Short Wavelength Analysis of the Evolution of Perturbations in a Two-component Cosmological Fluid

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Abstract

The equations describing a two-component cosmological fluid with linearized density perturbations are investigated in the small wavelength or large $k$ limit. The equations are formulated to include a baryonic component, as well as either a hot dark matter (HDM) or cold dark matter (CDM) component. Previous work done on such a system in static spacetime is extended to reveal some interesting physical properties, such as the Jeans wavenumber of the mixture, and resonant mode amplitudes. A WKB technique is then developed to study the expanding universe equations in detail, and to see whether such physical properties are also of relevance in this more realistic scenario. The Jeans wavenumber of the mixture is re-interpreted for the case of an expanding background spacetime. The various modes are obtained to leading order, and the amplitudes of the modes are examined in detail to compare to the resonances observed in the static spacetime results. It is found that some conclusions made in the literature about static spacetime results cannot be carried over to an expanding cosmology.

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I. INTRODUCTION

The analysis of cosmological perturbations in the Newtonian limit is a well studied problem in theories of structure formation, and it may be supposed that there is little left to learn from this theory. Most of the effort has gone into the study of the one-component cosmological fluid equations, and the results have been well expounded in many standard texts [1, 2, 3, 4, 5]. There is, however, still a wealth of problems remaining in the detailed analysis of two-component cosmological fluids and their linearized gravitational perturbation modes. In particular, if pressure effects are included so that the Jeans instability becomes an issue, the equations present a considerable analytic challenge, and a range of new physical effects become apparent. Some of these effects have been studied in the contrived case of a static spacetime background [6, 7]. In this scenario there is no expansion, so that the mathematics is considerably simplified, and solutions can easily be found. This is useful to gain some qualitative idea about physical phenomena observable, but to gain a true picture in a cosmological context, the expanding background spacetime given by the Friedmann-Robertson-Walker cosmologies is required.

There have been a variety of studies of the multi-component cosmological fluid equations, ranging from some relatively specific applications under certain cosmological scenarios [8, 9], to a broad mathematical study and classification [10]. A discussion of the application and validity of some of the equations mentioned in these previous studies, together with the solution of an unsolved set of two-component post-recombination equations, has recently been undertaken by the authors [11]. The system of equations described the interaction between a dark matter and baryonic component in the Newtonian regime (density fluctuations on scales well within the Hubble radius). A series expansion of the solutions for small wavenumber \( k \) (large scales) was presented. This allowed comparison with some of the previous work, in particular with the Meijer G-function classifications given by [10]. This region of \( k \)-space is also interesting because it is the region in which the Jeans instability is known to occur.

In this paper we wish to complete this study by examining the large \( k \) asymptotic region of the solutions. Such a study is worthwhile, in order to make contact with the static spacetime results of [7]. Although not realistic as cosmological solutions, these results displayed a number of little known physical phenomena associated with the linearized modes, which we wish to expand on here. The techniques required to analyze the expanding universe solutions
are also of interest in their own right mathematically, where a generalized WKB method will be expounded. It is possible to make a comparison with the work done in cosmological plasma physics in an Einstein-deSitter background [13, 14]. This is interesting because of the mathematically very similar form of fluid equations for both type of systems, which is due to the similarity of the electromagnetic and gravitational forces. Thus mathematical techniques employed in the analysis of plasma equations will be useful in this paper, and give clues as to how to proceed with some challenging mathematical analysis of gravitational density perturbation modes.

The paper is to be organized as follows. The relevant equations will be introduced in Section 2. The discussion will then be focused in Section 3, by reconsidering the two-component modes in a static spacetime. This investigation is by necessity of a qualitative nature, but gives a useful introduction to the concepts and interesting physical effects not found in the standard one-component analysis. The work of [7] will also be extended. The expanding universe baryonic and dark matter equations will then be considered in Section 4. The short wavelength (WKB) approximation will be utilized to complete the study of these equations initiated in [11]. The relevance of the previous work on static spacetime systems is revealed through this analysis. This will allow meaningful conclusions to be drawn about this whole area of study, and point to where promising future work may lie. These aspects are discussed in Section 5.

II. THE GOVERNING EQUATIONS

A broad survey of the Newtonian cosmological perturbation equations under various cosmological scenarios was given in [11]. This paper also gave a detailed derivation of the equations of interest for present analysis. We will not repeat such a detailed discussion here, but directly introduce the relevant equations.

The starting point is the equations for an $n$-component system of nonrelativistic species, as derived in all the standard texts. Given a density perturbation $\delta_i$ of the $i$-th component of the mass density $\rho_i$:

$$\delta_i(r, t) = \frac{\delta \rho_i}{\rho_i},$$

(2.1)
it may be decomposed into its Fourier plane wave modes with wave vector \( \mathbf{k} \)
\[
\delta_i(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int \delta_{\mathbf{k}i}(t) \exp(-i\mathbf{k} \cdot \mathbf{r}) d^3r.
\] (2.2)

Here \( \mathbf{r} \) is the physical spatial coordinate, and \( t \) is cosmic time. Using the Eulerian equations of motion describing a perfect fluid, a set of coupled second order equations for the Fourier modes \( \delta_i(t) \) (where we now drop the subscript \( k \)) are achieved:
\[
\frac{d^2 \delta_i}{dt^2} + 2 \dot{a} \frac{d\delta_i}{dt} + v_i^2 k^2 \delta_i = 4\pi G \sum_{i=1}^{n} \rho_i \delta_i, \quad i = 1, 2, \ldots n.
\] (2.3)

Overdots will denote derivatives with respect to \( t \).

The above equations contain the universe expansion factor \( a \) and sound velocities \( v_i \). We use the expression “sound velocity” fairly loosely. The parameters \( v_i \) could also denote a general velocity dispersion for a collisionless fluid. To be able to solve the equations, the time dependence of the physical variables needs to be made explicit. We will adopt the convention that barred variables will denote comoving quantities, independent of time. Thus the definition of comoving wave number \( \bar{k} = ak \) arises naturally. An assumption must also be made about the scaling of the sound velocities \( v_i \). In the post-recombination era, the adiabatic speed of sound follows the behavior \( v \propto a^{-1} \), so we will introduce the time independent quantities \( \bar{v}_i \equiv av_i \). The total background energy density \( \rho_0 \) can also be made independent of time by the definition \( \bar{\rho}_0 \equiv a^3 \rho_0 \). This enables us to introduce the useful parameter \( \epsilon_i = \rho_i/\rho_0 \), the fraction of mass density contributed by species \( i \).

We will not go into detail on how the equations are transformed into their simplest form here (instead see [11]), but briefly describe the important points. The equations are first transformed so that \( a \) is the only explicit temporal variable. This allows parameters specifying particular large scale cosmological dynamics to enter the equations, namely the cosmological constant \( \Lambda \) and spatial curvature \( k_c \). In this paper, a background Einstein deSitter cosmology will be employed \((k_c = 0, \Lambda = 0)\). Although this model has been ruled out with high confidence by current observations, it is sufficient for our purposes. We wish to study some important physical process without additional complications.

It is found that the following important parameters arise:
\[
\bar{k}^2_B = \frac{4\pi G \bar{\rho}_0}{\bar{v}^2_B}, \quad \bar{k}^2_D = \frac{4\pi G \bar{\rho}_0}{\bar{v}^2_D}.
\] (2.4)

They resemble the comoving Jeans wavenumbers for each component taken separately. The strict Jeans one-component wavenumbers are given by replacing \( \rho_0 \) with \( \rho_i \) in (2.4) (see
next section). The wavelength parameters defined above indicate whether gravity $\tilde{k}_i > \tilde{k}$ or pressure support $\tilde{k}_i < \tilde{k}$ dominate the dynamics, and thus whether the region of $k$-space under consideration is Jeans unstable. It may also be noted that the relation $\epsilon_B + \epsilon_D = 1$ holds.

It is useful to define the parameters

$$K_B = \frac{\tilde{k}}{\bar{k}_B}, \quad K_D = \frac{\tilde{k}}{\bar{k}_D},$$

(2.5)

for a clear dimensionless partitioning of parameter space. $K_i < 1$ corresponds to the Jeans unstable region in the single component analog of the equations, and $K_i > 1$ to the acoustic region. The cosmological fluid equations are finally written in terms of the variable $\chi = a^{-1/2}$, to give the canonical form of the system of differential equations to be studied in this paper:

$$\delta_B'' + 6 \left( K_B^2 - \frac{\epsilon_B}{\chi^2} \right) \delta_B - \frac{6 \epsilon_D}{\chi^2} \delta_D = 0,$$

(2.6)

$$\delta_D'' + 6 \left( K_D^2 - \frac{\epsilon_D}{\chi^2} \right) \delta_D - \frac{6 \epsilon_B}{\chi^2} \delta_B = 0.$$

(2.7)

The prime denotes differentiation with respect to $\chi$.

These equations bear a strong resemblance to the equations of an electron-proton cosmological plasma studied in [14] [equations (4.8) and (4.9) of that paper]. Considering the mathematical similarity between the electromagnetic and gravitational forces, this was to be expected. A manifestation of this fact is the close resemblance between the dispersion relation for the simple one-component Jeans instability and Langmuir modes. The techniques employed in [14] will be adopted and developed further to the current problem. In particular, some general WKB techniques will be extended. This will also indicate further results obtainable in cosmological plasma physics.

We now digress to an analysis of the static spacetime perturbation equations to introduce some new physical phenomena, which are to be scrutinized for their applicability in an expanding universe.
III. THE STATIC TWO-COMPONENT PROBLEM

A. Eigenvalues and Eigenvectors

The static spacetime results for a two-component fluid are well understood, though receive little attention in standard linearized structure formation theory, which aims to produce the power spectrum of density perturbations. We will extend the current results to facilitate understanding the general expanding universe scenario later. This section aims to develop some concepts in a relatively simple setting. Previous work on the static problem has been done in [6, 7]. We will in particular rely quite heavily on the notation and results of [7] in this section. Although the solutions are unrealistic as an application to cosmology, they display some similar qualitative features, and allow an exposition of the basic physical ideas without the complication of spacetime expansion being introduced. The static nature of the spacetime simplifies the mathematics greatly, and is thus useful in understanding the general problem.

In this section, there is no need to refer to barred (comoving) physical quantities, and all physical variables may be assumed to be constant in time, unless otherwise specified. With the expansion parameter $a$ set equal to unity, the general fluid equations (2.3) may be written as

\[\ddot{\delta}_D + (v_D^2 k^2 - W_D)\delta_D - W_B \delta_B = 0,\]  
\[\ddot{\delta}_B + (v_B^2 k^2 - W_B)\delta_B - W_D \delta_D = 0,\]  
with an overdot denoting differentiation with respect to $t$, and $W_i = 4\pi G\rho_i$. A study of the behavior of the solutions to these equations is most readily undertaken by reducing the system to a first order autonomous dynamical system, undertaken in [7]. To analyse the dynamical system, a solution needs to be found for the state vector

\[\mathbf{x} = (x_1, x_2, x_3, x_4)^T \equiv (\dot{\delta}_D, \delta_D, \dot{\delta}_B, \delta_B)^T.\]  

We just state the results here.

The most important feature discovered in [7] was the existence of a parameter dependent critical point of the dynamical system given by

\[k^2 = k_M^2 \equiv k_B^2 + k_D^2 = \frac{W_B}{v_B^2} + \frac{W_D}{v_D^2}.\]  

6
where \( k_B \) and \( k_D \) have been defined in terms of the density and velocity parameters of each matter component, and are slightly different from \( \bar{k}_B \) and \( \bar{k}_D \) defined in \( \text{(2.4)} \). The special value of the wavenumber \( k_M \), may be thought of as the Jeans wavenumber of a two-component fluid (the mixture wavenumber). It comprises the Jeans wavenumbers of each fluid taken separately, but it is possible to show that \( k = k_M \) is the only physical quantity which indicates an instability—both \( k = k_D \) and \( k = k_B \) have no such interpretation for the coupled two-component case.

With solutions of the form

\[
x(t) = \sum_{i=1}^{4} \alpha_i \exp(\lambda_i t) \xi_i,
\]

where the \( \alpha_i \) are amplitude functions dependent on \( k \) and determined by initial conditions, the solutions for the eigenvalues \( \lambda_i \) and eigenvectors \( \xi_i \) of the dynamical system are respectively

\[
\begin{align*}
\lambda_1 &= -\lambda_2 = \frac{1}{\sqrt{2}} \sqrt{f + \sqrt{f^2 + 4g}} \\
\lambda_3 &= -\lambda_4 = \frac{1}{\sqrt{2}} \sqrt{f - \sqrt{f^2 + 4g}}
\end{align*}
\]

with

\[
\begin{align*}
f(k) &= W_B + W_D - k^2(v_B^2 + v_D^2), \\
g(k) &= k^2(W_Bv_D^2 + W_Dv_B^2) - k^2v_B^2v_D^2,
\end{align*}
\]

and

\[
\xi_i = (\beta_i \lambda_i, \beta_i, \lambda_i, 1)^T, \quad i = 1, 2, 3, 4,
\]

with

\[
\begin{align*}
\beta_1 &= \beta_2 = \frac{1}{2W_D} (h + \sqrt{h^2 + 4W_BW_D}) \\
\beta_3 &= \beta_4 = \frac{1}{2W_D} (h - \sqrt{h^2 + 4W_BW_D})
\end{align*}
\]

and

\[
h(k) = W_D - W_B + k^2(v_B^2 - v_D^2).
\]

For calculational purposes, we note that

\[
h^2 + 4W_BW_D = f^2 + 4g.
\]

An examination of the real and imaginary parts of the \( \lambda_i \) will show that the \( \lambda_1 \) and \( \lambda_2 \) modes represent acoustic oscillations for \( k > k_M \) and growing and decaying modes for \( k < k_M \). The
\(\lambda_3\) and \(\lambda_4\) modes however always represent acoustic oscillations. The exponential nature of the solutions indicate that the growing and decaying modes do not have the typical power law behavior exhibited by expanding universe solutions, however the solutions exhibit the correct qualitative behavior in the regions below and above the critical point given by \(k = k_M\).

To gain a feel for the properties of the above eigenvalues and eigenvectors, which allows us to make direct contact with the physics of the solutions, we study the quantities in various asymptotic regimes, and examine some plots. This will be effectively facilitated if the quantities are reparameterized in terms of some dimensionless variables. We need only consider the eigenvalues \(\lambda_1\) and \(\lambda_3\). To indicate the nature of the dark matter, the sound velocities may be coalesced into the single variable

\[
V^2 = \frac{v^2_B}{v^2_D}. \tag{3.13}
\]

Then \(V \ll 1\) corresponds to HDM while \(V \gg 1\) corresponds to CDM. We also introduce the quantities \(\epsilon_D\) and \(\epsilon_B\) as used elsewhere in the paper. In this context, they may be defined as

\[
\epsilon_D = \frac{W_D}{W_B + W_D}, \quad \epsilon_B = \frac{W_B}{W_B + W_D}. \tag{3.14}
\]

We also parameterize the wavenumber dependence in units of the mixed Jeans wavenumber; thus we define

\[
K_M = \frac{k}{k_M}. \tag{3.15}
\]

It then follows that the eigenvalues may be written as

\[
\lambda_{1,3} = \frac{1}{\sqrt{2}}(W_B + W_D)^{1/2} \left\{ 1 - K^2_M - \left(\epsilon_D V^2 + \frac{\epsilon_B}{V^2}\right) K^2_M \right. \\
\left. \pm \left[ \left(1 - K^2_M - \left(\epsilon_D V^2 + \frac{\epsilon_B}{V^2}\right) K^2_M\right)^2 + 4 \left(\frac{\epsilon_B}{V} + \epsilon_D V\right)^2 K^2_M (1 - K^2_M) \right]^{1/2} \right\}^{1/2}. \tag{3.16}
\]
These expressions may be expanded for small and large $K_M$. The results are:

$$\lambda_1 \sim (W_B + W_D)^{1/2} \left[ 1 + \frac{1}{2V^2}(-\epsilon_B + \epsilon_B^2 - V^2) + 2\epsilon_B\epsilon_D V^2 - \epsilon_D V^4 + \epsilon_D^2 V^4)K_M^2 + \cdots \right], \quad K_M \ll 1 \tag{3.17}$$

$$\sim iK_M^{-\frac{1}{\sqrt{2}}}(W_B + W_D)^{1/2} \left\{ 1 + \epsilon_B/V^2 + \epsilon_D V^2 - \left[ (1 + \epsilon_B/V^2 + \epsilon_D V^2)^2 - 4(\epsilon_B/V + \epsilon_D V^2)^2 \right]^{1/2} \right\}^{1/2}, \quad K_M \gg 1, \tag{3.18}$$

$$\lambda_3 \sim iK_M(W_B + W_D)^{1/2}(\epsilon_B/V^2 + \epsilon_D V^2), \quad K_M \ll 1 \tag{3.19}$$

$$\sim iK_M^{-\frac{1}{\sqrt{2}}}(W_B + W_D)^{1/2} \left\{ 1 + \epsilon_B/V^2 + \epsilon_D V^2 + \left[ (1 + \epsilon_B/V^2 + \epsilon_D V^2)^2 - 4(\epsilon_B/V + \epsilon_D V^2)^2 \right]^{1/2} \right\}^{1/2}, \quad K_M \gg 1. \tag{3.20}$$

These expansions confirm the earlier statement, whereby the $\lambda_3$ (and equivalently $\lambda_4$) modes display acoustic oscillations at all wavelengths, whereas the $\lambda_1$ (and equivalently $\lambda_2$) modes undergo a Jeans instability to growing (decaying) modes for $K_M < 1$. It is also evident that at very large wavenumbers (small scales) the acoustic oscillations have a very large frequency, growing in proportion to the wavenumber, whereas for very low wavenumbers the $\lambda_3$ modes behave in a very slowly varying oscillatory manner, the frequency again being proportional to the wavenumber. In this regime the $\lambda_1$ and $\lambda_2$ modes comprise exponentially growing or decaying perturbations over an almost wavenumber independent timescale, approximately equal to $(W_B + W_D)^{1/2}$. These properties are illustrated in Fig.1, where the absolute values of the eigenvalues are plotted as a function of $K_M$ for a variety of H/CDM scenarios. The values $\epsilon_B = 0.1$ and $\epsilon_D = 0.9$ have been used in the plots, which is a fairly typical proportion of baryonic and dark matter mass density expected in the Universe.

**B. Interesting Scales**

It is evident that at certain scales the eigenvalues undergo some qualitatively interesting changes, which have been marked on the plots by some arrows. For wavenumbers around $K_M = 1$, the $\lambda_1$ eigenvalue drops very quickly to zero, indicating the Jeans instability, but the $\lambda_3$ eigenvalue displays uniform behavior in this region. There is another interesting scale in the $K_M < 1$ region for small $V$. The physical motivation for this scale was discussed in [7]. It corresponds to a critical wavenumber $k_C$, defined to be when the frequencies of each
component taken separately coincide, i.e. when
\[ v_B^2 k^2 - W_B = v_D^2 k^2 - W_D. \]  (3.21)

The wavenumber \( k = k_C \) is consequently given by
\[ k_C = \left( \frac{W_D - W_B}{v_D^2 - v_B^2} \right)^{1/2}. \]  (3.22)

It is interesting that the importance of this scale is only apparent for small \( V \), where a sudden increase in the magnitude of \( \lambda_1 \) and decrease in the magnitude of \( \lambda_3 \) is apparent. For all \( V \gtrsim 1 \), the plots would be almost identical to the displayed plots of \( V = 1000 \) in Fig.1.

To gain a better understanding of this behavior, it is useful to convert \( k_C \) into units of \( k_M \), which are the plotting units of all the figures. Thus
\[ K_{MC} \equiv \frac{k_C}{k_M} = \left( \frac{\epsilon_D - \epsilon_B}{\epsilon_D - \epsilon_B - \epsilon_D V^2 + \epsilon_B/V^2} \right)^{1/2}. \]  (3.23)

It is interesting to compare this quantity to the individual Jeans instability scales for each fluid taken separately:
\[ K_{MD} \equiv \frac{k_D}{k_M} = \left( \frac{\epsilon_D}{\epsilon_D/\epsilon_B + \epsilon_D} \right)^{1/2}, \]  (3.24)
\[ K_{MB} \equiv \frac{k_D}{k_M} = \left( \frac{\epsilon_B}{\epsilon_B + \epsilon_D V^2} \right)^{1/2}. \]  (3.25)

A better qualitative feel for these scales is facilitated by considering their expansions in the HDM and CDM regimes. For HDM, with \( V \ll 1 \) we find
\[ K_{MC}^2 = V^2 \left( \frac{\epsilon_D}{\epsilon_B} - 1 \right) \left[ 1 - \left( \frac{\epsilon_D}{\epsilon_B} - 1 \right) V^2 + O(V^4) \right], \]  (3.26)
\[ K_{MD}^2 = V^2 \left( \frac{\epsilon_D}{\epsilon_B} - 1 \right) \left[ 1 - \frac{\epsilon_D}{\epsilon_B} V^2 + O(V^4) \right], \]  (3.27)
\[ K_{MB}^2 = 1 - \frac{\epsilon_D}{\epsilon_B} V^2 + O(V^4). \]  (3.28)

From this it may be concluded that if

1. \( \epsilon_D \gg \epsilon_B \), then \( K_{MC} \sim K_{MD} \),

2. \( \epsilon_D \gtrsim \epsilon_B \), then \( K_{MC} \ll 1 \),

3. \( \epsilon_B > \epsilon_D \), then \( K_{MC} \) is imaginary (no physical significance).
This last point is also borne out by the original definition, where it is seen that for \( k_C \) to be real, the dominant component must also be the hotter component. For CDM, with \( V \gg 1 \) the corresponding relations are given by

\[
\begin{align*}
K_{MC}^2 &= V^{-2} \left( \frac{\epsilon_B}{\epsilon_D} - 1 \right) \left[ 1 - \left( \frac{\epsilon_B}{\epsilon_D} - 1 \right) V^{-2} + O(V^{-4}) \right], \\
K_{MD}^2 &= 1 - \frac{\epsilon_B}{\epsilon_D} V^{-2} + O(V^{-4}), \\
K_{MB}^2 &= V^{-2} \frac{\epsilon_B}{\epsilon_D} \left[ 1 - \frac{\epsilon_B}{\epsilon_D} V^{-2} + O(V^{-4}) \right].
\end{align*}
\]

(3.29) (3.30) (3.31)

This shows that if

1. \( \epsilon_B \gg \epsilon_D \), then \( K_{MC} \sim K_{MB} \),

2. \( \epsilon_B \gtrsim \epsilon_D \), then \( K_{MC} \ll 1 \),

3. \( \epsilon_D > \epsilon_B \), then \( K_{MC} \) is imaginary (no physical significance).

The position of the arrows in Fig.1 bear out the above relations, as do the arrows in Fig.2 and Fig.3 to be discussed more below.

In conclusion, the fact that all eigenvalues have been plotted for the values \( \epsilon_B = 0.1 \) and \( \epsilon_D = 0.9 \) means that the scale \( k_C \) is only physically relevant for HDM. This is why all plots for \( V \gtrsim 1 \) (CDM) are so similar. Given that the real Universe is now considered almost certainly CDM dominated, it is doubtful as to whether this potentially interesting physical effect given by the equations has any discernible effect on structure formation scenarios in pure CDM models. A H+CDM model may give similar interesting results as discussed here. This however is a three-component problem beyond the scope of this paper, but may be considered a worthwhile topic of research for future work in this area.

C. Other Qualitative Behavior

We briefly consider some other features of the solutions, to help better understand the mathematical properties. The qualitative features we wish to explore are the same for both the \( \lambda_1 \) and \( \lambda_3 \) modes, so we will concentrate only on the \( \lambda_1 \) modes here. The eigenvalue \( \lambda_1 \) is plotted for different values of \( \epsilon_B \) and \( \epsilon_D \) in Fig.2 and Fig.3. An interesting feature of these figures is that there is a one-to-one correspondence between each of the four plots in one
figure to a particular plot in the other figure, yet each plot corresponds to different physical parameters in each figure. This property highlights the symmetry of the eigenvalues. If the original analytic expression (3.16) for $\lambda_1$ is examined, it is clear that the expression retains an identical form if $\epsilon_D$ and $\epsilon_B$ are interchanged together with $V$ and $1/V$. Real values of $K_{MC}$ have also been marked in. They indicate when the critical scale $k_C$ is physically relevant. Related to this property is the fact that in Fig.2 all plots for $\epsilon_D \lesssim 0.1$ and $\epsilon_B \gtrsim 0.9$ are almost identical to the values of $\epsilon_D = 0.1$ and $\epsilon_B = 0.9$. Analogously, in Fig.3 the same may be said for all plots $\epsilon_D \gtrsim 0.9$ and $\epsilon_B \lesssim 0.1$.

We now turn to study the behavior of $\beta_1$ and $\beta_3$, which give an indication of the relative proportion of baryonic and dark matter in each of the modes [see for example the $x_2$ and $x_4$ components of the eigenvectors in Eq.(3.9)]. In dimensionless variables, $\beta_1$ and $\beta_3$ may be written as

$$
\beta_{1,3} = \frac{1}{2} \left\{ 1 - \frac{\epsilon_B}{\epsilon_D} + \left( \frac{\epsilon_B}{\epsilon_D} - \frac{\epsilon_B}{\epsilon_D V^2} + V^2 - 1 \right) K_M^2 
\pm \left[ \frac{1}{\epsilon_D} + 2 \left( 1 - \frac{\epsilon_B}{\epsilon_D} \right) \left( \frac{\epsilon_B}{\epsilon_D} - \frac{\epsilon_B}{\epsilon_D V^2} + V^2 - 1 \right) K_M^2 
+ \left( \frac{\epsilon_B}{\epsilon_D} - \frac{\epsilon_B}{\epsilon_D V^2} + V^2 - 1 \right)^2 K_M^4 \right]^{1/2} \right\}.
$$

(3.32)

It is obvious from inspection that for all wavenumbers $\beta_1$ and $\beta_3$ are real valued and $\beta_1 > 0$, $\beta_3 < 0$. The almost symmetrical nature of the quantities are well illustrated in Fig.4 for the opposing cases of H/CDM. Of interest here again is the scale $K_{MC}$, around which all the $\beta_i$ undergo an abrupt change. Fig.4 shows that either $\beta_1$ or $\beta_3$ will dominate very rapidly for increasing wavenumber, depending on the value of $V$. Stated more specifically, for the HDM scenario ($V \ll 1$), baryons will dominate the $\lambda_1$ modes and dark matter will dominate the $\lambda_3$ modes, and vice versa for the CDM scenario ($V \gg 1$). We refrain from examining the asymptotics of $\beta_1$ and $\beta_3$ now, and leave that task to when we derive analogous expressions in the expanding universe scenario. The asymptotics will confirm the present qualitative discussion.

D. Initial Conditions and Amplitudes

Having completed a study of the behavior of the general solutions, we now turn to consider the effect of initial conditions. The study of the amplitudes of the various modes was studied
with particular interest in [7], where the presence of a resonance was discovered at the scale $k_C$. We extend that work here to consider the amplitude functions at a wider range of scales, and for CDM as well. This will be of relevance when the expanding Universe scenario is analyzed in the next section, where the $K_M > 1$ range of scales needs to be considered. It is also of course of relevance from the fact that the Universe is believed to be CDM dominated.

The amplitudes are $k$-dependent functions, which also depend on the various constants in the problem. Consider the initial conditions at some time $t_0$ given by the constants $x_i(t_0) = x_{i0}$. Here and henceforth, any variable subscripted with a 0 (possibly together with other subscripts) denotes that quantity evaluated at $t = t_0$. As a reasonable simplifying assumption, the perturbations are assumed to start from rest, so that $x_{10} = x_{30} = 0$. The matter density perturbations may then be written in the form

$$\delta_D(\tau) = x_{20}[\zeta_1(e^{\lambda_1\tau} + e^{-\lambda_1\tau}) + \zeta_2 \cos(i\lambda_3\tau)], \quad (3.33)$$

$$\delta_B(\tau) = x_{40}[\zeta_3(e^{\lambda_1\tau} + e^{-\lambda_1\tau}) + \zeta_4 \cos(i\lambda_3\tau)], \quad (3.34)$$

where the time origin has been shifted by the definition $\tau = t - t_0$. The amplitude functions $\zeta_i$ are constructed out of the mode eigenvector functions $\beta_i$, as well as the ratio of initial densities $Q_0 = x_{40}/x_{20}$. They are

$$\zeta_1 = \frac{\beta_1 1 - Q_0\beta_3}{2 \beta_1 - \beta_3}, \quad \zeta_2 = \frac{\beta_3 Q_0\beta_1 - 1}{\beta_1 - \beta_3}, \quad (3.35)$$

$$\zeta_3 = \frac{1 Q_0^{-1} - \beta_3}{2 \beta_1 - \beta_3}, \quad \zeta_4 = \frac{\beta_1 - Q_0^{-1}}{\beta_1 - \beta_3}. \quad (3.36)$$

Of particular interest is the fact that some of the $\zeta_i$ display a resonance around the scale $k_C$. This scale has of course previously shown its significance in the behavior of the eigenvalues and the $\beta_i$. The fact that $k_C$ defines the scale at which the collapse times of the components taken separately coincide indicates that a resonance may well be expected to occur at this scale. The behavior of the various amplitudes over a wide range of scales, and in both the HDM and CDM scenarios are illustrated in Figs.5–12. The distinguishing feature of all the plots of HDM amplitudes is the rapid change of the functions around the scale $k_C$.

The analytic properties of $\zeta_1$ and in particular $\zeta_3$ are discussed at some length in [7]. Under some restrictive conditions (only HDM and certain initial values of $Q_0$) it was shown
that \( \xi_1 \) would not obtain a resonance, whereas \( \xi_3 \) would for
\[
Q_0 < \frac{1}{2} \left( \frac{v_D^2}{v_B^2} - \frac{W_D v_B^2}{W_B v_D^2} \right).
\] (3.37)

In contrast no resonances are observed in the CDM scenario, but the amplitudes still undergo a rapid change around the scale \( K_{MB} \). Both \( K_M = K_{MB} \) and \( K_M = K_{MD} \) are always less than \( K_M = 1 \), so that no significant behavior occurs in the long wavelength limit, a fact of some importance in later work.

**IV. THE SHORT WAVELENGTH APPROXIMATION IN THE EXPANDING UNIVERSE**

**A. Matrix Formulation**

We are now prepared to tackle the most general equations formulated for the current problem, given by (2.6) and (2.7). The solution to this system of equations cannot be classified by known analytic functions, so approximation schemes need to be implemented. This paper investigates a short wavelength approximation, which would be expected to probe the acoustic regime of the modes. A WKB type method may be employed for this. This is interesting, because through the derivation the explicit physical approximations required and type of solutions obtainable will naturally arise as a consequence of the method. A WKB method for coupled systems of equations in a cosmological plasma setting was expounded in [14]. We will further develop that method here for the current system, which is more complicated than anything considered previously.

To begin with, the equations must be reduced to a first order system. Thus as in the static case (3.3), we define
\[
\mathbf{x} = (x_1, x_2, x_3, x_4)^T \equiv (\delta_D', \delta_D, \delta_B', \delta_B)^T.
\] (4.1)

The system may be written in matrix form
\[
\mathbf{x}' = \mathbf{T}\mathbf{x},
\] (4.2)
with the definition

\[
T = \begin{bmatrix}
0 & -6K_D^2 + \frac{6\epsilon_D}{\chi^2} & 0 & \frac{6\epsilon_B}{\chi^2} \\
1 & 0 & 0 & 0 \\
0 & \frac{6\epsilon_D}{\chi^2} & 0 & -6K_B^2 + \frac{6\epsilon_B}{\chi^2} \\
0 & 0 & 1 & 0 \\
\end{bmatrix}.
\] (4.3)

The idea behind the method is to attempt to remove the coupling between equations as much as possible, hopefully relegating it to some lower order, which can then be dealt with by a suitable approximation. To this end we define the new matrices \( A \) and \( f \) such that

\[
x = A f.
\] (4.4)

\( A \) is chosen appropriately in order to diagonalise \( T \). Then (4.2) may formally be written as

\[
f' = A^{-1} T A f - A^{-1} A' f, \quad \det A \neq 0.
\] (4.5)

To diagonalise \( T \), we must first find its eigenvalues and eigenvectors. The structure of \( T \) is very similar to its static spacetime counterpart, so the four eigenvalues also have the form given by (3.6). In the present case however, \( f \) and \( g \) are functions of \( \chi \) and \( \bar{k} \), defined as

\[
f(\chi, \bar{k}) = \frac{6}{\chi^2} - 6(K_B^2 + K_D^2), \quad (4.6)
\]

\[
g(\chi, \bar{k}) = \frac{36}{\chi^2}(K_D^2\epsilon_B + K_B^2\epsilon_D) - 36K_B^2K_D^2. \quad (4.7)
\]

Note that unlike the static spacetime results, where \( f \) contained an expression of the form \( \epsilon_B + \epsilon_D = 1 \). The eigenvectors \( \xi_i \) corresponding the eigenvalues \( \lambda_i \) are also identical in structure to their static spacetime counterparts (3.9). In this case the functions \( \beta_i \) are given by

\[
\beta_1 = \beta_2 = \frac{1}{2} \frac{\chi^2}{6\epsilon_D} \left( h + \sqrt{h^2 + 4 \frac{36\epsilon_B\epsilon_D}{\chi^4}} \right), \quad (4.8)
\]

\[
\beta_3 = \beta_4 = \frac{1}{2} \frac{\chi^2}{6\epsilon_D} \left( h - \sqrt{h^2 + 4 \frac{36\epsilon_B\epsilon_D}{\chi^4}} \right), \quad (4.9)
\]

with

\[
h(\chi, \bar{k}) = \frac{6}{\chi^2}(\epsilon_D - \epsilon_B) + 6(K_B^2 - K_D^2). \quad (4.10)
\]
It is worthwhile to point out here for the sake of calculations that
\[ S \equiv \sqrt{h^2 + \frac{36\epsilon_B \epsilon_D}{\chi^4}} = \sqrt{f^2 + 4g}. \] (4.11)

The eigenvectors may be used to form the diagonalising matrix
\[ A = (\xi_1, \xi_2, \xi_3, \xi_4), \] (4.12)
whose inverse exists. This enables the formal equation (4.13) to be written explicitly:
\[
\begin{pmatrix}
  f'_1 \\
  f'_2 \\
  f'_3 \\
  f'_4
\end{pmatrix} =
\begin{pmatrix}
  \lambda_1 & 0 & 0 & 0 \\
  0 & -\lambda_1 & 0 & 0 \\
  0 & 0 & \lambda_3 & 0 \\
  0 & 0 & 0 & -\lambda_3
\end{pmatrix}
\begin{pmatrix}
  f_1 \\
  f_2 \\
  f_3 \\
  f_4
\end{pmatrix} -
\begin{pmatrix}
  \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \\
  \epsilon_2 & \epsilon_1 & \epsilon_4 & \epsilon_3 \\
  \epsilon_5 & \epsilon_6 & \epsilon_7 & \epsilon_8 \\
  \epsilon_6 & \epsilon_5 & \epsilon_8 & \epsilon_7
\end{pmatrix}
\begin{pmatrix}
  f_1 \\
  f_2 \\
  f_3 \\
  f_4
\end{pmatrix}. \] (4.13)

We have introduced eight new parameters here, all of which can be written in terms of \( \lambda_1, \lambda_3, \beta_1 \) and \( \beta_3 \). In what follows, let us use the shorthand
\[
\beta_{13} \equiv \beta_1 - \beta_3 = \frac{\chi^2}{6\epsilon_D} S
= \frac{1}{\epsilon_D} \sqrt{1 + 2\chi^2(\epsilon_D - \epsilon_B)(K_B^2 - K_D^2) + \chi^4(K_B^2 - K_D^2)^2}. \] (4.14)

The new parameters are defined by
\[
\begin{align*}
\epsilon_1 &= \frac{\beta'_1}{\beta_{13}} + \frac{\lambda'_1}{2\lambda_1}, \quad (4.15) \\
\epsilon_2 &= -\frac{\lambda'_1}{2\lambda_1}, \quad (4.16) \\
\epsilon_3 &= \frac{\beta'_3}{2\beta_{13}} \left( 1 + \frac{\lambda_3}{\lambda_1} \right), \quad (4.17) \\
\epsilon_4 &= \frac{\beta'_3}{2\beta_{13}} \left( 1 - \frac{\lambda_3}{\lambda_1} \right), \quad (4.18) \\
\epsilon_5 &= -\frac{\beta'_1}{2\beta_{13}} \left( 1 + \frac{\lambda_1}{\lambda_3} \right), \quad (4.19) \\
\epsilon_6 &= -\frac{\beta'_1}{2\beta_{13}} \left( 1 - \frac{\lambda_1}{\lambda_3} \right), \quad (4.20) \\
\epsilon_7 &= -\frac{\beta'_3}{\beta_{13}} + \frac{\lambda'_3}{2\lambda_3}, \quad (4.21) \\
\epsilon_8 &= -\frac{\lambda'_3}{2\lambda_3}. \quad (4.22)
\end{align*}

Let us consider the meaning of (4.13) more closely. Using the terminology of WKB theory, the first matrix on the right hand side will give us the leading order “control factor”—the fastest varying part of the solution, typically an exponential factor. This factor may indicate rapid oscillations for imaginary $\lambda_i$, or rapid growth or decay for real $\lambda_i$. The second matrix contains a collection of parameters, which determine further slowly varying behavior. For this to be true, the condition $\epsilon_i \ll \lambda_j$, $\forall i, j$ must hold. It is then possible to show that the four equations all decouple to leading order, and WKB solutions may be written down. The proof of this is quite involved, but it is worthwhile to pursue. A bonus of the proof is that through a careful consideration of the approximations required, some instructive physics is learnt along the way.

B. The WKB Approximation Criteria

In an attempt to decouple the equations, we need to consider more carefully the various criteria which constitute the condition $\epsilon_i \ll \lambda_j$, $\forall i, j$, which allow the WKB method to work. To begin with, we will assume the $\lambda_i$’s are of about the same order. Although their magnitude varies greatly over various scales, an examination of Fig.1 shows this assumption to hold fairly well in general. When we find, through the reasoning which follows, the precise scales of interest for a WKB approximation, we will see that this assumption is justified post facto. The essence of the WKB approximation is to assume

$$\left| \frac{\lambda_i'}{\lambda_i} \right| \ll |\lambda_i|,$$  \hspace{1cm} (4.23)

that is, the eigenvalues vary slowly over the timescale which they define. In the current context, this corresponds physically to many oscillations in a universe expansion time for the acoustic region of $k$-space, or a far shorter collapse time than a universe expansion time for perturbations in the region where modes are Jeans unstable. In what follows, exactly which type of modes (acoustic or collapse) do fall into the category defined by (4.23) will become apparent.

From the results of static spacetime, we suspect that a Jeans instability must exist, and in such a region, it should follow that one of the $\lambda_i$ be zero, rendering (4.23) false. Consequently, we need to find the critical points of the $\lambda_i$, dependent on the wavenumber
A consideration of the equation $\lambda_1^2(\chi) = 0$ gives a solution for a “critical time” $\chi = \chi_c$:

$$\chi_c^2 \equiv \frac{\epsilon_B}{K_B^2} + \frac{\epsilon_D}{K_D^2}. \quad (4.24)$$

It turns out that $\lambda_3$ however has no such time.

Let us examine the behavior of $\lambda_1(\chi)$ around $\chi \sim \chi_c$ more closely. We set $\chi = \chi_c^2(1 + \epsilon)$ for a small parameter $\epsilon$, and expand $\lambda_1$ in powers of $\epsilon$. It turns out that

$$f + \sqrt{f^2 + 4g} = -12\epsilon K_B^2 K_D^2 \frac{K_B^2 \epsilon_B + K_D^2 \epsilon_D}{K_D^2 \epsilon_B + K_B^2 \epsilon_D} + O(\epsilon^2), \quad (4.25)$$

so that $\lambda_1 \propto \sqrt{-\epsilon}$ for $\chi \sim \chi_c$. This dependence of $\lambda_1$ on $\epsilon$ gives a clear picture of how $\lambda_1$ changes around the critical point. For $\chi > \chi_c$, $\epsilon > 0$ and $\lambda_1$ is imaginary. This corresponds to acoustic oscillations, in the stable part of $k$-space. For $\chi < \chi_c$, $\epsilon < 0$ and $\lambda_1$ is real. This corresponds to an unstable part of $k$-space, so that $\chi_c$ is an indication of the transition through the Jeans instability. The time parameters may be defined so that initial time corresponds to $a_0 = 1$. This is because the explicit magnitude of $a_0$ is not determined by cosmology. By the definition $\chi = a^{-1/2}$, it is clear that $\chi$ begins at 1 and decreases with increasing time. This gives us two ways of looking at the Jeans instability. One way is to consider the instability at a particular instant in time. For a particular time $\chi$, a subset of modes will be unstable for values of $k$ for which $\chi_c(k) > \chi$ (we stress that $\chi_c$ is a function of $k$). We may then consider what occurs as these modes evolve through time from this particular instant. The critical time $\chi_c$ is fixed for any one mode, so that the modes which were originally acoustic will become unstable as $\chi \to \chi_c^+$. Consequently more and more modes pass through the instability as the Universe evolves. The physical wavenumber $k$ is of course dependent on time, thus the dependence of the instability on a time $\chi_c$ shows the inextricable link between the wavenumber and time.

We wish to relate these concepts back to the result discussed for static spacetime, and so must ask how the critical time $\chi_c$ is related to the critical wavenumber $k_M$ of the mixture of components. In static spacetime we defined

$$K_M^2 = \frac{k^2}{W_B/v_B^2 + W_D/v_D^2} \quad (4.26)$$

as the dimensionless parameter, indicating the relation of a mode to the instability at $K_M =$
1. To place this quantity in an expanding Universe context, the substitutions
\[
W_B \rightarrow \frac{6\epsilon_B}{\chi^2}, \quad v_B^2 k^2 \rightarrow 6K_B^2,
\]
\[
W_D \rightarrow \frac{6\epsilon_D}{\chi^2}, \quad v_D^2 k^2 \rightarrow 6K_D^2
\]
are required. This gives \(K_M\) the following form:
\[
K_M^2 = \frac{\chi^2}{\epsilon_B/K_B^2 + \epsilon_D/K_D^2} = \frac{\chi^2}{\chi_c^2}. \tag{4.27}
\]
It is explicitly seen here that the scale of instability changes with time, as was explained above. The analogy with the one-component case discussed in [11] may be made here, where solutions were found in terms of the one-component Jeans wavenumber \(K_J a^{-1/2}\). With \(\chi = a^{-1/2}\), we see that the quantity \(\chi_c^{-1}\) in the two-component case is the exact analogy of \(K_J\) for the one-component case.

Now that we have determined that \(\lambda_1\) approaches zero in a particular region, it becomes clear that the WKB approximation will not be valid in this region \(\chi \sim \chi_c\), from the condition (4.23). Let us examine \(\lambda_1'/\lambda_1^2\) in more detail, to determine its behavior over the whole of \(k\)-space. An explicit evaluation using the definitions of \(f\) and \(g\) from (4.6) and (4.7) gives
\[
\frac{\lambda_1'}{\lambda_1^2} = -\frac{3}{(\lambda_1 \chi)^3}(1 + F), \tag{4.28}
\]
where
\[
F = \frac{1 + (\epsilon_D - \epsilon_B)(K_B^2 - K_D^2)\chi^2}{\epsilon_D \beta_{13}}. \tag{4.29}
\]
In the ensuing discussion we use the quantity \(\bar{k}\) to describe the comoving wavenumber dependence of the quantities involved. If we denote the numerator of (4.29) by \(N\), it is simple to see that \(N^2 < \epsilon_D \beta_{13}^2\) for all \(\bar{k}\) [see (4.14)], so that \(|F| \leq 1\). We also consider the limits of \(F\) for large and small \(\bar{k}\). For \(\bar{k} \rightarrow 0\), \(F \rightarrow 1\) and for \(\bar{k} \rightarrow \infty\), \(F \rightarrow \epsilon_D - \epsilon_B\) if \(K_B > K_D\), and \(F \rightarrow \epsilon_B - \epsilon_D\) if \(K_D > K_B\). This means that asymptotically \(F\) is independent of \(\bar{k}\), and since \(|F| \leq 1\) for all \(\bar{k}\), \(\lambda_1'/\lambda_1^2\) does not change sign. It now becomes clear that the magnitude of \(\lambda_1'/\lambda_1^2\) is mainly dependent on the factor \((\lambda_1 \chi)^{-3}\). We already know that as \(\bar{k} \rightarrow \bar{k}_M\), \(\lambda_1 \rightarrow 0\) so that in this region it has been confirmed that \(\lambda_1'/\lambda_1^2 \gg 1\). Suitable regions where WKB might be valid must be sought far from the neighborhood \(\bar{k} \sim \bar{k}_M\). The dependence of \(\lambda_1'/\lambda_1^2\) on \(\lambda_1 \chi\) leads us to suggest the WKB criterion in the amended form \(\lambda_1 \chi \gg 1\). This makes good physical sense, because if \(\lambda_1\) is considered a frequency-inverse of a dynamical collapse
time, the WKB criteria requires that a large number of oscillations/significant change in $\delta_i$ occurs during an expansion time.

A similar analysis needs to be performed for $\lambda_3$. We find
\[
\frac{\lambda_3'}{\lambda_3^2} = -\frac{3}{(\lambda_3 \chi)^3}(1 - F).
\] (4.30)
This too demands the criterion $\lambda_3 \chi \gg 1$. To find how $\lambda_3$ behaves, consider firstly the $\bar{k} \to 0$ limit. This limit gives $\lambda_3 \to 0$, so this region of $k$-space is clearly inappropriate for WKB analysis. We note, however, that since $\lambda_3 \neq 0$ at $\bar{k} = \bar{k}_M$, this region should be checked more closely. After a careful examination of $\lambda_3$:

\[
\lambda_3 \chi = \sqrt{3} \sqrt{1 - (K_B^2 + K_D^2) \chi^2 - \epsilon_D \beta_{13}},
\] (4.31)
it becomes apparent that $\lambda_3 \chi \gg 1$ only when both $K_B \gg 1$ and $K_D \gg 1$. This corresponds to the region $\bar{k} \gg \bar{k}_M$, and so the region $\bar{k} \sim \bar{k}_M$ must be excluded from consideration as well. For $\lambda_3$, we are only left with the region $\bar{k} \gg \bar{k}_M$ as fulfilling the WKB criterion. For completeness, the same reasoning should also be applied to $\lambda_1$. The $\bar{k} \to 0$ limit applied to $\lambda_1$ gives
\[
\frac{3}{(\lambda_1 \chi)^3} \to \frac{1}{2\sqrt{6}},
\] (4.32)
which is not much less than 1, as is required to define it as a region amenable to WKB analysis. Thus the $\bar{k} \ll \bar{k}_M$ region is inappropriate for $\lambda_1$ as well.

In conclusion, $\bar{k} \gg \bar{k}_M$ is the only region for which the WKB approximation holds. We may summarize the methods available to analyze the two-component problem in various regions of $k$-space by the following classification:

$\bar{k} \gg \bar{k}_M$ The WKB method will give acoustic oscillations for all modes, with the rapidly varying part of the solution taking the form $\exp(\pm i|\lambda_i|\chi)$.

$\bar{k} \ll \bar{k}_M$ A Frobenius expansion (small parameter expansion) of the solutions needs to be developed. Some growing and decaying modes following a power law behavior may be expected for solutions corresponding to $\lambda_1$, while some low frequency acoustic oscillations may be expected for solutions corresponding to $\lambda_3$.

$\bar{k} \sim \bar{k}_M$ This region of parameter space is not accessible to analytic solution. Some numerics will be required to investigate this interesting region.
We will continue with the WKB analysis in this paper, and show how the equations (4.13) decouple. An investigation of the other regions of $k$-space are taken up in [11].

We still need to check how $\beta_1$ and $\beta_3$ vary, to ensure that all the $\epsilon_i$ are small. In particular, we need to consider whether relations of the form

$$\frac{\beta_i'}{\beta_{13}} \ll \lambda_j, \ i, j = 1, 3$$

are true. The analysis proceeds very similarly to that described above for the derivatives of $\lambda_1$ and $\lambda_3$, and a full description will not be given here. As a brief example, it can be shown that

$$\frac{\beta_1'}{\beta_{13} \lambda_1} \sim \frac{1}{\lambda_1 \chi}$$

for $\bar{k} \gg \bar{k}_M$, once again fulfilling the criterion $\lambda_1 \chi \gg 1$. Other cases follow similarly. Given that the relations (4.33) do hold, we can finally make the important statement that $\epsilon_i \ll \lambda_j, \forall i, j$ if $\bar{k} \gg \bar{k}_M$.

C. The Solutions

Now that we have worked out the region of $k$-space in which the WKB method produces valid leading order solutions, we proceed to derive these solutions by decoupling the equations. To illustrate how the equations (4.13) decouple, we begin with an example. Taking the top row of the matrix equation, the following suggestive equation for $f_1$ may be written:

$$f_1' - (\lambda_1 - \epsilon_1)f_1 = -\epsilon_2 f_2 - \epsilon_3 f_3 - \epsilon_4 f_4.$$  

(4.35)

This may be treated as a first order inhomogeneous ordinary differential equation (ODE) for $f_1$. The homogeneous portion has a simple solution

$$f_1 \sim c_1 \exp \left[ \int_{\chi_0}^{\chi} (\lambda_1 - \epsilon_1) d\chi \right],$$

(4.36)

with a constant of integration $c_1$. This may be considered to be a first approximation to the solution, though it remains to be shown that it is the full leading order result. To evaluate the inhomogeneous portion of the solution of (4.35), we require the first approximations for
the other $f_i$ as well. A corresponding analysis to that illustrated for $f_1$ yields

\begin{align}
   f_2 & \sim c_2 \exp \left[ - \int_{\chi_0}^{\chi} (\lambda_1 + \epsilon_1) d\chi \right], \\
   f_3 & \sim c_3 \exp \left[ \int_{\chi_0}^{\chi} (\lambda_3 - \epsilon_7) d\chi \right], \\
   f_4 & \sim c_4 \exp \left[ - \int_{\chi_0}^{\chi} (\lambda_3 + \epsilon_7) d\chi \right].
\end{align}

When these are substituted into (4.35), we achieve the rather complicated result

\begin{align}
f_1 & \sim \exp \left[ \int_{\chi_0}^{\chi} (\lambda_1 - \epsilon_1) d\chi \right] \left\{ c_{11} + c_{12} \int_{\chi_0}^{\chi} d\chi \epsilon_2 \exp \left[ -2 \int_{\chi_0}^{\chi} \lambda_1 d\chi \right] \\
   & \quad + c_{13} \int_{\chi_0}^{\chi} d\chi \epsilon_3 \exp \left[ \int_{\chi_0}^{\chi} (-\lambda_1 + \lambda_3 + \epsilon_1 - \epsilon_7) d\chi \right] \\
   & \quad + c_{14} \int_{\chi_0}^{\chi} d\chi \epsilon_4 \exp \left[ \int_{\chi_0}^{\chi} (-\lambda_1 - \lambda_3 + \epsilon_1 - \epsilon_7) d\chi \right] \right\}.
\end{align}

There are a number of integrals present here which need to be estimated to determine how the approximation is to proceed.

As an example of a generic type of integral to evaluate, consider

\begin{align}
I & = \int_{\chi_0}^{\chi} d\chi \epsilon_2 \exp \left[ -2 \int_{\chi_0}^{\chi} \lambda_1 d\chi \right].
\end{align}

Since the region of interest is $\bar{k} \gg \bar{k}_M$, $\lambda_1$ is imaginary, and consequently the definition of the real function $\psi(\chi) \equiv -i\lambda(\chi)$ is useful. We integrate $I$ by parts to obtain

\begin{align}
I & = \frac{\lambda_1^2}{4\lambda_1^2} \exp \left[ -2i \int_{\chi_0}^{\chi} \psi(\chi) d\chi \right] \left|_{\chi_0}^{\chi} \right. - \frac{1}{4} \int_{\chi_0}^{\chi} d\chi \left( \frac{d \lambda_1^2}{d\chi} \right) \exp \left[ -2i \int_{\chi_0}^{\chi} \psi(\chi) d\chi \right].
\end{align}

It is our aim to show that all corrections to the leading order results (4.36)–(4.39) are of $O(\epsilon)$, where $\epsilon$ generically denotes any of the small quantities $\epsilon_i, i = 1, 2, \ldots 8$. It is already obvious that the first term in (4.42) is of $O(\epsilon)$, because it consists of a $\lambda_1^2/\lambda_1^2$ term multiplied by a phase factor. We may constrain the second term by

\begin{align}
\left| \int_{\chi_0}^{\chi} d\chi \left( \frac{d \lambda_1^2}{d\chi} \right) \exp \left[ -2i \int_{\chi_0}^{\chi} \psi(\chi) d\chi \right] \right| \leq \int_{\chi_0}^{\chi} d\chi \left| \left( \frac{d \lambda_1^2}{d\chi} \right) \right| \left| \frac{\lambda_1^2}{\lambda_1^2} \right| \left| \lambda_1^2 \right|.
\end{align}

This shows that indeed $I \sim O(\epsilon)$.

The same type of analysis may be performed for the other more complicated integrals. In general, integration by parts is involved, and the terms may be shown to be bounded by
some expressions of the form $\lambda_i'/\lambda_j^2$, or $\beta_i'/(\beta_{13}\lambda_j)$. It is found that in the WKB approximation, (4.36) is the correct leading order term, and the integrals arising from considering the inhomogeneous portion of the ODE (4.35) are all of $O(\varepsilon)$ below this leading order term. Similar correction integrals to those found in (4.40) may be written down for $f_2$, $f_3$ and $f_4$, all of which take the same generic form as those evaluated in the $f_1$ case. A lengthy analysis will show that all the corrections are of $O(\varepsilon)$, so we have shown that to leading order we may effectively neglect the off-diagonal $\varepsilon_i$ in (4.13). In conclusion, the full leading order WKB solution to $f$ is given by (4.36)–(4.39).

These solutions contain two important terms in the integrals. The $\lambda_1$ and $\lambda_3$ terms represent the rapidly varying oscillatory portion of the solutions, i.e. the control factor mentioned earlier. To obtain the true frequencies, these integrals need to be converted to integrals over $t$. We expect the $\varepsilon_1$ and $\varepsilon_7$ terms to represent some slowly varying time dependent amplitude. To reveal the time dependent structure of the solutions more explicitly, the integrals of $\varepsilon_1$ and $\varepsilon_7$ need to be evaluated. This is facilitated greatly by the relation

$$\beta_1\beta_3 = -\frac{\varepsilon_B}{\varepsilon_D}.$$ (4.44)

For $G_1$ consider the integral

$$\int_{\chi_0}^{\chi} \frac{\beta'_1}{\beta_1 - \beta_3} d\chi = \int_{\chi_0}^{\chi} \frac{\beta_1\beta'_1}{\beta_1^2 + \varepsilon_B/\varepsilon_D} d\chi = \frac{1}{2} \log \left( \frac{\beta_1^2 + \varepsilon_B}{\varepsilon_D} \right) \bigg|_{\chi_0}^{\chi}. \quad (4.45)$$

Using (4.44) once more we find

$$\int_{\chi_0}^{\chi} \frac{\beta'_1}{\beta_1 - \beta_3} d\chi = \frac{1}{2} \log \left( \frac{\beta_1(\chi)}{\beta_1(\chi_0)} - \beta_3(\chi_0) \right). \quad (4.46)$$

For notational expedience, we define the tilde quantities

$$\tilde{\beta}_i(\chi) = \frac{\beta_i(\chi)}{\beta_i(\chi_0)}, \quad i = 1, 3,$$ (4.47)

$$\tilde{\beta}_{13}(\chi) = \frac{\beta_1(\chi) - \beta_3(\chi)}{\beta_1(\chi_0) - \beta_3(\chi_0)}, \quad (4.48)$$

$$\tilde{\lambda}_i(\chi) = \frac{\lambda_i(\chi)}{\lambda_i(\chi_0)}, \quad i = 1, 3,$$ (4.49)

with the obvious property that $\tilde{f}(\chi_0) = 1$, for any quantity $f(\chi)$. This allows us to write the integrals as follows:

$$- \int_{\chi_0}^{\chi} \varepsilon_1 d\chi = \frac{1}{2} \log \left( \frac{\tilde{\beta}_{13}}{\beta_1\lambda_1} \right), \quad (4.50)$$

$$- \int_{\chi_0}^{\chi} \varepsilon_7 d\chi = \frac{1}{2} \log \left( \frac{\tilde{\beta}_{13}}{\beta_3\lambda_3} \right). \quad (4.51)$$
Here the integral involving $\epsilon_7$ was calculated using identical techniques as just illustrated for $\epsilon_1$.

At last the final form of leading order solution for $f$ may be written down, with the explicit time dependent amplitude and rapidly varying oscillatory part:

$$f_{1,2} \sim c_{1,2} \left( \frac{\tilde{\beta}_{13}}{\beta_1 \lambda_1} \right)^{1/2} \exp \left( \pm \int_{x_0}^{x} \lambda_1 d\chi \right),$$  \hspace{1cm} (4.52)

$$f_{3,4} \sim c_{3,4} \left( \frac{\tilde{\beta}_{13}}{\beta_3 \lambda_3} \right)^{1/2} \exp \left( \pm \int_{x_0}^{x} \lambda_3 d\chi \right),$$  \hspace{1cm} (4.53)

We are now in a position to recover the original physical state vector $\mathbf{x}$ by multiplying these auxiliary functions by the matrix $\mathbf{A}$, as given in the original definition (4.4). The matrix $\mathbf{A}$ contains $\lambda$'s and $\beta$'s, which are not tilde quantities. The full general solution to (2.6) and (2.7) is finally given by

$$\begin{bmatrix}
\delta_D' \\
\delta_D \\
\delta_B' \\
\delta_B
\end{bmatrix} \sim c_1 \left( \frac{\tilde{\beta}_{13}}{\beta_1 \lambda_1} \right)^{1/2} \xi_1 \exp \left( i \int_{x_0}^{x} |\lambda_1| d\chi \right)
+ c_2 \left( \frac{\tilde{\beta}_{13}}{\beta_1 \lambda_1} \right)^{1/2} \xi_2 \exp \left( -i \int_{x_0}^{x} |\lambda_1| d\chi \right)
+ c_3 \left( \frac{\tilde{\beta}_{13}}{\beta_3 \lambda_3} \right)^{1/2} \xi_3 \exp \left( i \int_{x_0}^{x} |\lambda_3| d\chi \right)
+ c_4 \left( \frac{\tilde{\beta}_{13}}{\beta_3 \lambda_3} \right)^{1/2} \xi_4 \exp \left( -i \int_{x_0}^{x} |\lambda_3| d\chi \right).$$  \hspace{1cm} (4.54)

Here the $\xi_i$ are the eigenvectors defined by (3.9). In summary, this leading order solution represents acoustic oscillations in the short wavelength limit, defined as

$$k^2 \gg k_M^2 = \frac{4\pi G \rho_B}{v_B^2} + \frac{4\pi G \rho_D}{v_D^2},$$  \hspace{1cm} (4.55)

which is a time dependent quantity. Equivalently, we may view the limit as given by

$$\frac{\epsilon_B}{K_B^2} + \frac{\epsilon_D}{K_D^2} \gg \chi^2.$$  \hspace{1cm} (4.56)

A Jeans instability will not be evident for the solutions in this region of $k$-space, but the time dependence will mean that the period of oscillation becomes longer until a point is reached
at which the WKB approximation is no longer accurate, and the solutions as displayed are not realistic representations of the underlying physics. Then different approximations need to be considered. The methods involved are discussed in detail in [11].

D. Relative Amplitudes of the Solutions

The slowly varying time dependent amplitudes of the solutions (4.54) show how either dark matter or baryons dominate various modes, depending on whether HDM or CDM is being considered. This feature was noticed in the static case, and we now demonstrate it more fully through asymptotic analysis in the expanding universe scenario. The information is contained in the eigenvectors $\xi_i$, which give the relative amplitudes. It can be seen directly from (3.9) that

$$ \frac{\delta_B}{\delta_D} \propto \frac{1}{\beta_1} = \frac{\epsilon_D}{\epsilon_B} \beta_3 $$

(4.57)

for the $\lambda_1$ modes, and

$$ \frac{\delta_B}{\delta_D} \propto \frac{1}{\beta_3} = -\frac{\epsilon_D}{\epsilon_B} \beta_1 $$

(4.58)

for the $\lambda_3$ modes. At first glance this may appear a little surprising, as there seems to be an asymmetry in the solutions. If the indices are interchanged $D \leftrightarrow B$, the amplitudes do not appear to be the same, yet all such an interchange is doing is swapping the order the equations are written down in.

As an illustration for the $\lambda_1$ modes

$$ \begin{bmatrix} \delta_D \\ \delta_B \end{bmatrix} \propto \begin{bmatrix} \beta_1 \\ 1 \end{bmatrix} $$

with $\beta_1 = \frac{\chi^2}{12\epsilon_D}(h + S)$, (4.59)

whereas after the interchange $D \leftrightarrow B$

$$ \begin{bmatrix} \delta_B \\ \delta_D \end{bmatrix} \propto \begin{bmatrix} \beta_1^* \\ 1 \end{bmatrix} $$

with $\beta_1^* = \frac{\chi^2}{12\epsilon_B}(-h + S)$. (4.60)

The new quantity $\beta_1^*$ is defined as being the new form of $\beta_1$ after the interchange has been made. The square root term $S$, defined in (4.11) is invariant under the interchange, whereas an examination of (4.10) shows that under the interchange $h \rightarrow -h$. It is however simple to show that $\beta_1 \beta_1^* = 1$, which is to be expected from the symmetry of the differential
equations. The same result holds for the $\lambda_3$ mode, where it can be shown that $\beta_3 \beta_3^* = 1$ for the quantities

$$\beta_3 = \frac{\chi^2}{12\epsilon_D} (h - S), \quad \beta_3^* = \frac{\chi^2}{12\epsilon_B} (-h - S).$$  \hfill (4.61)

We now examine the behavior of the amplitudes more carefully, to find the dominant components of matter. A useful large expansion parameter in the analysis will be the quantity $y \equiv |K_B^2 - K_D^2| \chi^2$. Let us also introduce the notation $\sigma = \text{sign}(K_B^2 - K_D^2) = \pm 1$, and expand the eigenvalues in $y$. For the $\lambda_1$ modes we may write

$$\beta_1 = \frac{y}{2\epsilon_D} \left[ \sigma + \frac{\epsilon_D - \epsilon_B}{y} + \sqrt{1 + 2\sigma(\epsilon_D - \epsilon_B)y^{-1} + y^{-2}} \right],$$  \hfill (4.62)

from which the following results may be deduced:

$$\beta_1 = \begin{cases} \frac{y}{\epsilon_D} \left[ 1 + \frac{\epsilon_D - \epsilon_B}{y} + O(y^{-2}) \right], & K_B > K_D \ (\sigma = 1) \\ \frac{y}{1 - (\epsilon_D - \epsilon_B)^2} \left[ 1 + \frac{\epsilon_D - \epsilon_B}{y} + O(y^{-2}) \right], & K_D > K_B \ (\sigma = -1) \end{cases}.$$  \hfill (4.63)

This indicates that for CDM ($K_B > K_D$)

$$\beta_1 \propto (K_B^2 - K_D^2) \chi^2,$$

and the dark matter oscillations are dominant in the $\lambda_1$ modes, although as time increases they become less so. On the other hand for HDM ($K_D > K_B$)

$$\beta_1 \propto \frac{1}{(K_D^2 - K_B^2) \chi^2},$$

and the baryon oscillations dominate, and their dominance increases with time. The $\lambda_3$ modes display complementary behavior:

$$\beta_3 = \begin{cases} \frac{-1 + (\epsilon_D - \epsilon_B)^2}{4\epsilon_D y} \left[ 1 + \frac{\epsilon_B - \epsilon_D}{y} + O(y^{-2}) \right], & K_B > K_D \ (\sigma = 1) \\ \frac{-y}{\epsilon_D} \left[ 1 - \frac{\epsilon_B - \epsilon_D}{y} + O(y^{-2}) \right], & K_D > K_B \ (\sigma = -1) \end{cases}.$$  \hfill (4.64)

In this case for CDM

$$\beta_3 \propto -\frac{1}{(K_B^2 - K_D^2) \chi^2},$$

so that the baryon oscillations dominate, and the dominance increases with time, while for HDM

$$\beta_3 \propto -(K_D^2 - K_B^2) \chi^2,$$
which shows that dark matter oscillations dominate, but the dominance decreases with time. In summary, in a CDM scenario \((K_D > K_B)\) baryons dominate the \(\lambda_3\) mode and dark matter dominates the \(\lambda_1\) mode, whereas the situation is opposite for HDM.

The apparent asymmetry of these results with respect to interchange of subscripts \(B \leftrightarrow D\) can be easily explained by taking into account the relations (4.60)–(4.61). It was shown that after an interchange \(B \leftrightarrow D\), the new \(\beta\)'s are given by

\[
\beta_1^* = -\beta_3 \quad \text{and} \quad \beta_3^* = -\beta_1. \tag{4.65}
\]

This is the correct way to view the asymptotic forms for the \(\beta\)'s after an interchange, rather than by a direct swapping of subscripts. By deriving the asymptotic results, information has been lost and direct swapping is no longer valid. As an example, for \(K_B > K_D\)

\[
\beta_1 \sim (K_B^2 - K_D^2)\chi^2 \rightarrow \beta_1^* = -\beta_3 \sim \frac{1}{(K_B^2 - K_D^2)\chi^2}, \tag{4.66}
\]

and the expected result \(\beta_1\beta_1^* = 1\) holds.

Now that we have gained some insight into the behavior of the modes in a generic sense, let us be a little more specific and investigate the full leading order solutions under some tighter physical constraints to see some more physical effects emerge.

**E. Initial Conditions and Resonances**

We attempt to reproduce the resonances described in the static spacetime case by imposing some initial conditions on our solutions, and eliminating the arbitrary constants of integration \(c_i, i = 1, \ldots, 4\), using a similar procedure to that employed previously. In the static spacetime scenario we just stated the results, but here we explicitly go through the derivation. The general solution may be expediently written in the form

\[
x_i(\chi) = \sum_{j=1}^{4} c_j v_{(j)i}(\chi) \exp \left( \int_{\chi_0}^{\chi} \lambda_j d\chi \right). \tag{4.67}
\]

The amplitudes are given by the vectors

\[
v_{(1),(2)} = \left( \frac{\beta_{13}}{\beta_1 \lambda_1} \right)^{1/2} \xi_{1,2}, \quad v_{(3),(4)} = \left( \frac{\beta_{13}}{\beta_3 \lambda_3} \right)^{1/2} \xi_{3,4}. \tag{4.68}
\]

At \(\chi = \chi_0\), \(v_{(j)}(\chi_0) = \xi_j(\chi_0)\), and an equation for the initial conditions \(x_{i0}\) is obtained:

\[
x_{i0} = \sum_{j=1}^{4} c_j \xi_{(j)i}(\chi_0). \tag{4.69}
\]
Note that henceforth any variable subscripted with a 0 (possibly together with other subscripts) denotes that quantity evaluated at $\chi = \chi_0$. As a reasonable simplifying assumption, we once again take $x_{10} = x_{30} = 0$, i.e. the perturbations start from rest. This gives a simple algebraic system for the $c_i$, with the solution

$$
\begin{bmatrix}
  c_1 \\
  c_2 \\
  c_3 \\
  c_4
\end{bmatrix} = 
\begin{bmatrix}
  1 - Q_0 \beta_{30} \\
  \frac{x_{20}}{2\beta_{130}} (1 - Q_0 \beta_{30}) \\
  Q_0 \beta_{10} - 1 \\
  Q_0 \beta_{10} - 1
\end{bmatrix},
$$

(4.70)

where the ratio of initial conditions $Q_0 = x_{40}/x_{20}$ is used once more.

When the above expressions for the $c_i$ are substituted back into the general solution (4.54), a simplified result follows:

$$
\begin{bmatrix}
  \delta_D' \\
  \delta_D \\
  \delta_B'
\end{bmatrix} \sim \frac{x_{20}}{\beta_{130}} (1 - Q_0 \beta_{30}) \left(\frac{\bar{\beta}_{13}}{\beta_1 \lambda_1}\right)^{1/2} \begin{bmatrix}
  i \beta_1 \lambda_1 \sin \\
  \beta_1 \cos \\
  i \lambda_1 \sin \\
  \cos
\end{bmatrix} \left(\int_{\chi_0}^{\chi} |\lambda_1| d\chi\right) \\
+ \frac{x_{20}}{\beta_{130}} (Q_0 \beta_{10} - 1) \left(\frac{\bar{\beta}_{13}}{\beta_3 \lambda_3}\right)^{1/2} \begin{bmatrix}
  i \beta_3 \lambda_3 \sin \\
  \beta_3 \cos \\
  i \lambda_3 \sin \\
  \cos
\end{bmatrix} \left(\int_{\chi_0}^{\chi} |\lambda_3| d\chi\right).
$$

(4.71)

Let us concentrate in particular on the $x_2$ and $x_4$ components, which describe the actual matter content of the Universe, and may have some interesting implications for structure formation results. We may write the solutions in a form analogous to the static spacetime results:

$$
\begin{align*}
\delta_D(\chi) &= x_{20} \left[ \zeta_1 \cos \left(\int_{\chi_0}^{\chi} |\lambda_1| d\chi\right) + \zeta_2 \cos \left(\int_{\chi_0}^{\chi} |\lambda_3| d\chi\right) \right], \\
\delta_B(\chi) &= x_{40} \left[ \zeta_3 \cos \left(\int_{\chi_0}^{\chi} |\lambda_1| d\chi\right) + \zeta_4 \cos \left(\int_{\chi_0}^{\chi} |\lambda_3| d\chi\right) \right].
\end{align*}
$$

(4.72, 4.73)
The amplitudes are given by the expressions

\begin{align}
\zeta_1 &= \beta_1 \frac{1 - Q_0 \beta_{30}}{\beta_{130}} \left( \frac{\bar{\beta}_{13}}{\beta_1 \lambda_1} \right)^{1/2}, \\
\zeta_2 &= \beta_3 \frac{Q_0 \beta_{10} - 1}{\beta_{130}} \left( \frac{\bar{\beta}_{13}}{\beta_3 \lambda_3} \right)^{1/2}, \\
\zeta_3 &= \frac{Q_0^{-1} - \beta_{30}}{\beta_{130}} \left( \frac{\bar{\beta}_{13}}{\beta_1 \lambda_1} \right)^{1/2}, \\
\zeta_4 &= \frac{\beta_{10} - Q_0^{-1}}{\beta_{130}} \left( \frac{\bar{\beta}_{13}}{\beta_3 \lambda_3} \right)^{1/2}.
\end{align}

(4.74) (4.75) (4.76) (4.77)

The immediate obvious differences for these amplitudes with the static spacetime results are:

1. Missing factors of \( \frac{1}{2} \) for \( \zeta_1 \) and \( \zeta_3 \), because the solutions are now all of a cosine form, rather than real exponentials (which is due to the fact that we are considering the \( \bar{k} \gg \bar{k}_M \) region).

2. The amplitudes are all time varying.

3. The amplitudes contain extra tilde factors which equal one at the initial time, but in general contain other time dependent terms not present even as constant factors in the static spacetime amplitudes.

For \( \chi = \chi_0 \) the amplitudes correspond exactly to those of the static spacetime amplitudes, and Figs.5–12 are accurate representations of the amplitudes over a wide range of \( k \). As time increases, the amplitudes tend to grow, but retain the same qualitative shape with the most marked features still occurring around the scale \( K_M = K_{MC} \). Since the WKB solutions are only valid for \( k \gg k_M \), this interesting region of \( k \)-space does not apply, and the \( \zeta_i \) as presently defined should not be extrapolated to have any meaning around \( K_M = K_{MC} \). In the \( K_M \gg 1 \) region the \( \zeta_i \) show no significant behavior, just tending to constant values close to zero or one. In particular no resonances are apparent. Thus the potentially interesting resonant features discovered by de Carvalho and Macedo do not apply in the physically more realistic expanding universe scenario. Any significant effects would have to be sought from the small \( k \) solutions presented in 11. 
In addition to these statements, it must be added that the $\lambda_i$ eigenvalues characterizing the modes in the large $k$ region do not have any physical significance in the small $k$ region either. This was already apparent in the small $k$ expansions for the one-component solutions discussed in [11]. The characteristic Jeans dispersion relation

$$\omega = \sqrt{v_s^2 k^2 - 4\pi G \rho_0}. \quad (4.78)$$

is not apparent in the small $k$ expansions presented in [11], and likewise, the $\lambda_i$ found in the present paper are not apparent in the general solutions given by Eqs.(5.3)–(5.10) of [11]. With these facts in mind, the discussion of the physical information we are able to extract from the WKB solutions at present is complete.

V. CONCLUSIONS AND FURTHER WORK

The structure and behavior of the eigenvalues and eigenvectors of two-component cosmological density perturbations have been studied in great detail in this paper. We have reviewed the previous work done in a static spacetime background, and produced further results in this simple context. This has enabled the far more difficult expanding universe problem to be tackled.

The WKB method employed has produced the full leading order behavior of all the modes in the Einstein-deSitter expanding Universe scenario. These solutions represent acoustic oscillations for wavelengths much smaller than the Jeans scale. The Jeans scale of the mixture has arisen in a natural way out of the analysis of the eigenvalues obtained through the WKB method, with some interesting interpretation. It is now a straightforward task to adapt the methods developed here to study a variety of further cosmological plasma modes. The ion-sound and two-component Langmuir oscillations would follow directly from the results presented here, and more complicated modes involving magnetic fields could also be obtained by similar procedures.

We have also obtained the time- and $k$-dependent amplitudes of the modes in a fairly general setting (the one restriction being initial perturbations beginning from rest). These results have shown that the amplitudes are very constant in the region of interest. The existence of resonances in the amplitudes found for static spacetime results do not apply here, as all resonances occurred for wavenumbers far smaller than $k_M$. Thus a resonant
amplitude cannot be viewed as a mechanism for producing structures of a preferred scale in a two-component model. The eigenvalues derived in this paper do also not have any direct physical interpretation around the Jeans scale, or for small $k$ expansions of the solutions of Eqs. (2.6) and (2.7). Thus the results obtained in this paper must be considered to be restricted to the parameter regions considered here.

It may be interesting to investigate different models such as a three-component HDM+CDM+baryon fluid, or models involving a cosmological constant (especially given the weight of current observations [18], [19], [20]). The analytics would become considerably more complicated, but some other interesting resonant scales may be found with a direct implication for structure formation.

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FIG. 1: The eigenvalues for HDM and CDM in the static universe scenario.

FIG. 2: The eigenvalue $\lambda_1$ for a range of $\epsilon_D$ and $\epsilon_B$, with $V = 0.001$ in the static universe scenario.
FIG. 3: The eigenvalue $\lambda_1$ for a range of $\epsilon_D$ and $\epsilon_B$, with $V = 1000$ in the static universe scenario.

FIG. 4: The eigenvector amplitude functions $\beta_1$ and $\beta_3$ for HDM and CDM in the static universe scenario.
FIG. 5: The $k$-dependent amplitudes of the $\lambda_1$ dark matter modes in a HDM universe for a range of initial conditions $Q_0 = x_{40}/x_{20}$. 
FIG. 6: The $k$-dependent amplitudes of the $\lambda_1$ dark matter modes in a CDM universe for a range of initial conditions $Q_0 = x_{40}/x_{20}$. 

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FIG. 7: The $k$-dependent amplitudes of the $\lambda_3$ dark matter modes in a HDM universe for a range of initial conditions $Q_0 = x_{40}/x_{20}$. 
FIG. 8: The $k$-dependent amplitudes of the $\lambda_3$ dark matter modes in a CDM universe for a range of initial conditions $Q_0 = x_{40}/x_{20}$. 
FIG. 9: The $k$-dependent amplitudes of the $\lambda_1$ baryonic modes in a HDM universe for a range of initial conditions $Q_0 = x_40/x_{20}$.
FIG. 10: The $k$-dependent amplitudes of the $\lambda_1$ baryonic modes in a CDM universe for a range of initial conditions $Q_0 = x_{40}/x_{20}$. 
FIG. 11: The $k$-dependent amplitudes of the $\lambda_3$ baryonic modes in a HDM universe for a range of initial conditions $Q_0 = x_{40}/x_{20}$.
FIG. 12: The $k$-dependent amplitudes of the $\lambda_3$ baryonic modes in a CDM universe for a range of initial conditions $Q_0 = x_{40}/x_{20}$. 

\[ \kappa \]