INTEGRAL $u$-DEFORMED INVOLUTION MODULES

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ABSTRACT. Let $(W, S)$ be a Coxeter system and $*$ an automorphism of $W$ with order $\leq 2$ and $S^* = S$. Lusztig and Vogan [11, 14] have introduced a $u$-deformed version $M_u$ of Kottwitz’s involution module over the Iwahori–Hecke algebra $\mathcal{H}_u(W)$ with Hecke parameter $u^2$, where $u$ is an indeterminate. Lusztig has proved that $M_u$ is isomorphic to the left $\mathcal{H}_u(W)$-submodule of $\mathcal{H}_u$ generated by $X_u := \sum_{w \in W} w^{-\ell(w)} T_w$, where $\mathcal{H}_u$ is the vector space consisting of all formal (possibly infinite) sums $\sum_{x \in W} c_x T_x$ ($c_x \in \mathbb{Q}(u)$ for each $x$). He speculated that one can extend this by replacing $u$ with any $\lambda \in \mathbb{C} \setminus \{0, 1, -1\}$. In this paper, we give a positive answer to his speculation for any $\lambda \in K \setminus \{0, 1, -1\}$ and any $W$, where $K$ is an arbitrary ground field.

1. INTRODUCTION

Let $(W, S)$ be a fixed Coxeter system and $*$ a fixed automorphism of $W$ with order $\leq 2$ and such that $S^* = S$. That is, $s^* \in S$ for any $s \in S$. Let $\ell : W \to \mathbb{N}$ be the usual length function on $W$. If $w \in W$ then by definition

$$\ell(w) := \min\{k \mid w = s_{i_1} \cdots s_{i_k} \text{ for some } s_{i_1}, \ldots, s_{i_k} \in S\}.$$

**Definition 1.1.** We define $I_s := \{w \in W \mid w^* = w^{-1}\}$. The elements of $I_s$ will be called twisted involutions relative to $*$.

Let $u$ be an indeterminate over $\mathbb{Q}$ (the field of rational numbers).

**Definition 1.2 (11, 18).** Let $\mathcal{H}_u := \mathcal{H}_u(W)$ be the associative unital $\mathbb{Q}(u)$-algebra with a $\mathbb{Q}(u)$-basis $\{T_w \mid w \in W\}$ and multiplication defined by

$$T_w T_{w'} = T_{ww'}, \text{ if } \ell(w w') = \ell(w) + \ell(w');$$

$$(T_s + 1)(T_s - u^2) = 0 \text{ if } s \in S.$$

We call $\mathcal{H}_u(W)$ the Iwahori–Hecke algebra over $\mathbb{Q}(u)$ associated to $(W, S)$ with Hecke parameter $u^2$.

Let $A := \mathbb{Z}[u, u^{-1}]$ be the ring of Laurent polynomials on $u$. Let $\mathcal{H}_{A,u}$ be the $A$-subalgebra of $\mathcal{H}_u$ generated by $\{T_w \mid w \in W\}$. Then $\mathcal{H}_{A,u}$ is a natural $A$-form of $\mathcal{H}_u$ and isomorphic to the abstract $A$-algebra defined by the same generators and relations as in Definition 1.2. For any field $K$ and any $\lambda \in K^\times$, there is a unique ring homomorphism $\phi : A \to K$ satisfying that $\phi(u) = \lambda$. We define $\mathcal{H}_\lambda := K \otimes_A \mathcal{H}_u$ and call $\mathcal{H}_\lambda$ the specialized Iwahori–Hecke algebra associated to $(W, S)$ with Hecke parameter $\lambda^2$.

Let $M_u$ be a $\mathbb{Q}(u)$-linear space with a $\mathbb{Q}(u)$-basis $\{a_z \mid z \in I_s\}$.

**Lemma 1.3 (11, 14).** There is a unique $\mathcal{H}_u$-module structure on $M_u$ such that for any $s \in S$ and any $w \in I_s$,

$$T_s a_w = u a_w + (u + 1)a_{sw} \text{ if } sw = ws^* > w;$$

$$T_s a_w = (u^2 - u - 1)a_w + (u^2 - u)a_{sw} \text{ if } sw = ws^* < w;$$

$$T_s a_w = a_{sws^*} \text{ if } sw \neq ws^* > w;$$

$$T_s a_w = (u^2 - 1)a_w + u^2 a_{sws^*} \text{ if } sw \neq ws^* < w.$$

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When $u$ is specialized to $1$, the module $M_u$ degenerates to the involution module introduced more than fifteen years ago by Kottwitz [9]. Kottwitz found the module by analyzing Langlands theory of stable characters for real groups. He gave a conjectural description of it (later established by Casselman) in terms of the Kazhdan–Lusztig left cell representations of the Weyl group $W$. One interesting fact about the module $M_u$ is that if $W$ is of finite classical type then any irreducible representation $V$ appears as a component of $M_u$ if and only if $V$ is a special irreducible representation of $W$ in the sense of [10]. For this reason, we call $M_u$ the $u$-deformed involution modules.

In a series of papers [11], [12], [13], [14], Lusztig and Vogan have studied the $u$-deformed involution modules systematically. A bar invariant canonical basis for $M_u$ and certain coefficient polynomials $P^\mu_{\nu,w}$ were introduced, which can be regarded as some twisted analogue of the classical well-known Kazhdan–Lusztig basis and Kazhdan–Lusztig polynomials (8).

Let $\mathcal{H}(u)$ (resp., $\mathcal{H}$) be the free $A$-module (resp., the $\mathbb{Q}(u)$-vector space) consisting of all formal (possibly infinite) sums $\sum_{x \in W} c_x T_x$, where $c_x \in A$ (resp., $c_x \in \mathbb{Q}(u)$) for each $x \in W$.

**Definition 1.4.** ([12]) We define

$$X_0 := \sum_{x \in W, x^* = x} u^{-\ell(x)} T_x \in \mathcal{H}(u) \subseteq \mathcal{H}.$$

**Theorem 1.5** ([13]). The map $\mu : M_u \rightarrow \mathcal{H}$ which sends $a_1$ to $X_0$ can be extended uniquely to a left $\mathcal{H}$-module isomorphism $M_u \cong \mathcal{H} X_0$.

Note that the above theorem was proved in [7] by the first author and Jing Zhang in the special case when $W = S_n$ and $*$ = $id$.

**Definition 1.6.** We define

$$A_{\pm 1} := \mathbb{Z}[u, u^{-1}, (u + 1)^{-1}, (u - 1)^{-1}].$$

Let $M_{A,u}$ be the free $A$-submodule of $M_u$ generated by $\{a_z | z \in I_1\}$. By Lemma [13] it is clear that $M_{A,u}$ naturally becomes a left $\mathcal{H}(u)$-module. We set

$$M_{A_{\pm 1},u} := A_{\pm 1} \otimes_A M_{A,u}, \quad \mathcal{H}_{A_{\pm 1},u} := A_{\pm 1} \otimes_A \mathcal{H}(u), \quad \mathcal{H}_{A_{\pm 1},u} := A_{\pm 1} \otimes_A \mathcal{H}(u).$$

For any ring homomorphism $\phi : A_{\pm 1} \rightarrow K$ with $\lambda = \phi(u) \in K \setminus \{0, 1, -1\}$, we define

$$M_\lambda := K \otimes_{A_{\pm 1}} M_{A_{\pm 1},u}, \quad \mathcal{H}_\lambda := K \otimes_{A_{\pm 1}} \mathcal{H}(u), \quad \mathcal{H}_\lambda := K \otimes_{A_{\pm 1}} \mathcal{H}(u).$$

If $W$ is finite, then $\mathcal{H}_{A_{\pm 1},u} = \mathcal{H}_{A_{\pm 1},u}$, $\mathcal{H}_u = \mathcal{H}_u$ and $\mathcal{H}_\lambda = \mathcal{H}_\lambda$. Note that $\mathcal{H}_{A,u}$ (resp., $\mathcal{H}_{A_{\pm 1},u}$) is a free $A$-module (resp., $A_{\pm 1}$-module) with basis $\{T_w | w \in W\}$. For simplicity, we shall often abbreviate $1_K \otimes_A T_w$ and $1_K \otimes_{A_{\pm 1}} T_w$ as $T_w$.

By some calculations in small ranks, Lusztig has speculated in [13] §4.10 that Theorem [13] might be extended to the setting of specialized version $\mathcal{H}_\lambda$ of $\mathcal{H}$ for arbitrary $\lambda \in \mathbb{C} \setminus \{0, 1, -1\}$ when $W$ is finite. Therefore, it is natural to make the following conjecture.

**Conjecture 1.7.** Let $K$ be a field and $\lambda \in K \setminus \{0, 1, -1\}$. Let $(W, S)$ be an arbitrary Coxeter system.

1. The map $\mu$ restricts to a left $\mathcal{H}_{A_{\pm 1},u}$-module isomorphism $M_{A_{\pm 1},u} \cong \mathcal{H}_{A_{\pm 1},u} X_0$.
2. For any ring homomorphism $\phi : A_{\pm 1} \rightarrow K$ with $\lambda = \phi(u)$, the map which sends $1_K \otimes_{A_{\pm 1}} a_1$ to $1_K \otimes_{A_{\pm 1}} X_0$ can be extended uniquely to a well-defined left $\mathcal{H}_\lambda$-module isomorphism $M_\lambda \cong \mathcal{H}_\lambda(1_K \otimes_{A_{\pm 1}} X_0)$. 
(3) For any ring homomorphism φ : A_{±1} → K with λ = φ(u), the canonical map

\[ \iota_K : K \otimes A_{±1} \mathcal{H}_{A_{±1}} X_0 \rightarrow \mathcal{H}_\lambda (1_K \otimes A_{±1} X_0) \]

\[ r \otimes h X_0 \mapsto (r \otimes h) (1_K \otimes A_{±1} X_0), \quad \forall r \in K, h \in A_{±1}, \]

is a left \mathcal{H}_\lambda-module isomorphism.

(4) If \( W \) is finite, then \( \mathcal{H}_{A_{±1}} X_0 \) is a pure and free \( A_{±1} \)-submodule of \( \mathcal{H}_{A_{±1}} \).

The purpose of this paper is to give a proof of the above conjecture and thus give a positive answer to Lusztig’s speculation. As an application of our main result, we obtain a new integral basis for the module \( M_u \) and for the module \( \mathcal{H}_{A_{±1}} X_0 \), see Corollary 2.21 and Corollary 2.28.

2. PROOF OF CONJECTURE \[\text{(1.7)}\]

The purpose of this section is to give a proof of Conjecture \[\text{(1.7)}\].

**Definition 2.1** \[\text{(4)}\]. For any \( w \in I_0 \) and \( s \in S \), we define

\[ s \prec w := \begin{cases} sw & \text{if } sw = ws^*; \\ sws^* & \text{if } sw \neq ws^*. \end{cases} \]

For any \( w \in I_0 \) and \( s_{i_1}, \ldots, s_{i_k} \in S \), we define

\[ s_{i_1} \prec s_{i_2} \prec \cdots \prec s_{i_k} \prec w := s_{i_1} \prec (s_{i_2} \prec \cdots \prec (s_{i_k} \prec w) \cdots). \]

It is clear that \( s \prec w \in I_0 \) whenever \( w \in I_0 \) and \( s \in S \). Furthermore, \( \prec \) is in general not associative.

**Definition 2.2** \[\text{(2.4)}\]. Let \( w \in I_0 \). If \( w = s_{i_1} \prec s_{i_2} \prec \cdots \prec s_{i_k} \prec 1 \), where \( k \in \mathbb{N}, s_{i_j} \in S \) for each \( j \), then \( (s_{i_1}, \ldots, s_{i_k}) \) is called an \( I_0 \)-expression for \( w \). Such an \( I_0 \)-expression for \( w \) is reduced if its length \( k \) is minimal.

We regard the empty sequence ( ) as a reduced \( I_0 \)-expression for \( w = 1 \). Let “\( \leq \)” be the Bruhat partial ordering on \( W \) defined with respect to \( S \) (cf. [6]). We write \( u < w \) if \( u \leq w \) and \( u \neq w \). It follows by induction on \( \ell(w) \) that every element \( w \in I_0 \) has a reduced \( I_0 \)-expression.

**Lemma 2.3** \[\text{(4, 5)}\]. Let \( w \in I_0 \). Any reduced \( I_0 \)-expression for \( w \) has a common length. Let \( \rho : I_0 \rightarrow \mathbb{N} \) be the map which assigns \( w \in I_0 \) to this common length. Then \( (I_0, \prec) \) is a graded poset with rank function \( \rho \). Moreover, if \( s \in S \) then \( \rho(s \prec w) = \rho(w) + 1 \), and \( \rho(s \prec u) = \rho(u) \) if and only if \( \ell(sw) = \ell(w) - 1 \).

**Corollary 2.4** \[\text{(7 Corollary 2.6)}\]. Let \( w \in I_0 \) and \( s \in S \). Suppose that \( sw \neq ws^* \). Then \( \ell(sw) = \ell(w) + 1 \) if and only if \( \ell(ws^*) = \ell(w) + 1 \), and \( \ell(s \prec w) = \ell(w) + 2 \). The same is true if we replace “\( + \)” by “\( \cdot \)”.

**Lemma 2.5**. Let \( z \in I_0 \). Let \( (s_{i_1}, \ldots, s_{i_k}) \) (where \( k \in \mathbb{N} \)) be an arbitrary reduced \( I_0 \)-expression of \( z \). Then there exist \( s_{i_{k+1}}, s_{i_{k+2}}, \ldots, s_{i_1}, s_{i_2} \in S \) and integers \( k \leq t_k \leq t_{k-1} \leq \cdots \leq t_1 = r \) such that \( r = \ell(z) \), and for each \( 1 \leq a \leq k, \)

\[ z_a := s_{i_a} \times s_{i_{a+1}} \times \cdots \times s_{i_k} \times 1 = s_{i_a} s_{i_{a+1}} \cdots s_{i_k} s_{i_{k+1}} \cdots s_{i_{t_a}}, \]

\[ \rho(z_a) = k - a + 1, \quad \ell(z_a) = t_a - a + 1. \]

In particular, \( s_{i_1} s_{i_{k+1}} \cdots s_{i_k} \) is a reduced expression of \( z_a, s_{i_1} \cdots s_{i_k} \) is a reduced expression and \( (i_{k+1}, i_{k+2}, \ldots, i_r) \) is uniquely determined by the reduced \( I_0 \)-expression \( (s_{i_1}, \ldots, s_{i_k}) \).

**Proof.** This follows from Lemma 2.3 and Corollary 2.4 and an induction on \( k \). \( \square \)

**Definition 2.6**. For each \( z \in I_0 \) and each reduced \( I_0 \)-expression \( \sigma = (s_{i_1}, \ldots, s_{i_k}) \) of \( z \), we define

\[ \sigma_z := s_{i_1} \cdots s_{i_k} \in W. \]
In particular, we have $\sigma_1 = 1$ and $\rho(z) = \ell(\sigma_z)$ for any $z \in I_*$. In general, $\sigma_z$ depends on the choice of the reduced $I_*$-expression $(s_{i_1}, \cdots, s_{i_k})$ of $z$.

**Definition 2.8** ([5] Proposition 2.5, [13] Proposition 2.2). Let $w \in I_*$ and $(s_{i_1}, \cdots, s_{i_k})$ be a reduced $I_*$-expression of $w$. We define

$$w_0 := w, \quad w_t := s_{i_t} \cdot w_{t-1}, \quad \text{for } 1 \leq t \leq k.$$ 

Define $\ell^* : I_* \to \mathbb{N}$ by

$$\ell^*(w) := \# \{1 \leq t \leq k \mid s_{i_t} w_t = w_t s_{i_t} \}.$$ 

The notation $\ell^*(w)$ we used here was denoted by $\ell^*(w)$ in [4] and [5], and was denoted by $\phi$ in [13] §1.5. By [5], $\ell^*(w)$ depends only on $w$ but not on the choice of the reduced $I_*$-expression $(s_{i_1}, \cdots, s_{i_k})$ of $w$.

**Lemma 2.9** ([3] Theorem 4.8], [5] §2.2). Let $w \in I_*$. Then $\rho(w) = (\ell(w) + \ell^*(w))/2$.

Let $\tau : \mathcal{A} \to \mathcal{A}$ be the ring involution such that $\tau^n = (-u)^{-n}$ for any $n \in \mathbb{Z}$. Let $\epsilon : I_* \to \{1, -1\}, \quad z \mapsto (-1)^{\rho(z)}$, $\forall z \in I_*$. By Lemma 2.9 our $\epsilon$ coincides with the function $\epsilon$ defined in [13] §1.5.

**Definition 2.10** ([13] §1.1). Let $\{L^z_x \mid z \in I_*, x \in W\}$ be a set of uniquely determined polynomials in $\mathbb{Z}[u]$ such that

$$T_x a_1 = \sum_{z \in I_*} L^z_x a_z, \quad \forall x \in W.$$ 

**Definition 2.11** ([13] §1.6]). For $x \in W$, $z \in I_*$, we set

$$\tilde{L}^z_x := (-1)^{\ell(x)} \epsilon(z) L^z_x.$$ 

**Lemma 2.12** ([13] §1.7, the 5th line above §1.8]). For $z \in I_*$, we have

$$\mu(a_z) = \sum_{x \in W} \tilde{L}^z_x T_x.$$ 

Note that there is a typo in the identity on $\mu(a_z)$ in [13] §1.7, the 5th line above §1.8]. The element $T_x$ in the right hand should be replaced by $T_x$.

**Proposition 2.13.** Let $z \in I_*$ and $\sigma = (s_{i_1}, \cdots, s_{i_k})$ be a reduced $I_*$-expression of $z$. Let $(k_{k+1}, \cdots, k_r)$ be the unique $(r-k)$-tuple determined by this reduced $I_*$-expression as described in Lemma 2.5. Then $z = s_{i_1} \cdots s_{i_k} \cdot 1 = s_{i_1} \cdots s_{i_k} s_{k+1} \cdots s_r$ with $k = \rho(z) - \ell(z)$, where $s_{i_1}, \cdots, s_{i_r}$, $s_r \in S$, and $L^z_{\sigma^e} = (u+1)^{\epsilon(z)}$, and $L^\sigma_w \neq 0$ only if $\rho(w) < \rho(z)$ and there exists a reduced $I_*$-expression $\sigma^o$ of $w$ such that $\sigma^o_w < \sigma_z$. Moreover, $L^\sigma_w \in u\mathbb{Z}[u]$ if $w \neq z$.

**Definition 2.14.** Let $z \in I_*$ and $\sigma = (s_{i_1}, \cdots, s_{i_k})$ be a reduced $I_*$-expression of $z$. We define

$$I_*(\prec_{\sigma} z) := \left\{ w \in I_* \mid \rho(w) < \rho(z) \text{ and there exists a reduced } I_*-\text{expression } \sigma^o \text{ of } w \text{ such that } \sigma^o_w < \sigma_z \right\}.$$ 

Then Proposition 2.13 is equivalent to the following identity:

\begin{equation}
T_{\sigma} a_1 = (u+1)^{\ell(z)} a_z + \sum_{w \in I_*(\prec_{\sigma} z)} L^\sigma_w a_w, \quad L^\sigma_w \in u\mathbb{Z}[u].
\end{equation}

**Proof of Proposition 2.13.** Let $z \in I_*$. We prove the proposition by induction on $\rho(z)$. If $\rho(z) = 0$, then $\varepsilon = 1$, $\sigma_z = 1$ and $T_1 a_1 = a_1$.

Let $k \in \mathbb{N}^*$. Suppose that the statement holds when $\rho(z) < k$. Let $z \in I_*$ with $\rho(z) = k$. We follow the notation and hypothesis in Lemma 2.5 and Definition 2.6. Then $z = s_{i_1} \cdots s_{i_k} \cdot 1 = s_{i_1} \cdots s_{i_k} s_{k+1} \cdots s_r$ with $k = \rho(z)$, $r = \ell(z)$ for some $s_{i_1}, \cdots, s_{i_r} \in S$. By definition, $\sigma_z = s_{i_1} \cdots s_{i_k}$. Let $x' = s_{i_1} \sigma_z = s_{i_2} \cdots s_{i_k}$. Note that
\((s_1, \ldots, s_k)\) is a reduced \(I_\sigma\)-expression implies that \(\sigma' := (s_{i_2}, \ldots, s_{i_k})\) is a reduced \(I_\sigma\)-expression of \(z' := s_{i_2} \wedge \cdots \wedge s_{i_k} \wedge 1\), then \(\rho(z') = k - 1\) and \(x' = \sigma_{z'}\) in the notation of Definition 2.6. Now \(x' < \sigma_z\) and

\[
\begin{align*}
T_{\sigma_z}a_1 &= T_{a_1} = T_{\sigma_z}a_1 \\
&= T_{a_1} \left( (u + 1)^{\ell_\sigma}a_z + \sum_{z'' \in I_\sigma(\langle \sigma, z' \rangle)} L_{z''}a_z \right), \\
&= (u + 1)^{\ell_\sigma(z')}T_{a_1}a_{z'} + \sum_{z'' \in I_\sigma(\langle \sigma, z' \rangle)} L_{z''}a_{z''}.
\end{align*}
\]

We consider the first term in the above identity. There are two possibilities:

**Case 1.** If \(s_{i_1}z' = z's_{i_1}^\ast\), then \(z = s_{i_1} \wedge z' = s_{i_1}z'\). Thus

\[
T_{a_1}a_{z'} = ua_{z'} + (u + 1)a_{s_{i_1}z'} = ua_{z'} + (u + 1)a_z,
\]

where \(\ell_\sigma(z) = \ell_\sigma(z') + 1\) and \(z' \in I_\sigma(\langle \sigma, z \rangle)\), as required.

**Case 2.** If \(s_{i_1}z' \neq z's_{i_1}^\ast\), then \(z = s_{i_1} \wedge z' = s_{i_1}z's_{i_1}^\ast\). Thus

\[
T_{a_1}a_{z'} = a_{s_{i_1}z's_{i_1}^\ast} = a_z,
\]

where \(\ell_\sigma(z) = \ell_\sigma(z')\), as required.

Therefore, it remains to consider the term \(T_{a_1}a_{z''}\) for each \(z'' \in I_\sigma(\langle \sigma, z' \rangle)\). We know that \(\sigma_z = s_{i_1}x' = s_{i_1} \cdots s_{i_k}\) and \(\sigma_{z'} = x' = s_{i_1} \cdots s_{i_k}\) are reduced expressions. Combining our assumption \(z'' \in I_\sigma(\langle \sigma, z' \rangle)\) and Lemma 1.3, together we can deduce that \(T_{a_1}a_{z''}\) is a \(\mathbb{Z}[u]\)-linear combination of some \(a_w\) with \(w \in I_\sigma(\langle \sigma, z \rangle)\). Therefore, we get that

\[
T_{\sigma_z}a_1 = (u + 1)^{\ell_\sigma(z)}a_z + \sum_{w \in I_\sigma(\langle \sigma, z \rangle)} L_w^\sigma a_w,
\]

where \(L_w^\sigma \in u\mathbb{Z}[u]\) for each \(w \in I_\sigma(\langle \sigma, z \rangle)\). This completes the proof of the proposition.

Note that \(w \in I_\sigma(\langle \sigma, z \rangle)\) implies that \(\rho(w) < \rho(z)\).

**Corollary 2.16.** Let \(z \in I_\sigma\) and \(\sigma, \hat{\sigma}\) be two reduced \(I_\sigma\)-expressions of \(z\). Then

\[
T_{\sigma_z}a_1 = (u + 1)^{\ell_\sigma(z)}a_z + \sum_{w \in I_\sigma(\langle \sigma, z \rangle)} L_w^\sigma a_w, \quad L_w^\sigma \in u\mathbb{Z}[u],
\]

\[
T_{\sigma_z}a_1 \equiv T_{\hat{\sigma}_z}a_1 \pmod{\sum_{\rho(w) < \rho(z)} u\mathbb{Z}[u]a_w}.\]

**Corollary 2.17.** For each \(z \in I_\sigma\), we fix a reduced \(I_\sigma\)-expression \(\sigma_z\) of \(z\) and define \(\sigma_{z_1}\) as in Definition 2.6. Then the map \(\sigma_z : z \mapsto \sigma_{z_1}\) defines an injection from \(I_\sigma\) into \(W\). In other words, \(\sigma_{z_1} = \sigma_{z_2}\) if and only if \(z_1 = z_2\).

**Proof.** This follows from Proposition 2.7.3.

**Definition 2.18.** We define

\[
\mathcal{A}^{-1} := \mathbb{Z}[u, u^{-1}, (u + 1)^{-1}], \quad \mathcal{A}_1 := \mathbb{Z}[u, u^{-1}, (u - 1)^{-1}].
\]

**Corollary 2.19.** For each \(z \in I_\sigma\) we fix a reduced \(I_\sigma\)-expression \(\sigma_z\) of \(z\) and define \(\sigma_z\) as in Definition 2.6. Then

\[
a_z = \frac{1}{(u + 1)^{\ell_\sigma(z)}} T_{\sigma_z}a_1 + \sum_{w \in I_\sigma(\langle \sigma, z \rangle)} \xi_z^w T_{\sigma_w}a_1,
\]

where \(\xi_z^w \in \mathcal{A}_1\) and \(\mathcal{A}_1 \subset \mathcal{A}^{-1}\).
where for each $w \in I_\star$, $\xi^w \in A_{-1}$. In particular,
\[
a_z = \frac{1}{(u+1)^{\ell(z)}} T_{\sigma_z} a_1 + \sum_{\substack{w \in I_\star \\ \rho(w) < \rho(z)}} \xi^w T_{\sigma_w} a_1
\]
\[
= \frac{1}{(u+1)^{\ell(z)}} T_{\sigma_z} a_1 + \sum_{\substack{w \in I_\star \\ \ell(z) < \ell(\sigma_z)}} \xi^w T_{\sigma_w} a_1.
\]

**Proof.** This follows from Proposition 2.13 and Corollary 2.16. \hfill \Box

Let $M_{A,u}$ be the free $A$-submodule of $M_u$ generated by $\{a_z \mid z \in I_\star\}$. It is clear that $M_{A,u}$ naturally becomes a left $\mathcal{H}_{A,u}$-module. We set
\[
M_{A_{-1},u} := A_{-1} \otimes_A M_{A,u}, \quad M_{A_1,u} := A_1 \otimes_A M_{A,u}.
\]

For each $z \in I_\star$, we identify $1_{A_{-1}} \otimes_A a_z$, $1_{A_1} \otimes_A a_z$, $1_{A_{-1}} \otimes_A a_z$ and $1_{Q(u)} \otimes_A a_z$ with $a_z$.

**Corollary 2.21.** For each $z \in I_\star$, we fix a reduced $I_\star$-expression $\sigma$ of $z$ and define $\sigma_z$ as in Definition 2.6. Then the elements in the following set
\[
\{T_{\sigma_z} a_1 \mid z \in I_\star\}
\]
form an $A_{-1}$-basis of $M_{A_{-1},u}$, an $A_{1}$-basis of $M_{A_{1},u}$ and a $Q(u)$-basis of $M_u$. The same is true if one replaces $A_{-1}$ with any field $K$ and $u$ with any $\lambda \in K^\times$ whenever there is a ring homomorphism $\phi : A_{-1} \to K$ with $\lambda = \phi(u)$.

**Proof.** This follows from Proposition 2.13 and (2.15). \hfill \Box

**Corollary 2.23.** For each $z \in I_\star$, we fix a reduced $I_\star$-expression $\sigma$ of $z$ and define $\sigma_z$ as in Definition 2.6. We have that
\[
\mathcal{H}_{A_{-1},u} X_0 = A_{-1} \cdot \text{Span}\{T_{\sigma_z} X_0 \mid z \in I_\star\}.
\]
In particular,
\[
\mathcal{H}_{A_{1},u} X_0 = A_{1} \cdot \text{Span}\{T_{\sigma_z} X_0 \mid z \in I_\star\},
\]
and the map $\mu \downarrow_{M_{A_{-1},u}} : M_{A_{-1},u} \to \mathcal{H}_{A_{-1},u} X_0$ is surjective.

**Proof.** This follows from Corollary 2.21 and the surjectivity of $\mu \downarrow_{M_{A_{-1},u}} : M_{A_{-1},u} \to \mathcal{H}_{A_{-1},u} X_0$. \hfill \Box

For any field $K$ and any ring homomorphism $\phi : A_{\pm 1} \to K$ with $\lambda = \phi(u)$, we define
\[
\mu_K : M_X \to \mathcal{H}_{\lambda}(1_{K} \otimes_A X_0)
\]
to be the composition of the following surjection
\[
\text{id}_{K} \otimes_A \mu \downarrow_{M_{A_{\pm 1},u}} : M_X = K \otimes_{A_{\pm 1}} M_{A_{\pm 1},u} \to K \otimes_{A_{\pm 1}} \mathcal{H}_{A_{\pm 1},u} X_0
\]
with the canonical surjective homomorphism
\[
\iota_X : K \otimes_{A_{\pm 1}} \mathcal{H}_{A_{\pm 1},u} X_0 \to \mathcal{H}_{\lambda}(1_{K} \otimes_{A_{\pm 1}} X_0)
\]
introduced in Conjecture 1.7. By definition, we know that $\mu_K$ is surjective.

**Proposition 2.24.** Let $K$ be a field. For any ring homomorphism $\phi : A_{\pm 1} \to K$ with $\lambda = \phi(u)$, the elements in the following set
\[
\{Y_{K,z} := \mu_K (1_{K} \otimes_{A_{\pm 1}} a_z) \in \mathcal{H}_{\lambda}(1_{\otimes_{A_{\pm 1}}} X_0) \mid z \in I_\star\}
\]
form a $K$-basis of $\mathcal{H}_{\lambda}(1_{K} \otimes_{A_{\pm 1}} X_0)$. In particular, $\mu_K$ is a left $\mathcal{H}_{\lambda}$-module isomorphism. Furthermore, the elements in the following set
\[
\{Y_z := \mu(a_z) \in \mathcal{H}_{A_{\pm 1},u} X_0 \mid z \in I_\star\}
\]
form an $A_{\pm 1}$-basis of $\mathcal{H}_{A_{\pm 1},u} X_0$. 

Proof. We consider the first part of the proposition. Recall that $\mu_K$ is surjective. Since 
$\{1_K \otimes A_{z_1}, a_z \mid z \in I_1\}$ is a $K$-basis of $M_\lambda$, it suffices to show that the elements in the
subset (2.25) are $K$-linearly independent. Note that the assumption $\lambda \neq 0$, $-1$ is used here
to ensure that $\mu_K$ is surjective (by Corollary 2.21 and Corollary 2.23).
Suppose that the elements in the subset (2.25) are $K$-linearly dependent. That says, we
can find an positive integer $m$ and
$$\{z_1, z_2, \cdots, z_m\} \subseteq I_1,$$
such that $Y_{K, z_1}, \cdots, Y_{K, z_m}$ are $K$-linearly dependent. For each $z \in \{z_1, \cdots, z_m\}$, we fix
a reduced $I_1$-expression of $z$ and define $\sigma_z$ as in Definition 2.6. Without loss of generality,
we can assume that $\rho(z_1) \leq \rho(z_2) \leq \cdots \leq \rho(z_m)$. Equivalently, $\ell(\sigma_z_1) \leq \ell(\sigma_z_2) \leq \cdots \leq \ell(\sigma_{zm})$. Furthermore, we can find an integer $n \geq m$ and a finite subset \{w_{m+1}, \cdots, w_n\} of $W \setminus \{\sigma_{z_1}, \cdots, \sigma_{zm}\}$ such that for each $1 \leq j \leq m$,
\[
(2.27) \quad Y_{z_j} = \mu(a(z_j)) = \sum_{i=1}^{m} L_{z_j}^{\sigma_z_i} T_{\sigma_z_i} + \sum_{k=m+1}^{n} \tilde{L}_{w_k} T_{w_k}.
\]
By Definition 2.10, Definition 2.11, Lemma 2.12 and Proposition 2.13
\[
(Y_{z_1}, Y_{z_2}, \cdots, Y_{z_m}) = (T_{\sigma_z_1}, T_{\sigma_z_2}, \cdots, T_{\sigma_z_m}, T_{w_{m+1}}, T_{w_{m+2}}, \cdots, T_{w_n}) D_1 A_u D_2,
\]
where $D_1$ is the following $n \times n$ diagonal matrix:
$$D_1 = \text{Diag}((-1)^{\ell(\sigma_{z_1})}, \cdots, (-1)^{\ell(\sigma_{zm})}, (-1)^{\ell(w_{m+1})}, \cdots, (-1)^{\ell(w_n)}),$$
$D_2$ is the following $n \times n$ diagonal matrix:
$$D_2 = \text{Diag}(e(z_1), \cdots, e(z_m)),$$
and $A_u$ is the following $n \times m$ matrix in $M_{n \times m}(\mathbb{Z}[u^{-1}])$:
$$A_u = \begin{pmatrix}
(1 - u^{-1})^{\ell(z_1)} & 0 & & \\
& (1 - u^{-1})^{\ell(z_2)} & & \\
& & \ddots & \\
& & & (1 - u^{-1})^{\ell(z_m)}
\end{pmatrix},$$
such that the top $m \times m$ submatrix is a lower triangular matrix with diagonal elements
given by \{(1 - u^{-1})^{\ell(z_1)}, \cdots, (1 - u^{-1})^{\ell(z_m)}\}.

By assumption $\lambda = \phi(u) \not\in \{0, 1, -1\}$. We define $A_\lambda := A_u^{-1}$. Then
\[
(1_K \otimes A_{z_1}, Y_{z_1}, \cdots, 1_K \otimes A_{z_m}, Y_{z_m}) = \\
(1_K \otimes A_{z_1}, T_{\sigma_z_1}, \cdots, 1_K \otimes A_{z_m}, T_{\sigma_z_m}, 1_K \otimes A_{z_{m+1}}, T_{w_{m+1}}, \cdots, 1_K \otimes A_{z_n}, T_{w_n}) D_1 A_\lambda D_2,
\]
By the above discussion and the assumption that $\lambda \neq 1$ we can see that rank $A_\lambda = m$.
Note that $\{1_K \otimes A_{z_1}, T_w \mid w \in W\}$ is a subset of $K$-linearly independent elements in $\mathcal{H}_\lambda$.
Since $D_1, D_2$ are invertible, it follows that
$$\{Y_{K, z_1} = 1_K \otimes A_{z_1}, Y_{z_1}, \cdots, Y_{K, z_m} = 1_K \otimes A_{z_m}, Y_{z_m}\}$$
is a set of $K$-linearly independent elements in $\mathcal{H}(1_K \otimes A_{z_1}, X_0) \subseteq \mathcal{H}_\lambda$. We get a contradiction. In particular, this implies that $\mu_K$ is a left $\mathcal{H}_\lambda$-module isomorphism. This proves the first part of the proposition.
Finally, taking \( K = \mathbb{Q}(u) \) we see that \( \mu_{\mathbb{Q}(u)} \) is an isomorphism by the first part of the proposition which we have just proved. This further implies that \( \text{id}_{\mathbb{Q}(u)} \otimes A_{\pm 1, u} \downarrow M_{A_{\pm 1, u}} \) is an isomorphism. Since
\[
\mathbb{Q}(u) \otimes A_{\pm 1, u}, \text{ Ker } \mu \downarrow M_{A_{\pm 1, u}} \subseteq \text{ Ker(} \text{id}_{\mathbb{Q}(u)} \otimes A_{\pm 1, u} \downarrow M_{A_{\pm 1, u}} \text{)} = \{0\},
\]

it follows that \( \text{ Ker } \mu \downarrow M_{A_{\pm 1, u}} = 0 \). Hence \( \mu \downarrow M_{A_{\pm 1, u}} \) is an isomorphism and the elements in (2.26) form an \( A_{\pm 1} \)-basis of \( \mathcal{H}_{\pm 1, u}X_0 \). This proves the second part of the proposition and hence we complete the proof of the proposition. \( \square \)

Proof of Conjecture 1.7: (1) and (2) follows from Proposition 2.24. Now (3) follows from Proposition 2.13 and Proposition 2.12. It remains to consider (4). For this purpose, we assume that \( X_0 \in \mathcal{H}_{\pm 1, u} \).

By (2.27) and Proposition 2.13, we easily see that the elements in the following set
\[
\{ T_w, X_0 \mid z \in I \}
\]
form an \( A_{\pm 1} \)-basis of \( \mathcal{H}_{\pm 1, u}X_0 \). This implies that \( \mathcal{H}_{\pm 1, u}X_0 \) is a pure and free \( A_{\pm 1} \)-submodule of \( \mathcal{H}_{\pm 1, u} \). This completes the proof of Conjecture 1.7.

Corollary 2.28. The elements in the following set
\[
T_\zeta, X_0 \mid z \in I \}
\]
form an \( A_{\pm 1} \)-basis of \( \mathcal{H}_{\pm 1, u}X_0 \). The same is true if one replaces \( A_{\pm 1} \) with any field \( K \) and \( u \) with any \( \lambda \in K^* \) whenever there is a ring homomorphism \( \phi : A_{\pm 1} \rightarrow K \) with \( \lambda = \phi(u) \).

Proof. This follows from Corollary 2.21 Corollary 2.23 and Conjecture 1.7 (which we have just proved). \( \square \)

By Lemma 2.12
\[
\mu(a_z) = \sum_{x \in W} \tilde{L}_x^z T_x,
\]
where \( \tilde{L}_x^z \in \mathbb{Z}[\mu^{-1}] \). Following [13] Theorem 0.2(b)], we define \( n_x^z := \tilde{L}_x^z \downarrow_{\mu^{-1}w} \in \mathbb{Z} \). Then Lusztig has proved in [13] Theorem 0.2(c)] that there is a unique surjective function \( \pi : W \rightarrow I_x \) such that for any \( x \in W, z \in I, \) we have \( n_x^z = 1 \) if \( z = \pi(x) \); and \( n_x^z = 0 \) if \( z \neq \pi(x) \).

Our next result shows that the map \( \sigma_x \) introduced in Corollary 2.17 is a right inverse of \( \pi \).

Corollary 2.30. Let \( \sigma_x : I_x \hookrightarrow W \) be the injection defined in Corollary 2.17 Then \( \pi \circ \sigma_x = \text{id}_{I_x} \).

Proof. Let \( z \in I_x \). Following [13] §1.8], we use \( \{ T_w \mid w \in W \} \) to denote the standard basis of the specialization \( \mathcal{H}_0 \) of \( \mathcal{H}_u \) at \( u := 0 \), and use \( M_0 \) to denote the specialization of \( M \) at \( u := 0 \). Then \( M_0 \) is a \( \mathbb{Q} \)-space with basis \( \{ a_w \mid w \in I \} \) and with \( \mathcal{H}_0 \)-module structure given by
\[
\begin{align*}
T_w a_w &= a_w \quad \text{if } sw = ws^* > w; \\
T_w a_w &= a_{ws^*} \quad \text{if } sw \neq ws^* > w; \\
T_w a_w &= -a_w \quad \text{if } sw < w,
\end{align*}
\]
where \( s \in S, w \in I_x \).

Setting \( u := 0 \) on both sides of (2.15), we get that
\[
T_w a_w = a_w.
\]
On the other hand, since \( \tilde{L}^{x}_{\sigma z} := (-1)^{\ell(\sigma z)}e(x)L^{x}_{\sigma z} \) for any \( x \in I_{\ast} \), setting \( u := 0 \) in \( L^{x}_{\sigma z} \) is equivalent to setting \( u^{-1} := 0 \) in \( \tilde{L}^{x}_{\sigma z} \). We can deduce from [13] Theorem 0.2(c) that 
\[
\pi \circ \sigma_{z}(z) = \pi(\sigma_{z}) = z.
\]
Note that \((-1)^{\ell(\sigma z)}e(z) = (-1)^{\rho(z)}(-1)^{\rho(z)} = 1\). This completes the proof of the corollary. \( \square \)

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