A) = (C = C \boxtimes \text{Vec} \xrightarrow{\iota} C \boxtimes (C_A^0)^\text{rev} \cong Z(C_A) \xrightarrow{\text{Forg}} C_A),$$

which shows that $R_A = \iota^a \circ R$ where $R$ is the right adjoint functor of $\text{Forg} : Z(C_A) \to C_A$ and $\iota^a$ is the right adjoint of the inclusion $\iota : C \hookrightarrow C \boxtimes (C_A^0)^\text{rev}$. Then $A = R_A(1) = \iota^a(R(1))$ is a subalgebra of $R(1)$ as being the largest sub-object of $Z(C_A)$ that belongs to $C$. Then any unitary subalgebra of $A$ in $C$ is also a subalgebra of $R(1)$ in $Z(C_A)$ and therefore étale by Theorem 1.1. □

Remark 4.14. It is interesting to investigate if the result of Theorem 1.2 can be deduced by directly investigating the semisimplicity of the corresponding module category $C_M$. However, note that our proof gives a new perspective on the bijective correspondence from [6]. It shows that at least in the pseudo-unitary case, analogue to the group representations situation, fusion subcategories can be obtained via trivial actions or corresponding kernels of objects. Another natural question
at this stage is if the above result holds without the pseudo-unitarity assumption on the category or if one can remove the non-degeneracy assumption in the above statement.

4.4. A diagonal formula for $\lambda_D$. In this subsection we assume that $\mathcal{C}$ is pseudo-unitary. For a unitary subalgebra $L$ of $A$ in $\mathcal{Z}(\mathcal{C})$ we let $\mathcal{D} = \mathcal{S}_L$. Clearly, the algebra inclusion $\iota : L \hookrightarrow A$ is an inclusion in the module category $\mathcal{Z}(\mathcal{C})_L$. Since $L$ is an étale algebra it follows by [6, Theorem 3.2] that $\mathcal{Z}(\mathcal{C})_L$ is a semisimple category. The semisimplicity of this category implies that $\iota$ admits a retract $\pi : L \to A$ in the same category. For the rest of this section we assume that $\pi$ is such a retract.

It is easy to check that $\mathcal{F}_\lambda(\Lambda) = \frac{1}{\dim(\mathcal{C})} \ell_1$. Since $\ell_1 = \sum_{j \in J} \sum_{s \in \mathcal{M}_j} F_{ss}^j$ follows that

$$ \dim(\mathcal{C}) \Lambda = \sum_{j \in J, s \in \mathcal{M}_j} C_{ss}^j. $$

(4.15)

Applying $\pi_\ell^e$ to the Equation (4.15) it follows that

$$ \pi_\ell^e(\Lambda) = \frac{1}{\dim(\mathcal{C})} \left( \sum_{j \in J_L} \sum_{s \in \mathcal{L}_j} C_{ss}^j \right) $$

(4.16)

Lemma 4.17. With the above chosen map $\pi$ one has

$$ \pi_\ell^e(\Lambda) = \frac{1}{\dim(D)} \ell_0. $$

(4.18)

Proof. Since by construction $\pi_\ell^e(\Lambda) := 1 \xrightarrow{\mathcal{A}} A \xrightarrow{\pi} L$ is a morphism of right $L$-modules in $\mathcal{C}$ it follows by Lemma 3.21 that there is $\alpha \in \mathbb{K}$ such that $\pi_\ell^e(\Lambda) = \alpha \ell_0$. Equation (4.16) implies that $\dim(\mathcal{C})\alpha \ell_0 = \sum_{j \in J_L} \sum_{s \in \mathcal{L}_j} C_{ss}^j$. Applying $\ell_1$ to this equation one has $\dim(\mathcal{C})\alpha = \dim(L)$ which implies that $\alpha = \frac{\dim(L)}{\dim(C)} = \frac{1}{\dim(D)}$.

Remark 4.19. Equation (4.16) implies now that

$$ \ell_0 = \dim(D) \pi_\ell^e(\Lambda) = \frac{\dim(D)}{\dim(C)} \left( \sum_{j \in J_L} \sum_{s \in \mathcal{L}_j} C_{ss}^j \right) $$

(4.20)

Proposition 4.21. With the above notations it follows that

$$ \lambda_D = \sum_{j \in J_L} \sum_{s \in \mathcal{L}_j} F_{ss}^j. $$

(4.22)

Proof. Since $\ell_D := \mathcal{F}_\lambda^{-1}(\lambda_D)$ one has by Equation (3.59) that

$$ \ell_D = \sum_{j \in J_L} \sum_{s \in \mathcal{L}_j} C_{ss}^j + \frac{1}{\dim(S_L)} \left( \sum_{j \in J_L} \sum_{s \in \mathcal{L}_j, t \in \mathcal{L}_j} \beta_{st}^j C_{ss}^j \right). $$

(4.23)
Using Lemma 3.47 it follows that
\[(4.24)\quad \pi_1^e(\ell_D) = \sum_{j \in J_L} \sum_{s \in L_j} C_{ss}^j + \frac{1}{\dim(S_L)} \left( \sum_{j \in J_L, s \in L_j, t \notin L_j} \beta_{st}^j C_{st}^j \right).\]

On the other hand \( \ell_0 = \pi_1^e(\ell_0) = \pi_1^e(\sum_{i \in B_0} E_i) = \dim(D) \dim(C) \pi_1^e(\ell_D). \)

Comparing Equations (4.20) and (4.24) it follows that \( \beta_{st}^j = 0 \) for \( s \neq t \). \( \square \)

4.5. On the lattice of fusion subcategories. In this subsection we prove Theorem 1.3. It follows from the next three results presented below. For two subalgebras of the adjoint algebra \( A \) we denote by \( LM \) the image of \( L \otimes M \) under the multiplication \( m: A \otimes A \to A \). Since \( A \) is commutative it follows that \( LM = ML \) and moreover \( ML \) is a subalgebra of \( A \). Note that \( LM \) is the smallest unitary subalgebra of \( A \) containing both \( L \) and \( M \).

**Proposition 4.25.** Let \( C \) be a pseudo unitary fusion category. With the above notations one has
\( S_L \cap S_M = S_{LM} \).

**Proof.** Corollary 3.60 implies that \( S_{LM} \subseteq S_L \cap S_M \) since \( L, M \subseteq A \). On the other hand, if \( V \in S_L \cap S_M \), then \( LM \) acts trivially on \( V \), by the associativity of the action \( \alpha_V \) of \( A \) on \( V \). This shows that \( S_{LM} \supseteq S_L \cap S_M \). \( \square \)

**Lemma 4.26.** Let \( C \) be a pseudo-unitary fusion category and \( L, M \) be two unitary subalgebras of \( A \). Then:
\[(4.27)\quad \FPdim(S_L \cup S_M) \geq \FPdim(S_{LM}).\]

**Proof.** Suppose \( S_L \cup S_M = S_P \) for some subalgebra \( P \subseteq A \). One has \( S_L \subseteq S_P \) which implies that \( P \subseteq L \) since \( \beta \) is a reversing order bijection. Similarly one deduces that \( P \subseteq M \), and therefore \( P \subseteq L \cap M \).

Since \( A \) is a semisimple object, similarly to Corollary 3.60 one can choose a decomposition of \( A \) from equation (3.23) such that
\[ L = \bigoplus_{j \in J_L, s \in L_j^1} C_s^{(j)} \quad M = \bigoplus_{j \in J_M, s \in L_j^2} C_s^{(j)} \quad P = \bigoplus_{j \in J_P, s \in L_j^3} C_s^{(j)}.\]

for some subsets \( L_j^i \subseteq M_j = \{0, 1, \ldots, m_j\} \) for \( i = 1, 2, 3 \). Note also that
\[ L \cap M = \bigoplus_{j \in J_L \cap J_M, s \in L_j^1 \cap L_j^2} C_s^{(j)}.\]

Since \( P \subseteq L \cap M \) it follows that \( J_P \subseteq J_L \cap J_M \) and \( L_j^3 \subseteq L_j^1 \cap L_j^2 \) for any \( j \in J_P \).
By Equation (4.22) one has

\[(4.28)\]
\[
\lambda_{S_L} = \sum_{j \in J_L} \sum_{s \in \mathcal{L}_j^1} F_{ss}^j, \quad \lambda_{S_M} = \sum_{j \in J_M} \sum_{s \in \mathcal{L}_j^2} F_{ss}^j, \quad \lambda_{S_{L \cap M}} = \sum_{j \in J_L \cap J_M} \sum_{s \in \mathcal{L}_j^1 \cap \mathcal{L}_j^2} F_{ss}^j.
\]

Also

\[
\lambda_{S_P} = \sum_{j \in J_P} \sum_{s \in \mathcal{L}_j^3} F_{ss}^j.
\]

By Equation (3.29) and Equation (3.28) one has

\[
\tau(F_{ss}^j) = \frac{1}{n_j} = \frac{\dim(C_j)}{\dim(C)} > 0
\]
since $C$ is pseudo-unitary. It follows that

\[
\tau(\lambda_{S_P}) = \sum_{j \in J_P} \sum_{s \in \mathcal{L}_j^3} \tau(F_{ss}^j) = \sum_{j \in J_P} \frac{|\mathcal{L}_j^3|}{n_j}.
\]

Similarly one deduces that

\[
\tau(\lambda_{S_{L \cap M}}) = \sum_{j \in J_L \cap J_M} \frac{|\mathcal{L}_j^1 \cap \mathcal{L}_j^2|}{n_j}.
\]

Since $J_P \subseteq J_L \cap J_M$ and for any $j \in J_P$ one has $\mathcal{L}_j^3 \subseteq \mathcal{L}_j^1 \cap \mathcal{L}_j^2$ it follows that

\[(4.29)\]
\[
\tau(\lambda_{S_P}) \leq \tau(\lambda_{S_{L \cap M}}).
\]

On the other hand, since $\lambda_{S_P} = \frac{1}{\dim(S_P)}(\sum_{\chi_i \in \text{Irr}(S_P)} d_i \chi_i)$ we have that
\[
\tau(\lambda_{S_P}) = \frac{1}{\dim(S_P)} \text{ since } \tau(\chi_i) = 0 \text{ if } \chi_i \neq \epsilon_1. \text{ Similarly } \tau(\lambda_{S_{L \cap M}}) = \frac{1}{\dim(S_{L \cap M})}.
\]

Then Equation (4.29) can be written as

\[
\frac{1}{\dim(S_{L \cap M})} \geq \frac{1}{\dim(S_P)}.
\]

which proves the lemma. \qed

**Proposition 4.30.** Let $C$ be a pseudo unitary fusion category. With the above notations one has

\[
\mathcal{S}_L \lor \mathcal{S}_M = \mathcal{S}_{L \cap M}
\]

**Proof.** Since $\mathcal{S}_L \subseteq \mathcal{S}_{L \cap M}$ and $\mathcal{S}_M \subseteq \mathcal{S}_{L \cap M}$ it follows that $\mathcal{S}_L \lor \mathcal{S}_M \subseteq \mathcal{S}_{L \cap M}$. The equality follows from the Lemma above. \qed
Proposition 4.31. With the above notations one has

$$\text{FPdim}(LM) \leq \frac{\text{FPdim}(L)\text{FPdim}(M)}{\text{FPdim}(L \cap M)}.$$  

Moreover one has equality if and only if $S_L S_M = S_M S_L$.

Proof. From [12, Lemma 3.2] one has

$$\text{FPdim}(S_L S_M) = \frac{\text{FPdim}(S_L)\text{FPdim}(S_M)}{\text{FPdim}(S_L \cap S_M)}$$

which gives that

$$\text{FPdim}(S_L S_M) = \frac{\text{FPdim}(C)\text{FPdim}(LM)}{\text{FPdim}(L)\text{FPdim}(M)}$$

Since $S_L S_M \subseteq S_L \lor S_M$, one has $\text{FPdim}(S_L S_M) \leq \text{FPdim}(S_L \lor S_M)$. Note that

$$\text{FPdim}(S_L \lor S_M) = \text{FPdim}(S_L \cap M) = \frac{\text{FPdim}(C)}{\text{FPdim}(L \cap M)}.$$  

Thus

$$\frac{\text{FPdim}(C)}{\text{FPdim}(L \cap M)} \geq \frac{\text{FPdim}(C)\text{FPdim}(LM)}{\text{FPdim}(L)\text{FPdim}(M)}$$

which gives the above inequality. One has equality above if and only if $S_L S_M = S_L \lor S_M$ which is equivalent to $S_L S_M = S_M S_L$.

□

Corollary 4.32. If $C$ is a pseudo-unitary fusion category with a commutative Grothendieck ring then: One has that

$$\text{FPdim}(LM) = \frac{\text{FPdim}(L)\text{FPdim}(M)}{\text{FPdim}(L \cap M)}$$

for any tow uniray subalgebras $L$ and $M$ of $A$.

Proof. It is like Maschke’s theorem: □

4.6. Examples from semisimple Hopf algebras. Let $H$ be a semisimple Hopf algebra and let $C = \text{Rep}(H)$ be the fusion category of its finite dimensional representations over $\mathbb{k}$. As in [13, Section 3.7] we identify the left monoidal center of $C$ with the category $\mathcal{Z}(C) \simeq_{H}^{H} \mathcal{YD}$ of left-left Yetter-Drinfeld modules of $H$. The objects of the former category are pairs $(V, \delta)$ where $V$ is a left $H$-module and $\delta : V \to H \otimes V$ defines a left $H$-comodules structure on $V$ such that

$$(hv)_{-1} \otimes (hv)_{0} = h_{1}v_{-1}S(h_{3}) \otimes h_{2}v_{0}$$

for any $h \in H$ and any $v \in V$. A morphism in the category $\mathcal{YD}$ is a linear map which is simultaneously a morphism of left $H$-modules.
and of left $H$-comodules. The left half-braiding of $V \in \mathcal{YD}_H$ can be written as

$$c_{V,M} : V \otimes M \to M \otimes V, \ v \otimes m \mapsto v_{-1}m \otimes v_0$$

for any object $M \in \text{Rep}(H)$. The forgetful functor

$$F : \mathcal{YD}_H \to \mathcal{H} \text{-mod}$$

admits a right adjoint

$$R : \mathcal{H} \text{-mod} \to \mathcal{YD}_H$$

which can be written as $R(M) = H \otimes M$ with

$$h.(l \otimes m) = h_1 l S(h_3) \otimes h_2 m, \ \delta(h \otimes m) = h_1 \otimes (h_2 \otimes m).$$

Thus, with the above notations, the Hopf comonad $Z := FR$ can be written as $Z(M) = F(R(M)) = H \otimes M$ with the left $H$-module structure defined above. Then the left half-braiding of $R(M)$ becomes:

$$c_{Z(M),X} : Z(M) \otimes X \to X \otimes Z(M), \ (h \otimes m) \otimes x \mapsto h_1 x \otimes (h_2 \otimes m)$$

for any $X \in \text{Rep}(H)$. In particular one has that $A := R(1)$ is the pair $(H_{\text{ad}}, \Delta)$ where $H_{\text{ad}}$ coincides to $H$ as vector spaces, and the structure as left $H$-module on $H_{\text{ad}}$ is given by the left adjoint action $h.a = h_1 a S(h_2)$. Moreover the left comodule structure on $H_{\text{ad}}$ is given by the comultiplication $\Delta : H_{\text{ad}} \to H \otimes H_{\text{ad}}$. The multiplication and the unit of $R(1)$ are given the usual multiplication and unit of $H$. The universal dinatural projections from Subsection 2.1 are given by:

$$\pi_{1;M} : H_{\text{ad}} \to M \otimes M^*, \ h \mapsto \sum_i hm_i \otimes m_i^*,$$

where $m_i$ is a $k$-linear basis on $M$ and $\{m_i^*\}$ is the corresponding dual basis in $M^*$. In this case the natural action of $A = H_{\text{ad}}$ from Equation (2.7) can be written as $\alpha_M : A \otimes M \to M, \alpha_M(a \otimes m) = a.m,$ for each object $M \in \text{Rep}(H)$. Moreover, by considering the standard pivotal structure on $\text{Rep}(H)$, it follows that $\text{CF}(\mathcal{C}) = C(H)$, the character ring of $H$, and $\text{CE}(\mathcal{C}) = Z(H)$, the center of $H$.

In this case, unitary subalgebras of $A \in Z(\mathcal{C})$ are in bijection with the left normal coideal subalgebras of $H$. Recall that $L \subseteq H$ is a left normal coideal subalgebra of $H$ if it is a subalgebra of $H$ with $\Delta(L) \subseteq H \otimes L$ and $L$ is closed under the left adjoint action.

Moreover, for a semisimple Hopf algebra $H$ it follows that $\beta(L) = S_L = C_L^{\text{triv}} = \text{Rep}(H/L)$ is the fusion subcategory of $\text{Rep}(H)$ consisting of those $H$-modules receiving trivial action from $L$. The above equalities in this case were previously shown in [4].
The character space \( \text{CF}(L) := \text{Hom}_H(L, 1) \) defined in Section 3 can be described by \( \mu \in \text{CF}(L) \iff \mu(h_1 l S(h_2)) = \epsilon(h) \mu(l) \), for all \( h \in H \) and \( l \in L \). Note that there is an inclusion of

\[
\text{CF}(L) := \text{Hom}_H(L, 1) \hookrightarrow \text{Hom}_L(L, 1) = C(L),
\]

into the character space (trace space) \( C(L) \) of left \( L \)-modules.

Denote by \( \text{Res}_A^L : H \mod \rightarrow L \mod \) the restriction functor of \( H \)-modules to the subalgebra \( L \). This functor induces a map \( (\text{Res}_A^L)_* : \text{CF}(C) \rightarrow C(L) \) at the level of characters. It is easy to see that the image of this map is \( \text{CF}(L) \hookrightarrow C(L) \).

Note also that the central space \( \text{CE}(L) \) becomes in this case \( \text{CE}(L) = \text{Hom}_H(1, L) = L \cap Z(H) \) and the inclusion \( \iota^e : \text{CE}(L) \rightarrow \text{CE}(C) \) is the usual space inclusion \( L \cap Z(H) \hookrightarrow Z(H) \). Note also that in this case \( \ell_D \) is a multiple of the integral \( \Lambda_L \) defined in [3].

References

[1] A. Bruguières and S. Natale. Exact sequences of tensor categories. *International Mathematics Research Notices*, 24:5644–5705, 2011.

[2] A. Bruguières and A. Virelizier. Quantum double of Hopf monads and categorical centers. *Transactions of the American Mathematical Society*, 364(3):1225–1279, 2012.

[3] S. Burciu. Kernels of representations and coideal subalgebras of Hopf algebras. *Glasgow Mathematical Journal*, 54:107–119, 2012.

[4] S. Burciu. Normal coideal subalgebras of semisimple Hopf algebras. *Journal of Physics: Conference Series*, page 012004, 2012.

[5] S. Burciu. Conjugacy classes and centralizers for pivotal fusion categories. *Monatshefte für Mathematik*, 193(2):13–46, 2020.

[6] A. Davydov, M. Müger, D. Nikshych, and V. Ostrik. The Witt group of non-degenerate braided fusion categories. *Journal für die reine und angewandte Mathematik*, 677:135–177, 2013.

[7] A. Davydov, D. Nikshych, and V. Ostrik. On the structure of the Witt group of braided fusion categories. *Selecta Math. (N.S.)*, 19(1):237–269, 2013.

[8] B. Day and R. Street. Centres of monoidal categories of functors in algebra, geometry and mathematical physics. *Contemp. Math. Amer. Math. Soc., Providence, RI*, 431(2):187–202, 2007.

[9] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. Tensor categories, volume 205. *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, 2015.

[10] P. Etingof, D. Nikshych, and V. Ostrik. On fusion categories. *Annals of Mathematics*, 162:581–642, 2005.

[11] P. Etingof and V. Ostrik. Finite tensor categories. *Moscow Mathematical Journal*, 4(3):627–654, 2004.

[12] S. Gelaki. Exact factorizations and extensions of fusion categories. *J. Algebra*, 480, 2017.

[13] C. Kassel. Quantum groups. *Graduate Texts in Mathematics*, Springer Verlag, 1995.
[14] S. Mac Lane. Categories for the working mathematician, volume 5. Graduate Texts in Mathematics, second edition, Springer-Verlag, New York, 1998.

[15] B. Pareigis. On braiding and Dyslexia. J. Alg., 171(2):413–425, 1995.

[16] S. Sakalos. On categories associated to a Quasi-Hopf algebra. Communications in Algebra, 45(2):722–748, 2017.

[17] K. Shimizu. On unimodular finite tensor categories. Internat. Math. Res. Not., 2017(1):277–322, 2017.

[18] K. Shimizu. The monoidal center and the character algebra. Journal of Pure and Applied Algebra, 221(9):2338–2371, 2017.