Lagrangian Formulation, Conservation Laws, Travelling Wave Solutions: A Generalized Benney-Luke Equation

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Abstract: The aim of this paper is to find the Noether symmetries of a generalized Benney-Luke equation. Thereafter, we construct the associated conserved vectors. In addition, we search for exact solutions for the generalized Benney-Luke equation through the extended tanh method. A brief observation on equations arising from a Lagrangian density function with high order derivatives of the field variables, is also discussed.

Keywords: Noether symmetries; conservation laws; Benney-Luke equation; extended tanh method

1. Introduction

In 1964, D.J. Benney and J.C. Luke obtained the following equation [1]

$$u_{tt} - u_{xx} + \alpha u_{xxxx} - \beta u_{xxtt} + u_{t} u_{xx} + 2 u_{x} u_{xt} = 0. \quad (1)$$

Here $\alpha, \beta$ are positive constants. Benney-Luke Equation (1) models waves propagating on the surface of a fluid in a shallow channel of constant depth taking into consideration the surface tension effect. The Benney-Luke equation and its generalizations have been extensively investigated [2–6]. The approaches used in the investigation of nonlinear evolution equations include stability analysis, Cauchy problem, existence and analyticity of solutions, etc. We refer the interested reader to references [7–12] and references therein. However in this present work, our goal is to compute conservation laws and exact solutions of Equation (1).

The organization of this paper is twofold. Firstly, we compute conservation laws of Equation (1) by employing Noether’s theorem. Secondly, we search for exact solutions of (1) by using the extended tanh method.

2. Noether Symmetries and Conservation Laws

In this section, we recall some essential notions on Noether symmetries and the associated conservation laws. The interested reader is referred to the cited paper for details [13,14].

Let us consider the vector field

$$X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}, \quad (2)$$

with the second-order prolongation

$$X^{[2]} = X + \xi_{t} \frac{\partial}{\partial u_{t}} + \xi_{x} \frac{\partial}{\partial u_{x}} + \xi_{xx} \frac{\partial}{\partial u_{xx}} + \xi_{tx} \frac{\partial}{\partial u_{tx}}, \quad (3)$$

where the coefficients of the partial derivatives are defined in [13,14].

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The Euler-Lagrange operator is defined by
\[ \frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial t} - D_x \frac{\partial}{\partial x} + D_x^2 \frac{\partial}{\partial u_x^2} + D_t D_x \frac{\partial}{\partial u_{tx}} + \cdots. \] (4)

### 2.1. Euler-Lagrange Equation

Let us consider a partial differential equation
\[ \Psi(t, x, u, u_t, u_x, u_{tx}, u_{xx}, \ldots) = 0. \] (5)

A function \( L(t, x, u, u_t, u_x, u_{tx}, u_{xx}, \ldots) \) is called a Lagrangian of Equation (5) if satisfies the Euler-Lagrange equation given by
\[ \frac{\delta L}{\delta u} = 0. \] (6)

### 2.2. Noether Symmetry

The vector field \( X \), defined in Equation (2), is a Noether symmetry corresponding to the Lagrangian \( L(t, x, u, u_t, u_x, u_{tx}, u_{xx}, \ldots) \) of Equation (5) if there exist the gauge terms \( B_1(t, x, u) \) and \( B_2(t, x, u) \) such that
\[ X^2(L) + \{D_t(\tau) + D_x(\xi)\}L = D_t(B_1) + D_x(B_2). \] (7)

### 2.3. Noether’s Theorem

To every Noether symmetry \( X \) corresponding to the Lagrangian \( L \) of Equation (6), there corresponds a vector \( T = (T^1, T^2) \) with components
\[ T^1 = L \tau + (\eta - u_t \tau - u_x \xi) \left( \frac{\partial L}{\partial u_t} \right) - D_t \left( \frac{\partial L}{\partial u_{tt}} \right) + (\zeta_1 - u_{tt} \tau - u_{tx} \xi) \left( \frac{\partial L}{\partial u_{tt}} \right) \]
\[ + (\zeta_2 - u_{xt} \tau - u_{xx} \xi) \left( \frac{\partial L}{\partial u_{tx}} \right) - B_1, \] (8)
\[ T^2 = L \xi + (\eta - u_t \tau - u_x \xi) \left( \frac{\partial L}{\partial u_x} \right) - D_t \left( \frac{\partial L}{\partial u_{tx}} \right) - D_x \left( \frac{\partial L}{\partial u_{xx}} \right) \]
\[ + (\zeta_2 - u_{xt} \tau - u_{xx} \xi) \left( \frac{\partial L}{\partial u_{xx}} \right) - B_2. \] (9)

### 2.4. Conservation Laws

We observe that the generalized Benney-Luke Equation (1) has a variational structure.

**Lemma 1.** The generalized Benney-Luke Equation (1) satisfies the Euler-Lagrange equations with the Lagrangian function defined by
\[ L = \frac{1}{2} u_x^2 - \frac{1}{2} u_t^2 + \frac{1}{2} a u_{xx}^2 - \frac{1}{2} b u_{tx}^2 - \frac{1}{2} u_t u_{xx}^2. \] (10)

**Proof.** Substituting \( L \) from Equation (10) into Equation (7) and splitting the monomials yields a linear overdetermined system of partial differential equations. Solving the resulting system of PDEs, Equation (7) admits only the variational Noether symmetries: time translation, space translation, and phase translation. Thus, the application of Noether’s theorem leads to the nontrivial conserved vector associated with three Noether point symmetries,
\[
T_1^1 = \frac{1}{2} u_x^2 + \frac{1}{2} u_t^2 + \frac{1}{2} \alpha u_{xx}^2 + \frac{1}{2} \beta u_{tx}^2,
\]
\[
T_2^1 = -u_t u_x + u_t^2 u_x - \beta u_t u_{tx} + \alpha u_t u_{xxx} - \alpha u_{tx} u_{xx},
\]
\[
T_1^2 = u_x u_t + \frac{1}{2} u_t^3 + \beta u_{tx} u_{xx},
\]
\[
T_2^2 = -\frac{1}{2} u_x^2 - \frac{1}{2} u_t^2 - \frac{1}{2} \alpha u_{xx}^2 - \frac{1}{2} \beta u_{tx}^2 + \frac{1}{2} u_t u_x^2 - \beta u_t u_{tx} + \alpha u_{xx} u_{xxx},
\]
\[
T_3^1 = -u_t - \frac{1}{2} u_{tx}^2,
\]
\[
T_3^2 = u_x - u_t u_x + \beta u_{tx} - \alpha u_{xxx},
\]
respectively. □

**Note 1:** It is worth pointing out that for equations that admit Lagrangian density depending on higher order derivatives of the field variables, the computations become a bit thorny, this is not because of conceptual complications, but somewhat due to notational difficulties. One must be very observant in expanding the multiple summations to avoid double counting of the appropriate jet coordinates. Nevertheless, variables such as \(u_{tx}\) or \(u_{xt}\) referred to as mixed derivatives are the same [14]. If one performs some double counting of the jet coordinates then this will leads to incorrect conserved vectors [14].

We now examine and discuss the physical meaning of the derived conservation laws. To begin with, conservation laws are of undisputed significance. They are the keystone for every essential theory of many physical phenomena. In this studying, we notice that the conserved vectors (11) yield a conserved energy because it arises from the time translation symmetry which leaves the Lagrangian (10) invariant. We also notice that the conserved vectors (12) yield a conserved momentum because it arises from the space translation symmetry which leaves the Lagrangian (10) invariant. We further observe that the Lagrangian (10) is invariant under the phase translation symmetry and this yields to charge conserved vectors.

3. Exact Solutions Using the Extended Tanh Method

In this section, we use the extended tanh function method introduced by Wazwaz [15]. We use the following ansatz:

\[
u(x,t) = F(z), \quad z = x - \omega t.
\]

Making use of (14), Equation (1) is reduced to the following nonlinear ordinary differential equation:

\[
\alpha F'''(z) - \beta \omega^2 F'''(z) + \omega^2 F''(z) - F''(z) - 3\omega F(z) F''(z) = 0.
\]

The basic idea in this method is to assume that the solution of (15) can be written in the form

\[
F(z) = \sum_{i=-M}^M A_i H(z)^i,
\]

where \(H(z)\) satisfies an auxiliary equation, for example the Riccati equation

\[
H'(z) = 1 - H^2(z),
\]

whose solution is given by

\[
H(z) = \tanh(z).
\]
The positive integer $M$ is determined by the homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (15). Here $A_i$ are parameters to be determined. In our case, the balancing procedure gives $M = 1$ and so the solutions of (15) are of the form

$$F(z) = A_{-1} H^{-1} + A_0 + A_1 H.$$  \hfill (19)

Substituting (19) into (15) and making use of the Riccati Equation (17) and then equating the coefficients of the functions $H^i$ to zero, we obtain an algebraic system of equations in terms of $A_i$. Solving the resultant system of algebraic equations, with the aid of Mathematica, leads to the following three cases (see Figures 1–3):

**Case 1**

$$\omega = k,$$ \hfill (20a)

$$A_{-1} = 0,$$ \hfill (20b)

$$A_1 = -\frac{-4k\alpha + 4k\beta}{4\alpha - 1};$$ \hfill (20c)

**Case 2**

$$\omega = k,$$ \hfill (21a)

$$A_{-1} = -\frac{-4k\alpha + 4k\beta}{4\alpha - 1},$$ \hfill (21b)

$$A_1 = 0;$$ \hfill (21c)

**Case 3**

$$\omega = p,$$ \hfill (22a)

$$A_{-1} = -\frac{-4p\alpha + 4p\beta}{16\alpha - 1},$$ \hfill (22b)

$$A_1 = -\frac{-4p\alpha + 4p\beta}{16\alpha - 1},$$ \hfill (22c)

where $k$ and $p$ are any roots of $(4\beta - 1)k^2 - 4\alpha + 1 = 0$ and $(-1 + 16\beta)p^2 - 16\alpha + 1 = 0$ respectively. As a result, a solution of (1) is

$$u(x, t) = A_{-1} \coth(z) + A_0 + A_1 \tanh(z),$$ \hfill (23)

where $z = x - \omega t$.

Figure 1. Evolution of the solution of (20) for case 1.
Figure 2. Evolution of the solution of (21) for case 2.

Figure 3. Evolution of the solution of (22) for case 3.

Note 2: It should be noted that the three cases given by the extended tanh method rise to kink solutions [15–17].

4. Concluding Remarks

The Noether symmetries of a generalized Benney-Luke equation were computed. Thereafter, we constructed the associated conservation laws. Moreover, we derived exact solutions for the generalized Benney-Luke equation via the extended tanh method. An observation on equations arising from a Lagrangian density function with high order derivatives of the field variables was also elaborated. It is expected that the derived conservation laws and exact solutions computed in this paper can be used as yardsticks for numerical simulations of the underlying equation and these will be reported elsewhere.

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