HILBERT TRANSFORMS ALONG DOUBLE VARIABLE
FRACTIONAL MONOMIALS

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Abstract. In this paper, we obtain the $L^2(\mathbb{R}^2)$ boundedness and single annulus $L^p(\mathbb{R}^2)$ estimate for the Hilbert transform $H_{\alpha,\beta}$ along double variable fractional monomial $u_1(x_1)[t]^{\alpha} + u_2(x_1)[t]^{\beta}$

$$H_{\alpha,\beta}f(x_1,x_2) := \text{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u_1(x_1)[t]^{\alpha} - u_2(x_1)[t]^{\beta}) \frac{dt}{{t}},$$

with the bounds are independent of the measurable function $u_1$ and $u_2$. At the same time, we also obtain the $L^p(\mathbb{R})$ boundedness of the corresponding Carleson operator

$$C_{\alpha,\beta}f(x) := \sup_{\nu_1,\nu_2 \in \mathbb{R}} \left| \text{p.v.} \int_{-\infty}^{\infty} e^{i\nu_1|t|^{\alpha}+i\nu_2|t|^{\beta}} f(x-t) \frac{dt}{{t}} \right|,$$

where $|t|^{\alpha}$ stands for either $|t|^{\alpha}$ or $\text{sgn}(t)|t|^{\alpha}$, $|t|^{\beta}$ stands for either $|t|^{\beta}$ or $\text{sgn}(t)|t|^{\beta}$ and $\alpha, \beta, p \in (1, \infty)$.

1. Introduction. For any two given measurable function $u_1 : \mathbb{R} \to \mathbb{R}$ and $u_2 : \mathbb{R} \to \mathbb{R}$, the Hilbert transform $H_{\alpha,\beta}$ along double variable fractional monomial $u_1(x_1)[t]^{\alpha} + u_2(x_1)[t]^{\beta}$ is defined by

$$H_{\alpha,\beta}f(x_1,x_2) := \text{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u_1(x_1)[t]^{\alpha} - u_2(x_1)[t]^{\beta}) \frac{dt}{{t}},$$

where $|t|^{\alpha}$ stands for either $|t|^{\alpha}$ or $\text{sgn}(t)|t|^{\alpha}$, $|t|^{\beta}$ stands for either $|t|^{\beta}$ or $\text{sgn}(t)|t|^{\beta}$, $\alpha, \beta \in (1, \infty)$. Here and hereafter, p.v. denotes the principal-value integral and all of the operators in this paper original defined in the space of Schwartz functions. In [10], Guo et al. obtained the $L^p(\mathbb{R}^2)$ boundedness of the Hilbert transform $H_{\alpha}$ along variable fractional monomial $u(x_1)[t]^{\alpha}$ is defined by

$$H_{\alpha}f(x_1,x_2) := \text{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u(x_1)[t]^{\alpha}) \frac{dt}{{t}},$$

for any given $p \in (1, \infty)$. In [21], the author and Li considered a more generalized case that variable fractional monomial $u(x_1)[t]^{\alpha}$ in $H_{\alpha}$ is replaced by variable plane
we also established the $L^p(\mathbb{R}^2)$ boundedness of $H_\gamma$ for all $p \in (1, \infty)$ with $\gamma$ satisfying some conditions. The homogeneous curve $[t]^{\alpha}$ with $\alpha \in (1, \infty)$ is the model curve in [21]. However, the curve of the simple linear combination $[t]^{\alpha} + [t]^{\beta}$ does not satisfy our conditions on $\gamma$, where $\alpha \neq \beta$ and $\alpha, \beta \in (1, \infty)$. Thus, we hope to establish the $L^p(\mathbb{R}^2)$ boundedness of $H_\gamma$ for this curve. To this aim, we consider a wider operator (1). But the method that $L^p(\mathbb{R}^2)$ estimate for $H_\gamma$ in [21] no longer valid for the operator $H_{\alpha,\beta}$, since we always use the function $n_3 : \mathbb{R} \to \mathbb{Z}$ which is defined in (37) to connect $u_2(x_1)$ and $[t]^{\beta}$, and also be used to decompose our operator into a low frequency part and a high frequency part. But $n_3 : \mathbb{R} \to \mathbb{Z}$ can not connect $u_1(x_1)$ and $[t]^{\alpha}$, it furthermore implies that some estimate for parameter $\sigma_m^{(2)}$ similarly to ([21], (3.43)) can not been obtained, if we use the shifted maximal operator to form a pointwise estimate for taking average along double variable fractional monomial $u_1(x_1)[t]^{\alpha} + u_2(x_1)[t]^{\beta}$. Therefore, we consider a weaker result in this paper and establish the $L^2(\mathbb{R}^2)$ boundedness and single annulus $L^p(\mathbb{R}^2)$ estimate for the Hilbert transform $H_{\alpha,\beta}$ in (1), which is stated as follows:

**Theorem 1.1.** Let $u_1 : \mathbb{R} \to \mathbb{R}$ and $u_2 : \mathbb{R} \to \mathbb{R}$ be measurable function. Let $H_{\alpha,\beta}$ be given by (1). Then there exists a positive constant $C$ such that

$$
\|H_{\alpha,\beta}f\|_{L^2(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)}
$$

with the bound $C$ independent of $u_1$ and $u_2$, where $\alpha, \beta \in (1, \infty)$.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a non-negative smooth function supported in

$$
\{t \in \mathbb{R} : \frac{1}{2} \leq |t| \leq 2\}
$$

that satisfies $\sum_{z \in \mathbb{Z}} \psi(t) = 1$ for any $t \neq 0$, where $\psi(t) := \psi(2^{-l}t)$. For any $l \in \mathbb{Z}$, let $P_l$ denotes the Littlewood-Paley operator in the second variable associated with $\psi_l$. That is

$$
P_l f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1, x_2 - z) \psi_l(z) \, dz.
$$

(4)

Our single annulus $L^p(\mathbb{R}^2)$ estimate for the Hilbert transform $H_{\alpha,\beta}$ is

**Theorem 1.2.** Let $u_1 : \mathbb{R} \to \mathbb{R}$ and $u_2 : \mathbb{R} \to \mathbb{R}$ be measurable function. Let $H_{\alpha,\beta}$ be given by (1) and $p \in (1, \infty)$. Then there exists a positive constant $C$ such that

$$
\|H_{\alpha,\beta}P_l f\|_{L^p(\mathbb{R}^2)} \leq C\|P_l f\|_{L^p(\mathbb{R}^2)}
$$

uniformly in $l \in \mathbb{Z}$ with the bound $C$ independent of $u_1$ and $u_2$, where $\alpha, \beta \in (1, \infty)$.

Throughout this paper, we denote by $C$ a positive constant that is independent of the main parameters involved, but whose value may differ from line to line. For two real functions $f$ and $g$, the symbol $f \lesssim g$ or $g \gtrsim f$ means that $f \leq Cg$, if $f \lesssim g \lesssim f$, we then write $f \approx g$.

The interest of the above study can be traced back to the Hilbert transform $\mathcal{H}$ along curve $\gamma(t)$

$$
\mathcal{H}f(x_1, x_2) := \text{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - \gamma(t)) \frac{dt}{t},
$$

(5)
which has been extensively studied and the reader may look in [2, 3, 5, 15, 19, 20]. Furthermore, the $L^p(\mathbb{R}^2)$ boundedness of the following directional Hilbert transform $\mathbb{H}_\lambda$ along curve $\gamma(t)$ defined for a fixed direction $(1, \lambda)$ as

$$
\mathbb{H}_\lambda f(x_1, x_2) := \mathrm{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - \lambda \gamma(t)) \frac{dt}{t} \quad (6)
$$

can be established easily by Hilbert transform $\mathbb{H}$. On the other hand, we may conclude that

$$
\sup_{\lambda \in \mathbb{R}} \| \mathbb{H}_\lambda f \|_{L^p(\mathbb{R}^2)} \leq C \| f \|_{L^p(\mathbb{R}^2)}
$$

for all $p \in (1, \infty)$. By linearization, the $L^p(\mathbb{R}^2)$ boundedness of the corresponding maximal operator $\sup_{\lambda \in \mathbb{R}} | \mathbb{H}_\lambda f(x_1, x_2) |$ is tantamount to the $L^p(\mathbb{R}^2)$ estimate for

$$
H_U f(x_1, x_2) := \mathrm{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - U(x_1, x_2) \gamma(t)) \frac{dt}{t} \quad (7)
$$

and the bound must be independent of the measurable function $U$. However, the $L^p(\mathbb{R}^2)$ boundedness of $H_U$ cannot hold if we only assume $U$ is a measurable function, see [10]. Therefore, let $p \in (1, \infty)$, we cannot establish

$$
\left\| \sup_{\lambda \in \mathbb{R}} | \mathbb{H}_\lambda f \| \right\|_{L^p(\mathbb{R}^2)} \leq C \| f \|_{L^p(\mathbb{R}^2)}.
$$

Instead of this, Theorem 1.1 implies that

$$
\left\| \sup_{\lambda_1, \lambda_2 \in \mathbb{R}^2} \left\| p.v. \int_{-\infty}^{\infty} f(\cdot - 1, \cdot - 2 - \lambda_1 [t]\alpha - \lambda_2 [t]\beta) \frac{dt}{t} \right\|_{L^2(\mathbb{R}^2)} \right\|_{L^2(\mathbb{R}^2)} \leq C \| f \|_{L^2(\mathbb{R}^2)},
$$

which squeezes the supremum between the two $L^2$ norms. Here and hereafter, $\cdot_1$ and $\cdot_2$ denote the first variable $x_1$ and the second variable $x_2$, respectively.

The Carleson operator $C_{\alpha, \beta}$ appears naturally in the proof of the $L^2(\mathbb{R}^2)$ boundedness of the Hilbert transform $H_{\alpha, \beta}$ in (1), which is defined by

$$
C_{\alpha, \beta} f(x) := \sup_{N_1, N_2 \in \mathbb{R}} \left| p.v. \int_{-\infty}^{\infty} e^{iN_1 [t]^{\alpha} + iN_2 [t]^{\beta}} f(x - t) \frac{dt}{t} \right| \quad (8)
$$

The original Carleson operator $C$ is defined by

$$
C f(x) := \sup_{N \in \mathbb{R}} \left| p.v. \int_{-\infty}^{\infty} e^{iN t} f(x - t) \frac{dt}{t} \right| \quad (9)
$$

Carleson in [4] showed that $C$ maps $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$, and furthermore point out that the Luzin conjecture established. Hunt in [12] obtained its $L^p(\mathbb{R})$ boundedness for all $p \in (1, \infty)$. For more details on the subject we refer the reader to [6, 14, 17]. Guo in [8] proved the $L^p(\mathbb{R})$ boundedness of $C_{N, \varepsilon_1}^{\text{even}}$ and $C_{N, \varepsilon_2}^{\text{odd}}$ for all $p \in (1, \infty)$, where

$$
C_{N, \varepsilon_1}^{\text{even}} f(x) := \sup_{N \in \mathbb{R}} \left| p.v. \int_{-\infty}^{\infty} e^{iN |t|^{\varepsilon_1}} f(x - t) \frac{dt}{t} \right|, \quad \varepsilon_1 \in \mathbb{R}, \varepsilon_1 \neq 1, \quad (10)
$$

and

$$
C_{N, \varepsilon_2}^{\text{odd}} f(x) := \sup_{N \in \mathbb{R}} \left| p.v. \int_{-\infty}^{\infty} e^{iN \text{sgn}(t)|t|^{\varepsilon_2}} f(x - t) \frac{dt}{t} \right|, \quad \varepsilon_2 \in \mathbb{R}, \varepsilon_2 \neq 0. \quad (11)
$$
In [18], Stein and Wainger obtained the $L^p(\mathbb{R})$ boundedness of Carleson operator $\mathcal{C}_p$, which is defined by

$$\mathcal{C}_p f(x) := \sup_{P \in \mathcal{P}_d} \left| \text{p.v.} \int_{-\infty}^{\infty} e^{iP(t)} f(x - t) \frac{dt}{t} \right|,$$  \hspace{1cm} (12)

where $d \in \mathbb{N} \setminus \{1\}$ and $\mathcal{P}_d$ is the class of all real-coefficient polynomials $P$ with $\deg(P) \leq d$ and the restriction that the first-order terms vanish. Therefore, which further implies that the $L^p(\mathbb{R})$ boundedness of

$$\mathcal{C}_{m,n} f(x) := \sup_{N_1, N_2 \in \mathbb{R}} \left| \text{p.v.} \int_{-\infty}^{\infty} e^{iN_1 t^m + iN_2 t^\alpha} f(x - t) \frac{dt}{t} \right|,$$ \hspace{1cm} (13)

with $m, n \in \mathbb{N} \setminus \{1\}$. The Carleson operator $\mathcal{C}_{\alpha,\beta}$ in (8) is a nature extension of $\mathcal{C}_{m,n}$ in (13). Our $L^p(\mathbb{R})$ estimate for the Carleson operator $\mathcal{C}_{\alpha,\beta}$ is

**Theorem 1.3.** Let $u_1: \mathbb{R} \to \mathbb{R}$ and $u_2: \mathbb{R} \to \mathbb{R}$ be measurable function. Let $\mathcal{C}_{\alpha,\beta}$ be given by (8) and $p \in (1, \infty)$. Then there exists a positive constant $C$ such that

$$\|\mathcal{C}_{\alpha,\beta} f\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^p(\mathbb{R})}$$

with the bound $C$ independent of $u_1$ and $u_2$, where $\alpha, \beta \in (1, \infty)$.

This paper is structured as follows. In Section 2, we want to proof of Theorem 1.3 which gives the $L^2(\mathbb{R}^2)$ boundedness of (1), it is Theorem 1.1. Section 2 is devote to establishing the single annulus $L^p(\mathbb{R}^2)$ estimate for (1) for all $p \in (1, \infty)$, that is Theorem 1.2.

2. Proof of Theorem 1.1 and Theorem 1.3. From [16], the $L^2(\mathbb{R}^2)$ boundedness of the Hilbert transform $H_{\alpha,\beta}$ in (1), i.e. Theorem 1.1, is equivalent to the $L^2(\mathbb{R}^2)$ boundedness of the Carleson operator $\mathcal{C}_{\alpha,\beta}$ in (8) by the Fourier transform and Plancherel’s formula. Therefore, we have

$$\|H_{\alpha,\beta}\|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} \leq \sup_{\lambda \in \mathbb{R}} \|S_{\lambda}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})},$$

where

$$S_{\lambda} f(x) := \text{p.v.} \int_{-\infty}^{\infty} e^{-i\lambda u_1(x) |t|^\alpha - i\lambda u_2(x) |t|^\beta} f(x - t) \frac{dt}{t}.$$  \hspace{1cm} (13)

For the Carleson operator $\mathcal{C}_{\alpha,\beta}$ in (8), by linearizing the supremum, which $L^p(\mathbb{R})$ boundedness is equivalent to the $L^p(\mathbb{R})$ boundedness of

$$\mathcal{C}_{\alpha,\beta} f(x) = \text{p.v.} \int_{-\infty}^{\infty} e^{iu_1(x) |t|^\alpha + iu_2(x) |t|^\beta} f(x - t) \frac{dt}{t}$$ \hspace{1cm} (14)

with the bound must be independent of the measurable function $u_1$ and $u_2$. Here and hereafter, we rewrite the Carleson operator $\mathcal{C}_{\alpha,\beta}$ in (8) as (14). Thus, it is enough to prove of Theorem 1.3. The strategy of our proof is split our operator into two parts. The first part can be controlled by some operators which are bounded on $L^p(\mathbb{R})$, the second part is decomposed as a series of operators and all of these operators have a decay $L^p(\mathbb{R})$ estimate.

**Proof of Theorem 1.3.** For the case that $\alpha = \beta$, it has been proofed by Guo in [8]. It remains to consider the case that $\alpha \neq \beta$. Recall that the non-negative smooth function $\psi: \mathbb{R} \to \mathbb{R}$ is supported in the set $\{ t \in \mathbb{R} : \frac{1}{2} \leq |t| \leq 2 \}$ that satisfies
holds for any $x \in \mathbb{R}$. Let

$$C_{\alpha,\beta,k}f(x) := \int_{-\infty}^{\infty} e^{iu_1(x)[t]^{\alpha} + iu_2(x)[t]^{\beta}} f(x - t) \psi_k(t) \frac{dt}{t}.$$  

We can split the Carleson operator $C_{\alpha,\beta}$ into the following two parts:

$$C_{\alpha,\beta}f(x) = \sum_{k \leq n_1(x) - 1} C_{\alpha,\beta,k}f(x) + \sum_{k \geq n_1(x)} C_{\alpha,\beta,k}f(x) =: C^{(1)}_{\alpha,\beta}f(x) + C^{(2)}_{\alpha,\beta}f(x). \quad (16)$$

For the low frequency part $C^{(1)}_{\alpha,\beta}f$, let $\phi_1(t) := \sum_{k \leq n_1(x) - 1} \psi_k(t)$, we can write

$$C^{(1)}_{\alpha,\beta}f(x) = \text{p.v.} \int_{|t| \leq 2n_1(x)} e^{iu_1(x)[t]^{\alpha} + iu_2(x)[t]^{\beta}} f(x - t) \phi_1(t) \frac{dt}{t}$$

$$= \text{p.v.} \int_{|t| \leq 2n_1(x)} \left[ e^{iu_1(x)[t]^{\alpha} + iu_2(x)[t]^{\beta}} - e^{iu_1(x)[t]^{\alpha}} \right] f(x - t) \phi_1(t) \frac{dt}{t}$$

$$+ \text{p.v.} \int_{|t| \leq 2n_1(x)} e^{iu_1(x)[t]^{\alpha}} f(x - t) \phi_1(t) \frac{dt}{t}$$

$$=: T_1f(x) + T_2f(x).$$

For the former part $T_1f$, from (15) and $\beta > 1$, we have

$$T_1f(x) = \text{p.v.} \int_{|t| \leq 2n_1(x)} \left[ e^{iu_2(x)[t]^{\beta}} - 1 \right] e^{iu_1(x)[t]^{\alpha}} f(x - t) \phi_1(t) \frac{dt}{t} \quad (17)$$

$$\leq \int_{|t| \leq 2n_1(x)} |f(x - t)| |u_2(x)| \frac{|t|^{\beta}}{|t|} \phi_1(t) dt$$

$$\leq \frac{1}{2n_1(x)^{\beta}} 2^{n_1(x)(\beta - 1)} \int_{|t| \leq 2n_1(x)} |f(x - t)| dt \lesssim Mf(x).$$

Here and hereafter, $M$ denotes the Hardy Littlewood maximal operator.

For the latter part $T_2f$, by the linearization, we obtain that

$$|T_2f(x)| = \left| \int_{|t| \leq 2n_1(x)} e^{iu_1(x)[t]^{\alpha}} f(x - t) \frac{\phi_1(t) - 1}{t} dt \right| \quad (18)$$

$$\leq \int_{2n_1(x) - 1 \leq |t| \leq 2n_1(x)} |f(x - t)| \left| \frac{\phi_1(t) - 1}{t} \right| dt + C^*_\alpha f(x)$$

$$\leq \frac{1}{2n_1(x) - 1} \int_{|t| \leq 2n_1(x)} |f(x - t)| dt + C^*_\alpha f(x) \lesssim Mf(x) + C^*_\alpha f(x).$$

Here $C^*_\alpha$ is the maximal truncated Carleson operator along fractional monomial $[t]^{\alpha}$ given

$$C^*_\alpha f(x) := \sup_{N \in \mathbb{R}, \varepsilon > 0} \left| \text{p.v.} \int_{|t| < \varepsilon} e^{iN[t]^{\alpha}} f(x - t) \frac{dt}{t} \right|.$$
Therefore,
\[ C_{\alpha,\beta}^{(1)} f(x) \lesssim Mf(x) + C_{\alpha}^* f(x). \]
From the following Proposition 1, \( C_{\alpha}^* \) is bounded on \( L^p(\mathbb{R}) \). This, combined with the well-known \( L^p(\mathbb{R}) \) boundedness of \( M \), we may conclude that
\[ \|C_{\alpha,\beta}^{(1)} f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})} \]
for any given \( p \in (1, \infty) \).

For the high frequency part \( C_{\alpha,\beta}^{(2)} f \) which can be written as a series of operators, i.e.
\[ C_{\alpha,\beta}^{(2)} f(x) = \sum_{k \geq 0} \int_{-\infty}^{\infty} e^{iu_1(x)[t]^\alpha + iu_2(x)[t]^\beta} f(x - t) \psi_{k+1}(x)(t) \frac{dt}{t} =: \sum_{k \geq 0} S_{\alpha,\beta,k} f(x). \]
For each fixed \( k \geq 0 \), we first show that
\[ |S_{\alpha,\beta,k} f(x)| \leq \int_{2^{k+1}(x)-1 \leq |t| \leq 2^{k}(x)+1} |f(x - t)| \frac{|\psi_{k+1}(x)(t)|}{|t|} dt \]
\[ \leq \frac{1}{2^{k}(x)+1} \int_{|t| \leq 2^{k}(x)+1} |f(x - t)| dt \lesssim Mf(x). \]
Since the \( L^p(\mathbb{R}) \) boundedness of \( M \) for all \( p \in (1, \infty) \), it follows the trivial bound
\[ \|S_{\alpha,\beta,k} f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}. \]
From (15), we have
\[ |u_2(x)| 2^{(n_1(x)+k)\beta} \geq \frac{1}{2^{(n_1(x)+1)\beta}} 2^{(n_1(x)+k)\beta} = 2^{(k-1)\beta}. \]
From this and ([11], lemma 2.1), there exists a positive constant \( \varepsilon_0 \) such that
\[ \|S_{\alpha,\beta,k} f\|_{L^2(\mathbb{R})} \lesssim 2^{-\varepsilon_0 k} \|f\|_{L^2(\mathbb{R})}. \]
By interpolation and sum up \( k \geq 0 \) we assert that
\[ \|C_{\alpha,\beta}^{(2)} f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})} \]
for any given \( p \in (1, \infty) \), which completes the proof of Theorem 1.3. \( \square \)

**Proposition 1.** Let \( \gamma \in C^3(\mathbb{R}) \) be either odd or even, with \( \gamma(0) = \gamma'(0) = 0 \), and convex on \((0, \infty)\), satisfying
(i) \( \frac{\gamma''(t)}{\gamma(t)} \) is decreasing and bounded by a constant \( C_1 \) from above on \((0, \infty)\),
(ii) there exists positive constant \( C_2 \) such that \( \frac{\gamma''(t)}{\gamma(t)} \leq C_2 \) on \((0, \infty)\),
(iii) there exists a positive constant \( C_3 \) such that \( |\frac{\gamma''(t)}{\gamma(t)}| \geq \frac{C_3}{t^2} \) on \((0, \infty)\),
(iv) \( \frac{\gamma''(t)}{\gamma(t)} \) is strictly monotone or equals to a constant on \((0, \infty)\).
Then, for the maximal truncated Carleson operator along general curve \( \gamma \) which is defined by
\[ C_{\gamma}^* f(x) := \sup_{N \in \mathbb{R}, \varepsilon > 0} \left| \text{p. v.} \int_{|t| < \varepsilon} e^{iN\gamma(t)} f(x - t) \frac{dt}{t} \right|, \]
there exists a positive constant \( C \) such that
\[ \|C_{\gamma}^* f\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^p(\mathbb{R})} \]
holds for any given \( p \in (1, \infty) \).
Proof of Proposition 1. Linearizing the supremum, the estimate (23) is equivalent to establish
\[ \|C_{u,\varepsilon,\gamma}f\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})} \]
with the bound must be independent of the measurable functions \( u \) and \( \varepsilon \), where
\[ C_{u,\varepsilon,\gamma}f(x) := \text{p.v.} \int_{|t|<\varepsilon(x)} e^{iu(x)\gamma(t)} f(x-t) \frac{dt}{t}. \]
It is easy to check that \( \gamma \) is increasing on \((0, \infty)\) and \( \lim_{t \to \infty} \gamma(t) = \infty \), we can define \( n_2 : \mathbb{R} \to \mathbb{Z} \) such that
\[ \frac{1}{(2^{(n_2(x)+1)})} \leq |u(x)| \leq \frac{1}{\gamma(2^{n_2(x)})} \]
holds for all \( x \in \mathbb{R} \).
If \( \varepsilon(x) \leq 2^{n_2(x)} \), it is easy to see that \( \frac{\gamma(t)}{|t|} \) is increasing on \((0, \infty)\). This, Combined with the fact that \( \gamma \) is either odd or even, and \( \gamma \) is increasing on \((0, \infty)\), linearizing the supremum, we therefore have the estimate
\[ C_{u,\varepsilon,\gamma}f(x) = \text{p.v.} \int_{|t|<\varepsilon(x)} (e^{iu(x)\gamma(t)} - 1) f(x-t) \frac{dt}{t} + \text{p.v.} \int_{|t|<\varepsilon(x)} f(x-t) \frac{dt}{t} \]
\[ \leq \int_{|t|<\varepsilon(x)} |f(x-t)||u(x)| \frac{\gamma(|t|)}{|t|} \frac{dt}{t} + \mathcal{H}^* f(x) \]
\[ \leq \int_{|t|<2^{n_2(x)}} |f(x-t)| \frac{1}{\gamma(2^{n_2(x)})} \gamma(2^{n_2(x)}) |t|^2 dt + \mathcal{H}^* f(x) \leq M f(x) + \mathcal{H}^* f(x), \]
where \( \mathcal{H}^* \) is the maximal truncated Hilbert transform. Let \( p \in (1, \infty) \), we know that both \( M \) and \( \mathcal{H}^* \) are bounded on \( L^p(\mathbb{R}) \), from which we deduce
\[ \|C_{u,\varepsilon,\gamma}f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}. \]
If \( \varepsilon(x) > 2^{n_2(x)} \), we decompose \( C_{u,\varepsilon,\gamma} \) into two parts, i.e.
\[ C_{u,\varepsilon,\gamma}f(x) = \text{p.v.} \int_{|t|<2^{n_2(x)}} e^{iu(x)\gamma(t)} f(x-t) \frac{dt}{t} \]
\[ + \text{p.v.} \int_{2^{n_2(x)} \leq |t|<\varepsilon(x)} e^{iu(x)\gamma(t)} f(x-t) \frac{dt}{t} \]
\[ = C_1f(x) + C_2f(x), \]
For \( C_1f \), similarly to (25), we have
\[ \|C_1f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}. \]
For \( C_2f \), define \( m_1 : \mathbb{R} \to \mathbb{Z} \) such that for all \( x \in \mathbb{R} \),
\[ 2^{m_1(x)} \leq \varepsilon(x) \leq 2^{m_1(x)+1}. \]
We further decompose \( C_2f \) as
\[ C_2f(x) = \int_{2^{n_2(x)} \leq |t|<2^{m_1(x)}} e^{iu(x)\gamma(t)} f(x-t) \frac{dt}{t} + \int_{2^{m_1(x)} \leq |t|<\varepsilon(x)} e^{iu(x)\gamma(t)} f(x-t) \frac{dt}{t} \]
\[ = C_{2,1}f(x) + C_{2,11}f(x), \]

Recall that the non-negative smooth function \( \psi : \mathbb{R} \to \mathbb{R} \) is supported in the set \( \{ t \in \mathbb{R} : \frac{1}{2} \leq |t| \leq 2 \} \) and satisfies \( \Sigma_{t \in \mathbb{Z}} \psi(t) = 1 \) for any \( t \neq 0 \), where \( \psi(t) = \psi(2^{-l}t) \). We rewrite \( C_{2,l}f \) essentially as

\[
C_{2,l}f(x) = \sum_{k=0}^{m_1(x)-n_2(x)} \int_{-\infty}^{\infty} e^{iu(x)\gamma(t)} f(x-t)\psi_k(t) \frac{dt}{t}
\]

\[
= \sum_{k=0}^{m_1(x)-n_2(x)} \int_{-\infty}^{\infty} e^{iu(x)\gamma(t)} f(x-t)\psi_{k+n_2(x)}(t) \frac{dt}{t}
\]

where \( S_k \) can also be found in ([21], P.9). From ([21], (2.11)), we have that there exists a positive constant \( \omega_0 \) such that

\[
\|S_kf\|_{L^2(\mathbb{R})} \lesssim 2^{-\omega_0k}\|f\|_{L^2(\mathbb{R})}
\]

holds for any \( k \geq 0 \). Similarly to (20), we have

\[
\|S_kf\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}
\]

and

\[
\|C_{2,l}f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}
\]

for any given \( p \in (1, \infty) \). By a strategy of interpolation between (31) and (32), combining (33), we have

\[
\|C_{2}f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.
\]

Therefore, we conclude that

\[
\|C_{u,\gamma}f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}
\]

for any given \( p \in (1, \infty) \). This finishes the proof of Proposition 1. \( \square \)

3. Proof of Theorem 1.2. From [7] and [9], we know that the single annulus \( L^p(\mathbb{R}^2) \) estimate plays an important role in obtaining the \( L^p(\mathbb{R}^2) \) boundedness of the Hilbert transform along variable curve. There are many other works about single annulus \( L^p(\mathbb{R}^2) \) estimate; see, for example, [1, 13]. In this section, we establishing the single annulus \( L^p(\mathbb{R}^2) \) estimate for \( H_{\alpha,\beta} \).

Proof of Theorem 1.2. We can reduce our attention to the case that \( l = 0 \) by an anisotropic scaling

\[
x_1 \to x_1, \ x_2 \to 2^{-l}x_2.
\]

Let \( p \in (1, \infty) \), it is enough to prove that

\[
\|H_{\alpha,\beta}P_0f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.
\]

In deed, let \( \rho \) be a non-negative smooth function supported in the set

\[
\{ \xi \in \mathbb{R} : \frac{1}{4} \leq |\xi| \leq 4 \}
\]

and \( \rho = 1 \) on the set \( \{ \xi \in \mathbb{R} : \frac{1}{2} \leq |\xi| \leq 2 \} \), and let

\[
P_0f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1, x_2-s)\hat{\rho}(s) \, ds.
\]

Similarly to (35), we can obtain that

\[
\|H_{\alpha,\beta}\hat{P}_0f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.
\]

By Fourier transform, it is easy to see that

\[
P_0\hat{P}_0f = \hat{P}_0f.
\]

Therefore,

\[
\|H_{\alpha,\beta}P_0f\|_{L^p(\mathbb{R})} \lesssim \|P_0f\|_{L^p(\mathbb{R})}
\]
for all \( p \in (1, \infty) \), this is the desired estimate. Now, we show (35), let us set
\[
H_{\alpha, \beta} P_0 f(x_1, x_2) := \int_{-\infty}^{\infty} P_0 f(x_1 - t, x_2 - u_1(x_1)[t]^\alpha - u_2(x_1)[t]^\beta) \psi_k(t) \frac{dt}{t}.
\]
We define \( n_3 : \mathbb{R} \to \mathbb{Z} \) such that
\[
\frac{1}{2(n_3(x_1)+1)^\beta} \leq |u_2(x_1)| \leq \frac{1}{2n_3(x_1)^\beta}
\]
holds for all \( x_1 \in \mathbb{R} \). We decompose the function \( H_{\alpha, \beta} P_0 f \) into the following two parts:
\[
H_{\alpha, \beta} P_0 f(x_1, x_2) = \sum_{k \leq n_3(x_1)-1} H_{\alpha, \beta, k} P_0 f(x_1, x_2) + \sum_{k \geq n_3(x_1)} H_{\alpha, \beta, k} P_0 f(x_1, x_2)
\]
(38)
For \( H_{\alpha, \beta, 0} f \). Let \( \sum_{k \leq n_3(x_1)-1} \psi_k(t) =: \phi_3(t) \). We have that
\[
H_{\alpha, \beta, 0} P_0 f(x_1, x_2) = \text{p.v.} \int_{-\infty}^{\infty} P_0 f(x_1 - t, x_2 - u_1(x_1)[t]^\alpha - u_2(x_1)[t]^\beta) \phi_3(t) \frac{dt}{t}.
\]
(39)
Let
\[
\hat{H} P_0 f(x_1, x_2) := \text{p.v.} \int_{-\infty}^{\infty} P_0 f(x_1 - t, x_2 - u_1(x_1)[t]^\alpha) \phi_3(t) \frac{dt}{t}.
\]
Then
\[
\hat{H} P_0 f(x_1, x_2) = \text{p.v.} \int_{|t| \leq 2n_3(x_1)} P_0 f(x_1 - t, x_2 - u_1(x_1)[t]^\alpha) (\phi_3(t) - 1) \frac{dt}{t}
\]
(40)
\[
+ \text{p.v.} \int_{|t| \leq 2n_3(x_1)} P_0 f(x_1 - t, x_2 - u_1(x_1)[t]^\alpha) \phi_3(t) \frac{dt}{t}
\]
\[
\leq \int_{2n_3(x_1)-1 \leq |t| \leq 2n_3(x_1)} |P_0 f(x_1 - t, x_2 - u_1(x_1)[t]^\alpha)| \left| \phi_3(t) - 1 \right| \frac{dt}{t} + \mathcal{H}^*_\alpha P_0 f(x)
\]
\[
\leq M_\alpha P_0 f(x) + \mathcal{H}^*_\alpha P_0 f(x),
\]
where \( M_\alpha \) denotes the \textit{maximal operator} along variable fractional monomial \( u(x_1)[t]^\alpha \), which is defined by
\[
M_\alpha f(x_1, x_2) := \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x_1 - t, x_2 - u(x_1)[t]^\alpha)| \, dt,
\]
\( \mathcal{H}^*_\alpha \) denotes the \textit{truncated Hilbert transform} along variable fractional monomial \( u(x_1)[t]^\alpha \), which is defined by
\[
\mathcal{H}^*_\alpha f(x) := \int_{|t| < e(x_1)} f(x_1 - t, x_2 - u(x_1)[t]^\alpha) \frac{dt}{t}.
\]
Let \( p \in (1, \infty) \), from ([10], Corollary 1.3), we have
\[
\|M_\alpha P_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|P_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}
\]
(41)
with the bound independent of \( u \). From the following Proposition 2, we have
\[
\|\mathcal{H}^*_\alpha P_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|P_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}
\]
(42)
with the bound independent of $u$ and $\varepsilon$. Therefore,
\[ \| \tilde{H}P_0 f \|_{L^p(\mathbb{R}^2)} \lesssim \| P_0 f \|_{L^p(\mathbb{R}^2)} \lesssim \| f \|_{L^p(\mathbb{R}^2)}. \] (43)

Now we look at the difference between $H^{(1)}_{\alpha,\beta} P_0 f$ and $\tilde{H}P_0 f$ given
\[ \text{p.v.} \int_{|t| \leq 2^{n_3(x_1)}} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u_1(x_1)[t]^\alpha - z) \left[ \hat{\psi}(z - u_2(x_1)[t]^\beta) - \hat{\psi}(z) \right] dz \phi_3(t) \frac{dt}{|t|}. \] (44)

Since $|t| \leq 2^{n_3(x_1)}$, from (15), we obtain the key restriction that $|u_2(x_1)[t]^\beta| \leq 2^{n_3(x_1)} = 1$. We apply the mean value theorem to obtain
\[ |\hat{\psi}(z - u_2(x_1)[t]^\beta) - \hat{\psi}(z)| \leq \sum_{m \in \mathbb{Z}} \frac{1}{(|m| - 1)^2} \chi_{[m,m+1]}(z) |u_2(x_1)[t]^\beta|. \]

Since $\sum_{m \in \mathbb{Z}} \frac{1}{(|m| - 1)^2} \leq 1$, it is enough to consider the $L^p(\mathbb{R}^2)$ boundedness of the operator, for any fixed $m \in \mathbb{Z}$,
\[ K_m f(x_1, x_2) := \int_{m}^{m+1} \text{p.v.} \int_{|t| \leq 2^{n_3(x_1)}} |f(x_1 - t, x_2 - u_1(x_1)[t]^\alpha - z)||u_2(x_1)[t]^\beta| \phi_3(t) \frac{dt}{|t|} dz \] (45)

and the bound independent of $u_1$, $u_2$ and $m$. By Minkowski’s inequality and notice that $\beta > 1$, it follows that
\[ \| K_m f \|_{L^p(\mathbb{R}^2)} \lesssim \| f \|_{L^p(\mathbb{R}^2)} \] (46)

From (43) and (46), we have
\[ \| H^{(1)}_{\alpha,\beta} P_0 f \|_{L^p(\mathbb{R}^2)} \lesssim \| f \|_{L^p(\mathbb{R}^2)}. \]

For $H^{(2)}_{\alpha,\beta} P_0 f$, let $f := P_0 f$, then we can write $H^{(2)}_{\alpha,\beta} f$ as
\[ H^{(2)}_{\alpha,\beta} f(x_1, x_2) = \sum_{k \geq 0} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u_1(x_1)[t]^\alpha - u_2(x_1)[t]^\beta) \psi_{m_3(x_1) + k}(t) \frac{dt}{t}. \]
By Minkowski’s inequality

\[
\left\| \int_{-\infty}^{\infty} f_1(t) dt \right\|_{L_p(\mathbb{R}^2)} \leq \int_{-\infty}^{\infty} \left( \int_{|t| \leq \epsilon(x_1)} |f_1(t)|^2 dt \right)^{\frac{p}{2}} dt
\]

(47)

Similarly to (24), we define

\[
\mathcal{H}_x f(x_1, x_2) := \int_{|t| \leq \epsilon(x_1)} f(x_1 - t, x_2 - u(x_1)\gamma(t)) \frac{dt}{t}
\]

and any given \( p \in (1, \infty) \), there exists a positive constant \( C \) such that

\[
\| \mathcal{H}_x f \|_{L_p(\mathbb{R}^2)} \leq C \| P_t f \|_{L_p(\mathbb{R}^2)}
\]

(49)

holds for any \( P_t f \) which is defined as (4).

**Proof of Proposition 2.** Similarly to Theorem 1.3, let \( p \in (1, \infty) \), it suffices to proof that

\[
\| \mathcal{H}_x^* P_0 f \|_{L_p(\mathbb{R})} \lesssim \| P_0 f \|_{L_p(\mathbb{R})}
\]

Let

\[
\mathcal{H}_x^* P_0 f(x_1, x_2) := \int_{|t| \leq \epsilon(x_1)} P_0 f(x_1 - t, x_2 - u(x_1)\gamma(t)) \psi_k(t) \frac{dt}{t}
\]

Similarly to (24), we define \( n_4 : \mathbb{R} \to \mathbb{Z} \) such that

\[
\frac{1}{\gamma(2n_4(x_1)+1)} \leq |u(x_1)| \leq \frac{1}{\gamma(2n_4(x_1))}
\]

(50)

holds for all \( x_1 \in \mathbb{R} \).

If \( \epsilon(x_1) \leq 2n_4(x_1) \), we first consider

\[
\mathcal{H}_x^* f(x_1, x_2) := \int_{|t| \leq \epsilon(x_1)} f(x_1 - t, x_2) \frac{dt}{t},
\]

which can be seemed as truncated Hilbert transform. Therefore, we have

\[
\| \mathcal{H}_x^* P_0 f \|_{L_p(\mathbb{R}^2)} \lesssim \| P_0 f \|_{L_p(\mathbb{R}^2)} \lesssim \| f \|_{L_p(\mathbb{R}^2)}.
\]

(51)
Similarly to (44) through (46), we may obtain the $L^p(\mathbb{R}^2)$ boundedness of the difference between $H^*_1 P_0 f$ and $H^*_2 P_0 f$. Therefore,

$$\|H^*_1 P_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}.$$ 

If $2^{n_4(x_1)} < \varepsilon(x_1)$, we decompose

$$H^*_1 P_0 f(x_1, x_2) = \sum_{k \leq n_4(x_1)} H^*_{\gamma, k} P_0 f(x_1, x_2) + \sum_{k \geq n_4(x_1)} H^*_{\gamma, k} P_0 f(x_1, x_2)$$

(52)

$$= H^*_{\gamma, \gamma} P_0 f(x_1, x_2) + H^*_{I, \gamma} P_0 f(x_1, x_2).$$

For $H^*_{I, \gamma} P_0 f$. Let $\sum_{k \leq n_4(x_1)} \psi_k(t) =: \phi_4(t)$, it follows that

$$H^*_{I, \gamma} P_0 f(x_1, x_2) = p.v. \int_{|t| \leq 2^{n_4(x_1)}} P_0 f(x_1 - t, x_2 - u(x_1)\gamma(t)) \phi_4(t) \frac{dt}{t}.$$ 

(53)

Let us consider an approximate operator

$$\tilde{H} P_0 f(x_1, x_2) := p.v. \int_{|t| \leq 2^{n_4(x_1)}} P_0 f(x_1 - t, x_2) \phi_4(t) \frac{dt}{t}.$$ 

Similarly to (40), we have

$$\tilde{H} P_0 f(x_1, x_2) \lesssim M_1 P_0 f(x_1, x_2) + H^*_1 P_0 f(x_1, x_2).$$ 

(54)

Here and hereafter, $M_1$ denotes the Hardy-Littlewood maximal operator applied in the first variable, $H^*_1$ denotes the maximal truncated Hilbert transform applied in the first variable. Since both $M_1$ and $H^*_1$ are known to be bounded on $L^p(\mathbb{R}^2)$, from (54) we may conclude that

$$\|\tilde{H} P_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|P_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$ 

(55)

for any given $p \in (1, \infty)$. As before, we may obtain the $L^p(\mathbb{R}^2)$ boundedness of the difference between $H^*_1 P_0 f$ and $\tilde{H} P_0 f$. Therefore, we have

$$\|H^*_1 P_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|P_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}.$$ 

For $H^*_I P_0 f$, let $f := P_0 f$, we see that

$$H^*_I P_0 f(x_1, x_2) = \sum_{k \geq 2^0} \int_{|t| < \varepsilon(x_1)} f(x_1 - t, x_2 - u(x_1)\gamma(t)) \psi_{n_2(x_1) + k}(t) \frac{dt}{t}.$$ 

Define $m_2 : \mathbb{R} \to \mathbb{Z}$ such that for all $x_1 \in \mathbb{R},$

$$2^{m_2(x_1)} \leq \varepsilon(x_1) \leq 2^{m_2(x_1) + 1}.$$ 

(56)

We decompose $H^*_I P_0 f$ essentially into

$$H^*_I P_0 f(x_1, x_2) = \sum_{k = 0}^{m_2(x_1) - n_4(x_1)} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u(x_1)\gamma(t)) \psi_{n_2(x_1) + k}(t) \frac{dt}{t}$$

(57)

$$+ \int_{2^{m_2(x_1)} \leq |t| \leq \varepsilon(x_1)} f(x_1 - t, x_2 - u(x_1)\gamma(t)) \psi_{n_2(x_1) + k}(t) \frac{dt}{t}$$

$$= \mathcal{T}_1 f(x_1, x_2) + \mathcal{T}_2 f(x_1, x_2)$$

By Minkowski’s inequality, similarly to (47), for any given $p \in (1, \infty)$, we have

$$\left\| \int_{-\infty}^{\infty} f(-t, -u(-1)\gamma(t)) \psi_{n_4(-1) + k}(t) \frac{dt}{t} \right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}.$$ 

(58)
From (31), we have that there exists a positive constant $\omega_0$ such that, for any $k \geq 0$,
\[
\left\| \int_{-\infty}^{\infty} f(1 - t, 2 - u(1)\gamma(t))\psi_{n_k}(1) + k(t) \frac{dt}{t} \right\|_{L^2(\mathbb{R}^2)} \lesssim 2^{-\omega_0 k}\|f\|_{L^p(\mathbb{R}^2)}.
\] (59)

By interpolation between (58) and (59), which leads to
\[
\|T_1 f\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}
\]
for any given $p \in (1, \infty)$. Similarly to (47), for any given $p \in (1, \infty)$, we have
\[
\|T_2 f\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}.
\]
This finishes the proof of Proposition 2.

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