LONG-TIME DYNAMICS OF 2D DOUBLE-DIFFUSIVE CONVECTION: ANALYSIS AND/OOF NUMERICS

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ABSTRACT. We consider a two-dimensional model of double-diffusive convection and its time discretisation using a second-order scheme which treat the nonlinear term explicitly (backward differentiation formula with a one-leg method). Uniform bounds on the solutions of both the continuous and discrete models are derived (under a timestep restriction for the discrete model), proving the existence of attractors and invariant measures supported on them. As a consequence, the convergence of the attractors and long time statistical properties of the discrete model to those of the continuous one in the limit of vanishing timestep can be obtained following established methods.

1. Introduction

The phenomenon of double-diffusive convection, in which two properties of a fluid are transported by the same velocity field but diffused at different rates, often occur in nature [12]. Perhaps the best known example is the transport throughout the world’s oceans of heat and salinity, which has been recognised as an essential part of climate dynamics [17,23]. In contrast to simple convections (cf. [2]), double-diffusive convections support a richer set of physical regimes, e.g., a stably stratified initial state rendered unstable by diffusive effects. Although in this paper we shall be referring to the oceanographic case, the mathematical theory is essentially identical for astrophysical [15,18] and industrial [3] applications.

In this paper, we consider a two-dimensional double-diffusive convection model, which by now-standard techniques [20] can be proved to have a global attractor and invariant measures supported on it, and its temporal discretisation. We use a backward differentiation formula for the time derivative and a fully explicit one-leg method [8] for the nonlinearities, resulting in an accurate and efficient numerical scheme. Of central interest, here and in many practical applications, is the ability of the discretised model to capture long-time behaviours of the underlying PDE. This motivates the main aim of this article: to obtain bounds necessary for the convergence of the attractor and associated invariant measures of the discretised system to those of the continuous system. We do this using the framework laid down in [21,22], with necessary modifications for our more complex model.

For motivational concreteness, one could think of our system as a model for the zonally-averaged thermohaline circulation in the world’s oceans. Here the physical axes correspond to latitude and altitude, and the fluid is sea water whose internal motion is largely driven by density differentials generated by the temperature $T$ and salinity $S$, as well as by direct wind forcing on the surface. Both $T$ and $S$ are also

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driven from the boundary—by precipitation/evaporation and ice melting/formation for the salinity, and by the associated latent heat release and direct heating/cooling for the temperature. Physically, one expects the boundary forcing for $T$, $S$ and the momentum to have zonal (latitude-dependent) structure, so we include these in our model. Furthermore, one may also wish to impose a quasi-periodic time dependence on the forcing; although this is eminently possible, we do not do so in this paper to avoid technicalities arising from time-dependent attractors.

Taking as our domain $D_* = [0, L_*] \times [0, H_*]$ which is periodic in the horizontal direction, we consider a temperature field $T_*$ and a salinity field $S_*$, both transported by a velocity field $v_* = (u_*, w_*)$ which is incompressible, $\nabla_* \cdot v_* = 0$, and diffused at rates $\kappa_T$ and $\kappa_S$, respectively,

\begin{align}
\partial T_*/\partial t_* + v_* \cdot \nabla_* T_* &= \kappa_T \Delta_* T_* \\
\partial S_*/\partial t_* + v_* \cdot \nabla_* S_* &= \kappa_S \Delta_* S_*.
\end{align}

Here the star denotes dimensional variables. Taking the Boussinesq approximation and assuming that the density is a linear function of $T_*$ and $S_*$, which is a good approximation for sea water (although not for fresh water near the freezing point), the velocity field evolves according to

\begin{align}
\partial v_*/\partial t_* + v_* \cdot \nabla_* v_* + \nabla_* p_* &= \kappa_v \Delta_* v_* + (\alpha_T T_* - \alpha_S S_*) e_z
\end{align}

for some positive constants $\alpha_T$ and $\alpha_S$.

Our system is driven from the boundary by the heat and salinity fluxes (which could be seen to arise from direct contact with air and latent heat release in the case of heat, and from precipitation, evaporation and ice formation/melt in the case of salinity),

\begin{align}
\partial T_*/\partial n_* = Q_{T*} \quad \text{and} \quad \partial S_*/\partial n_* = Q_{S*}.
\end{align}

Here $n_*$ denotes the outward normal, $n_* = z_*$ at the top boundary and $n_* = -z_*$ at the bottom boundary. We also prescribe a wind-stress forcing,

\begin{align}
\partial u_*/\partial n_* = Q_{u*}
\end{align}

along with the usual no-flux condition $w_* = 0$ on $z_* = 0$ and $z_* = H_*$. 

Largely following standard practice, we cast our system in non-dimensional form as follows. Using the scales $\tilde{t}$, $\tilde{l}$, $\tilde{T}$ and $\tilde{S}$, we define the non-dimensional variables $t = t_* / \tilde{t}$, $x = x_* / \tilde{l}$, $v = v_* \tilde{l} / \tilde{t}$, $T = T_* / \tilde{T}$ and $S = S_* / \tilde{S}$, in terms of which our system reads

\begin{align}
\tilde{p}^{-1} (\partial_t v + v \cdot \nabla v) &= -\nabla p + \Delta v + (T - S)e_z \\
\partial_t T + v \cdot \nabla T &= \Delta T \\
\partial_t S + v \cdot \nabla S &= \beta \Delta S.
\end{align}

To arrive at this, we have put $\tilde{l} = H_*$ and taken the thermal diffusive timescale for

\begin{align}
\tilde{t} = l^2 / \kappa_T,
\end{align}

as well as scaled the dependent variables as

\begin{align}
\tilde{T} = p \tilde{t} / (\alpha_T \tilde{l}^2) \quad \text{and} \quad \tilde{S} = p \tilde{t} / (\alpha_S \tilde{l}^2),
\end{align}
where the non-dimensional Prandtl number and diffusivity ratio (also known as the Lewis number in the engineering literature) are
\begin{equation}
\text{Pr} = \frac{\kappa_v}{\kappa_T} \quad \text{and} \quad \beta = \frac{\kappa_T}{\kappa_S}.
\end{equation}

Another non-dimensional quantity is the domain aspect ratio \( \xi = L_x / \bar{L} \). The surface fluxes are non-dimensionalised in the natural way: \( Q_T = \rho Q_{T*} / (\alpha T \bar{T}^2) \), \( Q_S = \rho Q_{S*} / (\alpha S \bar{T}^2) \) and \( Q_u = Q_{u*} \bar{L} \).

For clarity and convenience, keeping in mind the oceanographic application, we assume that the fluxes vanish on the bottom boundary \( z = 0 \),
\begin{equation}
Q_u(x, 0) = Q_T(x, 0) = Q_S(x, 0) = 0.
\end{equation}

For boundedness of the solution in time, the net fluxes must vanish, so (1.9) then implies that the net fluxes vanish on the top boundary \( z = 1 \),
\begin{equation}
\int_0^\xi Q_u(x, 1) \, dx = \int_0^\xi Q_T(x, 1) \, dx = \int_0^\xi Q_S(x, 1) \, dx = 0.
\end{equation}

These boundary conditions can be seen to imply that the horizontal velocity flux is constant in time, which we take to be zero, viz.,
\begin{equation}
\int_0^1 u(x, z, t) \, dz = \int_0^1 u(x, z, 0) \, dz \equiv 0 \quad \text{for all } x \in [0, \xi].
\end{equation}

For some applications (e.g., the classical Rayleigh–Bénard problem), the fluxes on the bottom boundary may not vanish, which must then be balanced by the fluxes on the top boundary,
\begin{equation}
\int_0^\xi [Q_T(x, 1) - Q_T(x, 0)] \, dx = 0
\end{equation}
and similarly for \( Q_u \) and \( Q_S \). With some modifications (by subtracting background profiles from \( u, T \) and \( S \)), the analysis of this paper also apply to this more general case. This involves minimal conceptual difficulty but adds to the clutter, so we do not treat this explicitly here.

Defining the vorticity \( \omega := \partial_x w - \partial_z u \), the streamfunction \( \psi \) by \( \Delta \psi = \omega \) with \( \psi = 0 \) on \( \partial \mathcal{D} \) (this is consistent with (1.11)), and the Jacobian \( \partial(f, g) := \partial_x f \partial_z g - \partial_x g \partial_z f = -\partial(g, f) \), our system reads
\begin{equation}
\begin{align*}
p^{-1} \{ \partial_\xi \omega + \partial(\psi, \omega) \} &= \Delta \omega + \partial_x T - \partial_z S \\
\partial_t T + \partial(\psi, T) &= \Delta T \\
\partial_t S + \partial(\psi, S) &= \beta \Delta S.
\end{align*}
\end{equation}

The boundary conditions are
\begin{equation}
\partial_z T = Q_T, \quad \partial_z S = Q_S, \quad \omega = Q_u \quad \text{and} \quad \psi = 0 \quad \text{on } \partial \mathcal{D}.
\end{equation}

In the rest of this paper, we will be working with (1.13)–(1.14) and its discretisation. We assume that \( \omega, T \) and \( S \) all have zero integral over \( \mathcal{D} \) at \( t = 0 \). Thanks to the no-net-flux condition (1.10), this persists for all \( t \geq 0 \).

Another dimensionless parameter often considered in studies of (single-species) convection is the Rayleigh number \( Ra \). When the top and bottom temperatures are held at fixed values \( T_1 \) and \( T_0 \), \( Ra \) is proportional to \( T_0 - T_1 \). The relevant parameters in our problem would be \( Ra_T \propto |Q_T|_{L^2(\partial \mathcal{D})} \) and \( Ra_S \propto |Q_S|_{L^2(\partial \mathcal{D})} \), but we will not consider them explicitly here; see, e.g., (2.11) in [1]. For notational
We do not provide details on the convergence of the global attractors and long time statistical properties. Such kind of convergence can be obtained by following established methods once we have the uniform estimates derived here. See [9] for the convergence of the global attractors and [22] for the convergence of long time statistical properties.

The rest of this paper is structured as follows. In section 2 we review briefly the properties of the continuous system, setting up the scene and the notation for its discretisation. Next, we describe the time discrete system and derive uniform bounds for the solution. In the appendix, we present an alternate derivation of its discretisation. Next, we describe the time discrete system and derive uniform bounds for the solution. In the appendix, we present an alternate derivation of the boundedness results in [22], without using Wente-type estimates but requiring slightly more regular initial data.

2. Properties of the continuous system

In this section, we obtain uniform bounds on the solution of our system and use them to prove the existence of a global attractor $\mathcal{A}$. For the single diffusion case (of $T$ only, without $S$), this problem has been treated in [5] which we follow in spirit, though not in detail in order to be closer to our treatment of the discrete case.

We start by noting that the zero-integral conditions on $\omega$, $T$ and $S$ imply the Poincaré inequalities
\begin{equation}
|\omega|_{L^2(D)}^2 \leq c_0 |\nabla \omega|_{L^2(D)}^2,
\end{equation}
as well as the equivalence of the norms
\begin{equation}
|\omega|_{H^1(D)} \leq c |\nabla \omega|_{L^2(D)},
\end{equation}
with analogous inequalities for $T$ and $S$. The boundary condition $\psi = 0$ implies that (2.1)–(2.2) also hold for $\psi$, while an elliptic regularity estimate [6, Cor. 8.7] implies that
\begin{equation}
|\nabla \psi|_{L^2(D)}^2 \leq c_0 |\omega|_{L^2(D)}^2.
\end{equation}
Following the argument in [7], this also holds for functions, such as our $T$ and $S$, with zero integrals in $D$.

Let $\Omega$ be an $H^2$ extension of $Q_u$ to $\hat{D}$ (further requirements will be imposed below) and let $\hat{\omega} := \omega - \Omega$; we also define $\Delta \hat{\psi} := \hat{\psi}$ and $\Delta \hat{\Omega} := \Omega$ with homogeneous boundary conditions. Now $\hat{\omega}$ satisfies the homogeneous boundary conditions $\hat{\omega} = 0$ on $\partial \hat{D}$, and thus the Poincaré inequality (2.1)–(2.2). Furthermore, let $T_Q \in \hat{H}^2(D)$ be such that $\partial_z T_Q = Q_T$ on $\partial D$ (with other constraints to be imposed below) and let $\hat{T} := T - T_Q$; analogously for $S_Q$ and $\hat{S} := S - S_Q$. We note that since both $\hat{T}$ and $\hat{S}$ have zero integrals over $D$, they satisfy the Poincaré inequality (2.1)–(2.2).

We start with weak solutions of (1.13). For conciseness, unadorned norms and inner products are understood to be $L^2(D)$, $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(D)}$. With $\hat{\omega}$, $\hat{T}$ and $\hat{S}$ as defined above, we have
\begin{equation}
\begin{align*}
\partial_t \hat{\omega} + \partial_t (\hat{\psi} + \hat{\omega}) &= \partial_t \hat{\Omega} + \partial_t (\hat{\psi} + \hat{\omega}) \\
\partial_t \hat{T} + \partial_t (\hat{\psi} + \hat{T}) &= \partial_t (\hat{\psi} + \hat{T}) \\
\partial_t \hat{S} + \partial_t (\hat{\psi} + \hat{S}) &= \partial_t (\hat{\psi} + \hat{S})
\end{align*}
\end{equation}
On a fixed time interval $[0, T_*)$, a weak solution of (2.4) are
\[ \dot{\omega} \in C^0(0, T_*; L^2(\mathcal{D})) \cap L^2(0, T_*; H^1_0(\mathcal{D})) \]
(2.5) \[ \dot{T} \in C^0(0, T_*; L^2(\mathcal{D})) \cap L^2(0, T_*; H^1(\mathcal{D})) \]
\[ \dot{S} \in C^0(0, T_*; L^2(\mathcal{D})) \cap L^2(0, T_*; H^1(\mathcal{D})) \]
such that, for all $\dot{\omega} \in H^1_0(\mathcal{D})$, $\dot{T}, \dot{S} \in H^1(\mathcal{D})$, the following holds in the distributional sense,
\[
\frac{d}{dt}(\dot{\omega}, \hat{\omega}) + (\partial(\Psi + \hat{\psi}, \Omega + \hat{\omega}), \hat{\omega}) + p\{(\nabla \Omega + \nabla \hat{\omega}, \nabla \hat{\omega}) - (\partial_x T_Q + \partial_z \hat{T}, \hat{\omega}) + (\partial_x S_Q + \partial_z \hat{S}, \hat{\omega})\} = 0
\]
(2.6) \[
\frac{d}{dt}(\dot{T}, \hat{T}) + (\partial(\Psi + \hat{\psi}, T_Q + \hat{T}), \hat{T}) + (\nabla \hat{T}, \nabla \hat{T}) - (\Delta T_Q, \hat{T}) = 0
\]
\[
\frac{d}{dt}(\dot{S}, \hat{S}) + (\partial(\Psi + \hat{\psi}, S_Q + \hat{S}), \hat{S}) + \beta(\nabla \hat{S}, \nabla \hat{S}) - \beta(\Delta S_Q, \hat{S}) = 0.
\]
The existence of such solutions can be obtained by standard methods, so we do not do so explicitly here.

Next, we derive $L^2$ inequalities for $T$, $S$ and $\omega$. Multiplying (2.4) by $\dot{\omega}$ in $L^2(\mathcal{D})$ and noting that $(\partial(\psi, \hat{\omega}), \hat{\omega}) = 0$, we find
\[
\frac{1}{2} \frac{d}{dt}|\dot{\omega}|^2 + p|\nabla \dot{\omega}|^2 = -(\partial(\Psi, \Omega), \dot{\omega}) - (\partial(\hat{\psi}, \Omega), \dot{\omega}) + p\{(\Delta \Omega, \dot{\omega}) + (\partial_x T, \dot{\omega}) - (\partial_x S, \dot{\omega})\}.
\]
We bound the rhs as
\[
|\Delta \Omega, \dot{\omega}| = |\nabla \Omega| |\nabla \dot{\omega}| \leq \frac{1}{2} |\nabla \dot{\omega}|^2 + 2 |\nabla \Omega|^2
\]
\[
|\partial_x T, \dot{\omega}| = |\partial_x \omega| |T| \leq \frac{1}{8} |\nabla \dot{\omega}|^2 + 2 |T|^2 \leq \frac{1}{8} |\nabla \dot{\omega}|^2 + 4c_0|\nabla \dot{T}|^2 + 4 |T_Q|^2
\]
\[
|\partial_x S, \dot{\omega}| = |\partial_x \omega| |S| \leq \frac{1}{8} |\nabla \dot{\omega}|^2 + 2 |S|^2 \leq \frac{1}{8} |\nabla \dot{\omega}|^2 + 4c_0|\nabla \dot{S}|^2 + 4 |S_Q|^2.
\]
and the “nonlinear” terms as
\[
|\partial(\psi, \hat{\omega}), \Omega| \leq c|\nabla \psi|_{L^\infty} |\nabla \dot{\omega}|_{L^2} |\Omega|_{L^2} \leq \frac{c_1}{2} |\Omega|_{L^2} |\nabla \dot{\omega}|^2
\]
(2.8) \[
|\partial(\Omega, \hat{\omega}), \dot{\omega}| \leq c|\nabla \psi|_{L^\infty} |\nabla \dot{\omega}|_{L^2} |\Omega|_{L^2} \leq \frac{p}{8} |\nabla \dot{\omega}|^2 + \frac{c}{p} |\nabla \psi|_{L^\infty}^2 |\Omega|^2.
\]
This brings us to
\[
\frac{d}{dt}|\dot{\omega}|^2 + (p - c_1 |\Omega||\nabla \dot{\omega}|^2 \leq 4pc_0(|\nabla \dot{T}|^2 + |\nabla \dot{S}|^2)
\]
(2.9) \[ + \frac{c}{p} |\nabla \psi|_{L^\infty}^2 |\Omega|^2 + 4p(|\nabla \Omega|^2 + |T_Q|^2 + |S_Q|^2).
\]

Now for $\dot{S}$, we multiply (1.13), or equivalently,
(2.10) \[
\partial_t \dot{S} + \partial(\psi, \dot{S} + S_Q) = \beta(\Delta \dot{S} + \Delta S_Q),
\]
by $\dot{S}$ in $L^2(\mathcal{D})$ and use $(\partial(\psi, \dot{S}), \dot{S}) = 0$ to find
\[
\frac{1}{2} \frac{d}{dt}|\dot{S}|^2 + \beta|\nabla \dot{S}|^2 = -(\partial(\Psi, S_Q), \dot{S}) - (\partial(\hat{\psi}, S_Q), \dot{S}) + \beta(\Delta S_Q, \dot{S}).
\]
The last term on the rhs requires some care,
\begin{equation}
|\Delta S_Q| \leq |(Q, \hat S)|_{L^2(\partial D)} + |(\nabla S_Q, \nabla \hat S)|
\leq c |Q|_{H^{-1/2}(\partial D)} |\hat S|_{H^{1/2}(\partial D)} + |\nabla Q| \cdot |\nabla \hat S|
\leq \frac{1}{\beta} |\nabla \hat S|^2 + c (|\nabla S_Q|^2 + |Q|^2_{H^{-1/2}(\partial D)})
\leq \frac{1}{\beta} |\nabla \hat S|^2 + c |\nabla S_Q|^2
\end{equation}
where we have used the trace theorem for the second and last inequalities. We note that $|\nabla S_Q|_{L^2(\partial D)}$ ultimately depends on $|Q|_{H^{-1/2}(\partial D)}$ plus the constraint (2.17) below. Bounding the “nonlinear” terms as
\begin{equation}
|\langle \partial (\Psi, \hat S), S_Q \rangle| \leq c |\nabla \Psi|_{L^\infty} |\nabla \hat S|_{L^2} |S_Q|_{L^2} \leq \frac{\beta}{8} |\nabla \hat S|^2 + \frac{c}{\beta} |\nabla \Psi|^2_{L^\infty} |S_Q|^2
\end{equation}
\begin{equation}
|\langle \partial (\hat \psi, \hat S), S_Q \rangle| \leq c |\nabla \hat \psi|_{L^\infty} |\nabla \hat S|_{L^2} |S_Q|_{L^2} \leq \frac{\beta}{8} |\nabla \hat S|^2 + \frac{c}{\beta} |\nabla \hat \psi|^2_{L^\infty} |S_Q|^2,
\end{equation}
we arrive at
\begin{equation}
\frac{d}{dt} |\hat S|^2 + \beta |\nabla \hat S|^2 \leq \frac{c}{\beta} |\nabla \hat \psi|^2_{L^\infty} |S_Q|^2 + \frac{c}{\beta} |\nabla \Psi|^2_{L^\infty} |S_Q|^2 + c |\nabla S_Q|^2.
\end{equation}
Analogously, we have for $\hat T$,
\begin{equation}
\frac{d}{dt} |\hat T|^2 + |\nabla \hat T|^2 \leq c |\nabla \hat \psi|^2_{L^\infty} |T_Q|^2 + c |\nabla \Psi|^2_{L^\infty} |T_Q|^2 + c |\nabla T_Q|^2.
\end{equation}

Adding $8pc_0$ times (2.14) and $8pc_0/\beta$ times (2.17) to (2.9), we find
\begin{equation}
\frac{d}{dt} \left( |\hat \omega|^2 + 8pc_0 |\hat T|^2 + \frac{8pc_0}{\beta} |\hat S|^2 \right) + 4pc_0 (|\nabla \hat T|^2 + |\nabla \hat S|^2)
\leq cp |\nabla \Psi|^2_{L^\infty} \left( \frac{|\Omega|^2}{p^2} + |T_Q|^2 + |S_Q|^2/\beta^2 \right)
+ cp \left( |\nabla \Omega|^2 + |\nabla T_Q|^2 + |\nabla S_Q|^2 \right).
\end{equation}

If we now choose $\Omega$, $T_Q$ and $S_Q$ such that
\begin{equation}
|\Omega|_{L^2} \leq p/(8c_1), \ |T_Q|^2_{L^2} \leq 1/(8c_2) \text{ and } |S_Q|^2_{L^2} \leq \beta^2/(8c_2),
\end{equation}
(given the BC (1.14), this can always be done at the price of making $\nabla \Omega$, $\nabla T_Q$ and $\nabla S_Q$ large) we obtain the differential inequality
\begin{equation}
\frac{d}{dt} \left( |\hat \omega|^2 + 8pc_0 |\hat T|^2 + \frac{8pc_0}{\beta} |\hat S|^2 \right) + \frac{p}{2} |\nabla \hat \omega|^2 + 4pc_0 (|\nabla \hat T|^2 + |\nabla \hat S|^2) \leq ||F||^2,
\end{equation}
with $||F||^2$ denoting the purely “forcing” terms on the rhs of (2.16). Integrating this using the Gronwall lemma, we obtain the uniform bounds, with $|\hat U|^2 = |\hat \omega|^2 + 8pc_0 |\hat T|^2 + 8pc_0 |\hat S|^2/\beta$,
\begin{equation}
|\hat U(t)|^2 \leq e^{-\lambda t} |\hat U(0)|^2 + ||F||^2 / \lambda
\end{equation}
valid for all $t \geq 0$, for some $\lambda(\pi) > 0$. It is clear from (2.19) that we have an absorbing ball, i.e. $|U(t)|^2 \leq M_0(Q; \pi)$ for all $t \geq t_0(|U(0)|; \pi)$. 

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On to $H^1$, we multiply (2.24) by $-\Delta \hat{\omega}$ in $L^2$ to find

\[ \frac{1}{2} \frac{d}{dt} |\nabla \hat{\omega}|^2 + p |\Delta \hat{\omega}|^2 = - (\partial(\nabla \psi, \hat{\omega}), \nabla \hat{\omega}) + (\partial(\psi, \Omega), \Delta \hat{\omega}) \]

\[ - p (\Delta \Omega, \Delta \hat{\omega}) - p (\partial_x T, \Delta \hat{\omega}) + p (\partial_x S, \Delta \hat{\omega}). \]

Bounding the linear terms in the obvious way, and the nonlinear terms as

\[ |(\partial(\nabla \psi, \hat{\omega}), \nabla \hat{\omega})| \leq c |\nabla \hat{\omega}|^2 |\nabla^2 \psi|_{L^2} \leq c |\nabla \hat{\omega}| |\Delta \hat{\omega}| |\Delta \psi| \]

\[ \leq \frac{p}{\delta} |\Delta \hat{\omega}|^2 + \frac{c}{p} |\nabla \hat{\omega}|^2 (|\hat{\omega}|^2 + |\Omega|^2) \]

\[ |(\partial(\psi, \Omega), \Delta \hat{\omega})| \leq \frac{p}{\delta} |\Delta \hat{\omega}|^2 + \frac{c}{p} |\nabla \Omega|^2 (|\nabla \hat{\omega}|^2 + |\nabla \Omega|^2) \],

we find

\[ \frac{d}{dt} |\nabla \hat{\omega}|^2 + p |\Delta \hat{\omega}|^2 \leq \frac{c}{p} |\nabla \hat{\omega}|^2 (|\hat{\omega}|^2 + |\Omega|^2) + \frac{c}{p} |\nabla \psi|^2_{L^2} |\nabla \Omega|^2 + 8p (|\nabla T|^2 + |\nabla S|^2 + |\nabla S_Q|^2 + |\Delta \Omega|^2). \]

Since $\hat{\omega}$, $\hat{T}$ and $\hat{S}$ have been bounded uniformly in $L^2_t H^1_x$ in (2.19b), we can integrate (2.21) using the uniform Gronwall lemma to obtain a uniform bound for $|\nabla \hat{\omega}|^2$.

\[ |\nabla \hat{\omega}(t)|^2 \leq M_1(\cdots) \quad \text{and} \quad \int_t^{t+1} |\Delta \hat{\omega}(t')|^2 \, dt' \leq \tilde{M}_1(\cdots). \]

Similarly, multiplying (2.19) by $-\Delta \hat{S}$ in $L^2$, we find

\[ \frac{1}{2} \frac{d}{dt} |\nabla \hat{S}|^2 + \beta |\Delta \hat{S}|^2 = - \beta (\Delta S_Q, \Delta \hat{S}) \]

\[ - (\partial(\nabla \psi, \hat{S}), \nabla \hat{S}) + (\partial(\psi, S_Q), \Delta \hat{S}). \]

Bounding as we did for $\hat{\omega}$, we arrive at

\[ \frac{d}{dt} |\nabla \hat{S}|^2 + \beta |\Delta \hat{S}|^2 \leq 8\beta |\Delta S_Q|^2 \]

\[ + \frac{c}{\beta} |\nabla \hat{S}|^2 (|\hat{\omega}|^2 + |\Omega|^2) + \frac{c}{\beta} |\nabla S_Q|^2 (|\nabla \hat{\omega}|^2 + |\nabla \Omega|^2) \],

which can be integrated using the uniform Gronwall lemma to obtain

\[ |\nabla \hat{S}(t)|^2 \leq M_1(\cdots) \quad \text{and} \quad \int_t^{t+1} |\Delta \hat{S}(t')|^2 \, dt' \leq \tilde{M}_1(\cdots). \]

Obviously one has the analogous bound for $\hat{T}$,

\[ |\nabla \hat{T}(t)|^2 \leq M_1(\cdots) \quad \text{and} \quad \int_t^{t+1} |\Delta \hat{T}(t')|^2 \, dt' \leq \tilde{M}_1(\cdots). \]

These bounds allow us to conclude (2.20) the existence of a global attractor $\mathcal{A}$ and of an invariant measure $\mu$ supported on $\mathcal{A}$. Given a continuous functional $\Phi$, its long-time average satisfies

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \Phi(\hat{S}(U_0)) \, dt = \int_H \Phi(U) \, d\mu(U). \]
where $U(t) = S(t)U_0$ is the solution of (1.13) with initial data $U_0$. It is known that $\mathcal{A}$ is unique while $\mu$ may depend on the initial data $U_0$ and the definition of the generalised limit $\lim$.

Due to the boundary conditions, one cannot simply multiply by $\Delta^2 \dot{\omega}$, etc., to obtain a bound in $H^2$, but following [22], §6.2, one takes time derivative of (1.13) and uses the resulting bound on $|\partial_t \omega|$ to bound $|\Delta \omega|$, etc. We shall not do this explicitly here, although similar ideas are used for the discrete case below (proof of Theorem 2).

3. Numerical scheme: boundedness

Fixing a timestep $k > 0$, we discretise the system (1.13) in time by the following two-step explicit–implicit scheme,

$$
\frac{3\omega^{n+1} - 4\omega^n + \omega^{n-1}}{2k} + \partial_t(2\psi^n - \psi^{n-1}, 2\omega^n - \omega^{n-1}) = p \{ \Delta \omega^{n+1} + \partial_x T^{n+1} - \partial_x S^{n+1} \}
$$

$$
\frac{3T^{n+1} - 4T^n + T^{n-1}}{2k} + \partial_t(2\psi^n - \psi^{n-1}, 2T^n - T^{n-1}) = \Delta T^{n+1}
$$

$$
\frac{3S^{n+1} - 4S^n + S^{n-1}}{2k} + \partial_t(2\psi^n - \psi^{n-1}, 2S^n - S^{n-1}) = \beta \Delta S^{n+1},
$$

(3.1)

plus the boundary conditions (1.14). Writing $U^n = (\omega^n, T^n, S^n)$, we assume that the second initial data $U^1$ has been obtained from $U^0$ using some reasonable one-step method, but all we shall need for what follows is that $U^1 \in H^1(\Omega)$. The time derivative term is that of the backward differentiation formula (BDF) and the explicit nonlinear term is sometimes known as a “one-leg method” [8, (V.6.6)]. This results in a method that is essentially explicit yet second order in time, and as we shall see below, preserves the important invariants of the continuous system.

Subject to some restrictions on the timestep $k$, we can obtain uniform bounds and absorbing balls for the solution of the discrete system analogous to those of the continuous system. Our first result is the following:

**Theorem 1.** With $Q \in H^{3/2}(\partial\Omega)$, the scheme (3.1) defines a discrete dynamical system in $H^1(\Omega) \times H^1(\Omega)$. Assuming $U^0, U^1 \in H^1(\Omega)$ and the timestep restriction given in (3.2) below,

$$
k \leq k_1([U^0]_{H^1}, |U^1|_{H^1}; |Q|_{H^{3/2} (\partial\Omega)}; \pi),
$$

the following bounds hold

$$
|U^n|_{L^2}^2 \leq 40 e^{-\nu nk/4} ([U^0]_{L^2}^2 + |U^1|_{L^2}^2) + M_0([Q]_{H^{3/2}(\partial\Omega)}; \pi)
$$

$$
+ c([Q]_{H^{-1/2}(\partial\Omega)}; \pi) k e^{-\nu nk/4} ([U^0]_{H^1}^2 + |U^1|_{H^1}^2),
$$

(3.3)

$$
|U^n|_{H^1}^2 \leq N_1(nk; [U^0]_{H^1}, |U^1|_{H^1}, |Q|_{H^{3/2}(\partial\Omega)}, \pi) + M_1([Q]_{H^{3/2}(\partial\Omega)}; \pi),
$$

(4.4)

where $\nu(\pi) > 0$ and $N_1(t; \cdots) = 0$ for $t \geq t_1([U^0]_{H^1}, |U^1|_{H^1}; Q, \pi)$.

We note that the last term in (3.3) has no analogue in the continuous case; we believe this is an artefact of our proof, but have not been able to circumvent it. Unlike in [22], $H^2$ bounds do not follow as readily due to the boundary conditions, so we proceed by first deriving bounds for $|U^{n+1} - U^n|$, using an approach inspired by [19] §6.2. We state our result without the transient terms:
Theorem 2. Assume the hypotheses of Theorem 1. Then for sufficiently large nk, one has
\begin{equation}
|\omega^{n+1} - \omega^n|^2 + |T^{n+1} - T^n|^2 + |S^{n+1} - S^n|^2 \leq k^2 M_6(|Q|_{H^{3/2}(\partial \Omega)}; \pi).
\end{equation}

Furthermore, for large nk the solution is bounded in $H^2$ as
\begin{equation}
|\Delta \omega^n|^2 + |\Delta T^n|^2 + |\Delta S^n|^2 \leq M_2(|Q|_{H^{3/2}(\partial \Omega)}; \pi).
\end{equation}

We remark that these difference and $H^2$ bounds require no additional hypotheses on $Q$, indicating that Theorem 1 may be sub-optimal. We also note that using the same method (and one more derivative on $Q$) one could bound $|U^{n+1} - U^n|_{H^1}$ and $|U^n|_{H^3}$, although we will not need these results here.

Following the approach of [22], these uniform bounds (along with the uniform convergence results that follow from them) then give us the convergence of long-time statistical properties of the discrete dynamical system (3.1) to those of the continuous system (1.1).
Let $\hat{\omega}^n := \omega^n - \Omega$, $\hat{T}^n := T^n - T_Q$ and $\hat{S}^n := S^n - S_Q$ be defined as in the continuous case, i.e. $\Omega$, $T_Q$, $S_Q \in H^2(D)$ satisfying the boundary conditions $\Omega = Q_u$, $\partial_z T_Q = Q_T$ and $\partial_z S_Q = Q_S$, and the constraint \((3.29)\), which is essentially \((2.17)\). The scheme \((3.1)\) then implies
\[
\frac{3\hat{\omega}^{n+1} - 4\hat{\omega}^n + \hat{\omega}^{n-1}}{2k} + \partial(2\psi^n - \psi^{n-1}, 2\hat{\omega}^n - \hat{\omega}^{n-1} + \Omega) = p\{\Delta\hat{\omega}^{n+1} + \Delta\Omega + \partial_x T^{n+1} - \partial_x S^{n+1}\}
\]
\[(3.12)\]
\[
\frac{3\hat{T}^{n+1} - 4\hat{T}^n + \hat{T}^{n-1}}{2k} + \partial(2\psi^n - \psi^{n-1}, 2\hat{T}^n - \hat{T}^{n-1} + T_Q) = \Delta\hat{T}^{n+1} + \Delta T_Q
\]
\[
\frac{3\hat{S}^{n+1} - 4\hat{S}^n + \hat{S}^{n-1}}{2k} + \partial(2\psi^n - \psi^{n-1}, 2\hat{S}^n - \hat{S}^{n-1} + S_Q) = \beta(\Delta\hat{S}^{n+1} + \Delta S_Q)
\]
where we have kept some $\psi^n$, $T^n$ and $S^n$ for now. We start by deriving difference inequalities for $\hat{\omega}$, $T^n$ and $\hat{S}^n$. In order to bound terms of the form $|\nabla \hat{\psi}^n|_{L^\infty} \leq c |\hat{\omega}^n|_{H^{1/2}}$, we assume for now the uniform bound
\[(3.13)\]
\[|\hat{\omega}^n|_{H^{1/2}} \leq k^{-1/2} M_\omega \cdots \text{ for all } n = 0, 1, 2, \cdots\]
where $M_\omega$ will be fixed in \((3.31)\) below. We also assume for clarity that $k \leq 1$.

Multiplying \((3.12)\) by $2k\hat{\omega}^{n+1}$ in $L^2(D)$ and using \((3.10)\), we find
\[(3.14)\]
\[\|
\]
\[\hat{\omega}^n, \hat{\omega}^{n+1}\|_{L^\infty}^2 - \nu k |\hat{\omega}^{n+1}|^2 + 2pk |\nabla \hat{\omega}^{n+1}|^2 + \frac{|(1 + \nu k)\hat{\omega}^{n+1} - 2\hat{\omega}^n + \hat{\omega}^{n-1}|^2}{2(1 + \nu k)}
\]
\[\leq 2k \partial(2\psi^n - \psi^{n-1}, \hat{\omega}^{n+1}, (1 + \nu k)\hat{\omega}^{n+1} - 2\hat{\omega}^n + \hat{\omega}^{n-1})
\]
\[+ 2k \partial(2\psi^n - \psi^{n-1}, \hat{\omega}^{n+1}, \Omega) + 2k \partial(\hat{\Psi}, \hat{\omega}^{n+1}, \Omega)
\]
\[+ 2pk \{ (\Delta\Omega, \hat{\omega}^{n+1}) + (\hat{\omega}^{n+1}, \partial_x T^{n+1}) - (\hat{\omega}^{n+1}, \partial_x S^{n+1}) \}.
\]
where $\nu > 0$ will be set below. We bound the last terms as in the continuous case,
\[2 |(\partial_x T^{n+1}, \hat{\omega}^{n+1})| \leq \frac{1}{8} |\nabla \hat{\omega}^{n+1}|^2 + 16c_0 |\nabla \hat{T}^{n+1}|^2 + 16 |T_Q|^2
\]
and the previous one as
\[2 |(\partial_x S^{n+1}, \hat{\omega}^{n+1})| \leq \frac{1}{8} |\nabla \hat{\omega}^{n+1}|^2 + 16c_0 |\nabla \hat{S}^{n+1}|^2 + 16 |S_Q|^2
\]
and
\[(3.15)\]
\[2 |(\partial\Psi, \hat{\omega}^{n+1}, \Omega)| \leq \frac{p}{8} |\nabla \hat{\omega}^{n+1}|^2 + \frac{c}{p} |\nabla \Psi|_{L^\infty}^2 |\Omega|^2,
\]
Taking $\nu = p/(8c_0)$ for now, we can bound the second term in \((3.14)\) using the third. Using \((3.13)\), we then bound the first nonlinear term as
\[2 |(\partial(2\psi^n - \psi^{n-1}, \hat{\omega}^{n+1})|, (1 + \nu k)\hat{\omega}^{n+1} - 2\hat{\omega}^n + \hat{\omega}^{n-1})|
\]
\[(3.16)\]
\[\leq \frac{p}{8} |\nabla \hat{\omega}^{n+1}|^2 + \frac{c}{p} |2\nabla \psi^n - \nabla \psi^{n-1}|_{L^\infty}^2 |(1 + \nu k)\hat{\omega}^{n+1} - 2\hat{\omega}^n + \hat{\omega}^{n-1}|^2
\]
\[\leq \frac{p}{8} |\nabla \hat{\omega}^{n+1}|^2 + c_3 (k^{-1/2} M_\omega + |\nabla \Psi|_{L^\infty}^2) |(1 + \nu k)\hat{\omega}^{n+1} - 2\hat{\omega}^n + \hat{\omega}^{n-1}|^2.
\]
Recalling that the validity of (3.8) and (3.9) demands \( k \leq 1/\nu \), which we henceforth assume, we have \( 2(1 + \nu k) \leq 4 \). This then implies that \( k \) times the last term in (3.16) can be majorised by the fourth term in (3.14) if \( k \) is small enough that

\[
(3.17) \quad c_3 k^{1/2} (M_\omega + |\nabla \Psi|_{L^\infty}) \leq p.
\]

All this brings us to [cf. (2.4)]

\[
(3.18) \quad \left[ \hat{\omega}^n, \hat{\omega}^{n+1} \right]_{\nu k} + p k |\nabla \hat{\omega}^{n+1}|^2 \leq \frac{\left[ \hat{\omega}^{n-1}, \hat{\omega}^n \right]_{2k}}{1 + \nu k} + \frac{ck}{p} \left( |\nabla \hat{\omega}^{n-1}|^2 + |\nabla \hat{\omega}^n|^2 \right) + 16 c_0 p k \left( |\nabla \hat{T}^{n+1}|^2 + |\nabla \hat{S}^{n+1}|^2 \right) + c k |\nabla \Psi|^2 |\nabla \Omega|^2.
\]

For \( \hat{S}^n \), we multiply (3.12b) by \( 2k \hat{S}^{n+1} \) in \( L^2 (\mathcal{D}) \) and use (3.10) to find

\[
(3.19) \quad \nu = \min \left\{ \min \{p^2, \beta^2, 1\} / (8c_0) \right\}
\]

\[
(3.20) \quad k \leq \min \left\{ \min \{p^2, \beta^2, 1\} / (c_3 M_\omega + c_3 |\nabla \Psi|_{L^\infty}^2), 1/\nu \right\},
\]

we arrive at

\[
(3.21) \quad \left[ \hat{S}^n, \hat{S}^{n+1} \right]_{\nu k} + \nu k |\nabla \hat{S}^{n+1}|^2 \leq \frac{\left[ \hat{S}^{n-1}, \hat{S}^n \right]_{2k}}{1 + \nu k} + \frac{ck}{p} \left( |\nabla \hat{\omega}^{n-1}|^2 + |\nabla \hat{\omega}^n|^2 \right) |\nabla \Psi|^2 |\nabla \Omega|^2
\]

Similarly, for \( \hat{T}^n \) we have

\[
(3.22) \quad \left[ \hat{T}^n, \hat{T}^{n+1} \right]_{\nu k} + k |\nabla \hat{T}^{n+1}|^2 \leq \frac{\left[ \hat{T}^{n-1}, \hat{T}^n \right]_{2k}}{1 + \nu k} + c k (|\nabla \hat{\omega}^{n-1}|^2 + |\nabla \hat{\omega}^n|^2) |\nabla \Psi|^2 |\nabla \Omega|^2 + c k |\nabla \Psi|_{L^\infty}^2 |\nabla \Omega|^2 + c k |\nabla \Omega|^2 |\nabla \Psi|^2.
\]

Adding \( 16 p c_0 \) times (3.22) and \( 16 p c_0 / \beta \) times (3.21) to (3.18), and writing

\[
(3.23) \quad \left[ \hat{U}^n, \hat{U}^{n+1} \right]_{\nu k} := \left[ \hat{\omega}^n, \hat{\omega}^{n+1} \right]_{\nu k} + 16 p c_0 |\nabla \hat{T}^{n+1}|^2_{\nu k} + 16 p c_0 |\nabla \hat{S}^{n+1}|^2_{\nu k} / \beta,
\]

we have

\[
(3.24) \quad \left[ \hat{U}^n, \hat{U}^{n+1} \right]_{\nu k} + p k (|\nabla \hat{\omega}^{n+1}|^2 + 8 c_0 |\nabla \hat{T}^{n+1}|^2 + 8 c_0 |\nabla \hat{S}^{n+1}|^2 / \beta) \leq \frac{\left[ \hat{U}^{n-1}, \hat{U}^n \right]_{2k}}{1 + \nu k} + k \|F_1\|^2 (|\nabla \hat{\omega}^{n-1}|^2 + |\nabla \hat{\omega}^n|^2) + k \|F_2\|^2
\]

where

\[
(3.25) \quad \|F_1\|^2 := c_4 p (|\Omega|^2 / p^2 + |T_Q|^2 + |S_Q|^2 / \beta^2)
\]

\[
\|F_2\|^2 := |\nabla \Psi|_{L^\infty}^2 \|F_1\|^2 + c p (|\nabla T_Q|^2 + |\nabla S_Q|^2 + |\nabla \Omega|^2).
\]
In order to integrate this difference inequality, we consider a three-term recursion of the form
\[(3.26)\quad x_{n+1} + \mu y_{n+1} \leq (1 + \delta)^{-1}x_n + \varepsilon y_n + \varepsilon y_{n-1} + r_n.\]
For \(\mu > 0, \delta \in (0, 1)\) and \(\varepsilon \in (0, \mu/8)\), we have
\[(3.27)\quad x_n + \mu y_n \leq \frac{x_{n-m} + \mu y_{n-m}}{(1 + \delta)^m} + \frac{\varepsilon y_{n-m-1}}{(1 + \delta)^{m-1}} + \sum_{j=1}^{m} \frac{r_{n-j}}{(1 + \delta)^{n-j}}\]
(which follows readily by induction) and in particular
\[(3.28)\quad x_{n+1} + \mu y_{n+1} \leq \frac{x_1 + \mu y_0}{(1 + \delta)^n} + \frac{\varepsilon y_0}{(1 + \delta)^{n-1}} + \sum_{j=1}^{n} \frac{r_j}{(1 + \delta)^{n-j}}.\]
In order to apply the bound \((3.28)\) of \((3.26)\) to \((3.24)\), we demand that \(\Omega, T_Q\) and \(S_Q\) be small enough that
\[(3.29)\quad |\Omega|_{L^2}^2 \leq \frac{p^2}{(32c_4)}, \quad |T_Q|_{L^2}^2 \leq 1/(32c_4) \quad \text{and} \quad |S_Q|_{L^2}^2 \leq \beta^2/(32c_4).\]
We note that, up to parameter-independent constants, these conditions are identical to those in the continuous case \((2.17)\). Using the fact that \((1 + x)^{-1} \leq \exp(-x/2)\) for \(x \in (0, 1)\), we integrate \((3.24)\) to find a bound uniform in \(t_n = nk\),
\[\begin{align*}
\hat{U}_n, \hat{U}^{n+1}_\nu \leq \frac{\mu k}{2} |\nabla \omega^{n+1}|^2 \\
\leq e^{-\mu nk/2} \{ |\hat{U}^0, \hat{U}^{1}_\nu|^2_\nu + \mu k (|\nabla \omega^0|^2 + |\nabla \omega^1|^2) + \frac{2}{\nu} \|F_2\|^2.\]
\end{align*}\]
Using \((3.28), (3.29), (3.30)\) follows.

The hypothesis \((3.13)\) can now be recovered by interpolation,
\[|\hat{\omega}^n|_{H^{1/2}} \leq c |\hat{\omega}_n| |
abla \hat{\omega}^n| \leq c |\hat{U}^{n-1}, \hat{U}^n|_{\nu k} |
abla \hat{\omega}^n| \leq c (\mu k)^{-1/2} \{ |\hat{U}^0, \hat{U}^{1}_\nu|^2_\nu + \mu k (|\nabla \omega^0|^2 + |\nabla \omega^1|^2) + 2 \|F_2\|^2/\nu\}
\]
and replacing \([|\hat{U}^0, \hat{U}^{1}_\nu|^2_\nu\) by its sup over \(\nu k \in (0, 1)\). Summing \((3.24)\) and using \((3.29)\), we find (discarding terms on the lhs)
\[\begin{align*}
k \sum_{j=n+1}^{n+m} \frac{\mu k}{2} |\nabla \omega^j|^2 + 8c_0 |\nabla \hat{\omega}^j|^2 + \frac{8c_0}{\beta} |\nabla \hat{\omega}^j|^2 \\
\leq |\hat{U}^{n-1}, \hat{U}^{n+1}_\nu|^2_\nu + 2k \|F_1\|^2 (|\nabla \omega^{n-1}|^2 + |\nabla \omega^n|^2) + mk \|F_2\|^2.
\end{align*}\]

From \((3.29), (3.30), (3.32)\), it is clear that there exists a \(t_0(|\nabla U^0|, |\nabla U^1|, Q; \pi)\) such that, whenever \(nk \geq t_0\),
\[(3.33)\quad |\hat{U}^n|^2 \leq M_0(Q; \pi) \quad \text{and} \quad k \sum_{j=n}^{n+1/k} |\nabla \hat{U}^j|^2 \leq \tilde{M}_0(Q; \pi).
\]
We redefine \(M_0\) and \(\tilde{M}_0\) to bound \(|\hat{U}^n|^2\) and \(\sum_j |\nabla \hat{U}^j|^2\) as well.
On to $H^1$, we multiply $\eqref{3.12a}$ by $-2k\Delta \tilde{\omega}^{n+1}$ in $L^2$ to get
\[\begin{aligned}
[\nabla \omega^n, \nabla \omega^{n+1}]_{L^2} - \nu k |\nabla \omega^{n+1}|^2 + \frac{|(1 + \nu k)\nabla \omega^{n+1} - 2\nabla \omega^n + \nabla \omega^{n-1}|^2}{2(1 + \nu k)}
\end{aligned}\]
\[= \frac{|\nabla \omega^{n-1}, \nabla \omega^n|_{L^2}^2}{1 + \nu k} - 2pk |\Delta \tilde{\omega}^{n+1}|^2 - 2k (\partial (2\psi^n - \psi_0^n), \nabla \omega^{n+1}), (1 + \nu k)\nabla \omega^{n+1} - 2\nabla \omega^n + \nabla \omega^{n-1})
\]
\[\begin{aligned}
&- 2k (\partial (2\psi^n - \psi_0^n), \nabla \omega^{n-1}, 2\omega^n - \omega^{n-1}), \nabla \omega^{n+1})
\end{aligned}\]
\[\begin{aligned}
&- 2k (\partial (\nabla \psi, 2\omega^n - \omega^{n-1}, \nabla \omega^{n+1}) + 2k (\partial (2\psi^n - \psi_0^n, \Omega), \Delta \omega^{n+1})
\end{aligned}\]
\[\begin{aligned}
&+ 2pk (\partial_x T^{n+1} - \partial_x T^{n+1} - \Delta \Omega, \Delta \tilde{\omega}^{n+1})
\end{aligned}\]

Labelling the “nonlinear” terms by $①, \ldots, ④$, we bound them as
\[\begin{aligned}
① &\leq ck |2\nabla \psi^n - \nabla \psi_0^n|_{L^\infty} |\nabla^2 \omega^{n+1}|_{L^2} (1 + \nu k)\nabla \omega^{n+1} - 2\nabla \omega^n + \nabla \omega^{n-1}|_{L^2}
\end{aligned}\]
\[\begin{aligned}
&\leq \frac{p k}{8} |\Delta \tilde{\omega}^{n+1}|^2 + \frac{c k}{4p} |M_\omega + |\nabla \psi||_{L^\infty} |\nabla ((1 + \nu k)\nabla \omega^{n+1} - 2\nabla \omega^n + \nabla \omega^{n-1})|^2
\end{aligned}\]
\[\begin{aligned}
② &\leq ck |2\omega^n - \omega^{n-1}|_{L^4} |\nabla^2 \omega^{n+1}|_{L^2} |2\nabla \omega^n - \omega^{n-1}|_{L^4}
\end{aligned}\]
\[\begin{aligned}
&\leq \frac{p k}{8} |\Delta \tilde{\omega}^{n+1}|^2 + \frac{c k}{p} |2\omega^n - \omega^{n-1}|^2 |2\nabla \omega^n - \nabla \omega^{n-1}|^2
\end{aligned}\]
\[\begin{aligned}
③ &\leq ck |\Omega||_{L^\infty} |\nabla^2 \omega^{n+1}|_{L^2} |2\omega^n - \omega^{n-1}|_{L^2}
\end{aligned}\]
\[\begin{aligned}
&\leq \frac{p k}{8} |\Delta \tilde{\omega}^{n+1}|^2 + \frac{c k}{p} |\Omega||_{L^\infty} |2\omega^n - \omega^{n-1}|^2
\end{aligned}\]
\[\begin{aligned}
④ &\leq ck |2\nabla \psi^n - \nabla \psi_0^n|_{L^\infty} |\nabla \Omega||_{L^2} |\Delta \tilde{\omega}^{n+1}|_{L^2}
\end{aligned}\]
\[\begin{aligned}
&\leq \frac{p k}{8} |\Delta \tilde{\omega}^{n+1}|^2 + \frac{c k}{p} |\nabla \Omega||^2 |2\nabla \omega^n - \omega^{n-1}|^2
\end{aligned}\]

Bounding the linear term in the obvious fashion and again using $\eqref{3.19} - \eqref{3.20}$, we arrive at
\[\begin{aligned}
[\nabla \omega^n, \nabla \omega^{n+1}]_{L^2}^2 + pk |\Delta \omega^{n+1}|^2
\end{aligned}\]
\[\begin{aligned}
\leq \begin{cases}
\begin{aligned}
&[\nabla \omega^{n-1}, \nabla \omega^n]_{L^2}^2 [1 + c p^{-1} k (M_0 + |\nabla \Omega||^2)] + 8pk ([|\nabla T^{n+1}|^2 + |\nabla S^{n+1}|^2])
\end{aligned}
\end{cases}
\end{aligned}\]
\[\begin{aligned}
&+ c p^{-1} k (M_0 |\Omega||^2_{L^2} + |\nabla \Omega||^2 |\nabla \psi||_{L^2}^2) + 8pk (|\Delta \Omega||^2 + |\nabla TQ|^2 + |\nabla SQ|^2)
\end{aligned}\]
valid for large times $nk \geq t_0$.

Noting that, for $x_n \geq 0$, $r_n \geq 0$ and $b > 0$,
\[\begin{aligned}
x_{n+1} \leq (1 + b) x_n + r_n \quad \Rightarrow \quad x_{n+m} \leq (1 + b)^m (x_n + \sum_{j=n+1}^{m} r_j)
\end{aligned}\]
we can obtain a uniform $H^1$ bound from $\eqref{3.33}$ and $\eqref{3.35}$ as follows. Borrowing an argument from $[1]$, we conclude from $\eqref{3.33}$ that there exists an $n_\star \in \{n + [1/k], \ldots, n + [2/k] - 1\}$ such that
\[\begin{aligned}
|\nabla \omega^{n_\star}||^2 + |\nabla \tilde{\omega}^{n_\star+1}|^2 \leq c M_0(Q; \pi) \Rightarrow \begin{cases}
\begin{aligned}
&|\nabla \omega^{n_\star}, \nabla \tilde{\omega}^{n_\star+1}|_{L^2}^2 \leq c_5 M_0.
\end{aligned}
\end{cases}
\end{aligned}\]
(In other words, in any sequence of non-negative numbers, one can find two consecutive terms whose sum is no greater than four times the average.) Taking $n_\star \in \{[t_0/k], \ldots, ([t_0 + 1)/k] - 1\}$ and integrating $\eqref{3.35}$ using $\eqref{3.36}$ with $m = [2/k]$ and $\eqref{3.33}$ to bound the $|\nabla T^n|^2$ and $|\nabla S^n|^2$ on the rhs, we find
\[\begin{aligned}
|\nabla \omega^n, \nabla \tilde{\omega}^{n+1}|_{L^2}^2 \leq M_1(Q; \pi)
\end{aligned}\]
for all $n \in \{n_*, \ldots, n_* + [2/k] - 1\}$. We then find a $n_{**} \in \{n_* + [1/k], \ldots, n_* + [2/k] - 1\}$ that satisfies (3.37) and repeat the argument to find that (3.38) also holds for all $n \in \{n_*, \ldots, n_{**} + [2/k] - 1\}$. Since $n_{**} \geq n_* + [1/k]$, with each iteration we increase the time of validity of (3.38) by at least 1 using no further assumptions, implying that (3.38) in fact holds for all $n \geq n_*$, i.e. whenever $nk \geq t_0 + 1$.

Similarly for $\tilde{S}^n$, we multiply (3.12) by $-2\delta \tilde{S}^{n+1}$ in $L^2$ to find after a similar computation

\[
\| \tilde{\nabla} \tilde{S}^n, \tilde{\nabla} \tilde{S}^{n+1} \|_{L^2}^2 + \beta k |\Delta \tilde{S}^{n+1}|^2 \leq \| \tilde{\nabla} \tilde{S}^{n-1}, \tilde{S}^n \|_{L^2}^2 (1 + ck\beta^{-1}M_0) \\
+ \frac{ck}{\beta} (M_0 + |\nabla \Psi|)^2 (|\nabla \tilde{\omega}|^2 + |\nabla \tilde{\omega}^n|^2 + |\nabla \tilde{\omega}^{n+1}|^2) \\
+ \frac{ck}{\beta} M_0 |\nabla \Sigma|^2 + 8\beta |\delta \tilde{S}^n|^2.
\]

Arguing as we did with $\tilde{\omega}^n$, we conclude that (redefining $M_1$ as needed) one has

\[
\| \tilde{\nabla} \tilde{S}^n, \tilde{\nabla} \tilde{S}^{n+1} \|_{L^2}^2 \leq M_1(Q; \pi) \quad \text{whenever } nk \geq t_0 + 1.
\]

Obviously the same bound applies to $\tilde{T}^n$,

\[
\| \tilde{\nabla} \tilde{T}^n, \tilde{\nabla} \tilde{T}^{n+1} \|_{L^2}^2 \leq M_1(Q; \pi) \quad \text{whenever } nk \geq t_0 + 1.
\]

As with $M_0$, we redefine $M_1$ to bound $\| \tilde{\nabla} \omega^n, \tilde{\nabla} \omega^{n+1} \|_{L^2}^2$, etc., as well as $\| \tilde{\nabla} \tilde{\omega}^n, \tilde{\nabla} \tilde{\omega}^{n+1} \|_{L^2}^2$.

---

**Proof of Theorem 2.** Let $\delta U^n := U^n - U^{n-1} = \hat{U}^n - \hat{U}^{n-1}$. We first prove that $|\delta U^n|^2 \leq kM$ for all large $n$, and then use this result to prove (3.3).

Writing $3\omega^{n+1} - 4\omega^n + \omega^{n-1} = 3\delta \omega^{n+1} - \delta \omega^n$ and using the identity

\[
2(3\delta \omega^{n+1} - \delta \omega^n, \delta \omega^{n+1}) = 3|\delta \omega^{n+1}|^2 - \frac{1}{3}|\delta \omega^n|^2 + \frac{1}{3}|3\delta \omega^{n+1} - \delta \omega^n|^2,
\]

we multiply (3.14) by $4k\delta \omega^{n+1}$,

\[
3|\delta \omega^{n+1}|^2 + \frac{1}{3}|3\delta \omega^{n+1} - \delta \omega^n|^2 = \frac{1}{3}|\delta \omega^n|^2 \\
+ 4k(|\Delta \omega^{n+1}, \delta \omega^{n+1}| + 4k(\partial \omega T^{n+1} - \partial \omega S^{n+1}, \delta \omega^{n+1}) \\
- 4k(\partial (2\psi^n - \psi^{n-1}, 2\omega^n - \omega^{n-1}), \delta \omega^{n+1})).
\]

For the dissipative term, we integrate by parts using the fact that $\delta \omega^{n+1} = 0$ on the boundary to write it as

\[
-2(\Delta \omega^{n+1}, \delta \omega^{n+1}) = |\nabla \omega^{n+1}|^2 - |\nabla \omega^n|^2 + |\nabla \delta \omega^{n+1}|^2.
\]

We bound the nonlinear term as

\[
4|\partial (2\psi^n - \psi^{n-1}, 2\omega^n - \omega^{n-1}), \delta \omega^{n+1})| \\
\leq c|2\nabla \psi^n - \nabla \psi^{n-1}|_{L^\infty} |2\nabla \omega^n - \nabla \omega^{n-1}|_{L^2} |\delta \omega^{n+1}|_{L^2} \\
\leq \frac{1}{8}|\delta \omega^{n+1}|^2 + c|2\nabla \omega^n - \nabla \omega^{n-1}|^4.
\]
Bounding the buoyancy terms by Cauchy–Schwarz, we arrive at
\begin{equation}
2|\delta \omega^{n+1}|^2 + \frac{1}{3} |3\delta \omega^{n+1} - \delta \omega^n|^2 + 2p k |\nabla \omega^{n+1}|^2 + 2p k |\nabla \delta \omega^{n+1}|^2
\leq \frac{1}{3} |\delta \omega^n|^2 + 2p k |\nabla \omega^n|^2 + ck^2 |2\nabla \omega^n - \nabla \omega^{n-1}|^4
+ c p k^2 (|\partial_x T^{n+1}|^2 + |\partial_x S^{n+1}|^2)
\leq \frac{1}{3} |\delta \omega^n|^2 + c(\pi)(k M_1 + k^2 M^2_1).
\end{equation}

(3.46)

It is now clear that, since \(\delta \omega^1\) is bounded in \(L^2\), we have for large \(nk\)
\begin{equation}
|\delta \omega^n|^2 \leq k c(\pi)(M_1 + k M^2_1).
\end{equation}

(3.47)

Similarly for \(\hat{S}^n\), we multiply (3.12) by \(4k\delta \hat{S}^{n+1}\) to find
\begin{equation}
3|\delta \hat{S}^{n+1}|^2 + \frac{1}{3} |3\delta \hat{S}^{n+1} - \delta \hat{S}^n|^2 = \frac{1}{4} |\delta \hat{S}^n|^2 + 4k\beta (\Delta \hat{S}^{n+1} + \Delta S_Q, \delta \hat{S}^{n+1})
- 4k (\partial(2\psi^n - \psi^{n-1}, \hat{S}^{n+1} - \hat{S}^{n-1} + S_Q), \delta \hat{S}^{n+1}).
\end{equation}

(3.48)

Bounding the nonlinear term as we did for \(\omega^n\),
\begin{equation}
4 |(\partial(2\psi^n - \psi^{n-1}, \hat{S}^{n+1} - \hat{S}^{n-1} + S_Q), \delta \hat{S}^{n+1})|
\leq \frac{1}{3} |\delta \hat{S}^{n+1}|^2 + c|2\nabla \omega^n - \nabla \omega^{n-1}|^2(|2\nabla \hat{S}^{n+1} - \nabla \hat{S}^{n-1}|^2 + |\nabla S_Q|^2),
\end{equation}

(3.49)

and the linear terms as we did with \(\omega^n\), we arrive at
\begin{equation}
2|\delta \hat{S}^{n+1}|^2 + \frac{1}{3} |3\delta \hat{S}^{n+1} - \delta \hat{S}^n|^2 + 2\beta k |\nabla \hat{S}^{n+1}|^2 + 2\beta k |\nabla \delta \hat{S}^{n+1}|^2
\leq \frac{1}{3} |\delta \hat{S}^n|^2 + 2\beta k |\nabla \hat{S}^n|^2 + ck^2 |2\nabla U^n - \nabla U^{n-1}|^4 + c(\beta) k^2 (|\nabla S_Q|^4 + |\Delta S_Q|^2),
\end{equation}

(3.50)

whence
\begin{equation}
|\delta \hat{S}^n|^2 \leq k c(\pi)(M_1 + k M^2_1) \quad \text{for large } nk.
\end{equation}

(3.51)

Obviously a similar bound holds for \(\delta T^n\), so we conclude that
\begin{equation}
|\delta U^n|^2 \leq k c(\pi)(M_1 + k M^2_1) =: k \tilde{M}_4 \quad \text{for large } nk.
\end{equation}

(3.52)

By taking difference of (3.14), we find
\begin{equation}
\frac{3\delta \omega^{n+1} - 4\delta \omega^n + \delta \omega^{n-1}}{2k} + \partial(2\psi^{n-1} - \psi^n - 2\delta \omega^n - \delta \omega^n)
+ \partial(2\delta \psi^n - \delta \psi^n, \delta \omega^n - \omega^n) = p \{\Delta \delta \omega^{n+1} + \partial_x \delta T^{n+1} - \partial_x \delta S^{n+1}\}.
\end{equation}

(3.53)

Multiplying this by \(2k\delta \omega^{n+1}\) and using (3.10), we have
\begin{equation}
[\delta \omega^n, \delta \omega^{n+1}]^2_{\nu_k} - \nu_k |\delta \omega^{n+1}|^2 + \frac{|(1 + \nu k)\delta \omega^{n+1} - 2\delta \omega^n + \delta \omega^{n-1}|^2}{2(1 + \nu k)} + k I
\end{equation}

(3.54)

Here \(I = I_1 + I_2\) denotes the nonlinear terms, which we bound as
\begin{equation}
|I_1| \leq c |2\nabla \psi^{n-1} - \nabla \psi^n - \nabla \omega^{n-1}|_{L^2} |\nabla \delta \omega^{n+1}|_{L^2}|2\delta \omega^n - \delta \omega^{n-1}|_{L^2}
\leq \frac{p}{8} |\nabla \delta \omega^{n+1}|^2 + \frac{c}{p} |2\nabla \omega^{n-1} - \nabla \omega^n|^2 |2\delta \omega^n - \delta \omega^{n-1}|^2
\end{equation}

(3.55)

\begin{equation}
|I_2| \leq c |2\nabla \delta \psi^n - \nabla \delta \psi^{n-1}|_{L^2} |2\omega^n - \omega^{n-1}|_{L^2} |\nabla \delta \omega^{n+1}|_{L^2}
\leq \frac{p}{8} |\nabla \delta \omega^{n+1}|^2 + \frac{c}{p} |2\delta \omega^n - \delta \omega^{n-1}|^2 |2\nabla \omega^n - \nabla \omega^{n-1}|^2.
\end{equation}
Bounding the linear terms as
\[
|\partial_x \delta T^{n+1} - \partial_x \delta S^{n+1}, \delta \omega^{n+1}| \leq \frac{1}{4} |\nabla \delta \omega^{n+1}|^2 + 2 |\delta T^{n+1}|^2 + 2 |\delta S^{n+1}|^2
\]
and using (3.52), we obtain
\[
[\delta \omega^n, \delta \omega^{n+1}]_{v,k}^2 + pk |\nabla \delta \omega^{n+1}|^2 \leq \frac{1}{1 + \nu_k} [\delta \omega^{n-1}, \delta \omega^n]_{v,k}^2 + k^2 c(\pi) \delta \omega^n (1 + M_1).
\]
Integrating this and the analogous expressions for \( \delta T^n \) and \( \delta S^n \), we obtain (3.5) for \( nk \) large.

To prove (3.6), we note that (3.1b) implies
\[
|\Delta T^{n+1}| \leq |\partial (2\psi^n - \psi^{n-1}, 2\omega^n - \omega^{n-1})| + \frac{|3\delta T^{n+1} - \delta T^n|}{2k} \leq c |2\nabla \omega^n - \nabla \omega^{n-1}| |2\nabla T^n - \nabla T^{n-1}| + \frac{3|\delta T^{n+1}| + |\delta T^n|}{2k}.
\]
Since the right-hand side has been bounded (independently of \( k \) for the first term and by \( Mk \) for the second) on the attractor \( A_k \), it follows that \( |\Delta T^n| \) is uniformly bounded on \( A_k \) as well. Clearly similar \( H^2 \) bounds also hold for \( S^n \) and \( \omega^n \), proving (3.6) and the Theorem.

**Appendix A. 2d Navier–Stokes equations**

In this appendix we present an alternate derivation of the boundedness results in [22], without using the Wente-type estimate of [14] but requiring slightly more regular initial data. In principle these could be obtained following the proofs of Theorems 1 and 2 above, but the computation is much cleaner in this case (mostly due to the periodic boundary conditions) so we present it separately.

The system is the 2d Navier–Stokes equations
\[
\frac{3\omega^{n+1} - 4\omega^n + \omega^{n-1}}{2k} + \partial (2\psi^n - \psi^{n-1}, 2\omega^n - \omega^{n-1}) = \mu \Delta \omega^{n+1} + f^n
\]
with periodic boundary conditions. It is clear that \( \omega^n \) has zero integral over \( \mathbb{D} \), and we define \( \psi^n \) uniquely by the zero-integral condition. These imply (2.11)–(2.2), which we will use below without further mention. Assuming that the initial data \( \omega^0, \omega^1 \in H^{1/2} \) (in fact, we only need \( H^\epsilon \) for any \( \epsilon > 0 \), but will write \( H^{1/2} \) for concreteness), we derive uniform bounds for \( \omega^n \) in \( L^2, H^1 \) and \( H^2 \).

Assuming for now the uniform bound
\[
|\omega^n|_{H^{1/2}}^2 \leq k^{-1/2} M_\omega (\cdots) \quad \text{for } n \in \{2, 3, \cdots\},
\]
we multiply (A.1) by $2k\omega^{n+1}$ in $L^2$, use (3.10) and estimate as before,
\[
\left[\omega^n, \omega^{n+1}\right]_{v_k}^2 - \nu k |\omega^{n+1}|^2 + 2\mu k |\nabla \omega^{n+1}|^2 + \frac{|(1 + \nu k)\omega^{n+1} - 2\omega^n + \omega^{n-1}|^2}{2(1 + \nu k)}
\]
\[
= \frac{[\omega^{n-1}, \omega^n]_{v_k}^2}{1 + \nu k} + 2k (f^n, \omega^{n+1})
\]
\[
\leq \frac{[\omega^{n-1}, \omega^n]_{v_k}^2}{1 + \nu k} + \frac{\mu k}{\mu} |\nabla \omega^{n+1}|^2 + \frac{ck}{\mu} |f^n|_H^{-1}
\]
\[
+ \frac{ck}{\mu} |2\nabla \psi^n - \nabla \psi^{n-1}|_{L^\infty}^2 |(1 + \nu k)\omega^{n+1} - 2\omega^n + \omega^{n-1}|^2,
\]
giving (as before, we require $k \leq 1/\nu$)
\[
(A.4) \quad [\omega^n, \omega^{n+1}]_{v_k}^2 - \nu k |\omega^{n+1}|^2 + \frac{3\mu k}{2} |\nabla \omega^{n+1}|^2 \leq \frac{[\omega^{n-1}, \omega^n]_{v_k}^2}{1 + \nu k} + \frac{ck}{\mu} |f^n|_H^{-1}
\]
\[
+ |(1 + \nu k)\omega^{n+1} - 2\omega^n + \omega^{n-1}|^2 (c_3 k^{1/2} M_\omega / \mu - \frac{1}{4}).
\]
Setting $\nu = \mu/(2c_0)$ and imposing the timestep restriction
\[
(A.5) \quad k \leq k_0 := \min\{\mu^2 / (4c_3 M_\omega)^2, 1/\nu\},
\]
this gives
\[
(A.6) \quad [\omega^n, \omega^{n+1}]_{v_k}^2 + \mu k |\nabla \omega^{n+1}|^2 \leq \frac{[\omega^{n-1}, \omega^n]_{v_k}^2}{1 + \nu k} + \frac{ck}{\mu} |f^n|_H^{-1}.
\]
Integrating using the Gronwall lemma, we arrive at the $L^2$ bound
\[
[A.7] \quad [\omega^{n+1}, \omega^{n+2}]_{v_k}^2 + \mu k |\nabla \omega^{n+2}|^2 \leq e^{-\nu nk/2} [\omega^0, \omega^1]_{v_k}^2 + \frac{c}{\mu^2} \sup_j |f^j|_H^{-1}
\]
\[
\leq [\omega^0, \omega^1]_{v_k}^2 + c \frac{\mu k}{\mu^2} \sup_j |f^j|_H^{-1} =: M_0.
\]
The hypothesis (A.2) is now recovered by interpolation as before,
\[
[A.8] \quad |\omega^n|_H^{1/2} \leq c |\omega^n| |\nabla \omega^n| \leq c |[\omega^{n-1}, \omega^n]|_{v_k} |\nabla \omega^n|
\]
\[
\leq c (\mu k)^{-1/2} (|\omega^0, \omega^1|_{v_k}^2 + (1/\mu + 1/\mu^2) \sup_j |f^j|_H^{-1}).
\]
Summing (A.6), we find
\[
(A.9) \quad \mu k \sum_{j=n+1}^{n+1/|k|} |\nabla \omega^j|^2 \leq [\omega^{n-1}, \omega^n]_{v_k}^2 + c \mu \sup_j |f^j|_H^{-1}.
\]
It is clear that both bounds (A.7) and (A.9) can be made independent of the initial data for sufficiently large time, $nk \geq t_0 (\omega^0, \omega^1; f, \mu)$.
For the $H^1$ estimate, we multiply (A.1) by $-2k\Delta \omega^{n+1}$ in $L^2$ and use (3.10). Writing the nonlinear term as
\[
N_1 := (\partial(2\psi^n - \psi^{n-1}, 2\omega^n - \omega^{n-1}), \Delta \omega^{n+1})
\]
\[
(A.10) \quad - (\partial(2\nabla \psi^n - \nabla \psi^{n-1}, \nabla \omega^{n+1}), 2\omega^n - \omega^{n-1})
\]
\[
- (\partial(\psi^n - \psi^{n-1}, \nabla \omega^{n+1}), \nabla ((1 + \nu k)\omega^{n+1} - 2\omega^n + \omega^{n-1}))
\]
and bounding the terms as
\[
\begin{aligned}
|N_1| &\leq c|2\omega^n - \omega^{n-1}|_{L^2} |\nabla^2 \omega^{n+1}|_{L^2} |2\omega^n - \omega^{n-1}|_{L^2} \\
&\quad + c |2\nabla \psi^n - \nabla \psi^{n-1}|_{L^\infty} |\nabla^2 \omega^{n+1}|_{L^2} |\nabla((1 + \nu k)\omega^{n+1} - 2\omega^n + \omega^{n-1})|_{L^2}
\end{aligned}
\]
(A.11)

\[
\leq \frac{\mu}{2} |\Delta \omega^{n+1}|^2 + \frac{c}{\mu} |2\omega^n - \omega^{n-1}|^2 |2\nabla \omega^n - \nabla \omega^{n-1}|^2 \\
+ \frac{ck^{-1/2}}{\mu} M_\omega |\nabla((1 + \nu k)\omega^{n+1} - 2\omega^n + \omega^{n-1})|^2,
\]

we find the differential inequality, using the bound (A.9),
\[
\begin{aligned}
[\nabla \omega^n, \nabla \omega^{n+1}]_{\nu k}^2 + m_k |\Delta \omega^{n+1}|^2 &\leq [\nabla \omega^{n-1}, \nabla \omega^n]_{\nu k}^2 (1 + ck M_0/\mu) \\
+ |\nabla((1 + \nu k)\omega^{n+1} - 2\omega^n + \omega^{n-1})|^2 \left(c k^{1/2} M_\omega/\mu - \frac{1}{4}\right) + ck |f^n|^2/\mu.
\end{aligned}
\]
(A.12)

Using the earlier timestep restriction (A.5), we can suppress the second term on the r.h.s. Thanks to (A.9), for any \( n \in \{0, 1, \ldots\} \) we can find \( n_* \in \{n, \ldots, n + [1/k]\} \) such that \([\nabla \omega^n, \nabla \omega^{n+1}]_{\nu k}^2 \leq c(\nabla_0, 1, 1, 1, 1, \mu, \sup_j |f|)_{\nu k}^2\). Arguing as before, we can use this to integrate (A.12) to give us a uniform \( H^1 \) bound
\[
[\nabla \omega^n, \nabla \omega^{n+1}]_{\nu k} \leq M_1(\nabla \omega^0, 1, 1, 1, 1, \mu, \sup_j |f|) \quad \text{valid for all } n \in \{0, 1, \ldots\}.
\]
(A.13)

Moreover, \( M_1 \) can be made independent of the initial data \([\nabla \omega^0, 1, 1, 1, 1, \mu, \sup_j |f|] \) for sufficiently large \( n \); in fact, we do not even need \( \omega^0, \omega^1 \in H^1 \), although we still need them to be in \( H^2 \) for the timestep restriction (A.5). Summing (A.12) and using (A.13), we find
\[
\mu k \sum_{j=n+1/k}^{n+1} |\Delta \omega^j|^2 \leq M_1(\sup_j |f|; \mu) \quad \text{for all } nk \geq t_1(\omega^0, \omega^1, f; \mu).
\]
(A.14)

Similarly, for the \( H^2 \) estimate, we multiply (A.1) by \( 2k \Delta^2 \omega^{n+1} \) in \( L^2 \) and write the nonlinear term as
\[
N_2 := (\partial(\psi^n - \psi^{n-1}, 2\omega^n - \omega^{n-1}), \Delta^2 \omega^{n+1})
\]
(A.15)

\[
= - (\partial(2\nabla \psi^n - \nabla \psi^{n-1}, 2\omega^n - \omega^{n-1}), \Delta \omega^{n+1}) \\
- (\partial(2\nabla \psi^n - \psi^{n-1}, 2\omega^n - \nabla \omega^{n-1}), \nabla \Delta \omega^{n+1}).
\]

Bounding this as
\[
\begin{aligned}
|N_2| &\leq c|2\omega^n - \omega^{n-1}|_{L^\infty} |\nabla \omega^n - \nabla \omega^{n-1}|_{L^2} |\Delta \omega^{n+1}|_{L^2} \\
&\quad + c |2\nabla \psi^n - \nabla \psi^{n-1}|_{L^\infty} |\nabla \omega^n - \nabla \omega^{n-1}|_{L^2} |\Delta \omega^{n+1}|_{L^2}
\end{aligned}
\]
(A.16)

\[
\leq \frac{\mu}{2} |\Delta \omega^{n+1}|^2 + \frac{c}{\mu} |2\nabla \omega^n - \nabla \omega^{n-1}|^2 |\Delta \omega^{n+1}|_{\nu k}^2,
\]

we arrive at the differential inequality
\[
[\Delta \omega^n, \Delta \omega^{n+1}]_{\nu k}^2 + \mu k |\nabla \Delta \omega^{n+1}|^2 \leq [\Delta \omega^{n-1}, \Delta \omega^n]_{\nu k}^2 (1 + \nu k M_1/\mu) + ck |\nabla f^n|^2/\mu.
\]
(A.17)

As with (A.12), this can be integrated to obtain the uniform bound
\[
[\Delta \omega^n, \Delta \omega^{n+1}]_{\nu k}^2 \leq M_2(\sup_j |f|; \mu)
\]
valid whenever \( nk \geq t_2(\omega^0, \omega^1, f; \mu) \).

To bound the difference \( \delta \omega^n := \omega^n - \omega^{n-1} \), we write (A.3) as
\[
\frac{3\delta \omega^{n+1} - \delta \omega^n}{2k} + (\partial(\psi^n - \psi^{n-1}, 2\omega^n - \omega^{n-1}) = \mu \Delta \omega^{n+1} + f^n.
\]
(A.19)
Multiplying by $4k\delta\omega^{n+1}$ and using (3.42) and (3.44), we find
\[
3|\delta\omega^{n+1}|^2 + \frac{1}{2}|\delta\omega^{n+1} - \delta\omega^{n}|^2 = \frac{1}{2}|\delta\omega^{n}|^2
\]
(A.20)
\[
+ 2\mu k|\nabla\omega^{n}|^2 - 2\mu k|\nabla\omega^{n+1}|^2 - 2\mu k|\nabla\omega^{n+1}|^2
\]
\[
- 4k(\partial(2\psi^n - \psi^{n-1}, 2\omega^n - \omega^{n-1}), \delta\omega^{n+1}) + 4k(f^n, \delta\omega^{n+1}).
\]
Bounding the nonlinear term and suppressing harmless terms, we arrive at
\[
2|\delta\omega^{n+1}|^2 \leq \frac{1}{4}|\delta\omega^{n}|^2 + 2\mu k|\nabla\omega^{n}|^2
\]
(A.21)
\[
+ ck^2|2\nabla\psi^n - \nabla\psi^{n-1}|_2 \leq |2\nabla\omega^n - \nabla\omega^{n-1}|^2 + \frac{ck^2}{\mu}f^n|H^{-1}.
\]
Since the r.h.s. has been bounded uniformly for large $nk$, we conclude that
\[
(A.22) \quad |\delta\omega^n|^2 \leq kM_0(f, \mu)
\]
for $nk$ sufficiently large. Arguing as in (3.53), we can improve the bound on $|\delta\omega^n|$ to $O(k)$.

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