SYMMETRIES IN M THEORY:
MONSTERS, INC.

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Abstract We will review the algebras which have been conjectured as symmetries
in M-theory. The Borcherds algebras, which are the most general Lie
algebras under control, seem natural candidates.

Little is known about M-theory except that its low effective action
is described by eleven-dimensional supergravity [1]. The dimensional
reduction of this supergravity theory on a n-dimensional torus Tⁿ was
shown to be invariant under the split form Eₙ(n|n) of the complex ex-
tensional Lie algebra En [2]. The split form of a complex Lie algebra
is defined by the restriction of the field of coefficients from complex to
real numbers and in M-theory, it has been conjectured that we must
replace the real field by the ring of integers in the Cartan Weyl basis.
[3]. These arithmetic groups, called U-dualities, have been extensively
used to compute some non-perturbative contributions in string theory.
The well-known example is the tₘtₙRₜ term in IIB superstring which
can be computed exactly using supersymmetry constraints and the mod-
ular group SL(2, Z). The solution involves non-holomorphic Eisenstein
forms.

Using a rich connection between particular complex algebraic sur-
faces, called del Pezzo surfaces, and field theories, extending the work

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[4] presented by A. Neitzke in this volume, we have shown that the U-duality algebras can be enlarged into some Borcherds (super)algebras [5]. This class of algebras, which contains the Kac-Moody algebras, was introduced by Borcherds [6] in order to prove the Moonshine conjecture which states that the characters of representations of a sporadic group, called the Monster, are modular forms under Hecke subgroups of $SL(2, \mathbb{Z})$.

In our correspondence, the simple roots of the Borcherds algebra are associated to a del Pezzo surface $X^C$, they span the full Picard group $\text{Pic}(X^C)$ and the cone containing the positive roots is defined as the convex hull of the rational (i.e. of vanishing virtual genus) divisor classes of non negative degree (one must also exclude one half of the degree zero hyperplane). We can then read off the intersection matrix $A_{ij} = \alpha_i . \alpha_j$ and a $\mathbb{Z}_2$-graduation by $\text{grad} (\alpha_i) = -K. \alpha_i \mod 2$. However, it turns out that whenever a fermionic root of square $-1$ appears it should be viewed as an $\text{sl}(1\vert 1)$ superroot, i.e. it should have zero (intersection)-Cartan-Killing norm. The corresponding modified intersection matrix will be minus our Cartan matrix $a_{ij}$. This symmetric matrix $a_{ij}$ satisfies the defining properties of a Borcherds superalgebra:

\begin{align*}
(i) & \quad a_{ij} \leq 0 \quad \text{if } i \neq j \\
(ii) & \quad \frac{2a_{ij}}{a_{ii}} \in \mathbb{Z} \quad \text{if } a_{ii} > 0, \quad \text{for } \text{grad}(\alpha_i) = 0 \\
(iii) & \quad \frac{a_{ij}}{a_{ii}} \in \mathbb{Z} \quad \text{if } a_{ii} > 0, \quad \text{for } \text{grad}(\alpha_i) = 1
\end{align*}

The Borcherds superalgebra associated to the matrix $a_{ij}$ is generated by its Cartan subalgebra, the positive and negative generators satisfying the Chevalley-Serre (and superJacobi) relations. The Dynkin diagram of the real split Borcherds superalgebra corresponding to eleven-dimensional supergravity compactified on a $n$-torus ($n > 3$) is given in the figure 1. $\beta$ is a fermionic simple root of vanishing norm and $\alpha_i$ are simple bosonic roots of norm 2. The roots $(\alpha_i, \beta)$ for $i \geq 1$ define a $\text{sl}(n\vert 1)$ superalgebra and the roots $(\alpha_i)$ define the Dynkin diagram of the U-duality group $E_{(n\vert n)}$.

In [7], we have shown that the non-split form of these Borcherds superalgebras, which appear in supergravity theories with less than 32 supersymmetries, correspond also to possibly singular real del Pezzo surfaces. The real structure of the surface is a conjugation which preserves the intersection product and the canonical class $-K$, it is identified with the conjugation $\sigma$ defining the real form of the Borcherds algebra and defines a Satake superdiagram, which is a bicoloured Dynkin diagram with black vertices for the anti-invariant roots under $\sigma$. 
Note that the Borcherds (super)algebras have already appeared in superstring theory [9, 10, 11, 12]. Indeed, the physical states of a (super)string fully compactified on a Lorentzian lattice define a Borcherds (super)algebra. For example, if we compactify the superstring on the torus $T^9.1$ we obtain the Fake Monster superalgebra which contains the root lattice of $E_{10}$ and a $Z_2$-orbifolding gives the Monster superalgebra [13, 14]. It is important to observe that if we compactify the heterotic string on a torus the Narain lattice is not generally Lorentzian; the spectrum of perturbative 1/2-BPS states defines a vertex algebra and forms a Generalized Borcherds algebra which has not been studied in Mathematics [9]. The S-dual Lie algebra of the perturbative 1/2-BPS states on $T^4$, corresponding to 1/2-BPS D-branes of IIA on $K3$, can be obtained as the vertex algebra associated to the Picard group of $K3$. This is called the Nakajima construction [15].

Recently, $E_{10}$ (and $E_{11}$), introduced by one of us as candidate U-duality groups of eleven-dimensional supergravity compactified to one (and zero dimension), have been conjectured to be symmetries of M-theory in eleven dimension [16, 17]. Actually the Weyl group of $E_{10}$ appears when one studies the chaotic behaviour of M-theory near a cosmological singularity. The oscillatory evolution of generic cosmological solutions of Einstein’s D-dimensional gravity with several other fields coming from string theory or M-theory can be approximated by a flow in a relativistic billiard defined by the fundamental Weyl chamber of some Kac-Moody algebra. The solutions are chaotic if the corresponding Kac-Moody algebra is hyperbolic [18]. All these algebras suggest a new formulation of M-theory which should incorporate the fields and their duals in a symmetric way [19].

In the next part of this review, we will show that the classical equation of motion for the 3-form $A_3$ in M-theory [19] can be obtained using the superalgebra $osp(1|2)$ [5], which is generated by the following relations
for the positive generators:
\[
\{e_{\alpha_0}, e_{\alpha_0}\} = -e_{\alpha_1}, \quad [e_{\alpha_0}, e_{\alpha_1}] = 0, \quad [e_{\alpha_1}, e_{\alpha_1}] = 0.
\]  
(1.4)

Let us introduce the following nonlinear “potential” differential form:
\[
\mathcal{V} = e^{A(3)} e_{\alpha_0} e^{\tilde{A}(6)} e_{\alpha_1}.
\]  
(1.5)

The Grassmann angle $A(3)$ (resp. $\tilde{A}(6)$) is a 3-form (resp. 6-form) coupled to the M2- (resp. M5-) brane and defined on an eleven dimensional manifold $X$.

By an elementary calculation one checks that the field strength $\mathcal{G} = d\mathcal{V} \mathcal{V}^{-1}$, satisfying the Maurer-Cartan equation $d\mathcal{G} = \mathcal{G} \wedge \mathcal{G}$, following from (1.5) is given by
\[
\mathcal{G} = (dA(3)) e_{\alpha_0} + (d\tilde{A}(6) - \frac{1}{2} A(3) \wedge dA(3)) e_{\alpha_1},
\]  
(1.6)

Now, we introduce the self-duality equation: $S\mathcal{G} = *\mathcal{G}$ where $S$ is an operator which exchanges $e_{\alpha_0}$ with $e_{\alpha_1}$. Using (1.6), we obtain
\[
*S F(4) = F(7).
\]  
(1.7)

By taking the exterior derivative of this equation, we obtain the equation of motion of the 3-form in eleven-dimensional supergravity. This equation can be shown to be invariant under the Borel subgroup of a supergroup $OSP(1|2)$. Indeed, an element $\Lambda$ of this Borel subgroup satisfying $d\Lambda = 0$ acts on $\mathcal{V}$ on the right: $\mathcal{V}' = \mathcal{V} \Lambda$. The field strength is unchanged and the self-duality equation is preserved.

This construction can be generalized for the other toroidal compactifications of that theory and finite subsuperalgebras of the above Borcherds superalgebras preserve the equations of motion for the various $p$-forms in supergravity theories.

In [20] and recently in [21], it has been shown that the superalgebra (1.4) implies nonlinear relations of the type $\frac{t_{\alpha_0}}{2\pi} \frac{t_{\alpha_0}}{2\pi} = \frac{t_{\alpha_1}}{2\pi}$ where $t_{\alpha_0}$ (resp. $t_{\alpha_1}$) is the tension of the M2 brane (resp. M5 brane). Then using Dirac’s quantization condition the tensions can be computed. These relations can be generalized for our Borcherds superalgebras. This analysis suggest also that our Borcherds superalgebras must be broken in M-theory into an arithmetic supergroup with angles given by forms in $H^*(X, \mathbb{Z})$.

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